Noncommutative Symmetries and Stability of Black Ellipsoids in Metric–Affine and String Gravity

Sergiu I. Vacaru *, and Evghenii Gaburov †

* Centro Multidisciplinar de Astrofisica - CENTRA, Departamento de Fisica, Instituto Superior Tecnico, Av. Rovisco Pais 1, Lisboa, 1049-001, Portugal

and

† Department of Physics and Astronomy, University of Leicester, University Road, Leicester, LE1 7RH, UK

October 14, 2004

Abstract

We construct new classes of exact solutions in metric–affine gravity (MAG) with string corrections by the antisymmetric $H$–field. The solutions are parametrized by generic off–diagonal metrics possessing noncommutative symmetry associated to anholonomy framerelations and related nonlinear connection (N–connection) structure. We analyze the horizon and geodesic properties of a class of off–diagonal metrics with deformed spherical symmetries. The maximal analytic extension of ellipsoid type metrics are constructed and the Penrose diagrams are analyzed with respect to adapted frames. We prove that for small deformations (small eccentricities) there are such metrics that the geodesic behaviour is similar to the Schwarzschild one. We conclude that some static and stationary ellipsoid configurations may describe black ellipsoid objects. The new class of spacetimes do not possess Killing symmetries even in the limits to the general relativity and, in consequence, they are not prohibited by black hole uniqueness theorems. Such static ellipsoid (rotoid) configurations are compatible with the cosmic censorship criteria. We study the perturbations of two classes of static black ellipsoid solutions of four dimensional gravitational field equations. The analysis is performed in the approximation of small eccentricity deformations of the Schwarzschild solution. We conclude that such anisotropic black hole objects may be stable with respect to the perturbations parametrized by the Schrödinger equations in the framework of the one–dimensional inverse scattering theory. We emphasize that the anholonomic frame method of generating exact solutions is a general one for off–diagonal metrics (and linear and nonlinear connections) depending on 2–3 variables, in various types of gravity theories.

*E-mail address: vacaru@fisica.ist.utl.pt, sergiu–vacaru@yahoo.com

†e-mail: eg35@leicester.ac.uk
Pacs: 04.50.+h, 04.20.JB, 02.40.-k,
MSC numbers: 83D05, 83C15, 83E15, 53B40, 53C07, 53C60

## Contents

1. Introduction ........................................... 3

2. Anholonomic Frames and Off–Diagonal Metrics ........ 5

3. Anholonomic Noncommutative Symmetries ............... 10

4. 4D Static Black Ellipsoids in MAG and String Gravity 15
   4.1 Anholonomic deformations of the Schwarzschild metric 15
   4.2 Black ellipsoids and anistropic cosmological constants 18
   4.3 Analytic extensions of black ellipsoid metrics .......... 21
   4.4 Geodesics on static polarized ellipsoid backgrounds 23

5. Perturbations of Anisotropic Black Holes ............... 26
   5.1 Metrics describing anisotropic perturbations ........... 26
   5.2 Axial metric perturbations .............................. 29
   5.3 Polar metric perturbations .............................. 33
   5.4 The stability of polarized black ellipsoids ............ 36

6. Two Additional Examples of Off–Diagonal Exact Solutions 38
   6.1 Anholonomic ellipsoidal shapes .......................... 38
   6.2 Generalization of Canfora–Schmidt solutions .......... 39

7. Outlook and Conclusions ................................ 41
1 Introduction

In the past much effort has been made to construct and investigate exact solutions of gravitational field equations with spherical/cylindrical symmetries and/or with time dependence, parametrized by metrics diagonalizable by certain coordinate transforms. Recently, the off–diagonal metrics were considered in a new manner by diagonalizing them with respect to anholonomic frames with associated nonlinear connection structure \[1, 2, 3, 4\]. There were constructed new classes of exact solutions of Einstein’s equations in three (3D), four (4D) and five (5D) dimensions. Such solutions posses a generic geometric local anisotropy (e.g. static black hole and/or cosmological solutions with ellipsoidal or toroidal symmetry, various soliton–dilaton 2D and 3D configurations in 4D gravity, and wormholes and flux tubes with anisotropic polarizations and/or running constants with different extensions to backgrounds of rotation ellipsoids, elliptic cylinders, bipolar and toroidal symmetry and anisotropy).

A number of ansatz with off–diagonal metrics were investigated in higher dimensional gravity (see, for instance, the Salam, Strathee, Percacci and Randjbar–Daemi works \[5\]) which showed that off–diagonal components in higher dimensional metrics are equivalent to including \[U(1), SU(2)\] and \[SU(3)\] gauge fields. There are various generalizations of the Kaluza–Klein gravity when the compactifications of off–diagonal metrics are considered with the aim to reduce the vacuum 5D gravity to effective Einstein gravity and Abelian or non–Abelian gauge theories. There were also constructed 4D exact solutions of Einstein equations with matter fields and cosmological constants like black torus and black strings induced from some 3D black hole configurations by considering 4D off–diagonal metrics whose curvature scalar splits equivalently into a curvature term for a diagonal metric together with a cosmological constant term and/or a Lagrangian for gauge (electromagnetic) field \[6\].

We can model certain effective (diagonal metric) gravitational and matter fields interactions for some particular off–diagonal metric ansatz and redefinitions of Lagrangians. However, in general, the vacuum gravitational dynamics can not be associated to any matter field contributions. This holds true even if we consider non–Riemannian generalizations from string and/or metric–affine gravity (MAG) \[7\]. In this work (being the third partner of the papers \[8, 9\]), we prove that such solutions are not with usual Killing symmetries but admit certain anholonomic noncommutative symmetries and preserve such properties if the constructions are extended to MAG and string gravity (see also \[10\] for extensions to complex and/or noncommutative gravity).

There are constructed the maximal analytic extension of a class of static metrics with deformed spherical symmetry (containing as particular cases ellipsoid configurations). We analyze the Penrose diagrams and compare the results with those for the Reissner–Nordstrom solution. Then we state the conditions when the geodesic congruence with 'ellipsoid' type symmetry can be reduced to the Schwarzschild configuration. We argue that in this case we may generate some static black ellipsoid solutions which, for corresponding parametrizations of off–diagonal metric coefficients, far away from the horizon, satisfy the asymptotic conditions of the Minkowski spacetime.

For the new classes of ”off–diagonal” spacetimes possessing noncommutative symmetries, we extend the methods elaborated to investigate the perturbations and stability of black hole metrics. The theory of perturbations of the Schwarzschild spacetime black holes was initiated
in Ref. [11], developed in a series of works, e.g., Refs [12, 13], and related [14] to the theory of inverse scattering and its ramifications (see, for instance, Refs. [15]). The results on the theory of perturbations and stability of the Schwarzschild, Reissner–Nordstrom and Kerr solutions are summarized in a monograph [16]. As alternative treatments of the stability of black holes we cite Ref. [17].

Our first aim is to investigate such off–diagonal gravitational configurations in MAG and string gravity (defined by anholonomic frames with associated nonlinear connection structure) which describe black hole solutions with deformed horizons, for instance, with a static ellipsoid hypersurface. The second aim is to study perturbations of black ellipsoids and to prove that there are such static ellipsoid like configurations which are stable with respect to perturbations of a fixed type of anisotropy (i.e. for certain imposed anholonomic constraints). The main idea of a such proof is to consider small (ellipsoidal, or another type) deformations of the Schwarzschild metric and than to apply the already developed methods of the theory of perturbations of classical black hole solutions, with a re–definition of the formalism for adapted anholonomic frames.

We note that the solutions defining black ellipsoids are very different from those defining ellipsoidal shapes in general relativity (see Refs. [18]) associated to some perfect–fluid bodies, rotating configurations or to some families of confocal ellipsoids in Reimannian spaces. Our black ellipsoid metrics are parametrized by generic off–diagonal ansatz with anholonomically deformed Killing symmetry and not subjected to uniqueness theorems. Such ansatz are more general than the class of vacuum solutions which can not be written in diagonal form [19] (see details in Refs. [20, 10]).

The paper is organized as follows: In Sec. 2 we outline the necessary results on off–diagonal metrics and anholonomic frames with associated nonlinear connection structure. We write the system of Einstein–Proca equations from MAG with string corrections of the antisymmetric $H$–tensor from bosonic string theory. We introduce a general off–diagonal metric ansatz and derive the corresponding system of Einstein equations with anholonomic variables. In Sec. 3 we argue that noncommutative anholonomic geometries can be associated to real off–diagonal metrics and show two simple realisations within the algebra for complex matrices. Section 4 is devoted to the geometry and physics of four dimensional static black ellipsoids. We illustrate how such solutions can be constructed by using anholonomic deformations of the Schwarzschild metric, define analytic extensions of black ellipsoid metrics and analyze the geodesic behaviour of the static ellipsoid backgrounds. We conclude that black ellipsoid metrics posses specific noncommutative symmetries. We outline a perturbation theory of anisotropic black holes and prove the stability of black ellipsoid objects in Sec. 5. Then, in Sec. 6 we discuss how the method of anholonomic frame transforms can be related solutions for ellipsoidal shapes and generic off–diagonal solutions constructed by F. Canfora and H. -J. Schmidt. We outline the work and present conclusions in Sec. 7.

There are used the basic notations and conventions stated in Refs. [8, 9].
2 Anholonomic Frames and Off–Diagonal Metrics

We consider a 4D manifold $V^{3+1}$ (for MAG and string gravity with possible torsion and nonmetricity structures [7, 8, 9]) enabled with local coordinates $u^\alpha = (x^i, y^a)$ where the indices of type $i, j, k, ...$ run values 1 and 2 and the indices $a, b, c, ...$ take values 3 and 4; $y^3 = v = \varphi$ and $y^4 = t$ are considered respectively as the ”anisotropic” and time like coordinates (subjected to some constraints). It is supposed that such spacetimes can also admit nontrivial torsion structures induced by certain frame transforms.

The quadratic line element

$$ds^2 = g_{\alpha\beta} (x^i, v) du^\alpha du^\beta,$$

is parametrized by a metric ansatz

$$g_{\alpha\beta} = \begin{bmatrix}
g_1 + w_1^2 h_3 + n_1^2 h_4 & w_1 w_2 h_3 + n_1 n_2 h_4 & w_1 h_3 & n_1 h_4 \\
w_1 w_2 h_3 + n_1 n_2 h_4 & g_2 + w_2^2 h_3 + n_2^2 h_4 & w_2 h_3 & n_2 h_4 \\
w_1 h_3 & w_2 h_3 & h_3 & 0 \\
n_1 h_4 & n_2 h_4 & 0 & h_4
\end{bmatrix},$$

with $g_i = g_i(x^i), h_a = h_{ai}(x^k, v)$ and $n_i = n_i(x^k, v)$ being some functions of necessary smoothly class or even singular in some points and finite regions. The coefficients $g_i$ depend only on ”holonomic” variables $x^i$ but the rest of coefficients may also depend on one ”anisotropic” (anholonomic) variable $y^3 = v$; the ansatz does not depend on the time variable $y^4 = t$; we shall search for static solutions.

The spacetime may be provided with a general anholonomic frame structure of tetrads, or vierbiends,

$$e_\alpha = A_{\alpha}^\beta (u^\gamma) \partial/\partial u^\beta,$$

satisfying some anholonomy relations

$$e_\alpha e_\beta - e_\beta e_\alpha = w^{\gamma}_{\alpha\beta} (u^\epsilon) e_\gamma,$$

where $w^{\gamma}_{\alpha\beta} (u^\epsilon)$ are called the coefficients of anholonomy. A ’holonomic’ frame, for instance, a coordinate frame, $e_\alpha = \partial_\alpha = \partial/\partial u^\alpha$, is defined as a set of tetrads satisfying the holonomy conditions

$$\partial_\alpha \partial_\beta - \partial_\beta \partial_\alpha = 0.$$

We can ’effectively’ diagonalize the line element (1),

$$\delta s^2 = g_1(dx^1)^2 + g_2(dx^2)^2 + h_3(\delta v)^2 + h_4(\delta y^4)^2,$$

with respect to the anholonomic co–frame

$$\delta^\alpha = (d^i = dx^i, \delta^a = dy^a + N_i^a dx^i) = (d^i, \delta v = dv + w_i dx^i, \delta y^4 = dy^4 + n_i dx^i)$$

which is dual to the anholonomic frame

$$\delta_\alpha = (\delta_i = \partial_i - N_i^a \partial_a, \partial_b) = (\delta_i = \partial_i - w_i \partial_3 - n_i \partial_4, \partial_3, \partial_4),$$

5
where $\partial_i = \partial/\partial x^i$ and $\partial_b = \partial/\partial y^b$ are usual partial derivatives. The tetrads $\delta_a$ and $\delta^a$ are anholonomic because, in general, they satisfy the anholonomy relations (11) with some non-trivial coefficients,

$$w^a_{ij} = \delta_i N^a_j - \delta_j N^a_i, \quad w^b_{ia} = - \partial a N^b_i.$$

The anholonomy is induced by the coefficients $N^3_i = w_i$ and $N^4_i = n_i$ which "elongate" partial derivatives and differentials if we are working with respect to anholonomic frames. This results in a more sophisticated differential and integral calculus (as in the tetradic and/or spinor gravity), but simplifies substantially the tensor computations, because we are dealing with diagonalized metrics. In order to construct exact 'off-diagonal' solutions with 4D metrics depending on 3 variables $(x^k, v)$ it is more convenient to work with respect to anholonomic frames (7) and (8) for diagonalized metrics (3) than to consider directly the ansatz (11) (12) (13) (14).

There is a 'preferred' linear connection constructed only from the components $(g_i, h_a, N^b_k)$, called the canonical distinguished linear connection, which is similar to the metric connection introduced by the Christoffel symbols in the case of holonomic bases, i.e. being constructed only from the metric components and satisfying the metricity conditions. It is parametrized by the coefficients, $\Gamma^\alpha_{\beta\gamma} = (L^i_{jk}, L^a_{bk}, C^a_{jc}, C^a_{bc})$ stated with respect to the anholonomic frames (7) and (8) as

$$L^i_{jk} = \frac{1}{2} g^{in} (\delta_k g_{nj} + \delta_j g_{nk} - \delta_n g_{jk}),$$

$$L^a_{bk} = \partial_b N^a_k + \frac{1}{2} h^{ac} (\delta_k h_{bc} - h_{dc} \partial_b N^c_k - h_{db} \partial_c N^d_k),$$

$$C^a_{jc} = \frac{1}{2} g^{jk} \partial_c g_{jk}, \quad C^a_{bc} = \frac{1}{2} h^{ad} (\partial_c h_{db} + \partial_b h_{dc} - \partial_d h_{bc}),$$

computed for the ansatz (2). This induces a linear covariant derivative locally adapted to the nonlinear connection structure (N-connection, see details, for instance, in Refs. (21), (11) (20)). By straightforward calculations, we can verify that for $D_\alpha$ defined by $\Gamma^\alpha_{\beta\gamma}$ with the components (9) the condition $D_\alpha g_{\beta\gamma} = 0$ is satisfied.

We note that on (pseudo) Riemannian spaces the N-connection is an object completely defined by anholonomic frames, when the coefficients of tetradic transform (4), $A^a_{\beta\gamma} (u^\tau)$, are parametrized explicitly via certain values $(N^a_i, \delta^a_i, \delta^a_k)$, where $\delta^a_i$ and $\delta^a_k$ are the Kronecker symbols. By straightforward calculations we can compute (see, for instance Ref. (22)) that the coefficients of the Levi-Civita metric connection

$$\Gamma^\nabla_{\alpha\beta\gamma} = g (\delta_\alpha, \nabla_\gamma \delta_\beta) = g_{\alpha\tau} \Gamma^\nabla_{\beta\gamma},$$

associated to a covariant derivative operator $\nabla$, satisfying the metricity condition $\nabla_\gamma g_{\alpha\beta} = 0$ for $g_{\alpha\beta} = (g_{ij}, h_{ab})$,

$$\Gamma^\nabla_{\alpha\beta\gamma} = \frac{1}{2} [\delta_\beta g_{\alpha\gamma} + \delta_\gamma g_{\alpha\beta} - \delta_\alpha g_{\gamma\beta} + g_{\alpha\tau} w^\tau_{\beta\gamma} + g_{\beta\tau} w^\tau_{\gamma\alpha} - g_{\gamma\tau} w^\tau_{\alpha\beta}],$$

are given with respect to the anholonomic basis (6) by the coefficients

$$\Gamma^\nabla_{\beta\gamma} = \left( L^i_{jk}, L^a_{bk}, -\frac{\partial N^a_k}{\partial y^b}, C^a_{jc} + \frac{1}{2} g^{ik} \nabla^a_j h_{ca}, C^a_{bc} \right),$$

and

$$\Gamma^\nabla_{\beta\gamma} = \left( L^i_{jk}, L^a_{bk}, -\frac{\partial N^a_k}{\partial y^b}, C^a_{jc} + \frac{1}{2} g^{ik} \nabla^a_j h_{ca}, C^a_{bc} \right),$$

6
where \( \Omega^a_{jk} = \delta_k N^a_j - \delta_j N^a_k \). The anholonomic frame structure may induce on (pseudo) Riemannian spacetimes nontrivial torsion structures. For instance, the canonical connection (9), in general, has nonvanishing torsion components

\[
T^i_{ja} = -T^i_{aj} = C^i_{ja}, \quad T^a_{jk} = -T^a_{kj} = \Omega^a_{kj}, \quad T^a_{bk} = -T^a_{kb} = \partial_b N^a_k - L^a_{bk}.
\]

This is a "pure" anholonomic frame effect. We can conclude that the Einstein theory transforms into an effective Einstein–Cartan theory with anholonomically induced torsion if the general relativity is formulated with respect to general anholonomic frame bases. In this paper we shall also consider distortions of the Levi–Civita connection induced by nonmetricity.

A very specific property of off–diagonal metrics is that they can define different classes of linear connections which satisfy the metricity conditions for a given metric, or inversely, there is a certain class of metrics which satisfy the metricity conditions for a given linear connection. This result was originally obtained by A. Kawaguchi [23] (Details can be found in Ref. [21], see Theorems 5.4 and 5.5 in Chapter III, formulated for vector bundles; here we note that similar proofs hold also on manifolds enabled with anholonomic frames associated to a N–connection structure.)

The Levi–Civita connection does not play an exclusive role on non–Riemannian spaces. For instance, the torsion on spaces provided with N–connection is induced by anholonomy relation and both linear connections (9) and (11) are compatible with the same metric and transform into the usual Levi–Civita coefficients for vanishing N–connection and "pure" holonomic coordinates (see related details in Refs. [8, 9]). This means that to an off–diagonal metric we can associated different covariant differential calculi, all being compatible with the same metric structure (like in noncommutative geometry, which is not a surprising fact because the anolonomic frames satisfy by definition some noncommutative relations (11)). In such cases we have to select a particular type of connection following some physical or geometrical arguments, or to impose some conditions when there is a single compatible linear connection constructed only from the metric and N–coefficients.

The dynamics of generalized Finsler–affine string gravity is defined by the system of field equations (see Proposition 3.1 in Ref. [9])

\[
\hat{\mathcal{R}}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \hat{\mathcal{R}} = \kappa \left( \Sigma^{[\phi]}_{\alpha\beta} + \Sigma^{[m]}_{\alpha\beta} + \Sigma^{[T]}_{\alpha\beta} \right),
\]

\[
\hat{\mathcal{D}}_\nu \mathcal{H}^\mu = \mu^2 \phi^\mu,
\]

\[
\hat{\mathcal{D}}^\nu \mathcal{H}_{\nu\lambda\rho} = 0
\]

for

\[
\mathcal{H}_{\nu\lambda\rho} = \tilde{Z}_{\nu\lambda\rho} + \tilde{H}_{\nu\lambda\rho}
\]

being the antisymmetric torsion field

\[
\mathcal{H}_{\nu\lambda\rho} = \delta_\nu \mathcal{B}_{\lambda\rho} + \delta_\rho \mathcal{B}_{\nu\lambda} + \delta_\lambda \mathcal{B}_{\nu\rho}
\]

of the antisymmetric \( \mathcal{B}_{\lambda\rho} \) in bosonic string theory (for simplicity, we restrict our considerations to the sigma model with \( H \)–field corrections and zero dilatonic field). The covariant derivative \( \hat{\mathcal{D}}_\nu \) is defined by the coefficients (9) (we use in our references the "boldfaced" indices when it
is necessary to emphasize that the spacetime is provided with N–connection structure. The distorsion $\hat{Z}_{\nu\lambda\rho}$ of the Levi–Civita connection, when

$$\Gamma_{\beta\gamma}^{\tau} = \Gamma_{\nabla\beta\gamma}^{\tau} + \hat{Z}_{\beta\gamma}^{\tau},$$

from (13) is defined by the torsion $\hat{T}$ with the components computed for $\hat{D}$ by applying the formulas (12),

$$\hat{Z}_{\alpha\beta} = \delta_{\beta}^{\gamma} \hat{T}_{\alpha}^{\gamma} - \delta_{\alpha}^{\gamma} \hat{T}_{\beta}^{\gamma} + \frac{1}{2} \left( \delta_{\alpha}^{\gamma} \delta_{\beta}^{\gamma} \hat{T}_{\gamma}^{\gamma} \right) \delta_{\gamma},$$

see Refs. [8, 9] on definition of the interior product "|" and differential forms like $\hat{T}_{\gamma}$ on spaces provided with N–connection structure. The tensor $H_{\nu\mu} \equiv \hat{D}_{\nu}\phi_{\mu} - \hat{D}_{\mu}\phi_{\nu} + w_{\mu\nu}\phi_{\gamma}$ is the field strengths of the Abelian Proca field $\phi^{\alpha}$, with $\mu, \tilde{\kappa} = \text{const},$

$$\Sigma_{\alpha\beta}^{[\phi]} = H_{\alpha}^{\mu} H_{\beta\mu} - \frac{1}{4} g_{\alpha\beta} H_{\mu\nu} H_{\mu\nu} + \mu^{2} \phi_{\mu} \phi_{\beta} - \frac{\mu^{2}}{2} g_{\alpha\beta} \phi_{\mu} \phi_{\mu},$$

where the source

$$\Sigma_{\alpha\beta}^{[T]} = \Sigma_{\alpha\beta}^{[\phi]} \left( \hat{T}, H_{\nu\lambda\rho} \right)$$

contains contributions of the torsion fields $\hat{T}$ and $H_{\nu\lambda\rho}$. The field $\phi_{\alpha}$ with certain irreducible components of torsion and nonmetricity in MAG, see [7] and Theorem 3.2 in [9].

Our aim is to elaborate a method of constructing exact solutions of equations (13) for vanishing matter fields, $\Sigma_{\alpha\beta}^{[m]} = 0$. The ansatz for the field $\phi_{\mu}$ is taken in the form

$$\phi_{\mu} = \left[ \phi_{i} \left( x^{k} \right), \phi_{a} = 0 \right]$$

for $i, j, k \ldots = 1, 2$ and $a, b, \ldots = 3, 4$. The Proca equations $\hat{D}_{\nu} H_{\nu\mu} = \mu^{2} \delta_{\mu}^{\nu}$ for $\mu \to 0$ (for simplicity) transform into

$$\partial_{1} \left[ (g_{1})^{-1} \partial_{i}^{1} \phi_{k} \right] + \partial_{2} \left[ (g_{2})^{-1} \partial^{2} \phi_{k} \right] = L_{k}^{j} \partial_{i} \phi_{j} - L_{j}^{i} \partial_{j} \phi_{k}.$$

Two examples of solutions of this equation are considered in Ref. [9]. In this paper, we do not state any particular configurations and consider that it is possible always to define certain $\phi_{i} \left( x^{k} \right)$ satisfying the wave equation (14). The energy–momentum tensor $\Sigma_{\alpha\beta}^{[\phi]}$ is computed for one nontrivial value

$$H_{12} = \partial_{1} \phi_{2} - \partial_{2} \phi_{1}.$$ 

In result, we can represent the source of the fields $\phi_{k}$ as

$$\Sigma_{\alpha\beta}^{[\phi]} = \left[ \Psi_{2} \left( H_{12}, x^{k} \right), \Psi_{2} \left( H_{12}, x^{k} \right), 0, 0 \right].$$

The ansatz for the $H$–field is taken in the form

$$H_{\nu\lambda\rho} = \hat{Z}_{\nu\lambda\rho} + \hat{H}_{\nu\lambda\rho} = \lambda_{[H]} \sqrt{|g_{\alpha\beta}|} \varepsilon_{\nu\lambda\rho}$$

where $\varepsilon_{\nu\lambda\rho}$ is completely antisymmetric and $\lambda_{[H]} = \text{const}$. This ansatz satisfies the field equations $\hat{D}^{\nu} H_{\nu\lambda\rho} = 0$ because the metric $g_{\alpha\beta}$ is compatible with $\hat{D}$. The values $\hat{H}_{\nu\lambda\rho}$ have to
be defined in a form to satisfy the condition (15) for any \( \hat{Z}_{\nu\lambda\rho} \) derived from \( g_{\alpha\beta} \) and, as a consequence, from (9) and (12), for instance, to compute them as

\[
\hat{H}_{\nu\lambda\rho} = \lambda_{[H]} \sqrt{|g_{\alpha\beta}|} \varepsilon_{\nu\lambda\rho} - \hat{Z}_{\nu\lambda\rho}
\]

for defined values of \( \hat{Z}_{\nu\lambda\rho}, \lambda_{[H]} \) and \( g_{\alpha\beta} \). In result, we obtain the effective energy–momentum tensor in the form

\[
\Sigma_{a[\phi]} + \Sigma_{a[H]} = \left[ \Upsilon_2 (x^k) + \frac{\lambda_{[H]}^2}{4}, \Upsilon_2 (x^k) + \frac{\lambda_{[H]}^2}{4}, \Upsilon_2 (x^k) + \frac{\lambda_{[H]}^2}{4} \right].
\]

(16)

For the source (16), the system of field equations (13) defined for the metric (5) and connection (9), with respect to anholonomic frames (6) and (7), transform into a system of partial differential equations with anholonomic variables [1, 2, 20], see also details in the section 5.3 in Ref. [9],

\[
R_1 = R_2 = -\frac{1}{2g_1 g_2} \left[ \frac{g_2}{g_1^2} - \frac{g_2}{g_1} + \frac{g_2' g_2}{g_2 g_1} + \frac{g_1'}{g_2} \right] = -\frac{\lambda_{[H]}^2}{4},
\]

(17)

\[
R_3 = R_4 = -\frac{\beta}{2h_3 h_4} = -\frac{1}{2h_3 h_4} \left[ h_4^{**} - h_4^* \left( \ln \sqrt{|h_3 h_4|} \right)^* \right] = -\frac{\lambda_{[H]}^2}{4} - \Upsilon_2 (x^k),
\]

(18)

\[
R_{3i} = -w_i \frac{\beta}{2h_4} - \frac{\alpha_i}{2h_4} = 0,
\]

(19)

\[
R_{4i} = -\frac{h_4}{2h_3} \left[ n_i^{**} + \gamma n_i^* \right] = 0,
\]

(20)

where

\[
\alpha_i = \partial_i h_4^* - h_4^* \partial_i \ln \sqrt{|h_3 h_4|}, \quad \gamma = 3h_4^*/2h_4 - h_3^*/h_3,
\]

and the partial derivatives are denoted \( g_1^* = \partial g_1/\partial x^1, g_1' = \partial g_1/\partial x^2 \) and \( h_3^* = \partial h_3/\partial v \). We can additionally impose the condition \( \delta_i N_j = \hat{\delta}_j N_i \) in order to have \( \Omega_{jk}^n = 0 \) which may be satisfied, for instance, if

\[
w_1 = w_1 (x^1, v), \quad n_1 = n_1 (x^1, v), \quad w_2 = n_2 = 0,
\]

or, inversely, if

\[
w_1 = n_1 = 0, \quad w_2 = w_2 (x^2, v), \quad n_2 = n_2 (x^2, v).
\]

In this paper we shall select a class of static solutions parametrized by the conditions

\[
w_1 = w_2 = n_2 = 0.
\]

(22)

The system of equations (17)–(20) can be integrated in general form [9, 20]. Physical solutions are selected following some additional boundary conditions, imposed types of symmetries, nonlinearities and singular behaviour and compatibility in the locally anisotropic limits with some well known exact solutions.
Finally, we note that there is a difference between our approach and the so-called "tetradic" gravity (see basic details and references in [22]) when the metric coefficients \( g_{\alpha\beta}(u) \) are substituted by tetradic fields \( e^a_\alpha(u^n) \), mutually related by formula \( g_{\alpha\beta} = \epsilon_{\alpha\beta}^{\gamma} e^\gamma_a u_a \) with \( \eta_{\alpha\beta} \) chosen, for instance, to be the Minkowski metric. In our case we partially preserve some metric dynamics given by diagonal effective metric coefficients \( (g_i, h_a) \) but also adapt the calculus to tetrads respectively defined by \( (N^a_i, \delta^j_i, \delta^a_b) \), see (6) and (7). This substantially simplifies the method of constructing exact solutions and also reflects new type symmetries of such classes of metrics.

3 Anholonomic Noncommutative Symmetries

The nontrivial anholonomy coefficients, see (4) and (8) induced by off–diagonal metric (1) (and associated N–connection) coefficients emphasize a kind of Lie algebra noncommutativity relation. In this section, we analyze a simple realizations of noncommutative geometry of anholonomic frames within the algebra of complex \( k \times k \) matrices, \( M_k(\mathbb{C}, u^a) \) depending on coordinates \( u^a \) on spacetime \( V^{n+m} \) connected to complex Lie algebras \( SL(k, \mathbb{C}) \) (see Ref. [10] for similar constructions with the group \( SU_k \)).

We consider matrix valued functions of necessary smoothly class derived from the anholonomic frame relations (4) (being similar to the Lie algebra relations) with the coefficients (8) induced by off–diagonal metric terms in (2) and by N–connection coefficients \( N^a_i \). We use algebras of complex matrices in order to have the possibility for some extensions to complex solutions and to relate the constructions to noncommutative/complex gravity). For commutative gravity models, the anholonomy coefficients \( w^\gamma_{\alpha\beta} \) are real functions but there are considered also complex metrics and tetrads related to noncommutative gravity [24].

Let us consider the basic relations for the simplest model of noncommutative geometry realized with the algebra of complex \( (k \times k) \) noncommutative matrices \( M_k(\mathbb{C}) \). Any element \( M \in M_k(\mathbb{C}) \) can be represented as a linear combination of the unit \( (k \times k) \) matrix \( I \) and \((k^2 - 1)\) hermitian traceless matrices \( q_\underline{\alpha} \) with the underlined index \( \underline{\alpha} \) running values 1, 2, ..., \( k^2 - 1 \), i. e.

\[
M = \alpha I + \sum \beta^\underline{\alpha} q_\underline{\alpha}
\]

for some constants \( \alpha \) and \( \beta^\underline{\alpha} \). It is possible to chose the basis matrices \( q_\underline{\alpha} \) satisfying the relations

\[
q_\underline{\alpha} q_\underline{\beta} = \frac{1}{k} \rho_{\underline{\alpha}\underline{\beta}} I + Q^\gamma_{\underline{\alpha}\underline{\beta}} q_\underline{\gamma} - \frac{i}{2} f^\gamma_{\underline{\alpha}\underline{\beta}} q_\underline{\gamma},
\]

where \( i^2 = -1 \) and the real coefficients satisfy the properties

\[
Q^\gamma_{\underline{\alpha}\underline{\beta}} = Q^\gamma_{\underline{\beta}\underline{\alpha}}, \quad Q^\gamma_{\underline{\alpha}\underline{\alpha}} = 0, \quad f^\gamma_{\underline{\alpha}\underline{\beta}} = -f^\gamma_{\underline{\beta}\underline{\alpha}}, \quad f^\gamma_{\underline{\gamma}\underline{\alpha}} = 0
\]

with \( f^\gamma_{\underline{\alpha}\underline{\beta}} \) being the structure constants of the Lie group \( SL(k, \mathbb{C}) \) and the Killing–Cartan metric tensor \( \rho_{\underline{\alpha}\underline{\beta}} = f^\gamma_{\underline{\alpha}\underline{\gamma}} f^\gamma_{\underline{\gamma}\underline{\beta}} \). This algebra admits a formalism of interior derivatives \( \partial_{\underline{\alpha}} \) defied by relations

\[
\partial_{\underline{\alpha}} q_\underline{\beta} = ad(i q_{\underline{\gamma}}) q_\underline{\beta} = i[q_{\underline{\beta}}, q_{\underline{\alpha}}] = f^\underline{\beta}_{\underline{\alpha}\underline{\gamma}} q_{\underline{\gamma}}
\]
and

\[ \partial_\alpha \partial_\beta - \partial_\beta \partial_\alpha = f^\gamma_{\alpha \beta} \partial_\gamma \]

(25)

(the last relation follows the Jacobi identity and is quite similar to (4) but with constant values \( f^\gamma_{\alpha \beta} \)).

Our idea is to associate a noncommutative geometry starting from the anholonomy relations of frames (4) by adding to the structure constants \( f^\gamma_{\alpha \beta} \) the anholonomy coefficients \( w_{\alpha \gamma} \) (we shall put the label \([N]\) if would be necessary to emphasize that the anholonomic coefficients are induced by a non-linear connection. Such deformed structure constants consist from \( N \)-connection coefficients \( N^\alpha_i \) and their first partial derivatives, i.e. they are induced by some off–diagonal terms in the metric (2) being a solution of the gravitational field equations.

We emphasize that there is a rough analogy between formulas (25) and (4) because the anholonomy coefficients do not satisfy, in general, the condition \[ w_{\gamma \tau \alpha} = 0 \]. There is also another substantial difference because the anholonomy relations are defined for a manifold of dimension \( n + m \), with Greek indices \( \alpha, \beta, ... \) running values from 1 to \( n + m \) but the matrix noncommutativity relations are stated for traceless matrices labeled by underlined indices \( \underline{\alpha}, \underline{\beta}, \) running values from 1 to \( k^2 - 1 \). It is not possible to satisfy the condition \( k^2 - 1 = n + m \) by using integer numbers for arbitrary \( n + m \). We may extend the dimension of spacetime from \( n + m \) to any \( n' \geq n \) and \( m' \geq m \) when the condition \( k^2 - 1 = n' + m' \) can be satisfied by a trivial embedding of the metric (2) into higher dimension, for instance, by adding the necessary number of unities on the diagonal and writing

\[
\hat{g}_{\underline{\alpha} \underline{\beta}} =
\begin{bmatrix}
1 & ... & 0 & ... & 0 \\
... & ... & ... & ... & ... \\
0 & ... & 1 & ... & 0 \\
0 & ... & 0 & g_{ij} + N^\gamma_i N^\delta_j h_{ab} & N^\gamma_i h_{ae} \\
0 & ... & 0 & N^\epsilon_j h_{be} & h_{ab}
\end{bmatrix}
\]

and \( e_{\underline{\alpha}} = \delta_{\underline{\alpha}} = (1, 1, ..., e_{\underline{\alpha}}) \). For simplicity, we preserve the same type of underlined Greek indices, \( \underline{\alpha}, \underline{\beta}, ... = 1, 2, ..., k^2 - 1 = n' + m' \).

The anholonomy coefficients \( w^{[N]}_{\alpha \beta} \) can be extended with some trivial zero components and for consistency we rewrite them without labeled indices, \( w^{[N]}_{\alpha \beta} \rightarrow W^{[N]}_{\underline{\alpha} \underline{\beta}} \). The set of anholonomy coefficients \( w^{[N]}_{\alpha \beta} \) may result in degenerated matrices, for instance for certain classes of exact solutions of the Einstein equations. So, it would not be a well defined construction if we shall substitute the structure Lie algebra constants directly by \( w^{[N]}_{\alpha \beta} \). We can consider a simple extension \( w^{[N]}_{\alpha \beta} \rightarrow W^{[N]}_{\underline{\alpha} \underline{\beta}} \) when the coefficients \( W^{[N]}_{\underline{\alpha} \underline{\beta}}(u^\underline{\alpha}) \) for any fixed value \( u^{[N]} = u^{[N]}_0 \) would be some deformations of the structure constants of the Lie algebra \( SL (k, \mathbb{C}) \), like

\[
W^{[N]}_{\underline{\alpha} \underline{\beta}} = f^{[N]}_{\underline{\alpha} \underline{\beta}} + w^{[N]}_{\underline{\alpha} \underline{\beta}},
\]

being nondegenerate.

Instead of the matrix algebra \( M_k(\mathbb{C}) \), constructed from constant complex elements, we have also to introduce dependencies on coordinates \( u^{\underline{\alpha}} = (0, ..., u^{\alpha}) \), for instance, like a trivial
matrix bundle on $V^{n'+m'}$, and denote this space $M_k(\mathbb{C}, u^2)$. Any element $B (u^2) \in M_k(\mathbb{C}, u^2)$ with a noncommutative structure induced by $W_{\underline{\alpha} \underline{\beta}}$ is represented as a linear combination of the unit $(n' + m') \times (n' + m')$ matrix $I$ and the $[(n' + m')^2 - 1]$ hermitian traceless matrices $q_\underline{\alpha}$ with the underlined index $\underline{\alpha}$ running values $1, 2, \ldots, (n' + m')^2 - 1$,

$$B (u^2) = \alpha (u^2) I + \sum \beta (u^2) q_\underline{\beta} (u^2)$$

under condition that the following relation holds:

$$q_\underline{\alpha} (u^2) q_\underline{\beta} (u^2) = \frac{1}{n' + m'} \rho_{\underline{\alpha} \underline{\beta}} (u^2) + Q_{\underline{\alpha} \underline{\beta}}^\gamma (u^2) q_\underline{\gamma} (u^2) - \frac{i}{2} W_{\underline{\alpha} \underline{\beta}}^\gamma q_\underline{\gamma} (u^2)$$

with the same values of $Q_{\underline{\alpha} \underline{\beta}}^\gamma$ from the Lie algebra for $SL(k, \mathbb{C})$ but with the Killing–Cartan like metric tensor defined by anholonomy coefficients, i. e. $\rho_{\underline{\alpha} \underline{\beta}} (u^2) = W_{\underline{\alpha} \underline{\beta}}^\gamma (u^2) W_{\underline{\gamma} \underline{\delta}} (u^2)$. For complex spacetimes, we shall consider that the coefficients $N_\underline{\alpha}$ and $W_{\underline{\alpha} \underline{\beta}}$ may be some complex valued functions of necessary smooth (in general, with complex variables) class. In result, the Killing–Cartan like metric tensor $\rho_{\underline{\alpha} \underline{\beta}}$ can be also complex.

We rewrite (1) as

$$e_\underline{\alpha} e_\underline{\beta} - e_\underline{\beta} e_\underline{\alpha} = W_{\underline{\alpha} \underline{\beta}}^\gamma e_\underline{\gamma}$$

being equivalent to (25) and defining a noncommutative anholonomic structure (for simplicity, we use the same symbols $e_\underline{\alpha}$ as for some ‘N–elongated’ partial derivatives, but with underlined indices). The analogs of derivation operators (24) are stated by using $W_{\underline{\alpha} \underline{\beta}}^\gamma$:

$$e_\underline{\alpha} q_\underline{\beta} (u^2) = ad [i q_\underline{\alpha} (u^2)] q_\underline{\beta} (u^2) = i \left[ q_\underline{\alpha} (u^2) q_\underline{\beta} (u^2) \right] = W_{\underline{\alpha} \underline{\beta}}^\gamma q_\underline{\gamma}$$

The operators (28) define a linear space of anholonomic derivations satisfying the conditions (27), denoted $AderM_k(\mathbb{C}, u^2)$, elongated by N–connection and distinguished into irreducible h– and v–components, respectively, into $e_\underline{h}$ and $e_\underline{v}$, like $e_\underline{h} = \left( e_B = \partial_B - N_B^a e_a, e_a = \partial_a \right)$. The space $AderM_k(\mathbb{C}, u^2)$ is not a left module over the algebra $M_k(\mathbb{C}, u^2)$, which means that there is a substantial difference with respect to the usual commutative differential geometry where a vector field multiplied on the left by a function produces a new vector field.

The duals to operators (28), $e^\underline{\mu}$, found from $e^\underline{\mu} (e_\underline{\alpha}) = \delta^\underline{\mu}_\underline{\alpha}$, define a canonical basis of 1–forms $e^\underline{\mu}$ connected to the N–connection structure. By using these forms, we can span a left module over $M_k(\mathbb{C}, u^2)$ following $q_\underline{\alpha} e^\underline{\mu} (e_\underline{\beta}) = q_\underline{\alpha} \delta^\underline{\beta}_\underline{\mu} = q_\underline{\alpha} \delta^\underline{\beta}_\underline{\mu}$. For an arbitrary vector field

$$Y = Y^\underline{\alpha} e_\underline{\alpha} \rightarrow Y^\underline{\alpha} e_\underline{\alpha} = Y^\underline{\alpha} e_\underline{\alpha} + Y^\underline{\alpha} e_\underline{\alpha},$$

it is possible to define an exterior differential (in our case being N–elongated), starting with the action on a function $\varphi$ (equivalent, a 0–form),

$$\delta \varphi (Y) = Y \varphi = Y^\underline{\alpha} \delta^\underline{\alpha} \varphi + Y^\underline{\alpha} \partial^a \partial_\underline{\alpha} \varphi$$

when

$$(\delta I) (e_\underline{\alpha}) = e_\underline{\alpha} I = ad (i e_\underline{\alpha}) I = i [e_\underline{\alpha}, I] = 0, \ i. e. \ \delta I = 0,$$
\[ \delta q_\mu(e_\alpha) = e_\alpha(e_\mu) = i[e_\mu, e_\alpha] = W^2_{\alpha\mu}e_\gamma. \]  

(29)

Considering the nondegenerated case, we can invert (29) as to obtain a similar expression with respect to \( e_\mu \),

\[ \delta(e_\alpha) = W^\gamma_{\alpha\mu}e_\gamma. \]  

(30)

from which a very important property follows by using the Jacobi identity, \( \delta^2 = 0 \), resulting in a possibility to define a usual Grassman algebra of \( p \)-forms with the wedge product \( \wedge \) stated as

\[ e_\mu \wedge e_\nu = \frac{1}{2} (e_\mu \otimes e_\nu - e_\nu \otimes e_\mu). \]

We can write (30) as

\[ \delta(e_\alpha) = -\frac{1}{2} W^\alpha_{\beta\mu}e^\beta e^\mu \]

and introduce the canonical 1–form \( e = q_\alpha e^\alpha \) being coordinate–independent and adapted to the N–connection structure and satisfying the condition \( \delta e + e \wedge e = 0 \).

In a standard manner, we can introduce the volume element induced by the canonical Cartan–Killing metric and the corresponding star operator \( \star \) (Hodge duality). We define the volume element \( \sigma \) by using the complete antisymmetric tensor \( \epsilon_{\alpha_1\alpha_2...\alpha_{k-1}} \) as

\[ \sigma = \frac{1}{[(n'+m')^2 - 1]!} \epsilon_{\alpha_1\alpha_2...\alpha_{n'}\alpha_{n'+m'}} e^{\alpha_1} \wedge e^{\alpha_2} \wedge ... \wedge e^{\alpha_{n'+m'}} \]

to which any \((k^2 - 1)\)-form is proportional \((k^2 - 1 = n' + m')\). The integral of such a form is defined as the trace of the matrix coefficient in the from \( \sigma \) and the scalar product introduced for any couple of \( p \)-forms \( \varpi \) and \( \psi \)

\[ (\varpi, \psi) = \int (\varpi \wedge \star \psi). \]

Let us analyze how a noncommutative differential form calculus (induced by an anholonomic structure) can be developed and related to the Hamiltonian classical and quantum mechanics and Poisson bracket formalism:

For a \( p \)-form \( \varpi^{[p]} \), the anti–derivation \( i_Y \) with respect to a vector field \( Y \in \text{Ader} M_k(\mathbb{C}, u^\alpha) \) can be defined as in the usual formalism,

\[ i_Y \varpi^{[p]}(X_1, X_2, ..., X_{p-1}) = \varpi^{[p]}(Y, X_1, X_2, ..., X_{p-1}) \]

where \( X_1, X_2, ..., X_{p-1} \in \text{Ader} M_k(\mathbb{C}, u^\alpha) \). By a straightforward calculus we can check that for a 2–form \( \Xi = \delta e \) one holds

\[ \delta \Xi = \delta^2 e = 0 \text{ and } L_Y \Xi = 0 \]

where the Lie derivative of forms is defined as \( L_Y \varpi^{[p]} = (i_Y \delta + \delta i_Y) \varpi^{[p]} \).

The Hamiltonian vector field \( H_{[\varphi]} \) of an element of algebra \( \varphi \in M_k(\mathbb{C}, u^\alpha) \) is introduced following the equality \( \Xi (H_{[\varphi]}, Y) = Y \varphi \) which holds for any vector field. Then, we can
define the Poisson bracket of two functions (in a quantum variant, observables) \( \varphi \) and \( \chi \),

\[ \{ \varphi, \chi \} = \Xi \left( H[\varphi], H[\chi] \right) \]

when

\[ \{ e_\alpha, e_\beta \} = \Xi \left( e_\alpha, e_\beta \right) = i [ e_\alpha, e_\beta ] . \]

This way, a simple version of noncommutative classical and quantum mechanics (up to a factor like the Plank constant, \( \hbar \)) is proposed, being derived by anholonomic relations for a certain class of exact 'off–diagonal' solutions in commutative gravity.

In order to define the algebra of forms \( \Omega^* [M_k(\mathbb{C}, u^\alpha)] \) over \( M_k(\mathbb{C}, u^\alpha) \) we put \( \Omega^0 = M_k \) and write

\[ \delta \varphi (e_\alpha) = e_\alpha (\varphi) \]

for every matrix function \( \varphi \in M_k(\mathbb{C}, u^\alpha) \). As a particular case, we have

\[ \delta q^\alpha (e_\beta) = - W^\alpha_{\beta \gamma} q^\gamma \]

where indices are raised and lowered with the anholonomically deformed metric \( \rho^\alpha_{\beta \gamma}(u^\Delta) \). This way, we can define the set of 1–forms \( \Omega^1 [M_k(\mathbb{C}, u^\alpha)] \) to be the set of all elements of the form \( \varphi \delta \beta \) with \( \varphi \) and \( \beta \) belonging to \( M_k(\mathbb{C}, u^\alpha) \). The set of all differential forms define a differential algebra \( \Omega^* [M_k(\mathbb{C}, u^\alpha)] \) with the couple \( (\Omega^* [M_k(\mathbb{C}, u^\alpha)], \delta) \) said to be a differential calculus in \( M_k(\mathbb{C}, u^\alpha) \) induced by the anholonomy of certain exact solutions (with off–diagonal metrics and associated N–connections) in a gravity theory.

We can also find a set of generators \( e^\alpha \) of \( \Omega^1 [M_k(\mathbb{C}, u^\alpha)] \), as a left/ right –module completely characterized by the duality equations \( e^\mu (e_\alpha) = \delta^\mu_\alpha I \) and related to \( \delta q^\alpha \),

\[ \delta q^\alpha = W^\alpha_{\beta \gamma} q^\beta q^\gamma \] and \( e^\mu = q^\mu \delta q^\mu \).

Similarly to the formalism presented in details in Ref. [27], we can elaborate a differential calculus with derivations by introducing a linear torsionless connection

\[ D e^\mu = - \omega^\mu_\gamma \otimes e^\gamma \]

with the coefficients \( \omega^\mu_\gamma = - {1 \over 2} W^\mu_{\gamma \beta} e^\beta \), resulting in the curvature 2–form

\[ R^\mu_\gamma = {1 \over 8} W^\mu_{\beta \gamma} W^\beta_{\alpha \rho} e^\alpha \wedge e^\rho \].

This is a surprising fact that 'commutative' curved spacetimes provided with off–diagonal metrics and associated anholonomic frames and N–connections may be characterized by a non-commutative 'shadow' space with a Lie algebra like structure induced by the frame anholonomy. We argue that such metrics possess anholonomic noncommutative symmetries and conclude that for the 'holonomic' solutions of the Einstein equations, with vanishing \( w^\tau_{\alpha \beta} \), any associated noncommutative geometry or \( SL(k, \mathbb{C}) \) decouples from the off–diagonal (anholonomic) gravitational background and transforms into a trivial one defined by the corresponding structure constants of the chosen Lie algebra. The anholonomic noncommutativity and the related
differential geometry are induced by the anholonomy coefficients. All such structures reflect a specific type of symmetries of generic off–diagonal metrics and associated frame/ N–connection structures.

Considering exact solutions of the gravitational field equations, we can assert that we constructed a class of vacuum or nonvacuum metrics possessing a specific noncommutative symmetry instead, for instance, of any usual Killing symmetry. In general, we can introduce a new classification of spacetimes following anholonomic noncommutative algebraic properties of metrics and vielbein structures (see Ref. [28, 10]). In this paper, we analyze the simplest examples of such spacetimes.

4 4D Static Black Ellipsoids in MAG and String Gravity

We outline the black ellipsoid solutions [29, 30] and discuss their associated anholonomic noncommutative symmetries [10]. We note that such solutions can be extended for the (anti) de Sitter spaces, in gauge gravity and string gravity with effective cosmological constant [31]. In this paper, the solutions are considered for ‘real’ metric–affine spaces and extended to nontrivial cosmological constant. We emphasize the possibility to construct solutions with locally ”anisotropic” cosmological constants (such configurations may be also induced, for instance, from string/brane gravity).

4.1 Anholonomic deformations of the Schwarzschild metric

We consider a particular case of effectively diagonalized (5) (and corresponding off–diagonal metric ansatz (1)) when

\[ \delta s^2 = \left[ -\left( 1 - \frac{2m}{r} + \frac{\varepsilon}{r^2} \right)^{-1} dr^2 - r^2 q(r) d\theta^2 \right. \]

\[ \left. - \eta_3 (r, \varphi) r^2 \sin^2 \theta d\varphi^2 + \eta_4 (r, \varphi) \left( 1 - \frac{2m}{r} + \frac{\varepsilon}{r^2} \right) \delta t^2 \right] \]

where the ”polarization” functions \( \eta_{3,4} \) are decomposed on a small parameter \( \varepsilon \), \( 0 < \varepsilon \ll 1 \),

\[ \eta_3 (r, \varphi) = \eta_{3[0]} (r, \varphi) + \varepsilon \lambda_3 (r, \varphi) + \varepsilon^2 \gamma_3 (r, \varphi) + ..., \]

\[ \eta_4 (r, \varphi) = 1 + \varepsilon \lambda_4 (r, \varphi) + \varepsilon^2 \gamma_4 (r, \varphi) + ..., \]

and

\[ \delta t = dt + n_1 (r, \varphi) dr \]

for \( n_1 \sim \varepsilon ... + \varepsilon^2 \) terms. The functions \( q(r), \eta_{3,4} (r, \varphi) \) and \( n_1 (r, \varphi) \) will be found as the metric will define a solution of the gravitational field equations generated by small deformations of the spherical static symmetry on a small positive parameter \( \varepsilon \) (in the limits \( \varepsilon \to 0 \) and \( q, \eta_{3,4} \to 1 \) we have just the Schwarzschild solution for a point particle of mass \( m \)). The metric (31) is a
particular case of a class of exact solutions constructed in [11, 21, 20]. Its complexification by complex valued N–coefficients is investigated in Ref. [10].

We can state a particular symmetry of the metric (31) by imposing a corresponding condition of vanishing of the metric coefficient before the term $\delta t^2$. For instance, the constraints that

$$\eta_4(r, \varphi) \left(1 - \frac{2m}{r} + \frac{\varepsilon}{r^2}\right) = 1 - \frac{2m}{r} + \varepsilon \frac{\Phi_4}{r^2} + \varepsilon^2 \Theta_4 = 0,$$

$$\Phi_4 = \lambda_4 \left(r^2 - 2mr\right) + 1$$

$$\Theta_4 = \gamma_4 \left(1 - \frac{2m}{r}\right) + \lambda_4,$$

define a rotation ellipsoid configuration if

$$\lambda_4 = \left(1 - \frac{2m}{r}\right)^{-1} \left(\cos \varphi - \frac{1}{r^2}\right)$$

and

$$\gamma_4 = -\lambda_4 \left(1 - \frac{2m}{r}\right)^{-1}.$$

In the first order on $\varepsilon$ one obtains a zero value for the coefficient before $\delta t^2$ if

$$r_+ = \frac{2m}{1 + \varepsilon \cos \varphi} = 2m[1 - \varepsilon \cos \varphi],$$

which is the equation for a 3D ellipsoid like hypersurface with a small eccentricity $\varepsilon$. In general, we can consider arbitrary pairs of functions $\lambda_4(r, \theta, \varphi)$ and $\gamma_4(r, \theta, \varphi)$ (for $\varphi$–anisotropies, $\lambda_4(r, \varphi)$ and $\gamma_4(r, \varphi)$) which may be singular for some values of $r$, or on some hypersurfaces $r = r(\theta, \varphi)$ ($r = r(\varphi)$).

The simplest way to define the condition of vanishing of the metric coefficient before the value $\delta t^2$ is to choose $\gamma_4$ and $\lambda_4$ as to have $\Theta = 0$. In this case we find from

$$r_{\pm} = m \pm m \sqrt{1 - \varepsilon \frac{\Phi_4}{m^2}} = m \left[1 \pm \left(1 - \varepsilon \frac{\Phi_4}{2m^2}\right)^{1/2}\right]$$

where $\Phi_4(r, \varphi)$ is taken for $r = 2m$.

For a new radial coordinate

$$\xi = \int dr \sqrt{1 - \frac{2m}{r} + \frac{\varepsilon}{r^2}},$$

and

$$h_3 = -\eta_3(\xi, \varphi)r^2(\xi) \sin^2 \theta, \quad h_4 = 1 - \frac{2m}{r} + \frac{\varepsilon \Phi_4}{r^2},$$

when $r = r(\xi)$ is inverse function after integration in (36), we write the metric as

$$ds^2 = -d\xi^2 - r^2(\xi) q(\xi) d\theta^2 + h_3(\xi, \theta, \varphi) \delta \varphi^2 + h_4(\xi, \theta, \varphi) \delta t^2,$$

$$\delta t = dt + n_1(\xi, \varphi) d\xi,$$

16
where the coefficient \( n_1 \) is taken to solve the equation (20) and to satisfy the (22). The next step is to state the conditions when the coefficients of metric (31) define solutions of the Einstein equations. We put \( g_1 = -1, g_2 = -r^2(\xi) q(\xi) \) and arbitrary \( h_3(\xi, \theta, \varphi) \) and \( h_4(\xi, \theta) \) in order to find solutions the equations (17)–(19). If \( h_4 \) depends on anisotropic variable \( \varphi \), the equation (18) may be solved if

\[
\sqrt{|\eta_3|} = \eta_0 \left( \sqrt{|\eta_4|} \right)^* \quad (39)
\]

for \( \eta_0 = \text{const.} \). Considering decompositions of type (32) we put \( \eta_0 = \eta/\varepsilon \) where the constant \( \eta \) is taken as to have \( \sqrt{|\eta_3|} = 1 \) in the limits

\[
\varepsilon \to 0 \quad \Rightarrow \quad \frac{1}{\eta} = \text{const.} \quad (40)
\]

These conditions are satisfied if the functions \( \eta_{3[0]}, \lambda_{3,4} \) and \( \gamma_{3,4} \) are related via relations

\[
\sqrt{|\eta_{3[0]}|} = \frac{\eta}{2} \lambda_4^*, \lambda_3 = \eta \sqrt{|\eta_{3[0]}|} \gamma_4^*
\]

for arbitrary \( \gamma_3 (r, \varphi) \). In this paper we select only such solutions which satisfy the conditions (39) and (40).

For linear infinitezimal extensions on \( \varepsilon \) of the Schwarzschild metric, we write the solution of (20) as

\[
n_1 = \varepsilon \hat{n}_1 (\xi, \varphi)
\]

where

\[
\hat{n}_1 (\xi, \varphi) = n_{1[1]} (\xi) + n_{1[2]} (\xi) \int d\varphi \eta_3 (\xi, \varphi) / \left( \sqrt{|\eta_4 (\xi, \varphi)|} \right)^3, \eta_4^* \neq 0; \quad (41)
\]

\[
= n_{1[1]} (\xi) + n_{1[2]} (\xi) \int d\varphi \eta_3 (\xi, \varphi), \eta_4^* = 0;
\]

\[
= n_{1[1]} (\xi) + n_{1[2]} (\xi) \int d\varphi / \left( \sqrt{|\eta_4 (\xi, \varphi)|} \right)^3, \eta_3^* = 0;
\]

with the functions \( n_{k[1,2]} (\xi) \) to be stated by boundary conditions.

The data

\[
g_1 = -1, g_2 = -r^2(\xi) q(\xi), \quad (42)
\]

\[
h_3 = -\eta_3(\xi, \varphi)r^2(\xi) \sin^2 \theta, \quad h_4 = 1 - \frac{2m}{r} + \frac{\Phi_4}{r^2},
\]

\[
w_{1,2} = 0, n_1 = \varepsilon \hat{n}_1 (\xi, \varphi), n_2 = 0,
\]

for the metric (31) define a class of solutions of the Einstein equations for the canonical distinguished connection (9), with non–trivial polarization function \( \eta_3 \) and extended on parameter \( \varepsilon \) up to the second order (the polarization functions being taken as to make zero the second order coefficients). Such solutions are generated by small deformations (in particular cases of rotation ellipsoid symmetry) of the Schwarzschild metric.

We can relate our solutions with some small deformations of the Schwarzschild metric, as well we can satisfy the asymptotically flat condition, if we chose such functions \( n_{k[1,2]} (x^i) \) as \( n_k \to 0 \) for \( \varepsilon \to 0 \) and \( \eta_3 \to 1 \). These functions have to be selected as to vanish far away from the horizon, for instance, like \( \sim 1/r^{1+\tau}, \tau > 0 \), for long distances \( r \to \infty \).
4.2 Black ellipsoids and anistropic cosmological constants

We can generalize the gravitational field equations to the gravity with variable cosmological constants $\lambda_{[u]} (u^a)$ and $\lambda_{[v]} (u^a)$ which can be induced, for instance, from extra dimensions in string/brane gravity, when the non-trivial components of the Einsein equations are

$$R_{ij} = \lambda_{[b]} (x^1) g_{ij} \text{ and } R_{ab} = \lambda_{[v]} (x^k, v) g_{ab}$$

(43)

where Ricci tensor $R_{\mu\nu}$ with anholonomic variables has two nontrivial components $R_{ij}$ and $R_{ab}$, and the indices take values $i, k = 1, 2$ and $a, b = 3, 4$ for $x^i = \xi$ and $y^3 = v = \varphi$ (see notations from the previous subsection). The equations (43) contain the equations (17) and (18) as particular cases when $\lambda_{[b]} (x^1) = \frac{\lambda_{[b]}}{4}$ and $\lambda_{[v]} (x^k, v) = \frac{\lambda_{[v]}}{4} + \Upsilon_2 (x^k)$.

For an ansatz of type (3)

$$\delta s^2 = g_1(dx^1)^2 + g_2(dx^2)^2 + h_3 (x^i, y^3) (\delta y^3)^2 + h_4 (x^i, y^3) (\delta y^4)^2,$$

$$\delta y^3 = dy^3 + w_1 (x^k, y^3) dx^i, \quad \delta y^4 = dy^4 + n_1 (x^k, y^3) dx^i,$$

(44)

the Einstein equations (43) are written (see [20] for details on computation)

$$R_1^1 = R_2^2 = -\frac{1}{2g_1g_2} [g_2^{**} - \frac{g_1^{**}}{2g_1} - \frac{(g_2')^2}{2g_2} + g_1'' - \frac{g_1'g_2'}{2g_1} - \frac{(g_1')^2}{2g_1}] = \lambda_{[b]} (x^k),$$

(45)

$$R_3^3 = R_4^4 = -\frac{\beta}{2h_3h_4} = \lambda_{[v]} (x^k, v),$$

(46)

$$R_{3i} = -w_i \frac{\beta}{2h_4} - \frac{\alpha_i}{2h_4} = 0,$$

(47)

$$R_{4i} = -\frac{h_4}{2h_3} [n_i^{**} + \gamma n_i^*] = 0.$$  

(48)

The coefficients of equations (43) - (48) are given by

$$\alpha_i = \partial_i h_4^* - h_4^* \partial_i \ln \sqrt{|h_3/h_4|}, \quad \beta = h_4^{**} - h_4^* [\ln \sqrt{|h_3/h_4|}]^*, \quad \gamma = \frac{3h_4^*}{2h_4} - \frac{h_3^*}{h_3}. $$

(49)

The various partial derivatives are denoted as $a^* = \partial a/\partial x^1, a' = \partial a/\partial x^2, a^* = \partial a/\partial y^3$. This system of equations can be solved by choosing one of the ansatz functions (e.g. $g_1 (x^i)$ or $g_2 (x^i)$) and one of the ansatz functions (e.g. $h_3 (x^i, y^3)$ or $h_4 (x^i, y^3)$) to take some arbitrary, but physically interesting form. Then, the other ansatz functions can be analytically determined up to an integration in terms of this choice. In this way we can generate a lot of different solutions, but we impose the condition that the initial, arbitrary choice of the ansatz functions is “physically interesting” which means that one wants to make this original choice so that the generated final solution yield a well behaved metric.

In this subsection, we show that the data (12) can be extended as to generate exact black ellipsoid solutions with nontrivial polarized cosmological constant which can be imbedded in string theory. A complex generalization of the solution (12) is analyzed in Ref. [10] and the case locally isotropic cosmological constant was considered in Ref. [31].
At the first step, we find a class of solutions with \( g_1 = -1 \) and \( g_2 = g_2(\xi) \) solving the equation (45), which under such parametrizations transforms to
\[
g_2^{**} - \frac{(g_2^*)^2}{2g_2} = 2g_2\lambda[\eta](\xi). \tag{50}
\]
With respect to the variable \( Z = (g_2)^2 \) this equation is written as
\[
Z^{**} + 2\lambda[\eta](\xi) Z = 0
\]
which can be integrated in explicit form if \( \lambda[\eta](\xi) = \lambda[\eta]_0 = \text{const} \),
\[
Z = Z_0 \sin \left( \sqrt{2\lambda[\eta]_0} \xi + \xi_0 \right),
\]
for some constants \( Z_0 \) and \( \xi_0 \) which means that
\[
g_2 = -Z_0^2 \sin^2 \left( \sqrt{2\lambda[\eta]_0} \xi + \xi_0 \right) \tag{51}
\]
parametrize in 'real' string gravity a class of solution of (45) for the signature \((-,-,-,+)\). For \( \lambda[\eta] \to 0 \) we can approximate \( g_2 = r^2(\xi)q(\xi) = -\xi^2 \) and \( Z_0^2 = 1 \) which has compatibility with the data (42). The solution (51) with cosmological constant (of string or non–string origin) induces oscillations in the "horizontal" part of the metric written with respect to N–adapted frames. If we put \( \lambda[\eta](\xi) \) in (50), we can search the solution as \( g_2 = u^2 \) where \( u(\xi) \) solves the linear equation
\[
u^{**} + \frac{\lambda[\eta](\xi)}{4} u = 0.
\]
The method of integration of such equations is given in Ref. [32]. The explicit forms of solutions depends on function \( \lambda[\eta](\xi) \). In this case we have to write
\[
g_2 = r^2(\xi)q[\eta](\xi) = u^2(\xi). \tag{52}
\]
For a suitable smooth behaviour of \( \lambda[\eta](\xi) \), we can generate such \( u(\xi) \) and \( r(\xi) \) when the \( r = r(\xi) \) is the inverse function after integration in (36).

The next step is to solve the equation (50),
\[
h_4^{**} - h_4^* [\ln \sqrt{|h_3h_4|}]^* = -2\lambda[\eta](x^k,v)h_3h_4.
\]
For \( \lambda = 0 \) a class of solution is given by any \( \hat{h}_3 \) and \( \hat{h}_4 \) related as
\[
\hat{h}_3 = \eta_0 \left( \sqrt{|\hat{h}_4|} \right)^2
\]
for a constant \( \eta_0 \) chosen to be negative in order to generate the signature \((-,-,-,+)\). For non–trivial \( \lambda \), we may search the solution as
\[
h_3 = \hat{h}_3(\xi,\varphi) f_3(\xi,\varphi) \quad \text{and} \quad h_4 = \hat{h}_4(\xi,\varphi), \tag{53}
\]
\[19\]
which solves (16) if \( f_3 = 1 \) for \( \lambda_{[\nu]} = 0 \) and
\[
f_3 = \frac{1}{4} \left[ \int \frac{\lambda_{[\nu]} h_3 h_4}{h_4} d\varphi \right]^{-1} \quad \text{for } \lambda_{[\nu]} \neq 0.
\]

Now it is easy to write down the solutions of equations (17) (being a linear equation for \( w_i \)) and (18) (after two integrations of \( n_i \) on \( \varphi \)),
\[
w_i = \varepsilon \hat{w}_i = -\alpha_i / \beta,
\]
were \( \alpha_i \) and \( \beta \) are computed by putting (53) into corresponding values from (19) (we chose the initial conditions as \( w_i \to 0 \) for \( \varepsilon \to 0 \)) and
\[
n_1 = \varepsilon \hat{n}_1 (\xi, \varphi)
\]
where the coefficients
\[
\hat{n}_1 (\xi, \varphi) = n_{[1]} (\xi) + n_{[2]} (\xi) \int d\varphi \ f_3 (\xi, \varphi) \eta_3 (\xi, \varphi) / \left( \sqrt{|\eta_4 (\xi, \varphi)|} \right)^3, \eta_4^* \neq 0; \quad (55)
\]
\[
= n_{[1]} (\xi) + n_{[2]} (\xi) \int d\varphi \ f_3 (\xi, \varphi) \eta_3 (\xi, \varphi), \eta_4^* = 0;
\]
\[
= n_{[1]} (\xi) + n_{[2]} (\xi) \int d\varphi / \left( \sqrt{|\eta_4 (\xi, \varphi)|} \right)^3, \eta_3^* = 0;
\]
with the functions \( n_{k[1,2]} (\xi) \) to be stated by boundary conditions.

We conclude that the set of data \( g_1 = -1 \), with non-trivial \( g_2 (\xi) \), \( h_3, h_4, w_i \) and \( n_1 \) stated respectively by (51), (53), (54), (55) we can define a black ellipsoid solution with explicit dependence on polarized cosmological "constants" \( \lambda_{[\mu]} (x^1) \) and \( \lambda_{[\nu]} (x^k, v) \), i.e. a metric (14).

Finally, we analyze the structure of noncommutative symmetries associated to the (anti) de Sitter black ellipsoid solutions. The metric (14) with real and/or complex coefficients defining the corresponding solutions and its analytic extensions also do not posses Killing symmetries being deformed by anholonomic transforms. For this solution, we can associate certain noncommutative symmetries following the same procedure as for the Einstein real/complex gravity but with additional nontrivial coefficients of anholonomy and even with nonvanishing symmetries of the nonlinear connection curvature, \( \Omega^3_{[1]} = \delta_1 N_3^2 - \delta_2 N_1^1 \). Taking the data (54) and (55) and formulas (1), we compute the corresponding nontrivial anholonomy coefficients
\[
w^{[N]}_{31} = -w^{[N]}_{13} = \partial n_1 (\xi, \varphi) / \partial \varphi = n_1^* (\xi, \varphi),
\]
\[
w^{[N]}_{12} = -w^{[N]}_{21} = \delta_1 (\alpha_2 / \beta) - \delta_2 (\alpha_1 / \beta)
\]
for \( \delta_1 = \partial / \partial \xi - w_1 \partial / \partial \varphi \) and \( \delta_2 = \partial / \partial \theta - w_2 \partial / \partial \varphi \), with \( n_1 \) defined by (55) and \( \alpha_{1,2} \) and \( \beta \) computed by using the formula (19) for the solutions (53). We have a 4D exact solution with nontrivial cosmological constant. So, for \( n + m = 4 \), the condition \( k^2 - 1 = n + m \) can not satisfied by any integer numbers. We may trivially extend the dimensions like \( n' = 6 \) and \( m' = 2 \) and for \( k = 3 \) to consider the Lie group \( SL (3, \mathbb{C}) \) noncommutativity with corresponding values of \( Q^2_{\alpha \beta} \) and structure constants \( f^2_{\alpha \beta} \) see (23). An extension \( w^{[N]}_{\alpha \beta} \rightarrow W^{[N]}_{\alpha \beta} \)
may be performed by stating the N–deformed "structure" constants (26), $\mathcal{W} \gamma^\alpha_{\beta \gamma} = f^\gamma_{\alpha \beta} + w^{[N]}_{\gamma \beta} \partial_\beta$, with nontrivial values of $w^{[N]}_{\gamma \beta}$ given by (56). We note that the solutions with nontrivial cosmological constants are with induced torsion with the coefficients computed by using formulas (12) and the data (52), (53), (54) and (55).

4.3 Analytic extensions of black ellipsoid metrics

For the vacuum black ellipsoid metrics the method of analytic extension was considered in Ref. [29, 30]. The coefficients of the metric (31) (equivalently (38)) written with respect to the anholonomic frame (6) has a number of similarities with the Schwarzschild and Reissner–Nördstrom solutions. The cosmological "polarized" constants induce some additional factors like $q^{[u]}(\xi)$ and $f_3(\xi, \varphi)$ (see, respectively, formulas (52) and (53)) and modify the N–connection coefficients as in (54) and (55). For a corresponding class of smooth polarizations, the functions $q^{[u]}$ and $f_3$ do not change the singularity structure of the metric coefficients. If we identify $\varepsilon$ with $e^2$, we get a static metric with effective "electric" charge induced by a small, quadratic on $\varepsilon$, off–diagonal metric extension. The coefficients of this metric are similar to those from the Reissner–Nördstrom solution but additionally to the mentioned frame anholonomy there are additional polarizations by the functions $q^{[u]}$, $h_{3[0]}$, $f_3$, $\eta_{3,4}$, $w_i$ and $n_1$. Another very important property is that the deformed metric was stated to define a vacuum, or with polarized cosmological constant, solution of the Einstein equations which differs substantially from the usual Reissner–Nördstrom metric being an exact static solution of the Einstein–Maxwell equations. For the limits $\varepsilon \to 0$ and $q, f_3, h_{3[0]} \to 1$ the metric (31) transforms into the usual Schwarzschild metric. A solution with ellipsoid symmetry can be selected by a corresponding condition of vanishing of the coefficient before the term $\delta t$ which defines an ellipsoidal hypersurface like for the Kerr metric, but in our case the metric is non–rotating. In general, the space may be with frame induced torsion if we do not impose constraints on $w_i$ and $n_1$ as to obtain vanishing nonlinear connection curvature and torsions.

The analytic extension of black ellipsoid solutions with cosmological constant can be performed similarly both for anholonomic frames with induced or trivial torsions. We note that the solutions in string theory may contain a frame induced torsion with the components (12) (in general, we can consider complex coefficients, see Ref. [10]) computed for nontrivial $N^3_i = -\alpha_i / \beta$ (see (54)) and $N^1_i = \tilde{e} n_1(\xi, \varphi)$ (see (55)). This is an explicit example illustrating that the anholonomic frame method can be applied also for generating exact solutions in models of gravity with nontrivial torsion. For such solutions, we can perform corresponding analytic extensions and define Penrose diagram formalisms if the constructions are considered with respect to N–elongated vierbeins.

The metric (44) has a singular behaviour for $r = r_\pm$, see (35). The aim of this subsection is to prove that this way we have constructed a solution of the Einstein equations with polarized cosmological constant. This solution possess an "anisotropic" horizon being a small deformation on parameter $\varepsilon$ of the Schwarzschild’s solution horizon. We may analyze the anisotropic horizon’s properties for some fixed "direction" given in a smooth vicinity of any values $\varphi = \varphi_0$ and $r_+ = r_+(\varphi_0)$. The final conclusions will be some general ones for arbitrary $\varphi$ when the explicit values of coefficients will have a parametric dependence on angular
coordinate $\varphi$. The metrics (31), or (32), and (11) are regular in the regions I ($\infty > r > r_+^\Phi$), II ($r_+^\Phi > r > r_+^2$) and III ($r_-^\Phi > r > 0$). As in the Schwarzschild, Reissner–Nordstrom and Kerr cases these singularities can be removed by introducing suitable coordinates and extending the manifold to obtain a maximal analytic extension \[33, 34\]. We have similar regions as in the Reissner–Nordstrom space–time, but with just only one possibility $\varepsilon < 1$ instead of three relations for static electro–vacuum cases ($e^2 < m^2, e^2 = m^2, e^2 > m^2$; where $e$ and $m$ are correspondingly the electric charge and mass of the point particle in the Reissner–Nordstrom metric). So, we may consider the usual Penrose’s diagrams as for a particular case of the Reissner–Nordstrom space–time but keeping in mind that such diagrams and horizons have an additional polarizations and parametrization on an angular coordinate.

We can proceed in steps analogous to those in the Schwarzschild case (see details, for instance, in Ref. [37]) in order to construct the maximally extended manifold. The first step is to introduce a new coordinate

$$r^\parallel = \int dr \left(1 - \frac{2m}{r} + \frac{\varepsilon}{r^2}\right)^{-1}$$

for $r > r_+^1$ and find explicitly the coordinate

$$r^\parallel = r + \frac{(r_+^1)^2}{r_+^1 - r_+^1} \ln(r - r_+^1) - \frac{(r_+^1)^2}{r_+^1 - r_+^1} \ln(r - r_-^1),$$

(57)

where $r_+^1 = r_+^\Phi$ with $\Phi = 1$. If $r$ is expressed as a function on $\xi$, than $r^\parallel$ can be also expressed as a function on $\xi$ depending additionally on some parameters.

Defining the advanced and retarded coordinates, $v = t + r^\parallel$ and $w = t - r^\parallel$, with corresponding elongated differentials

$$\delta v = \delta t + dr^\parallel$$

and

$$\delta w = \delta t - dr^\parallel$$

the metric (38) takes the form

$$\delta s^2 = -r^2(\xi)q^{[u]}(\xi)d\theta^2 - \eta_3(\xi, \varphi_0) f_3(\xi, \varphi_0)^2(\xi) \sin^2 \theta \delta \varphi^2 + \left(1 - \frac{2m}{r(\xi)} + \frac{\varepsilon \Phi_4(r, \varphi_0)}{r^2(\xi)}\right) \delta v \delta w,$$

(58)

where (in general, in non–explicit form) $r(\xi)$ is a function of type $r(\xi) = r(r^\parallel) = r(v, w)$. Introducing new coordinates $(v'', w'')$ by

$$v'' = \arctan \left[\exp \left(\frac{r_+^1 - r_+^1}{4(r_+^1)^2} v\right)\right], w'' = \arctan \left[-\exp \left(-\frac{r_+^1 + r_+^1}{4(r_+^1)^2} w\right)\right]$$

Defining $r$ by

$$\tan v'' \tan w'' = -\exp \left[\frac{r_+^1 - r_+^1}{2(r_+^1)^2} r\right] \sqrt{\frac{r - r_+^1}{(r - r_+^1)\chi}}, \chi = \left(\frac{r_+^1}{r_+^1}\right)^2$$

and multiplying (58) on the conformal factor we obtain

$$\delta s^2 = -r^2 q^{[u]}(r) d\theta^2 - \eta_3(r, \varphi_0) f_3(r, \varphi_0) r^2 \sin^2 \theta \delta \varphi^2$$

$$+ 64 \frac{(r_+^1)^4}{(r_+^1 - r_+^1)^2} (1 - \frac{2m}{r(\xi)} + \frac{\varepsilon \Phi_4(r, \varphi_0)}{r^2(\xi)}) \delta v'' \delta w'' ,$$

(59)
As particular cases, we may choose \( \eta_3(r, \varphi) \) as the condition of vanishing of the metric coefficient before \( \delta v'' \delta w'' \) will describe a horizon parametrized by a resolution ellipsoid hypersurface. We emphasize that quadratic elements (58) and (59) have respective coefficients as the metrics investigated in Refs. [29, 30] but the polarized cosmological constants introduce not only additional polarizing factors \( q^{[vi]}(r) \) and \( f_3(r, \varphi_0) \) but also elongate the anholonomic frames in a different manner.

The maximal extension of the Schwarzschild metric deformed by a small parameter \( \varepsilon \) (for ellipsoid configurations treated as the eccentricity), i.e. the extension of the metric (44), is defined by taking (59) as the metric on the maximal manifold on which this metric is of smoothly class \( C^2 \). The Penrose diagram of this static but locally anisotropic space–time, for any fixed angular value \( \varphi_0 \) is similar to the Reissner–Nordstrom solution, for the case \( e^2 \to \varepsilon \) and \( e^2 < m^2 \) (see, for instance, Ref. [37]). There are an infinite number of asymptotically flat regions of type I, connected by intermediate regions II and III, where there is still an irremovable singularity at \( r = 0 \) for every region III. We may travel from a region I to another ones by passing through the ’wormholes’ made by anisotropic deformations (ellipsoid off–diagonality of metrics, or anholonomy) like in the Reissner–Nordstrom universe because \( \sqrt{\varepsilon} \) may model an effective electric charge. One can not turn back in such a travel. Of course, this interpretation holds true only for a corresponding smoothly class of polarization functions. For instance, if the cosmological constant is periodically polarized from a string model, see the formula (50), one could be additional resonances, aperiodicity and singularities.

It should be noted that the metric (59) can be analytic everywhere except at \( r = r_1^- \). We may eliminate this coordinate degeneration by introducing another new coordinates

\[
v'' = \arctan \left[ \exp \left( \frac{r_+ - r_1^-}{2n_0(r_1^-)^2} v \right) \right], w'' = \arctan \left[ -\exp \left( \frac{-r_+ + r_1^-}{2n_0(r_1^-)^2} w \right) \right],
\]

where the integer \( n_0 \geq (r_1^-)^2/(r_1^+)^2 \). In these coordinates, the metric is analytic everywhere except at \( r = r_1^+ \) where it is degenerate. This way the space–time manifold can be covered by an analytic atlas by using coordinate carts defined by \((v'', w'', \theta, \varphi)\) and \((v'', w'', \theta, \varphi)\). Finally, we note that the analytic extensions of the deformed metrics were performed with respect to anholonomic frames which distinguish such constructions from those dealing only with holonomic coordinates, like for the usual Reissner–Nördstrom and Kerr metrics. We stated the conditions when on ’radial’ like coordinates we preserve the main properties of the well know black hole solutions but in our case the metrics are generic off–diagonal and with vacuum gravitational polarizations.

### 4.4 Geodesics on static polarized ellipsoid backgrounds

We analyze the geodesic congruence of the metric (44) with the data (42) modified by polarized cosmological constant, for simplicity, being linear on \( \varepsilon \), by introducing the effective Lagrangian (for instance, like in Ref. [16])

\[
2L = g_{\alpha \beta} \frac{\delta u^\alpha}{ds} \frac{\delta u^\beta}{ds} = -\left( 1 - \frac{2m}{r} + \frac{\varepsilon}{r^2} \right)^{-1} \left( \frac{dr}{ds} \right)^2 r^2 q^{[vi]}(r) \left( \frac{d\theta}{ds} \right)^2 - r^2 \eta_3(r, \varphi) f_3(r, \varphi) r^2 \sin^2 \theta \left( \frac{d\varphi}{ds} \right)^2 - \left( 1 - \frac{2m}{r} + \frac{\varepsilon \Phi_4}{r^2} \right) \left( \frac{dt}{ds} + \varepsilon \bar{n}_1 \frac{dr}{ds} \right)^2, \tag{60}
\]
for \( r = r(\xi) \).

The corresponding Euler–Lagrange equations,

\[
\frac{d}{ds} \frac{\partial L}{\partial \dot{u}^a} - \frac{\partial L}{\partial u^a} = 0
\]

are

\[
\frac{d}{ds} \left[ -r^2 q^{[w]}(r) \frac{d\theta}{ds} \right] = -\eta_3 f_3 r^2 \sin \theta \cos \theta \left( \frac{d\varphi}{ds} \right)^2,
\]

\[
\frac{d}{ds} \left[ -\eta_3 f_3 r^2 \frac{d\varphi}{ds} \right] = -(\eta_3 f_3) \frac{r^2}{2} \sin^2 \theta \left( \frac{d\varphi}{ds} \right)^2 + \frac{\varepsilon}{2} \left( 1 - \frac{2m}{r} \right) \left[ \frac{\Phi_4}{r^2} \left( \frac{dt}{ds} \right)^2 + \hat{n}_1 \frac{d\xi}{ds} \right]
\]

and

\[
\frac{d}{ds} \left[ (1 - \frac{2m}{r} + \varepsilon \Phi_4/r^2) \left( \frac{dt}{ds} + \varepsilon \hat{n}_1 \frac{d\xi}{ds} \right) \right] = 0,
\]

where, for instance, \( \Phi_4^* = \partial \Phi_4/\partial \varphi \) we have omitted the variations for \( d\xi/ds \) which may be found from (60). The sistem of equations (60)–(62) transform into the usual system of geodesic equations for the Schwarzschild space–time if \( \varepsilon \to 0 \) and \( q^{[w]}, \eta_3, f_3 \to 1 \) which can be solved exactly [16]. For nontrivial values of the parameter \( \varepsilon \) and polarizations \( \eta_3, f_3 \) even to obtain some decompositions of solutions on \( \varepsilon \) for arbitrary \( \eta_3 \) and \( n_{1[1,2]} \), see (38), is a cumbersome task. In spite of the fact that with respect to anholonomic frames the metrics (38) and/or (44) has their coefficients being very similar to the Reissner–Nordstom solution. The geodesic behaviour, in our anisotropic cases, is more sophisticate because of anholonomy, polarization of constants and coefficients and "elongation" of partial derivatives. For instance, the equations (61) state that there is not any angular on \( \varphi \), conservation law if \( (\eta_3 f_3)^* \neq 0 \), even for \( \varepsilon \to 0 \) (which holds both for the Schwarzschild and Reissner–Nordstom metrics). One follows from the equation (62) the existence of an energy like integral of motion, \( E = E_0 + \varepsilon E_1 \), with

\[
E_0 = \left( 1 - \frac{2m}{r} \right) \frac{dt}{ds}, \quad E_1 = \frac{\Phi_4}{r^2} \frac{dt}{ds} + \left( 1 - \frac{2m}{r} \right) \hat{n}_1 \frac{d\xi}{ds}.
\]

The introduced anisotropic deformations of congruences of Schwarzschild’s space–time geodesics mantain the known behaviour in the vecinity of the horizon hypersurface defined by the condition of vanishing of the coefficient \( (1 - 2m/r + \varepsilon \Phi_4/r^2) \) in (59). The simplest way to prove this is to consider radial null geodesics in the "equatorial plane", which satisfy the condition (60) with \( \theta = \pi/2, d\theta/ds = 0, d^2\theta/ds^2 = 0 \) and \( d\varphi/ds = 0 \), from which follows that

\[
\frac{dr}{dt} = \pm \left( 1 - \frac{2m}{r} + \varepsilon \Phi_0 \right) \left[ 1 + \varepsilon \hat{n}_1 d\varphi \right].
\]

The integral of this equation, for every fixed value \( \varphi = \varphi_0 \) is

\[
t = \pm \int_0^{\varphi_0} + \varepsilon \int \frac{\Phi_4(r, \varphi_0) - 1}{2 \left( r^2 - 2mr \right)^2} - \hat{n}_1(r, \varphi_0) \right] dr.
\]

24
where the coordinate $r^\parallel$ is defined in equation (57). In this formula the term proportional to $\varepsilon$ can have non–singular behaviour for a corresponding class of polarizations $\lambda_4$, see the formulas (33). Even the explicit form of the integral depends on the type of polarizations $\eta_3 (r, \varphi_0), f_3 (r, \varphi_0)$ and values $n_{1,2} (r)$, which results in some small deviations of the null–geodesics, we may conclude that for an in–going null–ray the coordinate time $t$ increases from $-\infty$ to $+\infty$ as $r$ decreases from $+\infty$ to $r_1^+$, decreases from $+\infty$ to $-\infty$ as $r$ further decreases from $r_1^+$ to $r_1^-$, and increases again from $-\infty$ to a finite limit as $r$ decreases from $r_1^-$ to zero.

We have a similar behaviour as for the Reissner–Nordstrom solution but with some additional anisotropic contributions being proportional to $\varepsilon$. Here we also note that as $dt/ds$ tends to $+\infty$ for $r \to r_1^+ + 0$ and to $-\infty$ as $r \to r_1^- + 0$, any radiation received from infinity appear to be infinitely red–shifted at the crossing of the event horizon and infinitely blue–shifted at the crossing of the Cauchy horizon.

The mentioned properties of null–geodesics allow us to conclude that the metric (31) (equivalently, (38)) with the data (42) and their maximal analytic extension (59) really define a black hole static solution which is obtained by anisotropic small deformations on $\varepsilon$ and renormalization by $\eta_3 f_3$ of the Schwarzchild solution (for a corresponding type of deformations the horizon of such black holes is defined by ellipsoid hypersurfaces). We call such objects as black ellipsoids, or black rotoids. They exists in the framework of general relativity as certain solutions of the Einstein equations defined by static generic off–diagonal metrics and associated anholonomic frames or can be induced by polarized cosmological constants. This property disinguishes them from similar configurations of Reissner–Norstrom type (which are static electrovacuum solutions of the Einstein–Maxwell equations) and of Kerr type rotating configurations, with ellipsoid horizon, also defined by off–diagonal vacuum metrics (here we emphasized that the spherical coordinate system is associated to a holonomic frame which is a trivial case of anholonomic bases). By introducing the polarized cosmological constants, the anholonomic character of N–adapted frames allow to construct solutions being very different from the black hole solutions in (anti) de Sitter spacetimes. We selected here a class of solutions where cosmological factors correspond to some additional polarizations but do not change the singularity structure of black ellipsoid solutions.

The metric (31) and its analytic extensions do not posses Killing symmetries being deformed by anholonomic transforms. Nevertheless, we can associate to such solutions certain noncommutative symmetries (10). Taking the data (42) and formulas (8), we compute the corresponding nontrivial anholonomy coefficients

$$w^{[N]}_{a\beta} = -w^{[N]}_{24} = \partial n_2 (\xi, \varphi) / \partial \varphi = n^* \partial_2 (\xi, \varphi)$$

with $n_2$ defined by (12). Our solutions are for 4D configuration. So for $n + m = 4$, the condition $k^2 - 1 = n + m$ can not satisfied in integer numbers. We may trivially extend the dimensions like $n' = 6$ and $m' = 2$ and for $k = 3$ to consider the Lie group $SL (3, \mathbb{C})$ noncommutativity with corresponding values of $Q_{a\beta}$ and structure constants $f^{g}_{a\beta}$, see (23). An extension $w^{[N]}_{a\beta} \to W^{[N]}_{a\beta}$ may be performed by stating the N–deformed "structure" constants (26). $W^{[N]}_{a\beta} = f^{g}_{a\beta} + w^{[N]}_{a\beta}$, with only two nontrivial values of $w^{[N]}_{a\beta}$ given by (63). In a similar manner we can compute the anholonomy coefficients for the black ellipsoid metric with cosmological constant contributions (14).
5 Perturbations of Anisotropic Black Holes

The stability of black ellipsoids was proven in Ref. [36]. A similar proof may hold true for a class of metrics with anholonomic noncommutative symmetry and possible complexification of some off–diagonal metric and tetrads coefficients [10]. In this section we reconsider the perturbation formalism and stability proofs for rotoid metrics defined by polarized cosmological constants.

5.1 Metrics describing anisotropic perturbations

We consider a four dimensional pseudo–Riemannian quadratic linear element

\[ ds^2 = \Omega(r, \varphi) \left[ -\left(1 - \frac{2m}{r} + \frac{\varepsilon}{r^2}\right)^{-1} dr^2 - r^2 q^{[v]}(r)d\theta^2 - \eta_3^{[v]}(r, \theta, \varphi)r^2 \sin^2 \theta \delta \varphi^2 \right] + \left[1 - \frac{2m}{r} + \frac{\varepsilon}{r^2}\eta(r, \varphi)\right] \delta t^2, \]

(64)

\[ \eta_3^{[v]}(r, \theta, \varphi) = \eta_3(r, \theta, \varphi)f_3(r, \theta, \varphi) \]

with

\[ \delta \varphi = d\varphi + \varepsilon w_1(r, \varphi)dr, \quad \delta t = dt + \varepsilon n_1(r, \varphi)dr, \]

where the local coordinates are denoted \( u = \{u^\alpha = (r, \theta, \varphi, t)\} \) (the Greek indices \( \alpha, \beta, ... \) will run the values 1,2,3,4), \( \varepsilon \) is a small parameter satisfying the conditions \( 0 \leq \varepsilon \ll 1 \) (for instance, an eccentricity for some ellipsoid deformations of the spherical symmetry) and the functions \( \Omega(r, \varphi), q(r), \eta_3(r, \theta, \varphi) \) and \( \eta(\theta, \varphi) \) are of necessary smooth class. The metric (64) is static, off–diagonal and transforms into the usual Schwarzschild solution if \( \varepsilon \to 0 \) and \( \Omega, q^{[v]}, \eta_3^{[v]} \to 1 \). For vanishing cosmological constants, it describes at least two classes of static black hole solutions generated as small anhlonomic deformations of the Schwarzschild solution [1, 2, 29, 30, 31, 36] but models also nontrivial vacuum polarized cosmological constants.

We can apply the perturbation theory for the metric (64) (not paying a special attention to some particular parametrization of coefficients for one or another class of anisotropic black hole solutions) and analyze its stability by using the results of Ref. [16] for a fixed anisotropic direction, i.e. by imposing certain anholonomic frame constraints for an angle \( \varphi = \varphi_0 \) but considering possible perturbations depending on three variables \( (u^1 = x^1 = r, u^2 = x^2 = \theta, u^4 = t) \). We suppose that if we prove that there is a stability on perturbations for a value \( \varphi_0 \), we can analyze in a similar manner another values of \( \varphi \). A more general perturbative theory with variable anisotropy on coordinate \( \varphi \), i.e. with dynamical anholonomic constraints, connects the approach with a two dimensional inverse problem which makes the analysis more sophisticate. There have been not elaborated such analytic methods in the theory of black holes.

It should be noted that in a study of perturbations of any spherically symmetric system and, for instance, of small ellipsoid deformations, without any loss of generality, we can restrict our considerations to axisymmetric modes of perturbations. Non–axisymmetric modes of perturbations with an \( e^{i n \varphi} \) dependence on the azimuthal angle \( \varphi \) (\( n \) being an integer number)
can be deduced from modes of axisymmetric perturbations with \( n = 0 \) by suitable rotations since there are not preferred axes in a spherically symmetric background. The ellipsoid like deformations may be included into the formalism as some low frequency and constrained perturbations.

We develop the black hole perturbation and stability theory as to include into consideration off–diagonal metrics with the coefficients polarized by cosmological constants. This is the main difference comparing to the paper \[36\]. For simplicity, in this section, we restrict our study only to fixed values of the coordinate \( \varphi \) assuming that anholonomic deformations are proportional to a small parameter \( \varepsilon \); we shall investigate the stability of solutions only by applying the one dimensional inverse methods.

We state a quadratic metric element

\[
\begin{align*}
 ds^2 &= -e^{2\mu_1}(du^1)^2 - e^{2\mu_2}(du^2)^2 - e^{2\mu_3}(\delta u^3)^2 + e^{2\mu_4}(\delta u^4)^2, \\
 \delta u^3 &= d\varphi - q_1 dx^1 - q_2 dx^2 - \omega dt, \\
 \delta u^4 &= dt + n_1 dr
\end{align*}
\]  

(65)

where

\[
\begin{align*}
 \mu_\alpha(x^k, t) &= \mu^{(\varepsilon)}_\alpha(x^k, \varphi_0) + \delta \mu^{(\varepsilon)}_\alpha(x^k, t), \\
 q_\iota(x^k, t) &= q^{(\varepsilon)}_\iota(r, \varphi_0) + \delta q^{(\varepsilon)}_\iota(x^k, t), \quad \omega(x^k, t) = 0 + \delta \omega^{(\varepsilon)}(x^k, t)
\end{align*}
\]  

(66)

with

\[
\begin{align*}
 e^{2\mu_1^{(\varepsilon)}} &= \Omega(r, \varphi_0)(1 - \frac{2m}{r} + \frac{\varepsilon}{r^2})^{-1}, \quad e^{2\mu_2^{(\varepsilon)}} = \Omega(r, \varphi_0)q^{[v]}(r)r^2, \\
 e^{2\mu_3^{(\varepsilon)}} &= \Omega(r, \varphi_0)r^2 \sin^2 \theta \eta^{[v]}_3(r, \varphi_0), \quad e^{2\mu_4^{(\varepsilon)}} = 1 - \frac{2m}{r} + \frac{\varepsilon}{r^2} \eta(r, \varphi_0),
\end{align*}
\]  

(67)

and some non–trivial values for \( q^{(\varepsilon)}_\iota \) and \( \varepsilon n_1 \),

\[
\begin{align*}
 q^{(\varepsilon)}_\iota &= \varepsilon w_\iota(r, \varphi_0), \\
 n_1 &= \varepsilon \left( n_{1[1]}(r) + n_{1[2]}(r) \int_0^{\varphi_0} \eta_3(r, \varphi) d\varphi \right).
\end{align*}
\]

We have to distinguish two types of small deformations from the spherical symmetry. The first type of deformations, labeled with the index (\( \varepsilon \)) are generated by some \( \varepsilon \)-terms which define a fixed ellipsoid like configuration and the second type ones, labeled with the index (\( \varsigma \)), are some small linear fluctuations of the metric coefficients.

The general formulas for the Ricci and Einstein tensors for metric elements of class (65) with \( w_\iota, n_1 = 0 \) are given in \[16\]. We compute similar values with respect to anholonomic frames, when, for a conventional splitting \( u^\alpha = (x^i, y^a) \), the coordinates \( x^i \) and \( y^a \) are treated respectively as holonomic and anholonomic ones. In this case the partial derivatives \( \partial / \partial x^i \) must be changed into certain ’elongated’ ones

\[
\begin{align*}
 \frac{\partial}{\partial x^1} &\rightarrow \frac{\delta}{\partial x^1} = \frac{\partial}{\partial x^1} - w_1 \frac{\partial}{\partial \varphi} - n_1 \frac{\partial}{\partial t}, \\
 \frac{\partial}{\partial x^2} &\rightarrow \frac{\delta}{\partial x^2} = \frac{\partial}{\partial x^2} - w_2 \frac{\partial}{\partial \varphi},
\end{align*}
\]

27
In the ansatz (65), the anholonomic contributions of \( W_i \) are included in the coefficients \( q_i(x^k,t) \). For convenience, we give present below the necessary formulas for \( R_{\alpha\beta} \) (the Ricci tensor) and \( G_{\alpha\beta} \) (the Einstein tensor) computed for the ansatz (65) with three holonomic coordinates \((r, \theta, \varphi)\) and on anholonomic coordinate \( t \) (in our case, being time like), with the partial derivative operators

\[
\partial_1 \to \delta_1 = \frac{\partial}{\partial r} w_1 \frac{\partial}{\partial \varphi}, \quad \partial_2 = \frac{\partial}{\partial \theta} - n_1 \frac{\partial}{\partial t}, \quad \delta_2 = \frac{\partial}{\partial \theta} - w_2 \frac{\partial}{\partial \varphi}, \quad \delta_3 = \frac{\partial}{\partial \varphi},
\]

and for a fixed value \( \varphi_0 \).

A general perturbation of an anisotropic black-hole described by a quadratic line element (65) results in some small quantities of the first order \( \omega \) and \( q_i \), inducing a dragging of frames and imparting rotations, and in some functions \( \mu_\alpha \) with small increments \( \delta \mu_\alpha \), which do not impart rotations. Some coefficients contained in such values are proportional to \( \varepsilon \), another ones are considered only as small quantities. The perturbations of metric are of two generic types: axial and polar one. We shall investigate them separately in the next two subsection after we shall have computed the coefficients of the Ricci tensor.

We compute the coefficients of the the Ricci tensor as

\[
R^\alpha_{\beta\gamma\alpha} = R_{\beta\gamma}
\]

and of the Einstein tensor as

\[
G_{\beta\gamma} = R_{\beta\gamma} - \frac{1}{2} g_{\beta\gamma} R
\]

for \( R = g^{\beta\gamma} R_{\beta\gamma} \). Straightforward computations for the quadratic line element (65) give

\[
R_{11} = -e^{-2\mu_1} [\delta_1^2 (\mu_3 + \mu_4 + \mu_2) + \delta_1 \mu_3 \delta_1 (\mu_3 - \mu_1) + \delta_1 \mu_2 \delta_1 (\mu_2 - \mu_1) + \delta_1 \mu_4 \delta_1 (\mu_4 - \mu_1)] - e^{-2\mu_2} [\delta_2^2 \mu_1 + \delta_2 \mu_1 \delta_2 (\mu_3 + \mu_4 + \mu_1 - \mu_2)] +
\]

\[
e^{-2\mu_4} [\delta_4 \mu_1 + \delta_4 \mu_1 \delta_4 (\mu_3 - \mu_4 + \mu_1 + \mu_2)] - \frac{1}{2} e^{2(\mu_3 - \mu_1)} [e^{-2\mu_2} Q_{12}^2 + e^{-2\mu_4} Q_{14}^2],
\]

\[
R_{12} = -e^{-\mu_1 - \mu_2} [\delta_2 \delta_1 (\mu_3 + \mu_2) - \delta_2 \mu_1 \delta_1 (\mu_3 + \mu_1) - \delta_1 \mu_2 \partial_4 (\mu_3 + \mu_1) + \delta_1 \mu_3 \delta_2 \mu_3 + \delta_1 \mu_4 \delta_2 \mu_4 + \frac{1}{2} e^{2\mu_3 - 2\mu_4 - \mu_1 - \mu_2} Q_{14} Q_{24},
\]

\[
R_{31} = -\frac{1}{2} e^{2\mu_3 - 4\mu_4 - \mu_2} [\delta_2 (e^{3\mu_3 + \mu_4 - \mu_1 - \mu_2} Q_{21}) + \partial_4 (e^{3\mu_3 - \mu_4 + \mu_2 - \mu_1} Q_{41})],
\]

\[
R_{33} = -e^{-2\mu_1} [\delta_1^2 \mu_3 + \delta_1 \mu_3 \delta_1 (\mu_3 + \mu_4 + \mu_2 - \mu_1)] -
\]

\[
e^{-2\mu_2} [\delta_2^2 \mu_3 + \delta_2 \mu_3 \delta_2 (\mu_3 + \mu_4 - \mu_2 + \mu_1)] + \frac{1}{2} e^{2(\mu_3 - \mu_1 - \mu_2)} Q_{12}^2 +
\]

\[
e^{-2\mu_4} [\delta_4 \mu_3 + \delta_4 \mu_3 \delta_4 (\mu_3 - \mu_4 + \mu_2 + \mu_1)] - \frac{1}{2} e^{2(\mu_3 - \mu_4)} [e^{-2\mu_2} Q_{24}^2 + e^{-2\mu_4} Q_{14}^2],
\]

\[
R_{41} = -e^{-\mu_1 - \mu_4} [\partial_4 \delta_1 (\mu_3 + \mu_2) + \delta_1 \mu_3 \partial_4 (\mu_3 - \mu_1) + \delta_1 \mu_2 \partial_4 (\mu_2 - \mu_1) - \delta_1 \mu_4 \partial_4 (\mu_3 + \mu_2)] + \frac{1}{2} e^{2\mu_3 - 4\mu_4 - 2\mu_2} Q_{12} Q_{34},
\]

\[
28
\]
\[ R_{43} = -\frac{1}{2} e^{2\mu_3 - \mu_1 - \mu_2} [\delta_1 (e^{3\mu_3 - \mu_1 - \mu_2} Q_{14}) + \delta_2 (e^{3\mu_3 - \mu_1 - \mu_2} Q_{24})], \]

\[ R_{44} = -e^{-2\mu_4} \left[ \delta_{44}^2 (\mu_1 + \mu_2 + \mu_3) + \partial_4 \mu_3 \partial_4 (\mu_3 - \mu_4) + \partial_4 \mu_1 \partial_4 (\mu_1 - \mu_4) + \partial_4 \mu_2 \partial_4 (\mu_2 - \mu_4) \right] + e^{-2\mu_1} [\delta_{14}^2 \mu_4 + \delta_1 \mu_1 \delta_1 (\mu_3 + \mu_4 - \mu_1 + \mu_2)] +
\]
\[ e^{-2\mu_2} [\delta_{22}^2 \mu_4 + \delta_2 \mu_2 \delta_2 (\mu_3 + \mu_4 - \mu_1 + \mu_2)] - \frac{1}{2} e^{2(\mu_3 - \mu_4)} [e^{-2\mu_1} Q_{14}^2 + e^{-2\mu_2} Q_{24}^2], \]

where the rest of coefficients are defined by similar formulas with a corresponding changing of indices and partial derivative operators, \( R_{22}, R_{42} \) and \( R_{32} \) is like \( R_{11}, R_{41} \) and \( R_{31} \) with with changing the index \( 1 \to 2 \). The values \( Q_{ij} \) and \( Q_{i4} \) are defined respectively

\( Q_{ij} = \delta_j q_i - \delta_i q_j \) and \( Q_{i4} = \partial_4 q_i - \delta_i \omega. \)

The nontrivial coefficients of the Einstein tensor are

\[ G_{11} = -e^{-2\mu_4} \left[ \delta_{22}^2 (\mu_3 + \mu_4) + \delta_2 (\mu_3 + \mu_4) \delta_2 (\mu_4 - \mu_2) + \delta_2 \mu_3 \delta_2 \mu_3 \right] -
\]
\[ e^{-2\mu_1} [\delta_{14}^2 (\mu_3 + \mu_4) + \partial_4 \mu_3 \partial_4 (\mu_3 - \mu_4) + \delta_4 \mu_3 \partial_4 \mu_3] +
\]
\[ e^{-2\mu_1} [\delta_{14}^2 (\mu_3 + \mu_4) + \delta_1 (\mu_3 + \mu_2) + \delta_1 \mu_3 \delta_1 \mu_2] -
\]
\[ \frac{1}{4} e^{2\mu_3} [e^{-2(\mu_1 + \mu_2)} Q_{12}^2 - e^{-2(\mu_1 + \mu_4)} Q_{14}^2 + e^{-2(\mu_2 + \mu_3)} Q_{24}^2], \]

\[ G_{33} = e^{-2\mu_4} \left[ \delta_{11}^2 (\mu_4 + \mu_2) + \delta_1 \mu_4 \delta_1 (\mu_4 - \mu_1 + \mu_2) + \delta_1 \mu_2 \delta_1 (\mu_2 - \mu_1) \right] +
\]
\[ e^{-2\mu_2} [\delta_{22}^2 (\mu_4 + \mu_1) + \delta_2 (\mu_4 - \mu_2 + \mu_1) + \delta_2 \mu_1 \delta_2 (\mu_1 - \mu_2)] -
\]
\[ e^{-2\mu_1} [\delta_{14}^2 (\mu_1 + \mu_2) + \partial_4 \mu_1 \partial_4 (\mu_1 - \mu_4) + \delta_4 \mu_1 \partial_4 \mu_2 + \delta_4 \mu_1 \partial_4 \mu_4] +
\]
\[ 3 \frac{1}{4} e^{2\mu_3} [e^{-2(\mu_1 + \mu_4)} Q_{12}^2 - e^{-2(\mu_1 + \mu_4)} Q_{14}^2 - e^{-2(\mu_2 + \mu_3)} Q_{24}^2], \]

\[ G_{44} = e^{-2\mu_1} [\delta_{11}^2 (\mu_3 + \mu_2) + \delta_1 \mu_3 \delta_1 (\mu_3 - \mu_1 + \mu_2) + \delta_1 \mu_2 \delta_1 (\mu_2 - \mu_1)] -
\]
\[ e^{-2\mu_2} [\delta_{22}^2 (\mu_3 + \mu_1) + \delta_2 (\mu_3 - \mu_2 + \mu_1) + \delta_2 \mu_1 \delta_2 (\mu_1 - \mu_2)] - \frac{1}{4} e^{2(\mu_3 - \mu_1 - \mu_2)} Q_{12}^2
\]
\[ + e^{-2\mu_4} [\partial_4 \mu_3 \partial_4 (\mu_1 + \mu_2) + \partial_4 \mu_1 \partial_4 \mu_2] - \frac{1}{4} e^{2(\mu_3 - \mu_4)} [e^{-2\mu_1} Q_{14}^2 - e^{-2\mu_2} Q_{24}^2]. \]

The component \( G_{22} \) is to be found from \( G_{11} \) by changing the index \( 1 \to 2 \). We note that the formulas (69) transform into similar ones from Ref. [33] if \( \delta_2 \to \partial_2 \).

### 5.2 Axial metric perturbations

Axial perturbations are characterized by non–vanishing \( \omega \) and \( q_i \) which satisfy the equations

\[ R_{3i} = 0, \]
see the explicit formulas for such coefficients of the Ricci tensor in \[\text{(68)}\]. The resulting equations governing axial perturbations, \(\delta R_{31} = 0, \delta R_{32} = 0\), are respectively

\[
\begin{align*}
\delta_2 \left( e^{3\mu_i} + \frac{\mu_i^2}{2} - \mu_i - \nu_i - \mu_{\nu_i} \right) Q_{12} &= -e^{3\mu_i - \mu_i^2} - \mu_i + \mu_{\nu_i} \partial_4 Q_{14}, \\
\delta_1 \left( e^{3\mu_i} + \frac{\mu_i^2}{2} - \mu_i - \nu_i - \mu_{\nu_i} \right) Q_{12} &= e^{3\mu_i - \mu_i^2 + \mu_i + \nu_i} \partial_4 Q_{24},
\end{align*}
\]

where

\[
Q_{ij} = \delta_i q_j - \delta_j q_i, \quad Q_{i4} = \partial_4 q_i - \delta_i \omega
\]

and for \(\mu_i\) there are considered unperturbed values \(\mu_i^{(e)}\). Introducing the values of coefficients \[\text{[66]}\] and \[\text{[67]}\] and assuming that the perturbations have a time dependence of type \(\exp(i \sigma t)\) for a real constant \(\sigma\), we rewrite the equations \[\text{(70)}\]

\[
\frac{1 + \varepsilon \left( \frac{1}{\Delta} + 3r^2 \phi / 2 \right)}{r^4 \sin^3 \theta (\eta_3^{[v]})^{3/2}} \partial_2 Q^{(\eta)} = -i \sigma \delta_r \omega - \sigma^2 q_1, \tag{72}
\]

\[
\frac{\Delta}{r^4 \sin^3 \theta (\eta_3^{[v]})^{3/2}} \delta_1 \left\{ Q^{(\eta)} \left[ 1 + \frac{\varepsilon}{2} \left( \frac{\eta - 1}{\Delta} - r^2 \phi \right) \right] \right\} = i \sigma \delta_\theta \omega + \sigma^2 q_2 \tag{73}
\]

for

\[
Q^{(\eta)}(r, \theta, \varphi_0, t) = \Delta Q_{12} \sin^3 \theta = \Delta \sin^3 \theta (\partial_2 q_1 - \delta_1 q_2), \Delta = r^2 - 2mr,
\]

where \(\phi = 0\) for solutions with \(\Omega = 1\) and \(\phi(r, \varphi) = \eta_3^{[v]}(r, \theta, \varphi) \sin^2 \theta\), i.e. \(\eta_3(r, \theta, \varphi) \sim \sin^{-2} \theta\) for solutions with \(\Omega = 1 + \varepsilon\).

We can exclude the function \(\omega\) and define an equation for \(Q^{(\eta)}\) if we take the sum of the \[\text{(72)}\] subjected by the action of operator \(\partial_2\) and of the \[\text{(73)}\] subjected by the action of operator \(\delta_1\). Using the relations \[\text{(71)}\], we write

\[
r^4 \partial_1 \left\{ \frac{\Delta}{r^4 (\eta_3^{[v]})^{3/2}} \left[ \delta_1 \left[ Q^{(\eta)} + \frac{\varepsilon}{2} \left( \frac{\eta - 1}{\Delta} - r^2 \phi \right) \right] \right] \right\} + \sin^3 \theta \partial_2 \left[ \frac{1 + \varepsilon \left( \frac{1}{\Delta} + 3r^2 \phi / 2 \right)}{\sin^3 \theta (\eta_3^{[v]})^{3/2}} \partial_2 Q^{(\eta)} \right] + \frac{\sigma^2 r^4}{\Delta (\eta_3^{[v]})^{3/2}} Q^{(\eta)} = 0.
\]

The solution of this equation is searched in the form \(Q^{(\eta)} = Q + \varepsilon Q^{(1)}\) which results in

\[
r^4 \partial_1 \left( \frac{\Delta}{r^4 (\eta_3^{[v]})^{3/2}} \partial_1 Q \right) + \sin^3 \theta \partial_2 \left( \frac{1}{\sin^3 \theta (\eta_3^{[v]})^{3/2}} \partial_2 Q \right) + \frac{\sigma^2 r^4}{\Delta (\eta_3^{[v]})^{3/2}} Q = \varepsilon A(r, \theta, \varphi_0), \tag{74}
\]

where

\[
A(r, \theta, \varphi_0) = r^4 \partial_1 \left( \frac{\Delta}{r^4 (\eta_3^{[v]})^{3/2}} n_1 \right) \frac{\partial Q}{\partial t} - r^4 \partial_1 \left( \frac{\Delta}{r^4 (\eta_3^{[v]})^{3/2}} \partial_1 Q^{(1)} \right) - \sin^3 \theta \partial_2 \left[ \frac{1 + \varepsilon \left( \frac{1}{\Delta} + 3r^2 \phi / 2 \right)}{\sin^3 \theta (\eta_3^{[v]})^{3/2}} \partial_2 Q^{(1)} - \frac{\sigma^2 r^4}{\Delta (\eta_3^{[v]})^{3/2}} Q^{(1)} \right],
\]

30
with a time dependence like $\exp[i\sigma t]$

It is possible to construct different classes of solutions of the equation (74). At the first step we find the solution for $Q$ when $\varepsilon = 0$. Then, for a known value of $Q(r, \theta, \varphi_0)$ from

$$Q^{(n)} = Q + \varepsilon Q^{(1)},$$

we can define $Q^{(1)}$ from the equations (72) and (73) by considering the values proportional to $\varepsilon$ which can be written

$$\partial_1 Q^{(1)} = B_1 (r, \theta, \varphi_0),
\partial_2 Q^{(1)} = B_2 (r, \theta, \varphi_0).$$

The integrability condition of the system (75), $\partial_1 B_2 = \partial_2 B_1$ imposes a relation between the polarization functions $\eta_3, \eta, w_1$ and $n_1$ (for a corresponding class of solutions). In order to prove that there are stable anisotropic configurations of anisotropic black hole solutions, we may consider a set of polarization functions when $A(r, \theta, \varphi_0) = 0$ and the solution with $Q^{(1)} = 0$ is admitted. This holds, for example, if

$$\Delta n_1 = n_0 r^4 (\eta_3^{[v]})^{3/2}, \ n_0 = \text{const.}$$

In this case the axial perturbations are described by the equation

$$(\eta_3^{[v]})^{3/2} r^4 \partial_1 \left( \frac{\Delta}{r^4 (\eta_3^{[v]})^{3/2}} \partial_1 Q \right) + \sin^3 \theta \delta_2 \left( \frac{1}{\sin^3 \theta} \delta_2 Q \right) + \frac{\sigma^2 r^4}{\Delta} Q = 0$$

which is obtained from (74) for $\eta_3^{[v]} = \eta_3^{[v]} (r, \varphi_0)$, or for $\phi(r, \varphi_0) = \eta_3^{[v]} (r, \theta, \varphi_0) \sin^2 \theta$.

In the limit $\eta_3^{[v]} \to 1$ the solution of equation (76) is investigated in details in Ref. [16]. Here, we prove that in a similar manner we can define exact solutions for non–trivial values of $\eta_3^{[v]}$. The variables $r$ and $\theta$ can be separated if we substitute

$$Q(r, \theta, \varphi_0) = Q_0(r, \varphi_0) C_{l+2}^{-3/2}(\theta),$$

where $C_n^\nu$ are the Gegenbauer functions generated by the equation

$$\left[ \frac{d}{d\theta} \sin 2\nu \theta \frac{d}{d\theta} + n(n+2\nu) \sin 2\nu \theta \right] C_n^\nu(\theta) = 0.$$

The function $C_{l+2}^{-3/2}(\theta)$ is related to the second derivative of the Legendre function $P_l(\theta)$ by formulas

$$C_{l+2}^{-3/2}(\theta) = \sin^3 \theta \frac{d}{d\theta} \left[ \frac{1}{\sin \theta} \frac{dP_l(\theta)}{d\theta} \right].$$

The separated part of (76) depending on radial variable with a fixed value $\varphi_0$ transforms into the equation

$$(\eta_3^{[v]})^{3/2} \Delta \frac{d}{dr} \left( \frac{\Delta}{r^4 (\eta_3^{[v]})^{3/2}} \frac{dQ_0}{dr} \right) + \left( \sigma^2 - \frac{\mu^2 \Delta}{r^4} \right) Q_0 = 0,$$

(77)
where $\mu^2 = (l - 1)(l + 2)$ for $l = 2, 3, \ldots$. A further simplification is possible for $\eta_3^{[v]} = \eta_3^{[v]}(r, \varphi_0)$ if we introduce in the equation a new radial coordinate

$$r_\# = \int (\eta_3^{[v]})^{3/2}(r, \varphi_0)r^2dr$$

and a new unknown function $Z^{(n)} = r^{-1}Q_0(r)$. The equation for $Z^{(n)}$ is a Schrödinger like one-dimensional wave equation

$$\left(\frac{d^2}{dr^2 \#} + \frac{\sigma^2}{(\eta_3^{[v]})^{3/2}}\right)Z^{(n)} = V^{(n)}Z^{(n)} \tag{78}$$

with the potential

$$V^{(n)} = \frac{\Delta}{r^5(\eta_3^{[v]})^{3/2}} \left[ \mu^2 - r^4 \frac{d}{dr} \left( \frac{\Delta}{r^4(\eta_3^{[v]})^{3/2}} \right) \right] \tag{79}$$

and polarized parameter

$$\tilde{\sigma}^2 = \sigma^2/(\eta_3^{[v]})^{3/2}.$$

This equation transforms into the so-called Regge–Wheeler equation if $\eta_3^{[v]} = 1$. For instance, for the Schwarzschild black hole such solutions are investigated and tabulated for different values of $l = 2, 3$ and $4$ in Ref. [16].

We note that for static anisotropic black holes with nontrivial anisotropic conformal factor, $\Omega = 1 + \varepsilon \ldots$, even $\eta_3$ may depend on angular variable $\theta$ because of condition that $\phi(r, \varphi_0) = \eta_3^{[v]}(r, \theta, \varphi_0)\sin^2 \theta$ the equation transforms directly in (78) with $\mu = 0$ without any separation of variables $r$ and $\varphi$. It is not necessary in this case to consider the Gegenbauer functions because $Q_0$ does not depend on $\theta$ which corresponds to a solution with $l = 1$.

We may transform (78) into the usual form,

$$\left(\frac{d^2}{dr^2_*} + \sigma^2\right)\tilde{V}^{(n)} = \tilde{V}^{(n)}Z^{(n)}$$

if we introduce the variable

$$r_* = \int dr_\#(\eta_3^{[v]})^{-3/2}(r_\#, \varphi_0)$$

for $\tilde{V}^{(n)} = (\eta_3^{[v]})^{3/2}V^{(n)}$. So, the polarization function $\eta_3^{[v]}$, describing static anholonomic deformations of the Schwarzschild black hole, "renormalizes" the potential in the one-dimensional Schrödinger wave-equation governing axial perturbations of such objects.

We conclude that small static "ellipsoid" like deformations and polarizations of constants of spherical black holes (the anisotropic configurations being described by generic off-diagonal metric ansatz) do not change the type of equations for axial perturbations: one modifies the potential barrier,

$$V^{(-)} = \frac{\Delta}{r^3}\left[ (\mu^2 + 2) r - 6m \right] \longrightarrow V^{(n)}$$

and re-defines the radial variables

$$r_* = r + 2m \ln (r/2m - 1) \longrightarrow r_*(\varphi_0)$$

with a parametric dependence on anisotropic angular coordinate which is caused by the existence of a deformed static horizon.
\section*{5.3 Polar metric perturbations}

The polar perturbations are described by non-trivial increments of the diagonal metric coefficients, \( \delta \mu_\alpha = \delta \mu_\alpha^{(e)} + \delta \mu_\alpha^{(c)} \), for

\[
\mu_\alpha^{(e)} = \nu_\alpha + \delta \mu_\alpha^{(e)}
\]

where \( \delta \mu_\alpha^{(c)}(x^k, t) \) parametrize time depending fluctuations which are stated to be the same both for spherical and/or spheroid configurations and \( \delta \mu_\alpha^{(e)} \) is a static deformation from the spherical symmetry. Following notations \((66)\) and \((67)\) we write

\[
e^{v_1} = r / \sqrt{\Delta}, e^{v_2} = r \sqrt{|q^{[v]}(r)|}, e^{v_3} = rh_3 \sin \theta, e^{v_4} = \Delta / r^2
\]

and

\[
\delta \mu_1^{(e)} = -\frac{\varepsilon}{2} (\Delta^{-1} + r^2 \phi), \delta \mu_2^{(e)} = \delta \mu_3^{(e)} = -\frac{\varepsilon}{2} r^2 \phi, \delta \mu_4^{(e)} = \frac{\varepsilon \eta}{2 \Delta}
\]

where \( \phi = 0 \) for the solutions with \( \Omega = 1 \).

Examining the expressions for \( R_{4i}, R_{12}, R_{33} \) and \( G_{ij} \) (see respectively \((68)\) and \((69)\) ) we conclude that the values \( Q_{ij} \) appear quadratically which can be ignored in a linear perturbation theory. Thus the equations for the axial and the polar perturbations decouple. Considering only linearized expressions, both for static \( \varepsilon \)-terms and fluctuations depending on time about the Schwarzschild values we obtain the equations

\[
\begin{align*}
\delta_1 (\delta \mu_2 + \delta \mu_3) + (r^{-1} - \delta_1 \mu_4) (\delta \mu_2 + \delta \mu_3) - 2r^{-1} \delta_1 \mu_1 &= 0 \quad (\delta R_{41} = 0), \\
\delta_2 (\delta \mu_1 + \delta \mu_3) + (\delta \mu_2 - \delta \mu_3) \cot \theta &= 0 \quad (\delta R_{42} = 0), \\
\delta_2 \delta_1 (\delta \mu_3 + \delta \mu_4) - \delta_1 (\delta \mu_2 - \delta \mu_3) \cot \theta - (r^{-1} - \delta_1 \mu_4) \delta_2 (\delta \mu_4) - (r^{-1} + \delta_1 \mu_4) \delta_2 (\delta \mu_1) &= 0 \quad (\delta R_{42} = 0), \\
\delta_1 [\delta_1 (\delta \mu_3)] - 2r^{-1} \delta_1 \mu_1 (r^{-1} + 2 \delta_1 \mu_4) - 2e^{-2t/\mu_4} \partial_4 [\partial_4 (\delta \mu_3)] + r^{-2} [\delta_2 (\delta \mu_3)] + \delta_2 (2 \delta_3 \mu_3 + \delta \mu_4 + \delta \mu_1 - \delta \mu_2) \cot \theta + 2 \delta_2 \mu_2 &= 0 \quad (\delta R_{33} = 0), \\
e^{-2t/\mu_4} [r^{-1} \delta_1 (\delta \mu_4) + (r^{-1} + \delta_1 \mu_4) \delta_1 (\delta \mu_2 + \delta \mu_3) - 2r^{-1} \delta_1 \mu_1 (r^{-1} + 2 \delta_1 \mu_4)] - e^{-2t/\mu_4} \partial_4 [\partial_4 (\delta \mu_3 + \delta \mu_2)] + r^{-2} [\delta_2 (\delta \mu_3)] + \delta_2 (2 \delta_3 \mu_3 + \delta \mu_4 - \delta \mu_2) \cot \theta + 2 \delta_2 \mu_2 &= 0 \quad (\delta G_{11} = 0).
\end{align*}
\]

The values of type \( \delta \mu_\alpha = \delta \mu_\alpha^{(e)} + \delta \mu_\alpha^{(c)} \) from \((60)\) contain two components: the first ones are static, proportional to \( \varepsilon \), and the second ones may depend on time coordinate \( t \). We shall assume that the perturbations \( \delta \mu_\alpha^{(c)} \) have a time–dependence \( \exp[\sigma t] \) so that the partial time derivative "\( \partial_4 \)" is replaced by the factor \( i \sigma \). In order to treat both type of increments in a similar fashion we may consider that the values labeled with \( (\varepsilon) \) also oscillate in time like \( \exp[\sigma^{(e)} t] \) but with a very small (almost zero) frequency \( \sigma^{(e)} \rightarrow 0 \). There are also actions of "elongated" partial derivative operators like

\[
\delta_1 (\delta \mu_\alpha) = \partial_1 (\delta \mu_\alpha) - \varepsilon n_1 \partial_4 (\delta \mu_\alpha).
\]

To avoid a calculus with complex values we associate the terms proportional \( \varepsilon n_1 \partial_4 \) to amplitudes of type \( \varepsilon i n_1 \partial_4 \) and write this operator as

\[
\delta_1 (\delta \mu_\alpha) = \partial_1 (\delta \mu_\alpha) + \varepsilon n_1 \sigma (\delta \mu_\alpha).
\]
For the "non-perturbed" Schwarzschild values, which are static, the operator $\delta_1$ reduces to $\partial_1$, i.e. $\delta_1 v_\alpha = \partial_1 v_\alpha$. Hereafter we shall consider that the solution of the system (80) consists from a superposition of two linear solutions, $\delta \mu_\alpha = \delta \mu_\alpha^{(\varepsilon)} + \delta \mu_\alpha^{(\varsigma)}$, the first class of solutions for increments will be provided with index ($\varepsilon$), corresponding to the frequency $\sigma^{(\varepsilon)}$ and the second class will be for the increments with index ($\varsigma$) and correspond to the frequency $\sigma^{(\varsigma)}$. We shall write this as $\delta \mu_\alpha^{(A)}$ and $\sigma^{(A)}$ for the labels $A = \varepsilon$ or $\varsigma$ and suppress the factors $\exp[\sigma^{(A)} t]$ in our subsequent considerations. The system of equations (80) will be considered for both type of

and reduce the system of equations (80) to

\begin{align}
\delta_1 (N^{(A)} - L^{(A)}) &= (r^{-1} - \partial_1 v_4) N^{(A)} + (r^{-1} + \partial_1 v_4) L^{(A)}, \\
\delta_1 L^{(A)} + (2r^{-1} - \partial_1 v_4) N^{(A)} &= - \left[ \delta_1 X^{(A)} + (r^{-1} - \partial_1 v_4) X^{(A)} \right],
\end{align}

and

\begin{align}
2r^{-1} \delta_1 (N^{(A)} - l(l+1)r^{-2}e^{-2\nu_4} N^{(A)}) - 2r^{-1} (r^{-1} + 2 \partial_1 v_4) L^{(A)} - 2(r^{-1} + \partial_1 v_4) \delta_1 \left[ (N^{(A)} + (l-1)(l+2)V^{(A)}/2) - (l-1)(l+2)r^{-2}e^{-2\nu_4} (V^{(A)} - L^{(A)}) \right] - 2 \sigma^{(A)} e^{-4\nu_4} \left[ L^{(A)} + (l-1)(l+2)V^{(A)}/2 \right] &= 0,
\end{align}

where we have introduced new functions

\begin{equation}
X^{(A)} = \frac{1}{2} (l-1)(l+2)V^{(A)}
\end{equation}

and considered the relation

\begin{equation}
T^{(A)} - V^{(A)} + L^{(A)} = 0 \quad (\delta R_{42} = 0).
\end{equation}

We can introduce the functions

\begin{align}
\tilde{L}^{(A)} &= L^{(A)} + \varepsilon \sigma^{(A)} \int n_1 L^{(A)} \, dr, \quad \tilde{N}^{(A)} = N^{(A)} + \varepsilon \sigma^{(A)} \int n_1 N^{(A)} \, dr, \\
\tilde{T}^{(A)} &= N^{(A)} + \varepsilon \sigma^{(A)} \int n_1 N^{(A)} \, dr, \quad \tilde{V}^{(A)} = V^{(A)} + \varepsilon \sigma^{(A)} \int n_1 V^{(A)} \, dr,
\end{align}

for which

\begin{align}
\partial_1 \tilde{L}^{(A)} &= \delta_1 (L^{(A)}), \quad \partial_1 \tilde{N}^{(A)} = \delta_1 \left( N^{(A)} \right), \quad \partial_1 \tilde{T}^{(A)} = \delta_1 \left( T^{(A)} \right), \quad \partial_1 \tilde{V}^{(A)} = \delta_1 \left( V^{(A)} \right),
\end{align}

and, this way it is possible to substitute in (82) and (83) the elongated partial derivative $\delta_1$ by the usual one acting on "tilded" radial increments.

34
By straightforward calculations (see details in Ref. [16]) one can check that the functions
\[ Z^{(+)}(A) = r^2 \frac{6mX^{(A)} / r(l-1)(l+2) - L^{(A)} }{ r(l-1)(l+2)/2 + 3m } \]
satisfy one-dimensional wave equations similar to (78) for \( Z^{(\eta)} \) with \( \eta_3 = 1 \), when \( r_\ast = r_\ast \),
\[ \left( \frac{d^2}{dr_*^2} + \sigma_{(A)}^2 \right) \tilde{Z}^{(+)}(A) = V^{(+)} Z^{(+)}(A), \]  
\[ \tilde{Z}^{(+)}(A) = Z^{(+)}(A) + \varepsilon \sigma_{(A)} \int n_1 Z^{(+)}(A) \, dr; \]
where
\[ V^{(+)} = \frac{2\Delta}{r^5 [r(l-1)(l+2)/2 + 3m]^2} \times \left\{ 9m^2 \left[ \frac{r}{2} (l-1)(l+2) + m \right] \right\} \]
\[ + \frac{1}{4} (l-1)^2 (l+2)^2 r^3 \left[ 1 + \frac{1}{2} (l-1)(l+2) + \frac{3m}{r} \right]. \]
For \( \varepsilon \to 0 \), the equation (85) transforms in the usual Zerilli equation [40, 16].
To complete the solution we give the formulas for the "tilded" \( L-, X- \) and \( N- \) factors,
\[ \tilde{L}^{(A)} = \frac{3m}{r^2} \tilde{\Phi}^{(A)} - \frac{(l-1)(l+2)}{2r} \tilde{Z}^{(+)}(A), \]
\[ \tilde{X}^{(A)} = \frac{(l-1)(l+2)}{2r} (\tilde{\Phi}^{(A)} + \tilde{Z}^{(+)}(A)), \]
\[ \tilde{N}^{(A)} = \left( m - \frac{m^2 + r^4 \sigma_{(A)}^2}{r - 2m} \right) \tilde{\Phi}^{(A)} r^2 - \frac{(l-1)(l+2)r}{2(l-1)(l+2) + 12m} \frac{\partial \tilde{Z}^{(+)}(A)}{\partial r_\ast} \]
\[ - \frac{r(l-1)(l+2)}{[r(l-1)(l+2) + 6m]^2} \times \left\{ \frac{12m^2}{r} + 3m(l-1)(l+2) + \frac{r}{2} (l-1)(l+2) [(l-1)(l+2) + 2] \right\}, \]
where
\[ \tilde{\Phi}^{(A)} = (l-1)(l+2) e^{\nu_4} \int \frac{e^{-\nu_4} \tilde{Z}^{(+)}(A)}{(l-1)(l+2)r + 6m} \, dr. \]
Following the relations (84) we can compute the corresponding "untilded" values and put them in (81) in order to find the increments of fluctuations driven by the system of equations (80). For simplicity, we omit the rather compersome final expressions.

The formulas (87) together with a solution of the wave equation (85) complete the procedure of definition of formal solutions for polar perturbations. In Ref. [16] there are tabulated the data for the potential (86) for different values of \( l \) and \((l-1)(l+2)/2\). In the anisotropic case the explicit form of solutions is deformed by terms proportional to \( \varepsilon n_1 \sigma \). The static ellipsoidal like deformations can be modeled by the formulas obtained in the limit \( \sigma(\varepsilon) \to 0 \).
The problem of stability of anholonomically deformed Schwarzschild metrics to external perturbation is very important to be solved in order to understand if such static black ellipsoid like objects may exist in general relativity and its cosmological constant generalizations. We address the question: Let be given any initial values for a static locally anisotropic configuration confined to a finite interval of \( r_* \), for axial perturbations, and \( r_* \), for polar perturbations, will one remain bounded such perturbations at all times of evolution? The answer to this question is to obtained similarly to Refs. [16] and [36] with different type of definitions of functions \( g Z^{(n)}(\eta) \) and \( Z^{(\pm)}_{(A)}(A) \) for different type of black holes.

We have proved that even for anisotropic configurations every type of perturbations are governed by one dimensional wave equations of the form

\[
\frac{d^2 Z}{d\rho^2} + \sigma^2 Z = V Z \tag{88}
\]

where \( \rho \) is a radial type coordinate, \( Z \) is a corresponding \( Z^{(n)} \) or \( Z^{(\pm)}_{(A)} \) with respective smooth real, independent of \( \sigma > 0 \) potentials \( V^{(n)} \) or \( V^{(\pm)} \) with bounded integrals. For such equations a solution \( Z(\rho, \sigma, \varphi_0) \) satisfying the boundary conditions \( Z \to e^{i\sigma\rho} + R(\sigma)e^{-i\sigma\rho} \) \( (\rho \to +\infty) \) and \( Z \to T(\sigma)e^{i\sigma\rho} \) \( (\rho \to -\infty) \) (the first expression corresponds to an incident wave of unit amplitude from \( +\infty \) giving rise to a reflected wave of amplitude \( R(\sigma) \) at \( +\infty \) and the second expression is for a transmitted wave of amplitude \( T(\sigma) \) at \( -\infty \)), provides a basic complete set of wave functions which allows to obtain a stable evolution. For any initial perturbation that is smooth and confined to finite interval of \( \rho \), we can write the integral

\[
\psi(\rho, 0) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} \hat{\psi}(\sigma, 0) Z(\rho, \sigma) d\sigma
\]

and define the evolution of perturbations,

\[
\psi(\rho, t) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} \hat{\psi}(\sigma, 0) e^{i\sigma t} Z(\rho, \sigma) d\sigma.
\]

The Schrodinger theory guarantees the conditions

\[
\int_{-\infty}^{+\infty} |\psi(\rho, 0)|^2 d\rho = \int_{-\infty}^{+\infty} |\hat{\psi}(\sigma, 0)|^2 d\sigma = \int_{-\infty}^{+\infty} |\psi(\rho, 0)|^2 d\rho,
\]

from which the boundedness of \( \psi(\rho, t) \) follows for all \( t > 0 \).

In our consideration we have replaced the time partial derivative \( \partial/\partial t \) by \( i\sigma \), which was represented by the approximation of perturbations to be periodic like \( e^{i\sigma t} \). This is connected with a time–depending variant of (88), like

\[
\frac{\partial^2 Z}{\partial t^2} = \frac{\partial^2 Z}{\partial \rho^2} - V Z.
\]
Multiplying this equation on $\partial Z / \partial t$, where $\overline{Z}$ denotes the complex conjugation, and integrating on parts, we obtain
\[
\int_{-\infty}^{+\infty} \left( \frac{\partial \overline{Z}}{\partial t} \frac{\partial^2 Z}{\partial t^2} + \frac{\partial Z}{\partial \rho} \frac{\partial \overline{Z}}{\partial t \partial \rho} + V Z \frac{\partial \overline{Z}}{\partial t} \right) d\rho = 0
\]
providing the conditions of convergence of necessary integrals. This equation added to its complex conjugate results in a constant energy integral,
\[
\int_{-\infty}^{+\infty} \left( \left| \frac{\partial Z}{\partial t} \right|^2 + \left| \frac{\partial Z}{\partial \rho} \right|^2 + V |Z|^2 \right) d\rho = \text{const},
\]
which bounds the expression $|\partial Z / \partial t|^2$ and excludes an exponential growth of any bounded solution of the equation. We note that this property holds for every type of "ellipsoidal" like deformation of the potential, $V \to V + \varepsilon V^{(1)}$, with possible dependencies on polarization functions as we considered in (79) and/or (86).

The general properties of the one–dimensional Schrodinger equations related to perturbations of holonomic and anholonomic solutions of the Einstein equations allow us to conclude that there are locally anisotropic static configurations which are stable under linear deformations.

In a similar manner we may analyze perturbations (axial or polar) governed by a two–dimensional Schrodinger wave equation like
\[
\frac{\partial^2 Z}{\partial t^2} = \frac{\partial^2 Z}{\partial \rho^2} + A(\rho, \varphi, t) \frac{\partial^2 Z}{\partial \varphi^2} - V(\rho, \varphi, t) Z
\]
for some functions of necessary smooth class. The stability in this case is proven if exists an (energy) integral
\[
\int_0^\pi \int_{-\infty}^{+\infty} \left( \left| \frac{\partial Z}{\partial t} \right|^2 + \left| \frac{\partial Z}{\partial \rho} \right|^2 + \left| A \frac{\partial Z}{\partial \rho} \right|^2 + V |Z|^2 \right) d\rho d\varphi = \text{const}
\]
which bounds $|\partial Z / \partial t|^2$ for two–dimensional perturbations. For simplicity, we omitted such calculus in this work.

We emphasize that this way we can also prove the stability of perturbations along ”anisotropic” directions of arbitrary anholonomic deformations of the Schwarzschild solution which have non–spherical horizons and can be covered by a set of finite regions approximated as small, ellipsoid like, deformations of some spherical hypersurfaces. We may analyze the geodesic congruence on every deformed sub-region of necessary smoothly class and proof the stability as we have done for the resolution ellipsoid horizons. In general, we may consider horizons of with non–trivial topology, like vacuum black tori, or higher genus anisotropic configurations. This is not prohibited by the principles of topological censorship if we are dealing with off–diagonal metrics and associated anholonomic frames. The vacuum anholonomy in such cases may be treated as an effective matter which change the conditions of topological theorems.
6 Two Additional Examples of Off–Diagonal Exact Solutions

There are some classes of exact solutions wich can be modeled by anholonomic frame transforms and generic off–diagonal metric ansatz and related to configurations constructed by using another methods [18, 19]. We analyze in this section two classes of such 4D spacetimes.

6.1 Anholonomic ellipsoidal shapes

The present status of ellipsoidal shapes in general relativity associated to some perfect–fluid bodies, rotating configurations or to some families of confocal ellipsoids in Reimannian spaces is examined in details in Ref. [18]. We shall illustrate in this subsection how such configurations may be modeled by generic off–diagonal metrics and/or as spacetimes with anisotropic cosmological constant. The off–diagonal coefficients will be subjected to certain anholonomy conditions resulting (roughly speaking) in effects similar to those of perfect–fluid bodies.

We consider a metric ansatz with conformal factor like in (64)

$$\delta s^2 = \Omega(\theta, \nu) \left[ g_1 d\theta^2 + g_2(\theta) d\varphi^2 + h_3(\theta, \nu) \delta\nu^2 + h_4(\theta, \nu) \delta t^2 \right],$$

$$\delta\nu = d\nu + w_1(\theta, \nu) d\theta + w_2(\theta, \nu) d\varphi,$$

$$\delta t = dt + n_1(\theta, \nu) d\theta + n_2(\theta, \nu) d\varphi,$$

where the coordinate \((x^1 = \theta, x^2 = \varphi)\) are holonomic and the coordinate \(y^3 = \nu\) and the timelike coordinate \(y^4 = t\) are 'anisotropic' ones. For a particular parametrization when

$$\Omega = \Omega[0](\nu) = v(\rho) \rho^2, \quad g_1 = 1,$$

$$g_2 = g_2[0] = \sin^2 \theta, \quad h_3 = h_3[0] = 1, \quad h_4 = h_4[0] = -1,$$

$$w_1 = 0, \quad w_2 = w_2[0](\theta) = \sin^2 \theta, \quad n_1 = 0, \quad n_2 = n_2[0](\theta) = 2R_0 \cos \theta$$

and the coordinate \(\nu\) is defined related to \(\rho\) as

$$d\nu = \int \left| \frac{f(\rho)}{v(\rho)} \right|^{1/2} \frac{d\rho}{\rho},$$

we obtain the metric element for a special case spacetimes with co–moving ellipsoidal symmetry defined by an axially symmetric, rigidly rotating perfect–fluid configuration with confocal inside ellipsoidal symmetry (see formula (4.21) and related discussion in Ref. [18], where the status of constant \(R_0\) and functions \(v(\rho)\) and \(f(\rho)\) are explicitly defined).

By introducing nontrivial ”polarization” functions \(q^{[\nu]}(\theta)\) and \(\eta_{3,4}(\theta, \nu)\) for which

$$g_2 = g_2[0]q^{[\nu]}(\theta), \quad h_{3,4} = \eta_{3,4}(\theta, \nu)h_{3,4}[0]$$

we can state the conditions when the ansatz [18] defines a) an off–diagonal ellipsoidal shape or b) an ellipsoidal configuration induced by anisotropically polarized cosmological constant.
Let us consider the case a). The Theorem 2 from Ref. [20] and the formula (72) in Ref. [9] (see also the Appendix in [11]) states that any metric of type (89) is vacuum if $\Omega^{p_1/p_2} = h_3$ for some integers $p_1$ and $p_2$, the factor $\Omega = \Omega_{[1]}(\theta, \nu)\Omega_{[0]}(\nu)$ satisfies the condition

$$\partial_i\Omega - (w_1 + \zeta_i) \partial_\nu \Omega = 0$$

for any additional deformation functions $\zeta_i(\theta, \nu)$ and the coefficients

$$g_1 = 1, g_2 = g_2[0]q^{[v]}(\theta), \ h_3,4 = \eta_{3,4}(\theta, \nu)h_{3,4[0]}, w_1(\theta, \nu), n_1(\theta, \nu)$$

(91) satisfy the equations (17)–(20). The procedure of constructing such exact solutions is very similar to the considered in subsection 4.1 for black ellipsoids. For anholonomic ellipsoidal shapes (the are characterized by nontrivial anholonomy coefficients (8) and respectively induced noncommutative symmetries) we have to put as “boundary” condition in integrals of type (11) just to have $n_1 = 0$, $n_2 = n_{2[0]}(\theta) = 2R_0\cos\theta$ from data (90) in the limit when dependence on “anisotropic” variable $\nu$ vanishes. The functions $w_i(\theta, \nu)$ and $n_i(\theta, \nu)$ must be subjected to additional constraints if we want to construct ellipsoidal shape configurations with zero anholonomically induced torsion (12) and N–connection curvature, $\Omega^{a}_{jk} = \delta_k N^a_j - \delta_j N^a_k = 0$.

b) The simplest way to construct an ellipsoidal shape configuration induced by anisotropic cosmological constant is to find data (91) solving the equations (92) following the procedure defined in subsection 4.2. We note that we can solve the equation (50) for $g_2 = g_2(\theta) = \sin^2 \theta$ with $q^{[v]}(\theta) = 1$ if $\lambda_{[b]0} = 1/2$, see solution (51) with $\xi \to \theta$. For simplicity, we can consider that $\lambda_{[v]} = 0$. Such type configurations contain, in general, anholonomically induced torsion.

We conclude, that by using the anholonomic frame method we can generate ellipsoidal shapes (in general, with nontrivial polarized cosmological constants and induced torsions). Such solutions are similar to corresponding rotation configurations in general relativity with rigidly rotating perfect-fluid sources. The rough analogy consists in the fact that by certain frame constraints induced by off–diagonal metric terms we can model gravitational–matter like metrics. In previous section we proved the stability of black ellipsoids for small excentricities. Similar investigations for ellipsoidal shapes is a task for future (because the shapes could be with arbitrary excentricity). In Ref. [18], there were discussed points of matchings of locally rotationally symmetric spacetimes to Taub–NUT metrics. We emphasize that this topic was also specifically elaborated by using anholonomic frame transforms in Refs. [4].

### 6.2 Generalization of Canfora–Schmidt solutions

In general, the solutions generated by anholonomic transforms cannot be reduced to a diagonal transform only by coordinate transforms (this is stated in our previous works [1] 2 3 4 10 20 29 30 31 36 10, see also Refs. [48] for modelling Finsler like geometries in (pseudo) Riemannian spacetimes). We discuss here how 4D off–diagonal ansatz (2) generalize the solutions obtained in Ref. [19] by a corresponding parametrization of coordinates as $x^1 = x, x^2 = t, y^3 = \nu = y$ and $y^4 = p$. If we consider for (2) (equivalently, for (5) ) the non-trivial data

$$g_1 = g_{1[0]} = 1, \ g_2 = g_{2[0]}(x^1) = -B(x)P(x)^2 - C(x),$$

$$h_3 = h_{3[0]}(x^1) = A(x) > 0, \ h_4 = h_{4[0]}(x^1) = B(x),$$

$$w_i = 0, n_1 = 0, n_2 = n_{2[0]}(x^1) = P(x)/B(x)$$

(92)
we obtain just the ansatz (12) from Ref. [19] (in this subsection we use a different label for coordinates) which, for instance, for $B + C = 2, B - C = \ln |x|, P = -1/(B - C)$ with $e^{-1} < \sqrt{|x|} < e$ for a constant $e$, defines an exact 4D solution of the Einstein equation (see metric (27) from [19]). By introducing 'polarization' functions $\eta_k = \eta_k(x^i)$ [when $i, k, \ldots = 1, 2$] and $\eta_a = \eta_a(x^i, \nu)$ [when $a, b, \ldots = 3, 4$] we can generalize the data (92) as to have

$$g_k(x^i) = \eta_k(x^i)g_{k[0]}, \ h_a(x^i, \nu) = \eta_a(x^i, \nu)h_{a[0]}$$

and certain nontrivial values $w_i = w_i(x^i, \nu)$ and $n_i = n_i(x^i, \nu)$ solving the Einstein equations with anholonomic variables (17)–(20). We can easily find new classes of exact solutions, for instance, for $\eta_1 = 1$ and $\eta_2 = \eta_2(x^1)$. In this case $g_1 = 1$ and the function $g_2(x^1)$ is any solution of the equation

$$g_{2}^{\bullet \bullet} - \frac{(g_{2}^{\bullet})^{2}}{2g_{2}} = 0 \tag{93}$$

(see equation (60) for $\lambda_{[\nu]} = 0$), $g_{2}^{\bullet} = \partial g_{2}/\partial x^1$ which is solved as a particular case if $g_2 = (x^1)^2$. This impose certain conditions on $\eta_2(x^1)$ if we want to take $g_{2[0]}(x^1)$ just as in (92). For more general solutions with arbitrary $\eta_k(x^i)$, we have to take solutions of equation (17) and not of a particular case like (93).

We can generate solutions of (18) for any $\eta_a(x^i, \nu)$ satisfying the condition (82), $\sqrt{|\eta_3|} = \eta_0 \left(\sqrt{|\eta_1|}\right)^{\ast}, \eta_0 = const$. For instance, we can take arbitrary $\eta_4$ and using elementary derivations with $\eta_4^{\ast} = \partial \eta_4/\partial \nu$ and a nonzero constant $\eta_0$, to define $\sqrt{|\eta_3|}$. For the vacuum solutions, we can put $u_i = 0$ because $\beta = \alpha_i = 0$ (see formulas (18) and (21)). In this case the solutions of (19) are trivial. Having defined $\eta_{4}(x^i, \nu)$ we can integrate directly the equation (20) and find $n_i(x^i, \nu)$ like in formula (41) with fixed value $\varepsilon = 1$ and considering dependence on all holonomic variables,

$$n_i(x^k, \nu) = n_{i[1]}(x^k) + n_{i[2]}(x^k) \int d\nu \ \eta_3(x^k, \nu) / \left(\sqrt{|\eta_4(x^k, \nu)|}\right)^{3}, \eta_4^{\ast} \neq 0; \tag{94}$$

$$n_i(x^k, \nu) = n_{i[1]}(x^k) + n_{i[2]}(x^k) \int d\nu \ \eta_3(x^k, \nu), \eta_4^{\ast} = 0;$$

$$= n_{i[1]}(x^k) + n_{i[2]}(x^k) \int d\nu / \left(\sqrt{|\eta_4(x^k, \nu)|}\right)^{3}, \eta_3^{\ast} = 0.$$

These values will generalize the data (92) if we identify $n_{1[1]}(x^k) = 0$ and $n_{1[1]}(x^k) = n_{2[0]}(x^1) = P(x)/B(x)$. The solutions with vanishing induced torsions and zero nonlinear connection curvatures are to be selected by choosing $n_i(x^k, \nu)$ and $\eta_3(x^k, \nu)$ (or $\eta_4(x^k, \nu)$) as to reduce the canonical connection (9) to the Levi–Civita connection (as we discussed in the end of Section 2).

The solution defined by the data (92) is compared in Ref. [19] with the Kasner diagonal solution which define the simplest models of anisotropic cosmology. The metrics obtained by F. Canfora and H.-J. Schmidt (CS) is generic off–diagonal and can not written in diagonal form by coordinate transforms. We illustrated that the CS metrics can be effectively diagonalized with respect to N–adapted anholonomic frames (like a more general ansatz (2) can be reduced to (5)) and that by anholonomic frame transforms of the CS metric we can
generate new classes of generic off–diagonal solutions. Such spacetimes may describe certain models of anisotropic and/or inhomogeneous cosmologies (see, for instance, Refs. [48] were we considered a model of Friedman–Robertson–Walker metric with ellipsoidal symmetry). The anholonomic generalizations of CS metrics are with nontrivial noncommutative symmetry because the anholonomy coefficients \(\mathbf{58}\) (see also \(\mathbf{20}\)) are not zero being defined by nontrivial values \(\mathbf{94}\).

7 Outlook and Conclusions

The work is devoted to investigation of a new class of exact solutions in metric–affine and string gravity describing static back rotoid (ellipsoid) and shape configurations possessing hidden noncommutative symmetries. There are generated also certain generic off–diagonal cosmological metrics.

We consider small, with nonlinear gravitational polarization, static deformations of the Schwarschild black hole solution (in particular cases, to some resolution ellipsoid like configurations) preserving the horizon and geodesic behaviour but slightly deforming the spherical constructions. It was proved that there are such parameters of the exact solutions of the Einstein equations defined by off–diagonal metrics with ellipsoid symmetry constructed in Refs. [1, 2, 20, 29, 30, 36] as the vacuum solutions positively define static ellipsoid black hole configurations.

We illustrate that the new class of static ellipsoidal black hole solutions posses some similarities with the Reissner–Nordstrom metric if the metric’s coefficients are defined with respect to correspondingly adapted anholonomic frames. The parameter of ellipsoidal deformation results in an effective electromagnetic charge induced by off–diagonal vacuum gravitational interactions. We note that effective electromagnetic charges and Reissner–Nordstrom metrics induced by interactions in the bulk of extra dimension gravity were considered in brane gravity [42]. In our works we proved that such Reissner–Nordstrom like ellipsoid black hole configurations may be constructed even in the framework of vacuum Einstein gravity. It should be emphasized that the static ellipsoid black holes posses spherical topology and satisfy the principle of topological censorship [39]. Such solutions are also compatible with the black hole uniqueness theorems [33]. In the asymptotical limits at least for a very small eccentricity such black ellipsoid metrics transform into the usual Schwarzschild one. We have proved that the stability of static ellipsoid black holes can be proved similarly by considering small perturbations of the spherical black holes [29, 30] even the solutions are extended to certain classes of spacetimes with anisotropically polarized cosmological constants. (On the stability of the Schwarzschild solution see details in Ref. [16].)

The off–diagonal metric coefficients induce a specific spacetime distorsion comparing to the solutions with metrics diagonalizable by coordinate transforms. So, it is necessary to compare the off–diagonal ellipsoidal metrics with those describing the distorted diagonal black hole solutions (see the vacuum case in Refs. [44] and an extension to the case of non–vanishing electric fields [45]). For the ellipsoidal cases, the distorsion of spacetime can be of vacuum origin caused by some anisotropies (anholonomic constraints) related to off–diagonal terms. In the case of ”pure diagonal” distorsions such effects follow from the fact that the vacuum
Einstein equations are not satisfied in some regions because of presence of matter.

The off–diagonal gravity may model some gravity–matter like interactions like in Kaluza–Klein theory (for some particular configurations and topological compactifications) but, in general, the off–diagonal vacuum gravitational dynamics can not be associated to any effective matter dynamics. So, we may consider that the anholonomic ellipsoidal deformations of the Schwarzschild metric are some kind of anisotropic off–diagonal distortions modeled by certain vacuum gravitational fields with the distortion parameteres (equivalently, vacuum gravitational polarizations) depending both on radial and angular coordinates.

There is a common property that, in general, both classes of off–diagonal anisotropic and “pure” diagonal distorsions (like in Refs. [44]) result in solutions which are not asymptotically flat. However, it is possible to find asymptotically flat extensions even for ellipsoidal configurations by introducing the corresponding off–diagonal terms (the asymptotic conditions for the diagonal distortions are discussed in Ref. [45]; to satisfy such conditions one has to include some additional matter fields in the exterior portion of spacetime).

We analyzed the conditions when the anholonomic frame method can model ellipsoid shape configurations. It was demonstrated that the off–diagonal metric terms and respectively associated nonlinear connection coefficients may model ellipsoidal shapes being similar to those derived from solutions with rotating perfect fluids (roughly speaking, a corresponding frame anholonomy/ anisotropy may result in modeling of specific matter interactions but with polarizations of constants, metric coefficients and related frames).

In order to point to some possible observable effects, we note that for the ellipsoidal metrics with the Schwarzschild asymptotics, the ellipsoidal character could result in some observational effects in the vicinity of the horizon (for instance, scattering of particles on a static ellipsoid; we can compute anisotropic matter accretion effects on an ellipsoidal black hole put in the center of a galactic being of ellipsoidal or another configuration). A point of further investigations could be the anisotropic ellipsoidal collapse when both the matter and spacetime are of ellipsoidal generic off–diagonal symmetry and/or shape configurations (former theoretical and computational investigations were performed only for rotoids with anisotropic matter and particular classes of perturbations of the Schwarzschild solutions [46]). For very small eccentricities, we may not have any observable effects like perihelion shift or light bending if we restrict our investigations only to the Schwarzschild–Newton asymptotics.

We present some discussion on mechanics and thermodynamics of ellipsoidal black holes. For the static black ellipsoids with flat asymptotics, we can compute the area of the ellipsoidal horizon, associate an entropy and develop a corresponding black ellipsoid thermodynamics. This can be done even for stable black torus configurations. But this is a very rough approximation because, in general, we are dealing with off–diagonal metrics depending anisotropically on two/three coordinates. Such solutions are with anholonomically deformed Killing horizons and should be described by a thermodynamics (in general, both non-equilibrium and irreversible) of black ellipsoids self–consistently embedded into an off–diagonal anisotropic gravitational vacuum. This is a ground for numerous new conceptual issues to be developed and related to anisotropic black holes and the anisotropic kinetics and thermodynamics [2] as well to a framework of isolated anisotropic horizons [47] which is a matter of our further investigations. As an example of a such new concept, we point to a noncommutative dynamics which can be associated to black ellipsoids.
We emphasize that it is a remarkable fact that, in spite of appearance complexity, the perturbations of static off–diagonal vacuum gravitational configurations are governed by similar types of equations as for diagonal holonomic solutions. Perhaps in a similar manner (as a future development of this work) by using locally adapted ”N–elongated” partial derivatives we can prove stability of very different classes of exact solutions with ellipsoid, toroidal, dilaton and spinor–soliton symmetries constructed in Refs. [1, 2, 20, 29, 30, 36]. The origin of this mystery is located in the fact that by anholonomic transforms we effectively diagonalized the off–diagonal metrics by ”elongating” some partial derivatives. This way the type of equations governing the perturbations is preserved but, for small deformations, the systems of linear equations for fluctuations became ”slightly” nondiagonal and with certain tetradic modifications of partial derivatives and differentials.

It is known that in details the question of relating the particular integrals of such systems associated to systems of linear differential equations is investigated in Ref. [16]. For anholonomic configurations, one holds the same relations between the potentials $\tilde{V}(\eta)$ and $V(\varphi)$ and wave functions $Z^{(\eta)}$ and $Z^{(+)}$ with that difference that the physical values and formulas where polarized by some anisotropy functions $\eta_3(r, \theta, \varphi), \Omega(r, \varphi), q(r), \eta(r, \varphi), w_1(r, \varphi)$ and deformed on a small parameter $\varepsilon$. It is not clear that a similar procedure could be applied in general for proofs of stability of ellipsoidal shapes but it would be true for small deformations from a supposed to be stable primordial configuration.

We conclude that there are static black ellipsoid vacuum configurations as well induced by nontrivially polarized cosmological constants which are stable with respect to one dimensional perturbations, axial and/or polar ones, governed by solutions of the corresponding one–dimensional Schrodinger equations. The problem of stability of such objects with respect to two, or three, dimensional perturbations, and the possibility of modeling such perturbations in the framework of a two–, or three–, dimensional inverse scattering problem is a topic of our further investigations. The most important problem to be solved is to find a geometrical interpretation for the anholonomic Schrodinger mechanics of stability to the anholonomic frame method and to see if we can extend the approach at least to the two dimensional scattering equations.

Acknowledgements

The work is partially supported by a NATO/Portugal fellowship at CENTRA, Instituto Superior Tecnico, Lisbon. The author is grateful for support to the organizers of the International Congress of Mathematical Physics, ICMP 2003, and of the Satellite Meeting OPORTO 2003. He would like to thank J. Zsigrai for valuable discussions.

References

[1] S. Vacaru, JHEP 0104 (2001) 009.

[2] S. Vacaru, Ann. Phys. (NY) 290 (2001) 83.
[3] S. Vacaru and D. Singleton, Class. Quant. Grav. **19** (2002) 3583; 2793; J. Math. Phys. **43** (2002) 2486; S. Vacaru, D. Singleton, V. Botan and D. Dotenco, Phys. Lett. **B 519** (2001) 249.

[4] S. Vacaru and O. Tintareanu-Mircea, Nucl. Phys. **B 626** (2002) 239; S. Vacaru and F. C. Popa, Class. Quant. Grav. **18** (2001) 4921.

[5] A. Salam and J. Strathdee, Ann. Phys. (NY) **141** (1982) 316; R. Percacci and S. Randjbar-Daemi, J. Math. Phys. **24** (1983) 807.

[6] J. P. S. Lemos, Class. Quant. Grav. **12**, (1995) 1081; Phys. Lett. **B 352** (1995) 46; J. P. S. Lemos and V. T. Zanchin, Phys. Rev. **D54** (1996) 3840; P.M. Sa and J.P.S. Lemos, Phys. Lett. **B 423**(1998) 49; J. P. S. Lemos, Phys. Rev. **D57** (1998) 4600; Nucl. Phys. **B600** (2001) 272.

[7] F. M. Hehl, J. D. Mc Grea, E. W. Mielke, and Y. Ne’eman. Phys. Rep. **258**, 1 (1995); F. Gronwald and F. W. Hehl, in: *Quantum Gravity*, Proc. 14th School of Cosmology and Gravitation, May 1995 in Erice, Italy. Eds.: P. G. Bergmann, V. de Sabbata and H. -J. Treder (World Sci. Publishing, River Edge NY, 1996), 148–198; T. Dereli and R. W. Tucker, Class. Quantum Grav. **12**, L31 (1995); T. Dereli, M. Onder and R. W. Tucker, Class. Quantum Grav. **12**, L251 (1995); T. Dereli, M. Onder, J. Schray, R. W. Tucker and C. Wang, Class. Quantum Grav. **13**, L103 (1996); Yu. N. Obukhov, E. J. Vlachynsky, W. Esser and F. W. Hehl, Phys. Rev. D **56**, 7769 (1997).

[8] S. Vacaru, Generalized Finsler Geometry in Einstein, String and Metric–Affine Gravity, hep-th/0310132.

[9] S. Vacaru, E. Gaburov and D. Gontsa, A Method of Constructing Off–Diagonal Solutions in Metric–Affine and String Gravity, [hep-th/0310133](http://arxiv.org/abs/hep-th/0310133).

[10] S. Vacaru, Exact Solutions with Noncommutative Symmetries in Einstein and Gauge Gravity, [gr-qc/0307103](http://arxiv.org/abs/gr-qc/0307103).

[11] T. Regge and J. A. Wheeler, Phys. Rev. **108** (1957) 1063.

[12] C. V. Vishveshwara, Phys. Rev. D **1** (1970) 2870; J. M. Bardeen and W. H. Press, J. Math. Phys. **14** (1972) 7; S. Chandrasekhar, Proc. Roy. Soc. (London) A**343** (1975) 289; B. C. Xanthopoulos, Proc. Roy. Soc. (London) A**378** (1981) 61.

[13] J. L. Friedman, Proc. Roy. Soc. (London) A**335** (1973) 163;

[14] P. Deift and E. Trubowitz, Commun. Pure and Appl. Math. **32** (1979) 121.

[15] L. D. Faddeev, Soviet. Phys. Dokl. **3** (1958) 747; R. M. Miura, C. S. Gardner and M. D. Kruskal, J. Math. Phys. **9** (1968) 1204.

[16] S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Oxford University Press, 1992).
[17] V. Moncrief, Ann. Phys. (NY) 88 (1973) 323; R. M. Wald, J. Math. Phys. 20 (1979) 1056.

[18] J. Zsigrai, Class. Quant. Grav. 20 (2003) 2855; S. Chandrasekhar, Ellipsoidal Figures of Equilibrium (New York, Dover Publication Inc., 1987); A. Krasinski, Ann. Phys. (NY) 112 (1978) 22; I. Racz, Class. Quant. Grav. 9 (1992) L93; I. Racz and M. Suveges, Mod. Phys. Lett. A 13 (1998) 1101.

[19] F. Canfora and H. -J. Schmidt, Vacuum Solutions Which Cannot be Written in Diagonal Form, gr-qc/0305107.

[20] S. Vacaru, A New Method of Constructing Black Hole Solutions in Einstein and 5D Dimension Gravity, hep-th/0110250; S. Vacaru and E. Gaburov, Anisotropic Black Holes in Einstein and Brane Gravity, hep-th/0108065.

[21] R. Miron and M. Anastasiei, The Geometry of Lagrange Spaces: Theory and Applications (Dordrecht: Kluwer, 1994).

[22] C. W. Misner, K. S. Thorne and J. A. Wheeler, Gravitation (Freeman, 1973).

[23] A. Kawaguchi, Akad. Netensch. Amsterdam. Proc. 40 (1937) 596–601.

[24] A. H. Chamseddine, G. Felder and J. Frohlich, Commun. Math. Phys. 155 (1993) 205; A. H. Chamseddine, J. Frohlich and O. Grandjean, J. Math. Phys. 36 (1995) 6255; A. H. Chamseddine, Commun. Math. Phys. 218 (2001) 283; E. Hawkins, Commun. Math. Phys. 187 (1997) 471; D. Kastler, Commun. Math. Phys. 166 (1995) 633; T. Kopf, Int. J. Mod. Phys. A13 (1998) 2693; W. Kalau and M. Walze, J. Geom. Phys. 16 (1995) 327; G. Landi and C. Rovelli, Phys. Rev. Lett. 78 (1997) 3051; I. Vancea, Phys. Rev. Lett. 79 (1997) 3121; G. Landi, Nguyen A. V. and K. C. Wali, Phys. Lett. B326 (1994) 45; S. Majid, Int. J. Mod. Phys. B 14 (2000) 2427; J. W. Moffat, Phys. Lett. B 491 (2000) 345.

[25] M. Dubois–Violette, R. Kerner and J. Madore, J. Math. Phys. 31 (1990) 316; 323.

[26] M. Dubois–Violette, R. Kerner and J. Madore, Phys. Lett. B217 (1989) 485; Class. Quant. Grav. 7 (1989) 1709; J. Madore and J. Mourad, Class. Quant. Grav. 10 (1993) 2157; J. Madore, T. Masson and J. Mourad, Class. Quant. Grav. 12 (1995) 1249.

[27] J. Madore, An Introduction to Noncommutative Geometry and its Physical Applications, LMS lecture note Series 257, 2nd ed. (Cambridge University Press, 1999).

[28] S. Vacaru, Phys. Lett. B 498 (2001) 74.

[29] S. Vacaru, Int. J. Mod. Phys. D12 (2003) 479.

[30] S. Vacaru and D. E. Goksel, Horizons and Geodesics of Black Ellipsoids with Anholonomic Conformal Symmetries; in “Progress in Mathematical Physics”, Ed. Frank Columbus (Nova Science Publishers, NY, 2003), gr-qc/0206015.
[31] S. Vacaru, (Non) Commutative Finsler Geometry from String/M–Theory, hep-th/0211068.

[32] E. Kamke, Differentialgleichungen, Losungsmethoden und Lonsungen: I. Gewohnliche Differentialgleichungen (Leipzig, 1959).

[33] J. C. Graves and D. R. Brill, Phys. Rev. 120 (1960) 1507.

[34] B. Carter, Phys. Lett. 21 (1966) 423.

[35] A. H. Chamsedine, Phys. Lett. B 504 (2001) 33.

[36] S. Vacaru, Int. J. Mod. Phys. D12 (2003) 461.

[37] S. W. Hawking and G. F. R. Ellis, The Large Scale Structure of Space–Time (Cambridge University Press, 1973).

[38] R. H. Boyer and R. W. Lindquist, J. Math. Phys. 8 (1967) 265.

[39] S. W. Hawking, Commun. Math. Phys. 25 (1972) 152.

[40] F. J. Zerilli, Phys. Rev. Lett. 24 (1970) 2141.

[41] S. W. Hawking, Commun. Math. Phys. 25 (1972), 152; G. J. Galloway, K. Seheich, D. M. Witt and E. Woolgar, Phys. Rev. D 60 (1999) 104039.

[42] R. Maartens, Geometry and Dynamics of the Brane–World, gr-qc/0101059; N. Dahich, R. Maartens, P. Papadopoulos and V. Rezania, Phys. Lett. B 487 (2000) 1; C. Germani and R. Maartens, Phys. Rev. D64 (2001) 124010.

[43] W. Israel, Phys. Rev. 164 (1967) 1771; B. Carter, Phys. Rev. Lett. 26 (1967) 331; D. C. Robinson, Phys. Rev. Lett. 34 (1975) 905; M. Heusler, Black Hole Uniqueness Theorems (Cambridge University Press, 1996).

[44] L. Mysak and G. Szekeres, Can. J. Phys. 44 (1966) 617; R. Geroch and J. B. Hartle, J. Math. Phys. 23 (1982) 680.

[45] S. Fairhurst and B. Krishnan, Int. J. Mod. Phys. 10 (2001) 691.

[46] S. L. Shapiro and S. A. Teukolsky, Phys. Rev. Lett. 66 (1991) 944; J. P. S. Lemos, Phys. Rev. D 59 (1999) 044020.

[47] A. Ashtekar, C. Beetle and S. Fairhurst, Class. Quant. Grav. 17 (2000) 253; Class. Quant. Grav. 16 (1999) L1; A. Ashtekar, S. Fairhurst and B. Krishnan, Phys. Rev. D 62 (2000) 1040025; A. Ashtekar, C. Beetle, O. Dreyer et all. Phys. Rev. Lett. 5 (2000) 3564.

[48] S. Vacaru and H. Dehnen, Gen. Rel. Grav. 35 (2003) 209; H. Dehnen and S. Vacaru, Gen. Rel. Grav. 35 (2003) 807.