FINITE LINEAR GROUPS, LATTICES, AND PRODUCTS OF ELLIPTIC CURVES

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Abstract. Let $V$ be a finite dimensional complex linear space and let $G$ be an irreducible finite subgroup of $\text{GL}(V)$. For a $G$-invariant lattice $\Lambda$ in $V$ of maximal rank, we give a description of structure of the complex torus $V/\Lambda$. In particular, we prove that for a wide class of groups, $V/\Lambda$ is isogenous to a self-product of an elliptic curve, and that in many cases $V/\Lambda$ is isomorphic to a product of mutually isogenous elliptic curves with complex multiplication. We show that there are $G$ and $\Lambda$ such that the complex torus $V/\Lambda$ is not an abelian variety, but one can always replace $\Lambda$ by another $G$-invariant lattice $\Delta$ such that $V/\Delta$ is a product of mutually isogenous elliptic curves with complex multiplication. We amplify these results with a criterion, in terms of the character and the Schur $\mathbb{Q}$-index of $G$-module $V$, of the existence of a nonzero $G$-invariant lattice in $V$.

1. Introduction

This paper arose from the following observation made in [Po]. Let $V$ be a complex linear space of nonzero dimension $n < \infty$. Let $G \subset \text{GL}(V)$ be a finite irreducible reflection group, i.e., a subgroup of $\text{GL}(V)$ generated by (complex) reflections, and let $\Lambda$ be a $G$-invariant lattice in $V$ of rank $2n$ (hereinafter a lattice is a discrete additive subgroup of a complex or real linear space). All finite reflection groups are classified in [ST] (see also [C], [Po]), and all lattices invariant with respect to them are classified in [Po]. A posteriori, it follows from this classification of lattices that the complex torus $V/\Lambda$ is in fact an abelian variety. Moreover, this classification implies that $V/\Lambda$ is isogenous to a self-product of an elliptic curve, and if $G$ is not complexification of the Weyl group of an irreducible root system, then $V/\Lambda$ is a product of mutually isogenous elliptic curves with complex multiplication.
Our original goal was to give an a priori, independent of the classification of invariant lattices proof of these properties of $V/\Lambda$. On this way, we found out that in fact they hold for invariant lattices of a much wider class of irreducible finite subgroups of $GL(V)$ than that of reflection groups. We give a description of $V/\Lambda$ for every irreducible finite subgroup $G$ of $GL(V)$ and $G$-invariant lattice $\Lambda$ in $V$ of rank $2n$: the $G$-modules $V$ with Schur $\mathbb{Q}$-index 1 are considered in Theorem 3.1, and the other $G$-modules $V$ in Theorem 4.1. In particular, we prove that in the majority of cases (but not in all) $V/\Lambda$ is an abelian variety; moreover, in many cases $V/\Lambda$ is isogenous to a self-product of an elliptic curve or even isomorphic to a product of mutually isogenous elliptic curves with complex multiplication. We show (Theorem 4.1 and Example 4.3) that $G$ and $\Lambda$ such that the complex torus $V/\Lambda$ is not an abelian variety do exist, but one can always replace $\Lambda$ by another $G$-invariant lattice $\Delta$ such that $V/\Delta$ is a product of mutually isogenous elliptic curves with complex multiplication (Theorem 4.6). We amplify these results with a criterion (in terms of the character and the Schur $\mathbb{Q}$-index of $G$-module $V$) of the existence of a nonzero $G$-invariant lattice $\Lambda$ in $V$ (Theorem 2.12); the latter appears to be equivalent to the existence of $\Lambda$ of rank $2n$.

Our approach hinges on the representation theory of finite groups. However, for reflection groups, there is another, geometric approach. It gives a key to the classification of all invariant lattices and may be useful for solving other problems (for instance, it follows from the classification of invariant lattices of reflection groups that elliptic curves arising from $V/\Lambda$ for nonreal reflection groups are very specific: their endomorphism rings may be only the orders in $\mathbb{Q}(\sqrt{-d})$ for $d = 1, 2, 3,$ and 7, see [Po]; it would be interesting to find an a priori explanation of this phenomenon). Because of this reason, for reflection groups, we give in the last section a second, geometric proof of our result on the structure of $V/\Lambda$ (Theorem 5.3 and its proof).

Notation and terminology.

$\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are respectively the ring of integers, the field of rational numbers, the field of real numbers, and the field of complex numbers.

$\mathbb{H}$ is the Hamiltonian quaternion $\mathbb{R}$-algebra ($\frac{-1 - i}{\sqrt{2}}$) (see, e.g., [Pi, §1.6]).

$\mathbb{F}_q$ is the finite field that consists of $q$ elements.

The identity map of a set $S$ is denoted by $id_S$.

$\mathbb{Z}[S]$ is the subring of $\mathbb{C}$ generated by a subset $S$ of $\mathbb{C}$.
For a subring $A$ of $\mathbf{C}$ and a subset $P$ of a linear space $W$ over $\mathbf{C}$, the $A$-submodule of $W$ generated by $P$ is denoted by $AP$.

$\text{Tr}(P) := \{ \text{tr} (g) \mid g \in P \}$ where $P$ is a subset of $\text{End}_{\mathbf{C}}(V)$.

$M_r(R)$ is the algebra of $r \times r$-matrices over a ring $R$ (associative and with identity element).

$I_r$ is the identity matrix of $M_r(R)$.

$R^d$ is the space of column vectors over $R$ of height $d$.

If $G$ is a finite subgroup of $\text{GL}(V)$, then $\chi_{G,V}$ (respectively, $\text{Schur}_{G,V}$) is the character (respectively, the Schur index with respect to $\mathbf{Q}$) of $G$-module $V$.

$Z_G$ is the center of $\mathbf{Q}$-algebra $\mathbf{Q}G$.

The field generated over $\mathbf{Q}$ by $\text{Tr}(G)$ is denoted by $\mathbf{Q}(\chi_{G,V})$. If $\mathbf{Q}(\chi_{G,V})$ is $\mathbf{Q}$ (respectively, an imaginary quadratic number field), then the character $\chi_{G,V}$ is called rational (respectively, imaginary quadratic).

The $G$-module $V$ is called orthogonal (respectively, symplectic) if there is a symmetric (respectively, skew-symmetric) nondegenerate $G$-invariant bilinear form $V \times V \to \mathbf{C}$.

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2. Some generalities

Let $G$ be an irreducible finite subgroup of $\text{GL}(V)$. By Burnside’s theorem, the irreducibility of $G$ is equivalent to the equality

$$CG = \text{End}_{\mathbf{C}}(V).$$  \hfill (1)

Since $G$ is a finite group, $\mathbf{Q}G$ is a finite-dimensional $\mathbf{Q}$-algebra. The existence of natural epimorphism of the group algebra of $G$ over $\mathbf{Q}$ to $\mathbf{Q}G$ implies that $\mathbf{Q}G$ is a semisimple $\mathbf{Q}$-algebra. Clearly, $Z_G$ is either a (number) field or a product of (number) fields. Since $G \subset \mathbf{Q}G$, the elements of $Z_G$ commute with $G$ and hence, by (1), with $\text{End}_{\mathbf{C}}(V)$. This implies that

$$Z_G \subset \mathbf{C} \text{id}_V.$$  \hfill (2)

Hence $Z_G$ is a field. In turn, this implies that $\mathbf{Q}G$ is a simple $\mathbf{Q}$-algebra and therefore a central simple $Z_G$-algebra. The latter means that there is a central division $Z_G$-algebra $D$ and an integer $r > 0$ such that

$$\mathbf{Q}G \simeq M_r(D)$$  \hfill (isomorphism of $Z_G$-algebras).
Below we shall naturally identify $\text{Id}_V$ with $C$, and $Z_G$ with the corresponding subfield of $C$. The above notation and conventions are kept throughout the whole paper.

**Lemma 2.1.** The natural $C$-algebra homomorphism

$$
\psi: \mathbb{Q}G \otimes_{Z_G} C \longrightarrow CG = \text{End}_C(V)
$$

is an isomorphism. In particular,

$$
\dim_{Z_G}(\mathbb{Q}G) = n^2.
$$

Proof. Since $\mathbb{Q}G \otimes_{Z_G} C$ is a simple $C$-algebra, $\psi$ is injective. On the other hand, (1) implies that $\psi$ is surjective. □

**Corollary 2.2.** $n^2 = r^2 \dim_{Z_G} D$.

Proof. This follows from (3) and (5). □

**Lemma 2.3.** $Z_G = \mathbb{Q}(\chi_{G,V})$.

Proof. This is well known (see, e.g., [D, Lemma 24.7]). □

Recall (see, e.g., [CR, (70.4)]) that $\text{Schur}_{G,V} = \min[K : \mathbb{Q}(\chi_{G,V})]$ with the minimum taken over all subfields $K$ of $C$ such that the linear group $G$ is defined over $K$. The latter means that there exists a $G$-invariant $K$-form of the $C$-linear space $V$, i.e., a $K$-linear subspace $L$ of $V$ such that $\dim_K(L) = n$ and $CL = V$. It is known (see, e.g., [CR, (70.13)]) that

$$
\text{Schur}_{G,V} = \sqrt{\dim_{\mathbb{Q}(\chi_{G,V})} D}.
$$

(6)

In particular, (6), Corollary 2.2, and Lemma 2.3 imply that

$$
\text{Z}_G\text{-algebras } \mathbb{Q}G \text{ and } M_n(\text{Z}_G) \text{ are isomorphic } \iff \text{Schur}_{G,V} = 1.
$$

(7)

Clearly, $\text{Z}_G$ is an order in $\mathbb{Q}G$; in particular, $\text{Z}_G$ is a free $\mathbb{Z}$-module of rank $\dim_{\mathbb{Q}}(\mathbb{Q}G)$. It is clear as well that $\text{Z}_G \cap \text{Z}_G$ is an order in $\text{Z}_G$; in particular, it is a free $\mathbb{Z}$-module of rank $\dim_{\mathbb{Q}} \text{Z}_G$.

**Lemma 2.4.** ([Po, Section 3.1]) If there exists a nonzero $G$-invariant lattice $\Lambda$ in $V$, then $\text{rk}(\Lambda) = n$ or $2n$.
Proof. Since $\Lambda$ is $G$-invariant, the $\mathbb{C}$-linear subspaces $\mathbb{C}\Lambda = \mathbb{R}\Lambda + i\mathbb{R}\Lambda$ and $\mathbb{R}\Lambda \cap i\mathbb{R}\Lambda$ in $V$ are $G$-invariant as well. The irreducibility of $G$ then implies that $\mathbb{C}\Lambda = V$ and

\[ \mathbb{R}\Lambda \cap i\mathbb{R}\Lambda = \{0\} \text{ or } V. \]

Since $\dim_{\mathbb{R}} \mathbb{R}\Lambda = \dim_{\mathbb{R}} i\mathbb{R}\Lambda = \text{rk}(\Lambda)$, in the first case we obtain $2n = \dim_{\mathbb{R}} V = \dim_{\mathbb{R}} \mathbb{R}\Lambda + \dim_{\mathbb{R}} i\mathbb{R}\Lambda = 2\text{rk}(\Lambda)$, so $\text{rk}(\Lambda) = n$. In the second case, $2n = \dim_{\mathbb{R}} V = \dim_{\mathbb{R}} \mathbb{R}\Lambda = \text{rk}(\Lambda)$. \qed

Lemma 2.5. If there exists a nonzero $G$-invariant lattice $\Lambda$ in $V$, then $\chi_{G,V}$ is either rational or imaginary quadratic.

Proof. Pick a nonzero element $v \in \Lambda$. Since $\Lambda$ is $G$-invariant, it is also $\mathbb{Z}G$-invariant, and in particular, $\mathbb{Z}G \cap \mathbb{Z}G$-invariant. From (2) we then deduce that $(\mathbb{Z}G \cap \mathbb{Z}G)v \subseteq \Lambda \cap \mathbb{C}v$; whence $\mathbb{Z}G \cap \mathbb{Z}G$ is a nonzero lattice in $\mathbb{C}$. Therefore $\dim_{\mathbb{Q}} \mathbb{Z}G \leq 2$, i.e., $\mathbb{Z}G$ is either $\mathbb{Q}$ or a quadratic number field. Since the orders of every real quadratic number field are not discrete in $\mathbb{C}$ (see, e.g., [BS, Ch. II, §7]), the claim now follows from Lemma 2.3. \qed

Lemma 2.6. Suppose that $\chi_{G,V}$ is not rational. If there exists a nonzero $G$-invariant lattice $\Lambda$ in $V$, then

(i) $\mathbb{Z}G$ is an imaginary quadratic number field;
(ii) $\text{Schur}_{G,V} = 1$;
(iii) $\text{rk}(\Lambda) = 2n$.

Proof. Since $\chi_{G,V}$ is not rational, (i) readily follows from Lemmas 2.3 and 2.5. The $\mathbb{Q}$-linear space $\mathbb{Q}\Lambda$ carries a natural structure of $\mathbb{Z}G$-linear space, and (i) implies that $\dim_{\mathbb{Z}G}(\mathbb{Q}\Lambda) = \dim_{\mathbb{Q}}(\mathbb{Q}\Lambda)/2 = \text{rk}(\Lambda)/2$. Since $\mathbb{Q}\Lambda$ is $\mathbb{Q}G$-stable, we get a $\mathbb{Z}G$-algebra homomorphism $\varphi : \mathbb{Q}G \to \text{End}_{\mathbb{Z}G}(\mathbb{Q}\Lambda)$. Since $\varphi(1) = \text{id}_{\mathbb{Q}\Lambda}$ and $\mathbb{Q}G$ is simple, $\varphi$ is injective. This and (5) then imply

\[ n^2 = \dim_{\mathbb{Z}G}(\mathbb{Q}G) \leq (\dim_{\mathbb{Z}G}(\mathbb{Q}\Lambda))^2 = (\text{rk}(\Lambda)/2)^2 \leq n^2. \tag{8} \]

From (8) we deduce that (iii) holds and $\varphi$ is an isomorphism. The latter property and (7) clearly imply (ii). \qed

Corollary 2.7 (E. B. Vinberg). Suppose that the $G$-module $V$ is not self-dual. If there exists a nonzero $G$-invariant lattice $\Lambda$ in $V$, then the conclusions of Lemma 2.6 hold.
Proof. Since $V$ is not self-dual, the character $\chi_{G,V}$ is not real valued and therefore is not rational. The claim now follows from Lemma 2.6. □

**Lemma 2.8.** Suppose that the greatest common divisor of the integers $\ker(u)$, where $u$ runs through $Q^G$, is equal to 1. Then $\text{Schur}_{G,V} = 1$.

Proof. Since $G$ is irreducible and the elements of $\text{Tr}(G)$ are integral algebraic numbers, [CR, 30.10], the assumption of lemma implies by [V, Lemma 3] and Lemma 2.3 that $G$ is defined over $Z_G$, whence the claim by (7). □

**Lemma 2.9.** If $\chi_{G,V}$ is real valued, then $\text{Schur}_{G,V} \leq 2$.

Proof. This is the result of R. Brauer and A. Speiser, [Br] (see also [Fi], [Be], and [Fe, Corollary 2.4, p. 277]). □

**Lemma 2.10.** Suppose that $\chi_{G,V}$ is rational and $\text{Schur}_{G,V} \neq 1$. Then $D$ is a quaternion $Q$-algebra, $n = 2r$, and exactly one of the following two possibilities holds:

(i) the $G$-module $V$ is orthogonal and $D$ is indefinite, i.e., $D \otimes Q R$ is $R$-isomorphic to $M_2(R)$;

(ii) the $G$-module $V$ is symplectic and $D$ is definite, i.e., $D \otimes Q R$ is $R$-isomorphic to $H$.

Proof. Since $\chi_{G,V}$ is rational and $\text{Schur}_{G,V} \neq 1$, Lemmas 2.3 and Lemma 2.9 imply that $Z_G = Q$ and $\text{Schur}_{G,V} = 2$. By (6), this implies that $\dim_Q(D) = 4$. Since $D$ is a division $Q$-algebra, the latter equality implies that $D$ is a quaternion $Q$-algebra (see, e.g., [Pi, §13.1]). Corollary 2.2 now implies that $n^2 = 4r^2$, i.e., $n = 2r$.

Since the $R$-algebra $QG \otimes Q R$ is simple, the natural surjection $QG \otimes Q R \rightarrow RG$ is an isomorphism of $R$-algebras. Clearly, the linear group $G$ is defined over $R$ if and only if $RG$ is isomorphic to $M_n(R)$. Taking into account the $R$-algebra isomorphisms

$$QG \otimes Q R \cong M_{n/2}(D) \otimes Q R \cong M_{n/2}(D \otimes Q R),$$

from this we deduce that $G$ is defined over $R$ if and only if $D$ indefinite. Since a self-dual simple $G$-module is defined over $R$ if and only if it is orthogonal (see, e.g., [Se, Section 13.2]), this completes the proof. □

**Theorem 2.11.** Suppose that $n$ is even and there exists a quaternion $Q$-algebra $H$ such that the $Q$-algebras $QG$ and $M_{n/2}(H)$ are isomorphic. Then
(i) for each imaginary quadratic subfield $F$ of $H$, there exists a $G$-invariant lattice $\Lambda^{(F)}$ in $V$ of rank $2n$ such that the complex torus $V/\Lambda^{(F)}$ is isomorphic to a product of mutually isogenous elliptic curves with complex multiplication by an order of $F$;

(ii) there exists a $G$-invariant lattice $\Lambda$ in $V$ of rank $2n$ such that the $\mathbb{Q}$-algebras $\text{End}(V/\Lambda) \otimes \mathbb{Q}$ and $M_{n/2}(H)$ are isomorphic.

Proof. Since the center of $M_{n/2}(H)$ is $\mathbb{Q}$ (embedded by $a \mapsto a \cdot I_{n/2}$), we have $Z_G = \mathbb{Q}$. (9)

Fix an isomorphism of $\mathbb{Q}$-algebras $\tau: \mathbb{Q}G \longrightarrow M_{n/2}(H)$.

The $\mathbb{Q}$-linear space $H^{n/2}$ carries a natural structure of left $M_{n/2}(H)$-module. Fix in $H^{n/2}$ a nonzero $\tau(\mathbb{Z}G)$-stable finitely generated additive subgroup $\Pi$ (for instance, take $\Pi = \tau(\mathbb{Z}G) \cdot v$ for a nonzero vector $v \in H^{n/2}$). Since $H^{n/2}$ is a $\mathbb{Q}$-linear space, $\Pi$ is a free abelian group of finite rank and every its basis consists of linearly independent elements of $H^{n/2}$. Since the linear subspace $\mathbb{Q}\Pi$ of $H^{n/2}$ is stable with respect to $\mathbb{Q}\tau(\mathbb{Z}G) = \tau(\mathbb{Q}G) = M_{n/2}(H)$, we have $\mathbb{Q}\Pi = H^{n/2}$. Hence the natural map is an isomorphism of $\mathbb{Q}$-linear spaces

$$\Pi \otimes \mathbb{Q} \overset{\sim}{\longrightarrow} H^{n/2}. \quad (10)$$

The $\mathbb{Q}$-linear space $H^{n/2}$ carries also a natural structure of right $H$-module. The structures of left $M_{n/2}(H)$-module and right $H$-module on $H^{n/2}$ yield the $\mathbb{Q}$-linear embeddings of respectively $M_{n/2}(H)$ and $H$ into $\text{End}_{\mathbb{Q}}(H^{n/2})$. By Wedderburn’s theorem (see, e.g., [L1, Ch. XVII, §3, Corollary 3]), the images of these embeddings are the centralizers of each other in $\text{End}_{\mathbb{Q}}(H^{n/2})$.

Put $\Pi_R := \Pi \otimes \mathbb{R}$ and $H_R := H \otimes_{\mathbb{Q}} \mathbb{R}$. We naturally identify $\Pi$ with $\Pi \otimes 1$ in $\Pi_R$, and $H$ with $H \otimes 1$ in $H_R$. By (10), the following isomorphisms of $\mathbb{R}$-linear spaces hold:

$$\Pi_R \simeq H^{n/2} \otimes_{\mathbb{Q}} \mathbb{R} \simeq H^{n/2}_R. \quad (11)$$

From (11) we deduce that $\Pi_R$ is a $2n$-dimensional $\mathbb{R}$-linear space that carries the natural structures of left $M_{n/2}(H_R)$-module and right $H_R$-module, and $\Pi$ is a lattice in $\Pi_R$ of rank $2n$. These structures yield the $\mathbb{R}$-linear embeddings

$$\iota_l: M_{n/2}(H_R) \hookrightarrow \text{End}_{\mathbb{R}}(\Pi_R), \quad \iota_r: H_R \hookrightarrow \text{End}_{\mathbb{R}}(\Pi_R)$$
whose images are the centralizers of each other in $\text{End}_R(\Pi_R)$.

Since the elements of $G$ are invertible in $QG$, composition of the following embeddings of $Q$-algebras

$$QG \xrightarrow{\tau} M_{n/2}(H) \xrightarrow{id} M_{n/2}(H_R) \xrightarrow{\iota_l} \text{End}_R(\Pi_R)$$

(12)

embeds $G$ into the group of invertible elements of $\text{End}_R(\Pi_R)$. This defines an $R$-linear action of $G$ on $\Pi_R$. By construction, the lattice $\Pi$ in $\Pi_R$ is $G$-invariant.

We want to define on the real linear space $\Pi_R$ a structure of complex linear space in such a way that the algebra of its $C$-linear transformations contains $\iota_l(M_{n/2}(H_R))$. In order to do this, choose an element $c \in H_R$ with $c^2 = -1$: using that $R$-algebra $H_R$ is isomorphic to either $H$ or $M_2(R)$ (see, e.g., [Pi, §§1.7, 13.2]), it is easy to see that such $c$ exists and, moreover, since $(aca^{-1})^2 = -1$ for every invertible element $a$ of $H_R$, the set of such $c$ is uncountable (the latter fact will be used below). Define now the complex structure on $\Pi_R$ by letting $\iota_r(c)$ be the multiplication by $i$. Let $V_c$ be the $n$-dimensional complex linear space defined by this complex structure on $\Pi_R$. Since $\iota_r(c)$ commutes with the elements of $\iota_l(M_{n/2}(H_R))$, we have

$$\iota_l(M_{n/2}(H_R)) \subset \text{End}_C(V_c).$$

(13)

From (12) and (13) we deduce that the action of $G$ on $\Pi_R$ defined above is $C$-linear with respect to this complex structure. Also, (12) and (13) clearly yield a nonzero homomorphism of $C$-algebras $QG \otimes Q C \rightarrow \text{End}_C(V_c)$ that endows $V_c$ with a nonzero structure of $QG \otimes Q C$-module of $C$-dimension $n$. On the other hand, $V$ is a nontrivial $QG \otimes Q C$-module of $C$-dimension $n$ as well. Notice now that there is a unique (up to isomorphism) $QG \otimes Q C$-module of $C$-dimension $n$ since, by (9) and Lemma 2.1, the $C$-algebra $QG \otimes Q C$ is isomorphic to $M_n(C)$. This implies that there is an isomorphism of $QG \otimes Q C$-modules

$$\nu_c: V_c \longrightarrow V.$$  

(14)

It follows from $G \subset QG \subset QG \otimes Q C$ that $\nu_c$ is an isomorphism of $G$-modules.

Determine now the structure of the endomorphism algebra of complex torus $V_c/\Pi$, i.e., the $Q$-algebra

$$\text{End}^0(V_c/\Pi) := \text{End}(V_c/\Pi) \otimes Q.$$  

Recall from [OZ] that the Hodge algebra $\text{HDG}(V_c/\Pi)$ of $V_c/\Pi$ is the smallest $Q$-subalgebra $B$ of $\text{End}_Q(\Pi \otimes Q) \subset \text{End}_R(\Pi_R)$ such that $\iota_r(c) \in RB$ and $\text{End}^0(V_c/\Pi)$
coincides with the centralizer of \(B\) in \(\text{End}_Q(\Pi \otimes Q)\). Since \(c \in H_R\), we conclude that \(\text{HDG}(V_c/\Pi) \subset \iota_r(H)\). Since the centralizer of \(\iota_r(H)\) in \(\text{End}_Q(\Pi \otimes Q)\) is \(\iota_l(M_{n/2}(H))\), this implies that \(\text{End}^0(V_c/\Pi) \supset \iota_l(M_{n/2}(H))\). As \(c \notin R \cdot 1\), we have \(\dim_Q \text{HDG}(V_c/\Pi) \geq 2\). Hence, since \(H\) is a quaternion \(Q\)-algebra, \(\text{HDG}(V_c/\Pi)\) is either \(\iota_r(H)\) or \(\iota_r(F)\) where \(F\) is a quadratic subfield of \(H\). If \(\text{HDG}(V_c/\Pi) = \iota_r(F)\), then by dimension reason,

\[
F \otimes_Q R = R \cdot 1 + R \cdot c \simeq C, \tag{15}
\]

and therefore \(F\) is an imaginary quadratic field. If \(\text{HDG}(V_c/\Pi) = \iota_r(H)\), then \(\text{End}^0(V_c/\Pi)\) is the centralizer of \(\iota_r(H)\), i.e., \(\text{End}^0(V_c/\Pi) = \iota_l(M_{n/2}(H))\).

Prove now that when \(c\) varies, all possibilities for \(\text{HDG}(V_c/\Pi)\) do occur, i.e.,

(a) if \(F\) is an imaginary quadratic subfield of \(H\), then for some \(c\),

\[
\iota_r(F) = \text{HDG}(V_c/\Pi); \tag{16}
\]

(b) there exists \(c\) such that \(\text{End}^0(V_c/\Pi) = \iota_l(M_{n/2}(H))\).

First, if \(F\) is as in (a), then \(F \otimes_Q R \simeq C\) and therefore there is an element \(c_F \in F \otimes_Q R \subset H_R\) such that \(c_F^2 = -1\) (in fact, there are exactly two such elements). Clearly, then (16) holds for \(c = c_F\). This proves (a).

Second, notice that clearly the set of imaginary quadratic subfields of \(H\) is at most countable. For every such field \(F\), the intersection of \(F \otimes_Q R \simeq C\) with the set \(S := \{c \in H_R \mid c^2 = -1\}\) consists of two elements. Since \(S\) is uncountable, this implies that there exists \(c_0 \in H_R\) such that \(c_0^2 = -1\) and \(c_0\) does not lie in \(F \otimes_Q R\) for every imaginary quadratic subfield \(F\) of \(H\). Hence \(\text{HDG}(V_{c_0}/\Pi) = \iota_r(H)\). This proves (b).

Determine now the structure of \(V/\Pi\) in case when \(\text{HDG}(V_c/\Pi) = \iota_r(F)\) where \(F\) is an imaginary quadratic subfield of \(H\). The definition of Hodge algebra implies that \(\iota_r(F) \subset \text{End}^0(V_c/\Pi)\), i.e., \(Q\Pi\) is \(\iota_r(F)\)-stable. From (15) and the definition of \(V_c\) we deduce that

\[
\iota_r(F \otimes_Q R) = C \cdot \text{id}_{V_c}.
\]

Since \(Q\Pi\) is \(\iota_r(F)\)-stable, there is an order \(O'\) in \(F\) such that \(\Pi\) is \(\iota_r(O')\)-stable. This endows \(\Pi\) with a structure of \(O'\)-module. By a theorem of Z. I. Borevich and D. K. Faddeev [BF1] (see also [BF2], [BF3], [Sc, Satz 2.3]), this \(O'\)-module splits into a direct sum of \(n\) submodules of rank 1,

\[
\Pi = \Gamma_1 \oplus \ldots \oplus \Gamma_n. \tag{17}
\]
Clearly, each $\Gamma_j \otimes \mathbb{R}$ is a one-dimensional $\mathbb{C}$-linear space and (17) implies that

$$V_c/\Pi \text{ is isomorphic to } \prod_{j=1}^{n}(\Gamma_j \otimes \mathbb{R})/\Gamma_j.$$ 

Every $(\Gamma_j \otimes \mathbb{R})/\Gamma_j$ is an elliptic curve with complex multiplication by $\mathcal{O}'$. Therefore these curves are mutually isogenous.

To complete the proof, it only remains to remark that due to the existence of isomorphism (14), the complex tori $V_c/\Pi$ and $V/\nu_c(\Pi)$ are isomorphic. So in the above cases (a) and (b), putting respectively $\nu_c(\Pi) := \Lambda(F)$ and $\Lambda$, we obtain respectively the proofs of statements (i) and (ii) of the theorem. □

We can now give a criterion of the existence of a nonzero $G$-invariant lattice.

**Theorem 2.12.** (A) The following properties are equivalent:

(a) there is a nonzero $G$-invariant lattice in $V$;

(b) there is a $G$-invariant lattice in $V$ of rank $2n$;

(c) one of the following conditions hold:

(i) $\text{Schur}_{G,V} = 1$ and $\chi_{G,V}$ is either rational or imaginary quadratic;

(ii) $\text{Schur}_{G,V} = 2$ and $\chi_{G,V}$ is rational.

(B) A $G$-invariant lattice $\Lambda$ in $V$ of rank $n$ exists if and only if $G$ is defined over $\mathbb{Q}$, i.e., $\text{Schur}_{G,V} = 1$ and $\chi_{G,V}$ is rational. For such $\Lambda$ and every nonreal $c \in \mathbb{C}$, the additive subgroup $\Lambda + c\Lambda$ of $V$ is a $G$-invariant lattice in $V$ of rank $2n$.

**Proof.** (A) The equivalence of (a) and (b) follows from (B) that is proved below.

Assume now that (a) holds. If $\chi_{G,V}$ is not rational, then Lemmas 2.3 and 2.6 imply that $\text{Schur}_{G,V} = 1$ and $\chi_{G,V}$ is imaginary quadratic. If $\chi_{G,V}$ is rational, then Lemma 2.9 yields $\text{Schur}_{G,V} \leq 2$. This proves that (a) implies (c).

Conversely, assume that (c) holds. If (ii) is fulfilled, then (3), Lemma 2.10, and Theorem 2.11 imply that (a) holds. Consider now the case when (i) is fulfilled.

If $\chi_{G,V}$ is rational, then by definition of the Schur index, $G$ is defined over $\mathbb{Q}$. It is known (see, e.g., [CR, (73.5)]) that then $G$ is defined over $\mathbb{Z}$, i.e., there exists a basis $e_1, \ldots, e_n$ in $V$ such that the lattice

$$\mathbb{Z}e_1 + \ldots + \mathbb{Z}e_n$$

is $G$-invariant. Thus in this case (a) holds as well.

Finally, assume that $\chi_{G,V}$ is imaginary quadratic. Since $\text{Schur}_{G,V} = 1$, there exists a $G$-invariant $\mathbb{Q}(\chi_{G,V})$-form $L$ of $V$. Let $\mathcal{O}$ be the maximal order of $\mathbb{Q}(\chi_{G,V})$.
Take any nonzero vector \( v \in L \) and let \( \Lambda \) be the submodule of \( \mathcal{O}\)-module \( L \) generated by the \( G \)-orbit of \( v \),
\[
\Lambda := \sum_{g \in G} \mathcal{O}g(v).
\]
(19)
Since \( \mathcal{O} \) is a Dedekind ring (see, e.g., [CR, §18]) and \( \Lambda \) is a finitely generated torsion free \( \mathcal{O}\)-module, the latter is isomorphic to a direct sum of some fractional ideals \( \mathcal{I}_1, \ldots, \mathcal{I}_d \) of \( \mathbb{Q}(\chi_{G,V}) \) (see, e.g., [CR, (22.5)]). Hence there are linearly independent over \( \mathcal{O} \) vectors \( v_1, \ldots, v_d \in L \) such that
\[
\Lambda = \mathcal{I}_1 v_1 + \ldots + \mathcal{I}_d v_d.
\]
(20)
Since the fraction field of \( \mathcal{O} \) is \( \mathbb{Q}(\chi_{G,V}) \), vectors \( v_1, \ldots, v_d \) are linearly independent over \( \mathbb{Q}(\chi_{G,V}) \) as well, and since \( L \) is a \( \mathbb{Q}(\chi_{G,V}) \)-form of \( V \), they are linearly independent over \( \mathbb{C} \). Notice now that since \( \mathbb{Q}(\chi_{G,V}) \) is an imaginary quadratic number field, all its fractional ideals are lattices (of rank 2) in \( \mathbb{C} \). This and (20) imply now that \( \Lambda \) is a nonzero lattice in \( V \). On the other hand, (19) clearly implies that \( \Lambda \) is \( G \)-invariant. Hence (a) holds. This completes the proof that (c) implies (a).

(B) We have already proven that if \( \text{Schur}_{G,V} = 1 \) and \( \chi_{G,V} \) is rational, then there exists a \( G \)-invariant lattice of rank \( n \), namely, lattice (18). Conversely, let \( \Lambda \) be a \( G \)-invariant lattice of rank \( n \). Since \( C \Lambda = V \) because of the irreducibility of \( G \), the equality \( \text{rk}(\Lambda) = \dim_{\mathbb{C}} V \) implies that every basis \( e_1, \ldots, e_n \) of the \( \mathbb{Z} \)-module \( \Lambda \) is a basis of the \( \mathbb{C} \)-linear space \( V \). Hence \( \mathbb{Q} \Lambda \) is a \( G \)-invariant \( \mathbb{Q} \)-form of \( V \), i.e., \( G \) is defined over \( \mathbb{Q} \). Therefore in this case we have
\[
\Lambda + c \Lambda = (\mathbb{Z} + c \mathbb{Z}) e_1 \oplus \ldots \oplus (\mathbb{Z} + c \mathbb{Z}) e_n.
\]
(21)
The condition \( c \notin \mathbb{R} \) implies that \( \mathbb{Z} + c \mathbb{Z} \) is a lattice of rank 2 in \( \mathbb{C} \), and then from (21) we deduce that \( \Lambda + c \Lambda \) is a lattice of rank \( 2n \), which is clearly \( G \)-invariant. This completes the proof. \( \square \)

Corollary 2.13. If \( \text{Schur}_{G,V} \geq 3 \), then there are no nonzero \( G \)-invariant lattices in \( V \).

Corollary 2.14. Suppose that the greatest common divisor of the integers \( \ker(u) \), where \( u \) runs through \( \mathbb{Q}G \), is equal to 1. Then

(i) a nonzero \( G \)-invariant lattice in \( V \) exists if and only if \( \chi_{G,V} \) is either rational or imaginary quadratic;

(ii) a \( G \)-invariant lattice in \( V \) of rank \( n \) exists if and only if \( \chi_{G,V} \) is rational.

Proof. This follows from Lemma 2.8 and Theorem 2.12. \( \square \)
Lemma 2.15. Let $\Lambda$ and $\Lambda'$ be lattices of rank $2n$ in $V$ such that $\Lambda' \subseteq \Lambda$.

(a) The following properties are equivalent:
   (i) $V/\Lambda$ is an abelian variety;
   (ii) $V/\Lambda'$ is an abelian variety.

(b) If (i) and (ii) hold, then the abelian varieties $V/\Lambda$ and $V/\Lambda'$ are isogenous.

Proof. (a) Let (i) holds. This means that $V/\Lambda$ admits a polarization $\Psi$, i.e., there exists a positive-definite Hermitian form $\Psi : V \times V \to \mathbb{C}$ such that its imaginary part assumes integer values on $\Lambda \times \Lambda$ (see, e.g. [M]). Since $\Lambda' \subseteq \Lambda$, the same $\Psi$ is a polarization for $V/\Lambda'$; whence (ii).

Conversely, let (ii) holds and let $\Psi'$ be a polarization for $V/\Lambda'$. Since $\text{rk}(\Lambda) = \text{rk}(\Lambda')$, we have

$$[\Lambda : \Lambda'] < \infty.$$  \hspace{1cm} (22)

Then clearly, $[\Lambda : \Lambda']^2 \cdot \Psi'$ is a polarization for $V/\Lambda$; whence (i).

(b) The claim follows from inequality (22) meaning that $V/\Lambda$ is the quotient of $V/\Lambda'$ by a finite subgroup. \hfill \Box

Lemma 2.16. If an abelian variety is isogenous to a self-product of an elliptic curve with complex multiplication, then, in fact, it is isomorphic to a product of mutually isogenous elliptic curves with complex multiplication.

Proof. This is proved in [K] (see also [SM], [Sc]). \hfill \Box

3. The case of $\text{Schur}_{G,V} = 1$

The following theorem gives a description of $V/\Lambda$ in case when $\text{Schur}_{G,V} = 1$.

Theorem 3.1. Suppose that $\text{Schur}_{G,V} = 1$. If there exists a nonzero $G$-invariant lattice $\Lambda$ in $V$, then the following properties hold:

(i) $Z_G$ is either $\mathbb{Q}$ or an imaginary quadratic number field.

(ii) $\text{rk}(\Lambda) = n$ or $2n$.

(iii) If $\text{rk}(\Lambda) = n$, then $Z_G = \mathbb{Q}$ and $G$ is defined over $\mathbb{Q}$.

(iv) Suppose that $\text{rk}(\Lambda) = 2n$. Let $\mathcal{O}$ be the maximal order in $Z_G$. Fix a $Z_G$-algebra isomorphism (existing by (7))

$$\tau : M_n(Z_G) \cong QG.$$  \hspace{1cm} \footnotesize{\text{(7)}}

Then there is a lattice $\Lambda'$ in $V$ that enjoys the following properties:

(iv1) $\Lambda' \supseteq \Lambda$;
(iv₂) \( \Lambda' \) is \( \tau(M_n(\mathcal{O})) \)-invariant;

(iv₃) there exists a lattice \( \Gamma \) of rank 2 in \( \mathbb{C} \) and a \( \mathbb{C} \)-linear isomorphism \( \nu : \mathbb{C}^n \xrightarrow{\sim} V \) such that \( \Omega \Gamma = \Gamma \) and \( \nu(\Gamma^n) = \Lambda' \);

(iv₄) the complex torus \( V/\Lambda \) is an abelian variety isogenous to a self-product of an elliptic curve;

(iv₅) if \( \mathbb{Z} \Gamma \) is an imaginary quadratic number field, then \( V/\Lambda \) is isomorphic to a product of mutually isogenous elliptic curves with complex multiplication by \( \mathcal{O} \).

Proof. Lemmas 2.5 and 2.3 imply (i), Lemma 2.4 implies (ii), and Theorem 2.12 and Lemma 2.3 imply (iii). Assume now that \( \text{rk}(\Lambda) = 2n \) and prove (iv).

Clearly, the map \( \mathcal{O} \otimes \mathbb{Q} \to \mathbb{Z} \Gamma \), \( a \otimes r \mapsto ar \), is a ring isomorphism that yields the ring isomorphism

\[
M_n(\mathcal{O}) \otimes \mathbb{Q} \xrightarrow{\sim} M_n(\mathbb{Z} \Gamma), \quad z \otimes r \mapsto zr.
\]

Thus \( M_n(\mathcal{O}) \) is an order in the \( \mathbb{Q} \)-algebra \( M_n(\mathbb{Z} \Gamma) \); whence \( \tau(M_n(\mathcal{O})) \) is an order in the \( \mathbb{Q} \)-algebra \( \mathbb{Q} \Gamma \). Since \( \mathbb{Z} \Gamma \) is an order in \( \mathbb{Q} \Gamma \) as well, \( \mathbb{Z} \Gamma \cap \tau(M_n(\mathcal{O})) \) is a subgroup of finite index in \( \tau(M_n(\mathcal{O})) \). Since \( \Lambda \) is \( \mathbb{Z} \Gamma \)-invariant, this entails that there are only finitely many sets of the form \( z(\Lambda) \), where \( z \in \tau(M_n(\mathcal{O})) \). Every such set \( z(\Lambda) \) is a finitely generated additive subgroup in \( V \), and \( [z(\Lambda) : (\Lambda \cap z(\Lambda))] < \infty \). This implies that the sum of these subgroups,

\[
\Lambda' := \sum_{z \in \tau(M_n(\mathcal{O}))} z(\Lambda),
\]

is a \( \tau(M_n(\mathcal{O})) \)-invariant lattice in \( V \) containing \( \Lambda \) as a subgroup of finite index.

The \( \tau(M_n(\mathcal{O})) \)-module \( \Lambda' \) is faithful. Indeed, notice that since \( \Lambda' \) is \( \tau(M_n(\mathcal{O})) \)-invariant and \( \tau(M_n(\mathcal{O})) \) is an order in \( \mathbb{Q} \Gamma \), the \( \mathbb{Q} \)-linear subspace \( \mathbb{Q} \Lambda' \) in \( V \) is \( \mathbb{Q} \Gamma \)-invariant. Therefore if \( z\Lambda' = 0 \) for \( z \in \tau(M_n(\mathcal{O})) \), then \( z \) lies in the kernel of natural \( \mathbb{Q} \)-algebra homomorphism \( \mathbb{Q} \Gamma \to \text{End}_\mathbb{Q}(\mathbb{Q} \Lambda') \). Since \( \mathbb{Q} \Gamma \) is a simple \( \mathbb{Q} \)-algebra, this kernel is trivial. Thus \( z = 0 \); whence the faithfulness.

By construction, (iv₁) and (iv₂) hold. We are going to prove that (iv₃), (iv₄), and (iv₅) hold as well.

Tensoring \( \tau \) by \( \mathbb{C} \) over \( \mathbb{Z} \Gamma \), we obtain a \( \mathbb{C} \)-algebra isomorphism

\[
M_n(\mathcal{O}) \otimes \mathbb{Z} \Gamma \xrightarrow{\sim} \mathbb{Q} \Gamma \otimes_{\mathbb{Z} \Gamma} \mathbb{C}.
\]

Thus

\[
M_n(\mathcal{O}) = M_n(\mathbb{Z} \Gamma) \otimes_{\mathbb{Z} \Gamma} \mathbb{C} \xrightarrow{\sim} \mathbb{Q} \Gamma \otimes_{\mathbb{Z} \Gamma} \mathbb{C}.
\]
On the other hand, by Lemma 2.1, we have the \( \mathbb{C} \)-algebra isomorphism (4).

Composing isomorphisms (24) and (4), we get a \( \mathbb{C} \)-algebra isomorphism

\[
\tau_\mathbb{C} : M_n(\mathbb{C}) \to \text{End}_\mathbb{C}(V).
\]

Consider the coordinate \( \mathbb{C} \)-linear space \( \mathbb{C}^n \) endowed with the natural structure of left \( M_n(\mathbb{C}) \)-module. Since \( \mathbb{C}^n \) is the unique (up to isomorphism) left \( M_n(\mathbb{C}) \)-module of \( \mathbb{C} \)-dimension \( n \), there is a \( \mathbb{C} \)-linear isomorphism \( \varphi : \mathbb{C}^n \to V \) such that

\[
\varphi(v(y)) = \tau_\mathbb{C}(y)(\varphi(v)) \quad \text{for all } v \in \mathbb{C}^n, y \in M_n(\mathbb{C}).
\]

Clearly,

\[
\Lambda'' := \varphi^{-1}(\Lambda')
\]

is an \( M_n(\mathcal{O}) \)-invariant lattice of rank 2\( n \) in \( \mathbb{C}^n \). Let \( e_1, \ldots, e_n \) be the standard basis in \( \mathbb{C}^n \). Since \( \Lambda'' \) is \( M_n(\mathcal{O}) \)-invariant, it is easily seen that there exists a lattice \( \Gamma \) in \( \mathbb{C} \) such that

\[
\mathcal{O}\Gamma = \Gamma \quad \text{and} \quad \Lambda'' = \Gamma e_1 + \ldots + \Gamma e_n.
\]

Since \( \text{rk}(\Lambda'') = 2n \), we deduce from the second equality in (25) that \( \text{rk}(\Gamma) = 2 \), and the complex torus \( V/\Lambda' \) is isomorphic to the self-product of elliptic curve \( \mathbb{C}/\Gamma \), hence is an abelian variety. Since \( \Lambda \) is a subgroup of finite index in \( \Lambda' \), Lemma 2.15 implies that \( V/\Lambda \) is an abelian variety as well and \( V/\Lambda \) and \( V/\Lambda' \) are isogenous. This proves (iv3) and (iv4).

Now assume that \( Z_G \) is an imaginary quadratic field. Then \( \mathcal{O} \neq \mathbb{Z} \), and the first equality in (25) yields that the elliptic curve \( \mathbb{C}/\Gamma \) has complex multiplication. Hence \( V/\Lambda \) is isogenous to a self-product of an elliptic curve with complex multiplication. Now (iv5) follows from Lemma 2.16.

Combining Theorem 3.1 with the results of Section 2, we obtain the following applications.

**Theorem 3.2.** Suppose that \( \chi_{G,V} \) is not rational. If there exists a nonzero \( G \)-invariant lattice \( \Lambda \) in \( V \), then

1. \( Z_G \) is an imaginary quadratic number field;
2. \( \text{rk}(\Lambda) = 2n \);
3. \( V/\Lambda \) is isomorphic to a product of mutually isogenous elliptic curves with complex multiplication by \( Z_G \).

**Proof.** This follows from the combination of Lemma 2.6 and Theorem 3.1.
Corollary 3.3 (E. B. Vinberg). Suppose that the $G$-module $V$ is not self-dual. If there exists a nonzero $G$-invariant lattice $\Lambda$ in $V$, then $\text{rk}(\Lambda) = 2n$ and $V/\Lambda$ is isogenous to a self-product of an elliptic curve with complex multiplication.

Proof. This immediately follows from Theorem 3.2. □

Theorem 3.4. Suppose that the greatest common divisor of the integers $\ker(u)$, where $u$ runs through $\mathbb{Q}G$, is equal to 1. If there exists a nonzero $G$-invariant lattice $\Lambda$ in $V$, then the conclusions of Theorem 3.1 hold true.

Proof. This follows from the combination of Lemma 2.8 and Theorem 3.1. □

Remark 3.5. Since $\dim_{\mathbb{C}} \ker(0) = n$ and $\dim_{\mathbb{C}}(\ker(\text{id}_V - r)) = n - 1$ for any reflection $r \in \text{GL}(V)$, the condition that the greatest common divisor of the integers $\ker(u)$, where $u$ runs through $\mathbb{Q}G$, is equal to 1, always holds for every reflection group $G$. By Theorem 3.4, this implies that the conclusion of Theorem 3.1 holds true for every reflection group $G$ that admits a nonzero invariant lattice.

Example 3.6. Suppose that $n$ is a positive integer, $S$ is a $(n + 1)$-element set, $G$ is a doubly transitive permutation group of $S$ and $V$ is the $n$-dimensional $\mathbb{C}$-vector space of complex-valued functions $f : S \to \mathbb{C}$ with $\sum_{s \in S} f(s) = 0$. Clearly, the natural linear representation of $G$ in $V$ is faithful and defined over $\mathbb{Q}$; in particular, its character is rational. It is well-known that the $G$-module $V$ is (absolutely) simple (see, for instance, [Se, Sect. 2.3, Ex. 2]). Clearly, the Schur index is 1. It follows from Theorem 2.12 that there exist $G$-invariant lattices in $V$ of rank $n$ and $2n$. It follows from Theorem 3.1 that if $\Lambda$ is a $G$-invariant lattice of rank $2n$ in $V$ then $V/\Lambda$ is isogenous to a self-product of an elliptic curve.

Example 3.7. Let $G$ be the simple group $L_3(5) := \text{PSL}_3(\mathbb{F}_5)$. Then there exists a simple complex $G$-module $V$ such that $\dim_{\mathbb{C}}(V) = 124$, $Z_G = \mathbb{Q}(\chi_{G,V}) = \mathbb{Q}(\sqrt{-1})$ and Schur$_{G,V} = 1$, see [Fe, p. 283]. By Theorem 2.12, there are $G$-invariant lattices of rank 248 in $V$. It follows from Theorem 3.1 that if $\Lambda$ is a $G$-invariant lattice of rank 248 in $V$ then $V/\Lambda$ is isomorphic to a product of mutually isogenous elliptic curves with complex multiplication by $\mathbb{Q}(\sqrt{-1})$.

Example 3.8. Let $p$ be an odd prime that is congruent to 3 modulo 4, $r$ a positive integer and $q = p^{2r-1}$. Let $G$ be the group $\text{SL}_2(\mathbb{F}_q)$. Then there exists a faithful simple complex $G$-module $V$ such that $\dim_{\mathbb{C}}(V) = (q - 1)/2$, $Z_G = \mathbb{Q}(\chi_{G,V}) = \mathbb{Q}(\sqrt{-p})$. For instance, if $q = 5$ then $\chi_{G,V} = \sqrt{-1}$ and $Z_G = \mathbb{Q}(\sqrt{-1})$. This $G$-module has $2^{14}$ $G$-invariant lattices of rank $2^{14}$ in $V$. Note that for $r = 3$ the group $G$ admits a $G$-invariant lattice of rank 128, which is isogenous to a self-product of an elliptic curve with complex multiplication by $\mathbb{Q}(\sqrt{-1})$.
\(\mathbb{Q}(\sqrt{-q}) = \mathbb{Q}(\sqrt{-p})\) and Schur\(_{G,V} = 1\), see [J, p. 4], [Fe, p. 284–285]. By Theorem 2.12, there are \(G\)-invariant lattices of rank \(q-1\) in \(V\). It follows from Theorem 3.1 that if \(\Lambda\) is a \(G\)-invariant lattice of rank \(q-1\) in \(V\) then \(V/\Lambda\) is isomorphic to a product of mutually isogenous elliptic curves with complex multiplication by \(\mathbb{Q}(\sqrt{-p})\).

4. The case of Schur\(_{G,V} \neq 1\)

The following theorem gives a description of \(V/\Lambda\) in case when Schur\(_{G,V} \neq 1\) (i.e., in fact, when Schur\(_{G,V} = 2\), by Theorem 2.12).

**Theorem 4.1.** Suppose that Schur\(_{G,V} \neq 1\). If there exists a nonzero \(G\)-invariant lattice \(\Lambda\) in \(V\), then the following properties hold:

(i) \(\chi_{G,V}\) is rational (hence the \(G\)-module \(V\) is either orthogonal or symplectic).

(ii) \(n\) is even.

(iii) There exists a quaternion \(\mathbb{Q}\)-algebra \(H\) such that the \(\mathbb{Q}\)-algebras \(M_{n/2}(H)\) and \(\mathbb{Q}G\) are isomorphic.

(iv) \(\text{rk}(\Lambda) = 2n\).

(v) Fix an order \(\mathcal{O}\) in \(H\). Then there exists a two-dimensional complex torus \(T\) that enjoys the following properties:

\(v_1\) \(V/\Lambda\) is isogenous to a self-product of \(T\);

\(v_2\) there exists a ring embedding \(\mathcal{O} \rightarrow \text{End}(T)\);

\(v_3\) if the \(G\)-module \(V\) is orthogonal, then \(H\) is indefinite, and \(T\) and \(V/\Lambda\) are abelian varieties;

\(v_4\) if the \(G\)-module \(V\) is symplectic, then \(H\) is definite, and \(T\) and \(V/\Lambda\) either are not abelian varieties or are isomorphic to the products of mutually isogenous elliptic curves with complex multiplication.

**Proof.** Theorem 2.12, Lemma 2.10, and Lemma 2.4 imply (i), (ii), (iii), and (iv). Fix a \(\mathbb{Q}\)-algebra isomorphism

\[\tau : M_{n/2}(H) \xrightarrow{\sim} \mathbb{Q}G\]

and an order \(\mathcal{O}\) in \(H\). Both \(\mathbb{Z}G\) and \(\tau(M_{n/2}(\mathcal{O}))\) are the orders in \(\mathbb{Q}\)-algebra \(\mathbb{Q}G\). The same argument as in the part of proof of Theorem 3.1 related to formula (23) shows that there are only finitely many sets of the form \(z(\Lambda)\), where \(z \in \tau(M_{n/2}(\mathcal{O}))\), and the sum

\[\Lambda' := \sum_{z \in \tau(M_{n/2}(\mathcal{O}))} z(\Lambda) \subset V\]
is a $\tau(M_{n/2}(O))$-invariant lattice in $V$ containing $\Lambda$ as a subgroup of finite index and faithful as $\tau(M_{n/2}(O))$-module.

Tensoring $\tau$ by $\mathbb{C}$ over $\mathbb{Q}$ and composing with (4), we get a $\mathbb{C}$-algebra isomorphism

$$\tau_{\mathbb{C}}: M_{n/2}(H) \otimes_{\mathbb{Q}} \mathbb{C} \isom \text{End}_{\mathbb{C}}(V).$$

Since the $\mathbb{C}$-algebras $H \otimes_{\mathbb{Q}} \mathbb{C}$ and $M_2(\mathbb{C})$ are isomorphic, we may (and shall) fix a $\mathbb{C}$-algebra isomorphism

$$\kappa: H \otimes_{\mathbb{Q}} \mathbb{C} \isom M_2(\mathbb{C}).$$

It induces the $\mathbb{C}$-algebra isomorphism

$$\kappa_{n/2} : M_{n/2}(H) \otimes_{\mathbb{Q}} \mathbb{C} = M_{n/2}(H \otimes_{\mathbb{Q}} \mathbb{C}) \isom M_{n/2}(M_2(\mathbb{C})).$$

Hence we obtain a $\mathbb{C}$-algebra isomorphism

$$\tau_{\mathbb{C}} \circ \kappa_{n/2}^{-1} : M_{n/2}(M_2(\mathbb{C})) \isom \text{End}(V).$$

Consider the coordinate $\mathbb{C}$-linear space $\mathbb{C}^n$ presented as the direct sum of $n/2$ copies of $\mathbb{C}^2$,

$$\mathbb{C}^n = (\mathbb{C}^2)^{n/2} = \mathbb{C}^2 \oplus \ldots \oplus \mathbb{C}^2,$$

and endowed with the natural structure of left $M_{n/2}(M_2(\mathbb{C}))$-module. Since $(\mathbb{C}^2)^{n/2}$ is the unique (up to isomorphism) left $M_{n/2}(M_2(\mathbb{C}))(\simeq M_n(\mathbb{C}))$-module of $\mathbb{C}$-dimension $n$, there is a $\mathbb{C}$-linear isomorphism $\varphi: (\mathbb{C}^2)^{n/2} \isom V$ such that

$$\varphi(y(v)) = (\tau_{\mathbb{C}} \circ \kappa_{n/2}^{-1})(y)(\varphi(v)) \text{ for all } v \in (\mathbb{C}^2)^{n/2}, y \in M_{n/2}(M_2(\mathbb{C})).$$

We identify $H$ with $H \otimes_{\mathbb{Q}} 1$. Then clearly,

$$\Lambda'': = \varphi^{-1}(\Lambda')$$

is a $M_{n/2}(\kappa(O))$-invariant lattice of rank $2n$ in $(\mathbb{C}^2)^{n/2}$. Using the $M_{n/2}(\kappa(O))$-invariance of $\Lambda''$, it is easy to see that there exists a lattice $\Gamma$ in $\mathbb{C}^2$ such that

$$\kappa(O) \Gamma = \Gamma \text{ and } \Lambda'' = \Gamma^{n/2} = \Gamma \oplus \ldots \oplus \Gamma. \quad (26)$$

Since $\text{rk}(\Lambda'') = \text{rk}(\Lambda') = \text{rk}(\Lambda) = 2n$, we deduce from (26) that $\text{rk}(\Gamma) = 4$. Hence $T := \mathbb{C}^2/\Gamma$ is a 2-dimensional complex torus and we have the ring embedding

$$\mathcal{O} \isom \kappa(O) \hookrightarrow \text{End}(T). \quad (27)$$

Clearly, the complex torus $(\mathbb{C}^2)^{n/2}/\Lambda'' \simeq (\mathbb{C}^2/\Gamma)^{n/2} = T^{n/2}$ is isomorphic to $V/\Lambda'$. Since $\Lambda$ is a subgroup of finite index in $\Lambda'$, the complex torus $V/\Lambda$ is isogenous to $T^{n/2}$. This proves (v1) and (v2).
Suppose that $H$ is indefinite, i.e., by Lemma 2.10, the $G$-module $V$ is orthogonal. It then follows from [L2, Theorem 4.3, p. 152] that $T$ is an abelian surface. Now Lemma 2.15 implies that $V/\Lambda$ is an abelian variety. This proves (v3).

Assume now that $H$ is indefinite, i.e., by Lemma 2.10, the $G$-module $V$ is symplectic. By (27), the $\mathbb{Q}$-algebra $\text{End}_0^0(T) := \text{End}(T) \otimes \mathbb{Q}$ contains a subalgebra isomorphic to $\mathcal{O} \otimes \mathbb{Q} \cong H$ and therefore is noncommutative.

Suppose that $T$ is an abelian surface. Using tables in [O], one may then easily verify that $T$ is not simple. On the other hand, if $T$ is isogenous to a product of two non-isogenous elliptic curves, then $\text{End}_0^0(T)$ is commutative and cannot contain a subalgebra isomorphic to $H$. If $T$ is isogenous to a square of an elliptic curve without complex multiplication, then $\text{End}_0^0(T)$ is isomorphic to $M_2(\mathbb{Q})$ and hence cannot contain such a subalgebra as well. It follows that $T$ is isogenous to a square of an elliptic curve with complex multiplication. This implies that $V/\Lambda$ is isogenous to a self-product of an elliptic curve with complex multiplication, and hence, by Lemma 2.16, that $V/\Lambda$ is isomorphic to a product of mutually isogenous elliptic curves with complex multiplication.

Now assume that $T$ is not an abelian surface. Then $T_{n/2}$ is not an abelian variety because every polarization on $T_{n/2}$ obviously induces a polarization on (say, the first factor) $T$. It then follows from Lemma 2.15 that $V/\Lambda$ is not an abelian variety as well. This proves (v4). □

The following examples show that both possibilities in conclusion (v4) of Theorem 4.1 may indeed occur. In particular, there are finite irreducible groups $G$ and $G$-invariant lattices $\Lambda$ in $V$ such that $\text{rk}(\Lambda) = 2n$ and the complex torus $V/\Lambda$ is not an abelian variety.

**Example 4.2.** First, we can use that $H$ contains an imaginary quadratic subfield $F$ (see below the proof of Theorem 4.6). Then by Theorem 2.11, for $\Lambda = \Lambda^{(F)}$, the torus $V/\Lambda$ is isomorphic to a product of mutually isogenous elliptic curves with complex multiplication.

One can also give a more concrete example not exploiting Theorem 2.11. Let $G$ be the image of a (unique up to isomorphism) irreducible 2-dimensional complex representation of the quaternion group. Fixing a basis in the representation space $V$, we may (and shall) identify $V$ with $\mathbb{C}^2$ and $G$ with the matrix group

$$\left\{ \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \pm \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \pm \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \pm \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \right\}. \tag{28}$$
It is known, [CR, § 70], that Schur$_{G,V} = 2$. The $G$-module $C^2$ is symplectic and (28) clearly yields that the lattice $\Lambda := \{(a, b) \in C^2 \mid a, b \in \mathbb{Z} + i\mathbb{Z}\}$ is $G$-stable and $C^2/\Lambda$ is isomorphic to the square of the elliptic curve $C/(\mathbb{Z} + i\mathbb{Z})$ with complex multiplication by $\mathbb{Z}[i]$. □

Example 4.3. Consider the order $\Lambda := \mathbb{Z}1 + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}k$ in the Hamiltonian quaternion $\mathbb{Q}$-algebra $H := \left(\frac{-1}{\mathbb{Q}}\right) = \mathbb{Q}1 + \mathbb{Q}i + \mathbb{Q}j + \mathbb{Q}k$. It is a lattice of rank 4 in the underlying 4-dimensional real linear space $V$ of the quaternion $\mathbb{R}$-algebra $H = \mathbb{R}1 + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$. The quaternion group $G := \{\pm 1, \pm i, \pm j, \pm k\}$ acts $\mathbb{R}$-linearly and faithfully on $V$ by left multiplication in $H$. This action is irreducible and $\Lambda$ is $G$-invariant. Pick any element $c \in H$ such that $c^2 = -1$, and endow $V$ with a structure of 2-dimensional complex linear space $V_c$ defining multiplication by $i$ as right multiplication by $c$ in $H$. Since left and right multiplications commute, the action of $G$ on $V_c$ is $\mathbb{C}$-linear (and irreducible). Thus we may (and shall) view $G$ as an irreducible group of complex linear transformations of $V_c$.

Assume now that $c \notin \mathbb{R}F$ for any imaginary quadratic subfield $F$ of $H$ (such $c$’s do exist, see the proof of Theorem 2.11). Consider the endomorphism ring of 2-dimensional complex torus $V/\Lambda$,

$$\text{End}(V/\Lambda) := \{u \in \text{End}_{\mathbb{C}}(V) \mid u(\Lambda) \subseteq \Lambda\}.$$ 

Our assumption on $c$ implies that every $\mathbb{Q}$-linear endomorphism of $H$ that becomes an element of $\text{End}_{\mathbb{C}}(V)$ after the extension of scalars from $\mathbb{Q}$ to $\mathbb{R}$ must commute with right multiplication by every element of $H$, and therefore is left multiplication by an element of $H$. Hence $\text{End}(V/\Lambda)$ consists of all $u \in H$ with $u \cdot \Lambda \subseteq \Lambda$. It follows that $\text{End}(V/\Lambda)$ coincides with the set of left multiplications by elements of $\Lambda$ and therefore $\text{End}(V/\Lambda) \simeq \Lambda$. Notice that $H$ is a definite quaternion $\mathbb{Q}$-algebra. Thus the endomorphism ring of two-dimensional complex torus $V/\Lambda$ is an order in a definite quaternion $\mathbb{Q}$-algebra. But it is known, [O], that there are no complex abelian surfaces whose endomorphism ring is an order in a definite quaternion $\mathbb{Q}$-algebra. This implies that $V/\Lambda$ is not an abelian variety.

Note that if $c \in \mathbb{R}F$ for an imaginary quadratic subfield $F$ of $H$, then the endomorphism algebra of $V/\Lambda$ is isomorphic to $H \otimes_{\mathbb{Q}} F \simeq M_2(F)$ and therefore $V/\Lambda$ is isogenous to a square of an elliptic curve with complex multiplication by an order of $F$. □
Example 4.4. Here is another example of the outcome (v) of Theorem 4.1. Let $p$ be an odd prime, $r$ a positive integer and $q = p^{2r}$.

Let $G$ be the group $\text{SL}_2(F_q)$. Then there exists a faithful simple complex $G$-module $V$ such that $\dim_{\mathbb{C}}(V) = (q - 1)/2$, $\chi_{G,V}$ is rational, $\text{Schur}_{G,V} = 2$, and the quaternion $\mathbb{Q}$-algebra $H$ from Theorem 4.1 (iii) is ramified exactly at $p$ and $\infty$, see [J, p. 4], [Fe, p. 284–285]. In particular, $H$ is definite. By Theorem 2.12, there are $G$-invariant lattices $\Lambda$ of rank $q - 1$ in $V$. □

Example 4.5. Here is an example of the outcome (v) of Theorem 4.1. Let $G$ be the simple group $HJ$. Then there exists a simple complex $G$-module $V$ such that $\dim_{\mathbb{C}}(V) = 336$, $\chi_{G,V}$ is rational, $\text{Schur}_{G,V} = 2$, and the quaternion $\mathbb{Q}$-algebra $H$ from Theorem 4.1 (iii) is indefinite, see [Fe, p. 283]. By Theorem 2.12, there is a $G$-invariant lattice $\Lambda$ of rank 672 in $V$. □

According to Example 4.3, in general there exist $G$-invariant lattices $\Lambda$ such that the complex torus $V/\Lambda$ is not an abelian variety. However it appears that one can always replace $\Lambda$ by another $G$-invariant lattice $\Delta$ such that $V/\Delta$ is an abelian variety. More precisely, the following statement holds true.

Theorem 4.6. The following properties are equivalent.

(i) there exists a nonzero $G$-invariant lattice in $V$;

(ii) there exists a $G$-invariant lattice $\Delta$ in $V$ such that $V/\Delta$ is isomorphic to a product of mutually isogenous elliptic curves with complex multiplication.

Proof. Assume that (i) holds. Then Theorems 2.12 and 3.1 reduce proving (ii) to the cases when $\chi_{G,V}$ is rational and $\text{Schur}_{G,V}$ is 1 or 2.

Consider the case when $\chi_{G,V}$ is rational and $\text{Schur}_{G,V} = 1$. Let $\mathcal{O}$ be an order in an imaginary quadratic number field. We have $\mathcal{O} = \mathbb{Z} + c\mathbb{Z}$ for some non-real $c \in \mathbb{C}$. Theorem 2.12 then implies that (21) is a $G$-invariant lattice of rank $2n$; denote it by $\Delta$. By construction, $\mathbb{C}/\mathcal{O}$ is an elliptic curve with complex multiplication by $\mathcal{O}$, and (21) implies that $V/\Delta$ is isomorphic to $(\mathbb{C}/\mathcal{O})^n$. Thus in this case (ii) holds.

Consider now the case when $\chi_{G,V}$ is rational and $\text{Schur}_{G,V} = 2$. Lemma 2.10 and Theorem 2.11 then reduce proving (ii) to showing that every quaternion $\mathbb{Q}$-algebra $H = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$ contains an imaginary quadratic subfield. But the latter property indeed holds, since the maximal subfields of $H$ are precisely (up to isomorphism) the fields $\mathbb{Q}(\sqrt{ar_1^2 + br_2^2 - abr_3^2})$, where $r_1, r_2, r_3 \in \mathbb{Q}$ and $r_1^2 + r_2^2 + r_3^2 \neq 0$ (see, e.g., [Pi, §13.1, Exercise 4]). This completes the proof. □
5. Quotients modulo invariant lattices of reflection groups

In this section we give another, geometric proof of Theorem 3.1 for reflection groups (regarding the first proof see Remark 3.5). It provides a more precise information on invariant lattices.

Recall that an element $r \in \text{GL}(V)$ is called a (complex) reflection if

(i) the order of $r$ is finite;

(ii) $\dim_{\mathbb{C}}(\ker(\text{id}_V - r)) = n - 1$.

For such $r$, the linear subspace

$$l_r := (\text{id}_V - r)(V)$$

is one-dimensional, $r$-invariant, and $r$ acts on it as scalar multiplication by a root of unity $\theta_r \neq 1$.

**Remark 5.1.** This implies that the assumptions and conclusions of Lemma 2.8 and Corollary 2.14 hold if $G$ is a reflection group.

It is well known, [ST] (see also [C], [Po]), that every finite irreducible reflection group in $V$ is generated by $n$ or $n + 1$ reflections and, in the last case, it contains an irreducible reflection subgroup generated by $n$ reflections. Therefore describing invariant lattices of finite irreducible reflection groups in $V$, we may consider only the groups generated by $n$ reflections. Let $G$ be such a group, and let $r_1, \ldots, r_n$ be a system of reflections generating $G$. We put $l_{r_j} = l_j$, $\theta_{r_j} = \theta_j$.

Since $G$ is finite, we may (and shall) fix a $G$-invariant positive definite Hermitian inner product $\langle \cdot, \cdot \rangle$ on $V$. For every reflection $r \in G$, fix a vector $e_r \in l_r$ of length 1. We then have

$$r(v) = v - (1 - \theta_r)\langle v, e_r \rangle e_r, \quad v \in V.$$  \hfill (30)

We put $e_j := e_{l_j}$.

Denote by $\mathcal{L}$ the set of all the lines $l_r$ where $r$ runs through all the reflections in $G$. Since $G$ is irreducible, the mutual fixed point set of $r_1, \ldots, r_n$ is $\{0\}$; whence

$$V = l_1 \oplus \ldots \oplus l_n.$$  \hfill (31)

Let $\Lambda$ be a $G$-invariant lattice in $V$. We put

$$\Lambda^0 := \sum_{l \in \mathcal{L}} \Lambda_l, \quad \text{where } \Lambda_l := \Lambda \cap l,$$

and $\Lambda_j := \Lambda_{l_j}$. Then $\Lambda^0$ is a subgroup of $\Lambda$, hence $\Lambda^0$ is a lattice in $V$ as well.

Throughout this section we keep the above notation.
Lemma 5.2. ([Po, Section 4.1]) The following properties hold:

(i) \( \Lambda^0 \) is \( G \)-invariant;
(ii) \([\Lambda : \Lambda^0] < \infty\);
(iii) \( \Lambda^0 = \Lambda_1 + \ldots + \Lambda_n \);
(iv) if \( \text{rk} (\Lambda) = 2n \), then \( \text{rk} (\Lambda^0) = 2n \) and \( \text{rk} (\Lambda_j) = 2 \) for every \( j \).

**Proof.** Clearly, \( \mathcal{L} \) is \( G \)-invariant; whence (i). Consider the linear operator

\[
s := (\text{id}_V - r_1) + \ldots + (\text{id}_V - r_n).
\]

If \( v \in \ker(s) \), then (33), (30), and (31) imply \( \langle v, e_1 \rangle = \ldots = \langle v, e_n \rangle = 0 \). Hence \( v = 0 \), i.e., \( s \) is nondegenerate. Therefore \( \text{rk}(\Lambda) = \text{rk}(s(\Lambda)) \). But (33), (29), and \( G \)-invariance of \( \Lambda \) imply \( s(\Lambda) \subseteq \Lambda_1 + \ldots + \Lambda_n \subseteq \Lambda^0 \subseteq \Lambda \); whence (ii).

By [Po, Section 3.2], every reflection in \( G \) is conjugate to a power of some \( r_j \). Hence for every \( l \in \mathcal{L} \), there are \( g \in G \) and an integer \( j \in [1, n] \) such that \( g(l) = l_j \). Therefore \( g(\Lambda_i) \subseteq \Lambda_j \). On the other hand, (29) implies that \( \Lambda_1 + \ldots + \Lambda_n \) is invariant with respect to every \( \text{id}_V - r_j \), hence is \( G \)-invariant. This and (32) entails (iii).

By (ii), if \( \text{rk}(\Lambda) = 2n \), then \( \text{rk}(\Lambda^0) = 2n \). In turn, by (iii) and (31), the latter equality implies \( \text{rk}(\Lambda_j) = 2 \) for every \( j \). This proves (iv). \( \square \)

Theorem 5.3. Let \( \text{rk}(\Lambda) = 2n \). Then:

(i) \( V/\Lambda^0 \) is an abelian variety isomorphic to a product of mutually isogenous elliptic curves;
(ii) \( V/\Lambda \) is an abelian variety isogenous to \( V/\Lambda^0 \), and hence, by (i), isogenous to a self-product of an elliptic curve;
(iii) if \( G \) is not the complexification of the Weyl group of an irreducible root system, then \( V/\Lambda \) is isomorphic to a product of mutually isogenous elliptic curves with complex multiplication.

**Proof.** Since \( \Lambda^0 \subseteq \Lambda \), Lemma 5.2 (iv) implies that \( V/\Lambda, V/\Lambda^0 \) are complex tori and every \( l_j/\Lambda_j \) is an elliptic curve. From (31) and Lemma 5.2 (iii) we deduce that

\[
V/\Lambda^0 \text{ is isomorphic to } l_1/\Lambda_1 \times \ldots \times l_n/\Lambda_n,
\]

hence, in particular, \( V/\Lambda^0 \) is an abelian variety. By Lemma 2.15, this implies that \( V/\Lambda \) is an abelian variety as well and

abelian varieties \( V/\Lambda \) and \( V/\Lambda^0 \) are isogenous.
By (29), we have \((\text{id}_V - r_k)(l_j) \subseteq l_k\) for every \(j\) and \(k\). The dimension reason then implies that the \(\mathbb{C}\)-linear map
\[
(\text{id}_V - r_k)\mid_{l_j} : l_j \to l_k \quad (36)
\]
is either 0 or an isomorphism. In the latter case, (36) induces an isomorphism of elliptic curves
\[
l_j/\Lambda_j \cong l_k/(\text{id}_V - r_k)(\Lambda_j). \quad (37)
\]
On the other hand, since \(\Lambda\) is \(G\)-invariant, \((\text{id}_V - r_k)(\Lambda_j) \subseteq \Lambda_k\). Hence if (36) is an isomorphism, then \(l_k/(\text{id}_V - r_k)(\Lambda_j)\) and \(l_k/\Lambda_k\) are isogenous elliptic curves. In this case, by (37), \(l_j/\Lambda_j\) and \(l_k/\Lambda_k\) are isogenous elliptic curves as well. Take now into account that, by (30), the map (36) is an isomorphism if and only if \(e_j\) and \(e_k\) are not orthogonal, and, since \(G\) is irreducible, every pair of vectors from the sequence \(e_1, \ldots, e_n\) can be included in a subsequence in which every two neighboring elements are not orthogonal. Hence
\[
\text{elliptic curves } l_j/\Lambda_j \text{ and } l_k/\Lambda_k \text{ are isogenous for every } j, k. \quad (38)
\]
The proofs of (i) and (ii) now follow from (35), (34), and (38).

To prove (iii), notice that (30) implies
\[
(\text{id}_V - r_{j_1})(\text{id}_V - r_{j_m})(\text{id}_V - r_{j_{m-1}}) \cdots (\text{id}_V - r_{j_2})(e_{j_1}) = c_{j_1 \ldots j_m}e_{j_1}, \quad \text{where}
\]
\[
c_{j_1 \ldots j_m} := \langle e_{j_1}, e_{j_2} \rangle \langle e_{j_2}, e_{j_3} \rangle \cdots \langle e_{j_{m-1}}, e_{j_m} \rangle \langle e_{j_m}, e_{j_1} \rangle \prod_{t=1}^m (1 - \theta_{j_t}). \quad (39)
\]
From (39) and (29) we deduce
\[
\text{tr}(\text{id}_V - r_{j_1})(\text{id}_V - r_{j_m})(\text{id}_V - r_{j_{m-1}}) \cdots (\text{id}_V - r_{j_2}) = c_{j_1 \ldots j_m}. \quad (40)
\]
Additivity of \(\text{tr}\) implies that \(\mathbb{Z}[\text{Tr}(G)] = \mathbb{Z}[\text{Tr}(\mathbb{Z}G)]\). Since \(\text{id}_V - r_1, \ldots, \text{id}_V - r_n\) generate the ring \(\mathbb{Z}G\), the monomials \((\text{id}_V - r_{j_1}) \cdots (\text{id}_V - r_{j_m})\) generate \(\mathbb{Z}G\) as a \(\mathbb{Z}\)-module. This and (40) entail that \(\mathbb{Z}[\text{Tr}(G)] = \mathbb{Z}[\ldots, c_{j_1 \ldots j_m}, \ldots]\), whence
\[
\mathbb{Q}(\chi_{G,V}) = \mathbb{Q}(\ldots, c_{j_1 \ldots j_m}, \ldots). \quad (41)
\]
Suppose that \(G\) is not the complexification of the Weyl group of an irreducible root system. Then \(\chi_{G,V}\) is not rational. Indeed, otherwise Remark 5.1 and Theorem 2.12 would imply that \(G\) is complexification of a finite real \(n\)-dimensional irreducible reflection group that has an invariant lattice of rank \(n\), and it is well known that such a real group is the Weyl group of an irreducible root system, [Bo].

Since \(\chi_{G,V}\) is not rational, (41) yields the existence of \(j_1, \ldots, j_m\) such that
\[
c_{j_1 \ldots j_m} \notin \mathbb{Q}. \quad (42)
\]
Since $\Lambda$ is $G$-invariant, (39) implies that $c_{j_1\ldots j_m}$ is a multiplier of $\Lambda_{j_1}$, i.e.,

$$c_{j_1\ldots j_m} \cdot \Lambda_{j_1} \subseteq \Lambda_{j_1}.$$  \hfill (43)

Properties (42) and (43) imply that $l_{j_1}/\Lambda_{j_1}$ is an elliptic curve with complex multiplication. But (35), (34), and (38) imply that $V/\Lambda$ is isogenous to $(l_{j_1}/\Lambda_{j_1})^n$. Thus $V/\Lambda$ is isogeneous to a self-product of an elliptic curve with complex multiplication. Now (iii) follows from Lemma 2.16. \hfill $\square$

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