Nonlinear Realization of a Dynamical Poincaré Symmetry
by a Field-dependent Diffeomorphism

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Abstract

We consider a description of membranes by (2, 1)-dimensional field theory, or alternatively a description of irrotational, isentropic fluid motion by a field theory in any dimension. We show that these Galileo-invariant systems, as well as others related to them, admit a peculiar diffeomorphism symmetry, where the transformation rule for coordinates involves the fields. The symmetry algebra coincides with that of the Poincaré group in one higher dimension. Therefore, these models provide a nonlinear representation for a dynamical Poincaré group.

Submitted to Annals of Physics
MIT CTP# 2717

I. INTRODUCTION

When a dynamical system is invariant under some transformation, typically the dynamical variables transform in a naturally evident, linear fashion. However, it may happen that in special circumstances, there exist unexpected invariances, frequently called “dynamical symmetries,” and the relevant transformation law for the dynamical variables is intricate and nonlinear. An example is given by the motion of a particle in the $1/r$ potential. There is obvious invariance against the $O(3)$ group of rotation transformations, which rotate the particle position variable $r$. Additionally, there is a hidden $O(3, 1)$ or $O(4)$ symmetry, which acts in an unexpected manner on the dynamical variables.

In this paper we shall present a field theoretic instance of this phenomenon. We shall consider a family of field Lagrangians, which arise in various physical contexts. We shall show that these theories possess a hidden Poincaré invariance, where the Poincaré transformations are defined in one higher dimension, and act nonlinearly on the dynamical variables of the model.

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†This work is supported in part by funds provided by the U.S. Department of Energy (D.O.E.) under contract #DE-FC02-94ER40818.
II. DYNAMICAL MODEL

Consider the following field Lagrangian, defined on a $d$-dimensional space $\{r\}$, and describing first-order evolution in time $\{t\}$

$$L = \int d^d r \left( \dot{\theta} \dot{\rho} - \frac{1}{2} \rho \nabla \theta \cdot \nabla \theta - V(\rho) \right)$$  \hspace{1cm} (1)

The fields depend on time and space: $\rho(t, r)$, $\theta(t, r)$; an over-dot always indicates differentiation with respect to the time argument, the gradient is always with respect to spatial arguments. $V$ is an unspecified $\theta$-independent potential that provides interactions for the free Lagrangian

$$L_0 = \int d^d r \left( \dot{\theta} \dot{\rho} - \frac{1}{2} \rho \nabla \theta \cdot \nabla \theta \right)$$  \hspace{1cm} (2)

The symplectic structure indicates that $\theta$ and $\rho$ are canonically conjugate [1]. The Euler–Lagrange equations read

$$\dot{\rho} = -\nabla \cdot (\rho \nabla \theta)$$ \hspace{1cm} (3a)

$$\dot{\theta} = -\frac{1}{2} (\nabla \theta)^2 + f(\rho)$$ \hspace{1cm} (3b)

$$f(\rho) \equiv -\frac{\delta}{\delta \rho} \int d^d r V$$ \hspace{1cm} (3c)

where the “force” term $f(\rho)$ is absent in the free case.

One encounters the dynamics described by $L_0$ and $L$ in diverse branches of physics: fluid motion with no vorticity and pressure determined by $V$ (irrotational and isentropic motion) [2], quantum mechanics in a hydrodynamical formulation [3,4], membrane theory [5], and dimensional reduction of relativistic scalar field theory [6]. Here is one instructive derivation of $L_0$ [4].

In the free particle Lagrangian (mass set to unity)

$$L_{\text{free particle}} = \frac{1}{2} \sum_i v_i^2(t)$$ \hspace{1cm} (4)

we replace the discrete summation with a continuum integration and introduce a density $\rho$ and a current $j$,

$$j(t, r) = v(t, r) \rho(t, r)$$ \hspace{1cm} (5)

linked by a continuity equation

$$\dot{\rho} + \nabla \cdot j = 0 \hspace{1cm} (6)$$

Evidently (4) becomes $\frac{1}{2} \int d^d r v^2 \rho = \frac{1}{2} \int d^d r j^2 / \rho$, and (5) is implemented by a Lagrange multiplier $\theta$.

$$L'_0 = \int d^d r \left\{ \frac{1}{2} j^2 / \rho + \theta(\dot{\rho} + \nabla \cdot j) \right\}$$ \hspace{1cm} (7)

Since the symplectic term does not contain $j$, it may be eliminated by solving for it [4].

$$j = \rho \nabla \theta$$ \hspace{1cm} (8)
Eqs. (5) and (8) exhibit the irrotational character of the velocity ($\nabla \times \mathbf{v} = 0$) whose potential is $\theta$ ($\mathbf{v} = -\nabla \theta$), while substitution of (8) into (7) gives $L_0$ of (2). $V$, whose form in (1) is arbitrary, provides an isentropic pressure potential for the irrotational fluid motion described by $L_0$.

To see more clearly the connection with the equations of isentropic fluid mechanics, when the velocity field $\mathbf{v}$ is irrotational ($\nabla \times \mathbf{v} = 0$), we note first that Eq. (3a) is just the continuity equation. Moreover by taking the gradient of (3b), we find

$$\dot{\mathbf{v}} + \mathbf{v} \cdot \nabla \mathbf{v} = f'(\rho) \nabla \rho$$

This is Euler’s equation, provided we identify $f'(\rho) \nabla \rho = -(1/\rho) \nabla (\text{pressure})$. But such an identification is always possible for isentropic motion (Kelvin’s theorem), where pressure is a function of $\rho$ only. In that case $(\partial V/\partial \rho) = -f(\rho)$ coincides with the enthalpy, and $[\rho(\partial^2 V/\partial \rho^2)]^{1/2}$ is the sound speed.

For an alternative derivation of the Lagrangian (1) from a different framework, we begin with the (linear or nonlinear) Schrödinger theory Lagrangian

$$L_S = \int d^d r \left\{ i \psi^* \dot{\psi} - \frac{1}{2} (\nabla \psi^*) \cdot (\nabla \psi) - \bar{V}(\psi^* \psi) \right\}$$

with $\bar{V}$ determining any nonlinear interactions. The substitution

$$\psi = \rho^{1/2} e^{i\theta}$$

produces $L$ of (1) (apart from a total time derivative) with $V$ fixed at

$$V(\rho) = \bar{V}(\rho) + \frac{1}{8} (\nabla \rho)^2$$

This is the hydrodynamical form of the Schrödinger theory [3,4]: the kinetic term is $L_0$, but there is also a further nontrivial “interaction,” even in the absence of $\bar{V}$. The same result is obtained from a dimensional reduction of a scalar relativistic field theory [3].

Finally, we recall that a gauge-fixed formulation of a membrane in (3,1) Minkowski space-time [5] again leads to (1), with $d = 2$, and a specific potential of strength $g$

$$V(\rho) = \frac{g}{\rho}$$

[One obtains the same result for a “d-brane” moving in $(d + 1, 1)$ space-time.]

III. OBVIOUS SYMMETRIES OF THE MODEL

From its derivation, it is clear that (1) and (3) (with obvious restriction on $V$) possess the Galileo symmetry. For completeness, we list here the generators of the infinitesimal transformations, as integrals of the appropriate densities; also we specify the action of the finite transformation (parameterized by $\omega$) on the fields: $\rho \rightarrow \rho_\omega, \theta \rightarrow \theta_\omega$, by presenting formulas for $\rho_\omega(t, \mathbf{r})$ and $\theta_\omega(t, \mathbf{r})$ in terms of $\rho(t, \mathbf{r})$ and $\theta(t, \mathbf{r})$. One verifies that the generators are time independent according to the equations of motion (3) and this furthermore implies that the transformed fields $\rho_\omega$ and $\theta_\omega$ also solve (3).
• Time, space translation
  – Energy
  \[ H = \int d^d r \mathcal{E}, \quad \mathcal{E} = \frac{1}{2} \rho \nabla \theta \cdot \nabla \theta + V(\rho) = \frac{1}{2} j^2 / \rho + V(\rho) \] (14)
  – Momentum
  \[ P = \int d^d r \mathcal{P}, \quad \mathcal{P} = \rho \nabla \theta = j \] (15)

• Space rotation
  – Angular momentum
  \[ J^{ij} = \int d^d r (r^i \mathcal{P}^j - r^j \mathcal{P}^i) \] (16)

With these space-time transformations \( \rho_\omega, \theta_\omega \) are obtained from \( \rho, \theta \) by respectively translating the time, space arguments and by rotating the spatial argument.

• Galileo boost
  – Boost generator
  \[ B = t P - \int d^d r r \rho \] (17a)

The boosted fields are
\[
\rho_\omega(t, r) = \rho(t, r - \omega t) \]
\[
\theta_\omega(t, r) = \theta(t, r - \omega t) + \omega \cdot r - \omega^2 t / 2 \] (17c)

The inhomogeneous terms in \( \theta_\omega \) are recognized as the well-known Galileo 1-cocycle, compare (11). Also they ensure that the transformation law for \( \mathbf{v} = \nabla \theta \)
\[
\mathbf{v}(t, r) \rightarrow \mathbf{v}_\omega(t, r) = \mathbf{v}(t, r - \omega t) + \omega \] (17d)

is appropriate for a co-moving velocity. Furthermore, knowledge about the Galileo 2-cocycle leads us to examine the \( \mathbf{P}, \mathbf{B} \) bracket, and its extension exposes another conserved generator, arising from an invariance against translating \( \theta \) by a constant; this just reflects the phase arbitrariness in (11).

• Phase symmetry
  – Charge
  \[ N = \int d^d r \rho \] (18a)
  \[ \rho_\omega = \rho \] (18b)
  \[ \theta_\omega = \theta - \omega \] (18c)
IV. CONNECTION WITH POINCARÉ SYMMETRY

It is well known that a Poincaré group in \((d+1,1)\) dimensions possesses the above extended Galileo group as a subgroup. \[7\] This is seen by identifying selected light-cone components of the Poincaré generators \(P^\mu, M^{\mu\nu}\) with the Galileo generators,

\[
P^\mu = (P^-, P^+, P^i) = (H, N, P^i) \tag{19}
\]
\[
M^{\mu\nu} = (M^{+-}, M^{-i}, M^{+i}, M^{ij}) \tag{20}
\]

where the \(\pm\) components of tensors are defined by

\[
T(\pm) = \frac{1}{\sqrt{2}}(T^{(0)} \pm T^{(d+1)}) \tag{21}
\]

But the Lorentz generators \(M^{+-}\) and \(M^{-i}\) have no Galilean counterparts.

A remarkable fact, first observed in Ref. \[5\] and then again in simplified form in Ref. \[6\], is that in the model \(\text{(1)}\) with \(V(\rho) = g/\rho\) [as in the membrane application, eq. \(\text{(13)}\)] one can define quantities that can be set equal to the generators missing from the identification \(\text{(20)}\), namely \(M^{+-}\) and \(M^{-i}\). This holds for arbitrary interaction strength \(g\); setting it to zero allows the same construction for the free Lagrangian \(\text{(2)}\). We wish to determine what role the additional generators have for the models \(\text{(1)}\) and \(\text{(2)}\).

V. ADDITIONAL SYMMETRIES

We observe that the free action \(I_0 = \int dt L_0\), as well as the interacting one

\[
I_g = \int dt \, d^d r \left( \theta \dot{\rho} - \frac{1}{2} \rho \nabla \theta \cdot \nabla \theta - g/\rho \right) \tag{22}
\]

are invariant against time rescaling \(t \to e^\omega t\), which is generated by

\[
D = t H - \int d^d r \, \rho \dot{\theta} \tag{23a}
\]

Fields transform according to

\[
\rho(t, r) \to \rho_\omega(t, r) = e^{-\omega} \rho(e^\omega t, r) \tag{23b}
\]
\[
\theta(t, r) \to \theta_\omega(t, r) = e^{\omega} \theta(e^\omega t, r) \tag{23c}
\]

The dilation generator \(D\) is identified with \(M^{+-}\). It is straightforward to verify from \(\text{(2)}\) that \(D\) is indeed time independent.

More intricate is a further, obscure symmetry whose generator can be identified with \(M^{-i}\). Consider the field-dependent coordinate transformations, implicitly defined by

\[
t \to T(t, r) = t + \frac{1}{2} \omega \cdot (r + R(t, r))
\]
\[
r \to R(t, r) = r + \omega \theta(T, R) \tag{24}
\]

with Jacobian \(|J|\)

\[
J = \det \begin{pmatrix}
    \frac{\partial T}{\partial t} & \frac{\partial T}{\partial R^i} & \frac{\partial T}{\partial r^j} \\
    \frac{\partial T}{\partial R^i} & \frac{\partial T}{\partial R^j} & \frac{\partial T}{\partial r^j} \\
    \frac{\partial T}{\partial t} & \frac{\partial T}{\partial R^i} & \frac{\partial T}{\partial r^j}
\end{pmatrix} = \left(1 - \omega \cdot \nabla \theta(T, R) - \frac{1}{2} \omega^2 \dot{\theta}(T, R)\right)^{-1} \tag{25}
\]
The transformation parameter $\omega$ has dimensions of inverse velocity. When fields are taken to transform according to

$$
\rho(t, r) \rightarrow \rho_\omega(t, r) = \rho(T, R) \frac{1}{|J|} 
$$

(26a)

$$
\theta(t, r) \rightarrow \theta_\omega(t, r) = \theta(T, R) 
$$

(26b)

one verifies that $I_g$ and $I_0$ are invariant. This is readily seen for the interaction term

$$
g \int dt \, d^4r \, \frac{1}{\rho(t, r)} \rightarrow g \int dt \, d^4r \, \frac{|J|}{\rho(T, R)} = g \int dT \, d^4R \, \frac{1}{\rho(T, R)} .
$$

(27)

To establish invariance of $I_0$, it is useful to write it first as

$$
I_0 = - \int dt \, d^4r \, \rho(\dot{\theta} + \frac{1}{2} \nabla \theta \cdot \nabla \theta)
$$

$$
\rightarrow - \int dt \, d^4r \, \rho(T, R) \left\{ \frac{\partial}{\partial t} \theta(T, R) + \frac{1}{2} \frac{\partial}{\partial r^i} \theta(T, R) \frac{\partial}{\partial r^i} \theta(T, R) \right\}
$$

(28)

The desired result follows once it is realized that the quantity in curly brackets equals $J^2(\theta(T, R) + \frac{1}{2} (\nabla \theta(T, R))^2)$. The transformations (26) are generated by

$$
G = \int d^4r \ (r \mathcal{E} - \frac{1}{2} \rho \nabla \theta^2)
$$

$$
= \int d^4r \ (r \mathcal{E} - \theta \mathcal{P})
$$

(29)

which is time independent according to (3).

While we have no insight about the geometric aspects of this peculiar symmetry, the following remarks may help achieve some transparency.

Observe that the Galileo generators can be expressed in terms of $\rho$ and $j$ or $\rho$ and $v = j/\rho = \nabla \theta$. Consequently, they are also defined for velocity fields with vorticity, $(\nabla \times v \neq 0)$, and provide well-known constants of motion for the (isentropic) Euler equations [2]. However, the velocity potential $\theta$ is needed to form $D$ and $G$, which therefore have a role only in vortex-free motion (with a specific potential or no potential).

For gauge-fixed membrane theory in (3, 1)-dimensional space-time, the dynamical variables are the membrane’s transverse coordinates $r^i$, $(i = 1, 2)$, which are functions of time $t$ and of two parametric variables $\phi^i$, $(i = 1, 2)$. The quantity $1/\rho$ arises as the Jacobian for the transformation $(t, \phi) \rightarrow (t, r(t, \phi))$ [3]. Therefore it is natural that under the further transformation $(t, r) \rightarrow (T(t, r), R(t, r))$, $1/\rho$ acquires the Jacobian of that transformation. Presumably transformations (27), (28) reflect a residual invariance of gauge-fixed membrane theory, but the reason for the specific form (26) of the transformation is not apparent. From the identification with Poincaré generators, we see that $\rho$ is the $\mathbb{P}^+$ density and $\theta$ is in some sense like $x^-$, and indeed in membrane theory $\theta$ coincides with $x^-(t, \phi) \rightarrow x^-(t, r)$.

Note that for all the transformations that are identified with Lorentz transformations, viz. (16), (17), (23) and (24) – (26), the following relation holds between new (capitalized) and old (lower-case) coordinates

$$
2 T \theta(T, R) - R^2 = 2 t \theta_\omega(t, r) - r^2
$$

(30)

The naturalness of this relation is recognized once it is appreciated that in light-cone components $x^\mu x^\rho = 2 x^+ x^- - x^i x^i$.

In applications to fluid mechanics our transformation generates nontrivial solutions of Euler’s equations, as we now explain.
VI. TRANSFORMING EXPPLICIT SOLUTIONS

In order to gain insight into the peculiar diffeomorphism transformations (26), we consider its effect on some explicit solutions to eqs. (3), in the free ($V = 0$) and interacting ($V = g/\rho$) cases.

A. No interaction, $V = 0$

With $V = 0$, eq. (31) is solved by

$$\theta(t, r) = \frac{1}{2} \frac{r^2}{t}$$  \hspace{1cm} (31a)

which, apart from selecting an origin in time and space and presenting a rotation and boost invariant profile, is also invariant against time rescaling (23) and the unconventional diffeomorphism (24)–(26). The fluid moves with a velocity unaffected by boosts (17d)

$$v = \frac{r}{t}.$$  \hspace{1cm} (31b)

The density is not determined, since the solution of the continuity equation (31c) in $d$ spatial dimensions involves an arbitrary function of $t/r$, and of the angles specifying $r$

$$\rho(t, r) = \frac{f(t/r, \hat{r})}{r^d}.$$  \hspace{1cm} (31c)

With $\theta$ as in (31a), the coordinate transformations (24) take the explicit forms

$$T(t, r) = tr^2 \omega,$$
$$R(t, r) = r r_\omega,$$
$$J = r^2 \omega,$$
$$r_\omega \equiv \hat{\omega} + \frac{\omega r}{2 t}.$$  \hspace{1cm} (32)

so that, as stated, the transformed $\theta_\omega(t, r)$ coincides with $\theta(t, r)$, while the transformed density becomes a different function of $t/r$,

$$\rho_\omega(t, r) = \frac{f(t/r, \hat{r})}{r^d \omega}.$$  \hspace{1cm} (33)

This coincides with $\rho(t, r)$ for the special choice $f(t/r, \hat{r}) \propto (t/r)^{2+d}$ in (31c), which provides a density profile that is invariant under the diffeomorphism (24)–(26).

One may construct other “free” solutions, for which $\theta$ remains unchanged under (24), (25), (26), while $\rho$ involves arbitrary functions. Also there are free solutions that respond nontrivially to the transformations. We do not pursue any of this any further here, rather we examine the “interacting” problem.
B. With interaction, $V = g/\rho$

Remarkably, a solution of the form (31a) also works in the presence of interactions. The profiles

$$\theta(t, r) = -\frac{r^2}{2(d-1)t}$$

$$\rho(t, r) = \sqrt{\frac{2g}{d}}(d-1)\frac{|t|}{r}$$

solve (3) with $V = g/\rho$; evidently $d$ must be greater than 1, and $g$ positive. [The positivity requirement is natural, in view of the fact that the sound speed for our model is $[\rho(\partial^2 V/\partial \rho^2)]^{1/2} = \sqrt{2g/\rho}$.] This solution is time-rescaling invariant, but no longer boost nor diffeomorphism invariant. The velocity flow is given by

$$v = -\frac{r}{(d-1)t}$$

while the current reads

$$j = \mp \sqrt{\frac{2g}{d}} \hat{r}.$$ (34d)

where the sign is determined by the sign of $t$.

At $d = 1$ we can find time-rescaling invariant solutions for $g > 0$

$$\theta(t, x) = \frac{1}{2k^2t} \sinh^2 kx, \quad -\frac{1}{2k^2t} \cosh^2 kx$$

$$\rho(t, x) = \frac{\sqrt{2g} k |t|}{\sinh^2 kx}, \quad \frac{\sqrt{2g} k |t|}{\cosh^2 kx}$$

while for $g < 0$ one gets

$$\theta(t, x) = \frac{1}{2k^2t} \sin^2 kx, \quad \frac{1}{2k^2t} \cos^2 kx$$

$$\rho(t, x) = \frac{\sqrt{2 |g| k |t|}}{\sin^2 kx}, \quad \frac{\sqrt{2 |g| k |t|}}{\cos^2 kx}.$$ (35)

Here $k$ is an arbitrary, positive integration constant. (The above are general solutions that preserve the scale of time, since a second integration constant is the origin of $x$.)

We now exhibit the form of the solutions when the field-dependent, coordinate transformations (24)–(26) are carried out. For simplicity we discuss only the $d = 2$ (membrane) case, and take $t > 0$. The new coordinates are determined by the old ones by (24), (25), and (34a).

$$T = \frac{3}{4}t + \frac{1}{2} \omega \cdot r \pm \frac{1}{4} \sqrt{(t + 2\omega \cdot r)^2 - 2\omega^2 r^2}$$

$$R = r + \frac{\omega}{2\omega^2} \left[-t - 2\omega \cdot r \pm \sqrt{(t + 2\omega \cdot r)^2 - 2\omega^2 r^2}\right]$$

$$\frac{1}{J} = 1 + \frac{\omega \cdot R}{T} - \frac{\omega^2 R^2}{4T^2}.$$ (37c)
After \( \theta \) and \( \rho \) are transformed according to the rules (26), it is noticed that expressions are simplified by performing the Galileo boost \( r \rightarrow r - \omega t / \omega^2 \), according to the rules (17). (This precludes taking the limit \( \omega \rightarrow 0 \).) Also, time is rescaled according to (23), with \( t \rightarrow \sqrt{2} t \). Finally, we define \( \omega / \omega^2 = c \), which has dimension of velocity, and then the transformed profiles provide two solutions, depending on the sign of the square root

\[
\begin{align*}
\theta_c(t, r) &= \pm \sqrt{2 (c \cdot r)^2 - c^2 r^2 - c^4 t^2} \\
\rho_c(t, r) &= \frac{\sqrt{2 g}}{c^2} \left[ \frac{2 (c \cdot r)^2 - c^2 r^2 - c^4 t^2}{r^2 \mp 2t \sqrt{2 (c \cdot r)^2 - c^2 r^2 - c^4 t^2}} \right]^{1/2}.
\end{align*}
\] (38a, 38b)

The velocity is

\[
\begin{align*}
v_c(t, r) &= \pm \frac{2c (c \cdot r) - rc^2}{\sqrt{2 (c \cdot r)^2 - c^2 r^2 - c^4 t^2}}
\end{align*}
\] (38c)

and the current reads

\[
\begin{align*}
j_c(t, r) &= \pm \sqrt{2 g} \frac{2 \hat{c} (\hat{c} \cdot r) - r}{\left[ r^2 \mp 2t \sqrt{2 (\hat{c} \cdot r)^2 - c^2 r^2 - c^4 t^2} \right]^{1/2}}.
\end{align*}
\] (38d)

Note that \( c \) may be replaced by \( ic \), and \( \rho_c \) by \( -\rho_c \), to obtain another solution.

In the Figures we exhibit the profiles of the interacting solutions. We plot the original and transformed densities, and the transformed currents \( j_c = \rho_c \nabla \theta_c \), in terms of the variables \( r/t \ (t > 0) \). Without loss of generality, \( c \) is taken along the \( x \)-axis, and its magnitude is incorporated in the dimensionless ratio \( r/ct \). The original density possesses a singularity at the origin; in the transformed solutions the singularity is present only with the upper (negative) sign in the bracketed expression of (38b), where its denominator vanishes at \( r^2 = (\hat{c} \cdot r)^2 = 2c^2 t^2 \). The transformed currents exhibit a similar singularity. In the physical region the argument of the square root must be positive, \( 2(\hat{c} \cdot r)^2 - r^2 - c^2 t^2 \geq 0 \).
FIG. 1. The original density $\rho(t, r)/\sqrt{2g}$.

FIG. 2. The transformed density $\rho_c(t, r)/\sqrt{2g}$, with the upper sign.
FIG. 3. The transformed density $\rho_c(t, r)/\sqrt{2g}$, with the lower sign. The envelope defining the physical region is at $x^2 - y^2 = c^2t^2$.

FIG. 4. The transformed current $j_c(t, r)/\sqrt{2g}$, with the upper signs. The envelope defining the physical region is at $x^2 - y^2 = c^2t^2$. 
FIG. 5. The transformed current $j_c(t, r)/\sqrt{2g}$, with the lower signs. The envelope defining the physical region is at $x^2 - y^2 = c^2 t^2$. 
REFERENCES

[1] L.D. Faddeev and R. Jackiw, Phys. Rev. Lett. 60, 1692 (1988).
[2] L. Landau and E. Lifschitz, Fluid Mechanics, 2nd ed. (Pergamon, Oxford UK, 1987).
[3] E. Madelung, Z. Phys. 40, 322 (1926); E. Merzbacher, Quantum Mechanics, 3rd ed. (Wiley, New York, 1998).
[4] For recent work see N. Ogawa, preprint, hep-th/9801115.
[5] M. Bordemann and J. Hoppe, Phys. Lett. B317, 315 (1993).
[6] A. Jevicki, Phys. Rev. D 57, 5955 (1998).
[7] L. Susskind, Phys. Rev. 165, 1535 (1968).