Quantization of Non-Critical Bosonic Open String Theory

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The theory of relativistic strings is considered in frames of Hamiltonian formalism and Dirac’s quantization procedure. A special gauge fixing condition is formulated, related with the world sheet of the string in Lorentz-invariant way. As a result, a new set of Lorentz-invariant canonical variables is constructed, in which a consistent quantization of bosonic open string theory could be done in Minkowski space-time of dimension \( d = 4 \). The obtained quantum theory possesses spin-mass spectrum with Regge-like behavior.

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1 Introduction

The model of relativistic string has been proposed by Nambu, Hara and Goto [1–3] in the early 1970s for the description of the internal structure of strongly interacting particles, the hadrons. The initial purpose was the explanation of spin-mass spectra of the hadrons, as well as certain experimentally established properties of their interaction amplitudes [4–6]. Nowadays the construction of the string-inspired models of the hadrons is continued [7–14]. The idealized theoretical model was equipped by necessary physical details: quarks at the ends of the string, carrying masses, electric charges, spin and other quantum numbers; different mechanisms of the string breaking, responsible for the decays of the hadrons. The progress of this construction has been reported in the book [15]. Meanwhile the development of string theory selected a different way. It was early recognized [16–19] that quantization of Nambu-Goto model possesses anomalies at the physical dimension of the space-time \( d = 4 \), and is free of anomalies at \( d = 26 \). It was also noticed that inclusion of additional fermionic degrees of freedom to the theory [20, 21] cancels the anomaly at less value of dimension \( d = 10 \). Later this approach was combined with the other idea: some of dimensions were considered as coordinates on a compact manifold of physically small size. Further developments, excellently reviewed in [22, 23], have included more complex mathematical structures to the theory, considered multi-dimensional extensions of the strings (p-branes), as well as their supersymmetric analogs, and formed a powerful direction of modern theoretical physics, providing a basis for the construction of Grand Unification Theory. However, this approach cannot be immediately applied in the models of hadrons, where the subject of consideration is bosonic 4-dimensional Nambu-Goto string, while the introduction of extra dimensions and additional degrees of freedom changes this system essentially.

Nowadays a lot of work has been done in the field of non-critical string theory, studying the possibilities for string quantization beyond the critical dimension \( d = 26 \). Polyakov [24] used the path integration technique to construct the quantum string theory at arbitrary \( d \). It has been shown in this paper that at \( d = 26 \) the theory admits oscillator representation, while at \( d \neq 26 \) the theory becomes equivalent to a non-linear field theory. In operator formalism the problem was considered by Rohrlich [25]. In this paper a consistent quantum theory was constructed for Nambu-Goto string at arbitrary \( d \). For this purpose a special Lorentz-invariant parametrization of the world sheet (time-like gauge) was used, with the gauge axis directed along total momentum \( P_\mu \). Lorentz-invariant parametrizations were also used in papers [12, 26–29] presenting other approaches to quantization of non-critical string theory.

This paper extends the approaches of [12, 25–29] to the gauge of a special form, which can be considered as Lorentz-invariant modification of light cone one. The paper is organized as follows. Section 2 introduces general concepts and a formalism, used for the elimination of anomalies from the Lorentz group in string theory. This formalism is a combination of general methods, comprehensively de-
scribed in [26, 30, 31], which in fact can be applied to any theory. Wherever possible, we provide the mathematical details of the methods and the description of their physical meaning. In Sections 3 and 4 we investigate the algebraic and geometric properties of the constructed mechanics. Section 5 is devoted to its quantization. The obtained results are discussed in Section 6.

2 General concepts

Quantization [30, 32] is a linear mapping $f \rightarrow \hat{f}$ of a pre-defined set of classical variables to a set of operators, satisfying the following correspondence principle: $i\hbar \{f, g\} \rightarrow \{\hat{f}, \hat{g}\}$. Here $\{\cdot, \cdot\}$ is Poisson bracket, $[,]$ is commutator, $\hbar$ is Planck’s constant. Additionally it is required that real-valued dynamical variables should be represented by Hermitian operators, the unity should be preserved: $1 \rightarrow 1$, the space of states should be positively defined and irreducible under the action of the constructed operators (does not have a smaller subspace invariant under their action).

It is shown in [32] that the correspondence principle cannot be satisfied for all dynamical variables in the theory. This property is related with ordering ambiguities for non-commuting operators. Indeed, let’s consider two sets of classical variables, related by some non-linear transformation: $a_i \leftrightarrow b_i$. If the correspondence principle is satisfied for the set $a_i$: $\{a_i, a_j\} \rightarrow \{\hat{a}_i, \hat{a}_j\}$, then the commutator of operators $\{\hat{b}_i(\hat{a}_i), \hat{b}_j(\hat{a}_j)\}$, represented as non-linear functions of $\hat{a}$, can contain ordering ambiguities, which create anomalous terms absent in Poisson bracket $\{b_i, b_j\}$. In spite of the fact that a contribution of such terms is suppressed by additional factor $\hbar$, their occurrence can lead to serious problems in quantum theory, especially if $b_i$ represent the generators of symmetries of the system. On the other hand, if the correspondence principle is satisfied for the set $b_i$, the commutator $\{\hat{b}_i, \hat{b}_j\}$ is postulated directly from Poisson bracket $\{b_i, b_j\}$ and has no anomalies. The anomalies can appear in the commutator $\{\hat{a}_i(\hat{b}_i), \hat{a}_j(\hat{b}_j)\}$. The common practice is to select a convenient basis of independent variables, where the quantization is performed, trying to keep all generators of symmetries simple functions of independent variables and thus to avoid the occurrence of anomalies in their commutators.

To make this general consideration more concrete, let $a_i$ represent the set of oscillator variables of string theory in the light cone gauge, and $b_i = (M_{\mu\nu}, \xi_i)$, where $M_{\mu\nu}$ are the generators of Lorentz group and $\xi_i$ is a set of variables complementing $M_{\mu\nu}$ to the full phase space. In selection of such a set one needs to take care that $\xi_i$ will have simple Poisson brackets with $M_{\mu\nu}$ and among themselves. Local existence of such variables is provided by Darboux theorem [31], while the determination of the global structure for their region of variation is generally a complex task [26]. Further, performing the quantization of string theory in variables $a_i$, we obtain well known anomaly in the commutator $[M_{\mu\nu}(a_i), M_{\rho\sigma}(a)]$. On the other hand, performing the quantization of string theory in variables $b_i$, we do not have the anomaly in $[M_{\mu\nu}, M_{\rho\sigma}]$. Anomalies can appear in the commutator $[a_i(b), a_j(b)]$. However, the point is that there is no more necessity to use the variables $a_i$ in the theory, because the role of internal variables is now played by $\xi_i$, whose commutator does not have the anomalies. Also, if such necessity would appear, the anomaly in $[a_i(b), a_j(b)]$ does not lead to any further problems in the theory. Even if the anomalies would be excluded both from $[a_i, a_j]$ and $[b_i, b_j]$, one can always find the third set of variables $c_i$, highly non-linear in terms of independent ones that their commutator $[c_i, c_j]$ will have the anomalies. As we already mentioned, the correspondence principle cannot be satisfied for all variables in the theory, according to the theorem [32].

It becomes clear that the anomalies essentially depend on the choice of canonical variables in which the quantization is performed. This property was previously discussed in [26]. Actually, this fact is well known and even presented in the textbooks on string theory, e.g. in [23] on p.157: “Should one not try to use a different representation of the string operators so as to avoid the central charge? Again, it might very well be possible to construct such a representation and, if so, it is very likely that the resulting quantum theory would be very different from the one explained here. It could be that this yet-to-be-constructed theory would possess an intrinsic interest of its own (e.g., through the occurrence of infinite-dimensional representations of the Lorentz algebra). Moreover, because this theory would not be based on the use of oscil-
lator variables, it might be more easily extendable to higher-dimensional objects, such as the membrane." However, an opinion can be often encountered that anomalies in string theory are stable, i.e. can be transferred from one sector to the other one, but cannot be completely removed. Really this no-go theorem has never been proven. It is only proven that string theory in the oscillator representation, formed by Fourier coefficients of the world sheet expansion, has similar anomalies both in covariant and non-covariant approach. It is not proven that anomaly is present in every possible representation. The paper [33] can be considered as an attempt to build such a proof using geometrical quantization technique. The geometrical quantization [32] is an implementation of canonical procedure by means of differential geometrical constructions available in the classical theory. Most of the elements of this technique are coordinate independent. However, it was pointed out in [32] that as a whole it is not coordinate independent due to the choice of irreducible component in the space of states, known also as polarization. Holomorphic polarization chosen in [33] makes the theory equivalent to the oscillator one, reproducing the anomaly inherent to this representation. Again, it has not been proven that the anomaly is present in every polarization. It gradually becomes clear that the above mentioned no-go theorem is not satisfied, because the canonical quantization in the variables \((M_{\mu\nu}, \xi_i)\) immediately excludes the anomaly from the Lorentz group. Such a set was explicitly constructed in the paper [26]. An alternative set will be constructed in the given paper using a different representation.

Quantization of theories with gauge symmetries has own peculiarities. On the classical level such theories have Dirac’s constraints \(L_n\), which generate gauge symmetries via Poisson brackets [30]. There are two main approaches for quantization of such theories. In the first one, called covariant quantization, the constraints are imposed on the state vectors \(L_n \Psi = 0\), thus selecting gauge invariant wave functions in the space of states. If the algebra \([L_n, L_m]\) has the anomaly, it cannot be implemented by means of canonical formalism, but its certain extensions can be used, such as implementation only a half of constraints in the theory [23]. In the second approach the gauge fixing conditions are imposed in addition to the primary constraints, which select one representative from each gauge orbit. In this case the whole set of constraints belongs to the 2nd class in Dirac’s terminology [30], the mechanics should be reduced to their surface by corresponding redefinition of Poisson brackets, and after that can be quantized. An intermediate position is taken by the “reduced phase space” formalism [34], where the role of the phase space is played by the factor-space with respect to the group of gauge symmetry. In frames of this formalism the gauge fixation corresponds to a choice of basis in the space of gauge invariants. Indeed, in string theory the world sheet can be parameterized by a temporal component of coordinate along some axis \(n_\mu\). Light-like vector \(n_\mu\) corresponds to the light cone gauge, while \(n_\mu = P_\mu/\sqrt{P^2}\), where \(P_\mu\) is total momentum, is Rohrlich’s gauge [25]. The oscillator variables are Fourier coefficients

\[
a_k = \int d\sigma \, Q'(\sigma) e^{ik\sigma},
\]

where \(Q'(\sigma) = x'(\sigma) + p(\sigma)\) and \(\sigma\) is the light cone or Rohrlich’s gauge parameter: \(\sigma = \pi(nQ)/(nP)\).

Obviously, the same variables can be written in the form of parametric invariants

\[
a_k = \int dQ \, e^{ik\pi(nQ)/(nP)}. \tag{1}
\]

For the light cone gauge this formula gives well known DDF variables [35]. In terms of these variables the world sheet can be reconstructed (up to translations). Thus, the gauge fixation does not make the theory restricted or weak in some other sense, but provides the particular choice of basis in the space of gauge invariants. When \(n_\mu\) is changed, the basis \(a_k(n_\mu)\) is changed accordingly. As we already know, such change influences the anomalies, which therefore depend on the selected gauge.

This effect can be also understood using the following geometrical consideration. In the standard light cone gauge the vector \(n_\mu\) is non-dynamical, e.g. \(n_\mu = (1, 1, 0, 0)\). Because the Lorentz transformations change the position of the world sheet with respect to this axis, they are followed by reparametrizations of the world sheet. On quantum level the reparametrization group has anomaly, which appears also in Lorentz group and violates Lorentz covariance of the theory. This is
a main problem of string theory in standard light cone gauge. On the other hand, the Rohrlich’s gauge relates $n_\mu$ with the world sheet itself. As a result, the Lorentz generators transform $n_\mu$ and the world sheet simultaneously, without reparametrizations. The same property holds if one relates light-like axis $n_\mu$ with the world sheet, thus constructing Lorentz-invariant light cone gauge. In the paper [26] the $(M_{\mu\nu}, \xi_i)$ set was constructed with the aid of Rohrlich’s parametrization, while in the present paper we will use for this purpose the Lorentz-invariant light cone gauge. Below we present a general formalism used to construct various gauges of this kind and will select a particular one in the next Section. We will see also that the mechanism of anomaly cancellation from the Lorentz group is actually the same as in [26], namely – transition to such canonical basis, that the Lorentz group generators $M_{\mu\nu}$ will become simple functions of independent variables.

The construction of Lorentz-invariant light cone gauge uses the following general scheme, see fig.1. We consider a group $G$ of gauge transformations of the world sheet, induced by different choices of the gauge axis $n_\mu$ on the light cone. It is a finite-dimensional subgroup in a group of general reparametrizations of the world sheet. Standard fixation of the light cone gauge corresponds to a selection of one point in $G$, performed in Lorentz non-invariant way. The first step of our construction is a recovery of $G$-symmetry in the classical mechanics. The gauge axis $n_\mu$ becomes a dynamical variable, and the gauge group $G$ is canonically implemented by means of explicitly constructed Hamiltonian generators. Already at this step the Lorentz group generators $M_{\mu\nu}$ are constructed, which are gauge equivalent to the standard ones and transform the world sheet together with the gauge axis, rigidly attached to it. It is shown by explicit computation, that on quantum level the algebra of $M_{\mu\nu}$ has no anomalies. At this step the quantum anomaly is moved to the gauge group $G$. The second step consists in a choice of a new representative on the gauge orbit of $G$, this time related with the world sheet in Lorentz invariant way. We select a special representative, using gauge fixing conditions of Lorentz-invariant Abelian type, which lead to algebraically and geometrically simple mechanics. At the third step the gauge is finally fixed to this representative, and the anomalous gauge degree of freedom is eliminated.

![Fig.1. Lorentz-invariant fixation of the light cone gauge.](image)

Implementation of this program includes the following components.

**Geometrical description of the world sheet [26,36].** Let’s introduce a function, related with string’s coordinates and momenta by expressions

$$Q_\mu(\sigma) = x_\mu(\sigma) + \int_0^\sigma d\tilde{\sigma} \, p_\mu(\tilde{\sigma}),$$

$$x_\mu(\sigma) = (Q_\mu(\sigma) + Q_\mu(-\sigma))/2,$$

$$p_\mu(\sigma) = (Q_\mu(\sigma) + Q_\mu(-\sigma))/2$$

($x, p$ are even functions of $\sigma$). In terms of oscillator variables, commonly used in string theory:

$$Q_\mu(\sigma) = X_\mu + \frac{P_\mu}{\pi} \sigma + \frac{1}{\sqrt{\pi}} \sum_{n \neq 0} \frac{a_n^\mu}{i^n} \cos n\sigma.$$

The curve, defined by the function $Q_\mu(\sigma)$ (further called supporting curve) has the following properties: (i) the curve is light-like: $Q^2(\sigma) = 0$, this property is equivalent to Virasoro constraints on oscillator variables; (ii) the curve is periodical: $Q(\sigma + 2\pi) = Q(\sigma) = 2P$; (iii) the curve coincides with the world line of one string end: $x(0, \tau) = Q(\tau)$; the world line of another end is the same curve, shifted onto the semi-period: $x(\pi, \tau) = Q(\pi + \tau) = P$; (iv) the whole world sheet is reconstructed by this curve as follows:

$$x(\sigma, \tau) = (Q(\sigma_1) + Q(\sigma_2))/2, \sigma_{1,2} = \tau \pm \sigma;$$

(v) Poisson brackets for the function $Q_\mu(\sigma)$ are:

$$\{Q_\mu(\sigma), Q_\nu(\tilde{\sigma})\} = -2g_{\mu\nu}\tilde{\vartheta}(\sigma - \tilde{\sigma});$$

(vi) Virasoro constraints generate reparametrizations of supporting curve: $H = \int d\sigma F(\sigma)Q^2(\sigma)/4$, $\dot{Q}_\mu(\sigma) = \{Q_\mu(\sigma), H\} = F(\sigma)Q'_\mu(\sigma)$. Hamiltonian with $F = 1$ generates uniform shifts of the argument $Q(\sigma) \rightarrow Q(\tau + \sigma)$, corresponding to the evolution of the string in conformal parametrization on the world sheet: $(\dot{x} \pm x')^2 = 0$. Here $\vartheta(\sigma) = $
where $\sigma / 2\pi + 1/2$, $[x]$ is integer part of $x$, the derivative $\delta(\sigma) = \Delta(\sigma)$ is periodical delta-function; $P_\mu$ is total momentum of the string.

These properties are proven in [26] and by the other method in [36].

Mechanics in center-of-mass frame [26].

Let’s introduce orthonormal tetrad of vectors: $N_\mu^\alpha$, where $N_\mu^\alpha N_\mu^\beta = g^{\alpha\beta}$ and $N_\mu^0 = P_\mu / \sqrt{P^2}$. Let’s decompose the supporting curve by this tetrad:

$$Q_\mu(\sigma) = N_\mu^\alpha Q^\alpha(\sigma).$$

Remark: following to [26], it is assumed that variables $N_\mu^\alpha$ depend only on total momentum $P_\mu$ and constant non-dynamical vectors. Particularly, the space-like components $N_\mu^i$ can be explicitly constructed in terms of time-like vector $P_\mu$ and constant linearly independent space-like vectors $V_\mu^i$ by application of Gram-Schmidt orthonormalization procedure to $(P_\mu, V_\mu^i)$. After that, the dynamical action of Lorentz group cannot transform $N_\mu^\alpha$ as Lorentz vectors, but creates more complicated transformation law, see [27]. On the other hand, the performed substitution obviously does not change the Poisson brackets $\{M_{\mu\nu}, M_{\rho\sigma}\}$, $\{Q_\mu, M_{\rho\sigma}\}$, provided that Poisson brackets of new canonical variables $Q^\alpha$ are derived from old ones $Q_\mu$ using standard formalism, see Appendix 1. Therefore, the variables $Q_\mu$ and $M_{\mu\nu}$, being expressed in terms of new canonical variables $Q^\alpha$, are still transformed as Lorentz tensors. Further details of this technique are available in [26, 27].

The projection of supporting curve to CMF, given by space-like components $Q^i(\sigma)$, is a closed curve. The total length of this curve equals $2\sqrt{P^2}$. The total area vector for the oriented surface pulled on this curve is independent on the surface and equals $2\tilde{S}$, where $\tilde{S}$ is orbital moment of the string in CMF. These properties are also proven in [26, 36].

Lorentz-invariant light cone gauge [29].

Virasoro constraints generate reparametrizations of supporting curve, and the gauge fixing conditions to Virasoro constraints select particular parametrization on this curve. In principle, any parametrization can be constructed, introducing any necessary dynamical variables, but only some of them lead to simple Hamiltonian mechanics. Usually the following parametrization is selected: $\sigma = \pi(nQ)/(nP) + \text{Const}$, i.e. for the parameter $\sigma$ a component of $Q_\mu$ in the direction $n_\mu$ is used. The vector $n_\mu$ is called gauge axis. As we have mentioned above, the choice of the gauge axis in the direction of total momentum: $n_\mu = P_\mu / \sqrt{P^2}$ corresponds to time-like Rohrlich’s gauge [25], while the choice of the gauge axis in light-like direction: $n^2 = 0$ is called light cone gauge. In the last case the common procedure is to consider light-like vectors $n_\pm = (1, \pm \sqrt{2}, 0, 0, 1)$, $n^2 = 0$, $n_+ n_- = 1$, define the components of $Q'_\mu$ along $n_\pm$ as $Q'_\pm = (n_\mp Q')$, and the space-like component, orthogonal to $n$, as $Q' \perp$, select the gauge axis in the direction $n_- : Q'_- = P_+/\pi$, and using the condition $Q'^2 = 2Q'_+ Q'_- - Q'^2_\perp = 0$, finally express the remaining component in terms of $P$ and $Q'_\perp : Q'_\perp = \pi(Q'_\perp)^2 / 2P_+$. Then, substituting Fourier expansion for $Q'_\perp$, we can express the whole mechanics in terms of total momentum, mean coordinate and Fourier coefficients (transverse oscillator variables).

Let’s introduce the following parametrization on the supporting curve:

$$Q^\alpha(\sigma) = Q^\alpha(0) + \int_0^\sigma d\sigma' a^\alpha(\sigma'),$$

$$a^\alpha(\sigma) = \frac{\pi}{2\sqrt{P^2}} \left( \frac{P_\mu}{\pi^2} + |a(\sigma)|^2 \right),$$

$$\bar{a}(\sigma) = \frac{\pi}{2} \sum_{n \neq 0} a_n e^{-in\sigma}$$

$$+ \frac{\pi}{2\sqrt{P^2}} \left( \frac{P_\mu}{\pi^2} - |a(\sigma)|^2 \right) e_3,$$

where $a(\sigma) = \sqrt{\frac{\pi}{2}} \sum_{n \neq 0} a_n e^{-in\sigma}$ and $e_3$ is an orthonormal basis in CMF. Here we easily recognize the light cone gauge $Q^+ (\sigma) \equiv n_\mu^0 Q_\mu (\sigma) = Q^+ (0) + \frac{1}{\sqrt{2}} \sqrt{\frac{P}{\pi^2}} \sigma$ with the gauge axis $n_\mu = \frac{1}{\sqrt{2}} (N_\mu^0 - N_\mu^1 e_3)$. The difference from standard approach is that $n_\mu$
is now dynamical variable, because we do not fix \( \vec{e}_3 \) to a constant vector. Using this parametrization, we obtain the following mechanics:

\[
P_\mu, Z_\mu + \text{infinite set of oscillators } a_k, a_k^* + \text{the top } \vec{e}_i, \vec{S},
\]

with Poisson brackets, derived in Appendix 1:

\[
\{Z_\mu, P_\nu\} = g_{\mu\nu}, \\
\{a_k, a_k^*\} = i k \delta_{kn}, k, n \in \mathbb{Z}\setminus\{0\}, \\
\{S^i, S^j\} = -\epsilon^{ijk} S^k, \{S^i, e_n^j\} = -\epsilon^{ijk} e_n^k.
\]

Here \( Z_\mu = \frac{1}{2\sqrt{P_\mu^2}} \int_0^{2\pi} d\sigma\, a^0(\sigma)(Q_\mu(\sigma) - \frac{e}{\pi} - 1) P_\mu + \frac{1}{2} \epsilon^{ijk} \Gamma_{ij}^k \) is Newton-Wigner mean coordinate \([26]\), conjugated to \( P_\mu; \Gamma_{ij}^k = N_{i}^{j} \partial N_{j}^{i} / \partial P_{\mu} \) are Christoffel symbols and \( \vec{S} = -\frac{1}{2} \int_0^{2\pi} d\sigma\, \int_0^{\sigma} d\sigma' \vec{a}(\sigma) \times \vec{a}(\sigma') \) is an orbital moment of the string in CMF.

The mechanics is restricted by four constraints of the 1st class:

\[
\{\chi_0, \chi_i\} = 0, \{\chi_i, \chi_j\} = \epsilon_{ijk} \chi_k,
\]

which include mass shell condition and requirements of the form “spin of the top is equal to the spin of the string”:

\[
\chi_0 = \frac{e^2}{2\pi} - L_0 = 0, \quad L_0 = \sum_{n \neq 0} a_n^* a_n, \\
\chi_3 = S_3 - A_3 = 0, \quad A_3 = \sum_{n \neq 0} \frac{1}{2} a_n^* a_n, \\
\chi_+ = S_+ - A_+ = 0, \quad \chi_- = S_- - A_- = 0, \\
\chi_\pm = \chi_1 \pm i \chi_2, \quad S_\pm = S_1 \pm i S_2, \\
A_+ = \sqrt{\frac{2}{\pi}} \sum_{k,n,k+n \neq 0} \frac{1}{2} a_k^* a_n^* a_{k+n}^* \\
A_- = \sqrt{\frac{2}{\pi}} \sum_{k,n,k+n \neq 0} \frac{1}{2} a_k a_n a_{k+n},
\]

where \( S_1 = S^k e_k^i \) is a projection of \( \vec{S} \) onto \( \vec{e}_i \). According to Dirac’s theory of constrained mechanical systems \([30]\), the constraints of the 1st class generate gauge transformations:

- \( \chi_0 \) generates phase shifts of oscillator variables \( E_0 : a_n \rightarrow a_n e^{-i \alpha} \) and translations of mean coordinate \( Z \rightarrow Z + P\tau / \pi \); these transformations shift the argument \( Q(\sigma) \rightarrow Q(\sigma + \tau) \); equivalently they produce a reparametrization of the world sheet, related with the evolution of the string;

- \( \chi_3 \) generates phase shifts \( R_3 : a_n \rightarrow a_n e^{-i \alpha} \) and rotations of \( \vec{e}_{1,2} \) about \( \vec{e}_3 \): \( \vec{e}_i \rightarrow R(\vec{e}_3, -\alpha) \vec{e}_i \), \( i = 1, 2 \); these transformations preserve \( Q(\sigma) \) and points on the world sheet;

- \( \chi_{1,2} \) generate rotations of basis \( \vec{e}_i \) about axes \( \vec{e}_{1,2} \) and certain non-linear transformations of oscillator variables; these transformations change the direction of the gauge axis and perform corresponding reparametrizations of the supporting curve and the world sheet.

The obtained parametrization of the world sheet differs from the standard light cone gauge by the introduction of six variables \( (\vec{e}_i, \vec{S}) \), forming the mechanics of the top. It also includes three constraints of the 1st class, which directly eliminate three degrees of freedom \( (\vec{S} = \vec{A}) \), and generate gauge transformations, identifying three remaining ones (physical observables are independent on the choice of \( \vec{e}_i \)). According to Dirac’s theory of constraints \([30]\), such implementation leads to equivalent mechanical system. At this point we have recovered the gauge symmetry \( G \), mentioned in Section 2 in the classical mechanics. It is the group of the world sheet reparametrizations, related with a different choices of the gauge axis. Now the gauge axis is a dynamical variable, presented by the unit norm 3-vector \( \vec{e}_3 \) in the center-of-mass frame, and by the above defined light-like 4-vector \( n_\mu^- \) in Minkowski space. \( G \)-symmetry is generated canonically by the constraints \( \chi_i \), with factorization by the trivial subgroup \( R_3 \), and possesses the group manifold \( G = SO(3) / SO(2) = S^2 \).

**Lorentz group generators** \([26]\) are given by expression

\[
M_{\mu\nu} = \int_0^{\pi} d\sigma (x_\mu p_\nu - x_\nu p_\mu) = \\
= X_\mu P_\nu - X_\nu P_\mu + \epsilon_{ijk} N_{i}^{\mu} N_{j}^{\nu} S^k, \\
X_\mu = Z_\mu - \frac{1}{2} \epsilon_{ijk} \Gamma_{ij}^k S^k,
\]

they generate Lorentz transformations of a coordinate frame \( (N_{\mu}^0, N_{\mu}^k e^k) \), by which the configuration is decomposed with scalar coefficients. Thus, \( M_{\mu\nu} \) generate “rigid” Lorentz transformations of the world sheet, not changing its parametrization. Lorentz generators are in involution with constraints: \( \{M_{\mu\nu}, \chi_{0,i}\} = 0 \). This fact means simul-
taneously that the generators of gauge transformations $\chi_{0,i}$ are Lorentz-invariant and generators of Lorentz group $M_{\mu\nu}$ are gauge-invariant.

Lorentz generators are simple functions of variables $(Z, P, \vec{S})$, which in our approach are independent, i.e. their quantum commutators are postulated directly from Poisson brackets. In [27] it has been shown by direct calculation that under these conditions the quantum commutators $[M_{\mu\nu}, M_{\rho\sigma}]$ are anomaly free. We present this computation in Appendix 2. This result is not surprising, because after the performed formal substitutions the generators of rotations in CMF became independent canonical variables. At this point the anomaly is not removed from the theory, but transferred from the Lorentz group to the gauge group $G$. Indeed, the constraints $\chi_{i}$ are cubic in terms of oscillator variables and in quantization their algebra acquires exactly the same anomaly that earlier was in Lorentz group. Now we should do the second step of the diagram fig.1: select the gauge fixing conditions, which will eliminate the gauge freedom associated with $G$-symmetry. The gauge fixing conditions of the type $\vec{e}_{3} = (1, 0, 0)$ would return us immediately to the standard light cone gauge, with the anomaly in Lorentz group. We will use the alternative gauge fixing conditions, which relate $\vec{e}_{3}$ with respect to other dynamical vectors in the system, and introduce Lorentz-invariant parametrization on the world sheet.

At first, we will represent the straight-line string solution [27] in the constructed light cone gauge. For this solution the projection of supporting curve to CMF is a circle, and the string itself has a form of a straight segment rotating about its middle at constant angular velocity. According to [27], this solution allows anomaly-free quantization at arbitrary dimension $d \geq 3$. It belongs to a border of classical Regge-plot $P^{2}/2\pi \geq S$, and in quantum theory corresponds to a leading Regge-trajectory $P^{2}/2\pi = S$. In our system of variables this solution corresponds to a single excited mode $a_{1}$, if the gauge axis $\vec{e}_{3}$ is directed along the spin $\vec{S}$:

$$a_{n} = 0, \ n \neq 1, \ S_{\pm} = 0, \ \frac{P^{2}}{2\pi} = S = S_{3} = |a_{1}|^{2}. \quad (7)$$

Further we will refer to this case as northern pole solution (where the circle $\vec{Q}(\sigma)$ defines the equator). There is a gauge-equivalent one-modal solution with exited $(a_{-1})$-mode and $\vec{e}_{3}$ opposite to $\vec{S}$ (southern pole solution). Other directions of gauge axis for the straight-line string give infinitely-modal solutions. Later we will use the solution [7] to study the structure of general theory in its vicinity.

**Gauge fixing conditions** we propose have a form

$$a_{s} + a_{-s} = 0, \ a_{s}^{*} + a_{-s}^{*} = 0 \quad (8)$$

for some $s > 0$. The straight-line string solution [7] satisfies these conditions at $s > 1$. Later, in Section 4 we will show that conditions [8] can be imposed on any solution of string theory.

The gauge fixing conditions [8] are preserved by transformations $R_{3}$. They are not preserved by $E_{0}$, however, there is a remainder of $E_{0}$-symmetry, discrete transformation

$$D_{2s}: \ a_{n} \to a_{n}e^{-in\pi/s}$$

preserving [8]: $a_{s} + a_{-s} = 0 \to -a_{s} - a_{-s} = 0$. Therefore, $R_{3}, D_{2s}$-symmetries are present in the theory after gauge fixation.

The procedure of gauge fixation, described in Appendix 1, is the third step on the diagram fig.1. It results to the same canonical basis as [4], but without $a_{\pm s}$ oscillators. This exclusive property (simplicity of Poisson brackets) follows from the fact that two gauge fixing conditions [8] are in involution with each other: $\{a_{s} + a_{-s}, a_{s}^{*} + a_{-s}^{*}\} = 0$, i.e. generate Abelian group of transformations. By this fact our approach differs from non-Abelian gauges, considered in [37] Part I, which possess complicated Poisson brackets. Further we refer to the constructed parametrization as Lorentz-invariant Abelian light cone gauge (lia-leg).

The oscillators $a_{\pm s}$ now become dependent variables, whose expressions should be found from $\chi$-constraints. The contribution of $a_{\pm s}$ in $A_{3}$ vanishes: $\frac{1}{2}(a_{s}^{2} - |a_{-s}|^{2}) = 0$, as a result, $a_{\pm s}$-terms drop out from $\chi_{3} = S_{3} - A_{3}$. This result follows from the fact that gauge fixing conditions [8] are preserved by transformation $R_{3}$ and therefore are in involution with $\chi_{3}$. The gauge fixing conditions [8] are not in involution with $\chi_{0}$, and $a_{\pm s}$-terms in $L_{0}$ do not vanish: $|a_{s}|^{2} + |a_{-s}|^{2} = 2|a_{1}|^{2}$. Poisson brackets of [8] with $\chi_{\pm}$ also do not vanish. We conclude that [8] are gauges for $(\chi_{\pm}, \chi_{0})$, and $a_{\pm s}$ should be determined from these three constraints.
3 Algebraic properties of lia-lcg

Isolating contribution of \(a_{\pm s}\)-oscillators in \((8)\) and using the relation \(a_{-s} = -a_s\), we have

\[
a_s^2 d + \frac{1}{2} a_s a_s^* d + a_s f + a_s^* g + \Sigma_+ - \sqrt{\frac{p^2}{2\pi}} S_+ = 0,
\]

\[
a_s^{2s} d^* + \frac{1}{2} a_s a_s^* d + a_s f^* + a_s g^* + \Sigma_+ - \sqrt{\frac{p^2}{2\pi}} S_+ = 0,
\]

\[
p^2 = L_0^{(s)} + 2a_s a_s^*,
\]

where

\[
d = d_+ - d_-, \quad f = f_+ - f_-, \quad g = g_- - g_+, 
\]

\[
d_+ = a_{2s}^*/s, \quad d_- = a_{-2s}/s, \quad L_0^{(s)} = \sum_{k} a_k^* a_k, 
\]

\[
g_- = \sum_{k} \left( \frac{1}{s} + \frac{1}{k} \right) a_k a_{s-k}, \quad g_+ = \sum_{k} \left( \frac{1}{s} + \frac{1}{k} \right) a_k a_{-s-k}, 
\]

\[
f_+ = \sum_{k} \left( \frac{1}{s} + \frac{1}{k} \right) a_k a_{s+k}, 
\]

\[
 f_- = \sum_{k} \left( \frac{1}{s} + \frac{1}{k} \right) a_k a_{s-k}, 
\]

\[
 \Sigma_+ = \sum_{k} a_k^* a_k + n_2, \quad \Sigma_- = \sum_{k} a_k^* a_{-k} + n_2, 
\]

\[
 \Sigma^* = \Sigma_+, 
\]

Here in the sums \(\sum_{k}^\prime\) terms with \(a_0^{(s)}\) and \(a_s^{(s)}\) are excluded. Introducing denotations

\[
\lambda = \sqrt{\frac{p^2}{2\pi}}, \quad n_s = (\lambda^2 - L_0^{(s)})/2,
\]

\[
k = \Sigma_+ + \frac{1}{2} n_s d^* - \lambda S_-,
\]

we can treat \((8)\) as overdetermined polynomial system for \((a_s, a_s^*)\) at given values of coefficients \((d, f, g, k; n_s)\):

\[
a_s^2 d + a_s f + a_s^* g + k = 0, \quad (11)
\]

\[
a_s^{2s} d^* + a_s^* f^* + a_s g^* + k^* = 0, 
\]

\[
a_s a_s - n_s = 0. 
\]

In [37] this system was solved analytically using the technique of Groebner’s basis [39]. The result comprises a polynomial condition of consistency for the system \((11)\) and rational expressions for \(a_s\). The condition of consistency is the Dirac’s constraint, remaining after imposition of two gauge fixing conditions \((8)\) to three original constraints \(\chi_0, \chi_\pm\), it is a polynomial equation of 8th order in \(\lambda = \sqrt{\frac{p^2}{2\pi}}\), playing a role of new mass shell condition. Being used as Hamiltonian, it generates correct string evolution, consisting of shifts \(\sigma \rightarrow \sigma + \tau\) and such reparametrizations that keep gauge fixing conditions \((8)\) permanently satisfied.

Further analysis of this mechanics can be found in [37].

For the construction of quantum theory it’s more convenient to apply a different form of the mechanics, using expansion series in the vicinity of straight-line string \((7)\).

Solutions in the vicinity of straight-line string

Computing the Jacobian of the system \((9)\) on the straight-line string’s northern pole solution \((7)\), we see that this system is non-degenerate at \(s = 2\). In this case in the vicinity of \((7)\) it has a unique solution, representable as \(C^\infty\)-smooth analytical function of coefficients of the system. The solution is given by series:

\[
a_2 = \sum_{n=1} P_n a_1^n, \quad (12)
\]

where \(P_n\) are polynomials of \(a_1, \Sigma_-, S_-, d, f, g, g'\), their conjugates and \(\gamma, \gamma^{-1}\). Here \(g' = g_- - a_1^2\), \(g' = g_+ - g_-\). Explicit expressions for the first three polynomials \(P_n\) are given in Appendix 3, their general properties are studied in [37]. For the purposes of further consideration it is convenient to extract from \(a_2\) a common phase multiplier \(a_1^*/|a_1|^2\) and to define \(a_2 = a_2 \cdot a_1^2 / |a_1|^2\), so that \(|a_2| = |a_2|\)

\[
a_2 = \sum_{n=1} P_n a_1^n |a_1|^{4n-2}, \quad \frac{p^2}{2\pi} = L_0^{(2)} + 2|a_2|^2. \quad (13)
\]

4 Geometric properties of lia-lcg

In this Section we describe in more details the transformations, generated by constraints \(\chi_0, \chi_i\). At first, we introduce several definitions.

Let’s consider in 3D space: smooth closed curve \(\tilde{Q}(\sigma)\) with marked point \(O\) and unit norm vector \(\vec{e}_3\). Let’s introduce variables

\[
\vec{a}_n = \frac{1}{2\sqrt{2\pi}} \int d\tilde{Q}(\sigma) \cdot \exp \left( \frac{2\pi i}{L_t} \left( L(\sigma) - (\tilde{Q}(\sigma) - \tilde{Q}(0))\vec{e}_3 \right) \right), \quad (14)
\]

where \(L(\sigma)\) is a length of arc between points \(O\) and \(\tilde{Q}(\sigma)\) along the curve, \(L_t\) is total length of
the curve. Two properties obviously follow from the definition: \( \vec{a}_{-n} = \vec{a}_n^* \), \( \vec{a}_0 = 0 \). Let’s decompose vectors \( \vec{a}_n \) into the components, parallel and orthogonal to \( \vec{e}_3 \): 
\[
\vec{a}_n = a_n \vec{e}_3 + \vec{a}_{n\perp},
\]
and denote real and imaginary parts of \( \vec{a}_{n\perp} \) as \( \vec{q}_n \) and \( \vec{p}_n \): 
\[
\vec{a}_{n\perp} = \vec{q}_n + i\vec{p}_n.
\]
Let’s fix some \( n = s > 0 \) and write \( \vec{q}_s = \vec{q} \) and \( \vec{p}_s = \vec{p} \). Functions \( \vec{q}(\vec{e}_3), \vec{p}(\vec{e}_3) \) define smooth vector fields on unit sphere of \( \vec{e}_3 \) (tangent to the sphere). Due to topological “hedgehog” theorem, these fields have singular points on the sphere, where corresponding field vanishes, e.g. \( \vec{q} = 0 \).

Remark: the curve \( \vec{Q} \) is a projection of supporting curve to CMF. Gauge axis \( \vec{e}_3 \) relates the following parametrization to this curve:
\[
\sigma = \frac{2\pi}{L_1} (L(\sigma) - Q_3(\sigma) + Q_3(0)),
\]
where \( Q_3 = \vec{Q} \vec{e}_3 \). Now we recognize in \([14]\) Fourier modes of function \( \vec{a}(\sigma) = \vec{Q}(\sigma) \), where \( \sigma \) is leg-parameter \([15]\). This expression is written in parametric-invariant form, as circulation integral. Such form of definition is also known in string theory as DDF variables \([35]\). Vectors \( \vec{a}_{n\perp} \) are related with earlier introduced oscillator variables \( a_n \) as follows:
\[
\begin{align*}
\vec{a}_{n\perp} &= a_{n1} \vec{e}_1 + a_{n2} \vec{e}_2, \\
a_{n1} &= (a_n + a_{-n})/2, \quad a_{n2} = i(a_n - a_{-n})/2.
\end{align*}
\]

**Fig.2.** Definition of vector fields \( \vec{q}(\vec{e}_3), \vec{p}(\vec{e}_3) \).

From the above definitions the following properties become clear: \( \chi_3 \) generates transformations, preserving \( \vec{e}_3 \) and \( \vec{a}_{n\perp} \); \( \chi_{1,2} \) generate rotations of \( \vec{e}_3 \) and associated changes of \( \vec{a}_{n\perp} \) according to the formula \([14]\). Gauge fixing conditions \([8]\) correspond to a singular point \( \vec{q} = 0 \). Evolution, generated by \( \chi_0 \), is represented as phase rotations \( \vec{a}_{n\perp} \to \vec{a}_{n\perp} e^{-i\lambda t} \), equivalent to a motion of vectors \( \vec{q} \) and \( \vec{p} \) along the ellipse, shown at fig.2 on the right. During the evolution a vector \( \vec{a} = \vec{q} \times \vec{p} \) is preserved, and singular points \( \vec{q} = 0 \) move along zero-level curves of a function \( F(\vec{e}_3) = \vec{a} \vec{e}_3 = 0 \).

On these curves the ellipse shown at fig.2 right degenerates to a segment.

From here we see that gauge fixing conditions \([8]\) can be imposed on any solution of string theory. Namely, these conditions can be satisfied for any curve \( \vec{Q}(\sigma) \), directing the light cone gauge axis \( \vec{e}_3 \) to a singular point \( \vec{q}(\vec{e}_3) = 0 \) of a vector field on the sphere, constructed in terms of this curve. Therefore, \( \vec{e}_3 \) is implicitly expressed in terms of \( \vec{Q}(\sigma) \), which is uniquely related with the variables of original description: coordinates and momenta \( x(\sigma), p(\sigma) \). The vectors \( \vec{e}_{1,2} \) are not fixed and can freely rotate about \( \vec{e}_3 \). This gauge transformation is generated by the constraint \( \chi_3 \), which can be preserved in the theory, because its non-linearity is insufficient to create any anomalies.

Now let’s return to the consideration of singular points \( \vec{q}(\vec{e}_3) = 0 \). In general position the vector fields on the sphere have even number of singular points, which is \( \geq 2 \). We remind that non-degenerate singular points of 2-dimensional vector fields are \([40]\):

**Fig.3.** Non-degenerate singular points of 2-dimensional vector field.

**Definition 1:** For each type index of singularity is defined as algebraic number of rotations of vector \( \vec{q}(\vec{e}_3) \), when \( \vec{e}_3 \) passes around singular point \( (> 0 \), if directions of rotations of \( \vec{q} \) and \( \vec{e}_3 \) coincide; \( < 0 \) otherwise). For (center, node, focus) it is \( +1 \), for saddle \( -1 \).

The sum of all indices is equal to Euler characteristic of the surface, defined as (num. of vertices) 
\(-\) (num. of edges) \(+\) (num. of faces) for any tessellation of the surface. For the sphere this characteristic equals \( 2 \), thus generic vector field on a sphere has 2 singular points of index \( +1 \) and arbitrary number of self-compensating pairs \( (+1, -1) \). In special cases there can be degenerate singularities of more complex form, e.g. multi-saddles obtained by a fusion of several saddles.

Presence of several singular points \( \vec{q}(\vec{e}_3) = 0 \) leads to the fact that an orbit of the gauge group generated by \( \chi \)-constraints intersects the surface of gauge fixing conditions \([8]\) in several points of
the phase space. Because these points are gauge equivalent, the mechanics possesses discrete gauge symmetry. This phenomenon encountered in the theory of non-Abelian gauge fields, where it has been studied by V.N. Gribov [38]. In full generality this question was addressed in the paper [34]. We will call such equivalent points Gribov's copies. The following topological invariant can be introduced for Gribov’s copies.

Definition 2: let the phase space $M$ be a smooth orientable manifold. Let the orbit of gauge group $G$ and the surface of gauge fixing condition $F$ be its smooth orientable submanifolds with $\text{dim}(G) = \text{codim}(F)$. Let $P$ be the point of their transversal intersection. This means that in point $P$ the tangent spaces to $G$ and $F$ span the tangent space to $M$. Let $\bar{\tau}(F, P), \bar{\tau}(G, P)$ and $\bar{\tau}(M, P)$ be the bases in the tangent spaces, defining the orientation of $F$, $G$ and $M$, evaluated in point $P$. The index of intersection of $F$ and $G$ in point $P$ is defined as a number $\nu$, equal to $(+1)$ if the basis $(\bar{\tau}(F, P), \bar{\tau}(G, P))$ has the same orientation as the basis $\bar{\tau}(M, P)$, and equal to $(-1)$ if the orientations are opposite.

It has been shown in [37] that for the Gribov's copies of string theory the definitions 1 and 2 coincide.

In lia-lcg the copies comprise different sets of oscillator variables $\{a_n\}$, which reproduce the same but differently parameterized curves $\bar{Q}(\sigma)$. As a result, Gribov’s copies correspond to discrete reparametrizations of the world sheet. To identify the equivalent states, in classical theory the phase space should be factorized with respect to this symmetry. Analogous procedures can be applied in quantum mechanics, e.g. by constructing irreducible representations for operators possessing this discrete symmetry and formulating respective selection rules. However, on the quantum level the discrete non-linear reparametrizations of the world sheet can be violated by anomalies, excluding this symmetry from the theory.

The explicit formulae for the vector fields are given by (14). There is also an alternative definition:

$$\bar{a}_n = \frac{1}{2\sqrt{2\pi}} \oint d\bar{Q}(\sigma) \cdot \exp \left[ \frac{2\pi in}{L} \left( L(\sigma) - \frac{\bar{Q}(\sigma) - \bar{X}\bar{e}_3}{\bar{X} - \bar{Q}(0)}\bar{e}_3 \right) \right],$$

where $\bar{X} = \oint dL(\sigma)\bar{Q}(\sigma)/L$, defines an average position of the curve $\bar{Q}(\sigma)$ and coincides with the definition of mean coordinate (6) projected to CMF. The difference of (14) and (16) consists in $\bar{e}_3$-dependent phase factor $e^{in\varphi(\bar{e}_3)}$, $\varphi(\bar{e}_3) = 2\pi(\bar{X} - \bar{Q}(0))\bar{e}_3/L_t$. Because the evolution of $\bar{a}_n$ is the phase rotation, the phase factor actually introduces a difference of “local time” for the evolution of vector fields on the sphere. This factor preserves the orbits of Gribov’s copies and changes their evolution parameter from lcg’s (15) to the natural one (length of the curve). The definition (16) is more convenient to describe the evolution of Gribov’s copies. In [37] the structure of Gribov’s copies for nearly straight solution has been investigated using computer visualization of the vector fields and analytical criteria. It is shown, that for the straight-line string $\bar{Q}_0(\sigma) = (\cos \sigma, \sin \sigma, 0)$ the vector field $\bar{q}_{\perp}(\bar{e}_3)$ has the following singular points: at $s = 1$ two nodes on the equator; at $s = 2$ saddles on the northern and southern poles and four nodes on the equator; at $s > 2$ multi-saddles on the northern and southern poles and 2s nodes on the equator. During the evolution the nodes move along the equator, while the singular points in the poles stay fixed and saddle patterns rotate around them. After a small deformation of the string from straight configuration $\bar{Q}(\sigma) = \bar{Q}_0(\sigma) + \delta\bar{Q}(\sigma)$ the nodes move along a common trajectory in the vicinity of the equator; at $s = 2$ the saddles move in small loops near the poles; at $s > 2$ the multi-saddles are unfolded to $(s - 1)$ non-degenerate saddles moving near the poles. After the lapse of time $\Delta\tau_s = \pi/s = (\text{period of evolution})/2s$ the vector field reverses its direction and the pattern of singularities returns to the initial state ($D_{2s}$-symmetry). During this time the equatorial singularities move to the neighbor ones, pole singularities at $s = 2$ perform one revolution along the loops.
5 Quantum mechanics

Canonical operators
\[ [Z_\mu, P_\nu] = -ig_{\mu\nu}, \]
\[ [a_k, a_n^+] = k\delta_{kn}, \quad k, n \neq 0, \pm s, \] (17)
\[ [S^i, S^j] = i\epsilon_{ijk}S^k, \quad [S^i, e_j^k] = i\epsilon_{ikl}e_l^j, \]
\[ [S_i, S_j] = -i\epsilon_{ijk}S_k, \quad [S_i, e_j^k] = -i\epsilon_{ijk}e_k^j, \]
\[ [S^i, S_j] = 0, \quad e_i^k e_j^k = \delta_{ij}, \quad S_i = e_i^j S^j. \]

The space of states is a direct product of three components:

Space of functions \( \Psi(P) \) with the definition \( Z = -i\partial/\partial P \).

Fock space with a vacuum
\[ a_k|0\rangle = 0, \quad k > 0, \quad a_k^+|0\rangle = 0, \quad k < 0 \] (18)
and states created from vacuum by operators
\[ |\{n_k\}\rangle = \prod_{k>0,k\neq s} \frac{1}{\sqrt{k^{nk}n_k!}} (a_k^+)^{n_k} \cdot \prod_{k<0,k\neq -s} \frac{1}{\sqrt{(-k)^{nk}n_k!}} (a_k)^{n_k}|0\rangle. \]

For instance, we will write \(|1\rangle|2_1\rangle = \frac{1}{\sqrt{2}}a_1^+a_2^+|0\rangle\)

etc. So defined state vectors have positive norm. Occupation numbers
\[ n_k = \frac{1}{|k|} : a_k^+a_k : = \frac{1}{|k|} \cdot \left\{ \begin{array}{ll}
  a_k^+a_k, & k > 0 \\
  a_k^+a_k, & k < 0
\end{array} \right. \quad (0, 1, 2, \ldots) \]

Remark: one-component oscillator variables \( a_k \)
used throughout this paper are related with commonly applied two-component oscillators \( a_{k1,2} \) by the formulæ of Section 4 which on quantum level become:
\[ a_k = a_{k1} - ia_{k2}, \quad a_k^+ = a_{-k1} + ia_{-k2}. \]
The inverse formulæ are \( a_{k1} = (a_k + a_k^+)/2, \quad a_{k2} = i(a_k - a_k^+)/2 \). Here taking two sets of oscillator variables \( a_{ki}, \quad i = 1, 2 \) with \( a_{ki} = a_{-k} \) we construct one set without this property. Usage of one-component oscillators simplifies the algebra. It’s easy to verify that the definition of vacuum is equivalent to a standard one \( a_{ki}|0\rangle = 0, k > 0 \) and the states \( |\{n_k\}\rangle \) are the linear combinations of \( \prod_{k>0}(a_{k1}^+)^{n_{k1}}(a_{k2}^+)^{n_{k2}}|0\rangle \).

Quantum top: the space of states is formed by functions \( \Psi(e), \quad e_i^j \in SO(3) \). For the rotation group two representations are possible: single- and double-valued [41]. Spin is defined as differential operator \( S^3 = -i\epsilon_{ijk}e_j^k \partial/\partial e_i^k \), while the projection of spin onto the coordinate system \( \vec{e}_i \) is \( S_i = i\epsilon_{ijk}e_j^k \partial/\partial e_i^k \). Operator \( S^3 \) generates the rotation of the coordinate system \( \vec{e}_i \) in external space, while \( S_i \) generates the rotations about the axes \( \vec{e}_i \). These transformations act on different indices in \( e_j^k \). They commute, therefore \( S^3 \) and \( S_3 \) are simultaneously observable. Matrix elements of spin components do not depend on the representation of the algebra and have well known form [43]:
\[ (S(S_3 - 1)S^3|S_+SS_3S^3) = \sqrt{S(S_3 + 1) - S_3(S_3 - 1)}, \]
\[ (S(S_3 + 1)S^3|S_-SS_3S^3) = \sqrt{S(S_3 + 1) - S_3(S_3 + 1)}, \]
all other elements vanish. Concrete representation of quantum top is described by Wigner’s functions [42, 43]:
\[ |SS_3S^3\rangle = D^S_{S_3S_3}(\vec{e}), \quad S_3, S^3 = -S, -S + 1 \ldots S. \]
Here \( S \) characterizes the eigenvalue of Casimir operator \( \vec{S}^2 = S^iS^i = S_iS_i = S(S+1) \), commuting with all spin components. \( S \) is integer for single-valued representation of \( SO(3) \) and half-integer for double-valued one. Further constraint \( S_3 = A_3 \in \mathbb{Z} \) selects only integer \( S \)-values. The paper [37] Part III explains in more details, why in this problem only integer values of \( S \) are available. The constraints, being functions of \( S_i \) and oscillator variables, commute with \( \vec{S}^2 \), as a result, the determination of mass spectrum can be performed separately for each \( S \) value.

Lorentz group generators are directly defined by their classical expressions [38]:
\[ X_{\mu\nu} = X_{\mu|\nu} + \epsilon_{ijk}N_\mu^i N_\nu^j S^k, \]
where the square brackets denote antisymmetrization of indices. The generators are Hermitian operators, forming closed algebra of Lorentz group, see Appendix 2. Acting on the state vectors, the second term in this expression generates the rotations of \( \vec{e}_i \) in the argument of the wave function, while the first term generates Lorentz transformations of \( P_\mu \), associated changes of CMF axes \( N_\mu^i \) and certain rotations of the basis \( \vec{e}_i \) in CMF.
Constraints $\chi_3|\Psi\rangle = H|\Psi\rangle = 0$ are imposed on the states according to Dirac’s theory of constrained dynamical systems [30]. Here $\chi_3 = S_3 - A^3_3$, 

$$A^3_3 = \sum_{k \neq 0, \pm s} \frac{1}{k} : a^+_k a_k : = \sum_{k \neq 0, \pm s} \text{sign} k \cdot n_k,$$

$$L^{(s)}_0 = \sum_{k \neq 0, \pm s} : a^+_k a_k : = \sum_{k \neq 0, \pm s} |k| \cdot n_k,$$

and the Hamiltonian $H$ is a mass shell condition constructed below. The operators $\chi_3$ and $H$ are Hermitian. For the operators entering in Hamiltonian: $d_\pm, f_\pm, g_\pm, \Sigma_\pm$ and their conjugates we reserve a term elementary operators. They can be defined by their classical expressions [10], with only replacement of complex conjugation to Hermitian one. These definitions have no ordering ambiguities. For the states with the finite number of occupied modes the matrix elements $\langle \{n^\prime_k\}|op|\{n_k\}\rangle$ of elementary operators $op$ are described by finite sums.

Symmetries $R_3, D_{2s}$ in quantum theory are generated by operators

$$R_3 = e^{-i\chi_3 \alpha}, \quad D_{2s} = e^{iL_0^{(s)} \pi/s},$$

where tilde indicates that the symmetry acts in the space of internal variables, not including the mean coordinate $X$ (due to omission of $P^2$ in the generator).

It’s convenient to introduce the notion of $(\Delta S_3, \Delta A_3, \Delta L_0)$-charges for operators satisfying the relations: $[S_3, op] = \Delta S_3 \cdot op$, $[A^3_3, op] = \Delta A_3 \cdot op$, $[L^{(s)}_0, op] = \Delta L_0 \cdot op$, i.e. the operators, which increase or decrease the quantum numbers $(S_3, A^3_3, L^{(s)}_0)$ by $(\Delta S_3, \Delta A_3, \Delta L_0)$ units:

$$op|S_3, A^3_3, L^{(s)}_0\rangle = |S_3 + \Delta S_3, A^3_3 + \Delta A_3, L^{(s)}_0 + \Delta L_0\rangle.$$

Particularly, $\Delta S_3(a_n) = 0, \quad \Delta A_3(a_n) = -1, \quad \Delta L_0(a_n) = -n$. Hermitian conjugation of operators reverses the sign of their charges. The elementary operators have $\Delta S_3 = 0$ and definite $(\Delta A_3, \Delta L_0)$-charges, see Table 1. The discrete symmetry $D_{2s}$ is also characterized by the charge $Q = L^{(s)}_0 \mod 2s$, so that the subspaces with given $Q$ are eigenspaces $D_{2s}(Q) = e^{iQ/2} Q\rangle$, and $D_{2s}^{-}$ symmetric operators keep these subspaces invariant.

For spin components the commutation relations [17] correspond to raising/lowering operators $S^\pm = S_1 \pm i S_2, \quad [S^3, S^\pm] = \pm S^\pm, \quad [S^+, S^-] = 2 S^3$, $S_\mp = S_1 \mp i S_2, \quad [S_3, S_\pm] = \mp S_\pm, \quad [S^+, S^-] = -2 S_3$, i.e. $\Delta S^3(S^\pm) = \pm 1, \Delta S_3(S_\pm) = \mp 1$. In further mechanics only low-index operators $S_l$ will participate. Linear combinations $I_\pm = \lambda_1 S_\pm + \lambda_2 \Sigma_\pm$ do not have definite $\Delta S_3$ and $\Delta A_3$ charges, but have definite $\Delta \chi_3 = \Delta S_3 - \Delta A_3 = \mp 1$. Because the classical symmetries are defined by polynomial structure, they are preserved on the quantum level. Particularly, quantum Hamiltonian possesses $R_3, D_{2s}$-symmetries, as a result, its non-zero matrix elements form blocks, located on $(\chi_3 Q)$-diagonal: $\langle \chi_3 = 0, Q|H|\chi_3 = 0, Q\rangle$.

Quantum Hamiltonian

Further we fix $s = 2$. Quantum analog of (13) is constructed as follows:

$$\alpha_2 = \sum_{n \geq 1} \tilde{n}_1^{-2n+1} : P_n :, \quad (19)$$

where $\tilde{n}_1 = a^+_1 a_1 + c_1$. Here we introduce a constant term $c_1$, whose contribution vanishes on classical level (in the limit of large occupation numbers $a^+_1 a_1$), and add analogous terms in quantum definition $\gamma = (L_0^{(2)} + c_2)^{-1/2}$ and in the definition of mass shell condition, which we fix as follows:

$$\frac{P^2}{2\pi} = L_0^{(2)} + 2\alpha_2 a^+_2 + c_3. \quad (20)$$

The operator $\alpha_2$ has charges $(\Delta \chi_3 = -1, \Delta Q = 0)$, while for $P^2$ $(\Delta \chi_3 = 0, \Delta Q = 0)$. Polynomials $P_n$ are defined by expressions of Appendix 3, where we substitute the definitions [13] of elementary operators and fix the ordering, shown in Table 1. This ordering puts $L_0^{(2)}$-raising elementary operators to the right from $L_0^{(2)}$-raising ones, thus providing better convergence properties for the expansion [19]. Particularly, the matrix elements of $\alpha_2$ between the states with finite $L_0^{(2)}$ are given by finite sums, and large $n$ terms of [13] contribute only to the matrix elements with large $L_0^{(2)}$, due to the following property [37]:

$$\langle L_0^{(2)} = N_1, S | P_n : | L_0^{(2)} = N_2, S \rangle = 0, \quad \text{if} \quad n > \min\{1+(N_1+N_2)/2, (4+2(N_1+N_2)+4S)/5\}.$$
In the products of spin components $S_{\pm}$, commuting with elementary operators, we select the ordering $S_+S_-$, so that $S_3$-raising operator $S_+$ stands on the right and annihilates the states with maximal spin projection $S_3 = S$. We remind that classically $S_3 = S = P^2/2\pi$ corresponds to one-modal solution, straight-line string, associated with the leading Regge trajectory.

The introduced ordering possesses the other feature convenient for computations: the matrix elements of the normally ordered operators restricted to finite-dimensional subspaces, behave regularly when the dimension of subspaces increases. Namely, the matrix in larger space includes the matrix in smaller space as an exact submatrix. Let $op_i(N) = \langle L_0^{(2)} \leq N, S | op_i | L_0^{(2)} \leq N, S \rangle$ be restrictions of elementary operators $op_i$ to finite-dimensional space $L_0^{(2)} \leq N$, at given $S$. Let $P(op_i)$ be $L_0^{(2)}$-normally ordered polynomial of $op_i$. Then (i) at $N_1 < N_2$ the matrix $P(op_i(N_1))$ is a sub-matrix of $P(op_i(N_2))$ and (ii) $P(op_i(N)) = \langle L_0^{(2)} \leq N, S | P(op_i) | L_0^{(2)} \leq N, S \rangle$. Without normal ordering this simple behavior would be violated.

Practically, we compute the matrix elements of elementary operators in $(L_0^{(2)} \leq N, S)$-subspace. The dimension of this subspace is rapidly increasing with $N, S$, e.g. $dim = 281216$ for $N = 20$, $S = 6$, so that the elementary operators are represented as matrices of very large size ($dim \times dim$). The matrices have noticeable block structure, corresponding to their $(\Delta S_3, \Delta A_3, \Delta L_0)$-charge properties. As a result, non-zero matrix elements of $a_2$, necessary for computation of $P^2$, are located in the blocks $\langle S_3 = A_3^{(2)}, Q | a_2 | S_3 = A_3^{(2)} + 1, Q \rangle$, while $P^2$ itself is located in $\langle S_3 = A_3^{(2)}, Q | P^2 | S_3 = A_3^{(2)}, Q \rangle$.

In addition to these properties we use the fact that elementary matrices inside $(S_3, A_3^{(2)}, L_0^{(2)})$-blocks are very sparse (at large $N, S$ their non-zero content occupies less than 1% of the blocks), and implement special algorithms for sparse block matrix computations, described in more details in [37]. Finally, we determine the spectrum of $P^2/2\pi$ up to the values $N = 20, S = 6$ and the number of terms in expansion is $n = 3$. The resulting spectrum $(P^2/2\pi, S)$ is shown on fig.6. The spectrum has common features with the upper part of $(L_0^{(2)}, A_3^{(2)})$ spectrum, shown on fig.5. The beginning of the spectrum $(P^2/2\pi, S)$ consists of three almost linear Regge trajectories. There is a 2-unit gap between the first and the second trajectories. The third trajectory starts at $S = 1$ level. For the next trajectories the degenerate states of fig.5 become splitted on fig.6. The states at $(P^2/2\pi, S) = (3, 1)$ and $(4, 0)$ comprise two numerically close pairs with $P^2/2\pi = 3, 3.0046$ and $P^2/2\pi = 4, 4.0066$, while the other states on fig.6 are non-degenerate (not counting trivial degeneracy for the upper-index $S^3 = -S...S$ and direction of $P_\mu$). The spectrum is computed for the values $c_1 = 2, c_2 = 4, c_3 = 0$. Smaller values of $c_1, c_2$ correspond to higher non-linearities in the spectrum, while larger values of $c_1, c_2$ make the spectrum more linear and closer to the spectrum of $(L_0^{(2)}, A_3^{(2)}) \geq 0$). Further for clarity we fix $c_3 = 0$.

**Table 2: eigenvectors with $P^2/2\pi \in \mathbb{Z}$.**

| $P^2/2\pi = \Gamma_0^{(2)}$ | $S = S_3 = A_3^{(2)}$ | $\{n_k\}$ |
|-----------------------------|------------------------|-------------|
| 0                           | 0                      | $\{0\}$    |
| 2                           | 0                      | $\{1_{1-1}\}$ |
| 4                           | 0                      | $\{2_{2-1}\}$ |
| 1                           | 1                      | $\{1_{1}\}$  |
| 3                           | 1                      | $\{1_{3}\}$  |

The following properties of the spectrum have been found in [37]:

1) for all $c_{1,2} > 0$ the states from Table 2 are annulled by $a_2^\dagger$ and have integer-valued $P^2/2\pi$.
2) let’s keep in $a_2^\dagger$ only the leading (1/$\tilde{n}_1$)-term: $a_2^\dagger | \tilde{n}_1 \rangle = (-\Sigma_+ + S_-/\gamma)/\tilde{n}_1$. For all $c_{1,2} > 0$ the states from the first two Regge-trajectories are an-
nulated by $\alpha_2^+|_{n=1}$. In this approximation the first two Regge-trajectories have integer-valued $P^2/2\pi$ and are linear: $P^2/2\pi = S + k$, $k = 0, 2$.

3) In the limit $1 << c_1^2 << c_2 << c_1^4$ the spectrum of $(P^2/2\pi, S)$ tends to the spectrum of $(L_0^{(2)}, A_3^{(2)} \geq 0)$.

**Remark:** The first Regge trajectory corresponds to the straight-line string’s northern pole solution in the vicinity of the northern pole solution. Classically there are gauge equivalent solutions near the southern pole and the equator, which possess the same $(P^2, S)$. In quantum mechanics we do not see these additional solutions in the spectrum. Particularly, the first Regge trajectory is non-degenerate. We conclude, that the equivalence between these solutions is lost in quantum mechanics. The spectra for additional solutions can be shifted to the region of large masses, or the quantum expansions can even diverge on these solutions. Indeed, the classical solutions in the vicinity of the northern pole possess large $n_1$ and small $n_k$, $k \in Z \{0, 1, \pm 2\}$, providing the convergence for $(1/n_1$)-expansion. In quantum mechanics the convergence of expansion is supported by the finite number of occupied modes in $(L_0^{(2)} \leq N)$-spaces and the normal ordering of operators. For the solutions near the southern pole $n_1 \to 0$, and the usage of $(1/n_1)$-expansions is problematic. One can use $(1/n_{-1})$-series to construct a definition of mass shell condition, alternative to $(20)$, however these definitions will substantially differ on the quantum level and in fact will create two distinct theories. For the solutions on the equator infinite number of oscillator modes are excited, and for these solutions the convergence of the expansions is not guaranteed neither on classical nor on quantum level. This argumentation explains why the usage of $(1/n_1)$-series in the vicinity of the northern pole solution preserves only one Gribov’s copy in the quantum theory.

### 6 Discussion

In this paper we have constructed the quantum theory of open Nambu-Goto string in the spacetime of dimension $d = 4$. The general approach is the selection of the light-cone gauge with the gauge axis related in Lorentz-invariant way with the world sheet. In this approach the Lorentz group transforms the world sheet together with the gauge axis and is not followed by reparametrizations. As a result, the theory becomes free of anomalies in Lorentz group and in the group of internal symmetries of the system. The constructed quantum theory possesses spin-mass spectrum with Regge-like behavior.

Certain problems are still present in this theory, which however do not hinder its implementations, e.g. for the construction of string models of the hadrons. The results, produced by the theory, are influenced by ordering of operators and other details of quantization procedure. The theory does not contain algebraic anomalies, but possesses the
features, which can be called *spectral anomalies*. Particularly, Hamiltonian $P^2/2\pi$, classically generating $2\pi$-periodic evolution, in quantum theory is influenced by ordering rules and does not have strictly equidistant spectrum. This fact does not create problems for the hadronic models, where this spectrum is subjected to phenomenological corrections and experimentally is not strictly equidistant as well. The theory also possesses a topological defect, appearing as a discrete gauge symmetry, identifying the points in the phase space (Gribov’s copies). This classical symmetry is related with discrete non-linear reparametrizations of the world sheet and is not preserved on the quantum level. In our construction we use the expansion series in the vicinity of one Gribov’s copy, by these means distinguishing it in the quantum theory. We have also shown that the leading term of the expansion, which has a minimal ordering ambiguity and is easier for computation, is sufficient to reproduce Regge behavior of the spectrum. Therefore, practically one can keep this term to describe the main effect and include further terms in the form of phenomenological corrections, together with the contributions of other nature [7–15]: gluonic tube thickness, quark masses and charges, spin-orbital interaction, etc.

In conclusion we perform the comparison of the obtained spectrum with the results of other existing approaches to non-critical quantization of string theory. This problem was previously solved for certain submanifolds in the phase space of open string at $d = 4$, which represent the world sheets of a special form, i.e. particular types of string motion.

The first example is given by the above mentioned straight-line string solution [27], whose spin-mass spectrum consists of a single leading Regge trajectory, see fig.7a. In [12] this quantum model was extended by the spin and electric charges of the quarks, describing well the experimentally observed states in the spectrum of light mesons, and some of their radiative transitions.

The straight-line string can be considered as essentially one-parametric solution, where the only parameter is the length of the string. The paper [28] considered a generalization of this mechanics, and shown that an arbitrary two-parametric family of string motion, containing the straight-line string solution as a subset, after a definite weak

topological restriction, admits anomaly free quantization. Its spectrum, presented at fig.7d, contains infinite number of Regge trajectories. All the states in this spectrum are non-degenerate (have multiplicity one).

Fig.7. Spin-mass spectra for the theories of restricted types of string motion, admitting anomaly free quantization at $d = 4$: (a) straight-line string [27], (b) 2-parametric string [28], (c) axially symmetric string [29], (d) stratum $\nu_0 = 1$ in the phase space [26].
The paper [29] has considered a special type of string motion, where the string in the center-of-mass frame possesses an axial symmetry. This type of motion already includes an infinite number of degrees of freedom. It allows anomaly free quantization in Lorentz-invariant light cone gauge, related with the axis of symmetry. Corresponding spectrum, shown at fig. 7, consists of the infinite number of Regge trajectories with non-trivial multiplicities.

The paper [26] has considered the motions of a general form, and obtained spin-mass spectrum for a special infinite-dimensional subset ($\nu_0 = 1$ stratum), possessing a simple topological structure. Its spectrum (fig. 7) as well contains the infinite number of Regge trajectories, with the multiplicities constant along the trajectories, and 2-unit gap between the first two trajectories.

The presented spectra possess evident structural similarities. The discrepancies are caused by the fact that the approaches [26–29] consider different types of motion, and as well by deviations $\sim \hbar$ inherent to quantization procedure. Conceptually, these theories are different implementations of the same underlying ideas, leading to the main common result: the absence of quantum anomalies in Lorentz group at physical dimension of the space-time.

**Appendix 1:**

symplectic structure of the phase space.

The formalism of symplectic geometry [27, 31] is convenient to describe the structure of the phase space in Hamiltonian dynamics with constraints. The phase space is a smooth manifold, endowed by a closed non-degenerate differential 2-form $\Omega = \frac{1}{2} \omega_{ij} dX^i \wedge dX^j$ (in some local coordinates $X^i$, $i = 1, \ldots, 2n$). Poisson brackets are defined by the form as $\{X^i, X^j\} = \omega^{ij}$, where $||\omega^{ij}||$ is inverse to $||\omega_{ij}||$: $\omega_{ij} \omega^{jk} = \delta^i_k$. Let’s consider a surface in the phase space, given by the 2nd class constraints: $\chi_\alpha(X) = 0$ ($\alpha = 1, \ldots, r$), $det ||\chi_\alpha, \chi_\beta|| \neq 0$. Reduction on this surface consists in the substitution of its explicit parametrization $X^i = X^i(u^a)$ ($a = 1, \ldots, 2n - r$) into the form:

$$\Omega = \frac{1}{2} \Omega_{ab} du^a \wedge du^b, \quad \Omega_{ab} = \frac{\partial X^i}{\partial u^a} \frac{\partial X^j}{\partial u^b}. $$

Here $det ||\Omega_{ab}|| \neq 0$. Matrix $||\Omega_{ab}||$, inverse to $||\Omega_{ab}||$, defines Poisson brackets on the surface: $\{u^a, u^b\} = \Omega^{ab}$. This method is equivalent to commonly used Dirac brackets’ formalism [30]:

$$\{u^a, u^b\}^D = \{u^a, u^b\} - \{u^a, \chi_\alpha\} \Pi^{\alpha \beta} \{\chi_\beta, u^b\} = \Omega^{ab},$$

where $||\Pi^{\alpha \beta}||$ is inverse to $||\Pi_{\alpha \beta}||$: $\Pi_{\alpha \beta} = \{\chi_\alpha, \chi_\beta\}$. In string theory canonical Poisson brackets $\{x_\mu(\sigma), p_\nu(\tilde{\sigma})\} = g_{\mu\nu} \delta(\sigma - \tilde{\sigma})$ correspond to symplectic form $\Omega = \int_0^1 d\sigma \delta p_\mu(\sigma) \wedge \delta x_\mu(\sigma)$. By a substitution of expressions for $x_\mu(\sigma), p_\mu(\sigma)$ in terms of $Q_\mu(\sigma)$, given by eqs. (4), it is transformed to the form

$$\Omega = \frac{1}{2} dP_\mu \wedge dQ_\mu(0) +$$

$$+ \frac{1}{4} \int_0^{2\pi} d\sigma \delta Q_\mu'(\sigma) \wedge \delta Q_\mu(\sigma),$$

and by substitution of light cone parametrization (9):

$$\Omega = dP_\mu \wedge dZ_\mu + \sum_{k \neq 0} \frac{1}{2k} \delta a_k^* \wedge da_k +$$

$$+ \frac{1}{2} d\tilde{e}_i \wedge (\tilde{S} \times \tilde{e}_i). \quad (21)$$

Inverting the coefficient matrix of this form (in the presence of orthonormality constraints $\tilde{e}_i \tilde{e}_j = \delta_{ij}$), we obtain Poisson brackets (11). Further, substituting the gauge fixing conditions (5) to the form (21), we see that $a_\pm$ terms cancel each other: $\frac{1}{16}(da_s^* \wedge da_s - da_s^* \wedge da_s) = 0$. After the reduction we obtain the same canonical basis as (11), but with $\{a_k, a_s^*\} = ik \delta_{kn}, \ k, n \in \mathbb{Z} \setminus \{0, \pm s\}$. To study this property in more detail, we introduce the variables $q_1 = \text{Re}(a_s + a_s) / 2, \ q_2 = -\text{Im}(a_s + a_s) / 2, \ p_1 = \text{Im}(a_s - a_s) / 2, \ p_2 = \text{Re}(a_s - a_s) / 2$, and rewrite $a_\pm$-terms of the symplectic form as $\frac{1}{2}(dp_1 \wedge dq_1 - dp_2 \wedge dq_2)$. Gauge fixing conditions (5) are rewritten to $q_1 = q_2 = 0$, now we see that the variables $p_{1,2}$, canonically conjugated to $q_{1,2}$, drop out from the symplectic form. $p_{1,2}$ should be expressed in terms of other dynamical variables from $\chi$-constraints and, independently on the complexity of these expressions, Poisson brackets of the mechanics remain simple.

**Appendix 2:**

absence of anomalies in $[M_{\mu \nu}, M_{\rho \sigma}]$.

Representing $M_{\mu \nu} = \frac{Z_{[\mu \rho]} - \frac{1}{2} \epsilon_{ijk} \Gamma_{ij}^{\rho \sigma \kappa} S^{\kappa}}{Z_{[\rho \sigma]} + G_{\mu \nu}(P) S^{\kappa}}$ and writing the commutator $[Z_{[\mu \rho]} + G_{\mu \nu}(P) S^{\kappa}, Z_{[\rho \sigma]} + G_{\mu \nu}(P) S^{\kappa}]$, we get

$$\frac{1}{2} \Omega_{ab} du^a \wedge du^b, \quad \Omega_{ab} = \frac{\partial X^i}{\partial u^a} \frac{\partial X^j}{\partial u^b}.$$
$G_n^\mu(P)S^n)$, we see that the term $[Z_\mu P_\nu, Z_\rho P_\sigma]$ coincides with the commutator of Lorentz generators in a theory of point-like particles, which is free of any anomalies; in the term $[G_k^\mu (P)S^k, G_n^\rho (P)S^n]$ the only non-commuting variables are $[S^k, S^n] = i\epsilon_{knm}S^m$, as a result we obtain the expression of the form $GGS$, including only commuting variables, and the result coincides (up to the multiplier $i$) with the Poisson bracket of the corresponding classical variables; in the term $[Z_\mu P_\nu, G_n^\rho (P)S^n]$ and the remaining one, related with this term by the replacements ($\mu \leftrightarrow \rho, \nu \leftrightarrow \sigma, n \leftrightarrow k$), the only non-commuting variables are $[Z_\mu, G_n^\rho (P)]$, and the computation of this commutator gives the derivative $\partial G_n^\rho (P)/\partial P^\mu$, thus we obtain the expression $(\partial G)PS$, which includes the commuting operators and again coincides with the Poisson bracket. Therefore, considered commutator has no anomalous terms and its computation actually coincides with the computation of the corresponding Poisson bracket. Note that this bracket necessarily represents canonical relations for the Lorentz algebra:

$$\{M_{\mu \nu}, M_{\rho \sigma}\} = g_{\rho \mu}M_{\sigma \mu} + g_{\sigma \nu}M_{\mu \rho},$$

because the transformations we performed consist only in the change of canonical basis $a_i \rightarrow b_i$ in the phase space, which preserves the original Poisson bracket $\{M_{\mu \nu}, M_{\rho \sigma}\}$. This fact was also proven by direct computation in [27].

**Appendix 3:** coefficients of expansion [12].

\[
P_1 = -\Sigma_+ + S_+ / \gamma,
\]

\[
P_2 = -a_1^2 f^* S_\Sigma - a_2 a_3 f^* g^* S_\Sigma - a_1 g^* S_+ / \gamma - a_3^2 g^* S_+ / \gamma,
\]

\[
P_3 = -a_1^2 f^* g^* S_\Sigma - a_1^2 a_2 f^* f^* S_\Sigma - a_1^4 g^* S_\Sigma - a_1^2 a_3^2 d^* S_\Sigma - S_+ / 2 + a_1^4 f^* g^* S_\Sigma / \gamma + a_1^2 a_2^2 f^* g^* S_\Sigma / \gamma + 2 a_1^4 d^* S_\Sigma - a_1^2 S_\Sigma / \gamma / 2 - a_1^2 a_2^2 d^* S_\Sigma + S_+ / 2 - a_1^4 S_\Sigma - 2 / \gamma - a_1^2 a_3^2 f^* f^* S_\Sigma / \gamma + a_1^2 a_3^2 g^* S_+ / \gamma + a_1^2 a_3^2 d^* S_\Sigma - S_\Sigma / 2 + a_1^2 a_2^2 S_\Sigma - S_\Sigma / 2 - a_1^2 a_3^2 S_\Sigma - S_\Sigma / 2 + a_1^2 a_3^2 S_\Sigma / \gamma.
\]

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