PRIME GEODESIC THEOREM FOR THE MODULAR SURFACE

MUHAREM AVDISPAHIĆ

Abstract. Under the generalized Lindelöf hypothesis, the exponent in the error term of the prime geodesic theorem for the modular surface is reduced to $\frac{5}{8} + \epsilon$ outside a set of finite logarithmic measure.

1. Introduction

Let $\Gamma = \text{PSL}(2, \mathbb{Z})$ be the modular group and $\mathcal{H}$ the upper half-plane equipped with the hyperbolic metric. The norms $N(P_0)$ of primitive conjugacy classes $P_0$ in $\Gamma$ are sometimes called pseudo-primes. The length of the primitive closed geodesic on the modular surface $\Gamma \backslash \mathcal{H}$ joining two fixed points, which are the same for all representatives of $P_0$, equals $\log(N(P_0))$. The statement about the number $\pi_\Gamma(x)$ of classes $P_0$ such that $N(P_0) \leq x$, for $x > 0$, is known as the prime geodesic theorem, PGT.

The main tool in the proof of PGT is the Selberg zeta function, defined by

$$Z_\Gamma(s) = \prod_{\{P_0\}} \prod_{k=0}^{\infty} (1 - N(P_0)^{-s-k}), \quad \text{Re}(s) > 1,$$

and meromorphically continued to the whole complex plane.

The relationship between the prime geodesic theorem and the distribution of zeros of the Selberg zeta function resembles to a large extent the relationship between the prime number theorem and the zeros of the Riemann zeta.

However, the function $Z_\Gamma$ satisfies the Riemann hypothesis. It is an outstanding open problem whether the error term in the prime geodesic theorem is $O(x^{\frac{5}{8}+\epsilon})$ as it would be the case in the prime number theorem once the Riemann hypothesis be proved.

The obstacles in establishing an analogue of von Koch’s theorem [13, p. 84] in this setting comes from the fact that $Z_\Gamma$ is a meromorphic function of order 2, while the Riemann zeta is of order 1.

In the case of Fuchsian groups $\Gamma \subset \text{PSL}(2, \mathbb{R})$, the best estimate of the remainder term in PGT is still $O\left(\frac{x^{\frac{5}{8}}}{\log x}\right)$ obtained by Randol [18] (see also [7], [1] for different proofs). We note that its analogue $O\left(x^{\frac{d-1}{2}} (\log x)^{-1}\right)$ is valid also for strictly hyperbolic manifolds of higher dimensions, where $d_0 = \frac{d-1}{2}$ and $d \geq 3$ is the dimension of a manifold [3, Theorem 1].

The attempts to reduce the exponent $\frac{5}{8}$ in PGT were successful only in special cases. The chronological list of improvements for the modular group $\Gamma = \text{PSL}(2, \mathbb{Z})$

2010 Mathematics Subject Classification. 11M36, 11F72, 58J50.

Key words and phrases. Prime geodesic theorem, Selberg zeta function, modular group.
includes $\frac{35}{44} + \epsilon$ (Iwaniec [15]), $\frac{7}{10} + \epsilon$ (Luo and Sarnak [17]), $\frac{71}{102} + \epsilon$ (Cai [8]) and the present $\frac{57}{53} + \epsilon$ (Soundararajan and Young [19]).

Iwaniec [14] remarked that the generalized Lindelöf hypothesis for Dirichlet $L$-functions would imply $\frac{2}{3} + \epsilon$.

We proved [2] that $\frac{2}{3} + \epsilon$ is valid outside a set of finite logarithmic measure. In the present note, we relate the error term in the Gallagherian PGT on $\text{PSL}(2, \mathbb{Z})$ to the subconvexity bound for Dirichlet $L$-functions. This enables us to replace $\frac{2}{3} + \epsilon$ by $\frac{5}{8} + \epsilon$ under the generalized Lindelöf hypothesis. More precisely, the main result of this paper is the following theorem.

**Theorem.** Let $\Gamma = \text{PSL}(2, \mathbb{Z})$ be the modular group, $\epsilon > 0$ arbitrarily small and $\theta$ be such that

$$L \left( \frac{1}{2} + it, \chi_D \right) \ll (1 + |t|)^A |D|^\theta + \epsilon$$

for some fixed $A > 0$, where $D$ is a fundamental discriminant. There exists a set $B$ of finite logarithmic measure such that

$$\pi_\Gamma(x) = \int_0^x \frac{dt}{\log t} + O \left( x^{\frac{5}{8} + \frac{\theta}{4} + \epsilon} \right) \quad (x \to \infty, x \notin B).$$

Inserting the Conrey-Iwaniec [9] value $\theta = \frac{1}{6}$ into Theorem, we obtain

**Corollary 1.**

$$\pi_\Gamma(x) = \text{li}(x) + O \left( x^{\frac{5}{8} + \epsilon} \right) \quad (x \to \infty, x \notin B).$$

Any improvement of $\theta$ immediately results in the obvious improvement of the error term in PGT. Taking into account that the Lindelöf hypothesis allows $\theta = 0$, we get

**Corollary 2.** Under the Lindelöf hypothesis,

$$\pi_\Gamma(x) = \text{li}(x) + O \left( x^{\frac{5}{8} + \epsilon} \right) \quad (x \to \infty, x \notin B).$$

**Remark 1.** The obtained exponent for strictly hyperbolic Fuchsian groups is $\frac{5}{8} + \epsilon$ outside a set of finite logarithmic measure [3] and coincides with the above mentioned Luo-Sarnak unconditional result for $\Gamma = \text{PSL}(2, \mathbb{Z})$. In the case of a cocompact Kleinian group or a noncompact congruence group for some imaginary quadratic number field, the respective Gallagherian bound is $\frac{7}{10} + \epsilon$ [4].

2. Preliminaries.

The motivation for Theorem comes from several sources, including Gallagher [11], Iwaniec [15] and Balkanova and Frolenkov [6].

Recall that $\pi_\Gamma(x) = \text{li}(x) + O \left( x^{\frac{5}{8} + \frac{\theta}{4} + \epsilon} \right)$ is equivalent to $\psi_\Gamma(x) = x + O \left( x^{\frac{5}{8} + \frac{\theta}{4} + \epsilon} \right)$, where $\psi_\Gamma(x) = \sum_{N(P_0)^\frac{\theta}{4} \leq x} \log N(P_0)$ is the $\Gamma$ analogue of the classical Chebyshev function $\psi$.

Under the Riemann hypothesis, Gallagher improved von Koch’s remainder term in the prime number theorem from $\psi(x) = x + O \left( x^{\frac{1}{2}}(\log x)^2 \right)$ to $\psi(x) = x + O \left( x^{\frac{1}{2}}(\log \log x)^2 \right)$ outside a set of finite logarithmic measure.

Following Koyama [16], we shall apply the next lemma [10] due to Gallagher to our setting.
Lemma A. Let $A$ be a discrete subset of $\mathbb{R}$ and $\eta \in (0,1)$. For any sequence $c(\nu) \in \mathbb{C}$, $\nu \in A$, let the series

$$S(u) = \sum_{\nu \in A} c(\nu) e^{2\pi i \nu u}$$

be absolutely convergent. Then

$$\int_{-U}^{U} |S(u)|^2 \, du \leq \left( \frac{\pi \eta}{\sin \pi \eta} \right)^2 \int_{-\infty}^{\infty} \left| \frac{U}{\eta} \sum_{t \leq \nu \leq t + \frac{1}{n}} c(\nu) \right|^2 \, dt.$$

Iwaniec established the following explicit formula with an error term for $\psi_\Gamma$ on $\Gamma = \text{PSL}(2,\mathbb{Z})$.

Lemma B. For $1 \leq T \leq \frac{1}{(\log x)^2}$, one has

$$\psi_\Gamma(x) = x + \sum_{|\gamma| \leq T} \frac{x^{\rho}}{\rho} + O \left( \frac{x}{T} (\log x)^4 \right),$$

where $\rho = \frac{1}{2} + i \gamma$ denote zeros of $Z_\Gamma$.

Recently, O. Balkanova and D. Frolenkov have proved the following estimate.

Lemma C.\[ \sum_{|\gamma| \leq Y} x^{i \gamma} \ll \max \left( x^{\frac{1}{2} + \frac{3}{2} \theta}, x^2 \right) \log^3 Y, \]

$$\sum_{|\gamma| \leq Y} x^{i \gamma} \ll Y \log^2 Y \text{ if } Y > \frac{2^{\frac{1}{2} + \frac{3}{2} \theta}}{\kappa(x)},$$

where $\rho = \frac{1}{2} + i \gamma$ are the zeros of $Z_\Gamma$, $\theta$ is the subconvexity exponent for Dirichlet $L$–functions, and $\kappa(x)$ is the distance from $\sqrt{x} + \frac{1}{\sqrt{x}}$ to the nearest integer.

3. Proof of Theorem.

Inserting $T = \frac{1}{(\log x)^2}$ into Lemma we obtain

(1)\[ \psi_\Gamma(x) = x + \sum_{|\gamma| \leq T} \frac{x^{\rho}}{\rho} + O \left( \frac{x}{T} (\log x)^4 \right). \]

We would like to bound the expression $\sum_{|\gamma| \leq T} \frac{x^{\rho}}{\rho}$, where $Y \in (0, T)$ is a parameter to be determined later on.

Let $n = |\log x|$ and $B_n = \left\{ x \in [e^n, e^{n+1}) : \left| \sum_{|\gamma| \leq T} \frac{x^{\rho}}{\rho} \right| > x^Y \right\}$. Looking at the logarithmic measure of $B_n$, we get

\[ \mu^* B_n = \int_{\mathbb{R}} \frac{dx}{x} = \int_{A_n} \frac{dx}{x} + \int_{\mathbb{R} \setminus A_n} \frac{dx}{x} \leq \int_{e^n}^{e^{n+1}} \left| \sum_{|\gamma| \leq Y} \frac{x^{\gamma}}{\rho} \right|^2 \, dx \]

\[ \leq \frac{1}{e^{2n} \varepsilon Y} \int_{e^n}^{e^{n+1}} \left| \sum_{|\gamma| \leq Y} \frac{x^{\gamma}}{\rho} \right|^2 \, dx. \]
After substitution $x = e^{n} \cdot e^{2\pi \left( u + \frac{1}{n} \right)}$, the last integral becomes
\[
2\pi \int_{-\rho}^{\rho} \left| \sum_{|\gamma| \leq T} \frac{e^{(n+\frac{1}{2})i\gamma}}{\rho} e^{2\pi i u \gamma} \right|^2 du.
\]

Applying Lemma [A] with $\eta = U = \frac{1}{2\pi}$ and $c_{\gamma} = \frac{(n+\frac{1}{2})i\gamma}{\rho}$ for $|\gamma| \leq T$, $c_{\gamma} = 0$ otherwise, we get
\[
\left( \int_{-\rho}^{\rho} \left| \sum_{|\gamma| \leq T} \frac{e^{(n+\frac{1}{2})i\gamma}}{\rho} e^{2\pi i u \gamma} \right|^2 du \right)^{\frac{1}{2}} \leq \left( \frac{\frac{1}{2}}{\sin \frac{1}{4}} \right)^{2+\infty} \left( \int_{-\infty}^{+\infty} \sum_{|\gamma| \leq T} \frac{1}{|\rho|} \right)^{2} dt.
\]

Note that $\sum_{t<\gamma \leq t+1} \frac{1}{|\rho|} = O(1)$ since $\{ \gamma : t < |\gamma| \leq t+1 \} = O(t)$ by the Weyl law.
Thus,
\[
\int_{-\infty}^{+\infty} \left( \sum_{t<\gamma \leq t+1} \frac{1}{|\rho|} \right)^{2} dt = O \left( \int_{0}^{Y} dt \right) = O(Y).
\]

The relations (2), (3) and (4) imply $\mu^{*}B_{n} \ll \frac{Y}{\log Y} = \frac{1}{\rho}$. Hence, the set $B = \cup_{\gamma} B_{n}$ has a finite logarithmic measure.

For $x \notin B$, we have $\left| \sum_{|\gamma| \leq Y} \frac{x^{\gamma}}{n} \right| \leq x^{e}Y^{\frac{1}{4}}$, i.e.
\[
\sum_{|\gamma| \leq Y} \frac{x^{\gamma}}{n} \leq x^{e+\varepsilon}Y^{\frac{1}{4}}.
\]

Now, we rely on Lemma [C] to estimate $\left| \sum_{Y<|\gamma| \leq T} \frac{x^{\gamma}}{n} \right|$. Let us put $S(x, T) = \sum_{|\gamma| \leq T} x^{\gamma}$. By Abel’s partial summation, we have
\[
\sum_{Y<|\gamma| \leq T} \frac{x^{\gamma}}{n} \propto \frac{S(x, T)}{2 + iT} - \frac{S(x, Y)}{2 + iY} + i \int_{Y}^{T} \frac{S(x, u)}{Y} \left( \frac{1}{2 + iu} \right)^{2} du.
\]

Multiplying the last relation by $x^\frac{1}{2}$ and recalling that Lemma [C] yields $\sum_{|\gamma| \leq Y} x^{\gamma} \ll x^{\frac{1}{2}+\varepsilon + \frac{3}{4} + \varepsilon} Y^{\frac{1}{4}}$ for $Y < T = \frac{x^\frac{1}{2}}{(\log x)^2}$, we get
\[
\left| \sum_{Y<|\gamma| \leq T} \frac{x^{\gamma}}{n} \right| \ll \frac{x^{\frac{1}{2}+\varepsilon + \frac{3}{4} + \varepsilon}}{T^{\frac{1}{2}}} + \frac{x^{\frac{1}{2}+\varepsilon + \frac{3}{4} + \varepsilon} Y^{\frac{1}{4}}}{Y^{\frac{1}{2}}} + \int_{Y}^{T} \frac{x^{\frac{1}{2}+\varepsilon + \frac{3}{4} + \varepsilon} u^{\frac{1}{2}}}{u^{2}} du \ll \frac{x^{\frac{1}{2}+\varepsilon + \frac{3}{4} + \varepsilon}}{Y^\frac{1}{2}}.
\]

Combining (5) and (6), we see that the optimal choice for the parameter $Y$ is $Y \approx x^{\frac{1}{2}+\varepsilon}$. Then, $\sum_{|\gamma| \leq T} \frac{x^{\gamma}}{n} = O \left( x^{\frac{1}{2}+\varepsilon}Y^{\frac{1}{4}} \right) = O \left( x^{\frac{1}{2}+\varepsilon + \frac{3}{4} + \varepsilon} \right)$ for $x \notin B$.
The relation (11) becomes
\[ \psi_\Gamma (x) = x + O \left( x^{\frac{5}{8} + \epsilon} \right) \quad (x \to \infty, \ x \notin B), \]
as asserted.

References

[1] Avdispahić, M. “On Koyama’s refinement of the prime geodesic theorem.” Proc. Japan Acad. Ser. A 94, no. 3 (2018), 21–24.
[2] Avdispahić, M. “Gallagherian PGT on PSL(2, Z).” Funct. Approximatio. Comment. Math. doi:10.7169/facm/1686
[3] Avdispahić, M. “Prime geodesic theorem of Gallagher type.” arXiv:1701.02115
[4] Avdispahić, M. “On the prime geodesic theorem for hyperbolic 3-manifolds.” Math. Nachr. (to appear; cf. arXiv:1705.05626).
[5] Avdispahić, M., and Dž. Gušić. “On the error term in the prime geodesic theorem.” Bull. Korean Math. Soc. 49, no. 2 (2012), 367–372.
[6] Balkanova, O., and D. Frolenkov. “Bounds for the spectral exponential sum.” arXiv:1803.04201.
[7] Buser, P. Geometry and spectra of compact Riemann surfaces, Progress in Mathematics, Vol. 106, Birkhäuser, Boston-Basel-Berlin, 1992.
[8] Cai, Y. “Prime geodesic theorem.” J. Théor. Nombres Bordeaux 14, no. 1 (2002), 59–72.
[9] Conrey, J. B. and H. Iwaniec. “The cubic moment of central values of automorphic L-functions.” Ann. of Math. (2) 151, no. 3 (2000), 1175–1216.
[10] Gallagher, P. X. “A large sieve density estimate near \( \sigma = 1. \)” Invent. Math. 11 (1970), 329–339.
[11] Gallagher, P. X. “Some consequences of the Riemann hypothesis.” Acta Arith. 37 (1980), 339–343.
[12] Hejhal, D. A. The Selberg trace formula for PSL(2, R). Vol I, Lecture Notes in Mathematics, Vol 548, Springer, Berlin, 1976.
[13] Ingham, A. E. The distribution of prime numbers, Cambridge University Press, 1932.
[14] Iwaniec, H. “Non-holomorphic modular forms and their applications.” In Modular forms (Durham, 1983), 157–196, Ellis Horwood Ser. Math. Appl.: Statist. Oper. Res., Horwood, Chichester, 1984.
[15] Iwaniec, H. “Prime geodesic theorem.” J. Reine Angew. Math. 349 (1984), 136–159.
[16] Koyama, S. “Refinement of prime geodesic theorem.” Proc. Japan Acad. Ser A Math. Sci. 92, no. 7 (2016), 77–81.
[17] Luo, W. and P. Sarnak. “Quantum ergodicity of eigenfunctions on PSL_2(Z)\backslash H^2.” Inst. Hautes Études Sci. Publ. Math. no. 81 (1995), 207–237.
[18] Randol, B. “On the asymptotic distribution of closed geodesics on compact Riemann surfaces.” Trans. Amer. Math. Soc. 233 (1977), 241–247.
[19] Soundararajan, K. and M. P. Young. “The prime geodesic theorem.” J. Reine Angew. Math. 676 (2013), 105–120.

University of Sarajevo, Department of Mathematics, Zmaja od Bosne 33-35, 71000 Sarajevo, Bosnia and Herzegovina
E-mail address: mavdispa@pmf.unsa.ba