SPECIFICATION AND THERMODYNAMIC PROPERTIES OF NON-AUTONOMOUS DYNAMICAL SYSTEMS

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Abstract. In this paper, the topological aspects of the thermodynamical formalism for non-autonomous dynamical systems given by sequences of continuous maps on a compact metric space are studied. We show that each non-autonomous dynamical system of surjective maps with the specification property has positive topological entropy, every point is an entropy point; in particular, it is topologically chaotic. Moreover, we introduce a class of non-autonomous dynamical systems enjoying the specification and the shadowing properties. Furthermore, we prove that the topological entropy is an upper bound for the exponential growth rate of periodic points for uniformly expanding non-autonomous dynamical systems. Finally, we introduce a special class of continuous potentials such that their topological pressures can be computed as a limit at a definite size scale for any non-autonomous dynamical system enjoying the ∗-expansive property.

1. Introduction

The theory of thermodynamic formalism was brought from statistical mechanics to dynamical systems by the works of Sinai, Ruelle and Bowen in the 1970s [8, 9, 29, 30, 31]. Entropy, Gibbs Measures and equilibrium states are the main object of study in thermodynamic formalism. For uniformly hyperbolic systems Bowen presented a complete description of thermodynamic formalism [8]. To establish the thermodynamic formalism of a hyperbolic set he uses a Markov partition as a tool. A Markov partition creates a symbolic dynamic for the system and allows to apply standard techniques from symbolic dynamics as well. For the non-uniformly hyperbolic case, a general theory of thermodynamic formalism, despite substantial progress by several authors, is far from being complete.

Recently, there have been major efforts in establishing a general theory of systems with time-dependent dynamical laws, often called non-autonomous, see [2, 15, 16, 17, 18, 19, 25]. Although some authors try to extend the previous results from thermodynamic formalism to non-autonomous dynamical systems, a global theory is still out of reach. Our main goal in this paper is to describe the topological aspects of the thermodynamic formalism for non-autonomous dynamical systems.

In the theory of dynamical systems, topological entropy is a nonnegative extended real number measuring the complexity of a topological dynamical system. Topological entropy was first introduced by Adler, Konhelm and McAndrew, based on open covers for continuous maps in compact topological spaces [1] so-called autonomous dynamical

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systems. In 1970, Bowen gave another definition based on separated and spanning sets for uniformly continuous maps in metric spaces [7], and this definition is equivalent to Adler’s definition for continuous maps in compact metric spaces. Topological entropy has close relationships with many important dynamical properties, such as chaos, Lyapunov exponents, the growth of the number of periodic points and so on. Moreover, positive topological entropy have remarkable role in the characterization of the dynamical behaviors, for instance, Downarowicz proved that positive topological entropy implies chaos DC2 [10]. Thus, a lot of attention has been focused on computations and estimations of topological entropy of an autonomous dynamical system and many good results have been obtained [4, 5, 6, 7, 11, 12, 14, 22, 24, 26]. In 1996, Kolyada and Snoha extended the concept of topological entropy for autonomous systems to non-autonomous systems, based on open covers, separated sets and spanning sets, and obtained a series of important properties of these systems [19]. Recently, Kawan introduced the notion of metric entropy for non-autonomous dynamical systems which is related via a variational inequality to the topological entropy of non-autonomous dynamical systems as defined by Kolyada and Snoha [15, 17].

For semigroup actions, some authors provided conditions such that the dynamic has positive topological entropy. Shao et al. [13] have given an estimation of lower bound of topological entropy for coupled-expanding systems associated with transition matrices in compact Hausdorff spaces. Rodrigues and Varandas showed that any finitely generated continuous semigroup action on a compact metric space with the strong orbital specification property has positive topological entropy [27]. Also they show that if each element of the semigroup action is local homeomorphism and semigroup enjoying the strong orbital specification property, then every point is an entropy point. Entropy points are those that their local neighborhoods reflect the complexity of the entire dynamical system from the viewpoint of entropy theory.

In this paper, we try to extend these results to non-autonomous dynamical systems. We show that any non-autonomous dynamical system of surjective maps satisfying the specification property has positive topological entropy and every point is an entropy point. Moreover, we introduce a class of non-autonomous dynamical systems having the specification property. More precisely, we show that uniformly expanding non-autonomous dynamical systems enjoy the specification property. Also, we prove that the topological entropy is an upper bound for the exponential growth rate of periodic points of uniformly expanding non-autonomous dynamical systems.

The definition of topological pressure based on separated sets for autonomous dynamical systems were introduced by Ruelle [28] and later other definitions of topological pressure based on open covers and spanning sets given by Walters [36]. In 2008, Huang et al. [13] extended these definitions to non-autonomous dynamical systems, based on open covers, separated sets and spanning sets. Recently, Kawan introduced the notion of metric pressure for a non-autonomous dynamical system which is related via a variational inequality to the topological pressure of non-autonomous dynamical systems [16].

The topological pressure can be computed as the limiting complexity of the dynamical system as the size scale approaches zero. Thus, some authors provided conditions so that the topological pressure of the dynamical system can be computed as a limit at a definite
size scale. For instance, Rodrigues and Varandas showed that the topological pressure of any continuous potential that satisfies the uniformly bounded variation condition can be computed as a limit at a definite size scale for any finitely generated continuous semigroup action on a compact metric space with the $*$-expansive property [27]. In this paper, we try to extend these results to non-autonomous dynamical systems. More precisely, we show that the topological pressure of any continuous potential satisfying the uniformly bounded variation condition can be computed as a limit at a definite size scale for any non-autonomous dynamical system enjoying the $*$-expansive property.

This is how the paper is organized: In Section 2, we present an overview of the main concepts and introduce notations, moreover, we give some related terminology. In Section 3, we characterize entropy points of non-autonomous dynamical systems having the specification property and show that any non-autonomous dynamical system of surjective maps enjoying the specification property has positive topological entropy, every point is an entropy point; in particular, it is topologically chaotic. Uniformly expanding non-autonomous dynamical systems are discussed in Section 4. We show that each uniformly expanding NDS satisfies the shadowing and the specification properties. Moreover, we prove that every point of any uniformly expanding NDS is an entropy point, moreover it has positive topological entropy. Additionally, the NDS is topologically mixing and topologically chaotic. We also assert that for uniformly expanding NDSs the topological entropy is an upper bound on the growth rate of the number of periodic points. Finally, the notion of $*$-expansive NDS is introduced. We show that the topological pressure of a $*$-expansive NDS can be computed as the topological complexity that is observable at a definite size scale.

2. Preliminaries

A non-autonomous dynamical system or an NDS for short, is a pair $(X_{1,\infty}, \varphi_{1,\infty})$, where $X_{1,\infty} = (X_n)_{n=1}^{\infty}$ is a sequence of sets and $\varphi_{1,\infty} = (\varphi_n)_{n=1}^{\infty}$ is a sequence of maps $\varphi_n : X_n \to X_{n+1}$. By $(X_{n,\infty}, \varphi_{n,\infty})$, we denote the pair of shifted sequences $X_{n,\infty} = (X_{n+k})_{k=0}^{\infty}$, $\varphi_{n,\infty} = (\varphi_{n+k})_{k=0}^{\infty}$ and we use similar notation for other sequences associated with an NDS. If all the set $X_n$ are compact metric spaces and all the $\varphi_n$ are continuous, we say that $(X_{1,\infty}, \varphi_{1,\infty})$ is a topological NDS. Here, we assume that all the sets $X_n$ are equal to the set $M$ and we abbreviate $(X_{1,\infty}, \varphi_{1,\infty})$ by $(M, \varphi_{1,\infty})$.

The time evolution of the system is defined by composing the maps $\varphi_n$ in the obvious way. In general, we define

$$\varphi^n_k := \varphi_{k+n-1} \circ \ldots \circ \varphi_k \text{ for } k, n \in \mathbb{N}, \text{ and } \varphi^0_k := id_M. \tag{2.1}$$

We also put $\varphi^{-n}_k := (\varphi^n_k)^{-1}$, which is only applied to subsets $A \subset M$. The trajectory of a point $x \in M$ is the sequence $(\varphi^n(x))_{n=0}^{\infty}$.

Throughout this paper we work with topological NDSs and use NDS instead of topological NDS for simplicity. Otherwise, we express it with details.

**Definition 2.1.** Consider an NDS $(M, \varphi_{1,\infty})$. Based on [33], a point $x_0 \in M$ is said to be periodic point of $(M, \varphi_{1,\infty})$ if the orbit of $x_0$ is periodic, i.e. there exists an integer $k > 0$ such that $\varphi^{i+j}_1(x_0) = \varphi^i_1(x_0)$, for every $i \in \mathbb{N}$ and $0 \leq j < k$. The set of all periodic points of $(M, \varphi_{1,\infty})$ is denoted by $\text{Per}(M, \varphi_{1,\infty})$. Similarly, a point $x_0 \in M$ is said to be a
fixed point of the NDS \((M, \varphi_{1,\infty})\) if \(\varphi_n(x_0) = x_0\) for all \(n \geq 1\), the set of all fixed points of \((M, \varphi_{1,\infty})\) is denoted by \(\text{Fix}(M, \varphi_{1,\infty})\).

**Definition 2.2.** Following [23], we say that an NDS \((M, \varphi_{1,\infty})\) has the specification property if for every \(\delta > 0\) there is \(N = N(\delta) \in \mathbb{N}\) such that for any \(x_1, x_2, \ldots, x_s \in M\) with \(s \geq 2\) and any sequence \(0 = j_1 \leq k_1 < j_2 \leq k_2 < \ldots < j_s \leq k_s\) of integers with 
\[ j_{n+1} - k_n \geq N \text{ for } n = 1, \ldots, s - 1, \]
there is a point \(x \in M\) such that for each \(1 \leq m \leq s\) and any \(j_m \leq i \leq k_m\), one has 
\[ d(\varphi^i(x), \varphi^j(x_m)) \leq \delta. \]

For an NDS \((M, \varphi_{1,\infty})\) with compact metric space \((M, d)\), we define Bowen-metrics 
\[ d_{k,n}(x,y) := \max_{0 \leq i \leq n} d(\varphi^i_k(x), \varphi^i_k(y)) \text{ for } k \in \mathbb{N} \text{ and } n \geq 0 \quad (2.2) \]
on \(M\). Also, for any \(k \in \mathbb{N}, n \geq 0, x \in M\) and \(\epsilon > 0\), we define 
\[ B(x, \varphi^n_k, \epsilon) := B(x, k, n, \epsilon) := \{ y \in M : d_{k,n}(x,y) < \epsilon \}, \quad (2.3) \]
which is called a dynamical \((n + 1)\)-ball with initial state \(k\).

Fix an NDS \((M, \varphi_{1,\infty})\) and \(k \geq 1\). Based on [19], for the NDS \((M, \varphi_{k,\infty})\), the topological entropy \(h_{\text{top}}(M, \varphi_{k,\infty})\) is defined as follows. A family \(\mathcal{A}\) of subsets of \(M\) is called a cover (of \(M\)) if their union is all of the space \(M\). For open covers \( \mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \) of \(M\) we denote 
\[ \bigvee_{i=1}^n \mathcal{A}_i = \mathcal{A}_1 \vee \mathcal{A}_2 \vee \ldots \vee \mathcal{A}_n = \{ A_1 \cap A_2 \cap \ldots \cap A_n : A_i \in \mathcal{A}_i \text{ for } i = 1, \ldots, n \}. \]

Note that \(\bigvee_{i=1}^n \mathcal{A}_i\) is also an open cover. For an open cover \(\mathcal{A}\) we denote \(\varphi^{-n}_i(\mathcal{A}) = \{ \varphi^{-n}_i(A) : A \in \mathcal{A} \}\) and \(\mathcal{A}_i^n = \bigvee_{j=0}^{n-1} \varphi^{-j}_i(\mathcal{A})\). For each \(j\), \(\varphi^{-j}_i(\mathcal{A})\) is an open cover, so \(\mathcal{A}_i^n\) is also an open cover. Next, we denote by \(\mathcal{N}(\mathcal{A})\) the smallest possible cardinality of a subcover chosen from \(\mathcal{A}\).

Then 
\[ h(M, \varphi_{k,\infty}, \mathcal{A}) := \limsup_{n \to \infty} \frac{1}{n} \log \mathcal{N}(\mathcal{A}_i^n) \]
is said to be the topological entropy of the NDS \((M, \varphi_{k,\infty})\) on the cover \(\mathcal{A}\). The topological entropy of the NDS \((M, \varphi_{k,\infty})\) is defined by 
\[ h_{\text{top}}(M, \varphi_{k,\infty}) := \sup \{ h(M, \varphi_{k,\infty}, \mathcal{A}) : \mathcal{A} \text{ is a open cover of } M \}. \]

For introducing entropy points we need the following extension of the definition of topological entropy. Let \(Y\) be a nonempty subset of \(M\). The set \(Y\) may not be compact or may not exhibit any kind of invariance with respect to \(\varphi_{k,\infty}\). If \(\mathcal{A}\) is a cover of \(M\) we denote by \(\mathcal{A}|_Y\) the cover \(\{ A \cap Y : A \in \mathcal{A} \}\) of the set \(Y\). Then we define the topological entropy of the NDS \((M, \varphi_{k,\infty})\) on the set \(Y\) by 
\[ h_{\text{top}}(Y, \varphi_{k,\infty}) := \sup \{ h(Y, \varphi_{k,\infty}, \mathcal{A}) : \mathcal{A} \text{ is a open cover of } M \} \]
where 
\[ h(Y, \varphi_{k,\infty}, \mathcal{A}) := \limsup_{n \to \infty} \frac{1}{n} \log \mathcal{N}(\mathcal{A}_i^n|_Y). \]

Now, we consider the other definition for topological entropy of the NDS \((M, \varphi_{k,\infty})\) which is based on separated sets and spanning sets.
A subset $E$ of the space $M$ is called $(n, \varepsilon; \varphi_{k,\infty})$-separated if for any two distinct points $x, y \in E$, $d_{k,n}(x, y) > \varepsilon$. A set $F \subset M$ $(n, \varepsilon; \varphi_{k,\infty})$-spans another set $K \subset M$ provided that for each $x \in K$ there is $y \in F$ for which $d_{k,n}(x, y) \leq \varepsilon$. For a subset $Y$ of $M$ we define $s_n(Y; \varphi_{k,\infty}; \varepsilon)$ as the maximum cardinality of an $(n, \varepsilon; \varphi_{k,\infty})$-separated set in $Y$ and $r_n(Y; \varphi_{k,\infty}; \varepsilon)$ as the smallest cardinality of a set in $Y$ which $(n, \varepsilon; \varphi_{k,\infty})$-spans $Y$. If $Y = M$ we sometime suppress $Y$ and shortly write $s_n(\varphi_{k,\infty}; \varepsilon)$ and $r_n(\varphi_{k,\infty}; \varepsilon)$. Following [19] as in the autonomous case, it can be shown that

\[
\hat{h}_{\text{top}}(Y, \varphi_{k,\infty}) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log s_n(Y; \varphi_{k,\infty}; \varepsilon) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log r_n(Y; \varphi_{k,\infty}; \varepsilon). \tag{2.4}
\]

An autonomous dynamical system $(X, f)$ is called topologically chaotic if $\hat{h}_{\text{top}}(f) > 0$, however one could consider also a non-autonomous dynamical system $(M, \varphi_{1,\infty})$ with $\hat{h}_{\text{top}}(M, \varphi_{1,\infty}) > 0$ to be topologically chaotic but we give another definition that given by Kolyada and Snoha in [19].

Let an NDS $(M, \varphi_{1,\infty})$ and open cover $\mathcal{A}$ of $M$ be given, then by [19, Lemma 4.5] the limit

\[
h^*(M, \varphi_{\infty}, \mathcal{A}) := \lim_{n \to \infty} h(M, \varphi_{n,\infty}, \mathcal{A})
\]

exists. The quantity $h^*(M, \varphi_{\infty}, \mathcal{A})$ is said to be the asymptotical topological entropy of the sequence $\varphi_{1,\infty}$ on the cover $\mathcal{A}$. Put

\[
h^*(M, \varphi_{\infty}) := \sup_{\mathcal{A}} h^*(M, \varphi_{\infty}, \mathcal{A})
\]

where the supremum is taken over all open covers $\mathcal{A}$ of $M$. In [19], Kolyada and Snoha showed that

\[
h^*(M, \varphi_{\infty}) = \lim_{n \to \infty} h_{\text{top}}(M, \varphi_{n,\infty})
\]

\[
= \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{k} \log \log s_k(\varphi_{n,\infty}; \varepsilon)
\]

\[
= \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{k} \log r_k(\varphi_{n,\infty}; \varepsilon).
\]

The quantity $h^*(M, \varphi_{\infty})$ will be said to be the asymptotical topological entropy of the sequence $\varphi_{1,\infty}$. (In the notations $h^*(M, \varphi_{\infty}, \mathcal{A})$ and $h^*(M, \varphi_{\infty})$ we use the symbol $\varphi_{\infty}$ instead of more precise $\varphi_{1,\infty}$, because these quantities do not depend on whether we consider $\varphi_{1,\infty}$ or $\varphi_{i,\infty}$ for another $i \in \mathbb{N}$.)

**Definition 2.3.** The NDS $(M, \varphi_{1,\infty})$ is said to be topologically chaotic if it has positive asymptotical topological entropy, that is $h^*(M, \varphi_{\infty}) > 0$.

3. THE SPECIFICATION PROPERTY AND TOPOLOGICAL ENTROPY OF NDSs

The notion of entropy is one of the most important objects in dynamical systems, either as a topological invariant or as a measure of complexity of the dynamics. Several notions of entropy had been introduced for non-autonomous dynamical systems in an attempt to describe its dynamical characteristics. In this section, we characterize entropy points of non-autonomous dynamical systems with the specification property and show that any non-autonomous dynamical system of surjective maps enjoying the specification property
has positive topological entropy and every point is an entropy point (Theorems 3.2 and 3.3).

3.1. Specification and entropy points. In [3], Bishi introduced entropy points for finitely generated pseudogroup actions. Also, Rodrigues and Varandas in [27] defined entropy points for finitely generated group actions. Here, we extend the definition of entropy points to non-autonomous dynamical systems.

A topological NDS $\{M, \varphi_{1,\infty}\}$ admits an entropy point $x_0$ if for any open neighbourhood $U$ of $x_0$ the equality $h_{top}(M, \varphi_{1,\infty}) = h_{top}(\text{cl}(U), \varphi_{1,\infty})$ holds. Entropy points are those for which local neighborhoods reflect the complexity of the entire dynamical system in the context of topological entropy.

In [3], Bishi proved that any finitely generated pseudogroup of homeomorphisms on a compact metric space admits an entropy point. In the following theorem, we show that any non-autonomous dynamical system given by a sequence of local homeomorphisms on a compact metric space admits an entropy point which its proof is based on [3, Theorem 2.5].

**Theorem 3.1.** For any NDS $\{M, \varphi_{1,\infty}\}$, where $\varphi_{1,\infty}$ given by a sequence of local homeomorphisms $\varphi_n$ on a compact metric space $M$, there exists a point $x_0$ and an arbitrary small open neighbourhood $U$ of $x_0$ such that

$$h_{top}(M, \varphi_{1,\infty}) = h_{top}(\text{cl}(U), \varphi_{1,\infty}).$$

In particular, the NDS $\{M, \varphi_{1,\infty}\}$ admits an entropy point.

**Proof.** The result is obvious if $h_{top}(M, \varphi_{1,\infty}) = 0$, because in this case every point is entropy point. Assume that $h_{top}(M, \varphi_{1,\infty}) > 0$ and denote by $B^k(x)$ a closed ball in $M$, centered at $x$ of radius $r = 1/k$. Using the compactness of $M$, let

$$M \subseteq B^k(x_1) \cup B^k(x_2) \cup \ldots \cup B^k(x_m)$$

for some points $x_1, x_2, \ldots, x_m \in M$. By definition of $(n, \epsilon; \varphi_{1,\infty})$-separated sets we have

$$s_n(\varphi_{1,\infty}; \epsilon) \leq s_n(B^k(x_1); \varphi_{1,\infty}; \epsilon) + s_n(B^k(x_2); \varphi_{1,\infty}; \epsilon) + \ldots + s_n(B^k(x_m); \varphi_{1,\infty}; \epsilon).$$

Notice that for any positive integer $n$ there exists $i(n, \epsilon) \in \mathbb{N}$ such that

$$s_n(B^k(x_{i(n, \epsilon)}); \varphi_{1,\infty}; \epsilon) = \max\{s_n(B^k(x_j); \varphi_{1,\infty}; \epsilon) : j = 1, 2, \ldots, m\}.$$

Therefore, $s_n(\varphi_{1,\infty}; \epsilon) \leq m.s_n(B^k(x_{i(n, \epsilon)}); \varphi_{1,\infty}; \epsilon)$.

Choose an increasing sequence of integers $\{n_j\}_{j \in \mathbb{N}}$ such that the sequence

$$\left\{\frac{1}{n_j} \log s_{n_j}(\varphi_{1,\infty}; \epsilon)\right\}_{j \in \mathbb{N}}$$

tends to

$$\limsup_{n \to \infty} \frac{1}{n} \log s_n(\varphi_{1,\infty}; \epsilon)$$

whenever $j \to \infty$. At least one element of the set $\{B^k(x_1), B^k(x_2), \ldots, B^k(x_m)\}$ appears infinitely many times in the infinite sequence $\{B^k(x_{i(n, \epsilon)})\}_{j \in \mathbb{N}}$, say $B^k(x_{i^*})$. The ball $B^k(x_{i^*})$ certainly depends on $\epsilon$, therefore we write $B^k(x_{i^*}) = B^k(x_{i^*}(\epsilon))$. Again choosing
a subsequence of the sequence \( \{n_j\}_{j \in \mathbb{N}} \), for simplicity denote it again by \( \{n_j\}_{j \in \mathbb{N}} \), we may assume that
\[
B^k(x_{i(n_j, \epsilon)}) = B^k(x_{i(\cdot, \epsilon)})
\]
for any \( j \in \mathbb{N} \). It yields
\[
\lim_{j \to \infty} \frac{1}{n_j} \log s_{n_j}(\varphi_{1,\infty}; \epsilon) \leq \lim_{j \to \infty} \frac{1}{n_j} \log m + \lim_{j \to \infty} \frac{1}{n_j} \log s_{n_j}(B^k(x_{i(n_j, \epsilon)}); \varphi_{1,\infty}; \epsilon)
\]
\[
= 0 + \lim_{j \to \infty} \frac{1}{n_j} \log s_{n_j}(B^k(x_{i(\cdot, \epsilon)}); \varphi_{1,\infty}; \epsilon)
\]
\[
= \lim_{j \to \infty} \frac{1}{n_j} \log s_{n_j}(B^k(x_{i(\cdot, \epsilon)}); \varphi_{1,\infty}; \epsilon).
\]

Now, take sequence \( \{\epsilon_p\}_{p \in \mathbb{N}} \) of positive real numbers, convergence to zero. At least one ball of the set \( \{B^k(x_1), B^k(x_2), \ldots, B^k(x_m)\} \), say \( B^k(x_*) \), appears infinitely many times in the infinite sequence \( \{B^k(x_{i_p(\epsilon_p)})\}_{p \in \mathbb{N}} \), so taking a subsequence \( \{\epsilon_p\}_{p \in \mathbb{N}} \) we get the equality \( B^k(x_{i_p(\epsilon_p)}) = B^k(x_*) \), which holds for any \( l \in \mathbb{N} \). By above equation we conclude that
\[
h_{\text{top}}(M, \varphi_{1,\infty}) = \lim_{l \to \infty} \lim_{j \to \infty} \frac{1}{n_j} \log s_{n_j}(\varphi_{1,\infty}; \epsilon_p)
\]
\[
\leq \lim_{l \to \infty} \lim_{j \to \infty} \frac{1}{n_j} \log s_{n_j}(B^k(x_*)\varphi_{1,\infty}; \epsilon_p)
\]
\[
= h_{\text{top}}(B^k(x_*), \varphi_{1,\infty}).
\]
The inequality \( h_{\text{top}}(B^k(x_*), \varphi_{1,\infty}) \leq h_{\text{top}}(M, \varphi_{1,\infty}) \) is obvious. Hence, we have
\[
h_{\text{top}}(B^k(x_*), \varphi_{1,\infty}) = h_{\text{top}}(M, \varphi_{1,\infty}).
\]
The proof is complete. \( \square \)

In the next theorem we show that for any non-autonomous dynamical system of surjective maps enjoying the specification property, every point is an entropy point.

**Theorem 3.2.** Let \((M, \varphi_{1,\infty})\) be an NDS of surjective maps that satisfies the specification property, then every point in \( M \) is an entropy point.

**Proof.** By [19, Lemma 4.5] we have \( h_{\text{top}}(M, \varphi_{1,\infty}) \leq h_{\text{top}}(M, \varphi_{k,\infty}) \) for all \( k \geq 1 \). Also, by definition and equation (2.4) we have
\[
h_{\text{top}}(M, \varphi_{k,\infty}) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s_n(\varphi_{k,\infty}; \epsilon).
\]

Now we show that for any \( z \in M \) and any open neighborhood \( V \) of \( z \) we have \( h_{\text{top}}(M, \varphi_{1,\infty}) = h_{\text{top}}(\text{cl}(V), \varphi_{1,\infty}) \). For \( \delta > 0 \) define \( W_{\delta} := \{y \in V : d(y, \partial V) > \frac{\delta}{4}\} \).

Now, fix \( \epsilon > 0 \) such that the open set \( W_{\epsilon} \) be nonempty. Assume that
- \( N(\frac{\epsilon}{4}) \geq 1 \) is given by the specification property,
- \( E := \{w_1, w_2, \ldots, w_l\} \subseteq M \) is a maximal \((n, \epsilon; \varphi_{N(\frac{\epsilon}{4})+1,\infty})\)-separated set,
- \( E' = \{w'_1, w'_2, \ldots, w'_l\} \subseteq M \) is a preimage set of \( E \) under \( \varphi_{1}^{N(\frac{\epsilon}{4})} \), i.e., \( \varphi_{1}^{N(\frac{\epsilon}{4})}(w'_i) = w_i \) for \( 1 \leq i \leq l \),
\[ y \in W_\epsilon, \text{ (notice that } W_\epsilon \neq \emptyset). \]

Let \( j_1 = k_1 = 0, j_2 = N(\frac{\epsilon}{4}) \) and \( k_2 = N(\frac{\epsilon}{4}) + n \). By definition of the specification property for each \( w'_i \in E' \) by taking \( x_1 = y \) and \( x_2 = w'_i \), there exists \( y_i \in B(y, \frac{\epsilon}{4}) \) such that \( \varphi_1^{N(\frac{\epsilon}{4})}(y_i) \in B(\varphi_1^{N(\frac{\epsilon}{4})}(w'_i), N(\frac{\epsilon}{4}) + 1, n, \frac{\epsilon}{4}) = B(w_i, N(\frac{\epsilon}{4}) + 1, n, \frac{\epsilon}{4}). \) Since \( E := \{w_1, w_2, \ldots, w_l\} \subseteq M \) is a maximal \((n, \epsilon; \varphi_{N(\frac{\epsilon}{4})+1,\infty})\)-separated set, then the set \( \{y_i\}_{i=1}^l \subseteq \text{cl}(V) \) is a \((N(\frac{\epsilon}{4}) + n, \frac{\epsilon}{2}; \varphi_{1,\infty})\)-separated set. So
\[
s_{N(\frac{\epsilon}{4})+n}(cl(V); \varphi_{1,\infty}; \frac{\epsilon}{2}) \geq s_n(\varphi_{N(\frac{\epsilon}{4})+1,\infty}; \epsilon).
\]

Thus, we have
\[
\limsup_{n \to \infty} \frac{1}{N(\frac{\epsilon}{4}) + n} \log s_{N(\frac{\epsilon}{4})+n}(cl(V); \varphi_{1,\infty}; \frac{\epsilon}{2}) \geq \limsup_{n \to \infty} \frac{1}{N(\frac{\epsilon}{4}) + n} \log s_n(\varphi_{N(\frac{\epsilon}{4})+1,\infty}; \epsilon)
\]
\[
= \limsup_{n \to \infty} \frac{1}{n} \log s_n(\varphi_{N(\frac{\epsilon}{4})+1,\infty}; \epsilon).
\]
This implies that
\[
h_{\text{top}}(M, \varphi_{1,\infty}) \geq h_{\text{top}}(cl(V), \varphi_{1,\infty}) \geq h_{\text{top}}(M, \varphi_{N(\frac{\epsilon}{4})+1,\infty}) \geq h_{\text{top}}(M, \varphi_{1,\infty}).
\]

Hence we have \( h_{\text{top}}(M, \varphi_{1,\infty}) = h_{\text{top}}(cl(V), \varphi_{1,\infty}) \), that is, every point in \( M \) is an entropy point. The proof is complete. \( \square \)

Hence, in surjective NDSs enjoying the specification property, local neighborhoods reflect the complexity of the entire dynamical system from the viewpoint of entropy theory.

### 3.2. Specification and positive topological entropy.

In this subsection we show that any non-autonomous dynamical system of surjective maps satisfying the specification property has positive topological entropy. In the other words, the specification property is enough to guarantee that any non-autonomous dynamical system of surjective maps has positive topological entropy.

**Theorem 3.3.** Let \((M, \varphi_{1,\infty})\) be a topological NDS of surjective maps which satisfies the specification property. Then, it has positive topological entropy, i.e., \( h_{\text{top}}(M, \varphi_{1,\infty}) > 0 \).

**Proof.** By the definition of topological entropy we know that
\[
h_{\text{top}}(M, \varphi_{1,\infty}) = \lim \limsup_{\epsilon \to 0} \frac{1}{n} \log s_n(\varphi_{1,\infty}; \epsilon)
\]
where the limit can be replaced by \( \sup_{\epsilon > 0} \). Hence, it is enough to prove that there exists \( \epsilon > 0 \) small so that
\[
\limsup_{n \to \infty} \frac{1}{n} \log s_n(\varphi_{1,\infty}; \epsilon) > 0.
\]

Let \( \epsilon > 0 \) be small and fixed so that there are at least two distinct \( 2\epsilon \)-separated points \( x_1, y_1 \subseteq M \), i.e., \( d(x_1, y_1) > 2\epsilon \). Take \( N(\frac{\epsilon}{4}) \geq 1 \) given by the specification property. Moreover, take \( j_1 = k_1 = 0, j_2 = k_2 = N(\frac{\epsilon}{4}) \) and consider preimages \( x_2 \) of \( x_1 \) and \( y_2 \) of \( y_1 \) under \( \varphi_1^{N(\frac{\epsilon}{4})} \), that is, \( \varphi_1^{N(\frac{\epsilon}{4})}(x_2) = x_1 \) and \( \varphi_1^{N(\frac{\epsilon}{4})}(y_2) = y_1 \). By applying the specification property for pairs \((x_1, x_2), (x_1, y_2), (y_1, x_2)\) and \((y_1, y_2)\), there are the points \( x_{1,i} \subseteq B(x_1, \frac{\epsilon}{2}) \) and \( y_{1,i} \subseteq B(y_1, \frac{\epsilon}{2}) \), for \( i = 1, 2 \), such that
It is clear that the set \( \{x_{1,1}, x_{1,2}, y_{1,1}, y_{1,2}\} \) is \((N(\frac{\epsilon}{2}), \epsilon); \varphi_{1}\) separated. In particular, it follows that \( s_{N(\frac{\epsilon}{2})}(\varphi_{1}; \epsilon) \geq 2^2 \).

Next, we take \( j_3 = k_3 = 2N(\frac{\epsilon}{2}) \) and consider preimages \( x_3 \) of \( x_1 \) and \( y_3 \) of \( y_1 \) under \( \varphi^{2N(\frac{\epsilon}{2})}_{1} \), that is, \( \varphi^{2N(\frac{\epsilon}{2})}_{1}(x_3) = x_1 \) and \( \varphi^{2N(\frac{\epsilon}{2})}_{1}(y_3) = y_1 \). By applying the specification property for triples \( (x_1, x_2, x_3), (x_1, x_2, y_3), (x_1, y_2, x_3), (x_1, y_2, y_3), (y_1, y_2, x_3), (y_1, y_2, y_3) \) and \( (y_1, x_2, x_3) \), there are the points \( x_{1,j} \in B(x_{1, \frac{\epsilon}{2}}) \) and \( y_{1,j} \in B(y_{1, \frac{\epsilon}{2}}), j = 1, \ldots, 4 \), for which the following hold:

- \( \varphi_{1}^{N(\frac{\epsilon}{2})}(x_{1,1}), \varphi_{1}^{2N(\frac{\epsilon}{2})}(x_{1,1}) \in B(x_{1, \frac{\epsilon}{2}}) \), and \( \varphi_{1}^{N(\frac{\epsilon}{2})}(x_{1,4}), \varphi_{1}^{2N(\frac{\epsilon}{2})}(x_{1,4}) \in B(y_{1, \frac{\epsilon}{2}}) \);
- \( \varphi_{1}^{N(\frac{\epsilon}{2})}(x_{1,2}) \in B(x_{1, \frac{\epsilon}{2}}) \), and \( \varphi_{1}^{2N(\frac{\epsilon}{2})}(x_{1,2}) \in B(y_{1, \frac{\epsilon}{2}}) \);
- \( \varphi_{1}^{N(\frac{\epsilon}{2})}(x_{1,3}) \in B(y_{1, \frac{\epsilon}{2}}) \), and \( \varphi_{1}^{2N(\frac{\epsilon}{2})}(x_{1,3}) \in B(x_{1, \frac{\epsilon}{2}}) \);
- \( \varphi_{1}^{N(\frac{\epsilon}{2})}(y_{1,1}), \varphi_{1}^{2N(\frac{\epsilon}{2})}(y_{1,1}) \in B(y_{1, \frac{\epsilon}{2}}) \), and \( \varphi_{1}^{N(\frac{\epsilon}{2})}(y_{1,4}), \varphi_{1}^{2N(\frac{\epsilon}{2})}(y_{1,4}) \in B(x_{1, \frac{\epsilon}{2}}) \);
- \( \varphi_{1}^{N(\frac{\epsilon}{2})}(y_{1,2}) \in B(y_{1, \frac{\epsilon}{2}}) \) and \( \varphi_{1}^{2N(\frac{\epsilon}{2})}(y_{1,2}) \in B(x_{1, \frac{\epsilon}{2}}) \);
- \( \varphi_{1}^{N(\frac{\epsilon}{2})}(y_{1,3}) \in B(x_{1, \frac{\epsilon}{2}}) \) and \( \varphi_{1}^{2N(\frac{\epsilon}{2})}(y_{1,3}) \in B(y_{1, \frac{\epsilon}{2}}) \).

It is clear that the set \( \{x_{1,1}, x_{1,2}, x_{1,3}, x_{1,4}, y_{1,1}, y_{1,2}, y_{1,3}, y_{1,4}\} \) is \((2N(\frac{\epsilon}{2}), \epsilon); \varphi_{1}\) separated. In particular, it follows that \( s_{2N(\frac{\epsilon}{2})}(\varphi_{1}; \epsilon) \geq 2^3 \).

Now, let \( n = dN(\frac{\epsilon}{2}) \) where \( d \in \mathbb{N} \). Taking \( j_1 = k_1 = 0, j_2 = k_2 = N(\frac{\epsilon}{2}), j_3 = k_3 = 2N(\frac{\epsilon}{2}), \ldots, j_d = k_d = (d-1)N(\frac{\epsilon}{2}), j_{d+1} = k_{d+1} = dN(\frac{\epsilon}{2}) \) and consider the preimages \( x_i \) of \( x_1 \) and \( y_i \) of \( y_1 \) under \( \varphi^{(i-1)N(\frac{\epsilon}{2})}_{1} \) for \( i = 2, \ldots, d+1 \), that is, \( \varphi^{(i-1)N(\frac{\epsilon}{2})}_{1}(x_i) = x_1 \) and \( \varphi^{(i-1)N(\frac{\epsilon}{2})}_{1}(y_i) = y_1 \). By repeating the previous reasoning, it follows that \( s_{dN(\frac{\epsilon}{2})}(\varphi_{1}; \epsilon) \geq 2^{d+1} \). Thus,

\[
\limsup_{n \to \infty} \frac{1}{n} \log s_n(\varphi_{1}; \epsilon) \geq \limsup_{d \to \infty} \frac{1}{dN(\frac{\epsilon}{2})} \log s_{dN(\frac{\epsilon}{2})}(\varphi_{1}; \epsilon) \geq \limsup_{d \to \infty} \frac{1}{dN(\frac{\epsilon}{2})} \log 2^{d+1} = \frac{\log 2}{N(\frac{\epsilon}{2})}.
\]

This proves that the entropy is positive and hence the proof of the theorem is completed.

As a direct consequence of Theorem 3.3 and [19, Lemma 4.5] we have the next corollary which say that every surjective NDS satisfying the specification property is topologically chaotic.

**Corollary 3.4.** Let \((M, \varphi_{1})\) be an NDS of surjective maps that satisfies the specification property. Then, \((M, \varphi_{1})\) has positive asymptotical topological entropy. Hence, the NDS \((M, \varphi_{1})\) is topologically chaotic.
As you have seen before, surjective NDSs satisfying the specification property, local neighborhoods reflect the complexity of the entire dynamical system from the viewpoint of entropy theory. Also, by Theorem 3.3, surjective NDSs enjoying the specification property have positive topological entropy. Thus, by Theorem 3.2, surjective NDSs that satisfy the specification property, local neighborhoods have positive topological entropy. More precisely, we have the following corollary.

**Corollary 3.5.** Let \((M, \varphi_{1,\infty})\) be an NDS of surjective maps satisfying the specification property. Then \(h_{\text{top}}(\text{cl}(U), \varphi_{1,\infty}) > 0\) for any \(x \in M\) and any open neighborhood \(U\) of \(x\).

## 4. Uniformly Expanding NDSs

Based on [34], a continuous transformation \(f : M \to M\) in a compact metric space \(M\) is an *expanding map* if there exist constants \(\sigma > 1\) and \(\rho > 0\) such that for every \(p \in M\) the image of the ball \(B(p, \rho)\) contains a neighborhood of the closure of \(B(f(p), \rho)\) and

\[
d(f(x), f(y)) \geq \sigma d(x, y), \text{ for every } x, y \in B(p, \rho).
\]

(4.1)

Let \(M\) be a compact Riemannian manifold and \(f : M \to M\) be a map of class \(C^1\) such that for some \(\sigma > 1\) one has that

\[
\|Df(x)v\| \geq \sigma \|v\|
\]

(4.2)

for every \(x \in M\) and every \(v \in T_x M\), where \(T_x M\) is the tangent bundle at the point \(x\). Then it is easily seen that \(f\) is an expanding map; see Example 11.2.1. of [34].

An NDS \((M, \varphi_{1,\infty})\) given by a sequence of \(C^1\) local diffeomorphisms \(\varphi_n\) is *expanding* if for every \(n \in \mathbb{N}\), the mapping \(\varphi_n\) is an expanding map with the expansion factor \(\sigma_n > 1\), this means that

\[
\|D\varphi_n(x)v\| \geq \sigma_n \|v\|
\]

(4.3)

for every \(x \in M\) and every \(v \in T_x M\), where \(T_x M\) is the tangent bundle at the point \(x\).

We recall the next statement from [34]. Let \(f\) be an expanding local diffeomorphism of class \(C^r\), \(r \geq 1\), on \(M\) with the expansion factor \(\sigma > 1\). Then, by [34, Lemma. 11.1.3], there exists \(\rho > 0\) such that, for any pre-image \(x\) of any point \(y \in M\), there exists a map \(h : B(y, \rho) \to M\) of class \(C^r\) such that \(f \circ h = \text{id}, h(y) = x\) and

\[
d(h(y_1), h(y_2)) \leq \sigma^{-1}d(y_1, y_2) \text{ for every } y_1, y_2 \in B(y, \rho).
\]

(4.4)

Mappings \(h\) as in the statement are called *inverse branches* of the local diffeomorphism \(f\) and the constant \(\rho\) is called an *injectivity constant*. Moreover, the conclusion (4.4) implies that the inverse branches are contractions, with a uniform contraction rate. Here, we will extend the usual definition of a uniformly expanding map to NDSs.

**Definition 4.1.** An NDS \((M, \varphi_{1,\infty})\) is called *uniformly expanding* if it satisfies the following conditions:

1) \((M, \varphi_{1,\infty})\) is an expanding NDS given by a sequence of expanding maps \(\varphi_n\) with expansion factors \(\sigma_n\) and injectivity constants \(\rho_n\);

2) there exists a uniform bound \(\sigma > 1\) on expansion factors \(\sigma_n\), i.e. \(\sigma_n \geq \sigma\) for each \(n \in \mathbb{N}\);

3) there exists (an injectivity constant) \(\rho > 0\) such that \(\rho_n \geq \rho\), for each \(n \in \mathbb{N}\).
Remark 4.2. By the compactness of $M$, it is easily seen that if an NDS $(M, \varphi_{1,\infty})$ is uniformly expanding with the expansion factor $\sigma > 1$, then there exists some constant $\Gamma > 1$ such that for all $n \in \mathbb{N}$, one has $\|\varphi_n\| \leq \Gamma$.

Let $(M, \varphi_{1,\infty})$ be a uniformly expanding NDS given by a sequence of expanding $C^1$ local diffeomorphisms $\varphi_n$ on a compact connected smooth manifold $M$ with a uniform expansion factor $\sigma > 1$ and injectivity constant $\rho > 0$. By definition, for every $i$, the restriction of $\varphi_i$ to each ball $B(x, \rho)$ of radius $\rho$ is injective and its image contains the closure of $B(\varphi_i(x), \rho)$. Thus, the restriction $\varphi_i$ to $B(x, \rho) \cap \varphi_i^{-1}(B(\varphi_i(x), \rho))$ is a diffeomorphism onto $B(\varphi_i(x), \rho)$, we denote by

$$h_{i,x} : B(\varphi_i(x), \rho) \to B(x, \rho)$$

the inverse branch of $\varphi_i$ at $x$. It is clear that $h_{i,x}(\varphi_i(x)) = x$ and $\varphi_i \circ h_{i,x} = id$. The second condition of Definition 4.1 implies that $h_{i,x}$ is a $\sigma^{-1}$-contraction:

$$d(h_{i,x}(z), h_{i,x}(w)) \leq \sigma^{-1}d(z, w) \text{ for every } z, w \in B(\varphi_i(x), \rho).$$

More generally, for any $k, n \geq 1$, we call the inverse branch of $\varphi_k^n$ at $x$ the composition

$$h_{k,x}^n := h_{k,x} \circ h_{k+1,\varphi_k^n(x)} \circ h_{k+2,\varphi_k^n(x)} \circ \cdots \circ h_{k+n-1,\varphi_k^n(x)} : B(\varphi_k^n(x), \rho) \to B(x, \rho).$$

Observe that $h_{k,x}^n(\varphi_k^n(x)) = x$ and $\varphi_k^n \circ h_{k,x}^n = id$. Moreover, for each $0 \leq j \leq n$ we have

$$\varphi_k^{j} \circ h_{k,x}^n = h_{k+j,\varphi_k^j(x)}^{n-j} \quad \text{and} \quad h_{k+j,\varphi_k^j(x)}^{n-j} : B(\varphi_k^j(x), \rho) \to B(\varphi_k^{j}(x), \rho).$$

Hence,

$$d(\varphi_k^{j} \circ h_{k,x}^n(z), \varphi_k^{j} \circ h_{k,x}^n(w)) \leq \sigma^{-n}d(z, w)$$

for every $z, w \in B(\varphi_k^n(x), \rho)$ and every $0 \leq j \leq n$.

Our aim of this section is to show that any uniformly expanding NDS satisfies the specification and the shadowing properties. Thus, by Theorems 3.3 and 3.2, any uniformly expanding NDS has positive topological entropy and every point is an entropy point.

4.1. Uniformly Expanding NDSs and Shadowing Property. Our aim of this subsection is to prove Theorem 4.3 which says that any uniformly expanding NDS has the shadowing property.

For an NDS $(M, \varphi_{1,\infty})$, the finite or infinite sequence $\{x_1, x_2, x_3, \cdots\} \subseteq M$ is called a $\delta$-pseudo orbit for some $\delta > 0$, if $d(\varphi_i(x_i), x_{i+1}) < \delta$, for all $i \geq 1$.

The NDS $(M, \varphi_{1,\infty})$ has the shadowing property if for every $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that for every $\delta$-pseudo orbit $\{x_1, x_2, x_3, \cdots\} \subseteq M$ there exists $y \in M$ such that for all $i \geq 1$, $d(\varphi_{i+1}^{-1}(y), x_i) < \epsilon$. In this case we say that the point $y \in M$ shadows the sequence $\{x_1, x_2, x_3, \cdots\} \subseteq M$.

The NDS $(M, \varphi_{1,\infty})$ is periodic if there is a $k \geq 1$ such that $\varphi_n = \varphi_{n+k}$ for every $n \geq 1$. The smallest $k$ with this property is called the period of $(M, \varphi_{1,\infty})$. Also, a $\delta$-pseudo orbit $\{x_1, x_2, x_3, \cdots\} \subseteq M$ is periodic if there is a $k \geq 1$ such that $x_n = x_{n+k}$ for every $n \geq 1$. The smallest $k$ with this property is the period of $\delta$-pseudo orbit $\{x_1, x_2, x_3, \cdots\} \subseteq M$.

It is clear that every orbit is a $\delta$-pseudo orbit, for every $\delta > 0$. For uniformly expanding NDSs we have a kind of converse: every pseudo-orbit is uniformly close to (we say that it is shadowed by) some orbit of the NDS:
Theorem 4.3. Let \((M, \varphi_{1,\infty})\) be a uniformly expanding NDS given by a sequence of expanding \(C^1\) local diffeomorphisms \(\varphi_n\) on a compact connected smooth manifold \(M\) with the uniform expansion factor \(\sigma > 1\) and injectivity constant \(\rho > 0\). Then, the NDS \((M, \varphi_{1,\infty})\) has the shadowing property. If in the definition of shadowing property \(\varepsilon\) is small enough, so that \(2\varepsilon \leq \rho\), then the point \(x\) is unique. Additionally, if both the pseudo orbit and the NDS \((M, \varphi_{1,\infty})\) are periodic of the same period, then \(x\) is a periodic point.

Proof. Let \(\varepsilon > 0\). It is no restriction to suppose that \(\varepsilon\) is less than \(\rho\). Fix \(\delta > 0\) so that \(\sigma^{-1}\varepsilon + \delta < \varepsilon\). Consider \(\delta\)-pseudo orbit \(\{x_1, x_2, x_3, \ldots\} \subseteq M\). For each \(n \geq 1\), let \(h_{n,x_n} : B(\varphi_n(x_n), \rho) \to B(x_n, \rho)\) be the inverse branch of \(\varphi_n\) at \(x_n\). The property (4.6) ensures that

\[
h_{n,x_n}(\text{cl}(B(\varphi_n(x_n), \varepsilon))) \subseteq \text{cl}(B(x_n, \sigma^{-1}\varepsilon)) \quad \text{for every} \quad n \geq 1.
\] (4.9)

Since \(d(x_n, \varphi_{n-1}(x_{n-1})) < \delta\) and \(\sigma^{-1}\varepsilon + \delta < \varepsilon\), it follows that

\[
h_{n,x_n}(\text{cl}(B(\varphi_n(x_n), \varepsilon))) \subseteq \text{cl}(B(\varphi_{n-1}(x_{n-1}), \varepsilon)) \quad \text{for every} \quad n \geq 1,
\] (4.10)

because for \(z \in h_{n,x_n}(\text{cl}(B(\varphi_n(x_n), \varepsilon)))\) we have

\[
d(z, \varphi_{n-1}(x_{n-1})) \leq d(z, x_n) + d(x_n, \varphi_{n-1}(x_{n-1})) \leq \sigma^{-1}\varepsilon + \delta < \varepsilon.
\]

We may consider the composition \(h^n = h_{1,x_1} \circ \cdots \circ h_{n-1,x_{n-1}} \circ h_{n,x_n}\) of inverse branches, then by (4.10), the compact subsets \(K_n := h^n(\text{cl}(B(\varphi_n(x_n), \varepsilon)))\) are nested. Take \(x\) in the intersection. For every \(n \geq 1\), we have that \(x \in K_n\) and so \(\varphi_1^{-n}(x)\) belongs to

\[
\varphi_1^{-1}(K_n) = \varphi_1^{-1}(h^n(\text{cl}(B(\varphi_n(x_n), \varepsilon)))) = h_{n,x_n}(\text{cl}(B(\varphi_n(x_n), \varepsilon))).
\]

By this fact and (4.9), one has that \(d(\varphi_1^{-1}(x), x_n) \leq \sigma^{-1}\varepsilon < \varepsilon\) for every \(n \geq 1\). Consequently, the NDS \((M, \varphi_{1,\infty})\) satisfies the shadowing property.

The proof of the other statements in the theorem is simple. Indeed, if \(x'\) is another point as in the conclusion of the proposition then

\[
d(\varphi_1^{-1}(x), \varphi_1^{-1}(x')) \leq d(\varphi_1^{-1}(x), x_n) + d(x_n, \varphi_1^{-1}(x')) < 2\varepsilon \quad \text{for every} \quad n \geq 1.
\]

Since \(2\varepsilon\) is an expansivity constant, it follows that \(x = x'\). Moreover, if the pseudo orbit \(\{x_1, x_2, x_3, \ldots\}\) and the NDS \((M, \varphi_{1,\infty})\) are periodic with the same period \(k\), then

\[
d(\varphi_1^{-1}(\varphi_1^k(x)), x_n) = d(\varphi_1^{-1}(\varphi_1^k(x)), x_n) = d(\varphi_1^{n+k-1}(x), x_{n+k}) < \varepsilon \quad \text{for every} \quad n \geq 1.
\]

By uniqueness, it follows \(\varphi_1^k(x) = x\). Since \(\varphi_n = \varphi_{n+k}\) for each \(n \geq 1\), then \(\varphi_1^{ik+j}(x) = \varphi_1^j(x)\), for every \(i \in \mathbb{N}\) and \(0 \leq j < k\). Hence, \(x\) is a periodic point. The proof is complete.

We say that the NDS \((M, \varphi_{1,\infty})\) has the \(h\)-shadowing property if every finite pseudo orbit shadows by a true orbit; i.e., for every \(\varepsilon > 0\) there is \(\delta = \delta(\varepsilon) > 0\) such that for every \(\delta\)-pseudo orbit \(\{x_1, x_2, \ldots, x_n\} \subseteq M\) there is \(y \in M\) such that,

\[
d(\varphi_1^{-i}(y), x_i) < \varepsilon \quad \text{for all} \quad 1 \leq i \leq n - 1 \quad \text{and} \quad \varphi_1^{-1}(y) = x_n.
\]

In [23, Thm.2.8], the authors proved that any locally expanding NDS satisfies the \(h\)-shadowing property. Hence, we have the following corollary.
Corollary 4.4. Let \((M, \varphi_{1,\infty})\) be a uniformly expanding NDS given by a sequence of expanding \(C^1\) local diffeomorphisms \(\varphi_n\) on a compact connected smooth manifold \(M\) with the uniform expansion factor \(\sigma > 1\) and injectivity constant \(\rho > 0\). Then, the NDS \((M, \varphi_{1,\infty})\) satisfies the \(h\)-shadowing property.

4.2. Uniformly Expanding NDSs and Specification Property. Our aim of this subsection is to prove Theorem 4.7 which stress that any uniformly expanding NDS enjoys the specification property. The following two lemmas will be instrumental in the proof of Theorem 4.7 below.

Lemma 4.5. Let \((M, \varphi_{1,\infty})\) be a uniformly expanding NDS given by a sequence of expanding \(C^1\) local diffeomorphisms \(\varphi_n\) on a compact connected smooth manifold \(M\) with the uniform expansion factor \(\sigma > 1\) and injectivity constant \(\rho > 0\). Then for every \(x \in M\), \(k \in \mathbb{N}, n \geq 0\) and \(0 < \epsilon \leq \rho\) we have \(\varphi^n_k(B(x, k, n, \epsilon)) = B(\varphi^n_k(x), \epsilon)\), where \(B(x, k, n, \epsilon)\) is the dynamical \((n+1)\)-ball with initial state \(k\) around \(x\) of radius \(\epsilon\) given by (2.3).

Proof. Let \(B(x, k, n, \epsilon)\) be a dynamical \((n+1)\)-ball with initial state \(k\) around \(x\). We prove that \(\varphi^n_k(B(x, k, n, \epsilon)) = B(\varphi^n_k(x), \epsilon)\). The inclusion \(\varphi^n_k(B(x, k, n, \epsilon)) \subseteq B(\varphi^n_k(x), \epsilon)\) is an immediate consequence of the definition of dynamical ball. To prove the converse, consider the inverse branch \(h^n_{k,x}: B(\varphi^n_k(x), \rho) \rightarrow B(x, \rho)\). Given any \(y \in B(\varphi^n_k(x), \epsilon)\), let \(z = h^n_{k,x}(y)\). Then, \(\varphi^n_k(z) = y\) and, by relation (4.8) we have

\[
d(\varphi^j_k(z), \varphi^j_k(x)) \leq \sigma^{j-n}d(\varphi^0_k(z), \varphi^0_k(x)) \leq d(y, \varphi^0_k(x)) < \epsilon
\]

for every \(0 \leq j \leq n\). This shows that \(z \in B(x, k, n, \epsilon)\). The proof is complete. \(\Box\)

The following formulation of the topologically exact property is now folklore and we omit its proof.

Lemma 4.6. Let \((M, \varphi_{1,\infty})\) be a uniformly expanding NDS given by a sequence of expanding \(C^1\) local diffeomorphisms \(\varphi_n\) on a compact connected smooth manifold \(M\). Then for any \(\delta > 0\) there exists \(n = n(\delta) \in \mathbb{N}\) so that \(\varphi^n_k(B(x, \delta)) = M\) for every \(x \in M\) and every \(k \in \mathbb{N}\).

Notice that lemma 4.6 also implies that each expanding \(C^1\) local diffeomorphisms \(\varphi\) on a compact connected smooth manifold \(M\) is surjective.

Theorem 4.7. Let \((M, \varphi_{1,\infty})\) be a uniformly expanding NDS given by a sequence of expanding \(C^1\) local diffeomorphisms \(\varphi_n\) on a compact connected smooth manifold \(M\) with the uniform expansion factor \(\sigma > 1\) and injectivity constant \(\rho > 0\). Then \((M, \varphi_{1,\infty})\) satisfies the specification property.

Proof. The proof of the theorem can be followed from the previous two lemmas. Fix \(\delta > 0\), without loss of generality we assume that \(\delta < \rho\). Let \(N = N(\delta)\) be given by Lemma 4.6. Now, suppose the points \(x_1, x_2, \ldots, x_s\) with \(s \geq 2\) and a sequence \(0 = j_1 \leq k_1 < j_2 \leq k_2 < \ldots < j_s \leq k_s\) of integers with \(j_{n+1} - k_n \geq N\) for \(n = 1, \ldots, s-1\) are given. By Lemma 4.5 we have

\[
\varphi^{k_i-j_i}_{j_i+1}(B(\varphi^{j_i}(x_i), j_i + 1, k_i - j_i, \delta)) = B(\varphi^{k_i-j_i}_{j_i+1}(\varphi^{j_i}_1(x_i)), \delta) \text{ for } 1 \leq i \leq s
\]
and by Lemma 4.6 we have
\[ \varphi_{k_i+1}^{j_{i+1}-k_i}(B(\varphi_{j_{i+1}}^{k_i-j_i}(\varphi_{i}^{j}(x_i)), \delta)) = M \quad \text{for } i = 1, \ldots, s - 1. \] (4.12)

Equations (4.11) and (4.12) implies that for given \( \bar{x}_s \in B(\varphi_{j_1}^{s_1}(x_s), j_1 + k_s - j_s, \delta) \), one has \( \bar{x}_s = \varphi_{k_s+1}^{j_s-k_s-1}(\bar{x}_{s-1}) \), with \( \bar{x}_{s-1} \in B(\varphi_{j_{s-1}+1}^{k_s-j_{s-1}-1}(\varphi_{1}^{j_{s-1}}(x_{s-1})), \delta) \), and then \( \bar{x}_s = \varphi_{j_s+1}^{j_s-k_s-1} \circ \varphi_{j_{s-1}+1}^{k_s-j_{s-1}-1}(\bar{x}_{s-1}) \), for some \( \bar{x}_{s-1} \in B(\varphi_{1}^{j_{s-1}}(x_{s-1}), j_{s-1} + 1, k_{s-1} - j_{s-1}, \delta) \). By induction, there exists \( \bar{x}_1 \in B(\varphi_1^{j_1}(x_1), j_1 + 1, k_1 - j_1, \delta) \), such that
\[ \bar{x}_i = \varphi_{k_i+1}^{j_i-k_i-1} \circ \varphi_{j_{i-1}+1}^{k_{i-1}-j_{i-1}-1} \circ \ldots \circ \varphi_{k_1+1}^{j_1-k_1}(\bar{x}_1) \quad \text{for } i = 2, \ldots, s. \] (4.13)

Now, by equation (4.13) it is enough to take \( x = \bar{x}_1 \), then \( x \) satisfies the definition of the specification property. \( \square \)

As a direct consequence of Theorems 4.7, 3.3, 3.2 and Corollary 3.4 one can conclude the following corollary.

**Corollary 4.8.** Let \( (M, \varphi_{1,\infty}) \) be a uniformly expanding NDS given by a sequence of expanding \( C^1 \) local diffeomorphisms \( \varphi_n \) on a compact connected smooth manifold \( M \) with the uniform expansion factor \( \sigma > 1 \) and injectivity constant \( \rho > 0 \). Then the NDS \( (M, \varphi_{1,\infty}) \) has positive topological entropy, every point is an entropy point; in particular, it is topologically chaotic.

An NDS \( (M, \varphi_{1,\infty}) \) is **topologically mixing** if for any pair of nonempty open sets \( U, V \subseteq M \), there is \( N \geq 0 \) such that for all \( n \geq N \), \( \varphi_n^j(U) \cap V \neq \emptyset \), also a topological NDS \( (M, \varphi_{1,\infty}) \) is **distal** at the point \( x \in M \) if for every \( y \neq x \), \( \liminf_{n \to \infty} d(\varphi_n^j(x), \varphi_n^j(y)) > 0 \). Finally, the topological NDS \( (M, \varphi_{1,\infty}) \) is **distal** if it is distal at any point \( x \in M \).

In [23], the authors proved that if in the NDS \( (M, \varphi_{1,\infty}) \), \( \varphi_{1,\infty} \) is a sequence of surjective maps and \( (M, \varphi_{1,\infty}) \) has the specification property, then it is topologically mixing. Also, they asserted that if \( M \) is a complete metric space with more than two elements and \( (M, \varphi_{1,\infty}) \) has the specification property, then \( (M, \varphi_{1,\infty}) \) is not distal at any point of \( M \). By this facts and Theorem 4.7 we have the following corollary.

**Corollary 4.9.** Let \( (M, \varphi_{1,\infty}) \) be a uniformly expanding NDS given by a sequence of expanding \( C^1 \) local diffeomorphisms \( \varphi_n \) on a compact connected smooth manifold \( M \) with the uniform expansion factor \( \sigma > 1 \) and injectivity constant \( \rho > 0 \). Then, we have the following statements:

1) If \( \varphi_{1,\infty} \) is a sequence of surjective maps, then the NDS \( (M, \varphi_{1,\infty}) \) is topologically mixing.

2) If \( M \) has more than two elements, then the NDS \( (M, \varphi_{1,\infty}) \) is not distal at any point of \( M \).

4.3. **Growth rate of periodic points.** In this section, we prove that for uniformly expanding NDSs the topological entropy is an upper bound on the rate of growth of the number of periodic points. For an NDS \( (M, \varphi_{1,\infty}) \), denote by \( \text{Per}_n(M, \varphi_{1,\infty}) \) the set of all periodic points of period \( n \) and let \( N(\text{Per}_n(M, \varphi_{1,\infty})) \) be its cardinality.
**Definition 4.10.** An NDS ($M, \varphi_{1,\infty}$) is called expansive, if there exists $\delta > 0$ (called an expansive constant) such that for any $x, y \in M$ with $x \neq y$, $d(\varphi_1^n(x), \varphi_1^n(y)) > \delta$ for some $n \geq 1$. Equivalently, if for $x, y \in M$, $d(\varphi_1^n(x), \varphi_1^n(y)) \leq \delta$ for all $n \geq 0$, then $x = y$.

**Theorem 4.11.** Let $(M, \varphi_{1,\infty})$ be a uniformly expanding NDS given by a sequence of expanding $C^1$ local diffeomorphisms $\varphi_i$ on a compact connected smooth manifold $M$ with the uniform expansion factor $\sigma > 1$ and injectivity constant $\rho > 0$. Then

$$\limsup_{n \to \infty} \frac{1}{n} \log \mathcal{N}(\text{Per}_n(M, \varphi_{1,\infty})) \leq h_{\text{top}}(M, \varphi_{1,\infty}).$$

**Proof.** Let NDS $(M, \varphi_{1,\infty})$ satisfies the assumption of the proposition. By Propositions 7.12 and 7.14 of [16], the NDS $(M, \varphi_{1,\infty})$ is expansive and $\rho$ is a constant of expansivity for it. Let $A$ be any open cover of $M$ with $\text{diam}(A) < \rho$. We claim that every element of $A^n$ contains at most one point of $\text{Per}_n(M, \varphi_{1,\infty})$. Indeed, if $x, y \in \text{Per}_n(M, \varphi_{1,\infty})$ are in the same element of $A^n$ then we have

$$d(\varphi_1^n(x), \varphi_1^n(y)) < \sigma_1^{-1} \cdots \sigma_{n-1}^{-1} \rho < \rho$$

for $i = 0, 1, \cdots, n-1$. Because, by definition for each $A \in A^n$ we have

$$\text{diam}(A) \leq \sigma_1^{-1} \sigma_2^{-1} \cdots \sigma_{n-1}^{-1} \text{diam}(A) < \rho$$

where $\sigma_i$ is the expansion factor of $\varphi_i$. Since $\varphi_1^{n+j}(x) = \varphi_1^j(x)$ and $\varphi_1^{n+j}(y) = \varphi_1^j(y)$ for each $i \geq 0$ and $0 \leq j \leq n - 1$, it follows that $d(\varphi_1^n(x), \varphi_1^n(y)) < \rho$ for every $i \geq 0$. By expansivity, this implies that $x = y$, which proves our claim. It follows that

$$\limsup_{n \to \infty} \frac{1}{n} \log \mathcal{N}(\text{Per}_n(M, \varphi_{1,\infty})) \leq \limsup_{n \to \infty} \frac{1}{n} \log \mathcal{N}(A^n) = h(M, \varphi_{1,\infty}, A).$$

Taking the limit when the diameter of $A$ goes to zero, we get the conclusion of the proposition. \hfill $\square$

## 5. Thermodynamical Properties of NDSs

For non-autonomous dynamical systems, the notion of topological pressure was introduced by Huang et al. [13]. Based on [13], one can define the topological pressure of an NDS $(M, \varphi_{1,\infty})$ analogously to the topological pressure of an autonomous dynamical system $(M, \varphi)$ ([35]). Indeed, for $\varphi_1 = \varphi_2 = \cdots = \varphi$ we get the classical definition.

Let $C(M, \mathbb{R})$ be the space of real-valued continuous functions of $M$. For $\psi \in C(M, \mathbb{R})$ and $i, n \in \mathbb{N}$ we denote $(S_{i,n} \psi)(x) = \sum_{j=0}^{n-1} \psi(\varphi_1^j(x)).$

**Definition 5.1.** For any continuous observable $\psi \in C(M, \mathbb{R})$, we define the topological pressure of an NDS $(M, \varphi_{1,\infty})$ with respect to $\psi$ by

$$P_{\text{top}}(\varphi_{1,\infty}, \psi) := \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log P_n(\varphi_{1,\infty}, \psi, \epsilon),$$

where

$$P_n(\varphi_{1,\infty}, \psi, \epsilon) := \sup_{E} \left\{ \sum_{x \in E} e^{(S_{1,n} \psi)(x)} \right\}$$

and the supremum is taken over all the sets $E$ that are $(n, \epsilon; \varphi_{1,\infty})$-separated sets.
It is clear that \( P_{\text{top}}(\varphi_{1,\infty}, 0) = h_{\text{top}}(M, \varphi_{1,\infty}) \).

The topological pressure can also be defined in terms of spanning sets and sequences of open covers. For the sake of completeness, given \( \epsilon > 0 \), we take
\[
Q_n(\varphi_{1,\infty}, \psi, \epsilon) := \inf \left\{ \sum_{x \in E} e^{S_{1,\infty}(x)} : E \text{ is a } (n, \epsilon; \varphi_{1,\infty})\text{-spanning set for } M \right\}.
\]

Let \( A \) be an open cover of \( M \). We put
\[
q_n(\varphi_{1,\infty}, \psi, A) := \inf \left\{ \sum_{B \in B} \inf_{x \in B} e^{S_{1,\infty}(x)} : B \text{ is a finite subcover of } \bigcup_{j=0}^{n-1} \varphi_1^{-j}(A) \right\}
\]
and put
\[
p_n(\varphi_{1,\infty}, \psi, A) := \inf \left\{ \sum_{B \in B} \sup_{x \in B} e^{S_{1,\infty}(x)} : B \text{ is a finite subcover of } \bigcup_{j=0}^{n-1} \varphi_1^{-j}(A) \right\}
\]
Then by [13, proposition 2.1] we have
\[
P_{\text{top}}(\varphi_{1,\infty}, \psi) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log Q_n(\varphi_{1,\infty}, \psi, \epsilon) = \lim_{k \to \infty} \limsup_{n \to \infty} \frac{1}{n} \log q_n(\varphi_{1,\infty}, \psi, A_k) = \lim_{k \to \infty} \limsup_{n \to \infty} \frac{1}{n} \log p_n(\varphi_{1,\infty}, \psi, A_k),
\]
where \( A_k \) is a sequence of open covers with \( \text{diam}(A_k) \to 0 \).

For our purpose, we provide another formula to compute the topological pressure using open covers. Given \( \epsilon > 0 \), \( i, n \in \mathbb{N} \), we say that an open cover \( U \) of \( M \) is a \((i, n, \epsilon)\)-cover if any open set \( U \in U \) has \( d_{i,n}\)-diameter smaller than \( \epsilon \), where \( d_{i,n} \) is the Bowen-metric introduced in (2.2). To obtain another characterization of the topological pressure using open covers, we need continuous potentials satisfying a regularity condition. Given \( \epsilon > 0 \), \( i, n \in \mathbb{N} \) and \( \psi \in C(M, \mathbb{R}) \) we define the variation of \( (S_{1,n} \psi) \) on dynamical balls of radius \( \epsilon \) by
\[
\text{Var}_{i,n}(\psi, \epsilon) := \sup_{d_{i,n}(x, y) < \epsilon} \left| (S_{1,n} \psi)(x) - (S_{1,n} \psi)(y) \right|.
\]
We say that \( \psi \) has uniform bounded variation on dynamical balls of radius \( \epsilon \) if there exists \( C > 0 \) so that
\[
\sup_n \text{Var}_{1,n}(\psi, \epsilon) \leq C.
\]
The potential \( \psi \) has the uniformly bounded variation property whenever there exists \( \epsilon > 0 \) so that \( \psi \) has uniform bounded variation on dynamical balls of radius \( \epsilon \). In what follows, we prove that Hölder potentials have uniform bounded variation for uniformly expanding NDSs.

**Lemma 5.2.** Let \((M, \varphi_{1,\infty})\) be a uniformly expanding NDS given by a sequence of expanding \( C^1 \) local diffeomorphisms \( \varphi_n \) on a compact connected smooth manifold \( M \) with the uniform expansion factor \( \sigma > 1 \) and injectivity constant \( \rho > 0 \). Then any Hölder continuous observable \( \psi : M \to \mathbb{R} \) satisfies the uniformly bounded variation property.
Proof. Let $\psi$ be $(K, \alpha)$-Hölder. By Lemmas 4.5 and 4.6, for any $0 < \epsilon < \frac{C}{2}$, $n \in \mathbb{N}$ and $x, y \in M$ with $d_{1,n}(x, y) < \epsilon$ we have

$$\begin{align*}
| (S_{1,n}\psi)(x) - (S_{1,n}\psi)(y) | &= | \sum_{j=0}^{n-1} \psi(\varphi_j^1(x)) - \sum_{j=0}^{n-1} \psi(\varphi_j^1(y)) | \\
&\leq \sum_{j=0}^{n-1} K d(\varphi_j^1(x), \varphi_j^1(y)) \leq \sum_{j=0}^{n-1} K \sigma^{(j-n)\alpha} d(\varphi_j^1(x), \varphi_j^1(y)) ^\alpha \\
&= \frac{K}{1 - \sigma^{-\alpha}} \epsilon^{\alpha}.
\end{align*}$$

This proves the lemma. \hfill \square

**Proposition 5.3.** Let $\psi : M \to \mathbb{R}$ be a continuous map satisfying the uniformly bounded variation property. Then the topological pressure of the NDS $(M, \varphi_{1,\infty})$ with respect to $\psi$ satisfies

$$P_{\text{top}}(\varphi_{1,\infty}, \psi) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \left( \inf_{U} \sum_{U \in \mathcal{U}} e^{(S_{1,n}\psi)(U)} \right),$$

where the infimum is taken over all the open covers $U$ of $M$ such that $U$ is a $(1, n, \epsilon)$-open cover and $(S_{1,n}\psi)(U) = \sup_{x \in U} (S_{1,n}\psi)(x)$.

**Proof.** Take $\epsilon > 0$ and $n \in \mathbb{N}$. Let $P_n(\varphi_{1,\infty}, \psi, \epsilon)$ be given by the definition of the topological pressure. For simplicity, we denote $\inf_{U} \sum_{U \in \mathcal{U}} e^{(S_{1,n}\psi)(U)}$ by $C_n(\varphi_{1,\infty}, \psi, \epsilon)$, where the infimum is taken over all the open covers $\mathcal{U}$ of $M$ such that $\mathcal{U}$ is a $(1, n, \epsilon)$-open cover. Given a $(n, \epsilon; \varphi_{1,\infty})$-maximal separated set $E$, it follows that $\mathcal{U} := \{ B(x, 1, n, \epsilon) \}_{x \in E}$ is a $(1, n, 2\epsilon)$-open cover. By the uniformly bounded variation property we have

$$\begin{align*}
(S_{1,n}\psi)(B(x, 1, n, \epsilon)) &= \sup_{z \in B(x, 1, n, \epsilon)} (S_{1,n}\psi)(z) \leq (S_{1,n}\psi)(x) + C
\end{align*}$$

for some constant $C > 0$, depending only on $\epsilon$. Consequently,

$$\lim_{n \to \infty} \frac{1}{n} \log C_n(\varphi_{1,\infty}, \psi, 2\epsilon) \leq \limsup_{n \to \infty} \frac{1}{n} \log P_n(\varphi_{1,\infty}, \psi, \epsilon). \quad (5.1)$$

On the other hand, if $\mathcal{U}$ is $(1, n, \epsilon)$-open cover, for any $(n, \epsilon; \varphi_{1,\infty})$-separated set $E$, we have that $\mathcal{N}(E) \leq \mathcal{N}(\mathcal{U})$, since the diameter of any $U \in \mathcal{U}$ in the metric $d_{1,n}$ is less than $\epsilon$. By the uniformly bounded variation property, we get that

$$\limsup_{n \to \infty} \frac{1}{n} \log P_n(\varphi_{1,\infty}, \psi, \epsilon) \leq \limsup_{n \to \infty} \frac{1}{n} \log C_n(\varphi_{1,\infty}, \psi, \epsilon). \quad (5.2)$$

Now, combining equations (5.1) and (5.2), we get that

$$\begin{align*}
\limsup_{n \to \infty} \frac{1}{n} \log P_n(\varphi_{1,\infty}, \psi, \epsilon) &\leq \limsup_{n \to \infty} \frac{1}{n} \log C_n(\varphi_{1,\infty}, \psi, \epsilon) \\
&\leq \limsup_{n \to \infty} \frac{1}{n} \log P_n(\varphi_{1,\infty}, \psi, \frac{\epsilon}{2}).
\end{align*}$$

This completes the proof. \hfill \square
By the previous results, the topological pressure of an NDS can be computed as the limiting complexity of the NDS as the size scale \( \epsilon \) approaches zero. In what follows, we will be mostly interested in providing conditions for the topological pressure of an NDS to be computed as a limit at a definite size scale. Hence, we give the following definition.

**Definition 5.4.** Given \( \delta > 0 \), the NDS \( (M, \varphi_{1,\infty}) \) is \( \delta^* \)-expansive if for any \( \gamma > 0 \) and any \( x, y \in M \) with \( d(x, y) \geq \gamma \), there exists \( k_0 \geq 1 \) (depending on \( \gamma \)) such that \( d_{i,n}(x, y) > \delta \) for each \( i, n \in \mathbb{N} \) with \( n \geq k_0 \). Also, an NDS is called \( * \)-expansive if it is \( \delta^* \)-expansive for some \( \delta > 0 \).

In the rest of this section, we prove that the topological pressure of a \( * \)-expansive NDS can be computed as the topological complexity that is observable at a definite size scale. More precisely, we have the following theorem.

**Theorem 5.5.** Assume that the NDS \( (M, \varphi_{1,\infty}) \) is \( \delta^* \)-expansive for some \( \delta > 0 \). Then, for every continuous potential \( \psi : M \to \mathbb{R} \) with the uniformly bounded variation property and every \( 0 < \epsilon < \delta \),

\[
P_{\text{top}}(\varphi_{1,\infty}, \psi) = \limsup_{n \to \infty} \frac{1}{n} \log P_n(\varphi_{1,\infty}, \psi, \epsilon) = \limsup_{n \to \infty} \frac{1}{n} \log \left( \sup_{E} \sum_{x \in E} e^{(S_{1,n}\psi)(x)} \right),
\]

where the supremum is taken over all the sets \( E \) that are \((n, \epsilon; \varphi_{1,\infty})\)-separated sets.

**Proof.** Since \( M \) is compact and \( \psi : M \to \mathbb{R} \) is continuous, without loss of generality, we assume that \( \psi \) is non-negative. Fix \( \gamma \) and \( \epsilon \) with \( 0 < \gamma < \epsilon < \delta \). We use the definition of topological pressure in terms of separated sets to get the following inequality

\[
\limsup_{n \to \infty} \frac{1}{n} \log P_n(\varphi_{1,\infty}, \psi, \gamma) \geq \limsup_{n \to \infty} \frac{1}{n} \log P_n(\varphi_{1,\infty}, \psi, \epsilon).
\]

Hence, it suffices to show that

\[
\limsup_{n \to \infty} \frac{1}{n} \log P_n(\varphi_{1,\infty}, \psi, \gamma) \leq \limsup_{n \to \infty} \frac{1}{n} \log P_n(\varphi_{1,\infty}, \psi, \epsilon).
\]

By definition of \( \delta^* \)-expansivity, for any two distinct points \( x, y \in M \) with \( d(x, y) \geq \gamma \), there exists \( k_0 \geq 1 \) (depending on \( \gamma \)) such that \( d_{i,n}(x, y) > \delta \) for each \( i, n \in \mathbb{N} \) with \( n \geq k_0 \). Take \( n, k \geq k_0 \). Given any \((n, \gamma; \varphi_{1,\infty})\)-separated set \( E \), we claim that the set \( E \) is \((n+k, \epsilon; \varphi_{1,\infty})\)-separated. In fact, given \( x, y \in E \) there exists a \( 0 \leq j \leq n \) so that \( d(\varphi_{1}^{j}(x), \varphi_{1}^{j}(y)) \geq \gamma \). Using that \( n+k-j \geq k_0 \) and definition of \( \delta^* \)-expansivity, it follows that \( d_{j+1,n+k-j}(\varphi_{1}^{j}(x), \varphi_{1}^{j}(y)) > \delta > \epsilon \). This implies that \( d_{1,n+k}(x, y) > \epsilon \). Hence, \( E \) is \((n+k, \epsilon; \varphi_{1,\infty})\)-separated set. This proves the claim. Now, using that \( \psi \) is non-negative,

\[
e^{(S_{1,n}\psi)(x)} = e^{(S_{1,n}\psi)(x)} e^{(S_{n+1,k}\psi)(\varphi_{1}^{j}(x))} \geq e^{(S_{1,n}\psi)(x)},
\]

which implies that

\[
P_n(\varphi_{1,\infty}, \psi, \gamma) \leq P_{n+k}(\varphi_{1,\infty}, \psi, \epsilon).
\]
Hence, we have
\[
\limsup_{n \to \infty} \frac{1}{n} \log P_n(\varphi_{1,\infty}, \psi, \gamma) \leq \limsup_{n \to \infty} \frac{1}{n+k} \log P_{n+k}(\varphi_{1,\infty}, \psi, \epsilon) \leq \limsup_{n \to \infty} \frac{1}{n} \log P_n(\varphi_{1,\infty}, \psi, \epsilon).
\]
as we wanted to prove. \qed

**Remark 5.6.** We observe that in view of the previous characterization given in Proposition 5.3, the same result as Theorem 5.5 also holds if we consider open covers instead of separated sets. More precisely, assume that the NDS \((M, \varphi_{1,\infty})\) is \(\delta^s\)-expansive for some \(\delta > 0\). Then, for every continuous potential \(\psi : M \to \mathbb{R}\) with the uniformly bounded variation property and every \(0 < \epsilon < \delta\),
\[
P_{\text{top}}(\varphi_{1,\infty}, \psi) = \limsup_{n \to \infty} \frac{1}{n} \log \left( \inf_{U} \sum_{U \in \mathcal{U}} e^{\psi(S_{1,n})(U)} \right),
\]
where the infimum is taken over all open covers \(\mathcal{U}\) of \(M\) such that \(\mathcal{U}\) is a \((1, n, \epsilon)\)-open cover and \((S_{1,n})(U) = \sup_{x \in U} \psi(S_{1,n})(x)\).

**Lemma 5.7.** Let \((M, \varphi_{1,\infty})\) be a uniformly expanding NDS given by a sequence of expanding \(C^1\) local diffeomorphisms \(\varphi_n\) on a compact connected smooth manifold \(M\) with the uniform expansion factor \(\sigma > 1\) and injectivity constant \(\rho > 0\). Then \((M, \varphi_{1,\infty})\) is \(\ast\)-expansive.

**Proof.** By assumption, all inverse branches of \(\varphi_n\), for each \(n \geq 1\), are defined in balls of radius \(\rho\) and they are \(\sigma^{-1}\) contraction. Take \(\delta = \frac{\rho}{2}\). Given \(\gamma > 0\) take \(k_0 \geq 1\) (depending on \(\gamma\)) so that \(\sigma^{-k_0} \delta < \gamma\). We claim that for any \(x, y \in M\) with \(d(x, y) > \gamma\) and \(i, n \in \mathbb{N}\) with \(n \geq k_0\) we have \(d_{i,n}(x, y) > \delta\). Assume, by contradiction, that there exist \(i, n \in \mathbb{N}\) with \(n \geq k_0\) such that \(d_{i,n}(x, y) < \delta\). Then, \(d_{i,j}(x, y) \leq \sigma^{j-n} d_{i,n}(x, y)\) for every \(0 \leq j \leq n\) and so \(d(x, y) \leq \sigma^{-n} d_{i,n}(x, y) < \sigma^{-n} \delta \leq \sigma^{-k_0} \delta < \gamma\), which is a contradiction. This finishes the proof of the lemma. \qed

The next result is a consequence of the previous lemma and Theorem 5.5.

**Corollary 5.8.** Let \((M, \varphi_{1,\infty})\) be a uniformly expanding NDS given by a sequence of expanding \(C^1\) local diffeomorphisms \(\varphi_n\) on a compact connected smooth manifold \(M\). Then, for every continuous potential \(\psi : M \to \mathbb{R}\) with the uniformly bounded variation property and every \(0 < \epsilon < \delta\),
\[
P_{\text{top}}(\varphi_{1,\infty}, \psi) = \limsup_{n \to \infty} \frac{1}{n} \log P_n(\varphi_{1,\infty}, \psi, \epsilon) = \limsup_{n \to \infty} \frac{1}{n} \log \left( \sup_{E} \sum_{x \in E} e^{\psi(S_{1,n})(x)} \right),
\]
where the supremum is taken over all the sets \(E\) that are \((n, \epsilon; \varphi_{1,\infty})\)-separated sets.

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