Strictly respecting the Einstein equations, and supposing space-time is a medium, we derive the deformation of this medium by gravity. We derive the deformation in case of infinite plane, Robertson-Walker manifold, Schwarzschild manifold and gravitational waves. Some singularities are removed or changed. We call this procedure renormalization of gravity. We show that some results following from the classical gravity must be modified.

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I. INTRODUCTION

The term renormalization is usually considered as the theoretical procedure that changes some fundamental constants as charge, mass and others in such a way that they are in the final form finite and in agreement with experiment. While the renormalization in the quantum field theory can be in a certain sense mastered, the renormalization procedure in a gravity theory is in no case the closed problem. In the Einstein theory of gravity, one method of the renormalization consists for instance in introducing the cosmological constant which renormalizes the energy-momentum tensor and therefore modify the solutions of Einstein equations.

In this article we introduce the special restriction on the metrical tensor which modifies the consequences of the Einstein equations. Since the restriction condition has physical meaning, it means that the consequences of this restriction have also physical meaning.

We know that the main steps in the development of the theoretical view on the space time was, first, the isolation of space from time in the Newton period of physics, and later, the unification of space and time in the special relativity. The second step was performed by Lorentz who replaced the Galileian transformation by the so called Lorentz transformation where space and time are not isolated and which was used by Einstein in creation of the special theory of relativity.

The symbiosis of space and time was expressed elegantly by Minkowski who introduced the quasieuclidean space-time element where the space and time are in unity. Minkowski expressed it by his space-time element $ds$ the square of which is
\[
    ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2. \tag{1}
\]

Finally, Einstein, in his gravity theory, replaced the Minkowski space-time element by the four-dimensional Riemann manifold where the square of the space-time element was of the form:

\[
    ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{2}
\]

where \( g_{\mu\nu} \) is so called the metrical tensor. General relativity created by Einstein in 1915 is based on this curved manifold and Einstein equations are differential equations for determination of this metrical tensor.

In this article we suppose that the Einstein equations are correct. These equations are differential equations for determination of the metrical tensor \( g_{\mu\nu} \) and their form are as follows:

\[
    R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu}, \tag{3}
\]

where we here do not consider the additional term with the cosmological constant which was later introduced by Einstein in order to regularize some solutions of his equations. Einstein theory realizes in the mathematical form the Riemann fundamental idea that space-time is some form of matter. In other words, there exists in some sense ether. It means that space is in no case empty and forms some medium. To deny such medium means ultimately to assume that empty space has no physical qualities whatever which is in contradiction with the general theory of relativity where space is endowed with physical qualities. According to the general theory of relativity empty space is unthinkable. In empty space there not only would be no propagation of light, but also no possibility of existence for standards of space and time. If we suppose space-time is a medium, then, it consequently means that gravitational waves following from the Einstein theory are quite simply the vibration of space-time itself.

However, Einstein gives no explanation of the origin of the metrics, or, metrical tensor. He "derived" only the nonlinear equation \( \text{(2)} \) for the metrical tensor and never explained what is microscopical origin of the metric of space-time. Einstein supposed that it is adequate that the metric follows from differential equations as their solution. However, the metric has an microscopical origin similarly to situation where the phenomenological thermodynamics has also the microscopical and statistical origin.

Here we ask a question, what is the microscopical origin of the metric of space-time. We postulate that the origin of metric is the deformation of space-time continuum. We use as an analog the mechanics of continuum and we apply it to the space-time medium. The similar approach can be found in the Tartaglia article, \( \text{(3)} \) who also has considered space-time as a medium, however, without respecting the Einstein Equations.

In order to explain our ideas let us consider, first, some continous body in some three dimensional Cartesian coordinate system. The mathematical description of deformation is as follows \( \text{(4)} \).

The coordinate of the arbitrary point of the body let be given by radiusvector \( x(x^1, x^2, x^3) \), where \( x^i \) are the Cartesian coordinates. After some deformation the new radiusvector let be \( x' \). The relative displacement is then \( u \) where

\[
    u = x' - x. \tag{4}
\]

The quantity \( u \) is so called vector of deformation and obviously

\[
    dx'^i = dx^i + du^i. \tag{5}
\]
While the infinitesimal distance in the nondeformed body is
\[ dl^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 = \delta_{ik} dx^i dx^k, \]
for the deformed body we have
\[ dl'^2 = (dx'^1)^2 + (dx'^2)^2 + (dx'^3)^2 = (dx^i + du^i)^2 = (dx^i)^2 + 2dx^i du^i + (du^i)^2. \]
Or, with \( du^i = \frac{\partial u^i}{\partial x^k} dx^k \)
\[ dl'^2 = dl^2 + u^{ik} dx^i dx^k = (\delta^{ik} + u^{ik}) dx^i dx^k, \]
where
\[ u^{ik} = \left( \frac{\partial u^i}{\partial x^k} + \frac{\partial u^k}{\partial x^i} + \frac{\partial u^i}{\partial x^k} \frac{\partial u^k}{\partial x^i} \right). \]

The last equations are defined for dimension D=3 and Euclidean space. The analogical equations are evidently valid for Euclidean space with dimension D = 4. In case of the quasi-euclidean space time with the dimension D = 4, we can obviously write
\[ ds'^2 = (\eta_{\mu\nu} + u_{\mu\nu}) dx^\mu dx^\nu; \quad \mu, \nu = 0, 1, 2, 3; \quad x^0 = ct, \]
where
\[ u_{\mu\nu} = \left( \frac{\partial u^\mu}{\partial x^\nu} + \frac{\partial u^\nu}{\partial x^\mu} + \frac{\partial u^\mu}{\partial x^\alpha} \frac{\partial u^\nu}{\partial x^\alpha} \right) = \partial \mu u_\nu + \partial \nu u_\mu + \partial \mu u_\alpha \partial \nu u_\alpha + \partial \mu u_\alpha \partial \nu u_\alpha \]
and
\[ \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \]

In such a way, we have for the squared space-time element
\[ ds'^2 = g_{\mu\nu} dx^\mu dx^\nu, \]
where
\[ g_{\mu\nu} = (\eta_{\mu\nu} + u_{\mu\nu}). \]

Equation (14) with definition (11) excludes gravity with the nonsymmetrical tensor \( g_{\mu\nu} \). Or, the renormalized gravity is symmetrical.

Instead of work with the metrical tensor \( g_{\mu\nu} \), we can work with the tensor of deformation \( u_{\mu\nu} \) and we can consider the general theory of relativity as the four-dimensional theory of some real deformable medium as a partner of the metrical theory.

II. THE NONRELATIVISTIC LIMIT

The Lagrange function of a point particle with mass \( m \) moving in a potential \( \varphi \) is given by the following formula [3]:
\[ L = -mc^2 + \frac{mv^2}{2} - m\varphi. \]
Then, for a corresponding action we have

\[ S = \int L dt = -mc \int dt \left( c - \frac{v^2}{2c} + \frac{\varphi}{c} \right), \tag{16} \]

which ought to be compared with \( S = -mc \int ds \). Then,

\[ ds = \left( c - \frac{v^2}{2c} + \frac{\varphi}{c} \right) dt. \tag{17} \]

With \( d\mathbf{x} = v dt \) and neglecting higher derivative terms, we have

\[ ds^2 = (c^2 + 2\varphi)dt^2 - d\mathbf{x}^2 = \left( 1 + \frac{2\varphi}{c^2} \right) c^2 dt^2 - d\mathbf{x}^2. \tag{18} \]

The metric determined by this \( ds^2 \) can be obviously related to the \( u_\alpha \) as follows:

\[ g_{00} = 1 + 2\partial_0 u_0 + \partial_0 u^\alpha \partial_0 u_\alpha = 1 + 2\frac{\varphi}{c^2}. \tag{19} \]

We can suppose that the time shift caused by the potential is small and therefore we can neglect the nonlinear term in the last equation. Then we have

\[ g_{00} = 1 + 2\partial_0 u_0 = 1 + \frac{2\varphi}{c^2}. \tag{20} \]

The elementary consequence of the last equation is

\[ \partial_0 u_0 = \frac{\partial u_0}{\partial (ct)} = \frac{\varphi}{c^2}. \tag{21} \]

or,

\[ u_0 = \frac{\varphi}{c} t + \text{const.} \tag{22} \]

Using \( u_0 = g_{00} u^0 \), or, \( u^0 = g_{00}^{-1} u_0 = \frac{\varphi}{c} \), we get with

\[ u^0 = ct' - ct, \tag{23} \]

the following result (putting the integration constant to zero)

\[ t' = t \left( 1 + \frac{\varphi}{c^2} \right), \tag{24} \]

which is the Einstein formula relating time in the zero gravitational field to time in the gravitational potential \( \varphi \). The shift of light frequency corresponding to the gravitational potential is, as follows \([5]\).

\[ \omega = \omega_0 \left( 1 + \frac{\varphi}{c^2} \right). \tag{25} \]

So, we have seen that the red shift follows from our approach immediately, without application of the Einstein equations. Now, let us approach the analysis of the physical situations characterized by the metrics determined by the Einstein equations. First, let us consider the metric of the infinite plane.
III. THE INFINITE PLANE

An infinite plane sheet with homogenous mass distribution \( \mu \) per unit area generates the following metrics on it [6]:

\[
\begin{equation}
    ds^2 = \left(1 + \frac{8\pi G \mu z}{c^2}\right) c^2 dt^2 - dx^2 - dy^2 - \frac{dz^2}{1 + \frac{8\pi G \mu}{c^2} z}.
\end{equation}
\]

(26)

The corresponding matrix notation of the metric following from the last equation is:

\[
    g_{\mu\nu} = \begin{pmatrix}
        (1 + Az) & 0 & 0 & 0 \\
        0 & -1 & 0 & 0 \\
        0 & 0 & -1 & 0 \\
        0 & 0 & 0 & \left(\frac{-1}{1+Az}\right)
    \end{pmatrix}; \quad A = \frac{8\pi G \mu}{c^2}.
\]

(27)

The gravitational potential corresponding to the metric \( g_{\mu\nu} \) is \( \varphi = 4\pi G \mu z \) because of equation (19).

It can be observed the singular plane \( z = -c^2/8\pi G \mu \) in the last metric, which has evidently no physical meaning. At the same time the metric is not symmetrical with regard to the \( z \)-plane. It is not the goal of this article to solve this specific problem.

It is evident that in case of the infinite plane \( u^0 \) depends only on the time component and on the \( z \) component. In this case of the infinite plane must be \( u^1 = u^2 = 0 \) and \( u^3 \) is dependent only on the \( z \) component. So, we write:

\[
    u^0 = u^0(t, z), \quad u^1 = u^2 = 0, \quad u^3 = u^3(z).
\]

(28)

By the method analogical to the one in the preceding chapter we get the following equations for the determination of the \( u \)-components:

\[
    2\partial_0 u_0 + \partial_0 u^0 \partial_0 u_0 = \frac{8\pi G \mu z}{c^2}
\]

and

\[
    -1 + 2\partial_3 u_3 + \partial_3 u_3 \partial_3 u_3 = -\frac{1}{1 + \frac{8\pi G \mu}{c^2} z}.
\]

(30)

Her we can also suppose that the gravitational field generated by the infinite plane is weak and therefore we can neglect the nonlinear terms in the last equations. In such a way we have:

\[
    2\partial_0 u_0 = \frac{8\pi G \mu z}{c^2} = Az; \quad A = \frac{8\pi G \mu}{c^2}
\]

(31)

and

\[
    -1 + 2\partial_3 u_3 = -(1 + Az)^{-1} \approx -1 + Az.
\]

(32)

Using the same procedure as in the preceding chapter, we get in our linear approximation for the time shift from eq. (29)

\[
    t' = t (1 + Az/2) = t \left(1 + \frac{\varphi}{c^2}\right),
\]

(33)

where \( \varphi \) is the gravitational potential corresponding to the infinite plane. The solution of the equation (32) is (with \( \partial_3 = \partial/\partial z \))

\[
    u_3 = \frac{A}{4} z^2,
\]

(34)
where we have put the integration constant to zero.

Since

$$u_3 = g_{33} u^3 = g_{33} (z' - z),$$  

we have

$$z' - z = g^{-1}_{33} \frac{A}{z} z^2 \approx - \frac{A z^2}{4},$$  

or,

$$z' = z \left(1 - \frac{\varphi}{2z^2}\right),$$  

where $$\varphi = 4\pi G \mu z$$ is the gravitational potential corresponding to the infinite plane. The integration constant was put to zero with regard to the boundary condition applied at $$z = 0$$.

We can see that in case of the gravitating plane there exist not only the time shift but also the $$z$$-coordinate shift. This effect is not involved in the Will monography [7] on gravitational experiments and it can be considered as the additional effect in the general relativity.

IV. THE ROBERTSON-WALKER METRIC

This metric is defined by the following squared space-time element:

$$ds^2 = c^2 dt^2 - R^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right]; \quad k = -1, 0, 1.$$  

The element $$ds^2$$ has the corresponding form in the $$u$$-variables:

$$ds^2 = g_{00} c^2 dt^2 - \left[ u_{rr} dr^2 + u_{\theta\theta} d\theta^2 + u_{\phi\phi} d\phi^2 \right],$$  

where the nondiagonal elements are equal to zero. According to Landau [4], $$u_{rr} = (\partial/\partial r) u_r$$, or,

$$u_{rr} = \frac{\partial}{\partial r} u_r = \frac{R^2(t)}{1 - kr^2},$$  

or [8],

$$u_r = r' - r = R^2(t) \frac{1}{2\sqrt{k}} \ln \left| \frac{1 + r\sqrt{k}}{1 - r\sqrt{k}} \right|.$$  

The RW metrics with $$k = -1$$ is called hyperbolic universe, or open universe, the RW metrics with $$k = 0$$ is called flat universe and the RW metrics with $$k = 1$$ is so called elliptical cosmology, or closed cosmology. However, we easily see that the last equation for $$k = -1$$ is not physically meaningful.

At point $$k = 0$$ there is no singularity. It can be easily seen using substitution $$\varepsilon = \sqrt{kr}$$. Then, for $$\varepsilon > 0$$ and $$\varepsilon \to 0$$, we have from eq. (41): $$J = (1/2\sqrt{k}) \ln[(1 + r\sqrt{k})/(1 - r\sqrt{k})] = (r/2\varepsilon) \ln[(1 + \varepsilon)/(1 - \varepsilon)] = (r/2\varepsilon)[2(\varepsilon + \varepsilon^3/3 + ...)]\varepsilon^2 < 1$$. And we see that $$J(\varepsilon \to 0) = r$$.

It means that only $$k = 0$$ and $$k = 1$$ has physical meaning and so the cosmology is flat or elliptical. In other words RW universe is flat or closed. So, in such a way the so called renormalization of the RW cosmology involves the restriction of parameters $$k$$ to $$k = 0$$ and $$k = 1$$, which is not involved in the original approach to cosmology.
V. THE SCHWARZSCHILD METRIC

The squared space-time element of the Schwarzschild space-time is of the form:

\[ ds^2 = \left(1 - \frac{r_g}{r}\right)c^2dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}} - r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad r_g = \frac{2MG}{c^2}. \]  (42)

It means that for the t-components we have:

\[ g_{00} = 1 + 2\partial_0 u_0 = \left(1 - \frac{r_g}{r}\right), \quad u_{rr} = \frac{\partial}{\partial r} u_r = \frac{1}{1 - \frac{r_g}{r}}. \]  (43)

After elementary integration we get for the t-displacement

\[ u^0 = c(t' - t) = -\frac{r_g}{2r}ct + \text{const}, \]  (44)

or, (we put the constant to zero)

\[ t' = \left(1 - \frac{r_g}{2r}\right) = \left(1 + \frac{\varphi}{c^2}\right); \quad \varphi = -\frac{GM}{r}. \]  (45)

corresponds exactly to the formula of the time dilation with the Newton potential \( \varphi = -\frac{GM}{r} \).

For r-displacement it is

\[ u_r = r' - r = \int \frac{rdr}{r - r_g} = r + r_g \ln |r - r_g| + \text{const}. \]  (46)

We observe that the left and right limit of \( u_r \) for \( r \to r_g \) are identical in contradistinction with the limit in the original Schwarzschild metric. In other words,

\[ \lim_{r \to r_g^+} \ln |r - r_g| = \lim_{r \to r_g^-} \ln |r - r_g|. \]  (47)

While the singularity in the original metric is hyperbolical, the singularity after the \( u \)-mechanism is only logarithmical. So, the renormalization in this case is soft.

VI. THE WEAK GRAVITATIONAL WAVES

The weak gravitational field is defined by the metric

\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \]  (48)

where

\[ |h_{\mu\nu}| \ll 1. \]  (49)

It is possible to show that in this case the equation for these weak gravitational waves is the linear wave equation.

\[ \Box^2 h_{\mu\nu} = 0 \]  (50)

with additional constrain following from the conservation of laws,

\[ \partial_{\mu} h^\mu_{\nu} = \frac{1}{2} \partial_{\nu} h^\sigma_{\sigma}. \]  (51)

The gravitational waves in empty space are determined as a solution of the wave equation and has the well known form:
\[ h_{\mu\nu} = a_{\mu\nu} \cos(kx - \omega t), \]  

(52)

where \( a_{\mu\nu} \) is the tensor amplitude and it means that the differential equations for the \( u_\mu \)-functions must be related to \( h_{\mu\nu} \) in the following form:

\[ a_{\mu\nu} \cos(kx - \omega t) = \partial_\mu u_\nu + \partial_\nu u_\mu, \]  

(53)

where we have neglected the the nonlinear \( u \)-term on the right side of the last equation, because it is of the second order and it cannot be in principle related in this approximation to the terms of the first order.

The solution of the last equation forms simple mathematical problem which can be solved easily supposing

\[ u_\mu = u_\mu(x, y, z, t) \]  

(54)

Let us look for the wave propagating in direction \( x^3 = z \) with \( k \equiv (0, 0, k) \). In this case, it is well known that only the nonzero term \( h_{\alpha\beta} \) are the following ones:

\[ a_{11} = -a_{22}; \quad a_{12} = a_{21} \]  

(55)

Then, using relations (55) in eq. (53) we get

\[ A \cos(kz - \omega t) = 2 \frac{\partial u_1}{\partial x}; \quad A = a_{11} \]  

(56)

\[ -A \cos(kz - \omega t) = 2 \frac{\partial u_2}{\partial y}; \quad A = a_{11} \]  

(57)

\[ B \cos(kz - \omega t) = \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}; \quad B = a_{12}. \]  

(58)

The solution of the last equations can be obtained by the elementary integration in the following form (with \( \phi = (kz - \omega t) \)):

\[ u_1 = \frac{A}{2} x \cos \phi + \alpha(y, z, t); \quad u_2 = -\frac{A}{2} y \cos \phi + \beta(x, z, t), \]  

(59)

where \( \alpha \) and \( \beta \) are functions which can be determined from inserting eq. (59) to the eq. (58). Or,

\[ \frac{\partial \alpha}{\partial y} + \frac{\partial \beta}{\partial x} = B \cos \phi, \]  

(60)

and the solution of the last equation is of the form:

\[ \alpha = \frac{B}{2} y \cos \phi; \quad \beta = \frac{B}{2} x \cos \phi \]  

(61)

Now, we can write the four vector corresponding to the weak gravitational wave spreading in the direction of the z-axis in the form:

\[ u = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 \\ A x + B y \\ -A y + B x \\ 0 \end{pmatrix} \cos \phi; \quad \phi = (kz - \omega t) \]  

(62)

The constants \( A \) and \( B \) must be chosen in such a way that \( A = B \) because of the equivalence of the of \( x \)-coordinates with \( y \)-coordinates in free space.
VII. DISCUSSION

We have defined general relativity and gravitation as a deformation of a medium called space-time. We have used specific equation which relates Einstein metric to the displacement of points of the medium and applied it to the some gravitating systems, such as infinite plane desk, Schwarzschild universe, Robertson walker universe, and gravitational waves. It is evident that our method can be also applied to the further systems, such as the rotating plane desk, anti de Sitter space-time, the Reissner-Nordstrom space-time, the Newmann et al. space-time, the Kerr space-time and so on. Such application can be considered as interesting problems and will have certainly meaning in the future gravitational physics.

The future measurement of the gravitational waves, performed by the LIGO project [9], VIRGO project [10], GEO 600 project [11] and TAMA 300 project [12] will evidently have also the positive impact on the development of gravitational physics.

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