Proton rms-radii from low-q power expansions?

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Several recent publications claim that the proton charge rms-radius resulting from the analysis of electron scattering data restricted to low momentum transfer agrees with the radius determined from muonic hydrogen, in contrast to the radius resulting from analyses of the full (e,e) data set which is 0.04 fm larger. Here we show why these publications erroneously arrive at the low radii.

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Introduction. The determination of the rms-radius \( R \) of the proton charge distribution has recently attracted much attention. While standard analyses of electron-proton scattering data yield 0.879 ± 0.009 fm \(^1\), the Lamb shift measurement in muonic hydrogen gave 0.8409 ± 0.0004 fm \(^2\); this represents a \( \approx 5\sigma \) discrepancy. The radii from electron scattering near 0.88 fm come from analyses that fit with excellent \( \chi^2 \) the world cross section and polarization transfer data up to large momentum transfer \( q \). 5 fm\(^{-1}\) to 12 fm\(^{-1}\) \(^3\). \(^4\). \(^5\). Recently, 3 publications \(^6\) \(^7\) \(^8\) which restrict the analysis to the low-q data, with \( q_{\text{max}} = 0.7,0.9 \) and 1.6 fm\(^{-1}\) respectively, find \( R \) in the 0.84 fm neighborhood, i.e. compatible with the radius from muonic hydrogen. In this paper, we show why these analyses, which yield values of \( R \approx 0.04 \text{ fm} \) lower than refs. \(^6\) \(^7\) have led to erroneously low values.

Power series expansion. In terms of the electric Sachs form factor \( G_e(q) \) the proton charge rms-radius \( R \) is defined via the slope of \( G_e(q^2) \) at \( q^2 = 0 \). It therefore seems natural to parameterize \( G(q) \) in a power series

\[
G_e(q) = 1 + q^2 a_2 + q^4 a_4 + q^6 a_6 + \ldots
\]

where \( R^2 = -6a_2 \). Non-relativistically, \( a_4 = \langle r^4 \rangle /120 \) and \( a_6 = -\langle r^6 \rangle /5040 \) are given by the higher moments of the charge density distribution. The rationale behind an analysis restricted to data with low maximum momentum transfer \( q_{\text{max}} \): at low enough \( q \) the terms proportional to \( q^{2n} \) with \( n > 1 \) (or in some cases \( n > 2 \)) can be neglected, so a linear (quadratic) fit of the data in terms of powers of \( q^2 \) should suffice. Low order (one parameter) fits in terms of derived functions as e.g. a dipole, \( G(q) = 1/(1+q^2 b_2)^2 \), follow the same rationale, although these parameterizations do implicitly contain higher \( q^{2n} a_{2n} \) contributions as fixed by the analytical shape of the parameterization.

Problems with expansions of the proton form factors in terms of \( q^{2n} \) have been recognized earlier \(^1\) \(^2\). Due to the peculiar shape of the proton form factor — approximately a dipole — and the peculiar shape of the corresponding charge density — approximately an exponential — the moments \( \langle r^{2n} \rangle \) for \( n \geq 2 \) grow unusually fast with increasing order \( n \). In the form factor \( G(q) \) the moments \( \langle r^{2n} \rangle \) are tightly coupled and give contributions of alternating signs. In an expansion with small \( n (n = 1,2) \) the values found for \( \langle r^{2n} \rangle \) depend on the maximum \( n \) and the value of the maximum momentum transfer \( q_{\text{max}} \) employed, and always yield too small \( \langle r^2 \rangle \). This has recently been shown by Kraus et al. \(^3\) who quantitatively demonstrate the pitfalls of fits with low order power series by analyzing pseudo-data generated with known \( R \). They show that e.g. a linear fit in \( q^2 \) with \( q_{\text{max}} = 0.7 \text{ fm}^{-1} \) as employed in \(^4\) \(^5\) produces a value of \( R \) which is low by 0.04 fm.

This result of Kraus et al. can qualitatively be understood. When terminating the series eq.(1) with the \( q^2 \)-term, one implicitly posits \( \langle r^4 \rangle = 0 \). As \( \langle r^2 \rangle \approx 0.7 \text{ fm}^2 \) this implies a charge density that is positive at small \( r \) (charge proton +e), but has a negative tail at large \( r \); due to the larger weight in the \( r^4 \)-term the tail can reduce \( \langle r^4 \rangle \) to 0. This negative tail of course also affects \( \langle r^2 \rangle \), and leads to the systematically low values of \( R \). The same happens \( \text{mutatis mutandis} \) with truncations at higher order \(^3\).

The second, obvious, problem with very low \( q \): the finite size effect (FSE) \( 1 - G_e(q) \) decreases like \( q_{\text{max}} \). Already at the \( q \approx 0.8 \text{ fm}^{-1} \) of maximal sensitivity of the data to \( R \) (see below) the FSE \( \approx q^2 R^2 /6 \) amounts to 0.09 only. The smallness of the FSE emphasizes that fits used to extract \( R \) must reach the minimal \( \chi^2_{\text{min}} \), achievable, a visually good fit is not enough: a change of \( R \) of 1% corresponds to a systematic change of \( G_e \) of only 0.0015 (0.17% of \( G_e \)), a difference that is far below the resolution of typical plots of \( G_e(q) \) \(^4\) \(^5\) \(^6\) \(^7\).

The sensitivity of the data to \( R \) is shown in Fig.1 which results from a notch test employing SOG fits of the world data (for recent reference to notch tests see \(^8\)). When exploiting only part of the range of \( q \leq 1.5 \text{ fm}^{-1} \), one loses part of the experimental information on \( R \); analyses which limit the data to e.g. \( 0.8 \text{ fm}^{-1} \) as done in refs. \(^9\) \(^10\) then ignore half of the data sensitive to \( R \). Restriction to a subset of the world data only amplifies this problem.

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Contribution of higher moments. For a more detailed discussion of the problems with eq.(1), we start from the values of $a_2, a_4, \ldots$ determined by Bernauer et al. \cite{15} via a power-series fit (with a $\chi^2$ as low as a spline fit) to the Mainz data for $q_{\text{max}} = 5 \text{fm}^{-1}$. One might hope that, due to the large $q_{\text{max}}$ and the high order $2n = 20$ employed, the values of the lowest moments of interest here should not be affected seriously by the above-mentioned problems \cite{12}. Fig. 2 shows the percent contribution of the $a_4$ to $a_{10}$ terms to the FSE. Also indicated is the uncertainty in the FSE due to a (very optimistic) uncertainty of 0.2% in the experimental $G_e(q)$.

This figure shows several features:

1. At the q’s used in the ‘low-q fits’ referred to above, with $q_{\text{max}} = 0.72 - 0.9 \text{fm}^{-1}$, the contribution of the $q^2$-term to the FSE $\approx q^2 R^2/6$ amounts to 10–15% at the upper limit of the $q$-range where FSE is most sensitive to $R$. This shows immediately and without further calculation that neglecting this contribution in a linear fit in terms of $q^2$ must yield a value of $R^2$ which is low by a comparable percentage.

2. Even the contribution of the $q^6$-term is not entirely negligible (15% of the $q^4$-term at $q = 0.9 \text{fm}^{-1}$): when attempting to determine $a_4$ from a fit quadratic in $q^2$ a wrong value results if the contribution of the $q^6$-term is not accounted for.

3. Restriction of $q_{\text{max}}$ to extremely low values, such as to justifiably neglect the $q^2$-term and maintain an accuracy of 1% in $R$, would require $q_{\text{max}} < 0.35 \text{fm}^{-1}$. At these values of $q$, the FSE is $< 0.015$, and the typical error bars of $G_e(q)$ would yield huge uncertainties in the FSE contribution, hence $R^2$ (see dashed curve).

Fig. 2 makes it obvious that the low-q fits of refs. \cite{3, 10}, which neglect the $q^4$-contribution, must find wrong values for $R$ due to the omitted $q^4$ term (for a quantitative analysis see below). Fig. 2 also shows, without further calculation, that for $q \leq 1.6 \text{fm}^{-1}$ the information content of the data is 4-5 parameters (moments) which hardly can be represented correctly by a one-parameter form-factor such as employed by Horbatsch+Hessels \cite{11} (for a quantitative discussion see below).

Higher moments from world data. As was pointed out in \cite{12} and quantitatively demonstrated in \cite{13}, the determination of the lowest moments via a power-series fit is not very reliable and for the higher $n$ dependent on the cut-off in $n$. We therefore have made an independent determination.

We use the world data up to the maximum momentum transfer available for $G_e, 10 \text{fm}^{-1}$ (not including the data of ref. \cite{15} which show systematic differences \cite{3}). This data set, which comprises 603 cross sections and polarization transfer points, is corrected for 2-photon exchange effects \cite{10} and fitted with a Fourier transform of Laguerre functions of order 11 for both $G_e(q)$ and $G_m(q)$. Laguerre functions \cite{1} are particularly well suited as

- They provide an orthonormal basis which makes multi-parameter fits very efficient (even if the polynomials are not strictly orthogonal over the limited q-range of the data).
- They have a controlled behavior at large radii $r$ due to the $e^{-r}$ weight function, a consideration which is particularly important \cite{20} when addressing higher moments (an aspect shared with the parameterizations of the Vector Dominance Model VDM).
- They provide values for the moments insensitive to the

\footnote{For similar expansions see \cite{15, 19}}
cutoff in the number of terms employed; the moments \( \langle r^{2n} \rangle \) are given by the lowest \( 2n + 3 \) coefficients.

The set of data can be reproduced with a \( \chi^2 \) of 542 with 548 degrees of freedom when the normalizations of the individual data sets are floated. When keeping the normalizations at their measured values, and \textit{without} increasing the error bars due to systematic error of the normalizations, the \( \chi^2 \) amounts to 783 with 580 degrees of freedom. These \( \chi^2 \) values are excellent given a set of data measured over some 50 years. The results for \( \langle r^4 \rangle \) are 2.01 ± 0.05 (1.99) fm\(^4\). The quality of the fit and the values of the moments are very close to the ones obtained using SOG [21] (\( \langle r^4 \rangle = 2.03 \)) or a VDM-type parameterization (\( \langle r^4 \rangle = 2.01 \)). We have verified that a variation of \( q_{\text{max}} \) between 7 and 12 fm\(^{-1}\) and a variation of \( n \) between 10 and 13 changes \( \langle r^4 \rangle \) by \( \lesssim 0.03 \) fm\(^4\). Distler et al. [22] obtained 2.59±0.19 ± 0.04 from a mix of two form factor parametrizations fit separately to low-\( q \) [15] and high-\( q \) [23] data. With these preliminaries we are in the position to quantitatively discuss the recent low-\( q \) fits.

\textbf{Fits to very-low \( q \) data.} Higinbotham et al. [10] perform a linear fit in \( q^2 \) to a subset of the data available, the form factors of Mainz80+Saskatoon74 [21] [22]. For their highest \( q_{\text{max}} \) of 0.9 fm\(^{-1}\), \( q \) \textit{rms} fit, which yields the result with the smallest uncertainty, \( R \approx 0.844 \pm 0.014 \) fm. From this the authors conclude that \( R \) agrees with the value of 0.84 fm from muonic hydrogen. When repeating exactly the same analysis, but adding in the \( q^4 \) and \( q^6 \) contributions using the higher moments from the fit to the high-\( q \) data, one finds a reduced \( \chi^2 \) (\textit{i.e.} \( \chi^2 \) per degree of freedom) which is 11% smaller and a radius \( R \) of 0.899 fm. This \( R \) disagrees with the muonic value, and agrees with the above-cited \( R \)’s in the 0.88 fm region.

Higinbotham et al. also perform a fit quadratic in \( q^2 \), and find a radius of 0.873±0.039 fm. This agrees with the radii in the 0.88 fm region, although, as the authors want to see it, the value is “within one \( \sigma \) of the muonic result”. The uncertainty of ±0.039 fm illustrates the large error bars resulting from the restriction of the analysis to a fraction of the \( q \)-region sensitive to \( R \) (see Fig.1) and the large uncertainty of \( \langle r^4 \rangle \) due to the truncation in \( q \). When using, instead of the \( \langle r^4 \rangle = 1.32 \pm 0.96 \) of Higinbotham et al., the value 2.01 ± 0.05 we know from the fit to the high-\( q \) data, the result for \( R \) becomes 0.901 fm, with a smaller error bar of 0.010 fm.

Griffioen et al. [3] analyze part of the cross sections of [3] for \( q < 0.72 \) fm\(^{-1}\) using eq.(1) including terms up to \( a_4 \). They use a low-\( q \) parameterization for \( G_m/G_e \) and take the shortcut of ignoring the free relative normalizations of the individual data set [6]. They find an \( R \) of 0.850 ± 0.019 fm and conclude that this value is consistent with the muonic hydrogen result of 0.84 fm. Repeating their fit, but using the \( a_4 \), \( a_n \)-values as given by simple models for the proton charge density (uniform, exponential, gaussian) which all produce the same \( \chi^2 \); the resulting \( R \)-values are linearly correlated with \( a_4 \). Extrapolating these values linearly to the value of \( a_4 \) given by the fit to high-\( q \) data yields \( R = 0.876 \pm 0.008 \) fm, again in agreement with the \( R \)’s in the 0.88 fm region.

The bottom line: all the low-\( q \) fits of refs. [3] [10] yield radii in the 0.88 fm region once the higher moments of the charge density — which are non-zero but ignored (or poorly fixed in the low-\( q \) fits due to the truncation of the series in \( n \) of \( q_{\text{max}} \)) — are properly accounted for.

\textbf{Fits to not-so-low \( q \) data.} Horbatsch and Hessels [11] employ the cross sections of ref.[7] up to a \( q_{\text{max}} \) of 1.6 fm\(^{-1}\). They parameterize the form factors via a 1-parameter dipole expression for both \( G_e \) and \( G_m \). Their fit yields a reduced \( \chi^2 \) of 1.11, and a \( (\text{charge}) \) \( \chi^2 \)-radius \( R = 0.842 \pm 0.002 \) fm. From this, together with other fits which yield radii near 0.89 fm, the authors conclude that \( R \) is in the range 0.84 – 0.89 fm, \textit{i.e.} could be compatible with the radius from muonic hydrogen.

Fig.2 shows that for \( q_{\text{max}} = 1.6 \) fm\(^{-1}\) the moments up to at least \( 2n = 10 \) are important to get the full FSE. It is highly unlikely that the one-parameter dipole contains the mix of \( q^{2n} \)-terms for \( 2n = 4...10 \) appropriate for the proton. Indeed, expansion of the dipole in terms of powers of \( q^2 \) shows that the numerically largest difference to the power-series fit of [12] results from the contribution of the \( \langle r^4 \rangle \) term. This difference in \( \langle r^4 \rangle \) alone would lead, at the \( q = 0.85 \) fm\(^{-1}\) of maximal sensitivity to \( R \), to a difference \( \Delta G_e \) of 0.0081 corresponding to 9.5% in the FSE, hence \( R^2 \) (causing the systematic deviations just visible in Fig.3 of [11]). The same consideration applies to the parameterization of \( G(q) \) as a \( (\text{one-parameter}) \) linear function \( 1 - cz \) with \( z = (\sqrt{t_e} - t - \sqrt{t_c})/(\sqrt{t_e} - t + \sqrt{t_c}) \) and \( t = -q^2 \). The lacking flexibility of the fit function, causing systematic differences between data and fit and a \( \chi^2 \) larger than the one of already published fits, also affects the results from the high-\( q \) fits of [3] [10].

For the fits of Horbatsch and Hessels it is not practical to correct for the effect upon \( R \) of the incorrect higher \( q^{2n} \)-terms as we did above for the analyses of refs. [3] [10]: too many terms \( 2n = 4...10 \) would contribute. In order to demonstrate the importance of their effect we rather quote the result of a Laguerre-function fit (4 terms each for \( G_e \) and \( G_m \)) to exactly the same data, yielding a lower reduced \( \chi^2 \) of 1.045 and a \( (\text{charge}) \) \( \chi^2 \)-radius \( R = 0.884 \pm 0.016 \) fm. Due to the lacking flexibility the parameterization of Horbatsch+Hessels has a \( \chi^2 \) that is
higher by 50% From such a “fit”, that is some $7 \sigma$’s away from a genuine best-fit, one obviously cannot get a significant value for $R$.

**Conclusion.** The moments $\langle r^{2n} \rangle$ of the proton for $n > 1$ are there, and they are known to be large. Ignoring their strong correlation with $R$ [9,11] leads to wrong results for the proton rms-radius.

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