WEAKLY ASYMMETRIC NON-SIMPLE EXCLUSION PROCESS AND THE KARDAR-PARISI-ZHANG EQUATION

∗AMIR DEMBO AND †LI-CHENG TSAI

Abstract. We analyze a class of non-simple exclusion processes and the corresponding growth models by generalizing Gärtner’s Cole-Hopf transformation. We identify the main non-linearity and eliminate it by imposing a gradient type condition. For hopping range at most 3, using the generalized transformation, we prove the convergence of the exclusion process toward the Kardar-Parisi-Zhang (KPZ) equation. This is the first universality result concerning interacting particle systems in the context of KPZ universality class. While this class of exclusion processes are not explicitly solvable, we obtain the exact one-point limiting distribution for the step initial condition by using the previous result of [1] and our convergence result.

1. Introduction

There is much interest in randomly growing interfaces, whose macroscopic behavior is usually described by non-linear partial differential equations. To describe the random growth mechanism, a noise term is introduced to the equation. A paradigm of such modeling equations is the KPZ equation [13], which in one space dimension is

\[ \partial_T H = \frac{1}{2} \partial^2_X H - \frac{1}{2} (\partial_X H)^2 + W, \quad H = H_T(X), \]

where \( W \) is the space-time white noise: \( \mathbb{E}(W_T(X)W_S(X')) = \delta(T-S)\delta(X-X') \). Employing the non-rigorous renormalization argument of [7], Kardar, Parisi, and Zhang [13] showed that the scaling exponents of the fluctuation of \( H \), space \( X \), and time \( T \) follow a 1 : 2 : 3 ratio, which signifies a new universality class—the KPZ universality class. This universality class describes various phenomena including paper wetting, crack formation, and burning fronts. See [6, Section 1.1.2] and the references therein. Moreover, it connects various models describing other phenomenon including directed last passage percolation, directed polymer in a random media, and polynuclear growth. Recently, there has been intensive mathematical research on instances of explicitly solvable models in this universality class [4, 5, 11, 12, 17]. See also [6] and the references therein. They all confirm the 1 : 2 : 3 scaling exponents and have limiting one-point statistics related to random matrix theory, for example the GUE or GOE Tracy-Widom distribution. Exact one-point distribution of the KPZ equation starting from the narrow wedge initial condition is proven by [1], and independently obtained by [19]. This distribution converges to the GUE Tracy-Widom distribution as \( T \to \infty \) under the 1 : 2 : 3 scaling.

The quadratic term in (1.1) plagues rigorous mathematical treatment. Indeed, generic solutions to stochastic differential equations driven by the white noise exhibit non-differentiability. Hence, \((\partial_X H)^2\) alone does not have a mathematical meaning. Rather, the correct mathematical interpretation is the Cole-Hopf transformation to the stochastic heat equation (SHE)

\[ H_T(X) = -\log Z_T(X), \]
Figure 1. Non-simple exclusion process.

(1.3) \[ \partial_T Z = \frac{1}{2} \partial^2_X Z + Z W. \]

Since the SHE is linear, traditional stochastic calculus applies. Further, formally $HT(X)$ of (1.2) satisfies (1.1), so we define (1.1) by (1.2) and (1.3). This Hope-Cole transformation dates back to [10, 13]. A more comprehensive approach of defining (1.1) can be found in [9].

As we shall see, the Cole-Hopf solution of (1.1) also captures the asymptotic fluctuations of certain exclusion processes. Specifically, let $r_1, \ldots, r_m$ be parameters satisfying

(1.4) \[ r_k > 0, \quad r_1 + \ldots + r_m = 1. \]

For $k = 1, \ldots, m$, let $\gamma^k \in \mathbb{R}$, and

(1.5) \[ q_k^+ = \frac{1}{2} r_k \left( 1 - \gamma_k \varepsilon \right), \quad q_k^- = \frac{1}{2} r_k \left( 1 + \gamma_k \varepsilon \right). \]

Consider the finite range exclusion process with hopping rate $\{q_k^+\}_{k=\pm 1, \ldots, \pm m}$. That is, particles on the half integer lattice $\mathbb{Z} := \{ \frac{1}{2} \} + \mathbb{Z}$ wait an exponential time with parameter one and then attempt to hop $k$ units with probability $q_k^+$, under the constraint that a hop is prohibited when the destination site is already occupied. The case $m = 1$ is called the simple exclusion process. Let $\eta_t(y)$ be the occupation variable in spin convention at position $y \in \mathbb{Z}$ and time $t \in [0, \infty)$

(1.6) \[ \eta_t(y) := \begin{cases} 1, & \text{when the site } y \text{ is occupied at time } t, \\ -1, & \text{otherwise}. \end{cases} \]

We associate a height function $h$ by doing discrete integration of $\eta$, that is

(1.7) \[ h_t(x) = -N(t) + \begin{cases} \sum_{0 < y < x} \eta_t(y), & \text{when } x \geq 0, \\ -\sum_{x < y < 0} \eta_t(y), & \text{when } x < 0, \end{cases} \]

where $N(t)$ is the net flow of particles through $x = 0$ during the time interval $[0, t]$, counting right moving particles as positive. The height $h$ represents the interface growing under the following dynamics: when a particle hops $k$ unit to the left(right), it adds(removes) a box at each of the $k$ sites along the way, see Figure 1.

Fix any $\lambda > 0$. Define

(1.8) \[ Z^\varepsilon_t(x) := \exp \left( -\lambda \varepsilon^2 h_t(x) + \varepsilon \nu_t \right). \]
where $\nu_\varepsilon$ is defined later in (1.13). For simple ($m = 1$) exclusion, Gärtner [8] shows that for
\begin{equation}
(1.9) \quad r_1 = 1, \quad \gamma_1^\varepsilon = \frac{\varepsilon^{2\lambda_1/2} - 1}{\varepsilon^{1/2} (\varepsilon^{2\lambda_1/2} + 1)} = \lambda + \mathcal{O}(\varepsilon), \quad \nu_\varepsilon = \varepsilon^{-1} \left(1 - (4q_1^2q_1^0)^{1/2}\right),
\end{equation}
the drift term in the microscopic dynamical equation of $Z^\varepsilon_t$ is a discrete Laplacian, that is, $Z^\varepsilon_t$ satisfies a discrete stochastic heat equation (dSHE). Hereafter $\mathcal{O}(\varepsilon)$ stands for a generic function satisfying $\sup_{x \in (0,1)} |\mathcal{O}(\varepsilon)| \varepsilon^{-1} < \infty$.

**Remark 1.1.** Actually, in [8] $\gamma_1^\varepsilon$ is fixed and $\lambda$ is chosen as $\lambda = \frac{1}{2} \log \frac{q_0}{q_1}$. This is equivalent to the middle equation of (1.9). Here we choose to fix $\lambda$ and vary $\gamma_1^\varepsilon$ since it is more convenient for notations in our case. Straightforward calculation shows that when $m = 1$ the values of $\nu_\varepsilon$ defined by (1.13) and (1.9) differ by $\mathcal{O}(\varepsilon^3)$, and at the time scaling $t = \varepsilon^{-2} T$ this difference of $\mathcal{O}(\varepsilon^3)$ is negligible for the asymptotic behavior.

By choosing the parameters according to (1.9), [8] linearizes the drift term by matching three non-degenerated identities. Going beyond simple exclusion ($m > 1$), we encounter $2^m - 1 > 3$ identities, so this type of reasoning fails. Nevertheless, when $m > 1$ we show that for any given $r_k > 0$, $k = 1, \ldots, m$, any choice of
\begin{equation}
(1.10) \quad \gamma_k^\varepsilon - \lambda \left(\frac{2}{r_k} \sum_{k' = k+1}^m \frac{k' - k}{k} r_{k'} + 1\right) = \mathcal{O}(\varepsilon)
\end{equation}
eliminates the major non-linearity in drift of the microscopic dynamics, and the dynamical equation becomes approximately a dSHE. Since $r_k > 0$, $k = 1, \ldots, m$, are arbitrary, we have retained $m$ degrees of freedom in choosing parameters out of the $2m$ parameters $\{q_k\}$. To be more specific, we choose $\gamma_k^\varepsilon$ and $r_k$ in such a way that reduces the difference between the microscopic drift and
\begin{equation}
(1.11) \quad \Delta_t := 2^{-1} \sum_{k=1}^m \gamma_k^\varepsilon \left(\Delta_k Z^\varepsilon_t\right)(x),
\end{equation}
where $\Delta_t f(x) := f(x + k) + f(x - k) - 2 f(x)$. By (1.8) we thus have
\begin{equation}
(1.12) \quad \frac{\Delta_t}{Z^\varepsilon_t(x)} = \frac{1}{2} \sum_{k=1}^m \gamma_k^\varepsilon \left[\exp \left(- \lambda \varepsilon^{1/2} \sum_{y \in (x, x+k)} \eta(y)\right) + \exp \left(\lambda \varepsilon^{1/2} \sum_{y \in (x-k, x)} \eta(y)\right) - 2\right].
\end{equation}
We Taylor-expand (1.12), and match the $\eta$-linear and $\eta$-nonlinear terms with the $\eta$-linear and $\eta$-nonlinear terms in the microscopic drift. The matching of $\eta$-linear and $\eta$-quadratic terms require that for $j = 1, \ldots, m$,
\begin{equation}
(1.13) \quad \varepsilon^{\frac{1}{2}} \left(A^\varepsilon v^\varepsilon\right)_j = \varepsilon^{\frac{1}{2}} 4^{-1} \sum_{k=j}^m r_k (u(\varepsilon) + \varepsilon \gamma_k^\varepsilon v(\varepsilon)) + \mathcal{O}(\varepsilon^2),
\end{equation}
\begin{equation}
(1.14) \quad \varepsilon (B^\varepsilon v^\varepsilon)_j = \varepsilon 4^{-1} r_j (\gamma_j^\varepsilon u(\varepsilon) - v(\varepsilon)) + \mathcal{O}(\varepsilon^2),
\end{equation}
where $A^\varepsilon$ and $B^\varepsilon$ are the $m$-dimensional square matrices
\begin{equation}
(1.15) \quad A^\varepsilon_{jk} := \frac{\lambda}{2} + \varepsilon \lambda^\varepsilon \left(\frac{3k - 2 - j - 1}{12}\right),
\end{equation}
\begin{equation}
(1.16) \quad B^\varepsilon_{jk} := \lambda^2 (k - j) + j^{-1},
\end{equation}
and $u(\varepsilon)$ and $v(\varepsilon)$ are the analytic functions
\begin{equation}
(1.17) \quad u(\varepsilon) := \varepsilon^{-\frac{1}{2}} \sinh(2\lambda \varepsilon^{\frac{1}{2}}), \quad v(\varepsilon) := \varepsilon^{-1} (\cosh(2\lambda \varepsilon^{\frac{1}{2}}) - 1),
\end{equation}
\begin{equation}
(1.18) \quad u(0) := \lim_{\varepsilon \to 0} u(\varepsilon) = 2\lambda, \quad v(0) := \lim_{\varepsilon \to 0} v(\varepsilon) = 2\lambda^2.
\end{equation}
We prove in Appendix A the following lemma
Lemma 1.2. There exists some explicit $r_k^* \in \mathbb{R}$ such that any $\gamma^\varepsilon$ of the form (1.10) and any $\gamma^\varepsilon$ of the form (1.11) satisfy the equation (1.13).

Then, by (1.10) and (1.17), we have

\begin{equation}
\nu_\varepsilon := \sum_{k=1}^m 4^{-1} k r_k^*(\gamma^\varepsilon \mu(\varepsilon) - \nu(\varepsilon)) + \lambda^2 \sum_{k=1}^m r_k^* [k + \varepsilon (12)^{-1}(6k^2 - 5k)^2].
\end{equation}

Then, by (1.10) and (1.17), we have $\nu_\varepsilon = 2^{-1} \lambda^2 \sum_{k=1}^m r_k k^2 + O(\varepsilon)$.

Lemma 1.2. There exists some explicit $r_k^* \in \mathbb{R}$ such that any $\gamma^\varepsilon$ of the form (1.10) and any $\gamma^\varepsilon$ of the form (1.11) satisfy the equation (1.13).

The constant $\nu_\varepsilon$ in (1.8) is used to balance the constant coming from the microscopic dynamical equation and (1.11). It turns out to be

\begin{equation}
\nu_\varepsilon := \sum_{k=1}^m 4^{-1} kr_k (\gamma_k^\varepsilon u(\varepsilon) - \nu(\varepsilon)) + \lambda^2 \sum_{k=1}^m r_k^* [k + \varepsilon (12)^{-1}(6k^2 - 5k)^2].
\end{equation}

Then, by (1.1.10) and (1.1.17), we have $\nu_\varepsilon = 2^{-1} \lambda^2 \sum_{k=1}^m r_k k^2 + O(\varepsilon)$.

Lemma 1.2. There exists some explicit $r_k^* \in \mathbb{R}$ such that any $\gamma^\varepsilon$ of the form (1.10) and any $\gamma^\varepsilon$ of the form (1.11) satisfy the equation (1.13).

We say a process $Z_0(\cdot)$ is a mild solution to the SHE if

\begin{equation}
Z_1(X) = \int_\mathbb{R} P_T(X - X') Z_0(X') dX' + \int_0^T \int_\mathbb{R} P_{T-S}(X-X') Z_S(X') W(dX' dS),
\end{equation}

where $P_T(X) := (2\pi T)^{-1/2} \exp(-X^2/2T)$ is the heat kernel, and the stochastic integral is in the Itô sense. For the existence, uniqueness, continuity, and positivity of solutions to (1.19), see [6 Proposition 2.5] and [2, 3, 16, 24].

Let $\alpha$ denote the diffusivity of the symmetric part of the hopping rates, that is

\begin{equation}
\alpha := \sum_{k=1}^m k^2 r_k,
\end{equation}

and extend $Z_1(x)$ to $x \in \mathbb{R}$ by a linear interpolation. Consider the scaled field

\begin{equation}
Z_1^\varepsilon(X) := Z_1^\varepsilon(\varepsilon^{-1} \beta X),
\end{equation}

where

\begin{equation}
\beta := \alpha^{-1} \lambda^{-1}, \quad \beta' := \lambda^{-2}.
\end{equation}

Let $\|f_t(x)||_1 := [E(|f_t(x)|^l)]^{1/l}$ denote the $L^1$-norm over randomness, $l \geq 1$. Our main result is

**Theorem 1.3.** Let $Z_T(X)$ be the mild solution to the SHE starting from a $C(\mathbb{R})$-valued process $Z_0(X)$. Suppose $Z_0^\varepsilon(\cdot)$ weakly converges to $Z_0(\cdot)$, the hopping range $m$ is at most 3, and for any $u \in (0, 1/2)$ there exist finite $C = C(u)$ and $\alpha_0 = \alpha_0(u)$ such that

\begin{align}
(1.23a) & \quad \|Z_0^\varepsilon(x)\|_1 \leq e^{\alpha_0 \varepsilon|x|x} C, \\
(1.23b) & \quad \|Z_0^\varepsilon(x) - Z_0^\varepsilon(x')\|_1 \leq (\varepsilon|x-x'|)^{u} e^{\alpha_0 \varepsilon|x|x' + |x'|} C.
\end{align}

Then $Z_1^\varepsilon(\cdot)$ weakly converges (under the Skorokhod topology of $D([0, \infty), C(\mathbb{R}))$) to $Z(\cdot)$.

Utilizing Gärtner’s transformation (1.9), Bertini and Giacomin [3] proved Theorem 1.3 for the special case of the simple exclusion process. In contrast with the simple exclusion process, non-simple exclusion processes are not explicitly solvable. Nevertheless, by the uniqueness of the SHE, the class of exclusion processes we consider all share the same scaling exponent and limiting distribution. That is, we prove the first universality result concerning the fluctuation of interacting particle systems in the context of KPZ universality class.

The one point distribution of the solution $Z_T(X)$ to (1.19) for the delta initial condition $Z_0(\cdot) = \delta_0(\cdot)$ has been derived and proven by Amir-Corwin-Quastel [1], based on previous work of Tracy and Widom [22] (see also [20, 21, 23]). Specifically, put $\mathcal{F}(T, X) = \log Z_T(X) - \log P_T(X)$. The law of $\mathcal{F}(T, X)$ is given as in [1, Theorem 1.1]. By using Theorem 1.3 in a way similar to [1] and using the uniqueness of (1.19), we obtain
Theorem 1.4. Consider, in terms of occupation variable \( \eta \), the step initial condition
\[
\eta_0(y) = \mathbbm{1}_{(0, \infty)}(y) - \mathbbm{1}_{(-\infty, 0)}(y).
\]
For each fixed \((T, X) \in (0, \infty) \times \mathbb{R}\),
\[
F^\varepsilon_T(X) := -\lambda h_{\beta \varepsilon^{-2} T}(\beta' \varepsilon^{-1} X) + \nu \varepsilon^{-1} \beta T + \log \frac{\lambda \beta'}{2 \varepsilon^2} - \log P_T(X)
\]
converges in distribution to \( \mathcal{F}(T, X) \) as \( \varepsilon \to 0 \).

While there is no known exact formula for correlation functions or moments for non-simple exclusion processes, Theorem 1.3 implies exact limiting statistics for the step initial condition by providing convergence toward the SHE. In the context of KPZ universality Theorem 1.4 is the first exact limiting statistics result derived out of the limiting stochastic partial differential equation, and not by explicitly solving the model in question.

Since we are at the fluctuation scale, where the time span is of \( O(\varepsilon^{-2}) \), the nonlinearity of the microscopic drift causes much roughness, and the contribution of \( \eta \)-quadratic and \( \eta \)-cubic terms in the drift are not even uniformly (in \( \varepsilon \)) bounded. While for \( m = 1 \) the choice \((1.10)\) of parameters eliminates all nonlinear terms in the drift, for \( m > 1 \) we need to match the nonlinear terms in the drift with the nonlinear term in \((1.11)\) slightly better than \( O(\varepsilon^2) \). We achieve this goal by translating the \( \eta \)-quadratic and \( \eta \)-cubic terms. This matching by translation causes error terms of gradient type, which after being averaged over large space is much more controllable. The assumption \( m \leq 3 \) in Theorem 1.3 is used to ensure an \( \eta \)-cubic term is of gradient type, see Remark 2.4.

Remark 1.5. While our argument of converting terms into gradient terms requires the assumptions \((1.10)\) and \( m \leq 3 \), we predict that Theorem 1.3 with the scaling constants \((1.21)\) and \((1.22)\) holds even without these assumptions.

After we match the \( \eta \)-quadratic and \( \eta \)-cubic terms, there remains error terms (whose contributions are uniformly bounded) in the drift and in the quadratic variation of the martingale. For \( m = 1 \), [3] uses an ergodic type analysis to control the quadratic variation of the martingale, which applies only to a specific \( \eta \)-quadratic error term. Since these error terms are uniformly bounded, as sketched in [18, Proposition 3.28] (for the simple exclusion process on the torus \( \mathbb{Z}/N\mathbb{Z} \)), one expects a more general approach using replacement lemmas via the standard one-block and two-blocks estimates.

We give a complete proof in Section 4 adopting this general approach. Due to the boundary effect of entropy production on \( \mathbb{Z} \), we obtain a weaker bound of the Dirichlet form and our replacement lemma differs from the usual one, see Remark 2.4. The one-block and two-blocks estimates for the simple exclusion process on \( \mathbb{Z}/\mathbb{Z} \) has been done in [15].

We next outline this paper. In Section 2 we examine the microscopic dynamics of \( Z_i^\varepsilon \). Under the assumptions \((1.10)\) and \( m \leq 3 \) we obtain an approximated DSHE. In Sections 3 we prove Theorem 1.3 by establishing the following two propositions, which immediately imply Theorem 1.3.

Proposition 1.6. Assume that the initial condition satisfies \((1.23a)\) and \((1.23b)\), then the laws of \( \{Z^\varepsilon\}_{\varepsilon} \) on \( D([0, \infty), C(\mathbb{R})) \) is tight. Moreover, limit points of the law of \( \{Z^\varepsilon\}_{\varepsilon} \) concentrate on \( C([0, \infty), C(\mathbb{R})) \).

Proposition 1.7. The law of any limit point \( Z \) of \( \{Z^\varepsilon\}_{\varepsilon} \) is a solution to \((1.19)\) for the initial condition \( Z_0 \) as defined in Theorem 1.3.

Proposition 1.6 is proven by applying the Hölder continuity of the microscopic heat kernel to the DSHE, and Proposition 1.7 is proven by applying the martingale problem method. The key to the proof of Proposition 1.7 is Lemma 2.5 which assures certain fluctuation fields weakly converge to zero at the hydrodynamical scale. Lemma 2.5 is proven by standard hydrodynamic-limit type analysis using the one-block and two-blocks estimates, which we do in Section 4. In section 5 we establish the small-time estimates Lemma 5.1 for the step initial condition \((1.24)\), which allows us to apply Theorem 1.3 to conclude Theorem 1.4 for the step initial condition.
Aside from these contributions, there is a continuous growth of \( Z \) following infinite (2.5) and \( dZ \) (2.4), where the height function is defined, and use \( y, y', y_1 \) to denote points of \( \mathbb{L} = \mathbb{Z} + \frac{1}{2} \), where the particles are. For positive integers \( i, j, k \in \mathbb{N} \), we adopt the notation \( i := i - \frac{1}{2}, j := j - \frac{1}{2}, \) and \( k := k - \frac{1}{2} \).

We first derive the governing stochastic differential equation of \( Z^\varepsilon_t \). Let

\[
Q^\varepsilon_t(y) : k = \pm 1, \ldots, \pm m, y \in \mathbb{L}
\]

be mutually independent (in \( k \) and \( y \)) Poisson processes in \( t \), with each \( Q^\varepsilon_t(y) \) having rate \( q^\varepsilon_k \). Each \( Q^\varepsilon_t(y) \) counts the total number of attempted hops from \( y \) to \( y + k \) within the time interval \( [0, t] \). Put \( \mathcal{F}_t := \sigma \{ Q^\varepsilon_t(y) : s \in [0, t], y \in \mathbb{L}, |k| \leq m \} \) and let \( a_{y_1 \rightarrow y_2} \) be the indicator of allowed hops from \( y_1 \) to \( y_2 \), that is

\[
a_{y_1 \rightarrow y_2}(\eta) := \frac{1 + \eta(y_1) - \eta(y_2)}{2} \in \{0, 1\}.
\]

From the dynamics of \( h \) described in the introduction, we know that the height at \( x \) increases by two (a block) when a particle hops across \( x \) from left to right. By the definition (1.8) of \( Z^\varepsilon_t \), during \([t, t + dt]\) the total contribution to \( dZ^\varepsilon_t(x) \) of hops to the right is

\[
\left( e^{2\lambda \varepsilon^{1/2}} - 1 \right) Z^\varepsilon_t(x) \sum_{k=1}^{m} \sum_{x-k<y<x} a_{y-y+k}(\eta) dQ^\varepsilon_t(y).
\]

Similarly, the contribution of hops to the left to \( dZ^\varepsilon_t(x) \) is

\[
\left( e^{-2\lambda \varepsilon^{1/2}} - 1 \right) Z^\varepsilon_t(x) \sum_{k=1}^{m} \sum_{x<y<x+k} a_{y-y-k}(\eta) dQ^{-\varepsilon}_t(y).
\]

Aside from these contributions, there is a continuous growth of \( Z^\varepsilon_t \) due to \( \exp(\nu_c t) \). Gathering the preceding contributions together, and separating drifts and martingale in each \( dQ^\varepsilon_t \), we obtain the following infinite (\( x \in \mathbb{Z} \)) system of stochastic differential equations

\[
dZ^\varepsilon_t = (\Omega^\varepsilon + \varepsilon \nu_c) Z^\varepsilon_t dt + Z^\varepsilon_t dM^\varepsilon_t,
\]

where \( M^\varepsilon_t(x) \) is a martingale in \( t \) for each \( x \in \mathbb{Z} \), given by

\[
dM^\varepsilon_t(x) = \left( e^{2\lambda \varepsilon^{1/2}} - 1 \right) \sum_{k=1}^{m} \sum_{x-k<y<x} a_{y-y+k}(\eta) (dQ^\varepsilon_t(y) - q^\varepsilon_k dt)
\]

(2.5)

\[
+ \left( e^{-2\lambda \varepsilon^{1/2}} - 1 \right) \sum_{k=1}^{m} \sum_{x<y<x+k} a_{y-y-k}(\eta) (dQ^{-\varepsilon}_t(y) - q^{-\varepsilon}_k dt),
\]

and

\[
\Omega^\varepsilon_t(x) = \left( e^{2\lambda \varepsilon^{1/2}} - 1 \right) \sum_{k=1}^{m} q^\varepsilon_k \sum_{x-k<y<x} a_{y-y+k}(\eta)
\]

(2.6)

\[
+ \left( e^{-2\lambda \varepsilon^{1/2}} - 1 \right) \sum_{k=1}^{m} q^{-\varepsilon}_k \sum_{x<y<x+k} a_{y-y-k}(\eta).\]
For the rest of this section we focus on the drift term $\Omega^\varepsilon + \varepsilon \nu_\varepsilon$. When $m = 1$ under the choice of $G$, the drift is merely a discrete Laplacian

$$\Omega^\varepsilon + \varepsilon \nu_\varepsilon = 2^{-1} \tilde{r}_1^2 \Delta_1 \tilde{Z}^\varepsilon,$$

where $\tilde{r}_1 = r_1(1 - \varepsilon (\gamma_1^2)^2)^{\frac{1}{2}}$. For $m > 1$, we aim at linearizing the drift term, to which end we expand the difference $(\Omega^\varepsilon + \varepsilon \nu_\varepsilon) - \Delta$ in $\varepsilon$, where $\Delta$ is defined as in (1.11), and choose $\gamma_k^2$ and $\tilde{r}_k^2$ as (1.10) and (1.17) to reduce this difference.

Combining (2.3) and (2.0), we group terms into $\eta$-linear terms, $\eta$-quadratic terms, and constants. The $\eta$-linear terms always appear in a symmetric form:

$$E^\eta_t(x) := \eta_t(x - \tilde{j}) - \eta_t(x + \tilde{j}).$$

Similarly, fixing $k$ and summing over $y \in (x - k, x)$ or $y \in (x, x + k)$ in (2.3), all $\eta$-quadratic terms we get are of the form

$$Q_t^k(x) := \sum_{y_1 < x < y_2 \atop y_2 - y_1 = k} \eta_t(y_1) \eta_t(y_2).$$

Note that $-Q_t^k(x)$ counts the number of all possible hops of distant $k$ across $x$. After summing over $k$, we get from (2.6) that

$$\Omega^\varepsilon + \varepsilon \nu_\varepsilon = \varepsilon^2 \Omega^{\text{lin}}_t + \varepsilon \Omega^{\text{qd}}_t + (\nu_\varepsilon - \nu'_\varepsilon)\varepsilon,$$

where

$$\Omega^{\text{lin}}_t(x) := \sum_{j,k=1}^m \mathbb{1}_{j \leq k} \rho^j_k E_t^j(x), \quad \Omega^{\text{qd}}_t(x) := \sum_{k=1}^m \sigma^k E_t^k(x), \quad \nu'_\varepsilon := \sum_{k=1}^m k \sigma^k.$$

and using (1.5) we have

$$\rho^j_k := 4^{-1} \varepsilon^{-1/2} (\varepsilon^{2\lambda e^{1/2}} - 1) q^j_k + (1 - \varepsilon^{-2\lambda e^{1/2}}) q^j_k \varepsilon, \quad \sigma^k := 4^{-1} \varepsilon^{-1} (\varepsilon^{2\lambda e^{1/2}} - 1) q^k + (1 - \varepsilon^{-2\lambda e^{1/2}}) q^k \varepsilon.$$

for $u(\varepsilon)$ and $v(\varepsilon)$ as in (1.10).

Next we Taylor-expand (1.12). To accommodate the time evolution up to $\varepsilon^{-2\tilde{T}}$, we need to match (2.0) with (1.12) slightly better than $O(\varepsilon^2)$. Specifically, we neglect terms of the form $\varepsilon^2 \varepsilon G_t$, where $\varepsilon G_t$ is a linear combination of uniformly vanishing and weakly vanishing terms defined as following, and we also neglect terms of the form $\varepsilon G_t$, where $G_t$ is a gradient term defined as following.

**Definition 2.1.** We say an $\mathcal{F}_t$-adapted process $\varepsilon E_t(x)$ is weakly vanishing if

$$\sup_{\varepsilon \in (0,1)} \sup_{t,x} \|\varepsilon E_t(x)\|_\infty < \infty,$$

and for any compactly supported continuous $\phi$ defined on $\mathbb{R}$, any $T > 0$, $n = 1,2$,

$$\varepsilon^2 \int_0^{\varepsilon^{-2T}} \left( \varepsilon^2 \sum_{x} \phi(\varepsilon x) E_s(x) Z^\varepsilon_s(x)^n \right) ds \to 0.$$

We say an $\mathcal{F}_t$-adapted process $\varepsilon E_t(x)$ is uniformly vanishing if $\lim_{\varepsilon \to 0} \sup_{t,x} \|\varepsilon E_t(x)\|_\infty = 0.$

A process $G_t(x)$ is a gradient term if $G_t = \sum_{|k| \leq m} \nabla_k (E_t^k Z_t^k)$, where each $E_t^k$ is a linear combination of uniformly vanishing and weakly vanishing terms.

Indeed, when integrating a gradient term against a smooth test function as done in (2.14), by summation by parts we can move the discrete gradient to the test function and gain a factor of $\varepsilon$. Hence, we should think of a gradient term as carrying a factor of $\varepsilon$. 

In the sequel, we use $\mathcal{E}_t$ to denote a generic term that is a linear combination of uniformly vanishing and weakly vanishing terms, and use $\mathcal{G}_t$ to denote a generic gradient term. Our goal is to show

**Proposition 2.2.** For $m \leq 3$, under the choice of parameters,

\begin{equation}
(2.15) \quad dZ_t^\varepsilon = \frac{1}{2} \sum_{k=1}^{m} \tilde{r}_k^\varepsilon \Delta_k Z_t^\varepsilon dt + Z_t^\varepsilon dM_t^\varepsilon + (\varepsilon^2 \mathcal{E}_t + \varepsilon \mathcal{G}_t) Z_t^\varepsilon dt.
\end{equation}

To this end, we start by proving

**Proposition 2.3.** For any $\tilde{r}_k^\varepsilon \in \mathbb{R}$,

\begin{equation}
(2.16) \quad Z_t^\varepsilon \tilde{\Delta} = \left( \varepsilon^2 \sum_{j=1}^{m} (A^\varepsilon)_{ij} \mathcal{L}^\varepsilon_{ij} + \varepsilon \sum_{j=1}^{m} (B^\varepsilon)_{ij} \mathcal{Q}^\varepsilon_{ij} + \varepsilon \nu'' + \varepsilon^2 \tilde{C}_t + \varepsilon^2 \mathcal{E}_t \right) Z_t^\varepsilon + \varepsilon \mathcal{G}_t,
\end{equation}

where $A^\varepsilon$ and $B$ are defined as in and (1.14) and (1.15), and

Here $\tilde{C}_t$ denotes a generic $\eta$-cubic term of the form

\begin{equation}
(2.17) \quad \sum_{0<i<j<k \leq m} \alpha_{i,j,k}^\varepsilon \eta_i(x+i) \eta_j(x+j) \eta_k(x+k) - \eta_i(x-k) \eta_j(x-j) \eta_k(x-i),
\end{equation}

where $\alpha_{i,j,k}^\varepsilon$ are deterministic and bounded.

**Remark 2.4.** In Proposition 2.4, we will show that for $m \leq 3$, $\varepsilon^2 \tilde{C}_t$ is of the form $\varepsilon^2 (\mathcal{G}_t + \varepsilon^2 \mathcal{E}_t)$, hence negligible. While this is not true when $m > 3$, we conjecture that even then the overall contribution of $\varepsilon^2 \tilde{C}$ is still negligible in the limit as $\varepsilon \to 0$.

The following lemma, whose proof is deferred to Section 4, is needed for proving Proposition 2.3.

**Lemma 2.5.** For any fixed distinct $(y_1, \ldots, y_n) \in \mathbb{L}^l$, $n_0 = 1, \ldots, 4$, the term

\begin{equation}
(2.18) \quad \Phi_t(x) := \prod_{i=1}^{n_0} \eta_t(x+y_i)
\end{equation}

is weakly vanishing.

**Remark 2.6.** Since $\Phi_t(x) = \pm 1$, it does not vanish uniformly. Yet we expect it to vanish weakly because we at near equilibrium fluctuation. More precisely, for any $\alpha \in [0, 1]$, consider the product measure $\pi_\alpha$ on $\{ \pm 1 \}^L$ of the i.i.d. Bernoulli measures $\pi_\alpha(1) = a$, $\pi_\alpha(-1) = 1 - a$, which is an invariant measure of exclusion processes. The initial condition (1.23a) corresponds to fluctuations near $\nu_{1/2}$. Since $\nu_{1/2}(\prod_{i=1}^{n} \eta_t(x+y_i)) = 0$, we expect $\prod_{i=1}^{n} \eta_t(x+y_i)$ to be small after being averaged over a large space-time section.

**Proof of Proposition 2.3.** Taylor-expand the exponential functions in (1.12) up to the fourth order to get $\sum_{n=1}^{4} \varepsilon^n D_n + \varepsilon^4 R$, where $D_n$ is a linear combination of terms of the form $- \sum_{y \in \{x+k\}} \eta_y y^n$ and $R$ is a uniformly bounded remainder. Generically, $D_1$ consists of $\eta$-linear terms, $D_2$ consists of $\eta$-quadratic terms, $D_3$ consists of $\eta$-cubic terms, $D_4$ consists of $\eta$-quartic terms, respectively, but since $\eta_y y^n = 1$ we also get constants in $D_2$, $\eta$-linear terms in $D_3$, and $\eta$-quadratic terms in $D_4$. By Lemma 2.5, the non-constant terms of $D_4$ are weakly vanishing, and clearly $\varepsilon^4 R$ is uniformly vanishing. Hence the sum of all non-constant terms in last two terms $\varepsilon^2 (D_4 + \varepsilon^4 R)$ of the Taylor-expansion is of the type $\varepsilon^2 \mathcal{E}_t$. (We will repeatedly use this fact in this...
By Taylor-expanding the exponential function to the first order we obtain
\[
(Z_t^\eta)^{-1} \Delta = \varepsilon^+ D_{\text{lin}} + \varepsilon D_{\text{pol}} + \varepsilon^+ D_{\text{cub}} + \nu_{\varepsilon} \nabla^2 + \varepsilon^2 E,
\]
where
\[
(2.19) \quad D_{\text{lin}}^n(x) := \sum_{j,k=1}^m \left( \frac{\lambda}{2} + \varepsilon \lambda^2 \frac{3k-2}{12} \right) \mathbb{1}_{|j-k|} f_k L_i \eta_{\varepsilon}(x),
\]
\[
(2.20) \quad D_{\text{pol}}^n(x) := \frac{\lambda^2}{2} \sum_{k=1}^m \sum_{(y_1,y_2) \in T_k^+(x) \cup T_k^-(x)} \eta_{\varepsilon}(y_1) \eta_{\varepsilon}(y_2),
\]
\[
(2.21) \quad D_{\text{cub}}^n(x) := \frac{\lambda^3}{2} \sum_{k=1}^m \sum_{(y_1,y_2,y_3) \in T_k^+(x)} \eta_{\varepsilon}(y_1) \eta_{\varepsilon}(y_2) \eta_{\varepsilon}(y_3) - \sum_{(y_1,y_2,y_3) \in T_k^-(x)} \eta_{\varepsilon}(y_1) \eta_{\varepsilon}(y_2) \eta_{\varepsilon}(y_3),
\]
and
\[
T_k^n(x) := \{(y_1,\ldots,y_n) : y_1 < \ldots < y_n \in \mathbb{L} \cap (x,x+k)\},
\]
\[
T_k^n(x^-) := \{(y_1,\ldots,y_n) : y_1 < \ldots < y_n \in \mathbb{L} \cap (x-k,x)\}.
\]
The terms $D_{\text{pol}}$ is the sum of the signs of all possible hops within $(x,x+k)$ and within $(x-k,x)$, and $D_{\text{cub}}$ is the difference of two sums, consisting of signs corresponding to non-degenerated (distinct coordinates) cubic terms within $(x,x+k)$ and within $(x-k,x)$. The factor $3k-2$ in (2.19) counts the number of degenerated cubic terms with one coordinate being equal to a given value, the factor $k$ in (2.10) counts the total number of degenerated quadratic terms, and the factor $6k-5$ in (2.10) counts the number of of degenerated quartic terms such that two of its coordinates take the same value and the other two coordinates also take the same value (which can be the same or different from the value of the previous two coordinates).

The quadratic term $D_{\text{pol}}$ corresponds to hops not across $x$, whereas $D_{\text{pol}}$ corresponds to hops across $x$. Hence, we match $D_{\text{pol}}$ with $D_{\text{pol}}$ by translating the center point $x$ to lie between the two particles. That is, we rewrite a generic term $Q := \eta_{\varepsilon}(y_1) \eta_{\varepsilon}(y_2) \eta_{\varepsilon}(y_3) Z_t^+(x,i) + \eta_{\varepsilon}(y_1) \eta_{\varepsilon}(y_2) \eta_{\varepsilon}(y_3) (1 - Z_t^+(x,i)/Z_t^{-}(x,i))$, of $D_{\text{pol}}$ as
\[
(2.22) \quad Q = \eta_{\varepsilon}(y_1) \eta_{\varepsilon}(y_2) Z_t^+(x,i) + \eta_{\varepsilon}(y_1) \eta_{\varepsilon}(y_2) Z_t^+(x,1 - Z_t^+(x,i)/Z_t^{-}(x,i)),
\]
and choose $i \in I(y_1,y_2) := \mathbb{Z} \cap (y_1 - x, y_2 - x)$. Since, the first term of (2.22) is a translation of $\eta_{\varepsilon}(y_1-i) \eta_{\varepsilon}(y_2-i) \eta_{\varepsilon}(y_3) Z_t^+(x)$, we write it as the sum of a gradient term and a generic term $\eta_{\varepsilon}(y_1-i) \eta_{\varepsilon}(y_2-i) \eta_{\varepsilon}(y_3) \eta_{\varepsilon}(y_4) I_i^\alpha(x)$ of $D_{\text{pol}}$. By (2.23), $I_i^\alpha(x) = \eta_{\varepsilon}(y_1,\ldots, y_4)$ is an exponential function of the height difference.

By Taylor-expanding the exponential function to the first order we obtain $\eta_{\varepsilon}$-linear terms and some remainders. The $\eta_{\varepsilon}$-linear terms combined with $\eta_{\varepsilon}(y_1) \eta_{\varepsilon}(y_2)$ yield $\eta_{\varepsilon}$-cubic terms, except when the $\eta_{\varepsilon}$-linear term coincides with $\eta_{\varepsilon}(y_1)$, where we have $\eta_{\varepsilon}(y_2)$. Thus we obtain
\[
(2.23) \quad Q = G_t(x) + \left( \eta_{\varepsilon}(y_1-i) \eta_{\varepsilon}(y_2-i) + \lambda \varepsilon^+ c_t^{(0)} + \varepsilon E_t(x) \right) Z_t^+(x),
\]
where $c_t^{(0)}$ is a sum of $\eta_{\varepsilon}$-cubic terms, and $i \in I(y_1,y_2)$. Notice that $Q_i^t(x)$, as defined in (2.23), is the sum of $\eta_{\varepsilon}(y_1-i) \eta_{\varepsilon}(y_2-i)$ over $i \in I(y_1,y_2)$, where $y = y_2 - y_1$ is fixed. Hence, by summing (2.23) over $i \in I(y_1,y_2)$, we obtain
\[
(2.24) \quad j \eta_{\varepsilon}(y_1) \eta_{\varepsilon}(y_2) Z_t^+(x) = G_t(x) + \left( Q_i^t(x) + j \lambda \varepsilon^+ c_t^{(0)} + \varepsilon E_t(x) \right) Z_t^+(x),
\]
where $c_t$ is a sum of $\eta_{\varepsilon}$-cubic terms. Applying the same reasoning to the mirror image $(y_2',y_1')$ of $(y_1,y_2)$ with respect to $x$, namely $(y_2',y_1') = (2x - y_2, 2x - y_1)$, we get
\[
(2.25) \quad j \eta_{\varepsilon}(y_2') \eta_{\varepsilon}(y_1') Z_t^+(x) = G_t(x) + \left( Q_i^t(x) - j \lambda \varepsilon^+ c_t^{(0)} - \varepsilon E_t(x) \right) Z_t^+(x),
\]
where $C'_1$ is a sum of $\eta$-cubic terms, which, by symmetry, is the mirror image of $C_t$ with respect to $x$ (that is, $C_t - C'_1$ is of the form $\tilde{C_t}$ as in (2.17)). Combining (2.24) and (2.25) we obtain
\begin{equation}
2^{-1}(\eta(y_1)\eta(y_2) + \eta(y'_2)\eta(y'_1)) = \mathcal{G}_t(x) + (j^{-1} \mathcal{Q}_j(x) - 2^{-1} \lambda \varepsilon \tilde{\varepsilon} L^{\eta^2}(x) - \varepsilon \tilde{\varepsilon} \tilde{C}_t(x) + \varepsilon \mathcal{E}_t(x)) \mathcal{Z}_t(x).
\end{equation}

Since the set $\mathcal{I}^+_t(x)$ is the mirror image of $\mathcal{I}^+_t(x)$ with respect to $x$, we can apply (2.20) to (2.20).

After rearranging the sum over $y_1$ and $y_2$, we obtain
\begin{equation}
D_{t}^{\text{lin}} = \lambda^2 \sum_{j,k=1}^{m} \mathcal{Q}_j(x) \mathcal{L}_{j}^2(k-j) + j^{-1} - 2^{-1} \varepsilon \tilde{\varepsilon} \lambda \sum_{j,k=1}^{m} \tilde{\mathcal{R}}_j(k-j-1) \mathcal{L}_{j}^2 + \mathcal{G}_t + \varepsilon \tilde{\varepsilon} \tilde{C}_t.
\end{equation}

Combining the second term of (2.27) with $D_{t}^{\text{lin}}$, we obtain
\begin{equation}
\varepsilon \tilde{\tilde{\varepsilon}} D_{t}^{\text{lin}} + \varepsilon D_{t}^{\text{quad}} = \varepsilon \tilde{\tilde{\varepsilon}} \sum_{j=1}^{m} (A^\varepsilon \mathcal{R}_j^\varepsilon_{j} \mathcal{L}_{j}^2) + \varepsilon \sum_{j=1}^{m} (B^\varepsilon \mathcal{R}_j) \mathcal{Q}_j.
\end{equation}

To conclude the proof, it thus suffices to show that $D_{t}^{\text{cub}}$ is of the form $\varepsilon \tilde{\varepsilon} \mathcal{E}_t + \mathcal{G}_t$. To this end, we employ a translation similar to (2.22). For each cubic term $\eta(y_1)\eta(y_2)\eta(y_3)Z_t(x)$, $(y_1, y_2, y_3) \in \mathcal{I}^+_t(x)$, translate the center from $x$ to $x + k$ to get
\begin{equation}
\eta(y_1)\eta(y_2)\eta(y_3)Z_t(x) = \mathcal{G}_t(x) + \eta(y_1 - k)\eta(y_2 - k)\eta(y_3 - k)Z_t(x) + \eta(y_1)\eta(y_2)\eta(y_3)Z_t(x)(1 - Z_t(x)/Z_t(x + k)).
\end{equation}

For the last term in (2.29), using (1.8) and Taylor-expansion to the first order, we turn it into the form $\varepsilon \tilde{\varepsilon} \mathcal{E}_t$. Since $\mathcal{I}^+_t(x) \to \mathcal{I}^+_t(x)$: $(y_1, y_2, y_3) \mapsto (y_1 - k, y_2 - k, y_3 - k)$ is a bijection, the sum of $\eta(y_1 - k)\eta(y_2 - k)\eta(y_3 - k)$ over $\mathcal{I}^+_t(x)$ in (2.29) matches the sum over $\mathcal{I}^+_t(x)$ in (2.21).

Consequently, $D_{t}^{\text{cub}} = \varepsilon \tilde{\varepsilon} \mathcal{E}_t + \mathcal{G}_t$, as claimed.

**Proposition 2.7.** For $m \leq 3$, $\tilde{C}_t = \varepsilon \tilde{\varepsilon} \mathcal{E}_t + \mathcal{G}_t$.

**Proof.** Clearly, $\tilde{C}_t = 0$ when $m = 1, 2$, whereas when $m = 3$ (2.17) consists of the single term corresponding to $i = 1$, $j = 2, k = 3$. Since $(x + \frac{1}{2}, x + \frac{1}{2}, x + \frac{1}{2})$ is a translation by $3$ of $(x - \frac{1}{2}, x - \frac{1}{2}, x - \frac{1}{2})$, the argument in the last paragraph of the proof of Proposition 2.3 also applies to $\tilde{C}$, concluding the proof.

We now combine Proposition 2.4 and 2.7 to prove Proposition 2.2.

**Proof of Proposition 2.4.** First, comparing (1.8) with (2.10) and (2.11), we find that the constants always match, that is $\nu_{\varepsilon} = \nu_{\varepsilon} + \nu_{\varepsilon}'$. Next, by (2.9) and (2.28), the equation (1.13) implies
\begin{equation}
\varepsilon \tilde{\varepsilon} D_{t}^{\text{lin}} + \varepsilon D_{t}^{\text{quad}} = \varepsilon \tilde{\varepsilon} \Omega_{t}^\varepsilon + \varepsilon \Omega_{t}^\varepsilon + O(\varepsilon^2) R,
\end{equation}

where $R$ is a linear combination of $\eta$-linear and $\eta$-quadratic terms, which by Lemma 2.5 is weakly vanishing. Hence the remainder $O(\varepsilon^2) R$ is of the desired form $\varepsilon^2 \mathcal{E}_t$. Proposition 2.2 now follows from Proposition 2.3 and 2.7.

3. **Convergence to the SHE**

Let $p^t$ be the kernel of the following semi-discrete heat equation
\begin{equation}
\frac{d}{dt}p^t(x) = \frac{1}{2} \sum_{k=1}^{m} \tilde{\mathcal{R}}_k \Delta_k p^t(x),
\end{equation}

$p^0(x) = \mathbb{1}_{(0)}(x)$. 

Let * denote the convolution of two functions on \( \mathbb{Z} \), that is \((f * g)(x) := \sum_{x'} f(x - x')g(x')\). We rewrite the DSHE (2.15) as the following integrated form:

\[
(3.2) \quad Z_t^x = p_t^x * Z_0^x + \int_0^t p_{t-s}^x * (Z_s^x dM_s^x) + \int_0^t \varepsilon \partial p_{t-s}^x * (\mathcal{E}_s Z_s^x) ds + \sum_{|k| \leq m} \int_0^t \varepsilon \nabla p_{t-s}^x * (\mathcal{E}_s^k Z_s^x) ds,
\]

where we applied summation by parts to the last term. Here, as in Definition 2.1 each \( \mathcal{E}^k \) is a linear combination of uniformly vanishing and weakly vanishing terms.

### 3.1. Tightness

In this section we prove Proposition 1.6. The key to the proof is the Hölder estimates of \( Z_t^x \) given in Proposition 3.2 and Corollary 3.3 whose proofs require the following

**Lemma 3.1.** Given any deterministic function \( f_s(x, x') : [0, \infty) \times \mathbb{Z}^2 \rightarrow \mathbb{R} \), let

\[
\bar{f}_s(x, x') := \sup \{ |f_{s'}(x, x')f_{s'}(x, x' + j)| : |s| \leq |s'| + 1, |j| < m \}.
\]

For any \( n \in \mathbb{N} \) we have

\[
\left\| \int_0^t \sum_{x'} f_s(x, x')Z_s^x(x')dM_s^x(x') \right\|^{2n}_2 \leq C \varepsilon \int_t^{t'} \sum_{x'} \bar{f}_s(x, x') \left\| Z_s^x(x')^2 \right\|_{2n} ds.
\]

**Proof.** Fix \( t \) and let \( R(x) := \int_t^{t'} \sum_{x'} f_s(x, x')Z_s^x(x')dM_s^x(x') \). By the Burkholder-Davis-Gundy inequality,

\[
\|R(x)^2\|_n \leq C\|\langle R(x), R(x) \rangle_t\|_n,
\]

where \( \langle \cdot, \cdot \rangle \) denotes the quadratic variation. Let \( T_{(s_1, s_2)}(x) \) be the (random) set of all \( s \in (s_1, s_2] \) at which a particle hops across the site \( x \). Since the Poisson processes as defined in (2.15) are mutually independent, using (2.23) we have

\[
\langle R(x), R(x) \rangle_t = \sum_{x'} \sum_{s \in T(x') (s_1, s_2)} f_s(x, x')f_s(x, x' + l)(e^{2\varepsilon(x', l)\lambda t} - 1)^2 Z_s^x(x')Z_s^x(x' + l),
\]

where \( T'_t(x) := T_{(t, t')}(x) \cap T_{(t, t')}(x + l) \) is the set of \( s \in (s_1, s_2] \) at which a particle hops across both the site \( x \) and \( x + l \), and \( x(l, t) = 1 \) when the particle hops to the right, \( x(l, t) = -1 \) when the particle hops to the left. (Note that the sum of \( l \) goes over \(|l| < m \) since \( T_t (x) = \emptyset \) when \(|l| \geq m \).) Next we partition \([0, \infty)\) into subintervals \( T_i := [i, i + 1) \). Using \(|e^{\pm \lambda t} - 1| \leq C\varepsilon t^{1/2} \), and replacing \( f_s \) and \( Z_s^x \) by their supremum over \( T_i \), we have

\[
(3.4) \quad \|R(x, R(x))_t\|_n \leq C \varepsilon \sum_{i \in [t]} \sum_{x'} \bar{f}_s(x, x') \sup_{s \in T_i} \max_{|l| < m} Z_s^x(x' + l)^2.
\]

Since \(|h(i + l) - h(i)| \leq l| \), from the definition (1.8) of \( Z_t^x \),

\[
(3.5) \quad \max_{|l| < m} Z_s^x(x' + l)^2 \leq C\varepsilon t^{1/2} Z_s^x(x')^2.
\]

Next, since each hop across \( x \) increases or decreases \( Z_t^x(x) \) by a factor of \( e^{2\lambda x_{t-1}^{1/2}} \),

\[
(3.6) \quad \sup_{s \in T_i} Z_s^x(x')^2 \leq e^{4\lambda t^{1/2}N_i(x')}Z_t^x(x')^2 \quad \text{and} \quad Z_s^x(x')^2 \leq e^{4\lambda t^{1/2}N_i(x')} \inf_{s \in T_i} Z_t^x(x')^2,
\]

where \( N_i(x') \) is the number of hops across \( x' \) during the time interval \( T_i \), that is

\[
(3.7) \quad N_i(x') : = \# T_i(x'),
\]

which is stochastically bounded by a Poisson random variable with rate \( \sum_{k=1}^m kr_k \). Take the \( L^n \)-norm of (3.6) using the independence of \( N_i(x') \) and \( Z_s^x(x') \) and the bound \( \mathbb{E}(e^{\varepsilon h_1^{1/2}N_i(x')} - 1) \leq C(n) \). We then deduce

\[
(3.8) \quad \left\| \sup_{s \in T_i} Z_s^x(x')^2 \right\|_n \leq C \left\| \inf_{s \in T_i} Z_s^x(x')^2 \right\|_n \leq C \left\| T_i^{-1} \int_{T_i} Z_s^x(x')^2 ds \right\|_n \leq C \int_{T_i} \| Z_s^x(x')^2 \|_n dx.
\]
Combining (3.3), (3.4), (3.5), and (3.8), we conclude the lemma.

For process \( f(\cdot) \) over \( Z, a \geq 0, l \in \mathbb{N} \), define the following norm
\[
|f|_{a,l} := \sup_x |f(x)| e^{-a|x|}.
\]
Note that \( |f|^2_{a,l} = |f|^2_{a,2l} \) and for any \( x, x' \) we have
\[
|f(x')|_l \leq e^{a|x|} e^{a|x-x'|} |f|_{a,l}.
\]

**Proposition 3.2.** For any \( u \in (0, 1), a \geq 0, j \in \mathbb{N}, t, t' \in [0, T \varepsilon^{-2}] \),
\[
|Z^u_0(x) - Z^u_0(x')|_{2j} \leq C(a,u) (|Z^u_0|_{a,j} + N_{a,j,u/2}),
\]
where
\[
N_{a,j,v} := \sup_{x \neq x'} |e^{-|x-x'|} - e^{-a|x+|x'|}|.
\]

**Proof.** Let \( u \in (0, 1) \) be given and fixed, so that we do not specify the dependence of \( C \) on \( u \) and \( a \). Let \( I_1, I_2, I_3, I_4 \) denote the first, second, third, fourth terms on the RHS of (3.11), respectively.

We first prove (3.11). Using the readily verified inequality \((\sum_{i=1}^4 b_i)^2 \leq 4 \sum_{i=1}^4 b_i^2\), we get
\[
|(Z^u_0)^2|_{2a,2j} \leq 4(|Z^u_0|_{2a,j} + I_1^2 + I_2^2 + I_3^2 + I_4^2),
\]
For \( I_1 \) we have \( I_1^2 = \| p_t^i \|_{2a,j} \leq (p_t^i \| Z^u_0 \|_{2a,j})^2 \), which by using (3.10) for \( l = 2j \) and (A.20) yields \( I_1^2 \leq C \| Z^u_0 \|_{2a,j} \). Next, apply Lemma 3.1 to \( I_3 \) for \( s_1(x,x') = p^i_{l-s}(x-x') \), and then use (A.6) and (A.23). We then obtain
\[
\| I_3 \| \leq C \varepsilon \int_0^t (t-s)^{-\frac{5}{2}} (p^i_{l-s} \| Z^u_0 \|_{2a,j}) \mathrm{d}s.
\]
By applying (3.10) for \( l = 2j \) and (A.20), we bound the convolution in (3.15) by \( C e^{2a|x|} \| Z^u_0 \|_{2a,j} \), yielding
\[
\| I_3 \| \leq C e^{2a|x|} \| Z^u_0 \|_{2a,j} \]
As for \( I_3 \) and \( I_4 \), by Definition 2.1 we have \( \sup_{x\in\mathbb{R}} \| E(x) \|_{\infty}, \sup_{x\in\mathbb{R}} \| E^k(x) \|_{\infty} \leq C \), yielding
\[
\| I_4 \| \leq C \sum_{k \leq m} \left( \int_0^t \varepsilon \| \nabla \phi^k \|_{2a,j} \right)^2.
\]
Further apply the Cauchy-Schwartz inequality to get
\[
(3.17a) \quad \| I_4 \| \leq C \left( \int_0^t \varepsilon^2 \| \phi^k \|_{2a,j} + 1 \right)^2,
\]
\[
(3.17b) \quad \| I_4 \| \leq C \sum_{|k| \leq m} \left( \int_0^t \varepsilon \| \nabla \phi^k \|_{2a,j} + 1 \right)^2.
\]
where 1 denotes the constant function \( f(x) \equiv 1 \). The first integral in (3.17a) is clearly bounded by \( \hat{T} \), and by (A.21) for \( v = 1 \) the first integral in (3.17b) is bounded by \( C \hat{T}^{-\frac{7}{2}} \). Hence
\[
(3.18) \quad \| I_3 \| \leq C \int_0^t \varepsilon^2 \| \phi^k \|_{2a,j} \| Z^u_0 \|_{2a,j} \]
By using (A.21) for where
\[ \text{Applying (A.20) and (A.21) for} \]
\[ \text{Combining the preceding estimates of } |I^2_{t,2a,j}|, i = 1, \ldots, 4, \text{ we arrive at the following inequality} \]
\[ \text{where } f_\varepsilon(s) := \varepsilon^{-1/2} + \varepsilon^2. \text{ Iterate (3.20) to get} \]
\[ \text{Since } \nabla \text{ we have} \]
\[ \text{Iterate (3.20) to get} \]
\[ \text{This together with (A.20) implies} \]
\[ \text{Next, similar to (3.15), applying Lemma 3.1 to } \|\nabla I_2^2\|_j \text{ for } f_s(x,y) = \nabla p_{t-s}(x-y), \text{ and using} \]
\[ \|\nabla I_2^2\|_j \leq C \varepsilon^{1/2} \int_0^t |\nabla p_{t-s}(x-y)| + |\nabla p_{t-s}| \text{ and then combining (A.0) and (A.24), we get} \]
\[ \text{Using (A.20), (3.10), and (3.11), and calculating the time integral, we then turn (3.22) into} \]
\[ \text{For } \nabla I_3 \text{ and } \nabla I_4, \text{ similar to (3.14) we have} \]
\[ \text{By using (A.21) for } v = u \text{ and (A.22) for } v = u \text{ to estimates the first integral in (3.24) and (3.23), we further obtain} \]
\[ \text{Using (A.20), (A.21) for } v = 1, (3.10), \text{ and (3.11), we then bound } \|\nabla I_3\|_j^2 \text{ and } \|\nabla I_4\|_j^2 \text{ by} \]
\[ \|\nabla I_4\|_j^2 \leq C |n\varepsilon|^{u} \sum_{|k|\leq m} \int_0^t \varepsilon(|\nabla k p_{t-s}| + |\nabla k p_{t-s}|) * (Z_s^2)_{2a,j} \text{ds.} \]

We then prove (3.33). Without lost of generality, assume $t' > t$. It suffices to show for $i = 1, \ldots, 4$,
For $i = 1$, using the semi-group properties $p_t^p = p_{t - t}^p \ast p_t^p$ and $\sum_{x_1} p_{t - t}^p(x_1) = 1$ we have

\begin{equation}
(1) v(x) - (1) s(x) = \sum_{x_1} p_{t - t}^p(x - x_1)((1) t(x_1) - (1) s(x)).
\end{equation}

By (3.21), we have

\begin{equation}
\| (1) t(x_1) - (1) s(x) \|_{2j} \leq (\varepsilon|x - x_1|)^{u/2} e^a|x - x_1| e^{2a|x|CN_0,2j,u/2}.
\end{equation}

Combining (3.29) with (A.20) and (3.28), we conclude (3.27) for $i = 1$.

For $i = 2$, write $(1) t - (1) s$ as the sum of $I_{21} := \int_0^t p_{t - s}^p \ast Z_s^\varepsilon \, dM_s^\varepsilon$ and $I_{22} := \int_0^t (p_{t - s}^p - p_{t - s}^\varepsilon) \ast Z_s^\varepsilon \, dM_s^\varepsilon$. Similar to (3.16), applying Lemma (3.11), (A.20), and (A.23) to $I_{21}$, we bound $\| I_{21} \|^2_{2j}$ by

\begin{equation}
C \varepsilon \int_0^t (t - s)^{-\frac{1}{2}} e^{2a|x|} \| (Z_s^\varepsilon)^2 \|^2_{2a,2j} \, ds.
\end{equation}

By (3.11) and $t - t \leq \varepsilon^{-2} \tilde{T}$, we further bound this integral by $C [\varepsilon \tilde{T} (t - t) \tilde{T} e^{2a|x|} \| Z_0^{\varepsilon} \|^2_{2a,2j}]^2$. Similarly, applying Lemma (3.1) and (A.6) to $I_{22}$, using $\| (p_{t - s}^p - p_{t - s}^\varepsilon)^2 \leq \| p_{t - s}^p - p_{t - s}^\varepsilon \| (p_{t - s}^p + p_{t - s}^\varepsilon)$, and then combining (A.6) for $v = \tilde{T}$, (A.9), and (3.11), we bound $\| I_{22} \|^2_{2j}$ by $C [\varepsilon \tilde{T} (t - t) \tilde{T} e^{2a|x|} \| Z_0^{\varepsilon} \|^2_{2a,2j}]^2$.

The estimates for $\| I_{21} \|^2_{2j}$ and $\| I_{21} \|^2_{2j}$ conclude (3.27) for $i = 2$.

For $i = 3$, write $(1) t - (1) s$ as the sum of $I_{31} := \int_0^t \varepsilon^2 p_{t - s}^p \ast \varepsilon_s Z_s^\varepsilon \, ds$ and $I_{32} := \varepsilon^2 \int_0^t (p_{t - s}^p - p_{t - s}^\varepsilon) \ast \varepsilon_s Z_s^\varepsilon \, ds$. Applying (A.20) and (3.11), we bound $\| I_{32} \|^2_{2j}$ by a constant multiple of

\begin{equation}
\varepsilon \tilde{T} e^{2a|x|} \| Z_0^{\varepsilon} \|^2_{2a,2j}. 
\end{equation}

For $I_{32}$, similar to (3.17), by introducing a weight $g_s(x) := e^{x/|a|^{1/2}}$ in the Cauchy-Schwartz inequality, we bound $\| I_{32} \|^2_{2j}$ by

\begin{equation}
C \int_0^t \varepsilon^2 \delta((p_{t - s}^p - p_{t - s}^\varepsilon) \ast g_{t - s}) \delta((p_{t - s}^p + p_{t - s}^\varepsilon) \ast g_{t - s}) \ast \| (Z_s^\varepsilon)^2 \|^2_{2j} \, ds.
\end{equation}

For the first integral, using (A.7) for $v = \tilde{T}$, we bound it by $C \varepsilon \tilde{T} \varepsilon \tilde{T} e^{2a|x|} \| Z_0^{\varepsilon} \|^2_{2a,2j}$. For the second integral, combining (A.20) and (3.11) we bound it by $C e^{2a|x|} \| Z_0^{\varepsilon} \|^2_{2a,2j}$. Hence $\| I_{32} \|^2_{2j}$ is bounded by $C \varepsilon \tilde{T} \varepsilon \tilde{T} e^{2a|x|} \| Z_0^{\varepsilon} \|^2_{2a,2j}$.

Finally, for $i = 4$, we similarly define $I_{41} := \int_0^t \varepsilon \nabla_k p_{t - s}^p \ast \varepsilon_s Z_s^\varepsilon \, ds$ and $I_{42} := \int_0^t (\nabla_k p_{t - s}^p - \nabla_k p_{t - s}^\varepsilon) \ast \varepsilon_s Z_s^\varepsilon \, ds$. For $I_{41}$, applying (A.21) for $v = 1$ and (3.11) we bound $\| I_{41} \|^2_{2j}$ by a constant multiple of

\begin{equation}
\varepsilon \tilde{T} e^{2a|x|} \| Z_0^{\varepsilon} \|^2_{2a,2j}.
\end{equation}

For $I_{42}$, similar to (3.30), we bound $\| I_{42} \|^2_{2j}$ by a constant multiple of

\begin{equation}
\sum_{|k| \leq m} \varepsilon \delta((\nabla_k p_{t - s}^p - \nabla_k p_{t - s}^\varepsilon) \ast g_{t - s}) \ast \| (Z_s^\varepsilon)^2 \|^2_{2j} \, ds.
\end{equation}

For the first integral, using (A.8) for $v = \tilde{T}$, we bound it by $C \varepsilon \tilde{T} \varepsilon \tilde{T} e^{2a|x|} \| Z_0^{\varepsilon} \|^2_{2a,2j}$. For the second integral, combining (A.21) and (3.11) we bound it by $C e^{2a|x|} \| Z_0^{\varepsilon} \|^2_{2a,2j}$. Hence $\| I_{42} \|^2_{2j}$ is bounded by $C \varepsilon \tilde{T} \varepsilon \tilde{T} e^{2a|x|} \| Z_0^{\varepsilon} \|^2_{2a,2j}$.

Our estimates for $I_{41}$ and $I_{42}$ conclude (3.27) for $i = 4$.

Proposition 3.2 immediately implies following corollary

\begin{corollary}
Under the assumptions (1.23a) and (1.23b), for all $u \in (0,1)$ we have the following estimates
\begin{align}
&\| Z_t^\varepsilon \|_{14} \leq C(u) e^{a_0|x|}, \\
&\| Z_t^\varepsilon(x) - Z_t^\varepsilon(x') \|_{14} \leq C(u) (\varepsilon|x - x'|) \varepsilon \varepsilon \tilde{T} e^{a_0(\varepsilon|x + |x'|)},
\end{align}
\end{corollary}
Proof of Proposition 7.4 This proposition is the generalization of the first half (tightness) of [3, Theorem 3.3] to \( m > 1 \). The original proof by [3] for \( m = 1 \) actually applies to all processes satisfying the conclusions of [3, Lemma 4.1, 4.5-4.7], whose proofs relies only on the conclusions of [3, Lemma 4.1-4.3] for \( p > 12 \) and the fact that \( N_t(x') \), as defined in [3, (3.7)], is stochastically bounded by a Poisson random variable with a fixed rate. (Specifically, the assumption \( p > 12 \) is used in [3, (4.60)].) In our case, \( Z^\infty \) satisfies (3.31), (3.32), and (3.33), which correspond to the conclusions of [3, Lemma 4.1, 4.2, 4.3] for \( p = 14 > 12 \), respectively, and \( N_t(y) \) is stochastically bounded by a Poisson random variable with rate \( \sum_{k=1}^m r_k k \), as shown in the proof of Lemma 3.1. \( \square \)

3.2. Convergence. In this section we prove Proposition 7.7. To this end, we first obtain an expression of the predictable quadratic variation of \( M^\infty \).

Proposition 3.4. \( d(M^\infty(x), M^\infty(x')) = 0 \) vanishes unless \( |x - x'| < m \), and for \( |x - x'| < m \)

\[
\left\| Z^\infty_t(x) - Z^\infty_t(x') \right\|_{14} \leq C(u)(1 \vee |t' - t|^{4/5}) \varepsilon \tilde{e}^{2a_0 |x|},
\]

where \( a_0 \) is the same as in (1.23a) and (1.23b).

Proof. By the independence of the Poisson processes \( Q^\infty_t(y) \), we have

\[
\left\langle Q^\infty_t^{y_2 - y_1}(y_1), Q^\infty_t^{y_2 - y_1}(y_1) \right\rangle_t = \int_0^t \mathbb{1}_{(y_1,y_2)=(y_1,y_2 t)} q^\infty_{y_2 - y_1} ds.
\]

Rewrite (2.3) as

\[
dM^\infty_t(x) = \sum_{y_1,y_2} \left( e^{2 \text{sign}(y_2 - y_1) \lambda e^{1/2} - 1} \right) a_{y_1 \rightarrow y_2}(\eta_t) dQ^\infty_t^{y_2 - y_1}(y_1) - q^\infty_{y_2 - y_1} dt,
\]

where the sum is taken over all hops \( y_1 \rightarrow y_2 \) across \( x \). We then take the product of \( dM^\infty(x) \) and \( dM^\infty(x') \) using (3.35) and (3.36) to get

\[
d\langle M^\infty(x), M^\infty(x') \rangle = \sum_{y_1,y_2} \left( e^{2 \text{sign}(y_2 - y_1) \lambda e^{1/2} - 1} \right)^2 a_{y_1 \rightarrow y_2}(\eta_t) q^\infty_{y_2 - y_1} dt.
\]

Here the sum is taken over all hops \( y_1 \rightarrow y_2 \) that cross both \( x \) and \( x' \), hence is nonzero only when \( |x - x'| < m \), where by putting \( (y_1,y_2) = (x \pm j, x \pm j \mp k) \) we get

\[
d\langle M^\infty(x), M^\infty(x') \rangle = \sum_{k=1}^m \sum_{j=1}^{k \wedge (k+1)} \left( e^{-2 \lambda e^{1/2} - 1} \right)^2 a_{x+j \rightarrow x+2j-k}(\eta_t) q^\infty_{j-k} dt.
\]

Taylor-expanding \( e^{\pm 2 \lambda e^{1/2} - 1} \) to the first order and using (2.3a), we obtain

\[
d\langle M^\infty(x), M^\infty(x') \rangle = \varepsilon \sum_{k=1}^m \sum_{j=1}^{k \wedge (k+1)} \left( \lambda^2 (q^k \bar{q}^k) + w_1 \right) dt,
\]

where \( w_1 \) is a sum of \( \eta \)-linear and \( \eta \)-quadratic terms and \( w_2 \) is uniformly vanishing. By (1.8) and Taylor expansion to the first order, we have \( Z^\infty_t(x') = Z^\infty_t(x)(1 + w_3) \), for some uniformly vanishing \( w_3 \). Clearly \( w_1 w_3 \) and \( w_2 w_3 \) are uniformly vanishing, and by Lemma 2.5 \( w_1 \) is weakly vanishing. Multiplying (3.38) by \( Z^\infty_t(x)Z^\infty_t(x') = Z^\infty_t(x)^2(1 + w_3) \) and using \( q^k \bar{q}^k = r_k \), we conclude the proof. \( \square \)
We next use a martingale problem to prove Proposition 1.7.

**Definition 3.5.** Let \(Z(\cdot)\) be a \(C([0,\infty), C(\mathbb{R}))\)-valued process such that given any \(\overline{T} > 0\), there exists \(\overline{A} \geq 0\) such that

\[
\sup_{T \in [0, \overline{T}]} \sup_{X} e^{-A|X|} \mathbb{E}(Z_T(X)^2) < \infty.
\]

The process \(Z(\cdot)\) solves the martingale problem with initial condition \(Z_0\) if \(Z_0 = Z_0\) in distribution and

\[
(\phi, Z_T) - (\phi, Z_0) - \frac{1}{2} \int_0^T (\phi'', Z_S) dS,
\]

\[
N_T(\phi)^2 - \int_0^T (\phi^2, Z_S^2) dS,
\]

are local martingale for any \(\phi \in C_c^\infty(\mathbb{R})\), where \((\phi, \psi) := \int_{\mathbb{R}} \phi(x) \psi(x) dx\).

**Proof of Proposition 1.7.** Recall from [3] Proposition 4.11 that for any initial condition \(Z_0\) satisfying

\[(3.40) \quad \|Z_0(X)\|_2 \leq Ce^{A|X|}, \text{ for some } A > 0,
\]

the martingale problem Definition 3.5 has a unique solution, which coincides with the law of the solution to (1.19) with initial condition \(Z_0\). Consequently, it suffices to show that (3.40) holds and any limit point \(\tilde{Z}\) of \(\{Z^\varepsilon\}\) solve the martingale problem Definition 3.5 starting from \(Z_0\). By passing to the relative subsequence we assume \(Z^\varepsilon(\cdot) \Rightarrow Z(\cdot)\).

Clearly, (3.40) and (3.39) hold because of (1.23) and (3.31), respectively. Let \(\tilde{Z}_T(X) := Z^\varepsilon_{T\cdot}(e^{-1}X) = Z^\varepsilon_{T\cdot}(e^{-1}X)\). By the change of variables \((T, X) \mapsto (\beta^{-1}T, (\beta')^{-1}X)\), it suffices to show that

\[
N_T(\phi) := (\phi, \tilde{Z}_T) - (\phi, \tilde{Z}_0) - \frac{\alpha}{2} \int_0^T (\phi'', \tilde{Z}_S) dS,
\]

\[
\Lambda_T(\phi) := N_T(\phi)^2 - \alpha \lambda^2 \int_0^T (\phi^2, \tilde{Z}_S^2) dS,
\]

are local martingales. Let \(a\) be an arbitrary positive number, \(0 \leq S \leq S' \leq \overline{T}\), and let \(f : D([0, S], C(\mathbb{R})) \to \mathbb{R}\) be bounded and continuous with respect to the Skorokhod topology. Associate with any process \(A_T\) with the stopped process

\[
A^T_T := A^T_{T \wedge T_a}, \quad T_a := \inf \{T \geq 0 : |A_T| > a\}.
\]

It suffices to prove

\[
\mathbb{E}((NS_x(\phi)^a - NS_x(\phi)^a)f(N_x)) = 0, \quad \mathbb{E}((\Lambda_S(\phi)^a - \Lambda_S(\phi)^a)f(N_x)) = 0.
\]

To this end, set \((\phi, Z^\varepsilon_x) := \varepsilon \sum \phi(\varepsilon X) Z^\varepsilon_i(x), L^\varepsilon := \sum_{k=1}^m \varepsilon k \Delta_k\), and

\[
N^\varepsilon_T(\phi) := (\phi, Z^\varepsilon_{T\cdot}) - (\phi, Z^\varepsilon_0) - \frac{1}{2} \int_0^{\varepsilon^{-2}T} (L^\varepsilon \phi, Z^\varepsilon) ds,
\]

\[
\Lambda^\varepsilon_T(\phi) := N^\varepsilon_T(\phi)^2 - \varepsilon^2 \alpha \lambda^2 \int_0^{\varepsilon^{-2}T} (\phi^2, (Z^\varepsilon)^2) ds.
\]

By the definition of weak convergence, using \(\sum_{k=1}^m k^2 r_k \rightarrow \alpha\) (by (1.17) and (1.20)) and using localization by \(T_a\) to guarantee the boundedness of \(N^\varepsilon(\phi)^a\) and \(\Lambda^\varepsilon(\phi)^a\), we get

\[
\mathbb{E}((NS_x(\phi)^a - NS_x(\phi)^a)f(N_x)) = \lim_{\varepsilon \to 0} \mathbb{E}((N^\varepsilon_x(\phi)^a - N^\varepsilon_x(\phi)^a)f(N^\varepsilon_x)),
\]

\[
\mathbb{E}((\Lambda_S(\phi)^a - \Lambda_S(\phi)^a)f(N_x)) = \lim_{\varepsilon \to 0} \mathbb{E}((\Lambda^\varepsilon_x(\phi)^a - \Lambda^\varepsilon_x(\phi)^a)f(N^\varepsilon_x)).
\]

We next show that \(N^\varepsilon(\phi)\) and \(\Lambda^\varepsilon(\phi)\) are approximated by the martingales

\[
\tilde{N}^\varepsilon := \int_0^T (\phi, Z^\varepsilon dM^\varepsilon) ds, \quad (\tilde{N}^\varepsilon)^2 - \langle \tilde{N}^\varepsilon \rangle = 0.
\]
respectively, and that the RHS of (3.43) are zero. To this end, we integrate (2.15) to get

\[(3.44)\quad N_{\varepsilon}^{z-\lambda}(\phi) - \bar{N}_{\varepsilon}^{z} = \varepsilon^{2} \int_{0}^{t} (\phi, \mathcal{E}_{s})_{\varepsilon} ds + \sum_{|k| \leq m} \varepsilon \int_{0}^{t} (\nabla_{k} \phi, \mathcal{E}_{s}^{z})_{\varepsilon} ds.\]

Let \(N_{1}^{z}\) and \(N_{2}^{z}\) denote the first and second terms on the RHS of (3.44), respectively. By Taylor-expanding \(\nabla_{k} \phi\) to the first order, we rewrite \(N_{2}^{z}\) as the sum of

\[N_{21}^{z} := \sum_{|k| \leq m} \varepsilon^{2} \int_{0}^{t} (k \phi', \mathcal{E}_{s}^{k} Z_{s}^{z})_{\varepsilon} ds \quad \text{and} \quad N_{22}^{z} := \varepsilon^{3} \int_{0}^{t} \sum_{|k| \leq m} (\phi, \mathcal{E}_{s}^{k} Z_{s}^{z})_{\varepsilon} ds,
\]

where \(\phi_{k}(x)\) is a bounded (in \(x\) and \(\varepsilon\)) function. By Proposition 3.3,

\[\langle \bar{N}_{\varepsilon}^{z} \rangle_{t} = \varepsilon^{2} \int_{0}^{t} \sum_{|l| < m} \phi(x)\phi(x+l)\lambda^{2}(\alpha_{t} + \mathcal{E}_{s}(x)) Z_{s}^{z}(x)^{2} ds.
\]

By \(\phi(x)\phi(x+l) = \phi(x)^{2} + O(\varepsilon l)\) and \(\sum_{|l| < m} \alpha_{t} = \alpha\) (by (1.20) and (3.33)), we further write \(\bar{N}_{\varepsilon}^{z}\) as the sum of

\[\bar{A}_{\varepsilon}^{z} := \alpha^{2} \varepsilon^{2} \int_{0}^{t} (\phi^{2}, (Z_{s}^{z})^{2}) ds, \quad \Lambda_{t}^{z} := \varepsilon^{2} \int_{0}^{t} (\phi^{2}, \mathcal{E}_{s} Z_{s}^{z})_{\varepsilon} ds,
\]

\[A_{\varepsilon}^{z} := \varepsilon^{3} \int_{0}^{t} (\psi^{\varepsilon}, \mathcal{E}_{s}^{z} Z_{s}^{z})_{\varepsilon} ds,
\]

where \(\psi^{\varepsilon}(x)\) is a bounded (in \(x\) and \(\varepsilon\)). Hence

\[(3.45) \quad \Lambda_{\varepsilon}^{z} \phi = \left(\bar{N}_{\varepsilon}^{z} + (N_{1}^{z})_{t} + (N_{2}^{z})_{t}\right)_{\varepsilon}^{2} - \Lambda_{t}^{z} = (\bar{N}_{\varepsilon}^{z})^{2} - (\bar{N}_{\varepsilon}^{z})_{t} + R_{t}^{z},
\]

where

\[R_{t}^{z} := (A_{t}^{z})_{t} + (A_{t}^{z})_{t} + 2((N_{1}^{z})_{t} + (N_{2}^{z})_{t})\bar{N}_{\varepsilon}^{z} + ((N_{1}^{z})_{t} + (N_{2}^{z})_{t})^{2}.
\]

The proof is completed upon showing

\[(3.46) \quad \lim_{\varepsilon \to 0} \|\bar{N}_{\varepsilon}^{z}\|_{1} = 0, \quad \lim_{\varepsilon \to 0} \|N_{1}^{z}\|_{1} = 0, \quad \lim_{\varepsilon \to 0} \|N_{2}^{z}\|_{1} = 0, \quad \lim_{\varepsilon \to 0} \|R_{1}^{z}\|_{1} = 0.
\]

Since \(\phi^{\varepsilon}\) and \(\psi^{\varepsilon}\) are bounded, by (3.31) we have

\[(3.47) \quad \|\bar{N}_{\varepsilon}^{z}\|_{4}, \|\bar{(N_{2}^{z})}_{t}\|_{4}, \|\Lambda_{\varepsilon}^{z}\|_{4} \leq C,
\]

\[(3.48) \quad \|\bar{N}_{\varepsilon}^{z}\|_{2}^{2} \leq C, \quad \|\bar{(N_{2}^{z})}_{t}\|_{2} \leq C\varepsilon, \quad \|\bar{\Lambda}_{\varepsilon}^{z}\|_{2} \leq C\varepsilon.
\]

The bound (3.47) implies the uniformly integrability of \(\{(N_{1}^{z})_{t}\}_{\varepsilon}, \{(N_{2}^{z})_{t}\}_{\varepsilon}, \text{and} \{(\Lambda_{t}^{z})_{t}\}_{\varepsilon}\), which together with Lemma 2.5 implies

\[(3.49) \quad \lim_{\varepsilon \to 0} \|\bar{N}_{\varepsilon}^{z}\|_{2} = 0, \quad \lim_{\varepsilon \to 0} \|\bar{(N_{2}^{z})}_{t}\|_{2} = 0, \quad \lim_{\varepsilon \to 0} \|\bar{\Lambda}_{\varepsilon}^{z}\|_{1} = 0.
\]

Combining (3.48) and (3.49) we prove (3.36). \(\square\)

4. REPLACEMENT LEMMA

We first recall some basic notions of continuous time Markov processes associated with exclusion processes.

As mentioned in Remark 2.6 for any \(a \in [0, 1]\), the product measure \(\nu_{a}\) is an invariant measure of exclusion process. Let \(D := \{\pm 1\}^{\mathbb{Z}}\) be equipped with the corresponding cylindrical \(\sigma\)-algebra \(\mathcal{G}_{\infty}\) and the probability measure \(\nu := \nu_{\frac{1}{2}}\), let \(\Lambda_{n} := (-n, n) \cap \mathbb{L}\) be the \(n\)-th interval around 0, and let
\{G_n\} be the filtration corresponding to the restriction to \{±1\}^\Lambda_n of functions \(g : D \to \mathbb{R}\). For a function \(g : D \to \mathbb{R}\) define
\[
(\sigma_{y, y+k} g)(\eta) := g(\eta_{y+k}), \quad \eta_{y_1, y_2}(y) := \begin{cases} 
\eta(y_2), & \text{when } y = y_1, \\
\eta(y_1), & \text{when } y = y_2, \\
\eta(y), & \text{otherwise},
\end{cases}
\]
Recall from (2.3), \(a_{y_1, y_2}\) is the indicator function for allowed hops. Let
\begin{align}
(4.1) & \quad c_{y_1, y_2}^\varepsilon := \left(q_{y_2 - y_1}^\varepsilon, a_{y_1, y_2} + q_{y_1, y_2}^\varepsilon a_{y_2, y_1}\right) \mathbb{1}_{\{0 < |y_1 - y_2| \leq m\}}, \\
(4.2) & \quad c_{y_1, y_2}^\infty := \lim_{\varepsilon \to 0} c_{y_1, y_2}^\varepsilon = r_{|y_1 - y_2|}(|a_{y_1, y_2} + a_{y_2, y_1}|) \mathbb{1}_{\{0 < |y_1 - y_2| \leq m\}}.
\end{align}
The Markov generator of the exclusion process is
\[
L^\varepsilon := \sum_{y_1 < y_2} L^\varepsilon_{y_1, y_2}, \quad L^\varepsilon_{y_1, y_2} g := c_{y_1, y_2}^\varepsilon((\sigma_{y_1, y_2} g) - g).
\]
Let \(\mu_{t, \varepsilon}\) denote the law of the exclusion process on \(D\) at time \(t\), and \(f_{t, \varepsilon} := \frac{d\mu_{t, \varepsilon}}{dt}\). Recall that a function \(g_t(\eta) : D \times [0, \infty) \to \mathbb{R}\) (respectively \(g : D \to \mathbb{R}\)) is cylinder if there exists \(n\) such that for all \(t, g_t \in G_n\) (respectively \(g \in G_n\)). By the forward Kolmogorov equation, for any cylinder \(g_t\),
\[
(4.4) \quad \frac{d}{dt} \mathbb{E}_\nu(f_{t, \varepsilon} g_t) = \mathbb{E}_\nu(f_{t, \varepsilon} \partial_t g_t) + \mathbb{E}_\nu(f_{t, \varepsilon} L^\varepsilon g_t).
\]
For cylinder \(g\), define
\begin{align}
(4.5) & \quad D^\varepsilon_{y_1, y_2}(g) := \frac{1}{2} \mathbb{E}_\nu\left(c_{y_1, y_2}^\varepsilon((\sigma_{y_1, y_2} g)^\frac{1}{2} - g^\frac{1}{2})^2\right), \\
(4.6) & \quad D^\infty_{y_1, y_2}(g) := \frac{1}{2} \mathbb{E}_\nu\left(c_{y_1, y_2}^\infty((\sigma_{y_1, y_2} g)^\frac{1}{2} - g^\frac{1}{2})^2\right), \\
(4.7) & \quad D^\varepsilon(g) := \sum_{y_1 < y_2} D^\varepsilon_{y_1, y_2}(g).
\end{align}
Note that the sum \((4.7)\) is finite since \(g\) is cylinder. For each \(y_1, y_2\), the Dirichlet forms \(\{4.5\}\) and \(\{4.6\}\) are a convex and lower-semicontinuous function of \(g\) (see [14] Theorem A.1.10.2 and [14] Corollary A.1.10.3]). We have the identity (see [14] Theorem A.1.9.)
\[
(4.8) \quad \mathbb{E}_\nu(g^\frac{1}{2} L^\varepsilon g^\frac{1}{2}) = -D^\varepsilon(g).
\]
For \(0 < |y_1 - y_2| \leq m\), since \(c_{y_1, y_2}^\varepsilon + c_{y_2, y_1}^\varepsilon = r_{|y_1 - y_2|}(a_{y_1, y_2} + a_{y_2, y_1})\) and \(\sigma_{y_1, y_2} g = g\) unless the hop \(y_1 \to y_2\) or the hop \(y_2 \to y_1\) is allowed, we have
\[
(4.9) \quad D_{y_1, y_2}^\infty(g) = r_{|y_1 - y_2|} \mathbb{E}_\nu\left((\sigma_{y_1, y_2} g)^\frac{1}{2} - g^\frac{1}{2}\right)^2.
\]
Since \(\lim_{\varepsilon \to 0} q_{k, \varepsilon}^\infty = r_{|k|} > 0\), for all \(\varepsilon\) small enough we have
\[
(4.10) \quad 2D_{y_1, y_2}^\infty(g) \leq D_{y_1, y_2}^\varepsilon(g).
\]
Let \(g^n := \mathbb{E}_\nu(g^{\varepsilon_n})\). For a probability density function \(g\) define the \(n\)-th entropy as \(H^\varepsilon_n(g) := \mathbb{E}_\nu(g^n \log(g^n))\). Let \(D_n := \{\pm 1\}^{\Lambda_n}\). Since \#\(D_n = 2^{2n}\), we have the crude bounds
\begin{align}
(4.11) & \quad H^\varepsilon_n(g) \leq 2n \log 2, \\
(4.12) & \quad D^\varepsilon(g^n) \leq Ce^n C.
\end{align}
Next we gives a bound on the Dirichlet form:

**Lemma 4.1.** We have the estimate
\[
(4.13) \quad D^\varepsilon\left(t^{-1} \int_0^t f_{t, \varepsilon}(s)ds\right) \leq C + Ct^{-1} n.
\]
Remark 4.2. For exclusion processes on the torus \( \mathbb{Z}/N\mathbb{Z} \), one get the bound \( Ct^{-1}n \) instead of \( (4.13) \). We get the extra constant from the boundary effect of \( \Lambda_n \). Since \( t^{-1}n = O(\varepsilon) \) under the scaling \( n = \varepsilon^{-1}R \), this extra constant aggravates \( (4.13) \). Consequently, in Lemma 4.3, the radius of averaging we obtain is of the mesoscopic scale \( \varepsilon^{-\frac{1}{2}} \), not the macroscopic scale \( \varepsilon^{-1} \), which is the scale of standard replacement lemmas. While a more careful analysis might remove the extra constant, \( (4.13) \) and Lemma 4.3 suffice for proving Lemma 2.5.

Proof of Proposition 4.7. Take the time derivative of \( H^n_t(f_{t,\varepsilon}) \) using (4.3) to get

\[
\frac{d}{dt} H^n_t(f_{t,\varepsilon}) = \mathbb{E}_\nu \left( \partial_t f^n_{t,\varepsilon} / f^n_{t,\varepsilon} \right) + \mathbb{E}_\nu \left( f^n_{t,\varepsilon} L^\varepsilon \left( \log f^n_{t,\varepsilon} \right) \right).
\]

By the tower property of \( \sigma \)-algebras, the first expectation is \( \mathbb{E}_\nu \left( \partial_t f^n_{t,\varepsilon} \right) = d1/dt = 0 \). Applying the definition \( (4.3) \) of \( L^\varepsilon \) and the inequality \( \log(b/a) \leq 2a^{-\frac{1}{2}}(b^{-\frac{1}{2}} - a^{-\frac{1}{2}}) \) (which holds for all non-negative \( a, b \)), we obtain

\[
\frac{d}{dt} H^n_t(f_{t,\varepsilon}) \leq \sum_{y_1 < y_2} \mathbb{E}_\nu \left[ 2c_{y_1, y_2} f^n_{t,\varepsilon} \left( f^n_{t,\varepsilon} - f^n_{t,\varepsilon} \right)^{-\frac{1}{2}} \left( \sigma_{y_1, y_2} f^n_{t,\varepsilon} \right)^{-\frac{1}{2}} \right].
\]

Write \( f^{n+m}_{t,\varepsilon} (f^n_{t,\varepsilon})^{-\frac{1}{2}} \) as the sum of \( (f^n_{t,\varepsilon})^{-\frac{1}{2}} \) and \( (f^{n+m}_{t,\varepsilon} - f^n_{t,\varepsilon}) (f^n_{t,\varepsilon})^{-\frac{1}{2}} \). Using (4.8), we have

\[
\frac{d}{dt} H^n_t(f_{t,\varepsilon}) \leq -2D(f^n_{t,\varepsilon}) + D',
\]

where

\[
D' := 2 \sum_{y_1 < y_2} \mathbb{E}_\nu \left[ c_{y_1, y_2} (f^{n+m}_{t,\varepsilon} - f^n_{t,\varepsilon}) (f^n_{t,\varepsilon})^{-\frac{1}{2}} \left( \sigma_{y_1, y_2} f^n_{t,\varepsilon} \right)^{-\frac{1}{2}} \right].
\]

Note that in (4.15) we need only to sum over

\[
\partial \Lambda_n := \{(y_1, y_2) : y_1 < y_2, |y_1 - y_2| \leq m, \text{ exactly one of } y_1, y_2 \in \Lambda_n\}.
\]

Indeed, when \( y_1, y_2 \in (\Lambda_n)^c \), we have \( \sigma_{y_1, y_2} f^n_{t,\varepsilon} - f^n_{t,\varepsilon} = 0 \), and when \( y_1, y_2 \in \Lambda_n \), by the tower property of \( \sigma \)-algebras we can replace \( f^{n+m}_{t,\varepsilon} \) in (4.15) by \( f^n_{t,\varepsilon} \). Applying the inequality \( ab \leq (a^2 R^{-1} + Rb^2)2^{-1} \), we further obtain, for any \( R > 0 \),

\[
D' \leq \frac{1}{2R} \sum_{(y_1, y_2) \in \partial \Lambda_n} D'_y (f^n_{t,\varepsilon}) + D'' \leq \frac{1}{2R} D^n + D'',
\]

where

\[
D' := \frac{R}{2} \sum_{(y_1, y_2) \in \partial \Lambda_n} \mathbb{E}_\nu \left[ c_{y_1, y_2} \left( \sigma_{y_1, y_2} f^n_{t,\varepsilon} \right)^{\frac{1}{2}} - (f^n_{t,\varepsilon})^{\frac{1}{2}} \right]^2 f^n_{t,\varepsilon}^{-1}.
\]

Since \( |c_{y_1, y_2}| \leq 1 \) and \( (a - b)^2b^{-1} \leq 2(a^2 + b^2)b^{-1} \), the random variable in (4.18) is bounded by

\[
2\left[ (f^n_{t,\varepsilon} / f^n_{t,\varepsilon}) (f^n_{t,\varepsilon} + f^n_{t,\varepsilon}) \right].
\]

Since \( 1 = \mathbb{E}_\nu \left[ f^n_{t,\varepsilon} / f^n_{t,\varepsilon} | \mathcal{G}_n \right] (\eta) \) per \( \eta \) is the equally weighted average of the \( 2^n \) values that \( f^n_{t,\varepsilon} \) can take, it follows that \( f^n_{t,\varepsilon} \leq 2^{2m} \). Hence \( D'' \leq CR(\# \partial \Lambda_n) \leq C(R) \). Combining this with (4.14) and (4.17), we obtain

\[
\frac{d}{dt} H^n_t(f_{t,\varepsilon}) + 2D^n (f^n_{t,\varepsilon}) \leq \frac{1}{2R} D^n + C(R).
\]

Consider (4.19) for \( n = n + j, j \in \{0, 1, \ldots\} \). Integrating in time, multiplying by \( e^{-bj} \), and summing over \( j \in \{0, 1, \ldots\} \), we obtain

\[
-\sum_{j=0}^{\infty} e^{-bj} H^n_t(f_{t,\varepsilon}) + 2D'' \leq \frac{e^{mb}}{2R} D'' + C(R)t,
\]
Lemma 4.3. \[ \text{mesoscopic average.} \]

For \( (4.22) \) have one such where \( (4.23) \) we rewrite the expectation in \( (4.27) \) as

\[ X \]

Note that by \( (4.11) \) and \( (4.12) \), the sums of \( (4.20) \) and \( (4.21) \) are finite for all large enough \( b \). Fix one such \( b \), choose \( R \) so that \( e^{mb}(2R)^{-1} \leq 1 \), yielding \( 2D''_0 - e^{mb}(2R)^{-1} D''_m \leq D''_0 \). By \( (4.11) \) we have \( \sum_{j=0}^{\infty} e^{-bj} H^n_i((t, \varepsilon)) \leq Cn \). Thus, we obtain

\[ \int_0^t D'(f_{s, \varepsilon})ds \leq D''_0 \leq Cn + Ct. \]

Finally, using the convexity of \( D' \) we conclude the proof.

Recall that \( \Psi : D \rightarrow \mathbb{R} \) is Lipschitz if there exists \( l \) and \( C \) such that for all \( \eta \) and \( \xi \),

\[ |\Psi(\eta) - \Psi(\xi)| \leq C \sum_{|y|<l} |\eta(y) - \xi(y)|. \]

Let \( f_{s, \varepsilon} := (t-s)^{-1} f_s \) denote the average over the time interval \( (s, t) \), and for any \( P \subset \mathbb{Z} \) or \( P \subset \mathbb{L} \), let \( \sum_P := (\# P)^{-1} \sum_P \) denote the average over \( P \). We have the readily verified inequality

\[ \sum_{|x| \leq n} f(x) - \sum_{|x'| \leq j} \sum_{|x-x'| \leq n} f(x + x') \leq \frac{j}{n} \sum_{|x-n| \leq j} f(x) + \sum_{|x+n| \leq j} f(x). \]

For \( g : D \rightarrow \mathbb{R} \), put \( (\tau_s g)(\eta) := g(\tau_s \eta) \), where \( \tau_s \eta)(x) := \eta(x + i) \).

Next we prove a replacement lemma that allows us to replace the microscopic average of \( \Psi \) by a mesoscopic average.

**Lemma 4.3.** For any Lipschitz cylinder function \( \Psi \), any \( T, R > 0 \), \( T_0 \geq 0 \), \( X_0 \in \mathbb{R} \),

\[ \lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \mathbb{E}_\nu \left( \int_{T_0}^{T_0 + T} \sum_{|x| \leq \varepsilon^{-1} X_0} \sum_{|x| \leq \varepsilon^{-1} R} \sum_{|y| \leq \varepsilon^{-1} n} \frac{\tau_x V_{\delta x - \frac{1}{2}}(\eta_{\delta x - \frac{1}{2}})}{\mathbb{E}_\nu(\Psi)} \right) = 0, \]

where

\[ V_n(\eta) := \left| \sum_{|x| \leq n} (\tau_x \Psi)(\eta) - \bar{\Psi}(\eta^n(0)) \right|, \]

\[ \eta^n(x) := \sum_{|y-x| \leq n} \eta(y), \]

\[ \bar{\Psi}(a) := \mathbb{E}_\nu(\Psi), \] which is a function on \([0, 1]\).

**Proof.** Since we impose no assumption on the initial condition, without lost of generality we assume \( X_0 = T_0 = 0 \), and let \( \Psi \) depends only on coordinates of \( \Lambda_{n_0} \). Let

\[ f_{T, \varepsilon} := \int_0^{\varepsilon^{-2} T} f_{t, \varepsilon} dt, \quad N_\varepsilon := \varepsilon^{-1} R + \delta \varepsilon^{-\frac{1}{2}} + n_0. \]

We rewrite the expectation in \( (4.23) \) as

\[ \mathbb{E}_\nu \left[ \sum_{|x| \leq \varepsilon^{-1} R} (\tau_x V_{\delta x - \frac{1}{2}}) f_{T, \varepsilon} \right]. \]
We next use (4.28) to reduce (4.23) to the one-block and two-blocks estimates. In the definition (4.24) of $V_n$, add and subtract $V^{(1)} := \sum_{|x| \leq \delta^{-1/2}} \tau_x V_t(\eta)$ to get $V_{\delta^{-1/2}} \leq V^{(1)} + V^{(2)} + V^{(3)}$, where

\begin{align*}
V^{(2)} := \sum_{|x| \leq \delta^{-1/2}} \left| \tilde{\Psi}(\eta'(x)) - \tilde{\Psi}(\eta'(0)) \right|, \\
V^{(3)} := \sum_{|x| \leq \delta^{-1/2}} \left| \Psi(\tau_x \eta) - \sum_{|x| \leq \delta^{-1/2}} \sum_{|x'| \leq \delta^{-1/2}} \Psi(\tau_{x+x'} \eta) \right|.
\end{align*}

By [14, Corollary 2.3.6] (which applies to any Lipschitz cylinder $\Psi$ and $\tilde{\Psi}$ as in (4.26) as long as $\nu_a \leq \nu_{a'}$ for $a < a'$), $\Psi$ is Lipschitz, yielding

\[ V^{(2)} \leq C \sum_{|x| \leq \delta^{-1/2}} \left| \eta'(x) - \eta'(0) \right|. \]

By (4.22), we get $V^{(3)} \leq C \frac{l}{\delta^2} \sup \eta |\Psi| \leq Cl^2 \delta^{-1} \epsilon^2$ (cylinder functions are bounded). Therefore,

\[
\lim_{\delta \to 0} \lim_{\epsilon \to 0} \mathbb{E}_\nu \left[ \sum_{|x| \leq \epsilon^{-1} R} (\tau_x V_{\delta^{-1/2}}) f_{T,\epsilon}^{N} \right] \leq C \lim_{l \to \infty} \lim_{\epsilon \to 0} \mathbb{E}_\nu \left[ \sum_{|x| \leq \epsilon^{-1} (R+1)} (\tau_x V_t) f_{T,\epsilon}^{N} \right]
\]

\[ + C \lim_{l \to \infty} \lim_{\delta \to 0} \lim_{\epsilon \to 0} \sup \mathbb{E}_\nu \left[ \sum_{|x| \leq \epsilon^{-1} R} f_{T,\epsilon}^{N} |\eta'(x + x') - \eta'(x)| \right]. \]

We thus reduce (4.23) to the following one-block estimate (4.31) and two-blocks estimate (4.32). □

**Proposition 4.4.** For any $R, T > 0$,

\[
\lim_{l \to \infty} \lim_{\epsilon \to 0} \mathbb{E}_\nu \left[ \sum_{|x| \leq \epsilon^{-1} R} (\tau_x V_t) f_{T,\epsilon}^{N} \right] = 0.
\]

**Proposition 4.5.** For any $R, T > 0$,

\[
\lim_{l \to \infty} \lim_{\delta \to 0} \lim_{\epsilon \to 0} \sup \mathbb{E}_\nu \left[ \sum_{|x| \leq \epsilon^{-1} R} f_{T,\epsilon}^{N} |\eta'(x + x') - \eta'(x)| \right] = 0.
\]

**Proof of Proposition 4.4.** For any probability density function $g$ on $\mathcal{D}$ define

\[
\bar{g} := \sum_{|x| \leq \epsilon^{-1} R} \tau_x g,
\]

\[
D^\infty_{l}(g) := \sum_{y_1, y_2 \in \Lambda_l} D^\infty_{y_1, y_2}(g).
\]

Note that $\bar{g}^l = \bar{g}^l$, and that $D^\infty_{l}$ is convex and lower-semicontinuous. Since $\nu$ is translation invariant and $V_{l-n_0} \in \mathcal{G}_l$, we replace $l$ by $l - n_0$ in (4.31) and rewrite the expectation as

\[
\mathbb{E}_\nu \left[ V_{l-n_0} f_{T,\epsilon}^{N} \right] = \mathbb{E}_\nu \left[ V_{l-n_0} f_{T,\epsilon}^{N} \right].
\]

Since the collection of all probability density functions on $\mathcal{D}_l := \{ \pm 1 \}^\Lambda_l$ is compact,

\[
\lim_{l \to \infty} \lim_{\epsilon \to 0} \mathbb{E}_\nu \left[ V_{l-n_0} f_{T,\epsilon}^{N} \right] = \lim_{l \to \infty} \mathbb{E}_\nu \left[ V_{l-n_0} f_{T,\epsilon}^{N} \right].
\]
where $f_\infty^l := \lim_{t \to 0} \bar f_{T,z}^l$, which is also a probability density function on $D_l$. Indeed, $\bar f^l$ is an average in expectation of $\tilde f^N_z$, and by (4.38) $\tilde f^N_z$ is an average over space of $f^N_z$. Using the convexity of $D_l^\infty$ twice, we have

\begin{equation}
(4.37) \quad D_l^\infty(\bar f_{T,z}^l) \leq D_l^\infty(\tilde f_{T,z}^N) \leq \sum_{|x| \leq \varepsilon R} D_l^\infty(\tau_x \tilde f_{T,z}^N).
\end{equation}

Since for $\varepsilon$ small enough $N_\varepsilon > \varepsilon - 1 R + l + m$, by (4.10) and Lemma 4.1 we further get

\begin{equation}
(4.38) \quad D_l^\infty(\bar f_{T,z}^l) \leq C l \varepsilon D^\nu(f_{T,z}^l) \leq C \varepsilon l.
\end{equation}

Using (4.38), and the lower-semicontinuity of $D_l^\infty$, we obtain $D_l^\infty(f_\infty^l) = 0$.

Since $r_1 > 0$, $D_l^\infty(f_\infty^l) = 0$ implies $\sigma_{y_1,y_2} f_\infty^l = f_\infty^l$, for all $y_1, y_2 \in \Lambda_l$ with $|y_1 - y_2| = 1$. We then deduce that $f_\infty^l$ is a function of $\sum_{|y| < l} \eta(y) = 2l \eta^l(0)$, and since $f_\infty^l$ is a probability density function, we have

\begin{equation}
(4.39) \quad \sum_{|y| \leq l} f_\infty^l(2i) \mathbb{P}_\nu(\eta^l(0) = i/l) = \mathbb{E}_\nu(f_\infty^l) = 1.
\end{equation}

Using (4.39) we bound the expectation of (4.36) as

\begin{equation}
\mathbb{E}_\nu [V_{i-n_0} f_\infty^l] = \sum_{|y| \leq l} f_\infty^l(i) \mathbb{E}_\nu(V_{i-n_0} | \eta^l(0) = i/l) \mathbb{P}_\nu(\eta^l(0) = i/l) \leq \max_{|i| \leq l} \mathbb{E}_\nu(V_{i-n_0} | \eta^l(0) = i/l).
\end{equation}

It suffices to show

\begin{equation}
(4.40) \quad \lim_{l \to \infty} \max_{|i| \leq l} \mathbb{E}_\nu(V_{i-n_0} | \eta^l(0) = i/l) = 0.
\end{equation}

When $\eta^l(0) = i/l$, we have $\tilde \Psi(\eta^l(0)) = \mathbb{E}_{\nu_{y_0}^l}(\Psi) := \psi(i)$, and by (4.22) we further get, for any $l' \leq l - n_0$,

\begin{equation}
\begin{aligned}
V_{i-n_0} &= \sum_{|x| \leq l - n_0} \tau_x \Psi + \psi(i) \\
&\leq \sum_{|x| \leq l - n_0} \tau_x V_{l'} + C_l^l V_{l'} \leq \sum_{|x| \leq l - n_0 - l'} \tau_x V_{l'} + C_l^l V_{l'}.
\end{aligned}
\end{equation}

Since $\nu$ is translation invariant, for each $|x| \leq l - l' - n_0$ we have $\mathbb{E}_\nu(\tau_x V_{l'} | \eta^l(0) = i/l) = \mathbb{E}_\nu(V_{l'} | \eta^l(0) = i/l)$. Therefore, for any $l' > 0$

\begin{equation}
\lim_{l \to \infty} \max_{|i| \leq l} \mathbb{E}_\nu(V_{i-n_0} | \eta^l(0) = i/l) \leq \lim_{l \to \infty} \max_{|i| \leq l} \mathbb{E}_\nu(V_{l'} | \eta^l(0) = i/l).
\end{equation}

Indeed, $V_{l'} \in G_{l'+n_0}$. For any $\xi \in D_{l+n_0}$, let $u$ be the number of its coordinates taking the value $+1$, and let $v = 2l + 2n_0 - u$ be the number of its coordinates taking the value $-1$. Then we have

\begin{equation}
F_{l}^{u,v}(\tau_x \Psi) = \prod_{j_1=0}^{u-1} \left( a - \frac{j_1}{2l} \right) \prod_{j_2=0}^{v-1} \left( 1 - a - \frac{j_2}{2l} \right) \prod_{j_3=0}^{u+v-1} \left( 1 - \frac{j_3}{2l} \right).
\end{equation}

For each $u$ and $v$, $\lim_{l \to \infty} F_{l}^{u,v}(\tau_x \Psi) = a^u(1-a)^v$ uniformly for $a \in [0,1]$. Thus for any $l' > 0$,

\begin{equation}
\lim_{l \to \infty} \max_{|i| \leq l} \mathbb{E}_\nu(V_{l'} | \eta^l(0) = i/l) \leq \sup_{a \in [0,1]} \mathbb{E}_{\nu_a}(\left| \sum_{|x| \leq l'} \tau_x \Psi - \mathbb{E}_{\nu_a}(\Psi) \right|).
\end{equation}
Finally, since \( \nu_a \) is the product of i.i.d. measures and \( \Psi \) is cylinder,
\[
\lim_{l \to \infty} \sup_{\alpha \in [0,1]} \mathbb{E}_{\nu_a} \left[ \left( \sum_{|x| \leq l} \tau_x \Psi - \mathbb{E}_{\nu_a} (\Psi) \right)^2 \right] = 0,
\]
concluding (4.40).

**Proof of Proposition 4.5.**

By (4.22), we have
\[
\begin{align*}
\eta^{x \pm \frac{1}{2}} (x) &= \sum_{|x'| = \pm \frac{1}{2}} \eta^I (x + x') + O(l(\delta \varepsilon^{-\frac{1}{2}} - 1)).
\end{align*}
\]
For each \( x' \), the contribution of those \( x'' \) with \( |x' - x''| \leq 2l \) to \( \left( \sum_{|x''| \leq \delta \varepsilon^{-\frac{1}{2}}} \eta^I (x + x'') \right) \) is of \( O(l(\delta \varepsilon^{-\frac{1}{2}} - 1)) \). Thus we reduce (4.32) to showing
\[
\lim_{l \to \infty} \lim_{\delta \to 0} \lim_{\varepsilon \to 0} \sup_{2l < |j| \leq \delta \varepsilon^{-\frac{1}{2}}} \mathbb{E}_\nu \left[ \sum_{|x| \leq \varepsilon^{-1} R} f_{T,\varepsilon}^{N_a} \eta^I (x + j) - \eta^I (x) \right] = 0.
\]

Let \( G'_l \) be the \( \sigma \)-algebra corresponding to the restriction to \( \Lambda_l \cup (j + \Lambda_l) \), and let
\[
g^{l,j}: = \mathbb{E}_\nu [\sigma(G_l, G'_l)].
\]
Similar to (4.35), we rewrite the expectation of (4.41) as \( \mathbb{E}_\nu [(\bar{f}_{T,\varepsilon})^{l,j} \eta^I (j) - \eta^I (0)] \). Since \( |j| > 2l \), \( (\bar{f}_{T,\varepsilon})^{l,j} \) is a probability density function on the configuration of two disjoint intervals \( \Lambda_l \cup (j + \Lambda_l) \). By translating the interval \( (j + \Lambda_l) \) to \( (2l + \Lambda_l) \), we obtain a probability density function \( (\bar{f}_{T,\varepsilon})^{l,j} \) on \( D'_l := \{ \pm \}^{\Lambda_l \cup (2l + \Lambda_l)} \). We further write (4.41) as \( \mathbb{E}_\nu [(\bar{f}_{T,\varepsilon})^{l,j} \eta^I (l) - \eta^I (0)] \). Similar to (4.36), by taking limits in \( \mathbb{E}_\nu [(\bar{f}_{T,\varepsilon})^{l,j} \eta^I (l) - \eta^I (0)] \) we reduce (4.41) to showing
\[
\lim_{l \to \infty} \mathbb{E}_\nu (f_{\infty} \eta^I (l) - \eta^I (0)) = 0,
\]
where \( f_{\infty} \) is the limiting probability density function on \( D'_l \)
\[
f_{\infty} := \lim_{l \to \infty} \lim_{\delta \to 0} \lim_{\varepsilon \to 0} \sup_{2l < |j| \leq \delta \varepsilon^{-\frac{1}{2}}} (\bar{f}_{T,\varepsilon})^{l,j}.
\]

For any probability density function \( g \) on \( D'_l \), define
\[
\bar{D}^\infty (g) := D^\infty (g) + D^\infty (\tau_{2l} g) + \tilde{\Delta} (g),
\]
\[
\tilde{\Delta} (g) := r_1 \mathbb{E}_\nu \left[ \left( (\sigma_{\frac{1}{2} + \frac{1}{2} + 2l}^2 - g^{\frac{1}{2}} \right)^2 \right].
\]

The Dirichlet forms \( D^\infty (\cdot) \) and \( D^\infty (\tau_{2l} \cdot) \) correspond to hops within \( \Lambda_l \) and within \( 2l + \Lambda_l \), respectively, and \( \tilde{\Delta} \) corresponds to hops between \( \frac{1}{2} \) and \( \frac{1}{2} + 2l \). Similar to (4.37) and (4.38), we have \( D^\infty (f_{\infty}) = 0 \) and \( D^\infty (\tau_{2l} f_{\infty}) = 0 \). As for \( \tilde{\Delta} \), we have
\[
\tilde{\Delta} (f_{T,\varepsilon})^{l,j} = r_1 \mathbb{E}_\nu \left[ \left( (\sigma_{\frac{1}{2} + \frac{1}{2} + j} (\bar{f}_{T,\varepsilon})^{l,j} - ((\bar{f}_{T,\varepsilon})^{l,j})^{\frac{1}{2}} \right)^2 \right].
\]
Without lost of generality, assume \( j > 0 \). Since for \( y_1 < y_2 \) the swap \( \sigma_{y_1,y_2} \) can be decomposed as
\[
\sigma_{y_1,y_2}: = \sigma_{y_1,y_1+1} \cdots \sigma_{y_2,y_1+1} \cdots \sigma_{y_1+2,y_1+1} \sigma_{y_1+1,y_1+1} \sigma_{y_1,y_1+1} g,
\]
we have
\[
\sigma_{\frac{1}{2} + \frac{1}{2} + j} g - g = \sum_{i=1}^{j} (\sigma_{\frac{1}{2} + \frac{1}{2} + j} g_i - g_i) + \sum_{i=1}^{j-1} (\sigma_{\frac{1}{2} + \frac{1}{2} + j} g_i - g_i).
where each \(g_i \) and \(\tilde{g}_i \) is a swapped \(g \) (i.e. \(\sigma_{y_{1}} \sigma_{y_{2}} \sigma_{y_{1} \cdots} g \)). Furthermore, since \(\nu\) is invariant under swapping, \(g, g_i, \) and \(\tilde{g} \) are equal in distribution under \(\nu\). Hence, applying the Cauchy-Schwartz inequality to (4.44) and using (4.9), we obtain

\[
\tilde{\Delta}((\tilde{f}_{T,\varepsilon})^{(i,j)}) \leq C\varepsilon \sum_{i=1}^{j} D_{i+1}^{\infty}(\tilde{f}_{T,\varepsilon})^{(i,j)}.
\]

By using the convexity of \(D_{i+1}^{\infty} \) similar to (4.37), we further get

\[
\tilde{\Delta}((\tilde{f}_{T,\varepsilon})^{(i,j)}) \leq C\varepsilon^{2} D^{x}(f_{T,\varepsilon}^{N}) \leq C\varepsilon^{2} \varepsilon.
\]

Since \(|j| \leq \delta \varepsilon^{-\frac{1}{2}} \), using the lower-semicontinuity of \(\tilde{\Delta} \), we conclude \(\tilde{\Delta}(f_{\infty}) = 0 \), yielding \(\tilde{D}_{i}^{\infty}(f_{\infty}) = 0\).

From \(\tilde{D}_{i}^{\infty}(f_{\infty}) = 0\) we deduce that \(f_{\infty}^{i} \) is a function of the total number of particles in \(\Lambda_{t} \cup (2l+\Lambda_{t})\). Hence similar to (4.40) we can decomposed the expectation of (4.43) according to \(\sum_{y \in \Lambda_{t} \cup (2l+\Lambda_{t})} \eta(y)\), and use the same argument following (4.40) to conclude (4.36).

**Proof of Lemma 2.5.** Given \(T > 0, n = 1, 2, \phi \in C_{c}^{0}([-R,R] \mathbb{R}) \), and \(\Phi_{t}(x)\) as defined in (2.18), it suffices to show that \(\lim_{\varepsilon \to 0} \mathbb{E}(|U_{\varepsilon}|) = 0\), where

\[
U_{\varepsilon} := \int_{0}^{T \varepsilon^{-2}} \sum_{|x| \leq \varepsilon^{-1} R} \Phi_{t}(x) Z_{g}^{n}(x) \phi(\varepsilon x) ds.
\]

For the Lipschitz cylinder function \(\Psi(\eta) = \prod_{l=1}^{\infty} \eta \) we have \(\Phi_{t}(x) = (T_{x} \Psi)(\eta_{t})\). Clearly, \(T_{x} \Psi(a) = a^{n_{0}}\).

Using (4.22) and \(\|\phi\|_{\infty} < \infty\), we obtain \(|U_{\varepsilon}| \leq C(|U_{1}^{\varepsilon,\delta}| + |U_{2}^{\varepsilon,\delta}|)\), where

(4.45) \[
U_{1}^{\varepsilon,\delta} := \int_{0}^{T \varepsilon^{-2}} \sum_{|x| \leq \varepsilon^{-1} R} \left( \sum_{0 \leq x' \leq 2 \varepsilon^{-1/2}} (T_{x} \Psi)(\eta_{t})(Z_{g}^{n}(x+x') \phi(\varepsilon(x+x'))) \right) ds,
\]

\[
U_{2}^{\varepsilon,\delta} := \delta^{-1} \varepsilon^{2} \sum_{|x| \leq \varepsilon^{-1} R} \int_{0}^{T \varepsilon^{-2}} Z_{g}^{n}(x) ds.
\]

By (4.31), \(\lim_{\varepsilon \to 0} \mathbb{E}(|U_{2}^{\varepsilon,\delta}|) = 0\), for any \(\delta > 0\).

The proof is completed upon showing \(\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \mathbb{E}(|U_{1}^{\varepsilon,\delta}|) = 0\), which we next do. Add and subtract \(Z_{n}(x) \phi(\varepsilon x) \) inside the parentheses of (4.45), add and subtract \(Z_{n}(x) \phi(\varepsilon x) \) inside the bracket of (4.45), and make the change of variable \(x + \delta \varepsilon^{-\frac{1}{2}} \rightarrow x\). We then get \(|U_{1}^{\varepsilon,\delta}| \leq U_{11}^{\varepsilon,\delta} + U_{12}^{\varepsilon,\delta} + U_{13}^{\varepsilon,\delta}\), where

\[
(\tilde{U}_{11}^{\varepsilon,\delta})_{o}(x) := ||\Psi||_{\infty} \sum_{|x'| \leq \varepsilon^{-1/2}} \left( Z_{g}^{n}(x+x') \phi(\varepsilon(x+x')) - Z_{g}^{n}(x-\delta \varepsilon^{-\frac{1}{2}}) \phi(\varepsilon(x-\delta \varepsilon^{-\frac{1}{2}})) \right),
\]

\[
(\tilde{U}_{12}^{\varepsilon,\delta})_{o}(x) := (T_{x} V_{\delta \varepsilon^{-1/2}})(\eta_{t}) Z_{g}^{n}(x-\delta \varepsilon^{-\frac{1}{2}}) ||\phi||_{\infty},
\]

\[
(\tilde{U}_{13}^{\varepsilon,\delta})_{o}(x) := \eta_{t}^{\delta \varepsilon^{-\frac{1}{2}}(x)} ||Z_{g}^{n}(x-\delta \varepsilon^{-\frac{1}{2}}) \phi||_{\infty}.
\]
By (3.32) and the continuity of $\phi$, $\lim_{\varepsilon \to 0} \mathbb{E}\left(U_{12}^{\varepsilon, \delta}\right) = 0$, for any $\delta > 0$. Applying the Cauchy-Schwartz inequality to $U_{12}^{\varepsilon, \delta}$, we obtain

$$\left|\mathbb{E}(U_{12}^{\varepsilon, \delta})\right|^2 \leq \mathbb{E}\left(\int_0^{T\varepsilon^{-2}} \sum_{|x - \delta \varepsilon^{-\frac{1}{2}}| \leq \varepsilon^{-1}R} (\tau_x V_{\delta \varepsilon^{-1/2}})^2(\eta_s)\,ds\right)$$

(4.46)

$$\mathbb{E}\left(\int_0^{T\varepsilon^{-2}} \sum_{|x - \delta \varepsilon^{-\frac{1}{2}}| \leq \varepsilon^{-1}R} Z_s^\varepsilon(x - \delta \varepsilon^{-\frac{1}{2}})^2\,ds\right).$$

By (4.24), $V_n$ is bounded (in $n$) for any bounded $\Psi$, so we can replace $(\tau_x V_{\delta \varepsilon^{-1/2}})^2(\eta_s)$ in (4.46) by $(\tau_x V_{\delta \varepsilon^{-1/2}})(\eta_s)$. By Lemma 4.3, the first expectation of (4.46) goes to zero under the iterated limit (lim$_{\delta \to 0}$lim$_{\varepsilon \to 0} \mathbb{E}(U_{12}^{\varepsilon, \delta}) = 0$). Finally, for $U_{13}$, given any $|x - \delta \varepsilon^{-1/2}| \leq \varepsilon^{-1}R$ and $s \in [0, \varepsilon^{-2}T]$, define

$$D_1(s, x) := \{|Z_s^\varepsilon(x - \delta \varepsilon^{-\frac{1}{2}}) - Z_s^\varepsilon(x + \delta \varepsilon^{-\frac{1}{2}})| \leq \varepsilon^\frac{1}{4}\},$$

$$D_2(s, x) := \{|Z_s^\varepsilon(x - \delta \varepsilon^{-\frac{1}{2}}) > \varepsilon^\frac{1}{4}\}. $$

By (3.32) for $u = \frac{1}{2}$ and the Markov inequality we have $\mathbb{P}(D_1(s, x)) \leq C \varepsilon^\frac{1}{4} + \frac{1}{4}$. Applying the Cauchy-Schwartz inequality and (4.25), we get

$$\mathbb{E}(\bar{U}_{13}^{\varepsilon, \delta}(s) \mathbb{I}_{D_1(s, x)}) \leq \mathbb{P}(D_1(s, x)) \frac{1}{4} \|Z_s^\varepsilon(x)^n\|_2 \leq C \varepsilon^\frac{1}{4}.$$

Indeed, for $D_2(s, x)$ we have

$$\mathbb{E}(\bar{U}_{13}^{\varepsilon, \delta}(s) \mathbb{I}_{D_2(s, x)}) \leq C \varepsilon^\frac{1}{4}.$$

From the definitions (3.8) and (4.25), we have

$$\eta_s^{\delta \varepsilon^{-\frac{1}{2}}}(x) = -\frac{1}{2\sqrt{\alpha \lambda}} \log \frac{Z_s^\varepsilon(x + \delta \varepsilon^{-\frac{1}{2}})}{Z_s^\varepsilon(x - \delta \varepsilon^{-\frac{1}{2}})}.$$

Since on $D_1(s, x) \cap D_2(s, x)$ we have

$$\frac{|Z_s^\varepsilon(x - \delta \varepsilon^{-\frac{1}{2}}) - Z_s^\varepsilon(x + \delta \varepsilon^{-\frac{1}{2}})|}{Z_s^\varepsilon(x - \delta \varepsilon^{-\frac{1}{2}})} \leq \varepsilon^\frac{1}{4},$$

applying the inequality $|\log(1 + t)| \leq 2|t|$, which holds for $|t| < \frac{1}{2}$, we bound $(\bar{U}_{13}^{\varepsilon, \delta}(s) \mathbb{I}_{D_2(s, x)} \cap D_2(s, x))$ by $C(\delta)\varepsilon^\frac{1}{4} Z_s^\varepsilon(x) - \varepsilon^{-\frac{1}{4}})^n$. Further applying (4.34), we get

$$\mathbb{E}(\bar{U}_{13}^{\varepsilon, \delta}(s) \mathbb{I}_{D_2(s, x) \cap D_3(s, x)}) \leq C(\delta)\varepsilon^\frac{1}{4} \leq C(\delta)\varepsilon^\frac{1}{4}.$$

Combining (4.37), (4.38), and (4.39), we obtain $\lim_{\varepsilon \to 0} \mathbb{E}(U_{13}^{\varepsilon, \delta}) = 0$ for any $\delta > 0$, concluding the proof.

5. Exact statistics.

In this section we consider the step initial condition (1.24). Instead of (1.21), we use the following modified scaled field

$$\bar{Z}_T^\varepsilon(X) := \beta \lambda(2\varepsilon^\frac{1}{4})^{-1} Z_T^\varepsilon(X),$$

(5.1)

where $\beta$ and $\beta'$ are as in (1.22). The extract factor of $\beta \lambda(2\varepsilon^\frac{1}{4})^{-1}$ ensures that $\bar{Z}_T^\varepsilon(\cdot)$ converges to $\delta_0(\cdot)$. Since $P_T(X)$, as defined in (1.25), satisfies $P_T(X) e^{F(X)} = 2 \bar{Z}_T^\varepsilon(X)$ and $P_T(X) e^{g(T, X)} = Z_T^\varepsilon(X)$, to prove Theorem 1.4 it suffices to show the convergence of $\bar{Z}_T^\varepsilon(\cdot)$ to the solution of (1.19) for the initial condition $\delta_0(\cdot)$. However, Theorem 1.3 does not apply directly to the initial condition...
From Lemma 5.1, conditions (1.23a) and (1.23b) hold for $Z_1$. We circumvent this problem following the same argument of [1]. First we show the following holds.

**Lemma 5.1.** For the step initial condition (1.24), given any $j \in \mathbb{N}$, $\delta > 0$, $u \in (0, 1)$, we have

\[
\|e^{-\frac{t}{\delta}}Z_{\varepsilon-x}\|_{L_j} \leq C(j, \delta),
\]

\[
\|e^{-\frac{t}{\delta}}(Z_{\varepsilon-x} - Z_{\varepsilon-x}(x'))\|_{L_j} \leq (\varepsilon|\varepsilon-x'|)^{2} C(j, \delta, u).
\]

From Lemma 5.1, conditions (1.23a) and (1.23b) hold for $Z_{\varepsilon}$, for any $\delta > 0$. We then apply Theorem 1.3 to conclude $Z_{\varepsilon}(\cdot) \Rightarrow Z(\cdot)$ on $[\delta, \infty) \times \mathbb{R}$, where $Z_{T}(X)$ satisfying the SHE (1.19) on $[\delta, \infty) \times \mathbb{R}$.

Next, the extension argument in [1] Section 3 extends $Z_{T}(X)$ to $(T, X) \in (0, \infty) \times \mathbb{R}$, yielding $Z_{\varepsilon}(\cdot) \Rightarrow Z(\cdot)$ on $(0, \infty) \times \mathbb{R}$, and

\[
Z_{T}(X) = \int_{\mathbb{R}} P_{T-\delta}(X-X')Z_{\delta}(X')dX' + \int_{T}^{\infty} \int_{\mathbb{R}} P_{T-S}(X-X')Z_{S}(X')W(dX'dS),
\]

for any $\delta > 0$. The proof is then completed upon showing the following

**Lemma 5.2.** When $\delta \to 0$, the RHS of (5.4) converges weakly to

\[
P_{T}(X) + \int_{0}^{T} \int_{\mathbb{R}} P_{T-S}(X-X')Z_{S}(X')W(dX'dS).
\]

Now we prove Lemma 5.1 and 5.2.

**Proof of Lemma 5.1.** Let $I_1, I_2, I_3, I_4$ denote the first, second, third, fourth terms on the RHS of (5.4), respectively; and let $J_i := \varepsilon^{-\frac{1}{2}}I_i$, $i = 1, \ldots, 4$. Note that $J_1$ is deterministic since $Z_{0}$ is. By (3.14), we have

\[
\|e^{-\frac{t}{\delta}}Z_{\varepsilon}^2\|_{J_i} \leq 4 \left( (J_1)^2 + \|J_2\|_j + \|J_3\|_j + \|J_4\|_j \right).
\]

Since $Z_{\varepsilon}^2(\cdot)$ approximates the delta function, we have

\[
\varepsilon^{\frac{1}{2}} \sum_{x} Z_{0}^{2}(x) \leq C.
\]

Since $J_1 = p_{t}^{*} \ast e^{-\frac{t}{\delta}}Z_{0}^{2}$, using (A.20) we have $J_1 \leq C(\varepsilon^2 t)^{-\frac{1}{2}}$. From (3.15), (3.18), and (3.19), we obtain

\[
\|J_2\|_j \leq C \int_{0}^{t} \varepsilon^{2} s^{-\frac{1}{2}} p_{t-s}^{2} * \|e^{-\frac{t}{\delta}}Z_{\varepsilon}^2\|_{j} ds.
\]

\[
\|J_3\|_j \leq C \int_{0}^{t} \varepsilon^{2} s^{-\frac{1}{2}} \nabla p_{t-s}^{2} * \|e^{-\frac{t}{\delta}}Z_{\varepsilon}^2\|_{j} ds,
\]

\[
\|J_4\|_j \leq C \int_{0}^{t} \varepsilon \sum_{|k| \leq m} \|\nabla k \nabla p_{t-s}^{2} * \|e^{-\frac{t}{\delta}}Z_{\varepsilon}^2\|_{j} ds.
\]

Combining the preceding estimates of $\|J_i\|_j$, $i = 1, \ldots, 4$, we arrive at the following inequality

\[
\|e^{-\frac{t}{\delta}}Z_{\varepsilon}^2\|_{j} \leq C g_{t}^{2} + C \int_{0}^{t} f_{t-s}^{2} * \|e^{-\frac{t}{\delta}}Z_{\varepsilon}^2\|_{j} ds,
\]

where

\[
g_{t} := (\varepsilon^2 t)^{-\frac{1}{2}} \left( p_{t}^{*} \ast Z_{0}^{2} \right), \quad f_{t} := \varepsilon t^{\frac{1}{2}} p_{t}^{2} + \sum_{|k| \leq m} \varepsilon |\nabla k p_{t}^{2}|.
\]

After iteration we get

\[
\|e^{-\frac{t}{\delta}}Z_{\varepsilon}^2\|_{j} \leq C g_{t} + \sum_{n=1}^{\infty} \frac{C^n}{n!} \int_{\Delta_n(t)} f_{s_1}^{2} \star \cdots \star f_{s_n}^{2} \ast g_{s_{n+1}} ds_1 \cdots ds_n,
\]
where $\Delta_n(t) := \{(s_1, \ldots, s_{n+1}) : s_1 + \ldots + s_{n+1} = t\}$. Using (5.6), (A.23), (A.24), (A.20), and (A.21), we have

$$g_t^c \leq C(\varepsilon^2 t)^{-1}, \quad f_t^c \leq C(\varepsilon t^{-1} + \varepsilon^2 t^{-\frac{1}{2}}),$$

$$\sum_x g_t^c(x) \leq Ce^{-2t^{-\frac{1}{2}}}, \quad \sum_x f_t^c(x) \leq C\left((\varepsilon^2 t)^{-\frac{1}{2}} + \varepsilon^2\right).$$

For $(s_1, \ldots, s_{n+1}) \in \Delta_n(t)$, we must have $s_{i*} \geq t(n+1)^{-1}$ for some $i* \in \{1, \ldots, n+1\}$. Hence, using (5.9) and (5.10), we bound the integrand in (5.8) by

$$\sum_{i=1}^{n} C^n \left( \prod_{i \neq i*, n} \left( \varepsilon s_i^{-\frac{1}{2}} + \varepsilon^2 \right) \right) \left( \varepsilon^{-2}(s_{n+1})^{-\frac{1}{2}} \right) \left( \varepsilon t^{-1}(n+1) + \varepsilon^2 t^{-\frac{1}{2}}(n+1)^{\frac{1}{2}} \right)

+ C^n \left( \prod_{i=1}^{n} \left( \varepsilon s_i^{-\frac{1}{2}} + \varepsilon^2 \right) \right) \left( \varepsilon^2 t(n+1)^{-1} \right)^{-1}.$$ 

For $t \in [\varepsilon^{-2}\delta/2, \varepsilon^{-2}\hat{T}]$, integrating in time, we bound the integral in (5.8) by $n^2 C(\delta)^n$. Hence

$$\|(\varepsilon^{-\frac{1}{2}}Z^\varepsilon_t)^2\|_j \leq C(\delta), \text{ when } t \geq \frac{\delta \varepsilon^{-2}}{2}.$$ 

This proves (5.2). Similarly, summing over $x$ in (5.8) by using (5.10), we then get

$$\sum_x \| (\varepsilon^{-\frac{1}{2}}Z^\varepsilon_t(x))^2 \|_j = \sum_x \| (\varepsilon^{-\frac{1}{2}}Z^\varepsilon_t(x))\|_{2j}^2 \leq C(\varepsilon^{-2}t^{-\frac{1}{2}} + 1).$$

Next we prove (5.3). Put $n = x' - x$ so that $Z^\varepsilon_t(x') - Z^\varepsilon_t(x) = \nabla_n Z^\varepsilon_t(x)$. First we have

$$\|\varepsilon^{-\frac{1}{2}}\nabla_n Z^\varepsilon_{x-\hat{n}}\|_{2j} \leq \sum_{i=1}^{4} \|\nabla_n J_i \|_{2j}.$$ 

For $i = 1$ we have $\|\nabla_n J_i \|_{2j} \leq |\varepsilon^{-1}\nabla_n p^\varepsilon_{x-\hat{n}}| \cdot \varepsilon^\frac{1}{2} Z^\varepsilon_0$. Applying (5.6) and (A.24) for $v = \frac{\varepsilon}{\hat{T}}$ we get $|\nabla_n p^\varepsilon_{x-\hat{n}}| \cdot Z^\varepsilon_0 \leq C(\delta)\|\varepsilon n\|^\frac{1}{2}$. Next, by (3.22), (3.23), and (3.26), we have

$$\|\nabla_n (J_2) \|_{2j} \leq C\varepsilon \int_0^{\varepsilon^{-2}\delta} |n|^{\mu} (t - s)^{-\frac{1}{2} + a}\left( p^\varepsilon_{[t-s]} + \tau n p^\varepsilon_{[t-s]} \right) \cdot \|\varepsilon^{-\frac{1}{2}}Z^\varepsilon_s\|_{2j}^2 ds,$$

$$\|\nabla_n (J_3) \|_{2j} \leq C|n|^{\mu} \int_0^{\varepsilon^{-2}\delta} \varepsilon^2(p^\varepsilon_{[t-s]} + \tau n p^\varepsilon_{[t-s]}) \cdot \|\varepsilon^{-\frac{1}{2}}Z^\varepsilon_s\|_{2j}^2 ds,$$

$$\|\nabla_n (J_4) \|_{2j} \leq C|n|^{\mu} \sum_{|k| \leq m} \int_0^{\varepsilon^{-2}\delta} \varepsilon(|\nabla_k p^\varepsilon_{[t-s]}| + |\tau n \nabla_k p^\varepsilon_{[t-s]}|) \cdot \|\varepsilon^{-\frac{1}{2}}Z^\varepsilon_s\|_{2j}^2 ds.$$ 

To bound the terms in (5.13), we divide the time integrals into an integral over $(0, \delta \varepsilon^{-2}/2)$ and an integral over $(\delta \varepsilon^{-2}/2, \delta \varepsilon^{-2})$. For the integral over $(0, \delta \varepsilon^{-2}/2)$ apply the inequality

$$\sum_{x'} |a(x')b(x')| \leq \sup_{x'} |a(x')| \sum_{x'} |b(x')|$$

with $b = \|\varepsilon^{-\frac{1}{2}}Z^\varepsilon_s\|_{2j}$ and $a$ being the rest of the integrand. By using (A.6), (A.24), and (5.12), we bound the integrals by $C(\delta)|n|^{\mu/2}$. Similarly, for the integral over $(\delta \varepsilon^{-2}/2, \delta \varepsilon^{-2})$ apply (5.14) with $a = \|\varepsilon^{-\frac{1}{2}}Z^\varepsilon_s\|_{2j}$ and $b$ being the rest of the integrand. By using (A.20), (A.21), and (5.11), we bound the integrals by $C(\delta)|n|^{\mu/2}$. Hence $\|\varepsilon^{-\frac{1}{2}}\nabla_n (J_i) \|_{2j} \leq C(\delta)|n|^{\mu/2}$, for $i = 2, 3, 4$, concluding (5.3). \qed
Proof of Lemma 5.2. Let \( A_\delta, B_\delta \) denote the first and second terms on the RHS of (5.4), respectively, and let \( A, B \) denote the first and second term of (5.3), respectively. By the Itô isometry,
\[
E \left( \int_0^\delta \int_\mathbb{R} P_{T-S} (X - X') Z_S (X') W (dX dS) \right)^2 = \int_0^\delta \int_\mathbb{R} P_{T-S} (X - X')^2 E (Z_S (X'))^2 dX' dS.
\]
Using the boundedness of \( P_{T-S} (X - X')^2 \) and (5.12), we further bound this expression by
\[
(5.15) \quad C \lim_{\varepsilon \to 0} \int_0^\delta \varepsilon \sum_{x'} E \left( \varepsilon^{-\frac{1}{2}} Z_{\beta_\varepsilon - 2S} (\beta' x') \right)^2 dS \leq C \lim_{\varepsilon \to 0} \int_0^\delta \varepsilon S^{-\frac{1}{2}} dS = C \delta^{\frac{1}{2}}.
\]
Since the RHS of (5.15) converges to zero as \( \delta \to 0 \), \( B_\delta \) weakly converges to \( B \).

Let \( (K_\varepsilon^\gamma)_T (X) := \lambda \beta^{-1} (J_{\varepsilon})_{\beta_{\varepsilon - 2T} (\beta' x^{-1} X)} \). By definition \( A_\delta \) is the weak limit of
\[
\sum_{i = 1}^4 \lim_{\varepsilon \to 0} P_{T-\delta} \ast (K_\varepsilon^\gamma)_{\delta}.
\]

Since \( Z_{\delta} (\cdot) \) approximates the delta function, by (A.5) and (1.22) we have \( (K_\varepsilon^\gamma)_{\delta} (X) \xrightarrow{\varepsilon \to 0} P_{\beta} (X) \), yielding \( \lim_{\varepsilon \to 0} P_{T-\delta} \ast (K_\varepsilon^\gamma)_{\delta} = P_{T} \). Next, using (5.7), (5.12), (A.20), and the boundedness of \( P_{T-\delta} \), for \( i = 2, 3, 4 \) we have
\[
\lim_{\delta \to 0} \left\| \lim_{\varepsilon \to 0} P_{T-\delta} \ast (K_\varepsilon^\gamma)_{\delta} \right\|_2 \leq C \lim_{\delta \to 0} \lim_{\varepsilon \to 0} \varepsilon \sum_{x'} \left\| ((J_{\varepsilon})_{\varepsilon-2\delta} (x')) \right\|_2 = 0,
\]

concluding the proof. \( \square \)

APPENDIX A.

Proof of Lemma 1.3. We first solve the equation (1.13) without the \( O(\varepsilon^2) \), that is
\[
(A.1) \quad A^\varepsilon \tilde{r}^\varepsilon = 4^{-1} (u (\varepsilon) A - \varepsilon v (\varepsilon) A \gamma^\varepsilon),
B\tilde{r}^\varepsilon = 4^{-1} (u (\varepsilon) R \gamma^\varepsilon - v (\varepsilon) r),
\]

where \( A \) and \( R \) are the \( m \)-dimensional square matrices \( A_{jk} := \mathbb{I}_{\{j \leq k\}} \) and \( R := \mathbb{I}_{\{j = k\}} r_k \), and \( r := \{r_1, \ldots, r_m\} \). Since \( A^\varepsilon \) is invertible for all \( \varepsilon \) small enough, by multiplying the first equation of (A.1) by \( (A^\varepsilon)^{-1} \) and substituting it into the second equation, we arrive at the following equivalent equations
\[
(A.2) \quad \left( R + \varepsilon \frac{v (\varepsilon)}{u (\varepsilon)} B (A^\varepsilon)^{-1} A \right) \gamma^\varepsilon = \left( \frac{v (\varepsilon)}{u (\varepsilon)} + B (A^\varepsilon)^{-1} A \right) r,
\]

where \( R \) has full rank, (A.2) has a unique solution \( (\gamma (\varepsilon), \tilde{r} (\varepsilon)) \). Moreover, since \( (A^\varepsilon)^{-1} \), \( u (\varepsilon), v (\varepsilon) \) are \( C^\infty \) in \( \varepsilon \) for small enough \( \varepsilon \) (even at \( \varepsilon = 0 \)), and \( u (0) = 2 \lambda > 0, \gamma (\varepsilon) \) and \( \tilde{r} (\varepsilon) \) are also \( C^\infty \) in \( \varepsilon \). By solving (A.2) at \( \varepsilon = 0 \) (notice that \( A^\varepsilon = 1/2 A \) at \( \varepsilon = 0 \)), we obtain
\[
\gamma_k (0) = \lambda \left( \frac{2}{r_k} \sum_{k' = k+1}^m \frac{k' - k}{k} r_{k'} + 1 \right), \quad \tilde{r}_k (0) = r_k.
\]

By choosing \( \gamma_k := \frac{d \gamma_k}{d \varepsilon} (0) \), for any given \( \gamma^\varepsilon \) and \( \tilde{r}^\varepsilon \) of the form (1.10) and (1.17), we have \( \gamma^\varepsilon - \gamma (\varepsilon) = O(\varepsilon) \) and \( \tilde{r}^\varepsilon - \tilde{r} (\varepsilon) = O(\varepsilon^{\frac{3}{2}}) \). Tracking the coefficients multiplying \( \gamma_k^\varepsilon \) and \( \tilde{r}_k^\varepsilon \) in (1.13), we find that the difference of \( O(\varepsilon^2) \) between \( \gamma^\varepsilon \) and \( \gamma (\varepsilon) \) and the difference of \( O(\varepsilon^{\frac{3}{2}}) \) between \( \tilde{r}^\varepsilon \) and \( \tilde{r} (\varepsilon) \) will only contribute \( O(\varepsilon^2) \) to (1.13), concluding the proof. \( \square \)
Next, we provide some estimates of the semi-discrete heat kernel $p^\varepsilon$, as defined in (A.1). Solving (A.1) by Fourier series, we have

$$
\tag{A.3} p^\varepsilon(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ix\theta} \exp\left[-t\phi(\theta)\right] d\theta,
$$

where $\phi(\theta) := \sum_{k=1}^{m} (1 - \cos(k\theta))r^\varepsilon_k$.

From (A.4) and (A.1) we have

$$
\tag{A.4} 0 < c_0 \leq \phi(\theta) \theta^{-2} \leq C < \infty, \text{ for all } \theta \in [-\pi, \pi] \setminus \{0\}.
$$

Indeed,

$$
\varepsilon^{-1} p^\varepsilon(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ix\theta} \exp\left[-T\sum_{k=1}^{m} r^\varepsilon_k (1 - \cos(\varepsilon^{-1}))\right] d\theta.
$$

For each $T > 0$, by (A.4), the integrand is bounded by $e^{-c_0 T \theta^2}$. Hence by the dominated convergence theorem we have, for any $(T, X) \in (0, \infty) \times \mathbb{R}$,

$$
\tag{A.5} \lim_{\varepsilon \to 0} \varepsilon^{-1} p^\varepsilon(x) = P_{\alpha T}(X),
$$

where $\alpha$ is defined as in (A.20).

**Proposition A.1.** Given any $b \geq 0$, for any $|k| \leq m$, $n \in \mathbb{Z}$, $v \in [0, 1]$, $0 \leq t < t' < \infty$, and $x \in \mathbb{Z}$, we have

$$
\tag{A.6} p^\varepsilon_t(x) \leq e^{C(t'-t)} p^\varepsilon_t(x),
$$

$$
\tag{A.7} |p^\varepsilon_t(x) - p^\varepsilon_t(x)| \leq (1 + t^{-\frac{1}{2}-\nu}) (t' - t)^v C,
$$

$$
\tag{A.8} |\nabla_k p^\varepsilon_t(x) - \nabla_k p^\varepsilon_t(x)| \leq (1 + t^{-1-\nu}) (t' - t)^v C,
$$

$$
\tag{A.9} p^\varepsilon_t(x) \leq C(b)(1 + t^{-\frac{1}{2}}) e^{-b|x|/\sqrt{1+t^{-1/2}}},
$$

$$
\tag{A.10} |\nabla_n p^\varepsilon_t(x)| \leq C(b)(1 + t^{-\frac{3}{2}+\nu}) |n|^v e^{-b|x|/\sqrt{1+t^{-1/2}}},
$$

$$
\tag{A.11} |\nabla_n \nabla_k p^\varepsilon_t(x)| \leq C(b)(1 + t^{-\frac{5}{2}+\nu}) |n|^v e^{-b|x|/\sqrt{1+t^{-1/2}}}.
$$

**Proof.** The estimate (A.6) is derived the same way as (A.5).

Next we prove prove (A.7). Take the difference $p^\varepsilon_t(x) - p^\varepsilon_t(x)$ using (A.3), and then use (A.4) and the readily verified identity

$$
\tag{A.12} 1 - e^{-a} \leq 1 + a^v
$$

(which holds for $a \geq 0$) to bound the integrand by $C[1 + ((t'-t)\theta^2)^v] e^{-c_0 t^2 \theta^2}$. This expression can be further bounded by

$$
\tag{A.13} C(t'-t)^v (1 + \theta^{2a}) e^{-c_0 t^2 \theta^2},
$$

because $|\theta| \leq \pi$. To get (A.7), First use the bound $C(t'-t)^v$ in (A.13) and integrate over $\theta$, and then use the bound $C(t'-t)^v \theta^{2v} e^{-c_0 t^2 \theta^2}$ in (A.13) and integrate over $\mathbb{R}$. Similarly, modifying (A.3) by taking the discrete gradient, we get (A.8) through the same reasoning.

To prove (A.9), (A.10), and (A.11) we derive another integral expression of $p^\varepsilon_t$. First by making the change of variable $z = e^{i\theta}$ in (A.3), we turn the integral over $\theta \in [-\pi, \pi]$ into a contour integral of $e^{-ix\theta - t\phi(z)}$ along $\{ |z| = 1 \} \subset C$, where $\psi(z) := \sum_k r^\varepsilon_k [1 - (z + z^{-1})^2]$. Since this integrand is holomorphic on $C \setminus \{0\}$, we can deform the original contour $\{ |z| = 1 \}$ and integrate along $\{ |z| = 1 + \delta \}$, where

$$
\tag{A.14} \delta := e^{b \text{sign}(x) (1 + t^{-1/2})} - 1.
$$
Making another change of variable \((1+\delta)e^{-i\theta} = z\) in this new contour integral, we obtain
\[
(A.15) \quad \mathbf{p}_{T}^{e}(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (1+\delta)^{-x} e^{-iz\theta} \exp \left[-t\psi((1+\delta)e^{i\theta})\right] d\theta.
\]
Next, by the definition of \(\psi\) and \(\phi\) we have
\[
(A.16) \quad \psi((1+\delta)e^{i\theta}) = \phi(\theta) + i\delta \sum_{k=1}^{m} \tilde{\gamma}_{k} \sin(k\theta) + \frac{\delta^{2}}{2(1+\delta)} \sum_{k=1}^{m} \tilde{\gamma}_{k} e^{-ik\theta}.
\]
Combining (A.4), (A.13), and (A.16), and integration over \([-\pi, \pi]\), we bound \(\mathbf{p}_{T}^{e}(x)\) by \(C(1+\delta)^{-x}(1\wedge t^{-\frac{1}{2}})e^{C\delta^{2}t}\). By (A.14) and (A.12), we have \(\delta^{2}t \leq C(b)\) and \((1+\delta)^{-x} = e^{-b|x|/\sqrt{1+t^{-1}}/2}\), concluding (A.9).

Next, we turn to (A.10). Modifying (A.15) by taking the discrete gradient \(\nabla_{n}\), we get
\[
(A.17) \quad \nabla_{n} \mathbf{p}_{T}^{e}(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} S(\theta) \exp \left[-t(1-\psi((1+\delta)e^{i\theta}))\right] d\theta,
\]
where \(S(\theta) := (1+\delta)^{-x+n-1} e^{-i(x+n)\theta} (1+\delta)^{-x} e^{-i2\theta}\). Write \(S(\theta)\) as the sum of \((1+\delta)^{-x} e^{-i(x+n)\theta} ((1+\delta)^{-n}-1)\) and \((1+\delta)^{-x} e^{-i\theta} (e^{-ib\theta}-1)\). By (A.14) and (A.12), we have
\[
|1+\delta|^{-n} - 1 \leq C(b)|n(1\wedge t^{-1/2})|^{v},
\]
yielding,
\[
(A.18) \quad |S(\theta)| \leq C(b)|n|^{v}(1\wedge t^{-v/2} + |\theta|^{v}) e^{-b|x|/\sqrt{1+t^{-1/2}}},
\]
Combining this with (A.17), and (A.16), and integrating over \(\theta\), we obtain (A.10).

As for (A.10), similar to (A.10), and integrating over \(\theta\), we obtain (A.11).

Proposition A.1 immediately implies the following corollary

**Corollary A.2.** Given any \(b \geq 0\), for any \(|k| \leq m\), \(v \in [0,1]\), \(t \in [0, T] e^{-2}\), \(x \in \mathbb{Z}\), and \(j \in \mathbb{N}\), we have
\[
(A.20) \quad \sum_{x} \mathbf{p}_{T}^{e}(x) e^{b|x|} \leq C(b),
\]
\[
(A.21) \quad \sum_{x} |\nabla_{n} \mathbf{p}_{T}^{e}(x)| \leq C(b)t^{-\frac{1}{2}} |n|^{v},
\]
\[
(A.22) \quad \sum_{x} |\nabla_{n} \nabla_{k} \mathbf{p}_{T}^{e}(x)| \leq C(b)t^{-\frac{1}{2}} |n|^{v},
\]
\[
(A.23) \quad \sup_{x} \mathbf{p}_{T}^{e}(x) e^{b|x|} \leq t^{-\frac{1}{2}} C(b),
\]
\[
(A.24) \quad \sup_{x} |\nabla_{n} \mathbf{p}_{T}^{e}(x)| e^{b|x|} \leq C(b)(1\wedge t^{-\frac{1}{2}})|n|^{v}.
\]
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