Coset Character Identities in Superstring Compactifications

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Abstract
We apply the coset character identities (generalization of Jacobi’s abstruse identity) to compact and noncompact Gepner models. In the both cases, we prove that the partition function actually vanishes due to the spacetime supersymmetry. In the case of the compact models and discrete parts of the noncompact models, the partition function includes the expected vanishing factor. But the character identities used to the continuous part of the noncompact models suggest that these models have twice as many supersymmetry as expected. This fact is an evidence for the conjecture that the holographically dual of the string theory on an actually singular Calabi-Yau manifold is a super conformal field theory. The extra SUSY charges are interpreted as the superconformal S generators.
1 Introduction

The supersymmetric compactification of the string theory is an important problem in both the phenomenological and the formal sense. The internal space of the compactification should be a manifold of a special holonomy if we want to preserve some of the spacetime supersymmetry.

From the point of view of the worldsheet conformal field theory, the spacetime supersymmetry implies that the toroidal partition function should vanish. For example, if we consider the string theory on the flat spacetime, toroidal partition function vanishes because of the Jacobi’s abstruse identity: the relation among Jacobi’s theta functions

$$\theta_3(\tau)^4 - \theta_4(\tau)^4 - \theta_2(\tau)^4 = 0.$$ 

Let us use the affine so(8) characters $\chi^{so(8)}$. The subscript “$s$” (bas, vec, spi, cos) respectively expresses the basic, spinor, vector, and cospinor representation of so(8). Then the Jacobi’s abstruse identity can be written as

$$\chi^{so(8)}_{vec}(\tau) = \chi^{so(8)}_{spi}(\tau).$$

This equation shows that the number of the bosons (in the vector representation of affine so(8)) is equal to the number of the fermions (in the spinor representation of affine so(8)) in each mass level. This is the result of the spacetime supersymmetry.

Then, how about the compactification with a manifold of nontrivial special holonomy $G_{hol}$? In [1], it has been proposed that the coset CFT so(8)/$G_{hol}$ is essential to the spacetime supersymmetry. Especially, the character identity used to show that the partition function vanishes is

$$\chi^{so(8)/G_{hol}}_{vec,\lambda}(\tau) = \chi^{so(8)/G_{hol}}_{spi,\lambda}(\tau), \quad (1.1)$$

where $\lambda$ is a representation of level 1 affine $G_{hol}$ and $\chi^{so(8)/G_{hol}}_{spi,\lambda}(\tau)$ is a coset character. Each character identity of the case $G_{hol} = su(2), su(3), G_2$ has been found previously in [2–5]. Many other character identities related to SU(2)/U(1) coset models have been found in [6].

In this paper, we consider the noncompact Gepner models [7–12]. The partition functions of the models in [7,8,10,11] have been already shown to vanish in each paper. But, the models in [9,12] have not yet been shown to have vanishing partition functions. In this paper, we show these partition functions actually vanish because of the character identity (1.1).

Among these models, the continuous part of the noncompact Gepner models includes larger coset models than expected from their holonomies, as we show in this paper. That
is, the partition function of the noncompact Gepner models of an ALE compactification includes the affine \( \text{so}(8) \) itself, the partition function of a Calabi-Yau 3-fold compactification includes the coset \( \text{so}(8)/\text{su}(2) \), and the partition function of a Calabi-Yau 4-fold compactification includes the coset \( \text{so}(8)/\text{su}(3) \). This fact is an evidence for the conjecture that holographically dual field theory of the string theory on an actually singular Calabi-Yau manifold is a super conformal field theory \[13, 14\]. Generally, a superconformal theory has the superconformal \( S \) generators besides the ordinary \( Q \) generators. The extra spacetime supercharges in the singular Calabi-Yau models correspond to the superconformal \( S \) generators of the dual superconformal field theory.

The organization of this paper is as follows. In section 2, we review the character identities. In section 3, we treat the compact Gepner models. We show that a compact Gepner model includes the appropriate coset models. In section 4, we treat the noncompact Gepner models. The last section is devoted to conclusions and discussions.

## 2 Character identities

In this section, we review the character identities of eq.(1.1) and their properties.

The coset CFT \( \text{so}(8)/G_{\text{hol}} \) is essential to the spacetime supersymmetry of a superstring compactification with a manifold of special holonomy \( G_{\text{hol}} \). The character of this coset CFT is defined by the branching relation

\[
\chi_{\text{so}(8)} = \sum_{\lambda} \chi_{s,\lambda} \chi_{G_{\text{hol}}}. \tag{2.1}
\]

In order to express the character identities shortly, we define the vanishing functions \( \xi_{G_{\text{hol}}}^{G_{\text{hol}}} \) as

\[
\xi_{G_{\text{hol}}}^{G_{\text{hol}}} = \chi_{\text{vec},\lambda}^{\text{so}(8)/G_{\text{hol}}} - \chi_{\text{spi},\lambda}^{\text{so}(8)/G_{\text{hol}}}. \tag{2.2}
\]

This function \( \xi_{G_{\text{hol}}}^{G_{\text{hol}}} \) express the difference between the number of bosons and the number of fermions. By using this function \( \xi_{G_{\text{hol}}}^{G_{\text{hol}}} \), the character identity (1.1) can be written as \( \xi_{G_{\text{hol}}}^{G_{\text{hol}}} = 0 \). In order to consider the flat case as in the same manner, it is useful to denote \( \xi^{(1)} := \chi_{\text{vec}}^{\text{so}(8)} - \chi_{\text{spi}}^{\text{so}(8)} \). The Jacobi’s abstruse identity can be written as \( \xi^{(1)} = 0 \). With these notations and the branching relation (2.1), we obtain the equation

\[
\xi^{(1)} = \sum_{\lambda} \xi_{G_{\text{hol}}}^{G_{\text{hol}}} \chi_{G_{\text{hol}}}^{G_{\text{hol}}}. \tag{2.2}
\]

The left hand side of this equation vanishes due to the Jacobi’s abstruse identity and this is an evidence for the character identities \( \xi_{G_{\text{hol}}}^{G_{\text{hol}}} = 0 \).
The modular transformation laws of these functions are necessary in order to construct modular invariant partition functions. The modular properties of $\xi^{(1)}$ become

$$\xi^{(1)}(\tau + 1) = e^{\left[\frac{1}{3}\right]} \xi^{(1)}(\tau), \quad \xi^{(1)}(-1/\tau) = \xi^{(1)}(\tau),$$

where $e[x] := \exp(2\pi ix)$. By using these formulae, the modular properties of $\xi^{G_{\text{hol}}}$ can be read from the branching relation (2.2) and they become

$$\xi^{G_{\text{hol}}}(\tau + 1) = e^{\left[\frac{1}{3} - h_\lambda + c_{G_{\text{hol}}}/24\right]} \xi^{G_{\text{hol}}}(\tau), \quad \xi^{G_{\text{hol}}}(-1/\tau) = \sum_{\lambda'} S^{G_{\text{hol}}}_{\lambda'\lambda} \xi^{G_{\text{hol}}}(\tau),$$

where $h_\lambda$ is the conformal dimension of the representation $\lambda$, $c_{G_{\text{hol}}}$ is the central charge of the level 1 affine $G_{\text{hol}}$, and $S^{G_{\text{hol}}}_{\lambda'\lambda}$ is the modular S matrix of the level 1 affine $G_{\text{hol}}$.

Note that the relation (1.1) is satisfied not for all $G_{\text{hol}}$ and embeddings. It is clear that when $\chi^{\text{so}(8)}_{\text{vec},\lambda} \neq 0$ and $\chi^{\text{so}(8)}_{\text{spl},\lambda} = 0$ or vice versa from the selection rules, the relation (1.1) is not satisfied.

In this paper, we treat the cases $G_{\text{hol}} = \text{su}(2)$, $\text{su}(3)$, and $G_2$. In these cases, the relation (1.1) is satisfied. Let us see the relation explicitly for each case.

### 2.1 $\text{su}(2)$ holonomy case

The integrable highest weight representations of the level 1 affine $\text{su}(2)$ are the basic representation and the fundamental representation. We denote these two representations by $a = 0, 1$, respectively. Then, the vanishing function $\xi^{\text{su}(2)}_a$ becomes

$$\xi^{\text{su}(2)}_a = \frac{1}{\eta^2} \sum_{s \in \mathbb{Z}_4} (-1)^s \Theta_{s,2} \Theta_{s+2a+1,2} \Theta_{s+a,1}. \quad (2.3)$$

$\xi^{\text{su}(2)}_a = 0$ is the theta identity found in [2].

Actually, $\xi^{\text{su}(2)}_a$ is related to $\xi^{\text{su}(3)}_a$. We can prove the identity for the $\text{su}(2)$ holonomy by using the result of the identity for the $\text{su}(3)$ holonomy. We mention this proof in the next subsection after we see the form of the $\xi^{\text{su}(3)}_a$.

### 2.2 $\text{su}(3)$ holonomy case

The integrable highest weight representations of the level 1 affine $\text{su}(3)$ are the basic, the fundamental, and the conjugate fundamental representation. We denote these three representations by $a = 0, 1, -1$, respectively. The explicit form of the $\xi^{\text{su}(3)}_a$ can be written as

$$\xi^{\text{su}(3)}_a = \frac{1}{\eta^2} \sum_{s \in \mathbb{Z}_4} (-1)^s \Theta_{6+4a-3s,6} \Theta_{s,2}. \quad (2.4)$$
The identity $\xi_{a}^{\text{su}(3)} = 0$ is found in [3,4].

We can prove the character formula $\xi_{a}^{\text{su}(3)} = 0$ by explicit calculation. By using the product formula (A.2) and $\Theta_{m,k}(\tau) = \Theta_{-m,k}(\tau)$, $\xi_{a}^{\text{su}(3)}$ can be written as

$$
\xi_{a}^{\text{su}(3)} = \frac{1}{\eta^{2}} \sum_{s \in \mathbb{Z}} (-1)^{s} \Theta_{6-4a+3s,6} \Theta_{s,2} = \frac{1}{\eta^{2}} \sum_{r \in \mathbb{Z}} \Theta_{-12+8a+24r,96} \sum_{s \in \mathbb{Z}} (-1)^{s} \Theta_{-2+4(s+r-a+2),8}.
$$

In this equation, the second sum vanishes because of the equation

$$
\sum_{s \in \mathbb{Z}} (-1)^{s} \Theta_{-2+4(s+r-a+2),8} = (-1)^{r-a}(\Theta_{-2,8} - \Theta_{2,8} + \Theta_{-6,8} - \Theta_{6,8}) = 0.
$$

We can conclude that the identity $\xi_{a}^{\text{su}(3)} = 0$ is actually satisfied.

As mentioned in the previous subsection, we can write the relation between $\xi_{a}^{\text{su}(3)}(\tau)$, $a = -1,0,1$ and $\xi_{a}^{\text{su}(2)}(\tau)$, $a = 0,1$ as

$$
\xi_{a}^{\text{su}(2)} = \sum_{b=-1,0,1} \frac{\Theta_{3a+2b+1,3}}{\eta} \xi_{b}^{\text{su}(3)}.
$$

By using this formula and the proven identity $\xi_{a}^{\text{su}(3)}(\tau) = 0$, we can prove the character identity $\xi_{a}^{\text{su}(2)}(\tau) = 0$.

The $\xi_{a}^{\text{su}(3)}$ and the $\xi_{a}^{G_{2}}$ are also related. We express this relation in the next subsection.

2.3 $G_{2}$ holonomy case

The integrable highest weight representations of the level 1 affine $G_{2}$ are the basic and the fundamental representation. We denote these representations by $a = 0,1$, respectively. The explicit form of the $\xi_{a}^{G_{2}}$ can be written as

$$
\xi_{0}^{G_{2}} = \chi_{1/2}^{\text{Ising}} \lambda_{0}^{\text{Tri}} + \chi_{3/2}^{\text{Ising}} \lambda_{7/16}^{\text{Tri}} - \chi_{1/16}^{\text{Ising}} \lambda_{1/2}^{\text{Tri}},
$$

$$
\xi_{1}^{G_{2}} = \chi_{3/5}^{\text{Ising}} \lambda_{1/2}^{\text{Tri}} + \chi_{10}^{\text{Ising}} \lambda_{1/16}^{\text{Tri}} - \chi_{3/80}^{\text{Ising}} \lambda_{1/10}^{\text{Tri}}.
$$

where the $\chi^{\text{Ising}}$'s and $\chi^{\text{Tri}}$'s are the Virasoro characters of the Ising and tricritical Ising model, respectively. The explicit forms of these characters are shown in the Appendix [A.3]. The identity $\xi_{a}^{G_{2}} = 0$ is found in [1].

As mentioned in the previous subsection, $\xi_{a}^{G_{2}}$ and $\xi_{a}^{\text{su}(3)}$ are related by the equations

$$
\xi_{0}^{\text{su}(3)} = \xi_{0}^{G_{2}} C_{0}^{3\text{-Potts}} + \xi_{1}^{G_{2}} C_{5/2}^{3\text{-Potts}},
$$

$$
\xi_{\pm 1}^{\text{su}(3)} = \xi_{0}^{G_{2}} C_{2/3}^{3\text{-Potts}} + \xi_{1}^{G_{2}} C_{1/15}^{3\text{-Potts}},
$$

where $C_{3\text{-Potts}}$'s are the $W_{3}$ characters of the 3-state Potts model. The explicit forms of these characters are shown in the Appendix [A.3].
3 Compact Gepner models

In this section, for a warming up, we show that the partition functions of the compact Gepner models for Calabi-Yau 3-fold compactifications include $\xi_{\text{su}(3)}$. This fact is shown in \cite{3} by spectral flow method. In this paper, we use another method of construction — beta method \cite{15}.

The Gepner’s construction is as follows. We will work in the lightcone gauge. Then, the total central charge should be 12. We consider the model with 4-dimensional flat spacetime, then we need $c = 9$ theory for the internal space. The direct product of $\mathcal{N} = 2$ minimal models is used to this internal space. The total theory in lightcone gauge includes $R$ of the $\mathcal{N} = 2$ minimal models (internal space), and 2 pairs of free bosons and fermions (transverse directions of spacetime)

$$\mathbb{R}^2 \times M_{N_1} \times \cdots \times M_{N_R},$$

where $M_N$ stands for the level $(N - 2)$ $\mathcal{N} = 2$ minimal model. Since the total central charge should be $c = 12$, the following relation holds

$$1 + \sum_{j=1}^{R} \frac{N_j - 2}{N_j} = 4.$$

Now, let us construct the modular invariant partition function, by following \cite{15}. Let us define the characters of the total theory as

$$\chi^{\lambda,\mu}(\tau, z) := \chi^{\text{so}(2)}_{s_0}(\tau, z) \prod_{j=1}^{R} \chi^{(N_j);\ell_j, s_j}(\tau, z),$$

where $\chi^{\text{so}(2)}_{s_0}(\tau, z)$ is the character of the affine $\text{so}(2)$ constructed from the two free fermions of transverse directions of spacetime, and each $\chi^{(N_j);\ell_j, s_j}(\tau, z)$ is the contribution of the $j$th minimal model. $\lambda$ and $\mu$ are vectors of labels defined as

$$\lambda := (\ell_1, \ldots, \ell_R),$$
$$\mu := (s_0; s_1, \ldots, s_R; m_1, \ldots, m_R).$$

Next we introduce an inner product between two $\mu$’s as

$$\mu \cdot \mu' := -\sum_{j=0}^{R} \frac{s_j s'_j}{4} + \sum_{j=1}^{R} \frac{m_j m'_j}{2N_j}.$$ We also introduce $\beta$ vectors, which is the same type vector as $\mu$, defined as

$$\beta_0 := (1; 1, \ldots, 1; 1, \ldots, 1),$$
$$\beta_j := (2; 0, \ldots, 0; 2; 0, \ldots, 0; 0, \ldots, 0), \quad (j = 1, \ldots, R).$$
By using these notations, the GSO condition (the total U(1) charge is to be odd integer) and the condition of spin structure (all sub-theories are to be in NS sector, or all to be in R sector) can be written as

\[ 2\beta_0 \cdot \mu \in 2\mathbb{Z} + 1, \quad \beta_j \cdot \mu \in \mathbb{Z}, \quad j = 1, \ldots, R. \tag{3.1} \]

We can only use the modules satisfying the condition (3.1) to construct the partition function. We call this constraint (3.1) “beta constraint”.

Let us define the “orbit” of the character \( F_{\lambda}^{\mu}(\tau, z) \) for \( \mu \)'s satisfying the beta constraint (3.1) by the equation

\[
F_{\mu}(\tau, z) := \sum_{b_0 \in \mathbb{Z}_2K, b_j \in \mathbb{Z}_2} (-1)^{s_0 + b_0\lambda_\mu + \sum_{j=0}^R b_j\beta_j}(\tau, z),
\]

where \( K := \text{lcm}(N_j, 2) \).

By using this \( F_{\mu}^{\lambda} \), the partition function can be written as

\[
Z(\tau, \bar{\tau}) = \frac{1}{\sqrt{\tau_2|\eta|^2}} \times \frac{1}{4^{R^2}K} \sum_{\lambda, \mu} |F_{\mu}^{\lambda}(\tau, 0)|^2,
\]

where the factor \( \frac{1}{\sqrt{\tau_2|\eta|^2}} \) is the contribution from the two free bosons of flat spacetime. The sum \( \sum_{\lambda, \mu} \) means that the labels satisfy the beta constraint (3.1). Since there is spacetime supersymmetry, this partition function should vanish. For the partition function to vanish, the orbit \( F_{\mu}^{\lambda}(\tau, 0) \) should also vanish.

Because this Gepner model is a solvable realization of a Calabi-Yau compactification, we expect the \( F_{\mu}^{\lambda} \) is decomposed by \( \xi^{\text{su}(3)}_a \)'s, which mean, for some functions \( F_{\mu,a}^{\lambda}(\tau) \), we can write

\[
F_{\mu}^{\lambda}(\tau, z) = \sum_{a \in \mathbb{Z}_3} F_{\mu,a}^{\lambda}(\tau)\xi^{\text{su}(3)}_a(\tau, z), \tag{3.2}
\]

where \( z \) dependent \( \xi^{\text{su}(3)}_a(\tau, z) \) is defined simply as

\[
\xi^{\text{su}(3)}_a(\tau, z) = \frac{1}{\eta^2} \sum_{s \in \mathbb{Z}_4} (-1)^s \Theta_{6+4a-3s,6}(\tau, z)\Theta_{s,2}(\tau, z).
\]

It reduces to the definition (2.4) when we set \( z = 0 \).

Now, let us show the decomposition (3.2) and calculate the branching function \( F_{\mu,a}^{\lambda}(\tau) \). To do this, the product formula of the multiple theta functions is useful. This formula
can be written as

$$\prod_{j=1}^{R} \Theta_{m_j, k_j}(\tau, z) = \sum_{r \in \mathbb{Z}_k} B_{\{m_j\};\{k_j\}}^r(\tau) \Theta_{\sum_j m_j + 2r \cdot k}(\tau, z),$$

$$B_{\{m_j\};\{k_j\}}^r(\tau) := \sum_{\{n_j\}} \delta_{r-\sum_j k_j n_j, 0} q^\sum_j k_j \left( n_j + \frac{m_j}{2k_j} \right)^2 - \frac{4}{\text{Re}} \left[ \sum_j (m_j + 2k_j n_j) \right]^2,$$

where \( q := e[\tau] \). By using the formula (3.3) and formulas in appendix A, we can show the decomposition (3.2) is actually correct and the branching function \( F_{\lambda, \mu, a}^\lambda(\tau) \)'s are calculated concretely.

$$F_{\mu, a}^\lambda(\tau) = \eta(\tau) \sum_{b_j \in \mathbb{Z}_2} \alpha_{\mu + \sum_j b_j \beta_j, a - s_0 - 2 \sum_j b_j + 1}^\lambda(\tau),$$

$$\alpha_{\mu, v}^\lambda(\tau) := \frac{1}{2} \sum_{\{r_j\}} \sum_{r_0 \in \mathbb{Z}} \prod_{j=1}^{R} c_{m_j - s_j - 4r_j}^{(N_j - 2)}(\tau) \sum_{\{p_j\}} \sum_{u \in \mathbb{Z}_K} B_{\{2KQ_j(m_j, s_j)\};\{2KJ_j(N_j - 2)\}}^{2Ku} \left( \tau \right) \times \delta_2^{\sum_j Q_j(m_j, s_j) + 4u - s_0 - 2(6v_0 + 2v + 1), 0},$$

$$\sum_{\{r_j\}} := \prod_{j=1}^{R} \sum_{r_j \in \mathbb{Z}_{N_j - 2}}, \quad \sum_{\{p_j\}} := \prod_{j=1}^{R} \sum_{p_j \in \mathbb{Z}_{J_j}}, \quad J_j := \frac{N_j}{R},$$

$$Q_j(m_j, s_j) := m_j/N_j - s_j/2 + 2r_j + 2p_j(N_j - 2).$$

The detailed calculations are shown in appendix B.

We have shown that the Gepner model includes the \( \text{so}(8)/\text{su}(3) \) CFT in the appropriate form. Especially, by using the character identity \( \xi_{a}^{\text{su}(3)}(\tau, 0) = 0 \) and the decomposition (3.2), we have shown the identity \( F_{\mu}^\lambda(\tau, 0) = 0 \), and the partition function actually vanishes as expected.

We apply the same procedure to the models of \( [9, 12] \) in the next section.

## 4 Noncompact Gepner models

Let us proceed to the noncompact Gepner models. Each of the noncompact Gepner models includes the continuous part and discrete part due to its noncompactness.

We consider each parts below, and show their partition functions actually vanishes by using the character identities. For the continuous part, it seems that they have twice as many supersymmetry charges as expected from their holonomies. On the other hand, the discrete part seems to have just the same amount of the supersymmetry as expected.
4.1 Continuous part of the noncompact Calabi-Yau 4-fold compactifications

We consider the continuous part of a noncompact Calabi-Yau 4-fold compactification in this subsection. We show the partition function includes the $\xi^{\text{su}(3)}$, which suggests this compactification has twice as many supersymmetry charges as expected for the Calabi-Yau 4-fold.

First, we construct the partition function by the beta method, following [9]. In order to save the notations, we use the same notations of the beta methods for the various compactifications.

The total theory is the direct product of the \(N=2\) Liouville model, and \(R\) of minimal models with levels \((N_j - 2), \ j = 1, \ldots, R\). We define the integer \(K\) as \(K = \text{lcm}(N_j, 2)\), and the integer \(J\) as \(Q^2 = \frac{2K}{J}\), where \(Q\) is the Liouville background charge. The total character becomes

\[
\chi^\lambda_\mu(\tau, z) = \chi^\text{so}(2)_{s_0}(\tau, z) \frac{\Theta_{m_0, KJ}(\tau, 2z/K)}{\eta(\tau)} \prod_{j=1}^{R} \chi^{(N_j)}_{m_j, s_j}(\tau, z),
\]

where \(\chi^\text{so}(2)\) is the character of 2 fermions in the \(N=2\) Liouville model, and \(\Theta_{m_0, KJ}(\tau, 2z/K)\) is the character of the \(S^1\) boson in \(N=2\) Liouville. We also define the inner product between two \(\mu\) vectors as

\[
\mu \cdot \mu' = -\frac{s_0 s_0'}{4} - \sum_{j=1}^{R} \frac{s_j s_j'}{4} - \frac{m_0 m_0'}{2KJ} + \sum_{j=1}^{R} \frac{m_j m_j'}{2N_j}.
\]

We use the beta vectors to construct the modular invariant partition function. The beta vectors are defined as

\[
\beta_0 := (1; 1, \ldots, 1; -J; 1, \ldots, 1),
\]

\[
\beta_j := (2; 0, \ldots, 0, 2, 0, \ldots, 0; 0; 0, \ldots, 0), \quad (j = 1, \ldots, R).
\]

The GSO condition, and the spin structure condition can be written as

\[
2\beta_0 \cdot \mu \in 2\mathbb{Z} + 1, \quad \beta_j \cdot \mu \in \mathbb{Z}, \quad (j = 1, \ldots, R).
\]

Let us call the above constraint “the beta constraint”. The orbits are defined for \(\mu\) vectors which satisfy the beta constraint (4.1) as

\[
F^\lambda_\mu(\tau, z) := \sum_{b_0 \in \mathbb{Z}_{2K}, b_j \in \mathbb{Z}_2} \chi^\lambda_{\mu+b_0, b_0} \prod_{j=1}^{R} \theta_j(\tau, z)(-1)^{s_0 + b_0}.
\]
Using these notations, we can write down the modular invariant partition function

\[ Z = \frac{1}{\sqrt{\tau_2}} \eta(\tau) \frac{1}{4R^2 K} \sum_{\lambda,\mu}^{\text{beta}} \left| F^\lambda_\mu(\tau) \right|^2, \]

where the sum \( \sum_{\lambda,\mu}^{\text{beta}} \) is taken under the beta constraint (4.1). This partition function can be checked to be actually modular invariant [1].

The spacetime supersymmetry implies \( Z = 0 \) or equivalently \( F^\lambda_\mu(\tau, 0) = 0 \) for \( \lambda, \mu \) satisfying the beta constraint. We prove this identity by showing the decomposition

\[ F^\lambda_\mu(\tau, z) = \sum_{a \in \mathbb{Z}_3} F^\lambda_\mu,a(\tau) \xi^\text{su(3)}(\tau, z), \]  

(4.2)

is possible for some branching function \( F^\lambda_\mu,a(\tau) \).

We can show the decomposition is actually possible and obtain the branching function concretely

\[ F^\lambda_\mu,a(\tau) = \eta(\tau) \sum_{b_j \in \mathbb{Z}_2} \alpha^\lambda_\mu + \sum_{j=1}^R b_j \beta_j, a-s_0-2 \sum_{j=1}^R b_j + 1(\tau), \]

\[ \alpha^\lambda_\mu(\tau) := \frac{1}{2} \sum_{\{r_j\},\{p_j\}} \sum_{v_0 \in \mathbb{Z}_2} \left( \prod_{j=1}^R C_{\kappa_j}^{(N_j-2),K_j}(\tau) \right) \sum_{u \in \mathbb{Z}_{2KU}} B^{2KU}_{2\kappa_j}(\tau) \delta_{2 \sum_{j=0}^R Q_j + 4u-s_0-2(6v_0+2v+1),0}, \]

where the \( B^{2KU}_{2\kappa_j}(\tau) \)'s are the branching functions of theta functions defined by eq.(3.3). We also use the notations

\[ \kappa_j := 2KJ_j(N_j - 2), \quad \kappa_0 := 4KJ, \]

\[ Q_j := \frac{m_j}{N_j} \frac{s_j}{2} + 2r_j + 2p_j(N_j - 2), \quad Q_0 := \frac{m_0}{K} + 2Jp_0, \quad (j = 1, \ldots, R), \]

\[ \sum_{\{r_j\},\{p_j\}} = \sum_{p_0 \in \mathbb{Z}_2} \prod_{j=1}^R \left( \sum_{r_j \in \mathbb{Z}_{N_j-2}} \sum_{p_j \in \mathbb{Z}_{J_j}} \right), \quad J_j = K/N_j. \]

The decomposition (4.2) and the identity \( \xi^\text{su(3)}(\tau) = 0 \) lead to \( F^\lambda_\mu(\tau) = 0 \) and \( Z = 0 \).

Actually, the partition function includes \( \xi^\text{su(3)}(\tau) \) as shown eq.(4.2) and this fact suggests that this compactification have the same amount of supersymmetry as Calabi-Yau 3-fold compactifications, which is the twice as large amount as the Calabi-Yau 4-fold compactifications. This is an evidence that the holographic dual of this singular compactification is a super conformal field theory [14]. The extra SUSY generators are superconformal S generators in the superconformal field theory.
4.2 Continuous part of the noncompact Calabi-Yau 3-fold compactifications

We can perform the similar decomposition for the continuous part of a noncompact Calabi-
Yau 3-fold compactification.

First, we construct the modular invariant partition functions by the beta method. The
characters, $\mu$ vectors and the inner product between them are defined as

$$K := \text{lcm}(N_j, 2), \quad J := \frac{KQ^2}{2},$$

$$\chi_{\mu}^\lambda(\tau, z) = \chi_{s_{-1}}^{so(2)}(\tau, z)\chi_{s_0}^{so(2)}(\tau, z)\frac{\Theta_{m_0, KJ}(\tau, 2z/K)}{\eta(\tau)}\prod_{j=1}^R \chi_{m_j}^{(N_j); \ell_j, s_j}(\tau, z),$$

$$\lambda := (\ell_1, \ldots, \ell_R),$$

$$\mu := (s_{-1}, s_0; s_1, \ldots, s_R; m_0; m_1, \ldots, m_R),$$

$$\mu \cdot \mu' := -\frac{s_{-1}s_{-1}'}{4} - \frac{s_0s_0'}{4} - \sum_{j=1}^R \frac{s_js_j'}{4} - \frac{m_0m_0'}{2KJ} + \sum_{j=1}^R \frac{m_jm_j'}{2N_j}.$$ 

Here, the $\chi_{s_{-1}}^{so(2)}(\tau, z)$ is the character of two free fermions in transverse spacetime directions, and $\chi_{s_0}^{so(2)}(\tau, z)$ is the character of the two free fermions in $\mathcal{N} = 2$ Liouville theory.

The beta vectors are also defined as

$$\beta_0 := (1, 1; 1, \ldots, 1; -J; 1, \ldots, 1),$$

$$\beta_{-1} := (2, 2; 0, \ldots, 0; 0; 0, \ldots, 0),$$

$$\beta_j := (0, 2; 0, \ldots, 0; 2, 0, \ldots, 0; 0; 0, \ldots, 0), \quad (j = -1, 1, \ldots, R).$$

The beta constraint can be written as

$$2\beta_0 \cdot \mu \in 2\mathbb{Z} + 1, \quad \beta_j \cdot \mu \in \mathbb{Z}, \quad (j = -1, 1, \ldots, R).$$

The orbits and the partition functions can be written as

$$F_{\mu}^{\lambda}(\tau, z) := \sum_{b_0 \in \mathbb{Z}_2K, b_{-1} \in \mathbb{Z}_2, b_j \in \mathbb{Z}_2} \chi_{\mu+b_0\beta_0+b_{-1}\beta_{-1}+\sum_{j=1}^R b_j\beta_j}^{\lambda}(\tau, z)(-1)^{s_0+b_0},$$

$$Z(\tau, \bar{\tau}) = (\sqrt{\tau_2}|\eta(\tau)|^2)^{-3} \frac{1}{4R^4K} \sum_{\lambda, \mu} \text{beta} |F_{\mu}^{\lambda}(\tau, 0)|^2,$$

where $(\sqrt{\tau_2}|\eta(\tau)|^2)^{-3}$ is the contribution from the two free bosons in the transverse spacetime directions and the linear dilaton in the $\mathcal{N} = 2$ Liouville theory.
The spacetime supersymmetry implies $Z = 0$ and $F^\lambda_\mu(\tau, 0) = 0$. We prove this fact by showing the decomposition

$$F^\lambda_\mu(\tau, z) = \sum_{a \in \mathbb{Z}_2} F^\lambda^a_{\mu,a}(\tau) \xi^\text{su}(2)_a(\tau, z),$$

for some branching function $F^\lambda_{\mu,a}(\tau)$. Here we use the $z$ dependent $\xi^\text{su}(2)_a(\tau, z)$ defined as

$$\xi^\text{su}(2)_a(\tau, z) : = \sum_{s \in \mathbb{Z}_4} \chi^{\text{so}(2)}_s(\tau, z) \chi^{\text{so}(2)}_{2a+s}(\tau, z) \frac{\Theta_{a+s+1,1}(\tau, 2z)}{\eta(\tau)} (-1)^s.$$

These $\xi^\text{su}(2)_a(\tau, z)$ reduce to the functions defined in eq. (2.3) when we set $z = 0$.

The branching functions are obtained as follows.

$$F^\lambda_{\mu,a}(\tau) = \sum_{\{b_j\} \in \mathbb{Z}_2} \sum v \in \mathbb{Z}_2 \alpha^\lambda_{\mu,v}(\tau) \sum_{R} B_{2K_j \{\kappa_j\};\{\kappa_j\}}^u(\tau),$$

where the $B_{2K_j \{\kappa_j\};\{\kappa_j\}}^u(\tau)$'s are the branching functions of theta functions defined by eq.(3.3). We also use the notations

$$\kappa_j := 2K J_j(N_j - 2), \quad \kappa_0 := 4K J, \quad Q_j := \frac{m_j}{N_j} - \frac{s_j}{2} + 2r_j + 2p_j(N_j - 2), \quad Q_0 := \frac{m_0}{K} + 2Jp_0, \quad (j = 1, \ldots, R).$$

The decomposition (4.2) and the identity $\xi^\text{su}(2)(\tau) = 0$ lead to $F^\lambda_\mu(\tau) = 0$ and $Z = 0$.

In the case of the continuous part in a noncompact Calabi-Yau 3-fold compactification, the result suggests that this system has the same amount of supersymmetry as the Calabi-Yau 2-fold compactifications. The extra supersymmetry charges seem to correspond to superconformal S generators in the dual superconformal field theory.

### 4.3 Comments on the continuous part of the noncompact Calabi-Yau 2-fold compactifications and $G_2$ compactifications

As for the continuous part of a noncompact Calabi-Yau 2-fold compactification (an ALE compactification), it is shown in [8, 11] that the partition functions of all models of this
type includes the \( \xi^{(1)} \) and vanish due to the Jacobi’s abstruse identity itself. This result shows that the models of this type have the same amount of the supersymmetry as the flat 10-dimensional model, that is, twice as many supersymmetry as the ALE compactification. This fact is consistent with the conjecture that an actually singular ALE compactification is holographically dual to a 6 dimensional superconformal field theory.

We also comment about the noncompact \( G_2 \) holonomy models obtained in \cite{footnote_G2}. The partition functions of these models include the \( \xi^{su(3)} \). This fact suggests that these models have twice as many supercharges as expected for a \( G_2 \) compactification. For this reason, these actually singular models should be also holographically dual to 3-dimensional superconformal theories.

4.4 The discrete part of the ALE compactifications

In the noncompact theory, besides the continuous part of the spectrum which we treat in previous subsections, there is the discrete part of the spectrum. In this subsection, we consider the discrete part of the spectrum of ALE models \cite{footnote_ALE}.

We use the character of the \( SL(2,\mathbb{R})/U(1) \) Kazama-Suzuki model. The character of the discrete series of the \( SL(2,\mathbb{R})/U(1) \) Kazama-Suzuki model summed through the \( SL(2,\mathbb{R}) \) spectral flow orbit can be written as

\[
\tilde{\chi}^{(N)}_{m,\ell, s}(N) = \sum_{r \in \mathbb{Z}} \tilde{c}_{m-s-4r}(\tau) \Theta_{-2m-Ns+4Nr,2N}(\tau, z/N),
\]

where \( \tilde{c}^{(k)}(\tau) \) is the “\( SL(2,\mathbb{R}) \) string function”. The explicit form of this string function is written in \cite{footnote_ALE}, but we do not need it here.

Again, we construct the partition function by the beta method. We use the following notations

\[
K := \text{lcm}(N, 2), \quad J := K/N,
\]

\[
\chi^\lambda_\mu = \chi^{so(2)}_{s-1}(\tau, z) \chi^{so(2)}_{s}(\tau, z) \chi^{(N)\ell_1, s_1}(\tau, z) \chi^{(N)\ell_2, s_2}(\tau, z),
\]

\[
\lambda := (\ell_1, \ell_2), \quad \mu := (s_1, s_0; s_1, s_2; m_1, m_2),
\]

\[
\mu \cdot \mu' = -\frac{s_{-1}s'_{-1}}{4} - \frac{s_0s'_0}{4} - \frac{s_1s'_1}{4} - \frac{s_2s'_2}{4} - \frac{m_1m'_1}{2N} + \frac{m_2m'_2}{2N},
\]

\[
\beta_0 := (1, 1; 1, 1; 1, 1), \quad \beta_{-1} := (2, 2; 0; 0, 0),
\]

\[
\beta_1 := (0, 2; 2, 0; 0, 0), \quad \beta_2 := (0, 2; 0, 2; 0, 0),
\]
The orbit and the partition function can be written as

\[ F_\mu^\lambda := \sum_{b_0 \in \mathbb{Z}_2, \ b_{-1}, b_1, b_2 \in \mathbb{Z}_2} \chi_\lambda \mu + b_0 \beta_{b_1 + b_1 \beta + b_2} (\tau, z) (-1)^{s_0 + b_0}, \]

\[ Z = (\sqrt{\tau} |\eta(\tau)|^2)^{-2} \frac{1}{2^{2^{3/2} K}} \sum_{\lambda, \mu} |F_\mu^\lambda|^2. \]

We can again show \( Z = 0, \ F_\mu^\lambda = 0 \) by showing the decomposition

\[ F_\mu^\lambda (\tau, z) = \sum_{a \in \mathbb{Z}_2} F_{\mu,a}^\lambda (\tau) e^{su(2)} (\tau, z). \]

The \( F_{\mu,a}^\lambda (\tau) \) can be written as

\[ F_{\mu,a}^\lambda (\tau) = \sum_{\{b_j\}} \sum_{v \in \mathbb{Z}_2} G_{\mu + \sum_{j=1}^R \beta_j v} (\tau) \delta_{2a - (s_1 - s_0 - 2 \sum_{j=1}^R b_j)} \]

\[ \alpha_{\mu,v}^\lambda (\tau) := \frac{1}{2} \sum_{\{r_j\}, \{p_j\}} \sum_{v_0 \in \mathbb{Z}} \left( e^\ell_1_{-m_1 - s_1 - 4r_1} (\tau) e^\ell_2_{m_2 - s_2 - 4r_2} (\tau) \right) \]

\[ \times \sum_{u \in \mathbb{Z}_2} B_{2^{2KJ} \{2K\}}^{2Ku \{\kappa_j\}} (\tau) \delta_{2(Q_1 + Q_2) - s_1 - s_0 - 2(4v_0 + 2v + 1), 0}. \]

Here we use the notations

\[ \kappa_1 := 2KJ(N + 2), \quad \kappa_2 := 2KJ(N - 2), \]

\[ Q_1 := -\frac{m_1}{N} - \frac{s_1}{2} + 2r_1 + 2p_1(N + 2), \quad Q_2 := \frac{m_2}{N} - \frac{s_2}{2} + 2r_2 + 2p_2(N - 2). \]

In contrast to the continuous part, the discrete part seems to have just the same amount of the supersymmetry as expected for an ALE compactification. This is because the discrete part exists only in the theory of deformed singularity, and the holographic dual theory is relevantly perturbed and does not have conformal symmetry \[16, 17\].

5 Conclusion

In this paper, we show the partition functions of several supersymmetric string models include the coset characters with their vanishing forms. We have shown that all the partition functions of the supersymmetric models treated here do vanish.

The compact Gepner models and the discrete part of the noncompact Gepner models include the appropriate characters with their supersymmetry. But the partition functions of continuous part of noncompact Gepner models look as if they have twice as many supersymmetry as expected. We claim that this fact shows the holographic dual of an actually singular compactification is a super conformal field theory, and the extra supercharges
correspond to superconformal S generators in the holographic dual superconformal field theory.

On the other hand, the discrete part exist only in the deformed singularity, and the holographic dual theory is no longer conformally invariant and has renormalization group flow. This is why the partition function of the discrete part includes just the same amount of the supersymmetry as expected.

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A Theta functions and characters

In this appendix A, we collect several notations and summarize properties of theta functions. We use the following notations in this paper;

\[ e[x] := \exp(2\pi i x), \quad \delta_m \mod N := \begin{cases} 1 & (m \equiv 0 \mod N), \\ 0 & \text{(others)}, \end{cases} \]

where \( m \) and \( N \) are integers. For a function \( f(\tau, z) \), we sometimes use abbreviated notation \( f := f(\tau) := f(\tau, z = 0) \).

A.1 Theta functions

A set of SU(2) classical theta functions are defined as

\[ \Theta_{m,k}(\tau, z) = \sum_{n \in \mathbb{Z}} q^{k(n+m)^2} y^{k(n+m)}, \]

with \( q := e[\tau], y := e[z] \). We often uses the formulas for integer \( m, k, p \)

\[ \Theta_{m/p,k/p}(\tau, z) = \sum_{t \in \mathbb{Z}} \Theta_{m+2kt,pk}(\tau, z/p). \tag{A.1} \]

We also use the following product formula of the theta functions

\[ \Theta_{m_1,k_1}(\tau, z_1)\Theta_{m_2,k_2}(\tau, z_2) = \sum_{r \in \mathbb{Z}_{k_1+k_2}} \Theta_{m_2k_1-m_1k_2+2k_1k_2, k_1+k_2}(\tau, u)\Theta_{m_1+m_2+2k_2r, k_1+k_2}(\tau, v), \]

\[ u = \frac{z_2 - z_1}{k_1 + k_2}, \quad v = \frac{k_1z_1 + k_2z_2}{k_1 + k_2}. \tag{A.2} \]

The Jacobi’s theta functions are also defined in our convention

\[ \theta_1(\tau, z) := i \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2}(n-\frac{1}{2})^2} y^{n-\frac{1}{2}}, \theta_2(\tau, z) := \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}(n-\frac{1}{2})^2} y^{n-\frac{1}{2}}, \]
\[ \theta_3(\tau, z) := \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}n^2} y^n, \quad \theta_4(\tau, z) := \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2}n^2} y^n. \]

The above two kinds of theta functions are related through a set of linear transformations

\[ 2\Theta_{0,2}(\tau, z) = \theta_3(\tau, z) + \theta_4(\tau, z), \quad 2\Theta_{1,2}(\tau, z) = \theta_2(\tau, z) + i\theta_1(\tau, z), \]
\[ 2\Theta_{2,2}(\tau, z) = \theta_3(\tau, z) - \theta_4(\tau, z), \quad 2\Theta_{3,2}(\tau, z) = \theta_2(\tau, z) - i\theta_1(\tau, z). \]

The Dedekind \( \eta \) function is represented as an infinite product

\[ \eta(\tau) := q^{\frac{1}{12}} \prod_{n=1}^{\infty} (1 - q^n). \]
A.2 \( \mathcal{N} = 2 \) minimal models

The unitary minimal models of the \( \mathcal{N} = 2 \) superconformal algebra is labeled by an integer \( k = 1, 2, 3, \ldots \). Instead of \( k \) we mainly use the “dual Coxeter number” of ADE classification \( N = k + 2 \). The Verma module of the level \((N - 2)\) minimal model is labeled by a set of three integers \((\ell, m, s)\)

\[
\ell = 0, 1, \ldots, N - 2, \quad m \in \mathbb{Z}_{2N}, \quad s \in \mathbb{Z}_4,
\]

\[
\ell + m + s \equiv 0 \mod 2, \quad (\ell, m, s) \cong (N - 2 - \ell, m + N, s + 2).
\]

We introduce a character \( \chi^{(N);\ell,s}_m(\tau, z) \) of a Verma module \((\ell, m, s)\) in the level \((N - 2)\) minimal model. The form of the character \( \chi^{(N);\ell,s}_m(\tau, z) \) can be written as

\[
\chi^{(N);\ell,s}_m(\tau, z) = \sum_{r \in \mathbb{Z}_{N-2}} c^{(N-2);\ell}_m(\tau, z),
\]

where \( c^{(N-2);\ell}_m \)'s are the string functions of the level \((N - 2)\) affine SU(2), defined by the branching relation

\[
\chi^{SU(2),(N-2)}_\ell(\tau, z) = \sum_{m \in \mathbb{Z}_{N-2}} c^{(N);\ell}_m(\tau) \Theta_{2m+N(-s+4r),2N(N-2)}(\tau, z/N),
\]

In this equation, \( \chi^{SU(2),(k)}_\ell(\tau, z) \)'s are the level \( k \) affine SU(2) characters expressed by the Weyl-Kac formula

\[
\chi^{SU(2),(k)}_\ell(\tau, z) = \frac{\Theta_{\ell+1,k+2}(\tau, z) - \Theta_{-\ell-1,k+2}(\tau, z)}{\Theta_{1,2}(\tau, z) - \Theta_{-1,2}(\tau, z)}.
\]

\[\text{A.3 Virasoro minimal models}\]

The unitary minimal models are labeled by an integer \( m \) \((m = 3, 4, 5, \ldots)\). Its central charge is given by a formula

\[
c = 1 - \frac{6}{m(m+1)}.
\]

The Verma modules of each minimal model is classified by integers \( r, s \) in the regions

\[
r = 1, 2, \ldots, m - 1, \quad s = 1, 2, \ldots, m, \quad \text{with} \ ms < (m+1)r.
\]

The conformal dimension of the primary field is specified by the set \((r, s)\) and is evaluated as

\[
h_{r,s} = \frac{(m + 1)r - ms)^2 - 1}{4m(m + 1)}.
\]
The characters of these minimal models can be expressed for the primary field labelled by \((r, s)\)

\[ \chi_{r,s}^{(m)} = \frac{1}{\eta(\tau)} \{ \Theta_{(m+1)r-ms,m(m+1)}(\tau) - \Theta_{(m+1)r+ms,m(m+1)}(\tau) \}. \]

We use \(m = 3, 4, 5\) minimal models in this paper. The details of properties of these models are listed in the following table:

- **Ising model** \((c = \frac{1}{2})\)
  
  \[ h_{1,1} = 0, \ h_{2,1} = \frac{1}{2}, \ h_{1,2} = \frac{1}{16}. \]
  
  We write the Virasoro characters for this model as \(\chi_{\text{Ising}}^{h_{r,s}}\).

- **Tricritical Ising model** \((c = \frac{7}{10})\)
  
  \[ h_{1,1} = 0, \ h_{2,1} = \frac{7}{16}, \ h_{1,2} = \frac{1}{10}, \ h_{1,3} = \frac{3}{5}, \ h_{2,2} = \frac{3}{80}, \ h_{3,1} = \frac{3}{2}. \]
  
  We write the Virasoro characters of this model as \(\chi_{\text{Tri}}^{h_{r,s}}\).

- **3-state Potts model** \((c = \frac{4}{5})\)
  
  \[ h_{1,1} = 0, \ h_{2,1} = \frac{2}{5}, \ h_{3,1} = \frac{7}{5}, \ h_{1,3} = \frac{2}{3}, \ h_{4,1} = 3, \ h_{2,3} = \frac{1}{15}. \]
  
  The notation \(\chi_{3}\text{-Potts}^{h_{r,s}}\) is used for Virasoro characters for this Potts model. But we mainly use \(W_3\) characters constructed from those of the Potts model

  \[
  C_{0}^{3\text{-Potts}} = \chi_{0}^{3\text{-Potts}} + \chi_{3}^{3\text{-Potts}}, \quad C_{2/5}^{3\text{-Potts}} = \chi_{2/5}^{3\text{-Potts}} + \chi_{7/5}^{3\text{-Potts}}, \quad C_{2/3}^{3\text{-Potts}} = \chi_{2/3}^{3\text{-Potts}}, \quad C_{1/15}^{3\text{-Potts}} = \chi_{1/15}^{3\text{-Potts}}.
  \]

  The standard modular invariant partition function of the 3-state Potts model can be described by using these \(W_3\) characters \(C^{3\text{-Potts}}\)’s

  \[ Z = |C_{0}^{3\text{-Potts}}|^2 + |C_{2/5}^{3\text{-Potts}}|^2 + 2|C_{2/3}^{3\text{-Potts}}|^2 + 2|C_{1/15}^{3\text{-Potts}}|^2. \]

**A.4 Level 1 \(so(2r)\) WZW models**

We denote the character of the level 1 affine \(so(2r)\) as \(\chi_{s}^{so(2r)}(\tau), \ s = 0, 1, 2, 3\). The explicit forms are given as

\[
\chi_{0}^{so(2r)}(\tau) = \frac{1}{2\eta(\tau)^r}(\theta_3(\tau)^r + \theta_4(\tau)^r), \quad \chi_{2}^{so(2r)}(\tau) = \frac{1}{2\eta(\tau)^r}(\theta_3(\tau)^r - \theta_4(\tau)^r),
\]

\[
\chi_{1}^{so(2r)}(\tau) = \chi_{3}^{so(2r)}(\tau) = \frac{1}{2\eta(\tau)^r} \theta_2(\tau)^r.
\]

We sometimes use the \(z\) dependent character of the level 1 affine \(so(2)\)

\[
\chi_{s}^{so(2)}(\tau, z) = \frac{\Theta_{s,2}(\tau, z)}{\eta(\tau)}.
\]
B Detailed calculations

We show here the detailed calculations of the decomposition, in the case of the compact Gepner model treated in section 3. We use here the notation of the beta method in section 3. The other models treated in this paper can be calculated in almost the same manner.

First we show the product formula of multiple theta functions

\[
\prod_{j=1}^{R} \Theta_{m_j,k_j}(\tau, z) = \sum_{\{n_j\}} q \prod_{j} k_j \left(n_j + \frac{m_j}{2\pi i} \right)^2 y \prod_{j} k_j \left(n_j + \frac{m_j}{2\pi i} \right)^2 = \sum_{\{n_j\}} q \prod_{j} k_j \left(n_j + \frac{m_j}{2\pi i} \right)^2 y \prod_{j} k_j n_j + \frac{1}{2} \sum_{j} m_j.
\]

If we insert the identity

\[
1 = \sum_{n \in \mathbb{Z}, \ r \in \mathbb{Z}_k} \delta_{kn+r-\sum_{j} k_j n_j, 0}, \quad k := \sum_{j=1}^{R} k_j,
\]

then the product becomes

\[
\prod_{j=1}^{R} \Theta_{m_j,k_j}(\tau, z) = \sum_{r \in \mathbb{Z}_k} \sum_{n \in \mathbb{Z}} \delta_{kn+r-\sum_{j} k_j n_j, 0} q \prod_{j} k_j \left(n_j + \frac{m_j}{2\pi i} \right)^2 y \left(n + \frac{\sum_{j} m_j + 2r}{2k} \right)^2 \prod_{j} k_j \left(n_j + \frac{m_j}{2\pi i} \right)^2 y \left(n + \frac{\sum_{j} m_j + 2r}{2k} \right)^2.
\]

If we shift \(n_j \rightarrow n_j + n\), \(B_{\{m_j\};\{k_j\}}^{r}(\tau)\) can be rewritten as

\[
B_{\{m_j\};\{k_j\}}^{r}(\tau) := \sum_{\{n_j\}} q \prod_{j} k_j \left(n_j + \frac{m_j}{2\pi i} \right)^2 y \left(n + \frac{\sum_{j} m_j + 2r}{2k} \right)^2 \prod_{j} k_j \left(n_j + \frac{m_j}{2\pi i} \right)^2 y \left(n + \frac{\sum_{j} m_j + 2r}{2k} \right)^2.
\]

This \(B_{\{m_j\};\{k_j\}}^{r}(\tau)\) is actually independent of \(n\). Then, the product becomes

\[
\prod_{j=1}^{R} \Theta_{m_j,k_j}(\tau, z) = \sum_{r \in \mathbb{Z}_k} B_{\{m_j\};\{k_j\}}^{r}(\tau) \Theta_{\sum_{j} m_j + 2r, k}(\tau, z).
\]

This is the product formula (3.3). If \(K\) is a common divisor of \(\{k_j\}\),

\[
B_{\{m_j\};\{k_j\}}^{r}(\tau) = 0, \quad \text{if } r \not\equiv 0 \mod K.
\]

Then the product formula becomes

\[
\prod_{j=1}^{R} \Theta_{m_j,k_j}(\tau, z) = \sum_{r \in \mathbb{Z}_{k/K}} B_{\{m_j\};\{k_j\}}^{Kr}(\tau) \Theta_{\sum_{j} m_j + 2Kr, k}(\tau, z).
\]
Note that for any $a \in \mathbb{R}$ the relation
\[ B^{r}_{(m_{j}+k_{j},a);(k_{j})} = B^{r}_{(m_{j});(k_{j})}, \]
is satisfied.

Let us proceed to the branching function of the orbit. The character of a $\mathcal{N} = 2$ minimal model can be written as
\[ \chi_{m}^{(N_{j});\ell_{j},s_{j}}(\tau, z) = \sum_{r_{j} \in \mathbb{Z}_{N_{j}}-2} c_{(m_{j}-s_{j}-4r_{j})}^{(N_{j}-2);\ell_{j}}(\tau) \Theta_{2m_{j}-N_{j}s_{j}+4N_{j}r_{j}, 2N_{j}(N_{j}-2)}(\tau, z/N_{j}) \]
\[ = \sum_{r \in \mathbb{Z}_{N_{j}-2}} c_{(m_{j}-s_{j}-4r)}^{(N_{j}-2);\ell_{j}}(\tau) \sum_{p_{j} \in \mathbb{Z}_{j}} \Theta_{2K[m_{j}/N_{j}-s_{j}/2+2r_{j}+2p_{j}(N_{j}-2)], 2KJ_{j}(N_{j}-2)}(\tau, z/K). \]

Here, we use the formula (A.1) The total character becomes
\[ \chi_{\mu}^{(N_{j});\ell_{j},s_{j}}(\tau, z) = \chi_{s_{0}}^{(N_{j});\ell_{j},s_{j}}(\tau, z) \prod_{j=1}^{R} \chi_{s_{j}}^{(N_{j});\ell_{j},s_{j}}(\tau, z) \]
\[ = \chi_{s_{0}}^{(N_{j});\ell_{j},s_{j}}(\tau, z) \prod_{\{r_{j}\}}^{R} c_{(m_{j}-s_{j}-4r_{j})}^{(N_{j}-2);\ell_{j}}(\tau) \prod_{\{p_{j}\}}^{R} \Theta_{2K[m_{j}/N_{j}-s_{j}/2+2r_{j}+2p_{j}(N_{j}-2)], 2KJ_{j}(N_{j}-2)}(\tau, z/K) \]

If we use the product formula of multiple theta functions and $\sum_{j=1}^{R} 2K J_{j}(N_{j}-2) = 6K^{2}$, the total character becomes
\[ \chi_{\mu}^{(N);\ell_{j},s_{j}}(\tau, z) = \chi_{s_{0}}^{(N_{j});\ell_{j},s_{j}}(\tau, z) \prod_{\{r_{j}\}}^{R} c_{(m_{j}-s_{j}-4r_{j})}^{(N_{j}-2);\ell_{j}}(\tau) \prod_{\{p_{j}\}}^{R} \sum_{u \in \mathbb{Z}_{2K}} B_{\{2K[m_{j}/N_{j}-s_{j}/2+2r_{j}+2p_{j}(N_{j}-2)], 2KJ_{j}(N_{j}-2)\}}^{2Ku}(\tau) \times \Theta_{2K \sum_{j}[m_{j}/N_{j}-s_{j}/2+2r_{j}+2p_{j}(N_{j}-2)]+4Ku, 6K^{2}}(\tau, z/K) \]

Let us define $Q_{j}(m_{j}, s_{j}) = m_{j}/N_{j} - s_{j}/2 + 2r_{j} + 2p_{j}(N_{j}-2)$, then this is the charge contribution of the $j$th minimal model modulo 2. We can calculate the sum
\[ \sum_{a_{0} \in \mathbb{Z}_{K}/2} \chi_{\mu+4a_{0}\beta_{0}}^{(N);\ell_{j},s_{j}} = \frac{1}{2} \sum_{a_{0} \in \mathbb{Z}_{K}} \chi_{\mu+4a_{0}\beta_{0}}^{(N);\ell_{j},s_{j}} \]
\[ = \frac{1}{2} \chi_{s_{0}}^{(N);\ell_{j},s_{j}}(\tau, z) \prod_{\{r_{j}\}}^{R} c_{(m_{j}-s_{j}-4r_{j})}^{(N_{j}-2);\ell_{j}}(\tau) \sum_{\{p_{j}\}}^{R} \sum_{u \in \mathbb{Z}_{2K}} \Theta_{2K \sum_{j}[m_{j}/N_{j}-s_{j}/2+2r_{j}+2p_{j}(N_{j}-2)]+12Ku, 4Ku, 6K^{2}}(\tau, z/K) \]
\[ = \frac{1}{2} \chi_{s_{0}}^{(N);\ell_{j},s_{j}}(\tau, z) \prod_{\{r_{j}\}}^{R} c_{(m_{j}-s_{j}-4r_{j})}^{(N_{j}-2);\ell_{j}}(\tau) \sum_{\{p_{j}\}}^{R} \sum_{u \in \mathbb{Z}_{2K}} B_{\{2KQ_{j}(m_{j}, s_{j})\}}^{2Ku}(\tau) \times \Theta_{2K \sum_{j}[m_{j}/N_{j}-s_{j}/2+2r_{j}+2p_{j}(N_{j}-2)]+4Ku, 6K^{2}}(\tau, z), \]
where we use the formula (A.1). Due to the GSO condition, the sum of the charge
\[ \sum_j Q_j(m_j, s_j) - s_0/2 = (\text{odd}), \]
and the following identity is satisfied.
\[ 1 = \sum_{v_0 \in \mathbb{Z}, v \in \mathbb{Z}_3} \delta_2 \sum_j Q_j(m_j, s_j) + 4u - s_0 - 2(6v_0 + 2v + 1),.0. \]

Using these formulae, we obtain the sum
\[
\sum_{a_0 \in \mathbb{Z}_{K/2}} \chi^\lambda_{\mu+4a_0\beta_0} = \sum_{v \in \mathbb{Z}_3} \alpha^\lambda_{\mu,v}(\tau) \chi_{s_0}^{so(2)} \Theta_{s_0+4v+2.6}(\tau, z),
\]
\[
\alpha^\lambda_{\mu,v}(\tau) := \frac{1}{2} \sum_{\{r_j\}} \sum_{v_0 \in \mathbb{Z}_3} \prod_{j=1}^R c_{m_j-s_j-4r} (\tau) \sum_{u \in \mathbb{Z}_{3K}} B_{1}^{2Ku} \{2KQ_j(m_j, s_j); (2KJ_j(N_j-2)) (\tau) \}
\times \delta_2 \sum_j Q_j(m_j, s_j) + 4u - s_0 - 2(6v_0 + 2v + 1),.0.
\]

This \( \alpha^\lambda_{\mu,v}(\tau) \) satisfies the relation
\[
\alpha^\lambda_{\mu+c_0\beta_0,v} = \alpha^\lambda_{\mu,v} + \alpha^\lambda_{0}, \quad \text{for } c_0 \in \mathbb{Z}.
\]

By using this relation, we obtain the sum
\[
\sum_{c_0 \in \mathbb{Z}_4} \sum_{a_0 \in \mathbb{Z}_{K/2}} (-1)^{c_0+s_0} \chi^\lambda_{\mu+c_0\beta_0+4a_0\beta_0}(\tau, z) = \sum_{a \in \mathbb{Z}_3} \alpha^\lambda_{\mu,a-s_0+1}(\tau) \xi_{a}^{\text{su}(3)}(\tau, z),
\]
and the decomposition of orbit becomes
\[
F^\lambda_{\mu}(\tau, z) = \sum_{a \in \mathbb{Z}_3} F^\lambda_{\mu,a}(\tau) \xi_{a}^{\text{su}(3)}(\tau, z),
\]
\[
F^\lambda_{\mu,a}(\tau) := \eta(\tau) \sum_{b_j \in \mathbb{Z}_2} \chi^\lambda_{\mu+\sum_j b_j\beta_j, a-s_0-2\sum_j b_j+1}(\tau).
\]

Also in the case of the noncompact Gepner models, we can decompose the orbit in almost the same manner.
References

[1] K. Sugiyama and S. Yamaguchi, “Cascade of special holonomy manifolds and heterotic string theory”, Nucl. Phys. B622 (2002) 3–45, [hep-th/0108219].

[2] A. Bilal and J.-L. Gervais, “New critical dimensions for string theories”, Nucl. Phys. B284 (1987) 397.

[3] T. Eguchi, H. Ooguri, A. Taormina and S.-K. Yang, “Superconformal algebras and string compactification on manifolds with SU(n) holonomy”, Nucl. Phys. B315 (1989) 193.

[4] D. Kutasov, “Some properties of (non)critical strings”, [hep-th/9110041]

[5] T. Eguchi and Y. Sugawara, “CFT description of string theory compactified on non-compact manifolds with $G_2$ holonomy”, Phys. Lett. B519 (2001) 149–158, [hep-th/0108091].

[6] P. C. Argyres, K. R. Dienes and S. H. H. Tye, “New Jacobi like identities for $Z_K$ parafermion characters”, Commun. Math. Phys. 154 (1993) 471–508, [hep-th/9201078].

[7] S. Mizoguchi, “Modular invariant critical superstrings on four-dimensional Minkowski space × two-dimensional black hole”, JHEP 04 (2000) 014, [hep-th/0003053].

[8] T. Eguchi and Y. Sugawara, “Modular invariance in superstring on Calabi-Yau n-fold with A-D-E singularity”, Nucl. Phys. B577 (2000) 3–22, [hep-th/0002100].

[9] S. Yamaguchi, “Gepner-like description of a string theory on a non-compact singular Calabi-Yau manifold”, Nucl. Phys. B594 (2001) 190–208, [hep-th/0007069].

[10] S. Mizoguchi, “Noncompact Gepner models for type II strings on a conifold and an ALE instanton”, [hep-th/0009240]

[11] M. Naka and M. Nozaki, “Singular Calabi-Yau manifolds and ADE classification of CFTs”, Nucl. Phys. B599 (2001) 334–360, [hep-th/0010002].

[12] S. Yamaguchi, “Noncompact Gepner models with discrete spectra”, Phys. Lett. B509 (2001) 346–354, [hep-th/0102176].
[13] S. Gukov, C. Vafa and E. Witten, “CFT’s from Calabi-Yau four-folds”, Nucl. Phys. B584 (2000) 69–108, [hep-th/9906070].

[14] A. Giveon, D. Kutasov and O. Pelc, “Holography for non-critical superstrings”, JHEP 10 (1999) 035, [hep-th/9907178].

[15] D. Gepner, “Space-time supersymmetry in compactified string theory and superconformal models”, Nucl. Phys. B296 (1988) 757.

[16] A. Giveon and D. Kutasov, “Little string theory in a double scaling limit”, JHEP 10 (1999) 034, [hep-th/9909110].

[17] A. Giveon and D. Kutasov, “Comments on double scaled little string theory”, JHEP 01 (2000) 023, [hep-th/9911039].