Bounds for metric dimensions of generalized neighborhood corona graphs

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1. Introduction

Metric dimension is one of the concepts largely found in graph theory. It was previously introduced in two separate independent works. In 1975, Slater [1] was the first to introduce the concept of metric dimensions by using the terms “set of locations” and “reference set”. Separately, Harary and Melter [2] also introduced metric dimensions in 1976.

The concept was then developed into problems of determining metric dimensions of given graphs and characterizing graphs which have specific metric dimensions. Chartrand et al. [3] characterized all graphs with n vertices which have metric dimensions one, n – 1, or n – 2. Meanwhile, Jannesari and Omoomi [4] depicted all graphs with n vertices having metric dimensions n – 3. Furthermore, Khuller et al. [5] found some necessary requirements for graph to have metric dimensions two, and also determined metric dimensions for tree graphs and grid graphs.

On topic of metric dimensions of graph corona operation, Iswadi et al. [6] and Yero et al. [7] independently studied metric dimensions for corona graphs. Several studies then expand the topic to include several variations. Fernau [8] further expand the work to include adjacency, local, and local adjacency metric dimension, while Peterin [9] obtained edge metric dimension for corona graphs. In addition, Rinurwati et al. [10] analyzed metric dimensions for edge-corona graphs.

A variation of corona product graphs—i.e., neighborhood corona—was developed by Gopalapillai [11]. On another hand, there was also a generalization of corona product introduced by Laali et al. [12]. In this paper, we combine both variations to obtain generalized neighborhood corona product. We aim to discover bounds for metric dimensions of generalized neighborhood corona product of graphs. In terms of method, we employed concept of adjacency metric. Our work covered graphs in form G\( \star \)\( \bigwedge_{i=1}^{n} H_i \) where all graphs are non-trivial, and G is connected (\( H_i \) are not necessary to be connected). The lower bound that we found applies for arbitrary choice of \( G, H_1, H_2, ..., H_{|V(G)|} \), while the upper bound applies for cases where G has no false twin vertices. We then prove a characteristic which shows sufficient condition to achieve the aforementioned lower bound.

2. Preliminary notes

We use terms and notations in Chartrand et al. [13], with some modifications on symbols used. For every graph G, we use notation \( V(G) \) as vertices set and \( E(G) \) as edges set. Two vertices \( u \) and \( v \) in a graph \( G \), are called adjacent or neighbor if there is an edge \( uv \in E(G) \). The open neighborhood \( N(v; G) \) is set of neighbors of vertices \( v \) on graph \( G \). Degree of vertex \( v \in V(G) \) is the cardinality of the open neighborhood of \( v \) in \( G \), \( \deg(G, v) \). Vertex with degree one is called leaf. If two distinct vertices have same open neighborhood, then they are referred as false twin.

The dominating set of \( G \), \( S \subseteq V(G) \), is vertices subset of \( G \) such that each vertex in \( V(G) \) – \( S \) is adjacent to one or more vertices in \( S \).

2.1. Metric dimensions

Given a connected graph, a metric function can be assigned for the set of vertices. Metric of two vertices \( u, v \in V(G) \), \( d(u, v; G) \) is defined as...
minimum length of paths on $G$ that start on vertex $u$ and end on vertex $v$ or vice versa.

The concept of metric dimension revolves around finding subset of the set of vertices in graphs so that (metric) representation of each vertex is unique. For that purpose, we need to define a representation mapping. A formal definition is given as follows:

**Definition 2.1.1 [4]**

Given an ordered subset $S = \{s_1, s_2, ..., s_k\} \subseteq V(G)$, for any vertex $v \in V(G)$, the representation from $v$ to $S$ is a vector $(d(v, s_1; G), d(v, s_2; G), ..., d(v, s_k; G))$.

Set $S$ is called the resolving set for $G$ if every vertex in $G$ has a unique representation to $S$. The resolving set for $G$ with minimum cardinality is called the base for $G$. Cardinality-base for $G$ is called metric dimension, denoted with $\dim(G)$.

If two vertices $u, v \in V(G)$ have different representations to $S$, we refer it as $S$ distinguishes $u$ and $v$. According to Chartrand et al. [3], if $u \in S$ or $v \in S$, then it is guaranteed that $S$ distinguishes $u$ and $v$. Therefore, to find out whether $S$ is a resolving set or not, we only need to analyze whether it distinguishes any pair of vertices in $V(G) - S$.

Another variant of metric dimension concept was introduced by Jannesari and Omoom [14] as adjacency metric dimension. Rather than measuring a length of path connecting any two vertices, the concept of adjacency metric only acknowledges whether a pair of vertices are adjacent or not. Adjacency metric and adjacency dimensions are defined as follows:

**Definition 2.1.2 [14]**

Let $H$ be an arbitrary graph. Let $u$ and $v$ be two vertices in $H$. Adjacency metric of $u$ and $v$, $d_A(u, v; H)$, is defined as

$$d_A(u, v; H) = \begin{cases} 0 & \text{if } u = v \\ 1 & \text{if } u, v \text{ are adjacent} \\ 2 & \text{otherwise} \end{cases}$$

**Definition 2.1.3 [14]**

Given an ordered subset $S = \{s_1, s_2, ..., s_k\} \subseteq V(H)$, for any vertex $v \in V(H)$, the adjacency representation from $v$ to $S$ is a vector $(d_A(v, s_1; H), d_A(v, s_2; H), ..., d_A(v, s_k; H))$.

Set $S$ is called the adjacency resolving set for $H$ if every vertex in $H$ has a unique adjacency representation to $S$. The adjacency resolving set for $H$ with minimum cardinality is called the adjacency-base for $H$. Cardinality of adjacency-base for $H$ is called adjacency metric dimension, denoted with $\dim_A(H)$.

### 2.2. Neighborhood corona operation

Neighborhood corona graph was introduced by Gopalapillai [11]. He defined the graph operation as follows:

**Definition 2.2.1 [11]**

Let $G$ and $H$ be two graphs. Let $V(G) = \{g_1, g_2, ..., g_{|V(G)|}\}$ while $H_1, H_2, ..., H_{|V(G)|}$ be copy of graph $H$. Neighborhood corona graph $G[H]$ is obtained by taking graph $G, H_1, H_2, ..., H_{|V(G)|}$ and making all vertices in $V(H_1)$ adjacent with vertices in the open neighborhood $N(g; G)$, $\forall g \in \{1, 2, ..., |V(G)|\}$.

Following similar idea of Laali et al [12], we can define a generalization to neighborhood corona operation as follows:

**Definition 2.2.2**

Let $H_1, H_2, ..., H_{|V(G)|}$ be simple graphs and let $V(G) = \{g_1, g_2, ..., g_{|V(G)|}\}$. The generalized neighborhood corona graph $G[H]$ is the graph obtained by taking graph $G, H_1, H_2, ..., H_{|V(G)|}$ and making all vertices in $V(H_1)$ adjacent with vertices in open neighborhood $N(g; G)$, $\forall g \in \{1, 2, ..., |V(G)|\}$.

Since the discussion in this paper is about metric dimension of generalized neighborhood corona graph, from this point onward, we always assume $G[H]$ is a connected graph. This directly implies that $G$ is required to be connected non-trivial graph, while $H_i$ are not necessarily connected.

By the definition of generalized neighborhood corona graphs, we arrived at the following observations:

**Observation 2.2.1**

On a generalized neighborhood corona graph $G[H_i]$, if $u$ is a vertex in $G$ and $v$ is a vertex in $H_i$, then $u$ is adjacent with $v$ in $G[H_i]$ if and only if $u$ is a neighbor of $g$ in $G$.

**Observation 2.2.2**

On a generalized neighborhood corona graph $G[H_i]$, if $u$ and $v$ are two vertices that belong to a different copy of $H(u \in V(H_1); v \in V(H_2)$, with $i \neq j$, then $d(u, v; G[H_i]) > 1$.

### 3. Main results

This research results in determination of the lower and upper bounds for metric dimension of generalized neighborhood corona graphs. Furthermore, we have proven a characteristic which shows sufficient condition to achieve the lower bound.

#### 3.1. Bounds for metric dimensions of $G[H_i]$,

In this part, some lemmas about metric-similarities are proved. Then, we provide lower bounds of $\dim(G[H_i])$ for arbitrary graphs $G, H_1, H_2, ..., H_{|V(G)|}$ and upper bounds of $\dim(G[H_i])$ for G graph with no false twin vertices. As mentioned previously in introduction section, we only discuss cases where both $G$ and $H$ are non-trivial graphs.

**Lemma 3.1.1**

Let $i \in \{1, 2, ..., |V(G)|\}$, and $u, v$ be two distinct vertices that belong to $H_i$. Then, metric similarity $d(u, v; G[H_i]) = d_A(u, v; H_i)$ holds.

**Proof**

If $u$ and $v$ are adjacent in $H_i$, it follows that both vertices are neighbors in $G[H_i]$, thus $d(u, v; G[H_i]) = 1$. Conversely, if $u$ and $v$ are not adjacent in $H_i$, both of them are not neighbors in $G[H_i]$. Since $G$ is connected and nontrivial, we can take another vertex $g \in N(g; G)$. Vertex $g$ is adjacent to both vertices $u$ and $v$ in graph $G[H_i]$ resulting in $d(u, v; G[H_i]) = 2$. These results are equal to $d_A(u, v; H_i)$ as stated in Definition 2.2.1.

**Corollary 3.1.1**

If a subset $SCV(G[H_i])$ distinguishes adjacency representation of two vertices $u, v \in V(H_i)$, then $S$ distinguishes the representation of $u$ and $v$ in neighborhood corona graph $G[H_i]$.

**Lemma 3.1.2**

Let $u, v, w$ be three vertices in graph $G[H_i]$. For $i \in \{1, 2, ..., |V(G)|\}$, if $u$ and $v$ belong to vertices set $V(H_i)$ and $w$ does not belong to vertices set $V(H_i)$ then the following equality holds

$$d(u, w; G[H_i]) = d(v, w; G[H_i])$$

**Proof**

Notice that any path that connects $u$ and $w$ must pass through a vertex in open neighborhood $N(g; G)$. Similarly, any path that connects $v$ and $w$ must also pass through a vertex in open neighborhood $N(g; G)$. Both $u$ and $v$ have distance of one to any vertex in $N(g; G)$, thus both must have same distance to vertex $w$.

We found lower bound of $G[H_i]$ as given in Theorem 3.1.1 as follows:

**Theorem 3.1.1**

Metric dimension of generalized neighborhood corona graphs $G[H_i]$ satisfy the following lower bound condition:
is graph without false twin vertices, we can take a vertex distinguished by vertices in the following upper bound condition:

$$\sum_{i=1}^{\lvert V(G) \rvert} \dim_A(H_i) \leq \dim(G \star \bigcup_{i=1}^{\lvert V(G) \rvert} H_i)$$

**Proof**

This theorem is proved by showing that resolving set of $G \star \bigcup_{i=1}^{\lvert V(G) \rvert} H_i$ has at least $\sum_{i=1}^{\lvert V(G) \rvert} \dim_A(H_i)$ vertices. Suppose $S$ is a resolving set. Take any $i \in \{1, 2, \ldots, \lvert V(G) \rvert\}$ and observe vertices in $H_i$. According to Lemma 3.1.2, any two random vertices in $V(H_i) \subseteq V(G \star \bigcup_{i=1}^{\lvert V(G) \rvert} H_i)$ can only be distinguished by vertices in $S \cap V(H_i)$.

Lemma 3.1.1 stated that for two vertices of $V(H_i)$, distance in $G \star \bigcup_{i=1}^{\lvert V(G) \rvert} H_i$ is equal to adjacency metric in $H_i$. Therefore, we need at least $\dim_A(H_i)$ vertices in $S \cap V(H_i)$. Therefore, $S$ must have at least $\sum_{i=1}^{\lvert V(G) \rvert} \dim_A(H_i)$ vertices, which confirms our theorem. □

**3.1.2 Corollary**

Metric dimension of neighborhood corona graph $G \star H$ satisfies the following lower bound condition:

$$\dim_A(G \star H) \geq |V(G)| \times (\dim(G) - 1)$$

**Proof**

This theorem is proved by showing existence of resolving set with cardinality $|V(G)| \times (\dim(G) - 1)$. For $i \in \{1, 2, \ldots, |V(G)|\}$, let $S_i$ be an adjacency-base of $H_i$. Take $S = V(G) \cup \bigcup_{i=1}^{|V(G)|} S_i$. We will show that $S$ is a resolving set. Take two vertices $u$ and $v$ in $V(G \star \bigcup_{i=1}^{|V(G)|} H_i)$. There are two possibilities:

1. $u$ and $v$ belong to the same copy of $H$

   Let $u, v \in V(H_i)$, with $i \in \{1, 2, \ldots, |V(G)|\}$. Since $S_i$ is an adjacency-base of $H_i$, based on Corollary 3.1.1, we have $S_i$ distinguishes $u$ and $v$ in graph $G \star \bigcup_{i=1}^{|V(G)|} H_i$. Therefore, $S$ distinguishes $u$ and $v$.

2. $u$ and $v$ belong to different copies of $H$

   Let $u \in V(H_i)$, $v \in V(H_j)$, and $i, j \in \{1, 2, \ldots, |V(G)|\}$, with $i \neq j$. Since $G$ is graph without false twin vertices, we can take a vertex $g_k \in V(G) \subseteq S$ which is adjacent to exactly one of either $g_i$ or $g_j$. By Observation 2.3.1, $g_k$ has different distance to $u$ and $v$, thus $S$ distinguishes $u$ and $v$. □

**3.1.3 Corollary**

If $G$ is a graph without false twin vertices, then metric dimension of neighborhood corona graph $G \star H$ satisfies the following upper bound condition:

$$\dim(G \star H) \leq |V(G)| \times (\dim(H) + 1)$$

**3.2 Characteristic for metric dimension of $G \star \bigcup_{i=1}^{|V(G)|} H_i$**

After finding lower and upper bounds of $\dim(G \star \bigcup_{i=1}^{|V(G)|} H_i)$, we will take a look at a characteristics for $\dim(G \star \bigcup_{i=1}^{|V(G)|} H_i)$. We will show a sufficient condition for the metric dimension to reach lower bound in Theorem 3.1.1.

**Theorem 3.2.1**

Let $G$ be a graph with no false twin and no leaf vertex, and there exists a subset of $V(H_i)$ which is both adjacency-base and dominating set simultaneously for all $i \in \{1, 2, \ldots, |V(G)|\}$, then metric dimension of graph $G \star \bigcup_{i=1}^{|V(G)|} H_i$ satisfies

$$\dim(G \star \bigcup_{i=1}^{|V(G)|} H_i) = \sum_{i=1}^{|V(G)|} \dim_A(H_i)$$

**Proof**

It has been proven on Theorem 3.1.1 that $\sum_{i=1}^{|V(G)|} \dim_A(H_i) \leq \dim(G \star \bigcup_{i=1}^{|V(G)|} H_i)$. Therefore, we only need to show existence of resolving set with cardinality $\sum_{i=1}^{|V(G)|} \dim_A(H_i)$.

For $i \in \{1, 2, \ldots, |V(G)|\}$, let $S_i$ be subset of $V(H_i)$ which is adjacency-base and dominating set simultaneously. We define $S = \bigcup_{i=1}^{|V(G)|} S_i$, hence $|S| = \sum_{i=1}^{|V(G)|} \dim_A(H_i)$. We will prove that $S$ is resolving set as follows. Take two vertices $u, v \in V(G \star \bigcup_{i=1}^{|V(G)|} H_i)$.

1. $u$ and $v$ both belong to $G$

   Since $G$ has no false twin, it is guaranteed that there exists a $j \in \{1, 2, \ldots, |V(G)|\}$ with $g_i$ in exactly one of $N(u, G)$ or $N(v, G)$. Without losing generality, let $g_i \in N(u, G)$ and $g_j \notin N(v, G)$. We take a vertex $w \in S_j$. Since $g_j$ is adjacent with $u$ and not adjacent with $v$, by Observation 2.2.1, it has different distance to $u$ and $v$. Therefore, $u$ and $v$ have different representation to $S$.

2. $u$ belongs to $G$, $v$ belongs to a copy of $H$

   Let $u = g_i \in V(G)$, with $i \in \{1, 2, \ldots, |V(G)|\}$. There are two sub-cases: a. $v$ belong to the copy of $H$ that corresponds to $u$

   Choose a $j \in \{1, 2, \ldots, |V(G)|\}$ such that $g_j \in N(u, G)$. Then, take a vertex $w \in S_j \subseteq V(H_j)$. By Observation 2.2.1 and Observation 2.2.2, we obtain $d(u, w; G \star \bigcup_{i=1}^{|V(G)|} H_i)$ is greater than one.

3. $u$ and $v$ belong to same copy of $H$

   Since $G$ has no leaf vertex. Since there are more than one vertex in open neighborhood $N(u, G)$, we can take a $k \in \{1, 2, \ldots, |V(G)|\}$ such that $g_k \in N(u, G)$ and $g_k \notin N(v, G)$. Take another vertex $w \in S_k \subseteq V(H_k)$. By Observation 2.2.1 and Observation 2.2.2, we obtain $d(u, w; G \star \bigcup_{i=1}^{|V(G)|} H_i)$ is greater than one.

In both sub-cases, we can see that $u$ and $v$ have different representation to $S$.

4. $u$ and $v$ belong to different copy of $H$

   Let $u \in V(H_i)$, and $v \in V(H_j)$, with $i, j \in \{1, 2, \ldots, |V(G)|\}$ Take another vertex $w \in S_i$. Since $S_i$ is also dominating set of $H_i$, we know $u$ and $w$ are adjacent in $G \star \bigcup_{i=1}^{|V(G)|} H_i$. On other hand, since $v$ and $w$ belong to a different copy of $H$, they must be not adjacent in $G \star \bigcup_{i=1}^{|V(G)|} H_i$. Therefore, $S$ distinguishes $u$ and $v$.

   We arrived at a conclusion that $S$ resolves $G \star \bigcup_{i=1}^{|V(G)|} H_i$. Thus, the proof is complete. □
4. Conclusion

We studied metric dimensions of generalized neighborhood corona operation over non-trivial graphs. For arbitrary graphs \( G, H_1, H_2, \ldots, H_{|V(G)|} \), we found lower bound as follows: \( \sum_{i=1}^{V(G)} \dim_A(H_i) \leq \dim(G \star_{i=1}^{V(G)} H_i) \). For graph \( G \) with no false twin vertices, we found upper bound: \( \dim(G \star_{i=1}^{V(G)} H_i) \leq |V(G)| + \sum_{i=1}^{V(G)} \dim_A(H_i) \). We also proved a characteristic: if \( G \) has no false twin nor leaf vertex and each \( H_i \) has an adjacency-base that is also dominating set, then \( \dim(G \star_{i=1}^{V(G)} H_i) = \sum_{i=1}^{V(G)} \dim_A(H_i) \).

5. Discussion

Since we use generalized version of neighborhood corona operation, our result has benefit of a wider inclusion range of graphs. However, there are still limitation to our work. For example, we have not yet discovered whether upper bound given on Theorem 3.1.2 is strict. We left this problem open for future research(s). We also encourage future researchers to develop more characteristics for metric dimensions of generalized neighborhood corona operation of graphs.

Declarations

Author contribution statement

Rinurwati: Conceived and designed the experiments; Contributed reagents, materials, analysis tools or data; Wrote the paper.

S. E. Setiawan: Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data.

Slamin: Analyzed and interpreted the data.

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No data was used for the research described in the article.

Declaration of interests statement

The authors declare no conflict of interest.

Additional information

No additional information is available for this paper.

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