A locus problem solved through methods from the Theory of Mechanisms

I Popescu¹, A E Romanescu² and L Sass²

¹ Departament of Mechanical Engineering, Academy of Technical Sciences, Bucuresti, 26, Bd. Dacia, 010413, Roumanie
² Faculty of Mechanical Engineering, University of Craiova, 107, Calea Bucuresti, 200478, Roumanie

E-mail: romanescualinaelena@yahoo.com

Abstract. The specialty literature provides many mathematic problems dealing with graphical or analytical solutions for loci. Many mathematic curves are actually loci of particular points. In this paper with present the results of our researches related to solving a problem of locus by using methods from the Theory of Mechanisms. The locus studied in this paper is placed on the side of an anti-parallelogram. The equivalent mechanism for which we made a synthesis is of type R-RRR-RRP-PRP (it has a leading element and three dyads). The locus consists in a curve with a “banana” shape, distorted at one at its heads, modifying its sizes when changing the length of the mechanism’s crank while preserving its shape. Two solutions were found for the analyzed problem consisting in identical but displaced curves.

1. Introduction

Many mathematic curves were studied since ancient times. Many old statues and bas-reliefs, as well as remaining ornaments carved in stone or jewels come to certify the above. In time mathematic studies addressed those curves, determining their geometric properties, graphic methods to plot them and other details. Other curves were derived as a consequence and in time they became so common that many studies were focused on them. In general the mathematicians limited their efforts to the determination of curves’ properties, in few cases even conceiving graphical methods to plot them. The books of Geometry contain many (usually simple) loci. For example in [1] some examples of classical loci are given. There were obtained ordinary geometrical elements like lines and circles, secants, tangents, perpendicular lines etc. Usually mathematic details are also accompanying them. Mechanisms are approached in [2]. They generate ellipses, parables, hyperboles, cissoids and pedals of curves. The mechanisms’ synthesis is accomplished by starting from loci and from the properties of the corresponding curves.

A conicograph mechanism obtained when starting from a locus problem is presented in [3]. The obtained mechanism has 7 elements and 10 couples of class V. It consists in a leading element with rotation movement, two dyads of type RPR and a dyad of type PRP. Examples related to plotting of ellipses, parables and hyperboles are provided. In [4] Robert Ferréol, Jacques Mandonnet and Alain Esculier give examples of various curves which represent loci of certain points. Also are provided animations of the lines generating curves through certain points. Patents relying on loci can be found too. An example in this sense is [5], where a new profile for the gears teeth are provided.
Many mathematic problems for geometric loci are solved graphically or analytical in [6]. Simple problems are used as start points and more difficult problems are afterward approached. Movements of some mechanisms used to plot geometric curves are treated in [7].

2. The locus problem
We consider the anti-parallelogram ABCD and a point E on CD. From E the line EF is plotted (F belongs to the line AF perpendicular on AD). The locus of the point G is to be determined, where G represents the intersection of the lines BC and EF, figure 1.

![Figure 1. Geometric construction](image)

3. Study of the equivalent mechanism
The articulated anti-parallelogram mechanism ABCD was built firstly, figure 2. A point E is belonging to CD is considered and a line EF is plotted. F belongs to the line AF, which becomes the ordinate while AD represents the abscissa. In order to force the movement of F along a fixed line, while preserving constant values for the lengths AB, BC, CD, AD, DE and FE, a coulisse and a rotation couple are placed in F. In order to find the point where BC intersects EF, two coulisses are placed in G (they are related by means of a rotation couple). In this way we obtained an equivalent mechanism of the locus problem.

The structural schema of the mechanism from figure 2 is depicted by figure 3. One can see that the mechanism is of type R-RRR-RRP-PRP (it has a leading element and three dyads).

![Figure 2. Equivalent mechanism of type R – RRR – RRP - PRP](image)

![Figure 3. The mechanism structure](image)
4. Relations for calculus
Starting from figure 2 we can write the following equations by using the method of contours for the
cinematic analysis of mechanisms:
\[
\begin{align*}
\begin{cases}
x_c &= AB \cos \varphi + BC \cos \alpha = x_D + DC \cos \psi \\
y_c &= AB \sin \varphi + BC \sin \alpha = DC \sin \psi \\
x_E &= x_D + DE \cos \psi \\
y_E &= DE \sin \psi \\
x_F &= x_E + EF \cos \gamma = 0 \\
y_F &= y_E + EF \sin \gamma \\
x_G &= AB \cos \varphi + BG \cos \alpha = x_E + EG \cos \gamma \\
y_G &= AB \sin \varphi + BG \sin \alpha = y_E + EG \sin \gamma
\end{cases}
\end{align*}
\]
(1)

By using equation (1) we can solve the Kinematics of the contour ABCD. The lengths of elements
are known and we must determine the angles \( \alpha \) and \( \psi \). Equation (2) can be used to determine
the position of E on CD providing that the length ED is known. \( x_E \) and \( y_E \) are obtained from equation (3).

From the equation (3) of \( x_F \) we can get \( \cos \gamma \). In order to obtain \( y_F \) we must calculate \( \sin \gamma \). A
square root is obtained, with the signs \( \pm \). Therefore two positions for the curve to be determined as locus
are obtained. Further on, from equation (4) we can determine the variable lengths BG and EG and
finally the coordinates of the point G.

5. Results
The following values were used as initial data for the mechanism’s elements: \( AB=22; BC=80; DC=AB; \)
\( x_D=BC; AD=BC; DE=DC/2; EF=95 \).

Figure 4 depicts the cinematic chain ABCD and the trajectory of the center of the rod BC, which
falls into the category “lemniscate of Bernoulli”, represented here in order to see a specific locus of this
cinematic chain.

![Figure 4](image)

**Figure 4.** Cinematic chain ABCD and the trajectory of the center of the rod BC

For \( q=1 \) there were obtained the locus from figure 5 and the values of the races from figure 6.
This is a curve not known from Geometry. It has a turning point in a specific area and exhibits other
properties to be studied with methods from Differential Geometry.

Such curve can be used, for example, in technical applications - e.g. for tracing operations or for
trajectories imposed by certain points from equipment in the textile industry.
Figure 5 depicts the successive positions of this cinematic chain. The successive positions of the entire mechanism are given in figure 6.

![Figure 5](image1.png)  
**Figure 5.** Successive positions of this cinematic chain ABCD

![Figure 6](image2.png)  
**Figure 6.** Successive positions of the mechanism

Figure 7 depicts the mechanism in a certain position, along with the trajectories of the centers of the rods BC and EF. One can see that the trajectory of the center of the EF rod is curve resembling to an ellipse.

![Figure 7](image3.png)  
**Figure 7.** The mechanism in a certain position and the trajectories of the centers of the rods BC and EF

![Figure 8](image4.png)  
**Figure 8.** The mechanism in the second position

Because the algebraic system provides two solutions, we also considered the alternative with opposite sign (-). We obtained figure 8, where the trajectory of the center of the rod E is similar to that corresponding to the positive sign. Its position now is under the abscissa. The images from figure 7 and figure 8 provide information on the positions of the curves obtained as locus for G, when comparing to the trajectory of the center of the parallelogram’s rod.

The curve of intersection between the rods – that is the locus to be determined, corresponding to the point G, is represented by figure 9. It is a rod curve of “banana” shape, with a small distortion at its upper head. Mathematically, the tracing point falls outside the rods (the coulisse 6 can exceed the length EF and the coulisse 7 can exceed the length BC – possible trajectories of coulisses being along the rods’ extensions). When the studies are concerned only with the mechanism, disregarding the locus problem, restrictions can be imposed such as to prevent the intersection point to fall outside the rod’s lengths. In this case the crank AB can no longer describe full rotations.
When considering the solution of the algebraic system with the sign (-), for EF=95 one can obtain a curve as that from figure 9 and figure 10, placed now overhead the abscissa, figure 10.

Figure 11 provides the variations of the intersection point’s coordinates for the curve from figure 9, (that is for the locus). One can notice a certain symmetry of these curves along some sub-ranges, reaching the limit values for $\varphi =50^\circ$. The curves are continuous, without any jumps or discontinuities.

The curves described above are associated to the data considered in the initial stage for the mechanism. Modifications of data result into similar curves, with different sizes.

Figure 12 provides examples for them. When evaluating visually these figures, knowing that the figures are resized in order to fit their place in page, one should consider that the length of the abscissa from figure 12, figure 13, and figure 14 is identical to that from figure 9.
6. Conclusions
A locus problem was solved by using the method of mechanisms’ theory. A mechanism with three dyads of type R-RRR-RRP-PRP was obtained. The points from the rods describe curves like lemniscate or ellipses and the locus was found to be a curve with a shape of banana, with a distortion at its upper head.

When resizing the mechanism, similar curves were obtained. They have different sizes. The second solution of the involved algebraic system provides the same curves, shifted with respect to the first solution.

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