KERNEL THEOREMS AND NUCLEARITY IN IDEMPOTENT MATHEMATICS.
AN ALGEBRAIC APPROACH

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Dedicated in dear memory of F.A. Berezin

Abstract

In the framework of idempotent mathematics, analogs of the classical kernel theorems of L. Schwartz and A. Grothendieck are studied. Idempotent versions of nuclear spaces (in the sense of A. Grothendieck) are discussed. The so-called algebraic approach is used. This means that the basic concepts and results (including those of “topological” nature) are simulated in purely algebraic terms.

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Introduction

Idempotent mathematics is based on replacing the usual numerical fields with idempotent semifields and semirings. In other words, the usual arithmetic operations are replaced by a new set of basic associative operations (a new addition $\oplus$ and a new multiplication $\odot$) and all the semifield axioms or semiring axioms hold; moreover the new addition is idempotent, i.e. $x \oplus x = x$ for every element $x$ of the corresponding semiring, see, e.g., [1]–[10]. A typical example is the semifield $\mathbb{R}_{\text{max}} = \mathbb{R} \cup \{-\infty\}$ known as the Max-Plus algebra. This semifield consists of all real numbers and an additional element $0 = -\infty$. This element $0$ is the zero element in $\mathbb{R}_{\text{max}}$ and the basic operations are defined by the formulas $x \oplus y = \max\{x, y\}$ and $x \odot y = x + y$; the identity (or unit) element $1$ coincides with the usual zero $0$. Similarly the semifield

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\( \mathbb{R}_{\min} = \mathbb{R} \cup \{+\infty\} \) is defined. It consists of all real numbers and an additional element \( 0 = +\infty \); the basic operations are \( \oplus = \min \) and \( \odot = + \). Of course, the semifields \( \mathbb{R}_{\max} \) and \( \mathbb{R}_{\min} \) are isomorphic. Many nontrivial examples of idempotent semirings and semifields can be found, e.g., in [3]–[10].

Linear algebra over idempotent semirings was constructed by many authors starting from S.C. Kleene, N.N. Vorobjev, B.A. Carre, R.A. Cunningham-Green and others. Basic concepts and results of idempotent analysis and functional analysis are established by V.P. Maslov and his collaborators, see, e.g., [1]–[14]; in particular, important results and applications of idempotent analysis are due to V.N. Kolokoltsov (see, e.g., [3]–[6], [9], [13]–[15]). An important development of these ideas is due to the French mathematicians M. Akian, G. Cohen, S. Gaubert, J.-P. Quadrat and others, see, e.g., [9, 15, 16] and the survey [10]. A remarkable paper of O. Viro [17] was a starting point for idempotent geometry; now this subject is usually called tropical geometry. The subject is very popular and substantial contributions to its development are due to many authors (M. Kontsevich, G. Mikhalkin, M. Kapranov, I. Itenberg, V. Kharlamov, E. Shustin, B. Sturmfels and others). There are other areas of idempotent mathematics, e.g., idempotent interval analysis [18, 19]. Concerning the further development of the subject and its history, see, e.g., the survey [10].

Mathematics over idempotent semirings can be treated as a result of a dequantization of the traditional mathematics (over fields) and as its peculiar “shadow.” This “shadow” stands in the same relation to the traditional mathematics as does classical physics to quantum theory (see details in [7, 8, 10]). There is a heuristic correspondence between important, interesting, and useful constructions and results of the traditional mathematics over fields and similar constructions and results over idempotent semifields and semirings [7]. In many respects idempotent mathematics is simpler than the traditional one. However, the transition from traditional concepts and results to their idempotent versions is often nontrivial.

The aim of the paper is to describe idempotent versions of the classical kernel theorems of L. Schwartz and A. Grothendieck (see, e.g., [21, 22]). In other words, we describe conditions under which, in idempotent functional analysis, linear operators have integral representations in terms of the idempotent integral of V.P. Maslov. We define the notion of nuclear idempotent semimodule similar to the notion of nuclear space (in the sense of A. Grothendieck) in the traditional analysis (see, e.g., [21, 22]). Moreover,
we give a rather explicit description of semimodules of functions for which our kernel theorem is true. We use the so-called algebraic approach; this means that the basic concepts and results (including those of topological nature) are simulated in purely algebraic terms; in our subsequent papers a topological approach will be also used. This paper continues the series of publications on idempotent functional analysis [8, 11, 12] and we use the notation and terminology defined in those articles. Some of our results on idempotent semimodules of functions depend on semimodules’ imbeddings into the corresponding function spaces. In this case sometimes we discuss nonlinear (in the sense of idempotent mathematics) mappings and functionals. There are other results which are invariant with respect to such embeddings and can be rewritten in an invariant form.

For some concrete idempotent semimodules consisting of continuous or bounded functions, concrete kernel theorems are presented in [1]–[5], [13], [23]–[25]. A limiting case for kernel theorems is a result on integral representations of linear functionals; results of this kind are presented in [1]–[6], [8, 12, 13], [23]–[30]. A generalization of these results and their unification will be examined in a separate paper. In [31], for the idempotent semimodule of all bounded functions with values in $\mathbb{R}_{\min}$, there was posed a problem of describing the class of subsemimodules where the corresponding kernel theorem holds as well as of describing the corresponding integral representations of linear operators. In the present paper a very general case of semimodules over boundedly complete idempotent semirings is examined. Some of the results obtained here can be regarded as possible versions of an answer to the problem posed in [31]. Basic results of this paper were announced in [32].

1 Idempotent semimodules

1.1 Notation and basic terminology

In the present paper we use terminology and notation from [8, 11, 12]. Recall that an idempotent semigroup is an additive semigroup with commutative addition $\oplus$ such that for every element $x$ the equality $x \oplus x = x$ holds. Every idempotent semigroup will be treated as an ordered set with respect to the following (partial) standard order. $x \preceq y$ if and only if $x \oplus y = y$. It is easy to see that this order is well-defined and $x \oplus y = \sup\{x, y\}$,
that is every idempotent semigroup is an upper semilattice \(^2\) with respect to the standard order (see, e.g., [33]). For an arbitrary subset \(X\) of an idempotent semigroup we set \(\oplus X = \text{sup} \, X\) and \(\wedge X = \text{inf} \, X\) if \(\text{sup} \, X\) and \(\text{inf} \, X\) exist; in particular, we suppose that \(X\) is bounded from above or below respectively. An idempotent semigroup is called \(b\)-complete (or boundedly complete) if every its subset that is bounded from above (including the empty set) has the least upper bound. In particular, every \(b\)-complete idempotent semigroup contains a zero element (denoted \(0\)) which coincides with \(\oplus \emptyset\), where \(\emptyset\) is the empty set. Thus every nonempty subset of this semigroup is bounded from below by the zero \(0\). So every nonempty subset of the \(b\)-complete semigroup is bounded if and only if this subset is bounded from above. A homomorphism \(g\) for \(b\)-complete idempotent semigroups is called a \(b\)-homomorphism, if \(g(\oplus X) = \oplus g(X)\) for every subset \(X\) bounded from above.

An idempotent semiring is an idempotent semigroup endowed with an associative multiplication \(\circ\) with an identity element (denoted \(1\)) such that the corresponding distribution laws (left and right) hold. An idempotent semiring \(K\) is \(b\)-complete if it is a \(b\)-complete idempotent semigroup and the following “infinite” distribution laws hold: \(k \circ \oplus X = \oplus (k \circ X)\) and \((\oplus X) \circ k = \oplus (X \circ k)\) for every \(k \in K\) and every bounded subset \(X\) of \(K\).

A commutative idempotent semiring is called an \(idempotent semifield\) if every its nonzero element is invertible. A semifield is a \(b\)-complete idempotent semiring if and only if it is a \(b\)-complete idempotent semigroup with respect to the operation \(\oplus\) [8]. The algebra \(\mathbb{R}_\text{max}\) (or \(\mathbb{R}_\text{min}\)) described in the Introduction is an example of a \(b\)-complete semifield. Another example is the semifield \(\mathbb{Z}_\text{max}\) consisting of integer elements of \(\mathbb{R}_\text{max}\) and the element \(0 = -\infty\) with operations induced from \(\mathbb{R}_\text{max}\).

If we add a maximal element \(+\infty\) to \(\mathbb{R}_\text{max}\) and extend the operations in an obvious way, we shall get a new idempotent semiring (not a semifield!) \(\hat{\mathbb{R}}_\text{max} = \mathbb{R}_\text{max} \cup \{+\infty\}\); the semiring \(\hat{\mathbb{Z}}_\text{max} = \mathbb{Z}_\text{max} \cup \{+\infty\}\) is defined similarly. These semirings are not only \(b\)-complete but complete as partially ordered sets (with respect to the standard order \(\leq\)) and the passage from \(\mathbb{R}_\text{max}\) and \(\mathbb{Z}_\text{max}\) to \(\hat{\mathbb{R}}_\text{max}\) and \(\hat{\mathbb{Z}}_\text{max}\) is the so-called normal completion in the sense of [33]. Square matrices of the size \(n \times n\) with entries from a \(b\)-complete idempotent semiring \(K\) form a \(b\)-complete noncommutative semiring \(\text{Mat}_n(K)\) with re-

\(^2\)Let us remind that a partially ordered set \(V\) is called an \textit{upper semilattice} if for arbitrary elements \(x, y\) of \(V\) there exists their least upper bound \(\sup \{x, y\}\) in \(V\).
spect to the operations of addition and multiplication of matrices generated by the operations in $K$.

An idempotent semimodule over an idempotent semiring $K$ is an additive commutative idempotent semigroup $V$, with the addition operation denoted by $\oplus$ and the zero element denoted by $0$, such that a (left) multiplication $k \odot x$ is defined for all $k \in K$ and $x \in V$ in such a way that the usual rules are satisfied: $a \odot (b \odot x) = (a \odot b) \odot x$, $(a \oplus b) \odot x = a \odot x \oplus b \odot x$, $a \odot (x \oplus y) = a \odot x \oplus a \odot y$, $0 \odot x = 0$, $1 \odot x = x$ for all $a, b \in K$ and $x, y \in V$.

In what follows we shall suppose that all the semigroups, semirings, semifields, and semimodules are idempotent unless otherwise specified.

A semimodule $V$ over a $b$-complete semiring $K$ is $b$-complete if it is a $b$-complete semigroup and the following infinite distribution laws hold: $k \odot \oplus X = \oplus (k \odot X)$ for every $k \in K$ and every bounded subset $X \subset V$ as well as $(\oplus Q) \odot v = \oplus (Q \odot v)$ for every bounded subset $Q$ of $K$ and every element $v \in V$ (see [8], definition 4.3).

A homomorphism $g : V \to W$ for $b$-complete semimodules $V$ and $W$ is called a $b$-homomorphism or a $b$-linear mapping (operator) if $g(\oplus X) = \oplus g(X)$ for every bounded subset $X \subset V$ (a more general definition for the case of incomplete semimodules see in [8]); of course, here $g(a \odot x) = a \odot g(x)$ for all $a \in K$, $x \in V$. Homomorphisms taking their values in the basic semiring $K$ (treated as a semimodule over itself) are called linear functionals. Of course, $b$-linear functionals are linear functionals such that the corresponding homomorphisms are $b$-homomorphisms.

In idempotent mathematics, $b$-linear operators and functionals can be regarded as analogs of traditional (semi)continuous linear operators and functionals in classical analysis, see [8].

Let $V$ and $W$ be $b$-complete semimodules over a $b$-complete semiring $K$. Denote by $L_b(V, W)$ the set of all $b$-linear mappings from $V$ to $W$. It is easy to check that $L_b(V, W)$ is an idempotent semigroup with respect to the pointwise addition of operators; the composition (product) of $b$-linear operators is also a $b$-linear operator and therefore the set $L_b(V, V)$ is an idempotent semiring with respect to these operations, see, e.g., [8]. The following proposition can be treated as a version of the Banach-Steinhaus theorem in idempotent analysis.

**Proposition 1** Suppose that $S$ is a subset in $L_b(V, W)$ and the set $\{g(v) \mid g \in S\}$ is bounded in $W$ for every element $v \in V$; thus the element $f(v) =$
\[ \bigoplus_{g \in S} g(v) \] exists because the semimodule \( W \) is \( b \)-complete. Then the mapping \( v \mapsto f(v) \) is a \( b \)-linear operator, i.e. an element of \( L_b(V, W) \). The subset \( S \) is bounded; moreover, \( \sup S = f \).

**Proof.** It is easy to check that \( f \) is a homomorphism \( V \to W \). For every bounded subset \( V \) in \( W \) the following equations hold: \( f(\bigoplus X) = \bigoplus_{g \in S} g(\bigoplus X) = \bigoplus_{g \in S} \bigoplus_{v \in X} g(v) = \bigoplus_{v \in X} f(v) = \bigoplus f(X) \). This means that \( f \) is a \( b \)-linear operator. By our construction, \( f = \sup S \) in \( L_b(V, W) \). \( \square \)

**Corollary 1** The set \( L_b(V, W) \) is a \( b \)-complete idempotent semigroup with respect to the (idempotent) pointwise addition of operators. If \( V = W \), then \( L_b(V, V) \) is a \( b \)-complete idempotent semiring with respect to the operations of pointwise addition and composition of operators.

**Corollary 2** A subset \( S \) is bounded in \( L_b(V, W) \) if and only if the set \( \{ g(v) \mid g \in S \} \) is bounded in the semimodule \( W \) for every element \( v \in V \).

These corollaries can be easily deduced from Proposition 1 and the basic definitions.

A subset of an idempotent semimodule is called a *subsemimodule* if it is closed under addition and multiplication by scalar coefficients. A subsemimodule \( V \) of a \( b \)-complete semimodule \( W \) is *\( b \)-closed* if \( V \) is closed with respect to summing of all subsets of \( V \) that are bounded in \( W \). A subsemimodule of a \( b \)-complete semimodule is called a *\( b \)-subsemimodule* if the corresponding embedding is a \( b \)-homomorphism. It is easy to see that each \( b \)-closed subsemimodule is a \( b \)-subsemimodule but the converse is not true (see Section 1.2). The main feature of \( b \)-subsemimodules is that restrictions of \( b \)-linear operators and functionals to these semimodules are \( b \)-linear.

The following definitions are very important for our aims. Suppose that \( W \) is an idempotent \( b \)-complete semimodule over a \( b \)-complete idempotent semiring \( K \) and \( V \) is a subset of \( W \) such that \( V \) is closed under multiplication by scalar coefficients and is an upper semilattice with respect to the order induced from \( W \). Let us define an addition operation in \( V \) by the formula \( x \oplus y = \sup \{ x, y \} \), where \( \sup \) means the least upper bound in \( V \). If \( K \) is a semifield, then \( V \) is a semimodule over \( K \) with respect to this addition.

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For an arbitrary \( b \)-complete semiring \( K \) we shall say that \( V \) is a \emph{quasisubsemimodule} of \( W \) if \( V \) is a semimodule with respect to this addition (this means that the corresponding distribution laws hold).

A quasisubsemimodule \( V \) of an idempotent \( b \)-complete semimodule \( W \) is called a \( \wedge \)-\emph{subsemimodule} if it contains \( 0 \) and is closed under the operations of taking infima (greatest lower bounds) in \( W \). It is easy to check (e.g., using lemma 2.1 in [8]), that each \( \wedge \)-subsemimodule is a \( b \)-complete semimodule.

Note that quasisubsemimodules and \( \wedge \)-subsemimodules may fail to be subsemimodules because only the order is induced and not the corresponding addition (see Example 6 below).

Following [8], we say that idempotent semimodules over semirings are \emph{idempotent spaces}. In idempotent mathematics, such spaces are analogs of traditional linear (vector) spaces over fields. In a similar way we use the corresponding terms like \( b \)-spaces, \( b \)-subspaces, \( b \)-closed subspaces, \( \wedge \)-subspaces etc.

Numerous examples of idempotent semimodules and spaces can be found in [8]; see also [3]–[9], [11], [14]–[16], [25]. Some examples are presented below, see, e.g., Section 3.

### 1.2 Functional semimodules

Let \( X \) be an arbitrary nonempty set and \( K \) be an idempotent semiring. By \( K(X) \) denote the semimodule of all mappings (functions) \( X \to K \) endowed with the pointwise operations. By \( K_b(X) \) denote the subsemimodule of \( K(X) \) consisting of all bounded mappings. If \( K \) is a \( b \)-complete semiring, then \( K(X) \) and \( K_b(X) \) are \( b \)-complete semimodules. Note that \( K_b(X) \) is a \( b \)-subsemimodule but not a \( b \)-closed subsemimodule of \( K(X) \). For any point \( x \in X \), by \( \delta_x \) denote the functional on \( K(X) \) that maps \( f \) to \( f(x) \). It can be easily checked that the functional \( \delta_x \) is \( b \)-linear on \( K(X) \).

We shall say that any quasisubsemimodule of \( K(X) \) is an (idempotent) \emph{functional semimodule} on the set \( X \). An idempotent functional semimodule in \( K(X) \) is called \emph{\( b \)-complete} if it is a \( b \)-complete semimodule.

A functional semimodule \( V \subset K(X) \) is called a \emph{functional \( b \)-semimodule} if it is a \( b \)-subsemimodule of \( K(X) \); a functional semimodule \( V \subset K(X) \) is called a \emph{functional \( \wedge \)-semimodule} if it is a \( \wedge \)-subsemimodule of \( K(X) \).

In general, a functional of the form \( \delta_x \) on a functional semimodule is not even linear, much less \( b \)-linear (see Example 6 below). However, the following
proposition holds, which is a direct consequence of our definitions.

**Proposition 2** An arbitrary $b$-complete functional semimodule $W$ on a set $X$ is a $b$-subsemimodule of $K(X)$ if and only if each functional of the form $\delta_x$ (where $x \in X$) is $b$-linear on $W$.

**Proof.** Both statements mean that for each set of functions $S \subset W$ bounded in $W$ the formula $(\oplus S)(x) = (\oplus S(x))$ holds for all $x \in X$. □

**Example 1.** The semimodule $K_b(X)$ (consisting of all bounded mappings from an arbitrary set $X$ to a $b$-complete idempotent semiring $K$) is a functional $\land$-semimodule. Hence it is a $b$-complete semimodule over $K$. Moreover, $K_b(X)$ is a $b$-subsemimodule of the semimodule $K(X)$ consisting of all mappings $X \to K$.

**Example 2.** If $X$ is a finite set consisting of $n$ elements ($n > 0$), then $K_b(X) = K(X)$ is an “$n$-dimensional” semimodule over $K$ and it is denoted by $K^n$. In particular, $\mathbb{R}_\text{max}^n$ is an idempotent space over the semifield $\mathbb{R}_\text{max}$ and $\hat{\mathbb{R}}^n_\text{max}$ is a semimodule over the semiring $\hat{\mathbb{R}}_\text{max}$. Note that $\hat{\mathbb{R}}^n_\text{max}$ can be treated as a space over the semifield $\mathbb{R}_\text{max}$. For example, the semiring $\hat{\mathbb{R}}_\text{max}$ can be treated as a space (semimodule) over $\mathbb{R}_\text{max}$.

**Example 3.** Let $X$ be a topological space. Denote by $USC(X)$ the set of all upper semicontinuous functions taking their values in $\mathbb{R}_\text{max}$. By definition, a function $f(x)$ is upper semicontinuous if the set $X_s = \{x \in X \mid f(x) \geq s\}$ is closed in $X$ for every element $s \in \mathbb{R}_\text{max}$ (see, e.g., [8], section 2.8). If a collection $\{f_\alpha\}$ consists of upper semicontinuous (e.g., continuous) functions and $f(x) = \inf_\alpha f_\alpha(x)$, then $f(x) \in USC(X)$. It is easy to check that $USC(X)$ has a natural structure of an idempotent space over $\mathbb{R}_\text{max}$. Moreover, $USC(X)$ is a functional $\land$-space on $X$ and $b$-space. The subspace $USC(X) \cap K_b(X)$ of $USC(X)$ consisting of bounded (above) functions has the same properties.

**Example 4.** Suppose that $X$ is a partially ordered set and $K$ is the semifield $\mathbb{R}_\text{max}$ (or $\mathbb{Z}_\text{max}$). Denote by $N(X)$ the set of all nonincreasing functions defined on $X$ and taking their values in $K$. It is easy to check that $N(X)$ has a natural structure of a space over $K$; moreover, $N(X)$ is a functional
$b$-space and a functional $\wedge$-space on $X$. The space $N(X) \cap K_b(X)$ has the same properties.

**Example 5.** Let $\mathbb{C}$ be the set of all complex numbers with a singled out axis $\mathbb{R}$ of all real numbers. Denote by $N_{\mathbb{R}}(\mathbb{C})$ the set of all functions $\mathbb{C} \to K$ nonincreasing along the real axis $\mathbb{R}$, where $K = \mathbb{R}_{\text{max}}$ or $K = \mathbb{Z}_{\text{max}}$. It is easy to check that $N_{\mathbb{R}}(\mathbb{C})$ (as well as $N_{\mathbb{R}}(\mathbb{C}) \cap K_b(\mathbb{C})$) has a natural structure of a space over $K$ and this space is a functional $b$-space and a functional $\wedge$-space on $X$. This is an example of a natural “intersection” of space structures.

**Example 6.** Note that an idempotent functional semimodule (and even a functional $\wedge$-semimodule) on a set $X$ is not necessarily a subsemimodule of $K(X)$. The simplest example is the functional space (over $K = \mathbb{R}_{\text{max}}$) Conc($\mathbb{R}$) consisting of all concave functions on $\mathbb{R}$ with values in $\mathbb{R}_{\text{max}}$. Recall that a function $f$ belongs to Conc($\mathbb{R}$) if and only if the subgraph of this function is convex, i.e., the formula $f(ax + (1-a)y) \geq af(x) + (1-a)f(y)$ is valid for $0 \leq a \leq 1$. The basic operations with $0 \in \mathbb{R}_{\text{max}}$ can be defined in an obvious way. If $f, g \in \text{Conc}(\mathbb{R})$, then denote by $f \oplus g$ the sum of these functions in Conc($\mathbb{R}$). The subgraph of $f \oplus g$ is the convex hull of the subgraphs of $f$ and $g$. Thus $f \oplus g$ does not coincide with the pointwise sum (i.e., $\max\{f(x), g(x)\}$).

## 2 Integral representations of linear operators

### 2.1 Integral representations of linear operators in functional semimodules

Suppose $W$ is an idempotent $b$-complete semimodule over a $b$-complete semiring $K$ and $V \subset K(X)$ is a $b$-complete functional semimodule on $X$. A mapping $A : V \to W$ is called an *integral operator* or an operator with an *integral representation* if there exists a mapping $k : X \to W$, called the *integral kernel* (or *kernel*) of the operator $A$, such that

$$Af = \bigoplus_{x \in X} f(x) \odot k(x). \quad (1)$$

In idempotent analysis, the right hand side of formula (1) is often written as $\int_X f(x) \odot k(x)dx$ (see [1]–[7]). Regarding the kernel $k$ it is supposed that
the set \( \{ f(x) \odot k(x) \mid x \in X \} \) is bounded in \( W \) for all \( f \in V \) and \( x \in X \). We denote the set of all functions with this property by \( \text{kern}_{V,W}(X) \). In particular, if \( W = K \) and \( A \) is a functional, then this functional is called integral. Thus each integral functional can be presented in the form of a “scalar product” \( f \mapsto \int_X f(x) \odot k(x) \, dx \), where \( k(x) \in K(X); \) in idempotent analysis this situation is standard, see, e.g., [1]–[10].

Note that the functional of the form \( \delta_y \) (where \( y \in X \)) is a typical integral functional; in this case \( k(x) = 1 \) if \( x = y \) and \( k(x) = 0 \) otherwise.

We call a functional semimodule \( V \subset K(X) \) nondegenerate if for each point \( x \in X \) there exists a function \( g \in V \) such that \( g(x) = 1 \), and admissible if for each function \( f \in V \) and each point \( x \in X \) such that \( f(x) \neq 0 \) there exists a function \( g \in V \) such that \( g(x) = 1 \) and \( f(x) \odot g \preceq f \).

Note that all idempotent functional semimodules over semifields are admissible (it is sufficient to set \( g = f(x)^{-1} \odot f \)).

**Proposition 3** Denote by \( X_V \) the subset of \( X \) defined by the formula \( X_V = \{ x \in X \mid \exists f \in V : f(x) = 1 \} \). If the semimodule \( V \) is admissible, then the restriction to \( X_V \) defines an imbedding \( i : V \to K(X_V) \) and its image \( i(V) \) is admissible and nondegenerate.

If a mapping \( k : X \to W \) is a kernel of a mapping \( A : V \to W \), then the mapping \( k_V : X \to W \) which is equal to \( k \) on \( X_V \) and equal to \( 0 \) on \( X - X_V \) is also a kernel of the mapping \( A \).

A mapping \( A : V \to W \) is integral if and only if the mapping \( i_{-1}A : i(A) \to W \) is integral.

**Proof.** If the semimodule \( V \) is admissible, then it is easy to see that \( f(x) = 0 \) for all \( f \in V \) and \( x \in X - X_V \). All the statements of Proposition 3 can be easily deduced from this assertion. □

In what follows, \( K \) always denotes a fixed \( b \)-complete idempotent semiring. We shall discuss semimodules over this semiring. If an operator has an integral representation, this representation may not be unique. However, if the semimodule \( V \) is nondegenerate, then the set of all kernels of a fixed integral operator is bounded with respect to the natural order in the set of all kernels and is closed under the supremum operation applied to its arbitrary subsets. In particular, any integral operator defined on a nondegenerate functional semimodule has a unique maximal kernel.
An important point is that an integral operator is not necessarily $b$-linear and even linear except when $V$ is a $b$-subsemimodule of $K(X)$ (see Proposition 4 below).

If $W$ is a functional semimodule on a nonempty set $Y$, then the integral kernel $k$ of an operator $A$ can be naturally identified with a function on $X \times Y$ defined by the formula $k(x, y) = (k(x))(y)$. This function will also be called the integral kernel (or kernel) of the operator $A$. As a result, the set $\ker_{V,W}(X)$ is identified with the set $\ker_{V,W}(X, Y)$ of all mappings $k : X \times Y \to K$ such that for every point $x \in X$ the mapping $k_x : y \mapsto k(x, y)$ lies in $W$ and for every $v \in V$ the set $\{v(x) \circ k_x | x \in X\}$ is bounded in $W$. Accordingly, the set of all integral kernels of $b$-linear operators can be embedded to $\ker_{V,W}(X, Y)$.

If $V$ and $W$ are functional $b$-semimodules on $X$ and $Y$, respectively, then the set of all kernels of $b$-linear operators can be identified with $\ker_{V,W}(X, Y)$ (see Proposition 4 below) and the following formula holds:

$$Af(y) = \bigoplus_{x \in X} f(x) \circ k(x, y) = \int_{X} f(x) \circ k(x, y)dx. \quad (2)$$

This formula coincides with the usual definition of an operator’s integral representation. Note that formula (1) can be rewritten in the form

$$Af = \bigoplus_{x \in X} \delta_x(f) \circ k(x). \quad (3)$$

**Proposition 4** An arbitrary $b$-complete functional semimodule $V$ on a nonempty set $X$ is a functional $b$-semimodule on $X$ (i.e., a $b$-subsemimodule of $K(X)$) if and only if all integral operators defined on $V$ are $b$-linear.

**Proof.** Let $W$ be an arbitrary $b$-complete semimodule. If $V$ is a functional $b$-semimodule on $X$, then every operator $\Delta_{x,w} : V \to W$ of the form $f \mapsto \delta_x(f) \circ w$, where $x \in X$, $w \in W$ is $b$-linear by virtue of Proposition 2. By definition each integral operator is a sum of operators of this type; so every integral operator is $b$-linear by virtue of Proposition 1 and its corollaries. On the other hand, if each integral operator defined on $V$ is $b$-linear, then all the functionals of the form $\delta_x$ are $b$-linear because these functionals are integral operators from $V$ into $K$. From this it follows that $V$ is a functional $b$-semimodule on $X$ because of Proposition 2. So the proposition is proved. \qed
The following concept (definition) is especially important for our aims. Let $V \subset K(X)$ be a $b$-complete functional semimodule over a $b$-complete idempotent semiring $K$. We shall say that the kernel theorem holds for the semimodule $V$ if each $b$-linear mapping from $V$ into an arbitrary $b$-complete semimodule over $K$ has an integral representation.

**Theorem 1** Suppose that a $b$-complete semimodule $W$ over a $b$-complete semiring $K$ and an admissible functional $\land$-semimodule $V \subset K(X)$ are given. Then each $b$-linear operator $A : V \to W$ has an integral representation of the form (1). In particular, if $W$ is a functional $b$-semimodule on a set $Y$, then the operator $A$ has an integral representation of the form (2). So for the semimodule $V$ the kernel theorem holds.

**Proof.** Denote by $X_V$ the subset of $X$ defined by the formula $X_V = \{ x \in X \mid \exists f \in V : f(x) = 1 \}$ (see Proposition 3). If $x \in X_V$, then we set $d_x = \land \{ f \in V \mid f(x) = 1 \}$. By our construction we have $f(x) \circ d_x \equiv f$. The semimodule $V$ is admissible and it is a $\land$-semimodule, so $d_x(x) = 1$. From this we can easily deduce that $f = \bigoplus_{x \in X_V} f(x) \circ d_x$. Then $A(f) = \bigoplus_{x \in X_V} f(x) \circ A(d_x)$, that is the mapping $x \mapsto k(x)$ defined by the formulas $k(x) = A(d_x)$ for all $x \in X_V$ and $k(x) = 0$ for each $x \notin X_V$ is a kernel of the operator $A$. The theorem is proved. □

**Remark 1.** If in the framework of Theorem 1 the semimodule $V$ is nondegenerate, then the function $x \mapsto d_x$ is the maximal integral kernel of the identity operator $id : V \to V$.

Indeed, if under the conditions of Theorem 1 the semimodule $V$ is nondegenerate, then all the functions of the form $x \mapsto d_x$ belong to $V$. So if $k : X \to V$ is an integral kernel of the identity operator $id : V \to V$, then $d_y = \bigoplus_{x \in X} d_y(x) \circ k(x) \equiv d_y(y) \circ k(y)$ for each $y \in X$. So we have $d_y \equiv k(y)$ for each $y \in X$ because $d_y(y) = 1$, as was to be proved.

**Remark 2.** Examples of admissible functional $\land$-semimodules (and $\land$-spaces) appearing in Theorem 1 are presented above in the end of Section 1. Thus for these functional semimodules and spaces $V$ over $K$ the kernel theorem holds and each $b$-linear operator mapping $V$ into an arbitrary $b$-complete semimodule $W$ over $K$ has an integral representation (2). Recall that each functional space over a $b$-complete semifield is admissible, see above.
2.2 Integral representations of b-nuclear operators

Let us introduce some important definitions. Suppose that $V$ and $W$ are $b$-complete semimodules. A mapping $g : V \rightarrow W$ is called one-dimensional (or a mapping of the rank 1) if it is of the form $v \mapsto \phi(v) \odot w$, where $\phi$ is a $b$-linear functional on $V$ and $w \in W$. A mapping $g$ is called $b$-nuclear if it is a sum of a bounded set of one-dimensional mappings. Each one-dimensional mapping is $b$-linear because the functional $\phi$ is $b$-linear, so every $b$-nuclear operator is $b$-linear (see Corollary 1 above). Of course, $b$-nuclear mappings are closely related with tensor products of idempotent semimodules, see [11].

By $\phi \odot w$ we shall denote the one-dimensional operator $v \mapsto \phi(v) \odot w$. In fact this is an element of the corresponding tensor product.

Using Proposition 1 and its corollaries, it is easy to check that the following proposition holds.

**Proposition 5** The composition (product) of a $b$-nuclear and a $b$-linear mapping or of a $b$-linear and a $b$-nuclear mapping is a $b$-nuclear operator.

**Proof.** It is obvious from our definitions that the composition of a one-dimensional and a $b$-linear mapping or of a $b$-linear and a one-dimensional mapping is a one-dimensional mapping. So it is sufficient to note that the decomposition with a $b$-linear operator transforms each sum of operators to a sum of operators. $\square$

**Theorem 2** Suppose that $W$ is a $b$-complete semimodule over a $b$-complete semiring $K$ and $V \subset K(X)$ is a functional $b$-semimodule. If every $b$-linear functional on $V$ is integral, then any $b$-linear operator $A : V \rightarrow W$ has an integral representation if and only if it is $b$-nuclear.

**Proof.** Note that all functionals of the form $\delta_x$ (for $x \in X$) are $b$-linear because $V$ is a functional $b$-semimodule, see Proposition 2 above. Each $b$-linear functional is integral in our case, so every $b$-nuclear operator is a sum of a collection of one-dimensional operators of the form $k_w(x) \delta_x \odot w$, where $k_w \in K(X), w \in W$. Therefore, each $b$-nuclear operator is of the form $\bigoplus_{x \in X, w \in W} k_w(x) \delta_x \odot w$, where $k_w \in K(X)$, i.e., this operator has an integral representation with the kernel $k(x) = \bigoplus_w k_w(x) \odot w$.
On the other hand, each integral operator with a kernel \( k : X \to W \) can be presented in the form \( \bigoplus_{x \in X} A_x \), where the one-dimensional operator \( A_x \) is defined by the formula \( A_x = \delta_x \odot k(x) \), so it is \( b \)-nuclear. □

### 2.3 The \( b \)-approximation property and \( b \)-nuclear semimodules

We shall say that a \( b \)-complete semimodule \( V \) has the \( b \)-approximation property if the identity operator \( \text{id} : V \to V \) is \( b \)-nuclear (for a treatment of the approximation property for locally convex spaces in the traditional functional analysis see [21, 22]).

Let \( V \) be an arbitrary \( b \)-complete semimodule over a \( b \)-complete idempotent semiring \( K \). We call this semimodule a \( b \)-nuclear semimodule if any \( b \)-linear mapping of \( V \) to an arbitrary \( b \)-complete semimodule \( W \) over \( K \) is a \( b \)-nuclear operator. Recall that, in the traditional functional analysis, a locally convex space is nuclear if and only if all continuous linear mappings of this space to any Banach space are nuclear operators, see [21, 22].

Using Propositions 1 and 5, Corollary 1, and the fact that every mapping will not be changed after (left or right) multiplication by the identity operator, from our basic definitions it is easy to deduce the following proposition.

**Proposition 6** Let \( V \) be an arbitrary \( b \)-complete semimodule over a \( b \)-complete semiring \( K \). The following statements are equivalent:

1. the semimodule \( V \) has the \( b \)-approximation property;

2. each \( b \)-linear mapping from \( V \) to an arbitrary \( b \)-complete semimodule \( W \) over \( K \) is \( b \)-nuclear;

3. each \( b \)-linear mapping from an arbitrary \( b \)-complete semimodule \( W \) over \( K \) to the semimodule \( V \) is \( b \)-nuclear.

**Proof.** If the identity operator \( \text{id} : V \to V \) is \( b \)-nuclear and \( f : V \to W \) is an arbitrary \( b \)-linear mapping, then from the equality \( f = f \circ \text{id} \) and Proposition 5 we deduce that the operator \( f \) is \( b \)-nuclear. On the other hand, if each \( b \)-linear mapping from \( V \) to an arbitrary \( b \)-complete semimodule \( W \) over \( K \) is \( b \)-nuclear, then this is true for the identity mapping \( \text{id} : V \to V \). Therefore, statements (1) and (2) are equivalent. The equivalence of statements (1) and (3) can be proved similarly. □
Corollary 3 An arbitrary $b$-complete semimodule over a $b$-complete semiring $K$ is $b$-nuclear if and only if this semimodule has the $b$-approximation property.

Recall that, in the traditional functional analysis, any nuclear space has the approximation property but the converse statement is not true.

Some concrete examples of $b$-nuclear spaces and semimodules are described in Examples 1, 2, 4, and 5 (see above in the end of Section 1). Important $b$-nuclear spaces and semimodules are described below in Section 3 (e.g., Lipschitz spaces and semi-Lipschitz semimodules over commutative semirings). It is easy to show that the idempotent spaces $USC(X)$ and $Conc(\mathbb{R})$ (see Examples 3 and 6) are not $b$-nuclear (however, for these spaces the kernel theorem is true). The reason is that these spaces are not functional $b$-spaces and the corresponding $\delta$-functionals are not $b$-linear (and even linear).

2.4 Kernel theorems for functional $b$-semimodules

Let $V \subset K(X)$ be a $b$-complete functional semimodule over a $b$-complete semiring $K$. Recall that for $V$ the kernel theorem holds if each $b$-linear mapping of this semimodule to an arbitrary $b$-complete semimodule over $K$ has an integral representation.

Theorem 3 Suppose that a $b$-complete semiring $K$ and a nonempty set $X$ are given. The kernel theorem holds for any functional $b$-semimodule $V \subset K(X)$ if and only if each $b$-linear functional on $V$ is integral and the semimodule $V$ is $b$-nuclear, i.e., it has the $b$-approximation property.

Proof. The theorem follows from Theorem 2 and Proposition 6. □

Corollary 4 If for a functional $b$-semimodule the kernel theorem holds, then this semimodule is $b$-nuclear.

Note that the possibility to get an integral representation of a functional means that it is possible to decompose it into a sum of functionals of the form $\delta_x$. The following example demonstrates that it is not always possible to have an integral representation of a $b$-linear functional; moreover, this depends on embeddings of the semimodule to $K(X)$. On the other hand, the
b-approximation property (b-nuclearity) is invariant with respect to isomorphism of semimodules.

Example 7. Suppose that $K = \mathbb{R}_{\max}$, $X = \mathbb{R}$ and the function $f \in K(X)$ is defined by the formula $f(x) = -x$, $x \in \mathbb{R}$. Denote by $V$ the subsemimodule of $K(X)$ consisting of all functions of the form $a \odot f \oplus b$ for $a, b \in K$. It is easy to see that $V$ is a b-subsemimodule of $K(X)$ and the mapping $(a, b) \mapsto a \odot f \oplus b$ is an isomorphism of the b-space $\mathbb{R}_{\max} \times \mathbb{R}_{\max}$ onto $V$. So the mapping $\phi$ transforming $a \odot f \oplus b$ to $b$ is a b-linear functional. Let us show that $\phi$ has no integral representations.

Indeed, let $k : X \to K$ be an integral kernel of the functional $\phi$. For an arbitrary $v = a \odot f \oplus 1 \in V$ and $x \in X$ we have $0 = 1 = \phi(v) = \bigoplus_{y \in X} v(y) \odot k(y) \succ v(x) \odot k(x) = \max(a - x, 0) + k(x)$, i.e., $0 \succ \max(a - x, 0) + k(x)$, so $k(x) \preceq -\max(a - x, 0)$. The number $a$ can be chosen arbitrarily great, so $k(x) = 0$ for all $x \in X$, i.e. $k = 0$. But this is impossible because a nonzero functional is not able to have the zero integral kernel. Therefore, the functional $\phi$ has no integral kernels, as was to be proved.

Remark 3. The semimodule presented in Example 7 is naturally isomorphic to the semimodule $K(\{x, y\})$ of all functions defined on the two-point set $\{x, y\}$ and the isomorphism $K(\{x, y\}) \to V$ is defined by the formula $g \mapsto g(x) \odot f + g(y)$, where $g \in K(\{x, y\})$. So the functional $\phi$ described in Example 7 coincides with $\delta_y$ and it is integral (with the kernel $\delta_y$) in $K(\{x, y\})$. So we see that the property to be integral is not an “intrinsic” property of a functional but depends on its imbedding to a semimodule of functions.

Theorem 3a Suppose that a b-complete semiring $K$ and a nonempty set $X$ are given. The kernel theorem holds for a functional b-semimodule $V \subset K(X)$ if and only if the identity operator $id : V \to V$ is integral.

Proof. It follows from the obvious fact that the composition (product) of any integral operator with each b-linear operator is an integral operator. Indeed, suppose that $A$ is a b-linear operator transforming $V$ to a semimodule $W$ and $k : X \to V$ is an integral kernel of the operator $id$. Then $f = \bigoplus_{x \in X} f(x) \odot k(x)$ for each $f \in V$, so $(Af)(x) = \bigoplus_{x \in X} f(x) \odot A(k(x))$. Thus the mapping $x \mapsto A(k(x))$ is a kernel of $A$. The converse statement is trivial:
the identity operator is integral if the kernel theorem holds. □

3 A description of functional b-semimodules for which the kernel theorem holds

Suppose that $X$ is a nonempty set and $K$ is an idempotent semiring. We shall say that a function $d$ defined on $X \times X$ and taking its values in $K$ is a semimetric on $X$ with values in $K$ if

$$d(x, y) = \bigoplus_{z \in X} d(x, z) \odot d(z, y),$$

(4)

where $x, y \in X$. We shall say that this semimetric is symmetric if $d(x, y) = d(y, x)$ for all $x, y \in X$. If $d(x, x) = 1$ for all $x \in X$, then the semimetric is reflexive; in this case the condition (4) is equivalent to the triangle inequality:

$$d(x, y) \triangleright d(x, z) \odot d(z, y),$$

where $x, y, z \in X$.

**Example 8.** Let $r = r(x, y)$ be a metric on $X$. Then the function $d(x, y) = -r(x, y)$ is a reflexive symmetric semimetric on $X$ with values in $\mathbb{R}_{\text{max}}$.

Let us present an example of a nonsymmetric and nonreflexive semimetric on the set of real numbers.

**Example 9.** Suppose $X = \mathbb{R}$, $K = \mathbb{R}_{\text{max}}$. Set $d(x, y) = 1$ if $y < x$ and $d(x, y) = 0$ if $x \leq y$. Then $d(x, y)$ is a nonsymmetric semimetric and $d(x, x) = 0$ for all $x \in X$.

Now we introduce our basic definitions for this section. For a semimetric $d$ on $X$ with values in a $b$-complete semiring $K$, we define $b$-closed functional semimodules $\text{Lip}(X, d)$ and $\text{lip}(X, d)$ on $X$ by means of the formulas

$$\text{Lip}(X, d) = \{ f \in K(X) | f(x) \triangleright f(y) \odot d(y, x) \text{ for all } x, y \in X \},$$

$$\text{lip}(X, d) = \{ f \in K(X) | f(x) = \bigoplus_{y \in X} f(y) \odot d(y, x) \text{ for all } x \in X \}.$$

The functional semimodules of the form $\text{lip}(X, d)$ will be called semi-Lipschitz semimodules.
It follows from the definition that \( \text{lip}(X, d) \) is a \( b \)-closed subsemimodule of \( \text{Lip}(X, d) \). It is easy to see that in the case of a reflexive semimetric \( \text{lip}(X, d) \) and \( \text{Lip}(X, d) \) coincide. But in general this is not true. In particular, in the situation of Example 9 the space \( \text{Lip}(X, d) \) consists of all nonincreasing functions on \( \mathbb{R}_{\max} \) whereas \( \text{lip}(X, d) \) is the space of all lower semicontinuous functions\(^3\) belonging to \( \text{Lip}(X, d) \).

**Example 10.** Let \( X \) be a nonempty metric space with a fixed metric \( r \). Denote by \( \text{lip}(X) \) the set of all functions defined on \( X \), taking their values in \( \mathbb{R}_{\max} \), and satisfying the following *Lipschitz condition*:

\[
| f(x) \odot (f(y))^{-1} | = | f(x) - f(y) | \leq r(x, y),
\]

where \( x, y \) are arbitrary elements of \( X \). The set \( \text{lip}(X) \) consists of continuous real-valued functions (but not all of them!) and (by definition) the function which is equal to \(-\infty = 0\) at each point \( x \in X \). It is easy to check that \( \text{lip}(X) = \text{lip}(X, d) \), where \( d(x, y) = -r(x, y) \) (see Example 8 above); so \( \text{lip}(X) \) has a structure of an idempotent space over the semifield \( \mathbb{R}_{\max} \). Spaces of the form \( \text{lip}(X) \) are said to be *Lipschitz spaces* (see also [8], example 2.9.12). These spaces are \( b \)-nuclear and for each Lipschitz space the kernel theorem holds (see Theorem 4 below).

For \( x \in X \) denote by \( d_x \) the function on \( X \) defined by the formula \( d_x : y \mapsto d(x, y) \). By virtue of the equality (4) all functions of the form \( d_x \) are elements of the space \( \text{lip}(X, d) \). Denote by \( \text{lip}_0(X, d) \) the subsemimodule of \( \text{lip}(X, d) \) generated by these functions (i.e. consisting of all their finite linear combinations).

**Proposition 7** Suppose that a semimetric \( d \) takes its values in a \( b \)-complete semiring \( K \). Each \( b \)-subsemimodule \( V \) of \( \text{lip}(X, d) \) such that \( V \supset \text{lip}_0(X, d) \) is a lower ideal in \( \text{lip}(X, d) \) in the sense of the lattice theory [33]. This means that if a function \( f \) is an element of \( V \), then \( V \) contains all the elements of \( \text{lip}(X, d) \) majorized\(^4\) by \( f \). In particular, \( V \) is closed in \( \text{lip}(X, d) \) (but not

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\(^3\)Lower semicontinuous functions taking their values in \( \mathbb{R}_{\max} \) are defined in the same way as in Example 3 (see Section 1.2 above), but \( X_s \) is defined as the set \( \{ x \in X | f(x) \leq s \} \); see also [8].

\(^4\)Recall that an element \( g \) is majorized by \( f \) if \( g \preceq f \).
necessarily in $K(X)$ under the operation of taking infima (the greatest lower bounds) of arbitrary nonempty subsets.

**Proof.** Suppose $f \in V$, $g \in \text{lip}(X,d)$ and $g \preceq f$. Then the set of functions $S = \{ g(x) \odot d_x | x \in X \} \subset \text{lip}_0(X,d) \subset V$ is bounded in $V$ because all elements of this set are majorized by $f$. Using our definition of $\text{lip}(X,d)$, for each $g \in \text{lip}(X,d)$ we have $g(x) = \bigoplus_{y \in X} g(y) \odot d_y$, i.e. $g = \bigoplus_{y \in X} g(y) \odot d_y$ and $g = \bigoplus S$ is an element of $V$. □

**Theorem 4** Suppose that a $b$-complete semiring $K$ and a nonempty set $X$ are given and $V$ is a nondegenerate functional $b$-subsemimodule of $K(X)$. For $V$ the kernel theorem holds if and only if there exists a semimetric $d$ on $X$ such that $V$ is a $b$-subsemimodule of $\text{lip}(X,d)$ and $V \supset \text{lip}_0(X,d)$. In particular, if $V$ is a $b$-closed subsemimodule of $K(X)$, then $V = \text{lip}(X,d)$.

**Proof.** Suppose that the kernel theorem holds for $V$ and $d$ is a kernel of the identity operator $id : V \to V$. Then we can use $d$ as a semimetric on $X$. Indeed, for each $x \in X$, the function $d_x : y \mapsto d(x,y)$ is an element of $V$ by the kernel definition and $d_x = id(d_x) = \bigoplus_{z \in X} d_x(z) \odot d_z$. This can be rewritten in the form $d(x,y) = \bigoplus_{z \in X} d(x,z) \odot d(z,y)$ for each $y \in X$ because $V$ is a functional $b$-semimodule. So $d$ is a semimetric on $X$ and $\text{lip}_0(X,d) \subset V$. Then for each $f \in V$ we have $f = id(f) = \bigoplus_{z \in X} f(z) \odot d_z$; so $f(y) = \bigoplus_{z \in X} f(z) \odot d(z,y)$ for each $y \in X$. Thus $V \subset \text{lip}(X,d)$.

On the other hand if $V$ is a $b$-subsemimodule of $\text{lip}(X,d)$ and $V$ contains $\text{lip}_0(X,d)$, then each function of the form $d_z$ belongs to $\text{lip}_0(X,d) \subset V$. So, by our definition of $\text{lip}(X,d)$, the corresponding semimetric $d$ is a kernel of the identity mapping $id : V \to V$ and $d \in \text{kern}_V(X)$. □

**Corollary 5** Lipschitz spaces are nuclear. Nondegenerate semi-Lipschitz semimodules over $b$-complete semirings are nuclear.

Note that Corollary 5 is also true for degenerate admissible semi-Lipschitz semimodules over commutative semirings (see Remark 5 below).

**Remark 4.** If (under the conditions of Theorem 4) the $b$-semimodule $V$ is admissible and it is a $\wedge$-semimodule, then the semimetric $d$ constructed for
the proof of Theorem 4 is reflexive. In particular, if the $b$-semimodule $V$ is nondegenerate, then the maximal kernel of the identity operator (this integral kernel is constructed for the proof of Theorem 4) is a reflexive semimetric.

**Remark 5.** Suppose that a functional $b$-semimodule $V \subset K(X)$ is degenerate but admissible (for example, $V$ is admissible automatically if $K$ is a semifield). Then an analog of the Theorem 4 is true for the semimodule consisting of all restrictions from $V$ to the set $X_V = \{x \in X| (\exists f \in V): f(x) = 1\}$, see Proposition 3.

### 4 Integral representations of operators in abstract idempotent semimodules

In this section we examine the following problem: when a $b$-complete idempotent semimodule $V$ over a $b$-complete semiring is isomorphic to a functional $b$-semimodule $W$ such that the kernel theorem holds for $W$.

Suppose that $V$ is a $b$-complete idempotent semimodule over a $b$-complete semiring $K$ and $\phi$ is a $b$-linear functional defined on $V$. We call this functional a $\delta$-*functional* if there exists an element $v \in V$ such that

$$\phi(w) \odot v \preceq w$$

for each element $w \in V$. It is easy to see that every functional of the form $\delta_x$ is a $\delta$-functional in this sense (but the converse is not true in general).

Denote by $\Delta(V)$ the set of all $\delta$-functionals on $V$. Denote by $i_\Delta$ the natural mapping $V \to K(\Delta(V))$ defined by the formula

$$(i_\Delta(v))(\phi) = \phi(v)$$

for all $\phi \in \Delta(V)$. We shall call an element $v \in V$ *pointlike* if there exists a $b$-linear functional $\phi$ such that $\phi(w) \odot v \preceq w$ for all $w \in V$. The set of all pointlike elements of $V$ will be denoted by $P(V)$. Recall that by $\phi \odot v$ we denote the one-dimensional operator $w \mapsto \phi(w) \odot v$.

The following statement is an obvious consequence of our definitions (including the definition of the standard order) and idempotency of our addition.
Remark 6. If a one-dimensional operator $\phi \odot v$ appears in a decomposition of the identity operator on $V$ into a sum of one-dimensional operators, then $\phi \in \Delta(V)$ and $v \in P(V)$.

Denote by $id$ and $Id$ the identity operators on $V$ and $i_\Delta(V)$, respectively.

**Proposition 8**

1) If the operator $id$ is $b$-nuclear, then $i_\Delta$ is an embedding and the operator $Id$ is integral.

2) If the operator $i_\Delta$ is an embedding and the operator $Id$ is integral, then the operator $id$ is $b$-nuclear.

**Proof.** Statement 2) is obvious, so it is sufficient to prove statement 1). Then (by the condition of this statement) the operator $id$ is a sum of one-dimensional operators, i.e. $id = \bigoplus_{\phi \in \Delta} \phi \odot w_\phi$ for some collection of elements $w_\phi \in V$. In this case we have $Id = \bigoplus_{\phi \in \Delta} \delta_\phi \odot i_\Delta(w_\phi)$, i.e. the operator $Id$ is integral, as was to be proved. □

The following result is a direct consequence of Proposition 8, Theorem 3a, and Proposition 6.

**Theorem 5** A $b$-complete idempotent semimodule $V$ over a $b$-complete idempotent semiring $K$ is isomorphic to a functional $b$-semimodule for which the kernel theorem holds if and only if the identity mapping on $V$ is a $b$-nuclear operator, i.e. $V$ is a $b$-nuclear semimodule.

The following proposition shows that, in a certain sense, the imbedding $i_\Delta$ is a universal representation of a $b$-nuclear semimodule in the form of a functional $b$-semimodule for which the kernel theorem holds.

**Proposition 9** Let $K$ be a $b$-complete idempotent semiring, $X$ a nonempty set, and $V \subset K(X)$ a functional $b$-semimodule on $X$ for which the kernel theorem holds. Then there exists a natural mapping $i : X \to \Delta(V)$ such that the corresponding mapping $i_* : K(\Delta(V)) \to K(X)$ is an isomorphism of $i_\Delta(V)$ onto $V$.

**Proof.** The imbedding $i : X \to \Delta(V)$ is defined by the formula $i : x \mapsto \delta_x, x \in X$. This definition is correct by virtue of Proposition 1. It is easy to
check that the mapping $i_* : K(\Delta(V)) \to K(X)$ is an isomorphism of $i_\Delta(V)$ onto $V$. □

**Remark 7.** For the sake of simplicity, we treat only $b$-complete semimodules in this paper. Using the procedure of bounded completion and operating in the spirit of the paper [8], it is possible to extend a considerable part of the definitions and results to the case of incomplete semimodules over incomplete semirings.

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**References**

[1] V.P. Maslov, *Asymptotic methods for solving pseudodifferential equations*. Nauka, Moscow, 1987 (in Russian).

[2] V.P. Maslov, *Méthodes opératorielles*. MIR, Moscow, 1987.

[3] V.P. Maslov, V.N. Kolokoltsov, *Idempotent analysis and its application in optimal control*. Nauka, Moscow, 1994 (in Russian).

[4] V.N. Kolokoltsov, V.P. Maslov, *Idempotent Analysis and Applications*. Kluwer Acad. Publ., Dordrecht, 1997.

[5] V.P. Maslov, S.N. Samborskiı, Eds., *Idempotent Analysis*. Adv. Sov. Math. 13, Amer. Math. Soc., Providence, R.I., 1992.

[6] J. Gunawardena, Editor, *Idempotency*. Publ. of the I. Newton Institute, Cambridge, Cambridge Univ. Press, 1998.

[7] G.L. Litvinov, V.P. Maslov, *Correspondence Principle for Idempotent Calculus and Some Computer Applications*. (IHES/M/95/33). Institut des Hautes Études Scientifiques, Bures-sur-Yvette, 1995. See also in [6], p.420–443. E-print arXiv:math.GM/0101021 (http://arXiv.org).

[8] G.L. Litvinov, V.P. Maslov, G.B. Shpiz, *Idempotent functional analysis. An algebraic approach*. – Mathematical Notes 69, no. 5 (2001), 696–729; see also E-print arXiv:math.FA/0009128 (http://arXiv.org) (2000).
[9] G.L. Litvinov, V.P. Maslov (Eds.), *Idempotent Mathematics and Mathematical Physics*. Contemporary Mathematics, Vol. 377, Amer. Math. Soc., Providence, RI, 2005.

[10] G.L. Litvinov, *The Maslov dequantization, idempotent and tropical mathematics: a brief introduction*. – Journal of Mathematical Sciences (New York); see also E-print arXiv:math.GM/0507014 (http://arXiv.org) (2005).

[11] G.L. Litvinov, V.P. Maslov, G.B. Shpiz, *Tensor products of idempotent semimodules. An algebraic approach*. – Mathematical Notes 65, no. 4 (1999), 479–489; see also E-print arXiv:math.FA/0101153.

[12] G.L. Litvinov, V.P. Maslov, G.B. Shpiz, *Linear functionals on idempotent spaces: an algebraic approach*. – Doklady Mathematics 58, no. 3 (1998), 389–391; see also E-print arXiv:math.FA/0012268 (http://arXiv.org).

[13] V.N. Kolokoltsov, V.P. Maslov, *The general form of endomorphisms in the space of continuous functions with values in a numerical semiring with idempotent addition*. – Soviet Math. Doklady 36, no. 1 (1987), 55–59.

[14] V.N. Kolokoltsov, *Idempotent structures in optimization*. – Journal Math. Sci. 104, no. 1 (2001), 847–880.

[15] M. Akian, S. Gaubert, and V. Kolokoltsov, *Set coverings and invertibility of functional Galois connections*. – In: [9], 19–52.

[16] G. Cohen, S. Gaubert, and J.-P. Quadrat, *Duality and separation theorems in idempotent semimodules*. – Linear Algebra and its Applications 379 (2004), 395–422; see also E-print arXiv:math.FA/0212294 (http://arXiv.org).

[17] O. Viro, *Dequantization of real algebraic geometry on a logarithmic paper*. – In: 3rd European Congress of Mathematics, Barcelona, 2000; see also E-print arXiv:math.AG/0005163 (http://arXiv.org).

[18] G.L. Litvinov, A.N. Sobolevskii, *Exact interval solutions of the discrete Bellman equation and polynomial complexity of problems in interval idempotent linear algebra*. – Doklady Mathematics 62, no. 2 (2000), 199–201; see also E-print arXiv:math.LA/0101041 (http://arXiv.org).
[19] G.L. Litvinov, A.N. Sobolevskii, *Idempotent interval analysis and optimization problems*. – Reliable Computing 7, no. 5 (2001), 353–377; see also E-print arXiv:math.SC/0101080 (http://arXiv.org).

[20] L. Schwartz, *Théorie des noyaux*. Proc. of the Intern. Congress of Math., 1950, 1, 220–230.

[21] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*. – Mem. Amer. Math. Soc. 16, Providence (R.I.), 1955.

[22] H.H. Schäfer, *Topological vector spaces*. The Macmillan Company, New York and Collier-Macmillan Ltd, London, 1996.

[23] P.I. Dudnikov, S.N. Samborski, *Endomorphisms of semimodules over semirings with idempotent operation*. – Math. USSR Izvestiya 38, no. 1 (1992), 91–105.

[24] M.A. Shubin, *Algebraic remarks on idempotent semirings and the kernel theorem in spaces of bounded functions*. – In: V.P. Maslov and S.N. Samborski, Eds., *Idempotent analysis*, Adv. Soviet Math. 13, Amer Math. Soc., Providence R.I., 1992, 151–166.

[25] I. Singer, *Some relations between linear mappings and conjugations in idempotent analysis*. – J. Math. Sci. 115, no. 5 (2003), 2610–2630.

[26] G. Choquet, *Theory of capacities*. – Ann. Inst. Fourier 5 (1955), 131–295.

[27] W. Bryc, *Large deviations by the asymptotic value method*. – In: M. Pynsky (ed.), Diffusion Processes and Related Problems in Analysis, Birkhäuser, 1990, 447–472.

[28] A. Puhalskii, *Large deviations of semimartingales via convergence of the predictable characteristics*. – Stochastics 49 (1994), 27–85.

[29] V. Breier, O. Gulinsky, *Large deviations on infinite dimensional spaces*. – Preprint, Moscow, 1996 (in Russian).

[30] O.V. Gulinsky, *The principle of the largest terms and large deviations for a class of nonlinear functionals*, Preprint, 2001.
[31] J. Gunawardena, An Introduction to Idempotency. – In: [6], 1–49.

[32] G.L. Litvinov, G.B. Shpiz, Nuclear semimodules and kernel theorems in idempotent analysis. An algebraic approach. – Doklady Mathematics 66, no. 2 (2002), 197–199; see also arXiv:math.FA/0206026 (http://arXiv.org).

[33] G. Birkhoff, Lattice theory, 3rd edition. – Amer. Math. Soc. Colloquium Publications, vol XXV, Amer. Math. Soc., Providence R.I., 1967.