SCATTERING AND BLOW-UP CRITERIA FOR 3D CUBIC FOCUSING NONLINEAR INHOMOGENEOUS NLS WITH A POTENTIAL

QING GUO, HUA WANG AND XIAOHUA YAO

ABSTRACT. In this paper, we consider the 3d cubic focusing inhomogeneous nonlinear Schrödinger equation with a potential

\[ iu_t + 
\Delta u - Vu + |x|^{-b}|u|^2 u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3, \]

where \(0 < b < 1\). We first establish global well-posedness and scattering for the radial initial data \(u_0\) in \(H^1(\mathbb{R}^3)\) satisfying \(M(u_0)^{3-\epsilon} E(u_0)^{\epsilon} < E\) and \(\|u_0\|_{L^2}^{2(1-\epsilon)} \|H^\epsilon u_0\|_{L^2}^{\epsilon} < \mathcal{K}\) provided that \(V\) is repulsive, where \(E\) and \(\mathcal{K}\) are the mass-energy and mass-kinetic of the ground states, respectively. Our result extends the results of Hong [19] and Farah-Guzmán [11] with \(b \in (0, \frac{1}{2})\) to the case \(0 < b < 1\). We then obtain a blow-up result for initial data \(u_0\) in \(H^1(\mathbb{R}^3)\) satisfying \(M(u_0)^{3-\epsilon} E(u_0)^{\epsilon} < E\) and \(\|u_0\|_{L^2}^{2(1-\epsilon)} \|H^\epsilon u_0\|_{L^2}^{\epsilon} > \mathcal{K}\) if \(V\) satisfies some additional assumptions.

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1. Introduction

In this paper, we consider a 3d cubic focusing inhomogeneous NLS with a potential (INLS\(_V\))

\[
\begin{cases}
    iu_t - \Delta u + |x|^{-b}|u|^2 u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3, \\
u(0, x) = u_0(x) \in H^1(\mathbb{R}^3),
\end{cases}
\]

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where \( u : I \times \mathbb{R}^3 \to \mathbb{C} \) is a complex-valued function, \( 0 < b < 1 \), \( H = H_0 + V \), \( H_0 = -\Delta \). Here \( V : \mathbb{R}^3 \to \mathbb{R} \) is a real-valued short range potential with a small negative part, more precisely,

\[
V \in K_0 \cap L^2
\]

and

\[
\|V_-\|_K < 4\pi,
\]

where the potential class \( K_0 \) is the closure of bounded compactly supported functions with respect to the global Kato norm

\[
\|V\|_K = \sup_{x \in \mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|V(y)|}{|x - y|} \, dy
\]

and \( V_-(x) = \min\{V(x), 0\} \) is the negative part of \( V \). In the case \( V = 0 \) and \( b = 0 \), Holmer-Roudenko [18] and Duyckaerts-Homer-Roudenko [9] obtained the sharp criteria for global well-posedness and scattering in terms of conservation laws of the equation (1.1), where blow up result requires initial data is radial. Subsequently, for \( b = 0 \), Hong [19] established an analogous global well-posed and scattering result provided that \( V \) satisfies (1.2) and (1.3), \( V \geq 0, x \cdot \nabla V \leq 0 \) and \( |x| |\nabla V| \in L^2 \). However, he cannot give any blow up result. More recently, for \( V = 0 \), Farah-Guzán [11] and Dinh [7] extended the scattering result and the blow up result obtained by Holmer-Roudenko [18] to \( 0 < b < \frac{1}{2} \) and \( 0 < b < 1 \) under the radial assumption on the initial data \( u_0 \), respectively.

The mainly part of this paper is devoted to get a similar criteria for global well-posedness and scattering for (1.1) with the radial data \( u_0 \) under the similar condition on \( V \) as that in [19] and over the wider interval \( 0 < b < 1 \). Additionally, we further give a non-scattering or blow-up result based on the method of Du-Wu-Zhang [8] under some additional assumptions on \( V \).

Before the statement of our results, we briefly review some related results for the general \( \text{INLS}_V \) equation

\[
\begin{cases}
\ iu_t - Hu + g(x)|u|^{p-1}u = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}^N, \\
\ u(0, x) = u_0(x),
\end{cases}
\]

For \( p = 1 + \frac{4}{N} \), several authors have investigated critical mass blow-up solutions. For example, Banica-Carles-Duyckaerts [1] showed the existence of critical mass blow up solutions if \( V \in C^\infty(\mathbb{R}^N, \mathbb{R}) \) and \( g \in C^\infty(\mathbb{R}^N, \mathbb{R}) \) is sufficiently flat at a critical point. When \( V \equiv 0 \) and \( g \in C^\infty(\mathbb{R}^N, \mathbb{R}) \) is positive and bounded, Merle [24] and Raphaël-Szeftel [25] derived conditions on \( g \) for existence/nonexistence of minimal mass blow-up solutions. In the above works, \( V(x) \) and \( g(x) \) are both smooth. While Combet-Genoud [3] studied the classification of minimal mass blow-up solutions in the case \( V \equiv 0 \) and \( g(x) = |x|^{-b} \) with \( 0 < b < \min\{2, N\} \), \( N \geq 1 \). Besides, when \( V(x) = -\frac{1}{|x|^c} \) with \( 0 < c < \frac{(N-2)p}{2}, N \geq 3 \), Csobo-Genoud [4] constructed and classified finite time blow-up solutions at the minimal mass threshold.

Next we recall some well-posedness and scattering results for \( V(x) \equiv 0 \) and \( g(x) = |x|^{-b} \) with \( 0 < b < \min\{2, N\} \). One can easily see that the equation (1.4) is invariant under the scaling transformation \( u(t, x) = \lambda^{\frac{-2+b}{N}} u(\lambda^2 t, \lambda x) \), which also leaves the norm of the homogeneous Sobolev space \( \dot{H}^s(\mathbb{R}^N) \) invariant, where \( s_c = \frac{N}{2} - \frac{2-b}{p-1} \). So we call that the equation (1.4) is mass-supercritical.
and energy-subcritical for \(1 + p_s < p < 1 + p^*\) (i.e., \(0 < s_c < 1\)), where

\[
p^* = \begin{cases} \frac{4 - 2b}{N - 2}, & N \geq 3, \\ \infty, & N = 1, 2, \end{cases} \quad p_s = \frac{4 - 2b}{N}.
\]

Energy-criticality appears with the power \(p = 1 + \frac{4 - 2b}{N - 2}\) (i.e., \(s_c = 1\)) and mass-criticality with power \(p = 1 + \frac{4 - 2b}{N} \) (i.e., \(s_c = 0\)). Genoud-Stuart [15], using the abstract theory developed by Cazenave [2], showed that (1.4) is locally well-posed in \(H^1(\mathbb{R}^N)\) if \(1 < p < 1 + p^*\) and globally if \(1 < p < 1 + p_s\) for any initial data and \(1 + p_s \leq p < 1 + p^*\) for small initial data. Recently, Guzmán [16] gave an alternative proof of these results using the contraction mapping principle based on the Strichartz estimates.

When \(p = 1 + p_s\), Genoud [14] showed that (1.4) is global well-posed in \(H^1(\mathbb{R}^N)\) if \(u_0 \in H^1(\mathbb{R}^N)\) and

\[
||u_0||_{L^2} < ||Q_m||_{L^2},
\]

where \(Q_m\) is the ground state solution of the nonlinear elliptic equation

\[
\Delta Q_m - Q_m + |\cdot|^{-b}||Q_m||^{\frac{4 - 2b}{N}} Q_m = 0.
\]

On the other hand, Combet and Genoud [3] obtained the classification of minimal mass blow-up solutions for (1.4) with \(p = 1 + p_s\). When \(1 + p_s < p < 1 + p^*\), Farah [10] proved that (1.4) is globally well-posed in \(H^1(\mathbb{R}^N)\), \(N \geq 3\), assuming that \(u_0 \in H^1(\mathbb{R}^N)\),

\[
E_0(u_0)^s \cdot M(u_0)^{1 - s} < E_0(Q) \cdot M(Q)^{1 - s},
\]

and

\[
||\nabla u_0||_{L^2}^{2s} ||u_0||_{L^2}^{2(1 - s)} < ||\nabla Q||_{L^2}^{2s} ||Q||_{L^2}^{2(1 - s)},
\]

where \(E_0\) and \(M\) are a functional in (1.8) and (1.9), and \(Q\) is unique positive radial solution of the elliptic equation

\[
\Delta Q - Q + |\cdot|^{-b}||Q||^{p - 1} Q = 0.
\]

Farah [10] also considers the case

\[
||\nabla u_0||_{L^2}^{2s} ||u_0||_{L^2}^{2(1 - s)} > ||\nabla Q||_{L^2}^{2s} ||Q||_{L^2}^{2(1 - s)},
\]

which combined with (1.5) implies that the solution blows up in finite time if \(u_0\) satisfies \(|\cdot|u_0 \in L^2\). In the radial case for \(u_0\), Dinh [7] removed the the condition \(|\cdot|u_0 \in L^2\).

Moreover, Farah-Guzmán [11, 12] established scattering in the case that \(1 + p_s < p < 1 + 2^*, 0 < b < \min\{\frac{3}{2}, 1\}\) and \(u_0\) is radial, where

\[
2^* = \begin{cases} p^*, & N = 2, N \geq 4, \\ 3 - 2b, & N = 3. \end{cases}
\]

We note that, for \(N = 3\), the authors imposed an extra assumption, namely, \(1 + p_s < p < 1 + (3 - 2b)\) (when \(p = 3\), \(0 < b < \frac{1}{2}\)). Then they raised a question whether scattering holds under the condition \(1 + p_s < p < 1 + p^* = 1 + (4 - 2b)\). One purpose of this present paper is to give an affirmative answer to this question when \(N = 3\) and \(p = 3\). More precisely, we prove that scattering is true when \(0 < b < 1\).

When \(b = 0\), \(V(x)\) is inverse-square potential, Killip-Murphy-Visan-Zheng [23] established the sharp criteria for the global well-posedness and scattering in terms of conservation laws of (1.1). And Hong [19] established a similar result for real-valued short range and repulsive potential for
the equation (1.1) as mentioned above. In view of these results, we are further aimed at extending Hong’s result to $0 < b < 1$.

Under the assumptions (1.2) and (1.3), the Cauchy problem for (1.1) is locally well-posed in $H^1(\mathbb{R}^3)$ (see local theory Lemma 2.2). Moreover, the $H^1$ solution obeys the mass and energy conservation laws,

$$M(u) = \int_{\mathbb{R}^3} |u(x)|^2 dx,$$

and the energy is defined by

$$E(u) = E_V(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u(x)|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} V(x)|u(x)|^2 dx - \frac{1}{4} \int_{\mathbb{R}^3} |x|^{-b}|u(x)|^4 dx.$$

When $V$ vanishes, we just replace $E(u)$ by $E_0(u)$.

To state our main results, we need to introduce some notation as follows:

$$\mathcal{E} = \begin{cases} M(Q)^{1-\kappa} E_0(Q)^{\kappa}, & \text{if } V \geq 0, \\ M(Q)^{1-\kappa} E(Q)^{\kappa}, & \text{if } V < 0, \end{cases}$$

$$\mathcal{K} = \begin{cases} ||Q||_{L^2}^{2(p-1)} \||\nabla Q||_{L^2}^{2p}, & \text{if } V \geq 0, \\ ||Q||_{L^2}^{2(p-1)} \||H^{\frac{3}{2}} Q||_{L^2}^{2p}, & \text{if } V < 0, \end{cases}$$

where $Q$ is the ground state for the elliptic equation (1.7) with $p = 3$ and $Q$ solves the elliptic equation

$$(-\Delta + V)Q + w_0^2 Q - |x|^{-b} |Q|^2 Q = 0, \quad w_0 = \frac{\sqrt{1 - b}}{\sqrt{3 + b}} ||H^{\frac{3}{2}} Q||_{L^2}$$

(see Lemma 3.3 for details). It follows from Remark 3.2 and Lemma 3.3 in section 3 and (4.6), (4.7), (4.9) and (4.10) in section 4 that

$$\mathcal{E} = \left(\frac{s_c}{3 + b}\right)^{\kappa} \mathcal{K} \quad \text{and} \quad \mathcal{K} = \frac{4}{(3 + b)C_{GN}} = \frac{4}{(3 + b)C_{rad}^{rad}}$$

where $C_{GN}$ and $C_{rad}^{rad}$ are the sharp constants in the Gagliardo-Nirenberg inequalities with the potential $V$, respectively. It is worth pointing out that under out assumption (1.2), $C_{GN} = C_{rad}^{rad}$, while, in [23], $C_{rad}^{rad} < C_{GN}$ when $V(x) = \frac{a}{|x|^a}$ with $a > 0$. On the other hand, we will see in section 3 that $C_{GN} = C_{rad}^{rad}$ never be attained when $V$ is nonnegative and not zero a.e. on $\mathbb{R}^3$, which is another different phenomenon from the inverse-square-potential case ($V(x) = \frac{a}{|x|^a}$ with $a > 0$); while $C_{GN} = C_{rad}^{rad}$ can be reached by $Q$ when $V_{-} = 0$. (One can find more details in section 3.)

Our first result provides criteria for global well-posedness in terms of the mass-energy $\mathcal{E}$ and a critical number $\mathcal{K}$, which is involved with the kinetic energy.

**Theorem 1.1.** Suppose that $V$ is radially symmetric and satisfies (1.2) and (1.3), and $0 < b < 1$. We assume that

$$M(u_0)^{1-\kappa} E(u_0)^{\kappa} < \mathcal{E}.$$  

Let $u(t)$ be the solution to (1.1) with initial data $u_0 \in H^1(\mathbb{R}^3)$.

(i) If

$$\|u_0\|_{L^2}^{2(p-1)} \|H^{\frac{3}{2}} u_0\|_{L^2}^{2p} < \mathcal{K},$$
then \( u(t) \) exists globally in time, and
\[
\|u_0\|_{L^2}^{2(1-s)c}\|H^{s}u(t)\|_{L^2}^{2s} < \mathcal{K}, \forall t \in \mathbb{R}.
\]
(ii) If
\[
\|u_0\|_{L^2}^{2(1-s)c}\|H^{s}u_0\|_{L^2}^{2s} > \mathcal{K},
\]
then
\[
\|u_0\|_{L^2}^{2(1-s)c}\|H^{s}u(t)\|_{L^2}^{2s} > \mathcal{K}
\]
during the maximal existence time.

**Remark 1.2.** (i) Theorem 1.1 also holds provided that nonnegative \( V \) satisfies
\[
V, \nabla V \in L^\delta + L^{\infty}
\]
for some \( \delta \geq \frac{1}{2} \), or
\[
V \in L^\delta + L^{\infty}
\]
for some \( \delta > \frac{1}{2} \). If \( V \) satisfies (1.15), then the local wellposedness is true by Remark 4.4.8 in [2], Remark 2) on page 103 in [26] and Corollary 1.6 in [16], where the contraction mapping principle based on the Strichartz estimates is used. If \( V \) satisfies (1.16), then the local wellposedness is true by Theorem 4.3.1 in [2] and Theorem K.1 and Lemma K.2 in [15], where the abstract theory developed by Cazenave is used.

(ii) The radial condition on \( V \) in Theorem 1.1 is only used in the case that the initial data \( u_0 \) is radial, which will be applied to the following scattering result.

Another result is to show that the global solutions in Theorem 1.1 also scatters provided that \( u_0 \) is radial, \( V \) is repulsive and \( 0 < b < 1 \).

**Theorem 1.3.** Let \( V \) be radially symmetric and satisfy (1.2) and (1.3), and assume that \( x\nabla V(x) \leq 0, |x||\nabla V| \in L^\frac{3}{2} \) and \( 0 < b < 1 \). If \( u_0 \) is radial data in \( H^1(\mathbb{R}^3) \) and satisfies
\[
M(u_0)^{1-s}E(u_0) < E
\]
and
\[
\|u_0\|_{L^2}^{2(1-s)c}\|H^{s}u_0\|_{L^2}^{2s} < \mathcal{K}
\]
then \( u(t) \) scatters in \( H^1(\mathbb{R}^3) \). That is, there exists \( \phi_\infty \in H^1(\mathbb{R}^3) \) such that
\[
\lim_{t \to \pm \infty} \|u(t) - e^{itA}\phi_\infty\|_{H^1(\mathbb{R}^3)} = 0.
\]

**Remark 1.4.** In the defocusing case and without potentials, Dinh [6] obtained scattering in \( H^1(\mathbb{R}^N) \) provided that \( N \geq 4, 0 < b < 2, 1 + p_* < p < 1 + p^*, \) or \( N = 3, 0 < b < 1, 1 + \frac{2b}{3} < p < 1 + (3 - 2b), \) or \( N = 2, 0 < b < 1, 1 + p_* < p < 1 + p^* \). It is easy to see that when \( N = 3 \) and \( p = 3, b \) still satisfies \( 0 < b < \frac{1}{6} \). By small modifications of the proofs of Theorem 1.3, one can also obtain scattering for 3d cubic defocusing INLS\( V \)
\[
\frac{iu_t - H u - |x|^{-b}|u|^2 u = 0,}{u(0, x) = u_0(x) \in H^1(\mathbb{R}^3),}
\]
(1.17)
provided that \( u_0 \) is radial, \( 0 < b < 1 \) and the confining part of the potential \( (x \cdot \nabla V(x))_+ = \max\{x \cdot \nabla V(x), 0\} \) is small, precisely,

\[
\|(x \cdot \nabla V(x))_+\|_K < 8\pi.
\]

In other words, our result extends the result of Dinh \[6\] with \( 0 < b < \frac{1}{2} \) into \( 0 < b < 1 \) in the case of the radial data. For details of the proof, one can also refer to the one of Theorem B.1 in Hong \[19\].

Finally, we turn to establish the blow-up criterion. To this end, we need introduce another functional associated with the called Virial type identity.

\[
K(u) = \int |\nabla u|^2 dx - \frac{1}{2} \int (x \cdot \nabla V)|u|^2 dx - \frac{3}{4} b \int |x|^{-b}|u|^4 dx.
\]

It follows from Remark 1.2 (i) that Theorem 1.1 holds provided that nonnegative \( V \in L^\delta \) for some \( \delta > \frac{3}{2} \). Under some additional assumptions on \( V \), that is, \( x \cdot \nabla V \in L^\delta \) and \( V \) satisfies the following (1.20), we apply the method of Du-Wu-Zhang \[8\] to obtain a blow-up result, which will be stated as follow.

**Theorem 1.5.** Suppose that nonnegative \( V \), \( x \cdot \nabla V \in L^\delta \) for some \( \delta > \frac{1}{2} \) and \( V \) satisfies

\[
(1.20) \quad x \cdot \nabla V \leq 0, \quad \text{and} \quad 2V + x \cdot \nabla V \geq 0.
\]

We assume that \( 0 < b < 1 \) and

\[
M(u_0)^{1-\varepsilon} E(u_0)^{\varepsilon} < \mathcal{E}.
\]

Let \( u \in C([0, T_{max}), H^1(\mathbb{R}^3)) \) be the solution to (1.1) with initial data \( u_0 \in H^1(\mathbb{R}^3) \). If

\[
(1.21) \quad \|u_0\|_{L^2}^{2(1-\varepsilon)} \|H^{\frac{1}{2}} u_0\|_{L^2}^{\frac{2}{\varepsilon}} > \mathcal{K},
\]

then one of the following two statements holds true:

(i) \( T_{max} < \infty \), and

\[
\lim_{t \uparrow T_{max}} \|\nabla u(t)\|_{L^2} = \infty.
\]

(ii) \( T_{max} = \infty \), and there exists a time sequence \( \{t_n\} \) such that \( t_n \rightarrow \infty \), and

\[
\lim_{t_n \uparrow T_{max}} \|\nabla u(t_n)\|_{L^2} = \infty.
\]

**Remark 1.6.** If \( V \) is radial, the condition (1.20) implies that \( |x|^{-2} \leq |V(x)| \) for large \( |x| \), which deduces that \( V \not\in L^7 \). So we don’t give the blow up result under the condition (1.2) in this paper.

Actually, the proof of Theorem 1.5 can be obtained by the following result.

**Theorem 1.7.** Under the same assumptions as in Theorem 1.5, if there exists \( \beta_0 < 0 \) such that there holds

\[
(1.22) \quad \sup_{t \in [0, T_{max})} K(u(t)) \leq \beta_0 < 0,
\]

then there exists no global solution \( u \in C([0, T_{max}), H^1(\mathbb{R}^3)) \) with

\[
(1.23) \quad \sup_{t \in \mathbb{R}^+} \|\nabla u(t, \cdot)\|_{L^2} < \infty.
\]
This present paper is organized as follows. We fix notations at the end of section 1. In section 2, we establish Strichartz type estimates, upon which we obtain linear scattering, local theory, the small data scattering and the perturbation theory. The variational structure of the ground state of an elliptic problem is given in section 3. In section 4, we prove a dichotomy proposition of global well-posedness versus blowing up, which yields the comparability of the total energy and the kinetic energy. The concentration compactness principle is used in section 5 to give a critical element, which yields a contradiction through a virial-type estimate in section 6, concluding the proof of Theorem 1.3. In the last section, we use the localized virial identity to give the proofs of Theorem 1.5 and Theorem 1.7.

Notations:

We fix notations used throughout the paper. In what follows, we write \( A \lesssim B \) to signify that there exists a constant \( C \) such that \( A \leq C B \), while we denote \( A \sim B \) when \( A \lesssim B \sim A \).

Let \( L^q = L^q(\mathbb{R}^N) \) be the usual Lebesgue spaces, and \( L^q L^r_x \) or \( L^q(I, L^r_x) \) be the space of measurable functions from an interval \( I \subset \mathbb{R} \) to \( L^r_x \) whose \( L^q_I L^r_x \)-norm \( \|u\|_{L^q_I L^r_x} \) is finite, where

\[
\|u\|_{L^q_I L^r_x} = \left( \int_I \|u(t)\|_{L^r_x}^q \, dt \right)^{\frac{1}{q}}.
\]

(1.24)

When \( I = \mathbb{R} \) or \( I = [0, T] \), we may use \( L^q_t L^r_x \) instead of \( L^q_I L^r_x \), respectively. In particular, when \( q = r \), we may simply write them as \( L^q_t \) or \( L^q_{T,x} \), respectively.

Moreover, the Fourier transform on \( \mathbb{R}^N \) is defined by

\[
\hat{f}(\xi) = (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-ix\cdot\xi} f(x) \, dx.
\]

For \( s \in \mathbb{R} \), define the inhomogeneous Sobolev space by

\[
H^s(\mathbb{R}^N) = \{ f \in S'(\mathbb{R}^N) : \int_{\mathbb{R}^N} (1 + |\xi|^2)^{\frac{s}{2}} |\hat{f}(\xi)|^2 \, d\xi < \infty \}
\]

and the homogeneous Sobolev space by

\[
\dot{H}^s(\mathbb{R}^N) = \{ f \in S'(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\xi|^2 |\hat{f}(\xi)|^2 \, d\xi < \infty \},
\]

where \( S'(\mathbb{R}^N) \) denotes the space of tempered distributions.

Given \( p \geq 1 \), let \( p' \) be the conjugate of \( p \), that is \( \frac{1}{p} + \frac{1}{p'} = 1 \).

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2. Preliminaries

We start in this section with recalling the Strichartz estimates and norm equivalence established by Hong [19]. We say a pair \((q, r)\) is Schrödinger admissible, or \( L^2 \)-admissible, if \( 2 \leq r \leq \infty \) and

\[
\frac{2}{q} + \frac{3}{r} = \frac{3}{2}.
\]
We say that a pair \((q, r)\) is \(\dot{H}^s\)-admissible and denote it by \((q, r) \in \Lambda_s\) if \(0 \leq s < 1\), \(\frac{6}{r} \leq r \leq 6\) and
\[
\frac{2}{q} + \frac{3}{r} = \frac{3}{2} - s
\]
Correspondingly, we call the pair \((q', r')\) dual \(\dot{H}^s\)-admissible, denoted by \((q', r') \in \Lambda'_s\), if \((q, r) \in \Lambda_{-s}\), \((\frac{6}{r} - 2)^2 \leq r \leq 6\) and \((q', r')\) is the conjugate exponent pair of \((q, r)\). In particular, \((q, r) \in \Lambda_0\) is just a \(L^2\)-admissible pair.

We define the Strichartz norm by
\[
\|u\|_{S(L^2, I)} := \sup_{(q, r); L^2, I \text{-admissible}} \|u\|_{L^2(I, L^r')}
\]
and its dual norm by
\[
\|u\|_{S'(L^2, I)} := \inf_{(q, r); L^2, I \text{-admissible}} \|u\|_{L^r(I, L^2')}
\]
We also define the exotic Strichartz norm by
\[
\|u\|_{S(\dot{H}^s, I)} := \sup_{(q, r) \in \Lambda_s} \|u\|_{L^2(I, L^r')}
\]
and its dual norm by
\[
\|u\|_{S'(\dot{H}^s, I)} := \inf_{(q, r) \in \Lambda_s} \|u\|_{L^r(I, L^2')}
\]
Combining the results obtained by [20] and [13], the following Strichartz estimates and Kato inhomogeneous Strichartz estimates on \(I = [0, T]\) are true: If \(V\) satisfies (1.2) and (1.3), then
\[
(2.1) \quad \left\| e^{-it\mathcal{H}} f + \int_0^t e^{-i(t-s)\mathcal{H}} F(s, \cdot) ds \right\|_{S(\dot{H}^s, I)} \leq \|f\|_{\dot{H}^s} + \|F\|_{S'(\dot{H}^{-s}, I)}.
\]
If the time interval \(I\) is not specified, we take \(I = \mathbb{R}\), and \(S(\dot{H}^s, I)\) can be abbreviated as \(S(\dot{H}^s)\), similarly for \(S'(\dot{H}^{-s}, I)\).

In addition, in order to establish local theory, the two norm equivalent relations between the standard Sobolev norms and the Sobolev norms associated with \(H\) are needed: If \(V\) satisfies (1.2) and (1.3), then
\[
(2.2) \quad \|\dot{H}^s f\|_{L^2} \sim \|\dot{H}^s_0 f\|_{L^2} \sim \|\|\nabla\|^s f\|_{L^2} \text{ and } \|(1 + H)^s f\|_{L^2} \sim \|(1 + H_0)^s f\|_{L^2} \sim \|\|\nabla\|^s f\|_{L^2}
\]
where \(s \in [0, 2]\) and \(1 < r < \frac{3}{s} \).

As a simple application of (2.2), the following linear scattering result holds.

**Lemma 2.1.** [19] (i) If \(V\) satisfies (1.2) and (1.3), then for any given \(\phi \in L^2(\mathbb{R}^3)\), there exist \(\phi^\pm\) such that
\[
(2.3) \quad \lim_{t \to \pm \infty} \|e^{-it\mathcal{H}} \phi - e^{-it\mathcal{H}} \phi^\pm\|_{L^2(\mathbb{R}^3)} = 0.
\]
(ii) If further assume that \(\nabla V \in L^2\), then for any given \(\phi \in H^1(\mathbb{R}^3)\), there exist \(\phi^\pm\) such that
\[
(2.4) \quad \lim_{t \to \pm \infty} \|e^{-it\mathcal{H}} \phi - e^{-it\mathcal{H}} \phi^\pm\|_{H^1(\mathbb{R}^3)} = 0.
\]

We note that the statement and the proof of the following local theory are similar to those for (INLSO) (see Corollary 1.6 in Guzmán [16]). The only difference in the proof is that the norm equivalence is used in several steps.
**Lemma 2.2.** If $V$ satisfies (1.2) and (1.3), and $u_0 \in H^1(\mathbb{R}^3)$, then initial value problem (1.1) INLS$_V$ is locally well-posed in $H^1(\mathbb{R}^3)$ and

$$u \in C([0, T], H^1(\mathbb{R}^3)) \cap L^6([-T, T], H^1(\mathbb{R}^3))$$

for any $(q, r)$ $L^2$-admissible.

Before we show the small data scattering theory, we shall rely on the Sobolev inequality (see Theorem $B'$ in Stein-Weiss [27]) to get three crucial estimates.

**Lemma 2.3.** Let $u : I \times \mathbb{R}^3 \to \mathbb{C}$ be a complex function, then the following estimates hold.

(i) 
(2.5) \[ \|\nabla (|x|^{-b}|u|^2 u)\|_{S'(L^2, L^2)} \lesssim \|\nabla^{\alpha}|u|^2\|_{L^s(\mathbb{R}^3)} \|\nabla u\|_{L^2(\mathbb{R}^3)} \|\nabla u\|_{S'(L^2, L^2)} \lesssim \|\nabla u\|_{L^2(\mathbb{R}^3)} \|u\|_{L^2(\mathbb{R}^3)} \|u\|_{L^2(\mathbb{R}^3)}, \]

(ii) 
(2.6) \[ \|x^{-b}|u|^2 u\|_{S'(L^2, L^2)} \lesssim \|\nabla^{\alpha}|u|^2\|_{L^s(\mathbb{R}^3)} \|\nabla u\|_{L^2(\mathbb{R}^3)} \|\nabla u\|_{S'(L^2, L^2)} \lesssim \|\nabla u\|_{L^2(\mathbb{R}^3)} \|u\|_{L^2(\mathbb{R}^3)} \|u\|_{L^2(\mathbb{R}^3)} \]

and

(iii) \[ \|x^{-b}|u|^2 u\|_{S'(H^{s-1}, L^2)} \lesssim \|\nabla^{\alpha}|u|^2\|_{L^s(\mathbb{R}^3)} \|\nabla u\|_{L^2(\mathbb{R}^3)}^2 \|u\|_{L^2(\mathbb{R}^3)}^2 \]

**Proof.** We first recall the Sobolev inequality.

**Lemma 2.4.** Let $1 < p \leq q' < \infty$, $N \geq 1$, $0 < s < N$, and $\alpha, \beta \in \mathbb{R}$ obey the conditions

\[ \alpha > -\frac{N}{p'}, \]
\[ \beta > -\frac{N}{q'}, \]
\[ \alpha + \beta \leq 0 \]

and the scaling condition

\[ \alpha + \beta - N + s = -\frac{N}{p'} - \frac{N}{q'}, \]

Then for any $u : \mathbb{R}^N \to \mathbb{C}$, we have

(2.8) \[ \|x^\beta u\|_{L^p(\mathbb{R}^N)} \lesssim_{\alpha, \beta, p, q, s, t} \|x^\alpha |\nabla|^s u\|_{L^p(\mathbb{R}^N)}. \]

Next we give the proof of (2.5). Using Leibnitz rule gives

(2.9) \[ \|\nabla (|x|^{-b}|u|^2 u)\|_{S'(L^2, L^2)} \lesssim \|x^{-b-1}|u|^2 u\|_{S'(L^2, L^2)} \|x^{-b-1}|u|^2 u\|_{L^2(\mathbb{R}^3)}. \]

To control $\|x^{-b-1}|u|^2 u\|_{S'(L^2, L^2)}$, it follows from the definition of $S'(L^2, L^2)$ and Hölder inequality that

(2.10) \[ \|x^{-b-1}|u|^2 u\|_{S'(L^2, L^2)} \lesssim \|x^{-b-1}|u|^2 u\|_{L^2(\mathbb{R}^3)} \]
\[ \lesssim \|x^{-s}|u|_{L^2} \|x^{-s}|u|_{L^2} \|u|_{L^2} \|u|_{L^2}. \]

Using Hardy inequality yields

(2.11) \[ \|x^{-s}|u|_{L^2} \leq \|\nabla^{\alpha}|u|^2\|_{L^2} , \]
and using (2.8) with $\beta = -s_c$, $q' = 3$, $\alpha = 0$, $s = 1$ and $p = \frac{6}{1+q}$ gives
\begin{equation}
|||x||^{-s} u||^2_{L^{p}_t L^{6}_x} \leq |||\nabla u||^2_{L^{p}_t L^{\frac{6}{1+q}}_x}.
\end{equation}
(2.12)

Substituting (2.11) and (2.12) in the (2.10), using Hölder inequality in the time variable $t$ and noting that (4, 3), $\left(\frac{4}{3}, \frac{6}{3+q}\right) \in \Lambda_0$ and $\left(\frac{4}{3q}, 6\right) \in \Lambda_{s_c}$, we have
\begin{equation}
|||x||^{-b} |u|^2 u||_{S(L^{2},J)} \leq \left||||x||^{-b} u||_{L^{p}_t L^{6}_x} |||\nabla u||_{L^{p}_t L^{\frac{6}{1+q}}_x} |||u||_{L^{p}_t L^{\frac{6}{1+q}}_x} \right|
\leq |||\nabla x ||^{s_c} u||_{L^{p}_t L^{\frac{6}{1+q}}_x} |||\nabla u||_{L^{p}_t L^{\frac{6}{1+q}}_x} |||u||_{L^{p}_t L^{\frac{6}{1+q}}_x} \right|
\leq |||\nabla x ||^{s_c} u||_{S(L^{2},J)} |||\nabla u||_{S(L^{2},J)} |||u||_{S(H^{s_c},J)} \right|
\leq |||\nabla u||_{S(L^{2},J)} |||u||_{S(H^{s_c},J)} \right|
\end{equation}
(2.13)
where in the last step we have used the interpolation.

To control $|||x||^{-b} \nabla(|u|^2 u)||_{S(L^{2},J)}$, we apply Leibnitz rule and Hölder inequality to get
\begin{equation}
|||x||^{-b} \nabla(|u|^2 u)||_{S(L^{2},J)} \leq \left||||x||^{-b} u^* u^* \nabla u^* \right|
\leq \left||||x||^{-b} u^* ||_{L^{p}_t L^{6}_x} |||u^*||_{L^{p}_t L^{\frac{6}{1+q}}_x} |||\nabla u^*||_{L^{p}_t L^{\frac{6}{1+q}}_x} \right|
\end{equation}
(2.14)
where $u^*$ is either $u$ or $\bar{u}$. By (2.8) with $\beta = -b$, $q' = \frac{6}{3+q}$, $\alpha = 0$, $s = s_c = \frac{1+b}{2} \frac{1}{p}$ and $p = 3$, we have
\begin{equation}
|||x||^{-b} u||_{L^{p}_t L^{\frac{6}{1+q}}_x} \leq \left|||\nabla u||_{L^{p}_t L^{\frac{6}{1+q}}_x} \right|
\end{equation}
(2.15)
Substituting (2.15) in the (2.14), using Hölder inequality in the time variable $t$ and noting that (4, 3), $\left(\frac{4}{3}, \frac{6}{3+q}\right) \in \Lambda_0$ and $\left(\frac{4}{3q}, 6\right) \in \Lambda_{s_c}$, we have
\begin{equation}
|||x||^{-b} \nabla(|u|^2 u)||_{S(L^{2},J)} \leq \left||||x||^{-b} u||_{L^{p}_t L^{6}_x} |||\nabla u||_{L^{p}_t L^{\frac{6}{1+q}}_x} |||u||_{L^{p}_t L^{\frac{6}{1+q}}_x} \right|
\leq \left||||x||^{-b} u||_{L^{p}_t L^{6}_x} |||\nabla u||_{L^{p}_t L^{\frac{6}{1+q}}_x} |||u||_{L^{p}_t L^{\frac{6}{1+q}}_x} \right|
\leq \left||||x||^{-b} u||_{S(L^{2},J)} |||\nabla u||_{S(L^{2},J)} |||u||_{S(H^{s_c},J)} \right|
\leq \left||||x||^{-b} u||_{S(L^{2},J)} |||\nabla u||_{S(L^{2},J)} |||u||_{S(H^{s_c},J)} \right|
\end{equation}
(2.16)
where in the last step we have used the interpolation.

Putting (2.9), (2.13) and (2.16) together, we complete the proof of (2.5).

From the process for (2.14)-(2.16), we easily obtain that
\begin{equation}
|||x||^{-b} |u|^2 u||_{S(L^{2},J)} \leq \left||||x||^{-b} u||_{S(L^{2},J)} |||\nabla u||_{S(L^{2},J)} |||u||_{S(H^{s_c},J)} \right|
\end{equation}
(2.17)
Finally, we turn to the estimate of (2.7). we apply Leibnitz rule and Hölder inequality to get
\begin{equation}
|||x||^{-b} |u|^2 u||_{S(H^{s_c},J)} \leq \left||||x||^{-b} u^* u^* \right|
\end{equation}
(2.18)
By (2.8) with \( \beta = -b, \quad q' = \frac{4}{d_1 - 1}, \quad \alpha = 0, \quad s = s_c = \frac{3d_1}{2d_1 - 2}, \quad p = 2, \) we have

\[
\| |x|^{-b} u \|_{L^{\infty}_{t,x}} \leq \| |\nabla|^s u \|_{L^2_x}.
\]

(2.19)

Substituting (2.19) in the (2.18), using Hölder inequality in the time variable \( t \) and noting that 
\((\infty, 2) \in \Lambda_0 \) and \(( \frac{d}{d - 1}, 6) \in \Lambda_s, \) we have

\[
\| |x|^{-b} u \|^2 \|_{S(H^{\infty}, L^2)} \leq \left\| \| |\nabla|^s u \|_{L^2_x} \|_{L^\infty_t} \right\|_{L^2_t} \| u \|_{L^2_t L^\infty_x} \leq \| |\nabla|^s u \|_{S(L^2, L^\infty)}^2 \| u \|_{S(H^{\infty}, L^2)}^2.
\]

(2.20)

\[\boxed{\Box}\]

**Proposition 2.5.** If \( V \) satisfies (1.2) and (1.3). Assume \( u_0 \in H^1(\mathbb{R}^n) \) and \( \| u_0 \|_{H^1} \leq A. \) Then there exists \( \delta_d > 0 \) such that \( \| e^{-itH} u_0 \|_{S(H^{\infty})} \leq \delta_d, \) then there exists a unique global solution \( u \) of (1.1) with initial data \( u_0 \) such that

\[
\| u \|_{S(H^{\infty})} \leq 2\| e^{-itH} u_0 \|_{S(H^{\infty})}, \quad \| (\nabla) u \|_{S(L^2)} \leq 2\| u_0 \|_{H^1}.
\]

(2.21)

**Proof.** For \( M = c\| u_0 \|_{H^1}, \) we define a map as

\[
\Phi(v) = e^{-itH} u_0 + i \int_0^t e^{-i(t-s)H} |x|^{-b} |v|^2 v(s) ds
\]

(2.22)

and a set as

\[
B = \{ v : \| v \|_{S(H^{\infty})} \leq 2\| e^{-itH} u_0 \|_{S(H^{\infty})}, \quad \| (\nabla) v \|_{S(L^2)} \leq 2M \}
\]

(2.23)

equipped with the metric

\[
d(u, v) = \| u - v \|_{S(L^2)} + \| u - v \|_{S(H^{\infty})}
\]

By Strichartz estimates (2.1) with \( s = 0 \) and the norm equivalence (2.2),

\[
\| (\nabla) e^{-itH} u_0 \|_{S(L^2)} \sim \|(1 + H)^{\frac{d}{2}} e^{-itH} u_0 \|_{S(L^2)} \leq \|(1 + H)^{\frac{d}{2}} u_0 \|_{S(L^2)} \sim \| u_0 \|_{H^1}.
\]

(2.24)

By the Kato Strichartz estimates (2.1) and (2.7),

\[
\| \int_0^t e^{-i(t-s)H} |x|^{-b} |v|^2 v(s) ds \|_{S(H^{\infty})} \leq \| |x|^{-b} |v|^2 \|_{S(H^{\infty})}
\]

(2.25)

By the Kato Strichartz estimates (2.1), the norm equivalence (2.2), (2.5) and (2.6),

\[
\| (\nabla) \int_0^t e^{-i(t-s)H} |x|^{-b} |v|^2 v(s) ds \|_{S(L^2)} \sim \| (1 + H)^{\frac{d}{2}} \int_0^t e^{-i(t-s)H} |x|^{-b} |v|^2 v(s) ds \|_{S(L^2)}
\]

(2.26)

\[\boxed{\Box}\]
Therefore, we obtain that
\[
\|\Phi(v)\|_{S(I^\omega)} \leq \|e^{-itH}u_0\|_{S(I^\omega)} + c\|\|\nabla\|_{S(L^2)}|v|\|^2_{S(I^\omega)}
\]
(2.27)
\[
= \|e^{-itH}u_0\|_{S(I^\omega)} + 8c^2\|u_0\|_H^2\|e^{-itH}u_0\|_{S(I^\omega)}^2
\]
and
\[
\|\nabla\Phi(v)\|_{S(L^2)} \leq c\|u_0\|_H + c\|\nabla\|_{S(L^2)}|v|\|_{S(I^\omega)}
\]
(2.28)
\[
\leq c\|u_0\|_H + 8c^3\|u_0\|_H^2\|e^{-itH}u_0\|_{S(I^\omega)}.
\]
Now we let
\[
\delta_{ad} = \min\left\{\frac{1}{16c^2\|u_0\|_H^2}, \frac{1}{8c^3\|u_0\|_H}\right\}.
\]
Then \(\Phi(v) \in B\). The contraction property can be obtained by similar arguments. Therefore, by the contraction mapping theorem, \(\Phi(v)\) has a unique fixed point \(u \in B\), which is a global solution of (1.1). \(\square\)

Now we turn to the following scattering result, which can be combined with Proposition 2.5 to get a scattering result of small data.

**Proposition 2.6.** If \(V\) satisfies (1.2) and (1.3). Let \(u(t) \in C(\mathbb{R}, H^1(\mathbb{R}^3))\) be a radial solution of (1.1) such that \(\sup_{t \in \mathbb{R}} \|u(t)\|_H < B\), where \(B > 0\). If \(\|u\|_{S(I^\omega)} < \infty\), then \(u(t)\) scatters in \(H^1(\mathbb{R}^3)\). That is, there exists \(\phi^* \in H^1(\mathbb{R}^3)\) such that
\[
\lim_{t \to \pm \infty} \|u(t) - e^{-itH}\phi^*\|_{H^1(\mathbb{R}^3)} = 0.
\]

**Proof.** First, we claim that
\[
\|\nabla u\|_{S(L^2)} < \infty.
\]
(2.29)
Indeed, since \(\|u\|_{S(I^\omega)} < \infty\), given \(\delta > 0\), we can decompose \(0, \infty)\) into \(n\) intervals \(I_j = [t_j, t_{j+1})\) such that \(\|u\|_{S(I^\omega, I_j)} < \delta\) for all \(j = 1, \ldots, n\). On the time interval \(I_j\) we consider the integral equation
\[
u(t) = e^{-i(t-t_j)H}u(t_j) + \int_{t_j}^t e^{-i(s-t)H}(|x|^{-b}|u|^2u)ds.
\]
(2.30)
It follows from the Strichartz estimates (2.1) and the norm equivalence (2.2) that
\[
\|u\|_{S(L^2, I_j)} \leq c\|u(t_j)\|_{L^2} + c\|\nabla u\|_{S(L^2, I_j)}
\]
(2.31)
and
\[
\|\nabla u\|_{S(L^2, I_j)} \leq c\|\nabla u\|_{L^2} + c\|\nabla (|x|^{-b}|u|^2u)\|_{S(L^2, I_j)}.
\]
(2.32)
From (2.6) and (2.5), we have
\[
\|\nabla (|x|^{-b}|u|^2u)\|_{S(L^2)} \leq c\|\nabla u\|_{S(L^2)}^2 \|u\|_{S(L^2)}^{2-2b} \|u\|_{S(I^\omega)}
\]
(2.33)
and
\[
\|\nabla u\|_{S(L^2, I_j)} \leq c\|\nabla u\|_{S(L^2)} \|u\|_{S(I^\omega)}^{1-b} \|u\|_{S(I^\omega)}^{1-2b} \|u\|_{S(I^\omega)}.
\]
(2.34)
Thus, using (2.31), (2.32) and the last two estimates we get
\[
\|u\|_{S(L^2, I_j)} \leq cB + c\delta\|\nabla u\|_{S(L^2)} \|u\|_{S(L^2)}^{2-2b}.
\]
(2.35)
and
\[
\|\nabla u\|_{S([t^2, t^2])} \leq cB + c\delta\|\nabla u\|_{S([t^2, t^2])}^1 \|u\|_{S([t^2, t^2])}^1.
\]

Adding (2.35) and (2.36) and using the Young inequality give
\[
\|(\nabla)u\|_{S([t^2, t^2])} \leq cB + c\delta\|(\nabla)u\|_{S([t^2, t^2])}^2.
\]

Taking \(\delta > 0\) sufficiently small, we obtain
\[
\|(\nabla)u\|_{S([t^2, t^2])} \leq cB.
\]

Then by summing over the \(n\) intervals, we conclude the proof of (2.29).

Now we claim that
\[
\phi^\pm : = u_0 + i \int_0^{\pm\infty} e^{itH}(|x|^{-\beta}|u|^2u)(s)ds
\]
\[
= u_0 + i \lim_{t \to \pm\infty} \int_0^t e^{itH}(|x|^{-\beta}|u|^2u)(s)ds
\]
exists in \(H^1\). Indeed, using the norm equivalence (2.2), the Strichartz estimates (2.1) and the estimates (2.33) and (2.34) gives
\[
\left\| \int_{r_1}^{r_2} e^{itH}(|x|^{-\beta}|u|^2u)(s)ds \right\|_{H^1} \sim \left\| (1 + H)^{\frac{1}{2}} \int_{r_1}^{r_2} e^{itH}(|x|^{-\beta}|u|^2u)(s)ds \right\|_{L^2}.
\]

as \(t_1, t_2\) tend to \(\pm\infty\).

Hence, \(\phi^\pm\) is well defined. Then, repeating the above estimates again, we obtain that
\[
\left\| u(t) - e^{-it\phi^\pm} \right\|_{H^1} = \left\| \int_{r_1}^{r_2} e^{itH}(|u|^2u)(s)ds \right\|_{H^1}
\]
\[
\leq \left\| (\nabla)u\right\|_{S([t^2, t^2])}^2 \left\| u \right\|_{S(H^\infty, [t^2, t^2])} \to 0,
\]
as \(t\) tends to \(\pm\infty\).

Finally, we state the useful perturbation lemmas including short time one and long time one. Let’s first look at short time one.

**Lemma 2.7.** If \(V\) satisfies (1.2) and (1.3). Let \(I \subset \mathbb{R}\) be a time interval containing zero and let
\[
\bar{u} = \bar{u}(t, x) \in H^1\end{array}
\]
defined on \(I \times \mathbb{R}^3\) be a radial solution to
\[
i\bar{u}_t - H\bar{u} + |x|^{-\beta}\bar{u}^2\bar{u} = \epsilon,
\]
with radial initial data \(\bar{u}_0 \in H^1(\mathbb{R}^3)\) satisfying
\[
\sup_{t \in I} \|\bar{u}(t)\|_{H^1} \leq M \text{ and } \|\bar{u}\|_{S(H^\infty, I)} \leq \epsilon,
\]
for some positive constant \(M\) and some small \(\epsilon > 0\).

Let \(u_0 \in H^1(\mathbb{R}^3)\) satisfy
\[
\|e^{-itH}(u_0 - \bar{u}_0)\|_{S(H^\infty, I)} \leq \epsilon \text{ and } \|u_0 - \bar{u}_0\|_{H^1} \leq M', \text{ for } M' > 0.
\]
In addition, assume the following conditions

\[ (2.44) \quad \|\langle \nabla \rangle e\|_{S'(L^2;L^2)} + \|e\|_{S'(H^{-\infty};L^1)} \leq \varepsilon \]

There exists \( \varepsilon_0(\varepsilon, M) > 0 \) such that if \( \varepsilon < \varepsilon_0 \), then there is a unique solution \( u \) to (1.1) (INS) with initial data \( u_0 \) at the time \( t = 0 \) satisfying

\[ (2.45) \quad \|u\|_{S(H^{\infty};L^2)} \leq \varepsilon. \]

and

\[ (2.46) \quad \|\langle \nabla \rangle u\|_{S(L^2;L^2)} \leq c(M, M'). \]

Proof. We use the following claim (we will show it later): there exists \( \varepsilon_0 \) sufficiently small such that if \( \|\tilde{u}\|_{S(H^{\infty};L^2)} \leq \varepsilon_0 \) then

\[ (2.47) \quad \|\langle \nabla \rangle \tilde{u}\|_{S(L^2;L^2)} \leq M. \]

we may assume, without loss of generality, that \( 0 = \inf I \). Let us first prove the existence of a solution \( v \) for the following value problem

\[ (2.48) \begin{cases} 
iv_t - Hw + D(x, \tilde{u}, w) + e = 0, \\
w(0, x) = w_0(x) = u_0(x) - \tilde{u}_0(x),
\end{cases} \]

where \( D(x, \tilde{u}, w) = |x|^{-\rho}(\tilde{u} + w)^2(\tilde{u} + w) - |\tilde{u}|^2\tilde{u}. \)

To this end, let

\[ (2.49) \quad G(w) = e^{-itH}w_0 + i \int_0^t e^{-i(t-s)H}(D(x, \tilde{u}, w) + e)(s)ds \]

and define

\[ (2.50) \quad B_{\rho,K} = \{ w \in C(I, H^1(\mathbb{R}^3)) : \|w\|_{S(H^{\infty};L^2)} \leq \rho, \|\langle \nabla \rangle w\|_{S(L^2;L^2)} \leq K \}. \]

For a suitable choice of the parameter \( \rho > 0 \) and \( K > 0 \), we need to show that \( G \) in (2.49) defines a contraction on \( B_{\rho,K} \). Indeed, applying Strichartz inequalities (1.1) and the norm equivalence (1.2), we have

\[ (2.51) \quad \|G(w)\|_{S(H^{\infty};L^2)} \leq \|e^{itH}w_0\|_{S(H^{\infty};L^2)} + \|D(\cdot, \tilde{u}, w)\|_{S'(H^{-\infty};L^1)} + \|e\|_{S'(H^{-\infty};L^1)}, \]

\[ (2.52) \quad \|G(w)\|_{S(L^2;L^2)} \leq \|w_0\|_{L^2} + \|D(\cdot, \tilde{u}, w)\|_{S'(L^2;L^1)} + \|e\|_{S'(L^2;L^1)}, \]

and

\[ (2.53) \quad \|\langle \nabla \rangle G(w)\|_{S(L^2;L^2)} \leq \|\langle \nabla \rangle w_0\|_{L^2} + \|\langle \nabla \rangle D(\cdot, \tilde{u}, w)\|_{S'(L^2;L^1)} + \|\langle \nabla \rangle e\|_{S'(L^2;L^1)}. \]

On the other hand, since

\[ (2.54) \quad \|\tilde{u} + w\|^2(\tilde{u} + w) - |\tilde{u}|^2\tilde{u} \leq |\tilde{u}|^2|w| + |w|^3, \]

we get

\[ (2.55) \quad \|D(\cdot, \tilde{u}, w)\|_{S'(H^{-\infty};L^1)} \leq \|\|x|^{-\rho}|\tilde{u}|^2|w|\|_{L^1} + \|\|x|^{-\rho}|w|^3\|_{L^1}, \]

which implies, by (2.7), that

\[ (2.56) \quad \|D(\cdot, \tilde{u}, w)\|_{S'(H^{-\infty};L^1)} \leq (\|\langle \nabla \rangle \tilde{u}\|_{S(L^2;L^2)}\|\tilde{u}\|_{S(H^{\infty};L^2)} + \|\langle \nabla \rangle w\|_{S(L^2;L^2)}\|w\|_{S(H^{\infty};L^2)})\|w\|_{S(H^{\infty};L^2)}. \]

The same argument and (2.6) yield that

\[ (2.57) \quad \|D(\cdot, \tilde{u}, w)\|_{S(L^2;L^2)} \leq (\|\langle \nabla \rangle \tilde{u}\|_{S(L^2;L^2)}\|\tilde{u}\|_{S(H^{\infty};L^2)} + \|\langle \nabla \rangle w\|_{S(L^2;L^2)}\|w\|_{S(H^{\infty};L^2)})\|w\|_{S(L^2;L^2)}. \]
Now we estimate $\|\nabla D(\cdot, \tilde{u}, w)\|_{L^2(I; J)}$. Since

$$\|\nabla D(x, \tilde{u}, w)\| \leq |x|^{-b-1}(|\tilde{u}|^2 + |w|)w + |x|^{-b}(|\tilde{u}|^2 + |w|)\nabla w + E,$$

where $E \leq |x|^{-b}(|\tilde{u}| + |w|)|w|\nabla \tilde{u}$, then (2.13) and (2.16) leads to

$$\|\nabla D(\cdot, \tilde{u}, w)\|_{L^2(I; J)} \leq (|\nabla|^2 \tilde{u})_{L^2(I; J)} + |\nabla|^2 w_{L^2(I; J)} + |\nabla|^2 w_{L^2(I; J)} + \|\nabla\|_{L^2(I; J)}\|\nabla w\|_{L^2(I; J)}$$

(2.58)

$$+ (|\nabla|^2 \tilde{u})_{L^2(I; J)} + |\nabla|^2 w_{L^2(I; J)} + \|\nabla\|_{L^2(I; J)}\|\nabla w\|_{L^2(I; J)}.$$

Hence, if $w \in B_{\rho,K}$, it follows from (2.56) and (2.57) that

$$\|D(\cdot, \tilde{u}, w)\|_{L^2(I; J)} \leq (M\epsilon + K\rho)\rho$$

and

$$\|D(\cdot, \tilde{u}, w)\|_{L^2(I; J)} \leq (M\epsilon + K\rho)K.$$

Furthermore, (2.59) and (2.47) implies that

$$\|\nabla D(\cdot, \tilde{u}, w)\|_{L^2(I; J)} \leq (M\epsilon + K\rho)\rho + (M + K)\rho M.$$

Therefore, we deduce by (2.51) and (2.52), together with (2.60) and (2.61), that

$$\|G(w)\|_{L^2(I; J)} \leq c\epsilon + c\rho \epsilon$$

and

$$\|G(w)\|_{L^2(I; J)} \leq CM' + c\epsilon + cAK,$$

where we also used the hypothesis (2.43) and (2.44) and $A = M\epsilon + K\rho$.

We also have, using (2.53) and (2.62),

$$\|\nabla G(w)\|_{L^2(I; J)} \leq CM' + c\epsilon + cAK + c\rho M,$$

where $B = M + K$. Choosing $c\epsilon = \frac{\epsilon_0}{2}$, $cM' = \frac{K}{2}$ and $\epsilon_0$ sufficiently small such that

$$cA < \frac{1}{3} \text{ and } c(\epsilon + B\rho M) < \frac{K}{3},$$

we obtain

$$\|G(w)\|_{L^2(I; J)} \leq \rho \text{ and } \|\nabla G(w)\|_{L^2(I; J)} \leq K.$$

The above calculations establish that $G$ is well defined on $B_{\rho,K}$. The contraction property can be obtained by similar arguments. Hence, by the contraction mapping theorem, we obtain a unique solution $w$ on $I \times \mathbb{R}^3$ such that

$$\|w\|_{L^2(I; J)} + \|\nabla w\|_{L^2(I; J)} \leq M'.$$

Finally, it is easy to see that $u = \tilde{u} + w$ is a solution to (1.1) satisfying (2.45) and (2.46).

To complete the proof we now show (2.47). Indeed, using the same argument as before, we have

$$\|\nabla \tilde{u}\|_{L^2(I; J)} \leq \|\nabla \tilde{u}\|_{L^2(I; J)} + \|\nabla (|x|^{-b} |\tilde{u}|^2)\|_{L^2(I; J)} + \|\nabla \tilde{u}\|_{L^2(I; J)}$$

$$\leq M + \|\nabla \tilde{u}\|_{L^2(I; J)} + \|\nabla \tilde{u}\|_{L^2(I; J)} + \|\nabla \tilde{u}\|_{L^2(I; J)}$$

$$\leq M + \epsilon + \epsilon_0(\|\nabla \tilde{u}\|_{L^2(I; J)} + \|\nabla \tilde{u}\|_{L^2(I; J)}^2).$$

(2.69)
and

\[
\|\tilde{u}\|_{s(\mathbb{L}^2, J)} \leq \|u_0\|_{\mathbb{L}^2} + \|\mathcal{K} \psi \tilde{u}\|_{s(\mathbb{L}^2, J)} + \|e\|_{s(\mathbb{L}^2, J)}
\]

\[
\leq M + \|\nabla \tilde{u}\|_{s(\mathbb{L}^2, J)}^2 + M^2 + \|\tilde{u}\|_{s(\mathbb{L}^2, J)}^2 + \|e\|_{s(\mathbb{L}^2, J)}^2
\]

Adding (2.69) and (2.70) gives

\[
\|\nabla \tilde{u}\|_{s(\mathbb{L}^2, J)} + \|\tilde{u}\|_{s(\mathbb{L}^2, J)} \leq M + \epsilon + \epsilon_0(\|\nabla \tilde{u}\|_{s(\mathbb{L}^2, J)} + \|\tilde{u}\|_{s(\mathbb{L}^2, J)})^2.
\]

Therefore, choosing \(\epsilon_0\) sufficiently small, we conclude the proof of (2.47). \(\square\)

**Remark 2.8.** From the above lemma, we also have the following estimates.

\[(2.71) \quad \|D(\cdot, \tilde{u}, w)\|_{s(\mathbb{H}^{-1}, J)} \leq C(M, M')\epsilon^2\]

and

\[(2.72) \quad \|\nabla D(\cdot, \tilde{u}, w)\|_{s(\mathbb{L}^2, J)} \leq C(M, M')\epsilon\]

In the sequel, we prove the long-time perturbation result.

**Lemma 2.9.** If \(V\) satisfies (1.2) and (1.3). Let \(I \subset \mathbb{R}\) be a time interval containing zero and let

\[\tilde{u} = \tilde{u}(t, x) \in H^1\]

be a radial solution to

\[\tilde{u}_t + \tilde{H} \tilde{u} - |x|^{-\delta} |\tilde{u}|^2 \tilde{u} = \epsilon,\]

with radial initial data \(\tilde{u}_0 \in H^1(\mathbb{R}^3)\) satisfying

\[(2.73) \quad \sup_{t \in I} \|\tilde{u}(t)\|_{H^1} \leq M \quad \text{and} \quad \|\tilde{u}\|_{s(\mathbb{H}^\infty, I)} \leq L\]

for some positive constants \(M\) and \(L\).

Let \(u_0 \in H^1(\mathbb{R}^3)\) satisfy

\[(2.74) \quad \|e^{-i\tilde{H}}(u_0 - \tilde{u}_0)\|_{s(\mathbb{H}^\infty, J)} \leq \epsilon \quad \text{and} \quad \|u_0 - \tilde{u}_0\|_{H^1} \leq M',\]

for some positive constant \(M'\) and some \(0 < \epsilon < \epsilon_1 = \epsilon_1(M, M', L)\). In addition, assume the following conditions

\[(2.75) \quad \|\nabla e\|_{s(\mathbb{L}^2, J)} + \|e\|_{s(\mathbb{H}^\infty, J)} \leq \epsilon.\]

There exists \(\epsilon_0(M, M') > 0\) such that if \(\epsilon < \epsilon_0\), then there is a unique solution \(u\) to (1.1) \((\text{INS}_V)\) on \(I \times \mathbb{R}^3\) with initial data \(u_0\) at the time \(t = 0\) satisfying

\[(2.76) \quad \|u - \tilde{u}\|_{s(\mathbb{H}^\infty, J)} \leq c(M, M', L)\epsilon\]

and

\[(2.77) \quad \|u\|_{s(\mathbb{H}^\infty, J)} + \|\nabla \|_{s(\mathbb{L}^2, J)} \leq c(M, M', L).\]

**Proof.** First observe that since \(\|\tilde{u}\|_{s(\mathbb{H}^\infty, J)} \leq L\), given \(\epsilon < \epsilon_0(M, 2M')\), we can partition \(I\) into \(n = n(L, \epsilon)\) intervals \(I_j = [i_j, i_{j+1})\) such that for each \(j\), the quantity \(\|\tilde{u}\|_{s(\mathbb{H}^\infty, I_j)} \leq \epsilon\). Note that \(M'\) is being replaced by \(2M'\), as the \(H^1\) norm of the difference of two different initial data may increase in each iteration.
Again, we may assume, without loss of generality, that \(0 = \inf I\). Let \(w\) be defined by \(u = \tilde{u} + w\), then \(w\) solves (2.48) with initial time \(t_j\). Thus, the integral equation in the interval \(I_j = [t_j, t_{j+1})\) reads as follows

\[
w(t) = e^{-\hat{H}(t-t_j)}w(t_j) + \int_{t_j}^t e^{-\hat{H}(s-t)}(D(x, \tilde{u}, w) + e(s))ds,
\]

where \(D(x, \tilde{u}, w) = |x|^{-b}((\tilde{u} + w)^2(\tilde{u} + w) - |\tilde{u}|^2\tilde{u})\).

Thus, choosing \(\epsilon_1\) sufficiently small (depending on \(n, M\) and \(M'\)), we may apply short-time perturbation lemma 2.7 to obtain for each \(0 \leq j < n\) and all \(\epsilon < \epsilon_1\),

\[
\|u - \tilde{u}\|_{S(H^{\nu, I_j})} \leq c(M, M', J)\epsilon \tag{2.78}
\]

and

\[
\|(\nabla)w\|_{L^2(I_j)} \leq c(M, M', J). \tag{2.79}
\]

provided that we can show

\[
\|e^{-\hat{H}(s-t_j)}w(t_j)\|_{S(H^{\nu, I_j})} \leq c(M, M', J)\epsilon \leq \epsilon_0 \text{ and } \|u(t_j) - \tilde{u}(t_j)\|_{H^{\nu}} \leq 2M'. \tag{2.80}
\]

For each \(0 \leq j < n\). Indeed, by the Strichartz estimates (2.1), we have

\[
\|e^{-\hat{H}(s-t_j)}w(t_j)\|_{S(H^{\nu, I_j})} \leq \|e^{-\hat{H}w_0}\|_{S(H^{\nu, I_j})} + \|D(\cdot, \tilde{u}, w)\|_{S'(H^{-\nu, I_j})} + \|e\|_{S'(H^{-\nu, I_j})},
\]

which implies by (2.71) that

\[
\|e^{-\hat{H}(s-t_j)}w(t_j)\|_{S(H^{\nu, I_j})} \leq \epsilon + \sum_{k=0}^{j-1} c(k, M, M')\epsilon^2. \tag{2.82}
\]

Similarly, it follows from Strichartz estimates (1.1) and (2.72) that

\[
\|u(t_j) - \tilde{u}(t_j)\|_{H^{\nu}} \leq \|u_0 - \tilde{u}_0\|_{H^{\nu}} + \|(\nabla)e\|_{S'(L^2, I_j)} + \|(\nabla)D(\cdot, \tilde{u}, w)\|_{S'(L^2, I_j)}
\]

\[
\leq M' + \epsilon + \sum_{k=0}^{j-1} c(k, M, M')\epsilon \tag{2.83}
\]

Taking \(\epsilon_1 = \epsilon_1(M, M', L)\) sufficiently small, we see that (2.80) holds and so it implies (2.78) and (2.79).

Finally, summing them over all subintervals \(I_j\) we obtain (2.76) and (2.77). \(\square\)

### 3. Sharp Gagliardo-Nirenberg Inequality

In this section, we will find a maximizer or maximizing sequence of the nonlinear functional

\[
J_\nu(u) = \frac{\|\|x|^{-b}|u|^4\|_{L^1}}{\|u\|^b_{L^2} + \int_{\mathbb{R}^3} \|\nabla u\|^2 dx + \int_{\mathbb{R}^3} V|u|^2 dx}^{\frac{2-b}{2}}.
\]

To make this precise, we define

\[
C_{GN}^{rad} = \sup\{J_\nu(u) : u \in H^1(\mathbb{R}^3), u \text{ is radial and nonzero}\}
\]

and

\[
C_{GN} = \sup\{J_\nu(u) : u \in H^1(\mathbb{R}^3), u \text{ is nonzero}\}.
\]
It’s known from [10] that for \( V = 0 \), \( J_0(u) \) attains its maximum \( J_0 \) at \( u = Q(x) \geq 0 \), which solves the equation (1.7) with \( p = 3 \), and

\[
J_0 = J_0(Q) = \frac{\|\|x|^{-\frac{1}{2}}Q \|^4_{L^4}}{\|Q\|_{L^2}^{3+b}}
\]

which together with the identities

\[
\|\nabla Q\|^2_{L^2} = \frac{3 + b}{1 - b} \|Q\|^2_{L^2}, \quad \|\|x|^{-b}Q\|^4_{L^4} = \frac{4}{3 + b} \|\nabla Q\|^2_{L^2}, \quad E_0(Q) = \frac{4c}{3 + b} \|\nabla Q\|^2_{L^2},
\]

implies that the best constant of the Gagliardo-Nirenberg inequality

\[
\|\|x|^{-b}Q\|^4_{L^4} \leq C_{GN}^0 \|Q\|^{3+b}_{L^2} \|\nabla Q\|_{L^2}^3
\]

is

\[
C_{GN}^0 = J_0 = \frac{4}{(3 + b)\|Q\|^{2(1-s)}_{L^2} \|\nabla Q\|^2_{L^2}}.
\]

**Lemma 3.1.** If \( V \geq 0 \), then \( \{Q_n(x)\}_{n=1}^{\infty} \) is a maximizing sequence for \( J_V(u) \), where \( Q_n(x) = \frac{1}{n^\frac{3}{2}} Q\left(\frac{x}{n}\right) \).

**Proof.** It follows from (3.2), (3.4) and (3.5) that

\[
J_0(Q) \geq J_0(u).
\]

On the one hand,

\[
\lim_{n \to \infty} J_V(Q_n) = J_0(Q),
\]

which follows from

\[
\int_{\mathbb{R}^n} n^2 V(nx) Q(x)^2 \, dx \leq \|V\|_{L^\frac{1}{2}} \|Q\|^2_{L^2} \leq \|V\|_{L^\frac{1}{2}} \|\nabla Q\|_{L^2}^2
\]

and

\[
J_V(Q_n) = \frac{\|\|x|^{-\frac{1}{2}}Q \|^4_{L^4}}{\|Q\|_{L^2}^{3+b} \left(\|\nabla Q\|^2_{L^2} + \int_{\mathbb{R}^n} n^2 V(nx) Q^2 \, dx\right)^{\frac{b}{2}}}
\]

On the other hand, for \( V \geq 0 \), it is easy to see that for any \( u \in H^1 \),

\[
J_0(u) \geq J_V(u)
\]

Putting (3.6), (3.7) and (3.9) together yields that for any \( u \in H^1 \)

\[
\lim_{n \to \infty} J_V(Q_n) \geq J_V(u)
\]

Thus, we get our desired result. \( \square \)

**Remark 3.2.** (i) It follows from Lemma 3.1 that there hold that \( J_0(Q) = \lim_{n \to \infty} J_V(Q_n) \geq J_V(u) \) for any \( u \), which implies that with the same Gagliardo-Nirenberg constant (\( C_{GN} = C_{GN}^0 \) (3.5)), there holds the following sharp inequality:

\[
\|\|x|^{-b}Q\|^4_{L^4} \leq C_{GN}^0 \|Q\|^{3+b}_{L^2} \|\nabla Q\|^3_{L^2}
\]

\[
\|\|x|^{-b}Q\|^4_{L^4} \leq C_{GN}^0 \|Q\|^{3+b}_{L^2} \|\nabla Q\|^3_{L^2}
\]

\[
\|\|x|^{-b}Q\|^4_{L^4} \leq C_{GN}^0 \|Q\|^{3+b}_{L^2} \|\nabla Q\|^3_{L^2}
\]
In the case when $V$ is nonnegative and not zero a.e. on $\mathbb{R}^3$, the constant $C_{GN} = C_{GN}^{0}$ can never be attained. In fact, if not, then there exists some $\tilde{Q} \in H^1(\mathbb{R}^3)$ such that $C_{GN} = J_V(\tilde{Q}) < J_0(\tilde{Q}) = \tilde{C}_{GN} = C_{GN}$, which is a contradiction.

(ii) Since $\{Q_n(x)\}_{n=1}^\infty$ is a radial sequence, the arguments in Lemma 3.1 and (i) still work for radial functions. So we can find that $C_{GN}^{rad} = C_{GN}^{0} = C_{GN}$ and it is never attained, which is different from the case that $V$ is an inverse-square potential $\frac{1}{|x|^2}$ (see [23]), where $a > \frac{1}{2}$. In [23], the authors showed that $C_{GN}^{rad}$ can be attained but $C_{GN}$ cannot be and $C_{GN}^{rad} < C_{GN}$ provided that $a > 0$.

When $V \neq 0$, we also have that $C_{GN}^{rad} = C_{GN}$ which can be attained further. More precisely, we have the following.

**Lemma 3.3.** If $V$ is radially symmetric and $V \neq 0$, then the sharp constant $C_{GN}^{rad} = C_{GN}$ can be attained by a radially symmetric function $Q$, that is, there exists a maximizer $Q$ for $J_V(u)$, where $Q$ solves the elliptic equation

\[
(-\Delta + V)Q + w_3^2 Q - |x|^{-\beta} |Q|^2 Q = 0, \quad w_Q = \frac{\sqrt{1 - b} \||H^+ Q||_{L^2}}{\sqrt{3 + b} \||Q||_{L^2}}.
\]

Moreover, $Q$ satisfies the Pohozaev identities,

\[
||H^+ Q||_{L^2} = \frac{3 + b}{1 - b} w_Q \||Q||_{L^2}, \quad |||x|^{-\beta} |Q|^4||_{L^1} = \frac{4}{1 - b} w_Q^2 \||Q||_{L^2}^2.
\]

**Proof.** Set

\[ I(u) = \int_{\mathbb{R}^3} |x|^{-\beta} |u(x)|^4 \, dx. \]

Let $\{u_n\}_{n=1}^\infty \subset H^1(\mathbb{R}^3)$ be a maximizing sequence associated to $J_V(u)$. By Schwarz symmetrization, we can assume that $\{u_n\}_{n=1}^\infty$ is radial and radially non-increasing for all $n$. For each $n$, we choose $\alpha_n, r_n > 0$ such that

\[ ||\alpha_n u_n(\frac{\cdot}{r_n})||_{L^2}^2 = \alpha_n^2 r_n^3 ||u_n||_{L^2}^2 = 1 \]

and

\[ ||H^+ \alpha_n r_n^{-1} u_n(\frac{\cdot}{r_n})||_{L^2}^2 = \alpha_n^2 r_n ||H^+ u_n||_{L^2}^2 = 1 \]

where $H = -\Delta + \frac{1}{r^2} V(\frac{x}{r})$. Since $J_V(\alpha u) = J_V(u)$, replacing $\{u_n\}_{n=1}^\infty$ by $\{\alpha_n u_n\}_{n=1}^\infty$, we may assume that $||u(\frac{\cdot}{r_n})||_{L^2}^2 = 1$ and $||H^+ \alpha_n r_n^{-1} u_n(\frac{\cdot}{r_n})||_{L^2}^2 = 1$. Set $\tilde{u}_n = u_n(\frac{\cdot}{r_n})$. Then $\{\tilde{u}_n\}_{n=1}^\infty$ is a bounded sequence in $H^1$, since by the norm equivalence,

\[ ||\tilde{u}_n||_{L^2}^2 = 1, \quad ||\nabla \tilde{u}_n||_{L^2}^2 \sim ||H^+_{\tilde{u}_n} \tilde{u}_n||_{L^2}^2 = 1. \]

Therefore, there exists some $\tilde{u} \in H^1(\mathbb{R}^3)$ such that, up to a subsequence, $\tilde{u}_n \rightharpoonup \tilde{u}$ weakly in $H^1(\mathbb{R}^3)$. Furthermore, $\tilde{u}$ is nonnegative, spherically symmetric, radially non-increasing, and with some $r_0 \in (0, +\infty)$:

\[ ||\tilde{u}||_{L^2} \leq 1, \quad ||H^+_{\tilde{u}_n} \tilde{u}_n||_{L^2}^2 = \left( \int_{\mathbb{R}^3} |\nabla \tilde{u}_n|^2 + \frac{1}{r_0^2} V(\frac{\cdot}{r_0}) |\tilde{u}_n|^2 \, dx \right)^{\frac{1}{2}} \leq 1. \]
Indeed, if we suppose that \( r_n \to 0 \) or \( r_n \to +\infty \), then by the "free" Gagliardo-Nirenberg inequality and the assumption, 
\[
\|x|^{-b}Q\|_{L^4}^4 \geq \lim_{n \to \infty} \frac{\|x|^{-b}|u_n|\|_{L^4}^4}{\|u_n\|_{L^2}^{1+b}|\nabla u_n|_{L^2}^{3+b}} = \lim_{n \to \infty} \frac{\|x|^{-b}|\tilde{u}_n|\|_{L^4}^4}{\|\tilde{u}_n\|_{L^2}^{1+b}|\nabla \tilde{u}_n|_{L^2}^{3+b}} = C_{GN},
\]
with \( Q \) is the ground state of the free equation. On the other hand, since \( V_\gamma \neq 0 \), then there exists some \( x^* \in \mathbb{R}^3 \) and a small \( \epsilon > 0 \) such that 
\[
\int_{\mathbb{R}^3} V(x)Q^2 \left( \frac{x-x^*}{\epsilon} \right) dx < 0.
\]
Hence,
\[
C_{GN} \geq J_V \left( Q \left( \frac{x-x^*}{\epsilon} \right) \right) > \frac{\|x|^{-b}Q(\frac{x-x^*}{\epsilon})\|_{L^4}^4}{\|Q(\frac{x-x^*}{\epsilon})\|_{L^2}^{1+b}|\nabla Q(\frac{x-x^*}{\epsilon})|_{L^2}^{3+b}} = \frac{\|x|^{-b}|Q\|_{L^4}^4}{\|Q\|_{L^2}^{1+b}|\nabla Q|_{L^2}^{3+b}},
\]
contradicting (3.14).

In this stage, we set \( \psi(x) = \tilde{u}(r_0x) \) and obtain that
\[
C_{GN} = \lim_{n \to \infty} J_V(u_n) = \lim_{n \to \infty} \frac{\|x|^{-b}|u_n|\|_{L^4}^4}{\|u_n\|_{L^2}^{1+b}|\nabla u_n|_{L^2}^{3+b}} = \lim_{n \to \infty} \frac{\|x|^{-b}|\tilde{u}_n|\|_{L^4}^4}{\|\tilde{u}_n\|_{L^2}^{1+b}|\nabla \tilde{u}_n|_{L^2}^{3+b}} 
\leq \frac{\|x|^{-b}|\tilde{u}|\|_{L^4}^4}{\|\tilde{u}\|_{L^2}^{1+b}|\nabla \tilde{u}|_{L^2}^{3+b}} = \frac{\|x|^{-b}|\psi|\|_{L^4}^4}{\|\psi\|_{L^2}^{1+b}|\nabla \psi|_{L^2}^{3+b}} \leq C_{GN}.
\]
Therefore, we actually obtain that \( \tilde{u}_n \to \tilde{u} \) and \( r_n \to r_0 \) which give then \( u_n \to \psi \) in \( H^1 \), attaining \( C_{GN} \).

Now that \( \psi \) is a maximizer of \( J_V(u) \). Then, it solves the Euler-Lagrange equation equivalently,
\[
(H\psi + \frac{1-b}{3+b} \|H\psi\|_{L^2}^2 \psi - \frac{4}{3+b} \|H\psi\|_{L^2}^2 |x|^{-b}|\psi|^2 \psi) \cdot \psi = 0
\]
for all \( v \in H^1 \). We set
\[
Q = \frac{2}{\sqrt{3+b}} \frac{\|H\psi\|_{L^2}^2}{\|x|^{-b}|\psi|^4 L^2} \psi.
\]
Then \( Q \) is a weak solution to the ground state equation (3.12).

Let’s turn to the proof of (3.13). Formally, multiplying (3.12) by \( Q \) and \( x \cdot \nabla Q \), separately, integrating in \( x \) and applying integration by parts, we get
\[
\|H^\ast Q\|_{L^2}^2 + w_Q^2 \|Q\|_{L^2}^2 - \|x|^{-b}Q\|_{L^4}^4 = 0
\]
and
\[
\|H^\ast Q\|_{L^2}^2 + 3w_Q^2 \|Q\|_{L^2}^2 - \frac{3-b}{2} \|x|^{-b}Q\|_{L^2}^4 + \int_{\mathbb{R}^3} (2V + x \cdot \nabla V) |Q|^2 dx = 0
\]
The rigorous proof relies on the standard approximating method. Solving the simultaneous equations (3.15) and (3.16) in \( \|H^\ast Q\|_{L^2}^2 \) and \( \|x|^{-b}Q\|_{L^2}^4 \) gives
\[
\|H^\ast Q\|_{L^2}^2 = \frac{3+b}{1-b} w_Q^2 \|Q\|_{L^2}^2 + \frac{2}{1-b} \int_{\mathbb{R}^3} (2V + x \cdot \nabla V) |Q|^2 dx,
\]
and
\[
\|x|^{-b}\|Q|^4\|_{L^4} = \frac{4}{1-b} w_Q^2 \|Q\|^2_{L^2} + \frac{2}{1-b} \int_{\mathbb{R}^3} (2V + x \cdot \nabla V)|Q|^2 \, dx.
\]
Substituting
\[
w_Q = \sqrt{1-b} \frac{\|H^{1/2}Q\|_{L^2}}{\sqrt{3+b} \|Q\|_{L^2}}
\]
in (3.15) and (3.16) yields that
\[
\frac{2}{1-b} \int_{\mathbb{R}^3} (2V + x \cdot \nabla V)|Q|^2 \, dx = 0.
\]
Then (3.18) and (3.19) imply that (3.13) is true. □

4. Criteria for global well-posedness

In this section we first give a criterion for the global well-posedness, i.e., Theorem 1.1, and then obtain properties of such global solutions. For one’s convenience, we restate Theorem 1.1 as follows.

**Theorem 4.1.** Suppose that \( V \) is radially symmetric and satisfies (1.2) and (1.3), and \( 0 < b < 1 \). We assume that
\[
M(u_0)^{1-s_c} E(u_0)^{s_c} < E.
\]
Let \( u(t) \) be the solution to (1.1) with initial data \( u_0 \in H^1(\mathbb{R}^3) \).

(i) If
\[
\|u_0\|_{L^2}^{2(1-s_c)} \|H^{1/2} u_0\|_{L^2}^{2s_c} < \mathcal{K},
\]
then \( u(t) \) exists globally in time, and
\[
\|u_0\|_{L^2}^{2(1-s_c)} \|H^{1/2} u(t)\|_{L^2}^{2s_c} < \mathcal{K}, \forall t \in \mathbb{R}.
\]
(ii) If
\[
\|u_0\|_{L^2}^{2(1-s_c)} \|H^{1/2} u_0\|_{L^2}^{2s_c} > \mathcal{K},
\]
then
\[
\|u_0\|_{L^2}^{2(1-s_c)} \|H^{1/2} u(t)\|_{L^2}^{2s_c} > \mathcal{K}
\]
during the maximal existence time.

**Proof.** If \( V \geq 0 \), it follows from (3.3) and (3.5) that
\[
\mathcal{E} = M(Q)^{1-s_c} E_0(Q)^{s_c} = \left( \frac{s_c}{3+b} \right)^{s_c} \|Q\|_{L^2}^{2(1-s_c)} \|\nabla Q\|_{L^2}^{2s_c} = \left( \frac{s_c}{3+b} \right)^{s_c} \mathcal{K}
\]
and
\[
C_{GN} = \frac{4}{(3+b)\|Q\|_{L^2}^{2(1-s_c)} \|\nabla Q\|_{L^2}^{2s_c}} = \frac{4}{(3+b)\mathcal{K}}.
\]
If \( V \leq 0 \), using Pohozaev identities (3.13), we have
\[
E(Q) = \frac{1}{2} \|H^{1/2}Q\|_{L^2}^2 - \frac{1}{4} \|x|^{-b}\|Q|^4\|_{L^4} = \frac{s_c}{3+b} \|H^{1/2}Q\|_{L^2}^2.
\]
which implies that

\[ E = M(Q)^{1-\kappa} E(Q)^\kappa = \left(\frac{S_c}{3 + b}\right)^\kappa \|Q\|_{L^2}^{2(1-\kappa)} \|H^s Q\|_{L^2}^{2\kappa} = \left(\frac{S_c}{3 + b}\right)^\kappa \mathcal{K}. \]  

Using Pohozaev identities (3.13) again, we have

\[ C_{GN} = \frac{\|x^{-b} \|Q\|_{L^2}^4}{\|Q\|_{L^2}^{1-\kappa} \|H^s Q\|_{L^2}^{2\kappa + b}} = \frac{4}{(3 + b)\|Q\|_{L^2}^{2(1-\kappa)} \|H^s Q\|_{L^2}^{2\kappa}} = \frac{4}{(3 + b)\mathcal{K}}. \]

Then, it follows from the Gagliardo-Nirenberg inequality and the energy law that

\[ E > M(u_0)^{1-\kappa} E(u_0)^\kappa = M(u_0)^{1-\kappa} E(u(t))^\kappa \]

\[ = \|u_0\|_{L^2}^{2(1-\kappa)} (\frac{1}{2} \|H^s u(t)\|_{L^2}^2 - \frac{1}{4} \|x^{-b} u(t)\|_{L^2}^{4\kappa})^\kappa \]

\[ \geq \|u_0\|_{L^2}^{2(1-\kappa)} (\frac{1}{2} \|H^s u(t)\|_{L^2}^2 - \frac{C_{GN}}{4} \|u_0\|_{L^2}^{1-\kappa} \|H^s u(t)\|_{L^2}^{2\kappa + b})^\kappa \]

\[ = \left(\frac{1}{2}\|u_0\|_{L^2}^{2(1-\kappa)} \|H^s u(t)\|_{L^2}^2 - \frac{1}{(3 + b)\mathcal{K}} \|u_0\|_{L^2}^{2(1-\kappa)} \|H^s u(t)\|_{L^2}^{2\kappa + b}\right)^\kappa \]

which is continuous. Therefore, we conclude that either \( g(t) < \mathcal{K}^{\frac{1}{\kappa}} \) or \( g(t) > \mathcal{K}^{\frac{1}{\kappa}} \).

The next two lemmas provide some additional properties for the solution \( u \) under the hypotheses (4.1) and (4.2) of Theorem 4.1. These lemmas will be needed in the proof of Theorem 1.3 through a virial-type estimate, which will be established in the last two sections.

**Lemma 4.2.** In the situation (i) of Theorem 4.1, if \( u \) is a solution of the problem (1.1) with radial initial data \( u_0 \), then the following statements hold

\[(i)\]

\[ 2E(u_0) \leq \|H^s u(t)\|_{L^2}^2 < \frac{3 + b}{S_c} E(u_0), \quad \forall t \in \mathbb{R}. \]

\[(ii)\]

\[ \|u_0\|_{L^2}^{2(1-\kappa)} \|H^s u(t)\|_{L^2}^{2\kappa} < \omega \mathcal{K}, \quad \forall t \in \mathbb{R}, \]

where \( \omega = \frac{M(u_0)^{1-\kappa} E(u_0)^\kappa}{\mathcal{K}} \).

\[(iii)\]

\[ 8\|\nabla u(t)\|_{L^2}^2 - 2(3 + b)\|x^{-b} u(t)\|_{L^2}^4 > 8(1 - \omega)\|\nabla u\|_{L^2}^2 \sim E(u_0), \quad \forall t \in \mathbb{R}. \]

**Proof.** (i) By the energy conservation law, we obtain

\[ E(u_0) = E(u(t)) = \frac{1}{2} \|H^s u(t)\|_{L^2}^2 - \frac{1}{4} \|x^{-b} u(t)\|_{L^2}^4 \]
By the Gagliardo-Nirenberg inequality (with $C_{\text{GN}} = \frac{1}{(3 + b)^{3/2}}$) and (4.3), we obtain

\[
\|x^{l-b}|u(t)|^4\|_{L^2} \leq \frac{4}{(3 + b)^{3/2}}\|u(t)|_{L^2}^{1-b}\|H^\sigma u(t)\|_{L^2}^{3+b} = \frac{4}{(3 + b)^{3/2}}\|u(t)|_{L^2}^{2(l+\sigma)}\|H^\sigma u(t)\|_{L^2}^{2\sigma}\|H^\sigma u(t)\|_{L^2}^2 < \frac{4}{(3 + b)^{3/2}}\|H^\sigma u(t)\|_{L^2}^2,
\]

(4.16)

which combing with (4.15) gives

\[
E(u_0) = E(u(t)) > \frac{1}{2}\|H^\sigma u(t)\|_{L^2}^2 - \frac{1}{(3 + b)^{3/2}}\|H^\sigma u(t)\|_{L^2}^2 = \frac{s_c}{3 + b}\|H^\sigma u(t)\|_{L^2}^2.
\]

On the other hand, it is obviously that

\[
E(u_0) = E(u(t)) \leq \frac{1}{2}\|H^\sigma u(t)\|_{L^2}^2.
\]

Connecting (4.17) with (4.18) gives (4.12).

(ii) By the second inequality in (i),

\[
\|H^\sigma u(t)\|_{L^2}^{2\sigma} < \left(\frac{3 + b}{s_c}\right)^\sigma E(u_0)^\sigma.
\]

Multiplying both sides of (4.19) by $M(u_0)^{1-x_c} = \|u_0\|_{L^2}^{2(1-x_c)}$ and using (4.6) and (4.9) yield that

\[
\|u_0\|_{L^2}^{2(1-x_c)}\|H^\sigma u(t)\|_{L^2}^{2\sigma} < \left(\frac{3 + b}{s_c}\right)^\sigma E(u_0)^\sigma \|u_0\|_{L^2}^{2(1-x_c)}
\]

\[
= \left(\frac{3 + b}{s_c}\right)^\sigma E(u_0)^\sigma \|u_0\|_{L^2}^{2(1-x_c)}
\]

\[
\leq \omega K.
\]

(iii) If $V \geq 0$, using the “free” Gagliardo-Nirenberg inequality, (4.13), $\|H^\sigma u(t)\|_{L^2}^2 \geq \|\nabla u(t)\|_{L^2}^2$, the equivalence norm (2.2) and (4.12) successively gives

\[
8\|\nabla u(t)\|_{L^2}^2 - 2(3 + b)||x^{-b}|u(t)|^4\|_{L^2} \geq 8\|\nabla u(t)\|_{L^2}^2 - \frac{8}{K}\|u(t)|_{L^2}^{2(1-x_c)}\|\nabla u(t)\|_{L^2}^{2(1+x_c)}
\]

\[
\geq 8\|\nabla u(t)\|_{L^2}^2 - \frac{8}{K}\|u(t)|_{L^2}^{2(1-x_c)}\|H^\sigma u(t)\|_{L^2}^{2\sigma}\|\nabla u(t)\|_{L^2}^2
\]

\[
> 8(1 - \omega)\|\nabla u(t)\|_{L^2}^2 \sim \|H^\sigma u(t)\|_{L^2}^2 \sim E(u_0).
\]

If $V \leq 0$, using Gagliardo-Nirenberg inequality, (4.13), $\|H^\sigma u(t)\|_{L^2}^2 \leq \|\nabla u(t)\|_{L^2}^2$, the equivalence norm (2.2) and (4.12) successively gives

\[
8\|\nabla u(t)\|_{L^2}^2 - 2(3 + b)||x^{-b}|u(t)|^4\|_{L^2} \geq 8\|\nabla u(t)\|_{L^2}^2 - \frac{8}{K}\|u(t)|_{L^2}^{2(1-x_c)}\|H^\sigma u(t)\|_{L^2}^{2\sigma}\|\nabla u(t)\|_{L^2}^2
\]

\[
\geq 8\|\nabla u(t)\|_{L^2}^2 - \frac{8}{K}\|u(t)|_{L^2}^{2(1-x_c)}\|H^\sigma u(t)\|_{L^2}^{2\sigma}\|\nabla u(t)\|_{L^2}^2
\]

\[
> 8(1 - \omega)\|\nabla u(t)\|_{L^2}^2 \sim \|H^\sigma u(t)\|_{L^2}^2 \sim E(u_0).
\]

$\square$
Finally, we give the result about existence of wave operators, which will be used to established the scattering theory. Before stating it, we need the lemma established in [11].

**Lemma 4.3.** Let $0 < b < 1$. If $f$ and $g \in H^1(\mathbb{R}^3)$ then there exists some $\frac{12}{13} < r < 6$ such that

(i) $||x^{-b}f||_{L^r}^r \leq c ||f||_{L^r}^3 ||g||_{L^r} + c ||f||_{L^r}^3 ||g||_{L^r};$

(ii) $||x^{-b}f||_{L^r}^r \leq c ||f||_{H^1}^3 ||g||_{H^1};$

(iii) $\lim_{|\delta| \to 0} ||x^{-b}U(t)f||_{L^1}^2 = 0.$

**Proposition 4.4.** If $V$ is radially symmetric and satisfies (1.2) and (1.3), $V \geq 0$ or $V \leq 0$, and $0 < b < 1$. Suppose radial function $\psi^* \in H^1(\mathbb{R}^3)$ and

\[
\frac{1}{2} ||H^2\psi^*||_{L^2}^2 ||\psi^*||_{L^2}^{2(1-r)} < \mathcal{E}.
\]

Then there exists a radial function $v_0 \in H^1(\mathbb{R}^3)$ such that the solution $v$ of (1.1) with initial data $v_0$ obeys the assumptions (4.1) and (4.2) and satisfies

\[
\lim_{t \to +\infty} ||v(t) - e^{itH}\psi^*||_{H^1(\mathbb{R}^3)} = 0.
\]

Moreover,

\[
||v(t)||_{S(\mathbb{R}^4)} < \infty \text{ and } ||(\nabla)v||_{S(L^2)} < \infty.
\]

**Proof.** Similar to the proof of the small data theory Proposition 2.5, we can solve the integral equation

\[
v(t) = e^{-itH}\psi^* - i \int_t^\infty e^{-i(t-s)H}(|x|^{-b}|\psi|^2(t))ds
\]

for $t \geq T$ with $T$ large.

In fact, there exists some large $T$ such that $||e^{-itH}\psi^*||_{S(\mathbb{R}^4,[T,\infty))} \leq \delta_{sd}$, where $\delta_{sd}$ is defined by Proposition 2.5. Then, the same arguments as used in Proposition 2.5 give a solution $v \in C([T, \infty), H^1)$ of (4.25). Moreover, we also have

\[
||v||_{S(\mathbb{R}^4,[T,\infty))} \leq 2\delta_{sd}, \quad \text{and} \quad ||(\nabla)v||_{S(L^2,[T,\infty))} \leq 2||\psi^*||_{H^1}.
\]

Thus by (2.1), (2.2), (2.13), (2.14) and (2.17),

\[
||v - e^{-itH}\psi^*||_{L^1_T, L^1_x}^2 = \|\int_t^\infty e^{-it-sH}(|x|^{-b}|\psi|^2(t))ds\|_{L^1_T, L^1_x}^2
\]

\[
\leq \|\nabla(|x|^{-b}|u|^2(t))\|_{L^1_T, L^1_x}^2 \leq \|\nabla|u|^2\|_{L^1_T, L^1_x} \|u\|_{L_T^\infty L^1_x} \|\nabla|u|^2\|_{L^\infty_T L^2_x} \|\nabla|u|^2\|_{L^2_T L^\infty_x}
\]

\[
\leq ||\psi^*||_{H^1}^2 \delta_{sd}.
\]

Since $\delta_{sd} > 0$ is arbitrarily small, this proves that

\[
||v - e^{-itH}\psi^*||_{L^1_T, L^1_x} \to 0 \quad \text{as} \quad T \to \infty,
\]

which implies $v(t) - e^{-itH}\psi^* \to 0$ in $H^1(\mathbb{R}^3)$ as $t \to +\infty$. Hence, we have

\[
\lim_{t \to +\infty} ||u(t)||_{L^2_\infty} = ||\psi^*||_{L^2_\infty}
\]
and
\[ \lim_{t \to +\infty} \|H^s u(t)\|_{L_x^2} = \|H^s \psi^+\|_{L_x^2}. \]
By the mass conservation, we have \( \|u(t)\|_{L_x^2} = \|u(T)\|_{L_x^2} \) for all \( t \geq T \). So from (4.29), we deduce
\[ \|u(T)\|_{L_x^2} = \|\psi^+\|_{L_x^2}. \]

On the other hand, it follows from Lemma 4.3
\[ \|x^{-b}|u(t)|^4\|_{L_x^2} \leq \|x^{-b}|u(t) - e^{-itH} \psi^+|^4\|_{L_x^2} + \|x^{-b}|e^{-itH} \psi^+|^4\|_{L_x^2}, \]
which goes to zero as \( t \to +\infty \), by (4.28) and Lemma 4.3, i.e.,
\[ \lim_{t \to +\infty} \|x^{-b}|u(t)|^4\|_{L_x^2} = 0. \]

Thus combining (4.30) and (4.32), we obtain that
\[ M(v(T))^{2-s} E(v(T))^{1-s} = \lim_{t \to +\infty} M(v(t))^{2-s} E(v(t))^{1-s} = \|\psi^+\|_{L_x^2}^{2(2-s)} \frac{1}{2^s} \|H^s \psi^+\|_{L_x^2}^{2s} < \mathcal{E}. \]
Moreover, we note that
\[ \lim_{t \to +\infty} \|v(t)\|_{L_x^2}^{2(2-s)} \|H^s v(t)\|_{L_x^2}^{2s} = \|\psi^+\|_{L_x^2}^{2(2-s)} \|H^s \psi^+\|_{L_x^2}^{2s} < 2^{-s} \mathcal{E} = \left( \frac{2^s \mathcal{E}}{3 + b} \right)^{s} \mathcal{C} \mathcal{K} < \mathcal{K} \]

Hence, for sufficiently large \( T \), \( v(T) \) satisfies (4.1) and (4.2), which implies that \( v(t) \) is a global solution in \( H^1_x(\mathbb{R}^3) \). Thus, we can evolve \( v(t) \) from \( T \) back to the initial time \( 0 \). By the same way, we can show (4.23) for the negative time. In addition, (4.26) combined with local theory implies (4.24).

\[ \square \]

5. Existence and compactness of a critical element

**Definition 5.1.** We say that \( SC(u_0) \) holds if for radial \( u_0 \in H^1(\mathbb{R}^3) \) satisfying \( \|u_0\|_{L_x^2}^{2(1-s)} \|H^s u_0\|_{L_x^2}^{2s} < \mathcal{K} \) and \( E(u_0)^{s} M(u_0)^{1-s} < \mathcal{E} \), the corresponding solution \( u \) of (1.1) with the maximal interval of existence \( I = (-\infty, +\infty) \) satisfies
\[ \|u\|_{S(H^s)} < +\infty. \]

We first claim that there exists \( \delta > 0 \) such that if \( E(u_0)^{s} M(u_0)^{1-s} < \delta \) and \( \|u_0\|_{L_x^2}^{2(1-s)} \|H^s u_0\|_{L_x^2}^{2s} < \mathcal{K} \), then (5.1) holds. In fact, by the definition of norm \( \|\cdot\|_{S(H^s)} \), Sobolev embedding, the Strichartz estimate (see (2.1) with \( s = 0 \)), the norm equivalence (2.2) and (4.12), we have
\[ \|e^{-itH} u_0\|_{S(H^s)}^2 \leq \|H^s e^{-itH} u_0\|_{S(L^2)}^2 \leq \|H^s u_0\|_{L_x^2}^2 \sim \|\nabla^s u_0\|_{L_x^2}^2 \]
\[ \leq \|u_0\|_{L_x^2}^{2(1-s)} \|\nabla u_0\|_{L_x^2}^{2s} \sim \|u_0\|_{L_x^2}^{2(1-s)} \|H^s u_0\|_{L_x^2}^{2s} \sim E(u_0)^{s} M(u_0)^{1-s}. \]
Hence, it follows from Proposition 2.5 that (5.1) holds for all sufficiently small \( \delta > 0 \), which implies scattering by Proposition 2.6.

Now for each \( \delta > 0 \), we define the set \( S_\delta \) to be the collection of all such initial data in \( H^1 \):

\[
S_\delta = \{ u_0 \in H^1(\mathbb{R}^3) : E(u_0)^{1/2} M(u_0)^{1/2} < \delta \} \quad \text{and} \quad \|u_0\|^2_{L^2} \|H^{1/2} u_0\|^2_{L^2} < \mathcal{K} \Rightarrow (5.1) \text{ holds.}
\]

We also define

\[
E_c = \sup\{ \delta : u_0 \in S_\delta \Rightarrow SC(u_0) \text{ holds} \}.
\]

If \( E_c = E \), then we are done. Thus we assume now

\[
E_c < E.
\]

Our goal in this section is to show the existence of an \( H^1 \)- solution \( u_c \) of (1.1) with the initial data \( u_{c,0} \) such that

\[
\|u_{c,0}\|^{2(1-\kappa)}_{L^2}\|H^{1/2} u_{c,0}\|^{2\kappa}_{L^2} < \mathcal{K},
\]

and \( SC(u_{c,0}) \) does not hold. Moreover, we show that if \( \|u_c\|_{S(\mathcal{H}^\infty)} = \infty \), then \( K = \{ u_c(x,t) | t \in \mathbb{R} \} \) is precompact in \( H^1(\mathbb{R}^3) \).

Prior to fulfilling our main task, we first state the linear profile decomposition associated with a perturbed linear propagator \( e^{itH_{\phi_n}} \), with

\[
H_{\phi_n} = -\Delta + V_{\phi_n}, \quad V_{\phi_n}(x) = \frac{1}{r_n^2} V_{\phi_n}\left(\frac{x}{r_n}\right),
\]

which was established by Hong [19] in the case \( b = 0 \). The profile decomposition associated with the free linear propagator \( e^{it\Delta} \) [9, 18] was established by using the concentration compactness principle in the spirit of Keraani [22] and Kenig and Merle [21].

**Proposition 5.2.** If \( V \) is radial and satisfies (1.2), (1.3) and \( |x| \|\nabla V\| \in L^2 \). Let \( \phi_n(x) \) be radial and uniformly bounded in \( H^1(\mathbb{R}^3) \), and \( r_n = 1, r_n \to 0 \) or \( r_n \to \infty \). Then for each \( M \) there exists a subsequence of \( \phi_n \), which is denoted by itself, such that the following statements hold.

1. For each \( 1 \leq j \leq M \), there exists (fixed in \( n \)) a profile \( \psi_j(x) \) in \( H^1(\mathbb{R}^3) \) and a sequence (in \( n \)) of time shifts \( t_j \), and there exists a sequence (in \( n \)) of remainders \( W_n^M(x) \) in \( H^1(\mathbb{R}^3) \) such that

\[
\phi_n(x) = \sum_{j=1}^{M} e^{it_j H_{\phi_n}} \psi_j(x) + W_n^M(x).
\]

2. The time sequences have a pairwise divergence property, i.e., for \( 1 \leq j \neq k \leq M \),

\[
\lim_{n \to +\infty} |t_j^n - t_k^n| = +\infty.
\]

3. The remainder sequence has the following asymptotic smallness property:

\[
\lim_{M \to +\infty} \left[ \lim_{n \to +\infty} \|e^{-it H_{\phi_n}} W_n^M\|_{S(\mathcal{H}^\infty)} \right] = 0.
\]

4. For each fixed \( M \), we have the asymptotic Pythagorean expansion as follows

\[
\|\phi_n\|_{L^2}^2 = \sum_{j=1}^{M} \|\psi_j\|_{L^2}^2 + \|W_n^M\|_{L^2}^2 + o_n(1),
\]
(5.12) \[ \|H_{r_n}^M \phi_n\|_{L^2}^2 = \sum_{j=1}^{M} \|H_{r_n}^j \psi_j\|_{L^2}^2 + \|H_{r_n}^M W_n\|_{L^2}^2 + o_n(1), \]

where \(o_n(1) \to 0\) as \(n \to +\infty\).

**Proof.** The proof is similar to that of Proposition 5.1 in [19]. Let’s first consider the case \(r_n \to 0\) or \(r_n \to \infty\). According to the profile decomposition associated with free propagator, there exists a subsequence of \(\phi_n\), which is still denoted by itself, such that

(5.13) \[ \phi_n(x) = \sum_{j=1}^{M} e^{-i\Delta \lambda \psi_j(x)} + W_n^M(x) \]

satisfying the properties in Proposition 5.2 with \(V = 0\).

In order to get the form of (5.8), we can rewrite (5.13) as

(5.14) \[ \phi_n(x) = \sum_{j=1}^{M} e^{iH_n \psi_j(x)} + \tilde{W}_n^M(x), \]

where

(5.15) \[ \tilde{W}_n^M(x) = W_n^M(x) + \sum_{j=1}^{M} \left( e^{-i\Delta \lambda \psi_j(x)} - e^{iH_n \psi_j(x)} \right). \]

Now we start to verify that (5.14) satisfies the properties (5.9)-(5.12). It’s obvious that (5.9) is true, so let’s look at (5.10). We note that \(u(t) = e^{i\lambda t} u_0\) solves the integral equation

(5.16) \[ e^{i\lambda t} u_0 = e^{i\lambda t - V} u_0 - i \int_{0}^{t} e^{i\lambda t - (V + a \int_{s}^{t})} \psi(x) \, ds. \]

Applying the formula (5.16) to \(u_0 = W_n^M\) yields that

(5.17) \[ \|e^{-i\lambda t} W_n^M\|_{S(H^\infty)} \leq \|e^{i\lambda t} W_n^M\|_{S(H^\infty)} + \| e^{i\lambda t - (V + a \int_{s}^{t})} \psi(x) \, ds \|_{S(H^\infty)} \]

as \(n \to \infty\) and \(M \to \infty\).

Also applying (5.16), we obtain

(5.18) \[ \|e^{-i\lambda t} \psi(x) - e^{i\lambda t} \psi_j(x)\|_{S(H^\infty)} \]

\[ = \| \int_{-\infty}^{t} e^{-i\lambda t + c \int_{s}^{t}} \psi(x) \, ds \|_{S(H^\infty)} \]

as \(n \to \infty\), where the last step follows from

(5.19) \[ \|V_{r_n} e^{i\lambda t} \psi_j\|_{L^2} \leq \|V_{r_n}\|_{L^\infty} \|e^{i\lambda t} \psi_j\|_{L^2} \leq \|V\|_{L^\infty} \|\psi_j\|_{H^2}. \]

and the condition \(r_n \to 0\) or \(\infty\). Thus \(\tilde{W}_n^M(x)\) in (5.15) satisfies the property (5.10).
To get (5.11), it suffices to prove

\[(5.20) \quad \|\tilde{W}_n^M\|_{L^2}^2 = \|W_n^M\|_{L^2}^2 + o_n(1).\]

It follows from the expression of \(\tilde{W}_n^M(x)\) (5.15) that

\[(5.21) \quad \|\tilde{W}_n^M\|_{L^2}^2 = \|W_n^M\|_{L^2}^2 + 2 \sum_{j=1}^{M} \langle W_n^M, e^{-it_j\Delta} \psi_j - e^{it_jH_0} \psi_j \rangle + 2 \sum_{k \neq j} \langle e^{-it_k\Delta} \psi_k - e^{it_kH_0} \psi_k, e^{-it_j\Delta} \psi_j - e^{it_jH_0} \psi_j \rangle + \sum_{j=1}^{M} \|e^{-it_j\Delta} \psi_j - e^{it_jH_0} \psi_j\|_{L^2}^2,
\]

from which, we only need to show that

\[(5.22) \quad \|e^{-it_j\Delta} \psi_j - e^{it_jH_0} \psi_j\|_{L^2} \to 0,
\]
as \(n \to \infty\).

In fact,

\[(5.23) \quad \|e^{-it_j\Delta} \psi_j - e^{it_jH_0} \psi_j\|_{L^2} = \| \int_{-\tau}^{0} e^{-i(t_j+s)H_0} (V_n e^{i\xi\Delta} \psi_j) ds \|_{L^2} \leq \|V_n e^{i\xi\Delta} \psi_j\|_{L^2} \to 0,
\]
as \(n \to \infty\), where the last step follows from

\[(5.24) \quad \|V_n e^{i\xi\Delta} \psi_j\|_{L^2} \leq \|V_n\|_{L^2} \|e^{i\xi\Delta} \psi_j\|_{L^2} \leq \|V\|_{L^2} \|\psi_j\|_{L^2}.
\]

and the condition \(r_n \to 0\) or \(\infty\). Thus, we complete the proof of (5.11).

Now we turn to (5.12). Since

\[(5.25) \quad \|H^\perp_{r_n} f_n\|_{L^2}^2 = \|\nabla f_n\|_{L^2}^2 + \langle V_n, f_n, f_n \rangle
\]

and

\[\langle V_n, f_n, f_n \rangle \leq \|V_n\|_{L^2} \|f_n\|_{L^2} \leq \|V\|_{L^2} \|\nabla f_n\|_{L^2}^2,
\]

we have

\[(5.26) \quad \|H^\perp_{r_n} f_n\|_{L^2}^2 = \|\nabla f_n\|_{L^2}^2 + o_n(1),
\]

provided that \(\|\nabla f_n\|_{L^2}\) is uniformly bounded. Hence, applying (5.25) with \(\phi_n, \psi\) and \(\tilde{W}_n^M\) and using the asymptotic Pythagorean expansion associated with the free linear propagator, we find that (5.12) can be deduced from the following expression

\[(5.27) \quad \|\nabla \tilde{W}_n^M\|_{L^2}^2 = \|\nabla W_n^M\|_{L^2}^2 + o_n(1).
\]

As in the proof of (5.20), it suffices to prove

\[(5.28) \quad \|\nabla (e^{-it_j\Delta} \psi_j - e^{it_jH_0} \psi_j)\|_{L^2} \to 0,
\]
as \( n \to \infty \). Indeed, using (2.1) and (2.2), we have
\[

\| \nabla (e^{-it\Delta}\psi_j - e^{itH_0}\psi_j) \|_{L^2} \leq \|H^\frac{1}{2} \int_{-t_0}^0 e^{-it\Delta - i\lambda_0 H_0} (V_r e^{itH_0} \psi_j) ds \|_{L^2} \\
\leq \| \nabla (V_r e^{itH_0} \psi_j) \|_{L^2 \to L^\infty} \to 0,
\]
(5.28)

as \( n \to \infty \), where the last step follows from
\[

\| \nabla (V_r e^{it\Delta} \psi_j) \|_{L^2 \to L^\infty} \leq \| |x| \nabla V_r \|_{L^\infty} \| |x|^{-1} e^{it\Delta} \psi_j \|_{L^2 \to L^\infty} + \| V_r \|_{L^2} \| \nabla e^{it\Delta} \psi_j \|_{L^2 \to L^\infty}
\]
(5.29)

Now let’s consider the other case \( r_n = 1 \). Using (5.13) again gives
\[

\phi_n(x) = \sum_{j=1}^M e^{-it\Delta} \psi_j(x) + W^M_n(x).
\]
(5.30)

If \( t_j \to \infty \), by Lemma 2.1, there exists \( \tilde{\psi}^j \in H^1(\mathbb{R}^3) \) such that \( \| e^{-it\Delta} \psi_j - e^{itH_0} \tilde{\psi}^j \|_{H^1} \to 0 \). If, on the other hand, \( t_j = 0 \), we set \( \tilde{\psi}^j = \psi^j \). To sum up, in either case, we obtain a new profile \( \tilde{\psi}^j \) for the given \( \psi^j \) such that
\[

\| e^{-it\Delta} \psi_j - e^{itH} \tilde{\psi}^j \|_{H^1} \to 0, \text{ as } n \to +\infty.
\]
(5.31)

In order to get the form of (5.10), we can rewrite (5.8) as
\[

\phi_n(x) = \sum_{j=1}^M e^{itH} \tilde{\psi}^j(x) + \tilde{W}^M_n(x),
\]
(5.32)

where
\[

\tilde{W}^M_n(x) = W^M_n(x) + \sum_{j=1}^M \left( e^{-it\Delta} \psi_j(x) - e^{itH} \tilde{\psi}^j(x) \right)
\]
(5.33)

Here we only give the proof of (5.10), since all the proofs of (5.11)-(5.12) can be obtained by following the same argument in the case when \( r_n \to 0 \) or \( \infty \) and using (5.31). Indeed, (5.17) with \( r_n = 1 \) is still valid, which yields
\[

\lim_{M \to +\infty, n \to +\infty} \| e^{itH} W^M_n \|_{L^2(\mathbb{R}^3)} = 0.
\]
(5.34)

And using Sobolev embedding, the norm equivalence (2.2), the Strichartz estimate (2.1) with \( s = 0 \) and (5.31), we have
\[

\| e^{-itH} (e^{-it\Delta} \psi_j(x) - e^{itH} \tilde{\psi}^j(x)) \|_{L^2(\mathbb{R}^3)} \leq \| H^{\frac{1}{2}} e^{itH} (e^{-it\Delta} \psi_j(x) - e^{itH} \tilde{\psi}^j(x)) \|_{L^2(\mathbb{R}^3)}
\]
\[
\leq \| H^{\frac{1}{2}} e^{itH} (e^{-it\Delta} \psi_j(x) - e^{itH} \tilde{\psi}^j(x)) \|_{L^2}
\leq \| e^{-it\Delta} \psi_j(x) - e^{itH} \tilde{\psi}^j(x) \|_{H^1} \to 0,
\]
(5.35)

as \( n \to \infty \). Putting (5.34) and (5.35) together gives (5.10), that is,
\[

\lim_{M \to +\infty, n \to +\infty} \| e^{-itH} \tilde{W}_n \|_{L^2(\mathbb{R}^3)} = 0.
\]
(5.36)
Remark 5.3. We claim that
\[(5.37) \quad \lim_{M_n \to \infty} ||W_n^M||_{L^p} = 0.\]
where \(2 < p < 6.\)

Indeed, when \(r_n \to 0\) or \(\infty\), it follows from Remark 6.2 in [11] (that is, (5.37) holds when \(V = 0\)) and (5.15) that it suffices to show that

\[(5.38) \quad ||e^{-it\Delta} \psi_j - e^{it\Delta H_n} \psi_j||_{L^p} \to 0,\]
as \(n \to \infty\), which is implied by Sobolev embedding, (5.22) and (5.27).

Similarly, when \(r_n = 1\), by Remark 6.2 in [11] again and (5.33), we only prove that

\[(5.39) \quad ||e^{-it\Delta} \psi_j(x) - e^{it\Delta H} \tilde \psi_j(x)||_{L^p} \to 0,\]
as \(n \to \infty\), which is implied by Sobolev embedding and (5.31).

It follows from (5.37) and Lemma 4.3 that

\[(5.40) \quad \lim_{M_n \to \infty} |||x|^{-b}||W_n^M||_{L^p_x} = 0.\]

Next, we shall use Proposition 5.2 and Remark 5.3 to establish the energy pythagorean expansion.

Lemma 5.4. In the situation of Proposition 5.2, we have

\[(5.41) \quad E_{V_n}(\phi_n) = \sum_{j=1}^{M} E_{V_n}(e^{it\Delta} \psi_j) + E_{V_n}(W_n^M) + o_n(1).\]

Proof. According to (5.11) and (5.12), it suffices to establish for all \(M \geq 1,
\[(5.42) \quad |||x|^{-b}||\phi_n^n||_{L^p_x} = \sum_{j=1}^{M} |||x|^{-b}||e^{it\Delta H_n} \psi_j^n||_{L^p_x} + |||x|^{-b}||W_n^M||_{L^p_x} + o_n(1).\]

For arbitrary small \(\varepsilon > 0\), let \(\psi \in C_c^{\infty}\) such that \(||\psi_j - \tilde \psi_j||_{H^1} \leq \frac{\varepsilon}{\sqrt{t}}\). Replacing \(\psi\) by \(\psi_j\) in (5.42) with \(o(\varepsilon)\)-error. one may assume that \(\psi_j^n \in C_c^{\infty} \).

First, we observe that

\[(5.43) \quad \sum_{j=1}^{M} |||x|^{-b}||e^{it\Delta H_n} \psi_j^n||_{L^p_x} = \sum_{j=1}^{M} |||x|^{-b}||e^{it\Delta H_n} \tilde \psi_j^n||_{L^p_x} + o_n(1).\]

Indeed, each cross term of its left hand side is of the form

\[(5.44) \quad \int_{\mathbb{R}} |x|^{-b} e^{it\Delta H_n} \tilde \psi_j^n \bar{e^{it\Delta H_n} \tilde \psi_j^n} e^{it\Delta H_n} \tilde \psi_j^n e^{it\Delta H_n} \tilde \psi_j^n \bar{e^{it\Delta H_n} \tilde \psi_j^n} \bar{e^{it\Delta H_n} \tilde \psi_j^n} dx.\]

By (5.9), there is at least one \(j_k\) such that \(t_k \to \infty\), for example, \(t_k \to \infty\). Using Lemma 4.3, the dispersive estimate, Sobolev embedding and the norm equivalence, we have that (5.44) is
bounded by
\begin{equation}
\left(\|e^{it\Delta}u_n\|_{L^2} + \|e^{it\Delta}u_n\|_{L^2}ight) \prod_{k=2,3,4} \|e^{it\Delta}u_n\|_{L^2} \\
= \left(\|e^{it\Delta}u_n\|_{L^2} + \|e^{it\Delta}u_n\|_{L^2}ight) \prod_{k=2,3,4} \|e^{it\Delta}u_n\|_{L^2} \\
\leq \left(\frac{|t|^{\frac{1}{2}} - \frac{1}{2} + |t|^{\frac{1}{2}}}{L^2} \right) \|\phi^j\|_{L^2} + \|\phi^j\|_{L^2} \|\phi^j\|_{L^2} \|\phi^j\|_{L^2} \to 0,
\end{equation}

where \(\frac{12}{3n} < r < 12\). Thus, (5.44) tends to zero as \(n \to 0\). It follows from (5.40) that, for any \(\varepsilon > 0\), there exists \(M_1 \gg 1\) such that \(\|x|^{-b}|W_n^2\|_{L^1} \leq \varepsilon\) for all sufficiently large \(n\). Hence, we obtain
\begin{equation}
\|x|^{-b}|\phi_0^j|\|_{L^1} = \sum_{j=1}^{M_1} \|x|^{-b}|e^{it\Delta}u_n|\|_{L^1} + O(\varepsilon) + o_n(1)
\end{equation}
(5.46)

\begin{equation}
= \sum_{j=1}^{M_1} \|x|^{-b}|e^{it\Delta}u_n|\|_{L^1} + \|x|^{-b}|W_n^2 - W_n^2|\|_{L^1} + O(\varepsilon) + o_n(1)
\end{equation}
(5.47)

\begin{equation}
= \sum_{j=1}^{M_1} \|x|^{-b}|e^{it\Delta}u_n|\|_{L^1} + \|x|^{-b}|W_n^2|\|_{L^1} + O(\varepsilon) + o_n(1),
\end{equation}

\(\square\)

**Proposition 5.5.** If \(V\) is radial and satisfies (1.2) and (1.3), \(V \geq 0\) or \(V \leq 0\), and \(0 < b < 1\), there exists a radial \(u_c\) in \(H^1(\mathbb{R}^3)\) with
\begin{equation}
E(u_c,0)^{1/2}M(u_c,0)^{1/2} = E_c < E,
\end{equation}
(5.48)

\begin{equation}
\|u_c,0\|_{L^2}^{2(1-b)}\|H^\perp u_c,0\|_{L^2}^{2b} < K,
\end{equation}

such that if \(u_c\) is the corresponding solution of (1.1) with the initial data \(u_c,0\), then \(u_c\) is global and \(\|u_c\|_{S(\mathcal{H}^V)} = \infty\).

**Remark 5.6.** When \(V = 0\), using the same argument as that of Proposition 6.4 in [11], combined with our new estimates (2.5)-(2.7) established in the present paper, we actually extend the result obtained in [11] to the more general case \(0 < b < 1\), to get the following statement: Let \(V = 0\) and \(u_0 \in H^1\) be radial and \(0 < b < 1\). Suppose that (1.5) and (1.6) are satisfied, then the solution \(u\) of (1.1) is global in \(H^1(\mathbb{R}^3)\) and scattering both forward and backward in time.

**Proof.** By the assumption (5.5) and the definition of \(\mathcal{E}_c\), we can find a sequence of solutions \(u_n(t)\) = INLS\(_V\) in of (1.1) with initial data \(u_{n,0}\) such that
\begin{equation}
M(u_{n,0})^{1/2} E(u_{n,0})^{1/2} \downarrow \mathcal{E}_c,
\end{equation}
(5.49)

\begin{equation}
\|u_{n,0}\|_{L^2}^{2(1-b)}\|H^\perp u_{n,0}\|_{L^2}^{2b} < K
\end{equation}
(5.50)

and
\begin{equation}
\|u_n\|_{S(\mathcal{H}^V)} = \infty.
\end{equation}
(5.51)
Note that it’s not obvious for uniform boundedness of \( \|u_{n,0}\|_{H^1} \) because of the shortness of scaling invariance for the equation (1.1). Hence, the first step is to show that \( \|u_{n,0}\|_{H^1} \) is uniformly bounded, which can be obtained from the fact that passing to a subsequence,

\[
(5.52) \quad r_n = \|u_{n,0}\|_{L^2}^{\frac{1}{2}} \sim 1.
\]

Indeed, by the norm equivalence, we have

\[
\|u_{n,0}\|_{H^1}^2 = \|u_{n,0}\|_{L^2}^2 + \|
abla u_{n,0}\|_{L^2}^2
\]

\[
(5.53) \sim \|u_{n,0}\|_{L^2}^2 + \|H^\frac{1}{2} u_{n,0}\|_{L^2}^2 < r_n^{-2\varepsilon} + \mathcal{K} \approx r_n^{2(1-\varepsilon)}.
\]

Let (5.52) be false, then we may assume that \( r_n \to 0 \) or \( +\infty \). Next, we shall apply the linear profile decomposition and the perturbation lemma to get a contradiction. To this end, we define

\[
\tilde{u}_n(x,t) = \frac{1}{r_n} u_n(x, t, x/r_n),
\]

and

\[
\tilde{u}_{n,0}(x) = \frac{1}{r_n} u_{n,0}(x/r_n).
\]

Hence, \( \tilde{u}_n \) is the solution to the initial value problem

\[
(5.54) \begin{cases}
\tilde{u}_n + H_p \tilde{u}_n - |x|^{-\delta} |\tilde{u}_n|^2 \tilde{u}_n = 0, \\
\tilde{u}_n(0) = \tilde{u}_{n,0},
\end{cases}
\]

and \( \|\tilde{u}_{n,0}\|_{H^1} \) is uniformly bounded, which follows from

\[
(5.55) \quad \|\tilde{u}_{n,0}\|_{L^2}^2 = r_n^{-2\varepsilon} \|u_{n,0}\|_{L^2}^2 = 1
\]

and

\[
(5.56) \quad \|
abla \tilde{u}_{n,0}\|_{L^2}^2 \sim \|H^\frac{1}{2} \tilde{u}_{n,0}\|_{L^2}^2 = r_n^{b-1} \|H^\frac{1}{2} u_{n,0}\|_{L^2}^2
\]

\[
= \|u_{n,0}\|_{L^2}^2, \quad \|H^\frac{1}{2} u_{n,0}\|_{L^2}^2 < \mathcal{K}^{-\frac{1}{2}}.
\]

Therefore, we apply Proposition 5.2 to \( \tilde{u}_{n,0} \) to get

\[
(5.57) \quad \tilde{u}_{n,0}(x) = \sum_{j=1}^M e^{i\epsilon j H_n} \psi^j(x) + W^M_n(x).
\]

Then by (5.41), we have further that

\[
(5.58) \quad \sum_{j=1}^M \lim_{n \to \infty} E_{V_n}(e^{i\epsilon j H_n} \psi^j) + \lim_{n \to \infty} E_{V_n}(W^M_n) = \lim_{n \to \infty} E_{V_n}(\tilde{u}_{n,0}).
\]

Since also by the profile expansion, we have

\[
(5.59) \quad 1 = \|\tilde{u}_{n,0}\|_{L^2}^2 = \sum_{j=1}^M \|\psi^j\|_{L^2}^2 + \|W^M_n\|_{L^2}^2 + o_n(1),
\]

\[
(5.60) \quad \|H^\frac{1}{2} \tilde{u}_{n,0}\|_{L^2}^2 = \sum_{j=1}^M \|H^\frac{1}{2} e^{i\epsilon j H_n} \psi^j\|_{L^2}^2 + \|H^\frac{1}{2} e^{i\epsilon j H_n} W^M_n\|_{L^2}^2 + o_n(1),
\]

\[
1 = \|\tilde{u}_{n,0}\|_{L^2}^2 = \sum_{j=1}^M \|\psi^j\|_{L^2}^2 + \|W^M_n\|_{L^2}^2 + o_n(1),
\]

\[
\|H^\frac{1}{2} \tilde{u}_{n,0}\|_{L^2}^2 = \sum_{j=1}^M \|H^\frac{1}{2} e^{i\epsilon j H_n} \psi^j\|_{L^2}^2 + \|H^\frac{1}{2} e^{i\epsilon j H_n} W^M_n\|_{L^2}^2 + o_n(1),
\]
Since from (4.12), each energy is nonnegative, and then
\[
\lim_{n \to \infty} E_{V_n}(e^{it_H \psi}) \leq \lim_{n \to \infty} E_{V_n}(\bar{u}_{n,0}) = \lim_{n \to \infty} M(u_{n,0})e^{-c_n E(u_{n,0})} = \mathcal{E}_c^{-1} < \mathcal{E}_c^+.
\]
(5.61)

For a given $j$, if $|r_j| \to +\infty$, we may assume $t_{n,j} \to +\infty$ or $t_{n,j} \to -\infty$ up to a subsequence. In this case, by (5.59) and (5.61) with $V = 0$ and using Lemma 4.3 (iii), we have
\[
\frac{1}{2}||\nabla \psi||_{L^2}||\psi||_{L^2}^\perp < \mathcal{E}_c^+.
\]
(5.62)

If we denote by INLS$_0(t)\phi$ a solution of (1.1) with $V = 0$ and initial data $\phi$, then we get from the existence of wave operators (Proposition 4.4 with $V = 0$) that there exists $\tilde{\psi}^j$ such that
\[
\|\text{INLS}_0(-t_j^i)\tilde{\psi}^j - e^{-it_j^i \Delta} \psi\|_{H^1} \to 0, \quad \text{as} \quad n \to +\infty
\]
(5.63)

and
\[
||\text{INLS}_0(t)\tilde{\psi}^j||_{S(H^\omega)} < \infty \quad \text{and} \quad ||(\nabla)\text{INLS}_0(t)\tilde{\psi}^j||_{S(L^2)} < \infty.
\]
(5.64)

If, on the other hand, $t_{n,j}^i = 0$, we set $\tilde{\psi}^j = \psi^j$. To sum up, in either case, we obtain a $\tilde{\psi}^j$ for the given $\psi^j$ such that (5.63) and (5.64) are true.

In order to use the perturbation theory to get a contradiction, we set
\[
v^j(t) = \text{INLS}_0(t)\tilde{\psi}^j, \quad v_n(t) = \sum_{j=1}^{M} v^j(t-t_{n,j}^i), \quad \bar{v}_n(t) = \text{INLS}_0v_n(0).
\]

We will prove successively the following three claims to get a contradiction.

Claim 1. There exists a large constant $A_0$ and $M_0$ independent of $M$ such that there exists $n_0 = n_0(M)$ such that for $n \geq n_0$,
\[
||\bar{v}_n||_{S(H^\omega)} \leq A_0, \quad ||\bar{v}_n||_{L^2_x H^1_x} \leq M_0
\]
(5.65)

Indeed, using (5.9), (5.63) and Lemma 4.3 (ii), we have that
\[
E_0(v_n(0)) = \sum_{j=1}^{M} E_0(v^j(-t_{n,j}^i)) + o_n(1) = \sum_{j=1}^{M} E_0(e^{-it_j^i \Delta} \psi^j) + o_n(1)
\]
(5.66)

By (5.22), (5.28), Lemma 4.3 (ii), the assumption $r_n \to 0$ or $\infty$ and Lemma 5.4, we have
\[
\sum_{j=1}^{M} E_0(e^{-it_j^i \Delta} \psi^j) \leq \sum_{j=1}^{M} E_{V_n}(e^{it_j^i H_{n,0}} \psi^j) + o_n(1)
\]
(5.67)

Collecting (5.66) and (5.67) gives
\[
E_0(v_n(0)) \leq r_n^{2(\gamma-1)} E(u_{n,0}) + o_n(1).
\]
(5.68)

Similarly, we have
\[
M(v_n(0)) \leq M(\bar{u}_{n,0}) + o_n(1) = r_n^{2\gamma} M(u_{n,0}) + o_n(1)
\]
(5.69)

and
\[
||\nabla v_n(0)||_{L^2} \leq ||H^{\perp}_{x} \bar{u}_{n,0}||_{L^2_x} + o_n(1) = r_n^{\gamma-1} ||H^{\perp}_{x} u_{n,0}||_{L^2_x} + o_n(1).
\]
(5.70)
Hence, (5.68)-(5.70) imply for large $n$,

\[(5.71)\hspace{1em} M(v_n(0))^{\frac{1-s}{1-s}}E_0(v_n(0))^{\frac{1}{s}} \leq M(u_n(0))^{\frac{1-s}{1-s}}E(u_n(0))^{\frac{1}{s}} + o_n(1) = E_c + o_n(1) < E\]

and

\[(5.72)\hspace{1em} \|v_n(0)\|^{2(1-s)}_L^2 \|\nabla v_n(0)\|^{2s}_L^2 \leq \|u_n(0)\|^{2(1-s)}_L^2 \|H^\perp u_n(0)\|^{2s}_L^2 + o_n(1) < K.\]

Moreover, we have

\[(5.73)\hspace{1em} E \leq M(Q)^{\frac{1-s}{1-s}}E_0(Q)^{\frac{1}{s}} \text{ and } K \leq \|Q\|^{2(1-s)}_L^2 \|\nabla Q\|^{2s}_L^2.\]

Indeed, if $V \geq 0$, (5.73) is trivial. If $V \leq 0$, by the Gagliardo-Nirenberg inequality and the Pohozaev identities,

\[(5.74)\hspace{1em} \frac{4}{(3+b)\|Q\|^{2(1-s)}_L^2 \|H^\perp Q\|^{2s}_L^2} \geq \frac{\|x^{-b}|Q|^{4}_L\|^{2}_L^4}{\|Q\|^{2(1-b)}_L^2 \|H^\perp Q\|^{2s}_L^2} \geq \frac{\|x^{-b}|Q|^{4}_L\|^{2}_L^4}{(3+b)\|Q\|^{2(1-s)}_L^2 \|\nabla Q\|^{2s}_L^2}.

Thus, we obtain

\[(5.75)\hspace{1em} \|Q\|^{2(1-s)}_L^2 \|H^\perp Q\|^{2s}_L^2 \leq \|Q\|^{2(1-s)}_L^2 \|\nabla Q\|^{2s}_L^2,

which, by (3.3) and (4.8), implies that

\[(5.76)\hspace{1em} E = E(Q)^{\frac{1-s}{1-s}}M(Q)^{\frac{1}{s}} \leq \left(\frac{s_c}{3+b}\right)^{\frac{1}{s}} \|Q\|^{2(1-s)}_L^2 \|\nabla Q\|^{2s}_L^2 = M(Q)^{\frac{1-s}{1-s}}E_0(Q)^{\frac{1}{s}}.

Thus, we obtain (5.73).

Putting together (5.71)-(5.73), we deduce that

\[M(v_n(0))^{\frac{1-s}{1-s}}E_0(v_n(0))^{\frac{1}{s}} < M(Q)^{\frac{1-s}{1-s}}E_0(Q)^{\frac{1}{s}}\]

and

\[\|v_n(0)\|^{2(1-s)}_L^2 \|\nabla v_n(0)\|^{2s}_L^2 < \|Q\|^{2(1-s)}_L^2 \|\nabla Q\|^{2s}_L^2.\]

Hence, it follows from Remark 5.6 that (5.65) is true.

Claim 2. There exists a large constant $A_1$ and $M_1$ independent of $M$ such that there exists $n_1 = n_1(M)$ such that for $n \geq n_1$,

\[(5.77)\hspace{1em} \|v_n\|_{L^\infty(L^\infty)} \leq A_1, \|(|\nabla|)v_n\|_{L^1(L^2)} \leq M_1.

In fact, we note that

\[(5.78)\hspace{1em} i\partial_t v_n + \Delta v_n + |x|^{-b} |v_n|^2 v_n = e_n,

where

\[(5.79)\hspace{1em} e_n = |x|^{-b} \left( \sum_{j=1}^{M} v^j(t - t_n^j)^2 \sum_{j=1}^{M} v^j(t - t_n^j) - \sum_{j=1}^{M} |v^j(t - t_n^j)|^2 |v^j(t - t_n^j)|^2 \right).

It is clear that

\[(5.80)\hspace{1em} |e_n| \leq c \sum_{k \neq j} \sum_{k \neq j} |x|^{-b} |v^j(t - t_n^j)| |v^k(t - t_n^k)|^2.\]
Since, for \( j \neq k \), \(|r_n - r^j_n| \to +\infty\), it follows from (2.7) and the dominated convergence theorem that
\[
\|e_n\|_{S^1(H^\infty)} \to 0 \quad \text{as} \quad n \to \infty. \tag{5.81}
\]
Next, we prove
\[
\|e_n\|_{S^1(L^2)} \to 0 \quad \text{as} \quad n \to \infty. \tag{5.82}
\]
Indeed, using (5.80) again, we estimate
\[
\|e_n\|_{S^1(L^2)} \leq c \sum \sum_{k \neq j} \|x|^{-b}|v^j(t - t^j_n)|v^k(t - t^k_n)|^2\|_{L^2_{t,r} L^2_r}. \tag{5.83}
\]
Using (5.9) and (2.6) and the dominated convergence theorem yields (5.82).

Finally, we prove
\[
\|\nabla e_n\|_{S^1(L^2)} \to 0 \quad \text{as} \quad n \to \infty. \tag{5.84}
\]
Note that
\[
\nabla e_n = \nabla(|x|^{-b}) \left(f(v_n) - \sum_{j=1}^M f(\nu_j(t - t^j_n))\right) + |x|^{-b} \nabla \left( f(v_n) - \sum_{j=1}^M f(\nu_j(t - t^j_n))\right) \equiv I_1 + I_2, \tag{5.85}
\]
where \( f(v) = |v|^2 v \). First, we consider \( I_1 \).
\[
\|I_1\|_{S^1(L^2)} \leq c \sum \sum_{k \neq j} \|x|^{-b-1}|v^j(t - t^j_n)|v^k(t - t^k_n)|^2\|_{L^2_{t,r} L^2_r}. \tag{5.86}
\]
By (2.13), we deduce that \( \|x|^{-b-1}|v^j(t - t^j_n)|v^k(t - t^k_n)|^2\|_{L^2_{t,r} L^2_r} \) is finite, then by the same argument as before, we have
\[
\|I_1\|_{S^1(L^2)} \to 0 \quad \text{as} \quad n \to \infty. \tag{5.87}
\]
On the other hand, observe that
\[
\|I_2\|_{S^1(L^2)} \leq c \sum \sum_{k \neq j} \|x|^{-b}|v^j(t - t^j_n)|v^k(t - t^k_n)|\|\nabla v^j(t - t^j_n)|v^k(t - t^k_n)|\|_{L^2_{t,r} L^2_r}. \tag{5.88}
\]
From the proof of (2.16) and by an analogous argument as before, we deduce that
\[
\|I_2\|_{S^1(L^2)} \to 0 \quad \text{as} \quad n \to \infty. \tag{5.89}
\]
Putting (5.87) and (5.89) together gives (5.84).

Applying (5.81), (5.82), and (5.84) with (5.65) to Lemma 2.9 with \( V = 0 \), gives (5.77).

Claim 3. There exists a large constant \( A_2 \) independent of \( M \) such that there exists \( n_2 = n_2(M) \) such that for \( n \geq n_2 \),
\[
\|\tilde{u}_n\|_{S^1(H^\infty)} \leq A_2. \tag{5.90}
\]
To see this, we note that
\[
i\partial_t v_n - H_{v_n} + |x|^{-b}|v_n|^2 v_n = \tilde{v}_n, \tag{5.91}
\]
where
\[
\tilde{v}_n = e_n - V_{v_n}. \tag{5.92}
\]
We will use the perturbation theory to get (5.90). To this end, we should control the following four terms, that is,

\begin{align}
(5.93) & \quad \|\tilde{u}_{n,0} - v_n(0)\|_{H^\infty}, \quad \|e^{-itH_n} (\tilde{u}_{n,0} - v_n(0))\|_{S'(H^\infty)}, \\
(5.94) & \quad \|\tilde{e}_n\|_{S'(H^\infty)}, \quad \|\langle\nabla\rangle e_n\|_{S(L^2)}.
\end{align}

From (5.57) and the definition of $v_n(t)$, we have

\begin{equation}
(5.95) \quad \tilde{u}_{n,0} - v_n(0) = W_n^M + \sum_{j=1}^{M} (e^{it\tilde{H}_n} \psi^j - \psi^j(-t_n^j)).
\end{equation}

As $\|\tilde{u}_{n,0}\|_{H^1}$ is uniformly bounded,

\begin{equation}
(5.96) \quad \|W_n^M\|_{H^1} \text{ is uniformly bounded too.}
\end{equation}

From the triangle inequality, (5.22), (5.27) and (5.63), it follows that

\begin{equation}
(5.97) \quad \|e^{it\tilde{H}_n} \psi^j - \psi^j(-t_n^j)\|_{H^1} \to 0 \text{ as } n \to 0,
\end{equation}

which combining with (5.96) implies that

\begin{equation}
(5.98) \quad \|\tilde{u}_{n,0} - v_n(0)\|_{H^1} \text{ is uniformly bounded.}
\end{equation}

Let $\epsilon_0 = \epsilon_0(A_2, n)$ be a small number given in Lemma 2.9. By (5.10), taking $M$ large enough such that there exists $n_3 = n_3(M)$ satisfying

\begin{equation}
(5.99) \quad \|e^{-it\tilde{H}_n} W_n^M\|_{S'(H^\infty)} < \frac{\epsilon_0}{2}
\end{equation}

for all $n \geq n_3$. Next we turn to the estimate of

\begin{equation}
(5.100) \quad \|e^{it\tilde{H}_n} (e^{it\tilde{H}_n} \psi^j - \psi^j(-t_n^j))\|_{S'(H^\infty)}
\end{equation}

for each $j$. From Strichartz estimates, (5.97), it follows that there exists $n_4 = n_4(M)$ such that for each $j$ and $n \geq n_4$

\begin{equation}
(5.101) \quad \|e^{it\tilde{H}_n} (e^{it\tilde{H}_n} \psi^j - \psi^j(-t_n^j))\|_{S'(H^\infty)} < \frac{\epsilon_0}{2M}.
\end{equation}

From (5.99) and (5.101), it follows that

\begin{equation}
(5.102) \quad \|e^{it\tilde{H}_n} (\tilde{u}_{n,0} - v_n(0))\|_{S'(H^\infty)} < \epsilon_0
\end{equation}

for all $n \geq \max\{n_1, n_4\}$.

Similar to the proof of (5.18), (5.22) and (5.27), by using (5.77), we have that both $\|V_n v_n\|_{S'(H^\infty)}$ and $\|\langle\nabla\rangle V_n v_n\|_{S(L^2)}$ go to zero as $n \to \infty$, which together with $\lim_{n \to \infty} \|e_n\|_{S'(H^\infty)} = 0$ and $\lim_{n \to \infty} \|\langle\nabla\rangle e_n\|_{S'(L^2)} = 0$ gives

\begin{equation}
(5.103) \quad \lim_{n \to \infty} \|\tilde{e}_n\|_{S'(H^\infty)} = \lim_{n \to \infty} \|\langle\nabla\rangle \tilde{e}_n\|_{S'(L^2)} = 0.
\end{equation}

Applying Lemma 2.9 with (5.98), (5.102), (5.103) and (5.77), we get (5.90).

By scaling, we have

\begin{equation}
(5.104) \quad \|u_n\|_{S'(H^\infty)} = \|\tilde{u}_n\|_{S'(H^\infty)} \leq A_2,
\end{equation}

contradicting (5.51). So $\|u_{n,0}\|_{H^1}$ is uniformly bounded.

The next step is to extract $u_{c,0}$ from a bounded sequence $\{u_{n,0}\}_{n=1}^{\infty}$. We omit the proof because it is similar to the proof of Proposition 6.4 in [11]. Indeed, it suffices to replace $e^{-it\tilde{H}}$ and $\nabla$ by $e^{-itH}$.
Proof. By Theorem 5.5, we can obtain the following results of precompactness and uniform localization of the minimal blow-up solution, the proof of which is standard and we omit it here.

**Proposition 5.7.** Let \( u_c \) be as in Proposition 5.5. Then

\[ K = \{ u_c(t) \mid t \in \mathbb{R} \} \subset H^1(\mathbb{R}^3) \]

is precompact in \( H^1(\mathbb{R}^3) \).

**Corollary 5.8.** Let \( u \) be a solution of \( (1.1) \) such that \( K = \{ u(t) \mid t \in \mathbb{R} \} \) is precompact in \( H^1(\mathbb{R}^3) \). Then for each \( \epsilon > 0 \), there exists \( R > 0 \) independent of \( t \) such that

\[ \left( 5.105 \right) \int_{|x|>R} |\nabla u(x,t)|^2 + |u(x,t)|^2 + |u(x,t)|^4 + |u(x,t)|^6 dx \leq \epsilon. \]

6. **Scattering result**

In this section, we prove the following rigidity statement and finish the proof of Theorem 1.3.

**Theorem 6.1.** If \( V \) is radial and satisfies \((1.2)\) and \((1.3)\), \( x \cdot \nabla V \leq 0, |x||\nabla V| \in L^7, \) and \( 0 < b < 1. \) Suppose that \( u_0 \in H^1(\mathbb{R}^3) \) is radial, \( M(u_0)^{1-b} E(u_0)^{1-b} < \mathcal{E} \) and \( ||u_0||_{L^2}^{1-b} ||H^2 u_0||_{L^2}^{1-b} \leq \mathcal{K} \). Let \( u \) be the corresponding solution of \((1.1)\) with initial data \( u_0 \). If \( K_+ = \{ u(t) : t \in [0, \infty) \} \) is precompact in \( H^1(\mathbb{R}^3) \), then \( u_0 \equiv 0. \) The same conclusion holds if \( K_- = \{ u(t) : t \in (-\infty, 0] \} \) is precompact in \( H^1(\mathbb{R}^3) \).

**Proof.** By Theorem 1.1, we have that \( u \) is global in \( H^1(\mathbb{R}^3) \) and

\[ \left( 6.1 \right) \|u(t)\|_{L^2}^{1-b} \|H^2 u(t)\|_{L^2}^{1-b} \leq \|Q\|^{1-b} \|\nabla Q\|_{L^2}^{1-b} \]

We first define

\[ \left( 6.2 \right) M_a(t) = 2 \int_{\mathbb{R}^3} \partial_j a \operatorname{Im}(\bar{u}\partial_j u) dx, \]

where \( a \in C_c^\infty(\mathbb{R}^3) \). Following the computation of Lemma 5.3 in Tao, Visan and Zhang [28] (see also Lemma 4.1 in [5]) yields

\[ \left( 6.3 \right) M'_a(t) = 2 \int_{\mathbb{R}^3} \left( 2 \partial_j a \operatorname{Re}(\partial_j \bar{u} \partial_j u) - \frac{1}{2} \Delta^2 a |u|^2 \right) dx \]

\[ - \int_{\mathbb{R}^3} \Delta a |x|^{-b} |u|^d dx + \int_{\mathbb{R}^3} \nabla a \cdot \nabla |x|^{-b} |u|^d dx \]

\[ - 2 \int_{\mathbb{R}^3} \nabla a \cdot \nabla V |u|^2 dx, \]

Take a radially symmetric function \( \phi \in C_c^\infty \) such that \( \phi(x) = |x|^2 \) for \( |x| \leq 1 \) and \( \phi(x) = 0 \) for \( |x| \geq 2 \), and define \( a(x) = R^2 \phi(R^{-1} x) \). By the repulsiveness assumption on the potential \( V \), direct
computation gives
\[ M'_a(t) = 8 \int_{\mathbb{R}^3} |\nabla u|^2 dx - 2(3 + b) \int_{\mathbb{R}^3} |x|^{-b} |u|^4 dx - 4 \int_{\mathbb{R}^3} x \cdot \nabla V |u|^2 dx + (\text{Remainder}) \]
\[ \leq 8 \int_{\mathbb{R}^3} |\nabla u|^2 dx - 2(3 + b) \int_{\mathbb{R}^3} |x|^{-b} |u|^4 dx + (\text{Remainder}), \]
where
\[ (\text{Remainder}) = 4 \text{Re} \int_{\mathbb{R}^3} \left( \frac{x}{R} \left( \frac{x}{R} \right)^3 \right) \partial_x u^2 dx + 4 \sum_{j \neq k} \text{Re} \int_{\mathbb{R}^3} \left( \frac{x}{R} \right) \partial_k u_j \partial_x u_{j, k} dx \]
\[ - \frac{1}{R^2} \int_{\mathbb{R}^3} \nabla \left( \frac{x}{R} \right) |u|^2 dx + R \int_{\mathbb{R}^3} \nabla (|x|^{-b}) \cdot (\nabla \left( \frac{x}{R} \right) |u|^4 dx \]
\[ + \int_{\mathbb{R}^3} \left( \nabla V - R \nabla \left( \frac{x}{R} \right) \nabla \phi \right) |u|^2 dx. \]
Since \( \phi(x) \) is radial and \( \phi(x) = |x|^2 \) if \( |x| \leq 1 \), the sum of all terms in the definition of (Remainder) integrating over \( |x| \leq R \) is zero. Indeed, for the first three terms and the last term, this is clear by the definition of \( \phi(x) \). In the fourth term, we calculate that
\[ 2 \int_{|x| \leq R} \nabla (|x|^{-b}) \cdot x |u|^4 dx = 2 \int_{|x| \leq R} b |x|^{-b} |u|^4 dx. \]
Adding to the fifth term (also integrating over \( |x| \leq R \)), we get zero since \( \Delta \phi = 6 \) if \( |x| \leq R \). Therefore, from Corollary 5.8, we can infer that (Remainder) \( \to 0 \) as \( R \to \infty \) uniformly in \( t \in [0, \infty) \). In fact,
\[ (\text{Remainder}) \leq \int_{|x| \leq R} |\nabla u|^2 dx + \int_{|x| \geq R} |x|^{-b} |u|^4 dx + \frac{1}{R^2} \int_{|x| \leq R} |u|^2 dx + \int_{|x| \geq R} \|x\| \|\nabla V\|_{L^2} \|u\|_{L^4}^2 \rightarrow 0. \]
Putting (6.4), (6.7), (4.14) and (4.12) together and using the norm equivalence yield that there exists some constant \( \delta_0 > 0 \) such that
\[ M'_a(t) \geq \delta_0 \int_{\mathbb{R}^3} |\nabla u|^2 dx. \]
Thus, we have
\[ M_a(0) - M_a(t) \geq \delta_0 t \int_{\mathbb{R}^3} |\nabla u|^2 dx. \]
On the other hand, by the definition of \( M_a(t) \), the norm equivalence an (6.1), we should have
\[ |M_a(t)| \leq R \|u\|_{L^2} |\nabla u|_{L^2} \leq R \|u\|_{L^2} |H^\perp u|_{L^2} \leq cR, \]
which is a contradiction for \( t \) large unless \( u_0 = 0 \). \( \square \)

Now, we can finish the proof of Theorem 1.3.

**The Proof of Theorem 1.3.** In view of Proposition 5.7, Theorem 6.1 implies that \( u_c \) obtained
in Proposition 5.5 cannot exist. Thus, there must holds that \( E_c = E \), which combined with Proposition 2.6 implies Theorem 1.3. \( \square \)

7. Blow-up Criteria

We finally consider the blow-up in finite or infinite time following the idea from Du-Wu-Zhang [8].

**Proof of Theorem 1.7** Assume the contrary, then we have
\[
\bar{C}_0 = \sup_{t \in \mathbb{R}^n} \|\nabla u(t)\|_{L^2} < \infty.
\]
Consider the local Virial identity and let
\[
I(t) = \int \phi(x)\|u(t, x)\|^2 d\mathbf{x},
\]
then by straight computation, we get

**Lemma 7.1.** For any \( \phi \in C^4(\mathbb{R}^3) \),
\[
\left\{ \begin{array}{l}
I'(t) = 2Im \int \nabla \phi \cdot \nabla u\bar{u} d\mathbf{x}; \\
I''(t) = \int 4Re \nabla \bar{u} \nabla \phi \nabla u d\mathbf{x} + \int 2\nabla \phi \cdot \nabla |u|^2 + \Delta \phi |x|^{-b}|u|^4 - \nabla \phi \cdot \nabla (|x|^{-b}) |u|^4 d\mathbf{x} - \int \Delta^2 \phi |u|^2 d\mathbf{x}.
\end{array} \right.
\]

If \( \phi \) is radial, one may find that
\[
I'(t) = 2Im \int \left( \frac{\phi'(r)}{r} \right) \frac{x \cdot \nabla u}{r} \bar{u} d\mathbf{x},
\]
\[
I''(t) = 4 \int \frac{\phi'(r)}{r} |\nabla u|^2 d\mathbf{x} + 4 \int \left( \frac{\phi''(r)}{r^2} - \frac{\phi'(r)}{r^3} \right) |x \cdot \nabla u|^2 d\mathbf{x}
- 2 \int \phi'(r) x \cdot \nabla |u|^2 d\mathbf{x} - \int \left( \frac{2 + b}{r} \phi'(r) \right) |x|^{-b}|u|^4 d\mathbf{x} - \int \Delta^2 \phi |u|^2 d\mathbf{x}.
\]

**L^2 estimate in the exterior ball**

Fix some large constant \( R > 0 \), which will be decided later. We choose \( \phi \) in (7.11) such that
\[
\phi = \begin{cases} 
0, & 0 \leq r \leq \frac{R}{2}; \\
1, & r \geq R,
\end{cases}
\]
and
\[
0 \leq \phi \leq 1, \quad 0 \leq \phi' \leq \frac{4}{R}.
\]
By (7.12), there holds that
\[
I(t) = I(0) + \int_0^t I'(\tau) d\tau \leq I(0) + t\|\phi\|_{L^\infty} \sup_{s \in [0, t]} (\|u(s)\|_{L^2} \|\nabla u(s)\|_{L^2})
\leq \int_{|x| \geq \frac{R}{2}} |u_0|^2 d\mathbf{x} + \frac{4m_0 c_0 t}{R},
\]
where \( m_0 = \|u_0\|_{L^2} \). Observe that
\[
\int_{|x| \leq \frac{R}{2}} |u_0|^2 \,dx = \mathcal{O}(1),
\]
and
\[
\int_{|x| \geq R} |u(t,x)|^2 \,dx \leq I(t).
\]
To sum up, we obtain the following estimate.

**Lemma 7.2.** Given \( \eta_0 > 0 \), then for any
\[
t \leq \frac{\eta_0 R}{4m_0 \tilde{C}_0},
\]
we have that
\[
\int_{|x| \geq \frac{R}{2}} |u(t,x)|^2 \,dx \leq \eta_0 + \mathcal{O}(1).
\]

**Localized Virial identity**

We rewrite \( I''(t) \) as
\[
I''(t) = 8K(u) + R_1 + R_2 + R_3 + R_4,
\]
where
\[
R_1 = 4 \int \left( \frac{\phi'}{r} - 2 \right) |\nabla u|^2 \,dx + 4 \int \left( \frac{\phi''}{r^2} - \frac{\phi'}{r^3} \right) |x \cdot \nabla u|^2 \,dx,
\]
\[
R_2 = - \int \left( \phi'' + \frac{2 - b}{r} \phi'(r) - (6 - 2b) \right) |x|^{-b} |u|^4 \,dx,
\]
\[
R_3 = -2 \int \left( \frac{\phi'}{r} - 2 \right) (x \cdot \nabla V) |u|^2 \,dx,
\]
\[
R_4 = - \int \Delta^2 \phi |u|^2 \,dx.
\]
Indeed, \( R_j(j = 1, \ldots, 4) \) are the error terms from the localization. At this stage, we choose \( \phi \) such that
\[
0 \leq \phi \leq r^2, \quad \phi'' \leq 2, \quad \phi^{(4)} \leq \frac{4}{R^2},
\]
and
\[
\phi = \begin{cases} r^2, & 0 \leq r \leq R; \\ 0, & r \geq 2R \end{cases}
\]
to get the following result.

**Lemma 7.3.** There exist two constants \( \tilde{C} = \tilde{C}(m_0, \tilde{C}_0) > 0 \) and \( \theta_0 > 0 \), such that
\[
I''(t) \leq 8K(u(t)) + \tilde{C} \|u\|^{\theta_0}_{L^2(|x| > R)}.
\]

**Proof.** First, we claim \( R_1 \leq 0 \). For this purpose, we divide \( \mathbb{R}^3 \) into two parts:
\[
\left\{ \frac{\phi''}{r^2} - \frac{\phi'}{r^3} \leq 0 \right\} \cup \left\{ \frac{\phi''}{r^2} - \frac{\phi'}{r^3} > 0 \right\}.
\]
If \( \frac{\phi''}{r} - \frac{\phi'}{r} \leq 0 \) it is obviously true since \( \phi' \leq 2r \). If \( \frac{\phi''}{r} - \frac{\phi'}{r} > 0 \), since \( \phi'' \leq 2 \), it holds that
\[
R_1 \leq 4 \int \left( \frac{\phi'}{r} - 2 \right) |\nabla u|^2 \, dx + 4 \int \left( 2 - \frac{\phi'}{r} \right) |x \cdot \nabla u|^2 \, dx = 0.
\]

Next, since
\[
supp(\phi'' + \frac{2 - b}{r} \phi'(r) - (6 - 2b)) \subset [R, \infty),
\]
we can prove that
\[
R_2 \leq C(\eta_0, \tilde{C}_0) \|u\|_{L^2(\{x \mid |x| > R\})}^2.
\]
By the properties of \( \phi, R_4 \leq CR^{-2} \|u\|_{L^2(\{x \mid |x| > R\})}^2 \). Finally, by the assumption (1.20), \( x \cdot \nabla V \leq 0 \), and we obtain \( R_3 \leq 0 \).

Combining all the above estimates, there holds that for \( R > 1 \),
\[
I'(t) \leq 8K(u) + \tilde{C}(\eta_0^{\theta_0/2} + o_R(1)),
\]
where \( \theta_0 \in (0, 1) \), \( \tilde{C} > 0 \) depending on \( \eta_0 \) and \( C_0 \) and the lemma is proved.

Applying Lemma 7.2 and 7.3, we find that for any \( t \leq T := \eta_0 R/(4m_0 \tilde{C}_0), \)
\[
I''(t) \leq 8K(u) + \tilde{C}(\eta_0^{\theta_0/2} + o_R(1)).
\]
Integrating from 0 to \( T \), from the assumption (1.22) we get that
\[
I(T) \leq I(0) + I'(0)T + \int_0^T \left( 8K(u(s)) + \tilde{C}\eta_0^{\theta_0/2} + o_R(1) \right) \, ds
\]
\[
\leq I(0) + I'(0)T + (8\theta_0 + \tilde{C}\eta_0^{\theta_0/2} + o_R(1)) \frac{T^2}{2}.
\]
If we choose \( \eta_0 \) to satisfy that
\[
\tilde{C}\eta_0^{\theta_0/2} = -\beta_0,
\]
and take \( R \) large enough, then for \( T = \eta_0 R/(4m_0 \tilde{C}_0) \), we obtain that
\[
I(T) \leq I(0) + I'(0)\eta_0 R/(4m_0 \tilde{C}_0) + o_R R^2,
\]
where the constant
\[
o_0 = \beta_0 \eta_0^{\theta_0/2}/(4m_0 \tilde{C}_0)^2 < 0
\]
is independent of \( R \).

Now we claim that
\[
I(0) = o_R(1) R^2, \quad I'(0) = o_R(1) R.
\]
In fact,
\[
I(0) \leq \int_{|x| < \sqrt{R}} |x|^2 |u_0|^2 \, dx + \int_{R |x| < 2R} |x|^2 |u_0|^2 \, dx
\]
\[
\leq Rm_0^2 + R^2 \int_{|x| > \sqrt{R}} |u_0|^2 \, dx = o_R(1) R^2.
\]
A similar argument can be used to obtain the second estimate and then proved (7.14).

Together (7.13) with (7.14), by choosing \( R \) large enough, we obtain that
\[
I(T) \leq o_R(1) R^2 + o_0 R^2 \leq \frac{1}{2\alpha_0} R^2.
\]
Since \( \alpha_0 < 0 \), we get then \( I(T) < 0 \), which contradicts with the definition of \( I \), and the proof of Theorem 1.7 is completed.

\[ \square \]

We finally finish the proof of Theorem 1.5.

**Proof of Theorem 1.5** Firstly, we need to show that the assumption (4.1) implies (1.22). Indeed, first by Theorem 4.1, we know that (4.5) holds for any \( t \in [0, T_{\max}) \). Then by the definition of \( K(u) \) (1.19), we get

\[ (7.15) \quad K(u) = (3 + b)E(u) - \frac{1 + b}{2} |H^2 u|_{L^2}^2 - \frac{1}{2} \int (2V + x \cdot \nabla u)^2 \, dx. \]

Then by (4.5) and the assumption \( 2V + x \cdot \nabla V \geq 0 \) in (1.20), one obtains that

\[ K(u(t)) < 0, \quad \text{for any} \ t \in [0, T_{\max}). \]

Now we claim that there exists some \( \delta_0 > 0 \) such that for any \( t \in [0, T_{\max}) \),

\[ (7.16) \quad K(u) < -\delta_0 |H^2 u|_{L^2}^2. \]

Indeed, if on the contrary, there exists some time sequence \( \{t_n\} \subset [0, T_{\max}) \) such that

\[ -\delta_n \frac{1 + b}{2} |H^2 u|_{L^2}^2 < K(u(t_n)) < 0, \]

where \( \delta_n \to 0 \) as \( n \to \infty \). Then by (7.15),

\[ E(u(t_n)) \geq \frac{1}{3 + b} \left( K(u(t_n)) + \frac{1 + b}{2} |H^2 u|_{L^2}^2 \right) > (1 - \delta_n) \frac{1 + b}{2(3 + b)} |H^2 u|_{L^2}^2. \]

Therefore, we obtain

\[
M(u(t_n))^{1-v} E(u(t_n))^{v} \\
> M(u(t_n))^{1-v} (1 - \delta_n)^v \left( \frac{1 + b}{2(3 + b)} \right)^v |H^2 u|_{L^2}^2 \\
> (1 - \delta_n)^v \left( \frac{1 + b}{2(3 + b)} \right)^v \mathcal{K} = (1 - \delta_n)^v \omega,
\]

contradicting (4.1) and (7.16) holds. Finally, since by (4.5), the Kinetic \( |H^2 u|_{L^2}^2 > \epsilon_0 \) with some positive constant \( \epsilon_0 > 0 \), then we immediately obtain (1.22) and Theorem 1.5 is proved by Theorem 1.7.

\[ \square \]

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QING GUO, COLLEGE OF SCIENCE, MINZU UNIVERSITY OF CHINA, BEIJING, 100081, P.R. CHINA
E-mail address: guqing0117@163.com

HUA WANG, SCHOOL OF MATHEMATICS AND STATISTICS AND HUBEI PROVINCE KEY LABORATORY OF MATHEMATICAL PHYSICS, CENTRAL CHINA NORMAL UNIVERSITY, WUHAN, 430079, P.R. CHINA
E-mail address: wanghua_math@126.com

XIANGHUA YAO, SCHOOL OF MATHEMATICS AND STATISTICS AND HUBEI PROVINCE KEY LABORATORY OF MATHEMATICAL PHYSICS, CENTRAL CHINA NORMAL UNIVERSITY, WUHAN, 430079, P.R. CHINA
E-mail address: yaoxiaohua@mail.ccnu.edu.cn