Consistency relations between the source terms in the second-order Einstein equations for cosmological perturbations

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Abstract
In addition to the second-order Einstein equations on four-dimensional homogeneous isotropic background universe filled with the single perfect fluid, we also derived the second-order perturbations of the continuity equation and the Euler equation for a perfect fluid in gauge-invariant manner without ignoring any mode of perturbations. The consistency of all equations of the second-order Einstein equation and the equations of motion for matter fields is confirmed. Due to this consistency check, we may say that the set of all equations of the second-order are self-consistent and they are correct in this sense.

1 Introduction
The general relativistic second-order cosmological perturbation theory is one of topical subjects in the recent cosmology. By the recent observation\cite{1}, the first order approximation of the fluctuations of our universe from a homogeneous isotropic one was revealed. The observational results also suggest that the fluctuations of our universe are adiabatic and Gaussian at least in the first order approximation. We are now on the stage to discuss the deviation from this first order approximation from the observational\cite{2} and the theoretical side\cite{3} through the non-Gaussianity, the non-adiabaticity, and so on. To carry out this, some analyses beyond linear order are required. The second-order cosmological perturbation theory is one of such perturbation theories beyond linear order.

In this article, we confirm the consistency of all equations of the second-order Einstein equation and the equations of motion for matter fields, which are derived in Refs.\cite{4,5}. Since the Einstein equations include the equation of motion for matter fields, the second-order perturbations of the equations of motion for matter fields are not independent equations of the second-order perturbation of the Einstein equations. Through this fact, we can check whether the derived equations of the second order are self-consistent or not. This confirmation implies that the all derived equations of the second order are self-consistent and these equations are correct in this sense.

2 Metric perturbations
The background spacetime for the cosmological perturbations is a homogeneous isotropic background spacetime. The background metric is given by

\[ g_{ab} = a^2 \left\{ -(d\eta)_a (d\eta)_b + \gamma_{ij}(dx^i)_a(dx^j)_b \right\}, \]

where \( \gamma_{ab} := \gamma_{ij}(dx^i)_a(dx^j)_b \) is the metric on the maximally symmetric three-space and the indices \( i, j, k, ... \) for the spatial components run from 1 to 3. On this background spacetime, we consider the perturbative expansion of the metric as \( g_{ab} = g_{ab} + \lambda h_{ab} + \frac{\lambda^2}{2} l_{ab} + O(\lambda^3) \), where \( \lambda \) is the infinitesimal parameter for perturbation and \( h_{ab} \) and \( l_{ab} \) are the first- and the second-order metric perturbations, respectively. As shown in Ref.\cite{6}, the metric perturbations \( h_{ab} \) and \( l_{ab} \) are decomposed as

\[ h_{ab} =: H_{ab} + \mathcal{L}_X g_{ab}, \quad l_{ab} =: \mathcal{L}_Y + 2\mathcal{L}_X h_{ab} + (\mathcal{L}_Y - \mathcal{L}_X^2) g_{ab}, \]

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where $\mathcal{H}_{ab}$ and $\mathcal{L}_{ab}$ are the gauge-invariant parts of $h_{ab}$ and $l_{ab}$, respectively. The components of $\mathcal{H}_{ab}$ and $\mathcal{L}_{ab}$ can be chosen so that

$$
\mathcal{H}_{ab} = a^2 \left\{ -2 (1) \Phi (d\eta)_a (d\eta)_b + 2 (1) \nu_i (d\eta)_a (dx^i)_b + \left( -2 (1) \gamma_{ij} + (1) \chi_{ij} \right) (dx^i)_a (dx^j)_b \right\}, \\
\mathcal{L}_{ab} = a^2 \left\{ -2 (2) \Phi (d\eta)_a (d\eta)_b + 2 (2) \nu_i (d\eta)_a (dx^i)_b + \left( -2 (2) \gamma_{ij} + (2) \chi_{ij} \right) (dx^i)_a (dx^j)_b \right\}.
$$

In Eqs. (3) and (4), the vector-mode $\nu_i$ and the tensor-mode $\chi_{ij}$ ($p = 1, 2$) satisfy the properties

$$
D^i (p) \nu_i = \gamma^{ij} D_p (p) \nu_j = 0, \quad \chi^{ij} = 0, \quad D^i (p) \chi_{ij} = 0,
$$

where $\gamma^{ij}$ is the inverse of the metric $\gamma_{ij}$.

### 3 Background, First-, and Second-order Einstein equations

The Einstein equations of the background, the first order, and the second order on the above four-dimensional homogeneous isotropic universe are summarized as follows.

The Einstein equations for this background spacetime filled with a perfect fluid are given by

$$
(0) E^{(1)}_i := \mathcal{H}^2 + K - \frac{8\pi G}{3} a^2 \epsilon = 0, \quad (0) E^{(2)} := 2 \partial_\eta \mathcal{H} + \mathcal{H}^2 + K + 8\pi Ga^2 p = 0,
$$

where $\mathcal{H} = \partial_\eta a/a$, $K$ is the curvature constant of the maximally symmetric three-space, $\epsilon$ and $p$ are energy density and pressure, respectively.

On the other hand, the second-order perturbations of the Einstein equation are summarized as

$$
(2) E^{(1)}_i := -3 \mathcal{H} \partial_\eta + \Delta + 3K \left( \frac{2}{3} \Phi \right) - 3 \mathcal{H}^2 - 4 \pi Ga^2 \mathcal{E} - \Gamma_0 = 0, \\
(2) E^{(2)} := \left( \partial_\eta^2 + 2 \partial_\eta \partial_i - K \left( \frac{1}{3} \Delta \right) \right) \Phi + \left( \mathcal{H} \partial_\eta + 2 \partial_\eta \mathcal{H} + \mathcal{H}^2 + \frac{1}{3} \Delta \right) \Phi, \\
-4 \pi Ga^2 \mathcal{P} \mathcal{F} \Gamma_k = 0, \\
(2) E^{(3)} := \left( \frac{1}{2} \Gamma_{ik} \right) + 3 \left( \Delta + 3K \right)^{-1} \left( \Delta^{-1} D^i D^j \Gamma_{ij} - \frac{1}{3} \Gamma_k \right) = 0, \\
(2) E^{(4)}_i := \partial_\eta D_i \Phi + \mathcal{H} D_i \Phi - \frac{1}{2} D_i \Delta^{-1} D^k \Gamma_{jk} + 4 \pi Ga^2 (\epsilon + p) D_i \nu = 0, \\
(2) E^{(5)}_i := (\Delta + 2K) \nu_i + 2 \left( \Gamma_i - D_i \Delta^{-1} D^k \Gamma_{jk} \right) - 16 \pi Ga^2 (\epsilon + p) \mathcal{V}_i = 0, \\
(2) E^{(6)}_i := \partial_\eta \left( a^2 \nu_i \right) - 2(a^2 \Delta + 2K)^{-1} \left( D_i \Delta^{-1} D^k D^l \Gamma_{kl} - D^k \Gamma_{ik} \right) = 0, \\
(2) E^{(7)}_{ij} := \left( \partial^2_\eta + 2 \partial_\eta \partial_i + 2K - \Delta \right) \gamma^{ij} - 2 \Gamma_{ij} + \frac{2}{3} \gamma_{ij} \Gamma_k^k + 3 \left( D_i D_j - \frac{1}{3} \gamma_{ij} \Delta \right) \left( \Delta + 3K \right)^{-1} \left( \Delta^{-1} D^k D^l \Gamma_{kl} - \frac{1}{3} \Gamma_k \right) \\
+ 4 \left( D_i (\Delta + 2K)^{-1} D_j - D_j (\Delta + 2K)^{-1} D^k \Gamma_{jk} \right) = 0,
$$

where we denote $\Gamma_i^j = \gamma^{jk} \Gamma_{jk}$. In these equations, $\mathcal{E}$ and $\mathcal{P}$ are the second-order perturbations of the energy density and the pressure, respectively. Further, $D_i \nu^i$ and $\mathcal{V}_i$ are the scalar- and the vector-parts of the perturbations.
of the spatial components of the covariant fluid four-velocity, in these equations. $\Gamma_0$, $\Gamma_i$, and $\Gamma_{ij}$ are the collections of the quadratic terms of the linear-order perturbations in the second-order Einstein equations and these can be regarded as the source terms in the second-order Einstein equations. The explicit form of these source terms are given in Refs. [3, 7]. First-order perturbations of the Einstein equations are given by the replacements $\Phi \rightarrow \Phi', \Psi \rightarrow \Psi', \nu_i \rightarrow \nu_i', \chi_{ij} \rightarrow \chi_{ij}', \mathcal{E} \rightarrow \mathcal{E}', \mathcal{P} \rightarrow \mathcal{P}', D_i \rightarrow D_i', v_i \rightarrow v_i', \mathcal{V}_i \rightarrow \mathcal{V}_i'$, and $\Gamma_0 = \Gamma_i = \Gamma_{ij} = 0$.

4 Consistency with the equations of motion for matter field

Now, we consider the second-order perturbation of the energy continuity equation and the Euler equations. In terms of gauge-invariant variables, the second-order perturbations of the energy continuity equation and the Euler equation for a single perfect fluid are given by [5]

$$a^{(2)}c^{(p)}_0 := \partial_\eta \mathcal{E} + 3\mathcal{H} \mathcal{E} + (\mathcal{P} - \mathcal{E}) + (\epsilon + p) \left( \Delta \partial_\eta \mathcal{E} - 3\partial_\eta \mathcal{P} - \Xi \right) = 0, \quad (14)$$

$$a^{(2)}c^{(p)}_i := (\epsilon + p) \left\{ (\partial_\eta + \mathcal{H}) \left( D_i \mathcal{E} + \mathcal{V}_i \right) + D_i \mathcal{P} + \partial_\eta p \left( D_i \partial_\eta \mathcal{E} - \Xi \right) = 0, \quad (15)$$

where $\Xi_0$ and $\Xi^{(p)}_i$ are the collection of the quadratic terms of the linear order perturbations and its explicit forms are given in Ref. [5, 7].

To confirm the consistency of the background and the perturbations of the Einstein equation and the energy continuity equation [14], we first substitute the second-order Einstein equations [19, 20, and 11] into Eq. (14). For simplicity, we first impose the first-order version of Eq. (9) on all equations. Then, we obtain

$$4\pi Ga^2 \Xi^{(2)}_0 = -\partial_\eta \left( \mathcal{E}^{(2)}_0 - \mathcal{H} \mathcal{E}^{(2)} + 3\mathcal{H} \mathcal{E}^{(2)} + D^i \mathcal{P}^{(1)} \right) + \frac{3}{2} \left( \mathcal{E}^{(2)}_0 + (\mathcal{P} - \Xi) \right) \partial_\eta \mathcal{E} = 0. \quad (17)$$

This equation shows that the second-order perturbation [14] of the energy continuity equation is consistent with the second-order and the background Einstein equations if the equation

$$4\pi Ga^2 \Xi_0 + (\partial_\eta + \mathcal{H}) \Gamma_0 + \frac{1}{2} \mathcal{H} \Gamma_k^k - \frac{1}{2} D^k \Gamma_k = 0 \quad (16)$$

is satisfied under the background, the first-order Einstein equations. Actually, through the background Einstein equations [19] and the first-order version of the Einstein equations [2, 3, 20], we can easily see that Eq. (17) is satisfied under the Einstein equations of the background and of the first order [7].

Next, we consider the second-order perturbations of the Euler equations. For simplicity, we first impose the first-order version of Eq. (9) on all equations, again. Through the background Einstein equations [19] and the Einstein equations of the second order [2, 3, 10], we can obtain

$$8\pi Ga^2 \mathcal{E}^{(2)}_i = -8\pi Ga^3 \mathcal{E}^{(0)} - D_i \mathcal{E}^{(2)} + D_i \mathcal{E}^{(2)} + \mathcal{H} \mathcal{E}^{(2)} - 2D_i \mathcal{E}^{(0)} \mathcal{E}^{(2)} + \frac{1}{2} \mathcal{H} \mathcal{E}^{(2)} + (\partial_\eta + 2\mathcal{H}) \Gamma_j - D_j \Gamma_j + \frac{1}{2} \mathcal{H} \mathcal{E}^{(2)} + (\partial_\eta + 2\mathcal{H}) \Gamma_j - D_j \Gamma_j = 0 \quad (19)$$

This equation shows that the second-order perturbations of the Euler equations is consistent with the Einstein equations of the background and the second order if the equation

$$\left( \partial_\eta + 2\mathcal{H} \right) \Gamma_j - D_j \Gamma_j - 8\pi Ga^2 \Xi^{(p)}_j = 0 \quad (18)$$

is satisfied.
is satisfied under the Einstein equations of the background and the first order. Actually, we can easily confirm Eq. (19) due to the background Einstein equations and the first-order perturbations of the Einstein equations[7], and implies that the second-order perturbation of the Euler equation is consistent with the set of the background, the first-order, and the second-order Einstein equations.

The consistency of equations for perturbations shown here is just a well-known result, i.e., the Einstein equation includes the equations of motion for matter field due to the Bianchi identity. However, the above verification of the identities (17) and (19) implies that our derived second-order perturbations of the Einstein equation, the equation of continuity, and the Euler equation are consistent. In this sense, we may say that the derived second-order Einstein equations, especially, the derived formulae for the source terms $\Gamma_0$, $\Gamma_1$, $\Gamma_{ij}$, $\Xi_0$, and $\Xi_i$ in Ref. [7] are correct.

5 Summary

In summary, we show the all components of the second-order perturbation of the Einstein equation without ignoring any modes of perturbation in the case of a perfect fluid. The derivation is based on the general framework of the second-order gauge-invariant perturbation theory developed in Refs. [8]. In this formulation, any gauge fixing is not necessary and we can obtain any equation in the gauge-invariant form which is equivalent to the complete gauge fixing. In other words, our formulation gives complete gauge fixed equations without any gauge fixing. Therefore, equations which are obtained in gauge-invariant manner cannot be reduced without physical restrictions any more. In this sense, these equations are irreducible. This is one of the advantages of the gauge-invariant perturbation theory.

We have also checked the consistency of the set of equations of the second-order perturbation of the Einstein equations and the evolution equation of the matter field in the cases of a perfect fluid. Therefore, in the case of the single matter field, we may say that we have been ready to clarify the physical behaviors of the second-order cosmological perturbations. The physical behavior of the second-order perturbations in the universe filled with a single matter field will be instructive to clarify those of the second-order perturbations in more realistic cosmological situations. We leave these issues as future works.

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