Illiquidity and Insolvency: a Double Cascade Model of Financial Crises

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Abstract

In the aftermath of the interbank market collapse of 2007-08, the scope of systemic risk research has broadened to encompass a wide range of channels, notably asset correlations, default contagion, illiquidity contagion, and asset firesales. In current models of systemic risk, two facets of contagion, namely funding illiquidity and insolvency, are treated as two distinct and separate phenomena. The main goal of the double cascade model we introduce is to integrate these two facets as two faces of the same coin. In a default cascade, insolvency of a given bank will create a shock to the asset side of the balance sheet of each of its creditor banks. Under some circumstances, such “downstream” shocks can cause further insolvencies that may build up to create a global insolvency cascade. On the other hand, in a stress cascade, illiquidity that hits a given bank will create a shock to the liability side of the balance sheet of each of its debtor banks. Under some circumstances, such “upstream” shocks can cause further illiquidity stresses that may build up to create a global illiquidity cascade.

Our paper will introduce a deliberately simplified network model of insolvency and illiquidity that can quantify how illiquidity or default of one bank influences the overall level of liquidity stress and default in the network. Under an assumption we call “locally tree-like independence”, we derive large-network asymptotic cascade formulas. Results of numerical experiments then demonstrate that these asymptotic formulas agree qualitatively with Monte Carlo results for large finite networks, and quantitatively except when the system is placed in an exceptional “knife-edge” configuration. These experiments illustrate clearly our main conclusion that in financial
networks, the average default probability is inversely related to strength of banks’ stress response and therefore to the overall level of stress in the network.

**Key words:** Systemic risk, banking network, contagion, random graph, default, funding liquidity, credit risk, financial mathematics

**AMS Subject Classification:** 05C80, 91B30, 91B70, 91G40

### 1 Introduction

Since the banking crisis of 2007-08, the financial systemic risks transmitted through interbank exposures are now seen to involve not only cascades of defaulting banks but cascading of illiquid banks as well. A well-developed strand of literature on default cascade models, starting with Eisenberg and Noe [2001] and reviewed in Upper [2011], is based on the picture that insolvency of a given bank, defined as a bank whose net worth becomes non-positive, will generate shocks to the asset side of the balance sheet of each of its creditor banks. Under some circumstances, such “downstream” shocks can cause further insolvencies that may build up to create a global insolvency cascade. More recently, after remarking on the observed “freezing” of interbank lending around the time of the Lehman collapse, papers by Gai and Kapadia [2010b], Gai et al. [2011] and Lee [2013] adopt variations of an idea that funding illiquidity of a bank\(^1\) can also be transmitted contagiously through interbank exposures. They argue that an illiquid “stressed” bank whose liquid assets are insufficient to cover demands on its liabilities is likely to reduce its interbank lending, thereby creating shocks to the liability side of its debtor banks’ balance sheets. Under some circumstances, such “upstream” shocks can cause further illiquidity stress that may build up to create a global illiquidity cascade.

The purpose of the present paper is to construct a deliberately simplified cascade model of systemic risk, integrating both sides, illiquidity and insolvency, of a single coin. Only in a double cascade model such as ours can one frame the important question: What effect does a bank’s behavioural response to liquidity stress have on the probable level of eventual defaults in entire system? One would expect that a bank that reacts to stress by shrinking its own interbank lending, thereby inflicting liquidity shocks to its debtor banks, will at the same time protect itself from eventual default. We will measure the average strength of this stress reaction by a parameter \(\lambda\), and will find that in general, the overall level of default in the system is **negatively** related to \(\lambda\), and hence to the overall level of stress in the system. To arrive at this conclusion, we will address a number of issues. First, we will adopt a stylized model of an individual bank balance sheet similar to that in Gai and Kapadia [2010b] and Lee [2013]. In this balance sheet model, we identify the bank equity or net worth

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\(^1\)Funding illiquidity, being the insufficiency of liquid assets to cover a run on liabilities, is distinct from market illiquidity, where assets become difficult to sell due to an oversupply in the market. See Brunnermeier and Pedersen [2009] for a detailed analysis of these concepts.
as a default buffer and the liquid assets as a stress buffer. Secondly, three possible states of a bank, namely the normal state, the stressed state and the insolvent state, are defined by positivity conditions on the bank’s buffers. Thirdly, we will model the behavioural response of a bank when it finds itself in the stressed or insolvent states. Importantly, a stressed bank will recall a fraction of its interbank assets to raise additional liquidity and to reduce its exposure to defaulting counterparties. Such a reaction also forces its debtor banks to raise cash, and if forced to raise too much, these debtor banks will themselves become stressed. Therefore it is intuitively natural that the overall level of stress in the network goes up with the stress parameter $\lambda$ and down with the overall level of default.

The present paper continues a recent strand of literature, extending the economic interbank framework proposed by Eisenberg and Noe [2001], that focuses on deliberately simplified models of random financial networks. These papers aim to determine the key parameters that most impact the level of systemic risk in synthetic networks. They share an outlook in common with the wider network science literature, reviewed in the books Newman [2010], Jackson [2008], that understanding of complex phenomena can proceed by the numerical and analytic study of simple “toy models” chosen carefully to exhibit key features. Learning the true impact of various possible mechanisms in such toy models will provide a valuable guide to understanding more realistic models that try to represent the intricacies of observed financial systems. Nier et al. [2007] uses Monte Carlo methods to highlight how the key network parameters for a stylized random network of 25 banks can influence the total number of defaults in a nonlinear, indeed sometimes nonmonotonic, fashion. The paper of Gai and Kapadia [2010a], and its extension Hurd and Gleeson [2011], adapt the Watts [2002] network model of information cascades to the context of financial systems, deriving both analytical and Monte Carlo results showing the dependence of the default cascade on different structural parameters. May and Arinaminpathy [2010] present analytical formulas for the NYYA and GK models based on a mean field approximation that can explain some of the main properties of the graphs found in those papers. Gai and Kapadia [2010a], Gai et al. [2011] and Lee [2013] provide stylized models of liquidity shocks that are mathematically similar to the default model of Gai and Kapadia [2010a], but with cascades that flow in the reverse direction from creditor to debtor. Amini et al. [2012] develop a simple but general analytical criterion for resilience to default contagion in random financial networks, based on an asymptotic analysis of default cascades in heterogeneous networks.

In order to focus on pure contagion effects, these toy models explicitly rule out a large range of complex mechanisms that have been explored in the economics and finance literature. Diamond and Rajan [2005] investigate the role of bankers in intermediating between lenders and borrowers to explain their liquidity, solvency and behaviour. The impact of financial cascades on the non-financial economy, and the consequent feedback into the financial markets through “firesales” of assets have been extensively studied, for example in Cifuentes et al. [2005] and Adrian and Shin [2010], and it is known that these effects will amplify any cascade after it takes hold in the
network. Battiston et al. [2012] model the robustness dynamics of banks linked by their interbank exposures, highlighting the feedback effects of a “financial accelerator” mechanism on the level of systemic risk. Roukny et al. [2013] further develop this idea, investigating the systemic impact of network topologies and different levels of market illiquidity and applying this methodology to the Italian interbank money market from 1999 to 2011. As an example of a recent investigation into real world networks, we mention Kok and Montagna [2013] who develop an agent-based multi-layered interbank network model based on a sample of large EU banks.

The key technical innovation of the present paper is to introduce a model framework that will allow a mathematical analysis of the intertwining of stress and default cascades, each acting in opposing directions. The present paper provides a full mathematical treatment of a flexible family of synthetic networks, with arbitrary degree and edge distributions, random stress and default buffers and random interbank exposures, using techniques developed in Hurd and Gleeson [2013] to extend the “mean field” theory widely applied in network science. The main result of the paper, Theorem 1, provides an exact asymptotic analysis of default and stress probabilities at each step of the double cascade. The accuracy of these asymptotic approximations for a finite number $N$ of banks is not determined by our analysis, which means the usefulness of these results for finite networks requires extensive testing by Monte Carlo simulation.

Section 2 of the paper provides the network framework and assumptions underlying the balance sheet structure of banks, the timing of the crisis and bank behaviour. These assumptions lead to rules for the transmission of shocks through the double stress/default cascade, including the conditions for banks to become stressed or defaulted, that are provided in Section 3. Section 4 develops our main theorem, which yields an explicit asymptotic analysis of default and stress probabilities in large heterogeneous networks with random connectivity, balance sheets and interbank exposures. A key technical assumption needed to prove this result is a condition we call the locally tree-like independence (LTI) property. Section 5 provides a parallel development of default and stress probabilities for cascades on finite “real-world” networks, where it is assumed that the graph of interbank connections is known explicitly, but balance sheets and exposure sizes are still random. Several representative financial experiments are reported in Section 6. First we summarize experiments that verify the main theorem by direct comparison of large $N$ analytics to Monte Carlo simulation results. Secondly, we investigate the relationship between the stress response parameter and the level of stress and default, verifying our assertion that average stress probability increases and average default probability decreases as $\lambda$ increases. A final experiment demonstrates the usefulness of our analysis in a highly heterogeneous specification of the model that is consistent with known heuristics of financial networks, such as fat-tailed degree and exposure distributions, and the latest stress testing data on 90 large banks in the EU system. We observed that the network in this specification is very resilient, and only an extremely large shock to the average default buffer size will trigger a cascade of defaulted and stressed banks. The first
main conclusion of the paper, discussed in Section 7, is that the analytic asymptotic results on default and stress probabilities that stem from the main theorem, when used carefully, are consistent with results from Monte Carlo simulations on finite random graphs. A second conclusion is that stress and default are inversely related: as banks respond to stress more vigorously, creating more network stress, they protect the network from default.

2 Cascade Assumptions

Our assumptions are designed to lead to a simple network model admitting double cascades of illiquidity and insolvency that can be computed efficiently under a wide range of initial conditions, and can be justified by financial considerations. Many realistic effects are therefore left out. Banks respond mechanistically according to simple stylized rules rather than optimizing their behaviour. In common with many cascade models in the literature, we focus on contagion alone by ruling out non-contagion channels of systemic risk such as asset firesales, and ignore external cash flows, interest payments and price changes for the duration of the cascade. We make an assumption of zero-recovery on interbank assets on the default of a counterparty. The inclusion of these excluded realistic effects should be considered in future developments.

The network consists of a collection of $N$ banks, each structured in a similar manner and labelled by numbers from the set $\mathcal{N} = \{1, 2, \ldots, N\}$. The subset of debtor banks of a bank $v \in \mathcal{N}$ is called the in-neighbourhood of $v$, denoted by $\mathcal{N}_v^-$, and the in-degree is the number of debtors $j_v = |\mathcal{N}_v^-|$. Similarly the bank’s creditor banks form a subset called the out-neighbourhood $\mathcal{N}_v^+$ of $v$ whose size $k_v = |\mathcal{N}_v^+|$ is called the out-degree. The information of interbank counterparties is identifiable as a directed graph, i.e. a set of nodes or vertices with a collection of directed arrows, called edges, between pairs of nodes. Each debtor-creditor pair $v, w$ with $w \in \mathcal{N}_v^+$ is denoted by an arrow or edge $\ell = (vw)$ pointing from $v$ to $w$. The type $(j_v, k_v)$ of a bank $v$ is its in-degree $j_v$ and out-degree $k_v$. The type $(k_\ell, j_\ell)$ of an edge $\ell = (vw)$ is the out-degree $k_\ell = k_v$ of the debtor bank $v$ and the in-degree $j_\ell = j_w$ of the creditor bank $w$.

**Balance Sheet Assumptions:** The asset and liability classes of a stylized bank have the following characteristics:

1. Liquid assets of total value $A^L$ (held, for example, in the form of cash and risk-free government bonds); the loan book, composed of illiquid loans to the exterior of the network, with total value $A^F$; the unsecured loans to its debtor banks with total value $A^{IB}$.

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\(^2\)The convention that arrows point from debtors to creditors means that default shocks propagate in the downstream direction. Confusingly, much of the systemic risk literature uses the reverse convention that arrows point from creditors to debtors.
2. The liabilities consist of bank deposits with a total value $L^D$, and unsecured interbank loans with total value $L^{IB}$.

3. The bank equity or net worth has value $\Delta = \max(A^F + A^{IB} + A^L - L^D - L^{IB}, 0)$ that is held only by investors external to the network.

4. By limited liability, no bank may have negative equity.

5. At the onset of the crisis, interbank entries are valued as if all bank counterparties are solvent.

When the equity $\Delta_v$ of bank $v$ is zero, the assumption of limited liability is taken to mean that the bank must default, forcing its liquidation. Thus $\Delta_v$ can be interpreted as the bank’s default buffer that must always be kept positive to avoid default.

Banks are also concerned about the possibility of runs on their liabilities, and try to keep a positive buffer of liquid assets $A^L$, which we will henceforth call the stress buffer $\Sigma$, from which to pay creditors’ demands. A bank $v$ with $\Sigma_v$ fully depleted to zero will be called stressed or illiquid. This does not mean the bank is unable to meet the creditors’ demands (which under our assumptions would imply that the bank defaults). Rather it means that the bank is experiencing a significant degree of stress in meeting these demands and must turn to other assets, specifically interbank assets, to realize the needed cash.

The interbank liabilities $L^{IB}$ and assets $A^{IB}$ decompose into bilateral interbank exposures. For any bank $v$ and one of its creditors $w \in N_v^+$, we denote by $\Omega_{vw}$ the total exposure of $w$ to $v$. Then we have the constraints

$$A_v^{IB} = \sum_{w \in N_v^-} \Omega_{vw}; \quad L_v^{IB} = \sum_{w \in N_v^+} \Omega_{vw}.$$

Figure 1 illustrates the balance sheet of a bank $v \in N$.

Prior to the onset of the crisis, all banks are assumed to be in the normal state. Then, on day $n = 0$, a collection of banks, possibly all banks, are assumed to experience initial shocks. Two kinds of initial shocks are possible. First, an asset shock causes a drop in the mark-to-market value of the fixed asset portfolio, and reduces the default buffer. If the downward asset shock leads to $\Delta \leq 0$, the bank must default. The second kind of shock is a demand shock by the external depositors that reduces the stress buffer. If the demand shock leads to $\Sigma \leq 0$, the bank becomes stressed.

**Crisis Timing Assumptions:**

Prior to the crisis, all banks are in the normal state, neither stressed nor insolvent.

1. The crisis commences on day 0 after initial shocks trigger the default or stress of one or more banks;

2. Balance sheets are recomputed daily on a mark-to-market basis;

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3Treating cascade steps as daily is simply an aid to understanding. An acceptable alternative interpretation would be to treat the entire cascade as taking place instantaneously.
3. Banks respond daily on the basis of their newly computed balance sheets;
4. All external cash flows, interest payments, and asset and liability price changes are ignored throughout the crisis.

The daily response of banks is also governed by simple rules.

**Bank Behaviour Assumptions:** On each day of the crisis:
1. An insolvent bank, characterized by $\Delta = 0$, is forced by the regulator into receivership. At this moment, each of its creditor banks are obliged to write down their defaulted exposures to zero thereby experiencing a *solvency shock*.
2. Solvency shocks reduce a bank’s default buffer $\Delta$.
3. A stressed bank, defined to be a non-defaulted bank with $\Sigma = 0$, reacts at the moment it becomes stressed by reducing its interbank assets $A^{IB}$ to $(1 - \lambda)A^{IB}$ where $\lambda$ taken to be a fixed constant across all banks during the crisis. It does so by terminating a fraction $\lambda$ of its interbank loans, thereby transmitting a *stress shock* to the liabilities each of its debtor banks.
4. A newly defaulted bank also triggers maximal stress shocks (i.e. with $\lambda = 1$) to each of its debtor banks as its bankruptcy trustees recall all its interbank loans, reducing $A^{IB}$ to 0;
5. Stress shocks reduce any bank’s stress buffer $\Sigma$.
6. Stressed banks remain stressed until the end of the cascade or until they default.
Remark 1. The stress parameter $\lambda$ was introduced by Gai et al. [2011] to simplify banks’ response to liquidity stress. Unlike default, a bank is free to set its response to liquidity: both $\lambda$ and the size of the stress buffer $\Sigma$ are its own policy decisions. In a more realistic and complex model, $\lambda$ would be endogenously determined for each bank, reflecting the cumulative demands on its liabilities.

One consequence of these assumptions is that fixed assets $A^F$ and external deposits $L^D$ remain constant after the onset of the crisis. The dynamics of the cascade is completely determined by a reduced set of balance sheet data that consists of the collection of buffers $\Delta, \Sigma$ and interbank exposures $\Omega$.

In this paper, we shall apply the cascade rules to a financial network in a stochastic initial state. Such a stochastic initial state is described by the following random elements: the interbank links which form a directed random graph $\mathcal{E}$ on the set of banks $v \in \mathcal{N}$; the random buffers $\Delta_v, \Sigma_v$ for nodes $v \in \mathcal{N}$; and the random exposures $\Omega_{\ell v}$ for edges $\ell = (vw) \in \mathcal{E}$. We will be specific about the distributional properties of the random variables $(\mathcal{N}, \mathcal{E}, \Delta, \Sigma, \Omega)$ in subsequent sections.

3 Stochastic Double Cascade Dynamics

We now specify the precise dynamics of the double cascade on a random financial network that on day $n = 0$ of the crisis is characterized by the reduced set of data consisting of a random directed graph $\mathcal{E} \subset \mathcal{N} \times \mathcal{N}$, random buffers $\Delta_v, \Sigma_v$ for each bank (node) $v \in \mathcal{N}$ and random exposure sizes $\Omega_{\ell}$ for each edge $\ell \in \mathcal{E}$.

The set $\mathcal{D}_n$ contains all the defaulted banks after $n$ cascade steps, the set $\mathcal{S}_n$ comprises the undefaulted banks that are under stress after $n$ steps, and $\mathcal{D}_n^c \cap \mathcal{S}_n^c$ contains the remaining undefaulted, unstressed banks. In our model, banks do not recover from either default or stress during the crisis, so the sequences $\{\mathcal{D}_n\}_{n \in \mathbb{N}}$ and $\{\mathcal{D}_n \cup \mathcal{S}_n\}_{n \in \mathbb{N}}$ are non-decreasing. We use the notation that an event defined by some condition $P$ is written $\{P\}$, for example $\{v \in \mathcal{D}_n\}$, and the indicator random variable for that event is written $\mathbbm{1}_{\{P\}}$.

To say that a bank $v$ is defaulted at step $n$ means that default shocks to step $n - 1$ exceed its default buffer:

$$\mathcal{D}_n := \begin{cases} \{\Delta_v = 0\} & \text{for } n = 0, \\ \{v \mid \sum_{w \in \mathcal{N}_v^-} \Omega_{wv} \xi_{wv}^{(n-1)} \geq \Delta_v\} & \text{for } n \geq 1, \end{cases}$$

(1)

where the random variables $\xi$ indicate the fractional sizes of the various default shocks impacting $v$. Similarly, to say that a bank $v$ is stressed at step $n$ means both that it is not yet defaulted and the stress shocks to step $n - 1$ exceed the stress buffer, i.e.

\footnote{For any set $\mathcal{B}$, $\mathcal{B}^c$ denotes its complement.}
\[ S_n = D_n^c \cap \tilde{S}_n \] where

\[
\tilde{S}_n := \begin{cases} 
\{ \Sigma_v = 0 \} & \text{for } n = 0 \, , \\
\{ v \mid \sum_{w \in \mathcal{N}_v^+} \Omega_{vw} \zeta_{vw}^{(n-1)} \geq \Sigma_v \} & \text{for } n \geq 1 \, , 
\end{cases}
\] (2)

where the random variables \( \zeta \) indicate the fractional sizes of the stress shocks impacting \( v \).

Accounting for the fact that when \( v \) becomes stressed it reduces its interbank exposures, one can see that for \( n \geq 1 \) the fractions \( \xi^{(n)} \) are given recursively by

\[
\xi^{(n)}_{vw} := \begin{cases} 
\xi^{(n-1)}_{vw} & \text{when } w \in D_{n-1} \cup D_n^c \\
1 & \text{when } w \in D_n \setminus D_{n-1} \text{ and } v \in \tilde{S}_{n-1}^c \\
1 - \lambda & \text{when } w \in D_n \setminus D_{n-1} \text{ and } v \in \tilde{S}_{n-1} 
\end{cases}
\] (3)

with the initial values \( \xi^{(0)}_{vw} = \mathbb{1}_{\{w \in D_0\}} \). Similarly, accounting for the assumption that defaulted creditors have a maximal impact on the bank’s stress buffer, whereas stressed creditors only shock its stress buffer by a portion \( \lambda \) of the interbank exposure, one has for \( n \geq 0 \)

\[
\zeta^{(n)}_{vw} := \begin{cases} 
0 & \text{when } w \in \tilde{S}_n^c \cap D_n^c \\
\lambda & \text{when } w \in \tilde{S}_n \cap D_n^c \\
1 & \text{when } w \in D_n
\end{cases}
\] (4)

The most serious difficulty we will overcome in the subsequent probabilistic analysis stems from the intertwining of upstream and downstream shocks when banks default. Similar to a situation discussed in Hurd and Gleeson [2013], the separation of upstream and downstream shocks can eventually be ensured by considering properties that hold “without regarding” (WOR) a fixed bank. To this end, we will say “bank \( v \) is defaulted at step \( n \), without regarding a bank \( w \in \mathcal{N}_v^- \)”, and write \( \{ v \in D_n \not\rightarrow w \} \), if the default condition is true without including an in-link from \( w \) to \( v \). That is, for any \( w \in \mathcal{N} \)

\[
D_n \not\rightarrow w := \begin{cases} 
\{ \Delta_v = 0 \} \cap \mathcal{N}_w^+ & \text{for } n = 0 \, , \\
\{ v \in \mathcal{N}_w^+ \mid \sum_{w' \in \mathcal{N}_v^- \setminus w} \Omega_{w'v} \zeta_{w'v}^{(n-1)} \geq \Delta_v \} & \text{for } n \geq 1 \, .
\end{cases}
\] (5)

Using this definition, we can equivalently express the stress sets as \( S_n = D_n^c \cap \tilde{S}_n \) where

\[
\tilde{S}_n := \begin{cases} 
\{ \Sigma_v = 0 \} & \text{for } n = 0 \, , \\
\{ v \mid \sum_{w \in \mathcal{N}_v^+} \Omega_{vw} \tilde{\zeta}_{vw}^{(n-1)} \geq \Sigma_v \} & \text{for } n \geq 1 \, .
\end{cases}
\] (6)
with
\[ \xi_{wv}^{(n-1)} := \begin{cases} 0 & \text{when } w \in \hat{S}_{n-1} \cap \mathcal{D}_{n-1} \not\in v, \\ \lambda & \text{when } w \in \hat{S}_{n-1} \cap \mathcal{D}_{n-1} \not\in v, \\ 1 & \text{when } w \in \mathcal{D}_{n-1} \not\in v. \end{cases} \] (7)

The reason is that for \( v \in \mathcal{D}_{n} \), the \( \not\in v \) condition is redundant and thus \( \mathcal{D}_{n} \cap \hat{S}_{m} = \mathcal{D}_{n} \cap \hat{S}_{m} \) for any \( m \leq n \).

We need to check that the definition (3) of \( \xi \) remains essentially unaffected if \( \hat{S} \) is replaced by \( \hat{S} \) on the right hand side, i.e. that \( \xi \) can change only in a way that leaves the default set unchanged. This is true because the sets \( \hat{S}_{m} \cap \mathcal{D}_{m} \) and \( \hat{S}_{m} \cap \mathcal{D}_{m} \) are equal, while the sets \( \hat{S}_{m} \cap \mathcal{D}_{m} \) cannot be larger than the sets \( \hat{S}_{m} \cap \mathcal{D}_{m} \). Thus the fraction \( \xi_{wv} \) won’t change if \( v \) hasn’t yet defaulted, and may change but cannot decrease if \( v \) has already defaulted.

The next two sections are devoted to the probabilistic analysis of the double cascade in two distinct settings: the case of infinite networks and the case of finite real-world networks.

4 Large Networks

Our model of a random financial network has three layers of structure: the skeleton graph (a random directed graph \((N, E)\)); the buffer random variables \( \Delta_v, \Sigma_v, v \in N \) and the exposure weights \( \Omega_{\ell}, \ell \in E \).

The skeleton graph \((N, E)\) is a random assortative directed configuration graph as characterized by a construction developed in Hurd and Gleeson [2011] that extends the well-known configuration random graph model of Bollobás [1980]. Assortativity is the property that the joint (in, out) edge degree distribution, defined below by a matrix \( Q \), has a positive Pearson correlation. We extend the graph specification to allow assortativity because several studies of real financial networks, notably Bech and Atalay [2010], highlight the fact and relevance of their observed disassortative nature.

Definition 1. The infinite directed configuration graph model is the limit of any infinite sequence \((\Omega_N, F_N, P)\) of random graphs of expected size \(N\), and asymptotically consistent with compatible probability laws \(P, Q\) for the degree types of nodes and edges.

More specifically:

1. For degrees \( j, k \), \( P_{jk} := \mathbb{P}[N_{jk}] \) is the asymptotic probability of a random node having type \((j, k)\). This distribution has marginals \( P^+_k := \sum_j P_{jk}, P^-_j := \sum_k P_{jk} \) and mean in and out degree \( z = \sum_j jP^-_j = \sum_k kP^+_k \).

2. For degrees \( j, k \), \( Q_{kj} := \mathbb{P}[E_{kj}] \) is the asymptotic probability of a type \((k, j)\) edge. This distribution has marginals which are subject to compatibility constraints:

\[ Q^+_k := \sum_j Q_{kj} = kP^+_k / z, \quad Q^-_j := \sum_k Q_{kj} = jP^-_j / z. \]
3. The possible in and out degrees form finite sets $\mathcal{J} = \text{supp}(P^-), \mathcal{K} = \text{supp}(P^+)$.

4. For each $N = 1, 2, \ldots$ we have a random directed graph such that: the number of nodes of degree type $(j, k) \in \mathcal{J} \times \mathcal{K}$ is a random integer $d_{jk}^N$ with expected value $NP_{jk}$; the number of edges of degree type $(k, j)$ is a random integer $e_{kj}^N$ with expected value $NQ_{kj}$.

5. For each feasible realization (of the integer random variables $\{d_{jk}^N, e_{kj}^N\}$), the assortative configuration graph construction of Hurd and Gleeson [2011] is applied.

The non-negative random variables $\Delta_v$ have point masses at $x = 0$ that represent their initial default probability $p_0^v$. We assume that the distribution functions of $\Delta_v$ depend only on the type $(j, k)$, and have the following form:

$$D_{jk}(x) = \mathbb{P}[\Delta_v \leq x | v \in \mathcal{N}_{jk}] : \frac{d}{dx} D_{jk}(x) := p_0^{jk} \delta_0(x) + d_{jk}(x) .$$

where $d_{jk}(x) \geq 0$ is a specified function with $\int_0^\infty d_{jk}(x) dx = 1 - p_0^{jk}$. Similarly, $\Sigma_v$ has a point mass at $x = 0$ that represents this bank’s initial stress probability $q_0^v$ and a distribution function that depends only on its node type $(j, k)$. Thus the stress buffer distribution functions of nodes $v \in \mathcal{N}_{jk}$ have the following form:

$$S_{jk}(x) = \mathbb{P}[\Sigma_v \leq x, v \notin \mathcal{D}_0 | v \in \mathcal{N}_{jk}] : \frac{d}{dx} S_{jk}(x) := q_0^{jk} \delta_0(x) + s_{jk}(x) .$$

where $s_{jk}(x) \geq 0$ is a specified function with $\int_0^\infty s_{jk}(x) dx = 1 - p_0^{jk} - q_0^{jk}$. The edge weight random variables $\Omega_\ell$ are positive (i.e. there is zero probability to have a zero weight) and have distributions that depend only on the edge type $(k, j)$. These can be specified by the distribution functions

$$W_{kj}(x) = \mathbb{P}[\Omega_\ell \leq x | \ell \in \mathcal{E}_{kj}] : \frac{d}{dx} W_{kj}(x) = w_{kj}(x) .$$

Finally, conditional on the random skeleton graph $(\mathcal{N}, \mathcal{E})$, the collection of random variables $\{\Delta_v, \Sigma_v, \Omega_\ell\}$ is assumed to be mutually independent.

It was argued in Hurd and Gleeson [2011] that the above construction of a probability measure on random networks implies a property called locally tree-like independence (LTI) extending the locally tree-like property of random graphs that cycles of any fixed finite length occur in an infinite configuration graph only with zero probability. The probabilistic analysis to follow rests on this extended type of independence:

**The locally tree-like independence (LTI) property:** Consider a double cascade model on $\mathcal{N}$ defined by a collection of random variables $(\mathcal{N}, \mathcal{E}, \Delta, \Sigma, \Omega)$. Let $\mathcal{N}_1, \mathcal{N}_2 \subset \mathcal{N}$ be any two subsets that share exactly one node $\mathcal{N}_1 \cap \mathcal{N}_2 = \{v\}$ and let $X_1, X_2$ be any pair of random variables where for each $i = 1, 2$, $X_i$ is determined by the information on $\mathcal{N}_i$. Then, conditioned on information located at the node $v$, that is conditioned on $\Delta_v, \Sigma_v, j_v$ or $k_v$, $X_1$ and $X_2$ are independent.\(^5\)

\(^5\)More precisely, to any subset of nodes $\mathcal{N}' \subset \mathcal{N}$ we associate the sigma-algebra $\mathcal{G}'$ generated by the balance sheets and degrees of nodes in $\mathcal{N}'$ and edges in $\mathcal{N}' \times \mathcal{N}'$. Let the sigma-algebras corresponding to $\mathcal{N}_1, \mathcal{N}_2, \{v\}$ be denoted $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_v$. Then $\mathcal{G}_1$ and $\mathcal{G}_2$ are statistically independent, conditioned on $\mathcal{G}_v$. 

11
We are now in a position to derive exact formulas for the probabilistic cascade dynamics in infinite networks. Define
\begin{align}
p_{jk}^{(n)} &= \mathbb{P}[v \in \mathcal{D}_n|v \in \mathcal{N}_{jk}], \\
q_{jk}^{(n)} &= \mathbb{P}[v \in \mathcal{S}_n|v \in \mathcal{N}_{jk}], \\
\hat{p}_{jk}^{(n)} &= \mathbb{P}[v \in \mathcal{D}_n \mathbin{\mathbb{R}} w|v \in \mathcal{N}_{jk} \cap \mathcal{N}_{w}^+], \\
\hat{q}_{jk}^{(n)} &= \mathbb{P}[v \in \hat{\mathcal{S}}_n|v \in \mathcal{N}_{jk}].
\end{align}

We will shortly find we are able to compute \( p_{jk}^{(n)} , q_{jk}^{(n)} , \hat{p}_{jk}^{(n)} \), plus an additional quantity
\[ t_{jk}^{(n)} = \mathbb{P}\left[\xi_{wv}^{(n)} = 1|(w,v) \in \mathcal{E}_{kj}\right], \tag{12} \]
recursively over \( n \). These ingredients, when carefully combined with the LTI property, will enable us to account for the awkward fact that banks that default after being stressed give reduced default shocks. It is also convenient to define
\begin{align}
p_k^{(n)} &= \mathbb{P}[v \in \mathcal{D}_n|k_v = k] = \sum_j p_{jk}^{(n)} P_{j|k}, \\
\hat{p}_j^{(n)} &= \mathbb{P}[v \in \mathcal{D}_n \mathbin{\mathbb{R}} w|j_v = j, v \in \mathcal{N}_w^+] = \sum_k \hat{p}_{jk}^{(n)} P_{k|j}, \\
\hat{q}_j^{(n)} &= \mathbb{P}[v \in \hat{\mathcal{S}}_n|j_v = j] = \sum_k \hat{q}_{jk}^{(n)} P_{k|j},
\end{align}
where \( P_{j|k} := \frac{r_{jk}}{r_k} , P_{k|j} := \frac{r_{jk}}{r_j} \).

The computations that follow will rely on two important facts. The first is that if \( X,Y \) are two independent random variables with probability density functions (PDFs) \( f_X(x) = F_X'(x), f_Y(y) = F_Y'(y) \), then
\[
\mathbb{P}[X \geq Y] = \mathbb{E}\left[\mathbbm{1}_{\{X \geq Y\}}\right] = \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbbm{1}_{\{X \geq Y\}} f_X(x) f_Y(y) \, dx \, dy \\
= \int_{\mathbb{R}} F_Y(x) f_X(x) \, dx = \left\langle F_Y, f_X \right\rangle. \tag{14}
\]
In general, the Hermitian inner product on \( \mathbb{R} \) is defined as \( \langle f, g \rangle = \int_{-\infty}^{\infty} \bar{f}(x) g(x) \, dx \), but here, both operands are real functions and the conjugate operator disappears. A second fact is that if \( X_1, X_2, \cdots, X_n \) are \( n \) independent random variables with PDFs \( f_{X_i} \), then the PDF of the sum \( X = X_1 + X_2 + \cdots + X_n \) is the convolution
\[
f_X = f_{X_1} \ast f_{X_2} \ast \cdots \ast f_{X_n} = \oplus_{k=1}^n f_{X_k}, \tag{15}
\]
where the convolution product of two functions is the function defined by \( (f \ast g)(x) = \int_{\mathbb{R}} f(y)g(x-y) \, dy \). For convolution powers, we write \( \oplus_{k=1}^n f_X = f_X^{\oplus n} \).

Our main theorem gives recursive formulas for \( n \geq 1 \) starting with \( p_{jk}^{(0)} , q_{jk}^{(0)} , \hat{p}_{jk}^{(0)} , \hat{q}_{jk}^{(0)} = \hat{p}_k^{(0)} \) determined by the initial shock and stress probabilities.
Theorem 1. For any $n \geq 1$, suppose $p^{(n-1)}_{jk}, q^{(n-1)}_{jk}, \tilde{p}^{(n-1)}_{jk}, t^{(n-1)}_{kj}$ are known. Define the PDFs

$$g^{(n-1)}_j(x) = \sum_{k'} \left[ (1 - p^{(n-1)}_{k'}) \delta_0(x) + t^{(n-1)}_{kj} w_{k'j}(x) \right] + (p^{(n-1)}_{k'} - t^{(n-1)}_{kj}) \cdot \frac{1}{1 - \lambda} w_{k'j}(x/(1 - \lambda)) \cdot Q_{k'|j}, \quad (16)$$

$$h^{(n-1)}_k(x) = \sum_{j'} \left[ (1 - q^{(n-1)}_{j'}) (1 - \tilde{p}^{(n-1)}_{j'}) \delta_0(x) + \tilde{p}^{(n-1)}_{j'} w_{kj}(x) \right] + q^{(n-1)}_{j'} (1 - \tilde{p}^{(n-1)}_{j'}) \cdot \frac{1}{\lambda} w_{kj}(x/\lambda) \cdot Q_{j'k}, \quad (17)$$

where we use (13) to compute $p^{(n-1)}_{k'}, q^{(n-1)}_{j'}, \tilde{p}^{(n-1)}_{j'}$ and define $Q_{kj} = \frac{Q_{kj}}{Q_{j}}, Q_{jk} = \frac{Q_{jk}}{Q_{k}}$.

The quantities $p^{(n)}_{jk}, q^{(n)}_{jk}, \tilde{p}^{(n)}_{jk}, t^{(n)}_{kj}$ are then given by

$$p^{(n)}_{jk} = \left\langle D_{jk}, \left(g^{(n-1)}_j \right)^{\otimes j} \right\rangle, \quad (18)$$

$$\tilde{p}^{(n)}_{jk} = \left\langle D_{jk}, \left(g^{(n-1)}_j \right)^{\otimes j-1} \right\rangle, \quad (19)$$

$$q^{(n)}_{jk} = \left\langle S_{jk}, \left(h^{(n-1)}_k \right)^{\otimes k} \right\rangle, \quad (20)$$

$$t^{(n)}_{kj} = t^{(n-1)}_{kj} + (p^{(n-1)}_{kj} - p^{(n-1)}_{k})(1 - q^{(n-1)}_{j}) . \quad (21)$$

Moreover, the stress probabilities $q^{(n)}_{jk}$ are determined by

$$1 - q^{(n)}_{jk} - p^{(n)}_{jk} = \mathbb{P}[v \in \mathcal{S}_n^c \cap \mathcal{D}_n^c | v \in \mathcal{N}_{jk}] = \left( 1 - q^{(n)}_{jk} \right) \left( 1 - \left\langle D_{jk}, \left(g^{(n-1)}_j \right)^{\otimes j} \right\rangle \right) , \quad (22)$$

where

$$\tilde{g}^{(n-1)}_j(x) = \sum_{k'} \left[ (1 - p^{(n-1)}_{k'}) \delta_0(x) + p^{(n-1)}_{k'} w_{k'j}(x) \right] \cdot Q_{k'|j} \quad (23)$$

Proof: To verify (18) we use (1) and (14) to give the formula

$$p^{(n)}_{jk} = \mathbb{P}[\Delta_v \leq \sum_{w \in \mathcal{N}_v^-} \Omega_{wv} \xi^{(n-1)}_{wv} | v \in \mathcal{N}_{jk}] = \left\langle D_{jk}, \left(g^{(n-1)}_j \right)^{\otimes j} \right\rangle .$$

To verify (16) we note that

$$g^{(n-1)}_j(x) = \sum_{k'} Q_{k'|j} \frac{d}{dx} \mathbb{P}[\Omega_{wv} \xi^{(n-1)}_{wv} \leq x | v \in \mathcal{N}_{jk}, w \in \mathcal{N}_v^- , k_w = k'] .$$
Under the conditions \( v \in \mathcal{N}_{jk}, w \in \mathcal{N}^-_v, k_w = k' \), the events 
\( \{ \xi_{uw}(n-1) = 0 \}, \{ \xi_{uw}(n-1) = 1 \}, \{ \xi_{uw}(n-1) = 1 - \lambda \} \) have conditional probabilities 
\( 1 - p_k(n-1), t_{k'j}(n-1), (p_{k'}(n-1) - t_{k'j}(n-1)) \) respectively and the three events are conditionally independent of \( \Omega_{uw} \) by the LTI property. Also by the LTI property, the collection of random variables \( \Omega_{uw}, \xi_{uw}(n-1) \) for different \( w \in \mathcal{N}^-_v \) are mutually conditionally independent, justifying the use of formula (14).

To verify (19), we use (5) instead of (1) and follow these same steps. To verify (20), we use (6), (11) and (14) to give the formula 
\[
\hat{q}^{(n)}_{jk} = \mathbb{P}[\Sigma_v \leq \sum_{w \in \mathcal{N}^+_v} \Omega_{vw} \hat{z}^{(n)}_{vw} \in \mathcal{N}_{jk}] = \left( S_{jk}, \left( h_k^{(n-1)}(x) \right) \right)^{\otimes k},
\]
where 
\[
h_k^{(n-1)}(x) = \sum_{j'} \left[ \frac{d}{dx} \mathbb{P}[\Omega_{vw} \hat{z}^{(n-1)}_{vw} \leq x | v \in \mathcal{N}_{jk}, w \in \mathcal{N}^+_v, j_w = j'] \right] Q_{j'k}.
\]

To verify (17), note that under the conditions \( k_w = k, w \in \mathcal{N}^+_v, j_w = j' \), the events 
\( \{ \hat{z}_{vw}^{(n-1)} = 0 \}, \{ \hat{z}_{vw}^{(n-1)} = 1 \}, \{ \hat{z}_{vw}^{(n-1)} = \lambda \} \) are equivalent to the events 
\( \{ w \in \hat{S}_{n-1} \cap \mathcal{D}_{n-1} \setminus \mathcal{D}_{n-1} \}, \{ w \in \mathcal{D}_{n-1} \setminus \mathcal{D}_{n-1} \}, \{ w \in \mathcal{S}_{n-1} \} \) and hence have conditional probabilities \( (1 - \hat{q}_{j'}^{(n-1)}(1 - \hat{p}_{j'}^{(n-1)})), \hat{p}_{j'}^{(n-1)}, \hat{q}_{j'}^{(n-1)}(1 - \hat{p}_{j'}^{(n-1)}) \) respectively. To verify (21), we apply the LTI property again to compute the conditional probability of the disjoint union \( \{ \xi_{uw}(n) = 1 \} \cup \{ w \in \mathcal{D}_{n-1} \cap \mathcal{D}_{n-1} \setminus \mathcal{D}_{n-1} \} \) defined by (3).

Finally, to verify (22), we note that \( \hat{S}_{n} \cap \mathcal{D}_{n} = \hat{S}_{n} \cap \{ \Delta_v > \sum_{w \in \mathcal{N}^-_v} \Omega_{uw} \mathbb{I}_{\{ w \in \mathcal{D}_{n-1} \}} \} \).

By LTI, these last two events are independent conditioned on \( v \in \mathcal{N}_{jk} \), and the required formula results by following the steps taken to prove (18).

\[ \square \]

5 Real-World Networks

The goal of the present section is to derive approximate probabilistic formulas describing the double cascade on a real-world network where the skeleton graph is actually known (deterministic) and finite, while the buffers and weights are random. This analysis will allow us to address systemic risk in tractable models of real observed financial networks, without the need for Monte Carlo simulations.

Let \( A = A_{uv}, v, u \in \mathcal{N} \) be the nonsymmetric adjacency matrix of the fixed directed graph \( \mathcal{E} \). We number the nodes in \( \mathcal{N} \) by \( v = 1, 2, \ldots, N \) and the links by \( \ell = 1, 2, \ldots, L \) where \( L = \sum_{1 \leq v, u \leq N} A_{uv} \). The buffer random variables \( \Delta_v, \Sigma_v \) at each node are assumed to have a mass \( p_v^{(0)}, q_v^{(0)} \) at 0 (representing the initial default and stress probabilities) and continuous support with density functions \( d(x), s_v(x) \) on the positive reals. The edge weights \( \Omega_{\ell}, \ell \in \mathcal{E} \) have continuous support with densities \( w_{\ell}(x) \) on the positive reals but no mass at 0. The random variables \( \{ \Delta_v, \Sigma_v, \Omega_{\ell} \}, v \in \mathcal{N}, \ell \in \mathcal{E} \) are assumed to be an independent collection.
The aim of this section is to use the LTI property as an approximation to derive approximate formulas for the marginal likelihoods \( p_v^{(\infty)} \) and \( q_v^{(\infty)} \) for the eventual default and stress of individual nodes, as well as the possibility to compute formulas for more detailed systemic quantities. This approximation is not exact whenever there are cycles in the skeleton graph. In general, when the skeleton graph is a single random realization from a configuration graph ensemble, we expect the LTI approximation to get better with increasing \( N \). The LTI property will be exactly true in the special case of skeleton graphs that are trees: This fact will be used in Section 6.3 to verify the consistency of our numerical implementations.

We now present an approximate analysis, paralleling the \( N = \infty \) analysis of the previous section, of the sequence of probabilities

\[
\begin{align*}
\hat{p}_v^{(n)} &= \mathbb{P}[v \in D_n], \\
\hat{q}_v^{(n)} &= \mathbb{P}[v \in S_n], \\
\hat{p}_{wv}^{(n)} &= \mathbb{P}[v \in D_n \oplus w], \\
\hat{q}_{wv}^{(n)} &= \mathbb{P}[v \in \hat{S}_n],
\end{align*}
\]

(25)

for each node \( v \) or edge \( wv \). For the same reason as before we need in addition to track

\[
\hat{t}_{wv}^{(n)} = \mathbb{P}[\hat{e}_{wv}^{(n)} = 1 | v \in \mathcal{N}_w^+] .
\]

(26)

Inductively, we have

\[
\begin{align*}
\hat{p}_v^{(n)} &= \left\langle \frac{1}{D_v, \oplus v' \in \mathcal{N}_v^+} \left( g_{v'v}^{(n-1)} \right) \right\rangle, \\
\hat{p}_{wv}^{(n)} &= \left\langle \frac{1}{D_v, \oplus v' \in \mathcal{N}_v^w \setminus w} \left( g_{v'v}^{(n-1)} \right) \right\rangle, \\
\hat{q}_v^{(n)} &= \left\langle \frac{1}{S_v, \oplus v' \in \mathcal{N}_v^w} \left( h_{wv}^{(n-1)} \right) \right\rangle, \\
\hat{t}_{wv}^{(n)} &= \hat{t}_{wv}^{(n-1)} + (\hat{p}_{wv}^{(n)} - \hat{p}_{wv}^{(n-1)}) (1 - \hat{q}_{wv}^{(n-1)}), \\
\hat{q}_{wv}^{(n)} &= 1 - \hat{p}_v^{(n)} - \left( 1 - \hat{q}_v^{(n)} \right) \left\langle \frac{1}{D_v, \oplus v' \in \mathcal{N}_v^w} \left( g_{v'v}^{(n-1)} \right) \right\rangle .
\end{align*}
\]

(27) - (31)

The PDFs can be computed using the LTI approximation by following the logic of the proof of Theorem 1:

\[
\begin{align*}
g_{wv}^{(n-1)}(x) &= (1 - p_v^{(n-1)}) \delta_0(x) + t_{wv}^{(n-1)} w_{wv}(x) \\
&\quad + (p_v^{(n-1)} - t_{wv}^{(n-1)}) \cdot \frac{1}{1 - \lambda} w_{wv}(x/1 - \lambda), \\
h_{wv}^{(n-1)}(x) &= (1 - q_v^{(n-1)}) (1 - \hat{p}_{wv}^{(n-1)}) \delta_0(x) + \hat{p}_{wv}^{(n-1)} w_{wv}(x) \\
&\quad + q_v^{(n-1)} (1 - \hat{p}_{wv}^{(n-1)}) \cdot \frac{1}{\lambda} w_{wv}(x/\lambda), \\
g_{wv}^{(n-1)}(x) &= (1 - p_v^{(n-1)}) \delta_0(x) + p_{wv}^{(n-1)} w_{wv}(x).
\end{align*}
\]

(32) - (34)
6 Numerical Experiments

In this section, we report briefly on numerical experiments that illustrate the methods developed in this paper. Firstly, we aim to convince the reader that the LTI method correctly computes the double cascade in stylized networks with large values of $N$. Secondly, we will show how the LTI method can lead to answers to questions about the nature of systemic risk. Thirdly, we shall show how the method performs in a challenging stylized network specified to reflect the complex characteristics of a 2011 dataset on the network of 90 most systemically important banks in the European Union.

To crossvalidate the LTI method, we developed a Monte Carlo (MC) implementation of the double cascade model, and compared the final fraction of defaulted and stressed nodes generated using the LTI and MC computations. Before we present the results of the comparison, we mention two modifications that are necessary to implement these methods. First, implementation of the LTI method uses an FFT method sketched in Appendix A, which requires that the edge weight and buffer random variables take values on a common discrete lattice $\{m\delta\}_{m=0,1,\ldots,M-1}$. The second modification is that the MC implementation generated finite configuration graphs with $N = 20000$ nodes, with the specified $P$ and $Q$ matrices, rather than the infinite configuration graphs assumed in the LTI method.

6.1 Experiment 1: Verifying the LTI Method

This experiment aims to verify that the LTI method performs as expected when applied to a stylized financial network whose specification is similar to that given in Gai and Kapadia [2010a]. It consists of a random directed Poisson skeleton graph with mean degree $z = 10$, where each node $v$ can be viewed as a bank with a default buffer $\Delta_v = 0.04$ and stress buffer $\Sigma_v = 0.02$. Unlike the deterministic weights used in Gai and Kapadia [2010a], the edge weight $\Omega_\ell$ of an edge $\ell$, representing the exposure between two banks, is taken from a log normal distribution with mean $\mu_\ell = 0.2j_\ell^{-1}$, and standard deviation $0.383\mu_\ell$. Note that this specification makes the exposure size dependent on the lending bank. An initial shock is applied to the network that causes each bank to default with a 1% probability.

We compare the final fraction of defaulted bank and stressed banks as computed using MC simulation with 1000 realizations and the LTI analytic formulas. Directed Poisson random graphs are particularly amenable to study by Monte Carlo: To generate a random graph of size $N$ with mean degree $z > 0$ from this class, one simply selects directed edges independently from all $N(N - 1)$ potential edges, each with probability $p = z/(N - 1)$. The resultant bi-degree distribution is a product of independent binomials, $\mathbb{P}[v \in \mathcal{N}_{jk}] = \text{Bin}(N - 1, p, j) \times \text{Bin}(N - 1, p, k)$, which for large $N$ nodes is approximately a product of independent Poisson($z$) distributions.

---

6The mean degree is the average number of “in” edges per node, which equals the average number of “out” edges per node.
Figure 2 plots the results as functions of the stress response parameter $\lambda$, with error bars that represent the $10^{th}$ and $90^{th}$ percentile of the MC result. It shows the expected agreement between MC and LTI analytics, with discrepancies that can be attributed to finite $N$ effects present in the MC simulations.

One fundamental property of our model is clearly shown in this experiment: Stress and default are negatively correlated. This fact can be explained by the stress reaction which enables banks to react to liquidity shocks before they default, by reducing their interbank exposures. This response creates yet more stress, but leads to a more resilient network. The “knife-edge” property of default cascades is also clearly shown: In the model parametrization we chose, a very small increase in $\lambda$ dramatically alters the stability of the network. We also note that MC error bars are very large near the knife-edge.

![Figure 2: A comparison of the mean default (light/red) and stress (dark/blue) cascade size in Experiment 1 as computed by Monte Carlo (crosses) and LTI analytics (solid lines). Error bars indicate the $10^{th}$ and $90^{th}$ percentiles of the MC result.](image)

### 6.2 Experiment 2: A Stylized Poisson Network

The next experiment focuses again on Poisson networks, with the aim to better understand the effects of various parameters on network resilience. In general, we continue to find confirmation that the LTI results accurately reflect observations
from MC simulations.

6.2.1 Experiment 2A: Effects of Default and Stress Buffers

We consider how the parametrized financial network of Experiment 1 in a default-susceptible state with \( \lambda = 0.25 \) can be made resilient to random shocks by varying either the default buffers away from 0.04 or the stress buffers away from 0.02.

Figure 3(a) imagines what would happen if regulators had required all banks to have higher default buffers, without any change in their stress behaviour. We observe a very fast transition to a stable network as \( \Delta \) increases over the interval \([0.04, 0.045]\). This knife-edge property is observable in both the LTI analytics and in the MC simulations. Note again that the MC error bars, representing the 10\textsuperscript{th} and 90\textsuperscript{th} percentiles, become very large near the knife-edge.

If banks with \( \Delta = 0.04 \) reduce their stress buffers below 0.02, they will react more quickly to stress shocks: This can also reduce default cascade risk in the network. Figure 3(b) shows that such a change dramatically reduces the average default cascade size in the network. Taken together, these two plots show that there may be many different approaches to dealing with network resilience.

Figure 3: The results of Experiment 2A, showing the effects on the default resilience of a Poisson network when the default buffer (left) and stress buffer (right) are varied away from their benchmark values. Here \( \lambda = 0.25 \) and other parameters are chosen as in Experiment 1.

6.2.2 Experiment 2B: Effects of Graph Connectivity and Stress Response

Aside from mandating changes to the behaviour of FIs during or prior to a crisis, regulators can also influence the shape of the financial network as a whole. Experiment 2B aims to demonstrate the importance of the skeleton graph itself, so we
observed the systemic risk in a directed Poisson network as a function of the connectivity parameter $z$ and $\lambda$, the stress response. In our simple model specification, the mean degree $z$ is the only parameter that controls the shape of the skeleton graph, whereas in a more realistic modelling approach the skeleton graph may have many more parameters.

In this experiment, we increased the model complexity by assuming each node to have a random default buffer taken from a log normal distribution with mean 0.18 and standard deviation 0.18, and a stress buffer from an independent log normal distribution with mean 0.12 and standard deviation 0.12. The edge weights $\Omega_\ell$ come from a lognormal distribution with mean and standard deviation proportional to $(j_\ell k_\ell)^{-0.5}$, with the average edge weight on the entire network equal to 1. Once again we apply an initial shock so that each FI has 1% chance of defaulting initially.

Figure 4(a) shows a surface plot of the mean default cascade size in the network as a function of $z$ and $\lambda$. Figure 4(b) shows the mean stress cascade size of the network. For clarity of the graphics, we show LTI analytics only: the Monte Carlo results not shown continue to agree with LTI analytics. Again, in these plots we notice the strong anti-correlation between stress and default probabilities, and the effect of increasing the stress response. It is also interesting to observe that the final level of stress is not monotonic in the connectivity parameter $z$.

6.3 Experiment 3: A Real-World Network with 90 Nodes

While the above experiments on hypothetical financial networks demonstrate the range of options available in our framework, we are of course interested in having a
rough picture of the systemic risk of actual financial networks. In Experiment 3, we computed cascade dynamics on a single realization of a 90 node graph that aims to capture stylized features of the European Union network, using both the “real-world” LTI analytic method of Section 5 and a Monte Carlo method. As a preliminary step (not reported), we validated the consistency of the LTI analysis by verifying that the two methods agree as expected on a number of tree networks, using the fact observed earlier that the LTI analytic formulas of Section 5 are exact for real-world tree networks.

Numerous studies of real-world financial networks, notably Bech and Atalay [2010] and Cont et al. [2010], have observed their highly heterogeneous structure and concluded that in and out degrees have fat tailed distributions, as do exposure sizes, and presumably, buffers. Our schematic model of 90 EU banks was designed to capture these basic statistical features, and as well to fit aggregated statistics from data published on the 2011 ECB stress testing of systemically important banks in the European Union. The detailed construction of our stylized network, with specifications for buffer and exposure distributions and the skeleton graph shown in Figure 5, is given in Appendix B.

Figure 5: A representation of the undirected skeleton graph of the 90 bank network of Experiment 3. The nodes are plotted with total degree decreasing in the counterclockwise direction, with the maximally connected bank being the rightmost node.

The first part of the experiment determined the cascade distribution that results by shocking one bank of the EU network at random to default. Figure 6(a) shows the mean default and stress cascade sizes as they depend on the stress response parameter $\lambda$. This graph shows no evidence of cascading, demonstrating that the
MC and analytic computations agree that the EU network in June 2011 was resilient to such a shock.

To move the EU network to a knife-edge situation where a large scale double cascade can be triggered by a single bank default, we found it was necessary to imagine a dire crisis where prior to the default shock, the core T1 capital of all institutions (i.e. their default buffers) has been decimated to 10% of their initial amount. Figure 6(b) shows the intermediate size of the cascades that result, as a function of the stress response parameter $\lambda$. In this graph we see limitations of the LTI approximation that we presume are due to finite size effects and the extra difficulty in computing accurately near knife-edge situations. In this specification of the network, while the LTI and MC methods agree that there is an intermediate size cascade, they disagree strongly on the actual mean default cascade size.

Figure 6: The results of Experiment 3 that show the effects of changing the stress response $\lambda$ in a stylized finite network. The left plot corresponds to parameters that represent the EU financial system at the time of the 2011 stress testing exercise. The right plot shows the same system where a dire pre-shock crisis has decimated the banks’ default buffers.

Finally, in real-world networks, we can get a picture of which banks are most susceptible to the default of the single bank. Figure 7 shows the eventual stress and default probabilities bank by bank, in increasing order of the LTI estimated default probability. We see that LTI analytics and MC agree quite well on the ordering of the banks, on the eventual probability of stress, but not well on the eventual probability of default.

7 Conclusions

Our double cascade model, perhaps the first in the economics literature, is a natural extension of the previous systemic risk research that studies deliberately simplified
models which build in either default or stress cascades, but not both. Only by combining the default and stress mechanisms into a single model can one measure the intuitively obvious effect of banks using the stress response to reduce their risk of default.

Developing a feasible and reliable computation framework for a model as complex as our double cascade model poses many challenges. We have demonstrated how computations can be done by two complementary approaches: the Monte Carlo (MC) method and the locally tree-like independence (LTI) analytic method. In general, having these two approaches allows us to independently cross validate both methods, increasing confidence in the results one obtains.

It is surprising that the double cascade assumptions of Sections 2 and 3, which intertwine upstream and downstream shocks, are consistent with the LTI asymptotic analytics underlying Theorem 1 in Section 4. We have shown that careful use of the “without regarding” (WOR) conditions can unravel the dependence in this type of feedback.

As to the pros and cons of MC versus LTI, we mention some of the key issues. To counter the natural flexibility of MC methods, the LTI method, where it applies, adds the possibility to better understand the flow of the cascade. For example, using LTI one can determine sensitivities to changing parameters through explicit differentiation. A con for the MC method is that simulating general assortative
configuration models has not been well-studied in the literature, while the LTI method handles this generality without difficulty. On the other hand, relevant random graphs such as preferential attachment models have a straightforward MC simulation algorithm, but are not LTI. Another pro for MC is that the LTI approximation is uncontrolled for finite $N$ configuration graphs, meaning we can only learn how accurate it is by comparing to MC results. Through experience, we are learning rules of thumb for when LTI gives acceptable results, for example when $N$ is large and the cascade is far from critical. Apart from the graph generation step, MC is usually easier to program than LTI. But in some situations where the MC method leads to unacceptably long run times, LTI can be computed in seconds. Ultimately what is important is that both methods have complementary strengths and weaknesses, and when used in combination we can arrive at robust and reliable conclusions about a wide range of network effects.

Future theoretical work must proceed by improving and extending both the MC and LTI methods. For example, we need better MC algorithms for simulating different random graph models and we would like to find computable analytical methods for networks that are not LTI.

Many promising specifications of financial networks remain to be investigated using our techniques. While the systemic importance of parameters such as network connectivity, mean buffer strength, and the size of the interbank sector have been studied previously, other parameters such as the stress response, the buffer and exposure variances, and graph assortativity, remain almost completely unexplored. The effect of market illiquidity and asset fire sales has been omitted from the present paper, but its impact on the cascade and consequently the greater economy merits careful investigation. Financial network databases, and the statistical methods for matching such data to the model, are still in an underdeveloped state, but are needed to tie down the wide range of parameters in our model. Planned future investigations of the double cascade model developed in this paper will hopefully uncover further interesting and unexpected systemic phenomena.

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A Discrete Probability Distributions and the Fast Fourier Transform

Numerical implementation of these models follows the methods outlined in Hurd and Gleeson [2013]. In this section, we analyze the case where the random variables \( \{\Delta v, \Sigma v, \Omega \ell\} \) all take values in a specific finite discrete set \( M = \{0, 1, \ldots, (M - 1)\} \) with a large value \( M \). In such a situation, the convolutions in (15) can be performed exactly and efficiently by use of the discrete Fast Fourier Transform (FFT).

Let \( X, Y \) be two independent random variables with probability mass functions (PMF) \( p_X, p_Y \) taking values on the non-negative integers \( \{0, 1, 2, \ldots\} \). Then the random variable \( X + Y \) also takes values on this set and has the probability mass function (PMF) \( p_{X+Y} = p_X * p_Y \) where the convolution of two functions \( f, g \) is defined to be

\[
(f * g)(n) = \sum_{m=0}^{n} f(m)g(n - m).
\]

Note that \( p_{X+Y} \) will not necessarily have support on the finite set \( M \) if \( p_X, p_Y \) have support on \( M \). This discrepancy leads to the difficulty called “aliasing”.

We now consider the space \( \mathbb{C}^M \) of \( \mathbb{C} \)-valued functions on \( M = \{0, 1, \ldots, M - 1\} \). The discrete Fourier transform, or fast Fourier transform (FFT), is the linear mapping \( F: a = [a_0, \ldots, a_{M-1}] \in \mathbb{C}^M \rightarrow \hat{a} = F(a) \in \mathbb{C}^M \) defined by

\[
\hat{a}_k = \sum_{l \in M} \zeta_{kl}a_l , k \in M .
\]

where the coefficient matrix \( Z = (\zeta_{kl}) \) has entries \( \zeta_{kl} = e^{-2\pi i kl/M} \). The “inverse FFT” (IFFT), is given by the map \( a \rightarrow \tilde{a} = G(a) \) where

\[
\tilde{a}_k = \frac{1}{M} \sum_{l \in M} \bar{\zeta}_{kl}a_l , k \in M .
\]

If we let \( \bar{a} \) denote the complex conjugate of \( a \), we can define the Hermitian inner product between

\[
\langle a, b \rangle := \sum_{m \in M} \bar{a}_m b_m .
\]

We also define the convolution product of two vectors:

\[
(a * b)(n) = \sum_{m \in M} a(m) b(n - m \text{ modulo } M), \quad n \in M .
\]
Note that this agrees with (35) if and only if the sum of the supports of \( a \) and \( b \) is in \( \mathcal{M} \). Otherwise the difference is called an aliasing error: our numerical implementations reduce or eliminate aliasing errors by taking \( M \) sufficiently large.

The following identities hold for all \( a, b \in \mathbb{C}^M \): (i) Inverse mappings: 
\[
    a = \mathcal{G}(\mathcal{F}(a)) = \mathcal{F}(\mathcal{G}(a));
\]
(ii) Conjugation: 
\[
    \mathcal{G}(a) = \frac{1}{\pi} \mathcal{F}(\bar{a});
\]
(iii) Parseval Identity: 
\[
    \langle a, b \rangle = \frac{1}{M} \langle \hat{a}, \hat{b} \rangle;
\]
(iv) Convolution Identities: 
\[
    \hat{a} \cdot \hat{b} = \mathcal{F}(\hat{a} \cdot \hat{b}) = \mathcal{F}(\hat{a} \cdot \hat{b}) = \mathcal{F}(\hat{a} \cdot \hat{b}),
\]
where \( \cdot \) denotes the component-wise product.

As an example to illustrate how the above formulas help, we observe that a typical formula (18) can be computed instead by the formula
\[
    p_{jk}^{(n)} = \frac{1}{M} \langle \mathcal{F}(D), \left( g_j^{(n-1)} \right)^j \rangle,
\]
where \( D = \mathcal{F}(D) \), \( g_j^{(n-1)} = \mathcal{F}(g_j^{(n-1)}) \) and the power is the component-wise vector multiplication. Such FFT-based formulas can be computed systematically, very efficiently, if the discrete probability distributions for \( \Delta, \Sigma, \Omega \) are initialized in terms of their Fourier transforms.

### B EU Network Construction

In 2011, the European Banking Authority (EBA) reported the interbank exposures of a selection of European banks, as well as other information, such as Core Tier 1 Capital and RWA (Risk Weighted Assets). Their publicly available data set\(^7\) contains information about 90 medium-large European banks.

In this section we explain how we built the synthetic network in Experiment 3 that mimics stylized facts of the real EU network, using this EBA data as a source of relevant information about banks’ balance sheets and incorporating general topological network properties observed by other authors, in particular the papers Cont et al. [2010] and Bech and Atalay [2010]. Motivated by the ubiquitous relevance of networks with fat-tailed degree distributions, we built the skeleton graph using the method described in Bollobás et al. [2003]. Let \( \alpha, \beta, \gamma, \delta_{in}, \delta_{out} \) be non-negative real numbers with \( \alpha + \beta + \gamma = 1 \). This preferential attachment method grows directed networks from a finite initial “seed graph” using three rules:

1. With probability \( \alpha \), add a new vertex \( v \) together with an edge from \( v \) to an existing vertex \( w \), where \( w \) is chosen according to \( j_w + \delta_{in} \) (that is, with probability proportional to \( j_w + \delta_{in} \)).
2. With probability \( \beta \), add an edge from an existing vertex \( v \), chosen according to \( k_v + \delta_{out} \), to an existing vertex \( w \), chosen according to \( j_w + \delta_{in} \).
3. With probability \( \gamma \), add a new vertex \( v \) together with an edge from an existing vertex \( w \) to \( v \), where \( w \) is chosen according to \( k_w + \delta_{out} \).

\(^7\)Their data is found at http://www.eba.europa.eu/EU-wide-stress-testing/2011/2011-EU-wide-stress-test-results.aspx.
This method leads to scale-free directed graphs with fat-tailed degree distributions
\( P_j^- \sim j^{-\tau_-} \) and \( P_k^+ \sim k^{-\tau_+} \) with Pareto exponents
\[
\frac{1}{\tau_- - 1} = \frac{\alpha + \beta}{1 + \delta_{in}(\alpha + \gamma)}, \quad \frac{1}{\tau_+ - 1} = \frac{\beta + \gamma}{1 + \delta_{out}(\alpha + \gamma)}. \tag{36}
\]
The five parameters were determined to be \( \alpha = 0.169, \beta = 0.662, \gamma = 0.169, \delta_{in} = \delta_{out} = 4.417 \) by the following conditions. First we assumed Pareto exponents \( \tau_- = \tau_+ = 4 \) (ensuring finiteness of certain moments) and that \( \alpha = \gamma \). For various values of \( \alpha \), we generated 1000 samples of \( N = 1000 \) networks for a range of \( \alpha \) values. For each realized network sample, we selected the subnetwork of the 90 most connected nodes and calculated its mean degree \( z \). Finally we selected the \( \alpha \) value which provided a mean degree \( z \) closest to 10, which is a consensus view of the average number of counterparties in large banking networks. These parameters were then used to produce the skeleton graph of Experiment 3 by generating a final sample of a \( N = 1000 \) scale free graph, and retaining the most connected subnetwork of 90 nodes.

By taking to fix the default buffer distribution within a conditional log-normal parametric family. Similarly, we use sample moments of the reported aggregated interbank liabilities to evaluate the distribution of interbank assets through \( E[A_{IB}^v] \) and \( E[(A_{IB}^v)^2] \). We assumed an LTI compatible specification of the buffer and exposure random variables as log-normally distributed conditionally on the network topology:
\[
\Delta_v = (k_vj_v)^{\beta_1} \exp[a_1 + b_1 X_v], \tag{37}
\]
\[
\Sigma_v = \frac{2}{3} (k_vj_v)^{\beta_1} \exp[a_1 + b_1 \tilde{X}_v], \tag{38}
\]
\[
\Omega_v = (k_vj_v)^{\beta_2} \exp[a_2 + b_2 X_{\ell}], \tag{39}
\]
where the collection \{\( X_v, \tilde{X}_v, X_{\ell} \)\} consists of independent standard normal random variables. To fix the parameter values, we arbitrarily set \( \beta_1 = 0.3 \) and \( \beta_2 = -0.2 \) with the rationale that the default buffer should increase with bank connectivity, while a larger number of counterparties should imply lower average bilateral exposures. The reported Core Tier 1 Capital was taken as a proxy for the default buffers, and thus we matched the first and second sample moments \( E[\Delta_v] \) and \( E[\Delta_v^2] \) using equation (37). Since we found no proxy in the data for the stress buffers \( \Sigma \), we arbitrarily selected the same parameters as for \( \Delta \), but with a prefactor \( 2/3 \). Finally, matching equation (39) with sample moments \( E[A_{IB}^v] \) and \( E[(A_{IB}^v)^2] \), from the aggregated interbank exposure data, gives us enough equations to determine the full list of parameters:\( \beta_1 = 0.3, \ a_1 = 8.03, \ b_1 = 0.9, \ \beta_2 = -0.2, \ a_2 = 8.75, \ b_2 = 1.16. \)

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