THERMODYNAMIC FORMALISM OF INTERVAL MAPS FOR
UPPER SEMI-CONTINUOUS POTENTIALS: MAKAROV AND
SMIRNOV’S FORMALISM

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ABSTRACT. In this paper, we study the thermodynamic formalism of interval maps \( f \) with sufficient regularity, for a subclass \( \mathcal{U} \) composed of upper semi-continuous potentials which includes both Hölder and geometric potentials. We show that for a given \( u \in \mathcal{U} \) and negative values of \( t \), the pressure function \( P(f, -tu) \) can be calculated in terms of the corresponding hidden pressure function \( \tilde{P}(f, -tu) \). Determination of the values \( t \in (-\infty, 0) \) at which \( P(f, -tu) \neq \tilde{P}(f, -tu) \) is also characterized explicitly. These results generalize the recent studies on the absence of phase transitions for the Hölder continuous potentials in [LRL13a, Theo B], and develop a real version of Makarnov-Smirnov’s formalism for geometric potentials, in parallel to the complex version shown in [MS00, Theo A, B]. Moreover, our results also provide a new and simpler proof (using [Rue92, Coro6.3]) of the original Makarnov-Smirnov’s formalism in the complex setting, under an additional assumption about non-exceptionality, i.e., [MS00, Theo 3.1].

1. Introduction

1.1. Thermodynamic formalism and Hyperbolicity. Thermodynamic formalism can be interpreted as a set of ideas and techniques derived from statistical mechanics. A systematic study of thermodynamic formalism of uniform hyperbolic smooth dynamical systems originates from the pioneer works of Sinai, Bowen and Ruelle [Bow75, Rue76, Sin72], and has various applications from dimension theory to number theory. One of the outstanding result is that the uniform hyperbolicity implies the real analyticity of the topological pressure function with Hölder continuous potentials, and thus yields the absence of phase transitions.

Recently, the extension of thermodynamic formalism results to one dimensional non-uniform hyperbolic \(^1\) dynamical systems has attracted great interests. Various direct or indirect approaches have been proposed to compensate the lack of uniform hyperbolicity in the dynamical system. Perhaps one of the most promising and direct methods is via the demonstration of the hyperbolicity in the potential. Suppose \( X \) is a compact metric space, and a continues map \( T: X \to X \). A upper semi-continuous potential \( \phi: X \to \mathbb{R} \cup \{-\infty\} \) is said to be hyperbolic, if there is an integer \( n \geq 1 \) such that the function \( S_n(\phi) := \sum_{j=0}^{n-1} \phi \circ T^j \) satisfies

\[
\sup_{X} \frac{1}{n} S_n(\phi) < P(T, \phi), \text{ where } P(T, \phi) \text{ is the topological pressure.}
\]

\(^1\) The non-uniform hyperbolicity is usually caused by the existence of critical points and neutral cycles.

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The notation of hyperbolic potential has been extensively used in a number of works include the studies of Hölder continuous potentials in both complex rational maps [Hay99, DPU96, DU91, Prz90, IRRL12] and real interval maps with sufficient regularity [LRL13b, LRL13a]. Let us mention that these studies also include the classical result of Lasota and Yorke [LY73] where $f$ is assumed to be piecewise $C^2$ and uniformly expanding, and $\phi = -\log |f'|$. Comparing with the inducing schemes approaches (see the survey [IT13, SUZ11] and the references therein), demonstrations of the hyperbolicity turn out to possess two advantages. On one hand, it is shown in [LRL13a, Theo A] that every Hölder continuous potential is hyperbolic provided that the interval map $f$ satisfies the weak regularity assumptions, that is all periodic points of $f$ are hyperbolic repelling and for every critical value of $f$,

$$\lim_{n \to \infty} |Df^n(v)| = +\infty.$$ 

We also refer to [IRRL12] for a complex version of the above property. On the other hand, hyperbolicity of the potentials directly yields that the corresponding transfer operators is bounded and quasi-compact, and admit spectral gaps in certain regular Banach spaces (e.g., Sobolev spaces [MS00] in the complex setting or Keller’s spaces [Kel85] in the real setting). By standard arguments (see e.g., [DU91, HR92, HK82, Prz90]) these properties imply the followings:

(a) the equilibrium state is unique and non-atomic, and has strong statistical properties (e.g., exponentially mixing, satisfying central limit theory and level-1 large deviation principles etc);

(b) the topological pressure is real analytic, and no phase transition appears.

Hyperbolicity of the potentials also plays an important role in studies on geometric potentials (i.e., $\phi := -t \log |Df|$, $t \in \mathbb{R}$). Note that every upper semi-continuous potential $\phi : X \to \mathbb{R} \cup \{-\infty\}$ has the following agreeable property (shown in Lemma 6.7): $\phi$ is hyperbolic if and only if

$$\sup_{\nu \in M(X,f)} \int \phi d\nu < P(T,\phi),$$

where $M(X,f)$ is the set of Borel invariant probability measures.

Put

$$\chi_{\text{inf}} := \inf_{\nu \in M(X,f)} \int \log |Df| d\nu, \quad \chi_{\text{sup}} := \sup_{\nu \in M(X,f)} \int \log |Df| d\nu,$$

and define

$$t_- := \inf \left\{ t \in \mathbb{R} : P(f, -t \log |Df|) + t\chi_{\text{sup}}(f) > 0 \right\};$$

$$t_+ := \sup \left\{ t \in \mathbb{R} : P(f, -t \log |Df|) + t\chi_{\text{inf}}(f) > 0 \right\}.$$

In the view of (1.2), $\log |Df|$ is hyperbolic between $t_-$ and $t_+$. Using this fact, Przytycki and Rivera-Leteier showed the real analyticity of the pressure function $P(f, -t \log |Df|)$ at $(t_, t_+)$ for complex rational maps [PRL11] and real interval maps [PRL13] respectively. We also highlight the work of Makarov and Smirnov [MS00]. They established an elegant expression of the pressure function at $t < 0$ in terms of hidden pressure and Lyapunov exponent of periodic points for arbitrary rational maps [MS00, Theo A], and provided an explicit characterization [MS00, Theo B] at which the number $t_-$ is finite. Their methods involve a refine study on the hyperbolicity of the potential, and are closed related to a former paper by
Ruelle [Rue92] (see also an unpublished note from Smirnov [Smi99]). To simplify the notation, we use a convention Makarnov-Smirnov’s formalism to stand for the statements of Theorem A and Theorem B in [MS00] henceforth. In this paper, we study further about the thermodynamic formalism for a subclass of upper semi-continuous potentials. In particular, we will generalize the studies on the hyperbolicity of Hölder continuous potentials, and obtain an analogous Makarov-Smirnov’s formalism, but in the setting of interval maps with sufficient regularity. We stress that the proof in [MS00, Theo A, B] using the Sobolev spaces to create spectral gap of the corresponding transfer operator can not be directly applied to our interval setting, as it heavily relies on the fact that a non-constant complex rational map is open, i.e., the open sets has open images. However, interval maps are not open in general, so the corresponding transfer operator is usually not invariant even under the space of continuous functions (see [RL15, Ex2.2]). Instead, our methodology will be closely related to the recent results developed in [IRRL12, LRL13b, LRL13a]. The difference on the proofs between real and complex setting will be discussed in §8.

Let us proceed with the exact definitions and statements.

1.2. Precise statement. We begin by briefly introducing the main objects. Let $I$ be a compact interval in $\mathbb{R}$. For a differentiable map $f : I \to I$, a point of $I$ is critical if the derivative of $f$ vanishes at it. We denote by $\text{Crit}(f)$ the set of critical points of $f$. We also denote by $J(f)$ the Julia set, which is the complement of the largest open subset of $I$ on which the family of iterates of $f$ is normal. Let $\text{Crit}'(f) := \text{Crit}(f) \cap J(f)$. Let also $\text{Per}(f)$ be the set of periodic points.

In what follows, we denote by $\mathcal{A}$ the collection of all non-injective differentiable maps $f : I \to I$ such that

- The critical set is finite;
- $Df$ is Hölder continuous;
- Each critical point $c \in \text{Crit}(f)$ is non-flat, i.e., there exist a number $\ell_c \geq 1$ and diffeomorphisms $\varphi$ and $\psi$ with $D\varphi, D\psi$ Hölder continuous, such that $\varphi(c) = \psi(f(c)) = 0$, and such that on a neighborhood of $c$ on $I$, we have
  $$|\psi \circ f| = \pm|\varphi|^{\ell_c};$$
  (Such $\ell_c$ is usually called the local order of the critical point).
- The Julia set $J(f)$ is completely invariant \footnote{In contrast with complex rational maps, the Julia set of an interval map might not completely invariant. However, it is possible to make an arbitrarily small smooth perturbation of $f$ outside a neighborhood, so that the Julia set of the perturbed map is completely invariant, and coincides with $J(f)$ correspondingly. We refer to [dMvS93] for more detailed background on the theory of Julia set in the real setting.}, i.e., $f(J) = f^{-1}(J) = J$, and contains at least two points;
- Every point $p \in \text{Per}(f)$ is hyperbolic repelling, i.e., every periodic point $p$ of periodicity $N$ has $|(f^N)'(p)| > 1$;
- $f$ is topologically exact on the Julia set $J(f)$, i.e., for each open set $U \in J(f)$, there exists an integer $n \geq 0$ such that $J(f) \subset f^n(U)$.

It is clear that the set $\mathcal{A}$ contains the family of smooth non-degenerated interval maps which are topologically exact on the $J(f)$ and have no neutral cycles. Note
that every map \( f \in \mathcal{A} \), the Julia set \( J(f) \) has no isolated point, and contains all interesting part of the dynamics.

Throughout the rest of the paper, for each \( f \in \mathcal{A} \), we restrict the action of \( f \) to its Julia set \( f|_{J(f)} : J(f) \to J(f) \).

On the other hand, let us recall some basic setting of thermodynamic formalism, referring the interested reader to [Kel98] or [PU11] for more information.

Let \((X, d)\) be a compact metric space, and \( T : X \to X \) be a continuous map with topological entropy \( h_{\text{top}}(T) < \infty \). Denote by \( \mathcal{M}(X) \) the space of Borel probability measures on \( X \) endowed with the weak* topology, and let \( \mathcal{M}(T, X) \) denote the subset of \( T \)-invariant ones. For each measure \( \nu \in \mathcal{M}(T, X) \), denote by \( h_{\nu}(T) \) the measure-theoretic entropy of \( \nu \).

Given a upper semi-continuous\(^3\) function \( \phi : X \to \mathbb{R} \cup \{-\infty\} \), the free energy of \( T \) for the potential \( \phi \) is defined as

\[
\mathcal{F}_{\nu}(T, \phi) := h_{\nu}(T) + \int_X \phi d\nu.
\]

The pressure function \( P(T, \phi) \) can be defined by means of the variational principle:

\[
P(T, \phi) := \sup_{\nu \in \mathcal{M}(T, X)} \mathcal{F}_{\nu}(T, \phi).
\]

An equilibrium state of \( T \) for the potential \( \phi \) is a measure \( \nu \in \mathcal{M}(T, X) \) which satisfies \( \mathcal{F}_{\nu}(T, \phi) = P(T, \phi) \). If the function \( \mu \to h_{\mu} \) upper semi-continuous under the weak* topology, then such equilibrium states exist.

The hidden pressure function \( \widetilde{P}(T, \phi) \) is obtained by restricting the admissible measures to the set \( \widetilde{\mathcal{M}} \) of all invariant non-atomic measures:

\[
\widetilde{P}(T, \phi) := \sup_{\nu \in \widetilde{\mathcal{M}}} \mathcal{F}_{\nu}(T, \phi).
\]

Since \( \widetilde{\mathcal{M}}(T, \phi) \) is not compact under weak* topology, the hidden pressure may or may not attain its supremum. If the former case occurs, we say a non-atomic measure \( \nu \in \widetilde{\mathcal{M}}(T, X) \) is a hidden equilibrium state if \( \mathcal{F}_{\nu}(T, \phi) = \widetilde{P}(T, \phi) \).

Given an interval map \( f : J(f) \to J(f) \) in \( \mathcal{A} \), denote by \( \text{USC}(J(f)) \) the set of all upper semi-continuous functions from \( J(f) \) to \( \mathbb{R} \cup \{-\infty\} \). In this paper, we are particularly interested in a subclass \( \mathcal{U} \subset \text{USC}(J(f)) \)

\[
\mathcal{U} := \left\{ u \in \text{USC}(J(f)) : u(x) = g(x) + \sum_{c \in \text{Crit}(f)} b_u(c) \log |x - c|, \right. \\
\left. \text{with } g \text{ Hölder continuous, and } b_u(c) \geq 0 \right\}
\]

It is clear that every Hölder or geometric potential belongs to the set \( \mathcal{U} \). For each \( u \in \mathcal{U} \), and \( \nu \in \mathcal{M}(f, J(f)) \), put \( \theta_{\nu} := \int_{J(f)} u d\nu \). In particular for each periodic point \( a \in \text{Per}(f) \) of periodicity \( n \), put \( \theta_{\nu} : = \frac{1}{n} \sum_{j=0}^{n-1} u \circ f^j(a) \), and

\[
\theta_{\max} := \sup \{ \theta_{\nu} : a \in \text{Per}(f) \}.
\]

Consider \( \theta_{\nu}, \theta_{a}, \theta_{\max} \) with \( u \) replaced by \( \log |Df| \). These respectively give \( \chi_{\nu}, \chi_{a}, \chi_{\max} \), which are called Lyapunov exponent (point-wise maximum).

\(^3\)A function \( \phi : X \to \mathbb{R} \cup \{-\infty\} \) is upper semi-continuous if the sets \( \{ y \in X : \phi(y) < c \} \) are open for each \( c \in \mathbb{R} \). Since \( X \) is compact, \( \sup \phi < +\infty \).
Our first main result concerning the behavior of the pressure function $P(f, -tu)$ for $t < 0$ is the following.

**Theorem A.** Let $f : J(f) \to J(f)$ be an interval map in $\mathcal{A}$, and $u : J(f) \to \mathbb{R} \cup \{-\infty\}$ be a upper semi-continuous potential in $\mathcal{U}$. If the following hypothesis satisfies,

(*) For each $t < 0$, if $\bar{P}(f, -tu)$ attains the supremum then every hidden equilibrium state $\mu$ of $f$ for the potential $-tu$ has strictly positive Lyapunov exponent.

then the hidden pressure function $\bar{P}(f, -tu)$ is real analytic on $(-\infty, 0)$, and

(1.4) \[ P(f, -tu) = \max\{\bar{P}(f, -tu), -t\theta_{\max}\}, \forall t < 0. \]

Analogous to the complex rational maps, the pressure function $P(f, -tu)$ may or may not be analytic, and the first possibility is easier to construct. If $P(f, -tu)$ is not real analytic on $(-\infty, 0)$, We say that $f$ has a phase transition. Our second main result gives an explicit characterization on the appearance of phase transitions. To state this, we will need the following definition. For each upper semi-continuous function $u \in \mathcal{U}$, put

(1.5) \[ \Lambda(u) := \{c \in \text{Crit}'(f) : b_u(c) > 0\}, \]

and $u$ is said to be exceptional for $f$, if there is a non-empty forward invariant finite subset $\Sigma \subset J(f)$, satisfying

(1.6) \[ f^{-1}(\Sigma) \setminus \Sigma \subset \Lambda(u). \]

Such finite set $\Sigma$ is called a $\Lambda(u)$-exceptional set, or simply an exceptional set. If no such exceptional set exists, then the potential $u$ is said to be non-exceptional. Analogous to that in the complex setting, denote by $\Sigma_{\max}$ the union of all exceptional sets. We highlight that $\Sigma_{\max}$ will be shown as an exceptional set and contains at least one periodic orbit in Corollary 2.4. Therefore, we can define

\[ \theta_* := \max\{\theta_a : a \in \Sigma_{\max} \cap \text{Per}(f)\}, \chi_* := \max\{\chi_a : a \in \Sigma_{\max} \cap \text{Per}(f)\}. \]

In fact, we will provide a universal bound on the cardinality of $\Sigma_{\max}$ (see Corollary 2.4 and the remarks 2.5 and 2.6 afterwards) in terms of the cardinality of $\text{Crit}'(f)$ for every map $f$ in $\mathcal{A}$. This is a more refined estimation than that in [PRL13], and can be viewed as a parallel property to that in the complex setting: every exceptional set of a rational function has at most 4 elements [MS96, MS00, GPRRL13]. We believe this refined estimation will have its independent interest.

The second main result is as follows.

**Theorem B.** Let $f : J(f) \to J(f)$ be an interval map in $\mathcal{A}$, and $u : J(f) \to \mathbb{R} \cup \{-\infty\}$ be a upper semi-continuous potential in $\mathcal{U}$. If hypothesis (*) satisfies, then the pressure function $P(f, -tu)$ has a phase transition on $(-\infty, 0)$ if and only if $u$ is exceptional and

\[ \theta_* > \sup\{\theta_\nu : \nu \in \mathcal{M}(f, J(f)), \nu(\Sigma_{\max}) = 0\}. \]
1.3. Reductions. Before stating our strategy of the proof, we discuss a few interesting corollaries derived from Theorem A and Theorem B.

On one hand, reducing the potential \( u \) to be a Hölder continuous potential, we have

**Corollary 1.1.** Let \( f : J(f) \to J(f) \) be an interval map in \( A \), and \( -tu : J(f) \to \mathbb{R} \) be a hyperbolic Hölder continuous potential for every \( t < 0 \), then \( P(f, -tu) \) has no phase transitions and equals to \( \tilde{P}(f, -tu) \) on \( (-\infty, +\infty) \).

This corollary is actually the main theorem shown in \([LRL13a, Theo B]\), and we state below the proof briefly. Firstly, note that by our choice of \( f \in A \), the function \( \mu \to h_\mu(f) \) is upper semi-continuous, so the equilibrium states of \( f \) for \( -tu \) with \( t < 0 \) always exists. Note also every measure supported on a periodic orbit has positive Lyapunov exponent, thus hypothesis (*) is equivalent to the assumption that every equilibrium state has positive Lyapunov exponent. On the other side, following from \([Li14, Theo 1]\), the latter assumption is equivalent to the assumption that the potential \( -tu \) is hyperbolic. Secondly, for every Hölder continuous potential \( u \), we have \( \Lambda(u) = \emptyset \), and thus \( u \) is non-exceptional. Therefore, Theorem A and Theorem B yield Corollary 1.1 for \( t \leq 0 \). As readers will notice later, since \( -tu \) is also Hölder continuous for the positive values of \( t \), the same proof also works for the remaining part. Hence we obtain Corollary 1.1 completely. We highlight that the non-flatness hypothesis of critical points in the definition of class \( A \) is not needed for the validity of Corollary 1.1, but it will be fundamental for the proof of the general upper semi-continuous potential in \( U \).

On the other hand, reducing the potential \( u \) to be the geometric potential \( -t \log |Df| \), we will show the hypothesis (*) is automatically satisfied for every \( t < 0 \) (see Lemma 6.2). Thus Theorem A and Theorem B are reduced to the Makarov-Smirnov’s formalism for interval maps. Form now on, put \( P(t) \equiv P(f, -t \log |Df|) \) and \( \tilde{P}(t) \equiv P(f, -t \log |Df|) \) for every \( t < 0 \).

**Corollary 1.2 (A real version of Makarov-Smirnov’s formalism).** Let \( f : J(f) \to J(f) \) be an interval map in \( A \), then

(a) the hidden pressure \( \tilde{P}(t) \) is real analytic on \( (-\infty, 0) \), and

\[
P(t) = \max\{\tilde{P}(t), -t \chi_{\max}\}, \quad \forall t < 0;
\]

(b) \( P(t) \) has a phase transition on \( (-\infty, 0) \), if and only if \( \log |Df| \) is exceptional and satisfies

\[
\chi_* > \sup\{\chi_\nu : \nu \in \mathcal{M}(f, J(f)), \ \nu(\Sigma_{\max}) = 0\}.
\]

In particular, if \( \log |Df| \) is non-exceptional, then \( P(t) \) equals to \( \tilde{P}(t) \) for every \( t < 0 \), and has no phase transition on \( (-\infty, 0)^4 \).

In what follows, we consider only the proof of Corollary 1.2 directly. One of our novelties in our proof is we show that for every geometric potential, the non-exceptionality implies the its hyperbolicity. This fact allows some of the arguments or statements shorter and simpler. As readers will notice, there is no difficulty in

\[4\text{The last assertion in Statement (b) is also proved by [IT10, Theo B] and [GPR14, §2.2], but it seems that our hypothesis is weaker and our approach is very different from theirs.}\]
extending our proof of Corollary 1.2 to general upper semi-continuous potential in $\mathcal{U}$ and obtaining Theorem A and Theorem B.

1.4. Proof strategy. Let us describe briefly about our proof strategy of Corollary 1.2. The proof actually unifies and adapts several machineries used in [MS00, IRRL12, LRL13b, LRL13a]. To be more precise, we first take a co-homology transformation on the geometric potential and identify the so-called “Key Lemma” as follows.

**Theorem C** (Key Lemma). Let $f : J(f) \to J(f)$ be an interval map in $\mathcal{A}$, then there exist $G, -h \in \mathcal{U}$ such that

$$G = \log |f'| + h \circ f - h,$$

and such that

(a) $\bar{P}(f, -tG) = \bar{P}(t), \forall t < 0$;

(b) For each ergodic invariant probability measure $\mu$ with strictly positive Lyapunov exponent, there is a full $\mu$-measure subset $X \subset J(f)$ such that

$$\limsup_{n \to \infty} \frac{1}{n} \log \sum_{y \in f^{-n}(x_0)} \exp(S_n(-tG)(y)) > \int_{J(f)} -tGd\mu, \forall x_0 \in X, \forall t < 0.$$

**Remark 1.3.** We remark that if further assuming $\log |Df|$ is non-exceptional, then $G = \log |Df|$ and $h = 0$, and the non-flatness hypothesis of the critical points in the definition of class $\mathcal{A}$ is not required for the validity of Key Lemma.

Back to the strategy description, on one hand, Statement (a) transfers the studies on the pressure information from $\log |Df|$ to the new upper semi-continuous potential $G$. On the other side, Statement (b) is used to deduce the hyperbolicity of $G$. After obtaining the hyperbolicity, we can use the Patterson-Sullivan method to construct a non-atomic conformal measure, and can apply a certain Keller’s space to create a spectral gap for the corresponding transfer operator induced by the new potential $G$. These properties will be the main ingredients on deducing the assertions of Corollary 1.2.

We also remark that some regularity restrictions about the potentials are required on applying the Patterson-Sullivan construction and Keller’s result. In fact, this is the part in the proof where the potential $u$ is required to be in the subclass $\mathcal{U}$. Currently, we are unable to obtain Corollary 1.2 for arbitrary upper semi-continuous potentials.

1.5. Main ideas of the proofs and organization of the paper. The rest of the paper is organized as follows. In section 2, we discuss a few properties about the normality and their relations, which are basic to the argument that follow. Section 3, 4 and 5 are devoted to the proof of Key Lemma. In Section 3, we study in detail about the co-homology transformation. Based on this, we provide an explicit construction on the new potential $G$, and prove Statement (a). In Section 4, a refined iterated multi-valued function system is constructed in order to prove Statement (b) for the case where $\mu$ is non-atomic. The rest case where $\mu$ is atomic is dealt in Section 5. We use Key Lemma in Section 6 to show the hyperbolicity of $G$.

As a contrast, it is however unknown whether there is an analogous complex version of Theorem A and Theorem B, although the original Makarov-Smirnov’s formalism in the complex setting is obtained. The reasons will be explained in §8.
the new potential $G$, and construct a non-atomic conformal measure. These results are used in Section 7 to construct certain Keller’s spaces, where the corresponding transfer operators acting on have spectral gaps. Thus the hidden pressure function has no phase transition. In the final section, we discuss a few relations with the Makarov-Smirnov’s formalism in the complex setting under the view of our Key Lemma. In particular, we reprove [MS00, Theo 3.1] solely via $BV_2$ functional spaces, and conjecture that our result might be useful on reobtaining the original Makarov-Smirnov’s formalism by an alternative proof without introducing Sobolev spaces, provide that we can obtain a complex version of our Theorem A and B. The appendix is devoted to provide some basic background on Keller’s spaces which are introduced by [Kel85] (see also [RL15]).

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2. Normality

The main goal of this section is to show Proposition 2.1 below, which will be of great importance in the proof of Key Lemma substantially.

Given an interval map $f \in \mathcal{A}$, and a subset $\Lambda \subseteq \text{Crit}'(f)$, a point $x \in J(f)$ is $\Lambda$-normal or simply normal, if for any integer $n \geq 1$, there is a pre-image $y$ of $x$ by $f^n$, such that

\begin{equation}
\{y, f(y), \ldots, f^{n-1}(y)\} \cap \Lambda = \emptyset.
\end{equation}

Otherwise, this point $x$ is said to be $\Lambda$-abnormal or simply abnormal.

**Proposition 2.1.** Let $f : J(f) \to J(f)$ be an interval map in $\mathcal{A}$, and $\Lambda$ be a subset of $\text{Crit}'(f)$. If every non-empty forward invariant finite set $\Sigma$ satisfies

\begin{equation}
f^{-1}(\Sigma) \setminus \Sigma \not\subseteq \Lambda,
\end{equation}

then for each point $x \in J(f)$, there is an integer $N \geq 0$ such that $f^N(x) \not\in \Lambda$ and $f^N(x)$ is $\Lambda$-normal.

In addition, if further assuming $x$ is periodic, then $x$ itself is neither in $\Lambda$ nor $\Lambda$-normal.

The proof of Proposition 2.1 depends on several Lemmas, and will be given in the end of this section.

For brevity, given a map $\pi : \Xi \to \hat{\Xi}$, denote by $\mathfrak{K}$ the rank 1 pre-image set, $\mathfrak{K} := \{\xi \in \Xi : \sharp \pi^{-1}(\xi) = 1\}$, where $\sharp A$ means the cardinality of a finite set $A$.

**Lemma 2.2.** Let $\hat{\Xi}$ be a finite set, $\Xi$ a subset of $\hat{\Xi}$, $\pi : \Xi \to \hat{\Xi}$ a map, and $\mathfrak{K}$ is the resulting rank 1 pre-image set, then

\begin{equation}
\sharp(\Xi \cap \pi(\Xi)) \leq 3\sharp(\Xi \setminus \pi(\Xi)) + \sharp(\mathfrak{K} \cap \pi(\mathfrak{K})).
\end{equation}

**Proof.** We proceed by induction on the cardinality of $\Xi$. The case $\sharp \Xi = 1$ is trivial. Let $n \geq 2$ be an integer such that the desired inequality is satisfied for every set of cardinality $n - 1$. Let $\Xi$ be a set of cardinality $n$, and let $\pi$ and $\mathfrak{K}$ be as in the statement of the lemma.
For each $\xi$ in $\Xi$ we define an integer $n(\xi)$ as follows: if $\xi$ is in $\hat{\Xi}\setminus\Xi$, or if $\xi$ is periodic for $\pi$, put $n(\xi) = 0$; if $\xi$ is in $\Xi$ and is not periodic for $\pi$, let $n(\xi)$ be the least integer $n \geq 1$ such that $\pi^n(\xi)$ is in $\hat{\Xi}\setminus\Xi$ or is periodic for $\pi$. Moreover, put
\[ n(\Xi) := \max \{ n(\xi) : \xi \in \Xi \} . \]

If $n(\Xi) = 0$, then every point of $\Xi$ is periodic for $\pi$, so $\pi(\Xi) = \Xi$, and therefore $\hat{\Xi} = \Xi$. So the desired inequality is trivial in this case.

Suppose $n(\Xi) \geq 1$, let $\tilde{\xi}_0$ in $\Xi$ be such that $n(\tilde{\xi}_0) = n(\Xi)$, and put $\tilde{\xi} := \pi(\tilde{\xi}_0)$. There are 3 cases.

Case 1. $n(\Xi) = 1$. Then $\tilde{\xi}$ is in $\hat{\Xi}\setminus\Xi$, or is periodic for $\pi$. In both cases the set
\[ \Delta := \{ \xi \in \pi^{-1}(\tilde{\xi}) : n(\xi) = 1 \} \]
is nonempty and disjoint from $\pi(\Xi)$. If $\Delta$ is equal to $\Xi$, then $\pi(\Xi) \cap \Xi$ is empty and the desired inequality is trivial in this case. So we assume that the set $\hat{\Xi} := \Xi \setminus \Delta$ is nonempty. Put
\[ \hat{\pi} := \pi|_{\hat{\Xi}} \text{ and } \hat{\mathbf{X}} := \{ \xi \in \hat{\Xi} : \hat{\pi}^{-1}(\xi) = 1 \} . \]

Clearly, we have
\[ \hat{\Xi} \cap \hat{\pi}(\hat{\Xi}) = \Xi \cap \pi(\Xi) . \]

On the other hand, if $\tilde{\xi}$ is in $\hat{\Xi}\setminus\Xi$, then $\hat{\mathbf{X}} = \hat{\mathbf{X}}$, and if $\tilde{\xi}$ is periodic for $\pi$, then $\hat{\mathbf{X}} = \hat{\mathbf{X}} \cup \{ \xi \}$, and therefore
\[ \hat{\mathbf{X}} \cap \hat{\pi}(\hat{\mathbf{X}}) = \left( \hat{\mathbf{X}} \cup \{ \tilde{\xi} \} \right) \cap \left( \pi(\hat{\mathbf{X}}) \cup \{ \pi(\tilde{\xi}) \} \right) \subset \left( \hat{\mathbf{X}} \cap \pi(\hat{\mathbf{X}}) \right) \cup \{ \xi, \pi(\tilde{\xi}) \} \]
In both cases we have
\[ \sharp(\hat{\mathbf{X}} \cap \hat{\pi}(\hat{\mathbf{X}})) \leq \sharp(\mathbf{X} \cap \pi(\mathbf{X})) + 2. \]
So, by the induction hypothesis and the fact that $\Delta$ is nonempty and disjoint from $\pi(\Xi)$, we have
\[ \sharp(\Xi \cap \pi(\Xi)) = \sharp(\hat{\Xi} \cap \hat{\pi}(\hat{\Xi})) \leq 3\sharp(\hat{\Xi} \setminus \hat{\pi}(\hat{\Xi})) + \sharp(\hat{\mathbf{X}} \cap \hat{\pi}(\hat{\mathbf{X}})) \leq 3\sharp(\Xi \setminus \pi(\Xi)) - \sharp(\Delta) + \sharp(\mathbf{X} \cap \pi(\mathbf{X})) + 2 \leq 3\sharp(\Xi \setminus \pi(\Xi)) + \sharp(\mathbf{X} \cap \pi(\mathbf{X})) - \sharp(\Delta) . \]
This implies the desired inequality for $\pi$, and completes the proof of the induction step in the case $n(\Xi) = 1$.

Case 2. $n(\Xi) \geq 2$ and $\tilde{\xi} \notin \hat{\mathbf{X}}$. In this case the set $\pi^{-1}(\tilde{\xi})$ is disjoint from $\pi(\Xi)$ and the set $\hat{\Xi} := \Xi \setminus \pi^{-1}(\tilde{\xi})$ is nonempty. On the other hand, if we put
\[ \hat{\pi} := \pi|_{\hat{\Xi}} \text{ and } \hat{\mathbf{X}} := \{ \xi \in \hat{\Xi} : \hat{\pi}^{-1}(\xi) = 1 \} , \]
then
\[ \hat{\pi}(\hat{\Xi}) = \pi(\Xi) \setminus \{ \tilde{\xi} \} \text{ and } \hat{\mathbf{X}} = \hat{\mathbf{X}} . \]
Therefore we have
\[ \hat{\Xi} \cap \hat{\pi}(\hat{\Xi}) = (\Xi \cap \pi(\Xi)) \setminus \{ \tilde{\xi} \} , \]
\[ \hat{\Xi} \setminus \hat{\pi}(\hat{\Xi}) = \left( \Xi \setminus \pi^{-1}(\tilde{\xi}) \right) \setminus \left( \pi(\Xi) \setminus \{ \tilde{\xi} \} \right) = \left( (\Xi \setminus \pi(\Xi)) \cup \{ \tilde{\xi} \} \right) \setminus \pi^{-1}(\tilde{\xi}) , \]
and therefore
\[ \sharp \left( \hat{\Xi} \setminus \hat{\pi}(\hat{\Xi}) \right) = \sharp(\Xi \setminus \pi(\Xi)) - (\sharp(\pi^{-1}(\tilde{\xi}) - 1) . \]
So, by the induction hypothesis we have
\[
\sharp(\Xi \cap \pi(\Xi)) - 1 = \sharp (\hat{\Xi} \cap \hat{\pi}(\hat{\Xi})) \\
\leq 3\sharp (\hat{\Xi} \setminus \hat{\pi}(\hat{\Xi})) + \sharp (\hat{\Xi} \cap \hat{\pi}(\hat{\Xi})) \\
= 3\sharp (\Xi \setminus \pi(\Xi)) - 3(\sharp \pi^{-1}(\xi) - 1) + \sharp (\hat{\Xi} \cap \hat{\pi}(\hat{\Xi})) \\
\leq 3\sharp (\Xi \setminus \pi(\Xi)) + \sharp (\hat{\Xi} \cap \hat{\pi}(\hat{\Xi})) - 3
\]
This implies the desired inequality for \(\pi\), and completes the proof of the induction step in the case \(n(\Xi) \geq 2\) and \(\xi\) is not in \(\hat{\Xi}\).

**Case 3.** \(n(\xi) \geq 2\) and \(\xi \in \hat{\Xi}\). In this case \(\hat{\xi}_0\), defined above, is the unique element of \(\pi^{-1}(\xi)\). Note that \(\pi^{n(\hat{\xi}_0)}(\hat{\xi}_0)\) is in \(\hat{\Xi} \setminus \Xi\) or is periodic for \(\pi\). In particular, it is not in \(\hat{\Xi}\). It follows that there is at least an integer \(n \geq 2\) such that \(\pi^n(\hat{\xi}_0)\) is not in \(\hat{\Xi}\). Put
\[
\Delta := \{\pi^j(\hat{\xi}_0) : j \in \{0, 1, \ldots, n-1\}\},
\hat{\Xi} := (\Xi \setminus \pi(\Xi)) \setminus (\Delta \setminus \{\hat{\xi}_0\}),
\hat{\pi} := \{\xi \in \hat{\Xi} : \sharp \pi^{-1}(\xi) = 1\},
\]
and note that
\[
\hat{\Xi} \cap \hat{\pi}(\hat{\Xi}) = (\Xi \cap \pi(\Xi)) \setminus (\Delta \setminus \{\hat{\xi}_0\}),
\hat{\Xi} \setminus \hat{\pi}(\hat{\Xi}) = (\Xi \setminus \pi(\Xi)) \setminus \{\hat{\xi}_0\},
\]
and
\[
(\hat{\Xi} \cap \hat{\pi}(\hat{\Xi})) \cap \Delta = \{\pi^j(\hat{\xi}_0) : j \in \{2, \ldots, n-1\}\}.
\]
Suppose \(n = n(\hat{\xi}_0)\). If \(\hat{\Xi}\) is empty, then we have
\[
\Xi \cap \pi(\Xi) = \hat{\Xi} \cap \pi(\hat{\Xi}) = \Delta \setminus \{\hat{\xi}_0\},
\]
and the desired inequality holds in this case. If \(\hat{\Xi}\) is nonempty, then we have
\[
\hat{\Xi} = \hat{\Xi} \setminus (\Delta \setminus \{\hat{\xi}_0\}), \quad \text{and} \quad \hat{\Xi} \cap \hat{\pi}(\hat{\Xi}) = (\hat{\Xi} \cap \pi(\hat{\Xi})) \setminus (\Delta \setminus \{\hat{\xi}_0\})
\]
So, by the induction hypothesis we have
\[
\sharp(\Xi \cap \pi(\Xi)) - (n - 1) = \sharp (\hat{\Xi} \cap \hat{\pi}(\hat{\Xi})) \\
\leq 3\sharp (\hat{\Xi} \setminus \hat{\pi}(\hat{\Xi})) + \sharp (\hat{\Xi} \cap \hat{\pi}(\hat{\Xi})) \\
= 3\sharp (\Xi \setminus \pi(\Xi)) - 3(n - 1) + \sharp (\hat{\Xi} \cap \hat{\pi}(\hat{\Xi})) - (n - 1) \\
\leq 3\sharp (\Xi \setminus \pi(\Xi)) + \sharp (\hat{\Xi} \cap \hat{\pi}(\hat{\Xi})) - 4(n - 1).
\]
This implies the desired inequality when \(n = n(\hat{\xi}_0)\).

It remains to consider the case \(n \leq n(\hat{\xi}_0) - 1\). In this case \(\hat{\Xi}\) is nonempty, and we have
\[
\hat{\Xi} \subset (\hat{\Xi} \cap \hat{\Xi}) \cup \{\pi^n(\hat{\xi}_0)\},
\]
and
\[
\hat{\Xi} \cap \hat{\pi}(\hat{\Xi}) \subset (\hat{\Xi} \cap \pi(\hat{\Xi}) \cap \pi(\Xi)) \cup \{\pi^n(\hat{\xi}_0), \pi^{n+1}(\hat{\xi}_0)\}
\]
\[= (\hat{\Xi} \cap \pi(\hat{\Xi}) \setminus \{\pi^j(\hat{\xi}_0) : j \in \{2, \ldots, n-1\}\}) \cup \{\pi^n(\hat{\xi}_0), \pi^{n+1}(\hat{\xi}_0)\}
\]
and therefore
\[
\sharp (\hat{\Xi} \cap \hat{\pi}(\hat{\Xi})) \leq \sharp (\hat{\Xi} \cap \pi(\hat{\Xi})) - n + 4.
\]
So, by the induction hypothesis we have
\[ \sharp(\Xi \cap \pi(\xi)) - (n - 1) = \sharp(\Xi \cap \pi(\xi)) \]
\[ \leq 3\sharp(\Xi \cap \pi(\xi)) + \sharp(\pi(\Xi)) \]
\[ \leq 3\sharp(\Xi \setminus \pi(\Xi)) - 3 + \sharp(\pi(\Xi)) \cdot n \]
\[ \leq 3\sharp(\Xi \cap \pi(\Xi)) + \sharp(\pi(\Xi)) - (n - 1). \]

This implies the desired inequality for \( \pi \), and completes the proof of the induction step in the case \( n(\Xi) \geq 2 \) and \( \xi \) is in \( \pi \). The proof of the lemma is thus complete.

\[ \square \]

**Lemma 2.3.** Let \( f : J(f) \to J(f) \) be a continuous interval map with \( \pi \) the resulting rank 1 pre-image set. Put
\[ a_{\max} := \max_{x \in J(f)} \{ f(x) \} \quad \text{and} \quad a_{\min} := \min_{x \in J(f)} \{ f(x) \}. \]

If \( f \) is topological exact on \( J(f) \), then we have
\[ f(\pi) \cap \pi \subseteq \{ a_{\max}, a_{\min}, f(a_{\max}), f(a_{\min}) \}. \]

**Proof.** We prove the lemma by contradiction. Suppose on the contrast that there is a point \( x \in (f(\pi) \cap \pi) \setminus \{ a_{\max}, a_{\min}, f(a_{\max}), f(a_{\min}) \} \). By the definition of \( \pi \), there exist a unique pre-image \( y \in \pi \setminus \{ a_{\max}, a_{\min} \} \) with \( f(y) = x \), and a unique pre-image \( z \) with \( f(z) = y \). We first show that \( x, y, z \) are pairwise distinct. Otherwise, if say \( x = y \) then \( x \) is a fixed point and its unique pre-image is itself. Hence \( \cup_{i=1}^{n} f^{-i}(x) = \{ x \} \), which is a contradiction to the topological exactness. The remaining cases “\( y \neq z \)” and “\( x \neq z \)” are similarly supplied.

Next, we show that
\[ (a) \quad x \text{ must be an extreme value of both } f|_{[0,y]} \text{ and } f|_{[y,1]}; \]
\[ (b) \quad x \text{ is a maximum (resp. minimum) of } f|_{[0,y]}, \text{ if and only if } x \text{ is a minimum (resp. maximum) of } f|_{[y,1]}. \]

For Statement (a), we only prove \( f \) is an extreme value of \( f|_{[0,y]} \), the other case is similarly supplied. Suppose on the contrast that \( x \) is neither a maximum nor a minimum at \( f|_{[0,y]} \). Using intermediate value theorem and the complete invariance of \( J(f) \), there is another point \( d \in J \) differs from \( y \) such that \( f(d) = f(y) = x \), which is a contradiction to \( x \) being rank 1-pre-image. For Statement (b), notice that \( x \notin \{ a_{\max}, a_{\min} \} \), so \( x \) is neither a maximum nor a minimum of both \( f|_{[0,y]} \) and \( f|_{[y,1]} \) simultaneously. Analogously, we have
\[ (a') \quad y \text{ must be an extreme value of both } f|_{[0,z]} \text{ and } f|_{[y,1]}; \]
\[ (b') \quad y \text{ a maximum (resp. minimum) of } f|_{[0,z]}, \text{ if and only if } y \text{ is a minimum (resp. maximum) of } f|_{[y,1]}. \]

In the rest of the proof, we will show that \( f \) has a proper forward invariant open subset in \( J(f) \). This is a contradiction to the topological exactness, and the desired assertion follows. In order to do that, we need to distinguish the order of \( x, y, z \) on the real line. There are two possibilities, namely Case “\( z < y \)” and Case “\( y < z \)”.

Without loss of generality, it is sufficient to prove Case “\( z < y \)”.

Using the hypothesis “\( z < y \)” , we have
\[ J = ([a_{\min}, z] \cap J) \cup ([z, y] \cap J) \cup ([y, a_{\max}] \cap J). \]
In the following, we distinguish two cases.

**Case 1.** \(x\) is a minimum of \(f|_{[0,y]}\). Thus \(x\) must be a maximum of \(f|_{[0,z]}\) and a maximum of \(f|_{[z,1]}\). This claim can be proved by contradiction. Otherwise, suppose \(y\) is a maximum of \(f|_{[0,z]}\) and a minimum of \(f|_{[z,1]}\). Due to our hypothesis \(z < y\), we have \([0,z] \subset [0,y]\) and \([y,1] \subset [z,1]\). However, the former statement yields \(x < y\), while the latter statement yields \(x > y\), which is evidently impossible. Therefore, we obtain the claim, and thus we have \(x < y\). In other words \(x\) must be outside \([y,a_{\text{max}}] \cap J\).

There are further two possibilities.

**Subcase 1.** \(x \in [z,y] \cap J\). Then \(x\) must be a minimum of \(f|_{[z,y]}\) and \(y\) must be a maximum of \(f|_{[z,y]}\). Hence,

\[
f([z,y] \cap J) \subseteq f([z,y]) \cap J = [x,y] \cap J \subseteq [z,y] \cap J.
\]

This implies that \([z,y] \cap J\) is a proper open subset of \(J\) forward invariant under \(f\);

**Subcase 2.** \(x \in [a_{\text{min}},z] \cap J\). Then \(y\) must be a minimum of \(f|_{[0,z]}\) and \(x\) must be a maximum of \(f|_{[y,1]}\). Hence,

\[
f^2([a_{\text{min}}, z] \cap J) \subseteq f([y,a_{\text{max}}] \cap J) \subseteq [a_{\text{min}}, x] \cap J \subseteq [a_{\text{min}}, z] \cap J.
\]

This also implies that \([a_{\text{min}}, z] \cap J\) is a proper open subset of \(J\) forward invariant under \(f^2\).

In both subcases, \(f\) has a proper open and forward invariant subset of \(J\).

**Case 2.** \(x\) is a maximum of \(f|_{[0,y]}\). Thus \(x\) must be a minimum of \(f|_{[y,1]}\). Followed by similarly arguments as above, \(y\) must be a maximum of \(f|_{[0,z]}\) and a minimum of \(f|_{[z,1]}\). Hence \(y < x\). In other words, \(x\) must be in \([y,a_{\text{max}}] \cap J\). Therefore,

\[
f([y,a_{\text{max}}] \cap J) \subseteq [x,a_{\text{max}}] \cap J \subseteq [y,a_{\text{max}}] \cap J.
\]

This implies that \([y,a_{\text{max}}] \cap J\) is a proper open subset of \(J\) forward invariant under \(f\).

As a conclusion, we show that \(f\) has a proper open forward invariant subset of \(J(\Lambda)\) in both cases, which is a contradiction to the topological exactness on \(J(\Lambda)\). This contradiction yields the desired assertion (2.4), and completes the proof of this lemma.

\[\square\]

We are now ready to prove Proposition 2.1.

**Proof of Proposition 2.1.** The proof is split into 2 parts. In part 1, put

\[
S_\Lambda := \{x \in J(f), x \text{ is } \Lambda\text{-abnormal}\},
\]

and we show that

\[
\sharp S_\Lambda \leq 3\sharp \Lambda + 4.
\]

In part 2, we use (2.6) to show the desired assertions of the Proposition 2.1.

1. We first proceed by contradiction to show that

\[
f^{-1}(S_\Lambda) \setminus S_\Lambda \subset \Lambda.
\]
Otherwise, there is a point \( x \in f^{-1}(S_\Lambda) \setminus S_\Lambda \), but \( x \notin \Lambda \). This implies that \( x \notin S_\Lambda \cup \Lambda \), but \( f(x) \in S_\Lambda \). However, this is a contradiction to the abnormality of \( x \). Thus we obtain (2.7).

On the other hand, put \( \Xi := S_\Lambda \cup f^{-1}(S_\Lambda) \). So

\[
S_\Lambda \subset \Xi \cap f(\Xi) \text{ and } \Xi \setminus f(\Xi) \subset f^{-1}(S_\Lambda) \setminus S_\Lambda.
\]

Recall \( \mathcal{X} \) to be the rank 1 pre-image set of \( f \). Therefore,

\[
\sharp S_\Lambda \leq \sharp(\Xi \cap f(\Xi)) \quad \text{(Using (2.8))}
\]

\[
\leq 3\sharp(\Xi \setminus f(\Xi)) + \sharp(\mathcal{X} \cap f^{-1}(\mathcal{X})) \quad \text{(Using Lemma 2.2)}
\]

\[
\leq 3\sharp f^{-1}(S_\Lambda) \setminus S_\Lambda \leq 3\sharp \Lambda + 4 \quad \text{(Using (2.8) and Lemma 2.3)}
\]

which complete the proof of (2.6).

2. In this part, we first show the following claim.

**Claim 1.** For every \( x \in J(f) \), there is a non-negative integer \( N \), such that \( f^N(x) \) is \( \Lambda \)-normal.

We proceed the proof of Claim 1 by contradiction. Suppose on the contrast that there is a point \( x \in J(f) \), such that the forward orbit \( O_x := \{ f^n(x) \}_{n=0}^{\infty} \subset S_\Lambda \). Following from (2.6) in Part 1, the orbit \( O_x \) must be pre-periodic and finite. Put

\[
A_x := \{ y \in S_\Lambda : \exists m \geq 0, \text{ such that } f^m(y) \in O_x \}.
\]

In order to obtain Claim 1, it is sufficient to verify that \( A_x \) is non-empty, forward invariant finite and satisfies

\[
f^{-1}(A_x) \setminus A_x \subset \Lambda.
\]

This is a contradiction to the hypotheses (2.2), and thus Claim 1 follows. The non-empty, forward invariance and finiteness of \( A_x \) are straightforward from its definition. To verify (2.9), suppose on the contrast that there is a point \( z \in f^{-1}(A_x) \setminus A_x \), but \( z \notin \Lambda \). This means that \( z \notin A_x \cup \Lambda \), but \( f(z) \notin A_x \). Hence \( z \notin S_\Lambda \), and thus \( f(z) \notin S_\Lambda \), which is evidently impossible. This contradiction completes the proof of (2.9), and thus completes Claim 1.

Next, we show that the integer \( N \geq 0 \) stated in Claim 1 can be further adapted such that \( f^N(x) \notin \Lambda \). There are two cases.

**Case 1:** \( x \) is pre-periodic. Then let integer \( N \geq 0 \) be the smallest integer such that \( f^N(x) \) being periodic. Note that \( \Lambda \subset \text{Crit}(f) \) and there is no periodic critical point in \( J(f) \), thus \( f^N(x) \notin \Lambda \). Moreover, we further claim that \( f^N(x) \) is \( \Lambda \)-normal. In fact, following from Claim 1, there is an integer \( \hat{N} \geq 0 \) such that \( f^{\hat{N}}(x) \) is periodic and \( \Lambda \)-normal. Then \( f^{\hat{N}+1}(x) \) is also \( \Lambda \)-normal. By periodicity, \( f^N(x) \) must be \( \Lambda \)-normal.

**Case 2:** \( x \) is non-periodic. Then there is a subsequence \( \{ n_i \} \) such that each \( f^{n_i}(x) \) is \( \Lambda \)-normal, and \( f^{n_i}(x) \neq f^{n_j}(x), \forall i \neq j \). By the finiteness of \( \Lambda \), there is an integer \( N \geq 0 \), such that \( f^N(x) \) is \( \Lambda \)-normal and not in \( \Lambda \).

In both cases, we obtain the desired assertions. Thus the proof of the Lemma is completed. \( \square \)
We close this section mentioning the corollary below that perhaps have independent interests. For each interval map \( f \in \mathcal{A} \), denote by \( \Sigma_{\text{max}} \) the maximum exceptional set,

\[
\Sigma_{\text{max}} := \bigcup \{ \Sigma \subset J(f) : \Sigma \text{ is Crit}'(f)\)-exceptional \}.
\]

**Corollary 2.4.** Let \( \mathcal{A}_n := \{ f \in \mathcal{A}, \text{ Crit}'(f) \leq n \} \), then

\[
\sup_{f \in \mathcal{A}_n} \sharp \Sigma_{\text{max}}(f) \leq 3n + 4.
\]

**Proof.** Given an interval map \( f \) in \( \mathcal{A}_n \), on one hand, it is straightforward to see that each \( \text{Crit}'(f) \)-exceptional set \( \Sigma \) is contained in \( S_{\text{Crit}'(f)} \). On the other hand, for every two \( \text{Crit}'(f) \)-exceptional sets \( \Sigma_1, \Sigma_2 \), we have

\[
f^{-1}(\Sigma_1 \cup \Sigma_2) \setminus (\Sigma_1 \cup \Sigma_2) = [f^{-1}(\Sigma_1) \cup f^{-1}(\Sigma_2)] \setminus (\Sigma_1 \cup \Sigma_2)
\]

\[
= (f^{-1}(\Sigma_1) \setminus (\Sigma_1 \cup \Sigma_2)) \cup (f^{-1}(\Sigma_2) \setminus (\Sigma_1 \cup \Sigma_2))
\]

\[
\subseteq (f^{-1}(\Sigma_1) \setminus \Sigma_1) \cup (f^{-1}(\Sigma_2) \setminus \Sigma_2) \subseteq \text{Crit}'(f).
\]

This implies that \( \Sigma_{\text{max}} \) is a \( \text{Crit}'(f) \)-exceptional set, and is contained in \( S_{\text{Crit}'(f)} \).

Using (2.6), we have

\[
\sup_{f \in \mathcal{A}_n} \sharp \Sigma_{\text{max}} \leq \sup_{f \in \mathcal{A}_n} \sharp S_{\text{Crit}'(f)} \leq 3 \sharp \text{Crit}'(f) + 4 = 3n + 4,
\]

as wanted. \( \Box \)

**Remark 2.5.** Other than the upper bound estimation, we also have

\[
3n - 1 \leq \sup_{f \in \mathcal{A}_n} \sharp \Sigma_{\text{max}}.
\]

The proof is actually based on the construction of a real polynomial associated by an admissible kneading data. Since this result is not used in the sequel, we omit the details on the proof.

**Remark 2.6.** In the view of estimations (2.11) and (2.12), Corollary 2.4 is a parallel property to that in the complex setting: every exceptional set of a rational function has at most 4 elements.

3. Co-homology

This section is devoted to the co-homologous transformation of \( \log |f'| \) as stated in Proposition 3.1 below. The idea ordinates from [MS96, MS00], and the construction is analogs to the constructions of the ramification function of Thurston mappings [DH93], (but in an inverse direction). On the other side, some special cases of Proposition 3.1 (e.g., Chebyshev Polynomials) are also discussed in [BK98].

To state Proposition 3.1, we recall \( \mathcal{U} \) the set of upper semi-continuous potentials in (1.3), and for each \( u \in \mathcal{U} \), recall \( \Lambda(u) \) the singular set defined in (1.5). Given a map \( f \in \mathcal{A} \), recall \( \Sigma_{\text{max}} \) the maximum exceptional set in (2.10). We highlight that the proof of Proposition 3.1 is more complicated than that for complex rational maps. This is due to the fact that the topological structure of \( \Sigma_{\text{max}} \) for interval maps is highly more sophisticated than that for the complex rational maps.
Proposition 3.1. Let \( f : J(f) \to J(f) \) be an interval map in \( \mathcal{A} \), then there exist \( G, -h \in \mathcal{U} \), such that

\[
G = \log |f'| + h \circ f - h,
\]

and such that

1. \( \tilde{P}(f, -tG) = \tilde{P}(t), \forall t < 0 \),

2. The singular set \( \Lambda(G) \) is a proper subset of \( \text{Crit}'(f) \), and \( G \) is non-exceptional.

Proof of Proposition 3.1. We split the proof into 3 parts. In part 1, we give the concrete formalism of \( h \), and show \(-h \in \mathcal{U}\). In part 2, we show that the new potential \( G \in \mathcal{U} \). Finally, we prove statements (a) and (b) in Part 3.

1. If \( \Sigma_{\text{max}} = \emptyset \), then \( \log |f'| \) has already satisfies the desired assertions (a) and (b), and there is nothing to prove. So, without loss of generality, we assume that \( \Sigma_{\text{max}} \neq \emptyset \). By Corollary 2.4, it follows that \( \Sigma_{\text{max}} \) is also a \( \text{Crit}'(f) \)-exceptional set, and \( \sharp \Sigma_{\text{max}} \leq 2 \sharp \text{Crit}'(f) + 4 \). Denote by \( \text{Crit}^*(\Sigma_{\text{max}}) := f^{-1}(\Sigma_{\text{max}}) \setminus \Sigma_{\text{max}} \), and thus \( \text{Crit}^*(\Sigma_{\text{max}}) \) is a subset contained in \( \text{Crit}'(f) \).

To define the function \( h \), we need the following definitions. For each \( x \in J(f) \), put

\[
\ell(x) := \begin{cases} 
\ell_x & \text{if } x \in \text{Crit}'(f); \\
1 & \text{otherwise.}
\end{cases}
\]

On the other hand, for each critical point \( \xi \in \text{Crit}^*(\Sigma_{\text{max}}) \), put \( n(\xi) \) be the minimal non-negative integer \( n \) such that \( f^n(\xi) \) is periodic. For each periodic cycle \( O \subset \Sigma_{\text{max}} \), put

\[
\hat{\alpha}(O) := \max \left\{ \prod_{j=0}^{n(\xi)-1} (\ell(f^j(\xi)))^{-1} \colon \xi \in \text{Crit}^*(\Sigma_{\text{max}}) \text{ and } f^n(\xi) \in O \right\}.
\]

Based on (3.3) and (3.4), we define a function \( \alpha : J(f) \to \mathbb{R} \) inductively by

\[
\alpha(\xi) := \begin{cases} 
\hat{\alpha}(O) - 1, & \text{if } \xi \in O \cap \Sigma_{\text{max}}; \\
(\alpha(f(\xi)) + 1)\ell(\xi) - 1, & \text{if } \xi \text{ nonperiodic in } \Sigma_{\text{max}}; \\
0 & \text{Otherwise.}
\end{cases}
\]

With this convention, put

\[
h(x) := \sum_{\xi \in \Sigma_{\text{max}}} \alpha(\xi) \log |x - \xi|, \forall x \in J(f).
\]

In the rest of Part 1, we show that \(-h \in \mathcal{U} \). This is equivalent to show that

\[
-1 < \alpha(\xi) \leq 0, \forall \xi \in \Sigma_{\text{max}}.
\]

There are two cases.
Case 1: $\xi$ is not periodic. Then there exist a $z \in \text{Crit}^*(\Sigma_{\max})$ and an integer $1 \leq i \leq n(z) - 1$ such that $\xi = f^i(z)$. Therefore,

$$0 < \alpha(\xi) + 1 = \left[ \prod_{j=0}^{n(z)-2-i} \ell(f^j(\xi)) \right] \cdot \hat{\alpha}(O) \ell(f^{n(z)-1-i}(\xi))$$

$$= \prod_{j=0}^{n(z)-1-i} \ell(f^j(\xi)) \cdot \hat{\alpha}(O)$$

$$\leq \prod_{j=0}^{i-1} (\ell(f^j(z)))^{-1} \quad \text{(Using (3.5))}$$

$$\leq 1. \quad \text{(Using non-flatness hypothesis)}$$

This directly implies $-1 < \alpha(\xi) \leq 0$.

Case 2: $\xi$ is periodic. Then $-1 < \alpha(\xi) = \hat{\alpha}(O) - 1 \leq 0$.

In both cases, we obtain the desired inequality (3.7), and thus $-h \in U$.

2. Put

(3.8) \hspace{1cm} G := \log |f'| + h \circ f - h.

We show that $G \in U$ in this part. In fact, the non-flatness hypothesis yields that for each $\xi \in f^{-1}(\Sigma_{\max})$, there is a Hölder continuous map $t_\xi(x)$ with

$$|f(x) - f(\xi)| = t_\xi(x)|x - \xi|^\ell(\xi), \quad \text{and} \quad \inf_{\xi \in f^{-1}(\Sigma_{\max})} \inf_{x \in J(f)} t_\xi(x) > 0.$$ 

Hence, there is also a Hölder continuous function $t(x) > 0$, $\forall x \in J(f)$ with

$$|f'(x)| = \prod_{c \in \text{Crit}(f)} |x - c|^\ell(c)^{-1} t(x).$$

Therefore,

$$h \circ f(x) = \sum_{\xi \in \Sigma_{\max}} \alpha(\xi) \log |f(x) - \xi|$$

$$= \sum_{\xi \in f^{-1}(\Sigma_{\max})} \alpha(f(\xi)) \log |f(x) - f(\xi)|$$

$$= \sum_{\xi \in f^{-1}(\Sigma_{\max})} \alpha(f(\xi)) \log (|x - \xi|^{\ell(\xi)} \cdot t_\xi(x))$$

$$= \sum_{\xi \in f^{-1}(\Sigma_{\max})} \alpha(f(\xi)) [\ell(\xi) \cdot \log |x - \xi| + \log t_\xi(x)],$$
and thus
\[ G(x) = \left( \sum_{\xi \in \text{Crit}(f)} (\ell(\xi) - 1) \log |x - \xi| + \log t(x) \right) \]
\[ + \left( \sum_{\xi \in f^{-1}(\Sigma_{\max})} \alpha(f(\xi)) \left[ \ell(\xi) \cdot \log |x - \xi| + \log t_{\xi}(x) \right] \right) - \left( \sum_{\xi \in \Sigma_{\max}} \log \alpha(\xi)|x - \xi| \right) \]
\[ = \left( \log t(x) + \sum_{\xi \in f^{-1}(\Sigma_{\max})} \alpha(f(\xi)) \log t_{\xi}(x) \right) \]
\[ + \left( \sum_{\xi \in f^{-1}(\Sigma_{\max})} \left[ (\ell(\xi) - 1) + \alpha(f(\xi))\ell(\xi) - \alpha(\xi) \right] \log |x - \xi| \right) \]
\[ + \left( \sum_{\xi \in \text{Crit}(f) \setminus f^{-1}(\Sigma_{\max})} (\ell(\xi) - 1) \log |x - \xi| \right) \]
\[ \equiv g(x) + \sum_{\xi \in f^{-1}(\Sigma_{\max})} b(\xi) \log |x - \xi| + \sum_{\xi \in \text{Crit}(f) \setminus f^{-1}(\Sigma_{\max})} (\ell(\xi) - 1) \log |x - \xi|, \]
with \( g(x) \equiv \log t(x) + \sum_{\xi \in f^{-1}(\Sigma_{\max})} \alpha(f(\xi)) \log t_{\xi}(x) \) and \( b(\xi) \equiv (\ell(\xi) - 1) + \alpha(f(\xi))\ell(\xi) - \alpha(\xi) \), \( \forall \xi \in f^{-1}(\Sigma_{\max}) \).

In the view of the formalism above, the Hölder continuity of \( g \) is obvious, and the non-flat hypothesis ensures \( \ell(\xi) - 1 \geq 0 \), \( \forall \xi \in \text{Crit}(f) \setminus f^{-1}(\Sigma_{\max}) \). Therefore, in order to show \( G \in U \), it is sufficient to show that
\[ \log t(x) + \sum_{\xi \in f^{-1}(\Sigma_{\max})} \alpha(f(\xi)) \log t_{\xi}(x) + \sum_{\xi \in \text{Crit}(f) \setminus f^{-1}(\Sigma_{\max})} (\ell(\xi) - 1) \log |x - \xi|, \]
\[ \equiv g(x) + \sum_{\xi \in f^{-1}(\Sigma_{\max})} b(\xi) \log |x - \xi| + \sum_{\xi \in \text{Crit}(f) \setminus f^{-1}(\Sigma_{\max})} (\ell(\xi) - 1) \log |x - \xi|, \]
with \( g(x) \equiv \log t(x) + \sum_{\xi \in f^{-1}(\Sigma_{\max})} \alpha(f(\xi)) \log t_{\xi}(x) \) and \( b(\xi) \equiv (\ell(\xi) - 1) + \alpha(f(\xi))\ell(\xi) - \alpha(\xi) \), \( \forall \xi \in f^{-1}(\Sigma_{\max}) \).

There are three cases.

**Case 1:** \( \xi \in \text{Crit}^{*}(\Sigma_{\max}) \equiv f^{-1}(\Sigma_{\max}) \setminus \Sigma_{\max} \). Then for each \( 0 \leq i \leq n(\xi) - 2 \), let \( z := f^{n(\xi) - 1}(\xi) \in f^{-1}(\mathcal{O}) \), it then follows from (3.4) and (3.5) that
\[ \alpha(f^{n(\xi) - (i+1)}(\xi)) = \left[ \hat{\alpha}(\mathcal{O}) \cdot \prod_{j=0}^{i} \ell(f^{i+j}(z)) \right] - 1 \]
\[ \geq \left[ \prod_{j=0}^{n(\xi) - 1} (\ell(f^j(\xi)))^{-1} \cdot \prod_{j=0}^{i} \ell(f^{i+j}(z)) \right] - 1 \]
\[ \geq \left[ \prod_{j=0}^{n(\xi) - 1 - (i+1)} (\ell(f^j(\xi)))^{-1} \right] - 1 \]
\[ = \left[ \prod_{j=0}^{n(\xi) - 1 - 2} (\ell(f^j(\xi)))^{-1} \right] - 1. \]

Hence when \( i = n(\xi) - 2 \), we have \( \alpha(f(\xi)) \geq \ell(\xi)^{-1} - 1 \). This means \( b(\xi) \equiv \ell(\xi)(\alpha(f(\xi)) + 1) - (\alpha(\xi) + 1) \geq \ell(\xi)(\ell(\xi)^{-1} - 1 + 1) - (0 + 1) = 0. \)
Case 2: \( \xi \in \Sigma_{\text{max}} \), and \( \xi \) is not periodic. Then (3.5) directly implies that
\[
b(\xi) = \ell(\xi)(\alpha(f(\xi)) + 1) - (\alpha(\xi) + 1) = 0
\]

Case 3: \( \xi \) is periodic. Then \( \ell(\xi) = 1 \), and (3.5) implies that
\[
b(\xi) = \ell(\xi)(\alpha(f(\xi)) + 1) - (\alpha(\xi) + 1) = \hat{\alpha}(O) - \hat{\alpha}(O) = 0.
\]
As a conclusion, we obtain (3.9) for all cases above. Thus, the new potential \( G \in U \).

3. In this part, we verify the statements (a) and (b). It is clear that \( h \) is finite outside \( \Sigma_{\text{max}} \). Thus, for each \( t < 0 \) and \( \mu \in \hat{M}(f, J(f)) \), we have
\[
\int_{J(f)} -tGd\mu = \int_{J(f)} -t\log |f'|d\mu,
\]
This directly yields \( \bar{P}(f, -tG) = \bar{P}(t) \), which completes the proof of Statement (a).

Next, we verify Statement (b). On one hand, based on the estimations above, we have \( b(\xi) = 0 \) for every periodic point \( \xi \in \Sigma_{\text{max}} \). On the other side, the maximality in the definition of \( \hat{\alpha}(O) \) yields that for each periodic cycle \( O \subseteq \Sigma_{\text{max}} \), there is at least a critical point \( \xi \in \text{Crit}^\ast(\Sigma_{\text{max}}) \) with \( f^n(\xi) \in O \) and
\[
\alpha(f(\xi)) = \ell(\xi)^{-1} - 1,
\]
this implies
\[
(3.10) \quad b(f^j(\xi)) = 0, \text{ } \forall j \in \{0, 1, \ldots, n(\xi) - 1\}.
\]
Thus, \( \Lambda(G) \) is a proper subset of \( \text{Crit}^\ast(f) \). By the maximality of \( \Sigma_{\text{max}} \), it is easy to see that each critical point \( \xi \in \text{Crit}^\ast(\Sigma_{\text{max}}) \) is \( \text{Crit}^\ast(f) \)-normal, so \( \xi \) is \( \Lambda(G) \)-normal. Therefore, for every finite subset \( \Sigma' \subseteq J(f) \) satisfying \( f(\Sigma') \subseteq \Sigma' \), the potential \( G \) is finite at some points in \( f^{-1}(\Sigma') \setminus \Sigma' \). Thus we obtain the desired (3.2) and hence Statement (b). The proof of this lemma is thus completed. \( \square \)

4. Iterated Multi-valued Function Systems

In this section, we construct an “Iterated multi-valued function system”. This is the main ingredient in the proof of Statement (b) of Key Lemma, and is stated as Proposition 4.1 below.

4.1. Iterated Multi-valued Function Systems. The machinery of “Iterated Multi-valued Function Systems” approach is introduced in [LRL13b], motivated by dealing with the situation where the invariant measure has zero Lyapunov exponent. It is a generalization of [IRRL12, Main theorem], and is based on a more general type of induced systems.

We follow up notations from [LRL13b, §3]. Given an interval map \( f \in \mathcal{A} \), a compact and connected subset \( B_0^6 \) of \( J(f) \), a sequence of multi-valued functions \( (\phi_i)_{i=1}^{+\infty} \) is an Iterated Multi-valued Function System (IMFS) generated by \( f \), if for each integer \( l \geq 1 \), there exist an integer \( m_l \geq 0 \) and a pull back \( W_l \) of \( B_0^6 \) by \( f^{m_l} \) contained in \( B_0 \), such that

---

6We mean there is a compact and connected subset \( B_i^6 \) in \( I \), such that \( B_0 = B_i^6 \cap J(f) \).
7A multi-valued function \( \phi : B \to W \) is a function which maps each point \( x \) of \( B \) to a non-empty subset \( \phi(x) \) of \( W \).
8For a subset \( V \subset J(f) \) and an integer \( m \geq 1 \), each connected component of \( f^{-m}(V) \) is a pull-back of \( V \) by \( f^m \).
\[ f^{m_l} \] has an onto property from \( W_l \) to \( B_0 \), i.e., \( f^{m_l}(W_l) = B_0 \);

\[ \phi_l = (f^{m_l}|_{W_l})^{-1} \]

With this convention, we say \( (\phi_l)_{l=1}^{\infty} \) is defined on \( B_0 \), with \( (m_l)_{l=1}^{\infty} \) as its time sequence.

Let \( (\phi_l)_{l=1}^{\infty} \) be an IMFS generated by \( f \) defined on \( B_0 \) with time sequence \( (m_l)_{l=1}^{\infty} \). For each integer \( n \geq 1 \), put \( \Omega_n := \{1, 2, \ldots \}^n \), and denote the space of all finite words in the alphabet \( \{1, 2, \ldots \} \) by \( \Omega^* := \bigcup_{n \geq 1} \Omega_n \). For every integer \( k \geq 1 \) and every \( l := l_1l_2\cdots l_k \in \Omega^* \), put

\[ |l| = k, \quad m_l = m_{l_1} + m_{l_2} + \cdots + m_{l_k}, \quad \text{and} \quad \phi_{m_l} = \phi_{l_1} \circ \phi_{l_2} \circ \cdots \circ \phi_{l_k}. \]

Note that for every \( x_0 \in B_0 \), and every pair of distinct words \( l \) and \( l' \) in \( \Omega^* \) satisfying \( m_l = m_{l'} \), we have:

\[ m_l \]

If the set \( \phi_{m_l}(x_0) \) and \( \phi_{m_{l'}}(x_0) \) intersect, then \( \phi_{m_l}(x_0) = \phi_{m_{l'}}(x_0) \).

The IMFS is said to be free, if there is an \( x_0 \in B_0 \) such that for every pair of distinct words \( l \) and \( l' \) in \( \Omega^* \) with \( m_l = m_{l'} \), the set \( \phi_{m_l}(x_0) \) and \( \phi_{m_{l'}}(x_0) \) are disjoint.

**Proposition 4.1.** Let \( f : J(f) \rightarrow J(f) \) be an interval map in \( \mathcal{A} \), and \( G \) be the resulting upper semi-continuous potential in \( \mathcal{U} \) given by Proposition 3.1. Let \( \mu \) be an ergodic measure in \( \mathcal{M}(f, J(f)) \) with positive Lyapunov exponent. Then there exists a subset \( X \) of \( J(f) \) of full measure with respect to \( \mu \), such that for every point \( x_0 \in X \), the following property holds: There exist a constant \( C > 0 \), a compact and connected subset \( B_0 \) of \( J(f) \) containing \( x_0 \), and a free IMFS \( (\phi_l)_{l=1}^{\infty} \) generated by \( f \) with time sequence \( (m_l)_{l=1}^{\infty} \), such that \( (\phi_l)_{l=1}^{\infty} \) is defined on \( B_0 \), and such that for every integer \( l \geq 1 \) and every \( z \in \phi_l(B_0) \), we have

\[ S_{m_l}(-tG)(z) \geq m_l \int_{J(f)} -tGd\mu - C, \quad \forall t < 0. \]

**Remark 4.2.** Analogous to that in the proof of \( \text{[IRRL12, Main Theorem]} \), the main step in the proof of Proposition 4.1 is the construction of an IFMS, based on a given invariant measure with strictly positive Lyapunov exponent. However, since the potential \( G \) take value \( -\infty \) at the singular set \( \Lambda(G) \), we need to modify the construction of the IMFS to bypass \( \Lambda(G) \) (so that we can obtain the constant \( C \) in (4.2)). In fact, the modification is closed related to Proposition 2.1 developed in §2. On the other hand, the IMFS is also a powerful tool on overcoming the difficulties (e.g., discovering an appropriate pull back and time sequence, on which the IMFS admits the onto property) arising from the fact that interval maps are not open maps in general.

**Remark 4.3.** Although Proposition 4.1 is written for interval maps in \( \mathcal{A} \), it also works well for the complex rational maps of degree at least 2 on the Riemann sphere. In fact, the proof will be even simpler since rational maps are open.

The proof of Proposition 4.1 depends on several lemmas and is given in the end of this section.
4.2. **Pesin’s theory.** Let us begin with some preliminaries related to Pesin’s theory. Recall first the definition of the natural extension of $f$ on $J(f)$. Let $\mathbb{Z}_-$ denote the set of all non-positive integers and endow

$$\mathbb{Z} := \{(z_n)_{n \in \mathbb{Z}_-} \in (J(f))^\mathbb{Z}_- : \text{for every } n \in \mathbb{Z}_-, f(z_{n-1}) = z_n\}$$

with the product topology. Define $F : \mathbb{Z} \to \mathbb{Z}$ by

$$F((\cdots, z_{-2}, z_{-1}, z_0)) = (\cdots, z_{-2}, z_{-1}, f(z_0)),$$

and $\pi : \mathbb{Z} \to J(f)$ by $\pi((z_n))_{n \in \mathbb{Z}_-} = z_0$. If $\mu$ is a Borel probability measure that is invariant and ergodic for $f$, then there exists a unique Borel probability measure $\nu$ on $\mathbb{Z}$ that is invariant and ergodic for $F$, and satisfies $\pi_*\nu = \mu$. We say $(\mathbb{Z}, F, \nu)$ is the natural extension of $(J(f), f, \mu)$.

Pointwise ergodic theorem yields the following lemma.

**Lemma 4.4.** [PRLS04, Lem1.3] Let $(\mathbb{Z}, \nu)$ be a probability space, and let $F : \mathbb{Z} \to \mathbb{Z}$ be an ergodic measure preserving transformation. Then for each function $\phi : \mathbb{Z} \to \mathbb{R}$ that is integrable with respect to $\nu$, there exists a subset $Z$ of $\mathbb{Z}$ such that $\nu(Z) = 1$, and such that for every $z \in Z$, we have

$$\limsup_{n \to \infty} \sum_{i=0}^{n-1} \left( \phi(F^i(z)) - \int_\mathbb{Z} \phi d\nu \right) \geq 0.$$

We also need a version of Ledrappier’s unstable manifold theorem from Dobbs.

**Lemma 4.5.** [Dob13] Let $f$ be an interval map in $\mathcal{A}$. Suppose measure $\mu \in \mathcal{M}(f, J(f))$ is ergodic and has $\chi_\mu > 0$. Denote by $(\mathbb{Z}, F, \nu)$ the natural extension of $(J(f), f, \mu)$. Then there exists a measurable function $\alpha$ on $\mathbb{Z}$ such that $0 < \alpha < 1/2$ almost everywhere with respect to $\nu$, and such that for $\nu$–almost every point $y \in \mathbb{Z}$ there exists a set $V_y$ contained in $\mathbb{Z}$ with the following properties:

(a) $y \in V_y$ and $\pi(V_y) = B(\pi(y), \alpha(y))$;
(b) For each integer $n \geq 0$, $f^n : \pi(F^{-n}(V_y)) \to V_y$ is diffeomorphic;
(c) For each $y' \in V_y$,

$$\sum_{i=0}^{+\infty} \left| \log |f'(\pi(F^{-i}(y)))| - \log |f'(\pi(F^{-i}(y')))| \right| < \log 2.$$

(d) For each $\eta > 0$ there is a measurable function $\theta$ on $\mathbb{Z}$ with $0 < \theta < +\infty$ almost everywhere with respect to $\nu$, such that

$$\frac{1}{\theta(y)} \exp(n(\chi_\nu - \eta)) \leq |(f^n)'(\pi(F^{-n}(y)))| \leq \theta(y) \exp(n(\chi_\nu + \eta)).$$

In particular,

$$|\pi(F^{-n}(V_y))| \leq 2\theta(y) \exp(-n(\chi_\nu - \eta)).$$

**Remark 4.6.** A stronger version (with the same proof) of Property (3) is possible: For each upper semi continuous potential $u \in \mathcal{U}$, there is a constant $C$ such that for each $y' \in V_y$, we have

$$\sum_{i=0}^{+\infty} |u(\pi(F^{-i}(y))) - u(\pi(F^{-i}(y')))| < C.$$
Lemma 4.7. Let \( f \) be an interval map in \( \mathcal{A} \), and \( G \) be the resulting upper semi continuous potential in \( \mathcal{U} \) given by Proposition 3.1. Let \( \mu \) be an ergodic measure in \( \mathcal{M}(f,J(f)) \) with \( X_0 > 0 \). Then there is a subset \( X^* \) of \( J(f) \) full measure with respect to \( \mu \) possessing the following property: For every point \( x \in X^* \), there exist \( \tilde{\rho}_x > 0 \), \( D > 0 \) and a strictly increasing sequence of positive integers \( (n_l)_{l=1}^{\infty} \), such that for every \( l > 0 \), we can choose a point \( x_{n_l} \in f^{-n_l}(x) \) and a connected component \( W_{n_l} \) of \( f^{-n_l}(B(x,\tilde{\rho}_x)) \) containing \( x_{n_l} \) so that:

(a) \( x_{n_l+1} \in f^{-(n_l+1)-n_l}(x_{n_l}) \);
(b) For every point \( y \in W_{n_l} \),

\[
S_{n_l}(-tG)(y) \geq n_l \int_{J(f)} -tGd\mu - D \forall t < 0;
\]

(c) \( \lim_{l \to \infty} |W_{n_l}| = 0. \)

Proof of Lemma 4.7. Let \( (\mathbb{Z},F,\nu) \) be the natural extension of \( (J(f),f,\mu) \), and thus \( \nu \) is also invariant and ergodic with respect to \( F^{-1} \). For each \( t < 0 \), by applying Lemma 4.4 for \( F^{-1} \) to the integrable function \( -tG \circ \pi \), there exists a subset \( Z \) of \( \mathbb{Z} \) of full measure with respect to \( \nu \), such that

\[
\limsup_{n \to \infty} \sum_{n=0}^{n-1} \left( -tG \circ \pi(F^{-i}(z_m)_{m \in \mathbb{Z}_-}) + \int_{\mathbb{Z}} tG \circ \pi d\nu \right) \geq 0, \forall (z_m)_{m \in \mathbb{Z}_-} \in Z.
\]

Taking a subset of \( Z \) of full measure with respect to \( \nu \) if necessary, by Lemma 4.5, there is a function \( \alpha : \mathbb{Z} \to (0,1/2) \) such that \( Z \) and \( \alpha \) satisfy all the assertions of Lemma 4.5. Define the set \( X^* := \pi(Z) \), then we have

\[
\mu(X^*) = \nu((\pi(Z))) \geq \nu(Z) = 1.
\]

It remains to verify that \( X^* \) satisfies the desired properties. Fix a point \( x \in X^* \), and choose a point \( (z_m)_{m \in \mathbb{Z}_-} \) such that \( \pi((z_m)_{m \in \mathbb{Z}_-}) = x \), let \( V_{z_m} \) be given by Lemma 4.5, and put \( \rho_x := \alpha((z_m)_{m \in \mathbb{Z}_-}) \). Moreover, for each integer \( j \geq 1 \), put

\[
y_j := \pi(F^{-j}(z_m)_{m \in \mathbb{Z}_-}) = z_j \in f^{-j}(x),
\]

and \( U_j := \pi(F^{-j}(V_{z_m})) \). Using Assertions (a) and (b) of Lemma 4.5, it follows that for every integer \( j \geq 1 \), \( U_j \) is the connect component of \( f^j(B(x,\rho_x)) \) containing \( y_j \), and \( f^j : U_j \to B(x,\rho_x) \) is diffeomorphic. Moreover, using Remark 4.6 and part (4) of Lemma 4.5, it follows that there exist \( C' > 0 \) and \( \lambda > 1 \), such that every \( n \leq 1 \), we have

\[
|U_n| \leq C'\lambda^{-n},
\]

and there is a constant \( \tilde{C} \), such that for every pair of \( x, y \in U_n \), we have

\[
|S_n(-tG)(x) - S_n(-tG)(y)| < -t\tilde{C}, \forall t < 0.
\]

Finally, fix a \( D' > 0 \), (4.4) yields that there is a strictly increasing sequence of positive integers \( (n_l)_{l=1}^{\infty} \) such that for every \( l \geq 0 \), we have

\[
\sum_{i=0}^{n_l-1} -tG \circ \pi(F^{-i}(z_m)_{m \in \mathbb{Z}_-}) \geq n_l \int_{\mathbb{Z}} -tG \circ \pi d\nu - D' = n_l \int_{J} -tGd\mu - D'.
\]

For each integer \( l \geq 1 \), put \( x_{n_l} := y_{n_l} \) and \( W_{n_l} := U_{n_l} \), then statement (a) and (c) are automatically satisfied from the definition of \( y_{n_l} \) and (4.5), and Statement...
(b) follows from (4.6) and (4.7) with \( D := D' - t\tilde{C} \). The proof of this lemma is thus completed. \( \square \)

The following three technical lemmas are also required.

**Lemma 4.8.** [LRL13b, Lem.3.2] Given an interval map \( f \) in \( \mathcal{A} \), there is an \( \varepsilon > 0 \) such that the following property holds. Let \( J_0 \) be an interval contained in \( I \) satisfying \( |J_0| \leq \varepsilon \), let \( n \geq 1 \) be an integer, and let \( J \) be a pull-back of \( J_0 \) by \( f^n \), whose closure is contained in the interior of \( I \). Suppose in addition that for each \( j \in \{1, \cdots, n\} \) the pull-back of \( J_0 \) by \( f^j \) containing in \( f^{n-j}(\tilde{J}) \) has length bounded from above by \( \varepsilon \). Then \( f^n(\partial J) \subset \partial J_0 \).

**Lemma 4.9.** [LRL13b, Lem.3.3] Let \( f \) be an interval map in \( \mathcal{A} \), and let \( a \) be a point of \( J(f) \) such that \( (a, +\infty) \) (resp. \( (-\infty, a) \)) intersects \( J(f) \). Then for every open interval \( U \) intersecting \( J(f) \), and every sufficient large integer \( n \geq 1 \), there is a point \( y \) of \( U \) in \( f^{-n}(a) \) such that for every \( \varepsilon > 0 \), the set \( f^n(B(y, \varepsilon)) \) intersects \( (a, +\infty) \) (resp. \( (-\infty, a) \)).

**Lemma 4.10.** [RL13b, Lem. A.2] Given an interval map \( f \) in \( \mathcal{A} \), then for every \( \kappa > 0 \), there is a \( \delta > 0 \) such that for every \( x \in J(f) \), every integer \( n \geq 1 \), and every pull-back \( W \) of \( B(x, \delta) \) by \( f^n \), we have \( |W| < \kappa \).

4.3. Proof of Proposition 4.1. We are ready to prove Proposition 4.1 in this subsection. The proof is divided into two parts. In Part (1) below we define a IMFS concretely. In Part (2), we show the IMFS is free and satisfies (4.2).

**Proof.** Let \( \varepsilon > 0 \) be the constant given by Lemma 4.8 and let \( \delta > 0 \) be the constant given by Lemma 4.10 for \( \kappa = \varepsilon \). Let \( X^* \) be the subset in \( J(f) \) given by Lemma 4.7, and put

\[
X := X^* \backslash \text{Per}(f).
\]

Since the measure \( \mu \) is ergodic and not supported on a periodic orbit, the set \( X \) has full measure for \( \mu \). Fix a point \( x_0 \in X \), which is not an end point of \( I \).

1. Let \( \bar{\rho}_{x_0}, (n_i)_{i=1}^{+\infty}, (x_i)_{i=1}^{+\infty} \) and \( (W_i)_{i=1}^{+\infty} \) be given by Lemma 4.7 with a point \( x_0 \in X \) which is not an end point of \( I \). Fix \( \rho \in (0, \min\{\delta, \bar{\rho}_{x_0}, \text{dist}(x_0, \partial I)\}) \). Taking a subsequence if necessarily, and suppose

\[
\lim_{l \to \infty} x_{n_l} = b.
\]

Using Lemma 4.7, it follows that for each \( t < 0 \), there is a constant \( D > 0 \), such that for every \( z \in W_{n_l} \), we have

\[
S_{n_l}(-tG)(z) \geq n_l \int_{J(f)} -tGd\mu - D
\]

On the other side, Proposition 3.1 yields that for every non empty forward invariant finite set \( \Sigma \), we have \( f^{-1}(\Sigma) \backslash \Sigma \not\subseteq \Lambda(G) \). Therefore, Proposition 2.1 implies that there exists an integer \( N \geq 0 \) such that \( b_N := f^N(b) \) is \( \Lambda(G) \)-normal and \( b_N \notin \Lambda(G) \), and thus

\[
\lim_{l \to \infty} x_{n_l - N} = \lim_{l \to \infty} f^N(x_{n_l}) = f^N(\lim_{l \to \infty} x_{n_l}) = f^N(b) = b_N.
\]
So for each $\bar{\rho} > 0$, the point $x_{n_l-N} \in [b_N - \bar{\rho}, b_N]$ for infinitely many $l$. Meanwhile, using the normality of $b_N$ and the finiteness of the critical set, we have

(a) there is an integer $\tilde{M} > 0$, and $b_N'$ such that $f^{\tilde{M}}(b_N') = b_N$ and $f^{j}(b_N') \notin \Lambda(G), \forall j = 0, 1, \cdots, \tilde{M}$;

(b) every pre-image of $b_N'$ is not in $\text{Crit}'(f)$.

By statement (a), there exist $\bar{\rho} > 0$ and a closed one side subinterval of $B(b_N, \bar{\rho})$ (say $(b_N' - \bar{\rho}, b_N')$) intersecting with $\text{Crit}(f)$ and satisfying

$$f^{M}([b_N' - \bar{\rho}, b_N']) = [b_N - \bar{\rho}, b_N], \quad f^{j}([b_N' - \bar{\rho}, b_N']) \cap \Lambda(G) = \emptyset, \forall j = 0, 1, \cdots, \tilde{M}.$$  

This implies there exist points $x_{n_l-N+\tilde{M}} \in [b_N' - \bar{\rho}, b_N']$ such that $f^{\tilde{M}}(x_{n_l-N+\tilde{M}}) = x_{n_l-N}$, for infinitely many $l$.

By our choice, $x_0$ is not pre-periodic, so $x_0$ is not in the boundary of a periodic Fatou component. Hence, we can assume that there are two disjoint open intervals $U_0$ and $U_1$ in $(x_0 - \rho, x_0)$, each of them intersecting $\text{Crit}(f)$. Note that $f$ is topological exact on $\text{Crit}(f)$, and $(b_N' - \bar{\rho}, b_N')$ interest $\text{Crit}(f)$, so by applying Lemma 4.9 for $a = b_N'$, we conclude that there exist an integer $\tilde{M} > 1$, and two distinct points $\omega_0, \omega_1$ in $\tilde{U}_0$ and $\tilde{U}_1$, respectively. $f^{\tilde{M}}(\omega_0) = f^{\tilde{M}}(\omega_1) = b_N'$, such that for every $\varepsilon' > 0$, both sets $f^{\tilde{M}}(B(\omega_0, \varepsilon'))$ and $f^{\tilde{M}}(B(\omega_1, \varepsilon'))$ intersect with $(-\infty, b_N')$.

On the other hand, due to Statement (b), we can reduce the value of $\bar{\rho}$ (and so $\tilde{\rho}$ is reduced correspondingly), such that the pull backs $U_0, U_1$ of $(b_N' - \bar{\rho}, b_N')$ by $f^{M}$ has the properties: $\omega_i \in U_i \subset \tilde{U}_i, \forall i = 0, 1$ and

$$f^{j}(U_i) \cap \text{Crit}'(f) = \emptyset, \forall i = 0, 1 \text{ and } j = 0, 1, \cdots, \tilde{M}. 

So, $U_0$ and $U_1$ are disjoint and are contained in $(x_0 - \rho, x_0)$, and the points $x_{n_l-N+\tilde{M}}$ is contained in both $f^{M}(U_0)$ and $f^{M}(U_1)$ for infinitely many $l$.

Let $M := \tilde{M} + \tilde{M}$. The above discussions together with (4.10) yield that

$$x_{n_l-N} \text{ is contained in both } f^{M}(U_0) \text{ and } f^{M}(U_1), \text{ for infinitely many } l.$$

In addition, using that $\lim_{l \to \infty} |W_{n_l-N}| = 0$ and taking a subsequence if necessarily, then for every $l > 0$, we have the following properties: $n_{l+1} - n_l \geq M$, the point $x_{n_l-N}$ is contained in both $f^{M}(U_0)$ and $f^{M}(U_1)$, the length $|W_{n_l-N}| < \varepsilon$, and the pull back $W_{n_l-N}$ of $B(x_0, \rho)$ by $f^{n_l-N}$ containing $x_{n_l-N}$ is contained in $[b_N - \bar{\rho}, b_N]$. By interchanging $\omega_0$ and $\omega_1$ and taking a subsequence if necessarily, we can assume that every $l$, the point $f^{n_{l+1}-n_l-M-N}(x_{n_{l+1}})$ is not contained in $U_0$. For each $l$, choose the pull back $W_{n_l-N+M}$ of $W_{n_l-N} \cap [b_N - \bar{\rho}, b_N]$ by $f^{M}$ that contains a point $x'_{n_l-N+M}$ of $f^{-M}(x_{n_l-N})$, and that is contained in $U_0$.

With this convention, we have $W_{n_l-N+M} \subset U_0 \subset B(x_0, \rho)$, so the closure of $W_{n_l-N+M}$ is contained in the interior of $I$. On the other hand, By the choice of $\rho$ and Lemma 4.10, the length of $f^{i}(W_{n_l-N+M})$ is less than $\varepsilon$, for every $i = 0, 1, \cdots, n_l - N + M$. So by Lemma 4.8, we have $f^{n_l-N+M}(\partial W_{n_l-N+M})$ is contained in $\partial B(x_0, \rho)$. Also, note that (4.12) yields that $f^{n_l-N+M}(W_{n_l-N+M})$ contains $x_0$, hence we have the set $f^{n_l-N+M}(W_{n_l-N+M})$ contains either $[x_0 - \rho, x_0]$ or $[x_0, x_0 + \rho]$. There are two probabilities.
Case 1: There are infinitely many \( l \) such that the set \( f_{n_{l+1} - 1}^{-1} (W_{n_{l} - N + M}) \) contains \([x_0 - \rho, x_0]\). We can assume this property holds for every \( l \), by taking a subsequence. Then, there is a pull back \( W_{n_{l} - N + M}'' \) of \([x_0 - \rho, x_0]\) by \( f_{n_{l} - N + M} \) that is contained in \( W_{n_{l} - N + M}' \), and such that
\[
f_{n_{l} - N + M} (W_{n_{l} - N + M}'') = [x_0 - \rho, x_0].
\]
With this convention, we have
\[
W_{n_{l} - N + M}'' \subseteq W_{n_{l} - N + M}' \subseteq U_0 \subseteq [x_0 - \rho, x_0].
\]
Put
\[
B_0 := [x_0 - \rho, x_0], \ M' := M, \text{ and } U'_0 := U_0.
\]
Case 2: For each \( l \), outside finitely many exceptions, the set \( f_{n_{l} - N + M} (W_{n_{l} - N + M}) \) contains \([x_0, x_0 + \rho]\). We can assume this property holds for every \( l \), by taking a subsequence. Note that \( x_0 \) is not in the boundary of a Fatou component, Lemma 4.9 and topological exactness yield that there exist an integer \( M \geq 1 \) and a pull back \( U'_0 \) of \( U_0 \) by \( f_{n_{l}} \) that is contained in \([x_0, x_0 + \rho]\), and such that \( x_{n_{l} - N + M}' \) is contained in \( f_{n_{l}}^{M} (U'_0) \) for infinitely many \( l \). Take a subsequence if necessarily, assume for every \( l \), we have
\[
n_{l+1} - n_l \geq M + M', \quad \text{and} \quad x_{n_{l} - N + M}' \text{ is contained in } f_{n_{l}}^{M} (U'_0).
\]
Since for each \( l \) the point \( f_{n_{l+1} - n_l - M} (x_{n_{l+1} - N}) \) is not in \( U'_0 \), it implies that the point \( f_{n_{l+1} - n_l - M - M} (x_{n_{l+1} - N}) \) is not in \( U'_0 \). For each \( l \), choose a pull back \( \bar{W}_{n_{l} - N + M + M} \) of \( W_{n_{l} - N + M}' \) by \( f_{n_{l}}^{M} \) contained in \( U'_0 \) and that contains a point \( \bar{x}_{n_{l} - N + M + M} \) of \( f_{n_{l}}^{M} (x_{n_{l} - N + M}') \). By Lemma 4.8, the set \( f_{n_{l} - N + M + M} (\partial \bar{W}_{n_{l} - N + M}') \) is contained in \( \partial B(x_0, \rho) \). On the other hand, since the set \( f_{n_{l} - N + M + M} (\partial \bar{W}_{n_{l} - N + M}') \) is contained in \( f_{n_{l} - N + M} (W_{n_{l} - N + M}') \). But \( f_{n_{l} - N + M} (W_{n_{l} - N + M}') \) does not contain \([x_0 - \rho, x_0]\), we conclude that \( f_{n_{l} - N + M + M} (\partial \bar{W}_{n_{l} - N + M}') \) contains \([x_0, x_0 + \rho]\). So there is a pull back \( \bar{W}_{n_{l} - N + M + M}'' \) of \([x_0, x_0 + \rho]\) by \( f_{n_{l} - N + M + M} \) that is contained in \( \bar{W}_{n_{l} - N + M + M}'' \), and such that \( f_{n_{l} - N + M + M} (\bar{W}_{n_{l} - N + M + M}'') = [x_0, x_0 + \rho] \). With this convention, we have
\[
W_{n_{l} - N + M + M}'' \subseteq \bar{W}_{n_{l} - N + M + M}'' \subseteq U'_0 \subseteq [x_0, x_0 + \rho].
\]
Put
\[
B_0 := [x_0, x_0 + \rho], \ M' := M + M
\]
In both cases, for every \( l \), put
\[
\phi_l := \left( f_{n_{l} - N + M'} (W_{n_{l} - N + M}'') \right)^{-1}.
\]
Then \( (\phi_l)_{l=1}^{+\infty} \) is an IMFS generated by \( f \), that is defined on \( B_0 \) with the time sequence \( (n_l)_{l=1}^{+\infty} := (n_{l+1} - N + M')_{l=1}^{+\infty} \). Moreover, \( n_{l+1} - n_l \geq M' \), \( W_{n_{l} - N + M}' \subset U'_0 \), and \( f_{n_{l+1} - n_l - M'} (x_{n_{l+1} - N}) \notin U'_0 \). The construction of \( (\phi_l)_{l=1}^{+\infty} \) is also illustrated in Figure 1.

2. To prove that the IMFS \( (\phi_l)_{l=1}^{+\infty} \) is free, we follow the same idea from [LRL13b]. Let \( k, k' \geq 1 \) be two integers and let
\[
l := l_1 l_2 \cdots l_k, \text{ and } l' := l'_1 l'_2 \cdots l'_{k'}.
\]
be two distinct words in $\Omega^*$ such that $m_\ell = m_{\ell'}$. Without loss of generality we assume that $l'_k \geq l_k + 1$, then

$$f^{m_\ell - m_k} (\phi_\ell(x_0)) = \phi_{l_k}(x_0) \subset W''_{n_1-N+M'} \subset U_0.$$ 

On the other side, we have

$$m_{\ell'} - m_k = n_{\ell'} - n_k \geq n_{k+1} - n_k \geq M',$$

and thus the set

$$f^{m_{\ell'} - m_k} (\phi_{\ell'}(x_0)) = f^{m_{\ell'} - m_k} (\phi_{\ell'}(x_0))$$

$$= f^{m_{\ell'} - m_k} (\phi_{\ell'}(x_0))$$

$$= f^{m_{\ell'} - m_k - M' + N} \left( (f^{m_{\ell'}_{\ell'}} |_{W_{\ell'}})^{-1}(x_0) \right)$$

\textbf{Figure 1.} Illustration of the IMFS in Case 1. Lines indicate iterations of the map $f$ while dashed lines indicate the construction of the IMFS.
contains the point
\[ f^{m_{k'}} - m_k - M' + N(x_{n_{k'}}) = f^{n_{k'} - n_k - M' + N(x_{n_{k'}})} = f^{n_{k+1} - n_k - M' + N(x_{n_{k+1}})}. \]

By the construction, this point is not contained in \( U_0 \), so the sets
\[ f^{m_{k}}(\phi_l(x_0)) \text{ and } f^{m_{k}}(\phi_{l'}(x_0)) \]
are distinct. This implies that the sets \( \phi_l(x_0) \) and \( \phi_{l'}(x_0) \) are different. By (4.1), they are actually disjoint. So the IMFS \( (\phi_l)_{l=1}^{\infty} \) is free.

Finally, we verify (4.2) below. Following from (4.10) and (4.11), we have
\[ f^{i}(W_{n_1 - N + M'}) \cap \Lambda(G) = \emptyset, \quad \forall i = 0, 1, \ldots, M'. \]
Hence, for each \( t < 0 \), the numbers
\[ C_1 := t \sup_{\xi \in U_{l=0}^{M'} f(W_{n_1 - N + M'})} \log |f(\xi)| < +\infty, \]
and
\[ C_2 := -t \sup_{x \in J(f)} \log |f'(x)| < +\infty, \]
Recall that for every \( l \geq 1 \) and \( z \in \phi_l(B_0) \) the point \( f^{M' - N}(z) \in W_{n_l} \). Hence,
\[
S_{n_l}(-tG)(z) = S_{M' - N}(-tG)(z) + S_{n_l}(-tG)(f^{M' - N}(z)) \\
= S_{M'}(-tG)(z) - S_{N}(-tG)(f^{M' - N}(z)) + S_{n_l}(-tG)(f^{M' - N}(z)) \\
\geq -M'C_1 + NC_2 + n_l \int_{J(f)} -tGd\mu - D \quad \text{(Using (4.8).)} \\
= n_l \int_{J(f)} -tGd\mu - D - \left( M'C_1 - NC_2 + (M' - N) \int_{J(f)} -tGd\mu \right). \\
\]
This implies the desired inequality (4.2) with
\[ C := D + \left( M'C_1 - NC_2 + (M' - N) \int_{J(f)} -tGd\mu \right) < +\infty. \]

\[ \square \]

5. **Proof of Key Lemma**

In this section, we will complete the proof of Key Lemma. In the view of Proposition 3.1, it is sufficient to show Statement (b). We will distinguish two cases, according as the measure with positive Lyapunov exponent is supported on a periodic orbit or not. The former case is stated as Lemma 5.1. Analogous to Proposition 4.1, Lemma 5.1 is an adaption of [IRRL12, Prop4.1] and [LRL13a, Lemm4.1], according to the obstacles stated in Remark 4.2. On the other hand, we remark that Lemma 5.1 works well also for rational maps of degree at least 2 on the Riemann sphere.
Lemma 5.1. Let $f$ be an interval map in $\mathcal{A}$, and $G$ be the upper semi-continuous potential in $\mathcal{U}$ given by Proposition 3.1, then for every hyperbolic repelling periodic point $x_0 \in J(f)$ of period $N$, we have

\begin{equation}
\limsup_{n \to \infty} \frac{1}{n} \sum_{y \in f^{-n}(x_0)} \exp(S_n(G)(y)) > \frac{1}{N} S_N(G)(x_0).
\end{equation}

Proof of Lemma 5.1. Analogous to that in [LRL13b, Lem4.1, 4.1], we split the proof into 2 parts. In part 1, we construct the induced map, and in part 2 we show (5.1) for the induced map.

1. Fix a repelling periodic point $x_0 \in J(f)$ of period $N$. Since $|(f^N)'(x_0)| > 1$, there is $\rho > 0$ and a local inverse $\phi$ of $f^{2N}$ defined on $B(x_0, \rho)$ with $\phi(x_0) = x_0$. Note that $f^{2N} \circ \phi$ is the identity map on $B(x_0, \rho)$. Hence $\phi'(x_0) > 0$, thus $\phi$ is increasing on $B(x_0, \rho)$ and $f^{2N}$ is also increasing on $\phi(B(x_0, \rho))$. Since $x_0 \in J(f)$, reducing $\rho$ if necessarily, we assume that $(x_0, x_0 + \rho/2)$ intersects with $J(f)$. On the other hand, Proposition 3.1 implies that for every non-empty finite forward invariant subset $\Sigma \subset J(f)$, we have $f^{-1}(\Sigma) \backslash \Sigma \not\subset \Lambda(G)$. Hence, by Proposition 2.1, $x_0$ itself is $\Lambda(G)$-normal, and is not in $\Lambda(G)$. This implies:

(a) there exist an integer $\hat{k}$, a point $a \in J(f)$, and an open one side subinterval (say $(a, a + \rho)$) such that $f^{2N\hat{k}}(a) = x_0, f^{2N\hat{k}}(a + \rho/2) = (x_0, x_0 + \rho/2)$, and

\begin{equation}
f^{2Nj}(a) \notin \Lambda(G), f^{2Nj}(a + \rho/2) \cap \Lambda(G) = \emptyset, \forall j = 0, 1, \cdots, \hat{k}.
\end{equation}

(b) every pre-image of $a$ is not in $\text{Crit}'(f)$.

Note that $f$ is topological exact on $J(f)$. Using Lemma 4.9 for the point $a$, it follows that there exist an integer $\hat{k} \leq 1$ and a point $z' \in (x_0, x_0 + \rho/2)$ such that $f^{2N\hat{k}}(z') = x_0$, and such that for every $\varepsilon > 0$, the set $f^{2N\hat{k}}(B(z', \varepsilon))$ intersects $(a, a + \rho/2)$. Fix $\varepsilon \in (0, |z' - x_0|)$ such that $f^{2N\hat{k}}(B(z', \varepsilon)) \subset (a, a + \rho/2)$, and such that

\begin{equation}
f^{2Nj}(B(z', \varepsilon)) \cap \text{Crit}'(f) = \emptyset, \forall j = 0, 1, \cdots, \hat{k}.
\end{equation}

Thus the closure of $B(z', \varepsilon)$ is contained in $(x_0, x_0 + \rho/2)$.

Let $k' := \hat{k} + \hat{k}$, and $W$ be the pull back of $f^{2N\hat{k}}(B(z', \varepsilon)) \cap [x_0, x_0 + \rho/2]$ by $f^{2Nk'}$ containing $z'$. Put $U_0 := \mathcal{V}^{k'}(f^{2Nk'}(W))$. Since both $f^{2Nk'}$ and $\mathcal{V}^{k'}$ are continuous, reducing $\varepsilon$ if necessarily, $U_0$ is disjoint from $W$. By our choice of $\phi$, it follows that

\begin{equation}
W \subset B(z', \varepsilon) \subset (x_0, x_0 + \rho), \text{ and } x_0 \in f^{2Nk'}(W) \subset [x_0, x_0 + \rho/2).
\end{equation}

Moreover, for every open set $U \subset [x_0, x_0 + \rho/2)$, we have

$$
\lim_{k \to \infty} \text{diam}(\phi^k(U)) = 0, \text{ and } \lim_{k \to \infty} \text{dist}(\phi^k(U), x_0) = 0.
$$

Thus, there is an integer $k_1 \geq 0$, such that

$$
U_1 := \phi^{k_1}(W) \subset f^{2Nk'}(W),
$$

and

\begin{equation}
\text{diam}(\phi^{k_1+k'}(f^{2Nk'}(W))) < \text{diam}(f^{2Nk'}(W)).
\end{equation}

Put $k_0 := k_1 + k'$, and $U_0 := \phi^{k_1}(U_0)$. Then we have

$$
k_0 \geq 1, U_0 \cap U_1 = \emptyset, \text{ and } U_1 \subset f^{2Nk'}(W).
$$

A REAL VERSION OF MAKAROV AND SMIRNOV'S FORMALISM 27
We also note that $f^{2Nk}(W)$ contains $x_0$, we have

$$U_0 = \phi^{k_1}(U_0') = \phi^{k_0}(2Nk'(W)) \subset f^{2Nk}(W).$$

We also note that

$$f^{2Nk_0}(U_0) = f^{2Nk}(W) = f^{2Nk_0}(U_0).$$

Put

$$U := U_1 \cup U_2, \quad \hat{f} := f^{2Nk_0}|U.$$

2. We will prove (5.1) in this part. Put $\hat{G} := \frac{1}{2Nk_0}S_{2Nk_0}(G)$ and for every integer $m \geq 1$, put

$$\hat{S}_m(\hat{G}) := \hat{G} + \hat{G} \circ \hat{f} + \cdots + \hat{G} \circ \hat{f}^{m-1}.$$

Note that to prove the Lemma, it is sufficient to show that

$$\limsup_{m \to \infty} \frac{1}{m} \log \sum_{y \in f^{-m}(x_0)} \exp(\hat{S}_m(\hat{G})(y)) > \hat{G}(y).$$

This is equivalent to show that the convergent radius of the series

$$(5.5) \Xi(s) := \sum_{n=0}^{+\infty} \left( \sum_{z \in f^n(x_0)} \exp(\hat{S}_n(\hat{G}(z))) \right) s^n$$

is strictly less than $\exp(-\hat{G}(x_0))$.

The proof of this fact above is analogous to the proof of [IRRL12, Prop4.1]. We include it for completeness.

Put $\hat{K} := \bigcap_{i=0}^{+\infty} \hat{f}^{-i}(U)$, and consider the itinerary map $\iota : \hat{K} \to \{0,1\}^\mathbb{N}$ defined so that for every $i \in \{1,2,\cdots\}$, the point $\hat{f}^i(z)$ is in $U_{\iota(z)_i}$. Since $\hat{f}$ maps each of the sets $U_0, U_1$ onto $f^{NK}(W)$, and both $U_0, U_1$ are contained in this set, for every integer $k \geq 0$, and every sequence $a_0, a_1, \cdots, a_k$ of elements of $\{0,1\}$, there is a point of $\hat{f}^{-k+1}(x_0)$ in the set

$$\hat{K}(a_0, a_1, \cdots, a_k) := \{ z \in \hat{K} : \text{ for every } i \in \{0,1,\cdots,k\}, \text{ we have } \iota(z)_i = a_i \}.$$ 

By our choice of $\phi$, it directly follows that there is a constant $\hat{C} > 0$ such that for every integer $k \geq 1$ and every point $z \in \hat{K}(0, \cdots, 0)$, we have

$$(5.6) \quad \hat{S}_k(\hat{G})(z) \geq k\hat{G}(x_0) - \hat{C}.$$

Meanwhile, due to (5.2) and (5.3), there is a sufficiently large constant $\hat{C}$ such that

$$(5.7) \quad \hat{G}(z) \geq \hat{G}(x_0) - \hat{C}, \quad \forall z \in U.$$

Next we show that for every $k \geq 0$, every sequence $a_0, a_1, \cdots a_k$ of elements of $\{0,1\}$ with $a_0 = 1$, and every point $x \in \hat{K}(a_0a_1\cdots a_k)$, then

$$(5.8) \quad \hat{S}_{k+1}(\hat{G})(x) \geq (k+1)\hat{G}(x_0) - 2(a_0 + a_1 + \cdots + a_k)\hat{C}.$$

In fact, put $l := a_0 + \cdots + a_k$, $i_{l+1} := k + 1$, and also put $0 = i_1 < i_2 < \cdots i_l \leq k$ be all integers such that $a_i = 1$. It follows from (5.6) and (5.7) that

$$\hat{S}_{i_{j+1}-i_j}(\hat{G})(\hat{f}^{i_j}(x)) \leq (i_{j+1} - i_j)(\hat{G})(x_0) - 2\hat{C}, \quad \forall j \in \{1, \cdots, l\}.$$
Summing over \( j \in \{1, 2, \cdots, l\} \), and we obtain desired inequality (5.8). Thus, if we put

\[
\Phi(s) := \sum_{k=1}^{\infty} \exp(k\hat{G}(x_0) - 2\hat{C})s^k,
\]

then each of the coefficients of

\[
\Upsilon(s) := \Phi(s) + \Phi(s)^2 + \cdots
\]

is less than or equal to the corresponding coefficients of \( \Xi \), and therefore the radius of convergence of \( \Xi \) is less than or equal to that of \( \Upsilon \). Since \( \Phi(s_0) \to -\infty \) as \( s \to 1 \), there is an \( s_0 \in (0, \exp(-\hat{G}(x_0))) \) such that \( \Phi(s_0) > 1 \). It follows that the radius of convergence of \( \Upsilon \), and hence that of \( \Xi \), is less than or equal to \( s_0 \), and therefore it is strictly less than \( \exp(-\hat{G}(x_0)) \). The proof of this lemma is thus completed. \( \square \)

Once Lemma 5.1 is proved, we follow the same strategy to deduce Key Lemma. We include the proof of the remaining part for completeness.

5.1. Proof of Key Lemma.

Proof of Key Lemma. If an ergodic measure \( \mu \in M(f, J(f)) \) is supported on a repelling periodic point, the desired inequality follows from Lemma 5.1.

Otherwise, the ergodic measure \( \mu \in M(f, J(f)) \) is not supported on a periodic orbit. By the topological exactness on \( J(f) \), and the ergodicity of \( \mu \), it follows that \( \mu \) is non-atomic and fully supported on \( J(f) \). By Proposition 4.1, there exist a constant \( C > 0 \), a connected and compact subset \( B_0 \) of \( J(f) \), and a free IMFS \( (\phi_k)_{k=1}^{+\infty} \) generated by \( f \) with a time sequence \( (m_k)_{k=1}^{+\infty} \) that is defined on \( B_0 \), such that for every \( k \geq 1 \) and every point \( y \in \phi_k(B_0) \), we have

\[
S_{m_k}(-tG)((y)) \geq m_k \int -tGd\mu - C, \quad \forall t < 0.
\]

Since the IMFS \( (\phi_l)_{l=1}^{+\infty} \) is free, there is a point \( x_0 \in B_0 \) such that for every \( \ell, \ell' \in \Omega^* \) with \( m_{\ell} = m_{\ell'} \), the sets \( \phi_{\ell}(x_0) \) and \( \phi_{\ell'}(x_0) \) are disjoint. Moreover, for every integer \( k \geq 1 \), every \( \ell := l_1 \cdots l_k \in \Omega^* \), every \( y_0 \in \phi_{\ell}(x_0) \), and every \( j \in \{1, \cdots, k - 1\} \), the point

\[
y_j := f^{m_{l_1} + m_{l_2} + \cdots + m_{l_j}}(y_0)
\]

is in \( \phi_{m_{j+1}}(B_0) \). Hence, followed by (5.9), given a \( t < 0 \), we have

\[
S_{m_\ell}(-tG)(y_0) = S_{m_{\ell_1}}(-tG)(y_0) + S_{m_{\ell_2}}(-tG)(y_1) + \cdots + S_{m_{\ell_k}}(-tG)(y_k) \geq \sum_{i=1}^{k} \left( m_{\ell_i} \int -tGd\mu - C \right) = m_\ell \int -tGd\mu - kC.
\]

This implies that for every \( \ell \in \Omega^* \), and every \( y_0 \in \phi_{\ell}(x_0) \), we have

\[
\exp(S_{m_\ell}(-tG)(y_0)) \geq \exp(m_\ell \int -tGd\mu) \exp(-||\ell||C).
\]

In addition, we put

\[
\Xi_n := \bigcup_{\ell \in \Omega^*: m_\ell = n} \phi_{\ell}(x_0), \quad \forall n \geq 1,
\]
then the radius of convergence of the series
\[
\Xi(s) := \sum_{n=1}^{\infty} \left( \sum_{y \in \Xi_n} \exp(S_n(-tG))(y) \right) s^n,
\]
is given by
\[
R := \left( \limsup_{n \to \infty} \left( \sum_{y \in \Xi_n} \exp(S_n(-tG))(y) \right)^{1/n} \right)^{-1},
\]
and in particular,
\[
\exp \left( -\limsup_{n \to \infty} \frac{1}{n} \log \sum_{y \in f^{-n}(x_0)} \exp(S_n - tG(y)) \right) \leq R.
\]
Hence, to complete the proof of Key Lemma, it is sufficient to prove \( R < \exp(-\int -tGd\mu) \).

Denote
\[
\Phi(s) := \sum_{l=1}^{\infty} \exp(-C) \exp \left( m_l \int -tGd\mu \right) s^{m_l}
\]
Followed by (5.10), and the fact that the IMFS \( (\phi_k)_{k=1}^{+\infty} \) is free, each of the coefficients of the series
\[
\Upsilon(s) := \sum_{i=1}^{+\infty} (\Phi(s))^i = \sum_{n=1}^{+\infty} \left( \sum_{l \in \Omega, m_l = n} \exp \left( m_l \int -tGd\mu \right) \exp(-l|C) \right) s^n,
\]
is less or equal to the corresponding coefficient of series \( \Xi \). So the radius of convergence of \( \Xi \) is less or equal to that of \( \Upsilon \). Note also that
\[
\lim_{s \to \exp(-\int -tGd\mu)^-} \Phi(s) = +\infty.
\]
Hence, there is an \( s_0 \in (0, \exp(-\int \psi d\mu)) \) such that \( \Phi(s_0) \leq 1 \) and thus we have
\[
R \leq s_0 < \exp(-\int -tGd\mu),
\]
which implies (1.7) and completes the proof of Key Lemma.

6. Hyperbolicity and the existence of a non-atomic conformal measure

In this section, we will use the Key lemma to prove the Hyperbolicity and the existence of a conformal measure for the new potential \(-tG\), with \( t < 0 \). Based on this, it allows us to use Keller’s results on showing the absence of a phase transition of the hidden pressure function in next Section. We begin with the definition of the terminology “conformal measure.”

Let \( f : J(f) \to J(f) \) be a continuous interval map. Given a Borel measurable function \( g : J(f) \to [0, +\infty) \), a Borel probability measure \( \mu \) on \( J(f) \) is \( g \)-conformal for \( f \), if for each Borel set \( A \subset J(f) \) on which \( f \) is injective, we have
\[
\mu(f(A)) = \int_A gd\mu.
\]
Proposition 6.1. Let \( f : J(f) \to J(f) \) be an interval map in \( \mathcal{A} \), and let \( G : J(f) \to \mathbb{R} \cup \{-\infty\} \) be the upper semi-continuous potential in \( \mathcal{U} \) given by Proposition 3.1. Then for each integer \( t < 0 \),

- \(-tG\) is hyperbolic;
- there is a non-atomic \( \exp(P(f, -tG) + tG) \)-conformal measure for \( f \), with the support equals to \( J(f) \).

The proof of Proposition 6.1 will be at the end of this section and requires a few Lemmas.

Lemma 6.2 (Positive Lyapunov exponent). Let \( f : J(f) \to J(f) \) be an interval map in \( \mathcal{A} \), let \( G : J(f) \to \mathbb{R} \cup \{-\infty\} \) be the upper semi-continuous potential in \( \mathcal{U} \) given by Proposition 3.1, and measure \( \mu \in \mathcal{M}(f, J(f)) \) be an equilibrium state of \(-tG\) with \( t < 0 \), then \( \chi_{\mu} > 0 \).

Proof. The proof is divided by two parts. In part 1, we show that \( \tilde{P}(t) > 0 \), \( \forall t < 0 \).

In part 2, we use (3.1) to show that each equilibrium state of \(-tG\), with \( t < 0 \) has positive Lyapunov exponent.

1. Since \( f \) is topologically exact on \( J(f) \), there are two disjoint closed subset \( A_1, A_2 \subset J(f) \) with non-empty interior such that \( f^{N_1}(A_1) = f^{N_2}(A_2) = J(f) \). Let \( N := \max\{N_1, N_2\} \), and thus \( f^N \) has a 2-horse shoe \( \{A_1, A_2\} \). Therefore, there is an \( f^N \)-invariant compact set \( K \subset J(f) \), such that \( f|_K \) is semi-conjugate to the full shifts on 2-symbols. Hence

\[
(6.1) \quad h_{\text{top}}(f) = \frac{1}{N} h_{\text{top}}(f^N) \geq \frac{1}{N} \log 2 > 0.
\]

On the other side, Since \( f \) is continuously differentiable, and every periodic point is hyperbolic repelling, it follows from [RL13a, Proposition A.1] that every the measure \( \eta \in \mathcal{M}(f, J(f)) \) has \( \chi_{\eta} \geq 0 \). Put measure \( \nu \in \mathcal{M}(f, J(f)) \) be a maximum entropy measure, i.e., \( h_{\nu}(f) = h_{\text{top}}(f) \), then (6.1) yields that constant 0 function is a hyperbolic potential. By [LRL13a, Theorem B], such maximum entropy measure \( \nu \) is unique and non-atomic, and thus

\[
(6.2) \quad \tilde{P}(t) \geq h_{\nu} - t\chi_{\nu} = h_{\text{top}}(f) - t\chi_{\nu} \geq \frac{1}{N} \log 2 > 0, \ \forall t \leq 0.
\]

2. Put a measure \( \mu \in \mathcal{M}(f, J(f)) \) be an equilibrium state of the new potential \( G \). It is sufficient to consider ergodic \( \mu \), the general case will be followed by ergodic decomposition. We distinguish two cases.

- \( \mu \) is atomic. Then the topological exactness of \( f \) and ergodicity of \( \mu \) yield that \( \mu \) must be supported on a periodic orbit. On the other hand, note that every periodic orbit are hyperbolic repelling. So we have \( \chi_{\mu} > 0 \).
- \( \mu \) is non-atomic. Then following from (3.1) and (6.2), we have

\[
P(f, -tG) = h_{\mu} - \int_{J(f)} tGd\mu
\]

\[
\geq \tilde{P}(f, -tG) = \tilde{P}(t) > 0, \ \forall t < 0.
\]

So either \( h_{\mu} > 0 \) or \( \int_{J(f)} Gd\mu > 0 \). If \( h_{\mu} > 0 \), then by Ruelle’s inequality, we have \( \chi_{\mu} > 0 \); else if \( \int_{J(f)} Gd\mu > 0 \), then \( \chi_{\mu} = \int_{J(f)} Gd\mu > 0 \).
In both cases, we have $\chi_\mu > 0$, as wanted. \hfill \Box

**Lemma 6.3.** [Zha15, Prop 0.4] Let $f : J(f) \to J(f)$ be an interval map in $\mathcal{A}$, and $u : J(f) \to \mathbb{R} \cup \{-\infty\}$ be a upper semi-continuous potential in $U$ with $\Lambda(u)$ the resulting singular set. If $u$ is hyperbolic and non-exceptional for $f$, then for every periodic point $x \in J(f)$, or every non-periodic point $x \in J(f) \setminus \bigcup_{i=-\infty}^{\infty} f^i(\Lambda(u))$, we have

$$P(f,u) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{y = f^{-n}(x)} \exp(S_n(u)(y)).$$

As mentioned in [Zha15], we have an inequality version of Lemma 6.3 below.

**Lemma 6.4.** [Zha15, Lemm 0.3] Let $f : J(f) \to J(f)$ be an interval map in $\mathcal{A}$, and $u : J(f) \to \mathbb{R} \cup \{-\infty\}$ be a upper semi-continuous potential in $U$, then for every point $x \in J(f)$, we have

$$P(f,u) \geq \limsup_{n \to \infty} \frac{1}{n} \log \sum_{y = f^{-n}(x)} \exp(S_n(u)(y)).$$

The following lemma is a consequence of a general method on construction of a conformal measure, which is usually known as the “Patterson-Sullivan method”. This method is introduced respectively by Sullivan [Sul83] in the setting of complex rational maps, and by Denker and Urbanski [DU91] in the setting of real interval maps.

**Lemma 6.5.** [LRL13a, Prop 3.2] Let $f : I \to I$ be a continuous interval map, let $X$ be a compact subset of $I$ that contains at least 2 points and satisfies $f^{-1}(X) \subset X$, and let $u : X \to \mathbb{R} \cup \{-\infty\}$ be a upper semi-continuous function in $U$. Assume that $f$ has no periodic critical point in $X$, and that there is a point of $X$ and an integer $N > 1$ such that the number

$$P_{x_0} := \limsup_{n \to +\infty} \frac{1}{n} \log \sum_{y \in f^{-n}(x_0)} \exp(S_n(u)(y))$$

satisfies $P_{x_0} > \sup_X \frac{1}{N} S_N(u)$. Then there is an atom-free $\exp(P_{x_0} - u)$—conformal measure for $f$. If addition $f$ is topologically exact on $X$, then the support of this conformal measure is equal to $X$.

**Remark 6.6.** Actually, Lemma 6.5 is proved in [LRL13a] by assuming the potential $u$ to be Hölder continuous, however, its proof works without change for $u$ being in $U$. In fact, the hypothesis $u \in U$ implies the following property:

- $\forall \mu \in \mathcal{M}$, $\forall A \subset J(f)$, with $\mu(\partial A) = \mu(\partial f(A))$, and $\overline{A} \subset I \setminus \{x| f \text{ not open at } x\}$
  $$\Rightarrow \mu(f(A)) = \int_A \exp(P(f,u) - u) d\mu.$$

This property is fundamental for the validity of the Patterson-Sullivan method.

We also need a simple lemma below, and include its short proof for completeness.
Lemma 6.7. For each interval map $f : J(f) \to J(f)$ in $\mathcal{A}$, and every upper semi-continuous function $\phi : J(f) \to \mathbb{R} \cup \{-\infty\}$, we have

\[(6.5) \limsup_{n \to \infty} \sup_{J(f)} \frac{1}{n} S_n(\phi) = \sup_{\nu \in M(f,J(f))} \int \phi d\nu.\]

Proof. On one hand, for each $\nu \in M(f,J(f))$, and each $n > 0$, we have

\[ \int S_n(\phi) d\nu = \sum_{k=0}^{n-1} \int \phi \circ f^k d\nu = n \int \phi d\nu. \]

So

\[ \sup_{\nu \in M(f,J(f))} \int \phi d\nu \leq \limsup_{n \to \infty} \sup_{J(f)} \frac{1}{n} S_n(\phi). \]

On the other hand, for each $n$, the upper semi-continuity of $\frac{1}{n} S_n(\phi)$ and the compactness of $J(f)$ yields that there is an $x_n \in X$ such that $\frac{1}{n} S_n(\phi)(x_n) = \sup_{J(f)} \frac{1}{n} S_n(\phi)$.

Put $\nu_n := \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x_0)}$, so $\nu_n(\phi) = \frac{1}{n} S_n(\phi) = \sup_{J(f)} \frac{1}{n} S_n(\phi)$. Choose a subsequence $(n_m)_{m=1}^{\infty}$ of positive integers such that

\[ \lim_{n \to \infty} \frac{1}{n_m} S_{n_m}(\phi(x_{n_m})) = \limsup_{n \to \infty} \frac{1}{n} S_n(\phi). \]

Taking a subsequence if necessary, we can further assume that $(\nu_{n_m})$ converges in the weak* topology to a measure $\nu \in M(J(f), f)$. Then we have

\[ \limsup_{n \to \infty} \frac{1}{n} S_n(\phi) = \lim_{m \to \infty} \frac{1}{n_m} S_{n_m}(\phi(x_{n_m})) \]

\[ = \lim_{m \to \infty} \int \phi d\nu_{n_m} \]

\[ \leq \int \phi d\nu \quad \text{(using the upper semi-continuity of } \phi\text{)} \]

\[ \leq \sup_{\nu \in M(J(f), f)} \int \phi d\nu. \]

Therefore, we obtain the desired assertion (6.5), and complete the proof of this lemma. \(\square\)

We are ready to prove the Proposition 6.1.

Proof of Proposition 6.1. 1. This part deals with the proof of Statement (a) on the hyperbolicity of $G$. We first prove the following Claim.

Claim: If $\mu \in M(J(f), f)$ is an equilibrium state, then the measure-theoretic entropy $h_\mu$ is strictly positive.

We prove the Claim by contradiction. Suppose there is an equilibrium state $\mu$ with $h_\mu = 0$, then

\[(6.6) \int G d\mu = P(f,G).\]

Replacing by an ergodic component if possible, we can further assume $\mu$ satisfying (6.6) is ergodic. Note also from Lemma 6.2 that $\chi_\mu > 0$. We distinguish two cases.
If \( \mu \) is atomic, then \( \mu \) supports on a periodic orbit \( O_\mu \). Let \( y \) be a periodic point on \( O_\mu \); If \( \mu \) is non-atomic, then \( \mu \) supports on \( J(f) \). Recall \( X \) the \( \mu \)-full measure set in Statement (b) in Key Lemma, and let \( y \) be a point inside the set \( (J(f) \setminus \bigcup_{i=-\infty}^{+\infty} f^i(\Lambda(G))) \cap X \). Such \( y \) exists, since the intersection is a \( \mu \)-full measure set.

In both cases, we have for each \( t < 0 \),

\[
P(f, -tG) = \limsup_{n \to \infty} \frac{1}{n} \log \sum_{y=f^{-n}(x)} \exp(S_n(-tG)(y)) \quad \text{(Using Lemma 6.4)}
\]

\[
> \int -tGd\mu \quad \text{(Using Key Lemma)}
\]

\[
= P(f, -tG) \quad \text{(By (6.6)).}
\]

This is evidently impossible, so we obtain the Claim.

Hence,

\[
P(f, -tG) > \sup_{\nu \in \mathcal{M}(J(f), f)} \int -tGd\nu \quad \text{(Using the claim in particular \( h_\mu > 0 \))}
\]

\[
= \limsup_{n \to \infty} \sup_{J(f)} \frac{1}{n} S_n(-tG) \quad \text{(Using Lemma 6.7).}
\]

In other word, there exists an integer \( n \geq 0 \) such that \( P(f, -tG) > \sup_{J(f)} \frac{1}{n} S_n(-tG) \), which means the potential \( -tG \) is hyperbolic.

2. This part deals with Statement (b) on the existence of a conformal measure. Note that for every point \( x_0 \in J(f) \setminus \bigcup_{i=-\infty}^{+\infty} f^i(\Lambda(G)) \), we have

\[
P_{x_0} = P(f, -tG) \quad \text{(Using Lemma 6.3)}
\]

\[
> \sup_{J(f)} \frac{1}{N} S_N(-tG) \quad \text{(Using hyperbolicity).}
\]

By applying Lemma 6.5 with \( X = J(f) \), there is a non-atomic \( \exp(P(f, -tG) + tG) \) conformal measure. Note also \( f \) is topologically exact on \( J(f) \), so the support of this conformal measure is equal to \( J(f) \). This proves the Statement (b), and thus completes the proof of Proposition 6.1.

\[
\square
\]

7. Makarov-Smirnov’s formalism for interval maps

In this section, we finish the proof of the interval version of Makarov-Smirnov’s formalism. The proof relies on showing the spectral gap for the corresponding transfer operator in a certain Keller’s space. A good survey of this methodology comes from Keller’s original paper [Kel85], and also from Rivera-Leterlier’s lecture note [RL15].

The following lemma is given by [Zha15].

**Lemma 7.1.** [Zha15, Lemm 0.6 and Prop 0.4] Let \( f : J(f) \to J(f) \) be an interval map in \( \mathcal{A} \), and \( u : J(f) \to \mathbb{R} \cup \{-\infty\} \) be a upper semi-continuous potential in \( \mathcal{U} \). For each \( t < 0 \), suppose \( -tu \) is non-exceptional and is hyperbolic with \( N := N(t) \)
being the integer such that \( \sup_{J(f)} \frac{1}{N} S_N(-tu) < P(f, -tu) \). Let \( \tilde{u} := \frac{1}{N} S_N(u) \), the following properties hold.

- \( \sup_{J(f)} -tu < P(f, -\tilde{u}), \exp(-tu) \) and \( \exp(-\tilde{u}) \) are Hölder continuous with the same Hölder exponent;
- \( P(f, -tu) = P(f, -\tilde{u}) \), \( -tu \) and \( -\tilde{u} \) share the same equilibrium states;
- \( -\tilde{u} \) is non-exceptional;
- There is a non-atomic \( \exp(P(f, -\tilde{u}) + tu) \)-conformal measure supported on \( J(f) \).

Before the proof the interval version of Makarov-Smirnov's formalism, it is important to obtain the following.

**Proposition 7.2.** Let \( f : J(f) \to J(f) \) be an interval map in \( \mathcal{A} \). Let \( u : J(f) \to \mathbb{R} \cup \{-\infty\} \) be an upper semi-continuous potential in \( \mathcal{U} \). If \( u \) is hyperbolic and non-exceptional, then for each \( t < 0 \), the following holds:

- (a) there is a unique non-atomic equilibrium state \( \mu \) of \( f \) for the potential \( -tu \). Moreover, \( \mu \) is fully supported on \( J(f) \), and has strictly positive entropy. In addition, \( \mu \) is exponentially mixing;
- (b) the pressure function \( P(f, -tu) \) is equal to \( \tilde{P}(f, -tu) \), and it is real analytic on \( (-\infty, 0) \).

**Proof.** Fix a negative value \( t \in (-\infty, 0) \), and let \( N := N(t) \) be the integer such that \( \sup_{J(f)} \frac{1}{N} S_N(-tu) < P(f, -tu) \). By the hypothesis on \( u \in \mathcal{U} \), the function \( \exp(-tu) \) is \( \alpha \)-Hölder continuous. By the hypothesis of the hyperbolicity and non-exceptionality, Proposition 6.1 gives that there exists a \( \exp(P(f, -tu) + tu) \)-conformal measure \( m \).

With this convention, put \( g_t := \exp(-tu - P(f, -tu)) \).

It is clear that \( g_t \) is \( \alpha \)-Hölder continuous, so \( g_t \) is of bounded \( 1/\alpha \) variation. On the other hand, put \( g_{t,N} \) as in (9.4), then \( \sup_{J(f)} g_{t,N} < 1 \). Therefore, for each \( \tilde{\alpha} \in (0, \alpha] \), the assertions of Corollary 9.1 hold, with \( p = 1/\tilde{\alpha} \). Let \( A \) be the constant given by Corollary 9.1, and consider the Keller’s space \( H^{\tilde{\alpha},1}(m) \). In the view of Corollary 9.1, it is sufficient to verify the uniqueness of the equilibrium state in Statement (a), and the Statement (b).

We first proceed the proof on the uniqueness of the equilibrium state. Put \( \tilde{u} := \frac{1}{N} S_N(u) \), then equilibrium state \( \nu \) of \( f \) for the potential \( -\tilde{u} \) has

\[
\begin{aligned}
h_{\nu}(f) &= P(f, -\tilde{u}) + \int_{J(f)} t\tilde{u} d\nu \\
&\geq P(f, -\tilde{u}) - \sup_{J(f)} (-\tilde{u}) > 0 \text{ (Using Lemma 7.1)}. 
\end{aligned}
\]

By Ruelle's inequity, taking an ergodic exponent if necessarily, we have \( \chi_{\nu} > 0 \). Note also \( f \) is topologically exact on \( J(f) \), then [Dob13b, Theo 6] yields that the equilibrium state \( \nu \) of \( f \) for the potential \( -\tilde{u} \) is unique. Applying Lemma 7.1 again, we get the uniqueness of the equilibrium state of \( f \) for the potential \( -tu \).

In the rest of the proof, we will prove Statement (b). In other words, we will show the integer \( N \) is actually independent of \( t \) in a neighborhood of \( t \). For each
\( \tau \in \mathbb{C} \), let \( \mathcal{L}_\tau \) be the operator defined in Part (3) of Corollary 9.1. For each \( \varepsilon \in \mathbb{R} \), put 
\[ u_\varepsilon := (-t + \varepsilon)u, \text{ and } g_{t+\varepsilon} := \exp(u_\varepsilon - P(f, u_\varepsilon)). \]

It is clear that when \( \varepsilon \) is sufficiently small, then \( u_\varepsilon \) is in \( \mathcal{U} \), and notice the pressure is continuous, so there is an \( \varepsilon_0 > 0 \), such that
\[ \sup_{J(f)} \frac{1}{N} S_N(\varepsilon u) < P(f, u_\varepsilon) - \sup_{J(f)} \frac{1}{N} S_N(-tu). \]

Therefore,
\[ \sup_{J(f)} \frac{1}{N} S_N(u_\varepsilon) \leq \sup_{J(f)} \frac{1}{N} S_N(-tu) + \sup_{J(f)} \frac{1}{N} (\varepsilon u) < P(f, u_\varepsilon). \]

So, from Part (1) of Corollary 9.1, we have \( \exp(P(f, u_\varepsilon)) \) is equal to the spectral radius \( \mathcal{L}_{g_{t+\varepsilon}} \). Moreover, from Part (3) of Corollary 9.1, the function \( \varepsilon \to \exp(P(f, u_\varepsilon)) \) is real analytic on \((-\varepsilon_0, \varepsilon_0)\). The proof of this lemma is thus completed. \( \square \)

We are finally ready to proof the interval version of Makarov-Smirnov’s formalism.

**Proof of Corollary 1.2.** For each interval map \( f : J(f) \to J(f) \) in \( \mathcal{A} \), it directly follows from the definitions of \( P, \tilde{P} \) that
\[ P(t) \geq \max\{\tilde{P}(t), -t\chi_{\max}\}, \forall t < 0. \]

On the other hand, Key Lemma implies that there exists a new potential \( G, -h \in \mathcal{U} \) such that \(-tG = -t \log |Df| - (th) \circ f - th\) and \( h \) only has poles in \( \Sigma_{\max} \), and such that
\[ (7.1) \quad \tilde{P}(t) = \tilde{P}(f, -tG), \forall t < 0. \]

So
\[ P(t) = \max\{\tilde{P}(t), -t\chi_{\max}\}, \forall t < 0. \]

Note also from Proposition 6.1, we have \(-tG\) is hyperbolic and non-exceptional for \( f \). Applying Proposition 7.2 with \( u = G \), we have
\[ \tilde{P}(t) = P(f, -tG) = P(f, -tG), \forall t < 0, \]
and \( P(f, -tG) \) is real analytic at \((-\infty, 0]\). Therefore we obtain Statement (a).

A phase transition occurs if and only if \( \chi_* > \tilde{P}'(-\infty) \). Since we obtain the spectral gap result in Keller’s space, followed by the same argument in [MS00, Rem 3.7], we have
\[ \tilde{P}'(-\infty) = \sup\{\chi_{\nu} : \nu \in \mathcal{M}(f, J(f)), \nu(\Sigma_{\max}) = 0\}. \]

So we obtain Statement (b). \( \square \)
8. Revisit Ruelle’s result and complex Makarov-Smirnov’s formalism

The new ideas in our results yield also some progresses in the complex setting. In this section, we will revisit Ruelle’s result and the complex version Makarov-Smirnov’s formalism, and their relations under the views of our Key Lemma. To simplify the notation, let $f$ be a rational map of degree at least two on the Riemann sphere, denote by $\mathcal{U}$ an analogous subclass (but in the complex setting) of upper semi-continuous potentials as in (1.3), and denote by $BV_2$ the Banach space of functions for which the second derivatives are complex measures throughout this section.

Recall that the proof on the Makarov-Smirnov’s formalism for complex rational maps is closely related to a previous fact (stated below) by Ruelle [Rue92].

**Fact 8.1.** [Rue92, Coro 6.3] If $\log u : \hat{\mathbb{C}} \to \mathbb{R} \cup \{-\infty\}$ is a hyperbolic upper semi-continuous potential with $u$ in the functional space $BV_2$, and satisfying a certain integrability condition stated in [Rue92, Coro6.3] (This condition implies that $u$ vanishes at all the critical points of $f$). Then the corresponding transfer operator $L_u$ is bounded and has a spectrum gap under the space of $BV_2$.

Based our Key Lemma and the above Fact 8.1, we can reprove the complex Makarov-Smirnov’s result under an addition assumption of non-exceptionality.

**Corollary 8.2.** If $\log |Df|$ is non-exceptional, then the transfer operator $L$ associated by the weight $|Df|^{-t}$ admits a spectral gap under $BV_2$ for each $t < 0$. Moreover, $\mathcal{P}(t)$ equals to $\tilde{\mathcal{P}}(t)$, and is real analytic on $(-\infty, 0)$, and admits a unique non-atomic equilibrium state for each $t \leq 0$.

**Proof.** Using Key Lemma and Proposition 6.1, the non-exceptionality hypothesis implies that $-t \log |Df|$ is hyperbolic for every $t < 0$. On the other side, Makarov and Smirnov [MS00, §1] state that the weight $|Df|^{-t}$ actually satisfies the integrability condition. Therefore, Fact 8.1 directly yields the statement of Corollary 8.2. $\square$

**Remark 8.3.** Compared to the proof of [MS00, Theo 3.1], this is a simpler proof in particular on the validity of the uniqueness of the equilibrium state. In fact, due to the hyperbolicity, the uniqueness is a directly consequence of the standard arguments from [HR92, HK82, Prz90].

However, the methods in the proof of Corollary 8.2 seem not sufficiently strong to deal with exceptional rational maps. Let us explain the obstacles more precisely as follows. Given a rational map $f$ on the Riemann space, denote by $k(c)$ the multiplicity of its critical point at $c$. Following from the discussion in [MS00, §4] (or §3), there exist a function $h \in -\mathcal{U}$ and a number $\tilde{\kappa} > 0$ with

$$\frac{\kappa}{1 - \kappa} := \min\{k(c) : c \in f^{-1}(a) \cap \text{Crit}(f), a \in \text{Per}(f) \cap \Sigma_{\text{max}}\},$$

so that for each $t < 0$, the weight $|Df|^{-t}$ can be replaced with a homologous weight

$$G_{\tilde{\kappa}, t} := |Df|^{-t} \left( \frac{h \circ f}{h} \right)^{\tilde{\kappa} t},$$

We highlight that our Key Lemma also works well for all rational maps under an even simpler proof.
and meanwhile $\log G_{\tilde{\kappa},t}$ belongs to $\mathcal{U}$ and is non-exceptional. Therefore, our Key Lemma yields that $\log G_{\tilde{\kappa},t}$ is also hyperbolic.

Unfortunately, the potential $\log G_{\tilde{\kappa},t}$ is beyond the hypothesis of Fact 8.1. We state the reasons as follows. Following from the construction, the new potential $G_{\tilde{\kappa},t}$ will not vanish at least one critical point. This means the integrability condition in Fact 8.1 is never satisfied, thus one cannot apply Fact 8.1 to $G_{\tilde{\kappa},t}$ to get a spectral gap.

As a comparison, recall that we use Lemma 9.1 on the Keller’s space to create spectral gap for the interval maps, and the hypothesis of Lemma 9.1 don’t require an analogous integrability condition. So we don’t have this obstacles in the real setting. This is the fundamental difference between real and complex setting, which illustrates the power of Keller’s spaces.

The above discussions naturally lead to the following problem.

**Problem 1.** *Given a rational map $f$ on the Riemann sphere, and a hyperbolic upper semi-continuous potential $\log u$ in $\mathcal{U}$, find a Banach space on which the transfer operator $L_u$ acts with a spectral gap.*

The difficulty will be of course in the situation where $u$ does not vanish at every critical point of $f$, and it is plausible to expect such Banach spaces have analogous properties to the Keller’s spaces for complex situation. This space seems to be a generalization of $BV_2$, and contains the Sobolev spaces of certain exponent. For example, it might be the spaces of complex functions of certain “generalized bounded variation”.

Positive outcomes of Problem 1 will yield a complex version of Theorem A and Theorem B. What is more important, it will provide a new (and perhaps simpler) proof of the orginal Makarov-Smirnov’s formalism for rational maps. To be more precise, the method deduced from Problem 1 will directly prove that the transfer operator $L_{G_{\kappa,t}}$ has a spectral gap. In comparison, Makarov and Smirnov in [MS00] use the spectrums of a sequence of transfer operators $\{L_{G_{\kappa,t}}\}$ with $\kappa < \tilde{\kappa}$ to approximate the spectrum of the transfer operator $L_{G_{\tilde{\kappa},t}}$. Actually, these authors need to consider $L_{G_{\kappa,t}}$ acting on the Sobolev spaces instead of applying Fact 8.1, since every potential $\log G_{\kappa,t}$ is no longer hyperbolic.

### 9. Appendix A. Keller’s space and spectral gaps

This appendix provides some basic ideas/background from [Kel85] and [LRL13a] on the Keller’s space which are required in the proof of Proposition 7.2.

#### 9.1. Keller’s space.

Let $X$ be a compact subset of $\mathbb{R}$ and $m$ be a Borel non-atomic probability measure on $X$. We consider the quotient space on the space of complex valued functions taking values on $X$, defined by agreement on a set of full measure with respect to $m$.

Denote by $d$ the pseudo-distance on $X$ defined by

$$d(x, y) := m(\{z \in X : x \leq z \leq y \text{ or } y \leq z \leq x\}).$$

Note for every $x \in X$ and every $\varepsilon > 0$, the set of ball

$$B(x, \varepsilon) := \{y \in X, d(x, y) < \varepsilon\}$$

has positive measure with respect to $m$. 
Given a measurable function \( h : X \to \mathbb{C} \) and \( \varepsilon > 0 \), for each \( x \in X \), let
\[
\text{osc}(h, \varepsilon, x) := \text{ess-sup}\{ |h(y') - h(y)| : y, y' \in B(x, \varepsilon) \}
\]
and
\[
\text{osc}_1(h, \varepsilon) := \int_X \text{osc}(h, x, \varepsilon) dm(x).
\]
Given \( A > 0 \), and for each \( \alpha \in (0, 1] \) and each \( h : X \to \mathbb{C} \), put
\[
(9.1) \quad \var_{\alpha, 1}(h) := \sup_{\varepsilon \in (0, A]} \frac{\text{osc}_1(h, \varepsilon)}{\varepsilon^\alpha} \quad \text{and} \quad \|h\|_{\alpha, 1} := \|h\|_1 + \var_{\alpha, 1}(h).
\]

Let
\[
(9.2) \quad H^{\alpha, 1}(m) := \{ \text{m-equivalence class of functions } h : X \to \mathbb{C}, \|h\|_{\alpha, 1} < +\infty \}.
\]
We remark here \( \var_{\alpha, 1}(h) \) and \( \|h\|_{\alpha, 1} \) only depend on the equivalence class of \( h \), and \( (H^{\alpha, 1}, \| \cdot \|_{\alpha, 1}) \) is a Banach space.

9.2. Transfer operator. Fix \( p \geq 1 \). A function \( h : X \to \mathbb{C} \) is of bounded \( p \)-variation, if
\[
\sup \left\{ \left\{ \sum_{i=1}^k |h(x_i) - h(x_{i-1})|^p \right\}^{\frac{1}{p}} : k \leq 1, \ x_0 < \cdots < x_k \in X \right\} < +\infty.
\]

Given \( g : X \to [0, +\infty) \) be a function of bounded \( p \)-variation, let the transfer operator \( L_g \) be the operator acting on the space
\[
\text{Eb}(X) := \{ h : X \to \mathbb{C}, \ |h| < +\infty \},
\]
according to the formula
\[
(9.3) \quad L_g(h)(x) := \sum_{y \in f^{-1}(x)} g(y) h(y).
\]
Such \( g \) is also called the weight function, or simply weight.

9.3. Spectral gap theorem. Given an interval map \( f \in \mathcal{A} \), if the potential \( \log g \) in \( \mathcal{U} \) is hyperbolic, then Keller’s spaces are appropriate for the corresponding transfer operation \( L_g \) on which it admits a quasi-compactness property. We state below more explicitly.

Lemma 9.1. [LRL13a, Coro4.4] Let \( f : J(f) \to J(f) \) be an interval map in \( \mathcal{A} \), \( g : J(f) \to [0, +\infty) \) be a weight function of bounded \( p \)-variation, and \( L_g \) be the transfer operator defined in (9.3). Suppose that there is an integer \( n \geq 1 \) such that the function
\[
(9.4) \quad g_n(x) := g(x) \cdots g(T^{n-1}(x))
\]
satisfies \( \sup_{J(f)} g_n < 1 \). Suppose also \( f \) admits a \( g^{-1} \)-conformal measure \( m \). Then we have the follow properties:

(a) The number 1 is an eigenvalue of \( L_g \) of algebraic multiplicity 1. Moreover, there are constant \( A > 0 \) and \( \rho \in (0, 1) \), such that the spectrum of \( L_g|_{H^{1/p, 1}(m)} \) is contained in \( B(0, \rho) \cup \{1\} \);
(b) There exists a unique equilibrium state \( \mu \) of \( f \) for the potential \( \log g \), and moreover \( \mu \ll m \).
There is a constant $C > 0$, such that for every bounded measurable function $\varphi : X \rightarrow \mathbb{C}$, and every function $\psi \in H^{1/p, 1}(m)$, the equilibrium state $\mu$ has the decay of correlation

$$C_n(\varphi, \psi) \leq C \|\varphi\|_{\infty} \|\psi\|_{1/p, 1}^n, \quad \forall n \geq 1;$$

(d) Given $\psi \in H^{1/p, 1}(m)$, for each $\tau \in \mathbb{C}$ the operator $L_\tau$ defined by

$$L_\tau := L_g(\exp(\tau \psi) \cdot h)$$

is invariant under $H^{1/p, 1}(m)$, and the restriction $L_\tau|_{H^{1/p, 1}(m)}$ is bounded. Moreover, the map $\tau \mapsto L_\tau|_{H^{1/p, 1}(m)}$ is real analytic in the sense of Kato on $\mathbb{C}$, and the spectral radius of $L_\tau|_{H^{1/p, 1}(m)}$ depends on a real analytic way on $\tau$ on a neighborhood of $\tau = 0$.

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