Solving Linear Algebra by Program Synthesis

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Abstract
We solve MIT’s Linear Algebra 18.06 course and Columbia University’s Computational Linear Algebra COMS3251 courses with perfect accuracy by interactive program synthesis. This surprisingly strong result is achieved by turning the course questions into programming tasks and then running the programs to produce the correct answers. We use OpenAI Codex with zero-shot learning, without providing any examples in the prompts, to synthesize code from questions. We quantify the difference between the original question text and the transformed question text that yields a correct answer. Since all COMS3251 questions are not available online the model is not overfitting. We go beyond just generating code for questions with numerical answers by interactively generating code that also results visually pleasing plots as output. Finally, we automatically generate new questions given a few sample questions which may be used as new course content. This work is a significant step forward in solving quantitative math problems and opens the door for solving many university level STEM courses by machine.

1 Introduction
Language models have vastly improved in recent years, with the advent of large-scale Transformer models such as GPT-3 (Brown et al., 2020) that perform well on question answering tasks. However, when it comes to answering quantitative problems such as word problems in mathematics or deduction from formal logic, these models show poor performance achieving accuracies close to random baselines (Hendrycks et al., 2020), failing on even the most simple questions such as computing the length of a vector.

Part of the challenge in finding a solution to quantitative problems is to have access to a working tree-like recursive memory. Quantitative problems often require building arithmetic expression trees that help in mathematical deduction. These kinds of trees are also common in program representation and program synthesis. With this insight, we study the efficacy of solving math problems, specifically problems from introductory level undergraduate Linear Algebra courses, by turning each problem into the task of writing a function or program to solve that question. This is done using OpenAI’s Codex (Chen et al., 2021), a foundation model trained on both text and code.

We demonstrate the surprisingly simple yet strong result that foundation models for program synthesis such as OpenAI Codex succeed in synthesizing correct code for solving such quantitative math problems. Surprisingly, we find that Codex not only synthesizes correct code for problems that expect numerical answers, but also generates code for questions that ask to plot solutions. We achieve perfect accuracy in solving undergraduate level Linear Algebra course problems, and validate that our results are not merely overfitting the training data by solving a new course which is not available online, and is therefore unseen during Codex training.

As an example, consider a moderately involved question from MIT’s Linear Algebra course 18.06, Question 1 in Chapter 7.3 of Gilbert Strang’s textbook (2016), as shown in Figure 1. To the best of our knowledge none of the state-of-the-art quantitative reasoning models correctly answers such questions. As shown in Figure 1, given the question as text, we run the question through Codex as is without any modification to generate a program and execute the synthesized program to generate the correct solution.

2 Related Work
There have been several recent works that attempt to improve quantitative reasoning in math problems. MathBERT (Peng et al., 2021), for instance, is a Transformer based pre-trained language model that uses symbol and operator trees as intermediary
Problem

Suppose AB holds these 2 measurements of 5 samples: \( AB = \{5,4,3,2,1\} \) and find its eigenvalues. What line through the origin is closest to the 5 samples in the columns of \( AB \)?

Another line of work has focused on solving math questions from a large database of questions collected from Chinese elementary school math classes. Techniques have included sequence-to-sequence and graph-to-tree Transformers which achieve around 80% on Math23k and MAWPS datasets (Koncel-Kedziorski et al., 2016; Li et al., 2019; Wang et al., 2019; Zhang et al., 2020; Li et al., 2020; Qin et al., 2020; Lan et al., 2021). Other work (Tsai et al., 2021) includes knowledge graphs of geometry formulas into sequence-to-tree transformers to improve performance on the geometry section of Math23k dataset, and MWP-BERT, which adds masked fine-tuning to the BERT model using a large corpus of over 100,000 math word problems achieves an impressive 96.2% accuracy on the Math23k dataset (Liang et al., 2021) of elementary school math problems.

For solving university level machine learning problems specifically, a recent approach (Tran et al., 2021) uses graph neural networks and Transformers to predict an expression tree from the input question to calculate the answer. This achieves over 95% accuracy on numerical machine learning exercises, which is above human performance; however only works on the specific course it is trained on.

Rather than building a custom-designed solution, our work explores the use of a foundation model such as Codex which is trained on both text and code. Any program may be represented as an abstract syntax tree and many questions may also be represented as expression trees. Bringing the question and answer into a common representation makes it easier to find a correct solution. The advantage of studying this pre-trained model is that it may be applied at scale to many different topics or subjects without additional training. Our work is the first to demonstrate perfect performance of interactively solving linear algebra problems at a university-level difficulty.

3 Methods

Here we describe our dataset, solution generation pipeline, and evaluation methodology. The key components leading to our success are:

- Program synthesis: Insight to use a program synthesis to generate a program, that has a built-in tree representation, that produces the solution to the given problem.
- Interactive workflow: We interactively work with Codex to produce both the correct result and visually pleasing plots as shown in Figure 2. We place the question in context by augmenting the question with definitions and information required for the solving the question, rephrase and simplify. See the Appendix for all the original and transformed questions.

3.1 Datasets

We use (i) problem exercises from Gilbert Strang’s Introduction to Linear Algebra textbook (2016), which is used for MIT’s Linear Algebra 18.06 course, and (ii) exercises given as homework problems in Columbia’s Computational Linear Algebra COMS3251 course, as two challenging real-world university-level datasets. Both courses have multivariable calculus as their prerequisites and are usually taken by second-year EE/CS undergraduate students. To keep things tractable, we select 3 to 4 random problems from each chapter of the textbook (for MIT 18.06) and from each topic (for COMS3251), resulting in two datasets of 30
3.2 Interactive Workflow

Our interactive workflow is illustrated in Figure 2. We begin with the original question from Strang’s book (Question 2b, Chapter 4.2), which we feed into Codex that generates a Python program that is then executed. In this example, the result is missing the projection solution. We therefore transform the question to explicitly ask for the projection and have Codex re-generate a program to get the correct answer. The answer also consists of a plot, however the zero projection vector is not visible in the plot since it is a point. We therefore add an additional task which is to plot the projection vector with a marker so that it becomes visible. Codex re-generates the code which is executed to yield both a correct answer and a visually pleasing plot. In all of our experiments we set Codex parameters to be the same fixed default values (using davinci-codex with temperature 0 and response length 200).

4 Results

4.1 Performance Evaluation

Our dataset includes 30 questions from MIT’s 18.06 and 30 questions from Columbia University’s COMS3251 and gets perfect accuracy on these courses (see Appendix for detailed input and the solution output for each question in the datasets). In contrast, GPT3 yields 0% accuracy. We would like to quantify the extent of human effort required for achieving these perfect results. We therefore measure the similarity between the original question text and the final programming prompt that results in a correct answer. As shown in Figure 3 we observe highly similar texts. Specifically 90% median similarity for Columbia University’s COMS3251 and 80% median similarity for MIT’s 18.06, computed using the cosine similarity between their language embeddings. This demonstrates that only minor changes are required for turning a question into a program task that results in a correct answer. As a baseline we also include the similarity among the different original questions in each course, to

Figure 2: Interactive workflow: (A) We begin with the original question. Codex generates a program which is executed. The result is missing the projection. (B) We transform the question and Codex generates a program again to get the correct answer, though the zero projection vector does not appear on the plot. (C) An additional task to plot the projection vector with a marker so that it is visible results in Codex generating modified code which is executed to yield a correct answer and visually pleasing result.
verify the validity of our metric.

### 4.2 Generating New Questions

We are able to generate new questions with ease. We prompt Codex by a set of \( n \) numbered questions on different topics, and synthesize question number \( n + 1 \). Table 1 shows eight new generated questions for each course.

### 4.3 Limitations

We currently do not handle input drawings or any visual elements as input. Extending our approach to handle such inputs by using a multi-modal text and vision Transformer would help solve many diverse types of mathematical problems. However, our methodology works well for numerical outputs and figures, our pipeline doesn’t yet handle solutions that require multi-line derivations or proofs. We currently often modify the original question manually to form a question for which Codex returns a program which solves the question correctly making our method interactive. We plan on training a Transformer, such as T5 (Raffel et al., 2020), for paraphrasing and performing this step automatically.

### 5 Conclusion

Our work is the first to solve linear algebra problems at a university-level difficulty. Our results open the door for solving other STEM courses, which has the potential to disrupt higher education by: (i) automatically learning all university level STEM courses, (ii) automatically grading course, and (iii) rapidly generating new course content.

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**Table 1: New questions generated from MIT Linear Algebra 18.06 questions and Computational Linear Algebra (COMS3251) questions, and the closest question among the existing questions.**

| ID | Course       | Auto-Generated question                                                                 | Closest question in the dataset                                                                 |
|----|--------------|----------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------|
| 1  | MIT 18.06    | Find the eigenvalues and eigenvectors of the matrix \( A = \begin{bmatrix} 1 & 1; 1 & 2; 3, 1,3,6 \end{bmatrix} \). | Find the eigenvalues of A and B (easy for triangular matrices) and \( A + B: A = \begin{bmatrix} 3; 0; 1; 1, B = \begin{bmatrix} 1; 1,0; 3 \end{bmatrix}, A+B = \begin{bmatrix} 4,1,1,4 \end{bmatrix} \). (Ch 6.1 Q5) |
| 2  | MIT 18.06    | Find a matrix A such that \( A^T \) is not invertible but \( A \) is invertible.        | Find a matrix that has \( A^T \) does not equal 0 but \( A^T = 0 \). (Ch 2.4 Q23b)               |
| 3  | MIT 18.06    | Find a basis for the nullspace of \( A = \begin{bmatrix} 1; 1,1; 1,1,1,1 \end{bmatrix} \). | Construct a 2 by 2 matrix whose nullspace equals its column space. This is possible. (Ch 3.2 Q20) |
| 4  | MIT 18.06    | Find a basis for the nullspace of \( A = \) unit vectors, all mutually perpendicular.  | Find A'A if the columns of A are unit vectors, all mutually perpendicular. (Ch 4.1 Q25)        |
| 5  | MIT 18.06    | What 2 by 2 matrix \( R \) rotates every vector through 90 degrees?                    | What 2 by 2 matrix \( R \) rotates every vector through 45 degrees? Example: the vector \( \begin{bmatrix} 1,0 \end{bmatrix} \) goes to \( \begin{bmatrix} sqrt(2)/2, sqrt(2)/2 \end{bmatrix} \). (Ch 2.1, Q21) |
| 6  | MIT 18.06    | The complete solution to \( A x = \begin{bmatrix} 1; 3 \end{bmatrix} \) is \( x = \begin{bmatrix} 1; 0 \end{bmatrix} + c[0;1] \). Find the nullspace of \( A \). | Construct a 2 by 2 matrix whose nullspace equals its column space. This is possible. (Ch 3.2 Q20) |
| 7  | MIT 18.06    | Find a matrix \( A \) that has a negative eigenvalue and is symmetric.                | Find a symmetric matrix \( \begin{bmatrix} 1,b,b \end{bmatrix} \) that has a negative eigenvalue. (Ch 6.4, Q9a) |
| 8  | MIT 18.06    | Find the best plane \( C +Dt +E t^2 \) to fit \( b = \begin{bmatrix} 1,2,3,4,5 \end{bmatrix} \) at times \( t = 0,1,2,3,4 \). | Find the best line \( C +Dt \) to fit \( b = 3, -1,0,0 \) at times \( t = -2,-1,0,1,2 \). (Ch 4.3, Q22) |
| 9  | COMS3251     | Find the eigenvalues of the matrix \( A = \begin{bmatrix} 1; 1,2,2,-2,4 \end{bmatrix} \). | Find the eigenvalues of \( \begin{bmatrix} 0,2,0,3,0,2,0,3 \end{bmatrix} \). |
| 10 | COMS3251     | Compute the determinant of the following matrix: \( \begin{bmatrix} -1,2, -2 \end{bmatrix} \). | Compute the determinant of the following matrix: \( \begin{bmatrix} 3,4,5,0,1,5,5,4,3 \end{bmatrix} \). |
| 11 | COMS3251     | Find the determinant of the following matrix: \( \begin{bmatrix} -1,2,0,-1,-1 \end{bmatrix} \). | Compute the determinant of the following matrix: \( \begin{bmatrix} 3,4,5,0,1,5,5,4,3 \end{bmatrix} \). |
| 12 | COMS3251     | Compute an LU decomposition of the matrix \( A = \begin{bmatrix} 1,2,0,-2 \end{bmatrix} \). | Find an LU decomposition of the following matrix: \( \begin{bmatrix} -1,2,0,3;3,-2,1 \end{bmatrix} \). |
| 13 | COMS3251     | Which of the following matrices is a left inverse to \( A = \begin{bmatrix} 1,2,3,4 \end{bmatrix} \)? (a) \( \begin{bmatrix} -1,0,2,-1,3,2,-3 \end{bmatrix} \), (b) \( \begin{bmatrix} -1,0,0.5 \end{bmatrix} \), (c) \( \begin{bmatrix} -1,2,-3,3,2,-3,3 \end{bmatrix} \), (d) None of the above. | Compute the inverse of the following matrix: \( \begin{bmatrix} 1,-2,-2 \end{bmatrix} \). |
| 14 | COMS3251     | Find a combination of the vectors \( \begin{bmatrix} 1; 2,3 \end{bmatrix}, \begin{bmatrix} 4,5;6 \end{bmatrix} \), and \( \begin{bmatrix} 7; 8,9 \end{bmatrix} \) that gives the vector \( \begin{bmatrix} 4; 3,2 \end{bmatrix} \). | Find a combination of the vectors \( \begin{bmatrix} 1; 2,3 \end{bmatrix}, \begin{bmatrix} 4,5;6 \end{bmatrix} \), and \( \begin{bmatrix} 7; 8,9 \end{bmatrix} \) that give the zero vector. |
| 15 | COMS3251     | What is the dimension of the subspace spanned by the following vectors? \( \begin{bmatrix} 1,2,3 \end{bmatrix}, \begin{bmatrix} 0,1,0 \end{bmatrix}, \begin{bmatrix} -1,2, -1,3,1 \end{bmatrix} \). | What is the dimension of the subspace spanned by the following vectors? \( \begin{bmatrix} 2,1,2,1 \end{bmatrix}, \begin{bmatrix} 0,1,0 \end{bmatrix}, \begin{bmatrix} -1,2, -1,3,1 \end{bmatrix} \). |
| 16 | COMS3251     | Find the projection matrix onto the column space of \( A = \begin{bmatrix} 1,2,3,4,5,6 \end{bmatrix} \). | Find the projection matrix onto the column space of \( A \begin{bmatrix} 3,6,6,4,8,8 \end{bmatrix} \). |
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A Computational Linear Algebra: Columbia University COMS3251

| Topic | Matrix Algebra |
|-------|----------------|
| Original question | Compute the following expression: \( ([−1, 0, 2; 0, 1, 4]) \ast ([−2, 1]):([3, 1; 0, 0; −2, −1] \ast ([5, −3] + [−4, 2]^\prime)), \) where \(^\prime\) means transpose, \(^\ast\) means matrix product, . means inner product. |
| Codex input | Compute the following expression: \((\text{transpose}([−1, 0, 2; 0, 1, 4])) \ast ([−2, 1]):([3, 1; 0, 0; −2, −1] \ast \text{transpose}([5, −3] + [−4, 2]))\) |
| Codex output | ```
import numpy as np

a = np.array([[−1, 0, 2], [0, 1, 4]])
b = np.array([−2], [1])
c = np.array([[3, 1], [0, 0], [−2, −1]])
d = np.array([5], [-3])
e = np.array([−4], [2])

print(np.dot(np.transpose(a), b))
print(np.dot(c, (d+e)))
``` |
| Solution | 4 |

COMS3251, Question 1: Original question, Codex input and output, and solution.

| Topic | Vectors, Lengths and Dot products |
|-------|----------------------------------|
| Original question | Compute the squared \(L_2\) norm of the vector \([1; −4; 2; 8; −1]\). |
| Codex input | Original question |
| Codex output | ```
import numpy as np
def compute_squared_L2_norm(vector):
    return np.sum(np.square(vector))

vector = np.array([1, −4, 2, 8, −1])
print(compute_squared_L2_norm(vector))
``` |
| Solution | 86 |

COMS3251, Question 2: Original question, Codex input and output, and solution.
### Solving Linear System of Equations

**Original question**
Find a solution to the following system of four equations:

\[
\begin{align*}
4x - 2y + 8z + w &= 3, \\
-8x + 10y + 3w &= -2, \\
3x - y + 10z + 5w + 1 &= 0, \\
2x + 2y + 9z - 2w &= 8.
\end{align*}
\]

**Codex input**
Write a program to find a solution to the system of four equations:

\[
\begin{align*}
4x - 2y + 8z + w &= 3, \\
-8x + 10y + 3w &= -2, \\
3x - y + 10z + 5w + 1 &= 0, \\
2x + 2y + 9z - 2w &= 8.
\end{align*}
\]

**Codex output**

```python
from sympy import *

x, y, z, w = symbols('x y z w')

eq1 = 4*x - 2*y + 8*z + w - 3
eq2 = -8*x + 10*y + 3*w + 2
eq3 = 3*x - y + 10*z + 5*w + 1
eq4 = 2*x + 2*y + 9*z - 2*w - 8

solve([eq1, eq2, eq3, eq4], [x, y, z, w])
```

**Solution**

\[
x = \frac{11}{4}, y = \frac{97}{44}, z = \frac{-4}{11}, w = \frac{-15}{22}
\]

### COMS3251, Question 3: Original question, Codex input and output, and solution.

**Codex input**
Let \( a = [10; 120] \) and \( b = [6; 140] \). Write a program to compute positive \( x \) and \( y \) such that \( x\cdot a + y\cdot b = [72; 1340] \).

**Codex output**

```python
a = np.array([10, 120])
b = np.array([6, 140])
c = np.array([72, 1340])

def clock_angle(hour, minute):
    hour_angle = (hour * 30) + (minute * 0.5)
    minute_angle = (minute * 6)
    angle = abs(hour_angle - minute_angle)
    return angle

print(clock_angle(1, 15))
```

**Solution**

\( x=3, y=7 \)

### COMS3251, Question 4: Original question, Codex input and output, and solution.

**Codex input**
At noon, the minute and the hour hands of an analog clock coincide. Write a program to compute the angle (in degrees) subtended by the hands of an analog clock at 1:15.

**Codex output**

```python
def clock_angle(hour, minute):
    hour_angle = (hour * 30) + (minute * 0.5)
    minute_angle = (minute * 6)
    angle = abs(hour_angle - minute_angle)
    return angle

print(clock_angle(1, 15))
```

**Solution**

52.5

### COMS3251, Question 5: Original question, Codex input and output, and solution.
| Topic | Matrix Algebra |
|-------|----------------|
| Original question | For what matrix $R$, the following matrix equation is satisfied? $R \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 1 & 3 & 3 & 1 \end{bmatrix}$ |
| Codex input | $A = \text{np.matrix}([[1,0,0,0,0],[1,1,0,0,0],[1,2,1,0,0],[1,3,3,1,0],[1,4,6,4,1]])$ $B = \text{np.matrix}([[1,0,0,0,0],[0,1,0,0,0],[0,1,1,0,0],[0,1,2,1,0],[0,1,3,3,1]])$ Solve $X \cdot A = B$ for $X$ |
| Codex output | # Solve $X \cdot A = B$ for $X$
$X = \text{np.linalg.solve}(A,B)$
print($X$) |
| Solution | $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$ |

COMS3251, Question 6: Original question, Codex input and output, and solution.

| Topic | Rank, Basis and Dimension |
|-------|---------------------------|
| Original question | Given a d-dimensional non-zero vector $v$, compute the rank of the matrix $v \cdot v^T$ |
| Codex input | Given a d-dimensional non-zero vector $v$, write a program to compute the rank of the matrix $v \cdot v^T \cdot \text{transpose}(v)$ |
| Codex output | import numpy as np
def rank(v):
    return np.linalg.matrix_rank(np.dot(v, v.T)) |
| Solution | 1 |

COMS3251, Question 7: Original question, Codex input and output, and solution.

| Topic | Four Fundamental Subspaces of a Matrix |
|-------|----------------------------------------|
| Original question | Compute the dimension of the null space of the following matrix. $\begin{bmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{bmatrix}$ |
| Codex input | Write a program that finds the null space of the matrix $A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{bmatrix}$. Use scipy.linalg.null_space. Get the shape[1] of the null space. |
| Codex output | import numpy as np
from scipy import linalg
A = np.array([[1,2,0,-1],[-2,-3,4,5],[2,4,0,-2]])
print(linalg.null_space(A))
print(linalg.null_space(A).shape[1]) |
| Solution | 2 |

COMS3251, Question 8: Original question, Codex input and output, and solution.

| Topic | Four Fundamental Subspaces of a Matrix |
|-------|----------------------------------------|
| Original question | Compute the dimension of the left null space of the following matrix. $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 2 & 4 & 0 & -2 \end{bmatrix}$ |
| Codex input | Compute the dimension of the left null space of the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 2 & 4 & 0 & -2 \end{bmatrix}$. Use scipy.linalg.null_space. Get the shape[1] of the null space. |
| Codex output | import numpy as np
from scipy import linalg
A = np.array([[1,2,3,4,5,2,4,0,-2]])
print(linalg.null_space(A))
print(linalg.null_space(A).shape[1]) |
| Solution | 3 |

COMS3251, Question 9: Original question, Codex input and output, and solution.
**Topic**: Orthogonality and Projections

**Original question**: Compute the orthogonal projection of the vector \([-6;4]\) onto the line spanned by the vector \([3;2]\).

**Codex input**: Compute the orthogonal projection of the vector \([-6;4]\) onto the line spanned by the vector \([3;2]\).

**Codex output**

```python
import numpy as np

# Define the two vectors
v1 = np.array([3, 2])
v2 = np.array([-6, 4])

# Compute the orthogonal projection of v2 onto v1 and print it out
proj = (np.dot(v1, v2) / np.dot(v1, v1)) * v1
print(proj)
```

**Solution**: \([-30/13;-20/13]\)

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**Topic**: Least Squares Approximation

**Original question**: Find the least squares solution to the following matrix equation. 
\[
\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} x = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}
\]

**Codex input**: Write a program that finds the least squares solution to the matrix equation \[
\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} x = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}
\]

**Codex output**

```python
import numpy as np

A = np.array([[0,1],[1,1],[2,1]])
b = np.array([6,0,0])
x = np.linalg.solve(A,b)
print(x)
```

**Solution**: \([-3;5]\)

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**Topic**: Eigenvectors and Eigenvalues

**Original question**: Compute the eigenvalue associated with with the eigenvector \([-684/721;228/721]\) of the matrix \([-6,3;4,5]\).

**Codex input**: Write a program that finds the eigenvalue associated with with the eigenvector \([-684/721;228/721]\) of the matrix \([-6,3;4,5]\).

**Codex output**

```python
import numpy as np

A = np.array([-6,3;4,5])
eigenvalues, eigenvectors = np.linalg.eig(A)
print(eigenvalues)
p
print(eigenvectors)
print(eigenvalues[0]*eigenvectors[:,0])
p
print(eigenvalues[1]*eigenvectors[:,1])
```

**Solution**: \(-7\)

---

**Topic**: Matrix Algebra

**Original question**: Compute the inverse of the following matrix: \([-1,-2;-2,0]\).

**Codex input**: Compute the inverse of the following matrix: \([-1,-2;-2,0]\).

**Codex output**

```python
import numpy as np

A = np.array([-1,-2; -2,0])
print(np.linalg.inv(A))
```

**Solution**: \([0,-1/2;-1/2,1/4]\)

---

COMS3251, Question 10: Original question, Codex input and output, and solution.

COMS3251, Question 11: Original question, Codex input and output, and solution.

COMS3251, Question 12: Original question, Codex input and output, and solution.

COMS3251, Question 13: Original question, Codex input and output, and solution.
### Determinants

**Original question:** Compute the determinant of the following matrix: \[
\begin{bmatrix}
3 & -4 & 5 \\
0 & -1 & -5 \\
5 & -4 & 3
\end{bmatrix}
\]

**Codex input:**

```python
import numpy as np
matrix = np.array([[3, -4, 5], [0, -1, -5], [5, -4, 3]])
print(np.linalg.det(matrix))
```

**Solution:**

56

---

### Special Matrices

**Original question:** Find the real numbers \(a\), \(b\) and \(c\) so that the following matrix is symmetric:

\[
\begin{bmatrix}
0 & a+b & c+2 \\
a & 2 & c \\
4 & a+b & 4
\end{bmatrix}
\]

**Codex input:**

Use sympy to find \(a\), \(b\), \(c\) so that 

\[
\begin{bmatrix}
0 & a+b & c+2 \\
a & 2 & c \\
4 & a+b & 4
\end{bmatrix} = \text{transpose}((0,a+b+2,a;2,4,a+b,4))
\]

**Codex output:**

```python
import sympy
a, b, c = sympy.symbols('a b c')
A = sympy.Matrix([[0, a + b, c + 2], [a, 2, c], [4, a + b, 4]])
B = A.transpose()
print(sympy.solve([A[i] - B[i] for i in range(3)], (a, b, c)))
```

**Solution:**

\(a=2\), \(b=0\), \(c=2\)

---

### Matrix Decomposition

**Original question:** Find an LU decomposition of the following matrix:

\[
\begin{bmatrix}
-1 & -1 & 2 \\
2 & 0 & 3 \\
-3 & 2 & -1
\end{bmatrix}
\]

**Codex input:**

```python
import numpy as np
A = np.array([[-1, -1, 2], [2, 0, 3], [-3, 2, -1]])
L = np.eye(3)
U = A
for i in range(3):
    for j in range(i+1, 3):
        L[j,i] = U[j,i]/U[i,i]
        U[j,:] = U[j,:] - L[j,i]*U[i,:]
print(L)
print(U)
```

**Solution:**

\[
\begin{bmatrix}
1 & 0 & 0 \\
-2 & 1 & 0 \\
3 & -5/2 & 1
\end{bmatrix} \times
\begin{bmatrix}
-1 & -1 & 2 \\
0 & -2 & -1 \\
0 & 0 & 2/3
\end{bmatrix}
\]

---

**Original question:** Find an QR decomposition of the following matrix:

\[
\begin{bmatrix}
1 & 0 & 2 \\
0 & 2 & 0 \\
0 & -1 & 1
\end{bmatrix}
\]

**Codex input:**

```python
import numpy as np
A = np.array([[1, 0, 2], [0, 2, 0], [0, -1, 1]])
Q, R = np.linalg.qr(A)
print(Q)
print(R)
```

**Solution:**

\[
\begin{bmatrix}
1 & 0 & 0 \\
2/\sqrt{5} & 1/\sqrt{5} & 0 \\
-1/\sqrt{5} & 2/\sqrt{5} & -1/\sqrt{5}
\end{bmatrix} \times
\begin{bmatrix}
1 & 0 & 2 \\
0 & \sqrt{5} & -1/\sqrt{5} \\
0 & 0 & 2/\sqrt{5}
\end{bmatrix}
\]

---

**Original question:** Find an QR decomposition of the following matrix:

\[
\begin{bmatrix}
1.0 & 0.0 & 0.2/sqrt(5) \\
1.0 & sqrt(5) & 0.0 & -1/sqrt(5) \\
2.0 & 2/sqrt(5) & 0.0 & 0.0 & 2/sqrt(5)
\end{bmatrix}
\]

**Codex input:**

```python
import numpy as np
A = np.array([[1.0, 0.0, 0.2/sqrt(5), 1.0], [1.0, sqrt(5), 0.0, -1/sqrt(5)], [2.0, 2/sqrt(5), 0.0, 0.0, 2/sqrt(5)]])
Q, R = np.linalg.qr(A)
print(Q)
print(R)
```

**Solution:**

\[
\begin{bmatrix}
1.0 & 0.2/\sqrt{5} & 0.1/\sqrt{5} & -0.0/\sqrt{5} \\
1.0 & 0.2/\sqrt{5} & 0.1/\sqrt{5} & -0.0/\sqrt{5}
\end{bmatrix} \times
\begin{bmatrix}
1.0 & 2.0 & 0.0 & 2.0 \\
0.0 & 0.0 & 2.0 & 0.0
\end{bmatrix}
\]
### Diagonalization and Eigenvectors/Eigenvalues

**Original question**
Diagonalize the following matrix: 
\[
\begin{bmatrix}
2 & -3 & 0 \\
0 & -1 & 0 \\
1 & 3 & 1
\end{bmatrix}
\]

**Codex input**
Write a program that diagonalizes the matrix 
\[
\begin{bmatrix}
2 & -3 & 0 \\
0 & -1 & 0 \\
1 & 3 & 1
\end{bmatrix}
\]

**Codex output**
```python
import numpy as np
A = np.array([[2, -3, 0], [0, -1, 0], [1, 3, 1]])
print(A)
eig_val, eig_vec = np.linalg.eig(A)
print(eig_val)
print(eig_vec)
D = np.diag(eig_val)
print(D)
V = eig_vec
print(V)
V_inv = np.linalg.inv(V)
print(V_inv)
print(np.dot(V, np.dot(D, V_inv)))
```

**Solution**
\[
\begin{bmatrix}
-1 & 0 & 1 \\
-1 & 0 & 0 \\
2 & 1 & 1
\end{bmatrix}\times
\begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 2
\end{bmatrix}\times
\begin{bmatrix}
0 & -1 & 0 \\
-1 & 3 & 1 \\
1 & -1 & 0
\end{bmatrix}
\]

**COMS3251, Question 18:** Original question, Codex input and output, and solution.

### Matrix Algebra

**Original question**
Compute the trace of the following matrix: 
\[
\begin{bmatrix}
3 & 8 & -2 \\
1 & 0 & 2 \\
-2 & -1 & 5
\end{bmatrix}
\]

**Codex input**
Compute the trace of the following matrix: 
\[
\begin{bmatrix}
3 & 8 & -2 \\
1 & 0 & 2 \\
-2 & -1 & 5
\end{bmatrix}
\]

**Codex output**
```python
import numpy as np
matrix = np.array([[3, 8, -2], [1, 0, 2], [-2, -1, 5]])
print(matrix.diagonal().sum())
```

**Solution**
8

**COMS3251, Question 19:** Original question, Codex input and output, and solution.

### Four Fundamental Subspaces of a Matrix

**Original question**
Which of the vectors \( v = [3; -3; 1] \), \( u = [1; -2; 1] \) is an element of the nullspace of the following matrix: 
\[
\begin{bmatrix}
1 & 2 & -3 \\
-1 & -1 & 0 \\
-2 & -3 & 3
\end{bmatrix}
\]

**Codex input**
Write a program that checks if a the vector is an element of the nullspace of a matrix Use the program to check if the vector \( v = [3; -3; 1] \) is an element of the nullspace of the matrix 
\[
\begin{bmatrix}
1 & 2 & -3 \\
-1 & -1 & 0 \\
-2 & -3 & 3
\end{bmatrix}
\]
Use the program to check if the vector \( u = [1; -2; 1] \) is an element of the nullspace of the matrix 
\[
\begin{bmatrix}
1 & 2 & -3 \\
-1 & -1 & 0 \\
-2 & -3 & 3
\end{bmatrix}
\]

**Codex output**
```python
import numpy as np
def is_nullspace(matrix, vector):
    return np.allclose(np.dot(matrix, vector), 0)

if __name__ == '__main__':
    matrix = np.array([[1, 2, -3], [-1, -1, 0], [-2, -3, 3]])
    vector = np.array([[3], [-3], [1]])
    print(is_nullspace(matrix, vector))
    print(is_nullspace(matrix, vector))
```

**Solution**
\( u = [1; -2; 1] \)

**COMS3251, Question 20:** Original question, Codex input and output, and solution.
| Topic | Four Fundamental Subspaces of a Matrix |
|-------|----------------------------------------|
| Original question | Find the nullity of the following matrix: 
\[
\begin{bmatrix} 3, -2, -1, 0, 2; 1, -2, 1, -2, 4; -4, 4, 0, 2, -6 \end{bmatrix}
\] |
| Codex input | Write a program to find the rank of the matrix \[
\begin{bmatrix} 3, -2, -1, 0, 2; 1, -2, 1, -2, 4; -4, 4, 0, 2, -6 \end{bmatrix}
\]
Write a program to find the nullity, the number of columns minus the rank |
| Codex output | ```python
import numpy as np

matrix =
np.array([[3, -2, -1, 0, 2],
[1, -2, 1, -2, 4],
[-4, 4, 0, 2, -6]])

print(np.linalg.matrix_rank(matrix))

print(len(matrix[0]) - np.linalg.matrix_rank(matrix))
``` |
| Solution | 3 |

COMS3251, Question 21: Original question, Codex input and output, and solution.
| Topic                  | Solving Linear System of Equations |
|-----------------------|------------------------------------|
| Original question     | Compute the reduced row echelon form of the following matrix: $[-1,2,1,0;2,1,0,-1;5,0,-2,6]$ |
| Codex input           | Write a program to compute the reduced row echelon form (rref) of a matrix Compute the rref of the matrix $[-1,2,1,0;2,1,0,-1;5,0,-2,6]$ |
| Codex output          | def rref(matrix):
|                       |     if not matrix: return
|                       |     num_rows = len(matrix)
|                       |     num_cols = len(matrix[0])
|                       |     # Start at the last column and work backwards.
|                       |     for col in range(num_cols - 1, -1, -1):
|                       |         # Find the row with the leading non-zero entry in this column.
|                       |             for row in range(num_rows):
|                       |                 if matrix[row][col] != 0: break
|                       |         # If there is no leading non-zero entry, then all entries are zero.
|                       |             if row == num_rows: continue
|                       |         # Swap the current row with the one that has the leading non-zero entry.
|                       |             matrix[row], matrix[num_rows - 1] = matrix[num_rows - 1], matrix[row]
|                       |         # Eliminate all other entries in this column.
|                       |             for r in range(num_rows):
|                       |                 if r == num_rows - 1: continue # skip pivot row;
|                       |                 already done above
|                       |                 # Eliminate current row of all other entries in this column.
|                       |                 multiplier = matrix[r][col] / matrix[num_rows - 1][col] # get multiplier to eliminate value at [r][c] from rest of row (except pivot)
|                       |                 for c in range(col, num_cols):
|                       |                     matrix[r][c] -= multiplier * matrix[num_rows - 1][c]
|                       |     return matrix
|                       | if __name__ == "__main__":
|                       |     print(rref([[1, 2, 3], [4, 5, 6], [7, 8, 9]]))
|                       | ******
|                       | Compute the rref of the matrix $[-1,2,1,0;2,1,0,-1;5,0,-2,6]$ ******
|                       | if __name__ == "__main__":
|                       |     print(rref([[-1, 2, 1, 0], [2, 1, 0, -1], [5, 0, -2, 6]]))
| Solution              | $[1,0,0,-2;0,1,0,3;0,0,1,-8]$ |

COMS3251, Question 22: Original question, Codex input and output, and solution.
| Topic | Orthogonality and Projections |
|-------|-------------------------------|
| Original question | For what value of \( a \) makes the following matrix orthogonal: \[
\begin{bmatrix}
0,0,1;\-1,0,0;0,a,0
\end{bmatrix}
\] |
| Codex input | \( \text{transpose}(A) \cdot A = I \) Using sympy, write a program that finds \( x \) such that the 3x3 matrix \( A = \begin{bmatrix} 0,0,1;\-1,0,0;0,0,0,a,0 \end{bmatrix} \) is orthogonal |
| Codex output | |
| | |
| | |
| | | from sympy import * |
| | | x = Symbol('x') |
| | | A = Matrix([[0,0,1],[-1,0,0],[0,x,0]]) |
| | | print(A.transpose()*A) |
| | | print(A.transpose()*A == eye(3)) |
| | | print(solve(A.transpose()*A - eye(3), x)) |
| Solution | \( a = -1 \) |

COMS3251, Question 23: Original question, Codex input and output, and solution.

| Topic | Basis, Dimension and Span |
|-------|---------------------------|
| Original question | What is the dimension of the subspace spanned by the following vectors? \[
\begin{bmatrix}
2,-1/2;1,1;4,4
\end{bmatrix}
\] |
| Codex input | In Linear Algebra, the vectors \[2,-1/2;1,1;4,4\] span a subspace. Write a program that finds the dimension of the subspace |
| Codex output | |
| | |
| | |
| | # Solution: |
| | # |
| | # The vectors \[2,-1/2;1,1;4,4\] span a subspace. |
| | # |
| | # The vectors \[2,-1/2;1,1;4,4\] are linearly independent. |
| | # |
| | # The dimension of the subspace is 2. |
| | # Solution |
| | # |
| | # We can use the 'subspace_basis' function from the 'linear_algebra' package to find a basis for the subspace. |
| | |
| | |
| | using LinearAlgebra |
| | |
| | # Define the vectors |
| | v1 = [2,-1/2] |
| | v2 = [1,1] |
| | v3 = [4,4] |
| | |
| | # Find the basis for the subspace |
| | subspace_basis(v1,v2,v3) |
| Solution | 2 |

COMS3251, Question 24: Original question, Codex input and output, and solution.
### Basis, Dimension and Span

| Topic | Basis, Dimension and Span |
|-------|---------------------------|
| Original question | What are the coordinates of the vector \([2; 1]\) in the following basis? \([2; -1/2], [1, 1]\) |
| Codex input | Use sympy to solve: \(2x + y = 2\), \(-0.5x + y = 1\) |
| Codex output | ```python from sympy import * x = Symbol('x') y = Symbol('y') z = Symbol('z') print(solve([2*x + y - 2, -0.5*x + y - 1], [x, y])) ``` |
| Solution | \([0.4, 1.2]\) |

**COMS3251, Question 25:** Original question, Codex input and output, and solution.

### Orthogonality and Projection

| Topic | Orthogonality and Projection |
|-------|-------------------------------|
| Original question | Find the projection matrix onto the column space of \(A \begin{bmatrix} 3, 6, 6; 4, 8, 8 \end{bmatrix}\). |
| Codex input | Write a program that finds the projection matrix onto the column space of \(A \begin{bmatrix} 3, 6, 6; 4, 8, 8 \end{bmatrix}\). |
| Codex output | ```python import numpy as np A = np.array([[3, 6, 6], [4, 8, 8]]) print(np.linalg.svd(A)[2][0]) ``` |
| Solution | \([9/25, 12/25; 12/25, 16/25]\) |

**COMS3251, Question 26:** Original question, Codex input and output, and solution.

### Linear Combination and Span

| Topic | Linear Combination and Span |
|-------|-----------------------------|
| Original question | Find a combination of the vectors \([1; 2; 3], [4, 5, 6], and [7; 8; 9]\) that give the zero vector. |
| Codex input | Write a program that finds the linear combination of the vectors \([1; 2; 3], [4, 5, 6], [7; 8; 9]\) that give \([0;0;0]\). Original question should state non-zero combination |
| Codex output | ```python import numpy as np a = np.array([1, 2, 3]) b = np.array([4, 5, 6]) c = np.array([7, 8, 9]) d = np.array([0, 0, 0])

def linear_combination(a, b):
    return a[0] * b[0] + a[1] * b[1] + a[2] * b[2]
def find_vector(a, b):
    return d - linear_combination(a, d) / linear_combination(a, a) * a - linear_combination(b, d) / linear_combination(b, b) * b

print("The vector that gives \([0;0;0]\) is: ", find_vector(a, c)) ``` |
| Solution | \([1, -2, 1]\), also \([0,0,0]\) |

**COMS3251, Question 27:** Original question, Codex input and output, and solution.
Find the eigenvalues of \([-0.2, 0.3; 0.2, -0.3]\).

\[ \text{import numpy as np} \]
\[ A = \text{np.array}([[-0.2, 0.3], [0.2, -0.3]]) \]
\[ \text{print} (\text{np.linalg.eigvals}(A)) \]

\([0, 1/2]\)

If \( A = [1, 2; -2, -3] \), compute \( A^4 \).

\[ \text{import numpy as np} \]
\[ A = \text{np.array}([[1, 2], [-2, -3]]) \]
\[ \text{print}(A) \]
\[ \text{print}(A**4) \]

\([-7, -8; 8, 9]\)

Give an example of two positive definite matrices \( A \) and \( B \), whose difference is not positive definite.

\[ \text{import numpy as np} \]
\[ \text{def generate_matrices(n):} \]
\[ \text{A = np.random.rand(n, n)} \]
\[ \text{B = np.random.rand(n, n)} \]
\[ \text{while np.linalg.det(A - B) > 0:} \]
\[ \text{B = np.random.rand(n, n)} \]
\[ \text{return A, B} \]
\[ A, B = \text{generate_matrices}(3) \]
\[ \text{print}(A) \]
\[ \text{print}(B) \]
\[ \text{print}(A - B) \]
\[ \text{print}(\text{np.linalg.det}(A - B)) \]

\( A = B = I \)
### B Introduction to Linear Algebra: MIT 18.06

| Topic | Vectors and Linear Combinations (1.1, Q3) |
|-------|------------------------------------------|
| Original question | If \( v+w = (5,1) \) and \( v-w=(1,5) \), compute and draw the vectors \( v \) and \( w \) |
| Codex input | \( v \) and \( w \) are 1 by 2 vectors such that \( v + w = (5,1) \) and \( v - w = (1,5) \). Write a program to find the vectors \( v \) and \( w \). Draw the vectors \( v \) and \( w \) |
| Codex output | ```python
# Solution
#
# We can solve this problem by using the following steps:
#
# 1. We know that \( v + w = (5,1) \) and \( v - w = (1,5) \)
# 2. We can subtract \( v - w \) from both sides to get \( v + w - (v-w) = (5,1) + (1,5) - ((1,5)-(5,1)) \)
# 3. This gives us \( 2v = (6,6) \) and hence \( v = (3,3) \)
# 4. Similarly we can find out that \( w = (-2,-2) \)

import matplotlib.pyplot as plt
import numpy as np
plt.quiver(0, 0, v[0], v[1], angles='xy', scale_units='xy', scale=1)
plt.quiver(0, 0, w[0], w[1], angles='xy', scale_units='xy', scale=1)
plt.xlim(-10, 10)
plt.ylim(-10, 10)
``` |
| Solution | This problem gives the diagonals \( v + w \) and \( v - w \) of the parallelogram and asks for the sides. The opposite of Problem 2. In this example \( v = (3, 3) \) and \( w = (2, -2) \). |

---

| Topic | Vectors and Linear Combinations (1.1, Q11) |
|-------|------------------------------------------|
| Original question | Four of the eight corners of a unit cube are \( (0,0,0) \), \( (1,0,0) \), \( (0,1,0) \), \( (0,0,1) \). Find the coordinates of the center point of the cube. |
| Codex input | Find the coordinates of the center point of the cube. |
| Codex output | ```python
def find_center(x, y, z):
    return (x+0.5, y+0.5, z+0.5)
print(find_center(0,0,0))
``` |
| Solution | \((0.5, 0.5, 0.5)\) |

---

MIT 18.06, Question 1: Original question, Codex input and output, and solution.

MIT 18.06, Question 2: Original question, Codex input and output, and solution.
### Topic
Vectors and Linear Algebra (1.2 Q28)

| Original question | If \(v = (1,2)\) draw all vectors \(w = (x,y)\) in the xy plane with \(\text{dot}(v,w) = x+ 2^*y = 5\). Why do those \(w\)'s lie along a line? Which is the shortest \(w\)? |
|---|---|
| Codex input | If \(v = (1,2)\) draw all vectors \(w = (x,y)\) in the xy plane with \(\text{dot}(v,w) = x+ 2^*y = 5\). Why do those \(w\)'s lie along a line? Which is the shortest \(w\)? |
| Codex output | ```python
import matplotlib.pyplot as plt
import numpy as np
x = np.linspace(-10, 10, 100)
y = 5 - x/2
plt.plot(x, y)
plt.show()
plt.arrow(0, 0, 1, 2)  # (x1, y1), (x2, y2)
plt.show()
``` |
| Solution | \((1, 2)\) |

**MIT 18.06, Question 3:** Original question, Codex input and output, and solution.

### Topic
Vectors and Linear Algebra (1.3 Q1)

| Original question | Find the linear combination \(3s_1 + 4s_2 + 5s_3 = b\). Then write \(b\) as a matrix-vector multiplication \(Sx\), with \(3, 4, 5\) in \(x\). Compute the three dot products (row of \(S\)) \(x\): \(s_1 = [1;1;1]\), \(s_2 = [0;1;1]\), \(s_3 = [0;0;1]\) |
|---|---|
| Codex input | Write a function to calculate the linear combination \(3s_1 + 4s_2 + 5s_3\). Let \(s_1 = [1;1;1]\), \(s_2 = [0;1;1]\), \(s_3 = [0;0;1]\) |
| Codex output | ```python
def linear_combination(s1, s2, s3):
    return 3*s1 + 4*s2 + 5*s3
s1 = np.array([1,1,1])
s2 = np.array([0,1,1])
s3 = np.array([0,0,1])
print(linear_combination(s1, s2, s3))
``` |
| Solution | \(3s_1 + 4s_2 + 5s_3 = (3, 7, 12)\). The same vector \(b\) comes from \(S\) times \(x = (3, 4, 5)\). |

**MIT 18.06, Question 4:** Original question, Codex input and output, and solution.

### Topic
Vectors and Linear Algebra (1.3 Q4)

| Original question | Find a combination \(x_1*w_1+x_2*w_2+x_3*w_3\) that gives the zero vector with \(x_1 = 1\)
\(w_1\) is the vector \((1;2;3)\) \(w_2\) is the vector \((4;5;6)\) \(w_3\) is the vector \((7;8;9)\) |
|---|---|
| Codex input | Write a program to find a combination \(y\) and \(z\) such that multiplying \(\text{np.array([1, y, z])}.T\) \(\text{np.array([\begin{bmatrix}1 & 2 & 3 \end{bmatrix}, \begin{bmatrix}4 & 5 & 6 \end{bmatrix}, \begin{bmatrix}7 & 8 & 9 \end{bmatrix}\])}\) = 0. Use \text{sympy} and \text{numpy}. |
| Codex output | ```python
import sympy as sp
import numpy as np
y = sp.Symbol('\ y')
z = sp.Symbol('\ z')
eq = np.array([1, y, z]).T @ np.array([\begin{bmatrix}1 & 2 & 3 \end{bmatrix}, \begin{bmatrix}4 & 5 & 6 \end{bmatrix}, \begin{bmatrix}7 & 8 & 9 \end{bmatrix}\])
sp.solve(eq, [y, z])
``` |
| Solution | The combination \(0w_1 + 0w_2 + 0w_3\) always gives the zero vector, but this problem looks for other zero combinations (then the vectors are dependent, they lie in a plane): \(w_2 = (w_1 + w_3)/2\) so one combination that gives zero is \(w_1 - 2w_2 + w_3 = 0\). |

**MIT 18.06, Question 5:** Original question, Codex input and output, and solution.
### Vectors and Linear Equations (2.1, Q21)

**Original question**: What 2 by 2 matrix $R$ rotates every vector through 45 degrees? Example: the vector $[1,0]$ goes to $[\sqrt{2}/2, \sqrt{2}/2]$.

**Codex input**: What 2 by 2 matrix $R$ rotates every vector through 45 degrees? Example: the vector $[1,0]$ goes to $[\sqrt{2}/2, \sqrt{2}/2]$.

**Codex output**:

```python
def rotate_45(vector):
    return np.matmul(np.array([[np.sqrt(2)/2, np.sqrt(2)/2], [-np.sqrt(2)/2, np.sqrt(2)/2]]), vector)

print(rotate_45([1,0]))
```

**Solution**: $R = (1/2)\begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}$

MIT 18.06, Question 6: Original question, Codex input and output, and solution.

---

### Elimination Using Matrices (2.3, Q17)

**Original question**: The parabola $y = a + bx + cx^2$ goes through the points $(x,y) = (1,4)$ and $(2,8)$ and $(3,14)$. Find and solve a matrix equation for the unknowns $(a,b,c)$.

**Codex input**: The parabola $y = a + bx + cx^2$ goes through the points $(x_1,y_1) = (1,4)$ and $(x_2,y_2) = (2,8)$ and $(x_3,y_3) = (3,14)$. $y_1 = a + b*x_1 + c*x_1^2$ $y_2 = a + b*x_2 + c*x_2^2$ $y_3 = a + b*x_3 + c*x_3^2$ Solve for $(a, b, c)$.

**Codex output**:

```python
import numpy as np
x1 = 1
x2 = 2
x3 = 3
y1 = 4
y2 = 8
y3 = 14

A = np.array([[1, x1, x1**2], [1, x2, x2**2], [1, x3, x3**2]])
b = np.array([y1, y2, y3])
print(np.linalg.solve(A, b))
```

**Solution**: $a = 2$, $b = -1$, and $c = 1$.

MIT 18.06, Question 7: Original question, Codex input and output, and solution.
| Topic | Rules for Matrix Operations (2.4, Q23b) |
|-------|---------------------------------------|
| Original question | Find a matrix that has $A^2$ does not equal 0 but $A^3 = 0$ |
| Codex input | Loop through random integer matrices with 0 until matrix $A$ is found. $A$ must satisfy conditions: $A^2$ not equals `np.zeros((2,2))` and $A^3$ equals `np.zeros((2,2))`. |
| Codex output | ```python
import numpy as np

def find_matrix_A(n):
    """
    Find matrix $A$ with nxn dimensions.
    """
    while True:
        A = np.random.randint(0,10,(n,n))
        if np.array_equal(A**2, np.zeros((n,n))) == False and np.array_equal(A*3, np.zeros((n,n))) == True:
            return A

def main():
    """
    Main function.
    """
    n = int(input("Enter n: "))
    A = find_matrix_A(n)
    print(A)

if __name__ == "__main__":
    main()
| Solution | $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ |

MIT 18.06, Question 8: Original question, Codex input and output, and solution.

| Topic | Rules for Matrix Operations (2.5, Q1) |
|-------|---------------------------------------|
| Original question | Find the inverses (directly or from the 2 by 2 formula) of $A$, $B$, $C$ : $A = \begin{bmatrix} 0 & 4 \\ 3 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 4 & 2 \end{bmatrix}$, and $C = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}$. |
| Codex input | Find the inverses of the matrices $A$, $B$, $C$ : $A = \begin{bmatrix} 0 & 4 \\ 3 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 4 & 2 \end{bmatrix}$, and $C = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}$. |
| Codex output | ```python
import numpy as np

A = np.array([[0,4],[3,0]])
B = np.array([[2,0],[4,2]])
C = np.array([[3,4],[5,7]])

print(np.linalg.inv(A))
print(np.linalg.inv(B))
print(np.linalg.inv(C))
| Solution | $\text{inv}(A) = \begin{bmatrix} 0 & 1/4 \\ 1/3 & 0 \end{bmatrix}$, $\text{inv}(B) = \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix}$, and $\text{inv}(C) = \begin{bmatrix} 7 & -4 \\ -5 & 3 \end{bmatrix}$. |

MIT 18.06, Question 9: Original question, Codex input and output, and solution.
### Rules for Matrix Operations (2.5, Q3)

**Original question**
Solve for the first column \((x, y)\) and second column \((t, z)\) of \(A^{-1}\):
\[
\begin{bmatrix}
10 & 20 \\
20 & 50
\end{bmatrix} @ \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix}
10 & 20 \\
20 & 50
\end{bmatrix} @ \begin{bmatrix} t \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]

**Codex output**
```python
import numpy as np
A = np.array([[10, 20], [20, 50]])
b = np.array([[1], [0]])
x = np.linalg.inv(A) @ b
print(x)
A = np.array([[10, 20], [20, 50]])
b = np.array([[0], [1]])
x = np.linalg.inv(A) @ b
print(x)
```

**Solution**
\([x; y] = [0.5; -0.2]\) and \([t; z] = [-0.2; 0.1]\) so \(A^{-1} = 1/10 \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}\).

---

### Inverse Matrices (2.5, Q11b)

**Original question**
Find singular matrices \(A\) and \(B\) such that \(A + B\) is invertible.

**Codex output**
```python
import numpy as np
import numpy.linalg as la

def find_singular_matrices(n):
    
    Find two matrices \(A\) and \(B\) such that \(A + B\) is invertible.
    
    A = np.random.rand(n, n)
    B = np.random.rand(n, n)
    while la.det(A + B) == 0:
        A = np.random.rand(n, n)
        B = np.random.rand(n, n)
    return A, B

def main():
    
    Test the \(\text{find singular matrices}\) function.
    
    A, B = find_singular_matrices(3)
    print(A)
    print(B)
    print(A + B)

if __name__ == "__main__":
    main()
```

**Solution**
\(A = [1, 0; 0, 0]\) and \(B = [0, 0; 0, 1]\)
| Topic                  | Nullspaces (3.2, Q20) |
|------------------------|------------------------|
| Original question      | Construct a 2 by 2 matrix whose nullspace equals its column space. This is possible. |
| Codex input            | Write a function that checks the nullspace of a matrix. Write a function that checks the column space of a matrix. Write a function to iteratively search for a 2 by 2 matrix np.array([[a, b], [c, d]]) and returns the values of the matrix whose nullspace equals its column space. |
| Codex output           | ```python
def check_nullspace(matrix):
    '''
    Checks the nullspace of a matrix.
    '''
    return np.linalg.matrix_rank(matrix)

def check_columnspace(matrix):
    '''
    Checks the column space of a matrix.
    '''
    return np.linalg.matrix_rank(matrix.T)

def iterative_search(matrix):
    '''
    Iteratively searches for a 2 by 2 matrix np.array([[a, b], [c, d]]) and returns the values of the matrix whose nullspace equals its column space.
    '''
    for a in range(1, 10):
        for b in range(1, 10):
            for c in range(1, 10):
                for d in range(1, 10):
                    if check_nullspace(np.array([[a, b], [c, d]])) == check_columnspace(np.array([[a, b], [c, d]])):
                        return np.array([[a, b], [c, d]])

print(iterative_search(np.array([[1, 2], [3, 4]])))
``` |
| Solution               | $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ |

MIT 18.06, Question 12: Original question, Codex input and output, and solution.
Show that $v_1, v_2, v_3$ are independent but $v_1, v_2, v_3, v_4$ are dependent. $v_1 = [1; 0; 0], v_2 = [1; 1; 0], v_3 = [1; 1; 1], v_4 = [2; 3; 4]$. Write a function to show that $v_1, v_2, v_3$ are linearly independent. In linear algebra write a function that checks if vectors $v_1, v_2, v_3, v_4$ are independent. $v_1 = (1, 0, 0), v_2 = (1, 1, 0), v_3 = (1, 1, 1), v_4 = (2, 3, 4)$ Use arrays for the vectors $v_1, v_2, v_3, v_4$ are check if they are independent.

Let $v_1 = [1; 0; 0], v_2 = [1; 1; 0], v_3 = [1; 1; 1], v_4 = [2; 3; 4]$. Write a function to show that $v_1, v_2,$ and $v_3$ are linearly independent. In linear algebra write a function that checks if vectors $v_1, v_2, v_3, v_4$ are independent. $v_1 = (1, 0, 0), v_2 = (1, 1, 0), v_3 = (1, 1, 1), v_4 = (2, 3, 4)$ Use arrays for the vectors $v_1, v_2, v_3, v_4$ are check if they are independent.

\[
\begin{bmatrix}
1 & 1 & 1 & 1; 0 & 1 & 1 & 0 & 1 & 1; 0 & 0 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
c_1
c_2
c_3
\end{bmatrix}
= 0
\]
gives $c_3 = c_2 = c_1 = 0$. So those 3 column vectors are independent. But
\[
\begin{bmatrix}
1 & 1 & 1 & 1; 0 & 1 & 1 & 0 & 1 & 4
\end{bmatrix}
c = [0; 0; 0]
\]
is solved by $c = (1, 1, -4, 1)$ then $v_1 + v_2 - 4v_3 + v_4 = 0$ (dependent).

Find $A'A$ if the columns of $A$ are unit vectors, all mutually perpendicular.

\[
A = np.array([\begin{bmatrix}1, 0, 0\end{bmatrix}, \begin{bmatrix}0, 1, 0\end{bmatrix}, \begin{bmatrix}0, 0, 1\end{bmatrix}])
\]

\[
\text{print} (\text{transpose} \cdot \text{multiply}(A))
\]
| Topic               | Projections (4.2, Q2b)                               |
|--------------------|-----------------------------------------------------|
| Original question  | Draw the projection of b onto a: b = [1; 1] and a = [1; -1]. |
| Codex input        | The vector b is [1; 1] The vector a is [1; -1] Plot the projection of b onto a |
| Codex output       | """
|                    | The vector b is [1; 1]                              |
|                    | The vector a is [1; -1]                             |
|                    | Plot the projection of b onto a                      |
|                    | """
|                    | import numpy as np                                  |
|                    | import matplotlib.pyplot as plt                    |
|                    | a = np.array([1, -1])                               |
|                    | b = np.array([1, 1])                                |
|                    | # Projection of b onto a                             |
|                    | proj_b_a = (np.dot(b, a) / np.dot(a, a)) * a         |
|                    | # Plot                                              |
|                    | plt.plot([0, a[0]], [0, a[1]], 'r', label='a')      |
|                    | plt.plot([0, b[0]], [0, b[1]], 'g', label='b')      |
|                    | plt.plot([0, proj_b_a[0]], [0, proj_b_a[1]], 'b',  |
|                    |       label='projection of b onto a')               |
|                    | plt.axis('equal')                                   |
|                    | plt.legend()                                        |
|                    | plt.grid()                                          |
|                    | plt.show()                                          |
|                    | """
|                    | Plot the projection with circle marker              |
|                    | """
|                    | # Plot                                              |
|                    | plt.plot([0, a[0]], [0, a[1]], 'r', label='a')      |
|                    | plt.plot([0, b[0]], [0, b[1]], 'g', label='b')      |
|                    | plt.plot([0, proj_b_a[0]], [0, proj_b_a[1]], 'bo', |
|                    |       label='projection of b onto a')               |
|                    | plt.axis('equal')                                   |
|                    | plt.legend()                                        |
|                    | plt.grid()                                          |
|                    | plt.show()                                          |
| Solution           | (0,0) with a corresponding plot.                    |

MIT 18.06, Question 15: Original question, Codex input and output, and solution.
| Topic | Projections (4.2, Q16) | Least Squares Approximations (4.3, Q22) |
|-------|------------------------|----------------------------------------|
| Original question | What linear combination of (1,2,-1) and (1,0,1) is closest to b=(2,1,1)? | Find the best line C+Dt to fit b=4.3,-1,0,0 at times t=-2,-1,0,1,2. |
| Codex input | Use scipy to optimize for x and y such that np.array([1,2,-1])*x+np.array([1,0,1])*y is closest to b=np.array([2,1,1]). | Find the best fit line for x=np.array([-2,-1,0,1,2]) and y=np.array([4,3,-1,0,0]). |
| Codex output | import numpy as np<br>from scipy.optimize import minimize<br>def objective(x):<br>\[\text{return } np.sum(np.square(np.array([1,2,-1])*x[0]+np.array([1,0,1])*x[1]-np.array([2,1,1])))\] | import numpy as np<br>import matplotlib.pyplot as plt<br>x = np.array([-2,-1,0,1,2])<br>y = np.array([4,3,-1,0,0])<br>plt.plot(x,y,'ro')<br>plt.show()<br># y = mx + c<br># m = (mean(x)\cdot mean(y) - mean(x\cdot y)) / (mean(x)^2 - mean(x^2))<br># c = mean(y) - m\cdot mean(x)<br>m = (np.mean(x)\cdot np.mean(y) - np.mean(x\cdot y)) / (np.mean(x)^2 - np.mean(x^2))<br>c = np.mean(y) - m\cdot np.mean(x)<br>print(m,c)<br>y_pred = m\cdot x + c<br>plt.plot(x,y,'ro')<br>plt.plot(x,y_pred)<br>plt.show() |
| Solution | (2, 1, 1) | The best line is b = 1 - t. |

MIT 18.06, Question 16: Original question, Codex input and output, and solution.

MIT 18.06, Question 17: Original question, Codex input and output, and solution.
**Topic** Properties of Determinants (5.1, Q1)

**Original question**
If a 4 by 4 matrix has det(A)=1/2, find det(2A), det(-1A), det(A²) and det(A⁻¹).

**Codex input**
The determinant of a 4 by 4 matrix A is 0.5. What is det(2A), det(-1A), det(A²), and det(A⁻¹)? Write a function to calculate the determinant of 2A is 16 times the determinant of A. Write a function to calculate the determinant of -1A is -1 raised to the fourth power times the determinant of A. Write a function to calculate the determinant of A² is half the determinant of A. Write a function to calculate the determinant of A⁻¹ is 1/the determinant of A.

**Codex output**

```python
def det2A(A):
    return 16*det(A)

def detMinus1A(A):
    return -1**4*det(A)

def detA2(A):
    return det(A)/2

def detAinv(A):
    return 1/det(A)
```

**Solution**

\[
\text{det}(2A) = 2^4 \times \text{det}(A) = 8,
\text{det}(-A) = (-1)^4 \times \text{det}(A) = 0.5,
\text{det}(A^2) = 0.25,
\text{det}(A^{-1}) = 2
\]

MIT 18.06, Question 18: Original question, Codex input and output, and solution.

**Topic** Cramer’s Rule, Inverses and Volumes (5.3, Q16a)

**Original question**
Find the area of the parallelogram with edges v=(3,2) and w=(1,4).

**Codex input**
Find the area of the parallelogram with edges v=(3,2) and w=(1,4).

**Codex output**

```python
# Solution
v = (3, 2)
w = (1, 4)
area = abs(v[0] * w[1] - v[1] * w[0])
print(area)
```

**Solution**
10

MIT 18.06, Question 19: Original question, Codex input and output, and solution.

**Topic** Introduction to Eigenvalues (6.1, Q5)

**Original question**
Find the eigenvalues of A and B (easy for triangular matrices) and A + B: A = \[
\begin{bmatrix}
3 & 0 \\
1 & 1
\end{bmatrix}
\], B = \[
\begin{bmatrix}
1 & 1 \\
0 & 3
\end{bmatrix}
\], A+B = \[
\begin{bmatrix}
4 & 1 \\
1 & 4
\end{bmatrix}
\].

**Codex input**
Find the eigenvalues of A and B (easy for triangular matrices) and A + B: A = \[
\begin{bmatrix}
3 & 0 \\
1 & 1
\end{bmatrix}
\], B = \[
\begin{bmatrix}
1 & 1 \\
0 & 3
\end{bmatrix}
\], A+B = \[
\begin{bmatrix}
4 & 1 \\
1 & 4
\end{bmatrix}
\].

**Codex output**

```python
import numpy as np
A = np.array([[3, 0], [1, 1]])
B = np.array([[1, 1], [0, 3]])
print(np.linalg.eigvals(A))
print(np.linalg.eigvals(B))
print(np.linalg.eigvals(A+B))
```

**Solution**
A and B have eigenvalues 1 and 3

MIT 18.06, Question 20: Original question, Codex input and output, and solution.
### Introduction to Eigenvalues (6.1, Q10)

**Original question**
Find the eigenvalues and eigenvectors for both of these Markov matrices $A$ and $A_{\text{inf}}$. 
$A = \begin{bmatrix} .6 & .2 \\ .4 & .8 \end{bmatrix}$. $A_{\text{inf}} = \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{bmatrix}$.

**Codex input**
Find the eigenvalues and eigenvectors for both of these Markov matrices $A$ and $A_{\text{inf}}$. 
$A = \text{np.array([[[.6, .2], [.4, .8]]])}$. $A_{\text{inf}} = \text{np.array([[[1/3, 1/3], [2/3, 2/3]]])}$.

**Codex output**
```
A = np.array([[.6, .2], [.4, .8]])
A_inf = np.array([[1/3, 1/3], [2/3, 2/3]])

eig_vals, eig_vecs = np.linalg.eig(A)
eig_vals_inf, eig_vecs_inf = np.linalg.eig(A_inf)

print(eig_vals)
print(eig_vecs)
print(eig_vals_inf)
print(eig_vecs_inf)
```

**Solution**
$A$ – Eigenvalues: 1 and 0.4, with eigenvectors $(1, 2)$ and $x_2 = (1, -1)$. 
$A_{\text{inf}}$ – Eigenvalues: 1 and $(0.4)^{100}$, same eigenvectors.

**MIT 18.06, Question 21:** Original question, Codex input and output, and solution.

---

### Symmetric Matrices (6.4, Q9a)

**Original question**
Find a symmetric matrix $\begin{bmatrix} 1,b;b,1 \end{bmatrix}$ that has a negative eigenvalue.

**Codex input**
$A = \begin{bmatrix} 1,b;b,1 \end{bmatrix}$ is a symmetric matrix Find a value of $b$ such that any one eigenvalue of $A$ is negative.

**Codex output**
```
A = [[1,b],[b,1]] is a symmetric matrix
Find a value of b such that any one eigenvalue of A is negative.

import numpy as np
import scipy.linalg as la

def find_b(n):
    A = np.array([[1,b],[b,1]])
eigvals = la.eigvals(A)
    if any(eigvals < 0):
        return b
    else:
        return find_b(n+1)

b = find_b(1)
print(b)
```

**Solution**
$b=2$

**MIT 18.06, Question 22:** Original question, Codex input and output, and solution.
### Positive Definite Matrices (6.5, Q3)

**Original question**
For which numbers $b$ is the following matrices positive definite? $S = \begin{bmatrix} 1 & b \\ b & 9 \end{bmatrix}$, $S = \begin{bmatrix} 2 & 4 \\ 4 & c \end{bmatrix}$, $S = \begin{bmatrix} c & b \\ b & c \end{bmatrix}$.

**Codex input**
For which numbers $b$ is the following matrices positive definite? $S = \begin{bmatrix} 1 & b \\ b & 9 \end{bmatrix}$, $S = \begin{bmatrix} 2 & 4 \\ 4 & c \end{bmatrix}$, $S = \begin{bmatrix} c & b \\ b & c \end{bmatrix}$. Write a program to search a value $x$ and $y$ for all three matrices to be positive definite: $\text{np.array}([[1,x],[x,9]])$, $\text{np.array}([[2,4],[4,y]])$, and $\text{np.array}([[y,x],[x,y]])$.

**Codex output**

```python
import numpy as np

def is_pos_def(x):
    return np.all(np.linalg.eigvals(x) > 0)

def main():
    b = np.arange(1,10)
    c = np.arange(1,10)
    for i in b:
        for j in c:
            S1 = np.array([[1,i],[i,9]])
            S2 = np.array([[2,4],[4,j]])
            S3 = np.array([[j,i],[i,j]])
            if is_pos_def(S1) and is_pos_def(S2) and
               is_pos_def(S3):
                print('S1 = {}, S2 = {}, S3 = {}
'.format(S1,S2,S3))

def main():
    b = np.arange(1,10)
    c = np.arange(1,10)
    for i in b:
        for j in c:
            S1 = np.array([[1,i],[i,9]])
            S2 = np.array([[2,4],[4,j]])
            S3 = np.array([[j,i],[i,j]])
            if is_pos_def(S1) and is_pos_def(S2) and
               is_pos_def(S3):
                print('S1 = {}, S2 = {}, S3 = {}
'.format(S1,S2,S3))

Solution
(a) $-3 < b < 3$, (b) $c > 8$, (c) $c > b$

MIT 18.06, Question 23: Original question, Codex input and output, and solution.

### Image processing by Linear Algebra (7.1, Q6)

**Original question**
Find the eigenvalues and the singular values of this 2 by 2 matrix $A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$. The eigenvectors $(1,2)$ and $(1,-2)$ of $A$ are not orthogonal.

**Codex input**
Write a function to find the eigenvalues and the singular values of this 2 by 2 matrix $A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$. The eigenvectors $(1,2)$ and $(1,-2)$ of $A$ are not orthogonal.

**Codex output**

```python
import numpy as np

A = np.array([[2,1],[4,2]])
eigenvalues, eigenvectors = np.linalg.eig(A)
print(eigenvalues)
print(eigenvectors)
singular_values = np.linalg.svd(A)
print(singular_values)
```

**Solution**
Eigenvalues are 4 and 0

MIT 18.06, Question 24: Original question, Codex input and output, and solution.
### Topic
Principal Component Analysis (7.3,Q1)

### Original question
Suppose A₀ holds these 2 measurements of 5 samples: A₀ = \([5,4,3,2,1;-1,1,0,1,-1]\). Find the average of each row and subtract it to produce the centered matrix A. Compute the sample covariance matrix S=AA'/(n-1) and find its eigenvalues. What line through the origin is closest to the 5 samples in the columns of A?

### Codex input
Suppose A₀ holds these 2 measurements of 5 samples: A₀ = \([5,4,3,2,1;-1,1,0,1,-1]\). Find the average of each row and subtract it to produce the centered matrix A. Compute the sample covariance matrix S=AA'/(n-1) and find its eigenvalues. What line through the origin is closest to the 5 samples in the columns of A?

```python
import numpy as np
A0 = np.array([[5, 4, 3, 2, 1], [-1, 1, 0, 1, -1]])
A = A0 - np.mean(A0, axis=1).reshape(2,1)
S = np.dot(A, A.T) / (5-1)
eigvals, eigvecs = np.linalg.eig(S)
print(eigvals)
print(eigvecs)
```

### Codex output
Suppose A₀ holds these 2 measurements of 5 samples: A₀ = \([5,4,3,2,1;-1,1,0,1,-1]\). Find the average of each row and subtract it to produce the centered matrix A. Compute the sample covariance matrix S=AA'/(n-1) and find its eigenvalues. What line through the origin is closest to the 5 samples in the columns of A?

```python
import numpy as np
A0 = np.array([[5, 4, 3, 2, 1], [-1, 1, 0, 1, -1]])
A = A0 - np.mean(A0, axis=1).reshape(2,1)
S = np.dot(A, A.T) / (5-1)
eigvals, eigvecs = np.linalg.eig(S)
print(eigvecs[:,0])
```

### Solution
The vertical line passing through the origin is closest to the five datapoints.

MIT 18.06, Question 25: Original question, Codex input and output, and solution.
| Topic | The Idea of a Linear Transformation (8.1, Q12) |
|-------|------------------------------------------------|
| Original question | Suppose a linear \( T \) transforms \((1,1)\) to \((2,2)\) and \((2,0)\) to \((0,0)\). Find \( T(v) \), when \( v=[2,2] \), \( v=(3,1) \), \( v=(-1,1) \), \( v=(a,b) \). |
| Codex input | Use a while True loop to find a random matrix \( M \) such that \( M\cdot\text{np.array([[1,1],[2,2]])} = \text{np.array([[2,2],[0,0]])} \) Find \( M\cdot\text{np.array([2,2])} \) Find \( M\cdot\text{np.array([3,1])} \) Find \( M\cdot\text{np.array([-1,1])} \) Use Sympy to find Find \( M\cdot\text{np.array([a,b])} \). |
| Codex output | 

```python
import numpy as np
import sympy as sp
while True:
    M = np.random.randint(1,10,(2,2))
    if np.array_equal(M.dot(np.array([[1,1],[2,2]])), np.array([[2,2],[0,0]])):
        print(M)
        print(M.dot(np.array([2,2])))
        print(M.dot(np.array([3,1])))
        print(M.dot(np.array([-1,1])))
        a,b = sp.symbols('a b')
        print(M.dot(np.array([a,b])))
        break
```
| Solution | \( T(v) = (4, 4); (2, 2); (2, 2); \) if \( v = (a, b) = b(1, 1) + a-b \) then \( T(v) = b(2, 2) + (0, 0) \) |

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| Topic | Linear Algebra in Probability and Statistics (12.1, Q2) |
|-------|------------------------------------------------|
| Original question | We know: \( 1/3 \) of all integers are divisible by \( 3 \) and \( 1/7 \) of integers are divisible by \( 7 \). What fraction of integers will be divisible by \( 3 \) or \( 7 \) or both ? |
| Codex input | We know: \( 1/3 \) of all integers are divisible by \( 3 \) and \( 1/7 \) of integers are divisible by \( 7 \). Write a program to calculate the fraction of integers will be divisible by \( 3 \) or \( 7 \) or both ? |
| Codex output | 

```python
# Solution:
# import fractions module
import fractions
# define a function to calculate the fraction of integers divisible by 3 or 7 or both
def divisible_by_3_or_7():
    # calculate the fraction of integers divisible by 3 or 7 or both
    return fractions.Fraction(1, 3) + fractions.Fraction(1, 7) - fractions.Fraction(1, 21)
# print the result
print(divisible_by_3_or_7())
```
| Solution | If we add \( 1/3 \) to \( 1/7 \) (fraction of integers divisible by 3 plus fraction divisible by 7) we have double counted the integers divisible by both 3 and 7. This is a fraction \( 1/21 \) of all integers (because these double counted numbers are multiples of 21). So the fraction divisible by 3 or 7 or both is \( 1/3+1/7-1/21+1/21+1/21=9/21=3/7 \) |

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MIT 18.06, Question 26: Original question, Codex input and output, and solution.

MIT 18.06, Question 27: Original question, Codex input and output, and solution.
Original question: Sample again from 1 to 1000 but look at the last digit of the sample squared. That square could end with $x = 0, 1, 4, 5, 6, \text{ or } 9$. What are the probabilities $p_0, p_1, p_4, p_5, p_6, p_9$?

Codex output:
```
def prob_last_digit(n):
    
    # generate n samples
    samples = np.random.randint(1, 1000, size=n)
    # square the samples
    samples = np.power(samples, 2)
    # get the last digit of each sample
    samples = samples % 10
    # count the number of times each digit appears
    counts = np.bincount(samples)
    # normalize the counts
    counts = counts / n
    return counts

# print the probabilities
print(prob_last_digit(1000000))
```

Solution: For numbers ending in $0, 1, 2, \ldots, 9$ the squares end in $x = 0, 1, 4, 9, 6, 5, 6, 9, 4, 1$. So the probabilities of $x = 0$ and $5$ are $p = 1/10$ and the probabilities of $x = 1, 4, 6, 9$ are $p = 1/5$. The mean is $m = \sum p_i x_i = (0+5/10+(1/5)*(1 + 4 + 6 + 9)) = 4.5 = \text{ same as before}$.
| Topic | Linear Algebra in Probability and Statistics (12.1, Q9) |
|-------|-------------------------------------------------------|
| Original question | Find the average $A_N$ of a million random 0-1 samples! What is $X = (A_N - 1/2)/(2\times\sqrt{N})$? |
| Codex input | Find the average $A_N$ of a million random 0-1 samples! What is $X = (A_N - 1/2)/(2\times\sqrt{N})$? |

**Codex output**

```python
import numpy as np
import matplotlib.pyplot as plt

def get_average(N):
    '''
    N: number of samples
    '''
samples = np.random.randint(0,2,N)
    return np.mean(samples)

def get_X(N):
    '''
    N: number of samples
    '''
    return (get_average(N) - 0.5)/(2*np.sqrt(N))

def get_X_list(N):
    '''
    N: number of samples
    '''
    X_list = []
    for i in range(N):
        X_list.append(get_X(N))
    return X_list

def get_X_mean(N):
    '''
    N: number of samples
    '''
    return np.mean(get_X_list(N))

def get_X_std(N):
    '''
    N: number of samples
    '''
    return np.std(get_X_list(N))

N = 1000000
print(get_X_mean(N))
print(get_X_std(N))
plt.hist(get_X_list(N), bins=100)
plt.show()
```

**Solution**

$X = (A_N - 1/2)/(2\sqrt{N}) = (A_N - 1/2)/2000$ for $N = \text{one million.}$

MIT 18.06, Question 30: Original question, Codex input and output, and solution.