TOTALLY REFLEXIVE MODULES OVER RINGS THAT ARE CLOSE TO GORENSTEIN

ANDREW R. KUSTIN AND ADELA VRACIU

ABSTRACT. Let $S$ be a deeply embedded, equicharacteristic, Artinian Gorenstein local ring. We prove that if $R$ is a non-Gorenstein quotient of $S$ of small colength, then every totally reflexive $R$-module is free. Indeed, the second syzygy of the canonical module of $R$ has a direct summand $T$ which is a test module for freeness over $R$ in the sense that if $\text{Tor}_1^R(T,N) = 0$, for some finitely generated $R$-module $N$, then $N$ is free.

In honor of Craig Huneke, on the occasion of his sixty-fifth birthday.

1. INTRODUCTION.

Let $R$ be a commutative Noetherian ring. A finitely generated $R$-module $M$ is called totally reflexive if there exists a doubly infinite complex of finitely generated free $R$-modules

$$F : \cdots \rightarrow F_{i+1} \rightarrow F_i \rightarrow F_{i-1} \rightarrow \cdots,$$

such that $M$ is isomorphic to the cokernel of $F_{i+1} \rightarrow F_i$, for some $i$, and such that both $F$ and the dual complex $\text{Hom}_R(F,R)$ are exact. Totally reflexive modules were introduced by Auslander and Bridger [1], who proved that $R$ is Gorenstein if and only if every finitely generated $R$-module has a finite resolution by totally reflexive $R$-modules. Over a Gorenstein ring, the totally reflexive modules are precisely the maximal Cohen-Macaulay modules; hence, in particular, every singular Gorenstein local ring has a non-free totally reflexive module.

Totally reflexive modules have been the object of extensive study ever since their introduction; yet it remains an open problem to determine conditions on a ring $R$ that are necessary and sufficient for the existence of a non-free totally reflexive $R$-module. We follow the lead of Takahashi [15] and call a commutative Noetherian local ring $G$-regular if every totally reflexive module over the ring is free. Regular local rings are trivial examples of $G$-regular local rings. Avramov and Martsinkovsky [2, 3.5.(2)] proved that every Golod local ring that is not a hypersurface is $G$-regular.

The main result of the paper is Theorem 7.5 where we prove that if $R$ is a non-Gorenstein quotient of small colength of a deeply embedded, equicharacteristic Artinian Gorenstein local ring, then $R$ is $G$-regular. Roughly speaking, a local ring $R$
is deeply embedded if $R = P/I$ for some regular local ring $(P, \mathfrak{M})$ and $I \subseteq \mathfrak{M}^{n_0}$ for some large integer $n_0$; see 2.5.(b) or [13, 4.1] for a more precise formulation.

We consider local rings $(R, \mathfrak{m}, k)$ of the form $R = S/(0 :_S K)$ where $(S, \mathfrak{n})$ is an Artinian Gorenstein local ring, and $K$ is an ideal of $S$. Notice that $K$ is also an $R$-module; furthermore, the canonical module of $R$, which is denoted by $\omega_R$, is equal to $K$. Sometimes we assume $S = P/a$, where $P$ is a standard graded polynomial ring and $a$ is a homogeneous ideal of $P$.

Our favorite method of showing that a ring $R$ is $G$-regular is to exhibit a proj-test module which is a direct summand of a syzygy of the canonical module of $R$. The finitely generated $R$-module $T$ is a proj-test module if the only finitely generated $R$-modules $N$ with $\text{Tor}_1^R(T, N) = 0$ are the projective $R$-modules; see 2.10. We begin by summarizing our results that establish the existence of direct summands of syzygies of the canonical module for certain classes of rings. See 2.2 for our use of Fitt's 4.2 for the proof of Theorem 1.1, 6.1 for the proof of Theorem 1.2, and 6.5 for the proof of Theorem 1.3.

**Theorem 1.1.** Let $(R, \mathfrak{m}, k)$ be a local ring of the form $R = S/(0 :_S K)$ where $(S, \mathfrak{n})$ is an Artinian Gorenstein local ring and $K$ is an ideal of $S$. If

\begin{equation}
\text{Fitt}_1^S(K) = n,
\end{equation}

and

\begin{equation}
\mathfrak{n}K :_S \mathfrak{n} \neq K :_S \mathfrak{n},
\end{equation}

then the second syzygy of the canonical module of $R$, viewed as an $R$-module, has a direct summand isomorphic to $R/\mathfrak{m}$.

**Theorem 1.2.** Let $P$ be a commutative Noetherian ring and $A$ and $B$ be ideals of $P$ with $A \subseteq B^2$, and $P/A$ and $P/B$ both Artinian Gorenstein local rings. Assume $B$ is generated by a regular $P$-sequence of length at least two.

Let $R$ be the ring $R = P/(A :_P B)$. Then the second syzygy of the canonical module of $R$, viewed as an $R$-module, has a direct summand isomorphic to $B/B^2$.

**Theorem 1.3.** Let $k$ be a field of arbitrary characteristic, $P = k[X_1, X_2, X_3, Y_1, \ldots, Y_s]$ be a standard-graded polynomial ring over $k$ in $3 + s$ variables for some nonnegative integer $s$, $B'$ be an ideal in $P$ which is generated by five linearly independent quadratic forms in the variables $X_1, X_2, X_3$, $B''$ be the ideal $(Y_1, \ldots, Y_s)$ of $P$, $B$ be the ideal $B' + B''$ of $P$, and $A \subseteq (P)_1^5$ be a homogeneous ideal of $P$ with $P/A$ an Artinian Gorenstein local ring. Assume $[\text{socle}(P/B)]_1 = 0$.

Let $R$ be the ring $R = P/(A :_P B)$. Then the second syzygy of the canonical module of $R$, viewed as an $R$-module, has a direct summand isomorphic to $B'/BB'$.

In Section 3 we prove that for each ring under consideration in the main result, Theorem 7.5, one of the above Splitting Theorems applies. We prove the Splitting Theorems in Sections 4, 5, and 6. In Section 7 we prove that the summands produced by the Splitting Theorems are indeed proj-test $R$-modules. We complete the proof by appealing to Observation 2.10.3.

The Splitting Theorems of Sections 4, 5, and 6 and the Test Module Theorems of Section 7 are of independent interest and are stated with more general hypotheses than those of the main result, Theorem 7.5.
2. Notation, conventions, and preliminary results.

In this paper $k$ is always a field. If $V$ is vector space over $k$, then $\dim_k V$ is the dimension of $V$ as a vector space over $k$.

2.1. Let $I$ be an ideal in a ring $S$, $N$ be an $S$-module, and $L$ and $M$ be submodules of $N$. Then

$$L : I M = \{ x \in I \mid x M \subseteq L \} \quad \text{and} \quad L :_I I = \{ m \in M \mid I m \subseteq L \}.$$ 

If $L$ is the zero module, then we also use “annihilator notation” to describe these “colon modules”; that is,

$$\text{ann}_S M = 0 :_S M.$$ 

2.2. If $M$ is a finitely generated module over a local ring $S$, and

$$S^h \otimes S^{\mu(M)} \rightarrow M \rightarrow 0$$

is a presentation of $M$ with $\mu(M)$ equal to the minimal number of generators of $M$, then we write $\text{Fitt}_S^1(M)$ for the ideal of $S$ generated by the entries of $\phi$. Notice that $\text{Fitt}_S^1(M)$ is equal to the usual Fitting ideal $\text{Fitt}_{\mu(M)-1}(M)$ of $M$ and does not depend on the choice of $\phi$.

2.3. If $M$ is a matrix (or a homomorphism of free $R$-modules), then $I_r(M)$ is the ideal generated by the $r \times r$ minors of $M$ (or any matrix representation of $M$). We denote the transpose of a matrix $M$ by $M^T$.

2.4. If $S$ is a ring and $M$ is an $S$-module, then let $\lambda_S(M)$ denote the length of $M$ as an $S$-module. If $R$ is a quotient of $S$, then the colength of $R$ as an $S$-module is

$$c_S(R) = \lambda_S(S) - \lambda_S(R).$$

2.5. “Let $(S, n, k)$ be a local ring” identifies $n$ as the unique maximal ideal of the commutative Noetherian local ring $S$ and $k$ as the residue class field $k = S/n$.

(a) The embedding dimension of $S$ is $\dim_k(n/n^2)$.

(b) The parameter $v(S)$ is defined by

$$v(S) = \inf \left\{ i \mid \dim_k(n^i/n^{i+1}) < \binom{n-1}{i} \right\},$$

where $n$ is the embedding dimension of $S$. (This notation is introduced in [13, (4.1.1)].) Notice that if $n$ is minimally generated by $x_1, \ldots, x_n$ and $i$ is an integer with $0 \leq i \leq v(S) - 1$, then the $\binom{n-1}{i}$ monomials of degree $i$ in the symbols $x_1, \ldots, x_n$ represent a basis for the $k$-vector space $n^i/n^{i+1}$.

(c) We say that a local ring $S$ is “deeply embedded” if $v(S)$ is large.

2.6. Gorenstein duality. If $(S, n, k)$ is an Artinian Gorenstein local ring, then $S$ is an injective $S$-module, $\text{Hom}_S(-, S)$ is an exact functor, and $\lambda_S(M) = \lambda_S(\text{Hom}_S(M, S))$ for all $S$-modules $M$ of finite length. Furthermore, if $A$ is an ideal of $S$, then $(0 :_S A)$ is the canonical module of $S/A$. (See, for example, [3, 4].) The following statements are immediate consequences of the above facts. We refer to them as Gorenstein duality.

**Proposition 2.6.1.** If $A$ and $B$ are ideals of an Artinian Gorenstein local ring $S$, then
We denote the second syzygy module of the 

Recall from Schanuel’s Lemma that the notion of second syzygy is well-defined up to formation of direct sum with a projective module. In other words, if \( M' \) and \( M'' \) are both second syzygy modules of \( M \), then there are projective \( R \)-modules \( P' \) and \( P'' \) with \( M' \oplus P' \) isomorphic to \( M'' \oplus P'' \).

As always, let \( k \) be a field.

(a) If \( M \) is a graded module, then we write \([M]_i\) to represent the component of \( M \) which consists of all homogeneous elements of degree \( i \).

(b) If \( Q \) is a non-negatively graded ring over a local ring \( ([Q]_0, m, k) \), then \( Q \) has a unique maximal homogeneous ideal \( \mathfrak{M} = mQ + \bigoplus_{1 \leq j} [Q]_j \). If \( M \) is a \( Q \)-module, then the socle of \( M \) is the vector space socle\( M = 0 :_M \mathfrak{M} \).

(c) Recall that if \( M \) is a finitely generated graded module over a graded polynomial ring \( P = k[x_1, \ldots, x_n] \), then the socle degrees of \( M \) may be read from the back twists in a minimal homogeneous resolution

\[
0 \to C_n = \bigoplus_{i=1}^s P(-\beta_i) \to \ldots \to C_0
\]

of \( M \) by free \( P \)-modules. Indeed, the computation of \( \operatorname{Tor}_n^P(M, k) \) in each coordinate yields a graded isomorphism

\[
\operatorname{socle} M \cong \bigoplus_{i=1}^s k\left( \sum_{j=1}^n \deg x_j - \beta_i \right).
\]

(d) A graded ring \( S = \bigoplus_{0 \leq i} [S]_i \) is called a standard-graded \( k \)-algebra, if \( S \) is a commutative ring with \([S]_0 = k\), \( S \) is generated as an \([S]_0\)-algebra by \([S]_1\), and \([S]_1 \) is finitely generated as an \([S]_0\)-module.

(e) If \( (S, n, k) \) is a local ring, then the associated graded ring of \( S \) is the standard-graded \( k \)-algebra

\[
S^g = \bigoplus_{i=0}^\infty n_i^{i+1}.
\]

Let \( S \) be a commutative Noetherian graded ring and \( M \) be a graded \( S \)-module.

(a) Define

\[
\text{topdeg}(M) = \inf\{\tau \mid [M]_j = 0 \text{ for all } j \text{ with } \tau < j\},
\]

\[
\text{maxgd}(M) = \inf\{\tau \mid S\left( \bigoplus_{j \leq \tau} [M]_j \right) = M\}.
\]
The expressions “topdeg” and “maxgd” are read “top degree” and “maximal generator degree”, respectively. If \( M \) is the zero module, then topdeg(\( M \)) and maxgd(\( M \)) are both equal to \(-\infty\).

(b) If \( S \) is non-negatively graded, \([S]_0\) is Artinian, and \( M \) is finitely generated, then

(i) the Hilbert function of \( M \) is the function \( HF_S(M, \_\_\_) \), from the set of integers to the set of non-negative integers, with \( HF_S(M, i) = \lambda_{[S]_0}([M]_i) \), and

(ii) the Hilbert series of \( M \) is the formal generating function

\[
HS_S(M, t) = \sum_{i\in \mathbb{Z}} HF_S(M, i)t^i.
\]

If \( M = S \), we often write \( HS(S, t) \) (or simply \( HS(S) \)) in place of \( HS_S(S, t) \).

2.10. Test modules and G-regularity. Observation 2.10.3 is the starting point for the present paper. The authors of [14] refer to [14, Lemma 3.2], which is a similar result, as “well-known by the experts”. The present version uses the notion of proj-test module. Related test modules have been used in [12, 5, 6].

Definition 2.10.1. Let \( R \) be a commutative Noetherian ring and \( T \) be a finitely generated \( R \)-module. Then \( T \) is called a proj-test module for \( R \) if the only finitely generated \( R \)-modules \( N \) with \( \text{Tor}_i^R(T, N) = 0 \) for all positive \( i \) are the projective \( R \)-modules.

Observation 2.10.2. If \((R, m, k)\) is an Artinian local ring with canonical module \( \omega_R \) and \( M \) is a finitely generated \( R \)-module, then \( \text{Ext}_R^i(M, R) \) and \( \text{Tor}_i^R(M, \omega_R) \) are Matlis duals of one another.

Proof. Recall that Matlis dual is the functor \((-)^\vee = \text{Hom}_R(\_\_, E)\), where \( E \) is the injective envelope of the \( R \)-module \( k \); furthermore, in the present situation, \( E \) and \( \omega_R \) are isomorphic \( R \)-modules. The ring \( R \) is complete; so, \( N^{\vee\vee} \cong N \) for every finitely generated \( R \)-module \( N \).

Let \( F \) be a resolution of \( M \) by finitely generated free \( R \)-modules. Observe that

\[
\text{Ext}_R^i(M, R) = H^i(\text{Hom}_R(F, R))
\]

\[
= H^i(\text{Hom}_R(F, \text{Hom}_R(\omega_R, \omega_R))) \quad \text{because } R^{\vee\vee} \cong R
\]

\[
\cong H^i(\text{Hom}_R(F \otimes_R \omega_R, \omega_R)) \quad \text{by the adjoint isomorphism theorem}
\]

\[
= H^i((F \otimes_R \omega_R)^\vee)
\]

\[
\cong (H_i(F \otimes_R \omega_R))^{\vee} \quad \text{because } (-)^\vee \text{ is exact}
\]

\[
= (\text{Tor}_i^R(M, \omega_R))^{\vee}.
\]

\[\square\]

Observation 2.10.3. Let \((R, m, k)\) be an Artinian local ring with canonical module \( \omega_R \). If \( T \) is a proj-test module for \( R \) and \( T \) is a summand of a syzygy of \( \omega_R \), then \( R \) is G-regular.

Proof. If \( X \) is a totally reflexive \( R \)-module, then, by definition, \( \text{Ext}_R^i(X, R) = 0 \) for all positive \( i \) and therefore, from Observation 2.10.2, \( \text{Tor}_i^R(X, \omega_R) = 0 \) for all positive \( i \). It follows that \( \text{Tor}_i^R(X, T) = 0 \) for all positive \( i \); and therefore \( X \) is a free \( R \)-module by the definition of proj-test module. \[\square\]
3. RINGS OF SMALL GORENSTEIN COLENGTH.

Let $S$ be an equicharacteristic Artinian Gorenstein local ring. Assume that the invariant $v(S)$, of 2.5.(b), is large. In this section we prove that if $R$ is a non-Gorenstein quotient of $S$ of small colength, then one of the Splitting Theorems of Section 1 applies to $R$.

**Theorem 3.1.** Let $(S, n, k)$ be an equicharacteristic Artinian Gorenstein local ring with embedding dimension at least two, and $R$ be the ring $R = S/J$ for some proper non-zero ideal $J$ of $S$. Assume that either $k$ denotes the minimal number of generators of $\mathfrak{n}/(0 :_S J)$.

- $1 \leq c_S(R) \leq 4$, or
- $c_S(R) = 5$ and $S$ is a standard-graded algebra over a field $[S]_0$.

If the parameter $v(S)$ is sufficiently large, then one of the Theorems 1.1, 1.2, or 1.3 applies to $R$. In particular, the following statements hold, where $k$ denotes the minimal number of generators of $\mathfrak{n}/(0 :_S J)$.

(a) If $k = 0$, then Theorem 1.1 applies to $R$.
(b) If $k = 1$, then Theorem 1.2 applies to $R$, provided $2c_S(R) \leq v(S)$.
(c) If $2 \leq k = c_S(R) - 1$, then Theorem 1.1 applies to $R$, provided $3 \leq v(S)$.
(d) If $k = 2$, $c_S(R) = 4$, and $n^3 \subseteq n(0 :_S J)$, then Theorem 1.2 applies to $R$, provided $5 \leq v(S)$.
(e) If $k = 2$, $c_S(R) = 4$, and $n^3 \not\subseteq n(0 :_S J)$, then Theorem 1.1 applies to $R$, provided $4 \leq v(S)$.
(f) If $k = 2$, $c_S(R) = 5$, and $S$ is a standard-graded algebra over a field $[S]_0$, then Theorem 1.1 applies to $R$, provided $4 \leq v(S)$.
(g) If $k = 3$, $c_S(R) = 5$, $3 \leq \maxgd(0 :_S J)$, and $S$ is a standard-graded algebra over a field $[S]_0$, then Theorem 1.1 applies to $R$, provided $3 \leq v(S)$.
(h) If $k = 3$, $c_S(R) = 5$, $2 \leq \dim_k \text{socle}(S/0 :_S J)$, and $S$ is a standard-graded algebra over a field $[S]_0$, then Theorem 1.1 applies to $R$, provided $3 \leq v(S)$.
(i) If $k = 3$, $c_S(R) = 5$, $\maxgd(0 :_S J) \leq 2$, $\dim_k \text{socle}(S/(0 :_S J)) = 1$, and $S$ is a standard-graded algebra over a field $[S]_0$, then Theorem 1.3 applies to $R$, provided $5 \leq v(S)$.

Assertions (a) – (e) of Theorem 3.1 are proven in 3.2; assertions (f) – (i) are proven in 3.6. First we make some remarks about the statement and introduce some notation.

**Remark 3.1.1.** The ring $S$ of Theorem 3.1. is equicharacteristic and complete; so, the Cohen structure theorem guarantees that $S$ contains a copy of the residue field $k$; that is, there is a commutative diagram

$$
\begin{array}{ccc}
k & \cong & S \\
\downarrow & & \downarrow \text{natural map} \\
S/\mathfrak{n}. & & 
\end{array}
$$

Let $n$ be the embedding dimension of $S$, $U$ be an $n$-dimensional vector space over $k$, $P$ be the polynomial ring $\text{Sym}_k^c(U)$, and $\mathfrak{m}$ be the maximal ideal of $P$ generated by $\text{Sym}_1^c(U)$. Fix a vector space isomorphism $U \cong \mathfrak{n}/\mathfrak{n}^2$. The ring $S$ is Artinian; so, the
inclusion $K \rightarrow S$ and the isomorphism $U \cong n/n^2$ combine to induce a surjection $\pi : P \twoheadrightarrow S$. Let $A$ be the kernel of $\pi$. No harm is done if we write $P/A = S$.

Recall from 2.5.(b) that $A \subseteq \mathfrak{m}^v(S)$; but $A \not\subseteq \mathfrak{m}^{v(S)+1}$.

**Remark 3.1.2.** In the situation of Theorem 3.1, let $K$ be the ideal $(0 :_S J)$ of $S$. Gorenstein duality (see 2.6) guarantees that

$$(0 :_S K) = J.$$

Observe that

(a) $K$ is an $R$-module,
(b) $K$ is the canonical module of $R$, and
(c) $\lambda_S(K) = c_S(R)$.

Assertion (a) is obvious. For (b), observe that $S$ and $R$ have the same Krull dimension and $S$ is a Gorenstein local ring which maps onto $R$. It follows that the canonical module of $R$ is

$$\omega_R = \text{Hom}_S(R,S) = \text{Hom}_S(S/J,S) = (0 :_S J) = K.$$

Assertion (c) holds because $\lambda_S(R) = \lambda_S(\omega_R) = \lambda_S(K) = \lambda_S(S) - \lambda_S(S/K)$; hence,

$$\lambda_S(S/K) = \lambda_S(S) - \lambda_S(R) = c_S(R).$$

**Remark 3.1.3.** In the situation of Theorem 3.1, let $B = \pi^{-1}(K)$ be the preimage of $K$ in $P$. It follows that

$$R = \frac{S}{J/(0 :_S K)} = \frac{P/A}{(A :_P B)/A} = \frac{P}{(A :_P B)}.$$

**Remark 3.1.4.** In the situation of Theorem 3.1, the inequality $k + 1 \leq c_S(R)$ always holds because

$$k = \dim_k \left( \frac{n}{K + n^2} \right) \leq \lambda_S(n/K) = \lambda_S(S/K) - 1 = c_S(R) - 1.$$

The final equality is established in Remark 3.1.2.(c).

**Remark 3.1.5.** The assertion of Theorem 3.1 does not hold if the ring $R$ is Gorenstein; and for this reason the hypotheses of Theorem 3.1 exclude $J = 0$, exclude $c_S(R) = 0$, and exclude the case where $n$ is a principal ideal. Indeed, if $J$ were equal to 0 or if $c_S(R)$ were equal to 0, then $R$ would be equal the Gorenstein ring $S$. Similarly, if $n$ were a principal ideal, then $S$ and $R$ would both be Gorenstein rings of the form $k[X_1]/(X_1^{N_1})$ for integers $N_1$ and $N_2$.

**Remark 3.1.6.** The rings of Theorem 3.1.(a) are called Teter rings. It was already shown in [14] that Teter rings are $G$-regular. In fact, the present paper is inspired by [14].

**Remark 3.1.7.** All cases with $1 \leq c_S(R) \leq 5$ are covered in (a)–(g) of Theorem 3.1 because once (a), (b), and (c) are established, then, it follows from Remark 3.1.4, that it is only necessary to consider $2 \leq k \leq c_S(R) - 2$ for $c_S(R)$ equal to 4 or 5.
3.2. Proof of assertions (a) – (e) of Theorem 3.1. Retain the notation \( P, U, \mathfrak{m}, A, K, \) and \( B \) which is introduced in Remarks 3.1.1, 3.1.2, and 3.1.3. Choose a minimal generating set \( x_1, \ldots, x_k, y_1, \ldots, y_s \) for \( n \) such that \( x_1, \ldots, x_k \) represents a minimal generating set for \( n/K \) and \( y_1, \ldots, y_s \) are in \( K \). Select \( Y_1, \ldots, Y_s, X_1, \ldots, X_k \in U \) to be preimages of \( y_1, \ldots, y_s, x_1, \ldots, x_k \), respectively. Observe that \( Y_1, \ldots, Y_s, X_1, \ldots, X_k \) is a basis for the \( k \) vector space \( U \); indeed, \( P \) is the polynomial ring

\[
P = k[Y_1, \ldots, Y_s, X_1, \ldots, X_k].
\]

Let \( (x) \) and \( (y) \) denote the ideals \( (x_1, \ldots, x_k) \) and \( (y_1, \ldots, y_s) \) of \( S \), respectively, and let \( d \) denote the top degree of the associated graded ring

\[
(S/K)^g = \bigoplus_{i=0}^{\infty} \frac{n^i + K}{n^{i+1} + K}
\]

of \( S/K \). In other words, \( d \) is the smallest positive integer with \( (x)^{d+1} \subseteq K \). There are strict inclusions

\[
(3.2.1) \quad K \subset K + (x)^d \subset \cdots \subset K + (x)^2 \subset K + (x) = n \subseteq S.
\]

Notice that each quotient of consecutive terms in (3.2.1) is annihilated by \( n \) and the Hilbert series of \((S/K)^g\) is

\[
\text{HS} \left( (S/K)^g, t \right) = \sum_{i=0}^{d} \dim_k \frac{K + (x)^i}{K + (x)^{i+1}} t^i,
\]

furthermore,

\[
k = \dim_k \frac{K + (x)}{K + (x)^2} = \text{the coefficient of } t \text{ in } \text{HS}((S/K)^g, t) \quad \text{and}
\]

\[
c_S(R) = \lambda_S(S/K) = \sum_{i=0}^{d} \dim_k \frac{K + (x)^i}{K + (x)^{i+1}} = \text{HS}((S/K)^g, 1).
\]

(a) The parameter \( k \) is equal to \( 0 \); consequently, \( n = K \) and \( y_1, \ldots, y_s \) is a minimal generating set for \( n = K \). The hypotheses that \( n \) is not principal and \( K \) is not zero (that is, \( J \) is a proper ideal of \( S \)) guarantee that \( 2 \leq s \). Therefore, each \( y_i \) appears in a Koszul relation on \( y_1, \ldots, y_s \). It follows that

\[
n \subseteq \text{Fitt}_1^S(n) = \text{Fitt}_1^S(K) \subseteq n;
\]

and therefore condition (1.1.1) of Theorem 1.1 is satisfied. Furthermore, condition (1.1.2) is satisfied because \( 1 \in (K : n) \), but \( 1 \not\in (nK : n) \). We have shown that if \( k = 0 \), then Theorem 1.1 applies to the ring \( R \).

(b) We assume that \( k = 1 \) and \( 2c_S(R) \leq \nu(S) \). It follows that each quotient of consecutive terms in (3.2.1) has length one; thus, \( d + 1 = c_S(R) \) and

\[
(3.2.2) \quad (y_1, \ldots, y_s, x_1^{d+1}) \subseteq K.
\]

On the other hand,

\[
\lambda_S(S/(y_1, \ldots, y_s, x_1^{d+1})) = d + 1 = c_S(R) = \lambda_S(S/K);
\]
and equality holds in (3.2.2). In this case, the ideal $B$ of $P$ from (3.1.3) is generated by a regular sequence. The regular sequence has length at least two because $n$ is not zero and not principal. Furthermore, the hypothesis that

$$2c_S(R) \leq v(S)$$

forces

$$A \subseteq \mathfrak{m}^{v(S)} \subseteq \mathfrak{m}^{2c_S(R)} = \mathfrak{m}^{2(d+1)} \subseteq B^2.$$  

All of the hypotheses of Theorem 1.2 are satisfied by the ring $R$.

(c) We assume that $2 \leq k = c_S(R) - 1$ and $3 \leq v(S)$. It follows from (3.2.1) that $K + (x)^2 = K$; hence,

$$(y) + (x)^2 \subseteq K.$$  

In fact, equality holds, because

$$\frac{n}{(y) + (x)^2} \text{ and } \frac{n}{K}$$

both have length $k$.

The hypothesis $3 \leq v(S)$ ensures that $K = (y) + (x)^2$ is minimally generated by $y_1, \ldots, y_s$, together with the $\binom{k+1}{2}$ monomials of degree two in $x_1, \ldots, x_k$.

3.2.3. The ideal $K$ has at least two minimal generators; so, any Koszul relation involving any $y_i$ (with $1 \leq i \leq s$) and any other minimal generator exhibits $y_i$ as an element of $\text{Fitt}_S^1(K)$.

Furthermore, the parameter $k$ is at least two; so, if $i$ and $j$ are arbitrary with

$$1 \leq i \neq j \leq k,$$

then the relation $x_i(x_j^2) - x_j(x_i x_j)$ exhibits as $x_i$ and $x_j$ elements of $\text{Fitt}_S^1 K$. It is now apparent that hypothesis (1.1.1) of Theorem 1.1 holds. It is clear that $x_1 n \subseteq K$. The hypothesis $3 \leq v(S)$ also ensures that $x_1 n \nsubseteq nK$. We conclude that both assumptions of Theorem 1.1 hold in this case.

3.2.4. The case $(k, c_S(R)) = (2, 4)$. We make some preliminary calculations before dividing this case into the two sub-cases (d) and (e).

Examine the filtration (3.2.1) in order to see that $HS((S/K)^g) = 1 + 2t + t^2$. It follows, in particular, that

$$(3.2.5) \quad (K + (x_1, x_2)^2) / K$$

is a one-dimensional $k$-vector space. It also follows that $(x_1, x_2)^3 \subseteq K$. The hypothesis $3 \leq v(S)$ ensures that the $k$-submodule of $S$ which is spanned by $x_1^2, x_1 x_2, x_2^2$ is a three-dimensional vector space, which we call $V$. Select a basis $q_0, q_1, q_2$ for $V$ so that $q_0$ represents a basis for (3.2.5) and $q_1$ and $q_2$ are in $K$. It follows that

$$(3.2.6) \quad (y_1, \ldots, y_5, q_1, q_2) + (x_1, x_2)^3 \subseteq K;$$

and therefore,

$$(3.2.7) \quad 3 = \lambda_S \left( \frac{n}{K} \right) \leq \lambda_S \left( \frac{n}{(y_1, \ldots, y_5, q_1, q_2) + (x_1, x_2)^3} \right).$$
On the other hand, the module on the right side of (3.2.7) has length at most three because this module is generated by the images of $q_0$, $x_1$, and $x_2$, and these generators give rise to a filtration of length three whose factors are vector spaces of dimension at most one. We conclude that equality holds in both (3.2.7) and (3.2.6); in particular,

$$\text{(3.2.8)} \quad (y_1, \ldots, y_s, q_1, q_2) + (x_1, x_2)^3 = K.$$ 

The hypothesis $4 \leq v(S)$ guarantees that the elements $y_1, \ldots, y_s, q_1, q_2$ are the beginning of a minimal generating set for $K$; furthermore,

$$\text{(3.2.9)} \quad (y) + (q_1, q_2) = K \iff n^3 \subseteq nK.$$ 

We consider two sub-cases.

(d) Assume $(k, c_S(R)) = (2, 4)$, $n^3 \subseteq nK$, and $5 \leq v(S)$. Continue the discussion of 3.2.4. It follows from (3.2.8) and (3.2.9) that

$$(x_1, x_2)^3 \subseteq (y_1, \ldots, y_s, q_1, q_2).$$

Recall the polynomial ring $P = \text{Sym}^k U$ with $P/A = S$ which was introduced Remark 3.1.1. Select $Y_1, \ldots, Y_s, X_1, X_2 \in U$ to be preimages of $y_1, \ldots, y_s, x_1, x_2$, respectively. Observe that $Y_1, \ldots, Y_s, X_1, X_2$ is a basis for the $k$ vector space $U$. Select $Q_1$ and $Q_2$ in the vector space $kX_1^2 \oplus kX_1X_2 \oplus kX_2^2$ to be preimages of $q_1$ and $q_2$, respectively. Notice that $B$, from Remark 3.1.3, is equal to $(Y_1, \ldots, Y_s, Q_1, Q_2) + A$.

Observe that $Q_1, Q_2$ is a regular sequence in $P$. Otherwise, $Q_1$ and $Q_2$ have a common factor and the $k$-submodule of $P$ which is spanned by

$$\{X_iQ_j \mid 1 \leq i, j \leq 2\}$$

is a vector space (which we call $V'$) of dimension at most 3. On the other hand, the image of the above $V'$ in $S$ contains the four dimensional vector space spanned by the monomials of degree 3 in $x_1, x_2$. (Keep in mind that the monomials in $y_1, \ldots, y_s, x_1, x_2$ of degree $i$ represent a basis for $n^i/n^{i+1}$ for $0 \leq i \leq v(S) - 1$.) This contradiction establishes the claim that $Q_1, Q_2$ is a regular sequence in $P$.

Now that we know that $Q_1, Q_2$ is a regular sequence, a quick calculation shows that

$$\mathfrak{m}^5 \subseteq (Y_1, \ldots, Y_s, Q_1, Q_2)^2.$$ 

Therefore the hypothesis that $5 \leq v(S)$ guarantees that

$$A \subseteq \mathfrak{m}^{v(S)} \subseteq \mathfrak{m}^5 \subseteq (Y_1, \ldots, Y_s, Q_1, Q_2)^2.$$ 

Thus, $B$ is generated by the regular sequence $Y_1, \ldots, Y_s, Q_1, Q_2$; this regular sequence has length at least two; and $A \subseteq B^2$. All of the hypotheses of Theorem 1.2 are in effect.

(e) Assume $(k, c_S(R)) = (2, 4)$, $n^3 \nsubseteq nK$, and $4 \leq v(S)$. Continue the discussion of 3.2.4. It follows from (3.2.8) and (3.2.9) that

$$\text{(3.2.10)} \quad (x_1, x_2)^3 \nsubseteq (y_1, \ldots, y_s, q_1, q_2).$$

We saw in the proof of (d) that the quadratic forms $Q_1, Q_2$ of $k[X_1, X_2]$ must have a common linear factor. In other words, there are homogeneous linear forms $L, L_1, L_2$ in $k[X_1, X_2]$ with $Q_1 = LL_1$ and $Q_2 = LL_2$. If $\ell_1$ and $\ell_2$ are the images in $S$ of $L_1$ and $L_2$, then $\ell_1q_2 = \ell_2q_1$ is a relation on a minimal generating set for $K$ that exhibits $\ell_1$
and \( \ell_2 \) as elements of \( \text{Fitt}^1_3(K) \). The linear forms \( L_1, L_2 \) must be linearly independent in \( P \); therefore, \( k(\ell_1, \ell_2) = k(x_1, x_2) \) as sub-vector spaces of \( S \); hence, \( x_1 \) and \( x_2 \) are in \( \text{Fitt}^1_3(K) \). Of course, the argument of (3.2.3) shows that all of the elements \( y_1, \ldots, y_s \) are in \( \text{Fitt}^1_3(K) \). This verifies that Hypothesis (1.1.1) in Theorem 1.1 holds for \( R \). We verify Hypothesis (1.1.2) by showing that \( q_0 \in K :S n \) but \( q_0 \not\in nK :S n \). The first assertion holds because

\[
q_0 n \subseteq (y_1, \ldots, y_s) + (x_1, x_2)^3 \subseteq K.
\]

For the second assertion, write \((x_1, x_2)^3 = (q_0, q_1, q_2)(x_1, x_2) \). The ambient hypothesis \( n^3 \not\subseteq nK \) guarantees that there is an element \( \ell \in kx_1 \oplus kx_2 \) with \( \ell q_0 \in K \not\in nK \). □

Assertions (f) – (i) of Theorem 3.1 are proven in 3.6. We first prove some preliminary results.

**Lemma 3.3.** Let \( B' \) be a homogeneous ideal of the standard-graded polynomial ring \( P' = k[X_1, X_2] \). If

(a) the embedding dimension of \( P'/B' \) is 2 and the length of \( P'/B' \) is 5, or

(b) the embedding dimension of \( P'/B' \) is 2, the length of \( P'/B' \) is 4, and the maximal generator degree of \( B' \) is at least 3,

then \( \text{Fitt}_{P'}(B') = (X_1, X_2) \).

**Proof.** (a) The Hilbert series of \( P'/B' \) is either \( 1 + 2t + 2t^2 \) or \( 1 + 2t + t^2 + t^3 \). We first assume that \( \text{HS}(P'/B') = 1 + 2t + 2t^2 \). Compare \( \text{HS}(P'/B') \) and \( \text{HS}(P') \) to see that \( \text{HS}(B') = t^2 + 4t^3 + \ldots \). It follows that \( B' \) is minimally generated by one quadratic form and two cubic forms. Furthermore, we observe that

\[
\text{socle}(P'/B') = [P'/B']_2 \cong k(-2)^2.
\]

The ideal \( B' \) of \( P' \) is perfect of grade two and the minimal homogeneous resolution of \( P'/B' \) by free \( P' \)-modules looks like

\[
0 \rightarrow P'(-4)^2 \overset{\phi}{\rightarrow} P'(-2)^1 \oplus P'(-3)^2 \rightarrow P'.
\]

(We used 2.8.(c).) The matrix \( \phi \) has the form

\[
(3.3.1) \quad \phi = \begin{bmatrix}
\phi_{1,1} & \phi_{1,2} \\
\phi_{2,1} & \phi_{2,2} \\
\phi_{3,1} & \phi_{3,2}
\end{bmatrix},
\]

where each \( \phi_{i,j} \) is a homogeneous form in \( P' \) and

\[
\text{deg} \phi_{i,j} = \begin{cases} 
2 & \text{if } i = 1 \\
1 & \text{if } 2 \leq i.
\end{cases}
\]

The Hilbert-Burch Theorem guarantees that \( B' \) is generated by the maximal order minors of \( \phi \); hence \( B' \) is contained in the ideal generated by the degree one entries of \( \phi \). The ideal \( B' \) has grade 2; thus, the degree one entries of \( \phi \) generate the entire ideal \((X_1, X_2)\).
We now assume that $\text{HS}(P'/B') = 1 + 2t + t^2 + t^3$. In this case, the Hilbert function of $B'$ is $2t^2 + 3t^3 + 5t^4 + \cdots$. The ideal $B'$ has two minimal quadratic generators; but these generators have a linear factor $L$ in common; consequently, the Hilbert function of the ideal generated by the two quadratics is $2t^2 + 3t^3 + 4t^4 + \cdots$. We conclude that $B'$ is minimally generated by two quadratic forms and a homogeneous form of degree 4. The element of $P'/B'$ represented by $L$ is in the socle of $P'/B'$ and $[P'/B']_3$ is contained in the socle of $P'/B'$. At this point we know that the minimal homogeneous resolution of $P'/B'$ by free $P'$-modules looks like

$$
\begin{align*}
0 & \rightarrow P'(-5) \xrightarrow{\phi} P'(-2) \\
& \quad \oplus P'(-4) \rightarrow P',
\end{align*}
$$

for some homogeneous free $P'$-module $F$. Rank considerations show that $F$ is zero. The matrix $\phi$ has the form (3.3.1), where each $\phi_{i,j}$ is a homogeneous form in $P'$, $\phi_{3,1} = 0$ and

$$
\text{deg} \phi_{i,j} = \begin{cases} 
1 & \text{if } (i,j) \text{ equals } (1,1), (2,1), \text{ or } (3,2) \\
2 & \text{if } (i,j) \text{ equals } (1,2) \text{ or } (2,2). 
\end{cases}
$$

Once again, the Hilbert-Burch Theorem guarantees that $B'$ is generated by the maximal order minors of $\phi$; hence $B' \subseteq (\phi_1, \phi_2, \phi_1)$. The ideal $B'$ still has grade 2; thus, $(\phi_1, \phi_2, \phi_1) = (X_1, X_2)$, and the proof is complete.

(b) The ring $P'/B'$ is standard-graded; consequently,

$$
[P'/B']_i = 0 \implies [P'/B']_{i+1} = 0.
$$

It follows that $\text{HS}(P'/B') = 1 + 2t + t^2$. The ideal $B'$ has two minimal generators of degree two and, by hypothesis, at least one minimal generator of degree at least three. The quadratic generators must have a linear factor $L$ in common as in the $\text{HS} = 1 + 2t + t^2 + t^3$ case. It follows follows that $B'$ has a minimal cubic generator and no further generators are needed. The element $L$ represents an element of degree one in the socle of $P'/B'$ and $[P'/B']_2$ is also contained in the socle. Rank considerations show that the socle can not be any larger than $k(-1) \oplus k(-2)$ as in the $\text{HS} = 1 + 2t + t^2 + t^3$ case. Thus, the minimal homogeneous resolution of $P'/B'$ by free $P'$-modules looks like

$$
\begin{align*}
0 & \rightarrow P'(-3) \xrightarrow{\phi} P'(-2) \\
& \quad \oplus P'(-4) \rightarrow P',
\end{align*}
$$

for some homogeneous free $P'$-module $F$. The matrix $\phi$ has the form (3.3.1), where each $\phi_{i,j}$ is a homogeneous form in $P'$, $\phi_{3,1} = 0$ and

$$
\text{deg} \phi_{i,j} = \begin{cases} 
1 & \text{if } (i,j) \text{ equals } (1,1), (2,1), \text{ or } (3,2) \\
2 & \text{if } (i,j) \text{ equals } (1,2) \text{ or } (2,2). 
\end{cases}
$$

Once again, the entries in the first column of $\phi$ must generate $(X_1, X_2)$. \qed
Lemma 3.4. Let $P'$ be a standard-graded polynomial ring over the field $k$ with maximal homogeneous ideal $\mathfrak{M}'$ and let $B'$ be an $\mathfrak{M}'$-primary homogeneous ideal of $P'$. If the top degree of $P'/B'$ is less than the maximal generator degree of $B'$, then

$$(B':_{p'}\mathfrak{M}') \neq (\mathfrak{M}'B':_{p'}\mathfrak{M}').$$

Proof. Let $d$ denote the top degree of $P'/B'$. First notice that maxgd($B'$) is equal to $d + 1$. Indeed, the inequality $d + 1 \leq$ maxgd($B'$) is part of the hypothesis. On the other hand, $[P']_{d+1} = [\mathfrak{M}']_{d+1} \subseteq B'$. (The equality holds because $P'$ is standard-graded.) It follows that if $p \in [P']_{d+\ell}$, for some $\ell$ with $2 \leq \ell$, then $p \in \mathfrak{M}'B'$ (again because $P'$ is standard graded) and $p$ is not a minimal generator of $B'$; in other words, maxgd($B'$) $\leq d + 1$.

Now observe that

$$[B']_{d}[\mathfrak{M}']_{1} \subseteq [B']_{d+1} = [\mathfrak{M}']_{d+1} = [\mathfrak{M}']_{d}[\mathfrak{M}']_{1}.$$ 

The strict inclusion occurs because $B'$ requires a minimal generator in degree $d + 1$. It follows that there exists an element $\theta$ of $[\mathfrak{M}']_{d}$ with $\theta[\mathfrak{M}']_{1} \not\subseteq [B']_{d}[\mathfrak{M}']_{1}$. This $\theta$ is in $(B':_{p'}\mathfrak{M}') \setminus (\mathfrak{M}'B':_{p'}\mathfrak{M}')$. \hfill $\square$

Lemma 3.5. Let $P$ be a standard-graded polynomial ring over the field $k$ with maximal homogeneous ideal $\mathfrak{M}$, and let $A \subseteq B$ be $\mathfrak{M}$-primary ideals of $P$, with $P/A$ a Gorenstein ring. Denote $P/A$ by $S$, $B/A$ by $K$, and $S/(0 :_S K)$ by $R$. Let $Y_1, \ldots, Y_s, X_1, \ldots, X_k$ be a basis for $[P]_1$ and $P'$ be the subring $k[X_1, \ldots, X_k]$ of $P$. Denote the maximal homogeneous ideal of $P'$ by $\mathfrak{M}'$. Suppose that $B = (Y_1, \ldots, Y_s, B'P)$ for some ideal $B'$ of $P'$. If

(a) $\text{Fitt}^{1}_{p_{B'}}B' = \mathfrak{M}'$,
(b) $\text{maxgd}(B') < \text{topdeg}(P'/B')$, and
(c) $\text{topdeg}(P'/B') + 1 \leq v(S)$,

then Theorem 1.1 applies to $R$.

Proof. Let $d$ denote the top degree of $P'/B'$ and $n$ denote the maximal homogeneous ideal of $S$. Hypothesis (b) enables us to apply Lemma 3.4 in order to learn that there is an element $\theta' \in [P]_{d}$ with $\theta'[\mathfrak{M}'] \subseteq B'$ but $\theta'[\mathfrak{M}']_{1} \not\subseteq [\mathfrak{M}'B']_{d+1}$. Observe that $\theta'$ is also in $[P]_{d}$ with

$$\theta'[\mathfrak{M}] \subseteq B \quad \text{but} \quad \theta'[\mathfrak{M}]_{1} \not\subseteq [\mathfrak{M}B]_{d+1}.$$ 

Apply the natural quotient map $\pi : P \rightarrow S$. Let $\theta \in [S]_{d}$ denote $\pi(\theta')$. Hypothesis (c) guarantees that

$$\pi : [\mathfrak{M}B]_{d+1} \rightarrow [nK]_{d+1}$$

is an isomorphism; thus $\theta \in [S]_{d}$ with

$$\theta[n] \subseteq K \quad \text{but} \quad \theta[n]_{1} \not\subseteq [nK]_{d+1},$$

and Hypothesis (1.1.2) of Theorem 1.1 holds for $R$.

Now we verify that Hypothesis (1.1.1) of Theorem 1.1 holds for $R$. Let $f_1, \ldots, f_{c_1}$ be a minimal homogeneous generating for $B'$ and

$$(P')^{c_2} \xrightarrow{\phi'} (P')^{c_1} \xrightarrow{f=[f_1, \ldots, f_{c_1}]} B'.$$
be a presentation of $B'$. It follows that

\[ P_{c_2+s_0+\left(\frac{2}{3}\right)} \xrightarrow{\phi} P_{c_1+s} \begin{bmatrix} f & Y_1 & \ldots & Y_s \end{bmatrix} \rightarrow B, \]

with

\[
\phi = \begin{bmatrix}
\phi' & Y_1I_{c_1} & Y_2I_{c_1} & \cdots & Y_5I_{c_1} \\
-f & -f & \cdots & -Y_2 \\
-f & Y_1 & \cdots & Y_3 \\
& \ddots & \ddots & \ddots \\
& & \cdots & Y_s \\
& & & -Y_s \\
& & & & Y_{s-1}
\end{bmatrix},
\]

is a presentation $B$ by free $P$-modules with $f_1, \ldots, f_{c_1}, Y_1, \ldots, Y_s$ a minimal generating set for $B$. Hypothesis (a) guarantees that $I_1(\phi') = \mathfrak{M}'$. It follows that

\[(3.5.1) \quad I_1(\phi) = \mathfrak{M}.\]

Apply $\otimes P S$ to obtain the complex

\[(3.5.2) \quad S_{c_2+s_0} \xrightarrow{\phi \otimes P S} S_{c_1+s} \begin{bmatrix} f & Y_1 & \ldots & Y_s \end{bmatrix} \otimes P S \rightarrow K.\]

Observe that $\pi_1(f_1), \ldots, \pi(f_{c_1}), \pi(Y_1), \ldots, \pi(Y_s)$ is still a minimal generating set for $K$ because Hypothesis (c) ensures that $v(S)$ is larger than the maximal generator degree of $K$. (See the proof of Lemma 3.4, if necessary.) The fact that (3.5.2) is a complex ensures that $I_1(\phi(\pi)) \subseteq \text{Fitt}_3^1(K)$. On the other hand, (3.5.1) ensures that $I_1(\phi(\pi))$, which is equal to $\pi(I_1(\phi))$, is equal to $\pi(\mathfrak{M}) = n$. Thus, $\text{Fitt}_3^1(K) = n$ and Hypothesis (1.1.1) of Theorem 1.1 holds for $R$. \hfill \square

3.6. Proof of assertions (f) – (i) of Theorem 3.1. Throughout this proof, the ring $S$ of Theorem 3.1 is standard-graded over $[S]_0 = k$, the polynomial ring $P$ of Remark 3.1.1 is standard-graded over $[P]_0 = k$, the ideals $K$ in $S$ of Remark 3.1.2 and $A$ and $B$ in $P$ of Remarks 3.1.1 and 3.1.3 are homogeneous. The ring $S/K$ is standard-graded and is equal to the associated graded ring $(S/K)^\#$.

Recall the elements $x_1, \ldots, x_k, y_1, \ldots, y_s$ in $S$ and the corresponding elements $X_1, \ldots, X_k, Y_1, \ldots, Y_s$ of $P$ from the beginning of 3.2. Let $P'$ be the subring

\[ P' = k[x_1, \ldots, x_k] \]

of $P$. Observe that $B = (Y_1, \ldots, Y_s) + B'P$ for some homogeneous ideal $B'$ of $P'$; furthermore, there is a natural isomorphism $(P'/B') \cong S/K$, which is induced by

\[(3.6.1) \quad P' \xrightarrow{\pi} P \xrightarrow{\pi} S.\]

(f) We assume that $(k, c_S(R)) = (2, 5)$ and $4 \leq v(S)$. There are two possible Hilbert series for $S/K$ which are consistent with the hypothesis $(k, c_S(R)) = (2, 5)$; that is, $\text{HS}(S/K)$ is equal to $1 + 2t + t^2 + t^3$ or $1 + 2t + 2t^2$. Recall the standard-graded polynomial $P' = k[x_1, X_2]$, the homogeneous ideal $K'$ of $P'$ and the isomorphism $P'/B' \cong S/K$ of (3.6.1). Apply Lemma 3.3 to see that $\text{Fitt}_P^1(B')$ is equal to the maximal homogeneous ideal $\mathfrak{M}' = (X_1, X_2)$ of $P'$. The proof of Lemma 3.3 identifies topdeg$(P'/B')$ and maxgd$(B')$ for both choices of $\text{HF}(P'/B')$. In each case, the inequality topdeg$(P'/B') < \text{maxgd}(B')$ holds; furthermore in each case topdeg$(P'/B')$
is at most 3. All of the hypotheses of Lemma 3.5 are in effect. It follows that Theorem 1.1 applies to $R$.

3.6.2. The case $(k,c_S(R)) = (3,5)$. Examine the filtration (3.2.1) in order to see that $\text{HS}(S/K) = 1 + 3t + t^2$. It follows, in particular, that

$$(3.6.3) \quad (K + (x_1,x_2,x_3)^2)/K$$

is a one-dimensional $k$-vector space and that $(x_1,x_2,x_3)^3 \subseteq K$. Thus,

$$K = (y_1,\ldots,y_5,q_1,\ldots,q_5) + (x_1,x_2,x_3)^3,$$

where $q_1,\ldots,q_5$ are linearly independent quadratic forms in $(x_1,x_2,x_3)$. Furthermore, $(y_1,\ldots,y_5,q_1,\ldots,q_5)$ is the beginning of a minimal generating set for $K$. We show that Hypothesis (1.1.1) of Theorem 1.1 is satisfied. Consider the ten products

$$(3.6.4) \quad \{x_1q_j \in S \mid i \in \{1,2\} \text{ and } 1 \leq j \leq 5\},$$

which is a subset of the ten-dimensional vector space generated by the monomials of degree 3 in $x_1,x_2,x_3$. (We have used the hypothesis that $3 \leq v(S)$.) The monomial $x_3^3$ is not in the vector space spanned by (3.6.4); so the elements of (3.6.4) must be linearly dependent. This gives rise to a non-trivial relation $\sum_i q_i \ell_i = 0$, with each $\ell_i$ a linear form in $x_1$ and $x_2$. The vector space generated by $\ell_1,\ldots,\ell_5$ must have dimension more than one because $q \ell = 0$ is not possible in $S$ with $q$ a non-zero quadratic form in $x_1,x_2,x_3$ and $\ell$ a non-zero linear form in $x_1,x_2,x_3$. (Again, we have used the hypothesis that $3 \leq v(S)$.) Thus, $(x_1,x_2) \subseteq \text{Fitt}_1^1(K)$. A permutation of the subscripts yields that $x_3$ is in $\text{Fitt}_1^1(K)$, as well. We conclude that Hypothesis (1.1.1) holds for $R$.

At this point we have identified the beginning, $y_1,\ldots,y_5,q_1,\ldots,q_5$, of a minimal generating set for $K$ and the beginning, $[S/K]_2$, of the socle of $S/K$. The exact argument we use to finish the proof depends on whether more generators are needed to generate all of $K$ and whether more socle elements are needed to generate all of $\text{socle}(S/K)$.

(g) Assume $(k,c_S(R)) = (3,5)$ and $3 \leq \text{maxgd}(K)$. We proved in 3.6.2 that Hypothesis (1.1.1) is satisfied. We now consider Hypothesis (1.1.2). The inclusion $(x_1,x_2,x_3)^3 \subseteq K$ guarantees that $(x_1,x_2,x_3)^2 \subseteq (K :_S n)$. On the other hand, the hypothesis that some element of $(x_1,x_2,x_3)^3$ is a minimal generator of $K$ ensures that

$$(x_1,x_2,x_3)^2 \not\subseteq (nK :_S n);$$

so, Hypothesis (1.1.2) is satisfied and Theorem 1.1 applies to $R$ in this case.

(h) Assume $(k,c_S(R)) = (3,5)$ and $2 \leq \dim_k \text{socle}(S/K)$. We proved in 3.6.2 that Hypothesis (1.1.1) is satisfied. We now consider Hypothesis (1.1.2). In this case, there must be an element $\ell$ of $[S]_1$ which represents an element of the socle of $S/K$. It is clear that $\ell \in (K :_S n)$. On the other hand, $\ell n$ contains some of the minimal quadratic generators of $K$; so $\ell \not\in (nK :_S n)$. Thus, Hypothesis (1.1.2) is satisfied and Theorem 1.1 applies to $R$. 


(i) Assume \((k, c_S(R)) = (3, 5)\), \(\text{maxgd}(K) \leq 2\), \(\dim_k \ker(S/K) = 1\), and \(5 \leq v(S)\). We use the notation introduced in 3.6.2. The hypotheses ensure that \(K\) is minimally generated by \(y_1, \ldots, y_5, q_1, \ldots, q_5\) and the socle of \(S/K\) is equal to the one-dimensional vector space \([S/K]_2\). The ideal \(B\) of Remark 3.1.3 is equal to 
\[
(Y_1, \ldots, Y_5, Q_1, \ldots, Q_5) + A,
\]
where \(P\) is the standard-graded polynomial ring \(k[y_1, \ldots, y_5, x_1, x_2, x_3]\) and \(Q_i\) is the quadratic form in \(x_1, x_2, x_3\) which represents \(q_i\). The hypothesis \(5 \leq v(S)\) implies that 
\[
A \subseteq M^{\nu(S)} \subseteq M^5 \subseteq (Y_1, \ldots, Y_5, Q_1, \ldots, Q_5),
\]
where \(M\) is the maximal homogeneous ideal of \(P\); consequently, 
\[
B = (Y_1, \ldots, Y_5, Q_1, \ldots, Q_5).
\]
Observe that \(A \subseteq M^5\), \(S = P/A\) is an Artinian Gorenstein local ring, and 
\[
[\ker(P/B)]_1 = [\ker(S/K)]_1 = 0.
\]
Theorem 1.3 applies to \(P/(A :_P B)\), which according to Remark 3.1.3, is equal to \(R\). This completes the proof of Theorem 3.1 \(\square\)

4. THE FIRST SPLITTING THEOREM.

Although not explicitly stated there, Theorem 1.1 was shown in [14]. The proof we give is essentially the dual of the earlier proof.

**Lemma 4.1.** Let \((S, n, k)\) be a local ring, \(J\) be an ideal of \(S\), and \(d : F \to G\) be a homomorphism of free \(S\)-modules of finite rank. Suppose that \(I_1(d) \subseteq n\) and 
\[
(J :_S n) \cdot I_1(d) \nsubseteq nJ.
\]
Then \(k\) is a direct summand of the \(S\)-module \(\ker(d \otimes_S \frac{J}{J})\).

**Proof.** Let \(R\) denote \(\frac{S}{J}\). If \(f\) is an element of \(F\), then let \(\tilde{f}\) denote the image of \(f\) in \(F \otimes_S R\), which is equal to \(\tilde{f}F\).

According to the hypothesis, there are elements \(s \in S, \theta \in F\), and \(\alpha \in G^*\) such that \(sn \subseteq J\) and \(s\alpha(d(\theta))\) is a minimal generator of \(J\). Observe first that \(\overline{s\theta}\) is in \(\ker(d \otimes_S R)\). Indeed, 
\[
d(\overline{s\theta}) = s\alpha d = snG \subseteq JG.
\]
Now observe that \(\overline{s\theta}\) is a minimal generator of \(\ker(d \otimes_S R)\). Otherwise, there are elements \(s_i \in n\) and \(\theta_i \in F\) with \(s\theta - \sum s_i \theta_i \in JF\) and \(d\theta_i \in JG\). In this case, 
\[
s\alpha(d(\overline{\theta})) = \alpha \left( \sum s_i d(\theta_i) + \text{an element of } nJ \cdot G \right)
\]
\[
= \alpha(\text{an element of } nJ \cdot G) \in nJ,
\]
which is a contradiction.

Let \(\overline{s\theta}, \phi_2, \ldots, \phi_m\) be a minimal generating set for \(\ker(d \otimes_S R)\). Observe that as an \(R\)-module, \(\ker(d \otimes_S R)\) is equal to the direct sum 
\[
R\overline{s\theta} \oplus R(\phi_2, \ldots, \phi_m).
\]
It is clear that \(R\overline{s\theta} + R(\phi_2, \ldots, \phi_m) = \ker(d \otimes_S R)\). Furthermore, if 
\[
\overline{r\theta} \in R(\phi_2, \ldots, \phi_m),
\]
for some $r$ in $R$, then the definition of the $\phi$'s ensures that $r$ is in the maximal ideal of $R$ and $rs\theta = 0$ in $F/JF$.

We have shown that $Rs\theta$ is a non-zero direct summand of $\ker(d \otimes S R)$. The proof is complete because $ns \subseteq J$; and therefore $Rs\theta$ is isomorphic to $k$. □

4.2. Proof of Theorem 1.1. We apply Lemma 4.1 to the Artinian Gorenstein local ring $(S, n, k)$. Take $J$ to be the ideal $(0 :_S K)$, which defines $R$, from the statement of Theorem 1.1, and take $d$ to be a minimal presentation matrix for $K$ as an $S$-module.

It follows that $I_1(d) = \text{Fitt}_1^S(K)$, see 2.2. According to Gorenstein duality, see 2.6, the hypothesis $(J :_S n) \cdot I_1(d) \not\subseteq nJ$ of Lemma 4.1 is equivalent to

\[
\begin{align*}
(0 :_S nJ) &\not\subseteq (0 :_S ((J :_S n) \cdot I_1(d))) \\
\iff & (0 :_S J) :_S n \not\subseteq (0 :_S (J :_S n)) :_S I_1(d) \\
\iff & (K :_S n) \not\subseteq (0 :_S (0 :_S nK)) :_S I_1(d) \\
\iff & (K :_S n) \not\subseteq (nK :_S I_1(d)).
\end{align*}
\]

In Theorem 1.1 the hypotheses $\text{Fitt}_1^S(K) = n$ and $K :_S n \not\subseteq nK :_S n$ are both in effect; hence Lemma 4.1 applies and the proof of Theorem 1.1 is complete. □

5. THE METHOD FOR THE SECOND AND THIRD SPLITTING THEOREMS.

The main results in this section are Lemmas 5.4 and 5.5. In each result we identify a particular summand of a second syzygy. These results are used in Section 6 to prove Theorems 1.2 and 1.3.

Setup 5.1. Let $P$ be a commutative Noetherian ring, $A \subseteq B$ be ideals of $P$ with $A \subseteq B^2$, and $\Delta$ be an element of $P$ with

(a) $(A :_P B) = (A, \Delta)$, and
(b) $(A :_P \Delta) = B$.

Let $S$ denote the ring $S = P/A$, and $R$ denote the ring $R = P/(A, \Delta)$. Let

- denote $- \otimes_P S$ and $\equiv$ denote $- \otimes_P R$.

We study $BS$ as an $R$-module.

Remark 5.2. If Setup 5.1 is in effect and $S$ is Artinian, Gorenstein, and local, then $BS$ is the canonical module of $R$ because $S$ and $R$ have the same Krull dimension and $S$ is a Gorenstein local ring which maps onto $R$. It follows that the canonical module of $R$ is

\[
\omega_R = \text{Hom}_S(R, S) = \text{Hom}_{P/A} \left( \frac{P}{(A, \Delta)}, \frac{P}{A} \right) = \frac{A :_P \Delta}{A} = \frac{B}{A} = BS.
\]

Assumption 5.3. In the notation of Setup 5.1, let

\[
B_2 \xrightarrow{b_2} B_1 \xrightarrow{b_1} B \rightarrow 0
\]

be a presentation of $B$ by free $P$-modules. Assume that there exists a $P$-module homomorphism $L : B_1 \rightarrow B_2$ with

\[
b_2 \circ L \equiv \Delta \cdot \text{id}_{B_1} \mod AB_1.
\]
Lemma 5.4. Assume that the notation and hypotheses of 5.1 and 5.3 are in effect. If \( B \) is generated by a regular sequence and \( B \cdot I_1(L) \subseteq (A, \Delta) \), then the \( R \)-module \( B/B^2 \) is a direct summand of the \( R \)-module \( \text{syz}^R_2(\text{BS}) \).

Remark. Our application of Lemma 5.5 occurs when \( B'' \) is generated by variables which are not involved in \( B' \). One can read the statement of Lemma 5.5 with this application in mind; indeed, the statement is meaningful (and easier to digest) if \( B'' \), \( B'_1 \), and \( B'_2 \) all are zero and \( b_1 = b'_1, b_2 = b'_2 \) and \( L = L' \).

Lemma 5.5. Assume that the notation and hypotheses of 5.1 and 5.3 are in effect. Suppose that there are free \( P \)-module decompositions of \( B_1 \) and \( B_2 \) of the form:

\[
B_1 = B'_1 \oplus B'''_1 \quad \text{and} \quad B_2 = B'_2 \oplus B'''_2
\]

and corresponding decompositions of the homomorphisms \( b_1 \), \( b_2 \), and \( L \) of the form:

\[
b_1 = [b'_1 \quad b'''_1], \quad b_2 = \begin{bmatrix} b'_2 & b'''_2 \\ 0 & b''''_2 \end{bmatrix}, \quad \text{and} \quad L = \begin{bmatrix} L' & 0 \\ 0 & L'' \end{bmatrix},
\]

where

\[
b'_1 : B'_1 \to B, \quad b'''_1 : B'''_1 \to B, \quad b'_2 : B'_2 \to B'_1, \quad b'''_2 : B'''_2 \to B'''_1,
\]

\[
b''''_2 : B''''_2 \to B'_1, \quad L' : B'_1 \to B'_2, \quad \text{and} \quad L'' : B''''_2 \to B''''_2
\]

are homomorphisms of free \( P \)-modules. Let \( B' \) and \( B'' \) be the ideals \( b'_1(B'_1) \) and \( b'_1(B'''_1) \) of \( P \), respectively. Suppose further that

(a) \( B \cdot I_1(L') \subseteq (A, \Delta) \),
(b) \( L' \circ b'_2 = \Delta \cdot \text{id}_{B'_2} \mod AB'_2 \),
(c) \( I_1(L' \circ b'''_2) \subseteq (A, \Delta) \), and
(d) the complex \( B'_2 \xrightarrow{b'_1} B'_1 \xrightarrow{b'_1} B' \to 0 \) is exact.

Then the \( R \)-module \( B'/BB' \) is a direct summand of the \( R \)-module \( \text{syz}^R_2(\text{BS}) \).

Lemmas 5.4 and 5.5 are proved in 5.10 and 5.11, respectively. We first identify \( \text{syz}^R_2(\text{BS}) \) in terms of homomorphisms than can be readily calculated from the given data; see (5.7.1). This part of the argument is fairly routine and uses Assumption 5.3 but does not use the extra hypotheses of either Lemma 5.4 or 5.5.

Further notation 5.6. Assume the notation and hypotheses of Setup 5.1 and Assumption 5.3. Let

\[
\begin{array}{ccccccccc}
\ldots & \longrightarrow & A_2 & \xrightarrow{a_2} & A_1 & \xrightarrow{a_1} & A & \longrightarrow & 0 \\
\downarrow & & & & \downarrow & & \downarrow & & \\
\ldots & \longrightarrow & B_2 & \xrightarrow{b_2} & B_1 & \xrightarrow{b_1} & B & \longrightarrow & 0
\end{array}
\]

be a commutative diagram of \( P \)-modules with exact rows and each \( A_i \) and \( B_i \) free. (Take the bottom row of (5.6.1) to be a Koszul complex on a regular sequence which generates \( B \), whenever this is possible.) The hypothesis that \( A \subseteq B^2 \) guarantees that we may choose \( c_1 : A_1 \to B_1 \) with

\[
(5.6.2) \quad c_1(A_1) \subseteq BB_1.
\]
Let $\mu : B_1 \to B_2$ be a $P$-module homomorphism with

$$(b_2 \circ \mu)(\beta_1 \wedge \beta'_1) = b_1(\beta_1) \cdot \beta'_1 - b_1(\beta'_1) \cdot \beta_1.$$

Define $\phi : A_1 \otimes B_1 \to B_2$ to be the composition

$$A_1 \otimes B_1 \xrightarrow{c_1 \otimes 1} B_1 \otimes B_1 \xrightarrow{\text{natural quotient map}} \wedge^2 B_1 \xrightarrow{\mu} B_2.$$

Let

$$\begin{array}{cccc}
B_3 & \oplus & B_2 & \oplus \\
\oplus & \delta_{2,\ell} & \oplus & \delta_{2,r} \\
A_2 & \rightarrow & B_1 & \rightarrow \\
\oplus & \oplus & \oplus & \oplus \\
A_1 \otimes B_1 & \oplus & A_1 \otimes B_1 & \oplus \\
\oplus & \oplus & \oplus & \oplus \\
B_1 & \oplus & A_1 & \oplus \\
\end{array}$$

represent the $P$-module homomorphisms

$$\delta_{2,\ell} = \begin{bmatrix} b_3 & c_2 \\ 0 & -a_2 \end{bmatrix}, \quad \delta_{2,r} = \begin{bmatrix} \phi \\ 0 \end{bmatrix}, \quad \delta_2 = \begin{bmatrix} \delta_{2,\ell} & \delta_{2,r} \end{bmatrix}, \quad \text{and} \quad \delta_1 = \begin{bmatrix} b_2 & c_1 \end{bmatrix}.$$

**Proposition 5.7.** If the notation and hypotheses of 5.6 are in effect, then the following statements hold.

(a) The homomorphisms

$$\begin{array}{cccc}
\bar{B}_3 & \oplus & \bar{B}_2 & \oplus \\
\oplus & \delta_{2,\ell} & \oplus & \delta_{2,r} \\
\bar{A}_2 & \rightarrow & \bar{B}_1 & \rightarrow \\
\oplus & \oplus & \oplus & \oplus \\
\bar{A}_1 \otimes \bar{B}_1 & \oplus & \bar{A}_1 \otimes \bar{B}_1 & \oplus \\
\oplus & \oplus & \oplus & \oplus \\
\bar{B}_1 & \oplus & \bar{A}_1 & \oplus \\
\end{array}$$

form an exact complex of $S$-modules.

(b) The homomorphisms

$$\begin{array}{cccc}
\bar{B}_3 & \oplus & \bar{B}_2 & \oplus \\
\oplus & \delta_{2,\ell} & \oplus & \delta_{2,r} \\
\bar{A}_2 & \rightarrow & \bar{B}_1 & \rightarrow \\
\oplus & \oplus & \oplus & \oplus \\
\bar{A}_1 \otimes \bar{B}_1 & \oplus & \bar{A}_1 \otimes \bar{B}_1 & \oplus \\
\oplus & \oplus & \oplus & \oplus \\
\bar{B}_1 & \oplus & \bar{A}_1 & \oplus \\
\end{array}$$

form an exact complex of $R$-modules and

$$\text{(5.7.1)} \quad \text{syz}^R_2(BS) = \text{ker} \delta_1 = \text{im} \delta_2.$$

(Recall from 2.7 that "syz\text{~}^R_2(BS)" is meaningful up to formation of direct sum with a $R$-projective module.)
Proof. In each case the indicated maps form a complex. We show that the complexes are exact. The mapping cone

\[(5.7.2) \quad \ldots \quad \xrightarrow{[b_4 \ c_3 \ 0 \ -a_3]} B_3 \oplus A_2 \xrightarrow{[b_3 \ c_2 \ 0 \ -a_2]} B_2 \oplus A_1 \xrightarrow{b_1} B_1 \xrightarrow{B/A} BS \rightarrow 0\]

of (5.6.1) is a resolution of $BS$ by free $P$-modules.

(a) Apply $- \otimes_P S$ to (5.7.2) to see that

\[
\frac{B_2 \oplus A_1}{\delta_1 = [b_2 \ c_1]} \xrightarrow{B_1 \delta_1 = [b_1]} BS \rightarrow 0
\]

is an exact sequence of $S$-modules. We compute $\ker(\delta_1)$. Suppose $\theta \in B_2 \oplus A_1$ and $\delta_1(\theta) = 0$ in $B_1$. In this case, $\delta_1(\theta) = \sum_i a_1(\alpha_{1,i}) \cdot \beta_{1,i}$ for some $\alpha_{1,i} \in A_1$ and $\beta_{1,i} \in B_1$. Observe that

\[
(\delta_1 \circ \delta_2, r)
\]

\[
\sum_i \alpha_{1,i} \otimes \beta_{1,i}
\]

\[
\equiv \sum_i \mu(c_1(\alpha_{1,i}) \wedge \beta_{1,i})
\]

\[
\equiv \sum_i \beta_1(\beta_{1,i}) \cdot \alpha_{1,i}
\]

\[
= \sum_i a_1(\alpha_{1,i}) \cdot \beta_{1,i} = \delta_1(\theta).
\]

Thus

\[
\theta - \delta_2, r \left( \sum_i \alpha_{1,i} \otimes \beta_{1,i} \right) \in \ker(\delta_1).
\]

We see from (5.7.2) that

\[
\ker(\delta_1) = \im\left( \begin{bmatrix} b_3 & c_2 \\ 0 & -a_2 \end{bmatrix} \right).
\]

It follows that $\overline{\theta}$ is in the image of $\overline{\delta_2, r}$. The completes the proof of (a).

(b) The proof of (b) is essentially the same as the proof of (a). The right exactness of tensor product yields the exact sequence

\[
\frac{B_2 \oplus A_1}{\delta_1 = [b_2 \ c_1]} \xrightarrow{B_1 \delta_1 = [b_1]} BS \rightarrow 0.
\]

We compute $\ker(\overline{\delta_1})$. Suppose $\theta \in B_2 \oplus A_1$ and $\delta_1(\theta) = 0$ in $B_1$. In this case, $\delta_1(\theta) \equiv \Delta \beta_1 \mod AB_1$ for some $\beta_1 \in B_1$. Apply Assumption 5.3 and observe that

\[
(\delta_1 \circ \delta_2, r)(\beta_1) = (b_2 \circ L)(\beta_1) \equiv \Delta \beta_1 \equiv \delta_1(\theta) \mod AB_1.
\]

Thus $\overline{\theta - \delta_2, r(\beta_1)} \in \ker(\overline{\delta_1}) = \im(\overline{\delta_2, r})$, by part (a); and therefore, $\overline{\theta}$ is in the image of $\overline{\delta_2}$. This completes the proof that the complex of (b) is exact.
The exact complex of (b) gives rise to the exact sequence of $R$-modules

$$0 \to \ker \delta_1 \to (\overline{B}_2 \oplus \overline{A}_1) \xrightarrow{\delta_1} \overline{B}_1 \rightarrow BS \rightarrow 0$$

with $\overline{B}_2 \oplus \overline{A}_1$ and $\overline{B}_1$ free. It follows from Definition 2.7 that $\ker(\delta_1) = \text{syz}_2^R(BS)$. The exactness of the complex of (b) also yields that $\ker(\delta_1) = \text{im}(\delta_2)$. Assertion (5.7.1) has been established and the proof of Proposition 5.7 is complete. □

**Proposition 5.8.** If the notation and hypotheses of Lemma 5.4 and Notation 5.6 are in effect, then the following statements hold:

(a) $\ker \overline{L} = \frac{BB_1}{(A;B)B_1}$, and

(b) $\text{im} \overline{L} \cong B/B^2$.

**Proof.** We prove (a) by showing that

$$(5.8.1) \quad \{ \beta_1 \in B_1 \mid L(\beta_1) \in (A;\Delta)B_2 \} = BB_1.$$  

The inclusion “⊇” is guaranteed by the hypothesis $B \cdot I_1(L) \subseteq (A;\Delta)$. We prove “⊆”. Let $\beta_1$ be an element of $B_1$ with $L(\beta_1) \in (A;\Delta)B_2$. It follows that there is an element $\beta_2 \in B_2$ with $L\beta_1 \equiv \Delta \beta_2 \mod AB_2$. Apply $b_2$ to both sides and then use Assumption 5.3 to obtain

$$\Delta \beta_1 \equiv (b_2 \circ L)(\beta_1) \equiv \Delta b_2(\beta_2) \mod AB_1.$$ 

It follows that $\Delta(\beta_1 - b_2(\beta_2)) \in AB_1$ and

$$\beta_1 - b_2(\beta_2) \in (A : p)B_1 = BB_1.$$  

The hypothesis that $B$ is generated by a regular sequence ensures that the bottom row of (5.6.1) is a Koszul complex; and therefore, $b_2(\beta_2) \in BB_1$. We conclude that $\beta_1 \in BB_1$; hence, (a) is established. The proof of (b) follows in a routine manner:

$$\text{im} \overline{L} \cong \frac{\overline{B}_1}{\ker \overline{L}} \cong \frac{B_1}{BB_1} \cong \frac{B}{B^2}.$$  

The middle isomorphism is due to (a) and the final isomorphism is induced by the surjection $b_1 : B_1 \rightarrow B$. □

**Proposition 5.9.** If the notation and hypotheses of Lemma 5.5 and Notation 5.6 are in effect, then the following statements hold:

(a) $\ker \overline{L} = \frac{b'_2(B'_1) + B \cdot B'_1}{(A:p)B_1}$, and

(b) $\text{im} \overline{L} \cong B'/BB'$.

**Proof.** The proof makes use of the following statements:

(c) $\{ \beta'_1 \in B'_1 \mid L'(\beta'_1) \in (A;\Delta)B'_2 \} = b'_2(B'_2) + B \cdot B'_1$, and

(d) $(A : p)B'_1 \subseteq b'_2(B'_2) + B \cdot B'_1$.

We prove (c). Hypotheses (b) and (a) of Lemma 5.5 ensure that

$$(A,\Delta)B'_2 \supseteq L'(b'_2(B'_2) + B \cdot B'_1);$$

and this establishes the inclusion “⊇”. Now, we prove “⊆”. Suppose $\beta'_1$ is in $B'_1$ with

$$L'(\beta'_1) \equiv \Delta \beta'_2 \mod AB'_2,$$
for some $\beta'_2 \in B'_2$. Use Hypothesis 5.5.(b) to write

$$L'(\beta'_1) \equiv (L' \circ b'_2)(\beta'_2) \mod AB'_2.$$  

Apply $b'_2$ to obtain

$$(b'_2 \circ L')(\beta'_1 - b'_2\beta'_2) \equiv 0 \mod AB'_1.$$  

Employ Assumption 5.3 to see that

$$\Delta(\beta'_1 - b'_2\beta'_2) \in AB'_1;$$

hence,

$$\beta'_1 - b'_2\beta'_2 \in (A : p \Delta)B'_1 = BB'_1,$$

by 5.1.(b), and $\beta'_1 \in b'_2(B'_2) + BB'_1$. This completes the proof of (c).

We prove (d). Recall from 5.1 that $(A : p B) = (A, \Delta)$ and $A \subseteq B$. Thus, it suffices to consider $\Delta B'_1$. Assumption 5.3 and the decompositions of Lemma 5.5 ensure that

$$\Delta \cdot B'_1 \equiv (b_2 \circ L)(B'_1) = (b'_2 \circ L')(B'_1) \subseteq b'_2(B'_2).$$

The proof of (a) is now straightforward:

$$\ker \overline{L} = \frac{\{B'_1 \in B'_1 \mid L'(\beta'_1) \in (A, \Delta)B'_2 \} + (A : p B)B'_1}{(A : p B)B'_1},$$

by (c),

$$= \frac{b'_2(B'_2) + B \cdot B'_1 + (A : p B)B'_1}{(A : p B)B'_1}.$$  

The proof of (b) is also straightforward:

$$\text{im} \overline{L} \cong \frac{B'_1}{\ker \overline{L}} = \frac{B'_1}{b'_2(B'_2) + B \cdot B'_1} \cong \frac{B'_1}{BB'_1}.$$  

The final isomorphism is induced by the surjection $b'_1 : B'_1 \longrightarrow B'$ and uses Hypothesis (d) of Lemma 5.5.

5.10. The proof of Lemma 5.4. We use all of notation and hypotheses of 5.1, 5.4, and 5.6. We proved in Proposition 5.8.(b) that

$$\text{im}(\overline{\delta^2_{2,r}}) \cong B/B^2.$$  

We complete the proof of Lemma 5.4 by showing that $\text{im}(\overline{\delta^2_{2,r}})$ is a direct summand of the $R$-module $\text{syz}_2^R(BS)$. We accomplish this goal by showing that

$$\text{im}(\overline{\delta^2_{2,l}}) \oplus \text{im}(\overline{\delta^2_{2,r}}) = \text{syz}_2^R(BS).$$  

We know from (5.7.1) that

$$\text{im}(\overline{\delta^2_{2,l}}) + \text{im}(\overline{\delta^2_{2,r}}) = \text{syz}_2^R(BS).$$  

We show that $\text{im}(\overline{\delta^2_{2,l}}) \cap \text{im}(\overline{\delta^2_{2,r}}) = (0)$. Suppose

$$\beta_1 \in B_1 \quad \text{and} \quad \theta \in B_3 \oplus A_2 \oplus (A_1 \otimes B_1)$$
with

\[(5.10.1) \quad \delta_{2,r}(\beta_1) - \delta_{2,\ell}(\theta) \quad \text{in} \quad (A, \Delta)B_2 \oplus (A, \Delta)A_1.\]

We complete the proof by showing that

\[(5.10.2) \quad L(\beta_1) \in (A, \Delta)B_2.\]

We are told in (5.10.1) that

\[
\begin{bmatrix}
L(\beta_1) \\
0
\end{bmatrix} - \delta_{2,\ell}(\theta) = \begin{bmatrix}
\Delta \beta_2 \\
\Delta \alpha_1
\end{bmatrix} + \begin{bmatrix}
an \text{element of } AB_2 \\
an \text{element of } AA_1
\end{bmatrix},
\]

for some \(\beta_2 \in B_2\) and \(\alpha_1 \in A_1\). Apply \(\delta_1 = [b_2 \ c_1]\) to see that

\[(5.10.3) \quad b_2L\beta_1 - \delta_1\delta_{2,\ell}(\theta) = \Delta b_2\beta_2 + \Delta c_1\alpha_1 + \text{an element of } AB_1.\]

We know from 5.7.(a) that \(\delta_1\delta_{2,\ell}(\theta) \in AB_1\). The choice \(c_1(A) \subseteq BB_1\), which was made in (5.6.2), guarantees that \(\Delta c_1\alpha_1 \subseteq \Delta BB_1 \subseteq AB_1\). Of course, Assumption 5.3 guarantees that \(b_2L\beta_1 \equiv \Delta \beta_1 \mod AB_1\). Thus, (5.10.3) yields

\[
\Delta(\beta_1 - b_2\beta_2) \in AB_1;
\]

and therefore, \(\beta_1 - b_2\beta_2 \in BB_1\). Thus,

\[
\beta_1 \in b_2(B_2) + BB_1 \subseteq BB_1.
\]

(The hypothesis that \(B\) is generated by a regular sequence ensures that the bottom row of (5.6.1) is a Koszul complex; and therefore, \(b_2(B_2) \subseteq BB_1\) and the final inclusion holds.) Apply the hypothesis \(BI_1(L) \subseteq (A, \Delta)\) to conclude \(L(\beta_1) \in (A, \Delta)B_2\), which establishes (5.10.2) and completes the proof. \(\square\)

**5.11. The proof of Lemma 5.5.** We use all of notation and hypotheses of 5.1, 5.5, and 5.6. We proved in Proposition 5.9.(b) that

\[
\text{im}(\overline{L}) \cong B'/BB'.
\]

We complete the proof of Lemma 5.5 by showing that \(\text{im}(\overline{L})\) is a direct summand of the \(R\)-module \(\text{syz}_2^R(BS)\). We accomplish this goal by showing that

\[
\left( \text{im}(\overline{\delta_{2,\ell}}) \oplus \text{im}(\overline{L'}) \right) \oplus \text{im}(\overline{L}) = \text{syz}_2^R(BS).
\]

We know from (5.7.1) that

\[
\left( \text{im}(\overline{\delta_{2,\ell}}) + \text{im}(\overline{L'}) \right) + \text{im}(\overline{L}) = \text{syz}_2^R(BS).
\]

We show that \(\left( \text{im}(\overline{\delta_{2,\ell}}) + \text{im}(\overline{L'}) \right) \cap \text{im}(\overline{L}) = (0)\). Suppose

\[
\beta' \in B', \quad \beta'' \in B'', \quad \text{and} \quad \theta \in B_3 \oplus A_2 \oplus (A_1 \otimes B_1)
\]

with

\[(5.11.1) \quad L'(\beta'_1) - L''(\beta''_1) - \delta_{2,\ell}(\theta) \quad \text{in} \quad (A, \Delta)B'_2 \oplus (A, \Delta)B''_2 \oplus (A, \Delta)A_1.\]

We complete the proof by showing that

\[(5.11.2) \quad L'(\beta'_1) \in (A, \Delta)B'_2.\]
We are told in (5.11.1) that
\[ L(β'_1 - β''_1) - δ_2(θ) = \begin{bmatrix} Δβ_2 \\ Δα_1 \end{bmatrix} + \text{an element of } AB_2, \]
for some $β_2 \in B_2$, and $α_1 \in A_1$. Apply $δ_1 = [b_2 \ c_1]$ to see that
\[ (b_2 \circ L)(β'_1 - β''_1) - δ_1δ_2(θ) = Δb_2β_2 + Δc_1α_1 + \text{an element of } AB_1. \]
We know from 5.7.(a) that $δβ$ for some $\beta$ for
\[ \text{and therefore,} \]
\[ \Delta(β'_1 - β''_1) - b_2β_2) ∈ AB_1; \]
and therefore,
\[ \beta'_1 - β''_1 - b_2β_2 ∈ (A : p Δ)B_1 = BB_1. \]
Separate (5.11.4) into the components $B'_1 ⊕ B''_1 = B_1$. Recall that $b_2$ has the form
\[ b_2 = \begin{bmatrix} b'_2 \\ b''_2 \end{bmatrix}; \]
and write $β_2 = β'_2 + b''_2$ with $β'_2 \in B'_2$ and $b''_2 \in B''_2$. Conclude that
\[ \begin{bmatrix} β'_1 \\ -β''_1 \end{bmatrix} - \begin{bmatrix} b'_2 \\ 0 \end{bmatrix} \begin{bmatrix} b''_2 \\ b''_2 \end{bmatrix} = \begin{bmatrix} β'_1 \\ -β''_1 \end{bmatrix} + \text{an element of } BB'_1. \]
It follows that
\[ β'_1 - b'_2β'_2 - b''_2β''_2 ∈ BB'_1. \]
Apply $L'$ and invoke Hypotheses (a), (b), and (c) of Lemma 5.5 to conclude $Lβ'_1$ is in $(A, Δ)B'_2$, which establishes (5.11.2) and completes the proof.

6. COMPLETE INTERSECTIONS AND FIVE QUADRATICS IN THREE VARIABLES.

In this section, we prove Theorems 1.2 and 1.3 which were promised in Section 1. The proof of Theorem 1.2 is based on Lemma 5.4 and is given in 6.1. The proof of Theorem 1.3 is given in 6.5 after we collect the results that ensure that the hypotheses of Lemma 5.5 are satisfied.

6.1. The proof of Theorem 1.2. We apply Lemma 5.4 and Remark 5.2. Gorenstein duality guarantees that $(A : p B)/A$ is a cyclic $P/A$-module. Let $Δ$ be an element of $P$ with $(A : p B) = (A, Δ)$. Gorenstein duality also guarantees that $(A : p Δ) = B$.

Let $θ_1, \ldots, θ_n$ be a regular sequence in $P$ which generates $B$ and let
\[ B_2 \xrightarrow{b_2} B_1 \xrightarrow{b_1} B \to 0 \]
be the beginning of the Koszul complex on this generating set. We are now in the situation of Setup 5.1. It remains to establish the existence of a homomorphism $L : B_1 → B_2$ which satisfies
(a) $(b_2 \circ L) ∋ Δ \cdot id_{B_1}$ mod $AB_1$, and
(b) $B \cdot L(B_1) ⊆ (A, Δ)B_2$. 
Property (b) is satisfied provided
\[ L(B_1) \subseteq \left( (A, \Delta) : B \right) B_2 = \left( (A : B) \right) B_2 = (A : B^2) B_2. \]
Consequently, it suffices to prove that
\[ (A : B) B_1 \subseteq b_2 \left( (A : B^2) B_2 \right) + AB_1. \]

We think of \( B_1 \) as \( P^n \) and of the surjection \( B_1 \to B \) as the map given by the matrix \([\theta_1, \ldots, \theta_n] \). We show that
\[
\begin{bmatrix}
(A : B) \\
0 \\
\vdots \\
0
\end{bmatrix}
\subseteq P^n
\]
is in \( b_2 \left( (A : B^2) B_2 \right) + AB_1 \). One completes the argument by symmetry. Every Koszul relation on \( \theta_1, \ldots, \theta_n \) is in the image of \( b_2 \); so, in particular, each column of
\[
\begin{bmatrix}
\theta_2 & \cdots & \theta_n \\
-\theta_1 I_{n-1}
\end{bmatrix}
\]
is in the image of \( b_2 \), where \( I_{n-1} \) is the \((n-1) \times (n-1)\) identity matrix. Let \( \Theta \) be an element of \( (A : B) \). We show that there exist elements \( r_2, \ldots, r_n \) in \( (A : \theta_1) \cap (A : B^2) \) with \( \sum_{i=2}^n r_i \theta_i \equiv \Theta \), mod \( A \). In other words, it suffices to show that
\[ \Theta \in \left( (A : \theta_1) \cap (A : B^2) \right) (\theta_2, \ldots, \theta_n) + A. \]

Let \( \Theta \) roam over \( (A : B) \). It suffices to show that
\[ (A : B) \subseteq (A : (\theta_1, B^2))(\theta_2, \ldots, \theta_n) + A. \]

By Gorenstein duality, it suffices to show
\[ (6.1.1) \quad (\theta_1, B^2) : (\theta_2, \ldots, \theta_n) \subseteq B. \]
The hypothesis that \( B \) is a complete intersection of grade at least two ensures that (6.1.1) holds and the proof is complete. \( \square \)

We turn our attention to the proof of Theorem 1.3. Some preliminary results are needed.

**Observation 6.2.** Let \( k \) be a field of arbitrary characteristic, \( P' \) be a standard-graded polynomial ring in three variables over \( k \), and \( B' \) be an ideal in \( P' \) which is generated by a 5-dimensional subspace of \([P']_2\). Assume that
\[ (6.2.1) \quad [\text{socle}(P'/B')]_1 = 0. \]
Then the ring \( P'/B' \) is Gorenstein and has Hilbert series \( HS(P'/B') = 1 + 5t + t^2 \); furthermore, the ideals \( (B')^2 \) and \( ([P']_1)^4 \) of \( P \) are equal.

**Proof.** Let \( \phi : [P']_2 \to k \) be a \( k \)-module homomorphism with \([B']_2 = \ker \phi \). If the characteristic of \( k \) is different than two, then use standard results about symmetric
bilinear forms over a field (essentially Gram-Schmidt orthogonalization) to choose
a basis \(x, y, z\) for \([P']_1\) so that the matrix
\[
(6.2.2) \quad T = \begin{bmatrix}
\phi(x^2) & \phi(xy) & \phi(xz) \\
\phi(xy) & \phi(y^2) & \phi(yz) \\
\phi(xz) & \phi(yz) & \phi(z^2)
\end{bmatrix}
\]
is a diagonal matrix. The non-degeneracy hypothesis (6.2.1) ensures that the entries
on the main diagonal of \(T\) are units in \(k\), say \(u_1, u_2, u_3\). The same conclusion holds
in characteristic two; but the argument is more subtle. The argument we offer was
taken from some expository notes written by Keith Conrad [7, Exercise 5.4].

Assume \(k\) has characteristic two. The non-degeneracy hypothesis (6.2.1) ensures
that there is an element \(z_1 \in [P']_1\) with \(z_1^2 \notin B'\). (Otherwise, if \(x_1, y_1, z_1\) is any basis
of \([P']_1\), then \(\phi(y_1z_1)x_1 + \phi(x_1z_1)y_1 + \phi(x_1y_1)z_1\) represents a non-zero element of
\(\text{socle}(P'/B')_1\).) If there is an element \(y_1\) with \(y_1z_1 \in B'\) and \(y_1^2 \notin B'\), then the usual
argument maybe used to complete the proof. On the other hand, if every element of \(B' : [P']_1\), \(z_1\) squares to an element of \(B'\), then let \(x_1, y_1\) be a basis of \(B' : [P']_1\), \(z_1\) with
\(\phi(x_1y_1) = 1\). Consider the basis
\[
x = \phi(z_1^2)x_1 + z_1, \quad y = (1 + \phi(z_1^2))x_1 + y_1 + z_1, \quad \text{and} \quad \z = x_1 + y_1 + z_1
\]
for \([P']_1\). Observe that the matrix (6.2.2) is a diagonal matrix. In this case also, we
take the entries on the main diagonal of \(T\) to be the units \(u_1, u_2, u_3\) of \(k\).

It is now clear that
\[
(6.2.3) \quad B' = (xy, xz, yz, u_2x^2 - u_1y^2, u_3y^2 - u_1z^2).
\]

It is also clear that the row vector of signed maximal order Pfaffians of the alternating
matrix
\[
(6.2.4) \quad b'_2 = \begin{bmatrix}
0 & u_3x & -u_1u_3y & -z & 0 \\
-u_3x & 0 & u_1u_3z & 0 & y \\
u_1u_3y & -u_1u_3z & 0 & u_3x & -u_2x \\
z & 0 & -u_3x & 0 & 0 \\
0 & -y & u_{3x} & 0 & 0
\end{bmatrix}
\]
is
\[
(6.2.5) \quad b'_1 = [u_3xy, u_2xz, yz, u_2u_3x^2 - u_1u_3y^2, u_2u_3y^2 - u_1u_3z^2].
\]
Observe that the image of \(b'_1\) is equal to \(B'\) and that \(B'\) has grade 3. It follows that
\[
(6.2.6) \quad 0 \rightarrow P'(-5) \xrightarrow{b'_1} P'(-3)^5 \xrightarrow{b'_2} (P')(-2)^5 \xrightarrow{b'_1} P'
\]
is a minimal homogeneous resolution of \(P'/B'\); hence, \(P'/B'\) is a Gorenstein ring
with Hilbert series \(1 + 5t + t^2\). One can check the assertion \(B'^2 = ([P']_1)^4\) by hand
or one can read the minimal homogeneous resolution
\[
0 \rightarrow P'(-6)^{10} \rightarrow P'(-5)^{24} \rightarrow P'(-4)^{15} \rightarrow P'
\]
of \(P'/B'^2\) from [11, page 36] in order to conclude that the socle degree of \(P'/B'^2\) is
three; thus, \(([P']_1)^4 \leq B'^2\). The inclusion \(B'^2 \subseteq ([P']_1)^4\) is obvious. \(\square\)
Lemma 6.3. Let $k$ be a field of arbitrary characteristic, $P = k[X_1, X_2, X_3, Y_1, \ldots, Y_s]$ be a standard-graded polynomial ring over $k$ in $3 + s$ variables for some nonnegative integer $s$, $B'$ be an ideal in $P$ which is generated by five linearly independent quadratic forms in the variables $X_1, X_2, X_3$. $B''$ be the ideal $(Y_1, \ldots, Y_s)$ of $P$, $B$ be the ideal $B' + B''$ of $P$, and $A \subseteq ([P]_1)^S$ be a homogeneous ideal of $P$ with $P/A$ an Artinian Gorenstein local ring. Assume $[\text{socle}(P/B)]_1 = 0$. Then there is a presentation

$$(6.3.1) \quad B_2 \xrightarrow{b_2} B_1 \xrightarrow{b_1} B \to 0$$

of $B$ by free $P$-modules, a $P$-module homomorphism $L : B_1 \to B_2$, and direct sum decompositions

$$B_1 = B'_1 \oplus B''_1, \quad B_2 = B'_2 \oplus B''_2, \quad b_1 = \begin{bmatrix} b'_1 & b''_1 \end{bmatrix}, \quad b_2 = \begin{bmatrix} b'_2 & b''_2 \\ 0 & b'_2 \end{bmatrix}, \quad \text{and} \quad L = \begin{bmatrix} L' & 0 \\ 0 & L'' \end{bmatrix},$$

which satisfy Assumption 5.3 and conditions (a), (b), (c), and (d) of Lemma 5.5.

Proof. Let $P'$ be the polynomial ring $k[X_1, X_2, X_3]$. View $P'$ as a subring of $P$. Notice that $P/B = P'/\langle P' \cap B' \rangle$; so we may apply Observation 6.2 and pick a basis $\{x, y, z\}$ of $[P']_1$ so that the generators of $\langle P' \cap B' \rangle$ are given in (6.2.3) and the minimal homogeneous resolution of $P'/\langle P' \cap B' \rangle$ by free $P'$-modules is given in (6.2.6) with $b'_1$ and $b'_2$ given in (6.2.5) and (6.2.4), respectively.

Apply the functor $P \otimes_{P'} -$ to (6.2.6) to obtain the minimal homogeneous resolution

$$\mathbb{B}' : \quad 0 \to B'_3 = P(-5) \xrightarrow{b'_1^T} B'_2 = P(-3)^5 \xrightarrow{b'_2} B'_1 = P(-2)^5 \xrightarrow{b'_1} B'_0 = P$$

of $P/B'$. (This establishes Hypothesis (d) of Lemma 5.5.) Let $B''_1$ be a free $P$-module of rank $s$, $b''_1 : B''_1 \to P$ the homomorphism

$$P(-1)^s \xrightarrow{[Y_1 \cdots Y_s]} P,$$

and $\wedge^* B''_1$ be the Koszul complex associated to $b''_1$. The minimal homogeneous resolution of $P/B$ is $\mathbb{B} \otimes_P \wedge^* B''_1$. In this language, the presentation (6.3.1) of $B$ by free $P$-modules is

$$B'_2 \otimes_P \wedge^0 B''_1 \oplus B'_1 \otimes \wedge^1 B''_1 \xrightarrow{b_2} B'_1 \otimes_P \wedge^0 B''_1 \oplus B'_0 \otimes \wedge^1 B''_1 \to B = B' + B'' \to 0,$$

where

$$b_2 = \begin{bmatrix} b'_2 & -1 \otimes b''_1 & 0 \\ 0 & 1 \otimes b''_1 & 1 \otimes b''_1 \end{bmatrix}.$$

To complete the promised decomposition of (6.3.1) let

$$(6.3.2) \quad B''_2 = B'_1 \otimes \wedge^1 B''_1 \oplus B'_0 \otimes \wedge^2 B''_1, \quad b''_2 = [-1 \otimes b''_1 \ 0] \quad \text{and} \quad b''_2 = [b'_1 \otimes 1 \ 1 \otimes b''_1].$$
The homogeneous ideals $A$ and $B$ satisfy $A \subseteq B$ and both ideals define Gorenstein rings. Gorenstein duality ensures the existence of a homogeneous element $\Delta$ of $P$ with

\[(A : p B) = (A, \Delta) \quad \text{and} \quad (A : p \Delta) = B.\]

Let $S$ denote $P/A$, and $-\hat{}$ denote $-\otimes_p S$. Let $\{x, y, z, Y_1, \ldots, Y_s\}$ denote the set of monomials in $P$ of degree $i$ in $x, y, z, Y_1, \ldots, Y_s$. The hypothesis $A \subseteq ([P]_1)^5$ guarantees that

$$\{m \mid m \in \{x, y, z, Y_1, \ldots, Y_s\}\}$$

is a basis for $[S]_i$ for $0 \leq i \leq 4$. Let $d$ denote the socle degree of $S$ and let $\alpha_1$ be an element of $[P]_d$ with the property that $\overline{\alpha_1}$ is a basis for the socle $[S]_d$ of $S$. For each integer $i$, with $0 \leq i \leq d$, the multiplication map

\[(6.3.4) \quad [S]_i \times [S]_{d-i} \to [S]_d\]

is a perfect pairing. For $1 \leq i \leq 3$, select $\{\alpha_m \in [P]_{d-i} \mid m \in \{x, y, z, Y_1, \ldots, Y_s\}\}$ so that

$$\{\overline{\alpha_m} \mid m \in \{x, y, z, Y_1, \ldots, Y_s\}\}$$

is a basis for $[S]_{d-i}$ which is dual to $\{m \mid m \in \{x, y, z, Y_1, \ldots, Y_s\}\}$ in the sense that if $m, m' \in \{x, y, z, Y_1, \ldots, Y_s\}$, then

$$\overline{m'} \overline{\alpha_m} = \begin{cases} \overline{\alpha_1}, & \text{if } m = m', \text{ and} \\ 0, & \text{if } m \neq m'. \end{cases}$$

It follows from (6.3.4) that if $m \in \{x, y, z, Y_1, \ldots, Y_s\}$ and $m' \in \{x, y, z, Y_1, \ldots, Y_s\}$ for some $i$ and $i'$ with $0 \leq i, i' \leq 3$, then

$$\overline{m'} \overline{\alpha_m} = \begin{cases} \overline{\alpha_{m''}}, & \text{if } m'm'' = m, \text{ and} \\ 0, & \text{if } m' \text{ does not divide } m. \end{cases}$$

The generators of $B$ are $Y_1, \ldots, Y_s$, together with the five quadratics given in (6.2.3). Use the perfect pairing (6.3.4), with $i = 2$, to observe that

$$[A : p B]_{d-2} = [A]_{d-2} + k(u_1 \alpha_{x^2} + u_2 \alpha_{y^2} + u_3 \alpha_{z^2});$$

hence, we may take the $\Delta$ of (6.3.3) to be

\[(6.3.5) \quad \Delta = u_1 \alpha_{x^2} + u_2 \alpha_{y^2} + u_3 \alpha_{z^2}.\]

Let $L'$ be the $5 \times 5$ matrix of Table 1 with entries in $P$. Recall the matrix $b'_2$ from (6.2.4). A straightforward calculation shows that $b'_2 L'$ and $L'b'_2$ are both congruent to $\Delta I_5$, modulo $A$. The conclusion $b'_2 L' \equiv \Delta I$ establishes half of Assumption 5.3. (The other half is studied at (6.3.7).) The conclusion $L'b'_2 \equiv \Delta I$ demonstrates that condition Lemma 5.5.(b) holds. Observe that

\[(6.3.6) \quad I_1(b''_2) \cdot S \subseteq B'' \cdot S \subseteq \text{ann}_S I_1(L');\]

and therefore condition Lemma 5.5.(c) holds. (The second inclusion is easily read from the definition of $L'$.) To prove condition Lemma 5.5.(a), it suffices to show
Let \( L' \) be the matrix
\[
\begin{bmatrix}
0 & \frac{-u_1}{u_3} \alpha_{x}^3 & 0 & u_2 \alpha_{x}^3 & 0 \\
0 & 0 & 0 & \frac{-u_1}{u_3} \alpha_{x}^3 & -u_1 \alpha_{x}^3 \\
\frac{-u_2}{u_1 u_3} \alpha_{y}^3 & \frac{1}{u_1} \alpha_{x}^3 & 0 & \frac{-u_1}{u_3} \alpha_{x}^3 & 0 \\
\frac{-u_1}{u_3} \alpha_{x}^3 & \frac{u_2}{u_3} \alpha_{y}^3 & \frac{u_2}{u_3} \alpha_{x y} & -u_1 u_2 \alpha_{x y} & -u_1 u_3 \alpha_{x y} \\
\frac{-u_1}{u_3} \alpha_{x}^3 & \frac{u_2}{u_3} \alpha_{y}^3 & \frac{u_2}{u_3} \alpha_{x x} & 0 & 0
\end{bmatrix}.
\]

**Table 1.** The matrix \( L' \) from the proof of Lemma 6.3.

that \( B^2 \cdot I_1(L') \cdot S = 0 \). On the other hand,
\[
B^2 \cdot I_1(L') \cdot S \subseteq (B^2 + B'')I_1(L') \cdot S = B^2 I_1(L') \cdot S \subseteq [S]_{4} \cdot [S]_{d-3} \cdot S \subseteq [S]_{d+1} \cdot S = 0.
\]

### 6.3.7. To complete the proof, we must establish the rest of Assumption 5.3; that is, we must exhibit a \( P \)-module homomorphism \( L'': B_1'' \rightarrow B_2'' \) with

\[
(3.6.8) \quad b_2 \begin{bmatrix} 0 \\ L'' \end{bmatrix} \equiv \begin{bmatrix} 0 \\ \Delta \cdot \text{id}_{B_1''} \end{bmatrix} \mod AB_1.
\]

The next calculation is the main step in that direction.

**Claim 6.3.9.** The element \( \Delta \) of \( P \) is in the ideal \( B' \cdot (A : p B'') + A \).

**Proof of Claim 6.3.9.** By Gorenstein duality, 2.6, it suffices to show that
\[
A : p (B' \cdot (A : p B'') + A) \subseteq A : p (A, \Delta).
\]

The module on the right is \( B \). The module on the left is
\[
(A : p (A : p B'')) : p B' = (A, B'') : p B'.
\]

To prove Claim 6.3.9 it suffices to show that

\[
(3.6.10) \quad (A, B'') : p B' \subseteq B.
\]

Let \( p \) be a homogeneous element in \( (A, B'') : p B' \). Write \( p = p' + p'' \) for homogeneous elements \( p' \) and \( p'' \), with \( p' \) in the subring \( k[x, y, z] \) of \( P \) and \( p'' \) in the ideal \( B'' \) of \( P \). Of course, \( p'' \) is in \( (A, B'') : p B' \) and \( p'' \in B \); consequently, \( p' \) is in \( (A, B'') : p B' \) and it suffices to prove that \( p' \in B \). We are told that \( p' B' \subseteq (A, B'') \). There are non-zero elements of
\[
[p' B' \cap k[x, y, z]_{\deg p'} + 2.
\]
On the other hand, the ideal \( A \) is contained in \( ([P]_1) \), by hypothesis, so
\[
([A, B'']) \cap k[x, y, z]_i = 0 \quad \text{for } i \leq 4.
\]
Therefore, \( 5 \leq \deg p' + 2 \); hence \( 3 \leq \deg p' \) and \( p' \in B' \subseteq B \); and this completes the proof of Claim 6.3.9. (Recall that the Hilbert series of \( P' / B' \) is given in Observation 6.2.)

Claim 6.3.9 guarantees that there is an element \( c \) in \( B' \) with
\begin{equation}
(6.3.11) \quad b'_1(c) \equiv \Delta \mod A \quad \text{and} \quad B''I_1(c) \subseteq A.
\end{equation}
Recall from (6.3.2) that \( B''_2 = (B'_1 \otimes B''_1) \oplus 2B''_1 \). Define \( L'' : B''_1 \rightarrow B''_2 \) by
\[
L''(\beta''_1) = c \otimes \beta''_1 \in B'_1 \otimes B''_1 \subseteq B''_1, \quad \text{for } \beta''_1 \in B''_1.
\]
Observe that
\[
b_2 \begin{bmatrix} 0 \\ L'' \end{bmatrix} (\beta''_1) = \begin{bmatrix} b'_2 & -1 \otimes b''_1 \\ 0 & b'_1 \otimes 1 & 1 \otimes b''_1 \end{bmatrix} \begin{bmatrix} 0 \\ c \otimes \beta''_1 \\ 0 \end{bmatrix} = \begin{bmatrix} -b''_1(\beta''_1) \cdot c \\ b'_1(c) \cdot \beta''_1 \\ 0 \end{bmatrix} \equiv \begin{bmatrix} 0 \\ 0 \end{bmatrix} \mod A.
\]

The final congruence is due to (6.3.11). This completes the proof of (6.3.8) and also the proof of Lemma 6.3.

**Remark 6.4.** We used Macaulay2, [9], to solve a system of 250 non-homogeneous linear equations in 300 unknowns in order to produce the matrix \( L' \) of Table 1. The matrix \( L' \) is \( 5 \times 5 \); so, it has 25 entries, and each entry is a linear combination of the ten elements \( \{ \alpha_m \mid m \in \binom{3}{3} \} \). Altogether, \( 25 \times 10 \) coefficients must be determined. Each entry of \( b'_2L' \) and each entry of \( L'b'_2 \) is a linear combination of the six elements \( \{ \alpha_m \mid m \in \binom{3}{2} \} \). Altogether \( (25 + 25) \times 6 \) equations must be solved.

It is amusing to notice that, according to the computer, the vector space of solutions of the corresponding homogeneous equations has dimension 86, over \( \mathbb{Q} \) and also over \( \mathbb{Z}/(2) \); so, there are many choices for \( L' \). The fact that we used the computer to identify one choice for \( L' \) is essentially irrelevant because it is easy to check by hand that the matrix of Table 1 does satisfy \( b'_2L' = L'b'_2 = \overline{\Delta}I \).

### 6.5. The proof of Theorem 1.3
Apply Lemma 6.3 to see that all of the hypotheses of Lemma 5.5 hold. Lemma 5.5 yields the result. \( \square \)

### 7. Test Modules

In Theorem 7.1 and Proposition 7.3 we prove that many modules are proj-test modules in the sense of Definition 2.10.1. The modules “\( B' / BB'' \)” of Theorem 1.3 are covered by Theorem 7.1 and Remark 7.2. The modules “\( B / B'' \)” of Theorem 1.2 are covered in Proposition 7.3 and Remark 7.4. We state and prove the main result in the paper as Theorem 7.5.

**Theorem 7.1.** Let \( (R, m, k) \) be a local Noetherian ring with \( 3 \leq v(R) \). If \( T \) is a finitely generated \( R \)-module with \( m^2T = 0 \), then \( T \) is a proj-test module for \( R \), in the sense of Definition 2.10.1.

**Proof.** Let \( x_1, \ldots, x_n \) be a minimal generating set for \( m \). The assumption about \( v(R) \) guarantees that the \( \binom{n+1}{2} \) monomials of degree 2 in the symbols \( x_1, \ldots, x_n \) represent a basis for the \( k \)-vector space \( m^2 / m^3 \); see 2.5(b).
Fix finitely generated $R$-modules $T$ and $M$ with $m^2 T = 0$ and $\text{Tor}_i^R(T, M) = 0$. Let

$$\mathcal{F} : \ldots \xrightarrow{d_2} R^{b_1} \xrightarrow{d_1} R^{b_0}$$

be the minimal resolution of $M$. The matrix $d_i$ is well-defined for all non-negative integers $i$; if necessary, $d_i$ is allowed to be the zero matrix. Furthermore, each entry of each $d_i$ is in $m$. Write

$$d_i = \sum_{j=1}^n D_{i,j} x_j,$$

where each $D_{i,j}$ is a $b_{i-1} \times b_i$ matrix with entries in $R$. Use $d_id_{i+1} = 0$ and the hypothesis $3 \leq v(R)$ to see that

(7.1.1) $D_{i,j} D_{i+1,j} \equiv 0 \mod m$, for all $j$, and

(7.1.2) $D_{i,j_1} D_{i+1,j_2} + D_{i,j_2} D_{i+1,j_1} \equiv 0 \mod m$, for all $j_1 \neq j_2$.

The hypotheses $m^2 T = 0$ and $\text{Tor}_i^R(T, M) = 0$ guarantee that

$$(mT) \otimes R^{b_i} \subseteq \ker(T \otimes d_i) = \im(T \otimes d_{i+1})$$

Fix a positive index $i$. If $u$ is an arbitrary vector in $(mT)^{b_i}$, then

(7.1.3) $u = \sum_{j_0=1}^n D_{i+1,j_0} x_{j_0} u_{j_0},$

for some $u_1, \ldots, u_n \in T^{b_{i+1}}$. Each $x_{j_0} u_{j_0}$ is in $(mT)^{b_{i+1}}$; therefore

$$x_{j_0} u_{j_0} \in \im(T \otimes d_{i+2}) \text{ and}$$

(7.1.4) $x_{j_0} u_{j_0} = \sum_{j_1=1}^n D_{i+2,j_1} x_{j_1} u_{j_1,j_0},$

for some $u_{j_1,j_0} \in T^{b_{i+2}}$. Combine (7.1.3) and (7.1.4) to obtain

$$u = \sum_{j_0=1}^n \sum_{j_1=1}^n D_{i+1,j_0} D_{i+2,j_1} x_{j_1} u_{j_1,j_0},$$

In its turn, $x_{j_1} u_{j_1,j_0} \in (mT)^{b_{i+2}} \subseteq \im(T \otimes d_{i+3})$. When the above procedure is iterated $k$ times, one obtains

(7.1.5) $u = \sum_{j_0=1}^n \sum_{j_1=1}^n \cdots \sum_{j_k=1}^n D_{i+1,j_0} D_{i+2,j_1} \cdots D_{i+k+1,j_k} x_{j_k} u_{j_k,\ldots,j_1,j_0},$

for some $u_{j_k,\ldots,j_1,j_0} \in T^{b_{i+k+1}}$.

Observe that if $n < k$, then

(7.1.6) $D_{i+1,j_0} D_{i+2,j_1} \cdots D_{i+k+1,j_k} x_{j_k} \equiv 0 \mod m$

for each such product which appears in (7.1.5). Indeed there must be two $j$'s which are the same, say $j_{i_1} = j_{i_2}$ for some $i_1$ and $i_2$ with $0 \leq i_1 < i_2 \leq k$. If $i_2 = i_1 + 1$, then equation (7.1.1) shows that (7.1.6) holds. If not, use equation (7.1.2) to bring the equal $j$'s closer together.
It now follows that \((mT)^{b_i} = 0\), and therefore \(mT = 0\). In other words, \(T\) is a vector space \(\bigoplus k\). The hypothesis that \(\text{Tor}_1(T, M)\) is zero now implies that \(\text{Tor}_1(k, M) = 0\); and therefore, \(M\) is a free \(R\)-module. \(\square\)

**Remark 7.2.** Theorem 7.1 may be applied to the module \(B'/BB'\) of Theorem 1.3. Indeed, \(B'/BB'\) is annihilated by \((X_1, X_2, X_3)^2\) because

\[(X_1, X_2, X_3)^2B' \subseteq (X_1, X_2, X_3)^2(X_1, X_2, X_3)^2 = (X_1, X_2, X_3)^4 = B'^2 \subseteq BB'.\]

The right most equality is established in Observation 6.2.

**Proposition 7.3.** Let \(P\) be a regular ring, \(M\) a maximal ideal of \(P\), \(I \subseteq M\) be an ideal of \(P\) which is generated by a regular sequence, and \(R = P/J\), where \(J\) is an \(M\)-primary ideal of \(P\) and \(v(R)\) is sufficiently large. Then \(T = R/1R\) is a proj-test module for \(R\) in the sense of Definition 2.10.1.

**Proof.** It does no harm to localize \(P\) at \(M\) and to inflate the residue field of \(P_M\) in order to assume that the inflated residue field is infinite; see, for example [10, Ch. 0, 10.3.1]. We alter the notation to make it less cumbersome. In the new notation, \((P, M, k)\) is a regular local ring with \(k\) infinite, \(I\) is an ideal of \(P\) which is generated by a regular sequence, and \(R\) is an Artinian quotient of \(P\) with \(v(R)\) sufficiently large.

The Artin-Rees Lemma guarantees the existence of an integer \(n_0\) such that

\[I \cap M^{n_0} \subseteq IM.\]

In this proof, we insist that \(n_0 < v(R)\).

Let \(M\) be a finitely generated \(R\)-module with \(\text{Tor}_i^R(M, T) = 0\) for all positive \(i\). The ring \(R\) and the \(R\)-module \(M\) both have depth zero; consequently, according to the Auslander-Buchsbaum formula, to show that \(M\) is a free \(R\)-module, it suffices to show that \(M\) has finite projective dimension as an \(R\)-module. We assume that \(M\) has infinite projective dimension and we draw a contradiction.

Let

\[F : \ldots \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0\]

be a collection of \(P\)-module homomorphisms of free \(P\)-modules with the property that \(F \otimes_P R\) is the minimal \(R\)-resolution of \(M\). The hypothesis about the vanishing of \(\text{Tor}_i^R(M, T)\) guarantees that \((F \otimes_P R) \otimes_R T = F \otimes_P T\) is the minimal resolution of \(M \otimes_R T\) by free \(T\)-modules.

The ring \(T\) is a complete intersection, the ring \(P\) is regular local, and the residue field \(k\) of \(P\) is infinite; hence, the Eisenbud-Peeva [8] theorems apply to \(F \otimes_P T\). In particular, if \(i\) is sufficiently large, then the syzygy module \(\text{im}(d_i \otimes_P T)\) in \(F \otimes_P T\) is a Higher Matrix Factorization module [8, Theorem 1.3.1]; and therefore, when suitable bases are chosen in \(F\), some of the entries of the product matrix \(d_{i+1}d_{i+2}\) are minimal generators of \(I\). The hypothesis that \(n_0 < v(R)\) guarantees that every minimal generator of \(I\) represents a non-zero element of \(R\). On the other hand, \(F \otimes_P R\) is a complex and \(d_{i+1}d_{i+2} \otimes_P R = 0\). We have reached a contradiction

\[0 \neq d_{i+1}d_{i+2} \otimes_P R = 0;\]

and this completes the proof. \(\square\)
Remark 7.4. Proposition 7.3 may be applied to the module $B/B^2$ of Theorem 1.2. Indeed, the fact that the ideal $B$ of $P$ is generated by a regular sequence ensures that $B/B^2$ is a free $P/B$ module of rank $c$, where $c$ is the minimal number of generators of $B$; hence, as an $R$-module,

$$\frac{B}{B^2} = \frac{B}{B^2} \otimes_R P = \left(\frac{P}{B}\right)^c \otimes_R P = \left(\frac{R}{BR}\right)^c.$$ 

Theorem 7.5 is the main result of the paper. Recall that if $R = S/J$ is a quotient of the ring $S$, then $c_S(R)$ is the length of $J$ as an $S$-module. Also, recall the parameter $v(S)$ from 2.5.(b).

Theorem 7.5. Let $(S, n, k)$ be an equicharacteristic Artinian Gorenstein local ring with embedding dimension at least two, and $R$ be the ring $R = S/J$ for some proper non-zero ideal $J$ of $S$. Assume that either

- $1 \leq c_S(R) \leq 4$, or
- $c_S(R) = 5$ and $S$ is a standard-graded algebra over a field $[S]_0$.

If the parameter $v(S)$ is sufficiently large, then $R$ is $G$-regular.

Proof. Apply Theorem 3.1 to see that one of the Theorems 1.1, 1.2, or 1.3 applies to $R$. These theorems ensure that either $k$, “$B/B^{2n}$”, or “$B'/BB'$” is a direct summand of $\text{syz}_2^R(\Omega_R)$. The module $k$ is the prototype of a proj-test module. It is shown in Theorem 7.1 (together with Remark 7.2) and Proposition 7.3 (together with Remark 7.4) that “$B/B^{2n}$” and “$B'/BB'$” are proj-test modules. Apply Observation 2.10.3 to complete the proof. □

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ANDREW R. KUSTIN, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTH CAROLINA, COLUMBIA, SC 29208, U.S.A.
E-mail address: kustin@math.sc.edu

ADELA VRACIU, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTH CAROLINA, COLUMBIA, SC 29208, U.S.A.
E-mail address: vraciu@math.sc.edu