Research Article

Paracontact Metric \((\kappa, \mu)\)-Manifold Satisfying the Miao-Tam Equation

Dehe Li\(^1\) and Jiabin Yin\(^2\)

\(^1\)School of Mathematics and Statistics, Anyang Normal University, 455000 Anyang, Henan, China
\(^2\)School of Mathematical Sciences, Xiamen University, 361005 Xiamen, Fujian, China

Correspondence should be addressed to Dehe Li; lidehehe@163.com

Received 5 October 2020; Accepted 10 April 2021; Published 20 April 2021

Academic Editor: Remi Léandre

Copyright © 2021 Dehe Li and Jiabin Yin. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this paper, we classified the paracontact metric \((\kappa, \mu)\)-manifold satisfying the Miao-Tam critical equation with \(\kappa > -1\). We proved that it is locally isometric to the product of a flat \((n + 1)\)-dimensional manifold and an \(n\)-dimensional manifold of negative constant curvature \(-4\).

1. Introduction

Inspired by the positive mass theorem and the variational characterization of Einstein metrics on a closed manifold, with an aim to find a proper concept of metrics that would sit between constant scalar curvature metrics and Einstein metrics, in [1], Miao and Tam studied the variational properties of the volume functional on the space of constant scalar curvature metrics with a prescribed boundary metric. Specifically, they derived the following sufficient and necessary condition for a metric to be a critical point:

**Theorem 1** (Theorem 5 in [1]). Let \(\Omega\) be a compact \(n\)-dimensional Riemannian manifold with smooth boundary \(\Sigma\), \(\gamma\) be a given metric on \(\Sigma\), and \(\mathcal{M}_K^\gamma\) be the space of metrics on \(\Omega\) which have constant scalar curvature \(K\) and have induced metric on \(\Sigma\) given by \(\gamma\). Let \(g \in \mathcal{M}_K^\gamma\) be a smooth metric such that the first Dirichlet eigenvalue of \((n - 1)\Delta_g + K\) is positive. Then, \(g\) is a critical point of the volume functional in \(\mathcal{M}_K^\gamma\) if and only if there is a smooth function \(\lambda\) on \(\Omega\) such that \(\lambda = 0\) on \(\Sigma\) and

\[-(\Delta_g \lambda) g + \nabla^2_g \lambda - \lambda \text{Ric}(g) = g,\]

where \(\Delta_g\) and \(\nabla^2_g\) are the Laplacian and Hessian operators with respect to \(g\), and \(\text{Ric}(g)\) is the Ricci curvature of \(g\).

For brevity, we call such critical metric as Miao-Tam critical metric and refer to equation (1) as the Miao-Tam equation. A fundamental property of a Miao-Tam critical metric is that its scalar curvature is a constant (see Theorem 7 in [1]). Some explicit examples of Miao-Tam critical metrics can be found in [1, 2], including not only the standard metrics on geodesic balls in space forms but the spatial Schwarzschild metrics and AdS-Schwarzschild metrics restricted to certain domains containing their horizon and bounded by two spherically symmetric spheres. In [2], the authors classified all Einstein and conformally flat Miao-Tam critical metrics. In fact, they proved that any connected, compact, Einstein manifold with smooth boundary satisfying Miao-Tam critical condition is isometric to a geodesic ball in a simply connected space form. And then several generalizations of this rigidity result were found by different authors, replacing the Einstein assumption by a weaker condition such as harmonic Weyl tensor [3], parallel Ricci tensor [4], or cyclic parallel Ricci tensor [5]. For Some other generalizations or rigidity results, we can refer to [6–10], etc.

Recently, some geometricians focus on the study of Miao-Tam equation within the framework of contact metric manifolds. In [11], the authors proved that a complete \(K\)-contact metric satisfying the Miao-Tam critical condition is isometric to a unit sphere \(S^{2m+1}\). Furthermore, they studied \((k, \mu)\)-contact metrics satisfying the Miao-Tam equation.
Moreover, the Miao-Tam equation within the framework of Kenmotsu and almost Kenmotsu manifolds was studied in [12], and it was proved that a Kenmotsu metric satisfying the Miao-Tam equation is Einstein. In addition, in [13], the authors studied the critical point equation on K-paracontact manifolds; especially, they proved that any K-paracontact manifolds satisfying the Miao-Tam equation must be Einstein. We also note that some geometric structures such as Ricci soliton were studied within the framework of paracontact metric $(\kappa, \mu)$-manifold (see [14]). In this direction, it is natural to study paracontact metric $(\kappa, \mu)$-manifold satisfying the Miao-Tam equation. In this paper, we will prove the following main result:

**Theorem 2.** Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a paracontact metric $(\kappa, \mu)$-manifold of dimensional $(2n + 1)$ with $\kappa > -1$. If $(g, \lambda)$ is a nonconstant solution of the Miao-Tam equation, then $M^{2n+1}$ is locally flat in dimension 3, and in higher dimensions $(n > 1)$, it is locally isometric to the product of a flat $(n + 1)$-dimensional manifold and an $n$-dimensional manifold of negative constant curvature equal to $-4$.

### 2. Preliminaries

In this section, we recall some basic definitions and facts on paracontact metric manifolds which we will use later. For more details and some examples, we refer to [15–26].

A $(2n + 1)$-dimensional smooth manifold $M^{2n+1}$ is said to have an almost paracontact structure $(\varphi, \xi, \eta)$, if it admits a $(1, 1)$-tensor field $\varphi$, a vector field $\xi$, and a 1-form $\eta$ satisfying the following conditions:

(i) $\eta(\xi) = 1$, $\varphi^2 = id - \eta \otimes \xi$

(ii) The tensor field $\varphi$ induces an almost paracomplex structure on each fiber of $\mathcal{D} = \text{Ker} (\eta)$, i.e., the eigen-distributions $\mathcal{D}^+$ and $\mathcal{D}^-$ of $\varphi$ corresponding to the eigenvalues 1 and $-1$, respectively, have same dimension $n$.

From the definition, it is easy to see that $\varphi \xi = 0$, $\eta \circ \varphi = 0$, and the endomorphism $\varphi$ have rank $2n$. An almost paracontact structure is said to be normal if and only if the tensor field $N_{\varphi} = [\varphi, \varphi] - 2d\eta \otimes \xi$ vanishes identically. If an almost paracontact manifold admits a pseudo-Riemannian metric $g$ such that

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y),$$

for all $X, Y \in \Gamma(TM)$, then we say that $M$ has an almost paracontact metric structure, and $g$ is called compatible metric. It follows that $\eta = g(\cdot, \xi)$ and $g(\cdot, \varphi \cdot) = -g(\varphi \cdot, \cdot)$. Notice that any such a pseudo-Riemannian metric is necessarily of signature $(n + 1, n)$.

If in addition $d\eta(X, Y) = g(X, \varphi Y)$ for all vector fields $X, Y$ on $M$, then the manifold $M^{2n+1}(\varphi, \xi, \eta, g)$ is said to be a paracontact metric manifold. In this case, $\eta$ becomes a contact form, i.e., $\eta \wedge (d\eta)^n \neq 0$, with $\xi$ its Reeb vector field. In a paracontact metric manifold, one defines two self-adjoint opera-
tors $h$ and $l$ by $h = 1/2 \mathcal{L}_\xi \varphi$ and $l = R(\cdot, \xi)\xi$, where $\mathcal{L}_\xi$ is the Lie derivative along $\xi$, and $R$ is the curvature tensor of $g$. It is known in [25] that the two operators $h$ and $l$ satisfy

$$Tr h = 0, h\xi = 0, l\xi = 0, h\varphi = -\varphi h.$$  \hspace{1cm} (3)

And there also holds

$$\nabla_\xi \varphi = -\varphi X + \varphi hX,$$  \hspace{1cm} (4)

$$\nabla_\xi h = \varphi h^2 - \varphi - \varphi l,$$  \hspace{1cm} (5)

where $\nabla$ is the Levi-Civita connection of the pseudo-Riemannian manifold $(M, g)$. Moreover, $h = 0$ if and only if $\xi$ is a Killing vector field, and in this case, the paracontact metric manifold $M$ is said to be a $K$-paracontact manifold. A normal paracontact metric manifold is said to be a paraSasakiian manifold.

The study of nullity conditions on paracontact geometry is the most interesting topics in paracontact geometry. Motivated by the relationship between contact metric and paracontact geometry, in [18], Cappelletti Montano et al. introduced the following.

**Definition 3.** A paracontact metric manifold $M^{2n+1}(\varphi, \xi, \eta, g)$ is said to be a paracontact metric $(\kappa, \mu)$-manifold, if its curvature tensor $R$ satisfies

$$R(X, Y)\xi = \kappa[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY],$$  \hspace{1cm} (6)

for all tangent vector fields $X, Y$ on $M$, where $\kappa, \mu$ are real constants.

On a paracontact metric $(\kappa, \mu)$-manifold $M^{2n+1}(\varphi, \xi, \eta, g)(n \geq 1)$, the following formulas are valid [18]:

$$h^2 = (1 + \kappa)\varphi^2,$$  \hspace{1cm} (7)

$$Q\xi = 2\mu\kappa\xi,$$  \hspace{1cm} (8)

where $Q$ is the Ricci operator associated with the Ricci tensor $Ric$.

Paracontact metric $(\kappa, \mu)$-spaces satisfy (7) but this condition does not give any type of restriction over the value of $\kappa$, unlike in contact metric geometry, because the metric of a paracontact metric manifold is not positive definite. However, The geometric behavior of the paracontact metric $(\kappa, \mu)$-manifold is different according $\kappa < -1, \kappa = 1$ and $\kappa > -1$. In particular, for the case $\kappa < -1$ and $\kappa > -1$, $(\kappa, \mu)$-nullity condition (7) determines the whole curvature tensor field completely. The case $\kappa = -1$ is equivalent to $h^2 = 0$ but not to $h = 0$, which is different from contact $(\kappa, \mu)$-space. Indeed, there are examples of paracontact metric $(\kappa, \mu)$-spaces with $h^2 = 0$ but $h = 0$, as was first shown in [18, 27, 28]. In this paper, we consider the paracontact metric $(\kappa, \mu)$-manifolds with the condition $\kappa > -1$. 
3. The Proof of Theorem 2

Before giving the proof of Theorem 2, we introduce some important lemmas which will be used later. First of all, we recall a basic fact about paracontact metric $(\kappa, \mu)$-manifold.

Lemma 4 (Corollary 4.14 in [18]). In any $(2n+1)$-dimensional paracontact metric $(\kappa, \mu)$-manifold $M^{2n+1}(\varphi, \xi, \eta, g)$ such that $\kappa > -1$, the Ricci operator $Q$ of $M$ is given by

$$QX = [2(1-n) + n\mu]X + [2n - 1 + \mu]hX + [2n + n(2\kappa - \mu)]\eta(X)\xi,$$  \hspace{1cm} (9)

for any vector field $X$. In particular, $(M, g)$ is $\eta$-Einstein if and only if $\mu = 2(1 - n)$, Einstein if and only if $\kappa = \mu = 0$ and $n = 1$ (in this case, the manifold is Ricci-flat). Further, the scalar curvature of $M$ is $2n(2(1-n) + \kappa + n\mu)$.

In the following, we consider paracontact metric $(\kappa, \mu)$-manifold satisfying the Miao-Tam equation.

Lemma 5. Let $(g, \lambda)$ be a nonconstant solution of the Miao-Tam equation on the $k$-dimensional semi-Riemannian manifold $M^k$ with scalar curvature $S$. Then, the curvature tensor $R$ can be expressed as

$$R(X, Y)D\lambda = (X\lambda)QY - (Y\lambda)QX + \lambda(Q\lambda)QX - \lambda(Q\lambda)(QX)X,$$  \hspace{1cm} (10)

for any vector field $X$ and $Y$ on $M$, where $f = -(\lambda S + 1)/(k - 1)$.

Proof. Tracing (1), we obtain

$$\Delta_{g, \lambda} \lambda = \frac{\lambda S + k}{k - 1}.$$  \hspace{1cm} (11)

Then, the Miao-Tam equation (1) can be exhibited as

$$\nabla_X D\lambda = \lambda QX + \xi,$$  \hspace{1cm} (12)

for any vector field $X$ on $M$, where $f = -(\lambda S + 1)/(k - 1)$. Taking the covariant derivative of (12) along an arbitrary vector field $Y$ on $M$, we obtain

$$\nabla_Y (\nabla_X D\lambda) = (Y\lambda)QX + \lambda(Q\lambda)QX + \lambda Q\nabla_Y X + (Y\lambda)QX + \lambda Q\nabla_Y X + (Y\lambda)QX + \lambda Q\nabla_Y X,$$  \hspace{1cm} (13)

Similarly, we have

$$\nabla_X (\nabla_Y D\lambda) = (X\lambda)QY + \lambda(Q\lambda)QY + \lambda Q\nabla_X Y + (X\lambda)QY + \lambda(Q\lambda)QY + \lambda Q\nabla_X Y + (X\lambda)QY + \lambda(Q\lambda)QY + \lambda Q\nabla_X Y,$$  \hspace{1cm} (14)

for any vector field $X$ and $Y$ on $M$. Comparing the preceding two equations and using (12) in the well-known expression of the curvature tensor $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$, we obtain the result.

Lemma 6. Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a paracontact metric $(\kappa, \mu)$-manifold of dimension $(2n + 1)$ with $\kappa > -1$, and $(g, \lambda)$ be a nonconstant solution of the Miao-Tam equation on $M^{2n+1}$. Then, we have

$$\mu(n + \kappa + 1) = 2\kappa.$$  \hspace{1cm} (15)

Proof. Firstly, taking covariant derivative of (8) along any vector field $X$, and using (4), we can obtain

$$(V\xi Q)\xi = Q(\varphi X - \varphi hX) - 2\mu(\varphi X - \varphi hX).$$  \hspace{1cm} (16)

Taking the inner product of (10) with $\xi$ and using (8) and (16), we have

$$g(R(X, Y)D\lambda, \xi) = 2\mu n[(X\lambda)\eta(Y) - (Y\lambda)\eta(X)] + \lambda g(Q\varphi X - \varphi hX, Y)$$

$$+ \lambda g(Q\varphi Y - \varphi hY, X) + 4\mu \kappa g(\varphi Y, X) + (X\xi)\eta(Y) - (Y\xi)\eta(X),$$  \hspace{1cm} (17)

where $f = -(\lambda S + 1)/(2n)$ (noting that the dimension of $M$ is $2n + 1$).

It follows from (6) that $R(\varphi X, \varphi Y)\xi = 0$. Then, replacing $X$ by $\varphi X$ and $Y$ by $\varphi Y$ in (17), respectively, we obtain

$$\lambda [Q\varphi + Q\varphi h + hQ\varphi - 4\mu \kappa]X = 0.$$  \hspace{1cm} (18)

Since $\lambda$ is nonconstant on $M$, it is easy to see that

$$(Q\varphi + Q\varphi h + hQ\varphi - 4\mu \kappa)X = 0.$$  \hspace{1cm} (19)

Replacing $X$ by $\varphi X$ in (9), we have

$$Q\varphi X = [2(1-n) + n\mu]Q\varphi X + [2(n-1) + \mu]h\varphi X.$$  \hspace{1cm} (20)

Then, the action of $h$ on the (20) gives

$$hQ\varphi X = [2(1-n) + n\mu]h\varphi X + (1+\kappa)[2(n-1) + \mu]h\varphi X,$$  \hspace{1cm} (21)

where we have used (7).

Operating (9) by $\varphi$, we have

$$\varphi QX = [2(1-n) + n\mu]\varphi X + [2(n-1) + \mu]h\varphi X.$$  \hspace{1cm} (22)

Replacing $X$ by $hX$ in (22) and using (7) again, we get

$$\varphi QhX = [2(1-n) + n\mu]h\varphi X + (1+\kappa)[2(n-1) + \mu]h\varphi X.$$  \hspace{1cm} (23)

Substituting equations (20)-(23) into (19) yields

$$\mu(n+\kappa+1) = 2\kappa,$$  \hspace{1cm} (24)

which completes the proof of Lemma 6.

Next, we will give the complete proof of Theorem 2.
Proof. Firstly, taking $X = \xi$ in (17) gives
\[
g(R(\xi, Y)\xi, D\lambda) = g(k(\eta(Y)\xi - Y) - \mu hY, D\lambda) = k(\xi\lambda)\eta(Y) - \kappa Y\lambda - \mu(hY)\lambda.
\] (25)

Putting $X = \xi$ in (6) and comparing with the foregoing equation, we obtain
\[
\kappa D\lambda + \mu hD\lambda = -2n\kappa(\xi\lambda)D\lambda - (k(\xi\lambda) + (\xi f))\xi + Df = 0.
\] (26)

Noting that the scalar curvature $S$ is a constant, it follows from $f = -(\lambda S + 1)/(2n)$ that
\[
2nDf = -SD\lambda.
\] (27)

Then, we can obtain from (26) and (27) that
\[
2n\kappa D\lambda + 2\mu hD\lambda - 4n^2\kappa(\xi\lambda)D\lambda - 2n(\kappa(\xi\lambda) + (\xi f))\xi - SD\lambda = 0.
\] (28)

On the one hand, taking $Y = \xi$ in (6), since $h\xi = 0$, it follows that
\[
R(X, \xi)\xi = \kappa[X - \eta(X)\xi] + \mu[hX - \eta(X)h\xi] = \kappa\rho^2 X + \mu hX,
\] (29)

which gives
\[
l = \kappa\rho^2 + \mu h.
\] (30)

Substituting (7) and (30) in (5), we get
\[
\nabla_X h = -\mu\rho h = \mu\rho\varphi.
\] (31)

On the other hand, we obtain from (12) and (8) that
\[
\nabla_X D\lambda = (2n\kappa\lambda + f)\xi.
\] (32)

Next, taking covariant derivative of (28) along $\xi$ and making use of (31) and (32), we have
\[
(2n\kappa + 4n^2\kappa - S)(2n\kappa\lambda + f)\xi + 2n\rho^2 h\varphi D\lambda - 4n^2\kappa(\xi\lambda)\xi - 2n\kappa(\xi f)\xi - 2n\xi(\xi f)\xi = 0.
\] (33)

Operating this equation by $\varphi$ shows
\[
2n\rho^2 hD\lambda = 0.
\] (34)

By the action of $h$ in (34), it follows from (7) that
\[
\mu^2(\kappa + 1)\rho^2 D\lambda = 0.
\] (35)

Since we assume that $\kappa > -1$, we divide it into two cases: Case (i): $\mu = 0$; case (ii): $\rho^2 D\lambda = 0$.

If case (i) occurs, it follows from Lemma 6 that $\kappa = 0$. Hence, the definition of paracorrect metric $(\kappa, \mu)$-manifold gives that $R(X, Y)\xi = 0$ for any vector field $X, Y$. From Theorem 3.3 of [26], $M^{2n+1}$ is locally flat in dimension 3, and in higher dimensions $(n > 1)$, it is locally isometric to the product of a flat $(n + 1)$-dimensional manifold and an $n$-dimensional manifold of negative constant curvature $-4$.

If case (ii) occurs, then $\rho^2 D\lambda = -D\lambda(\xi)\xi = 0$, i.e., $D\lambda = (\xi\lambda)\xi$. Differentiating this along an arbitrary vector field $X$ together with (4) implies that
\[
\nabla_X D\lambda = X(\xi\lambda)\xi - (\xi\lambda)(\varphi X - \varphi hX).
\] (36)

It follows from (12) that $g(\nabla_X D\lambda, Y) = g(\nabla_Y D\lambda, X)$, and then the foregoing equation shows that
\[
X(\xi\lambda)\eta(Y) - (\xi\lambda)(\varphi X - \varphi hX) = 0.
\] (37)

Replacing $X$ by $\varphi X$, $Y$ by $\varphi Y$, and noting that $d\eta$ is nonzero for any paracorrect metric manifolds, it follows that $\xi\lambda = 0$. Hence, $D\lambda = 0$, $\lambda$ is a constant, which gives a contradiction.

This completes the proof of Theorem 2.

Data Availability
No data were used to support this study.

Conflicts of Interest
The author(s) declare(s) that they have no conflicts of interest.

Acknowledgments
The first author was supported by grant of NSFC (No. 11801011) and the Key Scientific Research Projects of Colleges and Universities of Henan Province (No. 19B110001).

References
[1] P. Miao and L.-F. Tam, "On the volume functional of compact manifolds with boundary with constant scalar curvature," Calculus of Variations and Partial Differential Equations, vol. 36, no. 2, pp. 141–171, 2009.
[2] P. Miao and L.-F. Tam, "Einstein and conformally flat critical metrics of the volume functional," Transactions of the American Mathematical Society, vol. 363, no. 6, pp. 2907–2937, 2011.
[3] H. Baltazar, R. Batista, and K. Bezerra, "On the volume functional of compact manifolds with boundary with harmonic Weyl tensor," 2017, https://arxiv.org/abs/1710.06247.
[4] H. Baltazar and E. Ribeiro Jr., "Critical metrics of the volume functional on manifolds with boundary," Proceedings of the American Mathematical Society, vol. 145, no. 8, pp. 3513–3523, 2017.
[5] W. Sheng and L. Wang, "Critical metrics with cyclic parallel Ricci tensor for volume functional on manifolds with boundary," Geometriae Dedicata, vol. 201, no. 1, pp. 243–251, 2019.
[6] A. Barros, R. Diógenes, and E. Ribeiro Jr., "Bach-flat critical metrics of the volume functional on 4-dimensional manifolds..."
with boundary,” Journal of Geometric Analysis, vol. 25, no. 4, pp. 2698–2715, 2015.

[7] R. Batista, R. Diógenes, M. Ranieri, and E. Ribeiro Jr., “Critical metrics of the volume functional on compact three-manifolds with smooth boundary,” Journal of Geometric Analysis, vol. 27, no. 2, pp. 1530–1547, 2017.

[8] A. Barros and A. da Silva, “Rigidity for critical metrics of the volume functional,” Mathematische Nachrichten, vol. 292, no. 4, pp. 709–719, 2019.

[9] H. Baltazar, A. da Silva, and F. Oliveira, “Weakly Einstein critical metrics of the volume functional on compact manifolds with boundary,” Journal of Mathematical Analysis and Applications, vol. 487, no. 2, p. 124013, 2020.

[10] H. Baltazar and E. Ribeiro Jr., “Remarks on critical metrics of the scalar curvature and volume functionals on compact manifolds with boundary,” Pacific Journal of Mathematics, vol. 297, no. 1, pp. 29–45, 2018.

[11] D. S. Patra and A. Ghosh, “Certain contact metrics satisfying Miao-Tam critical condition,” Annales Polonici Mathematici, vol. 116, pp. 263–271, 2016.

[12] A. Ghosh and D. S. Patra, “Certain almost Kenmotsu metrics satisfying the Miao-Tam equation,” Universitatis Debrecenien-sis, vol. 93, pp. 107–123, 2018.

[13] A. Sarkar and G. G. Biswas, "Critical point equation on K -paracontact manifolds," Balkan Journal of Geometry and its Applications, vol. 25, pp. 117–126, 2020.

[14] D. S. Patra, "Ricci solitons and paracontact geometry," Mediterranean Journal of Mathematics, vol. 16, no. 6, pp. 1–13, 2019.

[15] C. L. Bejan and M. Crasmareanu, "Second order parallel tensors and Ricci solitons in $\mathbb{S}^3 \times \mathbb{S}^3$-dimensional normal para-contact geometry," Annals of Global Analysis and Geometry, vol. 46, no. 2, pp. 117–127, 2014.

[16] B. Cappelletti-Montano, A. Carriazo, and V. Martin-Molina, "Sasaki-Einstein and paraSasaki-Einstein metrics from $(\kappa,\mu)$-structures," Journal of Geometry and Physics, vol. 73, pp. 20–36, 2013.

[17] B. Cappelletti-Montano and L. Di Terlizzi, "Geometric structures associated to a contact metric $(\kappa,\mu)$-space," Pacific Journal of Mathematics, vol. 246, no. 2, pp. 257–292, 2010.

[18] B. Cappelletti-Montano, I. Küpeli Erken, and C. Murathan, "Nullity conditions in paracontact geometry," Differential Geometry and its Applications, vol. 30, no. 6, pp. 665–693, 2012.

[19] G. Calvaruso and D. Perrone, "Geometry of H-paracontact metric manifolds," Universitatis Debrecenien-sis, vol. 86, pp. 325–346, 2015.

[20] A. Ghosh and D. S. Patra, "The critical point equation and contact geometry," Journal of Geometry, vol. 108, no. 1, pp. 185–194, 2017.

[21] S. Ivanov, D. Vassilev, and S. Zamkovoy, "Conformal paracontact curvature and the local flatness theorem," Geometriae Dedicata, vol. 144, no. 1, pp. 79–100, 2010.

[22] V. Martin-Molina, "Local classification and examples of an important class of paracontact metric manifolds," Univerzitet u Nišu, vol. 29, pp. 507–515, 2015.

[23] V. Martin-Molina, "Paracontact metric manifolds without a contact metric counterpart," Taiwanese Journal of Mathematics, vol. 19, pp. 175–191, 2015.

[24] D. M. Venkatesha, "Certain results on K-paracontact and paraSasakian manifolds," Journal of Geometry, vol. 108, no. 3, pp. 939–952, 2017.

[25] S. Zamkovoy, "Canonical connections on paracontact manifolds," Annals of Global Analysis and Geometry, vol. 36, no. 1, pp. 37–60, 2009.

[26] S. Zamkovoy and V. Tsanov, "Non-existence of flat paracontact metric structures in dimension greater than or equal to five," Godishnik na Sofiyskiya Universitet Sv Kliment Ohridski, vol. 100, pp. 27–34, 2011.

[27] B. Cappelletti-Montano, "Bi-paracontact structures and Legendre foliations," Kodai Mathematical Journal, vol. 33, pp. 473–512, 2010.

[28] I. Kupeli Erken and C. Murathan, "A Complete Study of Three-Dimensional Paracontact $(\kappa,\mu,\nu)$-spaces," https://arxiv.org/abs/1305.1511.