Vanishing viscosity limits for the free boundary problem of compressible viscoelastic fluids with surface tension

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Received April 15, 2022; accepted June 27, 2022; published online October 27, 2022

Abstract  We consider the free boundary problem of compressible isentropic neo-Hookean viscoelastic fluid equations with surface tension. Under the physical kinetic and dynamic conditions proposed on the free boundary, we investigate the regularity of classical solutions to viscoelastic fluid equations in Sobolev spaces which are uniform in viscosity and justify the corresponding vanishing viscosity limits. The key ingredient of our proof is that the deformation gradient tensor in Lagrangian coordinates can be represented as a parameter in terms of the flow map so that the inherent structure of the elastic term improves the uniform regularity of normal derivatives in the limit of vanishing viscosity. This result indicates that the boundary layer does not appear in the free boundary problem of compressible viscoelastic fluids, which is different from the case studied by Mei et al. (2018) for the free boundary compressible Navier-Stokes system.

Keywords  free boundary, viscoelastic fluid, vanishing viscosity, compressible fluid, elastodynamics

MSC(2020)  35Q35, 35R35, 76A10, 76N10, 76N20

Citation  Gu X M, Mei Y. Vanishing viscosity limits for the free boundary problem of compressible viscoelastic fluids with surface tension. Sci China Math, 2023, 66: 1263–1300, https://doi.org/10.1007/s11425-022-1998-9

1 Introduction

1.1 Formulation in Eulerian coordinates

Consider the motion of compressible isentropic neo-Hookean viscoelastic fluids in a time-dependent domain Ωε(t), whose exterior is assumed to be the atmosphere with a given constant pressure pe > 0. The boundary of the domain Γε(t) is free to move and subject to physical kinetic and dynamic conditions under the effect of surface tension. The governing equations of such a free boundary problem can be written as follows:

\[
\begin{align*}
\frac{\partial}{\partial t} \rho^\varepsilon + \text{div}(\rho^\varepsilon u^\varepsilon) &= 0 \quad \text{in } \Omega^\varepsilon(t), \\
\frac{\partial}{\partial t}(\rho^\varepsilon u^\varepsilon) + \rho^\varepsilon u^\varepsilon \otimes u^\varepsilon + \nabla p(\rho^\varepsilon) - \text{div}T^\varepsilon &= \text{div}(\rho^\varepsilon F^\varepsilon F^\varepsilon T) \quad \text{in } \Omega^\varepsilon(t), \\
\frac{\partial}{\partial t}F^\varepsilon + u^\varepsilon \cdot \nabla F^\varepsilon &= \nabla u^\varepsilon F^\varepsilon \quad \text{in } \Omega^\varepsilon(t),
\end{align*}
\] (1.1)

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where \( \rho^\varepsilon \) is the density, \( \mathbf{u}^\varepsilon = (u_1^\varepsilon, \ldots, u_d^\varepsilon)^T \in \mathbb{R}^d \) is the velocity, and \( \mathbf{F}^\varepsilon = (F_{ij}^\varepsilon) \in M_{d \times d} \) is the deformation gradient. Here, \( \mathbf{T} \) denotes the transpose of the matrix. The pressure \( p(\rho^\varepsilon) \) only depends on the density \( \rho^\varepsilon \) in the isentropic case which is assumed here to satisfy

\[
p(\rho^\varepsilon) = A(\rho^\varepsilon)^\gamma, \quad \gamma > 1. \tag{1.2}
\]

For simplicity, we take \( A = 1 \) here and afterwards. The viscous stress tensor \( \mathbf{T}^\varepsilon \) is given by

\[
\mathbf{T}^\varepsilon = 2\mu\varepsilon \mathbf{S}\mathbf{u}^\varepsilon + \lambda\varepsilon\text{div}\mathbf{u}^\varepsilon \mathbf{I}, \tag{1.3}
\]

where \( \mathbf{S}\mathbf{u}^\varepsilon = (\nabla\mathbf{u}^\varepsilon + (\nabla\mathbf{u}^\varepsilon)^T)/2 \) is the symmetric part of \( \nabla\mathbf{u}^\varepsilon \), and \( \mu\varepsilon \) and \( \lambda\varepsilon \) are viscosity coefficients satisfying the physical constraints \( \mu > 0 \) and \( 2\mu + d\lambda > 0 \). To study this free boundary problem, we impose the following two boundary conditions on \( \Gamma^\varepsilon(t) \). On the one hand, the kinetic boundary condition, which states that the free boundary moves along the fluid particles, reads

\[
\mathbf{V}^\varepsilon(\Gamma^\varepsilon(t)) = \mathbf{u}^\varepsilon \cdot \mathbf{n}^\varepsilon \quad \text{on} \quad \Gamma^\varepsilon(t), \tag{1.4}
\]

where \( \mathbf{V}^\varepsilon(\Gamma^\varepsilon(t)) \) is the normal velocity of \( \Gamma^\varepsilon(t) \) and \( \mathbf{n}^\varepsilon \) is the outward unit normal vector of \( \Gamma(t) \). On the other hand, the dynamic boundary condition, when the surface tension is considered, can be written as

\[
(-p(\rho^\varepsilon)\mathbf{I} + \mathbf{T}^\varepsilon + (\rho^\varepsilon \mathbf{F}^\varepsilon \mathbf{F}^\varepsilon^T - \mathbf{I})\mathbf{n}^\varepsilon + p_n \mathbf{n}^\varepsilon = \sigma \mathbf{H}\mathbf{n}^\varepsilon) \quad \text{on} \quad \Gamma^\varepsilon(t), \tag{1.5}
\]

which represents the balance of stress tensors on both sides of the free boundary. Here, \( \mathbf{H} \) is the twice mean curvature of \( \Gamma^\varepsilon(t) \), \( \sigma > 0 \) is the surface tension coefficient, and \( p_n > 0 \) is a given constant pressure outside \( \Omega(t) \) which can guarantee the non-vacuum density of the viscoelastic fluid if the density is initially non-vacuum. We also impose the following initial data for (1.1):

\[
(\rho^\varepsilon, \mathbf{u}^\varepsilon, \mathbf{F}^\varepsilon)(x, t = 0) = (\rho_0^\varepsilon, \mathbf{u}_0^\varepsilon, \mathbf{F}_0^\varepsilon), \quad x \in \Omega^\varepsilon(0). \tag{1.6}
\]

When the fluid is inviscid, we can reduce the free boundary problem (1.1)-(1.6) to the one of the compressible isentropic inviscid elastic fluid, through formally taking \( \varepsilon \to 0 \), which is

\[
\begin{aligned}
\partial_t \rho + \text{div}(\rho \mathbf{u}) &= 0 \quad &\text{in} \quad \Omega(t), \\
\partial_t (\rho \mathbf{u}) + \text{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\rho) &= \text{div}(\rho \mathbf{F} \mathbf{F}^T) \quad &\text{in} \quad \Omega(t), \\
\partial_t \mathbf{F} + \mathbf{u} \cdot \nabla \mathbf{F} &= \nabla \mathbf{u} \mathbf{F} \quad &\text{in} \quad \Omega(t), \\
\mathbf{V}(\Gamma(t)) &= \mathbf{u} \cdot \mathbf{n} \quad &\text{on} \quad \Gamma(t), \\
(-p + (\rho \mathbf{F} \mathbf{F}^T - \mathbf{I}))\mathbf{n} + p_n \mathbf{n} &= \sigma \mathbf{H}\mathbf{n} \quad &\text{on} \quad \Gamma(t), \\
(\rho, \mathbf{u}, \mathbf{F})(x, t = 0) &= (\rho_0, \mathbf{u}_0, \mathbf{F}_0) \quad &\text{in} \quad \Omega(0).
\end{aligned} \tag{1.7}
\]

In this paper, we investigate the vanishing viscosity limits of the free boundary problem of compressible isentropic viscoelastic fluid equations (1.1)-(1.6), i.e., whether the solutions to (1.1)-(1.6) converge to the ones to the free boundary elastodynamic equation (1.7) modeling inviscid compressible isentropic flows as the viscosity tends to zero.

### 1.2 Motivations and related results

The vanishing viscosity limit for the viscous fluid is one of the most fundamental problems in the mathematical theory of fluid mechanics. It has been extensively studied by many mathematicians in various settings of domains and physical boundary conditions as well as different kinds of fluids. For the incompressible homogeneous Newtonian fluids, the justification of vanishing viscosity limits in the whole space has been proved in [21, 31, 41] for smooth solutions and in [3, 5, 6, 14] for irregular ones. However, in the presence of physical boundaries, such a problem becomes much more complicated due to the possible appearance of boundary layers. When the Dirichlet boundary condition is imposed, the boundary layer, as illustrated by the Prandtl theory, must appear and be very strong so that the verification
of vanishing viscosity limits for incompressible Newtonian fluids becomes one of the most challenging open problems in mathematical fluid mechanics. Some important progress on, but still far from being completely solved, this issue is made in [12,28,33,35,36,43] for the analytic initial data at least near the boundary, whereas, when the Navier-slip boundary condition is imposed, the boundary layer becomes weaker so that the vanishing viscosity limit holds even if the vorticity is produced at the boundary. We can refer to [4,13,19,20,29,50] and the references therein for considerable progress. For the compressible Newtonian fluids, the vanishing viscosity limit problems are discussed in [34,40,45–48,51]. Precisely, Xin and Yang [51] studied the vanishing viscosity limit of the linearized compressible Navier-Stokes system with the no-slip boundary condition in the 2-D half-plane. Wang and Williams [47] constructed a boundary layer solution to the compressible Navier-Stokes equations with Navier-slip boundary conditions in the 2-D half-plane. Wang et al. [46] obtained the uniform regularity for the solutions to the compressible Navier-Stokes system with general Navier-slip boundary conditions in 3-D domains with curvature; especially, the vanishing viscosity limit of the viscous solution to the corresponding inviscid one was also obtained with the rate of convergence in $L^\infty$. It is also shown that the boundary layer for density is weaker than the one for velocity fields. Very recently, Wang and Xie [44] justified the vanishing viscosity limit of solutions to the compressible viscoelastic flows under the no-slip boundary condition governed by the viscoelastic equations, based on the uniform conormal regularity estimates.

Although much literature exists for the vanishing viscosity limit problems on a fixed domain, few discussions are available for viscous surface waves. To our knowledge, the first result of the vanishing viscosity limit for viscous surface waves is due to Masmoudi and Rousset [30], who proved the local existence of solutions to the free boundary incompressible Navier-Stokes system with uniform regularity with respect to the viscosity in conormal Sobolev and Lipschitz spaces. Later, the vanishing viscosity and surface tension limits were established in [11,49] for the incompressible viscous surface waves with surface tension. For compressible surface waves, Mei et al. [32] established the uniform regularity of both density and velocity in conormal Sobolev and Lipschitz spaces and justified the vanishing viscosity and surface tension limits.

In the results mentioned above for the vanishing viscosity limits of viscous surface waves, we can only expect uniform Lipschitz bounds, but not higher Sobolev norms $H^k$ ($k \geq 2$), since a boundary layer generally appears near the free boundary for viscous surface waves and uniform bounds of many normal derivatives break down as viscosity tends to zero. This motivates us to study whether there exists a mechanism to prevent the appearance of boundary layers for viscous surface waves. Recently, Gu and Lei [15] proved the local well-posedness of the free boundary incompressible elastodynamics with surface tension by establishing uniform Sobolev regularity with respect to the viscosity of modified viscoelastic surface waves. This result indicates that the inherent structure of the elastic term on the boundary may prevent the appearance of boundary layers and enable us to justify the vanishing viscosity limit in standard Sobolev spaces. Our aim in this paper is further to discuss this smoothing effect of elasticity for compressible isentropic viscoelastic fluids in the limit of vanishing viscosity. Precisely, we strictly prove that the solutions to the free boundary problem (1.1)–(1.6) converge to the ones to (1.7) in the $H^3$-sense.

Let us also review here the well-posedness results of viscoelastic and elastic fluids. For the Cauchy problem, we refer to [2,24,26,27] for the local and global well-posedness of the incompressible viscoelastic fluid system. Sideris and Thomases [37,38] established the global well-posedness of the three-dimensional incompressible neo-Hookean elastodynamic system with small initial data. Taking advantage of the strong null structure in Lagrangian coordinates, Lei [23] obtained the two-dimensional result by using the Klainerman vector field and Alinhac’s ghost weight methods. We also refer to Cai et al. [1] for the vanishing viscosity limit for global-in-time solutions to incompressible viscoelasticity in $\mathbb{R}^2$. For free boundary problems, the local and global well-posedness results of incompressible viscoelastic flows in an infinite strip with surface tension are, respectively, established in [22] for large initial data and [52] for small ones. Recently, Di Iorio et al. [7–9] proved the existence of splash singularities for incompressible viscoelastic flows. The well-posedness theory of the free boundary problem for the incompressible neo-Hookean elastodynamics is more complicated. A priori estimates and local well-posedness were obtained in [16–18,25] under some special boundary conditions, such as $F^T n = 0$, $p = 0$ or $p = 0$, $(FF^T - I)n$
= (F^T - I)n = 0. Recently, Gu and Lei [15] established the local well-posedness result for free boundary problems being subject to the natural force balance law, which is consistent with (1.7) in the compressible case here. With regard to the compressible neo-Hookean elastodynamics, under the special boundary condition $F^T n = 0, p = 0$, Trakhinin [42] proved the local-in-time existence of a unique smooth solution to the free boundary problem by the Nash-Moser iteration approach, if the non-collinearity of $F$ or the Rayleigh-Taylor sign condition is satisfied. Very recently, Zhang [53] obtained the local well-posedness in Sobolev spaces by the combination of the classical energy method and the hyperbolic approach and also established the incompressible limit.

1.3 Reformulation in Lagrangian coordinates

To study the free boundary problem (1.1)–(1.6), we use the Lagrangian flow map to transform it into the corresponding problem in a fixed reference domain $\Omega$. Let $\eta^\varepsilon(x, t) \in \Omega^\varepsilon(t)$ be the “position” of the compressible viscoelastic fluid particle $x$ at time $t$, i.e.,

$$
\begin{align*}
\begin{cases}
\partial_t \eta^\varepsilon(x, t) = u(\eta^\varepsilon(x, t), t) & \text{for } t > 0, \ x \in \Omega, \\
\eta^\varepsilon(x, 0) = \eta^\varepsilon_0(x) & \text{for } x \in \Omega,
\end{cases}
\end{align*}
$$

where $\eta^\varepsilon_0$ is a diffeomorphism from the reference domain $\Omega$ to the initial domain $\Omega^\varepsilon(0)$ satisfying

$$
F^\varepsilon_0(\eta^\varepsilon_0) = \nabla \eta^\varepsilon_0.
$$

We set the following Lagrangian variables:

$$
f^\varepsilon(x, t) = \rho^\varepsilon(\eta^\varepsilon(x, t), t), \quad v^\varepsilon(x, t) = u^\varepsilon(\eta^\varepsilon(x, t), t), \quad G^\varepsilon(x, t) = F^\varepsilon(\eta^\varepsilon(x, t), t)
$$

and introduce

$$
A^\varepsilon = (\nabla \eta^\varepsilon)^{-T}, \quad J^\varepsilon = \det \nabla \eta^\varepsilon, \quad a^\varepsilon = JA^\varepsilon, \quad q(f^\varepsilon) = p(f^\varepsilon) + 1 - p_c.
$$

Then utilizing the chain rule and Einstein’s summation convention for repeated indices, we see that the free boundary problem (1.1)–(1.6) can be written by components of Lagrangian variables in $\Omega$ as follows:

$$
\begin{align*}
\begin{cases}
\partial_t \eta^\varepsilon_i = v^\varepsilon_i & \text{in } \Omega, \\
\partial_t f^\varepsilon + f^\varepsilon A^\varepsilon_{ij} \partial_j v^\varepsilon_k = 0 & \text{in } \Omega, \\
f^\varepsilon \partial_t v^\varepsilon_i + A^\varepsilon_{ij} \partial_j q - 2\varepsilon A^\varepsilon_{ij} \partial_j (S_{A^\varepsilon} v^\varepsilon)_{ik} - \lambda \varepsilon A^\varepsilon_{ij} \partial_j (\text{div} A^\varepsilon v^\varepsilon) - A^\varepsilon_{ij} \partial_j (f^\varepsilon G^\varepsilon_{ij} G^\varepsilon_{kj}) = 0 & \text{in } \Omega, \\
\partial_t G^\varepsilon_{ij} = A^\varepsilon_{ik} \partial_k v^\varepsilon_l G^\varepsilon_{lj} & \text{in } \Omega, \\
-q a^\varepsilon_{ij} N_j + f^\varepsilon G^\varepsilon_{ij} J^\varepsilon a^\varepsilon_{kl} N_l + 2\mu a^\varepsilon_{ij} a^\varepsilon_{kl} N_l + \lambda \text{div} A^\varepsilon(v^\varepsilon) a^\varepsilon_{ij} N_j = \sigma \sqrt{\varepsilon} \Delta_g(\eta^\varepsilon_i) & \text{on } \Gamma, \\
(f^\varepsilon, v^\varepsilon, G^\varepsilon, \eta^\varepsilon)(x)|_{t=0} = (\rho^\varepsilon_0(\eta^\varepsilon_0(x)), u^\varepsilon_0(\eta^\varepsilon_0(x)), F^\varepsilon_0(\eta^\varepsilon_0(x)), \eta^\varepsilon_0(x)) & \text{in } \Omega,
\end{cases}
\end{align*}
$$

where $(S_{A^\varepsilon})_{ik} = (A^\varepsilon_{ik} \partial_j v^\varepsilon_j + A^\varepsilon_{ij} \partial_j v^\varepsilon_k)/2$, $(\text{div} A^\varepsilon) = A^\varepsilon_{ik} \partial_k v^\varepsilon_i$, $N = (N_1, \ldots, N_d)^T$ is the unit outward normal vector to $\Gamma := \partial \Omega$, and $\Delta_g$ is the Laplacian-Beltrami operator on the curve $\eta^\varepsilon(t, \Gamma)$ given by

$$
\Delta_g(\eta^\varepsilon_i) = \frac{1}{\sqrt{g}} \partial_\alpha (\sqrt{g} g^{\alpha \beta} \partial_\beta \eta^\varepsilon_i), \quad g^{\alpha \beta} = \partial_\alpha \eta^\varepsilon_j \partial_\beta \eta^\varepsilon_j, \quad g = \det(g^{\alpha \beta}).
$$

Without loss of generality, we consider the two-dimensional problem (i.e., $d = 2$) and the reference domain is given by $\Omega = \mathbb{T} \times (0, 1)$, where $\mathbb{T}$ denotes the 1-torus. The boundary of $\Omega$ is then given by the horizontally flat bottom and top $\Gamma = \mathbb{T} \times \{0 \cup \{1\}\}$. In such a case, $\sqrt{\varepsilon} \Delta_g(\eta^\varepsilon_i) = \partial_1 (\frac{\partial \eta^\varepsilon_i}{\partial x_1})$ and

$$
N = \begin{cases}
e_2, & x_2 = 1, \\
-e_2, & x_2 = 0.
\end{cases}
$$
Furthermore, from Jacobi’s formula
\[ J_t^f = J^f A^T_{ij} \partial_i v_j^f = a^f_{ijk} \partial_j v_k^f, \] (1.11)
one can solve the first equation in (1.10) to obtain \( \partial_t (f^e J_t^e) = 0 \), which implies
\[ f^e = \rho_0 (J^e)^{-1} \] (1.12)
Next, since \( \partial_t A^f_{ij} = -A^f_{ik} \partial_i v_k^f A^f_{ij} \), it is not difficult to derive from the fourth equation of (1.10) that \( \partial_t (A^T G^e) = 0 \), which yields
\[ G^e = \nabla \eta^e (\nabla \eta_0^e)^{-1} \] (1.13)
Noting that \( \alpha^e \) is the cofactor of \( \nabla \eta^e \), we have the following Piola identity:
\[ \partial_t \alpha_{kl}^e = \partial_t (J^e A^e_{kl}) = 0, \] (1.14)
which implies that \( -A^e_{ik} \partial_i (f^e G^e_{ij} A^e_{kj}) = -(J^e)^{-1} \partial_t (A^e_{ik} J^e G^e_{ij} A^e_{kj}) \). Therefore, substituting (1.12) and (1.13) into the remaining equations in (1.10), we obtain
\[
\begin{cases}
\partial_t \eta^e_t = v^e_t & \text{in } \Omega, \\
\rho_0 \partial_t v^e + a^e_{ik} \partial_i q - 2 \mu \varepsilon_{ik} \partial_i (S_A v)^{\varepsilon}_{jk} - \lambda \varepsilon_{ij} \partial_j (\text{div}_A v)^{\varepsilon} - \partial_j (\rho_0 \partial_j \eta^e) = 0 & \text{in } \Omega, \\
-q a_2 + 2 \mu \varepsilon (S_A v)^{\varepsilon}_{jk} a^{\varepsilon}_{jk} + \lambda \varepsilon (\text{div}_A v)^{\varepsilon} a^{\varepsilon} + \rho_0 \partial_2 \eta^e = \sigma \partial_1 \left( \frac{\partial_1 \eta^e}{|\partial_1 \eta^e|} \right) & \text{on } \Gamma, \\
(v, \eta^e)(x)|_{t=0} = (u^0_0(\eta^e_0(x)), \eta^e_0(x)) & \text{in } \Omega.
\end{cases}
\] (1.15)
If we formally take \( \varepsilon = 0 \), then we have the following free boundary elastodynamic equations in Lagrangian coordinates:
\[
\begin{cases}
\partial_t \eta_i = v_i & \text{in } \Omega, \\
\rho_0 \partial_t v_i + a_{ik} \partial_i q - \partial_j (\rho_0 \partial_j \eta_i) = 0 & \text{in } \Omega, \\
-q a_2 + \rho_0 \partial_2 \eta_i = \sigma \partial_1 \left( \frac{\partial_1 \eta_i}{|\partial_1 \eta_i|} \right) & \text{on } \Gamma, \\
(v, \eta_i)(x)|_{t=0} = (u^0_0(\eta_i_0(x)), \eta_i_0(x)) & \text{in } \Omega.
\end{cases}
\] (1.16)
From now on, we focus on the free boundary viscoelastic and elastic fluid systems (1.15) and (1.16) in the Lagrangian coordinates.

1.4 Notation and function spaces

Before stating our main results, we introduce the following notation. We use Einstein’s summation convention for repeated indices throughout the paper. The bold fonts, such as \( \mathbf{u}, \mathbf{v}, \mathbf{F}, \mathbf{G}, \mathbf{A} \), are used to denote vectors or matrices, while the non-bold fonts, such as \( u_i, v_i, F_{ij}, G_{ij}, A_{ij} \), stand for corresponding elements. We use \( \partial \) : \( \partial_i, \partial_j \) to denote the tangential derivatives and \( \partial \) to denote \( \partial \) and \( \partial_2 \). The usual \( L^p \) spaces, Sobolev spaces \( W^{m,p} \) and \( H^m = W^{m,2} \) on both the domain \( \Omega \) and the boundary \( \Gamma \) are used. For notational simplicity, the norms of these spaces defined on \( \Omega \) are denoted by \( \| \cdot \|_{L^p}, \| \cdot \|_{W^{m,p}} \) and \( \| \cdot \|_m \), respectively, and the norms of these spaces defined on \( \Gamma \) are denoted by \( | \cdot |_{L^p}, | \cdot |_{W^{m,p}} \) and \( | \cdot |_m \), respectively. For real \( s \geq 0 \), the Hilbert space \( H^s(\Gamma) \) and the boundary norm \( | \cdot |_s \) (or \( | \cdot |_{H^s(\Gamma)} \) are defined by interpolation. The negative-order Sobolev space \( H^{-s}(\Gamma) \) is defined via duality: for real \( s \geq 0 \), \( H^{-s}(\Gamma) := [H^s(\Gamma)]' \). Moreover,
\[ \| u \|^2 \| H^m \| := \sum_{\ell \leq m} \| \partial^\ell_t u \|^2 \|_0 \]
is introduced for a function \( u \in L^2 \). We also introduce the spatial-temporal Sobolev norms of any function \( u \in L^2 \) on \( \Omega \) at the instantaneous time \( t \) as
\[ \| u(t) \|^2 \| H^m \| := \sum_{\ell \leq m} \| \partial^\ell u \|^2 m^{-\ell}, \quad \| u \|^2 \| H^{m,k} \| := \sum_{\ell \leq m-k} \| \partial^\ell \partial^m -\ell u \|^2 \|_0, \quad \| u \|^2 \| H^{m,\ell}_m \| := \| u \|^2 \| H^{m,\ell}_m \|. \]
and the spatial-temporal Sobolev norms on \( \Gamma \) as \( |u(t)|^2_{H^m} := \sum_{\ell \leq m} |\partial^\ell_x u|^2_{m-\ell} \). \( L^p_i(X) \) is used as the norm of the space \( L^p([0, t], X) \). \( C \) is used to denote a generic constant, which only depends on the domain \( \Omega \) and the boundary \( \Gamma \), and \( f \lesssim g \) is used to denote \( f \leq Cg \). \( P(\cdot) \) denotes a generic polynomial function of its arguments, and the polynomial coefficients are generic constants. We also use \( \int f \) and \( \int f \) as the integral abbreviations of \( \int x dx \) and \( \int dx \), respectively, for notational simplicity.

1.5 Main theorems

The aim of this paper is to get a local well-posedness result for classical solutions to (1.15) in a temporal interval independent of the viscosity \( \varepsilon \in (0, 1] \). This result also implies the local existence of strong solutions to the free boundary elastodynamic equation (1.16) with surface tension. To obtain the uniform regularity, we define the energy functional for the free boundary viscoelastic fluid equation (1.15) as

\[
\mathcal{E}^\varepsilon(t) = \|\eta^\varepsilon\|^2_{H^m(\Omega)}(t) + \|\nabla\eta^\varepsilon\|^2_{H^{m+1}(\Omega)}(t) + \|\partial^{m-1}\partial^2_t\eta^\varepsilon \cdot n^\varepsilon\|^2_{H^m(\Omega)}(t) + \varepsilon \|\nabla^2\eta^\varepsilon\|^2_{H^{m-1}(\Omega)}(t)
+ \int_0^t \left( \|\nabla\eta^\varepsilon\|^2_{H^m(\Omega)} + \|\partial^m_{t}\eta^\varepsilon\|^2_{H^m(\Omega)} + \|\partial_{\gamma}^m\eta^\varepsilon\cdot n^\varepsilon\|^2_{H^m(\Omega)} \right) dt.
\]

The corresponding energy for the free boundary elastodynamic equation (1.16) is

\[
\mathcal{E}(t) = \|\eta\|^2_{H^m(\Omega)}(t) + \|\nabla\eta\|^2_{H^{m+1}(\Omega)}(t) + \|\partial^{m-1}\partial^2_t\eta \cdot n\|^2_{H^m(\Omega)}(t)
+ \int_0^t \left( \|\nabla\eta\|^2_{H^m(\Omega)} + \|\partial^m_{t}\eta\|^2_{H^m(\Omega)} + \|\partial_{\gamma}^m\eta\cdot n\|^2_{H^m(\Omega)} \right) dt.
\]

We also require that the initial data satisfy the compatibility condition on the boundary, i.e., the higher-order temporal initial data \((\partial^\ell_t v^\varepsilon(0), \partial^\ell_t \eta(0), \partial^\ell_t q(0))\) can be defined by

\[
\partial^\ell_t v^\varepsilon(0) = \bar{\rho}^\varepsilon \partial^\ell_{t-1}(-a^\varepsilon_{ik} \partial_i q + 2\mu a^\varepsilon_{ik} \partial_i q_{\text{div}} + \lambda \varepsilon a^\varepsilon_{ik} \partial_i q_{\text{div}} + \bar{\rho}^\varepsilon \partial^\ell_{t} q_{\gamma}) \big|_{t=0},
\]

\[
\partial^\ell_t \eta(0) = \partial^\ell_{t-1} \eta(0), \quad \partial^\ell_t q(0) = \bar{\rho}^\varepsilon \partial^\ell_{t} (J^{-1} q) \big|_{t=0}
\]

inductively for \( \ell = 1, \ldots, m \). These data should satisfy

\[
\partial^\ell_t (-a^\varepsilon_{ik} \partial_i q_{\gamma} + 2\mu a^\varepsilon_{ik} \partial_i q_{\text{div}} + \lambda \varepsilon a^\varepsilon_{ik} \partial_i q_{\text{div}} + \bar{\rho}^\varepsilon \partial^\ell_{t} q_{\gamma}) \big|_{t=0} = \sigma \partial^\ell_{t} \left( \frac{\partial^2_{\gamma} \eta^\varepsilon}{\partial^2_{\gamma} \eta^\varepsilon} \right) \big|_{t=0}
\]

for \( \ell = 1, \ldots, m - 1 \).

The uniform regularity theorem is stated as follows.

**Theorem 1.1.** Let \( m \geq 4 \). Suppose that the initial data \((\rho_0^\varepsilon, \eta_0^\varepsilon, v_0^\varepsilon)\) satisfy the compatibility conditions (1.19), and have the following uniform bounds:

\[
0 < c_0 \leq \bar{\rho}^\varepsilon \leq C_0,
\]

\[
\|\eta_0^\varepsilon\|^2_{H^m} + \|\nabla\eta_0^\varepsilon\|^2_{H^{m+1}} + \|\nabla^m v_0^\varepsilon\|^2_{H^m} + \|\nabla^m \nabla \eta_0^\varepsilon\|^2_{H^m} + \|\partial^{m-1}\partial^2_t \eta_0^\varepsilon \cdot n_0^\varepsilon\|^2_{H^m} \leq C_0
\]

for some generic constants \( c_0 \) and \( C_0 \). Then there exist a \( T_0 > 0 \) independent of \( \varepsilon \) and a unique solution \((\eta^\varepsilon, v^\varepsilon)\) to the free boundary problem (1.15) on the time interval \([0, T_0]\) such that

\[
\sup_{t \in [0, T_0]} \mathcal{E}^\varepsilon(t) \leq C_1,
\]

where \( C_1 \) is a generic constant depending only on \( c_0 \) and \( C_0 \).

**Remark 1.1.** The regularity of solutions implies that the flow map \( \eta \) is at least Lipschitzian so that we can get the corresponding classical solutions in Eulerian coordinates.
Remark 1.2. The inherent structure of the elastic term enables us to perform the uniform regularity in standard Sobolev spaces, which indicates that the boundary layer does not appear in the free boundary problem of compressible viscoelastic fluids. This is different from the case studied by Mei et al. [32] for the free boundary compressible Navier-Stokes system.

Based on the uniform regularity in Theorem 1.1, we can justify the vanishing viscosity of the limit for the free boundary problem of compressible viscoelastic fluids and obtain the local existence of classical solutions to the free boundary elastodynamic equations.

**Theorem 1.2.** Under the assumptions of Theorem 1.1, if we further assume that there exists $(\rho_0, \eta_0, v_0)$ such that
\[
\lim_{\varepsilon \to 0} \|\bar{\rho}_0 - \rho_0\|_0 + \|\bar{\eta}_0 - \eta_0\|_0 + \|\bar{v}_0 - v_0\|_0 = 0,
\]
then there exists $(\eta, v)(t, \cdot)$ on the time interval $[0, T_0]$ such that
\[
\sup_{t \in [0, T_0]} E(t) \leq C_1
\]
and
\[
\lim_{\varepsilon \to 0} \sup_{t \in [0, T_0]} (\|\eta^\varepsilon(t) - \eta(t)\|_{H^{m-1}} + \|v^\varepsilon(t) - v(t)\|_{H^{m-2}}) = 0.
\]
Moreover, $(\eta, v)$ is the unique classical solution to the free boundary elastodynamic equation (1.16).

### 1.6 Sketches of the proofs

We give some comments on the difficulties and ideas in the proofs of main theorems. The crucial step is to derive uniform a priori estimates of local classical solutions to (1.15) in a small time interval independent of $\varepsilon$. Although the idea in [15] can be adopted here to deal with elastic stress in some sense, we have to propose some new ideas to handle the compressibility of the fluid.

First, the estimate of $J$ now depends on the regularity of $\eta$, and the pressure $q$ in the compressible fluid is no longer a Lagrangian multiplier, whose estimate follows from an elliptic equation with the Dirichlet or Neumann boundary condition for incompressible fluids. We derive the $L^\infty(\mathcal{H}^m)$ estimates of $J$ and $q$ from a geometric identity, the Gagliardo-Nirenberg-Moser-type inequality and the equation $q = \bar{\rho}_0 J^{-\gamma}$, which further give the $L^\infty(\mathcal{H}^{m-1})$ ones by the fundamental theorem of calculus (see Lemma 3.2).

Next, for the tangential derivative estimates, we use the $L^2$-type energy estimate to get
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} (\bar{\rho}_0 |\bar{\nabla} v|^2 + \bar{\rho}_0 |\bar{\nabla} \eta|^2) dx - \int_{\Omega} \bar{\nabla} q \cdot \bar{\nabla} v(a_{ik}\partial_k v_i) dx + \text{dissipative terms}
\]
\[
+ \int_{\Omega} \bar{\nabla} q a_{ik} - \bar{\nabla} (2\mu \varepsilon a_{k2} (S_A v)_i + \lambda \varepsilon a_{k2} (\text{div}_A v) + \bar{\rho}_0 \partial_2 \eta_i)) \bar{\nabla} v_i = \cdots.
\]

Using the transport equation satisfied by pressure in the compressible fluid, we can obtain
\[
- \int_{\Omega} \bar{\nabla} q \bar{\nabla} \eta(a_{ik}\partial_k v_i) dx = \frac{1}{2\gamma} \frac{d}{dt} \int_{\Omega} \bar{\rho}_0 J^{\gamma+1} |\bar{\nabla} q|^2 dx + \cdots,
\]
which provides the $L^2$-type energy of pressure. For the boundary integral in the tangential derivative estimates, we use (1.15) and adopt a similar idea as in [15] to have
\[
\int_{\Gamma} (\bar{\nabla} q a_{ik} - \bar{\nabla} (2\mu \varepsilon a_{k2} (S_A v)_i + \lambda \varepsilon a_{k2} (\text{div}_A v) + \bar{\rho}_0 \partial_2 \eta_i)) \bar{\nabla} v_i
\]
\[
= -\sigma \int_{\Gamma} \bar{\nabla} \left( \frac{\partial^2 \eta a_{k2}}{\partial \eta_1^2} \right) a_{ik} \bar{\nabla} v_i - \int_{\Gamma} \mathfrak{B} \bar{\nabla}^a a_{ik} \bar{\nabla} v_i + \cdots,
\]
where $\mathfrak{B} = q + \partial_1 \eta a_{k2}/|\partial_1\eta|^3$. Then the first term on the right-hand side of the above equation, which is related to the surface tension, provides the regularity of the boundary, i.e.,
\[
-\sigma \int_{\Gamma} \bar{\nabla} \left( \frac{\partial^2 \eta a_{k2}}{\partial \eta_1^2} \right) a_{ik} \bar{\nabla} v_i = \frac{\sigma d}{2} \int_{\Gamma} \left| \bar{\nabla} \partial_1 \eta \cdot n \right| + \cdots,
\]
while due to the fact that $a_2 = (-\partial_1 \eta_2, \partial_1 \eta_1)^T = \partial_1 \eta^\perp$ and the anti-symmetric properties of $\eta^\perp$ and $\eta$, the second term can be reduced to

$$-\int_{\Gamma} \mathcal{B} \partial^\alpha a_2 \partial^\alpha v_i = - \frac{1}{2} \frac{d}{dt} \int_{\Gamma} \mathcal{B} \partial^\alpha a_2 \partial^\alpha \eta_i + \frac{1}{2} \int_{\Gamma} \partial_1 \mathcal{B}(\partial^\alpha v_2 \partial^\alpha \eta_i - \partial^\alpha v_1 \partial^\alpha \eta_2) + \cdots.$$ 

For the second term on the right-hand side above in the case $\bar{\partial}^\alpha = \partial^\alpha m$, by using the decomposition (3.58), we can write it into a troublesome term of the form $\int_{\Gamma} f a_2 \partial^\alpha v_i$, whereas we cannot use the duality argument as in [15] to control it, because the term $\|a_{ij} \partial_j \partial^\alpha v_i\|_0$, which follows from the normal trace estimate $|\partial^\alpha v_1 a_{ij}| ⩽ \|\nabla \eta\|_{L^\infty(\Omega)} (\|\partial^\alpha v_i\|_0 + \|a_{ij} \partial_j \partial^\alpha v_i\|_0)$, cannot be bounded for compressible fluids. Instead, we use integration by parts to get

$$\int_0^t \int_{\Omega} f a_2 \partial^\alpha v_i dx_1 d\tau = \int_0^t \int_{\Omega} f a_{ij} \partial_j \partial^\alpha v_i dx_1 d\tau + \int_0^t \int_{\Omega} \partial_1 f a_{ij} \partial^\alpha v_i dx_1 d\tau = \int_0^t \int_{\Omega} f a_{ij} \partial_j \partial^\alpha \eta_i dx_1 + \cdots$$

and bound it by $M_0 + \delta \|\nabla \partial^\alpha \eta(t)\|_0^2 + P(\sup_{0 \leq t \leq \tau} \|\xi(t)\|_0)$.

Finally, for the normal derivative estimates, the approach used in [53] and based on the Hodge-type estimate and the structure of wave equations of enthalpy, will be invalid for the physical dynamic boundary condition (1.5) studied here, because the boundary conditions for the equations of vorticity and enthalpy become much worse due to the elastic term $\rho \mathbf{F}^T$ on the boundary. Moreover, we cannot directly use the same argument as in [15], in which the elastic term provides a Laplacian equation of $\eta$ with source terms involving only tangential derivatives. In our case, the elastic term provides a semi-linear elliptic equation of $\eta$ which is further reduced to the following equation:

$$-A_{ij} \partial_j^2 \eta_i - \mu \varepsilon a_{k2} a_{k2} \partial_j^2 v_i = (\mu + \lambda) \varepsilon a_{k2} a_{k2} \partial_j^2 v_j = \text{lower-order terms},$$

where $A_{ij} = \bar{\rho}_0 J \delta_{ij} + \gamma (\rho_0 J^{-1})^2 a_{k2} a_{k2}$ is positively symmetric. Then we can perform $L^2$-type energy estimates and control the normal derivatives inductively.

## 2 Preliminaries

In this section, we recall some inequalities and derive some identities and elementary estimates.

### 2.1 General inequalities

The following Gagliardo-Nirenberg-Moser-type inequality will be used repeatedly in this paper.

**Lemma 2.1.** The following product and commutator estimates hold:

(i) For any $m \in \mathbb{N}$, $|\alpha| + |\beta| = m$ and $g, h \in L^\infty(\Omega \times [0, t]) \cap L^2([0, t]; \mathcal{H}^m)$, it holds that

$$\|\partial^\alpha g \partial^\beta h\|_{L^2_\mathcal{L}(\Omega \times [0, t])} \lesssim \|g\|_{L^\infty_\mathcal{L}(\Omega \times [0, t])} \|h\|_{L^2_\mathcal{L}(\mathcal{H}^m)} + \|h\|_{L^\infty_\mathcal{L}(\mathcal{H}^m)} \|g\|_{L^2_\mathcal{L}(\mathcal{H}^m)}. \tag{2.1}$$

(ii) For any $m \in \mathbb{N}$, $1 \leq |\alpha| \leq m$, $g, h \in L^\infty(\Omega \times [0, t]) \cap L^2([0, t]; \mathcal{H}^{m-1})$ and $g \in L^2([0, t]; \mathcal{H}^m)$ such that $\partial g \in L^\infty(\Omega \times [0, t])$, it holds that

$$\|\partial^\alpha g h\|_{L^2_\mathcal{L}(\Omega \times [0, t])} \lesssim \|h\|_{L^2_\mathcal{L}(\mathcal{H}^{m-1})} \|\partial g\|_{L^\infty_\mathcal{L}(\mathcal{H}^m)} + \|\partial g\|_{L^\infty_\mathcal{L}(\mathcal{H}^m)} \|h\|_{L^2_\mathcal{L}(\mathcal{H}^{m-1})}, \tag{2.2}$$

where $[\partial^\alpha, g, h] := \partial^\alpha (gh) - g \partial^\alpha h$.

(iii) For any $m \in \mathbb{N}$, $2 \leq |\alpha| \leq 3$ and $g, h \in L^\infty(\Omega \times [0, t]) \cap L^2([0, t]; \mathcal{H}^{m-1})$, it holds that

$$\|\partial^\alpha g h\|_{L^2_\mathcal{L}(\Omega \times [0, t])} \lesssim \|h\|_{L^2_\mathcal{L}(\mathcal{H}^{m-2})} \|\partial g\|_{L^\infty_\mathcal{L}(\mathcal{H}^m)} + \|\partial h\|_{L^\infty_\mathcal{L}(\mathcal{H}^m)} \|\partial g\|_{L^2_\mathcal{L}(\mathcal{H}^{m-2})}, \tag{2.3}$$

where $[\partial^\alpha, g, h] := \partial^\alpha (gh) - \partial^\alpha g h - g \partial^\alpha h$.

**Lemma 2.2.** Let $g \in H^1([0, t]; L^2\mathcal{L})$. Then we have

$$\|g(t)\|_0^2 \lesssim t \|\partial_1 g\|_{L^2_\mathcal{L}(\Omega \times [0, t])}^2 + \|g(0)\|_0^2. \tag{2.4}$$
Proof. Since $g \in H^1([0, t]; L^2)$, we have $g(t, x) \in C([0, t]; L^2)$. The fundamental theorem of calculus gives
\[ g(t, x) = g(0, x) + \int_0^t g_t(\tau, x). \]
Then by applying the Minkowski inequality and Hölder’s inequality, one can get (2.4).

Lemma 2.3. We have the anisotropic Sobolev embedding
\[ \|g\|_{L^\infty(\Omega)}^2 \lesssim \|\nabla g\|_{H^{1}\tan}^2 + \|g\|_{H^{2}\tan}^2. \]  \hspace{1cm} (2.5)

As a consequence, we also have
\[ \|g\|_{L^\infty(t, x)}^2 \lesssim \|\nabla g\|_{L^\infty(t; H^{1}\tan)}^2 + \|g\|_{L^\infty(t; H^{2}\tan)}^2. \]  \hspace{1cm} (2.6)

We also use the following lemma.

Lemma 2.4. For $g \in H^1(\Gamma)$ and $h \in H^{1/2}(\Gamma)$ or $h \in H^{-1/2}(\Gamma)$, it holds that
\[ |gh|_{\frac{1}{2}} \lesssim |g|_{1} |h|_{\frac{1}{2}}, \quad |gh|_{-\frac{1}{2}} \lesssim |g|_{1} |h|_{-\frac{1}{2}}. \]  \hspace{1cm} (2.7)

Proof. It is direct to check that $|gh|_{s} \lesssim |g|_{1} |h|_{s}$ for $s = 0, 1$ with the help of the Sobolev embedding $|f|_{L^\infty} \lesssim |f|_{1}$. Then the estimate (2.7) follows from the interpolation. The second inequality follows from the dual estimate.

We recall the next lemma for the embedding of fractional Sobolev spaces (see [10] for the proof).

Lemma 2.5. Let $\Gamma$ be a bounded domain in $\mathbb{R}^d$. Then for any $g \in H^{s}(\Gamma)$, we have the following Sobolev embedding:
\[ |g|_{L^r} \lesssim |g|_{s}, \]  \hspace{1cm} (2.8)

for any $r \in [1, \frac{2d}{d-2s}]$, if $2s < d$, and for any $r \in [1, \infty)$, if $2s = d$. In particular, for $d = 1, 2$, we have
\[ |g|_{L^1} \lesssim |g|_{\frac{1}{2}}. \]  \hspace{1cm} (2.9)

2.2 Trace estimates

First, we have the trace estimate, whose proof was given in [15].

Lemma 2.6. For $g \in H^1(\Omega)$, it holds that
\[ |g|^2 \lesssim \|g\|_0^2 + \|g\|_0 \|\nabla g\|_0. \]  \hspace{1cm} (2.10)

Next, we have the following normal trace estimate.

Lemma 2.7. It holds that
\[ |\omega_i a_{ij}|_{-\frac{1}{2}} \lesssim \|\nabla \eta\|_{L^\infty} \|\omega\|_0 + \|a_{ij} \partial_j \omega_i\|_0. \]  \hspace{1cm} (2.11)

Proof. We refer to [49, Subsection 5.9].

We also need the following lemma.

Lemma 2.8. For any $g \in H^{\frac{1}{2}}(\Gamma)$, it holds that
\[ |\partial_1 g|_{-\frac{1}{2}} \lesssim |g|_{\frac{1}{2}}. \]  \hspace{1cm} (2.12)

2.3 Korn’s inequality

We refer to [15,30] for the following Korn-type inequality.

Lemma 2.9. For any $f \in H^1(\Omega)$, it holds that
\[ \|\nabla f\|_0^2 \lesssim P(\|\nabla \eta\|_0^2)(\|S_A(f)\|_0^2 + \|f\|_0^2). \]  \hspace{1cm} (2.13)
2.4 Geometric identities

Differentiating $J$, $A$ and $a$, we obtain

$$
\partial J = a_{ij} \partial_j \partial_i \eta, \quad \partial A_{kj} = -A_{kj} \partial_i \partial_i A_{ij}, \quad \partial a_{kj} = a_{ik} \partial_i \partial_i A_{kj} - a_{ki} \partial_i \partial_i A_{kj},
$$

(2.14)

3 Viscosity-independent a priori estimates

In this section, we derive the $\varepsilon$-independent estimate of smooth solutions to (1.15), which is stated in the following proposition.

**Proposition 3.1.** Let $(\eta^\varepsilon, \nu^\varepsilon)$ be a solution to (1.15). Then there exists a time $T$ independent of $\varepsilon$ such that

$$
\sup_{t \in [0,T]} \mathcal{E}^\varepsilon(t) \leq 2M_0,
$$

(3.1)

where $M_0 = P(\mathcal{E}(0))$.

The proof of the proposition can be divided into proofs of the following lemmas. For notational simplicity, we only address the superscript $\varepsilon$ of $\nu^\varepsilon, \eta^\varepsilon, a^\varepsilon$, etc., in the statements of lemmas but omit it in the proof without causing any confusion. Since

$$
\tilde{\rho}_0 \geq c_0 > 0, \quad \frac{1}{c_0} > J_0 \geq c_0 > 0
$$

(3.2)

for some $c_0 > 0$, we can assume that there exists a sufficiently small $T_\varepsilon$ such that for $t \in [0,T_\varepsilon]$,

$$
|J'(t) - J_0| \leq \frac{1}{8} c_0, \quad |\partial_j \eta^\varepsilon_i(t) - \partial_j \eta^\varepsilon_i| \leq \frac{1}{8} c_0.
$$

(3.3)

We now derive $\varepsilon$-independent estimates of smooth solutions $(\nu^\varepsilon, \eta^\varepsilon)$ to (1.15) under the a priori assumption (3.3). We start from the basic energy estimates as follows.

3.1 Basic energy estimates

**Lemma 3.1.** For any $t \in [0,T_\varepsilon]$, it holds that

$$
\|\nu^\varepsilon(t)\|_0^2 + \|\nabla \eta^\varepsilon(t)\|_0^2 + \|Q(f^\varepsilon(t))\|_{L^1} + \sigma \|\partial \eta^\varepsilon(t)\|_{L^1} \lesssim M_0 + T_\varepsilon.
$$

(3.4)

**Proof.** Taking the $L^2(\Omega)$ inner product of (1.15) with $v_i$ gives

$$
\frac{1}{2} \frac{d}{dt} \int_\Omega \tilde{\rho}_0 |v|^2 dx + \int_\Omega a_{ik} \partial_k q v_i dx - 2\mu \varepsilon \int_\Omega a_{ik} \partial_i \partial_j (S_A v)_{ik} v_i dx

- \lambda \varepsilon \int_\Omega a_{ij} \partial_j (A_{kl} \partial_k v_i) v_i dx - \int_\Omega \partial_j (\tilde{\rho}_0 \partial_j \eta_i) v_i dx = 0.
$$

By integration by parts and using (1.14), (1.15) and (1.11), we have

$$
\int_\Omega a_{ik} \partial_k q v_i dx - 2\mu \varepsilon \int_\Omega a_{ik} \partial_i (S_A v)_{ik} v_i dx - \lambda \varepsilon \int_\Omega a_{ij} \partial_j (A_{kl} \partial_k v_i) v_i dx - \int_\Omega \partial_j (\tilde{\rho}_0 \partial_j \eta_i) v_i dx

= - \int_\Omega a_{ik} \partial_i q v_i v_i - 2\mu \varepsilon \int_\Omega (S_A v)_{ik} a_{ik} \partial_k v_i dx + \lambda \varepsilon \int_\Omega J(A_{ij} \partial_j v_i) v_i dx + \int_\Omega \tilde{\rho}_0 \partial_j \eta_i \partial_j v_i dx

+ \int_\Omega (q a_{ik} - 2\mu \varepsilon (S_A v)_{ik} a_{ik} - \lambda \varepsilon A_{ik} \partial_i A_{kj} a_{ki} - \tilde{\rho}_0 \partial_2 \eta_i) v_i

= - \int_\Omega \sigma \partial_i \left( \frac{\partial \eta_i}{\partial \eta} \right) v_i - \int_\Omega J q dx + \frac{1}{2} \frac{d}{dt} \int_\Omega \tilde{\rho}_0 |\nabla \eta|^2 dx

+ 2\mu \varepsilon \int_\Omega (S_A v)_{ik} a_{ik} \partial_k v_i dx + \lambda \varepsilon \int_\Omega J(A_{ij} \partial_j v_i)^2 dx
$$
Lemma 3.2. Before performing the higher-order estimates, we first prove the following key lemma for the estimates of

By using (2.14), (3.3) and (2.1), one has

Proof. Noticing that by a straightforward calculation,

we have

Therefore, we obtain

which, following integration in time, yields

Noticing that by a straightforward calculation,

by $2\mu + 2\lambda > 0$ and $\mu > 0$, we have

Thus, in view of (3.2), (3.3) and Korn's inequality (2.13), (3.4) holds true.

3.2 Estimates of $J$ and $q$

Before performing the higher-order estimates, we first prove the following key lemma for the estimates of velocity divergence and pressure.

Lemma 3.2. For any $t \in [0, T_r]$ and $m \geq 3$, it holds that

\[ \|J\|^2_{L^2(\mathbb{H}^m)} + \|q\|^2_{L^2(\mathbb{H}^m)} \leq T_r P \left( \sup_{t \in [0, T_r]} \mathcal{E}^\varepsilon(t) \right). \]  

\[ \|J\|^2_{L^\infty(\mathbb{H}^{m-1})} + \|q\|^2_{L^\infty(\mathbb{H}^{m-1})} \leq M_0 + T_r^2 P \left( \sup_{t \in [0, T_r]} \mathcal{E}^\varepsilon(t) \right). \]  

Proof. By using (2.14), (3.3) and (2.1), one has

\[ \|J\|^2_{L^2(\mathbb{H}^m)} \lesssim \|J\|^2_{L^2(L^2)} + \sum_{|\alpha| \leq m-1} \|\partial^\alpha (a_{ij} \partial_j \eta_i)\|^2_{L^2(L^2)} \lesssim c_0 t + \sum_{|\beta| + |\gamma| = |\alpha| \leq m-1} \|\partial^\beta a_{ij} \partial^\gamma \partial_j \eta_i\|^2_{L^2(L^2)} \lesssim c_0 t + \|a_{ij}\|^2_{L^2(\mathbb{H}^m)}, \]  

\[ \|J\|^2_{L^2(\mathbb{H}^m)} \lesssim \|J\|^2_{L^2(L^2)} + \sum_{|\alpha| \leq m-1} \|\partial^\alpha (a_{ij} \partial_j \eta_i)\|^2_{L^2(L^2)} \lesssim c_0 t + \|a_{ij}\|^2_{L^2(\mathbb{H}^m)}, \]  

\[ \|\partial^\alpha \Phi\|^2_{L^2(\mathbb{H}^m)} \lesssim \|\partial^\alpha \Phi\|^2_{L^2(L^2)}, \]  

\[ \|\partial^\alpha \Phi\|^2_{L^2(\mathbb{H}^m)} \lesssim \|\partial^\alpha \Phi\|^2_{L^2(L^2)} + \sum_{|\beta| + |\gamma| = |\alpha| \leq m-1} \|\partial^\beta a_{ij} \partial^\gamma \partial_j \eta_i\|^2_{L^2(L^2)} \lesssim c_0 t + \|a_{ij}\|^2_{L^2(\mathbb{H}^m)}. \]
\[
\lesssim c_0 t + \|\nabla \eta \|^2_{L^\infty_t L^\infty_x} \lesssim T P \left( \sup_{t \in [0,T]} \mathcal{E}(t) \right),
\]
where we have used
\[
\|\nabla \eta \|_{L^\infty_t L^\infty_x} \lesssim \|\nabla \eta \|_{L^2_t L^2_x} \lesssim T^\frac{1}{2} \|\eta\|_{L^\infty_t L^\infty_x}.
\]
Since \( J \) is bounded from below and above in (3.3), direct calculations yield that for any \( s \in \mathbb{R} \),
\[
|\partial^s(J^s)| \lesssim \sum_{|\beta_1| + \cdots + |\beta_k| = |\alpha|} |\partial^{\beta_1} J \cdots \partial^{\beta_k} J|.
\]
Then it follows from the Minkowski inequality, Hölder’s inequality and Gagliardo-Nirenberg’s inequality in \( \Omega \times [0,t] \) that
\[
\|J^s\|_{L^2_t L^2_x} \lesssim \sum_{|\alpha| \leq m} \|\partial^\alpha(J^s)\|^2_{L^2_t L^2_x} \lesssim \sum_{|\alpha| \leq m} \sum_{|\beta_1| + \cdots + |\beta_k| = |\alpha|} \|\partial^{\beta_1} J \cdots \partial^{\beta_k} J\|^2_{L^2_t L^2_x} \lesssim \sum_{|\alpha| \leq m} \sum_{|\beta_1| + \cdots + |\beta_k| = |\alpha|} \prod_{i=1}^k \|\partial^{\beta_i} J\|^2_{L^2_t L^2_x} + \|J\|^2_{L^2_t L^2_x} \lesssim \|J\|^2_{L^2_t L^2_x} + T \|J\|^2_{L^2_t L^2_x} \lesssim T P \left( \sup_{t \in [0,T]} \mathcal{E}(t) \right),
\]
where \( r_i \) and \( \theta_i \) (\( i = 1, \ldots, k \)) satisfy
\[
\sum_{i=1}^k \frac{1}{r_i} = \frac{1}{2}, \quad \frac{1}{r_i} = \left( 1 - \frac{|\beta_i|}{3} + \left( \frac{1}{2} - \frac{|\alpha|}{3} \right) \frac{1}{\theta_i} + \frac{1 - \theta_i}{\infty} \right)
\]
so that \( \sum_{i=1}^k \theta_i = 1 \). By using (2.4), one can get
\[
\|J\|^2_{L^\infty_t L^\infty_x} \lesssim \|J_0\|^2_{L^\infty_x} + T \|\partial_t J\|^2_{L^2_t L^2_x} \lesssim M_0 + T^2 P \left( \sup_{t \in [0,T]} \mathcal{E}(t) \right).
\]
For the pressure \( q \), we get from (1.2) and (1.12) that
\[
q = Af^\gamma + 1 - p_c = A \tilde{p}^\gamma + 1 - p_c.
\]
Then it follows from (2.1) and (3.10) that
\[
\|q\|^2_{L^2_t L^\infty_x} \lesssim \|	ilde{p}\|^2_{L^\infty_t L^\infty_x} \|J^{-\gamma}\|^2_{L^2_t L^2_x} + \|J^{-\gamma}\|^2_{L^\infty_t L^\infty_x} \|	ilde{p}\|^2_{L^\infty_t L^\infty_x} + T \lesssim T P \left( \sup_{t \in [0,T]} \mathcal{E}(t) \right).
\]
By using (2.4), we have
\[
\|q\|^2_{L^2_t L^\infty_x} \lesssim \|q_0\|^2_{L^\infty_x} + T \|\partial_t q\|^2_{L^2_t L^2_x} \lesssim M_0 + T^2 P \left( \sup_{t \in [0,T]} \mathcal{E}(t) \right).
\]
This completes the proof. \( \square \)
3.3 Tangential derivative estimates with at least one spatial derivative

We now derive the higher-order estimates of tangential derivatives. We have to separate the estimate of the fully temporal derivative from other tangential ones since we lose some key estimates for the fully temporal derivative case. The following lemma shows the estimate of tangential derivatives with at least one spatial one.

Lemma 3.3. For any \( t \in [0, T] \), \( m \geq 4 \) and any \( \beta = (\beta_0, \beta_1) \) with \( 0 \leq \beta_0 + \beta_1 \leq m - 1 \), it holds that

\[
\|\bar{\rho}^\alpha\|_{L^2_t(H^{m-1}_{\text{tan}})}^2 + \|\bar{\eta}^\alpha\|_{L^2_t(H^{m-1}_{\text{tan}})}^2 + \|\nabla \bar{\eta}^\alpha\|_{L^2_t(H^{m-1}_{\text{tan}})}^2 + \sum_{|\beta| \leq m-1} |\bar{\rho}^\alpha \partial^2_{x_j} \bar{\rho}^\alpha a_{ij}|^2 + \varepsilon \|\nabla \bar{\eta}^\alpha\|_{L^2_t(H^{m-1}_{\text{tan}})}^2
\]

\[
\leq M_0 + \delta \sup_{t \in [0, T]} \mathcal{E}(t) + T \mathbb{E} \left( \sup_{t \in [0, T]} \mathcal{E}(t) \right).
\]

Proof. For \( 1 \leq |\alpha| \leq m \), where \( \alpha = (\alpha_0, \alpha_1) \) with \( \alpha_1 \geq 1 \), applying \( \partial^\alpha \) to the second equation of (1.15), and then taking the \( L^2(\Omega) \) inner product with \( \partial^\alpha v_1 \), we have

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \bar{\rho}^\alpha |\partial^\alpha v_1|^2 dx + \int_{\Omega} a_{ik} \partial_k \partial^\alpha g \partial^\alpha v_1 dx - 2\mu \varepsilon \int_{\Omega} \partial^\alpha (a_{ik} \partial_i (S_\alpha v_{ik})) \partial^\alpha v_1 dx
\]

\[
- \lambda \varepsilon \int_{\Omega} \partial^\alpha (a_{ij} \partial_j \text{div}_A(v)) \partial^\alpha v_1 dx - \int_{\Omega} \partial_j (\partial^\alpha (\bar{\rho}^\alpha \partial_j \eta_j)) \partial^\alpha v_1 dx
\]

\[
= - \int_{\Omega} \bar{\rho}^\alpha \partial^\alpha a_{ik} \partial^\alpha v_1 dx - \int_{\Omega} \bar{\rho}^\alpha \partial^\alpha a_{ik} \partial^\alpha v_1 dx + 2\mu \varepsilon \int_{\Omega} \partial^\alpha ((S_\alpha v_{ik}) a_{ik}) \partial^\alpha v_1 dx
\]

\[
+ \lambda \varepsilon \int_{\Omega} \partial^\alpha (a_{ij} \text{div}_A(v)) \partial_j \partial^\alpha v_1 dx + \int_{\Omega} \partial^\alpha (\bar{\rho}^\alpha \partial_j \eta_j) \partial_j \partial^\alpha v_1 dx
\]

\[
+ \int_{\Omega} \left( \partial^\alpha q a_{ij} - \partial^\alpha (2\mu \varepsilon a_{ik} (S_\alpha v_{ik}) + \lambda \varepsilon a_{ij} (\text{div}_A(v) + \bar{\rho}^\alpha \partial_j \eta_j)) \bar{\rho}^\alpha v_1 \right)
\]

By integration by parts and using (1.14), the third equation of (1.15) and (1.11), we have

\[
\int_{\Omega} a_{ik} \partial_k \partial^\alpha g \partial^\alpha v_1 dx - 2\mu \varepsilon \int_{\Omega} \partial^\alpha (a_{ik} \partial_i (S_\alpha v_{ik})) \partial^\alpha v_1 dx
\]

\[
- \lambda \varepsilon \int_{\Omega} \partial^\alpha (a_{ij} \partial_j \text{div}_A(v)) \partial^\alpha v_1 dx - \int_{\Omega} \partial_j (\partial^\alpha (\bar{\rho}^\alpha \partial_j \eta_j)) \partial^\alpha v_1 dx
\]

\[
= - \int_{\Omega} \bar{\rho}^\alpha \partial^\alpha a_{ik} \partial^\alpha v_1 dx - \int_{\Omega} \bar{\rho}^\alpha \partial^\alpha a_{ik} \partial^\alpha v_1 dx + 2\mu \varepsilon \int_{\Omega} \partial^\alpha ((S_\alpha v_{ik}) a_{ik}) \partial^\alpha v_1 dx
\]

\[
+ \lambda \varepsilon \int_{\Omega} \partial^\alpha (a_{ij} \text{div}_A(v)) \partial_j \partial^\alpha v_1 dx + \int_{\Omega} \partial^\alpha (\bar{\rho}^\alpha \partial_j \eta_j) \partial_j \partial^\alpha v_1 dx
\]

\[
+ \int_{\Omega} \left( \partial^\alpha q a_{ij} - \partial^\alpha (2\mu \varepsilon a_{ik} (S_\alpha v_{ik}) + \lambda \varepsilon a_{ij} (\text{div}_A(v) + \bar{\rho}^\alpha \partial_j \eta_j)) \bar{\rho}^\alpha v_1 \right)
\]

\[
= \frac{1}{2} \frac{d}{dt} \int_{\Omega} \bar{\rho}^\alpha \nabla \bar{\eta}^\alpha \cdot \bar{\eta}^\alpha dx - \int_{\Omega} \partial^\alpha q \partial^\alpha \partial_j J dx + 2\mu \varepsilon \int_{\Omega} \partial^\alpha J_\alpha (\partial^\alpha v_1)^2 dx + \lambda \varepsilon \int_{\Omega} J_\alpha (\partial^\alpha v_1)^2 dx
\]
\[ + \int_\Gamma (\bar{\partial}^\alpha q_{a12} - \bar{\partial}^\alpha (2\mu \varepsilon a_{k2}(S_A v)_{1k} + \lambda \varepsilon a_{a2}(\text{div}_A v) + \bar{\rho}_0 \partial_2 \eta_i))\bar{\partial}^\alpha v_i + R^2_q + R^3_q + \sum_{i=1}^4 R^4_q. \]

Moreover, (1.12) implies
\[ \partial_t J = \partial_t (\bar{\rho}_0 f^{-1}) = -\bar{\rho}_0 \frac{\partial f}{f^2} = -\bar{\rho}_0 \frac{\partial q}{q(f) f^2} = -\bar{\rho}_0 \frac{\partial q}{\gamma A f^{\gamma + 1}} = -\frac{\gamma + 1}{\gamma A \bar{\rho}_0} \partial_t q \]
so that
\[ -\frac{1}{2\gamma} \int_{1_\Omega} \bar{\rho}_0 |(\bar{\partial}^\alpha v)|^2 + |\bar{\partial}^\alpha \nabla \eta|^2 dx + \frac{1}{2\gamma} \int_{1_\Omega} \bar{\rho}_0^{-\gamma} J^{\gamma+1} |\bar{\partial}^\alpha q|^2 dx \]
\[ + 2\mu \varepsilon \int_{1_\Omega} J |S_A (\bar{\partial}^\alpha v)|^2 dx + \lambda \varepsilon \int_{1_\Omega} J (\text{div}_A (\bar{\partial}^\alpha v))^2 dx \]
\[ + \int_{1_\Omega} (\bar{\partial}^\alpha q_{a12} - \bar{\partial}^\alpha (2\mu \varepsilon a_{k2}(S_A v)_{1k} + \lambda \varepsilon a_{a2}(\text{div}_A v) + \bar{\rho}_0 \partial_2 \eta_i))\bar{\partial}^\alpha v_i \]
\[ = -\left( \sum_{i=1}^3 R^i_q + \sum_{i=1}^4 R^i_q + \sum_{i=1}^4 R^i_q \right). \]

We now estimate the terms \( R^i_q, R^i_q \) and \( R^i_q \) one by one. It follows from (2.2) that
\[ \left| \int_0^t R^i_q \right| \lesssim \left| |\partial^\alpha \nabla \eta_i|_{L^{\gamma}_t} \right| |\partial^\alpha v_i|_{L^{\gamma}_t} \]
\[ \lesssim \left( \left| |\partial \bar{\rho}_0|_{L^{\gamma}_t} \right| |\partial \nabla \eta_i|_{L^{\gamma}_t} + \left| |\partial \bar{\rho}_0|_{L^{\gamma}_t} \right| |\partial \bar{\rho}_0|_{L^{\gamma}_t} \right) \]
\[ \lesssim \left( \left| |v|_{L^{\gamma}_t} \right| t^{\frac{\gamma}{2}} \left| |\bar{\partial}^\alpha v|_{L^{\gamma}_t} \right| \lesssim T^{\frac{\gamma}{2}} \sup_{t \in [0,T]} \varepsilon^\gamma(t), \right. \]

where we have used \( \left| |\partial v|_{L^{\gamma}_t} \right| \lesssim \left| |\partial \nabla \eta|_{L^{\gamma}_t} \right| \) from the Sobolev embedding. Similarly, we have
\[ \left| \int_0^t R^2_q \right| \lesssim \left| |a_{k2}|_{L^{\gamma}_t(\Omega)} |\partial_k v_i|_{L^{\gamma}_t(\Omega)} \right| \left| |\partial^\alpha v_i|_{L^{\gamma}_t(\Omega)} \right| \lesssim T^{\frac{\gamma}{2}} \sup_{t \in [0,T]} \varepsilon^\gamma(t), \]

and
\[ \left| \int_0^t R^1_q \right| \lesssim \left| |\bar{\partial}^\alpha \nabla \eta_i|_{L^{\gamma}_t(\Omega)} \right| \left| |\partial \nabla \eta_i|_{L^{\gamma}_t(\Omega)} \right| \lesssim T^{\frac{\gamma}{2}} \sup_{t \in [0,T]} \varepsilon^\gamma(t), \]
where in (3.17), we have used \( \|\tilde{\rho}_0\|_{H^m} \leq C \) and
\[
\left| \int_{\Omega} [\bar{\partial}^\alpha, \tilde{\rho}_0] \partial_\eta(t) \partial_j \tilde{\partial}^\alpha \eta(t) dx \right|
= \int_{\Omega} \left| \sum_{|\beta| \geq 1, |\beta| + |\nu| = m} C_0^\alpha \bar{\partial}^\beta \tilde{\rho}_0 \bar{\partial}^\beta \eta \partial_\eta \partial_j \eta dx \right|
\leq \delta \|\tilde{\partial}^\alpha \nabla \eta(t)\|_0^2 + C_\delta (\|\tilde{\partial} \tilde{\rho}_0\|_\infty^2 \|\tilde{\partial}^{m-1} \nabla \eta(t)\|_0^2 + \|\tilde{\partial}^m \tilde{\rho}_0\|_\infty^2 \|\nabla \eta(t)\|_L^2)
+ C_\delta \sum_{|\beta| \geq 2, |\beta| + |\nu| = m} \|\tilde{\partial}^\beta \eta\|_L^2 \|\tilde{\partial}^\nu \nabla \eta(t)\|_L^2
\leq \delta \|\tilde{\partial}^\alpha \nabla \eta(t)\|_0^2 + C_\delta \|\nabla \eta(t)\|_{L^2}^2
\leq \delta \|\tilde{\partial}^\alpha \nabla \eta(t)\|_0^2 + C_\delta \|\nabla \eta(t)\|_{H^{m-1}}^2.
\]

In view of (3.2) and (3.3), it is obvious that
\[
\left| \int_0^T R_0^2 \right| \leq T \sup_{t \in [0,T]} C(t). \tag{3.18}
\]

It follows from (2.2) and (3.6) that
\[
\left| \int_0^T R_0^1 \right| \leq \|\tilde{\partial}^\alpha q\|_{L^2(L^2)} \|\tilde{\partial}^\alpha, \tilde{\rho}_0 \tilde{\partial}^\gamma J^{m+1}\|_{L^2(L^2)}
\leq \|\tilde{\partial}^\alpha q\|_{L^2(L^2)} \|\tilde{\partial}^\alpha \tilde{\rho}_0\|_{L^2(H^{m-1})} + \|\tilde{\partial} q\|_{L^2(H^{m-1})} \|\tilde{\partial} \tilde{\rho}_0\|_{L^2(H^{m-1})} + \|\tilde{\partial} \tilde{\rho}_0\|_{L^2(H^{m-1})} \|\tilde{\partial}^\alpha \tilde{\partial}^\gamma J^{m+1}\|_{L^2(L^2)}
\leq \|\tilde{\partial}^\alpha q\|_{L^2(L^2)} \|\tilde{\partial} \tilde{\rho}_0\|_{L^2(H^{m-1})} \|\tilde{\partial}^\gamma J^{m+1}\|_{L^2(L^2)} \leq T \frac{1}{2} P \left( \sup_{t \in [0,T]} C(t) \right). \tag{3.19}
\]

In view of (3.3), Sobolev interpolation inequalities and the estimate from (2.6)
\[
\|\tilde{\partial} \nabla \eta\|_{L^2(H^m)} \leq \|\tilde{\partial} \nabla \eta\|_{L^2(H^m)} + \|\tilde{\partial}^2 \nabla \eta\|_{L^2(H^m)}, \tag{3.20}
\]

one has
\[
\left| \int_0^T (R_2^1 + R_2^2) \right| \leq \epsilon \|\tilde{\partial}^\alpha a \|_{L^\infty(L^2)} \|\tilde{\partial} \tilde{\partial}^\alpha \nabla \eta\|_{L^2(L^2)}^2
\leq \epsilon \|\tilde{\partial} \tilde{\partial}^\alpha a \|_{L^\infty(L^2)} \|\tilde{\partial} \nabla \eta\|_{L^2(H^{m-1})} + \|\tilde{\partial} \tilde{\partial}^\alpha a \|_{L^\infty(L^2)} \|\tilde{\partial} \nabla \eta\|_{L^2(H^{m-1})} \|\tilde{\partial}^\alpha \nabla \eta\|_{L^2(L^2)}^2
\leq \epsilon \|\tilde{\partial} \tilde{\partial}^\alpha \nabla \eta\|_{L^2(L^2)}^2 + TP \left( \sup_{t \in [0,T]} C(t) \right), \tag{3.21}
\]

where we have used (3.9) and
\[
\|\tilde{\partial} \tilde{\partial}^\alpha \nabla \eta\|_{L^2(H^{m-1})} \leq \|\tilde{\partial} \tilde{\partial}^\alpha \nabla \eta\|_{L^2(H^{m-1})} \leq \|\nabla \eta\|_{L^2(H^m)}. \tag{3.22}
\]

Similarly, it follows from (2.2) and (2.20) that
\[
\left| \int_0^T (R_3^1 + R_3^2) \right| \leq \epsilon \|\tilde{\partial}^\alpha, A\|_{L^\infty(L^2)} \|\tilde{\partial} \nabla \eta\|_{L^2(L^2)} \|\tilde{\partial}^\alpha \tilde{\partial}^\gamma J^{m+1}\|_{L^2(L^2)} + \|\tilde{\partial} \tilde{\partial}^\alpha \nabla \eta\|_{L^2(L^2)}
\leq \epsilon \|\tilde{\partial}^\alpha, A\|_{L^\infty(L^2)} \|\tilde{\partial} \nabla \eta\|_{L^2(L^2)} \|\tilde{\partial}^\alpha \tilde{\partial}^\gamma J^{m+1}\|_{L^2(L^2)} + \epsilon \|\tilde{\partial} \tilde{\partial}^\alpha \nabla \eta\|_{L^2(L^2)} + TP \left( \sup_{t \in [0,T]} C(t) \right). \tag{3.23}
\]

Next, we estimate the boundary term \( R_0 \). Since
\[
\sigma \partial_i \left( \frac{\partial_1 \eta}{|\partial_1 \eta^2|} \right) = \sigma \frac{\partial_1^2 \eta \partial_2}{|\partial_1 \eta|^3},
\]
one gets from (1.15) that

\[
\mathcal{B} := \sigma \frac{\partial_t^2 \eta k a_k}{|\partial_1 \eta|^3} + q = 2 \mu \frac{a_k A_i \partial_i v_k a_k}{|\partial_1 \eta|^2} + \lambda \varepsilon \text{div}_A \mathbf{v} + \tilde{\rho}_0 J |\partial_1 \eta|^2. \quad (3.24)
\]

Then we can obtain from (1.15) and integration by parts that

\[
R_b = - \int_{\Gamma} \sigma \partial_{\sigma} \left( \frac{\partial_t^2 \eta k a_k}{|\partial_1 \eta|^3} \right) a_k a_2 \partial^\sigma \mathbf{v} - \int_{\Gamma} \mathcal{B} \partial_{\sigma} a_k a_2 \partial^\sigma \mathbf{v} - \int_{\Gamma} \left[ \partial_{\sigma} \mathcal{B} a_{k2} \right] \partial^\sigma \mathbf{v}
\]

\[
= - \sigma \int_{\Gamma} \frac{1}{|\partial_1 \eta|^3} \partial_t^2 \eta k a_k a_2 \partial^\sigma \mathbf{v} - \sigma \int_{\Gamma} \partial_1 \left( \frac{a_k a_2}{|\partial_1 \eta|^3} \right) \partial_t^2 a_k a_2 \partial^\sigma \mathbf{v}
\]

\[
= - \int_{\Gamma} \left[ \frac{1}{|\partial_1 \eta|^3} \partial_t^2 \eta k a_k a_2 \partial^\sigma \mathbf{v} + R^1_b + R^2_b \right]
\]

\[
= \int_{\Gamma} \frac{1}{|\partial_1 \eta|^3} \partial_t^2 \eta k a_k a_2 \partial^\sigma \mathbf{v} + \sigma \int_{\Gamma} \frac{1}{|\partial_1 \eta|^3} \partial_1 \left( \frac{a_k a_2}{|\partial_1 \eta|^3} \right) \partial_t^2 \eta k a_2 \partial^\sigma \mathbf{v} + R^1_b + \cdots + R^6_b
\]

\[
(3.25)
\]

Before controlling the terms \(R^i_b (i = 1, \ldots, 6)\), we first derive an estimate of \(\mathcal{B}\) as follows. By using (3.24), (2.1) and (3.3), one has

\[
\| \mathcal{B} \|^2_{L^2(T)} \lesssim \varepsilon^2 \left( \left\| \frac{a_k a_2 A_i \partial_i v_k + a_k}{|\partial_1 \eta|^2} \right\|^2_{L^2(T)} + \| \text{div}_A \mathbf{v} \|^2_{L^2(T)} \right) + \left\| \tilde{\rho}_0 J |\partial_1 \eta|^2 \right\|^2_{L^2(T)}
\]

\[
\lesssim \varepsilon^2 \left( \| A \nabla \mathbf{v} \|^2_{L^2(T)} + \| \nabla \mathbf{v} \|^2_{L^2(T)} \left( \| \partial_1 \eta \|^2_{L^2(T)} + \left\| \frac{1}{|\partial_1 \eta|^2} \right\|^2_{L^2(T)} \right) \right)
\]

\[
+ \| J \|^2_{L^2(T)} + \left\| \frac{1}{|\partial_1 \eta|^2} \right\|^2_{L^2(T)} + \| \tilde{\rho}_0 \|^2_{L^2(T)}
\]

\[
\lesssim P(\| \partial_1 \eta \|^2_{L^2(T)}) (\varepsilon^2 \| \nabla \mathbf{v} \|^2_{L^\infty} + 1) + \varepsilon^2 \| \nabla \mathbf{v} \|^2_{L^2(T)}
\]

\[
\lesssim P(\| \partial_1 \eta \|^2_{L^2(T)}) (\varepsilon^2 \| \nabla \mathbf{v} \|^2_{L^2(T)} + 1) \lesssim P \left( \sup_{t \in [0, T]} \mathcal{E}(t) \right). \quad (3.26)
\]

where we have used

\[
\| \partial_1 \eta \|_{L^2(T)} \lesssim \| \partial_1 \eta \|^2_{L^2(T)} + P(\| \partial_1 \eta \|^2_{L^2(T)}).
\]

Then it follows from (2.4) that

\[
\| \mathcal{B} \|_{L^2(T)} \lesssim M_0 + T \| \partial_1 \mathcal{B} \|_{L^2(T)} \lesssim M_0 + TP \left( \sup_{t \in [0, T]} \mathcal{B}(t) \right). \quad (3.27)
\]

As a consequence, the terms \(R^i_b (i = 1, \ldots, 6)\) can be bounded as follows. For \(R^1_b\), direct calculations yield that

\[
R^1_b = - \frac{d}{dt} \int_{T} \mathcal{B} \partial_t^2 a_k a_2 \partial^\sigma \mathbf{v} + \int_{T} \partial_1 \mathcal{B} \partial_t^2 a_k a_2 \partial^\sigma \mathbf{v} + \int_{T} \mathcal{B} \partial_t^2 \partial_t a_k a_2 \partial^\sigma \mathbf{v}
\]

\[
= - \frac{d}{dt} \int_{T} \mathcal{B} \partial_t^2 a_k a_2 \partial^\sigma \mathbf{v} + \int_{T} \partial_1 \mathcal{B} \partial_t^2 a_k a_2 \partial^\sigma \mathbf{v} + \int_{T} \partial_1 \mathcal{B} (\partial_t^2 \partial_t a_k a_2 \partial^\sigma \mathbf{v} - \partial_t^2 \partial_t a_k a_2 \partial^\sigma \mathbf{v})
\]
\[
\begin{align*}
\mathcal{B}(\partial^\alpha v_2 \partial^\alpha \partial_1 \eta_1 - \partial^\alpha v_1 \partial^\alpha \partial_1 \eta_2) \\
\underbrace{- R_0^1}_{R_0^1}
\end{align*}
\]
so that
\[
R_0^1 = \frac{1}{2} \frac{d}{dt} \int_\Gamma \mathcal{B} \partial^\alpha \partial_2 \partial^\alpha \eta_1 + \frac{1}{2} \int_\Gamma \partial_1 \mathcal{B} \partial^\alpha \partial_2 \partial^\alpha \eta_1 + \frac{1}{2} \int_\Gamma \partial_1 \mathcal{B} (\partial^\alpha v_2 \partial^\alpha \eta_1 - \partial^\alpha v_1 \partial^\alpha \eta_2). \tag{3.28}
\]
To estimate \( R_0^{1,1} \), we use the following identities:
\[
a_{i\alpha} v_{i\alpha} + \partial_1 \eta_1 \partial_1 \eta_2 = |\partial_1 \eta_1|^2 \delta_{ij}, \quad \partial^\alpha v_{i\alpha} = \partial^\alpha \partial_1 \eta_1 \partial_1 \eta_i, \quad \partial^\alpha v_{i\alpha} \partial_1 \eta_i = - \partial^\alpha \partial_1 \eta_1 a_{i\alpha}
\]
to write
\[
\begin{align*}
R_0^{1,1} & = \frac{1}{2} \frac{d}{dt} \int_\Gamma \mathcal{B} \partial^\alpha \partial_1 \eta_1 \partial_1 \eta_2 \partial^\alpha \eta_1 \partial^\alpha \eta_1 \\
& = \frac{1}{2} \frac{d}{dt} \int_\Gamma \mathcal{B} \partial^\alpha \partial_1 \eta_1 \partial_1 \eta_2 \partial^\alpha \eta_1 + \frac{1}{2} \frac{d}{dt} \int_\Gamma \mathcal{B} \partial^\alpha \partial_1 \eta_1 \partial_1 \eta_2 \partial^\alpha \eta_1 + \frac{1}{2} \frac{d}{dt} \int_\Gamma \mathcal{B} \partial^\alpha \partial_1 \eta_1 \partial_1 \eta_2 \partial^\alpha \eta_1 \\
& = \frac{d}{dt} \int_\Gamma \mathcal{B} \partial^\alpha \partial_1 \eta_1 \partial_1 \eta_2 \partial^\alpha \eta_1 + \frac{d}{dt} \int_\Gamma \mathcal{B} \partial^\alpha \partial_1 \eta_1 \partial_1 \eta_2 \partial^\alpha \eta_1.
\end{align*}
\]
Thus, by using (2.9), (2.10) and (3.26), we have
\[
\left| \int_0^t R_0^{1,1} \right| \lesssim \left| \int_\Gamma \mathcal{B} \partial^\alpha \partial_1 \eta_1 \partial_1 \eta_2 \partial^\alpha \eta_1 \partial^\alpha \eta_1 \right| + \left| \int_\Gamma \partial_1 \left( \frac{\mathcal{B} \partial^\alpha \partial_1 \eta_1 \partial_1 \eta_2}{|\partial_1 \eta_1|^2} \right) \partial^\alpha \eta_1 \partial^\alpha \eta_1 \right| \\
\lesssim M_0 + |\partial^\alpha \partial_1 \eta_1 \partial_1 \eta_2| |\partial^\alpha \eta_1| |\mathcal{B}| + |\partial_1 \mathcal{B}| + |\partial^\alpha \eta_1| + |\partial^\alpha \eta_1| L_1^1 \left| \begin{array}{c} \mathcal{B} \partial^\alpha \eta_1 \partial_1 \eta_2 \\ |\partial_1 \eta_1|^2 \end{array} \right| + |\mathcal{B}| |\partial^\alpha \eta_1| L_1^1 \left| \begin{array}{c} \mathcal{B} \partial^\alpha \eta_1 \partial_1 \eta_2 \\ |\partial_1 \eta_1|^2 \end{array} \right| \\
\lesssim M_0 + |\partial^\alpha \partial_1 \eta_1 \partial_1 \eta_2| |\partial^\alpha \eta_1| |\mathcal{B}| + |\partial_1 \mathcal{B}| + |\partial^\alpha \eta_1| + |\partial^\alpha \eta_1| L_1^1 \left| \begin{array}{c} \mathcal{B} \partial^\alpha \eta_1 \partial_1 \eta_2 \\ |\partial_1 \eta_1|^2 \end{array} \right| + |\mathcal{B}| |\partial^\alpha \eta_1| L_1^1 \left| \begin{array}{c} \mathcal{B} \partial^\alpha \eta_1 \partial_1 \eta_2 \\ |\partial_1 \eta_1|^2 \end{array} \right| \\
\lesssim M_0 + \delta \left( \sup_{t \in [0, T]} \mathcal{E}(t) + TP \left( \sup_{t \in [0, T]} \mathcal{E}(t) \right) \right). \tag{3.30}
\]
For \( R_0^{1,2} \), since \( \alpha > 1 \), it follows from the dual estimate, (2.7) and (3.26) that
\[
\left| \int_0^t R_0^{1,2} \right| \lesssim \int_0^t |\partial^\alpha \mathcal{B} \partial^\alpha \eta_1| \left| \begin{array}{c} \mathcal{B} \partial^\alpha \eta_1 \\ |\partial_1 \eta_1|^2 \end{array} \right| |\partial^\alpha \eta_1| \left| \begin{array}{c} \mathcal{B} \partial^\alpha \eta_1 \\ |\partial_1 \eta_1|^2 \end{array} \right| \\
\lesssim \int_0^t \left( \frac{|\mathcal{B}|}{|\partial_1 \eta_1|^2} + \frac{|\mathcal{B}|}{|\partial_1 \eta_1|^2} \right) |\partial^\alpha \eta_1| \left| \begin{array}{c} \mathcal{B} \partial^\alpha \eta_1 \\ |\partial_1 \eta_1|^2 \end{array} \right| \left| \begin{array}{c} \mathcal{B} \partial^\alpha \eta_1 \\ |\partial_1 \eta_1|^2 \end{array} \right| \\
\lesssim \left( \frac{|\mathcal{B}|}{|\partial_1 \eta_1|^2} + \frac{|\mathcal{B}|}{|\partial_1 \eta_1|^2} \right) \left( |\partial^\alpha \eta_1| \left| \begin{array}{c} \mathcal{B} \partial^\alpha \eta_1 \\ |\partial_1 \eta_1|^2 \end{array} \right| \left| \begin{array}{c} \mathcal{B} \partial^\alpha \eta_1 \\ |\partial_1 \eta_1|^2 \end{array} \right| \\
\lesssim T^{1/2} \left( \sup_{t \in [0, T]} \mathcal{E}(t) \right), \tag{3.31}
\]
which together with (3.30) implies that
\[
\left| \int_0^t R_0^{1,2} \right| \lesssim M_0 + \delta \left( \sup_{t \in [0, T]} \mathcal{E}(t) + T^{1/2} \left( \sup_{t \in [0, T]} \mathcal{E}(t) \right) \right). \tag{3.32}
\]
For \( R_0^2 \), it follows from the duality and the trace estimate that
\[
\left| \int_0^t R_0^2 \right| \lesssim \int_0^t \left( |\partial^\alpha \mathcal{B}, a_{i\alpha}| \left| \begin{array}{c} \mathcal{B}, a_{i\alpha} \end{array} \right| \left| \begin{array}{c} \mathcal{B}, a_{i\alpha} \end{array} \right| \right) \left| \begin{array}{c} v_1 \\ |\partial_1 \eta_1|^2 \end{array} \right| \left| \begin{array}{c} v_1 \\ |\partial_1 \eta_1|^2 \end{array} \right| \\
\lesssim \frac{1}{2} \left( |\partial^\alpha \mathcal{B}, a_{i\alpha}| \left| \begin{array}{c} \mathcal{B}, a_{i\alpha} \end{array} \right| \left| \begin{array}{c} \mathcal{B}, a_{i\alpha} \end{array} \right| \right) \left| \begin{array}{c} v_1 \\ |\partial_1 \eta_1|^2 \end{array} \right| \left| \begin{array}{c} v_1 \\ |\partial_1 \eta_1|^2 \end{array} \right| \lesssim T^{1/2} \left( \sup_{t \in [0, T]} \mathcal{E}(t) \right). \tag{3.33}
\]
where we have used the following estimate, which is obtained by using (2.1) and (3.26):

\[
\|\nabla [\bar{\partial}^{\alpha} \mathcal{B}, a_{k2}] \|_{L^2_\Omega(L^2)}^2 \lesssim \sum_{1 \leq |\beta| \leq m - 1, |\gamma| = m} \int_0^t \left( \| \bar{\partial}^{\beta} \nabla \mathcal{B} \bar{\partial}^{\gamma} \partial_1 \eta \|_{L^2_\Omega}^2 + \| \bar{\partial}^{\beta} \mathcal{B} \bar{\partial}^{\gamma} \partial_1 \eta \|_{L^2_\Omega}^2 \right),
\]

\[
\lesssim \| \nabla \mathcal{B} \|_{L^\infty_\tau} \| \nabla \partial_1 \eta \|_{L^2_\Omega(L^\infty_\tau)} + \| \partial_1 \eta \|_{L^\infty_\tau(J^1)} \| \nabla \mathcal{B} \|_{L^2_\Omega(J^1)}
\]

\[
\lesssim \| \nabla \mathcal{B} \|_{L^2_\Omega(J^1)} \| \nabla \partial_1 \eta \|_{L^2_\Omega(J^1)} \leq P \left( \sup_{t \in [0,T]} \mathcal{E}(t) \right).
\]

For \( R^3_k \), noticing \( \bar{\partial}^{\alpha} a_{k2} \partial_1 \eta \), we can write it as

\[
R^3_k = \sigma \int_\Gamma \frac{1}{|\partial_1 \eta|^3} \bar{\partial}^{\alpha} \partial_1 \eta \partial_1 a_{k2} \partial_1 \bar{\partial}^{\alpha} \eta \partial_1 \eta - \sigma \int_\Gamma \frac{1}{|\partial_1 \eta|^3} \partial_1 a_{k2} \partial_1 \bar{\partial}^{\alpha} \eta \partial_1 \eta \partial_1 \eta \bar{\partial}^{\alpha} \eta
\]

Then utilizing integration by parts in \( x_1 \), (2.9), (2.10) and (2.4), we have

\[
\left| \int_0^t R^3_{k1} \right| \lesssim \sigma \int_0^t \frac{1}{|\partial_1 \eta|^3} \bar{\partial}^{\alpha} \partial_1 \eta \partial_1 a_{k2} \partial_1 \bar{\partial}^{\alpha} \eta \partial_1 \eta \bigg|_{t=0}^t \|
\]

\[
\lesssim M_0 + \sigma |\bar{\partial}^{\alpha} \partial_1 \eta|_0 \left| \frac{\partial_1 a_{k2}}{|\partial_1 \eta|^3} \right|_{L^\infty_\tau} |\bar{\partial}^{\alpha} \partial_1 \eta|_{L^1_\tau(t)}
\]

\[
+ \sigma |\bar{\partial}^{\alpha} \eta|_0 \left| \frac{\partial_1 a_{k2}}{|\partial_1 \eta|^3} \right|_{L^\infty_\tau} |\bar{\partial}^{\alpha} \partial_1 \eta|_{L^1_\tau(t)} + \sigma |\partial_1 a_{k2}|_0 |\partial_1 \eta|_0 |\partial_1 a_{k2}|_0 |\partial_1 \eta|_{L^4_\tau(t)}
\]

\[
\lesssim M_0 + \sigma |\bar{\partial}^{\alpha} \eta|_0 \left| \frac{\partial_1 a_{k2}}{|\partial_1 \eta|^3} \right|_{L^\infty_\tau} |\bar{\partial}^{\alpha} \partial_1 \eta|_{L^1_\tau(t)} + \sigma |\bar{\partial}^{\alpha} \eta|_0 \left| \frac{\partial_1 a_{k2}}{|\partial_1 \eta|^3} \right|_{L^\infty_\tau} |\bar{\partial}^{\alpha} \partial_1 \eta|_{L^1_\tau(t)} + \sigma |\bar{\partial}^{\alpha} \partial_1 \eta|_0 |\partial_1 a_{k2}|_0 |\partial_1 \eta|_0
\]

\[
+ \sigma |\bar{\partial}^{\alpha} \partial_1 \eta|_0 |\partial_1 a_{k2}|_0 |\partial_1 \eta|_0 \left| \frac{\partial_1 a_{k2}}{|\partial_1 \eta|^3} \right|_{L^\infty_\tau} |\bar{\partial}^{\alpha} \partial_1 \eta|_{L^1_\tau(t)} + \sigma |\bar{\partial}^{\alpha} \partial_1 \eta|_0 |\partial_1 a_{k2}|_0 |\partial_1 \eta|_0
\]

\[
+ \delta |\bar{\partial}^{\alpha} \partial_1 \eta|_0 + \delta |\bar{\partial}^{\alpha} \partial_1 \eta|_0 + TP \left( \sup_{t \in [0,T]} \mathcal{E}(t) \right).
\]
\[
\begin{align*}
&\lesssim \|\tilde{\partial}^{3}, a_{k2}\|_{L_{T}^{2}(H^{1})}^{2}\|\tilde{\partial}^{3}v\|_{L_{T}^{2}(H^{1})}^{2} + \|\tilde{\partial}^{3}(\tilde{\partial}_{0}\tilde{\eta}_{1}J)\|_{L_{T}^{2}(H^{1})}^{2} + \|\tilde{\partial}^{3}(\tilde{\partial}_{0}\tilde{\eta}_{1}J)\|_{L_{T}^{2}(H^{1})}^{2} + 2\|\tilde{\partial}^{3}(\operatorname{div}A\tilde{\nu}\tilde{\eta}_{1}J)\|_{L_{T}^{2}(H^{1})}^{2} + 2\|\tilde{\partial}^{3}(\operatorname{div}A\tilde{\nu}\tilde{\eta}_{1}J)\|_{L_{T}^{2}(H^{1})}^{2} \\
&\lesssim \int_{0}^{t}\|\nabla\tilde{\eta}_{m}\|^{2}_{L_{T}^{2}} + \|\nabla\eta\|_{L_{T}^{2}}^{2} + \|\tilde{\partial}^{3}v\|^{2}_{L_{T}^{2}} + \|\tilde{\partial}^{3}v\|^{2}_{L_{T}^{2}} \lesssim P\left(\sup_{t\in[0,T]} \mathcal{E}(t)\right). \quad (3.35)
\end{align*}
\]

which together with (2.7) yields that for \(\alpha_{1} \geq 1\), one has
\[
\begin{align*}
\left|\int_{0}^{t} R_{b}^{1/2}\right| &\lesssim \int_{0}^{t} \|\tilde{\partial}^{3}\tilde{\eta}_{m}\|_{L_{T}^{2}(H^{1})}^{1/2} \lesssim \int_{0}^{t} \|\tilde{\partial}^{3}\tilde{\eta}_{m,1}\|_{L_{T}^{2}(H^{1})}^{1/2} + \int_{0}^{t} \|\tilde{\partial}^{3}\tilde{\eta}_{2}\|_{L_{T}^{2}(H^{1})}^{1/2} \\
&\lesssim T^{1/2} P\left(\sup_{t\in[0,T]} \mathcal{E}(t)\right). \quad (3.36)
\end{align*}
\]

It follows from (2.7) that
\[
\begin{align*}
\left|\int_{0}^{t} R_{b}^{1/4}\right| &\lesssim \int_{0}^{t} \|\tilde{\partial}^{3}\tilde{\eta}_{m}\|_{L_{T}^{2}(H^{1})}^{1/4} \|\tilde{\partial}^{3}\tilde{\eta}_{m,2}\|_{L_{T}^{2}(H^{1})}^{3/4} + \int_{0}^{t} \|\tilde{\partial}^{3}\tilde{\eta}_{m}\|_{L_{T}^{2}(H^{1})}^{1/4} \|\tilde{\partial}^{3}\tilde{\eta}_{m}\|_{L_{T}^{2}(H^{1})}^{3/4} \\
&\lesssim \int_{0}^{t} \|\tilde{\partial}^{3}\tilde{\eta}_{m}\|_{L_{T}^{2}(H^{1})}^{1/4} \|\tilde{\partial}^{3}\tilde{\eta}_{m}\|_{L_{T}^{2}(H^{1})}^{3/4} \lesssim T^{1/2} P\left(\sup_{t\in[0,T]} \mathcal{E}(t)\right) \quad (3.37)
\end{align*}
\]

and
\[
\begin{align*}
\left|\int_{0}^{t} R_{b}^{1/4}\right| &\lesssim \int_{0}^{t} \left[\|\tilde{\partial}^{3}\tilde{\eta}_{m}\|_{L_{T}^{2}(H^{1})}^{1/4} \|\tilde{\partial}^{3}\tilde{\eta}_{m}\|_{L_{T}^{2}(H^{1})}^{3/4}\right] \lesssim \int_{0}^{t} \|\tilde{\partial}^{3}\tilde{\eta}_{m}\|_{L_{T}^{2}(H^{1})}^{1/4} \|\tilde{\partial}^{3}\tilde{\eta}_{m}\|_{L_{T}^{2}(H^{1})}^{3/4} \lesssim T^{1/2} P\left(\sup_{t\in[0,T]} \mathcal{E}(t)\right). \quad (3.38)
\end{align*}
\]

Thus, (3.34)–(3.38) yield
\[
\begin{align*}
\left|\int_{0}^{t} R_{b}\right| &\lesssim M_{0} + \delta \sup_{t\in[0,T]} \mathcal{E}(t) + T^{1/2} P\left(\sup_{t\in[0,T]} \mathcal{E}(t)\right). \quad (3.39)
\end{align*}
\]

For \(R_{a}\), we can obtain from (2.7), the trace estimate and (2.3) that
\[
\begin{align*}
\left|\int_{0}^{t} R_{a}\right| &\lesssim \int_{0}^{t} \left[\|\tilde{\partial}^{3}\tilde{\eta}_{m}\|_{L_{T}^{2}(H^{1})}^{1/2} \|\tilde{\partial}^{3}\tilde{\eta}_{m}\|_{L_{T}^{2}(H^{1})}^{1/2}\right] \lesssim \int_{0}^{t} \|\tilde{\partial}^{3}\tilde{\eta}_{m}\|_{L_{T}^{2}(H^{1})}^{1/2} \|\tilde{\partial}^{3}\tilde{\eta}_{m}\|_{L_{T}^{2}(H^{1})}^{1/2} \lesssim T^{1/2} P\left(\sup_{t\in[0,T]} \mathcal{E}(t)\right). \quad (3.40)
\end{align*}
\]

The term \(R_{b}\) can be bounded by similar arguments as in (3.34) and (3.35) as follows:
\[
\begin{align*}
\left|\int_{0}^{t} R_{b}\right| &\lesssim \int_{0}^{t} \sigma\|\tilde{\partial}^{3}\tilde{\eta}_{m}\|_{L_{T}^{2}(H^{1})}^{1/2} \|\tilde{\partial}^{3}\tilde{\eta}_{m}\|_{L_{T}^{2}(H^{1})}^{1/2} \lesssim \int_{0}^{t} \|\tilde{\partial}^{3}\tilde{\eta}_{m}\|_{L_{T}^{2}(H^{1})}^{1/2} \|\tilde{\partial}^{3}\tilde{\eta}_{m}\|_{L_{T}^{2}(H^{1})}^{1/2} \lesssim \int_{0}^{t} \|\tilde{\partial}^{3}\tilde{\eta}_{m}\|_{L_{T}^{2}(H^{1})}^{1/2} \|\tilde{\partial}^{3}\tilde{\eta}_{m}\|_{L_{T}^{2}(H^{1})}^{1/2} \lesssim T^{1/2} P\left(\sup_{t\in[0,T]} \mathcal{E}(t)\right). \quad (3.41)
\end{align*}
\]

For \(R_{b}\), we utilize (3.35), (2.7) and (2.10) to yield that
\[
\begin{align*}
\left|\int_{0}^{t} R_{b}\right| &\lesssim \int_{0}^{t} \left[\|\tilde{\partial}^{3}\tilde{\eta}_{m}\|_{L_{T}^{2}(H^{1})}^{1/2} \|\tilde{\partial}^{3}\tilde{\eta}_{m}\|_{L_{T}^{2}(H^{1})}^{1/2}\right] \lesssim \int_{0}^{t} \|\tilde{\partial}^{3}\tilde{\eta}_{m}\|_{L_{T}^{2}(H^{1})}^{1/2} \|\tilde{\partial}^{3}\tilde{\eta}_{m}\|_{L_{T}^{2}(H^{1})}^{1/2} \lesssim \int_{0}^{t} \|\tilde{\partial}^{3}\tilde{\eta}_{m}\|_{L_{T}^{2}(H^{1})}^{1/2} \|\tilde{\partial}^{3}\tilde{\eta}_{m}\|_{L_{T}^{2}(H^{1})}^{1/2} \lesssim T^{1/2} P\left(\sup_{t\in[0,T]} \mathcal{E}(t)\right). \quad (3.42)
\end{align*}
\]

Therefore, plugging (3.25) into (3.13), integrating the resulted equation over \([0,t]\), substituting (3.14)–(3.17), (3.18), (3.19), (3.21), (3.23), (3.32), (3.33) and (3.39)–(3.42) into the resultant equation, and then using the estimate (3.5) and Korn’s inequality (2.13), we can complete the proof of this lemma. \(\square\)
3.4 Fully temporal derivative estimates

In view of the argument in the proof of Lemma 3.3, we use the duality argument taking advantage of the estimate \([|\tilde{\mathcal{D}}^m \eta|]_{2} \leq |\tilde{\mathcal{D}}^{m-1} \eta| \frac{1}{2}\) if \(|\alpha| = m\) and \(\alpha_{1} \geq 1\), which is invalid for \(|\tilde{\mathcal{D}}^m \eta| \frac{1}{2}\). Moreover, we lack the estimate of \(\tilde{\mathcal{D}}^m_{t} \eta \partial_{\nu} a_{k2}\) [\(L^2_{\gamma}(H^\frac{1}{2})\)] while the similar estimate (3.35) is the key to derive (3.42) in the proof of Lemma 3.3. To overcome these difficulties, we introduce the following Alinhac’s good unknowns:

\[
\mathcal{V}^c_{ij} = \partial^m_{t} v^c_{ij} - \partial^m_{t} \eta^c_{k} A^c_{ki} \partial_{ij} q^c, \quad \mathcal{Q}^c = \partial^m_{t} q^c - \partial^m_{t} \eta^c_{k} A^c_{ki} \partial_{ij} q^c,
\]

which can be used to cancel the nonlinear term in the proof of fully temporal derivative estimates. In order to derive the equations of \(\mathcal{V}^c\) and \(\mathcal{Q}^c\), we take the fully temporal derivatives of \(a^c_{ij} \partial_{ij} q^c\) and \(a^c_{ij} \partial_{ij} v^c_{ij}\) and get from (2.14) that

\[
\begin{align*}
\partial_{t}^m (a^c_{ij} \partial_{ij} q^c) &= a^c_{ij} \partial_{ij} \mathcal{Q}^c + [\partial^m_{t}, a^c_{ij}, \partial_{ij} q^c] + \mathcal{C}_{i}(q^c), \\
\partial_{t}^m (a^c_{ij} \partial_{ij} v^c_{ij}) &= a^c_{ij} \partial_{ij} \mathcal{V}^c_{ij} + [\partial^m_{t}, a^c_{ij}, \partial_{ij} v^c_{ij}] + \mathcal{C}_{i}(v^c_{ij}),
\end{align*}
\]

where \(\mathcal{C}_{i}(f)\) is given by

\[
\begin{align*}
\mathcal{C}_{i}(f) &= \partial^m_{t} \eta^c_{k} a^c_{ij} \partial_{ij} (A^c_{ki} \partial_{t} f) - J^{c} [\partial^m_{t} \eta^c_{k}, A^c_{ki} \partial_{t} \eta^c_{k} \partial_{t} f + [\partial^m_{t}, J^{c}] A^c_{ki} \partial_{t} f].
\end{align*}
\]

Then by applying \(\partial^m_{t}\) to (1.15) and the equation

\[
a_{ij} \partial_{ij} v_{1} = J_{i} = -\frac{J^{\gamma+1}}{\gamma \rho_{0}} \partial_{t} q,
\]

we have

\[
\begin{align*}
\tilde{\rho}_{0} \partial_{t} \mathcal{V}^c_{ij} + a^c_{ij} \partial_{ij} \mathcal{Q}^c - 2\mu_{0} \partial_{t} \partial^m_{t} ((S_{A} v)^c_{ij} \eta^c_{k}) - \lambda \partial_{t} \partial^m_{t} (a^c_{ij} (\text{div} A v)^c) - \partial_{t} (\tilde{\rho}_{0} \partial_{t} \partial^m_{t} \eta^c_{k}) \\
= -\tilde{\rho}_{0} \partial_{t} (\partial^m_{t} \eta^c_{k} A^c_{ki} \partial_{ij} v^c_{ij}) - [\partial^m_{t}, a^c_{ij}, \partial_{ij} q^c] - \mathcal{C}_{i}(q^c)
\end{align*}
\]

and

\[
\begin{align*}
a^c_{ij} \partial_{ij} \mathcal{V}^c_{ij} &= -\left[\frac{(J^{c})^{\gamma+1}}{\gamma (\rho_{0})^{\gamma}} \partial_{t} \mathcal{Q}^c + \frac{(J^{c})^{\gamma+1}}{\gamma (\rho_{0})^{\gamma}} \partial_{t} (\partial^m_{t} \eta^c_{k} A^c_{ki} \partial_{t} q^c) - \left[\partial^m_{t}, \frac{(J^{c})^{\gamma+1}}{\gamma (\rho_{0})^{\gamma}} \partial_{t} q^c\right] - [\partial^m_{t}, a^c_{ij}, \partial_{ij} q^c]\right] - \mathcal{C}_{i}(v^c_{ij}).
\end{align*}
\]

Before giving the full temporal derivative estimates, we first derive the following estimate of the commutators \(\mathcal{C}_{i}(q^c)\) and \(\mathcal{C}_{i}(v^c_{ij})\).

**Lemma 3.4.** For any \(m \geq 4\) and \(t \in [0, T^{c}]\), it holds that

\[
\int_{0}^{t} \left(\|\mathcal{C}_{i}(q^c)\|_{2}^{2} + \|\mathcal{C}_{i}(v^c_{ij})\|_{2}^{2}\right) \leq T^{c} P\left(\sup_{t \in [0, T^{c}]} \mathcal{E}^{c}(t)\right).
\]

**Proof.** We only need to give the estimate of \(\mathcal{C}_{i}(q)\) since the other is the same. By using (1.14), (2.1), the Sobolev embedding and (3.3), one has

\[
\begin{align*}
\|\partial^m_{t} \eta_{k} a_{ij} \partial_{ij} (A_{ki} \partial_{t} q)\|_{L^{2}_{\gamma}}^{2} &= \|\partial^m_{t} \eta_{k} [a_{ij} A_{ki} \partial_{t} q]\|_{L^{2}_{\gamma}}^{2} \\
&\leq \|\partial^m_{t} \eta_{k}\|_{L^{2}_{\gamma}}^{2} [a_{ij} A_{ki} \partial_{t} q]\|_{L^{2}_{\gamma}}^{2} + \|\partial^m_{t} \eta_{k}\|_{L^{2}_{\gamma}}^{2} [a_{ij} A_{ki} \partial_{t} q]\|_{L^{2}_{\gamma}}^{2} \\
&\leq \|\partial^m_{t} \eta_{k}\|_{L^{2}_{\gamma}}^{2} [\|\nabla q\|_{L^{2}_{\gamma}}^{2} + \|\nabla q\|_{L^{2}_{\gamma}}^{2}] [\|\nabla q\|_{L^{2}_{\gamma}}^{2} + \|\nabla q\|_{L^{2}_{\gamma}}^{2}]
\end{align*}
\]

Similarly, it follows from (3.3) and (2.2) that

\[
\|J[\partial^m_{t} \eta_{k}, A_{ki} \partial_{t} \eta_{k} \partial_{t} q]\|_{L^{2}_{\gamma}}^{2}
\]
By integration by parts and (3.46), one has

\[ \| \nabla q \|_{L^2}^2 + \| (\partial_t^m + A_1A_k) \nabla q \|_{L^2}^2 \leq \| \nabla q \|_{L^2}^2 + \| (\partial_t^m + A_1A_k) \nabla q \|_{L^2}^2 \]

This completes the proof.

Now, we derive the following estimate of fully temporal derivatives.

Lemma 3.5. For any \( m \geq 4 \) and \( t \in [0, T] \), it holds that

\[
\int_0^t \left( \| \partial_t^m \mathbf{v} \|_0^2 + \| \partial_t^{m-1} \mathbf{q} \|_0^2 + \| \nabla \partial_t^m \mathbf{q} \|_0^2 + \| \mathbf{q} \|_0^2 + \| \nabla \partial_t^m \mathbf{v} \|_0^2 \right) \, dt \leq M_0 + \delta \sup_{t \in [0, T]} \mathbf{E}(t) + \epsilon^2 \int_0^T \left( \int_\Omega \| \nabla \partial_t^m \mathbf{v} \|_0^2 \right) \, dt
\]

Therefore, combining the above estimates, we have

\[ \| \mathbf{C}(t) \|_{L^2} \leq TP \left( \sup_{t \in [0, T]} \mathbf{E}(t) \right) \]

This completes the proof.

Now, we derive the following estimate of fully temporal derivatives.
It follows from (3.46) that

\[
- \int_{\Omega} a_{ij} \partial_j \mathcal{Q} \mathcal{V} \, dx = \int_{\Omega} \frac{J^{\gamma+1}}{\gamma \rho_{0}} \partial_t \mathcal{Q} \mathcal{Q} \, dx - \int_{\Omega} \frac{J^{\gamma+1}}{\gamma \rho_{0}} \partial_t (\partial_t \eta_k \mathcal{A}_{ik} \partial q) \mathcal{Q} \, dx + \int_{\Omega} \frac{\partial_t^m}{\gamma \rho_{0}} \partial_t \eta_k \partial_q \mathcal{Q} \, dx \\
+ \int_{\Omega} \frac{[\partial_t^m, a_{ij}, \partial_j \mathcal{V}]}{\gamma \rho_{0}} \mathcal{Q} \, dx + \int_{\Omega} C_{(\mathcal{V})} \mathcal{Q} \, dx \\
= \frac{1}{2} \frac{d}{dt} \int_{\Omega} \left( \rho_{0} |\mathcal{V}|^2 + \frac{J^{\gamma+1}}{\gamma \rho_{0}} Q^2 + \rho_{0} |\partial_t \mathcal{V}|^2 \right) \, dx + 2 \mu \varepsilon \int_{\Omega} J |\mathcal{S}_{\mathcal{A}}(\partial_t^m \mathcal{V})|^2 \, dx + \lambda \varepsilon \int_{\Omega} (\partial_t^m \mathcal{V})^2 \, dx \\
+ \int_{\Omega} \left( a_{ij} \mathcal{Q} - \rho_{0} \partial_t^m \mathcal{V} \mathcal{V} - 2 \mu \varepsilon \partial_t^m (\mathcal{S}_{\mathcal{A}}(\mathcal{V}) a_{ij}) - \lambda \varepsilon \partial_t^m (a_{ij} \mathcal{A}_{ik} \partial q) \right) \mathcal{V} \, dx
\]  

Plugging (3.50) and (3.51) into (3.49) gives

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \left( \rho_{0} |\mathcal{V}|^2 + \frac{J^{\gamma+1}}{\gamma \rho_{0}} Q^2 + \rho_{0} |\partial_t \mathcal{V}|^2 \right) \, dx + 2 \mu \varepsilon \int_{\Omega} J |\mathcal{S}_{\mathcal{A}}(\partial_t^m \mathcal{V})|^2 \, dx + \lambda \varepsilon \int_{\Omega} (\partial_t^m \mathcal{V})^2 \, dx \\
+ \int_{\Omega} \left( a_{ij} \mathcal{Q} - \rho_{0} \partial_t^m \mathcal{V} \mathcal{V} - 2 \mu \varepsilon \partial_t^m (\mathcal{S}_{\mathcal{A}}(\mathcal{V}) a_{ij}) - \lambda \varepsilon \partial_t^m (a_{ij} \mathcal{A}_{ik} \partial q) \right) \mathcal{V} \, dx
\]  

\[
= \sum_{i=1}^{3} \mathcal{R}_i + \sum_{i=1}^{4} \mathcal{R}_i + \sum_{i=1}^{6} \mathcal{R}_i + \sum_{i=1}^{6} \mathcal{R}_i \tag{3.52}
\]

Similar to (3.25), it follows from the third equation of (1.15) that

\[
\mathcal{R}_b = -\sigma \int_{\Gamma} \partial_t^m \left( \frac{\partial_t \eta_k a_{ij}}{\rho_{0} \eta_t^3} \right) a_{ij} \mathcal{V} - \int_{\Gamma} \mathcal{B}_{a_{ij}} \mathcal{V} - \int_{\Gamma} \partial_t^m \left( \frac{\partial_t \eta_k a_{ij}}{\rho_{0} \eta_t^3} \right) a_{ij} \mathcal{V} - \int_{\Gamma} \mathcal{B}_{a_{ij}} \mathcal{V}
\]

\[
= -\sigma \int_{\Gamma} \partial_t^m \left( \frac{\partial_t \eta_k a_{ij}}{\rho_{0} \eta_t^3} \right) a_{ij} \mathcal{V} - \int_{\Gamma} \partial_t^m \left( \frac{\partial_t \eta_k a_{ij}}{\rho_{0} \eta_t^3} \right) a_{ij} \mathcal{V} - \int_{\Gamma} \mathcal{B}_{a_{ij}} \mathcal{V}
\]

\[
+ \sigma \int_{\Gamma} \partial_t^m \partial_t \eta_k a_{ij} \left( \frac{\partial_t \eta_k a_{ij}}{\rho_{0} \eta_t^3} \right) a_{ij} \mathcal{V} - \int_{\Gamma} \mathcal{B}_{a_{ij}} \mathcal{V} - \int_{\Gamma} \partial_t^m \partial_t \eta_k a_{ij} \left( \frac{\partial_t \eta_k a_{ij}}{\rho_{0} \eta_t^3} \right) a_{ij} \mathcal{V}
\]

\[
= \sigma \int_{\Gamma} \partial_t^m \partial_t \eta_k a_{ij} \left( \frac{\partial_t \eta_k a_{ij}}{\rho_{0} \eta_t^3} \right) a_{ij} \mathcal{V} - \int_{\Gamma} \mathcal{B}_{a_{ij}} \mathcal{V} - \int_{\Gamma} \partial_t^m \partial_t \eta_k a_{ij} \left( \frac{\partial_t \eta_k a_{ij}}{\rho_{0} \eta_t^3} \right) a_{ij} \mathcal{V}
\]

\[
- \sigma \left( \int_{\Gamma} \mathcal{B}_{a_{ij}} \mathcal{V} - \int_{\Gamma} \partial_t^m \left( \frac{\partial_t \eta_k a_{ij}}{\rho_{0} \eta_t^3} \right) a_{ij} \mathcal{V} \right)
\]

\[
- \sigma \int_{\Gamma} \partial_t^m \partial_t \eta_k a_{ij} \left( \frac{\partial_t \eta_k a_{ij}}{\rho_{0} \eta_t^3} \right) a_{ij} \mathcal{V}
\]
\[
\begin{align*}
\frac{\sigma}{2} \frac{d}{dt} \int_{\Gamma} \left[ \frac{|\partial_{\eta}^m \partial_{\eta_a} \phi_{k2}|^2}{|\partial_{\eta}^m|^2} - \int_{\Gamma} [\mathcal{B}\partial_{\eta_a}^m \phi_{k2} \mathcal{V}_i - \int_{\Gamma} (\psi_{a2} \mathcal{B}, a_{i2}) \mathcal{V}_i \\
- \int_{\Gamma} \left[ \partial_{\eta}^m, \partial_{\eta_a}^m \phi_{k2} \frac{|a_{i2}|}{|\partial_{\eta}^m|} \right] \phi_{k2} \mathcal{V}_i - \int_{\Gamma} \left( \frac{1}{2} \partial_{\eta}^m \left( \frac{1}{|\partial_{\eta}^m|} \right) + \partial_{\eta}^m J \right) \frac{|\partial_{\eta}^m \partial_{\eta_a} \phi_{k2}|^2}{|\partial_{\eta}^m|^2} \\
- \int_{\Gamma} \left( \frac{1}{2} \partial_{\eta}^m \left( \frac{a_{i2}}{|\partial_{\eta}^m|} \right) \partial_\eta^m \partial_{\eta_a} \phi_{k2} - \partial_\eta^m \partial_{\eta_a} \partial_{\eta}^m \partial_{\eta_a} \phi_{k2} \frac{|a_{i2}|}{|\partial_{\eta}^m|} \right) \mathcal{V}_i \\
- \sigma \int_{\Gamma} \frac{|\partial_{\eta}^m \partial_{\eta_a} \phi_{k2}|^2}{|\partial_{\eta}^m|^2} \partial_{\eta}^m \partial_{\eta_a} \phi_{k2} \partial_{\eta_a} \mathcal{V}_i \right) \mathcal{V}_i \\
= \frac{\sigma}{2} \frac{d}{dt} \int_{\Gamma} \left[ \frac{|\partial_{\eta}^m \partial_{\eta_a} \phi_{k2}|^2}{|\partial_{\eta}^m|^2} - \sum_{i=1}^{7} \mathcal{R}_{1,i} \right]
\end{align*}
\]

where we have used the fact that

\[
\partial_{\eta_a} \partial_{\eta}^m \partial_{\eta_a} \phi_{k2} + a_{i2} \partial_{\eta}^m \partial_{\eta_a} \partial_{\eta_a} \phi_{k2} = \partial_{\eta}^m J^{-1} a_{i2} \partial_{\eta}^m \eta_a.
\]

Substituting (3.53) into (3.52) gives

\[
\begin{align*}
&\frac{1}{2} \frac{d}{dt} \int_{\Omega} \left( \hat{\rho}_0 |v|^2 + J^{\eta+1} \gamma \hat{\rho}_0 Q^2 + \hat{\rho}_0 |\partial_{\eta}^m (\nabla \eta)|^2 \right) dx + \frac{\sigma}{2} \frac{d}{dt} \int_{\Gamma} \frac{|\partial_{\eta}^m \partial_{\eta_a} \phi_{k2}|^2}{|\partial_{\eta}^m|^2} \\
&+ 2\mu \varepsilon \int_{\Omega} (J |S_{A}(\partial_{\eta}^m v)|^2) dx + \lambda \varepsilon \int_{\Omega} (J (\text{div}_{A}(\partial_{\eta}^m v))^2) dx \\
&= \sum_{i=1}^{7} \mathcal{R}_{1,i} + \sum_{i=1}^{3} \mathcal{R}_{1,i} + \sum_{i=1}^{4} \mathcal{R}_{1,i} + \mathcal{R}_{1,2} + \mathcal{R}_{1,2} + \sum_{i=1}^{6} \mathcal{R}_{1,i}.
\end{align*}
\]

Next, we estimate boundary terms \( \mathcal{R}_{1,i} \) on the right-hand side of (3.54) one by one. For \( \mathcal{R}_{1,b} \), similar to (3.28), one has

\[
\mathcal{R}_{1,b} = \frac{1}{2} \frac{d}{dt} \int_{\Gamma} \mathcal{B} \partial_{\eta_a}^m \partial_{\eta_a}^m \eta_a - \frac{1}{2} \int_{\Gamma} \partial_{\eta}^m \mathcal{B} \partial_{\eta_a}^m \partial_{\eta_a}^m \eta_a - \int_{\Gamma} \mathcal{B} \partial_{\eta_a}^m \partial_{\eta_a} \partial_{\eta_a} \phi_{k2} \partial_{\eta_a} \mathcal{V}_i \\
- \frac{1}{2} \int_{\Gamma} \partial_{\eta_a} \mathcal{B} (\partial_{\eta_a}^m v_1 \partial_{\eta_a}^m \eta_a - \partial_{\eta_a}^m v_1 \partial_{\eta_a}^m \eta_a) \mathcal{V}_i.
\]

It follows from the same argument as in (3.30) that

\[
\int_0^T \mathcal{R}_{1,b} = M_0 + \hat{\delta} (|\partial_{\eta}^m \partial_{\eta_a} \phi_{k2}|_{\Omega}^2 + |\partial_{\eta}^m \nabla \eta|_{\Omega}^2) + TP \left( \sup_{t \in [0,T]} \mathcal{E}(t) \right).
\]

By using (2.7), (2.24) and (2.26), one has

\[
\int_0^T \mathcal{R}_{1,b} \leq \int_0^T \left[ \hat{\delta} (|\partial_{\eta}^m \partial_{\eta_a} \phi_{k2}|_{\Omega}^2 + |\partial_{\eta}^m \nabla \eta|_{\Omega}^2) + |\mathcal{B} A_{k2} \partial_{\eta_a} v_i|_{L^2(\Omega)} \right] \\
\leq \frac{|\partial_{\eta}^m |^2}{L^2(\Omega)} \left( \hat{\delta} (|\partial_{\eta}^m \mathcal{B} |_{L^2(\Omega)}) + |\mathcal{B} A_{k2} \partial_{\eta_a} v_i|_{L^2(\Omega)} \right) \\
\leq \frac{P \left( \sup_{t \in [0,T]} \mathcal{E}(t) \right)}{L^2(\Omega)}.
\]

Since \( \mathcal{R}_{1,b} \) involves only time derivatives of the highest order, we cannot use the duality argument as in (3.31) and (3.35). Instead, we use the decomposition

\[
- a_{i2} \partial_{\eta_a} \phi_{k2} + \partial_{\eta_a} \partial_{\eta_a} \phi_{k2} = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right).
\]
which is different from (3.29), and use integration by parts to write the troublesome term into an integral in $\Omega$, i.e.,

$$
\mathcal{R}^{1,3}_{b} = \frac{1}{2} \int_{\Gamma} \frac{\partial_{\eta} B}{[\partial \eta]} ( - a_{i2} \partial_{1} \eta_{j} + \partial_{1} \eta_{a_{2}} ) \partial_{\eta}^{m} v_{i} \partial_{\eta}^{m} \eta_{j} \\
= - \int_{\Omega} \frac{\partial_{\eta} B}{[\partial \eta]} \partial_{1} \eta_{j} \partial_{\eta}^{m} \eta_{a_{2}} \partial_{\eta}^{m} v_{i} + \frac{1}{2} \frac{d}{dt} \int_{\Gamma} \frac{\partial_{\eta} B}{[\partial \eta]} \partial_{1} \eta_{j} \partial_{\eta}^{m} \eta_{a_{2}} \partial_{\eta}^{m} \eta_{j} \\
- \frac{1}{2} \frac{d}{dt} \int_{\Gamma} \partial_{\eta} \left( \frac{\partial_{\eta} B}{[\partial \eta]} \partial_{1} \eta_{a_{2}} \right) \partial_{\eta}^{m} \eta_{j} \partial_{\eta}^{m} \eta_{j} \\
= - \int_{\Omega} \frac{\partial_{\eta} B}{[\partial \eta]} \partial_{1} \eta_{j} \partial_{\eta}^{m} \eta_{a_{2} \partial_{\eta}^{m} \eta_{a_{2}} \partial_{\eta}^{m} \eta_{i}} dx - \int_{\Omega} \partial_{\eta} \left( \frac{\partial_{\eta} B}{[\partial \eta]} \partial_{1} \eta_{j} \partial_{\eta}^{m} \eta_{a_{2}} \partial_{\eta}^{m} \eta_{j} \right) a_{k} \partial_{\eta}^{m} \eta_{i} \eta_{i} dx \\
+ \frac{1}{2} \frac{d}{dt} \int_{\Gamma} \frac{\partial_{\eta} B}{[\partial \eta]} \partial_{1} \eta_{j} \partial_{\eta}^{m} \eta_{a_{2} \partial_{\eta}^{m} \eta_{j}} \partial_{\eta}^{m} \eta_{j} \\
= - \frac{d}{dt} \int_{\Omega} \frac{\partial_{\eta} B}{[\partial \eta]} \partial_{1} \eta_{j} \partial_{\eta}^{m} \eta_{a_{2} \partial_{\eta}^{m} \eta_{a_{2}} \partial_{\eta}^{m} \eta_{i}} \partial_{\eta}^{m} \eta_{i} dx + \int_{\Omega} \partial_{\eta} \left( \frac{\partial_{\eta} B}{[\partial \eta]} \partial_{1} \eta_{j} \partial_{\eta}^{m} \eta_{a_{2} \partial_{\eta}^{m} \eta_{a_{2}}} \partial_{\eta}^{m} \eta_{j} \right) a_{k} \partial_{\eta}^{m} \eta_{i} \eta_{i} dx \\
- \frac{1}{2} \frac{d}{dt} \int_{\Gamma} \partial_{\eta} \left( \frac{\partial_{\eta} B}{[\partial \eta]} \partial_{1} \eta_{j} \partial_{\eta}^{m} \eta_{a_{2} \partial_{\eta}^{m} \eta_{j}} \partial_{\eta}^{m} \eta_{j} \right)
$$

(3.59)

By using Hölder’s inequality and (3.27), one has

$$
\left| \int_{0}^{t} \mathcal{R}^{1,3,4}_{b} \right| \leq \left| \int_{\Omega} \frac{\partial_{\eta} B}{[\partial \eta]} \partial_{1} \eta_{j} \partial_{\eta}^{m} \eta_{a_{2} \partial_{\eta}^{m} \eta_{j}} \partial_{\eta} dx \right|
$$

$$
\lesssim M_{0} + ||\partial_{\eta}^{m} \nabla \eta(t)||_{0} ||\partial_{\eta}^{m} \eta(t)||_{0} ||\partial_{\eta} B(t)||_{L^{\infty}}
$$

$$
\lesssim M_{0} + \delta ||\partial_{\eta}^{m} \nabla \eta(t)||_{0}^{2} + ||\partial_{\eta}^{m} \eta(t)||_{0}^{2} ||\partial_{\eta} B(t)||_{2}
$$

$$
\lesssim M_{0} + \delta ||\partial_{\eta}^{m} \nabla \eta(t)||_{0}^{2} + P \left( \sup_{t \in [0, T]} \mathcal{E}(t) \right)
$$

It follows from the Sobolev embedding theorem and (3.27) that

$$
\left| \int_{0}^{t} (\mathcal{R}^{1,3,2}_{b} + \mathcal{R}^{1,3,3}_{b}) \right| \lesssim \int_{0}^{t} ||\partial_{t} \partial_{\eta} B||_{L^{1}} ||\partial_{\eta}^{m} \eta||_{L^{1}} ||\partial_{\eta}^{m} \eta||_{0} + \int_{0}^{t} ||\partial_{t} \partial_{\eta} B||_{L^{\infty}} ||\partial_{\eta}^{m} \eta||_{0}^{2}
$$

$$
+ \int_{0}^{t} ||\partial_{t} \partial_{\eta} B||_{L^{\infty}} ||\partial_{\eta}^{m} \eta||_{L^{1}} ||\partial_{\eta}^{m} \eta||_{L^{1}} ||\partial_{\eta}^{m} \eta||_{0}
$$

$$
\lesssim \left( ||\partial_{t} \partial_{\eta} B||_{L^{2}(\mathcal{H}^{1})} + ||\nabla \eta||_{L^{2}(\mathcal{H}^{3})} ||\partial_{t} \partial_{\eta} B||_{L^{2}(\mathcal{H}^{1})} ||\partial_{\eta}^{m} \eta||_{L^{2}(\mathcal{H}^{1})}^{2} \right)
$$

$$
\lesssim P \left( \sup_{t \in [0, T]} \mathcal{E}(t) \right)
$$

In view of (3.3), (3.26) and (2.10), one has

$$
\left| \int_{0}^{t} \mathcal{R}^{1,3,4}_{b} \right| \lesssim \left| \int_{\Gamma} \frac{\partial_{\eta} B}{[\partial \eta]} \partial_{1} \eta_{j} \partial_{\eta}^{m} \eta_{a_{2} \partial_{\eta}^{m} \eta_{j}} \right|
$$

$$
\lesssim M_{0} + ||\partial_{t} \partial_{\eta} B||_{L^{1}} ||\partial_{\eta}^{m} \eta||_{L^{1}} ||\partial_{\eta}^{m} \eta||_{0} \lesssim M_{0} + ||\partial_{t} \partial_{\eta} B||_{L^{\infty}} ||\partial_{\eta}^{m} \eta||_{0}^{2} \lesssim M_{0} + ||\partial_{t} \partial_{\eta} B||_{L^{\infty}} ||\partial_{\eta}^{m} \eta||_{0}^{2} + ||\partial_{\eta}^{m} \eta||_{0}^{2}
$$

$$
\lesssim M_{0} + \delta ||\partial_{\eta}^{m} \nabla \eta(t)||_{0}^{2} + P \left( ||\eta||_{L^{\infty}}^{2} + ||\partial_{\eta} B||_{2}^{2} \right)
$$

$$
\lesssim M_{0} + \delta ||\partial_{\eta}^{m} \nabla \eta(t)||_{0}^{2} + TP \left( \sup_{t \in [0, T]} \mathcal{E}(t) \right)
$$
and
\[
\left| \int_0^t R_b^{1,3,5} \right| \leq \int_0^t (|\partial_t \partial_1 \mathcal{B}|_{L^4} + |\partial_t \mathcal{B} \partial_t \partial_1 \eta|_{L^4}) |\partial_t^{m_n} \eta|_{L^4} |\partial_t^{m_n} \eta|_0
\]
\[
\lesssim \int_0^t (|\partial_t \partial_1 \mathcal{B}|_{2} + |\partial_t \mathcal{B} \partial_t \partial_1 \eta|_{2}) |\partial_t^{m_n} \eta|_{2} |\partial_t^{m_n} \eta|_0
\]
\[
\lesssim \int_0^t (|\partial_t \partial_1 \mathcal{B}|_1 + |\partial_t \mathcal{B} \partial_t \partial_1 \eta|_1) |\partial_t^{m_n} \eta|_1 |\partial_t^{m_n} \eta|_{1}^2 |\partial_t^{m_n} \nabla \eta|_0^2 + |\partial_t^{m_n} \eta|_0)
\]
\[
\lesssim TP \left( \sup_{t \in [0, T]} \mathcal{E}(t) \right).
\]
As a consequence, we have
\[
\left| \int_0^t R_b^{1,3} \right| \lesssim M_0 + \delta |\partial_t^{m_n} \eta(t)|_0^2 + P \left( \sup_{t \in [0, T]} \mathcal{E}(t) \right),
\]
which, together with (3.60) and (3.57), yields that
\[
\left| \int_0^t R_b^5 \right| \lesssim M_0 + \delta |\partial_t^{m_n} \partial_t \partial_1 \eta(t)|_0^2 + |\partial_t^{m_n} \nabla \eta(t)|_0^2 + P \left( \sup_{t \in [0, T]} \mathcal{E}(t) \right).
\]
For $R_b^5$, we cannot use the duality argument as in (3.33) since it only involves time derivatives of the highest order. Instead, we use integration by parts to get
\[
R_b^5 = \int_{\Omega} [\partial_t^{m_n} \mathcal{B}, \eta_i] \partial_t V_i \, dx + \int_{\Omega} \partial_t \eta_i \partial_t^{m_n} \mathcal{B}, \eta_i \, dx
\]
\[
= \int_{\Omega} [\partial_t^{m_n} \mathcal{B}, \partial_t \eta_i] \partial_t V_i \, dx - \int_{\Omega} \partial_t [\partial_t^{m_n} \mathcal{B}, \partial_t \eta_i] \, dx + \int_{\Omega} \partial_t \eta_i \partial_t^{m_n} \mathcal{B}, \eta_i \, dx
\]
\[
- \int_{\Omega} [\partial_t^{m_n} \mathcal{B}, \partial_t \eta_i] \partial_t (\eta_i \mathcal{B}(\partial_t V)) \, dx
\]
\[
=: R_b^{2,1} + R_b^{2,1} + R_b^{2,3} + R_b^{2,4}.
\]
By using the Sobolev embedding, (3.27) and (2.4), one has
\[
\left| \int_0^t R_b^{2,1} \right| \lesssim \int_{\Omega} \left[ |\partial_t^{m_n} \mathcal{B}, \eta_i| + |\partial_t^{m_n} \partial_t \eta_i, \eta_i| \right] \, dx
\]
\[
\lesssim M_0 + \delta |\partial_t^{m_n-1} \mathcal{B}|_0 + |\partial_t \mathcal{B}|_{L^\infty} |\partial_t^{m_n-1} \nabla \eta|_0 |\partial_t^{m_n} \nabla \eta|_0
\]
\[
+ \sum_{l=2}^{m-2} |\partial_t^{m_n-l} \mathcal{B}|_{L^2} |\partial_t^{l} \nabla \eta|_{L^4} |\partial_t^{m_n} \eta|_0
\]
\[
\lesssim M_0 + \delta |\partial_t^{m_n} \nabla \eta(t)|_0^2 + C_l |\eta(t)|_{L^\infty} |\mathcal{B}(t)|^2_{L^2} + |\mathcal{B}|_{L^2}^2_{L^2}
\]
\[
\lesssim M_0 + \delta |\partial_t^{m_n} \nabla \eta(t)|_0^2 + TP \left( \sup_{t \in [0, T]} \mathcal{E}(t) \right).
\]
By using H"older’s inequality, one has
\[
\left| \int_0^t (R_b^{2,2} + R_b^{2,3}) \right| \lesssim \int_0^t \left( |||\partial_t^{m_n} \mathcal{B}, \nabla \eta|||_0 + |||\partial_t^{m_n} \mathcal{B}, \partial \nabla \eta|||_0 \right) \left( |||\partial_t^{m_n} \partial_t \eta|||_0 + |||\partial_t^{m_n} \eta|||_0 \right) \left( \nabla v \right)_{L^\infty}
\]
\[
\lesssim T^\frac{1}{2} \left( |||\partial_t^{m_n} \mathcal{B}, \nabla \eta|||_{L^2(L^2)} + |||\partial_t^{m_n} \mathcal{B}, \partial \nabla \eta|||_{L^2(L^2)} \right)
\]
\[
\times \left( |||\partial_t^{m_n} \eta|||_{L^2(L^2)} + |||\partial_t^{m_n} \eta|||_{L^2(H^m)} \right) \nabla v \left( L^2(H^m) \right)
\]
\[
\lesssim T^\frac{1}{2} P \left( \sup_{t \in [0, T]} \mathcal{E}(t) \right).
\]
where we have used
\[
|||\partial_t^{m_n} \mathcal{B}, \nabla \eta|||_{L^2(L^2)} \lesssim ||\mathcal{B}||_{L^\infty} \left( \nabla \eta \right)_{L^2(H^m)} + ||\partial \nabla \eta||_{L^\infty} \left( \mathcal{B} \right)_{L^2(H^m)}
\]
Similarly, 

\[
\left\| \partial_t^m B \right\|_{L^2_t(\mathbb{H}^{m-1})} \left\| \partial_t \mathbf{\nabla} \eta \right\|_{L^2_t(\mathbb{H}^{m-1})} \lesssim P \left( \sup_{t \in [0,T]} \mathcal{E}(t) \right),
\]

and

\[
\left\| \left[ \partial_t^m, B, \partial_t \mathbf{\nabla} \eta \right] \right\|_{L^2_t(L^2)} \lesssim \sum_{1 \leq \ell \leq m-1} \left\| \partial_t^{m-\ell} B \partial_t^\ell \partial_t \mathbf{\nabla} \eta \right\|_{L^2_t(L^2)} \lesssim \left\| \partial_t B \right\|_{L^\infty_t(\mathbb{H}^{m-1})} \left\| \partial_t \mathbf{\nabla} \eta \right\|_{L^2_t(\mathbb{H}^{m-1})} \left\| \partial_t^m B \right\|_{L^2_t(\mathbb{H}^{m-1})} \lesssim P \left( \sup_{t \in [0,T]} \mathcal{E}(t) \right).
\]

Similarly,

\[
\left| \int_0^T \mathcal{R}_b^{3,4} \right| \lesssim T^\frac{1}{4} P \left( \sup_{t \in [0,T]} \mathcal{E}(t) \right).
\]

As a consequence,

\[
\left| \int_0^T \mathcal{R}_b^2 \right| \lesssim M_0 + \delta \left\| \partial_t^m \mathbf{\nabla} \eta(t) \right\|_0^2 + T^\frac{1}{4} P \left( \sup_{t \in [0,T]} \mathcal{E}(t) \right).
\]

To control \( \mathcal{R}_b^3 \), we cannot use either the duality argument as in (3.40) due to loss of spatial derivatives, or the similar idea as in (3.62) since \( \left[ \partial_t^m, \partial_t^2 \eta_k, a_{k2}/|\partial_1 \eta|^3 \right] \) has one more derivative than \( \left[ \partial_t^m, B, a_{k2} \right] \). Instead, we deal with the troublesome term involving

\[
\partial_t^{m-1} \partial_t^2 \eta_k \partial_t(a_{k2}/|\partial_1 \eta|^3)
\]

through combining it with \( \mathcal{R}_q^3 \). In view of (3.24), we have

\[
\mathcal{R}_q^3 = -m \int \partial_t^m q \partial_t a_{ij} \partial_t^{m-1} \partial_t v_i dx + \sum_{2 \leq \ell \leq m-1} C_{\ell} \int \partial_t^{m-\ell} q \partial_t a_{ij} \partial_t^\ell \partial_t^{m-\ell} \partial_t v_i dx
\]

\[
+ \int_{\Omega} \left[ \partial_t^m, a_{ij}, \partial_t v_i \right] \partial_t^m \eta_k A_{k1} \partial_t \mathbf{\nabla} \eta dx
\]

\[
= -m \int \partial_t^m q \partial_t a_{k2} \partial_t^{m-1} v_i + m \int \partial_t \partial_t^m q \partial_t a_{ij} \partial_t^{m-1} v_i dx
\]

\[
- \sum_{2 \leq \ell \leq m-1} C_{\ell} \int \partial_t^{m-\ell} q \partial_t^\ell a_{ij} \partial_t^{m-\ell} \partial_t v_i dx + \int_{\Omega} \left[ \partial_t^m, a_{ij}, \partial_t v_i \right] \partial_t^{m} \eta_k A_{k1} \partial_t \mathbf{\nabla} \eta dx
\]

\[
= m \int \partial_t^m \left( \partial_t^2 \eta_k a_{k2} \right) \partial_t a_{k2} \partial_t^{m-1} v_i - m \int \partial_t \mathbf{\nabla} \partial_t a_{k2} \partial_t^{m-1} v_i
\]

\[
+ m \int \partial_t \partial_t^m q \partial_t a_{ij} \partial_t^{m-1} v_i - \sum_{2 \leq \ell \leq m-1} C_{\ell} \int \partial_t^{m-\ell} q \partial_t^\ell a_{ij} \partial_t^{m-\ell} \partial_t v_i dx
\]

\[
=: \mathcal{R}_q^{3,1} + \ldots + \mathcal{R}_q^{3,5},
\]

which, added with \( -\mathcal{R}_b^3 \), yields

\[
\mathcal{R}_b^3 + \mathcal{R}_q^{3,1} = m \int \partial_t \left( \frac{a_{k2}}{|\partial_1 \eta|^3} \right) \partial_t^{m-1} \partial_t^2 \eta_k a_{k2} v_i + \sum_{2 \leq \ell \leq m-1} C_{\ell} \int \partial_t^{m-\ell} q \partial_t^\ell \eta_k a_{k2} \partial_t^{m-\ell} v_i
\]

\[
+ m \int \partial_t a_{k2} \partial_t^{m-1} v_i \partial_t^{m} \left( \frac{a_{k2}}{|\partial_1 \eta|^3} \right) + m \int \partial_t \partial_t^m q \partial_t^2 \eta_k a_{k2} \partial_t^{m-1} v_i
\]

\[
= \frac{d}{dt} \int \partial_t \partial_t^m q \partial_t^{m-1} \eta_k a_{k2} \partial_t^{m-1} v_i - m \int \partial_t \partial_t^m q \partial_t^{m-1} \frac{a_{k2} v_i}{|\partial_1 \eta|^3} - m \int \partial_t \partial_t^m q \partial_t^{m-1} \frac{a_{k2} v_i}{|\partial_1 \eta|^3}
\]

\[
- m \int \partial_t \partial_t^m q \partial_t^{m-1} \frac{a_{k2} v_i}{|\partial_1 \eta|^3} - m \int \partial_t \partial_t^m q \partial_t^{m-1} \frac{a_{k2} v_i}{|\partial_1 \eta|^3}.
\]
+ \sum_{2 \leq \ell \leq m-1} C_{m}^{\ell} \int_{\Gamma} \frac{\partial_{t}^{\ell-m} \partial_{\eta} \partial_{t} \partial_{y}^{2} a_{k2}}{[\partial_{\eta} \partial_{y}]^{2}} a_{12} \nu_{1} + m \int_{\Gamma} \frac{\partial_{t}^{m} \frac{a_{k2}}{[\partial_{\eta} \partial_{y}]^{2}}}{\partial_{t}^{m} \partial_{\eta} \partial_{t} \partial_{y}^{m-1} v_{i}} \\
= m \int_{\Gamma} \left( \frac{a_{k2}}{[\partial_{\eta} \partial_{y}]^{2}} \right) \partial_{t}^{m-1} \partial_{\eta} a_{12} \partial_{y}^{m-1} \eta_{i} A_{j} \partial_{t} v_{i} \\
= R_{6}^{3.1} + \cdots + R_{6}^{3.7}.

(3.67)

By using Hölder’s inequality, (2.9), the trace theorem and (3.11), one has

\begin{align}
\left\| \int_{0}^{t} R_{6}^{3.1} \right\| \lesssim & M_{0} + |\partial_{t}^{m-1} \partial_{\eta} a_{k2}| |\partial_{t}^{m-1} v_{L}| |\partial_{t} \eta|_{L}^{4} \\
\lesssim & M_{0} + |\partial_{t}^{m-1} \partial_{\eta} a_{k2}| |\partial_{t}^{m} \eta|| |\partial_{t} \eta|_{1} \\
\lesssim & M_{0} + \delta \|\nabla \partial_{t}^{m} \eta\| + P \left( \sup_{t \in [0,T]} E(t) \right).
\end{align}

(3.68)

The term \( R_{6}^{3.2} \) vanishes when \( i = k \), but reduces to the following equation when \( i \neq k \):

\begin{align}
R_{6}^{3.2} &= -m \int_{\Gamma} \frac{1}{[\partial_{\eta} \partial_{y}]^{2}} \left( \partial_{t}^{m-1} \partial_{\eta}^{2} \partial_{t}^{m} v_{1} - \partial_{t}^{m-1} \partial_{\eta} \partial_{t}^{m} v_{2} \right) (\partial_{t} a_{12} a_{22} - a_{12} \partial_{t} a_{22}) \\
&= -m \int_{\Gamma} \frac{d}{dt} \left( \partial_{t} a_{12} a_{22} \partial_{t}^{m-1} \partial_{\eta} \partial_{t}^{m} \eta_{k} \right) + m \int_{\Gamma} \partial_{t}^{m-1} \partial_{t} a_{12} \partial_{t}^{m} \eta_{k} \partial_{t}^{m} v_{i} \\
&= R_{6}^{3.2.1} + R_{6}^{3.2.2} + R_{6}^{3.2.3},
\end{align}

where we have used the following identities:

\begin{align}
\partial_{t}^{m-1} \partial_{\eta}^{2} \partial_{t}^{m} v_{1} - \partial_{t}^{m-1} \partial_{\eta} \partial_{t}^{m} v_{2} &= -\partial_{t}^{m-1} \partial_{t} a_{12} \partial_{t}^{m} v_{k}, \\
\partial_{t} a_{12} a_{22} - a_{12} \partial_{t} a_{22} &= -\partial_{t} a_{12} \partial_{t} a_{22}, \quad \partial_{t}^{m} a_{12} \partial_{t}^{m} \eta_{k} = 0.
\end{align}

Then it follows from (2.7), (2.9), the trace theorem and (3.11) that

\begin{align}
\left\| \int_{0}^{t} R_{6}^{3.2.1} \right\| \lesssim & M_{0} + |\partial_{t}^{m-1} \partial_{\eta} a_{12} a_{22}| \frac{1}{2} |\partial_{t}^{m} \eta_{k}| \frac{1}{2} |\partial_{t} \eta|_{1} a_{12} |1 \\
\lesssim & M_{0} + |\partial_{t}^{m-1} a_{12} a_{22}| \frac{1}{2} |\partial_{t}^{m} \eta|_{1} \left( |\partial_{t}^{2} a_{12} a_{22}| + |\partial_{t} a_{12} a_{12} a_{22}| + |\partial_{t} a_{12} a_{12} a_{12}| \right) \\
\lesssim & M_{0} + \|\partial_{t}^{m-1} \partial_{t} \eta|| |\partial_{t}^{m} \eta|| \left( |\partial_{t}^{2} a_{12} a_{22}| + |\partial_{t} a_{12} a_{12} a_{12}| + |\partial_{t} a_{12} a_{12} a_{12}| \right) \\
\lesssim & M_{0} + \delta \|\nabla \partial_{t}^{m} \eta\|^{2} + P \left( \sup_{t \in [0,T]} E(t) \right).
\end{align}

Similarly, one has

\begin{align}
\left\| \int_{0}^{t} \left( R_{6}^{3.2.2} + R_{6}^{3.2.3} \right) \right\| \lesssim & \int_{0}^{t} \|\partial_{t}^{m-1} \partial_{t} \eta\|_{1} \|\partial_{t}^{m} \eta\|_{1} \left( |\partial_{t}^{2} a_{12} a_{22}| + |\partial_{t}^{2} a_{12} a_{12} a_{12}| + |\partial_{t}^{2} a_{12} a_{12} a_{12}| \right) \\
\lesssim & T \tilde{P} \left( \sup_{t \in [0,T]} E(t) \right).
\end{align}

Therefore, we have

\begin{align}
\left\| \int_{0}^{t} R_{6}^{3.2} \right\| \lesssim & M_{0} + \delta \|\nabla \partial_{t}^{m} \eta\| + P \left( \sup_{t \in [0,T]} E(t) \right).
\end{align}

(3.69)

It follows from (2.7), (2.9), the trace theorem and (3.11) that

\begin{align}
\left\| \int_{0}^{t} R_{6}^{3.3} \right\| \lesssim & \int_{0}^{t} \left( |\partial_{t}^{m-1} \partial_{t} \eta| \frac{1}{2} |\partial_{t}^{m} \eta| \frac{1}{2} |\partial_{t} v \partial_{t} v| + |\partial_{t}^{m-1} \partial_{t} a_{k2} a_{k2}| |\partial_{t}^{m-1} v_{L}| |\partial_{t}^{2} \partial_{t} \eta|_{L}^{4} \right)
\end{align}
\[
\begin{align*}
\lesssim & \int_0^t \left( |\partial_t^{m-1} \partial_t \eta_{11}| \| \partial_t^{m} \eta_{11} \|_1 (\| \partial_t \partial_t^2 v_1 \|_1 + \| \partial_t^2 \partial_t^2 v_1 \|_1) \right) \\
& + \int_0^t \left( |\partial_t^{m-1} \partial_t \eta_{12}| \| \partial_t^{m} \eta_{12} \|_1 (\| \partial_t \partial_t^2 v_2 \|_1 + \| \partial_t^2 \partial_t^2 v_2 \|_1) \right) \\
& + \int_0^t \left( |\partial_t^{m-1} \partial_t \eta_{21}| \| \partial_t^{m} \eta_{21} \|_1 (\| \partial_t \partial_t^2 v_3 \|_1 + \| \partial_t^2 \partial_t^2 v_3 \|_1) \right) \\
& + \int_0^t \left( |\partial_t^{m-1} \partial_t \eta_{22}| \| \partial_t^{m} \eta_{22} \|_1 (\| \partial_t \partial_t^2 v_4 \|_1 + \| \partial_t^2 \partial_t^2 v_4 \|_1) \right) \\
& \lesssim T^2 \left( \| \partial_t^{m-1} \partial_t \eta_{L^\infty_1(H^1)} \|_{L^\infty_t(H^1)} + \| \partial_t^{m-1} \partial_t \eta_{L^\infty_2(L^2)} \|_{L^\infty_t(L^2)} \right) \| \partial_t \partial_t^2 v_1 \|_{L^\infty_t(H^1)} \| \partial_t^2 \partial_t^2 v_1 \|_{L^\infty_t(H^1)} \\
\lesssim & T^2 \left( \sup_{t \in [0,T]} \mathcal{E}(t) \right) . 
\end{align*}
\]

(3.70)

For \( R_b^{3,4} \), we cannot use the same argument as in (3.70) since we cannot bound \( |\partial_t^m v|_1 \). Instead, we use integration by parts to get

\[
R_b^{3,4} = -m \int_\Gamma \partial_t \left( \frac{1}{|\partial_t \eta|^2} \right) |\partial_t^{m-1} \partial_t \eta_{i,k} a_{i,k} \partial_t^{m+1} \partial_t \eta_{a_{i,k}} | \\
- m \int_\Gamma \partial_t \left( \frac{1}{|\partial_t \eta|^2} \right) |\partial_t^{m-1} \partial_t \eta_{a_{i,k}} a_{i,k} \partial_t^{m+1} \partial_t \eta_{i,k} | \\
= -m \int_\Gamma \partial_t \left( \frac{1}{|\partial_t \eta|^2} \right) |\partial_t^{m-1} \partial_t \eta_{i,k} a_{i,k} \partial_t^{m+1} \partial_t \eta_{a_{i,k}} | \\
+ m \int_\Gamma \partial_t \left( \frac{1}{|\partial_t \eta|^2} \right) |\partial_t^{m-1} \partial_t \eta_{a_{i,k}} a_{i,k} \partial_t^{m+1} \partial_t \eta_{i,k} | \\
=: R_b^{3,4,1} + \cdots + R_b^{3,4,4} .
\]

Then we can obtain from H"older's inequality, (2.9) and the trace theorem that

\[
\begin{align*}
\left| \int_0^t R_b^{3,4,1} \right| & \lesssim M_0 + |\partial_t^{m} \partial_t \eta_{i,k} a_{i,k} |_{L^\infty_t} \| \partial_t \eta_{i,k} \|_{L^1_t} \\
& \lesssim M_0 + |\partial_t^{m} \partial_t \eta_{i,k} a_{i,k} |_{L^\infty_t} \| \partial_t \eta_{i,k} \|_{L^1_t} \\
& \lesssim M_0 + |\partial_t^{m} \partial_t \eta_{i,k} a_{i,k} |_{L^\infty_t} \| \partial_t \eta_{i,k} \|_{L^1_t} + P \left( \sup_{t \in [0,T]} \mathcal{E}(t) \right ) 
\end{align*}
\]

(3.71)

and

\[
\begin{align*}
\left| \int_0^t R_b^{3,4,2} \right| & \lesssim \left| \int_0^t \left( |\partial_t \partial_t \eta_{i,k} a_{i,k} |_{L^\infty_t} |\partial_t^{m} \partial_t \eta_{i,k} a_{i,k} |_{L^1_t} \right) \right| \\
& \lesssim |\partial_t^{m} \partial_t \eta_{i,k} a_{i,k} |_{L^\infty_t} \| \nabla \eta \|_{L^2_t(\mathbb{R}^{n-1})} \\
& \lesssim T^2 \left( \sup_{t \in [0,T]} \mathcal{E}(t) \right ) .
\end{align*}
\]

(3.72)

Moreover, it follows from H"older's inequality, (2.9), (2.7) and the trace theorem that

\[
\begin{align*}
\left| \int_0^t R_b^{3,4,3} \right| & \lesssim \left| \int_0^t \int_\Gamma \partial_t \left( \frac{1}{|\partial_t \eta|^2} \right) |\partial_t^{m-1} \partial_t \eta_{i,k} a_{i,k} \partial_t^{m+1} \partial_t \eta_{a_{i,k}} | \\
& + \int_0^t \int_\Gamma \partial_t \left( \frac{1}{|\partial_t \eta|^2} \right) |\partial_t^{m-1} \partial_t \eta_{a_{i,k}} a_{i,k} \partial_t^{m+1} \partial_t \eta_{i,k} | \\
& \lesssim \int_0^t \left( |\partial_t^{m} \partial_t \eta_{i,k} a_{i,k} |_{L^\infty_t} |\partial_t^{m} \partial_t \eta_{i,k} a_{i,k} |_{L^1_t} \right) \\
& + \int_0^t \left( |\partial_t^{m} \partial_t \eta_{a_{i,k}} a_{i,k} |_{L^\infty_t} |\partial_t^{m} \partial_t \eta_{a_{i,k}} a_{i,k} |_{L^1_t} \right) \\
& + \int_0^t \left( |\partial_t^{m} \partial_t \eta_{i,k} a_{i,k} |_{L^\infty_t} |\partial_t^{m} \partial_t \eta_{i,k} a_{i,k} |_{L^1_t} \right) \\
& + \int_0^t \left( |\partial_t^{m} \partial_t \eta_{a_{i,k}} a_{i,k} |_{L^\infty_t} |\partial_t^{m} \partial_t \eta_{a_{i,k}} a_{i,k} |_{L^1_t} \right)
\end{align*}
\]

(3.73)
\[
\int_0^t R_b^{3,4,4} \leq T^\frac{1}{2} P \left( \sup_{t \in [0,T]} \mathcal{E}(t) \right).
\]

To bound \( R_b^{3,4,4} \), similar to (3.62), we use integration by parts to reduce this boundary integral to the volume one as

\[
R_b^{3,4,4} = -m \int_\Omega \partial_t \left( \frac{1}{|\partial_t \eta|^3} a_{k2a_{ij}} \right) \partial_t^{m-1} \partial_t \eta \partial_t \eta \partial_t \eta dx + m \int_\Omega \partial_t \left( \frac{1}{|\partial_t \eta|^3} a_{k2a_{ij}} \right) \partial_t^{m-1} \partial_t \eta \partial_t \eta \partial_t \eta dx + m \int_\Omega \partial_t \partial_t \left( \frac{1}{|\partial_t \eta|^3} a_{k2a_{ij}} \right) \partial_t^{m-1} \partial_t \eta \partial_t \eta \partial_t \eta dx = -\frac{d}{dt} \int_\Omega \partial_t \left( \frac{1}{|\partial_t \eta|^3} a_{k2a_{ij}} \right) \partial_t^{m-1} \partial_t \eta \partial_t \eta \partial_t \eta dx + m \int_\Omega \partial_t \partial_t \left( \frac{1}{|\partial_t \eta|^3} a_{k2a_{ij}} \right) \partial_t^{m-1} \partial_t \eta \partial_t \eta \partial_t \eta dx + m \int_\Omega \partial_t \partial_t \left( \frac{1}{|\partial_t \eta|^3} a_{k2a_{ij}} \right) \partial_t^{m-1} \partial_t \eta \partial_t \eta \partial_t \eta dx\]

By using Hölder’s inequality, one has

\[
\int_0^t \left| R_b^{3,4,4,1} \right| \leq \int_\Omega \partial_t \left( \frac{1}{|\partial_t \eta|^3} a_{k2a_{ij}} \right) \partial_t^{m-1} \partial_t \eta \partial_t \eta \partial_t \eta dx \left| t \right|_0^t \leq M_0 + \left( \| \partial_t \partial_t^2 \eta \|_{L^4} + \| \partial_t \partial_t \eta \|_{L^4} \| \partial_t^{m-1} \partial_t \eta \|_{L^4} \| \partial_t \eta \|_{L^4} \right) + P \left( \sup_{t \in [0,T]} \mathcal{E}(t) \right).
\]

As a consequence, we obtain

\[
\int_0^t \left| R_b^{3,4,4,4} \right| \leq M_0 + \delta \| \partial_t^m \eta \|_0^2 + P \left( \sup_{t \in [0,T]} \mathcal{E}(t) \right). \tag{3.74}
\]

Plugging (3.71)–(3.74) yields that

\[
\int_0^t R_b^{3,4,4} \leq M_0 + \delta \| \partial_t^m \eta \|_0^2 + \| \partial_t^m \partial_t \eta \|_0^2 + P \left( \sup_{t \in [0,T]} \mathcal{E}(t) \right). \tag{3.75}
\]
The remaining terms $\mathcal{R}_b^{3.5}$, $\mathcal{R}_b^{3.6}$ and $\mathcal{R}_b^{3.7}$ are easy to be bounded by the similar argument used for $\mathcal{R}_b^{3.3}$ and $\mathcal{R}_b^{3.4}$. We omit the details and conclude that

$$\left| \int_0^t (\mathcal{R}_b^{3.5} + \mathcal{R}_b^{3.6} + \mathcal{R}_b^{3.7}) \right| \lesssim M_0 + \delta(\|\partial_t^n \nabla \eta\|_0^2 + |\partial_t^n \partial_1 \eta_k a_{i2} (t)|^2) + P \left( \sup_{t \in [0, T]} E(t) \right). \quad (3.76)$$

Substituting (3.68)–(3.70), (3.75) and (3.76) into (3.67) yields that

$$\left| \int_0^t (\mathcal{R}_b^3 + \mathcal{R}_b^{3.1}) \right| \lesssim M_0 + \delta(\|\partial_t^n \nabla \eta(t)\|_0^2 + |\partial_t^n \partial_1 \eta_k a_{i2} (t)|^2) + P \left( \sup_{t \in [0, T]} E(t) \right).$$

Moreover, it is not difficult to prove that

$$\left| \int_0^t (\mathcal{R}_b^{3.2} + \cdots + \mathcal{R}_b^{3.7}) \right| \lesssim P \left( \sup_{t \in [0, T]} E(t) \right).$$

So one gets

$$\left| \int_0^t (\mathcal{R}_b^3 + \mathcal{R}_b^{3.1}) \right| \lesssim M_0 + \delta(\|\partial_t^n \nabla \eta(t)\|_0^2 + |\partial_t^n \partial_1 \eta_k a_{i2} (t)|^2) + P \left( \sup_{t \in [0, T]} E(t) \right). \quad (3.77)$$

For $\mathcal{R}_b^4$, it follows from (2.10) that

$$\left| \int_0^t (\mathcal{R}_b^4) \right| \lesssim M_0 + \delta(\|\partial_t^n \nabla \eta(t)\|_0^2 + |\partial_t^n \partial_1 \eta_k a_{i2} (t)|^2) + P \left( \sup_{t \in [0, T]} E(t) \right). \quad (3.78)$$

For $\mathcal{R}_b^5$, by a tedious but not difficult calculation, we have

$$\mathcal{R}_b^5 = \int_\Gamma \frac{\partial_t^{m+1} a_{i2} \partial_t^2 e_{i2}}{|\partial_1 \eta|^3} \partial_1 \eta_k \partial_1 a_{i2} \partial_1 a_{i2} - \int_\Gamma \frac{\partial_t^{m} \partial_1 \eta_k \partial_1 a_{i2} \partial_1 a_{i2} \partial_t^2 e_{i2}}{|\partial_1 \eta|^3}$$

$$+ 3 \int_\Gamma \frac{\partial_t^{m} \partial_1 \eta_k \partial_1 a_{i2} \partial_1 a_{i2} \partial_t^2 e_{i2}}{|\partial_1 \eta|^3} - \frac{3}{3} \left[ \partial_t^{m-1}, \partial_1 a_{i2}, \frac{a_{i2}}{|\partial_1 \eta|^3} \right] a_{i2} \partial_t^2 e_{i2}$$

$$- 3 \int_\Gamma \frac{\partial_1 a_{i2} \partial_t^{m-1}}{|\partial_1 \eta|^3} \partial_t e_{i2} \partial_1 a_{i2} \partial_t e_{i2} \partial_1 \eta_k \partial_1 a_{i2} \partial_1 a_{i2} \partial_t e_{i2}$$

$$= \int_\Gamma \frac{\partial_t^2 e_{i2}}{|\partial_1 \eta|^3} \partial_t^{m+1} a_{i2} + \mathcal{R}_b^{5.2} + \mathcal{R}_b^{5.3},$$

where we have used (3.29) and the following identity:

$$\partial_t^{m} \partial_1 \eta_k \partial_1 a_{i2} \partial_1 a_{i2} \partial_t^2 e_{i2} = |\partial_1 \eta|^2 \partial_t^{m} \partial_1 \eta_k \partial_1 a_{i2}.$$
Thus, we obtain
\[
\left| \int_0^t \mathcal{R}_6^1 \right| \lesssim \delta \| \nabla t^n \eta \|^2_0 + P \left( \sup_{t \in [0,T]} \mathcal{E}(t) \right). \tag{3.79}
\]
For $\mathcal{R}_6^2$, it follows from (2.10) that
\[
\left| \int_0^t \mathcal{R}_6^2 \right| \lesssim \int_0^t \left| \partial_t^n \eta A_k \partial_q a_{ij} \partial_t^n v_i \right| + \left| \partial_t^n \eta \left( |\partial_1 \nabla v|_{L^\infty} + |\partial_2 \nabla v|_{L^\infty} |\nabla v|_{L^\infty} \right) \right| \lesssim P \left( \sup_{t \in [0,T]} \mathcal{E}(t) \right). \tag{3.80}
\]
For $\mathcal{R}_6^3$, we obtain from integration by parts and the Piola identity (1.14) that
\[
\mathcal{R}_6^3 = \int_\Omega \frac{d}{dt} \left( \partial_t^n \eta A_k \partial_q a_{ij} \partial_t^n v_i \right) - \int_\Omega \partial_t^n \eta A_k \partial_q a_{ij} \partial_t^n v_i dx + \int_\Omega \partial_t^n \eta \partial_q a_{ij} \partial_t^n v_i dx
\]
\[
- \int_\Omega \partial_t^n \eta \partial_q a_{ij} \partial_t^n v_i dx
\]
\[
= \int_\Omega \frac{d}{dt} \left( \partial_t^n \eta A_k \partial_q a_{ij} \partial_t^n v_i \right) - \int_\Omega \partial_t^n \eta \partial_q a_{ij} \partial_t^n v_i dx.
\]
(3.81)

By the interpolation inequality, one has
\[
\left| \int_0^t \mathcal{R}_6^{1,2} \right| \lesssim M_0 + (\| \nabla t^n \eta \|^2_0 + \| \partial_t^n \eta \|^2_0) (\| \nabla v \|^2_0 + \| \nabla q \|_0 + \| \nabla t^n \eta \|_0)
\]
\[
\lesssim \delta \| \nabla t^n \eta \|^2_0 + M_0 + P(\sup_{t \in [0,T]} \mathcal{E}(t)). \tag{3.82}
\]
In view of (3.3) and the Sobolev embedding, one has
\[
\left| \int_0^t \mathcal{R}_6^{2,3} \right| \lesssim \int_0^t \| \partial_t^n v \|_0 \| \nabla q \| L^\infty \| \partial_t^n \nabla \eta \|_0
\]
\[
+ \int_0^t \| \partial_t^n \eta \|_L^8 \| A \| \nabla q \| L^8 \| \partial_t^n \nabla \eta \|_0 + \| \partial_t^n v \|_0
\]
\[
+ \int_0^t \| \partial_t^n \eta \| L^8 \| A \| \| \partial_t^n \nabla \eta \|_0 + \| \partial_t^n v \|_0
\]
\[
\lesssim \| \nabla q \| \| \partial_t^n \eta \| L^8 \| \partial_t^n \nabla \eta \|_0 \| \partial_t^n \nabla \eta \|_0 + \| \partial_t^n \eta \| L^8 \| \partial_t^n \nabla \eta \|_0
\]
\[
\lesssim P \left( \sup_{t \in [0,T]} \mathcal{E}(t) \right). \tag{3.83}
\]
On the other hand,
\[
\left| \int_0^t \mathcal{R}_6^{3,4} \right| \lesssim \int_0^t \| \partial_t^n \eta \|_2^2 \| A_k \partial_q a_{ij} \partial_t \partial_t v_i \|_0 \lesssim \int_0^t \| \partial_t^n \eta \|_2^2 \| \partial_q a_{ij} \partial_t v_i \| H^1
\]
\[
\lesssim \| \partial_t^n \eta \|_2^2 \| \nabla q \| L^2 \| \nabla v \| L^2 \| \nabla v \| \lesssim P \left( \sup_{t \in [0,T]} \mathcal{E}(t) \right). \tag{3.84}
\]
Plugging (3.82)–(3.84) into (3.81) yields that
\[
\left| \int_0^t \mathcal{R}_6^4 \right| \lesssim \delta \| \nabla t^n \eta \|^2_0 + M_0 + P \left( \sup_{t \in [0,T]} \mathcal{E}(t) \right). \tag{3.85}
\]
Finally, the volume integral terms $R_{i0}^i$, $R_{i0}^c$, $R_{c0}^i$ and $R_{c0}^c$ can be controlled as follows. It is clear that

$$
\left| \int_0^t R_{i0}^i \right| \lesssim \int_0^t \left( \| \partial_\mu^m v \|^2_{L^2(t)} \right) + \int_0^t \left( \| \partial_\mu^m \eta \|^2_{L^2(t)} \right).
$$

It follows from (2.3), the Sobolev embedding and (3.6) that

$$
\left| \int_0^t R_{i0}^c \right| \lesssim \left( \| \partial_\mu^m, a_\mu, \partial_h q \|_{L^2(t)} \right) + \| \partial_\mu^m v \|^2_{L^2(t)} + \| \partial_\mu^m \eta \|^2_{L^2(t)} + \| \partial_\mu^m \eta \|^2_{L^2(t)}.
$$

It is obvious that

$$
\left| \int_0^t R_{c0}^i \right| \lesssim \left( \| \partial_\mu^m, a_\mu, \partial_h q \|_{L^2(t)} \right) + \| \partial_\mu^m v \|^2_{L^2(t)} + \| \partial_\mu^m \eta \|^2_{L^2(t)}.
$$

Therefore, integrating (3.54) with respect to time, substituting the above estimates at hand into the resultant equation, and using the estimate (3.5) and Korn’s inequality (2.13), we have

$$
\| \mathcal{V} \|_{L^2} + \| \mathcal{Q} \|_{L^2} + \| \nabla \partial_\mu^m \eta \|_{L^2} + \| \partial_\mu^m \partial \eta \|_{L^2} \leq M_0 + \delta \| \mathcal{E} \|_{L^2} + P \left( \sup_{t \in [0, T]} \mathcal{E}(t) \right).
$$
Squaring the above inequality and integrating over time once again yield
\[
\int_0^t \left( \|\nabla \delta^m \eta \|^2_0 + \|\nabla \delta^m \eta \|^4_0 + \|\partial_t \eta \delta_t \nabla \delta^m \eta \|^2_0 + \|\partial_t \eta \delta_t \nabla \delta^m \eta \|^4_0 + \|\partial_t \eta \delta_t \nabla \delta^m \eta \|^2_0 \right) \leq M_0 + TP \left( \sup_{t \in [0,T]} \mathcal{E}(t) \right).
\]

By the definitions of \( \mathcal{V} \) and \( \mathcal{Q} \) and the fundamental theorem of calculus, we obtain
\[
\int_0^t \left( \|\nabla \delta^m \eta \|^2_0 + \|\nabla \delta^m \eta \|^4_0 \right) \leq \int_0^t \left( \|\nabla \delta^m \eta \|^2_0 + \|\nabla \delta^m \eta \|^4_0 \right) + \int_0^t \|\partial_t \eta \delta_t \nabla \delta^m \eta \|^2_0 \leq M_0 + TP \left( \sup_{t \in [0,T]} \mathcal{E}(t) \right).
\]

Therefore, we complete the proof. \( \square \)

### 3.5 Normal derivative estimates

To derive the normal derivative estimates, we can first obtain from (1.15), (1.12) and (2.14) that
\[
\begin{align*}
- \rho_0 J \Delta \eta &- (\rho_0 J^{-1})^\gamma a_{ik} a_{rs} \partial_k \partial_s \eta \rho_0 - \mu e a_{ik} a_{kj} \partial_j \partial_k \partial_t \eta v_i - (\mu + \lambda) \varepsilon a_{ik} a_{ij} \partial_t \partial_h \eta v_i \\
&= - (\rho_0 J^{-1})^\gamma a_{ik} a_{kj} \partial_j \partial_k \partial_t \eta v_i - \rho_0 \partial_t \partial_h \eta v_i + \mu e J a_{ik} a_{kj} \partial_j \partial_k \partial_t \eta v_i \\
&+ \mu e J a_{ik} a_{kj} \partial_j \partial_k \partial_t \eta v_i + \lambda e J a_{ik} a_{kj} \partial_j \partial_k \partial_t \eta v_i,
\end{align*}
\]
where we have used the fact that \( \partial_k f = -\rho_0 J^{-2} a_{rs} \partial_r \partial_s \eta \rho_0 + \partial_h \rho_0 J^{-1} \). As a consequence, we have
\[
- A_{ij} \partial_j^2 \eta v_i - \mu e a_{ik} a_{kj} \partial_j^2 \eta v_i - (\mu + \lambda) \varepsilon a_{ik} a_{ij} \partial_j^2 \eta v_i =: \mathcal{F}_i + \mathcal{G}_i,
\]
where
\[
A_{ij} = \rho_0 J \delta_{ij} + (\rho_0 J^{-1})^\gamma a_{ij} a_{ij},
\]
\[
\mathcal{F}_i = \sum_{l \neq 2 \text{ or } j \neq 2} (\mu e a_{ik} a_{kj} \partial_j \partial_k \partial_t \eta v_i + (\mu + \lambda) \varepsilon a_{ik} a_{ij} \partial_t \partial_h \eta v_i) \\
+ \mu e J a_{ik} a_{kj} \partial_j \partial_k \partial_t v_i + \mu e J a_{ik} a_{kj} \partial_j \partial_k \partial_t \eta v_i + \lambda e J a_{ik} a_{kj} \partial_j \partial_k \partial_t \eta v_i
\]
and
\[
\mathcal{G}_i = (\rho_0 J^{-1})^\gamma (a_{12} a_{12} \partial_1^2 \eta \rho_0 + a_{12} a_{12} \partial_2^2 \eta \rho_0 + a_{12} a_{12} \partial_1 \partial_2 \eta \rho_0) \\
- \rho_0 \partial_t \partial_h \eta v_i - \rho_0 J \partial_1^2 \eta - (\rho_0 J^{-1})^\gamma a_{ik} \rho_0 J \partial_1 \partial_2 \eta v_i + J \nabla \rho_0 \cdot \nabla \eta.
\]
It is clear that
\[
\det A = \rho_0 J + (\rho_0 J^{-1})^\gamma (a_{12}^2 + a_{22}^2) > \rho_0 J > 0
\]
so that \( A \) is invertible. Moreover, it is obvious that \( A \) is also symmetric. These imply that we can estimate the normal derivatives of \( \eta \) by using (3.92), so we have the following lemma.

**Lemma 3.6.** For any \( t \in [0,T] \) and \( m \geq 4 \), it holds that
\[
\| \nabla^\beta \eta \|^2_0 + \| \nabla \eta^\beta \|^2_0 + \| \nabla \eta^\beta \|^4_0 + \| \nabla^2 \eta^\beta \|^2_0 + \| \nabla^2 \eta^\beta \|^4_0 + \| \nabla^2 \eta^\beta \|^2_0 \leq M_0 + \delta \sup_{t \in [0,T]} \mathcal{E}(t) + T_0 \sup_{t \in [0,T]} \mathcal{E}(t).
\]

**Proof.** Applying \( \delta^\beta \) with \( |\beta| \leq m - 1 \) to (3.92) yields
\[
- A_{ij} \delta^\beta \partial_j^2 \eta v_i - \mu e a_{ik} a_{kj} \delta^\beta \partial_j^2 \eta v_i - (\mu + \lambda) \varepsilon a_{ik} a_{ij} \delta^\beta \partial_j^2 \eta v_i
\]
Similarly, by using (2.4), (3.6) and (3.11), we have

\[ |\partial^3 \mathcal{A}_i \|_{L^2_t L^2_x} + \| \mu (\partial^3, a_k a_k) \|_{L^2_t L^2_x} (\mu + \lambda) \leq |\partial^3 \mathcal{A}_i |_{L^2_t L^2_x} + |\partial^3 \mathcal{F}_i + \partial^3 \mathcal{G}_i |_{L^2_t L^2_x} \]  

(3.94)

Then by taking the inner product between the above equation and \( -\partial^2_t \partial^2_x \eta \), one has

\[
\int_\Omega \partial^3 \mathcal{A}_i \|_{L^2_t L^2_x} + \| \mu \|_{L^2_t L^2_x} + \| \lambda \|_{L^2_t L^2_x} \leq \| \partial^3 \mathcal{A}_i \|_{L^2_t L^2_x} + \| \partial^3 \mathcal{F}_i + \partial^3 \mathcal{G}_i \|_{L^2_t L^2_x} \]  

(3.95)

It follows from the Sobolev embedding and Hölder’s inequality that

\[
\int_0^T \| \partial_t \partial^3 \mathcal{A}_i \|_{L^2_t L^2_x} + \| \partial^3 \mathcal{A}_i \|_{L^2_t L^2_x} \leq \sup_{t \in [0, T]} \mathcal{E}(t) \]  

(3.96)

It follows from (2.1), (3.10) and (3.6) that

\[
\| \partial^3 \mathcal{A}_i \|_{L^2_t L^2_x} \leq \| \partial^3 \mathcal{A}_i \|_{L^2_t L^2_x} + \| \partial^3 \mathcal{A}_i \|_{L^2_t L^2_x} + \| \partial^3 \mathcal{A}_i \|_{L^2_t L^2_x} \]  

(3.97)

and

\[
\| \partial^3 \mathcal{G}_i \|_{L^2_t L^2_x} \leq \| \partial^3 \mathcal{G}_i \|_{L^2_t L^2_x} + \| \partial^3 \mathcal{G}_i \|_{L^2_t L^2_x} + \| \partial^3 \mathcal{G}_i \|_{L^2_t L^2_x} \]  

(3.98)

Similarly, by using (2.1), (3.3) and (3.11), we have

\[
\| \partial^3 \mathcal{G}_i \|_{L^2_t L^2_x} \leq \| \partial^3 \mathcal{G}_i \|_{L^2_t L^2_x} + \| \partial^3 \mathcal{G}_i \|_{L^2_t L^2_x} + \| \partial^3 \mathcal{G}_i \|_{L^2_t L^2_x} \]  

(3.99)

and

\[
\| \partial^3 \mathcal{G}_i \|_{L^2_t L^2_x} \leq \| \partial^3 \mathcal{G}_i \|_{L^2_t L^2_x} + \| \partial^3 \mathcal{G}_i \|_{L^2_t L^2_x} + \| \partial^3 \mathcal{G}_i \|_{L^2_t L^2_x} \]  

(3.100)
Furthermore, by using (2.4), we have
\[\|J\nabla \tilde{\rho}_0\|^2_{L_t^2(H^{m-1})} + \|\nabla \eta\|^2_{L_t^2(H^{m-1})} \leq T\|\nabla \eta\|^2_{L_t^2(H^{m-1})} + \|J\|_{L_t^2(H^{m-1})} + \|\nabla \eta\|^2_{L_t^2(H^{m-1})} + \|\nabla \tilde{\rho}_0\|^2_{L_t^2(H^{m-1})} \leq T\frac{1}{2} P\left(\sup_{t \in [0,T]} \mathcal{E}(t)\right).\] (3.100)

Therefore, plugging (3.96)–(3.100) into (3.95) yields
\[\|\partial_t^2 \eta\|_{L_t^2(H^{m-1})}^2 + \|\sqrt{\varepsilon} \partial_t^2 \eta\|_{L_t^2(H^{m-1})}^2 \lesssim M_0 + \delta \sup_{t \in [0,T]} \mathcal{E}(t) + T\frac{1}{2} P\left(\sup_{t \in [0,T]} \mathcal{E}(t)\right).\] (3.101)

On the other hand, by taking the inner product between (3.94) and \(-\varepsilon \partial_t^2 \eta_t\), integrating over \(\Omega \times [0,t]\) and using (3.97)–(3.101), we can obtain
\[\varepsilon \|\partial_t^2 \eta\|^2_{L_t^2(H^{m-1})} \lesssim M_0 + \delta \sup_{t \in [0,T]} \mathcal{E}(t) + T\frac{1}{2} P\left(\sup_{t \in [0,T]} \mathcal{E}(t)\right).\] (3.102)

Next, for any \(|\beta| \leq m-1 - \ell\) and \(\ell \in \mathbb{N}\), applying \(-\partial_t \partial_t^2\) to (3.92), taking the inner product with \(-\partial_t \partial_t^2 \eta_t\), and using the same argument as in the proofs of (3.101) and (3.102), we can successively obtain the following estimates for \(\ell = 1, 2, \ldots, m-1\):
\[\|\partial_t^2 \eta\|_{L_t^2(H^{m-1})}^2 + \|\sqrt{\varepsilon} \partial_t^2 \eta\|_{L_t^2(H^{m-1})}^2 \lesssim M_0 + \delta \sup_{t \in [0,T]} \mathcal{E}(t) + T\frac{1}{2} P\left(\sup_{t \in [0,T]} \mathcal{E}(t)\right).\]

As a consequence, one has
\[\|\partial_t^2 \eta\|_{L_t^2(H^{m-1})}^2 + \|\sqrt{\varepsilon} \partial_t^2 \eta\|_{L_t^2(H^{m-1})}^2 \lesssim M_0 + \delta \sup_{t \in [0,T]} \mathcal{E}(t) + T\frac{1}{2} P\left(\sup_{t \in [0,T]} \mathcal{E}(t)\right),\]
which, together with (3.11) and (3.48), implies
\[\|\nabla \eta\|_{H^{m-1}}^2 + \|\nabla \eta\|_{H^{m-1}}^2 + \|\partial_t \eta\|_{H^{m-1}}^2 + \|\partial_t^2 \eta\|_{L_t^2(H^{m-1})}^2 \lesssim M_0 + \delta \sup_{t \in [0,T]} \mathcal{E}(t) + T\frac{1}{2} P\left(\sup_{t \in [0,T]} \mathcal{E}(t)\right).\]

Furthermore, by using (2.4), we have
\[\|\nabla \eta\|_{H^{m-1}}^2 + \|\nabla \eta\|_{H^{m-1}}^2 \lesssim \|\eta_0\|_{H^m}^2 + \|\nabla \eta_0\|_{H^{m-1}}^2 + T\|\partial_t \eta\|_{L_t^2(H^{m-1})}^2 + \|\partial_t^2 \eta\|_{L_t^2(H^{m-1})}^2 \lesssim M_0 + \delta \sup_{t \in [0,T]} \mathcal{E}(t) + T\frac{1}{2} P\left(\sup_{t \in [0,T]} \mathcal{E}(t)\right).\]
Therefore, we complete the proof of this lemma. \(\square\)

### 3.6 The proof of Proposition 3.1

We now collect the estimates derived above and verify the \textit{a priori} assumption (3.3). Indeed, we obtain from Lemmas 3.1, 3.3, 3.5 and 3.6 that
\[\sup_{t \in [0,T]} \mathcal{E}(t) \leq M_0 + \delta \sup_{t \in [0,T]} \mathcal{E}(t) + T\frac{1}{2} P\left(\sup_{t \in [0,T]} \mathcal{E}(t)\right).\]
As a result, it holds for any \(t \in [0,T]\) that
\[|J^\varepsilon(t) - J_0^\varepsilon| \leq \left|\int_0^t J_t^\varepsilon\right| \leq T\varepsilon^{\frac{1}{2}} \|J^\varepsilon\|_{L_t^2(L^\infty)} \lesssim T\varepsilon^{\frac{1}{2}} \sup_{t \in [0,T]} \mathcal{E}(t).\]
Similarly, we also have
\[|\partial_j \eta^\varepsilon(t) - \partial_j \eta_0^\varepsilon| \lesssim T\varepsilon^{\frac{1}{2}} \sup_{t \in [0,T]} \mathcal{E}(t).\]
Therefore, by taking \(\delta\) sufficiently small, there exists a \(T\) independent of \(\varepsilon\) such that (3.3) is satisfied and
\[\sup_{t \in [0,T]} \mathcal{E}(t) \leq 2M_0.\]
4 The proofs of theorems

4.1 The proof of Theorem 1.1

For any fixed $\varepsilon > 0$, we can construct the local classical solutions $(\eta^\varepsilon, q^\varepsilon, v^\varepsilon)$ to the free boundary problem of the compressible viscoelastic fluid system (1.15) by the approach similar to that in [52] if the initial data $(\overline{\eta}_0, \eta_0^0, v_0^0)$ satisfy (1.19)–(1.21). We omit the details here for simplicity. From the uniform estimates of $(\eta^\varepsilon, q^\varepsilon, v^\varepsilon)$ obtained in Proposition 3.1, we can find a $T_0 > 0$ independent of $\varepsilon$ such that $(\eta^\varepsilon, q^\varepsilon, v^\varepsilon)$ satisfy sup$_{t \in [0, T]} E^\varepsilon(t) \leq C_1$, which completes the proof of Theorem 1.1.

4.2 The proof of Theorem 1.2

It follows from Theorem 1.1 that $\eta^\varepsilon$ is uniformly bounded in $L^\infty(0, T_0; H^m)$, $\nabla \eta^\varepsilon$ is uniformly bounded in $L^2(0, T_0; H^m)$, and $\partial_t \eta^\varepsilon$ is uniformly bounded in $L^\infty(0, T_0; H^{m-1}) \cap L^2(0, T_0; H^{m-1})$. Then by using the Aubin-Lions compactness theorem (see [39]), we see that $\eta^\varepsilon$ is compact in $C([0, T_0]; H^{m-1})$. In particular, there exist a sequence $\varepsilon_n \to 0^+$ and an $\eta$ such that $\eta^{\varepsilon_n} \to \eta$ in $C([0, T_0]; H^{m-1})$ and $\eta^{\varepsilon_n} \to \eta$ in $C([0, T_0]; H^{m-2})$ as $\varepsilon_n \to 0^+$. The convergence allows us to pass the limit in (1.15) and obtain $(\eta, v)$ solving the elastodynamic equation (1.16). Moreover, by applying the lower semi-continuity of norms to bounds in Theorem 1.1, we can see that $(\eta, v)$ satisfies the following uniform regularity of

$$
\|\eta\|_{L^\infty(0, T_0; H^m(\Omega))} + \|\nabla \eta\|_{L^\infty(0, T_0; H^{m-1}(\Omega))} + \|\partial_t \eta \cdot n\|_{L^\infty(0, T_0; L^2(\Gamma))} + \|\nabla \eta\|_{L^2(0, T_0; H^{m-1}(\Omega))}$$
$$+ \|\partial_t \eta \cdot n\|_{L^4(0, T_0; L^2(\Gamma))} + \|\partial_t \eta \cdot n\|_{L^4(0, T_0; L^2(\Gamma))} + \|\partial_t \eta \cdot n\|_{L^4(0, T_0; L^2(\Gamma))} \leq C.
$$

By the standard energy method, it is not difficult to prove the uniqueness of the classical solution $(\eta, v)$ to (1.16) with the above regularity. Therefore, we infer that the whole families $(\eta^\varepsilon, v^\varepsilon)$ converge to $(\eta, v)$.

Acknowledgements Xumin Gu was supported by National Natural Science Foundation of China (Grant No. 12031006) and the Shanghai Frontier Research Center of Modern Analysis. Yu Mei was supported by National Natural Science Foundation of China (Grant No. 12101496) and the Fundamental Research Funds for the Central Universities (Grant No. G2021KY05101). The authors thank Professor Zhen Lei for many stimulating discussions.

References

1 Cai Y, Lei Z, Lin F H, et al. Vanishing viscosity limit for incompressible viscoelasticity in two dimensions. Comm Pure Appl Math, 2019, 72: 2063-2120
2 Chen Y M, Zhang P. The global existence of small solutions to the incompressible viscoelastic fluid system in 2 and 3 space dimensions. Comm Partial Differential Equations, 2016, 31: 1793-1810
3 Ciampa G, Crippa G, Spirito S. Strong convergence of the vorticity for the 2D Euler equations in the inviscid limit. Arch Ration Mech Anal, 2021, 240: 295-326
4 Clopeau T, Mikelic A, Robert R. On the vanishing viscosity limit for the 2D incompressible Navier-Stokes equations with the friction type boundary conditions. Nonlinearity, 1998, 11: 1625-1636
5 Constantin P, Drivas T D, Elgindi T. Inviscid limit of vorticity distributions in the Yudovich class. Comm Pure Appl Math, 2022, 75: 65–82
6 Constantin P, Wu J H. Inviscid limit for vortex patches. Nonlinearity, 1995, 8: 735–742
7 Di Iorio E, Marcati P, Spirito S. Splash singularities for a 2D Oldroyd-B model with nonlinear Piola-Kirchhoff stress. NoDEA Nonlinear Differential Equations, 2017, 24: 60
8 Di Iorio E, Marcati P, Spirito S. Splash singularity for a free-boundary incompressible viscoelastic fluid model. Adv Math, 2020, 368: 107124
9 Di Iorio E, Marcati P, Spirito S. Splash singularities for a general Oldroyd model with finite Weissenberg number. Arch Ration Mech Anal, 2020, 235: 1589–1660
10 Di Nezza E, Palatucci G, Valdinoci E. Hitchhiker’s guide to the fractional Sobolev spaces. Bull Sci Math, 2012, 136: 521–573
11 Elgindi T, Lee D. Uniform regularity for free-boundary Navier-Stokes equations with surface tension. J Hyperbolic Differ Equ, 2018, 15: 37–118
12 Fei M W, Tao T, Zhang Z F. On the zero-viscosity limit of the Navier-Stokes equations in $\mathbb{R}^3$ without analyticity. J Math Pures Appl (9), 2018, 112: 170–229
13 Filho M C L, Lopes H J N, Planas G. On the inviscid limit for two-dimensional incompressible flow with Navier friction condition. SIAM J Math Anal, 2005, 36: 1130–1141
14 Gallay T. Interaction of vortices in weakly viscous planar flows. Arch Ration Mech Anal, 2011, 200: 445–490
15 Gu X M, Lei Z. Local well-posedness of free-boundary incompressible elastodynamics with surface tension via vanishing viscosity limit. Arch Ration Mech Anal, 2022, 245: 1285–1338
16 Gu X M, Wang F. Well-posedness of the free boundary problem in incompressible elastodynamics under the mixed type stability condition. J Math Anal Appl, 2020, 482: 123529
17 Hao C C, Wang D H. A priori estimates for the free boundary problem of incompressible neo-Hookean elastodynamics. J Differential Equations, 2016, 261: 712–737
18 Hu X P, Huang Y T. Well-posedness of the free boundary problem for incompressible elastodynamics. J Differential Equations, 2019, 266: 7844–7889
19 Iftimie D, Planas G. Inviscid limits for the Navier-Stokes equations with Navier friction boundary conditions. Nonlinearity, 2006, 19: 899–918
20 Iftimie D, Sueur F. Viscous boundary layers for the Navier-Stokes equations with the Navier slip conditions. Arch Ration Mech Anal, 2011, 199: 145–175
21 Kato T. Nonstationary flows of viscous and ideal fluids in $\mathbb{R}^3$. J Funct Anal, 1972, 9: 296–305
22 Le Meur H V J. Well-posedness of surface wave equations above a viscoelastic fluid. J Math Fluid Mech, 2011, 13: 481–514
23 Lei Z. Global well-posedness of incompressible elastodynamics in two dimensions. Comm Pure Appl Math, 2016, 69: 2072–2106
24 Lei Z, Liu C, Zhou Y. Global solutions for incompressible viscoelastic fluids. Arch Ration Mech Anal, 2008, 188: 371–398
25 Li H, Wang W, Zhang Z F. Well-posedness of the free boundary problem in incompressible elastodynamics. J Differential Equations, 2019, 267: 6604–6643
26 Lin F H, Liu C, Zhang P. On hydrodynamics of viscoelastic fluids. Comm Pure Appl Math, 2005, 58: 1437–1471
27 Lin F H, Zhang P. On the initial-boundary value problem of the incompressible viscoelastic fluid system. Comm Pure Appl Math, 2008, 61: 539–558
28 Maekawa Y. On the inviscid limit problem of the vorticity equations for viscous incompressible flows in the half-plane. Comm Pure Appl Math, 2014, 67: 1045–1128
29 Masmoudi N, Rousset F. Uniform regularity for the Navier-Stokes equation with Navier boundary condition. Arch Ration Mech Anal, 2012, 203: 529–575
30 Masmoudi N, Rousset F. Uniform regularity and vanishing viscosity limit for the free surface Navier-Stokes equations. Arch Ration Mech Anal, 2017, 223: 301–417
31 McGrath F J. Nonstationary plane flow of viscous and ideal fluids. Arch Ration Mech Anal, 1968, 27: 329–348
32 Mei Y, Wang Y, Xin Z P. Uniform regularity for the free surface compressible Navier-Stokes equations with or without surface tension. Math Models Methods Appl Sci, 2018, 28: 259–336
33 Nguyen T T, Nguyen T T. The inviscid limit of Navier-Stokes equations for analytic data on the half-space. Arch Ration Mech Anal, 2018, 230: 1103–1129
34 Paddick M. The strong inviscid limit of the isentropic compressible Navier-Stokes equations with Navier boundary conditions. Discrete Contin Dyn Syst, 2016, 36: 2673–2709
35 Sammartino M, Caflisch R E. Zero viscosity limit for analytic solutions of the Navier-Stokes equation on a half-space. I. Existence for Euler and Prandtl equations. Comm Math Phys, 1998, 192: 433–461
36 Sammartino M, Caflisch R E. Zero viscosity limit for analytic solutions of the Navier-Stokes equation on a half-space. II. Construction of the Navier-Stokes solution. Comm Math Phys, 1998, 192: 463–491
37 Sideris T C, Thomases B. Global existence for three-dimensional incompressible isotropic elastodynamics via the incompressible limit. Comm Pure Appl Math, 2005, 58: 750–788
38 Sideris T C, Thomases B. Global existence for three-dimensional incompressible isotropic elastodynamics. Comm Pure Appl Math, 2007, 60: 1707–1730
39 Simon J. Compact sets in the space $L^p(0,T;B)$. Ann Mat Pura Appl (4), 1986, 146: 65–96
40 Sueur F. On the inviscid limit for the compressible Navier-Stokes system in an impermeable bounded domain. J Math Fluid Mech, 2014, 16: 163–178
41 Swann H S G. The convergence with vanishing viscosity of nonstationary Navier-Stokes flow to ideal flow in $\mathbb{R}_3$. Trans Amer Math Soc, 1971, 157: 373–397
42 Trakhinin Y. Well-posedness of the free boundary problem in compressible elastodynamics. J Differential Equations, 2018, 264: 1661–1715
43 Wang C, Wang Y X, Zhang Z F. Zero-viscosity limit of the Navier-Stokes equations in the analytic setting. Arch Ration Mech Anal, 2017, 224: 555–595
44 Wang D H, Xie F. Inviscid limit of compressible viscoelastic equations with the no-slip boundary condition. arXiv:
45 Wang Y. Uniform regularity and vanishing dissipation limit for the full compressible Navier-Stokes system in three dimensional bounded domain. Arch Ration Mech Anal, 2016, 221: 1345–1415
46 Wang Y, Xin Z P, Yong Y. Uniform regularity and vanishing viscosity limit for the compressible Navier-Stokes with general Navier-slip boundary conditions in three-dimensional domains. SIAM J Math Anal, 2015, 47: 4123–4191
47 Wang Y-G, Williams M. The inviscid limit and stability of characteristic boundary layers for the compressible Navier-Stokes equations with Navier-friction boundary conditions. Ann Inst Fourier (Grenoble), 2012, 62: 2257–2314
48 Wang Y-G, Xin Z P. Zero-viscosity limit of the linearized compressible Navier-Stokes equations with highly oscillatory forces in the half-plane. SIAM J Math Anal, 2005, 37: 1256–1298
49 Wang Y J, Xin Z P. Vanishing viscosity and surface tension limits of incompressible viscous surface waves. SIAM J Math Anal, 2021, 53: 574–648
50 Xiao Y L, Xin Z P. On the vanishing viscosity limit for the 3D Navier-Stokes equations with a slip boundary condition. Comm Pure Appl Math, 2007, 60: 1027–1055
51 Xin Z P, Yanagisawa T. Zero-viscosity limit of the linearized Navier-Stokes equations for a compressible viscous fluid in the half-plane. Comm Pure Appl Math, 1999, 52: 479–541
52 Xu L, Zhang P, Zhang Z F. Global solvability of a free boundary three-dimensional incompressible viscoelastic fluid system with surface tension. Arch Ration Mech Anal, 2013, 208: 753–803
53 Zhang J Y. Local well-posedness and incompressible limit of the free-boundary problem in compressible elastodynamics. Arch Ration Mech Anal, 2022, 244: 599–697