RESIDUES AND DIFFERENTIAL OPERATORS ON SCHEMES

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0. Introduction

Suppose $X$ is a finite type scheme over a field $k$, with structural morphism $\pi$. Consider the twisted inverse image functor $\pi^! : D_c^+(k) \to D_c^+(X)$ of Grothendieck Duality Theory (see [RD]). The residue complex $K_X$ is defined to be the Cousin complex of $\pi^! k$. It is a bounded complex of quasi-coherent $O_X$-modules, possessing remarkable functorial properties. In this paper we provide an explicit construction of $K_X$. This construction reveals some new properties of $K_X$, and also has applications in other areas of algebraic geometry.

Grothendieck Duality, as developed by Hartshorne in [RD], is an abstract theory, stated in the language of derived categories. Even though this abstraction is suitable for many important applications, often one wants more explicit information. Thus a significant amount of work was directed at finding a presentation of duality in terms of differential forms and residues. Mostly the focus was on the dualizing sheaf $\omega_X$, in various circumstances. The structure of $\omega_X$ as a coherent $O_X$-module and its variance properties are thoroughly understood by now, thanks to an extended effort including
[Kl], [KW], [Li], [HK1], [HK2], [LS] and [HS]. Regarding an explicit presentation of the full duality theory of dualizing complexes, there have been some advances in recent years, notably in the papers [Ye1], [SY], [Hu], [Hg] and [Sa].

In this paper we give a totally new construction of the residue complex $K_X$, when $k$ is a perfect field of any characteristic and $X$ is any finite type $k$-scheme. The main idea is the use of Beilinson Completion Algebras (BCAs), which were introduced in [Ye2]. These algebras are generalizations of complete local rings, and they carry a mixed algebraic-analytic structure. A review of BCAs and their properties is included in Section 1, for the reader’s convenience.

Given a point $x \in X$, the complete local ring $\hat{O}_{X,x} = O_{X,(x)}$ is a BCA, so according to [Ye2] it has a dual module $K(O_{X,(x)})$. This module is a canonical model for the injective hull of the residue field $k(x)$. If $(x, y)$ is a saturated chain of points (i.e. $y$ is an immediate specialization of $x$) then there is a BCA $O_{X,(x,y)}$ and homomorphisms $q : K(O_{X,(x)}) \to K(O_{X,(x,y)})$ and $Tr : K(O_{X,(x,y)}) \to K(O_{X,(y)})$. The dual modules $K(\cdot)$ and the homomorphisms $q$ and $Tr$ have explicit formulas in terms of differential forms and coefficient fields. Set $\delta(x,y) := Tr \circ q : K(O_{X,(x)}) \to K(O_{X,(y)})$. Define a graded quasi-coherent sheaf $\mathcal{K}_X$ by

$$\mathcal{K}_X^q := \bigoplus_{\dim \{x\} = -q} K(O_{X,(x)})$$

and a degree 1 homomorphism

$$\delta := (-1)^{q+1} \sum_{(x,y)} \delta(x,y).$$

It turns out that $(\mathcal{K}_X, \delta)$ is a residual complex on $X$, and it is canonically isomorphic to $\pi^! k$ in the derived category $D(X)$. Hence it is the residue complex of $X$, as defined in the first paragraph. The functorial properties of $\mathcal{K}_X$ w.r.t. proper and étale morphisms are obtained directly from corresponding properties of BCAs, and therefore are reduced to explicit formulas. All this is worked out in Sections 2 and 3.

An $O_X$-module $M$ has a dual complex $\text{Dual} M := \mathcal{H}om_{O_X}(M, \mathcal{K}_X)$. Suppose $d : M \to N$ is a differential operator (DO). In Theorem 4.1 we prove there is a dual operator $\text{Dual}(d) : \text{Dual} N \to \text{Dual} M$, which commutes with $\delta$. The existence of $\text{Dual}(d)$ does not follow from formal considerations of duality theory; it is a consequence of our particular construction using BCAs (but cf. Remarks 4.6 and 4.7). The construction also provides explicit formulas for $\text{Dual}(d)$ in terms of differential operators and residues, which are used in the applications in Sections 6 and 7.

Suppose $A$ is a finite type $k$-algebra, and let $D(A)$ be the ring of differential operators of $A$. As an immediate application of Theorem 4.1 we obtain a description of the opposite ring $D(A)^\circ$, as the ring of DOs on $\mathcal{K}_A$ which commute with $\delta$ (Theorem 4.8). In the case of a Gorenstein algebra it follows
that the opposite ring $D(A) \otimes_A D(A) \otimes_A \omega_A^{-1}$ (Corollary 4.9).

Applying Theorem 4.1 to the De Rham complex $\Omega_{X/k}$ we obtain the De Rham-residue complex $\mathcal{F}_X = \text{Dual } \Omega_{X/k}$. Up to signs this coincides with El-Zein’s complex $K_{X,*}$ of [EZ] (Corollary 5.9). The fundamental class $C_Z \in \mathcal{F}_X$, for a closed subscheme $Z \subset X$, is easily described in this context (Definition 5.10).

The construction above works also for a formal scheme $X$ which is of formally finite type over $k$, in the sense of [Ye3]. An example of such a formal scheme is the completion $X = Y/X$, where $X$ is a locally closed subset of the finite type $k$-scheme $Y$. Therefore we get a complex $\mathcal{F}_X = \text{Dual } \Omega_{X/k}$. When $X \subset \mathfrak{X}$ is a smooth formal embedding (see Definition 6.5) we prove that the cohomology modules $H^q(X, \mathcal{F}_X)$ are independent of $X$. This is done by analyzing the $E_1$ term of the niveau spectral sequence converging to $H^\cdot(X, \mathcal{F}_X)$ (Theorem 6.16). Here we assume char $k = 0$. The upshot is that $H^q(X, \mathcal{F}_X) = H^{\text{DR}}_q(X)$, the De Rham homology. There is an advantage in using smooth formal embeddings. If $U \to X$ is any étale morphism, then there is an étale morphism $\mathfrak{U} \to \mathfrak{X}$ s.t. $U = \mathfrak{U} \times_X X$, so $U \subset \mathfrak{U}$ is a smooth formal embedding. From this we conclude that $H^{\text{DR}}(-)$ is a contravariant functor on $X_{\text{et}}$, the small étale site. Previously it was only known that $H^{\text{DR}}(-)$ is contravariant for open immersions (cf. [BlO] Example 2.2).

Suppose $X$ is smooth, and let $\mathcal{H}^p_{\text{DR}}$ be the sheafification of the presheaf $U \mapsto H^p_{\text{DR}}(U)$ on $X_{\text{Zar}}$. Bloch-Ogus [BlO] give a flasque resolution of $\mathcal{H}^p_{\text{DR}}$, the arithmetic resolution. It involves the sheaves $i^*_{\mathfrak{X}} H^q_{k(x)/k}$ where $i_{\mathfrak{X}} : \{x\} \to X$ is the inclusion map. Our analysis of the niveau spectral sequence shows that the coboundary operator of this resolution is a sum of Parshin residues (Corollary 6.24).

Our final application of the new construction of the residue complex is to describe the intersection cohomology $D$-module $\mathcal{L}(X, Y)$, when $X$ is an integral curve embedded in a smooth $n$-dimensional variety $Y$ (see [BrK]). Again we assume $k$ has characteristic 0. In fact we are able to describe all coherent $D_Y$-submodules of $\mathcal{H}^{n-1}_X \mathcal{O}_Y$ in terms of the singularities of $X$ (Corollary 7.6). This description is an algebraic version of Vilonen’s work in [Vi], replacing complex analysis with BCAs and algebraic residues. It is our hope that a similar description will be found in the general case, namely $\dim X > 1$. Furthermore, we hope to give in the future an explicit description of the Cousin complex of $\text{DR } \mathcal{L}(X, Y) = \Omega^{\text{DR}}_{Y/k} \otimes \mathcal{L}(X, Y)$. Note that for $X = Y$ one has $\mathcal{L}(X, Y) = \mathcal{O}_X$, so this Cousin complex is nothing but $\mathcal{F}_X$.

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1. Review of Beilinson Completion Algebras

Let us begin by reviewing some facts about Topological Local Fields (TLFs) and Beilinson Completion Algebras (BCAs) from the papers [Ye1] and [Ye2].

A semi-topological (ST) ring is a ring $A$, with a linear topology on its underlying additive group, such that for every $a \in A$ the multiplication (on either side) $a : A \to A$ is continuous.

Let $K$ be a field. We say $K$ is an $n$-dimensional local field if there is a sequence of complete discrete valuation rings $O_1, \ldots, O_n$, where the fraction field of $O_1$ is $K$, and the residue field of $O_i$ is the fraction field of $O_{i+1}$.

Fix a perfect field $k$. A topological local field of dimension $n$ over $k$ is a $k$-algebra $K$ with structures of semi-topological ring and $n$-dimensional local field, satisfying the following parameterization condition: there exists an isomorphism of $k$-algebras $K \cong F((s_1, \ldots, s_n))$ for some field $F$, finitely generated over $k$, which respects the two structures. Here $F((s_n)) \cdots ((s_1))$ is the field of iterated Laurent series, with its inherent topology and valuation rings ($F$ is discrete). One should remark that for $n = 1$ we are in the classical situation, whereas for $n \geq 2$, $F((s_1, \ldots, s_n))$ is not a topological ring.

TLFs make up a category $\text{TLF}(k)$, where a morphism $K \to L$ is a continuous $k$-algebra homomorphism which preserves the valuations, and the induced homomorphism of the last residue fields is finite. Write $\Omega_{K/k}^{\text{sep}}$ for the separated algebra of differentials; with the parameterization above $\Omega_{K/k}^{\text{sep}} \cong K \otimes_F \Omega_{F[[s]]}/k$. Then there is a functorial residue map $\text{Res}_{L/K} : \Omega_{L/k}^{\text{sep}} \to \Omega_{K/k}^{\text{sep}}$ which is $\Omega_{K/k}^{\text{sep}}$-linear and lowers degree by $\dim L/K$. For instance if $L = K((t))$ then

$$\text{Res}_{L/K} \left( \sum_i t^i dt \wedge \alpha_i \right) = \alpha_{-1} \in \Omega_{K/k}^{\text{sep}}.$$  

(1.1)

TLFs and residues were initially developed by Parshin and Lomadze, and the theory was enhanced in [Ye1].

The notion of a Beilinson completion algebra was introduced in [Ye2]. A BCA is a semi-local, semi-topological $k$-algebra, each of whose residue fields $A/m$ is a topological local field. Again there is a parameterization condition: when $A$ is local, there should exist a surjection

$$F((\underline{s}))[[\underline{t}]] = F((\underline{s}))[[t_1, \ldots, t_m]] \to A$$

which is strict topologically (i.e. $A$ has the quotient topology) and respects the valuations. Here $F((\underline{s}))$ is as above and $F((\underline{s}))[[\underline{t}]]$ is the ring of formal power series over $F((\underline{s}))$. The notion of BCA is an abstraction of the algebra gotten by Beilinson’s completion, cf. Lemma 1.5.

There are two distinguished kinds of homomorphisms between BCAs. The first kind is a morphism of BCAs $f : A \to B$ (see [Ye2] Definition 2.5), and the category they constitute is denoted $\text{BCA}(k)$. A morphism is
continuous, respects the valuations on the residue fields, but in general is not a local homomorphism. For instance, the homomorphisms \( k \to k[[s, t]] \to k((s))[[[t]]] \to k((s))(((t))) \) are all morphisms. TLF\((k)\) is a full subcategory of BCA\((k)\), consisting of those BCAs which are fields.

The second kind of homomorphism is an \textit{intensification homomorphism} \( u : A \to \widehat{A} \) (see [Ye2] Definition 3.6). An intensification is flat, topologically étale (relative to \( k \)) and unramified \( (\text{in the appropriate sense}) \). It can be viewed as a sort of localization or completion. Here examples are \( k((s))[[[t]]] \to k((s))(((t))) \) and \( k(s, t) \to k((s))(((t))) \to k((s))((((t))) \).

Suppose \( f : A \to B \) is a morphism of BCAs and \( u : A \to \widehat{A} \) is an intensification. The Intensification Base Change Theorem ([Ye2] Theorem 3.8) says there is a BCA \( \widehat{B} = B \otimes_A^\wedge \widehat{A} \), a morphism \( \hat{f} : \widehat{A} \to \widehat{B} \) and an intensification \( v : B \to \widehat{B} \), with \( vf = \hat{f}u \). These are determined up to isomorphism and satisfy certain universal properties. For instance, \( k((s))[[[t]]] = k((s))([[t]]) \).

According to [Ye2] Theorem 6.14, every \( A \in \text{BCA}(k) \) has a dual module \( \mathcal{K}(A) \). The module \( \mathcal{K}(A) \) is a ST \( A \)-module. Algebraically it is an injective hull of \( A/\mathfrak{r} \), where \( \mathfrak{r} \) is the Jacobson radical. \( \mathcal{K}(A) \) is also a right \( \mathcal{D}(A) \)-module, where \( \mathcal{D}(A) \) denotes the ring of continuous differential operators of \( A \) (relative to \( k \)). For a ST \( A \)-module \( M \) let \( \text{Dual}_A M := \text{Hom}_A^{\text{cont}}(M, \mathcal{K}(A)) \).

The dual modules have variance properties w.r.t. morphisms and intensifications. Given a morphism of BCAs \( f : A \to B \), according to [Ye2] Theorem 7.4 there is an \( A \)-linear map \( \text{Tr}_f : \mathcal{K}(B) \to \mathcal{K}(A) \). This induces an isomorphism \( \mathcal{K}(B) \cong \text{Hom}_A^{\text{cont}}(B, \mathcal{K}(A)) \). Given an intensification homomorphism \( u : A \to \widehat{A} \), according to [Ye2] Proposition 7.2 there is an \( A \)-linear map \( q_u : \mathcal{K}(A) \to \mathcal{K}(\widehat{A}) \). It induces an isomorphism \( \mathcal{K}(\widehat{A}) \cong \widehat{A} \otimes_A \mathcal{K}(A) \). Furthermore \( \text{Tr} \) and \( q \) commute across intensification base change: \( \text{Tr}_{\widehat{B}/A} \circ q_{\widehat{B}/B} = q_{\widehat{A}/A} \circ \text{Tr}_{B/A} \).

In case of a TLF \( K \), one has \( \mathcal{K}(K) = \omega(K) = \Omega^{1, \text{sep}}_{K/k} \), where \( p = \text{rank} \Omega^{1, \text{sep}}_{K/k} \). For a morphism of TLFs \( f : K \to L \) one has \( \text{Tr}_f = \text{Res}_f \), whereas for an intensification \( u : K \to \widehat{K} \) the homomorphism \( q_u : \Omega^{1, \text{sep}}_{K/k} \to \Omega^{1, \text{sep}}_{\widehat{K}/k} \) is the canonical inclusion for a topologically étale extension of fields.

\textbf{Example 1.2.} Take \( L := k(s, t), \widehat{L} := k((s))(((t))), A := k((s))[[[t]]], \widehat{A} := k((s))(((t))), K := k((s))) \), and \( \widehat{K} := k((s)) \). The inclusions \( \widehat{L} \to \widehat{L}, K \to \widehat{K} \) and \( A \to \widehat{A} \) are intensifications, whereas \( K \to A \to \widehat{L} \) and \( K \to \widehat{A} \) are morphisms. Using the isomorphism \( \mathcal{K}(A) \cong \text{Hom}_K^{\text{cont}}(A, \Omega^{1, \text{sep}}_{K/k}) \) induced by \( \text{Tr}_{A/K} \), we see that for \( \alpha \in \Omega^{2, \text{sep}}_{L/k} \) the element \( \text{Tr}_{\widehat{L}/A}(\alpha) \in \mathcal{K}(A) \) is represented by the functional \( a \mapsto \text{Res}_{L/k}(a\alpha), a \in A \). Also for \( \phi \in \mathcal{K}(A) \) the element \( \widehat{\phi} = q_{\widehat{A}/A}(\phi) \in \mathcal{K}(\widehat{A}) \) is represented by the unique \( K \)-linear functional \( \widehat{\phi} : \widehat{A} \to \Omega^{1, \text{sep}}_{\widehat{K}/k} \) extending \( \phi \).
Remark 1.3. The proof of existence of dual modules with their variance properties in [Ye2] is straightforward, using Taylor series expansions, differential operators and the residue pairing.

Let \( A \) be a noetherian ring and \( p \) a prime ideal. Consider the exact functor on \( A \)-modules \( M \mapsto M(p) := \hat{A}_p \otimes_A M \). For \( M \) finitely generated we have \( M(p) \cong \lim_{\rightarrow i} M/p^i M_p \), and if \( M = \lim_{\rightarrow \alpha} M_\alpha \), then \( M(p) \cong \lim_{\rightarrow \alpha} (M_\alpha)(p) \). This was generalized by Beilinson (cf. [Be]) as follows.

Definition 1.4. Let \( M \) be an \( A \)-module and let \( \xi = (p_0, \ldots, p_n) \) be a chain of prime ideals, namely \( p_i \subset p_{i+1} \). Define the Beilinson completion \( M_\xi \) by recursion on \( n, n \geq -1 \).

1. If \( n = -1 \) (i.e. \( \xi = \emptyset \)), let \( M_\xi := M \) with the discrete topology.
2. If \( n \geq 0 \) and \( M \) is finitely generated, let
   \[
   M_{(p_0, \ldots, p_n)} := \lim_{\rightarrow i} (M_{p_0}/p_0^i M_{p_0})(p_1, \ldots, p_n).
   \]
3. For arbitrary \( M \), let \( \{ M_\alpha \} \) be the set of finitely generated submodules of \( M \), and let
   \[
   M_{(p_0, \ldots, p_n)} := \lim_{\alpha \rightarrow} (M_\alpha)(p_0, \ldots, p_n).
   \]

A chain \( \xi = (p_0, \ldots, p_n) \) is saturated if \( p_{i+1} \) has height 1 in \( A/p_i \).

Lemma 1.5. If \( \xi = (p, \ldots) \) is a saturated chain then \( A_\xi \) is a Beilinson completion algebra.

Proof. By [Ye1] Corollary 3.3.5, \( A_\xi \) is a complete semi-local noetherian ring with Jacobson radical \( p_\xi \), and \( A_\xi/p_\xi = K_\xi \), where \( K := A_\pi/p_\pi \). Choose a coefficient field \( \sigma : K \rightarrow \hat{A}_p = A(p) \). By [Ye1] Proposition 3.3.6, \( K_\xi \) is a finite product of TLFs, and by the proof of [Ye1] Theorem 3.3.8, \( \sigma \) extends to a lifting \( \sigma_\xi : K_\xi \rightarrow A_\xi \). Sending \( t_1, \ldots, t_m \) to generators of the ideal \( p \), we get a topologically strict surjection \( K_\xi[[t_1, \ldots, t_m]] \twoheadrightarrow A_\xi \).

2. Construction of the Residue Complex \( \mathcal{K}^X \)

Let \( k \) be a perfect field, and let \( X \) be a scheme of finite type over \( k \). By a chain of points in \( X \) we mean a sequence \( \xi = (x_0, \ldots, x_n) \) of points with \( x_{i+1} \in \overline{\{x_i\}} \).

Definition 2.1. For any quasi-coherent \( \mathcal{O}_X \)-module \( \mathcal{M} \), define the Beilinson completion \( \mathcal{M}_\xi \) by taking an affine open neighborhood \( U = \text{Spec} \, A \subset X \) of \( x_n \), and setting
   \[
   M_\xi := \Gamma(U, \mathcal{M})_\xi \text{ as in Definition 1.4.}
   \]

These completions first appeared as the local factors of Beilinson’s adeles in [Be], and were studied in detail in [Ye1].

According to Lemma 1.5, if \( \xi = (x_0, \ldots, x_n) \) is saturated, i.e. \( \overline{\{x_{i+1}\}} \subset \overline{\{x_i\}} \) has codimension 1, then \( \mathcal{O}_{X,\xi} \) is a BCA. We shall be interested in the covertex maps

\[
\partial^- : \mathcal{O}_{X,(x_0)} \rightarrow \mathcal{O}_{X,\xi} \\
\partial^+ : \mathcal{O}_{X,(x_n)} \rightarrow \mathcal{O}_{X,\xi}
\]
which arise naturally from the completion process (cf. [Ye1] §3.1).

**Lemma 2.2.** $\partial^+$ is flat, topologically étale relative to $k$, and a morphism in $\text{BCA}(k)$. $\partial^-$ is an intensification homomorphism.

**Proof.** By definition $\partial^- = \partial^1 \circ \cdots \circ \partial^n$ and $\partial^+ = \partial^0 \circ \cdots \circ \partial^0$, where $\partial^i : O_{X,\partial_i} \to O_{X,\xi}$ is the $i$-th coface operator. First let us prove that $\partial^0 : O_{X,\partial_0} \to O_{X,\xi}$ is a morphism of BCAs. This follows from [Ye1] Theorem 3.3.2 (d), since we may assume that $X$ is integral with generic point $x_0$. By part (b) of the same theorem, $\partial^n : O_{X,\partial_n} \to O_{X,\xi}$ is finitely ramified and radially unraveled (in the sense of [Ye2] Definition 3.1).

Now according to [Ye1] Corollary 3.2.8, $\partial^i : O_{X,\partial_i} \to O_{X,\xi}$ is topologically étale relative to $k$, for any $i$. We claim it is also flat. For $i = 0$, $O_{X,\partial_0} \to (O_{X,\partial_0})_{x_0} = (O_{X,x_0})_{\partial_0}$ is a localization, so it’s flat. The map from $(O_{X,x_0})_{\partial_0}$ to its $m_{x_0}$-adic completion $O_{X,\xi}$ is also flat (these rings are noetherian). For $i > 0$, by induction on the length of chains, $O_{X,\partial_0 \partial_i} \to O_{X,\partial_0}$ is flat, and hence so is $(O_{X,x_0})_{\partial_0 \partial_i} \to (O_{X,x_0})_{\partial_0}$. Now use [CA] Chapter III §5.4 Proposition 4 to conclude that $$O_{X,\partial_0} = \lim_{\leftarrow j} (O_{X,x_0}/m^j_{x_0})_{\partial_0} \to \lim_{\leftarrow j} (O_{X,x_0}/m^j_{x_0})_{\partial_0} = O_{X,\xi}$$

is flat. \qed

**Example 2.3.** Take $X = \mathbb{A}^2 := \text{Spec} k[s,t]$, $x := (0)$, $y := (t)$ and $z := (s,t)$. Then with $L := k(x) = O_{X,(x)}$, $\tilde{L} := k(x)(y) = O_{X,(x,y)}$, $A := O_{X,(y)}$, $\tilde{A} := O_{X,(y,z)}$, $K := k(y)$ and $\tilde{K} := k(y)(z)$ we are exactly in the situation of Example 1.2.

**Definition 2.4.** Given a point $x$ in $X$, let $K_X(x)$ be the skyscraper sheaf with support $\{x\}$ and group of sections $K(O_{X,(x)})$.

The sheaf $K_X(x)$ is a quasi-coherent $O_X$-module, and is an injective hull of $k(x)$ in the category $\text{Mod}(X)$ of $O_X$-modules.

**Definition 2.5.** Given a saturated chain $\xi = (x, \ldots, y)$ in $X$, define an $O_X$-linear homomorphism $\delta_\xi : K_X(x) \to K_X(y)$, called the coboundary map along $\xi$, by $$\delta_\xi : K(O_{X,(x)}) \xrightarrow{\partial_0} K(O_{X,\xi}) \xrightarrow{\text{Tr}_{\xi}} K(O_{X,(y)}).$$

Throughout sections 2 and 3 the following convention shall be used. Let $f : X \to Y$ be a morphism of schemes, and let $x \in X$, $y \in Y$ be points. We will write $x \, | \, y$ to indicate that $x$ is a closed point in the fiber $X_y := X \times_Y \text{Spec } k(y) \cong f^{-1}(y)$. Similarly for chains: we write $(x_0, \ldots, x_n) \, | \, (y_0, \ldots, y_n)$ if $x_i \, | \, y_i$ for all $i$.

**Lemma 2.6.** Suppose $x \, | \, y$. Then $f^* : O_{Y,(y)} \to O_{X,(x)}$ is a morphism of BCAs. If $f$ is quasi-finite then $f^*$ is finite, and if $f$ is étale then $f^*$ is an intensification.

**Proof.** Immediate from the definitions. \qed
Lemma 2.7. Suppose $f : X \to Y$ is a quasi-finite morphism. Let $\eta = (y_0, \ldots, y_n)$ be a saturated chain in $Y$ and let $x \in X$, $x \mid y_n$. Consider the (finite) set of chains in $X$:

$$\Xi := \{ \xi = (x_0, \ldots, x_n) \mid \xi \mid \eta \text{ and } x_n = x \}.$$ 

Then there is a canonical isomorphism of BCAs

$$\prod_{\xi \in \Xi} \mathcal{O}_{X, \xi} \cong \mathcal{O}_{Y, \eta} \otimes_{\mathcal{O}_{Y, (y_n)}} \mathcal{O}_{X, (x)}.$$ 

Proof. Set $\hat{X} := \text{Spec} \mathcal{O}_{X, (x)}$ and $\hat{Y} := \text{Spec} \mathcal{O}_{Y, (y_n)}$, so the induced morphism $\hat{f} : \hat{X} \to \hat{Y}$ is finite. Let us denote by $\xi, \hat{\xi}, \hat{\eta}$ variable saturated chains in $X, \hat{X}, \hat{Y}$ respectively. For any $\hat{\eta} \mid \eta$ one has

$$\prod_{\hat{\xi} \mid \hat{\eta}} \mathcal{O}_{\hat{X}, \hat{\xi}} \cong \mathcal{O}_{\hat{Y}, \hat{\eta}} \otimes_{\mathcal{O}_{Y, (y_n)}} \mathcal{O}_{X, (x)} \quad (2.8)$$

by [Ye1] Proposition 3.2.3; note that the completion is defined on any noetherian scheme. Now from ibid. Corollary 3.3.13 one has $\mathcal{O}_{X, \xi} \cong \prod_{\hat{\xi} \mid \xi} \mathcal{O}_{\hat{X}, \hat{\xi}}$, so taking the product over all $\xi \in \Xi$ and $\hat{\eta} \mid \eta$ the lemma is proved.

Definition 2.9. Given an étale morphism $g : X \to Y$ and a point $x \in X$, let $y := g(x)$, so $g^* : \mathcal{O}_{Y, (y)} \to \mathcal{O}_{X, (x)}$ is an intensification. Define

$$q_g : \mathcal{K}_Y(y) \to g_* \mathcal{K}_X(x)$$

to be the $\mathcal{O}_Y$-linear homomorphism corresponding to $q_{g^*} : \mathcal{K}(\mathcal{O}_{Y, (y)}) \to \mathcal{K}(\mathcal{O}_{X, (x)})$ of [Ye2] Proposition 7.2.

Proposition 2.10. Let $g : X \to Y$ be an étale morphism.

(a) Given a point $y \in Y$, the homomorphism $1 \otimes q_g : g^* \mathcal{K}_Y(y) \to \bigoplus_{x \mid y} \mathcal{K}_X(x)$ is an isomorphism of $\mathcal{O}_X$-modules.

(b) Let $\eta = (y_0, \ldots, y_n)$ be a saturated chain in $Y$. Then

$$(1 \otimes q_g) \circ g^*(\delta_\eta) = \left( \sum_{\xi \mid \eta} \delta_\xi \right) \circ (1 \otimes q_g)$$

as homomorphisms $g^* \mathcal{K}_Y(y_0) \to \bigoplus_{x_n \mid y_n} \mathcal{K}_X(x_n)$.

Proof. (a) Because $\mathcal{K}_Y(y)$ is an artinian $\mathcal{O}_{Y,y}$-module, and $g$ is quasi-finite, we find that

$$g^* \mathcal{K}_Y(y) = \bigoplus_{x \mid y} \mathcal{O}_{X, (x)} \otimes_{\mathcal{O}_{Y, (y)}} \mathcal{K}_Y(y).$$

Now use [Ye2] Proposition 7.2 (i).

(b) From Lemma 2.7 and from [Ye2] Theorem 3.8 we see that for every $\xi \mid \eta$, $f^* : \mathcal{O}_{Y, \eta} \to \mathcal{O}_{X, \xi}$ is both a finite morphism and an intensification. By the definition of the coboundary maps, it suffices to verify that the diagram
commutes. The left square commutes by [Ye2] Proposition 7.2 (iv), whereas the right square commutes by Lemma 2.7 and [Ye2] Theorem 7.4 (ii).

**Definition 2.11.** Let $f : X \to Y$ be a morphism between finite type $k$-schemes, let $x \in X$ be a point, and let $y := f(x)$. Define an $\mathcal{O}_Y$-linear homomorphism

$$\text{Tr}_f : f_* \mathcal{K}_X(x) \to \mathcal{K}_Y(y)$$

as follows:

(i) If $x$ is closed in its fiber $X_y$, then $f^* : \mathcal{O}_{Y,(y)} \to \mathcal{O}_{X,(x)}$ is a morphism in $\text{BCA}(k)$. Let $\text{Tr}_f$ be the homomorphism corresponding to $\text{Tr}_{f^*} : \mathcal{K}(\mathcal{O}_{X,(x)}) \to \mathcal{K}(\mathcal{O}_{Y,(y)})$ of [Ye2] Theorem 7.4.

(ii) If $x$ is not closed in its fiber, set $\text{Tr}_f := 0$.

**Proposition 2.12.** Let $f : X \to Y$ be a finite morphism.

(a) For any $y \in Y$ the homomorphism of $\mathcal{O}_Y$-modules

$$\bigoplus_{x | y} f_* \mathcal{K}_X(x) \to \mathcal{H}om_{\mathcal{O}_Y}(f_* \mathcal{O}_X, \mathcal{K}_Y(y))$$

induced by $\text{Tr}_f$ is an isomorphism.

(b) Let $\eta = (y_0, \ldots, y_n)$ be a saturated chain in $Y$. Then

$$\delta_\eta \text{Tr}_f = \text{Tr}_f \sum_{\xi | \eta} f_*(\delta_\xi) : \bigoplus_{x_0 | y_0} f_* \mathcal{K}_X(x_0) \to \mathcal{K}_Y(y_n).$$

The sums are over saturated chains $\xi = (x_0, \ldots, x_n)$ in $X$.

**Proof.** (a) By [Ye1] Proposition 3.2.3 we have $\prod_{x | y} \mathcal{O}_{X,(x)} \cong (f_* \mathcal{O}_X)(y)$. Now use [Ye2] Theorem 7.4 (iv).

(b) Use the same diagram which appears in the proof of Proposition 2.10, only reverse the vertical arrows and label them $\text{Tr}_f$. Then the commutativity follows from [Ye2] Theorem 7.4 (i), (ii).

In [Ye1] §4.3 the notion of a system of residue data on a reduced scheme was introduced.

**Proposition 2.13.** Suppose $X$ is a reduced scheme. Then $(\{\mathcal{K}_X(x)\}, \{\delta_\xi\}, \{\Psi_{\sigma^{-1}}\})$, where $x$ runs over the points of $X$, $\xi$ runs over the saturated chains in $X$, and $\sigma : k(x) \to \mathcal{O}_{X,(x)}$ runs over all possible coefficient fields, is a system of residue data on $X$. 
Lemma 4.3.3. Let \( \xi = (x, \ldots, y) \) be a saturated chain, and let \( \sigma : k(x) \to \mathcal{O}_{X,(x)} \) and \( \tau : k(y) \to \mathcal{O}_{X,(y)} \) be compatible coefficient fields. Denote also by \( \tau \) the composed morphism \( \partial^+ \sigma = k(y) \to \mathcal{O}_{X,\xi} \). Then we get a coefficient field \( \sigma_\xi : k(\xi) = k(x)\xi \to \mathcal{O}_{X,\xi} \) extending \( \sigma \), which is a \( k(y) \)-algebra map via \( \tau \). Consider the diagram:

\[
\begin{array}{cccccc}
\mathcal{K}(\mathcal{O}_{X,(x)}) & \overset{q}{\rightarrow} & \mathcal{K}(\mathcal{O}_{X,\xi}) & = & \mathcal{K}(\mathcal{O}_{X,\xi}) & \overset{\text{Tr}}{\rightarrow} \mathcal{K}(\mathcal{O}_{X,(y)}) \\
\psi_\sigma & & & \psi_\tau & & \psi_r \\
\text{Dual}_\sigma \mathcal{O}_{X,(x)} & \overset{q_\sigma}{\rightarrow} & \text{Dual}_\sigma \mathcal{O}_{X,\xi} & \overset{h}{\rightarrow} & \text{Dual}_\tau \mathcal{O}_{X,\xi} & \overset{\text{Tr}_r}{\rightarrow} \text{Dual}_\tau \mathcal{O}_{X,(y)}
\end{array}
\]

where for a \( k(\xi) \)-linear homomorphism \( \phi : \mathcal{O}_{X,\xi} \to \omega(k(\xi)) \), \( h(\phi) = \text{Res}_{k(\xi)/k(y)} \phi \) (cf. [Ye2] Theorem 6.14). The left square commutes by [Ye2] Proposition 7.2 (iii); the middle square commutes by ibid. Theorem 6.14 (i); and the right square commutes by ibid. Theorem 7.4 (i), (iii). But going along the bottom of the diagram we get \( \text{Tr}_r h\omega = \delta_{\xi,\sigma/\tau} \), as defined in [Ye1] Lemma 4.3.3.

Lemma 2.14. Let \( \xi = (x, \ldots, y) \) and \( \eta = (y, \ldots, z) \) be saturated chains in the scheme \( X \), and let \( \xi \cup \partial_0 \eta := (x, \ldots, y, \ldots, z) \) be the concatenated chain. Then there is a canonical isomorphism of BCAs

\[
\mathcal{O}_{X,\xi \cup \partial_0 \eta} \cong \mathcal{O}_{X,\xi} \otimes_{\mathcal{O}_{X,\eta}}^{(\wedge)} \mathcal{O}_{X,\eta}
\]

(intensification base change).

Proof. Choose a coefficient field \( \sigma : k(y) \to \mathcal{O}_{X,(y)} \). This induces a coefficient ring \( \sigma_\eta : k(\eta) \to \mathcal{O}_{X,\eta} \), and using [Ye2] Theorem 3.8 and [Ye1] Theorem 4.1.12 one gets

\[
\mathcal{O}_{X,\xi \cup \partial_0 \eta} \cong \mathcal{O}_{X,\xi} \otimes_{k(\eta)}^{(\wedge)} k(\eta) \cong \mathcal{O}_{X,\xi} \otimes_{\mathcal{O}_{X,(y)}}^{(\wedge)} \mathcal{O}_{X,\eta}.
\]

Lemma 2.15. 1. Let \( \xi = (x, \ldots, y) \) and \( \eta = (y, \ldots, z) \) be saturated chains and let \( \xi \cup \partial_0 \eta = (x, \ldots, y, \ldots, z) \). Then \( \delta_0 \delta_\xi = \delta_{\xi,\partial_0 \eta} \).

2. Given a point \( x \in X \) and an element \( \alpha \in K_X(x) \), for all but finitely many saturated chains \( \xi = (x, \ldots) \) in \( X \) one has \( \delta_\xi(\alpha) = 0 \).

3. (Residue Theorem) Let \( x, z \in X \) be points s.t. \( z \in \{x\} \) and \( \text{codim}(\{z\}, \{x\}) = 2 \). Then \( \sum_y \delta_{(x,y,z)} = 0 \).

Proof. Using Lemma 2.14 we see that part 1 is a consequence of the base change property of traces, cf. [Ye2] Theorem 7.4 (ii). Assertions 2 and 3 are local, by Proposition 2.10, so we may assume there is a closed immersion \( f : X \to A^n_k \). By Proposition 2.12, we can replace \( X \) with \( A^n_k \), and so assume that \( X \) is reduced. Now according to Proposition 2.13 and [Ye1] Lemma 4.3.19, both 2 and 3 hold.
Definition 2.16. For any integer $q$ define a quasi-coherent sheaf

$$K^q_X := \bigoplus_{\dim \{x\} = -q} K_X(x).$$

By Lemma 2.15 there is a $\mathcal{O}_X$-linear homomorphism

$$\delta := (-1)^{q+1} \sum_{(x,y)} \delta_{(x,y)} : K^q_X \to K^{q+1}_X,$$

satisfying $\delta^2 = 0$. The complex $(K^\cdot_X, \delta)$ is called the Grothendieck residue complex of $X$.

In Corollary 3.8 we will prove that $K^\cdot_X$ is canonically isomorphic (in the derived category $D(X)$) to $\pi^! k$, where $\pi : X \to \text{Spec } k$ is the structural morphism.

Remark 2.17. A heuristic for the negative grading of $K^\cdot_X$ and the sign $(-1)^{q+1}$ is that the residue complex is the "$k$-linear dual" of a hypothetical "complex of localizations" $\cdots \prod \mathcal{O}_{X,y} \to \prod \mathcal{O}_{X,x} \to \cdots$. Actually, there is a naturally defined complex which is built up from all localizations and completions: the Beilinson adeles $\mathbb{A}^\text{red}_X$ (cf. [Be] and [HY1]). $\mathbb{A}^\text{red}_X$ is a DGA, and $K^\cdot_X$ is naturally a right DG-module over it. See [Ye4], and cf. also Remark 6.22.

Definition 2.18. 1. Let $f : X \to Y$ be a morphism of schemes. Define a homomorphism of graded $\mathcal{O}_Y$-modules $\text{Tr}_f : f_* K^\cdot_X \to K^\cdot_Y$ by summing the local trace maps of Definition 2.11.

2. Let $g : U \to X$ be an étale morphism. Define $q_g : K^\cdot_X \to g_* K^\cdot_U$ by summing all local homomorphisms $q_g$ of Definition 2.9.

Theorem 2.19. Let $X$ be a $k$-scheme of finite type.

(a) $(K^\cdot_X, \delta)$ is a residual complex on $X$ (cf. [RD] Chapter VI §1).

(b) If $g : U \to X$ is an étale morphism, then $1 \otimes q_g : g^* K^\cdot_X \to K^\cdot_U$ is an isomorphism of complexes.

(c) If $f : X \to Y$ is a finite morphism, then $\text{Tr}_f : f_* K^\cdot_X \to K^\cdot_Y$ is a homomorphism of complexes, and the induced map

$$f_* K^\cdot_X \to \text{Hom}_{\mathcal{O}_Y}(f_* \mathcal{O}_X, K^\cdot_Y)$$

is an isomorphism of complexes.

(d) If $X$ is reduced, then $(K^\cdot_X, \delta)$ is canonically isomorphic to the complex constructed in [Ye1] §4.3. In particular, if $X$ is smooth irreducible of dimension $n$, there is a quasi-isomorphism $C_X : \Omega^n_{X/k} \to K^\cdot_X$.

Proof. Parts (b), (c), (d) are immediate consequences of Propositions 2.10, 2.12 and 2.13 here, and [Ye1] Theorem 4.5.2. (Note that the sign of $\delta$ in [Ye1] is different.) As for part (a), clearly $K^\cdot_X$ is a direct sum of injective hulls of all the residue fields in $X$, with multiplicities 1. It remains to prove that $K^\cdot_X$ has coherent cohomology. Since this is a local question, we can assume using part (b) that $X$ is a closed subscheme of $\mathbf{A}^n_k$. According to
parts (c) and (d) of this theorem and [Ye1] Corollary 4.5.6, $\mathcal{K}_X$ has coherent cohomology.

From part (b) of the theorem we get:

**Corollary 2.20.** The presheaf $U \mapsto \Gamma(U, \mathcal{K}_U)$ is a sheaf on $X_{et}$, the small étale site over $X$.

**Definition 2.21.** For an $\mathcal{O}_X$-module $\mathcal{M}$ define dual complex

$$\text{Dual}_X \mathcal{M} := \text{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{K}_X).$$

Observe that since $\mathcal{K}_X$ is complex of injectives the derived functor $\text{Dual}_X : D(X)^\circ \to D(X)$ is defined. Moreover, since $\mathcal{K}_X$ is dualizing, the adjunction morphism $1 \to \text{Dual}_X \text{Dual}_X$ is an isomorphism on $D^b(X)$. We shall sometimes write $\text{Dual} \mathcal{M}$ instead of $\text{Dual}_X \mathcal{M}$.

### 3. Duality for Proper Morphisms

In this section we prove that if $f : X \to Y$ is a proper morphism of $k$-schemes, then the trace map $\text{Tr}_f$ of Definition 2.11 is a homomorphism of complexes, and it induces a duality in the derived categories.

**Proposition 3.1.** Let $f : X \to Y$ be a proper morphism between finite type $k$-schemes, and let $\eta = (y_0, \ldots, y_n)$ be a saturated chain in $Y$. Then there exists a canonical isomorphism of BCAs

$$\prod_{\xi|\eta} \mathcal{O}_X,\xi \cong \prod_{x_0|y_0} \mathcal{O}_{X,(x_0)} \otimes_{\mathcal{O}_{Y,(y_0)}} \mathcal{O}_{Y,\eta},$$

where $\xi = (x_0, \ldots, x_n)$ denotes a variable chain in $X$ lying over $\eta$.

**Proof.** The proof is by induction on $n$. For $n = 0$ this is trivial. Assume $n = 1$. Let $Z := \{x_0\}_{\text{red}}$, so $O_{Z,x_1}$ is a 1-dimensional local ring inside $k(Z) = k(x_0)$. Considering the integral closure of $O_{Z,x_1}$ we see that $k((x_0,x_1)) = k(x_0,x_1) = k(x_0) \otimes O_{Z,(x_1)}$ is the product of the completions of $k(x_0)$ at all discrete valuations centered on $x_1 \in Z$ (cf. [Ye1] Theorem 3.3.2). So by the valuative criterion for properness we get

$$\prod_{(x_0,x_1)|(y_0,y_1)} k(x_0)(x_1) \cong \prod_{x_0|y_0} k(x_0) \otimes_{k(y_0)} k(y_0)(y_1).$$

For $i \geq 1$ the morphism of BCAs

$$\prod_{x_0|y_0} (\mathcal{O}_{X,x_0}/m_{x_0}^i) \otimes_{\mathcal{O}_{Y,(y_0)}} \mathcal{O}_{Y,(y_0,y_1)} \to \prod_{(x_0,x_1)|(y_0,y_1)} \mathcal{O}_{X,(x_0,x_1)/m_{x_0,x_1}^i}$$

is bijective, since both sides are flat over $\mathcal{O}_{X,x_0}/m_{x_0}^i$, and by equation (3.2) (cf. [Ye2] Proposition 3.5). Passing to the inverse limit in $i$ we get an isomorphism of BCAs

$$\prod_{x_0|y_0} \mathcal{O}_{X,(x_0)} \otimes_{\mathcal{O}_{Y,(y_0)}} \mathcal{O}_{Y,(y_0,y_1)} \cong \prod_{(x_0,x_1)|(y_0,y_1)} \mathcal{O}_{X,(x_0,x_1)}. $$
Now suppose \( n \geq 2 \). Then we get
\[
\prod_{x_0 | y_0} \mathcal{O}_{X,(x_0)} \otimes^{(\wedge)}_{\mathcal{O}_{Y,(y_0)}} \mathcal{O}_{Y,\eta}
\]
\[
\cong \prod_{(x_0,x_1),(y_0,y_1)} \mathcal{O}_{X,(x_0,x_1)} \otimes^{(\wedge)}_{\mathcal{O}_{Y,(y_0,y_1)}} \mathcal{O}_{Y,\eta}
\] (i)
\[
\cong \prod_{(x_0,x_1),(y_0,y_1)} \mathcal{O}_{X,(x_0,x_1)} \otimes^{(\wedge)}_{\mathcal{O}_{Y,\partial_0 \eta}} \mathcal{O}_{Y,\partial_0 \eta}
\] (ii)
\[
\cong \prod_{(x_0,x_1)} \mathcal{O}_{X,(x_0,x_1)} \otimes^{(\wedge)}_{\mathcal{O}_{X,(x_1)}} \mathcal{O}_{X,\partial_0 \xi}
\] (iii)
\[
\cong \mathcal{O}_{X,\xi}
\] (iv)
where associativity of intensification base change ([Ye2] Proposition 3.10) is used repeatedly; in (i) we use formula (3.3); in (ii) we use Lemma 2.14 applied to \( \mathcal{O}_{Y,\eta} \); in (iii) we use the induction hypothesis; and (iv) is another application of Lemma 2.14.

The next theorem is our version of [RD] Ch. VII Theorem 2.1:

**Theorem 3.4.** (Global Residue Theorem) Let \( f : X \to Y \) be a proper morphism between \( k \)-schemes of finite type. Then \( \text{Tr}_f : f_* \mathcal{K}_X \to \mathcal{K}_Y \) is a homomorphism of complexes.

**Proof.** Fix a point \( x_0 \in X \), and let \( y_0 := f(x_0) \). First assume that \( x_0 \) is closed in its fiber \( X_{y_0} = f^{-1}(y_0) \). Let \( y_1 \) be an immediate specialization of \( y_0 \). By Proposition 3.1 we have
\[
\prod_{x_1 | y_1} \mathcal{O}_{X,(x_0,x_1)} \cong \mathcal{O}_{X,(x_0)} \otimes^{(\wedge)}_{\mathcal{O}_{Y,(y_0)}} \mathcal{O}_{Y,(y_0,y_1)},
\]
so just as in Proposition 2.12 (b), we get
\[
\delta_{(y_0,y_1)} \text{Tr}_f = \sum_{x_1 | y_1} \text{Tr}_f \delta_{(x_0,x_1)} : f_* \mathcal{K}_X(x_0) \to \mathcal{K}_Y(y_1).
\]

Next assume \( x_0, y_0 \) are as above, but \( x_0 \) is not closed in the fiber \( X_{y_0} \). The only possibility to have an immediate specialization \( x_1 \) of \( x_0 \) which is closed in its fiber, is if \( x_1 \in X_{y_0} \) and \( Z := \{ x_0 \}_{\text{red}} \subset X_{y_0} \) is a curve. We have to show that
\[
\sum_{x_1 | y_0} \text{Tr}_f \delta_{(x_0,x_1)} = 0 : f_* \mathcal{K}_X(x_0) \to \mathcal{K}_Y(y_0).
\]

Since \( \mathcal{K}_Z(x_0) \subset \mathcal{K}_X(x_0) \) is an essential submodule over \( \mathcal{O}_{Y,y_0} \) it suffices to check (3.5) on \( \mathcal{K}_Z(x_0) \). Thus we may assume \( X = \{ x_0 \}_{\text{red}} \) and \( Y = \{ y_0 \}_{\text{red}} \).

After factoring \( X \to Y \) through a suitable finite radiciel morphism \( X \to \tilde{X} \), and using Proposition 2.12, we may further assume that \( K = k(Y) \to k(X) \) is separable. Now
\[
\text{Tr}_f \delta_{(x_0,x_1)} = \text{Res}_{k((x_0,x_1))/K} : \Omega^{n+1}_{k(X)/k} \to \Omega^n_{K/k},
\]
finite type \( k \) and let \( \text{Hom}^0 \) be coherent \( k \)-module. Assume \( 7.4 \) (i),(iv) we get

\[
\sum_{x_1 \in X} \text{Res}_{k((x_0,x_1))/K} = 0 : \Omega^1_{k(X)/k} \to K.
\]

Let \( K' \) be the maximal purely inseparable extension of \( K \) in an algebraic closure, and let \( X' := X \times_K K' \). So

\[
k((x_0,x_1)) \otimes_K K' \cong \prod_{(x'_0,x'_1)|(x_0,x_1)} k((x'_0,x'_1))
\]

where \( (x'_0,x'_1) \) are chains in \( X' \). According to [Ye1] Lemma 2.4.14 we may assume \( k = K = K' \). Since now \( K \) is perfect, we are in the position to use the well known Residue Theorem for curves (cf. [Ye1] Theorem 4.2.15). \( \square \)

**Corollary 3.6.** Let \( f : X \to Y \) be a morphism between \( k \)-schemes of finite type, and let \( Z \subset X \) be a closed subscheme which is proper over \( Y \). Then \( \text{Tr}_f : f_\ast \Gamma Z \mathcal{K}_X \to \mathcal{K}_Y \) is a homomorphism of complexes.

**Proof.** Let \( I \subset \mathcal{O}_X \) be the ideal sheaf of \( Z \), and define \( Z_n := \text{Spec} \mathcal{O}_X/I^{n+1}, n \geq 0 \). The trace maps \( \mathcal{K}_{Z_0} \to \cdots \to \mathcal{K}_{Z_n} \to \cdots \to \mathcal{K}_X \) of Proposition 2.12 induce a filtration by subcomplexes \( \Gamma Z \mathcal{K}_X = \bigcup_{n=0}^\infty \mathcal{K}_{Z_n} \). Now since each morphism \( Z_n \to Y \) is proper, \( \text{Tr}_f : f_\ast \mathcal{K}_{Z_n} \to \mathcal{K}_Y \) is a homomorphism of complexes. \( \square \)

**Theorem 3.7.** (Duality) Let \( f : X \to Y \) be a proper morphism between finite type \( k \)-schemes. Then for any complex \( \mathcal{M} \in D_+^b(X) \), the homomorphism

\[
\text{Hom}_{D^b(X)}(\mathcal{M}, \mathcal{K}_X) \to \text{Hom}_{D^b(Y)}(Rf_\ast \mathcal{M}, \mathcal{K}_Y)
\]

induced by \( \text{Tr}_f : f_\ast \mathcal{K}_X \to \mathcal{K}_Y \) is an isomorphism.

**Proof.** The proof uses a relative version of Sastry’s notion of “residue pairs” and “pointwise residue pairs”, cf. [Ye1] Appendix. Define a residue pair relative to \( f \) and \( \mathcal{K}_Y \), to be a pair \( (\mathcal{R}, t) \), with \( \mathcal{R} \) a residual complex on \( X \), and with \( t : f_\ast \mathcal{R} \to \mathcal{K}_Y \) a homomorphism of complexes, which represent the functor \( \mathcal{M} \mapsto \text{Hom}_{D^b(Y)}(Rf_\ast \mathcal{M}, \mathcal{K}_Y) \) on \( D_+^b(X) \). Such pairs exist; for instance, we may take \( \mathcal{R} \) to be the Cousin complex \( f^! \mathcal{K}_Y \) associated to the dualizing complex \( f^! \mathcal{K}_Y \) (cf. [RD] ch. VII §3, or ibid. Appendix no. 4).

A pointwise residue pair relative to \( f \) and \( \mathcal{K}_Y \), is by definition a pair \( (\mathcal{R}, t) \) as above, but satisfying the condition: for any closed point \( x \in X \), and any coherent \( \mathcal{O}_X \)-module \( \mathcal{M} \) supported on \( \{x\} \), the map \( \text{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{R}) \to \text{Hom}_{\mathcal{O}_Y}(f_\ast \mathcal{M}, \mathcal{K}_Y) \) induced by \( t \) is an isomorphism. By the definition of the trace map \( \text{Tr}_f \), the pair \( (\mathcal{K}_X, \text{Tr}_f) \) is a pointwise residue pair. In fact, \( k \to \mathcal{O}_{Y,(f(x))} \to \mathcal{O}_{X,(x)} \) are morphisms in \( \text{BCA}(k) \), and by [Ye2] Theorem 7.4 (i),(iv) we get

\[
\text{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{K}_X) \cong \text{Hom}_{\mathcal{O}_Y}(f_\ast \mathcal{M}, \mathcal{K}_Y) \cong \text{Hom}_k(\mathcal{M}_x, k).
\]
The proof of [Ye1] Appendix Theorem 2 goes through also in the relative situation: the morphism $\text{Tr}_f : f_*\mathcal{K}'_X \to \mathcal{K}'_Y$ in $\mathcal{D}(Y)$ corresponds to a morphism $\zeta : \mathcal{K}'_X \to \mathcal{R}'$ in $\mathcal{D}(X)$. But since both $\mathcal{K}'_X$ and $\mathcal{R}'$ are residual complexes, $\zeta$ is an actual, unique homomorphism of complexes (cf. [RD] Ch. IV Lemma 3.2). By testing on $\mathcal{O}_X$-modules $\mathcal{M}$ as above we see that $\zeta$ is indeed an isomorphism of complexes. So $(\mathcal{K}'_X, \text{Tr}_f)$ is a residue pair.

Let $\pi : X \to \text{Spec } k$ be the structural morphism. In [RD] §VII.3 we find the twisted inverse image functor $\pi^! : \mathcal{D}^+(k) \to \mathcal{D}^+(X)$.

**Corollary 3.8.** There is a canonical isomorphism $\zeta_X : \mathcal{K}'_X \simeq \pi^! k$ in $\mathcal{D}(X)$. It is compatible with proper and étale morphisms. If $\pi$ is proper then

$$\text{Tr}_\pi = \text{Tr}^\text{RD}_\pi R\pi_* (\zeta_X) : \pi_* \mathcal{K}'_X \to k$$

where $\text{Tr}^\text{RD}_\pi : R\pi_* \pi^! k \to k$ is the trace map of [RD] §VII.3.

**Proof.** The uniqueness of $\zeta_X$ follows from considering closed subschemes $i_Z : Z \hookrightarrow X$ finite over $k$. This is because any endomorphism $a$ of $\mathcal{K}'_X$ in $\mathcal{D}(X)$ is a global section of $\mathcal{O}_X$, and $a = 1$ iff $i_Z^* (a) = 1$ for all such $Z$. Existence is proved by covering $X$ with compactifiable (e.g. affine) open sets and using Theorem 3.7, cf. [Ye1] Appendix Theorem 3 and subsequent Exercise. In particular $\zeta_X$ is seen to be compatible with open immersions. Compatibility with proper morphisms follows from the transitivity of traces. As for an étale morphism $g : U \to X$, one has $g^* \mathcal{K}'_X \simeq \mathcal{K}'_U$ by Theorem 2.19 (b), and also $g^* \pi^! k = g^! \pi^! k = (\pi g)^! \pi^! k$. Testing the isomorphisms on subschemes $Z \subset U$ finite over $k$ shows that $g^*(\zeta_X) = \zeta_U$. \qed

### 4. Duals of Differential Operators

Let $X$ be a $k$-scheme of finite type, where $k$ is a perfect field of any characteristic. Suppose $\mathcal{M}, \mathcal{N}$ are $\mathcal{O}_X$-modules. By a differential operator (DO) $D : \mathcal{M} \to \mathcal{N}$ over $\mathcal{O}_X$, relative to $k$, we mean in the sense of [EGA] IV §16.8. Thus $D$ has order $\leq 0$ if $D$ is $\mathcal{O}_X$-linear, and $D$ has order $\leq d$ if for all $a \in \mathcal{O}_X$, the commutator $[D, a]$ has order $\leq d - 1$.

Recall that the dual of an $\mathcal{O}_X$-module $\mathcal{M}$ is $\text{Dual} \mathcal{M} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{K}'_X)$. In this section we prove the existence of the dual operator $\text{Dual}(D)$, in terms of BCAs and residues. This explicit description of $\text{Dual}(D)$ will be needed for the applications in Sections 5-7. For direct proofs of existence cf. Remarks 4.6 and 4.7.

**Theorem 4.1.** Let $\mathcal{M}, \mathcal{N}$ be two $\mathcal{O}_X$-modules, and let $D : \mathcal{M} \to \mathcal{N}$ be a differential operator of order $\leq d$. Then there is a homomorphism of graded sheaves

$$\text{Dual}(D) : \text{Dual}\mathcal{N} \to \text{Dual}\mathcal{M}$$

having the properties below:

(i) $\text{Dual}(D)$ is a DO of order $\leq d$.

(ii) $\text{Dual}(D)$ is a homomorphism of complexes.
(iii) **Functoriality:** if $E : \mathcal{N} \to \mathcal{L}$ is another DO, then $\text{Dual}(ED) = \text{Dual}(D) \text{Dual}(E)$.

(iv) If $d = 0$, i.e. $D$ is $\mathcal{O}_X$-linear, then $\text{Dual}(D)(\phi) = \phi \circ D$ for any $\phi \in \text{Hom}_{\mathcal{O}_X}(\mathcal{N}, \mathcal{K}_X)$.

(v) **Adjunction:** under the homomorphisms $\mathcal{M} \to \text{Dual} \text{Dual} \mathcal{M}$ and $\mathcal{N} \to \text{Dual} \text{Dual} \mathcal{N}$, one has $D \mapsto \text{Dual}(\text{Dual}(D))$.

**Proof.** By [RD] Theorem II.7.8, an $\mathcal{O}_X$-module $\mathcal{M}'$ is noetherian iff there is a surjection $\bigoplus_{i=1}^n \mathcal{O}_{U_i} \to \mathcal{M}'$, for some open sets $U_1, \ldots, U_n$. Here $\mathcal{O}_{U_i}$ is extended by 0 to a sheaf on $X$. One has $\mathcal{M} = \lim_{\alpha} \mathcal{M}_\alpha$, where $\{\mathcal{M}_\alpha\}$ is the set of noetherian submodules of $\mathcal{M}$. (We did not assume $\mathcal{M}, \mathcal{N}$ are quasi-coherent!) So

$$\text{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{K}_X) \cong \lim_{\leftarrow \alpha} \text{Hom}_{\mathcal{O}_X}(\mathcal{M}_\alpha, \mathcal{K}_X).$$

Since the sheaf $\mathcal{P}^d_{X/k}$ of principal parts is coherent, and $D : \mathcal{M}_\alpha \to \mathcal{N}$ induces

$$\bigoplus_{i=1}^n (\mathcal{P}^d_{X/k} \otimes \mathcal{O}_{U_i}) \to \mathcal{P}^d_{X/k} \otimes \mathcal{M}_\alpha \to \mathcal{N},$$

we conclude that the module $\mathcal{N}_\alpha := \mathcal{O}_X \cdot D(\mathcal{M}_\alpha) \subset \mathcal{N}$ is also noetherian. Therefore we may assume that both $\mathcal{M}, \mathcal{N}$ are noetherian.

We have

$$\text{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{K}_X) = \bigoplus_x \text{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{K}_X(x)),$$

and $\text{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{K}_X(x))$ is a constant sheaf with support $\{x\}$ and module

$$\text{Hom}_{\mathcal{O}_X, x}(\mathcal{M}_x, \mathcal{K}_X(x)) = \text{Hom}_A(M, \mathcal{K}(A)) = \text{Dual}_A M,$$

where $A := \hat{\mathcal{O}}_{X,x} = \mathcal{O}_{X,(x)}$ and $M := A \otimes \mathcal{M}_x$. Note that $M$ is a finitely generated $A$-module. $D : \mathcal{M}_x \to \mathcal{N}_x$ induces a continuous DO $D : M \to N = A \otimes \mathcal{N}_x$ (for the $\mathfrak{m}$-adic topology). According to [Ye2] Theorem 8.6 there is a continuous DO

$$\text{Dual}_A(D) : \text{Dual}_A N \to \text{Dual}_A M.$$

Properties (i), (iii), (iv) and (v) follows directly from [Ye2] Theorem 8.6 and Corollary 8.8. As for property (ii), consider any saturated chain $\xi = (x, \ldots, y)$. Since $\partial^- : \mathcal{O}_{X,(x)} \to \mathcal{O}_{X,\xi}$ is an intensification homomorphism, and since $\partial^+ : \mathcal{O}_{X,(y)} \to \mathcal{O}_{X,\xi}$ is a morphism in $\text{BCA}(k)$ which is also topologically étale, we see that property (ii) is a consequence of [Ye2] Thm. 8.6 and Cor. 8.12.

Let $\mathcal{D}_X := \text{Diff}_{\mathcal{O}_X/k}(\mathcal{O}_X, \mathcal{O}_X)$ be the sheaf of differential operators on $X$. By definition $\mathcal{O}_X$ is a left $\mathcal{D}_X$-module.

**Corollary 4.2.** If $\mathcal{M}$ is a left (resp. right) $\mathcal{D}_X$-module, then $\text{Dual} \mathcal{M}$ is a complex of right (resp. left) $\mathcal{D}_X$-modules. In particular this is true for $\mathcal{K}_X = \text{Dual} \mathcal{O}_X$. 
Corollary 4.3. Suppose $\mathcal{M}$ is a complex sheaves, where each $\mathcal{M}^q$ is an $\mathcal{O}_X$-module, and $d : \mathcal{M}^q \to \mathcal{M}^{q+1}$ is a DO. Then there is a dual complex Dual($\mathcal{M}$).

Specifically, $(\text{Dual } \mathcal{M}, D)$ is the simple complex associated to the double complex $(\text{Dual } \mathcal{M})^{p,q} := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}^{-p}, \mathcal{K}^q_X)$. The operator is $D = D' + D''$, where

$D' := (-1)^{p+q+1} \text{Dual}(d) : (\text{Dual } \mathcal{M})^{p,q} \to (\text{Dual } \mathcal{M})^{p+1,q}$,

$D'' := \delta : (\text{Dual } \mathcal{M})^{p,q} \to (\text{Dual } \mathcal{M})^{p,q+1}$.

It is well known that if char $k = 0$ and $X$ is smooth of dimension $n$, then $\omega_X = \Omega^n_{X/k}$ is a right $\mathcal{D}_X$-module.

Proposition 4.4. Suppose char $k = 0$ and $X$ is smooth of dimension $n$. Then $C_X : \Omega^n_{X/k} \to \mathcal{K}^{-n}_X$ (the inclusion) is $\mathcal{D}_X$-linear.

Proof. It suffice to prove that any $\partial \in T_X$ (the tangent sheaf), which we view as a DO $\partial : \mathcal{O}_X \to \mathcal{O}_X$, satisfies $\text{Dual}(\partial)(\alpha) = -L_\partial(\alpha)$, where $L_\partial$ is the Lie derivative, and $\alpha \in \Omega^l_{X/k}$. Localizing at the generic point of $X$ we get $\partial \in \mathcal{D}(k(X))$ and $\alpha \in \omega(k(X))$. Now use [Ye2] Definition 8.1 and Proposition 4.2.

Remark 4.5. Proposition 4.4 says that in the case char $k = 0$ and $X$ smooth, the $\mathcal{D}_X$-module structure on $\mathcal{K}^{-n}_X$ coincides with the standard one, which is obtained as follows. The quasi-isomorphism $C_X : \Omega^n_{X/k} \to \mathcal{K}^{-n}_X$ identifies $\mathcal{K}^{-n}_X$ with the Cousin complex of $\Omega^n_{X/k}$, which is computed in the category $\text{Ab}(X)$ (cf. [Ha] Section I.2). Since any $D \in \mathcal{D}_X$ acts $\mathbb{Z}$-linearly on $\Omega^n_{X/k}$, it also acts on $\mathcal{K}^{-n}_X$.

Remark 4.6. According to [Sai] there is a direct way to obtain Theorem 4.1 in characteristic 0. Say $X \subset Y$, with $Y$ smooth. Then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{K}^{-l}_X) \cong \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{M}, \mathcal{K}^{-l}_Y)$. Now by [Sai] §2.2.3 any DO $D : \mathcal{M} \to \mathcal{N}$ of order $\leq d$ can be viewed as

$D \in \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{D}^d_Y) \subset \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{D}_Y, \mathcal{N} \otimes_{\mathcal{O}_Y} \mathcal{D}_Y)$ (right $\mathcal{D}_Y$-modules). Since $\mathcal{K}^{-l}_Y$ is a $\mathcal{D}_Y$-module (cf. Remark 4.5), we get

$\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{M}, \mathcal{K}^{-l}_Y) \cong \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{D}_Y, \mathcal{K}^{-l}_Y)$

and so we obtain the dual operator Dual($D$). I thank the referee for pointing out this fact to me.

Remark 4.7. Suppose char $k = p > 0$. Then a $k$-linear map $D : \mathcal{M} \to \mathcal{N}$ is a DO over $\mathcal{O}_X$ iff it is $\mathcal{O}_X^{(p^n/k)}$-linear, for $n \gg 0$. Here $X^{(p^n/k)} \to X$ is the Frobenius morphism relative to $k$, cf. [Ye1] Theorem 1.4.9. Since Tr induces an isomorphism

$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{K}^{-l}_X) \cong \mathcal{H}om_{\mathcal{O}_X^{(p^n/k)}}(\mathcal{M}, \mathcal{K}^{-l}_X^{(p^n/k)})$

we obtain the dual operator Dual($D$).
Let us finish this section with an application to rings of differential operators. Given a finitely generated (commutative) $k$-algebra $A$, denote by $\mathcal{D}(A) := \text{Diff}_{A/k}(A, A)$ the ring of differential operators over $A$. Such rings are of interest for ring theorists (cf. [MR] and [HoSt]). It is well known that if $\text{char} \ k = 0$ and $A$ is smooth, then the opposite ring $\mathcal{D}(A)^\circ \cong \omega_A \otimes_A \mathcal{D}(A) \otimes_A \omega_A^{-1}$, where $\omega_A = \Omega^n_{A/k}$. The next theorem is a vast generalization of this fact.

Given complexes $M^\cdot, N^\cdot$ of $A$-modules let $\text{Diff}^\cdot_{A/k}(M^\cdot, N^\cdot)$ be the complex of $k$-modules which in degree $n$ is $\prod_p \text{Diff}^p_{A/k}(M^p, N^{p+n})$. Let $K^\cdot_A := \Gamma(X, K^\cdot_X)$ with $X := \text{Spec} A$. By Corollary 4.2, it is a complex of right $\mathcal{D}(A)$-modules.

**Theorem 4.8.** There is a natural isomorphism of filtered $k$-algebras

$$\mathcal{D}(A)^\circ \cong H^0 \text{Diff}^\cdot_{A/k}(K^\cdot_A, K^\cdot_A).$$

**Proof.** First observe that since DOs preserve support, $\text{Diff}^p_{A/k}(K^p_A, K^{p+n}_A) = 0$ for all $p$. This means that every local section $D \in H^0 \text{Diff}^\cdot_{A/k}(K^\cdot_A, K^\cdot_A)$ is a well defined DO $D : K^\cdot_A \to K^\cdot_A$ which commutes with the coboundary $\delta$. Applying Dual and taking 0-th cohomology we obtain a DO $D^\vee = H^0 \text{Dual}(D) : H^0 \text{Dual}K^\cdot_A \to H^0 \text{Dual}K^\cdot_A$.

But $H^0 \text{Dual}K^\cdot_A = A$, so $D^\vee \in \mathcal{D}(A)$. Finally according to Theorem 4.1 (v), $D = D^\vee \vee$ for $D \in H^0 \text{Diff}^\cdot_{A/k}(K^\cdot_A, K^\cdot_A)$ or $D \in \mathcal{D}(A)$. \qed

Recall that an $n$-dimensional integral domain $A$ is a Gorenstein algebra iff $\omega_A = H^{-n}K^\cdot_A \to K^\cdot_A[-n]$ is a quasi-isomorphism, and $\omega_A$ is invertible.

**Corollary 4.9.** If $A$ is a Gorenstein $k$-algebra, there is a canonical isomorphism of filtered $k$-algebras

$$\mathcal{D}(A)^\circ \cong \text{Diff}^\cdot_{A/k}(\omega_A, \omega_A) \cong \omega_A \otimes_A \mathcal{D}(A) \otimes_A \omega_A^{-1}.$$
Note that the double complex $\mathcal{F}_{X}$ is concentrated in the third quadrant of the $(p, q)$-plane.

**Proposition 5.2.** $\mathcal{F}_{X}$ is a right DG module over $\Omega_{X/k}^{p,q}$.

**Proof.** The graded module structure is clear. It remains to check that

$$D(\phi \cdot \alpha) = (D\phi) \cdot \alpha + (-1)^{p+q}\phi \cdot (d\alpha)$$

for $\phi \in \mathcal{F}_{X}^{p,q}$ and $\alpha \in \Omega_{X/k}^{p,q}$. But this is a straightforward computation using Theorem 4.1. \hfill \Box

**Proposition 5.3.** Let $g : U \to X$ be étale. Then there is a homomorphism of complexes $q_{g} : \mathcal{F}_{X} \to g_{*}\mathcal{F}_{U}$, which induces an isomorphism of graded sheaves $1 \otimes q_{g} : g_{*}\mathcal{F}_{X} \cong \mathcal{F}_{U}$.

**Proof.** Consider the isomorphisms $g^{*}\Omega_{X/k}^{p,q} \cong \Omega_{U/k}^{p,q}$ and $1 \otimes q_{g} : g^{*}\mathcal{K}_{X} \to \mathcal{K}_{U}$ of Theorem 2.19. Clearly $1 \otimes q_{g} : g^{*}\mathcal{F}_{X}^{p,q} \to \mathcal{F}_{U}^{p,q}$ is an isomorphism. In light of [Ye2] Theorem 8.6 (iv), $q_{g} : \mathcal{F}_{X} \to g_{*}\mathcal{F}_{U}$ is a homomorphism of complexes. \hfill \Box

Let $f : X \to Y$ be a morphism of schemes. Define a homomorphism of graded sheaves $\text{Tr}_{f} : f_{*}\mathcal{F}_{X} \to \mathcal{F}_{Y}$ by composing $f^{*} : \Omega_{Y/k}^{p,q} \to f_{*}\Omega_{X/k}^{p,q}$ with $\text{Tr}_{f} : f_{*}\mathcal{K}_{X} \to \mathcal{K}_{Y}$ of Definition 2.11.

**Proposition 5.4.** $\text{Tr}_{f}$ commutes with $D'$. If $f$ is proper then $\text{Tr}_{f}$ also commutes with $D''$.

**Proof.** Let $y \in Y$ and let $x$ be a closed point in $f^{-1}(y)$. Then $f^{*} : \mathcal{O}_{Y(y)} \to \mathcal{O}_{X(x)}$ is a morphism in $\text{BCA}(k)$. Applying [Ye2] Cor. 8.12 to the DOs

$$df^{*} = f^{*}d : \Omega_{Y/k}^{p}(y) \to \Omega_{X/k}^{p+1}(x)$$

we get a dual homomorphism

$$\text{Dual}_{f^{*}}(df^{*}) = \text{Dual}_{f^{*}}(f^{*}d) : \text{Dual}_{\mathcal{O}_{X(x)}}\Omega_{X/k(x)}^{p+1} \to \text{Dual}_{\mathcal{O}_{Y(y)}}\Omega_{Y/k(y)}^{p},$$

which equals both $\text{Tr}_{f}\text{Dual}_{X}(d)$ and $\text{Dual}_{Y}(d)\text{Tr}_{f}$. The commutation of $D''$ with $\text{Tr}_{f}$ in the proper case is immediate from Thm. 3.4. \hfill \Box

Of course if $f : X \to Y$ is a closed immersion, then $\text{Tr}_{f}$ is injective, and it identifies $\mathcal{F}_{X}$ with the subsheaf $\mathcal{Hom}_{\Omega_{Y/k}^{p,q}}(\Omega_{X/k}^{p}, \mathcal{F}_{Y})$ of $\mathcal{F}_{Y}$. Just as in Corollary 3.6 we get:

**Corollary 5.5.** Let $f : X \to Y$ be a morphism of schemes, and let $Z \subset X$ be a closed subscheme which is proper over $Y$. Then the trace map $\text{Tr}_{f} : f_{*}\Gamma_{Z}\mathcal{F}_{X} \to \mathcal{F}_{Y}$ is a homomorphism of complexes.
Suppose $X$ is an integral scheme of dimension $n$. The canonical homomorphism
\begin{equation}
C_X : \Omega^n_X/k \to \mathcal{K}^{-n}_X = k(X) \otimes_{O_X} \Omega^n_X/k
\end{equation}
can be viewed as a global section of $\mathcal{F}^{-n,-n}_X$.

**Lemma 5.7.** Suppose $X$ is an integral scheme. Then $D'(C_X) = D''(C_X) = 0$.

**Proof.** By [Ye1] Section 4.5, $D''(C_X) = \pm \delta(C_X) = 0$. Next, let $K := k(X)$. Choose $t_1, \ldots, t_n \in K$ such that $\Omega^1_{K/k} = \bigoplus K \cdot dt_i$. Taking products of the $d t_i$ as bases of $\Omega^{n-1}_K$ and $\Omega^n_K$, we see from [Ye2] Theorem 8.6 and Definition 8.1 that Dual$_K(C_X) = 0$.

**Proposition 5.8.** If $X$ is smooth irreducible of dimension $n$, then the DG homomorphism $\Omega^\cdot_X/k \to \mathcal{F}^\cdot[-2n], \alpha \mapsto C_X \cdot \alpha$, is a quasi-isomorphism.

**Proof.** First note that $D(C_X) = 0$, so this is indeed a DG homomorphism. Filtering these complexes according to the $p$-degree we reduce to looking at $\Omega^p_{X/k}[n] \to \mathcal{F}^p_{X}$. That is a quasi-isomorphism by Theorem 2.19 part d.

**Corollary 5.9.** The complex $\mathcal{F}^\cdot_X$ is the same as the complex $\mathcal{K}^\cdot_X$ of [EZ], up to signs and indexing.

**Proof.** If $X$ is smooth of dimension $n$ this is because $\mathcal{F}^\cdot_X \cong \bigoplus \Omega^p_{X/k}[n] \otimes \mathcal{K}^\cdot_X$ is the Cousin complex of $\bigoplus \Omega^p_{X/k}[p]$, and $D'$ is (up to sign) the Cousin functor applied to $d$. If $X$ is a general scheme embedded in a smooth scheme $Y$, use Proposition 5.4.

**Definition 5.10.** Given a scheme $X$, let $X_1, \ldots, X_r$ be its irreducible components, with their induced reduced subscheme structures. For each $i$ let $x_i$ be the generic point of $X_i$, and let $f_i : X_i \to X$ be the inclusion morphism. We define the fundamental class $C_X$ by:
\begin{equation}
C_X := \sum_{i=1}^r \text{length}(O_{X,x_i}) \text{Tr}_{f_i}(C_{X_i}) \in \mathcal{F}^\cdot_X.
\end{equation}

The next proposition is easily verified using Propositions 5.4 and 5.3. It should be compared to [EZ] Theorem III.3.1.

**Proposition 5.11.** For any scheme $X$, the fundamental class $C_X \in \Gamma(X, \mathcal{F}^\cdot_X)$ is annihilated by $D'$ and $D''$. If $X$ has pure dimension $n$, then $C_X$ has bidegree $(-n, -n)$. If $f : X \to Y$ is a proper, surjective, generically finite morphism between integral schemes, then $\text{Tr}_f(C_X) = \deg(f) C_Y$. If $g : U \to X$ is étale, then $C_U = q_g(C_X)$.

**Remark 5.12.** In [Ye4] it is shown that $\mathcal{F}^\cdot_X$ is a right DG module over the DGA of Beilinson adeles $\mathcal{A}^\cdot_X = \Delta^\cdot_{\text{red}}(\Omega^\cdot_{X/k})$. Now let $\mathcal{E}$ be a locally
free $\mathcal{O}_X$-module of rank $r$, and let $Z \subset X$ be the zero locus of a regular section of $\mathcal{E}$. According to the adelic Chern-Weil theory of [HY2] there is an adelic connection $\nabla$ on $\mathcal{E}$ such that the Chern form $c_r(\mathcal{E}; \nabla) \in \mathcal{A}^r_X$ satisfies $C_Z = \pm C_X \cdot c_r(\mathcal{E}; \nabla) \in \mathcal{F}_X$.

6. De Rham Homology and the Niveau Spectral Sequence

Let $X$ be a finite type scheme over a field $k$ of characteristic 0. In [Ye3] it is shown that if $X \subset \mathfrak{X}$ is a smooth formal embedding (see below) then the De Rham complex $\hat{\Omega}^\cdot_{\mathfrak{X}/k}$ calculates the De Rham cohomology $H^{\cdot}_{\text{DR}}(X)$. In this section we will show that the De Rham-residue complex $\mathcal{F}^\cdot_X$ of $X$ calculates the De Rham homology $H^{\cdot}_{\text{DR}}(X)$. This is done by computing the niveau spectral sequence converging to $H^{\cdot}(X, \mathcal{F}^\cdot_X)$ (Theorem 6.16). We will draw a few conclusions, including the contravariance of homology w.r.t. étale morphisms (Theorem 6.23). As a reference for algebraic De Rham (co)homology we suggest [Ha].

Given a noetherian adic formal scheme $\mathfrak{X}$ and a defining ideal $I \subset \mathcal{O}_{\mathfrak{X}}$, let $X_n$ be the (usual) noetherian scheme $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/I^{n+1})$. Suppose $f : \mathfrak{X} \to \mathfrak{Y}$ is a morphism between such formal schemes, and let $I \subset \mathcal{O}_X$ and $J \subset \mathcal{O}_Y$ be defining ideals such that $f^{-1}J \cdot \mathcal{O}_X \subset I$. Such ideals are always available. We get a morphism of (usual) schemes $f_0 : X_0 \to Y_0$.

**Definition 6.1.** A morphism $f : \mathfrak{X} \to \mathfrak{Y}$ between (noetherian) adic formal schemes is called formally finite type (resp. formally finite or formally proper) if the morphism $f_0 : X_0 \to Y_0$ is finite type (resp. finite or proper).

Obviously these notions are independent of the particular defining ideals chosen.

**Example 6.2.** If $X \to Y$ is a finite type morphism of noetherian schemes, $X_0 \subset X$ is a locally closed subset and $\mathfrak{X} = X_{/X_0}$ is the formal completion, then $\mathfrak{X} \to Y$ is formally finite type. Such a morphism is called algebraizable.

**Definition 6.3.** A morphism of formal schemes $\mathfrak{X} \to \mathfrak{Y}$ is said to be formally smooth (resp. formally étale) if, given a (usual) affine scheme $Z$ and a closed subscheme $Z_0 \subset Z$ defined by a nilpotent ideal, the map $\text{Hom}_\mathfrak{Y}(Z, \mathfrak{X}) \to \text{Hom}_\mathfrak{Y}(Z_0, \mathfrak{X})$ is surjective (resp. bijective).

This is the definition of formal smoothness used in [EGA] IV Section 17.1. We shall also require the next notion.

**Definition 6.4.** A morphism $g : \mathfrak{X} \to \mathfrak{Y}$ between noetherian formal schemes is called étale if it is of finite type (see [EGA] I §10.13) and formally étale.

Note that if $\mathfrak{Y}$ is a usual scheme, then so is $\mathfrak{X}$, and $g$ is an étale morphism of schemes.

**Definition 6.5.** A smooth formal embedding of $X$ (over $k$) is a closed immersion of $X$ into a formal scheme $\mathfrak{X}$, which induces a homeomorphism on
the underlying topological spaces, and such that \(X\) is of formally finite type and formally smooth over \(k\).

**Example 6.6.** If \(X\) is smooth over \(Y = \text{Spec} \, k\) and \(X_0, \mathfrak{X}\) are as in Example 6.2, then \(X_0 \subset \mathfrak{X}\) is a smooth formal embedding.

Let \(\xi = (x_0, \ldots, x_d)\) be a saturated chain of points in the formal scheme \(\mathfrak{X}\). Choose a defining ideal \(\mathcal{I}\), and let \(X_n\) be as above. Define the Beilinson completion \(\mathcal{O}_{X,\xi} := \lim_{\leftarrow n} \mathcal{O}_{X_n,\xi}\) (which of course is independent of \(\mathcal{I}\)).

**Lemma 6.7.** Let \(\mathfrak{X}\) be formally finite type over \(k\), and let \(\xi\) be a saturated chain in \(\mathfrak{X}\). Then \(\mathcal{O}_{X,\xi}\) is a BCA over \(k\). If \(\mathfrak{X} = X/X_0\), then \(\mathcal{O}_{X,\xi} \cong \mathcal{O}_{X,\xi}\).

**Proof.** First assume \(\mathfrak{X} = X/X_0\). Taking \(\mathcal{I}\) to be the ideal of \(X_0\) in \(X\), we have

\[
\mathcal{O}_{X,\xi} = \lim_{\leftarrow n} (\mathcal{O}_X/\mathcal{I}^n)_{\xi} \cong \lim_{\leftarrow m,n} \mathcal{O}_{X,\xi}/(\mathcal{I}^n \mathcal{O}_{X,\xi} + \mathfrak{m}_\xi^m)
\]

\[
\cong \lim_{\leftarrow m} \mathcal{O}_{X,\xi}/\mathfrak{m}_\xi^m = \mathcal{O}_{X,\xi}.
\]

Now by [Ye3] Proposition 1.20 and Lemma 1.1, locally there is a closed immersion \(\mathfrak{X} \subset \mathfrak{Y}\), with \(\mathfrak{Y}\) algebraizable (i.e. \(\mathfrak{Y} = Y/Y_0\)). So there is a surjection \(\mathcal{O}_{\mathfrak{Y},\xi} \to \mathcal{O}_{\mathfrak{X},\xi}\), and this implies that \(\mathcal{O}_{X,\xi}\) is a BCA.

One can construct the complexes \(\mathcal{K}_\mathfrak{X}\) and \(\mathcal{F}_\mathfrak{X}\) for a formally finite type formal scheme \(\mathfrak{X}\), as follows. Define \(\mathcal{K}_\mathfrak{X}(x) := \mathcal{K}(\mathcal{O}_{\mathfrak{X}(x)})\). Now let \((x, y)\) be a saturated chain. Then there is an intensification homomorphism \(\delta^- : \mathcal{O}_{\mathfrak{X}(x)} \to \mathcal{O}_{\mathfrak{X}(x,y)}\) and a morphism of BCAs \(\delta^+ : \mathcal{O}_{\mathfrak{X}(y)} \to \mathcal{O}_{\mathfrak{X}(x,y)}\). Therefore we get a homomorphism of \(\mathcal{O}_{\mathfrak{X}}\)-modules \(\delta(x,y) : \mathcal{K}_\mathfrak{X}(x) \to \mathcal{K}_\mathfrak{X}(y)\). Define a graded sheaf \(\mathcal{K}_\mathfrak{X} = \bigoplus_{x \in \mathfrak{X}} \mathcal{K}_\mathfrak{X}(x)\) on \(\mathfrak{X}\), as in §1. Let \(\hat{\Omega}_{X/k}\) be the complete De Rham complex on \(\mathfrak{X}\), and set \(\mathcal{F}_{\mathfrak{X}} := \text{Hom}_{\mathcal{O}_{\mathfrak{X}}}(\hat{\Omega}_{X/k}^{-p}, \mathcal{K}_\mathfrak{X}^q)\).

**Proposition 6.8.** Let \(\mathfrak{X}\) be a formally finite type formal scheme over \(k\).

1. \(\mathcal{F}_\mathfrak{X}\) is a complex.
2. If \(\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}\) is étale, then there is a homomorphism of complexes \(\mathfrak{q}_\mathfrak{f} : \mathcal{F}_\mathfrak{X} \to \mathfrak{f}_! \mathcal{F}_\mathfrak{Y}\) which induces an isomorphism of graded sheaves \(\mathfrak{q}_\mathfrak{f} : \mathcal{F}_\mathfrak{X} \cong \mathcal{F}_\mathfrak{Y}\).
3. If \(\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}\) is formally proper, then there is a homomorphism of complexes \(\mathfrak{q}_\mathfrak{f} : \mathcal{F}_\mathfrak{X} \to \mathfrak{f}_! \mathcal{F}_\mathfrak{Y}\).

**Proof.** 1. Let \(X_n \subset \mathfrak{X}\) be as before. Then one has \(\mathcal{F}_\mathfrak{X} = \bigcup \mathcal{F}_{X_n}\), so this is a complex.
2. Take \(U_n := \mathfrak{U} \times \mathfrak{X} X_n\); then each \(U_n \to X_n\) is an étale morphism of schemes, and we can use Proposition 5.3.
3. Apply Proposition 5.4 to \(X_n \to Y_n\).

**Proposition 6.9.** Assume \(\mathfrak{X} = Y/X\) for some smooth irreducible scheme \(Y\) of dimension \(n\) and closed set \(X \subset Y\). Then there is a natural isomorphism...
of complexes
\[(6.10) \quad \mathcal{F}_X \cong \bigwedge_X \mathcal{F}_Y.\]
Hence \(\mathcal{F}_X \cong \text{R}\bigwedge_X \Omega_{Y/k}[2n] \) in the derived category \(\mathcal{D}(\text{Ab}(Y))\), and consequently
\[H^{-q}(X, \mathcal{F}_X) \cong H_X^{2n-q}(Y, \Omega_{Y/k}) = H_q^{\text{DR}}(X).\]

Proof. The isomorphism (6.10) is immediate from Lemma 6.7. But according to Proposition 5.8, \(\mathcal{F}_Y\) is a flasque resolution of \(\Omega_{Y/k}[2n] \) in \(\text{Ab}(Y)\), and consequently \(H^{-q}(X, \mathcal{F}_X) \cong H_X^{2n-q}(Y, \Omega_{Y/k}) = H_q^{\text{DR}}(X)\). \(\square\)

We need some algebraic results, phrased in the terminology of [Ye1] §1. Let \(K\) be a complete, separated semi-topological (ST) commutative \(k\)-algebra, and let \(t = (t_1, \ldots, t_n)\) be a sequence of indeterminates. Let \(K[[t]]\) and \(K(t)\) be the rings of formal power series, and of iterated Laurent series, respectively. These are complete, separated ST \(k\)-algebras. Let \(T\) be the free \(k\)-module with basis \(\alpha_1, \ldots, \alpha_n\) and let \(\bigwedge_k T\) be the exterior algebra over \(k\).

Lemma 6.11. ("Poincaré Lemma") The DGA homomorphisms
\[\Omega_{K/k}^\ast, \text{sep} \rightarrow \Omega_{K[[t]]/k}^\ast, \text{sep}\]
and
\[\Omega_{K/k}^\ast, \text{sep} \otimes_k \bigwedge_k T \rightarrow \Omega_{K(t)/k}^\ast, \text{sep}, \quad \alpha_i \mapsto \text{dlog } t_i\]
are quasi-isomorphisms.

Proof. Since
\[\Omega_{K[[t]]/k}^\ast, \text{sep} \cong K[[t]] \otimes_{k[[t]]} \Omega_{K[[t]]/k}^\ast, \text{sep}\]
the homotopy operator ("integration") of the Poincaré Lemma for the graded polynomial algebra \(k[[t]]\) works here also.

For \(K(t)\) (i.e. \(n = 1\)) we have
\[\Omega_{K(t)/k}^\ast, \text{sep} \cong \Omega_{K[t]/k}^\ast, \text{sep} \oplus \Omega_{K[t^{-1}]/k}^\ast, \text{sep} \wedge \text{dlog } t\]
so we have a quasi-isomorphism. For \(n > 1\) use induction on \(n\) and the Künneth formula. \(\square\)

Lemma 6.12. Suppose \(A\) is a local BCA and \(\sigma, \sigma' : K \rightarrow A\) are two coefficient fields. Then
\[H(\sigma) = H(\sigma') : H\Omega_{K/k}^\ast, \text{sep} \rightarrow H\Omega_{A/k}^\ast, \text{sep}.\]

Proof. Choosing generators for the maximal ideal of \(A\), \(\sigma\) induces a surjection of BCAs \(\tilde{A} = K[[t]] \rightarrow A\). Denote by \(\tilde{\sigma} : K \rightarrow \tilde{A}\) the inclusion. The coefficient field \(\sigma'\) lifts to some coefficient field \(\tilde{\sigma}' : K \rightarrow \tilde{A}\). It suffices to show that \(H(\tilde{\sigma}) = H(\tilde{\sigma}')\). But by Lemma 6.11 both of these are bijective, and using the projection \(\tilde{A} \rightarrow K\) we see they are in fact equal. \(\square\)
Given a saturated chain $\xi = (x, \ldots, y)$ in $X$ and a coefficient field $\sigma : k(y) \to \mathcal{O}_{X(y)}$, there is the Parshin residue map

$$\text{Res}_{\xi,\sigma} : \Omega_{k(x)/k} \to \Omega_{k(y)/k}$$

cf. [Ye1] Definition 4.1.3. It is a map of DG $k$-modules of degree equal to $-(\text{length of } \xi)$.

**Proposition 6.13.** Let $\xi = (x, \ldots, y)$ be a saturated chain in $X$. Then the map of graded $k$-modules

$$\text{Res}_{\xi} := H(\text{Res}_{L/K,\sigma}) : H\Omega_{L/k} \to H\Omega_{K/k}$$

is independent of the coefficient field $\sigma$.

**Proof.** Say $\xi$ has length $n$. Let $L$ be one of the local factors of $k(\xi) = k(x)_\xi$, so it is an $n$-dimensional topological local field (TLF). Let $K := \kappa_n(L)$, the last residue field of $L$, which is a finite separable $k(y)$-algebra. Then $\sigma$ extends uniquely to a morphism of TLFs $\sigma : K \to L$, and it is certainly enough to check that

$$H(\text{Res}_{L/K,\sigma}) : H\Omega_{L/k} \to H\Omega_{K/k}$$

is independent of $\sigma$.

After choosing a regular system of parameters $\xi = (t_1, \ldots, t_n)$ in $L$ we get $L \cong K((\xi))$. According to Lemma 6.11, $H(\sigma)$ induces an isomorphism of $k$-algebras

$$H\Omega_{K/k} \otimes_k \bigwedge_k^p T \cong H\Omega_{L/k}^{\text{sep}}.$$  

But by Lemma 6.12 this isomorphism is independent of $\sigma$. The map (6.14) is $H\Omega_{K/k}$-linear, and it sends $\bigwedge_k^p T$ to 0 if $p < n$, and $d\log t_1 \wedge \cdots \wedge d\log t_n \mapsto 1$. Hence (6.14) is independent of $\sigma$.  

The topological space $X$ has an increasing filtration by families of supports $\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots$, where

$$X_q := \{ Z \subset X \mid Z \text{ is closed and } \dim Z \le q \}.$$  

We write $x \in X_q/X_{q-1}$ if $\overline{\{x\}} \in X_q - X_{q-1}$, and the set $X_q/X_{q-1}$ is called the $q$-skeleton of $X$. (This notation is in accordance with [BLO]; in [Ye1] $X_q$ denotes the $q$-skeleton.) The niveau filtration on $\mathcal{F}_X$ is $N_q \mathcal{F}_X := \bigcap_{x \in X_q} \mathcal{F}_X$. Let us write $X^q/X^{q+1} := X_{-q}/X_{-q-1}$ and $N^q := N_{-q}$, so $\{N^q \mathcal{F}_X\}$ is a decreasing filtration.

**Theorem 6.16.** Suppose $\text{char } k = 0$ and $X \subset X$ is a smooth formal embedding. Then in the niveau spectral sequence converging to $H(X, \mathcal{F}_X)$, the $E_1$ term is (in the notation of [ML] Chapter XI):

$$E_1^{p,q} = H^{p+q}_{X^p/X^{p+1}}(X, \mathcal{F}_X) \cong \bigoplus_{x \in X^p/X^{p+1}} H^{q-p} \Omega_{k(x)/k}.$$  

and the coboundary operator is $(-1)^{p+1} \sum_{(x,y)} \text{Res}_{x,y}$. 


Proof. We shall substitute indices \((p, q) \mapsto (-q, -p)\); this puts us in the first quadrant. Because \(F_X\) is a complex of flasque sheaves, one has

\[
E_1^{p-q} = H^{p-q}_{X/X_{q+1}}(X, F_X) \cong \bigoplus_{x \in X_{q}/X_{q-1}} H^{-p}Hom_{\mathcal{O}_X} \left( \Omega_{\mathcal{X}/k}^1, K_{\mathcal{X}}(x) \right)
\]

(the operator \(\delta\) is trivial on the \(q\)-skeleton). Fix a point \(x\) of dimension \(q\) and let \(B := O_{\mathcal{X},(x)}\). Then \(\Omega_{\mathcal{X}/k}^1 \cong \Omega_{B/k}^{1, \text{sep}}\) and by definition

\[
Hom_{\mathcal{O}_X} \left( \Omega_{\mathcal{X}/k}^1, K_{\mathcal{X}}(x) \right) \cong \text{Dual}_{B} \Omega_{B/k}^{1, \text{sep}}.
\]

Choose a coefficient field \(\sigma : K = k(x) \rightarrow B\). By [Ye2] Theorem 8.6 there is an isomorphism of complexes

\[
\Psi_\sigma : \text{Dual}_{B} \Omega_{B/k}^{1, \text{sep}} \cong \text{Dual}_\sigma \Omega_{B/k}^{1, \text{sep}} = \text{Hom}_{B/k}^{\text{cont}}(\Omega_{B/k}^{1, \text{sep}}, \omega(K)).
\]

Here \(\omega(K) = \Omega_{K/k}^{l}\) and the operator on the right is \(\text{Dual}_\sigma(d)\) of [Ye2] Definition 8.1.

According to [Ye3] §3, \(k \rightarrow B\) is formally smooth; so \(B\) is a regular local ring, and hence \(B \cong K[[t]]\). Put a grading on \(\Omega_{K[[t]]/k}\) by declaring \(\deg t_i = \deg d_{t_j} = 1\), and let \(V_i \subset \Omega_{K[[t]]/k}\) be the homogeneous component of degree \(l\). In particular \(V_0 = \Omega_{K/k}^1\). Since \(d\) preserves each \(V_i\), from the definition of \(\text{Dual}_\sigma(d)\) we see that

\[
\text{Dual}_\sigma \Omega_{B/k}^{1, \text{sep}} = \bigoplus_{l=0}^{\infty} \text{Hom}_{K}(V_l, \omega(K))
\]

as complexes. Because the \(K\)-linear homotopy operator in the proof of Lemma 6.11 also preserves \(V_l\) we get \(H\text{Hom}_{K}(V_l, \omega(K)) = 0\) for \(l \neq 0\), and hence

\[
H^{-p}\text{Dual}_\sigma \Omega_{B/k}^{1, \text{sep}} \cong H^{-p}\text{Hom}_{K}(\Omega_{K/k}^1, \omega(K)) \cong H^{-p-1}\Omega_{K/k}^{l}
\]

(cf. proof of Lemma 5.7).

It remains to check that the coboundary maps match up. Given an immediate specialization \((x, y)\), choose a pair of compatible coefficient fields \(\sigma : k(x) \rightarrow O_{\mathcal{X},(x)} = B\) and \(\tau : k(y) \rightarrow O_{\mathcal{X},(y)} = A\) (cf. [Ye1] Definition 4.1.5). Set \(\hat{B} := O_{\mathcal{X}(x, y)}\), so \(f : A \rightarrow \hat{B}\) is a morphism of BCAs. A cohomology class \(\phi \in H^{-p}\text{Dual}_B \Omega_{B/k}^{1, \text{sep}}\) is sent under the isomorphism (6.17) to the class \([\beta]\) of some form \(\beta \in \Omega_{k(x)/k}^{l}\), such that \(d\beta = 0\) and on \(\sigma(\Omega_{k(x)/k}^{l}) \subset \Omega_{B/k}^{l, \text{sep}}\), \(\phi\) acts like left multiplication by \(\beta\). So for \(\alpha \in \Omega_{k(y)/k}^{l}\),

\[
\text{Tr}_{\hat{A}/k(y)} \text{Tr}_{\hat{B}/\hat{A}} \phi f \tau(\alpha) = \text{Res}_{k((x,y))/k(y);\tau} (\beta \wedge \tau(\alpha)) = \text{Res}_{k((x,y))/k(y);\tau} (\beta) \wedge \alpha.
\]

This says that under the isomorphism

\[
H^{-p}\text{Dual}_A \Omega_{A/k}^{l, \text{sep}} \cong H^{-p-1}\Omega_{k(y)/k}^{l}
\]
the class \(\delta(x,y)\) is sent to \(\text{Res}(x,y)\) \(\beta\).

\[\text{Proof.}\]

Remark 6.18. Theorem 6.16, but with \(\text{R}_{\mathcal{X}}^{\mathcal{Y}}\Omega_{\mathcal{Y}/k}\) instead of \(\mathcal{F}_{\mathcal{X}}\) (cf. Proposition 6.9), was discovered by Grothendieck (see [Gr] Footnotes 8,9), and proved by Bloch-Ogus [BIO]. Our proof is completely different from that in [BIO], and in particular we obtain the formula for the coboundary operator as a sum of Parshin residues. On the other hand the proof in [BIO] is valid for a general homology theory (including \(l\)-adic homology). Bloch-Ogus went on to prove additional important results, such as the degeneration of the sheafified spectral sequence \(E^p_{r,q}\) at \(r = 2\), for \(X\) smooth.

The next result is a generalization of [Ha] Theorem II.3.2. Suppose \(X \subset \mathcal{Y}\) is another smooth formal embedding. By a morphism of embeddings \(f: \mathcal{X} \to \mathcal{Y}\) we mean a morphism of formal schemes inducing the identity on \(X\). Since \(f\) is formally finite, according to Proposition 6.8, \(\text{Tr}_f: \mathcal{F}_{\mathcal{X}} \to \mathcal{F}_{\mathcal{Y}}\) is a map of complexes in \(\text{Ab}(X)\).

Corollary 6.19. Let \(f: \mathcal{X} \to \mathcal{Y}\) be a morphism of embeddings of \(X\). Then \(\text{Tr}_f: \Gamma(X, \mathcal{F}_{\mathcal{X}}) \to \Gamma(X, \mathcal{F}_{\mathcal{Y}})\) is a quasi-isomorphism. If \(g: \mathcal{X} \to \mathcal{Y}\) is another such morphism, then \(\text{H}(\text{Tr}_f) = \text{H}(\text{Tr}_g)\).

\[\text{Proof.}\] \(\text{Tr}_f\) induces a map of niveau spectral sequences \(E^p_{r,q}(\mathcal{X}) \to E^p_{r,q}(\mathcal{Y})\). The theorem and its proof imply that these spectral sequences coincide for \(r \geq 1\), hence \(\text{H}(\text{Tr}_f)\) is an isomorphism. The other statement is proved like in [Ye3] Theorem 2.7 (cf. next corollary).

Corollary 6.20. The \(k\)-module \(\text{H}^q(X, \mathcal{F}_{\mathcal{X}})\) is independent of the smooth formal embedding \(X \subset \mathcal{X}\).

\[\text{Proof.}\] As shown in [Ye3], given any two embeddings \(X \subset \mathcal{X}\) and \(X \subset \mathcal{Y}\), the completion of their product along the diagonal \((\mathcal{X} \times_k \mathcal{Y})_{/X}\) is also a smooth formal embedding of \(X\), and it projects to both \(\mathcal{X}\) and \(\mathcal{Y}\). Therefore by Corollary 6.19, \(\text{H}^q(X, \mathcal{F}_{\mathcal{X}})\) and \(\text{H}^q(X, \mathcal{F}_{\mathcal{Y}})\) are isomorphic. Using triple products we see this isomorphism is canonical.

Remark 6.21. We can use Corollary 6.20 to define \(\text{H}^{\text{DR}}(X)\) if some smooth formal embedding exists. For a definition of \(\text{H}^{\text{DR}}(X)\) in general, using a system of local embeddings, see [Ye3] (cf. [Ha] pp. 28-29).

Remark 6.22. In [Ye4] it is shown that \(\mathcal{F}_{\mathcal{X}}\) is naturally a DG module over the adele-De Rham complex \(\mathcal{A}_{\mathcal{X}} = \mathcal{A}_{\text{red}}(\Omega_{\mathcal{X}/k})\), and this multiplication induces the cap product of \(\text{H}^{\text{DR}}(X)\) on \(\text{H}^{\text{DR}}(X)\).

The next result is new (cf. [BIO] Example 2.2):

Theorem 6.23. De Rham homology \(\text{H}^{\text{DR}}(-)\) is contravariant w.r.t. \(\text{étale}\) morphisms.

\[\text{Proof.}\] The “topological invariance of \(\text{étale}\) morphisms” (see [Mi] Theorem I.3.23) implies that the smooth formal embedding \(X \subset \mathcal{X}\) induces an “embedding of \(\text{étale}\) sites” \(X_{\text{et}} \subset \mathcal{X}_{\text{et}}\). By this we mean that for every \(\text{étale}\)
morphism $U \to X$ there is some étale morphism $\mathcal{U} \to \mathcal{X}$, unique up to isomorphism, s.t. $U \cong \mathcal{U} \times \mathcal{X} X$ (see [Ye3]). Then $U \subset \mathcal{U}$ is a smooth formal embedding. From Proposition 6.8 we see there is a complex of sheaves $\mathcal{F}_X$ on $X_{\text{et}}$ with $\mathcal{F}_X|_U \cong g^* \mathcal{F}_X$ for every $g : \mathcal{U} \to \mathcal{X}$ étale (cf. [Mi] Corollary II.1.6). But by Corollary 6.20, $H^{\text{DR}}(U) = H^*(U, \mathcal{F}_X)$. □

Say $X$ is smooth irreducible of dimension $n$. Define the sheaf $H^{\text{DR}}(X)_{\mathcal{O}_Y}$ on $X_{\mathcal{O}_Y}$ to be the sheafification of the presheaf $U \mapsto \mathcal{F}_X$. For any point $x \in X$ let $i_x : \{x\} \to X$ be the inclusion. Let $x_0$ be the generic point, so $X_0/X_{n-1} = \{x_0\}$. According to [BLO] there is an exact sequence of sheaves

$$0 \to H^{\text{DR}} \to i_{x_0}^* \mathcal{F}_X \to \cdots \to \bigoplus_{x \in X_0/X_{n-1}} i_x^* \mathcal{F}_X \to \cdots$$

called the arithmetic resolution. Observe that this is a flasque resolution.

**Corollary 6.24.** The coboundary operator in the arithmetic resolution of $H^{\text{DR}}(X)$ is

$$(-1)^{q+1} \sum_{(x,y)} \text{Res}(x,y)$$

where $\text{Res}(x,y)$ is the Parshin residue of Proposition 6.13.

**Proof.** Take $\mathcal{X} = X$ in Theorem 6.16, and use [BLO] Theorem 4.2. □

### 7. The Intersection Cohomology $\mathcal{D}$-module of a Curve

Suppose $Y$ is an $n$-dimensional smooth algebraic variety over $\mathbb{C}$ and $X$ is a subvariety of codimension $d$. Let $\mathcal{H}_X^d \mathcal{O}_Y$ be the sheaf of $d$-th cohomology of $\mathcal{O}_Y$ with support in $X$. According to [BrK], the holonomic $\mathcal{D}_Y$-module $\mathcal{H}_X^d \mathcal{O}_Y$ has a unique simple coherent submodule $\mathcal{L}(X, Y)$, and the De Rham complex $\mathcal{L}(X, Y) = \mathcal{L}(X, Y) \otimes \Omega^d_{Y_{\text{an}}}[n]$ is the middle perversity intersection cohomology sheaf $\mathcal{L}_C$. Here $Y_{\text{an}}$ is the associated complex manifold. The module $\mathcal{L}(X, Y)$ was described explicitly using complex-analytic methods by Vilonen [Vi] and Barlet-Kashiwara [BaK]. These descriptions show that the fundamental class $C_{X/Y}$ lies in $\mathcal{L}(X, Y) \otimes \Omega^d_{Y/k}$, a fact proved earlier by Kashiwara using the Riemann-Hilbert correspondence and the decomposition theorem of Beilinson-Bernstein-Deligne-Gabber (see [Br]).

Now let $k$ be any field of characteristic 0, $Y$ an $n$-dimensional smooth variety over $k$, and $X \subset Y$ an integral curve with arbitrary singularities. In this section we give a description of $\mathcal{L}(X, Y) \subset \mathcal{H}_X^{n-1} \mathcal{O}_Y$ in terms of algebraic residues. As references on $\mathcal{D}$-modules we suggest [Bj] and [Bo] Chapter VI.

Denote by $w$ the generic point of $X$. Pick any coefficient field $\sigma : k(w) \to \mathcal{O}_{Y,w} = \mathcal{O}_{Y,w}$. As in [Hu] Section 1 there is a residue map

$$\text{Res}_{w,\sigma}^{\text{lc}} : H^{n-1}_{w} \Omega^d_{Y/k} \to \Omega^1_{k(w)/k}$$
Theorem 7.1. Let \((lc)\) be a closed point and let \(a \in \mathcal{O}_{X,(x)}\). Then for a generalized fraction, with \(\sigma \in \mathcal{O}_{k(X)/k}\), we have

\[
\text{Res}_{w_\sigma} = \begin{cases} \alpha & \text{if } (i_1, \ldots, i_{n-1}) = (1, \ldots, 1) \\ 0 & \text{otherwise.} \end{cases}
\]

Let \(\pi : \tilde{X} \to X\) be the normalization, and let \(\tilde{w}\) be the generic point of \(\tilde{X}\). For any closed point \(\tilde{x} \in \tilde{X}\) the residue field \(k(\tilde{x})\) is étale over \(k\), so it lifts into \(\mathcal{O}_{\tilde{X},(\tilde{x})}\). Hence we get canonical morphisms of BCAs \(k(\tilde{x}) \to \mathcal{O}_{\tilde{X},(\tilde{x})} \to k(\tilde{w})(\tilde{x})\), and a residue map

\[
\text{Res}(\tilde{w}, \tilde{x}) : \Omega^1_{k(w)/k} \to \Omega^1_{k(\tilde{w})(\tilde{x})/k} \to k(\tilde{w}).
\]

Define

\[
\text{Res}_{\tilde{w}, \tilde{x}}^c = \begin{cases} \text{Res}_{w_\sigma} & \text{if } \sigma \in \mathcal{O}_{k(X)/k} \\ 0 & \text{otherwise.} \end{cases}
\]

We shall see later that \(\text{Res}_{\tilde{w}, \tilde{x}}^c\) is independent of \(\sigma\). Note that \(\mathcal{H}^n_{w\sigma} = (\mathcal{H}^n_{X})_{w}\).

**Theorem 7.1.** Let \(x \in X\) be a closed point and let \(\alpha \in (\mathcal{H}^n_{X})_{x}\). Then \(\alpha \in \mathcal{L}(X, Y)\) iff \(\text{Res}_{w, x}^{\mathcal{L}}(\alpha) = 0\) for all \(\alpha \in \mathcal{O}_{Y,k,x}\) and \(\tilde{x} \in \pi^{-1}(x)\).

This is our algebraic counterpart of Vilonen’s formula in [Vi]. The proof of the theorem appears later in this section.

Fix a closed point \(x \in X\). Write \(B := \mathcal{O}_{Y,(w,x)}\) and \(L := \prod_{\tilde{x} \in \pi^{-1}(x)} k(\tilde{x})\).

**Lemma 7.2.** There is a canonical morphism of BCAs \(L \to B\), and \(B \cong L((g))[f_1, \ldots, f_{n-1}]\) for indeterminates \(g, f_1, \ldots, f_{n-1}\).

**Proof.** Because \(\mathcal{O}_{\tilde{X},(\tilde{x})}\) is a regular local ring we get \(\mathcal{O}_{\tilde{X},(\tilde{x})} \cong k(\tilde{x})[g]\). It is well known (cf. [Ye1] Theorem 3.3.2) that \(k(w)(x) = k(w) \otimes \mathcal{O}_{\tilde{X},(\tilde{x})} \cong \prod_{\tilde{x} \in \pi^{-1}(x)} k(\tilde{w})(\tilde{x})\), hence \(k(w)(x) \cong L((g))\).

Choose a coefficient field \(\sigma : k(w) \to \mathcal{O}_{Y,(x)}\). It extends to a lifting \(\sigma(x) : k(w)(x) \to \mathcal{O}_{Y,(w,x)} = B\) (cf. [Ye1] Lemma 3.3.9), and \(L \to B\) is independent of \(\sigma\). Taking a system of regular parameters \(f_1, \ldots, f_{n-1} \in \mathcal{O}_{Y,w}\) we obtain the desired isomorphism.

The BCA \(A := \mathcal{O}_{Y,(x)}\) is canonically an algebra over \(K := k(x)\), so there is a morphism of BCAs \(L \otimes_K A \to B\). Define a homomorphism

\[
T_x : K(B) \xrightarrow{T_x} K(L \otimes_K A) \cong L \otimes_K K(A).
\]

Since \(A \to L \otimes_K A \to B\) are topologically étale (relative to \(k\)), it follows that \(T_x\) is a homomorphism of \(\mathcal{D}(A)\)-modules.

Define

\[
V(x) := \text{Coker } (K \to L).
\]
Observe that \( V(x) = 0 \) iff \( x \) is either a smooth point or a geometrically unibranch singularity. We have \( V(x)^* \subseteq L^* \), where \( (-)^* := \Hom_k(-, k) \).

The isomorphism \( L^* \cong L \) induced by \( \Tr_{L/k} \) identifies \( V(x)^* \cong \Ker(L \xrightarrow{\Tr} K) \).

Since \( \Omega^n_{Y/k} [n] \to \K_Y \) is a quasi-isomorphism we get a short exact sequence

\[
0 \to (\mathcal{H}^{n-1}_X \Omega^n_{Y/k})_x \to \K_Y(w) \xrightarrow{\delta} \K_Y(x) \to 0.
\]

Also we see that \( \K(A) = \K_Y(x) \cong \mathcal{H}^{n-1}_X \Omega^n_{Y/k} \). Now \( \K_Y(w) = \K(\mathcal{O}_Y(w)) \subset \K(B) \). Because the composed map

\[
\K_Y(w) \xrightarrow{T_x} L \otimes_K \K_Y(x) \xrightarrow{\Tr_{L/K} \otimes 1} \K_Y(x)
\]

coincides with \( \delta \), and by the sequence (7.3), we obtain a homomorphism of \( \mathcal{D}_{Y,x} \)-modules

\[
T_x : (\mathcal{H}^{n-1}_X \Omega^n_{Y/k})_x \to V(x)^* \otimes_K \mathcal{H}^{n-1}_X \Omega^n_{Y/k}.
\]

**Theorem 7.5.** The homomorphism \( T_x \) induces a bijection between the lattice of nonzero \( \mathcal{D}_{Y,x} \)-submodules of \( (\mathcal{H}^{n-1}_X \Omega^n_{Y/k})_x \) and the lattice of \( k(x) \)-submodules of \( V(x)^* \).

The proof of the theorem is given later in this section.

In order to globalize we introduce the following notation. Let \( Z \) be the reduced subscheme supported on the singular locus \( X_{\text{sing}} \), so \( \mathcal{O}_Z = \prod_{x \in X_{\text{sing}}} k(x) \). Then \( V := \bigoplus_{x \in X_{\text{sing}}} V(x) \) and \( \mathcal{H}^n_{Z} \mathcal{O}_Y = \bigoplus_{x \in X_{\text{sing}}} \mathcal{H}^n_x \mathcal{O}_Y \) are \( \mathcal{O}_Z \)-modules. Using \( \Omega^n_{Y/k} \otimes \) to switch between left and right \( \mathcal{D}_Y \)-modules, and identifying \( V(x)^* \cong V(x) \) by the trace pairing, we see that Theorem 7.5 implies

**Corollary 7.6.** The homomorphism of \( \mathcal{D}_Y \)-modules

\[
T := \sum_{x} T_x : \mathcal{H}^{n-1}_X \mathcal{O}_Y \to (\mathcal{H}^n_Z \mathcal{O}_Y) \otimes_{\mathcal{O}_Z} V
\]

induces a bijection between the lattice of nonzero coherent \( \mathcal{D}_Y \)-submodules of \( \mathcal{H}^{n-1}_X \mathcal{O}_Y \) and the lattice of \( \mathcal{O}_Z \)-submodules of \( V \).

Since \( \mathcal{H}^n_{\{x\}} \mathcal{O}_Y \) is a simple \( \mathcal{D}_Y \)-submodule, as immediate corollaries we get:

**Corollary 7.7.** \( \mathcal{H}^{n-1}_X \mathcal{O}_Y \) has a unique simple coherent \( \mathcal{D}_Y \)-submodule \( \mathcal{L}(X,Y) \), and the sequence

\[
0 \to \mathcal{L}(X,Y) \to \mathcal{H}^{n-1}_X \mathcal{O}_Y \xrightarrow{T} (\mathcal{H}^n_Z \mathcal{O}_Y) \otimes_{\mathcal{O}_Z} V \to 0
\]

is exact.

**Corollary 7.9.** \( \mathcal{H}^{n-1}_X \mathcal{O}_Y \) is a simple coherent \( \mathcal{D}_Y \)-module iff the singularities of \( X \) are all geometrically unibranch.

According to Proposition 5.11 the fundamental class \( C_{X/Y} \) is a double cocycle in \( \mathcal{H}\text{om}(\Omega^1_{Y/k}, \K^{-1}) \), so it determines a class in \( (\mathcal{H}^{n-1}_X \mathcal{O}_Y) \otimes_{\mathcal{O}_Y} \Omega^{n-1}_{Y/k} \)
Theorem 7.10. $C_{X/Y} \in \mathcal{L}(X,Y) \otimes_{\mathcal{O}_Y} \Omega^{n-1}_{Y/k}$.

This of course implies that if $\alpha_1, \ldots, \alpha_n$ is a local basis of $\Omega^{n-1}_{Y/k}$ and $C_{X/Y} = \sum a_i \otimes \alpha_i$, then any nonzero $a_i$ generates $\mathcal{L}(X,Y)$ as a $\mathcal{D}_Y$-module. The proof of the theorem is given later in this section.

Remark 7.11. As the referee points out, when $k = \mathbb{C}$, Corollary 7.6 follows easily from the Riemann-Hilbert correspondence. In that case we may consider the sheaf $\mathcal{V}$ on the analytic space $X^{\text{an}}$, given by $\mathcal{V} := \text{Coker}(\mathcal{C}_{X^{\text{an}}} \rightarrow \pi^* \mathcal{C}_{\tilde{X}^{\text{an}}})$. Now $\mathcal{I}_X^{\text{an}} \cong \pi^* \mathcal{C}_{\tilde{X}^{\text{an}}}[1]$. The triangle $\mathcal{V} \to \mathcal{C}_{X^{\text{an}}}[1] \to \mathcal{I}_X^{\text{an}}$ is an exact sequence in the category of perverse sheaves, and it is the image of (7.8) under the functor $\text{Sol} = R\text{Hom}_{\mathcal{D}_{Y^{\text{an}}}}((-)^{\text{an}}, \mathcal{O}_{Y^{\text{an}}}[n])$. Nonetheless ours seems to be the first purely algebraic proof Theorem 7.5 and its corollaries (but cf. next remark).

Remark 7.12. When $Y = \mathbb{A}^2$ (i.e. $X$ is an affine plane curve) and $k$ is algebraically closed, Corollary 7.9 was partially proved by S.P. Smith [Sm], using the ring structure of $\mathcal{D}(X)$. Specifically, he proved that if $X$ has unibranch singularities, then $\mathcal{H}_X^1 \mathcal{O}_Y$ is simple.

Example 7.13. Let $X$ be the nodal curve in $Y = \mathbb{A}^2 = \text{Spec } k[s,t]$ defined by $f = s^2(s+1) - t^2$, and let $x$ be the origin. Take $r := t/s \in \mathcal{O}_{Y,w}$, so $s = (r+1)(r-1)$. We see that $\tilde{X} = \text{Spec } k[r]$ and $r+1, r-1$ are regular parameters at $\tilde{x}_1, \tilde{x}_2$ respectively on $\tilde{X}$. For any coefficient field $\sigma$,

$$\text{Res}_{w,\sigma}^{\text{lc}}\left[ \frac{ds \wedge df}{f} \right] = \text{Res}_{w,\sigma}^{\text{lc}}\left[ \frac{-d(r+1) \wedge df}{(r+1)(r-1)f} \right] = \frac{-d(r+1)}{(r+1)(r-1)}$$

and hence

$$\text{Res}_{(w,\tilde{x}_1)}^{\text{lc}}\left[ \frac{ds \wedge df}{f} \right] = \text{Res}_{(w,\tilde{x}_1)}^{\text{lc}}\left[ \frac{-d(r+1)}{(r+1)(r-1)} \right] = 2.$$ 

Likewise $\text{Res}_{(w,\tilde{x}_2)}^{\text{lc}}\left[ \frac{ds \wedge df}{f} \right] = -2$. Therefore $\left[ \frac{ds \wedge df}{f} \right] \notin \mathcal{L}(X,Y) \otimes \Omega^2_{Y/k}$. The fundamental class is $C_{X/Y} = \left[ \frac{df}{f} \right]$, and as generator of $\mathcal{L}(X,Y) \otimes \Omega^2_{Y/k}$ we may take $\left[ \frac{ds \wedge df}{f} \right]$.

Before getting to the proofs we need some general results. Let $A$ be a BCA over $k$. The fine topology on an $A$-module $M$ is the quotient topology w.r.t. any surjection $\bigoplus A \to M$. The fine topology on $M$ is $k$-linear, making it a topological $k$-module (but only a semi-topological (ST) $A$-module). According to [Ye2] Proposition 2.11.c, $A$ is a Zariski ST ring (cf. ibid. Definition 1.7). This means that any finitely generated $A$-module with the fine topology is separated, and any homomorphism $M \to N$ between such modules is topologically strict. Furthermore if $M$ is finitely generated then it is complete, so it is a complete linearly topologized $k$-vector space in the sense of [Ko].
Lemma 7.14. Let $A$ be a BCA. Suppose $M$ is a countably generated $ST$ $A$-module with the fine topology. Then $M$ is separated, and any submodule $M' \subset M$ is closed.

Proof. Write $M = \bigcup_{i=1}^{\infty} M_i$ with $M_i$ finitely generated. Suppose we put the fine topology on $M_i$. Then each $M_i$ is separated and $M_i \to M_{i+1}$ is strict. By [Ye1] Corollary 1.2.6 we have $M \cong \lim_{i\to\infty} M_i$ topologically, so by ibid. Proposition 1.1.7, $M$ is separated. By the same token $M/M'$ is separated too, so $M'$ is closed.

Proposition 7.15. Let $A \to B$ be a morphism of BCAs, $N$ a finitely generated $B$-module with the fine topology, and $M \subset N$ a finitely generated $A$-module. Then the topology on $M$ induced by $N$ equals the fine $A$-module topology, and $M$ is closed in $N$.

Proof. Since $A$ is a Zariski ST ring we may replace $M$ by any finitely generated $A$ module $M' \subset M \subset N$. Therefore we can assume $N = BM$ and $M = \bigoplus_{n \in \text{Max} B} M \cap N_n$. So in fact we may assume $A,B$ are both local. Like in the proof of [Ye2] Theorem 7.4 we may further assume that res. dim($A \to B$) $\leq 1$.

Put on $M$ the fine $A$-module topology. Let $\tilde{N}_i := N/n^iN$ and $\tilde{M}_i := M/(M \cap n^iN)$ with the quotient topologies. We claim $\tilde{M}_i \to \tilde{N}_i$ is a strict monomorphism. This is so because as $A$-modules both have the fine topology, $\tilde{M}_i$ is finitely generated and $\tilde{N}_i$ is countably generated (cf. part 1 in the proof of [Ye1] Theorem 3.2.14). Just as in part 2 of loc. cit. we get topological isomorphisms $M \cong \lim_{i\to\infty} \tilde{M}_i$ and $N \cong \lim_{i\to\infty} \tilde{N}_i$, so $M \to N$ is a strict monomorphism. But $M$ is complete and $N$ is separated, so $M$ must be closed.

Proposition 7.17. Assume $k \to A$ is a morphism of BCAs. Then the residue pairing $\langle -,- \rangle_{A/k} : A \times K(A) \to k$, $\langle a, \phi \rangle_{A/k} = \text{Tr}_{A/k}(a\phi)$, is a topological perfect pairing.
Proof. We may assume $A$ is local. Then $A = \lim_{i \to} A/m^i$ and $K(A) = \lim_{i \to} K(A/m^i)$ topologically. Let $K \to A$ be a coefficient field, so both $A/m^i$ and $K(A/m^i) \cong \text{Hom}_K(A/m^i, \omega(K))$ are finite $K$-modules with the fine topology. By [Ye1] Theorem 2.4.22 the pairing is perfect. \qed

From here to the end of this section we consider an integral curve $X$ embedded as a closed subscheme in a smooth irreducible $n$-dimensional variety $Y$. Fix a closed point $x \in X$, and set $A := O_{Y,x}$ and $K := k(x)$. Choosing a regular system of parameters at $x$, say $\ell = (t_1, \ldots , t_n)$, allows us to write $A = K[[\ell]]$. Let $D(A) := \text{Diff}_{A/k}(A, A)$. Since both $K[\ell] \to A$ and $O_{Y,x} \to A$ are topologically étale relative to $k$, we have

$$D(A) \cong A \otimes_K K[\partial/\partial t_1, \ldots , \partial/\partial t_n] \cong A \otimes_{O_{Y,x}} D_{Y,x}$$

(cf. [Ye2] Section 4).

Define $B := O_{Y,w,x}$. Since $A \to B$ is topologically étale relative to $k$, we get a $k$-algebra homomorphism $D(A) \to D(B)$. In particular, $B$ and $K(B)$ are $D(A)$-modules. Define $L := \prod_{\bar{x} \in \pi^{-1}(x)} k(\bar{x})$ as before.

**Lemma 7.18.** The multiplication map $A \otimes_K L \to B$ is injective. Its image is a $D(A)$-submodule of $B$. Any $D(A)$-submodule of $B$ which is finitely generated over $A$ equals $A \otimes_K W$ for some $K$-submodule $W \subset L$.

**Proof.** By [Kz] Proposition 8.9, if $M$ is any $D(A)$-module which is finitely generated over $A$, then $M = A \otimes_K W$, where $W \subset M$ is the $K$-submodule consisting of all elements killed by the derivations $\partial/\partial t_i$. Note that $\Omega_{B/k}^1$ is free with basis $dt_1, \ldots , dt_n$. Thus it suffices to prove that

$$L = \{ b \in B \mid \frac{\partial}{\partial t_1} b = \cdots = \frac{\partial}{\partial t_n} b = 0 \} = \Pi^0 \Omega_{B/k}^{1,\text{sep}}.$$  

We know that $B \cong L((g))[[f_1, \ldots , f_{n-1}]]$, so $B$ is topologically étale over the polynomial algebra $k[g, f_1, \ldots , f_{n-1}]$ (relative to $k$), and hence $dg, df_1, \ldots , df_{n-1}$ is also a basis of $\Omega_{B/k}^{1,\text{sep}}$. It follows that $\Pi^0 \Omega_{B/k}^{1,\text{sep}} = L$. \qed

**Proof.** (of Theorem 7.5) Set $M := H^{n-1}_{X,N} \Omega^n_{Y/k}$ and define $M := A \otimes_{O_{Y,x}} M$. Tensoring the exact sequence (7.3) with $A$ we get an exact sequence of $D(A)$-modules

$$0 \to M \to \mathcal{K}(B) \xrightarrow{\delta} \mathcal{K}(A) \to 0.$$  

The proof will use repeatedly the residue pairing $(-, -)_{B/k} : B \times \mathcal{K}(B) \to k$. By definition of $\delta$ (cf. Definition 2.5 and [Ye2] Section 7) we see that $M = A^\perp$. Consider the closed $k$-submodules $A \subset A \otimes_K L \subset B$ (cf. Proposition 7.15). Applying Lemma 7.16 to them, and using $V(x) = L/K$ and $\mathcal{K}(A) \cong A^*$, we get an exact sequence of $D(A)$-modules

$$0 \to (A \otimes_K L)^\perp \to M \xrightarrow{T'} V(x) \otimes_K \mathcal{K}(A) \to 0.$$  

Keeping track of the operations we see that in fact $T' = T|_M$.  


Put the fine $A$-module topology on $M$ and $\mathcal{K}(A)$, so $M \to V(x)^* \otimes_K \mathcal{K}(A)$ is continuous. By [Ye1] Proposition 1.1.8, $M_x \to M$ is dense. Since $\mathcal{K}(A)$ is discrete we conclude that $M_x \to V(x)^* \otimes_K \mathcal{K}(A)$ is a surjection of $\mathcal{D}_{Y,x}$-modules. Thus any $K$-module $W \subset V(x)^*$ determines a distinct nonzero $\mathcal{D}_{Y,x}$-module $N_x \subset M_x$.

Conversely, say $N_x \subset M_x$ is a nonzero $\mathcal{D}_{Y,x}$-module. On any open set $U \subset Y$ s.t. $U \cap X$ is smooth the module $M|_U$ is a simple coherent $\mathcal{D}_U$-module (by Kashiwara’s Theorem it corresponds to the $\mathcal{D}_{X,Y}$-module $\Omega^1_{(X \cap U)/k}$). Therefore the finitely generated $\mathcal{D}_{Y,x}$-module $C$ defined by

$$0 \to N_x \to M_x \to C \to 0$$

is supported on $\{x\}$. It follows that $C \cong \mathcal{K}(A)^r$ for some number $r$. Tensoring (7.21) with $A$ we get an exact sequence of $\mathcal{D}(A)$-modules

$$0 \to N \to M \to C \to 0$$

with $N \subset M \subset \mathcal{K}(B)$. By faithful flatness of $\mathcal{O}_{Y,x} \to A$ we see that $N_x = M_x \cap N$.

We put on $M, N$ the topology induced from $\mathcal{K}(B)$, and on $C$ the quotient topology from $M$. Now $\mathcal{K}(B)$ has the fine $A$-module topology and it is countably generated over $A$ (cf. proof of Proposition 7.15), so by Lemma 7.14 both $M, N$ are closed in $\mathcal{K}(B)$. Using Lemma 7.16 and the fact that $M^\perp = A$ we obtain the exact sequence

$$0 \to A \to N^\perp \to C^* \to 0,$$

with $N^\perp \subset B$. We do not know what the topology on $C$ is; but it is a ST $A$-module. Hence the identity map $\mathcal{K}(A)^r \to C$ is continuous, and it induces an $A$-linear injection $C^* \to A^r$. Therefore $C^*$, and thus also $N^\perp$, are finitely generated over $A$. According to Lemma 7.18, $N^\perp = A \otimes_K W$ for some $K$-module $W$, $K \subset W \subset \mathcal{L}$. But $N$ is closed, so $N = (N^\perp)^\perp$.

**Proof.** (of Theorem 7.10) For each $\bar{x} \in \pi^{-1}(x)$ define a homomorphism

$$T_{(\bar{w}, \bar{x})} : \mathcal{K}(B) \xrightarrow{\text{Tr}} \mathcal{K}(L \otimes_K A) \cong L \otimes_K \mathcal{K}(A) \to k(\bar{x}) \otimes_K \mathcal{K}(A),$$

so $T_x = \sum T_{(\bar{w}, \bar{x})}$. From the proof of Theorem 7.5 we see that the theorem amounts to the claim that $T_{\bar{x}}(C_{X/Y}(\alpha)) = 0$ for every $\bar{x}$ and $\alpha \in \Omega^1_{X/k, x}$.

But $C_{X/Y}$ is the image of $C_X \in \mathcal{H}om(\Omega^1_{X/k}, \mathcal{K}(X/w))$, so we can reduce our residue calculation to the curve $\bar{X}$. In fact it suffices to show that for every $\alpha \in \Omega^1_{X/k, \bar{x}}$ one has $\text{Res}_{(\bar{w}, \bar{x})} \alpha = 0$. Since $\alpha \in \Omega^1_{\bar{X}/k, \bar{x}}$ this is obvious.

**Proof.** (of Theorem 7.1) According to [SY] Corollary 0.2.11 (or [Hu] Theorem 2.2) one has

$$\text{Res}^{lc}_{(\bar{w}, \bar{x})} = (1 \otimes \text{Tr}_{A/K})T_{(\bar{w}, \bar{x})} : \mathcal{H}_{w}^{-1} \Omega^0_{Y/k} \to k(\bar{x}),$$

which shows that $\text{Res}^{lc}_{(\bar{w}, \bar{x})}$ is independent of $\sigma$. Now use Theorem 7.5.
Problem 7.22. What is the generalization to \( \dim X > 1 \)? To be specific, assume \( X \) has only an isolated singularity at \( x \). Then we know there is an exact sequence

\[
0 \to \mathcal{L}(X, Y) \to \mathcal{H}^1_X \mathcal{O}_Y \xrightarrow{T} \mathcal{H}^0_{\{x\}} \mathcal{O}_Y \otimes_{k(x)} V(x) \to 0
\]

for some \( k(x) \)-module \( V(x) \). What is the geometric data determining \( V(x) \) and \( T \)? Is it true that \( T = \sum T_\xi \), a sum of “residues” along chains \( \xi \in \pi^{-1}(x) \), for a suitable resolution of singularities \( \pi: \tilde{X} \to X \)?

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