Optimal dividend problems for a jump-diffusion model with capital injections and proportional transaction costs

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Abstract In this paper, we study the optimal control problem for a company whose surplus process evolves as an upward jump diffusion with random return on investment. Three types of practical optimization problems faced by a company that can control its liquid reserves by paying dividends and injecting capital. In the first problem, we consider the classical dividend problem without capital injections. The second problem aims at maximizing the expected discounted dividend payments minus the expected discounted costs of capital injections over strategies with positive surplus at all times. The third problem has the same objective as the second one, but without the constraints on capital injections. Under the assumption of proportional transaction costs, we identify the value function and the optimal strategies for any distribution of gains.

Key words and phrases. Barrier strategy, dual model, HJB equation, jump-diffusion,
optimal dividend strategy, stochastic control.

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1 INTRODUCTION

For the optimal dividend problem, one may adopt the objective of maximizing the expectation of the discounted dividends until possible ruin. This problem was first addressed by De Finetti [16] who considered a discrete time risk model with step sizes ±1 and showed that the optimal dividend strategy is a barrier strategy. Miyasawa [21] generalized the model to the case that periodic gains of a company can take on values −1, 0, 1, 2, 3, · · · , and showed that the optimal dividend strategy of the generalized model is a barrier one. Subsequently, the problem of finding the optimal dividend strategy has attracted great attention in the literature of insurance mathematics. For nice surveys on this topic, we refer the reader to Avanzi [3] and Schmidli [22]. Besides insurance risk models, the optimal dividend problem in the so-called dual model has also been studied extensively in recent years. Among others, Avanzi et al. [6] discussed how the expectation of the discounted dividends until ruin can be calculated for the dual model when the gain amounts follow an exponential distribution or a mixture of exponential distributions, and showed how the exact value of the optimal dividend barrier can be determined; and Avanzi and Gerber [5] examined the same problem for the dual model that is perturbed by diffusion, and showed that the optimal dividend strategy in the dual model is also a barrier strategy. To make the problem more interesting, the issue of capital injections has also been considered in the study of optimal dividends in the dual model. Yao et al. [23] studied the optimal problem with dividend payments and issuance of equity in the dual model with proportional transaction costs, and derived the optimal strategy that maximizes the expected present value of dividend payments minus the discounted costs of issuing new equity before ruin. Yao et al. [24] considered the same problem with both fixed and proportional transaction costs. Dai et al. [14,15] investigated the same problem as in Yao et al. [23] for the dual model with diffusion with bounded gains and exponential gains,
respectively. Avanzi et al. [7] derived an explicit expression for the value function in the
dual model with diffusion when the gains distribution in a mixture of exponentials in the
presence of both dividends and capital injections. Specifically, they showed that barrier
dividend strategy is optimal, and conjectured that the optimal dividend strategy in the
dual model with diffusion should be the barrier strategy regardless of the distribution of
gains. Bayraktar et al. [11] examined the same cash injection problem, and used the fluctua-
tion theory of spectrally positive Lévy processes to show the optimality of the barrier
strategy for all positive Lévy processes. Bayraktar et al. [12] extended the study to the
case with fixed transaction costs. Other related work can be found in Yin and Wen [26],
Yin, Wen and Zhao [28], Avanzi et al. [8], Yao et al. [25] and Zhang [29].

In this paper, we provide a uniform mathematical framework to analyze the optimal
control problem with dividends and capital injections in the presence of proportional
transaction costs for the dual model with random return on investment. The associated
value function is defined as the expected present value of dividends minus costs of capital
injections until ruin. The rest of the paper is organized as follows. In Section 2, we give a
rigorous mathematical formulation of the problem. Section 3 works on the model without
capital injections, while Section 4 deals with the model with capital injections which never
goes bankrupt. Finally, we solve the general stochastic control problem in Section 5.

2 Problem formulation

Assume that the surplus generating process $P_t$ at time $t$ is given by

$$P_t = x - pt + \sigma_p W_{p,t} + \sum_{i=1}^{N_t} X_i, \quad t \geq 0, \tag{2.1}$$

where $x > 0$ is the initial assets, $p$ and $\sigma_p$ are positive constants, $\{W_{p,t}\}_{t \geq 0}$ is a standard
Brownian motion independent of the homogeneous compound Poisson process $\sum_{i=1}^{N_t} X_i$,
and $\{X_i\}$ is a sequence of independent and identically distributed random variables having
common distribution function $F$ with $F(0) = 0$. Let $\lambda$ be the intensity of the Poisson
process $N_t$. We assume throughout the paper that $E[X_i] < \infty$ and $\lambda E[X_i] - p > 0$. Here,
we consider the return on investment generating process
\[ R_t = rt + \sigma_R W_{R,t}, \quad t \geq 0, \quad (2.2) \]
where \( \{W_{R,t}\}_{t \geq 0} \) is another standard Brownian motion, and \( r \) and \( \sigma_R \) are positive constants. It is assumed that \( W_{p,t} \) and \( W_{R,t} \) are correlated in the way that
\[ W_{R,t} = \rho W_{p,t} + \sqrt{1 - \rho^2}W_{p,t}^0, \]
where \( \rho \in [-1,1] \) is constant, and \( W_{p,t}^0 \) is a standard Brownian motion independent of \( W_{p,t} \).

Define the risk process \( U_t \) as the total assets of the company at time \( t \), i.e., \( U_t \) is the solution to the stochastic differential equation
\[ U_t = P_t + \int_0^t U_s \, dR_s, \quad t \geq 0. \quad (2.3) \]
The solution to (2.3) is given by (see, e.g. Jaschke [19, Theorem 1])
\[ U_t = \mathcal{E}(R)_t \left( x + \int_0^t \mathcal{E}(R)^{-1}_s \, dP_s - \rho \sigma_p \sigma_R \int_0^t \mathcal{E}(R)^{-1}_s \, ds \right), \]
where
\[ \mathcal{E}(R)_t = \exp \{ (r - \frac{1}{2} \sigma^2_R)t + \sigma_R W_{R,t} \}. \]

Using Itô’s formula for semimartingale, one can show that the infinitesimal generator \( \mathcal{L} \) of \( U = \{U_t, t \geq 0\} \) is given by
\[ \mathcal{L}g(y) = (ry - p)g'(y) + \frac{1}{2} \left[ (\sigma_p + \rho \sigma_R y)^2 + \sigma^2_R (1 - \rho^2) y^2 \right] g''(y) + \lambda \int_0^\infty [g(y + z) - g(y)] F(dz). \quad (2.4) \]
The model (2.3) is a natural extension of the dual model in Avanzi and Gerber [5] and Avanzi et al. [6]. As was mentioned in Avanzi et al. [6], the dual model is appropriate for companies that have deterministic expenses and occasional gains whose amount and frequency can be modelled by the jump process \( \sum_{i=1}^{N_t} X_i \). For example, for companies such as pharmaceutical or petroleum companies, the jump could be interpreted as the net present value of future gains from an invention or discovery. Another example is the
venture capital investments or research and development investments. Venture capital funds screen out start-up companies and select some companies to invest in. When there is a technological breakthrough, the jump is generated. More examples can be found in Bayraktar and Egami [10] and Avanzi and Gerber [5].

In this paper, we denote by $L_t$ the cumulative amount of dividends paid up to time $t$ with $L_{0-} = 0$, and by $G_t$ the total amount of capital injections up to time $t$ with $G_{0-} = 0$. A dividend control strategy $\xi$ is described by the stochastic process $\xi = (L_t, G_t)$. A strategy is called admissible if both $L$ and $G$ are non-decreasing $\{\mathcal{F}_t\}$-adapted processes, and their sample paths are right-continuous with left limits. We denote by $\Xi$ the set of all admissible dividend policies. The risk process with initial capital $x \geq 0$ and controlled by a strategy $\xi$ is given by $U^\xi_t = \{U^\xi_{t-}, t \geq 0\}$, where $U^\xi_t$ is the solution to the stochastic differential equation

$$dU^\xi_t = dP_t + U^\xi_{t-}dR_t - dL^\xi_t + dG^\xi_t, \quad t \geq 0.$$  

Moreover, $L^\xi_t - L^\xi_{t-} \leq U^\xi_{t-}$ for all $t$. In words, the amount of dividends is smaller than the size of the available capitals. Let $\tau^\xi = \inf\{t \geq 0 : U^\xi_t = 0\}$ be the ruin time. Then, the associated performance function is given by

$$V(x; \xi) = E_x \left( \alpha \int_0^{\tau^\xi} e^{-\delta t} dL^\xi_t - \beta \int_0^{\tau^\xi} e^{-\delta t} dG^\xi_t \right), \quad (2.5)$$

where $\delta > 0$ is the discounted rate, $1 - \alpha$ ($0 < \alpha \leq 1$) is the rate of proportional costs on dividend transactions, $1 \leq \beta < \infty$ is the rate of proportional transaction costs of capital injections. The notation $E_x$ represents the expectation conditioned on $U^\xi_0 = x$ and the integral is understood pathwise in a Lebesgue-Stieltjes sense. Our aim is to find the value function

$$V_*(x) = \sup_{\xi \in \Xi} V(x; \xi), \quad (2.6)$$

and the optimal policy $\xi^* \in \Xi$ such that $V(x; \xi^*) = V_*(x)$ for all $x \geq 0$.

The study of optimal dividends has been around many years. The commonly-used approach to solving these optimal control problems is to proceed by guessing a candidate optimal solution, constructing the corresponding value function, and subsequently
verifying its optimality through a verification result. For the model of study, i.e., an upward jump-diffusion process with random return on investment, the optimal control problem remains to be solved. The problem of study can be seen as a natural extension of Bayraktar and Egami [10], and Avanzi, Shen and Wong [7]. In addition, one can see later that the method used in Bayraktar, Kyprianou and Yamazaki [11] cannot be applied to our model since their proof relies on certain characteristics of Lévy process. In order to solve the optimal control problem in this paper, we shall first consider two sub-optimal problems in the next two sections.

3 Optimal dividend problem without capital injections

In this section, we first consider the dividend problem without capital injections. We shall show that the barrier strategy solve the optimal dividend problem regardless of the jump distribution.

Let $\Xi_d = \{\xi_d = (L^{\xi_d}, G^{\xi_d}) : (L^{\xi_d}, G^{\xi_d}) \in \Xi \text{ and } G^{\xi_d} \equiv 0\}$. The associated controlled process is denoted by $U^{\xi_d} = \{U^{\xi_d}_t, t \geq 0\}$, where $U^{\xi_d}_t$ is the solution to the stochastic differential equation
\[
dU^{\xi_d}_t = dP_t + U^{\xi_d}_t dR_t - dL^{\xi_d}_t, \quad t \geq 0.
\]
and the value function is given by
\[
V_d(x) = \sup_{\xi_d \in \Xi_d} V(x; \xi_d) \equiv \sup_{\xi_d \in \Xi_d} E_x \left( \alpha \int_{0-}^{\tau_d} e^{-\delta t} dL^{\xi_d}_t \right), \quad x \geq 0, \quad (3.1)
\]
where $\tau_d = \inf \{t : U^{\xi_d}_t = 0\}$ is the time of ruin under the strategy $\xi_d$. We next identify the form of the value function $V_d$ and the optimal strategy $\xi^*_d$ such that $V_d(x) = V(x; \xi^*_d)$.

3.1 HJB equation and verification lemma

For notational convenience, denote $v(x) = V(x; \xi^*_d)$. If $v$ is twice continuously differentiable, then applying standard arguments from stochastic control theory (see Fleming and Soner [17]) or an approach similar to that in Azcue and Muler [9], we can show that
the value function fulfills the dynamic programming principle

\[ v(x) = \sup_{\xi_d \in \Xi} E_x \left( \int_0^{\tau_{\xi_d} \wedge T} e^{-\delta s} dL^\xi_s + e^{-\delta(\tau_{\xi_d} \wedge T)} v(U^\xi_{\tau_{\xi_d} \wedge T}) \right), \]

for any stopping time \( T \), and that the associated Hamilton-Jacobi-Bellman (HJB) equation is

\[ \max \{ \mathcal{L}v(x) - \delta v(x), \alpha - v'(x) \} = 0, \quad x > 0, \quad (3.2) \]

with \( v(0) = 0 \), where \( \mathcal{L} \) is the the extended generator of \( U \) defined in \( (2.4) \). The HJB equation \( (3.2) \) can also be obtained by the heuristic argument of Avanzi et al. \( [7] \).

**Lemma 3.1.** *(Verification Lemma)* Let \( v \) be a solution to \( (3.2) \). Then,

\[ v(x) \geq V_d(x), \quad x > 0, \]

for any admissible strategy \( \xi_d \in \Xi_d \), and thus

\[ v(x) \geq V_d(x). \]

**Proof.** For any admissible strategy \( \xi_d \in \Xi_d \), put \( \Lambda = \{ s : L^\xi_s \neq L^\xi_{s-} \} \). Applying Ito’s formula for semimartingale to \( e^{-\delta t} v(U^\xi_t) \) gives

\[
E_x \left[ e^{-\delta (t \wedge \tau_{\xi_d} - \tau_{\xi_d}^\xi)} v(U^\xi_{t \wedge \tau_{\xi_d} - \tau_{\xi_d}^\xi}) \right] = v(x) + E_x \int_0^{t \wedge \tau_{\xi_d}} e^{-\delta s} (\mathcal{L} - \delta) v(U^\xi_{s-}) ds \\
+ E_x \sum_{s \in \Lambda, s \leq t \wedge \tau_{\xi_d}} e^{-\delta s} \left\{ v(U^\xi_{s-}) - v(U^\xi_{s-}) \right\} \\
- E_x \int_0^{t \wedge \tau_{\xi_d} - \tau_{\xi_d}^\xi} e^{-\delta s} v'(U^\xi_{s-}) dL^\xi_{s,c}, \quad (3.3)
\]

where \( L^\xi_{s,c} \) is the continuous part of \( L^\xi_s \). From \( (3.2) \), we see that \( (\mathcal{L} - \delta) v(U^\xi_{s-}) \leq 0 \) and \( v'(x) \geq \alpha \). Thus, for \( s \in \Lambda, s \leq t \wedge \tau_{\xi_d} \),

\[
v(U^\xi_{s-}) - v(U^\xi_{s-}) \leq -\alpha (L^\xi_{s-} - L^\xi_{s-}). \quad (3.4)
\]

It follows from \( (3.3) \) and \( (3.4) \) that

\[
E_x \left[ e^{-\delta (t \wedge \tau_{\xi_d} - \tau_{\xi_d}^\xi)} v(U^\xi_{t \wedge \tau_{\xi_d} - \tau_{\xi_d}^\xi}) \right] \leq v(x) - \alpha E_x \int_0^{t \wedge \tau_{\xi_d} - \tau_{\xi_d}^\xi} e^{-\delta s} dL^\xi_s. \quad (3.5)
\]

Letting \( t \to \infty \) in \( (3.5) \) yields the result. \( \square \)

### 3.2 Construction of a candidate solution

It is assumed that dividends are paid according to the barrier strategy \( \xi_b \). Such a strategy has a level of barrier \( b > 0 \). When the surplus exceeds the barrier, the excess is
paid out immediately as dividends. Let $L_t^b$ be the total amount of dividends up to time $t$. The controlled risk process when taking into account of the dividend strategy $\xi_b$ is $U^b = \{U^b_t, t \geq 0\}$, where $U^b_t$ is the solution to the following stochastic differential equation

$$\text{d}U^b_t = \text{d}P_t + U^b_{t-}\text{d}R_t - \text{d}L^b_t, \quad t \geq 0.$$ 

Denote by $V_b(x)$ the expected discounted dividends function if the barrier strategy $\xi_b$ is applied, that is,

$$V_b(x) = \alpha E_x \left( \int_{T^b_x} e^{-\delta t} \text{d}L^b_t \right),$$

where $\delta > 0$ is the force of interest and $T^b_x = \inf\{t \geq 0 : U^b_t = 0\}$.

The following result shows that $V_b(x)$ as a function of $x$ satisfies an integro-differential equation with certain boundary conditions.

**Lemma 3.2.** For the risk process $U$ of (2.3) and the infinitesimal generator $L$ of (2.4), if $h_b(x)$ solves

$$L h_b(x) = \delta h_b(x), \quad 0 < x < b,$$

and $h_b(x) = h_b(b) + \alpha(x - b)$, for $x > b$, together with the boundary conditions

$$h_b(0) = 0, \quad h'_b(b) = \alpha,$$

then $h_b(x)$ coincides with $V_b(x)$ given by (3.6).

**Proof.** Applying Ito’s formula for semimartingale to $e^{-\delta t}h_b(U^b_{t-})$ gives

$$e^{-\delta t}h_b(U^b_{t-}) \quad - \quad h_b(U^b_{t-}) = \int_{0-}^{t-} e^{-\delta s} \text{d}N^b_s + \int_{0-}^{t-} e^{-\delta s} (L - \delta)h_b(U^b_{s-}) \text{d}s$$

$$\quad + \quad \sum_{s < t} \mathbf{1}_{\{|\triangle P_s| > 0\}} e^{-\delta s} \left\{ h_b(U^b_{s-} + \triangle P_s - \triangle L_s) - h_b(U^b_{s-} + \triangle P_s) \right\}$$

$$\quad - \quad \int_{0-}^{t-} e^{-\delta s} h'_b(U^b_{s-}) \text{d}L^c_s,$$

where $L^c_s$ is the continuous part of $L_s$, and

$$N^b_t = \sum_{s \leq t} \mathbf{1}_{\{|\triangle P_s| > 0\}} \left\{ h_b(U^b_{s-} + \triangle P_s) - h_b(U^b_{s-}) \right\}$$

$$\quad - \quad \int_{0}^{t} \int_{0}^{\infty} \left\{ h_b(U^b_{s-} + y) - h_b(U^b_{s-}) \right\} \Pi(dy) \text{d}s$$

$$\quad + \sigma \int_{0}^{t} h'_b(U^b_{s-}) \text{d}W_s.$$
Let $x > \text{capitals} x$.

To prove the lemma, we use arguments similar to those in Kulenko and Schmidli [20].

**Proof.**

We conjecture that the barrier strategy $\xi$.

Note that $P(\Delta L > 0, \Delta P < 0) = 0$ and that $U_s^b + \Delta P_s \geq U_s^b + \Delta P_s - \Delta L_s \geq b$ on $\{\Delta L > 0, \Delta P > 0\}$. Consequently,

$$\sum_{s<t} 1_{\{\Delta L_s > 0\}} e^{-\delta s} \left\{ h_b(U_s^b + \Delta P_s - \Delta L_s) - h_b(U_s^b + \Delta P_s) \right\}$$

$$= -\alpha \sum_{s<t} 1_{\{\Delta L_s > 0\}} e^{-\delta s} \Delta L_s.$$  

Note that $N_t^b$ is a local martingale, and

$$\int_{0^-}^{t^-} e^{-\delta s} h_b(U_s^b) dL_s = \int_{0^-}^{t^-} e^{-\delta s} h_b(U_s^b) dL_s = \alpha \int_{0^-}^{t^-} e^{-\delta s} h_b(0) dL_s.$$  

Thus, for any appropriate localization sequence of stopping times $\{t_n, n \geq 1\}$, we have

$$E_x(e^{-\delta(t_n \wedge T^b)} h_b(U_{t_n \wedge T^b}^b)) - E_x h_b(U_0^b) = -\alpha E_x \int_{0^-}^{t_n \wedge T^b} e^{-\delta s} dL_s. \quad \text{(3.8)}$$

Letting $n \to \infty$ in (3.8) yields the result. \hfill \Box

**Lemma 3.3.** $V_b(x)$ is a concave increasing function on $(0, \infty)$.

**Proof.**

To prove the lemma, we use arguments similar to those in Kulenko and Schmidli [20]. Let $x > 0, y > 0$, and $l \in (0, 1)$. Consider the strategies $L^x$ and $L^y$ for the initial capitals $x$ and $y$. Define $L_t = lL_t^x + (1 - l)L_t^y$. Then, $L_t = L_t^{lx+(1-l)y}$. Since the processes $\{P_t, t \geq 0\}$ and $\{R_t, t \geq 0\}$ have no negative jumps, we have $\tau_L = \tau_{L^x} \vee \tau_{L^y}$. It follows that

$$V_b(lx + (1 - l)y) = \alpha E_x \left( \int_{0^-}^{\tau_L^-} e^{-\delta t} dL_t \right)$$

$$= \alpha l E_x \left( \int_{0^-}^{\tau_L^-} e^{-\delta t} dL_t^x \right) + \alpha (1 - l) E_x \left( \int_{0^-}^{\tau_L^-} e^{-\delta t} dL_t^y \right)$$

$$\geq \alpha l E_x \left( \int_{0^-}^{\tau_L^-} e^{-\delta t} dL_t^x \right) + \alpha (1 - l) E_x \left( \int_{0^-}^{\tau_L^-} e^{-\delta t} dL_t^y \right)$$

$$= lV_b(x) + (1 - l)V_b(y),$$

and thus the concavity of $V_b$ follows. The increasingness of $V_b(x)$ is trivial. \hfill \Box

### 3.3 Verification of optimality

Define the barrier level by

$$b^* = \sup\{b \geq 0 : V_b'(b-) = \alpha\}.$$  

We conjecture that the barrier strategy $\xi_{b^*}$ is optimal.
Proposition 3.1. \( b^* = 0 \) if and only if \( \lambda \int_0^\infty yF(dy) \leq p \).

Proof. Here, we follow the approach of Yao et al. [23] to prove the proposition. Suppose that \( b^* = 0 \). Then, the associated value function is \( V_d(x) = ax \) which satisfies the HJB equation (3.2). As a result, we obtain \( (\Gamma - \delta)V_d(x) \leq 0 \) which in turn gives \( \lambda \int_0^\infty yF(dy) \leq p \). On the other hand, suppose that \( \lambda \int_0^\infty yF(dy) \leq p \). Then, \( w(x) = ax \) satisfies (3.2). By Lemma 3.1, we get \( w(x) = V_d(x) \). However, \( w(x) \leq V_d(x) \) since \( w(x) = ax \) is the performance function associated with the strategy that \( x \) is paid immediately as dividends. In this case, ruin occurs immediately. Thus, \( w(x) = V_d(x) \) and the optimal barrier level \( b^* = 0 \). □

Theorem 3.1. If \( \lambda \int_0^\infty yF(dy) > p \), then the function \( V_{b^*} \) defined in (3.6) satisfies

\[
V_{b^*}(x) = V_d(x), \quad x \geq 0,
\]

and the optimal barrier strategy \( \xi^*_d \) is the solution to

\[
dU_t^{\xi^*_d} = dP_t + U_t^{\xi^*_d} dR_t - dL_t^{\xi^*_d}, \quad t \geq 0,
\]

with the conditions

\[
U_t^{\xi^*_d} \leq b^*, \quad G_t^{\xi^*_d} \equiv 0, \quad \int_0^\infty 1_{\{U_t^{\xi^*_d} < b^*\}} dL_t^{\xi^*_d} = 0.
\]

Proof. Using the method of Avanzi and Gerber [5], it can be shown that \( V_{b^*}(x) \) is twice continuously differentiable at \( x = b^* \). Consequently, \( V_{b^*} \in C^2(\mathbb{R}_+) \). Note that \( (\mathcal{L} - \delta)V_{b^*}(x) = 0 \) and \( V_{b^*}'(x) \geq \alpha \) for \( x \in [0, b^*) \) due to the concavity of \( V_{b^*} \) on \([0, b^*)\).

Since \( V_{b^*}'(x) = \alpha(x - b^*) + V_{b^*}(b^*) \) for \( x \geq b^* \), we have

\[
(\mathcal{L} - \delta)V_{b^*}(x) = -\alpha + \alpha \int_0^\infty yF(dy) - \alpha(x - b^*) - \delta V_{b^*}(b^*)
\]

\[
< -\alpha + \alpha \int_0^\infty yF(dy) - \delta V_{b^*}(b^*)
\]

\[
= \lim_{x \to b^*^+} (\mathcal{L} - \delta)V_{b^*}(x) = \lim_{x \to b^*^-} (\mathcal{L} - \delta)V_{b^*}(x) = 0,
\]

because of the continuity of \( V_{b^*}', V_{b^*}'' \), and \( V_{b^*}'' \) at \( x = b^* \). Thus, the function \( V_{b^*} \) satisfies the HJB equation (3.2). Then, it follows from Lemma 3.1 that \( V_{b^*}(x) \geq V_d(x) \). However, \( V_{b^*}(x) \leq V_d(x) \) by definition, and hence \( V_{b^*}(x) = V_d(x) \). □
3.4 Two closed-form solutions

Owing to the complexity of the equation, the solution may not be available in explicit form in general. The following two examples show that one can derive closed-form solution in some special cases.

Example 3.1. Assume that \( r = 0 \) and \( \sigma_R = 0 \). Then, \( V_{b^*}(x) \) satisfies the following integro-differential equation

\[
\mathcal{A}V_{b^*}(x) = \delta V_{b^*}(x), \quad 0 < x < b^*,
\]

and

\[
V_{b^*}(x) = \alpha(x - b^*) + V_{b^*}(b^*), \quad x > b^*,
\]

with the boundary conditions

\[
V_{b^*}(0) = 0, \quad V_{b^*}'(x)|_{x=b^*} = \alpha,
\]

where

\[
\mathcal{A}g(x) = \frac{1}{2}\sigma^2 g''(x) - pg'(x) - \lambda g(x) + \lambda \int_0^\infty g(x+y)F(dy).
\]

Following the arguments of Laplace transform used in Yin, Wen and Zhao [28], one can show that the solution to (3.9)-(3.11) is given by

\[
V_{b^*}(x) = -\alpha Z^{(\delta)}(b^*-x) + \alpha \frac{E[X_1]}{\delta},
\]

and

\[
b^* = (Z^{(\delta)})^{-1}\left(\frac{E[X_1]}{\delta}\right),
\]

where

\[
Z^{(\delta)}(x) = 1 + \delta \int_0^x W^{(\delta)}(y)dy, \quad \overline{Z^{(\delta)}}(x) = \int_0^x Z^{(\delta)}(y)dy, \quad x \in \mathbb{R}.
\]

Here, \( W^{(\delta)} \) is the so-called \( \delta \)-scale function defined in the way that \( W^{(\delta)}(x) = 0 \) for all \( x < 0 \) and that its Laplace transform on \([0, \infty)\) is given by

\[
\int_0^\infty e^{-\theta x} W^{(\delta)}(x)dx = \frac{1}{\Psi(\theta) - \delta}, \quad \theta > \sup\{\theta \geq 0 : \Psi(\theta) = \delta\},
\]
where

\[ \Psi(\theta) = p\theta + \frac{1}{2}a^2\theta^2 + \lambda \int_0^\infty (e^{-\theta x} - 1) F(dx). \]

For further details, the reader is referred to Yin and Wen [26]. \qed

**Example 3.2.** Let \( \sigma_R = \sigma_p = 0 \). Assume that \( X_i \) is exponentially distributed with parameter \( \mu \). Then, by Theorem 3.1 and Lemma 3.2, it can be shown that \( V_{b^*}(x) \) satisfies the following integro-differential equation

\[ (rx - p)V'_{b^*}(x) + \lambda \mu \int_0^\infty V_{b^*}(x + z)e^{-\mu z}dz = (\lambda + \delta)V_{b^*}(x), \quad 0 < x < b^*, \quad (3.12) \]

and

\[ V_{b^*}(x) = \alpha(x - b^*) + V_{b^*}(b^*), \quad x > b^*, \quad (3.13) \]

with the boundary conditions

\[ V_{b^*}(0) = 0, \quad V_{b^*}'(x)|_{x=b^*} = \alpha. \quad (3.14) \]

From equation (3.12), we find that

\[ zg''(z) + \left(1 - \frac{\lambda + \delta}{r} - z\right)g'(z) + \frac{\delta}{r}g(z) = 0, \]

where

\[ g(z) = V_{b^*}(x), \quad z = \mu \left(x - \frac{p}{r}\right). \]

Note that this is Kummer’s confluent hypergeometric equation with the solution given by

\[ g(z) = C_1M \left(-\frac{\delta}{r}, 1 - \frac{\lambda + \delta}{r}, z\right) + C_2U \left(-\frac{\delta}{r}, 1 - \frac{\lambda + \delta}{r}, z\right), \]

where \( C_1 \) and \( C_2 \) are constants, and \( M(a, b, x) \) is the standard confluent hypergeometric function with \( U(a, b, x) \) being its second form; see, for example, Abramowitz and Stugen [1, pp. 504-505]. Then, it follows that

\[ V_{b^*}(x) = C_1M \left(-\frac{\delta}{r}, 1 - \frac{\lambda + \delta}{r}, \mu(x - \frac{p}{r})\right) + C_2U \left(-\frac{\delta}{r}, 1 - \frac{\lambda + \delta}{r}, \mu(x - \frac{p}{r})\right). \]

Using the boundary conditions (3.14) and the formulae

\[ M'(a, b, z) = \frac{a}{b}M(a + 1, b + 1, z), \quad U'(a, b, z) = -aU(a + 1, b + 1, z), \]

\[ V_{b^*}(x) = C_1M \left(-\frac{\delta}{r}, 1 - \frac{\lambda + \delta}{r}, \mu(x - \frac{p}{r})\right) + C_2U \left(-\frac{\delta}{r}, 1 - \frac{\lambda + \delta}{r}, \mu(x - \frac{p}{r})\right). \]
we obtain the coefficients

\[ C_1 = \frac{\alpha U(-\delta/r, 1 - \frac{\lambda + \delta}{r}, -\mu r)}{\Delta(b^*)}, \]

and

\[ C_2 = -\frac{\alpha M(-\delta/r, 1 - \frac{\lambda + \delta}{r}, -\mu r)}{\Delta(b^*)}, \]

where

\[ \Delta(b^*) = -\frac{\mu \delta}{r - \lambda - \delta} U \left(-\delta/r, 1 - \frac{\lambda + \delta}{r}, -\frac{\mu p}{r}\right) M \left(1 - \frac{\delta}{r}, 2 - \frac{\lambda + \delta}{r}, \mu (b^* - \frac{p}{r})\right) \]

\[ + \frac{\mu \delta}{r} M \left(-\frac{\delta}{r}, 1 - \frac{\lambda + \delta}{r}, -\frac{\mu p}{r}\right) U \left(1 - \frac{\delta}{r}, 2 - \frac{\lambda + \delta}{r}, \mu (b^* - \frac{p}{r})\right), \]

and \( b^* \) is the maximizer of term \( 1/\Delta(b) \) with respect to \( b \), i.e.,

\[ b^* = \text{argmax} \frac{1}{\Delta(b)}. \]

\[ \square \]

4 Optimal dividend problem with capital injections

In this section, we consider the optimal dividend problem with capital injections. The set of admissible strategies is given by

\[ \Xi_c = \{\xi_c = (L_{\xi_c}, G_{\xi_c}) : (L_{\xi_c}, G_{\xi_c}) \in \Xi \text{ and } U^\xi_t \geq 0\}. \]

The controlled surplus process \( U^\xi_t \) satisfies

\[ dU^\xi_t = dP_t + U^\xi_t dR_t - dL^\xi_t + dG^\xi_t, \quad t \geq 0, \]

and the value function is defined as

\[ V_c(x) = \sup_{\xi_c \in \Xi_c} V(x; \xi_c) = \sup_{\xi_c \in \Xi_c} E_x \left( \alpha \int_0^\infty e^{-\delta t} dL^\xi_t - \beta \int_0^\infty e^{-\delta t} dG^\xi_t \right), \quad x \geq 0. \quad (4.1) \]

Since the controlled surplus process always stays positive, the company will never go bankrupt. We shall identify the form of the value function \( V_c \) and the optimal strategy \( \xi^*_c \) such that \( V_c(x) = V(x; \xi^*_c) \).

4.1 HJB equation and verification lemma
Applying the techniques used in Section 3, we get the HJB equation and the verification Lemma.

\[
\max\{Lw(x) - \delta w(x), \ \alpha - w'(x), w'(x) - \beta\} = 0, \quad x \geq 0. \tag{4.2}
\]

**Lemma 4.1.** *(Verification Lemma)* Let \( w \) be a solution to (4.2). Then, \( w(x) \geq V(x; \xi_c) \) for any admissible strategy \( \xi_c \in \Xi_c \), and thus \( w(x) \geq V_c(x) \).

### 4.2 Construction of a candidate solution

We now construct a concave \( C^2 \) solution \( H \) to the HJB equation (4.2). Due to the effect of the discount factor, it is clear that the optimal strategy is the one that postpone capital injections as long as possible, i.e., we inject capital only when surplus become zero.

Consider the barrier strategy with the upper barrier \( B^* \) and the lower barrier 0, and the strategy \( \pi^* = (L_{t}^{\pi^*}, G_{t}^{\pi^*}) \) where \((U_{t}^{\pi^*}, L_{t}^{\pi^*}, x_{t}, G_{t}^{\pi^*})\) is a solution to the following system

\[
dU_{t}^{\pi^*} = dP_t + U_{t}^{\pi^*}dR_t - dL_{t}^{\pi^*} + dG_{t}^{\pi^*}, \tag{4.3}
\]

\[
0 \leq U_{t}^{\pi^*} \leq B^*, \quad t \geq 0, \tag{4.4}
\]

\[
L_{t}^{\pi^*, x} = \max(x - B^*, 0) + \int_{0^-}^{t} 1(U_{s}^{\pi^*} = B^*)dL_{s}^{\pi^*}, \quad t > 0, \tag{4.5}
\]

\[
G_{t}^{\pi^*, x} = \max\left(-\inf_{0 \leq s \leq t}(P_s - L_{s}^{\pi^*}), 0\right), \quad t > 0. \tag{4.6}
\]

**Lemma 4.2.** For the problem of (4.3)-(4.6), if \( H(x) \) solves

\[
\mathcal{L}H(x) = \delta H(x), \quad 0 < x < B^*,
\]

with \( H(x) = H(B^*) + \alpha(x - B^*) \) for \( x > B^* \) and the boundary conditions

\[
H'(0) = \beta, \quad H'(B^*) = \alpha,
\]

where the infinitesimal generator \( \mathcal{L} \) is given by (2.4), then \( H(x) \) is given by

\[
H(x) = V(x; \pi^*) \equiv E_x \left( \alpha \int_{0^-}^{\infty} e^{-\delta t}dL_{t}^{\pi^*, x} - \beta \int_{0^-}^{\infty} e^{-\delta t}dG_{t}^{\pi^*, x}\right), \quad x \geq 0. \tag{4.7}
\]

**Proof.** For the strategy \( \pi^* \), define \( \Lambda = \{s : L_{s}^{\pi^*, x} \neq L_{s}^{\pi^*, x} \} \). Let \( L_{t}^{\pi^*, x,c} \) be the continuous part of \( L_{t}^{\pi^*, x} \). Since the process is skip-free downward, \( G_{t}^{\pi^*, x} \) is continuous. In addition, we
see from (4.6) that $G_t^{π^*,x} ≥ 0$ and that the support of the Stieltjes measure $dG_t^{π^*,x}$ is contained in the closure of the set $\{ t : U_t^{π^*} = 0 \}$. Applying Ito’s formula for semimartingale to $e^{-δt}H(U_t^{π^*})$ gives

$$E_x[e^{-δt}H(U_t^{π^*})] = H(x) + E_x \int_0^t e^{-δs}(L - δ)H(U_s^{π^*})ds + E_x \sum_{s ∈ Λ, s ≤ t} e^{-δs}\{H(U_s^{π^*}) - H(U_s^{π^*})\} - E_x \int_0^t e^{-δs}H'(U_s^{π^*})dL_s^{π^*,x,c} + E_x \int_0^t e^{-δs}H'(U_s^{π^*})dG_s^{π^*,x}. \tag{4.8}$$

Note that $(L - δ)H(U_t^{π^*}) = 0$, and that

$$E_x \sum_{s ∈ Λ, s ≤ t} e^{-δs}\{H(U_s^{π^*}) - H(U_s^{π^*})\} = α \sum_{s ≤ t} e^{-δs}(L_s^{π^*,x} - L_s^{π^*,x}),$$

$$E_x \int_0^t e^{-δs}H'(U_s^{π^*})dL_s^{π^*,x,c} = E_x \int_0^t e^{-δs}H'(U_s^{π^*})dL_s^{π^*,x,c} = αE_x \int_0^t e^{-δs}dL_s^{π^*,x,c},$$

$$E_x \int_0^t e^{-δs}H'(U_s^{π^*})dG_s^{π^*,x} = E_x \int_0^t e^{-δs}H'(U_s^{π^*})dG_s^{π^*,x} = βE_x \int_0^t e^{-δs}dG_s^{π^*,x}. $$

Then, it follows that

$$E_x[e^{-δt}H(U_t^{π^*})] = H(x) - αE_x \int_0^t e^{-δs}dL_s^{π^*,x} + βE_x \int_0^t e^{-δs}dG_s^{π^*,x}. \tag{4.9}$$

Since $\lim_{t → ∞} E_x[e^{-δt}H(U_t^{π^*})] ≤ \lim_{t → ∞} E_x[e^{-δt}H(B^*)] = 0$, letting $t → ∞$ in (4.9) and using the monotone convergence theorem yield

$$H(x) = αE_x \int_0^∞ e^{-δs}dL_s^{π^*,x} - βE_x \int_0^∞ e^{-δs}dG_s^{π^*,x} = V(x; π^*).$$

□

**Lemma 4.3.** $V(x; π^*)$ is a concave increasing function on $(0, ∞)$.

**Proof.** Similar to the proof of Lemma 3.3, we use the arguments of Kulenko and Schmidli [20]. Let $x > 0$, $y > 0$, and $l ∈ (0, 1)$. Consider the strategies $(L_t^{π^*,x}, G_t^{π^*,x})$ and $(L_t^{π^*,y}, G_t^{π^*,y})$ for the initial capitals $x$ and $y$. Define $L_t = lL_t^{π^*,x} + (1 - l)L_t^{π^*,y}$ and
\[G_t = lG_t^{\pi^*,x} + (1 - l)G_t^{\pi^*,y}. \] Then, \(L_t = L_t^{\pi^*,x + (1-l)y}\). So, we have

\[
lx + (1 - l)y + \int_0^t \mathcal{E}(R)^{-1}_s dP_s - \rho \sigma_p \sigma_R \int_0^t \mathcal{E}(R)^{-1}_s ds
- \int_0^t \mathcal{E}(R)^{-1}_s (ldL^{\pi^*,x}_s + (1 - l)dL^{\pi^*,y}_s)
+ \int_0^t \mathcal{E}(R)^{-1}_s (ldG^{\pi^*,x}_s + (1 - l)dG^{\pi^*,y}_s)
= l \left\{ x + \int_0^t \mathcal{E}(R)^{-1}_s dP_s - \rho \sigma_p \sigma_R \int_0^t \mathcal{E}(R)^{-1}_s ds
- \int_0^t \mathcal{E}(R)^{-1}_s dL^{\pi^*,x}_s + \mathcal{E}(R)_t \int_0^t \mathcal{E}(R)^{-1}_s dG^{\pi^*,x}_s \right\}
+ (1 - l) \left\{ y + \int_0^t \mathcal{E}(R)^{-1}_s dP_s - \rho \sigma_p \sigma_R \int_0^t \mathcal{E}(R)^{-1}_s ds
- \int_0^t \mathcal{E}(R)^{-1}_s dL^{\pi^*,y}_s + \mathcal{E}(R)_t \int_0^t \mathcal{E}(R)^{-1}_s dG^{\pi^*,y}_s \right\} \geq 0.
\]

This shows that the strategy \((L_t, G_t)\) is admissible and that

\[G_t^{\pi^*,x + (1-l)y} \leq lG_t^{\pi^*,x} + (1 - l)G_t^{\pi^*,y}. \]

It follows that

\[
V(lx + (1 - l)y, \pi^*) = E \left( \alpha \int_0^\infty e^{-\delta t} dL^{\pi^*,x + (1-l)y}_t - \beta \int_0^\infty e^{-\delta t} dG^{\pi^*,x}_t \right)
\geq lE \left( \alpha \int_0^\infty e^{-\delta t} dL^{\pi^*,x}_t - \beta \int_0^\infty e^{-\delta t} dG^{\pi^*,x}_t \right)
+ (1 - l)E \left( \alpha \int_0^\infty e^{-\delta t} dL^{\pi^*,y}_t - \beta \int_0^\infty e^{-\delta t} dG^{\pi^*,y}_t \right)
= lV(x, \pi^*) + (1 - l)V(y, \pi^*),
\]

which implies the concavity of \(V\). The proof of increasingness of \(V(x, \pi^*)\) is routine. \(\square\)

### 4.3 Verification of optimality

Define the barrier level as

\[B^* = \sup \{ B \geq 0 : H'(B^-) = \alpha \}. \]

We conjecture that the barrier strategy \(\pi^*\) is optimal.
Theorem 4.1. The value function $H$ defined in (4.7) satisfies

$$H(x) = V_c(x) = \sup_{\xi \in \Xi} V_{\xi c}(x),$$

and the joint strategy $\pi^* = (L^*, G^*)$ is optimal, where $(L^*, G^*)$ is given by (4.5) and (4.6).

Proof. Note that $(\mathcal{L} - \delta)H(x) = 0$ and $\alpha \leq H'(x) \leq \beta$ for $x \in [0, B^*)$ due to the concavity of $H$ on $[0, B^*)$. For $x \geq B^*$ and $H(x) = \alpha(x - B^*) + H(B^*)$, we have

$$(\mathcal{L} - \delta)H(x) = -p\alpha + \alpha \int_0^\infty y\Pi(dy) - \alpha(x - B^*) - \delta H(B^*) < -p\alpha + \alpha \int_0^\infty y\Pi(dy) - \delta H(B^*) = \lim_{x \to B^+} (\mathcal{L} - \delta)H(x) = \lim_{x \to B^-} (\mathcal{L} - \delta)H(x) = 0.$$ 

Due to the continuity of $H$, $H'$ and $H''$ at $x = B^*$. Thus, the function $H$ satisfies the HJB equation (4.2). By Lemma 4.1, we get $H(x) \geq V_c(x)$. On the other hand, $H(x) \leq V_c(x)$. Thus, $H(x) = V_c(x)$. □

4.4 Two closed-form solutions

We now present two examples in which closed-form solution can be derived.

Example 4.1. Assume that $r = 0$ and $\sigma_R = 0$. Then, $H(x)$ satisfies the following integro-differential equation

$$\mathcal{A}H(x) = \delta H(x), \quad 0 < x < B^*, \quad (4.10)$$

and

$$H(x) = \alpha(x - B^*) + H(B^*), \quad x > B^*, \quad (4.11)$$

with the boundary conditions

$$H'(0) = \beta, \quad H'(B^*) = \alpha, \quad (4.12)$$

where

$$\mathcal{A}g(x) = \frac{1}{2}\sigma_p^2 p''(x) - pg'(x) - \lambda g(x) + \lambda \int_0^\infty g(x + y)F(dy).$$
Again, using the arguments of Laplace transform, one can show that the solution to (4.10) and (4.11) is given by

\[ H(x) = -a \overline{Z}^{(\delta)}(B^* - x) + \alpha \frac{E[X_1]}{\delta}, \]

and

\[ B^* = (Z^{(\delta)})^{-1} \left( \frac{\beta}{\alpha} \right), \]

where \( Z^{(\delta)}(x) \) and \( \overline{Z}^{(\delta)}(x) \) are defined in Example 3.1. In the case of \( \alpha = 1 \), these formulae were obtained in Bayraktar, Kyprianou and Yamazaki [11] by using the fluctuation theory of spectrally positive Lévy processes.

**Example 4.2.** Let \( \sigma_R = \sigma_p = 0 \). Assume that \( X_i \) is exponentially distributed with parameter \( \mu \). Then, by Theorem 4.1 and Lemma 4.2, \( H(x) \) satisfies the following integro-differential equation

\[ (rx - p)H'(x) + \lambda \mu \int_{0}^{\infty} H(x + z)e^{-\mu z}dz = (\lambda + \delta)H(x), \quad 0 < x < B^*, \quad (4.13) \]

and

\[ H(x) = \alpha(x - B^*) + H(B^*), \quad x > B^*, \quad (4.14) \]

with the boundary conditions

\[ H'(0) = \beta, \quad H'(B^*) = \alpha. \quad (4.15) \]

Repeating the steps in Example 3.2, we obtain

\[ H(x) = C_3 M \left( -\frac{\delta}{r}, 1 - \frac{\lambda + \delta}{r}, \mu(x - \frac{p}{r}) \right) + C_4 U \left( -\frac{\delta}{r}, 1 - \frac{\lambda + \delta}{r}, \mu(x - \frac{p}{r}) \right). \]

The constants \( C_3 \) and \( C_4 \) can be determined from the boundary conditions (4.15). Using the formulae

\[ M'(a, b, z) = \frac{a}{b} M(a + 1, b + 1, z), \quad U'(a, b, z) = -a U(a + 1, b + 1, z), \]

we get

\[ C_3 = \frac{\beta \Delta_1 - \alpha \Delta_2}{\Delta_1 \Delta_4 - \Delta_2 \Delta_3}. \]
and

\[ C_4 = \frac{\alpha \Delta_1 - \beta \Delta_3}{\Delta_1 \Delta_4 - \Delta_2 \Delta_3}, \]

where

\[ \Delta_1 = -\frac{\mu \delta}{r - \lambda - \delta} M \left( 1 - \frac{\delta}{r}, 2 - \frac{\lambda + \delta}{r}, -\frac{\mu p}{r} \right), \]

\[ \Delta_2 = \frac{\mu \delta}{r} U \left( 1 - \frac{\delta}{r}, 2 - \frac{\lambda + \delta}{r}, -\frac{\mu p}{r} \right), \]

\[ \Delta_3 = -\frac{\mu \delta}{r - \lambda - \delta} M \left( 1 - \frac{\delta}{r}, 2 - \frac{\lambda + \delta}{r}, \mu (B^* - \frac{p}{r}) \right), \]

\[ \Delta_3 = \frac{\mu \delta}{r} U \left( 1 - \frac{\delta}{r}, 2 - \frac{\lambda + \delta}{r}, \mu (B^* - \frac{p}{r}) \right). \]

Here, \( B^* \) is the unique solution to the following equation with respect to \( b \):

\[-\frac{\mu \delta}{r - \lambda - \delta} C_3 M \left( 1 - \frac{\delta}{r}, 2 - \frac{\lambda + \delta}{r}, \mu (b - \frac{p}{r}) \right) + \frac{\mu \delta}{r} U \left( 1 - \frac{\delta}{r}, 2 - \frac{\lambda + \delta}{r}, \mu (b - \frac{p}{r}) \right) = \alpha. \]

5 Solution to the problem without constraints

We now consider the control problem (2.6) without any restrictions on capital injections. In this case, ruin can occur and the time of ruin for a control strategy \( \xi \) is defined as

\[ \tau_\xi = \inf \{ t : U_t^\xi = 0 \}, \]

because of the diffusion and the skip-free downward surplus process. Then, it follows from (3.1), (4.1) and (2.5) that for all \( x \geq 0 \), \( V_\xi (x) \geq \max \{ V_d (x), V_c (x) \} \). We shall determine \( V_* \) and the optimal strategy \( \xi^* \) such that \( V_* (x) = V (x; \xi^*) \).

5.1 Verification lemma

For the control problem without any restrictions on capital injections, we get the following associated HJB equation:

\[ \max \{ \mathcal{L} v(x) - \delta v(x), \alpha - v'(x), v'(x) - \beta \} = 0, \quad x \geq 0, \]

with the boundary condition

\[ \max \{ -v(0), v'(0) - \beta \} = 0. \]
Lemma 5.1. (Verification Lemma) If \( v \) satisfies the HJB equation (5.1) with the boundary condition (5.2), then \( v(x) \geq V_\xi(x) \) for any admissible policy \( \xi \).

Proof. For any admissible strategy \( \xi \in \Xi \), put \( \lambda = \{ s : L_\xi^s \neq L_\xi^{s-} \} \). Applying Ito’s formula for semimartingale to \( e^{-\delta t} v(U_\xi^t) \) gives

\[
E_x[e^{-\delta(t\wedge\tau_\xi)} v(U_\xi^{t\wedge\tau_\xi-})] = v(x) + E_x \int_0^{t\wedge\tau_\xi-} e^{-\delta s} (\mathcal{L} - \delta) v(U_\xi^s) ds + E_x \sum_{s \in \lambda, s \leq t \wedge \tau_\xi} e^{-\delta s} \left\{ v(U_\xi^s) - v(U_\xi^{s-}) \right\} - E_x \int_0^{t\wedge\tau_\xi-} e^{-\delta s} v'(U_\xi^{s-}) dL_\xi^s + E_x \int_0^{t\wedge\tau_\xi-} e^{-\delta s} v'(U_\xi^s) dG_\xi^s, \tag{5.3}
\]

where \( L_\xi^{s-} \) is the continuous part of \( L_\xi^s \). We see from (5.1) that \( (\mathcal{L} - \delta) v(U_\xi^s) \leq 0 \) and \( \alpha \leq v'(x) \leq \beta \). Thus,

\[
E_x \int_0^{t\wedge\tau_\xi-} e^{-\delta s} v'(U_\xi^{s-}) dL_\xi^s \leq \beta E_x \int_0^{t\wedge\tau_\xi-} e^{-\delta s} dG_\xi^s, \tag{5.4}
\]

and for \( s \in \lambda, s \leq t \wedge \tau_\xi \),

\[
v(U_\xi^s) - v(U_\xi^{s-}) \leq -\alpha (L_\xi^s - L_\xi^{s-}). \tag{5.5}
\]

It follows from (5.3) and (5.5) that

\[
E_x[e^{-\delta(t\wedge\tau_\xi)} v(U_\xi^{t\wedge\tau_\xi-})] \leq v(x) - \alpha E_x \int_0^{t\wedge\tau_\xi-} e^{-\delta s} dL_\xi^s + \beta E_x \int_0^{t\wedge\tau_\xi-} e^{-\delta s} dG_\xi^s, \tag{5.6}
\]

Finally, by letting \( t \to \infty \) in (5.6) and noting that (by Fatou’s lemma)

\[
\liminf_{t \to \infty} E_x[e^{-\delta(t\wedge\tau_\xi)} v(U_\xi^{t\wedge\tau_\xi-})] \geq E_x[\liminf_{t \to \infty} e^{-\delta(t\wedge\tau_\xi)} v(U_\xi^{t\wedge\tau_\xi-})] \geq v(0) E_x[e^{-\delta \tau_\xi}] \geq 0,
\]

we prove the lemma. \( \square \)

5.2 Construction of a candidate solution

For any \( x \geq 0 \), we set our candidate strategy to be

\[
\xi^* = \begin{cases} 
\xi_1^*, & \text{if } V_{\xi_1}'(0) \leq \beta, \\
\xi_2^*, & \text{if } H(0) \geq 0,
\end{cases} \tag{5.7}
\]
and our candidate solution to be
\[ V_{\xi^*}(x) = \begin{cases} V_d(x), & \text{if } V_{b^*}'(0) \leq \beta, \\ V_c(x), & \text{if } H(0) \geq 0, \end{cases} \tag{5.8} \]
where \( V_d \) and \( V_c \) are given by (3.1) and (4.1), respectively, and \( V_{b^*} \) and \( H \) are given by (3.6) and (4.7), respectively.

### 5.3 Verification of optimality

**Theorem 5.1.** The value function \( V_{\xi^*} \) defined in (5.8) satisfies
\[ V_{\xi^*}(x) = V_*(x) = \sup_{\xi \in \Xi} V(x; \xi), \]
and the joint strategy \( \xi^* \) defined in (5.7) is optimal.

**Proof.** If \( V_{b^*}'(0) \leq \beta \), then \( V_{b^*} \) satisfies the equation (5.1) with the condition (5.2). Hence, \( V_{b^*}(x) \geq V_*(x) \). On the other hand, \( V_{b^*}'(x) = V(x; \xi^*_d) \leq V_d(x) \). It follows that \( V_{\xi^*}(x) = V_{b^*}(x) = V_d(x) \). The optimality of \( \xi^*_d \) is verified by Theorem 3.1. If \( H(0) \geq 0 \), then \( H \) satisfies the HJB equation (4.1), so that \( H(x) \leq V_c(x) \). Since \( H \) also satisfies the equation (5.1) with the condition (5.2), \( H(x) \geq V(x; \xi^*_c) \geq V_c(x) \). Hence, we have \( V_{\xi^*}(x) = V(x; \xi^*_c) = V_c(x) \). The optimality of \( \xi^*_c \) is verified by Theorem 4.1. \( \square \)

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