Josephson dynamics of a spin-orbit coupled Bose-Einstein condensate in a double well potential

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We investigate the quantum dynamics of an experimentally realized spin-orbit coupled Bose-Einstein condensate in a double well potential. The spin-orbit coupling can significantly enhance the atomic inter-well tunneling. We find the coexistence of internal and external Josephson effects in the system, which are moreover inherently coupled in a complicated form even in the absence of interatomic interactions. Moreover, we show that the spin-dependent tunneling between two wells can induce a net atomic spin current referred as spin Josephson effects. Such novel spin Josephson effects can be observable for realistically experimental conditions.

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I. INTRODUCTION

Based on the Berry phase effect [1, 2] and its non-Abelian generalization [3], the creation of synthetic gauge fields in neutral atoms by controlling atom-light interaction has attracted great interest in recent theoretical studies [4–18], and has been realized in spinor Bose-Einstein condensates (BECs) in the pioneering experiments of the NIST group [19, 20] and also in several subsequent experiments of other groups [21–23]. The neutral atoms in the generated effective Abelian and non-Abelian gauge fields behave like electrons in an electromagnetic field [19, 21, 22] or electrons with spin-orbit (SO) coupling [20]. Different from electrons that are fermions, the atoms with the synthetic SO coupling can be bosons and typically BECs. This bosonic counterpart of the SO coupled materials has no direct analog in solid-state systems and thus has received increasing attention [24–44] for different types of SO coupling, different internal atomic structures (pseudospin-1/2, spin-1 and spin-2 bosons, etc.), and different external conditions (homogeneous, trapped and rotated). These theoretical investigations focus mainly on the static properties of SO coupled BECs and have revealed rich phase diagrams of the ground-states [25–31, 33–35, 39–41]. However, to our knowledge, their dynamics has been less studied [24–42], where the SO coupled BECs are demonstrated to exhibit interesting relativistic dynamics, such as analogs of self-localization [42], Zitterbewegung [43] and Klein tunneling [44] under certain conditions.

On the other hand, quantum dynamics of a BEC in a double well potential has been widely investigated. In particular, the coherent atomic tunneling between two wells results in oscillatory exchange of the BEC, which is analogous to the Josephson effects (JEs) for neutral atoms [45–48]. The weakly interacting BECs provide a further context [46–48] for JEs in superconductor systems because they display a nonlinear generalization of typical d.c. and a.c. JEs and macroscopic quantum self-trapping, all of which have been observed in experiments [49–51]. Apart from the conventional single-species BECs [45–48], the Josephson dynamics of two-species BECs [52] and spinor BECs without SO coupling [53] have also been studied [54–57]; however, the dynamics of SO coupled BECs is yet to be explored.

In this paper, we investigate the dynamics of a specific SO coupled BEC, which was realized in the experiment of the NIST group, in a double well trapping potential.
We find that the SO coupling in the system contributes to and increases the atomic tunneling to a large extent, which can significantly enhance the atomic JEs. The full dynamics of the system contains both internal and external JEs, which are moreover inherently coupled in a complicated form even in the absence of interatomic interactions. We further demonstrate that the spin-dependent Josephson tunneling can lead to a net atomic spin current by varying conditions, which we refer to as spin Josephson effects. The predicted spin-Josephson currents are robust against the parameter adjustment and varying initial conditions, and can be observable in the SO coupled BECs under realistic experimental conditions.

The paper is organized as follows. In Sec. II we construct a model that can be used to study the quantum dynamics of a SO coupled BEC in a double well potential. Then, in Sec. III, the Josephson dynamics of the constructed system is investigated, with the complicatedly coupled internal and external JEs being addressed. In Sec. IV we demonstrate that the spin JEs exhibit in the system under realistic conditions. Brief discussions and a short conclusion are given in Sec. V.

II. MODEL

In a very recent experiment, the NIST group realized a synthetic SO coupling in the $^{87}\text{Rb}$ BEC, in which a pair of Raman lasers generate a momentum-sensitive coupling between two internal atomic states $^{20}$. In the bare pseudo-spin basis $|↑\rangle_b = |m_F = 0\rangle$ and $|↓\rangle_b = |m_F = -1\rangle$, the SO coupling is described by the single particle Hamiltonian given by $^{20}$

$$\hat{h} = \frac{p^2}{2m} + \frac{1}{2} \left( \frac{\delta}{\Omega} e^{-2ik_Lx} - \frac{\Omega e^{2ik_Lx}}{\Omega} \right),$$

where $p$ is the atomic momentum in the $xy$ plane, $m$ is the atomic mass, $\delta$ is the tunable detuning behaved as a Zeeman field, $k_L$ is the wave number of the Raman laser, and $\Omega$ is the Raman coupling strength. Such kind of SO coupling is equivalent to that of an electronic system with equal contribution from Rashba and Dresselhaus SO coupling, and thus it is effective just in one-dimension (1D). So we restrict our discussions in 1D and focus on the motion of atoms along the $x$ axis by freezing their $y$ and $z$ degrees of freedom.

To proceed further, we introduce the dressed pseudospins $|\uparrow\rangle = e^{-ik_Lx}|\uparrow\rangle_b$ and $|\downarrow\rangle = e^{ik_Lx}|\downarrow\rangle_b$, then the single-particle Hamiltonian in 1D (along $x$ axis) can be written as

$$\hat{h}_0 = \frac{\hbar^2 k_x^2}{2m} + \alpha \sigma_z + \frac{\Omega}{2} \sigma_x + \frac{\delta}{2} \sigma_z,$$

where $\sigma_z$ is the atomic wave vector operator, and $\alpha = E_r/k_L$ is the SO coupling strength with $E_r = \hbar^2 k_x^2/2m$ being the single-photon recoil energy. The dispersion relation of the single particle Hamiltonian $^{2}$ with $\delta = 0$ is $E_\pm(k_x) = \frac{\hbar^2 k_x^2}{2m} \pm \sqrt{4\alpha^2 k_x^2 + \Omega^2/4}$, which exhibits a structure of two branches. We are interested in the lower energy one $E_-(k_x)$. There is only one minimum in $k_x = 0$ for large Raman coupling $\Omega > 4E_r$, where the atoms of both atomic levels condense. However, the lower branch for $\Omega < 4E_r$ presents two minima for condensation of dressed pseudo-spin-up (left one) atoms and dressed pseudo-spin-down (right one) atoms, respectively. The Raman coupling and a small $\delta$ modulate the population of atoms in these two states $^{20}$. Here we focus on the later regime, i.e. $\Omega < 4E_r$, because such BEC with spin-separated and non-zero central momentum is more interesting in contrast to a regular BEC with zero central momentum.

To be clearer, we can rewrite the Hamiltonian $^{2}$ as

$$\hat{h}_0 = \left( \frac{H_\uparrow}{\Omega} - \frac{\Omega}{2} H_\downarrow \right),$$

where $H_\uparrow = \frac{\hbar^2}{2m}(\hat{k}_x^2 + 2k_L\hat{k}_z) + \frac{\delta}{2}$ and $H_\downarrow = \frac{\hbar^2}{2m}(\hat{k}_x - 2k_L\hat{k}_z) - \frac{\delta}{2}$. Since it is more straightforward to describe the system in terms of dressed pseudo-spin states compared with using bare ones, we will work in the dressed pseudo-spin space and simply refer to dressed pseudo-spin as spin for convenience hereafter. We also note that the parameters $k_L$, $\Omega$ and $\delta$ in the single-particle Hamiltonian can be tuned independently in a wide range $^{20}$, making the SO coupled BEC a suitable platform for investigating the Josephson dynamics in the presence of SO coupling.

Now we turn to consider such a SO coupled BEC in a double well potential denoted by $V(x)$ as shown in Fig. 1(a). Note that the double well potential here is assumed to be spin-independent. To investigate the dynamics of the system, we adopt the two-mode approximation $^{45–48}$ with the field operator

$$\hat{\Psi}_\sigma(x) \sim \hat{a}_{L\sigma} \psi_{L\sigma}(x) + \hat{a}_{R\sigma} \psi_{R\sigma}(x),$$

where $\psi_{L\sigma}(x)$ is the ground state wave function of the $j$ well ($j = L, R$) with spin $\sigma$ ($\sigma = \uparrow, \downarrow$), and $\hat{a}_{j\sigma}$ is the annihilation operator for spin $\sigma$ in the $j$ well, satisfying the bosonic commutation relationship $[\hat{a}_{j\sigma}, \hat{a}_{j'\sigma'}^\dagger] = \delta_{jj'} \delta_{\sigma\sigma'}$. The validity of the two-mode approximation holds under two conditions: the weak atomic interaction and small effective Zeeman splitting, as the atoms can not be pumped out of the lowest state of each well in this case. In the second quantization formalism, the total Hamiltonian reads

$$\hat{H} = \int dx \hat{\Psi}^\dagger(x) \left[ \hat{h}_0 + V(x) + \hat{h}_{\text{int}} \right] \hat{\Psi}(x),$$

where the two-component field operator $\hat{\Psi} = (\Psi_\uparrow, \Psi_\downarrow)^T$, and the interaction Hamiltonian $\hat{h}_{\text{int}}$ will be specified below. Substituting Eq. (4) into Eq. (5), one can rewrite
the total Hamiltonian as
\begin{equation}
\mathcal{H} = \sum_{j,\sigma} \epsilon_j \sigma \hat{a}_{j,\sigma}^\dagger \hat{a}_{j,\sigma} + \sum_{\sigma,\sigma'} \left( J_{\sigma \sigma'} \hat{a}_{j,\sigma}^\dagger \hat{a}_{R,\sigma'} + \text{h.c.} \right)
+ \frac{\Omega}{2} \sum_j \left( \hat{a}_{j,\uparrow}^\dagger \hat{a}_{j,\downarrow} + \text{h.c.} \right)
+ \frac{\delta}{2} \sum_j \left( \hat{a}_{j,\uparrow}^\dagger \hat{a}_{j,\downarrow} - \hat{a}_{j,\downarrow}^\dagger \hat{a}_{j,\uparrow} \right) + \mathcal{H}_{\text{int}},
\end{equation}
where \( \epsilon_j = \int dx \psi_j^\ast(x) \left[ \mathcal{H}_0 + V(x) \right] \psi_j(x) \) is the single-particle ground state energy in the \( j \) well with \( \omega_j \) being the harmonic frequency of this well, \( J_{\sigma \sigma'} = \int dx \psi_j^\ast(x) \mathcal{H}_{\sigma \sigma'}(x) \psi_{R,\sigma'}(x) \) and \( \mathcal{H}_{\sigma \sigma'} = \int dx \psi_j^\ast(x) \mathcal{H}_{\sigma \sigma'}(x) \psi_{R,\sigma'}(x) \) (with \( \sigma \) and \( \sigma' \) referring to different spins) are the tunneling terms shown in Fig. 1(b). In addition, the interaction Hamiltonian is given by
\begin{equation}
\mathcal{H}_{\text{int}} = \frac{1}{2} \sum_j \left( g_{L\sigma}^{(j)} \hat{a}_{j,\uparrow}^\dagger \hat{a}_{j,\downarrow}^\ast \hat{a}_{j,\downarrow} \hat{a}_{j,\uparrow} + g_{R\sigma}^{(j)} \hat{a}_{j,\uparrow}^\dagger \hat{a}_{j,\downarrow}^\ast \hat{a}_{j,\uparrow} \hat{a}_{j,\downarrow} \right)
+ 2 \xi^{(j)} \hat{a}_{j,\uparrow}^\dagger \hat{a}_{j,\downarrow}^\ast \hat{a}_{j,\downarrow} \hat{a}_{j,\uparrow},
\end{equation}
where \( g_{L\sigma}^{(j)} = \frac{2\xi^{(j)}}{m \omega_j} \int dx \psi_j^\ast(x)^2 |\psi_j(x)|^2 \) is the effective 1D interacting strength with \( a_{\sigma \sigma'} \) being the s-wave scattering length between spin \( \sigma \) and \( \sigma' \) and \( l_{\perp} \) being the oscillator length associated to a harmonic vertical confinement. Note that here we have ignored the inter-well atomic interactions because the s-wave scattering length (which is on the order of nanometers) is much smaller than the inter-well distance (which is on the order of micrometers). We have also dropped the inter-well coupling since its strength is exponentially smaller than the intra-well counterpart. The Hamiltonian (6) describes the dynamic process of the system schematically shown in Fig. 1(b).

For simplicity, we assume the double well potential to be symmetric as shown in Fig. 1(a), with each well having the same harmonic trapping frequency \( \omega \). Thus we have \( \epsilon_L = \epsilon_R \) and \( g_{L\sigma}^{(j)} = g_{R\sigma}^{(j)} \). Such kind of double well potential can be generated in experiments [51] with the form
\begin{equation}
V(x) = a(x^2 - b^2)^2,
\end{equation}
where the parameters \( a \) and \( b \) are both tunable in the experiments [51]. Expanding \( V(x) \) near \( x = \pm b \), one obtains its harmonic form as \( V^{(2)}(x) = \frac{1}{2}m\omega^2(x \pm b)^2 \) and thus \( a = m\omega^2/8b^2 \). The ground state wavefunctions of the BEC in each well potential with each spin can be approximated by its corresponding lowest energy single-particle wavefunction, which can be worked out by solving the equations \( \mathcal{H}_0 + \frac{1}{2}m\omega^2(x \pm b)^2 |\psi_j(x) = \epsilon_j |\psi_j(x) \) (here \( \pm \) are for \( j = L, R \), respectively). The results are [24]
\begin{align}
\psi_{L\uparrow} &= \phi_0^{(L)}(x)e^{-ikLx}, \\
\psi_{L\downarrow} &= \phi_0^{(L)}(x)e^{ikLx}, \\
\psi_{R\uparrow} &= \phi_0^{(R)}(x)e^{-ikLx}, \\
\psi_{R\downarrow} &= \phi_0^{(R)}(x)e^{ikLx},
\end{align}
where \( \phi_0^{(L)}(x) = \frac{1}{\sqrt{2}}\left( \begin{array}{c} 1 \\ i \end{array} \right) e^{-x^2/2} \) and \( \phi_0^{(R)}(x) = \frac{1}{\sqrt{2}}\left( \begin{array}{c} 1 \\ i \end{array} \right) e^{x^2/2} \).
the same wavelength $\lambda_L = 2\pi/k_L = 0.8 \mu m$ and recoil frequency $E_r/\hbar = 22.5 \text{ kHz}$ as those in the experiments [20, b and $l_0$ to be on the order of micrometers [51, 52]. In fact, in the regime of $b^2/l_0^2 \sim [0.5, 2]$, we find that
\[
|\xi_{SO}| \gtrsim 100 \max\{|\xi_T|, |\xi_V|\}.
\] (12)

Besides, one can check that $\xi_R \sim \Omega e^{-64}$ for $l_0 \sim 1 \mu m$ and $\lambda_L = 0.8 \mu m$, and thus the Raman-coupling induced tunneling is negligible in this system. The comparisons among $J^{(n)}$ for some typical parameters are shown in Fig. 2. In other words, we find that under realistic experiment conditions [20, 50], the spin-flipping tunneling induced by Raman coupling is negligible but the SO coupling induced tunneling term $J^{(SO)}$ dominates and moreover it greatly enhances atomic tunneling in this system. Thus we may rewrite the tunneling terms as
\[
\begin{align*}
J_{\uparrow \downarrow} &= J_{\downarrow \uparrow} \approx 0, \\
J_{\uparrow \uparrow} &\approx J_{\downarrow \downarrow} \approx J^{(SO)} = -\gamma E_r,
\end{align*}
\] (13)

where $\gamma = \exp(-b^2/l_0^2) \sim [0.1, 0.6]$. It is worthwhile to note that the new tunneling terms $J^{(SO)}$ and $J^{(R)}$ in this SO coupled system are both tunable, enabling us to study the interesting effects of SO coupling in the atomic inter-well tunneling. For instance, one can decrease the effective wave number in $x$ axis to the scale $k_L \sim 1/l_0$ so that $J^{(R)} \sim 0.37\Omega$ and then the Raman-coupling induced tunneling can revive. This can be achieved by adjusting the angle between the applying Raman lasers and the trapping potential or alternatively by using lasers with larger wavelength. In addition, in the same way one can tune the recoil energy $E_r$ to identify the enhancement of atomic tunneling due to the SO coupling (i.e. the effect of $J^{(SO)}$) in experiments. As a first step to investigate the system under current experiment conditions, we here concentrate on the tunneling regime governed by Eq. (13).

### III. FULL DYNAMICS OF THE SYSTEM

We are now in the position to investigate the quantum dynamics of the system constructed in the previous section. We first address the non-interacting case, i.e. $\mathcal{H}_{int} = 0$ in Eq. (6), in which the single-particle Hamiltonian is given by
\[
\mathcal{H}_0 \simeq J_{\uparrow \uparrow} (\hat{a}_{L \uparrow}^\dagger \hat{a}_{R \uparrow} + \hat{a}_{L \uparrow}^\dagger \hat{a}_{R \uparrow}^\dagger) + J_{\downarrow \downarrow} (\hat{a}_{L \downarrow}^\dagger \hat{a}_{R \downarrow} + \hat{a}_{L \downarrow}^\dagger \hat{a}_{R \downarrow}^\dagger)
\]
\[
+ \frac{\Omega}{2} \left( \hat{a}_{L \uparrow}^\dagger \hat{a}_{L \downarrow}^\dagger + \hat{a}_{L \uparrow} \hat{a}_{L \downarrow} + \hat{a}_{R \uparrow} \hat{a}_{R \downarrow} + \hat{a}_{R \uparrow}^\dagger \hat{a}_{R \downarrow}^\dagger \right)
\]
\[
+ \frac{\delta}{2} \left( \hat{a}_{L \uparrow}^\dagger \hat{a}_{L \downarrow}^\dagger - \hat{a}_{L \uparrow} \hat{a}_{L \downarrow} + \hat{a}_{R \uparrow} \hat{a}_{R \downarrow} - \hat{a}_{R \uparrow}^\dagger \hat{a}_{R \downarrow}^\dagger \right).
\] (14)

Here we have dropped the tunneling terms $J_{\sigma \bar{\sigma}}$ since these spin-flipping tunneling progresses can be negligible in the current experiment conditions [20, 50]. In order to study the dynamic properties of the system, we need to work with the equation of motion. The corresponding Heisenberg equations read
\[
i\hbar \frac{d}{dt} \hat{a}_{\sigma} = [\hat{a}_{\sigma}, \mathcal{H}_0] = J_{\sigma \bar{\sigma}} \hat{a}_{\bar{\sigma}} + \frac{\Omega}{2} \hat{a}_{\sigma} + (-1)^p \frac{\delta}{2} \hat{a}_{\sigma},
\] (15)

where $\sigma$ and $\bar{\sigma}$ refer to different spin, while $j$ and $\bar{j}$ to different wells, and $p = 0, 1$ are for $\sigma = \uparrow, \downarrow$ respectively. Using the mean-field approximation, one has $\hat{a}_{\sigma} \simeq \langle \hat{a}_{\sigma} \rangle = a_{\sigma}$ with $a_{\sigma}$ being complex numbers. Thus we can rewrite the equations of motion as
\[
i\hbar \frac{d}{dt} a_{\sigma} = J_{\sigma \bar{\sigma}} a_{\bar{\sigma}} + \frac{\Omega}{2} a_{\sigma} + (-1)^p \frac{\delta}{2} a_{\sigma}.
\] (16)

By defining a four-component wavefunction $\Phi = (a_{\sigma \uparrow}, a_{\sigma \downarrow}, a_{\bar{\sigma} \uparrow}, a_{\bar{\sigma} \downarrow})^T$, Eq. (16) is rewritten as $i\hbar \frac{d}{dt} \Phi = H_M \Phi$, where the Hamiltonian of the system is given by
\[
H_M = \begin{pmatrix} 
\delta & \Omega & J_{\uparrow \uparrow} & 0 \\
\Omega & -\delta & 0 & J_{\downarrow \downarrow} \\
J_{\uparrow \uparrow} & 0 & \frac{\delta}{2} & \frac{\Omega}{2} \\
0 & J_{\downarrow \downarrow} & -\frac{\delta}{2} & -\frac{\Omega}{2} 
\end{pmatrix}.
\] (17)

We now look into the JEs in this system. Let us further express $a_{\sigma}$ as $a_{\sigma} = \sqrt{N_{\sigma}} e^{i\theta_{\sigma}}$, where the particle numbers $N_{\sigma}$ and phases $\theta_{\sigma}$ are all time-dependent in general. According to Eq. (16), we can obtain
\[
i\hbar \frac{d}{dt} \frac{N_{\sigma}}{\sqrt{N_{\bar{\sigma}}}} - \hbar N_{\sigma} \dot{\theta}_{\sigma} = J_{\sigma \bar{\sigma}} \sqrt{N_{\sigma}} N_{\bar{\sigma}} e^{i(\theta_{\sigma} - \theta_{\bar{\sigma}})} + \frac{\Omega}{2} \sqrt{N_{\sigma}} N_{\bar{\sigma}} e^{i(\theta_{\sigma} - \theta_{\bar{\sigma}})} + (-1)^p \frac{\delta}{2} N_{\sigma}.
\] (18)

Separating the image and real parts of Eq. (18) yields two groups of equations as
\[
\begin{align*}
\dot{N}_{\sigma} &= \frac{2J_{\sigma \bar{\sigma}}}{\hbar} \sqrt{N_{\sigma} N_{\bar{\sigma}}} \sin(\theta_{\sigma} - \theta_{\bar{\sigma}}) + \frac{\Omega}{2} \sqrt{N_{\sigma} N_{\bar{\sigma}}} \sin(\theta_{\sigma} - \theta_{\bar{\sigma}}), \\
\dot{\theta}_{\sigma} &= \frac{J_{\sigma \bar{\sigma}}}{\hbar} \sqrt{\frac{N_{\sigma}}{N_{\bar{\sigma}}}} \cos(\theta_{\sigma} - \theta_{\bar{\sigma}}) + \frac{\Omega}{2} \sqrt{\frac{N_{\sigma}}{N_{\bar{\sigma}}}} \cos(\theta_{\sigma} - \theta_{\bar{\sigma}}) + (-1)^p \frac{\delta}{2} \frac{N_{\sigma}}{N_{\bar{\sigma}}}.
\end{align*}
\] (19)

Eq. (19) consists actually of eight coupled equations. To simplify these equations, we introduce $\phi_{\sigma} = \theta_{R \sigma} - \theta_{L \sigma}$ and $\rho_{\sigma} = N_{R \sigma} - N_{L \sigma}$ for the phase and particle number differences between two wells with the same spin $\sigma$, and
\[
\phi_j = \theta_{j \uparrow} - \theta_{j \downarrow} \quad \text{and} \quad \rho_j = N_{j \uparrow} - N_{j \downarrow} \quad \text{for the phase and particle number differences between two spins in the same well}.
\]
well $j$, respectively. Thus we can obtain
\[ \dot{\rho}_\sigma = L_1 \sin \phi_\sigma + \frac{1}{2} \sum_j (-1)^q L_2 \sin \phi_j, \]
\[ \dot{\rho}_j = \frac{1}{2} \sum_\sigma (-1)^p L_1 \sin \phi_\sigma + L_2 \sin \phi_j, \]
(20)
where $L_1 = -\frac{2J_\sigma}{\hbar} \sqrt{N_\sigma N_{\bar{\sigma}}}$, $L_2 = -\frac{2\Omega}{\hbar} \sqrt{N_j \bar{N}_{j'}}$, and $q = 0, 1$ for $j = R, L$, respectively ($p = 0, 1$ for $\sigma = \uparrow, \downarrow$, respectively). From the above Eq. (20), we find the coexistence of internal JE related to $\dot{\rho}_j(\phi_j)$ and external JE related to $\dot{\rho}_\sigma(\phi_\sigma)$. Moreover, the internal and external JEIs are inherently coupled in a more complicated form.

Before ending this section, we briefly discuss the weakly interacting cases, which have been assumed to meet the requirement of two-mode approximation. In this regime, the mean-field analysis still works well, and the dropped term $H_{int}$ can be taken into count within the previous discussions. This leads to two additional terms related to interactions into Eq. (19), and now the equations of motion are given by
\[ i\hbar \dot{a}_{j\sigma} = J_{\sigma\sigma} a_{j\sigma} + \frac{\delta}{2} a_{j\bar{\sigma}} + (1)^p \frac{\delta}{2} a_{\bar{\sigma}\sigma} + g_{\sigma\sigma} a_{j\sigma} \bar{a}_{j\bar{\sigma}} + g_{\bar{\sigma}\sigma} a_{\bar{\sigma}j} \bar{a}_{\bar{\sigma}j}, \]
(21)
where the interacting strength $g_{\sigma\sigma\bar{\sigma}}$ can be found as $g_{\sigma\sigma\bar{\sigma}} = \frac{\sqrt{\delta^2 + \delta^2}}{\sqrt{\pi} m_1^{1/2} \hbar}$. The estimation of the interaction energy and the Josephson dynamics in the presence of weak interactions will be presented in the next section.

**IV. JOSEPHSON EFFECTS IN WEAK RAMAN COUPLING REGIMES**

In the preceding section, we have shown that the SO coupled BEC in a double well potential exhibits the complicated coupled external and internal Josephson dynamics. We, in this section, consider a specific dynamic process of the system in the weak Raman coupling regime (i.e. $\Omega/E_r \ll 1$), where the external Josephson dynamic dominates. In fact, the manipulation and detection of the SO coupled BECs in this regime have been performed in experiments [20].

For the weak Raman coupling, we find that the ratio $\nu \equiv |J_{\sigma\sigma}|/\Omega$ can reach several hundreds from Eq. (13). Thus within the time scale $\tau \sim \hbar/\Omega \simeq 45$ ms for $\Omega = 0.001E_r$, one can ignore the effects of the spin-flipping tunneling, which leads to two external Josephson tunneling processes for different spins. The spins in this regime are conserved and then the total particle number of spin $\sigma N_{\sigma t} = N_{L\sigma} + N_{R\sigma}$ are time-independent constants. We assume $N_{\uparrow t} = N_{\downarrow t}$ for simplicity. The equations of motion (21) in this case can be rewritten as
\[ i\hbar \frac{d}{dt} \begin{pmatrix} a_{L\sigma} \\ a_{R\sigma} \end{pmatrix} = \begin{pmatrix} (-1)^p \frac{\delta}{2} + g_{\sigma\sigma} |a_{L\sigma}|^2 + g_{\bar{\sigma}\sigma} |a_{R\sigma}|^2 \\ J_{\sigma\sigma} \end{pmatrix} \begin{pmatrix} a_{L\sigma} \\ a_{R\sigma} \end{pmatrix} \begin{pmatrix} (-1)^p \frac{\delta}{2} + g_{\sigma\sigma} |a_{L\sigma}|^2 + g_{\bar{\sigma}\sigma} |a_{R\sigma}|^2 \\ J_{\sigma\sigma} \end{pmatrix} \]
imbalance of the atomic gas \( \Sigma_\sigma \).

The time evolution of spin-dependent population effects, which can be observable in experiments by measuring the time-evolution of spin-dependent population imbalance of the atomic gas \( \Sigma_\sigma \).

For weakly interacting cases, we have to estimate the interaction energy \( U_{\sigma\sigma} \) and \( U_{\sigma\bar{\sigma}} \), which should be \( U_{\sigma\sigma}, U_{\sigma\bar{\sigma}} \ll \hbar \omega \) due to the two-mode approximation. This requirement results in \( U_{\sigma\sigma} \approx U_{\sigma\bar{\sigma}} \ll 0.1 E_r \). In this regime, the two spin-Josephson tunneling processes are coupled via atomic interactions. To understand the effects of the interaction, we show in Fig. 5 some typical results of the time evolution of \( \Sigma_\sigma \) for the same initial conditions and parameters in Fig. 3. It clearly demonstrate that the modification of \( \Sigma_\sigma(t) \) due to atomic interactions is not significant and even very minor in some cases (such as the case for \( \Sigma_\sigma(0) = 0 \) and \( \delta = 0 \) in Fig. 3(b), 5(b)) since the interaction energy is small compared with the tunneling energy. Therefore the spin Josephson dynamics still exhibit a similar oscillatory feature in this regime.

To see more clearly the oscillatory properties of the spin JEs, we have numerically calculated the frequency spectra of the net spin currents \( I_{\sigma} \) for various conditions (such as those in Fig. 3 and Fig. 5). We find that the spectra for different cases exhibit a single peak centered at the slightly shifted frequency, as seen in Fig. 6. The single-peak feature shown in Fig. 6 implies that the spin current \( I_{\sigma}(t) \) can well be described by a \( \sin \)-function, while the weak interatomic interactions or the small Zeeman field can merely modify the period and amplitude of the current slightly. Thus we conclude that the spin JEs in this system are robust against the parameter adjustment and initial conditions.

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**FIG. 3:** (Color online) The time evolution of \( \Sigma_\sigma \) in the noninteracting limitation. In (a) and (b) we have \( \delta = 0 \) and \( J_{\uparrow\uparrow} = J_{\downarrow\downarrow} = -0.1 E_r \); in (c) and (d) we have \( \delta = 0.01 E_r \), \( J_{\uparrow\uparrow} = -0.905 E_r \), and \( J_{\downarrow\downarrow} = -0.105 E_r \). The initial conditions are \( \Sigma_\sigma(0) = -\Sigma_\sigma(0) = 0.3 \), \( \phi_\uparrow(0) = 0.5 \phi_\downarrow(0) = \pi/4 \) in (a) and (c); and \( \Sigma_\sigma(0) = \Sigma_\sigma(0) = 0 \), \( \phi_\uparrow(0) = \phi_\downarrow(0) = \pi/4 \) in (b) and (d).

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**FIG. 4:** (color online) Josephson currents in the noninteracting limit. The time evolution of spin-dependent atomic currents \( I_\sigma \), a net spin current \( I_a \) and a total atomic current \( I_a \) in (a) for the same conditions in Fig. 3(a); and in (b) for the some conditions in Fig. 3(b).

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**FIG. 5:** (Color online) The time evolution of \( \Sigma_\sigma \) in the weakly interacting regime with \( U_{\sigma\sigma} = 0.01 E_r \) and \( U_{\sigma\bar{\sigma}} = 0.011 E_r \). Other parameters and initial conditions in (a)-(d) are the same with those in Fig. 3(a)-(d), respectively.
atomic inter-well tunneling and consider only the internal dynamics in each single well. The atomic tunneling between two spins refers to spin-flipping is induced by the Raman coupling, and thus such Josephson tunneling is in the spin space. Considering the atomic gas condenses with a finite but opposite momentum for different spins, the internal JEs connect interesting quantum tunneling in the momentum space.

In summary, we have investigated the quantum dynamics of a SO coupled BEC in a symmetric double well potential. The SO coupling contributes to atomic tunneling between wells and significantly enhances JEs for realistic conditions. We have predicted a novel spin Josephson effects which can be observed in a practical experiment since all the required ingredients, including the SO coupled BECs and the tunable double well potential, have already been achieved in the previous experiments.

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V. DISCUSSION AND CONCLUSION

Before concluding, we briefly discuss another specific dynamic process of the system in the relatively strong Raman coupling regime, \( \Omega \gg |J_{\sigma\sigma}| \), which can be realized such as by tuning \( J_{\sigma\sigma} \sim -0.1E_r \) and \( \Omega \sim E_r \). In this regime, within the time scale \( \hbar/|J_{\sigma\sigma}| \) one can ignore the
A. Eckardt, M. Lewenstein, P. Windpassinger, and K. Sengstock, Science 333, 996 (2011).

[24] T. D. Stanescu, B. Anderson, and V. Galitski, Phys. Rev. A 78, 023616 (2008).

[25] C. Wang, C. Gao, C. M. Jian, and H. Zhai, Phys. Rev. Lett. 105, 160403 (2010).

[26] C. M. Jian and H. Zhai, Phys. Rev. B 84, 060508 (2011).

[27] T. L. Ho and S. Zhang, Phys. Rev. Lett. 107, 150403 (2011).

[28] C. J. Wu, I. Mondragon-Shem, and X. F. Zhou, Chin. Phys. Lett. 28, 097102 (2011).

[29] S. K. Yip, Phys. Rev. A 83, 043616 (2011).

[30] Z. F. Xu, R. Lü, and L. You, Phys. Rev. A 83, 053602 (2011).

[31] T. Kawakami, T. Mizushima, and K. Machida, Phys. Rev. A 84, 011607 (2011).

[32] E. van der Bijl and R.A. Duine, Phys. Rev. Lett. 107, 195302 (2011).

[33] S. Gopalakrishnan, A. Lamacraft, and P. M. Goldbart, Phys. Rev. A 84, 061604(R) (2011).

[34] S. Sinha, R. Nath, and L. Santos, Phys. Rev. Lett. 107, 270401 (2011).

[35] H. Hu, B. Ramachandhran, H. Pu, and X. J. Liu, Phys. Rev. Lett. 108, 010402 (2012).

[36] Q. Zhu, C. Zhang, and B. Wu, arXiv: 1109.5811.

[37] R. Barnett, S. Powell, T. Grass, M. Lewenstein, and S. Das Sarma, arXiv: 1109.4945.

[38] Y. Deng, J. Cheng, H. Jing, C.-P. Sun, and S. Yi, arXiv: 1110.0558.

[39] X.-Q. Xu and J. H. Han, Phys. Rev. Lett. 107, 200401 (2011).

[40] X. F. Zhou, J. Zhou, and C. J. Wu, Phys. Rev. A 84, 063624 (2011).

[41] J. Radić, T. Sedrakyan, I. Spielman, and V. Galitski, Phys. Rev. A 84, 063604 (2011).

[42] M. Merkl, A. Jacob, F. E. Zimmer, P. Öhberg, and L. Santos, Phys. Rev. Lett. 104, 073603 (2010).

[43] Y. Zhang, L. Mao, and C. Zhang, arXiv: 1102.4045 [Phys. Rev. Lett. (to be published)].

[44] D. W. Zhang, Z. Y. Xue, H. Yan, Z. D. Wang, and S. L. Zhu, arXiv: 1104.0444 [Phys. Rev. A (to be published)].

[45] J. Javanainen, Phys. Rev. Lett. 57, 3164 (1986).

[46] G. J. Milburn, J. Corney, E. M. Wright, and D. F. Walls, Phys. Rev. A 55, 4318 (1997).

[47] A. Smerzi, S. Fantoni, S. Giovanazzi, and S. R. Shenoy, Phys. Rev. Lett. 79, 4950 (1997); S. Raghavan, A. Smerzi, S. Fantoni, and S. R. Shenoy, Phys. Rev. A 59, 620 (1999).

[48] S. Giovanazzi, A. Smerzi, and S. Fantoni, Phys. Rev. Lett. 84, 4521 (2000).

[49] S. Levy, E. Lahoud, I. Shomroni, and J. Steinhauser, Nature 449, 579 (2007).

[50] M. Albiez, R. Gati, J. Fölling, S. Hunsmann, M. Cristiani, and M. K. Oberthaler, Phys. Rev. Lett. 95, 010402 (2005).

[51] L. J. LeBlanc, A. B. Bardon, J. McKeever, M. H. T. Estavour, D. Jervis, J. H. Thywissen, F. Piazza, and A. Smerzi, Phys. Rev. Lett. 106, 025302 (2011).

[52] G. Thalhammer, G. Barontini, L. De Sarlo, J. Catani, F. Minardi, and M. Inguscio, Phys. Rev. Lett. 100, 210402 (2008).

[53] M.-S. Chang, Q. Qin, W. Zhang, L. You, and M. S. Chapman, Nat. Phys. 1, 111 (2005).

[54] X.-Q. Xu, L.-H. Lu, and Y.-Q. Li, Phys. Rev. A 78, 043609 (2009).

[55] B. Sun and M. S. Pindzola, Phys. Rev. A 80, 033616 (2009).

[56] H. Pu, W. P. Zhang, and P. Meystre, Phys. Rev. Lett. 89, 090401 (2002).

[57] R. Qi, X.-L. Yu, Z. B. Li, and W. M. Liu, Phys. Rev. Lett. 102, 185301 (2009).

[58] C. Chin, R. Grimm, P. Julienne, and E. Tiesinga, Rev. Mod. Phys. 82, 1225 (2010).

[59] Y.-A Chen, S. Nascimbène, M. Aidelsburger, M. Atala, S. Trotzky, I. Bloch, Phys. Rev. Lett. 107, 210405 (2011).