Multi-Time Systems of Conservation Laws

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Abstract
Motivated by the work of P.L. Lions and J-C. Rochet [12], concerning multi-time Hamilton-Jacobi equations, we introduce the theory of multi-time systems of conservation laws. We show the existence and uniqueness of solution to the Cauchy problem for a system of multi-time conservation laws with two independent time variables in one space dimension. Our proof relies on a suitable generalization of the Lax-Oleinik formula.

1 Introduction
This paper introduces the theory of multi-time systems of conservation laws. Since to our knowledge nothing is already done in this direction, we first give the statement of the theory in Section 1.1. In order to show that the theory is well-introduced, we prove on the final section the solvability of the Cauchy problem for a system of multi-time conservation laws with two independent time variables in one space dimension. The solvability relies on a generalization of the Lax-Oleinik formula for two independent times, see Definition 3.2. Therefore, we exploit in this paper the explicit Lax formula (2.11) as solution for the multi-time Hamilton-Jacobi system (2.1), which concept was introduced by Rochet [18] in the context of mathematical economic problems.

Besides the philosophical question of the existence of multiple time dimensions, multi-time phenomena are rather common. For instance, the time schedule of networks in communication theory, as well, the traffic models with possibility to consider traffic jam leading to the use a different time scale. Indeed, processes that are assumed to start at the same configuration, and the utility function has to be solution of two different optimization problems, which coupling is just the initial data. In this direction, we address the work of Gu, Chung and Hui [8], which is related to traffic flow problems in inhomogeneous lattices.

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In fact, traffic flow seems to be one of the prelude sources of conservation laws, leading for instance to Burgers equation. Another source of interesting physical problems, where multi-time phenomena is present, comes from general relativity and electromagnetism. In this direction, we address the reader to the works of Neagu and Udriste [15] and Stickforth [19], the last one is concerned with the Kepler problem. Although, one of the most amazing example which leads to multiple dimensions, even more than two time scales, is given by the string theory, we address the reader the books of Steven [9] and Zwiebach [22]. Most of these physical problems are modelled by systems of conservation laws, here with two or more time independent scales. Finally, we have to mention that, one of the motivations to introduce multi-time conservation laws, comes from the Lions-Rochet’s paper [12], concerning multi-time Hamilton-Jacobi equations.

The mathematical theory of multi-time Hamilton-Jacobi equations was developed by P.L. Lions and J-C. Rochet [12]. In that paper Lions and Rochet showed the existence of solution for (2.7). Since then, many works have been done in the context of multi-time Hamilton-Jacobi equations to extend the results of Lions and Rochet. The existing literature goes in the direction to show existence and uniqueness of the solution for more general class of Hamiltonians and to give weaker regularity conditions on the initial data. For instance, Barles and Tourin [3] for Lipschitz initial-data, Plaskacz and Quincampoix [16] for initial-data bounded by a semi-continuous function, they present existence and uniqueness under the hypotheses (H1), (H2), (H3) in [3] and Assumption A in [16], see Remark 2.1. We address also the paper of Imbert and Volle [10], which consider a more general class of vectorial Hamilton-Jacobi equations.

For our multi-time conservation laws purpose, we were here more interested in explicit Lipschitz regular solutions for (2.7). Then, under the condition that the initial-data is Lipschitz and the Hamiltonians are convex and coercive, we give an explicit and new proof of existence for the multi-time Hamilton-Jacobi equations, using the the inf-convolution and Γ-convolution operations. We show that Lax formula (2.11) is a Lipschitz function, which solves the Cauchy problem (2.7), see Theorem 2.6. The same strategy used to prove Theorem 2.6 with small modifications, shows also that the Lax formula is a viscosity solution of (2.7) in the sense presented on Definition 2.7. Although the section on viscosity solutions of Hamilton-Jacobi equations gives known results in literature, here we organize the topics in order to give the correspondence with multi-time conservation laws. To make the paper complete on its on, we prefer to give statements and proofs, adapted to this context. By the doubling variables technic, we show that there exists at most one Lipschitz, bounded solution for (2.7), see Theorem 2.8. Hence the final Section 3 presents the existence and uniqueness solution to the Cauchy problem (3.28). First, we differentiate the Lax formula with respect to the spacial variable, and formally show that, it is the best candidate to solve (3.28). After that we establish in Lemma 3.1 a generalization of the Lax-Oleinik formula for multi-time variables. Then, we give in Definition 3.3 the exact notion of solution to (3.28), and prove the existence of an integral solution on Theorem 3.4. After that, by the BV regularity property obtained
by the Lax-Oleinik formula, we show that the integral solution is an entropy solution to the Cauchy problem (3.28) in the sense of Definition 1.2. Finally, we prove the uniqueness result on Theorem 3.6.

1.1 Statement of the theory

The aim of this section is to provide the basic theory for multi-time systems of conservation laws in multidimensional space dimensions. We are going to formulate the initial-value problem, where the systems of equations is complemented by an initial data, that is, the Cauchy problem.

Fix \( n, d \) and \( s \) be positive natural numbers. Let \( t_1, t_2, \ldots, t_n \) be \( n \)-time independent scales, and consider the points \( (t_1, \ldots, t_n, x_1, \ldots, x_d) \in \mathbb{R}^n \times \mathbb{R}^d \). In fact, for simplicity of exposition, and without loss of generality, we consider only two time scales. Moreover, we denote the spacial variable \((x_1, \ldots, x_d) = x\).

Let \( U \) be an open subset of \( \mathbb{R}^s \), usually called the set of states, where for each \((t_1, t_2, x)\)

\[ u(t_1, t_2, x) \in U, \quad (u = (u^1, \ldots, u^s)). \]

Now, let \( f_i : U \to (\mathbb{R}^s)^d \), \((i = 1, 2)\), be two smooth maps called flux functions. In general, we postulate that there exist at most \( f_i \)'s different flux functions as the number of time independent variables. Then, we are in position to establish the following multi-time system of conservation laws in general form

\[
\begin{align*}
\frac{\partial u^i}{\partial t_1} + \frac{\partial f_{1j}^i(u)}{\partial x_j} &= 0, \\
\frac{\partial u^i}{\partial t_2} + \frac{\partial f_{2j}^i(u)}{\partial x_j} &= 0,
\end{align*}
\]

where \((t_1, t_2, x) \in (0, \infty)^2 \times \mathbb{R}^d\), \(u(t_1, t_2, x) \in U\) is the unknown and \( f_1, f_2 \) are given. Moreover, we remark that the summation convention is used, that is, whenever an index is repeated once, and only once, a summation over the range of this index is performed.

**Definition 1.1.** The system \((1.1)\) is said to be hyperbolic, when for any \( u \in U \) and any direction \( \xi \in S^{d-1} \), each matrix

\[ A_{1k}^i := \frac{\partial f_{1j}^i(u)}{\partial u_k} \xi_j \quad \text{and} \quad A_{2k}^i := \frac{\partial f_{2j}^i(u)}{\partial u_k} \xi_j \quad (1 \leq i, k \leq s), \]

has \( s \) real eigenvalues \( \lambda_{i1}(u, \xi) \leq \lambda_{i2}(u, \xi) \leq \ldots \leq \lambda_{is}(u, \xi), \ (i = 1, 2) \) and is diagonalizable. Therefore, there exist \( 2s \) linearly independent right and left corresponding eigenvectors respectively \( r_i(u, \xi), l_i(u, \xi), \ (i = 1, 2), \) and

\[ A_i(u, \xi) \ r_i(u, \xi) = \lambda_i \ r_i(u, \xi) \quad \text{and} \quad l_i^T(u, \xi) \ A_i(u, \xi) = \lambda_i \ l_i(u, \xi). \]

Moreover, when the eigenvalues are all distinct the system \((1.1)\) is said strictly hyperbolic.
Hence, we formulate the Cauchy Problem: Find \( u(t_1, t_2, x) \in U \) be a function in \((0, \infty)^2 \times \mathbb{R}^d\), which satisfies the system (1.1) and moreover the initial data
\[
   u(0, 0, x) = u_0(x) \quad \text{for all} \ x \in \mathbb{R}^d, \quad (1.2)
\]
where \( u_0 : \mathbb{R}^d \to U \) is a given function.

Therefore, we have established the Cauchy problem (1.1)-(1.2) for multi-time systems of conservation laws in general form and so, many questions are in order at this point. First of all, one could ask if (1.1)-(1.2) is well-defined, since this problem seems to be overdetermined. In this direction, for \( d \) and \( s \) equals one, Lipschitz initial-data and smooth convex flux-functions, we show in Section 3 well-posedness to the Cauchy problem (1.1)-(1.2).

Last but not least, let us write \( y = (t_1, t_2, x) \) and for \( u(y) \in \mathbb{R} \), we define
\[
   F(u) := \begin{pmatrix} u & 0 & f_1(u) \\ 0 & u & f_2(u) \end{pmatrix}.
\]

Then, from equation (1.1) we have
\[
   \text{div}_y F(u) \equiv \frac{\partial F_{ij}(u)}{\partial y_j} = 0, \quad (i = 1, 2; j = 1, \ldots, d + 2). \quad (1.3)
\]

One could expect to apply the standard conservation laws theory. In this way, we have the following

**Definition 1.2.** A field \( q(u) \) is called a convex entropy flux associated with the conservation law (1.3), if there exists a continuous differentiable convex function \( \eta : \mathbb{R} \to \mathbb{R} \), such that
\[
   q_{ij}(\lambda) = \int_0^\lambda \partial_u \eta(s) \partial_u F_{ij}(s) \, ds, \quad \text{for each} \ \lambda \in \mathbb{R}.
\]

Moreover, a measurable and bounded scalar function \( u = u(y) \) is called an entropy solution of the conservation law (1.3) associated with a initial data \( u_0 \in L^\infty(\mathbb{R}^d) \), if the following entropy inequality
\[
   \iint_{\mathbb{R}^{d+2}} q_{ij}(u) \, \partial_{y_j} \phi \, dy \geq 0
\]
holds for each convex entropy flux \( q \) and all smooth test function \( \phi \) compactly supported in \((0, T)^2 \times \mathbb{R}^d\), for all \( T > 0 \), and also the initial data
\[
   \text{ess lim}_{t_1, t_2 \to 0^+} \int_{\mathbb{R}} |u(t_1, t_2, x) - u_0(x)| \, dx = 0. \quad (1.4)
\]

The main issue of the paper, it will be the existence and uniqueness result as mentioned before when \( s = d = 1 \). For that, we exploit the well known idea establish to study conservation laws (at least in one spatial dimension) from the Hamilton-Jacobi equations.
1.2 Functional notation and some results

Let \( f : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\} \). The Legendre-Fenchel conjugate of \( f \), that is, the function \( f^* : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\} \) is defined by the formula

\[
    f^*(x) := \sup_{y \in \mathbb{R}^d} \{ x \cdot y - f(y) \},
\]

where \( x \cdot y \) is the scalar product of vectors \( x, y \in \mathbb{R}^d \). We recall that, \( f^* \) is a convex function, even if \( f \) is not, and we put \( f^{**} = (f^*)^* \). If \( f \) is convex the Fenchel-Moreau theorem establishes an important duality result between \( f \) and its conjugate: if \( f \) is lower semicontinuous and convex then \( f^{**} = f \). In the following we consider proper functions. If \( f \) is coercive, i.e.

\[
    \lim_{\|x\| \to \infty} f(x) = +\infty,
\]

where \( \| \cdot \| \) is the Euclidean norm on \( \mathbb{R}^d \), then \( f^* \) is also coercive.

For a Lipschitz function \( f : \mathbb{R}^d \rightarrow \mathbb{R} \), we denote by \( \text{Lip}(f) \) the Lipschitz constant of \( f \), that is, for each \( x, y \in \mathbb{R}^d \),

\[
    |f(x) - f(y)| \leq \text{Lip}(f) \|x - y\|.
\]

Given \( f, g : \mathbb{R}^d \rightarrow \mathbb{R} \), we define (for a more general context, see Moreau [13])

\[
    f \nabla g : \mathbb{R}^d \rightarrow \mathbb{R} \quad \text{and} \quad f \square g : \mathbb{R}^d \rightarrow \mathbb{R},
\]

respectively the infimal-convolution (or inf-convolution) and gamma-convolution (or \( \Gamma \)-convolution) of \( f, g \), by

\[
    (f \nabla g)(x) = \inf_{y \in \mathbb{R}^d} \{ f(x - y) + g(y) \}
\]

and

\[
    (f \square g)(x) = (f^*(x) + g^*(x))^*.
\]

These operations are dual in the following sense

**Theorem 1.3.** Let \( f, g : \mathbb{R}^d \rightarrow \mathbb{R} \) be two convex functions. Then,

\[
    f \nabla g = f \square g.
\]

The proof could be seen at Rockafellar’s book [17], page 145, Theorem 16.4. In fact, there are also more general conditions on \( f \) and \( g \), such that these operations are identical, we address [13]. One recalls further that, infimal-convolution and gamma-convolution have the properties of commutativity and associativity.

Finally, just for completeness of the paper, let us recall the Moreau-Yosida approximation, which will be mentioned a posteriori. For each \( \tau > 0 \), the Moreau-Yosida approximation of \( f : \mathbb{R}^d \rightarrow \mathbb{R} \) is given by

\[
    f^\tau(x) := \inf_{y \in \mathbb{R}^d} \left\{ \frac{\|x - y\|^2}{2\tau} + f(y) \right\}.
\]
2 Multi-time Hamilton-Jacobi equations

We begin this section by looking to some interesting features of the multi-time Hamilton-Jacobi equations. For simplicity of explanation, we consider only two independent times. So, we will focus on the following problem: Find \( w : (0, \infty)^2 \times \mathbb{R}^d \to \mathbb{R}, \) satisfying

\[
\begin{align*}
    w_t + H_1(Dw) &= 0 \quad \text{in } (0, \infty)^2 \times \mathbb{R}^d, \\
    w_t + H_2(Dw) &= 0 \quad \text{in } (0, \infty)^2 \times \mathbb{R}^d, \\
    w(0, 0, x) &= g(x) \quad \text{on } \mathbb{R}^d,
\end{align*}
\]

where \( g : \mathbb{R}^d \to \mathbb{R} \) is a given initial datum and \( H_i : \mathbb{R}^d \to \mathbb{R} \) \((i = 1, 2)\) are given functions usually called Hamiltonians. Here, we are mostly interested in explicit solutions for \( (2.7) \) given by formulas with \( \mathbb{R}^d \) domains, since they will be exploited a posteriori in order to show solvability of multi-time conservation laws.

When \( t_1 = t_2 = t \) and hence \( H_1 = H_2 = H \), the system \( (2.7) \) turns to the usual Hamilton-Jacobi equations. In this context, we recall some well-known facts and discuss new viewpoints. We address, for instance, Alvarez, Barron and Ishii [1], Bardi and Evans [2], also Lions and Rochet [12], and references there in.

1. If \( H \) is convex and coercive, \( g \) is Lipschitz, then we have an explicit solution called the Lax formula, that is

\[
    w_L(t, x) = \inf_{y \in \mathbb{R}^d} \left\{ tH^* \left( \frac{x - y}{t} \right) + g(y) \right\}
    = \inf_{y \in \mathbb{R}^d} \left\{ (tH)^*(x - y) + g(y) \right\}
    = ((tH)^* \nabla g)(x).
\]

Therefore, the Lax formula is given by the inf-convolution operation.

2. If \( g \) is convex and \( H \) is at least continuous, satisfying

\[
    \lim_{\|p\| \to \infty} \frac{tH(p) + g^*(p)}{\|p\|} = \infty
\]

uniformly with respect to any bounded \( t \), then we have an explicit solution called the Hopf formula, that is

\[
    w_H(t, x) = (tH + g^*)^*(x),
\]

which is clearly a convex function.

These two formulas are well-known in the literature as Hopf-Lax formulas. In fact, there exists a standard habit to call Hopf-Lax formula undistinguishable between them, in spite they are not equal. For instance, a necessary condition
to have both formulas defined, it is that $H$ and $g$ should be convex (assuming that we have enough regularity). Moreover, for convex Hamiltonian the Hopf formula could be written as

$$w_H(x) = ((tH)^* \Box g)(x).$$

Hence by Theorem 1.3 we see that

$$w_L(x) = ((tH)^* \nabla g)(x) = ((tH)^* \Box g)(x) = w_H(x).$$

Consequently, $H$ and $g$ be convex are a necessary and sufficient condition to have $w_L = w_H$, besides that $H$ coercive is equivalent to condition (2.9).

Now, we turn back our attention to the (vectorial) multi-time Hamilton-Jacobi problem (2.7) and, hereafter we do not use the under scripts $L$ and $H$ respectively to Lax and Hopf formulas. Under the assumption that $g$ is convex, continuous on $\mathbb{R}^d$ and $H_i$ $(i = 1, 2)$ are continuous and satisfy (2.9), the Proposition 4 at Lions-Rochet’s paper [12], presents an explicit Hopf formula, that is to say

$$w(t_1, t_2, x) = (t_1 H_1 + t_2 H_2 + g^*)(x),$$

which solves (2.7) a.e. in $[0, T]^2 \times \mathbb{R}^d$, for $T > 0$. Although, they do not present in that paper an explicit Lax formula. Indeed, considering that $H_i$ $(i = 1, 2)$ are convex, $g$ is bounded and uniformly continuous, further $Dg$ is measurable and bounded or $H_i$ $(i = 1, 2)$ are coercive, they show on Proposition 5 the following

$$w(t_1, t_2, x) = S_{H_1}(t_1) S_{H_2}(t_2) g(x) = S_{H_2}(t_2) S_{H_1}(t_1) g(x),$$

which solves (2.7) a.e. and is Lipschitz on $\mathbb{R}^d \times [\varepsilon, T]^2$ for all $\varepsilon > 0$.

On the other hand, following our discussion above, we propose here to study the following (called) Lax formula, that is

$$w(t_1, t_2, x) = \inf_{y \in \mathbb{R}^d} \{(t_1 H_1 + t_2 H_2)^*(x - y) + g(y)\}, \quad (2.11)$$

where for our purposes, we assume that $g$ is Lipschitz in $\mathbb{R}^d$.

**Remark 2.1.** Some remarks are in order just now:

1. The regularity of $g$, i.e. Lipschitz continuous, is a natural assumption in order to show solvability of the multi-time system of conservation laws. In fact, this condition could be relaxed using the Moreau-Yosida approximation $q^\tau$ of $g$ and then, applying the same strategy used in Alvarez, Barron and Ishii [7].

2. The Lax formula (2.11) already appears, as well, in Imbert and Vollet’s paper, see [10] to study the vectorial Hamilton-Jacobi equations. Although, completed different from that paper, here we are interested to show existence, uniqueness of (2.7) and, further Lipschitz regularity of (2.11) in an explicit and computationally way, which it will be exploited in the multi-time conservation laws section.
3. If we agree with the notation \( w(t, x) = (S_H(t) g)(x) \) for (2.8), then we observe that
\[
w(t_1, t_2, x) = ((t_1 H_1 + t_2 H_2)^* \nabla g)(x) \\
= ((t_1 H_1)^* \Box (t_2 H_2)^* \nabla g)(x) \\
= ((t_1 H_1)^* \nabla (t_2 H_2)^* \nabla g)(x),
\]
which justifies the notation and commutativity in Proposition 5 at Lions and Rochet’s paper [12].

4. For simplicity, we sometimes denote \( t_1 H_1 + t_2 H_2 =: t \cdot H \) (as obvious notation) and, the Lax formula (2.11) becomes
\[
w(t_1, t_2, x) = ((t \cdot H)^* \nabla g)(x).
\]

5. Finally, we give respectively the hypotheses (H1) – (H3) on Barles and Tourin [3] and the Assumption A on Plaskacz and Quincampoix [16]:

(H1) For any \( R > 0 \), there exists a constant \( K_R > 0 \), such that
\[
|H_i(x, p)| \leq K_R \quad \text{in } \mathbb{R}^d \times \{|p| \leq R\}, i = 1, 2,
\]
\[
|D_p H_i(x, p)| \leq K_R (1 + |x|) \quad \text{a.e. in } \mathbb{R}^d \times \{|p| \leq R\}, i = 1, 2.
\]

(H2) \( H_1, H_2 \) are coercive uniformly with respect to \( x \in \mathbb{R}^d \).

(H3) \( H_1, H_2 \) are \( C^1 \) in \( \mathbb{R}^d \times \mathbb{R}^d \) and satisfy
\[
D_x H_1(x, p) D_p H_2(x, p) - D_x H_2(x, p) D_p H_1(x, p) = 0,
\]
for each \( x, p \in \mathbb{R}^d \). The equality above is always satisfied if \( H_1, H_2 \) do not depend on \( x \), further the Hamiltonians could be assumed locally Lipschitz.

Assumption A: \( H(u, p) = \tilde{H}(u, p) + \lambda(u) \), where \( \lambda(u) \) is a \( C^1 \) real scalar non-negative and non-increasing function, and \( \tilde{H} : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \), satisfy
\[
\tilde{H}(u, \cdot) \quad \text{is a concave and positively homogeneous,}
\]
\[
\tilde{H} (\cdot, p) \quad \text{is a non-increasing } C^1 \text{ function.}
\]

### 2.1 Existence

First, we show that the infimum in (2.11) is in fact a minimum, hence the infimal convolution is said exact. Moreover, \( w \) is a continuous function.

**Lemma 2.2.** Assume that \( g : \mathbb{R}^d \to \mathbb{R} \) is a Lipschitz continuous function, and let \( w \) be defined by (2.11). Then,
\[
w(t_1, t_2, x) = \min_{y \in \mathbb{R}^d} \{(t \cdot H)^*(x - y) + g(y)\}.
\]
Moreover, \( w \) is a continuous function.
Proof. By definition of infimum, there exists \( \{y_n\} \) on \( \mathbb{R}^d \) such that
\[
w(t_1, t_2, x) = \lim_{n \to \infty} \left\{ (t \cdot \mathbf{H})^*(x - y_n) + g(y_n) \right\}.
\]
If \( \{y_n\} \) has at least one convergent subsequence, we are done. Otherwise, \( \{y_n\} \) should be unbounded, which is not the case. Indeed, recall that \( H_i^* \) (i = 1, 2) are coercive, hence \( (t \cdot \mathbf{H})^* \) is also coercive. Therefore, there exist \( \lambda \) a non-negative real arbitrary number and a constant \( \beta \), such that, for \( n \) sufficiently large
\[
(t \cdot \mathbf{H})^*(x - y_n) \geq \lambda \|x - y_n\| - \beta - 1/n.
\]
Moreover, since the function \( g \) is Lipschitz continuous, we have
\[
g(y_n) \geq -\text{Lip}(g) \|y_n\| + g(0).
\]
Then, it follows by the above inequalities that
\[
(t \cdot \mathbf{H})^*(x - y_n) + g(y_n) \geq \lambda \|x - y_n\| - \text{Lip}(g) \|y_n\| + g(0) - \beta - 1/n
\]
\[
\geq \lambda (\|y_n\| - \|x\|) - \text{Lip}(g) \|y_n\| + g(0) - \beta - 1/n
\]
\[
\geq C \|y_n\| + g(0) - \beta - 1/n,
\]
where \( C \) is a positive constant (take \( \lambda > \text{Lip}(g) \)). Then, passing to the limit as \( n \to \infty \), we have a contradiction, since the infimum in (2.11) is finite. \( \square \)

The next lemma establish the semigroup property of the Lax formula.

Lemma 2.3. Let \( g : \mathbb{R}^d \to \mathbb{R} \) be a Lipschitz continuous function and \( w \) defined by (2.11). Then, for each \( 0 \leq s_i < t_i \), (i = 1, 2), and all \( x \in \mathbb{R}^d \), it follows that
\[
w(t_1, t_2, x) = \min_{y \in \mathbb{R}^d} \left\{ ((t - s) \cdot \mathbf{H})^*(x - y) + w(s_1, s_2, y) \right\}. \tag{2.12}
\]
Proof. The proof is a simple application of the inf-convolution and \( \Gamma \)-convolution operations. Indeed, we have
\[
w(t_1, t_2, x) = ((t_1 H_1 + t_2 H_2)^* \triangledown g)(x)
\]
\[
= (((t_1 - s_1) H_1 + s_1 H_1 + (t_2 - s_2) H_2 + s_2 H_2)^* \triangledown g)(x)
\]
\[
= \left( (t_1 - s_1) H_1 + (t_2 - s_2) H_2 \right)^* \square (s_1 H_1 + s_2 H_2)^* \triangledown g(x)
\]
\[
= \left( (t_1 - s_1) H_1 + (t_2 - s_2) H_2 \right)^* \triangledown (s_1 H_1 + s_2 H_2)^* \triangledown g(x),
\]
where we have used Theorem 1.3. \( \square \)

Now, we prove that \( w \) defined by (2.11) is a Lipschitz continuous function. Therefore, by Rademacher’s Theorem, see [6], differentiable almost everywhere in \( \mathbb{R}^d \) and for almost all \( t_1, t_2 > 0 \).
Lemma 2.4. The function \( w \) defined by (2.11) is Lipschitz in \([0, \infty)^2 \times \mathbb{R}^d \). Moreover, we have

\[
\lim_{t_1, t_2 \to 0} w(t_1, t_2, x) = g(x) \quad \text{on} \quad \mathbb{R}^d. \tag{2.13}
\]

Proof. 1. First, fix \( t_1, t_2 > 0 \) and \( x, x_0 \in \mathbb{R}^d \). Choose \( y \in \mathbb{R}^d \), such that

\[
w(t_1, t_2, x) = (t_1 H_1 + t_2 H_2)^* (x - y) + g(y).
\]

Thus we have

\[
w(t_1, t_2, x_0) - w(t_1, t_2, x) = \min_{z \in \mathbb{R}^d} \{ (t_1 H_1 + t_2 H_2)^* (x - z) + g(z) \}
- (t_1 H_1 + t_2 H_2)^* (x - y) - g(y)
\leq g(x_0 - x + y) - g(y) \leq \text{Lip}(g) \| x_0 - x \|,
\]

where we have used \( z = x_0 - x + y \). Now, reverting \( x_0 \) and \( x \) in the above, we obtain

\[
|w(t_1, t_2, x) - w(t_1, t_2, x_0)| \leq \text{Lip}(g) \| x - x_0 \|, \tag{2.14}
\]

that is, \( w(t_1, t_2, x) \) is Lipschitz with respect to the spacial variable \( x \in \mathbb{R}^d \).

2. Since \( g \) is Lipschitz continuous, for each \( x, y \in \mathbb{R}^d \), we have

\[
g(y) \geq g(x) - \text{Lip}(g) \| x - y \|.
\]

Therefore, by definition of \( w(t_1, t_2, x) \), we obtain

\[
g(x) - w(t_1, t_2, x) \leq \max_{y \in \mathbb{R}^d} \{ \text{Lip}(g) \| x - y \| - (t_1 H_1 + t_2 H_2)^* (x - y) \}
\leq \max_{z \in \mathbb{R}^d} \{ \max_{\xi \in B_{\text{Lip}(g)}(0)} z \cdot \xi - (t_1 H_1 + t_2 H_2)^* (z) \}
\leq \max_{\xi \in B_{\text{Lip}(g)}(0)} (t_1 H_1 + t_2 H_2)(\xi).
\]

On the other hand, taking \( x = y \) in the definition of \( w(t_1, t_2, x) \), it follows that

\[
w(t_1, t_2, x) - g(x) \leq (t_1 H_1 + t_2 H_2)^* (0).
\]

Consequently, we obtain

\[
\inf_{\xi \in \mathbb{R}^d} (t \cdot \mathbf{H})(\xi) \leq g(x) - w(t_1, t_2, x) \leq \max_{\xi \in B_{\text{Lip}(g)}(0)} (t \cdot \mathbf{H})(\xi). \tag{2.16}
\]

Furthermore, passing to the limit as \( t_1, t_2 \to 0 \), we obtain (2.13).

3. Finally, we show that \( w \) is Lipschitz continuous with respect to the time variables. Fix \( 0 < s_i < t_i \) (\( i = 1, 2 \)) and \( x \in \mathbb{R}^d \). By (2.14) for each \( t_1, t_2 \), we have

\[
\text{Lip}(w(t_1, t_2, \cdot)) \leq \text{Lip}(g).
\]

Then, we apply the semigroup property of the Lax formula given by Lemma 2.3 and, moreover proceed similarly as we have done in step 2 above. Hence the result follows. \( \square \)
To end up this section, let us show that \(2.11\) solves the multi-time Hamilton-Jacobi partial differential equation in \((2.7)\), wherever \(w\) is differentiable. One recalls that, the initial-data is shown by Lemma 2.4.

Lemma 2.5. Let \((t_1, t_2, x) \in (0, \infty)^2 \times \mathbb{R}^d\) be a differentiable point for the multitime Lax formula given by \((2.11)\). Then,

\[
\begin{align*}
\partial_{t_1} w(t_1, t_2, x) + H_1(Dw(t_1, t_2, x)) &= 0, \\
\partial_{t_2} w(t_1, t_2, x) + H_2(Dw(t_1, t_2, x)) &= 0.
\end{align*}
\]

Proof. Let us show the first differential equality, the second is similar. First, by the semigroup property, we have

\[w(t_1, t_2, x) \leq (t_1 - s_1)H_1(x - y) + w(s_1, t_2, y),\]  \tag{2.17}

where we have used \(0 < s_2 = t_2, 0 < s_1 < t_1\) and \(y \in \mathbb{R}^d\). Take \(\delta > 0, q \in \mathbb{R}^d\) fixed, and replace in \((2.17)\) \(s_1 \mapsto t_1, t_1 \mapsto t_1 + \delta, y \mapsto x\) and \(x \mapsto x + \delta q\), thus we have

\[w(t_1 + \delta, t_2, x + \delta q) - w(t_1, t_2, x) \leq \delta H_1^*(q).
\]

Then, dividing by \(\delta\) and letting to \(0^+\), we obtain

\[w_{t_1}(t_1, t_2, x) + q \cdot Dw(t_1, t_2, x) - H_1^*(q) \leq 0.
\]

Consequently, by the above inequality, it follows that

\[w_{t_1}(t_1, t_2, x) + \max_{p \in \mathbb{R}^d} \{p \cdot Dw(t_1, t_2, x) - H_1^*(p)\} \leq 0,
\]

which implies

\[w_{t_1}(t_1, t_2, x) + H_1(Dw(t_1, t_2, x)) \leq 0.
\]

Now choose \(z \in \mathbb{R}^d\) such that

\[w(t_1, t_2, x) = (t_1H_1 + t_2H_2)^*(x - z) + g(z).
\]

Fix \(\delta > 0\) and conveniently set \(t_1 = s_1 + \delta, y = \frac{t_1 - \delta}{t_1} x + \frac{\delta}{t_1} z\), so \(\frac{x - z}{t_1} = \frac{y - z}{s_1}\).

Therefore, by definition of \(w(s_1, t_2, y),\) we obtain

\[w(t_1, t_2, x) - w(s_1, t_2, y) \geq (t_1H_1 + t_2H_2)^*(x - z) - (s_1H_1 + t_2H_2)^*(y - z) \geq \delta H_1^*(\frac{x - z}{t_1}).
\]

Then, passing to the limit as \(\delta \to 0^+\) after divide by \(\delta\), we obtain

\[w_{t_1}(t_1, t_2, x) + \frac{x - z}{t_1} \cdot Dw(t_1, t_2, x) - H_1^*(\frac{x - z}{t_1}) \geq 0. \tag{2.18}
\]
Finally, we have by (2.18)
\begin{align*}
w_t_1(t_1, t_2, x) + H_1(Dw(t_1, t_2, x)) &= w_t_2(t_1, t_2, x) \\
&+ \max_{q \in \mathbb{R}^d} \{ q \cdot Dw(t_1, t_2, x) - H_1^*(q) \} \\
&\geq w_t_2(t_1, t_2, x) \\
&+ \frac{x - z}{t_1} \cdot Dw(t_1, t_2, x) - H_1^\ast \left( \frac{x - z}{t_1} \right) \geq 0.
\end{align*}

Consequently, we have proved in this section the following

**Theorem 2.6.** Let \( w \) be the Lax formula given by (2.11). Then, \( w \) is Lipschitz continuous, is differentiable a.e. in \((0, \infty)^2 \times \mathbb{R}^d\), and solves the multi-time Hamilton-Jacobi initial-value problem

\[
\begin{align*}
w_t_1 + H_1(Dw) &= 0 \quad \text{a.e. in } (0, \infty)^2 \times \mathbb{R}^d, \\
w_t_2 + H_2(Dw) &= 0 \quad \text{a.e. in } (0, \infty)^2 \times \mathbb{R}^d, \\
w(0, 0, x) &= g(x) \quad \text{on } \mathbb{R}^d.
\end{align*}
\]

**Definition 2.7.** A continuous function \( w : (0, \infty)^2 \times \mathbb{R}^d \rightarrow \mathbb{R} \) is called:

- **A viscosity subsolution of the initial-value problem** (2.7), provided
  \[
  w(0, 0, \cdot) = g(\cdot) \quad \text{on } \mathbb{R}^d
  \]
  and for each \( \phi \in C^1((0, \infty)^2 \times \mathbb{R}^d) \) if \( w - \phi \) has a local maximum in \((\tau_1, \tau_2, \xi) \in (0, \infty)^2 \times \mathbb{R}^d\), then
  \[
  \phi_1(\tau_1, \tau_2, \xi) + H_1(D\phi(\tau_1, \tau_2, \xi)) \leq 0,
  \]
  \[
  \phi_2(\tau_1, \tau_2, \xi) + H_2(D\phi(\tau_1, \tau_2, \xi)) \leq 0.
  \]

- **A viscosity supersolution of the initial-value problem** (2.7), provided
  \[
  w(0, 0, \cdot) = g(\cdot) \quad \text{on } \mathbb{R}^d
  \]
  and for each \( \phi \in C^1((0, \infty)^2 \times \mathbb{R}^d) \) if \( w - \phi \) has a local minimum in \((\tau_1, \tau_2, \xi) \in (0, \infty)^2 \times \mathbb{R}^d\), then
  \[
  \phi_1(\tau_1, \tau_2, \xi) + H_1(D\phi(\tau_1, \tau_2, \xi)) \geq 0,
  \]
  \[
  \phi_2(\tau_1, \tau_2, \xi) + H_2(D\phi(\tau_1, \tau_2, \xi)) \geq 0.
  \]
Moreover, $w$ is said a viscosity solution of \(2.7\), when it is both a viscosity supersolution and a viscosity subsolution of \(2.7\).

One observes that, with a similar strategy used before, it is not difficult to show that $w$ given by \(2.11\) is a viscosity subsolution and also a viscosity supersolution of \(2.7\). Then, by definition it is a viscosity solution of \(2.7\).

### 2.2 Uniqueness

In this section using the idea of doubling variables, see for instance Kruzkov [11], Crandall, Evans and Lions [5], we show the uniqueness of bounded Lipschitz solutions for the initial-value problem \(2.7\).

**Theorem 2.8.** Assume that the initial-data $g$ is a bounded Lipschitz function, $H_i$ ($i = 1, 2$) are convex and coercive. Then, there exists at most one Lipschitz, bounded viscosity solution of \(2.7\).

**Proof.** 1. Let $\alpha$ be a positive real number, defined as

$$\alpha := \sup_{[0, +\infty) \times \mathbb{R}^d} (w - \tilde{w}),$$

where $w$ and $\tilde{w}$ are two Lipschitz, bounded solutions of \(2.7\) with the same initial-data. Now, we choose $0 < \epsilon, \lambda_1, \lambda_2 < 1$ and define the function $\Theta$ as

$$\Theta(t_1, t_2, s_1, s_2, x, y) := w(t_1, t_2, x) - \tilde{w}(s_1, s_2, y) - \rho_{\epsilon, \lambda_1, \lambda_2}(t_1, t_2, s_1, s_2, x, y)$$

for each $t_i, s_i \geq 0$ ($i = 1, 2$) and $x, y \in \mathbb{R}^d$, where

$$\rho_{\epsilon, \lambda_1, \lambda_2}(t_1, t_2, s_1, s_2, x, y) := \frac{\lambda_1}{2} (t_1 + s_1) + \frac{\lambda_2}{2} (t_2 + s_2)$$

$$+ \epsilon^2 \left( (t_1 - s_1)^2 + (t_2 - s_2)^2 + \|x - y\|^2 \right) + \epsilon \left( \|x\|^2 + \|y\|^2 \right)$$

So, as

$$\lim_{\|(t_1, t_2, s_1, s_2, x, y)\| \to +\infty} \rho_{\epsilon, \lambda_1, \lambda_2}(t_1, t_2, s_1, s_2, x, y) = +\infty,$$

we have

$$\lim_{\|(t_1, t_2, s_1, s_2, x, y)\| \to +\infty} \Theta(t_1, t_2, s_1, s_2, x, y) = -\infty$$

and, as the function $\Theta$ is continuous in its domain, and it is proper (not identically $\pm \infty$), it there must be a point of maximum, i.e., there exists a point $(\hat{t}_1, \hat{t}_2, \hat{s}_1, \hat{s}_2, \hat{x}, \hat{y}) \in [0, +\infty)^4 \times \mathbb{R}^{2d}$, such that

$$\Theta(\hat{t}_1, \hat{t}_2, \hat{s}_1, \hat{s}_2, \hat{x}, \hat{y}) = \max_{[0, +\infty)^4 \times \mathbb{R}^d} \Theta(t_1, t_2, s_1, s_2, x, y). \quad (2.20)$$

2. From \(2.20\), the map
has a maximum in \((\hat{t}_1, \hat{t}_2, \hat{x})\). If we write \(\Theta\) as
\[
\Theta(t_1, t_2, \hat{s}_1, \hat{s}_2, x, \hat{y}) = w(t_1, t_2, x) - v(t_1, t_2, x),
\]
where
\[
v(t_1, t_2, x) := \bar{w}(\hat{s}_1, \hat{s}_2, \hat{y}) + \rho_{\epsilon, \lambda_1, \lambda_2}(t_1, t_2, \hat{s}_1, \hat{s}_2, x, \hat{y})
\]
then, \((w - v)\) has a maximum in \((\hat{t}_1, \hat{t}_2, \hat{x})\). Since \(w\) is a viscosity solution of (2.21), it follows that
\[
\begin{align*}
 v_1(\hat{t}_1, \hat{t}_2, \hat{x}) + H_1(Dv(\hat{t}_1, \hat{t}_2, \hat{x})) &\leq 0, \\
v_2(\hat{t}_1, \hat{t}_2, \hat{x}) + H_2(Dv(\hat{t}_1, \hat{t}_2, \hat{x})) &\leq 0.
\end{align*}
\]
Now, using the definition of \(v\) we obtain
\[
\begin{align*}
\frac{\lambda_1}{2} + \epsilon^2(\bar{t}_1 - \hat{s}_1) + H_1 \left( \frac{2}{\epsilon^2}(\bar{x} - \hat{y}) + 2\epsilon \hat{x} \right) &\leq 0, \\
\frac{\lambda_2}{2} + \epsilon^2(\bar{t}_2 - \hat{s}_2) + H_2 \left( \frac{2}{\epsilon^2}(\bar{x} - \hat{y}) + 2\epsilon \hat{x} \right) &\leq 0.
\end{align*}
\]
(2.21)

Analogously, the map
\[
(s_1, s_2, y) \mapsto -\Theta(\hat{t}_1, \hat{t}_2, s_1, s_2, \hat{x}, \hat{y})
\]
has a minimum in \((\hat{s}_1, \hat{s}_2, \hat{y})\). Writing \(-\Theta(\hat{t}_1, \hat{t}_2, s_1, s_2, \hat{x}, \hat{y})\) as
\[
-\Theta(\hat{t}_1, \hat{t}_2, s_1, s_2, \hat{x}, \hat{y}) := \bar{w}(s_1, s_2, y) - \hat{v}(s_1, s_2, y)
\]
hence \((\bar{w} - \hat{v})\) has a minimum in \((\hat{s}_1, \hat{s}_2, \hat{y})\), where
\[
\hat{v}(s_1, s_2, y) := w(\hat{t}_1, \hat{t}_2, \hat{x}) - \rho_{\epsilon, \lambda_1, \lambda_2}(\hat{t}_1, \hat{t}_2, s_1, s_2, \hat{x}, \hat{y}).
\]
Similarly to (2.21), we have
\[
\begin{align*}
-\frac{\lambda_1}{2} + \epsilon^2(\bar{t}_1 - \hat{s}_1) + H_1 \left( 2\epsilon^2(\bar{x} - \hat{y}) - 2\epsilon \hat{y} \right) &\geq 0, \\
-\frac{\lambda_2}{2} + \epsilon^2(\bar{t}_2 - \hat{s}_2) + H_2 \left( 2\epsilon^2(\bar{x} - \hat{y}) - 2\epsilon \hat{y} \right) &\geq 0.
\end{align*}
\]
(2.22)

3. Finally, making the difference between (2.22) and (2.21) with respect to the first line, we have
\[
\lambda_1 \leq H_1 \left( 2\epsilon^2(\bar{x} - \hat{y}) - 2\epsilon \hat{y} \right) - H_1 \left( 2\epsilon^2(\bar{x} - \hat{y}) + 2\epsilon \hat{x} \right).
\]
Since \(H_1\) is locally Lipschitz continuous (and the maximum point \((\hat{t}_1, \hat{t}_2, \hat{s}_1, \hat{s}_2, \hat{x}, \hat{y})\)
is attained in a compact ball), we have
\[
\lambda_1 \leq 2\epsilon \|\hat{y} + \hat{x}\|.
\]
(2.23)
At this point, we need an estimate of \(\|\hat{y} + \hat{x}\|\) to conclude that \(\lambda_1 = 0\), since \(\epsilon > 0\) is arbitrary. It will be obtained thanks to the definition of \(\rho_{\epsilon, \lambda_1, \lambda_2}\). In fact, we can fix \(0 < \epsilon, \lambda_1, \lambda_2 < 1\) so small that (2.19) implies

\[
\Theta(\hat{t}_1, \hat{t}_2, \hat{s}_1, \hat{s}_2, \hat{x}, \hat{y}) \geq \sup_{[0,T]^2 \times \mathbb{R}^d} \Theta(t_1, t_2, t_1, t_2, x, x) \geq \frac{\alpha}{2} \tag{2.24}
\]

Moreover, since

\[
\Theta(\hat{t}_1, \hat{t}_2, \hat{s}_1, \hat{s}_2, \hat{x}, \hat{y}) \geq \Theta(0, 0, 0, 0, 0, 0),
\]

it follows that

\[
\rho_{\epsilon, \lambda_1, \lambda_2}(\hat{t}_1, \hat{t}_2, \hat{s}_1, \hat{s}_2, \hat{x}, \hat{y}) \leq \|w(\hat{t}_1, \hat{t}_2, \hat{x}) - w(0, 0, 0)\| - \|\hat{w}(\hat{s}_1, \hat{s}_2, \hat{y}) - \hat{w}(0, 0, 0)\|.
\]

Since \(w\) and \(\hat{w}\) are bounded, we obtain as \(\epsilon \to 0^+\)

\[
|\hat{t}_1 - \hat{s}_1|, |\hat{t}_2 - \hat{s}_2|, \|\hat{x} - \hat{y}\| = O(\epsilon),
\]

\[
\epsilon (\|\hat{x}\|^2 + \|\hat{y}\|^2) = O(1). \tag{2.25}
\]

The last equation of (2.25) implies that

\[
\epsilon (\|\hat{x}\| + \|\hat{y}\|) = \epsilon^\frac{1}{3} \epsilon^\frac{2}{3} (\|\hat{x}\| + \|\hat{y}\|)
\]

\[
\leq \epsilon^\frac{1}{3} + C \epsilon^\frac{2}{3} (\|\hat{x}\|^2 + \|\hat{y}\|^2) \leq C \epsilon^\frac{2}{3}
\]

for some positive constant \(C\). To complete the proof, we use (2.20) in (2.23) and get

\[
\lambda_1 \leq 2C\epsilon^\frac{2}{3}.
\]

Similarly, we obtain that \(\lambda_2 = 0\), and this contradiction completes the proof. \(\square\)

**Remark 2.9.** Note that in the proof, the points \(\hat{t}_1, \hat{t}_2, \hat{s}_1, \hat{s}_2\) could be zero, and in that case, with respect to the time, the function \(\Theta\) would be constant. To see that this does not happen, we recall that

\[
\Theta(\hat{t}_1, \hat{t}_2, \hat{s}_1, \hat{s}_2, \hat{x}, \hat{y}) \leq \Theta(\hat{t}_1, \hat{t}_2, \hat{s}_1, \hat{s}_2, \hat{x}, \hat{y})
\]

and from this, we get

\[
w(\hat{t}_1, \hat{t}_2, \hat{x}) - \hat{w}(\hat{s}_1, \hat{s}_2, \hat{y}) - \rho_{\epsilon, \lambda_1, \lambda_2}(\hat{t}_1, \hat{t}_2, \hat{s}_1, \hat{s}_2, \hat{x}, \hat{y})
\]

\[
\geq w(\hat{t}_1, \hat{t}_2, \hat{x}) - \hat{w}(\hat{t}_1, \hat{t}_2, \hat{x}) - \rho_{\epsilon, \lambda_1, \lambda_2}(\hat{t}_1, \hat{t}_2, \hat{t}_1, \hat{t}_2, \hat{x}, \hat{x}).
\]

Therefore, we obtain

\[
\epsilon^{-2}(\hat{t}_1 - \hat{s}_1)^2 + (\hat{t}_2 - \hat{s}_2)^2 + \|\hat{x} - \hat{y}\|^2 \leq w(\hat{x}, \hat{t}_1, \hat{t}_2) - \hat{w}(\hat{y}, \hat{s}_1, \hat{s}_2)
\]

\[
- \frac{\lambda_1}{2}(\hat{t}_1 - \hat{s}_1) + \frac{\lambda_2}{2}(\hat{t}_2 - \hat{s}_2) + \epsilon(\hat{x} - \hat{y})(\hat{x} + \hat{y}).
\]
Then, by \(2.25\), \(2.26\) and the Lipschitz continuity of \(\hat{w}\), we have
\[
|\dot{s}_1 - \hat{s}_1|, |\dot{s}_2 - \hat{s}_2|, \|\dot{x} - \hat{y}\| = o(\epsilon).
\] (2.27)

Now, let \(\omega\) be the modulus of continuity of \(w\); that is,
\[
|w(t_1, t_2, x) - \hat{w}(s_1, s_2, y)| \leq \omega(|t_1 - s_1| + |t_2 - s_2| + \|x - y\|)
\]
for all \(x, y \in \mathbb{R}^n\), \(0 \leq t, s \leq T\), and \(\omega(r) \to 0\) as \(r \to 0\). Similarly, \(\hat{\omega}(\cdot)\) will denote the modulus of continuity of \(\hat{w}\). Then \(2.21\) implies
\[
\frac{\alpha}{2} \leq w(\dot{t}_1, \dot{t}_2, \dot{x}) - \hat{w}(\dot{s}_1, \dot{s}_2, \hat{y}) = [w(\dot{t}_1, \dot{t}_2, \dot{x}) - w(\dot{t}_1, 0, \dot{x})] + [w(\dot{t}_1, 0, \dot{x}) - w(0, 0, \dot{x})]
\]
\[
+ [w(0, 0, \dot{x}) - \hat{w}(0, 0, \dot{x})] + [\hat{w}(0, 0, \dot{x}) - \hat{w}(\dot{t}_1, 0, \dot{x})]
\]
\[
+ [\hat{w}(\dot{t}_1, 0, \dot{x}) - \hat{w}(\dot{t}_1, \dot{t}_2, \dot{x})] + [\hat{w}(\dot{t}_1, \dot{t}_2, \dot{x}) - \hat{w}(\dot{s}_1, \dot{s}_2, \hat{y})].
\]

Therefore, using \(2.25\), \(2.27\) and the initial condition, we have
\[
\frac{\alpha}{2} \leq \omega(\dot{t}_2) + \omega(\dot{t}_1) + \hat{\omega}(\dot{t}_1) + \hat{\omega}(\dot{t}_2).
\]

As \(\epsilon\) is a positive arbitrary number, we can take it so small as necessary to obtain
\[
\frac{\alpha}{4} \leq \omega(\dot{t}_2) + \omega(\dot{t}_1) + \hat{\omega}(\dot{t}_1) + \hat{\omega}(\dot{t}_2)
\]
and this implies for some constant \(\mu > 0\),
\[
\dot{t}_1, \dot{t}_2 \geq \mu > 0.
\]

Analogously, we have \(\dot{s}_1, \dot{s}_2 \geq \mu > 0\).

### 3 Multi-time conservation laws

Once we have establish existence and uniqueness for multi-time Hamilton-Jacobi system, we are going to use it in this section, in order to show solvability of the multi-time system of conservation laws. Therefore, we fixe \(d, s\) equals one and for given \(H_i\) (i=1,2) two smooth (uniformly) convex flux-function, we consider the following Cauchy problem: Find \(u : (0, \infty)^2 \times \mathbb{R} \to \mathbb{R}\), satisfying
\[
u_1 + \partial_x H_1(u) = 0 \quad \text{in } (0, \infty)^2 \times \mathbb{R},
\]
\[
u_2 + \partial_x H_2(u) = 0 \quad \text{in } (0, \infty)^2 \times \mathbb{R},
\]
\[
u(0, 0, x) = u_0(x) \quad \text{on } \mathbb{R},
\]
where \(u_0 \in L^\infty(\mathbb{R})\) is a given initial-data. With no loss of generality, we assume \(H_i(0) = 0\) (i = 1, 2). Following a usual strategy to 1D scalar conservation laws, we define
\[
g(x) := \int_0^x u_0(y) \, dy \quad (x \in \mathbb{R}),
\] (3.29)
thus $g$ is a Lipschitz function with $\text{Lip}(g) = \|u_0\|_\infty$, and recall the multi-time Lax formula given by (2.11). Thus by Theorem 2.6, $w$ solves the multi-time Hamilton-Jacobi system (2.7) and, if we assume that $w$ is smooth, then we can differentiate that system with respect to $x$, to deduce

$$
\begin{align*}
& w_{x_1} + \partial_x H_1(w_x) = 0 \quad \text{in } (0, \infty)^2 \times \mathbb{R}, \\
& w_{x_2} + \partial_x H_2(w_x) = 0 \quad \text{in } (0, \infty)^2 \times \mathbb{R}, \\
& w_x(0,0,x) = u_0(x) \quad \text{on } \mathbb{R}.
\end{align*}
$$

Now, setting $u = w_x$ we obtain that $u$ solves the system (3.28). Certainly, the computation is only formal, indeed, even that the function $w$ is differentiable a.e., we are not allowed to differentiate $H_1(w_x)$ with respect to $x$, similarly to $H_2$. Although, $u(t_1, t_2, x) : = \partial_x (\min_{y \in \mathbb{R}} \{(t_1 H_1 + t_2 H_2)^*(x - y) + g(y)\})$

seems to be the best candidate for a solution of the Cauchy problem (3.28). In fact, we will show that such function $u$ as defined above is a (weak integral) solution, but before, let’s us first show a more useful formula.

**Lemma 3.1.** (Multi-time Lax-Oleinik formula). Assume $H_i : \mathbb{R} \rightarrow \mathbb{R}$ ($i = 1, 2$) are smooth uniformly convex, $u_0 \in L^\infty(\mathbb{R})$ and $g$ is given by (3.29). Then, for each $t_1, t_2 > 0$, there exists for all but at most countably many values $x \in \mathbb{R}$, such that (3.31) has the following form

$$
(3.32)
$$

where the mapping $x \mapsto y(t_1, t_2, x)$ is nondecreasing. Moreover, for each $z > 0$

$$
(3.33)
$$

**Definition 3.2.** Equation (3.32) is called the multi-time Lax-Oleinik formula.

**Proof.** 1. Fix $t_1, t_2 > 0, x_1 < x_2$. There exists at least one point $y_1 \in \mathbb{R}$, such that

$$
(3.34)
$$

Now, we claim that, for each $y < y_1,$

$$
(t_1 H_1 + t_2 H_2)^*(x_2 - y_1) + g(y_1) < (t_1 H_1 + t_2 H_2)^*(x_2 - y) + g(y).
$$

Indeed, let $\tau \in (0, 1)$, given by

$$
\tau = \frac{y_1 - y}{(x_2 - x_1) + (y_1 - y)}.
$$

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and for convenience, we write

\[ x_2 - y_1 = \tau (x_1 - y_1) + (1 - \tau)(x_2 - y), \]
\[ x_1 - y = (1 - \tau)(x_1 - y_1) + \tau(x_2 - y). \]

Therefore, since \((H^*_i)^{\prime\prime} > 0\) \((i = 1, 2)\), it follows that

\[ (t \cdot H)^*(x_2 - y_1) < \tau(t \cdot H)^*(x_1 - y_1) + (1 - \tau)(t \cdot H)^*(x_2 - y), \]
\[ (t \cdot H)^*(x_1 - y) < (1 - \tau)(t \cdot H)^*(x_1 - y_1) + \tau(t \cdot H)^*(x_2 - y). \]

Then, combining the two above inequalities, we obtain

\[ (t \cdot H)^*(x_2 - y_1) + (t \cdot H)^*(x_1 - y) < (t \cdot H)^*(x_1 - y_1) + (t \cdot H)^*(x_2 - y). \] \hspace{1cm} (3.35)

Moreover, by the definition of \(w(t_1, t_2, x_1)\), we have

\[ -(t_1H_1 + t_2H_2)^*(x_1 - y) - g(y) \leq -(t_1H_1 + t_2H_2)^*(x_1 - y_1) - g(y_1). \] \hspace{1cm} (3.36)

Then, from (3.35) and (3.36)

\[ (t_1H_1 + t_2H_2)^*(x_2 - y_1) + g(y_1) < (t_1H_1 + t_2H_2)^*(x_2 - y) + g(y), \]

and so the claim is proved.

2. From the claim proved before, we observe that to compute the minimum below, i.e.

\[ \min_{y \in \mathbb{R}} \left\{ (t_1H_1 + t_2H_2)^*(x_2 - y) + g(y) \right\}, \]

we only need to consider those \(y \geq y_1\), where \(y_1\) satisfies (3.34). Therefore, for each \(t_1, t_2 > 0\) and \(x \in \mathbb{R}\), we could define the point \(y(t_1, t_2, x)\) equal to the smallest value of those points \(y\) giving the minimum of

\[ (t_1H_1 + t_2H_2)^*(x - y) + g(y). \]

Consequently, for each \(t_1, t_2 > 0\), the mapping \(x \mapsto y(t_1, t_2, x)\) is nondecreasing, thus continuous for all but at most countably many \(x \in \mathbb{R}\). Moreover, at o such point \(x\), the value \(y(t_1, t_2, x)\) is the unique those \(y\) yielding the minimum.

3. Since the function \(w\) is Lipschitz, thus differentiable a.e. and the mapping \(x \mapsto y(t_1, t_2, x)\) is monotone and so differentiable a.e. as well, given \(t_1, t_2 > 0\), for a.e. \(x \in \mathbb{R}\), the mappings

\[ x \mapsto (t_1H_1 + t_2H_2)^*(x - y(t_1, t_2, x)), \]
\[ x \mapsto g(y(t_1, t_2, x)) \]
are also differentiable for a.e. \( x \in \mathbb{R} \). Then, we have for such a differentiable point \( x \),

\[
    u(t_1, t_2, x) = \partial_x \left( (t_1 H_1 + t_2 H_2)^* (x - y(t_1, t_2, x)) + g(y) \right)
    = \left( (t_1 H_1 + t_2 H_2)^* \right)'(x - y(t_1, t_2, x)) \left( 1 - y_z(t_1, t_2, x) \right)
    + \partial_x \left( g(y(t_1, t_2, x)) \right).
\]

But since the mapping \( y \mapsto (t_1 H_1 + t_2 H_2)^* + g \) has a minimum at \( y = y(t_1, t_2, x) \), it follows that

\[
    -\left( (t_1 H_1 + t_2 H_2)^* \right)'(x - y(t_1, t_2, x)) y_z(t_1, t_2, x) + \partial_x \left( g(y(t_1, t_2, x)) \right) = 0,
\]

and thus we obtain (3.32).

4. Finally, by equation (3.32), the monotonicity of \( (t_1 H_1 + t_2 H_2)^* \)' and \( y(t_1, t_2, \cdot) \) as well, we have for each \( z > 0 \)

\[
    u(t_1, t_2, x) = ((t_1 H_1 + t_2 H_2)^*)'(x - y(t_1, t_2, x))
    \geq ((t_1 H_1 + t_2 H_2)^*)'(x - y(t_1, t_2, x + z))
    \geq ((t_1 H_1 + t_2 H_2)^*)'(x + z - y(t_1, t_2, x + z))
    - \text{Lip} \left( ((t_1 H_1 + t_2 H_2)^*)' \right) z
    = u(t_1, t_2, x + z) - \text{Lip} \left( ((t_1 H_1 + t_2 H_2)^*)' \right) z.
\]

Therefore, we obtain

\[
    u(t_1, t_2, x + z) - u(t_1, t_2, x) \leq \text{Lip} \left( ((t_1 H_1 + t_2 H_2)^*)' \right) z.
\]

\[\square\]

### 3.1 Existence

Now we are ready to show the solvability of the multi-time system of conservation laws in 1D for two independent times. First, let us define in which sense a bounded and measurable real function \( u \) defined in \((0, \infty)^2 \times \mathbb{R}\) is a weak (integral) solution of (3.28).

**Definition 3.3.** Given \( u_0 \in L^\infty(\mathbb{R}) \), a function \( u \in L^\infty((0, \infty)^2 \times \mathbb{R}) \) is said a weak integral solution of the Cauchy problem (3.28), if it satisfies

- Multi-time conservation laws: For all \( \varphi \in C^\infty_0((0, \infty)^2 \times \mathbb{R}) \)

\[
    \int_0^\infty \int_0^\infty \int_\mathbb{R} u(t_1) H_1(u) \varphi_{t_1} + \varphi_{t_2} \, dx \, dt_1 \, dt_2 = 0,
\]

(3.37)
\[
\int_0^\infty \int_0^\infty \int_{\mathbb{R}} (u \varphi_{t_2} + H_2(u) \varphi_x) \, dx \, dt_1 \, dt_2 = 0. \tag{3.38}
\]

- **Initial condition:** For any \( \gamma \in L^1(\mathbb{R}) \)

\[
\text{ess lim}_{t_1, t_2 \to 0^+} \int_{\mathbb{R}} \left( u(t_1, t_2, x) - u_0(x) \right) \gamma(x) \, dx = 0. \tag{3.39}
\]

**Theorem 3.4.** The function \( u \in L^\infty((0, \infty)^2 \times \mathbb{R}) \) given by Lemma 3.1, equation (3.32), is a weak solution of the Cauchy problem (3.28).

**Proof.** First, we define for \( t_1, t_2 > 0 \) and \( x \in \mathbb{R} \),

\[
w(t_1, t_2, x) = \min_{y \in \mathbb{R}} \{(t_1 H_1 + t_2 H_2)^*(x - y) + g(y)\},
\]

which by Theorem 2.6 is a Lipschitz continuous function, differentiable a.e in \((0, \infty)^2 \times \mathbb{R}\), and solves

\[
w_{t_1} + H_1(w_x) = 0 \quad \text{a.e. in } (0, \infty)^2 \times \mathbb{R},
\]
\[
w_{t_2} + H_2(w_x) = 0 \quad \text{a.e. in } (0, \infty)^2 \times \mathbb{R}, \tag{3.40}
\]
\[
w(0, 0, x) = g(x) \quad \text{on } \mathbb{R}.
\]

Now, we take \( \varphi \in C_0^\infty((0, \infty)^2 \times \mathbb{R}) \) multiply the first equation in (3.40) by \( \varphi_x \) and integrate over \((0, \infty)^2 \times \mathbb{R}\), to obtain

\[
\int_0^\infty \int_0^\infty \int_{\mathbb{R}} (w_{t_1} \varphi_x + H_1(w_x) \varphi_x) \, dx \, dt_1 \, dt_2 = 0.
\]

Then, we observe that

\[
\int_0^\infty \int_0^\infty \int_{\mathbb{R}} w_{t_1} \varphi_x \, dx \, dt_1 \, dt_2 = -\int_0^\infty \int_0^\infty \int_{\mathbb{R}} w \varphi_{t_1x} \, dx \, dt_1 \, dt_2
\]
\[
= \int_0^\infty \int_0^\infty \int_{\mathbb{R}} w_x \varphi_{t_1} \, dx \, dt_1 \, dt_2,
\]

where we are allowed to integrate by parts, since the mapping \( x \mapsto w(t_1, t_2, x) \) is Lipschitz continuous and then, absolutely continuous for each \( t_1, t_2 > 0 \). Moreover, for each \( t_2 > 0 \) and \( x \in \mathbb{R} \), the mapping \( t_1 \mapsto w(t_1, t_2, x) \) is also absolutely continuous. Therefore, we have

\[
\int_0^\infty \int_0^\infty \int_{\mathbb{R}} (w_x \varphi_{t_1} + H_1(w_x) \varphi_x) \, dx \, dt_1 \, dt_2 = 0,
\]

and by similarly argument, we obtain

\[
\int_0^\infty \int_0^\infty \int_{\mathbb{R}} (w_x \varphi_{t_2} + H_2(w_x) \varphi_x) \, dx \, dt_1 \, dt_2 = 0.
\]
Finally, we recall that $u = w_x$ a.e. as precisely defined by (3.32). Then, the Multi-time conservation laws condition at Definition 3.3 is satisfied.

To show the initial-condition, we apply the same strategy before and the result follows using (2.13).

3.2 Uniqueness

We show the existence of a weak integral solution $u$ to the problem (3.28), where $u$ is given by (3.32). Recall that, the integral solution is slight different from the entropy solution given by Definition 1.2 that is, a measurable and bounded function $u(t_1, t_2, x)$ is an entropy solution to (3.28), if for all entropy pair $(\eta(u), q_i(u))$ $(i = 1, 2)$, and for each $T > 0$, the following holds true

\[
\int_0^T \int_0^T \int_{\mathbb{R}} (\eta(u) \varphi_{t_1} + q_1(u) \varphi_x) \, dx \, dt_1 \, dt_2 \geq 0,
\]

\[
\int_0^T \int_0^T \int_{\mathbb{R}} (\eta(u) \varphi_{t_2} + q_2(u) \varphi_x) \, dx \, dt_1 \, dt_2 \geq 0,
\]

for each non-negative test function $\varphi \in C_0^\infty((0, T)^2 \times \mathbb{R})$, and also the initial-condition (3.39) is satisfied. It follows by (3.33) that, for each $t_1, t_2 \in (0, T)$ fixed, $u(t_1, t_2, \cdot)$ has locally bounded variation. Indeed, we know that for each $z > 0$

\[
\frac{u(t_1, t_2, x + z) - u(t_1, t_2, x)}{z} \leq c,
\]

where $c := \text{Lip}((t_1 H_1 + t_2 H_2)^\dagger)$. Let, $\tilde{u}(t_1, t_2, x) = u(t_1, t_2, x) - \tilde{c} x$, for $\tilde{c} > c$. Then, we have for each $z > 0$

\[
\tilde{u}(t_1, t_2, x + z) - \tilde{u}(t_1, t_2, x) < 0,
\]

that is, $\tilde{u}(t_1, t_2, \cdot)$ is a decreasing function and hence has locally bounded total variation. Since this is also true for $\tilde{c} x$, we obtain that $u(t_1, t_2, \cdot)$ has locally bounded variation. Therefore, the well-known theory of Vol'pert [20] allow us to apply the chain rule for $BV$ functions, and write for a.e. $x \in \mathbb{R}$, $i = 1, 2$

\[
\partial_x H_i(u(t_1, t_2, x)) = H'_i(u(t_1, t_2, x)) (u(t_1, t_2, x))_x,
\]

and thus since $u$ is an integral solution, we have in the sense of measures

\[
|u_{t_i}| \leq \max_{\xi \in B_{i+1}\infty(0)} |H'_i(\xi)| \, |u_x|,
\]

that is to say, $u_{t_1}, u_{t_2}$ are locally Radon measures.

Now, let $\eta$ be a smooth convex function. Again with no loss of generality we may as well also take $\eta(0) = 0$. Then, we multiply (3.43) by $\eta'(u)$, and apply
again the chain rule for BV functions to obtain in the measure sense
\[ \eta(u)_t + \partial_x q_1(u) = 0, \]
\[ \eta(u)_t + \partial_x q_2(u) = 0. \]  \( (3.44) \)

Consequently, it is not difficult to see that, the integral solution \( u \) is in fact an entropy solution, where it is crucial the estimate \( (3.43) \) in order to show the initial data \( (1.4) \). Moreover, by a standard approximation procedure, we may assume that the pair \( (\eta, q_i) \) \( (i = 1, 2) \) are the Kruzkov entropies, that is,
\[ \int_0^T \int_{\mathbb{R}} \left( |u - v| \varphi_{t_1} + \text{sgn}(u - v)(H_1(u) - H_1(v)) \varphi_x \right) dx dt_1 dt_2 = 0, \]  \( (3.45) \)
\[ \int_0^T \int_{\mathbb{R}} \left( |u - v| \varphi_{t_2} + \text{sgn}(u - v)(H_2(u) - H_2(v)) \varphi_x \right) dx dt_1 dt_2 = 0, \]  \( (3.46) \)
for each \( v \in \mathbb{R} \) fixed and all test function \( \varphi \in C^\infty_0((0, T)^2 \times \mathbb{R}) \). Therefore, we are in position to apply the doubling variables technic due to Kruzkov, see \( [11] \). In fact, this is nowadays a standard procedure, thus adapted to our case leads to the following result

**Lemma 3.5.** Let \( u \) and \( v \) be two entropy solutions to the problem \( (3.28) \) corresponding to initial data \( u_0, v_0 \) respectively. Then, we have the \( L^1 \)-contraction type inequalities
\[
\int_0^T \int_{B_R(0)} |u(t_1, \tau, x) - v(t_1, \tau, x)| \zeta_2(\tau) \, dx \, d\tau \leq \int_0^T \int_{B_R(0)} |u(0, \tau, x) - v(0, \tau, x)| \zeta_2(\tau) \, dx \, d\tau,
\]
\[
\int_0^T \int_{B_R(0)} |u(\tau, t_2, x) - v(\tau, t_2, x)| \zeta_1(\tau) \, dx \, d\tau \leq \int_0^T \int_{B_R(0)} |u(\tau, 0, x) - v(\tau, 0, x)| \zeta_1(\tau) \, dx \, d\tau,
\]  \( (3.47) \)
which holds for all ball \( B_R(0), R > 0 \) and almost all \( t_1, t_2 > 0 \), where for \( i = 1, 2, \zeta_i \in C^\infty_0(0, T), B_{R_i} = B_{R+M_it_i}(0), \) and \( M_i \) denotes the Lipschitz constant of \( H_i \).

**Theorem 3.6.** Let \( u \) and \( v \) be two entropy solutions to the problem \( (3.28) \) corresponding to initial data \( u_0, v_0 \) respectively. If \( u_0 = v_0 \) almost everywhere, then \( u = v \) almost everywhere.

**Proof.** For \( \delta > 0 \), we take \( \zeta_1(\tau) = \chi_{(0, \delta)}(\tau) \) in the second inequality of \( (3.47) \). Then, dividing by \( \delta \) both sides of the inequality, and passing to the limit as \( \delta \to 0^+ \), we obtain
\[
\int_{\mathbb{R}} |u(0, t_2, x) - v(0, t_2, x)| \, dx = 0.
\]
Similarly, for $\theta > 0$ sufficiently small, we take $
exists \zeta_2(\tau) = \chi(t_2 - \theta, t_2 + \theta)(\tau)$ in the first inequality of (3.47). Again dividing the inequality by $\theta$ and passing to limit as $\theta$ goes to $0^+$, the uniqueness result follows, that is, $u \equiv v$ almost everywhere. □

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