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Arcwise Connectedness of the Solution Sets for Generalized Vector Equilibrium Problems

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Abstract: In this research, by means of the scalarization method, arcwise connectedness results were established for the sets of globally efficient solutions, weakly efficient solutions, Henig efficient solutions and superefficient solutions for the generalized vector equilibrium problem under suitable assumptions of natural quasi cone-convexity and natural quasi cone-concavity.

Keywords: generalized vector equilibrium problem; solution set; arcwise connectedness; natural quasi cone-convexity; natural quasi cone-concavity

MSC: 49J40; 47J20

1. Introduction

Let \( K \) be a nonempty subset of a Banach space \( X \). Let \( Y \) be a normed vector space and \( F : X \times X \rightarrow 2^Y \) be a multi-valued mapping. We consider the following generalized vector equilibrium problem (GVEP): find \( \overline{x} \in K \), such that
\[
F(\overline{x}, y) \cap (-\Omega) = \emptyset, \quad \forall y \in K,
\]
where \( \Omega \cup \{0\} \) is a convex cone in \( Y \). (GVEP) can provide a unifying framework for dealing with an astonishing variety of problems, such as vector variational inequality, vector optimization, the vector complementary problem, the Nash equilibrium problem, the saddle point problem and fixed point problem, etc. (see, e.g., [1–4]).

For the (generalized) vector equilibrium problem, the study of connectedness about its solution sets has been paid close attention by many authors. One important reason is that it guarantees the possibility of continuous variation from one solution to another. Up to now, a large quantity of meaningful results have been given in the literature (see, e.g., [5–14]). By applying the scalarization technique, Gong [6] discussed the connectedness of the sets of Henig efficient solutions and weakly efficient solutions for vector equilibrium problems in normed spaces. In [7], Gong proposed various kinds of notions of efficient solutions to the vector equilibrium problem and gave many characterizations in the way of scalarization. With the help of these characterizations, he further proved several theorems about connectedness for these efficient solution sets. In [8], by using a different method of density, Gong and Yao discussed the connectedness of the efficient solution set for Ky Fan inequalities with monotone mappings. Later, Han and Huang [9] extended the density method used in [8] to investigate the connectedness of the efficient solution set for the generalized vector quasi-equilibrium problem. Chen et al. [10] investigated the connectedness of the sets of \( \varepsilon \)-weakly efficient solutions and \( \varepsilon \)-efficient solutions for vector equilibrium problems. In 2016, without the assumptions of monotonicity and compactness, Han and Huang [11] obtained some results concerning the connectedness for many different kinds of efficient solution sets of (GVEP). Recently, by using a new nonconvex separation theorem, Xu and Zhang [12] established the connectedness and the path connectedness results for the solution set of the strong vector equilibrium problem. By virtue of a density...
The dual cone of problems with respect to the addition-invariant set. Very recently, Shao et al. [15] did some research on the connectedness of solution sets for generalized symmetric Ky Fan inequality via free-disposal sets. However, as far as we know, there has been no paper dealing with arcwise connectedness for vector optimization problems. Peng et al. [14] also considered the connectedness and that is, $Y$ in $V$ that $V$ is a normed vector space with its topological dual $X$. From now on, unless stated otherwise, we always suppose that $X$ is a Banach space and $Y$ is a normed vector space with its topological dual $Y^*$. Let $K$ be a nonempty subset in $X$ and $C$ be a closed, convex and pointed cone in $Y$. Let $F : X \times X \to 2^Y$ be a multi-valued mapping. Denote by $\mathbb{R}_+$ the set of all non-negative numbers, i.e., $\mathbb{R}_+ = \{ x \in \mathbb{R} : x \geq 0 \}$. The dual cone of $C$ is defined by

$$C^* = \{ f \in Y^* : f(c) \geq 0, \forall c \in C \}.$$ 

Moreover, the quasi-interior of $C^*$ is defined as follows:

$$C^# = \{ f \in Y^* : f(c) > 0, \forall c \in C \setminus \{0\} \}.$$

We denote by $\text{int}D$, $\text{cl}(D)$, $\text{cone}(D)$ and $\text{co}(D)$ the interior, closure, cone hull and convex hull of a nonempty subset $D$, respectively. Let $B$ be a nonempty convex subset of the convex cone $C$. We note that $B$ is a base of $C$ if $C = \text{cone}(B)$ and $0 \notin \text{cl}(B)$. Clearly, $C^# \neq \emptyset$ if and only if $C$ has a base.

Take a base $B$ for the convex cone $C$ and let it be fixed. Define

$$C^A = \left\{ f \in C^# : \text{there exists } t > 0 \text{ such that } f(b) \geq t, \text{ for all } b \in B \right\}.$$ 

Obviously, $C^A \subseteq C^#$. Furthermore, we can conclude from the separation theorem of convex sets that $C^A \neq \emptyset$. On the other hand, noting that $0 \notin \text{cl}(B)$, we can take

$$\delta = \sup \{ t > 0 : (tU_Y) \cap B = \emptyset \}$$

and let $V_B = \frac{1}{\delta}tU_Y$, where $U_Y$ stands for the open unit ball in $Y$. Then, it is easy to see that $V_B$ is an open convex neighborhood of 0 in $Y$. For each convex neighborhood $U$ of 0 in $Y$ with $U \subseteq V_B$, we can check that the set $B + U$ is convex and $0 \notin \text{cl}(B + U)$. Hence, $C_U(B) = \text{cone}(B + U)$ is a convex pointed cone satisfying $C \setminus \{0\} \subseteq \text{int}C_U(B)$.

In this research, we denote by $W(K, F)$ the weakly efficient solution set for (GVEP), that is,

$$W(K, F) = \{ x \in K : F(x, y) \cap (-\text{int}C) = \emptyset, \forall y \in K \}.$$ 

In addition, we denote by $E(K, F)$ the efficient solution set for (GVEP), that is,

$$E(K, F) = \{ x \in K : F(x, y) \cap (-C \setminus \{0\}) = \emptyset, \forall y \in K \}.$$
Definition 1. For (GVEP), we call a vector \( x \in K \)
(i) a globally efficient solution if there exists a convex pointed cone \( P \) in \( Y \) with \( C \setminus \{0\} \subseteq \text{int} P \), such that

\[
F(x, y) \cap ((-P) \setminus \{0\}) = \emptyset, \text{ for all } y \in K.
\]

Let \( G(K, F) \) be the set composed of all globally efficient solutions;
(ii) a Henig efficient solution if there exists a neighborhood \( U \) of \( 0 \) with \( U \subseteq V_B \), such that

\[
F(x, y) \cap (-\text{int} C_U(B)) = \emptyset, \text{ for all } y \in K.
\]

Let \( H(K, F) \) be the set composed of all Henig efficient solutions;
(iii) a superefficient solution if for any neighborhood \( V \) of \( 0 \), there exists a neighborhood \( U \) of \( 0 \), such that

\[
\text{cone}(F(x, K)) \cap (U - C) \subseteq V.
\]

Let \( S(K, F) \) be the set composed of all superefficient solutions.

Given arbitrary \( f \in C^* \setminus \{0\} \), let

\[
Q(f) = \{ x \in K : f(F(x, y)) \subseteq \mathbb{R}_+, \text{ for all } y \in K \}.
\]

We call \( Q(f) \) the \( f \)-efficient solution set of (GVEP).

Definition 2. Let \( D \) be a nonempty subset in \( X \). A multi-valued mapping \( M : X \to 2^Y \) is called
(i) \((14)\) \( C \)-convex on \( D \) if \( \forall x_1, x_2 \in D, \forall t \in [0, 1], \) one has

\[
tM(x_1) + (1 - t)M(x_2) \subseteq M(tx_1 + (1 - t)x_2) + C;
\]

(ii) \((16)\) properly quasi \( C \)-convex on \( D \) if \( \forall x_1, x_2 \in D, \forall t \in [0, 1], \) one has

\[
\begin{align*}
\text{either} & \quad M(x_1) \subseteq M(tx_1 + (1 - t)x_2) + C, \\
\text{or} & \quad M(x_2) \subseteq M(tx_1 + (1 - t)x_2) + C;
\end{align*}
\]

(iii) natural quasi \( C \)-convex on \( D \) if \( \forall x_1, x_2 \in D, \forall t \in [0, 1], \) there exists \( \gamma \in [0, 1], \) such that

\[
\gamma M(x_1) + (1 - \gamma)M(x_2) \subseteq M(tx_1 + (1 - t)x_2) + C;
\]

(iv) natural quasi \( C \)-concave on \( D \) if \( \forall x_1, x_2 \in D, \forall t \in [0, 1], \) there exists \( \gamma \in [0, 1], \) such that

\[
M(tx_1 + (1 - t)x_2) \subseteq \gamma M(x_1) + (1 - \gamma)M(x_2) + C;
\]

(v) strictly natural quasi \( C \)-concave on \( D \) if \( \forall x_1, x_2 \in D \quad (x_1 \neq x_2), \forall t \in (0, 1), \) there exists \( \gamma \in [0, 1], \) such that

\[
M(tx_1 + (1 - t)x_2) \subseteq \gamma M(x_1) + (1 - \gamma)M(x_2) + \text{int} C.
\]

Remark 1. Obviously, strictly natural quasi \( C \)-concavity implies natural quasi \( C \)-concavity.

Remark 2. If \( M \) is \( C \)-convex or properly quasi \( C \)-convex, then it must be natural quasi \( C \)-convex. However, the converse is not true in general as shown by the following example.

Example 1. Let \( X = \mathbb{R}, Y = \mathbb{R}^2, D = [0, \frac{\pi}{4}] \) and \( C = \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \geq 0\} \). Define a multi-valued mapping \( M : X \to 2^Y \) as follows: for each \( x \in X \),

\[
M(x) = (\sin x, 1 - \sin x) + B_Y,
\]

where \( B_Y \) denotes the closed unit ball in \( Y \). It is easy to verify that \( M \) is natural quasi \( C \)-convex on \( D \). However, it is neither \( C \)-convex nor properly quasi \( C \)-convex on \( D \).
The next example indicates that the strictly natural quasi C-concavity can be satisfied easily.

**Example 2.** Let \( X = \mathbb{R}, Y = \mathbb{R}^2, D = [-1,1] \) and \( C = \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \geq 0\} \). Define a multi-valued mapping \( M : X \to 2^Y \) as follows: for each \( x \in X \),

\[
M(x) = \left(-x^2 + 1, 2 \cos x\right) + B_Y,
\]

where \( B_Y \) presents the closed unit ball in \( Y \). Notice that the two functions \(-x^2 + 1\) and \( \cos x \) are both strictly concave. Then, we can conclude easily that \( M \) has strictly natural quasi C-concavity on \( D \).

**Definition 3.** Let \( E \) be a linear space and \( D \subseteq E \) be a nonempty subset. A multi-valued mapping \( T : D \to 2^E \) is called a KKM mapping if for any finite subset \( \{x_1, x_2, \ldots, x_n\} \) in \( D \), one has

\[
\text{co}(\{x_1, x_2, \ldots, x_n\}) \subseteq \bigcup_{i=1}^{n} T(x_i).
\]

**Lemma 1.** (Fan-KKM Theorem) Let \( E \) be a Hausdorff topological vector space and \( D \subseteq E \) be a nonempty subset. Assume that \( T : D \to 2^E \) is a KKM mapping. If for any \( x \in D \), \( T(x) \) is closed, and for at least one point \( x \in D \), \( T(x) \) is compact, then \( \cap_{x \in D} T(x) \neq \emptyset \).

**Definition 4.** Let \( E \) and \( Z \) be two topological spaces. A multi-valued mapping \( \Phi : E \to 2^Z \) is called

(i) upper semicontinuous (u.s.c.) at \( x \in E \) if for any open set \( V \) in \( Z \) with \( \Phi(x) \subseteq V \), there exists a neighborhood \( U \) of \( x \), such that \( \Phi(U) \subseteq V \) for every \( x' \in U \);
(ii) lower semicontinuous (l.s.c.) at \( x \in E \) if for any open set \( V \) in \( Z \) with \( \Phi(x) \cap V \neq \emptyset \), there exists a neighborhood \( U \) of \( x \), such that \( \Phi(U) \cap V \neq \emptyset \) for every \( x' \in U \);
(iii) u.s.c. (resp. l.s.c.) on \( E \) if it is u.s.c. (resp. l.s.c.) at each point \( x \in E \);
(iv) continuous on \( E \) if it is both u.s.c. and l.s.c. on \( E \).

**Lemma 2.** Let \( E \) and \( Z \) be two topological spaces and \( \Phi : E \to 2^Z \) be a multi-valued mapping. Given a point \( x \in E \):

(i) (17) \( \Phi \) is l.s.c. at \( x \) if and only if for each \( y \in \Phi(x) \) and for each net \( \{x_n\} \) in \( E \) with \( x_n \to x \), there exists a net \( \{y_n\} \) with \( y_n \in \Phi(x_n) \), such that \( y_n \to y \);
(ii) (4) If \( \Phi(x) \) is compact, then \( \Phi \) is u.s.c. at \( x \) if and only if for each net \( \{x_n\} \) in \( E \) with \( x_n \to x \) and for each net \( \{y_n\} \) with \( y_n \in \Phi(x_n) \), there exist \( y \in \Phi(x) \) and a subnet \( \{y_{n_k}\} \) of \( \{y_n\} \), such that \( y_{n_k} \to y \).

The following lemma is of great importance for us to establish our main results.

**Lemma 3.** (18) Let \( E \) be a paracompact Hausdorff arcwise connected space and \( Z \) be a Banach space. Assume that \( T : E \to 2^Z \) is a multi-valued mapping, such that for any \( x \in E \), \( T(x) \) is nonempty, closed and convex. If \( T \) is l.s.c. on \( E \), then \( T(E) = \bigcup_{x \in E} T(x) \) is arcwise connected.

The next lemma gives scalar characterizations for the solution sets \( G(K, F) \), \( H(K, F) \), \( W(K, F) \) and \( S(K, F) \), which is similar to Theorem 2.1 of [7].

**Lemma 4.** (11) Assume that for each \( x \in K \), the set \( F(x, K) + C \) is convex. If \( C \) has a nonempty interior, i.e., \( \text{int} C \neq \emptyset \), then

(i) \( W(K, F) = \bigcup_{f \in C^* \setminus \{0\}} Q(f) \).
(ii) \( G(K, F) = \bigcup_{f \in C^*} Q(f) \).
(iii) \( H(K, F) = \bigcup_{f \in C^*} Q(f) \).
Moreover, if $C$ has a bounded base, then

(iv) $S(K, F) = \bigcup_{f \in \text{int}C} Q(f)$.

**Remark 3.** It is easy to see that all the sets $C^* \setminus \{0\}$, $C^A$, $C^q$ and $\text{int}C^*$ are convex. As a consequence, they are all arcwise connected.

### 3. Arcwise Connectedness

In this section, with the help of natural quasi cone-convexity and natural quasi cone-concavity, we shall establish arcwise connectedness results for various kinds of efficient solution sets of (GVEP).

**Lemma 5.** Let $K \subseteq X$ be a nonempty, compact and convex subset. Suppose that

(i) for every $x \in K$, $F(x, x) \subseteq C$;

(ii) for any $y \in K$, the mapping $F(\cdot, y)$ is l.s.c. on $K$;

(iii) for any $x \in K$, the mapping $F(x, \cdot)$ is natural quasi $C$-convex on $K$.

Then, for each $f \in C^* \setminus \{0\}$, the $f$-efficient solution set $Q(f)$ of (GVEP) is nonempty.

**Proof.** Take any $f \in C^* \setminus \{0\}$ and let it be fixed. Define a multi-valued mapping $T : K \to 2^K$ as follows:

$$T(y) = \{x \in K : f(F(x, y)) \subseteq \mathbb{R}_+\}, \ \forall y \in K.$$  

For any given $y \in K$, we have $T(y) \neq \emptyset$ as $y \in T(y)$. Furthermore, we can prove that $T(y)$ is closed. In fact, let $\{u_n\}$ be any sequence in $T(y)$ such that $u_n \to u_0 \in X$. Then, $u_0 \in K$ since $K$ is closed. Furthermore, by the lower semicontinuity of $F(\cdot, y)$ and Lemma 2, we have that for any $w \in F(u_0, y)$, there exists $w_n \in F(u_n, y)$ such that $w_n \to w$. It follows that $f(w_n) \to f(w)$. Noting that $f(w_n) \subseteq \mathbb{R}_+$, we can obtain $f(w) \in \mathbb{R}_+$. By the arbitrariness of $w$, we obtain $f(F(u_0, y)) \subseteq \mathbb{R}_+$. Hence, $u_0 \in T(y)$, which indicates that $T(y)$ is closed. Moreover, $T(y)$ is compact, as $K$ is compact.

Next, we shall prove that $T$ is a KKM mapping. In fact, suppose by contradiction that there exist $y_1, y_2, \ldots, y_n \in K$ and some $y_0 \in \text{co}(\{y_1, y_2, \ldots, y_n\})$, such that $y_0 \notin T(y_1)$, $i = 1, 2, \ldots, n$. As $K$ is convex, we know that $y_0 \in K$. Hence, for any $i \in \{1, 2, \ldots, n\}$, there must exist some $w_i \in F(y_0, y_i)$, such that $f(w_i) < 0$. As $y_0 \in \text{co}(\{y_1, \ldots, y_n\})$, there exist real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n \geq 0$ with $\sum_{i=1}^{n} \lambda_i = 1$, such that $y_0 = \sum_{i=1}^{n} \lambda_i y_i$. Then, by the natural quasi $C$-convexity of $F(y_0, \cdot)$, there exist real numbers $t_i \geq 0, i = 1, 2, \ldots, n$ with $\sum_{i=1}^{n} t_i = 1$, such that

$$\sum_{i=1}^{n} t_i F(y_0, y_i) \subseteq F(y_0, \sum_{i=1}^{n} \lambda_i y_i) + C = F(y_0, y_0) + C \subseteq C + C = C.$$  

From this, we can obtain $\sum_{i=1}^{n} t_i w_i \in C$. As $f \in C^* \setminus \{0\}$, we have

$$0 \leq f(\sum_{i=1}^{n} t_i w_i) = \sum_{i=1}^{n} t_i f(w_i) < 0.$$  

This is impossible. So $T$ is a KKM mapping.

Therefore, by applying the Fan-KKM theorem, we can obtain $\bigcap_{y \in K} T(y) \neq \emptyset$. This means $Q(f) \neq \emptyset$. \hfill \Box

**Lemma 6.** Let $K \subseteq X$ be a nonempty convex subset. Suppose that for each $y \in K$, $F(\cdot, y)$ is natural quasi $C$-concave on $K$. Then, for any $f \in C^* \setminus \{0\}$, the $f$-efficient solution set $Q(f)$ of (GVEP) is convex.

**Proof.** Let $f \in C^* \setminus \{0\}$ and $x_1, x_2 \in Q(f)$. For any $t \in [0, 1]$, set $x_t = tx_1 + (1-t)x_2$. Then $x_t \in K$ as $K$ is convex. Moreover, for each $i \in \{1, 2\}$,

$$f(F(x_t, y)) \subseteq \mathbb{R}_+, \ \forall y \in K.$$  

(1)
By the natural quasi C-concavity of $F(\cdot, y)$, there exists some $\lambda \in [0, 1]$, such that
\[
F(x, y) \leq \lambda F(x_1, y) + (1 - \lambda) F(x_2, y) + C.
\]

(2)

Notice that $f \in C^*\setminus\{0\}$. Then, by (1) and (2), we can obtain
\[
f(f(x_1, y)) \leq \lambda f(F(x_1, y)) + (1 - \lambda) f(F(x_2, y)) + f(C) \subseteq \mathbb{R}_+.
\]

It follows that $x_1 \in Q(f)$. Thus, $Q(f)$ is convex. □

Lemma 7. Let $K \subseteq X$ be a nonempty closed subset. Let $f \in C^*\setminus\{0\}$ be given. If for every $y \in K$, $F(\cdot, y)$ is l.s.c. on $K$, then the $f$-efficient solution set $Q(f)$ of (GVEP) is closed.

Proof. Let $\{x_n\}$ be any sequence in $Q(f)$, such that $x_n \to x_0 \in X$. Then $x_n \in K$ and
\[
f(f(x_n, y)) \subseteq \mathbb{R}_+, \quad \forall y \in K.
\]

By the closedness of $K$, we have $x_0 \in K$. For any given $y \in K$, by the lower semicontinuity of $F(\cdot, y)$ and Lemma 2, we know that for each $z \in F(x_0, y)$, there exists $z_n \in F(x_n, y)$, such that $z_n \to z$. Then, $f(z_n) \geq 0$ and $f(z_n) \to f(z)$. It follows that $f(z) \geq 0$. As $z$ is arbitrary, we can obtain $f(F(x_0, y)) \subseteq \mathbb{R}_+$. Furthermore, since $y$ is also arbitrary, we obtain $x_0 \in Q(f)$. This implies that $Q(f)$ is closed. □

Lemma 8. Let $K \subseteq X$ be a nonempty, convex and compact subset. Suppose that the following conditions hold:

(i) for every $x \in K$, $F(x, x) \subseteq C$;
(ii) for each $y \in K$, the mapping $F(\cdot, y)$ is strictly natural quasi C-concave on $K$;
(iii) for each $x \in K$, the mapping $F(x, \cdot)$ is natural quasi C-convex on $K$;
(iv) $F(\cdot, \cdot)$ is continuous on $K \times K$ with nonempty compact values.

Then, the $f$-efficient solution mapping $Q(\cdot)$ is l.s.c. on $C^*\setminus\{0\}$, where the space $C^*\setminus\{0\}$ is endowed with the normed topology.

Proof. Take arbitrary $f_0 \in C^*\setminus\{0\}$ and let it be fixed. We only need to show that $Q(\cdot)$ is l.s.c. at $f_0$. Indeed, suppose to the contrary that $Q(\cdot)$ is not l.s.c. at $f_0$. Then, there exist $x_0 \in Q(f_0)$ and a neighborhood $W_0$ of 0 in $X$, such that for any neighborhood $U(f_0)$ of $f_0$, there exists $f \in U(f_0)$ satisfying $(x_0 + W_0) \cap Q(f) = \emptyset$. This indicates that there exists a sequence $\{f_n\}$ with $f_n \to f_0$, such that
\[
(x_0 + W_0) \cap Q(f_n) = \emptyset, \quad \forall n \in \mathbb{N}_+.
\]

(3)

We consider two cases:

Case 1. $Q(f_0)$ is singleton. For each $n$, by Lemma 5, we know that $Q(f_n) \neq \emptyset$. Take arbitrary $x_n \in Q(f_n) \subseteq K$. By the compactness of $K$, we may assume that $x_n \to \bar{x} \in K$. We assert $\bar{x} \in Q(f_0)$. Indeed, suppose by contradiction that $\bar{x} \notin Q(f_0)$. Then, there must exist $g \in K$ and some $z \in F(\bar{x}, g)$, such that $f_0(z) < 0$. By the lower semicontinuity of $F(\cdot, \cdot)$ and Lemma 2, we know that there exists a sequence $\{z_n\}$ in $Y$ such that $z_n \in F(x_n, g)$ and $z_n \to z$. Since $f_n \to f_0$, we derive $f_n(z_n) \to f_0(z)$. Then, by the fact that $f_0(z) < 0$, we can obtain $f_n(z_n) < 0$ for $n$ large enough, which contradicts $x_n \in Q(f_n)$. Therefore, $\bar{x} \in Q(f_0)$. Since $Q(f_0)$ is singleton, we know $\bar{x} = x_0$ and so $x_n \to x_0$. It follows that $x_n \in x_0 + W_0$ for a large enough $n$. Hence, $x_n \in (x_0 + W_0) \cap Q(f_n)$ for a large enough $n$, which is a contradiction to (3).

Case 2. $Q(f_0)$ is not singleton. Then, it has at least two elements. Take arbitrary $x' \in Q(f_0)$ satisfying $x' \neq x_0$. For any given $y \in K$, since $x', x_0 \in Q(f_0)$, we have $x', x_0 \in K$ and
\[
f_0(F(x_0, y)) \subseteq \mathbb{R}_+ \quad \text{and} \quad f_0(F(x', y)) \subseteq \mathbb{R}_+.
\]
For each \( t \in (0, 1) \), by the strictly natural quasi C-concavity of \( F(\cdot, y) \), there exists \( \lambda \in [0, 1] \) satisfying
\[
F(tx' + (1-t)x_0, y) \subseteq \lambda F(x', y) + (1 - \lambda) F(x_0, y) + \text{int} C.
\]

As \( f_0 \in C^* \setminus \{0\} \), we can obtain
\[
f_0(F(tx' + (1-t)x_0, y)) \subseteq \lambda f_0(F(x', y)) + (1 - \lambda) f_0(F(x_0, y)) + f_0(\text{int} C) \subseteq \mathbb{R}_+^n, \tag{4}
\]
where \( \mathbb{R}_+ := \{ x \in \mathbb{R} : x > 0 \} \). For each \( t \in (0, 1) \), we set \( x(t) = tx' + (1-t)x_0 \). Then, \( x(t) \in K \) and \( x(t) \to x_0 \) as \( t \to 0 \). Thus, there exists \( t_0 \in (0, 1) \) such that \( x(t_0) \in x_0 + W_0 \). This, together with (3), indicates that \( x(t_0) \notin Q(f_n) \) for every \( n \in \mathbb{N}_+ \). It follows that for every \( n \), there exists \( y'_n \in K \) and some \( z_n' \in F(x(t_0), y'_n) \), such that \( f_n(z_n') < 0 \). Since \( K \) is compact, without loss of generality, we may assume \( y'_n \to y' \in K \). Then, by the upper semicontinuity of \( F(\cdot, \cdot) \) and Lemma 2, \( \{ z_n' \} \) has a subsequence converging to some point \( z' \in F(x(t_0), y') \). Without loss of generality, we may assume \( z_n' \to z' \). Noting that \( f_n \to f_0 \), we conclude \( f_n(z_n') \to f_0(z') \). Then, by the fact \( f_n(z_n') < 0 \), we can obtain \( f_0(z') \leq 0 \), which contradicts (4). \( \square \)

**Theorem 1.** Suppose that all the assumptions in Lemma 8 are satisfied. If for each \( x \in K \), the set \( F(x, K) + C \) is convex and \( \text{int} C \neq \emptyset \), then the weakly efficient solution set \( W(K, F) \) is arcwise connected.

**Proof.** It is clear that \( C^* \setminus \{0\} \) is a metric space (the metric is induced by the norm in \( Y^* \)). Then, \( C^* \setminus \{0\} \) is paracompact. By Remark 3, we know that \( C^* \setminus \{0\} \) is also arcwise connected. Furthermore, by Lemma 4, we have \( W(K, F) = \bigcup_{f \in C^* \setminus \{0\}} Q(f) \). By Lemmas 5–7, we know that for each \( f \in C^* \setminus \{0\} \), \( Q(f) \) is nonempty, convex and closed. In addition, by Lemma 8, we understand that \( Q(\cdot) \) is still l.s.c. on \( C^* \setminus \{0\} \), where \( C^* \setminus \{0\} \) is endowed with the normed topology. Hence, we conclude from Lemma 3 that \( W(K, F) \) is arcwise connected. \( \square \)

**Theorem 2.** Suppose that all the assumptions in Lemma 8 are satisfied. If for each \( x \in K \), the set \( F(x, K) + C \) is convex and \( C \) has a base \( B \), then the globally efficient solution set \( G(K, F) \) and the Henig efficient solution set \( H(K, F) \) are both arcwise connected.

**Proof.** By Lemma 4, we have \( G(K, F) = \bigcup_{f \in C^*} Q(f) \) and \( H(K, F) = \bigcup_{f \in C^*} Q(f) \). Then, by proceeding with the argument in the proof of Theorem 1, we can derive the conclusions.

**Theorem 3.** Suppose that all the assumptions in Lemma 8 are satisfied. If for each \( x \in K \), the set \( F(x, K) + C \) is convex and \( C \) has a bounded base, then the superefficient solution set \( S(K, F) \) is arcwise connected.

**Proof.** Notice that \( S(K, F) = \bigcup_{f \in \text{int} C^*} Q(f) \). Then, the conclusion can be proved with the argument in the proof of Theorem 1. \( \square \)

Finally, we give two examples to illustrate our main results of arcwise connectedness.

**Example 3.** Let \( X = \mathbb{R}, Y = \mathbb{R}^2, K = [-1, 1] \) and \( C = \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \geq 0 \} \). Let \( F : X \times X \to 2^Y \) be defined as follows: for each \( (x, y) \in X \times X \),
\[
F(x, y) = (f_1(x, y), f_2(x, y)) + B_Y,
\]
where
\[
f_1(x, y) = -x^2 + y^2 + 1, \quad f_2(x, y) = \cos \left( \frac{1}{2} x \right) + y + 2.
\]

Clearly, for each \( y \in K \), the two functions \( f_1(\cdot, y) \) and \( f_2(\cdot, y) \) are both strictly concave on \( K \). Then, we can conclude that the multi-valued mapping \( F(\cdot, y) \) is strictly natural quasi C-concave on \( K \). Similarly, it is clear that for each \( x \in K \), the two functions \( f_1(x, \cdot) \) and \( f_2(x, \cdot) \) are both convex on
K. Then, we can derive that the multi-valued mapping $F(x, \cdot)$ is $C$-convex on $K$. From this, we can verify easily that the set $F(x, K) + C$ is convex, and the mapping $F(x, \cdot)$ is natural quasi $C$-convex on $K$. The other conditions of Theorems 1–3 can be checked easily. Thus, by Theorems 1–3, we know that all the solution sets $G(K, F), H(K, F), W(K, F)$ and $S(K, F)$ are arcwise connected. In fact, by applying Lemma 4, we can calculate $G(K, F) = H(K, F) = W(K, F) = S(K, F) = [-1, 1]$. Clearly, these sets are all convex. As a consequence, they are arcwise connected.

Example 4. Let $X = \mathbb{R}, Y = \mathbb{R}^2, K = [0, \frac{\pi}{2}]$ and $C = \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \geq 0\}$. Let $F : X \times X \to 2^Y$ be defined as follows: for each $(x, y) \in X \times X$,

$$F(x, y) = (f_1(x, y), f_2(x, y)) + B_Y,$$

where

$$f_1(x, y) = \sin(x) + y^2 + 1, \quad f_2(x, y) = -x^2 + xy + 1.$$

Then, by a similar argument as that of Example 3, we can verify that all the conditions of Theorems 1–3 are satisfied. Thus, we conclude from Theorems 1–3 that all the solution sets $G(K, F), H(K, F), W(K, F)$ and $S(K, F)$ are arcwise connected. In fact, by virtue of Lemma 4, we can compute $G(K, F) = [0, \frac{\pi}{2}]$ and $H(K, F) = W(K, F) = S(K, F) = [0, \frac{\pi}{2}]$. It is clear that all these sets are convex. Thus, they are arcwise connected.

4. Conclusions

In this research, we prove the arcwise connectedness for several kinds of efficient solution sets for (GVEP) under suitable assumptions of natural quasi cone-convexity and natural quasi cone-concavity. Here, the scalar characterizations for these efficient solutions play important roles in the proofs. We also give examples to illustrate our main results. We note that the efficient solution set $E(K, F)$ of (GVEP) is not well behaved since the set $C \setminus \{0\}$ is neither open nor closed in general. Thus, how to explore the arcwise connectedness for the efficient solution set $E(K, F)$ is yet a challenging work.

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