DIVISION ALGEBRAS OF PRIME PERIOD $\ell \neq p$ OVER
FUNCTION FIELDS OF $p$-ADIC CURVES

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Abstract. Let $F$ be a field finitely generated and of transcendence degree one over a $p$-adic field, and let $\ell \neq p$ be a prime. Results of Merkurjev and Saltman show that $H^2(F, \mu_\ell)$ is generated by $\mathbb{Z}/\ell$-cyclic classes. We prove the “$\mathbb{Z}/\ell$-length” in $H^2(F, \mu_\ell)$ equals the $\ell$-Brauer dimension, which Saltman showed to be two. It follows that all $F$-division algebras of period $\ell$ are crossed products, either cyclic (by Saltman’s cyclicity result) or tensor products of two cyclic division algebras. Our result was originally proved by Suresh assuming $F$ contains $\mu_\ell$.

1. Introduction

It is widely believed that for a field $F$ and a prime-to-char$(F)$ number $n$, the $n$-torsion $H^2(F, \mu_n) = n\text{Br}(F)$ of the Brauer group of $F$ is generated by $\mathbb{Z}/n$-cyclic classes, i.e., the cup product map $H^1(F, \mu_n) \otimes \mathbb{Z}H^1(F, \mathbb{Z}/n) \rightarrow H^2(F, \mu_n)$ is surjective. This is true, for example, when $F$ contains the $n$-th roots of unity by Merkurjev-Suslin’s theorem ([9, Theorem 16.1]), when $n = 3$ by [9, Corollary 16.4], and when $n = 5$ by a recent result of Matzri ([7, Theorem 4.5]). When $H^2(F, \mu_n)$ is generated by $\mathbb{Z}/n$-cyclic classes we define the $\mathbb{Z}/n$-length $nL(F)$ in $H^2(F, \mu_n)$ to be the smallest number of $\mathbb{Z}/n$-cyclic classes needed to write any class, or $\infty$ if no such number exists. If $nL(F) = c$ then any class in $H^2(F, \mu_n)$ has index at most $n^c$, so that the $n$-Brauer dimension $n\text{Br.dim}(F)$ is (finite and) bounded by $c$. It is not known whether finite $n$-Brauer dimension implies finite $\mathbb{Z}/n$-length, or even whether finite $n$-Brauer dimension implies $H^2(F, \mu_n)$ is generated by cyclic classes.

We study this problem when $F$ is a field that is finitely generated and of transcendence degree one over the $p$-adic field $\mathbb{Q}_p$. In [10, Theorem 3.4], Saltman showed that then $n\text{Br.dim}(F)$ equals two, for any (prime-to-$p$) $n$. Recently Suresh showed that when $n = \ell$ is prime and $F$ contains $\mu_\ell$, the $\mathbb{Z}/\ell$-length in $H^2(F, \mu_\ell)$ is also two ([13, Theorem 2.4]). The assumption on roots of unity excludes important cases such as the rational function field $F = \mathbb{Q}_p(T)$ (if $\ell \neq 2, 3$ and $p \neq 1$ (mod $\ell$)). But in this case the cup product map is surjective by Merkurjev’s theorem [8, Theorem 2] and Saltman’s cyclicity result for classes of prime index [12, Theorem 5.1], so in any case $H^2(F, \mu_\ell)$ is known to be generated by $\mathbb{Z}/\ell$-cyclic classes.

We show that for a prime $\ell \neq p$ the $\mathbb{Z}/\ell$-length in $H^2(F, \mu_\ell)$ is two, hence that all $F$-division algebras of period $\ell$ and index $\ell^2$ decompose into two cyclic $F$-division algebras of index $\ell$. It follows immediately that all $F$-division algebras of period $\ell$ are (abelian) crossed products. Noncrossed products of larger $\ell$-power period exist by [3] and [4].
Our results rely heavily on Saltman’s degree-\(\ell\) cyclicity result and his hot point criterion [12 Corollary 5.2], and our lifting results from [4], which use the machinery of Grothendieck’s existence theorem. We show an \(F\)-division algebra \(\Delta\) of period \(\ell\) and index \(\ell^2\) is decomposable by explicitly constructing a tensor factor of degree \(\ell\), lifting a class constructed over the generic points of the closed fiber of a 2-dimensional model \(X/\mathbb{Z}_p\), as developed in [3]. Since the function fields of the closed fiber are global fields, we can use class field theory (esp. Grunwald-Wang’s theorem) to manipulate the lifted class so that it cancels the hot points of \(\Delta\), which implies it is part of a decomposition of \(\Delta\) by the hot point criterion. The cyclicity result then shows the remaining factor is cyclic. Suresh’s approach in [13] similarly cancels \(\Delta\)’s hot points using a tensor factor, but his tensor factor is constructed as a symbol algebra, which requires \(\mu_\ell \subset F\). There is no obvious way to get to the general case from that construction.

2. Background and Conventions

2.1. Brauer Group Conventions. In this paper an \(F\)-division algebra is a division ring that is central and finite-dimensional over \(F\). If \(D\) is an \(F\)-division algebra we write \([D]\) for the class of \(D\) in the Brauer group \(\text{Br}(F)\), \(\text{ind}(D)\) for the index or degree of \(D\), and \(\text{per}(D)\) for the period of \(D\). We say \(D\) is a crossed product if it contains a maximal subfield that is Galois over \(F\). See [1] for a discussion of crossed product and noncrossed product division algebras.

We write \(\text{H}^2(F, \mu_n) = \eta \text{Br}(F)\) for the \(n\)-torsion subgroup, where \(n\) is prime-to-\(\text{char}(F)\) and \(\mu_n\) is the group of \(n\)-th roots of unity. In the terminology of [1 Section 4], the \(n\)-Brauer dimension \(\eta \text{Br}.\text{dim}(F)\) of \(F\) is the smallest number \(d\) such that every class in \(\text{H}^2(F, \mu_n)\) has index dividing \(n^d\), or \(\infty\) if no such number exists. We say \(\text{H}^2(F, \mu_n)\) is generated by \(\mathbb{Z}/n\)-cyclic classes if the cup product map \(\text{H}^1(F, \mu_n) \otimes_{\mathbb{Z}} \text{H}^1(F, \mathbb{Z}/n) \to \text{H}^2(F, \mu_n)\) is surjective, and a class is \(\mathbb{Z}/n\)-cyclic if it has the form \((f) \cdot \theta\) for some \((f) \in \text{H}^1(F, \mu_n)\) and \(\theta \in \text{H}^1(F, \mathbb{Z}/n)\). If \(\text{H}^2(F, \mu_n)\) is generated by cyclic classes, the \(\mathbb{Z}/n\)-length \(\eta \text{L}(F)\) is the smallest number of \(\mathbb{Z}/n\)-cyclic classes needed to express an arbitrary class, or \(\infty\) if no such number exists. See [1] Section 3] for a discussion of known results regarding \(\mathbb{Z}/n\)-length, usually called “symbol length” when \(F\) contains an \(n\)-th root of unity.

It is clear that if \(\text{H}^2(F, \mu_n)\) is generated by cyclic classes and \(\eta \text{L}(F)\) is finite then \(\eta \text{Br}.\text{dim}(F) \leq \eta \text{L}(F)\). Conversely, as mentioned above, it is not known whether a finite Brauer dimension implies a finite \(\mathbb{Z}/n\)-length, or even that \(\text{H}^2(F, \mu_n)\) is generated by \(\mathbb{Z}/n\)-cyclic classes. However, when \(n = \ell\) Merkurjev proved that \(\text{H}^2(F, \mu_\ell)\) is generated by classes of index \(\ell\) ([8 Theorem 2]), hence for the fields considered in this paper \(\text{H}^2(F, \mu_\ell)\) is generated by \(\mathbb{Z}/\ell\)-cyclic classes by Saltman’s cyclicity result.

2.2. General Conventions. Let \(S\) be an excellent scheme and suppose \(n\) is invertible on \(S\). We write \(\mathbb{Z}/n(r)\) for the étale sheaf \(\mathbb{Z}/n\) twisted by an integer \(r\), and \(\text{H}^0(S, r) = \text{H}^0(S, \mathbb{Z}/n(r))\) for the étale cohomology group. If \(S = \text{Spec} \ A\) for a ring \(A\), we write \(\text{H}^0(A, r)\). If \(T\) is a subscheme of \(S\) we write \(\kappa(T)\) for its ring of meromorphic functions, which is the localization of \(O_T\) at all associated points. If \(T \rightarrow S\) is a morphism of schemes then the restriction \(\text{res}_{T|S} : \text{H}^0(T, r) \rightarrow \text{H}^0(S, r)\)
is defined, and we write $\beta_S = \text{res}_{T|S}(\beta)$, and if $S = \text{Spec} \ A$ we write $\beta_A$. If $Z \subset S$ is a subscheme we write $Z_T$ for the preimage $Z \times_S T$.

If $v$ is a valuation on a field $F$, we write $\kappa(v)$ for the residue field of the valuation ring $O_v$, and $F_v$ for the completion of $F$ with respect to $v$. If $v$ arises from a prime divisor $D$ on $S$, we write $v = v_D$, $\kappa(D)$, and $F_D$. If a set $\{v_i\}$ arises from a divisor $D = \sum_i D_i$, we write $F_D = \prod_i F_{D_i}$. Recall that if $F = (F,v)$ is a discretely valued field and $\alpha \in H^q(F,r)$, then $\alpha$ has a residue $\partial_v(\alpha)$ in $H^{q-1}(\kappa(v), r-1)$. More generally if $\xi$ is a generic point of a scheme $S$, $F = \kappa(\xi)$, and $\alpha \in H^q(S, r)$, then for each discrete valuation $v$ on $F$ we define

$$\partial_v(\alpha) \triangleq \partial_v(\alpha_F) \in H^{q-1}(\kappa(v), r-1)$$

We say $\alpha$ is unramified with respect to $v$ if $\partial_v(\alpha) = 0$, and in that case the value of $\alpha$ at $v$ is the element $\alpha(v) = \text{res}_{F/F_v}(\alpha) \in H^q(\kappa(v), r) \leq H^q(F_v, r)$ ([5, 7.13, p.19]).

If $v$ arises from a prime divisor $D$ on a scheme, we will substitute the notations $\partial_D$ and $\alpha(D)$. If $S$ is noetherian we write $D_\alpha$ for the ramification divisor of $\alpha$ on $S$, which is the sum of (finitely many) prime divisors on $S$ at which $\alpha$ ramifies.

2.3. Setup. In the following, $F$ will always be a finitely generated field extension of $\mathbb{Q}_p$ of transcendence degree one, $n$ will be a prime-to-$p$ number, and $X/\mathbb{Z}_p$ will be a connected regular (projective, flat) relative curve over $\mathbb{Z}_p$ with function field $F = K(X)$. Such a surface exists for any $F$ by a theorem of Lipman (see [6, Theorem 8.3.44]). We write $X_0 = X \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ for the closed fiber, $C = X_0_{\text{red}}$ for the reduced scheme underlying the closed fiber, $C_1, \ldots, C_m$ for the irreducible components of $C$, and $S$ for the set of singular points of $C$. We assume that $X_0$ has normal crossings, hence that each $C_i$ is regular, and at most two of them meet (transversally) at each singular point of $C$. This is permitted by embedded resolution of curves in surfaces (see [3, Theorem 9.2.26]).

We say an effective divisor $D$ on $X$ is horizontal if each of its irreducible components maps surjectively to $\text{Spec} \mathbb{Z}_p$. By [4, Proposition 2.4] there exists for each closed point $z \in X \setminus S$ a regular irreducible horizontal divisor $D \subset X$ that intersects $C$ transversally. Let $D_S$ denote the support of these lifts. We say a divisor $D$ is distinguished and write $D \subset D_S$ if it is reduced and supported in $D_S$. Each $D \subset D_S$ is a disjoint union of its irreducible components, each of which has a single closed point and meets $C$ transversally.

By weak approximation ([11, Lemma]), we may choose an element $\pi \in F$ such that $\text{div}(\pi) = C + E \subset X$, where $E$ is horizontal and avoids all closed points of any finite set containing $S$.

Suppose $\ell \neq p$ is prime, $\alpha \in H^2(F, \mu_\ell)$, and $D_\alpha \subset X$ has normal crossings. Following Saltman’s terminology in [12] we say $\alpha$ has a hot point $z$ on $X$ if (and only if) $z$ is a nodal point of $D_\alpha$, and if $D, D' \subset D_\alpha$ are the two irreducible components meeting transversally at $z$, then $\partial_D(\alpha)$ and $\partial_{D'}(\alpha)$ are unramified at $z$, and $\langle \partial_D(\alpha)(z) \rangle \neq \langle \partial_{D'}(\alpha)(z) \rangle$. By [12, Corollary 5.2], $\alpha$ has index $\ell$ if and only if $D_\alpha$ has no hot points (hot point criterion), and by [12, Theorem 5.1], if $\alpha$ has index $\ell$ then it is cyclic.

Theorem 2.4 ([4, Lemma 4.4, Theorem 4.6]). Assume the setup of (2.3).

a) There is a decomposition $H^1(O_{C,S}, \mathbb{Z}/n) \simeq (\mathbb{Z}/n)^{BC} \oplus \Gamma$, where $BC$ is the Betti number of the dual graph of $C$, and $\Gamma \leq H^1(\kappa(C), \mathbb{Z}/n)$ is the set of tuples $\theta_C = \ldots
(θ₁, . . . , θₘ) ∈ H¹(κ(C), Z/n) such that each θᵢ is unramified at each z ∈ S ∩ Cᵢ, and θₖ(z) = θ₁(z) = θ₂(z) ∈ H¹(κ(z), Z/n) whenever z ∈ Cᵢ ∩ Cₖ.

b) For q ≥ 0 and any integer r there is a map

\[ \lambda : H^q(\mathcal{O}_C, S) \to H^q(F, r) \]

and a commutative diagram

\[
\begin{array}{c}
H^q(\mathcal{O}_C, S) \\
\downarrow \text{res}_i \\
H^q(\kappa(C), r)
\end{array}
\begin{array}{c}
\text{inf} \\
\downarrow \\
\oplus_i H^q(F_C, r)
\end{array}
\]

such that if α_C ∈ H^q(\mathcal{O}_C, S, r) and α = \lambda(α_C) then:

i) α is defined at the generic points of C_i, and α(C_i) = \text{res}_i(α_C).

ii) The ramification locus of α (on X) is contained in \mathcal{D}_S.

iii) If D ∈ \mathcal{D}_S is prime and z = D \cap C, then \partial_D \cdot \lambda = \text{inf}_{κ(z)|κ(D)}(\partial_z(\theta_C)) \cdot \partial_z.

iv) If α_C is unramified at a closed point z, and D is any (horizontal) prime lying over z, then α is unramified at D, and has value α(D) = \text{inf}_{κ(z)|κ(D)}(α_C(z)).

3. Computations

We first construct the cyclic class γ ∈ H²(F, μₙ) using a lift from H¹(κ(C), Z/n).

**Lemma 3.1.** Assume the setup of Theorem 2.4. Suppose θ_C ∈ Γ ≤ H¹(\mathcal{O}_C, S, Z/n) lifts (θ₁, . . . , θₘ) ∈ H¹(κ(C), Z/n) as in Theorem 2.4(a), such that θ_C is unramified at all z ∈ E \cap C, with value of \text{res}_z(θ_C) = 0 (in addition to being unramified at S). Let γ = (π) · \lambda(θ_C) ∈ H²(F, μₙ). Then for any prime divisor D on X

\[ \partial_D(\gamma) = \begin{cases} \theta_i & \text{if } D = C_i \\ -\text{res}_{κ(z)|κ(D)}(\partial_z(θ_C)) \cdot (π) & \text{if } D ∈ \mathcal{D}_S \text{ and } z = D \cap C \\ 0 & \text{otherwise} \end{cases} \]

The ramification divisor D_γ has normal crossings, and consists of each C_i at which θ_i is nonzero, together with all D ∈ \mathcal{D}_S lifting z : \partial_z(θ_C) ≠ 0.

**Proof.** Set θ = \lambda(θ_C). For any prime divisor D on X we compute

\[ \partial_D(\gamma) = [v_D(π)θ - \partial_D(θ) \cdot (π) + v_D(π)\partial_D(θ) \cdot (-1)]_{F_D} \]

This element is in the subgroup H¹(κ(D), Z/n) ≤ H¹(F_D, Z/n). Let z = D \cap C. There are several cases to consider.

If D = C_i is an irreducible component of C then since E contains no components of C (by Theorem 2.4) we have v_D(π) = 1, and since θ_C ∈ H¹(\mathcal{O}_C, S, Z/n) we have \partial_D(θ) = 0 by Theorem 2.4(b)(ii). Therefore \partial_D(γ) = \text{res}_{F_D}(θ) = θ_i ∈ H¹(κ(C_i), Z/n).

If D is horizontal and runs through a point of S, then v_D(π) = 0 since E avoids S, and \partial_D(θ) = 0 by Theorem 2.4(b)(ii), hence \partial_D(γ) = 0.

If D is horizontal, avoids S, and v_D(π) ≠ 0, then D is a component of E, so by assumption, \partial_z(θ_C) = 0 for z ∈ E \cap C and θ_C(z) = 0. Thus \partial_D(θ) = 0 and \partial_D(γ) = 0 by Theorem 2.4(b)(iv). Therefore \partial_D(γ) = 0.
If $D$ is horizontal, avoids $S$, and $v_D(\pi) = 0$, then $\partial_D(\gamma) = -\partial_D(\theta)(\pi)$. If $D \notin \mathcal{R}_S$ this is zero by Theorem $2.4$ b)(ii). If $D \in \mathcal{R}_S$ then $\partial_D(\gamma) = -\inf_{\kappa(z)|\kappa(D)}(\partial_z(\theta_C)) \cdot (\pi)$ by Theorem $2.4$ b)(iii), and since $\pi$ is a local equation for $C$ at $z$ and $D$ intersects $C$ transversally at $z$, the image of $\pi$ in the local field $\kappa(D)$ is a uniformizer, hence $(\pi)$ has order $n$ in $H^1(\kappa(D), \mu_n)$. Thus if $D$ is horizontal then $\partial_D(\gamma)$ is nonzero if and only if $D \in \mathcal{R}_S$ and $\partial_z(\theta_C) \neq 0$, and then $\partial_D(\gamma) = -\inf_{\kappa(z)|\kappa(D)}(\partial_z(\theta_C)) \cdot (\pi)$.

We conclude the ramification divisor $D_\gamma$ of $\gamma$ consists of the components $C_i$ of $C$ for which $\theta_i$ is nonzero, together with the distinguished prime divisors $D \in \mathcal{R}_S$ lying over points $z$ at which $\theta_C$ is ramified. Since all such $D$ are regular and intersect $C$ transversally, $D_\gamma$ has normal crossings.

Next we show an $F$-division algebra of prime period $\ell \neq p$ and index $\ell^2$ is decomposable by constructing a cyclic factor using Lemma $3.1$ designed to cancel the division algebra's hot points.

**Theorem 3.2.** Let $F$ be a field finitely generated of transcendence degree one over $\mathbb{Q}_p$, and suppose $\Delta$ is an $F$-division algebra of prime period $\ell \neq p$ and index $\ell^2$. Then $\Delta$ is decomposable.

*Proof.* We may assume $\ell$ is odd, since if $\ell = 2$ the result is a classical theorem of Albert. Assume the setup $(2.3)$, let $\alpha = [\Delta] \in H^2(F, \mu_\ell)$, and let $D_\alpha$ be the ramification divisor of $\alpha$ on $X$. We may assume $D_\alpha \cup C$ has normal crossings and horizontal components contained in $\mathcal{R}_S$, and that we have an element $\pi \in F$ as in $(2.3)$ with $\text{div}(\pi) = C + E$, where $E$ is horizontal and avoids the nodal points of $D_\alpha \cup C$.

By Grunwald-Wang's theorem applied over the global fields $\kappa(C_i)$ there exist elements $\theta_i \in H^1(\kappa(C_i), \mathbb{Z}/\ell)$, $i = 1, \ldots, m$, with the following properties:

a) $\partial_\gamma(<\theta_i>) = 0$ when $z \in C_i$ is a singular point of $C \cup D_\alpha \cup E$.

b) $\theta_i(z) = \theta_j(z)$ whenever $z \in C_i \cap C_j$.

c) $\theta_i(z) = 0$ at all $z \in E \cap C_i$.

d) If $z \in D_\alpha \cap S$ then
   
   (i) $\langle \partial_i(\alpha)(z) - \theta_i(z) \rangle = \langle -\theta_i(z) \rangle$ if $z \in C_i$ is a curve point of $D_\alpha$;
   
   (ii) $\theta_i(z) = 0$ if $z \in C_i$ is a not-hot nodal point of $D_\alpha$;
   
   (iii) $\langle \partial_i(\alpha)(z) - \theta_i(z) \rangle = \langle \partial_j(\alpha)(z) - \theta_i(z) \rangle$ if $z \in C_i \cap C_j$ is a hot point of $\alpha$.

e) If $z \in D_\alpha \setminus S$ then
   
   (i) $\langle -\theta_i(z) \rangle = \langle \partial_D(\alpha)(z) \rangle$ if $z \in C_i$ is a curve point of $D_\alpha$;
   
   (ii) $\theta_i(z) = 0$ if $z \in C_i$ is a not-hot nodal point of $D_\alpha$;
   
   (iii) $\langle \partial_i(\alpha)(z) - \theta_i(z) \rangle = \langle \partial_D(\alpha)(z) \rangle$ if $z \in C_i \cap D$ is a hot point of $\alpha$.

Note that (d)(i,iii) and (e)(i,iii) makes sense since the given residues of $\alpha$ are unramified at the given $z$; (d)(i,iii) and (e)(iii) are possible since $\ell$ is odd; (c) does not conflict with (d)(i,iii) and (e)(i,iii) since $E$ avoids the nodal points of $D_\alpha \cup C$; and (b) does not conflict with (d)(ii,iii) and (e)(ii,iii) by symmetry.

The $\theta_i$ are unramified with equal values at all nodal points $z \in S$ by (a,b), so they glue together to produce an element $\theta_C \in F \leq H^1(Q_{C,S}, \mathbb{Z}/\ell)$ by Theorem $2.4$ a). Note that $\theta_C$ is nonzero by (d)(iii) (or (e)(iii)) since $\alpha$ has at least one hot point by the hot point criterion, and then $\theta_C(z) = \theta_i(z)$ is necessarily nonzero.
Let $\gamma_1 = (\pi) \cdot \lambda(\theta_C)$. Then $\gamma_1$ and $E$ satisfy the hypotheses of Lemma 5.1 by (c) and the assumptions on $E$, hence $D_{\gamma_1} \cup C$ has normal crossings and distinguished horizontal components, and since $\theta_C$ is ramified at all nodal points of $D_{\gamma_1}$, $\gamma_1$ has no hot points, hence it has index $\ell$ by [12 Corollary 5.2]. Write

\[
D_\alpha = C' + H \\
D_{\gamma_1} = C'' + H'
\]

where $C', C'' \subset C$, and $H, H' \subset \mathcal{D}_S$ are distinguished horizontal divisors. Set

\[\gamma_2 = \alpha - \gamma_1\]

We intend to show that $\gamma_2$ has index $\ell$. Since $\theta_C$ is unramified at all singular points of $D_\alpha \cup C$ by (a), $H'$ avoids all of these points by Lemma 5.1, hence $H \cap H' = \emptyset$. Evidently then $D_{\gamma_2} \subset C + H + H'$. Now $D_{\gamma_2}$ has normal crossings on $X$ since $D_\alpha \cup C$ and $D_{\gamma_1} \cup C$ both have normal crossings. Since $D_{\gamma_2}$ has normal crossings, $\gamma_2$ has index $\ell$ if and only if $\gamma_2$ has no hot points on $X$. For the following analysis, note the nodal points $S_{\gamma_2}$ of $D_{\gamma_2}$ are in $H'$, $S$, and $H$, and in the latter two cases $\theta_C$ is unramified at $z$, hence it has a value $\theta_C(z)$.

Suppose $z \in S_{\gamma_2}$ and $z \notin D_\alpha$. Then $z$'s status as a point of $D_{\gamma_2}$ (hot, not hot) is the same as its status as a point of $D_{\gamma_1}$, hence it is not hot by Lemma 3.1 (it is a “cold” point in the terminology of [12]).

Suppose $z \in S_{\gamma_2} \cap D_\alpha \cap H'$. Then $z \in C_i \cap D$ for some $C_i$ and some prime divisor $D \subset H'$, and $\theta_C$ is ramified at $z$ by Lemma 3.1. We have $\partial_D(\alpha) = 0$ since $D \notin D_\alpha$, and by Lemma 3.1

\[
\partial_\pi(\partial_D(\gamma_2)) = \partial_\pi(-\partial_D(\gamma_1)) = \partial_\pi(\partial_\pi(\theta_C) \cdot (\pi)) = v_\pi(\pi)\partial_\pi(\theta_C)
\]

where $\pi$ is the image of $\pi$ in $\kappa(D)$. Since $\text{div}(\pi)$ has normal crossings at $z$, $v_\pi(\pi) = 1$, hence $\partial_\pi(\partial_D(\gamma_2)) = \partial_\pi(\theta_C) \neq 0$. Since $\partial_\pi(\partial_D(\gamma_2)) \neq 0$, $z$ is not a hot point of $\gamma_2$.

Suppose $z \in S_{\gamma_2} \cap D_\alpha \cap S$. If $z$ is a curve point of $D_\alpha$ on $C_i$, i.e., $\partial_{C_i}(\alpha) = 0$ where $C_i$ is the other component of $C$ at $z$, then $\partial_{C_i}(\gamma_2)(z) = \partial_{C_i}(\alpha)(z) - \theta_C(z)$ and $\partial_{C_i}(\gamma_2)(z) = -\theta_C(z)$ by Lemma 3.1 and so $z$ is not a hot point of $\gamma_2$ by (d)(i). If $z$ is a nodal point of $D_\alpha$ on $C_i \cap C_j$, then $\theta_C(z) = 0$ if $z$ is not a hot point of $\alpha$ by (d)(ii), so the status of $z$ for $\gamma_2$ is the same as for $\alpha$ (not hot); otherwise $\langle \partial_{C_i}(\gamma_2)(z) \rangle = \langle \partial_{C_j}(\gamma_2)(z) \rangle$ by (d)(iii), hence $z$ is not a hot point for $\gamma_2$ in any case.

Suppose $z \in S_{\gamma_2} \cap C_i \cap H$. Assume $z \in C_i \cap D$ for a prime divisor $D \subset H$. Then $\partial_D(\gamma_1) = 0$ since $H \cap H' = \emptyset$. If $z$ is a curve point of $D_\alpha$, i.e., $\partial_{C_i}(\alpha) = 0$, then $\partial_{C_i}(\gamma_2)(z) = -\theta_C(z)$ and $\partial_D(\gamma_2) = \partial_D(\alpha)$, hence $\langle \partial_{C_i}(\gamma_2)(z) \rangle = \langle \partial_D(\gamma_2)(z) \rangle$ by (e)(i), so $z$ is not a hot point for $\gamma_2$. If $z$ is a not-hot nodal point of $D_\alpha$ then $\theta_C(z) = 0$ by (e)(ii), so the status of $z$ is unchanged (not hot) for $\gamma_2$. If $z$ is a hot point of $\alpha$ then $\langle \partial_{C_i}(\alpha)(z) - \theta_C(z) \rangle = \langle \partial_D(\alpha)(z) \rangle$ by (e)(iii), hence $z$ is not a hot point for $\gamma_2$. This completes the analysis. We conclude $\gamma_2$ has no hot points on $X$, hence $\gamma_1$ and $\gamma_2$ both have index $\ell$.

Let $\Delta_1$ and $\Delta_2$ be the $F$-division algebras underlying $\gamma_1$ and $\gamma_2$, respectively, so that $[\Delta] = [\Delta_1 \otimes_F \Delta_2]$. Since $\text{ind}(\Delta_1 \otimes_F \Delta_2) = \text{ind}(\Delta) = \ell^2$ and $\text{ind}(\Delta_i) = \ell$, it follows that $\Delta_1 \otimes_F \Delta_2$ is a division algebra, hence we have a decomposition $\Delta \simeq \Delta_1 \otimes_F \Delta_2$.  \(\square\)
As mentioned in [21], it is known that \( \ell \text{Br.dim}(F) = 2 \), but not known in general whether the \( \mathbb{Z}/\ell \)-length in \( H^2(F, \mu_\ell) \) is finite. We now have the following.

**Corollary 3.3.** Let \( F \) be a field finitely generated and of transcendence degree one over \( \mathbb{Q}_p \). Then every element of \( H^2(F, \mu_\ell) \) is a sum of two \( \mathbb{Z}/\ell \)-cyclic classes.

**Proof.** If \( \alpha \in H^2(F, \mu_\ell) \) then the index of \( \alpha \) is either \( \ell \) or \( \ell^2 \) by [10, Theorem 3.4]. If it is \( \ell \), then \( \alpha \) is already \( \mathbb{Z}/\ell \)-cyclic by [12, Theorem 5.1]. If it is \( \ell^2 \) then \( \alpha = \gamma_1 + \gamma_2 \) for classes \( \gamma_i \) in \( H^2(F, \mu_\ell) \) of index \( \ell \) by Theorem 3.2. These classes are again \( \mathbb{Z}/\ell \)-cyclic by Saltman’s theorem, and the result follows.

Saltman proved that all \( F \)-division algebras of prime degree \( \ell \) are cyclic crossed products in [12], and Suresh proved the prime period case when \( F \) contains the \( \ell \)-th roots of unity in [13]. We now have the prime period case in general:

**Corollary 3.4.** Let \( F \) be a field finitely generated and of transcendence degree one over \( \mathbb{Q}_p \), and let \( \Delta \) be a division algebra of prime period \( \ell \neq p \). Then \( \Delta \) is a crossed product.

**Proof.** The index of \( \Delta \) is either \( \ell \) or \( \ell^2 \) by [10, Theorem 3.4]. If it is \( \ell \), then \( \Delta \) is a cyclic crossed product by [12, Theorem 5.1]. If it is \( \ell^2 \) then \( \Delta = \Delta_1 \otimes_F \Delta_2 \) by Theorem 3.2 and each \( \Delta_i \) is cyclic by Saltman’s theorem. Let \( L_i/F \) be a cyclic Galois maximal subfield of \( \Delta_i \). Then \( L = L_1 \otimes_F L_2 \) is a commutative Galois subalgebra of \( \Delta \) of degree \( \ell^2 \), hence it is a Galois field extension of \( F \), since \( \Delta \) is a division algebra. Since \( L \) obviously splits \( \Delta \), it is a Galois maximal subfield of \( \Delta \), hence \( \Delta \) is a crossed product.

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