Solvability and Controllability Of Nonlinear Second - Order Control Systems Using Fixed Point Theorem

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Abstract
In this work, the nonlinear 2-order systems in Banach-spaces are introduced. Thence the sufficient conditions for existence and controllability of solutions for these systems are established using a strongly continuous cosine family of bounded linear operations on Banach-space and fixed-point theorem by Schauder. Moreover, examples which illustrate mathematical justification of above results are given.

Keywords: Controllability, Second–order system, Semigroup theory, Schauder fixedpoint theorem.

1. Introduction
The theory of control in infinite-dimensional spaces (IDS) has been emerging as an important area for applications only after well-developed semigroup theory was at hand. Many problems from life can be modeled by partial differential equations (PDEqs.), integral Eqs., or coupled ordinary and PDEqs., that can be described as 1 and 2-order DEqs. in IDS using semi-group of operators. Nonlinear Eqs. with and without delays, serve as an abstract formulation for many PEqs. which arise in problems connected with heat flow in materials with memory, viscoelasticity, and other physical phenomena. Thus, it’s important to study the existence of solutions
(ES) of such systems in IDS and its controllability. For the linear systems we can see in [1,2] the semi-group theory and controllability.

A strongly continuous (cont.*) cosine family of bounded linear operators on Banach-space(B-space) will be called a C₀- cosine. Now, let ζ(i), i ∈ ℝ(real numbers), be a C₀- cosine on B-space Ξ and its associated C₀- sine Š(i), i ∈ ℛ, which is defined by:

\[ Š(i)x = \int_0^i ζ(s)xds, x ∈ Ξ, i ∈ ℛ. \]

More details about C₀- cosine theory can be found in [11,12]. In this work, we will be consider the ES for the nonlinear 2-order problem (β₁) and its controllability.

\[
\frac{d}{dt}[\dot{z}(i) - Σ(i, z_i)] = A\dot{z}(i) + B\dot{u}(i) + H(i, \dot{z}_i, \dot{z}(i)), \dot{h}(i, \dot{z}_i, \dot{z}(i)), i ∈ (0, b),
\]

\[ \dot{z}_0 = \emptyset, \quad z'(0) = x_0 \]  \hspace{1cm} (β₁)

Where the values of the state \( z(\cdot) \) is given in the B-space Ξ, and the control function \( \dot{u}(\cdot) \) is given in \( L^2([0, b], \dot{U}) \), a B- space of admissible control functions, with \( \dot{U} \) a B-space. Here the linear operator \( A \) generates a C₀- cosine \( ζ(i), i ∈ ℛ \), on a B-space Ξ, and \( B \) is a bounded linear operator from \( \dot{U} \) into Ξ. The nonlinear operators \( Σ: J × C → Ξ, h: J × C \times Ξ → Ξ, H: J × C × Ξ \times Ξ → Ξ \) are all uniformly bounded cont*. operators, and \( Ξ \in C = C(J_0, Ξ) \), where \( J_0 = [−α, 0] \) and \( J = [0, b] \).

For study the ES of 1-order Eqs. in B-spaces, one can refer to [3-5]. Al-Jawari and Ibrahim [6,7] studied the controllability of control problem in quasi-B-spaces. When we investigated the nonlinear 2-order neutral Eqs. in abstract-spaces, and its solvability we see that they are little known. Recently, the existence problem of 2-order Volterra integrodifferential Eqs. in B-spaces established in [8]. The contrability 2-order neutral functional differential inclusions in B-spaces proved in [9]. The ES for semilinear 2-order delay DEqs. studied in [10]. Sometime the 2-order abstract DEqs. can be convert to 1-order Eqs., and then study the solvability of these systems. But in many problems it is utility to handle the 2-order abstract DEqs. directly. A useful tool for this study is the theory of C₀- cosine. The aim of this work is to study the existence and controllability of the nonlinear 2-order system (β₁) in B-spaces using fixed-point theorem (FPT) by Schauder.

2. Preliminaries

The basic tools used in solving nonlinear Eqs. are FPT and semigroup theory. Here we present the Schauder FPT [Th.2.1] and some results of a C₀- cosine.
Theorem 2.1 [13]: Let $A$ be a nonempty closed convex subset of a normed space $E$. And let $T:A \rightarrow B \subseteq A$. If $T$ is an $C^t$ operator and $B$ compact, then $T$ has a FP.

Now suppose the condition on $A$ as follows.

($\mathcal{A}_1$) $A$ is the infinitesimal generator (IG) of $\zeta(i), i \in \mathcal{R}$ on $B$-space $E$, and the adjoint operator $A^*$ is densely defined (i.e., $\mathcal{D}(A^*)E^*$ [14]).

The IG of $\zeta(i), i \in \mathcal{R}$ is the operator $A:E \rightarrow E$ defined by

$$A_0 x = \frac{d^2}{dt^2} \zeta(i)x|_{t=0} , \quad x \in \mathcal{D}(A),$$

where the domain of $A D(A) = \{x \in E : \zeta(i)x$ is twice $C^t$. differentiable in $i\}$.

Define $Y = \{x \in E : \zeta(i)x$ is once $C^t$. differentiable in $i\}$. To illustrate our main results, we need some lemmas.

Lemma 2.1 [11]: Suppose ($\mathcal{A}_1$) hold. Then

1) $\exists$ constant $c_1 \geq 1$ and $m \geq 0$ such that $\|\zeta(i)\| \leq c_1 e^{m|i|}$ and $\|\dot{\zeta}(i) - \dot{\zeta}(i')\| \leq c_1 \int_0^{i'} e^{m|s|} ds$ for $i, i' \in \mathcal{R}$.

2) $\dot{\zeta}(i) \subseteq Y$ and $\dot{\zeta}(i) \mathcal{Y} \subseteq \mathcal{D}(A)$ for $i \in \mathcal{R}$.

3) $\frac{d}{dt} \zeta(i)x = A\dot{\zeta}(i)x$ for $x \in Y$ and $i \in \mathcal{R}$.

4) $\frac{d^2}{dt^2} \zeta(i)x = A_0 (i)x$ for $x \in \mathcal{D}(A)$ and $i \in \mathcal{R}$.

Lemma 2.2 [11]: Suppose ($\mathcal{A}_1$) hold. Let $V: \mathcal{R} \rightarrow E$ s. t $V$ is $C^t$. differentiable and let $(i) = \int_0^i \dot{\zeta}(i-s)\nu(s)ds$. Then is twice $C^t$. differentiable and for $i \in \mathcal{R}, (i) \in D(A)$, $(i) = \int_0^i \zeta(i-s)\nu(s)ds$ and $(i) = A(i) + \nu(i)$.

3. EMS Problem ($\mathcal{B}_2$).

In the following we will study the existence of mild solutions(EMS) to the problem ($\mathcal{B}_2$), but without control variable. Thus here we shall consider the following problem:

$$\frac{d}{dt} [\dot{z}(i) - \Xi(i, \dot{z}(i))] = A\dot{z}(i) + H(i, \dot{z}(i), \dot{z}(i), h(i, \dot{z}(i), \dot{z}(i))) , i \in (0, b), \left\{ \begin{array}{l}
\dot{z}_0 = \phi, \\
\dot{z}(0) = x_0
\end{array} \right.$$

The purpose of this section is to study the EMS in a $B$- space using the Schauder FPT [Th.2.1].

Let $b > 0$ be a real number and $E$ be a $B$- space with norm $\| \|$. Let $C = \{ \phi: [-a, 0] \rightarrow E, \phi$ is $C^t$. function} with sup-norm $\| \phi \| = \sup \{ \| \phi(\theta) \| : -a \leq \theta \leq 0 \}$.
\( \theta \leq 0 \), then \((C, \| \cdot \|)\) is a B- space. Also for \( \bar{z} \in C([-a, b], \mathbb{E}) \), we have \( \bar{z}_i \in C \) for \( i \in J = [0, b] \), and \( \bar{z}_i(\theta) = \bar{z}(i + \theta) \) for \( \theta \in [-a, 0] \). Let \( C^1(J, \mathbb{E}) \) be a B- spaces with sup-norm \( \| \bar{z}^i \| = \sup \{ \| \bar{z}'(i) \| : 0 \leq i \leq b \} \). Here we suppose that the condition \((\mathbf{h}_1)\) on the operator \( \mathcal{A} \) hold. Therefore the MS of the problem \((b_2)\) defined as follows.

**Definition 3.1**: A MS of the problem \((b_2)\) is a cont\( ^{\alpha} \) function \( \bar{z} : [-a, b] \to \mathbb{E}, b > 0 \) such that \( \bar{z}_0 = \emptyset \) and its satisfies the following equation:

\[
\bar{z}(i) = \zeta(i) \varnothing(0) + \tilde{S}(i)[x_0 - \mathbb{E}(0, \varnothing)] + \int_0^i \zeta(i - s) \mathbb{E}(s, \bar{z}_s)ds
\]

\[
+ \int_0^i \tilde{S}(i - s) \mathbb{H}(s, \bar{z}_s, \bar{z}'(s), h(s, \bar{z}_s, \bar{z}'(s))) ds, i \in J
\]

We assume the following hypotheses for the problem \((b_2)\).

\((\mathbf{h}_2)\) \( \zeta(i), i > 0 \) is compact, [then \( \tilde{S}(i), i > 0 \) is also compact [12].

Let \( m_1 = \sup \{ \| \zeta(i) \| : i \in J \} \) and \( m_2 = \sup \{ \| \mathcal{A} \tilde{S}(i) \| : i \in J \} \)

\((\mathbf{h}_3)\) \( \mathbb{S} : J \times C \to \mathbb{E} \) is completely cont\( ^{\alpha} \). and for any bounded set \( M \) in \( C([-a, b], \mathbb{E}) \), the set \( \{ i \to \mathbb{S}(i, \bar{z}_i) : \bar{z}_i \in M \} \) is equi-cont\( ^{\alpha} \). in \( C([0, b], \mathbb{E}) \).

\((\mathbf{h}_4)\) \exists positive constants \( \ell_1 \) and \( \ell_2 \) s.t: \( \| \mathbb{S}(i, \varnothing) \| \leq \ell_1 \| \varnothing \| + \ell_2, i \in J, \varnothing \in C \)

\((\mathbf{h}_5)\) The nonlinear operators \( h \in C(J \times C \times \mathbb{E}, \mathbb{E}), \mathbb{H} \in C(J \times C \times \mathbb{E} \times \mathbb{E}, \mathbb{E}) \), for \( i \in J \) satisfy \( \| \mathbb{H}(i, \bar{z}_i, \bar{z}'(i), h(s, \bar{z}_s, \bar{z}'(i))) \| \leq \ell_3 \) where \( \ell_3 > 0 \) be a constant.

\((\mathbf{h}_6)\) \( m_1 \| \varnothing \| + m_2 \{ \| x_0 \| + \ell_1 \| \varnothing \| + 2 \ell_2 \} + m_1 \ell_1 \| \bar{z}_i \| b + m_1 \ell_3 b \leq k_1 \), where \( k_1 \) be a positive constant.

\((\mathbf{h}_7)\) \( m_2 \| \varnothing \| + m_1 \{ \| x_0 \| + \ell_1 \| \varnothing \| + \ell_2 \} + \ell_1 \| \bar{z}_i \| + \ell_2 + m_2 \{ \ell_2 b + \ell_1 \| \bar{z}_i \| b \} + m_1 \ell_3 b \leq k_2 \), where \( k_2 \) be a positive constant.

**Theorem 3.1**: Let the hypotheses \((\mathbf{h}_1)\) – \((\mathbf{h}_7)\) are satisfied, then the problem \((b_2)\), has at least one mild solution on \([-a, b]\).

Proof: Suppose \( Q \) be the space defined by \( Q = C([-a, b], \mathbb{E}) \cap C^1(J, \mathbb{E}) \) with the following norm \( \| \bar{z} \|_Q = \max \{ \| \bar{z} \|_a, \| \bar{z}' \|_0 \} \),

where \( \| \bar{z} \|_a = \sup \{ \| \bar{z}(i) \| : -a \leq i \leq b \}, \| \bar{z}' \|_0 = \sup \{ \| \bar{z}'(i) \| : 0 \leq i \leq b \} \), thus \((Q, \| \cdot \|_Q)\) is a B- space.

Define the operator \( \varphi : Q \to Q \) as follows:

\( (\varphi \bar{z})(i) = \varnothing(i), -a \leq i \leq 0 \)
\((\varphi \mathcal{Z})(i) = \zeta(i) \emptyset (0) + \mathcal{S}(i)[x_0 - \mathcal{S}(0, \emptyset)] + \int_0^i \zeta(i - s) \mathcal{S}(s, \mathcal{Z}_s) \, ds \\
+ \int_0^i \mathcal{S}(i - s) H(s, \mathcal{Z}_s, \mathcal{Z}'(s), h(s, \mathcal{Z}_s, \mathcal{Z}'(s))) \, ds, \quad i \in J \quad (1)\)

We shall to apply Th.2.1, to prove the operator \(\varphi\) has a FP in \(Q\) which it a MS of the problem (b2).

First for \(0 \leq i_1 \leq i_2 \leq b\) we have

\[
\begin{align*}
\|\mathcal{Z}(i)\| & \leq m_1 \|\emptyset\| + m_1 b \{\|x_0\| + \ell_1 \|\emptyset\| + 2 \ell_2\} + m_1 \mathcal{E}_1 \|\mathcal{Z}_i\| + m_1 \ell_3 b \leq k_1 \quad (by \ \mathcal{H}_e) \quad and \quad by: \\
\mathcal{Z}'(i) & = A \mathcal{S}(i) \emptyset (0) + \zeta(i)[x_0 - \mathcal{S}(0, \emptyset)] + \mathcal{S}(i, \mathcal{Z}_i) + \int_0^i A \mathcal{S}(i - s) \mathcal{S}(s, \mathcal{Z}_s) \, ds + \\
& + \int_0^i \zeta(i - s) H(s, \mathcal{Z}_s, \mathcal{Z}'(s), h(s, \mathcal{Z}_s, \mathcal{Z}'(s))) \, ds.
\end{align*}
\]

We obtain \(\|\mathcal{Z}'(i)\| \leq m_2 \|\emptyset\| + m_1 \{\|x_0\| + \ell_1 \|\emptyset\| + \ell_2\} + \ell_3 \|\mathcal{Z}_i\| + \ell_2 + m_2 \{\ell_2 b + \ell_1 \|\mathcal{Z}_i\| + m_1 \ell_3 b \leq k_2 \quad (by \ \mathcal{H}_7).\)

We shall now prove that the operator \(\varphi\) in (1) is a completely cont\(^s\) operator. Let \(Q_0 = \{\mathcal{Z} \in Q: \|\mathcal{Z}\|_Q \leq \ell\}\) for some \(\ell \geq 1\). Then \(Q_0\) is clearly a bounded, closed and convex subset of \(Q\) [13].

We first show that \(\varphi\) maps \(Q_0\) into an equi-cont\(^s\) family. Let \(\mathcal{Z} \in Q_0\) and \(i_1, i_2 \in J\).

Then, if \(0 \leq i_1 \leq i_2 \leq b\) we get that:

\[
\begin{align*}
\|&(\varphi \mathcal{Z})(i_1) - (\varphi \mathcal{Z})(i_2)\| \leq \|\zeta(i_1) - \zeta(i_2)\| \|\emptyset\| + \|\mathcal{S}(i_1) - \mathcal{S}(i_2)\| \|x_0 - \mathcal{S}(0, \emptyset)\| \\
&+ \int_0^{i_1} \zeta(i_1 - s) - \zeta(i_2 - s) \mathcal{S}(s, \mathcal{Z}_s) \, ds + \int_0^{i_2} \zeta(i_2 - s) \mathcal{S}(s, \mathcal{Z}_s) \, ds \\
&+ \int_0^{i_1} \mathcal{S}(i_1 - s) H(s, \mathcal{Z}_s, \mathcal{Z}'(s), h(s, \mathcal{Z}_s, \mathcal{Z}'(s))) \, ds \\
&+ \int_0^{i_2} \mathcal{S}(i_2 - s) H(s, \mathcal{Z}_s, \mathcal{Z}'(s), h(s, \mathcal{Z}_s, \mathcal{Z}'(s))) \, ds \\
&\leq \|\zeta(i_1) - \zeta(i_2)\| \|\emptyset\| + \|\mathcal{S}(i_1) - \mathcal{S}(i_2)\| \|x_0\| + \ell_1 \|\emptyset\| + \ell_2 \\
&+ \int_0^{i_1} \zeta(i_1 - s) - \zeta(i_2 - s) \|\mathcal{E}_1\| \mathcal{Z}_s \, ds + \int_0^{i_2} \zeta(i_2 - s) \|\mathcal{E}_1\| \mathcal{Z}_s \, ds + \int_0^{i_1} \|\zeta(i_1 - s)\| \|\ell_1\| \mathcal{Z}_s \, ds + \\
&+ \int_0^{i_1} \|\zeta(i_1 - s)\| \|\ell_2\| ds + \int_0^{i_1} \|\mathcal{S}(i_1 - s) - \mathcal{S}(i_2 - s)\| ds, \quad (2)
\end{align*}
\]

Also similarity

\[
\|(\varphi \mathcal{Z})'(i_1) - (\varphi \mathcal{Z})'(i_2)\| \leq \|A (\mathcal{S}(i_1) - \mathcal{S}(i_2)) \emptyset (0)\| + \|\zeta(i_1) - \zeta(i_2)\| \|x_0 - \mathcal{S}(0, \emptyset)\|
\]
Therefore the right hand sides of (2) and (3) are tend to zero as $i_2 - i_1 \to 0$ because they are independent of $y \in Q_0$. By $(\mathcal{H}_2)$ and since $\zeta(i)$, $\tilde{S}(i)$ are uniformly cont$^+$ for $i \in J$, then impl their continuity in the uniform operator topology.

Thus, $\varphi$ maps $Q_0$ into an equi-cont$^+$ family of functions, it is not difficult to see that the family $\varphi Q_0$ is uniformly bounded.

Next we show that the closure of $\varphi Q_0$ is compact. It suffices by the Arzela – Ascoli theorem to show that $\varphi Q_0$ into a precompact set in $E$ (because we have shown $\varphi Q_0$ is an equi-cont$^+$ collection).

Let $0 < i \leq b$ be fixed and $0 < \delta < i$ be a real number. For $\bar{z} \in Q_0$, define

$$
(\varphi_{\delta} \bar{z})(i) = \zeta(i) \phi(0) + \tilde{S}(i)[x_0 - \mathcal{E}(0, \varnothing)] + \int_{0}^{i-\delta} \zeta(i-s) \mathcal{E}(s, \bar{z}_s)d\mathcal{s}
$$

$$
+ \int_{0}^{i-\delta} \tilde{S}(i-s)H(s, \bar{z}_s, \bar{z}'(s), h(s, \bar{z}_s, \bar{z}'(s)))d\mathcal{s}, i \in J.
$$

By $(\mathcal{H}_2)$ we see that the set $P_{\delta} = \{(\varphi_{\delta} \bar{z})(i): \bar{z} \in Q_0\}$ is precompact in $E$ $\forall \delta, 0 < \delta < i$. Moreover, $\forall \bar{z} \in Q_0$, we obtain that

$$
\|(\varphi \bar{z})(i) - (\varphi_{\delta} \bar{z})(i)\|
$$

$$
\leq \int_{i-\delta}^{i} \|\zeta(i-s) \mathcal{E}(s, \bar{z}_s)\|d\mathcal{s}
$$

$$
+ \int_{i-\delta}^{i} \|\tilde{S}(i-s)H(s, \bar{z}_s, \bar{z}'(s), h(s, \bar{z}_s, \bar{z}'(s)))\|d\mathcal{s}.
$$
≤ \int_{-\delta}^{\delta} \left\| \varsigma(i-s) \right\| \left\{ \ell_1 \| \bar{z}_n \| + \ell_2 \right\} ds + \int_{-\delta}^{\delta} \left\| \bar{S}(i-s) \right\| \ell_2 ds \to 0, \text{as } \delta \to 0

And

\| (\varphi \bar{z})'(i) - (\varphi \bar{z}')'(i) \| \leq \| \varsigma(i, \bar{z}) - \varsigma(\delta) \varsigma(i - \delta, \bar{z}_{i-\delta}) \| \\
+ \int_{-\delta}^{\delta} \left\| A \bar{S}(i-s) \varsigma(s, \bar{z}_n) \right\| ds \\
+ \int_{-\delta}^{\delta} \left\| \varsigma(i-s) H(s, \bar{z}_n, \bar{z}'(s), h(s, \bar{z}_n, \bar{z}'(s))) \right\| ds \\
\leq \| \varsigma(i, \bar{z}) - \varsigma(\delta) \varsigma(i - \delta, \bar{z}_{i-\delta}) \| + \int_{-\delta}^{\delta} \left\| A \bar{S}(i-s) \right\| \left\{ \ell_1 \| \bar{z}_n \| + \ell_2 \right\} ds \\
+ \int_{-\delta}^{\delta} \left\| \varsigma(i-s) \right\| \ell_2 ds \to 0, \text{as } \delta \to 0

Thus, there are precompact sets arbitrarily close to the set \( P_{\delta} \). Hence the set \( \{ (\varphi \bar{z}) : \bar{z} \in Q_n \} \) is precompact in \( \mathcal{E} \).

Finally we want to show that \( \varphi : Q \to Q \) is \cont^t. Let \( \{ \bar{z}_n \}_{n=0}^{\infty} \subseteq Q \) with \( \bar{z}_n \to \bar{z} \). Then there is an integer \( k \) s.t \( \| \bar{z}_n(i) \| \leq k, \| \bar{z}'_n(i) \| \leq k \) for all \( n \) and \( i \in J \), where \( k = \max \{ k_1, k_2 \} \), so \( \| \bar{z}(i) \| \leq k, \| \bar{z}'(i) \| \leq k \) and \( \bar{z}, \bar{z}' \in Q \).

Since the operators \( \varsigma \) and \( H \) are uniformly bounded \cont^t. operators (see section 1), then by the dominated convergence theorem we get that

\[
\| \varphi \bar{z}_n - \varphi \bar{z} \| = \text{Sup}_{i \in J} \left\| \int_{0}^{1} \varsigma(i-s) \left[ \varsigma(s, \bar{z}_{n,s}) - \varsigma(s, \bar{z}_s) \right] ds \right\|
\]

\[
+ \int_{0}^{b} \left\| \bar{S}(i-s) \left[ H(s, \bar{z}_{n,s}, \bar{z}'_n(s), h(s, \bar{z}_{n,s}, \bar{z}'_n(s))) \right] \right\| ds \\
\leq \left\| \int_{0}^{b} \varsigma(i-s) \left[ \varsigma(s, \bar{z}_{n,s}) - \varsigma(s, \bar{z}_s) \right] \right\| ds \\
+ \int_{0}^{b} \left\| \bar{S}(i-s) \left[ H(s, \bar{z}_{n,s}, \bar{z}'_n(s), h(s, \bar{z}_{n,s}, \bar{z}'_n(s))) \right] \right\| ds \to 0 \text{ as } n \to \infty
\]

And

\[
\| (\varphi \bar{z}_n)' - (\varphi \bar{z})' \|
\]

\[
= \text{Sup}_{i \in J} \left\| \left[ \varsigma(i, \bar{z}_n) - \varsigma(i, \bar{z}_s) \right] + \int_{0}^{i} A \bar{S}(i-s) \left[ \varsigma(s, \bar{z}_{n,s}) - \varsigma(s, \bar{z}_s) \right] ds \right\|
\]
\[ + \int_{0}^{1} \xi(i-s)[H(s, \tilde{z}_{n}, \tilde{z}'_{n}(s), h(s, \tilde{z}_{n}, \tilde{z}'_{n}(s))) - \tilde{H}(s, \tilde{z}_{n}, \tilde{z}'(s), h(s, \tilde{z}_{n}, \tilde{z}'(s)))]ds \]
\[ \leq \left\| \mathcal{S}(i, \tilde{z}_{n}) - \mathcal{S}(i, \tilde{z}_{n}) + \int_{0}^{b} A\mathcal{S}(i-s)[\mathcal{S}(s, \tilde{z}_{n}) - \mathcal{S}(s, \tilde{z}_{n})] \right\| ds \]
\[ + \int_{0}^{b} \left\| \xi(i-s)[H(s, \tilde{z}_{n}, \tilde{z}'_{n}(s), h(s, \tilde{z}_{n}, \tilde{z}'_{n}(s))) - \tilde{H}(s, \tilde{z}_{n}, \tilde{z}'(s), h(s, \tilde{z}_{n}, \tilde{z}'(s)))] \right\| ds \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty \]

Therefore \( \psi \) is cont\(^{\ast} \), and this complete the proof that \( \psi \) is completely cont\(^{\ast} \).

Thus by Th.2.1, the operator \( \psi \) has a FP in \( Q \). This means that any FP of \( \psi \) is a MS of \( (b_2) \) on \([-a, b]\) satisfying \( (\psi \xi)(i) = \xi(i) \). Therefore the problem \( (b_2) \) has at least one MS on \([-a, b]\).

**Example 3.1**

Let us consider the following PDEq:
\[
\frac{\partial}{\partial t} (\tilde{z}_{i}(x, i) - \nabla_{1}(i, \tilde{z}_{i}(x, i - a))) = \tilde{z}_{xx}(x, i) + \int_{0}^{1} Y(s, \tilde{z}(x, s-a), \tilde{z}_{s}(x, s), \int_{0}^{s} Y_{2}(\tau, \tilde{z}(x, \tau - a), \tilde{z}_{\tau}(x, \tau))d\tau)ds \quad (4) \]
\[
\tilde{z}(0, i) = \tilde{z}(\pi, i) = 0, \text{ for } i > 0, \]
\[
\tilde{z}(x, i) = \varnothing(x, i), \text{ for } -a \leq i \leq 0, \]
\[
\tilde{z}_{i}(x, 0) = \tilde{z}_{1}(x), \text{ for } i \in J = [0, b], 0 < x < \pi, \]
where \( Y_{1}, Y_{2}, \varnothing \) are functions defined below and \( \varnothing \) is a cont\(^{\ast} \) function.

Let \( E = L^{2}[\alpha, \pi] \) and \( \mathcal{A} : E \rightarrow E \) be an operator defined as follows \( \mathcal{A} e = e'' \), \( e, e'' \in D(\mathcal{A}) \), where \( D(\mathcal{A}) = \{ e \in E : e, e'' \text{ are absolutely cont}^{\ast}, e'' \in E, e(0) = e(\pi) = 0 \} \).

Therefore, \( \mathcal{A} e = \sum_{n=1}^{\infty} -n^{2}(e_{n} e_{n})e_{n}, e_{n} \in D(\mathcal{A}) \), where \( e_{n}(i) = \sqrt{2/\pi} \sin ni, n = 1, 2, \ldots \) is the orthogonal set of eigenvalues of \( \mathcal{A} \).

It isn’t difficult to check that \( \mathcal{A} \) is the IG of a \( \zeta(s), s \in \mathcal{R} \), in \( E \) and is given by \( \zeta(s)e = \sum_{n=1}^{\infty} \cos ns(e_{n} e_{n})e_{n}, e_{n} \in E \). The associated sine family is given by \( \mathcal{S}(s)e = \sum_{n=1}^{\infty} \frac{1}{n} \sin si \ (e_{n} e_{n})e_{n}, e_{n} \in E \).

Let the operator \( \mathcal{S} : J \times C \rightarrow E \) be defined by \( \mathcal{S}(i, )x) = Y_{1}(i, )x) \), \( e \in C, x \in [0, \pi], \) where \( Y_{1} : J \times [0, \pi] \rightarrow [0, \pi] \) is completely cont\(^{\ast} \), and \( \exists \) positive constants \( \ell_{1} \) and \( \ell_{2} \) s. t \( \| Y_{1}(i, )\| \leq \ell_{1} \| \) + \( \ell_{2} \).
Let the operator $\mathcal{V}: \mathbb{C} \times \mathbb{C} \times \mathbb{E} \to \mathbb{E}$ and $\mathcal{H}: \mathbb{C} \times \mathbb{C} \times \mathbb{E} \times \mathbb{E} \to \mathbb{E}$ be defined as follows:

$$\mathcal{V}(\mathbb{C} \times \mathbb{C} \times \mathbb{E} \times \mathbb{E}) = \int_{0}^{1} \mathcal{Y}_{2}(\tau, \mathbb{C}(x, \tau - a), \mathbb{C}(x, \tau))d\tau,$$ 

and

$$\mathcal{H}(\mathbb{C} \times \mathbb{C} \times \mathbb{E} \times \mathbb{E}) = \int_{0}^{1} \mathcal{Y}(s, \mathbb{C}(x, s - a), \mathbb{C}(x, s), \mathbb{C}(x))d\mathbb{C} \text{ where } i \in J, x \in [0, \pi] \text{ and } \mathbb{C} \in \mathbb{E}, \text{ s.t these operator } \mathcal{V} \text{ and } \mathcal{H} \text{ are uniformly bounded conts. operators.}$$

Thus, with this choice of $\mathbb{C}$, $\mathcal{V}$ and $\mathcal{H}$ we see that (4) becomes an abstract formula (b2). Furthermore, all the conditions stated in the Th.3.1, are satisfied. Hence (4) has at least one mild solution on $[-a, b]$.

### 4. Controllability Results for (b2).

In the following we will study the controllability of mild solutions (CMS) for the problem (b1), i.e., the problem will be consider here is the problem (b2) with a control variable. And hence its an application of theorem 3.1.

The aim of this section is to study the CMS to problem (b1) in a B-space using the FPT by Schauder.

**Definition 4.1:** A MS of the problem (b1) is a conts. function $\mathbb{C}: [-a, b] \to \mathbb{E}$, $b > 0$

s.t $\mathbb{C} = \emptyset$ and it is satisfies the following equation.

$$\mathbb{C}(i) = \mathbb{C}(i)[x_{0} - \mathbb{C}(0, 0)] + \int_{0}^{i} \mathbb{C}(i - s)[B\mathbb{C}(s)]ds$$

$$(5)$$

**Definition 4.2:** The system (b1) is said to be controllable on the interval $J$ if $\forall 0 \in \mathbb{C}$

with $0 \in \mathbb{C}(\mathbb{C})$, $x_{0} \in Y$ and $\mathbb{C} \in \mathbb{E}$, $\exists \mathbb{C}(J, \mathbb{U})$ s.t the solution $\mathbb{C}(\cdot)$ of (b1), satisfies $\mathbb{C}(b) = \mathbb{C}_{1}$.

In order that the problem (b1) makes sense throughout this section and establish the controllability result, we need the following additional assumptions.

(a) $B\mathbb{C}(t)$ is conts. in and $\|B\| \leq c_{1}$ for some constant $c_{1} > 0$.

(b) Define the linear operator $V: L^{2}(J, \mathbb{U}) \to \mathbb{E}$ as follows:

$$V\mathbb{U} = \int_{0}^{b} \mathbb{U}(b - s)B\mathbb{U}(s)ds,$$

this operator induces a bounded invertible operator $\mathbb{U}: L^{2}(J, \mathbb{U})/\ker V \to \mathbb{E}$

s.t $\|\mathbb{U}^{-1}\| \leq c_{2}$ for some constant $c_{2} > 0$ (for constration $\mathbb{U}^{-1}$ see [15]). Also, here we instead the conditions (c) and (d) by the following conditions:
(\mathcal{H}_c) m_1 \|\phi(0)\| + m_1 b c + c_4 + m b [c_1 c_2 (\|z_1\| + m_1 \|\phi(0)\| + m_1 c_3 + c_4 + m_1 \ell_3 b) + \ell_3] \\
\leq c_5, \text{ where } c_3 = \|x_0\| + \ell_1 \|\phi\| + 2 \ell_2, c_4 = m_1 \ell_1 \|z_1\| b \text{ and } c_5 \text{ are positive constants.}

(\mathcal{H}_c) m_2 \|\phi(0)\| + m_1 c_3 + c_4 + m_2 b \{\ell_2 + \ell_1 \|z_1\|\} + m_1 b [c_1 c_2 (\|z_1\| + m_1 \|\phi(0)\| + m_1 c_3 + c_4 + m_1 \ell_3 b)] + \ell_3 \leq c_6, \text{ where } c_6 \text{ is a positive constant.}

**Theorem 4.1:** When the conditions (\mathcal{H}_1) – (\mathcal{H}_9), are satisfied, then the system (\mathcal{H}_1) is controllable on J.

Proof: By (\mathcal{H}_9), for an arbitrary function z(.), we define the control
\[ \hat{u}(i) = V^{-1} [z_i - \zeta(b) \phi(0) - \hat{S}(b) \{x_0 - \mathcal{E}(0, \emptyset)\} - \int_0^b \zeta(b - s) \mathcal{E}(s, \hat{z}_s) ds - \int_0^b \hat{S}(b - s) \{H(s, \hat{z}_s, \hat{z}'(s), h(s, \hat{z}_s, \hat{z}'(s)))\} ds](i) \]

Now, by using this control we will define the operator \( \Phi: \mathcal{Q} \rightarrow \mathcal{Q} \) as follows:

\[ (\Phi \hat{z})(i) = \emptyset(i), -a \leq i \leq 0, \]
\[ (\Phi \hat{z})(i) = \zeta(i) \phi(0) + \hat{S}(i) \{x_0 - \mathcal{E}(0, \emptyset)\} + \int_0^i \zeta(i - s) \mathcal{E}(s, \hat{z}_s) ds \]
\[ + \int_0^i \hat{S}(i - s) B V^{-1} [z_i - \zeta(b) \phi(0) - \hat{S}(b) \{x_0 - \mathcal{E}(0, \emptyset)\}] \]
\[ - \int_0^i \zeta(b - i) \mathcal{E}(i, \hat{z}_i) di - \int_0^b \hat{S}(b - s) \{H(s, \hat{z}_s, \hat{z}'(s), h(s, \hat{z}_s, \hat{z}'(s)))\} ds](s) ds \]
\[ + \int_0^i \hat{S}(i - s) \{H(s, \hat{z}_s, \hat{z}'(s), h(s, \hat{z}_s, \hat{z}'(s)))\} ds] ds, i \in J. \]

Therefore by Th.2.1, and by using the same manner used in the proof of Th. 3.1., it is not difficult to see that the operator \( \Phi \) has FP \( \hat{z}(t) \). This FP is then a solution of Eq(5) and satisfying \( (\Phi \hat{z})(i) = \hat{z}(i) \). Clearly \( (\Phi \hat{z})(b) = \hat{z}(b) = \hat{z}_1 \), which means that the control \( \hat{u} \) steer the system (\mathcal{H}_1) from the initial state \( \hat{z}_0 \) to \( \hat{z}_1 \) in time b.

Hence the system (\mathcal{H}_1) is controllable on J.

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