Abstract

In this work a new mechanics will be studied which is based on the hypothesis that the change of linear momentum of a particle happens as a discrete pulses. By using this hypothesis and by considering Newton’s relation between energy and momentum, and the law of mass and energy conservation as a priori, the Einstein dispersion relation can be derived as a zero approximation without using Lorentz transformations. Other terms will be derived as corrections to this relation. It will be shown that the effect of the corrections will be smaller and smaller with the increase of momentum. The work will offer an explanation of why the velocity of light seems to be constant regardless of the velocity of the source, and under which condition this will be changed. Also a prediction is made that faster than light transition could happen theoretically under certain conditions, and a nonzero mass photon can exist in nature. The work is purely classical in the sense that it doesn’t involve any uncertainty relations.

1 Introduction

One of the unanswered questions in modern physics is whether the photon has a tiny rest mass or not. From a classical point of view and without the notion of the theory of relativity it is totally unacceptable to assume that a particle like the photon has a zero rest mass. Such a suggestion would be rejected for the following reasons:
1. How come a particle that is carrying energy like any other particles does not have something which the rest of the particles have in common, that is a rest mass.

2. From the Newtonian Mechanical point of view, zero mass means that a particle will be accelerated to infinite velocity by any force no matter how small it is.

3. If a particle reaches a certain velocity it must be accelerated from rest to this velocity; it meaningless to say that a particle has no mass at rest, or, in other words has no existence at rest, and then when this nonexisting entity gain momentum it came to creation. Again this may be acceptable from a quantum a mechanical point of view, but not from classical point of view.

Even after Special Relativity there are scientists who thought that the photon may have a very small rest mass. To shed light on this point I will quote from a conversation between Feynman and a French professor [1]. The professor asked “Tell me Professor Feynman, how sure are you that the photon has no rest mass,”. Feynman answered “Well, it depends on the mass; evidently if the mass is infinitesimally small, so that it would have no effect whatsoever, I could not disprove its existence, but I would be glad to discuss the possibility that the mass is not certain definite size. The condition is that after I give you arguments against such mass” The professor then chose a mass of $10^{-6}$ electron mass, Feynman answered “If we agreed that the mass of the photon was related to the frequency as $\omega = \sqrt{k^2 + m^2}$, photons of different wave lengths would travel with different velocities. Then in observing the eclipsing double star, which is sufficiently far away, we would observe the eclipse in blue light and red light at different times. Since nothing like this is observed we can put an upper limit on the mass, which, if we do the numbers, turns out to be of the order of $10^{-9}$ electron mass”. The professor asked if it is possible that the photon has a mass of $10^{-12}$. Feynman answered “If the photon has a small mass equal for all photons, larger fractional differences from the massless behavior are expected as the wave length gets longer. So that from the sharpness of the known reflection of radar pulses, we can put an upper limit to the photon mass which is somewhat better than the eclipsing double star argument. It is turn out to be smaller than $10^{-15}$ electron mass”. After this the professor asked what if the mass $10^{-18}$ electron mass. Feynman answered “From field theory argument the potential should go as $exp(-mr)/r$. Then the earth has a static magnetic field, which is known to extend out into space for some distance, from the behavior of the cosmic ray of the order of few earth radii. But this means that the photon
mass must be of a size smaller than that corresponding to a decay length of the order of 8000 miles, or some, \(10^{-20}\) mass”.

But what if the mass of the photon is much smaller than that. In fact L. de Broglie noted that a rest mass a photon of order \(10^{-65}\) g would be impossible to detect.

Here it must be noted that, if there is really a photon with a rest mass no matter how small it is, then this will impose a change on the basic assumptions of the theory of special relativity because the second postulate stated that the velocity of light is always constant and it is equal to \(c\) no matter what is the velocity of the source [2]. Clearly that is not the case for a photon with a rest mass, because the velocity of light will depend on the velocity of the source as well as the light’s wave length. The second postulate only applied to photons with infinity energy since only then the velocity of light will be independent of the velocity of the source and it will be exactly \(c\), and practically there is no such photons.

1.1 The Purpose and Outline of this Work

The purpose of this work is to introduce a new scheme under which the relativistic energy momentum relation (Einstein Dispersion Relation) can be derived without using Lorentz transformations and without putting any limit on the maximum velocity with which particles can propagate, and to investigate the possibility of rest mass photons.

The treatment is based on the assumption that the exchange of momentum between particles is not a continuous process, but it happens in the form of discrete pulses. The derived dispersion relation is an approximation to Einstein dispersion relation (EDR) ; an approximation that is becoming very high with the increase of energies. It will be shown that under such an assumption it is possible for particles with nonzero mass to travel at the velocity \(c\) and faster, which means that it is possible for the photon to have a nonzero rest mass. It will also be shown that the new understanding will explain the results of the Michelson-Morley experiment. The constancy of the velocity of light will be a result and not a postulate like in the case of SR. However it will be shown that the principal of the constancy of the velocity of light can be violated for low energy photons.

The second section will discuss the basic postulates of the work, the third and fourth will show the steps to find the total energy of a particle and the correction terms to EDR as a power series. Section four will show a way to find the correction as functions of momentum. Section five will examine the convergence of the solution, section six will discuss the relation between velocity and momentum and between velocity and energy, and faster than
light transition, the last section will be for conclusions.

2 The Basic Postulates

In this work there are two postulates and one preposition.

2.1 First Assumption

Particles in nature are interacting through exchanging momentum. We are use to deal with momentum in classical mechanics and in most of the cases of quantum mechanics as a continuous quantity. The core idea of this work is to deal with linear momentum as a sum of vectors of small values each of which has a universal magnitude denoted by $p_s$, this make linear momentum in some sense a discrete vector entity. The first postulate stated that: 

$I$- The change of the linear momentum of a body or a particle with time has a minimal vector quantity of $\vec{p}_s$, with $p_s$ as a universal constant. To understand what exactly this postulate means let us consider the change of momentum, say from zero value to a value $\vec{p}$ during a time $\Delta t$. The interval $\Delta t$ can be divided in to smallest possible subintervals $\Delta t_1, \Delta t_2, ..., \Delta t_i$ such that in each of them the momentum is changing by the smallest possible value. Since according to the first postulate, the smallest possible change is $\vec{p}_s$, the momentum of a particle can be written as

$$\vec{p} = \sum_{l=1}^{i} \vec{p}_{s_l}$$

(1)

, where $\vec{p}_{s_1}$ is the change of momentum during $\Delta t_1$, $\vec{p}_{s_2}$ is the change of momentum during $\Delta t_2$, ...and so forth, and

$$|\vec{p}_{s_1}| = |\vec{p}_{s_2}| = ... = p_s$$

(2)

It is assumed here that the particle was initially at rest. Since the discrete nature of linear momentum is so far undetectable experimentally, this suggests that the value of $p_s$ is extremely small. It is so small that the momentum of the particles seems to be continuous.

2.2 Second Postulate

The equivalence between energy and mass can be reached without using relativity. In fact, a work by Poincaré [3] had led to such a relation by studying the momentum of radiation. Another work by Hasenöhrl [4],[5] can lead to the relation by studying the pressure inside a system composed of a
hollow enclosure filled with radiation. Braunbeck [6] had shown in 1937 that
the verification of the mass energy equivalence relation by experiment must
not be regarded as a theorem that can be derived from other principles of
less direct and less empirical evidence, but should be taken as a fundamental
principle. This point of view will be adopted here. The mass energy relation
in this work is a postulate (or a principle), but I stress here that in the mass
energy relation,
\[ E = mc^2 \] (3)
the constant \( c \) will not be defined as the velocity of light, it will be considered
only as a constant that has the units of velocity. It will be one of the results of
this work that particles with \( p \gg m_0c \) will travel with a velocity approaching
\( c \). The second postulate is: \textit{mass and energy are equivalent.}

2.3 Phenomenological preposition

The third postulate is supposed to specify the relation between the energy
change of a particle due to a vector change of one momentum pulse \( \vec{p}_s \) and the
momentum vector \( \vec{p} \). By knowing this relation, the dispersion relation can
be found after summing the values of energies due to a change of momentum
by \( i \)-pulses.

To show how this phenomenological relation has been suggested, consider
a particle that has a change of linear momentum equal to \( \vec{p}_s \). According to
Newton’s mechanics the change of energy due to this change of momentum
will be
\[ \Delta E = \frac{1}{m_o} \left( \vec{p}_{bf} \cdot \vec{p}_s + \frac{p_s^2}{2} \right) \] (4)
where \( \vec{p}_{bf} \) is the initial momentum vector of the particle, or it is the mo-
mentum vector before the change of momentum by \( \vec{p}_s \), and \( m_o \) is the rest
mass of the particle. In this work it will be purposed that Newton relation
is applicable but with a difference, according to the second postulate mass
and energy are equivalent therefore the kinetic mass must be used instead of
the rest mass. More about the preposition will be discussed in another work
under preparation [7]. Accordingly the change of the energy of a particle due
to a change of its momentum \( \vec{p}_{bf} \) by one momentum pulse is
\[ \Delta E = \left( \frac{\vec{p}_{bf} \cdot \vec{p}_s}{m} + \frac{p_s^2}{2m} \right) \] (5)

\[ m = \frac{E^{(t)}}{c^2} \] (6)
where $E(t)$ is the total energy of the particle. Relation (5) will be used for calculating the energy that is gained by the particle due to the $i^{th}$ momentum pulse.

According to the previous discussion the state of the proposition will be: 

*if a particle with momentum $p_{bf}$ changed its momentum by one momentum pulse then the change of energy due to that will be according to relation(5).*

In this work only an accelerated particle from rest will be studied, it will be assumed that the acceleration process will not cause any change in the inner state of particle, and therefore the rest mass will not change. If a force acting on a particle has fixed direction, then it would be reasonable to believe that all the momentum pulses will be pointing in the same direction of the accelerating force. Accordingly, for all momentum pulses, $\theta = 0$ in relation (5). The total linear momentum of the particle after the end of the action of the accelerating force will be:

$$p = ip_s$$

(7)

, and therefore the energy change due to the $i^{th}$ momentum pulse can be written in the following form

$$\Delta E_i = \left(\frac{(i - 1)p_s^2c^2}{E_i^{(t)}} + \frac{p_s^2c^2}{2E_i^{(t)}}\right) = \left(\frac{(i - \frac{1}{2})p_s^2c^2}{E_i^{(t)}}\right)$$

(8)

, where $E_i^{(t)}$ is the total energy of the particle due to $i$ momentum pulses, or

$$E_i^{(t)} = m_0c^2 + \sum_{l=1}^{i} \Delta E_l$$

(9)

Here it must be noted that quantities like $p_s$, $m_0$, $E^{(t)}$ ...etc. are measured relative to one fixed observer.

### 3 Calculating The Total Kinetic Energy

The total kinetic energy $K$ of a particle can be calculated by adding the kinetic energy gained from the first pulse plus the energy gained from the second pulse ... and so forth. Therefore $K$ can be expressed as

$$K(i) = \sum_{k=1}^{k=i} E_k$$

(10)

, where the delta was dropped from $\Delta E_k$ for short hand.
Expressing $E_k$ will become more problematic with the increasing of $k$. This can be shown by calculating $E_1$ and $E_2$ for a particle with rest mass $m_o$, for example, (8) will give

$$E_1 = \frac{1}{2} \left( -m_o c^2 \pm \sqrt{m_o^2 c^4 + 2p_s^2 c^2} \right)$$

(11)

The negative energy change solution will lead to a jump of the total energy to negative value due to a small change in momentum equal to $p_s$. The aspect of such solution will not be discussed in this work.

Here it is important to note that the positive solution of (11) is deviating clearly from the SR solution, but this will not contradict the experiment since there is no measurement to energies of particles with low value of momentum such as one or a few $p_s$. On the other hand, it will be shown that $E^{(t)}$ will approach the one of SR as the momentum increases.

The expression of $E_2$ will be found by using the expression of $E_1$, because $E^{(t)} = m_o c^2 + E_1 + E_2$. Therefore (8) and (11) give

$$E_2 = \frac{1}{2} \left[ -\frac{1}{2} \left( m_o c^2 + \sqrt{m_o^2 c^4 + 2p_s^2 c^2} \right) + \right. \right.$$

$$\left. \sqrt{\frac{m_o^2 c^4}{2} + \frac{13p_s^2 c^2}{2} + \frac{m_o c^2}{2} \sqrt{m_o^2 c^4 + 2p_s^2 c^2}} \right]$$

(12)

It is clear that calculating $K$ in this way will be unpractical. The fact that $p_s$ is very small will help in finding a series solution so that the expression of $E_1, E_2, ..., E_i$ can be written as a power series in $p_s$. For example, $E_1$ can be expanded as

$$E_1 = \frac{1}{2} \frac{p_s^2}{m_o} - \frac{1}{4} \frac{p_s^4}{m_o^3 c^2} + \frac{1}{4} \frac{p_s^6}{m_o^5 c^4} - \frac{5}{16} \frac{p_s^8}{m_o^7 c^6} + ...$$

(13)

and the expansion of $E_2$ will be

$$E_2 = \frac{3}{2} \frac{p_s^2}{m_o} - \frac{3}{8} \frac{p_s^4}{m_o^3 c^2} + \frac{87}{8} \frac{p_s^6}{m_o^5 c^4} - \frac{771}{16} \frac{p_s^8}{m_o^7 c^6} + ...$$

(14)

The kinetic energy for this case, where $i = 2$, is

$$K(2) = E_1 + E_2 = \frac{2}{m_o} \frac{p_s^2}{2} - \frac{13}{4} \frac{p_s^4}{m_o^3 c^2} + \frac{89}{8} \frac{p_s^6}{m_o^5 c^4} - \frac{97}{2} \frac{p_s^8}{m_o^7 c^6} + ...$$

(15)

For large $i$ this will also be an unpractical way of calculation. A rule must be found for the coefficients of the power series for $K$ for arbitrary values of $i$. To reach to this rule $K$ will be written as

$$K(i) = \sum_{l=1}^{i} E_l = \sum_{j=1}^{\infty} g_j(i) \frac{p_s^{2j}}{m_o^{2j-1} c^{2j-2}}$$

(16)
So the next step is to find \( g_j(i) \) which represents the coefficients of the expansion of \( K \) for arbitrary \( i \). To do that, (8) and (9) will be used to write the change in kinetic energy due to the \( i \) the pulse in terms of the total energy due to \( i - 1 \) pulses,

\[
E_i = \frac{1}{2} \left( -(m_o c^2 + \sum_{k=1}^{i-1} E_k) + \sqrt{(m_o c^2 + \sum_{k=1}^{i-1} E_k)^2 + 2p_s^2(2i - 1)c^2} \right) \tag{17}
\]

Expanding \( E_i \) as a power series in \( p_s \) gives the following expression

\[
E_i = \frac{1}{2} \left[ \sum_{j_0=1}^{\infty} \frac{p_s^{2j_0}(2c)^{j_0}[2i - 1]^{j_0}C(1/2, j_0)}{(m_o c^2 + \sum_{k=1}^{i-1} E_k)^{2j_0 - 1}} \right] \tag{18}
\]

From (16) it is easy to prove that

\[
\left( \sum_{k=1}^{i-1} E_k \right)^r = \sum_{j_1=1}^{j_1=\infty} \ldots \sum_{j_r=1}^{j_r=\infty} g_{j_1}(i - 1) \ldots g_{j_r}(i - 1) \frac{p_s^{2(j_1 + \ldots + j_r)}}{m_o^{2(j_1 + \ldots + j_r) - r} c^{2(j_1 + \ldots + j_r) - 2r}} \tag{20}
\]

After substituting with

\[
j_1 + \ldots j_r = J \tag{21}
\]

equation (20) can be written as

\[
\left( \sum_{k=1}^{i-1} E_k \right)^r = \sum_{j_r=1}^{J=\infty} \sum_{j_1=1}^{J=\infty} \frac{p_s^{2j}}{m_o^{2j_r - r} c^{2j_r - 2r}} G_{j_r}(i - 1) \tag{22}
\]

where \( G_{j_r}(i - 1) \) is defined by the following relation

\[
G_{j_r}(i - 1) = \sum_{j_1+\ldots+j_r=J} g_{j_1}(i - 1) \ldots g_{j_r}(i - 1) \tag{23}
\]

Substituting (22) into (19) will give

\[
E_i = \frac{1}{2} \left[ \sum_{j_0=1}^{J=\infty} \sum_{j_r=0}^{J=\infty} \sum_{r=0}^{r=J} \frac{p_s^{2(j_0 + J)}}{m_o^{2(j_0 + J) - r} c^{2(j_0 + J) - 2r}} [2i - 1]^{j_0} 2^{j_0} \right] \tag{24}
\]

\[\text{1Functions like } G_{j_r} \text{ and later } F_{j_r}(x) \text{ will not be named, they are just a substitution to make the equations more compact. These functions have nothing to do with tensors, and in general there is no use of any tensors in this work.}\]
where \( r = \text{co.} \) means that the summation will begin from \( r = 0 \) if \( J = 0 \) and begins from \( r = 1 \) for \( J \geq 1 \). By substituting

\[
j = J + j_0
\]

into (24), and using (16) the expression of \( K \) will be

\[
K(i) = \sum_{l=1}^{i} E_l = \sum_{j=1}^{\infty} g_j(i) \frac{p_j^2}{m_o^{2j-1}c^{2j-2}} = \sum_{l=1}^{i} \sum_{j=1}^{\infty} \frac{p_j^2}{m_o^{2j-1}c^{2j-2}} \sum_{j=0}^{J} \sum_{r=\text{co.}}^{r=J} C(1/2, j-J)C(-2j+2J+1, r)[2l-1]^{j-J}2^{j-J-1}G_{Jr}(l-1)
\]

Equating the coefficients of \( p_j^l \) in both sides of (26) gives

\[
g_j(i) = \sum_{l=1}^{i} \sum_{j=0}^{J-j-1} C(1/2, j-J)C(-2j+2J+1, r)[2l-1]^{j-J}2^{j-J-1}G_{Jr}(l-1)
\]

It is easy to see that any \( g_k(l-1) \) that appears on the right hand side of (27) with order \( k < j \), therefore \( g_j(i) \) will be expressed in terms of \( g_j's \) with lower orders. The calculations will involve the values of the binomial coefficients, and it also involves summations like \( \sum_{l=1}^{n} l^n \) \( (n = 0, 1, 2...) \), that demand using rules concerning the values of a series of integers in which Bernoulli numbers are used. No details will be mentioned here about these things since they are available in many calculus books, like [8], [9].

Equation (27) gives

\[
g_1(i) = \frac{i^2}{2}
\]

\[
g_2(i) = -\frac{i^4}{8} - \frac{i^3}{6} + \frac{i}{24}
\]

\[
g_3(i) = \frac{i^6}{16} + \frac{11i^5}{60} + \frac{i^4}{8} - \frac{i^3}{16} + \frac{i^2}{6} + \frac{i}{240}
\]

\[
g_4(i) = \frac{5i^8}{128} - \frac{103i^7}{560} - \frac{13i^6}{48} + \frac{37i^5}{960} + \frac{19i^4}{160} - \frac{13i^3}{384} - \frac{17i^2}{1016} + \frac{101i}{6720}
\]

\[
g_5(i) = \frac{7i^{10}}{256} + \frac{1823i^9}{10080} + \frac{1219i^8}{2880} + \frac{4079i^7}{13440} - \frac{1607i^6}{5760} - \frac{277i^5}{640}
\]

\[
g_6(i) = \frac{479i^4}{48} + \frac{1957i^3}{10080} + \frac{i^2}{18} + \frac{18i}{6720}
\]

\[
g_6(i) = -\frac{21i^{12}}{1024} - \frac{31373i^{11}}{177408} - \frac{2589i^{10}}{4480} - \frac{81199i^9}{107520} + \frac{2627i^8}{26880} + \frac{29983i^7}{295680}
\]

\[
+ \frac{1333i^6}{161280} - \frac{12436i^5}{53760} + \frac{19829i^4}{107520} + \frac{32353i^3}{8960} + \frac{963i^2}{295680}
\]

\[
+ \frac{12223i}{12223i}
\]

(33)
To see what these calculation may lead to, a digression would be useful at this point. Let us consider a sufficiently high value of \( i \) such that in all the equations from (28) to (35) only the terms with higher power of \( i \) will be dominating and the other terms can be neglected, then from (16) and knowing that \( p = ip_s \), the expression of \( K(p) \) will be

\[
K(p) \approx \frac{p^2}{2m_o} - \frac{p^4}{8m_o^3c^2} + \frac{p^6}{16m_o^5c^4} - \frac{5p^8}{128m_o^7c^6} + \frac{7p^{10}}{256m_o^9c^8} - \frac{21p^{12}}{1024m_o^{11}c^{10}} + \frac{33p^{14}}{2048m_o^{13}c^{12}} - \frac{429p^{16}}{32768m_o^{15}c^{14}} + \ldots
\]

(36)

The generating function for the above series is

\[
K(p) \approx \sqrt{m_o^2c^4 + p^2c^2 - m_o c^2}
\]

(37)

This is nothing but the kinetic energy that was expressed from SR by using Lorentz transformations.

To put things in more specific form, it is useful to write \( K(p) \) as a power series of functions

\[
K(p) = \sum_{j=0}^{\infty} \frac{p^j}{m_o^{j-1}c^{j-2}} f_j\left(\frac{p}{m_o c}\right) = \sum_{j=0}^{\infty} \frac{p^j}{m_o^{j-1}c^{j-2}} f_j(x)
\]

(38)

where

\[
x = \frac{p}{m_o c}
\]

(39)
By comparing (38) with (16) and by using the equations from (28) to (35), the functions $f_j$ can be expressed as power series, which are

\[
f_0 = \frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{16} - \frac{5x^8}{128} + \frac{7x^{10}}{256} - \frac{21x^{12}}{1024} + \frac{33x^{14}}{2048} - \frac{429x^{16}}{32768} + \ldots \tag{40}
\]

\[
f_1 = -\frac{x^3}{6} + \frac{11x^5}{60} - \frac{103x^7}{560} + \frac{1823x^9}{10080} - \frac{31373x^{11}}{177144} + \frac{265537x^{13}}{153756} - \frac{444679x^{15}}{2635776} + \ldots \tag{41}
\]

\[
f_2 = \frac{x^4}{8} - \frac{13x^6}{48} + \frac{1219x^8}{2880} - \frac{2589x^{10}}{4480} + \frac{1182109x^{12}}{1612800} - \frac{94478431x^{14}}{106444800} + \frac{244535971x^{16}}{234823680} + \ldots \tag{42}
\]

\[
f_3 = \frac{x}{24} - \frac{x^3}{16} + \frac{37x^5}{960} - \frac{4079x^7}{13440} - \frac{81199x^9}{107520} + \frac{29978387x^{11}}{21288960} - \frac{12578196061x^{13}}{230028800} + \frac{4127110357x^{15}}{1230028800} + \ldots \tag{43}
\]

\[
f_4 = -\frac{x^2}{16} + \frac{19x^4}{96} - \frac{1607x^6}{5760} + \frac{2627x^8}{26880} + \frac{280981x^{10}}{460800} - \frac{1372584929x^{12}}{638668800} + \frac{225909079003x^{14}}{46495088640} - \frac{156769939621x^{16}}{17220403200} + \ldots \tag{44}
\]

4 The Series Solution

The generating function must be found in order to explore the dispersion relation corrected to the $j$ order, that is because one of the aims of this work is to calculate corrections to EDR and find the effect of these corrections on the velocity. Finding the form of the power series of $f_j$ will lead to guess only $f_0$ which represents the kinetic energy term in EDR, and $f_1$, which is the first order correction. They have the following expressions

\[
f_0(x) = \sqrt{1 + x^2} - 1 \tag{45}
\]

\[
f_1(x) = \frac{\tan^{-1}(x) - x}{2\sqrt{1 + x^2}} \tag{46}
\]

The other series of $f_2, f_3, \ldots$ are too complicated to give any clue about their generating functions. Therefore it is important to find another technique that can express the functions $f_j$ directly. The following section will discuss such a technique that involves solving first order differential equations. The equations will become increasingly long and complicated with the increase of the order of correction. According to my experience the series solution is vital for checking the results.
5 The Generating Function Solution

The change of energy of a particle due to \((i + 1)p_s\) pulse can be written in terms of the total energy of the particle when \(p = ip_s\). This can be done by using (8), and (9) that give

\[
E_{i+1}(p + p_s) = \frac{1}{2} \left( -E(t)(p) + \sqrt{(E(t)(p))^2 + 2p_s^2c^2(2i + 1)} \right) \tag{47}
\]

By using a Taylor expansion it is easy to see that

\[
E_{i+1}(p + p_s) = E(t)(p + p_s) - E(t)(p) = (e^{p_sD} - 1)E(t)(p) \tag{48}
\]

where the operator \(D\) is defined by the following equation

\[
D = \frac{\partial}{\partial p} = \frac{1}{m_o c} \frac{\partial}{\partial x} \equiv \frac{1}{m_o c} \partial
\]

From (47), (48), (49), and (38), and after some elaboration, a useful expression can be reached, that is

\[
\sum_{i=1}^{\infty} \frac{p_i^l}{m_i^l c^l} \left[ \sum_{n+m+j+k=i} \frac{p_s}{m_o c} \frac{\partial n f_j(x)}{n!} \frac{\partial m f_k(x)}{m!} + \sum_{n+j=i} \frac{\partial n f_j(x)}{n!} \right] + \sum_{j+n+k=i} f_k(x) \frac{\partial n f_j(x)}{n!} = \frac{2xm_o c + p_s}{2m_o c} \frac{p_s}{m_o c}
\]

, where \(n, m \geq 1\). Taking the coefficients of \(p_s/m_o c\) on both sides of (50) will give a first order differential equation for \(f_0\), that is

\[
(1 + f_0) \frac{\partial f_0}{\partial x} = x \tag{51}
\]

Solving the above equation after applying the condition \(f_0(0) = 0\) gives exactly the same expression of (45) for \(f_0\). To express \(f_1\) the coefficients of \(p_s^2/m_o^2 c^2\) must be equated on both sides of (50) to get

\[
\left( \frac{\partial f_0}{\partial x} \right)^2 + (1 + f_0) \left( \frac{1}{2} \frac{\partial^2 f_0}{\partial x^2} + \frac{\partial f_1}{\partial x} \right) + f_1 \left( \frac{\partial f_0}{\partial x} \right) = \frac{1}{2} \tag{52}
\]

Substituting for \(f_0\) from (45) and after that solving the resultant first order differential equation under the condition \(f_1(0) = 0\) will give exactly the expression of \(f_1\) in (46). The process can be continued in similar way to find \(f_2, f_3, f_4, \ldots\). Because of the increasing complexity, there will be no mention
here of the details of calculating $f_2, f_3, f_4$. The results can be summarized as follows

$$f_0(x) = \sqrt{1 + x^2} - 1$$  \hspace{1cm} (53)

$$f_1(x) = \frac{\tan^{-1}(x) - x}{2\sqrt{1 + x^2}}$$  \hspace{1cm} (54)

$$f_2(x) = \frac{x^2(\tan^{-1}(x))^2}{8(1 + x^2)^{3/2}}$$  \hspace{1cm} (55)

$$f_3(x) = -\frac{1}{192} (1 + x^2)^{-5/2} \left( -5x - 3x^3 - (3 + 6x + 3x^2)\tan^{-1}(x) + 12x^3(\tan^{-1}(x))^2 + 12x^2(\tan^{-1}(x))^3 \right)$$  \hspace{1cm} (56)

$$f_4(x) = -\frac{1}{1152} (1 + x^2)^{-5/2} \left( 9x^2 - 3x^4 - 4x^6 + (54x + 33x^3 + 9x^5) \tan^{-1}(x) + (9 + 18x^2 - 27x^4)(\tan^{-1}(x))^2 - 96x^3(\tan^{-1}(x))^3 + (9x^4 - 36x^2)(\tan^{-1}(x))^4 \right)$$  \hspace{1cm} (57)

To check whether these results are correct or not, the Maclaurin expansion should be found for each of the $f_j$ functions written in the above equation. If the expansion coincide with the corresponding one in (40) to (44) then this will certify that the results are correct.

6 The Convergence of the Series of Functions

The series of $E^{(t)}(x)$ can be written as

$$E^{(t)}(x) = m_oc^2 + m_oc^2 f_0(x) + \sum_{j=1}^{\infty} u_j(x)$$  \hspace{1cm} (58)

, where the third term on the left hand side is a series of functions with

$$u_j(x) = \frac{p_s^j}{m_o^{j-1} c^{j-2}} f_j(x)$$  \hspace{1cm} (59)

In this section, the conditions for the convergence of the series $\sum u_j(x)$ will be discussed.

The expression given for the functions $f_1, ..., f_4$ in equations from (54) to (57) give no clue about a general rule that can be reached by mathematical induction to express $f_j$ for any value of $j$. The Weierstrass theorem will be
useful to verify the convergence for a case when the general term of the series is unknown \([10],[11]\). The state of the theorem is: Suppose \(\{u_n\}\) is a sequence of functions defined on \(E\), and suppose

\[|u_j(x)| \leq M_j, \quad (x \in E, j = 1, 2, 3...)\]  

(60)

then \(\sum u_j(x)\) converges uniformly on \(E\) if \(\sum M_j\) converges. For this case \(E\) is the closed interval \([0, \infty]\). By plotting the functions \(f_1, ..., f_4\) it is obvious that \(|f_j|\) are bounded on the interval \([0, \infty]\). The graphs give

\[
\sup_{x \in [0, \infty]} |f_1(x)| = 0.5 \\
\sup_{x \in [0, \infty]} |f_2(x)| = 0.0565 \\
\sup_{x \in [0, \infty]} |f_3(x)| = 0.0159 \\
\sup_{x \in [0, \infty]} |f_4(x)| = 0.0073
\]

(61) (62) (63) (64)

Equations from (61) to (64) together with (59) and (60) suggest more than one form of \(\{M_j\}\), such as:

\[
M_j = \frac{1}{2} \sum_{j=1}^{j=\infty} \frac{p^j_s}{m_0^{-1} c^{j-2}}
\]

(65)

\[
M_j = \frac{1}{2} \sum_{j=1}^{j=\infty} \frac{p^j_s}{m_0^{-1} c^{j-2} j}
\]

(66)

The \(\rho\)-test for both two expressions of (65) and (66) will give convergence under the condition

\[
m_0 c > p_s
\]

(67)

which means that the suggested mathematical treatment by series will give the correct result only if \(m_0 c > p_s\). One might say that the expression

\[
M_j = \frac{1}{2} \sum_{j=1}^{j=\infty} \frac{p^j_s}{m_0^{-1} c^{j-2} 2^{j-1}}
\]

(68)

also satisfies (60), and gives a more flexible result since it leads to the condition \(m_0 c > p_s/2\), but this wouldn’t be safe. Adopting such a result demands the knowledge of at least \(f_5\) and \(f_6\) in order to conform that (60) will be satisfied. Finding these functions demand doing a very long and tedious calculations. To calculate the dispersion relation for \(m_0 c < p_s\), the mathematical approach must be modified, that will not be discussed in this work, only for special case when the momentum is few \(p_s\)’s, that is when the velocity of the particle will be studied next section.
7  The Velocity of Particles

The relation between the velocity and momentum is important, it will be used to explain why the velocity of light seems to be independent on the velocity of the source, and under what conditions this will be changed.

The relation between the velocity and total energy of a particle will allow us to check whether the present treatment will lead to the same Einstein relation between kinetic mass, rest mass and velocity. It will be shown that this relation will be expressed as a zeroth approximation.

7.1  The Velocity as a Function of $p$

By definition, a velocity of a particle is related to its linear momentum by the following relation

$$\vec{v} = \frac{\vec{p}}{m}$$  \hspace{1cm} (69)

or

$$v = \frac{p}{m}$$  \hspace{1cm} (70)

where

$$m = \frac{E(t)}{c^2} = m_0 + \frac{K}{c^2}$$  \hspace{1cm} (71)

From (70) and (71), (38) can be written as

$$v = \frac{p}{m_0 + m_0 f_0(x) + \sum_{j=1}^{\infty} \frac{p^j}{m^{j-1}c^j} f_j(x)}$$  \hspace{1cm} (72)

Applying the binomial theorem to the term at the denominator in (72) gives

$$v = \frac{p}{m_0 + m_0 f_0(x) + \sum_{j=1}^{\infty} \sum_{j_1=1}^{\infty} \ldots \sum_{j_r=1}^{\infty} \frac{p^{j_1+\ldots+j_r}}{m_0^{j_1+\ldots+j_r} c^{j_1+\ldots+j_r}} f_{j_1} \ldots f_{j_r}}$$  \hspace{1cm} (73)

After substituting $j_1 + \ldots + j_r = j$, (73) can be written as

$$v = \sum_{j=0}^{\infty} \sum_{r=0}^{j} \frac{p^j}{m_0 c^{j-1}} C(-1, r) F_{jr}(x)$$  \hspace{1cm} (74)

, where

$$F_{jr}(x) = \frac{x}{(1 + f_0(x))^{r+1}} \sum_{j_1+\ldots+j_r=j} f_{j_1}(x) \ldots f_{j_r}(x)$$  \hspace{1cm} (75)
To get an iterated value for the velocity, \( v \) will be written as

\[
v(x) = \sum_{j=0}^{\infty} \frac{p_j}{m_o C^j} v_j(x)
\] (76)

where \( v_0 \) is the velocity corrected to the zero order, \( v_1 \) is the first order correction to the velocity as a function of momentum, \( v_2 \) is the second order correction,... and so forth. From (76), (74), the expression for \( v_j \) can be found, which is

\[
v_j(x) = \sum_{r=0}^{r=j} c \, C(-1, r) F_{jr}(x)
\] (77)

### 7.2 The velocity of Light

To see what will be the velocity for a particle with \( p \gg m_o c \), the functions \( F_{jr}(x) \) must be calculated, and then find \( v_j \) for \( j = 0, 1, 2, ... \). Using (75), (53),(54) and (55) give

\[
F_{00} = \frac{x}{(1 + x^2)^{\frac{7}{2}}} \quad F_{11} = \frac{x(tan^{-1}(x) - x)}{2(1 + x^2)^{\frac{5}{2}}}
\]

\[
F_{21} = \frac{x^3(tan^{-1}(x))^2}{8(1 + x^2)^{\frac{7}{2}}} \quad F_{22} = \frac{x(tan^{-1}(x) - x)^2}{4(1 + x^2)^{\frac{5}{2}}}
\]

...etc.

For the case \( p \gg m_o c \) (or \( x \to \infty \)) it is obvious from (78) that all the functions \( F_{jr} \to 0 \), except the function \( F_{00} \to 1 \). Therefore (77) will give \( v_j \to 0 \) for \( j = 1, 2, ... \) and \( v_0 \to c \), and directly (74) will give

\[
\lim_{x \to \infty} v = c
\] (79)

that coincides with experiment.

To prove that the velocity of light is independent of the velocity of the source a simple fact will be used. If the source moved away or approached the observer, the momentum of the photon will change, that is decrease in the first case and increase in the second case. This fact is applied not only to the emission of photons but to the emission of any particle. Performing any experiment to measure the effect of the velocity of the source on the velocity of light will not give any indication that the velocity of light is changing as long as the condition \( p \gg m_o c \) does not change. To see why, the derivative of \( v \) with respect to \( p \) will be found, it is

\[
\frac{dv}{dp} = \frac{dv}{dx} \frac{dx}{dp} = \sum_{j=0}^{j=\infty} \sum_{r=0}^{r=j} \frac{1}{m_o} C(-1, r) \frac{dF_{jr}(x)}{dx}
\] (80)
From (78) and by induction it can be proved that

$$\lim_{x \to \infty} \frac{dF_{jr}(x)}{dx} = 0$$

(81)

which means that there will be no change in the velocity as a result of a change in momentum as long as \( p \gg m_o c \). More precisely the change will be so small that it is beyond detection. This is a result and not a postulate as in the case of SR.

In this work the difference is that, in principle, there could be an experiment that measures a change in velocity of photons, but this experiment must involve a source moving away from the observer with extremely high velocity to produce a high Doppler shift for photons that have originally a very long wave length, such that \( x \) will not be big enough to make the right hand side of (80) approaching zero. Then the change of the velocity can be detected. No further specifications can be given about such an experiment since the rest mass of the photon is still unknown.

### 7.3 The Total Energy Velocity Relation

To find the relation between total energy and velocity, (9) will be written as

$$E^{(t)} = K + m_o c^2 = m_o c^2 + \sum_{j=0}^{j=\infty} \frac{p_j^j}{m_o c^{j-2}} f_j \left( \frac{E^{(t)} v}{m_o c^3} \right)$$

(82)

The above equation can be reached directly by substituting

$$x = \frac{p}{m_o c} = \frac{1}{m_o c} \left( \frac{E^{(t)} v}{c^2} \right)$$

(83)

The expansion of the velocity as a function of total energy is

$$v = \sum_{j=0}^{j=\infty} \frac{p_j^j}{m_o c^{j}} \beta_j c$$

(84)

, where

$$\beta_j = \frac{\tilde{v}_j}{c} \quad (j = 0, 1, 2\ldots)$$

(85)

, and \( \tilde{v}_j \) is the j-correction of the velocity as a function of energy. Writing the equations in terms of \( \beta \) will give the equations in a more compact form, as we will see later. To find a series solution, the following substitution will be made

$$x = \xi_0 + \xi_1 + \ldots + \xi_n + \ldots = \xi_0 + \sum_{k=1}^{\infty} \xi_k$$

(86)
where
\[ \xi_n = \frac{E(t) \beta_n}{m_o c^2} \frac{p_s}{m_o c^3} \]  
(87)

Applying the chain rule and using (86) gives
\[ \frac{\partial}{\partial \xi_k} = \frac{\partial x}{\partial \xi_k} \frac{\partial}{\partial x} = \frac{\partial}{\partial x} \]  
(88)

Here it must be noted that for \( x = \frac{p_s}{m c} \) the only solution for (86) and (87) is
\[ x = \xi_0 = \frac{E(t) \beta_0}{m_o c^2} \left( \frac{p_s}{m_o c} \right)^0 = \frac{E(t) v_0}{m_o c^3} = \frac{p_s}{m_o c} \]

Accordingly the Taylor series of (82) about \( \xi_0 \) for the several variables \( \xi_1, \xi_2, ..., \xi_k, ... \) will be
\[ E(t) = m_o c^2 + m_o c^2 f_0(\xi_0) + \sum_{j=1}^{\infty} \frac{p_s^j}{m_o c^{j-2}} f_j(\xi_0) + \sum_{j=1}^{\infty} \frac{p_s^j}{m_o c^{j-1}} c^{j-2} \]  
(89)

where the above equation was reached by arranging term with equal power of \( p_s \). Taking the coefficients of \( p_s^0 \) in (89) gives
\[ E(t) = m_o c^2 + m_o c^2 f_0(\xi_0) = \sqrt{m_o^2 c^4 + \frac{E(t)^2 v_0^2}{c^2}} \]  
(90)

The above relation gives exactly the Einstein relation between kinetic mass, rest mass, and velocity:
\[ \frac{E(t)}{c^2} = m = \frac{m_o}{\sqrt{1 - \frac{v_0^2}{c^2}}} \]  
(91)

Equation (91) gives
\[ v_0 = c \sqrt{1 - \frac{m_o^2 c^4}{E(t)^2}} \]  
(92)

Equating the coefficients of equal \( p_s^j \) on both sides of (89) for \( j = 1, 2, ... \) gives
\[ \tilde{v}_j = - \left[ \frac{E(t)}{m_o c^2} \left( \frac{\partial f_0}{\partial x} \right)_{x=\xi_0} \right]^{-1} \left( f_j(\xi_0) + \sum_{J=0}^{j-1} \sum_{l=1}^{J} \frac{1}{l!} \left( \frac{\partial f_j}{\partial x^l} \right)_{x=\xi_0} \right) \]  
(93)
The prime on the second integral means that the term with \( J = 0 \) and \( l = 1 \) is not included in the summation. The first correction to the velocity as a function of total energy is

\[
\tilde{v}_1 = c \left[ \frac{2E(t)}{m_o c^2} \sqrt{\frac{E(t)^2}{m_o c^4} - 1} \right]^{-1} \left( \frac{E(t)^2}{m_o c^4} - 1 - \tan^{-1} \left( \frac{E(t)^2}{m_o c^4} - 1 \right) \right)
\]

(94)

The plot of the above function against \( E(t)/m_o c^2 \) has a maximum value of \( \tilde{v}_{1\text{max}} \approx 0.1c \) at \( E(t)/m_o c^2 \approx 2.2 \) and the function rapidly approach zero as \( E(t)/m_o c^2 \to \infty \).

In any event, for particles with \( m_o c \gg p_s \) like electrons and protons the effect of this term on the value of the velocity will be very small, while for particles with small masses such that \( m_o c \sim p_s \) the correction has larger impact when \( E(t) \sim m_o c^2 \).

### 7.4 Faster Than Light Transition

One of the important result of this work is the possibility of having particles that can travel faster than light, but as will be shown later, these particles must have a small mass in order to reach such velocities. Moreover the momentum of these particles must be very low, so that the particle can exceed the velocity of light in a measurable value. The calculations in this section will be done by using exact relations and not approximate relations as it was done in the previous sections, that is because the energy can be calculated exactly by using (8) and (9) when the calculation involves only momenta with few \( p_s \)'s. It will be shown next how the velocity changes with the change of mass for a certain value of the momentum.

1. The case \( p = 1p_s \):

\[
E(t) = \frac{1}{2} \left( m_o c^2 + \sqrt{m_o^2 c^4 + 2p_s^2 c^2} \right)
\]

(95)

This gives

\[
v = \frac{p_s}{m} = \frac{2\delta}{1 + \sqrt{1 + 2\delta^2}} c
\]

(96)

where \( \delta = p_s/m_o c \).

It is easy to prove that the above relation will give \( v = c \) for \( \delta = 2 \) which means that a particle with momentum \( p_s \) and mass \( m_o = p_s/2c \) will have a velocity equal to \( c \). The velocity will increase with the mass decrease further until it reaches \( v = \sqrt{2}c \), which represent the upper bound of the velocity for any mass with momentum \( p_s \) no matter how small it is.
Figure 1: The plot of the velocity in the units of $c$ versus $\delta$ for the case $p = p_s$.

2. The case $p = 2p_s$:

$$v = 8\delta c \left[ 1 + \sqrt{1 + 2\delta^2} + \sqrt{2 + 26\delta^2 + 2\sqrt{1 + 2\delta^2}} \right]^{-1}$$  

(97)

The plot of $v$ against $\delta$ for this case shows that the velocity of a particle will increase with the decrease of the mass until it will reach $c$ at $m_o = p_s/(1.22)c$. It will increase further with the decrease of $m_o$ (increase of $\delta$) until it will reach an upper bound when $\delta \to \infty$, then

$$v = \frac{8c}{\sqrt{2(1 + \sqrt{13})}} \approx 1.23c$$  

(98)

3. The case $p = 3p_s$:

$$v = 24\delta c \left[ 1 + \sqrt{1 + 2\delta^2} + \sqrt{2 + 26\delta^2 + 2\sqrt{1 + 2\delta^2}} + \sqrt{4 + 188\delta^2 + 4\sqrt{1 + 2\delta^2} + 2(1 + \sqrt{1 + 2\delta^2})\sqrt{2 + 26\delta^2 + 2\sqrt{1 + 2\delta^2}}} \right]^{-1}$$  

(99)

The plot (will not be shown here) of $v$ against $\delta$ for this case shows that the velocity of a particle increases with the decrease of the mass until it reaches $c$ at $m_o = p_s/(0.89)c$. It increases further with the decrease of $m_o$ (increase of $\delta$) until it reaches an upper bound when $\delta \to \infty$. Then

$$\lim_{\delta \to \infty} v = \frac{24c}{\sqrt{2 + \sqrt{26} + \sqrt{188 + 2\sqrt{2\sqrt{26}}}}} \approx 1.157c$$  

(100)
Figure 2: The plot of the velocity in the units of $c$ versus $\delta$ for the case $p = 2p_s$

It is obvious that the value $\lim_{\delta \to \infty} v$ decreases with the increase of $p$, and it will be equal to $c$ for $p_s/m_0c \to \infty$, which means that the value that a particle can exceed the velocity of light is getting smaller and smaller with the increase of $p$. Therefore the faster than light transition will be discernable only at very low values of $p$, and when the particles involved have very small masses. For the first case the mass must be smaller than $p_s/2c$ for the second case it must be smaller than $p_s/(1.22)c$ and for the third case it must be smaller than $p_s/(0.89)c$.

8 Summary of Conclusions

This work has proved that it is possible to get the Einstein dispersion relation without using Lorenz transformations. The EDR appears as a zeroth approximation. The plot of the functions $f_0, f_1, f_2, f_3, f_4$ against $x$ shows that $\lim_{x \to \infty} f_0 = \infty$, and $\lim_{x \to \infty} f_1(x) = 1/2$ while $\lim_{x \to \infty} f_k(x) = 0$ for $k = 2, 3, \ldots$. This means that the EDR will be more and more accurate with the increase of momentum. The derived dispersion relation agrees with the one of SR for high energies. But for low energies and masses this will not be the case because these terms may have a considerable value. One of the difficulties here is the unknown value of the universal constant $p_s$. Because of that it is not possible to predict at what low energies the corrections are needed.

The first assumption of SR that the velocity of light is independent of
the velocity of the source is a result here that could be concluded. This new understanding lead to an experimental conditions in which the first assumption of SR can be violated. The violation can occur when the wave length of the photons is very long and the source is moving away from the observer with very high velocity.

Another important result is it that is permissible for nonzero mass particles to travel with a velocity of \( c \) and faster. This will solve the conceptual difficulty of zero mass photon since it is permitted to all particles to have a rest mass here including photons.

The faster than light transition is permitted here only under a condition that the masses of the particles must have a certain degree of smallness compared to \( p_s/c \). If there are no such particles in nature (including photons) then there will be no FTL. On the other hand, if FTL is detected then it is the proof of an existence of such particles with such small rest masses. The results of the previous section show that there will be no hope of detecting such transition for photons with short wave length (high momentum) because from (77) such a photon will travel with velocity very close to \( c \), the FTL could be detected if the experiment is designed to measure the velocity of photons with extremely long wave length. It has been shown that the largest possible velocity of a particle is \( \sqrt{2}c \) that is when \( p = p_s \) and the mass approaches zero.

Here I must admit that there is no usefulness of the new approach in applied physics unless faster than light transition is detected and measured experimentally, it would be hard to justify the replacement of one postulate of SR with three postulate if FLT will not be detected.

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