On the Collatz Problem

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Abstract

An attempt to come closer to a resolution of the Collatz conjecture is presented. The central idea is the formation of a tree consisting of positive odd numbers with number 1 as root. Functions for generating the tree from the root are presented and paths from nodes to the root are given by Collatz sequences. The Collatz problem is thus reduced to showing that all positive odd numbers are present in the tree.

Keywords: Collatz conjecture, divergence, cycles, countability

1. Introduction

The Collatz problem, or the Collatz conjecture, is an unsolved problem in mathematics [1]. It was first proposed in 1937 by the mathematician Lothar Collatz [2]. The problem has been studied extensively (see, e.g., [3] and [4]) and it is also known under other names, e.g., the 3x + 1 problem. Collatz problem can be stated in a few sentences, but notwithstanding a long history a final proof has not yet been given. The problem concerns sequences of numbers, called the Collatz sequences, that are generated using the following algorithm:

initialization: choose a positive integer n
print n
while n ≠ 1 do
  if n is even then
    divide n by 2 (n = n/2)
  else
    multiply n by 3 and add 1 (n = 3n + 1)
  end
print n

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For example, the Collatz sequence with the initial value $n = 9$ is:

$$9 \quad 14 \quad 7 \quad 22 \quad 11 \quad 34 \quad 17 \quad 52 \quad 26 \quad 13 \quad 40 \quad 20 \quad 10 \quad 5 \quad 16 \quad 8 \quad 4 \quad 2 \quad 1.$$ 

The question is whether the Collatz sequence, thus generated, will eventually reach the number 1 for all initial values of $n$. By numerical testing one can make plausible that the conjecture ought to be given the answer yes. In 1999 all values up to about $2.7 \times 10^{16}$ were found to eventually end in 1 [5].

2. Reformulation of the Problem

In this work modified Collatz sequences will be studied, where the even numbers from the original Collatz sequences have been excluded. The modified Collatz algorithm is:

```plaintext
initialization: choose an odd positive integer n
print n
while n ≠ 1 do
    multiply n by 3 and add 1 ($n = 3n + 1$)
    while n is even do
        divide n by 2 ($n = n/2$)
    end
    print n
end
```

If the initial value is $n = 9$ then the modified Collatz sequence is:

$$9 \quad 7 \quad 11 \quad 17 \quad 13 \quad 5 \quad 1,$$

i.e., identical to the previous sequence, but with the even numbers omitted. The exclusion of the even numbers does not change the Collatz problem, but only removes unnecessary noise. The main question still remains: Do all modified Collatz sequences eventually reach the number 1? If the following two statements are true, then the answer is yes:

1. The length of a modified Collatz sequence is limited, i.e., no infinite divergent sequence occurs.
2. The numbers in a modified Collatz sequence are unique, i.e., no cycle occurs.

3. Attempt to Solution

In this section an attempt is made to partly resolve Collatz problem by demonstrating that odd numbers can be arranged in a tree, where all node values are unique, the root value is equal to 1 and the paths from the nodes
to the root are given by modified Collatz sequences. If it can be proven that all odd numbers indeed can be arranged in such a tree, then the truth of the two statements in the previous section has been shown.

3.1. Construction of a Tree

A part of the tree, with some of the odd numbers, is shown in Figure 1. It is easy to realize that the only possible root value is 1 since this is the only odd number $x$ fulfilling the equality $(3x + 1)/2^n = x$ for a positive integer $n$. Further, it can easily be shown that the leaf nodes have values that are multiples of 3 since there is no odd number $x$ fulfilling the equality $(3x + 1)/2^n = 3m$ for any pair of positive integers $n$ and $m$. The inner nodes can be expressed as either $6k + 1$ ($k \geq 0$) or $6k - 1$ ($k \geq 1$), and they have infinitely many children.

Figure 1: The Collatz tree of odd numbers. The leaf nodes (ovals) are multiples of 3. The inner nodes can be expressed as either $6k + 1$ (triangles) or $6k - 1$ (rectangles).
The parent of a node can be calculated using the modified Collatz algorithm. For the determination of the infinitely many children of an inner node the following ansatz (which will be proven in the next section) will be used:

\[ f_{\pm}(k, n) = k \cdot 2^{2n+1} \pm \frac{1}{3}(2^{2n} - 1), \quad k \geq 0, n \geq 1, \]  

for inner nodes with values \(6k + 1\) and

\[ f_{-}(k, n) = k \cdot 2^{2n+2} - \frac{1}{3}(2^{2n+1} + 1), \quad k \geq 1, n \geq 0, \]  

for inner nodes with values \(6k - 1\). By applying the modified Collatz algorithm on \(f_{\pm}(k, n)\) and \(f_{-}(k, n)\) one can verify that \(6k + 1\) and \(6k - 1\) are obtained, respectively. For a node value given by either Equation 1 or 2, the argument \(k\) is related to the parent and \(n\) to the position among the siblings.

3.2. Parent to Child

We will now go on proving that for each positive odd number \(x\) there is one and only one pair of nonnegative integers \(k\) and \(n\) such that either \(x = f_{+}(k, n)\) or \(x = f_{-}(k, n)\). Let \(N, Z_{+}\) and \(O_{+}\) denote the sets of natural, positive and positive odd numbers, respectively. Thus, the following four statements have to be proven:

1. For \(k_1, k_2 \in N\) and \(n_1, n_2 \in Z_{+}\), \(f_{+}(k_1, n_1) = f_{+}(k_2, n_2)\) iff \(k_1 = k_2\) and \(n_1 = n_2\).
2. For \(k_1, k_2 \in Z_{+}\) and \(n_1, n_2 \in N\), \(f_{-}(k_1, n_1) = f_{-}(k_2, n_2)\) iff \(k_1 = k_2\) and \(n_1 = n_2\).
3. \(f_{+}(k_1, n_1) \neq f_{-}(k_2, n_2)\) \forall \(k_1, n_1 \in N\) and \(k_2, n_2 \in Z_{+}\).
4. \(\forall x \in O_{+}\) \exists a pair of nonnegative integers \(k\) and \(n\) such that either \(x = f_{+}(k, n)\) or \(x = f_{-}(k, n)\).

The first two statements can be verified by proofs by contradiction. Assuming \((k_1, n_1) \neq (k_2, n_2)\) the equalities in statement 1 and 2 leads to

\[ 2 \cdot 2^{2(n_1-n_2)}k_1 - 2k_2 \pm (2^{2(n_1-n_2)} - 1)/3 = 0, \]  

for \(n_1 \geq n_2\). With two even terms and one odd term for \(n_1 > n_2\), Equation 3 can never be fulfilled. When \(n_1 = n_2\), Equation 3 can be fulfilled, but this implies that \(k_1 = k_2\), thus a contradiction.
Table 1: The positive odd numbers generated from $f_+$ and $f_-$ for the smallest values of $n$. $p$ is given by the number of times $3y + 1$ can be divided by two ($\lfloor (3y + 1)/2^p \rfloor \in O_+$).

| $p$ | $x \in O_+$ | $x$ (binary) | $y \in O_+$ generated from $x$ |
|-----|--------------|--------------|--------------------------------|
| 1   | $f_-(1, 0) = 3$ | 11           | $f_-(k, 0) = f_-(1, 0) + 2^2(k - 1)$ |
| 2   | $f_+(0, 1) = 1$ | 001          | $f_+(k, 1) = f_+(0, 1) + 3k$ |
| 3   | $f_-(1, 1) = 13$ | 1101         | $f_-(k, 1) = f_-(1, 1) + 2^4(k - 1)$ |
| 4   | $f_+(0, 2) = 5$ | 00101        | $f_+(k, 2) = f_+(0, 2) + 2^5k$ |
| 5   | $f_-(1, 2) = 53$ | 110101       | $f_-(k, 2) = f_-(1, 2) + 2^6(k - 1)$ |
| 6   | $f_+(0, 3) = 21$ | 0010101      | $f_+(k, 3) = f_+(0, 3) + 2^7k$ |

To verify the third statement is straightforward, since the equality in statement 3 leads to

$$2^{2(n_1 - n_2)}k_1 - 2k_2 + (2^{2(n_1 - n_2) - 1} + 1)/3 = 0,$$

for $n_1 > n_2$. With two even terms and one nonzero odd term, Equation 4 cannot be fulfilled. Similar equalities are given when $n_1 < n_2$ and $n_1 = n_2$, respectively.

The fourth statement is somewhat harder to verify. In Table 1, the positive odd values generated from $f_+$ and $f_-$ for the smallest values of $n$ are given. Clearly, $f_-(1, n)$ determines the $2n + 2$ last bits of the numbers $f_-(k, n)$ and $f_+(0, n)$ the $2n + 1$ last bits of the numbers $f_+(k, n)$. In addition, no set of last bits is a postfix of any other set of last bits. Thus, $f_-(1, 0)$ generates half of the positive odd numbers, $f_+(0, 1)$ a quarter of the positive odd numbers, etc. The $n$ and $k$ values of a given positive odd number $x$ can be easily found by transforming $x$ to binary form. The number of rightmost bits including the first pair of identical bits gives $n$ (actually $2n + 2$ or $2n + 1$) and the remaining bits give $k - 1$ or $k$ (see Table 1). The conclusion is that all positive odd numbers are generated.

### 3.3. Divergence and Cycles

What has been proven so far is that from the root value 1, an infinite tree with unique positive odd node values can be generated using the functions $f_+$ and $f_-$. It is indeed a tree since cycles would indicate that there are nodes having more than one parent, which is impossible. The remaining question is whether all positive odd values are present in the tree. Some, actually infinitely many, could possibly organize themselves in unrooted trees.
(see Figure 2). An unrooted tree can be excluded since that would imply an uncountable number of nodes, which is impossible since the number of positive odd values is countable. The uncountability of the number of nodes in an unrooted tree can be motivated by the fact that each node has infinitely many siblings and infinitely many ancestors. It would require infinite sequences of natural numbers to enumerate the nodes in such a tree. Another unrooted structure, a graph with a cycle (see Figure 2), can not be excluded for the same reason since the number of nodes could still be countable. Thus, the first statement in Section 2 can be considered verified, but the second statement still has to be verified.

Figure 2: The positive odd numbers could possibly be organized in a rooted tree (left), in unrooted trees (middle) or in graphs with cycles (right). The middle alternative can be excluded.

Assume that the \( m \) positive odd values \( x_0, x_1, \ldots, x_{m-1} \in O_+ \) constitute an \( m \)-cycle, i.e., that there exist positive integers \( p_0, p_1, \ldots, p_{m-1} \in \mathbb{Z}_+ \), such that

\[
\begin{cases}
\frac{3x_{i+1}}{2^{p_i}} = x_{i+1}, & \text{for } i = 0, 1, \ldots, m - 2, \\
\frac{3x_{m-1} + 1}{2^{p_{m-1}}} = x_0.
\end{cases}
\]

Equation (5) is a system of linear equations and its solution is

\[
x_i = \frac{1}{2^{P_m} - 3^m} \sum_{j=0}^{m-1} 3^j 2^{P_m-1-j},
\]

where \( P_m = \sum_{l=0}^{m-1} p_l, 2^{P_m} - 3^m > 0 \) and

\[
P_j^i = \sum_{l=0}^{j-1} p_{(l+i) \mod m}.
\]
In a discussion divided into three cases, where the first two can be proven, it will be motivated that for \( p_0, \ldots, p_{m-1} \in \mathbb{Z}_+ \) the only positive integer solution to Equation 5 is \( x_0 = \cdots = x_{m-1} = 1 \) given for \( p_0 = \cdots = p_{m-1} = 2 \):

1. If \( p_0, \ldots, p_{m-1} \geq 2 \) then \( x_0, \ldots, x_{m-1} \leq 1 \).
2. If \( P_m \geq 2m \) then \( \exists i \in [0, m-1] \) such that \( x_i \leq 1 \).
3. If \( P_m < 2m \) then it is likely \( \exists i \in [0, m-1] \) such that \( x_i \in Q_+ \setminus O_+ \), where \( Q_+ \) is the set of positive rational numbers.

**Case 1:** Assume that \( p_0, \ldots, p_{m-1} \geq 2 \) and that \( p_i \leq p_0, \ldots, p_{m-1} \) then

\[
x_i = \frac{\sum_{j=0}^{m-1} 3j2^{P_{m-1-j}}}{2^{p_i} - 3} \leq \frac{\sum_{j=0}^{m-1} 3j2^{P_{m-1-j}}}{2^{p_i} + P_{m-1} - 3} = \frac{(2^{p_i} - 3) \sum_{j=0}^{m-1} 3j2^{P_{m-1-j}} + \sum_{j=1}^{m-2} 3j(2^{P_{m-1-j}} - 2^{P_{m-1-j}})}{\sum_{j=0}^{m-1} 3j2^{P_{m-1-j}}} \leq \frac{(2^{p_i} - 3) \sum_{j=0}^{m-1} 3j2^{P_{m-1-j}} + \sum_{j=1}^{m-2} 3j(2^{p_i} + P_{m-1-j} - 2^{P_{m-1-j}})}{2^{p_i} - 3} = \frac{1}{2^{p_i} - 3} \leq 1.
\tag{8}
\]

The same holds for the remaining \( m - 1 \) values in the \( m \)-cycle since

\[
x_{i+1} = \frac{3x_i + 1}{2^{p_i}} \leq 3 + 1 \leq \frac{4}{4} = 1.
\tag{9}
\]

**Case 2:** Assume that \( P_m = 2m \) and that at least one \( p \)-value is equal to 1 then the maximum minimum (maximin) value \( x_0 \) is given for the \( m \)-cycle with as few \( p \)-values equal to 1 as possible: \( p_0 = \cdots = p_{m-3} = 2, p_{m-2} = 1 \) and \( p_{m-1} = 3 \):

\[
x_0 = \frac{2^{2(m-2)+1} + 3 \cdot 2^{2(m-2)} + 3 \cdot 2^{2(m-3)} + \cdots + 3^{m-2}2^2 + 3^{m-1}}{4^m - 3^m} = \frac{2^{2(m-2)+1} + 3[4^{m-2} + 3 \cdot 4^{m-3} + \cdots + 3^{m-3} \cdot 4 + 3^{m-2}]}{4^m - 3^m} = \frac{2 \cdot 4^{m-2} + 3\cdot 4^{m-1} - 3^{m-1}}{4^m - 3^m} = 1 - \frac{1}{2 \cdot 4^m - 3^m} < 1.
\tag{10}
\]

Thus, at least one value is less than 1 in all \( m \)-cycles, where \( P_m = 2m \) and at least one \( p \)-value is equal to 1. This can be considered as the base case.
in a proof by induction, where the induction step consists of showing that the values in an \( m \)-cycle decrease when one of the \( p \)-values increases by 1: \( P'_m = P_m + 1 \), \( P'_j \leq P_j + 1 \) and

\[
x'_i = \sum_{j=0}^{m-1} 3j2^{P_{m-1-j}'} \leq \sum_{j=0}^{m-1} 3j2^{1+P_{m-1-j}} \leq \frac{2}{2 \cdot 2P_m - 3m} \sum_{j=0}^{m-1} 3j2^{P_{m-1-j}} \leq \frac{2(2P_m - 3m)}{2(2P_m - 3m)} \cdot x_i.
\]

Thus, by increasing one of the \( p \)-values the \( x \)-values will decrease and at least one of them will be less than 1.

**Case 3:** The first two cases excludes true \( m \)-cycles, i.e., cycles with \( m \) different odd values for \( m > 1 \). In addition, according to the discussion in Section 3.1 the values in the cycle should be inner-node values divisible by neither two nor three. The solution (Equation 6) to Equation 5 may be rewritten as

\[
x_i = \begin{cases} 
-1 + \frac{2 \cdot 3}{2P_m - 3m} \left[ \frac{1+2P_{l_i}}{3} \right] 2^{P_{m-1}-1} + \sum_{j=1}^{m-2} 3j^{1-2}2^{P_{m-1-j}} - 3^{m-2} \\
1 + \frac{2 \cdot 3}{2P_m - 3m} \left[ \frac{1-2P_{l_i}}{3} \right] 2^{P_{m-1}-1} + \sum_{j=1}^{m-2} 3j^{1-2}2^{P_{m-1-j}} + 2 \cdot 3^{m-2} 
\end{cases}
\]

for \( p_{l_i} \) odd and even, respectively, where \( l_i = (i+m-1) \mod m \). The two solutions in Equation 12 reflect the fact that the inner-node values are expressed as either \( 6k + 1 \) or \( 6k - 1 \), where \( k \) is a positive integer. As a matter of fact it is possible to factorize the numerator in the second term in Equation 12 even further if there are several consecutive \( p \)-values around index \( i \) that are all either 1 or 2:

\[
x_i = \begin{cases} 
-1 + \frac{2^{1+h_{l_i}}3^{l_i}}{2P_m - 3m} \left[ (1 + 2^{P_{l_i}(t_i)})2^{y_{l_i}(t_i)} + \sum_{j=1+t_i}^{m-2-h_{l_i}} 3^{j-t_i}2^{y_{l_i}(j)} - 3^{2i} \right] \\
1 + \frac{2^{1+h_{l_i}}3^{l_i}}{2P_m - 3m} \left[ (1 - 2^{P_{l_i}(t_i)})2^{y_{l_i}(t_i)-h_{l_i}} + \sum_{j=1+t_i}^{m-2-h_{l_i}} 3^{j-t_i}2^{y_{l_i}(j)-h_{l_i}} + 2 \cdot 3^{2i} \right] 
\end{cases}
\]
for \( p_{l_i(0)} \) equal to 1 and 2, respectively, where \( l_i(t_i) = (i + m - 1 - t_i) \mod m \), \( y_i(j) = P^i_{m-1-j} - 1 - h_i \) and \( z_i = m - 1 - h_i - t_i \). \( h_i \) and \( t_i \) are the number of consecutive identical \( p \)-values (equal to either 1 or 2) from index \( i \) and onwards and from index \((i - 1) \mod m\) and backwards, respectively. From the \( h_i \) \( p \)-values from index \( i \) and onwards the factors of value 2 from the numerator are extracted and from the \( t_i \) \( p \)-values from index \((i - 1) \mod m\) and backwards the factors of value 3 are extracted.

The discussion, so far, has concerned the numerator of the second term in Equations 12 and 13. The denominator, \( 2P^m - 3^m \equiv 2^{2m-n} - 3^m \), which is common to all \( x \)-values in an \( m \)-cycle,

- is larger than 0 (giving \( 1 \leq n < m(2 - \log_2 3) \approx 0.415m \)),
- consists of no factors equal to 2 or 3,
- can not be equal to 1, according to Catalan’s conjecture [6], and
- is larger than \( 2P^m e^{-P^m/10} \approx 1.81P^m \) for \( P_m \geq 28 \), according to Pillai [7].

Due to the properties of the denominator described above it is not likely that the denominator is a divisor of the numerator (which may contain several factors 2 and 3) for all \( x \)-values in an \( m \)-cycle.

More evidence of the unlikelihood of cycles is provided by the following numerical calculation. The maximum minimum (maximin) \( x \)-value for a given \( P_m \)-value is obtained for the \( m \)-cycle, which has the least number of \( p \)-values equal to 1. If \( P_m = 2m - n \) then the maximin \( x \)-value is obtained if \( m - n \) \( p \)-values are equal to 2 and \( n \) \( p \)-values are equal to 1, evenly distributed in the cycle. In Figure 3 the maximin values are presented for three given values of \( n \) (10, 16 and 22). Clearly the maximin value approaches 1 for large cycles. The largest maximin value is obtained for the smallest possible \( m \)-value for the given \( n \)-value (\( n < m(2 - \log_2 3) \approx 0.415m \)). In Figures 4 and 5 the largest \( n \)-value for a given \( m \)-value is used: \( n = \lfloor 0.41m \rfloor \) in Figure 4 and \( n = \lfloor (2 - \log_2 3)m \rfloor \) in Figure 5. Using several significant figures when approximating \( 2 - \log_2 3 \) has a dramatic effect on the maximin value. In Figure 4 only two significant figures are used and then the maximin value seem to converge to a value between 60 and 70 for large cycles. Considering the fact that \( x \)-values up to about \( 2.7 \times 10^{16} \) have been numerically tested not belonging to a cycle, it seems unlikely that cycles with \( P_m = 2m - \lfloor 0.41m \rfloor \) exist. Using more significant figures will allow larger maximin values, as is
Figure 3: The maximum minimum value of $m$-cycles with $P_m = 2m - n$.

Figure 4: The maximum minimum value of $m$-cycles with $P_m = 2m - \lfloor 0.41m \rfloor$.

demonstrated in Figure 5. But to reach values larger than $2.7 \times 10^{16}$ large cycles (i.e., large $m$-values) are required. This in turn implies a large value of $P_m$ and thus a large value of the denominator, which should be a divisor of a large number of numerators. This may be very unlikely.

4. Conclusions

By using the property of the set of positive odd numbers of being countable infinite it has been possible to partly resolve the Collatz conjecture. The only remaining issue is one particular case in the proof of the non-existence of cycles. An interesting continuation of this work would be to study gener-
Figure 5: The maximum minimum value of $m$-cycles with $P_m = 2m - \lfloor (2 - \log_2 3)m \rfloor$.

alizations of the Collatz problem, i.e., using other functions than $3n + 1$ and $n/2$ in the Collatz algorithm and to find applications of it.

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