REVERSE ORBIFOLD CONSTRUCTION AND UNIQUENESS OF
HOLOMORPHIC VERTEX OPERATOR ALGEBRAS

CHING HUNG LAM AND HIROKI SHIMAKURA

Abstract. In this article, we develop a general technique for proving the uniqueness of holomorphic vertex operator algebras based on the orbifold construction and its “reverse” process. As an application, we prove that the structure of a strongly regular holomorphic vertex operator algebra of central charge 24 is uniquely determined by its weight one Lie algebra if the Lie algebra has the type $E_{6,3}G_{2,1}^3$, $A_{2,3}^6$ or $A_{5,3}D_{4,3}A_{1,1}^3$.

1. Introduction

This article is a continuation of our program on classification of holomorphic vertex operator algebras (VOAs) of central charge 24. In 1993, Schellekens [Sc93] obtained a partial classification by determining possible Lie algebra structures for the weight one subspaces of holomorphic VOAs of central charge 24. There are 71 cases in his list but not all cases were constructed explicitly at that time. Recently, there is much progress towards the classification; all 71 cases have been constructed ([FLM88, DGM96, La11, LS12, LS16a, Mi13, SS16, EMS, LS16b, LL]) and Schellekens’ list has also been reproved in [EMS]. The main technique for construction is often referred as to “Orbifold construction”. When the central charge is 24, it is also conjectured that the Lie algebra structure of the weight one subspace will determine the holomorphic VOA structure uniquely. This would be an analogue of the uniqueness of the Niemeier lattices: the isometry class of a positive-definite even unimodular lattice of rank 24 is uniquely determined by the root system consisting of the norm 2 vectors. However, this conjecture has been proved only for the following 26 cases. In [DM04b], it was shown that if the weight one Lie algebra has Lie rank 24, then the holomorphic VOA is isomorphic to the lattice VOA associated with one of the 24 Niemeier lattices. For the other cases, the classification of holomorphic framed VOAs of central charge 24 in [LS15] implies that if the weight one Lie algebra

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1After we finished this work, 13 more cases have been proved in [KLL] by using the main idea of this article and one more case has been proved in [LL] by using mirror extension.
has the type $A_{1,2}^6$ or $E_{8,2}B_{8,1}$, then the holomorphic VOA structure of central charge 24 is unique. Indeed, in these two cases, the subVOA generated by the weight one Lie algebra is framed, and so is the corresponding holomorphic VOA.

In this article, we discuss a general technique for proving the uniqueness of holomorphic vertex operator algebras. The key idea is to reverse the orbifold construction based on the following simple observation:

Let $V$ be a (strongly regular) holomorphic VOA and let $\sigma$ be an order $n$ automorphism of $V$. Assume that $\sigma$ satisfies certain condition (see Condition (I) in Section 4.1) so that one can apply the $\mathbb{Z}_n$-orbifold construction to $V$ and $\sigma$ (cf. [EMS, M616]). Then the resulting holomorphic VOA $\tilde{V}_\sigma$ is a $\mathbb{Z}_n$-graded simple current extension of the $\sigma$-fixed-point subalgebra $V^\sigma$. The $\mathbb{Z}_n$-grading of $\tilde{V}_\sigma$ gives rise to an order $n$ automorphism $\tau$ of $\tilde{V}_\sigma$. Clearly, $(\tilde{V}_\sigma)^\tau = V^\sigma$. In addition, the $\mathbb{Z}_n$-orbifold construction associated with $\tilde{V}_\sigma$ and $\tau$ gives the original holomorphic VOA $V$ as a $\mathbb{Z}_n$-graded simple current extension of $(\tilde{V}_\sigma)^\tau$, that is, $(\tilde{V}_\sigma)^\tau \cong V$ (see Corollary 5.1). We call this procedure the reverse orbifold construction, which is called the inverse orbifold in [EMS, M616].

Now, we will explain our hypotheses for the uniqueness results. Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{p}$ a subalgebra of $\mathfrak{g}$. Let $n$ be a positive integer and let $W$ be a strongly regular holomorphic VOA of central charge $c$.

(i) For any holomorphic VOA $V$ of central charge $c$ with $V_1 \cong \mathfrak{g}$, there exists an order $n$ automorphism $\sigma$ of $V$ such that $\tilde{V}_\sigma \cong W$ and $V^\sigma_1 \cong \mathfrak{p}$.

(ii) Any order $n$ automorphism $\varphi$ of $W$ belongs to a unique conjugacy class if $(W^\varphi)_1 \cong \mathfrak{p}$ and $(W^\varphi)^\tau_1 \cong \mathfrak{g}$.

Under the above hypotheses, we show in Theorem 5.2 that any holomorphic VOA $V$ of central charge $c$ with $V_1 \cong \mathfrak{g}$ is isomorphic to $\tilde{W}_\varphi$. Indeed, Hypothesis (i) implies $\tilde{V}_\sigma \cong W$ and the reverse orbifold construction shows $\tilde{W}_\tau \cong V$. In addition, Hypothesis (ii) implies that $\tau$ and $\varphi$ are conjugate, and hence $V \cong \tilde{W}_\tau \cong \tilde{W}_\varphi$. In particular, the holomorphic VOA structure of central charge $c$ with weight one Lie algebra $\mathfrak{g}$ is unique.

The similar arguments have also been used in [LY07, LS15] for proving the uniqueness of certain framed VOAs. As an application, we prove the following main theorem:

**Theorem 1.1.** The structure of a strongly regular holomorphic vertex operator algebra of central charge 24 is uniquely determined by its weight one Lie algebra if the Lie algebra has the type $E_{6,3}G_{2,1}^3$, $A_{2,3}^6$ or $A_{5,3}D_{4,3}A_{1,1}^3$.

One of the reasons why we deal with the three Lie algebras above is that the corresponding holomorphic VOAs of central charge 24 were explicitly constructed in [M613, SS16] by applying the $\mathbb{Z}_3$-orbifold construction to suitable Niemeier lattice VOAs and order 3
automorphisms. Hence we can easily find automorphisms for the reverse orbifold constructions, and, indeed, they are inner. Another reason is that a dimension formula for the \( \mathbb{Z}_3 \)-orbifold construction will be established (see Section 4.2), which plays important roles in verifying Hypothesis (i).

The organization of the article is as follows. In Section 3, we prove the uniqueness of the conjugacy classes of certain order 3 automorphisms of the lattice VOAs associated with Niemeier lattices with root sublattices of type \( E_6^4 \) and \( D_4^6 \), which will prove Hypothesis (ii). The argument is based on the results in [Ka90] about conjugacy classes of automorphisms of finite order of simple Lie algebras and the structure of the automorphism group of a lattice VOA ([FLM88, DN99]). In Section 4, we review the general \( \mathbb{Z}_n \)-orbifold construction from [EMS, Mo16]. We also establish a dimension formula about weight one Lie algebras for \( \mathbb{Z}_3 \)-orbifold construction. This formula was mentioned in [Mo94] and will be useful in determining the Lie algebra structure of the resulting holomorphic VOA. In Section 5, we establish the main technique for the uniqueness of a holomorphic VOA using the reverse orbifold construction. In Sections 6, 7 and 8, we prove the uniqueness of holomorphic VOAs. For each case, we verify Hypothesis (i) using the results established in Section 4.

**Notations**

- \((\cdot|\cdot)\) positive-definite symmetric bilinear form of a lattice, or the normalized Killing form so that \((\alpha|\alpha) = 2\) for long roots \( \alpha \).
- \(\langle\cdot|\cdot\rangle\) the normalized symmetric invariant bilinear form on a VOA \( V \) so that \( \langle 1|1\rangle = -1 \), equivalently, \(\langle a|b\rangle 1 = a_{(1)} b \) for \( a, b \in V_1 \).
- \(\alpha_i\) a simple root of a root system.
- \(g^g\) the fixed-point subalgebra of \( g \) in a Lie algebra \( g \).
- \(L(0)\) the weight operator \( \omega_{(1)} \).
- \(L_g(k, 0)\) the simple affine VOA associated with simple Lie algebra \( g \) at level \( k \).
- \(L_g(k, \lambda)\) the irreducible \( L_g(k, 0) \)-module with highest weight \( \lambda \).
- \(\Lambda_i\) the fundamental weight with respect to simple root \( \alpha_i \).
- \(M^{(h)}\) the \( \sigma_h \)-twisted V-module constructed from a V-module \( M \) by Li’s \( \Delta \)-operator.
- \(Ni(R)\) a Niemeier lattice with root sublattice \( R \).
- \(\Pi(\lambda)\) the set of all weights of the irreducible module with highest weight \( \lambda \) over a simple Lie algebra.
- \(\sigma_h\) the inner automorphism \( \exp(-2\pi \sqrt{-1} h_{(0)}) \) of a VOA \( V \) associated with \( h \in V_1 \).
- \(U(1)\) a 1-dimensional abelian Lie algebra.
- \(V^g\) the set of fixed-points of an automorphism \( g \) of a VOA \( V \).
- \(X_n\) (the type of) a root system, a simple Lie algebra or a root lattice.
- \(X_{n,k}\) (the type of) a simple Lie algebra whose type is \( X_n \) and level is \( k \).
2. Preliminary

In this section, we will review some fundamental results about VOAs.

2.1. Vertex operator algebras and weight one Lie algebras. Throughout this article, all VOAs are defined over the field \( \mathbb{C} \) of complex numbers. We recall the notion of vertex operator algebras (VOAs) and modules from [Bo86, FLM88, FHL93].

A vertex operator algebra (VOA) \((V, Y, 1, \omega)\) is a \( \mathbb{Z} \)-graded vector space \( V = \bigoplus_{m \in \mathbb{Z}} V_m \) equipped with a linear map

\[
Y(a, z) = \sum_{i \in \mathbb{Z}} a(i) z^{-i-1} \in \text{End}(V)[[z, z^{-1}]], \quad a \in V
\]

and the vacuum vector \( 1 \) and the conformal vector \( \omega \) satisfying a number of conditions ([Bo86, FLM88]). We often denote it by \( V \). For \( a \in V \) and \( n \in \mathbb{Z} \), we often call \( a(n) \) the \( n \)-th mode of \( a \). Note that \( L(n) = \omega(n+1) \) satisfy the Virasoro relation:

\[
[L(m), L(n)] = (m-n)L(m+n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0} c \text{id}_V,
\]

where \( c \) is a complex number and is called the central charge of \( V \).

A linear automorphism of a VOA \( V \) is called an automorphism of \( V \) if

\[
g \omega = \omega \quad \text{and} \quad gY(v, z) = Y(gv, z)g \quad \text{for all} \ v \in V.
\]

Let \( \text{Aut}(V) \) be the group of all automorphisms of \( V \). A vertex operator subalgebra (or a subVOA) is a graded subspace of \( V \) which has a structure of a VOA such that the operations and its grading agree with the restriction of those of \( V \) and that they share the vacuum vector. When they also share the conformal vector, we call it a full subVOA. For \( g \in \text{Aut}(V) \), let \( V^g \) be the set of fixed-points of \( g \). Note that \( V^g \) is a full subVOA of \( V \).

For \( g \in \text{Aut}(V) \) of order \( n \), a \( g \)-twisted \( V \)-module \((M, Y_M)\) is a \( \mathbb{C} \)-graded vector space \( M = \bigoplus_{m \in \mathbb{C}} M_m \) equipped with a linear map

\[
Y_M(a, z) = \sum_{i \in (1/n)\mathbb{Z}} a(i) z^{-i-1} \in \text{End}(M)[[z^{1/n}, z^{-1/n}]], \quad a \in V
\]

satisfying a number of conditions ([DLM00]). We often denote it by \( M \). Note that an (untwisted) \( V \)-module is a 1-twisted \( V \)-module and that a \( g \)-twisted \( V \)-module is an (untwisted) \( V^g \)-module. Recall that \( L(0)v = kv \) if \( v \in M_k \). The number \( k \) is called the \( L(0) \)-weight of a homogeneous vector \( v \in M_k \). If \( M \) is irreducible, then there exists \( w \in \mathbb{C} \) such that \( M = \bigoplus_{m \in (1/n)\mathbb{Z} \geq 0} M_{w+m} \) and \( M_w \neq 0 \). The number \( w \) is called the lowest \( L(0) \)-weight of \( M \).

A VOA is said to be rational if its admissible module category is semisimple. A rational VOA is said to be holomorphic if it itself is the only irreducible module up to isomorphism.
A VOA is said to be of CFT-type if $V_0 = \mathbb{C}1$ (and hence $V_n = 0$ for $n < 0$), and is said to be $C_2$-cofinite if the codimension in $V$ of the subspace spanned by the vectors of form $u(-2)v$, $u,v \in V$, is finite. A module is said to be self-dual if its contragredient module is isomorphic to itself. A VOA is said to be strongly regular if it is rational, $C_2$-cofinite, self-dual and of CFT-type.

Let $V$ be a VOA of CFT-type. Then, the weight one subspace $V_1$ has a Lie algebra structure via the 0-th mode. Moreover, the $n$-th modes $v(n)$, $v \in V_1$, $n \in \mathbb{Z}$, define an affine representation of the Lie algebra $V_1$ on $V$. For a simple Lie subalgebra $\mathfrak{s}$ of $V_1$, the level of $\mathfrak{s}$ is defined to be the scalar by which the canonical central element acts on $V$ as the affine representation. When the type of the root system of $\mathfrak{s}$ is $X_n$ and the level of $\mathfrak{s}$ is $k$, we denote by $X_{n,k}$ the type of $\mathfrak{s}$. Assume that $V$ is self-dual. Then there exists a symmetric invariant bilinear form $\langle \cdot | \cdot \rangle$ on $V$, which is unique up to scalar ($[Li94]$). We normalize it so that $\langle 1 | 1 \rangle = -1$. Then for $a,b \in V_1$, we have $\langle a|b \rangle 1 = a(1)b$. For an element $a \in V_1$, $\exp(a(0))$ is an automorphism of $V$, which is called an inner automorphism. For a semisimple element $h \in V_1$, we often consider the inner automorphism $\sigma_h = \exp(-\frac{2\pi}{\sqrt{-1}} h(0))$ associated with $h$.

Assume that $V_1$ is semisimple. Let $\mathfrak{h}$ be a Cartan subalgebra of $V_1$. Let $\langle \cdot | \cdot \rangle$ be the Killing form on $V_1$. We identify the dual $\mathfrak{h}^*$ with $\mathfrak{h}$ via $\langle \cdot | \cdot \rangle$ and normalize $\langle \cdot | \cdot \rangle$ so that $\langle \alpha | \alpha \rangle = 2$ for any long root $\alpha \in \mathfrak{h}$. In this article, weights for $\mathfrak{h}$ are defined via $\langle \cdot | \cdot \rangle$, that is, the weight of a vector $v \in V$ for $\mathfrak{h}$ is $\lambda \in \mathfrak{h}$ if $x(0)v = (x|\lambda)v$ for all $x \in \mathfrak{h}$. Remark that for $h \in \mathfrak{h}$, $\sigma_h$ acts on a vector with the weight $\lambda$ as the scalar multiple by $\exp(-2\pi\sqrt{-1} h(0)|\lambda))$. The following lemma is immediate from the commutator relations of $n$-th modes (cf. [DM06 (3.2)])

**Proposition 2.1.** Let $\mathfrak{s}$ be a simple Lie subalgebra of $V_1$ with level $k$. Then $\langle \cdot | \cdot \rangle = k(\cdot | \cdot)$ on $\mathfrak{s}$.

Next we recall some results related to the Lie algebra $V_1$.

**Proposition 2.2** ([DM06, Theorem 1.1, Corollary 4.3]). Let $V$ be a strongly regular VOA. Then $V_1$ is reductive. Let $\mathfrak{s}$ be a simple Lie subalgebra of $V_1$. Then $V$ is an integrable module for the affine representation of $\mathfrak{s}$ on $V$, and the subVOA generated by $\mathfrak{s}$ is isomorphic to the simple affine VOA associated with $\mathfrak{s}$ at positive integral level.

**Proposition 2.3** ([DM04a (1.1), Theorem 3 and Proposition 4.1]). Let $V$ be a strongly regular holomorphic VOA of central charge 24. If the Lie algebra $V_1$ is neither $\{0\}$ nor abelian, then $V_1$ is semisimple, and the conformal vectors of $V$ and the subVOA generated by $V_1$ are the same. In addition, for any simple ideal of $V_1$ at level $k$, the identity

$$\frac{h^\vee}{k} = \frac{\dim V_1 - 24}{24}$$
holds, where $h^\vee$ is the dual Coxeter number.

**Proposition 2.4** ([DM04b, Corollary 1.4]). Let $V$ be a strongly regular holomorphic VOA of central charge 24. If the Lie rank of $V_1$ is 24, then $V$ is isomorphic to a Niemeier lattice VOA.

### 2.2. Δ-operator, simple affine VOAs and twisted modules

In this subsection, we recall the construction of certain twisted modules given by Li [Li96]. We also discuss the lowest $L(0)$-weight of such a twisted module over a simple affine VOA.

Let $V$ be a VOA of CFT-type. Let $h \in V_1$ such that $L(1)h = 0$ and $h_{(0)}$ acts semisimply on $V$. Note that if $V$ is self-dual, then $L(1)V_1 = 0$. Suppose that there exists a positive integer $n \in \mathbb{Z}_{>0}$ such that spectra of $h_{(0)}$ on $V$ belong to $(1/n)\mathbb{Z}$. Then $\sigma_h = \exp(-2\pi \sqrt{-1} h_{(0)})$ is an automorphism of $V$ with $\sigma_h^n = 1$.

Let $\Delta(h, z)$ be Li’s $\Delta$-operator defined in [Li96], i.e.,

$$\Delta(h, z) = z^h_{(0)} \exp \left( \sum_{i=1}^{\infty} \frac{h_{(i)}}{-i} (z)^{-i} \right).$$

**Proposition 2.5** ([Li96, Proposition 5.4]). Let $\sigma$ be an automorphism of $V$ of finite order and let $h \in V_1$ be as above such that $\sigma(h) = h$. Let $(M, Y_M)$ be a $\sigma$-twisted $V$-module and define $(M(h), Y_M(h)(\cdot, z))$ as follows:

- $M(h) = M$ as a vector space;
- $Y_M(h)(a, z) = Y_M(\Delta(h, z)a, z)$ for any $a \in V$.

Then $(M(h), Y_M(h)(\cdot, z))$ is a $\sigma_h\sigma$-twisted $V$-module. Furthermore, if $M$ is irreducible, then so is $M(h)$.

For a $\sigma$-twisted $V$-module $M$ and $a \in V$, we denote by $a_{(i)}^{(h)}$ the operator which corresponds to the coefficient of $z^{-i-1}$ in $Y_M(h)(a, z)$, i.e.,

$$Y_M(h)(a, z) = \sum_{i \in (1/n)\mathbb{Z}} a_{(i)}^{(h)} z^{-i-1} \quad \text{for} \ a \in V.$$

Assume that $V$ is self-dual. Set $L(h)(i) = \omega_{(i+1)}^{(h)}$. Then the $L(0)$-weights on $M(h)$ are given by

$$L(h)(0) = L(0) + h_{(0)} + \frac{\langle h|h \rangle}{2} \text{id}. \quad (2.1)$$

The following lemma is immediate from the equation above.

**Lemma 2.6.** Let $M$ be a $\sigma$-twisted $V$-module whose $L(0)$-weights belong to $\mathbb{Z}/3$. Let $h \in V_1$ such that $\sigma(h) = h$, $\langle h|h \rangle \in (2/3)\mathbb{Z}$, and $h_{(0)}$ is semisimple on $M$. Assume that the spectra of $h_{(0)}$ on $M$ belong to $\mathbb{Z}/3$. Then the $L(0)$-weights of the $\sigma_h\sigma$-twisted $V$-module $M(h)$ also belong to $\mathbb{Z}/3$. 

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For a simple Lie algebra \( \mathfrak{a} \) and a positive integer \( k \), let \( L_\mathfrak{a}(k,0) \) be the simple affine VOA associated with \( \mathfrak{a} \) at level \( k \). The following lemma was proved in [LS16a, Lemma 3.6] for a simple Lie algebra. One can easily generalize it to a semisimple Lie algebra.

**Lemma 2.7.** (cf. [LS16a, Lemma 3.6]) Let \( \mathfrak{g} \) be a semisimple Lie algebra and let \( \mathfrak{g} = \bigoplus_{i=1}^{t} \mathfrak{g}_i \) be the decomposition of \( \mathfrak{g} \) into the direct sum of simple ideals \( \mathfrak{g}_i \). Let \( k_i \) be a positive integer and let \( \lambda_i \) be a dominant integral weight of \( \mathfrak{g}_i \) (with respect to a fixed Cartan subalgebra). Let \( L_{\mathfrak{g}_i}(k_i,\lambda_i) \) be the irreducible \( L_{\mathfrak{g}_i}(k_i,0) \)-module with highest weight \( \lambda_i \). Let \( h \) be an element in the Cartan subalgebra such that

\[
(h|\alpha) \geq -1
\]

for any root \( \alpha \) of \( \mathfrak{g} \). Then the lowest \( L(0) \)-weight of \( \bigotimes_{i=1}^{t} L_{\mathfrak{g}_i}(k_i,\lambda_i)^{(h)} \) is equal to

\[
\ell + \sum_{i=1}^{t} \min\{(h_i|\mu) \mid \mu \in \Pi(\lambda_i)\} + \frac{\langle h|h \rangle}{2},
\]

where \( \ell \) is the lowest \( L(0) \)-weight of \( \bigotimes_{i=1}^{t} L_{\mathfrak{g}_i}(k_i,\lambda_i) \) and \( \Pi(\lambda_i) \) is the set of all weights of the irreducible \( \mathfrak{g}_i \)-module with the highest weight \( \lambda_i \).

3. **Conjugacy classes of the automorphism group of a lattice VOA**

In this section, we discuss the conjugacy classes of certain order 3 automorphisms of Niemeier lattice VOAs.

3.1. **Conjugation under the inner automorphism group of a simple Lie algebra.**

We first recall from [Ka90] some results about conjugacy classes of the automorphism group of a simple Lie algebra and study certain order 3 automorphisms.

**Proposition 3.1.** ([Ka90] Proposition 8.1) Let \( \mathfrak{g} \) be a simple finite-dimensional Lie algebra. Let \( \mathfrak{h} \) be its Cartan subalgebra and let \( \Pi = \{\alpha_1, \ldots, \alpha_N\} \) be a set of simple roots. Let \( \sigma \in \text{Aut}(\mathfrak{g}) \) be such that \( \sigma^m = 1 \). Then \( \sigma \) is conjugate to an automorphism of \( \mathfrak{g} \) of the form

\[
\exp\left(\frac{2\pi \sqrt{-1}}{m} h\right) \mu, \quad h \in \mathfrak{h}^\sigma,
\]

where \( \mu \) is (the standard lift of) a diagram automorphism preserving \( \mathfrak{h} \) and \( \Pi \), \( \mathfrak{h}^\sigma \) is the fixed-point set of \( \mu \) in \( \mathfrak{h} \), and \( (\alpha_i|h) \in \mathbb{Z}, \ i = 1, \ldots, N \).

**Remark 3.2.** In Proposition 3.1, \( \mu \) commutes with \( \exp(\text{ad}\frac{2\pi \sqrt{-1}}{m} h) \). In addition, since Cartan subalgebras of \( \mathfrak{g} \) are conjugate under the inner automorphism group, \( \sigma \) is conjugate to \( \exp(\text{ad}\frac{2\pi \sqrt{-1}}{m} h)\mu \) under the inner automorphism group (see the proof of [Ka90 Proposition 8.1]).
Lemma 3.3. Let \( g \) be a simple Lie algebra of type \( E_6 \) and let \( h \) be a Cartan subalgebra. Let \( \alpha_1, \alpha_2, \ldots, \alpha_6 \in h \) be simple roots of \( E_6 \) such that \( (\alpha_i|\alpha_i) = 2 \) for \( 1 \leq i \leq 6 \), \( (\alpha_i|\alpha_j) = -1 \) if two nodes \( \alpha_i \) and \( \alpha_j \), \( 1 \leq i \neq j \leq 6 \), are joined in the diagram \( B_4 \) and \( (\alpha_i|\alpha_j) = 0 \) otherwise. Let \( \Lambda_1, \ldots, \Lambda_6 \in h \) be the fundamental weights such that \( (\alpha_i|\Lambda_j) = \delta_{i,j} \). Let \( g \) be an order 3 automorphism of \( g \) such that the Lie algebra structure of the fixed-point subspace \( g^a \) of \( g \) is \( A_2^3 \). Then \( g \) is conjugate to \( \exp \left( \text{ad}_{\frac{2\pi\sqrt{-1}}{3}}\Lambda_4 \right) \) under the inner automorphism group.

\[
\begin{array}{ccccccc}
\alpha_1 & - & \alpha_3 & - & \alpha_4 & - & \alpha_6 \\
(3.1)
\end{array}
\]

Proof. By Proposition 3.1 and Remark 3.2, we may assume that \( g = \exp \left( \text{ad}_{\frac{2\pi\sqrt{-1}}{3}}h \right) \mu \) for some \( h \) and \( \mu \), satisfying the conditions of Proposition 3.1. Since the diagram automorphism group of the root system of type \( E_6 \) is a cyclic group of order 2 and the order of \( g \) is 3, we have \( \mu = 1 \). Set \( L = \bigoplus_{i=1}^{6} \mathbb{Z}\alpha_i \) and \( P = \bigoplus_{i=1}^{6} \mathbb{Z}\Lambda_i \). Then \( h \in P \). By the assumption, the Lie algebra structure of \( g^a \) is \( A_2^3 \). Since \( g \) acts identically on \( h \), \( h \) is also a Cartan subalgebra of \( g^a \). Hence the sublattice \( \{ \alpha \in L : (\alpha|h) \in 3\mathbb{Z} \} \) of \( L \) is isomorphic to the root lattice \( A_2^3 \). Here, we use the fact that the Weyl group \( W(L) \cong U_4(2) \) of \( L \) acts transitively on the set \( S \) of sublattices of \( L \) isomorphic to \( A_2^3 \). Indeed, the stabilizer \( W(L)_X \) of an element \( X \in S \) in \( W(L) \) is of shape \( 3^3:S_4 \) ([CCNPW85, p26]) and the cardinality of \( S \) is 40, which is equal to \( |W(L)/W(L)_X| \). Since a lift of any element in \( W(L) \) is inner in \( \text{Aut}(g) \), we may assume that the sublattice of \( L \) corresponding to \( g^a \) is \( \mathbb{Z}\langle \alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6, \theta \rangle \) up to conjugation by inner automorphisms, where \( \theta \) is the highest root. Then \( h = \pm \Lambda_4 \) modulo \( 3P \). Remark that \( \exp \left( \text{ad}_{\frac{2\pi\sqrt{-1}}{3}}\Lambda_4 \right) \) and \( \exp \left( \text{ad}_{\frac{-2\pi\sqrt{-1}}{3}}\Lambda_4 \right) \) are conjugate under the inner automorphism group. Indeed, there exists an element in \( W(L) \) which sends \( \{ \alpha_i : 1 \leq i \leq 6 \} \) to \( \{ -\alpha_i : 1 \leq i \leq 6 \} \) ([Hum72, Excercise 10.9]), and hence it also sends \( \Lambda_4 \) to \( -\Lambda_4 \). Thus we have proved this lemma.

Lemma 3.4. Let \( g \) be a simple Lie algebra of type \( D_4 \) and let \( h \) be a Cartan subalgebra. Let \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in h \) be simple roots of \( D_4 \) such that \( (\alpha_i|\alpha_i) = 2 \) for \( 1 \leq i \leq 4 \), \( (\alpha_i|\alpha_j) = -1 \) if two nodes \( \alpha_i \) and \( \alpha_j \), \( 1 \leq i \neq j \leq 4 \), are joined in the diagram \( B_3 \) and \( (\alpha_i|\alpha_j) = 0 \) otherwise. Let \( \Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4 \in h \) denote the fundamental weights such that \( (\alpha_i|\Lambda_j) = \delta_{i,j} \). Let \( \tau \) be the standard lift of the diagram automorphism of \( g \) of order 3 such that \( \tau(\alpha_1) = \alpha_3 \), \( \tau(\alpha_3) = \alpha_4 \), \( \tau(\alpha_4) = \alpha_1 \) and \( \tau(\alpha_2) = \alpha_2 \). Let \( g \) be an order 3 automorphism of \( g \).

(1) If the Lie algebra structure of the fixed-point subspace \( g^a \) of \( g \) is \( A_2 \), then \( g \) is conjugate to \( \exp \left( \text{ad}_{\frac{2\pi\sqrt{-1}}{3}}\Lambda_2 \right) \tau \) or \( \exp \left( \text{ad}_{\frac{2\pi\sqrt{-1}}{3}}\Lambda_2 \right) \tau^{-1} \) under the inner automorphism group.
(2) If the Lie algebra structure of the fixed-point subspace $g^\theta$ of $g$ is $A_3^3 U(1)$, then $g$ is conjugate to $\exp \left( \text{ad} \frac{2\pi \sqrt{-1}}{3} \Lambda_2 \right)$ under the inner automorphism group.

(3.2)

Proof. By Proposition 3.1 and Remark 3.2 we may assume $g = \exp \left( \text{ad} \frac{2\pi \sqrt{-1}}{3} h \right) \mu$ for some $h$ and $\mu$ satisfying the conditions in Proposition 3.1.

(1) If $\mu = 1$, then $h$ is also a Cartan subalgebra of $g^\theta$, and the Lie rank of $g^\theta$ is 4, which is a contradiction. Hence $\mu \neq 1$. Since $g$ is of order 3, so is $\mu$. Hence we may assume that $\mu = \tau$ (or $\tau^{-1}$) since the difference of $\mu$ and $\tau$ (or $\tau^{-1}$) is an inner automorphism. We treat the case where $\mu = \tau$ only since the case $\mu = \tau^{-1}$ is similar.

By the assumption on $h$, we have $h \in \mathbb{Z}\Lambda_2 \oplus \mathbb{Z}(\Lambda_1 + \Lambda_3 + \Lambda_4) = \mathfrak{h}^\tau \cap P$, where $P = \bigoplus_{i=1}^4 \mathbb{Z}\Lambda_i$. Since the $-1$-isometry belongs to the Weyl group, we may assume that $h \in \{0, \Lambda_2, \Lambda_1 + \Lambda_3 + \Lambda_4, \Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4, 2\Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4 \}$ modulo $3P$. If $h = 0$ or $\Lambda_1 + \Lambda_3 + \Lambda_4$, then the Lie algebra structure of $g^\theta$ is $G_2$, which is a contradiction. One can easily check that the other automorphisms $\{ \exp \left( \text{ad} \frac{2\pi \sqrt{-1}}{3} \tau \right) h \mid h \in \{ \Lambda_2, \Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4, 2\Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4 \} \}$ are conjugate under the inner automorphism group. Indeed, letting $f = \exp \left( \text{ad} \frac{2\pi \sqrt{-1}}{3} (-\Lambda_1 + \Lambda_3) \right)$, we have

$$f \exp \left( \text{ad} \frac{2\pi \sqrt{-1}}{3} (\Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4) \right) \tau f^{-1} = \exp \left( \text{ad} \frac{2\pi \sqrt{-1}}{3} \Lambda_2 \right) \tau,$$

$$f^{-1} \exp \left( \text{ad} \frac{2\pi \sqrt{-1}}{3} (2\Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4) \right) \tau f = \exp \left( \text{ad} \frac{2\pi \sqrt{-1}}{3} \Lambda_2 \right) \tau.$$

Hence we obtain (1).

(2) If $\mu \neq 1$, then the Lie algebra structure of $g^\theta$ is $A_2$ or $G_2$ by the argument of (1). Hence $\mu = 1$. Set $L = \bigoplus_{i=1}^4 \mathbb{Z}\alpha_i$ and $P = \bigoplus_{i=1}^4 \mathbb{Z}\Lambda_i$. Then $h \in P$. Remark that the Weyl group $W(L)$ of $L$ acts transitively on the set of sublattices of $L$ isomorphic to $A_3^3$. Indeed, such a sublattice can be realized as $\{ v \in L \mid (\alpha|v) = 0 \}$ for some root $\alpha \in L$, and $W(L)$ acts transitively on the set of all roots of $L$. Since any lift of an element in $W(L)$ is inner in $\text{Aut}(g)$, we may assume that the sublattice of $L$ corresponding to $g^\theta$ is $\mathbb{Z}(\alpha_1, \alpha_3, \alpha_4, \theta)$ up to conjugation by inner automorphisms, where $\theta$ is the highest root. Hence we have $h = \pm \Lambda_2$ modulo $3P$. Considering the conjugation by (a lift of) the $-1$-isometry if necessary, we have $h = \Lambda_2$. 

\[ \square \]
Remark 3.5. The conjugacy classes in Aut(\(g\)) of automorphisms of \(g\) of finite order are classified in [Ka90, Theorem 8.6]. One can prove Lemmas 3.3 and 3.4 by using a similar arguments as in [Ka90, Theorem 8.6].

Lemma 3.6. Let \(g_1, g_2, g_3\) be isomorphic reductive Lie algebras and set \(g = g_1 \oplus g_2 \oplus g_3\). Let \(g\) be an automorphism of \(g\) such that \(g(g_1) = g_2, g(g_2) = g_3\) and \(g(g_3) = g_1\). Let \(f\) be an inner automorphism of \(g\). If the order of \(fg\) is 3, then there exists an inner automorphism \(x\) of \(g\) such that \(xgx^{-1} = fg\).

Proof. For \(1 \leq i \leq 3\), let \(f_i\) be the inner automorphism of \(g_i\) such that \(f = f_1f_2f_3\). It follows from \((fg)^3 = 1\) and \(g^3 = 1\) that

\[
(f_1f_3)^{g^2} f_2^g = f_2f_1^{g^2} f_3^g = f_3f_2^{g^2} f_1^g = 1,
\]

where \(a^b = b^{-1}ab\). Set \(x = f_1f_2f_3 f_1^{g^2}\). Then one can directly check that \(xgx^{-1} = fg\). \(\square\)

3.2. Conjugation under the inner automorphism group of a VOA. In this subsection, we prove some general results on conjugation under the inner automorphism group of a VOA.

Lemma 3.7. Let \(V\) be a VOA of CFT-type. Assume that \(V_1\) is reductive and let \(h\) be a Cartan subalgebra of \(V_1\). Let \(g\) be an order 3 automorphism of \(V\) such that \(g(h) = h\). Let \(u \in h\) such that \(\sum_{i=0}^2 g^i(u) = 0\). Then \(\sigma_u g\) is conjugate to \(g\) under the inner automorphism group.

Proof. Since \(g\sigma_u = \sigma_g(u)g\) and \(\sigma_u^{-1} = \sigma_{-u}\), we have

\[
\sigma_{\frac{1}{2}(g^2(u)-u)} \sigma_u g(\sigma_{\frac{1}{2}(g^2(u)-u)})^{-1} = \sigma_{\frac{1}{2}(g^2(u)-u)+u+\frac{1}{2}(-u+g(u))} g = \sigma_{\frac{1}{2}(u+g(u)+g^2(u))} g = g
\]

as desired. \(\square\)

Proposition 3.8. Let \(V\) be a self-dual VOA of CFT-type and \(g\) an order 3 automorphism of \(V\). Assume that the Lie algebra \(V_1\) is reductive and let \(V_1 = \bigoplus_{i=1}^s g_i \oplus a\) be the decomposition into the direct sum of simple ideals \(g_i\) with level \(k_i\) and the radical \(a\). Then \(g(a) = a\) and \(g\) acts on \(\{g_i : 1 \leq i \leq s\}\) as a permutation. Without loss of generality, we may assume that there exists a non-negative integer \(t\) such that \(3t \leq s\) and

\[
g(g_i) = \begin{cases} g_{i+t} & \text{if } 1 \leq i \leq 2t, \\ g_{i-2t} & \text{if } 2t+1 \leq i \leq 3t, \\ g_i & \text{if } 3t+1 \leq i \leq s.
\end{cases}
\]

Then

\[
V_1^g = \bigoplus_{i=1}^t (g_i \oplus g_{i+t} \oplus g_{i+2t})^g \oplus \bigoplus_{i=3t+1}^s g_i^g \oplus a^g
\]

is a direct sum of ideals. Moreover
(1) For $1 \leq i \leq t$, we must have an equality of the levels $k_i = k_{i+t} = k_{i+2t}$. In addition, the Lie subalgebra $(\mathfrak{g}_i \oplus \mathfrak{g}_{i+t} \oplus \mathfrak{g}_{i+2t})^g$ is a simple ideal of $V_1^g$ isomorphic to $\mathfrak{g}_i$ and its level is $3k_i$.

(2) Let $3t + 1 \leq i \leq s$. Assume that the type of $\mathfrak{g}_i$ is $A_n$, $D_n$ or $E_n$. If the restriction $g|_{\mathfrak{g}_i}$ of $g$ to $\mathfrak{g}_i$ is inner, then the Lie ranks of $\mathfrak{g}_i$ and $\mathfrak{g}_i^g$ are the same, and the level of any simple ideal of $\mathfrak{g}_i^g$ is $k_i$.

(3) Let $3t + 1 \leq i \leq s$. If the restriction $g|_{\mathfrak{g}_i}$ of $g$ to $\mathfrak{g}_i$ is not inner, then $\mathfrak{g}_i$ has the type $D_{4,k_i}$, and $\mathfrak{g}_i^g$ has the type $A_{2,3k_i}$ or $G_{2,k_i}$.

Proof. Since the decomposition of $\mathfrak{g}$ into a direct sum of simple ideals and the radical is unique, $g$ preserves the radical and permutes simple ideals. Hence we obtain the former assertion.

(1) Set $\mathfrak{b} = \mathfrak{g}_i \oplus \mathfrak{g}_{i+k} \oplus \mathfrak{g}_{i+2k}$. Recall that the level $k_i$ of $\mathfrak{g}_i$ is the scalar by which the canonical central element of the affine representation of $\mathfrak{g}_i$ acts on $V$ via vertex operators. Hence the levels of $\mathfrak{g}_i$, $g(\mathfrak{g}_i)$ and $g^2(\mathfrak{g}_i)$ are the same. Clearly, the map $\nu : \mathfrak{g}_i \to \mathfrak{b}^g$, $x \mapsto x + g(x) + g^2(x)$ is an isomorphism of Lie algebras, and $\mathfrak{b}^g$ is an ideal of $\mathfrak{g}^g$. Let $\alpha$ be a long root of $\mathfrak{g}_i \subset \mathfrak{b}$ and let $\langle \cdot | \cdot \rangle_{\mathfrak{b}}$ be the normalized Killing form on $\mathfrak{b}$ such that $\langle \alpha|\alpha\rangle_{\mathfrak{b}} = 2$. Let $\langle \cdot | \cdot \rangle$ be the normalized invariant form on $V$ so that $\langle 1|1\rangle = -1$. Then by Proposition 2.1, $\langle \alpha|\alpha\rangle = k_i \langle \alpha|\alpha\rangle_{\mathfrak{b}} = 2k_i$. By the isomorphism $\nu$, $\nu(\alpha)$ is a long root of $\mathfrak{b}^g$, and $(\nu(\alpha)|\nu(\alpha))_{\mathfrak{b}^g} = 2$, where $\langle \cdot | \cdot \rangle_{\mathfrak{b}^g}$ is the normalized Killing form on $\mathfrak{b}^g$. Let $k$ be the level of $\mathfrak{b}^g$. Then by Proposition 2.1, $\langle \nu(\alpha)|\nu(\alpha)\rangle = k(\nu(\alpha)|\nu(\alpha))_{\mathfrak{b}^g} = 2k$. On the other hand, $\langle \nu(\alpha)|\nu(\alpha)\rangle = 3\langle \alpha|\alpha\rangle = 6k_i$. Hence $k = 3k_i$.

(2) By Proposition 3.1 and the assumption of (2), $g|_{\mathfrak{g}_i}$ is conjugate to $\exp \left( \frac{2\sqrt{3}}{3} h \right)$ for some element $h$ in the weight lattice of $\mathfrak{g}_i$. Hence $\mathfrak{g}_i$ and $\mathfrak{g}_i^g$ share a Cartan subalgebra, and the Lie ranks of $\mathfrak{g}_i$ and $\mathfrak{g}_i^g$ are the same. Let $\mathfrak{c}$ be a simple ideal of $\mathfrak{g}_i^g$. Since $\mathfrak{g}_i$ and $\mathfrak{g}_i^g$ share a Cartan subalgebra and the type of $\mathfrak{g}_i$ is $A_n$, $D_n$ or $E_n$, any (long) root of $\mathfrak{c}$ is also a (long) root of $\mathfrak{g}_i$. By Proposition 2.1, the levels of $\mathfrak{c}$ and $\mathfrak{g}_i$ are the same.

(3) By the assumption, $\mathfrak{g}_i$ has a diagram automorphism of order 3, and hence the type of $\mathfrak{g}_i$ must be $D_4$. In the proof of Lemma 3.4 (1), we have proved that the Lie algebra structure of $\mathfrak{g}_i^g$ is $A_2$ or $G_2$. Let $\alpha_1, \ldots, \alpha_4$ be simple roots of $D_4$ as in Lemma 3.4. If the type of $\mathfrak{g}_i^g$ is $A_2$, then by Lemma 3.4 (1), we may assume that $\alpha_1 + \alpha_3 + \alpha_4$ is a root of $\mathfrak{g}_i^g$ under conjugation by inner automorphisms. Hence the level of $\mathfrak{g}_i^g$ is $3k_i$ by the same argument as in the proof of (1). If the type of $\mathfrak{g}_i^g$ is $G_2$, then we may assume that $\alpha_2$ is a long root of $\mathfrak{g}_i^g$ up to conjugation by inner automorphisms. By Proposition 2.1, the level of $\mathfrak{g}_i^g$ is also $k_i$. □

3.3. Certain order 3 automorphisms of Niemeier lattice VOAs. In this subsection, we characterize certain conjugacy classes of the lattice VOAs associated with the Niemeier lattices $Ni(E_8^4)$ and $Ni(D_8^4)$.
Let $L$ be an even lattice with positive-definite symmetric bilinear form $(\cdot|\cdot)$. Let $L^*$ denote the dual lattice of $L$, namely, $L^* = \{v \in \mathbb{Q} \otimes_{\mathbb{Z}} L \mid (v|L) \subset \mathbb{Z}\}$. Let $Q$ be the root sublattice of $L$, the sublattice generated by norm 2 vectors of $L$. First, we prove some lemmas about lattice VOAs $V_L$, which will be used later. For the explicit construction of lattice VOAs, see [FLM88].

**Lemma 3.9.** Assume that the ranks of $L$ and $Q$ are the same. Then $\{g \in \text{Aut} (V_L) \mid g = 1 \text{ on } (V_L)_1 = \{\sigma_u \mid u \in Q^*/L^*\}\}.

**Proof.** Set $I(V_L) = \{g \in \text{Aut} (V_L) \mid g = 1 \text{ on } (V_L)_1\}$. Note also that $\sigma_u = id_V$ if $u \in L^*$. Since $(V_L)_1 = (V_Q)_1$, we have $I(V_L) \supset \{\sigma_u \mid u \in Q^*/L^*\}.$

Conversely, let $g \in I(V_L)$. By $(V_L)_1 = (V_Q)_1$, we have $g = 1$ on $(V_Q)_1$. Since $V_Q$ is generated by $(V_Q)_1$ as a VOA, $g = 1$ on $V_Q$. Note that $V_L$ is an $L/Q$-graded simple current extension of $V_Q$. By Schur’s lemma, $g$ acts on every irreducible $V_Q$-submodule $V_{\lambda+Q}$ of $V_L$ by a scalar. By the $L/Q$-grading of $V_L$, these scalars form a linear character of $L/Q$. Since $Q^*/L^* \rightarrow \text{Hom}(L/Q, \mathbb{C}^\times), \lambda + L^* \mapsto \sigma_{\lambda}$ is a group isomorphism, we have $g \in \{\sigma_u \mid u \in Q^*/L^*\}.$

Let $\hat{L} = \{\pm e^\alpha \mid \alpha \in L\}$ be the central extension of $L$ by $\langle -1 \rangle$ with the commutator relation $e^\beta e^\alpha = (-1)^{\langle \alpha|\beta \rangle} e^\alpha e^\beta$. Let Aut($\hat{L}$) be the set of all group automorphisms of $\hat{L}$. For $\varphi \in \text{Aut} (\hat{L})$, we define the group automorphism $\tilde{\varphi}$ of $L$ by $\varphi(e^\alpha) \in \{\pm e^{\tilde{\varphi}(\alpha)}\}, \alpha \in L$. Let $O(L)$ be the isometry group of $L$. Set

$$O(\hat{L}) = \{\varphi \in \text{Aut} (\hat{L}) \mid \tilde{\varphi} \in O(L)\}.$$

It was proved in [FLM88, Proposition 5.4.1] that there exists a short exact sequence

$$1 \rightarrow \text{Hom}(L, \mathbb{Z}_2) \rightarrow O(\hat{L}) \rightarrow O(L) \rightarrow 1.$$ (3.3)

Let $g \in O(L)$ be of order $n$. We call $\varphi \in O(\hat{L})$ a lift of $g$ if $\tilde{\varphi} = g$. A lift $\varphi \in O(\hat{L})$ of $g$ is called standard if $\varphi(e^\alpha) = e^\alpha$ for $\alpha \in L^\circ = \{v \in L \mid g(v) = v\}$. Note that a standard lift of $g$ always exists ([Le85]) and its order is $n$ or $2n$ (see [EMS, Mo16] for detail).

Let $V_L$ be the lattice VOA associated with $L$. Then $O(\hat{L})$ is a subgroup of Aut($V_L$) ([FLM88]). Let $K$ be the normal subgroup of Aut($V_L$) generated by inner automorphisms. By [DN99], Aut($V$) $= KO(\hat{L})$. Set $M = K \cap O(\hat{L})$. Since $M$ contains Hom($L, \mathbb{Z}_2$), we obtain Aut($V_L)/K \cong O(\hat{L})/M \cong O(L)/M$.

Let $N$ be a Niemeier lattice, an even unimodular lattice of rank 24, and assume that $N$ is not the Leech lattice. Let $Q$ be the root sublattice of $N$ and let $W \subset O(N)$ be the Weyl group of $Q$. Note that the ranks of $Q$ and $N$ are the same. It follows from [CS99] that there exists a subgroup $H$ of $O(N)$ such that $O(N) = W:H$. Note that $H$ is isomorphic to the automorphism group of the glue code $N/Q$. 

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Lemma 3.10. Let $N$ be a Niemeier lattice and let $Q$ be the root sublattice of $N$. Let $W$ be the Weyl group of $Q$. Then $\overline{M} = W$, or equivalently, $\text{Aut}(V_N)/K \cong O(N)/W \cong H$.

Proof. Since a lift of a reflection of $Q$ belongs to $K$ (cf. [Hum72]), we have $\overline{M} \supset W$.

Let $g_0 \in \overline{M}$ and let $g \in M$ such that $\bar{g} = g_0$ as defined in [3.3]. It follows from $g \in K$ that $g$ preserves every $V_Q$-submodule of $V_N$. On the other hand, the subgroup of $O(N)$ that preserves every coset in $N/Q$ is exactly $W$. Note that for $f \in O(\tilde{N})$ with $\tilde{f}(Q) = Q$, the map $f$ sends an irreducible $V_Q$-submodule $V_{\lambda+Q}$ of $V_N$ to $V_{\tilde{f}(\lambda)+Q}$. Hence $g$ is a lift of an element in $W$, that is, $g_0 \in W$. Thus $\overline{M} \subset W$. \hfill $\square$

3.3.1. Niemeier lattice with root lattice $E_6^4$. Let $N = Ni(E_6^4)$ be a Niemeier lattice with root sublattice $Q \cong E_6^4$. We now fix the glue code ([1012], [1120], [1201]) of $N$ as in [CS99] (see also [SS16, Appendix A.1]). Note that $E_6^4/E_6 \cong \mathbb{Z}_3$ and that $N/Q \cong \mathbb{Z}_3^2$.

Let $\varphi$ be an order 3 fixed-point-free isometry of the root lattice $E_6$. Let $\sigma_6$ be the order 3 isometry of $N$ defined by

\begin{equation}
\sigma_6 : (\gamma_1, \gamma_2, \gamma_3, \gamma_4) \mapsto (\varphi(\gamma_1), \gamma_4, \gamma_2, \gamma_3).
\end{equation}

Let $\sigma_6 \in O(\tilde{N})$ be a standard lift of $\bar{\sigma}_6$ (cf. [SS16 Appendix A] and [M13]). Note that the order of $\sigma_6$ is also 3 and that the fixed-point weight 1 subspace $(V_N^{\sigma_6})_1$ of $\sigma_6$ in $(V_N)_1$ has the Lie algebra structure $E_{6,3}A_{2,1}^3$ (see [M13]).

Proposition 3.11. Let $g$ be an order 3 automorphism of $V_N$. Assume the following:

1. the Lie algebra structure of the fixed-point weight 1 subspace $(V_N^{\sigma_6})_1$ is $E_{6,3}A_{2,1}^3$;
2. the weight one space of the irreducible $g$-twisted $V_N$-module is non-zero.

Then $g$ is conjugate to $\sigma_6$ by an element of $\text{Aut}(V_N)$.

Proof. Let $H$ be the automorphism group of the glue code $N/Q$ and let $K$ be the inner automorphism group of $V_N$. Since $H$ has the shape $2.S_4$, it has exactly one conjugacy class of order 3. By Lemma 3.10, there exists a group homomorphism $p : \text{Aut}(V_N) \to H \cong 2.S_4$ with the kernel $K$. By (3.4), the assumption (1) and Proposition 3.8 $p(g)$ is conjugate to $p(\sigma_6)$ by an element of $H$. Hence, we may assume that

\begin{equation}
g = x\sigma_6, \quad x \in K,
\end{equation}

at the beginning of the proof.

Let $(V_N)_1 \cong \bigoplus_{i=1}^4 g_i$ be the decomposition of $(V_N)_1$ into the direct sum of simple ideals, where $g_i$ is a simple Lie algebra of type $E_6$ at level 1. Note that every $g_i$ is preserved by elements of $K$. By (3.4) and (3.5), we have $g(g_1) = g_1, g(g_2) = g_3, g(g_3) = g_4$ and $g(g_4) = g_2$. Let $K_i$ be the subgroup of $\text{Aut}(V_N)$ generated by inner automorphisms of $g_i$. By Lemma 3.6 we may assume that $x\sigma_6 = \sigma_6$ on $\bigoplus_{i=2}^4 g_i$ up to conjugation by an element of $K_2K_3K_4$. Note that both the Lie algebra structures of the fixed-point
subspaces of $\sigma_6$ and $x\sigma_6$ in $\mathfrak{g}_1$ are $A_2^6$. By Lemma 3.3, we may assume that $x\sigma_6 = \sigma_6$ on $\mathfrak{g}_1$ up to conjugation by an element of $K_1$. Hence we have $x\sigma_6 = \sigma_6$ on $(V_N)_1$. By Lemma 3.9, $x = \sigma_u$ for some element $u \in Q^*/N(\cong \mathbb{Z}_2^6)$. Let $P$ be the orthogonal projection $P : \mathfrak{h} \to \{a \in \mathfrak{h} | \sigma_6(a) = a\}$ and set $u' = P(u)$. Then $\sum_{i=0}^{2} g^i(u - u') = 0$, and by Lemma 3.7 $x\sigma_6$ is conjugate to $\sigma_u\sigma_6$ by an element of $K$.

Recall that $Q^*/N$ is generated by [1000] and [0100] and that $P([1000]) = 0$ and $P([0100]) = (1/3)[0111]$. Hence we may assume $u' = 0$ or $u' = \pm(1/3)[0111]$.

We will show that $u' = 0$. Suppose, for a contradiction, that $u' = \pm(1/3)[0111]$. Then $\langle u'|u' \rangle = (u'|u') = 4/9$. Note that $\sigma_6(u') = u'$. Let $V_N[\sigma_6]$ be the irreducible $\sigma_6$-twisted $V_N$-module. It follows from $(V_N[\sigma_6])_1 \neq 0$ (cf. [Mi13, SS16]) that $L(0)$-weights of $V_N[\sigma_6]$ belong to $(1/3)\mathbb{Z}$. Since $(P(N)u') \subset (1/3)\mathbb{Z}$, the spectrum of $u'_0$ on $V_N[\sigma_6]$ belongs to $(1/3)\mathbb{Z}$ (cf. [SS16] (2.2.8)). Hence by (2.1), the $L(0)$-weights of irreducible $\sigma_\psi\sigma_6$-twisted $V_N$-module belong to $2/9 + (1/3)\mathbb{Z}$, which contradicts the assumption (2). Thus $u' = 0$ and we obtain this proposition.

3.3.2. Niemeier lattice with root lattice $D_4^6$. Let $N = \text{Ni}(D_4^6)$ be a Niemeier lattice with root sublattice $Q \cong D_4^6$. We now fix the glue code $N/Q$ generated by

$$[111111], [222222], [002332], [023230], [032023], [020233]$$

as in [CS99] (see also [SS16, Section 4.2]). Note that $D_4^*/D_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $N/Q \cong \mathbb{Z}_2^6$. Let $\sigma_2$ and $\sigma_4$ be order 3 isometries of $N$ defined by

$$\sigma_2 : (\gamma_1, \gamma_2, \ldots, \gamma_6) \mapsto (\varphi(\gamma_1), \varphi(\gamma_2), \ldots, \varphi(\gamma_6)),\tag{3.6}$$

$$\sigma_4 : (\gamma_1, \gamma_2, \ldots, \gamma_6) \mapsto (\psi(\gamma_1), \varphi(\gamma_2), \varphi^{-1}(\gamma_3), \gamma_6, \varphi^{-1}(\gamma_4), \psi(\gamma_5)),\tag{3.7}$$

where $\varphi$ is an order 3 fixed-point-free isometry of $D_4$ (not in the Weyl group) and $\psi$ is an order 3 isometry of $D_4$ in the Weyl group. Let $\sigma_2, \sigma_4 \in O(\hat{N})$ be standard lifts of isometries $\bar{\sigma}_2, \bar{\sigma}_4$ (cf. [SS16, Section 4.2] and [SS16, Section 6.2]), respectively. Note that $\sigma_2$ and $\sigma_4$ have also order 3 and that the fixed-point subspaces $(V_N^{\sigma_2})_1$ and $(V_N^{\sigma_4})_1$ have the Lie algebra structures $A_{2,3}^6$ and $A_{2,3}^2D_{4,3}A_{1,1}^3U(1)$, respectively.

**Proposition 3.12.** Let $g$ be an order 3 automorphism of $V_N$.  

1. If the Lie algebra structure of $(V_N^{\sigma_2})_1$ is $A_{2,3}^6$, then $g$ is conjugate to $\sigma_2$.  
2. If the Lie algebra structure of $(V_N^{\sigma_4})_1$ is $A_{2,3}^2D_{4,3}A_{1,1}^3U(1)$, then $g$ is conjugate to $\sigma_4$.

**Proof.** Let $H$ be the automorphism group of the glue code $N/Q$. Note that $H$ has the shape $3.S_6$ and that it has exactly three conjugacy classes of order 3, which are represented by an element in the center and elements with cycle shapes $3^1$ and $3^2$ (see [CCNPW85, p4]). Let $K$ be the inner automorphism group of $V_N$. By Lemma 3.10 there exists a group homomorphism $p : \text{Aut}(V_N) \to H \cong 3.S_6$ with the kernel $K$. Let $(V_N)_1 \cong \bigoplus_{i=1}^{6} g_i$.
be the decomposition of \((V_N)_1\) into the direct sum of simple ideals, where \(g_i\) is a simple Lie algebra of type \(D_4\). Note that every \(g_i\) is preserved by the action of \(K\). Let \(K_i\) be the subgroup of \(\text{Aut} (V_N)\) generated by inner automorphisms of \(g_i\).

(1) By the Lie algebra structure of \((V_N)_1\), and Proposition 3.8, we have \(g(\sigma_i) = g_i\) for \(1 \leq i \leq 6\). By the definition of \(\tilde{\sigma}_2\) (see (3.6)), \(\sigma_2\) also stabilizes all \(g_i\) and hence \(p(g)\) is conjugate in \(H\) to \(p(\sigma_2)\), an element in the center of \(H\). Hence, we may assume that

\[
g = x\sigma_2, \quad x \in K
\]

(3.8)

at the beginning of the proof. By Lemma 3.4 (1), we may assume that \(x\sigma_2 = \sigma_2\) on \(\bigoplus_{i=1}^{6} g_i\), up to conjugation by an element of \(K\). By Lemma 3.9, \(x = \sigma_u\) for some \(u \in Q^*/N\). Since \(\sigma_2\) is fixed-point-free on the canonical Cartan subalgebra of \((V_N)_1\), we have \(\sum_{i=0}^{2} g^i(u) = 0\). Applying Lemma 3.7 to this case, we show that \(x\sigma_2\) is conjugate to \(\sigma_2\), which proves (1).

(2) By (3.7), the Lie algebra structure of \((V_N')_1\) and Proposition 3.8, \(p(g)\) is conjugate in \(H\) to \(p(\sigma_4)\), an element with cycle shape \(3^1\) in \(H\). Hence, we may assume that

\[
g = x\sigma_4, \quad x \in K
\]

(3.9)

at the beginning of the proof. By (3.7) and (3.9), we have \(g(\sigma_i) = g_i\) for \(1 \leq i \leq 3\), \(g(g_4) = g_5\), \(g(g_5) = g_6\) and \(g(g_6) = g_4\). By Lemma 3.6, we may assume that \(x\sigma_4 = \sigma_4\) on \(\bigoplus_{i=4}^{6} g_i\) up to conjugation by an element of \(K_4K_5K_6\). By Lemma 3.4 (1) and (2), we also may assume that \(x\sigma_4 = \sigma_4\) on \(\bigoplus_{i=1}^{3} g_i\) up to conjugation by an element of \(K_1K_2K_3\). Hence \(x\sigma_4 = \sigma_4\) on \((V_N)_1\). By Lemma 3.9, \(x = \sigma_u\), for some \(u \in Q^*/N\). Since \(Q^*/N \cong \mathbb{Z}_2^6\), the order of \(\sigma_u\) is 1 or 2. Let us show that \(\sigma_u\sigma_4\) and \(\sigma_4\) are conjugate when \(\sigma_u\) has order 2 (cf. ISS15, Remark 4.2 (2)). Since \(\sigma_u\sigma_4\) has order 3, so does \((\sigma_u\sigma_4)^{-1} = \sigma_4^{-1}\sigma_u\). Hence

\[
(\sigma_u\sigma_4^{-1})(\sigma_4\sigma_u^{-1}) = \sigma_u(\sigma_u^{-1}\sigma_u)^3 = \sigma_u = \sigma_4.
\]

Thus we have the desired result. \(\square\)

4. \(\mathbb{Z}_n\)- orbifold construction and dimension formula

4.1. \(\mathbb{Z}_n\)- orbifold construction. In this subsection, we will review the \(\mathbb{Z}_n\)- orbifold construction associated with a holomorphic VOA and an automorphism of arbitrary finite order from EMS, Mo16.

Let \(V\) be a strongly regular holomorphic VOA. Let \(g\) be an order \(n\) automorphism of \(V\). For \(0 \leq i \leq n-1\), we denote the unique irreducible \(g^i\)-twisted \(V\)-module by \(V[g^i]\) (cf. DLM00, Theorem 1.2). Note that \(V[g^0] = V\). Moreover, there exists an action \(\phi_j : \langle g \rangle \to \text{Aut}_C(V[g^j])\) such that for all \(v \in V\) and \(i \in \mathbb{Z}\),

\[
\phi_j(g^i)Y_{V[g^j]}(v, z)\phi_j(g^i)^{-1} = Y_{V[g^j]}(g^iv, z).
\]
Note that such an action is unique up to a multiplication of an \( n \)-th root of unity. Set 
\[ \phi_0(g) = g \in \text{Aut}(V). \]
For \( 0 \leq j, k \leq n - 1 \), denote 
\[ W^{(j,k)} = \{ w \in V[g^j] \mid \phi_j(g)w = e^{(2\pi k\sqrt{-1})/n}w \}. \]

Let \( V^g \) be the fixed-point subspace of \( g \), which is a full subVOA of \( V \). Note that \( W^{(0,0)} = V^g \) and all \( W^{(j,k)} \)'s are irreducible \( V^g \)-modules (cf. [MT04, Theorem 2]). It was also shown recently in [CM, Mi15] that \( V^g \) is strongly regular. Moreover, any irreducible \( V^g \)-module is a submodule of \( V \) or \( V[g^i] \) for some \( i \), and there exist exactly \( n^2 \) non-isomorphic irreducible \( V^g \)-modules (see [DRX]). By calculating the \( S \)-matrix of \( V^g \), it was proved in [EMS, Mö16] that all irreducible \( V^g \)-modules are simple current modules. It implies that the set of isomorphism classes of irreducible \( V^g \)-modules, denoted by \( R(V^g) \), forms an abelian group of order \( n^2 \) under the fusion product. We often identify an element in \( R(V^g) \) with its representative irreducible \( V^g \)-module. Then \( R(V^g) = \{ W^{(j,k)} \mid 0 \leq j, k \leq n - 1 \} \).

In addition, we assume the following:

(I) For \( 1 \leq i \leq n - 1 \), the lowest \( L(0) \)-weight of \( V[g^i] \) belongs to \( (1/n)\mathbb{Z}_{>0} \).

Under the above assumption, it is proved in [EMS, Mö16] that the abelian group \( R(V^g) \) is isomorphic to \( \mathbb{Z}_n \times \mathbb{Z}_n \). Moreover, one can choose the \( \phi_i \)'s such that
- \( W^{(i,j)} \boxtimes_{V^g} W^{(k,\ell)} \cong W^{(i+k,j+\ell)} \), where \( \boxtimes_{V^g} \) is the fusion product of \( V^g \)-modules;
- the lowest \( L(0) \)-weight of \( W^{(i,j)} \) belongs to \( ji/n + \mathbb{Z} \).

**Theorem 4.1** ([EMS, Mö16]). The \( V^g \)-module
\[ \tilde{V}_g = \bigoplus_{i=0}^{n-1} W^{(i,0)} \]
has a strongly regular holomorphic VOA structure as a \( \mathbb{Z}_n \)-graded simple current extension of \( V^g \).

The construction of \( \tilde{V}_g \) is often called the \( \mathbb{Z}_n \)-orbifold construction associated with \( V \) and \( g \). Note that \( W^{(1,0)} \) is the unique irreducible \( V^g \)-submodule of \( V[g] \) with integral weights and \( \{ W^{(i,0)} \mid 0 \leq i \leq n - 1 \} \) is the subgroup of \( R(V^g) \) generated by \( W^{(1,0)} \). Hence \( \tilde{V}_g \) is uniquely determined by \( V \) and \( g \), up to isomorphism.

**Remark 4.2.** (1) When \( n \) is prime, \( W^{(i,0)} \) is just the subspace of \( V[g^i] \) with integral weights for \( 1 \leq i \leq n - 1 \).
(2) Let \( g' \) be an automorphism of \( V \) which is conjugate to \( g \). Then \( g' \) also satisfies Condition (I), and \( \tilde{V}_{g'} \) is isomorphic to \( \tilde{V}_g \) as a VOA.
(3) For a prime order inner automorphism \( g \), Theorem 4.1 can be proved by using the argument as in [DLM96]; see Appendix A for a proof.
4.2. Montague’s dimension formula. In this subsection, we will prove a dimension formula for the weight one Lie algebra, which was first mentioned in [Mo94].

Notation 4.3. Let $U = \bigoplus_{n=0}^{\infty} U_n$ be a VOA of central charge $c$. Let $f$ be an order 3 automorphism of $U$. Let $W = \bigoplus_{n=0}^{\infty} W_{\lambda+n/3}$ be an irreducible $U$-module or an irreducible $f$-twisted $U$-module, where $\lambda \in \mathbb{C}$ is the lowest $L(0)$-weight of $W$. The character of $W$ is defined by the formal series

$$Z_W(q) = q^{\lambda-c/24} \sum_{n=0}^{\infty} \dim W_n q^{n/3},$$

and the trace function of $f$ on $U$ is defined by the formal series

$$Z_U(f, q) = q^{-c/24} \sum_{n=0}^{\infty} \text{Tr} f|_{U_n} q^n,$$

where $q$ is a formal variable.

Assume that $U$ is strongly regular and take $q$ to be the usual local parameter at infinity in the upper half-plane $\mathbb{H} = \{\tau \in \mathbb{C} | \text{Im}(\tau) > 0\}$, i.e., $q = e^{2\pi \sqrt{-1} \tau}$. Since $U$ is strongly regular, both $Z_W(q)$ and $Z_U(f, q)$ converge to holomorphic functions in $\mathbb{H}$ ([Zh96, Theorem 4.4.1] and [DLM00, Theorem 1.3]). We often denote $Z_W(q)$ and $Z_U(f, q)$ by $Z_W(\tau)$ and $Z_U(f, \tau)$, respectively.

Now let $V$ be a strongly regular holomorphic VOA of central charge 24. Let $g$ be an inner automorphism of $V$ of order 3. Note that $g = \exp(-2\pi \sqrt{-1} h(0))$ for some semisimple element $h \in V_1$. It was proved in [DLM00, Theorem 1.2] that for $0 \leq i \leq 2$, $V$ possesses a unique irreducible $g^i$-twisted $V$-module $V[g^i]$ up to isomorphism.

Let $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ be the standard generators of $SL(2, \mathbb{Z})$. Notice that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ acts on $\mathbb{H}$ by $A : \tau \mapsto \frac{a\tau + b}{c\tau + d}$. By [Zh96, Theorem 5.3.3],

$$Z_V(A\tau) = Z_V(\tau)$$

for all $A \in SL(2, \mathbb{Z})$. Since $g$ is an inner automorphism, we can apply [KMa12, Theorem 1.4] to our setting and obtain

(i) $Z_V(g^i, S\tau) = Z_V[g^i](\tau)$ for $0 \leq i \leq 2$.

Remark 4.4. Formula (i) is also proved for a general automorphism $g$ in [EMS, M616].

In this subsection, we also assume the following for $i = 1, 2$:

(ii) $Z_V[g^i](\tau) \in q^{-2/3} \mathbb{Z}[q^{1/3}]$, i.e., the $L(0)$-weights of $V[g^i]$ belong to $(1/3)\mathbb{Z}_{>0}$.

The next lemma about the congruence subgroup

$$\Gamma_0(3) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \ \bigg| \ c \equiv 0 \pmod{3} \right\}$$
is well-known (e.g. see [Ap90, Theorem 4.3] and [Ha10, Proposition 1.23]).

**Lemma 4.5.** (1) $\Gamma_0(3)$ is generated by $T$ and $ST^3S$.

(2) $\Gamma_0(3)$ has two cusps, and they are represented by $i\infty$ and 0.

Recall that $V^g$, the set of fixed-points of $g$, is a full subVOA of $V$. Using the lemma above, one can prove the following lemma by the same argument as in [LS16a, Proposition 4.1]

**Proposition 4.6.** The character $Z_{V^g}(q)$ of $V^g$ converges to a holomorphic function in $\mathbb{H}$. Moreover, it is a modular function of weight 0 on the congruence subgroup $\Gamma_0(3)$.

Let $V[g^i]_Z$ be the subspace of $V[g^i]$ with integral $L(0)$-weights and set $\tilde{V} = V^g \oplus V[g]_Z \oplus V[g^2]_Z$. Note that

\begin{equation}
Z_{\tilde{V}}(\tau) = Z_{V^g}(\tau) + Z_{V[g]_Z}(\tau) + Z_{V[g^2]_Z}(\tau).
\end{equation}

The following equation was mentioned in [Mo94, (15)]:

**Proposition 4.7.** Let $V$, $\tilde{V}$ and $V^g$ be as above. Then

\[ Z_V(\tau) + Z_{\tilde{V}}(\tau) = Z_{V^g}(\tau) + Z_{V^g}(S\tau) + Z_{V^g}(ST\tau) + Z_{V^g}(ST^2\tau). \]

**Proof.** By definition,

\begin{equation}
Z_{V^g}(\tau) = \frac{1}{3} \left( Z_V(\tau) + Z_V(g, \tau) + Z_V(g^2, \tau) \right).
\end{equation}

Since $V$ is holomorphic, $Z_V(\tau)$ is invariant under the action of $SL(2, \mathbb{Z})$. By [4], we obtain

\begin{equation}
Z_{V^g}(S\tau) = \frac{1}{3} \left( Z_V(\tau) + Z_{V[g]}(\tau) + Z_{V[g^2]}(\tau) \right).
\end{equation}

It follows from $Z_V(T\tau) = Z_V(\tau)$ that for $i = 1, 2$

\begin{equation}
Z_{V^g}(ST^i\tau) = \frac{1}{3} \left( Z_V(\tau) + Z_{V[g]}(T^i\tau) + Z_{V[g^2]}(T^i\tau) \right).
\end{equation}

On the other hand, by [2], for $i = 1, 2$, we have

\begin{equation}
Z_{V[g^i]_Z}(\tau) = \frac{1}{3} \left( Z_{V[g^i]}(\tau) + Z_{V[g^i]}(T\tau) + Z_{V[g^i]}(T^2\tau) \right).
\end{equation}

Thus by [1], [4], [4.4] and [4.5], we have

\[ Z_{V^g}(\tau) + Z_{V^g}(S\tau) + Z_{V^g}(ST\tau) + Z_{V^g}(ST^2\tau) = Z_V(\tau) + Z_{\tilde{V}}(\tau). \]

\[ \square \]

We now prove the following dimension formula described in [Mo94, (73)].
Theorem 4.8. The following equation holds:

\[ \dim V_1 + \dim \tilde{V}_1 = 4 \dim (V^g)_1 - 36(\dim V[g]_{1/3} + \dim V[g^2]_{1/3}) - 12(\dim V[g]_{2/3} + \dim V[g^2]_{2/3}) + 24. \]

Proof. It is well-known (e.g. see [Ha10, Exercise 3.18]) that a Hauptmodul for \( \Gamma_0(3) \) is given by

\[ f(\tau) := \frac{\eta(\tau)^{12}}{\eta(3\tau)^{12}} = q^{-1} - 12 + \left( \frac{12}{2} \right) q + \ldots, \]

where \( \eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \) and \( q = e^{2\pi \sqrt{-1} \tau} \), is the Dedekind eta function. By Lemma 4.6, \( Z_{V^g}(\tau) \) is a modular function of weight 0 on \( \Gamma_0(3) \) and holomorphic in \( \mathbb{H} \). In particular, \( Z_{V^g}(\tau) \) is a rational function of \( f(\tau) \). In addition, since the set of all cusps of \( \Gamma_0(3) \) is \( \{0, i\infty\} \) and \( f(\tau) \to 0 \) as \( \tau \to 0 \) (see (4.6) below), \( Z_{V^g}(\tau) \) is a Laurent polynomial of \( f(\tau) \), i.e.,

\[ Z_{V^g}(\tau) = \sum_{n \in \mathbb{Z}} c_n f^n(\tau), \]

where only finitely many coefficients \( c_n \in \mathbb{C} \) are non-zero. It follows from \( Z_{V^g}(\tau) \in q^{-1} + \mathbb{Z}[[q]] \) that \( c_n = 0 \) if \( n > 1 \), and \( c_1 = 1 \). Since \( \eta(S\tau) = (-i\tau)^{1/2} \eta(\tau) \), we have

(4.6) \( f(S\tau) = 3^6 \frac{\eta(\tau)^{12}}{\eta(\tau/3)^{12}} = 3^6 \left( q^{1/3} + 12q^{2/3} + \ldots \right), \)

(4.7)

\[ f^{-1}(S\tau) = \frac{1}{3^6} \frac{\eta(\tau/3)^{12}}{\eta(\tau)^{12}} = \frac{1}{3^6} \left( q^{-1/3} - 12 + \ldots \right), \]

(4.8)

\[ f^{-2}(S\tau) = \frac{1}{3^{12}} \frac{\eta(\tau/3)^{24}}{\eta(\tau)^{24}} = \frac{1}{3^{12}} \left( q^{-2/3} - 24q^{-1/3} + 12 \cdot 21 + \ldots \right), \]

(4.9)

\[ f^{-3}(S\tau) = \frac{1}{3^{18}} \frac{\eta(\tau/3)^{36}}{\eta(\tau)^{36}} = \frac{1}{3^{18}} \left( q^{-1} - 36q^{-2/3} + 18 \cdot 33q^{-1/3} + 36^2 - \left( \frac{36}{3} \right) + \ldots \right), \]

which shows that \( f^n(S\tau) \in q^{n/3} \mathbb{Q}[[q^{1/3}]] \). By (4.3) and the assumption (iii), we have \( Z_{V^g}(S\tau) \in \frac{1}{3} q^{-1} + q^{-2/3} \mathbb{Z}[[q^{1/3}]] \). Hence \( c_n = 0 \) if \( n < -3 \), and \( c_{-3} = 3^{17} \). Thus

(4.10) \( Z_{V^g}(\tau) = f(\tau) + c_0 + c_{-1} f^{-1}(\tau) + c_{-2} f^{-2}(\tau) + 3^{17} f^{-3}(\tau) = q^{-1} + \dim (V^g)_1 + \ldots \).

Comparing the constant terms of the equation above, we have

(4.11) \( \dim (V^g)_1 = c_0 = 12 \).
By (4.6), (4.7), (4.8) and (4.10), we have
\[
Z_{V^g}(S\tau) = \frac{1}{3} q^{-1} + \left( \frac{c-2}{3 \cdot 12} - 12 \right) q^{-2/3} + \left( \frac{c-1}{3 \cdot 6} - \frac{24c-12}{3 \cdot 12} + \frac{1}{3} 18 \cdot 33 \right) q^{-1/3} \\
+ \left( c_0 - \frac{12c-1}{3 \cdot 6} + \frac{12 \cdot 21c-2}{3 \cdot 12} + \frac{1}{3} 36^2 - \left( \frac{36}{3} \right) \right) \ldots
\]
Hence, comparing the coefficients of \( q^{-2/3} \) and \( q^{-1/3} \) in (4.3), we have
\[
(4.12) \quad \frac{c-2}{3 \cdot 12} - 12 = \frac{1}{3} \left( \dim V[g]_1/3 + \dim V[g^2]_1/3 \right),
\]
\[
(4.13) \quad \frac{c-1}{3 \cdot 6} - \frac{8c-2}{3 \cdot 11} + 6 \cdot 33 = \frac{1}{3} \left( \dim V[g]_2/3 + \dim V[g^2]_2/3 \right).
\]
Comparing the constant terms of both sides of the equation in Proposition 4.7, we obtain the following desired equation:
\[
\dim V_1 + \dim \tilde{V}_1 = \dim (V^g)_1 + 3 \left( c_0 - \frac{4c-1}{3 \cdot 5} + \frac{4 \cdot 7c-2}{3 \cdot 10} + \frac{1}{3} 36^2 - \left( \frac{36}{3} \right) \right) \\
= 4 \dim (V^g)_1 - 36 \left( \dim V[g]_1/3 + \dim V[g^2]_1/3 \right) \\
- 12 \left( \dim V[g]_2/3 + \dim V[g^2]_2/3 \right) + 24.
\]
\[\square\]

Remark 4.9. After we obtained the results in this section, we noticed that a general dimension formula for orbifold construction associated with automorphisms of order 2, 3, 5, 7, and 13 was established in [Mö16].

5. REVERSE ORBIFOLD CONSTRUCTION AND UNIQUENESS OF A HOLOMORPHIC VOA

Let \( V \) be a strongly regular holomorphic VOA and let \( g \) be an order \( n \) automorphism of \( V \) satisfying Condition (I) in Section 4.1. Let \( W = \tilde{V}_g \) be the resulting holomorphic VOA by applying \( \mathbb{Z}_n \)-orbifold construction to \( V \) and \( g \). Let \( f \) be an order \( n \) automorphism of \( W \) associated with the \( \mathbb{Z}_n \)-grading of \( W \). Then \( W^f = V^g \) and, for \( 1 \leq i \leq n - 1 \), the irreducible \( f^i \)-twisted \( W \)-module is a direct sum of irreducible \( V^g \)-modules. Hence \( f \) also satisfies (I). By the uniqueness of the resulting holomorphic VOA by the orbifold construction, we obtain the following:

**Corollary 5.1** (cf. [EMS, Mö16]). The VOA \( \tilde{W}_f \) is isomorphic to \( V \).

Since the VOA \( \tilde{W}_f \cong V \) is essentially obtained by reversing the original orbifold construction, we call this procedure the *reverse orbifold construction*, which is called the inverse orbifold in [EMS, Mö16]. By using this corollary, we can prove the following theorem about the uniqueness of holomorphic VOAs.
Theorem 5.2. Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{p}$ a subalgebra of $\mathfrak{g}$. Let $n \in \mathbb{Z}_{>0}$ and let $W$ be a strongly regular holomorphic VOA of central charge $c$. Assume that for any strongly regular holomorphic VOA $V$ of central charge $c$ whose weight one Lie algebra is $\mathfrak{g}$, there exists an order $n$ automorphism $\sigma$ of $V$ such that the following conditions hold:

(a) $\mathfrak{g}^\sigma \cong \mathfrak{p}$;
(b) $\sigma$ satisfies Condition (I) and $\tilde{V}_\sigma$ is isomorphic to $W$.

In addition, we assume that any automorphism $\varphi \in \text{Aut}(W)$ of order $n$ satisfying (I) and the conditions (A) and (B) below belongs to a unique conjugacy class in $\text{Aut}(W)$:

(A) $(W^\varphi)_1$ is isomorphic to $\mathfrak{p}$;
(B) $(\tilde{W}_\varphi)_1$ is isomorphic to $\mathfrak{g}$.

Then any strongly regular holomorphic VOA of central charge $c$ with weight one Lie algebra $\mathfrak{g}$ is isomorphic to $\tilde{W}_\varphi$. In particular, such a holomorphic VOA is unique up to isomorphism.

Proof. Let $V$ be a strongly regular holomorphic VOA of central charge $c$ such that $V_1 \cong \mathfrak{g}$. By (b), $\tilde{V}_\sigma \cong W$; we identify $\tilde{V}_\sigma$ with $W$. Let $\tau$ be an order $n$ automorphism of $W$ associated with the $\mathbb{Z}_n$-grading of $W$ as the extension of $V^\sigma$. Then $(W^\tau)_1 \cong V_1^\sigma \cong \mathfrak{p}$ by (a). By Corollary 5.1, we obtain $\tilde{W}_\tau \cong V$. Hence $(\tilde{W}_\tau)_1 \cong V_1 = \mathfrak{g}$. Since $\tau$ satisfies (A) and (B), it is conjugate to $\varphi$. Thus $V \cong \tilde{W}_\tau \cong \tilde{W}_\varphi$. □

Remark 5.3. (1) In some cases, the condition (B) can be replaced by a weaker condition for the uniqueness of the conjugacy class of $\varphi$. For example, we will consider the following condition (B') in the later sections:

(B') $\dim W[\varphi]_1 \neq 0$.

(2) Later, we will choose $\sigma$ in Theorem 5.2 as an inner automorphism $\sigma_u$ for some $u$ in a Cartan subalgebra.

6. Uniqueness of a holomorphic VOA with the Lie algebra $E_{6,3}G_{2,1}^3$

In this section, we consider a holomorphic VOA $V$ of central charge 24 whose weight one Lie algebra has the type $E_{6,3}G_{2,1}^3$. By applying the $\mathbb{Z}_3$-orbifold construction to $V$ and certain inner automorphism of order 3, we show that one can obtain a holomorphic lattice VOA of central charge 24 associated with the Niemeier lattice $N(E_6^4)$. In addition, we prove that the structure of a strongly regular holomorphic vertex operator algebra of central charge 24 is uniquely determined by its weight one Lie algebra if the Lie algebra has the type $E_{6,3}G_{2,1}^3$.

6.1. Simple affine VOA of type $G_2$ at level 1. Let $\alpha_1$ and $\alpha_2$ be simple roots of type $G_2$ such that $(\alpha_1|\alpha_1) = 2/3$, $(\alpha_2|\alpha_2) = 2$ and $(\alpha_1|\alpha_2) = -1$. Let $\Lambda_1$ and $\Lambda_2$ be the fundamental weights with respect to $\alpha_1$ and $\alpha_2$, respectively. Note that $2(\Lambda_i|\alpha_j)/(\alpha_j|\alpha_j) = \delta_{i,j}$ for
$i, j \in \{1, 2\}$. Let $L_{g}(1, 0)$ be the simple affine VOA associated with the simple Lie algebra $g$ of type $G_{2}$ at level 1. It is well-known (cf. [FZ92]) that there exist exactly two (non-isomorphic) irreducible $L_{g}(1, 0)$-modules $L_{g}(1, 0)$ and $L_{g}(1, \Lambda_{1})$.

For a dominant integral weight $\lambda$ of $g$, set

$$n_{\Lambda_{1}}(\lambda) = \min\{(\Lambda_{1}|\mu) \mid \mu \in \Pi(\lambda)\}.$$ 

Here $\Pi(\lambda)$ denotes the set of all weights of the irreducible $g$-module with the highest weight $\lambda$. One can easily verify the following lemma.

**Lemma 6.1.** The values of $n_{\Lambda_{1}}(\lambda)$ are given as follows:

| Weight $\lambda$ | Lowest $L(0)$-weight of $L_{g}(1, \lambda)$ | $(\Lambda_{1}|\lambda)$ | $n_{\Lambda_{1}}(\lambda)$ |
|------------------|------------------------------------------|---------------------------|---------------------------|
| 0                | 0                                        | 0                         | 0                         |
| $\Lambda_{1}$    | $2/5$                                    | $2/3$                     | $-2/3$                    |

6.2. **Inner automorphism of a holomorphic VOA with Lie algebra $E_{6,3}G_{2,1}$.** Let $V$ be a strongly regular holomorphic VOA of central charge 24 whose weight one Lie algebra has the type $E_{6,3}G_{2,1}$. Let $V_{1} = \bigoplus_{i=1}^{4} g_{i}$ be the decomposition into the direct sum of simple ideals, where the type of $g_{1}$ is $E_{6,3}$, and the types of $g_{2}$, $g_{3}$ and $g_{4}$ are $G_{2,1}$. Let $\mathfrak{g}$ be a Cartan subalgebra of $V_{1}$. Then $\mathfrak{g} \cap g_{i}$ is a Cartan subalgebra of $g_{i}$. Let $U$ be the subVOA generated by $V_{1}$. Set $k_{1} = 3$ and $k_{i} = 1$ for $i = 2, 3, 4$. By Proposition 2.2 $U \cong \bigotimes_{i=1}^{4} L_{g_{i}}(k_{i}, 0)$.

Let

$$(6.1) \quad h = (0, \Lambda_{1}, \Lambda_{1}, \Lambda_{1}) \in \bigoplus_{i=1}^{4} (\mathfrak{g} \cap g_{i}).$$

Note that

$$(6.2) \quad (h|h) = 3 \times (\Lambda_{1}|\Lambda_{1})_{g_{2}} = 2.$$ 

Since $(\Lambda_{1}|\alpha) \geq -1$ for any root of the simple Lie algebra of type $G_{2}$, we have

$$(6.3) \quad (h|\alpha) \geq -1$$ 

for all roots $\alpha$ of the semisimple Lie algebra $V_{1}$.

**Lemma 6.2.** The order of $\sigma_{h}$ is 3 on $V$, and the $L(0)$-weights of the irreducible $\sigma_{h}$-twisted (resp. $\sigma_{h}^{-1}$-twisted) $V$-module $V^{(h)}$ (resp. $V^{(-h)}$) belong to $\mathbb{Z}/3$.

**Proof.** Since $(\Lambda_{1}|\lambda) \in \mathbb{Z}/3$ for any weight $\lambda$ of the simple Lie algebra of type $G_{2}$, we have $(\sigma_{h})^{3} = 1$. In addition, $\sigma_{h}$ is an automorphism of order 3 on $V_{1}$, so is it on $V$. The latter assertion follows from Lemmas 2.6 and 6.1 and [6.2].

**Proposition 6.3.** The lowest $L(0)$-weight of the irreducible $\sigma_{h}$-twisted (resp. $\sigma_{h}^{-1}$-twisted) $V$-module $V^{(h)}$ (resp. $V^{(-h)}$) is 1. In particular, $(V^{(h)})_{i} = 0$ for $i \in \{1/3, 2/3\}$. 


Proof. Let $M \cong \bigotimes_{i=1}^{4} L_{g_{i}}(k_{i}, \lambda_{i})$ be an irreducible $U$-submodule of $V$. Let $\ell$ and $\ell^{(h)}$ be the lowest $L(0)$-weights of $M$ and of $M^{(h)}$, respectively. It suffices to show that $\ell^{(h)} \geq 1$.

By (6.3), we can apply Lemma 2.7 to our case. By (2.2), (6.1) and (6.2), we have

\[(6.4) \quad \ell^{(h)} = \ell + \sum_{i=2}^{4} n_{\Lambda_{1}}(\lambda_{i}) + 1,\]

where $n_{\Lambda_{1}}(\cdot)$ is defined as in the previous subsection. If $(\lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4}) = (0, 0, 0, 0)$, then $\ell = 0$ and $n_{\Lambda_{1}}(0) = 0$, and hence $\ell^{(h)} = 1$. Assume $(\lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4}) \neq (0, 0, 0, 0)$. Then $\ell \geq 2$ and $n_{\Lambda_{1}}(\lambda_{i}) \geq n_{\Lambda_{1}}(\Lambda_{1}) = -2/3$ by Lemma 6.1. Hence

\[\ell^{(h)} \geq 2 + 3n_{\Lambda_{1}}(\Lambda_{1}) + 1 = 1.\]

One can also prove $\ell^{(-h)} \geq 1$ by the same argument. \hfill \Box

6.3. Identification of the Lie algebra: Case $E_{6,1}^{4}$. Thanks to Proposition 6.3, we can apply Theorem 4.1 to $V$ and $\sigma_{h}$ (see also Theorems 4.3 and 4.4), and obtain a strongly regular holomorphic VOA of central charge 24

\[\tilde{V} = V_{\sigma_{h}} = V_{\sigma_{h}}^{+} \oplus (V^{(h)})_{\mathbb{Z}} \oplus (V^{(-h)})_{\mathbb{Z}},\]

where $(V^{(\pm h)})_{\mathbb{Z}}$ are irreducible $V_{\sigma_{h}}$-submodules of $V^{(\pm h)}$ with integral weights (cf. Remark 4.2 (1)). By the definition of $h$, we obtain the following:

Proposition 6.4. The Lie algebra structure of $(V^{\sigma_{h}})_{1}$ is $E_{6,1}^{4}A_{2,1}^{3}$ and $\dim(V^{\sigma_{h}})_{1} = 102$.

Next we will identify the Lie algebra structure of $\tilde{V}_{1}$.

Proposition 6.5. The Lie algebra structure of $\tilde{V}_{1}$ is $E_{6,1}^{4}$. In particular, $\tilde{V}$ is isomorphic to the lattice VOA associated with the Niemeier lattice $Ni(E_{6}^{4})$.

Proof. By Proposition 2.3 the Lie algebra $\tilde{V}_{1}$ is semisimple. By Proposition 6.4, we have $\dim(V^{\sigma_{h}})_{1} = 102$, and by Proposition 6.3 we have $(V^{(\pm h)})_{i} = 0$ for $i = 1/3, 2/3$. By Theorem 4.8

\[\dim \tilde{V}_{1} = 4 \times \dim(V^{\sigma_{h}})_{1} - \dim V_{1} + 24 = 312;\]

hence we obtain the ratio $h^{\vee}/k = 12$ by Proposition 2.3. Since the level $k$ of any simple ideal is integral, the dual Coxeter number $h^{\vee}$ is a multiple of 12, which shows that possible types of simple ideals of $\tilde{V}_{1}$ are $A_{11,1}, C_{11,1}, D_{7,1}$ and $E_{6,1}$. It follows from $\dim \tilde{V}_{1} = 312$ that the Lie algebra structure of $\tilde{V}_{1}$ is $A_{11,1}D_{7,1}E_{6,1}$ or $E_{6,1}^{4}$. Since $\tilde{V}_{1}$ contains a Lie subalgebra of the type $E_{6,3}$, the Lie algebra structure of $\tilde{V}_{1}$ must be $E_{6,1}^{4}$. By Proposition 2.4, we have proved this proposition. \hfill \Box
6.4. Main theorem for the Lie algebra $E_{6,3}G_{2,1}$. Set $N = Ni(E_6^3)$ and let $\sigma_6$ be the order 3 automorphism of $V_N$ defined in Section 3.3.1. Then $\sigma_6$ satisfies Condition (I) in Section 4.1 and the holomorphic VOA $(\tilde{V}_N)_{\sigma_6}$ has the weight one Lie algebra of type $E_{6,3}G_{2,1}$ ([Mi13, SS16]). Note that the Lie algebra $(V_N^{\sigma_6})_1$ has the type $E_{6,3}A_{2,1}^3$. We now prove the main theorem in this section.

**Theorem 6.6.** Let $V$ be a strongly regular holomorphic VOA of central charge 24 whose weight one Lie algebra has the type $E_{6,3}G_{2,1}$. Then $V$ is isomorphic to the holomorphic VOA $(\tilde{V}_N)_{\sigma_6}$. In particular, the structure of a strongly regular holomorphic vertex operator algebra of central charge 24 is uniquely determined by its weight one Lie algebra if the Lie algebra has the type $E_{6,3}G_{2,1}$.

**Proof.** It suffices to verify the hypotheses in Theorem 5.2 for $g = E_{6,3}G_{2,1}$, $p = E_{6,3}A_{2,1}^3$, $n = 3$ and $W = V_{Ni(E_6^3)}$. Take $h$ as in (6.1) and set $\sigma = \sigma_h$. Then by Lemma 6.2, the order of $\sigma$ is 3. The former hypotheses (a) and (b) hold by Propositions 6.3, 6.4 and 6.5. Note that the eigenspaces of $\sigma$ on $g$ with eigenvalues $e^{\pm 2\pi \sqrt{-1}/3}$ are non-zero. The latter hypotheses about the uniqueness of the conjugacy class follows from Proposition 5.11. We remark that the assumption (2) in Proposition 3.11 (or (B') in Remark 5.3 (1)) is weaker than the assumption (B) in Theorem 5.2. \[\Box\]

7. Uniqueness of a holomorphic VOA with the Lie algebra $A_{6,2}^3$

In this section, we consider a holomorphic VOA $V$ of central charge 24 with the weight one Lie algebra $A_{6,2}^3$. By applying the $\mathbb{Z}_3$-orbifold construction to $V$ and certain inner automorphism of order 3, we show that one can obtain a holomorphic lattice VOA of central charge 24 associated with the Niemeier lattice $Ni(D_6^4)$. In addition, the structure of a strongly regular holomorphic vertex operator algebra of central charge 24 is uniquely determined by its weight one Lie algebra if the Lie algebra has the type $A_{6,2}^3$.

7.1. Lowest weights of irreducible modules over a simple Lie algebra of type $A_n$. Let $\Phi$ be a root system of type $A_n$ and let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be a set of simple roots in $\Phi$ such that $(\alpha_i | \alpha_j) = 2\delta_{i,j} - \delta_{|i-j|,1}$. Let $W$ be the Weyl group of $\Phi$. Let $\tau$ be the Dynkin diagram automorphism of $\Phi$ defined by $\tau(\alpha_i) = \alpha_{n-i+1}$, $1 \leq i \leq n$. Let $\gamma$ denote the $-1$-isometry of $\Phi$, that is, $\gamma(\alpha_i) = -\alpha_i$, $1 \leq i \leq n$. It is well-known (cf. [Hum72, Section 12.2, Exercise 13.5]) that $\tau$ and $\gamma$ do not belong to $W$ and that $\tau \gamma \in W$. Clearly, the order of $\tau \gamma$ is 2.

Let $\mathfrak{g}$ be a simple Lie algebra associated with $\Phi$ and let $\lambda$ be a dominant integral weight of $\Phi$ with respect to $\Delta$. Let $\Pi(\lambda)$ be the set of all weights of the irreducible $\mathfrak{g}$-module with highest weight $\lambda$. Let $\prec$ be the partial order on $\mathbb{R}\Phi(\cong \mathbb{R}^n)$ such that $x \prec y$ if $y - x$
is a non-negative $\mathbb{R}$-linear combination of simple roots. Note that $\mu \prec \lambda$ for any weight $\mu$ in $\Pi(\lambda)$.

**Lemma 7.1.** The lowest weight in $\Pi(\lambda)$ is $\tau\gamma(\lambda)$, that is, $\tau\gamma(\lambda) \prec \mu$ for any weight $\mu$ in $\Pi(\lambda)$.

*Proof.* By the definition of $\tau$ and $\gamma$, for $x, y \in \mathbb{R}\Phi$, $x \prec y$ if and only if $\tau\gamma(y) \prec \tau\gamma(x)$. It follows from $\tau\gamma \in W$ that $\tau\gamma$ stabilizes $\Pi(\lambda)$ (cf. [Hum72, (7) of the proof of Lemma 21.2]). Since $\lambda$ is the highest weight in $\Pi(\lambda)$, we obtain this lemma. \qed

The following corollary is immediate from the lemma above.

**Corollary 7.2.** Let $\Lambda$ be a $\mathbb{R}_{\geq 0}$-linear combination of fundamental weights. Then

$$\min\{(\Lambda|\mu) \mid \mu \in \Pi(\lambda)\} = (\Lambda|\tau\gamma(\lambda)).$$

### 7.2. Simple affine VOA of type $A_2$ at level 3.

Let $\alpha_1, \alpha_2$ be simple roots of type $A_2$ such that $(\alpha_1|\alpha_1) = (\alpha_2|\alpha_2) = 2$ and $(\alpha_1|\alpha_2) = -1$. Let $\Lambda_1$ and $\Lambda_2$ be the set of the fundamental weights with respect to $\alpha_1$ and $\alpha_2$, respectively. Let $L_\varnothing(3,0)$ be the simple affine VOA associated with the simple Lie algebra $\varnothing$ of type $A_2$ at level 3. It is well-known (cf. [FZ92]) that there exist exactly 10 (non-isomorphic) irreducible $L_\varnothing(3,0)$-modules $L_\varnothing(3,\lambda)$ with the highest weight $\lambda$, where $\lambda$ ranges over $\{0, \Lambda_i, 2\Lambda_i, 3\Lambda_i, \Lambda_1 + \Lambda_2, 2\Lambda_1 + \Lambda_2, \Lambda_1 + 2\Lambda_2 \mid i = 1, 2\}$.

For a dominant integral weight $\lambda$ of $\varnothing$, set

$$n_{\Lambda_1}(\lambda) = \min\{(\Lambda_1|\mu) \mid \mu \in \Pi(\lambda)\}.$$

By Corollary 7.2 and $\tau\gamma(\Lambda_1) = -\Lambda_2$, we have $n_{\Lambda_1}(\lambda) = (\Lambda_1|\tau\gamma(\lambda)) = -(\Lambda_2|\lambda)$.

**Lemma 7.3.** The values of $n_{\Lambda_1}(\lambda)$ are given as follows:

| Weight $\lambda$ | Lowest $L(0)$-weight of $L_\varnothing(3,\lambda)$ | $(\Lambda_1|\lambda)$ | $n_{\Lambda_1}(\lambda)$ |
|------------------|----------------------------------|-----------------|-----------------|
| 0                | 0                                | 0               | 0               |
| $\Lambda_1$      | $2/9$                            | $2/3$           | $-1/3$          |
| $\Lambda_2$      | $2/9$                            | $1/3$           | $-2/3$          |
| $2\Lambda_1$     | $5/9$                            | $4/3$           | $-2/3$          |
| $2\Lambda_2$     | $5/9$                            | $2/3$           | $-4/3$          |
| $\Lambda_1 + \Lambda_2$ | $1/2$                  | 1               | $-1$            |
| $3\Lambda_1$     | 1                                | 2               | $-1$            |
| $3\Lambda_2$     | 1                                | 1               | $-2$            |
| $2\Lambda_1 + \Lambda_2$ | $8/9$                  | $5/3$           | $-4/3$          |
| $\Lambda_1 + 2\Lambda_2$ | $8/9$                  | $4/3$           | $-5/3$          |
7.3. Inner automorphism of a holomorphic VOA with Lie algebra $A_{2,3}^6$. Let $V$ be a strongly regular holomorphic VOA of central charge 24 whose weight one Lie algebra has the type $A_{2,3}^6$. Let $V_1 = \bigoplus_{i=1}^6 g_i$ be the decomposition into the direct sum of simple ideals, where the type of $g_i$ is $A_{2,3}$ for $1 \leq i \leq 6$. Let $\mathfrak{H}$ be a Cartan subalgebra of $V_1$. Then $\mathfrak{H} \cap g_i$ is a Cartan subalgebra of $g_i$. Let $U$ be the subVOA generated by $V_1$. By Proposition 2.2, $U \cong \bigotimes_{i=1}^6 L_{g_i}(3,0)$.

Let $h = (\Lambda_1, 0, 0, 0, 0, 0) \in \bigoplus_{i=1}^6 (\mathfrak{H} \cap g_i)$. Note that

(7.1) \[ \langle h|h \rangle = 3 \times (\Lambda_1|\Lambda_1)_{|g_1} = 2 \]

and that for any root $\alpha$ of $V_1$, we have

(7.3) \[ (h|\alpha) \geq -1. \]

Now, let us recall the following theorems:

**Theorem 7.4.** ([KMi15, Theorem 2]) Let $P$ be a strongly regular holomorphic VOA and let $Q$ be a simple regular subVOA of $P$. Assume that the commutant subalgebra $Q^c$ is simple, regular, and that $(Q^c)^c = Q$. Then all irreducible $Q$-modules appear in $P$ as $Q$-submodules.

**Theorem 7.5.** ([HKL15, Theorem 3.5]) Let $W$ be a regular simple VOA of CFT-type. Assume that for any irreducible $W$-module, its lowest $L(0)$-weight is nonnegative except for $W$ itself. Let $Q$ be a $C_2$-cofinite, simple VOA of CFT-type containing $W$ as a full subVOA. Then $Q$ is rational.

Using the theorems above, we prove the following lemma:

**Lemma 7.6.** The order of $\sigma_h$ is 3 on $V$, and the $L(0)$-weights of the irreducible $\sigma_h$-twisted (resp. $\sigma_h^{-1}$-twisted) $V$-module $V^{(h)}$ (resp. $V^{(-h)}$) belong to $\mathbb{Z}/3$.

**Proof.** Let $M \cong \bigotimes_{i=1}^6 L_{g_i}(3, \lambda_i)$ be an irreducible $U$-submodule of $V$. By Lemma 7.3, $(h|(\lambda_1, \ldots, \lambda_6)) \in \mathbb{Z}/3$. Hence the order of $\sigma_h$ is 1 or 3 on $V$.

Let $Q = \langle g_1 \rangle_{\text{VOA}}$ be the subVOA of $V$ generated by $g_1$ and let $Q^c$ be the commutant subalgebra of $Q$ in $V$. Since the lowest $L(0)$-weight of any irreducible $Q$-module is less than 2 (see Lemma 7.3), we have $(Q^c)^c = Q$. Let $W = \langle g_i \mid 2 \leq i \leq 6 \rangle_{\text{VOA}}$ be the subVOA of $V$ generated by $\bigoplus_{i=2}^6 g_i$. Note that $W$ is regular, and the $L(0)$-lowest weight of any irreducible $W$-module is positive except for $W$ itself. Since $Q^c$ is an extension of $W$, $Q^c$ is $C_2$-cofinite by [ABD04]. In addition, $Q^c$ is simple (see [ACKL, Lemma 2.1]). Then
by Theorem 7.5, $Q^c$ is rational, and by [ABD04], $Q^c$ is regular. Applying Theorem 7.4 to $Q$, we know that every irreducible $Q$-submodule appears in $V$. Hence by Lemma 7.3, there exists a $Q \otimes W$-submodule with highest weight $(\lambda_1, \ldots, \lambda_6)$ such that $(h|\langle \lambda_1, \ldots, \lambda_6 \rangle) \notin \mathbb{Z}$, which shows that $\sigma_h \neq 1$ on $V$.

The latter assertion follows from Lemmas 2.6 and 7.3 and (7.2).

**Proposition 7.7.** The lowest $L(0)$-weight of the irreducible $\sigma_h$-twisted (resp. $\sigma_h^{-1}$-twisted) $V$-module $V^{(h)}$ (resp. $V^{(-h)}$) is 1. In particular, $(V^{(\pm h)})_i = 0$ for $i \in \{1/3, 2/3\}$.

**Proof.** Let $M \cong \bigotimes_{i=1}^6 L_{\ell_i}(3, \lambda_i)$ be an irreducible $U$-submodule of $V$. Let $\ell$ and $\ell^{(h)}$ be the lowest $L(0)$-weights of $M$ and of $M^{(h)}$, respectively. It suffices to show that $\ell^{(h)} \geq 1$. By (7.3), we can apply Lemma 2.7 to our case. By (2.2), (7.1) and (7.2), we have

$$\ell^{(h)} = \ell + n_{A_1}(\lambda_1) + 1,$$

where $n_{A_1}(\Lambda)$ is defined in the previous subsection. If $\lambda_i = 0$ for $1 \leq i \leq 6$, then $\ell = 0$ and $\ell^{(h)} = 1$. If $\lambda_i \neq 0$ for some $i$, then $\ell \geq 2$ and, by Lemma 7.3, $n_{A_1}(\lambda_1) \geq -2$. Hence $\ell^{(h)} \geq 1$.

One can also prove $\ell^{(-h)} \geq 1$ by the same argument. \hfill \Box

**7.4. Identification of the Lie algebra: Case $D_{4,1}^6$.** Thanks to Proposition 7.7, we can apply Theorem 4.1 to $V$ and $\sigma_h$ (see also Theorems A.3 and A.4), and obtain the strongly regular holomorphic VOA of central charge 24

$$\tilde{V} = \tilde{V}_{\sigma_h} = V^{\sigma_h} \oplus (V^{(h)})_\mathbb{Z} \oplus (V^{(-h)})_\mathbb{Z},$$

where $(V^{\pm h})_\mathbb{Z}$ are irreducible $V^{\sigma_h}$-submodules of $V^{(\pm h)}$ with integral weights (cf. Remark 4.2 (1)). Since $\sigma_h$ acts trivially on $V_1$, we have $(V^{\sigma_h})_1 = V_1$.

**Lemma 7.8.** The Lie algebra structure of $(V^{\sigma_h})_1$ is $A_{2,3}^6$ and $\dim(V^{\sigma_h})_1 = 48$.

Next we will identify the Lie algebra structure of $\tilde{V}_1$.

**Proposition 7.9.** The Lie algebra structure of $\tilde{V}_1$ is $D_{4,1}^6$. In particular, $\tilde{V}$ is isomorphic to the lattice VOA associated with the Niemeier lattice $Ni(D_{4}^6)$.

**Proof.** By Proposition 2.8, the Lie algebra $\tilde{V}_1$ is semisimple. By Proposition 7.8, we have $\dim(V^{\sigma_h})_1 = 48$, and by Proposition 7.7 we have $(V^{(\pm h)})_i = 0$ for $i = 1/3, 2/3$. By Theorem 4.8

$$\dim \tilde{V}_1 = 4 \times \dim(V^{\sigma_h})_1 - \dim V_1 + 24 = 168;$$

hence we obtain the ratio $h^{\vee}/k = 6$ by Proposition 2.8. It also follows Proposition 2.8 that $\tilde{V}_1$ is semisimple. Since the level $k$ of any simple ideal of $\tilde{V}_1$ is integral, its dual Coxeter number $h^{\vee}$ is a multiple of 6, which shows that the possible types of simple ideals of $\tilde{V}_1$ are $A_{5,1}, A_{11,1}, C_{5,1}, D_{4,1}, D_{7,2}, E_{6,2}$ and $E_{7,3}$. It follows from $\dim \tilde{V}_1 = 168$ that $\tilde{V}_1$ has
the type $A_{5,1}^4D_{4,1}, D_{4,1}^6, A_{5,1}E_{7,3}$ or $A_{5,1}C_{5,2}E_{6,2}$. Since $\tilde{V}$ is a simple current extension of $V^{\sigma_h}$ graded by $\mathbb{Z}_3$, there exists an order 3 automorphism of $\tilde{V}_1$ of which the fixed-point subspace has the Lie algebra structure $A_{2,3}^6$. Notice that a simple Lie algebra with non-inner order 3 automorphisms is of type $D_4$. One can easily see that the only possible type for $\tilde{V}_1$ is $D_{4,1}^6$. By Proposition 2.4, we have proved this proposition.

7.5. **Main theorem for the Lie algebra $A_{2,3}^6$.** Set $N = Ni(D_6^4)$ and let $\sigma_2$ be the order 3 automorphism of $V_N$ defined in Section 3.3. Then $\sigma_2$ satisfies Condition (I) in Section 4.1 and the holomorphic VOA $(\tilde{V}_N)_{\sigma_2}$ has the weight one Lie algebra of type $A_{2,3}^6$ (SS16). Note that the Lie algebra $(V_N^{\sigma_2})_1$ has also the type $A_{2,3}^6$. We now prove the main theorem in this section.

**Theorem 7.10.** Let $V$ be a strongly regular holomorphic VOA of central charge 24 such that the Lie algebra structure of $V_1$ is $A_{2,3}^6$. Then $V$ is isomorphic to the holomorphic VOA $(\tilde{V}_N)_{\sigma_2}$. In particular, the structure of a strongly regular holomorphic vertex operator algebra of central charge 24 is uniquely determined by its weight one Lie algebra if the Lie algebra has the type $A_{2,3}^6$.

**Proof.** It suffices to verify the hypotheses in Theorem 3.2 for $g = p = A_{2,3}^6$, $n = 3$ and $W = V_{Ni(D_6^4)}$. Take $h$ as in (7.1) and set $\sigma = \sigma_h$. Then by Lemma 7.6 the order of $\sigma$ is 3 on $V$. The former hypotheses (a) and (b) hold by Lemma 7.8 and Propositions 7.7 and 7.9. The latter hypotheses about the uniqueness of the conjugacy class follows from Proposition 3.12 (1).

8. **Uniqueness of a holomorphic VOA with the Lie algebra $A_{5,3}D_{4,3}A_{1,1}^3$.**

In this section, we consider a holomorphic VOA $V$ of central charge 24 whose weight one Lie algebra has the type $A_{5,3}D_{4,3}A_{1,1}^3$. By applying the $\mathbb{Z}_3$-orbifold construction to $V$ and certain inner automorphism of order 3, we show that one can obtain a holomorphic lattice VOA of central charge 24 associated with the Niemeier lattice $Ni(D_6^4)$). In addition, the structure of a strongly regular holomorphic vertex operator algebra of central charge 24 is uniquely determined by its weight one Lie algebra if the Lie algebra has the type $A_{5,3}D_{4,3}A_{1,1}^3$.

8.1. **Simple affine VOA of type $A_1$ at level 1.** Let $\alpha_1$ be a simple root of type $A_1$ such that $(\alpha_1|\alpha_1) = 2$. Then the fundamental weight is $\Lambda_1 = \alpha_1/2$. Let $L_{g}(1,0)$ be the simple affine VOA associated with the simple Lie algebra $g$ of type $A_1$ at level 1. There exist exactly 2 (non-isomorphic) irreducible $L_{g}(1,0)$-modules, $L_{g}(1,0)$ and $L_{g}(1,\Lambda_1)$.

**Lemma 8.1.** The lowest $L(0)$-weights of irreducible $L_{g}(1,0)$-modules are given as follows:
8.2. Simple affine VOA of type $A_5$ at level 3. Let $\alpha_1, \alpha_2, \ldots, \alpha_5$ be simple roots of type $A_5$ such that $(\alpha_i|\alpha_j) = 2\delta_{i,j} - \delta_{|i-j|,1}$ for $1 \leq i, j \leq 5$. Let $\Lambda_1, \Lambda_2, \ldots, \Lambda_5$ be the fundamental weights such that $(\Lambda_i|\alpha_j) = \delta_{i,j}, 1 \leq i, j \leq 5$. Let $L_g(3,0)$ be the simple affine VOA associated with the simple Lie algebra $g$ of type $A_5$ at level 3. It is well-known (cf. [FZ92]) that there exist exactly 56 (non-isomorphic) irreducible $L_g(3,0)$-modules. They are irreducible highest weight modules associated with dominant integral weights.

Set

$$\Lambda = \frac{2}{3} \Lambda_3 = \frac{1}{3}(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5).$$

Then $(\Lambda|\Lambda) = 2/3$. For a dominant integral weight $\lambda$ of $g$, set

$$n_\Lambda(\lambda) = \min\{(\Lambda|\mu) | \mu \in \Pi(\lambda)\}.$$ 

Here $\Pi(\lambda)$ denotes the set of all weights of the irreducible $g$-module with the highest weight $\lambda$. By Corollary 7.2 and $\tau\gamma(\Lambda_3) = -\Lambda_3$, we have $n_\Lambda(\lambda) = (\Lambda|\tau\gamma(\lambda)) = -(\Lambda|\lambda)$.

**Lemma 8.2.** The dominant integral weights, the lowest $L(0)$-weights of the associated $L_g(3,0)$-modules and the values of $(\Lambda|\lambda)$ and $n_\Lambda(\lambda)$ are given as follows:

| Weight $\lambda$ | Lowest $L(0)$-weight of $L_g(3,\lambda)$ | $(\Lambda|\lambda)$ | $n_\Lambda(\lambda)$ |
|------------------|------------------------------------------|---------------------|---------------------|
| 0                | 0                                        | 0                   | 0                   |
| $\Lambda_1$, $\Lambda_5$ | 35/108                                  | 1/3                 | $-1/3$              |
| $\Lambda_2$, $\Lambda_4$ | 14/27                                   | 2/3                 | $-2/3$              |
| $\Lambda_3$      | 7/12                                     | 1                   | $-1$                |
| $2\Lambda_1$, $2\Lambda_5$ | 20/27                                   | 2/3                 | $-2/3$              |
| $2\Lambda_2$, $2\Lambda_4$ | 32/27                                   | 4/3                 | $-4/3$              |
| $2\Lambda_3$     | 4/3                                      | 2                   | $-2$                |
| $\Lambda_1 + \Lambda_2$, $\Lambda_4 + \Lambda_5$ | 11/12                                  | 1                   | $-1$                |
| $\Lambda_1 + \Lambda_3$, $\Lambda_3 + \Lambda_5$ | 26/27                                  | 4/3                 | $-4/3$              |
| $\Lambda_1 + \Lambda_4$, $\Lambda_2 + \Lambda_5$ | 95/108                                 | 1                   | $-1$                |
| $\Lambda_1 + \Lambda_5$ | 2/3                                      | 2/3                 | $-2/3$              |
| $\Lambda_2 + \Lambda_3$, $\Lambda_3 + \Lambda_4$ | 131/108                                | 5/3                 | $-5/3$              |
| $\Lambda_2 + \Lambda_4$ | 10/9                                   | 4/3                 | $-4/3$              |
| $3\Lambda_1, 3\Lambda_5$ | 5/4                                       | 1                   | $-1$                |
| $3\Lambda_2, 3\Lambda_4$ | 2                                        | 2                   | $-2$                |
8.3. **Simple affine VOA of type $D_4$ at level 3.** Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be simple roots of type $D_4$ such that $(\alpha_i|\alpha_i) = 2$ for $1 \leq i \leq 4$, $(\alpha_i|\alpha_j) = -1$ if two nodes $\alpha_i$ and $\alpha_j$, $1 \leq i \neq j \leq 4$, are jointed in the diagram [3.2] and $(\alpha_i|\alpha_j) = 0$ otherwise. Let $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$ be the fundamental weights such that $(\alpha_i|\Lambda_j) = \delta_{i,j}$ for $1 \leq i, j \leq 4$. Let $L_\mathfrak{g}(3,0)$ be the simple affine VOA associated with the simple Lie algebra $\mathfrak{g}$ of type $D_4$ at level 3. It is well-known (cf. [FZ92]) that there exist exactly 24 (non-isomorphic) irreducible $L_\mathfrak{g}(3,0)$-modules. They are irreducible highest weight modules associated with dominant integral weights.

**Lemma 8.3.** The dominant integral weights and the lowest $L(0)$-weights of the associated $L_\mathfrak{g}(3,0)$-modules are given as follows:

| Weight $\lambda$ | Lowest $L(0)$-weight of $L_\mathfrak{g}(3, \lambda)$ |
|------------------|-------------------------------------------------|
| $0$              | $0$                                             |
| $\Lambda_i, \ i \in \{1, 3, 4\}$ | $7/18$                                         |
| $\Lambda_2$      | $2/3$                                           |
| $2\Lambda_i, \ i \in \{1, 3, 4\}$ | $8/9$                                          |
| $\Lambda_i + \Lambda_2, \ i \in \{1, 3, 4\}$ | $7/6$                                           |
8.4. Inner automorphism of a holomorphic VOA with Lie algebra $A_{5,3}D_{4,3}A_{1,1}^3$.
Let $V$ be a strongly regular holomorphic VOA of central charge 24 whose weight one Lie algebra has the type $A_{5,3}D_{4,3}A_{1,1}^3$. Let $V_1 = \bigoplus_{i=1}^5 g_i$ be the decomposition into the direct sum of simple ideals, where the types of $g_1$, $g_2$ and $g_i$ ($3 \leq i \leq 5$) are $A_{5,3}$, $D_{4,3}$ and $A_{1,1}$, respectively. Let $\mathfrak{H}$ be a Cartan subalgebra of $V_1$. Then $\mathfrak{H} \cap g_i$ is a Cartan subalgebra of $g_i$. Let $U$ be the subVOA generated by $V_1$. Set $k_i = 3$ for $i = 1, 2$ and $k_j = 1$ for $j = 3, 4, 5$. By Proposition 2.2, $U \cong \bigotimes_{i=1}^5 L_{g_i}(k_i, 0)$.

Let

\begin{equation}
(8.2) \quad h = (h_1, h_2, h_3, h_4, h_5) = \frac{2}{3}(\Lambda_3, 0, 0, 0, 0) \in \bigoplus_{i=1}^5 (\mathfrak{H} \cap g_i).
\end{equation}

One can easily check that

\begin{equation}
(8.3) \quad \langle h | h \rangle = 3 \times \left( \frac{2}{3} \Lambda_3 \left| \frac{2}{3} \Lambda_3 \right|_{g_1} \right)_{g_1}^2 = 2
\end{equation}

and that for all roots $\alpha$ of the semisimple Lie algebra $V_1$, we have

\begin{equation}
(8.4) \quad (h | \alpha) \geq -1.
\end{equation}

**Lemma 8.4.** The order of $\sigma_h$ is 3 on $V$, and the $L(0)$-weights of the irreducible $\sigma_h$-twisted (resp. $\sigma_h^{-1}$-twisted) $V$-module $V_h$ (resp. $V_{-h}$) belong to $\mathbb{Z}/3$.

**Proof.** It follows from (8.1) that $(h | \lambda) \in \mathbb{Z}/3$ for any weights $\lambda$ of the semisimple Lie algebra $V_1$. Hence $(\sigma_h)^3 = 1$. In addition, $\sigma_h$ is an automorphism of order 3 on $V_1$, so it is on $V$. The latter assertion follows from Lemmas 2.6 and 2.2 and (8.3). 

**Proposition 8.5.** The lowest $L(0)$-weight of the irreducible $\sigma_h$-twisted (resp. $\sigma_h^{-1}$-twisted) $V$-module $V_h$ (resp. $V_{-h}$) is 1. In particular, $(V^{(\pm h)})_i = 0$ for $i \in \{1/3, 2/3\}$.

**Proof.** Let $M \cong \bigotimes_{i=1}^5 L_{g_i}(k_i, \lambda_i)$ be an irreducible $U$-submodule of $V$. Let $\ell$ and $\ell^{(h)}$ be the lowest $L(0)$-weights of $M$ and of $M_h$, respectively. It suffices to show that $\ell^{(h)} \geq 1$. By (8.4), we can apply Lemma 2.7 to our case. By (2.2), (8.2) and (8.3), we have

$$
\ell^{(h)} = \ell + n_\Lambda(\lambda_1) + 1,
$$

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
$\Lambda_i + \Lambda_j$, $i, j \in \{1, 3, 4\}$, $i \neq j$ & $5/6$ \\
\hline
$3\Lambda_i$, $i \in \{1, 3, 4\}$ & $3/2$ \\
\hline
$\Lambda_1 + \Lambda_3 + \Lambda_4$ & $4/3$ \\
\hline
$2\Lambda_i + \Lambda_j$, $i, j \in \{1, 3, 4\}$, $i \neq j$ & $25/18$ \\
\hline
\end{tabular}
\end{table}
where $\Lambda = (2/3)\Lambda_3$ in $\mathfrak{g}_1$. If $\lambda_i = 0$ for $1 \leq i \leq 5$, then $\ell = 0$ and $\ell^{(h)} = 1$. Assume $\lambda_i \neq 0$ for some $1 \leq i \leq 5$. Then $\ell \geq 2$. Hence if $n_\Lambda(\lambda_1) \geq -2$, then $\ell^{(h)} \geq 1$. By Lemma 8.2, $n_\Lambda(\lambda_1) < -2$ if and only if

$$n_\Lambda(\lambda_1) = \begin{cases} 3\Lambda_3, 2\Lambda_3 + \Lambda_5, 2\Lambda_2 + 2\Lambda_4, 2\Lambda_3 + \Lambda_4, \Lambda_2 + 2\Lambda_3, \Lambda_2 + 3\Lambda_3 + \Lambda_4, & \text{for } 1 \leq i \leq 5. \end{cases}$$

and $-3 \leq n_\Lambda(\lambda_1) < -2$ for every $\lambda_1$ in (8.5). Clearly, $\ell$ is integral. Recall that $\ell$ is the sum of lowest $L(0)$-weights of $L_{\mathfrak{g}_i}(k_i, \lambda_i)$, $1 \leq i \leq 5$. By Lemma 8.2, if

$$\lambda_1 \in \{3\Lambda_3, 2\Lambda_2 + \Lambda_4, 2\Lambda_3 + \Lambda_4, \Lambda_2 + 2\Lambda_3, \Lambda_2 + 3\Lambda_3 + \Lambda_4\},$$

then the lowest $L(0)$-weight of $L_{\mathfrak{g}_i}(\lambda_1)$ is greater than 2, and hence $\ell \geq 3$. For the other $\lambda$, by Lemmas 8.1, 8.2 and 8.3, there are no weights $\lambda_i$ ($2 \leq i \leq 5$) such that the lowest $L(0)$-weight of $\bigotimes_{i=1}^5 L_{\mathfrak{g}_i}(k_i, \lambda_i)$ is 2. Hence, for $\lambda_1$ in (8.5), we have $\ell \geq 3$. Thus $\ell^{(h)} \geq 1$.

One can also prove $\ell^{(-h)} \geq 1$ by the same argument. \hfill \Box

8.5. Identification of the Lie algebra: Case $D_{4,1}^6$. Thanks to Proposition 8.6, we can apply Theorem 4.4 to $V$ and $\sigma_h$ (see also Theorems A.3 and A.4), and obtain the strongly regular holomorphic VOA of central charge 24

$$\tilde{V} = \tilde{V}_{\sigma_h} = V^{\sigma_h} \oplus (V^{(h)})_Z \oplus (V^{(-h)})_Z,$$

where $(V^{(\pm h)})_Z$ are irreducible $V^{\sigma_h}$-submodules of $V^{(\pm h)}$ with integral weights (cf. Remark 4.2 (1)). By the definition of $h$, we obtain the following:

Proposition 8.6. The Lie algebra structure of $(V^{\sigma_h})_1$ is $A_{2,3}^2 U(1)D_{4,3}A_{1,1}^3$ and $\dim(V^{\sigma_h})_1 = 54$.

Let us identify the Lie algebra structure of $\tilde{V}_1$.

Proposition 8.7. The Lie algebra structure of $\tilde{V}_1$ is $D_{4,1}^6$. In particular, $\tilde{V}$ is isomorphic to the lattice VOA associated with the Niemeier lattice $Ni(D_{4,1}^6)$.

Proof. By Proposition 2.3, the Lie algebra $\tilde{V}_1$ is semisimple. By Proposition 8.6, we have $\dim(V^{\sigma_h})_1 = 54$, and by Proposition 8.5 we have $(V^{(\pm h)})_i = 0$ for $i = 1/3, 2/3$. By Theorem 4.8

$$\dim \tilde{V}_1 = 4 \times \dim(V^{\sigma_h})_1 - \dim V_1 + 24 = 168;$$

hence we obtain the ratio $h^\vee / k = 6$ by Proposition 2.3. By the same argument as in the proof of Proposition 7.9, $\tilde{V}_1$ is of the type $A_{3,1}^1 D_{4,1}, D_{4,1}^6, A_{5,1} E_{7,3}$ or $A_{5,1} C_{5,2} E_{6,2}$. Since $\tilde{V}$ is a simple current extension of $V^{\sigma_h}$ graded by $\mathbb{Z}_3$, there exists an order 3 automorphism of the semisimple Lie algebra $\tilde{V}_1$ such that the fixed-point subspace has the Lie algebra structure $A_{2,3}^2 D_{4,3} U(1)A_{1,1}^3$. One can easily see that the only possible type of $\tilde{V}_1$ is $D_{4,1}^6$. By Proposition 2.3 we have proved this proposition. \hfill \Box
8.6. **Main theorem for the Lie algebra** $A_{5,3}D_{4,3}A_{1,1}^3$. Set $N = Ni(D_4^2)$ and let $\sigma_4$ be the order 3 automorphism of $V_N$ defined in Section 3.3.2. Then $\sigma_4$ satisfies Condition (I) in Section 4.1 and the holomorphic VOA $(\widetilde{V}_N)_{\sigma_4}$ has the weight one Lie algebra of type $A_{5,3}D_{4,3}A_{1,1}^3$ ([SS16]). Note that the Lie algebra $(V_{N,1})^{\sigma_4}$ has the type $A_{2,3}U(1)D_{4,3}A_{1,1}^1$. We now prove the main theorem in this section.

**Theorem 8.8.** Let $V$ be strongly regular holomorphic VOAs of central charge 24 such that the Lie algebra structures of $V_1$ is $A_{5,3}D_{4,3}A_{1,1}^3$. Then $V$ is isomorphic to the holomorphic VOA $(\widetilde{V}_N)_{\sigma_4}$. In particular, the structure of a strongly regular holomorphic vertex operator algebra of central charge 24 is uniquely determined by its weight one Lie algebra if the Lie algebra has the type $A_{5,3}D_{4,3}A_{1,1}^3$.

**Proof.** It suffices to verify the hypotheses in Theorem 5.2 for $g = A_{5,3}D_{4,3}A_{1,1}^3$, $p = A_{2,3}U(1)D_{4,3}A_{1,1}^1$, $n = 3$ and $W = V_{N,1}$. Take $h$ as in (8.2) and set $\sigma = \sigma_h$. Then the order of $\sigma$ is 3 by Lemma 8.4 and the former hypotheses (a) and (b) hold by Propositions 8.5, 8.6 and 8.7. The latter hypothesis about the uniqueness of the conjugacy class follows from Proposition 3.12 (2). $\square$

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**Appendix A.** $\mathbb{Z}_p$-orbifold construction associated with inner automorphisms

In this section, we review the $\mathbb{Z}_p$-orbifold construction associated with an inner automorphism of prime order $p$ (cf. [DLM96]). The case where $p = 2$ was done in [LS16a].

**Remark A.1.** After we obtained the results in this section, a general orbifold construction was established in [EMS] [Mö16] independently.

Let $V$ be a strongly regular VOA. Let $h \in V_1$ such that $\langle h|h \rangle \in 2\mathbb{Z}$ and $h(0)$ acts semi-simply on $V$. Let $p$ be a prime. Suppose that $g = \exp(-2\pi \sqrt{-1} h(0))$ has order $p$. Note that spectra of $h(0)$ on $V$ belong to $\frac{1}{p}\mathbb{Z}$.

Set $L = Zh$. For $r \in \mathbb{Z}$ and $\nu \in \mathcal{C}h$, we set

$$V^{(rh,\nu)} := \{ v \in V^{(rh)} \mid h^{(rh)}(0)v = \langle h|rh + \nu \rangle v \},$$

and we set $P = \{ \nu \in \mathcal{C}h \mid V^{(0,\nu)} \neq 0 \}$. Since $V$ is simple, there exists $s \in \mathbb{Q}h$ such that $P = \mathbb{Z}s$. It follows from $|g| = p$ that $\langle h|s \rangle \in \mathbb{Z} + \frac{1}{p}\mathbb{Z}$. For $i, j \in \{0, 1, \ldots, p - 1\}$, we set

$$U^{(ih,js)} := \bigoplus_{\beta \in pP + js} V^{(ih,\beta)}.$$
For $1 \leq r \leq p - 1$, the $g^r$-twisted module $V^{(rh)}$ has the decomposition:

$$V^{(rh)} = \bigoplus_{j=0}^{p-1} U^{(rh,js)} \quad \text{and} \quad (V^{(rh)})_\mathbb{Z} = U^{(rh,0)}.$$ 

Note that $U^{(0,0)} = V^g = \{ v \in V \mid g(v) = v \}$.

The following proposition is proved in [DLM96].

**Proposition A.2 ([DLM96, Theorem 3.21]).** There is an intertwining operator $\bar{Y}$ on the space

$$\bar{U} = \bigoplus_{i,j=0}^{p-1} U^{(ih,js)}$$

such that $(U, \bar{Y})$ is an abelian intertwining algebra and it satisfies the generalized Jacobi identity (see [DLM96, (3.88)])

\begin{equation}
\begin{aligned}
&\bar{C}((\lambda_i, \alpha), (\lambda_j, \beta)) z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) \left( \frac{z_1 - z_2}{z_0} \right) \eta((\lambda_i, \alpha), (\lambda_j, \beta)) \bar{Y}(u, z_1) \bar{Y}(v, z_2) w \\
&= \bar{C}((\lambda_i, \alpha), (\lambda_j, \beta)) z_0^{-1} \delta \left( \frac{z_2 - z_1}{z_0} \right) \left( \frac{z_2 - z_1}{z_0} \right) \eta((\lambda_i, \alpha), (\lambda_j, \beta)) \bar{Y}(v, z_2) \bar{Y}(u, z_1) w \\
&= z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) \left( \frac{z_1 - z_0}{z_2} \right) \eta((\lambda_i, \alpha), (\lambda_k, \gamma)) h(i, j, k) \bar{Y}(u, z_0) v, z_2) w,
\end{aligned}
\end{equation}

where $u \in U^{(\lambda_i, \alpha)}$, $v \in U^{(\lambda_j, \beta)}$, $w \in U^{(\lambda_k, \gamma)}$ and $\eta, \bar{C}$ and $h$ are defined as in [DLM96, Section 3].

**Theorem A.3.** Let $h \in V_1$ such that $\langle h \mid h \rangle \in 2\mathbb{Z}$ and $h_{(0)}$ acts semisimply on $V$. Suppose that $g = \exp(-2\pi \sqrt{-1} h_{(0)})$ has prime order $p$. Then the $V^g$-module $\bar{V} = V^g \oplus \bigoplus_{r=1}^{p-1} (V^{(rh)})_\mathbb{Z}$ forms a vertex operator algebra.

**Proof.** Since the subspace $\bar{V}$ of $\bar{U}$ is a $V^g$-module and is closed with respect to the operator $\bar{Y}$, it satisfies the axiom of a VOA except for the Jacobi identity. Recall the generalized Jacobi identity as stated in Proposition A.2 (cf. [DLM96, (3.88)]). It remains to check that the maps $\eta(\cdot, \cdot)$, $\bar{C}(\cdot, \cdot)$, $h(\cdot, \cdot, \cdot)$ are trivial on $\bar{V}$.

First we consider the map $\eta : (L/pL \times P/pP) \times (L/pL \times P/pP) \to (\mathbb{Z}/p\mathbb{Z})/2\mathbb{Z}$ defined in [DLM96, (3.19)], namely,

$$\eta((\lambda_i, \alpha), (\lambda_j, \beta)) = -\langle \lambda_i | \lambda_j \rangle - \langle \lambda_i | \beta \rangle - \langle \lambda_j | \alpha \rangle + 2\mathbb{Z}.$$ 

Since $\langle h \mid h \rangle \in 2\mathbb{Z}$, it is obvious that $\eta((ih, 0), (jh, 0)) \equiv 0 \mod 2\mathbb{Z}$ for any $0 \leq i, j \leq p - 1$.

Next we consider the map $\bar{C} : (L \times P) \times (L \times P) \to \mathbb{C}^*$ defined in [DLM96, (3.20) and (3.87)], where

$$\bar{C}((\lambda_i, \alpha), (\lambda_j, \beta)) = e^{i(\lambda_i | \beta) - (\lambda_j | \alpha)\pi \sqrt{-1}} C_1(\alpha, \beta).$$
By [DLM96] Remark 3.16, we may assume \( C_1 \equiv 1 \) on \( P \) since the rank of \( P \) is one. It is also clear that \( \tilde{C}((ih, 0), (jh, 0)) \equiv 1 \).

Finally, we consider the map \( h : \tilde{A} \times \tilde{A} \times \tilde{A} \rightarrow \mathbb{C}^* \) defined in [DLM96] (3.83), where
\[
h((\lambda_l, \alpha_1), (\lambda_j, \alpha_2), (\lambda_k, \alpha_3)) = e^{-(\lambda_l + \lambda_j - \lambda_l + j, \lambda_k)\pi \sqrt{-1}} C_1 (\lambda_l + \lambda_j - \lambda_l + j, \lambda_k)^2,
\]
\( \tilde{A} = (L \times P)/\{ (\alpha, -\alpha) \mid \alpha \in pL \} \) and \( \lambda_l + j \in \{ 0, h, \ldots, (p - 1)h \} \) so that \( \lambda_l + j = \lambda_l + \lambda_j \) (mod \( pL \)). Notice that \( \tilde{A} \) is identified with \( \{ (ih, \alpha) \mid i = 0, \ldots, p - 1, \alpha \in P \} \) and \( \tilde{U} \) is isomorphic to \( \bigoplus_{(\lambda, \alpha) \in \tilde{A}} V^{(\lambda, \alpha)} \) as a vector space. In our case, \( \lambda_l + \lambda_j - \lambda_l + j \in pZh \). Since \( \langle h|h \rangle \in 2\mathbb{Z} \), we have \( e^{-(\lambda_l + \lambda_j - \lambda_l + j, \lambda_k)\pi \sqrt{-1}} \equiv 1 \). As we mentioned above, \( C_1 \equiv 1 \). Hence \( h(\cdot, \cdot, \cdot) \equiv 1 \) on \( \tilde{A} \times \tilde{A} \times \tilde{A} \). Thus (A.1) is the usual Jacobi identity on \( \tilde{V} \).

**Theorem A.4.** (cf. [EMS]) Assume that \( V \) is holomorphic and that the lowest \( L(0) \)-weights of the \( g^r \)-twisted \( V \)-modules \( V^{(r)} \) are positive for \( 1 \leq r \leq p - 1 \). Then the VOA \( \tilde{V} \) constructed in Theorem A.3 is strongly regular and holomorphic.

**Proof.** By the assumption, \( \tilde{V} \) is of CFT-type. It was shown in [Mi15] that \( V^g \) is \( C_2 \)-cofinite. In addition, it was shown in [CM] that \( V^g \) is rational. By calculating the Verlinde formula, one can prove that any irreducible \( V^g \)-module is a simple current (cf. [EMS]). Hence \( \tilde{V} \) is a \( \mathbb{Z}_p \)-graded simple current extension of \( V^g \), and \( \tilde{V} \) is rational and \( C_2 \)-cofinite (see [LY08] Proposition 1 and reference therein). In addition, one can classify all irreducible \( \tilde{V} \)-modules ([LY08 Theorem 1]), and hence \( \tilde{V} \) is holomorphic. Clearly, \( \tilde{V} \) is self-dual.

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