Fluctuating spin and charge stripes in the two-dimensional Hubbard model in the thermodynamic limit

Peizhi Mai¹, Seher Karakuzu¹, Giovanni Balduzzi², Steven Johnston³, and Thomas A. Maier¹,*

¹Computational Sciences and Engineering Division, Oak Ridge National Laboratory, Oak Ridge, TN, 37831-6494, USA.
²Institute for Theoretical Physics, ETH Zurich, 8093 Zurich, Switzerland.
³Department of Physics and Astronomy, University of Tennessee, Knoxville, Tennessee 37996-1200, USA. and
*To whom correspondence should be addressed; E-mail: maierta@ornl.gov.
(Dated: June 4, 2021)

The high-temperature superconducting cuprates are governed by intertwined spin, charge, and superconducting orders. While various state-of-the-art numerical methods have demonstrated that these phases also manifest themselves in doped Hubbard models, they differ on which is the actual ground state. Finite cluster methods typically indicate that stripe order dominates while embedded quantum cluster methods, which access the thermodynamic limit by treating long-range correlations with a dynamical mean field, conclude that superconductivity does. Here, we report the observation of fluctuating stripe order persists in the thermodynamic limit. This discovery also provides a new opportunity to study the influence of fluctuating stripe correlations in the model’s pairing correlations within a unified numerical framework.

Introduction — A common element of strongly correlated materials is the existence of several nearly degenerate states, which compete or cooperate to produce novel phases of matter [1]. For example, in the high-temperature (high-\(T_c\)) superconducting cuprates, multiple experiments point to intertwined orders of spin and charge stripes, charge- and pair-density-waves, and unconventional superconductivity [2–5]. Understanding the relationships among these orders and how they shape the cuprate phase diagram is a central problem in condensed matter physics [6].

Addressing this question using nonperturbative methods remains challenging as even the simplest correlated electron models also contain near-degenerate orders, which can be difficult to discern from one another. For example, state-of-the-art numerical studies have identified a plethora of low-energy states in the single-band Hubbard and \(t-J\) models that contend for the ground state [7–21]. Methods like Hartree-Fock mean-field theory [7], density matrix renormalization group (DMRG) [8–11], density matrix embedding theory [8, 12], variational Monte Carlo [15], auxiliary field quantum Monte Carlo [8, 16], and infinite projected entangled-pair states [8, 17] tend to find static stripe order as the ground state, and recent determinant quantum Monte Carlo (DQMC) calculations [13, 14] have found evidence for fluctuating stripe order at finite temperatures. In contrast, quantum cluster methods like cellular dynamical mean-field theory [22] and the dynamical cluster approximation (DCA) [23, 24], which, unlike finite size cluster techniques, directly access the thermodynamic limit, typically find superconducting solutions with a \(d\)-wave symmetry [18–21]. While approximative calculations of this sort with large unit cells but correlations restricted to small clusters have found evidence of stripes [25–27], more reliable calculations with clusters large enough to accommodate the stripe periodicity have yet to find any indication of stripe-like solutions. This dichotomy has made it difficult to understand the relationships between the relevant orders since each method has its approximations, which introduce systematic errors that can bias towards particular solutions. To overcome this issue, it is desirable to identify a single framework capable of identifying the relevant states to avoid compounding systematic biases.

Here, we demonstrate that quantum Monte Carlo DCA [23] methods can resolve fluctuating stripes in the two-dimensional single-band Hubbard model. The QMC-based impurity solver captures the intra-cluster correlations exactly, while longer-range correlations are treated at a mean-field level that approximates the infinite system. The observation of fluctuating stripes with this method therefore provides crucial confirmation that fluctuating stripe order persists in the thermodynamic limit. It also fixes the discrepancy between finite-cluster and quantum cluster methods, thus allowing us to examine the influence of fluctuating stripe correlations on \(d\)-wave pairing in the single-band Hubbard model using a unified framework, provided the clusters are large enough to accommodate the relevant periodicity such that the spatial modulations are not averaged out by the DCA mean field.

Model and Methods — We consider the two-dimensional single-band Hubbard Hamiltonian defined
on a rectangular $N = N_x \times N_y$ lattice

$$H = -\sum_{i,j,\sigma} t_{ij} \left( c^\dagger_{i\sigma} c_{j\sigma} + \text{h.c.} \right) - \mu \sum_{i,\sigma} n_{i\sigma} + U \sum_{i} n_{i\uparrow} n_{i\downarrow}. \tag{1}$$

Here, $c^\dagger_{i\sigma}$ ($c_{i\sigma}$) creates (annihilates) a spin-$\sigma$ (=↑, ↓) electron on site $i$; $n_{i\sigma} = c^\dagger_{i\sigma} c_{i\sigma}$ is the number operator; $t_{ij}$ is the hopping integral between sites $i$ and $j$; $\mu$ is the chemical potential; and $U$ is the on-site Hubbard repulsion. Throughout, we restrict $t_{ij}$ to nearest-neighbor ($t$) and next-nearest-neighbor ($t'$) hopping only and set $U = 6t$ to facilitate comparisons to Ref. [13].

We solved Eq. (1) using DCA [23] and with a continuous-time QMC impurity solver [20], as implemented in the DCA++ code [28], and complementary DQMC calculations [29, 30]. To study the fluctuating spin stripes, we measured the equal-time staggered spin-spin correlation function $S^{stag}(\mathbf{r}) = (-1)^{r_x + r_y} S(\mathbf{r})$, where $\mathbf{r} = a(r_x, r_y)$ is the position of each atom on the square lattice with lattice constant $a$, $S(\mathbf{r}) = \frac{1}{N} \sum_{i} \langle \hat{S}_i^z \hat{S}_{i+r}^z \rangle$, and $\hat{S}_i^z = \frac{1}{2} \left( c_{i\uparrow}^\dagger c_{i\downarrow} - c_{i\downarrow}^\dagger c_{i\uparrow} \right)$ is the $z$-component of the local spin operator at site $i$.

The charge stripe correlations are assessed by measuring the density-density correlation function $N(\mathbf{r}) = \frac{1}{N} \sum_{i} \langle n_i n_{i+r} \rangle$, where $n_i = \sum_{\sigma} n_{i,\sigma}$ is the local density operator. The pairing tendencies are accessed by measuring the equal-time pairing correlation function in the $d$-wave channel $P_d(\mathbf{r}) = \frac{1}{N} \sum_{i} \langle \Delta_i^\dagger \Delta_{i+r} \rangle$, where $\Delta_i = c_{i,\uparrow} (c_{i+\hat{x},\downarrow} + c_{i-\hat{x},\downarrow} - c_{i+\hat{y},\downarrow} - c_{i-\hat{y},\downarrow})$ destroys a pair of electrons with $d$-wave symmetry. We also determined the structure of the pairing interaction by explicitly solving the Bethe-Salpeter equation (BSE) in the particle–particle singlet channel to obtain its leading eigenvalues and eigenvectors [19, 31]. Due to the large cluster sizes and the self-consistency loop, our DCA calculations are significantly more expensive than the corresponding DQMC calculations. For this reason, here we focus on an average density $\langle n \rangle = 0.8$, where we have observed strong stripe correlations.

Results — Figure 1 plots $S^{stag}(\mathbf{r})$ for several values of the next-nearest neighbor hopping $t'$, where we see clear evidence for fluctuating spin stripes in our DCA calculations (Panels A, B, D-F). Here, we employ large $16 \times 4$ clusters embedded in the DCA self-consistent mean-field, where we access temperatures as low as $T = 0.167t$ (inverse temperature $\beta = 6/t$). Since the staggered spin-spin correlation function imposes a sign flip on every other site, the positive blue regions in the middle of each panel represent short-range antiferromagnetic (AFM) correlations. In contrast, the negative red regions represent AFM regions but with a $\pi$ phase shift. As $t'/t$ decreases from positive to negative, red negative regions form on both sides of the central blue region, signaling the formation and growth of fluctuating AFM stripe correlations, similar to those observed in finite size DQMC calculations [13, 14]. In general, we find that the boundary between the red and blue regions mixes, suggesting that the stripes are incommensurate. We have observed...
similar correlations for different cluster sizes and geometries, including $8 \times 8$, $8 \times 6$ and $8 \times 4$ clusters (see Figs. 4 and the supplementary material).

For comparison, Fig. 1C shows $S_{\text{stag}}(r)$ obtained from a DQMC calculation at $T = 0.2t$ ($\beta = 5/t$), $t' = -0.25t$. The DQMC results are consistent with both the corresponding DCA results (Fig. 1B) and Ref. 13, which considered a slightly higher temperature and a slightly different density. Since DQMC treats the system exactly on an extended but finite cluster, one must perform a finite-size scaling analysis to access the thermodynamic limit. On the other hand, DCA accesses the thermodynamic limit by embedding its clusters in a dynamical mean field that approximates the rest of the system. Comparing Figs. 1B and 1C, we find that DCA predicts weaker stripe correlations compared to DQMC for the same $t'$, despite the lower temperature. This observation helps to explain why stripes have previously gone unobserved in quantum cluster approaches. The origin of the reduced correlations is unclear at this time. One possibility is that the correlations observed by DQMC would weaken as the cluster size increases. Another is that the mean field reduces the effective correlations in the DCA treatment of the problem. Nevertheless, the observation of fluctuating stripes with DCA provides crucial evidence that they persist in the thermodynamic limit.

Our calculations also find evidence for fluctuating charge stripes for the first time. Fig. 2 plots $N(r)$ for the same $16 \times 4$ DCA and DQMC simulations shown in Fig. 1. For $r = 0$, $N(0) = n - n^2 + 2(n_1 n_2)$, so the positive $r = 0$ correlation is due to the fact that the filling $n = 0.8 < 1$ and the double occupancy $n_1 n_2 > 0$. For $t' = 0.2t$ (panel F), $N(r \neq 0) < 0$ for all $r \neq 0$, reflecting a small but negative non-local charge correlation. For $t' \leq 0$, the charge correlations remain negative but decay on longer length scales and display some uni-directional behaviors. With increasing negative $t'$, we also observe weak indications of positive correlations on large length scales, but no modulated intensity as one might expect. The DQMC results for $t' = -0.25t$ in panel C display a pattern qualitatively similar to the corresponding DCA results. Comparing the charge patterns with the spin patterns in Fig. 1, one sees that the charge correlations have the largest negative values in the region corresponding to the central blue spin stripe, where the AFM correlations are strongest. This observation suggests that doped holes have a tendency to be pushed out of the regions with strong antiferromagnetic correlations.

The presence of the spin and charge stripes is more readily observed by examining the dynamical spin $S(Q, \omega)$ and charge $N(Q, \omega)$ structure factors, which are accessible in scattering experiments. Here, we consider the static limit ($\omega = 0$), which can be obtained by Fourier transforming the corresponding unequal-time correlation functions and integrating over imaginary time. Fig. 3 summarizes $S(Q, \omega = 0)$ along $Q = (Q_x, \pi)$ (top row) and $N(Q, \omega = 0)$ along $Q = (Q_x, 0)$ (bottom row) for different values of the next-nearest-neighbor hopping $t'$. When the spin stripe correlations are strong, they should manifest as incommensurate peaks in $S(Q, 0)$ centered at $(\pi \pm \delta_s, \pi)$ while the charge stripes should manifest as incommensurate peaks in $N(Q, 0)$ centered at $(\pm \delta_c, 0)$, where $\delta_s = 2\delta_c$. To check this, we fit the spectra with pairs of Lorentzian functions (plus a constant background) and extracted the corresponding values of $\delta_{s,c}$. In all cases, the structure factors are well repre-
FIG. 3: DCA results for the zero-frequency dynamical spin \( S(\mathbf{Q}, \omega = 0) \) (Panels A-E) and charge \( N(\mathbf{Q}, \omega = 0) \) (Panels F-J) structure factors, obtained on a 16 \( \times \) 4 cluster embedded in a dynamical mean field and with \( \langle n \rangle = 0.8 \). The top row shows the spin structure factors along \( \mathbf{Q} = (Q_x, \pi) \) for A \( t' = -0.3t \), B \( t' = -0.25t \), C \( t' = -0.2t \), D \( t' = 0 \), and E \( t' = 0.2t \). Each spectra is fit with a pair of Lorentzian functions centered at \( (\pi \pm \delta_s, \pi) \) plus a constant background. The bottom row shows the corresponding charge structure factors along \( \mathbf{Q} = (Q_x, 0) \). Each spectra is fit with a pair of Lorentzian functions centered at \( \mathbf{Q} = (\pm \delta_c, 0) \) plus a constant background. All results were obtained for \( T = 0.167t \) (\( \beta = 6/t \)). The ratio of the spin and charge incommensurability is given by \( r = \delta_s/\delta_c \) and panel K shows temperature evolution of the spin and charge incommensurability and \( r \) for \( t' = -0.3t \).

sent by fits, and the resulting ratio \( r = \delta_s/\delta_c \) is given in the top panels of Fig. 3, where we find \( 0.66 > r > 0.3 \). These values are close to the expected \( r = 0.5 \) result, for which the periodicity of the charge fluctuations is twice that of the spin fluctuations, with increasingly negative \( t' \).

To examine how the spin and charge stripes form, we also extracted the values of \( \delta_s \), \( \delta_c \), and \( \delta_s/\delta_c \) as a function of temperature for \( t' = -0.3t \), as shown in Fig. 3K. (The corresponding structure factor data are provided in the supplementary material.) We find that \( \delta_s \) remains relatively fixed as a function of temperature, while \( \delta_c \) appears to lock in to its value of \( \delta_c \approx 2\delta_s \) as the temperature is lowered. This result suggests that the charge stripes form after the spin stripes in the Hubbard model, and that the periodicity of the former locks into the value set by the latter.

Previous DCA studies have resolved a finite-temperature transition to the \( d \)-wave superconducting state \([18, 31]\). Now that DCA also finds evidence for fluctuating stripes, both in the spin and charge sector, it is natural to ask how they affect the formation of Cooper pairs. To answer this question, we now examine the equal-time \( d \)-wave pairing correlation function \( P_d(r, \tau = 0) \). Since a rectangular cluster geometry can mix pairing symmetries \([32]\), we first consider a square \( N = 8 \times 8 \) cluster to proceed without any bias. The real-space equal-time staggered spin correlation, density correlations, and \( d \)-wave pairing correlations are shown in Figs. 4A-D, 4E-H, and 4I-L, respectively, for different values of \( t' \) and \( T = 0.2t \). In all cases, the leading eigenvector of the BSE corresponds to a \( d \)-wave state.

Since the \( 8 \times 8 \) cluster no longer breaks \( C_4 \) symmetry, both the spin and charge stripe correlations now appear as superimposed vertical and horizontal stripes. As with Fig. 1, their strength increases as \( t' \) becomes more negative. Interestingly, as the spin stripe correlations grow, the pairing correlations also develop a real-space structure suggestive of a pair-density-wave \([33]\). For example, for \( t' = 0.2 \) (Fig. 4L), the real-space pairing correlations are mostly positive and extend nearly across the entire cluster. As \( t' \) is reduced, however, regions begin to emerge on short length scales that display a sign change (\( \pi \) phase shift) in the \( d \)-wave pairing correlations.

These observations suggest that modulations in the pair correlations develop together with the stripe correlations; however, this effect is difficult to disentangle from the changes in band structure that also occur as \( t' \) is varied \([31]\). We do not observe any unidirectional behavior in the pair density modulations, however, even when we break \( C_4 \) symmetry by using a rectangular cluster. For example, Fig. 5A compares the real-space pairing

\[
t' = -0.3t, r = 0.527
\]

\[
t' = -0.25t, r = 0.512
\]

\[
t' = -0.2t, r = 0.599
\]

\[
t' = 0, r = 0.663
\]

\[
t' = 0.2t, r = 0.306
\]
Moreover, by accessing these phases using an embedded cluster technique, we are able to examine the correlation function, while panels (C-K) show the corresponding real-space density-density correlation function, while panels (I-L) show the real-space pair-field correlation functions.

Conclusions — Our results demonstrate that fluctuating spin and charge stripe orders are a property of the doped single-band Hubbard model in the thermodynamic limit. Moreover, by accessing these phases using an embedded cluster technique, we are able to examine the correlations on a 16 x 4 cluster against the correlations obtained for the same parameters but on an 8 x 8 cluster (Fig. 5B). Here, we find that the real space pairing correlations are largely unaffected by the cluster geometry. We have also attempted to induce a unidirectional pair-density correlation by imposing a static spin stripe in our DQMC simulations; however, we find that the resulting pair correlations remain remarkably bi-directional (C4 symmetric), even under the application of unreasonably large spin stripe fields (see the Supplementary Material). These observations provide strong evidence that the modulations in the d-wave pair correlations we observe at this temperature are unrelated to the fluctuating spin and charge stripes.
ways in which the stripe fluctuations couple to superconducting correlations in the model. Our results suggest that fluctuating and static stripe orders are decoupled from the real-space pairing correlations at the temperatures accessible to our simulations and that the two phenomena are intertwined through a more indirect means.

Acknowledgements — We thank T. P. Devereaux, E. Huang, B. Moritz, and D. J. Scalapino for useful discussions. This work was supported by the Scientific Discovery through Advanced Computing (SciDAC) program funded by the U.S. Department of Energy, Office of Science, Advanced Scientific Computing Research and Basic Energy Sciences, Division of Materials Sciences and Engineering. This research used resources of the Oak Ridge Leadership Computing Facility, which is a DOE Office of Science User Facility supported under Contract DE-AC05-00OR22725. This manuscript has been authored by UT-Battelle, LLC under Contract No. DE-AC05-00OR22725 with the U.S. Department of Energy. The United States Government retains a non-exclusive, paid-up, irrevocable, world-wide license to publish or reproduce the published form of this manuscript, or allow others to do so, for United States Government purposes. The Department of Energy will provide public access to these results of federally sponsored research in accordance with the DOE Public Access Plan (http://energy.gov/downloads/doe-public-access-plan)

[1] E. Dagotto, Science 309, 257 (2005).
[2] J. M. Tranquada, B. J. Sternlieb, J. D. Axe, Y. Nakamura, and S. Uchida, Nature 375, 561 (1995).
[3] M. Fujita, H. Goka, K. Yamada, and M. Matsuda, Phys. Rev. Lett. 88, 167008 (2002).
[4] P. Abbamonte, A. Rusydi, S. Smadici, G. D. Gu, G. A. Sawatzky, and D. L. Feng, Nat. Phys. 1, 155 (2005).
[5] P. Choubey, S. H. Joo, K. Fujita, Z. Du, S. D. Edkins, M. H. Hamidian, H. Eisaki, S. Uchida, A. P. Mackenzie, J. Lee, J. C. S. Davis, and P. J. Hirschfeld, Proc. Natl. Acad. Sci. 117, 14805 (2020), https://www.pnas.org/content/117/26/14805.full.pdf.
[6] E. Fradkin, S. A. Kivelson, and J. M. Tranquada, Rev. Mod. Phys. 87, 457 (2015).
[7] J. Zaanen and O. Gunnarsson, Phys. Rev. B 40, 7391 (1989).
[8] B.-X. Zheng, C.-M. Chung, P. Corboz, G. Ehlers, M.-P. Qin, R. M. Noack, H. Shi, S. R. White, S. Zhang, and G. K.-L. Chan, Science 358, 1155 (2017).
[9] S. R. White and D. J. Scalapino, Phys. Rev. Lett. 91, 136403 (2003).
[10] H.-C. Jiang and T. P. Devereaux, Science 365, 1424 (2019).
[11] Y.-F. Jiang, J. Zaanen, T. P. Devereaux, and H.-C. Jiang, Phys. Rev. Research 2, 033073 (2020).
[12] B.-X. Zheng and G. K.-L. Chan, Phys. Rev. B 93, 035126 (2016).
[13] E. W. Huang, C. B. Mendel, H.-C. Jiang, B. Moritz, and T. P. Devereaux, npj Quantum Mater. 3, 22 (2018).
[14] E. W. Huang, C. B. Mendel, S. Liu, S. Johnston, H.-C. Jiang, B. Moritz, and T. P. Devereaux, Science 358, 1161 (2017).
[15] K. Ido, T. Ohgoe, and M. Imada, Phys. Rev. B 97, 045138 (2018).
[16] S. Sorella, “The phase diagram of the hubbard model by variational auxiliary field quantum monte carlo,” (2021), arXiv:2101.07045 [cond-mat.str-el].
[17] P. Corboz, T. M. Rice, and M. Troyer, Phys. Rev. Lett. 113, 046402 (2014).
[18] T. A. Maier, M. Jarrell, T. C. Schulthess, P. R. C. Kent, and J. B. White, Phys. Rev. Lett. 95, 237001 (2005).
[19] T. A. Maier, M. S. Jarrell, and D. J. Scalapino, Phys. Rev. Lett. 96, 047005 (2006).
[20] E. Gull, P. Werner, O. Parcollet, and M. Troyer, Europhys. Lett. 82, 57003 (2008).
[21] G. Sordi, P. Sémon, K. Haule, and A.-M. S. Tremblay, Phys. Rev. Lett. 108, 216401 (2012).
[22] G. Kotliar, S. Savrasov, G. Pállson, and G. Biroli, Phys. Rev. Lett. 87, 186401 (2001).
[23] M. Jarrell, T. Maier, C. Huscroft, and S. Moukouri, Phys. Rev. B 64, 195130 (2001).
[24] T. Maier, M. Jarrell, T. Pruschke, and M. H. Hettler, Rev. Mod. Phys. 77, 1027 (2005).
[25] M. Fleck, A. Lichtenstein, and E. Pavarini, Phys. Rev. Lett. 84, 4962 (2000).
[26] T. I. Vanhala and P. Törmä, Phys. Rev. B 97, 1 (2018), 1708.06749.
[27] S. S. Dash and D. Sénéchal, Phys. Rev. B 103, 1 (2021), 2008.08661.
[28] U. R. Hähner, G. Alvarez, T. A. Maier, R. Solcà, P. Staar, M. S. Summers, and T. C. Schulthess, Comput. Phys. Commun. 246, 106709 (2020).
[29] R. Blankenbecler, D. J. Scalapino, and R. L. Sugar, Phys. Rev. D 24, 2278 (1981).
[30] S. R. White, D. J. Scalapino, R. L. Sugar, E. Y. Loh, J. E. Gubernatis, and R. T. Scalettar, Phys. Rev. B 40, 506 (1989).
[31] P. Mai, G. Balduzzi, S. Johnston, and T. Maier, npj Quantum Mater. 6, 26 (2021).
[32] C.-M. Chung, M. Qin, S. Zhang, U. Schollwöck, and S. R. White (The Simons Collaboration on the Many-Electron Problem), Phys. Rev. B 102, 041106 (2020).
[33] D. F. Agterberg, J. S. Davis, S. D. Edkins, E. Fradkin, D. J. Van Harlingen, S. A. Kivelson, P. A. Lee, L. Radzihovsky, J. M. Tranquada, and Y. Wang, Annu. Rev. Condens. Matter Phys. 11, 231 (2020).
Fluctuating spin and charge stripes in the two-dimensional Hubbard model in the thermodynamic limit – Supplementary Material

Peizhi Mai\textsuperscript{1}, Seher Karakuzu\textsuperscript{1}, Giovanni Balduzzi\textsuperscript{2}, Steven Johnston\textsuperscript{3}, and Thomas A. Maier\textsuperscript{1,*}
\textsuperscript{1}Computational Sciences and Engineering Division, Oak Ridge National Laboratory, Oak Ridge, TN, 37831-6494, USA.
\textsuperscript{2}Institute for Theoretical Physics, ETH Zurich, 8093 Zurich, Switzerland.
\textsuperscript{3}Department of Physics and Astronomy, University of Tennessee, Knoxville, Tennessee 37996-1200, USA. and
\textsuperscript{*}To whom correspondence should be addressed; E-mail: maierta@ornl.gov.
(Dated: June 4, 2021)

THE SPIN STRIPE AT DIFFERENT CLUSTER SIZE

Figure 1 of the main text shows the spin stripe patterns calculated on 16 × 4 clusters embedded in the DCA mean field. Fig. S1 presents similar results obtained on smaller 8 × 4 and 8 × 6 clusters, which both show indications of spin stripe correlations. For example, the correlations observed in the 8 × 4 case resemble the ones found in the corresponding region of the 16 × 4 case (Fig. 1A). Similarly, the correlations observed in the 8 × 6 case are similar to the corresponding correlations found in the 8 × 8 case shown in Fig. 4A.

**FIG. S1**: The real-space equal-time staggered spin-spin correlation function of the singleband Hubbard model calculated on an 8 × 4 cluster (**A**) and an 8 × 6 cluster (**B**) at $t' = -0.3t$, $\langle n \rangle = 0.8$, and $T = 0.2t$ ($\beta = 5/t$). Both clusters show a spin stripe pattern.

THE EFFECTS OF DIFFERENT BOUNDARY CONDITIONS AND MEAN FIELDS ON THE SPIN CORRELATIONS

In this section, we explore the effect of different boundary conditions and the application of different mean fields on the spin correlations. Fig. S2 A presents the dynamical spin structure factor at zero frequency, calculated on a finite-size cluster using continuous-time auxiliary field quantum Monte-Carlo. In this case, we have not coupled the system to a mean-field and applied different boundary conditions to the cluster. Although translational symmetry is
FIG. S2: The dynamical spin structure factor at zero frequency $S(Q, \omega = 0)$ for a $16 \times 4$ cluster with different boundary conditions. Results are shown for a cluster (A) without a mean field and (B) embedded in a mean field. Here, “pbc” means periodic boundary condition in all directions; “obc” means open boundary condition in all directions; “open x” means obc along the $x$-axis and pbc along the $y$-axis; likewise, “open y” means obc along the $y$-axis and pbc along the $x$-axis. Finally, “MF$_y$” means that we have applied the DCA mean field only along the $y$-axis but not along the $x$-axis. The model parameters in both panels are $t' = -0.3t$, $\langle n \rangle = 0.8$, and $T = 0.22t$ ($\beta = 4.5/t$).

broken when the boundary is open in one direction, we assume that the off-diagonal $(k,k')$ elements are negligible and keep only the diagonal $(k,k)$ elements after Fourier transform. We observe that opening the boundary along the $y$-axis on a $16 \times 4$ cluster suppresses the spin stripe correlations while opening the boundary along the $x$-axis only has a marginal effect on our results.

Figure S2B explores the effect of the DCA mean-field and the effect of the boundary condition in the presence of the DCA mean-field. Here, a comparison between the blue and orange curves shows that the additional inclusion of the mean field along the $x$-axis in the full DCA result slightly weakens the stripe correlations. Comparing the green and orange curves indicates that opening the left and right boundaries while keeping periodic boundary conditions and the mean-field along the $y$-axis enhances the stripe correlations.

Figure S3 explores the effect of the mean field further. Panels A and (B,C) show $S_{\text{strain}}(r)$ for $t' = -0.25t$, $T = 0.22t$ ($\beta = 4.5/t$) with and without the mean field, respectively, on a cluster with periodic boundary condition. We can see that turning on the mean field weakens the stripe pattern, as indicated by the lighter red region, and expands the central blue region. Panel B and C are from finite-cluster CT-AUX (short for continuous time auxiliary field quantum Monte-carlo, the cluster solver for DCA) and DQMC calculations, respectively. Except for the time-discretization error in DQMC (CT-AUX does not have this error), both methods are numerically exact for a finite cluster. Therefore, the slight difference in the respective results can be attributed to the discrete-time error in DQMC. In panel D, the dip around $Q = (\pi, \pi)$ in the finite-cluster CT-AUX and DQMC curves also signals a stronger stripe correlation than the flat peak in the DCA curve.
To show how the spin and charge stripe form as the system cools down, we plot the zero-frequency dynamical spin and charge structure factors at different temperatures and $t' = -0.3t$ in Fig. S4. As $T$ decreases, $S(Q, \omega = 0)$ (panels A-D) gradually develops two peaks located at $(\pi, 0)$ and $(\pi + \delta_c, 0)$. This position is almost independent of temperature. A double peak structure also forms in $N(Q, \omega = 0)$ (panels E-H) as temperatures is lowered, this time centered at $(0, 0)$ and $(0 + \delta_c, 0)$. At high-temperature, $\delta_c = \pi$; however, as the temperature decreases, $\delta_c$ is reduced and the two peaks

| Temperature | $\beta$ | $r$ | Spin Correlation | Charge Correlation |
|-------------|---------|-----|------------------|-------------------|
| A           | 6       | 0.527 | [Graph]          |                   |
| B           | 5       | 0.542 | [Graph]          |                   |
| C           | 4       | 0.493 | [Graph]          |                   |
| D           | 3       | 0.436 | [Graph]          |                   |

**FIG. S4:** DCA results for the zero-frequency dynamical spin $S(Q, \omega = 0)$ ( Panels A-D) and charge $N(Q, \omega = 0)$ ( Panels E-H) structure factors, obtained on a 16 x 4 cluster embedded in a dynamical mean field. The top row shows the spin structure factors along $Q = (Q_x, \pi)$ for A $\beta = 6/t$, B $\beta = 5/t$, C $\beta = 4/t$, and D $\beta = 3/t$. Each spectrum is fit with a pair of Lorentzian functions centered at $(\pi, \pi)$ plus a constant background. The bottom row shows the corresponding charge structure factors along $Q = (Q_x, 0)$. Each spectrum is fit with a pair of Lorentzian functions centered at $Q = (\pm \delta_c, 0)$ plus a constant background. All results were obtained for $t' = -0.3t$ and $\langle n \rangle = 0.8$. 

**FIG. S3:** The real-space equal-time staggered spin-spin correlation function in a 16 x 4 cluster (A) embedded in a mean field and (B,C) without a mean field. Panel B and C shows the CT-AUX and DQMC results, respectively. Panel D presents the dynamical structure factor for all cases. The model parameters in both panels are $t' = -0.3t$, $\langle n \rangle = 0.8$, and $T = 0.22t$ ($\beta = 4.5/t$).
move towards each other until $\delta_c \approx 2\delta_s$ at the lowest temperatures. The measured $\delta_s$ and $\delta_c$ as a function of $T$ are presented in Fig. 3K of the main text.

**IMPOSING A SPIN STRIPE FIELD AND ITS IMPACT ON THE PAIRING CORRELATIONS**

To test the relationship between the stripe correlations and the pairing modulations, we imposed a unidirectional spin stripe by introducing a site- and spin-dependent site energy

$$\epsilon_{i,\sigma} = \pm A_{sf} \cos(\pi i_x / 4)(-1)^{i_x + i_y},$$  \hspace{1cm} (S1)

where the + and - signs correspond to the spin $\sigma = \uparrow$ and $\sigma = \downarrow$, respectively. We then solved 8x8 clusters using DQMC for $t' = -0.25t$, $T = 0.22t$ ($\beta = 4.5/t$), and $\langle n \rangle = 0.8$. Figs. S5A-C, D-F, and G-I show the resulting staggered AFM spin-spin, density-density, and pair-field correlation functions, respectively, for increasing values of $A_{sf}$. For $A_{sf} = 0$, we observe the same bidirectional fluctuating stripe and pair-density correlations observed with DCA (see Fig. 4). When $A_{sf} \neq 0$, a strong static spin stripe forms (Figs. S5B, C) while the charge correlations (Figs. S5E, F) are only slightly modified. At the same time, the modulations in the pairing correlations become somewhat unidirectional (Figs. S5H, I), with the pair-density modulations becoming stronger in the direction perpendicular to the spin stripes;
however, the degree of $C_4$ symmetry breaking is nowhere near as strong as the spin-spin correlations. We, therefore, conclude that the stripe and pair-density correlations are only weakly coupled at these temperatures.

**OUR CUSTOMIZED COLOR BAR**

The color bars we used throughout this paper are derived from a customized color map defined to highlight the differences in the correlations at small values. Our definition for this map is the same as the one used in Refs. [?] and [?]. It is defined using the RGB (Red, Green, Blue) scheme. First, we define a piecewise function: $f(x) = 1$ if $x < 0.5$; $f(x) = 1 - \sqrt{2x - 1}$ if $x > 0.5$, setting $x$ in the range $[0, 1]$. Then

$$
\begin{align*}
    r &= f(x), \\
    g &= \min(f(x), f(1 - x)), \\
    b &= f(1 - x),
\end{align*}
$$

(S2)

where $r$, $g$, $b$ stand for red, green, blue color, respectively.