QUANTITATIVE COMPARISONS OF MULTISCALE GEOMETRIC PROPERTIES

JONAS AZZAM AND MICHELE VILLA

ABSTRACT. We generalize some characterizations of uniformly rectifiable (UR) sets to sets whose Hausdorff content is lower regular (and in particular, do not need to be Ahlfors regular). For example, David and Semmes showed that, given an Ahlfors $d$-regular set $E$, if we consider the set $\mathcal{B}$ of surface cubes (in the sense of Christ and David) near which $E$ does not look approximately like a union of planes, then $E$ is UR if and only if $\mathcal{B}$ satisfies a Carleson packing condition, that is, for any surface cube $R$,

$$\sum_{Q \subseteq R, Q \in \mathcal{B}} (\text{diam} Q)^d \lesssim (\text{diam} R)^d.$$ 

We show that, for lower content regular sets that aren’t necessarily Ahlfors regular, if $\beta_E(R)$ denotes the square sum of $\beta$-numbers over subcubes of $R$ as in the Traveling Salesman Theorem for higher dimensional sets [AS18], then

$$\mathcal{H}^d(R) + \sum_{Q \subseteq R, Q \in \mathcal{B}} (\text{diam} Q)^d \sim \beta_E(R).$$

We prove similar results for other uniform rectifiability criteria, such as the Local Symmetry, Local Convexity, and Generalized Weak Exterior Convexity conditions.

En route, we show how to construct a corona decomposition of any lower content regular set by Ahlfors regular sets, similar to the classical corona decomposition of UR sets by Lipschitz graphs developed by David and Semmes.

CONTENTS

1. Introduction ................................................................. 2
  1.1. Background ....................................................... 2
  1.2. Outline .......................................................... 8
  1.3. Acknowledgements ............................................... 8
2. Preliminaries ............................................................ 8
  2.1. Notation .......................................................... 8

2010 Mathematics Subject Classification. 28A75, 28A78, 28A12.
Key words and phrases. Rectifiability, Traveling Salesman, beta numbers, coronizations, corona decompositions.
1. INTRODUCTION

1.1. Background. A set $E \subseteq \mathbb{R}^n$ is said to be $d$-rectifiable if it can be covered up to Hausdorff $d$-measure zero by Lipschitz images of $\mathbb{R}^d$. While classifying rectifiable sets is a classical problem dating back to Besicovitch, starting in the late 80’s, geometric measure theorists and harmonic analysts began to study rectifiability in a quantitative manner with an eye on applications to harmonic analysis, particularly singular integrals, analytic capacity, and harmonic measure.

Much of this work has focused on classifying when Ahlfors regular sets are uniformly rectifiable, which was initiated by David and Semmes in their seminal texts [DS91, DS93]. Recall that a set $E \subseteq \mathbb{R}^n$ is Ahlfors $d$-regular if there is $A > 0$ so that

$$r^d/A \leq \mathcal{H}^d(B(\xi, r) \cap E) \leq Ar^d$$

for $\xi \in E, r \in (0, \text{diam } E)$

and is uniformly rectifiable (UR) if it has $E$ has big pieces of Lipschitz images (BPLI), i.e. there are constants $L, c > 0$ so for all $\xi \in E$ and $r \in (0, \text{diam } E)$, there is an $L$-Lipschitz map $f : \mathbb{R}^d \to \mathbb{R}^n$ such that

$$\mathcal{H}^d(f(\mathbb{R}^d) \cap B(\xi, r)) \geq cr^d.$$
criteria for one characterization of uniform rectifiability than another (as in [HM14, NTV14, HLMN17]).

We will define some of these criteria here. Let $\mathcal{D}$ denote the Christ-David cubes for $E$ (see Theorem 2.2 below for their definition and the relevant notation we will use below). We say a family of cubes $\mathcal{C}$ satisfies a \textit{Carleson packing condition} if there is a constant $C$ so that for all $R \in \mathcal{D}$,

$$\sum_{Q \in \mathcal{C}} \ell(Q)^d \leq C \ell(R)^d.$$ 

By Theorem 2.2, for each cube $Q \in \mathcal{D}$, there is a ball $B_Q$ centered on and containing $Q$ of comparable size. Given two closed sets $E$ and $F$, and $B$ a set we denote

$$d_B(E, F) = \frac{2}{\text{diam } B} \max \left\{ \sup_{y \in E \cap B} \text{dist}(y, F), \sup_{y \in F \cap B} \text{dist}(y, E) \right\}$$

For $C_0 > 0$, and $\epsilon > 0$, let

$$\mathcal{BWGL}(C_0, \epsilon) = \{ Q \in \mathcal{D} | \ d_{C_0 B_Q}(E, P) \geq \epsilon \ \text{for all } d\text{-planes } P \}.$$ 

\text{BWGL} stands for the \textit{bilateral weak geometric lemma}. David and Semmes showed in [DS93] that $E$ is UR if and only if for every $C_0 \geq 1$ there is $\epsilon > 0$ sufficiently small so that $\mathcal{BWGL}(C_0, \epsilon)$ satisfies a Carleson packing condition (with constant depending on $\epsilon$).

Another important classification from [DS93] is \textit{bilateral approximation uniformly by planes} (BAUP): for $R \in \mathcal{D}$ and $\epsilon > 0$, let

(1.2) \hspace{1cm} \mathcal{BAUP}(C_0, \epsilon) = \{ Q \in \mathcal{D} | \ d_{C_0 B_Q}(E, U) \geq \epsilon, \ U \text{ is a union of } d\text{-planes} \}.

David and Semmes showed that $E$ is UR if and only if $\mathcal{BAUP}(C_0, \epsilon)$ satisfies a Carleson packing condition each $C_0 > 1$ and $\epsilon > 0$ small enough (depending on $C_0$). This was a key tool in [HLMN17] in showing that the weak-$A_\infty$ condition for harmonic measure implies UR, and also was key in Nazarov, Tolsa, and Volberg’s solution to David and Semmes’ conjecture in codimension 1 [NTV14].

The focus on Ahlfors regular sets is due to the fact that Hausdorff measure on the set is rather well behaved, and so techniques like stopping-time arguments on dyadic cubes in the Euclidean setting often translate over to this non-smooth setting. The motivation of the current paper, however, is to try and obtain similar estimates on multiscale geometry that exist for uniformly rectifiable sets, but instead for sets that are not Ahlfors regular. Not all quantitative results on rectifiability are in the Ahlfors regular setting. The classical example is the \textit{Analyst’s Traveling Salesman Theorem} stated below, which will serve as a model for the kind of results we are after.
For sets $E, B \subseteq \mathbb{R}^n$, define
\begin{equation}
\beta^d_{E,\infty}(B) = \frac{2}{\text{diam}(B)} \inf_L \sup \{ \text{dist}(y, L) | y \in E \cap B \}
\end{equation}
where $L$ ranges over $d$-planes in $\mathbb{R}^n$. Thus, $\beta^d_{E,\infty}(B) \text{diam}(B)$ is the width of the smallest tube containing $E \cap B$.

**Theorem 1.1.** (Jones: $\mathbb{R}^2$ [Jon90]; Okikiolu: $\mathbb{R}^n$ [Oki92], Schul [Sch07])

Let $n \geq 2$. There is a $C = C(n)$ such that the following holds. Let $E \subset \mathbb{R}^n$. Then there is a connected set $\Gamma \supseteq E$ such that
\begin{equation}
\mathcal{H}^1(\Gamma) \lesssim_n \text{diam } E + \sum_{Q \in \Delta} \beta^1_{E,\infty}(3Q)^2 \text{diam}(Q).
\end{equation}

Conversely, if $\Gamma$ is connected and $\mathcal{H}^1(\Gamma) < \infty$, then
\begin{equation}
\text{diam } \Gamma + \sum_{Q \in \Delta} \beta^1_{\Gamma,\infty}(3Q)^2 \text{diam}(Q) \lesssim_n \mathcal{H}^1(\Gamma).
\end{equation}

This was first shown by Jones in [Jon90] in the plane, then Okikiolu in $\mathbb{R}^n$ [Oki92], and finally in Hilbert space by Schul [Sch07], though the statement is different than above. There are also some partial and complete generalizations that hold for curves in other metric spaces [DS17, DS19, FFP07, LS16a, LS16b, Li19].

An analogue for $d$-dimensional of the second half of Theorem 1.1 is false due to Fang (see [AS18] for a proof). David and Semmes, however, coined a different definition of a $\beta$-number in terms of which they gave a classification of uniformly rectifiable sets. In [AS18], the first author and Schul altered their definition to get a version of Theorem 1.1 for higher dimensional sets, which we describe now.

For a set $E$, a ball $B$, and a $d$-dimensional plane $L$, define
\begin{equation}
\beta^{d,p}_{E, L}(B, L) = \left( \frac{1}{r_B^d} \int_0^1 \mathcal{H}^d_{\infty}(\{ x \in B \cap E | \text{dist}(x, L) > tr_B \}) t^{p-1} dt \right)^{\frac{1}{p}}
\end{equation}
where $r_B$ is the radius of $B$, and set
\[ \beta^{d,p}_{E, L}(B) = \inf \{ \beta^{d,p}_{E, L}(B, L) | L \text{ is a } d \text{-dimensional plane in } \mathbb{R}^n \}. \]

If $E$ is Ahlfors $d$-regular and we replace $\mathcal{H}^d_{\infty}$ with $\mathcal{H}^d$, this is the $\beta$-number David and Semmes used. However, the $d$-dimensional traveling salesman will be stated for lower content regular sets.
**Definition 1.2.** A set \( E \subseteq \mathbb{R}^n \) is said to be \((c, d)\)-lower content regular in a ball \( B \) if

\[
\mathcal{H}^d_s(E \cap B(x, r)) \geq cr^d \quad \text{for all } x \in E \cap B \text{ and } r \in (0, r_B).
\]

We can now state the result from [AS18]. It is phrased slightly differently from there, but we justify the reformulation in the appendix.

**Theorem 1.3.** Let \( 1 \leq d < n \) and \( E \subseteq \mathbb{R}^n \) be a closed set. Suppose that \( E \) is \((c, d)\)-lower content regular and let \( D \) denote the Christ-David cubes for \( E \). Let \( C_0 > 1 \). Then there is \( \epsilon > 0 \) small enough so that the following holds. Let \( 1 \leq p < p(d) \) where

\[
p(d) := \begin{cases} 
\frac{2d}{d-2} & \text{if } d > 2 \\
\infty & \text{if } d \leq 2
\end{cases}.
\]

For \( R \in D \), let

\[
\text{BWGL}(R) = \text{BWGL}(R, \epsilon, C_0) = \sum_{Q \subseteq R} \mathcal{L}(Q)^d.
\]

and

\[
\beta_{E,A,p}(R) := \ell(R)^d + \sum_{Q \subseteq R} \beta_{E,A,p}^d(ABQ)^2 \ell(Q)^d.
\]

Then for \( R \in D \),

\[
(1.7) \quad \mathcal{H}^d(R) + \text{BWGL}(R, \epsilon, C_0) \sim_{A,n,c,p,C_0} \beta_{E,A,p}(R).
\]

Since all these values are comparable for all admissible values of \( A \) and \( p \), below we will simply let

\[
\beta_E(R) := \beta_{E,3,2}.
\]

The presence of \( \text{BWGL}(R, \epsilon, C_0) \) may seem odd, but it disappears in some natural situations. It is just zero if \( E \) is \( \epsilon \)-Reifenberg flat, for example (c.f. [DT12] for this definition). When \( d = n - 1 \) and \( E \) satisfies Condition B, we have that for any cube \( R \subseteq E \),

\[
\text{BWGL}(R, \epsilon, C_0) \lesssim \mathcal{H}^d(R).
\]

In an upcoming paper, the second author will show that this same estimate occurs for the higher codimensional generalized Semmes surfaces introduced by David in [Dav88] (check there for these definitions). In these scenarios, we then have the more natural looking estimate (more closely resembling (1.4))

\[
\beta_E(R) \sim_{A,n,c,\epsilon} \mathcal{H}^d(R).
\]
Even in the general case, the higher dimensional Traveling Salesman Theorem above says that $BWGL(R, \epsilon, C_0)$ has some meaning if we compute the sum for a non-Ahlfors regular set: even though it does not necessarily satisfy a Carleson packing condition, it is comparable to the square sum of $\beta$'s for any lower regular set. This opens the question of whether the same holds for sums over other cube families for which a Carleson packing condition would characterize UR sets in the Ahlfors regular setting.

In this paper, we show this is indeed the case for a large class of the original UR characterizations developed by David and Semmes. A consequence of our results is the following (see Section 6 for its proof):

**Theorem 1.4.** Let $E \subseteq \mathbb{R}^n$ be a $(c, d)$-lower content regular set with Christ-David cubes $\mathcal{D}$. For $R \in \mathcal{D}$, define

$$BAUP(R, C_0, \epsilon) = \sum_{Q \subseteq R} \ell(Q)^d.$$  

For all $R \in \mathcal{D}$, $C_0 > 1$, and $\epsilon > 0$ small enough depending on $C_0$ and $c$,

$$H^d(R) + BAUP(R, C_0, \epsilon) \sim_{C_0, \epsilon, c} \beta_E(R).$$  

We mention one other geometric criteria studied by David and Semmes which we consider: The Local Symmetry’ (LS) property is defined as follows. Given $\epsilon > 0$, let $LS(R, \epsilon, \alpha)$ be the sum of $\ell(Q)^d$ over those cubes in $R$ for which there are $y, z \in B_Q \cap E$ so that $\text{dist}(2^\alpha y - z, E) \geq \epsilon r$.

**Theorem 1.5.** Let $E \subseteq \mathbb{R}^n$ be a $(c, d)$-lower content regular set and $\mathcal{D}$ its Christ-David cubes. Then for $\epsilon > 0$ small enough (depending on $c$), and $R \in \mathcal{D}$,

$$\beta_E(R) \sim_{c, \epsilon} H^d(R) + LS(R, \epsilon).$$  

This may be surprising, since the Local Symmetry condition is dimensionless, that is, the integer $d$ does not appear in the definition at all, and in fact it could be that, in the “good” cubes not featured in the sum, $E$ could be very not flat and quite close in the Hausdorff distance to a $(d + 1)$-dimensional cube, say, whereas the $\beta$-numbers measure the distance to a $d$-dimensional plane and would be large for these cubes. However, with the assumption that $H^d(R)$ is finite, this prevents there being too many cubes where $E$ is symmetric but looks like a $(d + 1)$-dimensional surface (and this is natural considering that the proof in [DS91] connecting LS to flatness of the set uses the Ahlfors regularity of the sets they consider).

Our method for extending these results is quite flexible: the other characterizations of UR for which we prove analogous statements like those are the Local Convexity (LCV) and Generalized Weak Exterior Convexity...
(GWEC) conditions, although one could also consider other suitable characterizations in [DS93] as well. In fact, our main result is a general test for when a geometric criteria that guarantees uniform rectifiability (like BAUP or BWGL) also implies a result of the form Theorem 1.4. Its statement is a bit lengthy to give here, so we postpone it to Section 4. Loosely speaking, by a geometric criteria \( \mathcal{P} \), we mean a way of splitting up the surface cubes of a set \( E \) into “good” and “bad” cubes, the good cubes being those cubes near which \( E \) satisfies some condition that is trivially satisfied for a \( d \)-dimensional plane, like being close in the Hausdorff distance to a plane or union of planes. We say it guarantees UR if, whenever we have an Ahlfors regular set, a Carleson packing condition on the bad cubes implies UR. Our result, Lemma 4.5 below, states that if we have a geometric criterion that guarantees UR and it is, in some sense, continuous in the Hausdorff metric, then a result like Theorem 1.4 hold with BAUP replaced by \( \mathcal{P} \).

The main lemma that we use may be of independent interest, and has a few forthcoming applications to other problems (see [Azz, Vil]). For the reader familiar with uniform rectifiability, this result says that we can perform a Coronization of lower regular sets by Ahlfors regular sets in a way similar to how David and Semmes construct Coronizations of uniformly rectifiable sets by Lipschitz graphs (see [DS91, Chapter 2]). For the definitions of Christ-David cubes and stopping-time regions, see Section 2.3.

**Main Lemma.** Let \( k_0 > 0, \tau > 0, d > 0 \) and \( E \) be a set that is \((c, d)\)-lower content regular. Let \( \mathcal{D}_k \) denote the Christ-David cubes on \( E \) of scale \( k \) and \( \mathcal{D} = \bigcup_{k \in \mathbb{Z}} \mathcal{D}_k \). Let \( Q_0 \in \mathcal{D}_0 \) and \( \mathcal{D}(k_0) = \bigcup_{k=0}^{k_0} \{ Q \in \mathcal{D}_k \mid Q \subseteq Q_0 \} \). Then we may partition \( \mathcal{D}(k_0) \) into stopping-time regions \( \text{Tree}(R) \) for \( R \) from some collection \( \text{Top}(k_0) \subseteq \mathcal{D}(k_0) \) with the following properties:

1. We have
   \[
   \sum_{R \in \text{Top}(k_0)} \ell(R)^d \lesssim_c d \mathcal{H}^d(Q_0).
   \]

2. Given \( R \in \text{Top}(k_0) \) and a stopping-time region \( \mathcal{T} \subseteq \text{Tree}(R) \) with maximal cube \( T \), let \( \mathcal{F} \) denote the minimal cubes of \( \mathcal{T} \) and
   \[
   d_{\mathcal{F}}(x) = \inf_{Q \in \mathcal{F}} (\ell(Q) + \text{dist}(x, Q)).
   \]

For \( C_0 > 4 \) and \( \tau > 0 \), there is a collection \( \mathcal{C} \) of disjoint dyadic cubes covering \( C_0 B_T \cap E \) so that
   \[
   E(\mathcal{T}) = \bigcup_{I \in \mathcal{C}} \partial_d I,
   \]

where \( \partial_d I \) denotes the \( d \)-dimensional skeleton of \( I \), then the following hold:
(a) $E(\mathcal{T})$ is Ahlfors regular with constants depending on $C_0, \tau, d,$ and $c$.

(b) We have the containment

$$C_0 B_T \cap E \subseteq \bigcup_{I \in \mathcal{C}} I \subseteq 2C_0 B_T.$$  

(c) $E$ is close to $E(\mathcal{T})$ in $C_0 B_T$ in the sense that

$$\text{dist}(x, E(\mathcal{T})) \lesssim \tau \inf_{x \in I} d_{E}(x) \quad \text{for all } x \in E \cap C_0 B_T.$$  

(d) The cubes in $\mathcal{C}$ satisfy

$$\ell(I) \sim \tau \inf_{x \in I} d_{E}(x) \quad \text{for all } I \in \mathcal{C}.$$  

The last inequality says that the cubes in $\mathcal{C}$ are distributed in a sort of Whitney fashion. In particular, if two cubes in $\mathcal{C}$ are adjacent, then they have comparable sizes.

Observe that the constants don’t depend on $k_0$. The presence of $k_0$ is an artifact of the proof, but in applications we will take $k_0 \to \infty$.

1.2. **Outline.** In Section 3, we prove the Main Lemma and show that a general lower regular set can be approximated by Ahlfors regular sets. In Section 4, we show how, if the sum of cubes where a geometric criteria like the BAUP is finite, then we can actually make these Ahlfors regular sets uniformly rectifiable. Using a result of David and Semmes, we know that the sum of $\beta$’s will be finite for these sets, and then that will imply the $\beta$’s for the original set are summable by approximation. After that, we apply our works to get results similar to the Traveling Salesman, but with BWGL replaced by other geometric criteria. In Section 5, we show the same result holds with BWGL replaced by the Local Symmetry and Local Convexity conditions. In Section 6, we consider the BAUP condition and prove Theorem 1.4, and in Section 7, we study the GWEC.

1.3. **Acknowledgements.** We’d like to thank Raanan Schul for his useful conversations and encouragement and Matthew Hyde for carefully proofreading the manuscript.

2. **Preliminaries**

2.1. **Notation.** We will write $a \lesssim b$ if there is $C > 0$ such that $a \leq Cb$ and $a \lesssim_t b$ if the constant $C$ depends on the parameter $t$. We also write $a \sim b$ to mean $a \lesssim b \lesssim a$ and define $a \sim_t b$ similarly.

For sets $A, B \subset \mathbb{R}^n$, let

$$\text{dist}(A, B) = \inf \{|x - y| \mid x \in A, y \in B\}, \quad \text{dist}(x, A) = \text{dist}(\{x\}, A),$$
and
\[ \text{diam } A = \sup \{|x - y| \mid x, y \in A \}. \]

2.2. Dyadic cubes. Let \( \mathcal{I} \) denote the dyadic cubes in \( \mathbb{R}^n \), and for \( k \in \mathbb{Z} \), let \( \mathcal{I}_k \) be those dyadic cubes of sidelength \( 2^{-k} \). Give a dyadic cube \( I_0 \), we will write \( \mathcal{I}(I_0) \) to denote the subfamily of dyadic cubes which are contained in \( I_0 \). Given some \( m \in \mathbb{Z} \), we set \( \mathcal{I}^m := \bigcup_{k=m}^{\infty} \mathcal{I}_k \), that is, \( \mathcal{I} \) is the family of dyadic cubes with side length at least \( 2^m \). We will also write \( \mathcal{I}^m(I_0) := \mathcal{I}^m \cap \mathcal{I}(I_0) \).

Finally, given a dyadic cube \( I \), we denote by \( n(I) \) the integer number so that \( \ell(I) = 2^n(I) \).

For a cube \( I \in \mathcal{I} \), we write \( \partial I \) to denote the \( d \)-dimensional skeleton of \( I \). For a dyadic cube \( I \), its \( d \)-dimensional skeleton is just the union of its \( d \)-dimensional faces.

**Remark 2.1.** We may also use the notation \( \mathcal{I}_m \) to mean the family of cubes with side length \( \ell(I) = 2^{-m} \).

2.3. Christ-David Cubes. We recall the following version of “dyadic cubes” for metric spaces, first introduced by David [Dav88] but generalized in [Chr90] and [HM12].

**Theorem 2.2.** Let \( X \) be a doubling metric space. Let \( X_k \) be a nested sequence of maximal \( \rho^k \)-nets for \( X \) where \( \rho < 1/1000 \) and let \( c_0 = 1/500 \).

For each \( n \in \mathbb{Z} \) there is a collection \( \mathcal{D}_k \) of “cubes,” which are Borel subsets of \( X \) such that the following hold.

1. For every integer \( k \), \( X = \bigcup_{Q \in \mathcal{D}_k} Q \).
2. If \( Q, Q' \in \mathcal{D} = \bigcup \mathcal{D}_k \) and \( Q \cap Q' \neq \emptyset \), then \( Q \subseteq Q' \) or \( Q' \subseteq Q \).
3. For \( Q \in \mathcal{D} \), let \( k(Q) \) be the unique integer so that \( Q \in \mathcal{D}_k \) and set \( \ell(Q) = 5\rho^{k(Q)} \). Then there is \( \zeta_Q \in X_k \) so that

(2.1) \[ B_X(\zeta_Q, c_0 \ell(Q)) \subseteq Q \subseteq B_X(\zeta_Q, \ell(Q)) \]

and \( X_k = \{ \zeta_Q \mid Q \in \mathcal{D}_k \} \).

For a cube \( Q \in \mathcal{D}_k \), we put

(2.2) \[ \text{Child}(Q) := \{ Q' \in \mathcal{D}_{k+1} \mid Q' \subset Q \} . \]
Definition 2.3. A collection $T \subseteq D$ is a stopping-time region or tree if the following hold:

1. There is a cube $Q(T) \in T$ that contains every cube in $T$.
2. If $Q \in T$, $R \in D$, and $Q \subseteq R \subseteq Q(T)$, then $R \in T$.
3. $Q \in T$ and there is $Q' \in \text{Child}(Q) \setminus T$, then $\text{Child}(Q) \subset T^c$.

3. Proof of the Main Lemma

Let $E$ and $Q_0$ be as in the Main Lemma. Notice that $Q_0$ is also a lower regular set, although it may not be closed, but we will not need that.

We split the proof into a few subsections.

3.1. Frostmann’s Lemma. The first step of the proof follows the proof Frostmann’s lemma, but with some extra care.

Let $I_0 = [0, 1]^n$. Without loss of generality, we assume that $Q_0 \subset I_0$ and that $\text{diam}(Q_0) \geq \ell(I_0)$.

For $k \in \mathbb{Z}$, let

$$I_k(Q_0) = \{I \in \mathcal{S} \mid I \cap Q_0 \neq \emptyset\}, \quad \mathcal{S}^k(Q_0) = \bigcup_{j=0}^k \mathcal{S}_j, \quad \mathcal{S}(Q_0) = \mathcal{S}^\infty(Q_0)$$

and

$$E_k = \bigcup_{I \in \mathcal{S}(Q_0)} I.$$

Let $m \in \mathbb{N}$ (we will choose it later). First let $\mu_m^n = \mathcal{H}^n|_{E_m} 2^{(n-d)m}$. In this way,

$$\mu_m^n(I) = \ell(I)^d \quad \text{for all } I \in \mathcal{S}(Q_0).$$

We define a set of cubes $\text{Bad}$ (which depends on $m$) as follows. First, we immediately add $\mathcal{S}(Q_0)$ to $\text{Bad}$. Next, for each $I \in \mathcal{S}_{m-1}(Q_0)$, if

$$\mu_m(I) > 2\ell(I)^d,$$

then we add $I$ to $\text{Bad}$ and define

$$\mu_{m-1}^{|I} = \ell(I)^d \frac{\mu_m^{|I}}{\mu_m(I)} < \frac{1}{2} \mu_m^{|I}.$$

Otherwise, we set

$$\mu_{m-1}^{|I} = \mu_m^{|I}.$$

Inductively, suppose we have defined $\mu_m^{k+1}$ for some integer $k < m$. For $I \in \mathcal{S}_k(E)$, if

$$\mu_m^{k+1}(I) > 2\ell(I)^d,$$

then we add $I$ to $\text{Bad}$ and define

$$\mu_{m-1}^{k+1} = \ell(I)^d \frac{\mu_m^{k+1}^{|I}}{\mu_m^{k+1}(I)} < \frac{1}{2} \mu_m^{k+1}^{|I}.$$
then place $I \in \text{Bad}$ and set

$$\mu_m^k|_I = \ell(I)^d \mu_{m+1}^k|_I < \frac{1}{2} \mu_{m+1}^k|_I. \tag{3.1}$$

Otherwise, we set

$$\mu_m^k|_I = \mu_m^{k+1}|_I. \tag{3.2}$$

Finally, we put $I_0 \in \text{Bad}$.

Given a cube $I \in \mathcal{I}$, recall that $n(I)$ is the integer such that $\ell(I) = 2^{-n(I)}$. Moreover, for $J \in \mathcal{I}_m(Q_0)$ (so that $\ell(J) \geq 2^{-m}$), we let $b(I)$ be the number of cubes from $\text{Bad}$ properly containing $J$. Now, again with $J \in \mathcal{I}_m(Q_0)$, let $I_0, ..., I_{b(I)} \in \text{Bad}$ be all the bad cubes containing $J$, so that $I_j \supset I_{j+1}$ (note that this is consistent with how we defined $I_0$ before). With this notation, we see that $b(I_j) = j$ for all $j$ and if a dyadic cube $I \in \text{Bad}$, then $I = I_{b(I)}$. Let now $I \in \text{Bad}$ and $J \in \mathcal{I}_m(Q_0)$ so that $J \subset I$. We write

$$\mu_m^{n(I)}(J) = \mu_m^{n(I)}(I)(\text{3.1}) < \frac{1}{2} \mu_m^{n(I)+1}(I) \tag{3.2} = \frac{1}{2} \mu_m^{n(I)+1}(J) < \ldots$$

$$\ldots < \frac{1}{2^{b(I)-b(J)}} \mu_m^{n(I+b)}(J) = \frac{\ell(J)^d}{2^{b(I)-b(J)}} \mu_m^m(J) = \frac{\ell(J)^d}{2^{b(I)-b(J)}},$$

Finally, observe that since $Q_0$ is $(c, d)$-lower content regular, if $J \cap Q_0 \neq \emptyset$ and $J \in \mathcal{I}_m(Q_0)$, then

$$\ell(J)^d \leq c \mathcal{H}_\infty^d(3J \cap Q_0) \leq \mathcal{H}_\infty^d(3J \cap Q), \tag{3.4}$$

and the cubes $\{3J \mid J \in \mathcal{I}_m(Q_0)\}$ have bounded overlap. Thus,

$$\sum_{I \in \text{Bad}} \ell(I)^d = \sum_{I \in \text{Bad}} \mu_m^{n(I)}(I) = \sum_{I \in \text{Bad}} \sum_{J \in \mathcal{I}_m(Q_0)} \mu_m^{n(I)}(J),$$

$$\leq \sum_{I \in \text{Bad}} \sum_{J \in \mathcal{I}_m(Q_0)} 2^{-b(J)+b(I)} \ell(J)^d$$

$$= \sum_{J \in \mathcal{I}_m(Q_0)} \ell(J)^d \sum_{I \in \text{Bad}, J \supset I} 2^{-b(J)+b(I)} \leq \sum_{J \in \mathcal{I}_m(Q_0)} \ell(J)^d$$

$$\leq c \sum_{J \in \mathcal{I}_m(Q_0)} \mathcal{H}_\infty^d(3J \cap Q_0)^d \leq \mathcal{H}_\infty^d(Q_0).$$

For $I \in \text{Bad}$, let

$$\mu_I := \mu_m^{n(I)}|_I.$$
Note that by construction, for each \( J \subseteq I \), we have that \( \mu^I(J) \leq 2\ell(J)^d \), and thus this also holds for all dyadic cubes \( J \), even when \( J \supseteq I \) or \( J \cap I = \emptyset \). In particular, since any ball \( B(x, r) \) can be covered by boundedly many dyadic cubes of size comparable to \( r \), we obtain that

\[
\mu^I(B(x, r)) \lesssim r^d \quad \text{for all } x \in \mathbb{R}^n, \ r > 0.
\]

Moreover,

\[
\mu^I(I) = \ell(I)^d.
\]

**Remark 3.1.** Ideally what we’d like to do at this stage is, for each \( I \in \text{Bad} \), find the maximal bad cubes \( I_j \in \text{Bad} \) properly contained in \( I \) and define a set like

\[
E_I = \bigcup_j \partial_d I_j
\]

where \( \partial_d J \) is the \( d \)-dimensional skeleton of a cube \( J \). Then one can use \( \mu^I \) to show that \( E_I \) is an Ahlfors regular set. However, the collection \( E_I \) will not be suitable for the applications we have in mind, since we need that the sizes of the cubes whose skeletons form \( E_I \) don’t vary too wildly (that is, adjacent cubes should have comparable sizes). This is why more work is needed.

3.2. **Trees.** For \( I \in \text{Bad} \), we will let Tree\((I)\) be those cubes in \( \mathcal{I} \) contained in \( I \) for which the smallest cube from \( \text{Bad} \) that they are properly contained in is \( I \), and we will let Stop\((I)\) be those cubes from \( \text{Bad} \) in Tree\((I)\) properly contained in \( I \).

**Remark 3.2.** Observe that Stop\((I) \subseteq \text{Tree}(I)\), and while the collections \( \{\text{Tree}(I) : I \in \text{Bad}\} \) do not form a disjoint partition of \( \mathcal{I}^m \), they do cover \( \mathcal{I}^m \), and they only intersect at the top cubes and stopped cubes.

**Lemma 3.3.** For \( I \in \text{Bad} \) and \( J \in \text{Stop}(I) \),

\[
2^{d-n-1} \ell(I)^d \leq \mu^I(J) \leq 2\ell(J)^d
\]

**Proof.** Note that by construction, for \( I \in \text{Bad} \), and because there are \( 2^n \)-dyadic cubes \( J \subseteq I \) with \( \ell(J) = \ell(I)/2 \),

\[
\mu^{n(I)+1}_m(I) = \sum_{J \in \mathcal{I}^{n(I)+1}} \mu^{n(I)+1}_m(J) \leq \sum_{J \in \mathcal{I}^{n(I)+1}} 2\ell(J)^d
\]

\[
= \sum_{J \in \mathcal{I}^{n(I)+1}} 2^{1-d} \ell(I)^d \leq 2^{n-d+1} \ell(I)^d
\]

and for \( J \in \text{Stop}(I) \),

\[
\mu^{n(I)+1}_m(J) = \mu^{n(J)}_m(J) = \ell(J)^d.
\]
Thus,
\[
2\ell(J)^d \geq \mu^I(J) = \mu^m_n(I)\frac{\ell(I)^d}{\mu^m_n(I)^{d-1}} \geq 2^{d-n-1}\ell(J)^d.
\]

Let \( M > 1 \), we will choose it later. For \( Q \in \mathcal{D}(k_0) \) and \( I \in \mathcal{I}(Q_0) \), we write \( Q \sim I \) if
\[
MB_Q \cap I \neq \emptyset \quad \text{and} \quad \rho \ell(I) \leq \ell(Q) < \ell(I),
\]
where \( \rho \) is as in Theorem 2.2. Observe that for \( m \) large enough,
\[
\{ I \in \mathcal{I}(Q_0) : I \sim Q \text{ for some } Q \in \mathcal{D}(k_0) \} \subseteq \mathcal{I}^m(Q_0).
\]

Indeed, If \( Q \in \mathcal{D}(k_0) \), this means \( \ell(I) \geq \rho \ell(Q) \geq 5\rho k_0^d > 2^{-m} \) for \( m \) large enough, and now we just recall Remark 3.2. We now perform the following stopping-time algorithm on the cubes \( \mathcal{D}(k_0) \). For \( R \in \mathcal{D}(k_0) \) contained in \( Q_0 \), we let \( \text{Stop}(R) \) denote the set of maximal cubes in \( R \) that are either in \( \mathcal{D}(k_0) \) or have a child \( Q \) for which there is \( I \in \text{Bad} \) such that \( Q \sim I \). Observe that if \( R \in \mathcal{D}(k_0) \), then \( \text{Stop}(R) = \{ R \} \). We then let \( \text{Tree}(R) \) be those cubes contained in \( R \) that are not properly contained in any cube from \( \text{Stop}(R) \), so in particular, \( \text{Stop}(R) \subseteq \text{Tree}(R) \). Let \( \text{Next}(R) \) be the children of cubes in \( \text{Stop}(R) \) that are also in \( \mathcal{D}(k_0) \) (so this could be empty).

Now let \( \text{Top}_0 = \{ Q_0 \} \), and for \( R \in \text{Top}_k \), we let
\[
\text{Top}_{k+1} = \bigcup_{R \in \text{Top}_k} \text{Next}(R),
\]
that is, \( \text{Top}_{k+1} \) are the children of the cubes in \( \text{Stop}(R) \) for each \( R \in \text{Top}_k \). Let
\[
\text{Top} = \bigcup_{k \geq 0} \text{Top}_k.
\]

Note that for each \( R \in \text{Top} \), if \( R_1 \) is its parent, then \( R_1 \in \text{Stop}(R') \) for some cube \( R' \), and so there is \( I_R \in \text{Bad} \) with \( I_R \sim R'' \) for some sibling \( R'' \in \text{Child}(R_1) \). In particular, the map \( R \mapsto I_R \) maps boundedly many cubes to one cube, and so
\[
\sum_{R \in \text{Top}} \ell(R)^d \lesssim_M \sum_{I \in \text{Bad}} \ell(I)^d \lesssim_N \mathcal{H}^d(Q_0).
\]

The collection \( \text{Top} \) is our desired collection and \( \{ \text{Tree}(R) \mid R \in \text{Top} \} \) are the desired stopping-time regions for the Main Lemma and (1.10) now follows from (3.10). It remains to verify items (2) of the Main Lemma, which will be the focus of the next two subsections. We will first need a lemma about our trees:
Lemma 3.4. Let $R \in \text{Top}$ and

$$S(R) := \{ I \in \mathcal{I}(Q_0) | Q \sim I \text{ for some } Q \in \text{Tree}(R) \}.$$ 

Then there is $N_0 \lesssim_{n,M} 1$ and $J_1(R), \ldots, J_{N_0}(R) \in \text{Bad}$ so that

$$S(R) \subseteq \text{Tree}(J_1(R)) \cup \cdots \cup \text{Tree}(J_{N_0}(R)).$$

Proof. Consider the cubes $I_1, \ldots, I_{N_0}$ in $\mathcal{I}(Q_0)$ of maximal size so that $I_j \sim R$ (note that $N_0$ here depends only on $n$ and $M$). Then for $m$ large enough, each $I_j$ is contained in $\text{Tree}(J_j)$ for some $J_j \in \text{Bad}$ by (3.9).

Now let $I \in S(R)$, so by definition there is $Q \in \text{Tree}(R)$ satisfying (3.8), then $I \subseteq I_j \subseteq J_j$ for some $j$. If $I \not\subseteq \text{Tree}(J_j)$, then there is $J \in \text{Stop}(J_i)$ so that $I \not\subseteq J \subseteq I_j \subseteq J_j$. Since $I_j \sim R$, $\ell(I_j) < \ell(R)$, so we must have $\ell(J) < \ell(R)$. Thus, if $Q'$ is the maximal ancestor of $Q$ with $\ell(Q') < \ell(J)$, then $\ell(Q') < \ell(R)$, and so $Q' \not\subseteq R$. Since $Q \in \text{Tree}(R)$, this implies $Q' \in \text{Tree}(R)$. Since $Q \sim I$ and $\rho(J) \leq \ell(Q') < \ell(J)$ by the maximality of $Q'$, we also have $Q' \sim J$. So the parent $Q'' \subseteq R$ of $Q'$ must be in $\text{Stop}(R)$, but this contradicts $Q' \sim J$. We let $J_i(R) = J_i$ and this proves the lemma.

3.3. Smoothing. We follow the “smoothing” process of David and Semmes (c.f. [DS91, Chapter 8]). Fix $0 < \tau < 1$. For a finite family of cubes $\mathcal{F} \subset \mathcal{D}$, define the following smoothing function: for a point $x \in \mathbb{R}^n$, set

$$(3.11) \quad d_\mathcal{F}(x) := \inf_{S \in \mathcal{F}} (\ell(S) + \text{dist}(x,S)),$$

and for a dyadic cube $I \in \mathcal{I}$,

$$(3.12) \quad d_\mathcal{F}(I) := \inf_{x \in I} d_\mathcal{F}(x) = \inf_{S \in \mathcal{F}} (\ell(S) + \text{dist}(I,S)).$$

We define $\mathcal{C}_\mathcal{F}$ to be the set of maximal cubes $I \in \mathcal{I}(Q_0)$ for which

$$(3.13) \quad \ell(I) < \tau d_\mathcal{F}(I).$$

The following lemmas are quite standard and appear in different forms depending on the scenario in which they are being applied (depending on between which kinds of cubes, dyadic or not, that $d_\mathcal{F}$ is computing), see for example [DS91, Lemma 8.7]. We include their proofs below for completeness.

Lemma 3.5. Let $I, I' \in \mathcal{I}$. Then,

$$(3.14) \quad d_\mathcal{F}(I) \leq 2\ell(I) + \text{dist}(I, I') + 2\ell(I') + d_\mathcal{F}(I').$$

Proof. Let $x, y \in I$ and $x', y' \in I'$. Let also $Q \in \mathcal{F}$; we have

$$(3.15) \quad d_\mathcal{F}(x) \leq |x - y| + |y - y'| + |y' - x'| + \text{dist}(x', Q) + \ell(Q),$$

and

$$(3.16) \quad d_\mathcal{F}(I) \leq \ell(I) + \text{dist}(I, I') + d_\mathcal{F}(I').$$

Therefore, $d_\mathcal{F}(I) \leq 2\ell(I) + \text{dist}(I, I') + 2\ell(I') + d_\mathcal{F}(I').$
simply by triangle inequality and the definition of $d_{\mathcal{F}}$. Clearly, $|y - y'| \leq \text{dist}(I, I')$; moreover, infimising first over all $Q \in \mathcal{F}$ and then over all $x' \in I$, we obtain (3.14).

Lemma 3.6. Let $I \in C_{\mathcal{F}}$; then

$$\frac{\tau}{2}d_{\mathcal{F}}(I) \leq \ell(I) < \tau d_{\mathcal{F}}(I).$$

Proof. By (3.13), $\ell(I) < \tau d_{\mathcal{F}}(I)$, and by definition it is a maximal cube satisfying this inequality. Hence if $\hat{I}$ is the parent of $I$, there is a point $z \in \hat{I}$ with $\tau d_{\mathcal{F}}(z) \leq 2\ell(I)$. The fact that $d_{\mathcal{F}}$ is 1-Lipschitz gives the remaining inequality.

The following lemma says that if two cubes in $C_{\mathcal{F}}$ are close to each other, then they have comparable size.

Lemma 3.7. Let $I, J \in C_{\mathcal{F}}$ and recall that $C_{\mathcal{F}}$ depends on a parameter $\tau$. Let $0 < \eta < 1$ be another small parameter. If

$$\eta^{-1}J \cap \eta^{-1}I \neq \emptyset,$$

for $\tau^{-1} > 2\sqrt{n}/\eta$,

$$\ell(I) \sim \ell(J).$$

Proof. It suffices to show that for all $y \in \eta^{-1}J$,

$$\tau^{-1}\ell(J) \sim d_{\mathcal{F}}(y)$$

Since $d_{\mathcal{F}}$ is 1-Lipschitz, we see that

$$|d_{\mathcal{F}}(J) - d_{\mathcal{F}}(y)| \leq \eta^{-1}\text{diam}(J) = \frac{\sqrt{n}}{\eta}\ell(J).$$

Hence if $\tau^{-1} > 2\sqrt{n}/\eta$,

$$d_{\mathcal{F}}(y) \geq d_{\mathcal{F}}(J) - \frac{\sqrt{n}}{\eta}\ell(J) \geq \left(\tau^{-1} - \frac{\sqrt{n}}{\eta}\right)\ell(J) \geq \frac{1}{2\tau}\ell(J)$$

On the other hand, again using the fact that $d_{\mathcal{F}}$ is 1-Lipschitz, we see that

$$d_{\mathcal{F}}(y) \lesssim (\eta^{-1} + \tau^{-1})\ell(J) \lesssim \tau^{-1}\ell(J).$$
3.4. Constructing an Ahlfors regular set with respect to a tree. Let $R \in \text{Top}$ and $T \subseteq \text{Tree}(R)$ be a stopping-time region, let $T$ denote the maximal cube in $T$, $\mathcal{T}$ be the set of minimal cubes of $T$ (that is, those cubes in $T$ that don’t properly contain another cube in $T$).

Observe that since all the cubes we are working with come from $D(k_0)$ and the number of these cubes in $Q_0$ is finite, the infimum $d_\mathcal{T}$ is attained, and so for each $I \in \mathcal{I}$ there is $Q_I \in F$ so that

$$d_\mathcal{T}(I) = \ell(Q_I) + \text{dist}(Q_I, I).$$

Let $C_0 > 4$ and set

$$\hat{T} = \bigcup \{Q \in D \mid \ell(Q) = \ell(T), \ Q \cap C_0 B_T \neq \emptyset\},$$

$$\mathcal{C} = \{I \in \mathcal{C}_\mathcal{T} \mid I \cap \hat{T} \neq \emptyset\},$$

and

$$\hat{E} := \bigcup_{I \in \mathcal{C}} \partial_d I.$$

This set $\hat{E}$ will be our desired $E(T)$ as in the statement of the Main Lemma (we just write $\hat{E}$ for short).

**Lemma 3.8.** For $m$ large enough,

$$\mathcal{C} \subseteq \mathcal{I}^m.$$  

**Proof.** Note that by (3.16), and because $Q_I \in D(k_0)$, for $I \in \mathcal{C}$,

$$\ell(I) \geq \frac{\tau}{2} d_\mathcal{T}(I) \geq \frac{\tau}{2} \ell(Q_I) \geq \frac{5\tau}{2} \rho^{k_0},$$

and for $\tau$ small enough,

$$\ell(I) < \frac{\tau}{2} d_\mathcal{T}(I) \leq \tau(\ell(T) + \text{dist}(I, T)) \leq \tau(C_0 + 1)\ell(T) < \frac{1}{5} \ell(Q_0) = 1.$$

Thus, (3.26) follows for $m$ large enough from these two inequalities.  

**Remark 3.9.** Note that we definitely don’t have that $\mathcal{C} \subseteq \mathcal{I}^m(Q_0)$, since some cubes in $\mathcal{C}$ are actually disjoint from $Q_0$. This will cause some difficulties later.

**Lemma 3.10.** Part (b) of the Main Lemma holds.

**Proof.** Firstly, as $C_0 B_T \cap E \subseteq \hat{T}$, we immediately have the first containment, so we just need to show the second containment.

Note that if $I \in \mathcal{C}$, then $I \cap Q \neq \emptyset$ for some $Q \in D$ with $\ell(Q) = \ell(T)$ and $Q \cap C_0 B_T \neq \emptyset$. Thus,

$$\text{dist}(I, T) \leq \text{dist}(Q, T) + \text{diam} Q \leq C_0 \ell(T) + 2\ell(T) < (C_0 + 2)\ell(T).$$
Thus,
\[ \text{diam } I = \sqrt{n} \ell(I) < \sqrt{n} \tau d_\mathcal{F}(I) \leq \sqrt{n} \tau \text{dist}(I, T) + \ell(T) < \sqrt{n} \tau (C_0 + 3) \ell(T) \]
so for \( \tau > 0 \) small, diam \( I \leq \frac{C_0}{2} \ell(T) \). Thus, \( I \subseteq (3C_0/2 + 2) B_T \subseteq 2C_0 B_T \), which proves the lemma.

**Lemma 3.11.** Part (c) of the Main Lemma holds.

**Proof.** Let \( x \in E \cap C_0 B_T \subseteq \hat{T} \). By part (b), there is \( I \) so that \( x \in \mathcal{C} \subseteq \mathcal{C}_\mathcal{F} \). By definition, \( \partial_d I \subseteq \hat{E} \), and so
\[ \text{dist}(x, \hat{E}) \leq \text{diam } I \leq \sqrt{n} \ell(I) < \sqrt{n} \tau d_\mathcal{F}(I) \leq \sqrt{n} \tau d_\mathcal{F}(x). \]

Moreover, (1.14) follows from (3.16). Thus, to prove the Main Lemma, all that remains to be shown is the following lemma.

**Lemma 3.12.** Part (a) of the Main Lemma holds, that is, the set \( \hat{E} \) is Ahlfors \( d \)-regular.

**Proof.** Let \( x \in \hat{E} \) and \( 0 < r < \text{diam } \hat{E} \leq 2C_0 \ell(T) \). We define
\[ \mathcal{C}(x, r) = \{ I \in \mathcal{C} \mid I \cap B(x, r) \neq \emptyset \}. \]

We split into three cases, and in each case we prove first the upper estimate for being Ahlfors regular and then the lower estimate.

**Case 1:** \( 2r \leq d_\mathcal{F}(x) \). Since \( d_\mathcal{F} \) is Lipschitz, this means \( d_\mathcal{F}(y) \geq d_\mathcal{F}(x) - |x - y| \), and so if \( I \in \mathcal{C}(x, r) \), \( y \in I \) is so that \( d_\mathcal{F}(I) = d_\mathcal{F}(y) \), and \( z \in I \cap B(x, r) \), then \( |z - y| \leq \text{diam } I = \sqrt{n} \ell(I) \), and so
\[ \ell(I) \overset{(3.16)}{=} \tau d_\mathcal{F}(I) = \tau d_\mathcal{F}(y) \geq \tau (d_\mathcal{F}(x) - |x - y|) \geq \tau (2r - |x - z| - |z - y|) \geq \tau (r - \sqrt{n} \ell(I)) \]
and so for \( \tau \ll \sqrt{n} \) we have \( \ell(I) \gtrsim \tau r \). This implies \#\( \mathcal{C}(x, r) \approx_{n, \tau} 1 \), and so it is not hard to show that
\[ \mathcal{H}^d(\hat{E} \cap B(x, r)) \approx_{n, \tau} 1. \]

**Case 2:** \( 8 \ell(T) > 2r > d_\mathcal{F}(x) \).

Before we proceed, we record a few estimates. First, for \( I \in \mathcal{C}(x, r) \), if \( 2r > d_\mathcal{F}(x) \),
\[ \tau^{-1} \ell(I) \overset{(3.16)}{<} d_\mathcal{F}(I) \leq d_\mathcal{F}(y) \leq d_\mathcal{F}(x) + |x - y| < 2r + r = 3r \]
Next, note that for all \( I \in \mathcal{C} \), \( \ell(Q_I) \leq d_\mathcal{F}(I) \). Let \( Q'_I \) be the largest cube in \( \mathcal{T} \) containing \( Q_I \), so that \( \ell(Q'_I) \leq d_\mathcal{F}(I) \).

**Lemma 3.13.** If \( x \in \hat{E} \) and \( d_\mathcal{F}(x) < 2r < 24 \ell(T) \), then
\[ \ell(Q'_I) \overset{\tau}{\approx} \ell(I). \]
Proof. If \( Q'_1 = T \), then \( Q_T = T \), so

\[
\ell(T) \leq d_\mathcal{F}(I) \quad \text{(3.27)}
\]
and so \( \ell(Q'_1) = \ell(T) \sim \ell(I) \). Otherwise, if \( \ell(Q'_1) < \ell(T) \), then \( \ell(Q'_1) \sim d_\mathcal{F}(I) \) \( (3.16) \) \( \ell(I) \) by maximality of \( Q'_1 \) (indeed, if \( \ell(Q'_1) < \rho d_\mathcal{F}(I) \), then its parent \( Q''_1 \) satisfies \( \ell(Q''_1) < d_\mathcal{F}(I) \) and \( Q''_1 \in \mathcal{T} \) since \( Q'_1 \subseteq T \), but this contradicts the maximality of \( Q'_1 \)). This proves the lemma. \( \square \)

Recall (3.26) and let

\[
\mathcal{C}_1(x, r) = \{ I \in \mathcal{C}(x, r) : I \cap Q_0 \neq \emptyset \} = \mathcal{C}(x, r) \cap \mathcal{I}_m(Q_0),
\]
\[
\mathcal{C}_2(x, r) = \mathcal{C}(x, r) \setminus \mathcal{C}_1(x, r).
\]

Lemma 3.14. If \( x \in \hat{E} \) and \( d_\mathcal{F}(x) < 2r < 24\ell(T) \), then

\[
\sum_{I \in \mathcal{C}_1(x, r)} \ell(I)^d \lesssim r^d. \tag{3.30}
\]

Proof. We need an estimate like \( \ell(I)^d \lesssim \mathcal{H}_\infty^d(I \cap Q_0) \), but this may not necessarily be true: of course \( I \cap Q_0 \neq \emptyset \) since \( I \in \mathcal{C}_1(x, r) \), but it could be that \( I \) only intersects \( Q_0 \) at a corner of \( I \) so \( \mathcal{H}_\infty^d(I \cap Q_0) \) could be very small compared to \( \ell(I)^d \). To overcome this, we associate to \( I \) a neighboring dyadic cube that does intersect \( E \) in a large set. Let \( \text{Nei}(I) \) be the set of dyadic cubes \( J \subseteq 3I \) with \( \ell(J) = \ell(I) \). Then

\[
\ell(I)^d \lesssim \mathcal{H}_\infty^d(3I \cap Q_0) \leq \sum_{J \in \text{Nei}(I)} \mathcal{H}_\infty^d(J \cap Q_0).
\]

Hence there is \( I' \in \text{Nei}(I) \) so that

\[
\mathcal{H}_\infty^d(I' \cap Q_0) \gtrsim \ell(I)^d.
\]

Since \( I' \subseteq 3I \), we know that

\[
\text{dist}(I', Q'_1) \leq \text{diam} I + \text{dist}(I, Q_1) \leq \sqrt{\tau} \ell(I) + d_\mathcal{F}(I) \lesssim \tau^{-1} \ell(I) \sim \ell(Q'_1).
\]

As \( \ell(I) \sim \ell(Q'_1) \), for \( M \gg \tau^{-1} \) large enough, \( MB_Q \cap I' \neq \emptyset \), and since \( Q'_1 \in \mathcal{T} \subseteq \text{Tree}(R) \) and \( I' \in \mathcal{I}_m(Q_0) \) (because \( I' \cap Q_0 \neq \emptyset \) and \( I \in \mathcal{I}_m \) by (3.26)), this implies \( I' \sim Q'_1 \), and so \( I' \in \mathcal{S}(R) \) (where \( \mathcal{S}(R) \) is as in Lemma 3.4). In particular, there is \( J_i = J_i(R) \) so that \( I' \in \text{Tree}(J_i) \) by Lemma 3.4. We will use this fact shortly, but we need one more estimate: We now claim that

\[
\sum_{I \in \mathcal{C}_1(x, r)} 1_{I'} \lesssim 1_{B(x, 2r)} \tag{3.31}.
\]

Indeed, if \( y \in I'_1 \cap \cdots \cap I'_\ell \) for some distinct \( I_1, \ldots, I_\ell \in \mathcal{C}_1(x, r) \), then the \( I_j \) are disjoint and \( y \in 3I_1 \cap \cdots \cap 3I_\ell \), so Lemma 3.7 implies they have
sizes all comparable to $I_1$ and are also contained in $9I_1$ (assuming $I_1$ is the largest). Thus if $|A|$ denotes the Lebesgue measure of a set $A$,

$$\ell|I_1| \sim \sum_{i=1}^{\ell} |I_i| = \left| \bigcup_{i=1}^{\ell} I_i \right| \leq |9I_1|$$

which implies $\ell \lesssim 1$, thus, $\sum_{I \in \mathcal{G}_1(x,r)} \mathbf{1}_I \lesssim 1$. Finally, note that

$$\text{diam} I = \sqrt{n\ell(I)} \overset{(3.16)}{<} \tau \sqrt{n d_\mathcal{F}(I)} \overset{(3.27)}{<} 3\sqrt{n\tau r}$$

and since $I$ and $I'$ touch, $\text{dist}(x, I') \leq \text{diam} I + r < (3\sqrt{n\tau} + 1)r$, so for $\tau > 0$ small enough, $I' \subseteq B(x, 2r)$. Thus, (3.31) follows.

Thus,

$$\mathcal{H}^d(\hat{E} \cap B(x, r)) \lesssim \sum_{I \in \mathcal{G}_1(x,r)} \ell(I)^d \lesssim \sum_{I \in \mathcal{G}_1(x,r)} \mathcal{H}_\infty^d(I' \cap Q_0)$$

$$\leq \sum_{I \in \mathcal{G}_1(x,r)} \sum_{i=1}^{N_0} \sum_{J \in \text{Stop}(J_i)} (\text{diam} J)^d$$

$$\overset{(3.7)}{\lesssim} \sum_{I \in \mathcal{G}_1(x,r)} \sum_{i=1}^{N_0} \sum_{J \in \text{Stop}(J_i)} \mu^J_i(J)$$

$$\leq \sum_{I \in \mathcal{G}_1(x,r)} \sum_{i=1}^{N_0} \mu^J_i(I') \overset{(3.31)}{\lesssim} \sum_{i=1}^{N_0} \mu^J_i(B(x, 2r)) \lesssim r^d.$$

This proves (3.30).

\[\square\]

**Lemma 3.15.** If $x \in \hat{E}$ and $d_\mathcal{F}(x) < 2r < 8\ell(T)$, then

$$\sum_{I \in \mathcal{G}_2(x,r)} \ell(I)^d \lesssim r^d. \tag{3.32}$$

**Proof.** For $I \in \mathcal{G}_2(x,r)$, let $\tilde{Q}_I$ denote the child of $Q'_I$ containing the center of $Q'_I$. We claim that the cubes $\{\tilde{Q}_I : I \in \mathcal{G}_2(x,r)\}$ have bounded overlap. Indeed, suppose there were $I_1, \ldots, I_\ell \in \mathcal{G}_2(x,r)$ distinct and a point

$$y \in \bigcap_{j=1}^{\ell} \tilde{Q}_{I_j}.$$
We can assume that \( \tilde{Q}_{I_1} \) is the largest, and since they are all cubes, this implies \( \tilde{Q}_{I_1} \supseteq \tilde{Q}_j \) for all \( j \). Since (3.33)
\[
\text{dist}(I_j, \tilde{Q}_{I_1}) \leq \text{dist}(I_j, \tilde{Q}_{I_1}) \leq \text{dist}(I_j, Q_{I_1}) \leq d_{\mathcal{D}}(I_j) \lesssim \tau^{-1} \ell(I_j)
\]
and the \( I_j \) are disjoint, and because \( \ell(I_j) \sim \ell(Q_{I_1}) \sim \ell(\tilde{Q}_{I_1}) \), for given \( \epsilon > 0 \), there can be at most boundedly many \( I_j \) (depending on \( \epsilon \) and \( \tau \)) for which \( \text{diam} I_j \geq \epsilon \ell(\tilde{Q}_{I_1}) \). For the rest of the \( j \), we have that
\[
\text{dist}(I_j, \tilde{Q}_{I_1}) \lesssim \tau^{-1} \ell(I_j) < \frac{\epsilon}{\tau} \ell(\tilde{Q}_{I_1}),
\]
so for \( \epsilon > 0 \) small enough, and recalling that \( \rho < c_0/2 \) in Theorem 2.2, this implies \( I_j \subseteq c_0 B_{Q_{I_1}} \). Since \( I_j \cap \tilde{T} \neq \emptyset \) and the balls \( \{c_0 B_Q : Q \in \mathcal{D}_k\} \) are disjoint for each \( k \) by Theorem 2.2, this means \( \emptyset \neq I_j \cap Q_{I_1} \subseteq I_j \cap Q_0 \), and so \( I_j \in \mathcal{C}_1(x, r) \), which is a contradiction since we assumed \( I_j \in \mathcal{C}_2(x, r) \). Thus, there are no other \( j \), and so \( \ell \lesssim \epsilon \). This finishes the proof that the sets \( \{\tilde{Q}_I : I \in \mathcal{C}_2(x, r)\} \) have bounded overlap.

Fix \( I \in \mathcal{C}_2(x, r) \) and let \( J \in \mathcal{C} \) so that \( J \cap \frac{\alpha}{2} B_{\tilde{Q}_I} \neq \emptyset \). Then \( \ell(J) < \tau d_{\mathcal{D}}(J) \leq \tau \ell(\tilde{Q}_I) \), so for \( \tau \) small enough, \( J \subseteq c_0 B_{\tilde{Q}_I} \). Thus, if
\[
\{J_I^{(i)}\}_{i=1}^{L_I} = \{J \in \mathcal{C} : J \cap \frac{c_0}{2} B_{\tilde{Q}_I} \neq \emptyset\},
\]
since the \( \tilde{Q}_I \) have bounded overlap, do so the cubes
\[
\{J_I^{(i)} : i = 1, \ldots, L_I, \ I \in \mathcal{C}_2(x, r)\}.
\]
For \( I \in \mathcal{C}(x, r) \) and \( i = 1, \ldots, L_I \),
\[
\text{dist}(I, J_I^{(i)}) \leq \text{dist}(I, \tilde{Q}_I) \leq \text{dist}(I, Q_I) \leq d_{\mathcal{D}}(I) < 2r,
\]
hence \( J_I^{(i)} \in \mathcal{C}_1(x, 3r) \). Now we have by our assumptions that
\[
d_{\mathcal{D}}(x) < 2r < 2 \cdot (3r) = 3 \cdot (2r) < 3 \cdot 8 \ell(T) = 24 \ell(T).
\]
Thus, (3.30) holds for \( 3r \) in place of \( r \), and so
\[
\sum_{I \in \mathcal{C}_2(x, r)} \ell(I)^d \sim \sum_{I \in \mathcal{C}_2(x, r)} \ell(\tilde{Q}_I)^d \sim_c \sum_{I \in \mathcal{C}_2(x, r)} \mathcal{H}_\infty^d \left( \frac{c_0}{2} B_{\tilde{Q}_I} \right)
\]
\[
\lesssim \sum_{I \in \mathcal{C}_2(x, r)} \sum_{J \cap \frac{c_0}{2} B_{\tilde{Q}_I} \neq \emptyset} \ell(J)^d = \sum_{I \in \mathcal{C}_2(x, r)} \sum_{i=1}^{L_I} \ell(J_I^{(i)})^d
\]
\[
\lesssim \sum_{J \in \mathcal{C}_1(x, 3r)} \ell(J)^d \lesssim r^d
\]
where we used the bounded overlap property in the penultimate inequality. □

Thus, combining the two previous lemmas, we have that

$$\mathcal{H}^d(\hat{E} \cap B(x, r)) \lesssim \sum_{i=1}^{2} \sum_{I \in \mathcal{C}(x, r)} \ell(I)^d \lesssim r^d.$$  

Now to complete the proof in this case, we need to show the reverse estimate. Let $I \in \mathcal{C}(x, r/2)$. Then (3.27) implies that for $\tau$ small enough, $I \subset B(x, r)$. Moreover, since $I \in \mathcal{C}$, $I \cap Q \neq \emptyset$ for some $Q \subseteq \hat{T}$ with $\ell(Q) = \ell(T)$. If $I \in \mathcal{C}(x, r/2)$ is the cube so that $x \in \partial_d I$, then for $\tau$ small,

$$\text{dist}(x, Q) \leq \text{diam} I \quad \text{(3.27)} \leq 3\sqrt{n} \tau r < \frac{r}{4}.$$  

Thus, there is $y \in Q \cap B(x, r/4)$, and so we can find a subcube $Q' \subseteq B(x, r/2) \cap Q$ containing $y$ so that $\ell(Q') \sim r$ and the cubes from $\mathcal{C}(x, r/2)$ cover $Q'$. Thus,

$$\mathcal{H}^d(B(x, r) \cap \hat{E}) \geq \sum_{I \in \mathcal{C}(x, r/2)} \mathcal{H}^d(\partial_d I) \sim \sum_{I \in \mathcal{C}(x, r/2)} \ell(I)^d \gtrsim \mathcal{H}^d(Q') \gtrsim \ell(Q')^d \sim r^d.$$  

**Case 3:** $2C_0 \ell(T) > r > 4\ell(T)$.

Note that by the previous case,

$$\mathcal{H}^d(B(x, r) \cap 2B_T \cap \hat{E}) \leq \mathcal{H}^d(2B_T \cap \hat{E}) \lesssim \ell(T)^d \lesssim r^d.$$  

So to prove upper regularity, we just need to verify

$$\mathcal{H}^d(B(x, r) \cap \hat{E} \setminus 2B_T) \lesssim r^d.$$  

If $I \cap B(x, r) \setminus 2B_T \neq \emptyset$, and if $y \in I \setminus 2B_T$,

$$\ell(I) \sim \tau d_{\mathcal{P}}(I) \geq \tau(d_{\mathcal{P}}(y) - \text{diam} I) \geq \tau \text{dist}(y, T) - \tau \sqrt{n} \ell(I) \geq \tau \ell(T) - \tau \sqrt{n} \ell(I)$$

and so for $\tau$ small enough,

$$\frac{\tau}{2} \ell(T) \leq \ell(I)$$

Moreover, since $I \cap B(x, r) \neq \emptyset$, $x \in \hat{E}$, and $T \subseteq \bigcup_{J \in \mathcal{C}, \mathcal{P}} J$,  

\[ \ell(I) < \tau d_x(I) \leq \tau(\ell(T) + \text{dist}(I, T)) \]
\[ < \tau(\ell(T) + \text{diam } I + 2r + \text{dist}(x, T)) \]
\[ < \tau(\ell(T) + \sqrt{n}\ell(I) + 4C_0\ell(T) + \text{diam } \mathring{E}) \]
\[ \lesssim \tau(\ell(T) + \ell(I)) \]

So for \( \tau > 0 \) small enough, we also have \( \ell(I) \lesssim \tau \ell(T) \), hence \( \ell(I) \sim \tau \ell(T) \). There can only be at most boundedly many disjoint cubes \( I \in \mathcal{C} \) with \( \ell(I) \sim \tau \ell(T) \), and so
\[ \mathcal{H}^d(\mathring{E} \cap B(x, r) \setminus 2B_T) \lesssim \ell(T)^d \sim r^d. \]

For the lower bound, if \( x \in \mathring{E} \cap 2B_T \), then \( r > 4\ell(T) \) implies by the previous case that
\[ \mathcal{H}^d(\mathring{E} \cap B(x, r)) \geq \mathcal{H}^d(\mathring{E} \cap 2B_T) \gtrsim \ell(T)^d \sim r^d. \]

Alternatively, if \( x \in \mathring{E} \setminus 2B_T \), then by the arguments above, if \( I \in \mathcal{C} \) contains \( x \), then \( \ell(I) \sim \tau \ell(T) \sim \tau r \), so for \( \tau \) small enough, \( I \subseteq B(x, r) \). Thus,
\[ \mathcal{H}^d(\mathring{E} \cap B(x, r)) \geq \mathcal{H}^d(\partial_d I) \sim \ell(I)^d \sim r^d. \]

This completes the proof.

\[ \square \]

This finishes the proof of the Main Lemma.

4. A GENERAL LEMMA ON QUANTITATIVE PROPERTIES

We now want to apply the approximation by Ahlfors regular sets obtained in the previous section to derive quantitative bounds on the sum of the \( \beta \) coefficients. The method we present is quite easy and general. The idea is the following: let us pick one of the quantitative properties described by David and Semmes. For example, the BAUP (which stands for bilateral approximation by union of planes) (see [DS93], II, Chapter 3), the GWEC (generalised weak exterior convexity) (see [DS93], II, Chapter 3), or the LS (local symmetry), see [DS91], Definition 4.2. On each cube \( R \in \text{Top} \), we run a stopping time on Tree(\( R \)) where we stop whenever we meet a cube which does not satisfy the chosen property. By doing so, we obtain a new tree and consequently a new approximating Ahlfors regular set. This time, however, this set will turn out to be uniformly rectifiable exactly because it approximates \( E \) at those scales where \( E \) is very well behaved.

Let us try to make all this precise.
Definition 4.1 (Quantitative property). By a quantitative property (QP) $\mathcal{P}$ of $E$ we mean a finite set of real numbers $\{p_1, \ldots, p_N\}$ together with two subsets of $E \times \mathbb{R}_+ = E \times (0, \infty)$

$$\mathcal{G}^\mathcal{P} = \mathcal{G}^\mathcal{P}(p_1, \ldots, p_N) \quad \text{and} \quad \mathcal{B}^\mathcal{P} = \mathcal{B}^\mathcal{P}(p_1, \ldots, p_N),$$

which depend on $\{p_1, \ldots, p_N\}$, such that

$$(4.1) \quad \mathcal{G}^\mathcal{P} \cup \mathcal{B}^\mathcal{P} = E \times \mathbb{R}_+ \quad \text{and} \quad \mathcal{G}^\mathcal{P} \cap \mathcal{B}^\mathcal{P} = \emptyset.$$

We will call $\{p_1, \ldots, p_N\}$ the parameters of $\mathcal{P}$.

If we want to specify the subset $E$ upon which we are applying a quantitative property $\mathcal{P}$, we may write, for example, $\mathcal{G}_E^\mathcal{P}$, or $\mathcal{B}_E^\mathcal{P}$. Let us give a few examples of quantitative properties described in the book [DS93]:

**BWGL**: The so-called ‘Bilateral Weak Geometric Lemma’ (BWGL) is a quantitative property. Given a real number $\epsilon > 0$, for each pair $(x, r) \in E \times \mathbb{R}_+$, BWGL asks whether there exists a plane $P$ so that $d_{B(x, r)}(E, P) < \epsilon$. If one such a plane exists, then we put $(x, r) \in \mathcal{G}^{\text{BWGL}}$; if not, then $(x, r) \in \mathcal{B}^{\text{BWGL}}$. This is clearly a partition of $E \times \mathbb{R}_+$ and so BWGL is a QP with parameter $\epsilon$.

**LS**: The ‘Local Symmetry’ (LS) property is defined as follows. Given $\epsilon > 0$, for each pair $(x, r) \in E \times \mathbb{R}_+$, we say $(x, r) \in \mathcal{B}^{\text{LS}}(\epsilon, \alpha)$ if there are $y, z \in B(x, r) \cap E$ so that $\text{dist}(2y - z, E) \geq \epsilon r$.

**LCV**: For the quantitative property ‘Local Convexity’ (LCV), we define $\mathcal{B}^{\text{LCV}}$ to be those $(x, r) \in E \times \mathbb{R}_+$ for which there are $y, z \in B(x, r) \cap E$ such that $\text{dist}((y + z)/2, E) \geq \epsilon r$.

**WCD**: Let two positive numbers $C_0$ and $\epsilon$ be given. The ‘Weak Constant Density’ (WCD) condition asks the following: for $(x, r) \in E \times \mathbb{R}_+$, does a measure $\mu_{x,r}$ exist, such that $\text{spt}(\mu_{x,r}) = E$;

* $\mu_{x,r}$ is Ahlfors $d$ -- regular with constant $C_0 \geq 1$;
* $|\mu_{x,r}(y, s) - s^d| \leq ct^d$ for all $y \in E \cap B(x, r)$ and $0 < s \leq r$.

If one such a measure $\mu_{x,r}$ exists, then we put $(x, r) \in \mathcal{G}^{\text{WCD}}(C_0^{-1}, \epsilon)$. If not, then $(x, r) \in \mathcal{B}^{\text{WCD}}(C_0^{-1}, \epsilon)$. This is clearly a partition of $E \times \mathbb{R}_+$ and so WCD is a QP with parameters $(C_0^{-1}, \epsilon)$.

**BP**: Let us give one more example. Let $1 \geq \theta > 0$ be a positive real number. The ‘Big Projection’ (BP) condition asks if for a pair $(x, r)$, there exists a $d$-dimensional plane $P$ such that $|\Pi_P(B(x, r) \cap E)| \geq \theta r^d$. 

where \( \Pi_P \) is the standard orthogonal projection onto \( P \) and \( | \cdot | \) is the \( d \)-dimensional Lebesgue measure on \( P \). We put \((x, r) \in \mathcal{B}_P^\theta(\theta)\) if this is the case; otherwise \((x, r) \in \mathcal{B}_P^\theta(\theta)\). Thus \( \mathcal{B}_P \) is a QP with parameter \( \theta > 0 \).

**Definition 4.2.** Fix a (small) parameter \( \epsilon_1 > 0 \) and two (large) constants \( C_1, C_2 \geq 1 \) and let \( \mathfrak{P} \) be a quantitative property with parameters \( \{p_1, \ldots, p_N\} \). We say that \( \mathfrak{P} \) is \((\epsilon_1, C_1, C_2)\)-continuous, if there exist positive constants \( 0 < c_1, \ldots, c_N < \infty \) depending on \( \epsilon_1 \) and \( C_1 \) such that the following holds. Let \( E_1 \) and \( E_2 \) be two subsets of \( \mathbb{R}^n \) and let \( B = B(x_B, r_B) \) be a ball so that 

\[ B \text{ is centered on } E_1; \]
\[ (x_B, r_B) \in \mathcal{B}_{E_1}^\mathfrak{P}(p_1, \ldots, p_N); \]
\[ d_{C_2}(E_1, E_2) < \epsilon. \]

If \( B' = B(x_{B'}, r_{B'}) \) is a ball so that 

\[ B' \text{ is centered on } E_2; \]
\[ C_2 B' \subset B; \]
\[ r_{B'} \geq \frac{r_B}{C_1}, \]

then

\[(x_{B'}, r_{B'}) \in \mathcal{B}_{E_2}^\mathfrak{P}(c_1 p_1, \ldots, c_N p_N). \]

**Remark 4.3.** In particular a continuous quantitative property is monotonic (or stable) in the following sense; take a set \( E \) and a ball \( B \) centered on \( E \) with \((x_B, r_B) \in \mathcal{B}_E^\mathfrak{P}(p_1, \ldots, p_N) \). If we assume that \( \mathfrak{P} \) is continuous and we take \( E_1 = E_2 = E \) in Definition 4.2, then we see that \((x_{B'}, r_{B'}) \in \mathcal{B}_E^\mathfrak{P}(c_1 p_1, \ldots, c_N p_N) \) whenever \( C_2 B' \subset B \) and \( r_{B'} \geq \frac{r_B}{C_1} \).

Let us look at our concrete examples of QP, and see whether they are continuous, and thus stable.

- One can quite easily check that BWGL, LS, LCV, and BP are stable quantitative properties.
- On the other hand, the WCD is not.

**Definition 4.4** (QP guaranteeing uniform rectifiability). We say a QP (with parameters \( p_1, \ldots, p_N \)) guarantees uniform rectifiability for Ahlfors \( d \)-regular sets with constant \( C_1 \) if, whenever \( A \) is Ahlfors \( d \)-regular with constant \( C_1 \) and 

\[ 1_{\mathcal{B}_{(p_1, \ldots, p_N)}^\mathfrak{P}} \frac{dr}{r} d\mathcal{H}^d |_A \text{ is a Carleson measure on } A \times R_+, \]

then \( A \) is a uniformly rectifiable set. Conversely, if \( A \) is uniformly rectifiable, then we say a QP (with parameters \( p_1, \ldots, p_N \)) is guaranteed by...
uniform rectifiability if the measure in (4.3) is a Carleson measure for the parameters \((p_1, \ldots, p_N)\).

Let us go back to our examples.

- In the two monographs [DS91] and [DS93], David and Semmes prove that the properties BWGL ([DS93], II.2, Proposition 2.2), and WCD are indeed examples of QP guaranteeing uniform rectifiability (see [DS93], I.2, Proposition 2.56, and [T], Theorem 1.1). To further comment on the remark above, consider BWGL: if an Ahlfors \(d\)-regular set \(A\) is uniformly rectifiable, then there exists a universal constant \(\epsilon_0 > 0\) so that for all \(0 < \epsilon < \epsilon_0\), we have that

\[
\int_{B \cap E} \int_0^R 1_{B_{\epsilon}(x,r)}(x,r) \frac{dr}{r} d\mathcal{H}^d(x) \leq C(\epsilon)r_B^d,
\]

for all balls \(B\) centered on \(E\) with \(r_B \leq \text{diam}(E)\). In general, one may have that \(C(\epsilon) \to \infty\) as \(\epsilon \to 0\). On the other hand, it suffices to find a sufficiently small \(\epsilon > 0\) for which (4.4) holds to prove that \(A\) is uniformly rectifiable.

- The property BP, on the other hand, does not guarantee uniform rectifiability. The standard 4-corner Cantor set is purely unrectifiable but still satisfy the Carleson measure condition above since it has large projections in some directions (although of course not many directions), see [Dav91, Part III Chapter 5].

Let now \(\mathcal{P}\) be a continuous quantitative property with parameters \(\{p_1, \ldots, p_N\}\). For a cube \(Q_0 \in \mathcal{D}\), we let

\[
\mathcal{D}^\mathcal{P}(Q_0) = \mathcal{D}^\mathcal{P}(Q_0, p_1, \ldots, p_N) := \left\{ Q \in \mathcal{D} \mid Q \subset Q_0; (\zeta_Q, \ell(Q)) \in \mathcal{D}^\mathcal{P}\right\};
\]

\[
\mathcal{G}^\mathcal{P}(Q_0) = \mathcal{G}^\mathcal{P}(Q_0, p_1, \ldots, p_N) := \mathcal{D}(Q_0) \setminus \mathcal{D}^\mathcal{P}.
\]

Thus we put

\[
\mathcal{P}(Q_0, p_1, \ldots, p_N) := \mathcal{P}(Q_0) := \sum_{Q \in \mathcal{D}^\mathcal{P}_{Q_0}} \ell(Q)^d.
\]

The following is the main result of this section. In later sections, we will show how the comparability results (Theorems 1.4 and 1.5) follow as corollaries.

**Lemma 4.5.** Let \(E \subset \mathbb{R}^n\) be a \((c,d)\)-lower content regular set, and let \(0 < \epsilon < 1\), \(C_0 \geq 1\), and \(C_1 > 4C_2/p\). There is \(C'_0\) depending on \(c\) so that the following holds. Let \(\mathcal{P}\) be a QP of \(E\) with parameters \(\{p_1, \ldots, p_N\}\) such
that
\( P \) is \((\epsilon, C_1, C_2)\)-continuous. with constants \( c_1, \ldots, c_N \).
\[ \text{(4.6)} \]

\( P \) guarantees (and is guaranteed by) UR for \( C_0'\)-Ahlfors \( d \)-regular sets
for parameters \( c_1 p_1, \ldots, c_N p_N \);
\[ \text{(4.7)} \]

Then for any \( Q_0 \in \mathcal{D} \)
\[ \beta_E(Q_0) \lesssim c_{C_1, \epsilon} \mathcal{H}^d(Q_0) + P(Q_0, c_1 p_1, \ldots, c_N p_N). \]
\[ \text{(4.9)} \]

The proof of Lemma 4.5 will take up the rest of this section. Let us get started by first modifying the tree structure of \( \text{Top}(k_0) \), as in the statement of the Main Lemma by introducing a further stopping condition which is related to the QP \( \Psi \). Let \( R \in \text{Top}(k_0) \) and \( R' \in \text{Tree}(R) \). Let \( \text{Stop}(R') \) be the maximal cubes in \( \text{Tree}(R) \) that are either in \( \text{Stop}(R) \) or contain a child in \( \mathcal{B}^\Psi(Q_0) \), and let \( \text{Tree}(R') \) be the subfamily of cubes \( Q \in \text{Tree}(R) \) contained in \( R \) that are not properly contained in a cube from \( \text{Stop}(R') \).

In other words, \( \text{Tree}(R') \) is a pruned version of \( \text{Tree}(R) \), where we cut whenever we found a cube \( Q \in \mathcal{B}^\Psi. \)

Let \( \text{Next}_0(R) = \{R\} \) and for \( j \geq 0 \), if we have defined \( \text{Next}_k(R) \), let
\[ \text{Next}_{j+1}(R) = \bigcup_{R' \in \text{Next}_j} \bigcup_{Q \in \text{Tree}(R')} \text{Child}(Q). \]

This process terminates at some integer \( K_R \) since \( \text{Tree}(R) \) is finite. Enumerate \( \text{Next}_j = \{Q^j_i\}_{i=1}^{\ell} \).

**Lemma 4.6.** Let \( R \in \text{Top}(k_0) \) and let \( 0 \leq j \leq K_R \) and \( 1 \leq i \leq i_j \). Then there exists a constant \( c_1 < 1 \) so that
\[ \sum_{Q \in \text{Tree}(Q^j_i)} \beta_E^d(3B_Q)\ell(Q)^d \leq C(c_1, \tau, n, C_0)\ell(Q^j_i)^d. \]
\[ \text{(4.10)} \]

To prove Lemma 4.6, we will need the following Lemma from [AS18].

**Lemma 4.7** ([AS18], Lemma). Let \( 1 \leq p < \infty \) and \( E_1, E_2 \) lower content \( d \)-regular subsets of \( \mathbb{R}^n \); let moreover \( x \in E_1 \) and choose a radius \( r > 0 \). Then if \( y \in E_2 \) is so that \( B(x, r) \subset B(y, 2r) \), we have
\[ \beta_{E_1}^p(x, r) \lesssim \beta_{E_2}^p(y, 2r) + \left( \frac{1}{r^d} \int_{E_1 \cap B(x, 2r)} \left( \frac{\text{dist}(y, E_2)}{r} \right)^p d\mathcal{H}_\infty^d(y) \right)^{\frac{1}{p}}. \]
\[ \text{(4.11)} \]
Let $R, j, i$ as above. Let $E_{Q^j_i} = E_{i,j}$ be the Ahlfors regular set obtained from the Main Lemma for $\tilde{\text{Tree}}(Q^j_i)$ and $d_{Q^j_i}$ be the function defined in (1.11), where, in this instance, $\mathcal{F} = \tilde{\text{Stop}}(Q^j_i)$ and $\mathcal{F} = \tilde{\text{Tree}}(Q^j_i)$. Specifically, for $C_0 > 4$, as in (3.23), we set

$$\hat{T}_{i,j} := \{Q \in \mathcal{D} | \ell(Q) = \ell(Q^j_i), Q \cap C_0 B_{Q^j_i} \neq \emptyset\};$$

following (3.24), we then put

$$C_{i,j} := \{I \in \mathcal{I} | I \cap \hat{T} \neq \emptyset \text{ and } I \text{ is maximal with } \ell(I) < \tau d_{Q^j_i}(I)\}.$$

Then

$$E_{i,j} := \bigcup_{I \in C_{i,j}} \partial_d I.$$ (4.12)

It follows from the Main Lemma that $E_{i,j}$ is Ahlfors $d$-regular.

**Lemma 4.8.** Let $k_0, \tau > 0, R, j$ and $i$ as above. Then $E_{i,j}$ is uniformly rectifiable.

We want to use the fact that $\mathbb{P}$ guarantees uniform rectifiability and that it is continuous. We will show that there exist constants

$$c_1, \ldots, c_N$$

such that the measure

$$1_{\mathbb{P}^1(c_1, p_1, \ldots, c_N p_N)}(x, r) \frac{dt}{t} d\mathcal{H}^d(x)$$ (4.13)

is Carleson on $E_{i,j} \times \mathbb{R}_+$. We test this measure on a ball $B$ centered on $E_{i,j}$ and with radius $r_B$. Note that

$$\int_{B \cap E_{i,j}} \int_0^{\eta r d_{Q^j_i}(x)} 1_{\mathbb{P}^1(c_1, p_1, \ldots, c_N p_N)}(x, r) \frac{dr}{r} d\mathcal{H}^d(x) \lesssim_{n, d} r_B^d.$$ (4.14)

holds automatically: indeed, for any $x \in E_{i,j}$ and whenever $0 < r \leq \eta r d_{Q^j_i}(x)$, $B(x, r) \cap E_{i,j}$ is just a finite union of $d$-dimensional planes, and the number of planes in this union is bounded above by a universal constant only depending on $n$ and $d$. Therefore $B(x, r) \cap E_{i,j}$ is uniformly rectifiable and thus (4.14) holds. Also, using the Ahlfors regularity of $E_{i,j}$, it is immediate to see that

$$\int_{B \cap E_{i,j}} \int_{\eta^{-1} d_{Q^j_i}(x)}^{\eta r d_{Q^j_i}(x)} 1_{\mathbb{P}^1(c_1, p_1, \ldots, c_N p_N)}(x, r) \frac{dr}{r} d\mathcal{H}^d(x) \lesssim_{\tau, \eta} r_B^d.$$ (4.15)
Let us check that
\[
\int_{B \cap E_{i,j}} \int_{\eta^{-1}d_{Q_i}(x)}^{\eta^{-1}d_{Q_i}(x)} 1_{\mathcal{G}_{(e_1,p_1,\ldots,e_N,p_N)}}(x, r) \frac{dr}{r} d\mathcal{H}^d(x) \lesssim_{\tau, \eta} r^d.
\]

**Lemma 4.9.** Let \((x, r) \in E_{i,j} \times \mathbb{R}_+\) be such that
\[
\eta^{-1}d_{Q_i}(x) \leq r \leq \tau \ell(Q_i).
\]

Then, for \(\eta > 0\) sufficiently small (depending only on \(n\)), there exists a cube \(P\) in \(\tilde{\text{Tree}}(Q_i)\) so that
\[B_P \subset B(x, r).\]

**Proof.** For this proof, we put \(Q = Q_i\). Let \(I_x\) be the cube in \(C_Q\) containing \(x\), so \(\ell(I_x) \sim \tau d_Q(x)\). Let \(P^*\) be the minimiser of \(d_Q(x)\). Note that
\[
dist(x, P^*) \leq d_Q(x) \leq \eta r.
\]

Let us look at two distinct cases.

**Case 1.** Suppose first that
\[
d_Q(x) = \ell(P^*) + dist(x, P^*) \leq 2\ell(P^*).
\]

Then we immediately obtain that
\[
\ell(P^*) \leq d_Q(x) \leq 2\ell(P^*)
\]
and therefore that
\[
\ell(P^*) \sim \tau^{-1}\ell(I_x).
\]

But (4.19) also implies that
\[
dist(x, P^*) \leq \ell(P^*)
\]
Now, because of the assumption (4.17), we see that (using also (4.20))
\[
r \geq \eta^{-1}d_Q(x) \geq \eta^{-1}d_Q(I_x) \sim \eta^{-1}\tau^{-1}\ell(I_x) \sim \eta^{-1}\ell(P^*),
\]
and so, because (4.21) and (4.18), we have for \(\eta\) small \(B_{P^*} \subset B(x, r)\).

**Case 2.** Suppose now that
\[
d_Q(x) = \ell(P^*) + dist(x, P^*) \leq 2 dist(x, P^*)
\]
Then we have
\[
dist(x, P^*) \sim d_Q(x) \leq C\eta r.
\]
Also, by (4.17), it holds that
\[
\ell(P^*) \leq d_Q(x) \leq \eta r.
\]
This implies, for \(\eta > 0\) sufficiently small, that also in this case we have \(B_{P^*} \subset B(x, r)\). \(\square\)
Lemma 4.10. There exist constants \((c_1, \ldots, c_N)\) such that the following holds. Let \((x, r) \in E_{i,j} \times \mathbb{R}_+\) be such that
\[
\eta^{-1}d_{Q_i^j}(x) \leq r \leq \tau \ell(Q_i^j).
\]
Then
\[
(x, r) \in \mathcal{H}^p_{E_{i,j}}(c_1p_1, \ldots, c_Np_N).
\]

Proof. We know from Lemma 4.9 that if \((x, r)\) satisfies (4.22), then there exists a cube \(P^* \in \text{Stop}(Q_i^j)\) such that \(B_{P^*} \subset B(x, r)\). Thus, there must exist an ancestor \(\hat{P}^* \in \text{Tree}(Q_i^j)\) of \(P^*\) so that
\[
\rho \ell(\hat{P}^*) \leq 4C_2r < \ell(\hat{P}^*),
\]
and thus so that \(B(x, C_2r) \subset B_{\hat{P}^*}\), and since \(C_1 > 4C_2/\rho\), we also have \(r \geq \ell(\hat{P}^*)/C_1\). But recall that if \(\hat{P}^* \in \text{Tree}(Q_i^j)\), then we must have, by definition, that \((\zeta_{\hat{P}^*}, \ell(\hat{P}^*)) \in \mathcal{H}^p(p_1, \ldots, p_N)\).

Let us check that
\[
d_{C_2B_{\hat{P}^*}}(E_{i,j}, E) < \tau.
\]
By (1.13), if \(y \in E \cap C_2B_{\hat{P}^*}\)
\[
dist(y, E_{i,j}) \lesssim \tau d_{Q_i^j}(y) \leq \tau(\text{dist}(y, \hat{P}^*) + \ell(\hat{P}^*)) \lesssim C_2\tau\ell(\hat{P}^*).
\]
That for any \(x \in E_{i,j} \cap C_2B_{\hat{P}^*}\) we have \(\text{dist}(x, E) \lesssim \tau\ell(\hat{P}^*)\) follows in the same way, since any such \(x\) is contained in a dyadic cube \(I\) touching \(E\) so that
\[
\ell(I) < \tau d_{Q_i^j}(I) \overset{(4.22)}{\leq} \eta \tau r \overset{(4.23)}{\leq} 8\eta \tau \ell(\hat{P}^*).
\]
Choosing \(\tau\) in the construction of \(E_{i,j}\) appropriately (depending on \(\epsilon\) and \(C_2\)), the lemma follows from the \((\epsilon, C_1, C_2)\)-continuity of \(\mathcal{H}^p\). \(\square\)

Proof of Lemma 4.8. We have shown that there exist constants \(c_1, \ldots, c_N\) such that, for any pair \((x, r) \in E_{i,j} \times \mathbb{R}_+\) with
\[
\eta^{-1}d_{Q_i^j}(x) \leq r \leq \tau \ell(Q_i^j)
\]
we have
\[
(x, r) \in \mathcal{H}^p_{E_{i,j}}(c_1p_1, \ldots, c_Np_N).
\]
Thus the integral in (4.16) equals to zero. Now, we also see that, trivially
\[
\int_{B \cap E_{i,j}} \int_{\frac{\text{diam}(E_{i,j})}{\eta \ell(Q_i^j)}} 1_{\mathcal{H}^p(c_1p_1, \ldots, c_Np_N)}(x, r) \frac{dr}{r} d\mathcal{H}^d(x) \lesssim_{\tau} r^d_B.
\]
This together with the previous estimates (4.14), (4.15) and (4.16) proves that the measure \(1_{\mathcal{H}^p(c_1p_1, \ldots, c_Np_N)}(x, r) \frac{dr}{\tau} d\mathcal{H}^d|_{E_{i,j}}(x)\) is a Carleson measure on \(E_{i,j} \times \mathbb{R}_+\); then, because \(\mathcal{H}^p\) guarantees uniform rectifiability with
the appropriate parameters and it is \((\epsilon, C_1, C_2)\)-continuous, \(E_{i,j}\) is uniformly rectifiable. Note that all the constants involved depend only on \(n, d, \tau, \eta\) (and \(c_0\)); in particular, they are all independent of \(Q_i\), \(R\) and \(k_0\).

**Proof of Lemma 4.6.** We want to apply Lemma 4.7 with \(E_1 = E, E_2 = E_{i,j}\) and \(p = 2\). For \(Q \in \text{Tree}(Q_i')\), recall that \(\zeta_Q\) denotes the center of \(Q\). By (1.13), we know that \(\text{dist}(z_Q, E_{i,j}) \lesssim \tau d_Q(z_Q) \leq \tau \ell(Q)\), and in particular, if we denote by \(x_Q'\) the point in \(E_{i,j}\) which is closest to \(x_Q\), we see that \(B_Q := B(\zeta_Q, \ell(Q)) \subset B(x_Q', 2\ell(Q)) =: B_Q'\). Hence for each cube \(Q \in \text{Tree}(Q_i')\) the hypotheses of Lemma 4.7 are satisfied and we may write

\[
\sum_{Q \in \text{Tree}(Q_i')} \beta_E^2 (3B_Q)^2 \ell(Q)^d \lesssim \sum_{Q \in \text{Tree}(Q_i')} \beta_{E_{i,j}}^2 (6B_Q')^2 \ell(Q)^d
\]

\[
+ \sum_{Q \in \text{Tree}(Q_i')} \left( \frac{1}{\ell(Q)^d} \int_{6B_Q \cap E} \left( \frac{\text{dist}(x, E_{i,j})}{\ell(Q)} \right)^2 \frac{d\mathcal{H}_d}{\ell(Q)} \right) := I_1 + I_2.
\]

We first look at \(I_1\). We apply Theorem 2.2 to \(E_{i,j}\); let us denote the cubes so obtained by \(\mathcal{D}_{E_{i,j}}\). Note that for each \(P \in \text{Tree}(Q_i')\) with \(P \in \mathcal{D}(k_0)\), \(x_P'\) belongs to some cube \(P' \in \mathcal{D}_{E_{i,j}}\) so that \(\ell(P') \sim \ell(P)\); hence there exists a constant \(C_1 \geq 1\) so that

\[
(4.26) \quad 6B_P' \subset C_1 B_{P'}.
\]

This in turn implies that \(\beta_{E_{i,j}}^{p,d} (6B_P')^2 \ell(P)^d \lesssim_{p,n,d,C_1,\beta_{E_{i,j}}} C_1 B_{P'}.\) Hence,

\[
(4.27) \quad \sum_{P \in \text{Tree}(Q_i')} \beta_{E_{i,j}}^{2,d} (6B_P')^2 \ell(P)^d \lesssim_{p,n,d,C_1} \sum_{P' \in \mathcal{D}_{E_{i,j}}} \beta_{E_{i,j}}^{2,d} (C_1 B_{P'}).^2 \ell(P')^d.
\]

Since \(E_{i,j}\) is uniformly rectifiable, we immediately have that \(I_1 \lesssim \ell(Q_i')^d\) by the main results of \([DS91]\) (in particular, see (C3) and (C6) in \([DS91,\text{Chapter 1}]\)).

Let us now worry about \(I_2\). We put

\[
\text{Approx}(Q_i') := \left\{ \text{maximal } S \in \mathcal{D}(k_0) \mid \text{there is } I \in \mathcal{E}_{i,j} \text{ s.t. } I \cap S \neq \emptyset \right\}
\]

\[
(4.28) \quad \text{and } \rho(I) \leq \ell(I) \leq \ell(S).
\]

It is clear that

\[
(4.29) \quad Q_i^j \subset \bigcup_{S \in \text{Approx}(Q_i')} S,
\]
since $\mathcal{C}_{i,j}$ covers $Q_i^j$. Now let $x \in Q_i^j$. We claim that there exists a cube $S \in \text{Approx}(Q_i^j)$ so that

$$\text{dist}(x, E_{i,j}) \leq C \ell(S). \quad (4.30)$$

By (4.29), we see that if $x \in Q_i^j$, then there exists an $S \in \text{Approx}(Q_i^j)$ so that $x \in S$. But, then, by definition, there exists an $I \in \mathcal{C}_{i,j}$ such that $\ell(I) \leq \ell(S)$ and $I \cap S \neq \emptyset$. Thus

$$\text{dist}(x, E_{i,j}) \leq \text{diam} I + \text{dist}(x, I) \lesssim \ell(S).$$

We now estimate $I_2$ as follows: first,

$$\frac{1}{\ell(Q)^d} \int_{6B_Q \cap E} \left( \frac{\text{dist}(x, E_{i,j})}{\ell(Q)} \right)^2 d\mathcal{H}^d(x) \lesssim \sum_{S \in \text{Approx}(Q_i^j)} \int_S \frac{\ell(S)^2}{\ell(Q)^{d+2}} d\mathcal{H}^d \lesssim \sum_{S \in \text{Approx}(Q_i^j)} \frac{\ell(S)^2 + d}{\ell(Q)^{d+2}}. \quad (4.31)$$

Hence we obtain that

$$I_2 \lesssim \sum_{Q \in \text{Tree}(Q_i^j)} \sum_{S \in \text{Approx}(Q_i^j)} \frac{\ell(S)^{d+2}}{\ell(Q)^2} \quad (4.32)$$

Note that the number of cubes $Q \in \tilde{\text{Tree}}(Q_i^j)$ which belong to a given generation and such that $S \cap 6B_Q \neq \emptyset$ is bounded above by a constant $C$ which depends on $n$. Indeed, if $S \cap 6B_Q \neq \emptyset$, then we must have that $\text{dist}(Q, S) \leq 6\ell(Q)$. Moreover, because $S \in \text{Approx}(Q_i^j)$ and using Lemma 3.6, we see that, if $I \in \mathcal{C}_{i,j}$ is so that $I \cap S \neq \emptyset$ and $\ell(S) \sim \ell(I)$ (as in (4.28)),

$$\ell(S) \sim \ell(I) \sim \tau d_{Q_i^j}(I) \lesssim \tau (\ell(Q) + \text{dist}(I, Q)) \leq \tau \ell(Q) + \tau (6\ell(Q) + 2\ell(S))$$

so for $\tau$ small enough, $\ell(S) \lesssim \ell(Q)$. Thus we can sum the interior sum in (4.32):

$$\sum_{Q \in \text{Tree}(Q_i^j)} \frac{1}{\ell(Q)^2} \lesssim \frac{1}{\ell(S)^2}.$$
Finally, we see that
\begin{equation}
I_2 \lesssim_{\tau,n} \sum_{\substack{S \in \text{Approx}(Q^j_i) \cap 6B_{Q^j_i}}} \frac{\ell(S)^{d+2}}{\ell(S)^2} = \sum_{\substack{S \in \text{Approx}(Q^j_i) \cap 6B_{Q^j_i}}} \ell(S)^d.
\end{equation}

Now, by definition of $\text{Approx}(Q^j_i)$, the last sum in (4.33) is bounded above by a constant times
\begin{equation}
\sum_{I \in C_{i,j}} \ell(I^d) \lesssim \mathcal{H}^d \left( \bigcup_{I \in C_{i,j}} \partial I \right) = \mathcal{H}^d(E_{i,j}) \lesssim \ell(Q^j_i)^d,
\end{equation}
where we also used the Ahlfors regularity of $E_{i,j}$. This proves (4.10). □

Proof of Lemma 4.5. Let $Q_0 \in \mathcal{D}$ as in the statement of the Lemma. Then we see that
\begin{equation}
\sum_{R \in \text{Top}(k_0)} \sum_{j=0}^{K_R} \sum_{Q \in \text{Next}_j(R)} \sum_{P \in \text{Tree}(Q)} \beta_E^{2d}(3B_P)^2 \ell(P)^d \lesssim_{\tau} \sum_{R \in \text{Top}(k_0)} \sum_{j=0}^{K_R} \sum_{Q \in \text{Next}_j(R)} \ell(Q)^d.
\end{equation}

Note that for $1 \leq j \leq K_R$, if $Q \in \text{Next}_j(R)$, then there is a sibling $Q'$ of $Q$ so that $(\zeta_{Q'}, \ell(Q')) \in \mathcal{B}^p$. Also recall that we put $\text{Next}_0(R) = \{ R \}$. Then any cube appearing in the sum (4.34), either belongs to $\text{Top}(k_0)$ (whenever it belongs to $\text{Next}_0(R)$), or is adjacent to a cube in $\mathcal{B}^p(Q_0, p_1, \ldots, p_N)$, as defined in (4.5). Thus we see that
\begin{equation}
(4.34) \lesssim_{\tau} \left( \sum_{R \in \text{Top}(k_0)} \ell(R)^d + \Psi(Q_0, p_1, \ldots, p_N) \right)^{(1.10)} \lesssim_{\tau} \mathcal{H}^d(Q_0) + \Psi(Q_0).
\end{equation}

Note that all these estimates were independent of $k_0$. Sending $k_0$ to infinity and recalling (A.3) (and recalling that $\ell(Q_0)^d \lesssim_c \mathcal{H}^d(Q_0)$) gives the estimate (4.9).

□

5. APPLICATIONS: THE DIMENSIONLESS QUANTITIES LS AND LCV

Here we give a proof of Theorem 1.5

Proof. First, it is not hard to show that there is $c > 0$ so that if $Q \in \mathcal{D}^{BWGL}_{E}(Q_0, c\epsilon)$, then for any children $Q'$ of $Q$, since
\[ \ell(Q') = \rho \ell(Q) < \frac{1}{4} \ell(Q), \]

we have \( Q' \in \mathcal{G}_E^{LS}(\epsilon) \). Using this fact, we get
\[
\mathcal{H}^d(R) + \text{LS}(R, \epsilon) \leq \mathcal{H}^d(R) + \text{BWGL}(R, c\epsilon) \lesssim \beta_E(R)
\]
and so we just need to prove the reverse inequality.

First we show that for all \( C > 1 \) and \( \epsilon > 0 \) is small depending on \( C \) and \( B \in \mathcal{G}_E^{LS}(\epsilon) \) and \( E' \) is another lower \( d \)-regular set so that \( d_{4B}(E, E') < \epsilon \), then any ball \( B' \) with \( 4B' \subseteq B \) centered on \( E' \) with \( r_{B'} \geq r_B/C \), we have that \( B' \in \mathcal{G}_E^{LS}(c\epsilon) \) for some \( c > 0 \), and so LS is \((\epsilon, C, 4)\)-continuous for all \( C > 1 \) and \( \epsilon > 0 \) sufficiently small depending on \( C \).

Let \( x', y' \in E' \cap B' \), then there are \( x, y \in E \) with \(|x - x'|, |y - y'| < 4\epsilon r_B \). For \( \epsilon > 0 \) small depending on \( C \), since \( r_{B'} \geq r_B/C \), \( x, y \in \frac{3}{2}B' \), and so \( 2x - y \in 3B' \subseteq B \). Since \( B \in \mathcal{G}_E^{LS}(\epsilon) \), there is \( \xi \in E \) so that \(|2x - y - \xi| < \epsilon r_B \). For \( \epsilon > 0 \) small enough, since \( 2x - y \in \frac{3}{2}B' \), \( \xi \in 4B' \subseteq B \), thus there is \( \xi' \in E' \) with \(|\xi - \xi'| < 4\epsilon r_B \). Thus,
\[
\text{dist}(2x' - y', E') \leq |2x' - y' - \xi| \leq |2x - y - \xi| + |x - x'| + |y - y'| + |\xi - \xi'| < 16\epsilon r_B.
\]

Hence, \( B' \in \mathcal{G}_E^{LS}(16\epsilon) \). Thus, for \( \epsilon > 0 \) small enough, Lemma 4.5 implies the second half of (5.1). This completes the proof.

Another dimensionless quantity is the LCV. This can be proven in much the same way, so we omit the proof.

**Theorem 5.1.** Let \( E \subseteq \mathbb{R}^n \) be a lower \( d \)-regular set and \( \mathcal{D} \) its Christ-David cubes. Then for \( \epsilon > 0 \) small enough, and \( R \in \mathcal{D} \),
\[
(5.1) \quad \beta_E(R) \sim \mathcal{H}^d(R) + \text{LCV}(R, \epsilon).
\]

**6. Application: the BAUP**

In this section, we show that we can apply Lemma 4.5 to the quantitative property BAUP (recall the definition (1.2)). Namely, we will show that BAUP is \((\epsilon, C_1, C_2)\)-continuous. That BAUP guarantees rectifiability is due to David and Semmes, see [DS93], Proposition 3.18.

Let \( \epsilon_0 > 0 \) and \( C_0 \geq 1 \) be given. Let us first define the actual partition that BAUP determines. We put
\[
\mathcal{G}^{\text{BAUP}}(\epsilon_0, C_0) = \mathcal{G}^{\text{BAUP}} := \left\{ (x, r) \in E \times R_+ \mid \text{there is a family } \mathcal{F} \text{ of } d\text{-planes s.t. } d_{B(x, Cr)}(E, \cup_{P \in \mathcal{F}} P) < \epsilon_0 \right\}
\]
\[
\mathcal{G}^{\text{BAUP}}(\epsilon_0, C_0) = \mathcal{G}^{\text{BAUP}} := E \times \mathbb{R}_+ \setminus \mathcal{G}^{\text{BAUP}}.
\]
Lemma 6.1. Let $\epsilon_0 > 0$, $C_0 \geq 1$, and consider the quantitative property BAUP with parameters $(\epsilon_0, C_0)$. If $C_1 \geq 1$, $C_2 > 2C_0$, $\epsilon_0$ is small enough (depending on $C_2$ and $C_1$), and $0 < \epsilon_1 \leq \epsilon_0$ then BAUP is $(\epsilon_1, C_1, C_2)$-continuous.

Proof. Let us consider two subsets $E_1, E_2$ or $\mathbb{R}^n$. From Definition 4.2, we take a ball $B = B(x_B, r_B)$ centered on $E_2$ and so that, first,

$$(x_B, r_B) \in \mathcal{G}_{E_1}^{\text{BAUP}}(\epsilon_0, C_0),$$

and second,

$$d_{C_2B}(E_1, E_2) < \epsilon_1,$$

where $C_2$ and $\epsilon_1 \leq \epsilon_0$ will be determined later with respect to $C_0$ and $\epsilon_0$. Thus, there is a union of $d$-dimensional planes $\mathcal{F}$ so that

$$d_{C_2B}(E_1, \mathcal{F}) < \epsilon_0.$$

Next, we consider a ball $B' = B(x_B', r_B')$ centered this time on $E_2$ with $C_2B' \subseteq B$ and so that $r_B' \geq \frac{C_0}{C_1}$. We want to show that for any such a ball $B'$,

$$d_{C_0B'}(E_2, \mathcal{F}) < c_1 \epsilon_0 r_B'.$$

for some constant $c_1$ to be determined. Let $y \in E_2 \cap C_0B'$. Since $2B' \subseteq C_2B' \subseteq C_2B$, we have $2C_0B' \subseteq C_0B \subseteq C_2B$, so we can use (6.1) to find an $x \in E_1$ so that $|x - y| < \epsilon_1 C_2 r_B$. Since $\epsilon_0 \leq 1$, $x \in E_1 \cap 2C_0B' \subseteq C_0B$, and because $(x_B, r_B) \in \mathcal{G}_{E_1}^{\text{BAUP}}(\epsilon_0, C_0)$, it holds that $\text{dist}(x, \mathcal{F}) < \epsilon_0 C_0 r_B$. Now, because $\epsilon_1 \leq \epsilon_0$, we have that

$$\sup_{y \in E_2 \cap C_0B'} \text{dist}(y, \mathcal{F}) \leq \epsilon_1 C_2 r_B + \epsilon_0 C_0 r_B \leq (2C_2 C_1) \epsilon_0 r_B'.$$

Next, for $q \in \mathcal{F} \cap C_0B'$, we look at $\text{dist}(q, E_2)$; note in particular that $q \in \mathcal{F} \cap C_0B$ and thus, because $d_{C_0B}(E_1, \mathcal{F}) < \epsilon_0$, there is an $x \in E_1$ with $|x - q| \leq \epsilon_0 C_0 r_B$. Moreover, choosing $C_2 > 2C_0$, since $\epsilon_0 \leq 1$, we also have that $x \in 2C_0B \subseteq C_2B$, and thus $\text{dist}(x, E_2) < C_2 \epsilon_1 r_B$. All in all, we obtain that

$$\sup_{q \in E_2 \cap C_0B'} \text{dist}(q, E_2) \leq |x - q| + \text{dist}(x, E_2) \leq C_0 \epsilon_0 r_B + C_2 \epsilon_1 r_B \leq (2C_2 C_1) \epsilon_0 r_B'.$$

This implies (6.2) with $c_1 = 2C_1 C_2$; thus BAUP is $(\epsilon_1, C_1, C_2)$-continuous, whenever $\epsilon_1 \leq \epsilon_0$, and $C_2$ is sufficiently large, with respect to the parameter $C_0$. \qed
We can now prove Theorem 1.4. Firstly, note that we immediately have
\[ \text{BAUP}(Q_0, C_0, \epsilon) \leq \text{BWGL}(Q_0, C_0, \epsilon) \lesssim \beta_E(Q_0). \]
Furthermore, since \( \text{BAUP}(C_0, \epsilon) \) guarantees and is guaranteed by UR for all \( \epsilon > 0 \) sufficiently small depending on \( C_0 \) by [DS93, Theorem III.3.18]. Since it is also \((\epsilon, C_1, C_2)\)-continuous for \( C_2 > 2C_0 \) and all \( C_1 \geq 1 \) and \( \epsilon > 0 \) sufficiently small, we have, for all \( C_0 \geq 1 \) and \( \epsilon \) small enough (depending on \( C_0 \))
\[ \beta_E(Q_0) \lesssim \mathcal{H}^d(Q_0) + \text{BAUP}(Q_0, C_0, \epsilon). \]

7. Application: The GWEC

Let us give one last example of quantitative property which can be handled within the framework of Lemma 4.5. For a parameter \( \epsilon_0 > 0 \), we put in \( \mathcal{B}^{\text{GWEC}} \) all the pairs \( (x, r) \in \mathbb{R} \times R_+ \) for which there exists an \( (n - d - 1) \)-dimensional sphere \( S \) satisfying the following three conditions.

(7.1) \( S \subset B(x, r) \) and \( \text{dist}(S, E) > \epsilon_0 r \);

(7.2) \( \{ y \in B(x, r) | \text{dist}(y, E) > \epsilon_0 r \} \);

(7.3) \( \text{ch}(S) \cap E \neq \emptyset \),

where \( \text{ch}(S) \) is the convex hull of \( S \). We then put
\[ \mathcal{G}^{\text{GWEC}}(\epsilon_0) := \mathbb{R} \times R_+ \setminus \mathcal{B}^{\text{GWEC}}(\epsilon_0). \]

We want to check that we can apply Lemma 4.5 with this quantitative property. That the GWEC guarantees uniform rectifiability is Theorem 3.28 in [DS93]. All that’s left to do is to prove that GWEC is continuous.

**Lemma 7.1.** The quantitative property GWEC with parameter \( \epsilon_0 > 0 \) is \((\epsilon_1, C_1, C_2)\)-continuous, for all \( C_1 \geq 3 \), for all \( C_2 \geq 1 \) and whenever \( \epsilon_1 \) is sufficiently small with respect to \( \epsilon_0, C_1, \) and \( C_2 \).

**Proof.** Let \( E_1 \) and \( E_2 \) be two subsets of \( \mathbb{R}^n \). Let \( B = B(x_B, r_B) \) be a ball centered on \( E_1 \) so that \( (x_B, r_B) \in \mathcal{G}^{\text{GWEC}}_{E_1}(\epsilon_0) \) and
\[ d_{C_2B}(E_1, E_2) < \epsilon_1 C_2 r_B. \]

We want to find a constant \( c_1 \) so that, for any ball \( B' = B(x'_B, r'_B) \) centered on \( E_2 \) and with \( 2B' \subset B \) and \( r'_B \geq r_B/C_1 \), we have that \( (x'_B, r'_B) \in \mathcal{G}^{\text{GWEC}}_{E_2}(c_1 \epsilon_0) \).

We argue by contradiction. Suppose that for some \( c_1 \) (to be determined), we can find a sphere \( S' \) as in (7.1), (7.2) and (7.3) for the ball \( B' \). We will construct a sphere \( S \) for \( B \) satisfying the same three conditions: this will contradict the hypothesis that \( B \) is a good ball.
Let \( \hat{y} \in E_2 \cap \operatorname{ch}(S') \); note that in particular \( \hat{y} \in B(x_B', r_B') \subset B \), and thus we can find a point \( \hat{x} \in E_1 \) with \(|\hat{y} - \hat{x}| < \epsilon_1 C_2 r_B \) (using (7.4)). If \( W' \) is the \((n-d)\)-dimensional plane which contains \( S' \), we put \( W = W' + (\hat{x} - \hat{y}) \). Hence we let \( S \) denote the sphere in \( W \) with center center(\( S' \)) + (\( \hat{x} - \hat{y} \)) and radius equal to that of \( S' \). We claim that \( S \) satisfies (7.1), (7.2) and (7.3) relative to the pair \((x_B, r_B)\). Note first that
\[
(7.5) \quad S \subset N_{2C_2 \epsilon_1 r_B}(S').
\]
We show that \( \operatorname{dist}(E_1, N_{2C_2 \epsilon_1 r_B}(S')) > \epsilon_0 r_B \). Let \( s' \in S' \) and \( y \in E_1 \) be closest to each other. Since \( s' \in B \), we must have \( y \in 2B \). Let \( s \in S \) be closest to \( s' \), so \(|s - s'| < \epsilon_1 C_2 r_B \). Let \( y' \in E_2 \) be closest to \( y \); then as \( y \in 2B \), \(|y - y'| < \epsilon_1 C_2 r_B \); then we have that
\[
\operatorname{dist}(E_1, N_{2C_2 \epsilon_1 r_B}(S')) = |y - s'| \geq |y' - s| - |s' - s| - |y - y'| \\
\geq \operatorname{dist}(E_2, S) - 2\epsilon_1 C_2 r_B \\
\geq c_1 \epsilon_0 r_B - 2\epsilon_1 C_2 r_B \\
\geq \frac{c_1}{C_1} \epsilon_0 r_B - 2\epsilon_1 C_2 r_B.
\]
Now, choosing \( \epsilon_1 \) small enough (depending on \( \epsilon_0 \)) and \( c_1 \) sufficiently large (depending on \( C_1 \)), it follows that
\[
\operatorname{dist}(E_1, S) \geq \operatorname{dist}(E_1, N_{2C_2 \epsilon_1 r_B}(S')) > \epsilon_0 r_B.
\]
This proves (7.1) for \((x_B, r_B)\).

We now need to show that we can contract \( S \) to a point inside the set \( \{ y \in B(x_B, r_B) \mid \operatorname{dist}(y, E_1) > \epsilon_0 r_B \} \). To see this, we use (7.5): if we denote by \( Q_t \) the contraction of \( S' \) to a point, then \( \operatorname{dist}(Q_t, E_2) > c_1 \epsilon_0 r_B \). Denote by \( \{T_t\}_{0 \leq t \leq 1} \) the homotopy \( T_t(x) = x + t(\hat{y} - \hat{x}) \), so that \( T_0(S) = S \), \( T_1(S) = S' \) and \( T_1(S') \) is a \((n-d-1)\)-dimensional sphere lying in the \( \operatorname{ch}(S \cup S') \). Then we see that \( T_1(S) \subset N_{2C_2 \epsilon_1 r_B}(S') \), so \( \operatorname{dist}(T_1(S), E_1) \geq \epsilon_0 r_B \). Thus, putting
\[
\tilde{T}_t(x) := \begin{cases} 
T_{2t}(x) & \text{for } 0 \leq t \leq \frac{1}{2} \\
Q_{2t-1} & \text{for } \frac{1}{2} \leq t \leq 1,
\end{cases}
\]
we see that \( \tilde{T}_t \) is the desired contraction; this settles (7.2). Moreover, (7.3) holds from the definition of \( S \). But this implies that \((x_B, r_B)\) belongs to \( \mathcal{G}_{E_1}^{\text{GWEC}}(\epsilon_0) \). This is impossible, and so no sphere \( S' \) satisfying (7.1) to (7.3) can exists, and therefore \((x_B', r_B') \in \mathcal{G}_{E_2}^{\text{GWEC}}(c_1 \epsilon_0) \) for \( c_1 \) appropriately chosen (depending on \( C_1 \)), and \( \epsilon_1 \) sufficiently small. \( \square \)

We can now apply Lemma 4.5 (and use the fact that \( \text{GWEC}(Q_0, \epsilon) \lesssim \text{BWGL}(Q_0, c\epsilon) \lesssim \beta_B(Q_0) \) for some \( c > 0 \)), to obtain the following corollary.
Corollary 7.2. Let $E$ be lower content regular, let $Q_0 \in \mathcal{D}$. Then
\[
\beta(Q_0) \sim \mathcal{H}^d(Q_0) + \text{GWEC}(Q_0, \epsilon).
\]

APPENDIX A. THE TRAVELING SALESMAN THEOREM

In this section we prove Theorem 1.1. We begin by recalling the original Traveling Salesman Theorem for higher dimensional sets from [AS18, Theorem 3.2 and 3.3].

Theorem A.1. Let $1 \leq d < n$ and $E \subseteq \mathbb{R}^n$ be a closed. Suppose that $E$ is $(c, d)$-lower content regular and let $D$ denote the Christ-David cubes for $E$. Let

(1) Let $C_0 > 1$ and $A > \max\{C_0, 10^5\} > 1$, $p \geq 1$, and $\epsilon > 0$ be given. For $R \in \mathcal{D}$, let
\[
\begin{align*}
\mathcal{H}^d(R) + \text{BWGL}(R, \epsilon, C_0) &\lesssim A^{n, c, C_0} \epsilon \ell(R)^d + \sum_{Q \subseteq R} \beta_E^d(ABQ)^2 \ell(Q)^d. \\
\end{align*}
\]
Furthermore, if the right hand side of (A.1) is finite, then $E$ is $d$-rectifiable.

(2) For any $A > 1$ and $1 \leq p < p(d)$, there is $C_0 \gg A$ and $\epsilon_0 = \epsilon_0(n, A, p, c) > 0$ such that the following holds. Let $0 < \epsilon < \epsilon_0$. Then
\[
\ell(R)^d + \sum_{Q \subseteq R} \beta_E^d(ABQ)^2 \ell(Q)^d \sim A^{n, c, \epsilon} \mathcal{H}^d(R) + \text{BWGL}(R, \epsilon, C_0)
\]
If we set
\[
\beta_{E,A,p}(R) := \ell(R)^d + \sum_{Q \subseteq R} \beta_E^d(ABQ)^2 \ell(Q)^d,
\]
we will now show
\[
\beta_{E,A,p}(R) \sim A_p \beta_{E,3,2}(R) =: \beta_E(R).
\]
Indeed, one can check that $\beta_E^d(3BQ) \lesssim A_{d,p} \beta_E^d(ABQ)^2$ [AS18, Lemma 2.11]. Moreover, note that for every $Q \subseteq R$, if $Q^N$ denotes the $N$th ancestor of $Q$, then there is $N$ so that $3B_{Q^N} \supseteq ABQ$. With these observations, we have
\[
\beta_{E,3,p}(R) \lesssim_{A,p} \beta_{E,A,p}(R) \lesssim_N \ell(R)^d + \sum_{Q^N \subseteq R} \beta_E^d(ABQ)^2 \ell(Q)^d \lesssim_p \beta_{E,3,p}(R)
\]
Furthermore, by [AS18, Lemma 2.13], we see that $\beta_{E}^{d,1} \lesssim \beta_{E}^{d,p}$ for all $p > 1$. Thus, by the Traveling Salesman Theorem, for $A \gg C_0 \gg 3$
\[
\beta_{E,3,p}(R) \lesssim \text{BWGL}(R, \epsilon, C_0) \lesssim \beta_{E,A,1}(R) \lesssim \beta_{E,3,1}(R) \lesssim \beta_{E,3,2}(R).
\]
This completes the proof of Theorem 1.1.


\section*{References}

[Azz] J. Azzam, Harmonic Measure and the Analyst’s Traveling Salesman Theorem \textit{Arxiv preprint} 2019. \url{https://arxiv.org/abs/1905.09057}

[AHM+17] J. Azzam, S. Hofmann, J.M. Martell, K. Nyström, and T. Toro. A new characterization of chord-arc domains. \textit{J. Eur. Math. Soc. (JEMS)}, 19(4):967–981, 2017.

[AS18] J. Azzam and R. Schul. An analyst’s traveling salesman theorem for sets of dimension larger than one. \textit{Math. Ann.}, 370(3-4):1389–1476, 2018.

[Chr90] M. Christ. A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral. \textit{Colloq. Math.}, 60/61(2):601–628, 1990.

[Dav88] G. David. Morceaux de graphes lipschitziens et intégrales singulières sur une surface. \textit{Rev. Mat. Iberoamericana}, 4(1):73–114, 1988.

[Dav91] G. David. \textit{Wavelets and singular integrals on curves and surfaces}, volume 1465 of \textit{Lecture Notes in Mathematics}. Springer-Verlag, Berlin, 1991.

[DS91] G. David and S. W. Semmes. Singular integrals and rectifiable sets in $\mathbb{R}^n$: Beyond Lipschitz graphs. \textit{Astérisque}, (193):152, 1991.

[DS93] G. David and S. W. Semmes. \textit{Analysis of and on uniformly rectifiable sets}, volume 38 of \textit{Mathematical Surveys and Monographs}. American Mathematical Society, Providence, RI, 1993.

[DS00] G. David and S. W. Semmes. \textit{Uniform rectifiability and quasiminimizing sets of arbitrary codimension}. American Mathematical Soc., 2000.

[DT12] G. David and T. Toro. \textit{Reifenberg parameterizations for sets with holes} American Mathematical Soc., 2012.

[DS17] G. C. David and R. Schul. The analyst’s traveling salesman theorem in graph inverse limits. \textit{Ann. Acad. Sci. Fenn. Math.}, 42(2):649–692, 2017.

[DS19] G. C. David and R. Schul. A sharp necessary condition for rectifiable curves in metric spaces. \textit{arXiv preprint arXiv:1902.04030}, 2019.

[FFP07] F. Ferrari, B. Franchi, and H. Pajot. The geometric traveling salesman problem in the Heisenberg group. \textit{Rev. Mat. Iberoam.}, 23(2):437–480, 2007.

[HLMN17] S. Hofmann, P. Le, J.M. Martell, and K. Nyström. The weak-$A_\infty$ property of harmonic and $p$-harmonic measures implies uniform rectifiability. \textit{Anal. PDE}, 10(3):513–558, 2017.

[HM14] S. Hofmann and J. M. Martell. Uniform rectifiability and harmonic measure I: Uniform rectifiability implies Poisson kernels in $L^p$. \textit{Ann. Sci. Éc. Norm. Supér. (4)}, 47(3):577–654, 2014.

[HM18] S. Hofmann and J. M. Martell. On quantitative absolute continuity of harmonic measure and big piece approximation by chord-arc domains. \textit{arXiv preprint arXiv:1712.03696}, 2018.

[HM12] T. Hytönen and H. Martikainen. Non-homogeneous $Tb$ theorem and random dyadic cubes on metric measure spaces. \textit{J. Geom. Anal.}, 22(4):1071–1107, 2012.

[Jon90] P. W. Jones. Rectifiable sets and the traveling salesman problem. \textit{Invent. Math.}, 102(1):1–15, 1990.

[Li19] Sean Li. Stratified $\beta$-numbers and traveling salesman in carnrot groups. \textit{arXiv preprint}, 2019. \url{https://arxiv.org/abs/1902.03268}
[LS16a] S. Li and R. Schul. The traveling salesman problem in the Heisenberg group: upper bounding curvature. *Trans. Amer. Math. Soc.*, 368(7):4585–4620, 2016.

[LS16b] S. Li and R. Schul. An upper bound for the length of a traveling salesman path in the Heisenberg group. *Rev. Mat. Iberoam.*, 32(2):391–417, 2016.

[NTV14] F. Nazarov, X. Tolsa, and A. Volberg. On the uniform rectifiability of AD-regular measures with bounded Riesz transform operator: the case of codimension 1. *Acta Math.*, 213(2):237–321, 2014.

[Oki92] K. Okikiolu. Characterization of subsets of rectifiable curves in $\mathbb{R}^n$. *J. London Math. Soc. (2)*, 46(2):336–348, 1992.

[Sch07] R. Schul. Subsets of rectifiable curves in Hilbert space—the analyst’s TSP. *J. Anal. Math.*, 103:331–375, 2007.

[T] X Tolsa. Uniform measures and uniform rectifiability. *J. London Math. Soc.*, 92(1):1-8, 2015.

[Vil] M. Villa. Sets with topology, the Analyst’s TST, and applications. *Arxiv preprint*, 2019. [https://arxiv.org/abs/1908.10289](https://arxiv.org/abs/1908.10289)

**School of Mathematics, University of Edinburgh, JCMB, Kings Buildings, Mayfield Road, Edinburgh, EH9 3JZ, Scotland.**

*E-mail address:* j.azzam "at" ed.ac.uk

**School of Mathematics, University of Edinburgh, JCMB, Kings Buildings, Mayfield Road, Edinburgh, EH9 3JZ, Scotland.**

*E-mail address:* m.villa-2 "at" sms.ed.ac.uk