A Simple and Computationally Trivial Estimator for Grouped Fixed Effects Models*

Martin Mugnier†

Abstract
This paper introduces a new fixed effects estimator for linear panel data models with clustered time patterns of unobserved heterogeneity. The method avoids non-convex and combinatorial optimization by combining a preliminary consistent estimator of the slope coefficient, an agglomerative pairwise-differencing clustering of cross-sectional units, and a pooled ordinary least squares regression. Asymptotic guarantees are established in a framework where $T$ can grow at any power of $N$, as both $N$ and $T$ approach infinity. Unlike most existing approaches, the proposed estimator is computationally straightforward and does not require a known upper bound on the number of groups. As existing approaches, this method leads to a consistent estimation of well-separated groups and an estimator of common parameters asymptotically equivalent to the infeasible regression controlling for the true groups. An application revisits the statistical association between income and democracy.

Keywords: panel data, time-varying unobserved heterogeneity, grouped fixed effects, agglomerative clustering

JEL Codes: C14, C23, C38.

*An earlier version of this paper was circulated under the title “Make the Difference! Computationally Trivial Estimators for Grouped Fixed Effects Models”. I thank Stéphane Bonhomme, Xavier D’Haultfoeuille, Elena Manresa, Pauline Rossi, Ao Wang, Martin Weidner, Andrei Zeleneev, and seminar participants at CREST, LMU Munich, UChicago, Oxford EET 2023, and Bristol ESG 2022 for helpful comments. This research is supported by the French National Research Agency grants ANR-17-CE26-0015-041, ANR-18-EURE-0005, ANR-11-LABX-0047, and the European Research Council grant ERC-2018-CoG-819086-PANEDA.

†Department of Economics, University of Oxford, martin.mugnier@economics.ox.ac.uk
1 Introduction

Suppose a sample of longitudinal or panel data \( \{(y_{it}, x_{it}) : 1 \leq i \leq N, 1 \leq t \leq T\} \) is observed, and consider a linear regression model with group fixed effects:

\[
\begin{cases}
  y_{it} = x_{it}' \beta + \alpha_{g_it} + v_{it}, \\
  \mathbb{E}[v_{it}] = \mathbb{E}[\alpha_{g_it}v_{it}] = \mathbb{E}[x_{itk}v_{it}] = 0,
\end{cases}
\]

where \( i \) denotes cross-sectional units, \( t \) denotes time periods, \( y_{it} \in \mathbb{R} \) is a dependent variable, and \( x_{it} \in \mathbb{R}^K \) is a vector of explanatory covariates uncorrelated with the zero-mean random variable \( v_{it} \in \mathbb{R} \) but possibly arbitrarily correlated with the unobserved group membership variable \( g_i \in \{1, \ldots, G\} \) and group-time-specific effect \( \alpha_{g_it} \in \mathbb{R} \).

This paper focuses on the estimation of and inference on the unknown slope parameter \( \beta \in \mathbb{R}^K \), group memberships \( g_i \in \{1, \ldots, G\} \), number of groups \( G \), and group-time effects \( (\alpha_{1t}, \ldots, \alpha_{Gt})' \in \mathbb{R}^G \), in an asymptotic framework such that \( N/T^\nu \to 0 \) for some constant \( \nu > 0 \), as \( N \) and \( T \) diverge while \( K \) and \( G \) remain fixed.

Special cases of factor models with factor loadings confined to a finite set, grouped fixed effects (GFE hereafter) provide a parsimonious yet flexible device to accommodate cross-sectional correlations and a few unrestricted trends of unobserved heterogeneity. Since their introduction (Hahn and Moon, 2010; Bonhomme and Manresa, 2015), GFE have gained considerable interest in both methodological and applied work (e.g., Su, Shi, and Phillips, 2016; Cheng, Schorfheide, and Shao, 2019; Bonhomme, Lamadon, and Manresa, 2019; Gu and Volgushev, 2019; Chetverikov and Manresa, 2021; Mugnier, 2022b; Janys and Siflinger, 2023).

Treating GFE as interactive fixed effects, however, leads to two main issues. First, parametric rate inference for the common parameters is generally not available when \( T \) grows very slowly with \( N \) (e.g., Bai, 2009; Moon and Weidner, 2019; Beyhum and Gautier, 2023). Second, although Higgins (2022) shows that parametric rate inference remains possible under some circumstances, his proposed method

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\(^1\)Note that \( \alpha_{g_it} = \lambda_i' F_t \) for any \( \lambda_i' = (c_1 1\{g_i = 1\}, \ldots, c_G 1\{g_i = G\}) \), \( F_t' = (\alpha_{1t}/c_1, \ldots, \alpha_{Gt}/c_G) \), and \( c = (c_1, \ldots, c_G)' \in \left( \mathbb{R} \setminus \{0\} \right)^G \). Thus \( \lambda_1/c, \ldots, \lambda_N/c \) lie in the finite set of vertices of the unit simplex of \( \mathbb{R}^G \). Reciprocally, if \( \bar{\lambda}_i \in \mathbb{R}^r \) for some \( r \in \mathbb{N} \setminus \{0\} \) with \( \left| \{\lambda_1, \ldots, \lambda_N\} \right| = \bar{G} \), then there exist \( \bar{g}(1, \ldots, \bar{g}_N)' \in \{1, \ldots, \bar{G}\}^N \) and \( \bar{a}(1, \ldots, \bar{a}_{GT})' \in \mathbb{R}^{\bar{G}T} \) such that \( \lambda_i' F_t = \bar{a}_{g_it} \).
solves a high-dimensional non-convex least-squares problem typically subject to local minima. Similarly, addressing the combinatorial nature of GFE introduces two main difficulties. First, GFE estimators, defined as pseudo joint maximum likelihood estimators optimizing over all possible partitions of cross-sectional units in $G$ groups, encounter a challenging non-convex and combinatorial optimization problem. If $P \neq NP$, exact solutions in polynomial time are unattainable for most real-world datasets of interest. Second, GFE estimators typically require $G$ or $G_{\text{max}} \geq G$ to be known to the analyst.

This paper provides a three-step estimation procedure free of these limitations. The first step leverages the low-rank factor structure of GFE to compute a computationally simple estimator of the slope coefficient. I focus on smooth and convex regularized nuclear norm estimation, but the interactive fixed effects literature provides many candidates further discussed. The second step examines the correlation in pairwise differences of residuals within triads of units to build distances between units that are shown to be asymptotically zero if and only if two units belong to the same groups, whenever group-time effects are well separated. An agglomerative clustering based on thresholding distances, in turn, is natural. The third step computes a pooled ordinary least squares regression (OLS) of the outcome on the covariates and interaction of time and estimated group dummies.

While the estimator combines existing disparate ideas from the panel data literature, this particular combination and its appealing large sample statistical properties are truly novel. Specifically, I show that model selection (i.e., estimation of $G$) can be achieved simultaneously without the need to choose a pre-specified range of misspecified models (an upper bound $G_{\text{max}}$) but simply merging close units at a rate governed by time dependence conditions. Since $G$ is unknown, regularization is needed and there is no free lunch. Bonhomme and Manresa (2015) discuss the use of AIC/BIC information criteria applied to a computationally difficult estimator but do not provide asymptotic guarantees within the asymptotic sequences considered. In contrast, I propose a computationally trivial data-driven procedure which is shown to be theoretically valid and performs relatively well in finite samples. Under standard conditions, I show that the first two steps achieve consistent estimation of well-separated groups. As similar methods, this leads to an estimator of common pa-
parameters asymptotically equivalent to the infeasible regression controlling for the true groups. Compared to the existing literature, the new approach is computationally trivial, generates the number of groups endogenously with a theoretically valid data-driven selection rule and is based on minimal assumptions on the covariates (sufficient conditional variation through a restricted eigenvalue condition). This generalization of Bonhomme and Manresa (2015) comes at the price of ruling out cross-sectional correlations in the error terms because the new estimator relies on triad-specific comparisons, instead of cross-sectional averages, to estimate the grouping structure.

While I focus on a simple linear model with a homogeneous slope to best convey the core proof ideas, the versatility of the new approach is illustrated by several extensions with a growing number of groups or covariates, heterogeneous slope coefficients (in time or units), or multiplicative network models (e.g., gravity trade equations) discussed in Appendix B. This paper does not address the question of conducting inference on groups. Dzemska and Okui (2018) provide pointwise valid inference methods given a preliminary estimator $\hat{\beta}$ is available. To this end, one can use the estimator proposed in the present paper.

I investigate the finite-sample performance of the new estimator across different Monte Carlo experiments calibrated to the empirical application. I compare the new estimator to (possibly misspecified) GFE estimators with several prespecified numbers of groups and the nuclear norm regularized (NN-reg) estimator. In a pure setting without covariates, the new method asymptotically outperforms all GFE estimators and achieves homogeneous clustering performance in terms of Precision, Recall, and Rand Index defined later. In a full setting with a scalar covariate, the new estimator significantly improves upon NN-reg in terms of bias and root mean square error. For $T \geq 20$, it achieves similar performance as the infeasible pooled OLS regression, and confidence intervals based on an estimator of the asymptotic variance attain their pre-specified 95% level.

Finally, I showcase the usefulness of the method by revisiting Bonhomme and Manresa (2015)’s study of the statistical association between income and democracy for a large panel of countries between 1970-2000. This association could be confounded by critical junctures in history that led to similar unobserved development paths for some countries and different for others. While
the authors find statistically significant (approximated) GFE estimates of the effect of lagged income per capita on a measure of democracy ranging between 0.061 and 0.089 and suggest that the number of groups is less than 10, I estimate 3 groups, an effect of 0.105 and a much higher cumulative income effect of 0.243 (v.s. 0.104 to 0.151). The preliminary regularized nuclear norm estimator delivers point estimates of 0.016 and 0.078 respectively.

**Related Literature.** This paper contributes to a vast literature on estimating interactive fixed effects panel models (e.g., Pesaran, 2006; Bai, 2009; Moon and Weidner, 2015, 2017; Bonhomme, Lamadon, and Manresa, 2022; Armstrong, Weidner, and Zeleneev, 2023; Beyhum and Gautier, 2023). Convex nuclear norm regularized estimators (e.g., Moon and Weidner, 2019; Chernozhukov, Hansen, Liao, and Zhu, 2019), though computationally simpler than non-convex least squares (e.g., Bai, 2009), may converge slower than parametric rates. I use bounds on the rate of convergence of such estimators derived in this literature to show how computationally simple estimators of the special case of grouped fixed effects models converge at a parametric rate. Chetverikov and Manresa (2021) independently used similar ideas, although they impose a factor structure on the covariates and a known upper bound on the number of groups to apply spectral clustering techniques. Differently, I only rely on a restricted eigenvalue condition and achieve model selection simultaneously. The regularization parameter is a data-driven substitute for a known upper bound $G_{\text{max}} \geq G$ and an AIC/BIC model selection criterion often used to select the number of factors in interactive fixed effects models (e.g., Bai, 2003).

This paper also contributes to a rapidly growing literature on estimating grouped fixed effects models. Bonhomme and Manresa (2015)’s GFE estimator, an extension of $k$-means clustering to handle covariates, solves an NP-hard optimization problem. Algorithms that provide fast solutions may not converge to its true value, and the same holds true for extensions and other non-convex estimators (e.g., Su, Shi, and Phillips, 2016; Ando and Bai, 2022; Mugnier, 2022b; Lumsdaine, Okui, and Wang, 2023). In contrast, the inferential theory developed in this paper is valid for a computationally simple estimator which substitutes a known upper bound on the number of groups with willingness to merge units into groups.
based on estimated pairwise distances. Similarly, because inference is on a true population parameter, my results contrast with Pollard (1981, 1982) which provides an asymptotic theory for the solution to the population \( k \)-means sum of squares problem in the cross-sectional case, i.e., only for a pseudo-true value. While the closest approach of Chetverikov and Manresa (2021)’s spectral and post-spectral estimators is computationally straightforward, its implementation requires prior knowledge about the number of groups (or a consistent estimator which is not provided by the authors) and the theoretical validity of the spectral (resp. post-spectral) estimator crucially rests upon a factor (resp. grouped) structure for the covariates. Such assumptions bring the model closer to random or correlated random effects in the spirit of Pesaran (2006) and could be restrictive in practice. Differently, I do not impose a model for the covariates (only sufficient variation) neither an upper bound for the number of groups. Similarly, Lewis, Melcangi, Pilossoph, and Toner-Rodgers (2022) recently introduced a fuzzy-clustering procedure that is shown to work well on simulations but whose large sample properties are not investigated and which requires the number of groups to be known to the researcher. Mehrabani (2023) adapts “sum-of-norm” convex clustering (e.g., Hocking, Joulin, Bach, and Vert, 2011; Tan and Witten, 2015) to a linear panel data model with latent group structure but time-constant group effects only. Extending his approach to accommodate time-varying effects seems difficult. Albeit close in spirit, the proposed procedure is different from the binary segmentation algorithm developed in Wang and Su (2021), or the pairwise comparisons method proposed in Krasnokutskaya, Song, and Tang (2022). A different approach would apply spectral clustering to some dissimilarity matrix (see, e.g., Ng, Jordan, and Weiss, 2002; von Luxburg, 2007; Chetverikov and Manresa, 2021; Brownlees, Guðmundsson, and Lugosi, 2022; Yu, Gu, and Volgushev, 2022). A disadvantage is that it adds one layer of complexity/tuning parameters: \( L \) eigenvectors of the dissimilarity matrix have to be computed and clustered (and \( L \) chosen), which is usually done by approximating a NP-hard \( k \)-means solution that this paper aims to avoid to ensure valid inference.

Finally, this paper establishes a connection between the mature statistical and operation research literature on clustering problems and the recent grouped fixed effects literature. Though agglomerative (hierarchical) clustering methods are well known
in the former, adapting them to the latter is relatively new (a few exceptions in different models being Vogt and Linton, 2016; Chen, 2019; Mammen, Wilke, and Zapp, 2022). The advantage lies in considerable gains of time and valid inference. I suspect the idea of breaking the high-dimensionality of $k$-means by considering agglomerative approaches in discrete econometric models to be very versatile and fruitful. Hence, this paper can be seen as the first application and analysis of an agglomerative clustering method to the econometric panel data model (1.1).\textsuperscript{2} The measure of pairwise distance used in the clustering step has already been employed in the mathematical statistics and econometric literature to study topological properties of the graphon (e.g., Zhang, Levina, and Zhu, 2017; Lovász, 2012; Zeleneev, 2020; Auerbach, 2022). More generally, dyad, triad, or tetrad comparisons have proven useful in a variety of other econometric contexts (see, e.g., Honoré and Powell, 1994; Graham, 2017; Charbonneau, 2017; Jochmans, 2017).

The rest of the paper is organized as follows. Section 2 introduces the three-step estimation procedure. Section 3 presents large sample properties including uniform consistency for the grouping structure and asymptotic normality at parametric rates. Section 4 reports the results of a small-scale Monte Carlo exercise calibrated to the empirical application of Section 5. Section 6 concludes. All proofs are in the Appendix. Additional material is in the Supplemental Material (Mugnier, 2022a), with section numbers S.1, etc.

\textbf{Notation.} \textsuperscript{1}{}\{\cdot\} denotes the indicator function. For any set $S \subset \mathbb{R}^p$ (for any $p \geq 1$), let $S^* \equiv S \setminus \{0\}$ and $|S|$ denote the cardinal of $S$. For all $\mathcal{I} \subset \mathbb{N}^*$, all $k \in \mathbb{N}^*$, let $\mathcal{P}_k(\mathcal{I})$ denote the set of subsets of $\mathcal{I}$ with cardinal $k$. The operators $\overset{p}{\rightarrow}$, $\overset{d}{\rightarrow}$, and plim denote respectively, convergence in probability, convergence in distribution, and probability limit. All vectors are column vectors. $\|\cdot\|$ denotes the Euclidean norm. For an $m \times n$ real matrix $A$ of rank $r$, I write the transpose $A'$, the Frobenius norm as $\|A\|_F \equiv \sqrt{\text{Tr}(A'A)}$, the spectral norm as $\|A\|_\infty \equiv \sigma_1(A)$, its max norm as

\begin{itemize}
\item[2] Since the first arXiv version of this paper, Freeman and Weidner (2023) also propose a hierarchical clustering algorithm but do not explicitly verify that it meets the approximation conditions for their large sample theory to be valid. Differently from $k$-means (e.g. Bonhomme, Lamadon, and Manresa, 2022; Graf and Luschgy, 2002), it seems not obvious when these conditions are met.
\end{itemize}
\[ \|A\|_{\max} \equiv \max_{i=1,\ldots,m; j=1,\ldots,n} |a_{ij}|, \text{ and its nuclear norm as } \|A\|_1 \equiv \sum_{i=1}^{r} \sigma_i(A), \text{ where } \sigma_1(A) \geq \cdots \geq \sigma_r(A) > 0 \text{ are the positive singular values of } A. \]

## 2 Three-step estimation

In this section, I introduce the three-step triad pairwise-differencing (TPWD) estimator \( \hat{\theta} \) for the parameter \( \theta \equiv (G, g_1, \ldots, g_N, \alpha_{11}, \ldots, \alpha_{GT}, \beta')' \in \Theta \) of model (1.1), where

\[ \Theta \equiv \bigcup_{g \in \mathbb{N}} \Theta_g, \Theta_g \equiv \{g\} \times \{1, \ldots, g\}^N \times \mathcal{A}^{gT} \times \mathcal{B} \text{ for some } \mathcal{B} \subset \mathbb{R}^K, \mathcal{A} \subset \mathbb{R}. \]

Sections 2.1-2.3 describe the three key ingredients of \( \hat{\theta} \): a preliminary consistent estimator of \( \beta \), a measure of pairwise distances between units, and an agglomerative clustering algorithm. Section 2.4 formally defines \( \hat{\theta} \).

### 2.1 Preliminary consistent estimation of the slope coefficient

The first ingredient is a preliminary consistent estimator \( \hat{\beta}^1 \) of \( \beta \):

\[ \|\hat{\beta}^1 - \beta\| = o_p(1) \text{ as } \min(N, T) \rightarrow \infty. \]

Intuitively, as covariates can be arbitrarily correlated with grouped fixed effects, the clustering problem is simpler for the approximate “pure” grouped fixed effects model:

\[ y_{it} - x_{it}' \hat{\beta}^1 = \alpha_{g,t} + v_{it} + o_p(1), \quad i = 1, \ldots, N, \ t = 1, \ldots, T, \]

as \( \min(N, T) \rightarrow \infty. \)

The interactive fixed effects literature provides various computationally simple estimators under different identifying assumptions (e.g., Moon and Weidner, 2019; Chernozhukov, Hansen, Liao, and Zhu, 2019; Beyhum and Gautier, 2019, 2023). Large sample properties developed in Section 3 do not depend on the rate of convergence of the computationally simple preliminary estimator which is generally less than \( \sqrt{NT} \). Only an upper bound on this rate must be known to implement the
method, which is the case for all the above-mentioned approaches under regularity conditions.

As an illustrative example, consider the nuclear norm regularized estimator. Let $Y \equiv (y_{it})_{i=1,...,N; t=1,...,T} \in \mathbb{R}^{N \times T}$, $X_k \equiv (x_{it,k})_{i=1,...,N; t=1,...,T} \in \mathbb{R}^{N \times T}$ for all $k \in \{1, \ldots, K\}$, and $v \cdot X \equiv \sum_{k=1}^{K} X_k v_k$ for all $v \in \mathbb{R}^K$. For all regularization parameter $\psi \in (0, \infty)$, let $Q_\psi$ denote the nuclear norm regularized concentrated objective function such that, for all $\beta \in \mathbb{R}^K$,

$$Q_\psi(\beta) \equiv \min_{\Gamma \in \mathbb{R}^{N \times T}} \left\{ \frac{1}{2NT} \|Y - \beta \cdot X - \Gamma\|_F^2 + \frac{\psi}{\sqrt{NT}} \|\Gamma\|_1 \right\}.$$

Define

$$\hat{\beta}_1(\psi) \in \arg \min_{\beta \in \mathbb{R}^K} Q_\psi(\beta). \quad (2.1)$$

Minimization problem (2.1) is convex and easily solvable with modern optimization techniques. Under regularity conditions, $\hat{\beta}_1(\psi)$ is unique and called the nuclear norm regularized estimator (see, e.g., Moon and Weidner, 2019). Instead of controlling for unobserved discrete trends $\Gamma = [\alpha_{g,t}]_{i,t}$, which leads to an NP-hard problem, the nuclear norm regularized estimator penalizes the nuclear norm (the sum of singular values) of an unrestricted matrix $\Gamma$ of individual trends. Under some assumptions (especially on $\psi$), the interactive fixed effects structure of grouped fixed effects is sufficient for $\hat{\beta}_1(\psi)$ to reach a convergence rate of at least $\sqrt{\min(N,T)}$ as proved in Section A. In the Monte Carlo experiments, the theoretically valid choice $\psi \equiv \log(\log(T))/\sqrt{16 \min(N,T)}$ is used.

If the researcher is willing to assume a known upper bound on the number of groups and a factor structure on the covariates, other examples include the correlated random effects estimators proposed in Chetverikov and Manresa (2021) or Armstrong, Weidner, and Zeleneev (2022)’s approach (which only requires an upper bound on $G$).

2.2 Pairwise distance between cross-sectional units

Equipped with a preliminary consistent estimator $\hat{\beta}_1$, the second ingredient is a measure of distance between any two units $i$ and $j$ informative of whether $i$ and $j$ belong to the same group or not. This can be achieved by using the linear (or multiplicative)
structure of the model. Define the first-step residual $\hat{v}_{it} \equiv y_{it} - x'_{it}\hat{\beta}$. An empirical distance $\hat{d}_\infty^2(i, j)$ between $i$ and $j$ is

$$
\hat{d}_\infty^2(i, j) \equiv \max_{k \in \{1, \ldots, N\} \setminus \{i, j\}} \left| \frac{1}{T} \sum_{t=1}^{T} (\hat{v}_{it} - \hat{v}_{jt}) \hat{v}_{kt} \right| .
$$

The distance $\hat{d}_\infty^2$ is borrowed from the statistical literature on graphon estimation (see, e.g., Lovász, 2012; Zeleneev, 2020; Auerbach, 2022). Other distances could be employed, e.g.,

$$
\hat{d}_2^2(i, j) \equiv \frac{1}{N} \sum_{k=1}^{N} \left| \frac{1}{T} \sum_{t=1}^{T} (\hat{v}_{it} - \hat{v}_{jt}) \hat{v}_{kt} \right| ,
$$

and I focus on one that works well in practice and turns out to be convenient to prove large sample properties (for instance, $\hat{d}_2^2$ would rule out asymptotically negligible groups to obtain consistent clustering estimation while $\hat{d}_\infty^2$ allows for such groups). I collect all pairwise distances in the symmetric dissimilarity matrix $\hat{D} \equiv (\hat{d}_\infty^2(i, j))_{(i, j) \in \{1, \ldots, N\}^2}$.

Why is the empirical distance $\hat{d}_\infty^2(i, j)$ informative about whether $g_i = g_j$? The intuition is as follows.\(^3\) Because $\hat{\beta} \xrightarrow{p} \beta$, it holds with probability approaching one that $\hat{v}_{it} \approx \alpha_{g_it} + v_{it}$. Under weak time dependence, tails, and cross-sectional independence restrictions on the error terms of the form $E[(v_{it} - v_{jt})v_{kt}] = 0$, it then holds “uniformly” over $i, j$, and $k \in \{1, \ldots, N\} \setminus \{i, j\}$ that

$$
\frac{1}{T} \sum_{t=1}^{T} (\hat{v}_{it} - \hat{v}_{jt}) \hat{v}_{kt} \approx \frac{1}{T} \sum_{t=1}^{T} (\alpha_{g_it} - \alpha_{g_jt}) \alpha_{gkt} .
$$

If $g_i = g_j$, then $\alpha_{g_it} - \alpha_{g_jt} = 0$ and

$$
\hat{d}_\infty^2(i, j) = \max_{k \in \{1, \ldots, N\} \setminus \{i, j\}} \left| \frac{1}{T} \sum_{t=1}^{T} (\hat{v}_{it} - \hat{v}_{jt}) \hat{v}_{kt} \right| \approx \max_{k \in \{1, \ldots, N\} \setminus \{i, j\}} \left| \frac{1}{T} \sum_{t=1}^{T} (\alpha_{g_it} - \alpha_{g_jt}) \alpha_{gkt} \right| = 0 .
$$

Reciprocally, if

$$
\hat{d}_\infty^2(i, j) = \max_{k \in \{1, \ldots, N\} \setminus \{i, j\}} \left| \frac{1}{T} \sum_{t=1}^{T} (\hat{v}_{it} - \hat{v}_{jt}) \hat{v}_{kt} \right| \approx 0 ,
$$

\(^3\)See also p.14 in Zeleneev (2020) in a network setting.
then necessarily \( g_i = g_j \). In fact, suppose that \( g_i \neq g_j \). Then, provided each group has at least two units asymptotically (which is weak) there exist \( k^*, l^* \in \{1, \ldots, N\} \setminus \{i, j\} \) such that \( g_{k^*} = g_i \) and \( g_{l^*} = g_j \). Equation (2.2) therefore implies

\[
\frac{1}{T} \sum_{t=1}^{T} (\hat{v}_{it} - \hat{v}_{jt}) \hat{v}_{k^*t} \approx \frac{1}{T} \sum_{t=1}^{T} (\alpha_{gt} - \alpha_{gt}) \alpha_{g_{k^*t}} \approx 0, \tag{2.3}
\]

\[
\frac{1}{T} \sum_{t=1}^{T} (\hat{v}_{it} - \hat{v}_{jt}) \hat{v}_{l^*t} \approx \frac{1}{T} \sum_{t=1}^{T} (\alpha_{gt} - \alpha_{gt}) \alpha_{g_{l^*t}} \approx 0. \tag{2.4}
\]

Differencing (2.3)-(2.4) and using that \( \alpha_{g_{k^*t}} = \alpha_{g_{l^*t}} \) and \( \alpha_{g_{l^*t}} = \alpha_{g_{l^*t}} \) yields

\[
\frac{1}{T} \sum_{t=1}^{T} (\alpha_{gt} - \alpha_{gt})^2 \approx 0,
\]

a contradiction if groups are “well separated”, i.e., if for all \((g, \tilde{g}) \in \{1, \ldots, G\}^2\) such that \( g \neq \tilde{g} \), there exists \( c_{g, \tilde{g}} > 0 \) such that

\[
\frac{1}{T} \sum_{t=1}^{T} (\alpha_{gt} - \alpha_{gt})^2 \geq c_{g, \tilde{g}} > 0.
\]

Section 3 formalizes this equivalence result by proving uniform asymptotic control of the remainders in the stochastic approximations.

### 2.3 Agglomerative clustering based on thresholding distances

Given a preliminary consistent estimator \( \hat{\beta}^1 \) and a distance \( \hat{d}^2_{\infty} \), the last ingredient is an agglomerative clustering algorithm that builds clusters from the dissimilarity matrix \( \hat{D} \equiv (\hat{d}^2_{\infty}(i, j))_{(i, j) \in \{1, \ldots, N\}^2} \). Section 3 (see also Lemma C.1) provides sufficient conditions under which any matrix \( \hat{W}(c_{NT}) \equiv 1\{\hat{D} \leq c_{NT}\} \) with \( c_{NT} \to 0 \) converges in max norm to \( W \equiv (1\{g_i = g_j\})_{(i, j) \in \{1, \ldots, N\}^2} \):

\[
\|\hat{W}(c_{NT}) - W\|_{\text{max}} = \max_{(i, j) \in \{1, \ldots, N\}^2} |\hat{W}_{ij}(c_{NT}) - W_{ij}| = o_p(1) \quad \text{as \( \min(N, T) \to \infty \).}
\]

In light of this result solving the problem of estimating a sparse (block diagonal) graphon, it is natural to propose agglomerative clustering algorithms based on merging units whose (weighted) pairwise distances fall below some threshold. I focus on the following intuitive and simple algorithm.

**Agglomerative Clustering Algorithm:**

Set the value of willingness-to-pair parameter \( c_{NT} \in (0, \infty) \). Then,
1. If there are no unclustered pairs \((i,j)\) such that \(\hat{d}_\infty(i,j) \leq c_{NT}\), assign each unclustered units to singleton groups.

2. Else:

2.1. Create a new group consisting of the pair(s) of units with the smallest pairwise distance. Let \(k \geq 2\) denote the number of such reference units.

2.2. Add to this group all units whose weighted pairwise distance to those \(k\) reference units falls below \(c_{NT}\).

2.3. Go to Step 1.

This procedure always produces \(\hat{G} \in \{1, \ldots, N\}\) non-empty groups.

### 2.4 A three-step triad pairwise-differencing estimator

The TPWD estimator \(\hat{\theta} \in \Theta\) for \(\theta \in \Theta\) is obtained as follows.

1. **Preliminary Slope Estimation Step:** Compute \(\hat{\beta}^1\) as in Section 2.1.

2. **Agglomerative Clustering Step:**

   2.a. Set \(c_{NT} \in (0, \infty)\).

   2.b. Compute \(\hat{D}\) as in Section 2.2.

   2.c. Compute \(\{\hat{g}_1, \ldots, \hat{g}_N\}\) and \(\hat{G} = |\{\hat{g}_1, \ldots, \hat{g}_N\}|\) as in Section 2.3.

3. **Projection Step:**

   Compute:

   \[
   \left(\hat{\beta}', \hat{\alpha}_{11}, \ldots, \hat{\alpha}_{G_T}\right) \in \arg\max_{(\beta', \alpha_{11}, \ldots, \alpha_{G_T}) \in B \times A} \sum_{i=1}^{N} \sum_{t=1}^{T} \left(y_{it} - x_{it}'\beta - \alpha_{g_{it}}\right)^2.
   \]

   Provided \(B = \mathbb{R}^K\) and \(A = \mathbb{R}\), the projection step to obtain \(\left(\hat{\beta}', \hat{\alpha}_{11}, \ldots, \hat{\alpha}_{G_T}\right)\) is a pooled OLS regression of \(y_{it}\) on \(x_{it}\) and the interactions of estimated groups and time dummies.

**Remark 1:** Consider a finite sample of fixed dimensions \(N\) and \(T\). If all random variables except group memberships are continuous, then to the extreme where \(c_{NT} \rightarrow\)
0, $\hat{G} \to N$ and each group contains a single unit. To the extreme where $c_{NT} \to +\infty$, $\hat{G} \to 1$ and a single group contains all units. Given the low CPU time of the method, an entire regularization path can be reported by the researcher by making $c_{NT}$ vary between these two regimes. In particular, the clustering step can be made all vectorized, greatly reducing computational burden compared to running loops.\footnote{MATLAB code is provided on the author’s website: https://martinmugnier.github.io/research.}

Remark 2: In Section 3, I provide theory for the choice of $c_{NT}$. In Section 3.3, I provide a simple data-driven selection rule that works well in finite samples. It is possible to further improve the finite sample performance by re-running the first step with $\tilde{v}_{it} = y_{it} - x_{it}'\tilde{\beta}$ in place of $\hat{v}_{it} = y_{it} - x_{it}'\hat{\beta}_1$ to obtain new $\hat{g}_1, \ldots, \hat{g}_N$, and then re-running the second and third steps and iterating again until some convergence criterion is achieved. The asymptotic results hold for any of the subsequent iterates and Monte Carlo performance suggest important gains of precision.

Remark 3: The computation routine requires $O(N^3T)$ operations, and it is quite fast compared to the NP-hard $k$-means problem that forms the basis of Bonhomme and Manresa (2015)’s GFE estimator. Whether the computational cost of an exact algorithm can be lowered to $O(N^2T)$ remains to the best of my knowledge an open problem. A current limitation of the method is rather the memory required to store triad differences of residuals, but it seems hardly possible to cluster units without any measure of distance between them. When unobserved heterogeneity is assumed to be time-constant, the $O(N^3T)$ computation cost can be reduced to $O(N^2T)$ and the preliminary estimator can be replaced with any standard differencing fixed effects estimator such as Arellano and Bond (1991). See also Wooldridge (2010). This particular case, as well as some extensions of model (1.1), is discussed in the Supplemental Material S.2.

### 3 Large sample properties

In this section, I derive the large sample properties of the TPWD estimator introduced in Section 2 under conditions similar to the existing literature. Consider the data
generating process:

\[ y_{it} = x'_{it} \beta^0 + \alpha^0_{gt} + v_{it}, \quad i = 1, \ldots, N, \ t = 1, \ldots, T, \]  

(3.1)

where \( g^0_i \in \{1, \ldots, G^0\} \) denotes group membership, and where the 0 superscripts refer to true parameter values. I consider asymptotic sequences \((N, T)\) where \( N \) and \( T \) diverge jointly to infinity which I denote \( \min(N, T) \to \infty \). The number of groups \( G^0 \) is fixed relative to \( N, T \) but unknown, and I defer the discussion on the case of an increasing sequence \( G^0 = G^0_{NT} \) to the Supplemental Material S.1.

3.1 Consistency of estimated group memberships

Consider the following assumptions.

**Assumption 1** \( \| \hat{\beta} - \beta^0 \| = O_p(r_{NT}) \) for some deterministic sequence \( r_{NT} \to 0 \) as \( N \) and \( T \) tend to infinity.

**Assumption 2** There exist constants \((C, \nu, \kappa) \in (0, +\infty)^2 \times (0, 1/2)\) such that

(a) \( N T^{-\nu} \to 0 \) as \( \min(N, T) \to \infty \).

(b) \( c_{NT} \to 0 \), \( c_{NT} r_{NT}^{-1} \to \infty \), and \( \Pr(c_{NT} T^{\alpha} \geq C) \to 1 \) as \( \min(N, T) \to \infty \).

**Assumption 3** There exist constants \( a, b, c, d, M > 0 \) and a sequence \( \tau(t) \leq e^{-at^d} \) such that:

(a) \( A \) is a compact subset of \( \mathbb{R} \).

(b) For all \((i,t) \in \{1, \ldots, N\} \times \{1, \ldots, T\}: \Pr(|v_{it}| > m) \leq e^{-(m/b)^{d_2}} \) for all \( m > 0 \).

(c) For all \((g, \tilde{g}) \in \{1, \ldots, G^0\}^2\) such that \( g \neq \tilde{g} \):

\[
\operatorname{plim}_{\min(N,T) \to \infty} T^{-1} \sum_{t=1}^{T} (\alpha^0_{gt} - \alpha^0_{\tilde{g}t})^2 = c_{g,\tilde{g}} \geq c.
\]

(d) For all \((i,j,k,g,\tilde{g}) \in \mathcal{P}_3(\{1, \ldots, N\}) \times \{1, \ldots, G^0\}^2\) such that \( g \neq \tilde{g} \), \( \{v_{it}\}_{t}, \ \{(v_{it} - v_{jt})v_{kt}\}_{t}, \ \{\alpha^0_{gt} - \alpha^0_{\tilde{g}t}\}_{t} \) and \( \{\alpha^0_{gt} - \alpha^0_{\tilde{g}t}v_{it}\}_{t} \) are strongly mixing processes with mixing coefficients \( \tau(t) \). Moreover, \( \mathbb{E}[v_{it}v_{jt}] = 0 \).
\[(e) \lim_{\min(N,T) \to \infty} \Pr(\min_{g \in \{1, \ldots, G^0\}} \sum_{i=1}^{N} 1\{g_i^0 = g\} \geq 2) = 1.\]

\[(f) \text{ As } N \text{ and } T \text{ tend to infinity:} \]

\[
\sup_{i \in \{1, \ldots, N\}} \Pr \left( \frac{1}{T} \sum_{t=1}^{T} \|x_{it}\|^2 \geq M \right) = o(T^{-\delta}) \text{ for all } \delta > 0.
\]

Assumption 1 requires \(\hat{\beta}^1\) to be consistent for \(\beta^0\) at a rate bounded by \(r_{NT}\). This rate can be slow. Examples of computationally simple estimators satisfying this condition are given in Sections 2.1 and A. Assumption 2 allows \(T\) to grow considerably more slowly than \(N\) (when \(\nu \gg 1\)). It requires that the tuning parameter decreases to zero, but not too fast, at a rate bounded below by \(T^{-1/2}\) and strictly slower than \(r_{NT}\). I use probability limit because I allow \(c_{NT}\) to be data-driven and hence random. Assumptions 3(a)-(b) and 3(d) collect standard moment, tail, and dependence conditions. They do not impose homoscedasticity but only require uniform bounds on the unconditional variances. Assumption 3(c) requires groups to be well-separated. Assumption 3(d) rules out cross-sectional correlation in the error term. Assumption 3(e) allows for asymptotically negligible groups but requires that each group has at least two members with probability approaching one. Assumption 3(f) is a slight reinforcement of Bonhomme and Manresa (2015)’s Assumption 2(e). It holds if covariates have bounded support or if they satisfy dependence and tail conditions similar to \(v_{it}\). All results below are understood up to group relabeling.

**Proposition 3.1 (Sup-Norm Classification Consistency)** Let Assumptions 1-3 hold. Then, as \(N\) and \(T\) tend to infinity,

\[
\max_{i \in \{1, \ldots, N\}} \left| \hat{\beta}_i - \beta_i^0 \right| \overset{p}{\to} 0, \tag{3.2}
\]

and

\[
\hat{G} - G^0 \overset{p}{\to} 0. \tag{3.3}
\]

### 3.2 Asymptotic distribution

The next assumption is useful to establish the asymptotic distribution of \(\hat{\beta}\) and \(\hat{\alpha}_{gt}\).

**Assumption 4**
(a) For all \( g \in \{1, \ldots, G^0\} \): \( \lim \inf_{N,T \to \infty} \frac{1}{N} \sum_{i=1}^{N} 1\{g^0_i = g\} = \pi_g > 0 \).

(b) For all \((i,j,t) \in \{1,\ldots,N\}^2 \times \{1,\ldots,T\}\): \( \mathbb{E}[x_{it}v_{ut}] = 0 \).

(c) There exist positive definite matrices \( \Sigma_\beta \) and \( \Omega_\beta \) such that
\[
\Sigma_\beta = \lim \inf_{N,T \to \infty} \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} (x_{it} - \bar{x}_{g^0_i t})(x_{it} - \bar{x}_{g^0_i t})',
\]
\[
\Omega_\beta = \lim \inf_{N,T \to \infty} \frac{1}{NT} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} \mathbb{E} \left[ v_{it}v_{js} (x_{it} - \bar{x}_{g^0_i t})(x_{js} - \bar{x}_{g^0_i s})' \right],
\]
where \( \bar{x}_{gt} \equiv \left( \sum_{i=1}^{N} 1\{g^0_i = g\} \right)^{-1} \sum_{i=1}^{N} 1\{g^0_i = g\} x_{it} \).

(d) As \( N \) and \( T \) tend to infinity: \( \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=1}^{T} (x_{it} - \bar{x}_{g^0_i t})v_{it} \overset{d}{\to} \mathcal{N}(0, \Omega_\beta) \).

(e) For all \((g,t) \in \{1, \ldots, G^0\} \times \{1, \ldots, T\}\):
\[
\lim \inf_{N,T \to \infty} \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbb{E} \left[ 1\{g^0_i = g\} 1\{g^0_j = g\} v_{it}v_{jt} \right] = \omega_{gt} > 0.
\]

(f) For all \((g,t) \in \{1, \ldots, G^0\} \times \{1, \ldots, T\}\): as \( N \) and \( T \) tend to infinity,
\[
\frac{1}{\sqrt{N}} \sum_{i=1}^{N} 1\{g^0_i = g\} v_{it} \overset{d}{\to} \mathcal{N}(0, \omega_{gt}).
\]

Assumption 4 ensures that the infeasible least squares estimator has a standard asymptotic distribution. Assumption 4(b) is satisfied if the \( x_{it} \) are strictly exogenous or predetermined and observations are independent across units. As a special case, lagged outcomes may thus be included in \( x_{it} \). The assumption does not allow for spatial lags such as \( y_{i-1} \).

**Corollary 3.2 (Asymptotic Distribution)** Let Assumptions 1-4 hold. Then, as \( N \) and \( T \) tend to infinity,
\[
\sqrt{NT}(\hat{\beta} - \beta^0) \overset{d}{\to} \mathcal{N} \left( 0, \Sigma_\beta^{-1}\Omega_\beta\Sigma_\beta^{-1} \right),
\]
and, for all \( t \):
\[
\sqrt{N}(\hat{\alpha}_{gt} - \alpha_{gt}^0) \overset{d}{\to} \mathcal{N} \left( 0, \frac{\omega_{gt}}{\pi_g^2} \right), \quad g = 1, \ldots, G^0,
\]
where \( \Sigma_\beta, \Omega_\beta, \omega_{gt}, \) and \( \pi_g \) are defined in Assumption 4.

Consistent plug-in estimates of the asymptotic variances can easily be constructed (see, e.g., Supplemental Material in Bonhomme and Manresa, 2015).
3.3 Choice of the tuning parameter

Assumption 2 provides theoretical guidance for choosing the tuning parameter $c_{NT}$. Because this guidance is based on asymptotic arguments, it has the convenience of not requiring any unknown constant but the drawback of depending on rates. As is the case with bandwidth selection in nonparametric density estimation, infinitely many different values consistent with the theory can be chosen in applications where only a finite sample is observed.

Below, I provide a theoretically valid data-driven selection rule that is shown to work well on the Monte Carlo experiments reported in Section 4. I assume that a consistent estimator $\hat{\beta}_1$ that converges at a rate of at least $\sqrt{\min(N,T)}$ is available. For instance, by Proposition A.1, the nuclear norm regularized estimator $\hat{\beta}_1(\psi)$ with $
abla = \log(\log(T))/\sqrt{16 \min(N,T)}$ verifies this assumption under weak conditions. The data-driven rule is as follows:

1. Compute $\hat{v}_{it} \equiv y_{it} - x_i'\hat{\beta}_1$.
2. Compute $\hat{\sigma}^1 \equiv \sqrt{\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \left( \hat{v}_{it} - \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \hat{v}_{it} \right)^2}$.
3. Set $c_{NT} \equiv \hat{\sigma}^1 \log(T)/\sqrt{T}$.

Theoretical validity follows from $\hat{\sigma}^1 = O_p(1)$ under the maintained assumptions, so that $c_{NT}$ verifies Assumption 2.

4 Monte Carlo simulations

In this section, I investigate the finite sample properties of the TPWD across two Monte Carlo experiments calibrated to the empirical application: a pure GFE model without covariates, and a full GFE model with a scalar covariate. I consider sample sizes $(N,T) \in \{90, 180\} \times \{7, 10, 20, 40\}$ and numbers of groups $G \in \{3, 4\}$. Mimicking the empirical results of Bonhomme and Manresa (2015), I let $\alpha_{1t} \equiv 1$, $\alpha_{2t} \equiv \frac{t-1}{T-1}$, $\alpha_{3t} \equiv 0$, and $\alpha_{4t} \equiv 1\{t \geq \lfloor T/2 \rfloor \} \frac{T-\lfloor T/2 \rfloor}{T-\lfloor T/2 \rfloor}$. In their application, the first group is referred to as “high democracy”, the second to “transition”, the third to “low
democracy”, and the fourth one to “late transition”. I consider balanced groups:
\[ g_i = 1 + \sum_{g=1}^{G-1} 1\{i > g \lfloor N/G \rfloor\}, \quad i = 1, \ldots, N, \]
and refer to Section S.3.1 for similar results with unbalanced groups. Similarly, I refer to Sections S.3.2-3.6 for additional results for models with more groups, lagged outcomes, unit-specific effects, higher signal-to-noise ratio, or time-invariant unobserved heterogeneity. For all experiments, results are averaged across 500 Monte Carlo samples.

4.1 Pure GFE model

First, consider a pure GFE model without covariates:
\[ y_{it} = \alpha_{g_i} + v_{it}, \quad i = 1, \ldots, N, \ t = 1, \ldots, T, \]
where \( v_{it} \sim N(0, (1/3)^2) \) across units and time periods. The signal-to-noise ratio is one. I evaluate the performance of the TPWD estimator with \( \hat{\beta}^1 = 0 \) and \( c_{NT} = \hat{\sigma}^1 \log(T)/\sqrt{T} \) by considering the following three criteria. First, I report the (average) root mean square errors (RMSE) of the estimated grouped fixed effects, which is measured by
\[
\text{RMSE}(\hat{\alpha}) = \frac{1}{500} \sum_{b=1}^{500} \sqrt{\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} (\hat{\alpha}_{\hat{g}_{it}} - \alpha_{g_{it}})^2}.
\]
Second, I report the average estimated number of group \( \hat{G} \). Third, I assess clustering accuracy using three metrics from the statistical learning literature on binary classification. Define
\[
FP \equiv \sum_{i<j} 1\{\hat{g}_i = \hat{g}_j\} 1\{g_i \neq g_j\},
\]
\[
TP \equiv \sum_{i<j} 1\{\hat{g}_i = \hat{g}_j\} 1\{g_i = g_j\},
\]
\[
FN \equiv \sum_{i<j} 1\{\hat{g}_i \neq \hat{g}_j\} 1\{g_i = g_j\},
\]
\[
TN \equiv \sum_{i<j} 1\{\hat{g}_i \neq \hat{g}_j\} 1\{g_i \neq g_j\}.
\]
The following three measures of estimated clustering accuracy are invariant to cluster relabelling:

\[
P = \frac{TP}{TP + FP},
\]

\[
R = \frac{TP}{TP + FN},
\]

\[
RI = \frac{TP + TN}{TP + FP + FN + TN}.
\]

The precision rate (P) measures the proportion of correctly matched pairs of units among all matched pairs of units. The recall rate (R) measures the proportion of correctly matched pairs of units among all pairs of units that belong to the same population cluster. The Rand Index (RI) measures the proportion of correct matching/non-matching decisions among all decisions made by the clustering algorithm.

Table 1 shows the results for the first two criteria and compares them to a Lloyd’s approximation of the GFE estimator with user-specified number of groups \( g \in \{2, 3, 10\} \) and 10,000 initializers. TPWD has the lowest RMSE in all settings for which \( T \geq 20 \). For settings with \( T < 20 \), TPWD does always better than mispecified GFE estimators and has a RMSE twice as large as the well-specified GFE. The number of groups is estimated consistently in less than 0.4s when \( N = 90 \), and less than 3s when \( N = 180 \). TPWD estimates are exact, which is not the case of the GFE estimator approximated by a Lloyd’s type algorithm (see MATLAB documentation on k-means).
Table 1: Estimation of the grouped fixed effects (pure GFE model)

| G  | N  | T  |  $\hat{G}$ | RMSE | CPU time | GFE $^G=2$ | GFE $^G=3$ | GFE $^G=10$ |
|----|----|----|-----------|------|----------|-----------|-----------|------------|
| 3  | 90 | 7  | 4.486    | 0.154 | 0.066    | 0.272     | 0.096     | 0.206      |
|    | 10 |    | 3.784    | 0.123 | 0.090    | 0.264     | 0.084     | 0.180      |
|    | 20 |    | 3.086    | 0.074 | 0.164    | 0.250     | 0.087     | 0.150      |
|    | 40 |    | 3.002    | 0.061 | 0.331    | 0.244     | 0.088     | 0.134      |
| 3  | 180| 7  | 5.450    | 0.145 | 0.521    | 0.271     | 0.077     | 0.189      |
|    | 10 |    | 4.270    | 0.106 | 0.683    | 0.261     | 0.059     | 0.160      |
|    | 20 |    | 3.246    | 0.059 | 1.249    | 0.246     | 0.061     | 0.127      |
|    | 40 |    | 3.008    | 0.043 | 2.427    | 0.241     | 0.073     | 0.107      |
| 4  | 90 | 7  | 4.744    | 0.176 | 0.065    | 0.265     | 0.128     | 0.215      |
|    | 10 |    | 3.946    | 0.176 | 0.085    | 0.250     | 0.131     | 0.192      |
|    | 20 |    | 3.576    | 0.165 | 0.173    | 0.236     | 0.124     | 0.157      |
|    | 40 |    | 3.652    | 0.156 | 0.352    | 0.233     | 0.126     | 0.135      |
| 4  | 180| 7  | 5.502    | 0.145 | 0.559    | 0.205     | 0.124     | 0.208      |
|    | 10 |    | 4.434    | 0.139 | 0.742    | 0.193     | 0.119     | 0.184      |
|    | 20 |    | 3.952    | 0.133 | 1.409    | 0.191     | 0.105     | 0.141      |
|    | 40 |    | 3.978    | 0.122 | 2.764    | 0.192     | 0.113     | 0.112      |

Notes: This table reports the estimated number of group ($\hat{G}$), the root mean square error (RMSE) and the execution time in seconds (CPU time) for the triad pairwise distance estimator (TPWD) with $\hat{\beta} = 0$ and $c_{NT} = \hat{\sigma}^2 \log(T)/\sqrt{T}$. It reports the RMSE for a Lloyd’s approximation to the grouped fixed effects estimator with user-specified number of groups $g \in \{2, 3, 10\}$ and 10,000 initializers (GFE $^G=g$). Results averaged across 500 Monte Carlo samples.

Table 2 shows the result for the last three criteria. The accuracy of TPWD classification in terms of Precision, Recall and Rand Index quickly improves with $T$, homogeneously across the three measures. For $G = 3$, the TPWD estimator performs always better in the limit than the GFE estimator, irrespective of the specified number of groups. This may suggest that Lloyd’s algorithm approximating the GFE estimator
does not return the global optimum. For $G = 4$ the performance of TPWD in terms of RI is close to the well-specified GFE estimator. When $g = 10$, the recall of the GFE estimator is quite low (less than 50%); when $g = 2$ the precision of the GFE estimator is quite low (always less than 60%). In contrast, the TPWD estimator achieves a data-driven balance between Precision and Recall.

Table 2: Classification accuracy (pure GFE model)

| $G$ | $N$ | $T$ | TPWD | GFE$^G=2$ | GFE$^G=3$ | GFE$^G=10$ |
|-----|-----|-----|------|-----------|-----------|-----------|
|     |     |     | P    | R        | RI        | P         | R        | RI        |
| 3   | 90  | 7   | 0.892| 0.834    | 0.913     | 0.568     | 0.911    | 0.746     | 0.957     | 0.967     | 0.973     | 0.964     | 0.364     | 0.788     |
| 10  | 0.930| 0.910| 0.948| 0.574     | 0.932     | 0.753     | 0.958     | 0.980     | 0.974     | 0.988     | 0.393     | 0.801     |
| 20  | 0.987| 0.986| 0.991| 0.586     | 0.977     | 0.768     | 0.940     | 0.979     | 0.963     | 0.999     | 0.420     | 0.811     |
| 40  | 1.000| 1.000| 1.000| 0.590     | 0.994     | 0.773     | 0.935     | 0.980     | 0.961     | 1.000     | 0.433     | 0.815     |
| 3   | 180 | 7   | 0.906| 0.812    | 0.910     | 0.571     | 0.905     | 0.745     | 0.974     | 0.975     | 0.983     | 0.968     | 0.355     | 0.784     |
| 10  | 0.946| 0.912| 0.954| 0.579     | 0.933     | 0.754     | 0.979     | 0.988     | 0.987     | 0.990     | 0.373     | 0.792     |
| 20  | 0.988| 0.985| 0.991| 0.591     | 0.982     | 0.771     | 0.961     | 0.986     | 0.976     | 1.000     | 0.408     | 0.805     |
| 40  | 1.000| 1.000| 1.000| 0.595     | 0.998     | 0.776     | 0.935     | 0.979     | 0.960     | 1.000     | 0.436     | 0.814     |
| 4   | 90  | 7   | 0.568| 0.792    | 0.803     | 0.384     | 0.931     | 0.621     | 0.616     | 0.946     | 0.841     | 0.733     | 0.366     | 0.814     |
| 10  | 0.563| 0.823| 0.800| 0.387     | 0.953     | 0.623     | 0.621     | 0.959     | 0.843     | 0.802     | 0.411     | 0.833     |
| 20  | 0.578| 0.856| 0.810| 0.390     | 0.981     | 0.623     | 0.627     | 0.985     | 0.848     | 0.905     | 0.484     | 0.8630    |
| 40  | 0.604| 0.853| 0.827| 0.398     | 0.992     | 0.632     | 0.625     | 0.991     | 0.847     | 0.968     | 0.538     | 0.884     |
| 4   | 180 | 7   | 0.716| 0.827    | 0.779     | 0.564     | 0.964     | 0.656     | 0.724     | 0.907     | 0.806     | 0.775     | 0.206     | 0.622     |
| 10  | 0.714| 0.908| 0.798| 0.569     | 0.985     | 0.663     | 0.726     | 0.929     | 0.814     | 0.832     | 0.227     | 0.637     |
| 20  | 0.706| 0.950| 0.802| 0.565     | 0.991     | 0.658     | 0.731     | 0.983     | 0.831     | 0.929     | 0.268     | 0.667     |
| 40  | 0.744| 0.960| 0.835| 0.565     | 0.999     | 0.658     | 0.724     | 0.981     | 0.824     | 0.988     | 0.315     | 0.695     |

Notes: This table reports the precision rate (P), recall rate (R), and Rand Index (RI) for the triad pairwise distance estimator (TPWD) with $\hat{\beta}^1 = 0$ and $c_{NT} = \hat{\sigma}^1 \log(T) / \sqrt{T}$, as well as for a Lloyd’s approximation to the grouped fixed effects estimator with user-specified number of groups $g \in \{2, 3, 10\}$ and 10,000 initializers (GFE$^{G=g}$). Results are averaged across 500 Monte Carlo samples.

4.2 Full GFE model

In this section, I turn to a full GFE model with a scalar covariate:

\[ y_{it} = x_{it} \beta + \alpha_{g_it} + v_{it}, \quad i = 1, \ldots, N, \ t = 1, \ldots, T, \]

where $\beta = 1$, $x_{it} = 0.5\alpha_{g_it} + u_{it}$, $u_{it} \sim \mathcal{N}(0, (1/3)^2)$, $v_{it} \sim \mathcal{N}(0, (1/3)^2)$, and $u_{it}$ and $v_{it}$ are mutually independent across units and time periods. The signal-to-noise ratio is one.
I evaluate the performance of the TPWD estimator using $\hat{\beta}^1(\psi)$ the nuclear-norm regularization described in Section 2.1 with $\psi = \log(\log(T))\sqrt{16 \min(N, T)}$ as a preliminary estimator and the data-driven rule $c_{NT} = \hat{\sigma}^1\log(T)/\sqrt{T}$. I report the TPWD estimate obtained after four iterations, where convergence is achieved. Beyond the root mean square error (RMSE) of the estimated grouped fixed effects, I report that of the estimated regression coefficient, which is measured by

$$\text{RMSE}(\hat{\beta}) = \sqrt{\frac{1}{500} \sum_{b=1}^{500} \|\hat{\beta}^{(b)} - \beta\|^2},$$

and the coverage rate of a 95%-level confidence interval for $\beta$ based on an estimation of the large-$T$ asymptotic variance of the TPWD estimator clustered at the unit level.

I compare the TPWD estimator to the nuclear-norm regularized estimator (NN-reg) with regularization parameter $\psi = \log(\log(T))\min(N, T)^{-1/2}/4$ and the infeasible pooled OLS regression with known group memberships (Infeasible OLS).

Table 3 shows the results in terms of bias, RMSE, and coverage of a 95%-level confidence interval (when available). For $G = 3$, the bias of the TPWD is estimator is only 10% to 0.1% of that of NN-reg, which converges very slowly but do not prevent TPWD from achieving high precision gains. Similarly, the TPWD coverage rate reaches its specified level for $T = 40$ and is already quite close for $T = 20$. The clustering task becomes more difficult with $G = 4$, and the price to pay for model selection is that bias exceeds variance in finite sample so that 95%-level confidence intervals have poor finite sample coverage (less than 50%). Under additional assumptions on the covariate structure and knowledge of the number of groups, one solution to obtain improved CI is to improve the rate of convergence of the preliminary estimator using the (post-)spectral estimator (e.g. Chetverikov and Manresa, 2021). If these assumptions are overly restrictive, and given the low bias, another possibility is to use the bootstrap.
Table 3: Estimation of the slope coefficient and grouped fixed effects
(full GFE model)

| G | N  | T  | Bias $\hat{\beta}$ | RMSE $\hat{\beta}$ | Bias $\hat{\beta}$ | RMSE $\hat{\beta}$ | $\alpha_{\hat{\beta}}$ | G  | Bias $\hat{\beta}$ | RMSE $\hat{\beta}$ | 95 | RMSE $\alpha_{\hat{\beta}}$ |
|---|-----|----|-------------------|-------------------|-------------------|-------------------|-------------------|----|-------------------|-------------------|----|-------------------|
| 3 | 90  | 7  | 0.550             | 0.561             | 0.084             | 0.123             | 0.560             | 0.178 | 4.494             | -0.001             | 0.002           | 0.953             | 0.061           |
|   | 10  | 0.448 | 0.451           | 0.060             | 0.095             | 0.638             | 0.144             | 3.718 | 0.000             | 0.001             | 0.971           | 0.061           |
|   | 20  | 0.306 | 0.313           | 0.013             | 0.045             | 0.878             | 0.083             | 3.102 | -0.001             | 0.001             | 0.972           | 0.061           |
|   | 40  | 0.234 | 0.234           | -0.001            | 0.000             | 0.944             | 0.061             | 3.010 | -0.001             | 0.000             | 0.972           | 0.061           |
|   | 180 | 7   | 0.551             | 0.557             | 0.068             | 0.089             | 0.466             | 0.160 | 5.360             | 0.002             | 0.001           | 0.965             | 0.044           |
|   | 10  | 0.443 | 0.449           | 0.045             | 0.071             | 0.638             | 0.123             | 4.244 | 0.001             | 0.001             | 0.972           | 0.043           |
|   | 20  | 0.299 | 0.300           | 0.008             | 0.032             | 0.894             | 0.060             | 3.206 | 0.000             | 0.000             | 0.976           | 0.043           |
|   | 40  | 0.219 | 0.226           | 0.000             | 0.000             | 0.952             | 0.043             | 3.006 | 0.000             | 0.000             | 0.976           | 0.043           |
| 4 | 90  | 7   | 0.552             | 0.563             | 0.112             | 0.138             | 0.396             | 0.197 | 4.620             | -0.002             | 0.002           | 0.950             | 0.071           |
|   | 10  | 0.450 | 0.452           | 0.121             | 0.138             | 0.258             | 0.195             | 3.772 | 0.000             | 0.001             | 0.966           | 0.070           |
|   | 20  | 0.308 | 0.315           | 0.109             | 0.123             | 0.144             | 0.178             | 3.454 | -0.001             | 0.001             | 0.964           | 0.070           |
|   | 40  | 0.235 | 0.237           | 0.097             | 0.105             | 0.050             | 0.168             | 3.552 | -0.001             | 0.000             | 0.974           | 0.071           |
| 4 | 180 | 7   | 0.498             | 0.504             | 0.059             | 0.071             | 0.506             | 0.155 | 5.514             | 0.002             | 0.001           | 0.962             | 0.051           |
|   | 10  | 0.397 | 0.404           | 0.067             | 0.078             | 0.352             | 0.148             | 4.388 | 0.001             | 0.001             | 0.968           | 0.049           |
|   | 20  | 0.267 | 0.268           | 0.068             | 0.071             | 0.126             | 0.141             | 3.930 | 0.000             | 0.000             | 0.976           | 0.050           |
|   | 40  | 0.195 | 0.202           | 0.059             | 0.063             | 0.088             | 0.130             | 3.962 | 0.000             | 0.000             | 0.977           | 0.050           |

Notes: This table reports results for the nuclear norm regularized (NN-reg) estimator with regularization parameter $\psi = \log(\log(T)) \min(N,T)^{-1/2}/4$; triad pairwise differencing estimator (TPWD) obtained after four iterations starting at NN-reg and setting $c = \sigma \log(T) / \sqrt{T}$; and infeasible pooled OLS regression with known group memberships (Infeasible OLS). .95 denotes coverage rate of a 95%-level confidence interval for $\beta$ based on large $N,T$ approximations and an estimator of the asymptotic variance clustered at the unit level. Results are averaged across 500 Monte Carlo samples.

Table 4 reports the results regarding clustering accuracy. For $G = 3$, the three measures are little affected by the slow rate of convergence of the preliminary consistent estimator, even for small values of $T$. For $G = 4$, Precision decreases significantly (between 0.53 and 0.72) while Recall remains at a high level (between 0.77 and 0.95). This mitigates the drop in the Rand Index.
Table 4: Classification accuracy of the TPWD estimator (full GFE model)

| $G$ | $N$  | $T$ | Precision | Recall | Rand Index |
|-----|------|-----|-----------|--------|------------|
| 3   | 90   | 7   | 0.836     | 0.798  | 0.881      |
|     | 10   |     | 0.886     | 0.884  | 0.923      |
|     | 20   |     | 0.974     | 0.975  | 0.983      |
|     | 40   |     | 1.000     | 1.000  | 1.000      |
| 3   | 180  | 7   | 0.872     | 0.796  | 0.894      |
|     | 10   |     | 0.916     | 0.894  | 0.938      |
|     | 20   |     | 0.987     | 0.984  | 0.990      |
|     | 40   |     | 1.000     | 1.000  | 1.000      |
| 4   | 90   | 7   | 0.533     | 0.771  | 0.779      |
|     | 10   |     | 0.529     | 0.812  | 0.775      |
|     | 20   |     | 0.549     | 0.869  | 0.788      |
|     | 40   |     | 0.571     | 0.867  | 0.803      |
| 4   | 180  | 7   | 0.703     | 0.816  | 0.767      |
|     | 10   |     | 0.696     | 0.901  | 0.782      |
|     | 20   |     | 0.688     | 0.947  | 0.785      |
|     | 40   |     | 0.720     | 0.953  | 0.815      |

**Notes:** This table reports results for the triad pairwise differencing estimator (TPWD) obtained after four iterations starting at the nuclear norm regularized estimator $\tilde{\beta}^1(\psi)$ with $\psi = \log(\log(T)) \min(N, T)^{-1/2}/4$ and setting $c_{NT} = \tilde{\sigma}_1 \log(T)/\sqrt{T}$. Results are averaged across 500 Monte Carlo samples.
5 Empirical application: income and (waves of) democracy

Characterizing the statistical relationship between income and democracy has been a longstanding problem in political science and economics (Lipset, 1959; Barro, 1999). Using panel data for $N = 90$ countries observed at $T = 7$ points in time over the period 1970-2000, Acemoglu, Johnson, Robinson, and Yared (2008) documented that the statistically significant positive effect of income on democracy disappears when including country fixed-effects in the regression. They argued that these results are consistent with countries having embarked on divergent paths of economic and political development at certain points in history, or critical junctures. Some of the examples they mention are the end of feudalism, the industrialization age, or the process of colonization. In this perspective, the fixed effects are meant to capture these highly persistent historical events. Bonhomme and Manresa (2015) proposed to test this assumption by using an approximation of the GFE estimator using alternative minimization and reporting the results for several numbers of groups. In particular, they considered a BIC criterion to perform model selection, unfortunately not shown to be consistent for the number of groups when $T$ grows slowly with $N$. They argued that the true number of groups should be less than 10, reporting statistically significant income effects between 0.061 and 0.089 (elasticity of a measure of democracy to lagged income per capita).

In this section, I revisit their study and consistently estimate the number of groups by applying the TPWD estimator to their preferred specification: a regression model of democracy (measured by the Freedom House indicator) on lagged democracy and lagged log-GDP per capita with unrestricted group-specific time patterns of heterogeneity $\alpha_{gt}$:

$democracy_{it} = \beta_1 democracy_{it-1} + \beta_2 \log GDP_{pc_{it-1}} + \alpha_{gt} + v_{it}.$

The data is obtained from the balanced subsample of Acemoglu, Johnson, Robinson, and Yared (2008). The preliminary estimator is a nuclear norm regularized (NN-reg) estimator with tuning parameter set to the

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5Available here: https://www.aeaweb.org/articles?id=10.1257/aer.98.3.808.
theoretically valid rule $\psi = \log(\log(T))/\sqrt{16 \min(N, T)}$ (results are not sensitive to alternative choices such as 3-fold cross-validation as long as the grid is sufficiently large and fine). The data-driven rule of the TPWD estimator is as in Section 3.3.

Figure 1 shows the regularization path of the TPWD estimator at the first four iterations: the estimated number of groups as a function of different values of the thresholding parameter (data-driven choices appear as vertical lines). Estimates values stabilize after iteration 9, and then alternate between the results of iterations 8 and 9 where only one unit switches from Group 1 to Group 2. The number of estimated groups moves from 2 to 3 at the third iteration and remains at $\hat{G} = 3$ for iterations 8 and 9.

Figure 1: TPWD regularization path

Notes: The solid blue, dashed green, dotted red, and dash-dotted magenta lines correspond to the 1st, 2nd, 3rd, and 4th iterations of the TPWD estimator with its thresholding parameter varying over a grid that contains 1,000 equispaced points between 0 and 0.176. Vertical lines represent the data-driven rule $c_{NT} = \hat{\sigma}^1 \log(7)/\sqrt{7}$ at each iteration.

Table 5 reports and compares NN-reg estimates, TPWD$^k_{it}$ estimates at iterations $k \in \{2, 3, 8, 9\}$ (results vary smoothly in between), and GFE$^G=g$ estimates with user-specified number of groups $g \in \{2, 3, 10\}$.
Table 5: Estimation of the effect of lagged democracy and lagged income on democracy

| Dependent variable: Democracy | NN-reg | TPWD$^{1\text{st}}$ | TPWD$^{2\text{nd}}$ | TPWD$^{3\text{rd}}$ | TPWD$^{4\text{th}}$ | GFE$^{G=2}$ | GFE$^{G=3}$ | GFE$^{G=10}$ |
|------------------------------|--------|---------------------|---------------------|---------------------|---------------------|----------------|----------------|----------------|
| $\hat{G}$                    | -      | 2                   | 2                   | 3                   | 3                   | -              | -              | -              |
| Lagged Democracy ($\beta_1$) | 0.800  | 0.691               | 0.679               | 0.580               | 0.570               | 0.601          | 0.407          | 0.277          |
|                             | (0.044)| (0.046)             | (0.050)             | (0.051)             | (0.041)             | (0.052)        | (0.052)        | (0.049)        |
| Lagged Income ($\beta_2$)    | 0.016  | 0.078               | 0.080               | 0.102               | 0.105               | 0.061          | 0.089          | 0.075          |
|                             | (0.013)| (0.013)             | (0.013)             | (0.014)             | (0.011)             | (0.011)        | (0.011)        | (0.008)        |
| Cumulative Income ($\frac{\beta_1}{1-\beta_1}$) | 0.078  | 0.252               | 0.248               | 0.244               | 0.243               | 0.152          | 0.151          | 0.104          |
|                             | (0.019)| (0.019)             | (0.014)             | (0.014)             | (0.021)             | (0.013)        | (0.013)        | (0.009)        |

Notes: Balanced sample of Acemoglu, Johnson, Robinson, and Yared (2008) with $N = 90$ countries and $T = 7$ time periods at the five-year frequency between 1970-2000. Democracy is measured as the Freedom House indicator of democracy. This table reports results for the nuclear norm regularized (NN-reg) estimator with regularization parameter $\psi = \log(\log(7))7^{-1/2}/4$, the triad pairwise differencing estimator (TPWD$^{k\text{th}}$) obtained after $k$ iterations starting at NN-reg and setting $c_{NT} = \tilde{\beta}^2 \log(7) / \sqrt{7}$, and Bonhomme and Manresa (2015)’s approximating Algorithm 2 for the GFE estimator with number of groups set to $g$ (GFE$^{G=g}$). When available, analytical standard errors based on large $N,T$ approximations and clustered at the country level are shown in parentheses.

First, while the nuclear norm is quite different from the GFE estimates, the 8th and 9th iterations (at which the sequence converges) of TPWD estimates of the lagged democracy and lagged outcome effects are in-between the NN-reg and the GFE with several pre-specified numbers of groups on average. After three iterations, TPWD converges to three estimated groups and delivers estimates closer to the GFE approximation with three groups, which confirms the intuition of Bonhomme and Manresa (2015). The income effect is statistically significant and similar to the GFE, but the cumulative income effect is significantly almost twice as large.

Estimated groups differ markedly from that reported by the GFE estimator with 3 groups (permanently low, permanently high, and transitioning democracy groups), which turn into slightly different estimates. Figures 2 plot the time effects at iteration 8 (left panel) and 9 (right panel). On the left panel, 79 countries exhibit a high and steadily increasing trend in democracy (Group 1), while Algeria, Cameroon, Gabon, Guinea, Iran, Mauritania, Peru, Singapore, the Syrian Arab Republic, and Tunisia exhibit a lower level of development that is increasing for the period 1970-1990 and decreasing on 1990-1995, and Nigeria is surprisingly volatile in terms of democracy (Group 3). On the right panel, Peru switches from Group 2 to Group 1, which does
not affect much the results.

Figure 2: Time effects $\tilde{\alpha}_{gt}$

(a) 8 iterations

(b) 9 iterations

Notes: TPWD estimates with NN-reg as a preliminary estimator and data-driven rule $c_{NT} = \tilde{\sigma}^1 \log(7)/\sqrt{T}$ after 8 iterations (left panel) and 9 iterations (right panel). The solid blue, dashed red, dash-dotted yellow lines correspond to Group 1 (“steady democracy”), Group 2 (“volatile”), and Group 3 (“highly volatile”) respectively.

6 Conclusion

Grouped fixed effects models are plagued with an underlying difficult combinatorial classification problem, rendering estimation and inference difficult. In this paper, I propose a novel strategy for constructive identification of all the model parameters including the number of groups. The method therefore simultaneously solve the estimation, classification, and model selection problem. The corresponding three-step estimator has a polynomial computational cost and is straightforward to implement: only smooth convex optimization and elementary arithmetic operations are required. It is based on thresholding a suitable pairwise differencing transformation of the regression equation and a preliminary off-the-shelf consistent estimator of the slope. Mild conditions are given under which the proposed estimator is uniformly consistent for the latent grouping structure and asymptotically normal as both dimensions
diverge jointly. Importantly, the number of groups is consistently estimated without any prior knowledge, and the time dimension can grow much more slowly than the cross-sectional dimension. The paper leaves a few questions unanswered. For instance: could the approach be fruitful to build a test for the grouping assumption? Could similar differencing ideas be applied to nonlinear structural models and/or potential outcome models? I leave these questions for further research.
Appendix

A  Sufficient conditions for consistency of the nuclear norm regularized estimator

Define $\gamma^0 \equiv (1 \{ g_i^0 = g \})_{i=1,...,N; g=1,...,G^0} \in \{0,1\}^{N \times G^0}$, $\alpha^0 \equiv (\alpha^0_{gt})_{t=1,...,T; g=1,...,G^0} \in \mathcal{A}^{T \times G^0}$, $x_k \equiv \text{vec}(X_k)$, and $x \equiv (x_1, \ldots, x_k)$.

Assumption 5

(a) As $N$ and $T$ tend to infinity: $\psi \to 0$ such that $\sqrt{\min(N,T)} \psi \to \infty$.

(b) Let $C \equiv \{ A \in \mathbb{R}^{N \times T} : \| M_{\gamma^0} A M_{\alpha^0} \|_1 \leq 3 \| A - M_{\gamma^0} A M_{\alpha^0} \|_1 \}$, where $M_B \equiv I - B(B'B)^{\dagger}B'$, $I$ is the identity matrix of appropriate dimensions, and $\dagger$ refers to the Moore-Penrose generalized inverse. There exists $\mu > 0$, independent from $N$ and $T$, such that for any $a \in \mathbb{R}^{NT}$ with $\text{mat}(a) \in C$, $a'M_a a \geq \mu a'a$ holds for $N,T$ sufficiently large.

(c) $\| (v_{it})_{i=1,...,N; t=1,...,T} \|_{\infty} = O_p \left( \sqrt{\max(N,T)} \right)$.

(d) As $N$ and $T$ tend to infinity: $\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} x_{it} x_{it}' \overset{P}{\to} \Sigma > 0$ and $\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=1}^{T} v_{it} x_{it} = O_p(1)$.

Assumption 5(b) is a restricted eigenvalue condition common in high-dimensional modelling (e.g., Bickel, Ritov, and Tsybakov, 2009). Sufficient conditions for Assumption 5(c) are given in Supplementary Appendix S.2 of Moon and Weidner (2017).

Proposition A.1 (Moon and Weidner (2019)) Let Assumption 5 hold. Then, as $N$ and $T$ diverge jointly to infinity, $\| \hat{\beta}^1(\psi) - \beta^0 \| = O_p(\psi)$.

Proof of Proposition A.1. This follows by Moon and Weidner (2019)'s Theorem 2 after noticing the interactive fixed effects structure of model (3.1) as in Footnote 1.
B Extensions

The three-step procedure underlying the TPWD estimator can be applied to several extensions of model (1.1). I briefly discuss a few of them in this section.

B.1 Linear models with heterogeneous slopes/asymptotically close groups

To allow for unit-specific effects and/or unit/time-specific slopes, one may work with time-differenced or demeaned equations and/or use a computationally simple first-step estimator well suited for heterogeneous/time-varying slope (e.g., Chernozhukov, Hansen, Liao, and Zhu, 2019). Group separation conditions need to be adjusted similarly to what is discussed in Bonhomme and Manresa (2015) ’s Supplemental Material, but the main arguments convey.

The approach of this paper allows for some degree of asymptotic closeness in group-specific effects. Suppose that for all \((g, \tilde{g}) \in \{1, \ldots, G^0\}^2\) such that \(g \neq \tilde{g}\):

\[
T^{-1} \sum_{t=1}^{T} (\alpha^0_{gt} - \alpha^0_{\tilde{g}t})^2 \geq \rho_{NT},
\]

for some deterministic sequence \(\rho_{NT} \to 0\) as \(\min(N,T) \to \infty\). Then, only the rate conditions in Assumption 2 have to be adapted to recover the true group memberships with probability approaching one. The asymptotic behaviour of the oracle OLS estimator, however, depends on this weak factor property.

B.2 Nonlinear multiplicative models for networks

Consider dyadic observations \(\{(y_{ij}, x_{ij}) : (i, j) \in \{1, \ldots, n\}^2, i \neq j\}\) for \(n\) agents such that

\[
y_{ij} = \varphi(x_{ij}; \beta_0)u_{ij}, \quad i \neq j, \tag{B.1}
\]

where \(\varphi : \mathbb{R}^K \to \mathbb{R}^+\) is a function known up to the finite-dimensional parameter vector \(\beta_0 \in \mathbb{R}^K\), and \(u_{ij} \in \mathbb{R}^+\) is a latent disturbance. Suppose

\[
u_{ij} = \alpha_i \gamma_j \omega_{g_i, h_j} \varepsilon_{ij}, \quad i \neq j, \tag{B.2}
\]
where $\alpha_i \in \mathbb{R}^{+*}$ and $\gamma_j \in \mathbb{R}^{+*}$ are permanent sender (exporter) and receiver (importer) unobserved effects, $g_i \in \{1, \ldots, G_0\}$ is an unobserved exporter-group membership variable, $G_0$ is the number of groups of exporters (considered exogenous and fixed), $h_j \in \{1, \ldots, H_0\}$ is an unobserved importer group membership variable, $H_0$ is the number of groups of importers (considered exogenous and fixed), $\omega_{g,h} \in \mathbb{R}^{+*}$ is a permanent unobserved effect affecting group $g$ exporting to group $h$, and $\varepsilon_{ij} \in \mathbb{R}^{+}$ is an idiosyncratic disturbance such that $\text{Pr}(\varepsilon_{ij} = 0) < 1$. Here, I treat $\alpha_i, \gamma_j, \omega_{g,h}, g^i, h^j$ as fixed, that is I condition on them.

Model (B.1) extends Jochmans (2017) by allowing for latent grouped interactions on top of standard sender and receiver effects. Such grouped effects might capture nonlinear latent measures of reciprocity between units $i$ and $j$: trade shocks that are shared by unobserved groups of exporters or groups of importers. Suppose interest lies in estimating $\beta_0, \alpha \equiv (\alpha_1, \ldots, \alpha_n), \gamma \equiv (\gamma_1, \ldots, \gamma_n), g \equiv (g_1, \ldots, g_n), h \equiv (h_1, \ldots, h_n)$, and $\Omega \equiv (\omega_{g,h})_{(g,h) \in [G_0] \times [H_0]}$ under the conditional mean restriction

$$E[\varepsilon_{ij} | x_{12}, \ldots, x_{n(n-1)}] = 1, \quad \forall i \neq j.$$  

(B.3)

Because $g, h, \alpha, \gamma$ and $\Omega$ are unobserved, one needs a normalization specified as:

$$\prod_{i=1}^{n} \alpha_i = \prod_{j=1}^{n} \gamma_j = 1.$$  

(B.4)

Section S.2.1 outlines identification arguments based on triad-differencing.

### B.3 Rubin-Holland Potential Outcomes

Consider a binary treatment $D_{it} \in \{0, 1\}$ and potential outcomes $Y_{it}(0)$ and $Y_{it}(1)$ if individual $i$ is non-treated or treated at period $t$ respectively. Suppose a researcher wants to learn about the treatment effect $Y_{it}(1) - Y_{it}(0)$, but she only observes $Y_{it}(D_{it})$ which is the fundamental problem of causal inference. Under a version of parallel trends holding within each unobserved group $g_i \in \{1, \ldots, N\}$, one strategy is to use

$$\sum_{i=1}^{n} \nu_i = \sum_{j=1}^{n} \xi_j = 0$$ where $\nu_i = \log(\alpha_i)$ and $\xi_j = \log(\gamma_j)$. The second equality is needed because of the introduction of the group effects. These normalization choices are arbitrary and one could alternatively assume $\alpha_1 = 1$ and $\gamma_1 = 1$ without affecting our results.

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The first equality is a standard choice in the literature Jochmans (2017); Dzemski (2019). It is equivalent to $\sum_{i=1}^{n} \nu_i = \sum_{j=1}^{n} \xi_j = 0$ where $\nu_i \equiv \log(\alpha_i)$ and $\xi_j \equiv \log(\gamma_j)$. The second equality is needed because of the introduction of the group effects. These normalization choices are arbitrary and one could alternatively assume $\alpha_1 = 1$ and $\gamma_1 = 1$ without affecting our results.
untreated periods to estimate the group memberships under possibly heterogeneous treatment effects by applying directly step 2 to differenced outcomes. Because $D_{it} = 0$ for this subsample, heterogeneous treatment effects do not perturbate the estimation of the latent group structure. With a sufficiently large number of pre-treatment outcomes, one consistently estimate the groups and can build counterfactual in post-treatment periods to identify a menu of average treatment effect. The theoretical analysis of this procedure generalizing de Chaisemartin and D’Haultfœuille (2020) and a large-scale application is work in progress.

C  Proofs of the results

C.1 Proof of Proposition 3.1

Let $W^0 \equiv (1\{g_i^0 = g_j^0\})_{i=1,\ldots,N; j=1,\ldots,N}$. Equations (3.2) and (3.3) are immediate corollaries of Lemma C.1 below.

Lemma C.1  Let Assumptions 1-3 hold. Then, as $N$ and $T$ tend to infinity,

$$\left\|\hat{W} - W^0\right\|_{\text{max}} = o_p(1).$$  \hfill (C.1)

Proof of Lemma C.1. Let $\epsilon, K_1 > 0$. By Assumption 1, there exists $K_2 > 0$ such that, letting $\mathcal{E}_{1NT} \equiv \left\{\left\|\hat{\beta}^1 - \beta^0\right\| > K_2 r_{NT}\right\}$, $\Pr(\mathcal{E}_{1NT}) < \epsilon$ for $N, T$ sufficiently large. Define $Z_{1NT}(i, j) \equiv \hat{W}_{ij}(1 - W^0_{ij})$, $Z_{2NT}(i, j) \equiv (1 - \hat{W}_{ij})W^0_{ij}$, and the probability events

$$\mathcal{E}_{2NT} \equiv \left\{\min_{g \in \{1, \ldots, G^0\}} \sum_{i=1}^N 1\{g_i^0 = g\} \geq 2\right\} \cap \{C \leq c_{NT} T^x \leq T^x C\} \cap \{c_{NT} r_{NT}^{-1} \geq K_1\}$$
$$\cap \{r_{NT} \leq \min(\eta/K_2, 1)\} \cap \{c_{NT} < c/72\},$$
where $\eta$ is defined in Equation (C.5) and $c, C$ in Assumptions 3(c) and 2(b), and $\mathcal{E}_{NT} \equiv \mathcal{E}_{1NT} \cap \mathcal{E}_{2NT}$. By the union bound, for $N, T$ sufficiently large,

$$\Pr \left( \max_{(i,j)\in\{1,\ldots,N\}^2} |\hat{W}_{ij} - W_{ij}^0| > 0 \right)$$

$$\leq \Pr(\mathcal{E}_{NT}^c) + \sum_{(i,j)\in\{1,\ldots,N\}^2} \Pr(\hat{W}_{ij} \neq W_{ij}^0, \mathcal{E}_{NT})$$

$$\leq \Pr(\mathcal{E}_{1NT}) + \Pr(\mathcal{E}_{2NT}) + \sum_{(i,j)\in\{1,\ldots,N\}^2} \Pr(\hat{W}_{ij} \neq W_{ij}^0, \mathcal{E}_{NT})$$

$$\leq 2\epsilon + \sum_{(i,j)\in\{1,\ldots,N\}^2} \Pr(Z_{1NT}(i,j) = 1, \mathcal{E}_{NT}) + \Pr(Z_{2NT}(i,j) = 1, \mathcal{E}_{NT}), \quad (C.2)$$

where the last inequality follows from $\lim_{\min(N,T)\to\infty} \Pr(\mathcal{E}_{2NT}^c) = 0$ by Assumptions 2(b) and 3(e). Below, I prove that, for $\ell \in \{1, 2\}$, and as $N$ and $T$ tend to infinity,

$$\max_{(i,j)\in\{1,\ldots,N\}^2} \Pr(Z_{\ell NT}(i,j) = 1, \mathcal{E}_{NT}) = o(N^2T^{-\delta}) \text{ for all } \delta > 0. \quad (C.3)$$

Equation (C.1) then follows by combining (C.2)-(C.3) with Assumption 2(a) and because $\epsilon$ is unrestricted.

1. First, I show (C.3) for $\ell = 1$.\footnote{Actually, I show the stronger result that the supremum is $o(T^{-\delta})$.} Let $(i, j) \in \{1, \ldots, N\}^2$ and $\delta > 0$.

$$Z_{1NT}(i,j) = 1\left\{ \max_{k \in \{1, \ldots, N\} \setminus \{i, j\}} \left| \frac{1}{T} \sum_{t=1}^{T} (\hat{v}_{it} - \hat{v}_{jt}) \hat{v}_{kt} \right| \leq c_{NT} \right\} 1\{g_i^0 \neq g_j^0\}. $$

If $G^0 = 1$, then almost surely $g_i^0 = g_j^0$ and $Z_{1NT}(i,j) = 0$, i.e., (C.3) holds. Else,

$$1\{Z_{1NT}(i,j) = 1, \mathcal{E}_{NT}\}$$

$$= 1\{\mathcal{E}_{NT}\} \times$$

$$\sum_{(g, \tilde{g})\in\{1,\ldots,G^0\}^2} 1\{g_i^0 = g\} 1\{g_j^0 = \tilde{g}\} 1\left\{ \max_{k \in \{1,\ldots,N\} \setminus \{i,j\}} \left| \frac{1}{T} \sum_{t=1}^{T} (\hat{v}_{it} - \hat{v}_{jt}) \hat{v}_{kt} \right| \leq c_{NT} \right\}.$$
\[ \mathcal{P}_2 (\{1, \ldots, N\} \setminus \{i, j\}) \text{ such that } g^0_{k^*(i,j,g^0_i)} = g^0_i \text{ and } g^0_{l^*(i,j,g^0_j)} = g^0_j. \] It follows that

\[ 1\{Z_{1NT}(i, j) = 1, \mathcal{E}_{NT}\} \leq 1\{\mathcal{E}_{NT}\} \times \sum_{(g, \tilde{g}) \in \{1, \ldots, G^0\}^2, \tilde{g} \neq \bar{g}} \left\{ \mathbf{1}\{g^0_i = g\} \mathbf{1}\{g^0_j = \tilde{g}\} \mathbf{1}\left\{ \frac{1}{T} \sum_{t=1}^{T} (\hat{v}_{it} - \hat{v}_{jt})(\hat{v}_{k^*(i,j,g^0_i)t} - \hat{v}_{l^*(i,j,g^0_j)t}) \leq c_{NT} \right\} \right\}. \]

where the first inequality uses the definition of the maximum and the second inequality follows from the triangle inequality. Since there is at most one pair \((g, \tilde{g}) \in \{1, \ldots, G^0\}^2\) such that \(g \neq \tilde{g}\) and \(1\{g^0_i = g\} 1\{g^0_j = \tilde{g}\} = 1\), developing
the product and using \(1\{|a| \leq b\} \leq 1\{a \leq b\}\) for any \((a, b) \in \mathbb{R} \times \mathbb{R}^*\) yields

\[
1\{Z_{NT}(i, j) = 1, \mathcal{E}_{NT}\} \leq 1\{\mathcal{E}_{NT}\} \times 
\]

\[
\max_{(g, \tilde{g}) \in \{1, \ldots, G^0\}} \frac{1}{T} \sum_{t=1}^{T} \left( \alpha^0_{gt} - \alpha^0_{g\tilde{t}} \right)^2 + \frac{1}{T} \sum_{t=1}^{T} \left( \alpha^0_{gt} - \alpha^0_{g\tilde{t}} \right) \left( v_{it} - v_{jt} + v_{k^*(i, j, g)t} - v_{l^*(i, j, \tilde{g})t} \right)
\]

\[
+ \frac{1}{T} \sum_{t=1}^{T} \left( \alpha^0_{gt} - \alpha^0_{g\tilde{t}} \right) \left( \beta^0 - \beta^1 \right)' \left( x_{it} - x_{jt} + x_{k^*(i, j, g)t} - x_{l^*(i, j, \tilde{g})t} \right)
\]

\[
+ \frac{1}{T} \sum_{t=1}^{T} \left( \beta^0 - \beta^1 \right)' \left( x_{it} - x_{jt} \right) \left( x_{k^*(i, j, g)t} - x_{l^*(i, j, \tilde{g})t} \right)
\]

\[
+ \frac{1}{T} \sum_{t=1}^{T} \left( v_{it} - v_{jt} \right) \left( v_{k^*(i, j, g)t} - v_{l^*(i, j, \tilde{g})t} \right)
\]

\[
+ \frac{1}{T} \sum_{t=1}^{T} \left( v_{k^*(i, j, g)t} - v_{l^*(i, j, \tilde{g})t} \right) \left( \beta^0 - \beta^1 \right)' \left( x_{it} - x_{jt} \right) \leq 2c_{NT}\}
\]

\[
= 1\{\mathcal{E}_{NT}\} \times \max_{(g, \tilde{g}) \in \{1, \ldots, G^0\}} \frac{1}{T} \left\{ A_T(i, j, g, \tilde{g}) \leq 2c_{NT} \right\}, \quad (C.4)
\]

where \(A_T(i, j, g, \tilde{g})\) is defined implicitly. Define

\[
B_T(i, j, g, \tilde{g}) \equiv \left| A_T(i, j, g, \tilde{g}) - \frac{1}{T} \sum_{t=1}^{T} \left( \alpha^0_{gt} - \alpha^0_{g\tilde{t}} \right)^2 \right.
\]

\[
- \frac{1}{T} \sum_{t=1}^{T} \left( \alpha^0_{gt} - \alpha^0_{g\tilde{t}} \right) \left( v_{it} - v_{jt} + v_{k^*(i, j, g)t} - v_{l^*(i, j, \tilde{g})t} \right)
\]

\[
- \frac{1}{T} \sum_{t=1}^{T} \left( v_{it} - v_{jt} \right) \left( v_{k^*(i, j, g)t} - v_{l^*(i, j, \tilde{g})t} \right) \right|.
\]

Let \(\pi \equiv \sup_{a \in \mathcal{A}} |a| < \infty\) by Assumption 3(a). It is easy to show using the Cauchy-
Schwarz inequality that

\[ B_T(i, j, g, \tilde{g}) \leq \|\hat{\beta}^1 - \beta^0\| \left\{ \frac{2\pi}{T} \sum_{t=1}^{T} \left( \|x_{it}\| + \|x_{jt}\| + \|x_{k^*(i, j, g) t}\| + \|x_{l^*(i, j, \tilde{g}) t}\| \right) \\
+ \frac{4\|\hat{\beta}^1 - \beta^0\|}{T} \sum_{t=1}^{T} \left( \|x_{it}\|^2 + \|x_{jt}\|^2 + \|x_{k^*(i, j, g) t}\|^2 + \|x_{l^*(i, j, \tilde{g}) t}\|^2 \right) \\
+ \left( \sqrt{\frac{1}{T} \sum_{t=1}^{T} v_{it}^2} + \sqrt{\frac{1}{T} \sum_{t=1}^{T} v_{jt}^2} \right) \sqrt{\frac{1}{T} \sum_{t=1}^{T} \|x_{k^*(i, j, g) t}\|^2 + \|x_{l^*(i, j, \tilde{g}) t}\|^2} \\
+ \left( \sqrt{\frac{1}{T} \sum_{t=1}^{T} v_{k^*(i, j, g) t}^2} + \sqrt{\frac{1}{T} \sum_{t=1}^{T} v_{l^*(i, j, \tilde{g}) t}^2} \right) \sqrt{\frac{1}{T} \sum_{t=1}^{T} \|x_{it}\|^2 + \|x_{jt}\|^2} \right\} . \]

By Assumption 3(b), there exists \( M^* > 0 \) such that \( \mathbb{E}[v_{it}^2] \leq M^* \) for all \( i, t \). Let \( \tilde{M} = \max(M, \max(M^*, 1)) \), where \( M \) is defined in Assumption 3(f), and \( \eta > 0 \) such that

\[ \eta \leq \min \left( 1, \frac{c}{24 \left( 2\pi 4\sqrt{\tilde{M}} + 8\tilde{M} + 4\sqrt{2}\tilde{M} \right)} \right), \quad \text{(C.5)} \]

where \( c \) is defined in Assumption 3(c). By definition of \( \mathcal{E}_{NT} \), \( \|\hat{\beta}^1 - \beta^0\| \leq \eta \) on \( \mathcal{E}_{NT} \). Using the Cauchy-Schwarz inequality again and \( \eta \leq 1 \) yields

\[ 1\{\mathcal{E}_{NT}\} B_T(i, j, g, \tilde{g}) \leq \eta \left\{ \frac{2\pi}{T} \sum_{t=1}^{T} \|x_{it}\|^2 + \|x_{jt}\|^2 + \|x_{k^*(i, j, g) t}\|^2 + \|x_{l^*(i, j, \tilde{g}) t}\|^2 \\
+ \frac{4\|\hat{\beta}^1 - \beta^0\|}{T} \sum_{t=1}^{T} \left( \|x_{it}\|^2 + \|x_{jt}\|^2 + \|x_{k^*(i, j, g) t}\|^2 + \|x_{l^*(i, j, \tilde{g}) t}\|^2 \right) \\
+ \left( \sqrt{\frac{1}{T} \sum_{t=1}^{T} v_{it}^2} + \sqrt{\frac{1}{T} \sum_{t=1}^{T} v_{jt}^2} \right) \sqrt{\frac{1}{T} \sum_{t=1}^{T} \|x_{k^*(i, j, g) t}\|^2 + \|x_{l^*(i, j, \tilde{g}) t}\|^2} \\
+ \left( \sqrt{\frac{1}{T} \sum_{t=1}^{T} v_{k^*(i, j, g) t}^2} + \sqrt{\frac{1}{T} \sum_{t=1}^{T} v_{l^*(i, j, \tilde{g}) t}^2} \right) \sqrt{\frac{1}{T} \sum_{t=1}^{T} \|x_{it}\|^2 + \|x_{jt}\|^2} \right\} \equiv C_T(i, j, g, \tilde{g}). \]
Plugging this upper bound into (C.4), I obtain
\[
1\{Z_{1NT}(i, j) = 1, \mathcal{E}_{NT}\}
\leq \max_{(g, \bar{g}) \in \{1, \ldots, G^0\}^2} \sum_{g \neq \bar{g}} \frac{1}{T} \sum_{t=1}^{T} (\alpha_{gt}^0 - \alpha_{\bar{g}t}^0)^2
\]
\[
+ \frac{1}{T} \sum_{t=1}^{T} (\alpha_{g0}^0 - \alpha_{\bar{g}0}^0) \left( v_{it} - v_{jt} + v_{k^*(i,j,g)t} - v_{l^*(i,j,\bar{g})t} \right) 
\]
\[
+ \frac{1}{T} \sum_{t=1}^{T} (v_{it} - v_{jt}) \left( v_{k^*(i,j,g)t} - v_{l^*(i,j,\bar{g})t} \right) \leq 2c_{NT} + C_T(i, j, g, \bar{g}).
\]

By the Cauchy-Schwarz inequality again, and because \(\tilde{M} \geq 1\), note the implication
\[
\frac{1}{T} \sum_{t=1}^{T} \left\| x_{it} \right\|^2 \leq \tilde{M} \iff \frac{1}{T} \sum_{t=1}^{T} \left\| x_{it} \right\| \leq \sqrt{\tilde{M}} \leq \tilde{M}.
\]

Using this result, the union bound, and some probability algebra, it follows that
\[
\Pr (Z_{1NT}(i, j) = 1, \mathcal{E}_{NT}) \leq \sum_{(g, \bar{g}) \in \{1, \ldots, G^0\}^2} \Pr \left( \frac{1}{T} \sum_{t=1}^{T} (\alpha_{gt}^0 - \alpha_{\bar{g}t}^0) v_{it} \leq -\frac{c}{12} + 2c_{NT} + \eta \left(2\pi4\sqrt{\tilde{M}} + 8\tilde{M} + 4\sqrt{2}\tilde{M}\right) \right)
\]
\[
+ 4G^0 (G^0 - 1) \left[ \sup_{g \neq \bar{g}} \Pr \left( \frac{1}{T} \sum_{t=1}^{T} (\alpha_{gt}^0 - \alpha_{\bar{g}t}^0)^2 \leq \frac{c}{2} \right) \right]
\]
\[
+ \sup_{g \neq \bar{g}} \Pr \left( \frac{1}{T} \sum_{t=1}^{T} (\alpha_{g0}^0 - \alpha_{\bar{g}0}^0) v_{it} \geq \frac{c}{12} \right)
\]
\[
+ \sup_{i \in \{1, \ldots, N\}, g \neq \bar{g}} \Pr \left( \frac{1}{T} \sum_{t=1}^{T} v_{it}^2 \geq \tilde{M} \right)
\]
\[
+ \sup_{(i,j,k) \in \mathcal{A}(\{1, \ldots, N\})} \Pr \left( \frac{1}{T} \sum_{t=1}^{T} (v_{it} - v_{jt}) v_{kt} \geq \frac{c}{12} \right) \right].
\]

First, I bound the terms with a supremum. By Assumption 3(c), it holds that
\[
\lim_{\min(N,T) \to \infty} \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}[(\alpha_{gt}^0 - \alpha_{\bar{g}t}^0)^2] = c_{g,\bar{g}} > c.
\]
So for \(N, T\) large enough, I have
\[
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \left[ (\alpha_{gt}^0 - \alpha_{\bar{g}t}^0)^2 \right] \geq \frac{2c}{3}.
\]

Applying Lemma B.5 in Bonhomme and Manresa (2015) to \(z_t = (\alpha_{gt}^0 - \alpha_{\bar{g}t}^0)^2 - \mathbb{E}[(\alpha_{gt}^0 - \alpha_{\bar{g}t}^0)^2]\), which satisfies appropriate mixing and tail conditions by Assumption 3(b) and (d), and taking \(z = c/6\) yields, as \(N, T\) tends to infinity,
\[
\Pr \left( \frac{1}{T} \sum_{t=1}^{T} (\alpha_{gt}^0 - \alpha_{\bar{g}t}^0)^2 \leq \frac{c}{2} \right) = o(T^{-\delta}),
\]
(C.7)
uniformly across \( g \) and \( \tilde{g} \). Similarly, applying Lemma B.5 to \( z_t = v_{it}^2 - \mathbb{E}[v_{it}]^2 \) and taking \( z = \tilde{M} - M^* \) yields

\[
\Pr \left( \frac{1}{T} \sum_{t=1}^{T} v_{it}^2 \geq \tilde{M} \right) = o(T^{-\delta}), \tag{C.8}
\]

uniformly across units \( i \). Note that \( \{v_{it}^2\}_t \) is strongly mixing as \( \{v_{it}\}_t \) is strongly mixing by Assumption 3(d). By Assumption 3(d), the process \( \{(\alpha_0^0 - \alpha_0^g) v_{it}\}_t \) has zero mean, and is strongly mixing with faster-than-polynomial decay rate. Moreover, for all \( i, t \) and \( m > 0 \),

\[
\Pr \left( \left| (\alpha_0^0 - \alpha_0^g) v_{it} \right| > m \right) \leq \Pr \left( |v_{it}| > \frac{m}{2a} \right),
\]

so \( \{(\alpha_0^0 - \alpha_0^g) v_{it}\}_t \) also satisfies the tail condition of Assumption 3(b), albeit with a different constant \( b' > 0 \) instead of \( b > 0 \). Lastly, applying Lemma B.5 from Bonhomme and Manresa (2015) again with \( z_t = (\alpha_0^0 - \alpha_0^g) v_{it} \) and taking \( z = c/12 \) yields

\[
\Pr \left( \left| \frac{1}{T} \sum_{t=1}^{T} (\alpha_0^0 - \alpha_0^g) v_{it} \right| \geq \frac{c}{12} \right) = o(T^{-\delta}) \tag{C.9}
\]

uniformly across \( i, g, \) and \( \tilde{g} \). An analogous reasoning yields

\[
\sup_{(i,j,k) \in \mathcal{P}_3(\{1,\ldots,N\})} \Pr \left( \left| \frac{1}{T} \sum_{t=1}^{T} (v_{it} - v_{jt}) v_{kt} \right| \geq \frac{c}{12} \right) = o(T^{-\delta}). \tag{C.10}
\]

Finally, because \( c_{NT} \leq c/72 \) on \( \mathcal{E}_{NT} \), a similar reasoning yields

\[
\Pr \left( \frac{1}{T} \sum_{t=1}^{T} (\alpha_0^g - \alpha_0^g) v_{it} \leq -\frac{c}{12} + 2c_{NT} + \eta \left( 2\pi 1 \mathbb{E}^2 \tilde{M} + 8\tilde{M} + 4\sqrt{2}\tilde{M} \right) \right)
\leq \Pr \left( \frac{1}{T} \sum_{t=1}^{T} (\alpha_0^g - \alpha_0^g) v_{it} \leq -\frac{c}{72} \right)
\leq o(T^{-\delta}), \tag{C.11}
\]

uniformly across \( g, \tilde{g} \), where I have used the value of \( \eta \) given in (C.5). Combining (C.6)-(C.11) and using Assumption 3(f) yields

\[
\sup_{(i,j) \in \{1,\ldots,N\}^2} \Pr (Z_{1NT}(i, j) = 1, \mathcal{E}_{NT}) = G^0 (1 - G^0) \times o_p(T^{-\delta}) = o_p(T^{-\delta}),
\]

i.e., (C.3) for \( \ell = 1 \) holds.
2. Second, I show (C.3) for \( \ell = 2 \). I now have

\[
\mathbb{1}\{Z_{2NT}(i, j) = 1, \mathcal{E}_{NT}\} \\
= \mathbb{1}\{\mathcal{E}_{NT}\} \mathbb{1}\left\{ \max_{k \in \{1, \ldots, N\} \setminus \{i, j\}} \left| \frac{1}{T} \sum_{t=1}^{T} (\tilde{v}_{it} - \tilde{v}_{jt}) \tilde{v}_{kt} \right| > c_{NT} \right\} \mathbb{1}\{g_i^0 = g_j^0\} \\
\leq \mathbb{1}\{\mathcal{E}_{NT}\} \mathbb{1}\left\{ \max_{k \in \{1, \ldots, N\} \setminus \{i, j\}} \left| \frac{1}{T} \sum_{t=1}^{T} (v_{it} - v_{jt}) v_{kt} + \frac{1}{T} \sum_{t=1}^{T} (v_{it} - v_{jt}) \alpha_0^0 g_{it} \right| \\
+ \frac{1}{T} \sum_{t=1}^{T} (\beta^0 - \beta^1)' (x_{it} - x_{jt}) (\beta^0 - \beta^1)' x_{kt} \\
+ \frac{1}{T} \sum_{t=1}^{T} (v_{it} - v_{jt}) (\beta^0 - \beta^1)' x_{kt} + \frac{1}{T} \sum_{t=1}^{T} \alpha_0^0 g_{kt} (\beta^0 - \beta^1)' (x_{it} - x_{jt}) \\
+ \frac{1}{T} \sum_{t=1}^{T} x_{kt} (\beta^0 - \beta^1)' (x_{it} - x_{jt}) \right| > c_{NT} \right\}.
\]

By the union bound, the triangle inequality, and the Cauchy-Schwarz inequality,

\[
\Pr (Z_{2NT}(i, j) = 1, \mathcal{E}_{NT}) \\
\leq (N - 2) \sup_{(i, j, k) \in \{1, \ldots, N\}^3} \left\{ \Pr \left( \left| \frac{1}{T} \sum_{t=1}^{T} (v_{it} - v_{jt}) v_{kt} \right| > \frac{c_{NT}}{10}, \mathcal{E}_{NT} \right) \\
+ \Pr \left( \frac{1}{T} \sum_{t=1}^{T} \left\| x_{it} \right\|^2 + \left\| x_{jt} \right\|^2 + \left\| x_{kt} \right\|^2 > \frac{c_{NT}}{10 \times 4K_2^2 \tau_{NT}^2}, \mathcal{E}_{NT} \right) \\
+ 4 \Pr \left( \left( \frac{1}{T} \sum_{t=1}^{T} v_{it}^2 \right)^{1/2} \left( \frac{1}{T} \sum_{t=1}^{T} \left\| x_{kt} \right\|^2 \right)^{1/2} > \frac{c_{NT}}{10K_2 r_{NT}}, \mathcal{E}_{NT} \right) \\
+ 2 \Pr \left( \frac{1}{T} \sum_{t=1}^{T} v_{it}^2 > \frac{c_{NT}}{10K_2 r_{NT} \times \pi} \right) \\
+ 2 \Pr \left( \frac{1}{T} \sum_{t=1}^{T} v_{it}^2 \left. \right| \frac{1}{T} \sum_{t=1}^{T} \left\| x_{kt} \right\|^2 > \frac{c_{NT}}{10K_2 r_{NT} \times \pi}, \mathcal{E}_{NT} \right) \right\}.
\]

Under the strong mixing and tail conditions given by Assumptions 3(b) and 3(d), and because \( K_1 \leq c_{NT}/r_{NT} \leq c_{NT}/r_{NT}^2 \) on \( \mathcal{E}_{NT} \), for \( K_1 \) sufficiently large, all noninitial probabilities in the above expression can be shown to be \( o(T^{-\delta}) \) for all \( \delta > 0 \), uniformly across \( (i, j, k) \), by similar arguments as in Step 1. For the first probability, a close inspection of the proof of Lemma B.5 in Bonhomme and Manresa (2015) reveals that,
by taking \( z_t = (v_{it} - v_{jt})v_{kt} \) and \( z = c_{NT}/6 \), and because \( c_{NT} \geq CT^{-\kappa} \) on \( E_{NT} \),

\[
\Pr \left( \left\| \frac{1}{T} \sum_{t=1}^{T} (v_{it} - v_{jt})v_{kt} \right\| \geq \frac{c_{NT}}{10} \right) \leq 4 \left( 1 + \frac{T^{1/2 - 2\kappa}}{C_1} \right)^{-(1/2)T^{1/2}} + C_2 T^{\kappa} \exp \left( -C_3 \left( T^{(1/2 - \kappa)/C_4} \right) \right), \tag{C.12}
\]

where \( C_1, C_2, C_3, \) and \( C_4 \) are positive constants that do not depend on \( i, j, k \). Since \( \kappa < 1/2 \), the upper bound is \( o_p(T^{-\delta}) \) for all \( \delta > 0 \). This shows \( (C.3) \) for \( \ell = 2 \). \( \square \)

### C.2 Proof of Corollary 3.2

Let \( \tilde{\beta} \) and \((\tilde{\alpha}_{11}, \ldots, \tilde{\alpha}_{G^0T})'\) denote the infeasible oracle estimators computed from a pooled OLS regression of \( y_{it} \) on \( x_{it} \) and the interactions of group and time indicators \( 1\{g_i^0 = 1\}, \ldots, 1\{g_i^0 = G^0\}, 1\{t = 1\}, \ldots, 1\{t = T\} \). By the same reasoning as in section S.A.1. in Bonhomme and Manresa (2015)’s Supplemental Material, it holds

\[
\sqrt{NT}(\tilde{\beta} - \beta^0) \overset{d}{\to} N \left( 0, \Sigma^{-1}_\beta \Omega^{-1} \Sigma^{-1}_\beta \right), \tag{C.13}
\]

and, for all \((g, t) \in \{1, \ldots, G^0\} \times \{1, \ldots, T\} \),

\[
\sqrt{N}(\tilde{\alpha}_{gt} - \alpha_{gt}^0) \overset{d}{\to} N \left( 0, \frac{\omega_{gt}}{\pi^2_g} \right). \tag{C.14}
\]

Without loss of generality, I assume that the labelling of predicted groups matches the true group labeling. By Proposition 3.1, for all \((g, t) \in \{1, \ldots, G^0\} \times \{1, \ldots, T\} \),

\[
\Pr \left( \{\hat{\alpha}_{gt} \neq \tilde{\alpha}_{gt}\} \cup \{\hat{\beta} \neq \tilde{\beta}\} \right) \leq \Pr (\hat{G} \neq G^0) + \Pr \left( \max_{i \in \{1, \ldots, N\}} |\hat{g}_i - g_i^0| > 0 \right) = o(1) + o(1) = o(1).
\]

Then, Eq. (3.5) follows from

\[
\left| \Pr \left( \sqrt{N}(\hat{\alpha}_{gt} - \alpha_{gt}^0) \leq a \right) - \Pr \left( \sqrt{N}(\hat{\alpha}_{gt} - \alpha_{gt}^0) \leq a \right) \right| \leq \left| \Pr \left( \sqrt{N}(\hat{\alpha}_{gt} - \alpha_{gt}^0) \leq a, \sqrt{N}(\tilde{\alpha}_{gt} - \alpha_{gt}^0) > a \right) \right| + \left| \Pr \left( \sqrt{N}(\tilde{\alpha}_{gt} - \alpha_{gt}^0) > a, \sqrt{N}(\tilde{\alpha}_{gt} - \alpha_{gt}^0) \leq a \right) \right| \leq \Pr (\hat{\alpha}_{gt} \neq \tilde{\alpha}_{gt}) + \Pr (\hat{\alpha}_{gt} \neq \tilde{\alpha}_{gt}) = o(1).
\]

for any \( a > 0 \). Eq. (3.4) follows from a similar argument.
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