Monte Carlo Simulations of 2-Dimensional Quantum Gravity Coupled to $c = 1$ Matter \(^{\ast}\)

Thomas Filk\(^{a}\), Mihail Marcu\(^{b}\) and Bernhard Scheffold\(^{a}\)

\(^{a}\)University of Freiburg, Department of Physics, Hermann-Herder-Str. 3, D-7800 Freiburg, Germany

\(^{b}\)School of Physics and Astronomy, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, 69978 Tel Aviv, Israel

We present results of a high precision Monte Carlo simulation of dynamically triangulated random surfaces (up to $\approx 34,000$ triangles) coupled to one scalar field ($c = 1$). The mean square extent has been measured for different actions to test the universality of the leading term as a function of the size of the surfaces. Furthermore, the integrated 2-point correlation function for vertex operators is compared with conformal field theory and matrix model predictions.

1. INTRODUCTION

This talk reports about the results of several projects, where – apart from the authors – also M. Agishtein, R. Ben-Av, I. Klebanov, A. A. Migdal and S. Solomon were involved. Additional details and even more precise data will be published soon.

$c = 1$ matter coupled to 2-dimensional quantum gravity seems to be a critical model in the sense that most approaches to the theory of matter coupled to gravity in 2 dimensions predict qualitatively different behaviour for $c < 1$ and $c > 1$. Furthermore, there are two different theoretical approaches (conformal field theory and matrix models) which give quite detailed predictions for this model.

The main aim of our high precision Monte Carlo simulations was to confirm those predictions where conformal field theory and matrix models agree, and to decide or clarify the situation where the two approaches differ.

2. DEFINITION OF THE MODEL AND THEORETICAL PREDICTIONS

Let $T_N$ denote the set of abstract triangulations of the 2-dimensional sphere with $N$ triangles. This is equivalent to the set of planar, regular graphs of degree 3 with $N$ vertices and without non-trivial 2-point subgraphs. Each graph can be characterized by its adjacency matrix: $C_{ij} = 1$ if $i$ is a neighbour of $j$ and $C_{ij} = 0$ otherwise. The partition function of the models we simulated is

$$Z_N = \sum_{T_N} \int dX_1 \ldots dX_N \exp \left( -\frac{1}{2} \sum_{i,j} C_{ij} E(X_i, X_j) \right).$$

$X_i$ is a real, scalar field attached to the faces of the triangulation. We mainly considered

$$E_2(X, Y) = \frac{1}{2} (X - Y)^2,$$ (2)

the discrete analog of the squared derivative, but for a check of universal properties of some of the quantities we also present data for

$$E_1(X, Y) = |X - Y|.$$ (3)

Both actions are expected to describe $c = 1$ matter coupled to quantum gravity.

The 2-point function for the tachyon with momentum $p$ is

$$G_N(p) = \left\langle \frac{1}{N^2} \sum_{i,j} e^{ip(X_i - X_j)} \right\rangle = \left\langle \frac{1}{N} \sum_i e^{ipX_i} \right\rangle^2.$$ (4)
From this one obtains the moments of $X$ by derivatives, e.g. the mean square extent

$$\langle X^2 \rangle_N = \frac{1}{N^2} \left\langle \sum_{i,j} (X_i - X_j)^2 \right\rangle$$  \hspace{1cm} (5)

$$= -\left. \frac{\partial^2}{\partial p^2} G_N(p) \right|_{p=0} .$$  \hspace{1cm} (6)

For these quantities exist theoretical predictions from conformal field theory [1,2,3,4,5] and from matrix models [6,7,8,9,10,11], which can be summarized in the following formulas:

1. Both approaches agree that  
   $$G(p) = \frac{z}{\sinh^2 \sqrt{z}} z = \frac{3}{2} \langle X^2 \rangle p^2 .$$  \hspace{1cm} (7)

2. Furthermore, both approaches predict the general dependence of the mean square extent on the number of triangles:
   $$\langle X^2 \rangle_N \propto (\ln N)^2 .$$  \hspace{1cm} (8)

This equation, which has first been conjectured in [12], implies that $G_N(p)$ becomes a function of $(p \ln N)$ only.

3. In order to fix the coefficient in (8) one has to normalize the field $X$. This is usually done by using the asymptotic behaviour of the 2-point function in flat space:
   $$\langle X_\sigma X_{\sigma'} \rangle \longrightarrow -\alpha' \ln |\sigma - \sigma'| \hspace{1cm} \text{for} \hspace{0.5cm} |\sigma - \sigma'| \rightarrow \infty .$$  \hspace{1cm} (9)

For the Gaussian action (8) this normalization is known:
   $$\alpha'(E_2) = 1/2\pi \approx 0.159 .$$  \hspace{1cm} (10)

To determine this factor for action $E_1$ of eq. (3) we performed a Monte Carlo simulation on a flat 2-dimensional lattice ($a 512 \times 512$ square lattice with periodic boundary conditions) and found
   $$\alpha'(E_1) \approx 0.134(2) .$$  \hspace{1cm} (11)

The conformal weight of the tachyon operator in flat space is
   $$\Delta_0(p) = \frac{\alpha'}{4} p^2 .$$  \hspace{1cm} (12)

Using the KPZ formula [2] to relate the weight of a conformal field in flat space with the one coupled to 2-dimensional gravity,
   $$\Delta(p) = \frac{\sqrt{1 - c + 24\Delta_0(p) - \sqrt{1 - c}}}{\sqrt{25 - c - \sqrt{1 - c}}} ,$$  \hspace{1cm} (13)

one can apply formal scaling arguments to obtain $G(p)$ as a function of the area $A$ [2,3,4,13]:
   $$G_A(p) \propto A^{1-2\Delta(p)} .$$  \hspace{1cm} (14)

For $c = 1$ this expression is non-analytic at $p = 0$, and we get a wrong minus sign if we try to obtain the mean square extent directly from this formula, according to eq. (6). For $p$ large however, equation (7), which has been derived in conformal field theory by introducing a cut-off [5], agrees with (14), provided that the mean square extent is given by
   $$\langle X^2 \rangle = \frac{\alpha'}{6} (\ln N)^2 .$$  \hspace{1cm} (15)

On the other hand, the mean square extent has been calculated in matrix models [10,11], yielding
   $$\langle X^2 \rangle = \frac{\alpha}{6} (\ln N)^2 .$$  \hspace{1cm} (16)

This calculation uses the action
   $$E_\alpha(X,Y) = \frac{|X - Y|}{\sqrt{\alpha}} .$$  \hspace{1cm} (17)

Matrix models, however, do not only generate graphs corresponding to triangulations but also graphs with nontrivial 2-point subgraphs. This might imply that the “mass” $1/\sqrt{\alpha}$ has to be renormalized before one can compare the results.

3. DATA AND RESULTS

We made simulations for $N = 140, 420, 1260, 3780, 11340$ and $34020$ triangles. The update of the scalar field was speeded up considerably by the use of the VMR (“valleys-to-mountains reflections) cluster algorithm [14]. For the update of the triangulation we used the triangle flip [15,16,17,18,19], which had to be supplemented by a local update procedure for the scalar field (we used both heat bath and Metropolis). Careful analysis of the autocorrelations revealed an
autocorrelation time of \( \approx 300 \) local sweeps (with a few clusters after each sweep) for the largest system, for which we performed about 200,000 sweeps.

No errorbars in the figures indicates that the errors are small compared to the symbols used to mark the data points. We should emphasize that the data for the simulations of action \( E_1 \) are less accurate since we have not finished our runs.

Figure 1. Mean square extent for \( E_2 = \frac{1}{2}(X - Y)^2 \)

\[ \langle X^2 \rangle \]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1}
\caption{Mean square extent for \( E_2 = \frac{1}{2}(X - Y)^2 \)}
\end{figure}

Fig. 2 shows the mean square extent for \( E_1 \) as a function of \( N \). A least square fit to

\[ \langle X^2 \rangle_N = \alpha''(\ln N)^2 + \beta(\ln N) + \gamma \]  

(18)
yields \( \alpha'' = 0.080(4) \). The value for \( N = 8 \) can be calculated analytically and has been added for curiosity.

Fig. 3 shows the corresponding data for action \( E_1 \), with a coefficient \( \alpha'' = 0.11(1) \) from a least square fit. While both actions quite nicely exhibit the \((\ln N)^2\) behavior, the coefficients are different, even if we take into account the normalization of the \( X \)-fields. This indicates that the coefficient might not be universal. The disagreement with the matrix model predictions \( [10] \) can be due to the subtraction of graphs with non-trivial 2-point subgraphs.

\[ \langle X^2 \rangle \]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2}
\caption{Mean square extent for \( E_1 = |X - Y| \)}
\end{figure}

Figs. 3 and 4 show \( G(p) \) for the actions \( E_2 \) \((N = 34020 \) triangles\) and \( E_1 \) \((N = 3780 \) triangles\) respectively, as well as a one-parameter fit with respect to \( \langle X^2 \rangle \) using the form \( [1] \). The universality of the scaling function is nicely confirmed. For smaller lattices the curves, as a function of \( p \ln N \), almost are on top of each other, which again implies the universality of eq. \( [3] \). Small deviations from the theoretical prediction are not visible in the figures, but can be seen by comparing the numbers. They are probably due to finite size effects. From the values of \( \langle X^2 \rangle \) obtained by fitting \( G(p) \), the coefficient of \((\ln N)^2\) is, consistently with the direct measurement presented before, 1.42 times larger for the action \( E_1 \) than for \( E_2 \). Whether we take into account the flat lattice normalization or not, this result indicates non-universality of \( \alpha'' \).

4. CONCLUSIONS

The Monte Carlo simulations we performed for the mean square extent and the tachyonic 2-point function \( G(p) \) of \( c = 1 \) matter coupled to 2-dimensional quantum gravity confirmed the universal scaling behavior for \( G(p) \). Furthermore, they confirmed the \( \langle X^2 \rangle \propto \ln^2(N) \) law and its universality. The coefficient in front of the \( \ln^2 \) however seems not to be universal. The contra-
diction with matrix model predictions for this coefficient might be due to a renormalization of the parameters in the matrix model. The discrepancy between two $c = 1$ scalar fields, which have the same conformal weight in flat space, can be a more serious problem. Presently, we perform several numerical checks to clarify these issues.

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