Desingularization of 3D steady Euler equations with helical symmetry

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Abstract
In this paper, we study desingularization of steady solutions of 3D incompressible Euler equation with helical symmetry in a general helical domain. We construct a family of steady helical Euler flows, such that the associated vorticities tend asymptotically to a helical vortex filament. The solutions are obtained by solving a semilinear elliptic problem in divergence form with a parameter

$$-\varepsilon^2 \text{div}(K_H(x) \nabla u) = f(u - q|\ln \varepsilon|) \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega.$$ 

By using the variational method, we show that for any $0 < \varepsilon < 1$, there exist ground states concentrating near minimum points of $q^2 \sqrt{\det(K_H)}$ as the parameter $\varepsilon \to 0$. These results show a striking difference with the 2D and the 3D axisymmetric Euler equation cases.

1 Introduction and main results

1.1 Introduction

The movement of incompressible Euler flow confined in a three-dimensional domain $D$ without external force is governed by the following system
In this paper, we are devoted to Euler Eqs. (1.1) with helical symmetry. To this end, we first define helical solutions and deduce the 2D vorticity Eq. of (1.2), see [12, 13]. Let $k > 0$. Define a one-parameter group $G_k = \{ H_\rho : \mathbb{R}^3 \to \mathbb{R}^3 \}$, where

$$H_\rho(x_1, x_2, x_3)^t = (x_1 \cos \rho + x_2 \sin \rho, -x_1 \sin \rho + x_2 \cos \rho, x_3 + k \rho)^t.$$ 

Here $A^t$ is the transposition of a matrix $A$. Let $R_\rho = \begin{pmatrix} \cos \rho & \sin \rho & 0 \\ -\sin \rho & \cos \rho & 0 \\ 0 & 0 & 1 \end{pmatrix}$ be the rotation with respect to $x_3$-axis. Define a vector field

$$\overrightarrow{\zeta} = (x_2, -x_1, k)^t.$$ 

Clearly $\overrightarrow{\zeta}$ is the field of tangents of symmetry lines of $G_k$.

To study helical solutions of (1.1), we need to define helical domains. We say $D \subseteq \mathbb{R}^3$ is a \textit{helical domain}, if $H_\rho(D) = D$ for any $\rho$. Let $\Omega = D \cap \{x \mid x_3 = 0\}$ be the section of $D$ over $x_1Ox_2$ plane. Then $D$ can be generated by $\Omega$ by letting $D = \bigcup_{\rho \in \mathbb{R}} H_\rho(\Omega)$. Throughout this paper, we always assume that $\Omega$ is a simply-connected bounded domain with $C^\infty$ boundary.

Now we define helical functions and helical vector fields. A scalar function $h$ is called a \textit{helical} function, if for any $\rho \in \mathbb{R}, x \in D$,

$$h(H_\rho(x)) = h(x).$$

A vector field $\mathbf{h} = (h_1, h_2, h_3)$ is called a \textit{helical} field, if for any $\rho \in \mathbb{R}, x \in D$,

$$\mathbf{h}(H_\rho(x)) = R_\rho \mathbf{h}(x).$$

Helical solutions of (1.1) are then defined as follows.

\textbf{Definition 1.1} A function pair $(\mathbf{v}, P)$ is called a \textit{helical} solution pair of (1.1), if $(\mathbf{v}, P)$ satisfies (1.1) and both vector field $\mathbf{v}$ and scalar function $P$ are helical.
Throughout this paper, helical solutions also need to satisfy the orthogonality condition:
\[ \mathbf{v} \cdot \zeta = 0. \]  
(1.5)

Under the condition (1.5), one can check that the vorticity field \( \mathbf{w} \) satisfies (see [13])
\[ \mathbf{w} = \frac{\omega}{k} \zeta, \]  
(1.6)

where \( \omega := w_3 = \partial_{x_1} v_2 - \partial_{x_2} v_1 \), the third component of vorticity field \( \mathbf{w} \), is a helical function. Moreover, \( \omega \) satisfies
\[ \partial_t \omega + (\mathbf{v} \cdot \nabla) \omega = 0. \]  
(1.7)

From (1.7) we deduce that, \( \omega \) satisfies a transport equation, which is very similar to the case of 2D Euler equations (see, e.g., [22, 25]). Moreover, for a solution \( \omega \) of (1.7), the vorticity field \( \mathbf{w} \) is determined by (1.6).

We now define a stream function and deduce the 2D vorticity Eq. of (1.2). Since \( \mathbf{v} \) is helical, one has
\[ x_2 \partial_{x_1} v_3 - x_1 \partial_{x_2} v_3 + k \partial_{x_3} v_3 = 0. \]  
(1.8)

The orthogonal condition implies that
\[ x_2 v_1 - x_1 v_2 + k v_3 = 0. \]  
(1.9)

Thus taking (1.8) and (1.9) into the incompressible condition, we have
\[ 0 = \partial_{x_1} v_1 + \partial_{x_2} v_2 + \partial_{x_3} v_3 = \frac{1}{k^2} \partial_{x_1} [(k^2 + x_1^2) v_1 - x_1 x_2 v_2] + \frac{1}{k^2} \partial_{x_2} [(k^2 + x_1^2) v_2 - x_1 x_2 v_1]. \]

Since \( \Omega \) is simply-connected, we can define a stream function \( \varphi : \Omega \rightarrow \mathbb{R} \) such that \( \partial_{x_2} \varphi \equiv \frac{1}{k^2} [(k^2 + x_1^2) v_2 - x_1 x_2 v_1] \), that is,
\[ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = - \frac{1}{k^2 + x_1^2 + x_2^2} \begin{pmatrix} x_1 x_2 & -k^2 - x_1^2 \\ k^2 + x_1^2 & -x_1 x_2 \end{pmatrix} \begin{pmatrix} \partial_{x_1} \varphi \\ \partial_{x_2} \varphi \end{pmatrix}. \]  
(1.10)

By the definition of \( \omega \) and (1.10), we get
\[ \omega = \partial_{x_1} v_2 - \partial_{x_2} v_1 = - (\partial_{x_1}, \partial_{x_2}) \begin{pmatrix} 1 \\ k^2 + x_1^2 + x_2^2 \end{pmatrix} \begin{pmatrix} k^2 + x_1^2 & -x_1 x_2 \\ -x_1 x_2 & k^2 + x_1^2 \end{pmatrix} \begin{pmatrix} \partial_{x_1} \varphi \\ \partial_{x_2} \varphi \end{pmatrix} = \mathcal{L}_H \varphi. \]  
(1.11)

Here the operator \( \mathcal{L}_H \) defined by
\[ \mathcal{L}_H \varphi := - \text{div}(K_H(x_1, x_2) \nabla \varphi) \]  
(1.12)

is a second-order elliptic operator of divergence type with the coefficient matrix
\[ K_H(x_1, x_2) = \frac{1}{k^2 + x_1^2 + x_2^2} \begin{pmatrix} k^2 + x_1^2 & -x_1 x_2 \\ -x_1 x_2 & k^2 + x_1^2 \end{pmatrix}. \]  
(1.13)

Clearly \( K_H \) is a positive-definite matrix satisfying
\[(\mathcal{K}1) \quad K_H \text{ is smooth, i.e., } (K_H(\cdot))_{ij} \in C^\infty(\overline{\Omega}) \text{ for } i, j = 1, 2. \]
\[(\mathcal{K}2) \quad \mathcal{L}_H \text{ is strictly elliptic. Indeed, two eigenvalues of } K_H \text{ are } \lambda_1 = 1, \lambda_2 = \frac{k^2}{k^2 + |x|^2}, \text{ i.e.,} \]
\[ \frac{k^2}{k^2 + |x|^2} |\zeta|^2 \leq (K_H(x)\zeta, \zeta) \leq |\zeta|^2, \quad \forall \ x \in \Omega, \ \zeta \in \mathbb{R}^2. \]
Taking (1.9) and (1.10) into (1.7), one has
\[
0 = \partial_t \omega + v_1 \partial_{x_1} \omega + v_2 \partial_{x_2} \omega + v_3 \partial_{x_3} \omega \\
= \partial_t \omega + \frac{1}{k^2} (v_1, v_2) \begin{pmatrix}
    k^2 + x_2^2 - x_1 x_2 \\
    -x_1 x_2
\end{pmatrix} \begin{pmatrix}
    \partial_{x_1} \omega \\
    \partial_{x_2} \omega
\end{pmatrix}
\]
\[
= \partial_t \omega - \frac{1}{k^2 (k^2 + x_1^2 + x_2^2)} (\partial_{x_1} \varphi, \partial_{x_2} \varphi) \begin{pmatrix}
    x_1 x_2 \\
    -k^2 - x_1^2 - x_1 x_2
\end{pmatrix} \begin{pmatrix}
    k^2 + x_2^2 - x_1 x_2 \\
    -x_1 x_2
\end{pmatrix} \begin{pmatrix}
    \partial_{x_1} \omega \\
    \partial_{x_2} \omega
\end{pmatrix}
\]
\[
\quad \quad \quad = \partial_t \omega + \nabla \omega \cdot \nabla \perp \varphi.
\] (1.14)

where \( \perp \) denotes the clockwise rotation through \( \pi/2 \).

For the boundary condition of \( \varphi \), from the fact that \( \mathbf{v} \) is a helical vector field and the domain \( D \) is helical, \( v_n \) is a helical function defined on \( \partial D \) and can be generated by \( v_n|_{\partial \Omega} \). Moreover, it follows from \( \mathbf{v} \cdot \mathbf{n} = v_n \) on \( \partial D \) that (see (2.66), [13])
\[
v_n|_{\partial \Omega} = \mathbf{v} \cdot \mathbf{n}|_{\partial \Omega} = \frac{1}{k^2} (v_1, v_2) \begin{pmatrix}
    k^2 + x_2^2 - x_1 x_2 \\
    -x_1 x_2
\end{pmatrix} \begin{pmatrix}
    n_1 \\
    n_2
\end{pmatrix}
\] (1.15)

Combining this with (1.10), one has
\[
v_n|_{\partial \Omega} = -\frac{1}{k^2 (k^2 + |x|^2)} (\partial_{x_1} \varphi, \partial_{x_2} \varphi) \begin{pmatrix}
    x_1 x_2 \\
    -k^2 - x_1^2 - x_1 x_2
\end{pmatrix} \begin{pmatrix}
    k^2 + x_2^2 - x_1 x_2 \\
    -x_1 x_2
\end{pmatrix} \begin{pmatrix}
    n_1 \\
    n_2
\end{pmatrix}
\] (1.15)

Here \( \mathbf{n} = (n_1, n_2) \) is the two-dimensional vector of the first two component of \( \mathbf{n} \) on \( \partial \Omega \), which is an outward normal of \( \partial \Omega \).

Thus, from (1.11), (1.14) and (1.15) the 2D vorticity equations of 3D Euler equations with helical symmetry is as follows
\[
\begin{align*}
\partial_t \omega + \nabla \omega \cdot \nabla \perp \varphi &= 0, \\
\omega &= \mathcal{L}_H \varphi, \\
\nabla \perp \varphi \cdot \mathbf{v}|_{\partial \Omega} &= v_n|_{\partial \Omega},
\end{align*}
\] (1.16)

where \( \mathcal{L}_H \) is defined by (1.12). For a solution pair \( (\omega, \varphi) \) to the 2D vorticity Eq. (1.16), one can recover the helical velocity field \( \mathbf{v} \) and vorticity field \( \omega \) of 3D Euler Eqs. (1.1) by using (1.10), (1.9), (1.4), (1.3) and (1.6).

When considering the case \( v_n \equiv 0 \), which corresponds to the impermeable boundary condition, (1.15) implies that \( \varphi \) is a constant on \( \partial \Omega \). Thus one can choose \( \varphi \) such that \( \varphi \equiv 0 \) on \( \partial \Omega \). The associated vorticity equations then become
\[
\begin{align*}
\partial_t \omega + \nabla \omega \cdot \nabla \perp \varphi &= 0, \\
\omega &= \mathcal{L}_H \varphi, \\
\varphi|_{\partial \Omega} &= 0.
\end{align*}
\] (1.17)

The research of 3D Euler equations with helical symmetry has received much attention in recent years. [13] first proved the global well-posedness of \( L^1 \cap L^\infty \) weak solutions to the Euler equation with helical symmetry without vorticity stretching (i.e., (1.17)). Note that this result corresponds to the classical Yudovich’s result [25], since the structure of vorticity Eq. (1.17) is similar to that of 2D Euler flows. As for the steady solution of 3D Euler equations with helical symmetry, nonlinear stability for stationary smooth Euler flows with helical symmetry is considered in [2] by using the direct method of Lyapunov. More results of the existence and regularity of Euler equation with helical symmetry can be found in [1, 4, 12, 19] for instance.
In this paper we are interested in vortex desingularization problem of steady 3D Euler equations with helical symmetry, that is, constructing a family of “true” solutions of steady Euler equations, such that the corresponding vorticity has a small cross-section and concentrates near a vortex filament. The research of this problem can be traced back to Helmholtz [15], who first studied the motion of the traveling vortex rings whose vorticities are supported in toroidal regions with a small cross-section. Then many articles considered this problem. As for the vortex being a tube with a small cross-section whose centerline is a straight line and a circle, which can be reduced to 2D Euler equation and 3D axisymmetric Euler equation respectively, results can be found in [3, 5, 6, 9, 11, 14, 23] and reference therein. However, for the case of Euler equations with helical symmetry, results seem to be few. Dávila et al. [10] constructed rotational-invariant Euler flows with helical symmetry in the whole space by considering

$$- \text{div}(K_H(x) \nabla u) = f_\epsilon \left( u - \alpha \left| \ln \epsilon \right| \frac{|x|^2}{2} \right) \quad \text{in } \mathbb{R}^2,$$

where $f_\epsilon(t) = \epsilon^2 e^t$. By using the Lyapunov-Schmidt reduction method, the authors proved that solutions will concentrate near a helix in the distributional sense, which satisfies the vortex filament conjecture, see [17, 18]. Rotational-invariant helical solutions in helical domains with bounded cross-section were constructed in [8]. For the problem of desingularization of steady Euler equations with helical symmetry, few results give us an answer.

Our goal in this article is to study vortex desingularization problem of steady Euler equations with helical symmetry in general helical domains. We will construct steady solutions of vorticity Eqs. (1.16) and (1.17), such that the associated vorticities have small cross-sections and concentrate near a single point as parameter changes. Accordingly, the vorticity field will concentrate near a one-dimensional helical filament. Both the case of $v_n \equiv 0$ and that of $v_n \not\equiv 0$ are considered. To get these results, we solve a semilinear elliptic equation in divergence form (see (2.1)). By studying the associated variational structure and using stream function method, we get the existence and limiting behavior of ground states of (2.1).

It should be noted that, Euler equations with helical symmetry can be regarded as the general case of 2D and 3D axisymmetric Euler equations. $k \to +\infty$ and $k = 0$ correspond to the 2D Euler equations and 3D axisymmetric Euler equations, respectively. However, there are striking differences between the helical symmetric case and the 2D and 3D axisymmetric cases. Note that the elliptic operator in 2D Euler equation is $-\Delta$, while in 3D axisymmetric case the operator is $-\frac{1}{b(x)} \nabla \cdot \left( \frac{1}{b(x)} \nabla \right)$ with $b(x) = x_1$, see [3, 11, 14, 23] for instance. In contrast to the 2D and 3D axisymmetric problem, the associated operator $L_H$ in vorticity Eq. (1.16) is a general elliptic operator in divergence form, which brings essential difficulty in studying the existence and asymptotic behavior of solutions. It seems impossible to reduce the second-order operator $L_H$ to the standard Laplacian by means of a single change of coordinates. Moreover, since the eigenvalues of $K_H$ are not the same, the limiting shape of support set of solutions is not a disc but an ellipse, which is totally different from the 2D and 3D axisymmetric cases. We will give qualitative estimates of the limiting location and diameter of the support set of ground states, see the proof of Theorem 1.5 in Sect. 3.

Now we introduce some notations. Define $\kappa(\omega) = \int_\Omega \omega(x) dx$ the circulation of the vorticity $\omega$. For two sets $A, B$, we define $\text{dist}(A, B) = \min_{x \in A, y \in B} |x - y|$ the distance between sets $A$ and $B$ and $\text{diam}(A)$ the diameter of the set $A$. Throughout the paper, the symbol $C$ denotes always a positive constant independent of $\epsilon$, which could be changed from one line to another.
Let us first consider desingularization of steady solutions of 3D Euler equations with helical symmetry under the impermeable boundary condition \( v_n \equiv 0 \). Since \((\omega, \varphi)\) is a steady solution, that is, the distribution of \(\omega, \varphi\) is independent of \(t\), by (1.17), \((\omega, \varphi)\) satisfies the steady vorticity equations

\[
\begin{align*}
\nabla \omega \cdot \nabla \varphi & = 0, \\
\omega & = \mathcal{L}_H \varphi, \\
\varphi|_{\partial \Omega} & = 0.
\end{align*}
\]

(1.18)

Formally, if

\[
\mathcal{L}_H \varphi = \omega = \frac{1}{\varepsilon^2} f(\varphi - \mu), \quad \varphi|_{\partial \Omega} = 0,
\]

for some function \(f\) and constants \(\varepsilon, \mu\), then (1.18) automatically holds. To conclude, it suffices to look for solutions of the Dirichlet problem of semilinear elliptic equations in divergence form

\[
\begin{align*}
\mathcal{L}_H \varphi(x) & = -\text{div} \cdot (K_H(x) \nabla \varphi(x)) = \frac{1}{\varepsilon^2} f(\varphi(x) - \mu), & x & \in \Omega, \\
\varphi(x) & = 0, & x & \in \partial \Omega.
\end{align*}
\]

(1.19)

Our first result is as follows.

**Theorem 1.2** For every \(k > 0, m > 0, 0 < \varepsilon < 1\), there exists a family of steady helical solution pairs \((v_\varepsilon, P_\varepsilon)(x, t) \in C^1(D \times \mathbb{R}^+)\) of Euler Eq. (1.1) such that the support set of \(\text{curl}v_\varepsilon\) is a topological helical tube and the associated vorticity-stream function pair \((\omega_\varepsilon, \varphi_\varepsilon)\) is a steady solution of vorticity Eq. (1.17). Moreover, there holds

1. \(v_\varepsilon \cdot n = 0\) on \(\partial D\).
2. The support set of \(\omega_\varepsilon\) is simply-connected and

\[
\lim_{\varepsilon \to 0} \frac{\ln \text{diam}(\text{supp}(\omega_\varepsilon))}{\ln \varepsilon} = 1.
\]

As a consequence, \(\lim_{\varepsilon \to 0} \text{diam}(\text{supp}(\omega_\varepsilon)) = 0\).

3. \(\lim_{\varepsilon \to 0} \text{dist}(\text{supp}(\omega_\varepsilon), x^*) = 0\), where \(x^* \in \overline{\Omega}\) satisfies \(|x^*| = \max_{\overline{\Omega}} |x|\).

4. \(\lim_{\varepsilon \to 0} \kappa(\omega_\varepsilon) = \frac{2k\pi m}{\sqrt{k^2 + |x^*|^2}}\).

To prove Theorem 1.2, we consider ground state solutions \(\varphi_\varepsilon\) of problems (1.19) with \(f(t) = t_+^p\) for \(p > 1\) and \(\mu = m \ln \frac{1}{\varepsilon}\) for a prescribed constant \(m > 0\), see the proof of Theorem 1.2 in Sect. 4.

**Remark 1.3** In [10], Dávila et al. constructed rotational-invariant solutions of (1.17) with angular velocity \(\alpha|\ln \varepsilon|\) in \(\mathbb{R}^2\). However, because of the choice of \(f_\varepsilon\), the support set of vorticity is still the whole plane. In contrast to [10], Theorem 1.2 shows the existence of steady solutions of (1.17) in a simply-connected bounded domain \(\Omega\) with \(C^\infty\) boundary, whose vorticity has small cross-section with non-vanishing circulation and shrinks to a point as \(\varepsilon \to 0\). Note that \(\Omega\) can be any simply-connected smooth bounded domain, such as a disk, an ellipse and a dumbbell-shaped domain.

**Remark 1.4** From Theorem 1.2, the limiting location \(x^*\) satisfies \(|x^*| = \max_{\overline{\Omega}} |x|\), which implies that the support set of the vorticity \(\omega_\varepsilon\) must concentrate near a point on the boundary.
of \( \Omega \). However in [10], the authors constructed helical symmetric solutions in the entire space, which obviously do not concentrate at the maximum radius of the region. This difference may come from the fact that the solutions in [10] and in Theorem 1.2 are two different solutions. Steady solutions of \((1.17)\) are constructed in Theorem 1.2, whereas rotational-invariant solutions of \((1.17)\) are constructed in [10].

Our second result is the desingularization of steady solutions of vorticity equations when the boundary is penetrable. Assume that \( v \cdot n = v_n \ln \frac{1}{\varepsilon} \) for some helical function \( v_n \neq 0 \). By \((1.16)\), steady solution pairs \((\omega, \varphi)\) satisfy

\[
\begin{cases}
\nabla \omega \cdot \nabla \varphi = 0, \\
\omega = \mathcal{L}_H \varphi, \\
\nabla \varphi \cdot v_{|\partial \Omega} = v_n |\partial \Omega \ln \frac{1}{\varepsilon}.
\end{cases} \tag{1.20}
\]

Suppose that \( q \in C^2(\Omega) \cap C^1(\overline{\Omega}) \) satisfies

\[
\begin{cases}
\mathcal{L}_H q = 0, \\
\nabla_q q \cdot v_{|\partial \Omega} = -v_n |\partial \Omega.
\end{cases} \tag{1.21}
\]

Note that for a solution \( q \) of \((1.21)\), \( q + C \) is also a solution for any constant \( C \). Thus one can always assume that \( \min_{\Omega} q > 0 \). Let \( u = \varphi + q \ln \frac{1}{\varepsilon} \). Then \( u \) satisfies

\[
\begin{cases}
\nabla \omega \cdot \nabla (u - q \ln \frac{1}{\varepsilon}) = 0, \\
\omega = \mathcal{L}_H u, \\
\nabla \mathcal{L} u \cdot v_{|\partial \Omega} = 0.
\end{cases} \tag{1.22}
\]

So if

\[
\mathcal{L}_H u = \omega = \frac{1}{\varepsilon^2} f(u - q \ln \frac{1}{\varepsilon}), \quad u_{|\partial \Omega} = 0,
\]

for some function \( f \) and constant \( \varepsilon \), then \((1.22)\) automatically holds. The solution pairs \((\omega, \varphi)\) of \((1.20)\) can then be obtained by letting \( \omega = \mathcal{L}_H u \) and \( \varphi = u - q \ln \frac{1}{\varepsilon} \).

Denote \( \det(K_H) \) the determinant of \( K_H \). Our second result is as follows.

**Theorem 1.5** Let \( k > 0, q > 0 \) satisfy \( \mathcal{L}_H q = 0 \) and \( v_n \) be a helical function defined on \( \partial D \) with \( v_n_{|\partial \Omega} = -\nabla_q q \cdot v_{|\partial \Omega} \). Then for every \( 0 < \varepsilon < 1 \), there exists a family of steady helical solution pairs \((v_\varepsilon, P_\varepsilon)(x, t) \in C^1(D \times \mathbb{R}^+)\) of Euler Eq. \((1.1)\) such that the support set of \( \text{curl} v_\varepsilon \) is a topological helical tube and the associated vorticity-stream function pair \((\omega_\varepsilon, \varphi_\varepsilon)\) is a solution of steady vorticity Eq. \((1.20)\). Moreover, the following conclusions hold

1. \( v_\varepsilon \cdot n = v_n \ln \frac{1}{\varepsilon} \) on \( \partial D \).
2. The support set of \( \omega_\varepsilon \) is simply-connected and

\[
\lim_{\varepsilon \to 0} \frac{\ln \text{diam}(\text{supp}(\omega_\varepsilon))}{\ln \varepsilon} = 1.
\]

3. \( \lim_{\varepsilon \to 0} \text{dist}(\text{supp}(\omega_\varepsilon), x^*) = 0 \), where \( x^* \in \overline{\Omega} \) is a minimum point of \( q^2 \sqrt{\det(K_H)} \), that is,

\[
q(x^*)^2 \sqrt{\det(K_H(x^*))} = \min_{\overline{\Omega}} q^2 \sqrt{\det(K_H)}.
\]

4. \( \lim_{\varepsilon \to 0} \int_{\Omega} \omega_\varepsilon(x) dx = 2\pi q(x^*) \sqrt{\det(K_H(x^*))} \).
Moreover, if $x^* \in \Omega$, then there exist $R_1, R_2 > 0$ independent of $\varepsilon$ such that

$$R_1 \varepsilon \leq \text{diam}(\text{supp}(\omega_\varepsilon)) \leq R_2 \varepsilon.$$  

Theorems 1.2 and 1.5 are proved via the so-called stream function method. The strategy is to consider ground state solutions of a Dirichlet problem of semilinear elliptic equations in divergence form

$$(\mathcal{P}) \quad \begin{cases} L_H u = -\text{div}(K_H(x) \nabla u) = \frac{1}{\varepsilon^2} \left( u - q \ln \frac{1}{\varepsilon} \right)_+^p, & x \in \Omega, \\ u = 0, & x \in \partial \Omega, \end{cases}$$

see (2.1) in Sect. 2. First, using the critical point theory, the existence of mountain pass solutions $u_\varepsilon$ of $(\mathcal{P})$ with critical value $c_\varepsilon$ of the corresponding variational functional is proved. By choosing proper test functions, we get the optimal upper bound of $c_\varepsilon$, from which we get the connectivity of the vortex core. Note that since eigenvalues of $K_H$ are different, one cannot use classical test functions to get accurate upper bounded of $c_\varepsilon$. The scaling parameters $l_1, l_2$ must be chosen appropriately, see Proposition 3.1. The boundedness of the energy of the vortex core is then obtained. Finally based on the classical estimates of capacity (see, e.g., [3, 11, 23]), we prove the lower bound of $c_\varepsilon$ and the limiting location of the vortex core. We will show that the concentration point of ground states $u_\varepsilon$ is a minimum point of $q^2 \sqrt{\det(K_H)}$, which is not known in the existing literature. To this end, the optimal upper and lower bounds of $c_\varepsilon$ must be obtained. Note that by using this method, we can get solutions of $(\mathcal{P})$ for any $\varepsilon \in (0, 1)$, rather than for small $\varepsilon$, and the support set of solutions is simply connected, see Proposition 3.2. Moreover, since minimum points of $q^2 \sqrt{\det(K_H)}$ on the boundary $\partial \Omega$, it is possible that solution $u_\varepsilon$ concentrates near the boundary. A direct consequence of Theorem 1.5 is that, if all the minimum points of $q^2 \sqrt{\det(K_H)}$ are on $\partial \Omega$, then ground states $u_\varepsilon$ of $(\mathcal{P})$ will concentrate near the boundary.

**Remark 1.6** The results of Theorem 1.5 can be regarded as a general result of classical planar vortex case (see [20, 23]) and the vortex ring case (see [11]). Note that the cases of planar vortices and vortex rings correspond to the coefficient matrix $K_H = Id$ and $\frac{1}{p} Id$, respectively. In [11], by considering solutions of

$$\begin{cases} -\text{div}(\frac{1}{p} \nabla u) = \frac{1}{\varepsilon^2} \left( u - q \ln \frac{1}{\varepsilon} \right)_+^{p-1}, & x \in \Omega, \\ u = 0, & x \in \partial \Omega, \end{cases}$$

where $b$ and $q$ are positive functions satisfying $-\text{div}(\frac{1}{p} \nabla q) = 0$, the authors constructed $C^1$ solutions $u_\varepsilon$ with non-vanishing circulation concentrating near a minimizer of $q^2/b$ as $\varepsilon \to 0$. Indeed, if we choose $K_H(x) = \frac{1}{p} Id$ in Theorem 1.5, then solutions will concentrate near minimizers of $q^2 \sqrt{\det(K_H)} = q^2/b$, which coincides with the results in [11].

**Remark 1.7** In fact, using the same proof as that in Sects. 2–3 it is possible to obtain the existence and asymptotic behavior of ground states of the Dirichlet problem of semilinear elliptic equations

$$\begin{cases} -\text{div}(K(x) \nabla u) = \frac{1}{\varepsilon^2} \left( u - q \ln \frac{1}{\varepsilon} \right)_+^p, & x \in \Omega, \\ u = 0, & x \in \partial \Omega, \end{cases}$$

where $K$ is a positive-definite matrix satisfying ($K_1$)-($K_2$). The key is to get the optimal upper bound of $c_\varepsilon$, see Proposition 3.1. Once the upper bound is obtained, we can repeat the proof in Sect. 3 to get the existence of ground state solutions concentrating near minimizers of $q^2 \sqrt{\det(K)}$. 

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This paper is organized as follows. In Sect. 2, we introduce the associated variational structure and prove the existence of mountain pass solutions of (2.1) for every \( \varepsilon \in (0, 1) \). Some fundamental properties which will be used in Sect. 3 are also proved. In Sect. 3 we prove the limiting behavior of \( u_\varepsilon \). The proof of Theorem 1.5 and Theorem 1.2 will be given in Sect. 4.

### 2 Variational problem

We now consider the following Dirichlet problem of semilinear elliptic equations

\[
\begin{align*}
\mathcal{L}_H u &= -\text{div}(K_H(x) \nabla u) = \frac{1}{\varepsilon^2} \left( u - q \ln \frac{1}{\varepsilon} \right)^p, \quad x \in \Omega, \\
u &= 0, \quad x \in \partial \Omega,
\end{align*}
\]

where \( p > 1, \Omega \subseteq \mathbb{R}^2 \) is bounded, \( K_H(x) = \frac{1}{k^2 + x_1^2 + x_2^2} \left( \frac{k^2 + x_2^2}{-x_1 x_2} - \frac{x_1 x_2}{k^2 + x_1^2} \right) \) and \( q \) is a function defined in \( \Omega \) satisfying

(Q1). \( q \in C^2(\Omega) \cap C^1(\overline{\Omega}) \) and \( q > 0 \) in \( \Omega \).

(Q2). \( q \) is a \( \mathcal{L}_H - \)harmonic function, i.e., \( \mathcal{L}_H q = -\text{div}(K_H(x) \nabla q) = 0 \).

Let \( (K_H(x)a,b) = \sum_{i,j=1}^2 (K_H)_{i,j}(x) a_i b_j \) for two vectors \( a, b \). Define

\[
\mathcal{H}(\Omega) = \left\{ u \in H^1_0(\Omega) \mid \int_{\Omega} (K_H(x) \nabla u | \nabla u) dx < +\infty \right\}
\]

with the norm

\[
||u||_{\mathcal{H}(\Omega)} := \left( \int_{\Omega} (K_H(x) \nabla u | \nabla u) dx \right)^{\frac{1}{2}}.
\]

Since \( K_H \) is a positive definite matrix with two positive eigenvalues \( \lambda_1 = 1 \) and \( \lambda_2 = \frac{k^2}{k^2 + x_1^2 + x_2^2} \), two norms \( || \cdot ||_{\mathcal{H}(\Omega)} \) and \( || \cdot ||_{H^1_0(\Omega)} \) are equivalent.

Define the associated energy functional of (2.1)

\[
I_\varepsilon(u) = \frac{1}{2} \int_{\Omega} (K_H(x) \nabla u | \nabla u) dx - \frac{1}{(p+1)\varepsilon^2} \int_{\Omega} \left( u - q \ln \frac{1}{\varepsilon} \right)^{p+1} dx, \quad \forall u \in \mathcal{H}(\Omega).
\]

By the definition of \( \mathcal{H} \), \( I_\varepsilon \) is a well-defined \( C^1 \) functional on \( \mathcal{H} \).

Define the Nehari manifold

\[
\mathcal{N}_\varepsilon = \{ u \in \mathcal{H}(\Omega) \setminus \{0\} \mid \langle I_\varepsilon'(u), u \rangle = 0 \}
\]

\[
= \left\{ u \in \mathcal{H}(\Omega) \setminus \{0\} \mid \int_{\Omega} (K_H(x) \nabla u | \nabla u) dx = \frac{1}{\varepsilon^2} \int_{\Omega} \left( u - q \ln \frac{1}{\varepsilon} \right)^p u dx \right\}.
\]

### 2.1 Existence of solutions

First, using the classical critical point theory, we get the existence of ground state solutions of (2.1).
Since the nonlinearity \( f(t, x) = \frac{1}{\varepsilon^2} ((t - q(x) \ln \frac{1}{\varepsilon})^p + \text{for } p > 1, I_\varepsilon(u) \) has a mountain pass geometry. Thus we can define the mountain pass value
\[
c_\varepsilon = \inf_{\gamma \in \mathcal{P}_\varepsilon} \max_{t \in [0, 1]} I_\varepsilon(\gamma(t)),
\]
where
\[
\mathcal{P}_\varepsilon = \{ \gamma \in C([0, 1], \mathcal{H}(\Omega)) \mid \gamma(0) = 0, I_\varepsilon(\gamma(1)) < 0 \}.
\]
Clearly, \( c_\varepsilon > 0 \). We have the following characterization of \( N_\varepsilon \) and the mountain pass value \( c_\varepsilon \), see [7, 16].

**Lemma 2.1** ([7], Theorem 1.3.7) For any \( u \in N_\varepsilon, u_+ \neq 0 \). For any \( u \in \mathcal{H}(\Omega) \) with \( u_+ \neq 0 \), there exists a unique \( t(u) > 0 \) such that \( t(u)u \in N_\varepsilon \). The value of \( t(u) \) is characterized by the identity
\[
I_\varepsilon(t(u)u) = \max\{I_\varepsilon(tu), t > 0\}. \tag{2.4}
\]
Moreover, there holds
\[
c_\varepsilon \leq \inf_{w \neq 0, w \in \mathcal{H}(\Omega)} \max_{t \geq 0} I_\varepsilon(tw) = \inf_{w \in N_\varepsilon} I_\varepsilon(w).
\]
Finally, if the mountain pass value \( c_\varepsilon \) is a critical value for \( I_\varepsilon \), then \( c_\varepsilon = \inf_{w \in N_\varepsilon} I_\varepsilon(w) \) is the least nontrivial critical value.

Using the mountain pass theorem, we get the existence of critical points of \( I_\varepsilon \) with critical value \( c_\varepsilon \).

**Proposition 2.2** \( I_\varepsilon \) has a critical point with mountain pass value \( c_\varepsilon \). Namely, one can find \( u_\varepsilon \in \mathcal{H}(\Omega) \) satisfying
\[
I_\varepsilon(u_\varepsilon) = c_\varepsilon, \quad I_\varepsilon'(u_\varepsilon) = 0. \tag{2.5}
\]
As a consequence, there holds \( c_\varepsilon = \inf_{w \neq 0, w \in \mathcal{H}(\Omega)} \max_{t \geq 0} I_\varepsilon(tw) = \inf_{w \in N_\varepsilon} I_\varepsilon(w). \)

**Proof** By standard mountain pass lemma (see, e.g., Sect. 1.4 in [24]), we can prove that there exists \( u_\varepsilon \) such that \( I_\varepsilon'(u_\varepsilon) = 0 \) and \( I_\varepsilon(u_\varepsilon) = c_\varepsilon \). So by Lemma 2.1, \( c_\varepsilon = \inf_{w \in N_\varepsilon} I_\varepsilon(w) \). \( \square \)

**Remark 2.3** The critical point \( u_\varepsilon \) obtained in Proposition 2.2 is a positive solution of (2.1), which is usually called the mountain pass solution.

### 2.2 Basic properties

First, we give some basic properties of \( I_\varepsilon \) and the operator \( L_H \) as follows.

**Lemma 2.4** For any \( u \in \mathcal{H}(\Omega) \),
\[
\left( \frac{1}{2} - \frac{1}{p + 1} \right) \|u\|_{\mathcal{H}(\Omega)}^2 \leq I_\varepsilon(u) - \frac{1}{p + 1} (I_\varepsilon'(u), u). \tag{2.6}
\]
Proof It follows from the definition of $I_\varepsilon$ and $I'_\varepsilon$ that

$$I_\varepsilon(u) - \frac{1}{p+1} \langle I'_\varepsilon(u), u \rangle = \frac{1}{2} ||u||_{\mathcal{H}(\Omega)}^2 - \frac{1}{(p+1)\varepsilon^2} \int_{\Omega} \left( u - q \ln \frac{1}{\varepsilon} \right)^{p+1} + \left( \frac{1}{p+1} ||u||_{\mathcal{H}(\Omega)}^2 - \frac{1}{(p+1)\varepsilon^2} \int_{\Omega} \left( u - q \ln \frac{1}{\varepsilon} \right)^p \right) u \geq \left( \frac{1}{2} - \frac{1}{p+1} \right) ||u||_{\mathcal{H}(\Omega)}^2.$$  

\[ \square \]

Lemma 2.5 For any $u \in \mathcal{H}(\Omega)$,

$$\int_{\Omega} (K_H(x) \nabla u | \nabla u) dx = \int_{\Omega} q^2 \left( K_H(x) \nabla \left( \frac{u^2}{q} \right) \right) dx. \tag{2.7}$$

Proof We first claim that

$$\left( K_H(x) \nabla q | \nabla \left( \frac{u^2}{q} \right) \right) = (K_H(x) \nabla u | \nabla u) - q^2 \left( K_H(x) \nabla \left( \frac{u}{q} \right) \right) \tag{2.8}.$$  

Indeed, we have

$$(K_H)_{11}(\partial_1 u)^2 - q^2(K_H)_{11} \left( \partial_1 \left( \frac{u}{q} \right) \right)^2 = (K_H)_{11} \left( \frac{2u}{q} \partial_1 \partial_1 q - \frac{u^2}{q^2} (\partial_1 q)^2 \right) = (K_H)_{11} \partial_1 q \partial_1 \left( \frac{u^2}{q} \right),$$

$$((K_H)_{12} + (K_H)_{21}) \partial_1 u \partial_2 u - q^2((K_H)_{12} + (K_H)_{21}) \partial_1 q \partial_2 \left( \frac{u^2}{q} \right) = 2(K_H)_{12} \frac{u}{q} (\partial_1 q \partial_2 u + \partial_2 q \partial_1 u) - 2(K_H)_{12} \frac{u^2}{q^2} \partial_1 q \partial_2 q,$$

$$= (K_H)_{12} \partial_1 q \partial_2 \left( \frac{u^2}{q} \right) + (K_H)_{21} \partial_2 q \partial_1 \left( \frac{u^2}{q} \right),$$

and

$$(K_H)_{22}(\partial_2 u)^2 - q^2(K_H)_{22} \left( \partial_2 \left( \frac{u}{q} \right) \right)^2 = (K_H)_{22} \left( \frac{2u}{q} \partial_2 u \partial_2 q - \frac{u^2}{q^2} (\partial_2 q)^2 \right)$$

$$= (K_H)_{22} \partial_2 q \partial_2 \left( \frac{u^2}{q} \right).$$

Adding up the above inequalities, we get (2.8).

Since $\mathcal{L}_H q = 0$ and $\frac{u^2}{q} \in H^1_0(\Omega)$, we have $\int_{\Omega} \left( K_H(x) \nabla q | \nabla \left( \frac{u^2}{q} \right) \right) dx = 0$. Integrating both sides of (2.8) over $\Omega$, we get (2.7).

\[ \square \]
3 Asymptotic behavior of $u_\varepsilon$

Now, we give the asymptotic behavior of mountain pass solutions $u_\varepsilon$ of (2.1). Let $x^* \in \overline{\Omega}$ be such that

$$q^2(x^*)\sqrt{\det(K_H(x^*))} = \min_{x \in \Omega} q^2(x)\sqrt{\det(K_H(x))}.$$ 

Let $0 < \varepsilon < 1$.

### 3.1 Upper bound of $c_\varepsilon$

First, by choosing proper competitors, we get the following upper bound of $c_\varepsilon$.

**Proposition 3.1** There holds

$$\limsup_{\varepsilon \to 0} \frac{c_\varepsilon}{\ln 1/\varepsilon} \leq \pi q^2\sqrt{\det(K_H(x^*))} = \min_{x \in \Omega} q^2\sqrt{\det(K_H(x))}.$$ 

Moreover, if $x^* \in \Omega$, then

$$c_\varepsilon \leq \pi \min_{x \in \Omega} q^2\sqrt{\det(K_H(x))} \ln \frac{1}{\varepsilon} + O(1).$$

**Proof** Let $U(x)$ be a $C^\infty$ radially symmetric function such that

$$\begin{cases}
U(x) \geq 0, & x \in B_1(0), \\
U(x) = \ln \frac{1}{|x|}, & x \in B_1(0)^c.
\end{cases}$$

For any $\tilde{x} \in \Omega$, we choose $\delta > 0$ sufficiently small and a truncation $\phi_\delta \in C^\infty_c(B_{2\delta}(0))$ such that

$$0 \leq \phi_\delta \leq 1 \text{ in } B_{2\delta}(0); \quad \phi_\delta = 1 \text{ in } B_\delta(0).$$

Since $K_H(\tilde{x})$ is positive-definite, one can choose an orthogonal matrix $P_{\tilde{x}}$ such that $P_{\tilde{x}}K_H(\tilde{x})P_{\tilde{x}}^t$ is a diagonal matrix. Define for any $x \in \Omega$

$$\tilde{K}_H(x) = P_{\tilde{x}}K_H(x)P_{\tilde{x}}^t.$$ 

Then $\tilde{K}_H(\tilde{x})$ is diagonal and $\det(\tilde{K}_H(\tilde{x})) = (\tilde{K}_H)_{11}(\tilde{K}_H)_{22}(\tilde{x}) = \det(K_H)(\tilde{x})$.

For any constants $l_1, l_2 > 0$ (which will be determined later), define

$$\hat{U}(x_1, x_2) = U\left(\frac{x_1}{l_1}, \frac{x_2}{l_2}\right).$$

So the support set of $\hat{U}_+$ is the ellipse $\left\{(x_1, x_2) \mid \frac{x_1^2}{l_1^2} + \frac{x_2^2}{l_2^2} \leq 1 \right\}$. For any set $A$, define

$$\hat{A} = \left\{(x_1, x_2) \mid \left(\frac{x_1}{l_1}, \frac{x_2}{l_2}\right) \in A \right\}.\text{ Let } \hat{\phi}_\delta(x_1, x_2) = \phi_\delta\left(\frac{x_1}{l_1}, \frac{x_2}{l_2}\right).\text{ Then } \text{supp}(\hat{\phi}_\delta) = \hat{B}_\delta \text{ and } \hat{\phi}_\delta \equiv 1 \text{ on } \hat{B}_\delta.$$

For any $r > 0$, define $P_{\tilde{x}}^{-1}\hat{B}_r(\tilde{x}) = \{P_{\tilde{x}}^{-1}x + \tilde{x} \mid x \in \hat{B}_r(0)\}$.

We define for any $\tau > 0$ a test function

$$u_\varepsilon^r = q(x)\left(\hat{U}\left(\frac{P_{\tilde{x}}(x - \tilde{x})}{\varepsilon}\right) + \ln \frac{\tau}{\varepsilon}\right)\hat{\phi}_\delta(P_{\tilde{x}}(x - \tilde{x})).$$

$$= \left[q \ln \frac{1}{\varepsilon} + q\left(\hat{U}\left(\frac{P_{\tilde{x}}(x - \tilde{x})}{\varepsilon}\right) + \ln \tau\right)\right]\hat{\phi}_\delta(P_{\tilde{x}}(x - \tilde{x})).$$
Then \( v_\varepsilon^\tau \in H_0^1(\Omega) \). Define
\[
g_\varepsilon(\tau) := \frac{1}{\ln \varepsilon} \langle I'_\varepsilon(v_\varepsilon^\tau), v_\varepsilon^\tau \rangle = \frac{1}{\ln \varepsilon} \left( \int_\Omega (K_H(x) \nabla v_\varepsilon^\tau | \nabla v_\varepsilon^\tau) dx - \frac{1}{\varepsilon^2} \int_\Omega \left( v_\varepsilon^\tau - q \ln \frac{1}{\varepsilon} \right)^p + v_\varepsilon^\tau dx \right).
\]

We now prove that there exists \( \tau_\varepsilon > 0 \) such that \( g_\varepsilon(\tau_\varepsilon) = 0 \), that is, \( v_\varepsilon^\tau \in \mathcal{N}_\varepsilon \). By Lemma 2.5,
\[
\int_\Omega (K_H(x) \nabla v_\varepsilon^\tau | \nabla v_\varepsilon^\tau) dx = \int_\Omega q^2 \left( K_H(x) \nabla \left( \frac{v_\varepsilon^\tau}{q} \right) | \nabla \left( \frac{v_\varepsilon^\tau}{q} \right) \right) dx
\]
\[
= \left( \int_{P_x^{-1} \hat{B}_{\varepsilon}(\tilde{x}) \setminus P_x^{-1} \hat{B}_{\varepsilon}(\tilde{x})} q^2 \left( K_H(x) \nabla \left( \frac{v_\varepsilon^\tau}{q} \right) | \nabla \left( \frac{v_\varepsilon^\tau}{q} \right) \right) dx \right)
\]
\[
= A_1 + A_2 + A_3.
\]

By the definition of \( v_\varepsilon^\tau \), we have
\[
A_1 \leq C \left( 1 + \ln \frac{\tau}{\delta} \right),
\]
where \( C \) is independent of \( \tau \). Since \( \varepsilon \) sufficiently small, we can assume that \( \varepsilon < \delta \), which implies that
\[
A_3 = \int_{P_x^{-1} \hat{B}_{\varepsilon}(\tilde{x}) \setminus P_x^{-1} \hat{B}_{\varepsilon}(\tilde{x})} q^2 \left( K_H(x) \nabla \hat{U} \left( \frac{P_x(x - \tilde{x})}{\varepsilon} \right) | \nabla \hat{U} \left( \frac{P_x(x - \tilde{x})}{\varepsilon} \right) \right)
\]
\[
= \int_{\hat{B}_{\varepsilon}(0)} q^2 (P_x^{-1} y + \tilde{x}) \left( K_H(\varepsilon P_x^{-1} y + \tilde{x}) \nabla \hat{U} \left( \frac{P_x x}{\varepsilon} \right) | \nabla \hat{U} \left( \frac{P_x x}{\varepsilon} \right) \right) dy
\]
\[
\Rightarrow \int_{\hat{B}_{\varepsilon}(0)} q^2 (\tilde{x}) (K_H(\tilde{x}) \nabla \hat{U}(y) | \nabla \hat{U}(y)) dy \quad \text{as} \quad \varepsilon \to 0.
\]
The convergence is uniform about \( \tau \).

As for \( A_2 \), we have
\[
A_2 = \int_{P_x^{-1} \hat{B}_{\varepsilon}(\tilde{x}) \setminus P_x^{-1} \hat{B}_{\varepsilon}(\tilde{x})} q^2 \left( K_H(x) \nabla \left( \frac{v_\varepsilon^\tau}{q} \right) | \nabla \left( \frac{v_\varepsilon^\tau}{q} \right) \right)
\]
\[
= \int_{P_x^{-1} \hat{B}_{\varepsilon}(0) \setminus P_x^{-1} \hat{B}_{\varepsilon}(0)} q^2 (P_x^{-1} x + \tilde{x}) \left( K_H(P_x^{-1} x + \tilde{x}) \nabla \hat{U} \left( \frac{X}{\varepsilon} \right) | \nabla \hat{U} \left( \frac{X}{\varepsilon} \right) \right)
\]

Note that for any vector \( a = (a_1, a_2) \),
\[
(\tilde{K}_H(x)a|a) = (\tilde{K}_H)_{11}(x)a_1^2 + ((\tilde{K}_H)_{12} + (\tilde{K}_H)_{21})(x) a_1 a_2 + (\tilde{K}_H)_{22}(x) a_2^2.
\]

Hence we have
\[
A_2 = \int_{\hat{B}_{\varepsilon}(0) \setminus \hat{B}_{\varepsilon}(0)} q^2 (\tilde{K}_H)_{11}(P_x^{-1} x + \tilde{x}) \partial_1 \hat{U} \left( \frac{X}{\varepsilon} \right) \partial_1 \hat{U} \left( \frac{X}{\varepsilon} \right) dx
\]
\[
+ \int_{\hat{B}_{\varepsilon}(0) \setminus \hat{B}_{\varepsilon}(0)} q^2 (P_x^{-1} x + \tilde{x})(\tilde{K}_H)_{12} + (\tilde{K}_H)_{21}) \partial_1 \hat{U} \left( \frac{X}{\varepsilon} \right) \partial_2 \hat{U} \left( \frac{X}{\varepsilon} \right) dx
\]
\[
+ \int_{\hat{B}_{\varepsilon}(0) \setminus \hat{B}_{\varepsilon}(0)} q^2 (\tilde{K}_H)_{22}(P_x^{-1} x + \tilde{x}) \partial_2 \hat{U} \left( \frac{X}{\varepsilon} \right) \partial_2 \hat{U} \left( \frac{X}{\varepsilon} \right) dx.
\]
Note that \( \nabla \hat{U}(x_1, x_2) = \left( -\frac{1}{l_1^2} \frac{x_1}{x_1^2 + x_2^2}, -\frac{1}{l_2^2} \frac{x_2}{x_1^2 + x_2^2} \right) \) on \( \hat{B}_1(0)^c \). Hence direct calculation yields

\[
\int_{\hat{B}_1(0) \setminus \hat{B}_e(0)} q^2(\tilde{K}_H)_{11}(P_{\tilde{x}}^{-1}x + \tilde{x}) \partial_1 \hat{U} \left( \frac{x}{\epsilon} \right) \partial_1 \hat{U} \left( \frac{x}{\epsilon} \right) \, dx
= \int_{\hat{B}_1(0) \setminus \hat{B}_e(0)} q^2(\tilde{K}_H)_{11}(\epsilon P_{\tilde{x}}^{-1}x + \tilde{x}) \partial_1 \hat{U}(x) \partial_1 \hat{U}(x) \, dx
= \frac{1}{l_1^4} \int_{\hat{B}_1(0) \setminus \hat{B}_e(0)} q^2(\tilde{K}_H)_{11}(\epsilon P_{\tilde{x}}^{-1}x + \tilde{x}) \frac{x^2}{(\frac{x_1}{l_1^2} + \frac{x_2}{l_2^2})^2} \, dx
= \frac{l_2}{l_1} \int_{\hat{B}_1(0) \setminus \hat{B}_e(0)} \frac{x^2}{(x_1^2 + x_2^2)^2} \, dx
= \frac{l_2}{l_1} \pi q^2(\tilde{K}_H)_{11}(\tilde{x}) \ln \frac{\delta}{\epsilon} + O(1),
\]

where we used \( K_H, q \in C^2 \) and Taylor expansion.

Similarly, we can get

\[
\int_{\hat{B}_1(0) \setminus \hat{B}_e(0)} q^2(\tilde{K}_H)_{22}(P_{\tilde{x}}^{-1}x + \tilde{x}) \partial_2 \hat{U} \left( \frac{x}{\epsilon} \right) \partial_2 \hat{U} \left( \frac{x}{\epsilon} \right) \, dx = \frac{l_1}{l_2} \pi q^2(\tilde{K}_H)_{22}(\tilde{x}) \ln \frac{\delta}{\epsilon} + O(1).
\]

(3.4)

By the definition of \( \tilde{K}_H \), we get \( (\tilde{K}_H)_{12} = (\tilde{K}_H)_{21} = 0 \), which implies that

\[
\int_{\hat{B}_1(0) \setminus \hat{B}_e(0)} q^2 \left( (\tilde{K}_H)_{12} + (\tilde{K}_H)_{21} \right) (P_{\tilde{x}}^{-1}x + \tilde{x}) \partial_1 \hat{U} \left( \frac{x}{\epsilon} \right) \partial_2 \hat{U} \left( \frac{x}{\epsilon} \right) \, dx
= \int_{\hat{B}_1(0) \setminus \hat{B}_e(0)} q^2 \left( (\tilde{K}_H)_{12} + (\tilde{K}_H)_{21} \right) (\tilde{x}) \frac{x^2}{(x_1^2 + x_2^2)^2} \, dx + O(1) = O(1).
\]

(3.5)

So by (3.3), (3.4) and (3.5) we have

\[
A_2 = \frac{l_2}{l_1} \pi q^2(\tilde{K}_H)_{11}(\tilde{x}) \ln \frac{1}{\epsilon} + \frac{l_1}{l_2} \pi q^2(\tilde{K}_H)_{22}(\tilde{x}) \ln \frac{1}{\epsilon} + O(1),
\]

(3.6)

where \( O(1) \) is some bounded quantity independent of \( \tau \).

Choosing \( \frac{l_2}{l_1} = \sqrt{\frac{(\tilde{K}_H)_{22}}{(\tilde{K}_H)_{11}}} (\tilde{x}) \) and by (3.1), (3.2) and (3.6), we get

\[
\int_{\Omega} (K_H(x) \nabla v_\tau^x \nabla v_\tau^x) \, dx = 2\pi q^2 \sqrt{(\tilde{K}_H)_{11}(\tilde{K}_H)_{22}}(\tilde{x}) \ln \frac{1}{\epsilon} + O(1) + C|\ln \tau|
\]

(3.7)

\[
= 2\pi q^2 \sqrt{\det(K_H)(\tilde{x})} \ln \frac{1}{\epsilon} + O(1) + C|\ln \tau|,
\]
from which we deduce that
\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \int_\Omega \left( v_\varepsilon^T - q(x) \ln \frac{1}{\varepsilon} \right)_+^p v_\varepsilon^T dx = \frac{1}{\varepsilon^2} \int_\Omega \left( v_\varepsilon^T - q(x) \ln \frac{1}{\varepsilon} \right)_+^p dx + \frac{1}{\varepsilon^2} \int_\Omega \left( v_\varepsilon^T - q(x) \ln \frac{1}{\varepsilon} \right)_+ q(x) \ln \frac{1}{\varepsilon} dx.
\]

By the definition of \( v_\varepsilon^T \), for \( \varepsilon \) sufficiently small and every \( x \in \Omega \), we have
\[
\left( v_\varepsilon^T(x) - q(x) \ln \frac{1}{\varepsilon} \right)_+ = q(x) \left( \hat{U} \left( \frac{P_x(x - \bar{x})}{\varepsilon} \right) + \ln \tau \right)_+.
\]

Hence we get
\[
\frac{1}{\varepsilon^2} \int_\Omega \left( v_\varepsilon^T - q(x) \ln \frac{1}{\varepsilon} \right)_+^p dx = \int_{\bar{B}_1(0)} q^{p+1}(\bar{x} + \varepsilon P_{\bar{x}}^{-1} y)(\hat{U}(y) + \ln \tau)_+^{p+1} dy
\]
\[
\Rightarrow q^{p+1}(\bar{x}) \int_{\bar{B}_1(0)} (\hat{U}(y) + \ln \tau)_+^{p+1} dy,
\]
and
\[
\frac{1}{\varepsilon^2} \int_\Omega \left( v_\varepsilon^T - q(x) \ln \frac{1}{\varepsilon} \right)_+ q(x) \ln \frac{1}{\varepsilon} dx \Rightarrow q^{p+1}(\bar{x}) \int_{\bar{B}_1(0)} (\hat{U}(y) + \ln \tau)_+^{p} dy.
\]

The convergences are uniform in any compact set of \( \tau \). By (3.9) and (3.10), we get
\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2 \ln \frac{1}{\varepsilon}} \int_\Omega \left( v_\varepsilon^T - q(x) \ln \frac{1}{\varepsilon} \right)_+^p v_\varepsilon^T dx = q^{p+1}(\bar{x}) \int_{\bar{B}_1(0)} (\hat{U}(y) + \ln \tau)_+^{p} dy. \tag{3.11}
\]

It follows from (3.8) and (3.11) that for any \( \tau > 0 \), \( \lim_{\varepsilon \to 0} g_\varepsilon(\tau) = g(\tau) \), where \( g \) is defined by \( g(\tau) = 2\pi q^2 \sqrt{det(KH)}(\bar{x}) - q^{p+1}(\bar{x}) \int_{\bar{B}_1(0)} (\hat{U}(y) + \ln \tau)_+^{p} dy \), and the convergence is uniform in any compact set of \( \tau \). Now it is not hard to prove that there exist two numbers \( \tau_1, \tau_2 > 0 \) such that \( g(\tau_1) < 0 < g(\tau_2) \). So for \( \varepsilon \) sufficiently small, we have \( g_\varepsilon(\tau_1) < 0 < g_\varepsilon(\tau_2) \), from which we deduce that, there exists \( \tau_\varepsilon \in (\tau_1, \tau_2) \) satisfying \( g_\varepsilon(\tau_\varepsilon) = 0 \). Then \( v_\varepsilon^T \in N_\varepsilon \).

Now by (3.8), (3.11) and \( \tau_\varepsilon \in (\tau_1, \tau_2) \), we can compute that
\[
\lim_{\varepsilon \to 0} \frac{1}{\ln \frac{1}{\varepsilon}} \int_\Omega (v_\varepsilon^T)_+ = \lim_{\varepsilon \to 0} \frac{1}{2} \ln \frac{1}{\varepsilon} \int_\Omega (KH(x) \nabla v_\varepsilon^T \nabla v_\varepsilon^T) - \lim_{\varepsilon \to 0} \frac{1}{(p+1)\varepsilon^2 \ln \frac{1}{\varepsilon}} \int_\Omega \left( v_\varepsilon^T - q(x) \ln \frac{1}{\varepsilon} \right)_+^{p+1} = \pi q^2 \sqrt{det(KH)(\bar{x})}.
\]

Taking the infimum over \( \bar{x} \in \Omega \), we get \( \limsup_{\varepsilon \to 0} \frac{\varepsilon}{\ln \frac{1}{\varepsilon}} \leq \pi q^2(x^*) \sqrt{det(KH(x^*))} = \pi \min_{x \in \Omega} q^2 \sqrt{det(KH)(x)} \).
If $x^* \in \Omega$, we can improve the above estimate as follows. Indeed, choosing $\bar{x} = x^*$, and using (3.7) and (3.9) again, we can get

$$
c_{k} \leq I_{e}(v_{e}^{\tau}) = \frac{1}{2} \int_{\Omega}(K_{H}(x)\nabla v_{e}^{\tau}|\nabla v_{e}^{\tau})dx - \frac{1}{(p+1)e^{2}} \int_{\Omega} \left( v_{e}^{\tau} - q - \frac{1}{e} \right)^{p+1}dx
$$

$$
= \pi q^{2} \sqrt{\text{det}(K_{H})}/\Omega(x^{*}) \ln \frac{1}{e} + o(1),
$$

where we used the fact that $\varepsilon < 1$ and again, we can get

$$
\text{Proposition 3.2} \quad \text{For every \( \varepsilon > 0 \), \( \partial \Omega \) is an open subset of \( \varepsilon \).}
$$

As a consequence, diam $A_{\varepsilon}$ tends to 0 as $\varepsilon \to 0$.

**Proof** Assume that $A_{\varepsilon}$ has two components $A_{1}, A_{2}$. We denote $\psi_{1} = (u_{\varepsilon} - q \ln 1/e)_{+} + \chi_{A_{1}}$. Since $\varepsilon < 1$ and $\min_{\Omega} q > 0$, we get $q \ln 1/e > 0$ on $\partial \Omega$, which implies that $A_{\varepsilon}$ has a positive distance from $\partial \Omega$ and $(u_{\varepsilon} - q \ln 1/e)_{+} \in H^{1}_{0}(\Omega)$. Thus using the fact that $A_{1}$ is open and lemma 12 in [11], we get $\psi_{1} \in H^{1}_{0}(\Omega)$. Let $\eta_{0} > 0$ to be determined later. Define $\tilde{w}_{\varepsilon}(s) = u_{\varepsilon} + s\psi_{1} - s\eta_{0}\psi_{2}$. Then $\tilde{w}_{\varepsilon}(s) \in H^{1}_{0}(\Omega)$ for $s \geq 0$ sufficiently small. By Proposition 2.2, $t_{0}$ is a maximum point of $I_{e}(t\tilde{w}_{\varepsilon})$ if and only if

$$
t_{0} \int_{\Omega}(K_{H}(x)\nabla \tilde{w}_{\varepsilon}|\nabla \tilde{w}_{\varepsilon})dx = \frac{1}{e^{2}} \int_{\Omega} \left( t_{0}\tilde{w}_{\varepsilon} - q - \frac{1}{e} \right)^{p} \tilde{w}_{\varepsilon}dx.
$$

Let $D(s, t) = t \int_{\Omega}(K_{H}(x)\nabla \tilde{w}_{\varepsilon}|\nabla \tilde{w}_{\varepsilon})dx - \frac{1}{e^{2}} \int_{\Omega} (t\tilde{w}_{\varepsilon} - q - \frac{1}{e})^{p} \tilde{w}_{\varepsilon}dx$. Then $D(0, 1) = 0$. Since $q > 0$ and $\varepsilon \in (0, 1)$, we have $q \ln 1/e > 0$, which implies that

$$\frac{\partial D(0, t)}{\partial t} \bigg|_{t=1} = \int_{\Omega}(K_{H}(x)\nabla u_{\varepsilon}|\nabla u_{\varepsilon})dx - \frac{p}{e^{2}} \int_{\Omega} (u_{\varepsilon} - q - \frac{1}{e})^{p-1}u_{\varepsilon}^{2}dx
$$

$$= \frac{1}{e^{2}} \int_{\Omega} \left( u_{\varepsilon} - q - \frac{1}{e} \right)^{p}u_{\varepsilon} - p \left( u_{\varepsilon} - q - \frac{1}{e} \right)^{p-1}u_{\varepsilon}^{2}dx < 0.$$

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Hence by the implicit function theorem, there is a function \( t = t(s) \) in the neighborhood of \( s = 0 \) such that \( t(0) = 1 \) and \( D(s, t(s)) = 0 \), which implies that, \( t(s)\bar{w}_\varepsilon \in N_{\varepsilon} \). Note that

\[
D_s(0, 1) = 2 \int_\Omega (K_H(x)\nabla u_e|\nabla (\psi_1 - \eta_0 \psi_2)) dx - \frac{1}{\varepsilon^2} \int_\Omega \left( u_e - q \ln \frac{1}{\varepsilon} \right)^p_+ (\psi_1 - \eta_0 \psi_2) dx
- \frac{1}{\varepsilon^2} \int_\Omega p \left( u_e - q \ln \frac{1}{\varepsilon} \right)^{p-1}_+ u_e (\psi_1 - \eta_0 \psi_2) dx
= -\frac{1}{\varepsilon^2} \int_\Omega \left( u_e - q \ln \frac{1}{\varepsilon} \right)^{p-1}_+ \left( (p-1)u_e + q \ln \frac{1}{\varepsilon} \right) (\psi_1 - \eta_0 \psi_2) dx.
\]

If we choose

\[
\eta_0 = \frac{\int_\Omega \left( u_e - q \ln \frac{1}{\varepsilon} \right)^{p-1}_+ (p-1)u_e + q \ln \frac{1}{\varepsilon} \psi_1 dx}{\int_\Omega \left( u_e - q \ln \frac{1}{\varepsilon} \right)^{p-1}_+ (p-1)u_e + q \ln \frac{1}{\varepsilon} \psi_2 dx} > 0,
\]

then by the chain rule, \( t'(0) = -\frac{D_s(0, 1)}{D_t(0, 1)} = 0 \), which implies that for \( s \) small \( t(s) = 1 + O(s^2) \).

We calculate \( I_\varepsilon(t(s)\bar{w}_\varepsilon) \). Since \( supp(\psi_1) \cap supp(\psi_2) = \emptyset \), we obtain

\[
\int_\Omega (K_H(x)\nabla \bar{w}_\varepsilon|\nabla \bar{w}_\varepsilon) dx
= \int_\Omega (K_H(x)\nabla u_e|\nabla u_e) dx + s^2 \int_\Omega (K_H(x)\nabla \psi_1|\nabla \psi_1) dx + \eta_0^2 s^2 \int_\Omega (K_H(x)\nabla \psi_2|\nabla \psi_2) dx
+ 2s \frac{1}{\varepsilon^2} \int_\Omega \left( u_e - q \ln \frac{1}{\varepsilon} \right)^p_+ \psi_1 dx - 2s \eta_0 \frac{1}{\varepsilon^2} \int_\Omega \left( u_e - q \ln \frac{1}{\varepsilon} \right)^p_+ \psi_2 dx.
\]

Since \( t(s) = 1 + O(s^2) \), we have

\[
\frac{1}{(p+1)\varepsilon^2} \int_\Omega \left( t(s)\bar{w}_\varepsilon - q \ln \frac{1}{\varepsilon} \right)^{p+1}_+ dx
= \frac{1}{(p+1)\varepsilon^2} \int_\Omega \left( t(s)u_e - q \ln \frac{1}{\varepsilon} \right)^{p+1}_+ dx + \frac{1}{\varepsilon^2} \int_\Omega \left( t(s)u_e - q \ln \frac{1}{\varepsilon} \right)^p_+ (s\psi_1 - \eta_0 s\psi_2) dx
+ \frac{p}{2\varepsilon^2} \int_\Omega \left( t(s)u_e - q \ln \frac{1}{\varepsilon} \right)^{p-1}_+ (s\psi_1 - \eta_0 s\psi_2)^2 dx + O(s^{2+\sigma})
= \frac{1}{(p+1)\varepsilon^2} \int_\Omega \left( t(s)u_e - q \ln \frac{1}{\varepsilon} \right)^{p+1}_+ dx + \frac{1}{\varepsilon^2} \int_\Omega \left( u_e - q \ln \frac{1}{\varepsilon} \right)^p_+ (s\psi_1 - \eta_0 s\psi_2) dx
+ \frac{p}{2\varepsilon^2} \int_\Omega \left( u_e - q \ln \frac{1}{\varepsilon} \right)^{p-1}_+ (s^2\psi_1^2 + \eta_0^2 s^2 \psi_2^2) dx + O(s^{2+\sigma}),
\]
Thus conclude that

\[ A \]

Direct calculation shows that

\[ \text{Note that} \]

\[ \text{Here we have used} \]

\[ \text{Moreover, we can prove that} \]

\[ \text{Thus} \]

\[ \text{By the strong maximum principle,} \]

\[ \text{So by Proposition 2.2,} \]

\[ \text{Note that} \]

\[ \text{Here we have used} \]

\[ \text{Moreover, we can prove that} \]

\[ \text{Thus} \]

\[ \text{So by Proposition 2.2,} \]

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\[ \text{Moreover, we can prove that} \]

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\[ \text{By the strong maximum principle,} \]

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\[ \text{Here we have used} \]

\[ \text{Moreover, we can prove that} \]

\[ \text{Thus} \]

\[ \text{By the strong maximum principle,} \]

\[ \text{So by Proposition 2.2,} \]

\[ \text{Note that} \]

\[ \text{Here we have used} \]

\[ \text{Moreover, we can prove that} \]

\[ \text{Thus} \]

\[ \text{By the strong maximum principle,} \]
where $\lambda_2(x) = \frac{k^2}{k^2 + |x|^2}$ is the smaller eigenvalue of $K_H$. Since $\mathbb{R}^2 \setminus \Omega$ is connected and unbounded, by the classical estimates of capacity (see [11, 23]), we have

$$\text{cap}(A_\varepsilon, \Omega) \geq \frac{2\pi}{\ln 16 \left(1 + \frac{2\text{dist}(A_\varepsilon, \partial \Omega)}{\text{diam}(A_\varepsilon)}\right)}.$$  

By Lemmas 2.4, 2.5 and Proposition 3.1, we have

$$\left(\ln \frac{1}{\varepsilon}\right)^2 \int_{\Omega \setminus A_\varepsilon} q^2 \left(K_H(x) \nabla \left(\frac{u_\varepsilon}{q \ln \frac{1}{\varepsilon}}\right)\right) \left|\nabla \left(\frac{u_\varepsilon}{q \ln \frac{1}{\varepsilon}}\right)\right| \, dx \leq \int_{\Omega} (K_H(x) \nabla u_\varepsilon \cdot \nabla u_\varepsilon) \, dx \leq \frac{2(p+1)}{p-1} I_\varepsilon(u_\varepsilon) \leq c \ln \frac{1}{\varepsilon}.$$  

Combining all these inequalities, we get $\lim_{\varepsilon \to 0} \frac{\text{diam}(A_\varepsilon)}{\text{dist}(A_\varepsilon, \partial \Omega)} = 0$.  

Define the energy of the vortex core $E_c(\varepsilon) = \int_{A_\varepsilon} (K_H(x) \nabla (u_\varepsilon - q \ln \frac{1}{\varepsilon})) \cdot \nabla (u_\varepsilon - q \ln \frac{1}{\varepsilon})) \, dx$. We will show that $E_c(\varepsilon)$ is uniformly bounded with respect to $\varepsilon$.

**Lemma 3.3** There holds for some $C$ independent of $\varepsilon$

$$\int_{A_\varepsilon} \left(K_H(x) \nabla \left(u_\varepsilon - q \ln \frac{1}{\varepsilon}\right)\right) \cdot \nabla \left(u_\varepsilon - q \ln \frac{1}{\varepsilon}\right) \, dx \leq C.$$  

**Proof** Direct calculation yields that

$$\int_{A_\varepsilon} \left(K_H(x) \nabla \left(u_\varepsilon - q \ln \frac{1}{\varepsilon}\right)\right) \cdot \nabla \left(u_\varepsilon - q \ln \frac{1}{\varepsilon}\right) \, dx = \frac{1}{\varepsilon^2} \int_{A_\varepsilon} \left(u_\varepsilon - q \ln \frac{1}{\varepsilon}\right)^p \, dx,$$

and

$$\int_{\Omega} (K_H(x) \nabla u_\varepsilon \cdot \nabla u_\varepsilon) \, dx - \int_{A_\varepsilon} \left(K_H(x) \nabla \left(u_\varepsilon - q \ln \frac{1}{\varepsilon}\right)\right) \cdot \nabla \left(u_\varepsilon - q \ln \frac{1}{\varepsilon}\right) \, dx = \frac{\ln \frac{1}{\varepsilon}}{\varepsilon^2} \int_{A_\varepsilon} \left(u_\varepsilon - q \ln \frac{1}{\varepsilon}\right)^p \, dx.$$  

By (3.12), (K2) and the classical Gagliardo-Nirenberg inequality, we get

$$\int_{A_\varepsilon} \left(K_H(x) \nabla \left(u_\varepsilon - q \ln \frac{1}{\varepsilon}\right)\right) \cdot \nabla \left(u_\varepsilon - q \ln \frac{1}{\varepsilon}\right) \, dx \leq \frac{C^*}{\varepsilon^2} \int_{A_\varepsilon} \left(u_\varepsilon - q \ln \frac{1}{\varepsilon}\right)^p \, dx \left(\int_{A_\varepsilon} \left|\nabla \left(u_\varepsilon - q \ln \frac{1}{\varepsilon}\right)\right|^2 \, dx\right)^{\frac{1}{2}} \leq \frac{C^*}{\sqrt{\inf_{\Omega \setminus A_\varepsilon} \lambda_2}} \int_{A_\varepsilon} \left(u_\varepsilon - q \ln \frac{1}{\varepsilon}\right)^p \, dx \left(\int_{A_\varepsilon} \left|K_H(x) \nabla \left(u_\varepsilon - q \ln \frac{1}{\varepsilon}\right)\right|^2 \, dx\right)^{\frac{1}{2}},$$

which implies that

$$\int_{A_\varepsilon} \left(K_H(x) \nabla \left(u_\varepsilon - q \ln \frac{1}{\varepsilon}\right)\right) \cdot \nabla \left(u_\varepsilon - q \ln \frac{1}{\varepsilon}\right) \, dx \leq C \left(\frac{1}{\varepsilon^2} \int_{A_\varepsilon} \left(u_\varepsilon - q \ln \frac{1}{\varepsilon}\right)^p \, dx\right)^2.$$  

By (3.13) and Proposition 3.1, we get

$$\frac{1}{\varepsilon^2} \int_{A_\varepsilon} \left(u_\varepsilon - q \ln \frac{1}{\varepsilon}\right)^p \, dx \leq \frac{C}{\ln \frac{1}{\varepsilon}} \int_{\Omega} (K_H(x) \nabla u_\varepsilon \cdot \nabla u_\varepsilon) \, dx \leq \frac{C}{\ln \frac{1}{\varepsilon}} I_\varepsilon(u_\varepsilon) \leq C.$$  

$\blacksquare$
Thus we get
\[
\int_{A_\varepsilon} \left( K_H(x) \nabla \left( u_\varepsilon - q \ln \frac{1}{\varepsilon} \right) \right) dx = \frac{1}{\varepsilon^2} \int_{A_\varepsilon} \left( u_\varepsilon - q \ln \frac{1}{\varepsilon} \right)^{p+1} dx \leq C.
\]

Using Lemma 3.3, we can get the lower bound of the diameter of the vortex core \( A_\varepsilon \) as follows.

**Lemma 3.4** There exists a constant \( R_1 > 0 \) independent of \( \varepsilon \) such that
\[
diam(A_\varepsilon) \geq R_1 \varepsilon.
\]

**Proof** By (3.12) and the Sobolev inequality, we have
\[
\int_{A_\varepsilon} \left( K_H(x) \nabla \left( u_\varepsilon - q \ln \frac{1}{\varepsilon} \right) \right) dx = \frac{1}{\varepsilon^2} \int_{A_\varepsilon} \left( u_\varepsilon - q \ln \frac{1}{\varepsilon} \right)^{p+1} dx
\]
\[
\leq \frac{C_s |A_\varepsilon|}{\varepsilon^2} \left( \int_{A_\varepsilon} \left| \nabla \left( u_\varepsilon - q \ln \frac{1}{\varepsilon} \right) \right|^2 dx \right)^{\frac{p+1}{2}}
\]
\[
\leq \frac{C_s |A_\varepsilon|}{(\inf_{\Omega_1} \lambda_2) \varepsilon^2} \left( \int_{A_\varepsilon} \left( K_H(x) \nabla \left( u_\varepsilon - q \ln \frac{1}{\varepsilon} \right) \right) dx \right)^{\frac{p+1}{2}},
\]
from which we deduce that
\[
\frac{|A_\varepsilon|}{\varepsilon^2} \geq C \left( \int_{A_\varepsilon} \left( K_H(x) \nabla \left( u_\varepsilon - q \ln \frac{1}{\varepsilon} \right) \right) dx \right)^{-\frac{p+1}{2}}.
\]
By Lemma 3.3, we conclude that \( |A_\varepsilon| \geq C \varepsilon^2 \). Thus we complete the proof by using the isoperimetric inequality \( |A_\varepsilon| \leq \pi \text{diam}(A_\varepsilon)^2/4 \).

**3.3 Asymptotic location of \( A_\varepsilon \)**

It follows from Proposition 3.2 that \( \lim_{\varepsilon \to 0} \text{diam}(A_\varepsilon) = 0 \), that is, the vortex core of \( u_\varepsilon \) will shrink to a single point \( \hat{x} \) as \( \varepsilon \to 0 \). We now prove that the limiting location of \( A_\varepsilon \) is a minimum point of \( q^2 \sqrt{\det(K_H)} \), by choosing test functions suitably and using the classical stream-function method.

**Proposition 3.5** There holds
\[
\lim_{\varepsilon \to 0} \text{dist}(A_\varepsilon, \hat{x}) = 0,
\]
where \( \hat{x} \) is a minimizer of \( q^2 \sqrt{\det(K_H)} \). As a consequence, there holds
\[
\lim_{\varepsilon \to 0} \frac{c_\varepsilon}{\ln \frac{1}{\varepsilon}} = \pi \min_{\Omega} q^2 \sqrt{\det(K_H)}. \tag{3.14}
\]
Proof It follows from (3.13) that
\[
\ln \frac{1}{\varepsilon^2} \int_{A_\varepsilon} \left( u_\varepsilon - q \ln \frac{1}{\varepsilon} \right)^p q \, dx
\]
\[
= \int_\Omega (K_H(x) \nabla u_\varepsilon \nabla q) \, dx - \int_{A_\varepsilon} \left[ K_H(x) \nabla \left( u_\varepsilon - q \ln \frac{1}{\varepsilon} \right) \right] \nabla \left( u_\varepsilon - q \ln \frac{1}{\varepsilon} \right) \, dx
\]
\[
= 2I_\varepsilon(u_\varepsilon) - \frac{p-1}{(p+1)\varepsilon^2} \int_{A_\varepsilon} \left( u_\varepsilon - q \ln \frac{1}{\varepsilon} \right)^{p+1} \, dx.
\]
Hence by Lemma 3.3, we get
\[
\frac{1}{\varepsilon^2} \int_{A_\varepsilon} \left( u_\varepsilon - q \ln \frac{1}{\varepsilon} \right)^p q \, dx \leq \frac{2c_\varepsilon}{\ln \frac{1}{\varepsilon}} + O \left( \frac{1}{\ln \frac{1}{\varepsilon}} \right) \leq C.
\]
(3.15)

For any \(0 < \tau < \sigma \leq 1\), define \(w^{\sigma,\tau}_\varepsilon := \min \left\{ \left( u_\varepsilon - q \ln \frac{1}{\varepsilon} \right)_+, 1 \right\} \in H^1_0(\Omega)\) and \(A^{\sigma}_\varepsilon := \{ x \in \Omega \mid u_\varepsilon(x) > q(x) \ln \frac{1}{\sigma} \}\). Then one computes directly that \(w^{\sigma,\tau}_\varepsilon \equiv 1\) on \(A^{\tau}_\varepsilon\) and \(\text{supp}(w^{\sigma,\tau}_\varepsilon) = A^{\sigma}_\varepsilon\).

We claim that for every \(\varepsilon \leq \tau\),
\[
\ln \frac{\sigma}{\tau} \int_\Omega (K_H(x) \nabla w^{\sigma,\tau}_\varepsilon \nabla w^{\sigma,\tau}_\varepsilon) \, dx = \frac{1}{\varepsilon^2} \int_\Omega \left( u_\varepsilon - q \ln \frac{1}{\varepsilon} \right)^p \, dx.
\]
(3.16)

Indeed, multiplying both sides of (2.1) by \(\phi = w^{\sigma,\tau}_\varepsilon q \in H^1_0(\Omega)\) and using integration by parts, we get
\[
\int_\Omega (K_H(x) \nabla u_\varepsilon \nabla (w^{\sigma,\tau}_\varepsilon q)) \, dx = \frac{1}{\varepsilon^2} \int_\Omega \left( u_\varepsilon - q \ln \frac{1}{\varepsilon} \right)^p \, dx.
\]

Direct computations show that
\[
\frac{1}{\varepsilon^2} \int_\Omega \left( u_\varepsilon - q \ln \frac{1}{\varepsilon} \right)^p \, dx = \frac{1}{\varepsilon^2} \int_\Omega \left( u_\varepsilon - q \ln \frac{1}{\varepsilon} \right)^p \, dx
\]
and
\[
\int_\Omega (K_H(x) \nabla u_\varepsilon \nabla (w^{\sigma,\tau}_\varepsilon q)) \, dx = \int_\Omega \left( K_H(x) \nabla \left( u_\varepsilon - q \ln \frac{1}{\varepsilon} \right) \nabla (w^{\sigma,\tau}_\varepsilon q) \right) \, dx
\]
\[
= \ln \frac{\sigma}{\tau} \int_\Omega (K_H(x) \nabla (w^{\sigma,\tau}_\varepsilon q) \nabla (w^{\sigma,\tau}_\varepsilon q))
\]
\[
+ \int_\Omega \left( K_H(x) \nabla \left( u_\varepsilon - q \ln \frac{1}{\varepsilon} - \frac{\sigma}{\tau} w^{\sigma,\tau}_\varepsilon q \right) \right) \nabla (w^{\sigma,\tau}_\varepsilon q)
\]
\[
= \ln \frac{\sigma}{\tau} \int_\Omega (K_H(x) \nabla (w^{\sigma,\tau}_\varepsilon q) \nabla (w^{\sigma,\tau}_\varepsilon q)) + \int_{A^{\tau}_\varepsilon} \left( K_H(x) \nabla \left( u_\varepsilon - q \ln \frac{1}{\varepsilon} \right) \right) \nabla q
\]
\[
= \ln \frac{\sigma}{\tau} \int_\Omega (K_H(x) \nabla (w^{\sigma,\tau}_\varepsilon q) \nabla (w^{\sigma,\tau}_\varepsilon q)) \, dx
\]
\[
= \ln \frac{\sigma}{\tau} \int_\Omega q^2 \nabla (w^{\sigma,\tau}_\varepsilon q) \, dx,
\]
where we have used the assumption \(\mathcal{L}_H q = 0\) and Lemma 2.5. Thus we get (3.16).
By the definition of capacity, (3.15) and (3.16), we get

\[
\cap(A^\tau_\varepsilon, \Omega) \leq \int_\Omega |\nabla w^{1, \tau}_\varepsilon|^2 \, dx \leq \frac{1}{\inf_\Omega q^2 \lambda_2} \int_\Omega q^2(K_H(x) \nabla w^{1, \tau}_\varepsilon |\nabla w^{1, \tau}_\varepsilon) \, dx \leq \frac{C}{\ln 1/\tau}.
\]

Using the capacity estimates in [11] again, we get

\[
\frac{2\pi}{\ln 16 \left(1 + \frac{2\text{dist}(A^\tau_\varepsilon, \partial\Omega)}{\text{diam}(A^\tau_\varepsilon)}\right)} \leq \cap(A^\tau_\varepsilon, \Omega),
\]

from which we deduce that,

\[
\frac{2\pi}{\ln 16 \left(1 + \frac{2\text{dist}(A^\tau_\varepsilon, \partial\Omega)}{\text{diam}(A^\tau_\varepsilon)}\right)} \leq \frac{C}{\ln 1/\tau}.
\]

So there exist constants \(C_1, C_2 > 0\) independent of \(\varepsilon, \tau\), such that for any \(0 < \tau < 1\) and \(\varepsilon \leq \tau\),

\[
\text{diam}(A^\tau_\varepsilon) \leq C_1 \tau C_2.
\]

We now claim that for any \(\delta > 0\), there exist \(\rho > 0\) and \(0 < \varepsilon_0 < \rho\), such that for any \(\varepsilon \in (0, \varepsilon_0)\) and \(x, y \in A^\rho_\varepsilon\),

\[
q(x)^2 \leq q(y)^2 (1 + \delta), \quad (3.18)
\]

and

\[
(K_H(x)\zeta |\zeta) \leq (1 + \delta)(K_H(y)\zeta |\zeta), \quad \forall \zeta \in \mathbb{R}^2. \quad (3.19)
\]

Indeed, since \(\inf_\Omega q > 0\) and \(q \in C^2(\Omega) \cap C^1(\overline{\Omega})\), it is easy to get (3.18). By (K2) and the regularity of \(K_H\), one can also get (3.19).

Thus taking \(\sigma = \rho, \tau = \varepsilon\) in (3.16), we get for any \(x_\varepsilon \in A_\varepsilon \subseteq A^\rho_\varepsilon\)

\[
\frac{1}{\varepsilon^2 \ln \frac{\rho}{\varepsilon}} \int_\Omega \left(u_\varepsilon - q \ln \frac{1}{\varepsilon}\right)^p \, dx \geq \int_\Omega q^2(K_H(x)\nabla w^{\rho, \varepsilon}_\varepsilon |\nabla w^{\rho, \varepsilon}_\varepsilon) \geq \frac{q^2(x_\varepsilon)}{(1 + \delta)^2} \int_\Omega (K_H(x_\varepsilon)\nabla w^{\rho, \varepsilon}_\varepsilon |\nabla w^{\rho, \varepsilon}_\varepsilon). \quad (3.20)
\]

Define a linear transformation matrix \(T_\varepsilon : \mathbb{R}^2 \rightarrow \mathbb{R}^2\) satisfying

\[
T_\varepsilon K_H(x_\varepsilon)T_\varepsilon^T = Id.
\]

Then \(|\text{det}(T_\varepsilon)| = |\text{det}(K_H)(x_\varepsilon)|^{-\frac{1}{2}}\). Let \(\Omega' = T_\varepsilon(\Omega), A'_\varepsilon = T_\varepsilon(A_\varepsilon)\). For any \(y = T_\varepsilon(x) \in \Omega'\), define \(\bar{w}^{\rho, \varepsilon}_\varepsilon(y) = w^{\rho, \varepsilon}_\varepsilon(x) = w^{\rho, \varepsilon}_\varepsilon(T_\varepsilon^{-1}(y))\). Since \(w^{\rho, \varepsilon}_\varepsilon \equiv 1\) on \(A'_\varepsilon\) and \(\bar{w}^{\rho, \varepsilon}_\varepsilon \in H^1_0(\Omega')\),
using the capacity estimates again we have
\[
\int_{\Omega} (K_H(x_\varepsilon) \nabla w_{\varepsilon}^{p,\varepsilon} | \nabla w_{\varepsilon}^{p,\varepsilon}) \, dx = \int_{\Omega'} (T_\varepsilon K_H(x_\varepsilon) T_\varepsilon^T \nabla \tilde{w}_{\varepsilon}^{p,\varepsilon} | \nabla \tilde{w}_{\varepsilon}^{p,\varepsilon}) | \det(T_\varepsilon^{-1})| \, dy \\
= \sqrt{\det(K_H(x_\varepsilon))} \int_{\Omega'} |\nabla \tilde{w}_{\varepsilon}^{p,\varepsilon}|^2 \, dy \\
\geq \sqrt{\det(K_H(x_\varepsilon))} \cdot \text{cap}(A'_\varepsilon, \Omega') \\
\geq \sqrt{\det(K_H(x_\varepsilon))} \cdot \frac{2\pi}{\ln 16 \left(1 + \frac{2 \text{dist}(A'_\varepsilon, \partial \Omega')}{\text{diam}(A'_\varepsilon)}\right)} \\
\geq \sqrt{\det(K_H(x_\varepsilon))} \cdot \frac{2\pi}{\ln 16 \left(1 + \frac{C_0 \text{dist}(A_\varepsilon, \partial \Omega)}{\text{diam}(A_\varepsilon)}\right)},
\]
(3.21)
for some $C_0 > 0$ independent of $\varepsilon$. Thus by (3.20) and (3.21), we get
\[
q^2 \sqrt{\det(K_H(x_\varepsilon))} \leq \frac{(1 + \delta)^2}{2\pi \ln \frac{p}{\varepsilon}} \ln 16 \left(1 + \frac{C_0 \text{dist}(A_\varepsilon, \partial \Omega)}{\text{diam}(A_\varepsilon)}\right) \left(\frac{1}{\varepsilon^2} \int_{\Omega} (u_{\varepsilon} - q \ln \frac{1}{\varepsilon})^p \, dx \right).
\]
(3.22)
Taking the limit superior in both sides of (3.22), using (3.15) and Proposition 3.1, we obtain that for any $\delta > 0$ and $x_\varepsilon \in A_\varepsilon$,
\[
\limsup_{\varepsilon \to 0} q^2 \sqrt{\det(K_H(x_\varepsilon))} \leq \frac{(1 + \delta)^2}{\ln 16 \left(1 + \frac{C_0 \text{dist}(A_\varepsilon, \partial \Omega)}{\text{diam}(A_\varepsilon)}\right)} \cdot 2\pi \min_{x \in \Omega} q^2 \sqrt{\det(K_H)(x)}.
\]
By Lemma 3.4 and $\text{dist}(A_\varepsilon, \partial \Omega) \leq \text{diam}(\Omega)$, we get
\[
\limsup_{\varepsilon \to 0} \frac{\ln 16 \left(1 + \frac{C_0 \text{dist}(A_\varepsilon, \partial \Omega)}{\text{diam}(A_\varepsilon)}\right)}{\ln \frac{p}{\varepsilon}} \leq \limsup_{\varepsilon \to 0} \frac{\ln 16 \left(1 + \frac{C}{p}\right)}{\ln \frac{p}{\varepsilon}} = 1.
\]
Hence we have
\[
\limsup_{\varepsilon \to 0} q^2 \sqrt{\det(K_H(x_\varepsilon))} \leq (1 + \delta)^2 \min_{x \in \Omega} q^2 \sqrt{\det(K_H)(x)}.
\]
By the arbitrariness of $\delta > 0$, we conclude that for any $x_\varepsilon \in A_\varepsilon$, $x_\varepsilon$ tends to $\hat{x}$, where $\hat{x}$ is a minimizer of $q^2 \sqrt{\det(K_H)}$.

Taking the limit inferior in both sides of (3.22), using (3.15), and by the arbitrariness of $\delta > 0$, we have
\[
\liminf_{\varepsilon \to 0} \frac{C_\varepsilon}{\ln \frac{1}{\varepsilon}} \geq \pi \min_{\Omega} q^2 \sqrt{\det(K_H)}.
\]
Combining this with Proposition 3.1, we get (3.14). The proof is thus complete.

We can then get estimates of the diameter of $A_\varepsilon$ as follows.

**Lemma 3.6** There holds
\[
\lim_{\varepsilon \to 0} \frac{\ln \left(\frac{\text{dist}(A_\varepsilon, \partial \Omega)}{\text{diam}(A_\varepsilon)}\right)}{\ln \frac{1}{\varepsilon}} = 1.
\]
(3.23)
Proof On the one hand, by Lemma 3.4, we have
\[ \limsup_{\varepsilon \to 0} \frac{\ln \text{dist}(A_\varepsilon, \partial \Omega_1)}{\ln \frac{1}{\varepsilon}} \leq \limsup_{\varepsilon \to 0} \frac{\ln C}{\ln \frac{1}{\varepsilon}} = 1. \]

On the other hand, taking the limit inferior in both sides of (3.22) and using (3.15) and Proposition 3.1, we get
\[ \frac{(1 + \delta)^2}{2\pi} \liminf_{\varepsilon \to 0} \frac{\ln 16}{2} \left(1 + \frac{C_0 \text{dist}(A_\varepsilon, \partial \Omega_1)}{\text{diam}(A_\varepsilon)}\right) \cdot 2\pi \min_{x \in \Omega} q^2 \sqrt{\det(K_H(x))} \geq \liminf_{\varepsilon \to 0} q^2 \sqrt{\det(K_H)}, \]

which implies that
\[ \frac{1}{(1 + \delta)^2} \leq \liminf_{\varepsilon \to 0} \frac{\ln 16}{2} \left(1 + \frac{C_0 \text{dist}(A_\varepsilon, \partial \Omega_1)}{\text{diam}(A_\varepsilon)}\right) = \liminf_{\varepsilon \to 0} \frac{\ln \text{dist}(A_\varepsilon, \partial \Omega_1)}{\ln \frac{1}{\varepsilon}}. \]

By the arbitrariness of \( \delta > 0 \), we have \( \liminf_{\varepsilon \to 0} \frac{\ln \text{dist}(A_\varepsilon, \partial \Omega_1)}{\ln \frac{1}{\varepsilon}} \geq 1 \). The proof is thus complete. \( \square \)

Remark 3.7 A direct consequence of Lemmas 3.6 and 3.4 is that for any \( \alpha \in (0, 1) \), there exists \( C_1, C_2 > 0 \) such that
\[ C_1 \varepsilon \leq \text{diam}(A_\varepsilon) \leq C_2 \varepsilon^\alpha. \]

When the limiting location \( \bar{x} \) of \( A_\varepsilon \) is on the boundary of \( \Omega \), such an estimate is optimal. Similar results have been found for 2D Euler equations and 3D axisymmetric equations, see [11, 20, 23] for example. However when \( \bar{x} \in \Omega \), we can improve estimates of \( \text{diam}(A_\varepsilon) \).

By Proposition 3.5, we show that the limiting location of \( A_\varepsilon \) is \( x^* \), where \( q^2 \sqrt{\det(K_H(x^*))} = \min_{x \in \Omega} q^2 \sqrt{\det(K_H)} \). Note that \( \kappa(\omega_\varepsilon) = \frac{1}{\varepsilon^2} \int_{A_\varepsilon} (u_\varepsilon - q \ln \frac{1}{\varepsilon} )^p \) is the circulation of \( \omega_\varepsilon = \frac{1}{\varepsilon^2} (u_\varepsilon - q \ln \frac{1}{\varepsilon} )^p \). The limit of \( \kappa(\omega_\varepsilon) \) can be obtained as follows.

Lemma 3.8 There holds
\[ \lim_{\varepsilon \to 0} \kappa(\omega_\varepsilon) = 2\pi q \sqrt{\det(K_H(x^*)}. \]

Proof It follows from (3.13), (3.12) and the definition of \( c_\varepsilon \) that
\[ \frac{1}{\varepsilon^2} \int_{A_\varepsilon} (u_\varepsilon - q \ln \frac{1}{\varepsilon} )^p = \int_{\Omega} (K_H \nabla u_\varepsilon | \nabla u_\varepsilon ) \]
\[ - \int_{A_\varepsilon} \left( K_H \nabla \left( u_\varepsilon - q \ln \frac{1}{\varepsilon} \right) | \nabla \left( u_\varepsilon - q \ln \frac{1}{\varepsilon} \right) \right) \]
\[ = 2c_\varepsilon - \frac{p - 1}{(p + 1)\varepsilon^2} \int_{A_\varepsilon} \left( u_\varepsilon - q \ln \frac{1}{\varepsilon} \right)^{p+1}. \]

By Lemma 3.3, we have
\[ \frac{1}{\varepsilon^2} \int_{A_\varepsilon} \left( u_\varepsilon - q \ln \frac{1}{\varepsilon} \right)^{p+1} \leq C. \]
So \( \frac{1}{\varepsilon^2} \int_{A_\varepsilon} \left( u_\varepsilon - q \ln \frac{1}{\varepsilon} \right)^p + q = \frac{2c_\varepsilon}{\ln \varepsilon} + O \left( \frac{1}{\ln \varepsilon} \right) \). By Proposition 3.5, we get

\[
\lim_{\varepsilon \to 0} \kappa(\omega_\varepsilon) = q(x^*)^{-1} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \int_{A_\varepsilon} \left( u_\varepsilon - q \ln \frac{1}{\varepsilon} \right)^p + q = q(x^*)^{-1} \lim_{\varepsilon \to 0} \frac{2c_\varepsilon}{\ln \varepsilon} = 2\pi q(x^*)\sqrt{\det(K_H(x^*))}.
\]

\( \Box \)

### 3.4 Further analysis when the limiting location of \( A_\varepsilon \) is in \( \Omega \)

When the limiting location of \( A_\varepsilon \) is in \( \Omega \), we can improve the results in Proposition 3.5 and Lemma 3.6 by giving more accurate estimates of lower bound of \( c_\varepsilon \) and upper bound of the diameter of \( A_\varepsilon \). Indeed, we have

**Proposition 3.9** If for any \( x_\varepsilon \in A_\varepsilon \), \( \lim_{\varepsilon \to 0} x_\varepsilon = x^* \in \Omega \), then

\[
\begin{align*}
\kappa(\omega_\varepsilon) &= q(x^*)^{-1} \kappa(\omega), \\
&= q(x^*)^{-1} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \int_{A_\varepsilon} \left( u_\varepsilon - q \ln \frac{1}{\varepsilon} \right)^p + q \\
&= q(x^*)^{-1} \lim_{\varepsilon \to 0} \frac{2c_\varepsilon}{\ln \varepsilon} = 2\pi q(x^*)\sqrt{\det(K_H(x^*))}.
\end{align*}
\]

Moreover, there exist \( R_1, R_2 > 0 \) independent of \( \varepsilon \) such that

\[
R_1 \varepsilon \leq \text{diam}(A_\varepsilon) \leq R_2 \varepsilon.
\]

**Proof** By (3.17), we get for any \( 0 < \tau < 1 \) and \( \varepsilon \leq \tau \),

\[
\begin{align*}
2\pi &\leq \frac{\ln 16}{\ln \varepsilon} \frac{1}{1 + \frac{2\text{dist}(A^*_\varepsilon, \partial \Omega)}{\text{diam}(A^*_\varepsilon)}} \\
&\leq C
\end{align*}
\]

So there exist \( C_0, \alpha_0 > 0 \) such that \( \frac{\text{dist}(A^*_\varepsilon, \partial \Omega)}{\text{diam}(A^*_\varepsilon)} \geq \frac{C_0}{\tau^{\alpha_0}} \), which implies that \( \text{diam}(A^*_\varepsilon) \leq C_1 \tau^{\alpha_0} \) for some \( C_1 > 0 \). That is, for any \( x, y \in A^*_\varepsilon \), \( |x - y| \leq C_1 \tau^{\alpha_0} \).

By \( q \in C^1(\Omega) \cap C^1(\partial \Omega), (K_H)_{ij} \in C^\infty(\Omega) \) for \( i, j = 1, 2 \), \( \inf_{\Omega} q > 0 \) and \( (K_2) \), similarly to the proof of (3.18) and (3.19), we can get that \( q^2 K_H \) is Dini-continuous uniformly in \( \Omega \), which means that, there exists a non-negative function \( \gamma(s) \), such that \( \int_0^{s_0} \gamma(s)ds < +\infty \) for some \( s_0 > 0 \) and

\[
q^2(x)(K_H(x)\xi|\xi) \leq (1 + \gamma(|x - y|))q^2(y)(K_H(y)\xi|\xi) \quad \forall x, y \in \Omega, \ \xi \in \mathbb{R}^2.
\]

Thus by (3.16) and (3.26), we get for any \( 0 < \tau < \sigma < 1, \varepsilon \leq \tau \) and \( x_\varepsilon \in A_\varepsilon \subset A^\sigma_\varepsilon \),

\[
\begin{align*}
\frac{1}{\varepsilon^2} \int_{\Omega} \left( u_\varepsilon - q \ln \frac{1}{\varepsilon} \right)^p + q &= \frac{1}{\varepsilon^2} \int_{A_\varepsilon} \left( u_\varepsilon - q \ln \frac{1}{\varepsilon} \right)^p + q \\
&\geq \ln \frac{\sigma}{\tau} \frac{1}{1 + \gamma(C_1 \sigma^{\alpha_0})} \int_{\Omega} q^2(x_\varepsilon)(K_H(x_\varepsilon)\nabla w^\sigma_{\varepsilon} \nabla w^\sigma_{\varepsilon})dx \\
&\geq \ln \frac{\sigma}{\tau} \frac{1}{1 + \gamma(C_1 \sigma^{\alpha_0})} \int_{\Omega} q^2(x_\varepsilon)(K_H(x_\varepsilon)\nabla w^\sigma_{\varepsilon} \nabla w^\sigma_{\varepsilon})dx,
\end{align*}
\]

which implies that

\[
\begin{align*}
\left( \ln \frac{\sigma}{\tau} \right)^2 \int_{\Omega} q^2(x_\varepsilon)(K_H(x_\varepsilon)\nabla w^\sigma_{\varepsilon} \nabla w^\sigma_{\varepsilon}) \\
&\leq \ln \frac{\sigma}{\tau} (1 + \gamma(C_1 \sigma^{\alpha_0})) \frac{1}{\varepsilon^2} \int_{\Omega} \left( u_\varepsilon - q \ln \frac{1}{\varepsilon} \right)^p + q.
\end{align*}
\]

\( \Box \)
Taking $\varepsilon = \tau_1 < \sigma_1 = \tau_2 < \sigma_2 = \tau_3 < \cdots < \sigma_k = \rho$, and summing (3.27) over $(j = 1, 2, \cdots, k)$, we get for any $x_{\varepsilon} \in A_{\varepsilon}$

$$
\left( \ln \frac{\rho}{\varepsilon} \right)^2 \int_{\Omega} q^2(x_{\varepsilon})(K_H(x_{\varepsilon})\nabla w_{\rho,\varepsilon}^\rho | \nabla w_{\rho,\varepsilon}^\rho)
\leq \left( \ln \frac{\rho}{\varepsilon} + \sum_{j=1}^k \gamma(C_1\sigma_{j}^{(0)}) \ln \frac{\sigma_j}{\tau_j} \right) \frac{1}{\varepsilon^2} \int_{\Omega} \left( u_{\varepsilon} - q \ln \frac{1}{\varepsilon} \right)^p q.
$$

By taking the limit of Riemann sums in the above inequality, we have

$$
\left( \ln \frac{\rho}{\varepsilon} \right)^2 \int_{\Omega} q^2(x_{\varepsilon})(K_H(x_{\varepsilon})\nabla w_{\rho,\varepsilon}^\rho | \nabla w_{\rho,\varepsilon}^\rho)
\leq \left( \ln \frac{\rho}{\varepsilon} + \int_{\rho}^{\rho_{\varepsilon}} \frac{\gamma(C_1\sigma_{0})}{\sigma} d\sigma \right) \frac{1}{\varepsilon^2} \int_{\Omega} \left( u_{\varepsilon} - q \ln \frac{1}{\varepsilon} \right)^p q.
$$

Since

$$
\int_{\rho}^{\rho_{\varepsilon}} \frac{\gamma(C_1\sigma_{0})}{\sigma} d\sigma = \int_{\rho_{\varepsilon}}^{\rho_{0}} \frac{\gamma(C_1\sigma_{0})}{\sigma} \frac{1}{\sigma_{0}} \frac{\sigma_{0}}{\sigma_{0}-1} d\sigma = \frac{1}{\sigma_{0}} \int_{C_1\rho_{\varepsilon}}^{C_1\rho_{0}} \frac{\gamma(\sigma)}{\sigma} d\sigma < +\infty,
$$

we get

$$
\int_{\Omega} q^2(x_{\varepsilon})(K_H(x_{\varepsilon})\nabla w_{\rho,\varepsilon}^\rho | \nabla w_{\rho,\varepsilon}^\rho)dx
\leq \left( \frac{1}{\ln \frac{\rho}{\varepsilon}} + \frac{C}{(\ln \frac{\rho}{\varepsilon})^2} \right) \frac{1}{\varepsilon^2} \int_{\Omega} \left( u_{\varepsilon} - q \ln \frac{1}{\varepsilon} \right)^p q dx,
$$

which is a refined version of (3.20). So repeating the proof of Proposition 3.5, we have

$$
q^2 \sqrt{\text{det}(K_H)(x_{\varepsilon})} \leq \ln 16 \left( 1 + \frac{C_{0\text{dist}(A_{\varepsilon}, \partial\Omega)}}{\text{diam}(A_{\varepsilon})} \right) \frac{1}{2\pi} \left( \frac{1}{\ln \frac{\rho}{\varepsilon}} + \frac{C}{(\ln \frac{\rho}{\varepsilon})^2} \right) \left( \frac{1}{\varepsilon^2} \int_{\Omega} \left( u_{\varepsilon} - q \ln \frac{1}{\varepsilon} \right)^p q dx \right),
$$

which improves (3.22).

Thus, taking (3.15) into (3.29) and using Proposition 3.1, we obtain

$$
\min_{\Omega} q^2 \sqrt{\text{det}(K_H)} \ln \frac{1}{\varepsilon}
\leq \ln 16 \left( 1 + \frac{C_{0\text{dist}(A_{\varepsilon}, \partial\Omega)}}{\text{diam}(A_{\varepsilon})} \right) \frac{1}{2\pi} \left( \frac{1}{\ln \frac{\rho}{\varepsilon}} + \frac{C}{(\ln \frac{\rho}{\varepsilon})^2} \right) \left( 2c_{\varepsilon} + O(1) \right)
\leq \ln 16 \left( 1 + \frac{C_{0\text{dist}(A_{\varepsilon}, \partial\Omega)}}{\text{diam}(A_{\varepsilon})} \right) \frac{1}{2\pi} \left( \frac{1}{\ln \frac{\rho}{\varepsilon}} + \frac{C}{(\ln \frac{\rho}{\varepsilon})^2} \right) \left( 2\pi \min_{\Omega} q^2 \sqrt{\text{det}(K_H)} \ln \frac{1}{\varepsilon} + O(1) \right).
$$

Direct computation shows that

$$
\ln \frac{C_{0\text{dist}(A_{\varepsilon}, \partial\Omega)}}{\text{diam}(A_{\varepsilon})} \geq \ln \frac{1}{\varepsilon} + O(1),
$$

which implies that $\text{diam}(A_{\varepsilon}) \leq R_2\varepsilon$ for some $R_2 > 0$. 

\( \square \)
Finally, by taking \( \text{diam}(A_\varepsilon) \geq R_1\varepsilon \) (see Lemma 3.4) into (3.29) and using (3.15), we get

\[
\begin{align*}
c_\varepsilon \geq & \pi \min_{\Omega} q^2 \sqrt{\det(K_H)} \frac{1}{\ln 16 \left( 1 + \frac{C_{\text{dist}}(A_\varepsilon, \partial \Omega_1)}{\text{diam}(A_\varepsilon)} \right) \left( \frac{1}{\ln \frac{\varepsilon}{\rho}} + \frac{C}{(\ln \frac{\varepsilon}{\rho})^2} \right)} \\ & \quad \times \ln \frac{1}{\varepsilon} + O(1) \\ \geq & \pi \min_{\Omega} q^2 \sqrt{\det(K_H)} \frac{1}{\ln \frac{\varepsilon}{\rho}} \left( \frac{1}{\ln \frac{\varepsilon}{\rho}} + \frac{C}{(\ln \frac{\varepsilon}{\rho})^2} \right) \ln \frac{1}{\varepsilon} + O(1) \\ \geq & \pi \min_{\Omega} q^2 \sqrt{\det(K_H)} \ln \frac{1}{\varepsilon} + O(1).
\end{align*}
\]

Combining this with Proposition 3.1, we get (3.25). The proof is thus complete. \( \square \)

### 4 Proof of Theorem 1.5 and 1.2

#### 4.1 Proof of Theorem 1.5

Based on proofs in Sects. 2.1, 3.1, 3.2, 3.3 and 3.4 we show that there exists a family of solutions \( u_\varepsilon \) of (2.1) concentrating near minimum points of \( q^2 \sqrt{\det(K_H)} \). Let \( \omega_\varepsilon = \mathcal{L}_H u_\varepsilon, \varphi_\varepsilon = u_\varepsilon - q \ln \frac{1}{\varepsilon} \). Then \( (\omega_\varepsilon, \varphi_\varepsilon) \) is the desired solution pair of (1.20) and we finish the proof of Theorem 1.5.

#### 4.2 Proof of Theorem 1.2

Based on results in Theorem 1.5, we now give the proof of Theorem 1.2. For every \( m > 0 \), let \( q(x) = m \). Then \( q \) satisfies \( \mathcal{L}_H q = 0 \). So by Theorem 1.5, there exist solutions \( u_\varepsilon \) of (1.19) with \( f(t) = t^p \) and \( \mu = m \ln \frac{1}{\varepsilon} \) concentrating near \( x^* \), which is a minimizer of \( q^2 \sqrt{\det(K_H)} \).

Since \( q^2 \sqrt{\det(K_H)}(x) = m^2 \left( \frac{k^2}{k^2 + |x|^2} \right)^{\frac{1}{2}} \), we get that \( x^* \in \Omega \) satisfies \( |x^*| = \max_{\Omega} |x| \). Let \( \omega_\varepsilon = \mathcal{L}_H u_\varepsilon \). By Lemma 3.8, the limit of circulation is

\[
\lim_{\varepsilon \to 0} \kappa(\omega_\varepsilon) = 2\pi q(x^*) \sqrt{\det(K_H(x^*))} = 2\pi m \cdot \left( \frac{k^2}{k^2 + |x^*|^2} \right)^{\frac{1}{2}} = \frac{2k\pi m}{\sqrt{k^2 + |x^*|^2}}.
\]

To conclude, \( (\omega_\varepsilon, u_\varepsilon) \) is the desired solution pair and the proof of Theorem 1.2 is complete.

**Remark 4.1** From the proof of Theorem 1.2, we know that the limiting location \( x^* \) of \( w_\varepsilon \) satisfies \( |x^*| = \max_{\Omega} |x| \). So \( x^* \) must be on the boundary of \( \Omega \). This implies that results in Sect. 3.4 can not hold. In this case, the optimal estimates of diameter of \( A_\varepsilon \) is Lemma 3.6, rather than Proposition 3.9.

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References

1. Abidi, H., Sakrani, S.: Global well-posedness of helicoidal Euler equations. J. Funct. Anal. 271(8), 2177–2214 (2016)
2. Benvenutti, M.: Nonlinear stability for stationary helical vortices. NoDEA Nonlinear Differ. Equ. Appl. 27(2), 15 (2020)
3. Berger, M.S., Fraenkel, L.E.: Nonlinear desingularization in certain free-boundary problems. Commun. Math. Phys. 77, 149–172 (1980)
4. Bronzi, A.C., Lopes Filho, M.C., Nussenzveig Lopes, H.J.: Global existence of a weak solution of the incompressible Euler equations with helical symmetry and $L^p$ vorticity. Indiana Univ. Math. J. 64(1), 309–341 (2015)
5. Cao, D., Liu, Z., Wei, J.: Regularization of point vortices for the Euler equation in dimension two. Arch. Ration. Mech. Anal. 212, 179–217 (2014)
6. Cao, D., Peng, S., Yan, S.: Planar vortex patch problem in incompressible steady flow. Adv. Math. 270, 263–301 (2015)
7. Cao, D., Peng, S., Yan, S.: Singularly Perturbed Methods for Nonlinear Elliptic Problems. Cambridge University Press, Cambridge (2021)
8. Cao, D., Wan, J.: Desingularization of rotational-invariant solutions to 3D Euler equation with helical symmetry, Preprint
9. Dávila, J., del Pino, M., Musso, M., Wei, J.: Gluing methods for vortex dynamics in Euler flows. Arch. Ration. Mech. Anal. 235(3), 1467–1530 (2020)
10. Dávila, J., del Pino, M., Musso, M., Wei, J.: Travelling helices and the vortex filament conjecture in the incompressible Euler equations. Calc. Var. Partial Differ. Equ. 61, 119 (2022)
11. de Valeriola, S., Van Schaftingen, J.: Desingularization of vortex rings and shallow water vortices by semilinear elliptic problem. Arch. Ration. Mech. Anal. 210(2), 409–450 (2013)
12. Dutrifoy, A.: Existence globale en temps de solutions héliocaidales des équations d’Euler. C. R. Acad. Sci. Paris Sér. I Math. 329(7), 653–656 (1999)
13. Ettinger, B., Titi, E.S.: Global existence and uniqueness of weak solutions of three-dimensional Euler equations with helical symmetry in the absence of vorticity stretching. SIAM J. Math. Anal. 41(1), 269–296 (2009)
14. Fraenkel, L.E., Berger, M.S.: A global theory of steady vortex rings in an ideal fluid. Acta Math. 132, 13–51 (1974)
15. Helmholtz, H.: On integrals of the hydrodynamics equations which express vortex motion. J. Reine Angew. Math. 55, 25–55 (1858)
16. Jeangean, L., Tanaka, K.: A remark on least energy solutions in $\mathbb{R}^N$. Proc. Am. Math. Soc. 131(8), 2399–2408 (2003)
17. Jerrard, R.L., Seis, C.: On the vortex filament conjecture for Euler flows. Arch. Ration. Mech. Anal. 224(1), 135–172 (2017)
18. Jerrard, R.L., Smets, D.: On the motion of a curve by its binormal curvature. J. Eur. Math. Soc. (JEMS) 17(6), 1487–1515 (2015)
19. Jiu, Q., Li, J., Niu, D.: Global existence of weak solutions to the three-dimensional Euler equations with helical symmetry. J. Differ. Equ. 262(10), 5179–5205 (2017)
20. Li, G., Yan, S., Yang, J.: An elliptic problem related to planar vortex pairs. SIAM J. Math. Anal. 36, 1444–1460 (2005)
21. Majda, A., Bertozzi, A.: Vorticity and Incompressible Flow. Cambridge University Press, Cambridge (2002)
22. Marchioro, C., Pulvirenti, M.: Mathematical Theory of Incompressible Nonviscous Fluids. Springer-Verlag, Berlin (1994)
23. Smets, D., Van Schaftingen, J.: Desingularization of vortices for the Euler equation. Arch. Ration. Mech. Anal. 198(3), 869–925 (2010)
24. Willem, M.: Minimax Theorems. In: Progress in Nonlinear Differential Equations and Their Applications. Birkhäuser Boston, Boston (1996)
25. Yudovich, V.I.: Non-stationary flow of an ideal incompressible fluid. USSR Comput. Math. Math. Phys. 3, 1407–1456 (1963)
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