Inverse construction of the $\Lambda$LTB model from a distance-redshift relation

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Abstract. Spherically symmetric dust universe models with a positive cosmological constant $\Lambda$, known as $\Lambda$-Lemaître-Tolman-Bondi ($\Lambda$LTB) models, are considered. We report a method to construct the $\Lambda$LTB model from a given distance-redshift relation observed at the symmetry center. The spherical inhomogeneity is assumed to be composed of growing modes. We derive a set of ordinary differential equations for three functions of the redshift, which specify the spherical inhomogeneity. Once a distance-redshift relation is given, with careful treatment of possible singular points, we can uniquely determine the model by solving the differential equations for each value of $\Lambda$. As a demonstration, we fix the distance-redshift relation as that of the flat $\Lambda$CDM model with $(\Omega_{\text{dis}m0}, \Omega_{\text{dis}\Lambda0}) = (0.3, 0.7)$, where $\Omega_{\text{dis}m0}$ and $\Omega_{\text{dis}\Lambda0}$ are the normalized matter density and the cosmological constant, respectively. Then, we construct the $\Lambda$LTB model for several values of $\Omega_{\Lambda0} := \Lambda/(3H_0^2)$, where $H_0$ is the present Hubble parameter observed at the symmetry center. We obtain void (over dense) structure around the symmetry center for $\Omega_{\Lambda0} < \Omega_{\text{dis}\Lambda0}(\Omega_{\Lambda0} > \Omega_{\text{dis}\Lambda0})$. We show the relation between the ratio $\Omega_{\Lambda0}/\Omega_{\text{dis}\Lambda0}$ and the amplitude of the inhomogeneity.

Keywords: dark energy theory, supernova type Ia - standard candles

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1 Introduction

The cosmological principle is one of the most fundamental principles in cosmology, and established as a useful and successful working hypothesis. However, it is still of interest to consider the largest amplitude of cosmological scale inhomogeneity which can be compatible with the latest observational data. In other words, observational tests of the cosmological principle with precision measurements may be interesting subjects in observational cosmology. Since the isotropy of our universe is strongly supported by the isotropy of the cosmic microwave background (CMB), in this paper, we focus on spherically symmetric universe models with an observer at the symmetry center.

Spherically symmetric inhomogeneous universe models have attracted much attention as alternative models to explain the apparent accelerated expansion of our universe without a cosmological constant [1–3]. After the compatibility with the CMB anisotropy was discussed in ref. [4], many observational constraints on the spherical inhomogeneity were discussed by using Type Ia supernovae data, CMB anisotropy, baryon acoustic oscillation and so on (see, e.g. ref. [5] for a recent detailed analysis). Those analyses, especially constraints from the kinematic Sunyaev-Zeldovich effect [6], revealed that the apparent accelerated expansion cannot be explained only by introducing spherical inhomogeneity without a cosmological constant if we assume the standard history of our universe before the photon last scattering. If we do not assume the inflationary paradigm and the standard thermal history before the photon last scattering, any observational quantity related to the photon last scattering cannot be predicted and used for testing the models. However, even such eccentric models would be tested by future precision observations of the late time expansion of our universe [7–11].

In this paper, we consider spherically symmetric dust universe models with a positive cosmological constant $\Lambda$, known as $\Lambda$-Lemaître-Tolman-Bondi (ALTB) models.\footnote{Although the LTB solution originally contains the cosmological constant as a parameter, we use the word “ALTB” throughout this paper to avoid misconceptions.}
consider the relation between spherical inhomogeneity in ΛLTB models and observables, one of useful strategies is the inverse construction of the model inhomogeneity starting from given observables. A pioneering work has been done by Mustapha, Hellaby and Ellis in ref. [12], where the angular diameter distance and the redshift-space mass density were supposed as the observables. The approach proposed in ref. [12] has been successfully performed in refs. [13–17] for specific situations. Another important approach was proposed by Iguchi, Nakamura and Nakao (INN) [18], where the inhomogeneity is assumed to be composed of growing modes. This assumption is often adopted to guarantee the compatibility with the inflationary paradigm. The INN approach has been solved in ref. [8] in the whole redshift range. Apart from these two approaches, there are also several related works on the inverse construction [2, 19–25].

In this paper, along the INN approach, we explicitly construct the ΛLTB models whose distance-redshift relation coincides with that in the flat ΛCDM model with $(Ω_{\text{m0}}^{\text{dis}}, Ω_{\Lambda 0}^{\text{dis}}) = (0.3, 0.7)$, where $Ω_{\text{m0}}^{\text{dis}}$ and $Ω_{\Lambda 0}^{\text{dis}}$ are the normalized matter density and the cosmological constant for the flat ΛCDM model, respectively. It should be noted that $Ω_{\text{m0}}^{\text{dis}}$ and $Ω_{\Lambda 0}^{\text{dis}}$ are the parameters characterizing the distance-redshift relation, and $Ω_{\Lambda 0}^{\text{dis}}$ is not necessarily equal to $Ω_{\Lambda 0} := Λ/(3H_0^2)$, where $H_0$ is the present Hubble parameter observed at the symmetry center. Since we add a parameter Λ to the case in the preceding works [8, 18, 26], we obtain an one-parameter family of solutions for a given set of values $(Ω_{\text{m0}}^{\text{dis}}, Ω_{\Lambda 0}^{\text{dis}})$. The one-parameter family can be characterized by the parameter $Ω_{\Lambda 0}$. The difference between $Ω_{\Lambda 0}$ and $Ω_{\Lambda 0}^{\text{dis}}$ can be regarded as a systematic error in estimation of Λ due to the spherical inhomogeneity as is discussed in refs. [27, 28]. In order to estimate the magnitude of the systematic error due to possible inhomogeneity, we calculate the amplitude of the inhomogeneity for several values of $Ω_{\Lambda 0}/Ω_{\Lambda 0}^{\text{dis}}$. The method used in this paper is similar to that in appendix of ref. [26], where a cosmological constant is not considered and the LTB solution can be described by a simple parametric form. In this paper, we work through all the complexity associated with a positive finite value of the cosmological constant (see ref. [29] for fast accurate evaluation of metric components). Similar analysis has been done in ref. [28] for perturbations on a homogeneous background.

In this paper, we use the geometrized units in which the speed of light and Newton’s gravitational constant are one, respectively.

2 Conditions to determine the ΛLTB model

2.1 ΛLTB model and the radial geodesic

We consider the Λ-Lemaître-Tolman-Bondi (ΛLTB) solution, whose line element is given by

\[ ds^2 = -dt^2 + \frac{(\partial_t R(t, r))^2}{1 - k(r)r^2} dr^2 + R^2(t, r)dΩ^2, \]  

(2.1)

where $k(r)$ is an arbitrary function of $r$ and $R(t, r)$ is the areal radius. The energy-momentum tensor is given by that of dust fluids:

\[ T^\mu_\nu = \rho(t, r)u^\mu u^\nu, \]  

(2.2)

where $\rho(t, r)$ is the mass density and $u^\mu = \delta^\mu_0$ is the 4-velocity of dust particles. From the Einstein equations, we obtain the following equation:

\[ (\partial_t R)^2 = -k(r)r^2 + \frac{2M(r)}{R} + \frac{1}{3}\Lambda R^2, \]  

(2.3)
where $M(r)$ is an arbitrary function of $r$. By using $M(r)$, we can write $\rho(t, r)$ as follows:

$$\rho(t, r) = \frac{1}{4\pi} \frac{\partial_r M(r)}{R^2 \partial_r R}.$$  

(2.4)

For convenience, we introduce the following functions:

$$m(r) := \frac{6M(r)}{r^3}, \quad S(t, r) := \frac{R(t, r)}{r}.$$  

(2.5)

Then, eq. (2.3) is written as follows:

$$(\partial_t S)^2 = f(r, S) := -k(r) + \frac{m(r)}{3S} + \frac{1}{3}\Lambda S^2.$$  

(2.6)

Eq. (2.6) can be integrated as

$$t - t_B(r) = \int_0^S \frac{dX}{\sqrt{f(r, X)}}$$  

(2.7)

with an arbitrary function $t_B(r)$, which is called the bang time function because the areal radius $R$ vanishes at $t = t_B(r)$. When we consider the case $k(r) =\text{const.}$ and $m(r) =\text{const.}$, small perturbative inhomogeneity associated with $t_B(r)$ is given by purely decaying modes in terms of cosmological perturbation theory. Therefore, in this paper, we simply assume that the bang time function is constant to guarantee the compatibility with the inflationary paradigm. The constant value of $t_B$ can be eliminated by the time shift degree of freedom, namely, we can set $t_B(r) = 0$.

We assume that the observer is at the symmetry center $r = 0$. Then, we consider the past light-cone emanated from the symmetry center expressed by a trajectory parametrized by the redshift as follows:

$$t = t_{lc}(z),$$  

(2.8)

$$r = r_{lc}(z).$$  

(2.9)

Hereafter, for notational simplicity, we often omit the subscript “lc”. The null geodesic equations in the ΛLTB solution is given as

$$\frac{dr}{dz} = \frac{1}{1 + z \sqrt{f + r \partial_r \partial_r S}},$$  

(2.10)

$$\frac{dt}{dz} = -\frac{1}{1 + z \sqrt{f + r \partial_r \partial_r S}}.$$  

(2.11)

### 2.2 Conditions to determine the arbitrary functions

The ΛLTB solution has three arbitrary functions $k(r), m(r)$ and $t_B(r)$. One of these functional degrees of freedom corresponds to the gauge degree of freedom associated with the choice of the radial coordinate $r$. We impose a gauge condition for the radial coordinate $r$ and require the distance-redshift relation coincides with a given function $D_A(z)$.

We fix the gauge by imposing the light-cone gauge condition given by

$$t(z) = t_0 - r(z),$$  

(2.12)
where $t_0$ is the present time at the central observer. From this condition, $t_{lc}$ can be trivially given by $r_{lc}$. Therefore, the remaining independent functions are $r_{lc}(z)$, $k(r_{lc}(z))$ and $m(r_{lc}(z))$. Combining the light-cone gauge condition and the geodesic equations, we obtain

$$
(r\partial_r S + S - \sqrt{1 - k r^2})|_{r=r_{lc}, \ t=t_{lc}} = 0.
$$

We consider eqs. (2.10), (2.11) and (2.13) as independent equations.

The angular diameter distance on the past light-cone is given by $R(t(z), r(z))$. We impose that the angular diameter distance coincides with a given function $D_A(z)$. In practice, we impose the following differential equation:

$$
\frac{dR}{dz} = \frac{dD_A}{dz}.
$$

In this paper, for demonstration, we use the distance-redshift relation in a flat $\Lambda$CDM universe instead of actual observational data. That is, we use the distance characterized by the cosmological parameters for the flat $\Lambda$CDM universe as follows:

$$
D_A(z) = D_{\Lambda\text{CDM}}(z; \Omega^{\text{dis}}_{m0}, \Omega^{\text{dis}}_{\Lambda0}),
$$

where $\Omega^{\text{dis}}_{m0}$ and $\Omega^{\text{dis}}_{\Lambda0}$ are the normalized matter density and the cosmological constant for the flat $\Lambda$CDM model. It should be noted that in a spherically symmetric inhomogeneous universe model, expansion rates to the radial direction and the transverse direction can be different from each other. In this paper, we define the present Hubble parameter $H_0$ as follows:

$$
H_0 := (\partial_t R/R)|_{z=0}.
$$

The normalization of the cosmological parameters, e.g. $\Omega^{\text{dis}}_{m0}$, is performed by using $H_0$. In our numerical calculations, we use the unit system given by $H_0 = 1$, and all dimensionful variables are normalized by $H_0$. Although we do not specify the value of $H_0$ in this paper, in practice, the value of $H_0$ should be fixed to the best fit observational value given by local $H_0$ observations.

### 3 Derivation of differential equations

The method used in this paper is an extension of the method in appendix of ref. [26]. A difference is in the treatment of the function $S(t, r)$ for which we do not have analytic expression in the case $\Lambda \neq 0$. Therefore, in the case $\Lambda = 0$ our method reduces to that in ref. [26]. Let us derive the differential equations to determine three arbitrary functions $r(z)$, $k(z)$ and $m(z)$. Differentiating eq. (2.3) with respect to $r$, we obtain

$$
\partial_t \partial_r S = \frac{1}{2} r^{-1/2} \left( -\partial_r k + \frac{S \partial_r m - m \partial_r S}{3S^2} + \frac{2}{3} \Lambda S \partial_r S \right).
$$

Multiplying $dr/dz$ to the above equation and using the null geodesic equations (2.10) and (2.11), we get the following differential equation:

$$
\left[ \left( -\frac{m}{3S^2} + \frac{2}{3} \Lambda S \right) \left( \sqrt{1 - kr^2} - S \right) + 2f \right] \frac{dr}{dz} - r \frac{dk}{dz} + \frac{r}{3S} \frac{dm}{dz} - \frac{2\sqrt{7} \sqrt{1 - kr^2}}{1 + z} = 0.
$$

-- 4 --
Differentiating eq. (2.7) with respect to $r$, we obtain

$$0 = \frac{\partial_r S}{\sqrt{f(r,S)}} - \frac{1}{2} \int_0^S f(r,X)^{-2/3} \left( -\partial_r k + \frac{\partial_r m}{3X} \right) dX. \quad (3.3)$$

Multiplying $dr/dz$ to the above equation and using eq. (2.13), we get the following differential equation:

$$0 = \frac{\sqrt{1 - kr^2} - S}{r \sqrt{f}} \frac{dr}{dz} - P \frac{dk}{dz} + Q \frac{dm}{dz}, \quad (3.4)$$

where

$$P := -\frac{1}{2} \int_0^S f(r,X)^{-3/2} dX, \quad (3.5)$$

$$Q := -\frac{1}{2} \int_0^S f(r,X)^{-3/2} X dX. \quad (3.6)$$

These integrals can be numerically evaluated.

Since $dR/dz$ is calculated as

$$\frac{dR}{dz} = \partial_t R \frac{dt}{dz} + \partial_r R \frac{dr}{dz} \left( -r \sqrt{f} + \sqrt{1 - kr^2} \right) \frac{dr}{dz},$$

from eq. (2.14), we get the following differential equation:

$$\frac{dr}{dz} = \frac{1}{-r \sqrt{f} + \sqrt{1 - kr^2}} \frac{dD_A(z)}{dz}. \quad (3.7)$$

We note that $(dD_A/dz)_{z=0}$ gives the inverse of the Hubble parameter associated with $D_A$ by definition. While $(dr/dz)_{z=0}$ gives the Hubble parameter defined by eq. (2.16) as follows:

$$\left( \frac{dr}{dz} \right)_{z=0} = \left( f^{-1/2} \right)_{z=0} = \left( \frac{1}{\partial_t S} \right)_{z=0} = \left( \frac{R}{\partial_t R} \right)_{z=0} = 1/H_0, \quad (3.9)$$

where we have used eqs. (2.5), (2.6), (2.10) and $R|_{z=0} = r$ from eq. (2.13). That is, we obtain $(dD_A/dz)_{z=0} = 1/H_0$.

Using eqs. (3.2), (3.4) and (3.8), we can derive a set of three differential equations for $r(z), k(z), m(z)$ as follows:

$$\frac{dr}{dz} = A(r,k,m) \frac{dD_A(z)}{dz}, \quad (3.10)$$

$$\frac{dk}{dz} = B(r,k,m) \frac{dr}{dz} + \frac{1}{3S} \frac{dm}{dz} + C(r,k,m), \quad (3.11)$$

$$\frac{dm}{dz} = D(r,k,m) \frac{dr}{dz} + \frac{3P}{Q} \frac{dk}{dz}, \quad (3.12)$$

where

$$A(r,k,m) = \frac{1}{-r \sqrt{f} + \sqrt{1 - kr^2}}, \quad (3.13)$$

$$B(r,k,m) = \frac{1}{r} \left[ \left( -\frac{m}{3S^2} + \frac{2}{3} \frac{AS}{S} \right) \left( \sqrt{1 - kr^2} - S \right) + 2f \right], \quad (3.14)$$

$$C(r,k,m) = -\frac{2 \sqrt{f} \sqrt{1 - kr^2}}{r(1 + z)}, \quad (3.15)$$

$$D(r,k,m) = -\frac{3}{rQ \sqrt{f}} \left( \sqrt{1 - kr^2} - S \right). \quad (3.16)$$
4 Regularity conditions and the solving method

In the differential equations (3.10), (3.11) and (3.12), there are two possible singular points (see also, e.g. refs. [30, 31]). One is at the center and the other is associated with the apparent horizon [32] satisfying \(dD_A/dz = 0\). At these points, we impose regularity conditions.

4.1 Regularity at the center

We expand \(r(z), k(z)\) and \(m(z)\) near the center as follows:

\[
\begin{align*}
  r(z) &= r_1 z + \frac{1}{2} r_2 z^2 + \mathcal{O}(z^3), \\
  k(z) &= k_0 + k_1 z + \mathcal{O}(z^2), \\
  m(z) &= m_0 + m_1 z + \mathcal{O}(z^2),
\end{align*}
\]

where we have assumed \(r = 0\) at \(z = 0\). The right-hand side of eq. (3.11) has the following term of the order \(z^{-1}\):

\[
\sqrt{-k_0 + \frac{m_0}{3} + \Lambda_0} \left( \sqrt{-k_0 + \frac{m_0}{3} + \frac{\Lambda}{3}} - H_0 \right) z^{-1}.
\]

For the regularity at the center, we require the coefficient of this term vanishes at the center. Then, we obtain the following condition:

\[
-k_0 + \frac{1}{3} m_0 + \frac{1}{3} \Lambda = H_0^2.
\]

This condition is consistent with the definition of \(H_0\). Therefore we have a constraint for the three parameters \(k_0/H_0^2\), \(m_0/H_0^2\) and \(\Lambda/H_0^2\). Hereafter, for convenience, let us consider \(m_0/H_0^2\) and \(\Lambda/H_0^2\) as the only independent parameters.

4.2 Regularity on the apparent horizon

At the point \(z = z_{ah}\) satisfying \(dD_A/dz = 0\), from eq. (3.10), we obtain \(dr/dz = 0\) unless \(1/A(r, k, m) = 0\). The point with \(dr/dz = 0\) causes a unphysical solution with divergent physical quantities in general. Therefore, we impose \(1/A(r, k, m) = 0\) at \(z = z_{ah}\) so that \(r(z)\) can be a monotone increasing function of \(z\). Let us consider the Taylor expansion near \(z = z_{ah}\) as follows:

\[
\begin{align*}
  r(z) &= r_{ah} + r_{ah1}(z - z_{ah}) + \mathcal{O}((z - z_{ah})^2), \\
  k(z) &= k_{ah} + k_{ah1}(z - z_{ah}) + \mathcal{O}((z - z_{ah})^2), \\
  m(z) &= m_{ah} + m_{ah1}(z - z_{ah}) + \mathcal{O}((z - z_{ah})^2).
\end{align*}
\]

From the equation \(1/A(r_{ah}, k_{ah}, m_{ah}) = 0\), we obtain the following equation:

\[
m_{ah} r_{ah}^3 = 3D_A(z_{ah}) - \Lambda D_A(z_{ah})^3.
\]

Since we can eliminate \(r_{ah}\) by using eq. (4.9), we consider \(k_{ah}\) and \(m_{ah}\) are the only independent parameters associated with boundary conditions at \(z = z_{ah}\). 




4.3 Newton-Raphson method

As is shown in the previous subsections, independent parameters are $m_0/H_0^2$, $k_{ah}/H_0^2$ and $m_{ah}/H_0^2$ for a fixed value of $\Lambda/H_0^2$. To determine these parameters, we adopt the following method. First, we solve the differential equations from the center to a middle point $z = z_m < z_{ah}$ using a trial value of $m_0/H_0^2$. Second, we solve the differential equations from the point $z = z_{ah}$ to the middle point using trial values of $k_{ah}/H_0^2$ and $m_{ah}/H_0^2$. Finally, we impose the smoothness conditions at the middle point $z = z_m$ as follows:

$$
    r_{m-0}(m_0) - r_{m+0}(k_{ah}, m_{ah}) = 0,
    k_{m-0}(m_0) - k_{m+0}(k_{ah}, m_{ah}) = 0,
    m_{m-0}(m_0) - m_{m+0}(k_{ah}, m_{ah}) = 0,
$$

where $X_{m-0}$ and $X_{m+0}$ are the values of $X$ at $z = z_m$ when we solve from the center and $z = z_{ah}$, respectively. These smoothness conditions can be regarded as three independent conditions for $m_0/H_0^2$, $k_{ah}/H_0^2$ and $m_{ah}/H_0^2$. Then, we search for the values of $m_0/H_0^2$, $k_{ah}/H_0^2$ and $m_{ah}/H_0^2$ by using the 3-dimensional Newton-Raphson method. After the convergence, the smoothness conditions are satisfied within the accuracy $\sim 10^{-10}$ in our numerical calculations. As far as we checked, $m_0/H_0^2$, $k_{ah}$ and $m_{ah}/H_0^2$ are uniquely determined if they exist for any value of $\Lambda/H_0^2$. Eventually, we can obtain a unique solution for each set of a value of $\Lambda/H_0^2$ and a distance-redshift relation $D_A(z)$. We have checked that the reconstructed $\Lambda$LTB models realize the input $D_A$ with the accuracy at the level of $10^{-8}$.

5 Solutions and density profile

We checked that our reconstruction reproduces the $\Lambda$CDM model for $\Omega_{\Lambda 0} = \Omega_{\Lambda}^{\text{dis}}$ within numerical precision. We also checked that our method reproduces the previous results given in refs. [8, 26]. In this section, as a demonstration, we consider the case $D_A(z) = D_{\Lambda\text{CDM}}(z; 0.3, 0.7)$. We define $R_\Lambda$ as

$$
    R_\Lambda := \frac{\Omega_{\Lambda 0}}{\Omega_{\Lambda}^{\text{dis}}},
$$

where $\Omega_{\Lambda 0} := \Lambda/(3H_0^2)$. We show $m(r(z))/H_0^2$ and $k(r(z))/H_0^2$ as functions of $z$ for several values of $R_\Lambda$ in figure 1.
In these ΛLTB models, we can evaluate the density distribution on the present time slice $t = t_0$, where $t_0$ can be evaluated by

$$t_0 = \int_0^1 \frac{dX}{\sqrt{f(0,X)}}. \quad (5.2)$$

In order to obtain $\rho(t_0, r)$ from eq. (2.4), we need to calculate $S(t_0, r)$ and $\partial_r S(t_0, r)$ as functions of $r$. $S(t_0, r)$ can be calculated by numerically solving eq. (2.7) with $t = t_0$. Then, we can obtain $\partial_r S(t_0, r)$ by numerically solving eq. (3.3). We note that the hypersurface $t = t_0$ is a spacelike hypersurface and the quantity $\rho(t_0, r)$ is not a direct observable. Observational aspects are discussed elsewhere, and we simply use $\rho(t_0, r)$ to demonstrate the inhomogeneity in this paper. Let us define the density fluctuation $\Delta_0$ as

$$\Delta_0(t_0, r_{lc}(z)) := \rho(t_0, r_{lc}(z)) - \rho(t_0, r_{lc}(z = 10)) \rho(t_0, r_{lc}(z = 10)). \quad (5.3)$$

It should be noted that, although we describe $\Delta_0$ as a function of $z$, $\Delta_0(t_0, r_{lc}(z))$ is defined on the spacelike surface $t = t_0$. The redshift $z$ is simply used to specify the radial coordinate $r$.

In figure 2, we show the density fluctuation $\Delta_0(t_0, r_{lc}(z))$ as a function of $z$ and the value of $\Delta_0(t_0, 0)$ for several values of $R_\Lambda$. As is shown in figure 2, we obtain void (over dense) structures for $R_\Lambda < 1(> 1)$ around the symmetry center. The magnitude of the density inhomogeneity $\Delta_0(t_0, 0)$ is roughly proportional to the value of $R_\Lambda$ and $\Delta_0(t_0, 0) \sim -0.25$ for $R_\Lambda = 0.8$.

6 Summary and discussion

In this paper, we have described technical details of the construction of the ΛLTB model for a given set of a distance-redshift relation and a value of $\Lambda$ with the bang time function being constant. It has been shown that we can obtain a unique ΛLTB model for each set of $\Lambda$ and a distance-redshift relation. As a demonstration, we have constructed the ΛLTB model whose distance-redshift relation is given by that in the flat ΛCDM model with the cosmological parameters $(\Omega_{m0}^{\text{dis}}, \Omega_{\Lambda0}^{\text{dis}}) = (0.3, 0.7)$. As is expected from previous works, we obtain void type structure for a smaller value of $\Lambda$ compared with $3\Omega_{\Lambda0}^{\text{dis}}H_0^2$. It should be noted that, while void type structure may reduce the value of $\Lambda$ compared with $3\Omega_{\Lambda0}^{\text{dis}}H_0^2$, possibility of over dense type structure should be also taken into account in a similar extent when we compare...
them with observations. We also note that void structure may not be necessary for a smaller value of $\Lambda$ in general situation [15].

The method of the inverse construction can be a complement to the conventional method in which LTB functions ($k(r)$, $M(r)$ and $t_B(r)$ in the text) are directly parametrized by using several parameters (see, e.g. ref. [5]). The models given by solving the inverse construction problem may be significantly different from the models given by the direct parametrization of the LTB functions. Therefore, it is important to combine the inverse construction method and analyses with observational data (see, e.g. ref. [33]). We will report the CMB and local Hubble parameter analysis combined with our inverse construction method elsewhere [34].

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