SUBCRITICAL GAUSSIAN MULTIPLICATIVE CHAOS IN THE WIENER SPACE: CONSTRUCTION, MOMENTS AND VOLUME DECAY

RODRIGO BAZAES\textsuperscript{1}, ISABEL LAMMERS\textsuperscript{2}, CHIRANJIB MUKHERJEE\textsuperscript{3}

Universität Münster

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Abstract: We construct and study properties of an infinite dimensional analog of Kahane’s theory of Gaussian multiplicative chaos \cite{14}. Namely, we consider a random field defined with respect to space-time white noise integrated w.r.t. Brownian paths in $d \geq 3$ and construct the infinite volume limit of the normalized exponential of this field, weighted w.r.t. the Wiener measure, in the entire weak disorder (subcritical) regime. Moreover, we characterize this infinite volume measure, which we call the subcritical GMC on the Wiener space, w.r.t. the mollification scheme in the sense of Shamov \cite{32} and determine its support by identifying its thick paths. This in turn implies that almost surely, the subcritical GMC on the Wiener space is singular w.r.t. the Wiener measure. We also prove, in the subcritical regime, existence of negative and positive ($L_p$ for $p > 1$) moments of the total mass of the limiting GMC, and deduce its Hölder exponents (small ball probabilities) explicitly. While the uniform Hölder exponent (the upper bound) and the pointwise scaling exponent (the lower bound) differ for a fixed disorder, we show that, as the disorder goes to zero, the two exponents agree, coinciding with the scaling exponent of the Wiener measure.

1. Introduction

In this article, we construct and study properties of an infinite dimensional analog of the Gaussian multiplicative chaos (GMC) measures, namely, the measures

$$
\mu_{\gamma}(d\omega) = \lim_{T \to \infty} \mu_{\gamma,T}(d\omega),
$$

where

$$
\mu_{\gamma,T}(d\omega) = \exp \left( \gamma H_T(\omega) - \frac{\gamma^2}{2} \mathbb{E}[H_T^2(\omega)] \right) \mathbb{P}_0(d\omega). \tag{1.1}
$$

Here $\mathbb{P}_0$ stands for the Wiener measure corresponding to Brownian paths $\omega : [0, \infty) \to \mathbb{R}^d$, and the Gaussian process $\{H_T(\omega)\}_{\omega \in \Omega}$, indexed by Brownian paths, is driven by a Gaussian space-time white noise $\dot{B}$ (under the probability measure $\mathbb{P}$) integrated w.r.t. the Brownian path:

$$
H_T(\omega) = H_T(\phi, \dot{B}, \omega) = \int_{\mathbb{R}^d} \int_0^T \phi(\omega_s - y) \dot{B}(s, y) ds dy,
$$

Here $\phi$ is a normalized mollifier. Developing this framework for GMC measures is quite natural and important in the field due to the relations to the continuous directed polymers as well as the multiplicative noise stochastic heat equation in $d \geq 3 \ \text{\cite{25}}$. 

\textsuperscript{1}Universität Münster, Einsteinstr. 62, Münster 48149, rbazaes@uni-muenster.de
\textsuperscript{2}Universität Münster, Einsteinstr. 62, Münster 48149, isabel.lammers@uni-muenster.de
\textsuperscript{3}Universität Münster, Einsteinstr. 62, Münster 48149, chiranjib.mukherjee@uni-muenster.de

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Our main goal is to develop an infinite dimensional analog of Kahane's theory \cite{14} of GMC (see below) in this setup. For this purpose, the first task would be to construct the infinite-volume measure \((\mu_\gamma)\) of the field \(H\). This is the content of our first main result – in Theorem 2.1 it is shown that, for any \(d \geq 3\) and in the entire uniform integrability (or weak disorder) regime \(\gamma \in (0, \gamma_c)\), the infinite volume GMC measure

\[
\mu_\gamma(d\omega) := \lim_{T \to \infty} \mu_{\gamma,T}(d\omega) \quad \mathbb{P}\text{-a.s.} \tag{1.3}
\]

exists, is non-trivial and non-atomic. This limit, which is taken w.r.t. the topology of weak convergence, also exists if the Gaussian noise \(\dot{\gamma}\) is replaced by a random environment with finite exponential moment. Subsequently, in Theorem 2.2 we identify the support of this limit, where it is shown that for almost every realization of \(\dot{\gamma}\), the value of the underlying field \(H_T(\omega)\) is atypically large – that is, for any \(\gamma \in (0, \gamma_c)\),

\[
\mu_\gamma \left( \omega : \lim_{T \to \infty} \frac{H_T(\omega)}{T} = \gamma(\phi \ast \phi)(0) \right) = 1 \quad \mathbb{P}\text{-a.s.} \tag{1.4}
\]

In other words, \(\hat{\mu}_\gamma\) is supported on high values of the field \(H_T(\omega)\) and every path \(\omega\) is “\(\gamma\)-thick w.r.t. \(\hat{\mu}_\gamma\)”. Consequently, \(\hat{\mu}_\gamma\) is also almost surely singular w.r.t. the Wiener measure \(\mathbb{P}_0\).

We next investigate the universality of the limit \(\hat{\mu}_\gamma\) by determining the role of the cut-off \(\phi\). Following Kahane’s construction of log-correlated GMC, one expects that, in the current infinite dimensional setup, a well defined limiting object should not depend so much on the choice of the mollifier. For this purpose, and to emphasize the role of \(\phi\), let us write \(\mu_{\gamma,H(\phi)}\) for the infinite-volume limit (which is almost surely a functional of the field \(H(\phi)\)). In this vein, we first show that a strict uniqueness can not hold in the current infinite dimensional setup – Proposition 2.4 implies that \(\mu_{\gamma,H(\phi)}(\cdot) \neq \mu_{\gamma,H(\phi')}(\cdot)\) unless \(\phi\) and \(\phi'\) are identically equal. Then the question naturally arises if one can determine to what degree the limit \(\mu_{\gamma,H(\phi)}\) depends on the mollifying scheme \(\phi\). In this regard, denote by \(P = \mathbb{P}B^{-1}\) the law of the white noise (a probability measure on tempered distributions \(S'(\mathbb{R}_+ \times \mathbb{R}^d)\)). We show in Theorem 2.5 that \(\mu_{\gamma,H(\phi)}\) is the unique measure such that the distribution of \(H_T(\phi)\) under \(\mu_{\gamma}(d\dot{\gamma}, d\omega)P(d\dot{\gamma})\) is the same as the distribution of \(H_T(\phi) + T(\phi \ast \phi)(0)\) under \(P \otimes \mathbb{P}_0\).

In other words, the only way to perturb linearly the distribution \(B\) with the test function

\[
(s,y) \mapsto \phi(\omega_y - y)
\]

is by using the limiting GMC measure \(\mu_{\gamma,H(\phi)}\). That is, the limit satisfies a “Cameron-Martin equation”

\[
\mu_{\gamma,H(\phi)}(v(\omega), d\omega) = e^{v(\omega)} \mu_{\gamma,H(\phi)}(d\omega) \mathbb{P}\text{-a.s.} \tag{1.5}
\]

for all deterministic \(v : \Omega \to \mathbb{R}\) so that the law of \(H(\phi) + v\) is absolutely continuous w.r.t. that of \(H(\phi)\). Thus, at very small scales, the infinite-volume limit still remembers how the field \(H\) was regularized, although it is conceivable that such a small dependence is expected in an infinite dimensional setup (cf. Remark 2.6). In other words, this limiting measure really could be thought of as a family \(\{\mu_{\gamma,H(\phi)}\}_\phi\) where each member of this family verifies the Cameron-Martin equation \(\mu_{\gamma,H(\phi)}(d\omega) = e^{v(\omega)} \mu_{\gamma,H(\phi)}(d\omega)\) for a fixed \(\phi\), as the field \(H\) (being a function of the noise \(\dot{B}\)) varies. This Cameron-Martin characterization (i.e., validity of \(\mu_{\gamma,H(\phi)}(d\omega) = e^{v(\omega)} \mu_{\gamma,H(\phi)}(d\omega)\)) is reminiscent of Shamov’s definition \cite{32} of finite-dimensional GMC, see Remark 2.6. Shamov’s argument shows that, in finite-dimensions and for log-correlated fields, the solution to the Cameron-Martin equation is unique (see also the book by Berestycki and Powell \cite{3} Sec. 3.4 where Shamov’s argument is revisited in a simpler way).

\footnote{As we will mention later, this regime is characterized by the uniform integrability of the martingale \(\{\mu_{\gamma,T}(\Omega)\}_{T > 0}\) when its almost sure limit \(\lim_{T \to \infty} \mu_{\gamma,T}(\Omega) > 0\) remains strictly positive.}
We then deduce some salient properties of \( \mu_{\gamma} \). In Theorem 2.7 we prove the existence of positive and negative moments of its total mass in the entire weak disorder regime, showing that for all \( \gamma \in (0, \gamma_c) \)

\[
\mu_{\gamma}(\Omega) \in L^p(\mathbb{P}) \quad \text{for some } p > 1, \quad \text{and} \\
\mu_{\gamma}(\Omega) \in L^{-q}(\mathbb{P}) \quad \text{for some } q > 0^2.
\]

Moreover, for \( \gamma \) even smaller (in the so-called \( L^2 \)-regime), it holds that \( \mu_{\gamma}(\Omega) \) has negative moments of all order. Finally, we deduce the volume decay or (uniform) Hölder exponents of the normalized GMC measure \( \hat{\mu}_{\gamma} \) – namely, in Theorem 2.9 we show that, for weak disorder, and \( \mathbb{P} \)-almost surely,

\[
\liminf_{\varepsilon \to 0} \varepsilon^2 \log \hat{\mu}_{\gamma}(\|\omega\| < \varepsilon) \geq -C_2, \quad \limsup_{\varepsilon \to 0} \sup_{\eta \in \Omega_0} \varepsilon^2 \log \hat{\mu}_{\gamma}(\|\omega - \eta\| < \varepsilon) \leq -C_1. \tag{1.6}
\]

Here, \( \Omega_0 \) is a subset of the paths carrying a (weighted) norm \( \|\cdot\| \) that makes \( \Omega_0 \) a Banach space with \( \mathbb{P}_0(\Omega_0) = 1 \), and \( C_1, C_2 \) are explicit constants for weak disorder. While for a fixed \( \gamma \in (0, \gamma_c) \), the constants \( C_1 \) and \( C_2 \) do not match (and, given their nature, they should not match, as will be explained below), the bounds given in (1.6) agree in the limit \( \gamma \to 0 \) and coincide with the scaling exponents of the Wiener measure \( \mathbb{P}_0 \) as well. This will also be shown in Theorem 2.9. Finally we mention that, as in Theorem 2.1 Theorem 2.7 and Theorem 2.9 hold also for continuous directed polymers in random environments with finite exponential moments, while for Theorem 2.2 and Theorem 2.5 we do require a Gaussian random environment. We also refer to Section 2.5 for the main ideas of proof.

In order to draw analogies, let us briefly recall Kahane’s theory of GMC for log-correlated fields on finite dimensional spaces. Given a domain \( D \subset \mathbb{R}^d \), a GMC is a rigorously defined version of the measures

\[
m_{\gamma,h}(dx) = \exp \left[ \gamma h(x) - \frac{\gamma^2}{2} \mathbb{E}[h^2(x)] \right] dx.
\]

Here, \( \{h(x)\}_{x \in D} \) is a log-correlated centered Gaussian field with \( \mathbb{E}[h(x)h(y)] = -\log |x-y| + O(1) \). The logarithmic divergence along the diagonal prevents to define \( h \) pointwise, and a regularization process becomes necessary to define \( h \), and consequently, the measures \( m_{\gamma,h} \), in a precise sense. Since the work of Kahane [14], there have been very important works in the field by Robert and Vargas [28], Duplantier and Sheffield [11], Shamov [32] and Berestycki [2], who showed that if \( (h_{\varepsilon})_{\varepsilon \in (0,1)} \) is a suitable approximation of \( h \), then as long as \( \gamma \in (0, \sqrt{2d}) \), which is known as the uniform integrability (or the subcritical) regime, then

\[
\lim_{\varepsilon \to 0} m_{\gamma,h_{\varepsilon}}(dx) = m_{\gamma,h}(dx) \quad \text{weakly and in } \mathbb{P}\text{-probability.}
\]

Moreover, \( m_{\gamma,h} \) is non-trivial, non-atomic and \( \mathbb{P} \)-a.s. and for a.e. \( x \in D \) sampled according to \( m_{\gamma,h} \), it holds that \( h_{\varepsilon}(x) \sim \gamma \log \left( \frac{1}{\varepsilon} \right) \) – that is, almost every point chosen via \( \mu_{\gamma,h} \) is \( \gamma \)-thick. Consequently, \( m_{\gamma,h} \) is singular w.r.t. the Lebesgue measure. We underline the analogies of these statements to (1.3) and (1.4). There have been notable instances where, using the scale-invariance of the logarithmic correlations, studying the positive and negative moments of the total mass \( m_{\gamma,h}(D) \) have been instrumental (see [12] [21] [30] and [3] Ch. 3.7-3.9). A related geometric behavior of \( m_{\gamma,h} \) is captured by the asymptotic behavior of the volume decay \( \log m_{\gamma,h}(B_{\varepsilon}(x)) \) as \( \varepsilon \downarrow 0 \). This is known as the (uniform) Hölder exponents of \( m_{\gamma,h} \) and is closely related to its multifractal behavior. In fact our bounds (1.6) for the infinite-volume GMC \( \mu_{\gamma} \) resemble a similar behavior of the GMC measures \( m_{\gamma,h} \). Indeed, recall that for \( m_{\gamma,h} \), the uniform Hölder exponent is given by \( d(1 - \frac{2}{\sqrt{2d}})^2 \), while its pointwise Hölder exponent is \( d + \gamma^2/2 \) (see [29] Sec. 4.1 for precise definitions). However, none of these exponents fully

\footnote{Here and through the sequel, we say for \( p > 0 \) that \( X \in L^{-p} \) if and only if \( X^{-p} \in L^1 \).}
captures its multifractal spectrum, which roughly says that if a point \( x \in D \) is \( \alpha \)-thick, then
\[
m_{\gamma,\alpha}(B_\varepsilon(x)) \sim C \varepsilon^{d+\frac{2}{\alpha} - 2\gamma}.
\]
Thus, the pointwise (resp. uniform) H\"older scaling exponents are the extremal values of the multifractal spectrum, and these therefore do not match for a fixed \( \gamma \) (but do so as \( \gamma \to 0 \)). We refer to the discussion below [1.6] again to underline the analogy to the H\"older exponents \( C_1, C_2 \) in our setup.

In summary, it is for the first time, to the best of our knowledge, that the existence and the above properties of \( \mu_\gamma \) in [1.3] have been established in the infinite-dimensional setup, and this is done in the entire uniform integrability regime for continuous directed polymers. While GMC for log-correlated fields are not formally related to the fields studied here, we find the point of view of the former quite appealing for studying the latter. By now there is also a rich theory of log-correlated GMC in the critical [10, 27] and supercritical regime [24, 6]. With this background, we expect the present results and the GMC point of view to be useful for further applications to the continuous directed polymer and the corresponding multiplicative noise stochastic PDEs in \( d \geq 3 \).

1.1 Setup and notation.

For a fixed dimension \( d \geq 1 \), let \( \Omega := C([0, \infty), \mathbb{R}^d) \) be the space of continuous functions from \([0, \infty)\) to \( \mathbb{R}^d \), endowed with the topology of uniform convergence on compact sets. We equip this space with the Wiener measure denoted by \( \mathbb{P} \), so that a typical path \( \omega = (\omega_s)_{s \in [0, \infty)} \in \Omega \) corresponds to a realization of a \( \mathbb{R}^d \)-valued Brownian motion starting at 0. Similarly, we denote by \( \mathbb{P}_x \) the Wiener measure corresponding to a Brownian motion starting at \( x \in \mathbb{R}^d \). Let \( (\mathcal{E}, \mathcal{F}, \mathbb{P}) \) be a complete probability space, so that \( \tilde{B} \) is a space-time white noise which is independent of the Wiener measure. More precisely, denote by \( \mathcal{S}(\mathbb{R}_+ \times \mathbb{R}^d) \) the space of rapidly decaying Schwartz functions on \( \mathbb{R}_+ \times \mathbb{R}^d \). Then \( \tilde{B} = \tilde{B}(f)_{f \in \mathcal{S}(\mathbb{R}_+ \times \mathbb{R}^d)} \) is a Gaussian process with mean zero and covariance
\[
\mathbb{E}[\tilde{B}(f)\tilde{B}(g)] = \int_0^\infty \int_{\mathbb{R}^d} f(s, y)g(s, y) \, dy \, ds
\]
\[
= \langle f, g \rangle_{L^2(\mathbb{R}_+ \times \mathbb{R}^d)}, \quad f, g \in \mathcal{S}(\mathbb{R}_+ \times \mathbb{R}^d).
\]

Here, we use the notation
\[
\tilde{B}(f) = \int_0^\infty \int_{\mathbb{R}^d} f(s, y)\tilde{B}(s, y) \, dy \, ds \quad \text{for any} \quad f \in \mathcal{S}(\mathbb{R}_+ \times \mathbb{R}^d).
\]

We can extend the integral to \( f \in L^2(\mathbb{R}_+ \times \mathbb{R}^d) \) via approximation. Indeed, if \( (f_n)_{n \in \mathbb{N}} \subset \mathcal{S}(\mathbb{R}_+ \times \mathbb{R}^d) \) approximates \( f \) in \( L^2(\mathbb{R}_+ \times \mathbb{R}^d) \) (such a sequence exists since \( \mathcal{S}(\mathbb{R}_+ \times \mathbb{R}^d) \) is dense in \( L^2(\mathbb{R}_+ \times \mathbb{R}^d) \)), then the sequence \( (\tilde{B}(f_n))_n \) is Cauchy in \( L^2(\mathbb{P}) \), since we have
\[
\mathbb{E}[(\tilde{B}(f_n) - \tilde{B}(f_m))^2] = \mathbb{E}[\tilde{B}(f_n - f_m)^2] = \|f_n - f_m\|_{L^2(\mathbb{R}_+ \times \mathbb{R}^d)}.
\]

Therefore, for every \( f \in L^2(\mathbb{R}_+ \times \mathbb{R}^d) \), we can define the \( L^2(\mathbb{P}) \)-limit
\[
\int_0^\infty \int_{\mathbb{R}^d} f(s, y)\tilde{B}(s, y) \, dy \, ds := \lim_{n \to \infty} \tilde{B}(f_n).
\]

For \( f \in L^2(\mathbb{R}_+ \times \mathbb{R}^d) \) and \( T > 0 \), we will write
\[
\int_0^T \int_{\mathbb{R}^d} f(s, y)\tilde{B}(s, y) \, dy \, ds := \int_0^\infty \int_{\mathbb{R}^d} \mathbb{1}_{[0,T]}(s, y)f(s, y)\tilde{B}(s, y) \, dy \, ds.
\]

By construction, for \( f \in L^2(\mathbb{R}_+ \times \mathbb{R}^d) \), the random variable \( \int_0^\infty \int_{\mathbb{R}^d} f(s, y)\tilde{B}(s, y) \, dy \, ds \) is Gaussian distributed with mean zero and variance \( \|f\|_{L^2(\mathbb{R}_+ \times \mathbb{R}^d)} \) and for \( f, g \in L^2(\mathbb{R}_+ \times \mathbb{R}^d) \), the covariance
$E[\hat{B}(f)\hat{B}(g)]$ is also given by (1.7). We also define the family of space-time shifts $\{\theta_{t,x} : t > 0, x \in \mathbb{R}^d\}$ acting on the white-noise environment (i.e., $\theta_{t,x}(\hat{B}(s,y)) = \hat{B}(t+s,x+y)$), and remind the reader that $\hat{B}$ is stationary under this action.

Next, let $\phi$ be a mollifier—that is, $\phi$ is a smooth, non-negative, spherically symmetric and compactly supported function $\phi : \mathbb{R}^d \to \mathbb{R}$ such that $\int_{\mathbb{R}^d} \phi(x)dx = 1$. We define the Gaussian field $(H_T(\omega))_{\omega \in \Omega}$ as the Itô integral

$$H_T(\omega) := \int_0^T \int_{\mathbb{R}^d} \phi(\omega_s - y) \hat{B}(s,y) dy ds, \quad \omega \in \Omega.$$  

In particular, $(H_T(\omega))_{\omega \in \Omega}$ is also a Gaussian process with mean zero and covariance

$$E[H_T(\omega)H_T(\omega')] = \int_0^T \int_{\mathbb{R}^d} \phi(\omega_s - y) \phi(\omega'_s - y) dy ds = \int_0^T (\phi \ast \phi)(\omega_s - \omega'_s) ds,$$

where $(f \ast g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y)dy$. Note that, for any $\omega \in \Omega$, $\text{Var}[H_T(\omega)] = T(\phi \ast \phi)(0) - \text{that is, for } T \gg 1$ large, the covariance of the field $(H_T(\omega))_{\omega \in \Omega}$ diverges along the diagonal.

Given any $\gamma > 0$ and $T > 0$, we define the random measure on the path space $\Omega$

$$\mu_{\gamma,T}(d\omega) := \exp \left( \gamma H_T(\omega) - \frac{\gamma^2}{2} \text{Var}(H_T(\omega)) \right) \mathbb{P}_0(d\omega)$$

$$= \exp \left( \gamma H_T(\omega) - \frac{\gamma^2}{2} T(\phi \ast \phi)(0) \right) \mathbb{P}_0(d\omega),$$

and in the sequel, we will refer to $\mu_{\gamma,T}$ as the finite-volume Gaussian multiplicative chaos on $\Omega$ or the continuous directed polymer. Also, we will write

$$\hat{\mu}_{\gamma,T}(A) := \frac{\mu_{\gamma,T}(A)}{\mu_{\gamma,T}(\Omega)}, \quad A \subset \Omega$$

for its normalized counterpart. Let $\mathcal{F}_T$ be the $\sigma$-algebra generated by the noise $\hat{B}$ until time $T$. Since $\hat{B}$ is smoothened only in space, the total mass, or the partition function,

$$\mu_{\gamma,T}(\Omega) = \int_\Omega \exp \left( \gamma H_T(\omega) - \frac{\gamma^2}{2} T(\phi \ast \phi)(0) \right) \mathbb{P}_0(d\omega)$$

is adapted to the filtration $(\mathcal{F}_T)_T$ and is a martingale. Our main results will hold in the uniform integrability or the entire weak-disorder phase $\gamma \in (0,\gamma_c)$ of the martingale $(\mu_{\gamma,T}(\Omega))_T$ determined by the critical value $\gamma_c \in (0,\infty)$ for $d \geq 3$ defined in [43]. We are now ready to state our main results.

2. Main results.

2.1 Infinite volume GMC on $\Omega$: existence, support and singularity. Our first main result provides the existence of the infinite-volume measure $\lim_{T \to \infty} \mu_{\gamma,T}(\cdot)$ on the path space.

**Theorem 2.1** (Existence). Fix $d \geq 3$ and $\gamma \in (0,\gamma_c)$. Then there exists a non-trivial measure $\mu_{\gamma}$ on $\Omega$ such that $\mathbb{P}$-a.s., $\mu_{\gamma,T}$ converges weakly to $\mu_{\gamma}$ as $T \to \infty$. Moreover, for any fixed Borel set $A \subset \Omega$ with $\mathbb{P}_0(A) > 0$, $\mu_{\gamma}(A) > 0$ $\mathbb{P}$-a.s. Similarly, there exists a probability measure $\tilde{\mu}_{\gamma}$ on $\Omega$ such that $\mathbb{P}$-a.s., the normalized approximations $\hat{\mu}_{\gamma,T}$ converge weakly to $\tilde{\mu}_{\gamma}$. Finally, the same conclusion holds if the noise $\hat{B}$ is replaced by any other random environment with finite exponential moments.
The next main result identifies the support of the limiting measure \( \hat{\mu}_\gamma \) in \( \Omega \). For any fixed realization of \( \dot{B} \) and \( a > 0 \), let

\[
T_a := \left\{ \omega \in \Omega : \lim_{T \to \infty} \frac{H_T(\omega)}{T(\phi \ast \phi)(0)} = a \right\}
\]

be the (random) set of \( a \)-thick paths \( \omega \in \Omega \). The next result shows that, for any \( \gamma \in (0, \gamma_c) \), the infinite volume GMC \( \mu_\gamma \) is supported only on \( \gamma \)-thick paths \( \omega \in \Omega \):

**Theorem 2.2** (Support, thick paths and singularity). Fix \( d \geq 3 \) and \( \gamma \in (0, \gamma_c) \). Let \( \hat{\mu}_\gamma \) be the measure defined in Theorem 2.1. Then we have

\[
\hat{\mu}_\gamma(T_\gamma) = 1 \quad \text{P-a.s.} \quad (2.1)
\]

In particular,

\[
\hat{\mu}_\gamma(T_c) = 0 \quad \text{P-a.s.,}
\]

and consequently, for P-a.e. realization of \( \dot{B} \), \( \hat{\mu}_\gamma \) is singular w.r.t. the Wiener measure \( \mathbb{P}_0 \).

### 2.2 Uniqueness.

We turn to the issue of determining to what extent the limiting measure \( \mu_\gamma \) depends on the choice of the mollification \( \phi \). To emphasize this dependence, we will write \( \mu_{\gamma, \phi, T} = \mu_{\gamma, \phi} \) for the approximating measure defined in (1.10) and \( \mu_{\gamma, \phi} = \lim_{T \to \infty} \mu_{\gamma, \phi, T} \) for its infinite volume limit constructed in Theorem 2.1. Before addressing the aforementioned issue, let us note the following immediate consequence of Theorem 2.2.

**Corollary 2.3.** Let \( \phi \) and \( \phi' \) be two mollifiers. Then the limiting measures \( \mu_{\gamma, \phi} \) and \( \mu_{\gamma, \phi'} \) are singular unless \((\phi \ast \phi)(0) = (\phi' \ast \phi')(0)\).

The next result shows that, even if \((\phi \ast \phi)(0) = (\phi' \ast \phi')(0)\), the two measures \( \mu_{\gamma, \phi} \) and \( \mu_{\gamma, \phi'} \) are different unless \( \phi = \phi' \).

**Proposition 2.4.** Let \( \phi \) and \( \phi' \) be two mollifiers. If \( \phi(\cdot) \not\equiv \phi'(\cdot) \), then \( \mu_{\gamma, \phi} \not\equiv \mu_{\gamma, \phi'} \).

While the limiting measures may not be exactly the same for different mollifiers, the following result identifies the limit \( \mu_{\gamma, \phi} \) in a unique manner and determines to what extent it depends on the mollification scheme \( \phi \). We introduce the following notation. Let \( P = \mathbb{P} \circ \dot{B}^{-1} \) be the law of the white noise. For a measure \( \nu \) on \( \Omega \), which may depend on \( \dot{B} \), set

\[
\mathbb{Q}_\nu(d\dot{B}, d\omega) := \nu(d\omega, \dot{B})P(d\dot{B}).
\]

**Theorem 2.5** (Characterization of \( \mu_\gamma \)). For a fixed \( \gamma > 0 \) and \( \phi \) as above, the (unnormalized) GMC measure \( \mu_{\gamma, \phi} \) is the unique measure such that the law of \( \dot{B} \) under \( \mathbb{Q}_{\mu_{\gamma, \phi}} \) is the same as the law of the (Schwartz) distribution

\[
\dot{B}_\phi(f) = \dot{B}(f) + \gamma \int_{\mathbb{R}_+ \times \mathbb{R}^d} f(s, y)\phi(\omega_s - y)dsdy, \quad f \in \mathcal{S}(\mathbb{R}_+ \times \mathbb{R}^d)
\]

under \( P \otimes \mathbb{P}_0 \). In other words, \( \mu_{\gamma, \phi} \) is the unique measure satisfying

\[
\mathbb{E}^P \left[ \int_\Omega \mu_{\gamma, \phi}(d\omega)F(\dot{B}, \omega) \right] = \mathbb{E}^{P \times \mathbb{P}_0} [F(\dot{B}_\phi, \omega)]. \quad (2.3)
\]

for any bounded measurable function \( F : \Omega \times \mathcal{E} \to \mathbb{R} \).
Remark 2.6. The theorem above asserts that the only way to perturb linearly the distribution \( \hat{B} \) with the test function
\[
(s, y) \mapsto \phi(\omega_y - y)
\]
is by using the measure \( \mu_{\gamma, \phi} \). In particular, \( \mu_{\gamma, \phi} \) is the unique measure such that the distribution of \( H_T \) under \( Q_{\mu_{\gamma, \phi}} \) is the same as the distribution of \( H_T + T(\phi \ast \phi)(0) \) under \( P \otimes P_0 \). Note that the characterization of \( \mu_{\gamma} \) provided by Theorem 2.5 is reminiscent of Shamov’s definition of a subcritical GMC [32]. Under this setup, a GMC over the (generalized) Gaussian field \( H \) is a random measure \( \mu = \mu_H \) (measurable w.r.t. \( H \)) on a measure space \( (X, \mathcal{G}) \) if for all deterministic \( v : X \to \mathbb{R} \) such that the law of \( H + v \) is absolutely continuous w.r.t. that of \( H \), then \( \mu_{H + v}(dx) = e^{v(x)} \mu_H(dx) \). The construction of such measure goes via approximating \( H \) by a sequence of fields \( (H_n) \) such that \( H_n \to H \) in a suitable sense. A difference to the present infinite dimensional setup is that the limiting field does depend on the mollifier \( \phi \), since for \( \omega \neq \omega' \in \Omega \),
\[
\lim_{T \to \infty} \mathbb{E}[H_T(\omega)H_T(\omega')] = \int_0^\infty (\phi \ast \phi)(\omega_s - \omega'_s)ds < \infty \quad \mathbb{P}^\otimes \Omega^2 \text{-a.s.}
\]

2.3 Moments of the GMC \( \mu_{\gamma} \). By the martingale convergence theorem, we always have \( \mathbb{E}[\mu_{\gamma}(\Omega)] = 1 \) for all \( \gamma \in (0, \gamma_c) \). Moreover, we also have

**Theorem 2.7** (Positive and negative moment). Fix \( d \geq 3 \) and \( \gamma \in (0, \gamma_c) \).

(i) There is some \( p > 1 \) such that
\[
\mu_{\gamma}(\Omega) \in L^p(P).
\]

(ii) There is some \( q > 0 \) such that
\[
\mu_{\gamma}(\Omega) \in L^{-q}(P).
\]

Moreover, if \( \gamma \in (0, \gamma_c) \) is chosen so that the martingale \( (\mu_{\gamma,T}(\Omega))_T \) is bounded in \( L^2(P) \), then
\[
\mu_{\gamma}(\Omega) \in L^{-q}(P) \quad \forall q \in (0, \infty).
\]

Remark 2.8. We can show [21] and [23] exactly in the same manner if \( \hat{B} \) is replaced by a random environment with finite exponential moments. In [8], it was shown (using Talagrand’s concentration inequality and for Gaussian random environment) that in the “\( L^2 \) region” (i.e., when \( \gamma \in (0, \gamma_c) \) is restricted so that the martingale \( (\mu_{\gamma,T}(\Omega))_T \) remains \( L^2(P) \) bounded) and for all \( q \in (-\infty, 0) \), \( (\mu_{\gamma,T}(\Omega))_T \) is \( L^q(P) \) bounded. It remains an open problem to extend [26] in the full weak disorder regime \( \gamma \in (0, \gamma_c) \). For negative moments of the total mass (or more generally, the tail distribution of the inverse of the partition function) of finite dimensional log-correlated GMC we refer to [28, 12, 21]. For this setup, the tail is extremely thin, and the log-Gaussian fluctuations are due to the variation of the averaged noise (this would correspond to the first term for the chaos expansion of the partition function). Subtracting this average, one obtains a much thinner tail, see [21, Theorem 4.5].

2.4 Volume decay of \( \tilde{\mu}_{\gamma} \). We now turn to the volume decay rate of \( GMC \)-balls, i.e., we determine asymptotic behavior of the quantities
\[
\log \tilde{\mu}_{\gamma}(\omega \in \Omega : \| \omega \| < \varepsilon) \quad \text{as} \quad \varepsilon \downarrow 0.
\]

Note that \( \Omega = C([0, \infty); \mathbb{R}^d) \) is not a Banach space. To define a suitable norm, we will consider a subset \( \Omega_0 \subset \Omega \) with a norm \( \| \cdot \| \) such that \( P_0(\Omega_0) = 1 \) and that \( (\Omega_0, \| \cdot \|) \) is a Banach space. For this purpose, we consider a class of maps \( g : (0, \infty) \to (0, \infty) \) satisfying
\[
\left( \inf_{0 < t < \infty} g(t) \right) \wedge \left( \inf_{0 < t < \infty} \frac{g(t)}{t} \right) > 0.
\]

Remark 5.2. The theorem above asserts that the only way to perturb linearly the distribution \( \hat{B} \) with the test function
\[
(s, y) \mapsto \phi(\omega_y - y)
\]
is by using the measure \( \mu_{\gamma, \phi} \). In particular, \( \mu_{\gamma, \phi} \) is the unique measure such that the distribution of \( H_T \) under \( Q_{\mu_{\gamma, \phi}} \) is the same as the distribution of \( H_T + T(\phi \ast \phi)(0) \) under \( P \otimes P_0 \). Note that the characterization of \( \mu_{\gamma} \) provided by Theorem 2.5 is reminiscent of Shamov’s definition of a subcritical GMC [32]. Under this setup, a GMC over the (generalized) Gaussian field \( H \) is a random measure \( \mu = \mu_H \) (measurable w.r.t. \( H \)) on a measure space \( (X, \mathcal{G}) \) if for all deterministic \( v : X \to \mathbb{R} \) such that the law of \( H + v \) is absolutely continuous w.r.t. that of \( H \), then \( \mu_{H + v}(dx) = e^{v(x)} \mu_H(dx) \). The construction of such measure goes via approximating \( H \) by a sequence of fields \( (H_n) \) such that \( H_n \to H \) in a suitable sense. A difference to the present infinite dimensional setup is that the limiting field does depend on the mollifier \( \phi \), since for \( \omega \neq \omega' \in \Omega \),
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\[
\left( \inf_{0 < t < \infty} g(t) \right) \wedge \left( \inf_{0 < t < \infty} \frac{g(t)}{t} \right) > 0.
\]
For a fixed $g$ as above, we set

$$
\Omega_0 = \Omega_0(g) := \left\{ \omega \in \Omega : \lim_{t \to \infty} \frac{|\omega_t|}{g(t)} = 0 \right\}.
$$

(2.9)

Since $P_0 \left( \lim_{t \to \infty} \frac{|\omega_t|}{g(t)} = 0 \right) = 1$, condition (2.8) assures that $P_0(\Omega_0) = 1$. Moreover, under the norm $\| \cdot \|$, defined as

$$
\| \omega \| := \sup_{t > 0} \frac{|\omega_t|}{g(t)} \quad \text{(finite for all } \omega \in \Omega_0 \text{ by (2.8))},
$$

(2.10)

$\Omega_0$ is a separable Banach space. From now on, we consider an arbitrary $g$ satisfying (2.8).

The next theorem consists of two parts. In the first one, we will determine the decay exponents (upper and lower estimates) for (2.7) valid in the whole weak disorder region $\gamma \in (0, \gamma_c)$. In the second one, we will show that, as $\gamma \to 0$, the upper and lower estimates coincide.

**Theorem 2.9 (Volume decay).** Fix $d \geq 3$.

(i) Given $\gamma \in (0, \gamma_c)$ and $g$ satisfying (2.8), there exists $r_0 > 0$ such that for all $r \in (0, r_0)$, there are explicit constants $0 < C_1 \leq C_2 < \infty$ (defined in (7.12) and (7.13)) fulfilling

$$
- C_2 \leq \liminf_{\varepsilon \to 0} \varepsilon^2 \log \mu_\gamma(\|\omega\| < r \varepsilon) \\
\leq \limsup_{\varepsilon \to 0} \varepsilon^2 \sup_{\eta \in \Omega_0} \log \mu_\gamma(\|\omega - \eta\| < r \varepsilon) \leq -C_1.
$$

(2.11)

Moreover, if $\gamma > 0$ is small enough, there are $p_0 > 1$ and $q_0 > 0$ such that $C_1, C_2$ can be chosen as

$$
C_1 := \frac{(p_0 - 1)}{4p_0 r^2} \int_0^\infty g^{-2}(t) dt - \frac{(p_0 + 1)}{2(p_0 - 1)} \gamma^2 (\phi \ast \phi)(0),
$$

$$
C_2 := \left( \frac{q_0 + 1}{q_0} \right) \left( \frac{j_{\frac{\gamma}{2}}^2}{2r^2} \int_0^\infty g^{-2}(t) dt + \frac{\gamma^2}{2} (\phi \ast \phi)(0) \frac{q_0^2 + 3q_0 + 1}{(q_0 + 1)^2} \right),
$$

(2.12)

where $j_{\frac{\gamma}{2}}$ is the smallest positive root of the Bessel function $J_{\frac{\gamma}{2}}$.

(ii) For every $\delta > 0$, there exists $0 < \gamma_\delta$ such that for all $0 < \gamma < \gamma_\delta$, the constants $C_1, C_2$ from part (i) can be chosen so that

$$
C_2 - C_1 < \delta.
$$

**Remark 2.10.** We first remark that instead of considering weighted norms on $\Omega_0$, it is also possible to obtain analogous estimates as in Theorem 2.9 on $\Omega$ by considering the sets $\{ \omega \in \Omega : \sup_{0 \leq t \leq r} |\omega_t| < \varepsilon \}$ as $\varepsilon \to 0$. Also, we note that the upper bound in (2.11) is uniform over shifted balls of a given radius, while the lower bound holds pointwise. In fact, a uniform lower bound over $\eta \in \Omega_0$ in Theorem 2.9 is not expected. Indeed, already for the Wiener measure $P_0$ (corresponding to the case $\gamma = 0$), for every function in the Cameron-Martin space, defined as the Hilbert space

$$
H^1 := \left\{ \eta \in \Omega_0 : \eta \in L^2([0, \infty); \mathbb{R}^d) \right\}
$$

with inner product induced by the norm $\| \eta \|_{H^1} := \| \eta \|_{L^2([0, \infty); \mathbb{R}^d)}$, we have

$$
P_0(\| \omega - \eta \| < r) \geq P_0(\| \omega \| < r)e^{- \frac{1}{2} \| \eta \|_{H^1}^2}.
$$

We refer to (3.2) in Theorem 3.1.
2.5 Key ideas of the proofs. We briefly outline the key ideas of the proof of our results. For the proof of Theorem 2.1, we have drawn inspiration from the simple and very elegant approach of Berestycki [2] for the construction of GMC for log-correlated fields on finite dimensional spaces. His route goes over a $L^2$ computation to prove convergence in the $L^2$ region, then using Girsanov’s formula to determine thick points (and show that every Liouville point $x \in D$ is $\gamma$-thick in the uniform integrability threshold), and finally removing points that are thicker than this prescribed value to show convergence of the measure in the entire uniform integrability region. In the current setup, we approach the uniform integrability phase directly via the martingale properties of $(\mu_{\gamma,T}(\Omega))_{T \geq 0}$ to show that for a fixed Borel set $A \subset \Omega$, $\lim_{T \to \infty} \mu_{\gamma,T}(A)$ exists $P$-a.s. and in $L^1(P)$. Together with a suitable tightness criteria valid in the Wiener space (see Lemma 4.3), we will deduce the weak convergence of $\mu_{\gamma,T}$ and also its normalization $\hat{\mu}_{\gamma,T}$. It might be worth pointing out that this approach is robust in the sense that it does not rely on the $L^2$ computation, and also it does not need Gaussianity of the environment for the existence of the limit in Theorem 2.1. On the other hand, the proofs of Theorems 2.2 and 2.5 (which from our point of view are important and of independent interest) do require the Cameron-Martin-Girsanov theorem (and therefore, the Gaussianity of the environment is necessary).

We also remark that a possible extension of our results could be provided if we replace the compactly supported function $\phi$ by a mollifier which has a power-law decay at infinity. Such a model for continuous directed polymer was studied by Lacoin [20, Theorem 1.2] where it was shown that a weak disorder phase (in the sense of uniform integrability) exists if the mollifier has a power-law decay at infinity with a certain exponent (the result there gives a sharp criterion). Thus the existence of the GMC could be established in this wider setup just by applying our proof, since uniform integrability seems to be the only important ingredient for this part.

To show Theorem 2.7, we take ideas from the model of discrete directed polymers in $Z^d$. For bounded environments, it has been shown very recently [13] that the polymer martingale is $L^p$ bounded in weak disorder, for some $p > 1$ and in $L^{-q}$ for some $q > 0$. We adapt this approach in the continuum, without requiring any boundedness assumption on the environment and building on the aforementioned martingale properties in the continuum. This argument is also robust and can be easily adapted to continuous directed polymers in environments that are not necessarily Gaussian. Finally, to show Theorem 2.9, we use our earlier result [5] as a guiding philosophy for treating finite-volume GMC $\tilde{\mu}_{\gamma,T}$. However, deducing the Hölder exponents for the infinite-volume limit $\tilde{\mu}_\gamma$ is more subtle. For this purpose, we need to additionally exploit the aforementioned moment estimates – the $L^p$ moment for $p > 1$ of $\nu_{\gamma}(\Omega)$ is used for the upper bound of Theorem 2.9 while the negative moments are needed for the corresponding lower bound. Together with this input, the stationarity of the white noise and a suitable application of the ergodic theorem, hold the key for the proof of Theorem 2.9.

Organization of the article: The rest of the article is organized as follows. Sections 3 and 4 constitute the proof of Theorem 2.1. Theorems 2.2 and 2.5 will be shown in Section 5. The moment estimates in Theorem 2.7 are shown in Section 6 while the Hölder exponents of the GMC measures will be derived in Section 7.

3. Martingale arguments.

Lemma 3.1. Let $(\mathcal{F}_T)$ be the $\sigma$-algebra generated by the noise up to time $T$. Then the following statements hold with respect to the filtration $(\mathcal{F}_T)_{T > 0}$:

- For every $\omega \in \Omega$, the process $H(\omega) = (H_T(\omega))_{T > 0}$ is a martingale. Its quadratic variation is given by
  \[ \langle H(\omega) \rangle_T = \mathbb{E}[H_T(\omega)^2] = \int_0^T (\phi \ast \phi)(0)ds = T(\phi \ast \phi)(0). \]
(b) For every \( \omega \in \Omega \) and \( \gamma > 0 \), the process \( \left( \int_A \exp \left\{ \gamma H_T(\omega) - \frac{\gamma^2}{2} T(\phi \star \phi)(0) \right\} \, dP_0(\omega) \right)_{T > 0} \) is a martingale.

(c) For every Borel set \( A \subset \Omega \) and \( \gamma > 0 \),

\[
\left( \int_A \exp \left\{ \gamma H_T(\omega) - \frac{\gamma^2}{2} T(\phi \star \phi)(0) \right\} \, dP_0(\omega) \right)_{T > 0}
\]

is a martingale.

**Proof.** Parts (a) and (b) are well-known. The statement in (c) follows by the lemma below together with part (b). \( \Box \)

**Lemma 3.2.** Let \( X \in L^1(\mathbb{P} \otimes \mathbb{P}_0) \) and \( \mathcal{G} \subset \mathcal{F} \) be a \( \sigma \)-algebra. Then

\[
\mathbb{E} \left[ \int_\Omega X(\omega, \cdot) \, dP_0(\omega) \bigg| \mathcal{G} \right] = \int_\Omega \mathbb{E} [X(\omega, \cdot) | \mathcal{G}] \, dP_0(\omega).
\]

**Proof.** Let

\[
\mathcal{H} := \left\{ X \in L^1(\mathbb{P} \otimes \mathbb{P}_0) : \text{equation (3.1) is satisfied for any } \sigma \text{-algebra } \mathcal{G} \subset \mathcal{F} \right\}.
\]

**Step 1:** We will first show that \( \mathcal{H} \) is a monotone class:

- Let \( X, Y \in \mathcal{H} \), and \( a > 0 \). Then

\[
\mathbb{E} \left[ \int_\Omega aX(\omega, \cdot) + Y(\omega, \cdot) \, dP_0(\omega) \bigg| \mathcal{G} \right] = a \mathbb{E} \left[ \int_\Omega X(\omega, \cdot) \, dP_0(\omega) \bigg| \mathcal{G} \right] + \mathbb{E} \left[ \int_\Omega Y(\omega, \cdot) \, dP_0(\omega) \bigg| \mathcal{G} \right]
\]

hence \( aX + Y \in \mathcal{H} \).

- Let \( X_1, X_2, \ldots \in \mathcal{H} \) be monotonically increasing, i.e. for every \( n \in \mathbb{N} \) and every \( \omega \in \Omega \), we have \( X_n(\omega, \cdot) \leq X_{n+1}(\omega, \cdot) \). Then for every \( \zeta \in \mathcal{E} \), by monotone convergence

\[
\int_\Omega \lim_{n \to \infty} X_n(\omega, \zeta) \, dP_0(\omega) = \lim_{n \to \infty} \int_\Omega X_n(\omega, \zeta) \, dP_0(\omega).
\]

On the other hand, for every \( \omega \in \Omega \), by monotone convergence of the conditional expectation, we have

\[
\mathbb{E} \left[ \lim_{n \to \infty} X_n(\omega, \cdot) \bigg| \mathcal{G} \right] = \lim_{n \to \infty} \mathbb{E} \left[ X_n(\omega, \cdot) | \mathcal{G} \right].
\]

These two observations and further usage of monotone convergence plus the fact that \( X_n \in \mathcal{H} \) for all \( n \in \mathbb{N} \) yield

\[
\mathbb{E} \left[ \int_\Omega \lim_{n \to \infty} X_n(\omega, \cdot) \, dP_0(\omega) \bigg| \mathcal{G} \right] = \lim_{n \to \infty} \mathbb{E} \left[ \int_\Omega X_n(\omega, \cdot) \, dP_0(\omega) \bigg| \mathcal{G} \right]
\]

\[
= \lim_{n \to \infty} \int_\Omega \mathbb{E} \left[ X_n(\omega, \cdot) | \mathcal{G} \right] \, dP_0(\omega)
\]

\[
= \int_\Omega \mathbb{E} \left[ \lim_{n \to \infty} X_n(\omega, \cdot) | \mathcal{G} \right] \, dP_0(\omega).
\]
Hence, \( \mathcal{H} \) is a monotone class.

**Step 2:** Next, we will show that \( \mathcal{H} \) contains all indicator functions of the form \( \mathbb{1}_{A \times B} \) for \( A \in \mathcal{B} \) and \( B \in \mathcal{F} \). To that end, fix such sets \( A \) and \( B \). Then

\[
\mathbb{E} \left[ \int_{\Omega} \mathbb{1}_{A \times B}(\omega, \cdot) \mathrm{d}\mathbb{P}_0(\omega) \mid \mathcal{G} \right] = \mathbb{E} \left[ \mathbb{P}_0(A) \mathbb{1}_B(\cdot) \mid \mathcal{G} \right] = \mathbb{P}_0(A) \mathbb{P}(B \mid \mathcal{G}).
\]

Similarly and by using independence, we obtain

\[
\int_{\Omega} \mathbb{E} \left[ \mathbb{1}_{A \times B}(\omega, \cdot) \mid \mathcal{G} \right] \mathrm{d}\mathbb{P}_0(\omega) = \int_{\Omega} \mathbb{E}(\omega) \mathbb{E} \left[ \mathbb{1}_B(\cdot) \mid \mathcal{G} \right] \mathrm{d}\mathbb{P}_0(\omega) \mathbb{P}_0(A) \mathbb{P}(B \mid \mathcal{G}).
\]

Therefore, \( \mathbb{1}_{A \times B} \in \mathcal{H} \).

**Step 3:** The same conclusion as Step 2 is true for the indicator function of \( (A \times B)^c = (A^c \times B^c) \cup (A^c \times B) \cup (A \times B^c) \) (here, \( \cup \) means that the union is disjoint). By the monotone class argument, it follows that \( \mathcal{H} \) contains all positive measurable functions and by linearity, also all \( L^1(\mathbb{P} \otimes \mathbb{P}_0) \)-functions. \( \square \)

### 3.1 Uniform integrability.

**Lemma 3.3.** If for some \( \gamma > 0 \), \( (\mu_{\gamma,T}(\Omega))_{T \geq 0} \) is uniformly integrable, then the same holds for all \( \gamma' < \gamma \).

**Proof.** Fix \( \gamma > 0 \) such that \( (\mu_{\gamma,T}(\Omega))_{T \geq 0} \) is uniformly integrable. Let \( \hat{B}, \hat{B}' \) be independent copies of the noise, and let \( \gamma' < \gamma \), so that \( \gamma' = c\gamma \) for some \( 0 < c < 1 \). To avoid ambiguities, we write \( \mu_{\gamma,T}(\Omega) = \mu_{\gamma,T}(\Omega, \hat{B}) \). Note that

\[
\mu_{\gamma',T}(\Omega, \hat{B}) = \mu_{c\gamma,T}(\Omega, \hat{B}) = \mathbb{E} \left[ \mu_{\gamma,T}(\Omega, c\hat{B} + \sqrt{1 - c^2}\hat{B}')|\hat{B} \right].
\] (3.2)

Since \( (\mu_{\gamma,T}(\Omega, \hat{B}))_{T \geq 0} \) is uniformly integrable, then by de La Vallée-Poussin’s theorem, there exists a function \( f : [0, \infty) \to [0, \infty) \) convex and increasing such that

\[
\lim_{x \to \infty} \frac{f(x)}{x} = \infty, \quad \text{and} \quad \sup_T \mathbb{E}[f(\mu_{\gamma,T}(\Omega, \hat{B}))] < \infty.
\]

By (3.2) and Jensen’s inequality,

\[
\mathbb{E}[f(\mu_{\gamma',T}(\Omega, \hat{B}))] = \mathbb{E} \left[ f \left( \mathbb{E} \left[ \mu_{\gamma,T}(\Omega, c\hat{B} + \sqrt{1 - c^2}\hat{B}')|\hat{B} \right] \right) \right] \leq \mathbb{E} \left[ f \left( \mu_{\gamma,T}(\Omega, c\hat{B} + \sqrt{1 - c^2}\hat{B}') \right) \right] = \mathbb{E}[f(\mu_{\gamma,T}(\Omega, \hat{B}))].
\]

It follows that \( \sup_T \mathbb{E}[f(\mu_{\gamma,T}(\Omega, \hat{B}))] < \infty \), which in turn implies the uniform integrability of \( (\mu_{\gamma',T}(\Omega, \hat{B}))_{T \geq 0} \). \( \square \)

The last lemma allows us to define the critical parameter

\[
\gamma_c = \gamma_c(d) = \sup \left\{ \gamma : \text{the martingale } \mu_{\gamma,T}(\Omega) \text{ is uniformly integrable} \right\}. \tag{3.3}
\]

In particular, if \( \gamma < \gamma_c \), then \( (\mu_{\gamma,T}(\Omega))_{T \geq 0} \) is uniformly integrable, while for \( \gamma > \gamma_c \), \( (\mu_{\gamma,T}(\Omega))_{T \geq 0} \) is not uniformly integrable. Since we will be dealing also with the normalized probability measure \( \bar{\mu}_{\gamma,T} \) from (1.10), we need to know whether the denominator vanishes or not at the limit. The next lemma tell us that in the uniform integrable phase, the limit \( \lim_{T \to \infty} \mu_{\gamma,T}(\Omega) \) is positive \( \mathbb{P} \)-a.s:
**Lemma 3.4.** If \((\mu_{\gamma,T}(\Omega))_{T \geq 0}\) is uniformly integrable, then \(\lim_{T \to \infty} \mu_{\gamma,T}(\Omega) > 0\) \(P\)-a.s.

**Proof.** First we note that \(E[\mu_{\gamma,T}(\Omega)] = 1\), so that \(P(\lim_{T \to \infty} \mu_{\gamma,T}(\Omega) > 0) > 0\). Moreover, the event

\[ A_\gamma := \{\mu_{\gamma,T}(\Omega) > 0\} \tag{3.4} \]

has probability 0 or 1 in virtue of Kolmogorov’s 0-1 law.

The non-triviality of \(\gamma_c\), i.e., that \(\gamma_c > 0\) for \(d \geq 3\) is implied by the existence of the so-called \(L^2\)-phase:

**Lemma 3.5.** For \(d \geq 3\) and \(\gamma > 0\) small enough, the martingale \((\mu_{\gamma,T}(\Omega))_{T \geq 0}\) is bounded in \(L^2(P)\). In particular, \(\gamma_c > 0\).

**Proof.** Let \(\omega'\) be an independent copy of \(\omega\) under \(P_0\). Denote by \(P_0^{\otimes 2}\) the product measure of \(P_0\). Then by definition and Fubini’s theorem,

\[
E[\mu_{\gamma,T}(\Omega)^2] = E\left[ \int_{\Omega^2} e^{\gamma(H_T(\omega) + H_T(\omega')) - \gamma^2 T(\phi*\phi)(0)} dP_0^{\otimes 2}(\omega, \omega') \right] 
= \int_{\Omega^2} dP_0^{\otimes 2}(\omega, \omega') E\left[ e^{\gamma(H_T(\omega) + H_T(\omega')) - \gamma^2 T(\phi*\phi)(0)} \right]. 
\]

Recall that for \(\omega, \omega' \in \Omega\), \(H_T(\omega) + H_T(\omega')\) is a Gaussian random variable with mean 0 and variance

\[
2T(\phi*\phi)(0) + 2 \int_0^T (\phi*\phi)(\omega_s - \omega'_s) ds,
\]

and so

\[
E\left[ e^{\gamma(H_T(\omega) + H_T(\omega')) - \gamma^2 T(\phi*\phi)(0)} \right] = e^{\gamma^2 \int_0^T (\phi*\phi)(\omega_s - \omega'_s) ds}.
\]

Thus,

\[
E[\mu_{\gamma,T}(\Omega)^2] = \int_{\Omega^2} dP_0^{\otimes 2}(\omega, \omega') e^{\gamma^2 \int_0^T (\phi*\phi)(\omega_s - \omega'_s) ds} = \int_{\Omega} dP_0(\omega) e^{\gamma^2 \int_0^T (\phi*\phi)(\sqrt{2} \omega_s) ds} 
\leq \int_{\Omega} dP_0(\omega) e^{\gamma^2 \int_0^\infty (\phi*\phi)(\sqrt{2} \omega_s) ds}.
\]

Since \(d \geq 3\) and \(\phi*\phi\) is bounded with compact support,

\[
I(\phi) := \sup_{x \in \mathbb{R}^d} \int_{\Omega} dP_0(\omega) \int_0^\infty (\phi*\phi)(\sqrt{2} \omega_s) ds < \infty,
\]

so that that for \(\gamma > 0\) small enough,

\[
\gamma^2 I(\phi) < 1.
\]

By Kahs’minski’s lemma [17] we deduce that

\[
\sup_T E[\mu_{\gamma,T}(\Omega)^2] \leq \sup_{x \in \mathbb{R}^d} \int_{\Omega} dP_x(\omega) e^{\gamma^2 \int_0^\infty \phi*\phi(\sqrt{2} \omega_s) ds} < \infty.
\]

**Remark 3.6.** By an application of Kahane’s inequality [15], in [25] it is proved additionally that \(\gamma_c < \infty\), and

\[(\mu_{\gamma,T}(\Omega))_{T \geq 0}\] is uniformly integrable \(\iff\ \lim_{T \to \infty} \mu_{\gamma,T}(\Omega) > 0\) \(P\)-a.s.,

strengthening Lemma 3.4 to an “if and only if” statement. These results are not required in the sequel. We also refer to [31] where the weak and strong disorder for the partition function as well as
concentration inequalities were obtained for the continuous directed polymer in a Gaussian random environment.

4. Proof of Theorem 2.1

The proof of Theorem 2.1 is split into two parts. At first, we will show that for a fixed Borel set $A \subset \Omega$, the sequence $(\mu_{\gamma,T}(A))_{T \geq 0}$ converges to some random variable. The second step will be to deduce that $\mu_{\gamma,T}$ and its normalized version converge weakly as a measure.

**Lemma 4.1.** Fix a Borel set $A \subset \Omega$. For $\gamma < \gamma_c$, the sequence $(\mu_{\gamma,T}(A))_{T \geq 0}$ converges in $L^1(\mathbb{P})$ and almost surely to a random variable $\mu_{\gamma}(A)$.

**Proof.** By Lemma 3.1, for any Borel set $A \subset \Omega$, 

$$
\mu_{\gamma,T}(A) = \int_A \exp \left\{ \gamma H_T(\omega) - \frac{\gamma^2}{2} T(\phi \ast \phi)(0) \right\} \, d\mathbb{P}_0(\omega)
$$

(4.1)

is a martingale with respect to the filtration $(\mathcal{F}_T)_{T}$. By our choice of $\gamma$, the sequence $(\mu_{\gamma,T}(A))_{T}$ is uniformly integrable and therefore the same is true for $(\mu_{\gamma,T}(A))_{T}$. By the martingale convergence theorem, the latter converges in $L^1(\mathbb{P})$ and almost surely to a random variable. We denote the limit by $\mu_{\gamma}(A)$.

□

**Remark 4.2.** By Lemma 3.4, we know that $\mu_{\gamma}(\Omega) > 0$ $\mathbb{P}$-a.s. Hence, we can replace $\mu_{\gamma,T}$ by its normalized version while preserving the almost sure convergence. Indeed, for any $A \in \mathcal{B}$, we have

$$
\hat{\mu}_{\gamma,T}(A) := \frac{\mu_{\gamma,T}(A)}{\mu_{\gamma,T}(\Omega)} \rightarrow \frac{\mu(A)}{\mu(\Omega)} \ \mathbb{P}\text{-a.s.}
$$

The next lemma provides a tightness criterion for a sequence of probability measures on the Wiener space.

**Lemma 4.3.** [18, Theorem. 18.13] A sequence $P_n$ of probability measures on $(\Omega, \mathcal{B})$ is tight if and only if the following two conditions hold:

(a) For any $T > 0$ and $\eta > 0$, there is $a > 0$ such that

$$
P_n \left( \left\{ \omega : \sup_{0 \leq t \leq T} |\omega(t)| > a \right\} \right) < \eta, \ \ n \geq 1.
$$

(b) For any $T > 0, \eta > 0$ and $\varepsilon > 0$, there is $\delta > 0$ such that

$$
P_n \left( \left\{ \omega : m_T(\omega, \delta) > \varepsilon \right\} \right) < \eta, \ \ n \geq 1,
$$

where $m_T(\omega, \delta) := \sup_{0 \leq s, t \leq T, |t-s| < \delta} |\omega(t) - \omega(s)|$.

**Lemma 4.4.** Let $\mu_{\gamma}$ be defined as in Lemma 4.1. If $(A_n)_n \subset \mathcal{B}$ is a sequence of subsets of $\Omega$ that is increasing in the sense that $A_n \subset A_{n+1}$ for all $n \geq 1$, then $\mathbb{P}$-a.s. it is true that

$$
\mu_{\gamma} \left( \bigcup_n A_n \right) = \lim_{n \to \infty} \mu_{\gamma}(A_n).
$$

(4.2)

An analogous statement holds for decreasing events.
Proof. First note that the limit on the right hand side in (4.2) exists since the sequence $(\mu_\gamma(A_n))_n$ is non-decreasing (by monotonicity of the limit and of $\mu_{\gamma,T}$ as a measure) and it is bounded by

$$\mu_\gamma \left( \bigcup_n A_n \right) \geq \lim_{n \to \infty} \mu_\gamma(A_n).$$

To prove almost sure equality, it suffices to show that their expectations coincide. By $L^1(P)$-convergence, the definition of the measure $\mu_{\gamma,T}$ and Fubini’s theorem, we obtain

$$E \left[ \mu_\gamma \left( \bigcup_n A_n \right) \right] = \lim_{T \to \infty} E \left[ \mu_{\gamma,T} \left( \bigcup_n A_n \right) \right]$$

$$= \lim_{T \to \infty} \int_{\bigcup_n A_n} \exp \left\{ \gamma H_T(\omega) - \frac{\gamma^2}{2} T(\phi \ast \phi)(0) \right\} dP_0(\omega)$$

$$= \lim_{T \to \infty} \int_{\bigcup_n A_n} E \left[ \exp \left\{ \gamma H_T(\omega) - \frac{\gamma^2}{2} T(\phi \ast \phi)(0) \right\} \right] dP_0(\omega)$$

$$= \lim_{T \to \infty} P_0 \left( \bigcup_n A_n \right) = \lim_{n \to \infty} P_0(A_n).$$

On the other hand, by monotone convergence, $L^1(P)$-convergence and the same argument as above, we obtain

$$E \left[ \lim_{n \to \infty} \mu_\gamma(A_n) \right] = \lim_{n \to \infty} E[\mu_\gamma(A_n)] = \lim_{n \to \infty} E \left[ \lim_{T \to \infty} \mu_{\gamma,T}(A_n) \right]$$

$$= \lim_{n \to \infty} \lim_{T \to \infty} E[\mu_{\gamma,T}(A_n)] = \lim_{n \to \infty} P_0(A_n).$$

This concludes the proof. \qed

Remark 4.5. If we normalize $\mu_\gamma(A)$, i.e. if we consider the random variables given by $\frac{\mu_\gamma(A)}{\mu_\gamma(\Omega)}$ for $A \in B$, then clearly the statement in Lemma 4.4 remains true.

Proposition 4.6. Suppose that for each Borel set $A \subset \Omega$, the sequence $\mu_{\gamma,T}(A)$ converges in $L^1(P)$ and almost surely to some random variable $\mu_\gamma(A)$ as $T \to \infty$. Then the sequence $(\mu_{\gamma,T})_{T \geq 0}$ converges weakly $P$-a.s. to a random measure $\mu_\gamma$. Similarly, the sequence of normalized measures $(\hat{\mu}_{\gamma,T})_{T \geq 0}$ converges weakly $P$-a.s. to a random probability measure $\hat{\mu}_\gamma$.

Proof. First we will introduce some notation. Define the sets

$$A_o := \{(x_1, y_1) \times \ldots \times (x_d, y_d) : x_i, y_i \in Q\},$$

$$A_c := \{[x_1, y_1] \times \ldots \times [x_d, y_d] : x_i, y_i \in Q\},$$

$$A_s := \{[x_1, y_1] \times \ldots \times [x_d, y_d] : x_i, y_i \in Q\},$$

and

$$A_o^1 := \left\{ \omega : \omega(t_1) \in A_1, \ldots, \omega(t_n) \in A_n \right\} : n \in \mathbb{N}, t_1 < \ldots < t_n \in \mathbb{Q}^+, A_1, \ldots, A_n \in A_o \right\}. \quad (4.4)$$
GMC AND CONTINUOUS DIRECTED POLYMER IN THE WIENER SPACE IN $d \geq 3$

Analogously define $X_1^c$ and $X_1^s$ and set $X_1 := X_1^1 \cup X_1^c \cup X_1^s$. Further, define

$$X_2 := \Omega \cup \left\{ \omega : \sup_{0 \leq t \leq T} |\omega(t)| > a \right\} \cup \left\{ \omega : m^T(\omega, \delta) > \varepsilon \right\} : T, \delta, \varepsilon \in \mathbb{Q}^+ \right\}, \text{ and } \mathcal{X} := X_1 \cup X_2.$$  \hspace{1cm} (4.5)

Then $\mathcal{X}$ is a countable subset of the Borel $\sigma$-algebra on $\Omega$. Since $\mathcal{X}$ is countable, by Lemma 4.1 (or more precisely, by the subsequent remark),

$$\hat{\mu}_{\gamma,T}(A) \overset{T \to \infty}{\to} \mu_{\gamma}(A)/m_{\gamma}(\Omega)$$

almost surely for all $A \in \mathcal{X}$ (simultaneously). We will use this observation to show that

$$\hat{\mu}_{\gamma,T} \overset{T \geq 0}{\text{is P-a.s. tight.}}$$

In order to show tightness, we will verify that the conditions of Lemma 4.3 are satisfied, for which we will first check condition (a) thereof. Note that for a fixed $s \in \mathbb{Q}^+$, the events

$$Q^+ \ni a \mapsto \left\{ \omega : \sup_{0 \leq t \leq s} |\omega(t)| > a \right\}$$

are decreasing. Hence, by Lemma 4.4 (and its the subsequent remark), we deduce that

$$\lim_{a \to \infty, a \in \mathbb{Q}^+} \frac{\mu_{\gamma}(\{\omega : \sup_{0 \leq t \leq s} |\omega(t)| > a\})}{\mu_{\gamma}(\Omega)} = 0 \quad \text{P-a.s.}$$

In particular, for fixed $s, \eta, \varepsilon \in \mathbb{Q}^+$, one can find an $a \in \mathbb{Q}^+$ such that P-a.s. we have

$$\frac{\mu_{\gamma}(\{\omega : \sup_{0 \leq t \leq s} |\omega(t)| > a\})}{\mu_{\gamma}(\Omega)} \leq \eta$$

Since

$$\lim_{T \to \infty} \frac{\hat{\mu}_{\gamma,T}(\{\omega : \sup_{0 \leq t \leq s} |\omega(t)| > a\})}{\mu_{\gamma}(\Omega)} = \frac{\mu_{\gamma}(\{\omega : \sup_{0 \leq t \leq s} |\omega(t)| > a\})}{\mu_{\gamma}(\Omega)} \leq \frac{\eta}{2} \quad \text{P-a.s.,}$$

we have

$$\hat{\mu}_{\gamma,T}(\{\omega : \sup_{0 \leq t \leq s} |\omega(t)| > a\}) \leq \eta, \quad \text{P-a.s.}$$

for some $T > 0$ large enough. Hence, we can choose $a$ large enough such that we can assure that the inequality holds for all $T > 0$. Therefore, condition (a) of Lemma 4.3 is satisfied. Condition (b) follows with a similar argumentation. Indeed, for fixed $\ell, \eta, \varepsilon \in \mathbb{Q}^+$, the events

$$Q^+ \ni \delta \mapsto \left\{ \omega : \sup_{0 \leq s, t \leq \ell, |t-s| \leq \delta} |\omega(t) - \omega(s)| > \varepsilon \right\}$$

are decreasing as $\delta \downarrow 0$. Again by Lemma 4.4, it holds that

$$\lim_{\delta \to 0, \delta \in \mathbb{Q}^+} \frac{\mu_{\gamma}(\{\omega : m^\ell(\omega, \delta) > \varepsilon\})}{\mu_{\gamma}(\Omega)} = 0 \quad \text{P-a.s.}$$

Thus, there exists a $\delta \in \mathbb{Q}^+$ such that

$$\lim_{T \to \infty} \frac{\hat{\mu}_{\gamma,T}(\{\omega : m^\ell(\omega, \delta) > \varepsilon\})}{\mu_{\gamma}(\Omega)} = \frac{\mu_{\gamma}(\{\omega : m^T(\omega, \delta) > \varepsilon\})}{\mu_{\gamma}(\Omega)} \leq \frac{\eta}{2} \quad \text{P-a.s.}$$

Hence, for some $T$ large enough we have

$$\hat{\mu}_{\gamma,T}(\{\omega : m^\ell(\omega, \delta) > \varepsilon\}) \leq \eta, \quad \text{P-a.s.}$$
If we now choose $\delta$ sufficiently small, we can assure that the inequality holds for all $k \geq 1$. We have shown that $\mathbb{P}$-a.s. conditions (a) and (b) of Lemma 4.3 are satisfied and therefore the sequence $(\hat{\mu}_{\gamma,T})_{T \geq 0}$ is $\mathbb{P}$-a.s. tight.

Therefore, for each subsequence of $(\hat{\mu}_{\gamma,T})_{T \geq 0}$ there is a further subsequence converging weakly to some random measure $\hat{\mu}_\gamma$. It remains to show that the limiting measure $\hat{\mu}_\gamma$ is uniquely determined (i.e., independent of the subsequence). We will use Portmanneau’s Theorem to show that for all Borel sets $A \subset \Omega$, it is true that any weak limit satisfies

$$\hat{\mu}_\gamma(A) = \frac{\mu_\gamma(A)}{\mu_\gamma(\Omega)}.$$

More precisely, it suffices to show that this is true for all sets $A \in \mathcal{X}_s^1$ (recall (4.1)). Hence, let $A \in \mathcal{X}_s^1$, i.e. $A$ is of the form $A = \{\omega : \omega(t_1) \in A_1, \ldots, \omega(t_n) \in A_n\}$ for some $n \in \mathbb{N}, t_1 < \ldots < t_n \in \mathbb{Q}^+$ and $A_1, \ldots, A_n \in \mathcal{A}_s$. Then by Lemma 4.3,

$$\frac{\mu_\gamma(A)}{\mu_\gamma(\Omega)} = \sup_{A_1', \ldots, A_n'} \frac{\mu_\gamma(\{\omega : \omega(t_1) \in A_1', \ldots, \omega(t_n) \in A_n'\})}{\mu_\gamma(\Omega)} \quad \text{P-a.s.,}$$

where the supremum is over $A_1', \ldots, A_n' \in \mathcal{A}_c$ (recall (4.3)) such that $A_i' \subset A_i$ for all $1 \leq i \leq n$. Similarly, we have

$$\frac{\mu_\gamma(A)}{\mu_\gamma(\Omega)} = \inf_{A_1', \ldots, A_n'} \frac{\mu_\gamma(\{\omega : \omega(t_1) \in A_1', \ldots, \omega(t_n) \in A_n'\})}{\mu_\gamma(\Omega)} \quad \text{P-a.s.,}$$

where the infimum is over all $A_1', \ldots, A_n' \in \mathcal{A}_o$ such that $A_i \subset A_i'$ for all $1 \leq i \leq n$. By Lemma 4.4 and Portmanneau’s Theorem, for any $A \in \mathcal{X}_s^1$ and any $A_1', \ldots, A_n' \in \mathcal{A}_o$ with $A_i \subset A_i'$ for all $1 \leq i \leq n$, $\mathbb{P}$-a.s., we have

$$\hat{\mu}_\gamma(\{\omega : \omega(t_1) \in A_1', \ldots, \omega(t_n) \in A_n'\}) \leq \liminf_{T \to \infty} \frac{\mu_\gamma(T(\{\omega : \omega(t_1) \in A_1', \ldots, \omega(t_n) \in A_n'\}))}{\mu_\gamma(T(\Omega))} = \lim_{T \to \infty} \frac{\mu_\gamma(T(\{\omega : \omega(t_1) \in A_1', \ldots, \omega(t_n) \in A_n'\}))}{\mu_\gamma(T(\Omega))} = \frac{\mu_\gamma(\{\omega : \omega(t_1) \in A_1', \ldots, \omega(t_n) \in A_n'\})}{\mu_\gamma(\Omega)}.$$

Since $A_1', \ldots, A_n'$ are arbitrary, by taking the infimum and using (4.7), we can deduce that $\mathbb{P}$-a.s., $\hat{\mu}_\gamma(A) \leq \frac{\mu_\gamma(A)}{\mu_\gamma(\Omega)}$. Proceeding analogously with sets $A_1', \ldots, A_n' \in \mathcal{A}_c$ such that $A_i' \subset A_i$ for all $1 \leq i \leq n$ and using (4.6), we obtain $\mathbb{P}$-a.s., $\hat{\mu}_\gamma(A) \geq \frac{\mu_\gamma(A)}{\mu_\gamma(\Omega)}$. Therefore, $\hat{\mu}_\gamma(A) = \frac{\mu_\gamma(A)}{\mu_\gamma(\Omega)}$ $\mathbb{P}$-a.s. as desired. The proof of the weak convergence for the unnormalized measures can be performed by the same methods, by noting that $\mu_\gamma(\Omega) > 0$ $\mathbb{P}$-a.s.

We can now complete the

**Proof of Theorem 2.1.** By Lemma 4.1, for each Borel set $A \subset \Omega$, the sequence $(\hat{\mu}_{\gamma,T}(A))$ converges $\mathbb{P}$-a.s. to $\frac{\mu_\gamma(A)}{\mu_\gamma(\Omega)}$ as $T \to \infty$. Hence, by Proposition 4.6, $\hat{\mu}_{\gamma,T}$ converges weakly $\mathbb{P}$-a.s. to the measure $\hat{\mu}_\gamma$. \hfill \Box

**Remark 4.7.** For discrete directed polymers, Comets and Yosida [9] construct a limiting measure $\mu_\infty$ on the $\sigma$-algebra generated by finite paths, such that, on this $\sigma$-algebra, $\mu_\infty(A) = \lim_{n \to \infty} \mu_n(A)$ (here, $\mu_n$ is the normalized polymer measure at time $n$). Indeed, the limiting measure can be identified explicitly as the law of a random, time-inhomogeneous Markov chain, and therefore the martingale

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convergence theorem guarantees the weak convergence of $\mu_n$ to $\mu_\infty$. In our setup, such limiting measure does not exist a priori, so that showing convergence over fixed sets does not guarantee weak convergence of $(\hat{\mu}_n)_T \geq 0$.

5. Uniqueness: Proofs of Theorem 2.4 and Theorem 2.5

We will now prove Theorem 2.5 which will imply Theorem 2.4. We conclude this section by providing the proof of Proposition 2.4.

5.1 Proof of Theorem 2.5

Recall that $H_T(\omega) = H_T(\omega, \hat{B})$ where $\hat{B}$ is the space-time white noise. Similarly, we write $\mu_\gamma(d\omega) = \mu_\gamma(d\omega, \hat{B})$. Given $T > 0$, set (recall the notation from (2.2))

$$Q_{\mu_\gamma}(d\hat{B}, d\omega) = \exp \left\{ \gamma H_T(\omega, \hat{B}) - \frac{\gamma^2}{2} T(\phi * \phi)(0) \right\} \mathbb{P}_0(d\omega) P(d\hat{B}).$$

This is a probability measure since $E \left[ \exp \left\{ \gamma H_T(\omega, \hat{B}) - \frac{\gamma^2}{2} T(\phi * \phi)(0) \right\} \right] = 1$. Similarly define $Q_{\mu_\gamma}(d\hat{B}, d\omega) = \mu_\gamma(d\omega, \hat{B}) P(d\hat{B})$. Before continuing, we show that

**Lemma 5.1.** $Q_{\mu_\gamma} \rightarrow Q_{\mu_\gamma}$ weakly as $T \rightarrow \infty$.

**Proof.** The proof follows a similar line of arguments as that of Theorem 2.4. We prove first that $(Q_{\mu_\gamma})_{T \geq 0}$ is tight. It is enough to verify that the marginals $Q_{\mu_\gamma}^{1, T}(\cdot) := Q_{\mu_\gamma}^{1, T}(\cdot \times S)'$ and $Q_{\mu_\gamma}^2(\cdot) := Q_{\mu_\gamma}^{1, T}(\Omega \times \cdot) = P(\cdot)$ are tight. We can again check the tightness of $Q_{\mu_\gamma}^1$ by applying Lemma 4.3 as in Proposition 4.6. On the other hand, since $S'$ is $\sigma$-compact (by the Banach-Alaoglu theorem), then $P$ is also tight. 

By the previous lemma, if $n \in \mathbb{N}$, $f_1, \ldots , f_n \in S(\mathbb{R}_+ \times \mathbb{R}^d)$ and $g : \Omega \rightarrow \mathbb{R}$ is bounded and continuous, i.e the map $(\omega, \hat{B}) \mapsto g(\omega) \hat{B}(f_1) \cdots \hat{B}(f_n)$ is a bounded continuous function on $\Omega \times S'$, then

$$\lim_{T \rightarrow \infty} E \left[ \hat{B}(f_1) \cdots \hat{B}(f_n) \int F(\omega) \exp \left\{ \gamma H_T(\omega, \hat{B}) - \frac{\gamma^2}{2} T(\phi * \phi)(0) \right\} \mathbb{P}_0(d\omega) \right]$$

$$= E \left[ \hat{B}(f_1) \cdots \hat{B}(f_n) \int F(\omega) \mu_\gamma(d\omega, \hat{B}) \right].$$

Now, conditioning $Q_{\mu_\gamma}(d\hat{B}, d\omega)$ on $\omega \in \Omega$, we obtain

$$Q_{\mu_\gamma}(d\hat{B} | \omega) = \exp \left\{ \gamma H_T(\omega, \hat{B}) - \frac{\gamma^2}{2} T(\phi * \phi)(0) \right\} P(d\hat{B}).$$

By the Cameron-Martin-Girsanov Theorem, we know that under the measure $Q_{\mu_\gamma}(d\hat{B} | \omega)$, $(\hat{B}(f))_f$ is a Gaussian process with the same covariance structure as in (1.7) and mean given by

$$\int \hat{B}(f) dQ_{\mu_\gamma}(d\hat{B} | \omega) = \gamma \text{Cov}(\hat{B}(f), H_T(\omega)) = \gamma \int_{\mathbb{R}_+ \times \mathbb{R}^d} 1_{[0,T]}(s) f(s, y) \phi(\omega s - y) ds dy.$$
Hence, the expression in (5.1) is equal to
\[
\lim_{T \to \infty} \int g(\omega) \int_{\Omega} \hat{B}(f_1) \cdot \ldots \cdot \hat{B}(f_n) d\mathbb{Q}_{\mu,\gamma,\tau}(d\hat{B}, \omega) d\mathbb{P}_0(d\omega)
\]
\[
= \lim_{T \to \infty} \int g(\omega) \int_{\Omega} \prod_{i=1}^{n} \left( \hat{B}(f_i) + \gamma \text{Cov} \left( \hat{B}(f_i), H_T(\omega) \right) \right) P(d\hat{B}) d\mathbb{P}_0(d\omega)
\]
\[
= \lim_{T \to \infty} \int g(\omega) \int_{\Omega} \prod_{i=1}^{n} \left( \hat{B}(f_i) + \gamma \int_{\mathbb{R}^+ \times \mathbb{R}^d} \mathbb{1}_{[0,T]}(s)f_i(s,y)\phi(\omega_s-y) ds dy \right) P(d\hat{B}) d\mathbb{P}_0(d\omega).
\]
Writing \( \mathbb{Q}_{\mathbb{P}_0}(d\hat{B}, d\omega) = \mathbb{P}_0(d\omega)P(d\hat{B}) \), and using (5.1), we deduce that
\[
\int g(\omega) \prod_{i=1}^{n} \left( \hat{B}(f_i) + \gamma \int_{\mathbb{R}^+ \times \mathbb{R}^d} f_i(s,y)\phi(\omega_s-y) ds dy \right) \mathbb{Q}_{\mathbb{P}_0}(d\hat{B}, d\omega)
\]
\[
= \int g(\omega) \prod_{i=1}^{n} \hat{B}(f_i) \mathbb{Q}(d\hat{B}, d\omega).
\]
Since \( g, f_1, \ldots, f_n \) are arbitrary, we conclude that the law of \( \hat{B} \) under \( \mathbb{Q} \) is the same as the law of \( \hat{B} \) defined by \( \hat{B}(f) := \hat{B}(f) + \gamma \int_{\mathbb{R}^+ \times \mathbb{R}^d} f(s,y)\phi(\omega_s-y) ds dy \) under \( \mathbb{Q}_{\mathbb{P}_0} \). The uniqueness is immediate from this construction. \( \square \)

5.2 Proof of Theorem 2.2. By Theorem 2.3, we know that the law of \( H_T \) under the measure \( \mathbb{Q}_{\mu,\gamma}(d\hat{B}, d\omega) = \mu,\gamma,\tau(\omega, \hat{B})P(d\hat{B}) \) is the same as the law of \( H_T + \gamma T(\phi \star \phi)(0) \) under \( P \otimes \mathbb{P}_0 \). Furthermore, for every \( \omega \in \Omega \), we have
\[
\lim_{T \to \infty} \frac{H_T(\omega)}{T} = 0 \quad \mathbb{P}\text{-a.s.}
\]
(since the stochastic integral is a martingale and its quadratic variation is equal to \( T(\phi \star \phi)(0) \) [23 Chapter 2, §6]), so that
\[
\lim_{T \to \infty} \frac{1}{T} (H_T + T(\phi \star \phi)(0)) = \gamma(\phi \star \phi)(0) \quad P \otimes \mathbb{P}_0\text{-a.s.}
\]
Hence,
\[
\lim_{T \to \infty} \frac{H_T}{T} = \gamma(\phi \star \phi)(0) \quad \mathbb{Q}\text{-a.s.,}
\]
so that \( \left( \lim_{T \to \infty} \frac{H_T}{T} \neq \gamma(\phi \star \phi)(0) \right) = 0 \quad \mathbb{P}\text{-a.s.} \quad \square \)

Proof of Proposition 2.4. Since for any \( A \subset \Omega \) Borel measurable, \( (\mu_{\gamma,\tau}(A))_{T \geq 0} \) and \( (\mu_{\gamma,\tau'}(A))_{T \geq 0} \) are uniformly integrable martingales converging respectively to \( \mu_{\gamma,\tau}(A) \) and \( \mu_{\gamma,\tau'}(A) \), we have \( \mathbb{P}\text{-a.s.} \) the following identifications:
\[
\mathbb{E}[\mu_{\gamma,\phi}(A)|\mathcal{F}_T] = \mu_{\gamma,\tau}(A), \quad \mathbb{E}[\mu_{\gamma,\phi'}(A)|\mathcal{F}_T] = \mu_{\gamma,\tau'}(A).
\]
Thus, if \( \mu_{\gamma,\phi}(A) = \mu_{\gamma,\phi'}(A) \) \( \mathbb{P}\text{-a.s.} \), then we deduce that for all \( T \geq 0 \), \( \mu_{\gamma,\tau}(A) = \mu_{\gamma,\tau'}(A) \) \( \mathbb{P}\text{-a.s.} \).

By choosing an appropriate countable collection of Borel sets \( A \in \Omega \) (such as the sets \( A_{s}^{1} \) that appear in the proof of Proposition 4.6) one can deduce that \( \mathbb{P} \otimes \mathbb{P}_0\text{-a.s.} \) for all \( T \in \mathbb{Q}^+ \) and assuming that \( \phi \star \phi(0) = \phi' \star \phi'(0) \),
\[
eq e^{\gamma H_{T,\phi} - \frac{T}{2} T \phi \star \phi(0)} = e^{\gamma H_{T,\phi'} - \frac{T}{2} T \phi' \star \phi'(0)} = e^{\gamma H_{T,\phi'} - \frac{T}{2} T \phi \star \phi(0)}.
\]
concluding that $\mathbf{P} \otimes \mathbb{P}_0$-a.s., $H_{T,\phi} = H_{T,\phi'}$. Note that that for $\omega \in \Omega$, the quadratic variation of $H_{T,\phi}(\omega) - H_{T,\phi'}(\omega)$ is equal to

$$2T \phi * \phi(0) - 2T \int_{\mathbb{R}^d} \phi(y) \phi'(y)dy = 2T \left( \int_{\mathbb{R}^d} \phi(y)^2dy - \int_{\mathbb{R}^d} \phi(y)\phi'(y)dy \right).$$

By Cauchy-Schwarz inequality, we know that the last display is equal to zero if and only if $\phi = \lambda \phi'$ for some $\lambda > 0$, but using that $\int_{\mathbb{R}^d} \phi(y)dy = \int_{\mathbb{R}^d} \phi'(y)dy = 1$, we deduce that $\lambda = 1$, so $\phi = \phi'$.

**Remark 5.2.** One may wonder if, using Theorem 2.2, for a fixed $\tau$, then by Fatou’s lemma we can deduce (6.2) for $\tau$.

Before proving the lemma, we introduce, for $\gamma > 0$, the stopping time

$$\tau_n := \inf\{T \geq 0 : H_{T,\phi}(\omega) \geq \gamma \},$$

where we remind the reader that $\theta_{t,x}$ is the space-time shift in the environment.

If $[0,T] \cap \{t_i\}_{n\in\mathbb{N}} = \{t_1, \cdots, t_n\}$, then using the convexity of $f$ and Jensen’s inequality,
where we used that 

Indeed,

Given \( \varepsilon > 0 \), let

\[
\begin{align*}
    f(x) := \left( \frac{x}{\varepsilon} - 1 \right) \land 1.
\end{align*}
\]

We note that \( f_{\varepsilon} \) is concave and for all \( x \geq 0 \),

\[
\mathbb{1}_{[\varepsilon, \infty)}(x) \geq f_{\varepsilon}(x) \geq \mathbb{1}(x)\mathbb{1}_{[2 \varepsilon, \infty)} - \mathbb{1}_{[0, \varepsilon]}(x).
\]

The proof is complete once we can find \( \varepsilon > 0 \) such that for all \( T > 0 \) and \( u > 1 \),

\[
P(M_T > u) \leq 2P(\mu_{\gamma,T}(\Omega) > u\varepsilon).
\]

Indeed,

\[
\mathbb{E}[M_T] = \int_0^1 P(M_T > u)du + \int_1^\infty P(M_T > u)du \leq 1 + \frac{2}{\varepsilon} \mathbb{E}[\mu_{\gamma,T}(\Omega)] = 1 + \frac{2}{\varepsilon}.
\]
For a fixed $u > 1$, recall the definition of $\tau = \tau_u$ from (6.1), so that $\mu_{\gamma, \tau}(\Omega) = u$. Hence, by Lemma 6.2 and Eq. (6.3)

$$P(\mu_{\gamma, T}(\Omega) > u \varepsilon) \geq P\left(\tau \leq T, \frac{\mu_{\gamma, T}(\Omega)}{\mu_{\gamma, \tau}(\Omega)} > \varepsilon\right) \geq E\left[f_{\varepsilon}(\frac{\mu_{\gamma, T}(\Omega)}{\mu_{\gamma, \tau}(\Omega)}), \tau \leq T\right] \geq P(\tau \leq T) E[f_{\varepsilon}(\mu_{\gamma, T}(\Omega))] \geq P(\tau \leq T) \inf_{T \geq 0} E[f_{\varepsilon}(\mu_{\gamma, T}(\Omega))].$$

(6.5)

We use again (6.3) to deduce

$$E\left[\inf_{T \geq 0} f_{\varepsilon}(\mu_{\gamma, T}(\Omega))\right] \geq E\left[\inf_{T \geq 0} 1_{\mu_{\gamma, T}(\Omega) \geq 2\varepsilon}\right] - E\left[\sup_{T \geq 0} 1_{\mu_{\gamma, T}(\Omega) \leq \varepsilon}\right] \geq P\left(\inf_{T \geq 0} \mu_{\gamma, T}(\Omega) \geq 2\varepsilon\right) - P\left(\inf_{T \geq 0} \mu_{\gamma, T}(\Omega) \leq \varepsilon\right).$$

To see the last equality, note that

$$\inf_{T \geq 0} 1_{\mu_{\gamma, T}(\Omega) \geq 2\varepsilon} = 1 \text{ if and only if } \inf_{T \geq 0} \mu_{\gamma, T}(\Omega) \geq 2\varepsilon$$

and

$$\sup_{T \geq 0} 1_{\mu_{\gamma, T}(\Omega) \leq \varepsilon} = 1 \text{ if and only if } \inf_{T \geq 0} \mu_{\gamma, T}(\Omega) \leq \varepsilon.$$ 

Letting $\varepsilon \to 0$ in the last display and noting that $P(\inf_{T \geq 0} \mu_{\gamma, T}(\Omega) = 0) = P(\mu_{\gamma}(\Omega) = 0) = 0$, we conclude that

$$\lim_{\varepsilon \to 0} E\left[\inf_{T \geq 0} f_{\varepsilon}(\mu_{\gamma, T}(\Omega))\right] \geq P\left(\inf_{T \geq 0} \mu_{\gamma, T}(\Omega) \geq 0\right) - P\left(\inf_{T \geq 0} \mu_{\gamma, T}(\Omega) = 0\right) = 1.$$ 

Thus, for $\varepsilon > 0$ small enough, and recalling the computations from (6.5), we conclude that

$$P(\mu_{\gamma, T}(\Omega) > u \varepsilon) \geq \frac{1}{2} P(\tau \leq T) = \frac{1}{2} P(M_T > u),$$

which is (6.4). □

Now we are ready to give the

**Proof of Theorem 2.7** Set $\gamma < \gamma_c$. For some fixed $u > 1$, to be determined later, we use again the stopping time $\tau = \tau_u$ defined in (6.1). For any $p > 1$ and $T > 0$,

$$E[\mu_{\gamma, T}(\Omega)^p] = E[\mu_{\gamma, T}(\Omega)^p, \tau > T] + E[\mu_{\gamma, T}(\Omega)^p, \tau \leq T] \leq u^p + u^p E\left[\left(\frac{\mu_{\gamma, T}(\Omega)}{\mu_{\gamma, \tau}(\Omega)}\right)^p, \tau \leq T\right] \leq u^p + u^p P(\tau \leq T) E[\mu_{\gamma, T}(\Omega)^p],$$

where in the last line we used Lemma 6.2. By Lemma 6.1

$$E[M_\infty] = \int_0^\infty P(M_\infty > u)du = 1 + \int_1^\infty P(M_\infty > u)du < \infty,$$
so that there exists some \( u > 1 \) satisfying \( P(M_\infty > u) \leq \frac{1}{2u} \). Since
\[
P(\tau \leq T) = P(M_T > u) \leq P(M_\infty > u) \leq \frac{1}{2u},
\]
we deduce from (6.6) the upper bound
\[
E[\mu_{\gamma,T}(\Omega)^p] \leq u^p + \frac{u^{p-1}}{2} E[\mu_{\gamma,T}(\Omega)^p].
\]
If we choose \( p > 1 \) satisfying \( u^p - 1 < 2 \), we conclude that for all \( T > 0 \),
\[
E[\mu_{\gamma,T}(\Omega)^p] \leq \frac{2u^p}{2-u^{p-1}}.
\]
We turn to the proof of (2.5) and (2.6). To show (2.5), we use the stopping time \( \tau = \tau_{1/u} \) for \( u > 1 \) (recall (6.1)). Proceeding as in (6.6), noting that \( x \mapsto x^{-q} \) is convex on \((0, \infty)\) and using Lemma 6.2, we have
\[
E[\mu_{\gamma,T}(\Omega)^{-q}] = E[\mu_{\gamma,T}(\Omega)^{-q}, \tau > T] + E[\mu_{\gamma,T}(\Omega)^{-q}, \tau \leq T] 
\leq u^q + u^q E\left[\left(\frac{\mu_{\gamma,T}(\Omega)}{\mu_{\gamma,T}(\Omega)}\right)^{-q}, \tau \leq T\right]
\leq u^q + u^q P(\tau \leq T) E[\mu_{\gamma,T}(\Omega)^{-q}].
\]
Since \( P(\tau \leq T) \leq P(\inf_{T \geq 0} \mu_{\gamma,T} \leq u^{-1}) \to 0 \) as \( u \to \infty \), we infer
\[
\sup_{T \geq 0} P(\tau \leq T) \leq \frac{1}{4}
\]
for \( u > 1 \) large enough. If \( q \in (0, 1) \) is chosen so that \( u^q \leq 2 \), from (6.7) we conclude that
\[
E[\mu_{\gamma,T}(\Omega)^{-q}] \leq 2 + \frac{E[\mu_{\gamma,T}(\Omega)^{-q}]}{2},
\]
and hence
\[
\sup_{T \geq 0} E[\mu_{\gamma,T}(\Omega)^{-q}] \leq 4.
\]
This finishes the proof of (2.5). To show (2.6), we appeal to [8, Theorem 1.3], where it was shown that in the “\( L^2 \)-region” (i.e., when the martingale \( (\mu_{\gamma,T}(\Omega))^T \) is bounded in \( L^2(\mathbb{P}) \)), for all \( q \in (0, \infty) \),
\[
\sup_{T \geq 0} E[\mu_{\gamma,T}(\Omega)^{-q}] < \infty.
\]
Since Theorem 2.4 holds in the entire weak disorder regime (and therefore, in particular in the \( L^2 \) region), the above estimate also implies (2.6) for all negative \( q \) for \( L^2 \) disorder. \( \square \)

7. Proof of Theorem 2.9

Recall the definition of \( (\Omega_0, \| \cdot \|) \) defined in Section 2.4. We will need the following estimate valid for the Wiener measure \( \mathbb{P}_0 \) on \( \Omega_0 \):

**Proposition 7.1.** [19, Theorem 1.4] Let \( j_{\frac{d}{2}} \) be the smallest positive root of the Bessel function \( J_{\frac{d}{2}} \). If \( g \) satisfies (2.8), then
\[
\lim_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}_0 (\| \omega \| < \varepsilon) = -\frac{j_{\frac{d}{2}}^2}{2} \int_0^\infty g^{-2}(t)dt.
\]
Additionally, we need an explicit formula for the *free energy* $\lim_{T \to \infty} \frac{1}{T} \log \mu_{\gamma,T}(\Omega)$ for all $\gamma > 0$. To describe this formula, we need some further definitions. Denote by $\mathcal{M}_1 = \mathcal{M}_1(\mathbb{R}^d)$ (resp. $\mathcal{M}_{\leq 1}$) the space of probability (resp. sub-probability) measures on $\mathbb{R}^d$, which acts as an additive group of translations on these spaces. Let $\tilde{\mathcal{M}}_1 = \mathcal{M}_1/\sim$ be the quotient space of $\mathcal{M}_1$ under this action, that is, for any $\mu \in \mathcal{M}_1$, its orbit is defined by $\tilde{\mu} = \{ \mu \ast \delta_x : x \in \mathbb{R}^d \} \in \tilde{\mathcal{M}}_1$. The quotient space $\tilde{\mathcal{M}}_1$ can be embedded in a larger space

$$\tilde{\mathcal{X}} := \left\{ \xi : \xi = \{ \tilde{\alpha}_i \}_{i \in I}, \alpha_i \in \mathcal{M}_{\leq 1}, \sum_{i \in I} \alpha_i(\mathbb{R}^d) \leq 1 \right\},$$

which consists of all empty, finite or countable collections of orbits from $\tilde{\mathcal{M}}_{\leq 1}$ whose masses add up to at most one. The space $\tilde{\mathcal{X}}$ and a metric structure there was introduced in [26], and it was shown that under that metric, $\tilde{\mathcal{M}}_1(\mathbb{R}^d)$ is densely embedded in $\tilde{\mathcal{X}}$ and any sequence in $\tilde{\mathcal{M}}_1(\mathbb{R}^d)$ converges along some subsequence to an element $\xi$ of $\tilde{\mathcal{X}}$ – that is, $\tilde{\mathcal{X}}$ is the *compactification* of the quotient space $\tilde{\mathcal{M}}_1(\mathbb{R}^d)$, see Appendix A for details. On the space $\tilde{\mathcal{X}}$ we define an energy functional

$$F_\gamma(\xi) = \frac{\gamma^2}{2} \sum_{\tilde{\alpha} \in \xi} \int_{\mathbb{R}^d \times \mathbb{R}^d} (\phi \ast \phi)(x_1 - x_2) \prod_{j=1}^2 \alpha(dx_j) \quad \forall \xi \in \tilde{\mathcal{X}},$$

and

$$\mathcal{E}_{F,\gamma}(\vartheta) = \int_{\tilde{\mathcal{X}}} F_\gamma(\xi) \vartheta(d\xi) \quad \vartheta \in \mathcal{M}_1(\tilde{\mathcal{X}}).$$

Here, $\mathcal{M}_1(\tilde{\mathcal{X}})$ denotes the space of probability measures on the space $\tilde{\mathcal{X}}$. There is an interesting connection between the structure of the space $\tilde{\mathcal{X}}$ and the solution of the variational problem $\sup_{\vartheta \in \mathcal{M}_1(\tilde{\mathcal{X}})} \mathcal{E}_{F,\gamma}(\vartheta)$: indeed, (i) there is a non-empty, compact subset $m_\gamma \subset \mathcal{M}_1(\tilde{\mathcal{X}})$ consisting of the maximizer(s) of the variational problem $\sup_{\vartheta \in m_\gamma} \mathcal{E}_{F,\gamma}(\vartheta)$, (ii) the maximizing set is a singleton $\delta_0 \in \tilde{\mathcal{X}}$ if $d \geq 3$ and small enough $\gamma$ (in particular, this is true if $\gamma \in (0, \gamma_c)$) and (iii) any maximizer assigns positive mass only to those elements of the compactification $\tilde{\mathcal{X}}$ whose total mass add up to one, see Proposition A.3 for details.

**Theorem 7.2.** [Eq. (4.12)] For any $d \in \mathbb{N}$ and $\gamma > 0$, it holds that

$$\lim_{T \to \infty} \frac{1}{T} \log \mu_{\gamma,T}(\Omega) = \lambda(\gamma) := - \sup_{\vartheta \in m_\gamma} \mathcal{E}_{F,\gamma}(\vartheta),$$

where $m_\gamma$ is the compact subset of $\mathcal{M}_1(\tilde{\mathcal{X}})$ defined in (A.3). Furthermore, $\lambda(\gamma) = 0$ if $d \geq 3$ and $\gamma \in (0, \gamma_c)$.

**Remark 7.3.** It is known that for any $d \geq 1$, $\gamma \mapsto \lambda(\gamma)$ is non-decreasing and continuous in $(0, \infty)$. Moreover, $\lambda(\gamma) > 0$ implies that $\lim_{T \to \infty} \mu_{\gamma,T}(\Omega) = 0$. Thus, if we set $\gamma_1 := \inf\{ \gamma > 0 : \lambda(\gamma) > 0 \}$, then $\gamma_1 \geq \gamma_c$ (recall Remark 3.6), and in fact equality is conjectured to be true.

**Proof of Theorem 2.9.**

**Proof of Part (i):** since $\gamma < \gamma_c$, we know that $P$-a.s.,

$$\varepsilon^2 \log \tilde{\mu}_\gamma(\|\omega - \eta\| < r\varepsilon) = \varepsilon^2 \log \mu_\gamma(\|\omega - \eta\| < r\varepsilon) - \varepsilon^2 \log \mu_\gamma(\Omega) \quad \forall \eta \in \Omega_0.$$

Hence, when letting $\varepsilon \to 0$, the second term vanishes, so we consider from now on only the asymptotic behavior of the first term

$$\varepsilon^2 \log \mu_\gamma(\|\omega - \eta\| < r\varepsilon).$$
Thus, an application of Proposition 7.1 leads to inequality (see (B.1) in Theorem B.1), for any \( \eta \in \Omega_0 \). By the Markov property,

\[
\log \mu_\gamma(\|\omega - \eta\| < r\varepsilon) = \log \lim_{s \to \infty} \mu_{\gamma,s+\varepsilon^{-2}}(\|\omega - \eta\| < r\varepsilon) \\
\leq \log \lim_{s \to \infty} \mathbb{E}_0 \left[ e^{\gamma H_{\varepsilon^{-2}}(\omega) - \frac{2p}{p-1} \gamma^2 \varepsilon^{-2} \phi(0)} \mu_{\gamma,s}(\Omega) \circ \theta_{\varepsilon^{-2},\omega_{\varepsilon^{-2}}} 1_{\|\omega - \eta\| < r\varepsilon} \right] \\
\leq \log \mathbb{E}_0 \left[ e^{\gamma H_{\varepsilon^{-2}}(\omega) - \frac{2p}{p-1} \gamma^2 \varepsilon^{-2} \phi(0)} \mu_\gamma(\Omega) \circ \theta_{\varepsilon^{-2},\omega_{\varepsilon^{-2}}} 1_{\|\omega - \eta\| < r\varepsilon} \right],
\]

where

\[
\|f\|_\varepsilon := \sup_{0 < t < \varepsilon^{-2}} \frac{|f(t)|}{g(t)},
\]

Since \( \gamma < \gamma_c \), by Part (i) of Theorem 2.7,

\[
\mu_\gamma(\Omega) \in L^{p_0}(\mathbb{P}) \quad \text{for some } p_0 > 1.
\]

For any \( 1 < p \leq p_0 \), we apply Hölder’s inequality to (7.4), so that

\[
\log \mu_\gamma(\|\omega - \eta\| < r\varepsilon) \leq \frac{p-1}{2p} \log \mathbb{E}_0 \left[ e^{\frac{2p}{p-1} \gamma H_{\varepsilon^{-2}}(\omega) - \frac{2p^2}{(p-1)^2} \gamma^2 \varepsilon^{-2} \phi(0)} \right] \\
+ \frac{p-1}{2p} \log \mathbb{P}_0(\|\omega - \eta\|_\varepsilon < r\varepsilon) \\
+ \frac{1}{p} \log \mathbb{E}_0 \left[ \mu_\gamma(\Omega) \circ \theta_{\varepsilon^{-2},\omega_{\varepsilon^{-2}}} \right] \\
= \frac{p-1}{2p} \log \mathbb{E}_0 \left[ e^{\frac{2p}{p-1} \gamma H_{\varepsilon^{-2}}(\omega) - \frac{2p^2}{(p-1)^2} \gamma^2 \varepsilon^{-2} \phi(0)} \right] \\
+ \frac{p-1}{2p} \log \mathbb{P}_0(\|\omega - \eta\|_\varepsilon < r\varepsilon) \\
+ \frac{1}{p} \log \mathbb{E}_0 \left[ \mu_\gamma(\Omega) \circ \theta_{\varepsilon^{-2},\omega_{\varepsilon^{-2}}} \right] + \frac{(p+1)}{2(p-1)} \gamma^2 \varepsilon^{-2} \phi \ast \phi(0).
\]

By Theorem 7.2, we can handle the first term in the last display on the right hand side above: indeed, for any \( \gamma > 0 \) and \( \mathbb{P} \)-a.s. we have

\[
\frac{p-1}{2p} \lim_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{E}_0 \left[ e^{\frac{2p}{p-1} \gamma H_{\varepsilon^{-2}}(\omega) - \frac{2p^2}{(p-1)^2} \gamma^2 \varepsilon^{-2} \phi(0)} \right] = \frac{p-1}{2p} \lambda \left( \frac{2p}{p-1} \gamma \right),
\]

We bound the second term in the last display of the right hand side of (7.7) as follows. By Anderson’s inequality (see (B.1) in Theorem B.1), for any \( \eta \in \Omega_0 \),

\[
\mathbb{P}_0(\|\omega - \eta\|_\varepsilon < r\varepsilon) \leq \mathbb{P}_0(\|\omega\|_\varepsilon < r\varepsilon).
\]

Thus, an application of Proposition 7.1 leads to

\[
\limsup_{\varepsilon \to 0} \sup_{\eta \in \Omega_0} \frac{p-1}{2p} \varepsilon^2 \log \mathbb{P}_0(\|\omega - \eta\|_\varepsilon < r\varepsilon) \leq - \frac{(p-1)}{4pr^2} \int_0^{\gamma} g^{-2}(t) dt.
\]

Thus, we concentrate on the third term of the last display of the (r.h.s.) of (7.7). Set

\[
f_\varepsilon := \mathbb{E}_0 \left[ \mu_\gamma(\Omega) \circ \theta_{\varepsilon^{-2},\omega_{\varepsilon^{-2}}} \right].
\]
Since $\hat{B}$ is stationary with respect to space-time shifts $(\theta_{t,x})_{t>0, x \in \mathbb{R}^d}$, then $f_\varepsilon$ is stationary and $\mathbb{E}[f_\varepsilon] = \mathbb{E}[\mu_\gamma(\Omega)^p] < \infty$. By the ergodic theorem for stationary processes\(^3\) that there is a (possibly random) $C = C(\hat{B})$ such that for all $\varepsilon > 0$ sufficiently small,

$$\mathbb{E}_0 \left[ \mu_\gamma^p(\Omega) \circ \theta_{\varepsilon^{-2} \omega_{\varepsilon^{-2}}} \right] \leq C \varepsilon^{-2} \quad \mathbb{P} \text{-a.s.},$$

and consequently,

$$\limsup_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{E}_0 \left[ \mu_\gamma^p(\Omega) \circ \theta_{\varepsilon^{-2} \omega_{\varepsilon^{-2}}} \right] \leq 0 \quad \mathbb{P} \text{-a.s.} \quad (7.11)$$

Combining (7.7)-(7.11), we obtain

$$\limsup_{\varepsilon \to 0} \varepsilon^2 \log \mu_\gamma(\|\omega\| < r \varepsilon) \leq \inf_{1 < p \leq p_0} \left[ \left( \frac{p + 1}{2(p - 1)} \right)^2 \phi \ast \phi(0) - \frac{(p - 1)}{4p^2} \int_0^{\infty} g^{-2}(t) dt + \frac{p - 1}{2p} \lambda \left( \frac{2p}{p - 1} \right) \right] + C_1(d, \gamma, g, r),$$

and observe that $C_1 > 0$ if we choose $r$ small enough.

**Lower bound:** Recall the norm $\| \cdot \|_\varepsilon$ from (7.5). For a fixed $\gamma < \gamma_c$, we apply part (ii) of Theorem 2.7 to find some $q_0 > 0$ such that for all $q > q_0$,

$$\mathbb{E}[\mu_\gamma(\Omega)^{-q}] < \infty. \quad (7.13)$$

By the Markov property and the reverse Hölder’s inequality,\(^4\)

$$\mu_\gamma(\|\omega\| < r \varepsilon) = \mathbb{E}_0 \left[ \left( e^{x H_{\varepsilon^{-2} - \frac{\varepsilon^2}{2} \phi \ast \phi(0)} 1_{\|\omega\| < r \varepsilon} \right) \left( \mu_\gamma(\Omega) \circ \theta_{\varepsilon^{-2} \omega_{\varepsilon^{-2}}} \right) \right] \geq \mathbb{E}_0 \left[ e^{x H_{\varepsilon^{-2} - \frac{\varepsilon^2}{2} \phi \ast \phi(0)} 1_{\|\omega\| < r \varepsilon} \right]^{-\frac{1}{q+1}} \mathbb{E}_0 \left[ \mu_\gamma(\Omega)^{-q} \circ \theta_{\varepsilon^{-2} \omega_{\varepsilon^{-2}}} \right]^{-\frac{1}{q}}. \quad (7.14)$$

Thus,

$$\varepsilon^2 \log \mu_\gamma(\|\omega\| < r \varepsilon) \geq \varepsilon^2 \left( \frac{q + 1}{q} \right) \log \mathbb{E}_0 \left[ e^{x H_{\varepsilon^{-2} - \frac{\varepsilon^2}{2} \phi \ast \phi(0)} 1_{\|\omega\| < r \varepsilon} \right] - \frac{1}{q} \varepsilon^2 \log \mathbb{E}_0 \left[ \mu_\gamma(\Omega)^{-q} \circ \theta_{\varepsilon^{-2} \omega_{\varepsilon^{-2}}} \right]$$

$$= - \frac{\gamma^2}{2(q + 1)} \phi \ast \phi(0) + \varepsilon^2 \left( \frac{q + 1}{q} \right) \log \mu_\gamma \left[ e^{x H_{\varepsilon^{-2}(\|\omega\| < r \varepsilon)} \right] - \frac{1}{q} \varepsilon^2 \log \mathbb{E}_0 \left[ \mu_\gamma(\Omega)^{-q} \circ \theta_{\varepsilon^{-2} \omega_{\varepsilon^{-2}}} \right]. \quad (7.15)$$

---

\(^3\)Here we invoke the ergodic theorem for general stationary processes which implies that the time averages of a stationary process evaluated w.r.t. a $L^1$ function $f$ converge almost surely to the conditional expectation of $f$ w.r.t. the invariant $\sigma$-algebra. In the present context, the function (7.10) is in $L^1(\mathbb{P})$, therefore the conditional expectation $C(\hat{B}) < +\infty$ almost surely. Moreover, for $\gamma \in (0, \gamma_c)$, $\mu_\gamma(\Omega) > 0$ almost surely w.r.t. $\mathbb{P}$, therefore $C(\hat{B}) > 0$ almost surely, hence log $C(\hat{B}) < \infty$, implying (7.11).

\(^4\) Recall that Hölder’s inequality implies that if $\theta > 1$ and $f, g$ are measurable functions satisfying $\|f\|_1 < \infty$ and $\|g\|_{\frac{1}{\theta - 1}} < \infty$, then $\|fg\|_1 \geq \|f\|_\frac{\theta}{\theta - 1}\|g\|_{\frac{1}{\theta - 1}}$. To deduce (7.14), we apply this inequality for $\theta = 1 + q$. 

---
In [5, Theorem 2.2], it was shown that for all $\gamma > 0$,
\[
\liminf_{\varepsilon \to 0} \varepsilon^2 \log \mu_{r, \varepsilon} \left( \|\omega\|_\varepsilon < r \varepsilon \right) \geq -\left( \frac{j^2 + 2}{2r^2} \int_0^\infty g^{-2}(t)dt + \frac{\gamma^2}{2}(\phi * \phi)(0) + \lambda(\gamma) \right).
\]
In particular,
\[
\liminf_{\varepsilon \to 0} \varepsilon^2 \left( \frac{q + 1}{q} \right) \log \mu_{r, \varepsilon} \left( \|\omega\|_\varepsilon < r \varepsilon \right)
\geq -\left( \frac{q + 1}{q} \right) \left( \frac{j^2 + 2}{2r^2} \int_0^\infty g^{-2}(t)dt + \frac{\gamma^2}{2}(\phi * \phi)(0) + \lambda(\gamma) \right)
\] (7.16)
Thus, using (7.13), the ergodic theorem and following exactly the same argument as for deducing (7.11), we obtain that
\[
\text{lim sup}_{\varepsilon \to 0} \frac{1}{q} \log \mathbb{E}_0 \left[ \mu_{r}(\Omega)^{-q} \circ \theta_{e}^{-2} \omega_{e}^{-2} \right] \leq 0.
\] (7.17)
Finally, combining (7.15)-(7.17), it holds $\mathbb{P}$-a.s.
\[
\liminf_{\varepsilon \to 0} \varepsilon^2 \log \mu_{r} \left( \|\omega\| < r \varepsilon \right)
\geq -\left( \frac{q + 1}{q} \right) \left( \frac{j^2 + 2}{2r^2} \int_0^\infty g^{-2}(t)dt + \frac{\gamma^2}{2}(\phi * \phi)(0) \right)
\] (7.18)
Clearly, $C_2 < \infty$. To extend the argument to the norm $\| \cdot \|$, we can apply again the reverse Hölder’s inequality to deduce, for any $p > 1$,
\[
\liminf_{\varepsilon \to 0} \varepsilon^2 \log \mu_{r} \left( \|\omega\| < \varepsilon \right) \geq p \liminf_{\varepsilon \to 0} \varepsilon^2 \log \mu_{r} \left( \|\omega\|_\varepsilon < \varepsilon \right) - (p - 1) \limsup_{\varepsilon \to 0} \varepsilon^2 \log \mu_{r} \left( \sup_{t, \varepsilon} \frac{\|\omega(t)\|}{g(t)} < \varepsilon \right)
\geq -pC_2 - (p - 1) \limsup_{\varepsilon \to 0} \varepsilon^2 \log \mu_{r} \left( \Omega \right)
= -pC_2.
\]
As $p > 1$ is arbitrary, we conclude the proof of (2.11). To deduce the constants in (2.12), note that for $\gamma > 0$ small enough, one can choose $p_0$ in (7.6) such that $\gamma \frac{2p_0}{p_0 - 1} < \gamma_c$. In particular, $\lambda(\gamma \frac{2p_0}{p_0 - 1}) = 0$ and hence, the calculations that lead to (7.12) imply
\[
\limsup_{\varepsilon \to 0} \varepsilon^2 \log \mu_{r} \left( \|\omega\| < r \varepsilon \right) \leq \frac{(p_0 + 1)}{2(p_0 - 1)} \gamma^2 \phi * \phi(0) - \frac{(p_0 - 1)}{4p_0 r^2} \int_0^\infty g^{-2}(t)dt.
\] (7.19)
Similarly, for any $\gamma < \gamma_c$, if $q_0 > 0$ satisfies $\frac{\gamma q_0}{q_0 + 1} < \gamma_c$, then $\lambda(\frac{\gamma q_0}{q_0 + 1}) = 0$ and, the same arguments that led to (7.18) give the lower bound
\[
\liminf_{\varepsilon \to 0} \varepsilon^2 \log \mu_{r} \left( \|\omega\| < r \varepsilon \right) \geq -\left( \frac{q_0 + 1}{q_0} \right) \left( \frac{j^2 + 2}{2r^2} \int_0^\infty g^{-2}(t)dt + \frac{\gamma^2}{2}(\phi * \phi)(0) \right).
\] (7.20)
Eqs. (7.19)-(7.20) imply (2.12), finishing the proof of part (i).
Proof of Part (ii): fix $\delta > 0$. We explain briefly how to adapt the previous arguments to obtain the decay estimates that match as $\gamma \to 0$. For a fixed $r > 0$, the lower bound is the same as in (7.18). To estimate the upper bound, we choose $p > 1$ and $q > 0$ large enough in such a way that

$$
\left( \frac{q + 1}{q} - \frac{1}{p} \right) \frac{j_{\gamma}^2}{2r^2} \int_0^\infty g^{-2}(t)dt < \frac{\delta}{2},
$$

and let $\gamma'$ small enough so that

$$(\mu_{\gamma,T}(\Omega))_{T \geq 0} \in L^{\frac{2p}{p-1}}(P) \quad \text{for all } \gamma < \gamma'.$$

By Hölder’s inequality,

$$
\varepsilon^2 \sup_{\eta \in \Omega_0} \log \mu_\gamma(\|\omega - \eta\| < \varepsilon) \leq \varepsilon^2 \frac{p-1}{2p} \log \mathbb{E}_0 \left[ \frac{e^{2\mu_{\gamma,H_{\varepsilon^{-2}}}(\omega)}}{e^{2\mu_{\gamma,H_{\varepsilon^{-2}}}(\omega)-\frac{2p^2}{(p-1)^2} \varepsilon^2 \phi \ast \phi(0)}} \right]
$$

$$+ \varepsilon^2 \sup_{\eta \in \Omega_0} \log \mathbb{P}_0(\|\omega - \eta\| < r\varepsilon)
$$

$$+ \varepsilon^2 \frac{p-1}{2p} \log \mathbb{E}_0 \left[ \frac{e^{\mu_{\gamma,H_{\varepsilon^{-2}}}(\omega)}}{\mu_{\gamma,H_{\varepsilon^{-2}}}(\omega) \ast \mu_{\gamma,H_{\varepsilon^{-2}}}(\omega)} + \frac{(p+1)}{2(p-1)} \varepsilon^2 \phi \ast \phi(0) \right] (7.22)
$$

Next, choose $\gamma''$ small enough so that

$$
\lim_{\varepsilon \to 0} \varepsilon^2 \frac{p-1}{2p} \log \mathbb{E}_0 \left[ \frac{e^{\mu_{\gamma,H_{\varepsilon^{-2}}}(\omega)}}{e^{\mu_{\gamma,H_{\varepsilon^{-2}}}(\omega)-\frac{2p^2}{(p-1)^2} \varepsilon^2 \phi \ast \phi(0)}} \right]
$$

$$= \frac{p-1}{2p} \lambda \left( \frac{2p}{p-1} \gamma \right) = 0
$$

as well as $\lambda \left( \frac{\gamma q}{q+1} \right) = 0$ for all $\gamma < \gamma''$ (such $\gamma''$ exists due to Theorem 7.2). According to (7.9), (7.11) and (7.22), we infer that

$$
\limsup_{\varepsilon \to 0} \sup_{\eta \in \Omega_0} \varepsilon^2 \sup_{\eta \in \Omega_0} \log \mu_\gamma(\|\omega\| < \varepsilon) \leq \frac{(p+1)}{2(p-1)} \gamma^2 \phi \ast \phi(0) - \frac{1}{2p} \frac{j_{\gamma}^2}{r^2} \int_0^\infty g^{-2}(t)dt. (7.23)
$$

Thus, if we choose $\gamma'''$ small enough satisfying

$$
\frac{(\gamma''')^2}{2} \phi \ast \phi(0) \left[ \frac{p+1}{p-1} + \frac{q^2 + 3q + 1}{q(q+1)} \right] < \frac{\delta}{2},
$$

and then setting $\gamma_\delta := \gamma' \wedge \gamma'' \wedge \gamma'''$, we deduce from (7.21) and (7.23) that

$$
C_2 - C_1 \leq \left( \frac{q+1}{q} \right) \left( \frac{j_{\gamma}^2}{2r^2} \int_0^\infty g^{-2}(t)dt + \frac{\gamma^2}{2} \phi \ast \phi(0) \frac{q^2 + 3q + 1}{(q+1)^2} \right)
$$

$$+ \frac{(p+1)}{2(p-1)} \gamma^2 \phi \ast \phi(0) - \frac{1}{2p} \frac{j_{\gamma}^2}{r^2} \int_0^\infty g^{-2}(t)dt
$$

$$= \left( \frac{q+1}{q} - \frac{1}{p} \right) \frac{j_{\gamma}^2}{2r^2} \int_0^\infty g^{-2}(t)dt + \frac{(\gamma''')^2}{2} \phi \ast \phi(0) \left[ \frac{p+1}{p-1} + \frac{q^2 + 3q + 1}{q(q+1)} \right]
$$

$$< \delta
$$

for all $\gamma < \gamma_\delta$, concluding the proof of the theorem. \qed
Appendix A. The space $\mathcal{X}$ and its energy functionals

We denote by $\mathcal{M}_1 = \mathcal{M}_1(\mathbb{R}^d)$ (resp., $\mathcal{M}_{\leq 1}$) the space of probability (resp., subprobability) distributions on $\mathbb{R}^d$ and by $\widetilde{\mathcal{M}}_1 = \mathcal{M}_1/\sim$ the quotient space of $\mathcal{M}_1$ under the action of $\mathbb{R}^d$ (as an additive group on $\mathcal{M}_1$), that is, for any $\mu \in \mathcal{M}_1$, its orbit is defined by $\tilde{\mu} = \{\mu * \delta_x : x \in \mathbb{R}^d\} \in \widetilde{\mathcal{M}}_1$. Then we define

$$\mathcal{X} = \left\{ \xi : \xi = \{\tilde{\alpha}_i\}_{i \in I}, \alpha_i \in \mathcal{M}_{\leq 1}, \sum_{i \in I} \alpha_i(\mathbb{R}^d) \leq 1 \right\}$$

(A.1)

to be the space of all empty, finite or countable collections of orbits of subprobability measures with total masses bounded by 1. Note that the quotient space $\widetilde{\mathcal{M}}_1(\mathbb{R}^d)$ is embedded in $\mathcal{X}$ – that is, for any $\mu \in \mathcal{M}_1(\mathbb{R}^d)$, $\tilde{\mu} \in \widetilde{\mathcal{M}}_1(\mathbb{R}^d)$ and the single orbit element $\{\tilde{\mu}\} \in \mathcal{X}$ belongs to $\mathcal{X}$ (in this context, sometimes we will write $\tilde{\mu} \in \mathcal{X}$ for $\{\tilde{\mu}\} \in \mathcal{X}$).

The space $\mathcal{X}$ also comes with a metric structure. If for any $k \geq 2$, $\mathcal{H}_k$ is the space of functions $h : (\mathbb{R}^d)^k \to \mathbb{R}$ which are invariant under rigid translations and which vanish at infinity, we define, for any $h \in \mathcal{H} = \bigcup_{k \geq 2} \mathcal{H}_k$, the functionals

$$J(h, \xi) = \sum_{\tilde{\alpha} \in \xi} \int_{(\mathbb{R}^d)^k} h(x_1, \ldots, x_k) \alpha(dx_1) \cdots \alpha(dx_k).$$

(A.2)

A sequence $\xi_n$ is said to converge to $\xi$ in the space $\mathcal{X}$ if

$$J(h, \xi_n) \to J(h, \xi) \quad \forall h \in \mathcal{H}.$$  

This leads to the following definition of the metric $D$ on $\mathcal{X}$. For any $\xi_1, \xi_2 \in \mathcal{X}$, set

$$D(\xi_1, \xi_2) = \sum_{r=1}^{\infty} \frac{1}{2^r} \frac{1}{1 + \|h_r\|_\infty} \left| J(h_r, \xi_1) - J(h_r, \xi_2) \right|$$

$$= \sum_{r=1}^{\infty} \frac{1}{2^r} \frac{1}{1 + \|h_r\|_\infty} \left| \sum_{\tilde{\alpha} \in \xi_1} \int_{(\mathbb{R}^d)^k} h_r(x_1, \ldots, x_k) \prod_{i=1}^{kr} \alpha(dx_i) - \sum_{\tilde{\alpha} \in \xi_2} \int_{(\mathbb{R}^d)^k} h_r(x_1, \ldots, x_k) \prod_{i=1}^{kr} \alpha(dx_i) \right|.$$  

The following result was proved in [26, Theorem 3.1-3.2].

Theorem A.1. We have the following properties of the space $\mathcal{X}$.

- $D$ is a metric on $\mathcal{X}$ and the space $\widetilde{\mathcal{M}}_1(\mathbb{R}^d)$ is dense in $(\mathcal{X}, D)$.
- Any sequence in $\mathcal{M}_1(\mathbb{R}^d)$ has a convergent subsequence with a limit point in $\mathcal{X}$. Thus, $\mathcal{X}$ is the completion and the compactification of the totally bounded metric space $\widetilde{\mathcal{M}}_1(\mathbb{R}^d)$ under $D$.
- Let a sequence $(\xi_n)_n$ in $\mathcal{X}$ consist of a single orbit $\gamma_n$ and $D(\xi_n, \xi) \to 0$ where $\xi = (\tilde{\alpha}_i)_i \in \mathcal{X}$ such that $\alpha_1(\mathbb{R}^d) \geq \alpha_2(\mathbb{R}^d) \geq \ldots \ldots$. Then given any $\varepsilon > 0$, we can find $k \in \mathbb{N}$ such that
  - $\sum_{i > k} \alpha_i(\mathbb{R}^d) < \varepsilon$ and we can write $\gamma_n = \sum_{i=1}^k \alpha_{n,i}^\gamma + \beta_n$, such that
    - for any $i = 1, \ldots, k$, there is a sequence $(\alpha_{n,i})_n \subset \mathbb{R}^d$ satisfying
      $$\alpha_{n,i} * \delta_{a_{n,i}} \Rightarrow \alpha_i \text{ with } \lim_{n \to \infty} \inf_{i \neq j} |a_{n,i} - a_{n,j}| = \infty.$$  
  - The sequence $\beta_n$ totally disintegrates, meaning that for any $r > 0$, $\sup_{x \in \mathbb{R}^d} \beta_n(B_r(x)) \to 0$. \hfill $\square$
Recall the definition of $F_\gamma : \tilde{X} \to \mathbb{R}$ from (7.2):

$$F_\gamma (\xi) = \frac{\gamma^2}{2} \sum_{i \in I} \int_{\mathbb{R}^d \times \mathbb{R}^d} V(x_1 - x_2) \prod_{j=1}^{2} \alpha_i (dx_j), \quad \xi = (\tilde{\alpha}_i)_{i \in I}. \quad (A.3)$$

Because of shift-invariance of the integrand in (A.3), $F_\gamma$ is well-defined on $\tilde{X}$. Moreover, we have

**Lemma A.2.** $F_\gamma$ is continuous and non-negative on $\tilde{X}$, and $F_\gamma (\cdot) \leq \frac{\gamma^2}{2} V(0)$.

**Proof.** For the continuity of $F_\gamma$, we refer to [20 Corollary 3.3]. Recall that $V = \phi \ast \phi$ and $\phi$ is rotationally symmetric. Hence, for any $\alpha \in \mathcal{M}_{\leq 1}(\mathbb{R}^d)$, by Cauchy-Schwarz inequality,

$$\int_{\mathbb{R}^{2d}} V(x_1 - x_2) \alpha(dx_1) \alpha(dx_2) = \int_{\mathbb{R}^d} \alpha(dx_1) \alpha(dx_2) \int_{\mathbb{R}^d} d\phi(x_1 - z) \phi(x_2 - z) \leq \int_{\mathbb{R}^d} \alpha(dx_1) \alpha(dx_2) \left[ \int_{\mathbb{R}^d} d\phi^2(x_1 - z) \right]^{1/2} \left[ \int_{\mathbb{R}^d} d\phi^2(x_2 - z) \right]^{1/2} \leq \alpha(\mathbb{R}^d) \| \phi \|_2^2. \quad (A.4)$$

Thus, $F_\gamma (\xi) \leq \frac{\gamma^2(\phi \ast \phi)(0)}{2} \sum_{i \in I} \alpha_i(\mathbb{R}^d)^2 \leq \frac{\gamma^2(\phi \ast \phi)(0)}{2}$ since, for $\xi = (\tilde{\alpha}_i)_{i \in I} \in \tilde{X}$, we have $\sum_{i \in I} \alpha_i(\mathbb{R}^d) \leq 1$. Moreover, since $V = \phi \ast \phi$ is non-negative, also $F_\gamma (\cdot) \geq 0$. \hfill \Box

For any $\alpha \in \mathcal{M}_{\leq 1}(\mathbb{R}^d)$, let

$$\mathcal{G}_t (\alpha) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \alpha(dx) E_{\xi} \left[ \mathbb{1}\{\omega_t \in dx\} \exp \left\{ \gamma \mathcal{H}_t (\omega) - \frac{\gamma^2}{2} t V(0) \right\} \right],$$

and note that for any $a \in \mathbb{R}^d$ and $t > 0$, $\mathcal{G}_t (\alpha_i) \overset{(d)}{=} \mathcal{G}_t (\alpha_i \ast \delta_a)$. Hence, we may define $\mathcal{G}_t, \mathcal{G}_{t, \alpha} : \tilde{X} \to \mathbb{R}$ as

$$\mathcal{G}_t (\xi) = \sum_{i} \mathcal{G}_t (\alpha_i), \quad \mathcal{G}_{t, \alpha} (\xi) = \mathcal{G}_t (\xi) + \mathbb{E} [ \mathcal{G}_t (\xi) ] \forall \xi = (\tilde{\alpha}_i)_{i \in I} \in \tilde{X}. \quad (A.5)$$

Next, for any $t > 0$, and for $\xi = (\tilde{\alpha}_i)_{i \in I} \in \tilde{X}$, we set

$$\alpha_i^{(t)} (dx) := \frac{1}{\mathcal{G}_t (\xi)} \int_{\mathbb{R}^d} \alpha_i(dx) E_{\xi} \left[ \mathbb{1}\{\omega_t \in dx\} \exp \left\{ \gamma \mathcal{H}_t (\omega) - \frac{\gamma^2}{2} t V(0) \right\} \right], \quad (A.6)$$

$$\xi^{(t)} := (\tilde{\alpha}_i^{(t)})_{i \in I} \in \tilde{X}.$$

Recall that $\mathcal{G}_t (\alpha_i) \overset{(d)}{=} \mathcal{G}_t (\alpha_i \ast \delta_a)$ and likewise, $(\alpha_i \ast \delta_a)^{(t)} (dx) \overset{(d)}{=} (\alpha_i^{(t)} \ast \delta_a)(dx)$. For any $\vartheta \in \mathcal{M}_1 (\tilde{X})$, then (A.3) further defines a transition kernel

$$\Pi_t (\vartheta, d\xi') = \int_{\tilde{X}} \pi_t (\xi, d\xi') \vartheta (d\xi) \quad \text{where} \quad \pi_t (\xi, d\xi') = \mathbb{P} [ \xi^{(t)} \in d\xi' | \xi ] \in \mathcal{M}_1 (\tilde{X}). \quad (A.7)$$

A discrete version of this transition kernel in $\mathbb{Z}^d$ (for discrete time random walks) was first introduced in [11] on the compactified space $\tilde{X}$.

**Lemma A.3.** The set

$$\mathbf{m}_t = \{ \vartheta \in \mathcal{M}_1 (\tilde{X}) : \Pi_t \vartheta = \vartheta \text{ for all } t > 0 \} \quad (A.8)$$

of fixed points of $\Pi_t$ is a non-empty, compact subset of $\mathcal{M}_1 (\tilde{X})$. 

Proof. Note that \( m_\gamma \neq \emptyset \), because \( \delta_0 \in m_\gamma \). Moreover, by the definition of the metric \( D \) on \( \tilde{X} \) and by the resulting convergence criterion determined by Theorem \( \text{[A.1]} \), the map \( \tilde{X} \ni \xi \mapsto \pi_t(\xi, \cdot) \) is continuous. This property, together with the compactness of \( \tilde{X} \) (and therefore also that of \( \mathcal{M}_1(\tilde{X}) \)), we have that \( \mathcal{M}_1(\tilde{X}) \ni \vartheta \mapsto \Pi_t(\vartheta, \cdot) \) is continuous too for any \( t > 0 \). It follows that \( m_\gamma \) is a closed subset of the compact metric space \( \mathcal{M}_1(\tilde{X}) \), implying the compactness of \( m_\gamma \). \( \square \)

The following proposition provides additional information about the maximizers of the continuous map \( \mathcal{E}_{F_\gamma} \), defined in \( \text{(7.2)} \) in the compact set \( m_\gamma \). Its proof can be found in \([5\), Proposition 4.5].

Proposition A.4. Fix \( d \in \mathbb{N} \) and \( \gamma > 0 \). Then the supremum (recall \( \text{(A.8)} \))

\[
\sup_{\vartheta \in m_\gamma} \mathcal{E}_{F_\gamma}(\vartheta) = \sup_{\vartheta \in m_\gamma} \int_{\tilde{X}} F_\gamma(\xi) \vartheta(d\xi) \tag{A.9}
\]

is attained, and we always have \( \sup_{m_\gamma} \mathcal{E}_{F_\gamma}(\cdot) \in [0, \gamma^2(\phi \ast \phi)(0)/2] \). Moreover, there exists \( \gamma_1 = \gamma_1(d) \) such that \( \gamma_1 > 0 \) if \( d \geq 3 \), and if \( \gamma \in (0, \gamma_1] \), then \( m_\gamma = \{ \delta_0 \} \) is a singleton consisting of the Dirac measure at \( \tilde{0} \in \tilde{X} \). Consequently, in this regime, \( \sup_{m_\gamma} \mathcal{E}_{F_\gamma}(\cdot) = 0 \). If \( \gamma > \gamma_1 \), then \( \sum_{i \in I} \sigma_i(\mathbb{R}^d) = 1 \) (i.e., any maximizer of \( \text{(A.9)} \) assigns positive mass only to those elements of \( \tilde{X} \) whose total mass add up to one).

Appendix B. Gaussian inequalities.

Theorem B.1. Let \( E \) be a separable Banach space, with \( X \) being a \( E \)-valued centered Gaussian space with law \( \mu \). Then the following hold:

(i) For any symmetric convex subset \( A \subset E \) and \( x \in E \),

\[
\mu(A + x) \leq \mu(A). \tag{B.1}
\]

(ii) If \( H_\mu \) is the Cameron-Martin space of \( \mu \), then for any \( r > 0 \), and \( \eta \in H_\mu \),

\[
e^{-\frac{1}{2}||\eta||^2_\mu} \mu(\omega \in E : ||\omega|| \leq r) \leq \mu(\omega \in E : ||\omega - \eta|| \leq r) \leq \mu(\omega \in E : ||\omega|| \leq r). \tag{B.2}
\]

Proof. \( \text{(B.1)} \) is Anderson’s inequality, which follows from log-concavity of Gaussian measures (see \([4\), Thm. 2.8.10], or \([22\), Thm. 2.13]). The upper bound of \( \text{(B.2)} \) follows from \( \text{(B.1)} \), while the lower bound follows from the Cameron-Martin formula

\[
\mu(A - \eta) = \int_A \exp \left( -\frac{1}{2}||\eta||^2_\mu + \langle \omega, \eta \rangle_\mu \right) \mu(d\omega), \quad A \subset E, \eta \in H_\mu,
\]

as well as Hölder’s inequality and the symmetry of \( \langle \omega, \eta \rangle_\mu \) on \( \{ \omega \in E : ||\omega|| \leq r \} \), see \([22\), Thm. 3.1]. \( \square \)

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\footnote{Let \( E^* \) denote the dual of \( E \), with \( S : E^* \to E \) being the map defined via the integral \( S\xi = \int_E \xi(x)\mu(dx) \) for \( \xi \in E^* \). Then \( H_\mu \subset E \) is the Hilbert space which is the completion of the range of \( S \) under the inner product \( \langle \xi, \xi' \rangle_\mu := \int \xi(x)\xi'(x)\mu(dx) \). This inner product induces a norm \( || \cdot ||_\mu \) on \( H_\mu \).}
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