Abstract. The Ellsberg and Machina paradoxes reveal that expected utility theory is problematical when real subjects take decisions under uncertainty. Suitable generalizations of expected utility exist which attempt to solve the Ellsberg paradox, but none of them provides a satisfactory solution of the Machina paradox. In this paper we elaborate a quantum model in Hilbert space describing the Ellsberg situation and also the Machina situation, and show that we can model the specific aspect of the Machina situation that is unable to be modeled within the existing generalizations of expected utility.

Keywords: Ellsberg paradox, Machina paradox, ambiguity aversion, quantum modeling.

1 Introduction

In economics, the predominant model of decision making is the Expected Utility Theory (EUT) [12]. Notwithstanding its simplicity, mathematical tractability and predictive success, the empirical validity of EUT at the individual level is questionable. Indeed, examples exist in the literature which show an inconsistency between real preferences and the predictions of EUT. These deviations were put forward by considering specific situations of uncertainty often commonly referred to now as paradoxes [34].

EUT was formally developed by von Neumann and Morgenstern [1]. They presented a set of axioms that allow to represent decision–maker preferences over the set of acts (functions from the set of states of the world into the set of consequences) by the functional $E_p u(\cdot)$, for some real–valued Bernoulli utility function $u$ on the set of consequences and an objective probability measure $p$ on the set of states of the nature. An important aspect of EUT concerns the treatment of uncertainty. Knight had highlighted the difference between risk and uncertainty reserving the term risk for ventures that can be described by known (or physical) probabilities, and the term uncertainty to refer to situations in which agents did not know the probabilities associated with each of the possible outcomes of an
Table 1. The payoff matrix for the Ellsberg paradox situation

| Act | red | yellow | black |
|-----|-----|--------|-------|
| f_1 | 12$ | 0$     | 0$    |
| f_2 | 0$  | 0$     | 12$   |
| f_3 | 12$ | 12$    | 0$    |
| f_4 | 0$  | 12$    | 12$   |

act [5]. However, probabilities in the von Neumann and Morgenstern modeling are *objectively* or, physically, given. Later, Savage extended EUT allowing agents to construct their own subjective probabilities when physical probabilities are not available [2]. Then according to Savage’s model, the distinction put forward by Knight seems to be irrelevant. Ellsberg’s experiments instead showed that Knightian’s distinction is empirically meaningful [3]. In particular, he presented the following experiment. Consider one urn with thirty red balls and sixty balls that are either yellow or black, the latter in unknown proportion. One ball will be drawn from the urn. Then, free of charge, a person is asked to bet on one of the acts $f_1$, $f_2$, $f_3$ and $f_4$ defined in Table 1.

When asked to rank these gambles most of the persons choose to bet on $f_1$ over $f_2$ and $f_4$ over $f_3$. This empirical result cannot be explained by EUT. In fact, we can see that individuals’ ranking of the sub–acts $[12$ on red; $0$ on black] versus $[0$ on red; $12$ on black] depends upon whether the event yellow yields a payoff of 0 or 12, contrary to what is suggested by the Sure–Thing principle, an important axiom of Savage’s model. Nevertheless, these choices have a direct intuition: $f_1$ offers the 12 prize with an objective probability of $1/3$, and $f_2$ offers the same prize but in an element of the subjective partition \{black, yellow\}. In the same way, $f_4$ offers the prize with an objective probability of $2/3$, whereas $f_3$ offers the same payoff on the union of the unambiguous event red and the ambiguous event yellow. Thus, in both cases the unambiguous bet is preferred to its ambiguous counterpart, a phenomenon called by Ellsberg *ambiguity aversion*.

After the work of Ellsberg many extensions of EUT have been developed to represent this kind of preferences, all replacing the Sure–Thing Principle by weaker axioms. The first extension is *Choquet Expected Utility*, also known as expected utility with *non–additive* probabilities [6]. This model considered a subjective non–additive probability (*capacity*) over the states of nature instead of a subjective probability. Thus, decision–makers could underestimate or overestimate probabilities in the Ellsberg experiment and the ambiguity aversion is equivalent to the convexity of the capacity (pessimistic beliefs). A second approach is the *Max – Min Expected Utility*, or expected utility with multi–prior [7]. In this case the lack of knowledge about the states of nature of the decision–maker cannot be represented by a unique probability measure, instead he or she thinks are relevant a set of probability measures, then an act $f$ is preferred to $g$ if $\min_{p \in P} E_p u(f) > \min_{p \in P} E_p u(g)$, where $P$ is a convex and closed set of additive probability measures. The ambiguity aversion is represented by the pessimistic beliefs of the agent which takes decisions considering the worst