Given a graph $G = (V, E)$ with costs on its edges, the minimum-cost edge cover problem consists of finding a subset of $E$ covering all vertices in $V$ at minimum cost. If $G$ is bipartite, this problem can be solved in time $O(|V|^3)$ via a well-known reduction to a maximum-cost matching problem on $G$. If in addition $V$ is a set of points on the Euclidean line, Collanino et al. showed that the problem can be solved in time $O(|V| \log |V|)$ and asked whether it can be solved in time $o(|V|^3)$ if $V$ is a set of points on the Euclidean plane. We answer this in the affirmative, giving an $O(|V|^{2.5} \log |V|)$ algorithm based on the Hungarian method using weighted Voronoi diagrams. We also propose some 2-approximation algorithms and give experimental results of our implementations.
further that the former constitute a matching $M$ of $G$, while each of the latter must be a minimum-cost edge incident to its assigned vertex. For each $v \in V$, define $d(v)$ to be the minimum cost among the edges incident to $v$ and, for each $u, v \in E$, define the reduced cost $c(uv) = d(u) + d(v) - d(uv)$. It follows that the cost $d(C)$ of $C$ equals $d(V) - c(M) = \sum_{v \in V} d(v) - \sum_{uv \in M} c(uv)$. Since the first term is constant, the cost $d(C)$ is minimized when the reduced cost $c(M)$ of $M$ is maximized. This $O(|V| + |E|)$ time transformation due to Geelen (see [5, page 165]) implies that one can solve the minimum-cost edge cover problem on general graphs in time $O(|V|^2|E|)$ using the blossom algorithm [7], and on bipartite graphs in time $O(|V||E|)$ using a well-known improvement of the Hungarian method [8].

When $G$ is bipartite, the minimum-cost edge cover problem has sometimes been studied under the name many-to-many matching. In particular, Collanino et al. studied this problem when $V$ is a set of points on the Euclidean line, that is, when the cost of edges between points in different partitions are implicitly given by the Euclidean lengths of the respective segments [9]. They showed that the problem can be solved in time $O(|V| \log |V|)$ and asked whether it could be solved in time $o(|V|^3)$ if $V$ were a set of points on the Euclidean plane. Our main result is an affirmative answer to this question, and throughout this paper we will focus on this version of the problem.

Our $O(|V|^{2.5} \log |V|)$ algorithm for this problem is based on the Hungarian method using weighted Voronoi diagrams. We develop our algorithm in Section 2. First we show that the use of reduced costs in the Hungarian method implies that the dual variables are nicely bounded. Then we modify the Hungarian method with reduced costs to avoid many dual updates. Finally, we adapt Vaidya’s geometric ideas for solving the minimum-cost bipartite perfect matching problem in order to solve our problem.

Although we succeeded in achieving an $o(|V|^3)$ algorithm to solve our problem, it may be impractical for large instances. Hence we consider approximation algorithms, which produce close to optimal solutions with lower complexity and usually are conceptually simpler and easier to implement. We direct the interested reader to a recent survey on exact and approximation algorithms for our and related problems [10]. In Section 3 we propose some 2-approximation algorithms for the problem, and in Section 4 we compare experimentally our implementations of the algorithms described in this paper. Finally, in Section 5 we present our conclusions and ideas for future work.

The results of this paper were partially presented at the 38th European Workshop on Computational Geometry (EuroCG 2022) [11].
2 A Subcubic Algorithm

Let $G$ be a complete bipartite graph with partition $R = \{r_1, \ldots, r_n\}$ and $B = \{b_1, \ldots, b_m\}$. Without loss of generality, we shall assume $n \geq m \geq 1$. For each $1 \leq i \leq n$ and $1 \leq j \leq m$, let $c(r_i b_j)$ be the cost of edge $r_i b_j$. The maximum-cost bipartite matching problem with cost $c$ can be formulated using variables $x_{ij}$ for each $1 \leq i \leq n$ and $1 \leq j \leq m$ as the (primal) linear program

$$
z = \max \sum_{i=1}^{n} \sum_{j=1}^{m} c(r_i b_j) x_{ij} \quad (1)$$

subject to

$$\sum_{j=1}^{m} x_{ij} \leq 1 \text{ for all } 1 \leq i \leq n \quad (2)$$

$$\sum_{i=1}^{n} x_{ij} \leq 1 \text{ for all } 1 \leq j \leq m \quad (3)$$

$$x_{ij} \geq 0 \text{ for all } 1 \leq i \leq n, 1 \leq j \leq m. \quad (4)$$

Its linear programming dual, with variables $\alpha_i$ for each $1 \leq i \leq n$ and $\beta_j$ for each $1 \leq j \leq m$, is given by

$$z = \min \sum_{i=1}^{n} \alpha_i + \sum_{j=1}^{m} \beta_j \quad (5)$$

subject to

$$\alpha_i + \beta_j \geq c(r_i b_j) \text{ for all } 1 \leq i \leq n, 1 \leq j \leq m \quad (6)$$

$$\alpha_i, \beta_j \geq 0 \text{ for all } 1 \leq i \leq n, 1 \leq j \leq m. \quad (7)$$

Note that any integral solution of the primal corresponds to a matching $M = \{r_i b_j : x_{ij} = 1\}$. Given a matching $M$, we say that it covers the ends of its edges. We say that the vertices not covered by $M$ are exposed. An edge $r_i b_j$ such that $\alpha_i + \beta_j = c(r_i b_j)$ is called an equality edge. In these terms, we can write the complementary slackness conditions at primal-dual optimality as:

**First condition** Each edge $r_i b_j \in M$ is an equality edge.

**Second condition** For each $1 \leq i \leq n$, $\alpha_i > 0$ implies $r_i$ is covered.

**Third condition** For each $1 \leq j \leq m$, $\beta_j > 0$ implies $b_j$ is covered.

We say that vertex $b_j$ is bad if it fails the last condition, that is, if $\beta_j > 0$ but it is exposed.
2.1 The Hungarian Method with Reduced Costs

We follow the well-known Hungarian method for maximum-cost bipartite matching \cite{13,14}, that is, we construct a series of integral primal solutions and dual solutions satisfying the first two complementary slackness conditions and, at the end, also the third complementary slackness condition (no bad vertices). Our starting primal solution is \( x_{ij} \leftarrow 0 \) for all \( 1 \leq i \leq n, j \leq m \) (the empty matching \( M \)), and our starting dual solution is \( \alpha_i \leftarrow 0 \) for all \( 1 \leq i \leq n \) and \( \beta_j \leftarrow \max\{c(r_i b_j) : 1 \leq i \leq n\} \) for all \( 1 \leq j \leq m \) (if \( \beta_j < 0 \), then \( \beta_j \leftarrow 0 \)).

A path is alternating if it consists of edges that are alternately in \( M \) and not in \( M \). An alternating path \( P \) is augmenting if it starts and ends in two distinct exposed vertices. Observe that augmenting paths must start and end on distinct partitions. They are called augmenting since \( M' = P \triangle M \) is a matching larger than \( M \). An alternating tree is a tree rooted at an exposed vertex in \( B \), such that all its paths from the root to its leaves are alternating and consist of equality edges. If any such path \( P \) is augmenting, then \( M' = P \triangle M \) also satisfies the first two complementary slackness conditions.

Each iteration of the Hungarian method consists of growing a forest of alternating trees, rooted at bad vertices and consisting of equality edges, until the amount of bad vertices decreases. The method stops when no bad vertices remain. There are at most \( m \) iterations. At the start of each iteration, let \( S \neq \emptyset \) be the set of bad vertices of \( B \), let \( F \leftarrow R \), let \( T \leftarrow \emptyset \), let \( b_s \in S \) be such that \( \beta_s \leq \beta_j \) for all \( b_j \in S \), and let \( \epsilon \leftarrow \beta_s > 0 \).

As long as \( F \) is non-empty, let \( r_i b_j \) be an edge that achieves the minimum dual slack as

\[
\delta \leftarrow \min\{\alpha_i + \beta_j - c(r_i b_j) : r_i \in F, b_j \in S\}
\]  

Four mutually exclusive cases may occur:

**Case 1 (\( \delta = 0 \) and \( r_i \) is exposed):** Edge \( r_i b_j \) is an equality edge which is added to the alternating forest. Furthermore, the path \( P \) from \( r_i \) to the root \( b_t \) of its alternating tree is an augmenting path. Let \( M \leftarrow M \triangle P \). Now \( b_t \) is not bad. Stop the iteration.

**Case 2 (\( \delta = 0 \) and \( r_i \) is covered):** In this case \( r_i \) is matched in \( M \) to \( b_k \notin S \). Edges \( b_k r_i \in M \) and \( r_i b_j \) are equality edges which are added to the alternating forest. Let \( F \leftarrow F \setminus \{r_i\} \), \( T \leftarrow T \cup \{r_i\} \), and \( S \leftarrow S \cup \{b_k\} \). If \( \beta_k < \epsilon \), then let \( s \leftarrow k \) and \( \epsilon \leftarrow \beta_k \). Continue with the iteration.

**Case 3 (\( \epsilon > \delta \)):** Let \( \alpha_i \leftarrow \alpha_i + \delta \) for each \( r_i \in T \), \( \beta_j \leftarrow \beta_j - \delta \) for each \( b_j \in S \), and \( \epsilon \leftarrow \epsilon - \delta \). This keeps all equality edges in the alternating forest and creates at least one equality edge. Continue with the iteration.
Lemma 1. During an execution of the Hungarian method with reduced costs, the dual variables satisfy $0 \leq \alpha_i \leq d(r_i)$ for all $1 \leq i \leq n$ and $0 \leq \beta_j \leq d(b_j)$ for all $1 \leq j \leq m$.

Proof. Recall that $0 \leq d(r_i) \leq d(r_i b_j)$ and $0 \leq d(b_j) \leq d(r_i b_j)$ for all $1 \leq i \leq n$ and all $1 \leq j \leq m$.

At the start of the method $\beta_j = \max\{c(r_i b_j) : 1 \leq i \leq n\}$. Since $c(r_i b_j) = d(b_j) - [d(r_i b_j) - d(r_i)] \leq d(b_j)$ for all $1 \leq i \leq n$, it follows that $\beta_j \leq d(b_j)$. Moreover, if $r_k \in R$ is closest to $b_j$, then $d(b_j) = d(r_k b_j)$ and $\beta_j \geq c(r_k b_j) = d(r_k) - [d(r_k b_j) - d(b_j)] = d(r_k) \geq 0$. Since $\beta_j$ never grows and never becomes negative, $0 \leq \beta_j \leq d(b_j)$ remains true during the execution of the algorithm.

At the start of the method $\alpha_i = 0$. Note that $\alpha_i$ can increase only when we consider an equality edge $r_i b_j$. In this case $\alpha_i = c(r_i b_j) - \beta_j = d(r_i) - [d(r_i b_j) - d(r_i b_j)]$.

Case 4 ($\delta \geq \epsilon$): Let $\alpha_i \leftarrow \alpha_i + \epsilon$ for each $r_i \in T$, $\beta_j \leftarrow \beta_j - \epsilon$ for each $b_j \in S$, and $\epsilon \leftarrow 0$. This keeps all equality edges in the alternating forest. Let $P$ be the path from $b_s$ to the root $b_t$ of its alternating tree. Let $M \leftarrow M \triangle P$. Now $b_t$ is not bad. Stop the iteration.

Computing $\delta$ once takes time $O(nm)$. During an iteration, this occurs $O(n)$ times: Cases 1 and 4 occur only once, and each time that Case 3 occurs it is followed by a Case 2, which decreases $|F|$. Hence, the total time spent computing $\delta$ is $O(n^2m)$. Case 1 occurs at most once and it takes time $O(|P|) \leq O(n + m)$). Case 2 occurs at most $n$ times, each one taking time $O(1)$, for a total time $O(n)$. Case 3 occurs at most $n$ times, each one taking time $O(n + m)$, for a total time $O(n(n + m))$. Case 4 occurs at most once and it takes time $O(n + m + |P|) = O(n + m)$. Hence, Case 3 is the slowest among all cases, but the real bottleneck during an iteration is the computation of $\delta$. Since there are $O(m)$ iterations, the total running-time is $O(n^2m^2)$. 

In our case, we start with an instance of the minimum-cost edge cover problem where, for each $1 \leq i \leq n$ and $1 \leq j \leq m$, the original cost of edge $r_i b_j$ is given by $d(r_i b_j) \geq 0$. Following the reduction, we compute the minimum cost of the edges incident to each vertex of $G$. That is, for each $1 \leq i \leq n$, let $d(r_i) = \min\{d(r_i b_j) : 1 \leq j \leq m\}$ and, for each $1 \leq j \leq m$, let $d(b_j) = \min\{d(r_i b_j) : 1 \leq i \leq n\}$.

Finally, for each $1 \leq i \leq n$ and $1 \leq j \leq m$, the reduced cost of edge $r_i b_j$ is given by $c(r_i b_j) = d(r_i) + d(b_j) - d(r_i b_j)$.

We prove now that, under these conditions, the dual variables remain within certain natural bounds during the execution of the Hungarian method. In particular, $\beta_j < 0$ cannot occur during its initialization, even though some $c(r_i b_j)$ might be negative.
\[
d(b_j) - \beta_j \leq d(r_i). \text{ Since } \alpha_i \text{ never decreases, } 0 \leq \alpha_i \leq d(r_i) \text{ remains true during the execution of the algorithm.} \]

2.2 Delayed Dual Updates

In order to accelerate Case 3, we want to avoid updating \( \alpha_i \) and \( \beta_j \) more than once during an iteration of the Hungarian method. To this end, we introduce weights \( w(r_i) \leftarrow d(r_i) - \alpha_i \) for each \( r_i \in R \) and \( w(b_j) \leftarrow d(b_j) - \beta_j \) for each \( b_j \in B \). We also introduce the total dual change \( \Delta \leftarrow 0 \), used at the end of an iteration to update \( \alpha_i \) and \( \beta_j \). Consider the following adaptation to the Hungarian method using delayed dual updates [10]:

As long as \( F \) is non-empty, let \( r_ib_j \) be an edge that achieves the minimum dual slack as

\[
\delta \leftarrow \min \{d(r_ib_j) - w(r_i) - w(b_j) : r_i \in F, b_j \in S\} - \Delta
\]  

(9)

Four mutually exclusive cases may occur:

Case 1 (\( \delta = 0 \) and \( r_i \) is exposed): Edge \( r_ib_j \) is an equality edge which is added to the alternating forest. Let \( F \leftarrow F \setminus \{r_i\} \), \( T \leftarrow T \cup \{r_i\} \), and \( w(r_i) \leftarrow w(r_i) + \Delta \). Furthermore, the path \( P \) from \( r_i \) to the root \( b_i \) of its alternating tree is an augmenting path. Let \( M \leftarrow M\Delta P \). Now \( b_i \) is not bad. Let \( \alpha_i \leftarrow d(r_i) - w(r_i) + \Delta \) for each \( r_i \in T \) and \( \beta_j \leftarrow d(b_j) - w(b_j) - \Delta \) for each \( b_j \in S \). Stop the iteration.

Case 2 (\( \delta = 0 \) and \( r_i \) is covered): In this case \( r_i \) is matched in \( M \) to \( b_k \notin S \). Edges \( b_kb_i \in M \) and \( r_ib_j \) are equality edges which are added to the alternating forest. Let \( F \leftarrow F \setminus \{r_i\} \), \( T \leftarrow T \cup \{r_i\} \), \( S \leftarrow S \cup \{b_k\} \). Let \( w(r_i) \leftarrow w(r_i) + \Delta \) and \( w(b_k) \leftarrow w(b_k) - \Delta \). If \( \beta_k < \epsilon \), then let \( s \leftarrow k \) and \( \epsilon \leftarrow \beta_k \). Continue with the iteration.

Case 3 (\( \epsilon > \delta \)): Let \( \Delta \leftarrow \Delta + \delta \) and \( \epsilon \leftarrow \epsilon - \delta \). This keeps all equality edges in the alternating forest and creates at least one equality edge. Continue with the iteration.

Case 4 (\( \delta \geq \epsilon \)): Let \( \Delta \leftarrow \Delta + \epsilon \) and \( \epsilon \leftarrow 0 \). This keeps all equality edges in the alternating forest. Let \( P \) be the path from \( b_i \) to the root \( b_i \) of its alternating tree. Let \( M \leftarrow M\Delta P \). Now \( b_i \) is not bad. Let \( \alpha_i \leftarrow d(r_i) - w(r_i) + \Delta \) for each \( r_i \in T \) and \( \beta_j \leftarrow d(b_j) - w(b_j) - \Delta \) for each \( b_j \in S \). Stop the iteration.

Note that, by Lemma 1, the weights \( w(r_i) \) and \( w(b_j) \) remain non-negative. Also note that each time that we evaluate (9), its right-hand side is identical to the right-hand side of (8), that is, \( \alpha_i + \beta_j - c(r_ib_j) = d(r_ib_j) - w(r_i) - w(b_j) - \Delta \) for all \( r_i \in F \) and \( b_j \in S \).
and $b_j \in S$. This can be proved by induction. Although the times for computing $\delta$ and of Cases 1, 2, and 4 do not change, the total time for Case 3 decreases to $O(n)$.

### 2.3 Weighted Voronoi Diagrams

From now on, we assume that $R$ and $B$ are sets of red and blue points in the plane, respectively, and that $d$ is given by the Euclidean distance between pairs of points.

We shall closely adapt Vaidya's $O(|V|^{5/2} \log |V|)$ algorithm for the minimum-cost bipartite perfect matching problem with Euclidean costs [17]. This algorithm has also been adapted to solve the transportation problem with Euclidean costs [1].

Recall that our problem of interest is a maximum-cost bipartite, non-necessarily perfect matching problem with non-Euclidean costs (in fact, some costs are positive and some others are negative). However, the original Euclidean costs together with Lemma 1 will allow us to compute $\delta$ using the same data structures as in [17].

We start with a geometric interpretation of the reduced cost $c$. Consider a circle $X$ of radius $d(r_i)$ centered at $r_i \in R$, a circle $Z$ of radius $d(b_j)$ centered at $b_j \in B$, and the segment $L$ joining $r_i$ and $b_j$. Since $c(r_i, b_j) = d(r_i) + d(b_j) - d(r_i, b_j)$, it follows that $X$ and $Z$ intersect if and only if $c(r_i, b_j) \geq 0$. Moreover, if we let $x = X \cap L$ and $z = Z \cap L$, then $|c(r_i, b_j)|$ is the length of the segment $xz$. Edges with $c(r_i, b_j) < 0$ are ignored by the Hungarian method: since $\alpha_i + \beta_j \geq 0 > c(r_i, b_j)$ they cannot be equality edges. See Figure 1.

![Figure 1: Geometric interpretation of reduced cost.](image)

Now, we give a geometric interpretation of the dual variables and the non-negative weights $w(r_i) = d(r_i) - \alpha_i$ and $w(b_j) = d(b_j) - \beta_j$ defined in the adapted Hungarian method. Let $X'$ be a circle of radius $w(r_i)$ centered at $r_i$ and $Z'$ be a circle of radius $w(b_j)$ centered at $b_j$. It follows that the interiors of $X'$ and $Z'$ do not intersect if and only if $d(r_i, b_j) - w(r_i) - w(b_j) \geq 0$. Moreover, if we let $x' = X' \cap L$ and $z' = Z' \cap L$, then $|d(r_i, b_j) - w(r_i) - w(b_j)|$ is the length of the segment $x'z'$. In particular, when the interiors of $X'$ and $Z'$ do not intersect, $\delta_{ij} = d(r_i, b_j) - w(r_i) - w(b_j)$ is the distance between the circles $X'$ and $Z'$. See Figure 2.
This implies that the computation of the minimum slack $\delta$ is equivalent to the computation of the minimum distance between pairs of circles with disjoint interiors, one centered at $r_i \in F$ with radius $w(r_i)$, the other centered at $b_j \in S$ with radius $w(b_j)$.

Let $P$ be a set of points in the plane with given weights $w(p) \geq 0$ for each $p \in P$. A weighted Voronoi diagram divides the plane into $|P|$ possibly empty regions $V_p$ for each $p \in P$ given by

$$V_p = \{q : d(qp) - w(p) \leq d(qp') - w(p') \text{ for all } p' \in P\}. \quad (10)$$

A weighted Voronoi diagram can be constructed and preprocessed in time $O(|P| \log |P|)$ so that, given any point $q$, it is possible to find $p \in P$ such that $q \in V_p$ in time $O(\log |P|)$ [6, 9]. In this case, we denote the point $p$ by nearest$(q,P)$ and the edge $qp$ by shortest$(q,P)$. Similarly, for $P_1, P_2 \subseteq P$, we denote by shortest$(P_1, P_2)$ the edge $\arg \min \{d(p_1p_2) - w(p_1) - w(p_2) : p_1 \in P_1, p_2 \in P_2\}$.

Recall that minimum (binary) heaps are simple implementations of priority queues such that: a minimum heap with $k$ elements can be created in time $O(k)$ and each of its operations (deletion of the element with minimum priority, insertion of an element, and the update of any priority) can be done in time $O(\log k)$.

Let $h = \lceil \sqrt{n} \rceil$ and consider an iteration of the Hungarian method with reduced costs. After computing the set $S$ of bad vertices, we partition it into $S_1$ and $S_2$, ensuring that $|S_2| \leq h$. We also partition $F$ into $F_1, \ldots, F_h$, ensuring that each part has cardinality $\leq h$. We compute the following data structures:

1. A weighted Voronoi diagram for $S_1$ with weights $w$. Since $|S_1| \leq m$, this takes time $O(m \log m)$.

2. A minimum heap $H_1$ containing the edge $r_ib_j = \text{shortest}(r_i, S_1)$ for each $r_i \in F$ with priority $d(r_ib_j) - w(r_i) - w(b_j)$. Each of the $n = |F|$ edges can be computed in time $O(\log m)$ using the weighted Voronoi diagram for $S_1$. The heap can be created in time $O(n)$. The total time is $O(n \log m)$.

Figure 2: Geometric interpretation of dual slack.
3. A weighted Voronoi diagram for each $F_1, \ldots, F_h$ with weights $w$. Since $|F_i| \leq h$, this takes time $O(h^2 \log h) = O(n \log n)$.

4. A minimum heap $H_2$ containing the edge $r_ib_j = \text{shortest}(b_j, F_k)$ for each $b_j \in S_2$ and each $F_k$ with priority $d(r_ib_j) - w(r_i) - w(b_j)$. Each of the $h|S_2| \leq h^2$ edges can be computed in time $O(\log h)$ using the weighted Voronoi diagram for $F_k$. The corresponding heap can be constructed in time $O(h^2) = O(n)$. The total time is $O(h^2 \log h) = O(n \log n)$.

We can compute $\delta$ and a corresponding edge $r_ib_j$ by examining the minimum heaps $H_1$ and $H_2$. This can be done in time $O(\log n)$. The previously computed data structures need to be updated as follows:

**Delete** $r_i$ from $F$: Assume $r_i \in F_k$. The weighted Voronoi diagram for $F_k$ needs to be recomputed. This takes time $O(h \log h) = O(\sqrt{n} \log n)$. We also need to recompute $\text{shortest}(b_j, F_k)$ for each $b_j \in S_2$ and update the minimum heap $H_2$. This also takes time $O(\sqrt{n} \log n)$.

**Insert** $b_j$ into $S_2$: We need to compute $\text{shortest}(b_j, F_k)$ for each $1 \leq k \leq h$ and insert it into the minimum heap $H_2$. This can be done in time $O(\sqrt{n} \log n)$.

**Flush** $S_2$: If $|S_2| = h$, then move all vertices from $S_2$ into $S_1$. We need to recompute $\text{shortest}(r_i, S_1)$ for each $r_i \in F$. This can be done in time $O(n \log m)$ using the weighted Voronoi diagram for $S_1$.

Since an iteration of the Hungarian method with reduced costs has $O(n)$ deletions and insertions, the total time spent in the first two of these updates is $O(n \sqrt{n} \log n)$. Since $S_2$ can reach size $h$ at most $h$ times during an iteration, the total time spent in the last update is also $O(n \sqrt{n} \log m)$. Since there are at most $m$ iterations, we obtain the following:

**Theorem 2.** The maximum-cost matching problem with reduced Euclidean costs on a complete bipartite graph $G = (V,E)$ can be solved in time $O(|V|^2.5 \log |V|)$.

An application of Geelen’s reduction gives us our main result:

**Theorem 3.** The minimum-cost edge cover problem with Euclidean costs on a complete bipartite graph $G = (V,E)$ can be solved in time $O(|V|^2.5 \log |V|)$.

### 3 Approximation Algorithms

Again, let $R = \{r_1, \ldots, r_n\}$ and $B = \{b_1, \ldots, b_m\}$ be sets of red and blue points in the plane, respectively, and let $G$ be the complete bipartite graph with partition
$R$ and $B$, with cost $d$ given by the Euclidean distance between pairs of points. In this section we describe some approximation algorithms for the minimum-cost edge cover problem on $G$. One of these algorithms is well known [15], while the others are introduced here.

A **star** is a tree of $G$ with at least two leaves and one internal vertex, referred as the **center** of the star. See Figure 3.

![Figure 3: Two stars (left and middle) and an isolated edge (right).](image)

Since a minimal edge cover cannot contain paths of more than two edges, it is easy to see that any minimal edge cover of $G$ is a forest of stars and isolated edges. Our algorithms try to make the most of this fact.

### 3.1 Nearest Neighbor Algorithms

The simplest approach to obtain a low-cost edge cover $C'$ for any general graph is, for each vertex $v$, add to $C'$ an edge of minimum cost $d(v)$ incident to $v$. This is called the **Nearest Neighbor** (NN) algorithm and has $O(|E|)$ time complexity. Even though $C'$ could have several redundant edges, it is a 2-approximation to an edge cover of minimum cost of the graph $G$ [15]. As the name suggests, in our version of the problem the NN algorithm joins each point of $G$ to the nearest point of the other color.

Figure 4 shows an example, based on two equilateral triangles of sides 1 and $\epsilon$, where the approximation ratio of the NN algorithm is tight. The NN algorithm joins $r_2$ and $b_2$ together, while $r_1$ is joined to $b_2$ and $b_1$ is joined to $r_2$, see Figure 4(b). This cover has cost $2 - \epsilon$, whereas the optimal cover (Figure 4(a)) has cost $1 + \epsilon$. So as $\epsilon \to 0$, we get the tight approximation ratio 2.

A slightly better algorithm is obtained by trying to reduce the number of redundant edges generated in the NN algorithm, we call it the **Best of Two** (BOT) algorithm and works as follows:

1. For each $r_i \in R$, add to the cover an edge of minimum cost $d(r_i)$ incident to $r_i$.

2. For each $b_j \in B$ left uncovered, add to the cover an edge of minimum cost $d(b_j)$ incident to $b_j$. Let $C_R$ be the resulting cover.
3. Similarly, let $C_B$ be the cover obtained by the previous steps, but reversing
the roles of $R$ and $B$.

4. Output the cover with the lowest cost between $C_R$ and $C_B$.

BOT works on any bipartite graph and has $O(|E|)$ time complexity. Clearly,
the obtained cover has cost lower or equal than that obtained by the NN algorithm,
thus BOT is also a 2-approximation algorithm. Furthermore, the same example of
Figure 4 shows that the approximation ratio is tight. The step 1. of BOT joins
both $r_1$ and $r_2$ to $b_2$, while step 2. joins $b_1$ to $r_2$. Step 3. selects the same edges of
the previous steps, though in different order, so $C_R = C_B$ and hence the resulting
cover has cost of $2 - \epsilon$, see Figure 4(b). The optimal cover has cost $1 + \epsilon$, so as
$\epsilon \to 0$, we get again the tight approximation ratio 2.

3.2 Greedy Algorithms

In this subsection, we introduce another approximation algorithm for the minimum-
cost edge cover problem on $G$, this time using the Greedy approach. This algorithm
is then taken as starting point for two progressive improvements. The resulting
greedy algorithms have slightly worse time complexities that the ones in Subsection
3.1 yet, in general they achieve edge covers of $G$ of lower cost.

We name our base algorithm Greedy Star Forest (GSF), which is similar
to Kruskal’s minimum spanning tree algorithm. We start with an empty cover $C'$
and process all the edges of $G$ in order of increasing cost. At each iteration, we take
the edge $uv$ being processed and check if one of the following cases applies, taken
in order:

**G-1** If the subgraph $C' \cup \{uv\}$ is a forest of stars and isolated edges then add $uv$
to $C'$.
**G-2** If \( u \) is uncovered and \( v \) is a leaf of some star with center \( c \), then \( uv \) is added to \( C' \) while \( vc \) is removed. See Figure 5(a).

**G-3** If \( u \) and \( v \) are leaves of two different stars (with centers \( a \) and \( c \) respectively) and \( d(uv) < d(au) + d(vc) \), then \( uv \) is added to \( C' \) while \( au \) and \( vc \) are removed from \( C' \). See Figure 5(b).

![Figure 5: Cases G-2 and G-3 of the GSF algorithm.](image)

If none of the cases above occurs, then the edge \( uv \) is discarded and we proceed to the next iteration.

Note that, in the cases G-1 and G-2 the set \( C' \) gets at least one new vertex covered, and in the case G-3 the cost of \( C' \) decreased. Either way, the current cover \( C' \) would still be a forest of stars and isolated edges.

To process the edges of \( G \) in increasing cost order, one can simply sort them in \( O(|E| \log |V|) \) time, and iterate over the resulting list. A union-find data structure can be used to maintain the forest \( C' \), including the vertices yet uncovered. Since the elements in the forest have each maximum height 2, each operation done to that data structure takes \( O(1) \) time, even that of removing an edge from \( C' \). Thus, once the edges are sorted, the rest of the algorithm runs in \( O(|E|) \) time, and therefore, the GSF algorithm has \( O(|E| \log |V|) \) time complexity.

The GSF algorithm can be improved by adding another case right after G-3. As before, \( uv \) is the edge being processed in this iteration:

**G-4** If \( ua, vc \in C' \) with \( ua \) being an isolated edge in \( C' \), and \( d(ua) + d(vc) > d(uv) + d(ac) \). Then \( ua \) and \( vc \) are removed from \( C' \) while \( uv \) and \( ac \) are added to \( C' \). See Figure 6.

Instead of G-4, we refer to this case as the quadrilateral case, since the involved edges form such a figure and we are keeping the pair of opposite sides whose sum of costs is minimum. Note that applying this case takes also \( O(1) \) time and afterwards \( C' \) is still a forest of stars and isolated edges.

**Theorem 4.** Both GSF and GSF with the quadrilateral improvement are 2-approximation algorithms to an edge cover of minimum cost of the graph \( G \).
Proof. GSF processes the edges of $G$ in increasing cost order, so each time a new vertex $u$ is covered in $C'$, the corresponding edge must have minimum cost $d(u)$. Now note that G-3 is the only case where $C'$ changes (decreasing its cost) but no new vertex is being covered. Then, a change in $C'$ in GSF is either the addition of an edge of cost $d(u)$ (part of a NN cover) or a decrease in its cost. Hence, the resulting cover has cost at most that obtained by the NN algorithm, and thus GSF is also a 2-approximation algorithm. Regarding the improvement, the case G-4 is similar to G-3. Therefore, GSF with the quadrilateral case improvement is also a 2-approximation algorithm.

The example of Figure 4 shows that the approximation ratio of the base GSF algorithm is tight. GSF first adds $r_2b_2$ to the cover, and then adds either $b_1r_2$ or $b_2r_1$ as both have the same cost. Either way, the case G-2 will be applied in the next iteration and remove $r_2b_2$, leaving both $b_1r_2$ and $b_2r_1$. Finally, $b_1r_1$ is discarded, see Figure 4(c). This cover has cost $2 - 2\epsilon$, whereas the optimal cover (Figure 4(a)) has cost $1 + \epsilon$, so as $\epsilon \to 0$, we get approximation ratio 2.

The last improvement to GSF is actually a post-process that can be applied to any edge cover $C'$ of $G$. At each iteration of this post-process, we search $C'$ for a leaf $u$ of some star such that $d(uc) > d(u)$, with $c$ the center of that star. In other words, we search for a leaf $u$ that is not joined to its nearest point of the other color. If no such $u$ exist, the process is over. Otherwise, we remove $uc$ from $C'$ and add an edge of minimum cost $d(u)$ incident to $u$ (join $u$ with its nearest point of the other color).

Note that, each iteration can be done in $O(|V|)$ time and at each one, all vertices remain covered and the cost of $C'$ decreases. Hence, this can occur only a finite number of times. However, it is not clear whether only polynomially many iterations can occur.
4 Computational experiments

4.1 Instances and implementations

In order to experimentally compare the algorithms described in this work, we developed implementations for them and created a set of test instances. Additionally, we computed the optimal solution for each instance using Gurobi [11]. Our programs were implemented in C++ and ran on a multiprocessor computer with two AMD Opteron 6174 CPUs clocked at 2.2 GHz. For the subcubic algorithm, we used the implementation of weighted Voronoi diagrams available in CGAL and the implementation of Fibonacci heaps available in Boost [12, 2]. All programs ran using a single thread of execution with the sole exception of Gurobi, which used the 24 threads provided by the hardware.

We implemented an instance generator to create the set of instances. The instance generator takes four parameters $n, m, a, s$ where $a$ is a character that determines the general structure of the instance and $s$ is an integer seed for the internal pseudo-random number generator. Let $t = 3\sqrt{n + m}$. If $a = R$, all points lie inside a square of side $t$ uniformly at random. If $a = P$, all points lie inside a circle of diameter $t$; the two components $0 \leq \rho \leq \frac{t}{2}$ and $0 \leq \theta < 360$ of the polar coordinates of each point are generated uniformly at random. If $a = E$, all points lie inside a circle of diameter $t$ such that $0 \leq \rho \leq \lambda$ for the red points and $\frac{t}{2} - \lambda \leq \rho \leq \frac{t}{2}$ for the blue points, where $\lambda = \frac{\log t}{2}$. See Figure 7.

The source code of our implementations, the set of instances, visualizations and a driver script are available at \url{https://github.com/rcc-uam/edge-cover}.

![Figure 7: Instances of type R, P, E for $n = m = 25$. The Voronoi diagram of $B$ is also shown.](image)

4.2 Experiments and results

In Table 1 we compare the performance of Gurobi against the Hungarian method, the subcubic algorithm and a non-subcubic variant of the latter that does not compute Voronoi diagrams and instead uses linear search for nearest point queries. For the Hungarian-like algorithms, we tested two versions: one where $S$ contains
the whole set of bad vertices from the start and one where $S$ starts with a single arbitrary bad vertex. All computations were performed using double precision floating point arithmetic.

| Instance | Gurobi | Hungarian | Subcubic | Non-Subcubic (w/o Voronoi) |
|----------|--------|-----------|----------|---------------------------|-------------------|-------------------|-------------------|-------------------|
|          |        | Any | All | Any | All | Any | All | Any | All |
| 25_25_E_0  | 0.1    | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 25_25_P_0  | 0.0    | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 25_25_R_0  | 0.0    | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 50_50_E_0  | 0.1    | 0   | 0.1 | 0   | 0.1 | 0   | 0.1 | 0   | 0.1 |
| 50_50_P_0  | 0.1    | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 50_50_R_0  | 0.1    | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 250_250_E_0 | 1.5   | 11  | 13.5 | 1.1 | 2.6 | 0   | 0.4 | 1.5 | 1.5 |
| 250_250_P_0 | 1.1   | 0   | 6.1 | 0.6 | 1.6 | 0   | 0.0 | 0.8 | 0.8 |
| 250_250_R_0 | 1.1   | 0   | 6.1 | 0.5 | 1.9 | 0.1 | 0.6 | 0.8 | 0.8 |
| 500_500_E_0 | 5.9   | 20.9 | 201.3 | 5.4 | 15.9 | 1.5 | 11.2 | 11.2 | 11.2 |
| 500_500_P_0 | 3.6   | 0.3 | 88.3 | 2.2 | 8.3 | 0.3 | 5.0 | 5.0 | 5.0 |
| 500_500_R_0 | 3.8   | 0.1 | 84.9 | 2.1 | TLE | 0   | 0.4 | 5.3 | 5.3 |
| 1250_1250_E_0 | 36.5 | 652.9 | TLE | 45.2 | RTE | 20.1 | 182.3 | 182.3 | 182.3 |
| 1250_1250_P_0 | 20.0 | 0.9 | TLE | 14.5 | TLE | 1.1 | 112.8 | 112.8 | 112.8 |
| 1250_1250_R_0 | 19.8 | 0.6 | TLE | 14.1 | TLE | 1.1 | 109.5 | 109.5 | 109.5 |
| 2500_2500_E_0 | 613.7 | TLE | TLE | 209.2 | RTE | 121.0 | TLE | 1178.3 | 1178.3 |
| 2500_2500_P_0 | 98.3 | 2.6 | TLE | 58.0 | TLE | 3.8 | 1140.3 | 1140.3 | 1140.3 |
| 2500_2500_R_0 | 98.4 | 2.7 | TLE | 58.4 | TLE | 3.7 | 1178.3 | 1178.3 | 1178.3 |

Table 1: Runtimes of the exact algorithms (in seconds) under a time limit of 1800 seconds. TLE stands for “Time Limit Exceeded” and RTE stands for “Runtime Error”, which happened due to loss of numeric precision during the computation of Voronoi diagrams.

Experimentally, instances of types R and P, which are those with points randomly scattered inside a square and a circle of area $\Theta(t^2)$, respectively, are solved much quicker than instances of type E. This is because the number of edges with positive reduced costs $c(uv) = d(u) + d(v) - d(uv)$ is expected to be linear, since the expected values of $d(u)$ and $d(v)$ are $O\left(\frac{t}{\sqrt{n+m}}\right)$ and $d(uv)$ is $\Theta(t)$ [19]. In contrast, all edges of instances of type E will have positive reduced costs since both $d(u)$ and $d(v)$ are almost equal to $d(uv)$ by design.

The versions of the Hungarian-like methods where $S$ starts with the whole set of bad vertices are experimentally much slower than the versions where $S$ starts with a single bad vertex. We conjecture that the Euclidean bipartite edge cover problem is solvable in time $O(nm)$ and, as such, that the versions where $S$ starts with the whole set of bad vertices incur in an unnecessary amount of computational overhead, while the versions where $S$ starts with a single bad vertex are fast due to
a yet unknown theoretical property.

The subcubic implementation is, experimentally, much slower than the non-subcubic variant that does not use Voronoi diagrams. In our tests, avoiding the computation of Voronoi diagrams is better in every situation. In the case of instances of type E, such avoidance brings just a minor improvement and we believe that the hidden constant factors inside the implementation of Voronoi diagrams are just too big. In contrast, such avoidance is overwhelmingly better in the case of instances of type R and P, where it is possible to find the optimal solution much earlier and where the time needed to compute the Voronoi diagrams is much greater than the total time of just a few nearest point searches.

In Table 2 we compare the effectiveness of the approximation algorithms against the fastest exact method. For the Greedy Star Forest algorithm, we test the baseline algorithm and a version that includes the quadrilateral case and the post-processing improvements, the latter being restricted to perform a linearithmic amount of work while searching and fixing leaves. In Figure 8 we visually compare the solutions computed by all algorithms for a test instance.

![Figure 8: Solutions for a subset of 25_25_R_0. Points that belonged to connected components appearing in all solutions were excluded. Some points were shifted for visualization purposes.](image_url)

For practical purposes, the runtimes of all the approximation algorithms are negligible. If any, we may say that the Greedy Star Forest algorithm is a bit slower than both the Nearest Neighbor and Best of Two algorithms, which are equally fast.
| Instance   | Fastest Exact | Nearest Neighbor | Best of Two | Greedy Star Forest |
|------------|---------------|------------------|-------------|--------------------|
|            |               |                  |             | Baseline | Improved |
| 25_25_E_0  | 215.3 s       | 164.8%           | 154.7%      | 101.2%   | 100.3%   |
| 25_25_P_0  | 52.7 s        | 129.6%           | 115.3%      | 105.8%   | 103.8%   |
| 25_25_R_0  | 62.7 s        | 115.0%           | 107.0%      | 105.3%   | 103.4%   |
| 50_50_E_0  | 633.8 s       | 178.5%           | 171.6%      | 100.8%   | 100.4%   |
| 50_50_P_0  | 130.4 s       | 120.7%           | 114.9%      | 108.8%   | 103.4%   |
| 50_50_R_0  | 171.9 s       | 115.7%           | 112.1%      | 106.6%   | 101.9%   |
| 250_250_E_0| 7616.9 s      | 187.7%           | 184.4%      | 100.2%   | 100.4%   |
| 250_250_P_0| 609.1 s       | 118.6%           | 113.5%      | 104.8%   | 102.0%   |
| 250_250_R_0| 723.0 s       | 122.2%           | 114.4%      | 103.2%   | 101.6%   |
| 500_500_E_0| 22096.7 s     | 190.8%           | 189.0%      | 100.1%   | 100.1%   |
| 500_500_P_0| 1243.6 s      | 119.7%           | 113.0%      | 103.4%   | 101.7%   |
| 500_500_R_0| 1572.4 s      | 120.3%           | 113.1%      | 103.3%   | 101.6%   |
| 1250_1250_E_0| 89184.7 s  | 194.4%           | 193.0%      | 100.1%   | 100.1%   |
| 1250_1250_P_0| 3091.9 s   | 120.3%           | 113.9%      | 103.4%   | 101.6%   |
| 1250_1250_R_0| 3688.4 s   | 120.3%           | 113.7%      | 103.6%   | 101.9%   |
| 2500_2500_E_0| 255344.5 | 195.5%           | 194.9%      | 100.1%   | 100.1%   |
| 2500_2500_P_0| 6170.7 s   | 120.2%           | 113.9%      | 103.8%   | 101.8%   |
| 2500_2500_R_0| 7380.7 s   | 121.1%           | 114.1%      | 103.9%   | 102.0%   |

Table 2: The optimal value of each instance, the runtime of the fastest exact algorithm, the optimality ratio (expressed as percentage) and the runtime of each approximation algorithm.
themselves. However, there are ample differences in the quality of the computed solutions. The Nearest Neighbor and Best of Two algorithms exhibit their worst case guarantees in instances of type E, while being not impressively good in the other types of instances. In the case of the Greedy Star Forest algorithm, both the baseline and the improved versions compute solutions that are almost optimal in all instances. The improved version yields better solutions than its baseline almost always, while being occasionally worse due to its greedy nature, yet by marginal amounts. We believe that the improved Greedy Star Forest algorithm may have a better approximation factor.

5 Conclusions and Further Work

In this work we obtained a subcubic algorithm to solve the Euclidean bipartite edge cover problem. The main tool that allowed us to adapt Vaidya’s use of weighted Voronoi diagrams was our Lemma 1 on the bounds of the dual variables with reduced costs. To the best of our knowledge, this result was previously unknown. In addition, we conjecture that the Euclidean bipartite edge cover problem can be solved in time $O(nm)$.

We also described three fast 2-approximation algorithms for the Euclidean bipartite edge cover problem, one previously known (NN) and two new (BOT and GSF). On the one hand, we showed that NN, BOT, and base GSF have a tight guarantee of 2, even on Euclidean instances. On the other hand, given our experimental results, we believe that our improved GSF may have a guarantee better than 1.5 on Euclidean instances, which would be better than the best known fast approximation algorithm for the minimum-cost edge cover problem (see Table 9.2 in [15]).

Acknowledgments

Rodrigo A. Castro, Marco A. Heredia, Jorge Urrutia, and Francisco J. Zaragoza are partially supported by Sistema Nacional de Investigadores, Consejo Nacional de Ciencia y Tecnología, México. José M. Díaz-Báñez and Inmaculada Ventura are partially supported by European Union’s Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement #734922 and Ministerio de Ciencia e Innovación CIN/AEI/10.13039/501100011033 (PID2020-114154RB-I00). Jorge Urrutia is partially supported by PAPIIT IN105221 Programa de Apoyo a la Investigación e Innovación Tecnológica, UNAM, Mexico.
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