A Lipschitz Refinement of the Bebutov–Kakutani Dynamical Embedding Theorem

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Abstract We prove that an \( \mathbb{R} \)-action on a compact metric space embeds equivariantly in the space of one-Lipschitz functions \( \mathbb{R} \to [0, 1] \) if its fixed point set can be topologically embedded in the unit interval. This is a refinement of the classical Bebutov–Kakutani theorem (1968).

Keywords Compact universal flow · Dynamical embedding · Lipschitz functions · Local section · Bebutov–Kakutani theorem

Mathematics Subject Classification 37B05 · 54H20

1 Introduction

The purpose of this short paper is to refine a classical theorem of Bebutov [2] and Kakutani [4] on dynamical systems. We call \((X, T)\) a \textit{flow} if \(X\) is a compact metric space and

\[ T : \mathbb{R} \times X \to X, \ (t, x) \mapsto T_t x \]
is a continuous action of \( \mathbb{R} \). We define \( \text{Fix}(X, T) \) (sometimes abbreviated to \( \text{Fix}(X) \)) as the set of \( x \in X \) satisfying \( T_t x = x \) for all \( t \in \mathbb{R} \). We define \( C(\mathbb{R}) \) as the space of continuous maps \( \varphi : \mathbb{R} \to [0, 1] \). It is endowed with the topology of uniform convergence over compact subsets of \( \mathbb{R} \), namely the topology given by the distance

\[
\sum_{n=1}^{\infty} 2^{-n} \max_{|t| \leq n} |\varphi(t) - \psi(t)|, \quad (\varphi, \psi \in C(\mathbb{R})).
\] (1.1)

The group \( \mathbb{R} \) continuously acts on it by the translation:

\[
\mathbb{R} \times C(\mathbb{R}) \to C(\mathbb{R}), \quad (s, \varphi(t)) \mapsto \varphi(t + s).
\] (1.2)

A continuous map \( f : X \to C(\mathbb{R}) \) is called an embedding of a flow \((X, T)\) if \( f \) is an \( \mathbb{R} \)-equivariant topological embedding. Bebutov [2] and Kakutani [4] found that the \( \mathbb{R} \)-action on \( C(\mathbb{R}) \) has the following remarkable “universality”:

**Theorem 1.1** (Bebutov–Kakutani) A flow \((X, T)\) can be equivariantly embedded in \( C(\mathbb{R}) \) if and only if \( \text{Fix}(X, T) \) can be topologically embedded in the unit interval \([0, 1]\).

The “only if” part is trivial because the set of fixed points of \( C(\mathbb{R}) \) is homeomorphic to \([0, 1]\). So the main statement is the “if” part.

Although the Bebutov–Kakutani theorem is clearly a nice theorem, it has one drawback: The space \( C(\mathbb{R}) \) is not compact (nor locally compact). So it is not a “flow” in the above definition. This poses the following problem:

**Problem 1.2** Is there a compact invariant subset of \( C(\mathbb{R}) \) satisfying the same universality?

The purpose of this paper is to solve this problem affirmatively. Let \( L(\mathbb{R}) \) be the set of maps \( \varphi : \mathbb{R} \to [0, 1] \) satisfying the one-Lipschitz condition:

\[
\forall s, t \in \mathbb{R} : \ |\varphi(s) - \varphi(t)| \leq |s - t|.
\] (1.3)

\( L(\mathbb{R}) \) is a subset of \( C(\mathbb{R}) \). It is compact with respect to the distance (1.1) by Ascoli–Arzela’s theorem. The \( \mathbb{R} \)-action (1.2) preserves \( L(\mathbb{R}) \). So it becomes a flow. Our main result is the following. This solves [3, Question 4.1].

**Theorem 1.3** A flow \((X, T)\) can be equivariantly embedded in \( L(\mathbb{R}) \) if and only if \( \text{Fix}(X, T) \) can be topologically embedded in the unit interval \([0, 1]\).

As in the case of the Bebutov–Kakutani theorem, the “only if” part is trivial because the fixed point set \( \text{Fix}(L(\mathbb{R})) \) is homeomorphic to \([0, 1]\). Since \( L(\mathbb{R}) \) is compact, it is a more reasonable choice of such a “universal flow”.

The proof of Theorem 1.3 is based on the techniques originally used in the proof of the Bebutov–Kakutani theorem (in particular, the idea of local section). A main new ingredient is the topological argument given in Sect. 2, which has some combinatorial flavor.

**Remark 1.4** Problem 1.2 asks us to find a universal flow smaller than \( C(\mathbb{R}) \). If we look for a universal flow larger than \( C(\mathbb{R}) \), then it is much easier to find an example. Let \( L^\infty(\mathbb{R}) \) be the set of \( L^\infty \)-functions \( \varphi : \mathbb{R} \to [0, 1] \). (We identify two functions which are equal to each other almost everywhere.) We consider the weak* topology on it. Namely a sequence \( \{\varphi_n\} \) in \( L^\infty(\mathbb{R}) \) converges to \( \varphi \in L^\infty(\mathbb{R}) \) if for every \( L^1 \)-function \( \psi : \mathbb{R} \to \mathbb{R} \)

\[
\lim_{n \to \infty} \int_{\mathbb{R}} \varphi_n(t) \psi(t) \, dt = \int_{\mathbb{R}} \varphi(t) \psi(t) \, dt.
\]
Then $L^\infty(\mathbb{R})$ is compact and metrizable by Banach–Alaoglu’s theorem and the separability of the space of $L^1$-functions, respectively. The group $\mathbb{R}$ acts continuously on it by translation. So it becomes a flow. Note that $\text{Fix}(L^\infty(\mathbb{R}))$ is homeomorphic to $[0, 1]$ and that the natural inclusion map $C(\mathbb{R}) \subset L^\infty(\mathbb{R})$ is an equivariant continuous injection. Then the Bebutov–Kakutani theorem implies the universality of $L^\infty(\mathbb{R})$: A flow $(X, T)$ can be equivariantly embedded in $L^\infty(\mathbb{R})$ if and only if $\text{Fix}(X, T)$ can be topologically embedded in $[0, 1]$.

### 2 Topological Preparations

Let $a$ be a positive number. We define $L[0, a]$ as the space of maps $\varphi : [0, a] \to [0, 1]$ satisfying

$$\forall s, t \in [0, a] : \ |\varphi(s) - \varphi(t)| \leq |s - t|. $$

$L[0, a]$ is endowed with the distance $\|\varphi - \psi\|_\infty = \max_{0 \leq t \leq a} |\varphi(t) - \psi(t)|$. We define $F_L[0, a] \subset L[0, a]$ as the space of constant functions $\varphi : [0, a] \to [0, 1]$, which is homeomorphic to $[0, 1]$.

Let $(X, d)$ be a compact metric space. We define $C(X, L[0, a])$ as the space of continuous maps $f : X \to L[0, a]$, which is endowed with the distance

$$\max_{x \in X} \|f(x) - g(x)\|_\infty. $$

**Lemma 2.1** Let $f \in C(X, L[0, a])$ and suppose there exists $0 < \tau < 1$ satisfying

$$\forall x \in X, \forall s, t \in [0, a] : \ |f(x)(s) - f(x)(t)| \leq \tau |s - t|. \quad (2.1) $$

Then for any $\delta > 0$ there exists $g \in C(X, L[0, a])$ satisfying

1. $\max_{x \in X} \|f(x) - g(x)\|_\infty < \delta.$
2. $g(x)(0) = f(x)(0)$ and $g(x)(a) = f(x)(a)$ for all $x \in X$.
3. $g(X) \cap F_L[0, a] = \emptyset.$

**Proof** We take $0 < b < c < a$ satisfying $b = a - c < \delta/4$. We take an open covering $\{U_1, \ldots, U_M\}$ of $X$ satisfying

$$\forall 1 \leq m \leq M : \ \text{diam} \ f(U_m) < \min\left(\frac{\delta}{4}, \frac{(1 - \tau)b}{2}\right). \quad (2.2) $$

We take a point $p_m \in U_m$ for each $m$. We choose a natural number $N$ satisfying

$$N > M, \quad \Delta = \frac{c - b}{N - 1} < \frac{\delta}{4}. $$

We divide the interval $[b, c]$ into $(N - 1)$ intervals of length $\Delta$:

$$b = a_1 < a_2 < \cdots < a_N = c, \quad a_{n+1} - a_n = \Delta \quad (\forall 1 \leq n \leq N - 1). $$

Set $A = \{a_1, \ldots, a_N\}$ and define a vector $e \in \mathbb{R}^A$ by $e = (1, 1, \ldots, 1)$. Notice that $f(p_m)|_A$ is an element of $[0, 1]^A$. Since $N > M$ we can choose $u_1, \ldots, u_M \in [0, 1]^A$ satisfying

1. $|f(p_m)(a_n) - u_m(a_n)| < \min(\delta/4, (1 - \tau)b/2)$ for all $1 \leq m \leq M$ and $1 \leq n \leq N$.
2. $|u_m(a_{n+1}) - u_m(a_n)| < \Delta$ for all $1 \leq m \leq M$ and $1 \leq n \leq N - 1$.
3. The $(M + 1)$ vectors $e, u_1, \ldots, u_M$ are linearly independent.

\[ \text{ Springer} \]
Let \( \{h_m\}_{m=1}^M \) be a partition of unity on \( X \) satisfying \( \text{supp} \ h_m \subset U_m \) for all \( m \). For \( x \in X \) we define a piecewise linear function \( g(x) : [0, a] \to [0, 1] \) as follows. (We set \( a_0 = 0 \) and \( a_{N+1} = a \).)

- \( g(x)(0) = f(x)(0) \) and \( g(x)(a) = f(x)(a) \).
- \( g(x)(a_n) = \sum_{m=1}^M h_m(x) u_m(a_n) \) for \( 1 \leq n \leq N \).
- We extend \( g(x) \) linearly. Namely, for \( t = (1 - \lambda)a_n + \lambda a_{n+1} \) with \( 0 \leq \lambda \leq 1 \) and \( 0 \leq n \leq N \) we set \( g(x)(t) = (1 - \lambda)g(a_n) + \lambda g(a_{n+1}) \).

\( \square \)

**Claim 2.2** \( g(x) \in L[0, a] \) and \( \|g(x) - f(x)\|_{\infty} < \delta \).

**Proof** For proving \( g(x) \in L[0, a] \) it is enough to show \( |g(x)(a_{n+1}) - g(x)(a_n)| \leq |a_{n+1} - a_n| \) for all \( 0 \leq n \leq N \). For \( 1 \leq n \leq N - 1 \), this is a direct consequence of the property (2) of \( u_m \). So we consider the case of \( n = 0 \). (The case of \( n = N \) is the same).

\[
|g(x)(b) - f(x)(0)| \leq \sum_{m=1}^M h_m(x)|u_m(b) - f(p_m)(b)| + \sum_{m=1}^M h_m(x)|f(p_m)(b) - f(x)(b)|
\]

\[
+ |f(x)(b) - f(x)(0)|.
\]

We apply to each term of the right-hand side the property (1) of \( u_m \), \( \text{diam} \ f(U_m) < (1 - \tau)b/2 \) in (2.2) and \( |f(x)(b) - f(x)(0)| \leq \tau b \) in (2.1) respectively. Then this is bounded by

\[
\frac{(1 - \tau)b}{2} + \frac{(1 - \tau)b}{2} + \tau b = b.
\]

This proves \( g(x) \in L[0, a] \).

Next we show \( |g(x)(a_n) - f(x)(a_n)| < \delta/2 \) for all \( 0 \leq n \leq N + 1 \). For \( n = 0, N + 1 \), this is trivial. For \( 1 \leq n \leq N \), we can bound \( |g(x)(a_n) - f(x)(a_n)| \) from above by

\[
\sum_{m=1}^M h_m(x)|u_m(a_n) - f(p_m)(a_n)| + \sum_{m=1}^M h_m(x)|f(p_m)(a_n) - f(x)(a_n)|
\]

\[
< \frac{\delta}{4} + \frac{\delta}{4} = \frac{\delta}{2} \quad \text{(by the property (1) of} \ u_m \ \text{and} \ \text{diam} \ f(U_m) < \frac{\delta}{4} \ \text{in} \ (2.2)\).
\]

Finally, let \( a_n < t < a_{n+1} \). We can bound \( |g(x)(t) - f(x)(t)| \) by

\[
|g(x)(t) - g(x)(a_n)| + |g(x)(a_n) - f(x)(a_n)| + |f(x)(a_n) - f(x)(t)|
\]

\[
< 2(a_{n+1} - a_n) + \frac{\delta}{2} \quad \text{(by} \ f(x), g(x) \in L[0, a])
\]

\[
< \delta \quad \text{(by} \ a_{n+1} - a_n \leq \max(b, \Delta) < \frac{\delta}{4} \).
\]

\( \square \)

For every \( x \in X \), the function \( g(x) : [0, a] \to [0, 1] \) is a non-constant function because

\[
g(x)|_{\Lambda} = \sum_{m=1}^M h_m(x) u_m \notin \mathbb{R}e \quad \text{(by the property (3) of} \ u_m).\]

This proves the statement. \( \square \)
We need two lemmas on linear algebra. For $u = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}$ we set

$$Du = (x_2 - x_1, x_3 - x_2, \ldots, x_{n+1} - x_n) \in \mathbb{R}^n.$$ 

**Lemma 2.3** Let $l \geq m + 1$ and set $e = (1, 1, \ldots, 1) \in \mathbb{R}^l$. The set of $(u_1, \ldots, u_m) \in \mathbb{R}^{l+1} \times \cdots \times \mathbb{R}^{l+1} = (\mathbb{R}^{l+1})^m$ such that

the vectors $e, Du_1, Du_2, \ldots, Du_m$ are linearly independent

(2.3)

is open and dense in $(\mathbb{R}^{l+1})^m$.

**Proof** The condition (2.3) defines a Zariski open set in $(\mathbb{R}^{l+1})^m$. So it is enough to show that the set is non-empty because a non-empty Zariski open set is always dense in the Euclidean topology. We set

$$u_i = (-1, \ldots, -1, 0, \ldots, 0), \quad (1 \leq i \leq m).$$

Then

$$Du_i = (0, \ldots, 0, 1, 0, \ldots, 0).$$

The vectors $e, Du_1, \ldots, Du_m$ are linearly independent. \hfill \Box

**Lemma 2.4** Let $n > l \geq 2m$. The set of $(u_1, \ldots, u_m) \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n = (\mathbb{R}^n)^m$ such that, for any integer $\alpha$ with $2 \leq \alpha \leq n - l + 1$,

$u_1|_l, u_1|^{\alpha+l-1}, u_2|_l, u_2|^{\alpha+l-1}, \ldots, u_m|_l, u_m|^{\alpha+l-1}$ are linearly independent in $\mathbb{R}^l$ (2.4)

is open and dense in $(\mathbb{R}^n)^m$. Here for $u_i = (x_{i1}, \ldots, x_{in})$

$$u_i|_l = (x_{i1}, \ldots, x_{il}), \quad u_i|^{\alpha+l-1} = (x_{i,\alpha}, \ldots, x_{i,\alpha+l-1}).$$

**Proof** The defined set $A = \cap_{\alpha=2}^{n-l+1} A_{\alpha}$, where

$$A_{\alpha} = \{(u_1, \ldots, u_m) \in (\mathbb{R}^n)^m \mid (2.4) \text{ is satisfied}\},$$

is a Zariski open set in $(\mathbb{R}^n)^m$. Hence it is enough to show that every $A_{\alpha}$ is non-empty. For each fixed $2 \leq \alpha \leq n - l + 1$, we define $u_i = (x_{i1}, \ldots, x_{in})$ $(1 \leq i \leq m)$ by

$$x_{ij} = 1 \quad (j = i, \alpha + l - i), \quad x_{ij} = 0 \quad \text{(otherwise)}.$$ 

Then it is direct to check that $(u_1, \ldots, u_m) \in A_{\alpha}$. One can also use a proof from [5, Lemma 5.5]. \hfill \Box

**Lemma 2.5** Let $f \in C(X, L[0, a])$ and suppose there exists $0 < \tau < 1$ satisfying (2.1). Then for any $\delta > 0$ there exists $g \in C(X, L[0, a])$ satisfying

1. $\max_{x \in X} \|f(x) - g(x)\|_{\infty} < \delta$.
2. $g(x)(0) = f(x)(0)$ and $g(x)(a) = f(x)(a)$ for all $x \in X$.
3. If $x, y \in X$ and $0 \leq \varepsilon \leq a/2$ satisfy

$$\forall t \in [0, a - \varepsilon] : g(x)(t + \varepsilon) = g(y)(t)$$

then $\varepsilon = 0$ and $d(x, y) < \delta$. \hfill \Box
Proof} Except for the use of the above two lemmas on linear algebra, the proof is close to Lemma 2.1. We take $0 < b < c < a$ with $b = a - c < \min(\delta/4, a/4)$. We take an open covering $\{U_1, \ldots, U_M\}$ satisfying $\text{diam}(U_n) < \delta$ and $\text{diam}(f(U_m)) < \min(\delta/4, (1 - \tau)b/2)$ for all $1 \leq m \leq M$. Take $p_m \in U_m$ for each $m$. Let $N \geq 2$ be a natural number and set $\Delta = (c - b)/(N - 1)$. We introduce a partition $b = a_1 < a_2 < \cdots < a_N = c$ by $a_n = b + (n - 1)\Delta$. We set $A = \{a_1, \ldots, a_N\}$ and $\Lambda = A \cap [b, a/4] = \{a_1, \ldots, a_L\}$. We also set $e = (1, 1, \ldots, 1) \in \mathbb{R}^L$. We choose $N$ sufficiently large so that

$$\Delta < \frac{\delta}{4}, \quad N > L \geq 2M.$$ 

Since $L \geq 2M \geq M + 1$, by using Lemmas 2.3 and 2.4, we can choose $u_1, \ldots, u_M \in [0, 1]^k$ satisfying

1. $|f(p_m)(a_n) - u_m(a_n)| < \min(\delta/4, (1 - \tau)b/2)$ for all $1 \leq m \leq M$ and $1 \leq n \leq N$.
2. $|u_m(a_{n+1}) - u_m(a_n)| < \Delta$ for all $1 \leq m \leq M$ and $1 \leq n \leq N - 1$.
3. Define $D_L u_m = (u_m(a_2) - u_m(a_1), \ldots, u_m(a_{L+1}) - u_m(a_L)) \in \mathbb{R}^L$. Then the $(M + 1)$ vectors $e, D_L u_1, \ldots, D_L u_M$ in $\mathbb{R}^L$ are linearly independent.
4. For any $\varepsilon > 0$ with $\varepsilon + \Lambda \subset A$,

$$u_1|_\Lambda, u_1|_{\varepsilon + \Lambda}, u_2|_\Lambda, u_2|_{\varepsilon + \Lambda}, \ldots, u_m|_\Lambda, u_m|_{\varepsilon + \Lambda}$$

are linearly independent in $\mathbb{R}^A$.

For $x \in X$ we define $g(x) : [0, a] \to [0, 1]$ in the same way as in the proof of Lemma 2.1. Namely, we set $g(x)(0) = f(x)(0)$, $g(x)(a) = f(x)(a)$ and $g(x)(a_n) = \sum_{m=1}^M h_m(x) u_m(a_n)$ for $1 \leq n \leq N$, where $\{h_m\}$ is a partition of unity satisfying $\text{supp} h_m \subset U_m$. We extend $g(x)$ to $[0, a]$ by linearity. It follows that $g(x) \in L[0, a]$ and $\|g(x) - f(x)\|_\infty < \delta$ as before. We need to check the property (3) of the statement. Suppose there exist $x, y \in X$ and $0 \leq \varepsilon \leq a/2$ satisfying $g(x)(t + \varepsilon) = g(y)(t)$ for all $0 \leq t \leq a - \varepsilon$.

First we show $\varepsilon + \Lambda \subset A$. Otherwise, $(\varepsilon + \Lambda) \cap A = \emptyset$. Then it follows from the piecewise linearity that the function $g(x)(t)$ becomes differentiable at every $t \in \Lambda$, which implies

$$g(y)(a_{n+1}) - g(y)(a_n) = g(y)(a_{n+2}) - g(y)(a_{n+1}) \quad (1 \leq n \leq L - 1),$$

and hence

$$\sum_{m=1}^M h_m(y) (u_m(a_{n+1}) - u_m(a_n)) = \sum_{m=1}^M h_m(y) (u_m(a_{n+2}) - u_m(a_{n+1})) \quad (1 \leq n \leq L - 1).$$

This means that $\sum_{m=1}^M h_m(y) D_L u_m \in \mathbb{R}^e$, which contradicts the property (3) of $u_m$. So we must have $\varepsilon + \Lambda \subset A$.

The equation $g(x)(t + \varepsilon) = g(y)(t)$ $(0 \leq t \leq a - \varepsilon)$ implies

$$\sum_{m=1}^M h_m(x) u_m|_{\varepsilon + \Lambda} = \sum_{m=1}^M h_m(y) u_m|_\Lambda.$$ 

It follows from the property (4) of $u_m$ that $\varepsilon = 0$ and $h_m(x) = h_m(y)$ for all $1 \leq m \leq M$. Then $x, y \in U_m$ for some $m$ and hence $d(x, y) \leq \text{diam} U_m < \delta$. \hfill $\square$
3 Proof of Theorem 1.3

Let \((X, T)\) be a flow. Set \(F = \text{Fix}(X, T)\). We define \(F_L = \text{Fix}(L(\mathbb{R}))\). Namely \(F_L\) is the space of constant maps \(\varphi : \mathbb{R} \to [0, 1]\), which is homeomorphic to \([0, 1]\). Suppose there exists a topological embedding \(h : F \to F_L\). We would like to show that there exists an equivariant embedding \(f : X \to L(\mathbb{R})\) with \(f \mid_F = h\). We define \(C_{T, h}(X, L(\mathbb{R}))\) as the space of equivariant continuous maps \(f : X \to L(\mathbb{R})\) satisfying \(f \mid_F = h\), which is endowed with the compact-open topology. For \(f \in C_{T, h}(X, L(\mathbb{R}))\) we define \(\text{Lip}(f)\) as the supremum of

\[
\frac{|f(x)(t) - f(x)(s)|}{|s - t|}
\]

over all \(x \in X\) and \(s, t \in \mathbb{R}\) with \(s \neq t\).

**Lemma 3.1** The space \(C_{T, h}(X, L(\mathbb{R}))\) is not empty. Moreover for any \(\delta > 0\) there exists \(f \in C_{T, h}(X, L(\mathbb{R}))\) satisfying \(\text{Lip}(f) \leq \delta\).

**Proof** Consider the map

\[
F \ni x \mapsto h(x)(0) \in [0, 1].
\]

By the Tietze extension theorem, we can extend this function to a continuous map \(h_0 : X \to [0, 1]\). Let \(\varphi : \mathbb{R} \to [0, 1]\) be a smooth function satisfying

\[
\int_{-\infty}^{\infty} \varphi(t) \, dt = 1, \quad \int_{-\infty}^{\infty} \varphi'(t) \, dt \leq \min(1, \delta).
\]

For \(x \in X\) we define \(f(x) : \mathbb{R} \to [0, 1]\) by

\[
f(x)(t) = \int_{-\infty}^{\infty} \varphi(t-s) h_0(T_s x) \, ds.
\]

Then \(|f(x)'(t)| \leq \min(1, \delta)\) and \(f = h\) on \(F\). Hence \(f \in C_{T, h}(X, L(\mathbb{R}))\) and \(\text{Lip}(f) \leq \delta\).

We borrow the next lemma from Auslander [1, p. 186, Corollary 6].

**Lemma 3.2** Let \(p \in X \setminus F\). There exist \(a > 0\) and a closed set \(S \subset X\) containing \(p\) such that the map

\[
[-a, a] \times S \to X, \quad (t, x) \mapsto T_t x
\]

is a continuous injection whose image contains an open neighborhood of \(p\) in \(X\). We call \((a, S)\) a local section around \(p\) and denote the image of (3.1) by \([-a, a] \cdot S\).

**Proof** We explain the proof for the convenience of readers. We can find \(c < 0\) and a continuous function \(h : X \to [0, 1]\) satisfying \(T_c p \notin \text{supp} \, h\) and \(h = 1\) on a neighborhood of \(p\). We define \(f : X \to \mathbb{R}\) by

\[
f(x) = \int_{c}^{0} h(T_t x) \, dt.
\]

We choose \(0 < a < |c|\) and a closed neighborhood \(A\) of \(p\) satisfying

\[
\bigcup_{|t| \leq a} T_t(A) \subset \{h = 1\}, \quad \bigcup_{|t| \leq a} T_{t+c}(A) \cap \text{supp} \, h = \emptyset.
\]
It follows that \( f( T_t x) = f(x) + t \) for any \( x \in A \) and \( |t| \leq a \). Set \( S = \{ x \in A | f(x) = f(p) \} \). Then \((a, S)\) becomes a local section. Indeed if \( x, y \in S \) and \( s, t \in [-a, a] \) satisfy \( T_s x = T_t y \), then \( s + f(p) = f(T_s x) = f(T_t y) = t + f(p) \) and hence \( s = t \) and \( x = y \). Thus the map (3.1) is injective. We take \( 0 < b < a \) and an open neighborhood \( U \) of \( p \) satisfying 
\[ \bigcup_{|t|<b} T_t(U) \subset A. \]
Then the set \([ -a, a] \cdot S \) contains
\[ \{ x \in U | -b < f(x) - f(p) < b \} \tag{3.2} \]
because if \( x \in U \) satisfies \( t \defeq f(x) - f(p) \in (-b, b) \) then \( f(T_{-t} x) = f(x) - t = f(p) \) (i.e. \( T_{-t} x \in S \)) and \( x = T_t(T_{-t} x) \in [-a, a] \cdot S \). The set (3.2) is an open neighborhood of \( p \).
\[ \square \]

**Lemma 3.3** For any point \( p \in X \setminus F \) there exists a closed neighborhood \( A \) of \( p \) in \( X \) such that the set
\[ G(A) = \left\{ f \in C_{T,h}(X, L(\mathbb{R})) | f(A) \cap F_L = \emptyset \right\} \tag{3.3} \]
is open and dense in the space \( C_{T,h}(X, L(\mathbb{R})) \).

**Proof** Take a local section \((a, S)\) around \( p \). For \( x \in X \) we define \( H(x) \subset \mathbb{R} \) (the set of hitting times) as the set of \( t \in \mathbb{R} \) satisfying \( T_t x \in S \). Any two distinct \( s, t \in H(x) \) satisfy \( |s-t| > a \). Notice that if \( x \in F \) then \( H(x) = \emptyset \). We denote by \( \text{Int} \left( [-a, a] \cdot S \right) \) the interior of \([ -a, a] \cdot S \). We choose a closed neighborhood \( 0 \) of \( p \) in \( S \) satisfying \( 0 \subset \text{Int} \left( [-a, a] \cdot S \right) \).

We define a closed neighborhood \( A \) of \( p \) in \( X \) by
\[ A = \bigcup_{|t| \leq a} T_t(0) = \bigcup_{|t| \leq a} T_t(0). \]
We choose a continuous function \( q : S \rightarrow [0, 1] \) satisfying \( q = 1 \) on \( 0 \) and \( \text{supp} q \subset \text{Int} \left( [-a, a] \cdot S \right) \).

The set \( G(A) \) defined in (3.3) is obviously open. So it is enough to prove that it is dense. Take \( f \in C_{T,h}(X, L(\mathbb{R})) \) and \( 0 < \delta < 1 \). By Lemma 3.1 we can find \( f_0 \in C_{T,h}(X, L(\mathbb{R})) \) satisfying \( \text{Lip}(f_0) \leq 1/2 \). We define \( f_1 \in C_{T,h}(X, L(\mathbb{R})) \) by
\[ f_1(x)(t) = (1 - \delta) f(x)(t) + \delta f_0(x)(t). \]
It follows \( \text{Lip}(f_1) \leq 1 - \frac{\delta}{2} < 1 \). We apply Lemma 2.1 to the map
\[ X \ni x \mapsto f_1(x)|_{[0,a]} \in L[0,a]. \]
Then we find \( g \in C(X, L[0,a]) \) satisfying
\begin{enumerate}
  \item \( |g(x)(t) - f_1(x)(t)| < \delta \) for all \( x \in X \) and \( 0 \leq t \leq a \).
  \item \( g(x)(0) = f_1(x)(0) \) and \( g(x)(a) = f_1(x)(a) \) for all \( x \in X \).
  \item \( g(x) \cap F_L[0,a] = \emptyset \).
\end{enumerate}
We set \( u(x)(t) = g(x)(t) - f_1(x)(t) \) for \( x \in X \) and \( 0 \leq t \leq a \). We define \( g_1 \in C_{T,h}(X, L(\mathbb{R})) \) as follows: Let \( x \in X \).

- For each \( s \in H(x) \), we set
  \[ g_1(x)(t) = f_1(x)(t) + q(T_s x) \cdot u(T_s x)(t-s) \quad \text{for} \ t \in [s, s+a]. \]
- For \( t \in \mathbb{R} \setminus \bigcup_{s \in H(x)} [s, s+a] \), we set \( g_1(x)(t) = f_1(x)(t) \).
This satisfies
\[ |g_1(x)(t) - f(x)(t)| \leq |g_1(x)(t) - f_1(x)(t)| + |f_1(x)(t) - f(x)(t)| \leq 3\delta \]
for all \( x \in X \) and \( t \in \mathbb{R} \). If \( x \in A \) then there exists \( s \in [-a, a] \) with \( T_s x \in A_0 \) and hence
\[ g_1(x)(s + t) = g(T_s x)(t) \quad \text{for} \quad t \in [0, a]. \]
It follows from the property (3) of \( g \) that the function \( g_1(x) \) is not constant. Thus \( g_1 \in G(A) \).
Since \( f \) and \( \delta \) are arbitrary, this proves that \( G(A) \) is dense in \( C_{T,h} (X, L(\mathbb{R})) \). \( \square \)

**Lemma 3.4** For any two distinct points \( p \) and \( q \) in \( X \setminus F \) there exist closed neighborhoods \( B \) and \( C \) of \( p \) and \( q \) in \( X \) respectively such that the set
\[
G(B, C) = \{ f \in C_{T,h} (X, L(\mathbb{R})) \mid f(B) \cap f(C) = \emptyset \}
\]
is open and dense in \( C_{T,h} (X, L(\mathbb{R})) \).

**Proof** Take local sections \((a, S_1)\) and \((a, S_2)\) around \( p \) and \( q \) respectively. We can assume that \([-a, a] \cdot S_1 \) and \([-a, a] \cdot S_2 \) are disjoint with each other. For \( x \in X \) we define \( H(x) \) as the set of \( t \in \mathbb{R} \) satisfying \( T_t x \in S_1 \cup S_2 \). We choose closed neighborhoods \( B_0 \) of \( p \) in \( S_1 \) and \( C_0 \) of \( q \) in \( S_2 \) respectively satisfying \( B_0 \subset \text{Int} \{ [a, a] \cdot S_1 \} \) and \( C_0 \subset \text{Int} \{ [-a, a] \cdot S_2 \} \). We take a continuous function \( \tilde{q} : X \rightarrow [0, 1] \) satisfying \( \tilde{q} = 1 \) on \( B_0 \cup C_0 \) and \( \text{supp} \tilde{q} \subset \text{Int} \{ [-a, a] \cdot S_1 \} \cup \text{Int} \{ [a, a] \cdot S_2 \} \). We define closed neighborhoods \( B \) and \( C \) of \( p \) and \( q \) respectively by
\[
B = \bigcup_{|t| \leq a/4} T_t(B_0), \quad C = \bigcup_{|t| \leq a/4} T_t(C_0).
\]

The set \( G(B, C) \) defined in (3.4) is open. We show that it is dense. Take \( f \in C_{T,h} (X, L(\mathbb{R})) \) and \( 0 < \delta < 1 \). We can assume that
\[
\delta < d(B_0, C_0) \overset{\text{def}}{=} \min_{x \in B_0, y \in C_0} d(x, y). \quad (3.5)
\]
We define \( f_1 \in C_{T,h} (X, L(\mathbb{R})) \) exactly in the same way as in the proof of Lemma 3.3. It satisfies \( \text{Lip}(f_1) \leq 1 - \delta/2 \) and \(|f(x)(t) - f_1(x)(t)| \leq 2\delta \) for all \( x \in X \) and \( t \in \mathbb{R} \).

We apply Lemma 2.5 to the map
\[
X \ni x \mapsto f_1(x)[0, a] \in L[0, a].
\]
Then we find \( g \in C(X, L[0, a]) \) satisfying

1. \(|g(x)(t) - f_1(x)(t)| \leq \delta \) for all \( x \in X \) and \( 0 \leq t \leq a \).
2. \( g(x)(0) = f_1(x)(0) \) and \( g(x)(a) = f_1(x)(a) \) for all \( x \in X \).
3. If \( x, y \in X \) and \( 0 \leq \epsilon \leq a/2 \) satisfy
   \[
   \forall t \in [0, a - \epsilon] : g(x)(t + \epsilon) = g(y)(t)
   \]
   then \( d(x, y) < \delta \).

We set \( u(x)(t) = g(x)(t) - f_1(x)(t) \) for \( x \in X \) and \( 0 \leq t \leq a \). We define \( g_1 \in C_{T,h} (X, L(\mathbb{R})) \) as before. Namely, for \( x \in X \),

- For each \( s \in H(x) \), we set
  \[
  g_1(x)(t) = f_1(x)(t) + \tilde{q}(T_s x) \cdot u(T_s x)(t - s) \quad \text{for} \quad t \in [s, s + a].
  \]
For \( t \in \mathbb{R} \setminus \bigcup_{s \in H(x)} [s, s + a] \), we set \( g_1(x)(t) = f_1(x)(t) \).

This satisfies \( |g_1(x)(t) - f(x)(t)| \leq |g_1(x)(t) - f_1(x)(t)| + |f_1(x)(t) - f(x)(t)| \leq 3\delta \).

We would like to show \( g_1(B) \cap g_1(C) = \emptyset \). Suppose \( x \in B \) and \( y \in C \) satisfy \( g_1(x) = g_1(y) \). There exist \( |s_1| \leq a/4 \) and \( |s_2| \leq a/4 \) satisfying \( T_{s_1}x \in B_0 \) and \( T_{s_2}y \in C_0 \). We can assume \( s_1 \leq s_2 \) without loss of generality. Set \( \varepsilon = s_2 - s_1 \in [0, a/2] \). We have

\[
\begin{align*}
g_1(x)(s_1 + t) &= g(T_{s_1}x)(t) \quad \text{and} \quad g_1(y)(s_2 + t) = g(T_{s_2}y)(t) \quad \text{for } t \in [0, a].
\end{align*}
\]

\( g_1(x) = g_1(y) \) implies that

\[
\begin{align*}
g(T_{s_1}x)(t + \varepsilon) &= g(T_{s_2}y)(t) \quad \text{for } t \in [0, a - \varepsilon].
\end{align*}
\]

It follows from the property (3) of \( g \) that \( d(T_{s_1}x, T_{s_2}y) < \delta \). Since \( \delta < d(B_0, C_0) \leq d(T_{s_1}x, T_{s_2}y) \), this is a contradiction. Therefore \( g_1(B) \cap g_1(C) = \emptyset \). This proves the lemma. \( \square \)

Now we can prove Theorem 1.3. Note that \( X \) and \( X \times X \) are hereditarily Lindelöf (that means that every open cover of a subspace has a countable subcover). Using these facts and applying Lemma 3.3 to each point in \( X \setminus F \) and Lemma 3.4 to every pair of distinct points in \( X \setminus F \), there exist countable families of closed sets \( \{A_n\}_{n=1}^{\infty}, \{B_n\}_{n=1}^{\infty} \) and \( \{C_n\}_{n=1}^{\infty} \) of \( X \setminus F \) such that

- \( X \setminus F = \bigcup_{n=1}^{\infty} A_n \) and \( (X \setminus F) \times (X \setminus F) \setminus \{(x, x) : x \in X\} = \bigcup_{n=1}^{\infty} B_n \times C_n \).
- \( G(A_n) \) are open and dense in the space \( C_{T, h}(X, L(\mathbb{R})) \) for all \( n \geq 1 \).
- \( G(B_n, C_n) \) are open and dense in the space \( C_{T, h}(X, L(\mathbb{R})) \) for all \( n \geq 1 \).

By the Baire category theorem, the set

\[
\bigcap_{n=1}^{\infty} G(A_n) \cap \bigcap_{n=1}^{\infty} G(B_n, C_n)
\]

is dense and \( G_\delta \) in \( C_{T, h}(X, L(\mathbb{R})) \). In particular it is not empty. Any element \( f \) in this set gives an embedding of the flow \( (X, T) \) in \( L(\mathbb{R}) \).

Remark 3.5 The proof of the Bebutov–Kakutani theorem in [1,4] used the idea of “constructing large derivative”. It is possible to prove Theorem 1.3 by adapting this idea to the setting of one-Lipschitz functions. But this approach seems a bit tricky and less flexible than the proof given above. The above proof possibly has a wider applicability to different situations (e.g. other function spaces).

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