MAXIMAL MULTIPLICATIVE PROPERTIES OF PARTITIONS

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Abstract. Extending the partition function multiplicatively to a function on partitions, we show that it has a unique maximum at an explicitly given partition for any \( n \neq 7 \). The basis for this is an inequality for the partition function which seems not to have been noticed before.

1. Introduction and statement of results

For \( n \in \mathbb{N} \), the partition function \( p(n) \) enumerates the number of partitions of \( n \), i.e., the number of integer sequences \( \lambda = (\lambda_1, \lambda_2, \ldots) \) with \( \lambda_1 \geq \lambda_2 \geq \ldots > 0 \) and \( \sum_{j \geq 1} \lambda_j = n \). It plays a central role in many parts of mathematics and has been for centuries an object whose properties have been studied in particular in combinatorics and number theory.

While explicit formulae for \( p(n) \) are known due to the work of Hardy, Ramanujan and Rademacher, and the recent work of Bruinier and the second author [1] on a finite algebraic formula, these expressions can be quite forbidding when one wants to check even simple properties. In a representation theoretic context, a question came up which led to the observation of surprisingly nice multiplicative behavior.

In this note, we show in Theorem 2.1 the following inequality:

For any integers \( a, b \) such that \( a, b > 1 \) and \( a + b > 9 \), we have \( p(a)p(b) > p(a + b) \).

This result allows us to study an “extended partition function”, which is obtained by defining for a partition \( \mu = (\mu_1, \mu_2, \ldots) \):

\[ p(\mu) = \prod_{j \geq 1} p(\mu_j). \]

Let \( P(n) \) denote the set of all partitions of \( n \). Here we determine the maximum of the partition function on \( P(n) \) explicitly; more precisely, we find in Theorem [2.1] that for \( n \neq 7 \), the maximal value

\[ \max p(n) = \max \{ p(\mu) \mid \mu \in P(n) \} \]

is attained at a unique partition of \( n \) of a very simple form, which depends on \( n \) modulo 4.

**Theorem 1.1.** Let \( n \in \mathbb{N} \). For \( n \geq 4 \) and \( n \neq 7 \), the maximal value \( \max p(n) \) of the partition function on \( P(n) \) is attained at the partition

\[
\begin{align*}
(4,4,4,\ldots,4,4) & \text{ when } n \equiv 0 \pmod{4} \\
(5,4,4,\ldots,4,4) & \text{ when } n \equiv 1 \pmod{4} \\
(6,4,4,\ldots,4,4) & \text{ when } n \equiv 2 \pmod{4} \\
(6,5,4,\ldots,4,4) & \text{ when } n \equiv 3 \pmod{4}
\end{align*}
\]

The second author thanks the NSF and the Asa Griggs Candler Fund for their generous support.
In particular, if \( n \geq 8 \), then

\[
\max_p(n) = \begin{cases} 
5^n & \text{if } n \equiv 0 \pmod{4}, \\
7 \cdot 5^{n-3} & \text{if } n \equiv 1 \pmod{4}, \\
11 \cdot 5^{n-6} & \text{if } n \equiv 2 \pmod{4}, \\
11 \cdot 7 \cdot 5^{n-11} & \text{if } n \equiv 3 \pmod{4}.
\end{cases}
\]

2. An analytic result on the partition function

The main result of this section is the following analytic inequality for the partition function \( p(n) \).

**Theorem 2.1.** If \( a, b \) are integers with \( a, b > 1 \) and \( a + b > 8 \), then

\[
p(a)p(b) \geq p(a + b).
\]

with equality holding only for \( \{a, b\} = \{2, 7\} \).

**Remark.** Of course, the inequality in Theorem 2.1 always fails if we take \( a = 1 \). The complete set of pairs of integers \( 1 < a \leq b \) for which the inequality fails is

\[
\{(2, 2), (2, 3), (2, 4), (2, 5), (3, 3), (3, 5)\},
\]

while for

\[
\{(2, 6), (3, 4)\}
\]

we have equality.

The main tool for deriving Theorem 2.1 is the following classical result of D. H. Lehmer [2].

**Theorem 2.2 (Lehmer).** If \( n \) is a positive integer and \( \mu = \mu(n) := \frac{1}{6} \sqrt{24n - 1} \), then

\[
p(n) = \frac{\sqrt{12}}{24n - 1} \left[ \left( 1 - \frac{1}{\mu} \right) e^\mu + \left( 1 + \frac{1}{\mu} \right) e^{-\mu} \right] + E(n),
\]

where we have that

\[
|E(n)| < \frac{\pi^2}{\sqrt{3}} \cdot \left[ \frac{1}{\mu^3} \sinh(\mu) + \frac{1}{6} - \frac{1}{\mu} \right].
\]

**Proof of Theorem 2.1.** By Theorem 2.2 it is straightforward to verify for every positive integer \( n \) that

\[
\frac{\sqrt{3}}{12n} \left( 1 - \frac{1}{\sqrt{n}} \right) e^{\mu(n)} < p(n) < \frac{\sqrt{3}}{12n} \left( 1 + \frac{1}{\sqrt{n}} \right) e^{\mu(n)}.
\]

We may assume \( 1 < a \leq b \); for convenience we let \( b = \lambda a \), where \( \lambda \geq 1 \). These inequalities immediately give:

\[
p(a)p(\lambda a) > \frac{1}{48\lambda a^2} \left( 1 - \frac{1}{\sqrt{a}} \right) \left( 1 - \frac{1}{\sqrt{\lambda a}} \right) e^{\mu(a) + \mu(\lambda a)},
\]

\[
p(a + \lambda a) < \frac{\sqrt{3}}{12(a + \lambda a)} \left( 1 + \frac{1}{\sqrt{a + \lambda a}} \right) e^{\mu(a + \lambda a)}.
\]
For all but finitely many cases, it suffices to find conditions on \(a > 1\) and \(\lambda \geq 1\) for which
\[
\frac{1}{48\lambda a^2} \left( 1 - \frac{1}{\sqrt{a}} \right) \left( 1 - \frac{1}{\sqrt{\lambda a}} \right) e^{\mu(a) + \mu(\lambda a)} > \frac{\sqrt{3}}{12(a + \lambda a)} \left( 1 + \frac{1}{\sqrt{a + \lambda a}} \right) e^{\mu(a + \lambda a)}.
\]
Since \(\lambda \geq 1\), we have that \(\frac{\lambda}{\lambda + 1} \geq 1/2\), and so it suffices to consider when
\[
e^{\mu(a) + \mu(\lambda a) - \mu(a + \lambda a)} > 2a \sqrt{3} \cdot S_a(\lambda),
\]
where
\[
S_a(\lambda) := \frac{1 + \frac{1}{\sqrt{a + \lambda a}}}{\left( 1 - \frac{1}{\sqrt{a}} \right) \left( 1 - \frac{1}{\sqrt{\lambda a}} \right)}.
\]
By taking the natural log, we obtain the inequality
\[
T_a(\lambda) > \log(2a \sqrt{3}) + \log(S_a(\lambda)),
\]
where we have that
\[
T_a(\lambda) := \frac{\pi}{6} \left( \sqrt{24a - 1} + \sqrt{24\lambda a - 1} - \sqrt{24(a + \lambda a) - 1} \right).
\]
We consider (2.1) and (2.3) as functions in \(\lambda \geq 1\) and fixed \(a > 1\). Simple calculations reveal that \(S_a(\lambda)\) is decreasing in \(\lambda \geq 1\), while \(T_a(\lambda)\) is increasing in \(\lambda \geq 1\). Therefore, (2.2) becomes
\[
T_a(\lambda) \geq T_a(1) > \log(2a \sqrt{3}) + \log(S_a(1)) \geq \log(2a \sqrt{3}) + \log(S_a(\lambda)).
\]
By evaluating \(T_a(1)\) and \(S_a(1)\) directly, one easily finds that (2.2) holds whenever \(a \geq 9\). To complete the proof, assume that \(2 \leq a \leq 8\). We then directly calculate the real number \(\lambda_a\) for which
\[
T_a(\lambda_a) = \log(2a \sqrt{3}) + \log(S_a(\lambda_a)).
\]
By the discussion above, if \(b = \lambda a \geq a\) is an integer for which \(\lambda > \lambda_a\), then (2.2) holds, which in turn gives the theorem in these cases. The table below gives the numerical calculations for these \(\lambda_a\). Only finitely many cases remain, namely the pairs of integers where \(2 \leq a \leq 8\) and \(1 \leq b/a \leq \lambda_a\). We computed \(p(a), p(b)\) and \(p(a + b)\) for these cases to complete the proof of the theorem.

\[\begin{array}{|c|c|}
\hline
a & \lambda_a \\
\hline
2 & 57.08\ldots \\
3 & 7.42\ldots \\
4 & 3.62\ldots \\
5 & 2.36\ldots \\
6 & 1.74\ldots \\
7 & 1.38\ldots \\
8 & 1.15\ldots \\
\hline
\end{array}\]

1 \leq b/a \leq \lambda_a. We computed \(p(a), p(b)\) and \(p(a + b)\) for these cases to complete the proof of the theorem. \(\square\)
3. The maximum property

Here we use the result in the previous section to prove Theorem 1.1.

Proof of Theorem 1.1. For the proof, we will need the partitions where $\text{maxp}(n)$ is attained for $n \leq 14$; these are given in Table 2 below (computed by Maple).

| $n$ | $p(n)$ | $\text{maxp}(n)$ | $\mu$   |
|-----|--------|------------------|---------|
| 1   | 1      | 1                | (1)     |
| 2   | 2      | 2                | (2)     |
| 3   | 3      | 3                | (3)     |
| 4   | 5      | 5                | (4)     |
| 5   | 7      | 7                | (5)     |
| 6   | 11     | 11               | (6)     |
| 7   | 15     | 15               | (7), (4,3) |
| 8   | 22     | 25               | (4,4)   |
| 9   | 30     | 35               | (5,4)   |
| 10  | 42     | 55               | (6,4)   |
| 11  | 56     | 77               | (6,5)   |
| 12  | 77     | 125              | (4,4,4) |
| 13  | 101    | 175              | (5,4,4) |
| 14  | 135    | 275              | (6,4,4) |

We see that the assertion holds for $n \leq 14$, and we may thus assume now that $n > 14$. Let $\mu = (\mu_1, \mu_2, \ldots) \in P(n)$ be such that $p(\mu)$ is maximal. If $\mu$ has a part $k \geq 8$, then by Theorem 2.1 and Table 2, replacing $k$ by the parts $\lfloor k/2 \rfloor, \lceil k/2 \rceil$ would produce a partition $\nu$ such that $p(\nu) > p(\mu)$. Thus all parts of $\mu$ are smaller than 8. Let $m_j$ be the multiplicity of a part $j$ in $\mu$. If $m_1 \neq 0$, then for $\nu = (\mu_1 + 1, \mu_2, \ldots)$ we have $p(\nu) > p(\mu)$. So $m_1 = 0$. If $m_2 \geq 2$, then replacing parts 2, 2 in $\mu$ by one part 4 gives a partition $\nu$ with $p(\nu) > p(\mu)$. So $m_2 \leq 1$. Similarly, the operations of replacing (3, 3) by a part 6, (5, 5) by the parts (6, 4), (6, 6) by the parts (4, 4, 4), and (7, 7) by the parts (6, 4, 4), respectively, show that we have $m_3, m_5, m_6, m_7 \leq 1$.

Now assume that $m_7 = 1$. As $n > 14$, one of $m_2, m_3, m_4, m_5, m_6$ is nonzero; now by Table 2, performing one of the following replacement operations

$$7, 2 \rightarrow 5, 4; 7, 3 \rightarrow 6, 4; 7, 4 \rightarrow 6, 5; 7, 5 \rightarrow 4, 4, 4; 7, 6 \rightarrow 5, 4, 4$$

gives a partition $\nu$ with $p(\nu) > p(\mu)$, a contradiction. Hence $m_7 = 0$.

Next assume that $m_6 = 1$. If $m_2 = 1$ or $m_3 = 1$, we can use the following operations that increase the $p$-value:

$$6, 2 \rightarrow 4, 4; 6, 3 \rightarrow 5, 4$$

By the choice of $\mu$, we conclude that $m_2 = 0 = m_3$. Thus in this case we have $\mu = (6, 4, 4, \ldots, 4, 4)$ or $\mu = (6, 5, 4, 4, \ldots, 4, 4)$, which is in accordance with the assertion.
Thus we may now assume \( m_6 = 0 \). Assume first that \( m_5 = 1 \). Note that we must have a part 4, since \( n > 14 \). If \( m_2 = 1 \) or \( m_3 = 1 \), we can increase the \( p \)-value by the replacements:

\[
5, 4, 2 \to 6, 5; 5, 3 \to 4, 4.
\]

By the choice of \( \mu \), this implies \( m_2 = 0 = m_3 \). Thus we have in this case \( \mu = (5, 4, 4, \ldots, 4, 4) \), again in accordance with the assertion.

Now we consider the case where also \( m_5 = 0 \). If \( m_2 = 1 \) or \( m_3 = 1 \), we use the replacements

\[
4, 2 \to 6; 4, 4, 3 \to 6, 5
\]

to get a contradiction. Hence \( m_2 = 0 = m_3 \) and we have the final case \( \mu = (4, 4, \ldots, 4, 4) \) occurring in the assertion.

The four types of partitions we have found occur at the four different congruence classes of \( n \mod 4 \); thus for each value of \( n \neq 7 \), we have found a unique partition \( \mu \) that maximizes the \( p \)-value and we are done.

\[\square\]

4. Concluding remarks

We note that recently also other multiplicative properties of the partition function have been studied. Originating in a conjecture by William Chen, DeSalvo and Pak have proved log-concavity for the partition function for all \( n > 25 \); indeed, they have shown that for all \( n > m > 1 \) the following holds:

\[
p(n)^2 > p(n - m)p(n + m).
\]

Note that the border case \( m = n \) (which is not covered here) is included in our results, for any \( n > 3 \).

The opposite inequality

\[
p(1)p(n) < p(n + 1)
\]

has an easy combinatorial proof by an injection \( P(n) \to P(n + 1) \). One may ask whether there is also a combinatorial argument for proving Theorem 2.1.

The behavior that we have seen here for the partition function \( p(n) \) seems to be shared also by the enumeration of other sets of suitably restricted partitions; work on this is in progress.

References

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