Electrical Conductivity of Fermi Liquids. II.
Quasiparticle Transport

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We develop a general theory of Fermi liquids to discuss the Kadowaki-Woods relation $A \propto \gamma^2$ between the coefficient $A$ of the $T^2$ term in the resistivity $\sigma^{-1} = AT^2$ and $\gamma$ of the specific heat $C = \gamma T$ at low temperatures. We derive a formula for the ratio $A/\gamma^2$ which is expressed as a product of two dimensionless parameters $\alpha$ and $F$, where $\alpha$ represents a coupling constant for quasiparticle scattering and $F$ is a geometric factor determined by the shape of the Fermi surface. Then we argue that the universal ratio $A/\gamma^2 = 1.0 \times 10^{-5} \mu \Omega \text{cm}(\text{mol K/mJ})^2$ observed in heavy fermion compounds is reproduced under the conditions $\alpha \sim 1$ and $F \sim 20$. The former is regarded as a universality of Fermi liquids in a strong coupling regime, and the latter is corroborated by evaluating $F$ definitely in simple cases. It is noted that the proportional relation is just an example of the universal phenomena to be expected for the whole class of strong coupling Fermi liquids.

KEYWORDS: Fermi Liquid Theory, electrical conductivity, Boltzmann equation, many-body effect, quasiparticle transport

§1. Introduction

In the previous paper, we discussed the many-body effect on the Drude weight on the basis of the Fermi liquid theory, and showed that the Drude weight is directly related not to the velocity $v^*_k$ but to the current $j^*_k$ of quasiparticle. It was stressed that the effective mass defined for $j^*_k$ is generally different from the thermal mass estimated from the $T$ linear coefficient of the specific heat. Although we may work with the zero temperature formalism to discuss the Drude weight and the other phenomena in a collisionless regime, the effect of quasiparticle damping should be taken into account in a hydrodynamic regime, for example, to investigate dc conductivity. In this article, we develop a general theory to discuss both regimes in an unified manner and derive a formula for the Kadowaki-Woods ratio $A/\gamma^2$ between the coefficient $A$ of the $T^2$ term in the resistivity $\sigma^{-1} = AT^2$.
and $\gamma$ of the specific heat $C = \gamma T$. Then we discuss the conditions for the Kadowaki-Woods relation $A \propto \gamma^2$ and estimate an universal value taken by the ratio $A/\gamma^2$.

A framework for this purpose was first provided by Éliashberg. His formalism is sophisticated and firmly established so that we take this as a starting point to discuss electrical properties of Fermi liquids. Therefore a large part of this paper is devoted to recapitulation of his formalism. This is necessary not only to introduce notations, but because Éliashberg mainly concentrated upon a microscopic theory of zero sound, while we are interested in a general theory of quasiparticle transport, especially the electrical conductivity. Thus in the first half of this paper we derive a linearized transport equation in a general but concise form, so as to serve our purpose of the latter sections. All the results will be expressed in terms of renormalized quantities characterizing quasiparticles, and then we discuss transport properties of Fermi liquids in general terms.

For heavy fermion compounds at low temperatures, it has been known that the ratio $A/\gamma^2$ takes a universal value $\approx 1.0 \times 10^{-5} \mu\Omega\text{cm}(\text{mol K/mJ})^2$. In view of this experimental fact, we must theoretically explain not only (i) the reason why the proportional relation $A \propto \gamma^2$ holds for a certain class of systems, but also (ii) the value of the universal ratio itself. These points have been discussed theoretically by many authors. However, the theories are more or less based on some specific models or approximations. Among them, the theories presented in refs. are not enough for us to study the point (ii) microscopically, as their results are indefinite up to a factor of order unity related to effectiveness of Umklapp processes responsible for a finite conductivity. Therefore we shall follow the microscopic theory of Yamada and Yosida developed on the basis of the periodic Anderson model, in which the effect of Umklapp processes is explicitly specified. Unlike ref., however, we try to keep the theory as general as possible. This is because we believe that such a universal relation as observed by Kadowaki and Woods should be explained within the context of general theory of Fermi liquids without making any specific assumptions. Moreover we take a method different from ref. to derive a formula for $A/\gamma^2$, thereby the fully renormalized three-point vertex function $\tilde{A}_p$ introduced by Yamada and Yosida is shown to be related to the deviation from local equilibrium quasiparticle distribution $\varphi_p \propto \delta\bar{n}_p^0$, (eqs. (3.15) and (4.2)). Once we obtain a definite general expression for $A/\gamma^2$, the observed universality suggests that it would give a reasonable estimate for the ratio even if it is applied for a simple model. As a matter of fact, we deal with some simple models to evaluate the ratio. At the outset, however, no assumption is made except for those assuring the applicability of the Fermi liquid theory, which are briefly sketched below.

We discuss a single-band electron system with the band dispersion $\varepsilon_p$ for the momentum $p$ and the velocity $v_p = \partial\varepsilon_p/\partial p$. We do not assume an explicit form of electron-electron interaction, except that it must conform to the basic hypothesis of the Fermi liquid theory: We assume a well-defined quasiparticle $p$ of the energy dispersion $\varepsilon_p^*$ which is formally obtained from non-interacting state
$\varepsilon_p$ by perturbation theory. The interaction between quasiparticles is described by the scattering amplitude $A_{pp'k} = A(p, p'; p + k, p - k)$ which also, in principle, is to be derived from a microscopic model. At absolute zero, $A(p_1, p_2; p_3, p_4)$ is assumed to be a real and regular function of $p_i \ (i = 1 \sim 4)$ around the Fermi surface, $\varepsilon^*_{p_i} = \mu$. As we discuss a linear response of the system to a macroscopic plane-wave perturbation with a sufficiently small wave number $k$ and frequency $\omega$, the energy dependence of $A_{pp'k}$ gives rise only to correction terms of higher order than we need. Thus the energy variables of $A_{pp'k}$ are fixed to take the Fermi energy $\mu$. The procedure we follow is a perturbation theory with respect to $k$ and $\omega$, i.e., at $T = 0$ only the terms linear in $k$ and $\omega$ will be retained; still the effect of electron-electron interaction is taken into account to an arbitrary order. At finite temperature, the quasiparticle scattering $A_{pp'k}$ induces a finite decay rate $\gamma^*_{p}(T)$ for the quasiparticle $p$. The imaginary part of the scattering amplitude is found to be the same order as $\gamma^*_{p}(T) \sim T^2$. Under the condition $|\varepsilon^*_p - \mu| \gg \gamma^*_p$, the quasiparticle peak in the one-particle Green’s function $G(p, \varepsilon)$ is approximated by the delta function $\delta(\varepsilon - \varepsilon^*_p + \mu)$ and the quasiparticle $p$ is well defined. Therefore, we have to be in a low temperature region $T \lesssim T_0$, where $T_0$ is roughly estimated by $|\varepsilon^*_p - \mu| \sim T \approx \gamma^*_p(T_0)$. We neglect the terms of order $O(\gamma^2_p)$. To sum up, we regard $k$, $\omega$ and $\gamma^*_p$ as the small parameters with which to develop a perturbation expansion of the electrical conductivity $\sigma(k, \omega)$, where the singular terms due to the quasiparticle poles are retained systematically.

In the next section, we follow Éliashberg to transform the Kubo formula of the electrical conductivity $\sigma_{\mu\nu}(k, \omega)$, expressed by temperature Green’s functions and vertex functions, into a form suitable for our purpose. In §3, we investigate an explicit form of the vertex functions to derive a set of formulas to calculate $\sigma_{\mu\nu}(k, \omega)$. One of our results is microscopic derivation of a linearized Boltzmann equation. In §3.1, the collisionless regime $\omega \gg \gamma^*_p$ is investigated, where, after elucidating the quantities defined in §2 in physical terms, we derive the formula for the optical conductivity obtained in the previous paper. In §3.2, the hydrodynamic regime $\omega \ll \gamma^*_p$ is investigated, where we derive the collision integral of the Boltzmann equation using results of Yamada and Yosida and estimate the low frequency limit of the incoherent part of the conductivity due to quasiparticle scattering. Using the result of §3.2, we derive a formula for the coefficient $A = \sigma(0)^{-1}/T^2$. Then in §4 we discuss the universality of the Kadowaki-Woods relation $A \propto \gamma^2$. With the results obtained for Fermi surfaces of simple examples, we estimate the ratio $A/\gamma^2$ and find that the observed ratio is reasonably reproduced for a strong coupling Fermi liquid. Discussion and conclusion are presented in §§5 and 6 respectively.
§2. Éliashberg Formalism

According to the linear response theory, the electrical conductivity $\sigma_{\mu\nu}(k, \omega)$ is given by

$$
\sigma_{\mu\nu}(k, \omega) = \frac{e^2}{\Omega} \sum_{p, p'} v_{\mu p} K_{pp'}^{R}(k, \omega) - K_{pp'}^{R}(k, 0) v_{\nu p'},
$$

(2.1)

where $\Omega$ is the volume of the system, and the function $K_{pp'}^{R}(k, \omega)$ is defined by

$$
K_{pp'}^{R}(k, i \omega_m) = K_{pp'}(k, \omega_m) \quad \text{for} \quad \omega_m > 0,
$$

(2.2)

$$
K_{pp'}(k, \omega_m) = \frac{1}{2} \int_{-1/T}^{1/T} e^{\omega_m \tau} K_{pp'}(k, \tau) d \tau,
$$

(2.3)

$$
K_{pp'}(k, \tau) = (T' e^{(H-\mu N)\tau} \hat{c}_{p'-k/2} \hat{c}_{p+k/2} e^{-(H-\mu N)\tau} \hat{c}_{p+k/2} \hat{c}_{p-k/2}).
$$

(2.4)

After the analytic continuation with respect to $\omega$, one finds:

$$
K^R(k) = -\frac{1}{4\pi i} \int_{-\infty}^{\infty} d\varepsilon \left[ \tanh \frac{\varepsilon - \omega/2}{2T} K_1(p, k) + \left( \tanh \frac{\varepsilon + \omega/2}{2T} - \tanh \frac{\varepsilon - \omega/2}{2T} \right) K_2(p, k) \right]
$$

(2.5)

$$
- \tanh \frac{\varepsilon + \omega/2}{2T} K_3(p, k),
$$

where

$$
K_i(p, k) = g_i(p, k) \left[ 1 + \frac{1}{4\pi i} \int_{-\infty}^{\infty} d\varepsilon' \sum_{k=1}^{3} T_{ik}(p, p'; k) g_k(p', k) \right],
$$

(2.6)

and

$$
g_1(p, k) = G^R(p + k/2) G^R(p - k/2),
$$

$$
g_2(p, k) = G^R(p + k/2) G^A(p - k/2),
$$

$$
g_3(p, k) = G^A(p + k/2) G^A(p - k/2).
$$

(2.7)

Hereafter we use the abbreviated notation such as

$$
p = (p, \varepsilon), \quad p' = (p', \varepsilon'), \quad k = (k, \omega).
$$

The retarded Green’s function $G^R(p)$ is given by

$$
G^R(p) = \frac{1}{\varepsilon - \varepsilon_p - \mu - \Sigma^R(p, \varepsilon)},
$$

(2.8)

where $\Sigma^R(p, \varepsilon)$ is the retarded selfenergy.

In eq. (2.3), $T_{ij}(p, p'; k)$ for $i, j = 1, 2, 3$ represent properly antisymmetrized vertex functions. Among them, $T_{22}(p, p'; k)$, shown in Fig. 1 together with $g_2(p, k)$ of eq. (2.7), plays an important role to determine quasiparticle transport properties. The function $T_{22}(p, p'; k)$ is expressed in terms of $\Gamma(p, p'; k)$, the four-point vertex function analytic in the region defined by

$$
\text{Im}(\varepsilon - \omega/2) < 0, \quad \text{Im}(\varepsilon + \omega/2) > 0,
$$

$$
\text{Im}(\varepsilon' - \omega/2) < 0, \quad \text{Im}(\varepsilon' + \omega/2) > 0.
$$

(2.9)
In fact, this region must be further divided into several portions in accordance with the analytic property of $\Gamma(p, p'; k)$. As a result, we obtain

$$T_{22}(p, p'; k) = \left( \tanh \frac{\epsilon' + \omega/2}{2T} - \tanh \frac{\epsilon' - \omega/2}{2T} \right) \Re \Gamma(p, p'; k)$$

$$+ \coth \frac{\epsilon' - \epsilon}{2T} \left( \Gamma^{II}(p, p'; k) - \Gamma^{III}(p, p'; k) \right) - \tanh \frac{\epsilon' - \omega/2}{2T} \Im \Gamma^{II}(p, p'; k)$$

$$+ \coth \frac{\epsilon' + \epsilon}{2T} \left( \Gamma^{III}(p, p'; k) - \Gamma^{IV}(p, p'; k) \right) - \tanh \frac{\epsilon' + \omega/2}{2T} \Im \Gamma^{IV}(p, p'; k),$$

where $\Gamma^{II}(p, p'; k)$, $\Gamma^{III}(p, p'; k)$ and $\Gamma^{IV}(p, p'; k)$ are analytic respectively in

$$\begin{align*}
\text{II} & : \Im(\epsilon - \epsilon') < 0, \quad \Im(\epsilon + \epsilon') > 0, \\
\text{III} & : \Im(\epsilon - \epsilon') > 0, \quad \Im(\epsilon + \epsilon') > 0, \\
\text{IV} & : \Im(\epsilon - \epsilon') > 0, \quad \Im(\epsilon + \epsilon') < 0.
\end{align*}$$

The imaginary part of $\Gamma(p, p'; k)$ has discontinuity at the boundary of the regions defined in eq. (2.11), so that the imaginary parts of $\Gamma^{II}(p, p'; k)$, $\Gamma^{III}(p, p'; k)$ and $\Gamma^{IV}(p, p'; k)$ are generally different, while the real part of them takes a common value $\Re \Gamma(p, p'; k)$. Accordingly, we wrote eq. (2.10) in such a way that the first line in the right-hand side is real but the second and third lines are pure imaginary. As for the functions $T_{ij}$ other than $T_{22}$, it is enough for us to note

$$T_{12}, T_{32}, T_{21}, T_{23} \propto \tanh \frac{\epsilon' + \omega/2}{2T} - \tanh \frac{\epsilon' - \omega/2}{2T} = O(\omega). \quad (\omega \to 0)$$

The energy $\epsilon_p^*$ and the decay rate $\gamma_p^*$ of the quasiparticle $p$ are defined by the real and imaginary part of

$$G^R(p, \epsilon_p^* - \mu - i\gamma_p^*)^{-1} = 0,$$
namely, by
\[ \varepsilon_p^* - \mu = z_p (\varepsilon_p + \text{Re}\Sigma(p,0) - \mu), \quad (2.14) \]
and
\[ \gamma_p^* = -z_p \text{Im}\Sigma^R(p,0), \quad (2.15) \]
where the renormalization factor \( z_p \) is given by
\[ z_p^{-1} = 1 - \frac{\partial}{\partial \varepsilon} \text{Re}\Sigma^R(p,\varepsilon) \bigg|_{\varepsilon=0}. \quad (2.16) \]
For \( \varepsilon \sim |\varepsilon_p^* - \mu| \gg \gamma_p^* \), we may write
\[ \text{Im} G^R(p) = -\pi z_p \delta(\varepsilon - \varepsilon_p^* + \mu), \quad (2.17) \]
and
\[ g_2(p,k) = \frac{2\pi i \varepsilon_p^2 \delta(\varepsilon - \varepsilon_p^* + \mu)}{\omega - v_p^* \cdot k + 2i\gamma_p^*}, \quad (2.18) \]
where
\[ v_p^* = \frac{\partial \varepsilon_p^*}{\partial p_\mu}. \quad (2.19) \]
These results are legitimately used in the following discussion, as we shall find that the energy region relevant to us is \( \varepsilon \sim |\varepsilon_p^* - \mu| \sim T \gg \gamma_p^* \). Equation (2.18) is valid as far as the linear terms with respect to \( \omega, k \) and \( \gamma_p^* \) are concerned. In effect, an essential singularity in the linear response to a macroscopic perturbation is brought about by the quasiparticle-quasihole pair propagator \( g_2(p,p';k) \). The function \( g_1(p,p';k) \) and \( g_3(p,p';k) \) are less singular than \( g_2(p,p';k) \), and these functions, combined with \( T_{ij} (i,j = 1 \text{ or } 3) \), give rise only to the vertex correction, \( v_{p\mu} \rightarrow \Lambda_{p\mu} \) in eq. (2.1). As a result, the conductivity of Fermi liquids is given by
\[ \sigma_{\mu\nu}(k,\omega) = \frac{e^2}{2\Omega} \int \frac{d^4p}{(2\pi)^4} \frac{1}{2T} \cosh^{-2} \frac{\varepsilon_p}{2T} \Lambda_{p\mu} g_2(p,k) \left( \Lambda_{\mu\nu} - \frac{i}{2} \int \frac{d^4p'}{(2\pi)^4} T_{22}(p,p';k) g_2(p',k) \Lambda_{p'\nu} \right). \quad (2.20) \]
In fact, by the close inspection of Eliashberg’s derivation, we can verify that the relation,
\[ z_p \Lambda_{p\mu} = v_{p\mu}^*, \quad (2.21) \]
holds for the vertex function \( \Lambda_{p\mu} \) under the assumption of the Fermi liquid theory to neglect the imaginary part of \( \Gamma(p,p';k) \). Noting
\[ \frac{1}{4T} \cosh^{-2} \frac{\varepsilon_p^* - \mu}{2T} = -\frac{\partial n_{p0}^0}{\partial \varepsilon_p^*}, \quad (2.22) \]
\[ n_{p0}^0 = n_0(\varepsilon_p^*) = \frac{1}{e^{(\varepsilon_p^* - \mu)/T} + 1}, \quad (2.23) \]
and using eqs. (2.20) and (2.21), we write the total current \( j_\mu(k,\omega) \) per volume as
\[ j_\mu(k,\omega) = \sum_\nu \sigma_{\mu\nu}(k,\omega) E_\nu = \frac{e}{\Omega} \sum_p \delta \bar{n}_p v_{p\mu}^*, \quad (2.24) \]
where

$$\delta n_p \equiv \frac{\partial n_p^0}{\partial \varepsilon_p^*} \phi_p,$$

and

$$\phi_p = -e \int_{-\infty}^{\infty} \frac{d\varepsilon}{2\pi} z_p^{-1} g_2(p, k) \left( \mathbf{A}_p \cdot \mathbf{E} - \frac{i}{2} \int \frac{d^4p'}{(2\pi)^4} T_{22}(p, p'; k) g_2(p', k) \mathbf{A}_{p'} \cdot \mathbf{E} \right).$$

The function $T_{22}(p, p'; k)$, by definition, is the vertex part which is connected to $g_2(p, k)$ on the left end and to $g_2(p', k)$ on the right end (Fig. 1). Thus, in general, $T_{22}(p, p'; k)$ may comprise the sector $g_i (i \neq 2)$ and $T_{ij} (i, j \neq 2)$ in between the $g_2$ functions on both ends. However, because of eq. (2.12), we may neglect the portions $T_{ij} (i, j \neq 2)$ which may appear in $T_{22}$ to the accuracy of order $O(\omega)$. Hence, for our purpose, it is convenient to introduce the vertex $T^{(0)}_{22}(p, p'; k)$ which is irreducible with respect to the section $g_2$:

$$T_{22}(p, p'; k) = T^{(0)}_{22}(p, p'; k) - \frac{i}{2} \int \frac{d^4p''}{(2\pi)^4} T^{(0)}_{22}(p, p''; k) g_2(p'', k) T_{22}(p'', p'; k),$$

Then, in terms of $T^{(0)}_{22}(p, p'; k)$, eq. (2.26) is cast into

$$\phi_p = -e \int_{-\infty}^{\infty} \frac{d\varepsilon}{2\pi} z_p^{-1} g_2(p, k) \mathbf{A}_p \cdot \mathbf{E} - \frac{i}{2} \int_{-\infty}^{\infty} \frac{d\varepsilon}{2\pi} \int \frac{d^4p'}{(2\pi)^4} z_p^{-1} g_2(p, k) T^{(0)}_{22}(p, p'; k) z_p \phi_{p'},$$

where $\phi_p$ is defined by

$$\varphi_p = \frac{\int_{-\infty}^{\infty} d\varepsilon}{2\pi} \phi_p.$$  

As the function $\varphi_p$ is always multiplied by $\partial n_p^0/\partial \varepsilon_p^*$, we may use eq. (2.18) in eq. (2.28). Noting that

$$\phi_p \propto g_2(p, k) \propto \delta(\varepsilon - \varepsilon_p^* + \mu),$$

we may put

$$\phi_p = 2\pi \delta(\varepsilon - \varepsilon_p^* + \mu) \varphi_p.$$ 

Therefore, from eq. (2.28) with eq. (2.18), we obtain

$$i(\omega - \mathbf{v}_p^* \cdot \mathbf{k}) \varphi_p = e \mathbf{v}_p^* \cdot \mathbf{E} + \frac{i}{2} \int \frac{d^4p'}{(2\pi)^3} \delta(\varepsilon' - \varepsilon_p^* + \mu) z_p z_p' T^{(0)}(p, p'; k) \bigg|_{\varepsilon = 0} \varphi_{p'} + 2\gamma_p^* \varphi_p,$$

for the momentum $p$ on the Fermi surface, $\varepsilon_p^* = \mu$. As the vertex functions which appear in the following section are all of the type $T^{(0)}_{22}(p, p'; k)$, this is simply written as $T^{(0)}(p, p'; k)$ without the subscripts.

§3. Four-Point Vertex Function

3.1 Real part of $T^{(0)}$: Collisionless Regime, $\omega \gg \gamma_p^*$

The four-point vertex function $T^{(0)}$ in eq. (2.30) is divided into a real and imaginary part. Since eq. (2.10) for $T$ and $\Gamma$ can be used as well for $T^{(0)}$ and $\Gamma^{(0)}$, using the first line of eq. (2.10) and
where \( \theta \) functions of \( \cos \). Then, using eq. (2.30) is written

\[
\frac{i}{2} \int \frac{d^4 p'}{(2\pi)^3} \delta(\varepsilon' - \varepsilon'' + \mu) z_p z_{p'} T^{(0)}(p, p'; k) \bigg|_{\varepsilon = 0} \varphi_{p'}
\]  

(3.1)

where the function \( A(p, p'; k) \) signifies the scattering amplitude of quasiparticles on the Fermi surface,

\[
A(p, p'; k) \equiv z_p z_{p'} \text{ Re } \Gamma^{(0)}(p, p'; k) \bigg|_{\varepsilon' = 0}.
\]  

(3.3)

To the accuracy of our concern, we may put \( k = 0 \) in the first term of the right-hand side of eq. (3.2). Consequently, in terms of the forward scattering amplitude,

\[
A(p, p') \equiv A(p, p'; k = 0),
\]  

(3.4)

eq (2.30) is written

\[
- i \omega \psi_p + i v^*_p \cdot k \varphi_p + e v^*_p \cdot E = I(\varphi),
\]  

(3.5)

where

\[
\psi_p = \varphi_p + \sum_{p'} A(p, p') \frac{\partial n^0}{\partial \varepsilon_{p'}} \varphi_{p'},
\]  

(3.6)

and

\[
I(\varphi) = \frac{1}{2} \int \frac{d^4 p'}{(2\pi)^3} \delta(\varepsilon' - \varepsilon'' + \mu) z_p z_{p'} \text{ Im } T^{(0)}(p, p'; k) \bigg|_{\varepsilon = 0} \varphi_{p'} - 2 \gamma_p^* \varphi_p.
\]  

(3.7)

In the following part of this subsection, we discuss the left-hand side of eq. (3.5), neglecting \( I(\varphi) \) of the right-hand side. As we shall see, this is justified when \( \omega \gg \gamma_p^* \), i.e., in the collisionless regime.

To see the physical meaning of \( \psi_p \) defined by eq. (3.6), let us define

\[
f(p, p') = z_p z_{p'} \text{ Re } \Gamma(p, p'; k) \bigg|_{k = 0}.
\]  

Then, using

\[
g_2(p, k) \simeq \frac{2\pi i z^2}{\omega} \delta(\varepsilon - \varepsilon_p^* + \mu), \quad (k = 0, \omega \gg \gamma_p^*)
\]  

(3.8)

from eq. (2.27), we obtain

\[
f(p, p') = A(p, p') - \sum_{p''} A(p, p'') \frac{\partial n^0}{\partial \varepsilon_{p''}} f(p'', p'), \quad (\omega \gg \gamma_p^*)
\]  

(3.9)

Therefore the function \( f(p, p') \), related with \( A(p, p') \) by eq. (3.3), is identified as the Landau function of the Fermi liquid theory. In particular, in an isotropic system, \( f(p, p') \) and \( A(p, p') \) are functions of \( \cos \theta = p \cdot p'/p^2 \), so that they may be expanded in spherical harmonics,

\[
A(p \sigma, p' \sigma') = \frac{\pi^2 \hbar^3}{\Omega m^* \mu_{\text{F}}} \sum_l (A_l^* + \sigma \cdot \sigma' A_l^0) P_l(\cos \theta),
\]

\[
f(p \sigma, p' \sigma') = \frac{\pi^2 \hbar^3}{\Omega m^* \mu_{\text{F}}} \sum_l (F_l^* + \sigma \cdot \sigma' F_l^0) P_l(\cos \theta),
\]
where
\[
\frac{m^* p_F}{2\pi^2 \hbar^3} = - \left. \frac{\partial n_0^p}{\partial \epsilon_p^*} \right|_{\bar{\epsilon} = \epsilon_p^*}.
\] (3.10)

Here we have explicitly introduced the spin indices. Then, eq. (3.9) gives
\[
\frac{A_l^s}{2l + 1} = \frac{F_l^s}{2l + 1}, \quad \frac{A_l^a}{2l + 1} = \frac{F_l^a}{2l + 1}.
\] (3.11)

For \(\delta n_p\) defined by
\[
\delta n_p = \frac{\partial n_0^p}{\partial \epsilon_p^*} \psi_p,
\] (3.12)
and \(\delta \bar{n}_p\), eq. (2.25), there are relations
\[
\delta n_p = \delta \bar{n}_p + \sum_{p'} A(p, p') \delta \bar{n}_{p'},
\] (3.13)
and
\[
\delta \bar{n}_p = \delta n_p - \sum_{p'} f(p, p') \delta n_{p'}.
\] (3.14)

which are obtained by using eqs. (3.9) and (3.9). Equation (3.14) suggests the following relations,
\[
\delta \bar{n}_p \equiv n_p - n_p^0, \quad \delta n_p \equiv n_p - n_p^0,
\] (3.15)
\[
\bar{n}_p^0 \equiv n^0(\bar{\epsilon}_p^*), \quad n_p^0 \equiv n^0(\epsilon_p^*),
\] (3.16)
\[
\bar{\epsilon}_p^* \equiv \epsilon_p^* + \sum_{p'} f(p, p') \delta n_{p'},
\] (3.17)

Therefore, \(\delta \bar{n}_p\) is regarded as the deviation from \(n_p^0\), the equilibrium distribution function for the local excitation energy \(\bar{\epsilon}_p^*\) and \(\delta n_p\) as the deviation from the ground state \(n_p^0\). Hence the physical meaning of \(\varphi_p\) and \(\psi_p\) is now clarified with the help of eqs. (2.25) and (3.12).

It is noted that the total current \(j_{\mu}(k, \omega)\) given in eq. (2.24) is written also as
\[
j_{\mu}(k, \omega) = \frac{e}{\Omega} \sum_p \delta n_p j_{p\mu}^*,
\] (3.18)
with
\[
j_{p\mu}^* \equiv v_{p\mu}^* - \sum_{p'} f(p, p') \frac{\partial n_0^p}{\partial \epsilon_p^*} v_{p'\mu}^*.
\] (3.19)

Hence \(j_{p\mu}^*\) is regarded as the current carried by the quasiparticle \(p\). For \(j_{p\mu}^*\) thus defined, using eq. (3.9), we find a relation
\[
v_{p\mu}^* \equiv j_{p\mu}^* + \sum_{p'} A(p, p') \frac{\partial n_0^p}{\partial \epsilon_p^*} j_{p'\mu}^*, \quad (\omega \gg \gamma_p^*)
\] (3.20)
Fig. 2. (a) The vertex function $A(p, p'; p' + k, p - k)$ and (b) the self-energy $\Sigma(p)$ which gives rise to the quasiparticle damping $\gamma_p^* \propto T^2$.

In particular, for $k = 0$ and $\omega \gg \gamma_p^*$, eq. (3.5) without $I(\varphi)$ gives

$$e v_p^* \cdot E = -i \omega \psi_p$$

$$= -i \omega \varphi_p + \sum_{p'} A(p, p') \frac{\partial n^0}{\partial \varepsilon_{p'}} (-i \omega \varphi_{p'}). \quad (3.21)$$

Comparing eq. (3.20) with eq. (3.21), we can immediately solve the latter for $\varphi_p$, that is, the solution is

$$-i \omega \varphi_p = e j_p^* \cdot E. \quad (3.22)$$

Therefore, from eqs. (2.24) and (2.25), we obtain

$$\sigma_{\mu\nu}(\omega) = \frac{i e^2}{\omega} \frac{1}{\Omega} \sum_{p'} v_{p\mu} j_{p\nu}^* \left( -\frac{\partial n^0}{\partial \varepsilon_{p}^*} \right). \quad (\omega \gg \gamma_p^*) \quad (3.23)$$

This is the result of our previous paper.\[3]

3.2 Imaginary part of $T^{(0)}$

3.2.1 Hydrodynamic regime, $\omega \ll \gamma_p^*$

In this subsection we shall investigate $I(\varphi)$ of eq. (3.7). Among other mechanisms which would contribute to $I(\varphi)$, we must at least take account of electron-electron interaction, which is responsible for making renormalized quasiparticles. To this end, it is convenient to define

$$|A(p, p'; p' + k, p - k)|^2 \equiv z_p z_{p'} z_{p' + k} z_{p - k} \left| \operatorname{Re} \Gamma \left( p - \frac{k}{2}, p' + \frac{k}{2}; k \right) \right|^2. \quad (3.24)$$

To estimate the term of order $O(T^2)$ for the decay rate $\gamma_p^*$, we have to consider the selfenergy shown in Fig. 2 (b), where all the energy variables of $A(p, p'; p' + k, p - k)$, Fig. 2 (a), are to be fixed at the
As for $\text{Im} T$ where we used eq. (2.17) and defined $\rho$ in eq. (2.11), we find $\text{Im} T$ discontinuity in the diagram for $\gamma$. Therefore the function $A(p, p'; p' + k, p - k)$ which appears in the following equations is to be regarded as the scattering amplitude of quasiparticle. Now it is straightforward to obtain

\[ 2\gamma_p^* = -2z_p \text{Im} \Sigma^R(p, 0) \]
\[ = 2z_p \int_{-\infty}^{\infty} \frac{d^4p'}{(2\pi)^4} \int_{-\infty}^{\infty} \frac{d^4k'}{(2\pi)^4} \left( \frac{\omega' + \tanh \omega'}{2T} \right) \left( \frac{\tanh \frac{\epsilon'}{2T} - \tanh \frac{\epsilon' + \omega'}{2T}}{\tanh \frac{\omega}{2T} + \tanh \frac{\omega'}{2T}} \right) \]
\[ \times \left| \text{Re} \Gamma \left( p - \frac{k}{2}, p', k \right) \right|^2 \text{Im} G^R(p - k') \text{Im} G^R(p') \text{Im} G^R(p' + k') \bigg|_{\epsilon=0} \]
\[ = 2 \int_{-\infty}^{\infty} \frac{d^4p'}{(2\pi)^4} \int_{-\infty}^{\infty} \frac{d^4k'}{(2\pi)^4} \left( \frac{\omega' - \tanh \omega'}{2T} \right) \left( \frac{\tanh \frac{\epsilon'}{2T} - \tanh \frac{\epsilon' + \omega'}{2T}}{\tanh \frac{\omega}{2T} + \tanh \frac{\omega'}{2T}} \right) \]
\[ \times |A(p, p'; p' + k', p - k')|^2 \pi^2 \rho_{p-k'}^*(\epsilon) \rho_{p'}^*(\epsilon') \rho_{p'+k'}^*(\epsilon' + \omega') \]
\[ = (\pi T)^2 \sum_{p', k'} |A(p, p'; p' + k', p - k')|^2 \rho_{p-k'}^*(0) \rho_{p'}^*(0) \rho_{p'+k'}^*(0), \] (3.25)

where we used eq. (2.17) and defined $\rho_p^*(\epsilon)$ by

\[ \rho_p^*(\epsilon) \equiv \delta(\epsilon - \omega^* + \mu). \] (3.26)

As for $\text{Im} T^{(0)}(p, p'; k)$, we may set $k = 0$ to the accuracy of our concern. To be consistent with the diagram for $\gamma$, we have to consider three diagrams shown in Fig. 3(a) $\sim$ (c). Considering discontinuity in $\text{Im} T^{(0)}(p, p')$ which may appear at the boundaries between the three regions defined in eq. (2.11), we find

\[ -\text{Im} \Gamma^{\text{III}}(p, p') = \text{Im} \Gamma^{\text{IV}}(p, p') = \text{Im} \Gamma^{\text{V}}(p, p') = \text{Im} \Gamma^{(a)}(p, p'), \]
\[ -\text{Im} \Gamma^{\text{II}}(p, p') = \text{Im} \Gamma^{\text{III}}(p, p') = \text{Im} \Gamma^{\text{IV}}(p, p') = \text{Im} \Gamma^{(b)}(p, p'), \]
\[ \text{Im} \Gamma^{\text{III}}(p, p') = \text{Im} \Gamma^{\text{III}}(p, p') = -\text{Im} \Gamma^{\text{IV}}(p, p') = \text{Im} \Gamma^{(c)}(p, p'). \] (3.27)
These functions $\Gamma^{(i)}(p, p')$ are related to $T^{(i)}(p, p')$ as $\Gamma(p, p')$ is to $T(p, p')$. Therefore, eq. (2.10) gives

$$\text{Im } T^{(i)}(p, p') = -2 \left( \coth \frac{\varepsilon'}{2T} - \tanh \frac{\varepsilon'}{2T} \right) \text{Im } \Gamma^{(i)}(p, p'), \quad (i = a, b) \quad (3.28)$$

and

$$\text{Im } T^{(c)}(p, p') = 2 \left( \coth \frac{\varepsilon' + \varepsilon}{2T} - \tanh \frac{\varepsilon'}{2T} \right) \text{Im } \Gamma^{(c)}(p, p'). \quad (3.29)$$

Hence, using

$$\text{Im } \Gamma^{(a)}(p, p') = \int \frac{d^4k'}{(2\pi)^4} \left( \tanh \frac{\varepsilon + \omega'}{2T} - \tanh \frac{\varepsilon' + \omega'}{2T} \right) \times \left| \text{Re } \Gamma \left( p + \frac{k'}{2}, p' + \frac{k'}{2}; k' \right) \right|^2 \text{Im } G^R(p + k') \text{Im } G^R(p' + k') \quad (3.30)$$

we obtain

$$\frac{1}{2} \int \frac{d^4p'}{(2\pi)^3} \delta(\varepsilon' - \varepsilon_{p'}^* + \mu) z_p z_{p'} \text{Im } T^{(a)}(p, p') \bigg|_{\varepsilon = 0} \varphi_{p'}$$

$$= 2 \int_{-\infty}^\infty \frac{d\varepsilon'}{2\pi} \int_{-\infty}^\infty \frac{d\omega'}{2\pi} \left( \coth \frac{\varepsilon'}{2T} - \tanh \frac{\varepsilon'}{2T} \right) \left( \tanh \frac{\varepsilon' + \omega'}{2T} - \tanh \frac{\omega'}{2T} \right) \times \sum_{p', k'} |A(p, p' + k'; p + k', p')|^2 \pi^3 \rho_{p'}^*(\varepsilon')\rho_{p+k'}^*(\omega')\rho_{p+k'}^*(\varepsilon' + \omega') \varphi_{p'}$$

$$= (\pi T)^2 \sum_{p', k'} \pi |A(p, p' + k'; p + k', p')|^2 \rho_{p-k'}^*(0)\rho_{p-k'}^*(0) \varphi_{p-k'}$$

$$= (\pi T)^2 \sum_{p', k'} \pi |A(p, p' + k'; p + k', p')|^2 \rho_{p-k'}^*(0)\rho_{p-k'}^*(0) \varphi_{p-k'} \quad (3.31)$$

Similarly, we get

$$\frac{1}{2} \int \frac{d^4p'}{(2\pi)^3} \delta(\varepsilon' - \varepsilon_{p'}^* + \mu) z_p z_{p'} \text{Im } T^{(b)}(p, p') \bigg|_{\varepsilon = 0} \varphi_{p'}$$

$$= (\pi T)^2 \sum_{p', k'} \pi |A(p, p' - k'; p + k', p - k')|^2 \rho_{p'}^*(0)\rho_{p-k'}^*(0) \rho_{p' + k'}^*(0) \varphi_{p' + k'}$$

$$= (\pi T)^2 \sum_{p', k'} \pi |A(p, p' + k'; p + k', p - k')|^2 \rho_{p' + k'}^*(0)\rho_{p-k'}^*(0) \rho_{p'}^*(0) \varphi_{p' + k'} \quad (3.32)$$

and

$$\text{Im } \Gamma^{(c)}(p, p') = - \int_{-\infty}^\infty \frac{d\omega'}{2\pi} \int \frac{d^3k'}{(2\pi)^3} \left( \tanh \frac{\varepsilon - \omega'}{2T} + \tanh \frac{\varepsilon' + \omega'}{2T} \right) \times \left| \text{Re } \Gamma \left( p - \frac{k'}{2}, p' + \frac{k'}{2}; k' \right) \right|^2 \text{Im } G^R(p - k') \text{Im } G^R(p' + k') \quad (3.33)$$

$$\frac{1}{2} \int \frac{d^4p'}{(2\pi)^3} \delta(\varepsilon' - \varepsilon_{p'}^* + \mu) z_p z_{p'} \text{Im } T^{(c)}(p, p') \bigg|_{\varepsilon = 0} \varphi_{p'}$$

$$= -2 \int_{-\infty}^\infty \frac{d\varepsilon'}{2\pi} \int \frac{d\omega'}{2\pi} \left( \coth \frac{\varepsilon'}{2T} - \tanh \frac{\varepsilon'}{2T} \right) \left( \tanh \frac{\varepsilon' + \omega'}{2T} - \tanh \frac{\omega'}{2T} \right)$$
\[ \times \sum_{p', k'} |A(p, p'; p' + k', p - k')|^2 \pi^3 \rho_{p'}^*(\varepsilon') \rho_{p-k'}^*(-\omega') \rho_{p'+k'}^*(\varepsilon' + \omega') \varphi_{p'} \]

\[ = - (\pi T)^2 \sum_{p', k'} \pi |A(p, p'; p' + k', p - k')|^2 \rho_{p'}^*(0) \rho_{p-k'}^*(0) \rho_{p'+k'}^*(0) \varphi_{p'} . \quad (3.34) \]

To take account of quasiparticle spin, we just have to use \( |A_{\sigma\sigma'}(p, p'; p' + k', p - k')|^2 \), which depends on the spin components \( \sigma \) and \( \sigma' \) of the incoming quasiparticles \( p \) and \( p' \), that is to say, \( |A|^2 \) is to be replaced by

\[ |A_{\uparrow\uparrow}|^2 + \frac{1}{2} |A_{\uparrow\downarrow}|^2 , \]

where the factor \( 1/2 \) in front of \( |A_{\uparrow\downarrow}|^2 \) is added not to overcount the identical quasiparticle states, namely, \( (p_1 \uparrow, p_2 \uparrow) \) and \( (p_2 \uparrow, p_1 \uparrow) \). As a result, with eqs. (3.25), (3.31), (3.32) and (3.34) substituted into eq. (3.7), we find

\[ I(\varphi) = - (\pi T)^2 \sum_{p', k'} W(p, p'; p' + k', p - k') \rho_{p'}^* \rho_{p-k'}^* \rho_{p'+k'}^*(\varphi_p + \varphi_{p'} - \varphi_{p'+k'} - \varphi_{p-k'}) , \quad (3.35) \]

where \( \rho_{p} \equiv \rho_{p}^*(0) \). We defined the transition probability for the binary collision of quasiparticles,

\[ 2W \equiv 2\pi \left( |A_{\uparrow\uparrow}|^2 + \frac{1}{2} |A_{\uparrow\downarrow}|^2 \right) . \quad (3.36) \]

In effect, eq. (3.35) is obtained from the classical formula,

\[ I = - \sum_{p', k'} W(p, p'; p' + k', p - k') \delta(\varepsilon_p^* + \varepsilon_{p'}^* - \varepsilon_{p'+k'}^* - \varepsilon_{p-k'}^*) \]

\[ \times [\bar{n}_p \bar{n}_{p'}(1 - \bar{n}_{p'+k'})(1 - \bar{n}_{p-k'}) - (1 - \bar{n}_p)(1 - \bar{n}_{p'})(1 - \bar{n}_{p'+k'}\bar{n}_{p-k'})] , \quad (3.37) \]

after linearization with respect to the deviation \( \delta \bar{n}_p \) from the local equilibrium;

\[ I \longrightarrow \frac{\partial n^0}{\partial \varepsilon_p^*} I(\varphi) . \]

To summarize, on the basis of the linear response theory and the finite temperature formalism, we derived eq. (3.35), the collision integral of the linearized Boltzmann equation \[ 3.3 \].

### 3.2.2 Incoherent absorption

As discussed in the previous paper\[ 1 \] the part in the Drude weight that is lost by Umklapp processes is transferred to the incoherent part \( \sigma_{\text{inc}}(\omega) \). Although the full \( \omega \)-dependence of \( \sigma_{\text{inc}}(\omega) \) cannot be generally described in the framework of the Fermi liquid theory, the low frequency limit of the part due to quasiparticle scattering can be derived as follows. To this end, we consider the correction of order \( O(\omega^2) \) which should be added to eq. \( 3.3 \).

The vertex functions \( T_{ij}(p, p'; k) \) of eq. (2.12) (i.e., for \( i \) or \( j \) = 1, 3) can be neglected since they should appear pairwise and thus if included give rise to terms of order \( O(\omega^3) \) at most. In eq. \( 2.18 \) for \( g_2(p, k) \), we have to consider the energy dependence of \( \gamma_p^* \):

\[ \gamma_p^*(\varepsilon) \equiv -z_p \text{Im} \Sigma^R(p, \varepsilon) . \]
Then, $\varepsilon$ in the delta function of eq. (2.18) should be replaced with $\varepsilon - \omega/2$. Accordingly, we must use $\gamma^*(\omega/2)$ and $\text{Im} T^{(0)}(p, p'; k)|_{\varepsilon' = \varepsilon = \omega/2}$ in place of $\gamma^*$ and $\text{Im} T^{(0)}(p, p'; k)|_{\varepsilon = 0}$ in eq. (3.4). On the other side, for $\text{Im} T^{(0)}$, the $\omega$-dependence occurs only in the hyperbolic functions of eqs. (3.28) and (3.29), because of the assumption $\text{Im} T^{(0)}(p, p'; k) = 0$ for $\omega = T = 0$. Thus as a result we find

$$I(\varphi) = -\left((\omega/2)^2 + (\pi T)^2\right) \sum_{p', k'} W_{pp'k'p'p_{k'+k'}}(\varphi_p + \varphi_p' - \varphi_{p'+k'} - \varphi_{p-k'}). \tag{3.38}$$

where $W_{pp'k} \equiv W(p, p'; p' + k', p - k')$. It is noted that this is obtained from $I$ of eq. (3.37) by formally using $\delta(\tilde{\varepsilon}^*_p + \tilde{\varepsilon}^*_p - \tilde{\varepsilon}^*_{k'} + \tilde{\varepsilon}^*_{k'} + \omega)$ instead of the delta function in eq. (3.37). The collision integral $I(\varphi)$ of eq. (3.38) is used to describe damping of zero sound. Therefore, our microscopic derivation proves that the quasiparticle part of the matrix element $W(p, p'; p' + k', p - k')$ of the zero sound damping coincides with the transition probability for the binary collision of quasiparticles.

To discuss the quasiparticle part of the optical conductivity, we assume $k = 0$ and $T = 0$. Then eq. (3.5) reads

$$-i\omega \psi_p + ev_p^* \cdot E = -\frac{\omega^2}{4} \sum_{p', k'} W_{pp'k'} \rho^*_p \rho^*_p \rho^*_{p-k'} \rho^*_{p'+k'}(\varphi_p + \varphi_p' - \varphi_{p'+k'} - \varphi_{p-k'}). \tag{3.39}$$

This is solved by successive approximation as a series in powers of $\omega$:

$$i\omega \psi_p = i\omega\psi_p^{(0)} + i\omega\psi_p^{(1)},
\quad i\omega \varphi_p = i\omega\varphi_p^{(0)} + i\omega\varphi_p^{(1)}, \tag{3.40}$$

where

$$i\omega\psi_p^{(0)} \sim O(1), \quad i\omega\psi_p^{(1)} \sim O(\omega), \quad \text{etc.}$$

The terms in this series satisfy the equations,

$$-i\omega\psi_p^{(0)} + ev_p^* \cdot E = 0, \tag{3.41}$$

and

$$-i\omega\psi_p^{(1)} = -\frac{\omega^2}{4} \sum_{p', k'} W_{pp'k'} \rho^*_p \rho^*_p \rho^*_{p-k'} \rho^*_{p'+k'}(\varphi_p^{(0)} + \varphi_p^{(0)} - \varphi_{p'+k'}^{(0)} - \varphi_{p-k'}^{(0)}). \tag{3.42}$$

As in eq. (3.22), the former gives

$$-i\omega\varphi_p^{(0)} = e\tilde{j}_p^* \cdot E. \tag{3.43}$$

Then, the latter gives

$$-i\omega\psi_p^{(1)} = \frac{i\omega}{4} \sum_{p', k'} \sum_{\mu} W_{pp'k'} \rho^*_p \rho^*_p \rho^*_{p-k'} \rho^*_{p'+k'}(\tilde{j}_p^* \mu + \tilde{j}_p^* \mu - \tilde{j}_p^* \mu - \tilde{j}_p^* \mu) E_\mu, \tag{3.44}$$

Therefore, we obtain

$$\sigma_{\mu\nu}(\omega) = \sigma_{\mu\nu}^{(0)} + \sigma_{\mu\nu}^{(1)}, \tag{3.45}$$

14
for which $\sigma_{\mu\nu}^{(0)}$ is given by eq. (3.23) and $\sigma_{\mu\nu}^{(1)}$ is

$$\sigma_{\mu\nu}^{(1)} = 2e^2 \sum_{p,p',k'} W_{pp'k'} \rho_{p'}^* \rho_{p-k'} \rho_{p-k'}^* \rho_{p-k'}^* \left( j_{pp'}^* (j_{pp'}^* + j_{pp'}^* - j_{pp'}^* - j_{pp'}^*) \right),$$

(3.46)

where we used eqs. (3.12) and (3.18) for $\psi_{1p}$, and

$$\frac{\partial n_{0p}}{\partial \epsilon_p^*} = \rho_{p'}^*.$$

Equation (3.46) indicates the incoherent part $\sigma_{\mu\nu}^{(1)}$ must vanish unless the current conservation is violated. Thus we obtain $\sigma_{\mu\nu}^{(1)} = 0$ in an isotropic system, as it should be the case.

§4. $T^2$ Contribution to the Electrical Resistivity

4.1 Formula

In this section, we discuss the dc conductivity due to quasiparticle transport at finite temperature. For $\omega = 0$ and $k = 0$, eq. (3.35) gives

$$ev_p^* \cdot E = -(\pi T)^2 \sum_{p',k'} W_{pp'k'} \rho_{p'}^* \rho_{p-k'} \rho_{p-k'}^* \rho_{p-k'}^* (\varphi_p + \varphi_{p'} - \varphi_{p'} - \varphi_{p-k'}).$$

(4.1)

Yamada and Yosida\(5\) were the first who derived this sort of equation on the basis of the linear response theory. In their work, the function $\varphi_p$ was defined after they derived the integral equation for the full vertex part $\tilde{\Lambda}_p$ that includes the effect of the section $g_2$ as well as $g_1$ and $g_3$; in our notation, they defined $\varphi_p$ by

$$\varphi_p \equiv \frac{- e \Lambda_p \cdot E}{2 \Delta_p},$$

(4.2)

where

$$\Delta_p = -\text{Im} \Sigma^R(p,0).$$

Eq. (4.2) is to be compared with eq. (2.26) by noting

$$g_2(p,0) = \frac{\pi z_p}{\Delta_p} \delta (\epsilon - \epsilon_p^* + \mu).$$

(4.3)

As discussed above, our derivation gives a definite interpretation of $\varphi_p$ as the deviation from the local equilibrium distribution function, eq. (2.25).

As is usually the case in solving the Boltzmann equation, we must assume a specific form for the function $\varphi_p$ to solve eq (4.1). In an isotropic system, as a scalar function of the vectors $p$ and $E$, the function $\varphi_p$ is assumed to be $\varphi_p \propto p \cdot E$. Then the right-hand side of eq. (4.1) identically vanishes owing to momentum conservation, indicating that the current persists even at finite temperatures\(5\).

In general, on physical grounds, we may assume

$$\varphi_p = -e\tau_p^* v_p^* \cdot E,$$
with $\tau_p^*$ as a function of $p$. However, for simplicity, neglecting the momentum dependence of $\tau_p^*$, we assume

$$\varphi_p = -e\tau_p^* v_p^* \cdot E.$$  \hspace{1cm} (4.4)

Then, the transport relaxation time $\tau_{tr}^*$ is obtained from eq. (4.1):

$$\frac{1}{\tau_{tr}^*} = (\pi T)^2 \sum_{p,p',k} W_{pp'k} \rho_p^* \rho_{p'}^* \rho_{p-k}^* \rho_{p'+k}^* v_{p\nu}^* (v_{p\nu}^* + v_{p'\nu}^* - v_{p'+k\nu}^* - v_{p-k\nu}^*) \sum_p \rho_p^* v_{p\nu}^* v_{p\nu}^*.$$  \hspace{1cm} (4.5)

It is stressed here that this result based on eq. (4.4) is approximate. One may follow a recent work of Maebashi and Fukuyama$^{14}$ to improve upon the approximation systematically. It is interesting to see that the velocity $v_p^*$ in eq. (4.5) take the place of the current $j_p^*$ of eq. (3.14) in the coherent regime. By eqs. (2.24) and (2.25), the conductivity $\sigma_{\nu\nu}(0)$ is written

$$\sigma_{\nu\nu}(0) = \frac{e^2}{\Omega} \sum_p \rho_p^* v_{p\nu}^* = \rho_{\nu\nu}(0)^{-1} = AT^2$$  \hspace{1cm} (4.6)

Hence, the coefficient $A$ of the electrical resistivity $R = \sigma_{\nu\nu}(0)^{-1} = AT^2$ is given by

$$\frac{2e^2 A}{\pi^2} = \frac{1}{\Omega} \sum_{p,p',k} W_{pp'k} \rho_{p}^* \rho_{p'}^* \rho_{p-k}^* \rho_{p'+k}^* v_{p\nu}^* (v_{p\nu}^* + v_{p'\nu}^* - v_{p'+k\nu}^* - v_{p-k\nu}^*) \sum_p \rho_p^* v_{p\nu}^* v_{p\nu}^*.$$  \hspace{1cm} (4.7)

In our simple cases discussed below, the normal processes which give finite contribution to the summation,

$$\sum_{p_1, p_2, p_3} \rho_{p_1} \rho_{p_2} \rho_{p_3} \delta_{p_1+p_2-p_3} \rho_{p_4} = \rho_{\nu\nu}(0)^{-1} = AT^2$$  \hspace{1cm} (4.8)

are those satisfying (i) $p_1 = p_3$, $p_2 = p_4$, (ii) $p_1 = p_4$, $p_2 = p_3$ or (iii) $p_1 + p_2 = p_3 + p_4 = 0$. Therefore, the normal processes do not contribute to $A$ because of the factor $v_{p\nu}^* + v_{p'\nu}^* - v_{p'+k\nu}^* - v_{p-k\nu}^* = 0$ in eq. (4.7). Non-zero resistivity is brought about by Umklapp processes$^{14}$ (Fig. 4). Moreover, eq. (4.7) shows that we must assume $W_{pp'k} \propto |A_{pp'k}|^2 \propto (\rho^*)^2$ so as to obtain the Kadowaki-Woods relation $A \propto (v^*)^{-2} \propto (\rho^*)^2$. In this respect, we can make an order of magnitude estimate of $\rho^* A_{pp'k}$ in terms of eq. (3.9), which shows that $\rho^* A(p, p')$ in fact approaches a constant independent of $\rho^*$ in the strong coupling regime $\rho^* \Omega f(p, p') \gg 1$. It is instructive to refer eq. (3.11) by which we obtain $A_i \approx \rho^* \Omega A(p, p') \lesssim O(1)$ for $F_i \gg 1$. Thus in this situation we are led to the Kadowaki-Woods relation $A \propto \gamma^2$ with $\gamma$ of the specific heat coefficient,

$$\gamma = \frac{2\pi^2}{3} \rho^*, \quad \rho^* = \frac{1}{\Omega} \sum_p \rho_p^*.$$  \hspace{1cm} (4.9)

The ratio $A/\gamma^2$ however may depend on the geometry of the Fermi surface relative to the Brillouin zone boundary. To give a rough estimate, neglecting the momentum dependence of $W_{pp'k}$ or $A_{pp'k}$,
we introduce the coupling constants $A^s$ and $A^a$ by
\[ 2\rho^*\Omega A_{\uparrow\downarrow} \equiv A^s - A^a, \quad 2\rho^*\Omega A_{\uparrow\uparrow} \equiv A^s + A^a. \]
(4.10)

Then, we obtain
\[ A = \frac{9\hbar a}{8e^2\pi} \alpha F = 8.5\alpha F \times 10^{-7} \mu\text{cm}(\text{mol K/mJ})^2, \quad (a = 4\text{Å}) \]
(4.11)

where
\[ \alpha = \frac{1}{4} \left( (A^s - A^a)^2 + \frac{1}{2}(A^s + A^a)^2 \right) \equiv \alpha_{\uparrow\downarrow} + \alpha_{\uparrow\uparrow}, \]
(4.12)
and
\[ F = \frac{1}{\Omega^3} \sum_{p,p',k} \rho_p \rho_{p-k} \rho_{p+k} \rho_{p'} \left(v_{p\nu}^* - v_{p-k\nu}^* - v_{p'k\nu}^* + v_{p\nu}^* - v_{p'k\nu}^* + v_{p-k\nu}^* \right) \left(1 + \frac{1}{\Omega} \sum_p \rho_p v_{p\nu}^* v_{p\nu}^* \right)^2. \]
(4.13)

The parameter $\alpha$ of eq. (4.12) is a coupling constant of quasiparticle interaction. On the other side, the parameter $F$ is invariant under the transformation,
\[ \varepsilon_p^* - \mu \rightarrow z(\varepsilon_p^* - \mu), \]
\[ v_p^* \rightarrow zv_p^*, \]
\[ \rho_p^* \rightarrow \rho_p^*/z. \]
(4.14)

Thus $F$ is regarded as a quantity characterizing a shape of the Fermi surface, so that we may disregard the superscript $*$ for the renormalized quantities in eq. (4.13). Equation (4.11) indicates that for the systems obeying the Kadowaki-Woods relation the product $\alpha F$ must be a universal constant.
4.2 Two-dimensional systems

To estimate \( F \) quantitatively, it is convenient to introduce a coordinate system on the Fermi surface. Although it is possible to treat an arbitrary case in principle, for simplicity, we consider two dimensional systems with a closed Fermi ‘surface’ in the Brillouin zone \((-\pi \leq k_x, k_y \leq \pi)\). Since then points on the Fermi surface are parameterized by the azimuthal angle \( \theta = \tan^{-1} \frac{k_y}{k_x} \), we define the function \( k_\theta \), which is determined so as to satisfy \( \varepsilon_k = \mu \), for \((k_x, k_y) = (k_\theta \cos \theta, k_\theta \sin \theta)\). In terms of the function \( k_\theta \), local density of states on the Fermi surface is given by

\[
\rho_\theta d\theta = \frac{1}{(2\pi)^2} \sqrt{k_\theta^2 + k^2} \rho_\theta d\theta,
\]

where

\[
\dot{k}_\theta = \frac{dk_\theta}{d\theta}, \quad v_\theta = \sqrt{v_x^2 + v_y^2}, \quad v_\mu = \frac{\partial \varepsilon_k}{\partial k_\mu}.
\]

Then, we may use

\[
\frac{1}{\Omega} \sum_k = \int \rho_\theta d\theta d\varepsilon,
\]

in which the integral over \( \varepsilon \) eliminates the delta function \( \rho_p(\varepsilon) = \delta(\varepsilon - \varepsilon_p + \mu) \). Thus we can use

\[
\frac{1}{\Omega^3} \sum_{k_1,k_2,k_3} \rho_{k_1} \rho_{k_2} \rho_{k_3} \rho_{k_4} \delta_{k_1+k_2-k_3-k_4} = \int d\theta_1 d\theta_2 d\theta_3 \rho_\theta_1 \rho_\theta_2 \rho_\theta_3 \rho_\theta_4,
\]

where \( k_4 = k_1 + k_2 - k_3 \). We assumed that \( k_1, k_2 \) and \( k_3 \) are on the Fermi surface in the first Brillouin zone, without loss of generality. The process in which to have \( k_4 \) outside the Brillouin zone is the Umklapp process giving rise to a finite contribution to \( F \). To eliminate the last delta function \( \rho_{k_4} \) in eq. (4.10), we shall regard \( \varepsilon_{k_4} \) as a function of \( \theta_3 \), i.e., \( \varepsilon_{k_4} \equiv \varepsilon(\theta_3) \), and define \( \bar{\theta}_3 \) as a solution of \( \varepsilon(\bar{\theta}_3) = \mu \), by which \( \bar{\theta}_3 \) is determined as a function of \( \theta_1 \) and \( \theta_2 \). Then, we may put

\[
\rho_{k_4} = \delta(\theta_3 - \bar{\theta}_3)/a_{\bar{\theta}_3}, \quad a_{\bar{\theta}_3} = \left| \frac{d\varepsilon(\bar{\theta}_3)}{d\bar{\theta}_3} \right|,
\]

and the delta function is eliminated by the integral over \( \theta_3 \). As a result, to evaluate eq. (4.13) we may use

\[
\frac{1}{\Omega^3} \sum_{k_1,k_2,k_3} \rho_{k_1} \rho_{k_2} \rho_{k_3} \rho_{k_4} \delta_{k_1+k_2-k_3-k_4} = \int d\theta_1 d\theta_2 d\theta_3 \rho_\theta_1 \rho_\theta_2 \rho_\theta_3 /a_{\bar{\theta}_3}.
\]

In this manner, we calculated the parameter \( F \) as a function of density \( n \) for (a) \( \varepsilon_k = -\cos k_x - \cos k_y \) for \( n < 1 \), and (b) \( \varepsilon_k = k_x^2 + k_y^2 \) for \( n < \pi/2 \) (Fig. 5.) The result of the case (a) alone may be obtained by using the recent results for the Hubbard model by Maebashi and Fukuyama. In effect, from the figure, we observe \( F \simeq 10 \) as far as Umklapp processes are effective. For small \( n \) (< \( n_c \)), there can be no Umklapp process available, so that we obtain \( F = 0 \) identically. It is elementary to estimate the critical concentration \( n_c \). Using \( k_4 = k_1 + k_2 - k_3 = 3k_1 \) for \( k_1 = k_2 = -k_3 \parallel k_4 \),
Fig. 5. $F$ as a function of $n$ for (a) $\varepsilon_k = -\cos k_x - \cos k_y$ ($n < 1$), and (b) $\varepsilon_k = k_x^2 + k_y^2$ ($n < \pi/2$). The function $F$ identically vanishes for $n/2 < n_c/2 \approx 0.2$ owing to the absence of Umklapp scattering.

Fig. 6. The parameters $F_\mu$ as a function of $n$ for the Fermi surface of $\varepsilon_k = -\cos k_x - \cos k_y - \tau \cos k_z$.

on the threshold of the Umklapp process $k_4 = 2\pi - k_1$, we obtain $k_1 = \pi/2$. This corresponds to $n_c/2 = \pi k_1^2/(2\pi)^2 = \pi/16 = 0.196$ for $\varepsilon_k = k^2$. This is consistent with the results shown in the figure.

4.3 Three-dimensional systems

Next we consider the Fermi surfaces of three dimensional systems which may be parameterized by cylindrical coordinates $k = (k \cos \theta, k \sin \theta, k_z)$, where the radial function $k$ is now regarded as a function of $\theta$ and $k_z$. Then we can proceed similarly as in the previous subsection, i.e., we use

$$\frac{1}{\Omega} \sum_k = \int \rho d\theta dk_z d\varepsilon,$$
where
\[
\rho d\theta dk_z = \frac{1}{(2\pi)^3} \sqrt{k^2(1 + \partial_z k^2) + \partial_y k^2} \frac{d\theta dk_z}{v},
\]
and
\[
v = \sqrt{v_x^2 + v_y^2 + v_z^2}, \quad v_\mu = \frac{\partial \varepsilon_k}{\partial k_\mu},
\]
We evaluated $F_\mu$ for the dispersion
\[
\varepsilon_k = -\cos k_x - \cos k_y - \tau \cos k_z,
\]
as a function of the density $n$ (Fig. 6) and the anisotropy $\tau$ (Fig. 7). From the figures we see $F_\mu \lesssim 20$, except for those in resistive directions such as $F_z$ for $\tau < 1$ and $F_x = F_y$ for $\tau > 1$.

To conclude this section, let us estimate the ‘universal’ ratio $A/\gamma^2$. As noted above, the two parameters $\alpha$ and $F_\mu$ should take constant values for the ratio $A_\mu/\gamma^2$ to be universal. For the former we may assume the strong coupling condition $\alpha \sim 1$ for the universality. For the latter, we assume $F = 10 \sim 20$ as a typical value indicated by the above figures. Then, for example, for $\alpha = 1$ (e.g., for $A^a = 1$ and $A^a = -1$) and $F = 20$, eq. (4.11) gives $A/\gamma^2 \simeq 1.7 \times 10^{-5} \mu\Omega\text{cm}(\text{mol K/mJ})^2$. This is reasonable as a rough estimate for the observed ratio $4 \times 10^{-5} \mu\Omega\text{cm}(\text{mol K/mJ})^2$.

§5. Discussion

In this article, on the basis of Eliaшкиб’s formalism and using the results of Yamada and Yosida, we formulated the Landau Fermi liquid theory microscopically in the form presented by Pines and Nozi`eres. In so doing our motivation was to eliminate microscopic quantities $\Sigma(p)$ and $\Gamma(p,p';k)$ in favor of $\varepsilon_k^*$ and $A_{pp'k}$; to derive general results one should not introduce approximations even if they are found to be physically appropriate. In a sense we must use $A_{pp'k}$ and $A(p,p') = A_{pp'k=0}$ for the theory to be definite or to avoid confusion, since
\(\Gamma(p, p') \equiv \Gamma(p, p'; p, p) \equiv \lim_{k \to 0} \Gamma(p, p'; k)\) is not well defined as it depends on the limiting procedure. The Landau parameters \(f(p, p')\), which are usually used to describe static properties of the system, are obtained from the forward scattering amplitudes \(A(p, p')\) by solving eq. (3.9).

In effect, for practical purposes, we do not have to know \(f(p, p')\) since physical quantities in the hydrodynamic regime are usually expressed in terms of \(A(p, p')\) in a direct manner. For example, the total magnetization \(M\) in the magnetic field \(H\) is given by

\[
M = \mu_B \sum_{p, \sigma = \pm} \sigma \delta n_{p\sigma},
\]

while the effect of \(H\) causes the deviation from the local equilibrium,

\[
\delta n_{p\sigma} = \mu_B \sigma H \left( -\frac{\partial n^0}{\partial \varepsilon^*_{p\sigma}} \right).
\]

Using eq. (3.13) to express \(\delta n_{p\sigma}\) in terms of \(\delta \bar{n}_{p\sigma}\), the susceptibility \(\chi_s\) is given by

\[
\chi_s = \frac{M}{\Omega H} = 2 \mu_B^2 \rho^*(1 - A_0^a),
\]

where

\[
2 \rho^* A_0^a = \frac{4}{\Omega} \sum_{p, p'} \rho_p^* \rho_{p'}^* A^a(p, p'),
\]

\[
2A^a(p, p') = A_{\uparrow\uparrow}(p, p') - A_{\uparrow\downarrow}(p, p').
\]

In particular, for an isotropic system, eq. (3.11) gives

\[
A_0^a = \frac{F_0^s}{1 + F_0^s},
\]

and we recover a well-known result,

\[
\chi_s = \frac{2 \mu_B^2 \rho^*}{1 + F_0^s}.
\]

Similarly, the charge susceptibility \(\chi_c\) is given by

\[
\chi_c = \frac{\partial n}{\partial \mu} = 2 \rho^*(1 - A_0^c),
\]

\[
2 \rho^* A_0^c = \frac{4}{\Omega} \sum_{p, p'} \rho_p^* \rho_{p'}^* A^c(p, p'),
\]

\[
2A^c(p, p') = A_{\uparrow\uparrow}(p, p') + A_{\uparrow\downarrow}(p, p').
\]

It is noted that part of Yamada and Yosida’s results\(^5\) for \(\chi_s\) and \(\chi_c\) due to \(f\) electrons in heavy electron systems can be cast into the above forms. In other words, eqs. (5.3) and (5.4) are derived microscopically by using the results of ref. \(^5\). The derivation is as follows. From eqs. (2.18) and (2.21) of ref. \(^5\) we get

\[
\chi_s = \frac{2 \mu_B^2}{\Omega} \sum_p z_p \rho_p^* \left( \tilde{\chi}_{\uparrow\uparrow}(p) + \tilde{\chi}_{\uparrow\downarrow}(p) \right).
\]
On the other side, from eqs. (3.2), (3.9) and (3.16) of ref. 5, we obtain

\[
\rho^* = \frac{1}{\Omega} \sum_p z_p \rho^*_p \tilde{\chi}_{\uparrow\uparrow}(p) + \frac{1}{\Omega} \sum_{p,p'} \rho^*_p \rho^*_p A_{\uparrow\uparrow}(p,p'),
\]

(5.10)

and

\[
\frac{1}{\Omega} \sum_p z_p \rho_p \tilde{\chi}_{\uparrow\downarrow}(p) = \frac{1}{\Omega} \sum_{p,p'} \rho_p \rho_{p'} A_{\uparrow\downarrow}(p,p'),
\]

(5.11)

respectively. By eliminating \(\tilde{\chi}_{\uparrow\uparrow}(p)\) and \(\tilde{\chi}_{\uparrow\downarrow}(p)\), we are led to eq. (5.3) with eqs. (5.4) and (5.5). The result for \(\chi_c\) can be derived similarly. In this formulation, the Wilson ratio \(R_W\) is simply expressed by the single parameter \(A_0^a\) as \(R_W = 1 - A_0^a\), instead of two parameters \(\chi_{\uparrow\downarrow}/\chi_{\uparrow\uparrow}\) and \(\delta_{\uparrow\uparrow}/\chi_{\uparrow\uparrow}\) as discussed below eq. (3.20) of ref. 5. This is because of our proper use of \(\rho^*\) for the specific heat coefficient \(\gamma\) without neglecting the momentum dependence of the selfenergy \(\Sigma(p)\).

As is clear from these examples and eq. (4.7), in all cases, the scattering amplitude \(A_{pp'k}\) enters in physical quantities as some kind of average over momentum variables at the Fermi level. Therefore the whole momentum-dependence of \(A_{pp'k}\) would not be determined solely by experiment, or the theory requires more information than experiment offers. An ultimate goal on the theoretical side would be to derive \(\varepsilon_k^*\) and \(A_{pp'k}\) by eqs. (2.14) and (3.24) from first principles, e.g., on the basis of the periodic Anderson Hamiltonian. As this is far from a trivial task, one has to regard these as parameters in practice, giving up explaining why and to what extent the quasiparticle mass of a certain system is heavily enhanced. Then, to sum up, all we can do is to obtain an order-of-magnitude estimate of the parameters \(A^s\) and \(A^a\) from, e.g., \(A^s_0\) and \(A^a_0\) of eqs. (5.3) and (5.6). Nevertheless, this is no way disappointing in view of the fact that there are a large class of systems among which \(\rho^*\) may vary so widely that the rough estimation can still make sense. To put it differently, one may consider that the observed universality in the ratio \(A/\gamma^2\) supports the assumption to neglect the momentum dependence of the scattering amplitude in eq. (4.10).

Keeping these points in mind, let us discuss factors which may spoil the agreement between theory and experiment on the ratio \(A/\gamma^2\), noted at the end of the previous section. (i) The lattice constant \(a\) is assumed to be \(a = 4\AA\); The error caused by this factor will not be appreciable. (ii) On the geometric factor \(F\): We estimated \(F\) only for several simple examples. In our theory, the assumption that \(F \lesssim 20\) will not depend on the detailed structure of the Fermi surface is crucial to reproduce the universal value for the ratio \(A/\gamma^2\). In principle, given the Fermi surface, the factor \(F\) defined in eq. (4.13) can be evaluated by parameterizing coordinates on the Fermi surface. Thus it is desirable to make sure that this is in fact the case for as many arbitrary but typical Fermi surfaces as possible. As examples for which \(F\) deviates from the universal value, we showed the results for anisotropic systems in the previous section. Moreover, we can imagine an exceptional case as for \(n = 1\) of the square lattice (Fig. 5(a)). In general, one can define the geometric factor \(F\) for the bands comprising degenerate orbitals. In this case, as a result of possible interband Umklapp processes, the geometric factor \(F\) may deviate from the universal value estimated in single band models with
large Fermi surfaces. For example, $F$ can become small in the case where intraband Umklapp processes are suppressed by a peculiar geometry of available phase space, e.g., when relevant Fermi surfaces are too small compared with the size of unit cell spanned by reciprocal wavevectors. This point may be relevant to the experimental fact that the ratio $A/\gamma^2$ for the transition metals such as Ni and Pd in the vicinity of ferromagnetic instability is more than an order of magnitude smaller than that for the heavy fermion compounds,$^6$ although it may be due to a relatively weak electron correlation effect, or due to small $\alpha$. In our formulation, the latter possibility may correspond to the viewpoint of Miyake et al.$^6$ (iii) We must have $\alpha \sim 1$: If we were to have $F < \sim 20$, then $\alpha \sim 1$ is required to explain the universality $A/\gamma^2 \simeq 10^{-5} \mu \Omega \text{cm}(\text{mol K/mJ})^2$, as observed experimentally. For this to be the case, we must be not only in the strong coupling (repulsive) regime where $2\rho^* \Omega f(p,p') \gg 1$ (i.e., $A^a \simeq 1$), but also well away from the Fermi liquid instabilities so that $|A^a| \sim 1$. When the quasiparticle interaction is attractive and the system is in the close vicinity of the Fermi liquid instability,$^8$ $|A(p,p')|^2$ can become quite large, and the ratio $A/\gamma^2$ can deviate from a universal value. In this regard, however, a note is in order. The parameter $A^a_0$ determined from $\chi$, for example, is defined by the forward scattering amplitude $A^a_0(p,p')$. Thus it may happen that the enhancement which give rise to a large $|A^a_0|$ is prominent only in the forward region $k \sim 0$ of the scattering amplitude $|A_{pp'k}|$. Then, the proximity to the instability does not necessarily imply a large resistivity since those processes with $k \sim 0$ do not contribute much to the resistivity.

With respect to the point (iii) noted above, well-known characteristics of the heavy fermion systems obeying the Kadowaki-Woods relation are regarded as general properties of stable strong-coupling Fermi liquids; that is, the charge susceptibility $\chi_c$, eq. (5.4), will be suppressed because of $A^a_0 \lesssim 1$, the Wilson ratio $R_W = 1 - A^a_0$, eq. (5.3), will take a value of order unity, and the system will possess well-defined collective modes, namely, zero sounds, because of $2\rho^* \Omega f(p,p') \gg 1$. In other words, the ratio $A/\gamma^2$ does not depend sensitively on the Wilson ratio as far as $R_W \sim 1$. These characteristics are implicit in ref.$^5$. Furthermore, we may say that these systems are in the regime well away from the Fermi liquid instability, including the metal-insulator transition: When $F^a_1 \gg 1$ as generally expected from $2\rho^* \Omega f(p,p') \gg 1$, the Drude weight $D$ is not reduced as much as expected from the heavily enhanced quasiparticle mass $m^*$ since$^9$

$$D \propto \frac{1}{m'} = \frac{1}{m^*} \left(1 + \frac{F^a_1}{3}\right) \gg \frac{1}{m^*}.$$  

In general, $D$ will never vanish as in liquid $^3$He, for which the ‘optical mass’ $m' = m$ is not renormalized at all while $m^*$ is heavily enhanced as the pressure is increased. Still it is remarked that, according to the above definition, $^3$He cannot be regarded as a stable Fermi liquid because of $F^a_0 \simeq -0.7$ or $A^a_0 \simeq -2$.

We saw above that the transport equation (3.5) involves the time derivative of $\delta n_p \propto \psi_p$, and the spatial derivative of $\delta n_p \propto \varphi_p$, while the collision integral, eq. (3.35), is written in terms of $\delta n_p$.
and $A_{pp'k}$. Therefore, in the static case $\omega = 0$, the effect of the quasiparticle interaction $A_{pp'k}$ does not appear in the transport equation except in the transition probability, eq. (3.36). Then we can discuss quasiparticle transport properties in general perspective on the assumption of the general validity of the Boltzmann equation.

According to the above discussion, the universal ratio $A/\gamma^2$ is due to the transition probability $W$, eq. (3.36), being proportional to $(\rho^*)^{-2}$, which we claim is realized in the strong coupling Fermi liquids well away from the Fermi liquid instability. In this case, the energy scale is set solely by the quasiparticle energy $\varepsilon^*_p$. For example, the decay rate $\gamma^*_p$ is given by

$$\gamma^*_p \simeq \rho^* (\pi T)^2.$$  

Thus, by the condition

$$|\varepsilon^*_p - \mu| \sim \pi T \gtrsim \gamma^*_p \simeq \rho^* (\pi T)^2,$$

the concept of well-defined quasiparticle is warranted for $T \lesssim T_0 \simeq 1/\pi \rho^*$. This is in contrast to the case in the vicinity of the Fermi liquid instability, where $T_0$ can be quite suppressed owing to an enhanced transition probability $W$. A general theory based on the Boltzmann equation states that universal relations must hold not only for the resistivity but for the other static transport coefficients: In fact, as in eq. (4.11), we can write universal relations for the viscosity $\eta$, the thermal conductivity $K$ and the spin-diffusion coefficient $D$ (Appendix). Here, however, we show them in a particular case of an isotropic system, for which we may refer the classical papers.20, 21)

For the results of the isotropic system, we use

$$\gamma = \left(\frac{\pi}{3n}\right)^{2/3} \frac{m^*}{h^2}, \quad n = \frac{k_F^3}{3\pi^2},$$

for $\gamma$ of the specific heat $C = \gamma T$ per particle, and define the ‘lattice constant’ $a$ by

$$\frac{4\pi}{3}a^3 n = 1.$$  

Then, under the assumption to neglect the momentum dependence of $A_{pp'k}$, it is straightforward to obtain the relations

$$KT^2 \gamma^2 = \frac{3^{2/3} \pi^{4/3} \rho}{4^{1/3} h a^{1/3}} \alpha^{-1},$$

$$\eta T^2 \gamma^2 = \frac{5h}{8a^3} \alpha^{-1},$$

$$DT^2 \gamma^3 = \frac{2^{1/3} \pi^{7/3} a^2}{3^{4/3} \hbar} \frac{\alpha_{\uparrow\downarrow}^{-1}}{1 - A_0^3}.$$  

The factor $1 - A_0^3$ in $D$ corresponds to that used in the susceptibility, eq. (5.3). The parameters $\alpha$ and $\alpha_{\uparrow\downarrow}$ are defined in eq. (4.12). Nevertheless, we must note that the averaged quantity $\alpha$ for $\sigma(0)T^2 \gamma^2$, eq. (4.11), and that for $KT^2 \gamma^2$, eq. (5.12), for example, should not be literally identified with each other, for they are not quite the same as their definitions use different types of average
of $A_{pp'/k}$. In a lattice system, a term due to Umklapp processes is added to $D$ unless $\alpha_{\uparrow\uparrow} = 0$. According to the exact solutions,² ²³ eq. (5.12) has to be multiplied by a factor $\sim 0.5$, and eqs. (5.13) and (5.14) by $\sim 0.8$. Therefore, we may write

$$KT\gamma^2 \sim \frac{0.8}{\hbar a} a^{-1},$$
$$\eta^2 \gamma^2 \sim \frac{0.5}{a^3} \hbar a^{-1},$$
$$DT^2 \gamma^3 \sim \frac{3a^2}{\hbar} \frac{\alpha_{\uparrow\downarrow}}{1 - A_0^2}.$$  (5.15)

These relations are approximate and might not be followed well by experiment as the relation for $A/\gamma^2$ is, since the kinetic coefficients which are mainly determined by normal processes may be found to depend sensitively on the detailed structure of $A_{pp'/k}$ and the Fermi surface. In this respect also, the relation between geometry of the Fermi surface and the scattering processes relevant to kinetic coefficients should be investigated specifically. It is interesting to note that we could estimate not only the Lorenz ratio $K/\sigma T$, which may take a constant value for a large class of systems (the Wiedemann-Franz law), but $A/\gamma^2$ and $KT\gamma^2$ separately. This is because of the universality due to the saturation of the coupling constant $A^s \simeq 1$ in the strong coupling regime, where the relevant energy scale is set solely by the density of states $\rho_k^s$ of quasiparticle, as mentioned above. Nonetheless, one should still keep in mind the point that the universality is concluded by the neglect of the angular dependence of the scattering amplitude and the detailed structure of the Fermi surface.

§ 6. Conclusion

To evaluate a universal value taken by the Kadowaki-Woods ratio $A/\gamma^2$ microscopically, we developed a theory of electrical conductivity of Fermi liquids, following Éliashberg² and using the results of Yamada and Yosida.⁵ We derived an equation to describe quasiparticle transport, by which our previous result for the Drude part, eq. (3.23), and Yamada-Yosida’s result, eq. (4.1), are reproduced in the coherent and hydrodynamic regime, respectively. Moreover, the low-frequency limit of the incoherent part due to quasiparticle was given in eq. (3.46). It was shown that some important results of Yamada and Yosida are interpreted exactly and comprehensively in a general context of the phenomenological Landau theory. So far this point seems not to be pointed out definitely, presumably because of the thorough use of bare microscopic quantities in the literature following ref. ⁶. In order to write the results just in terms of renormalized quantities, assumptions should not be made except those ensuring the validity of Fermi liquid concept. In particular, one should not make any specific assumption to derive a formula for the conductivity. From this point of view, the Boltzmann equation was obtained as a result. We consider that in a microscopic treatment it is important to distinguish what may be obtained phenomenologically from what is not. We were mainly concerned with the former.
To clarify the physics underlying the universality in the ratio $A/\gamma^2$, we expressed the ratio in a form proportional to the product of two factors $\alpha$ and $F$, and argued that the universal relation $A \propto \gamma^2$ is due to constant values taken by these factors. Moreover, we concluded that we must have $\alpha \sim 1$ as well as $F \sim 20$ to reproduce the observed universal ratio. The conjecture for the latter, related to a shape of Fermi surface, was borne out by evaluating $F$ explicitly in simple examples. The former, the condition for the coupling constant of quasiparticle interaction, was categorized as a universality of Fermi liquids in the strong coupling regime. In contrast with a customary view in the heavy fermion problem, we need not assume heavily enhanced effective mass of quasiparticle, $z_p^{-1} \gg 1$, so as to reproduce the relation $A \propto \gamma^2$. Needless to say, however, it may happen that the mass enhancement effect and the strong coupling condition $\alpha \sim 1$ are shown to be correlated to each other in a certain microscopic model. But we consider that this point, the universality in that particular model, is a secondary problem. It was claimed for the strong coupling Fermi liquids that the universal relation in terms of $\gamma$ may be concluded not only for $A$ of resistivity ($A/\gamma^2$=const.), but for any static kinetic coefficients, if one may disregard the effect due to specific geometric features of the Fermi surface as compared with the dominant effect due to the density of states $\rho^* \propto \gamma$. By way of illustration, we estimated the ratios $KT\gamma^2$, $DT^2\gamma^3$ and $\eta T^2\gamma^2$ for the thermal conductivity $K$, the spin-diffusion constant $D$ and the viscosity $\eta$.

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Appendix: Transport Coefficients

As in eq. (4.11), we can derive relations for the viscosity $\eta$, the thermal conductivity $K$, and the spin-diffusion coefficient $D$ on the basis of the Boltzmann equation. In the following expressions, $\alpha$, $\alpha_{\uparrow\downarrow}$ and $\alpha_{\uparrow\uparrow}$ are defined in eq. (4.12).

A.1 Viscosity

$$\frac{1}{\eta T^2 \gamma^2} = \frac{9}{4\pi} \alpha F_{\eta},$$

where

$$F_{\eta} = \frac{1}{\Omega^2} \sum_{p,p',k} \rho_p \rho_{p'} \rho_{p-k} \rho_{p'+k}\text{tr}P_p(P_p + P_{p'} - P_{p'+k} - P_{p-k})$$

and

$$(P_p)_{ik} \equiv p_i v_k - \frac{p \cdot v}{3} \delta_{ik}.$$
A.2 Thermal conductivity

\[ \frac{1}{K T \gamma^2} = \frac{27}{5 \pi^4} \alpha F_k, \]  

(A.3)

where

\[ F_k = \frac{1}{\Omega^2} \sum_{p,p',k} \rho_p \rho_{p'} \rho_{p-k} \rho_{p'+k} v_{p\nu} (v_{p\nu} - v_{p-k\nu}) \rho^4 \left( \frac{1}{\Omega} \sum_p \rho_p v_{p\nu} v_{p\nu} \right)^2. \]  

(A.4)

A.3 Spin-diffusion coefficient

\[ \frac{1}{D T^2 \gamma^3} = \frac{27}{16 \pi^3} (1 - A_0^1) (\alpha_{\uparrow\downarrow} F_d + \alpha_{\uparrow\uparrow} F), \]  

(A.5)

where

\[ F_d = \frac{1}{\Omega^2} \sum_{p,p',k} \rho_p \rho_{p'} \rho_{p-k} \rho_{p'+k} v_{p\nu} (v_{p\nu} + v_{p'\nu} - v_{p'+k\nu} - v_{p-k\nu}) \rho^4 \left( \frac{1}{\Omega} \sum_p \rho_p v_{p\nu} v_{p\nu} \right)^2, \]  

(A.6)

and \( F \) is defined in eq. (4.13).

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12. As the collision integral generally takes a finite value in the lattice system at finite temperature, the resulting Boltzmann equation breaks time reversal symmetry, despite the fact that we started with a microscopic model which has the symmetry. This is understood by noting (i) the system of electrons on a lattice is not closed in the presence of the Umklapp processes and (ii) we chose to use the retarded Green’s functions on physical grounds of causality.
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\( \varphi_{p' + k'} - \varphi_{p' - k'} \) in eq. (4.1) generally makes the coefficient \( A \) of \( \sigma^{-1} = AT^2 \) converge, with the exception of some singular cases as the system with a completely nested Fermi surface.

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