Radiation of mixed layer near-inertial oscillations into the ocean interior

J. Moehlis\textsuperscript{1}, Stefan G. Llewellyn Smith\textsuperscript{2}

February 3, 2022

\textsuperscript{1}Department of Physics, University of California, Berkeley, CA 94720
\textsuperscript{2}Department of Mechanical and Aerospace Engineering, University of California, San Diego 9500 Gilman Drive, La Jolla, CA 92093-0411

Abstract

The radiation from the mixed layer into the interior of the ocean of near-inertial oscillations excited by a passing storm in the presence of the beta effect is reconsidered as an initial-value problem. Making use of the fact that the mixed layer depth is much smaller than the total depth of the ocean, the solution is obtained in the limit of an ocean that is effectively infinitely deep. For a uniform initial condition, analytical results for the velocity, horizontal kinetic energy density and fluxes are obtained. The resulting decay of near-inertial mixed layer energy in the presence of the beta effect occurs on a timescale similar to that observed.

1 Introduction

There is much observational evidence, starting with Webster (1968) and Pollard and Millard (1970), that storms can excite near-inertial currents in the mixed layer of the ocean. This phenomenon is evident in observations from the Ocean Storms Experiment (D’Asaro et al. 1995, Levine and Zervakis 1995, Qi et al. 1995). Simple models which treat the mixed layer as a solid slab have been quite successful at explaining the process by which wind generates such currents (see, e.g., Pollard and Millard (1970), D’Asaro (1985)). A weakness of the model of Pollard and Millard (1970) is that it explains the decay of these currents with an arbitrary decay constant. Much subsequent work has attempted to determine the detailed characteristics of this decay, with possible mechanisms including nonlinear interactions which transfer energy to other frequencies (Henyey et al. 1986), turbulent dissipation (Hebert and Moun 1993), and the radiation of downward propagating near-inertial oscillations (NIOs) excited
by inertial pumping into the interior of the ocean (Gill 1984). The downward radiation of NIOs will be the focus of this paper.

Observations give a timescale for the decay of the energy deposited by the passing storm on the order of ten to twenty days (D'Asaro et al. 1995, Levine and Zervakis 1995, Qi et al. 1995). This timescale stands in contrast with estimates such as that by Gill (1984) that near-inertial currents decaying through the downward propagation of NIOs and with a horizontal length scale typical of the atmospheric forcing mechanism can remain in the mixed layer for longer than a year. To account for this difference, several mechanisms for the enhancement of vertical propagation of NIOs have been suggested. D'Asaro (1989) demonstrated that the \( \beta \)-effect causes a reduction of horizontal scales because the meridional wavenumber evolves according to \( l = l_0 - \beta t \), where \( l_0 \) is the initial wavenumber, and \( l < 0 \) corresponds to southward propagation; this accelerates the rate of inertial pumping of energy out of the mixed layer, thereby enhancing the decay. The decay is also enhanced through interaction with background geostrophic or quasigeostrophic flow (e.g. Balmforth et al. 1998, Balmforth and Young 1999, and van Meurs 1998).

This paper reconsiders the vertical propagation of near-inertial energy deposited into the mixed layer by a storm, in the presence of the \( \beta \)-effect, using a different approach from that of D'Asaro (1989). The analysis uses the formalism of Young and Ben Jelloul (1997) which is outlined in Section 2. In Section 3, a simplified model with three main assumptions is presented. First, the background flow is assumed to be constant in the zonal direction (i.e. independent of longitude with zero vorticity). Second, the buoyancy frequency is taken to be small in the mixed layer, and constant in the ocean interior (i.e. beneath the mixed layer). Third, it is assumed that the storm has moved very rapidly across the ocean and has created a horizontally uniform near-inertial current to the east concentrated within the mixed layer: it is the subsequent evolution of this motion that is examined. Section 4 uses the fact that the depth of the ocean is very much larger than the mixed layer depth to formulate and solve the model for an ocean which is effectively infinitely deep. Section 5 discusses the results and suggests directions for further investigation.

2 The NIO equation

We consider an ocean of infinite horizontal extent and depth \( D \), with the mixed layer comprising the region \(-H_{\text{mix}} < z < 0\), and the rest of the water column occupying \(-D < z < -H_{\text{mix}}\). The \( x \) and \( y \) axes are taken to point to the east and north, respectively. The buoyancy frequency \( N = N(z) \) is an arbitrary piecewise continuous function of depth \( z \).

Young and Ben Jelloul (1997) derive an evolution equation for a complex field \( A(x, y, z, t) \) which governs leading-order NIO motion in the presence of a steady barotropic background
flow and the $\beta$-effect:

$$LA_t + \frac{\partial(\psi, LA)}{\partial(x,y)} + i \frac{f_0}{2} \nabla^2 A + i \left( \beta y + \frac{1}{2} \zeta \right) LA = 0,$$

(1)

where

$$LA = \frac{\partial}{\partial z} \left( \frac{f_0^2}{N^2} \frac{\partial A}{\partial z} \right),$$

(2)

$\psi$ is the streamfunction for the background flow, $\zeta \equiv \nabla^2 \psi$ is the associated vorticity, and the Coriolis parameter is $f = f_0 + \beta y$. Here $\nabla$ is the horizontal gradient, and $\nabla^2 = \partial^2_x + \partial^2_y$. Subscripts denote partial differentiation. The NIO velocity field $(u, v, w)$, buoyancy $b$, and pressure $p$ are given by

$$u + iv = e^{-if_0 t} LA,$$

$$w = -\frac{1}{2} f_0^2 N^{-2} (A_{xz} - iA_{yz}) e^{-if_0 t} + c.c.,$$

$$b = \frac{i}{2} f_0 (A_{xz} - iA_{yz}) e^{-if_0 t} + c.c.,$$

$$p = \frac{i}{2} (A_x - iA_y) e^{-if_0 t} + c.c.$$

The buoyancy $b$ is related to the density $\rho$ by

$$\rho = \rho_0 \left[ 1 - \frac{1}{g} \int_0^z N^2(z') dz' - \frac{b}{g} \right],$$

where $\rho_0$ is the reference density at the top of the ocean. The pressure $p$ has been normalized by $\rho_0$.

The boundary conditions are that $A_z = 0$ at $z = 0$ and $z = -D$. This ensures that $w$ vanishes at the top and bottom of the ocean. Using these boundary conditions,

$$\int_{-D}^0 (u + iv) = 0.$$

(3)

Thus barotropic motion is not included in the analysis. However Gill (1984) has shown that the barotropic response to a storm is instantaneous and the associated currents are weak.

### 3 A Simplified Model

To simplify the analysis, we assume that $A$ and $\psi$ do not vary in the $x$-direction, and that $\zeta = 0$. The analysis thus neglects the effect of background barotropic vorticity but crucially keeps the $\beta$-effect. The buoyancy frequency profile is taken to be

$$N^2 = \epsilon^2 N_0^2, \quad -H_{mix} < z < 0,$$

$$N^2 = N_0^2, \quad -D < z < -H_{mix},$$

$$N^2 = N_0^2, \quad -H_{mix} < z < 0.$$
where $\epsilon \ll 1$. Finally, the storm is assumed to have produced an initial condition of a horizontally uniform near-inertial current to the east concentrated within the mixed layer. Instead of approaching this problem by use of an integral operator as in D’Asaro (1989) or by projecting onto normal modes (e.g., Gill 1984, Balmforth et al. 1998), the problem will be formulated as an initial value problem on a semi-infinite domain corresponding to an ocean that is effectively infinitely deep. In order to formulate the problem properly for this limit, this section considers an ocean of finite depth. In Section 4 the solution in the limit that the depth of the interior is much greater than the mixed layer depth will be found.

This formulation as a radiation problem which ignores the presence of the ocean bottom requires the projection of the initial condition to be spread across all the normal modes. This is certainly true for small mixed layer depths in the model of Gill (1984), as shown in Table 1 of that paper; also see Table 1 of Zervakis and Levin (1995). For deeper mixed layers, this is no longer true since half the initial energy becomes concentrated in the first two or three modes. However, as pointed in Section 7 of Gill (1984), the depth of the ocean “influences the rate of loss of energy by imposing modulations on the rate, but the average rate of loss is not affected very much by depth changes”. Hence the results presented here should be qualitatively relevant even when the continuum assumption is not valid.

### 3.1 Nondimensionalization

Quantities are nondimensionalized according to

\[ \hat{y} = y/Y, \quad \hat{z} = 1 + z/H_{\text{mix}}, \quad \hat{t} = \Omega t, \quad \hat{N} = N/N_0, \]

where

\[ Y \equiv \left( \frac{H_{\text{mix}}^2 N_0^2}{\beta f_0} \right)^{1/3}, \quad \Omega \equiv \left( \frac{\beta^2 H_{\text{mix}}^2 N_0^2}{f_0} \right)^{1/3}. \]

Typical values $\beta = 10^{-11} \text{ m}^{-1} \text{ s}^{-1}$, $H_{\text{mix}} = 100 \text{ m}$, $f_0 = 10^{-4} \text{ s}^{-1}$, $N_0 = 10^{-2} \text{ s}^{-1}$ give $Y = 10^5$ m and $\Omega = 10^{-6} \text{ s}^{-1}$. The relevant timescale is thus $\Omega^{-1} = 11.5 \text{ days}$. Also, the velocity and the field $A$ are nondimensionalized by

\[ (\hat{u}, \hat{v}) = \left( \frac{u, v}{U} \right), \quad \hat{A} = \frac{f_0^2}{UN_0^2 H_{\text{mix}}^2} A, \]

where $U$ is a characteristic value of the initial velocity.

The hats are now dropped for ease of notation. With this nondimensionalization, the buoyancy frequency profile is

\[ N^2 = \epsilon^2, \quad 0 < z < 1, \]

\[ N^2 = 1, \quad -H \equiv 1 - D/H_{\text{mix}} < z < 0, \]

\[ 4 \]

and the NIO equation (1), the boundary conditions, and initial condition become

\[ A_{zzt} + \frac{i}{2} N^2 A_{yy} + i y A_{zz} = 0, \]  
\[ A_z = 0, \quad z = -H, \quad z = 1, \]  
\[ A_{zz} = N^2 (u + iv), \quad t = 0. \]  

The requirement that \( u \) and \( v \) remain finite imply the jump conditions

\[ A_z|_{z=0^+} = \epsilon^2 A_z|_{z=0^-}, \quad A_{yy}|_{z=0^+} = A_{yy}|_{z=0^-}, \]  

where \( z = 0^+ \) and \( z = 0^- \) are the limits as \( z \to 0 \) from positive and negative \( z \) values, respectively.

This nondimensionalization allows some immediate conclusions to be drawn about the propagation of NIO energy downwards. Most importantly, if \( H_{\text{mix}} \) increases, then the timescale \( \Omega^{-1} \) decreases. Thus, assuming that the storm causes a uniform near-inertial current throughout the whole mixed layer, energy transfer will be faster for a deeper mixed layer. This confirms the results of Gill (1984), which associated the more efficient transfer with a larger projection of the initial velocity profile on the first vertical mode.

### 3.2 Boundary Condition at the Base of the Mixed Layer

Expanding \( A(y, z, t) = A_0(y, z, t) + \epsilon^2 A_2(y, z, t) + \mathcal{O}(\epsilon^4) \) for \( 0 < z < 1 \), (4) becomes at \( \mathcal{O}(\epsilon^0) \)

\[ A_{0zzt} + iy A_{0zz} = 0. \]

Integrating this subject to the boundary condition that \( A_z \) and thus \( A_{0z} \) vanishes at \( z = 1 \) implies that \( A_0 \) is independent of \( z \). At \( \mathcal{O}(\epsilon^2) \),

\[ A_{2zzt} + iy A_{2zz} + \frac{i}{2} A_{0yy} = 0, \]  

which may be integrated subject to the boundary condition that \( A_{2z} \) vanishes at \( z = 1 \) to give

\[ A_{2zt} + iy A_{2z} + \frac{i}{2} A_{0yy}(z - 1) = 0. \]

Evaluating at \( z = 0^+ \) and using \( A_{yy} = A_{0yy} + \mathcal{O}(\epsilon^2) \) and \( A_z = \epsilon^2 A_{2z} + \mathcal{O}(\epsilon^4) \),

\[ A_{zt} + iy A_z - \frac{i \epsilon^2}{2} A_{yy} = \mathcal{O}(\epsilon^4), \quad z = 0^+. \]

Finally, applying (7) gives the upper boundary condition for the NIO field in the ocean interior to leading order in \( \epsilon \):

\[ A_{zt} + iy A_z - \frac{i}{2} A_{yy} = 0 \quad z = 0^- . \]
Results obtained in the ocean interior using (9) are in fact leading-order solutions. We shall continue to use the notation $A$, even though it is really the leading-order term in the expansion.

### 3.3 Initial Condition

Suppose that in a short time compared with the NIO wave propagation time, the passing storm induces near-inertial currents in the mixed layer with a horizontal scale that is much larger than the one under consideration, and which can hence be taken to be uniform. For simplicity, the initial velocity (consistent with equation (3)) is assumed to be piecewise constant with depth:

\[
(u, v) = (1, 0) \quad 0 < z < 1,
\]
\[
= (-H^{-1}, 0), \quad -H < z < 0.
\]

The weak flow in the ocean interior is necessary to ensure that the flow has no barotropic component. Integrating equation (6) with respect to $z$ and using the boundary conditions (5) gives at $t = 0$

\[
A_z = \epsilon^2(z - 1), \quad 0 < z < 1, \quad (10)
\]
\[
A_z = -(z + H)/H, \quad -H < z < 0. \quad (11)
\]

### 4 Solution for an Infinitely Deep Ocean

The total depth of the ocean is typically on the order of a hundred times the depth of the mixed layer. Thus, the limit of infinite depth is considered. The initial condition is taken to be equation (11) with $H \to \infty$. The boundary condition for $z \to -\infty$ is taken to be $A_{zz} \to 0$, corresponding to the near-inertial velocities vanishing at infinite depth. Of course, this limit excludes the possibility of reflections off the bottom of the ocean which may be important. Finally, the boundary condition for $z = 0^-$ given by equation (5) is used. Hence the problem to be solved for the semi-infinite domain $z < 0$ becomes

\[
A_{ztt} + \frac{i}{2} A_{yy} + iy A_{zz} = 0, \quad z < 0,
\]
\[
A_{zt} + iy A_z - \frac{i}{2} A_{yy} = 0, \quad z = 0^-,
\]
\[
A_{zz} \to 0, \quad z \to -\infty,
\]
\[
A_z = -1, \quad t = 0.
\]
4.1 NIO velocity field

These equations may be solved using Laplace transforms. Here we present only the major
results; further details are given in Moehlis (1999). We make the transformations
\[ A(y, z, t) = e^{-iyt} \tilde{B}(z, T), \quad T \equiv t^3/3, \quad \alpha \equiv (1 + i)/2 \]
and define the Laplace transform of \( \tilde{B} \) by
\[ b(z, p) \equiv \mathcal{L}[\tilde{B}] \equiv \int_0^\infty \tilde{B}(z, T)e^{-pT}dT. \]
(12)

Then
\[ b(z, p) = -\frac{1}{\alpha} \frac{1}{\sqrt{p} + \alpha} \exp \left( \frac{\alpha z}{\sqrt{p}} \right). \]
(13)
This Laplace transform and its derivatives with respect to \( z \) must be inverted numerically
for the ocean interior (\( z < 0 \)). For the top of the ocean interior (\( z = 0^- \)) however, they may
be obtained in closed form. For example,
\[ A_{zz}(y, 0^-, t) = e^{-iyt} \left[ e^{it^3/6} \text{erfc} \left( \frac{1 + i}{2\sqrt{3}} t^{3/2} \right) - 1 \right]. \]
(14)

We now consider the back-rotated velocity \( A_{zz} = e^{if_0 t}(u + iv) \), which filters out purely
inertial motion at frequency \( f_0 \). Back-rotated velocities may be represented by hodographs
which show the vector \( (\text{Re}(A_{zz}), \text{Im}(A_{zz})) \) as curves parametrized by time. For \( f_0 > 0 \),
if these curves are traced out in a clockwise (counterclockwise) fashion, the corresponding
motion has frequency larger (smaller) than \( f_0 \). Figure 1 shows the back-rotated velocity
at different locations. A common characteristic is that the magnitude of the back-rotated
velocity starts at zero, reaches a peak value shortly after the storm, then decays away. The
depth dependence of the back-rotated velocity is seen by comparing Figure 1 (a) and (b),
where both have \( y = 0 \) and thus the same value of the Coriolis parameter \( f \). Qualitatively the
results are the same, but closer to the mixed layer the direction change of the back-rotated
velocity becomes slower, meaning that the frequency is closer to \( f_0 \). An idea of the latitudinal
dependence is seen by comparing Figure 1 (a,c,d): at \( y = 1 \) the hodograph is traced out in a
clockwise fashion as for \( y = 0 \), but at \( y = -2 \) it is traced out in a counterclockwise fashion.

4.2 Kinetic energy density and fluxes

The horizontal kinetic energy (HKE) per unit area contained within the mixed layer is
\[ \int_0^1 dz \left| \frac{A_{zz}}{N^2} \right|^2 \equiv \int_0^1 dz \left| \frac{A_{zz}}{\epsilon^2} \right|^2 = \int_0^1 dz |A_{zz}|^2. \]
Expanding \( \tilde{B}(z, T) = \tilde{B}_0(z, T) + \epsilon^2 \tilde{B}_2(z, T) + O(\epsilon^4) \) in the mixed layer, (8) may be used to show that
\[ pb_{2zz} - \tilde{B}_{2zz}(z, 0) - \frac{i}{2} b_0 = 0, \]
(15)
Figure 1: Back-rotated velocity for (a) \( z = -1, y = 0 \), (b) \( z = -0.5, y = 0 \), (c) \( z = -1, y = 1 \), and (d) \( z = -1, y = -2 \). The diamonds are drawn at \( t = 0, 5, 10, 15, 20 \).
where \( b_2 = \mathcal{L}[\tilde{B}_2] \) and \( b_0 = \mathcal{L}[\tilde{B}_0] \). The initial condition within the mixed layer is \( \tilde{B}_{2zz}(z, 0) = 1 \). Now \( A \) is continuous across \( z = 0 \), and \( \tilde{B}_0 \) is independent of \( z \) (see Section 3.2). Hence

\[
b_{2zz} = \frac{1}{p} - \frac{i}{2\alpha p} \frac{1}{\sqrt{p + \alpha}},
\]

which may be inverted to give

\[
A_{2zz}(y, t) = e^{-iyt}e^{\alpha^2i^3/3} \text{erfc} \left( \frac{\alpha}{\sqrt{3} t^{3/2}} \right).
\]

Therefore the HKE within the mixed layer is

\[
e_{\text{ML}} \equiv \left| \text{erfc} \left( \frac{1+i}{2\sqrt{3}} t^{3/2} \right) \right|^2.
\]

The time dependence of \( e_{\text{ML}} \) is shown in Figure 2. Asymptotic results from Abramowitz and Stegun (1972) for the complementary error function imply that

\[
e_{\text{ML}} \sim \left\{ \begin{array}{ll}
1 - \frac{2}{\sqrt{3\pi}} t^{3/2}, & t \ll 1, \\
\frac{6}{\pi t^3}, & t \to \infty.
\end{array} \right.
\]
Since the energy which leaves the mixed layer enters the interior of the ocean, this implies that for short times the energy in the interior increases like $t^{3/2}$. This does not contradict the result from D’Asaro (1989) that for short times the thermocline energy grows like $t^6$. That result assumes that the wind persists to generate a constant inertially oscillating velocity, and that there is no propagating inertial motion. Here, the wind has an instantaneous effect, causing an initial horizontally uniform inertial current, and propagating inertial motion is included fully.

Another quantity of interest is the flux of HKE. Using (4) and its complex conjugate gives

$$\frac{\partial}{\partial t} \text{HKE} = \frac{\partial}{\partial t} \left| A_{zz} \right|^2 = \frac{i}{2N^2} \frac{\partial}{\partial y} (A_{zz}^* A_y - A_{zz} A_y^*) + \frac{i}{2N^2} \frac{\partial}{\partial z} (A_{yz}^* A_y - A_{yz} A_y^*). \tag{17}$$

Assuming $A_{zz} A_y^* - A_{zz}^* A_y$ vanishes for $|y| \to \infty$ and using equation (5),

$$\frac{d}{dt} \int_{-H}^{-d} dz \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy |A_{zz}|^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_E(y,t; d) dx dy, \tag{18}$$

where

$$F_E(y,t; d) \equiv \frac{i}{2} (A_{yz}^* A_y - A_{yz} A_y^*)|_{z=-d} \tag{19}$$

gives the flux of HKE from the region $z > -d$ to the region $z < -d$. For this model, we consider the flux per unit area. Integrating (18) with respect to time shows that the quantity

$$E(t; d) \equiv \int_0^t F_E dt$$

gives the total amount of HKE which has penetrated into the region $z < -d$. Note that $E(t; d) \to 1$ corresponds to all the energy originally in the mixed layer having reached depths below $z = -d$. Results for $F_E(t; d)$ and $E(t; d)$ obtained by numerically inverting the appropriate Laplace transforms are shown in Figure 3. $F_E$ peaks at the nondimensionalized time $t \approx 0.62$; for the typical values quoted in Section 3.1, this corresponds to about a week after the storm. From Figure 3(b) and using the fact that whatever energy flows through $z = 0^-$ must have initially been in the mixed layer, we see that by $t = 1$ (about 11.5 days after the storm) nearly half of the energy associated with horizontal NIO currents caused by the storm has left the mixed layer; however, only about 38% of the total energy has penetrated below $z = -1$. By $t = 2$ (about 23 days after the storm), 82% of the total energy has left the mixed layer, but only 58% has penetrated below $z = -1$. Thus, at $t = 2$ nearly a quarter of the total energy is contained in the distance $H_{\text{mix}}$ immediately beneath the mixed layer. This is reminiscent of the accumulation of NIO energy below the mixed layer seen in Balmforth, Llewellyn Smith and Young (1998). This model thus gives reasonable estimates for the timescale for which the decay of NIO energy occurs: for example, D’Asaro et al.
Figure 3: (a) $F_E(t;d)$ and (b) $E(t;d)$ for different depths $d$ below the base of the mixed layer. These show instantaneous and time-integrated fluxes of HKE.

(1995) found that the mixed layer inertial energy was reduced to background levels by 21 days after the storm.

Figure 4 shows the vertical dependence of the HKE and $F_E$ at different times. As time increases the instantaneous distribution of HKE becomes more sharply peaked near the base of the mixed layer, but remains bounded (asymptotically approaching unity) because of energy conservation.

4.3 Large-time behavior

The asymptotic behavior of near-inertial properties may be derived using the method of steepest descents (see Moehlis 1999 for details). This shows that in the limit of large $\xi \equiv z^{2/3}t$, and along the “rays” $z = -\eta_0^3 t^3$, $u^2 + v^2 \sim \frac{2}{(1 + \eta_0^3) \pi \eta_0^3 t^3}$, $F_E \sim \frac{2\eta_0}{\pi(1 + \eta_0^3) t}$. $\eta_0$

A useful way to represent the asymptotic results is to write $\eta_0$ in terms of $z$ and $t$ and then draw contour plots of quantities of physical interest in the $(z,t)$ plane: this is shown in Figure 5. In the asymptotic limit for large $\xi$, with $z$ constant, $u^2 + v^2$ and $F_E$ decrease as time increases. Note that $\xi$ is large for sufficiently large $z$ and/or $t$.

Finally, Moehlis (1999) also obtained results for the vertical shear $u_z^2 + v_z^2$. To leading order in $\epsilon$, the vertical shear within the mixed layer is zero. The results for vertical shear for
Figure 4: Vertical profiles of (a) $u^2 + v^2$ and (b) $F_E(t, |z|)$ at $y = 0$ for different times showing the decay of energy from the mixed layer ($0 < z < 1$) and resultant behavior in the interior ($z < 0$). Note the different vertical scales.

the interior of the ocean lack physical realism because the model allows the shear to grow forever as a consequence of the initial infinite shear due to the discontinuity in the initial velocity profile.

5 Conclusion

A simplified model has been developed to examine the decay due to the $\beta$-effect of near-inertial currents excited in the mixed layer by a passing storm. This decay occurs due to the radiation of downward propagating NIOs into the interior of the ocean. The main assumptions of the model are that the background flow does not vary in the longitudinal direction and has no associated vorticity, that the ocean has a simple (piecewise constant) buoyancy frequency profile, and that the storm has moved very quickly over the ocean causing a horizontally uniform near-inertial current concentrated in the mixed layer. The $\beta$-effect is included in the analysis and is responsible for the radiation of NIOs. Because the depth of the mixed layer is much smaller than the total depth of the ocean, the problem is formulated in the limit of an effectively infinitely deep ocean; the resultant initial value problem is solved by Laplace transforms. Analytical results are given for the horizontal kinetic energy density
in the mixed layer, and results from the numerical inversion of the appropriate Laplace transforms are given for horizontal kinetic energy, energy flux, and back-rotated velocity. The asymptotic behavior is also investigated.

Although this simplified model cannot be expected to capture the full complexity of the aftermath of a storm passing the ocean, it does capture much of the observed behavior. Most importantly, in the presence of the $\beta$-effect the decay of near-inertial mixed layer energy is found to occur on the appropriate timescale (approximately twenty days), which confirms the analysis of D’Asaro (1989) and observations by D’Asaro et al. (1995), Levine and Zervakis (1995), and Qi et al. (1995). The main advantage of the approach described in this paper is that many aspects of the decay in the mixed layer are analytically obtained for all times, unlike D’Asaro (1989) which predicts the timescale for the decay in a short time limit or estimates it in terms of the time it takes normal modes to become out of phase (cf. Gill 1984). Extensions to a more realistic ocean and storm would involve including a more realistic buoyancy frequency profile (for example, the profile used by Gill 1984), considering the effect of different initial velocities (including both horizontal and vertical structure), and considering the effect of background flow. The study of all of these could use the same formalism of Young and Ben Jelloul (1997) and an approach similar to that presented here.

Figure 5: Contour plots of the asymptotic results for (a) $u^2+v^2$ and (b) $F_E$. Darker shading corresponds to smaller values.
Acknowledgments

The majority of this work was carried out at the 1999 Geophysical Fluid Dynamics program at the Woods Hole Oceanographic Institution. The authors would particularly like to thank W. R. Young for many useful discussions regarding this work.

References

[1] Abramowitz, M. and Stegun, I. A. (1972) Handbook of Mathematical Functions, Wiley Interscience Publications, 1046 pp.

[2] Balmforth, N. J., Llewellyn Smith, S. G. and Young, W. R. (1998) Enhanced dispersion of near-inertial waves in an idealized geostrophic flow. J. Mar. Res., 56:1–40.

[3] Balmforth, N. J. and Young, W. R. (1999) Radiative damping of near-inertial oscillations in the mixed layer. J. Mar. Res., 57:561–584.

[4] D’Asaro, E. A. (1985) The energy flux from the wind to near-inertial motions in the surface mixed layer. J. Phys. Oceanogr., 15:1043–1059.

[5] D’Asaro, E. A. (1989) The decay of wind-forced mixed layer inertial oscillations due to the $\beta$ effect. J. Geophys. Res., 94:2045–2056.

[6] D’Asaro, E. A., Eriksen, C. C., Levine, M. D., Niiler, P., Paulson, C. A., and van Meurs, P. (1995) Upper-ocean inertial currents forced by a strong storm. Part I: Data and comparisons with linear theory. J. Phys. Oceanogr., 25:2909–2936.

[7] Garrett, C. (1999) What is the “near-inertial” band and why is it different? Unpublished manuscript.

[8] Gill, A. E. (1984) On the behavior of internal waves in the wakes of storms. J. Phys. Oceanogr., 14:1129–1151.

[9] Hebert, D. and Moum, J. N. (1993) Decay of a near-inertial wave. J. Phys. Oceanogr., 24:2334–2351.

[10] Henyey, F. S., Wright, J. A., and Flatté, S. M. (1986) Energy and action flow through the internal wave field: an eikonal approach. J. Geophys. Res., 91:8487–8495.

[11] Levine, M. D. and Zervakis, V. (1995) Near-inertial wave propagation into the pycnocline during ocean storms: observations and model comparison. J. Phys. Oceanogr., 25:2890–2908.
[12] Moehlis, J. (1999) Effect of a simple storm on a simple ocean, in *Stirring and Mixing, 1999 Summer Study Program in Geophysical Fluid Dynamics*, Woods Hole Oceanogr. Inst. Unpublished manuscript.

[13] Pollard, R. T. and Millard, R. C. Jr. (1970) Comparison between observed and simulated wind-generated inertial oscillations. *Deep-Sea Res.*, 17:813–821.

[14] Qi, H., De Szoeke, R. A., Paulson, C. A., and Eriksen, C. C. (1995) The structure of near-inertial waves during ocean storms. *J. Phys. Oceanogr.*, 25:2853–2871.

[15] van Meurs, P. (1998) Interactions between near-inertial mixed layer currents and the mesoscale: the importance of spatial variabilities in the vorticity field. *J. Phys. Oceanogr.*, 28:1363–1388.

[16] Webster, F. (1968) Observation of inertial-period motions in the deep sea. *Rev. Geophys.*, 6:473–490.

[17] Young, W. R. and Ben Jelloul, M. (1997) Propagation of near-inertial oscillations through a geostrophic flow. *J. Mar. Res.*, 55:735–766.

[18] Zervakis, V. and Levine, M. D. (1995) Near-inertial energy propagation from the mixed layer: theoretical considerations. *J. Phys. Oceanogr.*, 25:2872–2889.