ROUTING IN AN UNCERTAIN WORLD: ADAPTIVITY, EFFICIENCY, AND EQUILIBRIUM

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Abstract. We consider the traffic assignment problem in nonatomic routing games where the players’ cost functions may be subject to random fluctuations (e.g., weather disturbances, perturbations in the underlying network, etc.). We tackle this problem from the viewpoint of a control interface that makes routing recommendations based solely on observed costs and without any further knowledge of the system’s governing dynamics – such as the network’s cost functions, the distribution of any random events affecting the network, etc. In this online setting, learning methods based on the popular exponential weights algorithm converge to equilibrium at an $O(1/\sqrt{T})$ rate: this rate is known to be order-optimal in stochastic networks, but it is otherwise suboptimal in static networks. In the latter case, it is possible to achieve an $O(1/T^2)$ equilibrium convergence rate via the use of finely tuned accelerated algorithms; on the other hand, these accelerated algorithms fail to converge altogether in the presence of persistent randomness, so it is not clear how to achieve the “best of both worlds” in terms of convergence speed. Our paper seeks to fill this gap by proposing an adaptive routing algorithm with the following desirable properties: (i) it seamlessly interpolates between the $O(1/T^2)$ and $O(1/\sqrt{T})$ rates for static and stochastic environments respectively; (ii) its convergence speed is polylogarithmic in the number of paths in the network; (iii) the method’s per-iteration complexity and memory requirements are both linear in the number of nodes and edges in the network; and (iv) it does not require any prior knowledge of the problem’s parameters.

1. Introduction

Transportation networks in major metropolitan areas carry several million car trips and commutes per day, giving rise to a chaotic and highly volatile environment for the average commuter. As a result, navigation apps like Google Maps, Waze and MapQuest have seen an explosive growth in their user base, routinely receiving upwards of $10^4$ routing requests per second during rush hour – and going up to $10^5$ queries/second in the largest cities in the US and China [10]. This vast number of users must be routed efficiently, in real-time, and without causing any “ex-post” regret at the user end; otherwise, if a user could have experienced better travel times along a non-recommended route, they would have no incentive to follow the app recommendation in the first place. In the language of congestion games

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this requirement is known as a “Wardrop equilibrium”, and it is typically represented as a high-dimensional vector describing the traffic flow along each path in the network [44]. Ideally, this traffic equilibrium should be computed before making a routing recommendation in order to minimize the number of disgruntled users in the system. In practice however, this is rarely possible: the state of the network typically depends on random, unpredictable factors that may vary considerably from one epoch to the next – e.g., due to road incidents, rain, fog and/or other weather conditions – so it is generally unrealistic to expect that such a recommendation can be made in advance. Instead, it is more apt to consider an online recommendation paradigm that unfolds as follows:

1. At each time slot \( t = 1, 2, \ldots, T \), a control interface – such as Google Maps – determines a traffic assignment profile and routes all demands received within this slot according to this profile.

2. The interface observes and records the travel times of the network’s users within the time slot in question; based on this feedback, it updates its candidate profile for the next epoch and the process repeats.

Of course, the crux of this paradigm is the optimization algorithm used to update traffic flow profiles from one epoch to the next. Any such algorithm would have to satisfy the following \textit{sine qua non} requirements:

i) \textit{Universal convergence rate guarantees in terms of the number of epochs}: The distance from equilibrium of the candidate traffic assignment after \( T \) epochs should be as small as possible in terms of \( T \). However, learning methods that are well-suited for rapidly fluctuating environments may be too slow in static environments (i.e., when cost functions do not vary over time); conversely, methods that are optimized for static environments may fail to converge in the presence of randomness. As a result, it is crucial to employ \textit{universal} algorithms that achieve the “best of both worlds” in terms of convergence speed.

ii) \textit{Fast convergence in terms of the size of the network}: The algorithm’s convergence speed must be at most polynomial in the size \( G \) of the underlying graph (i.e., its number of edges \( E \) plus the number of vertices \( V \)). In turn, since the number of paths \( P \) in the network is typically exponential in \( G \), the algorithm’s convergence speed must be at most polylogarithmic in \( P \).

iii) \textit{Scalable per-iteration complexity}: An algorithm can be implemented efficiently only if the number of arithmetic operations and total memory required at each iteration remains scalable as the network grows in size. In practice, this means that that the algorithm’s per-iteration complexity must not exceed \( \mathcal{O}(G) \).

iv) \textit{Parameter-freeness}: In most practical situations, the parameters of the model – e.g., the distribution of random events or the smoothness modulus of the network’s cost functions – cannot be assumed known, so any routing recommendation algorithm must likewise not require such parameters as input.

To put these desiderata in context, we begin below by reviewing the most relevant works in this direction.

\textbf{Related work.} The static regime of our model matches the framework of Blum et al. [8] who showed that a variant of the \textit{exponential weights} (ExpWeight) algorithm [3, 5, 28, 43] converges to equilibrium at an \( \mathcal{O} (\log P / \sqrt{T}) \) rate (in the sense of time averages). This result was subsequently extended to routing games with stochastic cost functions by Krichene et al. [21, 22], who showed that ExpWeight also enjoys an \( \mathcal{O} (\log P / \sqrt{T}) \) convergence rate to \textit{mean} Wardrop equilibria (again, in the Cesàro sense). However, if the learning rate of the
ExpWeight algorithm is not chosen appropriately in terms of $T$, the method may lead to non-convergent, chaotic behavior, even in symmetric congestion games over a 2-link Pigou network [37].

From an optimization viewpoint, nonatomic congestion games with stochastic cost functions correspond to convex minimization problems with stochastic first-order oracle feedback. In this setting, the $O(1/\sqrt{T})$ rate is, in general, unimprovable [9, 31], so the guarantee of Krichene et al. [22] is order-optimal. On the other hand, in the static regime, the $O(1/\sqrt{T})$ convergence speed of Blum et al. [8] is not optimal: since nonatomic congestion games admit a smooth convex potential – known as the Beckmann–McGuire–Winsten (BMW) potential [6] – this rate can be improved to $O(1/T^2)$ via the seminal “accelerated gradient” algorithm of Nesterov [30]. On the downside, if applied directly to our problem, the algorithm of Nesterov [30] has a catastrophic $\Theta(P)$ dependence on the number of paths; however, by coupling it with a “mirror descent” template in the spirit of [33, 45], Krichene et al. [22] proposed an accelerated method with an exponential projection step that is particularly well-suited for congestion problems. In fact, as we show in the sequel, it is possible to design an accelerated exponential weights method – which we call AcceleWeight – that achieves an $O(\log P/T^2)$ rate in static environments.

Importantly, despite its optimality in the static regime, AcceleWeight fails to converge altogether in stochastic problems; moreover, the method’s step-size must also be tuned with prior knowledge of the problem’s smoothness parameters (which are not readily available to the optimizer). The universal primal gradient descent (UPGD) algorithm of Nesterov [34] provides a work-around to resolve the latter issue, but it relies on a line-search mechanism that cannot be applied to stochastic problems, so it does not resolve the former. Instead, a partial solution for the stochastic case is achieved by the accelerated stochastic approximation (AC-SA) algorithm of Lan [25] which achieves order-optimal rates in both the static and stochastic regimes; however: a) the running time $T$ of the AC-SA algorithm must be fixed in advance as a function of the accuracy threshold required; and b) AC-SA further assumes full knowledge of the smoothness modulus of the game’s cost functions (which cannot be computed ahead of time).

To the best of our knowledge, the first parameter-free algorithm with optimal rate interpolation guarantees is the AcceleGrad algorithm of Levy et al. [27], which was originally developed for unconstrained problems (and assumes knowledge of a compact set containing a solution of the problem). The UnixGrad proposal of Kavis et al. [20] subsequently achieved the desired adaptation in constrained problems, but under the requirement of a bounded Bregman diameter. This requirement rules out the ExpWeight template (the simplex has infinite diameter under the entropy regularizer that generates the ExpWeight algorithm), so the convergence speed of UnixGrad ends up being polynomial in the number of paths in the network – and since the latter scales exponentially with the size of the network, UnixGrad is unsuitable for networks with more than 30 or so nodes.

Finally, another major challenge in terms of scalability is the algorithm’s per-iteration complexity, i.e., the memory and processing requirements for making a single update. In regard to this point, the per-iteration complexity of the UPGD, AC-SA, AcceleGrad and UnixGrad algorithms is linear in the number of paths $P$ in the network, and hence exponential in the size of the network. The only exception to this list is the (non-accelerated) ExpWeight algorithm, which can be implemented efficiently with $O(G)$ operations per iteration via a technique known as “weight-pushing” [18, 42]. However, the weight-pushing technique is by no means universal: it was specifically designed for the ExpWeight algorithm, and it is not applicable to any of the universal methods discussed above (precisely because of the acceleration mechanism involved).
**Our contributions.** In summary, all existing traffic assignment algorithms and methods fail in at least two of the axes mentioned above: either they are slow / non-convergent outside the specific regime for which they were designed, or they cannot be implemented efficiently (see also Table 1 for an overview). Consequently, our paper focuses on the following question:

*Is it possible to design a scalable, parameter-free traffic assignment algorithm which is simultaneously order-optimal in both static and stochastic environments?*

To provide a positive answer to this question, we take a two-step approach. Our first contribution is to design an adaptive exponential weights algorithm – dubbed AdaWeight – which is simultaneously order-optimal, in both $T$ and $P$, and in both static and stochastic environments. Informally, we have:

**Informal Version of Theorem 3.** The AdaWeight algorithm enjoys the following equilibrium convergence guarantees after $T$ epochs:

i) In static networks, AdaWeight converges to a Wardrop equilibrium at a rate of $O((\log P)^{3/2} / T^2)$.

ii) In stochastic networks, it converges to a mean Wardrop equilibrium at a rate of $O((\log P)^{3/2} / \sqrt{T})$.

In the above, the logarithmic dependency on $P$ ensures that the convergence speed of AdaWeight is polynomial – and in fact, subquadratic – in $G$. In this way, AdaWeight successfully solves the challenge of achieving optimal convergence rates in both static and stochastic environments, while remaining parameter-free and “anytime” (i.e., there is no need to tune the algorithm in terms of $T$). To the best of our knowledge, this is the first method that simultaneously achieves these desiderata.

On the other hand, a crucial drawback of AdaWeight is that it operates at the path level, i.e., it updates at each stage a state variable of dimension $P$; as a result, the algorithm’s per-iteration complexity is typically exponential in $G$. To overcome this, we propose an improved algorithm, which we call AdaLight (short for adaptive local weights algorithm), and which emulates the AdaWeight template with two fundamental differences: First, instead of maintaining a state variable per path, AdaLight maintains a state vector per node, with dimension equal to the node’s out-degree. Second, the recommended path for any particular user of the navigation interface is constructed “bottom-up” via a sequence of routing probabilities updated at each node, so there is no need to ever keep in memory (or update) a path variable.

This process looks similar to the weight-pushing technique of [18, 42] used to scale down the per-iteration complexity of the ExpWeight algorithm. However, the acceleration mechanism of AdaWeight involves several moving averages that cannot be implemented via weight-pushing; and since these averaging steps are downright essential for the universality of AdaWeight, they cannot be omitted from the algorithm. Instead, our solution for executing these averaging steps is based on the following observation: although the averaged recommendations profiles cannot be “weight-pushed”, the induced edge-load profiles (i.e., the mass of traffic induced on each edge) can. This observation allows us to introduce a novel dynamic programming subroutine – called PushPullMatch – which computes efficiently these averaged loads and then uses them to reconstruct a set of routing recommendations that are consistent with these loads.

In this way, by combining the AdaWeight blueprint with the PushPullMatch subroutine, AdaLight only requires $O(G)$ memory and processing power per update, all the
Table 1: Overview of related work in comparison to the AdaLight algorithm (this paper). For the purposes of this table, \( G \) refers to the size of the underlying graph while \( P \) refers to the number of relevant paths in the network (so \( P \) is typically exponential in \( G \)). The “anytime” property refers to whether the number of iterations \( T \) must be fixed at the outset and included as a parameter in the algorithm; if not, the algorithm is labeled “anytime”. Finally, the “parameter-agnostic” property refers to whether any other parameters – such as the Lipschitz modulus of the network’s cost functions – need to be known beforehand or not. All estimates are reported in the \( O(\cdot) \) sense.

|                  | ExpWeight | AcceleWeight | UnixGrad | UPGD | AC-SA | AdaLight |
|------------------|-----------|--------------|----------|------|-------|----------|
| Static           | \( \log P/\sqrt{T} \) | \( \log P/T^2 \) | \( P/T^2 \) | \( \log P/T^2 \) | \( \log P/T^2 \) | \( (\log P)^{3/2}/T^2 \) |
| Stochastic       | \( \log P/\sqrt{T} \) | \( \times \) | \( P/\sqrt{T} \) | \( \times \) | \( \log P/\sqrt{T} \) | \( (\log P)^{3/2}/\sqrt{T} \) |
| Anytime          | partially | \( \checkmark \) | \( \times \) | \( \times \) | \( \times \) | \( \checkmark \) |
| Param.-Agn.      | \( \times \) | \( \times \) | \( \checkmark \) | \( \checkmark \) | \( \times \) | \( \checkmark \) |
| Compl./Iter.     | \( G \) | \( P \) | \( P \) | \( P \) | \( P \) | \( G \) |

while retaining the sharp convergence properties of the AdaWeight mother scheme. In more detail, we have:

**Informal Version of Theorem 4.** The AdaLight algorithm converges to equilibrium at a rate of \( O((\log P)^{3/2}/T^2) \) in static environments and \( O((\log P)^{3/2}/\sqrt{T}) \) in stochastic environments; moreover, the total amount of arithmetic operations required per epoch is \( O(G) \) where \( G \) is the size of the network.

This result shows that AdaLight successfully meets the desiderata stated earlier, so AdaWeight might appear redundant. However, it is not possible to derive the equilibrium convergence properties of AdaLight without first going through AdaWeight, so, to better convey the ideas involved, we also describe the AdaWeight algorithm in detail (even though it is not scalable per se).

**Paper outline.** Our paper is structured as follows. In Section 2, we formally define our congestion game setup, the relevant equilibrium notions, and our learning model. Subsequently, to set the stage for our main results, we present in Section 3 the classic exponential weights algorithm as well as the AcceleWeight variant which achieves an \( O((\log P/T^2) \) convergence rate in static environments; both algorithms are non-adaptive, and they are used as a baseline for our adaptive results. Our analysis proper begins in Section 4, where we present the AdaWeight algorithm and its convergence analysis. Then, in Section 5, we present the local flow setup used to construct the AdaLight algorithm and prove its convergence guarantees. Finally, in Section 6, we report a series of numerical experiments validating our theoretical results in real-life transport networks.

2. Problem setup

We begin in this section by introducing the basic elements of our model.

2.1. The game. Building on the classic congestion framework of Beckmann et al. [6], we consider a class of nonatomic routing games defined by the following three primitives: (i) an underlying network structure; (ii) the associated set of traffic demands; and (iii) the network’s cost functions. The formal definition of each of these primitives is as follows:
1. **Network structure:** Consider a multi-graph $G = (V, \mathcal{E})$ with vertex set $V$ and edge set $\mathcal{E}$. The focal point of interest is a set of distinct origin-destination (O/D) pairs $(O^i, D^i) \in V \times V$ indexed by $i \in \mathcal{N} = \{1, \ldots, N\}$. For each $i \in \mathcal{N}$, we assume given a directed acyclic subgraph $\mathcal{G}^i = (V^i, \mathcal{E}^i)$ of $\mathcal{G}$ that determines the set of routing paths from $O^i$ to $D^i$, and we write $\mathcal{P}^i$ for the corresponding set of paths joining $O^i$ to $D^i$ in $\mathcal{G}^i$. For posterity, we will also write $\mathcal{P} = \bigcup_{i \in \mathcal{N}} \mathcal{P}^i$ for the set of all routing paths in the network, and $P = |\mathcal{P}|$ and $P^i = |\mathcal{P}^i|$ for the respective cardinalities; likewise, we will write $G^i = |\mathcal{V}^i| + |\mathcal{E}^i|$ for the size of $\mathcal{G}^i$, and $G = \sum_{i \in \mathcal{N}} G^i$ for the total size of the network.

2. **Traffic demands and flows:** Each pair $i \in \mathcal{N}$ is associated to a traffic demand $M^i > 0$ that is to be routed from $O^i$ to $D^i$ via $\mathcal{P}^i$; we also write $M_{\text{tot}} = \|M\|_1 = \sum_{i \in \mathcal{N}} M^i$ and $M_{\max} = \|M\|_\infty = \max_{i \in \mathcal{N}} M^i$ for the total and maximum traffic demand associated to the network’s O/D pairs respectively.

Now, to route this traffic, the set of feasible traffic assignment profiles – or flows – is defined as

$$\mathcal{F} := \{f \in \mathbb{R}_+^P : \sum_{p \in \mathcal{P}} f_p = M^i, i = 1, \ldots, N\}$$

i.e., as the product of scaled simplices $\mathcal{F} = \prod_i M^i \Delta(\mathcal{P}^i)$. In turn, each feasible flow profile $f \in \mathcal{F}$ induces on each edge $e \in \mathcal{E}$ the corresponding traffic load

$$\ell_e(f) = \sum_{p \in \mathcal{P}} 1_{\{e \in p\}} f_p$$

i.e., the accumulated mass of all traffic going through $e$. It is also worth noting here that the path index $p \in \mathcal{P}$ completely characterizes the O/D pair $i \in \mathcal{N}$ to which it belongs; when we want to make this relation explicit, we will write $f^i_p$ instead of $f_p$.

3. **Congestion costs:** The traffic routed through a given edge $e \in \mathcal{E}$ incurs a congestion cost depending on the total traffic on the edge and/or any other exogenous factors. Formally, we will collectively encode all exogenous factors in a state variable $\omega \in \Omega$ – the "state of the world" – that takes values in some ambient probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In addition, we will assume that each edge $e \in \mathcal{E}$ is endowed with a cost function $c_e : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ which determines the cost $c_e(\ell_e(f); \omega)$ of traversing $e \in \mathcal{E}$ when the network is at state $\omega \in \Omega$ and traffic is assigned according to the flow profile $f \in \mathcal{F}$. Analogously, the cost to traverse a path $p \in \mathcal{P}$ will be given by the induced path-cost function $c_p : \mathcal{F} \times \Omega \to \mathbb{R}_+$ defined as

$$c_p(f; \omega) = \sum_{e \in p} c_e(\ell_e(f); \omega).$$

In this general setting, the only assumption that we will make for the game’s cost functions is as follows:

**Assumption 1.** Each cost function $c_e(x; \omega)$, $e \in \mathcal{E}$, is measurable in $\omega$ and non-decreasing, bounded and Lipschitz continuous in $x$. Specifically, there exist $H > 0$ and $L > 0$ such that $c_e(x; \omega) \leq H$ and $|c_e(x; \omega) - c_e(x'; \omega)| \leq L|x' - x|$ for all $x, x' \in [0, M_{\text{tot}}]$, all $e \in \mathcal{E}$, and $\mathbb{P}$-almost all $\omega \in \Omega$.

Assumption 1 represents a very mild regularity requirement that is satisfied by most congestion models that occur in practice – including BPR, polynomial, or regularly-varying latency functions, cf. [8, 12, 26, 35, 36, 38] and references therein. For this reason, we will treat Assumption 1 as a standing, blanket assumption and we will not mention it explicitly in the sequel.
2.2. Regimes of uncertainty and examples. The advent of uncertainty in the game’s cost functions (as modeled by $\omega \in \Omega$) is an important element of our congestion game framework. To elaborate further on this, it will be convenient to define the mean cost function of edge $e \in \mathcal{E}$ as

$$\bar{c}_e(f) = \mathbb{E}_\omega[c_e(\ell_e(f); \omega)]$$

and consider the corresponding (random) fluctuation process

$$U_e(f; \omega) = c_e(\ell_e(f); \omega) - \bar{c}_e(f)$$

i.e., the deviation of the network’s cost functions at state $\omega \in \Omega$ from their mean value. The magnitude of these deviations may then be quantified by the randomness parameter

$$\sigma = \text{ess sup}_{\omega \in \Omega} \max_{f \in \mathcal{F}, e \in \mathcal{E}} |U_e(f; \omega)|$$

where $\text{ess sup}$ denotes the essential supremum over $\Omega$ with respect to $\mathbb{P}$ (so $\sigma \leq 2H$ by Assumption 1). Informally, larger values of $\sigma$ indicate a higher degree of randomness, implying in turn that the traffic assignment problem becomes more difficult to solve; on the other hand, if $\sigma = 0$, one would expect algorithmic methods to achieve better results. This distinction plays a key role in the sequel so we formalize it as follows:

**Definition 1 (Static and Stochastic Environments).** When $\sigma = 0$, we say that the environment is static; otherwise, if $\sigma > 0$, we say that the environment is stochastic.

The dependence of the network’s cost functions on exogenous random factors is the main difference of our setup with standard congestion models in the spirit of Beckmann et al. [6] and Nisan et al. [35]. For concreteness, we mention below two existing models that can be seen as special cases of our framework:

**Example 2.1 (Deterministic routing games).** The static regime described above matches the deterministic routing models of Blum et al. [8], Fischer and Vöcking [16] and Krichene et al. [23, 24]. In these models, the network only has a single (deterministic) state so, trivially, $\sigma = 0$.

**Example 2.2 (Noisy cost measurements).** To model uncertainty in the cost measurement process, Krichene et al. [21, 24] considered the case where the network’s cost functions are fixed, but cost measurements are only accurate up to a random, zero-mean error. In our framework, this can be modeled by simply assuming a family of cost functions of the form $c_e(f; \omega) = \bar{c}_e(f) + \omega_c$ with $\mathbb{E}[\omega_c] = 0$ for all $e \in \mathcal{E}$.

Besides these standard examples, our model is sufficiently flexible to capture other random factors such as weather conditions, traffic accidents, road incidents, etc. For instance, to model the difference between dry and wet weather, one can take $\Omega = \{\omega_{\text{dry}}, \omega_{\text{wet}}\}$ and consider a set of cost functions with $c_e(\ell_e(f); \omega_{\text{dry}}) < c_e(\ell_e(f); \omega_{\text{wet}})$ to reflect the fact that the congestion costs are higher when the roads are wet. However, to maintain the generality of our model, we will not focus on any particular application.

2.3. Notions of equilibrium. In the above framework, each state variable $\omega \in \Omega$ determines an instance of a routing game, defined formally as a tuple $\Gamma_\omega \equiv \Gamma(\mathcal{G}, \mathcal{N}, \mathcal{P}, c_\omega)$ where $c_\omega$ is shorthand for the network’s cost functions $\{c_e(\cdot; \omega)\}_{e \in \mathcal{E}}$ instantiated at $\omega$. Of course, in analyzing the game, each individual instance $\Gamma_\omega$ is meaningless by itself unless $\mathbb{P}$ assigns positive probability only to a single $\omega$. For this reason, we will instead focus on the mean game $\Gamma \equiv \Gamma(\mathcal{G}, \mathcal{N}, \mathcal{P}, \bar{c})$ which has the same network and routing structure as every $\Gamma_\omega$, $\omega \in \Omega$, but with congestion costs given by the mean cost functions

$$\bar{c}_p(f) = \mathbb{E}[c_p(f; \omega)] = \sum_{e \in \mathcal{E}} \mathbb{E}[c_e(\ell_e(f); \omega)].$$
Motivated by the route recommendation problem described in the introduction, we will focus on traffic assignment profiles where Wardrop’s unilateral optimality principle [44] holds on average, i.e., all traffic is routed along a path with minimal mean cost. Formally, we have:

**Definition 2** (Mean equilibrium flows). We say that \( f^* \in \mathcal{F} \) is a mean equilibrium flow if and only if \( \tilde{c}_p(f^*) \leq \tilde{c}_q(f^*) \) for all \( i \in \mathcal{N} \) and all \( p, q \in \mathcal{P}^i \) such that \( (f^*)_p^q > 0 \).

**Remark 1.** Definition 2 means that, on average, no user has an incentive to deviate from the recommended route; obviously, when the support of \( \mathcal{P} \) is a singleton, we recover the usual definition of a Wardrop equilibrium [6, 35, 44]. This special case will be particularly important and we discuss it in detail in the next section.

Importantly, the problem of finding an equilibrium flow of a (fixed) routing game \( \Gamma_\omega \) admits a potential function – often referred to as the Beckmann–McGuire–Winsten (BMW) potential [6, 14], specifically, for a given instance \( \omega \in \Omega \), the BMW potential is defined as

\[
\Phi_\omega(f) = \sum_{e \in \mathcal{E}} \int_0^{\ell_e(f)} c_e(x; \omega) \, dx \quad \text{for all } f \in \mathcal{F},
\]

(BMW)

and it has the fundamental property that \( \text{arg min} \Phi_\omega \) coincides with the set \( \text{Eq}(\Gamma_\omega) \) of equilibria of \( \Gamma_\omega \). In our stochastic context, a natural question that arises is whether this property can be extended to the mean game \( \Gamma \) where the randomness is “averaged over”. Clearly, the most direct candidate for a potential function in this case is the mean BMW potential

\[
\Phi(f) = \mathbb{E}[\Phi_\omega(f)] = \mathbb{E} \left[ \sum_{e \in \mathcal{E}} \int_0^{\ell_e(f)} c_e(x; \omega) \, dx \right].
\]

(8)

Since \( c_e \) is continuous and non-decreasing for all \( e \in \mathcal{E} \), the mean BMW potential \( \Phi \) is, in turn, continuously differentiable and convex on \( \mathcal{F} \). Of course, since the probability law \( \mathbb{P} \) is not known, we will not assume that \( \Phi \) (and/or its gradients) can be explicitly computed in general. Nevertheless, building on the deterministic characterization of Beckmann et al. [6], we have the following equivalence between mean equilibria and minimizers of \( \Phi \):

**Proposition 1.** \( \Phi \) is a potential function for the mean game \( \Gamma \equiv (\mathcal{G}, \mathcal{N}, \mathcal{P}, \mathbb{P}, \bar{c}) \); in particular, we have

\[
\frac{\partial \Phi}{\partial f_p} = \bar{c}_p(f) \quad \text{for all } f \in \mathcal{F} \text{ and all } p \in \mathcal{P}.
\]

(9)

Accordingly, a flow profile \( f^* \in \mathcal{F} \) is an equilibrium of the mean game \( \Gamma \) if and only if it is a minimizer of \( \Phi \) over \( \mathcal{F} \); more succinctly, the equilibrium set \( \text{Eq}(\Gamma) \) of \( \Gamma \) satisfies

\[
\text{Eq}(\Gamma) = \arg \min_{f \in \mathcal{F}} \Phi(f).
\]

**Proof.** First, apply the dominated convergence theorem (saying that gradients and expectations commute), we have \( \frac{\partial \Phi}{\partial f_p} = \mathbb{E} \left[ \frac{\partial \Phi_\omega}{\partial f_p} \right] = \mathbb{E}[\bar{c}_p(f; \omega)] = \bar{c}_p(f) \). The rest of the proof of Proposition 1 can be obtained by following [6, 40] for the mean game \( \Gamma \).

In view of Proposition 1, \( \Phi \) provides a natural merit function for examining how close a given flow profile \( f \in \mathcal{F} \) is to being an equilibrium. Formally, given a sequence of candidate flow profiles \( f^t \in \mathcal{F}, t = 1, 2, \ldots \), we define the associated equilibrium gap after \( T \) epochs as

\[
\text{Gap}(T) = \Phi(f^T) - \min \Phi.
\]

(10)

Clearly, the sequence in question converges to \( \text{Eq}(\Gamma) \) if and only if \( \text{Gap}(T) \to 0 \). Moreover, the speed of this convergence is also captured by \( \text{Gap}(T) \), so all our convergence certificates will be stated in terms of \( \text{Gap}(T) \).
2.4. Sequence of events. With all this in hand, the sequence of events in our model unfolds as follows:

i) At each time slot \( t = 1, 2, \ldots \), the navigation interface selects a traffic assignment profile \( f^t \in \mathcal{F} \) and it routes all demands received within this time slot according to \( f^t \).

ii) Concurrently, the state \( \omega^t \) of the network is drawn i.i.d. from \( \Omega \) according to \( P \).

iii) The interface observes the realized congestion costs \( C^t_e = c_e(\ell_e(f^t); \omega^t) \) along each edge \( e \in \mathcal{E} \); subsequently, it uses this information to update the routing profile \( f^t \), and the process repeats.

In what follows, we will analyze different learning algorithms for updating \( f^t \) in the above sequence of events, and we will examine their equilibrium convergence properties in terms of the equilibrium gap function (10). For posterity, we emphasize again that the optimizer has no prior knowledge of the environment, the distribution of random events, the network’s cost functions, etc., so we will pay particular attention to the dependence of each algorithm’s guarantees on these (otherwise unknown) parameters.

3. Non-adaptive methods

In the rest of our paper, we discuss a series of routing algorithms and their equilibrium convergence guarantees. To set the stage for our main contributions, we begin by presenting two learning methods that are order-optimal in the two basic regimes described in the previous section:

i) The “vanilla” exponential weights (ExpWeight) method of Blum et al. [8] for stochastic environments.

ii) An accelerated exponential weights algorithm – which we call AcceleWeight – for static environments.

Both methods rely crucially on the (rescaled) logit choice map \( \Lambda : \mathbb{R}^P \rightarrow \mathcal{F} \), given in components as

\[
\Lambda_p(w) = \frac{M^i \exp(w_p)}{\sum_{q \in p, i} \exp(w_q)} \quad \text{for all } w \in \mathbb{R}^P, p \in \mathcal{P}^i \text{ and } i \in \mathcal{N}.
\]  

(11)

For conciseness, we will also write \( C^t_p = \sum_{e \in p} C^t_e \) for the cost of path \( p \in \mathcal{P} \) at stage \( t \), and \( C^t = (C^t_p)_{p \in \mathcal{P}} \) for the profile thereof.

3.1. Exponential weights in stochastic environments. We begin by presenting the exponential weights algorithm of Blum et al. [8] in standard pseudocode form:

**Algorithm 1: Exponential weights (ExpWeight)**

1. **Initialize** \( w^1 \leftarrow 0 \)
2. for \( t = 1, 2, \ldots \) do
3.   set \( f^t \leftarrow \Lambda(w^t) \) and get \( C^t \leftarrow c(f^t, \omega^t) \) \hspace{1cm} // route and measure costs
4.   set \( w^{t+1} \leftarrow w^t - \gamma^t C^t \) \hspace{1cm} // update path scores

We then have the following equilibrium convergence result:

**Theorem 1.** Suppose that ExpWeight (Algorithm 1) is run for \( T \) epochs with a fixed learning rate \( \gamma^t = \sqrt{\log(M_{\max} P / M_{\text{tot}})} / (H \sqrt{T}) \) for any \( t \). Then the time-averaged flow
profile \( f^T = (1/T) \sum_{t=1}^T f^t \) enjoys the equilibrium convergence rate:

\[
\mathbb{E}[\text{Gap}(T)] \leq \frac{2M_{tot}H \sqrt{\log (M_{max}P/M_{tot})}}{\sqrt{T}} = O\left(\frac{(\log P)^{1/2}}{\sqrt{T}}\right).
\]  

(12)

The proof of Theorem 1 relies on standard techniques, so we omit it; for a series of closely related results, we refer the reader to Krichene et al. [21] and Shalev-Shwartz [41, Corollary 2.14 and Theorem 4.1]: What is more important for our purposes is that Theorem 1 confirms that EXPWEIGHT achieves an \( O(1/\sqrt{T}) \) convergence speed in the number of epochs \( T \), so it is order-optimal in stochastic environments [9, 31]. Moreover, the algorithm’s convergence speed is logarithmic in \( P \), and hence (at most) linear in the size \( G \) of the underlying network.

On the downside, the convergence rate obtained for Algorithm 1 concerns the time-averaged flow \( f^T \), not the actual recommendation, a distinction which is important for practical applications. Second, Theorem 1 is not an “anytime” convergence result in the sense that the algorithm’s learning rate \( \gamma^t \) must be tuned relative to \( T \) (the rate is not valid otherwise). This issue can be partially resolved either via a doubling trick [41] (at the cost of restarting the algorithm ever so often) or by using a variable learning rate of the form \( \gamma^t \propto 1/\sqrt{t} \) (at the cost of introducing an additional \( \log T \) factor in the algorithm’s convergence speed). However, in either case, the rate (12) only applies to the sequence of time averages, not the actual recommendations.

3.2. Accelerated exponential weights in static environments. We now turn our attention to the static regime, i.e., when there are no exogenous variations in the network’s cost functions (\( \sigma = 0 \)); in this case, it is reasonable to expect that the \( O(1/\sqrt{T}) \) convergence speed of EXPWEIGHT can be improved. Indeed, as we show below, the BMW potential of the game is Lipschitz smooth, so the optimal convergence speed in static environments is \( O(1/T^2) \) [9, 31].

Proposition 2. The BMW potential \( \Phi \) is Lipschitz smooth relative to the \( L^1 \) norm on \( F \). In particular, its smoothness modulus is \( \beta = KL \), where \( K = \max_{p \in P} \sum_{e \in e} \mathbb{I}_{\{e \in p\}} \) is the length of the longest path in \( P \).

Proof of Proposition 2. It suffices to show that the gradient \( \nabla \Phi \) of \( \Phi \) is Lipschitz continuous with respect to the (primal) \( L^1 \) norm on \( F \) and the (dual) \( L^\infty \) norm on \( \mathbb{R}^P \). Indeed, for all \( f, f' \in F \), combining Proposition 1 and Assumption 1 yields

\[
\|\nabla \Phi(f) - \nabla \Phi(f')\|_{\infty} = \max_{p \in P} |\bar{c}_p(f) - \bar{c}_p(f')| \\
\leq \max_{p \in P} \sum_{e \in E} L \ell_e(f) - \ell_e(f') | \\
\leq L \cdot \max_{p \in P} \sum_{e \in E} \mathbb{I}_{\{e \in p\}} \sum_{q \in P} \mathbb{I}_{\{e \in q\}} |f_q - f'_q| \\
\leq L \cdot \max_{p \in P} \sum_{e \in E} \mathbb{I}_{\{e \in p\}} |f_q - f'_q| \\
\leq L \cdot \max_{p \in P} \sum_{e \in E} \mathbb{I}_{\{e \in p\}} \cdot \sum_{q \in P} |f_q - f'_q| = KL||f - f'||_1 \\
\]  

[by Cauchy-Schwarz]

so our claim follows. \( \blacksquare \)

Given that \( \Phi \) is smooth, the iconic \( O(1/T^2) \) rate mentioned above can be achieved by the accelerated gradient algorithm of Nesterov [30]. However, if Nesterov’s algorithm were applied
“off the shelf”, the constants involved would be linear in $P$ (the problem’s dimensionality), and hence exponential in the size of the network. Instead, building on ideas by Allen-Zhu and Orecchia [1] and Krichene et al. [22], this problem can be overcome by combining the acceleration mechanism of Nesterov [30] with the dimensionally-efficient ExpWeight template. The resulting accelerated exponential weights (AcceleWeight) method proceeds in pseudocode form as follows:

**Algorithm 2: Accelerated exponential weights (AcceledWeight)**

**Input:** Smoothness parameter $\beta = KL$

1. Initialize $w^1 \leftarrow 0$; $f^0 \leftarrow 0$; $\alpha^0 \leftarrow 0$; $\gamma^0 \leftarrow (NM_{\max} \beta)^{-1}$

2. for $t = 1, 2, \ldots$ do

3. set $z^t \leftarrow \Lambda(w^t)$ \hspace{1cm} // exploratory flow obtained from path scores

4. set $f^t \leftarrow \alpha^t f^{t-1} + (1 - \alpha^t) z^t$ \hspace{1cm} // average with previous state

5. set $\gamma^t \leftarrow \gamma^{t-1} + \gamma^0 / 2 + \sqrt{\gamma^{t-1} \gamma^0} + (\gamma^0 / 2)^2$ \hspace{1cm} // update step-size

6. set $\alpha^t \leftarrow \gamma^{t-1} / \gamma^t$ \hspace{1cm} // update averaging weight

7. set $f^t \leftarrow \alpha^t f^t + (1 - \alpha^t) z^t$ and get $\bar{C}^t \leftarrow c(f^t, \omega^t)$ \hspace{1cm} // route and measure costs

8. set $w^{t+1} \leftarrow w^t - (1 - \alpha^t) \gamma^t \bar{C}^t$ \hspace{1cm} // update path scores

At a high level, AcceledWeight follows the acceleration template of Nesterov [30], but replaces Euclidean projections with the logit map (11). With this in mind, we obtain the following convergence guarantee:

**Theorem 2.** The sequence of flow profiles $f^1, f^2, \ldots$ generated by AcceledWeight (Algorithm 2) in a static environment ($\sigma = 0$) enjoys the equilibrium convergence rate

$$\text{Gap}(T) \leq \frac{4\beta N^2 M_{\max}^2 \log(M_{\max} P / M_{\text{tot}})}{(T - 1)^2} = \mathcal{O}\left(\frac{\log P}{T^2}\right).$$

The proof of Theorem 2 is based on techniques that are widely used in the analysis of accelerated methods so, to streamline our presentation, we relegate it to Appendix A.

Theorem 2 confirms that, in the static regime, AcceledWeight converges to equilibrium with an optimal $\mathcal{O}(1/T^2)$ convergence speed, as desired. For our purposes, it is more important to note that Theorem 2 confirms that, in static environments, accelerated exponential weights achieves a much faster convergence rate than ExpWeight. However, this improvement comes with several limitations: First and foremost, as we discuss in Section 6, AcceledWeight fails to converge altogether in the stochastic regime, so it is not a universal method (i.e., an algorithm that is simultaneously order-optimal in both static and stochastic environments). Second, to obtain the accelerated rate (13), the step-size of AcceledWeight must be tuned with prior knowledge of the problem’s smoothness modulus $\beta$ (which is not realistically available to the learner), so the method is not adaptive. These are both crucial limitations that we seek to overcome in Sections 4 and 5.

### 3.3. Per-iteration complexity of each method.

We close this section with a short discussion on the per-iteration complexity of ExpWeight and AcceledWeight. The key point of note is that Algorithms 1 and 2 both require $\mathcal{O}(P)$ time and space per iteration (i.e., in terms of arithmetic operations and memory storage). This is inefficient in large-scale networks where $P$ is exponentially large in $G$ (the size of the network), so it is not clear how to implement either of these algorithms in practice.

As discussed in Section 1, there exist variants of the exponential weights algorithm that can be implemented efficiently in $\mathcal{O}(G)$ time and space per iteration. This is accomplished via a dynamic programming procedure known as “weight-pushing” [18, 42]; however, this technique is limited to boosting the computation step involving the logit mapping (Line
3 of Algorithm 1 and Line 3 of Algorithm 2) and cannot efficiently execute other steps of ACCELWEIGHT that require $O(P)$ operations or storage bits (like the averaging steps involved in the acceleration mechanism). To maintain the flow of our discussion, we will revisit this issue in detail in Section 5.

4. ADAWEIGHT: ADAPTIVE LEARNING IN THE PRESENCE OF UNCERTAINTY

4.1. Adaptivity of ADAWEIGHT in optimizing equilibrium-convergence rates. To summarize the situation so far, we have seen that EXPWEIGHT attains an $O(\log(P)/\sqrt{T})$ rate, which is order-optimal in the stochastic case but suboptimal in static environments; by contrast, ACCELWEIGHT attains an $O(\log(P)/T^2)$ rate in static environment, but has no convergence guarantees in the presence of randomness and uncertainty. Consequently, neither of these algorithms meets our stated objective to concurrently achieve order-optimal guarantees in both the static and stochastic cases (and without requiring prior knowledge of the problem’s smoothness modulus).

To resolve this gap, we propose below an adaptive exponential weights method – ADAWEIGHT for short – which achieves these objectives by mixing the acceleration template of ACCELWEIGHT with the dual extrapolation method of Nesterov [32]. We present the pseudocode of ADAWEIGHT below:

Algorithm 3: Adaptive exponential weights (ADAWEIGHT)

1. Initialize $\alpha^0 \leftarrow 0$, $z^0 \leftarrow 0$, $\eta^1 \leftarrow 1$ and $w^1 \leftarrow 0$
2. for $t = 1, 2, \ldots$ do
3.   set $r^t \leftarrow \sum_{s=0}^{t-1} \alpha^s z^s$ // set an anchor primal point
4.   set $\hat{z}^t \leftarrow \Lambda(\eta^t w^t)$ // the test phase:
5.   set $\hat{f}^t \leftarrow (\alpha^t \hat{z}^t + r^t) / \sum_{s=0}^{t} \alpha^s$ and get $\hat{C}^t \leftarrow c(\hat{f}^t, \omega^t)$ // reweigh with anchor + query
6.   set $\hat{w}^t \leftarrow w^t - \alpha^t \hat{C}^t$ // exploratory score update
7.   set $z^t \leftarrow \Lambda(\eta^t \hat{w}^t)$ // the recommendation phase:
8.   set $f^t \leftarrow (\alpha^t z^t + r^t) / \sum_{s=0}^{t} \alpha^s$ and get $C^t \leftarrow c(f^t, \omega^t)$ // reweigh with anchor + route
9.   set $w^{t+1} \leftarrow w^t - \alpha^t C^t$ // update path scores
10. set $\eta^{t+1} \leftarrow 1 / \sqrt{1 + \sum_{s=0}^{t} \|\alpha^t (C^s - \hat{C}^s)\|_\infty^2}$ // update learning rate

The main novelty in the definition of the ADAWEIGHT algorithm is the introduction of two “extrapolation” sequences, $z^t$ and $\hat{w}^t$, that venture outside the convex hull of the generated primal (flow) and dual (score) variables respectively. These leading states are subsequently averaged, and the method proceeds with an adaptive step-size rule. In more details, ADAWEIGHT relies on three key components:

a) A dual extrapolation mechanism for generating the leading sequences in Lines 4 and 7; these sequences are central for anticipating the loss landscape of the problem.

b) An acceleration mechanism obtained from the $(\alpha^t)$-weighted average steps in Lines 5 and 8; in the analysis, $\alpha^t$ will grow as $t$, so almost all the weight will be attributed to the state closest to the current one.
c) An adaptive sequence of learning rates (cf. Line 10) in the spirit of Rakhlin and Sridharan [39], Kavis et al. [20] and Antonakopoulos et al. [2]. This choice is based on the ansatz that, if the algorithm encounters coherent gradient updates (which can only occur in static environments), it will eventually stabilize to a strictly positive value; otherwise, it will decrease to zero at a $\Theta(1/\sqrt{T})$ rate. This property is crucial to interpolate between the stochastic and static regimes.

The combination of the weighted average iterates and adaptive learning rate in AdaWeight is shared by the UnixGrad algorithm proposed by Kavis et al. [20], which also provides rate interpolation in constrained problems. However, UnixGrad requires the problem’s domain to have a finite Bregman diameter – and, albeit compact, the set of feasible flows $\mathcal{F}$ has an *infinite* diameter under the entropic regularizer that generates the ExpWeight template. Therefore, UnixGrad is not applicable to our routing games. This is the reason for switching gears to the “primal-dual” approach offered by the dual extrapolation template; this primal-dual interplay provides the missing link that allows AdaWeight to simultaneously enjoy order-optimal convergence guarantees in both settings while maintaining the desired polynomial dependency on the problem’s dimension.

In light of the above, our main convergence result for AdaWeight is as follows:

**Theorem 3.** Let $f^1, f^2, \ldots$, be the sequence of flows recommended by AdaWeight (Algorithm 3) running with $\alpha^t = t$ for any $t = 1, 2, \ldots$, then it enjoys the following equilibrium convergence rate:

$$
\mathbb{E}[\text{Gap}(T)] \leq 2\sqrt{2} A \sigma \sqrt{T} + \frac{16\beta \sqrt{NM_{\max}}A^{3/2} + B}{T^2} = \mathcal{O}\left((\log P)^{3/2} \left(\frac{\sigma}{\sqrt{T}} + \frac{1}{T^2}\right)\right). \tag{14}
$$

Specifically, in the static case, AdaWeight enjoys the sharper rate: $\text{Gap}(T) \leq \mathcal{O}\left((\log P)^{3/2}/T^2\right)$. In the above expressions, $\sigma$ is as defined in Equation (6) while $A$ and $B$ are positive constants given by $A := NM_{\max}[2 \log (PM_{\max}/M_{\text{tot}}) + 13] = \mathcal{O}(\log P)$ and $B := M_{\text{tot}} \log (PM_{\max}/M_{\text{tot}}) = \mathcal{O}(\log P)$.

Theorem 3 characterizes the convergence speed of AdaWeight according to the number of learning iterations $T$, the number of paths $P$ and the level of randomness $\sigma$ of the network’s states (as defined in (6)). In the stochastic environment, $\sigma \geq 0$ and hence, AdaWeight enjoys a convergence rate of order $\mathcal{O}\left((\log P)^{3/2}/\sqrt{T}\right)$. In the static case, $\sigma = 0$ and the convergence rate is accelerated to $\mathcal{O}\left((\log P)^{3/2}/T^2\right)$. In instances where the cost observations become more accurate over time, Theorem 3 also allows us to achieve a smooth trade-off in the convergence rate. For example, if $\sigma = \mathcal{O}(1/T^x)$ for some suitable $x$, AdaWeight’s rate of convergence carries a dependence of the order of $\mathcal{O}(T^{\max(1-x/2,-x)})$.

In view of these result, Theorem 3 confirms that AdaWeight satisfies the following desiderata: (i) it achieves simultaneously optimal guarantees in both stochastic and static environments in the number of learning iterations (i.e., $\mathcal{O}(1/\sqrt{T})$ and $\mathcal{O}(1/T^2)$ respectively); (ii) the derived rates maintain a polynomial dependency in terms of the network’s combinatorial primitives; and (iii) it requires no prior tuning by the learner. Moreover, unlike ExpWeight, the convergence of AdaWeight corresponds to an actual traffic flow profile that is implemented in epoch $t$ and not the average flow.

**The per-iteration complexities of AdaWeight.** The AdaWeight method, as presented in Algorithm 3, while having a simple presentation, takes $\mathcal{O}(P)$ time (and space) to complete each iteration. Therefore, implementing AdaWeight is inefficient. Although AdaWeight shares certain elements of the ExpWeight template, the naive application of weight-pushing fails to obtain an implementation of AdaWeight having a polynomial per-iteration complexity.
in the network’s size. This is due to the complicated averaging steps in ADAWEIGHT (Lines 7 and 8 of Algorithm 3). In Section 5, we re-discuss this in more details and we propose another algorithm, called ADAIGHT, that maintains all desired features of ADAWEIGHT and achieves an efficient per-iteration complexity (i.e., sub-quadratic in the size of the underlying graph).

4.2. Proof of Theorem 3. ADAWEIGHT is not merely a “convex combination” of the two non-adaptive optimal algorithms (ACCELWEIGHT and EXPWEIGHT). For this reason, the proof of Theorem 3 is technically involved. Particularly, the primal-dual averaging method that we use in ADAWEIGHT create sequences of filtration-dependent step-sizes; while this is the key element allowing ADAWEIGHT to achieve the best-of-both-worlds, previously known approaches in deriving the convergence analyses (as in non-adaptive algorithmic schemes – EXPWEIGHT and ACCELWEIGHT – and/or in UNIXGRAD) are not applicable in this case. To handle this new challenge, we propose a completely new way to treat the learning rate in order to make the derived bounds summable. To ensure the comprehensibility, in this section, we first present a sketch of proof explaining the high-level idea before presenting the complete proof with all technical details.

4.2.1. Sketch of proof (Theorem 3). Let \( \overline{R}_T(f^*) := \sum_{t=1}^{T} \alpha^t \langle \nabla \Phi(f^t), z^t - f^* \rangle \), the starting point of our proof is the following result:

\[
E[\text{Gap}(T)] = E[\Phi(f^T) - \Phi(f^*)] \leq 2 E[\overline{R}_T(f^*)] / T^2 \text{ for any } T.
\]  

(15) is a standard result in working with the \( \alpha^t \)-weighted averaging technique that also appears in several previous works [13, 20]. For the sake of completeness, we provide the proof of (15) in B.

Recall that as ADAWEIGHT is run, \( \nabla \Phi(f^t) \) is not observable and only the cost \( C_t \) is observed and used. Therefore, instead of \( \overline{R}_T(f^*) \), we now focus on the term \( R_T(f^*) := \sum_{t=1}^{T} \alpha^t \langle C^t, z^t - f^* \rangle \). In the stochastic case, we can prove that \( R_T(f^*) \) is an unbiased estimation of \( \overline{R}_T(f^*) \) (and in the static case, they coincide). Therefore, any upper-bound of \( R_T(f^*) \) can be translated via (15) into an upper-bound of the left-hand-sides of (14). The key question becomes “Which upper-bound of \( R_T(f^*) \) can be derived to guarantee the convergence rates in (14)?”

To answer the question above, we note that ADAWEIGHT is built on the dual extrapolation template with two phases: the test phase and the recommendation phase. This allows us to upper-bound \( R_T(f^*) \) in terms of the “distance” between the pivot primal points in these phases (i.e., \( \bar{z}^t \) and \( z^t \)) and the difference between the costs measured in these phases (i.e., \( \bar{C}^t \) and \( C^t \)). Particularly, we have

\[
R_T(f^*) \leq \sum_{t=1}^{T} g_{\text{primal}}(\eta_t^{t+1}) \| z^t - \bar{z}^t \|^2 + \sum_{t=1}^{T} g_{\text{dual}}(\eta_t^{t+1})(\alpha^t)^2 \| C^t - \bar{C}^t \|^2, \tag{16}
\]

where \( g_{\text{primal}}(\eta_t^{t+1}) \) and \( g_{\text{dual}}(\eta_t^{t+1}) \) are certain functions that also depend on other parameters of the game. Importantly, the terms in the right-hand-size of (16) are actually summable: they are bounded by the summation of two terms \( \sum_{t=1}^{T} g(\eta_t)(\alpha^t)^2 \| C^t - \bar{C}^t \|^2 \) and \( \sqrt{\sum_{t=1}^{T} (\alpha^t)^2 \| z^t - \bar{z}^t \|^2} \). Here, the first term arises when we bound \( \| z^t - \bar{z}^t \| \) using the smoothness of the BMW potential \( \Phi \). Moreover, by our special choice of adaptive sequence of learning rate \( \eta_t \), there exists \( T_0 \ll T \) such that only the first \( T_0 \) components in this summation are positive and hence, this summation is actually of order \( O(1) \) (i.e., it does not depend on \( T \)). The second term quantifies the error of the observed costs with respect to the actual gradients of the potential. This term is of order \( O(\sigma T^{3/2}) \)
when we choose α^t = t. Combine these results with (15), we obtain the convergence rates indicated in (14).

4.2.2. Proof with technical details (Theorem 3). In this section, we work with the entropy regularizer \( h(f) := \sum_{i \in \mathcal{N}} \sum_{p \in \mathcal{P}_i} f_p \log(f_p), \forall f \in \mathcal{F} \) which can be used to define the exponential weights template. Trivially, \( h \) is \( 1/\kappa \)-strongly convex on \( \mathcal{F} \) w.r.t the \( \| \cdot \|_1 \) norm, where we define \( \kappa := M_{\max N} \). We will also work with the Fenchel conjugate of \( h \), defined as \( h^*(w) := \max_{f \in \mathcal{F}} \langle f, w \rangle - h(f) \), and its Fenchel coupling, defined as \( F(f, w) = h(f) + h^*(w) - \langle f, w \rangle \) for any \( f \in \mathcal{F} \) and any \( w \in \mathbb{R}^P \). For the sake of brevity, we will also define \( \min h := \min_{f \in \mathcal{F}} h(f) \) and denote the \( \| \cdot \|_1 \)-diameter of \( \mathcal{F} \) by \( D := \max_{f, f' \in \mathcal{F}} \| f - f' \|_1 \). We also denote the Kullback-Leibler divergence between two flows \( f \) and \( f' \) by \( \text{KL}(f \| f') \). Note also that throughout this section, when we mention Algorithm 3, we understand that it is run with \( \alpha^t := t \) as chosen in Theorem 3.

First, we prove the following proposition (corresponding to Equation (16) in the proof sketch):

**Proposition 3.** Run Algorithm 3, for any \( f \in \mathcal{F} \), we have:

\[
\mathcal{R}_T(f^*) \leq h(f^*) - \min h + A_{\text{dual}} \sum_{t=1}^T (\alpha^t)^2 \eta_t + 1 \| C^t - \tilde{C}^t \|_\infty^2 - \sum_{t=1}^T \frac{1}{2 \kappa \eta_t^t + 1} \| z^t - \tilde{z}^t \|_1^2, \tag{17}
\]

where \( A_{\text{dual}} := h(f^*) - \min h + \kappa^2 + 2D^2 \).

Proof of Proposition 3. Consider an “intermediate” point \( y^t := \Lambda(\eta^t w^t + 1) \), we focus on the terms \( \alpha^t \langle C^t, y^t - f^* \rangle \) and \( \alpha^t \langle C^t, z^t - y^t \rangle \). These terms sum up to \( \alpha^t \langle C^t, z^t - f^* \rangle \) which defines \( \mathcal{R}_T(f^*) \). First, from the update rule of \( w^t \) in Algorithm 3, we have

\[
\alpha^t \langle C^t, y^t - f^* \rangle = \frac{1}{\eta_t} \langle \eta^t w^t - \eta^t w^t + 1, y^t - f^* \rangle = \frac{1}{\eta_t} \left[ -h(y^t) - h^*(\eta^t w^t + 1) + \langle f^*, \eta^t w^t + 1 - \eta^t w^t \rangle + \langle y^t, \eta^t w^t \rangle \right]
\]

\[
\leq \frac{1}{\eta_t} \left[ F(f^*, \eta^t w^t) - F(f^*, \eta^t w^t + 1) - \text{KL}(y^t \| z^t) \right].
\]

Here, the last inequality comes from the definition of Fenchel coupling and the fact that \( z^t = \Lambda(\eta^t w^t) \). Now, apply the three-point inequality with Bregman divergence (Lemma 3.1 of [11]) to the KL-divergence,

\[
\frac{1}{\eta_t^2} \text{KL}(y^t \| z^t) - \frac{1}{\eta_t^2} \text{KL}(z^t \| z^t) - \frac{1}{\eta_t^2} \text{KL}(y^t \| z^t) = \langle \nabla h(z) - \nabla h(z), z^t - y^t \rangle \geq \alpha^t \langle \tilde{C}^t, z^t - y^t \rangle.
\]

Combining the two inequalities derived above, we have:

\[
\mathcal{R}_T(f^*) = \sum_{t=1}^T \alpha^t \langle C^t, y^t - f^* \rangle + t \langle C^t, z^t - y^t \rangle
\]

\[
\leq \sum_{t=1}^T \frac{1}{\eta_t} \left[ F(f^*, \eta^t w^t) - F(f^*, \eta^t w^t + 1) \right] + \sum_{t=1}^T \alpha^t \langle C^t - \tilde{C}^t, z^t - y^t \rangle - \sum_{t=1}^T \frac{1}{\eta_t^2} \text{KL}(y^t \| z^t) - \sum_{t=1}^T \frac{1}{\eta_t^2} \text{KL}(z^t \| z^t).
\]

(18)

Now, we look for upper-bounds of the three terms in the right-hand-side of (18). First, we trivially have

\[
A_1 \leq \frac{1}{\eta_t} F(f^*, \eta^t w^t) + \left( \frac{1}{\eta_t^2 + 1} - \frac{1}{\eta_t^t} \right) | h(f^*) - \min h | = \frac{1}{\eta_t^2 + 1} | h(f^*) - \min h |.
\]

(19)
We aim to construct the upper-bounds the last two terms in the right-hand-side of (17) in the following way:

Second, for any $T$, from the Cauchy-Schwarz inequality and the fact that $\|f - f'\|_1 \|w - w'\|_\infty = \min_{x > 0} \left\{ \frac{1}{2}\|f - f'\|_1^2 + \frac{1}{\alpha^2} \|w - w'\|_\infty^2 \right\}$ for any $f, f', w, w' \in \mathbb{R}^d$, we have

$$
\alpha^t(C^t - \tilde{C}^t, y^t - z^t) \leq \|\alpha^t(C^t - \tilde{C}^t)\|_\infty \|y^t - z^t\|_1 \leq \frac{(\alpha^t)^2 \kappa \eta t + 1}{2} \|C^t - \tilde{C}^t\|_\infty^2 + \frac{1}{2\kappa \eta t + 1} \|y^t - z^t\|_1^2.
$$

(20)

Combine this with the strong convexity of $\tilde{h}$, we obtain:

$$
A_2 \leq \frac{\kappa}{2} \sum_{t=1}^T (\alpha^t)^2 \eta_{t+1} \|C^t - \tilde{C}^t\|_\infty^2 + \frac{1}{\kappa} \sum_{t=1}^T \left( \frac{1}{\eta_{t+1}} - \frac{1}{\eta_t} \right) \|y^t - z^t\|_1^2 \leq \frac{\kappa}{2} \sum_{t=1}^T (\alpha^t)^2 \eta_{t+1} \|C^t - \tilde{C}^t\|_\infty^2 + \frac{D_2^2}{2\kappa} \left( \frac{1}{\eta_{t+1}} - 1 \right).
$$

(21)

Third, recall the notion $D$ denoting the $\|\cdot\|_1$-diameter of $F$, we have

$$
A_3 \geq \sum_{t=1}^T \frac{1}{2\kappa \eta_t} \|z^t - \tilde{z}^t\|_1^2 = \sum_{t=1}^T \frac{1}{2\kappa \eta_{t+1}} \|z^t - \tilde{z}^t\|_1^2 - \frac{D_2^2}{2\kappa} \left( \frac{1}{\eta_{t+1}} - 1 \right).
$$

(22)

Moreover, from the update rule of $\eta_{t+1}$ and Lemma 2 of [20] (also presented in [27, 29]), we have:

$$
\frac{1}{\eta_{t+1}} = \sqrt{1 + \sum_{i=1}^T (\alpha^i)^2 \|C^i - \tilde{C}^i\|_\infty^2} \leq \sqrt{1 + \sum_{i=1}^T (\alpha^i)^2 \|C^i - \tilde{C}^i\|_\infty^2} + 1 = \sum_{i=1}^T (\alpha^i)^2 \eta_{t+1} \|C^i - \tilde{C}^i\|_\infty^2 + 1.
$$

Apply this inequality to (22) then apply it with (19), (21) into (18), we obtain precisely (17).

Now, let us define the noises of the observed costs $\tilde{C}^t$ and $C^t$, induced by the flow profiles $\tilde{f}^t$ and $f^t$, in comparison with the corresponding gradients of the BMW potential $\Phi$ as follows:

$$
\tilde{U}^t := \tilde{C}^t - \nabla \Phi(\tilde{f}^t) \quad \text{and} \quad U^t := C^t - \nabla \Phi(f^t).
$$

By the definition of $\sigma$ (cf. Equation (6)), we have $\sup \left\{ \mathbb{E} \left[ \|\tilde{U}^t\|_\infty^2 \right], \mathbb{E} \left[ \|U^t\|_\infty^2 \right] \right\} \leq \sigma^2$. From these definitions, we also deduce that $\mathbb{E} \left[ \tilde{U}^t | \mathcal{H}^{t-1} \right] = \mathbb{E} \left[ U^t | \mathcal{H}^{t-1} \right] = 0$ where $\mathcal{H}^{t-1} = \left\{ f^{t-1}, \tilde{f}^{t-1}, \omega^{t-1}, \ldots, f^1, \tilde{f}^1, \omega^1 \right\}$ is the filtration up to time epoch $t - 1$.

Equation (17) involves the difference of the actual gradients of the BMW potential $\nabla \Phi(f^t) - \nabla \Phi(\tilde{f}^t)$ and the difference of costs at $f$ and $\tilde{f}$, i.e., $C^t - \tilde{C}^t$ (in the static case, these differences are equal). For the sake of brevity, we define the following terms in order to analyze the gap between these differences:

$$
\kappa^t = \min \left\{ \|\nabla \Phi(f^t) - \nabla \Phi(\tilde{f}^t)\|_\infty^2, \|C^t - \tilde{C}^t\|_\infty^2 \right\} \quad \text{and} \quad \xi^t = \left[ C^t - \tilde{C}^t \right] - \left[ \nabla \Phi(f^t) - \nabla \Phi(\tilde{f}^t) \right].
$$

We aim to construct the upper-bounds the last two terms in the right-hand-side of (17) in terms of $\kappa^t$ and $\xi^t$. Particularly, from (17), we can prove the following proposition:

1Consider the function $\psi(x) = \frac{1}{2} \|f - f'\|_1^2 + \frac{1}{\kappa} \|w - w'\|_\infty^2$; then $x^* = \|f - f'\|_1 \|w - w'\|_\infty$ is the minimizer of $\psi$. 
Proposition 4. Run Algorithm 3 and define $\tilde{\eta}^t := 1/\sqrt{1 + 2 \sum_{s=1}^{t-1} (\alpha^s + 1)^2 \kappa^s}$, we have

$$R_T(f^*) \leq h(f^*) - \min h + A_{stock} \frac{2 \sqrt{2}}{\kappa} \left( \sum_{t=1}^{T} (\alpha^t)^2 \| z_t \|_2^2 \right)^{1/2} + \sum_{t=1}^{T} g(\tilde{\eta}^{t+1}) (\alpha^t)^2 \kappa^t, \quad (23)$$

where $A_{stock} := h(f^*) - \min h + \frac{\kappa^2 + 3 \beta^2}{2 \kappa}$ and $g(\tilde{\eta}^{t+1}) := 4 A_{stock} \tilde{\eta}^{t+1} - \frac{1}{8 \beta^2 \kappa^t}.

Proof of Proposition 4. First, by definitions of $\kappa^t$ and $\xi^t$, we have $\| C^t - \hat{C}^t \|_2^2 \leq 2 \kappa^t + 2 \| \xi^t \|_2^2$. Apply this and Lemma 2 of [20], we have

$$\sum_{t=1}^{T} (\alpha^t)^2 \eta^{t+1} \| C^t - \hat{C}^t \|_2^2 = \sum_{t=1}^{T} \frac{(\alpha^t)^2 \| C^t - \hat{C}^t \|_2^2}{1 + \sum_{s=1}^{t} \alpha^s \| C^s - \hat{C}^s \|_2^2} \leq 2 \sqrt{1 + \sum_{t=1}^{T} (\alpha^t)^2 \| C^t - \hat{C}^t \|_2^2} \leq 2 \sqrt{1 + \sum_{t=1}^{T} (\alpha^t)^2 \kappa^t} + 2 \sqrt{2 \sum_{t=1}^{T} (\alpha^t)^2 \| \xi^t \|_2^2} \leq 4 \sum_{t=1}^{T} \frac{(\alpha^t)^2 \kappa^t}{1 + 2 \sum_{s=1}^{t} (\alpha^s)^2 \kappa^t} + 2 \sum_{t=1}^{T} (\alpha^t)^2 \| \xi^t \|_2^2. \quad (24)$$

On the other hand, from the definition of $\tilde{\eta}^t$, we have $\frac{1}{\eta^t} \leq \frac{1}{\tilde{\eta}^t}$, $\forall t$ and hence,

$$\sum_{t=1}^{T} \frac{1}{\tilde{\eta}^{t+1}} \| z^t - \hat{z}^t \|_1^2 \leq \sum_{t=1}^{T} \left( \frac{1}{\eta^{t+1}} - \frac{1}{\tilde{\eta}^t} \right) \| z^t - \hat{z}^t \|_1^2 + \frac{1}{\tilde{\eta}^t} \| z^t - \hat{z}^t \|_1^2 \leq \left( \frac{1}{\eta^{t+1}} - 1 \right) \frac{D^2}{2} + \sum_{t=1}^{T} \frac{1}{\tilde{\eta}^t} \| z^t - \hat{z}^t \|_1^2.$$

Combine this with the fact that $z^t - \hat{z}^t = \frac{\alpha^t + 1}{\eta^{t+1}} \left( f^t - \hat{f}^t \right)$ (from Lines 5 and 8 of Algorithm 3) and choose an increasing sequence of $\alpha^t$ (such as $\alpha^t = t$), we have

$$- \sum_{t=1}^{T} \frac{1}{2 \kappa \tilde{\eta}^{t+1}} \| z^t - \hat{z}^t \|_1^2 \leq - \frac{1}{2 \kappa} \sum_{t=1}^{T} \frac{1}{\eta^{t+1}} \| z^t - \hat{z}^t \|_1^2 + \frac{D^2}{2 \kappa} \left( \frac{1}{\eta^{t+1}} - 1 \right) \leq \frac{1}{2 \kappa} \sum_{t=1}^{T} \frac{1}{\eta^{t+1}} \| f^t - \hat{f}^t \|_1^2 + \frac{D^2}{2 \kappa} \left( \frac{1}{\eta^{t+1}} - 1 \right) \leq \frac{1}{2 \kappa} \sum_{t=1}^{T} \frac{(\alpha^t + 1)^2}{\eta^{t+1}} \| \nabla \Phi(f^t) - \nabla \Phi(\hat{f}^t) \|_\infty^2 + \frac{D^2}{2 \kappa} \left( \frac{1}{\eta^{t+1}} - 1 \right) \leq \frac{1}{8 \beta^2 \kappa} \sum_{t=1}^{T} (\alpha^t)^2 \kappa^t + \frac{D^2}{2 \kappa} \left( \frac{1}{\eta^{t+1}} - 1 \right). \quad (25)$$

Apply (24) and (25) into (17), we obtain (23) and finish the proof of Proposition 4. ■

We now focus in the last term of (23). We denote $T_0 := \max \left\{ 1 \leq t \leq T : \tilde{\eta}^{t+1} \geq \left[ 32 \beta^2 \kappa A_{stock} \right]^{-1/2} \right\}$. Then for any $t \geq T_0$, we have $g(\tilde{\eta}^{t+1}) < 0$. In other words, in $\sum_{t=1}^{T} g(\tilde{\eta}^{t+1}) (\alpha^t)^2 \kappa^t$, only the
first $T_0$ components are positive. Therefore,
\[
\sum_{t=1}^{T} g(\tilde{\eta}^{t+1})(\alpha^t)^2\kappa_t \leq \sum_{t=1}^{T_0} g(\tilde{\eta}^{t+1})(\alpha^t)^2\kappa_t \leq \sum_{t=1}^{T_0} 4A_{\text{stoch}}\tilde{\eta}^{t+1}(\alpha^t)^2\kappa_t
\]
\[
= 4A_{\text{stoch}}\sum_{t=1}^{T_0} \frac{(\alpha^t)^2\kappa_t}{1 + 2\sum_{t=1}^{T_0}(\alpha^t)^2\kappa_t}
\]
\[
\leq 4A_{\text{stoch}}\cdot 2\sqrt{1 + 2\sum_{t=1}^{T_0}(\alpha^t)^2\kappa_t}
\]
\[
= \frac{8}{\tilde{\eta}_0} A_{\text{stoch}}
\]
\[
\leq 32\beta\sqrt{2\kappa}(A_{\text{stoch}})^{3/2}. \tag{26}
\]

Combine (26), (23) and the fact that $E[\|\xi_t\|_2^2] \leq 2E[\|U^t\|_2^2] + 2E[\|U^t\|_\infty^2] \leq 4\sigma^2$, we have
\[
E[\mathcal{R}_T(f^*)] \leq h(f^*) - \min h + \sigma A_{\text{stoch}}4\sqrt{2\sum_{t=1}^{T}(\alpha^t)^2} + 32\beta\sqrt{2\kappa}(A_{\text{stoch}})^{3/2}. \tag{27}
\]

Finally, in order to apply (15), we need to make the connection between $E[\mathcal{R}_T(f^*)]$ and $E[\mathcal{R}_T(f^*)]$. Particularly, we have:
\[
E[\mathcal{R}_T(f^*)] = E[\mathcal{R}_T(f^*)] - E\left[\sum_{t=1}^{T} \alpha^t\langle C^t - \nabla \Phi(f^t), z^t - f^*\rangle\right] = E[\mathcal{R}_T(f^*)] - E\left[\sum_{t=1}^{T} \alpha^t\langle U^t, z^t - f^*\rangle\right]
\]
\[
= E[\mathcal{R}_T(f^*)]. \tag{28}
\]

Here, the last equality comes from the fact that $E[\alpha^t\langle U^t, z^t - f^*\rangle] = E[\alpha^t\langle E[U^t|\mathcal{H}^t-1], z^t - f^*\rangle] = 0$ (by law of total expectation). Combine (28) and (27), then apply (15) with the choice of step-size $\alpha^t = t, \forall t$, we have:
\[
E[\Phi(f^T) - \Phi(f^*)] \leq \frac{1}{T^2} \left( h(f^*) - \min h + \sigma A_{\text{stoch}}4\sqrt{2T^{3/2}} + 32\beta\sqrt{2\kappa}(A_{\text{stoch}})^{3/2}\right). \tag{29}
\]

Finally, recall that $h(f^*) \leq M_{\text{tot}} \log(M_{\text{max}}), -\min h \leq M_{\text{tot}} \log(P/M_{\text{tot}}), \kappa := NM_{\text{max}}, D \leq 2M_{\text{tot}}, \sigma \leq 2H$ and the definition of $A_{\text{stoch}}$ (cf. Proposition 4), we have $A_{\text{stoch}} \leq \frac{1}{2}NM_{\text{max}} \left[ 2 \log \left( \frac{PM_{\text{max}}}{M_{\text{tot}}} \right) + 13 \right] := \frac{1}{2}A$ for the constant $A$ defined in Theorem 3. Plug these results into (29), we obtain (14) and conclude the proof.

5. AdaLight: Adaptive learning with efficient per-iteration complexity

In this section, our main focus is to design an algorithm that not only has the adaptive optimality of AdaWeight but also has a per-iteration complexity that is scalable in the network’s size. To do this, in Section 5.1, we first introduce an alternative routing paradigm, called local flows. In Section 5.2 and Section 5.3, we then propose the AdaLight algorithm and analyze its convergence properties and its per-iteration complexities.

In the sequel, we will additionally use the following set of notation. For any (directed) edge $e \in \mathcal{E}$, we denote by $v_e^+$ and $v_e^-$ the tail vertex and head vertex of $e$ respectively, i.e., $e$ goes from $v_e^+$ to $v_e^-$. For any vertex $v$ and any sub-graph $\mathcal{G}$, let $\text{In}_v$, and $\text{Out}_v$, respectively denote the set of incoming edges to $v$ and the set of outgoing edges from $v$, and let $\text{Child}_v$ and $\text{Parent}_v$ denote the set of its direct successors and the set of its direct predecessors. For
We define the local flow profiles inducing the same loads (and costs) as the logit-mapped flows. In Algorithm 3, it takes two major types of flows updating: the flows described in the previous section.

5.2. AdaLight algorithm. In this section, we propose a new equilibrium learning method called ADAIGHT. This method will be implemented in the local flow learning model described in the previous section.

To build ADAIGHT, our starting point is the AdaWeight method. In Algorithm 3, there are two major types of flows updating: the flows \( \tilde{z}^t \) and \( z^t \) outputting from the logit mapping \( \Lambda \), and the flows \( f^t \) and \( \tilde{f}^t \) obtained from averaging between two pre-computed flows. In Algorithm 3, it takes \( O(P) \) computations to update each of these flow profiles in each iteration. To improve the per-iteration complexity, we can leverage the weight-pushing technique [18, 42]: by assigning weights on edges and use dynamic programming principles, we can find the local flow profiles inducing the same loads (and costs) as the logit-mapped flows \( \tilde{z}^t \) and \( z^t \). However, the weight-pushing technique fails to derive local flow profiles
matching the averaged flows $f^t$ and $f^*$ of Algorithm 3. Particularly, weight-pushing two flows and then taking the average will not incur the same costs as implementing the average of the two flows. Recall that these averaging steps are the key elements allowing the adaptability of $\text{AdaWeight}$, the key challenge now is to implement efficiently these steps.

Facing up this challenge, we make the following observation: while the averaged flows of $\text{AdaWeight}$ are not “weight-pushable”, their induced loads are $\text{AdaWeight}$. In this perspective, there arise two new challenges. First, we need to efficiently compute the load profiles induced by the averaged flows (without explicitly computing these flows). To do this, we introduce a sub-routine called $\text{pulling-forward}$ – reflecting the fact that we start from the origin and work forward (unlike the classical weight-pushing that starts from the destination and works backward). Second, we need to derive local flow profiles that matches these averaged loads. We refer to this as $\text{matching-load}$. Note that $\text{pulling-forward}$ and $\text{matching-load}$ are completely novel contributions.

Taking an overall view, $\text{AdaLight}$ is a combination of the weight-pushing technique (to handle the logit-mapped flows) and the pulling-forward and matching-load procedures (to handle the averaged flows) into the $\text{AdaWeight}$ template. For the sake of conciseness, we first present a sub-algorithm in Section 5.2.1 and present a pseudo-code form of $\text{AdaLight}$ in Section 5.2.2.

5.2.1. $\text{PushPullMatch}$ sub-algorithm. In this section, we present a sub-algorithm, called $\text{PushPullMatch}$, that combines the three auxiliary routines: pushing-backward, pulling-forward and matching-load. In Algorithm 4, we give a pseudo-code form of $\text{PushPullMatch}$. It takes four inputs: an “anchor local-load” profile $r = (r^i_e)_{i \in \mathcal{N}, e \in \mathcal{E}^i}$, a “local-weight” profile $w = (w^i_e)_{i \in \mathcal{N}, e \in \mathcal{E}^i}$; and two scalar numbers $\alpha^i$ and $\sum_{s=0}^{t} \alpha^s$ (we use these notations to facilitate the presentation of $\text{AdaLight}$ in Section 5.2). When it finishes, Algorithm 4 outputs a local flow profile and updates the anchor local-load profile.

Particularly, when we run $\text{PushPullMatch}$, it executes the following three phases for each O/D pair $i$:

i) The pushing-backward phase: in this phase, we consider vertices in $\mathcal{G}^i$ one by one in a reversed topological order, i.e., we work backwardly from the destination to the origin. For each vertex $v$, we assign a score – called the backward score (denoted by $\text{BWscore}$ in Algorithm 4) – depending on the weights of $v$‘s outgoing edges (i.e., $w^i_e$, $\forall e \in \text{Out}_v^i$) and its children’s scores. Then, we “pushes” the backward score of $v$ to its parents so that, in their turns, we can compute their scores. Finally, based on the backward score of $v$, we compute a local flow $z^i_v \in \mathcal{X}_v^i$.

ii) The pulling-forward phase: in this phase, we consider vertices in $\mathcal{G}^i$ one by one in a topological order, i.e., we work forwardly from the origin to the destination. On each vertex $v$, we assign a score – called the forward score (denoted by $\text{FWscore}$ in Algorithm 4) – computed from the loads induced by $z$ on the incoming edges of $v$ that are “pulled” from $v$‘s parents. Then, for any $e \in \text{Out}_v^i$, we derive $\ell^i_e(z)$ and compute another term – called $\ell^i_v(x)$ – by taking the average of $\ell^i_e(z)$ and the anchor load $r^i_e$. Finally, we update the anchor load-profile $r$ (that will be used later in $\text{AdaLight}$).

iii) The matching-load phase: in this phase, we once again consider vertices in $\mathcal{G}^i$ in a topological order. For each $v$, we compute a local flow $x^i_v \in \mathcal{X}_v^i$ that matches precisely

---

2 As a side note on $\text{AcceleWeight}$ (cf. Section 3.2), a direct application of weight-pushing also fails to achieve an efficient implementation for the same reasons of that of $\text{AdaWeight}$. Note that we can use techniques that we introduce in $\text{AdaLight}$ to improve $\text{AcceleWeight}$ and achieve an efficient version. As $\text{AcceleWeight}$ is not the focus of our work, we omit the details.

3 Note that since $\mathcal{G}^i$ are DAGs, such an topological order exists and can be found in $O(|\mathcal{E}^i|)$ time.
We present a pseudo-code form of AdaLight method. The key difference that makes AdaLight stand out from AdaWeight is that it uses the sub-algorithm PushPullMatch to efficiently compute the corresponding local flows instead of working with the $P$-dimensional flow profiles as in AdaWeight. We present a pseudo-code form of AdaLight in Algorithm 5. Similar to AdaWeight, AdaLight follows two phases: the test phase and the recommendation phase:

1. In the **test phase**, we run PushPullMatch with the anchor load profile $r^i$ and the weight profile $\eta^i w^i$ as inputs (here, $\eta^i$ is a learning rate). At the end of the test phase, the cost $C_i$ at a “test” local flow is derived and used to update the test-weight profile $\tilde{W}^i$.

2. In the **recommendation phase**, we re-run PushPullMatch but this time, with the test weight $\eta^i \tilde{W}^i$ obtained previously in the test phase as input. Moreover, we will also use the output of PushPullMatch to update the anchor load-profile $r^i$.

### Algorithm 4: PushPullMatch($r, w, \alpha, \sum_{s=0}^l \alpha^s$)

**Input:** $r, w \in \mathbb{R}^{|V| \times \sum_{v} \text{dim}_v}$ and $\alpha, \sum_{s=0}^l \alpha^s > 0$

**Output:** $x \in \mathcal{X}$ and update $r$

1. **In parallel for all** $i \in \mathcal{N}$
   2. Fix an topological order $v_0, v_1, \ldots, v_{|V|}$ of the graph $G^i$ (such that $v_0 \equiv O^i$ and $v_{|V|} \equiv D^i$)
   3. **Pushing-backward phase**
      4. **for** $v = v_{|V|}, \ldots, v_0$ **do**
         5. **if** $v = D^i$ **then** set $\text{BWScore}_v \leftarrow 1$ // initialize BWscore at destination
         6. **else** set $\text{BWScore}_v \leftarrow \sum_{e \in \text{Out}_v} \text{BWScore}_{v_e} \cdot \exp(w^i_e)$ // compute BWscore based on BWscore of children
         7. set $z_{v,e} \leftarrow \exp(w^i_e) \text{BWScore}_{v_e} / \text{BWScore}_v$ **for** $e \in \text{Out}_v$ // compute pivot outgoing flow
   4. **Pulling-forward phase**
      5. **for** $v = v_0, \ldots, v_{|V|}$ **do**
         6. **if** $v = O^i$ **then** set $\text{FWScore}_v \leftarrow M_i$ // initialize FWscore at origin
         7. **else** set $\text{FWScore}_v \leftarrow \sum_{e \in \text{In}_v} \exp(z_{v,e})$ // compute FWscore from loads of incoming edges
         8. **for** $e \in \text{Out}_v$ **do**
            9. set $L_z^i(z) \leftarrow \text{FWScore}_v \cdot z_{v,e}$ // pull the loads forward
            10. set $L_z^i(x) \leftarrow [\alpha^i L_z^i(z) + r_z^i] / \sum_{s=0}^l \alpha^s$ // average $k^i$ pull forward
            11. set $r_z^i \leftarrow r_z^i + \alpha^i \cdot L_z^i(z)$ // update the anchor
      12. **Matching-load phase**
         13. **for** $v = v_0, \ldots, v_{|V|}$ **do**
            14. **if** $v = O^i$ **then** set $\mu^i_v(x) \leftarrow M_i$ // initialize mass arriving at origin
            15. **else** set $\mu^i_v(x) \leftarrow \sum_{e \in \text{In}_v} \ell^i_e(x)$ // compute mass that would arrive at $v$ if $x$ is implemented
            16. set $x_{v,e} \leftarrow L_z^i(x) / \mu^i_v(x)$ for $e \in \text{Out}_v$ // compute outgoing flow matching $\ell^i_e(x)$

the load $\ell^i_e(x), \forall e \in \text{Out}_v$ obtained in the pulling-forward phase. To do this, we need to compute $\mu^i_v(x)$ – the mass of $i$-type traffic arriving at $v$ induced by $x$ – this can be done by pulling the loads corresponding to $x^i$ from the parents of $v$. The local flow profiles $x^i, i \in \mathcal{N}$ constitute the output of PushPullMatch.
recommendation phase, we obtain the local flow profile \( x^t \), route the traffic accordingly, then update the weight profile \( w^t \) by using the incurred costs.

Finally, we observe that in each iteration of Algorithm 5, it is required to compute the term \( \max_{p \in \mathcal{P}} \left( \sum_{e \in p} (C_e^t - \tilde{C}_e^t) \right) \) for updating the learning rate \( \eta^t \). This can also be done in efficiently: for any \( i \in \mathcal{N} \), we solve a weighted-shortest-path problem on \( \mathcal{G}^i \) where each edge \( e \) is assigned with a weight \( x_e := - (C_e^t - \tilde{C}_e^t) \). For instance, this step can be done by the classical Bellman-Ford algorithm \([7, 17]\) that takes only \( \mathcal{O}(|\mathcal{V}|) \) rounds of computations.\(^4\)

---

**Algorithm 5: Adaptive local weights (AdaLight)**

1. Initialize \( w^1 = 0, \ r^1 = 0, \ \alpha^0 = 0 \) and \( \eta^1 = 1 \)
2. for \( t = 1, 2, \ldots \) do
   
   // Test phase
   
   3. set \( \tilde{x}^t \leftarrow \text{output of PushPullMatch}(r^t, \eta^t w^t, \alpha^t, \sum_{s=0}^{t-1} \alpha^s) \) // compute a test local flow
   
   4. get \( \tilde{C}_e^t \leftarrow c_e(t_e(x^t), \omega^t) \) for \( e \in \mathcal{E} \) // query the test local flow
   
   5. set \( \tilde{w}_e^{t+1} \leftarrow w_e^{t} - \alpha^t \tilde{C}_e^t \) for any \( i \in \mathcal{N} \) and \( e \in \mathcal{E}^i \) // update test weight

   // Recommendation phase

   6. set \( x^t, r^{t+1} \leftarrow \text{outputs of PushPullMatch}(r^t, \eta^t \tilde{w}^t, \alpha^t, \sum_{s=0}^{t-1} \alpha^s) \) // compute local flow
   
   7. Route according to \( x^t \) and get \( C_e^t \leftarrow c_e(t_e(x^t), \omega^t) \) for \( e \in \mathcal{E} \) // route, measure costs
   
   8. set \( w_e^{t+1} \leftarrow w_e^t - \alpha^t C_e^t \) for any \( i \in \mathcal{N} \) and \( e \in \mathcal{E}^i \) // update weight

   // Update the learning rate

   9. set \( \eta^{t+1} \leftarrow 1 / \sqrt{1 + \sum_{s=0}^{t} \alpha^s \max_{p \in \mathcal{P}} \left( \sum_{e \in p} (C_e^t - \tilde{C}_e^t) \right)^2} \)

---

5.3. AdaLight: convergence results and per-iteration complexities. The convergence properties of AdaLight is formally presented in the following theorem:

**Theorem 4.** Let \( x^1, x^2, \ldots \) be the sequence of local flow profiles recommended by AdaLight (i.e., Algorithm 5) running with \( \alpha^t = t, \forall t = 1, 2, \ldots \); then the following results hold:

i) The sequence of flow profiles \( f^1, \forall t = 1, 2, \ldots \) where \( f_p^{t+1} := M^t \prod_{e \in p} x_{e^+}^{t+1} \) for any \( i \in \mathcal{N} \) and \( p \in \mathcal{P} \) enjoys the following equilibrium convergence rate:

\[
\mathbb{E}[\text{Gap}(T)] \leq \mathcal{O} \left( (\log P)^{3/2} \left( \frac{\sigma}{\sqrt{T}} + \frac{1}{T^2} \right) \right),
\]

and specifically, in the static case, it enjoys the convergence rate \( \text{Gap}(T) \leq \mathcal{O}(\log(P)^{3/2}/T^2) \).

ii) Moreover, each iteration of AdaLight requires only an \( \mathcal{O}(|\mathcal{N}| |\mathcal{V}| |\mathcal{E}|) \) number of computations.

Result (i) of Theorem 4 shows that AdaLight also converges toward an equilibrium with the same rate as AdaWeight: an \( \mathcal{O}(\log P)^{3/2}/\sqrt{T} \) rate in the stochastic regime and an \( \mathcal{O}((\log P)^{3/2}/T^2) \) rate in the static regime. More precisely, we shall see in the proof of Theorem 4 that AdaLight recommends a sequence of local flows that induce the same BMW-potential values as the sequence of flows recommended by AdaWeight which, in turn, approaches the potential value at an equilibrium (see Proposition 5 below). Note that

\(^4\)There exist more complicated shortest-path algorithms that have better complexities (e.g., \([4, 15, 19]\)); however, the complication of these algorithms is beyond the purpose of our work and they do not improve the complexity of AdaLight in general. In this work, we only analyze AdaLight with the simple Bellman-Ford algorithm.
although the flow $f^t$ is mentioned in Theorem 4, it is never computed by ADA_LIGHT and it is not needed for routing the traffic in practice (only the local flow $x^t$ is needed).

Moreover, Result (ii) of Theorem 4 shows the main difference between using ADA_LIGHT and using AdaWEIGHT: the per-iteration (space and time) complexities of ADA_LIGHT are polynomial in terms of the network’s primitive parameters (numbers of O/D pairs, numbers of vertices and numbers of edges). Therefore, unlike AdaWEIGHT, ADA_LIGHT can run efficiently even in large networks.

5.4. Proof of Theorem 4.

First, we prove that $x^T$ and $f^T$ – as defined in Theorem 4 – induce the same costs. In fact, we can prove a stronger result as follows:

**Proposition 5.** Any local flow profile $x \in X$ and the flow profile $f \in F$ such that $f_p^i := M^i \prod_{e \in p} x_e^i$ induce the same load profiles, i.e., $\ell_e(x) = \ell_e(f)$ for any $e$.

**Proof.** Proof of Proposition 5. Fix an O/D pair $i \in \mathcal{N}$. Let $\mathcal{P}^O_v$ and $\mathcal{P}^v_D$, denotes the set of paths in $\mathcal{G}^i$ going from $O^i$ to $v$ and the set of paths going from $v$ to $D^i$ respectively. Let us denote $H^i_v := \sum_{p \in \mathcal{P}^O_v} \prod_{e \in p} x_e^i$ for any $v \in \mathcal{V} \setminus \{D^i\}$ and conventionally set $H^i_{D^i} = 1$. By induction (following any topological order of vertices), we can prove that $H^i_v = 1$ for any $v \in \mathcal{V}^i$. From this result, the definition of the load profile in (30) and the definition of $H^i_v$, the following equality holds for any $v$ and $e \in \text{Out}^i_v$:

$$f_e^i(x) = \mu_e^i(x) \cdot x_e^i \cdot H^i_{v^e} = M^i \sum_{q \in \mathcal{P}^O_v} \prod_{e^q \in q} x_{e^q}^i \cdot \sum_{p \in \mathcal{P}^v_D} \prod_{e^p \in p} x_{e^p}^i = M^i \sum_{p \in \mathcal{P}^O_v} \prod_{e \in p} x_e^i = \sum_{p \in \mathcal{P}^O_v} f_p^i = \ell_e^i(f).$$

Apply Proposition 5, since $x^T$ and its corresponding flow $f^T$ induce the same load profiles, by definition of the cost functions, they induce the same costs, i.e., $c(f^T, \omega) = c(x^T, \omega)$ for any state $\omega \in \Omega$.

Second, we prove that the recommendations of ADA_LIGHT and AdaWEIGHT coincide. Let us first focus on the recommendation phase of Algorithm 5. Particularly, assume that at time epoch $t$, the anchor local-load profile $r^t$ in Algorithm 5 matches the load induced by the anchor flow $r^t$ defined at Line 3 of Algorithm 3 and also assume that the local-weight $w^t$ used in Algorithm 5 and the weight $w^t$ defined at Line 6 of Algorithm 3 satisfy that $\sum_{e \in p} w^t_e = \tilde{w}^t_p$ for any $p \in \mathcal{P}$ and $i \in \mathcal{N}$. We will prove that the local flow profile $x^t$ output from PUSH_PULLMATCH($r^t, \eta^t \bar{w}^t, \alpha^t, \sum_{s=0}^{t} \alpha^s$) induces the same load profiles (and the same costs) as the flow profile $f^t$ in Algorithm 3.

To do this, we observe that the following equality holds true for any $i \in \mathcal{N}$ and $p \in \mathcal{P}^i$ and with the local flow profile $z$ computed in the pushing-backward phase of PUSH_PULLMATCH($r^t, \eta^t \bar{w}^t, \alpha^t, \sum_{s=0}^{t} \alpha^s$):

$$M^i \prod_{e \in p} z_{v_e^t, e}^i = M^i \prod_{e \in p} \exp\left(\frac{\text{BWscore}_v^i}{\text{BWscore}_v^t} \right) = \exp\left(\frac{\sum_{e \in p} \tilde{w}_e^t}{\text{BWscore}_O^t} \right) = \exp\left(\tilde{w}_p^t \right) = z_p^t.$$  

Here, the last equality comes directly from the update rule of $z^t$ in Line 7 of Algorithm 3.

Now, combine (31) with Proposition 5, the loads induced by $z$ of PUSH_PULLMATCH($r^t, \eta^t \bar{w}^t, \alpha^t, \sum_{s=0}^{t} \alpha^s$) are precisely the loads induced by $z^t$ in Algorithm 3. Moreover, by definition of local-load (cf. Section 5.1), we have that the term $f_e^i(z)$ computed at Line 11 of PUSH_PULLMATCH are precisely these loads.
Due to the arguments above and the assumption on the relation of \( r^i \) and \( r^{i*} \), the term \( f_i^t(x) \) computed at Line 12 of PushPullMatch is precisely the load induced by the averaged flow \( f^t \) computed at Line 8 of Algorithm 3. Finally, from the updating rule of the local flow profile \( x \) at Line 17 of PushPullMatch, we deduce that these loads match precisely with the loads induced by \( x \) (which is the output of PushPullMatch).

As a conclusion, since we set \( x^t \) as the output of PushPullMatch\((r^t, \eta^t \tilde{w}^t, \alpha^t, \sum_{s=0}^t \alpha^s)\), the loads (and hence, the costs) induced by \( x^t \) are precisely that of the flows \( f^t \) recommended by Algorithm 3. Using a similar line of arguments, we can also deduce that the local flow profile \( \tilde{x}^t \) computed in Algorithm 5 also matches the flow profile \( f^t \) computed in AdaWeight. For this reason, the assumptions we made on \( \tilde{w}^t \) and \( r^t \) in the above proof actually hold true.

To sum up, we have proved that the flow profiles defined in Theorem 4 have the same loads as the local flow profiles recommended by AdaLight that, in turn, coincide with the loads incurred by the recommended flows of AdaWeight. This leads to the fact that AdaLight (Algorithm 5) inherits the convergence rates of AdaWeight (Algorithm 3). This concludes the proof of Result (i) of Theorem 3.

Finally, we justify the per-iteration complexity of AdaLight. It is trivial to see that when PushPullMatch is run, at any vertex \( v \) and O/D pair \( i \), each of its phases only takes \( O(\max\{|\text{In}^i_v|, |\text{Out}^i_v|\}) \) rounds of computations where \( |\text{In}^i_v| \) and \( |\text{Out}^i_v| \) are the in-degree and out-degree of \( v \) in \( G^t \). As mentioned above, the learning rate update step can be done in \( O(|V|) \) time. We conclude that each iteration of AdaLight requires only an \( O(|V||V||E|) \) number of computations at each vertex.

\[ \Box \]

6. Numerical Experiments

In this section, we report the results of several numerical experiments that we conducted to justify the theoretical convergence results of AdaWeight and AdaLight. Particularly, in Section 6.1, we present the experiments on a toy example to highlight the advantages of AdaWeight / AdaLight over several benchmark algorithms. Then, in Section 6.2, we show the superiority in performance of AdaLight in real-world networks. The experiments with real-world data pose several additional computational challenges; we also discuss these challenges and provide quick-fix solutions. The codes of our experiments are available at dongquan-vu/Adaptive_Distributed_Routing.

6.1. Experiments on small-size networks. We first consider a toy-example of on a network with 4 vertices and 4 edges. Specifically, the edges in this network are arranged to form 2 parallel paths going from an origin vertex, namely \( O \), to a destination vertex, namely \( D \). At each time epoch, a traffic demand (i.e., inflow) of size \( M = 10 \) is sent from \( O \) to \( D \).

In the experiments presented below, we run AdaWeight, ExpWeight and AcceleWeight in \( T_{\text{max}} = 10e5 \) epochs. We record the values of the BMW potential \( c \) at outputs of each algorithm at each time \( T = 1, 2, \ldots, T_{\text{max}} \). We then identify the value \( c(f^*) \) which is the minimum among all \( c \)-value of flow profiles computed by these algorithms in all iterations; thus, \( f^* \) represents the equilibrium flow. In order to track down the convergence properties of these algorithms, we will compute and plot out the evolution (when \( T = 1, 2, \ldots, T_{\text{max}} \) of the gaps (cf. Equation (10)) of the flows derived from these algorithms that are analyzed in Theorem 1, Theorem 2 and Theorem 3 (we denote these gaps by \( \text{Gap}_{\text{AdaWeight}}(T) \), \( \text{Gap}_{\text{ExpWeight}}(T) \), and \( \text{Gap}_{\text{AcceleWeight}}(T) \) for short).

First, we consider a static environment. Particularly, the costs on edges are determined via fixed linear cost functions such that when being routed with the same load, one path has a higher cost than the other. We plot out the evolution of \( \text{Gap}_{\text{AdaWeight}}(T) \), \( \text{Gap}_{\text{ExpWeight}}(T) \), and \( \text{Gap}_{\text{AcceleWeight}}(T) \) in Fig. 1a. We observe that in this static setting, all these terms converge toward \( 0 \); in other words, all three algorithms converge towards...
equilibria of the game. Importantly, we observe that AdaWeight and AcceleWeight enjoy an accelerated convergence speed that is much faster than that of ExpWeight; this coincides with theoretical convergence results in the previous sections. Note that if AcceleWeight is run with a badly-tuned initial step-size $\gamma^0$ that is too small or too large (e.g., when $\gamma^0 = 1$ or $\gamma^0 = 1e-8$), it requires a long warming-up phase and might converge slowly. We also observe that in Fig. 1, $\text{Gap}_{\text{ExpWeight}}(T)$ does not seem to fluctuate as much as $\text{Gap}_{\text{AdaWeight}}(T)$ and $\text{Gap}_{\text{AcceleWeight}}(T)$, this comes from the fact that $\text{Gap}_{\text{ExpWeight}}(T)$ is computed from the time-averaged flow profiles of outputs of ExpWeight. We recall that these time-averaged flows are not the flows recommended by ExpWeight.

Second, we consider a stochastic setting. Particularly, the costs on edges are altered by adding noises generated randomly from a zero-mean normal distribution. To have a better representation of these uncertainties, we run each algorithm in 5 different instances of the noises’ layouts. We take the averaged results among these instances and plot out the evolution of $\text{Gap}_{\text{AdaWeight}}(T)$, $\text{Gap}_{\text{ExpWeight}}(T)$ and $\text{Gap}_{\text{AcceleWeight}}(T)$ in Fig. 1b. In this setting, we observe that while AdaWeight and ExpWeight converge toward equilibria, AcceleWeight fails to do so. Moreover, in this stochastic environment, the accelerated rate is no longer obtainable: $\text{AdaWeight}$ and $\text{ExpWeight}$ converge with the same rate (of order $O(1/\sqrt{T})$). This confirms the theoretical results of AdaWeight, ExpWeight and AcceleWeight.

6.2. Experiments on real-world datasets. In this section, we present several experiments using a real-world dataset collected and provided in [?] (with free license). This dataset contains the road networks of different cities in the world. The congestion model in this dataset is assumed to follow the BPR cost functions whose coefficients are estimated a priori. The purpose of our experiments is to measure the equilibrium convergence of our proposed methods with the presence of uncertainties and no knowledge on the cost functions.

Due to their per-iteration complexity, the naive implementations of ExpWeight, AcceleWeight and AdaWeight in Algorithms 1, 2, 3 take an extremely long running time when being run in these real-world networks. Therefore, to deal with these large-scale networks, we consider the distributed model (presented in Section 5.1) and run AdaLight in place of AdaWeight. As a benchmark, we consider a variant of ExpWeight implemented
with the classical weight-pushing technique. These implementations enjoy a per-iteration complexity that is polynomial in terms of the networks’ sizes.5

Before presenting the obtained experimental results (in Section 6.2.2), we first address a computational issue in using weight-pushing ideas in large-scale networks. As far as we know, this issue has not been reported in any previous works (prior to this work, weight-pushing is mostly analyzed theoretically and real-world implementations have not been provided). We formally address this computational issue and introduce a quick-fix solution in Section 6.2.1 (readers who are eager to see the numerical experiments can skip this section).

6.2.1. Computational issues of weight-pushing with large weights. The PushPullMatch sub-algorithm that we constructed in Section 5.2 involves the pushing-backward phase where a backward score (denoted by BWscore) is assigned on each vertex in the network. In theory, this score can be computed efficiently. In practice though, when the magnitude of costs values are large, the input weights of PushPullMatch is proportional to the negative accumulation of costs (that decreases quickly to \(-\infty\)) and hence, the computation of BWscore involves a division of two infinitesimally small numbers; this is often unsolvable by computers.

To resolve this issue, instead of keeping track of BWscore, we keep track of its logarithm. Particularly, fix an O/D pair \(i\), for each vertex \(v\) and each outgoing edge \(e \in \text{Out}_i\), let us denote \(\logscore_v := \log(BWscore_v)\) and \(\text{edgescore}_e = \logscore_v + w^i_e\), then from Line 5 of Algorithm 4, it can be computed by

\[
\logscore_v = \log \left[ \sum_{e \in \text{Out}_i} \exp(\logscore_v + w^i_e) \right] \\
= \max_{e' \in \text{Out}_i} (\text{edgescore}_{e'}) + \log \left[ \sum_{e \in \text{Out}_i} \exp\left(\text{edgescore}_e - \max_{e' \in \text{Out}_i} (\text{edgescore}_{e'})\right) \right].
\]

The expression in (32) allows computers to compute \(\logscore_v\) without any issues even when the magnitude of costs and \(w^i_e\) is large. Finally, the local flow profile at Line 6 of the pushing-backward phase of PushPullMatch can be computed as \(z^i_{v,e} = 1/\exp(\logscore_v - \text{edgescore}_e)\).

6.2.2. Experimental results. We consider several instances in the dataset [7] representing the urban traffic networks of different cities. The primitive parameters of these networks are summarized in Table 2. More information on these networks are given at https://github.com/bstabler/TransportationNetworks. Note that in previous work, this dataset is used mostly for social-costs optimization; in our knowledge, no work has used this dataset for equilibrium searching problems.

| Network name            | # of vertices | # of edges | # of O/D pairs | min demands | max demands |
|-------------------------|---------------|------------|----------------|-------------|-------------|
| SiouxFalls              | 24            | 76         | 528            | 100         | 4400        |
| Eastern-Massachusetts   | 74            | 258        | 1113           | 0.5         | 957.7       |
| Berlin-Friedrichshain   | 224           | 523        | 506            | 1.05        | 107.53      |
| Anaheim                 | 416           | 914        | 1406           | 1           | 2106.5      |

5In this section, to improve the visibility of numerical results, we choose to only use ExpWeight as the benchmark and do not report the performance of AccelWeight that does not guarantee an equilibrium convergence in stochastic settings.
For our experiments, in each network, we run \textsc{AdaLight} and \textsc{ExpWeight} in $T_{\text{max}} = 10^{4}$ time epochs.

Static environment. First, we consider the static setting where in each time epoch, for each network, the costs on edges are determined by a BPR function with coefficients given by the dataset. The results are reported in Fig. 2, particularly as follows:

i) In Fig. 2a, Fig. 2c, Fig. 2e and Fig. 2g, we plot out the evolution of $\text{Gap}_{\textsc{AdaLight}}(T)$ and $\text{Gap}_{\textsc{ExpWeight}}(T)$ in different networks. Again, both algorithms converge toward equilibria. Importantly, the results in these large-scale networks show the superiority of \textsc{AdaLight} (and \textsc{AdaWeight}) in the convergence speed in comparison with \textsc{ExpWeight}. We also make several side-comments as follows. First, the evolution of $\text{Gap}_{\textsc{AdaLight}}(T)$ appears to involve less fluctuations than that in the small-size network considered in Section 6.1. This phenomenon is trivially explicable: in the toy-example with only two paths, any small changes in the implemented flow profile might lead to a large impact on the costs; in real-world instances, moving a mass of traffic from one path to the others does not make so much differences in the total induced costs accumulated from a large number of paths. Second, as we consider networks with larger sizes, both \textsc{AdaLight} and \textsc{ExpWeight} require a longer warming-up phase (where $\text{Gap}(T)$ does not decrease significantly) before picking up good recommendations and eventually converge. Albeit the case, this warming-up phase is still reasonably small and it does not restrict the applicability of our proposed algorithms.

ii) To validate the rate of convergence (in terms of $T$), we plot out the evolution of the terms $T^2 \cdot \text{Gap}_{\textsc{AdaLight}}(T)$ and $T^2 \cdot \text{Gap}_{\textsc{ExpWeight}}(T)$ in Fig. 2b, Fig. 2d, Fig. 2f and Fig. 2h. In these plots, $T^2 \cdot \text{Gap}_{\textsc{AdaLight}}(T)$ approach a horizontal line; this confirms that the convergence speed of \textsc{AdaLight} (also of \textsc{AdaWeight}) are precisely in order $O(T^2)$. On the other hand, \textsc{ExpWeight} does not reach this convergence order (and hence, $T^2 \cdot \text{Gap}_{\textsc{ExpWeight}}(T)$ continue increasing).

Stochastic environment. In this setting, the cost of edges are altered by adding random noises generated from zero-mean normal distributions. For validation purposes, for each network, we run \textsc{AdaLight} and \textsc{ExpWeight} in 5 instances (of noises’ layout) and report the averaged results across these instances in Fig. 3.

i) In Fig. 3a, Fig. 3c, Fig. 3e and Fig. 3g, we plot out the evolution of $\text{Gap}_{\textsc{ExpWeight}}(T)$ and $\text{Gap}_{\textsc{AdaLight}}(T)$. These terms tend to zero as $T$ increases; this confirms that \textsc{ExpWeight} and \textsc{AdaLight} converge toward equilibria in this stochastic setting. Moreover, in this setting, the convergence rate of \textsc{AdaLight} (also of \textsc{AdaWeight}) and \textsc{ExpWeight} are of the same order.

ii) To justify the convergence speed of the considered algorithms, in Fig. 3b, Fig. 3d, Fig. 3f and Fig. 3h, we plot our the terms $\sqrt{T} \cdot \text{Gap}_{\textsc{AdaLight}}(T)$ and $\sqrt{T} \cdot \text{Gap}_{\textsc{ExpWeight}}(T)$. These terms approach horizontal lines as $T$ increases. This reaffirms the fact that the speed of convergence of \textsc{ExpWeight} and \textsc{AdaWeight} are $O(1/\sqrt{T})$ in the stochastic regime. This result is consistent with our theoretical results.

On the elapsed time of \textsc{AdaLight}. We end this section with an interesting remark. Even in the largest network instance in our experiments (Anaheim), it only takes \textsc{AdaLight} around 4 seconds to finish one round of learning iteration and to output a route recommendation (the computations are conducted on a machine with the following specs: Intel Core(TM) i7-9750H CPU 2.60GHz and 8GB RAM). For most of the applications in urban traffic routing, the scale of time between fluctuations of networks’ states is often much larger than this elapsed time (it might take hours or even days for a significant change to happen). This highlights the implementability and practicality of our proposed methods, even in networks with much larger sizes.
Figure 2: Convergence speed of AdaLight (AdaWeight) and ExpWeight in static environments.

Appendix A. Proof of Theorem 2
Our goal in this appendix is to prove the $\mathcal{O}(\log P/T^2)$ equilibrium convergence rate of ACCELEWEIGHT in static environments. In the following, we use the entropy regularizer $h$...
as defined in Section 4.2 and let \( f^t, z^t \) and \( \gamma^t \) be defined as per Algorithm 2. With all this in hand, our proof of Theorem 2 will proceed in two basic steps:

**Step 1: Establish an energy function.** Building on the analysis of accelerated mirror descent algorithms [1, 22], we will consider the energy function

\[
\Delta^t = \gamma^{t-1} \text{Gap}(t) + \text{KL}(\gamma^t \| z^t)
\]

(A.1)

where \( \text{Gap}(t) = \Phi(f^t) - \min \Phi \) and \( \text{KL}(\gamma^t \| z^t) \) denotes the Kullback–Leibler divergence between \( f^* \) and \( z^t \).

Our goal in the sequel will be to prove that \( \Delta^t \) is decreasing in \( t \). Indeed, from Proposition 2 and the strong convexity of \( h \), we have

\[
\Phi(f^{t+1}) \leq \Phi(z^t) + \langle \nabla \Phi(z^t), f^{t+1} - z^t \rangle + \frac{\beta}{2} \| f^{t+1} - z^t \|^2\]

\[
= \Phi(z^t) + \langle \nabla \Phi(z^t), f^{t+1} - z^t \rangle + \frac{\beta}{2} (1 - \alpha^t)^2 \| z^{t+1} - z^t \|^2\]

\[
\leq \Phi(z^t) + \langle \nabla \Phi(z^t), f^{t+1} - z^t \rangle + \beta(1 - \alpha^t)^2 \text{KL}(z^{t+1} \| z^t).\]

(2.2)

Moreover, from (A.2) and the convexity of \( \Phi \), for all \( f^* \in \mathcal{F}^* \) := arg min \( \Phi \), we have

\[
\text{Gap}(t+1) = \Phi(f^{t+1}) - [\alpha^t \Phi(f^t) + (1 - \alpha^t) \Phi(f^*)]
\]

\[
\leq \Phi(f^{t+1}) - \Phi(\hat{z}^t) + \langle \nabla \Phi(\hat{z}^t), \hat{z}^t - f^{t+1} \rangle - \langle \nabla \Phi(f^t), f^t - f^{t+1} \rangle + \beta(1 - \alpha^t)^2 \text{KL}(z^{t+1} \| z^t)\]

\[
=(1 - \alpha^t) \gamma^t \langle \nabla \Phi(\hat{z}^t), z^{t+1} - f^* \rangle + \beta(1 - \alpha^t)^2 \text{KL}(z^{t+1} \| z^t).\]

(A.3)

To proceed, let \( W^t = (1 - \alpha^t) \gamma^t \nabla \Phi(z^t) \). Then, from Proposition 1 and the update structure of Algorithm 2, we have:

\[
\langle \nabla h(z^t) - \nabla h(z^{t+1}), z^{t+1} - f^* \rangle
\]

\[
= \sum_{i \in N} \log \left( \frac{\sum_{q \in \mathcal{P}_i} z_i^t \exp \left( - \frac{W^t}{M} \right) \sum_{p \in \mathcal{P}_i} (z_i^{t+1} - f^*_p)}{M^t} \right) \sum_{p \in \mathcal{P}_i} (z_i^{t+1} - f^*_p) + \sum_{i \in N} \sum_{p \in \mathcal{P}_i} W^t_i (z_i^{t+1} - f^*_p)
\]

\[
= 0 + \langle W^t, z^{t+1} - f^* \rangle \tag{A.4}
\]

[since \( \sum_{p \in \mathcal{P}_i} z_i^{t+1} = \sum_{p \in \mathcal{P}_i} f^*_p = M^t \)]

\[
= (1 - \alpha^t) \gamma^t \langle \nabla \Phi(z^t), z^{t+1} - f^* \rangle.
\]

(A.4)

Thus, multiplying both sides of \( A.3 \) by \( \gamma^t \) and combining them with \( A.4 \), we obtain

\[
\gamma^t \text{Gap}(t+1) - \alpha^t \gamma^t \text{Gap}(t) \leq \langle \nabla h(z^t) - \nabla h(z^{t+1}), z^{t+1} - f^* \rangle + \gamma^t \kappa(1 - \alpha^t)^2 \text{KL}(z^{t+1} \| z^t)\]

\[
= \text{KL}(f^* \| z^t) - \text{KL}(f^* \| z^{t+1}) + \gamma^t \kappa(1 - \alpha^t)^2 - 1) \text{KL}(z^{t+1} \| z^t).\]

(A.5)

Now, by the update rule of \( \gamma^t \) in Line 5 of Algorithm 2, the choice of \( \gamma^0 \) in Line 1 and the update rule of \( \alpha^t \) in Line 6, we get

\[
\gamma^t \kappa(1 - \alpha^t)^2 = \gamma^0 \left( 1 - \frac{\gamma^{t-1}}{\gamma^t} \right)^2 = \frac{(\gamma^{t-1} - \gamma^t)^2}{\gamma^t \gamma^0} = 1.
\]

(A.6)

Therefore, the last term in (A.5) vanishes and we can rewrite (A.5) as

\[
\gamma^t \text{Gap}(t+1) + \text{KL}(f^* \| z^{t+1}) \leq \gamma^t \text{Gap}(t) + \text{KL}(f^* \| z^t).\]

(A.7)

This shows that \( \Delta^t \leq \Delta^t - \cdots \leq \Delta^1 \) (by convention, we set \( \gamma^{-1} = 0 \), i.e., \( \Delta^t \) is decreasing in \( t \), as claimed.}

\]
Step 2: Upper-bounding the equilibrium gap. By iterating (A.7), we readily obtain
\[ \gamma^{T-1} \text{Gap}(T) \leq \Delta^T \leq \cdots \leq \Delta^1 = \text{KL}(f^* \| z^1) \] (A.8)
and hence
\[ \text{Gap}(T) \leq \text{KL}(f^* \| z^1)/\gamma^{T-1} \] (A.9)
To finish the proof, we need to bound \( \gamma^{T-1} \) and \( \text{KL}(f^* \| f^1) \) from above.

We begin by noting that
\[ \sqrt{\kappa \beta \gamma^{T-1}} = \sqrt{\kappa \beta \gamma^T} \left( 1 - \frac{1}{\sqrt{\kappa \beta \gamma^T}} \right) = \sqrt{\kappa \beta \gamma^T} - \frac{1}{2} \] (A.10)
so \( \sqrt{\kappa \beta \gamma^T} \geq \sqrt{\kappa \beta \gamma^{T-1}} + 1/2 \). Therefore, telescoping this last bound, we get
\[ \sqrt{\kappa \beta \gamma^{T-1}} \geq \sqrt{\kappa \beta \gamma^0} + \frac{T-1}{2} = \frac{T+1}{2} \geq \frac{T}{2} \]
and hence
\[ \gamma^{T-1} > T^2/(4 \kappa \beta) > 0. \] (A.11)

Second, from the choice of \( w^0 \) and \( a^0 \) in Line 1 of Algorithm 2, it follows that \( f_p^1 = z_p^1 = M_p / P_i \) for all \( p \in P^i, i \in N \). We thus get \( \langle \nabla h(f^1), f^1 - f^* \rangle = 0 \) so \( \text{KL}(f^* \| f^1) = h(f^*) - h(f^1) \leq \max h - \min h \). Moreover, by a straightforward calculation, we get \( \max h = M_{\text{tot}} \log(M_{\text{max}}) \) and \( \min h = M_{\text{tot}} \log(P/M_{\text{tot}}) \), so
\[ \text{KL}(f^* \| f^1) \leq M_{\text{tot}} \log(P M_{\text{max}}/M_{\text{tot}}). \] (A.12)

Thus, combining Equations (A.8), (A.11) and (A.12) and recalling that \( \kappa := M_{\text{max}} N \) (i.e., the strongly-convexity constant of \( h \)) and the fact that \( M_{\text{tot}} = \sum_{i \in N} M^i \leq N M_{\text{max}} \) we finally obtain
\[ \text{Gap}(T) \leq \frac{4 \beta N M_{\text{max}} M_{\text{tot}} \log(P M_{\text{max}}/M_{\text{tot}})}{(T-1)^2} \leq \frac{4 \beta N M_{\text{max}}^2 \log(P M_{\text{max}}/M_{\text{tot}})}{(T-1)^2} \] (A.13)
and our proof of Theorem 2 is complete. \( \blacksquare \)

Appendix B. Proof of Equation (15)

Let us denote \( R^t := \sum_{s=1}^t \alpha^s \). By Line 8 of Algorithm 3, for any \( t \), we have:
\[ z^t = \frac{R^t}{\alpha^t} f^t - \frac{R^{t-1}}{\alpha^t} f^{t-1}. \] As a consequence,
\[ \sum_{t=1}^T \alpha^t \langle z^t - f^*, \nabla \Phi(f^t) \rangle = \sum_{t=1}^T \alpha^t \left( \frac{R^t}{\alpha^t} f^t - \frac{R^{t-1}}{\alpha^t} f^{t-1} - f^*, \nabla \Phi(f^t) \right) \]
\[ = \sum_{t=1}^T \left[ R^{t-1} \langle f^t - f^{t-1}, \nabla \Phi(f^t) \rangle + \alpha^t \langle f^t - f^*, \nabla \Phi(f^t) \rangle \right] \]
\[ \geq \sum_{t=1}^T R^{t-1} \left[ \Phi(f^t) - \Phi(f^{t-1}) \right] + \sum_{t=1}^T \alpha^t \left[ \Phi(f^t) - \Phi(f^*) \right] \]
\[ = \sum_{t=1}^T \alpha^t \left[ \Phi(f^T) - \Phi(f^*) \right]. \] (B.1)

Here, the last equality is achieved via telescopic sum. Now, when we choose \( \alpha^t = t \) (as indicated in Theorem 3), we notice that \( R^t > \frac{T^2}{2} \). Divide two sides of (B.1) by \( R^t \) and taking expectations, we obtain precisely (15). \( \blacksquare \)
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