Stability and intersection properties of solutions to the nonlinear biharmonic equation

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Abstract
We study the positive, regular, radially symmetric solutions to the nonlinear biharmonic equation $\Delta^2 \varphi = \varphi^p$. First, we show that there exists a critical value $p_c$, depending on the space dimension, such that the linearized operator has a negative eigenvalue, if $p < p_c$, but no negative spectrum, if $p \geq p_c$. Focusing on the supercritical case $p \geq p_c$, we then show that the graphs of no two solutions intersect one another.

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1. Introduction

In a recent paper [5], Strauss and the author derived a sharp condition for the instability of time-independent solutions of the nonlinear heat and wave equations

$$\partial_t^i u - \Delta u = |u|^p,$$

where $i = 1, 2$ and $p > 1$. To prove instability for a given time-independent solution $\varphi$, they had to analyse the spectrum of the linearized operator $-\Delta - p|\varphi|^{p-2}\varphi$, and they were able to carry out this task for all positive, radially symmetric solutions $\varphi$. In this paper, we extend the analysis in [5] and we begin a similar investigation for the fourth-order analogue

$$\partial_t^4 u + \Delta^2 u = |u|^p$$

that arises when one studies systems of second-order problems. In particular, we analyse the spectrum of the linearized operator for all positive, radially symmetric solutions $\varphi$ and we also address the intersection properties\(^1\) of such solutions.

\(^1\) Due to scaling, the existence of one solution implies the existence of infinitely many solutions.
When it comes to the fourth-order equation, the existence of time-independent solutions was studied in [1, 3, 8]. It is known that positive, regular, radially symmetric solutions of

$$\Delta^2 \psi(x) = \psi(x)^p, \quad x \in \mathbb{R}^n$$

exist when $n > 4$ and $p \geq \frac{n+4}{n-4}$, while no positive, regular solutions exist, otherwise. To address the stability of these solutions in the context of the time-dependent problem (1.2), one has to first analyse the spectrum of the linearized operator $\mathcal{L} = \Delta^2 - p\psi^{p-1}$. In this paper, we show that $\mathcal{L}$ has an unstable (i.e. negative) eigenvalue if and only if

$$p \cdot Q_4 \left( \frac{4}{p-1} \right) \leq Q_4 \left( \frac{n-4}{2} \right), \quad Q_4(\alpha) \equiv |x|^{\alpha+4} \Delta^4 |x|^{-\alpha}. \quad (1.4)$$

This provides a fourth-order analogue of the condition that arose in the second-order case [5], in which case the linearized operator has an unstable eigenvalue if and only if

$$p \cdot Q_2 \left( \frac{2}{p-1} \right) \leq Q_2 \left( \frac{n-2}{2} \right), \quad Q_2(\alpha) \equiv |x|^{\alpha+2} (\Delta) |x|^{-\alpha}. \quad (1.5)$$

Although originally stated in a different way, the last two conditions appeared in the work of Wang [7] for the second-order equation and Gazzola and Grunau [3] for the fourth-order one. Among other things, these authors studied the intersection properties of radially symmetric solutions, and they found that the above conditions play a crucial role in that context.

When it comes to the second-order equation $-\Delta \psi = \psi^p$, Wang [7] showed that the graphs of no two radially symmetric solutions intersect one another, if (1.5) holds, while the graphs of any two radially symmetric solutions intersect one another, otherwise. A similar dichotomy is expected to hold for the fourth-order equation as well; however the methods used for the second-order equation are no longer applicable. In this paper, we prove the first part of such a dichotomy by showing that the graphs of no two radially symmetric solutions of (1.3) intersect one another when (1.4) holds. In particular, our main result can be stated as follows.

**Theorem 1.** Let $n > 4$ and $p \geq \frac{n+4}{n-4}$. Let $\psi$ denote any one of the positive solutions of (1.3) whose existence is provided by lemma 5.

(a) Linear instability. If condition (1.4) fails to hold, then $\psi$ is linearly unstable in the sense that the linearized operator $\Delta^2 - p\psi^{p-1}$ has a negative eigenvalue.

(b) Linear stability. If condition (1.4) holds, then $\psi$ is linearly stable in the sense that the linearized operator $\Delta^2 - p\psi^{p-1}$ has no negative spectrum.

(c) Non-intersection. If condition (1.4) holds, then the graphs of no two positive $C^4$, radially symmetric solutions of (1.3) intersect one another.

The proofs of parts (a), (b) and (c) of theorem 1 appear in sections 2, 3 and 4, respectively; see proposition 6 for part (a), proposition 9 for part (b) and theorem 12 for part (c). See also lemma 10 for a simplified version of condition (1.4).

**Remark 2.** Following the completion of this paper, Ferrero et al [2] have shown that part (c) is sharp as well. If condition (1.4) fails to hold, that is, then each positive $C^4$, radially symmetric solution oscillates an infinite number of times around an explicit singular solution; see equation (4.3) and the statement of theorem 12.

**Remark 3.** When it comes to the second-order case (1.1), conclusions analogous to those of theorem 1 provide the key ingredients in the relevant nonlinear (in)stability results for the time-dependent problems [4, 5]. Nevertheless, the proofs of these results do not apply to the fourth-order case (1.2), so some new ideas are needed here.
2. Linear instability

The main result in this section is proposition 6, which gives a sufficient condition for linear instability. Although this condition will be simplified in section 4, it is much more convenient to initially state it in terms of the quartic polynomial

\[ Q_4(\alpha) \equiv |x|^{n+4} \Delta^2 |x|^{-n} = \alpha(\alpha + 2)(\alpha + 2 - n)(\alpha + 4 - n). \]  

(2.1)

This polynomial is closely related to Rellich’s inequality

\[ \int_{\mathbb{R}^n} (\Delta u)^2 \, dx \geq \frac{n^2(n - 4)^2}{16} \int_{\mathbb{R}^n} |x|^{-4} u^2 \, dx, \]  

(2.2)

which is valid for each \( u \in H^2(\mathbb{R}^n) \) and each \( n > 4 \). Namely, the constant that appears on the right-hand side is merely the unique local maximum value of \( Q_4 \), and it is known to be sharp in the following sense.

**Lemma 4.** Let \( n > 4 \) and let \( V \) be a bounded function on \( \mathbb{R}^n \) that vanishes at infinity. If there exists some \( \varepsilon > 0 \) such that

\[ V(x) \leq -(1 + \varepsilon) \cdot \frac{n^2(n - 4)^2}{16} \cdot |x|^{-4} \]

for all large enough \( |x| \), then the operator \( \Delta^2 + V \) has a negative eigenvalue.

For a proof of Rellich’s inequality (2.2) and lemma 4, we refer the reader to section II.7 in Rellich’s book [6].

We now use the previous lemma to address the linear instability of positive, regular solutions to (1.3). The known results on the existence of such solutions are summarized in our next lemma; see [8, 1, 3] for parts (a), (b) and (c), respectively.

**Lemma 5.** Let \( n \geq 1 \) and \( p > 1 \). Denote by \( Q_4 \) the quartic in (2.1).

(a) If either \( n \leq 4 \) or \( p \leq \frac{n+4}{n-4} \), then equation (1.3) has no positive \( C^4 \) solutions.

(b) If \( n > 4 \) and \( p = \frac{n+4}{n-4} \), then all positive \( C^4 \) solutions of (1.3) are of the form

\[ \varphi_\lambda(x) = \left( \frac{n}{n - 4}(n - 2)n(n + 2) \right)^{\frac{1}{p-1}} \cdot \left( \frac{\lambda}{\lambda^2 + |x - y|^2} \right)^{\frac{1}{p-1}} \]  

(2.3)

for some \( \lambda > 0 \) and some \( y \in \mathbb{R}^n \).

(c) If \( n > 4 \) and \( p > \frac{n+4}{n-4} \), then the positive \( C^4 \), radially symmetric solutions of (1.3) form a one-parameter family \( \{ \varphi_\alpha \}_{\alpha > 0} \), where each \( \varphi_\alpha \) is such that

\[ \varphi_\alpha(0) = \alpha, \quad \lim_{|x| \to \infty} |x|^4 \varphi_\alpha(x)^{p-1} = Q_4 \left( \frac{4}{p - 1} \right) > 0. \]  

(2.4)

**Proposition 6.** Let \( n > 4 \) and \( p \geq \frac{n+4}{n-4} \). Let \( Q_4 \) be the quartic in (2.1) and let \( \varphi \) denote any one of the solutions provided by lemma 5. Then \( \Delta^2 - p\varphi^{p-1} \) has a negative eigenvalue, if

\[ p \cdot Q_4 \left( \frac{4}{p - 1} \right) > Q_4 \left( \frac{n - 4}{2} \right) = \frac{n^2(n - 4)^2}{16}. \]  

(2.5)

In particular, it has a negative eigenvalue, if \( p = \frac{n+4}{n-4} \).

**Proof.** Suppose first that \( p > \frac{n+4}{n-4} \). Using part (c) of lemma 5 and our assumption (2.5), we can then find some small enough \( \varepsilon > 0 \) such that

\[ \lim_{|x| \to \infty} |x|^4 \varphi(x)^{p-1} = Q_4 \left( \frac{4}{p - 1} \right) > p^{-1}(1 + 2\varepsilon) \cdot \frac{n^2(n - 4)^2}{16}. \]
Since this implies that
\[ V(x) \equiv -p\varphi(x)^p - 1 < -(1 + \varepsilon) \cdot \frac{n^2(n - 4)^2}{16} \cdot |x|^{-4} \]
for all large enough $|x|$, the existence of a negative eigenvalue follows by lemma 4.

Suppose now that $p = \frac{n+4}{n-2}$. Then our assumption (2.5) automatically holds because
\[ \frac{n^2(n - 4)^2}{16} = Q_{4}\left(\frac{n - 4}{2}\right) = Q_{4}\left(\frac{4}{p - 1}\right) < p \cdot Q_{4}\left(\frac{4}{p - 1}\right) \]
for this particular case. According to part (b) of lemma 5, we also have
\[ \Delta^2 - p\varphi(x)^p = \Delta^2 - \lambda^4(n - 2)n(n + 2)(n + 4) \cdot (\lambda^2 + |x - y|)^{-4} \]
for some $\lambda > 0$ and some $y \in \mathbb{R}^n$. Thus, it suffices to check that the associated energy
\[ E(\zeta) = \int_{\mathbb{R}^n} (\Delta \zeta)^2 \, dx - \int_{\mathbb{R}^n} p\varphi^p \zeta^2 \, dx \]
is negative for some test function $\zeta \in H^2(\mathbb{R}^n)$. Let us then consider the test function
\[ \zeta(x) = (\lambda^2 + |x - y|)^{-\frac{n+2}{2}}. \]
Since $n > 4$, we have $\zeta \in H^2(\mathbb{R}^n)$, while a straightforward computation gives
\[ E(\zeta) = -8\lambda^4 n(n - 2)(n + 1) \int_{\mathbb{R}^n} (\lambda^2 + |x - y|)^{n+2} < 0. \]
This implies the presence of a negative eigenvalue and it also completes the proof.  

3. Linear stability

In this section, we address the linear stability of the solutions provided by lemma 5. First, we use an Emden–Fowler transformation to transform (1.3) into an ODE whose linear part has constant coefficients. Although this transformation is quite standard, the subsequent part of our analysis is not. The main result of this section is given in proposition 9.

**Lemma 7.** Let $n \geq 1$ and $p > 1$. Let $Q_4$ be the quartic in (2.1) and suppose $\varphi$ is a positive solution of the biharmonic equation (1.3). Setting $m = \frac{4}{p-1}$ for convenience, the function
\[ W(s) = e^{ms} \varphi(e^s) = r^m \varphi(r), \quad s = \log r = \log |x| \]
must then be a solution to the ordinary differential equation
\[ Q_4(m - \partial_s) W(s) = W(s)^p. \]

**Proof.** Since $\partial_s = e^{-s} \partial r$, a short computation allows us to write the radial Laplacian as
\[ \Delta = \partial_r^2 + (n - 1) r^{-1} \partial_r = e^{-2s} (n - 2 + \partial_s) \partial_s. \]

Using the operator identity $\partial_r e^{-k^2 s} = e^{-k^2} (\partial_r - k)$, one can then easily check that
\[ \Delta^2 e^{-ms} = e^{-4ms} Q_4(m - \partial_s) = e^{-mp}\varphi Q_4(m - \partial_s). \]
This also implies that $Q_4(m - \partial_s) W(s) = e^{mp}\Delta^2 \varphi(r^p) = W(s)^p$, as needed.

**Lemma 8.** Let $n > 4$ and $p > \frac{n+4}{n-2}$. Set $m = \frac{4}{p-1}$ and let $Q_4$ be the quartic in (2.1). Assuming that condition (1.4) holds, the polynomial
\[ P(\lambda) = Q_4(m - \lambda) - p Q_4(m) \]
must then have four real roots $\lambda_1, \lambda_2, \lambda_3 < 0 < \lambda_4$.  

\[ Q_4(m - \partial_s) W(s) = W(s)^p. \]
On the nonlinear biharmonic equation

Proof. Noting that \( Q_4 \) is symmetric about \( \frac{n-4}{2} \), we see that \( \mathcal{P} \) is symmetric about
\[
\lambda_* \equiv m - \frac{n-4}{2} = \frac{4}{p-1} - \frac{n-4}{2},
\]
where \( \lambda_* < 0 \) because \( p > \frac{n+4}{n-4} \) by assumption. Moreover, we have
\[
\lim_{\lambda \to \pm \infty} \mathcal{P}(\lambda) = +\infty, \quad \mathcal{P}(2\lambda_*) = \mathcal{P}(0) = (1 - p) \cdot Q_4(m) < 0
\]
because of (2.4), and we also have
\[
\mathcal{P}(\lambda_*) = Q_4 \left( \frac{n-4}{2} \right) - p \cdot Q_4 \left( \frac{4}{p-1} \right) \geq 0
\]
because of (1.4). This forces \( \mathcal{P}(\lambda) \) to have at least one root in each of the intervals
\[(\lambda_*, 2\lambda_*), \quad (2\lambda_*, \lambda_*), \quad [\lambda_*, 0), \quad (0, \infty).\]
In the case that \( \lambda_* \) itself happens to be a root, then it must be a double root by symmetry. In any case then, \( \mathcal{P}(\lambda) \) has three negative roots and one positive root, as needed. \( \blacksquare \)

Proposition 9. Let \( n > 4 \) and \( p > \frac{n+4}{n-4} \). Let \( Q_4 \) be the quartic in (2.1) and let \( \psi \) denote any one of the solutions provided by lemma 5. Assuming that (1.4) holds, one has
\[
|x|^p \psi(x)^{p-1} \leq Q_4 \left( \frac{4}{p-1} \right)
\]
for each \( x \in \mathbb{R}^n \), and the operator \( \Delta^2 - p\psi^{p-1} \) has no negative spectrum.

Proof. First, suppose that (3.4) does hold. Using our assumption (1.4), we then get
\[
-p\psi(x)^{p-1} \geq -p \cdot Q_4 \left( \frac{4}{p-1} \right) \cdot |x|^{-4} \geq -\frac{n^2(n-4)^2}{16} \cdot |x|^{-4}
\]
for each \( x \in \mathbb{R}^n \), so \( \Delta^2 - p\psi^{p-1} \) has no negative spectrum by Rellich’s inequality (2.2).

Let us now focus on the derivation of (3.4). Set \( m = \frac{4}{p-1} \) and consider the function
\[
W(s) = e^{ms} \psi(e^s) = r^m \psi(r), \quad s = \log r = \log |x|.
\]
Then \( W(s) \) is positive and it satisfies the equation
\[
Q_4(m - \partial_s) W(s) = W(s)^p
\]
by lemma 7. We note that \( s \) ranges over \((-\infty, \infty)\) as \( r \) ranges from 0 to \( \infty \), while
\[
\lim_{s \to -\infty} W(s) = \lim_{r \to 0^+} r^m \psi(r) = 0,
\]
The derivatives of \( W(s) \) must also vanish at \( s = -\infty \) because
\[
\lim_{s \to -\infty} W'(s) = \lim_{s \to 0^+} r^m \psi'(r) = 0,
\]
and so on. Using the fact that \( x \mapsto x^p \) is convex on \((0, \infty)\), we now find
\[
W(s)^p - Q_4(m) \frac{W(s)}{W(s)^{p-1}} \geq pQ_4(m) \cdot \left( W(s) - Q_4(m) \frac{W(s)}{W(s)^{p-1}} \right).
\]
Inserting this inequality in (3.5), we thus find
\[
Q_4(m - \partial_s) W(s) - Q_4(m) \frac{W(s)}{W(s)^{p-1}} \geq pQ_4(m) \cdot \left( W(s) - Q_4(m) \frac{W(s)}{W(s)^{p-1}} \right).
\]
To eliminate the constant term on the left-hand side, we change variables by
\[
Y(s) = W(s) - Q_4(m) \frac{1}{W(s)^{p-1}}.
\]
Then we can write equation (3.7) in the equivalent form
\[
Q_4(m - \partial_s) - pQ_4(m) Y(s) \geq 0.
\]
Invoking lemma 8, we now factor the last ODE to obtain
\[
(\partial_s - \lambda_1)(\partial_s - \lambda_2)(\partial_s - \lambda_3)(\partial_s - \lambda_4) Y(s) \geq 0
\]
for some \(\lambda_1, \lambda_2, \lambda_3 < 0 < \lambda_4\). Multiplying by \(e^{-\lambda_4 s}\) and integrating over \((-\infty, s)\), we get
\[
e^{-\lambda_4 s}(\partial_s - \lambda_2)(\partial_s - \lambda_3)(\partial_s - \lambda_4) Y(s) \geq 0
\]
because \(\lambda_1 < 0\). We ignore the exponential factor and use the same argument twice to get
\[
(\partial_s - \lambda_4) Y(s) \geq 0
\]
since \(\lambda_2, \lambda_3 < 0\) as well. Multiplying by \(e^{-\lambda_4 s}\) and integrating over \((s, +\infty)\), we then find
\[
e^{-\lambda_4 s}
\]
\[
W(s) - Q_4(m) \frac{1}{p - 1} \leq \lim_{s \to \infty} e^{-\lambda_4 s} W(s) - Q_4(m) \frac{1}{p - 1}.
\]
The limit on the right-hand side is zero because \(\lambda_4 > 0\) by above and since
\[
\lim_{s \to \infty} W(s) = \lim_{|x| \to \infty} |x|^{\frac{n}{p - 1}} \varphi(x) = Q_4(m) \frac{1}{p - 1}
\]
by (2.4). In particular, we may finally deduce the estimate
\[
W(s) \leq Q_4(m) \frac{1}{p - 1},
\]
which is precisely the desired estimate (3.4) because \(W(s) = |x|^{\frac{n}{p - 1}} \varphi(x)\) by above.

4. Intersection properties

In this section, we give a simplified version of condition (1.4) and we also address the intersection properties of the positive, regular, radially symmetric solutions to (1.3).

Lemma 10. Let \(n > 4\) and \(p \geq p_n \equiv \frac{\pi^2}{n - 2}\). Let \(Q_4\) be the quartic in (2.1).

If \(n \leq 12\), then condition (1.4) does not hold for any \(p \geq p_n\) whatsoever.

If \(n \geq 13\), on the other hand, then the equation
\[
p \cdot Q_4 \left( \frac{4}{p - 1} \right) = Q_4 \left( \frac{n - 4}{2} \right)
\]
has a unique solution \(p_c > p_n\), and condition (1.4) holds if and only if \(p \geq p_c\).

Proof. To say that condition (1.4) fails to hold is to say that
\[
Q(p) \equiv 16(p - 1)^4 \cdot \left[ Q_4 \left( \frac{n - 4}{2} \right) - p \cdot Q_4 \left( \frac{4}{p - 1} \right) \right]
\]
is negative. Let us now combine our definitions (2.1) and (4.1) to write
\[
Q(p) = n^2(n - 4)^2(p - 1)^4 - 2^7 p(p + 1) \left( (n - 4)p - n \right) \left( (n - 2)p - (n + 2) \right).
\]
Using this explicit equation, it is easy to see that
\[
Q(0) = n^2(n - 4)^2, \quad Q(1) = -2^{12}, \quad Q \left( \frac{n + 2}{n - 2} \right) = \frac{2^5 n^2(n - 4)^2}{(n - 2)^4},
\]
while a short computation gives
\[
Q(p_n) = Q \left( \frac{n + 4}{n - 4} \right) = -\frac{2^{15} n^2}{(n - 4)^4}.
\]
This forces \( Q(p) \) to have three real roots in the interval \((0, p_n)\), so the fourth root must also be real. To find its exact location, we compute

\[
\lim_{p \to \pm \infty} \frac{Q(p)}{p^4} = (n - 4) \cdot (n^3 - 4n^2 - 128n + 256) \tag{4.2}
\]

and we examine two cases.

\textbf{Case 1.} When \( 4 < n \leq 12 \), the limit in (4.2) is negative. Since \( Q(0) \) is positive by above, the fourth root lies in \((-\infty, 0)\), so \( Q(p) \) is negative for any \( p \geq p_n \) whatsoever.

\textbf{Case 2.} When \( n \geq 13 \), the limit in (4.2) is positive. Since \( Q(p_n) \) is negative by above, the fourth root \( p_c \) lies in \((p_n, \infty)\), so \( Q(p) \) is negative on \([p_n, p_c)\) and non-negative on \([p_c, \infty)\).

\begin{lemma}
Let \( n > 4 \) and \( p > \frac{n+4}{n-4} \). Set \( m = \frac{4}{p-1} \) and let \( Q_4 \) be the quartic in (2.1). Then

\[
R(\mu) = Q_4(m - \mu) - Q_4(m)
\]

has four real roots \( \mu_1 < \mu_2 < \mu_3 = 0 < \mu_4 \).
\end{lemma}

\begin{proof}
As in the proof of lemma 8, we exploit the fact that \( R(\mu) \) is symmetric about

\[
\mu_* \equiv m - \frac{n - 4}{2} = \frac{4}{p-1} - \frac{n - 4}{2} < 0.
\]

It is clear that \( \mu_3 = 0 \) is a root of \( R(\mu) \). Then \( \mu_2 = 2\mu_* < 0 \) must also be a root by symmetry. To see that a positive root \( \mu_4 > m \) exists, we note that

\[
\lim_{\mu \to +\infty} R(\mu) = +\infty, \quad R(m) = -Q_4(m) < 0
\]

by (2.4). Then \( \mu_1 = 2\mu_* - \mu_4 = \mu_2 - \mu_4 < \mu_2 \) must also be a root by symmetry.
\end{proof}

Finally, we address the intersection properties of the solutions provided by lemma 5 in the supercritical case \( p \geq p_c \). To this end, let us also introduce the function

\[
\Phi(x) = Q_4(m) \cdot |x|^{-m}, \quad m = \frac{4}{p-1}
\]

which is easily seen to be a singular solution of (1.3).

\begin{theorem}
Suppose that \( n \geq 13 \) and \( p \geq p_c \), where \( p_c \) is given by lemma 10. In other words, suppose that \( n > 4 \) and \( p > \frac{n+4}{n-4} \) and that (1.4) holds. If \( \psi_\alpha, \psi_\beta \) are any two of the solutions provided by lemma 5, then

(a) the graph of \( \psi_\alpha \) does not intersect the graph of the singular solution (4.3); (b) the graph of \( \psi_\alpha \) does not intersect the graph of \( \psi_\beta \), unless \( \alpha = \beta \).
\end{theorem}

\begin{proof}
To establish part (a), we have to show that

\[
|x|^m \psi_\alpha(x) < Q_4(m)^{\frac{1}{p-1}}
\]

for each \( x \in \mathbb{R}^n \). This amounts to a slight refinement of inequality (3.4) in proposition 9, as we now need the inequality to be strict. Let us then consider the function

\[
W(s) = r^m \psi_\alpha(r), \quad s = \log r = \log |x|, \quad m = \frac{4}{p-1}.
\]

Inequality (3.4) in proposition 9 reads

\[
W(s) \leq Q_4(m)^{\frac{1}{p-1}}, \quad s \in \mathbb{R}
\]

for each \( x \in \mathbb{R}^n \). This amounts to a slight refinement of inequality (3.4) in proposition 9, as we now need the inequality to be strict. Let us then consider the function

\[
W(s) = r^m \psi_\alpha(r), \quad s = \log r = \log |x|, \quad m = \frac{4}{p-1}.
\]

Inequality (3.4) in proposition 9 reads

\[
W(s) \leq Q_4(m)^{\frac{1}{p-1}}, \quad s \in \mathbb{R}
\]
and we now have to show that this inequality is actually strict. Since
\[ \lim_{s \to -\infty} W(s) = \lim_{r \to 0^+} r^m \phi_m(r) = 0 < Q_4(m) \frac{1}{p} \]
by (2.4), we do have strict inequality near \( s = -\infty \). Suppose equality holds at some point, and let \( s_0 \) be the first such point. Since \( W(s) \) reaches its maximum at \( s_0 \), we then have
\[ W(s_0) = Q_4(m) \frac{1}{p}, \quad W'(s_0) = 0, \quad W(s) < Q_4(m) \frac{1}{p} \tag{4.6} \]
for each \( s < s_0 \). When it comes to the interval \((-\infty, s_0)\), we thus have
\[ W(s) > Q_4(m) \frac{1}{p} > \frac{1}{p} Q_4(m) \tag{4.7} \]
by convexity. This is the same inequality as (3.6), except that the inequality is now strict. In particular, the argument that led us to (3.9) now leads us to a strict inequality
\[ (\partial_x - \lambda_1)(\partial_x - \lambda_2)(\partial_x - \lambda_3)(\partial_x - \lambda_4)Y(s) > 0, \quad s < s_0 \]
for some \( \lambda_1, \lambda_2, \lambda_3 < 0 < \lambda_4 \). Using the same argument as before, we get
\[ \partial_x - \mu_1)(\partial_x - \mu_2)(\partial_x - \mu_4)W'(s) < 0 \tag{4.8} \]
for some \( \mu_1 < \mu_2 < 0 < \mu_4 \). Once again, the last equation easily leads to
\[ \partial_x - \mu_4)W'(s) < 0 \]
because \( \lambda_3 < 0 \) as well. In view of definition (3.8) of \( Y(s) \), this actually gives
\[ W'(s_0) > \lambda_4 \left( W(s_0) - Q_4(m) \frac{1}{p} \right), \]
which is contrary to (4.6). In particular, the inequality in (4.5) must be strict at all points and the proof of part (a) is complete.

In order to prove part (b), we shall first show that
\[ W'(s) > 0, \quad s \in \mathbb{R}. \tag{4.7} \]
Using lemma 7 and the strict inequality in (4.5), we find that
\[ \left[ Q_4(m - \partial_x) - Q_4(m) \right]W(s) = W(s)^p - Q_4(m) W(s) < 0. \]
Invoking lemma 11, we now factor the left-hand side to get
\[ (\partial_x - \mu_1)(\partial_x - \mu_2)(\partial_x - \mu_4)W'(s) < 0 \]
for some \( \mu_1 < \mu_2 < 0 < \mu_4 \). Once again, the last equation easily leads to
\[ (\partial_x - \mu_4)W'(s) < 0 \tag{4.8} \]
because \( \mu_1, \mu_2 < 0 \). This makes \( W'(s) - \mu_4 W(s) \) decreasing for all \( s \), so the limit
\[ \lim_{s \to +\infty} W'(s) - \mu_4 W(s) \]
exists. Since \( W(s) \) tends to a finite limit as \( s \to +\infty \) by (2.4), it easily follows that \( W'(s) \) must approach zero as \( s \to +\infty \). Since \( \mu_4 > 0 \) by above, this gives
\[ \lim_{s \to +\infty} e^{-\mu_4 s}W'(s) = 0, \]
and then we may integrate (4.8) over \((s, +\infty)\) to deduce the desired (4.7).

Now, the inequality (4.7) we just proved can also be written as
\[ 0 < W'(s) = r \cdot \hat{\partial}_r [r^m \phi_m(r)] = r^m [m \phi_m(r) + r \phi'_m(r)] \tag{4.9} \]
in view of our definition (3.1). On the other hand, a scaling argument shows that the solutions provided by lemma 5 are subject to the relation
\[ \phi_m(r) = \alpha \phi_{\alpha} \left( \frac{r}{\alpha} \right). \tag{4.10} \]
Differentiating (4.10) and using (4.9), one now easily finds that \( \hat{\partial}_r \phi_m(r) > 0 \) for all \( \alpha, r > 0 \). In particular, the graphs of distinct solutions cannot really intersect, as needed.
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