Possible Structures of Sprites

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Abstract

Upon using the hydrodynamic analog we can derive some families of stationary Beltrami field-like solutions from the free Maxwell equations in vacuum. These stationary electromagnetic fields are helical and/or column-like once they are represented in a suitable frame of reference. Possible dendritic and jelly-fish-like patterns of sprites are demonstrated.

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The majority of the great discoveries in physics have been made by chance (electromagnetism, radioactivity, etc.) and only in some rare cases was the experiment preceded by theory (lasers, thermonuclear reactors). It seems likely that the problem of sprites belongs to the latter one. Since their discovery by Franz et. al. [1], sprites have now been observed over thunderstorms all over the world [2-4]. A spectacular manifestation of these sprites: the intense transient quasistatic electric (QE) fields of up to $\sim 1 \, \text{kV} \cdot \text{m}^{-1}$, which for positive (Cloud-to-Ground) CG discharges is directed downwards, can avalanche accelerate upward-driven runaway MeV electron beams, producing brief ($\sim 1 \, \text{ms}$) flashes of gamma radiation. These intense fields are large luminous discharges in the altitude range of $\sim 40 \, \text{km}$ to $90 \, \text{km}$, which are produced by the heating of ambient electrons for a few to tens of milliseconds following intense lightning flashes.

Thousands of positive ground flash-sprite associations have been identified through comparisons with video imaging/optical sensor verification of sprites. Beginning with suggestions by Wilson [5], the electrostatic field change of the lightning flash was sufficient to exceed the dielectric strength of the mesosphere and initiate the sprite. This Wilson mechanism for sprites initiated by conventional dielectric breakdown is polarity independent-positive and negative changes in charge moment change in excess of the threshold should be equally effective in the initiation of sprites. Sprites require a lowly or nonionized medium; they can propagate from the ionosphere downwards or from some lower base upwards or they can emerge at some immediate height and propagate upwards and downwards [6-7].

The first columniform sprites (c-sprites) reported [8] were vertical columns of light that extended from about 76 to 87 km and probably less than 1 km in diameter. According to Wescott et al. [8] the uncertainty in height determination for their c-sprite measurements was approximately 1 pixel or less than 1 km. The standard deviation of the heights for the 28 triangulated images was 1.9 km for the top and 1.4 km for the bottom. Additional observations of c-sprites, some of which may extend to lower altitudes, have been recently reported [9-10]. The bodies of c-sprites usually appear as nearly straight lines, in some cases bent slightly, and are composed of bright beads and dark regions.

Meanwhile, some researchers [7-8] described the phenomenon, called red Sprites, as a luminous column that stretches between 50 and 90 km, with peak luminosity in the vicinity of 70-80 km. The flashes have an average lifetime of a few milliseconds and an optical intensity of about 100 kR. Red Sprites are associated with the presence of massive thunderstorm
clouds, and among the most puzzling aspects of the observations is the presence of spatial structure in the emissions reported by Winckler et al. [7]. Vertical striations, denominated ‘tendrils’, with horizontal size of 1 km or smaller, often limited by the instrumental resolution, are apparent in the red Sprite emissions [9-10].

As one example, the key features of the previous model [9] are: (1) an ambient conductivity profile that falls between a measured nighttime and a measured daytime conductivity; (2) an aerosol reduced conductivity in a trail from a meteor that passed through some time during the evening, and (3) a cloud-to-ground (hereafter CG) lightning stroke, with sufficient charge transfer, subsequent to and occurring within an hour of the development of the reduced conductivity trail. This model predicts a temporally brief column of light resulting from the conventional breakdown of air in a strong electric field in the observed altitude range. The results we shall describe in this short paper might provide an explanation for the observed c-sprite phenomena.

The free Maxwell equations (FME) are

\[
\epsilon_{ijk}\partial_j E_k = -\frac{1}{c} \frac{\partial B_i}{\partial t}, \quad \epsilon_{ijk}\partial_j B_k = \frac{1}{c} \frac{\partial E_i}{\partial t},
\]

\[
\partial_i E_i = 0, \quad \partial_i B_i = 0,
\]

where \( \partial_i \equiv \partial/\partial x_i, \ x_i = (x, y, z), \) \( \epsilon_{ijk} \) is an antisymmetric tensor or symbol. We presume that the solutions of FME are

\[
E(\mathbf{r}, t) = \mathbf{e}(\mathbf{r})F(t), \quad \text{and} \quad B(\mathbf{r}, t) = \mathbf{b}(\mathbf{r})G(t).
\]  

(1)

In order to obtain solutions of this system with three constants only and to obtain sinusoidal solutions, we also presume that

\[
\frac{-dG(t)/dt}{F(t)} = \frac{dF(t)/dt}{G(t)} = \Omega.
\]  

(2)

Therefore, the general solutions of above system [11] are

\[
F(t) = C_0 \sin(\Omega t - \phi), \quad \text{and} \quad G(t) = C_0 \cos(\Omega t - \phi),
\]  

(3)

where \( C_0 \) and \( \phi \) are arbitrary constants. Now, equations for \( \mathbf{e} \) and \( \mathbf{b} \) become

\[
\nabla \times \mathbf{e} = \frac{\Omega}{c} \mathbf{b}, \quad \nabla \times \mathbf{b} = \frac{\Omega}{c} \mathbf{e}.
\]  

(4)
Above equations give
\[ \nabla \times \mathbf{A} = \frac{\Omega}{c} \mathbf{A}, \quad \text{with} \quad \mathbf{A} = e + b, \tag{5} \]
where \( e \) is a polar vector (linked to the electric field \( \mathbf{E} \) in space, \( \mathbf{E} = e \mathcal{E}(t) \)), \( b \) is an axial vector (linked to the magnetic field \( \mathbf{B} \) in space, \( \mathbf{B} = b \mathcal{G}(t) \)).

Now, from the vector analysis and the hydrodynamic analogy (e.g., for a velocity field \( \mathbf{u} \), we have the vorticity field \( \omega \) which is defined as \( \nabla \times \mathbf{u} \)), we know that
\[ (\nabla \times \mathbf{A}) \parallel \mathbf{A} \quad \text{or} \quad (\nabla \times \mathbf{A}) \times \mathbf{A} = 0, \tag{6} \]
here, \( \parallel \) means 'parallel to'. In fact, people termed \( \mathbf{L} = (\nabla \times \mathbf{A}) \times \mathbf{A} \) as a \textit{Lamb} vector [12-13].

In generalized fluid mechanics, we call the flow which satisfies
\[ \omega \times \mathbf{u} = 0 \tag{7} \]
a \textit{Beltrami} flow [11-14], sometimes a \textit{screw} motion or helical flow [11-13] (here, \( \omega \neq 0 \)).

With regard to these, vector fields satisfying Eq. (7) are helical or screw fields. It means \( \mathbf{A} \) or the combination of \( e \) and \( b \) is a helical or screw field. One class of these solutions can be obtained and expressed in cylindrical polar coordinates \((r, \theta, z)\) as
\[ u = \bar{A}_0 r \cos(\zeta), \quad v = \bar{A}_0 r \sin(\zeta), \quad w = -\frac{2\bar{A}_0}{k} \sin(\zeta), \tag{8} \]
where \( \zeta = k(z - c_0 t) = k z - 2\bar{\Omega} t \). When \( \mathbf{u} = (u, v, w) \equiv \mathbf{A} \) is expressed in a frame of reference rotating with the mean angular velocity \( \bar{\Omega} \) (\( \bar{v} = \bar{A}_0 r \sin(\zeta) \)), we have \( \nabla \times \mathbf{u} \equiv \omega = -k \mathbf{u} \) or \( \omega \times \mathbf{u} = 0 \). The stationary form of this solution could be similar to \( e + b \) (i.e., \( \mathbf{A} \)). We remind the readers that eqns. (8) describe a field that grows proportionally to the distance from the cylindrical axis. Therefore, in unbounded vacuum, as there is no dissipation, \( \bar{A}_0 r \sim O(1) \) as \( r \to \infty \) should be presumed otherwise they will give some non-localized electromagnetic field configurations with infinite energy. Under this consideration, once \( r \to \infty \), \( w \) is rather small (as \( \bar{A}_0 \to 0 \) if \( k \) is finite) and this (axial) component (related to magnetic field; will be illustrated below) behaves like a weak 'thread'! On the other hand, since \( \bar{A}_0 r \sim O(1) \), once \( r \to 0 \), there might be a singularity at \( r = 0 \) (\( \bar{A}_0 \) has been restricted for very larger \( r \)). This singularity resembles that of a free vortex (vortex core or the filament of a vortex tube). Note that, once the amplitude \( \bar{A}_0 \) being replaced by \( A(t) \) (with \( dA/dt = -\nu A k^2 \), \( \nu \) plays the role of damping, the solution of above equation easily reads \( A(t) = A(0)e^{-\nu k^2 t} \),
we then have \( u = A(t)r \cos(\zeta), \) \( \hat{v} = A(t)r \sin(\zeta), \) \( w = -2A(t) \sin(\zeta)/k. \) To demonstrate our solutions, we plot those stationary ones of them into following figures. We shall firstly present the electromagnetic field in vacuum. Parameters are selected as \( \bar{A}_0 = 0.45, k = 2.5; \) the maximum spatial range : \( 0 \leq r \leq 1, 0 \leq z \leq 2\pi. \) The vector field is represented in a rotating frame of reference (Fig. 1) with the mean angular velocity \( \bar{\Omega} = 5.0. \) We can observe the helical and the dendritic features of our solutions and the possible pattern of a sprite-like electromagnetic field.

The outer and inner isosurface plots (for the total modulus of \( \mathcal{A} \) and the modulus of \( ui_r + vi_\theta, \) respectively) are presented in Fig. 1. To make sure there are also possible sprite-like fields in our results as there are so many parameters needed for tuning and the 3D vector view is sometimes not comprehensive, thus we only show those isosurface plots below.

With intensive searching, we can find out, at least, for \( \bar{\Omega} = 5, \) \( \bar{A}_0 = 0.45, k = 1.5, \) there exists also possible column-like together with helical field for sprites. This pattern is shown in Fig. 2 for the same frame of reference (in a rotating frame of reference). The outer (green) surface is for the total modulus of \( \mathcal{A} \) which is already in a helical shape. The inner (yellow) surface is for the modulus of \( ui_r + vi_\theta \) which demonstrates the possible column-like channel. As parameters are tuned into \( \bar{\Omega} = 0.2, \bar{A}_0 = 1.5, k = 2.5, \) the dendritic features of sprites are illustrated clearly in Fig. 3. We can now easily capture the characteristics of helical or possible helix-like fields in Figs. 1, 2 and 3. These results confirm our present approach (cf. images or figures in [8-9]).

Note that, once we introduce

\[
e(r) = \frac{1}{2}[\mathcal{A}(r) - \mathcal{A}(-r)], \quad \mathbf{b}(r) = \frac{1}{2}[\mathcal{A}(r) + \mathcal{A}(-r)],
\]

(9)

considering the definition of Poynting vector \( \mathbf{S} = c(\mathbf{E} \times \mathbf{B})/(4\pi), \) we also can obtain the relevant results about \( \mathbf{S} \) (in fact, \( \mathbf{b}(r) \propto \mathbf{w} \) as evidenced from above equation). The calculation is now only related to \( \mathcal{A}(r) \times \mathcal{A}(-r) \) instead of \( \mathbf{e} \times \mathbf{b}. \) Meanwhile, as the energy density \( W \) is defined as \( (E^2 + B^2)/8\pi, \) we can directly interpret \( W \) from the magnitude or modulus of \( \mathcal{A} \) from Eq. (8). Meanwhile, from Eqs. (8) and (9), we can obtain the explicit expression of the electric (\( \mathbf{e}(r) \)) and magnetic (\( \mathbf{b}(r) \)) fields. We might have, the electromagnetic energy \( (|\mathbf{E}| \propto \alpha \gamma R_0^{-3}, \) \( \alpha = -(\Omega R_0/c) \cos(\Omega R_0/c) + \sin(\Omega R_0/c) \) and \( \gamma = \alpha - (\Omega^2 R_0^2/c^2) \cos(\Omega R_0/c)) \) within spheres of radius \( R_0, \) which is the solution of \( \tan(\Omega R_0/c) = \Omega R_0/c, \) and it is independent on the time.
However, once we selected parameters as $\hat{\Omega} = 5, \bar{A}_0 = 1.2, k = 0.8$, a single sprite appears (cf. Fig. 4 in [8] or Fig. 4a in [9] or Figs. 5 and 6 in [15]) which is shown in Fig. 4. The other interesting case is for $\hat{\Omega} = 0.6, \bar{A}_0 = 0.6, k = 0.15$ as shown in Fig. 5: a jelly-fish-like pattern with the downward tendrils-like feature appears.

To be specific, we like to make a final remark about the characteristic of our Beltrami fields: Eq. (8) especially when $k$ is rather small (solutions for $k = 0$ violate the Beltrami condition!). Once $k \to 0$, considering the limit of the $w$-component in the stationary case, $w \propto z$ with $z$ being finite. Under this situation, our solutions resemble those results for high-electric-field cases [15].

In a brief summary, the author likes to point out, the helical and column-like fields illustrated above are closely linked to the formation of sprites, however, there will be other unknown, say, relativistic, effects during the final stage in the formation of lightning. We shall consider other more complicated problems [16-21] in our future works.

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Fig. 1  Isosurface plots of the solution $(e + b)$ in a rotating frame of reference.

Possible helical and column-shape electromagnetic fields. $\hat{\Omega} = 5, \tilde{A}_0 = 0.45, k = 2.5$.

(the outer or green surface is for $|A|$ and the inner or yellow surface is for $|u_i r + v_i \theta|$)

This result resembles qualitatively that (cf. Fig. 4 in [15]) by Mironychev and Babich and those (say, cf. Fig. 1 in [8]) by Wescott et al. It has the dendritic type.
Fig. 2 Isosurface plots of the solution $(e + b)$ in a rotating frame of reference.
Possible column(sprite)-like and helical electromagnetic field. $\hat{\Omega} = 5, \hat{A}_0 = 0.45, k = 1.5$. 
Fig. 3  Isosurface plots of the solution (e + b) in a rotating frame of reference. Possible sprite- and column-like electromagnetic field. $\hat{\Omega} = 0.2, \bar{A}_0 = 1.5, k = 2.5$.

(the spherical or green surface is for $|A|$)
Fig. 4  Isosurface plots of the solution \((e + b)\) in a rotating frame of reference.
Possible sprite and column-like electromagnetic fields. \(\Omega = 5.0, A_0 = 1.2, k = 0.8\).

The column-like (yellow) isosurface is for \(e(r)\). This result resembles qualitatively that (cf. Figs. 5 and 6 in [15]) by Mironychev and Babich.
Fig. 5  Isosurface plots of the solution $(e + b)$ in a rotating frame of reference. Possible sprite-like electromagnetic field $\hat{\Omega} = 0.6$, $\hat{A}_0 = 0.6$, $k = 0.15$.

Possible jelly-fish-like pattern with the downward tendrils-like feature appears.