On the numerical solution of Burgers-Fisher equation by the Strang Splitting Method

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Abstract. In this paper, we apply the Strang splitting method to approximate the solution of Burgers-Fisher equation. This method offers a second-order accurate approximation to evolution equation by combining two non-commuting operators. This accuracy is achieved through a symmetric decomposition in which one operator is applied twice for half timesteps, and the other operator is applied once for a full timestep. The stability criteria is derived using the invariant region. The numerical results obtained are compared with the exact solution. The numerical error shows that the exact and the numerical solution are agrees with each other with a good accuracy.

1. Introduction
The Burgers-Fisher equation is a non linear partial differential equation that models various phenomena for example fluid dynamics, gas dynamics, heat transfers and etc. This equation is a mix of parabolic-hyperbolic type of nonlinear partial differential equation which consist of diffusion term, reaction term, and convection term. Many methods has been developed to find the solution of this problem. Ismail et. al [1], Wazwaz and Gorguis [2] studied Adomian decomposition method for Burgers-Huxley and Burgers-Fisher equations. Wazwaz [3] use the Tanh method to study the generalized forms of nonlinear heat conduction and Burgers-Fisher equations. In [4, 5] Javidi and Golbabai studied spectral collocation method and spectral domain decomposition method for the solution of the generalized Burger-Fisher equation. From the numerical approach many method has been used to solve this equation. Mickens [6] uses a non-standard finite difference scheme. Kaya and Sayed [7] introduced a numerical simulation and explicit solutions of the generalized Burgers-Fisher equation. Chandraker et. al. [8] use various finite difference approach and compare the results. We use the Strang Splitting Method (SSM) which developed by Strang [9] because of its advantageous in giving the second order accurate approximation. This method has been applied to solve several non-linear equation such as Benjamin-Bona-Mahoney equation [10], Korteweg-de Vries equation [11], Fishers equation [12], Vlasoc-type equation [13], and etc.

In this paper we derive the stability condition for the Finite difference method applied to Burger-Fisher equation using invariant region similar to the approach used in [14] for Fishers equation. We provide the scheme, application, and error calculation of the Strang Splitting Method that derived from applying the finite difference method to each term of the equation. Then the standard Euler method is applied to perform the time integration due to its simplicity.
Therefore the Strang splitting method reads as follows:

c(t) = (A + B)c(t), \quad t \in \mathbb{R}^+ \cup \{0\},
c(0) = c_0 \in X

(1)

hence the exact solution of eq. 1 is

c(t) > \text{arbitrary but fixed}

Let

\[ C \in \mathbb{X} \]

2. Outline of the Method

Let \( X \) be a Banach space, \( A, B : X \to X \) be a bounded linear operators. Consider the abstract Cauchy problem in \( X \)

\[ c'(t) = (A + B)c(t), \quad t \in \mathbb{R}^+ \cup \{0\}, \]
\[ c(0) = c_0 \in X \quad (1) \]

The stability condition of the finite difference method applied to the Burgers-Fisher equation and the application of the Strang Splitting Method are described in section 3 and 4. Finally the concluding remarks and some open problems are described in section 5.

3. Stability Analysis and Invariant Region

Let \( x^k = a + k \Delta x \) be a partition of spatial interval \([a, b]\) with \( \Delta x = \frac{b-a}{K} \) and \( k = 0, 1, 2, \ldots, K \). Consider the Burger-Fisher equation of the form

\[ u_t = u_{xx} + uu_x + u(1 - u), \quad t \in (0, T), \quad x \in (a, b) \]

(12)

Let \( u^n_k \) denote an approximation of \( u(t^n, x^k) \) and \( u_{xx} \approx \frac{u^n_{k+1} - 2u^n_k + u^n_{k-1}}{(\Delta x)^2} \) and \( u_x \approx \frac{u^n_{k+1} - u^n_k}{\Delta x} \). Then an explicit finite difference scheme for eq. 12 is

\[ u^{n+1}_k = (r + s)u^{n+1}_{k+1} + (1 - 2r - s)u^n_k + ru^n_{k-1} + hu^n_k(1 - u^n_k) \]

(13)
where \( r = \frac{h}{(\Delta x)^2} \) and \( s = \frac{h}{\Delta x} \). To derive the stability condition for scheme 13 we start by considering a fixed time \( t^n \) and assume that

\[
0 \leq u^n_k \leq 1 \quad \text{for} \quad k = 0, 1, 2, \ldots, K \tag{14}
\]

Now let us define the following auxiliary functions:

\[
A(u) = 2r + s + (1 - 2r - s)u + h\ u(1 - u)
\]

and

\[
B(u) = (1 - 2r - s)u + h\ u(1 - u).
\]

Using the scheme 13 and the assumptions 14 we have

\[
u^{n+1}_k \leq 2r + s + (1 - 2r - s)u^n_k + h\ u^n_k(1 - u^n_k) = A(u^n_k) \tag{15}
\]

and

\[
u^{n+1}_k \geq (1 - 2r - s)u^n_k + h\ u^n_k(1 - u^n_k) = B(u^n_k). \tag{16}
\]

Hence,

\[
A'(u) = B'(u) = (1 - 2r - s) + (1 - 2u)h \geq 1 - 2r - s - h > 0
\]

for all \( u \in [0, 1] \). From here we have the following stability condition for the scheme 13

\[
h < \frac{(\Delta x)^2}{2 + \Delta x + (\Delta x)^2}. \tag{17}
\]

Since \( A \) and \( B \) are strictly increasing functions, it follows from 15 and 16 that

\[
u^{n+1}_k \leq A(u^n_k) \leq A(1) = 1
\]

and that

\[
u^{n+1}_k \geq B(u^n_k) \geq B(0) = 0.
\]

Therefore we have

\[
0 \leq u^n_k \leq 1.
\]

Now by induction on the time level we have the following result.

**Theorem 1** Suppose \( u^n_k \) is generated by the scheme 13 and that the mesh parameters satisfy the condition 17. Furthermore, we assume that the initial condition satisfy

\[
0 \leq f(x) \leq 1 \quad \text{for} \quad x \in [0, 1]
\]

Then

\[
0 \leq u^n_k \leq 1
\]

for all \( k \) and \( n \geq 0 \).
4. Application and Result
In this section we apply the Strang Splitting Method described in scheme in eq. (6) - (11) and compare the result with the exact solution. Consider the Burger-Fishers equation of the form

\[ u_t = u_{xx} + uu_x + u(1 - u), \quad t \in (0, T), \quad x \in (-20, 20) \]  \hspace{1cm} (18)

for fix \( T > 0 \), with initial condition

\[ u(0, x) = \frac{1}{2} + \frac{1}{2} \tanh \left( \frac{1}{4} x \right) \quad \text{for} \quad x \in [-20, 20], \]  \hspace{1cm} (19)

left boundary condition

\[ u(t, -20) = \frac{1}{2} + \frac{1}{2} \tanh \left( \frac{1}{4} \left( -20 + \frac{5}{4} t \right) \right). \]  \hspace{1cm} (20)

and right boundary condition

\[ u(t, 20) = \frac{1}{2} + \frac{1}{2} \tanh \left( \frac{1}{4} \left( 20 + \frac{5}{4} t \right) \right). \]  \hspace{1cm} (21)

The exact solution for problem 18 - 21 is

\[ u(t, x) = \frac{1}{2} + \frac{1}{2} \tanh \left( \frac{1}{4} \left( x + \frac{5}{4} t \right) \right). \]  \hspace{1cm} (22)

Applying the Strang Splitting method we have the following scheme

\[ v^{n+1}_k = \frac{r}{2} v^n_{k+1} + (1-r) v^n_k + \frac{r}{2} v^n_{k-1} \]  \hspace{1cm} (23)
\[ v^n_k = z^n_k \]  \hspace{1cm} (24)
\[ w^{n+1}_k = w^n_k \left( w^n_{k+1} - w^n_k \right) + hw^n_k \left( 1 - w^n_k \right) \]  \hspace{1cm} (25)
\[ w^n_k = v^{n+1}_k \]  \hspace{1cm} (26)
\[ z^{n+1}_k = \frac{r}{2} z^n_{k+1} + (1-r) z^n_k + \frac{r}{2} z^n_{k-1} \]  \hspace{1cm} (27)
\[ z^n_k = w^{n+1}_k \]  \hspace{1cm} (28)

where \( v^n_k = u(0, x^k) \) and the solution of \( u(t^{n+1}, x^k) \) will be approximated by \( z^{n+1}_k \). The schemes (23) - (28) can be written in the form of matrix equation as

\[ v^{n+1} = \frac{r}{2} A v^n + \frac{h}{2} v^n + v^n \]  \hspace{1cm} (29)
\[ w^{n+1} = hw^n w^n - hw^n w^n + w^n \]  \hspace{1cm} (30)
\[ z^{n+1} = \frac{r}{2} A z^n + \frac{r}{2} z^n + z^n \]  \hspace{1cm} (31)

where,

\[ v^{n+1} = \begin{bmatrix} v^{n+1}_0 \\ v^{n+1}_1 \\ \vdots \\ v^{n+1}_K \end{bmatrix}, \quad v^n = \begin{bmatrix} v^n_0 \\ v^n_1 \\ \vdots \\ v^n_K \end{bmatrix}, \quad w^{n+1} = \begin{bmatrix} w^{n+1}_0 \\ w^{n+1}_1 \\ \vdots \\ w^{n+1}_K \end{bmatrix}, \quad w^n = \begin{bmatrix} w^n_0 \\ w^n_1 \\ \vdots \\ w^n_K \end{bmatrix}, \quad z^{n+1} = \begin{bmatrix} z^{n+1}_0 \\ z^{n+1}_1 \\ \vdots \\ z^{n+1}_K \end{bmatrix} \]
and

$$z^n = \begin{bmatrix} \tilde{z}_0^n \\ \tilde{z}_1^n \\ \vdots \\ \tilde{w}_K^n \end{bmatrix}.$$ 

The matrix $\tilde{w}^n$ is diagonal form of matrix $w^n$ that is

$$\tilde{w}^n = \begin{bmatrix} w_0^n & 0 & 0 & \cdots & 0 \\ 0 & w_1^n & 0 & \cdots & 0 \\ 0 & 0 & w_3^n & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & w_K^n \end{bmatrix}.$$ 

and $A$ is a discrete form of operator $\frac{\partial^2}{\partial x^2}$ through a center difference scheme that is

$$A = \begin{bmatrix} 1 & -2 & 1 & & & & \\ 1 & -2 & 1 & & & & \\ & & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 & \\ & & & 1 & -2 & 1 & \\ & & & & & 1 & -2 \end{bmatrix}. \quad (32)$$

The calculation start from $n = 0$ until $n = T$ by first set $v^0 = u^0$, where $u^0$ is the discrete form of $u(0, x)$ that is

$$u^0 = \begin{bmatrix} u_0^0 \\ u_1^0 \\ \vdots \\ u_K^0 \end{bmatrix}.$$ 

Then for eq. 30 and eq. 31 set $w^n = v^{n+1}$ and $z^n = w^{n+1}$ and the approximated value of $u(t^n, x)$ will be in the form of $z^n$.

We apply the scheme with $T = 15, h = 1 \times 10^{-3},$ and $\Delta x = 0.2$, therefore we have generate a matrix $z$ with the size $15000 \times 200$. We compare the resulted matrix $z$ with the exact solution matrix $u_e$ determined from equation (22) that have the same size with $z$ and plot the result for $t = 0, t = 2, t = 5, t = 7,$ and $t = 15$ as depicted in figure 1.

From Figure 1 we see that the approximate solution are agree with the exact solution (22). To confirm the result, for fix $t$, we calculate the error $\varepsilon$ with respect to the maximum norm that is

$$\varepsilon = \max_k \{|z(k) - u_e(k)|\}. \quad (33)$$

This norm takes the maximum value of elementwise difference of $z$ and $u_e$ for $k = 0, 1, 2, \ldots, K$. The error $\varepsilon$ for $t = 2, t = 5, t = 7,$ and $t = 15$ are listed in the table 1.

From Table 1 we see that the error are stable in order of $10^{-3}$. This error value are maintained until the last time step. Therefore we can conclude that the SSM can approximate the Burger-Fisher equation considerably good even with the value $\Delta x = 0.2$. 

5
Figure 1. Comparison of exact solution and SSM method for $\Delta x = 0.2$

Table 1. Errors by SSM with respect to the maximum norm

| $t$  | $\varepsilon$          |
|------|------------------------|
| 2    | $6.67935 \times 10^{-3}$|
| 5    | $6.68852 \times 10^{-3}$|
| 7    | $6.68860 \times 10^{-3}$|
| 15   | $6.68860 \times 10^{-3}$|

5. Discussion

The Strang Splitting Method has been applied to solve the Burger-Fisher equation. The stability condition for the finite difference scheme for this equation has been derived. The Strang Splitting
Method split the equation into two parts, that is linear and non linear part. This method gives the accuracy at the order of $10^{-3}$. Therefore we can conclude that this method is considerably good to approximate the Burger-Fisher equation. Based on our observations we presume that this method will be work better when the linear and nonlinear parts of the equation contribute considerably equal. However, when the nonlinear part dominate the problem, splitting the equation into linear and nonlinear part may not always work. Finding such equation and thinking of how to solve that will be an interesting topic for the future works.

References

[1] Ismail H N, Raslan K and Abd Rabboh A A 2004 *Applied mathematics and computation* 159 291–301
[2] Wazwaz A M and Gorguis A 2004 *Applied Mathematics and Computation* 154 609–620
[3] Wazwaz A M 2005 *Applied Mathematics and Computation* 169 321–338
[4] Javidi M 2006 *Applied Mathematics and Computation* 174 345–352
[5] Golbabai A and Javidi M 2009 *Chaos, Solitons & Fractals* 39 385–392
[6] Mickens R and Gumel A 2002 *Journal of sound and vibration* 257 791–797
[7] Kaya D and El-Sayed S M 2004 *Applied Mathematics and computation* 152 403–413
[8] Chandraker V, Awasthi A and Jayaraj S 2016 *Procedia Technology* 25 1217–1225
[9] Strang G 1968 *SIAM journal on numerical analysis* 5 506–517
[10] Gücüyenen N 2017 *Journal of Computational and Applied Mathematics* 318 616–623
[11] Gücüyenen N and Tanoğlu G 2011 *Applied Mathematics and Computation* 218 777–782
[12] Hamzah D A, Tuwankotta J M and Soeharyadi Y 2017 *Neural, Parallel, and Scientific Computations* 25 395–402
[13] Einkemmer L and Ostermann A 2014 *SIAM Journal on Numerical Analysis* 52 140–155
[14] Tveito A and Winther R 2004 *Introduction to partial differential equations: a computational approach* vol 29 (Springer Science & Business Media)