FUNCTIONAL LIMITS FOR “TIED DOWN” 
OCCUPATION TIME PROCESSES OF INFINITE 
ERGODIC TRANSFORMATIONS

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ABSTRACT. We prove functional, distributional limit theorems for the occupation times of pointwise dual ergodic transformations at “tied-down” times immediately after “excursions”. The limiting processes are tied down Mittag-Leffler processes and the transformations involved exhibit functional versions of the tied-down renewal properties in [AS19].

§0 Introduction

Overview.

For a pointwise dual ergodic measure preserving transformation (see p.6) with $\gamma$-regularly varying return sequence ($\gamma \in (0,1)$) the $\gamma$-Mittag-Leffler process ($m_\gamma$ on p.6) appears as a scaling limit of the normalized Birkhoff sums of non-negative, integrable functions ($R$ on p.7).

The tied down $\gamma$-Mittag-Leffler process $w_\gamma$ is a functional of $m_\gamma$, defined by $\Phi$ on p. 6 so that, in particular, 1 is a point of increase (i.e. a.s. $w_\gamma(t) < w_\gamma(1) \forall t < 1$).

In this paper we consider similar functionals of the Birkhoff sum processes: conditioning on arrival under the increment semiflow (p. 5) in a fixed, finite measure set at the final observation time. In particular, we show that their weighted scaling limits, conditioned thus, are tied-down Mittag-Leffler processes (Theorem U on p.11).

This kind of conditioning has been studied in the context of Markov processes (e.g. [DK63, Wen64, Lig68, BPY89, FPY93, Ber96, CUB11]).

2010 Mathematics Subject Classification. 37A40, 60F05.

Key words and phrases. infinite ergodic theory, pointwise dual ergodic, Darling-Kac theorem, Mittag Leffler processes, stable subordinator, rational weak mixing, local time process, conditional limit theorem, tied-down renewal theory.

©2019-21. Aaronson’s research was partially supported by ISF grant No. 1289/17. Sera’s research was partially supported by JSPS KAKENHI grants No. JP19J11798 and No. JP21J00015.
The tied-down Mittag-Leffler processes appear as local times of Markov bridges (see §6).

**Non-decreasing stochastic processes.**

Let $\mathcal{D}_{1,R}$ denote the collection of distribution functions of finite Borel measures on $[0, R]$ ($R \in \mathbb{R}_+$) and let $\mathcal{D}_{1,\infty}$ denote the distribution functions of Radon measures on $[0, \infty)$:

$$\mathcal{D}_{1,R} := \{ \xi : [0, R] \to [0, \infty) : \xi \text{ is non-decreasing, } \xi(x^+) = \xi(x) \ \forall \ x \in [0, R] \}$$

where $\xi(x^+) := \lim_{y \to x, \ y > x} \xi(y)$ ($0 \leq x < \infty$) & $\xi(x^-) := \lim_{y \to x, \ y < x} \xi(y)$ ($0 < x \leq \infty$); and let

$$\mathcal{D}_{1,\infty} := \{ \xi : [0, \infty) \to [0, \infty) : \xi|_{[0,R]} \in \mathcal{D}_{1,R} \ \forall \ R > 0 \}.$$

Equip $\mathcal{D}_{1,R}$ ($0 < R < \infty$) with the $\sigma$-algebra

$$\mathcal{B}_R := \sigma(\{ e_t : 0 \leq t \leq R \})$$

where for $\xi \in \mathcal{D}_{1,R}$, $e_t(\xi) := \xi(t)$

and equip $\mathcal{D}_{1,\infty}$ with $\mathcal{B}_\infty := \sigma(\{ e_t : t \in [0, \infty) \})$.

A non-decreasing stochastic process on $[0, R]$ is a random variable $Z$ on the measurable space $(\mathcal{D}_{1,R}, \mathcal{B}_R)$. It is a.s. continuous if a.s., $Z$ is continuous; and continuous in probability if $\forall \ t \in [0, R]$, $Z(s) \to Z(t)$ as $s \to t$ in probability; equivalently $Z(t-) = Z(t)$ a.s. $\forall \ t \in [0, R]$.

**Convergence.**

The Skorokhod $J_1$ topology on $\mathcal{D}_{1,R}$ ($R \leq \infty$) is generated by the Polish metric $J_1^{(R)}$ ([Sko56, EK86, Bil99]) defined for $R < \infty$ by

$$J_1^{(R)}(\xi, \eta) := \inf \{ \| \xi - \eta \circ \ell \|_{L^\infty([0,R])} + \| \log(\ell') \|_{L^\infty([0,R])} : \ell : [0, R] \to [0, R] \text{ an increasing bi-Lipschitz homeomorphism} \}$$

and for $R = \infty$ by

$$J_1^{(\infty)}(\xi, \eta) := \inf \{ \| \log(\ell') \|_{L^\infty([0,\infty))} + \int_0^\infty e^{-u}(\|\xi^{(u)} - \eta^{(u)} \circ \ell \|_{L^\infty([0,u])} \wedge 1)du : \ell : [0, \infty) \to [0, \infty) \text{ an increasing bi-Lipschitz homeomorphism} \}$$

where $x^{(u)}(t) := x(t \wedge u)$.

It follows that for $\xi_n, \xi \in \mathcal{D}_{1,\infty}$,

$$\xi_n \overset{J_1^{(\infty)}}{\to} \xi \iff \xi_n^{(u)} \overset{J_1^{(\infty)}}{\to} \xi^{(u)} \ \forall \ \text{continuity points } u \text{ of } \xi.$$

The distribution function (DF) topology on $\mathcal{D}_{1,R}$ is that inherited from the weak * topology on $C([0, R])^*$ for $R < \infty$ and from the weak
topology on $C_C([0,\infty))^*$ for $R = \infty$ where $C_C([0,\infty))$ denotes the space of continuous functions with compact support.

The spaces $(D_{1,R},\mathcal{DF})$ are Polish spaces and locally compact for $R < \infty$.

Throughout, given a Polish space $Y$,

- $C_B(Y)$ is the space of bounded, continuous, $\mathbb{R}$-valued functions equipped with the norm $\|g\|_{C_B} := \sup_{y \in Y} |g(y)|$,
- $B(Y) := \sigma(\{\text{open sets}\}) = \sigma(C_B(Y))$ is the Borel $\sigma$-algebra.

**Comparison: $\mathcal{DF}$ vs $J_1$ topologies.**

$\mathcal{DF}$ convergence is pointwise convergence at all continuity points of the limit and follows from $J_1$ convergence.

$\mathcal{DF}$ convergence to a continuous limit entails (by monotonicity) uniform convergence on bounded subintervals, whence $J_1$ convergence.

There are sequences converging $\mathcal{DF}$ (to discontinuous limits) which do not converge $J_1$.

As shown in [Sko56],

$$B(D_{1,R},J_1) = B(D_{1,R},\mathcal{DF}) = B_R.$$ 

**Distributional convergence.**

Given a Polish space $Y$, let

$$\text{RV}(Y) := \{\text{random variables on } Y\}$$
equipped with the topology of *weak convergence* so that

$$Z_n \xrightarrow{\text{RV}(Y)} Z \text{ if } \mathbb{E}(g(Z_n)) \xrightarrow{n \to \infty} \mathbb{E}(g(Z)) \forall g \in C_B(Y).$$

For $Z \in \text{RV}(Y)$ the *distribution* of $Z$ is that probability measure $\text{dist } Z \in \mathcal{P}(Y)$ so that

$$\mathbb{E}(g(Z)) = \int_Y gd(\text{dist } Z) \text{ for } g \in C_B(Y).$$

Note that for $R < \infty$, a collection $Z \subset \text{RV}(D_{1,R},\mathcal{DF})$ is *tight* (i.e. precompact) in $\text{RV}(D_{1,R},\mathcal{DF})$ iff $\{Z(R) : Z \in Z\}$ is precompact in $\text{RV}(\mathbb{R}^+_R) (R < \infty)$.

A collection $Z \subset \text{RV}(D_{1,\infty},\mathcal{DF})$ is tight when $\{Z(r) : Z \in Z\}$ is precompact in $\text{RV}(\mathbb{R}^+_r) \forall r > 0$. 
Convergence in the space $\text{RV}(D_{1,R}, \text{DF})$ is characterized by convergence of finite dimensional distributions (FDDs) (see [Kal02]):

FDD

$$Z_n \xrightarrow{\text{RV}(D_{1,R}, \text{DF})} n \to \infty \quad \iff \quad Z_n \xrightarrow{\text{FDD}} n \to \infty \quad \text{i.e.:}$$

$$(Z_n(t_1), Z_n(t_2), \ldots, Z_n(t_k), Z_n(R)) \xrightarrow{\text{RV}(\mathbb{R}^{k+1})} n \to \infty \quad (Z(t_1), Z(t_2), \ldots, Z(t_k), Z(R))$$

$\forall \ k \geq 1, \ (t_1, t_2, \ldots, t_k) \in (0, R)^k$ with $\mathbb{P}(Z(t_i) = Z(t_i^-)) = 1$; and

$$Z_n \xrightarrow{\text{RV}(D_{1,R}, \text{DF})} n \to \infty \quad \text{iff} \quad Z_n|_{[0,R]} \xrightarrow{\text{RV}(D_{1,R}, \text{DF})} n \to \infty \quad \forall \ R \in \mathbb{R}_+ \quad \text{with} \quad \mathbb{P}(Z(R) = Z(R^-)) = 1.$$

**Distributional convergence to an a.s. continuous limit.**

It follows from the comparison remarks above (p. 3) that

- convergence in $\text{RV}(D_{1,R}, J_1)$ implies FDD convergence; and,

- FDD convergence to an a.s. continuous limit $Z$ with $Z(0) = 0$ a.s. is sufficient for convergence in $\text{RV}(D_{1,R}, J_1)$ ([JS03, Thm. VI.3.37]).

The test functions used here to prove FDD convergence to an a.s. continuous limit include:

**Product functions.** A *product function* is a function $g_2 : D_{1,1} \to \mathbb{R}$ of form

$$g_2(\xi) = \left[ \prod_{\nu=1}^{N} g_\nu(\xi(t_\nu)) \right] h(\xi(1) - \xi(t_N))$$

where $t = (t_1, t_2, \ldots, t_N)$ with $0 \leq t_1 < t_2 < \cdots < t_N < 1$ and $g_1, \ldots, g_N, h \in C([0, \infty))$.

Denote $\Pi \coloneqq \{\text{product functions}\}$.

It is standard to show that $\Pi \subset C_{B,c}(D_{1,1})$, the collection of bounded, measurable functions on $D_{1,1}$ continuous at each $\xi \in D_{1,1} \cap C([0,1])$ (i.e. at each $\xi : [0,1] \to \mathbb{R}$ which is continuous and non-decreasing).

Moreover, for $Z_n$, $Z \in \text{RV}(D_{1,1})$,

$$\mathbb{E}(g(Z_n)) \xrightarrow{n \to \infty} \mathbb{E}(g(Z)) \quad \forall \ g \in \Pi$$

$$\implies \quad Z_n \xrightarrow{\text{FDD}} n \to \infty \quad \implies \quad Z.$$

\footnote{However convergence in $\text{RV}(D_{1,R}, J_1)$ is not in general characterized by convergence of FDDs even when the limit is continuous in probability as shown by the example in [Bil99, Problem 12.5].}
Self similarity, increment stationarity and their dynamics.

The space $D_{1,\infty}$ is naturally equipped with

- the increment semiflow $T_s : D_{1,\infty} \to D_{1,\infty}$ ($s \geq 0$) defined by $T_s Z(t) := Z(s + t) - Z(s)$;

and

- for $\gamma > 0$, the $\gamma$-scalings $\Delta_{a, \gamma} : D_{1,\infty} \to D_{1,\infty}$ ($a \geq 0$) defined by $\Delta_{a, \gamma} Z := \frac{1}{\alpha} Z \circ D_a$ with $D_a(t) := at$.

For $\gamma > 0$, we call the process $Z \in RV(D_{1,\infty})$ $\gamma$-self similar if $\Delta_{a, \gamma} Z = Z \forall a > 0$.

Equivalently, $(D_{1,\infty}, \text{dist } Z, \Delta_{a, \gamma})$ is a probability preserving transformation $\forall a > 0$.

Also, we’ll say that the process $Z \in RV(D_{1,\infty})$ has stationary increments if $T_s Z = Z \forall s > 0$.

Equivalently, $(D_{1,\infty}, \text{dist } Z, T_s)$ is a probability preserving transformation $\forall s > 0$.

Inverse process.

The inverse of $\xi \in D_{1,\infty}$ with $\xi(\infty-) = \infty$ is $I(\xi) := \xi^{-1} \in D_{1,\infty}$ defined by $\xi^{-1}(t) := \inf \{s > 0 : \xi(s) > t\}$.

If $Z \in RV(D_{1,\infty})$ is $\gamma$-self similar with $\gamma > 0$ and $Z(\infty-) = \infty$ a.s., then $Z^{-1}$ is $\frac{1}{\gamma}$-self similar.

Waiting times.

Define the waiting time functionals $G, D$ on $D_{1,\infty}$ as follows:

$G(\xi)(t) := \inf \{s \leq t : \xi(s) = \xi(t)\}$

so that $G : D_{1,\infty} \to D_{1,\infty}$; and

$D(\xi)(t) := \sup \{s \geq t : \xi(s) = \xi(t)\} \leq \infty$.

If $\xi(\infty-) = \infty$, then $D(\xi) \in D_{1,\infty}$.

Subordinators and Mittag-Leffler processes.

For $\gamma \in (0,1)$,

- the $\gamma$-stable subordinator is $\eta_{\gamma} \in RV(D_{1,\infty})$, $\frac{1}{\gamma}$-self similar with positive, stationary, independent $\gamma$-stable increments so that $\mathbb{E}(\eta_{\gamma}^{-1}(1)) = \mathbb{E}(\frac{1}{\eta_{\gamma}(1)\gamma}) = 1$;
the $\gamma$-Mittag-Leffler ($\gamma$-ML) process is $m_\gamma = \eta_\gamma^{-1}$ and is $\gamma$-self similar;

- the tied down $\gamma$-ML process is $w_\gamma \in \mathbb{RV}(D_{1,1})$ defined by

$$w_\gamma(t) := m_\gamma(g_{\gamma,1} t) = \Delta g_{\gamma,1,\gamma} m_\gamma(t)$$

where $g_{\gamma,1} := G(m_\gamma)(1)$.

These processes $w_\gamma$ for $0 < \gamma \leq \frac{1}{2}$ correspond to the local time at zero of the symmetric $\frac{1}{1-\gamma}$-stable bridge; and for $0 < \gamma < 1$, to the local time at zero of the Bessel bridge of dimension $2-2\gamma$ (see §6).

**Almost sure continuity of the ML processes.**

The stable subordinators $\eta_\gamma$ are strictly increasing a.s.; thus, a.s., $m_\gamma = \eta_\gamma^{-1}$ is continuous (see e.g. the remark on p.8 of [RY99]); as is $w_\gamma$ by $\otimes$.

**Pointwise dual ergodic transformations.**

Let $(X, m, T)$ denote a measure preserving transformation $T$ of the non-atomic, Polish measure space $(X, m)$ (where $m$ is a $\sigma$-finite, non-atomic measure defined on the Borel subsets $B(X)$ of $X$).

The associated transfer operator $\mathcal{T}: L^1(m) \to L^1(m)$ is the predual of $f \mapsto f \circ T$ ($f \in L^\infty(m)$), that is

$$\int_X \mathcal{T} f g dm = \int_X f g \circ T dm \text{ for } f \in L^1(m) \text{ & } g \in L^\infty(m).$$

The measure preserving transformation $(X, m, T)$ is called pointwise dual ergodic if there is a sequence $a(n) = a_n(T)$ (the return sequence of $(X, m, T)$) so that

$$PDE \quad \frac{1}{a(n)} \sum_{k=0}^{n-1} \mathcal{T}^k f \xrightarrow{n \to \infty} \int_X f dm \text{ a.e. } \forall f \in L^1(m).$$

Pointwise dual ergodicity entails

- **conservativity** (aka recurrence—no non-trivial wandering sets);
- **ergodicity** (no non-trivial invariant sets) and
- **rational ergodicity** as defined in [Aar97, §3.8] (see also ▲ on p.11).

**Stationary processes and skyscrapers.**

A (discrete) stationary process is a quadruple $(\Omega, \mu, \tau, \phi)$ where $(\Omega, \mu, \tau)$ is a probability preserving transformation and $\phi : \Omega \to \mathbb{K}$ (some metric space) is measurable. The stationary process $(\Omega, \mu, \tau, \phi)$ is called ergodic if $(\Omega, \mu, \tau)$ is an ergodic probability preserving transformation.
For example if \((X, m, T)\) is a conservative measure preserving transformation and
\[ \Omega \in \mathcal{F}_+ := \{ A \in \mathcal{B}(X) : 0 < m(A) < \infty \}, \]
the return time function to \(\Omega\) is \(\varphi = \varphi_\Omega : \Omega \to \mathbb{N}\) defined by \(\varphi(\omega) := \min\{ n \geq 1 : T^n\omega \in \Omega \} < \infty\) a.s. by conservativity.

The induced transformation on \(\Omega\) is \(T_\Omega : \Omega \to \Omega\) defined by \(T_\Omega(\omega) := T^{\varphi(\omega)}(\omega).\) As shown in \([\text{Kak43}]\), \((\Omega, m_\Omega, T_\Omega)\) is a probability preserving transformation which, in case \(X \cong \bigcup_{n \geq 0} T^{-n}\Omega\), is ergodic together with \((X, m, T)\). The \(\mathbb{N}\)-valued stationary process \((\Omega, m_\Omega, T_\Omega, \varphi_\Omega)\) is called the return time process to \(\Omega\).

As in \([\text{Kak43}]\), the skyscraper over the \(\mathbb{N}\)-valued stationary process \((\Omega, \mu, \tau, \phi)\) is the conservative measure preserving transformation \((X, m, T) = (\Omega, \mu, \tau)^\phi\) defined by
\[
X := \{ (\omega, n) \in \Omega \times \mathbb{N} : 1 \leq n \leq \varphi(\omega) \}, \quad m := \mu \times \#|_X \ &
T(\omega, n) := \left\{ \begin{array}{ll}
(\omega, n + 1) & n < \varphi(\omega) \\
(\tau(\omega), 1) & n = \varphi(\omega).
\end{array} \right.
\]

If \((X, m, T)\) is a conservative, ergodic, measure preserving transformation, \(\Omega \in \mathcal{B}(X), 0 < m(\Omega) < \infty\), then \((X, m, T)\) is a factor of \((\Omega, m_\Omega, T_\Omega)^\varphi\) (isomorphic if \((X, m, T)\) is invertible) and the return time process of \((\Omega, m_\Omega, T_\Omega)^\varphi\) to \(\Omega \times \{1\}\) is isomorphic with \((\Omega, \mu, \tau, \phi)\).

**Functional distributional limits.** Let \((X, m, T)\) be a pointwise dual ergodic measure preserving transformation with \(\gamma\)-regularly varying return sequence \(a(n) = a_\gamma(T) (0 < \gamma < 1)\).

By the functional Darling-Kac Theorem ([Bin71, OS15]),
\[
\Psi_{n, f} \xrightarrow{n \to \infty} m(f)m_\gamma \text{ in } (D_{1,\infty}, J_1) \forall f \in L_1^\gamma := \{ g \in L^1(m) : g \geq 0, \int_X gdm > 0 \}
\]
where \(\Psi_{n, f}(t) := \frac{S_{[nt]}(f)}{a(n)}\) with \(S_n(f) := \sum_{k=0}^{n-1} f \circ T^k\) is the normalized Birkhoff-sum step function and where \(\Rightarrow\) in \(\Xi^\gamma\) means distributional convergence in the metric space \(\Xi\) with respect to all \(m\)-absolutely continuous probabilities as obtained by Eagleson’s theorem ([Eag76, TZ06]).

By [TK10a, TK10b], if \((\Omega, \mu, \tau, \phi)\) is an exponentially continued fraction mixing, \(\mathbb{R}_+\)-valued stationary process, that is for some \(K^\gamma > 0, \theta \in (0, 1),\)
\[
|\mu(a \cap \tau^{-(n+k)}B) - \mu(a)\mu(B)| \leq K^\gamma \theta^n \mu(a)\mu(B) \quad \forall n, k \geq 1, a \in \sigma(\{ \phi \sigma^j \}^{k-1}_0), \quad B \in \mathcal{B}(\Omega);
\]
\[

\text{The return time process to } \Omega.
\]
\[
\text{As shown in } [\text{Kak43}], \(\Omega, m_\Omega, T_\Omega)\text{ is a probability preserving transformation which, in case } X \cong \bigcup_{n \geq 0} T^{-n}\Omega, \text{ is ergodic together with } (X, m, T). \text{ The } \mathbb{N}\text{-valued stationary process } (\Omega, m_\Omega, T_\Omega, \varphi_\Omega) \text{ is called the return time process to } \Omega. \]
\[
\text{If } (X, m, T) \text{ is a conservative, ergodic, measure preserving transformation, } \Omega \in \mathcal{B}(X), 0 < m(\Omega) < \infty, \text{ then } (X, m, T) \text{ is a factor of } (\Omega, m_\Omega, T_\Omega)^\varphi \text{ (isomorphic if } (X, m, T) \text{ is invertible) and the return time process of } (\Omega, m_\Omega, T_\Omega)^\varphi \text{ to } \Omega \times \{1\} \text{ is isomorphic with } (\Omega, \mu, \tau, \phi). \]
\[
\textbf{Functional distributional limits.} \text{ Let } (X, m, T) \text{ be a pointwise dual ergodic measure preserving transformation with } \gamma\text{-regularly varying return sequence } a(n) = a_\gamma(T) (0 < \gamma < 1). \]
\[
\text{By the functional Darling-Kac Theorem ([Bin71, OS15]),}
\]
\[
\Psi_{n, f} \xrightarrow{n \to \infty} m(f)m_\gamma \text{ in } (D_{1,\infty}, J_1) \forall f \in L_1^\gamma := \{ g \in L^1(m) : g \geq 0, \int_X gdm > 0 \}
\]
where \(\Psi_{n, f}(t) := \frac{S_{[nt]}(f)}{a(n)}\) with \(S_n(f) := \sum_{k=0}^{n-1} f \circ T^k\) is the normalized Birkhoff-sum step function and where \(\Rightarrow\) in \(\Xi^\gamma\) means distributional convergence in the metric space \(\Xi\) with respect to all \(m\)-absolutely continuous probabilities as obtained by Eagleson’s theorem ([Eag76, TZ06]).

By [TK10a, TK10b], if \((\Omega, \mu, \tau, \phi)\) is an exponentially continued fraction mixing, \(\mathbb{R}_+\)-valued stationary process, that is for some \(K^\gamma > 0, \theta \in (0, 1),\)
\[
|\mu(a \cap \tau^{-(n+k)}B) - \mu(a)\mu(B)| \leq K^\gamma \theta^n \mu(a)\mu(B) \quad \forall n, k \geq 1, a \in \sigma(\{ \phi \sigma^j \}^{k-1}_0), \quad B \in \mathcal{B}(\Omega);
\]
and \( \frac{1}{\mu([0, t])} \) is \( \gamma \)-regularly varying with \( 0 < \gamma < 1 \), then with
\[
a(t) := \frac{1}{\Gamma(1 - \gamma)\Gamma(1 + \gamma)\mu([\phi > t])},
\]
\( \Phi_n \xrightarrow{n \to \infty} \eta_\gamma \) in \((D_{1, \infty}, J_1)\)
where \( \Phi_n(t) := \frac{\phi_{[\xi_0, t]}(\xi)}{\phi_{[\xi_0, t]}(\xi)} \) with \( \phi_k := \sum_{j=0}^{k-1} \phi \circ \tau^j \).

In the sequel, we’ll prove tied down versions of (\( \clubsuit \)) for conservative, ergodic, measure preserving transformations having (among other properties) a return time process satisfying (\( \spadesuit \)).

§1 Results

Tied down Mittag-Leffler processes.

The following propositions give characterizations of \( w_\gamma \) defined by (\( \heartsuit \)) (on p. 6):

**Proposition C** (characterization of \( w_\gamma \))

For \( \gamma \in (0, 1) \),
\[
\mathbb{E}(h(w_\gamma)) = \mathbb{E}(\int_0^1 h(\Delta_{t, \gamma} m_\gamma) dm_\gamma(t))
\]
\( \forall \ h \in C_B(D_{1,1}) := C_B(D_{1,1}, J_1) \).

It follows by choosing \( h(\xi) = H(\xi(1)) \) \((H \in C_B(\mathbb{R}))\) in (\( \heartsuit \)) that
\[
\mathbb{E}(H(w_\gamma(1))) = \mathbb{E}(m_\gamma H(m_\gamma(1))) \quad \text{for } H \in C_B(\mathbb{R}).
\]

**Proposition D** (FDDs of \( w_\gamma \))

For \( g_\xi : D_{t,1} \to \mathbb{R}_+ \) a product function \(^2\) of form
\[
g_\xi(\xi) = g(\xi) h(\xi(1) - \xi(t_N)) \text{ where } g(\xi) := \prod_{\nu=1}^{N} g_\nu(\xi(t_\nu))
\]
with \( 0 \leq t_1 < t_2 < \cdots < t_N < 1 \) and \( g_1, \ldots, g_N, h \in C([0, \infty])_+ \),
\[\mathbb{E}(g_\xi(w_\gamma)) = e_\gamma(g_\xi)
\]
\[
:= \mathbb{E}\left( g(m_\gamma) 1_{[D(m_\gamma)(t_N) \leq 1]}(1 - D(m_\gamma)(t_N))^1 - \gamma h(1 - D(m_\gamma)(t_N))^{-\gamma} W_\gamma \right)
\]
where \( W_\gamma \perp m_\gamma \) and \( W_\gamma \overset{d}{=} w_\gamma(1) \).

\(^2\)see (\( \heartsuit \)) on p. 4
Examples: Renewal shifts.
Write $\mathcal{P}(\mathbb{N}) := \{\text{probability measures on } \mathbb{N}\}$.
For $f \in \mathcal{P}(\mathbb{N})$, the associated Bernoulli map is
$$(\Omega, p, S) := (\mathbb{N}^\mathbb{N}, f^\mathbb{N}, \text{shift}).$$
The renewal shift corresponding to $f$ is the Kakutani skyscraper
$$(X, m, T) := (\Omega, p, S)^\phi$$
where $\phi : \Omega \to \mathbb{N}$ is defined by $\phi(x) := x_1$.
The base of the skyscraper $(\Omega, p, S)^\phi$ is $\tilde{\Omega} := \Omega \times \{1\} \in \mathcal{B}(X)$. It is aka the recurrent event of the renewal shift and the corresponding distribution $f$ is known as its lifetime distribution.

The renewal sequence $(u(n) : n \geq 0)$ corresponding to $f \in \mathcal{P}(\mathbb{N})$ is
$$u(n) = u_f(n) := m(\tilde{\Omega} \cap T^{-n} \tilde{\Omega})$$
and satisfies $u(n) = \sum_{k=1}^{n} f_k u(n-k)$ where (here and throughout) $f_k := f(\{k\})$.
The renewal shift $(X, m, T)$ is a conservative, ergodic Markov shift, whence pointwise dual ergodic with return sequence $a_n(T) = a(n) = a_f(n) := \sum_{k=1}^{n} u(k)$.

Now let $\gamma \in (0, 1)$ and let $f \in \mathcal{P}(\mathbb{N})$, then by Karamata’s theorems
$$c(n) := p([\phi \geq n]) = \sum_{k \geq n} f_k$$
is $(-\gamma)$-regularly varying iff $a(n)$ is $\gamma$-regularly varying and in this case, $a(n) \sim \frac{1}{c(n)}^3$

The Strong Renewal Theorem
Let $f \in \mathcal{P}(\mathbb{N})$ and let $a(n)$ be $\gamma$-regularly varying with $\gamma \in (0, 1)$.

(i) ([GL63, CD19]) If $f$ is non-arithmetic and $a(n) \gg \sqrt{n}$, then
$$u(n) \sim \frac{\gamma a(n)}{n}.$$

(ii) ([CD19]) For each $\gamma$-regularly varying $a(n) \gg \sqrt{n}$, $\exists f \in \mathcal{P}(\mathbb{N})$
aperiodic with $a_f(n) \sim a(n)$ and for which (SRT) fails.

(iii) ([Don97, Thm. B] – see also [Gou11])
If $f$ is aperiodic and $f_n \ll \frac{c(n)}{n}$, then (SRT) holds.

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3 Here and throughout, for $a_n, b_n > 0$, $a_n \sim b_n$ means $\frac{a_n}{b_n} \sim c \in \mathbb{R}_+$.

4 Here and throughout, for $a_n, b_n > 0$, $a_n \ll b_n$ means $\lim_{n \to \infty} \frac{a_n}{b_n} < \infty$ and $a_n \sim b_n$
means $\lim_{n \to \infty} \frac{a_n}{b_n} \rightarrow 1$. 
For \((X, m, T) = (\Omega, p, S)\) a renewal shift, write
\[ s_n := S_n(1_\Omega) \quad \text{and} \quad \psi_n(t) := \frac{s_{nt}}{a(n)}. \]

**Proposition E**

Let \((X, m, T) = (\Omega, p, S)\) be a renewal shift with aperiodic lifetime distribution \(f \in \mathcal{P}(\mathbb{N})\) so that \(a(n)\) is \(\gamma\)-regularly varying with \(\gamma \in (0, 1)\) and satisfying \((\text{SRT})\), then for \(g \in C_B(D_{1,1})\),
\[
\mathbb{E} \left( \frac{1}{\nu(N)} \int_{\Omega \cap T^{-N} \Omega} g(\psi_N) \, dm \right) \xrightarrow[N \to \infty]{} \mathbb{E}(g(\nu)).
\]

Proposition E is a strengthening of the Strong Renewal Theorem as in [GL63, Don97, Gou11, CD19] and is a special case of Theorem U below on p.12.

For the general, renewal shift with regularly varying \(a(n)\), we have weighted Cesaro versions of \(\mathbb{E}\) as in Theorem U, \(\mathbb{F}\) & \(\mathbb{G}\).

**Gibbs-Markov sets and cylinders.**

A **Markov map** is a quarpule \((Y, \nu, \tau, \alpha)\) where \((Y, \nu, \tau)\) is a non-singular transformation of the Polish probability space \((Y, \nu)\) and \(\alpha \subset \mathcal{B}(Y)\) is a countable partition so that for each \(a \in \alpha\), \(\tau a \in \sigma(\alpha)\) and \(\tau: \alpha \to \tau a\) is nonsingular, invertible.

It follows that if \((Y, \nu, \tau, \alpha)\) is a Markov map, then so is \((Y, \nu, \tau^n, \alpha_n) \forall n \geq 1\) where \(\alpha_n := \bigvee_{k=0}^{n-1} \tau^{-k} \alpha\).

The transfer operators \(\tau^n: L^1(\nu) \leftrightarrow (n \geq 1)\) are given by
\[
\tau^n f = \sum_{a \in \alpha_n} 1_{\tau^n a} v'_a f \circ v_a
\]
where for \(a \in \alpha_n\), \(v_a : \tau^n a \to a\), \(v_a \circ \tau = \text{Id}\) on \(a\) and \(v'_a := \frac{dv_{\nu a}}{dv_{\nu}}\).

As in [AD01] (also [AS19]), a **Gibbs-Markov map** is a Markov map \((Y, \nu, \tau, \alpha)\) so that \(\inf_{a \in \alpha} \nu(\tau a) > 0\) and, for some \(\theta \in (0, 1)\)
\[
\sup_{n \geq 1, \ a \in \alpha_n, \ x, y \in \tau^n a} \frac{1}{\theta^{t(x, y)}} \left| \frac{v'_a(x)}{v'_a(y)} - 1 \right| < \infty
\]
where \(t(x, y) = \min\{n \geq 1: \alpha_n(x) \neq \alpha_n(y)\} \leq \infty\) with \(x \in \alpha_n(x) \in \alpha_n\).

If \((Y, \nu, \tau, \alpha)\) is a Gibbs-Markov map, then so is \((Y, \nu, \tau^n, \alpha_n) \forall n \geq 1\).

As in [AD01] (also [AS19]), the transfer operator \(\mathcal{T}\) acts quasiconactly on the space of \((\alpha, \theta)\)-Hölder functions on \(Y\); that is
\[
L_{\alpha, \theta} := \{ f: Y \to \mathbb{R}: D_{Y, \theta, \alpha}(f) := \sup_{x, y \in \Omega} \frac{|f(x) - f(y)|}{\theta^{t(x, y)}} < \infty \}
\]
equipped with the norm \(\|f\|_{L_{\alpha, \theta}} := \|f\|_1 + D_{Y, \theta, \alpha}(f)\).
Let \((X, m, T)\) be a conservative, ergodic, measure preserving transformation. We’ll call the set \(\Omega \in \mathcal{B}(X), \ 0 < m(\Omega) < \infty\) a Gibbs-Markov set if there is a countable partition \(\alpha \subset \mathcal{B}(\Omega)\) so that \((\Omega, m_\Omega, T_\Omega, \alpha)\) is a mixing Gibbs-Markov map and the first return time \(\varphi_\Omega: \Omega \to \mathbb{N}\) is \(\alpha\)-measurable. In this case, a \((n, \alpha)\)-cylinder \((n \geq 1)\) is a set of form

\[
a = \bigcap_{j=0}^{n-1} T_\Omega^{-j} a_j
\]

where \(a_0, \ldots, a_{n-1} \in \alpha\). The collection of \((n, \alpha)\)-cylinders is \(\alpha_n := \bigvee_{j=0}^{n-1} T_\Omega^{-j} \alpha\) and the collection of cylinders is

\[
\mathcal{C}_\alpha := \bigcup_{n=1}^{\infty} \alpha_n.
\]

**Ergodic functional tied-down renewal properties.**

These (i.e. \(\clubsuit\) and \(\heartsuit\) below) are functional analogues of the tied-down renewal mixing properties considered in [AS19].

If the conservative, ergodic, measure preserving transformation \((X, m, T)\) has a Gibbs-Markov set \(\Omega\), then it is pointwise dual ergodic, whence rationally ergodic (with the same return sequence); and weakly mixing iff \(\varphi_\Omega\) is non-arithmetic with respect to \(T_\Omega\). It is standard that these latter properties lift to the natural extension.

By [Aar77, Proposition 1.1], if \((X, m, T)\) is a rationally ergodic measure preserving transformation, then there is a dense, \(T\)-invariant, hereditary ring \(R(T) \subset \mathcal{F}_+\) (as on p. 7) and a sequence \(a(n) = a_n(T) > 0\) (the return sequence) so that

\[
\frac{1}{a(n)} \sum_{k=0}^{n-1} m(A \cap T^{-k} B) \xrightarrow{n \to \infty} m(A) m(B) \forall A, B \in R(T).
\]

A measure preserving transformation \((X, m, T)\) satisfying \(\clubsuit\) is called (in [Aar77]) weakly rationally ergodic.

Our main results are integrated, functional versions of [AS19, Theorem A].

In the following, \(\Psi_n(f) := \frac{s_{\text{att}}(f)}{a(n)} \ (f \in L^1_+)\).

**Theorem U**

- Suppose that \((X, m, T)\) is a pointwise dual ergodic conservative, ergodic, measure preserving transformation with \(a(n) = a_n(T) \gamma\)-regularly varying with \(0 < \gamma < 1\), then
with \( u(n) := \frac{\gamma(n)}{n} \),

\[
\frac{1}{a(n)} \sum_{k=1}^{n} \int_A g(\Psi_{k,f}) 1_B \circ T^k dm \xrightarrow{n \to \infty} m(A)m(B)\mathbb{E}(g(m(f)w_\gamma))
\]

\[\forall A, B \in R(T), \ f \in L^1_+, \ g \in C_B(D_{t,1}). \]

- If, in addition, \((X, m, T)\) is weakly mixing and has a Gibbs-Markov set \(\Omega \in \mathcal{F}_+\), then

\[
\frac{1}{a(n)} \sum_{k=1}^{n} |\tilde{T}^n(1_A g(\Psi_{k,f})) - u(k)m(A)\mathbb{E}(g(m(f)w_\gamma))| \xrightarrow{n \to \infty} 0 \quad \text{a.e.} \quad \forall A \in \mathcal{F}_+, \ f \in L^1_+, \ g \in C_B(D_{t,1}).
\]

- If, in addition \((\Omega, m_\Omega, T_\Omega, \alpha)\) satisfies

\[
(\text{OSRT}) \quad \frac{1}{u(n)} \tilde{T}^n(1_A) \xrightarrow{n \to \infty} m(A) \text{ uniformly on } \Omega \quad \forall A \in C_\alpha,
\]

then

\[
\frac{1}{u(n)} \tilde{T}^n(1_A g(\Psi_{n,1\Omega})) \xrightarrow{n \to \infty} \mathbb{E}(g(w_\gamma))m(A) \quad \text{uniformly on } \Omega
\]

\[\forall A \in C_\alpha, g \in C_B(D_{t,1}). \]

Remarks

(i) It is standard to show that if a non-invertible conservative, ergodic, measure preserving transformation satisfies \(\triangleright\), then so does its natural extension. Also, if \(T\) satisfies \(\triangleright\), then \(T\) and its natural extension also satisfy

\[
\frac{1}{a(n)} \sum_{k=1}^{n} |\int_{A_{\mathbb{R}(T)_k}} g(\Psi_{k,f}) dm - u(k)m(A)m(B)\mathbb{E}(g(m(f)w_\gamma))| \xrightarrow{n \to \infty} 0 \quad \forall A, B \in R(T), \ f \in L^1_+, \ g \in C_B(D_{t,1}).
\]

(ii) The condition \((\text{OSRT})\) for \((\Omega, m_\Omega, T_\Omega, \alpha)\) was shown in [MT12] for \(\frac{1}{2} < \gamma < 1\) (with no further assumptions) and in [Gou11] for \(0 < \gamma < 1\) under the additional assumption that the \(\alpha\)-measurable return time function \(\varphi : \Omega \to \mathbb{N}\) is aperiodic and \(m_\Omega(\lfloor \varphi = n \rfloor) \ll \frac{m_\Omega(\lfloor \varphi \geq n \rfloor)}{n}\).

Examples: Intermittent interval maps.

An intermittent interval map satisfying Thaler’s conditions as in [Tha83]:

- admits an absolutely continuous invariant measure with density continuous away from the indifferent fixed points ([Tha80]);
- admits wandering rates ([Tha83, \S3]).
is pointwise dual ergodic and has Gibbs-Markov sets ([Aar86, Theorem 3]). See also [Aar97, Chapter 4]. In particular, each map $T_\gamma : [0,1] \to [0,1]$ ($0 < \gamma < 1$) defined by

$$T_\gamma x = \begin{cases} 
  x(1 + (2x)^{1/\gamma}) , & 0 \leq x < \frac{1}{2}, \\
  2x - 1 , & \frac{1}{2} \leq x \leq 1
\end{cases}$$

satisfies Thaler’s conditions as above and indeed all the assumptions of Theorem U.\(^5\)

Thus if $h_\gamma$ is the $T_\gamma$-invariant density, then $\left( [0,1], \mu_\gamma, T_\gamma \right)$ is pointwise dual ergodic with $d\mu_\gamma(x) = h_\gamma(x) dx$.

It follows from the regularly varying expansion near the indifferent fixed point 0 that the invariant density $h_\gamma$ for $T_\gamma$ satisfies $h_\gamma(x) \propto x^{-1/\gamma}$ as $x \to 0$, whence the wandering rate $L_{T_\gamma}(n) \propto n^{1-\gamma}$ and the return sequence $a_n(T_\gamma) \propto n^\gamma$. Thus the functional Darling-Kac theorem holds.

The set $\Omega = \left( \frac{1}{2}, 1 \right]$ is a Gibbs-Markov set for each $T_\gamma$ with respect to the return time partitions $\alpha_\gamma : = \{ [\varphi_\gamma = n] \cap \Omega : n \geq 1 \}$ (where $\varphi_\gamma(x) : = \min\{ j \geq 1 : T_j^\gamma(x) \in \Omega \}$). Moreover $\mu_\Omega^\gamma([\varphi_\gamma = n]) \propto \frac{1}{n^{1+\gamma}}$ whence $(\Omega, \mu_\Omega^\gamma, (T_\gamma)_\Omega, a)$ satisfies (OSRT). Therefore each $\left( [0,1], \mu^\gamma, T_\gamma \right)$ satisfies all the assumptions of Theorem U.

**Outline of Proofs.**

The proofs of Propositions C, D, E (p.20) and ☠ (p.12) in Theorem U, are interwined and use

- Proposition 4 (p.13) which “converts” a distributional limit to a tied down limit; and
- Lemma 5 (p.17) which gives tied down limits for well behaved renewal shifts.

The statement ☺ in Theorem U will follow (in §3) from the Uniform GL Lemma ((UGL) on p. 21).\(^6\)

§2 **Tied down Mittag-Leffler limits**

**Proposition 4** Let $(X, m, T)$ be a weakly rationally ergodic measure preserving transformation with $\gamma$-regularly varying return sequence $a(n) = a_n(T)$ ($\gamma \in (0,1]$).

\(^5\) and are aka “LSV maps” having been considered in [LSV99] for the different parameters $\gamma > 1$.

\(^6\) “GL” stands for Garsia-Lamperti in honor of [GL63, Theorem 1.1] which inspired (UGL).

\(^7\) as in ☼ on p. 11
Suppose that
\[ \Psi_{n,f} \xrightarrow{\text{d}} m(f)m_\gamma \text{ in } (D_{t,\infty}, J_1) \forall f \in L^1(m) \]
where \( \Psi_{n,f}(t) := \frac{s_{[nt]}(f)}{a(n)} \) and \( m_\gamma \in \text{RV}(D_{t,\infty}) \) is a.s. continuous with \( m_\gamma(0) = 0 \), then
\[ \mathbb{E} h(w(m_\gamma)) = \mathbb{E} (\int_0^1 h(\Delta_{t,\gamma}m_\gamma)dm_\gamma(t)) \forall h \in C_B(D_{t,1}) \].

**Proof** We assume WLOG that \( g = g_t \in \Pi \) (as in \( \Psi \) on p. 4) with \( x \mapsto \log h(e^{x}) \& x \mapsto \log g_t(e^{x}) (1 \leq \nu \leq N) \) uniformly continuous.

1 We first show \( \Delta \) with \( A = \Omega \in R(T) \& f = 1_{\Omega} \). For convenience we assume \( m(\Omega) = 1 \). Fix \( 0 < \varepsilon < 1 \).

Writing \( s_n := S_n(1_\Omega) \& \psi_n(t) := \Psi_{n,1_\Omega}(t) = \frac{s_{[nt]}}{a(n)} \), we have uniformly in \( t \in [\varepsilon, 1] \), with \( k = [nt] \& u \in (0,1) \), that
\[ \psi_k(u) \sim \frac{s_{[ntu]}}{a(n)} \]
\[ = \frac{\psi_n(tu)}{t^\gamma} =: (\Delta_{t,\gamma} \psi_n)(u). \]

Next,
\[ \int_\varepsilon^1 g(\Delta_{t,\gamma} \psi_n) d\psi_n(t) \sim \sum_{\frac{n}{\varepsilon}(\varepsilon,1)}^k \int_\varepsilon^1 g(\Delta_{t,\gamma} \psi_n) d\psi_n(t) \]
\[ \sim \frac{1}{a(n)} \sum_{\frac{n}{\varepsilon}(\varepsilon,1)} g(\psi_k) 1_{\Omega} \circ T^k. \]

For \( \xi \in D_{t,1}, 0 < \varepsilon < 1 \), let
\[ \xi_{g,\varepsilon}(\xi) := \int_\varepsilon^1 g(\Delta_{t,\gamma} \xi) d\xi(t). \]

We claim that
If \( \xi_n, \xi \in D_{t,1} \) with \( \xi \) continuous and \( \xi_n \underset{n \to \infty}{\overset{\text{DF}}{\to}} \xi \), then for \( 0 < \varepsilon < 1 \),
\[ \xi_{g,\varepsilon}(\xi_n) \xrightarrow{n \to \infty} \xi_{g,\varepsilon}(\xi). \]

**Proof of** Fix \( 0 < \varepsilon < 1 \).
Since $\xi$ is continuous, so is each $\Delta_{t,\gamma}\xi \ (t \in (0, 1])$ whence the mapping $t \mapsto \Delta_{t,\gamma}\xi$ is continuous $[E, 1] \rightarrow (D_{t,1}, J_1)$. The collection $\{\Delta_{t,\gamma}\xi : \ t \in [E, 1]\}$ is therefore compact in $(D_{t,1}, J_1)$ and for each $\epsilon > 0$, $\exists \delta(\epsilon) > 0$ so that $z \in D_{1,1}, \ t \in [E, 1]: \ \sup_{s \in [0,1]} |z(s) - \Delta_{t,\gamma}\xi(s)| < \delta(\epsilon) \Rightarrow |g(z) - g(\Delta_{t,\gamma}\xi)| < \epsilon.$

Thus, if $z \in D_{1,1}$ and $\sup_{s \in [0,1]} |z(s) - \xi(s)| < \delta(\epsilon) E^\gamma$, then for each $t \in [E, 1]$, $s \in [0, 1]$, we have $|\Delta_{t,\gamma}z(s) - \Delta_{t,\gamma}\xi(s)| \leq \frac{1}{E^\gamma}|z(st) - \xi(st)| < \delta(\epsilon)$ whence $\sup_{t \in [E, 1]} |g(\Delta_{t,\gamma}z) - g(\Delta_{t,\gamma}\xi)| < \epsilon.$

Thus, since $\xi_n \xrightarrow{n \to \infty} \xi$ uniformly on $[0, 1]$, $\delta(E, n) := \sup_{t \in [E, 1]} |g(\Delta_{t,\gamma}\xi_n) - g(\Delta_{t,\gamma}\xi)| \xrightarrow{n \to \infty} 0$

and $\int_E^1 g(\Delta_{t,\gamma}\xi_n) d\xi_n(t) = \int_E^1 g(\Delta_{t,\gamma}\xi) d\xi_n(t) + \delta(E, n) \xi_n(1).$

Next, since $t \mapsto g(\Delta_{t,\gamma}\xi)$ is continuous $[E, 1] \rightarrow \mathbb{R}$, $\int_E^1 g(\Delta_{t,\gamma}\xi) d\xi_n(t) \xrightarrow{n \to \infty} \int_E^1 g(\Delta_{t,\gamma}\xi) d\xi(t).$

Thus $\mathcal{L}_{g,\xi}(\xi_n) \xrightarrow{n \to \infty} \mathcal{L}_{g,\xi}(\xi).$ \hspace{1cm}$\Box$

In view of $\mathcal{E}^\gamma$, we now have by Skorokhod’s Representation Theorem ([Bil99, Thm. 6.7]) and $\mathcal{E}^\gamma$ that $\forall \ 0 < \mathcal{E} < 1$, in $\mathcal{W}(D_{1,1}, J_1)$, $\frac{1}{a(n)} \sum_{n \in (E,1)}^\frac{1}{a(n)} g(\psi_k) 1_T \circ T^k \sim \mathcal{L}_{g,\xi}(\psi_n) \xrightarrow{n \to \infty} \mathcal{L}_{g,\xi}(m_\gamma).$

To complete the proof of $\mathcal{L}1$, we must establish the limits as $\mathcal{E} \rightarrow 0$. To this end we claim that $\mathbb{E}(m_\gamma(1)) \leq 1$. Indeed

$$1 \xrightarrow{n \to \infty} \int_\Omega^\frac{a_n}{a(n)} dm : \ \Omega \in R(T), \ m(\Omega) = 1$$

$$\geq \int_\Omega^\frac{a_n}{a(n) \wedge M} dm \ \forall \ M > 0,$$

$$\xrightarrow{n \to \infty} \mathbb{E}(m_\gamma(1) \wedge M) \xrightarrow{M \uparrow \infty} \mathbb{E}(m_\gamma(1)).$$

In view of this, since $m_\gamma$ is a.s. continuous & $m_\gamma(0) = 0$, by dominated convergence $\mathbb{E}(m_\gamma(E)) \xrightarrow{E \rightarrow 0^+} 0,$
whence
\[ \mathbb{E} \left( \int_0^\varepsilon g(\Delta_t \gamma_m) \, dm_t \right) \leq \|g\|_{C_B} \mathbb{E}(m_\gamma(\mathcal{E})) \xrightarrow{\varepsilon \to 0^+} 0. \]

Moreover, by the \( \gamma \)-regular variation of \( a(n) \),
\[ \frac{1}{a(n)} \sum_{k \in (0, \varepsilon)} \int_\Omega g(\psi_k) 1_{\Omega} \circ T^k \, dm \leq \|g\|_{C_B} \frac{a(n \varepsilon)}{a(n)} \sim \varepsilon \gamma \|g\|_{C_B}. \]

This proves \( \triangledown \) with \( A = \Omega \in R(T) \) & \( f = 1_\Omega \). \[ \triangledown \]

\[ \triangledown \] Next, we prove \( \triangledown \) for any \( A \), \( \Omega \in R(T) \) with \( f = 1_\Omega \).

Define \( G_N : \Omega \to \mathbb{R} \) by
\[ G_N := \frac{1}{a(N)} \sum_{k=1}^N g(\psi_k) 1_{\Omega} \circ T^k, \]
then
- \( \int_\Omega G_N \, dm \xrightarrow{N \to \infty} \mathbb{E}(g(w(m_\gamma))) \) by \( \triangledown \);
- \( G_N \leq \|g\|_{C_B} \frac{a(N \varepsilon)}{a(N)} \), whence \( \{G_N : N \geq 1\} \) is weakly, sequentially compact in \( L^1(F) \) \( \forall \ F \in R(T) \); and
- \( G_N - G_N \circ T \xrightarrow{N \to \infty} 0. \)

Fix \( F \in R(T) \), \( F \supset \Omega \cup A \). Suppose that \( G_{N_k} \xrightarrow{k \to \infty} H \) weakly in \( L^1(F) \), then \( H = H \circ T \) whence by ergodicity, \( H \) is constant and
\[ H = \int_\Omega H \, dm \leftarrow \int_\Omega G_{N_k} \, dm \xrightarrow{k \to \infty} \mathbb{E}(g(w(m_\gamma))). \]

Thus
\[ G_N \xrightarrow{N \to \infty} \mathbb{E}(g(w(m_\gamma))) \] weakly in \( L^1(F) \)
which establishes \( \triangledown \).

\[ \triangledown \] To finish, we show \( \triangledown \) for any \( A \), \( \Omega \in R(T) \) and \( f \in L^1(m)_+. \)

**Proof of \( \triangledown \)**

Fix \( A \), \( \Omega \in R(T) \) & \( f \in L^1(m)_+. \)

By the ratio ergodic theorem and Egorov’s theorem,
\[ \exists A_0 \in \mathcal{B}(A), \ A_0 \uparrow A \text{ so that for each } \nu \geq 1, \]
\[ \frac{S_\nu(f)}{s_\nu} \xrightarrow{\nu \to \infty} \frac{m(f)}{m(\Omega)} \text{ uniformly on } A_\nu. \]

\[ \frac{a_n}{b_n} \leq 1. \]
whence
\[
\sum_{k=1}^n 1_{\Omega} \circ T^k g(\Psi_{n,f}) \sim \sum_{k=1}^n 1_{\Omega} \circ T^k g\left(\frac{m(f)}{m(\Omega)} \psi_n\right) \text{ uniformly on } A_\nu.
\]
Thus
\[
\frac{1}{a(n)} \sum_{k=1}^n \int_{A_\nu \cap T^{-k} \Omega} g(\Psi_{n,f}) \, dm \approx \frac{1}{a(n)} \sum_{k=1}^n \int_{A_\nu \cap T^{-k} \Omega} g\left(\frac{m(f)}{m(\Omega)} \psi_n\right) \, dm \longrightarrow_{n \to \infty} m(A_\nu) m(\Omega) \mathbb{E}(g(m(f) \psi(\omega)))
\]
9 by \(\Psi\) and
\[
\left| \frac{1}{a(n)} \sum_{k=1}^n \int_{A \cap T^{-k} \Omega} g(\Psi_{n,f}) \, dm - m(A) m(\Omega) \mathbb{E}(g(m(f) \psi(\omega))) \right| \\
\lesssim \left\| g \right\|_{C_B} \left( \frac{1}{a(n)} \sum_{k=1}^n m(A \setminus A_\nu \cap T^{-k} \Omega) + m(A \setminus A_\nu) m(\Omega) \right) \\
\longrightarrow_{n \to \infty} 2 \left\| g \right\|_{C_B} m(A \setminus A_\nu) m(\Omega) \\
\longrightarrow_{\nu \to \infty} 0.
\]
This completes the proof of \(\Delta\) and Proposition 4. \(\Box\)

**Lemma 5**

Let \((X, m, T) = (\Omega, p, S)^{\phi}\) be a renewal shift with aperiodic lifetime distribution \(f \in \mathcal{P}(\mathbb{N})\) so that \(a(n)\) is \(\gamma\)-regularly varying with \(\gamma \in (0, 1)\) and satisfying (SRT) on p. 9, then
for \(g_L \in \Pi\) a product function as in \(\mathfrak{U}\) (p. 4),
\[
\exists \quad \frac{1}{u(N)} \int_{\Omega \cap T^{-N} \Omega} g_L(\psi_N) \, dm \longrightarrow_{N \to \infty} \mathbf{e}_\gamma(g_L).
\]
where \(\mathbf{e}_\gamma(g_L)\) denotes the RHS of \(\mathfrak{U}\) on p. 8.

**Continuity of functionals.**

The proofs below need [Ser20, Section 5]:

For \(\Phi = \mathcal{I}, \mathcal{G} \circ \mathcal{I}, \mathcal{D} \circ \mathcal{I}\) (as on p. 5),
\(\circ\) If \(\xi_n \overset{J_1}{\longrightarrow} \xi, \xi(0) = 0, \xi\) strictly increasing & \(\xi(\infty-\infty) = \infty,\)
then \(\Phi(\xi_n) \overset{J_1}{\longrightarrow} \Phi(\xi).\)

\(\text{9} \)where \(a_n \approx b_n\) means \(a_n - b_n \longrightarrow 0\)
Proof of Lemma 5.

Let
\[ \Pi_0 := \{ g_\pi \in \Pi : \ N \geq 1, \ \tau \in \mathbb{Q}_+^N, \ g_1, g_2, \ldots, g_N, h \in \mathcal{A} \} \]
with \( g_\pi \) & \( \Pi \) as in \( \mathcal{U} \) (p.4), and where \( \mathcal{A} \subset C([0, \infty], [0, 1]) \) is countable, dense, \( \mathbf{1} \in \mathcal{A} \) and \( g \in \mathcal{A} \Rightarrow 1 - g \in \mathcal{A} \).

We'll first establish that
\[ \lim_{N \to \infty} \frac{1}{u(N)} \int_{\Omega \cap T^{-n} \Omega} g_\pi(\psi_N) \, dm \geq c_\gamma(g_\pi) \ \forall \ g_\pi \in \Pi_0. \]
To this end, define \( Z_n(x) := \min \{ k \geq n + 1 : T^k x \in \Omega \} \), then
\[ D(\psi_n)(t) = \frac{1}{n} Z_{[nt]}. \]
By [TK10a], \( \psi_n \xrightarrow{\text{RV}(D_{1, \infty}, J_1)} \mathcal{A}_\gamma^{-1} \), whence by \( \ominus \) on p. 17:
\[ (\psi_n, D(\psi_n)(t_N)) \xrightarrow{n \to \infty} (\mathcal{A}_\gamma, D(\mathcal{A}_\gamma)(t_N)) \]
Next, writing \( \kappa := [nt_N] \),
\[ 1_{\Omega} g_\pi(\psi_n) 1_{\Omega} \circ T^n = \sum_{k=0}^n 1_{\Omega} g_\pi(\psi_n) 1_{[Z_n=k]} 1_{\Omega} \circ T^n = \sum_{k=0}^n \int_{\Omega} g_\pi(\psi_n) 1_{[Z_n=k]} 1_{\Omega} \circ T^n. \]
Fix \( \lambda \in (t_N, 1) \). Using the Markov property, we have
\[ \int_{\Omega \cap T^{-n} \Omega} g_\pi(\psi_n) \, dm = \sum_{k=0}^n \int_{\Omega} g_\pi(\psi_n) 1_{[Z_n=k]} \, dm \int_{\Omega} h(\frac{s_{n-k} \circ T^k}{a(n)}) 1_{\Omega} \circ T^{n-k} \, dm \geq \sum_{k+1 \leq k \leq n \lambda} \int_{\Omega} h(\frac{s_{n-k} \circ T^k}{a(n)}) \, dm =: I_n(\lambda). \]
To establish \( \dagger \), we'll show that \( \exists \ L : (t_N, 1) \to \mathbb{R}_+ \) so that
\[ L \xrightarrow{u(n)} L(\lambda) \xrightarrow{\lambda \to 1} c_\gamma(g_\pi). \]

Proof of \( \ddagger \)
By (SRT) on p. 9, \( m(\Omega \cap T^{-n} \Omega) = u(n) \sim \frac{\gamma a(n)}{n} \), whence by [AS19, lemma 2.1] and the “remarks about mixing” there,
\[ \int_{\Omega \cap T^{-n} \Omega} h(\frac{S_n(h)}{a(n)}) \, dm = \int_{\Omega \cap T^{-n} \Omega} h(\psi_n(1)) \, dm \sim \mathbb{E}(h(W_n)) \cdot u(n). \]
For $\rho > 1$, fix $K = K_\rho \geq 1$ so that
\[
\int_{\tilde{\Omega} \cap T^{-N}_{-}\tilde{\Omega}} h(\psi_N(1)) \, dm = \rho^{+1} E(h(W_\gamma)) \cdot u(N) \quad \forall \, N \geq K,
\]
then for fixed $\lambda \in (t_N, 1)$ & $n \geq \frac{K}{1-\lambda}$,
\[
I_n(\lambda) = \rho^{+1} \sum_{\kappa+1 \leq k \leq n\lambda} \int_{\tilde{\Omega}} g(\psi_n)1_{[Z_n=k]} dm \cdot E(h((1 - \frac{k}{n})^\gamma W_\gamma))u(n-k)
\]
\[
= \rho^{+1} \sum_{\kappa+1 \leq k \leq n\lambda} \int_{\tilde{\Omega}} g(\psi_n)1_{[Z_n=k]} dm \cdot (\frac{1}{1-\frac{k}{n^\gamma}})^{1-\gamma} E(h((1 - \frac{k}{n})^\gamma W_\gamma))u(n)
\]
for large $n$ with increased $\lambda$.

By $\natural$, on p. 18,
\[
\int_{\tilde{\Omega}} g(\psi_n)1_{[Z_n \leq n\lambda]}(\frac{1}{1-D(\psi_n)(t_N)})^{1-\gamma} \cdot E(h((1 - D(\psi_n)(t_N))^\gamma W_\gamma)) \, dm
\]
\[
\xrightarrow{n \to \infty} \underbrace{E(g(m_\gamma)1_{[D(m_\gamma)(t_N) \leq \lambda]}(\frac{1}{1-D(m_\gamma)(t_N)})^{1-\gamma} h((1 - D(m_\gamma)(t_N))^\gamma W_\gamma))}_{=: L(\lambda)}
\]
establishing $\natural$. $\Box$.

**Proof of $\natural$**

Evidently $\natural$ (on p. 18) holds $\forall \ g \in \Pi_1$ where
\[
\exists \Pi_1 := \{ \sum_{m=1}^M g^{(m)} : M \geq 1, \ g^{(1)}, g^{(2)}, \ldots g^{(M)} \in \Pi_0 \}
\]
with $\Pi_0$ as in $\natural$ (p.18) and where $e_\gamma(\sum_{m=1}^M g^{(m)}) := \sum_{m=1}^M e_\gamma(g^{(m)})$.

Moreover, $g_{\xi} \in \Pi_0 \Rightarrow 1 - g_{\xi} \in \Pi_1$ because, for $g_{\xi} \in \Pi_0$ and $\xi \in D_{1,1}$,
\[
1 - g_{\xi}(\xi) = 1 - \prod_{j=1}^N g_j \cdot h = \sum_{k=1}^N (1 - g_k) \prod_{k<j \leq N} g_j h + 1 - h
\]
with each $g_k := g_k(\xi(t_k))$, $1 - g_k$, $h := h(\xi(1) - \xi(t_N))$, $1 - h \in A$.

For $g_{t} \in \Pi_0$, we have on $\tilde{\Omega}$:
\[
e_\gamma(g_{t}) \leq \frac{1}{u(n)} \bar{T}_n^\gamma(1_{\tilde{\Omega}}g_{\xi}(\psi_n)) \quad \text{by $\natural$}
\]
\[
= \frac{1}{u(n)} \bar{T}_n^\gamma(1_{\tilde{\Omega}}) - \frac{1}{u(n)} \bar{T}_n^\gamma(1_{\tilde{\Omega}}(1 - g_{t}(\psi_n)))
\]
\[
= 1 - \frac{1}{u(n)} \bar{T}_n^\gamma(1_{\tilde{\Omega}}(1 - g_{t}(\psi_n))) \quad \text{by $\natural$}
\]
\[
\leq 1 - e_\gamma(1 - g_{t}) \quad \text{by $\natural$}
\]
\[
= e_\gamma(g_{t}).
\]

This proves $\natural$ and Lemma 5. $\Box$.
Proposition E follows from Lemma 5 and Proposition D.

**Proof of Propositions C and D**

Fix $\gamma \in (0, 1)$. Let, as in Lemma 5, $(X, m, T) = (\Omega, p, S)^\phi$ be a renewal shift with aperiodic lifetime distribution $f \in \mathcal{P}(\mathbb{N})$ so that $a(n)$ is $\gamma$-regularly varying with $\gamma \in (0, 1)$ and which satisfies (SRT).

Let

$$
\psi_n(t) := \frac{s_{[n]}(n)}{a(n)}, \quad s_n := S_n(1_{\bar{\Omega}}), \quad \text{and} \quad z_n := \max\{k \leq n : T^k x \in \bar{\Omega}\},
$$

then $\frac{z_n}{n} = G(\psi_n)(1)$.

By [TK10a, TK10b], (see on p. 8),

$$
\psi_n^{-1} \xrightarrow{\text{RV}(D_{t, \varpi}, J_1)} \eta_\gamma = m_\gamma^{-1}.
$$

Thus by $\bigcirc$ on p. 17, $(\psi_n, \frac{z_n}{n}) \xrightarrow{\text{RV}(D_{t, \varpi}, J_1 \times \mathbb{R}_+)} (m_\gamma, G(m_\gamma)(1))$ whence

$$
(\Delta \frac{z_n}{n}, \psi_n)_{1\leq n}[\frac{z_n}{n} > 0] \xrightarrow{\text{RV}(D_{t, 1}, J_1)} \Delta g(m_\gamma)(1)\gamma m_\gamma =: w_\gamma
$$

and for for $g_L \in \Pi_0$ as in $\bullet$ on p. 18,

$$
\mathbb{E}(g_L(m_\gamma)) \xrightarrow{n \to \infty} \int_{\bar{\Omega} \cap [\varphi \leq n]} g_L(\Delta \frac{z_n}{n}, \gamma \psi_n)dm
$$

$$
= \sum_{k=1}^n \int_{\bar{\Omega} \cap [\varphi \leq n]} g_L(\frac{a(k)}{a(n)} \gamma \psi_k)_{1\leq n}[\varphi > n-k] \cdot T^k dm
$$

$$
\sim \sum_{k=1}^n \int_{\bar{\Omega} \cap [\varphi \leq n]} g_L(\psi_k)_{1\leq n}[\varphi > n-k] \cdot T^k dm \cdot c(n-k)
$$

$$
= m(\bar{\Omega} \cap [\varphi \leq n]) e_\gamma(g_L)
$$

$$
\xrightarrow{n \to \infty} e_\gamma(g_L). \quad \bigcirc \quad \text{Propositions D & E.}
$$

To establish Proposition C, recall that $\psi_n \xrightarrow{\text{RV}(D_{t, \varpi}, J_1)} m_\gamma$, whence by Proposition 4 ($\triangle$ on p. 14), for $h \in C_B(D_{t, 1})$,

$$
\sum_{k=1}^n \int_{\bar{\Omega}} h(\psi_k)_{1\leq n}[\varphi > n-k] \cdot T^k dm \sim \mathbb{E}(h(w(m_\gamma))) a(n) \text{ as } n \to \infty,
$$

Functional limits for “tied down” occupation time processes
whereas by Proposition E,
\[ \int_{\Omega} h(\psi_k)1_{\Omega} \circ T^k dm \sim u(k)\mathbb{E}(h(\omega_\gamma)), \]
and (since \( a(n) \sim \sum_{k=1}^{n} u(k) \)),
\[ \omega_\gamma \overset{d.}{=} \omega(m_\gamma) \]
proving Proposition C. \( \Box \)

We conclude this section with the:

**Proof of Theorem U**

By the functional Darling-Kac theorem ([Bin71, OS15]), \( \mathbb{M} \) (on p. 14) is satisfied with \( m_\gamma = m_\gamma \). By Propositions 4 and C, \( \mathbb{E} \) holds with \( \omega_\gamma(m_\gamma) = \omega_\gamma \). This proves \( \mathbb{F} \). \( \Box \)

§3 Tied down occupation times

In this section, we prove Theorem U \( \mathbb{D} \) & \( \mathbb{E} \) (p.12).

We’ll use the following Uniform GL Lemma, which generalizes [GL63, Lemma 2.2.1].

**Uniform GL Lemma**

*For every* \( C = [c,d] \subset \mathbb{R}_+ \), \( \exists \Delta_n \downarrow 0 \) *so that for* \( h \in L_{\alpha,\theta}, g \in C_B(\mathbb{R}), g,h \geq 0 \), *on the Gibbs-Markov set* \( \Omega \),

\[ \frac{1}{u(n)} \mathbb{T}^n(hg(S_n(1_\Omega))) \geq m_\Omega(h)\mathbb{E}(g(W_\gamma)1_C(W_\gamma))) - \Delta_n \| g \|_{C_B} \| h \|_{L_{\alpha,\theta}}. \]

*where* \( W_\gamma \) *is as in* \( \mathbb{M} \) *on p.8.*

We’ll prove \( \mathbb{U} \mathbb{G} \mathbb{L} \) in the next section using “operator renewal theoretic” techniques as in [Sar02, Gou11, MT12].

We’ll also use the

**Gibbs Markov distortion lemma**

*Suppose that* \( (Y,\mu,\tau,\alpha) \) *is a Gibbs-Markov map (as on p.10), then* \( \exists M > 0 \) *so that*

\[ \mathbb{T}^n(F) = M^{n\alpha}F \|_{L^1} \] *on* \( Y \) \( \forall \ n \geq 1 \), \( F \in L^1(Y,\sigma(\alpha_n),\mu) \);\n
\[ \| \mathbb{T}^n(F) \|_{L_{\alpha,\theta}} \leq M F \|_{L^1} \] *\( \forall \ n \geq 1 \), \( F \in L^1(Y,\sigma(\alpha_n),\mu) \).
22: Functional limits for “tied down” occupation time processes

**Proof** By the Gibbs-Markov property (p.10), \( \exists C > 1, \ r \in (0,1) \) so that

\[
| \log v'_a(x) - \log v'_a(y) | \leq C r^{t(x,y)} \quad \forall \ n \geq 1, \ a \in \alpha_n, \ x, y \in \tau^n a
\]

whence

\[
v'_a(x) = C^{\pm 1} \frac{\mu(a)}{\mu(\tau^n a)} = C^{\prime \pm 1} \mu(a) \quad \forall \ a \in \alpha_n, \ x \in \tau^n a
\]

where \( C' = \min_{a \in \alpha_n} \mu(\tau a) \); and

\[
|v'_a(x) - v'_a(y)| \leq CC' \mu(a) r^{t(x,y)} \quad \forall \ a \in \alpha_n, \ x, y \in \tau^n a.
\]

Thus for \( F: \Omega \to \mathbb{R} \) \( \alpha_n \)-measurable, non-negative,

\[
\bar{\tau}^n(F)(x) = \sum_{a \in \alpha_n} F(a) v'_a(x) = C^{\prime \pm 1} \int_Y F d\mu \quad \&
\]

\[
\| \bar{\tau}^n(F) \|_{L_{\alpha,\theta}} \leq \sum_{a \in \alpha_n} |F(a)| v'_a \|_{L_{\alpha,\theta}} \leq 2CC' \sum_{a \in \alpha_n} \mu(a)|F(a)|
\]

where \( L_{\alpha,\theta} \) is as on p.10. \( \Box \)

**Proof of Theorem U continued.** \( \forall \) \( \alpha \) \& \[a\] \[a\]

Let \( \Omega \in \mathcal{B}(X) \) be the Gibbs-Markov set with \( m(\Omega) = 1 \) and return time function \( \varphi: \Omega \to \mathbb{N} \).

Let \( (\Omega, m_\Omega, T_\Omega, \alpha) \) be the corresponding Gibbs-Markov map and let

\[
\psi_n(t) := \frac{s(t)}{a(t)}, \ s_n := s_n(1_\Omega), \ Z_n := \min\{ k \geq n + 1 : T^k x \in \Omega \}.\]

It suffices to establish \( \forall \) \( g = g_\xi \in \Pi_0 \) (as defined by \( \bowtie \) on p.18). We claim that for this, it suffices to show \( \Theta \)

\[
\bar{T}^n(1_A g_\xi(\psi_n)) \geq \mathbb{E}(g_\xi(w_\gamma)) m(A) u(n)
\]

uniformly on \( \Omega \) \( \forall \ A \in \mathcal{C}_\alpha \) & \( g_\xi \in \Pi_0 \)

where \( g_\xi \in \Pi_0 \) (as defined by \( \bowtie \) on p.18).

Note that \( \Theta \) is an extension of \( \bowtie \) on p.18 and similarly extends to hold \( \forall \ g \in \Pi_1 \) (as defined by \( \bowtie \) on p.19).

Assume \( \Theta \) (whose proof is given on p.24), then \( \bowtie \) with \( f = 1_\Omega \) and \( A \in \mathcal{C}_\alpha \) follows from [Aar13, Prop. 4.2] via

\[
\frac{1}{a(N)} \sum_{n=1}^N \bar{T}^n(1_A g_\xi(\psi_n)) \xrightarrow{N \to \infty} \mathbb{E}(g_\xi(w_\gamma)) m(A)
\]

\( \mathrm{a.e. \ on} \ \Omega \ \forall \ A \in \mathcal{C}_\alpha \) & \( g_\xi \in \Pi_0 \).

To extend \( \bowtie \) to hold \( \forall \ f \in L^1_+ & \ A \in \mathcal{F}_+ \), apply the argument in [AS19, Proof of Theorem A: finish]. \( \Box \) \( \bowtie \)
Remark.

Suppose that \( t \in [0, 1] \mapsto f(t) \in L^1(m) \) and set \( \Psi_{n,t}(t) := \frac{S_{n|t|}(f(t))}{a(n)} \).

Variations of the argument in [AS19, Proof of Theorem A: finish] show that under the assumptions of Theorem U \( \cup \) and \( \odot \) respectively, we have for \( g \in \Pi \) that

\[
\frac{1}{a(n)} \sum_{k=1}^{n} \int_{A} g(\Psi_{k,t}) 1_B \circ T^k dm \xrightarrow{n \to \infty} m(A) m(B) E(g(m(\mathcal{f})w_{\gamma}))
\]

\( \forall A, B \in R(T) \)

where \( m(\mathcal{f})w_{\gamma} \in \mathcal{RV}(\mathbb{R}_{+}^{[0,1]}), \ t \mapsto m(\mathcal{f}(t))w_{\gamma}(t) \); and

\[
\frac{1}{a(n)} \sum_{k=1}^{n} |\tilde{T}^k(1_A g(\Psi_{k,t})) - u(k) m(A) E(g(m(\mathcal{f})w_{\gamma}))| \xrightarrow{n \to \infty} 0 \ a.e. \ \forall A \in \mathcal{F}_{+}.
\]

Proof of \( \bullet \) given \( \odot \)

Let \( g \in \Pi_0 \), then \( 1 - g \in \Pi_1 \) and by \( \odot \) (extended to \( \Pi_1 \)) and pointwise dual ergodicity, a.e. on \( \Omega \),

\[
E(g(w_{\gamma})) m(A) \leq \frac{1}{a(N)} \sum_{n=1}^{N} \tilde{T}^n(1_A g(\psi_{n}))
\]

\[
= \frac{1}{a(N)} \sum_{n=1}^{N} \tilde{T}^n(1_A - 1_A g(\psi_{n}))
\]

\[
\leq m(A) - E(1 - g(w_{\gamma})) m(A)
\]

\[
= E(g(w_{\gamma})) m(A),
\]

which implies \( \bullet \) for \( g \in \Pi_0 \).

Since \( \psi_{n} \in D_{\gamma,1} \) and \( w_{\gamma} \) is continuous a.s., the convergence \( \bullet \) also holds for \( g \in C_B(D_{\gamma,1}) \). \( \odot \)

Proof of Theorem U \( \Downarrow \) given \( \odot \)

Let \( g \in \Pi_0 \), then \( 1 - g \in \Pi_1 \). By \( \odot \) and (OSRT), for \( A \in \mathcal{C}_{\alpha} \), uniformly on \( \Omega \),

\[
E(g(w_{\gamma})) m(A) \leq \frac{1}{u(n)} \tilde{T}^n(1_A g(\psi_{n}))
\]

\[
= \frac{1}{u(n)} \tilde{T}^n(1_A) - \frac{1}{u(n)} \tilde{T}^n(1_A g(\psi_{n}))
\]

\[
\leq m(A) - E(1 - g(w_{\gamma})) m(A)
\]

\[
= E(g(w_{\gamma})) m(A).
\]
Therefore we obtain the desired uniform convergence for \( g \in \Pi_0 \). This convergence also holds for \( g \in C_B(D_{t,1}) \) since \( \mathbf{w}_\gamma \) is a continuous process.

Proof of 10

The rest of this section is devoted to a proof of \( \Theta \) given the uniform GL lemma which enables a suitable modification of the proof of \( \Psi \) (p. 18) without using a Markov property.

Again writing \( \kappa := \lfloor nt_N \rfloor \), fix \( A \in \alpha_J \), then

\[
1_A G_t^n(\psi_n) = \sum_{k=\kappa+1}^{n} 1_A g_t^n(\psi_n) 1_{[Z_k=k]} \\
= \sum_{k=\kappa+1}^{n} 1_A g_t^n(\psi_n) 1_{[Z_k=k]} 1_{\Omega} T^k h(\psi_n(1) - \psi_n(t_N)) \\
= \sum_{k=\kappa+1}^{n} 1_A g_t^n(\psi_n) 1_{[Z_k=k]} 1_{\Omega} T^k h(\frac{a(n-k)T^k}{a(n)});
\]

and

\[
\hat{T}^n(1_A g_t^n(\psi_n)) = \\
= \sum_{k=\kappa+1}^{n} \hat{T}^n(1_A g_t^n(\psi_n) 1_{[Z_k=k]} 1_{\Omega} T^k h(\frac{a(n-k)T^k}{a(n)})) \\
= \sum_{k=\kappa+1}^{n} \hat{T}^{n-k}(1_A g_t^n(\psi_n) 1_{[Z_k=k]} 1_{\Omega} h(\frac{a(n-k)T^k}{a(n-k)})).
\]

Next, for fixed \( \kappa + 1 \leq k \leq n \), define on \( \Omega \),

\[
G_{k,n} := \hat{T}^k(1_A g_t^n(\psi_n) 1_{[Z_k=k]}) \\
= \sum_{\ell=1}^{k} \hat{T}^\ell_t(1_A g_t^n(\psi_n) 1_{[\varphi_\ell=Z_k=k]});
\]

where \( \varphi_\ell := \sum_{k=0}^{\ell-1} \varphi_t \circ T^j_t \).

We claim that \( \exists C'' > 0 \) so that

\[
\|G_{k,n}\|_{L_{\alpha,\theta}} \leq C''\|g\|_{C_B m_\Omega([Z_k=k])} \quad \forall \ \kappa + 1 \leq k \leq n.
\]

Proof of \( \Psi \) For \( \kappa \) as above and \( \kappa + 1 \leq k \leq n \), \( \ell \leq k \), let

\[
F_{k,\ell,n} := 1_A g_t^n(\psi_n) 1_{[\varphi_\ell=Z_k=k]}
\]

10 on p. 22
is $\alpha$-measurable and by (p. 21), $G_{k,\ell,n} := \overline{T}_{\Omega}^k F_{k,\ell,n} \in L_{\alpha,\theta}$ (as on p.10) with
\[
\|G_{k,\ell,n}\|_{L_{\alpha,\theta}} \leq C'' m_{\Omega}(\|F_{k,\ell,n}\|) \leq C'' \|g\|_{C_B} m_{\Omega}(\|\varphi_{\ell} = Z_\kappa = k\|)
\]
where $\|g\|_{C_B} := \sup_{\xi \in D_{\gamma,R}} |g(\xi)|$. Thus
\[
\|G_{k,n}\|_{L_{\alpha,\theta}} \leq \sum_{\ell=1}^k \|G_{k,\ell,n}\|_{L_{\alpha,\theta}} \leq C'' \sum_{\ell=1}^k \|g\|_{C_B} m_{\Omega}(\|\varphi_{\ell} = Z_\kappa = k\|)
\]
\[
= C'' \|g\|_{C_B} m_{\Omega}(\|Z_\kappa = k\|)
\]
Fix $C = [c,d] \subset \mathbb{R}_+$. By (UGL) and $\therefore$,
\[
\frac{1}{u(n-k)} \overline{T}_{\Omega}^n (G_{k,n} \mathbf{1}_{\Omega} h(\frac{a(n-k)}{a(n)} s_{n-k} - n)) \geq m_{\Omega}(G_{n,k}) \mathbb{E}(h(\frac{a(n-k)}{a(n)} W_{\gamma}) 1_{C}(W_{\gamma})) - \Delta_{n-k} \|G_{n,k}\|_{L_{\alpha,\theta}} \|h\|_{C_B}
\]
\[
= \int_A g(\psi_n) 1_{[z_n=k]} dm \mathbb{E}(h(\frac{a(n-k)}{a(n)} W_{\gamma}) 1_{C}(W_{\gamma})) - \Delta_{n-k} C'' \|g\|_{C_B} m_{\Omega}(\|Z_\kappa = k\|).
\]
It follows that for fixed $t_N < \lambda < 1$,
\[
\overline{T}_{\Omega}^n (1_A g(\psi_n)) \geq \sum_{\kappa+1 \leq k \leq \lambda n} \int_A g(\psi_n) 1_{[z_n=k]} dm \cdot \mathbb{E}(h(\frac{a(n-k)}{a(n)} W_{\gamma}) 1_{C}(W_{\gamma})) u(n-k)
\]
\[
- C'' \|g\|_{C_B} \sum_{\kappa+1 \leq k \leq \lambda n} m_{\Omega}(\|Z_\kappa = k\|) u(n-k) \Delta_{n-k}.
\]
To continue, fix $\eta > 0$ and let $J = J(C, \lambda, \eta)$ be so that $\forall \ n \geq J, \ \kappa + 1 \leq k \leq \lambda n$,
\[
u(n-k) = (1 \pm \eta) \frac{u(n)}{(1-\frac{1}{\lambda})^{1-\gamma}} \& \ h(t \frac{a(n-k)}{a(n)}) = (1 \pm \eta) h(t(\frac{n-k}{n})^\gamma) \ \forall \ t \in C.
\]
For $n \geq J$,
\[
\sum_{\kappa+1 \leq k \leq \lambda n} m_{\Omega}(\|Z_\kappa = k\|) u(n-k) \Delta_{n-k} \leq (1 + \eta) u(n) \sum_{\kappa+1 \leq k \leq \lambda n} m_{\Omega}(\|Z_\kappa = k\|)(\frac{n}{n-k})^{1-\gamma} \Delta_{n-k}
\]
\[
\leq (1 + \eta) u(n)(\frac{1}{1-\lambda})^{1-\gamma} \Delta_{(1-\lambda)n}
\]
and
\[
\sum_{\kappa+1 \leq k \leq \lambda n} \int_A g(\psi_n) 1_{[z_n=k]} dm \cdot \mathbb{E}(h(\frac{a(n-k)}{a(n)} W_{\gamma}) 1_{C}(W_{\gamma})) u(n-k)
\]
\[
\geq (1-\eta)^2 u(n) \sum_{\kappa+1 \leq k \leq \lambda n} \int_A g(\psi_n) 1_{[z_n=k]} dm \cdot \mathbb{E}(h(\frac{n-k}{n})^{\gamma} W_{\gamma}) 1_{C}(W_{\gamma}))(\frac{n}{n-k})^{1-\gamma}.
\]
Moreover, with \((\psi_n, Z_\kappa) \perp W_\gamma\),

\[
\sum_{\kappa+1 \leq k \leq \lambda n} \int_A g(\psi_n) 1_{[Z_\kappa=k]} dm \cdot \mathbb{E}(h((\frac{n-k}{n})^\gamma W_\gamma) 1_C(W_\gamma))(\frac{n}{n-k})^{1-\gamma}
\]

\[
= \int_A g(\psi_n) \sum_{\kappa+1 \leq k \leq \lambda n} \left( 1_{[Z_\kappa=k]} \cdot \mathbb{E}(h((\frac{n-k}{n})^\gamma W_\gamma) 1_C(W_\gamma))(\frac{n}{n-k})^{1-\gamma} \right) dm
\]

\[
= \int_A \left( g(\psi_n) 1_{[Z_\kappa \leq \lambda n]} \mathbb{E}(h((\frac{n-Z_\kappa}{n})^\gamma W_\gamma) 1_C(W_\gamma))(\frac{n}{n-Z_\kappa})^{1-\gamma} \right) dm \cdot (\psi_n, Z_\kappa) \perp W_\gamma.
\]

Using ✩ on p.18 and Eagleson’s theorem ([Eag76]),

\[
\int_A \left( g(\psi_n) 1_{[Z_\kappa \leq \lambda n]} \mathbb{E}(h((\frac{n-Z_\kappa}{n})^\gamma W_\gamma) 1_C(W_\gamma))(\frac{n}{n-Z_\kappa})^{1-\gamma} \right) dm \xrightarrow{n \to \infty} m(A).
\]

Thus,

\[
\mathcal{T}^n(1_A g_\xi(\psi_n)) \geq m(A)\mathbb{E}(g_\xi(\psi_n)) u(n) \text{ uniformly on } \Omega. \quad \Box \Theta.
\]

**§4 Uniform LLTs & The Uniform GL Lemma**

In this section, we’ll prove the uniform GL lemma UGL (p. 21). Throughout the section, we use notations from [AS19, §3]

Let \((\Omega, \mu, \tau, \alpha)\) be a mixing, probability preserving Gibbs-Markov map as on p.10 and let \(\phi : \Omega \to \mathbb{N}\) be locally \(L_{\alpha,\theta}\), in the sense that \(\exists M > 0\) so that

\[
\forall A \in \alpha, \ 1_A \phi \in L_{\alpha,\theta} \& \|1_A \phi - \int_A \phi d\mu\|_{L_{\alpha,\theta}} \leq M;
\]

and also satisfy

\[
\mu([\phi > t]) \sim \frac{1}{\Gamma(1+\gamma)\Gamma(1-\gamma)} \frac{1}{a(t)}
\]

with \(a(t)\ \gamma\)-regularly varying with \(0 < \gamma < 1\).

As in Nagaev’s theorem ([Nag57], see also [AD01, Theorem 4.1] & the proof of Theorem 3.1 in [AS19]), we write:

\[
P_{\phi,t}(f) := \mathcal{T}(e(t\phi)f), \quad \mathcal{N}(t) = N_\phi(t) := N(P_{\phi,t}) \quad \lambda(t) = \lambda_\phi(t) := \lambda(P_{\phi,t}).
\]
(with \( e(t) := e^{2\pi it} \)).

By [AD01, theorem 6.1], under the above conditions,

\[
\frac{\phi_n}{a^{-1}(n)} \xrightarrow{n \to \infty} Z_\gamma := \eta_\gamma(1)
\]

where \( \phi_n := \sum_{k=0}^{n-1} \phi \circ \tau^k \); and

\[
\lambda(\frac{a^{-1}(n)}{n}) \xrightarrow{n \to \infty} \hat{f}_\gamma(t).
\]

We’ll need a uniform version of the lattice LLT.

As in [AS19, §3], suppose that \( \phi : \Omega \to \mathbb{N} \) is non-arithmetic, then either \( \phi \) is aperiodic, or \( \phi \) has a reduction

\[
\phi = F - F \circ \tau + p\psi + \xi
\]

where \( F : \Omega \to \mathbb{N}, \ F \in L_{\alpha,\theta}, \ p \in \mathbb{N}, \ p \geq 2, \ \xi \in [1, p-1] \cap \mathbb{N}, \ \gcd\{\xi, p\} = 1 \)
and \( \psi : \Omega \to \mathbb{Z} \) is locally \( L_{\alpha,\theta} \) and aperiodic.

Uniform conditional lattice LLT

Under the above conditions, \( \exists \Delta(N) \downarrow 0 \) so that if

\[
k_n \in \mathbb{Z}, \ \kappa_n := \frac{k_n}{a^{-1}(n)} \quad \forall \ n \geq 1,
\]

then \( \forall \ n \geq 1, \ h \in L_{\alpha,\theta} \):

- in case \( \phi \) is aperiodic:

\[
\|a^{-1}(n)\mathbb{Z}(h_{[\phi_n=k_n]}) - \mu(h)f_{Z_\gamma}(\kappa_n)\|_{L_{\alpha,\theta}} \leq \Delta(n)\|h\|_{L_{\alpha,\theta}};
\]

and, in case \( \phi \) has the reduction \( \Phi \),

\[
\|a^{-1}(n)\mathbb{Z}(h_{[\phi_n=k_n]}) - 1_p\mathbb{Z}(n\xi-k_n)f_{Z_\gamma}(\kappa_n)\mu(h)\|_{L_{\alpha,\theta}} \leq \Delta(n)\|h\|_{L_{\alpha,\theta}}.
\]

Remarks

The estimates are useful when \( (\kappa_n : n \geq 1) \) is bounded away from \( 0 \) & \( \infty \) (i.e. \( \inf_n f_{Z_\gamma}(\kappa_n) > 0 \)).

For estimates outside this range, see the “local large deviations” of [CD19, MT21] and the “extended LLTs” of [AT20] and references therein.

The periodic LLT \( \Delta \) was first proved in [She64] (in the independent case).

(i) **Proof of \( \square \)** By [AD01, Thm. 4.1], \( \exists \delta > 0, \ \theta \in (0,1), \ C > 1 \) such that \( \forall \ n \geq 1, \ h \in L_{\alpha,\theta} \):

\[
\|P_{\phi,t}^n h - \lambda(t)^n N(t)h\|_{L_{\alpha,\theta}} \leq C\|h\|_{L_{\alpha,\theta}} \theta^n \ \forall \ |t| \leq \delta,
\]

and

\[
\|P_{\phi,y}^n 1\|_{L_{\alpha,\theta}} \leq C\|h\|_{L_{\alpha,\theta}} \theta^n \ \forall \ \delta \leq |y| \leq \pi.
\]

Write \( a(t) = \frac{\ell(t)}{t(t)} \) with \( \ell(t) \) slowly varying at infinity.
Using [AD01, Thm. 5.1], by possibly shrinking $\delta > 0$, we can ensure in addition that

\[-\text{Re } \log \lambda(t) \geq \frac{c}{2a(\frac{1}{2})} \quad \forall \ |t| \leq \delta\]

and by slow variation of $\ell$, $\exists 0 < \epsilon = \epsilon(\delta)$ such that

\[\frac{\ell(\alpha^{-1}(n))}{\ell(\alpha^{-1}(n))} \geq |t|^\epsilon \quad \forall \ n \geq 1, \ |t| \leq \delta \alpha^{-1}(n).\]

The integrals below converge in $L_{\alpha,\theta}$ and we have, in $L_{\alpha,\theta}$,

\[2\pi a^{-1}(n) \tilde{\gamma}^n (h l_{[\phi_n=kn]}) = a^{-1}(n) \tilde{\gamma}^n \left( h \int_{-\pi}^{\pi} e^{-it\kappa_n} e^{it\phi_n} dt \right)\]

\[= a^{-1}(n) \int_{-\pi}^{\pi} e^{-it\kappa_n} \tilde{\gamma}^n (he^{it\phi_n}) dt \]

\[= a^{-1}(n) \int_{-\pi}^{\pi} e^{-it\kappa_n} P_{\phi,t}^n h dt \]

\[= a^{-1}(n) \int_{|t| \leq \delta} e^{-it\kappa_n} \lambda(t)^n N(t) h dt \pm 2\pi C \|h\|_{L_{\alpha,\theta}} a^{-1}(n)^\theta^n.\]

Next

\[a^{-1}(n) \int_{|t| \leq \delta} e^{-it\kappa_n} \lambda(t)^n N(t) h dt = \int_{-\delta a^{-1}(n)}^{\delta a^{-1}(n)} e^{-it\kappa_n} \lambda(\frac{t}{a^{-1}(n)})^n N(\frac{t}{a^{-1}(n)}) h dt\]

\[\& \quad 2\pi f_{Z,\gamma}(\kappa) = \int_{\mathbb{R}} \tilde{f}_{Z,\gamma}(t) e^{-\kappa t} dt,\]

so

\[a^{-1}(n) \int_{|t| \leq \delta} e^{-it\kappa_n} \lambda(t)^n N(t) h dt = 2\pi \mu(h) \tilde{f}_{Z,\gamma}(\kappa_n)\]

\[= \int_{|t| \leq \delta a^{-1}(n)} e^{-it\kappa_n} \left( \lambda(\frac{t}{a^{-1}(n)})^n N(\frac{t}{a^{-1}(n)}) h - \mu(h) \tilde{f}_{Z,\gamma}(t) \right) dt\]

\[+ \int_{|t| > \delta a^{-1}(n)} e^{-it\kappa_n} \tilde{f}_{Z,\gamma}(t) dt \mu(h)\]

\[=: I_n + II_n.\]

Now $|\tilde{f}_{Z,\gamma}(t)| = e^{-c|t|^\gamma}$ ($c > 0$) so

\[\|II_n\|_{L_{\alpha,\theta}} \leq \|h\|_{L_{\alpha,\theta}} \int_{|t| > \delta a^{-1}(n)} e^{-c|t|^\gamma} dt \underset{n \to \infty}{\longrightarrow} 0.\]

By [AD01, Theorems 2.4 & 5.1],

\[\|N(\frac{t}{a^{-1}(n)}) h - \mu(h)\|_{L_{\alpha,\theta}} \leq M m_{\Omega} (\|e^{\frac{t}{a^{-1}(n)}} - 1\|) \leq M'' \frac{1}{a(\frac{1}{2})} \|h\|_{L_{\alpha,\theta}}\]

\[\sim M'' \frac{1}{a(\frac{1}{2})} \|h\|_{L_{\alpha,\theta}} \sim M'' \frac{1}{n} \|h\|_{L_{\alpha,\theta}}\]
and

\[ \| I_n \|_{L_{\alpha, \theta}} = \int_{-\delta^{-1}(n)}^{\delta^{-1}(n)} \| \lambda \left( \frac{-t}{\alpha^{-1}(n)} \right) \|^n N \left( \frac{-t}{\alpha^{-1}(n)} \right) h - \mu(h) \|_{L_{\alpha, \theta}} dt \]

\[ \leq \int_{-\delta^{-1}(n)}^{\delta^{-1}(n)} |\lambda \left( \frac{-t}{\alpha^{-1}(n)} \right) \|^n N \left( \frac{-t}{\alpha^{-1}(n)} \right) h - \mu(h) \|_{L_{\alpha, \theta}} dt \]

\[ + |\mu(h)| \int_{-\delta^{-1}(n)}^{\delta^{-1}(n)} \left| \lambda \left( \frac{-t}{\alpha^{-1}(n)} \right) \right|^n - \|_{L_{\alpha, \theta}} dt \]

\[ \leq \| h \|_{L_{\alpha, \theta}} \left( \frac{M^n}{n} \int_{-\delta^{-1}(n)}^{\delta^{-1}(n)} |t|^\frac{1}{n} \left| \lambda \left( \frac{-t}{\alpha^{-1}(n)} \right) \right|^{\|n dt + \int_{-\delta^{-1}(n)}^{\delta^{-1}(n)} \left| \lambda \left( \frac{-t}{\alpha^{-1}(n)} \right) \right|^n - \|_{L_{\alpha, \theta}} dt \right). \]

By \( \phi \) (on p. 28), and regular variation of \( a^{-1} \), \( \exists \epsilon > 0 \) so that \( |\lambda \left( \frac{-t}{\alpha^{-1}(n)} \right) |^{n} \ll e^{-\epsilon |t|^\gamma} \).

Moreover, \( |\tilde{f}_{Z_n}(t)| \ll e^{-\epsilon |t|^\gamma} \) so

\[ \int_{-\delta^{-1}(n)}^{\delta^{-1}(n)} |t|^\frac{1}{n} \left| \lambda \left( \frac{-t}{\alpha^{-1}(n)} \right) \right|^{n} dt \leq M \int_{\mathbb{R}} |t|^\frac{1}{\gamma} e^{-\epsilon |t|^\gamma} dt < \infty, \text{ whence} \]

\[ \int_{-\delta^{-1}(n)}^{\delta^{-1}(n)} \left| \lambda \left( \frac{-t}{\alpha^{-1}(n)} \right) \right|^n - \|_{L_{\alpha, \theta}} dt \xrightarrow{n \to \infty} 0 \]

by dominated convergence. \( \triangleright \square. \)

(ii) Proof of \( \triangle \)

Recall from \( \bullet \) on p. 27 that \( \phi = p\psi + \xi + F - F \circ \tau \) where

- \( p, \xi \in \mathbb{N}, 1 \leq \xi < p; \gcd \{ p, \xi \} = 1 \);
- \( \psi : \Omega \to \mathbb{N} \) is aperiodic & \( F : \Omega \to \mathbb{Z} \) satisfies \( F \circ \tau - F \in p\mathbb{Z} \).

For \( k_n \in \mathbb{Z} \),

\[ \phi_n = k_n \iff p\psi_n = k_n - n\xi - F + F \circ \tau^n \]

and this entails \( k_n - n\xi \in p\mathbb{Z} \).

Thus,

---

\( ^{11} \)This proof is an modification of the proof of the lattice case of [AS19, Thm. 3.1].
\[
\tilde{\tau}_n(h_1[\phi_n=k_n])(x) = 1_p Z(n\xi - k_n) \int_T e(-k_n t)\tilde{\tau}_n(e(t\phi_n)h)(x) dt \\
= 1_p Z(n\xi - k_n) \int_T e(t(n\xi - k_n))\tilde{\tau}_n(e(pt\psi_n)e(t(F - F \circ \tau^n))h)(x) dt \\
= 1_p Z(n\xi - k_n) \int_T e(pt \cdot \frac{n\xi - k_n}{p})\tilde{\tau}_n(e(pt\psi_n)e(pt \cdot \frac{F - F \circ \tau^n}{p})h)(x) dt \\
= p1_p Z(n\xi - k_n) \int_T e(\frac{t}{p}(n\xi - k_n))\tilde{\tau}_n(e(t\psi_n)e(\frac{t}{p}(F - F \circ \tau^n))h)(x) dt \\
\vdash t \mapsto pt \text{ is measure preserving on } T; \\
= p1_p Z(n\xi - k_n) \int_T e(\frac{t}{p}(n\xi - k_n) - F(x))) P_n^{\tilde{\tau}_n}(e(\frac{t}{p})h)(x) dt.
\]

By the proof of \(\Box\),
\[
\| a^{-1}(n) \int_T e(\frac{t}{p}(n\xi - k_n) - F(x))) P_n^{\tilde{\tau}_n}(e(\frac{t}{p})h)(x) dt - \mu(h) f_{\frac{t}{p}}(\frac{\nu_n}{p}) \|_{L_{\alpha, \theta}} \leq \Delta(n) \| h \|_{L_{\alpha, \theta}}
\]
and \(\triangle\) follows from this. \(\checkmark\)

**Proof of the uniform GL lemma**

As in the proof of [AS19, Lemma 2.1(GL)], we assume \(WLOG\) that \(a^{-1}(n+1) - a^{-1}(n) \sim \frac{a^{-1}(n)}{a(n)}\) and set \(x_{\nu, n} := \frac{n}{a^{-1}(n)}\), obtaining (see [AS19])
\[
\Delta = \frac{1}{a^{-1}(n)} \sim \frac{\gamma_k}{n} \cdot (x_{k, n} - x_{k+1, n}) \sim \frac{\gamma_k}{n} \cdot \frac{x_{k, n} - x_{k+1, n}}{x_{k, n}} = u(n) \cdot \frac{x_{k, n} - x_{k+1, n}}{x_{k, n}}.
\]

It suffices to consider \(g \in C_B(\mathbb{R}_+\mathbb{R})\) so that \(log G : [c, d] \to \mathbb{R}\) is smooth where \(G(x) := g(\frac{1}{x})\).

Fixing \(h \in L_{T_{\Omega}}, h \geq 0\), we have that
\[
\frac{1}{u(n)} \tilde{T}_n^n(hg(S_{\nu, n}(\frac{\alpha}{a(n)}))) = \sum_{\nu=1}^{n} T_{\Omega}^\nu(h_1[\phi_n=\nu]g(\frac{\nu}{a(n)})) \\
\geq \frac{1}{u(n)} \sum_{1 \leq \nu \leq n, x_{\nu, n} \in [c, d]} T_{\Omega}^\nu(h_1[\phi_n=\nu]g(\frac{\nu}{a(n)})) \\
\geq \frac{1}{u(n)} \sum_{1 \leq \nu \leq n, x_{\nu, n} \in [c, d]} \frac{1}{a^{-1}(\nu)} \left[ 1_p Z(n - \nu \xi) m_{\Omega}(h) g(\frac{\nu}{a(n)}) f_{\gamma_n}(x_{\nu, n}) \right] \| g \|_{C_B} \| h \|_{L_{T_{\Omega}}(\Delta(\nu))} \] by \(\Delta\)
\[
= m_{\Omega}(h) \frac{1}{u(n)} \sum_{1 \leq \nu \leq n, x_{\nu, n} \in [c, d]} \frac{1}{a^{-1}(\nu)} 1_p Z(n - \nu \xi) g(\frac{\nu}{a(n)}) f_{\gamma_n}(x_{\nu, n}) \right] \| g \|_{C_B} \| h \|_{L_{T_{\Omega}}} \mathcal{E}_{n}^{(1)}
\]

where
\[
\mathcal{E}_{n}^{(1)} := \frac{1}{u(n)} \sum_{1 \leq \nu \leq n, x_{\nu, n} \in [c, d]} \frac{\Delta(\nu)}{a^{-1}(\nu)}
\]
with \(\Delta(\nu)\) as in \(\Box\) on p. 27.
To see that $E_n^{(1)} \xrightarrow{n \to \infty} 0$, we note that for $1 \leq \nu \leq n$,

$$x_{\nu,n} \in [c, d] \iff \nu \in [a(n)^\nu, a(n)^{\nu+1}].$$

Thus

$$E_n^{(1)} = \frac{1}{u(n)} \sum_{1 \leq \nu \leq n, \ x_{\nu,n} \in [c, d]} \frac{\Delta(\nu)}{a^{-1}(\nu)} \leq \frac{\Delta(a(n)^\nu)}{u(n)} \sum_{\nu \geq a(n)^\nu} \frac{1}{a^{-1}(\nu)} \leq \frac{\Delta(a(n)^{a(n)^{2\nu}})}{u(n)} \frac{1}{a^{-1}(a(n)^{2\nu})} \leq \Delta(a(n)^{a(n)^{2\nu}}) \xrightarrow{n \to \infty} 0.$$

Next, again as in the proof of [AS19, Lemma 2.1(GL)],

$$\frac{1}{u(n)} \sum_{1 \leq \nu \leq n, \ x_{\nu,n} \in [c, d]} \frac{1}{a^{-1}(\nu)} p_{Z}(n - \nu \xi) g(\frac{\nu}{a(n)}) f_{Z}(x_{\nu,n}) = \frac{1}{u(n)} \sum_{1 \leq k \leq n, \ x_{k,n} \in [c, d]} g(\frac{1}{a(n)} p_{Z}(x_{k,n}) \frac{1}{a^{-1}(k)}) 1_{[c, d]}(Z_{\gamma} g(Z_{\gamma}^{-1}) Z_{\gamma}^{-1}) \pm E_n^{(2)} \|g\|_{C_{\beta}}$$

where $E_n^{(2)} \to 0$.

Thus,

$$\frac{1}{u(n)} \tilde{F}^{n}(h g(S_{a(n)^{1\alpha}})) \geq m_{\Omega}(h) E(g(W_{\gamma} 1_{C(W_{\gamma})})) - (E_n^{(1)} + E_n^{(2)}) \|g\|_{C_{\beta}} \|h\|_{L_{a,\theta}} \mp$$

§5 Remarks on 1-self similar limit processes with stationary increments

Let $(\Omega, \mu, \tau)$ be a probability preserving transformation & let $\varphi : X \to \mathbb{R}_+$ be measurable and suppose that $\frac{\tilde{S}_{n}}{b(n)} \xrightarrow{RV(R_{+})} Y \in RV(R_{+})$ with $b: \mathbb{R}_{+} \to \mathbb{R}_{+}$ is 1-regularly varying.

As remarked in [AW18],

$$\Phi_n \xrightarrow{n \to \infty} Y$$

\[^{12}\text{Here, for } a_n, b_n > 0, \ a_n \asymp b_n \text{ means } a_n \ll b_n \text{ & } b_n \ll a_n \text{ where } \ll \text{ is as in the footnote on p. 9.}$$
Functional limits for “tied down” occupation time processes

in \((D_{\uparrow,\infty}, DF)\) where \(\Phi_n(t) := \frac{\varphi_n(t)}{b(n)} \& \Phi(t) := t\).

This follows from the 1-regular variation of \(b : \mathbb{R}_+ \to \mathbb{R}_+\) via [Ver85, Theorem 3.1].

Now let \((X, m, T)\) be a conservative, ergodic, measure preserving transformation and suppose that

\[
\frac{S_n(f)}{a(n)} \xrightarrow{b(n)} Z m(f) \quad \forall \ f \in L^1(m)_+
\]

where \(Z \in RV(\mathbb{R}_+)\) and \(a : \mathbb{R}_+ \to \mathbb{R}_+\) is 1-regularly varying. We claim that

\[
\Psi_{n,f} \xrightarrow{\Phi(n)} m(f) Z
\]

in \((D_{\uparrow,\infty}, DF)\) where \(\Psi_{n,f}(t) := \frac{S_n(f)}{a(n)}\).

**Proof of (III)** Fix \(\Omega \in B(X)\), \(m(\Omega) = 1\) and let \(\mu := m_\Omega\), \(\varphi : \Omega \to \mathbb{N}\), \(\varphi(x) := \min\{n \geq 1 : T^n x \in \Omega\}\) & \(\tau : \Omega \to \Omega\), \(\tau(x) := T^{\varphi(x)}(x)\).

By inversion \(\frac{\varphi_n}{b(n)} \xrightarrow{b(n)} Z^{-1} \in RV(\mathbb{R}_+)\) with \(b := a^{-1}\) and by (III),

\[
\Phi_n \xrightarrow{b(n)} Z^{-1} \text{ in } (D_{\uparrow,\infty}, DF),
\]

whence, again by inversion

\[
\Psi_{n,\mu} \xrightarrow{b(n)} Z \text{ in } (D_{\uparrow,\infty}, DF).
\]

Conclude to obtain (III) by applying the ratio ergodic theorem and the functional version of Eagleson’s theorem ([Eag76, TZ06].

Examples of conservative, ergodic, measure preserving transformations satisfying (III) are given in [AW18].

**Proposition 6** Suppose that \((X, m, T)\) is a weakly rationally ergodic measure preserving transformation \(^{13}\) so that

\[
\Phi_n(f) \xrightarrow{a(n)} m(f) \quad \forall \ f \in L^1(m)_+
\]

then

\[
\frac{1}{a(n)} \sum_{k=1}^{n} \int_A g(S_k(f)/a(k)) 1_B \circ T^k dm - g(m(f)) m(A \cap T^{-k}B) \]

\[
\xrightarrow{n \to \infty} 0 \quad \forall \ f \in L^1(m)_+, \ g \in C_B(\mathbb{R}_+), A, B \in R(T).
\]

\(^{13}\) as in \(\spadesuit\) on p.11
If, in addition, \((X, m, T)\) is RWM, then

\[
\mathbb{E} \left| \frac{1}{a(N)} \sum_{n=1}^{N} \int_{B \cap T^{-n}C} g\left( \frac{S_n(f)}{a(n)} \right) dm - m(B)m(C)g(m(f))u_n \right| \xrightarrow{N \to \infty} 0 \quad \forall \ B, C \in R(T), \ g \in C_B(\mathbb{R}_+) \quad \& \quad f \in L^1_+
\]

where \(a(n) \sim \sum_{k=1}^{n} u_k\).

Note that \(\mathbb{E}\) does not entail spectral weak mixing; and \(\mathbb{E}\) is the integrated, tied-down renewal mixing property in [AS19, Theorem 6.2] with \(\gamma = 1 \& W_1 \equiv 1\) when \(u_n \sim \frac{a(n)}{n}\).

**Proof of** \(\mathbb{E}\). By [Aar79, Theorem 4.1], \(a(n)\) is 1-regularly varying. Thus by Corollary 3 and Proposition 4,

\[
\frac{1}{a(n)} \sum_{k=1}^{n} \int_{A} g\left( \frac{S_k(f)}{a(k)} \right) 1_B \circ T^k dm \xrightarrow{n \to \infty} m(A)m(B)g(m(f))
\]

\(\forall \ A, \ B \in R(T), \ f \in L^1_+, \ g \text{ Riemann integrable on } \mathbb{R}_+\).

Fix \(f \in L^1_+(m), \ \epsilon \in (0, 1)\) and let \(g_{\epsilon} := 1_{(1-\epsilon)m(f),(1+\epsilon)m(f))}\), then writing \(s_n := S_n(f)\), we have

\[
\frac{1}{a(n)} \sum_{k=1}^{n} \int_{A} g_{\epsilon}\left( \frac{s_k}{a(k)} \right) 1_B \circ T^k dm \xrightarrow{n \to \infty} m(A)m(B) \quad \forall \ A, \ B \in R(T).
\]

For \(\epsilon > 0 \& x \in X\), let

\[K(\epsilon, x) := \{k \geq 1 : | \frac{s_k}{a(k)} - m(f) | < \epsilon m(f) \} \text{.}\]

By the above, for \(A, \ B \in R(T)\),

\[
\int_{A} \sum_{k \in [1,n] \cap K(\epsilon, x)} 1_B \circ T^k dm = o(a(n)) \text{ as } n \to \infty.
\]

Now let \(g \in C_B(\mathbb{R}_+)\), fix \(\Delta > 0\) and let \(\epsilon > 0\) be so that

\[g(y) = g(m(f)) \pm \Delta \quad \forall \ y \in \mathbb{R}_+, \ |y - m(f)| < \epsilon m(f)\text{.}\]

It follows that

\[
\sum_{k=1}^{n} \left| \int_{A} g\left( \frac{s_k}{a(k)} \right) 1_B \circ T^k dm - g(m(f))m(A \cap T^{-k}B) \right| = \sum_{k=1}^{n} \left| \int_{A \cap T^{-k}B} g\left( \frac{s_k}{a(k)} \right) - g(m(f))dm \right| \leq \int_{A} \left( \sum_{k=1}^{n} 1_B \circ T^k |g\left( \frac{s_k}{a(k)} \right) - g(m(f))| \right) dm.
\]
Now
\[ \sum_{k=1}^{n} 1_B \circ T^k |g(\frac{s_k}{a(k)}) - g(m(f))| \]
\[ = \left( \sum_{k \in [1,n] \cap K(\epsilon,x)} \mathbb{1}_B \circ T^k |g(\frac{s_k}{a(k)}) - g(m(f))| + \sum_{k \in [1,n] \setminus K(\epsilon,x)} \mathbb{1}_B \circ T^k |g(\frac{s_k}{a(k)}) - g(m(f))| \right) \]
\[ =: I_n + II_n; \]

and
\[ \int_A II_n dm \]
\[ \leq 2\|g\|_{C_B} \int_A \sum_{k \in [1,n] \cap K(\epsilon,x)} 1_B \circ T^k dm \]
\[ = o(a(n)) \text{ as } n \to \infty; \]

whereas
\[ I_n \leq \Delta \sum_{k \in [1,n] \cap K(\epsilon,x)} 1_B \circ T^k \]

and
\[ \int_A I_n dm \leq \Delta \sum_{k=1}^{n} m(A \cap T^{-k}B) \sim \Delta m(A)m(B)a(n). \]

This proves $\mathfrak{H}, \mathfrak{V}$

**Proof of $\mathfrak{Q}$**

Using RWM & $\mathfrak{F}$,
\[ \sum_{k=1}^{n} \left| \int_A g(\frac{s_k}{a(k)}) 1_B \circ T^k dm - g(m(f))u(k)m(A)m(B) \right| \]
\[ \leq \sum_{k=1}^{n} \left| \int_{A \cap T^{-k}B} (g(\frac{s_k}{a(k)}) - g(m(f))) dm \right| + \]
\[ + \left| g(m(f)) \sum_{k=1}^{n} |m(A \cap T^{-k}B) - u(k)m(A)m(B)| \right| \]
\[ = o(a(n)) \text{ as } n \to \infty. \quad \mathfrak{Q} \]

The following extends $\mathfrak{Q}$ in [AS19]

**Corollary 7** (discrete time tied-down renewal)
Suppose that \((\xi_1, \xi_2, \ldots)\) are \(\mathbb{N}\)-valued i.i.d. so that \(t(t) := \mathbb{E}(\xi \land t)\) is slowly varying, then with \(s_n := \sum_{k=1}^n \xi_k\),
\[
\frac{1}{a(n)^N} \sum_{n=1}^N \left| \sum_{k=1}^{\lfloor a(n)n \rfloor} g\left( \frac{k}{a(n)} - s \right) P\left( \{ s_k = n \} \right) - g(1) u(n) \right| \xrightarrow{n \to \infty} 0 \quad \forall \ g \in C_B(\mathbb{R}_+)
\]
where \(u(n) := \sum_{k=1}^n P\left( \{ s_k = n \} \right)\) and \(a(n) := \sum_{k=1}^n \frac{u(k)}{t(n)}\).

**Proof** Let \((X, m, T)\) be the corresponding renewal shift—a pointwise dual ergodic measure preserving transformation with \(a_n(T) \sim a(n)\). By the Darling Kac theorem \(\mbox{on p.32}\) holds and the result follows from Proposition 6. \(\Box\)

§6 Appendix

**Mittag-Leffler processes and local time processes.**

For \(I = [0, R]\) or \([0, \infty)\), let
\[
D(I) := \{ \omega: I \to \mathbb{R}: \forall \ t \in I \exists \omega(t^\pm) \land \omega(t^+) = \omega(t) \}
\]
where \(\omega(t^\pm) := \lim_{s \to t^\pm} \omega(s)\) and \(\omega(0^-) := \omega(0)\); the space of all càdlàg paths from \(I\) to \(\mathbb{R}\) and write \(D_R := D([0, R]) \land D_\infty := D([0, \infty))\).

For \(0 < \gamma \leq 1/2\), let \(X \in RV(D_\infty)\) be a symmetric \(1/(1 - \gamma)\)-stable Lévy process with \(X_0 = 0\).

As in [Ber96, Chapter V], denote by \(p_t(x) = \mathbb{P}[X_t \in dx]/dx\) its transition density and set \(c := 2\int_0^1 p_s(0)ds\), then \(X\) admits the local time process at 0, \(L \in RV(D_{1,\infty})\):
\[
L(t) = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_0^t \mathbb{1}\{ |X_s| < \varepsilon \} ds.
\]

By [Ber96, Proposition V.4], \(L = m_\gamma\).

As shown in [Ber96] (Theorem IV.4 and Proposition V.4), the closed support of the Stieltjes measure \(\mu_L\) on \([0, \infty)\) defined by \(L\) is \(\{ t \geq 0: X(t) = 0 \}\).

Thus a.s., the last visit time of \(X\) at 0 before time 1 is given by
\[
\mathcal{G} := \sup\{ 0 \leq t \leq 1: X_t = 0 \} = \sup([0, 1] \cap \text{supp}(\mu_L))
\]
\[
= \mathcal{G}(L)(1) = \mathcal{G}(m_\gamma)(1)
\]
where \(\mathcal{G}\) is as in §0.

By [Ber96, Theorem VIII.12] the symmetric \(1/(1 - \gamma)\)-stable Lévy bridge process (from 0 to 0 of length 1) \(X^{(br)} \in RV(D_1)\) is given by
\[
X^{(br)}(t) = \frac{X_{Gt}}{G^{1-\gamma}}.
\]
Moreover, $G$ is independent of $X^{(br)}$ and is $\text{Beta}(\gamma, 1-\gamma)$-distributed.

As before, the local time process $L^{(br)}(t) \in \text{RV}(D_{\gamma,0})$ of $X^{(br)}$ at 0 satisfies

$$L^{(br)}(t) \sim \frac{1}{c_\varepsilon} \int_0^t 1\{|X^{(br)}(s)| < \varepsilon\} ds$$

$$= \frac{1}{c_\varepsilon} \int_0^t 1\{|X(Gs)| < G^{1-\gamma}\varepsilon\} ds$$

$$= \frac{1}{Gc_\varepsilon} \int_0^{Gt} 1\{|X(u)| < G^{1-\gamma}\varepsilon\} du$$

changing variables

$$\xrightarrow{\varepsilon \to 0^+} \frac{m_\gamma(Gt)}{G\gamma} = \omega_\gamma(t),$$

where $\omega_\gamma$, the tied-down $\gamma$-ML process as in §0.

The process $\omega_\gamma = L^{(br)}$ is independent of $G$, since $L^{(br)}$ is a functional of $X^{(br)}$.

Similarly, for $0 < \gamma < 1$, the local time of the $(2-2\gamma)$-dimensional Bessel process is also supported on the closure of the zero set and equal distributionally to $m_\gamma$ ([MO69]). The $(2-2\gamma)$-dimensional Bessel bridge from 0 to 0 of length 1 is constructed e.g. in [CUB11, Theorem 3 and Section 2]. The local time of this bridge process is shown to be $\omega_\gamma$ in [BPY89].

**A proof of Proposition C via Bessel bridges.**

We’ll deduce Proposition C as a consequence of [FPY93, Proposition 2]. Following the notation there, for $r \in (0, \infty]$, let

$$D_{r,\infty} := \{ \omega \in D_r : \omega \geq 0 \}.$$ 

Defining $X_t : D_{r,\infty} \to [0, \infty)$ by $X_t(\omega) := \omega(t)$, set $\mathcal{F}_t = \sigma(X_s : 0 \leq s \leq t)$ and $\mathcal{F}_{\infty} = \sigma(X_s : s \geq 0)$.

Fix $0 < \gamma < 1$ and for $x \geq 0$, let $\mathbb{P}_x \in \mathcal{P}(D_{r,\infty})$ denote the law of a $(2-2\gamma)$-dimensional Bessel process on $[0, \infty)$ starting from $x$.

For $t > 0$ and $x, y \geq 0$, the $(x, t, y)$-bridge law $\mathbb{P}_{x,y}^t$ on $(D_{r,\infty}, \mathcal{F}_t)$ is the regular conditional distribution of $\mathbb{P}_{x,|x|}$ given $X_{-t} = y$ (see [FPY93, Section 2] and [CUB11, Sections 1 and 2]).

Denote the usual augmentation (as in [RY99]) by $\mathcal{F}^* = (\mathcal{F}_t^* : t \geq 0)$ which is the smallest right continuous filtration where each $\mathcal{F}_t^*$ is a completion of $\sigma(X_s : 0 \leq s \leq t)$ with respect to $(\mathbb{P}_x : x \geq 0)$ and let $(H_t : t \geq 0)$ be a non-negative predictable process and $(A_t : t \geq 0)$ be a continuous additive functional with respect to $(\mathcal{F}_t^* : t \geq 0)$.

\[\text{For more details about Bessel processes, bridges and their local time processes see [FPY93, PY97, RY99, RW00].}\]
Assuming $\mathbb{P}_x[A_t < \infty] = 1$, for any $x$ and $t$, [FPY93, Proposition 2] yields that

\[ \mathbb{E}_x \left[ \int_0^t H_s dA_s \right] = \mathbb{E}_x \left[ \int_0^t \mathbb{E}_{x,X_s}[H_s] dA_s \right], \]

where $\mathbb{E}_x$ and $\mathbb{E}_{x,y}^s$ denote the expectations with respect to $\mathbb{P}_x$ and $\mathbb{P}_{x,y}^s$, respectively.

The local time at 0: $L \in RV(D_{1,\infty})$ with $\mathbb{E}_0[L_1] = 1$ is a continuous additive functional and hence a predictable process.

In addition we see $\int_0^t 1_{X_s = 0} dL_s = L_t$ for any $t$. See [FPY93, Example] or [Ber96, Chapter IV].

Substituting $A_s = L_s$ into the $\mathbb{A}$, we obtain

\[ \mathbb{E}_x \left[ \int_0^t H_s dL_s \right] = \mathbb{E}_x \left[ \int_0^t \mathbb{E}_{x,0}[H_s] dL_s \right]. \]

Recall $\Delta_{s,\gamma} L = (L_{st}/s^\gamma : 0 \leq t \leq 1)$ and $L^{(1)} = (L_t : 0 \leq t \leq 1)$. Let $h : D_{1,1} \rightarrow \mathbb{R}$ be a bounded measurable functional.

To prove Proposition C, apply $\mathbb{B}$ with $x = 0$, $t = 1$ and $H_s = h(\Delta_{s,\gamma} L)$ and obtain

\[ \mathbb{E}_0 \left[ \int_0^1 h(\Delta_{s,\gamma} L) dL_s \right] = \mathbb{E}_0 \left[ \int_0^1 \mathbb{E}_{0,0}[h(\Delta_{s,\gamma} L)] dL_s \right] \quad \text{by $\mathbb{B}$} \]

\[ = \mathbb{E}_0 \left[ \int_0^1 \mathbb{E}_{0,0}[h(L^{(1)})] dL_s \right] \quad \text{by $\mathbb{B}$} \]

\[ = \mathbb{E}_0[L_1] \mathbb{E}_{0,0}[h(L^{(1)})] = \mathbb{E}_{0,0}[h(L^{(1)})]. \]

This completes the proof, since $L = m_\gamma$ under $\mathbb{P}_0$ & $L^{(1)} = w_\gamma$ under $\mathbb{P}_{1,0}^1$ as explained above. $\check{\square}$

**References**

[Aar77] Jon Aaronson. Rational ergodicity and a metric invariant for Markov shifts. *Israel J. Math.*, 27(2):93–123, 1977.

[Aar79] Jon Aaronson. On the pointwise ergodic behaviour of transformations preserving infinite measures. *Israel J. Math.*, 32(1):67–82, 1979.

[Aar86] Jon. Aaronson. Random $f$-expansions. *Ann. Probab.*, 14(3):1037–1057, 1986.

[Aar97] Jon Aaronson. *An introduction to infinite ergodic theory*, volume 50 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997.

[Aar13] Jon Aaronson. Rational weak mixing in infinite measure spaces. *Ergodic Theory and Dynamical Systems*, 33:1611–1643, 12 2013.

[AD01] Jon Aaronson and Manfred Denker. Local limit theorems for partial sums of stationary sequences generated by Gibbs-Markov maps. *Stoch. Dyn.*, 1(2):193–237, 2001.
[AS19] Jon. Aaronson and Toru Sera. Tied-down occupation times of infinite ergodic transformations. arXiv e-prints, page 1910.09846, 2019.
[AT20] Jon Aaronson and Dalia Terhesiu. Local limit theorems for suspended semiflows. Discrete Contin. Dyn. Syst., 40(12):6575–6609, 2020.
[AW18] Jon Aaronson and Benjamin Weiss. Distributional limits of positive, ergodic stationary processes and infinite ergodic transformations. Ann. Inst. Henri Poincaré Probab. Stat., 54(2):879–906, 2018.
[Ber96] Jean Bertoin. Lévy processes, volume 121 of Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 1996.
[Bil99] Patrick Billingsley. Convergence of probability measures. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons, Inc., New York, second edition, 1999. A Wiley-Interscience Publication.
[Bin71] N. H. Bingham. Limit theorems for occupation times of Markov processes. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, 17:1–22, 1971.
[BPY89] Martin Barlow, Jim Pitman, and Marc Yor. Une extension multidimensionnelle de la loi de l’arc sinus. In Séminaire de Probabilités, XXIII, volume 1372 of Lecture Notes in Math., pages 294–314. Springer, Berlin, 1989.
[CD19] Francesco Caravenna and Ron Doney. Local large deviations and the strong renewal theorem. Electron. J. Probab., 24:Paper No. 72, 48, 2019.
[CUB11] Loïc Chaumont and Gerónimo Uribe Bravo. Markovian bridges: weak continuity and pathwise constructions. Ann. Probab., 39(2):609–647, 2011.
[DK63] Meyer Dwass and Samuel Karlin. Conditioned limit theorems. Ann. Math. Statist., 34:1147–1167, 1963.
[Don97] R. A. Doney. One-sided local large deviation and renewal theorems in the case of infinite mean. Probab. Theory Related Fields, 107(4):451–465, 1997.
[Eag76] G. K. Eagleson. Some simple conditions for limit theorems to be mixing. Teor. Verojatnost. i Primenen., 21(3):653–660, 1976.
[EK86] Stewart N. Ethier and Thomas G. Kurtz. Markov processes, characterization and convergence. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons, Inc., New York, 1986.
[FPY93] Pat Fitzsimmons, Jim Pitman, and Marc Yor. Markovian bridges: construction, Palm interpretation, and splicing. In Seminar on Stochastic Processes, 1992 (Seattle, WA, 1992), volume 33 of Progr. Probab., pages 101–134. Birkhäuser Boston, Boston, MA, 1993.
[GL63] Adriano Garsia and John Lamperti. A discrete renewal theorem with infinite mean. Comment. Math. Helv., 37:221–234, 1962/1963.
[Gou11] Sébastien Gouëzel. Correlation asymptotics from large deviations in dynamical systems with infinite measure. Colloq. Math., 125(2):193–212, 2011.
[JS03] Jean Jacod and Albert N. Shiryaev. Limit theorems for stochastic processes, volume 288 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, second edition, 2003.
[Kak43] Shizuo Kakutani. Induced measure preserving transformations. *Proc. Imp. Acad. Tokyo*, 19:635–641, 1943.

[Kal02] Olav Kallenberg. *Foundations of modern probability*. Probability and its Applications (New York). Springer-Verlag, New York, second edition, 2002.

[Lig68] Thomas M. Liggett. An invariance principle for conditioned sums of independent random variables. *J. Math. Mech.*, 18:559–570, 1968.

[LSV99] Carlangelo Liverani, Benoît Saussol, and Sandro Vaienti. A probabilistic approach to intermittency. *Ergodic Theory Dynam. Systems*, 19(3):671–685, 1999.

[MO69] S. A. Molchanov and E. Ostrovskii. Symmetric stable processes as traces of degenerate diffusion processes. *Teor. Veroyatnost. i Primenen.*, 14:127–130, 1969. English translation: *Theor. Probability Appl.* 14 (1969), 128–131.

[MT12] Ian Melbourne and Dalia Terhesiu. Operator renewal theory and mixing rates for dynamical systems with infinite measure. *Invent. Math.*, 189(1):61–110, 2012.

[MT21] Ian Melbourne and Dalia Terhesiu. Analytic proof of multivariate stable local large deviations and application to deterministic dynamical systems. *arXiv e-prints*, page 2009.02514, 2021.

[Nag57] S. V. Nagaev. Some limit theorems for stationary Markov chains. *Teor. Veroyatnost. i Primenen.*, 2:389–416, 1957.

[OS15] Takashi Owada and Gennady Samorodnitsky. Functional central limit theorem for heavy tailed stationary infinitely divisible processes generated by conservative flows. *Ann. Probab.*, 43(1):240–285, 2015.

[PY97] Jim Pitman and Marc Yor. The two-parameter Poisson-Dirichlet distribution derived from a stable subordinator. *Ann. Probab.*, 25(2):855–900, 1997.

[RW00] L. C. G. Rogers and David Williams. *Diffusions, Markov processes, and martingales*. Vol. 2. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2000. Itô calculus, Reprint of the second (1994) edition.

[RY99] Daniel Revuz and Marc Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.

[Sar02] Omri Sarig. Subexponential decay of correlations. *Invent. Math.*, 150(3):629–653, 2002.

[Ser20] Toru Sera. Functional limit theorem for occupation time processes of intermittent maps. *Nonlinearity*, 33(3):1183–1217, feb 2020.

[She64] L. A. Shepp. A local limit theorem. *Ann. Math. Statist.*, 35:419–423, 1964.

[Sko56] A. V. Skorohod. Limit theorems for stochastic processes. *Teor. Veroyatnost. i Primenen.*, 1:289–319, 1956.

[Tha80] Maximilian Thaler. Estimates of the invariant densities of endomorphisms with indifferent fixed points. *Israel J. Math.*, 37(4):303–314, 1980.

[Tha83] Maximilian Thaler. Transformations on $[0, 1]$ with infinite invariant measures. *Israel J. Math.*, 46(1-2):67–96, 1983.
1. Marta Tyran-Kamińska. Convergence to Lévy stable processes under some weak dependence conditions. *Stochastic Process. Appl.*, 120(9):1629–1650, 2010.

2. Marta Tyran-Kamińska. Weak convergence to Lévy stable processes in dynamical systems. *Stoch. Dyn.*, 10(2):263–289, 2010.

3. M. Thaler and R. Zweimüller. Distributional limit theorems in infinite ergodic theory. *Probab. Theory Related Fields*, 135(1):15–52, 2006.

4. Wim Vervaat. Sample path properties of self-similar processes with stationary increments. *Ann. Probab.*, 13(1):1–27, 1985.

5. J. G. Wendel. Zero-free intervals of semi-stable Markov processes. *Math. Scand.*, 14:21–34, 1964.

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