ON THE RECURSIVE CONSTRUCTION OF INDECOMPOSABLE QUIVER REPRESENTATIONS

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Abstract. Fixed a root of a quiver, it is a very hard problem to construct all or even only one indecomposable representation with this root as dimension vector. We investigate two methods which can be used for this purpose. In both cases we get an embedding of the category of representations of a new quiver to the category of representations of the original one which increases dimension vectors. Thus it can be used to construct indecomposable representations of the original quiver recursively. Actually, it turns out that there is a huge class of representations which can be constructed using these methods.

1. Introduction

Fixed a quiver, Kac proved in [6] that the set of dimension vectors corresponding to the set of indecomposable representations coincides with the root system of the underlying graph of the quiver. Thus since the root system is independent of the orientation of the arrows, the question of the existence of indecomposables with this root as dimension vector is also independent. Actually, he also proved that there exists a certain parameter family of indecomposables where the number of parameters depends on the Euler form of the quiver. Fixed a root this raises the question which of the representations in the variety of representations are indecomposable. In general, this is very hard to decide. But in many cases it is possible to construct indecomposable representations recursively. Mostly the idea is to fix a set of indecomposables of smaller dimensions and to glue them appropriately in order to get indecomposable representations of greater dimension. In most cases the new indecomposables can be seen as the middle terms of certain exact sequences. Examples for this are Ringel’s reflection functor, see [10], and Schofield induction, see [13]. Also the methods presented in [8, Section 4] and [15] state a recursive construction. Even if it is rarely possible to construct all indecomposable representations of a fixed dimension with these methods, a first step would be to construct an indecomposable tree module for every root. Its existence is conjectured in [12] and proved in several cases. In fact, all of the mentioned constructions can be used to construct tree modules. The only conditions for it are that the smaller indecomposables are tree modules and that we choose tree-shaped bases of the groups of extensions between the respective representations. The existence of indecomposable tree modules is known for exceptional roots, see [11], for imaginary Schur roots, see [15], and for a lot of cases which are mostly obtained by the above mentioned constructions, see [16] as an example.

In this paper we investigate two methods which can be used to construct indecomposable representations. Both based on the same idea: fixing a tuple of representations we can consider the quiver where the vertices are in one-to-one correspondence with the representations and where the number of arrows is the dimension of the respective groups of extensions. This gives a functor from the representation category of the new quiver to the representation category of the original one. In general, it is not clear which properties this functor has. In most cases, it
does not preserve morphism spaces or even indecomposability. We consider two cases where
this functor is either a fully faithful embedding or at least preserves indecomposability.

In a sense the first case fills the gap to quivers with oriented cycles which are not covered
by [13]. More detailed, we fix a sequence of (possibly exceptional) Schur representations
such that there are pairwise no homomorphisms between them. Here the induced quiver
may have oriented cycles. This gives a huge number of examples which are not covered by
the methods mentioned above. Moreover, even in the cases which are also covered by other
functors or methods, it is often easier to control the recursive construction in this way.

The second method, which we investigate, can be seen as a generalization of some of
the functors considered by Ringel in [10]. In there, initially, an exceptional representation
is fixed. Then different categories as the induced perpendicular categories are considered.
For instance, fixed an indecomposable representation which has neither homomorphisms nor
extensions to the exceptional representation, this functor can be used to construct a new in-
decomposable representation by gluing the exceptional representation a prescribed number
of times. In a sense, this can be seen as a functor from the category of the real root represen-
tation of dimension $(1, d)$ of the quiver which has two vertices and $d$ arrows in one direction
to the category of representations of the original quiver. Here $d$ denotes the dimension of
the extension space. We generalize this to the effect that we allow exceptional sequences of
representations with pairwise vanishing homomorphism spaces instead of one exceptional
representation. Fixed an indecomposable representation in one of the corresponding perpen-
dicular categories, we can again consider the quiver as prescribed by the extension spaces.
In this situation we also obtain a functor from the category of representations of this quiver
to the one of representations of the original quiver. But here the only restriction we do have
for the dimension vectors is that the dimension at the sink or source corresponding to the
fixed indecomposable is one. This again gives a huge number of new examples.

We should mention that the existence of indecomposable tree modules for every root is
still an open problem. Actually it seems that “most of the cases” are already covered. But
it is hard to classify those roots which are not covered by at least one of the constructions
mentioned in the introduction. Even a counterexample which can not be constructed with
these methods is not known to me.

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2. Recollection and Notation

2.1. Representations of quivers. Let $k$ be an algebraically closed field. Let $Q = (Q_0, Q_1)$
be a quiver with vertices $Q_0$ and arrows $Q_1$ denoted by $\rho : i \rightarrow j$ for $i, j \in Q_0$. A vertex
$i \in Q_0$ is called sink if there does not exist an arrow $\rho : i \rightarrow j \in Q_1$. A vertex $j \in Q_0$ is called
source if there does not exist an arrow $\rho : i \rightarrow j \in Q_1$. In the following we only consider
quivers without loops, i.e. arrows $\rho : i \rightarrow i$.

Define the abelian group

$$
\mathbb{Z}Q_0 = \bigoplus_{i \in Q_0} \mathbb{Z}i
$$

and its monoid of dimension vectors $\mathbb{N}Q_0$.

A finite-dimensional $k$-representation of $Q$ is given by a tuple

$$
X = ((X_i)_{i \in Q_0}, (X_{\rho} : X_i \rightarrow X_j)_{\rho : i \rightarrow j \in Q_1})
$$
of finite-dimensional $k$-vector spaces and $k$-linear maps between them. The dimension vector $\dim X \in \mathbb{N}Q_0$ of $X$ is defined by $\dim X = \sum_{i \in Q_0} \dim_k X_i$. We denote by $\text{Rep}(Q)$ the category of finite-dimensional representations of $Q$. By $S_q$ we denote the simple representation corresponding to the vertex $q$.

Let $\alpha \in \mathbb{N}Q_0$ be a dimension vector. The variety $R_\alpha(Q)$ of $k$-representations of $Q$ of dimension vector $\alpha$ is defined as the affine $k$-space

$$R_\alpha(Q) = \bigoplus_{\rho : i \to j \in Q_1} \text{Hom}_k(k^{\alpha_i}, k^{\alpha_j}).$$

On $\mathbb{Z}Q_0$ we have a non-symmetric bilinear form, the Euler form, which is defined by

$$\langle \alpha, \beta \rangle = \sum_{i \in Q_0} \alpha_i \beta_i - \sum_{\rho : i \to j \in Q_1} \alpha_i \beta_j$$

for $\alpha, \beta \in \mathbb{Z}Q_0$.

For more details concerning the connection between roots and indecomposable representations we refer to [6, Section 2]. We only give a short summary. A dimension vector is called a root if there exists an indecomposable representation of this dimension. It is called Schur root if there exists a representation with trivial endomorphism ring. By $W(Q)$ we denote the Weyl group of the quiver $Q$. A root $\alpha \in \mathbb{N}Q_0$ is called real if we have $\alpha \in W(Q)Q_0$, i.e. $\alpha$ arises by reflecting a simple root. All the other roots are called imaginary. Recall that a root is real if and only if $\langle \alpha, \alpha \rangle = 1$ and imaginary if and only if $\langle \alpha, \alpha \rangle \leq 0$. In the following we do not always distinguish between a real root and the unique indecomposable representation of this dimension. If $\alpha$ is a real root of a quiver, we will denote the unique indecomposable representation of this dimension by $X_\alpha$.

Let $(\alpha, \beta) := \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle$ be the symmetrized Euler form. The fundamental domain $F(Q)$ of $\mathbb{N}Q_0$ is given by the dimension vectors $\alpha$ with connected support such that $\langle \alpha, q \rangle \leq 0$ for all $q \in Q_0$. Moreover, we have $\alpha \in W(Q)F(Q)$ for all imaginary roots $\alpha$.

For the remaining part of this section we assume that $Q$ has no oriented cycles. Let $X$ and $Y$ be two representations of a quiver $Q$. We consider the linear map

$$d_{X,Y} : \bigoplus_{i \in Q_0} \text{Hom}_k(X_i, Y_i) \to \bigoplus_{\rho : i \to j \in Q_1} \text{Hom}_k(X_i, Y_j)$$

defined by $d_{X,Y}(f_i)_{i \in Q_0} = (Y_{\rho} f_i - f_j X_{\rho})_{\rho : i \to j \in Q_1}$. Then we have $\ker(d_{X,Y}) = \text{Hom}(X, Y)$ and $\text{coker}(d_{X,Y}) = \text{Ext}(X, Y)$, see [9].

Recall that every morphism $f \in \bigoplus_{\rho : i \to j \in Q_1} \text{Hom}_k(X_i, Y_j)$ defines an exact sequence $E(f) \in \text{Ext}(X, Y)$ by

$$0 \to Y \to ((Y_i \oplus X_i)_{i \in Q_0}, \left(\begin{array}{cc} Y_{\rho} & f_{\rho} \\ 0 & X_{\rho} \end{array}\right)_{\rho \in Q_1}) \to X \to 0$$

with the canonical inclusion on the left hand side and the canonical projection on the right hand side. Then it is straightforward to check that two sequences $E(f)$ and $E(g)$ are equivalent if and only if $f - g \in \text{Im}(d_{X,Y})$.

By [9], for two representations $X, Y$ of $Q$ we have

$$\langle \dim X, \dim Y \rangle = \dim_k \text{Hom}(X, Y) - \dim_k \text{Ext}(X, Y)$$

and $\text{Ext}^i_k(X, Y) = 0$ for $i \geq 2$.

If some property is independent of the point chosen in some open subset $U$ of $R_\alpha(Q)$, following [13], we say that this property is true for a general representation of dimension vector $\alpha \in \mathbb{N}Q_0$. 
Theorem 2.1. follows: summands, extensions, images, kernels and cokernels. From [14, Theorems 2.3 and 2.4] it sequence.

of Rep(Q) has the same objects as C and a full subcategory S⊥ of Rep(Q) respectively such that these quivers have n−r vertices and no oriented cycles.

(2) There is an isometry with respect to the Euler form between the dimension vectors of Q⊥S (resp. Q(S⊥)) and the dimension vectors of S⊥ (resp. S⊥) given by \( \Phi((d_1, \ldots, d_{n-r})) = \sum_{i=1}^{n-r} d_i \beta_i \) where \( \beta_1, \ldots, \beta_{n-r} \) are the dimension vectors of the simple representations of the perpendicular categories.

For an exceptional sequence \((E_1, \ldots, E_r)\) we define \( C(E_1, \ldots, E_r) \) as the full subcategory of Rep(Q) which contains \( E_1, \ldots, E_r \) and which is closed under extensions, kernels of epimorphisms and cokernels of monomorphisms. Suppose that \( E = (E_1, \ldots, E_r) \) is a reduced exceptional sequence. Then [9, Lemma 2.35] we have that \( E_1, \ldots, E_r \) are the simple objects of \( C(E_1, \ldots, E_r) \). Thus, by Theorem 2.1 it follows that the category \( C(E_1, \ldots, E_r) \) is equivalent to the category of representations of the quiver \( Q' \) having \( r \) vertices \( q_1, \ldots, q_r \) and \( \text{ext}(e_i, e_j) = \dim \text{Ext}(E_i, E_j) \) arrows from \( q_i \) to \( q_j \). In the following we denote this quiver by \( Q(E) \).

Thus an immediate consequence of Theorem 2.1 is the following, see also [13, Section 2] and [2, Theorem 2.38]:

Corollary 2.2. Let \( \epsilon = (\epsilon_1, \ldots, \epsilon_r) \) be a reduced exceptional sequence. Then \( \alpha = \sum_{i=1}^{r} k_i \epsilon_i \) is a root of \( Q \) if and only if \( (k_1, \ldots, k_r) \) is a root of \( Q(\epsilon) \).

2.3. Ringel’s reflection functor. We review some of the results of [10]. Fixing an exceptional module \( S \) and a full subcategory \( \mathcal{C} \) of Rep(Q) we denote by \( \mathcal{C}/S \) the category which has the same objects as \( \mathcal{C} \) and the same maps modulo those factorizing through \( \bigoplus_{i=1}^{n} S \) for some \( n \in \mathbb{N} \). We define the following full subcategories of Rep(Q):

(1) \( \mathcal{M}^{-S} = \{ X \in \text{Rep}(Q) \mid \text{Hom}(X, S) = 0 \} \)
\[ M^{-S} = \{ X \in \text{Rep}(Q) \mid \text{Hom}(S,X) = 0 \} \]

(3) \( M^S \) as the category of representations \( X \in \text{Rep}(Q) \) with \( \text{Ext}(S,X) = 0 \) such that, moreover, there does not exist a direct summand of \( X \) which can be embedded into a direct sum of copies of \( S \).

(4) \( M_S \) as the category of representations \( X \in \text{Rep}(Q) \) with \( \text{Ext}(X,S) = 0 \) such that, moreover, no direct summand of \( X \) is a quotient of a direct sum of copies of \( S \).

Let \( X \in M^S \) and \( B := \{ \varphi_1, \ldots, \varphi_n \} \) be a basis of \( \text{Hom}(X,S) \). Following [10], there exists an exact sequence

\[ 0 \to X^{-S} \to X \to \bigoplus_{i=1}^n S \to 0 \]

induced by \( B \) such that the induced sequences \( e_1, \ldots, e_n \) form a basis of \( \text{Ext}(S,X^{-S}) \). Moreover, we have \( X^{-S} \in M^{-S} \). The other way around, if \( Y \in M^{-S} \) and \( \{ e_1, \ldots, e_n \} \) is a basis of \( \text{Ext}(S,Y) \) we have an induced sequence

\[ 0 \to Y \to Y^S \to \bigoplus_{i=1}^n S \to 0 \]

such that \( Y^S \in M^S \). We can proceed similarly for \( X \in M^S \) and \( Y \in M^{-S} \). Then we have the following theorem:

**Theorem 2.3.**

(1) There exists an equivalence of categories given by the functor \( F : M^S/S \to M^{-S}, X \mapsto X^{-S} \).

(2) There exists an equivalence of categories given by the functor \( G : M_S/S \to M_S^{-S}, X \mapsto X^{-S} \).

(3) There exist equivalences \( \Psi : M^S_{-S} \to M^S_{-S} \) and \( \Phi : M^S/S \to M^S_{-S} \) induced by composing the functors from above.

If \( \alpha = \text{dim} S \) we define \( M_\alpha := M^S/S \). We proceed in the same manner in the other three cases.

### 2.4. Tree modules

We introduce coefficient quivers and tree modules following [11], see also [15]. Let \( X \) with \( \text{dim} X = \alpha \) be a representation of \( Q \). A basis of \( X \) is a subset \( B \) of \( \bigoplus_{q \in Q_0} X_q \) such that

\[ B_q := B \cap X_q \]

is a basis of \( X_q \) for all vertices \( q \in Q_0 \). For every arrow \( \rho : i \to j \) we may write \( X_\rho \) as a \((\alpha_j \times \alpha_i)\)-matrix \( X_\rho;B \) with coefficients in \( k \) such that the rows and columns are indexed by \( B_j \) and \( B_i \), respectively.

**Definition 2.4.** The coefficient quiver \( \Gamma(X,B) \) of a representation \( X \) with a fixed basis \( B \) has vertex set \( B \) and arrows between vertices are defined by the condition: if \( (X_\rho;B)_{b,b'} \neq 0 \), there exists an arrow \( (\rho,b,b') : b \to b' \).

A representation \( X \) is called a tree module if there exists a basis \( B \) for \( X \) such that the corresponding coefficient quiver is a tree.

In order to construct a tree module and its coefficient quiver respectively, it is often useful to consider the universal covering quiver \( \tilde{Q} \) of the given quiver \( Q \). To do so, we use the notation of [15 Section 2].
Let $W_{Q} = \{ a, a^{-1} \mid a \in Q_{1} \}$ the set of words in $Q_{1}$. Every representation $\tilde{X}$ of $\tilde{Q}$ gives rise to a representation of $X$ of $Q$ in the following way:

$$X_{q} = \bigoplus_{w \in W_{Q}} \tilde{X}_{(q, w)}, \quad q \in Q_{0}$$

and $X_{a} : X_{s} \to X_{t}$ is defined by $X_{a}|_{X_{(q, w)}} = \tilde{X}_{a(q, w)}$. Now we can make use of the following result, see \cite[Lemma 3.5]{4}:

**Theorem 2.5.** If $\tilde{X}$ is an indecomposable representation of $\tilde{Q}$, the corresponding representation $X$ of $Q$ is also indecomposable.

Note that it is straightforward to check that every indecomposable tree module is already a representation of a connected component of the universal cover.

For the simple reflection $\tilde{s}_{q} \in W(\tilde{Q})$ at $q$ define $\tilde{s}_{q} := \prod_{w \in W(\tilde{Q})} \tilde{s}(q, w) \in W(Q)$ where $\tilde{s}(q, w)$ is the simple reflection at $(q, w)$. Note that the product can be assumed to be finite when applied to a dimension vector with finite support and, moreover, that $\tilde{s}(q, w) \cdot \tilde{s}(q, w') = \tilde{s}(q, w') \cdot \tilde{s}(q, w)$. Thus $\tilde{s}_{q}$ is independent of the chosen order. Now we can recursively define $\tilde{w}$ for every $w \in W(Q)$.

Recall the conjecture of Ringel in \cite{12} saying that there exists an indecomposable tree module for every root of a quiver without oriented cycles. The following statement assures that it is enough to show the existence of indecomposable tree modules for all roots of quivers which are trees:

**Proposition 2.6.** For every root $\alpha$ of $Q$ there exists a root $\tilde{\alpha}$ of $\tilde{Q}$ such that $\sum_{w \in W_{Q}} \tilde{\alpha}(q, w) = \alpha_{q}$.

**Proof.** The statement is clear for simple roots. If $\alpha$ is real, then there exists a $w \in W(Q)$ such that $\alpha = w \cdot e_{q}$ for a simple root $e_{q}$. Thus $\tilde{\alpha} = w \cdot e_{(q, 1)}$, where 1 denotes the empty word, satisfies the statement. Thus let $\alpha$ be an imaginary root. We can assume that $\alpha \in F(Q)$. Indeed, for a general imaginary root we can again apply reflections like we did in the case of real roots.

Every root in the fundamental domain is a Schur root because otherwise there exists a real Schur root $\beta$ in the canonical decomposition such that $s_{\beta}(\alpha) < \alpha$. Thus, since

$$s_{\beta}(\alpha) = \alpha - \sum_{i \in Q_{0}} \beta_{i}(\alpha, e_{i}) \beta < \alpha,$$

we have $(\alpha, e_{i}) > 0$ for at least one $i \in Q_{0}$.

By \cite[Theorem 3.19]{15} there exists an indecomposable tree module for every Schur root and thus for every root in the fundamental domain. But this is, as already mentioned, a representation of the universal covering quiver. $\square$

In the last part of this section, we want to discuss a question related to Ringel’s conjecture. We fix an indecomposable representation $X$ of a quiver $Q$ with $\alpha := \dim X$ and a basis $B$ of $X$. Then we denote by $a(X, B)$ the number of arrows of the coefficient quiver $\Gamma(X, B)$. There are two natural questions related to the conjecture of Ringel:

- What is $m(X) := \min\{ a(X, B) \mid B \text{ basis of } X \}$?
- What is $m(\alpha) := \max\{ m(X) \mid X \text{ indecomposable, } \dim X = \alpha \}$?
Since Kac’s Theorem predicts a $1-\langle \alpha, \alpha \rangle$-parameter family of indecomposable representation, the first guess could be that we have

$$\sum_{i \in Q_0} \alpha_i - 1 \leq m(X) \leq \sum_{i \in Q_0} \alpha_i - \langle \alpha, \alpha \rangle$$

and $m(\alpha) = \sum_{i \in Q_0} \alpha_i - \langle \alpha, \alpha \rangle$. Note that the first inequality trivially follows from the indecomposability of $X$.

For exceptional representations the inequalities clearly hold because they are tree modules by [11]. But the second inequality is already wrong for the Kronecker quiver $Q = K(2)$ and $\alpha = (2, 2)$. Indeed, up to isomorphism and permutation of the two linear maps, the indecomposables are given by

$$X(\lambda) := ((k^2, k^2), \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right), \left(\begin{array}{cc} \lambda & 1 \\ 0 & \lambda \end{array}\right)))$$

where $\lambda \in k$. Thus we actually have $m(X(\lambda)) = 5$ if $\lambda \neq 0$. But there seem to be many examples where the inequalities are true. One of these is obtained when considering the $n$-subspace quiver, i.e. $Q_0 = \{q_0, \ldots, q_n\}$ and $Q_1 = \{\rho_i : q_i \rightarrow q_0 \mid i = 1, \ldots, n\}$, and the root $(2, 1, \ldots, 1)$.

This raises the questions for which representations or roots these inequalities are true and for which roots do we have $m(\alpha) = \sum_{i \in Q_0} \alpha_i - \langle \alpha, \alpha \rangle$ respectively?

3. Recursive construction of indecomposable representations

In this section we present two methods which can be used to construct indecomposable representations recursively. As already mentioned in the introduction, both base on the same idea: we fix a tuple of representation and consider the quiver where the vertices are in one-to-one correspondence with the representation and where the number of arrows is equal to the dimension of the respective extension spaces. Under some additional conditions it can be seen that the resulting functor has good properties. In particular, it preserves indecomposability.

In a sense, the first case fills the gap to quivers with oriented cycles which are not covered by [13]. Moreover, the second case, which we investigate, includes some of the functors considered by Ringel in Theorem 2.3, see also Remark 3.12.

3.1. Indecomposable representations obtained by gluing Schur representations.

Let $Q$ be quiver without oriented cycles. Recall from Section 2.2 that, if we fix a reduced exceptional sequence $E$ in $Q$, the exceptional representations of this sequence are precisely the simple objects of the category $\langle -E \rangle^\perp$. This gives a functor from the category $\text{Rep}(Q(E))$ to $\text{Rep}(Q)$ which is a fully faithful embedding. We generalize this functor to the effect that we do not restrict to reduced exceptional sequences as the building blocks of our categories. We rather allow extensions in both directions whence not admitting homomorphisms. Moreover, we do not restrict to exceptional representations.

**Definition 3.1.** A sequence of Schur roots $\epsilon = (\epsilon_1, \ldots, \epsilon_r)$ such that $\text{hom}(\epsilon_i, \epsilon_j) = 0$ for $i \neq j$ is called an elementary sequence (of Schur roots). It is additionally called exceptional if all roots are exceptional.
We fix an elementary sequence $\epsilon = (\epsilon_1, \ldots, \epsilon_r)$. Then we choose Schur representations $E_1, \ldots, E_r$ with $\dim E_i = \epsilon_i$ with $\text{Hom}(E_i, E_j) = 0$ for $i \neq j$ and subsets

$$B_{ij} = \{\chi_{i,j}^{1}, \ldots, \chi_{n_{ij}}^{i,j}\} \subseteq \bigoplus_{\rho : q \rightarrow q' \in Q_i} \text{Hom}((E_i)_q, (E_j)_{q'})$$

such that the corresponding residue classes are a basis of $\text{Ext}(E_i, E_j)$. Recall that we can always assume that this basis is tree-shaped, i.e. every tuple $\chi_{i,j}^{l}$ has only one entry which is not zero, see for instance [15, Section 3.4]. This is very helpful in order to construct indecomposable tree modules.

As before we can derive a quiver $Q(\epsilon)$ from the elementary sequence having vertices $q_1, \ldots, q_r$ and $-\langle \epsilon_i, \epsilon_j \rangle = n_{ij}$ arrows from $q_i$ to $q_j$ for $i \neq j$. We denote these arrows by $\chi_{i,j}^{l}$ for $l = 1, \ldots, n_{ij}$ in the following.

We define a functor $F : \text{Rep}(Q(\epsilon)) \rightarrow \text{Rep}(Q)$ which will turn to be a fully faithful embedding. Thus it also preserves indecomposable representations. In order to prove this we mainly proceed along the lines of [8, Section 4]. In [8], W. Peternell restricts to the case where the quiver $Q(\epsilon)$ is bipartite, i.e. every vertex is either a sink or a source. In particular, in this case every representation of $Q$ is already given as the middle term of an exact sequence. In return, in [8] morphisms between certain representations are allowed and, moreover, the representations under consideration do not have to be Schurian.

Let $X$ be a representation of $Q(\epsilon)$ and denote $X_{q,i} := (E_i)_q \otimes_k X_q$. Define a representation $FX$ of $Q$ by the vector spaces

$$(FX)_q = \bigoplus_{i=1}^{r} X_{q,i} \text{ for all } q \in Q_0$$

and for $\rho : q \rightarrow q'$ we define linear maps $(FX)_\rho = \bigoplus_{i=1}^{r} X_{q,i} \rightarrow \bigoplus_{i=1}^{r} X_{q',i}$ by

$$((FX)_\rho)_{i,j} = (E_i)_\rho \otimes_k \text{id}_{X_q} : X_{q,i} \rightarrow X_{q',i}$$

and

$$((FX)_\rho)_{i,j} = \sum_{l=1}^{n_{ij}} (\chi_{j,i}^{l})_{\rho} \otimes_k X_{q,j}^{l,j} : X_{q,j} \rightarrow X_{q',i}$$

for $i \neq j$.

Let $(f_{q,i})_{i=1,\ldots,r} : X \rightarrow X'$ be a morphism. Then we define $Ff : FX \rightarrow FX'$ by

$$(Ff)_{q,i,j} := \begin{cases} \text{id}_{(E_i)_q} \otimes_k f_{q,i} : X_{q,i} \rightarrow X'_{q,j} & \text{if } i = j \\ 0 : X_{q,i} \rightarrow X'_{q,j} & \text{if } i \neq j \end{cases}$$

If $X$ is a representation of $Q(\epsilon)$, by $S_iX$ we denote the representation defined by

$$(S_iX)_{q,j} = \begin{cases} X_q & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}$$

Note that $FS_iX = E_i^{\dim X_q}$. Therefore, there exist isomorphisms

$$\Theta^{ij}_{X,X'} : \bigoplus_{\rho : q \rightarrow q_i} \text{Hom}_k(S_jX, S_iX') \cong \text{Ext}(E_j^{\dim X_q}, E_i^{\dim X'_q})$$

induced by $(\phi_i)_i \mapsto \sum_{l=1}^{n_{ij}} (\chi_{j,i}^{l})_{\rho} \otimes_k \phi_l)_{\rho \in Q_i} = F((\phi_l)_l)$. It is straightforward to check that $\Theta^{ij}$ is functorial in $X$ and $X'$. This means that, for a morphism $f = (f_i)_{i=1,\ldots,r} : X' \rightarrow Y$
with $X, Y, X' \in \text{Rep}(Q(\epsilon))$, we have $\Theta_{X,X'}^j(f_i \circ (\phi_l)_l) = F(f_i \circ (\phi_l)_l) = F(f_i) \circ F((\phi_l)_l)$. We obtain the analogue statement for morphisms $f : X \to Y$.

Since we have $\text{Hom}(E_i, E_i) = k$, the functor $F$ induces a fully faithful embedding $F|_{Q_i} : \text{Rep}(Q(\epsilon_i)) \to \text{add}(E_i)$ where here $\text{add}(E_i)$ is the full subcategory of $\text{Rep}(Q)$ whose objects consist of direct sums of $E_i$.

**Lemma 3.2.** Let $X$ and $X'$ be two representations of $Q(\epsilon)$. Let $r = 2$ and assume that $X_{\lambda_i}^{21} = X'_{\lambda_i}^{21} = 0$ for $l = 1, \ldots, n_{12}$. Then for every morphism $f : FX \to FX'$ we have

$$f = (f_q)_{q \in Q_0} = \begin{pmatrix} (f_q)_{1,1,1} & 0 \\ 0 & (f_q)_{2,2} \end{pmatrix}$$

with $f_{ii} = ((f_q)_{ii})_{q \in Q_0} \in \text{Hom}_Q(FS_{i}X, FS_{i}X')$.

**Proof.** Let $f : FX \to FX'$ be a morphism. Thus we have $(FX')_{\rho} \circ f_q = f_q \circ (FX)_{\rho}$ for every $\rho : q \to q'$. Since $X_{\lambda_i}^{21} = X'_{\lambda_i}^{21} = 0$ for $l = 1, \ldots, n_{12}$ by assumption, we have $((FX)_\rho)_{2,1} = 0$ and $((FX')_\rho)_{2,1} = 0$. Using $\text{Hom}_Q(E_1, E_2) = 0$ we obtain that $f$ is of the form

$$f = (f_q)_{q \in Q_0} = \begin{pmatrix} (f_q)_{1,1,1} & (f_q)_{1,2} \\ 0 & (f_q)_{2,2} \end{pmatrix}$$

with morphisms $f_{ii} \in \text{Hom}_Q(FS_{i}X, FS_{i}X')$ and linear maps

$$f_{1,2} := ((f_q)_{1,2})_{q \in Q_0} \in \bigoplus_{q \in Q_0} \text{Hom}_k(X_{q,2}, X'_{q,1}).$$

Moreover, we get

$$d_{FS_{2}X, FS_{1}X'}(f_{1,2}) = \begin{pmatrix} ((FX')_{\rho})_{1,1} \circ (f_q)_{1,2} - ((FX')_{\rho})_{1,2} \circ f_q \\ ((FX')_{\rho})_{1,2} \circ (f_q)_{1,2} - ((FX')_{\rho})_{2,2} \circ (f_q)_{2,2} \\ ((FX')_{\rho})_{2,2} \circ (f_q)_{2,2} \end{pmatrix}_{\rho : q \to q' \in Q_1}$$

where $d_{FS_{2}X, FS_{1}X'}$ is the map introduced in Section 2. In particular, the induced exact sequence splits. Since $F$ induces a fully faithful embedding when restricted to the quiver with one vertex, there exist morphisms $\varphi \in \text{Hom}_Q(S, X, S, X)$ such that $f_{ii} = F(\varphi_i)$. Using the functoriality of $\Theta$ for $(g_{\rho})_{\rho \in Q(\epsilon)} := ((\varphi)_q \circ X_{\rho} - X'_{\rho} \circ (\varphi_2)_{q})_{\rho \in Q(\epsilon)}$, we get

$$\Theta_{X,X'}^{j_2}(g_{\rho})_{\rho \in Q(\epsilon)} = (F(\varphi)_{q} \circ ((FX')_{\rho})_{1,2} - ((FX')_{\rho})_{2,2} \circ F(\varphi)_{q})_{\rho : q \to q' \in Q_1}$$

$$= ((f_q)_{1,1,1} \circ ((FX')_{\rho})_{1,2} - ((FX')_{\rho})_{2,2} \circ (f_q)_{2,2})_{\rho : q \to q' \in Q_1}$$

$$= d_{FS_{2}X, FS_{1}X'}(f_{1,2})$$

But since $\Theta$ is an isomorphism, this already means that $d_{FS_{2}X, FS_{1}X'}(f_{1,2}) = 0$ and thus $f_{1,2} \in \text{Hom}_Q(FX, FX')$. But since $\text{Hom}_Q(E_1, E_2) = 0$, it follows that $f_{1,2} = 0$.

**Theorem 3.3.** $F$ is a fully faithful embedding. In particular, $FX$ is indecomposable if and only if $X$ is indecomposable.

**Proof.** We first show that $F$ is faithful. Let $f : X \to X'$ be a morphism. We have to show that $(FX')_{\rho} \circ (Ff)_{q} = (Ff)_{q'} \circ (FX)_{\rho}$ for all $\rho : q \to q'$. Using that $f$ is a morphism and
Hom$_Q(E_i, E_j) = 0$ we obtain

$$((FX')_\rho \circ (Ff)_q)_{i,j} = \sum_{l=1}^{r} ((FX')_\rho)_{i,l} \circ ((Ff)_q)_{l,j} = ((FX')_\rho)_{i,j} \circ ((Ff)_q)_{j,j}$$

$$= \left( \sum_{l=1}^{n_{ji}} (\chi_{ji}^{\ell})_\rho \otimes_k X_{\chi_{ji}^{\ell}}' \right) = \text{id}_{(E_q)_q} \otimes_k f_{q,j}$$

$$= \sum_{l=1}^{n_{ji}} (\chi_{ji}^{\ell})_\rho \otimes_k f_{q,i} \circ X_{\chi_{ji}^{\ell}}'$$

$$= \text{id}_{(E_q)_q} \otimes_k f_{q,i} \circ \left( \sum_{l=1}^{n_{ji}} (\chi_{ji}^{\ell})_\rho \otimes_k X_{\chi_{ji}^{\ell}}' \right)$$

$$= ((Ff)_q)_{i,i} \circ ((FX)_\rho)_{i,j} = \sum_{l=1}^{r} ((Ff)_q')_{i,l} \circ ((FX)_\rho)_{l,j}$$

$$= ((Ff)_q') \circ (FX)_\rho)_{i,j}.$$ 

This shows that $F$ is faithful. Thus let $f : FX \to FX'$ be a morphism. It induces linear maps $(f_q)_{i,j} : X_{q,j} \to X_{q,i}$ for all $q \in Q_0$ and all $1 \leq i, j \leq r$ such that

$$\sum_{k=1}^{r} ((FX')_\rho)_{i,k} \circ (f_q)_{k,j} = \sum_{k=1}^{r} (f_q')_{i,k} \circ ((FX)_\rho)_{k,j}$$

for all $\rho \in Q_1$. Thus we get

$$e_{i,j} = ((FX')_\rho)_{i,i} \circ (f_q)_{i,j} - (f_q')_{i,j} \circ ((FX)_\rho)_{j,j}$$

$$= \sum_{k=1}^{r} (f_q')_{i,k} \circ ((FX)_\rho)_{k,j} - \sum_{k=1}^{r} ((FX')_\rho)_{i,k} \circ (f_q)_{k,j} \in \text{Hom}_k((FS_jX)_q, (FS_iX')_{q'}).$$

But this means that $((e_{i,j})_\rho)_{\rho \in Q_1} = d_{FS_iX, FS_jX'}((f_q)_{i,j})_{q \in Q_0}$. In particular, the induced exact sequence $((e_{i,j})_\rho)_{\rho \in Q_1} \in \text{Ext}(FS_jX, FS_iX')$ splits. Using Lemma 3.2 there exist isomorphisms $g_i \in \text{End}(FS_iX) = \text{End}(E_i^{\dim X_i})$ and $g_j \in \text{End}(E_j^{\dim X_j})$ such that $(g_j)_q' \circ (e_{i,j})_\rho \circ (g_i)^{-1} = 0$. Indeed, the isomorphism $g$ between the middle term of the exact sequence $E((e_{i,j})_\rho)_{\rho \in Q_1}$ and the middle term of the exact sequence $E(0)$ is a block matrix

$$g = \begin{pmatrix} g_j & 0 \\ 0 & g_i \end{pmatrix}.$$ 

This already means that we have $(e_{i,j})_\rho = 0$ for every $\rho \in Q_1$. Now since Hom$_Q(E_i, E_j) = 0$ for $i \neq j$, we have $f_{i,j} = 0$ for $i \neq j$. Thus it follows that $f_{i,i} \in \text{End}(FS_iX)$. Since $F$ induces a fully faithful embedding $F|_{q_0} : \text{Rep}(Q(e_1)) \to \text{add}(E_i)$, the claim follows.

**Example 3.4.**

Maybe the first non-trivial example is obtained when considering the 4-subspace quiver and the two exceptional representations $X_\alpha$ and $X_\beta$ where $\alpha = (1, 1, 1, 0, 0)$ and $\beta = (1, 0, 0, 1, 1)$. 
Remark 3.5.

A basis of \( \text{Ext}(X_\alpha, X_\beta) \) is for instance induced by \( e_{q_1,q_0} : (X_\alpha)_1 \to (X_\beta)_0, 1 \mapsto 1 \). A basis of \( \text{Ext}(X_\beta, X_\alpha) \) is for instance induced by \( e_{q_3,q_0} : (X_\beta)_3 \to (X_\alpha)_0, 1 \mapsto 1 \). Since the indecomposable representation of \( Q(\alpha, \beta) \) have dimension vectors \((n, n + 1), (n + 1, n)\) and \((n, n)\) for \( n \in \mathbb{N} \), in this way we can construct the real root representations of \( Q \) of dimensions \((2n + 1, n, n, n + 1, n + 1)\) and \((2n + 1, n + 1, n + 1, n, n)\) respectively. Moreover, we can construct a \( \mathbb{P}^1 \)-family of indecomposable representations of dimension \((2n, n, n, n, n)\). Note that this family depends on the chosen basis. Moreover, note that if \( X \) is a tree module of \( Q(\alpha, \beta) \), then \( FX \) is tree module of \( Q \). Moreover, this gives an embedding \( \text{Rep}(\tilde{A}_2) \to \text{Rep}(\tilde{D}_4) \) where \( \tilde{A}_2 \) and \( \tilde{D}_4 \) are considered with the mentioned orientations.

**Remark 3.5.**

For every root \( \alpha \) of a quiver with \( n \) vertices \( \{1, \ldots, n\} \) there exists the trivial exceptional sequence \((S_1, \ldots, S_n)\) such that \( \alpha = \sum_{i=1}^n \alpha_i \cdot \dim S_i \). A basic question is if one can find a non-trivial reduced exceptional (resp. exceptional elementary) sequence for every root of a (tree) quiver such that the root is a positive linear combination of the roots contained in the sequence. This would already imply the existence of indecomposable tree modules for every root. Indeed, we could start an induction on the set of roots of all quivers.

An algorithm to obtain a possibly trivial exceptional sequence for a root, which is no Schur root, is given as follows (recall that Schur roots are covered by [15] Section 3): we start with the canonical decomposition of \( \alpha \), say \( \alpha = \bigoplus_{i=1}^n \alpha_i \cdot \dim S_i \), see [13] and [2] for an algorithm to compute it and some additional properties. Then either \( \alpha_1 \) or \( \alpha_n \) is exceptional. We also know that \( \beta := \alpha - \sum_{i=2}^n \alpha_i \cdot \dim S_i \). Thus if \( \alpha_1 \) is exceptional, there exists a decomposition \( \beta = \sum_{i=2}^t \beta_i \) into exceptional roots \( \beta_1, \ldots, \beta_t \) such that \((\beta_1, \ldots, \beta_t)\) is a reduced exceptional sequence. Such a decomposition exists because we can always take the simple roots in \( \alpha_1 \). In particular, \((\alpha_1, \beta_1, \ldots, \beta_t)\) is an exceptional sequence. Note that we have \( \text{ext}(\beta, \alpha_1) = 0 \) because we started with the canonical decomposition. If there exists a \( \beta_i \) such that \( \text{hom}(\beta_i, \alpha_1) \neq 0 \), i.e. the sequence is not satisfying the requested properties, we can proceed in the same manner and decompose \( \gamma := \alpha_1 + \sum_{i=1}^{t-1} \beta_i \) into exceptional roots of \( \alpha_1 \). We proceed in the same manner until we obtain a reduced exceptional sequence. This algorithm terminates because at some point we obtain a reduced exceptional sequence. Indeed, because the roots are getting smaller in every step, we end up with the trivial exceptional sequence at some point.

Unfortunately, it seems that there does not always exist a non-trivial decomposition as Example 3.6 suggests. At least the algorithm from above is not applicable to obtain a non-trivial sequence in that case. The next question would be if there is an elementary exceptional sequence for every root.

**Example 3.6.**

Let \( Q \) be the 5-subspace quiver and the isotropic root \( \alpha = (10, 3, 3, 3, 3, 8) \). Then the canonical decomposition is

\[
\alpha = (6, 2, 2, 2, 2, 4) \oplus (1, 1, 0, 0, 0, 1) \oplus \ldots \oplus (1, 0, 0, 0, 0, 1, 1).
\]

In the first step we get an exceptional sequence

\[
((3, 1, 1, 1, 1, 0), S_5, (1, 1, 0, 0, 0, 1), \ldots, (1, 0, 0, 0, 0, 1, 1)).
\]
In the next step we obtain the sequence
\[((3, 1, 1, 1, 1, 0), (1, 1, 0, 0, 0, 0), \ldots, (1, 0, 0, 0, 1, 0), S_5).\]
Now it easy to check that we end up with the trivial decomposition. But we get an exceptional
elementary sequence/decomposition
\[\alpha = (1, 1, 0, 0, 0) + (1, 0, 0, 1, 1, 0) + (1, 1, 0, 0, 0, 1)^2 + \ldots + (1, 0, 0, 0, 1, 1)^2.\]
Thus we can construct indecomposables of dimension \(\alpha\) when constructing indecomposables
of the quiver
\[
\begin{tikzpicture}
  \draw (0,0) -- (1,1) -- (2,0) -- (1,-1) -- cycle;
  \draw (1,0) -- (2,1) -- (3,0) -- (2,-1) -- cycle;
  \draw (2,0) -- (3,1) -- (4,0) -- (3,-1) -- cycle;
\end{tikzpicture}
\]
with dimension vector as indicated.

We should mention that we can also construct it by successively applying Ringel’s reflection
functor and thus with the functor of Section 3.2.

**Example 3.7.**

But there are plenty of (even non-Schurian) roots for which the algorithm gives a non-
trivial decomposition. For instance if \(\alpha = (3, 2, 2, 1, 1)\) and \(Q\) is the 4-subspace quiver,
starting with the canonical decomposition \((2, 1, 1, 1, 1) \oplus (1, 1, 1, 0, 0)\) the algorithm terminates
with the non-trivial reduced exceptional sequence
\[((1, 0, 1, 0, 0), (1, 0, 0, 1, 1), S_1).\]

**Example 3.8.**

The indecomposable real root representation considered in [15, Example 4.1], which can
not be constructed using Ringel’s reflection functors, is also covered by the methods of this
section: consider the 8-subspace quiver and the real root \(\alpha = (48, 1, 1, 1, 15, 15, 18, 18, 46)\).
Denoting by \(e_i\) the simple root corresponding to \(q_i\), consider the elementary sequence
\[E := (\beta_1, \beta_2, \beta_3, \beta_4) := (e_0 + e_6 + e_7 + e_8, (2, 1, 1, 1, 0, 2, 2, 0), e_0 + e_4 + e_8, e_0 + e_5 + e_8).\]
Then we have \(\alpha = 16\beta_1 + 13\beta_2 + 15\beta_3 + 15\beta_4\) and the corresponding quiver is
\[
\begin{tikzpicture}
  \draw (0,0) -- (1,1) -- (2,0) -- (1,-1) -- cycle;
  \draw (1,0) -- (2,1) -- (3,0) -- (2,-1) -- cycle;
  \draw (2,0) -- (3,1) -- (4,0) -- (3,-1) -- cycle;
\end{tikzpicture}
\]
where the number in brackets indicates the number of arrows. Now the indecomposable
representation of dimension \((16, 1, 15, 15)\) yields the indecomposable of dimension \(\alpha\).
3.2. Extending representations by exceptional representations. The second functor we want to investigate can be seen as a generalization of parts of Theorem 3.10. The respective functors are obtained when considering the case of sequences of length two, see also Remark 3.12.

For an additive category $C$ we denote by $\mathcal{R}_C$ its radical. Let $E = (M_1, \ldots, M_r)$ be a sequence of indecomposable representations such that the following conditions are satisfied:

1. There exists at most one representation $M_t$ such that $\dim M_t$ is not Schurian and we have $t \in \{1, r\}$.
2. The representations $M_j, j \neq t$ are Schurian.
3. We have $\text{Hom}(M_i, M_j) = \text{Ext}(M_i, M_j) = 0$ for $i < j$.
4. If $i, j \neq t$ we have $\text{Hom}(M_j, M_i) = 0$ for $i < j$.

This means if $t = r$ we allow homomorphisms $M_r \rightarrow M_j$ for $j < t$ and and if $t = 1$ we allow homomorphisms $M_j \rightarrow M_1$ for $j > t$. Moreover, note that the representations $M_j$ do not have to be exceptional. Let $Q(E)$ be the quiver having vertices $q_1, \ldots, q_r$ and $\dim \text{Ext}(M_j, M_i) =: n_{ji}$ arrows from $q_j$ to $q_i$ for $i < j$. Note that condition (3) assures that $Q(E)$ has no oriented cycles.

We choose subsets $B_{ji} = \{\chi_{j,i}^{ji}, \ldots, \chi_{n_{ji}}^{ji}\}$ of $\bigoplus_{\rho \neq \rho' \in Q_1} \text{Hom}((M_i)_q, (M_j)_{q'})$ such that the corresponding residue classes build a (tree-shaped) basis of $\text{Ext}(M_j, M_i)$. Now the functor $F$ and the representations $S_i M$ are defined as in Section 3.1. Moreover, we again have isomorphisms $\Theta_{M_i M_j}^{ji}$. Here we get a fully faithful embedding $F|_{q_i} : \text{Rep}(Q(M_i)) \rightarrow \text{add}(M_i)/\mathcal{R}_{\text{add}(M_i)}$. Recall that, since $M_i$ is indecomposable, we have that $f \in \text{Hom}(M_i, M_i)$ is in the radical if and only if it is no isomorphism. As before one shows that $F$ is faithful.

We need the following lemma:

**Lemma 3.9.** Let $X$ and $Y$ be two representations with $\text{Hom}(X, Y) = 0$. Then every short exact sequence

$$0 \rightarrow X \xrightarrow{\gamma} Z \xrightarrow{\delta} Y \rightarrow 0$$

induces a ring homomorphism $\Phi : \text{End}(Z) \rightarrow \text{End}(X)$ with $\phi \in \ker(\Phi)$ if and only if $\phi$ factors over $Y$.

**Proof.** Let $\phi \in \text{End}(Z)$. Since $\text{Hom}(X, Y) = 0$, by the universal property of the kernel and cokernel respectively, $\phi$ induces two unique endomorphisms $\psi \in \text{End}(X)$ and $\pi \in \text{End}(Y)$. In particular, we get the following commutative diagram:

$$
\begin{array}{ccc}
0 & \xrightarrow{\gamma} & Z & \xrightarrow{\delta} & Y & \xrightarrow{\pi} & 0 \\
\downarrow{\psi} & & & & & & \\
0 & \xrightarrow{\gamma} & Z & \xrightarrow{\delta} & Y & \xrightarrow{\pi} & 0
\end{array}
$$

This defines a ring homomorphism $\Phi : \text{End}(Z) \rightarrow \text{End}(X)$. If $\psi = 0$, we have $\phi \circ \gamma = 0$. Again by the universal property of the cokernel, we obtain a morphism $\iota : Y \rightarrow Z$ such that $\phi = \iota \circ \delta$.

The other way around if $\phi = \iota \circ \delta$ for some $\iota : Y \rightarrow Z$, we obtain $\gamma \circ \psi = \phi \circ \gamma = \iota \circ \delta \circ \gamma = 0$. Since $\gamma$ is injective, we obtain that $\psi = 0$.

We also need the following lemma, see also [1] Corollary I.4.8:

**Lemma 3.10.** Let $Q$ be a quiver and let $X \in \text{Rep}(Q)$ be indecomposable and let $Y$ be a proper factor, representation of $X$. If $\phi \in \text{End}_Q(X)$ is a morphism factoring over $Y$, then $\phi$ is nilpotent.
Proof. Since $X$ is indecomposable, $\text{End}(X)$ is local. Thus we can write $\phi = \phi_d + \phi_n$ such that $\phi_d = \lambda \text{Id}_X$, $\lambda \in k$, and $\phi_n$ is nilpotent. Since $Y$ is a proper factor $\phi$ is not invertible and thus $\lambda = 0$.

With these properties in hand we get the following result:

**Theorem 3.11.** The following holds:

1. We have that $F$ is a faithful functor. If $\dim \text{Hom}(M_i, M_j) = \delta_{i,j}$, then $F$ is full and $FX$ is indecomposable if $X$ is indecomposable.
2. If $M_i$ is imaginary and $\dim X_{q_i} = 1$, we have that $FX$ is indecomposable whenever $X$ is indecomposable.

Proof. The first statement is actually a special case of Theorem 3.3. We may assume that $t = 1$. The case $t = r$ is obtained analogously. Let $X$ be a representation of $Q(E)$. Then there exists a short exact sequence

$$0 \rightarrow FX_1 \xrightarrow{\gamma} FX \xrightarrow{\delta} FX_2 \rightarrow 0$$

for some representation $X_2$ of the full subquiver $Q \setminus \{q_1\}$ of $Q$. Note that $X_1$ is the simple representation $S_{q_1}$ and we have $FX_1 = M_1$. Since we have $\text{Hom}_Q(M_1, M_2) = 0$, every morphism $f \in \text{End}(FX)$ is of the form

$$f = \begin{pmatrix} f_{11} & f_{12} \\ 0 & f_{22} \end{pmatrix}$$

such that we have $f_{11} \in \text{Hom}_Q(FX_1, FX_1)$, $f_{22} \in \text{Hom}_Q(FX_2, FX_2)$ and, moreover, $f_{12} \in \bigoplus_{q \in Q} \text{Hom}_k((FX_2)_q, (FX_1)_q)$. In particular, we get a commutative diagram

$$
\begin{array}{c}
0 \\ F X_1 \xrightarrow{\gamma} FX \xrightarrow{\delta} FX_2 \\
\downarrow f_{11} \quad \quad \quad \downarrow f \\
0 \\
F X_1 \xrightarrow{\gamma} FX \xrightarrow{\delta} FX_2
\end{array}
$$

inducing a ring homomorphism $\Phi : \text{End}(FX) \rightarrow \text{End}(FX_1) = \text{End}(M_1)$ by Lemma 3.9. By the first part of the theorem there exists a morphism $g_{22} \in \text{Hom}_Q(X_2, X_2)$ such that $f_{22} = F(g_{22})$. Moreover, there exists a morphism $g_{11} \in \text{Hom}_Q(X_1, X_1)$ and a morphism $f_{11} \in \mathcal{R}_{\text{add}(M_1)}$ such that $f_{11} = F(g_{11}) + \tilde{f}_{11}$. Since $f$ is a morphism, we obtain

$$
\begin{align*}
(d_{FX_2, FX_1}(f_{12}))_\rho &= (f_{12})_{q'} \circ (FX_2)_\rho - (FX_1)_\rho \circ (f_{12})_q \\
&= ((FX)_\rho)_{12} \circ (f_{22})_{q'} - (f_{11})_{q'} \circ ((FX_2)_\rho)_{12} \\
&= (FX)_\rho \circ (F(g_{22}))_q - (F(g_{11}) + \tilde{f}_{11})_{q'} \circ ((FX_2)_\rho)_{12}
\end{align*}
$$

for every $\rho : q \rightarrow q'$. If $f_{11} = 0$, i.e. $f \in \ker(\Phi)$, we have $g_{11} = 0 = \tilde{f}_{11}$. As in the proof of Theorem 3.3 the functoriality of $\Theta$ yields

$$
((FX)_\rho)_{12} \circ (F(g_{22}))_q = 0
$$

and thus $(X_\rho)_{12} \circ (g_{22})_q = 0 = (g_{11})_{q'} \circ (X_\rho)_{12}$. In particular, $g_{22}$ is already a morphism of $X$. Since $X$ is indecomposable, $g_{22}$ is nilpotent by Lemma 3.10. Thus $F(g_{22}) = f_{22}$ is nilpotent and, therefore, also $f$ is nilpotent. In summary we get an embedding

$$R := \text{End}_Q(FX)/\ker(\Phi) \hookrightarrow \text{End}(M_1)$$
where \( \text{End}(M_1) \) is local. Thus \( R \) is local because it is a subring of a local ring. Since \( \ker(\Phi) \) only consists of nilpotent elements, \( \text{End}(FX) \) is already local. Indeed, obviously every idempotent of \( \text{End}(FX) \) is either nilpotent or a unit. \( \square \)

**Remark 3.12.**

Let \( X \) be a representation of a quiver \( Q \) and \( S \) an exceptional representation such that \( \text{Hom}(X,S) = 0, \text{Hom}(S,X) = 0, \dim \text{Ext}(X,S) = n_1 \) and \( \dim \text{Ext}(S,X) = n_2 \). Then every indecomposable representation of

\[
\begin{array}{c}
\rho_1 \\
\vdots \\
\rho_{n_2}
\end{array}
\]

with dimension vector as indicated yields an indecomposable representation of dimension \( \dim X + n_2 \cdot \dim S \). In particular, we can construct the indecomposable representation obtained in Theorem 2.3 in this way, i.e. the middle term of the exact sequence

\[
0 \to X \to X^S \to \bigoplus_{i=1}^{n_2} S \to 0.
\]

Now we have \( \text{Ext}(S,X^S) = \text{Hom}(S,X^S) = 0 \) and since we have \( \text{Ext}(X^S,S) \cong \text{Ext}(X^S,S) \), every indecomposable representation of the quiver

\[
\begin{array}{c}
1 \\
\vdots \\
\rho_{n_1}
\end{array}
\]

with dimension vector as indicated yields an indecomposable representation of \( Q \) of dimension \( \dim X^S + n_1 \cdot \dim S \). In particular, we can construct the representation \( X^S \) obtained by the reflection functor.

**Remark 3.13.**

Maybe the main advantage of this functor in comparison to Ringel’s reflection functor is that we do not have to restrict to one exceptional representation which is glued to a fixed representation. In fact, we also allow extension between the (exceptional) representations which are glued.

**Example 3.14.**

Let \( Q \) be the 6-subspace quiver and \( \beta_3 = (1,1,0,0,0,1,0,0), \ldots, \beta_6 = (1,0,0,0,1,1,0,1), \beta_7 = S_6 \). The category \( \{\beta_3, \ldots, \beta_7\} \) is generated by the exceptional roots \( \beta_1 = (3,1,1,1,0,0) \) and \( \beta_2 = S_5 \). Thus every representation \( X \) of \( K(3) = (\{q_1, q_2\}, \{\alpha_i : q_1 \to q_2 \mid i = 1,2,3\}) \) of dimension \( (m,n) \) gives a representation in \( \{\beta_3, \ldots, \beta_7\} \) of dimension \( \beta_1^m + \beta_2^n \). Let \( m_i = \dim \text{Ext}(X_{\beta_i}, X) \) and consider the quiver

\[
\begin{array}{c}
X \\
\beta_3 (m_3) \\
\beta_4 (m_4) \\
\beta_7 (m_7) \\
\beta_5 (m_5) \\
\beta_6 (m_6)
\end{array}
\]
where the number in brackets indicates the number of arrows. Now every indecomposable representation of this quiver such that the source has dimension one, induces an indecomposable representation of the original quiver. Note that different representations $X$ of the same dimension can induce different quivers.

We can assume that the representation of dimension $\beta_1$ is given by the coefficient quiver

```
\begin{tikzpicture}[scale=0.5]
  \node (q1) at (0,0) {$q_1^1$};
  \node (q0) at (1,-1) {$q_0^1$};
  \node (q2) at (2,-2) {$q_0^2$};
  \node (q3) at (3,-3) {$q_3^3$};
  \node (q4) at (4,-4) {$q_4^4$};
  \node (q5) at (5,-5) {$q_5^5$};

  \draw[->] (q1) -- (q0);
  \draw[->] (q1) -- (q2);
  \draw[->] (q1) -- (q3);
  \draw[->] (q1) -- (q4);
  \draw[->] (q1) -- (q5);

  \node at (0,-6) {$q_1^1$};
  \node at (1,-7) {$q_0^1$};
  \node at (2,-8) {$q_0^2$};
  \node at (3,-9) {$q_0^3$};
  \node at (4,-10) {$q_3^4$};
  \node at (5,-11) {$q_4^4$};
\end{tikzpicture}
```

For instance, a basis of $\text{Ext}(X_{\beta_2}, X_{\beta_1})$ is induced by the linear maps

\[ f_i : (S_5)_5 \rightarrow (X_{\gamma_1})_0, 1 \mapsto q_0^i, \quad i = 1, 2, 3. \]

Let $(m, n) = (2, 3)$ and consider the representation of this dimension which has the coefficient quiver

```
\begin{tikzpicture}[scale=0.5]
  \node (f1) at (0,0) {$f_1$};
  \node (f2) at (1,1) {$f_2$};
  \node (f3) at (2,2) {$f_3$};

  \node at (0,-1) {$f_1$};
  \node at (1,-2) {$f_2$};
  \node at (2,-3) {$f_3$};
\end{tikzpicture}
```

Then it is straightforward to check that we have $\dim \text{Hom}(X_{\beta_i}, X) = 1$ for $i = 3, 4, 5$ and $\dim \text{Hom}(X_{\beta_6}, X) = 0$ and $\dim \text{Hom}(X_{\beta_7}, X) = 0$. Thus we obtain the quiver

```
\begin{tikzpicture}[scale=0.5]
  \node (beta3) at (0,0) {$\beta_3$};
  \node (beta4) at (1,1) {$\beta_4$};
  \node (beta5) at (2,2) {$\beta_5$};
  \node (beta6) at (3,3) {$\beta_6$};
  \node (beta7) at (4,4) {$\beta_7$};

  \draw[->] (beta3) -- (beta4) -- (beta5) -- (beta6) -- (beta7);
\end{tikzpicture}
```

Note that the greater the dimension of the homomorphism spaces the greater the dimension of the extension spaces. And thus the greater the number of indecomposables which can be constructed with the original representations as building blocks.

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