Upper Bound for Diameter of Cosmological Black Holes and Nonexistence of Black Strings

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September 30, 2016

Abstract
The diameter of the apparent horizon, defined by the distance between furthest points on the horizon, in spacetimes with a positive cosmological constant $\Lambda$ has been investigated. It is established that the diameter of the apparent horizon on the totally umbilic partial Cauchy surface cannot exceed $2\pi/\sqrt{3\Lambda}$. Then, it is argued that arbitrary long black strings cannot be formed in our universe.

1 Introduction
The general properties of black holes in classical general relativity have been extensively studied. In particular, the equilibrium problem of black holes in asymptotically flat spacetimes is highlighted by the uniqueness theorem for the Kerr-Newman solution [1]. The notion of black holes is extended to the cosmological setting, where the cosmological horizon appears due to presence of the positive cosmological constant. Since the equilibrium problem of such cosmological black holes is not established, they share a lot of beautiful properties with asymptotically flat black holes. In particular, many local results for the apparent horizon can be applied to the cosmological black holes. For example, the Hawking’s theorem [2] that asserts that the apparent horizon must be topological two-sphere also holds for black holes in spacetimes with a positive cosmological constant.

Cosmological black holes are fascinating in their own right. They admit an interesting exact solution representing dynamical collision of black holes in the Einstein-Maxwell equations with the positive cosmological term [3]. As recent cosmological observations strongly suggest that our present universe has a positive cosmological constant, it is very natural to seek for the general properties of cosmological black holes. Hence, the main concern in this article is the black holes in four-dimensional spacetimes with the positive cosmological constant $\Lambda$.

A remarkable property of the cosmological black hole is that the area of the black hole cannot exceed the value $4\pi/\Lambda$ [3, 4, 5]. Thus, the black holes in inflationary universe cannot grow unboundedly, and so much large black holes
cannot merge into one, or otherwise the naked singularity would be formed \[7\].

One might however expect that more precise geometrical information about the black hole horizon would be obtained from the knowledge of the appropriate length size of the black holes. For example, the area bound does not control the nonexistence of the black string solution, as we can consider very thin and long horizons with the area of horizon fixed.

It is a general belief that there are no black string solution in four-dimensional general relativity. This is supported by the absence of known exact solutions or numerical examples. A conclusive result excluding black strings, however, seems to be hardly known.

On the other hand, Thorne’s hoop conjecture \[8\] in four-dimensional general relativity can be seen as an implication for the nonexistence of such string-shaped black holes. It claims that the black hole horizon forms if and only if the mass \(M\) gets compacted into the region whose circumference \(C\) in every direction satisfies \(C \leq 4\pi M\). Then, the only-if part of the conjecture claiming that a realized horizon is subject to the above inequality seems to exclude arbitrary long horizons for the given gravitational mass. No counter example to the hoop conjecture has been reported, while it has been tested for various exact solutions to the Einstein equation, or numerically generated spacetimes \[9\]. Note however that we must appropriately define what is meant by mass, circumference, and horizon in the statement, when it is applied to the specific problem, since these notions are not specified there.

Nevertheless, the knowledge of a characteristic length scale of the horizon, combined with that of its topology and area, would provide with certain useful information about its geometric shape. Here, we focus on the specific length scale of the apparent horizons in cosmological spacetimes that is given by the intrinsic distance between furthest pair of points on the horizon, which is proposed as a definition of half the circumference in the Flanagan’s work \[10\] seeking for the rigorous formulation of the hoop conjecture, and it is also known as the diameter of compact manifolds in differential geometry.

In the following note, we point out that the diameter of the apparent horizon of the cosmological black hole on the totally umbilic partial Cauchy surface has the upper bound given by \(2\pi/\sqrt{3}\Lambda\). This seems to be the first conclusive example that excludes the existence of arbitrary long black strings in a certain class of cosmological spacetimes.

2 The upper bound for the diameter of the black hole horizon

Firstly, let us explain the general setting of the problem. Let \(M\) be the differentiable manifold endowed with the Lorentzian metric \(g_{ab}\) with the signature \((- , + , + , + )\). Let \(\Sigma\) be a partial Cauchy surface in \(M\), and let \(U^a\) be a future-pointing timelike unit vector field on a neighborhood \(\mathcal{U}\) of \(\Sigma\), which is normal
to $\Sigma$. The tensor field
\[ h_{ab} = g_{ab} + U_a U_b \]
on $\Sigma$ gives the Riemannian metric on $\Sigma$, when restricted to $\Sigma$. Since $U^a$ is orthogonal to $\Sigma$, it satisfies
\[ U[a \nabla_b U_c] = 0 \]
on $\Sigma$. Then, the covariant derivative of $U_a$ is decomposed as
\[ \nabla_a U_b = K_{ab} - U_a A_b, \]
on $\Sigma$, where
\[ K_{ab} := h^c_a \nabla_c U_b \]
gives the second fundamental form of $\Sigma$, and
\[ A_a := U^b \nabla_b U_a \]
is the acceleration vector of $U^a$. The restrictions of $K_{ab}$ and $A_a$ to $\Sigma$ are tensor fields on $\Sigma$, in the sense that these do not have a nonzero component tangent to $U^a$.

Let a closed 2-surface $H$ be an apparent horizon on $\Sigma$. We consider a deformation of $H$ by
\[ S : [-1/2, 1/2] \times H \to \Sigma; (\xi, x) \mapsto S_\xi(x), \]
such that $S_0 = i : H \hookrightarrow \Sigma$ is the inclusion map, and that $S_\xi$ is a surface outside $H$ for $\xi > 0$.

Let $N^a$ be the tangent vector field on $\text{Im}(S)$, which is the outward-pointing unit normal to $S_\xi$. We define the tensor field on $\text{Im}(S)$ as
\[ \gamma_{ab} := h_{ab} - N_a N_b, \]
which gives the induced Riemannian metric on $S_\xi$. The covariant derivative of $N_a$ is decomposed as
\[ \nabla_a N_b = \chi_{ab} + N_a \alpha_b, \]
where
\[ \chi_{ab} := \gamma^c_a \nabla_c N_b \]
gives the second fundamental form of $H$ as a surface in $\Sigma$, and
\[ \alpha_a := N^b \nabla_b N_a \]
is defined. The tensor fields $\chi_{ab}$ and $\alpha_a$ are regarded as those on $H$, when they are restricted on $H$. The normal vector field $N_a$ can be written as

$$N_a = f \partial_a \xi,$$

where the parameter $\xi$ of the deformation of $H$ is regarded as a function on $\text{Im}(S)$. Then, it holds

$$\alpha_a = - f^{-1} \partial_a f.$$

The second fundamental form $K_{ab}$ of $\Sigma$ is decomposed as

$$K_{ab} = \beta_{ab} + \zeta_a N_b + N_a \zeta_b + \mu N_a N_b,$$

where

$$\beta_{ab} := \gamma_a \gamma_b K_{cd},$$
$$\zeta_a := \gamma_a^b K_{bc} N^c,$$
$$\mu := K_{ab} N^a N^b$$

give tensor fields on $H$.

The light rays in $M$ emanating from $S_\xi$ are tangent to the null vector field

$$\ell^a := U^a + N^a$$

on $S_\xi$ (See Figure 1). The expansion $\theta$ of the vector field $\ell^a$ is defined as

$$\theta := \gamma^{ab} \nabla_a \ell_b = \beta + \chi,$$

where we abbreviate $\beta := \beta^a_a$, $\chi := \chi^a_a$. The apparent horizon $H$ is a marginally trapped surface, i.e., it holds

$$\theta = \beta + \chi = 0, \text{ on } H.$$
We write the vector field, which gives the deformation $S_\xi$ of $H$, as

$$X^a = f N^a.$$  

Then, we obtain the differential of $\theta$ as

$$f^{-1} X^a \partial_a \theta = N^a \partial_a (\beta + \chi).$$

(1)

Here, using the Codazzi equation for $\Sigma$

$$h^b_a R_{bc} U^c = \nabla_b K_a - \nabla_a K,$$

we obtain

$$R_{ab} N^a U^b = N^a \nabla^b K_a - N^a \nabla_a K$$

$$= \nabla_a \zeta^a - 2 \zeta_a \alpha^a + \mu \chi - \beta_{ab} \chi^{ab} - N^a \partial_a \beta,$$

or

$$\partial_N \beta = \nabla_a \zeta^a - 2 \zeta_a \alpha^a + \mu \chi - \beta_{ab} \chi^{ab} - R_{ab} N^a U^b,$$

(2)

where $K := K_a^a$ is defined and $\nabla_a$ denotes the covariant derivative on $H$.

From

$$h^b_{abcd} N^d = (\nabla_a \nabla_b - \nabla_b \nabla_a) N_c$$

$$= \nabla_a \chi_{bc} - \nabla_b \chi_{ac} + N_{ac} \alpha_b \alpha_c$$

$$- \alpha_a N_b \alpha_c + N_b \nabla_a \alpha_c - N_c \nabla_a \alpha_b,$$

it follows that

$$h^b_{ab} N^a N^b = - \chi_{ab} \chi^{ab} + \alpha_a \alpha^a + \nabla_a \alpha^a - N^a \partial_a \chi,$$

or

$$\partial_N \chi = - \chi_{ab} \chi^{ab} - f^{-1} \nabla_a \nabla^a f - R_{ab} N^a N^b.$$

(3)

holds.

Using Eqs. (2) and (3), the Eq. (1) becomes

$$f^{-1} \partial_X \theta = \nabla_a \zeta^a - 2 \zeta_a \alpha^a + \mu \chi - \beta_{ab} \chi^{ab} - R_{ab} N^a U^b$$

$$- \chi_{ab} \chi^{ab} - f^{-1} \nabla_a \nabla^a f - R_{ab} N^a N^b.$$

(4)

The Gauss equation for $\Sigma$

$$h^b_{abcd} = K_{ad} K_{bc} - K_{ac} K_{bd} + h^p_a h^q_b h^r_c h^s_d R_{pqrs}$$
leads to
\[ h^R = K_abK^{ab} - K^2 + R + 2R_{ab}U^aU^b. \] (5)

Also, the Gauss equation for \( H \) as a surface in \( \Sigma \)
\[ \hat{R}_{abcd} = \chi_{ac}\chi_{bd} - \chi_{ad}\chi_{bc} + h_a^p h_b^q h_c^r h_d^s R_{pqrs} \]
gives
\[ \hat{r} = \chi^2 - \chi_{ab} \chi^{ab} + h^R - 2h^R_{ab} N^a N^b. \] (6)

Eqs. (5) and (6) are put together into the form
\[ 2h^R_{ab} N^a N^b = \chi^2 - \chi_{ab} \chi^{ab} + h^R - 2h^R_{ab} N^a N^b. \]

Substituting this into Eq. (4), we obtain
\[ f^{-1} \partial_X \theta = -\nabla_a (\zeta^a - f^{-1} \nabla^a f) - (\zeta^a - f^{-1} \nabla^a f) (\zeta^a - f^{-1} \nabla^a f) \]
\[ - \frac{1}{2} \theta_{ab} \theta^{ab} + \frac{1}{2} (\mu - \chi) \theta + \frac{1}{2} \hat{r} - 8\pi G T_{ab} U^a U^b - \Lambda, \] (7)

where we define
\[ \theta_{ab} := \beta_{ab} + \chi_{ab}, \]
and the Einstein equation
\[ R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 8\pi G T_{ab} \]
is applied. Here and in what follows, we set the speed of light to unity.

For every deformation of \( H \), which is determined by the positive function \( f \) on \( H \), \( S_\xi \) should not be a trapped surface for \( \xi > 0 \), since \( H \) is the outermost trapped surface. This requirement leads to the nonnegativity of the principal eigenvalue of the elliptic operator associated with Eq. (7).

**Lemma 1** Under the dominant energy condition, the principal eigenvalue of the linear operator acting on the function on \( H \):
\[ A = -\nabla_a \nabla^a + 2\zeta^a \nabla_a + \frac{1}{2} \hat{r} - \Lambda + (\nabla_a \zeta^a) - \zeta_a \zeta^a \]
is nonnegative.

\(^1\)The elliptic operator introduced here may not be a symmetric operator (i.e. with a drift term), so that its eigenvalues may be complex numbers. It however has the real eigenvalue \( \lambda_1 \), called the principal eigenvalue, such that \( \lambda_1 < \text{Re}(\lambda) \) holds for every eigenvalue \( \lambda \in \mathbb{C} \), and that the corresponding eigenfunction is a positive function (See e.g. Ref. [11], Chap. 6.).
Proof. By definition, the dominant energy condition requires that $T_{ab}V^aW^a \geq 0$ holds for every pair of future pointing timelike vectors $(V^a, W^a)$. It follows that $T_{ab}U^aU^b \geq 0$ holds by continuity.

Let the real number $\lambda_1$ be the principal eigenvalue of $A$. Consider the deformation of $H$ in terms of the deformation vector $X^a = fN^a$, where the positive function $f$ is taken to be the corresponding eigenfunction $f$. Then, on the apparent horizon $H$, the Eq. (7) gives

$$\partial_X \theta = Af - \frac{1}{2} \theta_{ab}\theta^{ab}f^2 - 8\pi G T_{ab}U^aU^bf \leq \lambda_1 f.$$ 

It follows that $\lambda_1$ must be nonnegative, since otherwise we have $\partial_X \theta < 0$ at every point on $H$, to contradict the condition that $H$ is the outermost trapped surface on $\Sigma$.

In the following, we consider a specific class of partial Cauchy surfaces given by

$$K_{ab} = \frac{1}{3}K_{h_{ab}},$$

which we call the totally umbilic initial data $(\Sigma, h_{ab}, K_{ab})$. This restricted class of initial data is still allowed in a wide class of spacetimes, such as the Kastor-Traschen multi-black-hole spacetimes [3].

Here, we show that a characteristic length of the horizon must be not greater than the cosmological length scale, when $A$ is positive.

**Definition 2** For a closed 2-surface $S$ in $\Sigma$, the diameter of $S$ is defined by

$$\text{diam}(S) := \max \{ \text{dist}_S(p, q) | p, q \in S \},$$

where $\text{dist}_S(p, q)$ denotes the distance between $p$ and $q$ determined by the intrinsic geometry of $S$.

**Theorem 3** Let $(\Sigma, h_{ab}, K_{ab})$ be a totally umbilic initial data for the spacetime with the positive cosmological constant, and let $H$ be the apparent horizon in $\Sigma$. Under the dominant energy condition, the diameter of $H$ satisfies

$$\text{diam}(H) \leq \frac{2\pi}{\sqrt{3A}}.$$ \hspace{1cm} (8)

**Proof.** Take arbitrary pair of points $p, q$ on $H$. Let $\Gamma : [0, L] \rightarrow H$ be the curve in $H$ connecting $p$ and $q$, that minimizes the integral

$$I_\Gamma := \int_\Gamma f ds,$$
where \( f > 0 \) is the principal eigenfunction of the linear operator \( A \), which in the present case \( (\zeta^a = 0) \) takes the form

\[
A = -\nabla_a \nabla^a + \frac{1}{2} \gamma R - \Lambda.
\]

Let \( \nu^a \) be the unit vector field on the neighborhood of \( \Gamma \) in \( H \), which is normal to \( \Gamma \), and let \( q \) be a smooth real function on \( \Gamma \) vanishing at the endpoints \( p \) and \( q \). Now we consider the variation of \( I_f \) in terms of the deformation vector \( g\nu^a \).

The first variation of \( I_f \) becomes

\[
\delta I_f = \int_{\Gamma} \left( \nu^a \partial_a f + f \nabla_a \nu^a \right) g ds,
\]

so that

\[
\nabla_a \nu^a = -f^{-1} \partial_b f
\]

should hold on \( \Gamma \).

Since \( \Gamma \) minimizes \( I_f \), its second variation should be nonnegative. This can be computed as

\[
\delta^2 I_f = \int_{\Gamma} \left\{ -f \frac{d^2 g}{ds^2} - \frac{dg}{ds} \frac{df}{ds} + \left[ \nabla_a \nabla^a f - \frac{R}{2} f - \frac{d^2 f}{ds^2} - f (\nabla_a \nu^a)^2 \right] g \right\} ds.
\]

By Lemma 1, the inequality

\[
\nabla_a \nabla^a f - \frac{R}{2} f \leq -\Lambda f
\]

holds on \( H \). Then, we have

\[
\delta^2 I_f \leq \int_{\Gamma} \left( -fg \frac{d^2 g}{ds^2} - \frac{dg}{ds} \frac{df}{ds} - g^2 \frac{d^2 f}{ds^2} - \Lambda g^2 \right) ds
\]

\[
= \int_{\Gamma} f \left[ -2g \frac{d^2 g}{ds^2} - \left( \frac{dg}{ds} \right)^2 - \Lambda g^2 \right] ds.
\]

Now we take \( g = [\sin(\pi s/L)]^{2/3} \), where \( L \) denotes the length of \( \Gamma \). Then, the above inequality leads to

\[
\left( \frac{4\pi^2}{3L^2} - \Lambda \right) \int_{\Gamma} fg^2 ds \geq 0.
\]

Hence, we have

\[
L \leq \frac{2\pi}{\sqrt{3\Lambda}}.
\]

Since \( \text{diam}(H) \leq L \) holds by definition, the statement of the theorem immediately follows.
Although Theorem 3 does not make sense for $\Lambda = 0$, it is easy to obtain the version of Theorem 3 without the cosmological term, by slightly modifying the above proof.

**Theorem 4** Let $(\Sigma, h_{ab}, K_{ab})$ be a totally umbilic initial data for the Einstein equation

$$R_{ab} - \frac{1}{2} R g_{ab} = 8\pi G T_{ab}.$$ 

Let $\rho = T_{ab} U^a U^b$ and $J_a = h^{bc} T_{bc} U^b$ be the energy density and the energy flux of the matter field, respectively. If an apparent horizon $H$ on $\Sigma$ is located within the region in which

$$8\pi G (\rho - \sqrt{J_a J^a}) > c$$

holds for a positive constant $c$, then, the diameter of $H$ satisfies

$$\text{diam}(H) < \frac{2\pi}{\sqrt{3}c}.$$ 

**Sketch of a Proof.** This can be proved along similar lines to the reasoning of Lemma 1 and Theorem 3, noting that the condition on the energy current 4-vector implies that the inequality

$$8\pi G T_{ab} U^a \ell^b > c$$

holds on $H$, so that the linear operator

$$-\nabla_a \nabla^a + \frac{1}{2} \hat{R} - c$$

acting on the function on $H$ has the positive principal value.

3 Final Remarks

We consider the apparent horizon in spacetimes with a cosmological constant. Then, we show that the diameter of the horizon on the totally umbilic partial Cauchy surface has the upper bound given by $2\pi/\sqrt{3}\Lambda$ in terms of the standard variational technique. Since this upper bound depends only on the cosmological constant, it suggests the absence of arbitrary long black strings in the universe with a cosmological constant.

Though Theorem 3 puts restrict on the hoop length of the black hole horizons, it is not relevant for the Thorne’s hoop conjecture. In fact, the hoop conjecture with just that could tell nothing about the arbitrary long black strings, since it contains the gravitational mass scale in the inequality.
It would be better if the condition of the total umbilicity in Theorem 3 could be relaxed, since it is far from trivial if generic cosmological spacetimes admit such a time slicing.

It is also unclear whether the present diameter bound is the best one or not. Regarding the exact solutions, the supremum for the diameter of the apparent horizons among the Schwarzschild-de Sitter class is given by \( \pi/\sqrt{\Lambda} \), which is nearly 87% of \( 2\pi/\sqrt{3}\Lambda \). As one direction of the future work, it might be interesting to test the sharpness of the present diameter bound in terms of the numerical search of the apparent horizons for various initial data sets.

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