Conformal Galilean-type algebras, massless particles and gravitation

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Abstract

After defining conformal Galilean-type algebras for arbitrary dynamical exponent $z$ we consider the particular cases of the conformal Galilei algebra (CGA) and the Schrödinger Lie algebra (sch). Galilei massless particles moving with arbitrary, finite velocity are introduced

i) in $d = 2$ as a realization of the centrally extended CGA in 6 dimensional phase space,

ii) in arbitrary spatial dimension $d$ as a realization of the unextended $sch$ in $4d$ dimensional phase space.

A particle system, minimally coupled to gravity, shows, besides Galilei symmetry, also invariance with respect to arbitrary time dependent translations and to dilations with $z = (d + 2)/3$. The most important physical property of such a self-gravitating system is the appearance of a dynamically generated gravitational mass density of either sign. Therefore, this property may serve as a model for the dark sector of the universe. The cosmological solutions of the corresponding hydrodynamical equations show a deceleration phase for the early universe and an acceleration phase for the late universe. This paper is based, in large part, on a recent work with W.J. Zakrzewski: Can cosmic acceleration be caused by exotic massless particles?
arXiv: 0904.1375 (astro-ph.CO) [1].
1 Conformal Galilean-type algebras

We consider \((d + 1)\)-dimensional space-time.

1.1 Conformal subalgebra

For arbitrary dynamical exponent \(z\) we have in terms of differential operators

\[
H = \partial_t, \quad \text{Time translations}
\]

\[
D = t\partial_t + \frac{1}{z}x_i\partial_i \quad \text{Dilations}
\]

\[
K = t^2\partial_t + \frac{2}{z}tx_i\partial_i \quad \text{Expansions} \quad i = 1, 2, ..., d.
\]

Thus \(x_i\) and \(t\) transform differently with respect to:

Dilations

\[
x_i^* = \lambda^z x_i, \quad t^* = \lambda t, \quad \text{where } \lambda > 0,
\]

Expansions

\[
x_i^* = \left(\frac{1}{1 - kt}\right)^\frac{z}{2} x_i, \quad t^* = \frac{t}{1 - kt}, \quad \text{where } k \in \mathbb{R}^1.
\]

The operators \((H, D, K)\) form the \(O(2, 1)\) algebra (independent of \(z\))

\[
[D, H] = -H, \quad [K, H] = -2D, \quad [D, K] = K.
\]
1.2 (Enlarged) Galilean algebra

Define (disregard rotations):

\[ P_i = -\partial_i, \quad \text{Space translations} \]
\[ K_i = -t \partial_i, \quad \text{Boosts} \]

and the components of the enlargement

\[ F_i = -t^2 \partial_i \quad \text{Accelerations} \]

etc., together, forming the Galilei algebra (considering only nonvanishing Lie brackets)

\[ [H, K_i] = P_i \]

and its enlargements

\[ [H, F_i] = 2K_i \quad \text{etc.} \]

1.3 Mixed Lie brackets

Lie brackets between conformal and (enlarged) Galilei generators depend on \( z \). In particular:

\[ [D, P_i] = -\frac{1}{z} P_i, \quad [D, K_i] = (1 - \frac{1}{z}) K_i, \]
\[ [K, P_i] = -\frac{2}{z} K_i, \quad [K, K_i] = (1 - \frac{2}{z}) F_i, \]
\[ [K, F_i] = \ldots \text{etc,} \]

This infinite series terminates if \( z = \frac{2}{n} \) for \( n \in \mathbb{N} \).

Particular cases are:

1. \( z = 2 \) Schrödinger algebra (sch)

   We have no accelerations \( F_i \). For arbitrary \( d \) one central extension is possible

   \[ [P_i, K_j] = m\delta_{ij}, \quad m = \text{mass.} \]

2. \( z = 1 \) Conformal Galilei algebra (CGA) (enlarged by accelerations \( F_i \))

   We obtain \( m = 0 \) from the Jacobi identity

   \[ 0 = [H, [P_i, F_k]] + [P_i, [F_k, H]] + [F_k, [H, P_i]] \]
   \[ \downarrow \quad \downarrow \quad \downarrow \]
   \[ c\text{-number} \quad -2K_k \quad 0. \]

   This argument (for \( m = 0 \)) holds for \( z \neq 2 \), i.e, if accelerations are present.
Remark: CGA is the algebra obtained by means of the nonrelativistic contraction from the conformal Poincaré algebra [4].

For $d = 2$ we can introduce one central extension:

$$[K_i, K_j] = \theta \epsilon_{ij}.$$  

Introducing two additional central charges for the acceleration enlarged Galilean algebra leads to a onefold centrally extended, two parameter deformed CGA [5].

Remark: In the expansion-less case we have no accelerations. The algebra consisting of the Galilean generators and dilations closes for arbitrary $z$.

References for this section:

1. Algebra: Henkel [2]; Negro et al, [3]
2. Central extensions and dynamical realisations in $d = 2$; Lukierski, Stichel and Zakrzewski [4,5].

## 2 Galilean massless particles

### 2.1 Statement

The realisation of $[P_i, K_j] = 0$ (↔ $m = 0$) by means of $K_j = f_j$ (phase space variables) with $\frac{d}{dt} f_j = 0$ requires an enlarged phase space.

To show this consider a massless particle in $d$-dim space with position $x_i$ and canonical momentum $p_i$.

Then proceed in three steps:

1. **Conserved translations**: We get $[x_i, p_j] = \delta_{ij}$ and $\dot{p}_j = 0$.

   **Conserved boosts**: From $[x_i, K_j] = t \delta_{ij}$ we obtain $K_j = p_j t + q_j$

   with $\dot{q}_j = -p_j$ and $[q_j, x_i] = 0$.

   **The Requirement**

   $[p_i, K_j] = 0$ leads to $[q_j, p_i] = 0$

   *i.e* we obtain

   $$q_j \neq g_j(x_i, p_i)$$

Thus $\tilde{d} := \dim$ (phase space) $\geq 3d$.

(cp. to $\tilde{d} = 2d$ in standard case).

**Question**: What is $\tilde{d}_{\text{min}}$?
2. Next we introduce **velocities** \( y_i := [x_i, H] \).

Then, by applying the Jacobi identity, we obtain for translations: \([y_i, p_j] = 0\)

res. for boosts: \([y_i, K_j] = \delta_{ij}\) leading to \([y_i, q_j] = \delta_{ij}\)

and \(y_i \neq h_i(x_k, p_k)\) (otherwise we would get \([h_i, q_j] = 0\) due to \([q_j, p_i] = [q_j, x_i] = 0\)).

3. Now we have to distinguish between the cases \(d = 2\) and \(d \neq 2\).

a) In \(d = 2\) we can realise \([y_i, q_j] = \delta_{ij}\) by

\[
q_i = \theta \epsilon_{ij} y_j \quad \text{with} \quad [y_i, y_j] = \frac{\epsilon_{ij}}{\theta}
\]

(leading to \([K_i, K_j] = \theta \epsilon_{ij})\).

Thus we see that \(d_{min} = 6\).

Now let us derive the corresponding minimal 1st-order Lagrangian [4].

From \(\dot{q}_i = -p_i\) we obtain \(\dot{y}_i = \frac{1}{\theta} \epsilon_{ij} p_j\).

ie EOM are derived by means of PBs from \(H = p_i y_i\) being equivalent to

\[
L = p_i \dot{x}_i - \frac{\theta}{2} \epsilon_{ij} y_i \dot{y}_j - H.
\]

b) In \(d \neq 2\) (resp. \(\theta = 0\) for \(d = 2\)) we have to assume that \([y_i, q_j] = \delta_{ij}\) together with \([q_i, q_j] = 0\). The latter requirement follows from \([K_i, K_j] = 0\). Clearly the ansatz \(q_i = q_i(\vec{y})\) contradicts our assumption. Thus \(q_i\) must be independent of \(y_i\) and so the phase space = \(\{x_i, p_i, y_i, q_i\}\) where \(i = 1..d\) and so \(d_{min} = 4d\).

Now consider the corresponding minimal 1st-order Lagrangian.

The EOM are

\[
\dot{x}_i = y_i, \quad \dot{p}_i = 0, \quad \dot{q}_i = -p_i, \quad \dot{y}_i = ?
\]

The Poisson brackets are

\[
[x_i, p_j] = \delta_{ij}, \quad [y_i, q_j] = \delta_{ij}
\]

and they imply that

\[
H = p_i y_i + f(q_i).
\]

The minimal (parameter free) Lagrangian, obtained from \(H_{f=0}\) is then

\[
L_0 = p_i \dot{x}_i + q_i \dot{y}_i - p_i y_i.
\]
2.2 Conformal generators

a) $d = 2$ \cite{4}.

- **Dilation** (conserved) $D = tH - x_i p_i$
- **Expansion** (conserved) $K = -t^2 H + 2t D - 2 \theta \epsilon_{ij} x_i y_j$

We obtain $[D, P_i] = -P_i$.

ie we have $z = 1$ (CGA)

b) $d \neq 2$ (resp. $\theta = 0$ for $d = 2$)

- **Dilation** (conserved for any $z$) $D = tH - \frac{1}{z} x_i p_i + (1 - \frac{1}{z}) y_i q_i$
- **Expansion** We make the ansatz $K = -t^2 H + 2t D + \alpha x_i q_i$.

Then from $\frac{d}{dt} K = 0$ we get $\alpha = -1$ and so $z = 2$

**Remark:** Our massless particles move with arbitrary finite velocity. Therefore they are distinct from the Galilean massless particles introduced by Duval and Horvathy \cite{6} moving with infinite velocity.

3 Coupling to gravity

We start, for any $d$, with the parameter-free Lagrangian

$$L_0 = p_i \dot{x}_i + q_i \dot{y}_i - H$$

leading to the EOM

$$\ddot{x}_i = \ddot{y}_i = 0.$$  \hspace{1cm} (1)

We introduce a minimal coupling to the gravitational field $\phi(\bar{x}, t)$ $\bar{x} = \{x_i\}, \ (i = 1, ..d)$ in accordance with Einstein’s equivalence principle (“free falling elevator”).

As, at each fixed point $\bar{x}$, the gravitational force $-\partial_i \phi$ is equivalent to an acceleration $b_i(t)$ we see that the equation (1) has to be modified to

$$\ddot{x}_i(t) = -\partial_i \phi(\bar{x}(t), t)$$  \hspace{1cm} (2)

because (2) is invariant with respect to arbitrary time-dependent translations (cp. \cite{7})

$$x_i \rightarrow x'_i = x_i + a_i(t)$$

provided that $\phi$ transforms as

$$\phi(\bar{x}, t) \rightarrow \phi'(\bar{x}', t) = \phi(\bar{x}, t) - \ddot{a}_i x_i + h(t).$$

Note that (2) can be realised if we add to $L_0$ an interaction part

$$L_{int} = q_i \partial_i \phi.$$  

Then the EOM $\dot{p}_i = 0$ gets replaced by

$$\dot{p}_i = q_k \partial_k \partial_i \phi.$$
4 A self-gravitating fluid

4.1 Lagrange picture

We generalise the one-particle phase space coordinates $A_i (A_i \in (x_i, p_i, y_i, q_i))$ to the continuum labeled by $\vec{\xi} \in R^d$ (comoving coordinates)

$$A_i(t) \rightarrow A_i(\vec{\xi}, t).$$

The Lagrangian $L$ for the self-gravitating fluid becomes

$$L = \int d^d\xi \left[ L_0(A_i(\vec{\xi}, t)) + L_{int}(A_i(\vec{\xi}, t), \phi(\vec{x}(\vec{\xi}, t))) \right] + L_\phi$$

where the field part $L_\phi$ has the standard form

$$L_\phi = -\frac{1}{8\pi G} \int d^d x (\partial_i \phi)^2,$$

and $G$ is Newton’s gravitational constant.

4.2 Eulerian picture

We transform $A_i \in (p_i, y_i, q_i)$ from comoving coordinates $\vec{\xi}$ to fixed space coordinates $\vec{x} \in R^d$, i.e., $A_i(\vec{\xi}, t) \rightarrow A_i(\vec{x}, t)$ by (cp. [8])

$$A_i(\vec{x}, t)n(\vec{x}, t) = \int d^d\xi A_i(\vec{\xi}, t) \delta(\vec{x} - \vec{x}(\vec{\xi}, t)).$$

Here $n(\vec{x}, t)$ is the particle number density

$$n(\vec{x}, t) := \int d^d\xi \delta(\vec{x} - \vec{x}(\vec{\xi}, t)).$$

We apply these transformations to the Euler-Lagrange EOM obtained from $L$ in section 4.1 (def. $u_i(\vec{x}, t) := y(\vec{x}, t)$) and get the fluid dynamical EOM

$$\partial_t n + \partial_k (n u_k) = 0, \quad \text{Continuity equation}$$

$$D_t u_i =: -\partial_i \phi \quad \text{Euler equation} \quad \text{where} \quad D_t := \partial_t + u_k \partial_k \quad \text{(convective derivative)}$$

and

$$\Delta \phi = 4\pi G \partial_k(nq_k) \quad \text{Poisson equation}.$$
- sign leading to repulsive gravitation. This promotes the self-gravitating fluid to a possible candidate for the dark sector of the universe!

To get the $q_i$ we have to solve the additional EOM - due to the enlarged phase space -i.e.

$$D_t q_i = -p_i, \quad D_t p_i = q_k \partial_k \partial_i \phi.$$ 

4.3 Symmetries

Fluid dynamical EOM exhibit the following symmetries:

i) Clearly rotational symmetry

ii) Invariance with respect to infinitesimal time-dependent translations

$$\delta x_i = a_i(t) \quad \text{with}$$

$$\delta \phi = -\dot{a_i} x_i - a_k \partial_k \phi + h(t)$$

$$\delta u_i = \dot{a_i} - a_k \partial_k \phi.$$ 

All other fields transform as scalars, i.e.

$$\delta n = -a_k \partial_k n \quad (\text{etc})$$

(the same for $p_i$, resp. $q_i$).

Note that by a suitable choice of a time dependent translation one can pass to a Galilean frame in which the solution of the Poisson eq. is given by (in the $d=3$ case)

$$\phi(\vec{x}, t) = -G \int d^3 x' \frac{(\partial_i (n q_i))(\vec{x}', t)}{|\vec{x} - \vec{x}'|}.$$ 

Then the symmetry algebra becomes the expansion-less conformal Galilei algebra with $z = \frac{5}{3}$ ($z = \frac{d+2}{3}$ for any $d$).

Conserved Generators are:

$$P_i = \int d^3 x n p_i \quad \text{Linear momentum}$$

$$K_i = t P_i + Q_i \quad \text{Boost generators}$$

with

$$Q_i := \int d^3 x n q_i$$

Note that we still have

$$[P_i, K_j] = 0.$$
We also have

\[ \vec{J} = \int d^3x \, n \, (\vec{x} \wedge \vec{p} + \vec{u} \wedge \vec{q}) \]

Angular momentum

and Energy is given by

\[ H = \int d^3x \, p_i u_i - \frac{G}{2} \int d^3x \, d^3x' \frac{(\partial_i(nq_i))(\vec{x}, t)(\partial_j(nq_j))(\vec{x}', t)}{|\vec{x} - \vec{x}'|}. \]

We make the following ansatz for the dilation charge

\[ D = Ht + (1 - \frac{1}{z}) \int d^3x \, nq_i u_i - \frac{1}{z} \int d^3x \, n x_i p_i \]

which, from \( \frac{d}{dt}D = 0 \), gives \( z = \frac{5}{3} \).

**Remark:** Adding standard matter (e.g. cold dark matter cp. [1]) leads to the violation of dilation symmetry because the corresponding kinetic term would scale with \( z = 2 \) whereas the potential \( (1/r) \) term would scale with \( z = 1 \).

## 5 Cosmological solutions

We consider the self-gravitating fluid as a model for the dark sector of the universe. Hence we look for solutions of the fluid EOM satisfying the cosmological principle (the universe is supposed to be isotropic and homogeneous on large scales).

This gives us (in a suitable Galilean frame)

\[ n(\vec{x}, t) = n(t), \quad u_i(\vec{x}, t) = x_i H(t). \]

Here \( H(t) \) is the Hubble “parameter” given in terms of the cosmic scale factor \( a(t) \) by

\[ H = \frac{\dot{a}}{a} \]

and

\[ q_i(\vec{x}, t) = x_i g(a(t)). \]

We insert all this into the EOM and get:

i) From the continuity eq.

\[ \dot{n} + 3\frac{\dot{a}}{a}n = 0 \]

leading, after integration, to

\[ n(t) = \frac{D}{4\pi a^3(t)} \]

with \( D=\text{const} > 0 \).

ii) From the Euler eq.

\[ \phi(\vec{x}, t) = \frac{r^2}{2} \varphi(t) \]

with

\[ \varphi(t) = -\frac{\dot{a}}{a}. \]
iii) From the EOM for \( p_i \) and \( q_i \)
\[
\ddot{g} + 2\frac{\dot{a}}{a}\dot{g} = 0
\]
which, after integration, gives us
\[
\dot{g}(a(t)) = \frac{\beta}{a^2(t)}, \quad \beta = \text{const.}
\] (3)

iv) Putting all this into the Poisson eq. we get
\[
-\ddot{a} = 3GD\frac{a^2}{g(a)}
\] (4)

ie a Friedmann-like equation.

Using (3) we integrate (4) twice obtaining a cubic equation for \( g(a) \)
\[
g(g^2 + C_1) + C_0 \left(1 - \frac{a_t}{a}\right) = 0,
\] (5)
where \( C_{0,1} \) are integration constants and
\[
a_t := \frac{2\beta^2}{GDC_0}.
\]

If now we have \( C_{0,1} > 0 \) and we use \( a(t) \) as a measure of time (we have \( \dot{a} > 0 \), ie an expanding universe) we get from (5) that

- For \( a < a_t \) we have \( g(a) > 0 \) and so from (4) \( \ddot{a} < 0 \) ie we are in the deceleration phase of the early universe.

- For \( a > a_t \) we get from (5) \( g(a) < 0 \), hence \( \ddot{a} > 0 \) and we are in the acceleration phase of the late universe.

Hence \( a_t \) defines the point of transition from one phase to the other. This picture is consistent with astrophysical observations.

Remark: \( C_{0,1} > 0 \) is also necessary to obtain these results.

Furthermore, \( C_{0,1} = 0 \) would reproduce the scale invariant solution valid asymptotically at small \( t \) (then \( a(t) \sim t^{\frac{2}{3}} \)).

To prove this we start with the behaviour of the particle density \( n(\vec{x}, t) \) with respect to dilations
\[
[n(\vec{x}, t), D] = (t\partial_t + \frac{3}{5}x_i\partial_i + \frac{9}{5})n(\vec{x}, t)
\]
ie that the scale invariant solution for \( n(t) \) is given by
\[
n(t) \sim t^{-9/5}
\]
and from
\[
n(t) \sim a^{-3}(t)
\]
we obtain the stated result.
6 Final remarks

6.1 The main achievements of our model are

- Parameter free Lagrangian for a self-gravitating system of Galilean massless particles
- Dynamical generation of an active gravitational mass density of either sign
- Explanation of the deceleration phase of the early universe and of the acceleration phase of the late universe

In order to see clearly the relevance of our results we have to contrast our model with other attempts to explain the accelerated expansion of the universe (for details and references see [1]).

Other models, introduced in the framework of General Relativity, consist of two classes:

i) either one modifies the geometric part of the Einstein-Hilbert action,

or

ii) one modifies the matter part of this action by introducing either a positive cosmological constant or a dynamical model leading to negative pressure (scalar fields etc.).

But all these models contain at least one new parameter (in most cases even some unknown function). None of these models is derived from fundamental physics. This distinguishes other models from ours.

6.2 Open problems (cosmological solutions)

- Determination (or restriction), by physical arguments, of the a-priori unknown integration constants, arising from the additional phase space dimensions
- Comparison with astrophysical data
  But such a comparison meets the difficulty that the determination of cosmological parameters from observational data is at present mostly model dependent (cp. [9] for the case of $a_t$).

6.3 Open problems of a general nature

Relativistic generalisation of our model and its relation to the framework of General Relativity.

It is unlikely, if not impossible, to obtain our model as a nonrelativistic limit of a relativistic model. Massless relativistic particle models show conformal Poincaré symmetry leading, in the nonrelativistic limit, to conformal Galilei symmetry [4], i.e. $z = 1$. But
we have $z = 5/3$. In [1] we have speculated, that the relativistic generalization of our Galilean massless particles are tachyons. But it seems to me more likely that we meet here a situation being to Hořava gravity [10], where we have nonrelativistic symmetry in the ultraviolet (small $t$) and approach General Relativity only in the infrared limit (large $t$). To establish such a picture we have to modify our model in an appropriate manner. This is a challenge for further research.

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