Formulas for Calculating Composite Rod with Initial Curvature

G F Sidorov
Department of Construction Industry and Theory of Structures, South Ural State University, 76, Lenin Avenue, Chelyabinsk 454080, The Russian Federation
E-mail: sidorovgf@susu.ru

Abstract. The problem of stress distribution in the cross-section of an initially curved composite rod is solved in normal (master) coordinates by the method of partial stiffness. The correctness of the obtained solution is confirmed by comparison with other solutions, other methods and are special cases of a more general solution presented. The formulas to determine the center of stiffness and torsional (flexural) stiffness of the cross-section are derived. The formulas take into account the offset center of rigidity of the nonlinear nature of section deformation of composite materials as well as the case of a non-uniform distribution of the reinforcement material. One needs to take into account a changing moment of the «inertia» section when changing the position of the stiffness center. The paper shows the possibility to solve the problems of expansion of the discrete and continual static indeterminacy basing on the superposition principle and applying the method of partial stiffness without the need of using the equation systems. These circumstances offer an opportunity to address the task of optimization in a non-linear statement without iterative methods application. The obtained formulas allow producing an updated calculation of rod reinforced concrete structures in the field of curved interfaces of their elements.

1. Introduction
Calculations of reinforced concrete elements of building structures are regulated by building specifications [1]. The stiffness of cross-sections of reinforced concrete elements is determined according to the general rules of strength of materials for a conditionally resilient solid member. In case the flexures of reinforced concrete elements mainly depend on flexural deformations, the flexure values are determined by the curvatures of the elements with the use of the Mohr integral:

\[ f_m = \int M_x (1/r) \, dx, \]

where \( M_x \) is the moment of flexure in the cross-section \( x \), \((1/r) \) is the total curvature of the element in the cross-section \( x \) of the stress, under which the flexure is determined. The initial curvature is assumed to be equal to zero. In case the initial curvature is different from zero, an additional displacement of the neutral layer and the stress distribution across the rod cross-section takes place, which largely influences the accuracy of subsequent calculations of strength and stiffness. Building specifications neither consider nor regulate the influence of the initial curvature of a flexural reinforced concrete element. This paper considers the position of the neutral layer and the stress distribution across the rod cross-section.

The work is done at the Department of building production and theory of structures of South Urals State University within the framework of the project «5-100» increase the competitiveness of leading Russian universities among the world's leading scientific and educational centers.
2. Method of partial stiffness’s

Traditional methods of expansion of static indeterminacy in the rod theory require reference to systems of equations which interconnect conditions of equilibrium, displacement compatibility and consistency of constraint equations. Solution of higher order equation systems is not only labor-consuming, but it also deprives the solution of analytic properties. This circumstance takes on a dimension if the problem of strength optimization should be solved. In case bonds are defined by differential equations in partial derivatives [2] (theory of elasticity) or by transcendental equations [3-7] (nonlinear mechanics), deformable body mechanics comes to analytically undecidable mathematical problems.

Engineering practice of strength calculations is associated with using the hypotheses of plane stiff (Bernoulli) [8], nonplan stiff (Verkhovsky) [9], warping (Sidorov) [10] cross-sections, as well as the hypotheses of rigid normal (Kirchhoff) [11], etc. Sidestepping the problems of solving on the basis of equation systems, I devised a unitary engineering method of calculating forces and stresses, which is invariant to the index of static indeterminacy degree. That is the method of partial stiffness’s [12]. The method of partial stiffness’s is based on the application of Chales’ theorem [13,14] to a solid body, situated in vector space of stiffness’s, and is characterized by the following peculiarities:

1) the notions of center and principal axes of stiffness’s are introduced;
2) the notions of translational and torsional stiffness (integral and partial) are introduced;
3) the law of partial stiffness is formulated: the response value of a separate bond is proportional to the relationship of its partial stiffness to the integral one.

The following designations are used in formulas of the method of partial stiffness’s: \( \mathbf{r} \) – the radius-vector of the restraint point in the system of normal axes; \( \mathbf{e} \) – the unit vector of the bond; \( \mathbf{i} \) – the unit vector of the principal axis of stiffnesses; \( \alpha \) – the angle between the unit vector of the bond and the unit vector of the principal axis of stiffnesses; \( K_n \) – the bond stiffness.

The main formulas of the method of partial stiffness’s in vector [15] and scalar forms of notation are:

The condition for determining the center of stiffness’s:

\[
\sum_{n} K_n \left[ (e_n \times \mathbf{i}) r_n \right] e_n = 0;
\]

\[
\sum_{n} K_n r_n \sin \alpha_n = 0.
\]  

(1)

The condition for determining the principal axes of stiffness’s:

\[
\sum_{n} K_n \left( \mathbf{i} \times e_n \right) \left( \mathbf{i} \times e_n \right) = 0;
\]

\[
\sum_{n} \sin 2\alpha_n = 0.
\]  

(2)

The principal translational stiffness:

\[
K = \sum_{n} K_n \left( \mathbf{i} \times e_n \right) \left( \mathbf{i} \times e_n \right);
\]

\[
K = \sum_{n} K_n \cos^2 \alpha_n.
\]  

(3)

The principal torsional stiffness:

\[
II = \sum_{n} K_n \left[ (e_n \times \mathbf{i}) r_n \right]^2;
\]

\[
II = \sum_{n} K_n r_n^2 \sin^2 \alpha_n
\]  

(4)
Figure 1. Determining the center of stiffness of the cross-section of initially curved composite rod.

3. Derivation of formula for calculating normal stresses in reinforcing fibers and matrix of an initially curved composite rod with pure bending

According to the chart of rod approximation [16] in figure 1, the elementary bonds between the neighboring absolutely stiff cross-sections (the law of plane cross-sections) have the length $L$, which is proportional to the radius of curvature $\rho$:

$$\frac{L}{b} = \frac{\rho}{r} \Rightarrow L = b \frac{\rho}{r}$$

Here $b$ is the bond length in the segment with the least radius of curvature $r$. The stiffness of the elementary bond when extended or compressed is calculated [17]:

$$K_n = \frac{E_n \cdot dS}{L} = \frac{rE}{\rho b} dS$$

(5)

When cross-section material is uniform, the position of the center of stiffness is determined by switching in formula (1) from sums to quadrature’s [18]:

$$\sum_n K_n r_n \sin \psi_n = \frac{E_n}{b} \int_\gamma \left( \frac{\rho - \rho_k}{\rho} \right) dS = \frac{E_n}{b} \left( S - \rho_k \int_\gamma \frac{dS}{\rho} \right) = 0,$$

$$\rho_k = \frac{S}{\int_\gamma \frac{dS}{\rho}}.$$

(6)

For composite material, the part $kS$ (for $k \leq 1$) of the rod cross-section $S$ is filled with reinforcing fibers, the rest $(1-k)S$ is filled with matrix. Partial stiffness’s of reinforcing fibers and matrix should be considered separately:

$$E_v \int_{kS} \frac{\rho - \rho_k}{\rho} dS + E_m \int_{(1-k)S} \frac{\rho - \rho_k}{\rho} dS = 0,$$

where $E_v$, $E_m$ – the moduli of fiber stiffness and matrix stiffness respectively. Enabling this condition for the radius of curvature of the neutral layer $\rho_k$, which passes through the center of stiffness $k$ of the composite cross-section, we obtain the following:

$$\rho_k = \frac{S \left[ E_m + k (E_v - E_m) \right]}{E_v \int_{kS} \frac{dS}{\rho} + E_m \int_{(1-k)S} \frac{dS}{\rho}}$$

(7)
We determine the torsional stiffness of the cross-section by the formula (4), replacing sums with quadratures:

\[
\Pi_z = r \left[ \frac{1}{E_v} \int_{S} \left( \frac{\rho - \rho_k}{\rho} \right)^2 dS + \frac{1}{E_m} \int_{(1-k)S} \left( \frac{\rho - \rho_k}{\rho} \right)^2 dS \right] = \\
= rs \left\{ \left[ kE_v \rho_cv + (1-k)E_m \rho_cm - \rho_k \left[ kE_v + (1-k)E_m \right] \right] \right\} 
\]

(8)

here \( \rho_cv, \rho_cm \) – are the coordinates of the centers of areas (center of cross-section “gravity”), occupied by fibers and matrix coordinately. In case of uniform filling of the matrix with fibers, \( \rho_cv = \rho_cm = \rho_c \). In this case the expression for the torsional stiffness is simplified:

\[
\Pi_z = rs \left[ kE_v + (1-k)E_m \right] (\rho_c - \rho_k) 
\]

(9)

The formula of stresses in reinforcing fibers, which uniformly fill the cross-section of the initially curved rod, according to the law of partial stiffness’s is the following:

\[
\sigma_v = \frac{M \left( \rho - \rho_k \right)}{S \rho \left( \rho_c - \rho_k \right) \left[ k + (1-k) \frac{E_m}{E_v} \right]} 
\]

(10)

Transforming it into the coordinate system with reference to the neutral axis, introducing the designations \( e = \rho_c - \rho_k, y = \rho - \rho_k \), we reduce the formula to a classical form [8]:

\[
\sigma_v = \frac{M}{Se} \cdot \frac{y}{\rho_k + y} \cdot \frac{1}{k + (1-k) \frac{E_m}{E_v}} 
\]

(11)

when \( k = 1 \), this expression coincides with the calculation formula for a uniform rod of greater curvature [19, 20]:

\[
\sigma = \frac{M}{Se} \cdot \frac{y}{\rho_k + y} 
\]

(12)

In the general case, when reinforcing fibers are not uniformly arranged in matrix, the formula of reinforcement stresses in the reference system with respect to the curvature center of the rod for the reinforcement is the following:

\[
\sigma_v = \frac{M \left( \rho - \rho_k \right)}{S \rho \left\{ k \rho_cv + (1-k) \rho_cm \frac{E_m}{E_v} - \rho_k \left[ k + (1-k) \frac{E_m}{E_v} \right] \right\}} 
\]

(13)

For the matrix:

\[
\sigma_m = \frac{M \left( \rho - \rho_k \right)}{S \rho k \frac{E_v}{E_m} \left( \rho_cv - \rho_k \right) + (1-k) \left( \rho_cm - \rho_k \right)} 
\]

(14)

4. Conclusions
1. A mathematical model of the initially curved rod behavior was developed on the basis of an engineering approach, namely the method of partial stiffness’s. The model does not require systems of equations when static indeterminacy and allows obtaining calculation results in a formula form.
2. The analytical expressions for torsional (flexural) stiffness of composite cross-sections in sections with nonzero curvature were obtained. This is relevant for the calculation of stiffness and the determination of flexures in initially curved beams.

3. The formulas for calculating stresses in reinforcement and matrix of initially curved composite rods were derived. This is relevant for the calculation of strength of initially curved composite beams, arches, reinforced concrete plates and trusses with curvilinear transition elements.

Acknowledgments
The work was supported by Act 211 Government of the Russian Federation, contract № 02.A03.21.0011.

References
[1] 2003 SP 63.13330.2012 Concrete and reinforced concrete structures. Updated version of SNiP 52-01-2003 (Moscow) p 168
[2] Rabotnov Yu N 1988 Deformable body mechanics (Moscow: Nauka Publ.) p 712
[3] Shlychkov S V, Ivanov S P, Kuzovkov S G and Loskutov Yu V 2008 Calculation of Geometrically Nonlinear Structures Using Finite Element Method News of higher educational institutions. Technical science. Mechanical Engineering and Engineering Science 4 pp 145–152
[4] Simon E V 2012 Calculation of Geometrically Nonlinear Rod Systems in Mixed Form Internet-bulletin of VolgGASU. Polythematical series 3 Retrieved from http://vestnik.vgasu.ru/attachments/Simon-2012_3(23).pdf
[5] Ereemeev P G 2006 Features of design of unique large-span buildings Modern industrial and civil engineering 1 vol 2 pp 5–15
[6] Proenko A M 1982 The theory of elastic-perfectly-plastic systems (Moscow: Nauka Publ.) p 287
[7] Sventikov A A 1970 Geometrically Nonlinear Calculation of the Hanging Rod Designs. Part 2: The Matrix Calculation of Suspension Systems Construction and architecture 1 vol 1 pp 18–27
[8] Bernoulli J 1705 Veritable hypothese de la resistance des solides avec la demonstration de la courbure des corps qui font ressort Histoire de l’Academy des sciences de Paris pp 176–186
[9] Verkhovskiy A V 1971 Method of nonplane cross-sections (Gorkiy: Volgo-Vyatsk Publ.) p 248
[10] Sidorov G F 1989 Strength of materials. Experience of a deductive imparting of the course: teaching aid (Chelyabinsk: HVVAIU Publ.) p 177
[11] Kirchhoff G R 1850 Ueber das Gleichgewicht und die Bewegungeinerelastischen Scheibe Crelle Journal für die reine und angewandte Matematik Bd 40 pp 51–88
[12] Sidorov G F and Pozdnyshch E O 2011 Expansion of static indeterminacy by the method of partial stiffnesses Mekhanics vol 5 (Gomel) pp 239–244
[13] Markeev A P 1990 Theoretical mechanics: teaching aid (Moscow: Nauka Publ.) p 414
[14] Kilchevsky N A 1977 Course of theoretical mechanics (Moscow: Nauka Publ.) p 475
[15] Zeldovich Ya B and Myshcis A D 1968 Elements of applied mathematics (Moscow: Nauka Publ.) p 615
[16] Abovsky N P, Endqievsky L B, Savchenkov V I, Deruga A P and Reitman M I 1978 Selekted problems of building mechanics and theory elasticity (Moscow: Stroyizdat Publ.) p 188
[17] Chertov A G 1977 Units of physics quantities (Moscow: High School Publ.) p 286
[18] Prudnikov A P, Brychkov U A and Marichev O I 1981 Integrals and series (Moscow: Nauka Publ.) p 793
[19] Feodosev V I 2004 Strength of materials. Textbook (Moscow: N.E. Bauman’s MGTU Publ) p 592
[20] Birger I A and Panovko Ya G 1968 Durability, stability, oscillations. Reference book (Moscow, Machine Building Publ.) p 831