Decoherence in closed and open systems

Mario Castagnino
CONICET-Instituto de Astronomía y Física del Espacio
Casilla de Correos 67, Sucursal 28, 1428, Buenos Aires, Argentina

Roberto Laura
Facultad de Ciencias Exactas, Ingeniería y Agrimensura, Universidad Nacional de Rosario
Av. Pellegrini 250, 2000, Rosario, Argentina

Olimpia Lombardi
CONICET-Universidad Nacional de Quilmes
Rivadavia 2328, 6º Derecha, 1034, Buenos Aires, Argentina

A generalized formal framework for decoherence, that can be used both in open and closed quantum systems, is sketched. In this context, the relationship between the decoherence of a closed system and the decoherence of its subsystems is studied, and the corresponding decoherence times, \( t_{DU} \) for the closed system and \( t_{DS} \) for the open system, are defined: for macroscopic systems, \( t_{DU} \gg t_{DS} \). Finally, it is shown that the application of the new formal framework to a well-known model leads to physically adequate results.

PACS No. 03.65.Yz, 03.65.Db

I. INTRODUCTION

The peculiar features of quantum mechanics are mainly due to the principle of superposition and its consequence, the phenomenon of interference. Therefore, any attempt to explain how classicality emerges from quantum behavior must include two elements: a process through which interference vanishes, and a resulting superselection rule that precludes superpositions. Decoherence is the process that cancels interference and leads to the rule that selects the candidates for classical states.

Historically, decoherence was conceived in terms of explaining how a coherent pure state becomes a final stable decohered mixture with no interference terms. In other words, the task was to explain how the state of a quantum system goes from the frontier of the convex set of states to an interior point of the convex. On this basis, decoherence was studied in closed as in open systems. Schematically, three periods can be identified in the development of this general program:

• **First period: Closed systems** (van Kampen [1], van Hove [2], Daneri et al. [3]). In order to understand how classical macroscopic features arise from quantum microscopic behavior, "gross" observables are defined, and the states that are indistinguishable for a macroscopic observer are described by the same coarse-grained state \( \rho_G(t) \). When the evolution of \( \rho_G(t) \) (or of the expectation value of the gross observables) is studied, it is proved that \( \rho_G(t) \) reaches equilibrium in a relaxation time \( t_R \); therefore, \( \rho_G(t) \) decoheres in its own eigenbasis after a decoherence time \( t_D = t_R \).

This approach was rooted in the traditional study of irreversible processes. The main problem of this period was the fact that decoherence times computed with this primitive formalism turned out to be too long to account for experimental data (see [4]).

• **Second period: Open systems**. The open system \( S \) is considered in interaction with its environment \( E \), and the evolution of the reduced state \( \rho_S(t) = Tr_E(\rho(t)) \) is studied. The so-called "environment-induced decoherence (EID) approach" (Zeh [5], Zurek [6] [7]) proves that, since the states of \( E \) become rapidly orthogonal, the interference terms of \( \rho_S(t) \) rapidly vanish and \( \rho_S(t) \) decoheres in an adequate pointer basis after a very short decoherence time \( t_D \); this result solves the main problem of the first period.

This approach is rooted in the theory of quantum measurements, where the open system \( S \) interacts with a measurement apparatus \( M \), and the evolution correlates the states of both systems. In EID, \( E \) plays the role of \( M \), and it is said that \( E \) continuously measures \( S \). At present this formalism has been applied to a wide range of models, and its results have many experimental confirmations (see [8]). However, the EID approach still has to face three conceptual difficulties: (a) it cannot be applied to closed systems, in particular, to the universe; according to Zurek, the issue of the classicality of closed systems or of the universe as a whole cannot even be
both theories can be viewed as complementary and leading to compatible results. In particular, we will develop our argument by comparing SID and EID; we will show that our aim is to present a new conceptual perspective that will clarify some points that still remain rather obscure in the general theoretical framework where the problems of both kinds of approaches can be successfully faced. Our main contrary, formalisms originally devised to deal just with closed or open systems can be subsumed under a more formalisms dealing with different phenomena [16]. In this paper we will argue that this is not the case; on the contrary, formalisms originally devised to deal just with closed or open systems can be subsumed under a more general theoretical framework where the problems of both kinds of approaches can be successfully faced. Our main aim is to present a new conceptual perspective that will clarify some points that still remain rather obscure in the literature on the subject. In particular, we will develop our argument by comparing SID and EID; we will show that both theories can be viewed as complementary and leading to compatible results.

In spite of the fact that the theories of decoherence in closed and open systems coexist in the third period, in the literature both kinds of approaches are usually conceived as alternative scenarios for decoherence, or even as formalisms dealing with different phenomena [16]. In this paper we will argue that this is not the case; on the contrary, formalisms originally devised to deal just with closed or open systems can be subsumed under a more general theoretical framework where the problems of both kinds of approaches can be successfully faced. Our main aim is to present a new conceptual perspective that will clarify some points that still remain rather obscure in the literature on the subject. In particular, we will develop our argument by comparing SID and EID; we will show that both theories can be viewed as complementary and leading to compatible results.

On this basis, the paper is organized as follows. In Section II we will present a general framework for decoherence and show how SID and EID can be expressed in this theoretical context. Such a presentation will allow us to explain, in Section III, the relationship between decoherence in closed and open systems. Section IV is devoted to stress the compatibility between the results obtained with SID and EID, in particular with respect to the decoherence time. In Section V we will study in detail a well-known example, showing how some results not correctly interpreted in previous works can be understood from the new perspective. Finally, in Section VI we will draw our conclusions.

II. OBSERVABLES, EXPECTATION VALUES AND WEAK LIMITS

As emphasized by Omnés [17], decoherence is just a particular case of the general problem of irreversibility in quantum mechanics. The problem of irreversibility can be roughly expressed in the following terms. Since the quantum state \( \rho(t) \) follows an unitary evolution, it cannot reach a final equilibrium state for \( t \to \infty \). Therefore, if the non-unitary evolution towards equilibrium is to be accounted for, a further element has to be added to the unitary evolution. From the most general viewpoint, this element consists in the splitting of the maximal information about the system into a relevant part and an irrelevant part: whereas the irrelevant part is disregarded, the relevant part is retained and its evolution may reach a final equilibrium situation.

This broadly expressed idea can be rephrased in operators language. The maximal information about the system is given by the set of all its potentially possible observables. By selecting a particular subset \( \mathcal{O} \) of this set, we restrict the maximal information to a relevant part: the expectation values \( \langle O_R \rangle_{\rho(t)} \) of the observables \( O_R \in \mathcal{O} \) express the relevant information about the system. Of course, the decision about which observables are to be considered as relevant depends on the particular purposes in each situation; but without this restriction, irreversible evolutions cannot be described.

Since decoherence is an irreversible process, it must include the splitting of the whole set of observables into the relevant subset \( \mathcal{O} \) and the irrelevant subset. In fact, it is easy to see that the different approaches to decoherence, when considered from this point of view, always select a set \( \mathcal{O} \) of relevant observables in terms of which the time behavior of the system is described: gross observables in van Kampen [1], macroscopic observables of the apparatus in Daneri et al. [3], observables of the open system in EID [5] [6] [7], relevant observables in Omnés [17], van Hove observables in SID [15], etc.

Once the essential role played by the selection of the relevant observables is clearly understood, the phenomenon of decoherence can be explained in three general steps:

1. **First step:** The set \( \mathcal{O} \) of relevant observables is defined.

2. **Second step:** The expectation value \( \langle O_R \rangle_{\rho(t)} \), for any \( O_R \in \mathcal{O} \), is obtained. This step can be formulated in two different but equivalent ways:

   - \( \langle O_R \rangle_{\rho(t)} \) is computed as the expectation value of \( O_R \) in the unitarily evolving state \( \rho(t) \).
• A coarse-grained state \( \rho_G(t) \) is defined by \( \langle O_R \rangle_{\rho_G(t)} = \langle O_R \rangle_{\rho(t)} \) for any \( O_R \in \mathcal{O} \) (see Appendix A, eq.(A4)), and its non-unitary evolution (governed by a master equation) is computed.

3. **Third step:** It is proved that \( \langle O_R \rangle_{\rho(t)} = \langle O_R \rangle_{\rho_G(t)} \) reaches a final equilibrium value \( \langle O_R \rangle_{\rho_*} \):

\[
\lim_{t \to \infty} \langle O_R \rangle_{\rho(t)} = \lim_{t \to \infty} \langle O_R \rangle_{\rho_G(t)} = \langle O_R \rangle_{\rho_*}. \tag{1}
\]

This means that the coarse-grained state \( \rho_G(t) \) evolves towards a final equilibrium state (see Appendix A, eq.(A6)):

\[
\lim_{t \to \infty} \langle O_R \rangle_{\rho_G(t)} = \langle O_R \rangle_{\rho_G*}. \tag{2}
\]

The final equilibrium state \( \rho_* \) is obviously diagonal in its own eigenbasis, which turns out to be the final pointer basis. But, as follows from eq.(1), the unitarily evolving quantum state \( \rho(t) \) of the whole system has *only a weak limit*:

\[
W^- \lim_{t \to \infty} \rho(t) = \rho_* \tag{3}
\]

As a consequence, the coarse-grained state \( \rho_G(t) \) also has a weak limit, as follows from eq.(2):

\[
W^- \lim_{t \to \infty} \rho_G(t) = \rho_{G*} \tag{4}
\]

These weak limits mean that, although the off-diagonal terms of \( \rho(t) \) never vanish through the unitary evolution, the system decoheres *from an observational point of view*, that is, from the viewpoint given by any relevant observable \( O_R \in \mathcal{O} \).

From this general perspective, it turns out to be clear that decoherence is a coarse-grained process that leads to classicality in a coarse-grained sense. In fact, the phenomenon of interference is suppressed because the off-diagonal terms of \( \rho(t) \) and \( \rho_G(t) \) vanish from the viewpoint of the relevant observables, and the superselection rule that precludes superpositions retains the states defined by the corresponding pointer bases.

In the next subsections we will argue for the generality of this theoretical framework by applying it to the SID and the EID approaches. This will show that, in spite of the fact that SID deals with closed systems and EID describes open systems, the general concept of decoherence expressed by steps 1 to 3 lies behind both approaches.

### A. SID: decoherence in closed systems

In the SID approach, the three steps are explicit in the formalism. For conciseness, we will present the theory in the simplest case (for more general cases, see [15]).

1. **First step:** Let us consider a quantum system endowed with a Hamiltonian \( H \) with continuous spectrum: \( H|\omega\rangle = \omega|\omega\rangle, \omega \in [0, \infty) \). A generic observable reads

\[
O = \int_0^\infty \int_0^\infty \tilde{O}(\omega, \omega')|\omega\rangle \langle \omega'|d\omega d\omega' \tag{5}
\]

where \( \tilde{O}(\omega, \omega') \) is any kernel or distribution. The restriction in the set of observables is introduced by considering only the *van Hove operators*, whose components are given by

\[
\tilde{O}_R(\omega, \omega') = O(\omega)\delta(\omega - \omega') + O(\omega, \omega') \tag{6}
\]

where \( O(\omega, \omega') \) is a regular function. Then, the relevant observables have the following form:

\[
O_R = \int_0^\infty O(\omega)|\omega\rangle d\omega + \int_0^\infty \int_0^\infty O(\omega, \omega')|\omega, \omega'\rangle d\omega d\omega' \tag{7}
\]
Riemann-Lebesgue theorem can be applied to eq. (10).

where $|\omega\rangle = |\omega\rangle_{\omega}$ and $|\omega,\omega'\rangle = |\omega\rangle_{\omega'}$. These relevant observables belong to the van Hove space $\mathcal{O}_{VH}$, whose basis is $\{|\omega\rangle, |\omega,\omega'\rangle\}$.\footnote{This restriction on operators does not diminish the generality of SID, since the observables not belonging to the van Hove space are not experimentally accessible and, for this reason, in practice they are always approximated, with the desired precision, by regular observables for which the approach works satisfactorily (for a full argument, see [18]).} States $\rho$ are represented by linear functionals on $\mathcal{O}_{VH}$, that is, they belong to the dual space $\mathcal{O}_{VH}'$ and read

$$\rho = \int_0^\infty \rho(\omega)|\omega\rangle \langle \omega| + \int_0^\infty \int_0^\infty \rho(\omega,\omega')|\omega\rangle_{\omega'} \langle \omega'| \frac{d\omega}{d\omega'}$$

where $\{|\omega\rangle, |\omega,\omega'\rangle\}$ is the cobasis of $\{|\omega\rangle, |\omega,\omega'\rangle\}$, that is, the basis of $\mathcal{O}_{VH}'$. States must satisfy the usual requirements: $\rho(\omega)$ is real and positive and $\int_0^\infty \rho(\omega)d\omega = 1$. We also require that $\rho(\omega,\omega')$ be a regular function. Under these conditions, states belong to a convex set $S \subset \mathcal{O}_{VH}'$.

2. **Second step**: The expectation value of the observable $O_R \in \mathcal{O}_{VH}$ in the state $\rho \in S$ can be computed as the action of the functional $\rho$ on the operator $O_R$:

$$\langle O_R \rangle_\rho = \langle \rho | O_R \rangle = \int_0^\infty \rho^*(\omega)O(\omega) d\omega + \int_0^\infty \int_0^\infty \rho^*(\omega,\omega')O(\omega,\omega') d\omega d\omega'$$

where $\rho(\omega)$ and $O(\omega)$ are such that the first integral is well defined. The time evolution of this expectation value is given by

$$\langle O_R \rangle_{\rho(t)} = \int_0^\infty \rho^*(\omega)O(\omega) d\omega + \int_0^\infty \int_0^\infty \rho^*(\omega,\omega')O(\omega,\omega') e^{i\frac{\omega - \omega'}{2}t} d\omega d\omega'$$

3. **Third step**: Since the function $\rho^*(\omega,\omega')O(\omega,\omega')$ is regular (precisely, it is $L_1$ in variable $\nu = \omega - \omega'$), the Riemann-Lebesgue theorem can be applied to eq. (10).\footnote{The Riemann-Lebesgue theorem, which mathematically expresses the phenomenon of destructive interference, states that, if $f(\nu) \in L_1$, $f(\nu) \in L_1$,}

$$\lim_{t \to \infty} \langle O_R \rangle_{\rho(t)} = \int_0^\infty \rho^*(\omega)O(\omega) d\omega$$

This means that, for $t \to \infty$, the expectation value of any observable $O_R \in \mathcal{O}_{VH}$ in the state $\rho \in S$ can be computed as if the system were in a final stable state $\rho_*:

$$\lim_{t \to \infty} \langle O_R \rangle_{\rho(t)} = \langle O_R \rangle_{\rho_*}$$

where $\rho_* = \int_0^\infty \rho^*(\omega)|\omega\rangle d\omega$ has only singular diagonal terms in the eigenbasis of the Hamiltonian. This result can also be expressed as a weak limit:

$$W - \lim_{t \to \infty} \rho(t) = \rho_*$$

This means that the system decoheres in the eigenbasis of the Hamiltonian: these states are stationary and, therefore, completely robust.

Summing up, through steps 1 to 3 (see the coincidence between eqs. (12)-(13) and eqs. (1)-(3)), SID cancels interference and leads to the superselection rule that precludes superpositions.

More general models have been treated with the SID approach [15], and decoherence times have been computed [19]; the foundations of the theory have also been conceptually explained [18]. Although SID strictly applies in the continuous case, it also leads to approximate decoherence in quasi-continuous models, that is, discrete models where (i) the energy spectrum is quasi-continuous, i.e., has a small discrete energy spacing, and (ii) the functions of energy used in the formalism are such that the sums in which they are involved can be approximated by Riemann
integrals. These conditions are applied to a concrete example in [20] where it is shown that, in spite of the fact that, strictly speaking, a system with discrete spectrum never reaches equilibrium due to Poincaré recurrence, for times \( t \ll t_R \), where \( t_R \) is the recurrence time, the discrete spectrum can be approximated by a continuous spectrum when the involved functions satisfy the usual conditions of regularity and integrability. These conditions are rather weak: in fact, the overwhelming majority of the physical models studied in the literature on dynamics, thermodynamics, quantum mechanics and quantum field theory are quasi-continuous, and the well-known strategy for transforming sums in integrals is applied.\(^3\)

### B. EID: decoherence in open systems

In the case of the EID approach, steps 1 to 3 are usually not explicit in the formalism. However, the theory can be rephrased in such a way that it can be analyzed from the general framework introduced at the beginning of this section.

1. **First step:** Let us consider a closed system \( U \) that can be decomposed into a proper system \( S \) and its environment \( E \). Let \( \mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E \), where \( \mathcal{H}_S \) is the Hilbert space of \( S \) and \( \mathcal{H}_E \) the Hilbert space of \( E \). The corresponding von Neumann-Liouville space of \( U \) is \( \mathcal{L} = \mathcal{H} \otimes \mathcal{H} = \mathcal{L}_S \otimes \mathcal{L}_E \), where \( \mathcal{L}_S = \mathcal{H}_S \otimes \mathcal{H}_S \) and \( \mathcal{L}_E = \mathcal{H}_E \otimes \mathcal{H}_E \). A generic observable belonging to \( \mathcal{L} \) reads

\[
O = O_S \otimes O_E \in \mathcal{L}, \quad \text{with } O_S \in \mathcal{L}_S \text{ and } O_E \in \mathcal{L}_E
\]

\[\text{e.g., with coordinates } (O_{i\alpha j\beta}) = (O_{ij}O_{\alpha\beta}), \text{ where } i, j, \ldots \text{ are the indices corresponding to } \mathcal{H}_S, \text{ and } \alpha, \beta, \ldots \text{ are the indices corresponding to } \mathcal{H}_E.\]

The relevant observables are those having the following form:

\[
O_R = O_S \otimes I_E \in \mathcal{O}_R, \quad \text{with coordinates } (O_{ij}\delta_{\alpha\beta})
\]

where \( I_E \) is the identity operator in \( \mathcal{L}_E \). Therefore, \( \mathcal{O}_R \subset \mathcal{L} \) is the subset of the relevant observables, that is, those corresponding to the proper system \( S \).

2. **Second step:** The expectation value of any observable \( O_R \in \mathcal{O}_R \) in the state \( \rho \) of \( U \) reads

\[
\langle O_R \rangle \rho = Tr (\rho O_R) = \sum_{ij\alpha\beta} \rho^\ast_{ij\alpha\beta} O_{ij} \delta_{\alpha\beta} = \sum_{ij} O_{ij} \sum_{\alpha\beta} \rho^\ast_{ij\alpha\beta} \delta_{\alpha\beta} = \sum_{ij} O_{ij} \sum_{\alpha} \rho^\ast_{ij\alpha}
\]

If we define the reduced density operator \( \rho_S \) by tracing over the environmental degrees of freedom, we obtain

\[
\rho_S = Tr_E \rho \in \mathcal{L}^\prime_S, \quad \text{with coordinates } \left( \sum_{\alpha} \rho_{ij\alpha} \right) = (\rho_{ij})
\]

where \( \mathcal{L}^\prime_S \) is the dual space of \( \mathcal{L}_S \). Therefore, the expectation value \( \langle O_R \rangle_{\rho(t)} \) can be expressed as

\[
\langle O_R \rangle_{\rho(t)} = Tr (\rho(t) O_R) = Tr (\rho(t)(O_R \otimes I_E)) = Tr (\rho_S(t) O_S) = \langle O_S \rangle_{\rho_S(t)}
\]

3. **Third step:** The EID approach studies the time evolution of the reduced density operator \( \rho_S(t) \) governed by an effective master equation. For many physical models where the space \( \mathcal{O}_R \) has a finite number of dimensions, this approach shows that, for \( t \to \infty \), \( \rho_S(t) \) strongly reaches an equilibrium state \( \rho_{S*} \) (see Appendix A, eq.(A11)):

\[\text{If we have the sum:} \]

\[
\frac{2\pi\hbar}{L} \sum_0^p f_p
\]

where \( L \) is "the size of the box", from the uncertainty principle we can make \( \frac{2\pi\hbar}{L} \simeq \Delta p \) and, therefore, the strategy is:

\[
\sum_0^p f_p \Delta p \longrightarrow \int_0^p f(p) \, dp
\]
\[ \rho_S(t) \rightarrow \rho_{S*} \]  

Since \( \rho_{S*} \) is obviously diagonal in its eigenbasis, the system \( S \) decoheres in the eigenbasis of \( \rho_{S*} \), which turns out to be the final pointer basis. But if we take into account the definition of \( \rho_S \) as a partial trace (see eq.(17)), we can obtain the limit of the expectation values of eq.(18) as

\[ \lim_{t \to \infty} \langle O_S \rangle_{\rho_S(t)} = \lim_{t \to \infty} \langle O_R \rangle_{\rho(t)} = \langle O_S \rangle_{\rho_{S*}} = \langle O_R \rangle_{\rho*} \]  

where \( \rho* \) is such that \( \rho_{S*} \) results from the projection of \( \rho* \) onto \( O_R \) (see Appendix A). Therefore, for any observable \( O_R \subset O_R \),

\[ \lim_{t \to \infty} \langle O_R \rangle_{\rho(t)} = \langle O_R \rangle_{\rho*} \]  

This result can also be expressed as a weak limit:

\[ W - \lim_{t \to \infty} \rho(t) = \rho* \]  

Summing up, through steps 1 to 3, EID also suppresses interference and leads to the superposition rule that precludes superpositions.

If the just obtained eqs.(21) and (22) are compared with the corresponding eqs.(12) and (13) in the SID approach, the similarity between them can be easily verified. This shows that the EID approach can also be formulated from the viewpoint of the closed composite system \( U \) and, from this perspective, it can be explained in the context of the general framework introduced at the beginning of this section. In other words, the splitting of the closed system into a proper open system and an environment is just a way of selecting the relevant observables of the closed system. In fact, the environment may be external -such as particles if air of photons scattered of the system- or internal -such as collections of phonons or other internal excitations-; thus, the splitting of \( U \) consists in a decision about which degrees of freedom are of direct interest to the observer and which are irrelevant. Since the same system \( U \) can be decomposed in many different ways, there is nothing essential in such a decomposition: there is no need of an unequivocal criterion for placing the cut between "the" system and "the" environment. From this perspective, the essential physical fact is that, among all the possible decompositions of a closed system, there are some that lead to identify a subset of relevant observables for which the system decoheres.

The EID approach is usually applied to models with discrete energy spectrum. However, in most cases the sums introduced by the formalism are replaced by Riemann integrals because of the quasi-continuous character of the model under study (see, e.g., eq.(3-11) of [21]). Nevertheless, there seems to be particular examples where the EID formalism can be successfully applied for times \( t \ll t_R \), where \( t_R \) is the recurrence time, in spite of the fact that the conditions for quasi-continuity are not satisfied (see [16]).

### III. CLOSED AND OPEN SYSTEMS

Since we have showed that decoherence in open and closed systems can be understood in the context of a common general framework, now the relationship between both cases can be studied. In particular, we will explore under what conditions (i) the decoherence of a closed composite system implies the decoherence of any of its open subsystems, and (ii) the decoherence of the open subsystems implies the decoherence of the closed composite system. The results obtained from this analysis will point to the fact that the formalisms of decoherence for closed and open systems are complementary, and both cooperate in the understanding of the same physical phenomenon.

In order to develop our arguments, we will study the case of systems with discrete and finite spectra; the obtained results can be extended to the case of continuous spectrum under the assumption of the usual conditions of quasi-continuity. This strategy does not involve a loss of physical generality for the following reason. Theoretical results must always be tested by numerical simulations; then, continuous functions need to be approximated by discrete functions, and the numerical experiments are performed for a progressively increasing number \( N \) of degrees of freedom in order to simulate the continuous situation. In the case of testing decoherence results, such a procedure is completely reasonable from a physical point of view. In fact, let us consider a closed system \( U \) partitioned into two subsystems \( S_1 \) and \( S_2 \) such that \( U = S_1 \cup S_2 \). Let us call \( O_0 \) the set of the discrete observables of \( U \), and \( O_1 \) and \( O_2 \) the sets of discrete observables \( O_{R1} \) and \( O_{R2} \) that are relevant from the viewpoint of \( S_1 \) and \( S_2 \) respectively:

\[ O_{R1} = O_{S1} \otimes I_{S2} \in O_1 \subset O_0 \]

\[ O_{R2} = O_{S2} \otimes I_{S1} \in O_2 \subset O_0 \]  

For placing the cut between "the" system and "the" environment. From this perspective, the essential physical fact is that, among all the possible decompositions of a closed system, there are some that lead to identify a subset of relevant observables for which the system decoheres.
When $N$ is large enough, and under the conditions of quasi-continuity, the $U$-relevant observables belonging to the van Hove space $O_{VH}$ can be approximated by discrete observables belonging to $O_0$ because any distribution can always be approximated, in the context of integration, by a discrete function with the desired precision. As a consequence, numerical simulations will show that, if the system $U$ decoheres according to SID, it approximately decoheres for any discrete observable $O_U \in O_0$. Therefore, the assumption that the $S_1$-relevant observables and the $S_2$-relevant observables belong to the set of the $U$-relevant observables does not diminish the physical generality of our arguments.

A. From the closed system to its open subsystems

Let us consider the closed system $U$ partitioned into the two subsystems $S_1$ and $S_2$ (each subsystem can be thought as the environment of the other). If $\mathcal{H}$ is the Hilbert space of $U$ with non-prime dimension $N = nm$, with $n, m \in \mathbb{N}$, $\mathcal{H}$ can be decomposed as

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$$

(25)

where $\mathcal{H}_1$ and $\mathcal{H}_2$ are the Hilbert spaces of $S_1$ and $S_2$ respectively. Let $\{|i\rangle\}$ be a basis of $\mathcal{H}_1$ ($i = 1, ..., n$), and $\{|\alpha\rangle\}$ be a basis of $\mathcal{H}_2$ ($\alpha = 1, ..., m$). Then, a basis of $\mathcal{H}$ is

$$\{|i, \alpha\rangle\} = \{|i\rangle \otimes |\alpha\rangle\}$$

(26)

The coordinates of the observables $O_U \in O_0$, $O_{R1} \in O_1$ and $O_{R2} \in O_2$ are

$$\langle O_{ioj\beta} \rangle : \text{coordinates of } O_U$$

$$\langle O_{R1}^{ioj\beta} \rangle = O_{S1}^{i\beta} \delta_{\alpha\beta} : \text{coordinates of } O_{R1}$$

$$\langle O_{R2}^{ioj\beta} \rangle = \delta_{ij} O_{S2}^{\alpha\beta} : \text{coordinates of } O_{R2}$$

(27, 28, 29)

If we assume that the closed system $U$ decoheres, the expectation value of any observable $O_U \in O_0$ in the state $\rho_U(t)$ of $U$ reaches a final stable value (see eqs.(1) and (12)):

$$\lim_{t \to \infty} \langle O_U \rangle_{\rho_U(t)} = \langle O_U \rangle_{\rho_U*}$$

(30)

This means that the state $\rho_U(t)$ has a weak limit (see eqs.(3) and (13)):

$$W - \lim_{t \to \infty} \rho_U(t) = \rho_U*$$

(31)

and, as a consequence, each one of the components ($\rho_{ioj\beta}(t)$) of $\rho_U(t)$ reaches a final stable value ($\rho_{*ioj\beta}$); these values are the coordinates of $\rho_{*U}$:

$$\rho_{ioj\beta}(t) \rightarrow \rho_{*ioj\beta}$$

(32)

In particular,

$$\rho_{ioja}(t) \rightarrow \rho_{*ioja}$$

(33)

$$\rho_{i\alpha j\beta}(t) \rightarrow \rho_{*i\alpha j\beta}$$

(34)

By means of eq.(17), let us now define the reduced density operators $\rho_{S1}$ and $\rho_{S2}$ by tracing over the degrees of freedom corresponding to $S_2$ and $S_1$ respectively:

$$\rho_{S1} = Tr_{S2} \rho_U, \quad \text{with coordinates} \quad \sum_{\alpha} \rho_{ioja} = \rho_{ij}$$

(35)

$$\rho_{S2} = Tr_{S1} \rho_U, \quad \text{with coordinates} \quad \sum_{\beta} \rho_{ioj\beta} = \rho_{\alpha\beta}$$

(36)

From eqs.(33) and (34) we know that these coordinates also reach their final equilibrium values:
\[
\begin{align*}
\rho_{ij}(t) &= \sum_{\alpha} \rho_{i\alpha j\alpha}(t) \quad \rightarrow \quad (\rho_{*ij}) \\
\rho_{\alpha\beta}(t) &= \sum_{i} \rho_{i\alpha i\beta}(t) \quad \rightarrow \quad (\rho_{*\alpha\beta})
\end{align*}
\]

where \((\rho_{*ij})\) and \((\rho_{*\alpha\beta})\) can be conceived as the components of the final states \(\rho_{S1*}\) and \(\rho_{S2*}\) of \(S_1\) and \(S_2\) to which the reduced states \(\rho_{S1}\) and \(\rho_{S2}\) tend respectively for \(t \rightarrow \infty\) (see eq.(19)). Therefore, the subsystems \(S_1\) and \(S_2\) also decohere for their respective relevant observables (see eq.(20)):

\[
\begin{align*}
\lim_{t \rightarrow \infty} \langle O_{R1} \rangle_{\rho_{U}(t)} &= \lim_{t \rightarrow \infty} \langle O_{S1} \rangle_{\rho_{S1}(t)} = \langle O_{S1} \rangle_{\rho_{S11}}, \\
\lim_{t \rightarrow \infty} \langle O_{R2} \rangle_{\rho_{U}(t)} &= \lim_{t \rightarrow \infty} \langle O_{S2} \rangle_{\rho_{S2}(t)} = \langle O_{S2} \rangle_{\rho_{S22}}.
\end{align*}
\]

Let us note that the argument does not depend on the particular partition of \(U\) into \(S_1\) and \(S_2\). This means that, when the whole composite system decoheres according to SID, the subsystems will also decohere no matter how many degrees of freedom they have. In fact, if \(U\) is a system of \(N\) interacting oscillators, we can decide to split it into a single oscillator as \(S_1\) and the remaining \(N-1\) oscillators as \(S_2\): if \(U\) decoheres, \(S_1\) and \(S_2\) also decohere. This conclusion shows that it is not always necessary for the decoherence of an open system its interaction with an environment with many, potentially infinite, degrees of freedom: the decoherence of the whole composite system imposes a physical situation as strong as to lead to the decoherence of any of its subsystems.

**B. From the open subsystems to the closed system**

Let us consider again the closed system \(U\) partitioned into the subsystems \(S_1\) and \(S_2\) such that \(U = S_1 \cup S_2\), and whose Hilbert space \(\mathcal{H}\) can be decomposed as

\[
\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2
\]

where \(\mathcal{H}_1\) and \(\mathcal{H}_2\) are the Hilbert spaces of \(S_1\) and \(S_2\) respectively. If \(\{|i\}\) is a basis of \(\mathcal{H}_1\) \((i=1,\ldots,n)\) and \(\{|\alpha\}\) is a basis of \(\mathcal{H}_2\) \((\alpha = 1,\ldots,m)\), then a basis of \(\mathcal{H}\) is

\[
\{|A\}\} = \{|i,\alpha\}\} = \{|i\} \otimes \{\alpha\}\}
\]

For a different partition \(U = S'_1 \cup S'_2\), the Hilbert space \(\mathcal{H}\) of dimension \(N = nm = kl\), with \(n,m,k,l \in \mathbb{N}\), can be decomposed as

\[
\mathcal{H} = \mathcal{H}'_1 \otimes \mathcal{H}'_2
\]

where \(\mathcal{H}'_1\) and \(\mathcal{H}'_2\) are the Hilbert spaces of \(S'_1\) and \(S'_2\) respectively. If \(\{|I\}\) is a basis of \(\mathcal{H}'_1\) \((I = 1,\ldots,k)\) and \(\{|\Gamma\}\) is a basis of \(\mathcal{H}'_2\) \((\Gamma = 1,\ldots,l)\), then another basis of \(\mathcal{H}\) will be

\[
\{|A'\}\} = \{|I,\Gamma\}\} = \{|I\} \otimes \{\Gamma\}\}
\]

The change of basis from \(\{|A\}\) to \(\{|A'\}\) can be performed by a linear transformation \(A'_A = A'_i A'_a\) such that

\[
|A'\} = \sum_A A'_A |A\}
\]

If we want to prove that the closed composite system \(U\) decoheres, we have to find the \((nm)^2 - 1\) real numbers that define the \((nm)^2\) complex coordinates \((\rho_{*i\alpha j\beta})\) of the final state \(\rho_{U*}\) of \(U\).\(^4\) Let us assume that the open subsystems \(S_1\) and \(S_2\) respectively decohere for their relevant observables \(O_{R1} \in \mathcal{O}_1\) and \(O_{R2} \in \mathcal{O}_2\) \((or\ O_{S1} \text{ and } O_{S2})\) given

---

\(^4\)A state represented by an \(N \times N\) self-adjoint matrix has \(N - 1\) real diagonal independent coordinates and \(\frac{1}{2}(N^2 - N)\) complex off-diagonal coordinates. Therefore, the state is completely determined by \(N - 1 + 2 \left[\frac{1}{2}(N^2 - N)\right] = N^2 - 1\) real numbers.
by eqs. (23) and (24), and whose corresponding coordinates are given by eqs. (28) and (29). The decoherence of the subsystems means that (see eqs. (39) and (40))

\[
\lim_{t \to \infty} \langle O_{R_1} \rangle_{\rho_U(t)} = \langle O_{R_1} \rangle_{\rho_{S_1}} = \lim_{t \to \infty} \langle O_{S_1} \rangle_{\rho_{S_1}} = \langle O_{S_1} \rangle_{\rho_{S_1}} \\
\lim_{t \to \infty} \langle O_{R_2} \rangle_{\rho_U(t)} = \langle O_{R_2} \rangle_{\rho_{S_2}} = \lim_{t \to \infty} \langle O_{S_2} \rangle_{\rho_{S_2}} = \langle O_{S_2} \rangle_{\rho_{S_2}},
\]

(46)

(47)

where \( \rho_{S_1}(t) \) and \( \rho_{S_2}(t) \) are the reduced density operators defined in eqs. (35) and (36), and \( \rho_{S_1} \) and \( \rho_{S_2} \) are the final states to which the reduced states \( \rho_{S_1}(t) \) and \( \rho_{S_2}(t) \) tend respectively for \( t \to \infty \) (see eq. (19)). Therefore,

\[
\sum_i \rho_{*\alpha i \alpha} O_{i \alpha i \alpha}^{R_1} = \rho_{*\alpha i \alpha} S_{i \alpha i \alpha}^{S_1} \\
\sum_i \rho_{*\alpha i \beta} O_{i \alpha i \beta}^{R_2} = \rho_{*\alpha i \beta} S_{i \alpha i \beta}^{S_2}
\]

(48)

(49)

Eqs. (48) and (49) represent a system of \((n^2 - 1) + (m^2 - 1)\) real equations which, in general, is not enough to lead to the \((nm)^2 - 1\) real numbers required to define the \((nm)^2\) coordinates \( \rho_{*\alpha i \beta} \) of \( \rho_U \). Nevertheless, we can introduce a different partition of \( U: U = S'_1 \cup S'_2 \). If we repeat the argument for this new partition, we will obtain

\[
\sum_I \rho_{*I \Gamma I} O_{I \Gamma I}^{R_1} = \rho_{*I \Gamma I} S_{I \Gamma I}^{S_1} \\
\sum_I \rho_{*I \Gamma I} O_{I \Gamma I}^{R_2} = \rho_{*I \Gamma I} S_{I \Gamma I}^{S_2}
\]

(50)

(51)

which represent a system of \((k^2 - 1) + (l^2 - 1)\) real equations. Eqs. (50) and (51) are independent of eqs. (48) and (49) because the coefficients of the first equations are the \( O_{ij}^{S_1}, S_{ij}^{S_2} \), the coefficients of the last equations are the \( O_{ij}^{S_1}, S_{ij}^{S_2} \), and these groups of coefficients are not related by a linear transformation. Therefore, now we have obtained \( n^2 + m^2 + k^2 + l^2 - 4 \) real equations: if this number of equations is enough to obtain the \((nm)^2 - 1\) real numbers required to define the coordinates of \( \rho_U \), then we have proved the decoherence of the whole system \( U \) for the observables belonging to \( O_1 \cup O_2 \cup O'_1 \cup O'_2 \subset O_0 \). If the \( n^2 + m^2 + k^2 + l^2 - 4 \) real equations are not yet sufficient to define \( \rho_U \), we can introduce further partitions up to reach the necessary number of equations. \(^5\)

This argument can be used to study well-known models. For instance, let us consider a set of \( N \) oscillators, each one with its corresponding Hilbert space \( \mathcal{H}_i \) and the creation and annihilation operators \( a_i^\dagger, a_i; \) these operators define the bases of the \( \mathcal{H}_i \) from a common vacuum \( |0 \rangle \). In this case, the complete Hilbert space is \( \mathcal{H} = \bigotimes_i \mathcal{H}_i \). But if we make a Bogoliubov transformation:

\[
a_i^\dagger = \sum_j \alpha_i^\dagger a_j + \beta_i a_j^\dagger, \quad a_i = \sum_j \alpha_i a_j^\dagger + \beta_i^\dagger a_j
\]

(52)

the new operators \( a_i^\dagger, a_i \) define the corresponding bases of the new Hilbert spaces \( \mathcal{H}_i \) and, therefore, they introduce a different partition \( \mathcal{H} = \bigotimes_i \mathcal{H}_i \) of the whole composite system. \(^6\) In this physical example, the argument developed in this subsection can be easily applied.

Summing up, this argument shows that the decoherence of the subsystems of a closed system is not a phenomenon of a different nature than or independent of the decoherence of the whole composite system; on the contrary, there is a close relationship between both phenomena. Therefore, the decoherence of the whole closed system can be understood by studying the behavior of its subsystems.

\(^5\) Let us consider the \( n, m \) partitions corresponding to different rotated bases \{\(|A_i^\dagger \rangle\} \}. In this case, the necessary number of partitions is:

\[
N = \frac{(nm)^2 - 1}{n^2 + m^2 - 2} \approx \left( \frac{1}{n^2} + \frac{1}{m^2} \right)^{-1}
\]

when \( m \gg 1 \). If we keep \( n \) finite and make \( m \to \infty \), we just need a finite number of partitions \( N = n^2 \).

\(^6\) If \( \beta_i = 0 \), we remain in the same decomposition since \( a_i = \sum_j \alpha_i a_j \) produces just a change of basis in \( \mathcal{H}_i \).
IV. DECOHERENCE TIMES

In the previous sections we have showed how the SID and the EID approaches to decoherence can be understood from a general theoretical framework, and we have argued that there is a close link between the decoherence of a whole closed system and the decoherence of its subsystems. Clearly, if this is the case, there must be a meaningful relationship between the decoherence times of the whole system and of its subsystems. This is the point that we will address in this section.

A. EID: decoherence time in open systems

In several models studied by the EID approach, the decoherence time \( t_{DS} \) of an open subsystem \( S \) in interaction with its environment \( E \) turns out to be the relaxation time \( t_R \) of the whole system \( S \cup E \) multiplied by a macroscopicity coefficient. For instance, in eq.(47) of [21] or in eq.(3.136) of [8],

\[
    t_{DS} = \left( \frac{\lambda_{DB}}{L_0} \right)^2 t_R
\]

where \( \lambda_{DB} \) is the de Broglie length and \( L_0 \) is a macroscopic characteristic length. In turn, in page 51 of [21],

\[
    t_{DS} = \left( \frac{\Delta x}{2L_0} \right)^2 t_R
\]

where \( \frac{\Delta x}{L_0} \) is the ratio between a microscopic and a macroscopic characteristic lengths. In any case, \( t_D \) is extremely short since the macroscopicity ratios \( \frac{\lambda_{DB}}{L_0} \) or \( \frac{\Delta x}{L_0} \) are extremely small (e.g. \( 10^{-20} \), see [21]). Therefore, \( t_D \ll t_R \). 

B. SID: decoherence time in closed systems

In paper [19], we have computed the decoherence time \( t_{DU} \) of a closed system \( U \) in terms of the poles of the Hamiltonian resolvent and of the initial conditions; in particular, we have shown that, if the Hamiltonian and the initial conditions are trivial -that is, with just real poles-, the decoherence time is infinite. In the Appendix B of that paper, we have applied the method to a two-times evolution (application that can be easily generalized to a n-times evolution). In this subsection we will rephrase that appendix to show that, for an adequate choice of the interactions, the characteristic times of the two-times evolution are the decoherence time \( t_{DS} \) of the open proper system and the decoherence time \( t_{DU} \) of the closed composite system.

Let us consider a closed system \( U \) partitioned into an open system \( S \) and its environment \( E \), whose total Hamiltonian reads

\[
    H = H_0 + V = H_0 + \int_0^\infty \int_0^\infty [V^{(1)}(\omega,\omega') + V^{(2)}(\omega,\omega')] |\omega\rangle \langle \omega'| d\omega d\omega'
\]

where \( H_0 \) is the free trivial Hamiltonian of \( U \), \( \{ |\omega\rangle \} \) is its eigenbasis, \( V^{(1)} \) represents the interaction between \( S \) and \( E \), and \( V^{(2)} \) represents the interaction of the parts of the environment \( E \) among themselves.\(^7\) We will also assume that \( V^{(1)}(\omega,\omega') \gg V^{(2)}(\omega,\omega') \). This relationship holds in many cases of interest, e.g.:

- The Hamiltonian \( H \) given by the eq.(64) of the next section, where \( V^{(2)}(\omega,\omega') = 0 \).
- The Hamiltonian \( H \) given by the eq.(1) of paper [22], where the Hamiltonian of the proper system is \( \Omega |1\rangle \langle 1| \), the Hamiltonian of the environment is \( \int_0^\infty \omega |\omega\rangle \langle \omega'| d\omega \), and

\[
    \int_0^\infty \int_0^\infty V^{(1)}(\omega,\omega') d\omega d\omega' \to \int_0^\infty g(\omega)(|1\rangle \langle 1| + |\omega\rangle \langle \omega|) d\omega, \quad V^{(2)}(\omega,\omega') \to 0
\]

\(^7\)Eq. (55) is written with integrals for simplicity, but it could also be expressed with sums in a discrete case.
The Hamiltonian $H$ given by the eq. (2.1) of paper [23], where the Hamiltonian of the proper system is $\Omega a^\dagger a$, the Hamiltonian of the environment is $\int_0^\infty \omega b_k^\dagger b_k dk$, and

$$
\int_0^\infty \int_0^\infty V^{(1)}(\omega, \omega') \, d\omega d\omega' \to \int_0^\infty g(k)(a^\dagger b_k + b_k^\dagger a) \, dk, \quad V^{(2)}(\omega, \omega') \to 0
$$

(57)

a) First interaction: Since $V^{(1)} \gg V^{(2)}$, in a first step we can neglect $V^{(2)}$ and consider the Hamiltonian (see [19])

$$
H^{(1)} = H_0 + V_1 = \int_0^\infty \omega |\omega\rangle \langle \omega| \, d\omega + \int_0^\infty \int_0^\infty V^{(1)}(\omega, \omega') |\omega\rangle \langle \omega'| \, d\omega d\omega'
$$

(58)

The eigenbasis $\{ |\omega\rangle^{(1)} \}$ of $H^{(1)}$ is obtained as

$$
|\omega\rangle^{(1)} = |\omega\rangle + \frac{1}{\omega + i0 - H^{(1)}} V^{(1)} |\omega\rangle
$$

(59)

If we pre-multiply by $\langle \varphi|$

$$
\langle \varphi| |\omega\rangle^{(1)} = \langle \varphi| |\omega\rangle + \langle \varphi| \frac{1}{\omega + i0 - H^{(1)}} V^{(1)} |\omega\rangle
$$

(60)

we can compute the analytical continuation of eq. (60) in the lower half-plane:

$$
\langle \varphi| \langle z|^{(1)} = \langle \varphi| z \rangle + \langle \varphi| \frac{1}{z - H^{(1)}} V^{(1)} |z\rangle
$$

(61)

and obtain the complex poles of such an analytical continuation. An analogous procedure can be followed to obtain the complex poles of the initial condition $\rho_0$. On this basis, the decoherence time $t_D^{(1)}$ resulting from the first interaction turns out to be $t_D^{(1)} = \hbar/\gamma^{(1)}$, where $\gamma^{(1)}$ is the imaginary part of the pole closer to the real axis (see [19]). Therefore, for times $t \gg t_D^{(1)}$, the state $\rho(t)$ can be considered nearly diagonal for all practical purposes.\(^8\)

b) Second interaction: However, the state $\rho(t)$ has not completely decohered yet, because the interaction $V^{(2)}$, even if very small, is still present. Then, after the first period where $V^{(1)}$ is dominant, for times $t \gg t_D^{(1)}$, $V^{(2)}$ becomes relevant; in this situation, the total Hamiltonian can be written as

$$
H = H^{(1)} + V_2 = \int_0^\infty \omega |\omega\rangle^{(1)} \langle \omega|^{(1)}_\dagger \, d\omega + \int_0^\infty \int_0^\infty V^{(2)}(\omega, \omega') |\omega\rangle^{(1)} \langle \omega'|^{(1)}_\dagger \, d\omega d\omega'
$$

(62)

where $V^{(2)}(\omega, \omega')$ is $V^{(2)}(\omega, \omega')$ in the new basis $\{ |\omega\rangle^{(1)} \}$. Now, the eigenbasis $\{ |\omega\rangle^{+} \}$ of $H$ is obtained as

$$
|\omega\rangle^{+} = |\omega\rangle^{(1)} + \frac{1}{\omega + i0 - H^{(2)}} V^{(2)} |\omega\rangle^{(1)}
$$

By repeating the procedure applied in the case of the first interaction, we can compute the decoherence time $t_D^{(2)}$ obtained by taking into account the second interaction, which results $t_D^{(2)} = \hbar/\gamma^{(2)}$, where $\gamma^{(2)}$ is again the imaginary part of the pole closer to the real axis (see [19]). For times $t \gg t_D^{(2)}$, the state $\rho(t)$ can be considered completely diagonal.

c) Estimating and comparing results: As proved in paper [19], $\gamma^{(1)}$ and $\gamma^{(2)}$ are proportional to the corresponding interactions; therefore, the decoherence times $t_D^{(1)}$ and $t_D^{(2)}$ are proportional to the inverse of the interaction.

\(^8\)For a complete example of this point, see in paper [20] the exhaustive analysis of the way in which the Friedrich model reaches equilibrium in the discrete and in the continuous case. In particular, see figures 3-7 and the computation of the pole in eq. (39). Nevertheless, in that paper there is no reference to decoherence because at that time the present analysis of the problem was not yet developed.
If $V^{(1)}$ is a macroscopic interaction, then $t^{(1)}_D \approx 10^{-39} s$; in turn, if $V^{(2)}$ is a microscopic interaction, the decoherence time $t^{(1)}_D$ may be of the order of $10^{-15} s$ (see details in [19]). As expected, $t^{(1)}_D \ll t^{(2)}_D$.

When these general results are applied to our case, where a closed system $U$ is partitioned into a proper system $S$ and an environment $E$, they acquire a new meaning. In fact, since $V^{(1)}$ represents the interaction between $S$ and $E$, $t^{(1)}_D$ turns out to be the decoherence time $t_{DS}$ of the system $S$ in interaction with its environment $E$. In turn, since $V^{(2)}$ represents the interaction of the parts of the environment $E$ among themselves, when it is included in the total Hamiltonian (see eq.(62)), the time $t^{(2)}_D$ turns out to be the decoherence time $t_{DU}$ of the whole composite system $U = S \cup E$. As expected, $t_{DS} \ll t_{DU}$: in general, the time that a whole system needs to decohere is much longer than the decoherence time of a small subsystem strongly coupled with the rest of the degrees of freedom.

Summing up, from this general perspective we can describe a two-times process, with an extremely short decoherence time for the open subsystem $S$, and a long (even infinite, if $V^{(2)} = 0$) decoherence time for the whole closed system $U$.

V. A WELL-KNOWN MODEL

In this section we will apply the general theoretical framework just presented to an example extensively treated in the literature on decoherence. This task will allow us to draw certain conceptual conclusions that may remain obscure when the model is studied exclusively by means of numerical techniques.

Let us consider a system $S$ consisting in a single spin-1/2 particle $S_0$ (with Hilbert space $\mathcal{H}_0$), and its environment $E$ composed by a collection of $N$ spin-1/2 particles $S_i$ (with Hilbert spaces $\mathcal{H}_i$). If the free Hamiltonians of the system and the environment are assumed to be zero:

$$H_S = H_E = 0$$

the total Hamiltonian $H = H_S + H_E + H_{SE}$ of the composite system $U = S \cup E$ reads (see [6] [16])

$$H = H_{SE} = \frac{1}{2} \sum_{i=1}^{N} g_i (| \uparrow_i \rangle \langle \uparrow_i | - | \downarrow_i \rangle \langle \downarrow_i |) \bigotimes_{j \neq i} I_j$$

Let us consider a pure initial state of the system $U$:

$$| \psi_0 \rangle = (a | 0 \rangle + b | 1 \rangle) \bigotimes_{i=1}^{N} (\alpha_i | \uparrow_i \rangle + \beta_i | \downarrow_i \rangle)$$

where $\alpha_i$ and $\beta_i$ are aleatory coefficients such that $|\alpha_i|^2 + |\beta_i|^2 = 1$. The state $| \psi_0 \rangle$ evolves as

$$| \psi(t) \rangle = a | 0 \rangle | \mathcal{E}_0(t) \rangle + b | 1 \rangle | \mathcal{E}_1(t) \rangle$$

where

$$| \mathcal{E}_0(t) \rangle = | \mathcal{E}_1(-t) \rangle = \bigotimes_{i=1}^{N} (\alpha_i e^{i \epsilon_{\uparrow i} t/2} | \uparrow_i \rangle + \beta_i e^{-i \epsilon_{\downarrow i} t/2} | \downarrow_i \rangle)$$

The density matrix corresponding to this state will be $\rho(t) = | \psi(t) \rangle \langle \psi(t) |$.

Now we will analyze this model from the perspective given by steps 1 to 3 of Section II.

1. **First step:** The relevant observables $O_R \in \mathcal{O}$ for this case will be of the form (see eq.(18) of [16])

$$O_R = (s_0 | 0 \rangle \langle 0 | + s_1 | 1 \rangle \langle 1 | + s_{11} | 1 \rangle \langle 1 |) \bigotimes_{i=1}^{N} (\epsilon_{\uparrow i} | \uparrow_i \rangle \langle \uparrow_i | + \epsilon_{\downarrow i} | \downarrow_i \rangle \langle \downarrow_i | + \epsilon_{\uparrow \downarrow} | \uparrow_i \rangle \langle \downarrow_i | + \epsilon_{\downarrow \uparrow} | \downarrow_i \rangle \langle \uparrow_i |)$$

where $s_0$, $s_{11}$, $\epsilon^{(i)}_{\uparrow \downarrow}$, $\epsilon^{(i)}_{\downarrow \uparrow}$ are real numbers, and $s_0 = s_{11}^* \epsilon_{10}^{(i)} \epsilon_{10}^{(i)\ast}$ are complex numbers.

2. **Second step:** The expectation value of any observable $O_R \in \mathcal{O}$ in the state $| \psi(t) \rangle$ can be computed as

$$\langle O_R \rangle_{\psi(t)} = (|a|^2 s_0 + |b|^2 s_{11}) \Gamma_0(t) + 2 \text{Re} [ab^* s_{10} \Gamma_1(t)]$$

(69)
where

\[
\Gamma_0(t) = \prod_{i=1}^{N} \left[ |\alpha_i|^2 \epsilon_{\uparrow\uparrow}^{(i)} + |\beta_i|^2 \epsilon_{\downarrow\downarrow}^{(i)} + \alpha_i^* \beta_i \epsilon_{\uparrow\downarrow}^{(i)} e^{-ig_i t} + (\alpha_i^* \beta_i \epsilon_{\downarrow\uparrow}^{(i)})^* e^{ig_i t} \right]
\] (70)

\[
\Gamma_1(t) = \prod_{i=1}^{N} \left[ |\alpha_i|^2 \epsilon_{\uparrow\downarrow}^{(i)} e^{ig_i t} + |\beta_i|^2 \epsilon_{\downarrow\uparrow}^{(i)} e^{-ig_i t} + \alpha_i^* \beta_i \epsilon_{\uparrow\downarrow}^{(i)} + (\alpha_i^* \beta_i \epsilon_{\downarrow\uparrow}^{(i)})^* \right]
\] (71)

At this point, we will consider two particular cases:

**Case (a):** When \( \epsilon_{\uparrow\uparrow}^{(i)} = \epsilon_{\uparrow\downarrow}^{(i)} = 1 \) and \( \epsilon_{\downarrow\downarrow}^{(i)} = 0 \), the model is a typical example for EID, where the relevant observables are only those corresponding to the proper system \( S = S_0 \) strongly coupled with its environment. In fact, these relevant observables \( O_{R_0} \in O_0 \) read (see eq.(15))

\[
O_{R_0} = \left( \sum_{s,s'=0,1} s_{ss'} |s\rangle \langle s'| \right) \bigotimes_{i=1}^{N} I_i = O_{S_0} \bigotimes_{i=1}^{N} I_i
\] (72)

and their expectation values in the state \( \psi(t) \) of \( U \) result

\[
\langle O_{R_0} \rangle_{\psi(t)} = |a|^2 s_{00} + |b|^2 s_{11} + \text{Re}[ab^* s_{10} r(t)]
\] (73)

where

\[
r(t) = \langle E_1(t) | E_0(t) \rangle
\] (74)

and

\[
|r(t)|^2 = \prod_{i=1}^{N} (|\alpha_i|^4 + |\beta_i|^4 + 2|\alpha_i|^2|\beta_i|^2 \cos 2g_i t)
\] (75)

**Case (b):** However, we can also decide to "observe" just one particle \( S_j \) of the environment, that is, to consider the observables corresponding to \( S_j \) as the relevant ones. These relevant observables \( O_{R_j} \in O_j \) read (see eq.(15))

\[
O_{R_j} = I_0 \otimes O_{S_j} \bigotimes_{i \neq j} I_i
\] (76)

where

\[
O_{S_j} = \epsilon_{\uparrow\uparrow}^{(j)} |\uparrow_j\rangle \langle \uparrow_j | + \epsilon_{\downarrow\downarrow}^{(j)} |\downarrow_j\rangle \langle \downarrow_j | + \epsilon_{\uparrow\downarrow}^{(j)} |\uparrow_j\rangle \langle \downarrow_j | + \epsilon_{\downarrow\uparrow}^{(j)} |\downarrow_j\rangle \langle \uparrow_j | + \epsilon_{\uparrow\downarrow}^{(j)} |\uparrow_j\rangle \langle \uparrow_j | + \epsilon_{\downarrow\uparrow}^{(j)} |\downarrow_j\rangle \langle \downarrow_j |
\] (77)

and \( \epsilon_{\uparrow\uparrow}^{(j)} \), \( \epsilon_{\downarrow\downarrow}^{(j)} \), \( \epsilon_{\uparrow\downarrow}^{(j)} \) are now generic. In this case, the expectation value of \( O_{R_j} \) in the state \( \psi(t) \) reads

\[
\langle O_{R_j} \rangle_{\psi(t)} = \langle \psi(t) | O_{R_j} | \psi(t) \rangle = |a|^2 (|\alpha_i|^2 \epsilon_{\uparrow\uparrow}^{(j)} + |\beta_i|^2 \epsilon_{\downarrow\downarrow}^{(j)} + \alpha_j \beta_i \epsilon_{\uparrow\downarrow}^{(j)} e^{-ig_j t} + \alpha_j^* \beta_i^* \epsilon_{\downarrow\uparrow}^{(j)} e^{ig_j t}) + |b|^2 (|\alpha_i|^2 \epsilon_{\uparrow\downarrow}^{(j)} + |\beta_i|^2 \epsilon_{\downarrow\uparrow}^{(j)} + \alpha_j \beta_i \epsilon_{\uparrow\downarrow}^{(j)} e^{-ig_j t} + \alpha_j^* \beta_i^* \epsilon_{\downarrow\uparrow}^{(j)} e^{ig_j t})
\] (78)

3. **Third step:** The time evolution of the expectation values of the relevant observables can be computed in both cases:

**Case (a):** Since

\[
\max(|\alpha_i|^4 + |\beta_i|^4 + 2|\alpha_i|^2|\beta_i|^2 \cos 2g_i t) = 1
\] (79)

\[
\min(|\alpha_i|^4 + |\beta_i|^4 + 2|\alpha_i|^2|\beta_i|^2 \cos 2g_i t) = (2|\alpha_i|^2 - 1)^2
\] (80)

\(|\alpha_i|^4 + |\beta_i|^4 + 2|\alpha_i|^2|\beta_i|^2 \cos 2g_i t|\) is an aleatory number that, if \( t \neq 0 \), fluctuates between 1 and \((2|\alpha_i|^2 - 1)^2\). Then, from eq.(75) we can conclude that, for \( N \to \infty \),

\[
\lim_{t \to \infty} r(t) = 0
\] (81)
In this paper we have presented a common theoretical framework that encompasses both EID and SID, and probably other decoherence approaches. When it is accepted that the formalisms of decoherence for open and closed systems

Another decoherence approaches. When it is accepted that the formalisms of decoherence for open and closed systems

S decohere, then we can be sure that the whole composite system will neither decohere. This is precisely the case of to imply the decoherence of any of its subsystems. Therefore, if any subsystem of a closed composite system does not

end of Subsection III.A, the decoherence of the whole composite system imposes a physical condition strong enough to imply the decoherence of any of its subsystems. Therefore, if any subsystem of a closed composite system does not decohere, then we can be sure that the whole composite system will neither decohere. This is precisely the case of our model: since S decoheres but E does not decohere, the composite system U = S ∪ E cannot decohere, that is, it has an infinite decoherence time resulting from the trivial interaction among its parts.

Now, this result allows us to conceptually infer the behavior of the whole composite system. As we conclude at the end of Subsection III.A, the decoherence of the whole composite system imposes a physical condition strong enough to imply the decoherence of any of its subsystems. Therefore, if any subsystem of a closed composite system does not decohere, then we can be sure that the whole composite system will neither decohere. This is precisely the case of our model: since S decoheres but E does not decohere, the composite system U = S ∪ E cannot decohere, that is, it has an infinite decoherence time, t_{DS} ≪ t_{DU} = ∞.9

Summing up, the arguments presented in this section show that, when this well-known model is analyzed in the context of our theoretical framework, the results obtained in the special case can be viewed from a new general perspective. In particular, certain results computed by means of numerical techniques and which may seem puzzling when considered in isolation, turn out to be necessary conceptual consequences of the full understanding of the physical phenomenon.

VI. CONCLUSIONS

In this paper we have presented a common theoretical framework that encompasses both EID and SID, and probably other decoherence approaches. When it is accepted that the formalisms of decoherence for open and closed systems

\[ W - \lim_{t \to \infty} \rho(t) = \rho_0 \]

\[ \lim_{t \to \infty} \langle O_{R_0} \rangle_{\psi(t)} = \langle O_{R_0} \rangle_{\rho_0} = \lim_{t \to \infty} \langle O_{S_0} \rangle_{\rho_{S_0}}(t) = \langle O_{S_0} \rangle_{\rho_{S_0_0}} \]

\[ \rho_{S_0_0} = \begin{pmatrix} |a|^2 & 0 \\ 0 & |b|^2 \end{pmatrix} \]

This result shows that, as expected, the system S = S_0 in interaction with the environment E decoheres in the eigenbasis of \( \rho_{S_0_0} \).

Case (b): If we decide to "observe" the particle S_j of the environment, we have to consider the evolution of the expectation value of the corresponding relevant observables \( O_{R_j} \in O_j \). Eq.(78) shows that \( \langle O_{R_j} \rangle_{\psi(t)} \) just oscillates and, therefore, it has no limit for \( t \to \infty \). As a consequence, a generic particle S_j of the environment does not decohere. This result is completely foreseeable from a physical point of view: to the extent that the particles S_i of the environment E = \( \bigcup_j S_j \) are uncowpled to each other, they freely evolve; therefore, the environment composed by these freely evolving particles is unable to reach a final decohered state.

The results just obtained point to the fact that we can gain a better understanding of the behavior of a closed system by studying the behavior of its subsystems. In our case, the behavior of the whole system U can be completely described by analyzing only the observables corresponding to cases (a) and (b). In fact, the total Hamiltonian \( H \) of eq.(64) is not symmetric with respect to the particle S_0 and the generic particle S_j: whereas S_0 is coupled to all the S_i of the environment, the S_j do not interact among themselves and they are only coupled to S_0 (and this coupling vanishes when \( N \to \infty \)). Therefore, S = S_0 decoheres in interaction with the environment E in a finite decoherence time \( t_{DS} \), but the environment \( E \) does not decohere, that is, it has an infinite decoherence time resulting from the trivial interaction among its parts.

Now, this result allows us to conceptually infer the behavior of the whole composite system. As we conclude at the end of Subsection III.A, the decoherence of the whole composite system imposes a physical condition strong enough to imply the decoherence of any of its subsystems. Therefore, if any subsystem of a closed composite system does not decohere, then we can be sure that the whole composite system will neither decohere. This is precisely the case of our model: since S decoheres but E does not decohere, the composite system U = S ∪ E cannot decohere, that is, it has an infinite decoherence time, t_{DS} ≪ t_{DU} = ∞.9

Summing up, the arguments presented in this section show that, when this well-known model is analyzed in the context of our theoretical framework, the results obtained in the special case can be viewed from a new general perspective. In particular, certain results computed by means of numerical techniques and which may seem puzzling when considered in isolation, turn out to be necessary conceptual consequences of the full understanding of the physical phenomenon.

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In this paper we have presented a common theoretical framework that encompasses both EID and SID, and probably other decoherence approaches. When it is accepted that the formalisms of decoherence for open and closed systems

9When this well-known model is understood from our general perspective, the criticisms to SID presented in paper \[16\] vanish (for a detailed criticism of [16], see [24]). In fact, the model does not show that the destructive interference of the off-diagonal terms is not always efficient, as claimed, but it merely proves that the closed system does not decohere because the Hamiltonian of the environment is trivial, and this -physically obvious- fact is perfectly explained in the context of the SID approach.
cooperate in the understanding of the same physical phenomenon, the results obtained by means of the different theories of decoherence can be retained as relevant acquisitions: for instance, the large amount of experimental confirmations of EID (see [8]), or the complete description of the classical limit of quantum mechanics [25] and the study of the role of complexity in decoherence [26] in the case of SID, or the extremely short decoherence times computed by EID and SID [19].

In turn, from the new general perspective just proposed, the difficulties that the EID approach has to face (see Introduction) are not as serious as originally supposed. In fact,

- It can be explained that closed systems do decohere; moreover, the decoherence time of a closed system can be computed.
- The serious conceptual problem of deciding where to place the cut between "the" proper system S and "the" environment E in a composite system $U = S \cup E$ is dissolved, because the closed system U may be partitioned in several different ways: no one of them is the "true" or "correct" partition. When we want to understand the behavior of the whole system U, we have to select certain partitions to study the resulting subsystems: the choice of the relevant partitions is guided by the inspection of the total Hamiltonian.
- The final pointer bases in which the open and the closed systems decohere are well defined in the corresponding weak limits, with complete generality.

Finally, it is interesting to note that, since from this general perspective decoherence is conceived as a particular case of irreversible phenomena, the conclusions drawn in this paper may serve to illuminate some traditional issues of the problem of irreversibility. For instance, the old question about the criterion for selecting the relevant macroscopic variables of a system may receive a simple answer: we can choose different sets of macroscopic variables, but we have to study the behavior of more than one set if we want to reach the understanding of the behavior of the whole irreversibly evolving system.

VII. ACKNOWLEDGMENTS

We are very grateful to Roland Omnés and Maximilian Schlosshauer for many comments and criticisms. This research was partially supported by grants of the University of Buenos Aires, the CONICET and the FONCYT of Argentina.

APPENDIX A: COARSE-GRAINING AND PROJECTION

As it is well-known, a coarse-graining amounts to a projection whose action is to eliminate some components of the state vector corresponding to the finer description. If this idea is generalized, coarse-graining can be conceived as a projection that reduces the number of components of a generalized vector representing a state. In the light of this idea, in Section II we have argued that, for any observable $O_R$ belonging to the space $\mathcal{O}$ of relevant observables, the expectation value of $O_R$ in the state $\rho(t)$ can be expressed in terms of a coarse-grained state $\rho_G(t)$ such that

$$\langle O_R \rangle_{\rho(t)} = \langle O_R \rangle_{\rho_G(t)}.$$  \hfill (A1)

In this Appendix, we will prove that: (i) the $\rho_G(t)$ so defined is the result of the projection of the state $\rho(t)$ onto the space $\mathcal{O}$ of relevant observables, (ii) the final state $\rho_{G*}$ of $\rho_G(t)$ is the result of the projection of the final state $\rho_*$ of $\rho(t)$ onto $\mathcal{O}$, and (iii) when $\mathcal{O}$ has a finite number of dimensions, for $t \to \infty$, $\rho_G(t)$ tends to $\rho_{G*}$ not only in a weak sense but also in a strong sense.

1. Let us use the notation $\langle O \rangle_{\rho} = \langle \rho \vert O \rangle$. Let the basis of $\mathcal{O}$ be $\{\langle R_i \rangle\}$, and let us define a projector

$$\pi = \sum_{i=1}^n |O_{R_i}\rangle\langle \rho_i|, \quad (\rho_i|O_{R_j}\rangle = \delta_{ij}$$  \hfill (A1)

where the $\langle \rho_i \rangle$ are functionals defined by $\langle \rho_i \vert O_{R_j} \rangle = \delta_{ij}$.\footnote{If we are working in a finite dimensional space $\mathcal{O}$, we can choose $\langle R_j \rangle = |\alpha\rangle\langle \beta|$ with $\langle \alpha\vert \beta \rangle = \delta_{\alpha\beta}$ and, then, $\rho_i = |\alpha\rangle\langle \beta|$.} Obviously, $\pi^2 = \pi$. Then, we define

$$\rho_G(t) = (\rho(t)|\pi$$  \hfill (A2)
Now,
\[(\rho_G(t)|O_{Rj}) = (\rho(t)|\pi|O_{Rj}) = (\rho(t)|\sum_{i=1}^{n}|O_{Ri})(\rho_i|O_{Rj}) = (\rho(t)|\sum_{i=1}^{n}|O_{Ri})\delta_{ij} = (\rho(t)|O_{Rj})\] (A3)

Then, making linear combinations, we obtain
\[(\rho(t)|O_R) = (\rho_G(t)|O_R), \text{ if } |O_R| \in \mathcal{O}\] (A4)

(ii).- But now
\[(\rho_G(t)|O_{Rj}) = (\rho(t)|\pi|O_{Rj}) = \sum_{i=1}^{n}(\rho(t)|O_{Ri})(\rho_i|O_{Rj})\] (A5)

So, using eq.(1), we obtain
\[
\lim_{t \to \infty} (\rho_G(t)|O_{Rj}) = \lim_{t \to \infty} (\rho(t)|\pi|O_{Rj}) = \lim_{t \to \infty} \sum_{i=1}^{n}(\rho(t)|O_{Ri})(\rho_i|O_{Rj}) = \\
= \sum_{i=1}^{n}(\rho_*(|O_{Ri})(\rho_i|O_{Rj}) = (\rho_*|\pi|O_{Rj}) = (\rho_*|O_{Rj})\] (A6)

where we have defined
\[(\rho_*) = (\rho_*|\pi)\] (A7)

And from eq.(A6) we obtain
\[W - \lim_{t \to \infty} (\rho_G(t)| = (\rho_G*)\] (A8)

(iii).- In the special case that \(\mathcal{O}\) as a finite number \(n\) of dimensions, we can compute the finite number of coordinates of \((\rho_G(t)|\) and \((\rho_G*|):\] \[\rho_{Gi}(t) = (\rho_G(t)|O_{Ri}), \quad \rho_{Gi*} = (\rho_G*|O_{Ri})\] (A9)

So, from eq.(A6),
\[
\lim_{t \to \infty} \rho_{Gi}(t) = \rho_{Gi*}\] (A10)

These are simple limits of the coordinates. But, since \(\mathcal{O}\) has finite dimension, we obtain the strong limit
\[S - \lim_{t \to \infty} (\rho_G(t)| = (\rho_G*)\] (A11)

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11In fact, with the notation of the last footnote, \(\rho_{Gi}(t) = (\rho_G(t)|O_{Ri}) = Tr(\rho_G(t)O_{Ri}) = \langle \beta|\rho_G(t)|\alpha \rangle = \rho_{G\beta\alpha}(t)\).
