GLOBAL WELL-POSEDNESS OF THE $n$-DIMENSIONAL HYPER-DISSIPATIVE BOUSSINESQ SYSTEM WITHOUT THERMAL DIFFUSIVITY

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ABSTRACT. In this paper, we start to investigate the global existence and uniqueness of weak solutions of the $n$-dimensional ($n \geq 3$) hyper-dissipative Boussinesq system without thermal diffusivity in the periodic domain $\mathbb{T}^n$ with the initial data $u_0 \in L^2(\mathbb{T}^n)$ and $\theta_0 \in L^2(\mathbb{T}^n) \cap L^{\frac{2n}{n+2}}(\mathbb{T}^n)$. Then we focus on the vanishing thermal diffusion limit and obtain the convergent result in the sense of $L^2$-norm. Ultimately, we also prove the global regularity of this system in the case of 3-dimension.

1. Introduction. The Boussinesq equations are a set of partial differential equations describing large scale atmospheric and oceanic fluidity. We consider the following Boussinesq system without thermal diffusion in torus $\mathbb{T}^n$:

$$
\begin{align*}
\partial_t u + \nu (-\Delta)^{\alpha} u + u \cdot \nabla u + \nabla \Pi &= \theta e_n, &\text{in } \mathbb{T}^n \times [0, T], \\
\partial_t \theta + u \cdot \nabla \theta &= 0, &\text{in } \mathbb{T}^n \times [0, T], \\
\nabla \cdot u &= 0, &\text{in } \mathbb{T}^n \times [0, T], \\
(u, \theta)|_{t=0} &= (u_0, \theta_0), &\text{in } \mathbb{T}^n,
\end{align*}
$$

where $u = u(x, t)$ is the velocity and $\theta = \theta(x, t)$, $\Pi = \Pi(x, t)$, $e_n = (0, \cdots, 0, 1)$ denote the temperature, pressure and $n$-dimensional unit vector of torus $\mathbb{T}^n$, respectively. $\alpha > 0$ is the index of fractional derivatives, and $\nu > 0$ is the viscosity.
The operator \((-\Delta)_{\alpha} := \Lambda^{2\alpha}\) is defined by the Fourier series,
\[
\Lambda^{2\alpha} f(x) = \sum_{k \in \mathbb{Z}^n} |k|^{2\alpha} \hat{f}_k e^{ik \cdot x} \quad \text{when} \quad \hat{f}_k = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} e^{-ik \cdot x} f(x) dx,
\]
where \(\mathbb{Z}^n := \{ k \in \mathbb{Z} : |k| \neq 0 \} \).

In recent years, many significant works on the existence and uniqueness of the global weak solution of the Boussinesq equations with partial viscosity have been investigated. In the two-dimensional space, Hou and Li [13] demonstrated a unique global smooth solution of the Boussinesq system for the initial data \((u_0, \theta_0) \in H^s(\mathbb{R}^2) \times H^{s-1}(\mathbb{R}^2)\) with \(s \geq 3\). Applied spectral analysis on Stokes operator, L. He [5] showed the uniqueness of the Boussinesq equations for the initial data \((u_0, \theta_0) \in L^2(\Omega) \times L^2(\Omega)\). Following, Titi and co-authors [12] proved the uniqueness of the Boussinesq equations by the “stream-function” methods in the periodic domain for the initial data \((u_0, \theta_0) \in H^1(\mathbb{T}^2) \times L^2(\mathbb{T}^2)\). Recently, I. Kukavica and W. Wang [9] stated the global existence of solutions for the fractional Boussinesq system in \(W^{s,q}(\mathbb{R}^2) \times W^{s,q}(\mathbb{R}^2)\) by defining a generalized vorticity. More results about the existence and uniqueness of the global weak solution of the 2D Boussinesq system can be found in the references (see [3,4,10,11]) and references therein. In the case of higher-dimensional space, Qiu and Yu [8] proved that the 3D Boussinesq equations has a global existence and unique solution. R.H. Ji et.al [2] obtained a unique and global existence of weak solution to the \(n\)-dimensional Boussinesq system given the weaker initial data \(u_0 \in L^2(\mathbb{R}^n)\) and \(\theta_0 \in L^2(\mathbb{R}^n) \cap L^{\frac{4n}{n+2}}(\mathbb{R}^n)\) (where \(n \geq 2\)).

To improve regularity, Yamazaki [20] depicted the regular estimates of the \(n\)-dimensional Boussinesq equations for the initial data \((u_0, \theta_0) \in H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)\) with \(s \geq 2 + \frac{2}{n}\), in the case of \(\alpha \geq \frac{1}{2} + \frac{2}{n}\). It is mentioned that Ye [22] used interpolation inequality to prompt the regularity results to \(s > 1 + \frac{2}{n}\).

The work of this paper is divided into two parts. Firstly, we investigate the global existence and uniqueness of weak solutions and the regularity of the Boussinesq system (1.1)-(1.4). The second work researches the vanishing thermal diffusion limit problem of the following \(n\)-dimensional Boussinesq system
\begin{align}
\partial_t u^\kappa + \nu(-\Delta)^\alpha u^\kappa + u^\kappa \cdot \nabla u^\kappa + \nabla \Pi^\kappa &= \theta^\kappa e_n, \quad \text{in} \quad \mathbb{T}^n \times [0, T], \\
\partial_t \theta^\kappa - \kappa \Delta \theta^\kappa + u^\kappa \cdot \nabla \theta^\kappa &= 0, \quad \text{in} \quad \mathbb{T}^n \times [0, T], \\
\nabla \cdot u^\kappa &= 0, \quad \text{in} \quad \mathbb{T}^n \times [0, T], \\
(u^\kappa, \theta^\kappa)|_{t=0} &= (u_0^\kappa, \theta_0^\kappa), \quad \text{in} \quad \mathbb{T}^n,
\end{align}
where \(\kappa \geq 0\) is the molecular diffusion.

The global existence and unique solution of the weak solution of the Boussinesq system (1.5)-(1.8) was proved by Xiang and Yan in [19]. Study on the vanishing thermal diffusion limit problem, Jin and co-authors [7] discussed the vanishing thermal diffusion limit problem of the two-dimensional Boussinesq equations with a slip boundary conditions. Recently, Ji and co-authors [2] investigated the vanishing thermal diffusion limit problem of the \(n\)-dimensional Boussinesq equations with \(\alpha \geq \frac{1}{2} + \frac{2}{n}\) and obtained the convergence rate in the sense of \(L^2\)-norm.

We investigate the existence of the global weak solution of the Boussinesq equations (1.1)-(1.4) by the classical Galerkin approximation. The main motivation of our research is to prove the uniqueness of the weak solution of the non-diffusive generalized Boussinesq system when the initial data requirement is lowered. Inspired
by [2,5], we prove the uniqueness of the weak solution of the equations. We apply spectral theory to the Boussinesq equations to obtain the estimates
\[ \|2^{m(m-1)} P_m u\|_{L^1_T L^2} \|u(m)\| \leq C(T, \|u_0\|_{L^2}, \|\theta_0\|_{L^2}), \]  
(1.9) in Lemma 3.1. This is a triple norm, where \( P_m u \) represents the projection of \( u \) on the set spanned by these eigenvectors of Stokes whose eigenvalues lie between \( 2^{m-1} \) and \( 2^m \), and \( L^1_T \) represents the integral of the function from time 0 to \( T \) with respect to time \( t \). The greatest difficulty of inequality (1.9) is to estimate the term of \( \|P_m (u \cdot \nabla u)\|_{L^2(\mathbb{T}^n)} \). However, there is no Littlewood-Paley decomposition and Bony's paraproduct decomposition in the bounded domain. By using the new Bernstein type inequality not only the harmonic techniques can be avoided, but also the setting of the initial data can be reduced. Then, we carry out the variable transformation to transfer the nonlinear equation into a linear equation without using Osgood lemma and complete the proof by techniques of Yudovich (see [21]). Therefore we set up our main results: Theorem 2.5 and Theorem 2.6. At same time, we construct logarithmic inequality to get a norm estimate of the \( \|\nabla u\|_{L^\infty} \) and improve the regularity of the solutions obtained in Theorem 2.5 provided that \( n = 3 \).

The rest of the paper is organized as follows. Section 2 derives some technique lemmas to state the main results after briefly introducing some content of spectral analysis. we prove the global existence and uniqueness of weak solutions in Section 3. The vanishing thermal diffusion limit for the Boussinesq equations (1.5)-(1.8) is discussed in Section 4. The regularity of the solutions for the equations (1.1)-(1.4) are obtained in last Section.

2. Preliminaries and main results. In this section we will introduce some notations and preliminary results, and then state main theorems in this paper. Firstly, we express \( A \leq cB \) as \( A \lesssim B \), where \( c \) is a constant depending only on the dimension \( n \), the viscosity coefficient \( \nu \), and the initial data \( u_0 \) and \( \theta_0 \). Denote the space \( L^p(0; T; X) \) (where \( X \) is Banach space) with the norm \( \| \cdot \|_{L^p X} \), for example, the notation \( \|u\|_{L^1_T L^2} \) represents the norm of \( u \in L^1(0; T; L^2(\mathbb{T}^n)) \). We denote the bilinear commutator operator \( [A, B] = AB - BA \). For the completeness, we still introduce relevant results about spectral theory and refer to the reference [14].

Define the inhomogeneous Lebesgue Sobolev space \( H^s(\mathbb{T}^n) \) and the homogeneous Sobolev space \( \dot{H}^s(\mathbb{T}^n) \), respectively, by
\[
H^s(\mathbb{T}^n) = \left\{ f \in H^s(\mathbb{T}^n)^n : f(x) = \sum_{k \in \mathbb{Z}^n} \hat{f}_k e^{ik \cdot x}, \hat{f}_k = \bar{\hat{f}}_{-k}, \|f\|_{H^s}^2 < \infty \right\},
\]
\[
\dot{H}^s(\mathbb{T}^n) = \left\{ f \in H^s(\mathbb{T}^n)^n : \int_{\mathbb{T}^n} f \, dx = 0, \|f\|_{\dot{H}^s}^2 < \infty \right\},
\]
where
\[
\|f\|_{H^s}^2 = (2\pi)^n \sum_{k \in \mathbb{Z}^n} (1 + |k|^{2s})^2 |\hat{f}_k|^2, \quad \|f\|_{\dot{H}^s}^2 = (2\pi)^n \sum_{k \in \mathbb{Z}^n} |k|^{2s} |\hat{f}_k|^2.
\]
In particular, the spaces \( H^s(\mathbb{T}^n) \) and \( \dot{H}^s(\mathbb{T}^n) \) become the spaces \( L^2(\mathbb{T}^n) \) and \( \dot{L}^2(\mathbb{T}^n) \) when \( s = 0 \). Then we introduce
\[
\dot{L}^2_\sigma(\mathbb{T}^n) = \left\{ f \in \dot{L}^2(\mathbb{T}^n)^n : \nabla \cdot f = 0 \right\},
\]
where
\[ H^1(\mathbb{T}) = \left\{ f \in H^1(\mathbb{T})^n : \nabla \cdot f = 0 \right\}. \]

Let \( P \) be the Leray projector \( P : L^2(\mathbb{T})^n \to \dot{L}^2(\mathbb{T})^n \). Then we define the Stokes operator \( A \) by
\[ A : \mathcal{D}(A) \subset \dot{L}^2(\mathbb{T})^n \to \dot{L}^2(\mathbb{T})^n, \quad A = P(-\triangle), \]
where
\[ \mathcal{D}(A) = H^2(\mathbb{T})^n \cap H^1(\mathbb{T}). \]
The following result is well-known, and we can refer [14].

The vectors \( \omega_j \in \mathcal{D}(A) \) are the eigenfunctions of the Stokes operator \( A \) and \( \lambda_j \) is the eigenvalue corresponding to \( \omega_j \), that is \( A\omega_j = \lambda_j \omega_j \) with
\[ 0 < \lambda_1 \leq \cdots \leq \lambda_j \leq \lambda_{j+1} \leq \cdots \text{ and } \lim_{j \to \infty} \lambda_j = \infty. \]

Next we introduce the projection operators \( \{P_m\}_{m \geq 1} \) corresponding to above basis \( \{\omega_j\} \),
\[ P_m u = \begin{cases} \sum_{\lambda_j < 2}(u, \omega_j)\omega_j, & m = 1, \\ \sum_{2m-1 \leq \lambda_j < 2m}(u, \omega_j)\omega_j, & m > 1. \end{cases} \]
Then we introduce the operator \( A^\alpha \) by
\[ A^\alpha u = \sum_j \lambda_j^\alpha (u, \omega_j)\omega_j, \quad u \in \mathcal{D}(A^\alpha), \]
where
\[ \mathcal{D}(A^\alpha) = \left\{ u \in \dot{L}^2(\mathbb{T})^n | \sum_j \lambda_j^{2\alpha} |(u, \omega_j)|^2 < \infty \right\}. \]

**Lemma 2.1.** Let \( u \in \dot{L}^2(\mathbb{T}) \cap \dot{H}^k(\mathbb{T}), |k| \leq \alpha \) and \( p \in [1, \infty) \), then
\[ \|D^k P_m u\|_{L^p} \lesssim 2^\frac{k+\frac{\alpha}{2}}{2} \|P_m u\|_{L^2}. \]

**Proof.** First we show a very useful inequality in the paper. When \( m > 1 \),
\[ 2^{\alpha(m-1)}\|P_m u\|_{L^2} \leq (A^\alpha P_m u, P_m u) = \sum_{2^{m-1} \leq \lambda_j < 2^m} \lambda_j^\alpha |(u, \omega_j)|^2 \leq 2^{\alpha m} \|P_m u\|_{L^2}. \]
Namely,
\[ 2^{\frac{\alpha(m-1)}{2}}\|P_m u\|_{L^2} \leq \|A^\alpha P_m u\|_{L^2} \leq 2^{\frac{\alpha m}{2}} \|P_m u\|_{L^2}, \tag{2.1} \]
here we use the following facts that
\[ (A^\alpha P_m u, P_m u) = \|A^\alpha P_m u\|_{L^2}^2. \]
However, for the case of \( m = 1 \), we can only obtain
\[ \|A^\alpha P_m u\|_{L^2} \leq 2^{\frac{\alpha}{2}} \|P_1 u\|_{L^2}. \tag{2.2} \]
With the Gagliardo-Nirenberg inequality and inequality (2.1) and (2.2), we have the Bernstein type inequality,
\[ \|D^k P_m u\|_{L^p} \lesssim \|P_m u\|_{L^2}^{\frac{a-k-\frac{d}{2}+\frac{\alpha}{2}+\frac{n}{2}}{a}} \|A^\alpha P_m u\|_{L^2}^{\frac{k+\frac{\alpha}{2}+\frac{n}{2}}{a}} \lesssim 2^{\frac{k+\frac{\alpha}{2}}{2}} \|P_m u\|_{L^2}. \]

**Lemma 2.2.** Let \( u \in \dot{L}^2(\mathbb{T}) \cap \dot{H}^s(\mathbb{T}) \), suppose \( s > \frac{n}{2} + 1 \) and \( n \geq 2 \), then
\[ \|\nabla u\|_{L^\infty} \lesssim 1 + \|u\|_{L^2} + \|u\|_{\dot{H}^{1+\frac{n}{2}}} \log(2 + \|u\|_{\dot{H}^s}). \]
Proof. We make estimates as follows

\[ \| \nabla u \|_{L^\infty} \lesssim \| \nabla P_1 u \|_{L^\infty} + \sum_{m=2}^{N} \| \nabla P_m u \|_{L^\infty} + \sum_{m=N+1}^{\infty} \| \nabla P_m u \|_{L^\infty} \]

\[ \lesssim 2^{\frac{n}{2}} \| P_1 u \|_{L^2} + \sum_{m=2}^{N} 2^{\frac{n}{2} + \frac{m}{2}} \| \nabla P_m u \|_{L^2} + \sum_{m=N+1}^{\infty} 2^{\frac{n}{2} + (\frac{m}{2} + 1 - s) + \frac{m}{2}} \| P_m u \|_{L^2} \]

\[ \lesssim \| u \|_{L^2} + N \| \Lambda^{\frac{n}{2} + 1} u \|_{L^2} + 2^{\frac{n}{2} + (\frac{n}{2} + 1 - s)\| u \|_{\dot{H}^s}} \]

Using Lemma 2.1 and taking \( N = \left\lceil \frac{2}{s \log(2 + \| u \|_{\dot{H}^s})} \right\rceil \), then we obtain the result. \( \square \)

Lemma 2.3. (see [15].) Let \( s > 0, p \in (1, \infty), \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4} = \frac{1}{p}, \) \( p_2 \in [p, \infty) \) and \( p_1, p_4 \in [p, \infty] \). Assuming \( f, g \) be smooth functions satisfying \( \nabla f \in L^{p_3}, \Lambda^s f \in L^{p_3}, \Lambda^{s-1} g \in L^{p_2} \) and \( g \in L^{p_4} \). Then

\[ \| \Lambda^s (f g) - f \Lambda^s g \|_{L^p} \lesssim \| \nabla f \|_{L^{p_3}} \| \Lambda^{s-1} g \|_{L^{p_2}} + \| \Lambda^s f \|_{L^{p_3}} \| g \|_{L^{p_4}}. \]

Definition 2.4. Assume \((u_0, \theta_0) \in L^2(\mathbb{T}^n) \times L^2(\mathbb{T}^n), \alpha \geq \frac{1}{2} + \frac{n}{4} \) and \( T > 0, (u, \theta) \) is a weak solution pair of the Boussinesq equations (1.1)-(1.4) on the time interval [0, T] if it satisfies:

(i) \( u \in C_w([0, T]; L^2) \cap L^2([0, T]; \dot{H}^\alpha) \) and \( \theta \in C_w([0, T]; L^2) \cap L^\infty([0, T]; L^2); \)

(ii) For any \( \Psi \in \big[ C^\infty(\mathbb{T}^n \times [0, T]) \big]^n, \nabla \cdot \Psi = 0, \)

\[ - \int_0^T (\theta, \frac{d}{dt}\Psi) \text{d}s + \nu \int_0^T (\Lambda^s u, \Lambda^s \Psi) \text{d}s + \int_0^T (u \cdot \nabla u, \Psi) \text{d}s \]

\[ \leq (u_0, \Psi(0)) - (u(t), \Psi(t)) + \int_0^T (\varphi_0, \Psi) \text{d}s \]

and for any \( \Phi \in C^\infty(\mathbb{T}^n \times [0, T]), \)

\[ - \int_0^T (\theta, \frac{d}{dt}\Phi) \text{d}s + \int_0^T (\theta u, \nabla \Psi) \text{d}s \leq (\theta_0, \Phi(0)) - (\theta(t), \Phi(t)). \]

Using these technique lemmas, we obtain the following results in this paper.

Theorem 2.5. Let \( \alpha \geq \frac{1}{2} + \frac{n}{4} \) and \( n \geq 3 \). Assume \( u_0 \in L^2(\mathbb{T}^n) \) and \( \theta_0 \in L^2(\mathbb{T}^n) \cap L^{\frac{4n}{n+2}}(\mathbb{T}^n). \) Then there exists a unique weak solution \((u, \theta)\) of the equations (1.1)-(1.4). In addition,

\( u \in C_w([0, T]; L^2) \cap L^2([0, T]; \dot{H}^\alpha), \theta \in C_w([0, T]; L^2) \cap L^\infty([0, T]; L^2), \)

\( \| \Lambda^{(m-1)} \|_{P_m u} \|_{L^2} \|_{L^2(\mathbb{T}^n)} < \infty. \)

Theorem 2.6. Let \( \alpha \geq \frac{1}{2} + \frac{n}{4} \), \( n \geq 3 \) and \( T > 0 \). Suppose that the initial data \((u_0^0, \theta_0^0) \in L^2(\mathbb{T}^n) \times L^2(\mathbb{T}^n), (u_0^0, \theta_0^0) \in L^2(\mathbb{T}^n) \times L^2(\mathbb{T}^n) \) and \( \theta_0^0 \in L^{\frac{4n}{n+2}}(\mathbb{T}^n), \theta_0^0 \in L^{\frac{4n}{n+2}}(\mathbb{T}^n). \) Assume that \((u^0, \theta^0)\) and \((u^0, \theta^0)\) are solutions of the Boussinesq equations (1.5)-(1.8) with initial data \((u_0^0, \theta_0^0)\) and (1.1)-(1.4) with initial data \((u_0^0, \theta_0^0)\) on the interval [0, T], respectively. Then we have

\( \| u^0(t) - u^0(t) \|_{L^2} + \| \nabla \eta(t) - \nabla \eta(t) \|_{L^2} \leq C(T)\kappa, \)

for every \( t \in [0, T] \), where \( -\Delta \eta = \theta^0, -\Delta \eta = \theta^0. \)
Theorem 2.7. Let $\alpha \geq \frac{5}{4}$ and $\gamma > \frac{5}{2}$. Suppose $(u_0, \theta_0) \in \dot{H}^\gamma(T^3) \times \dot{H}^\gamma(T^3)$. Then there exists a unique solution $(u, \theta)$ of the equations (1.1)-(1.4). In addition, 

$$u \in L^\infty([0, T]; \dot{H}^\gamma(T^3)) \cap L^2([0, T]; \dot{H}^{\gamma + \alpha}(T^3)), \quad \theta \in L^\infty([0, T]; \dot{H}^\gamma(T^3)).$$

3. Proof of Theorem 2.5. By using a prior estimate, one has

$$\sup_{t \in [0, T]} \|(u, \theta)\|^2_{L^2} + \nu \int_0^T \|\Lambda^\alpha u\|^2_{L^2} \lesssim 1,$$

where $\|(u, \theta)\|^2_{L^2} = \|u\|^2_{L^2} + \|\theta\|^2_{L^2}$.

From the equation (1.2) note the following result

To prove the existence, we first consider the following approximation system:

$$\begin{align*}
\partial_t u_m + \nu(-\Delta)^\alpha u_m + u_m \cdot \nabla u_m + \nabla \Pi_m &= \theta_m e_n, \\
\partial_t \theta_m + u_m \cdot \nabla \theta_m &= 0, \\
\nabla \cdot u_m &= 0, \\
(u_m, \theta_m)|_{t=0} &= (u_{0,m}, \theta_{0,m}),
\end{align*}$$

where $u_m = \sum_{j=1}^m (u_j, \omega_j) \omega_j$, $\theta_m = \sum_{j=1}^m (\theta_j, \bar{\omega}_j) \bar{\omega}_j$ and $\{\bar{\omega}_j\}^m$ is the smooth orthogonal basis of $L^2$. By the argument of the classical Galerkin method, we deduce

$$u \in C_w([0, T]; L^2) \cap L^2([0, T]; H^\alpha), \quad \theta \in C_w([0, T]; L^2) \cap L^\infty([0, T]; L^2).$$

To proceed the proof smoothly, we have to prove a very helpful lemma:

**Lemma 3.1.** Let $n \geq 2$, $T > 0$, $q \geq 2$ and $\alpha \geq \frac{1}{2} + \frac{a}{q}$. Suppose $(u_0, \theta_0) \in L^2(T^n) \times L^2(T^n)$. If $(u, \theta)$ is the solution pair of the equation (1.1)-(1.4), then

$$\|\nabla u\|_{L^q T^\alpha} \lesssim \sqrt{q}.$$

**Proof.** Using $P_m$ on the equation (1.1) and taking the inner product with $P_m u$, one obtains

$$\begin{align*}
\frac{1}{2} \frac{d}{dt} \|P_m u\|^2_{L^2} + \nu \|\Lambda^\alpha P_m u\|^2_{L^2} \\
\leq \|P_m u\|_{L^2} \|P_m \theta\|_{L^2} + |(P_m (u \cdot \nabla u), P_m u)| \\
\leq \|P_m u\|_{L^2} \|P_m \theta\|_{L^2} + \|\nabla P_m u\|_{L^\infty} \|P_m u\|^2_{L^2}.
\end{align*}$$

Then by inequality (2.1), we have

$$\frac{d}{dt} \|P_m u\|_{L^2} + \nu 2^{\alpha(m-1)} \|P_m u\|_{L^2} \leq \|P_m \theta\|_{L^2} + \|\nabla P_m u\|_{L^\infty} \|P_m u\|_{L^2}. \quad (3.5)$$

Applied Lemma 2.1 and inequality (2.1), it is easy to see

$$\|\nabla P_m u\|_{L^\infty} \lesssim 2^{\frac{\alpha}{2}(1+\frac{a}{2})} \|P_m u\|_{L^2} \leq 2^{\frac{\alpha}{2}(1+\frac{a}{2}-\alpha+\frac{\alpha}{2})} \|\Lambda^\alpha u\|_{L^2},$$

thus

$$\|\nabla P_m u\|_{L^\infty} \lesssim 2^{m(\frac{1}{2}+\frac{a}{2}-\alpha)+\alpha} \|\Lambda^\alpha u\|_{L^2} \|\Lambda^\alpha P_m u\|_{L^2} \leq 2^{\alpha} \|\Lambda^\alpha u\|_{L^2} \|\Lambda^\alpha P_m u\|_{L^2}.$$

Integrating over time $[0, t]$ in (3.5), it follows

$$\|P_m u(t)\|_{L^2} + \nu \int_0^t 2^{\alpha(m-1)} \|P_m u(s)\|_{L^2} ds$$
\[ \leq \|P_m u_0\|_{L^2} + \int_0^t \|P_m \theta\|_{L^2} ds + 2^{\alpha} \int_0^t \|\Lambda^\alpha u\|_{L^2} \|\Lambda^\alpha P_m u\|_{L^2} ds. \]

From Minkowski’s inequality, one obtains

\[ \|P_m u\|_{L^2} \|\theta\|_{L^2} + \nu \int_0^T 2^{\alpha(m-1)} \|P_m u\|_{L^2} ds \|\theta\|_{L^2} \]

\[ \leq \|u_0\|_{L^2} + \int_0^T \|\theta_0\|_{L^2} ds + 2^{\alpha} \int_0^T \|\Lambda^\alpha u\|_{L^2}^2 ds. \]

Above inequality and H"older’s inequality imply that

\[ \int_0^t \|\nabla u\|_{L^2} ds \leq \int_0^t \sum_{m=1}^\infty \|\nabla P_m u\|_{L^2} ds \leq \sum_{m=1}^\infty \int_0^t 2^{3\alpha(1+n(\frac{1}{2} - \frac{1}{4}))} \|P_m u\|_{L^2} ds \]

\[ \leq \sum_{m=1}^\infty 2^{\frac{3}{2} + \frac{n}{4} - \frac{n}{q}} \int_0^t 2^{(\frac{3}{2} + \frac{n}{4})(m-1)} \|P_m u\|_{L^2} ds \]

\[ \leq \left( \sum_{m=1}^\infty 2^{1 + \frac{n}{4} - \frac{n}{q}} \right) \left[ \sum_{m=1}^\infty \left( \int_0^t 2^{(\frac{3}{2} + \frac{n}{4})(m-1)} \|P_m u\|_{L^2} ds \right)^{\frac{2}{3}} \right]^{\frac{3}{2}}, \]

Here we note that the facts

\[ \sum_{m=1}^\infty 2^{1 + \frac{n}{4} - \frac{n}{q}} \leq 2^{1 + \frac{n}{4}} \int_0^\infty 2^{-\frac{n}{4}(x-1)} dx \lesssim q. \]

Therefore, we conclude

\[ \int_0^t \|\nabla u\|_{L^2} ds \lesssim \sqrt{q}. \]

To check the uniqueness, we firstly assume that \((u_1, \theta_1)\) and \((u_2, \theta_2)\) are two pairs of solutions of the Boussinesq system on \([0, T]\). Denote \(\tilde{u} = u_1 - u_2, \tilde{\theta} = \theta_1 - \theta_2\), and define \(-\Delta \eta_i = \theta_i, (i = 1, 2), \tilde{\eta} = \eta_1 - \eta_2, \int T^\nu \eta_i dx = 0\). Each equation corresponds to a difference, it follows that

\[ \partial_t \tilde{u} + \nu (-\Delta)^\alpha \tilde{u} + u_2 \cdot \nabla \tilde{u} + \tilde{u} \cdot \nabla u_1 + \nabla \tilde{\Pi} = \tilde{\theta} e_n, \quad (3.6) \]

\[ \partial_t \tilde{\theta} + \tilde{u} \cdot \nabla \theta_1 + u_2 \cdot \nabla \tilde{\theta} = 0, \quad (3.7) \]

\[ \nabla \cdot \tilde{u} = 0, \quad (3.8) \]

\[ (\tilde{u}, \tilde{\theta})|_{t=0} = 0. \quad (3.9) \]

Taking the \(L^2\) inner product of the equation (3.6) with \(\tilde{u}\) and applying the equation (3.8). Then, we obtain

\[ \frac{1}{2} \frac{d}{dt} \|\tilde{u}\|_{L^2}^2 + \nu \|\Lambda^\alpha \tilde{u}\|_{L^2}^2 \leq \left| \int_{T^\nu} \tilde{\theta} e_n \cdot \tilde{u} dx \right| + \left| \int_{T^\nu} \tilde{u} \cdot \nabla u_1 \cdot \tilde{u} dx \right| = I_1 + I_2. \quad (3.10) \]

For \(I_1\), using Hölder’s inequality, Gagliardo-Nirenberg inequality and Young’s inequality, one obtains

\[ I_1 \leq \left| \int_{T^\nu} \nabla \tilde{\eta} \cdot \nabla \tilde{u} dx \right| \leq \|\nabla \tilde{\eta}\|_{L^2} \|\nabla \tilde{u}\|_{L^2} \lesssim \|\nabla \tilde{\eta}\|_{L^2} \|\Lambda^\alpha \tilde{u}\|_{L^2}^{\frac{1}{2}} \|\tilde{u}\|_{L^2}^{\frac{\alpha - 1}{2}} \]

\[ \lesssim \|\nabla \tilde{\eta}\|_{L^2} (\|\Lambda^\alpha \tilde{u}\|_{L^2} + \|\tilde{u}\|_{L^2}) \lesssim \|\nabla \tilde{\eta}\|_{L^2}^2 + \|\tilde{u}\|_{L^2}^2 + \frac{\nu}{6} \|\Lambda^\alpha \tilde{u}\|_{L^2}^2. \]
For $I_2$, making use of the Sobolev embedding $\dot{H}^\alpha(T^n) \hookrightarrow L^{4\alpha} \hookrightarrow \dot{H}^{\alpha-1}(T^n)$ and the Young’s inequality, it attains

$$I_2 \leq \int_{T^n} |\nabla u_1||\tilde{u}|^2 dx \leq \|\tilde{u}\|_{L^{4\alpha}} \|\nabla u_1\|_{L^{4\alpha}} \|\tilde{u}\|_{L^2} \lesssim \|\Lambda^\alpha \tilde{u}\|_{L^2} \|\Lambda^\alpha u_1\|_{L^2} \|\tilde{u}\|_{L^2} \lesssim \|u_1\|_{L^{4\alpha}}^2 \|\tilde{u}\|_{L^2}^2 + \lambda \|\Lambda^\alpha \tilde{u}\|_{L^2}^2.$$

Taking the $L^2$ inner product of the equation (3.7) with $\tilde{\eta}$, it is easy to see

$$\frac{1}{2} \frac{d}{dt} \|\nabla \tilde{\eta}\|_{L^2}^2 \leq \int_{T^n} \tilde{u} \cdot \nabla \theta \tilde{\eta} dx + \int_{T^n} u_2 \cdot \nabla \tilde{\theta} \tilde{\eta} dx = I_3 + I_4. \quad (3.11)$$

From the Sobolev embedding $\dot{H}^\alpha(T^n) \hookrightarrow L^{4\alpha}$ and the Young’s inequality, and the fact $\|\theta\|_{L^p(T^n)} \leq \|\theta_0\|_{L^p(T^n)}$, the term of $I_3$ can be bounded by

$$I_3 \leq \int_{T^n} \tilde{u} \cdot \theta \nabla \tilde{\eta} dx \leq \|\tilde{u}\|_{L^{4\alpha}} \|\theta_1\|_{L^{4\alpha}} \|\nabla \tilde{\eta}\|_{L^2} \lesssim \|\nabla \tilde{\eta}\|_{L^2} \lesssim \|\nabla \tilde{\eta}\|_{L^2}^2 + \frac{\lambda'}{6} \|\Lambda^\alpha \tilde{u}\|_{L^2}^2.$$

For $I_4$, using Hölder’s inequality with the index $q$ satisfying $2q - n > 0$, it is easy to obtain the following inequality

$$I_4 = \int_{T^n} u_2 \Delta \tilde{\eta} \cdot \nabla \tilde{\eta} dx \leq \int_{T^n} \nabla u_2 \nabla \tilde{\eta} \cdot \nabla \tilde{\eta} dx \lesssim \|\nabla u_2\|_{L^q} \|\nabla \tilde{\eta}\|_{L^{q'}} \|\nabla \tilde{\eta}\|_{L^2} \lesssim \|\nabla u_2\|_{L^q} \|\nabla \tilde{\eta}\|_{L^{q'}} \|\nabla \tilde{\eta}\|_{L^2}^{2-n}.$$

Let $\psi(t) = \|\tilde{u}, \nabla \tilde{\eta}(t)\|_{L^2}^2$, $\psi_\varepsilon(t) = \|\tilde{u}, \nabla \tilde{\eta}(t)\|_{L^2}^2 + \varepsilon$. Adding inequalities (3.10) and (3.11), we find

$$\frac{d}{dt} \psi_\varepsilon(t) \lesssim (1 + \|u_1\|_{L^{4\alpha}}^2) \psi_\varepsilon(t) + \|\theta_0\|_{L^2}^2 \|\nabla u_2\|_{L^q} \psi_\varepsilon^{1-\frac{n}{2q}}(t).$$

Multiplying both sides by $e^{-\frac{\lambda}{2} t} \int_0^t + \|u_1\|_{L^{4\alpha}} ds \psi_\varepsilon e^{\frac{-\lambda}{2} t} (t)$ of above inequality and setting $\phi_\varepsilon(s) = e^{-\frac{\lambda}{2} \int_0^s + \|u_1\|_{L^{4\alpha}} ds} \psi_\varepsilon(s)$ and $\phi(s) = e^{-\frac{\lambda}{2} \int_0^t + \|u_1\|_{L^{4\alpha}} ds} \psi(s)$, we obtain

$$\frac{d}{dt} \phi_\varepsilon(t) \lesssim \|\theta_0\|_{L^2}^2 \|\nabla u_2(s)\|_{L^q} \psi_\varepsilon^{1-\frac{n}{2q}}(t).$$

Integrating on the interval $[0, t]$, we have

$$\phi_\varepsilon(t) \lesssim \psi_\varepsilon^{1-\frac{n}{2q}} + \int_0^t \|\theta_0\|_{L^2}^2 \|\nabla u_2(s)\|_{L^q} \psi_\varepsilon^{1-\frac{n}{2q}} ds \Rightarrow \psi_\varepsilon^{\frac{1}{2}}.$$

Letting $\varepsilon \to 0$, we find

$$\phi(t) \lesssim \left( \|\theta_0\|_{L^2}^2 \int_0^t \|\nabla u_2(s)\|_{L^q} ds \right)^{\frac{2q}{2q}} \psi(t) \lesssim \left( \frac{1}{\sqrt{q}} \right)^{\frac{2q}{2q}} (t).$$

Thanks to the conclusion of Lemma 3.1, we get

$$\|\tilde{u}, \nabla \tilde{\eta}(t)\|_{L^2}^2 \lesssim \left( \frac{1}{\sqrt{q}} \right)^{\frac{2q}{2q}} (t).$$

Letting $q \to \infty$, we obtain $\|\tilde{u}, \nabla \tilde{\eta}(t)\|_{L^2} = (0, 0)$ on $[0, T]$. Then since $\tilde{\theta}$ is $L^2$ weak continuous in time, it implies that $\tilde{\theta} = 0$ on $[0, T]$. 
4. Proof of Theorem 2.6. Assume that \((u^0, \theta^0)\) and \((u^\infty, \theta^\infty)\) are solutions of the Boussinesq equations (1.1)-(1.4) and (1.5)-(1.8) on \([0, T]\), respectively. Then we prove that as \(\kappa \to 0\), solution \((u^\infty, \theta^\infty)\) converges to solution \((u^0, \theta^0)\) in some suitable space. Set \(\tilde{u} = u^\infty - u^0\), \(\tilde{\theta} = \theta^\infty - \theta^0\), \(\nabla \tilde{\Pi} = \nabla \Pi^\infty - \nabla \Pi^0\) and define

\[-\Delta \eta^\kappa = \theta^\infty, -\Delta \eta^0 = \theta^0, \quad \eta = \eta^\kappa - \eta^0, \quad \int_{\mathbb{T}^n} \eta^\kappa \, dx = 0, \quad \int_{\mathbb{T}^n} \eta^0 \, dx = 0.\]

Each equation corresponds to a difference. It follows that

\[
\begin{align*}
\partial_t \tilde{u} + \nu (-\Delta)^\alpha \tilde{u} + u^0 \cdot \nabla \tilde{u} + \tilde{u} \cdot \nabla u^0 + \tilde{u} \cdot \nabla \tilde{\Pi} = \tilde{\theta} e_n, \\
\partial_t \tilde{\theta} - \kappa \Delta \tilde{\theta} - \kappa \Delta \theta^0 + \tilde{u} \cdot \nabla \theta^0 + \tilde{u} \cdot \nabla \tilde{\theta} + u^0 \cdot \nabla \tilde{\theta} = 0,
\end{align*}
\]

for \(\kappa \in (0, 1)\), where \(\nu \geq 0\) is a viscosity parameter. For \(\kappa = 0\), the above system becomes the Boussinesq system (1.1)-(1.4) and (1.5)-(1.8). Letting \(\kappa \to 0\) we obtain the estimates.

Proof of Theorem 2.6. By the Gronwall’s inequality, it is easy to see

\[
\|\tilde{u}\|_{L^2}^2 + \|\tilde{\theta}\|_{L^2}^2 \lesssim \|\tilde{u}^0\|_{L^2}^2 + \|\tilde{\theta}^0\|_{L^2}^2 + \int_0^T \|\tilde{u}\|_{L^2}^2 + \|\tilde{\theta}\|_{L^2}^2 \, dt.
\]

This completes to prove the Theorem 2.6.
5. **Proof of Theorem 2.7.** In this section we improve the regularity of the solutions to equations (1.1)-(1.4) in the case of 3-dimension.

Applying the operator $\Lambda^{\alpha}$ on the equation (1.1) and taking the inner product with $\Lambda^{\alpha} u$, one infers

$$\frac{1}{2} \frac{d}{dt} \|\Lambda^{\alpha} u\|_{L^2}^2 + \nu \|\Lambda^{2\alpha} u\|_{L^2}^2 \leq \left| \int_{D^3} \Lambda^{\alpha} u \cdot \Lambda^{\alpha} u dx \right| + \left| \int_{D^3} \Lambda^{\alpha} u \cdot \nabla u \cdot \Lambda^{\alpha} u dx \right| = J_1 + J_2.$$

From Hölder’s inequality and Young’s inequality, $J_1$ can be bounded by

$$J_1 \leq \|\theta\|_{L^2} \|\Lambda^{2\alpha} u\|_{L^2} \lesssim \|\theta_0\|_{L^2}^2 + \frac{1}{4} \nu \|\Lambda^{2\alpha} u\|_{L^2}^2.$$

Thanks to Lemma 2.3, the Sobolev embedding $\dot{H}^{\alpha}(\mathbb{T}^3) \hookrightarrow L^{12}(\mathbb{T}^3)$ and the Young’s inequality, one obtains

$$J_2 \lesssim \|\Lambda^{\alpha} u \|_{L^2} \|\Lambda^{2\alpha} u\|_{L^2} \lesssim \|\nabla u\|_{L^2} \|\Lambda^{\alpha} u\|_{L^2} \lesssim \|\Lambda^{\alpha} u\|_{L^2}^2 + \frac{1}{4} \nu \|\Lambda^{2\alpha} u\|_{L^2}^2.$$

Then we have

$$\frac{d}{dt} \|\Lambda^{\alpha} u(t)\|_{L^2}^2 + \nu \int_0^t \|\Lambda^{2\alpha} u(s)\|_{L^2}^2 ds \leq C.$$

For the brevity, we focus on the case of $\alpha = \frac{1}{3}$ in the equation (1.1), then applying the operator $\Lambda^{\gamma}$ on the equation and taking the inner product with $\Lambda^{\gamma} u$, one infers

$$\frac{1}{2} \frac{d}{dt} \|\Lambda^{\gamma} u\|_{L^2}^2 + \nu \|\Lambda^{\gamma + \frac{3}{2}} u\|_{L^2}^2 \leq \|\Lambda^{\gamma} u \|_{L^2} \|\Lambda^{\gamma} \nabla u\|_{L^2} + \|\Lambda^{\gamma} u \|_{L^2} \|\nabla u\|_{L^2} \|\Lambda^{\gamma} \nabla u\|_{L^2} \lesssim \|\Lambda^{\gamma} u\|_{L^2}^2 + \|\Lambda^{\gamma + \frac{3}{2}} u\|_{L^2}^2 + \nu \|\Lambda^{\gamma + \frac{3}{2}} u\|_{L^2}^2.$$

Applying $\Lambda^{\gamma}$ on the equation (1.2) and taking the inner product with $\Lambda^{\gamma} \theta$, it implies

$$\frac{1}{2} \frac{d}{dt} \|\Lambda^{\gamma} \theta\|_{L^2}^2 \lesssim \|\nabla u\|_{L^2} \|\Lambda^{\gamma} \theta\|_{L^2} \|\Lambda^{\gamma} \theta\|_{L^2} + \|\Lambda^{\gamma} u\|_{L^2} \|\Lambda^{\gamma} \theta\|_{L^2} = J_3 + J_4.$$

For $J_3$, from Lemma 2.2, we obtain

$$J_3 \lesssim (1 + \|u\|_{L^2} + \|u\|_{\dot{H}^{\frac{3}{2}}} \log(2 + \|u\|_{\dot{H}^{\frac{3}{2}}})) \|\Lambda^{\gamma} \theta\|_{L^2}^2 \lesssim (1 + \|u\|_{\dot{H}^{\frac{3}{2}}} \log(2 + \|u\|_{\dot{H}^{\frac{3}{2}}})) \|\Lambda^{\gamma} \theta\|_{L^2}^2.$$

For $J_4$, adopting the method of Ye [22], applying interpolation and Young’s inequality multiple times, we find

$$J_4 \lesssim \|\theta\|_{L^2}^{\frac{2(\gamma + \frac{3}{2})}{8}} \|\Lambda^{\gamma + \frac{3}{2}} u\|_{L^2}^{\frac{4}{\gamma + \frac{3}{2}}} \|\Lambda^{\gamma + \frac{3}{2}} \theta\|_{L^2}^{\frac{4}{\gamma + \frac{3}{2}}} \lesssim \|\theta_0\|_{L^2} \|\Lambda^{3} u\|_{L^2} \|\Lambda^{\gamma + \frac{3}{2}} \theta\|_{L^2}^2 + \frac{1}{4} \|\Lambda^{\gamma + \frac{3}{2}} u\|_{L^2}^2.$$
Combining these two differential inequalities, we have
\[ \| \Lambda^\gamma \theta \|^2_{L^2} + \| \Lambda^{\frac{5}{2}} u \|_{L^2}^2 \leq \| \Lambda^{\frac{5}{2}} u \|_{L^2}^2 + \frac{\nu}{4} \| \Lambda^{\frac{5}{2}} u \|_{L^2}^2. \]

Applying the Gronwall’s inequality again, we have got
\[ \| (u, \theta) \|^2_{H^\gamma} + \nu \int_0^t \| u \|^2_{H^{\gamma + \frac{5}{4}}} \leq C(1 + \| u \|^2_{H^{\gamma + \frac{5}{4}}} (1 + \log(2 + \| (u, \theta) \|^2_{H^\gamma}))) \| (u, \theta) \|^2_{H^\gamma}. \]

Applying the Gronwall’s inequality again, we have got
\[ \| (u, \theta) \|^2_{H^\gamma} + \nu \int_0^t \| u \|^2_{H^{\gamma + \alpha}} ds \leq C. \]

This finishes the proof of Theorem 2.7.

REFERENCES

[1] H. Abidi and P. Zhang, On the global well-posedness of 2-D Boussinesq system with variable viscosity, Adv. Math., 305 (2017), 1202–1249.
[2] N. Boardman, R. H. Ji, H. Qiu and J. Wu, Uniqueness of weak solutions to the Boussinesq equations without thermal diffusion, Commun. Math. Sci., 17 (2019), 1595–1624.
[3] C. Cao and J. Wu, Global Regularity for the Two-Dimensional Anisotropic Boussinesq Equations with Vertical Dissipation, Archive for Rational Mechanics & Analysis, 208 (2013), 985–1004.
[4] D. Chae, Global regularity for the 2D Boussinesq equations with partial viscosity terms, Adv. Math., 203 (2006), 497–513.
[5] L. He, Smoothing estimates of 2D incompressible Navier-Stokes equations in bounded domains with applications, J. Funct. Anal., 262 (2012), 3430–3464.
[6] T. Hmidi and S. Keraani, On the global well-posedness of the Boussinesq system with zero viscosity, Indiana Univ. Math. J., 58 (2009), 1591–1618.
[7] L. Jin, J. Fan, G. Nakamura and Y. Zhou, Partial vanishing viscosity limit for the 2D Boussinesq system with a slip boundary condition, Bound. Value Probl., 2012 (2012), 20–24.
[8] Q. Jiu and H. Yu, Global well-posedness for 3D generalized Navier-Stokes-Boussinesq equations, Acta Mathematicae Applicatae Sinica, English Series, 32 (2016), 1–16.
[9] I. Kukavica and W. Wang, Global Sobolev persistence for the fractional Boussinesq equations with zero diffusivity, Pure Appl. Funct. Anal., 5 (2020), 27–45.
[10] I. Kukavica and W. Wang, Long time behavior of solutions to the 2D Boussinesq equations with zero diffusivity, J. Dyn. Differ. Equ., 32 (2020), 2061–2077.
[11] M. Lai, R. Pan and K. Zhao, Initial Boundary Value Problem for Two-Dimensional Viscous Boussinesq Equations, Arch. Ration. Mech. Anal., 199 (2011), 739–760.
[12] A. Larios, E. Lunasin and E. S. Titi, Global well-posedness for the 2D Boussinesq system with anisotropic viscosity and without heat diffusion, J. Differ. Equ., 9 (2013), 2636–2654.
[13] C. Li and T. Hou, Global well-posedness of the viscous Boussinesq equations, Discrete Contin. Dyn. S., 12 (2005), 1–12.
[14] J. Robinson, J. Rodrigo and W. Sadowski, The Three-Dimensional Navier-Stokes Equations: Classical Theory, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2016.
[15] K. Tosio and P. Gustavo, Commutator estimates and the Euler and Navier-Stokes equations, Commun. Pure. Appl. Math., 41 (1988), 891–907.
[16] C. Wang and Z. Zhang, Global well-posedness for the 2-D Boussinesq system with the temperature-dependent viscosity and thermal diffusivity, Adv. Math., 228 (2011), 43–62.
[17] J. Wu, X. Xu, L. Xue and Z. Ye, Regularity results for the 2D Boussinesq equations with critical or supercritical dissipation, Commun. Math. Sci., 14 (2016), 1963–1997.
[18] J. Wu, X. Xu and Z. Ye, The 2D Boussinesq equations with fractional horizontal dissipation and thermal diffusion, Journal De Mathématiques Pures Et Appliqués, 115 (2018), 187–217.
[19] Z. Xiang and W. Yan, Global regularity of solutions to the Boussinesq equations with fractional dissipation, Adv. Differ. Equ., 18 (2013), 1105–1128.
[20] K. Yamazaki, On the global regularity of N-dimensional generalized Boussinesq system, Appl. Math., 60 (2015), 109–133.
[21] V. Yudovich, Non-stationary flows of an ideal incompressible fluid, Ž. Vyčisl. Mat. i Mat. Fiz, 3 (1963), 1032–1066.
[22] Z. Ye, A note on global well-posedness of solutions to Boussinesq equations with fractional dissipation, *Acta Math. Sci.*, 35 (2015), 112–120.

[23] Z. Ye, Remarks on the improved regularity criterion for the 2D Euler-Boussinesq equations with supercritical dissipation, *Zeitschrift Für Angewandte Mathematik Und Physik*, 67 (2016), 149–156.

[24] Z. Ye, On global well-posedness for the 3D Boussinesq equations with fractional partial dissipation, *Appl. Math. Lett.*, 90 (2019), 1–7.

[25] X. Zhai, B. Dong and Z. Chen, Global well-posed for 2D Boussinesq system with the temperature-dependent viscosity and supercritical dissipation, *J. Differ. Equ.*, 267 (2019), 364–387.

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