EPSILON FACTORS OF SYMPLECTIC TYPE CHARACTERS IN THE
WILD CASE

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Abstract. By John Tate we can associate the epsilon factor with every multiplicative character of a local field. In this paper we determine the explicit signs of the epsilon factors for symplectic type characters of $K^\times$, where $K/F$ is a wildly ramified quadratic extension of a non-Archimedean local field $F$ of characteristic zero.

1. Introduction

Let $F$ be a non-Archimedean local field of characteristic zero, i.e., a finite extension of $p$-adic numbers field $\mathbb{Q}_p$, where $p$ is a prime. Let $K/F$ be a finite extension of the field $F$. Let $e_{K/F}$ be the ramification index of the extension $K/F$ and $f_{K/F}$ be the residue degree of $K/F$. The extension $K/F$ is called unramified if $e_{K/F} = 1$; equivalently $f_{K/F} = [K : F]$. The extension $K/F$ is called totally ramified if $e_{K/F} = [K : F]$; equivalently $f_{K/F} = 1$. Let $\text{Char}(\mathbb{F}_{q_F})$ be the characteristic of the residue field $\mathbb{F}_{q_F}$ of $F$, where $q_F$ is the cardinality of the residue field of $F$. If $\gcd(\text{Char}(\mathbb{F}_{q_F}), [K : F]) = 1$, then the extension $K/F$ is called tamely ramified, otherwise wildly ramified. A multiplicative character $\chi$ of $K^\times$ (where $K/F$ is a quadratic field extension) is said to be symplectic type if $\chi|_{F^\times} = \omega_{K/F}$, where $\omega_{K/F}$ is the quadratic character of $F^\times$ associated to $K^\times$ by class field theory.

In [8], Dipendra Prasad shows that the epsilon factor of a symplectic type character is always a sign and in unramified and totally tamely ramified cases, he gives explicit sign of symplectic type character (cf. [8], pp. 22-23) by using Deligne’s twisting formula for characters (cf. Corollary 2.3[3]). This explicit computation of the sign has many applications. For instance, in Theorem 1.2 of [8], Prasad generalizes the Tunnell Theorem (cf. [11]) in terms of the explicit signs of symplectic type characters. But in the wild case (that is, when $K/F$ is a quadratic wildly ramified extension), the determination of the explicit signs of epsilon factors of symplectic type characters were not known. In this paper, we determine the explicit signs of the epsilon factors of symplectic type characters in the wild case. The computation of this paper is based on the Lamprecht-Tate formula (cf. [1], Theorem 3.1).

For applying Lamprecht-Tate formula, we first need to know the conductor of characters. In the following theorem first we show that the conductor of a symplectic type character is either $2t + 1$ or an even $\geq 2(t + 1)$, where $t$ is the ramification break of the extension $K/F$ (or of the Galois group $\text{Gal}(K/F)$).

Theorem 1.1. Let $K/F$ be a quadratic wildly ramified extension of $F$ and $t$ be the ramification break of the extension $K/F$. Denote $\omega_{K/F}$ as the quadratic character of $F^\times$ associated to $K$ by

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class field theory. Let \( m \) be the conductor of symplectic type character \( \chi \). Then the conductor of \( \omega_{K/F} \) is \( t + 1 \) and \( m = 2t + 1 \) when \( m \) is odd and \( m \geq 2(t + 1) \) when \( m \) is even.

When the conductor of a symplectic type character is even, we have the following theorem.

**Theorem 1.2.** Let \( K/F \) be a wild quadratic extension of \( F \). Let \( \psi_F \) be a fixed nontrivial additive character of \( F \). Let \( \pi_K \) be a uniformizer of \( K \) and consider \( \pi_F = N_{K/F}(\pi_K) \) as a uniformizer of \( F \). Let \( \psi \) be a nontrivial additive character of \( K \) of the form \( \psi = c \cdot (\psi_F \circ Tr_{K/F}) \) with conductor \( 2l + 1 \), where \( c \in K^\times \) such that the trace \( Tr_{K/F}(c) = 0 \). Let \( \chi \) be a symplectic type character of \( K^\times \) with even conductor \( a(\chi) = 2d(d \geq 2) \). Then we have

\[
\epsilon(\chi, \psi) = \begin{cases} 
1 & \text{when } \chi(-1) = 1 \\
(-1)^d & \text{when } \chi(-1) = -1.
\end{cases}
\]

(1.1)

And when the conductor of symplectic type character is \( 2t + 1 \), we obtain the following theorem.

**Theorem 1.3.** Let \( K/F \) be a wild quadratic extension of \( F \). Let \( \psi_F \) be a fixed nontrivial additive character of \( F \). Let \( \pi_K \) be a uniformizer of \( K \) and consider \( \pi_F = N_{K/F}(\pi_K) \) as a uniformizer of \( F \). Let \( \psi \) be a nontrivial additive character of \( K \) of the form \( \psi = c \cdot (\psi_F \circ Tr_{K/F}) \) with conductor \( 2l + 1 \), where \( c \in K^\times \) such that the trace \( Tr_{K/F}(c) = 0 \). Let \( t \) be the ramification break of the extension \( K/F \). Let \( \chi \) be a symplectic type character with conductor \( a(\chi) = 2t + 1 \). Then we have

\[
\epsilon(\chi, \psi) = \begin{cases} 
G(Q) & \text{if } \chi(-1) = 1 \\
(-1)^t \cdot G(Q) & \text{if } \chi(-1) = -1
\end{cases}
\]

(1.2)

where

\[
G(Q) := q_K^{-1/2} \sum_{x \in P_K^{l + 1}} Q(x), \quad Q(x) := \chi^{-1}(1 + x)(c^{-1}\psi)(x),
\]

and \( \nu_K(c') = 2t + 2l + 2 \) such that

\[
\chi(1 + y) = \psi(y/c'), \quad \text{for all } y \in P_K^{l + 1}.
\]

2. Notations and Preliminaries

Let \( F \) be a non-Archimedean local field of characteristic zero. Let \( O_F \) be the ring of integers in local field \( F \) and \( P_F = \pi_F O_F \) is the unique prime ideal in \( O_F \) and \( \pi_F \) is a uniformizer, i.e., an element in \( P_F \) whose valuation \( v_F(\pi_F) = 1 \). The cardinality of the residue field of \( F \) is \( q_F \). Let \( U_F = O_F - P_F \) be the group of units in \( O_F \). Let \( P_F^i = \{ x \in F : v_F(x) \geq i \} \) and for \( i \geq 0 \) define \( U_F^i = 1 + P_F^i \) (with \( U_F^0 = U_F = O_F^\times \)).

We also consider \( a(\chi) \) as the conductor of nontrivial character \( \chi : F^\times \to \mathbb{C}^\times \), i.e., \( a(\chi) \) is the smallest integer \( \geq 0 \) such that \( \chi \) is trivial on \( U_F^{a(\chi)} \). We say \( \chi \) is unramified if the conductor of \( \chi \) is zero and otherwise ramified. If \( K/F \) is a quadratic field extension of \( F \), then we denote \( \omega_{K/F} \) as the quadratic character of \( F^\times \) associated to \( K \) by class field theory; i.e., it is a unique nontrivial character of \( F^\times /N_{K/F}(K^\times) \), where \( N_{K/F} \) denotes the norm map from \( K^\times \) to \( F^\times \).

The conductor of any nontrivial additive character \( \psi \) of the field \( F \) is an integer \( n(\psi) \) if \( \psi \) is trivial on \( P_F^{-n(\psi)} \), but nontrivial on \( P_F^{-n(\psi)-1} \).
2.1. **Epsilon factors.** For a nontrivial multiplicative character $\chi$ of $F^\times$ and nontrivial additive character $\psi$ of $F$, we have

$$\epsilon(\chi, \psi) = \chi(c) \frac{\int_{U_F} \chi^{-1}(x)\psi(x/c)dx}{\int_{U_F} \chi^{-1}(x)\psi(x/c)dx},$$

where the Haar measure $dx$ is normalized such that the measure of $O_F$ is 1 and where $c \in F^\times$ with $F$-valuation $n(\psi) + a(\chi)$. The above formula (2.1) can be modified (cf. [10], pp. 93-94) as follows:

$$\epsilon(\chi, \psi) = \chi(c) q_F^{-n(\chi)/2} \sum_{x \in \mathcal{U}_F/\mathcal{U}_F^{n(\chi)}} \chi^{-1}(x)\psi(x/c).$$

where $c = \pi_F^{a(\chi)+n(\psi)}$.

**Remark 2.1.** If $u \in U_F$ is a unit and replace $c = cu$ in equation (2.2), then we observe that $\epsilon(\chi, \psi)$ depends only on the exponent $\nu_F(c) = a(\chi) + n(\psi)$.

For this paper we need the following Lamprecht-Tate formula.

**Theorem 2.2 ([1], Theorem 3.1, Lamprecht-Tate formula).** Let $F$ be a non-Archimedean local field. Let $\chi$ be a character of $F^\times$ of conductor $a(\chi)$ and let $m$ be a natural number such that $2m \leq a(\chi)$. Let $\psi$ be a nontrivial additive character of $F$. Then there exists $c \in F^\times$, $\nu_F(c) = a(\chi) + n(\psi)$ such that

$$\chi(1 + y) = \psi(c^{-1}y) \quad \text{for all} \quad y \in P_F^a(x)^{-m},$$

and for such a $c$ we have:

$$\epsilon(\chi, \psi) = \chi(c) \cdot q_F^{-\frac{a(\chi)+n(\psi)-2m}{2}} \sum_{x \in \mathcal{U}_F^{m}/\mathcal{U}_F^{a(\chi)-m}} \chi^{-1}(x)\psi(c^{-1}x).$$

**Remark:** The assumption (2.3) is obviously fulfilled for $m = 0$ because then both sides are 1, and the resulting formula for $m = 0$ is the original formula (2.2) for abelian local constant.

**Corollary 2.3 (cf. [1], Corollary 3.2).** Let $\chi$ be a character of $F^\times$. Let $\psi$ be a nontrivial additive character of $F$.

1. When $a(\chi) = 2d (d \geq 1)$, we have

$$\epsilon(\chi, \psi) = \chi(c)\psi(c^{-1}).$$

2. When $a(\chi) = 2d + 1 (d \geq 1)$, we have

$$\epsilon(\chi, \psi) = \chi(c)\psi(c^{-1}) \cdot q_F^{-\frac{1}{2}} \sum_{x \in P_F^{d}/P_F^{d+1}} \chi^{-1}(1 + x)\psi(c^{-1}x).$$

3. Deligne’s twisting formula (cf. [3], Lemma 4.16): If $\alpha, \beta \in \hat{F}^\times$ with $a(\alpha) \geq 2 \cdot a(\beta)$, then

$$\epsilon(\alpha\beta, \psi) = \beta(c) \cdot \epsilon(\alpha, \psi).$$

Here $c \in F^\times$ with $F$-valuation $\nu_F(c) = a(\chi) + n(\psi)$, and in Case (1) and Case (2), $c$ also satisfies
\[
\chi(1 + x) = \psi\left(\frac{x}{c}\right) \text{ for all } x \in F^\times \text{ with } 2 \cdot \nu_F(x) \geq a(\chi).
\]

And in case (3), we have \(\alpha(1 + x) = \psi(x/c)\) for all \(\nu_F(x) \geq \frac{a(\alpha)}{2}\).

2.2. Ramification break. Let \(K/F\) be a Galois extension of \(F\) and \(G\) be the Galois group of the extension \(K/F\). For each \(i \geq -1\) we define the \(i\)-th ramification subgroup of \(G\) (in the lower numbering) as follows:

\[
G_i = \{\sigma \in G | \ v_K(\sigma(\alpha) - \alpha) \geq i + 1 \text{ for all } \alpha \in O_K\}.
\]

An integer \(t\) is called a ramification break or jump for the extension \(K/F\) or the ramification groups \(\{G_i\}_{i \geq -1}\) if

\[
G_t \neq G_{t+1}.
\]

We also know that there is a decreasing filtration (with upper numbering) of \(G\) and which is defined by the Hasse-Herbrand function \(\Psi = \Psi_{K/F}\) as follows:

\[
G^u = G_{\Psi(u)}, \text{ where } u \in \mathbb{R}, u \geq -1.
\]

Since by the definition of Hasse-Herbrand function, \(\Psi(-1) = -1, \Psi(0) = 0\), we have \(G^{-1} = G_{-1} = G\), and \(G^0 = G_0\). Thus a real number \(t \geq -1\) is called a ramification break for \(K/F\) or the filtration \(\{G^u\}_{u \geq -1}\) if

\[
G^t \neq G^{t+\varepsilon}, \text{ for all } \varepsilon > 0.
\]

When \(G\) is abelian, it can be proved (cf. Hasse-Arf theorem, [4], p. 91) that the ramification breaks for \(G\) are integers. But in general, the set of ramification breaks of a Galois group of a local fields is countably infinite and need not consist of integers.

If \(K/F\) is quadratic extension, it can be proved that there exists a unique ramification break \(t\) for which we have

\[
G_t = G \text{ and } G_{t+1} = \{1\}.
\]

When \(K/F\) is unramified (resp. totally tamely ramified), the ramification jump is \(t = -1\) (resp. jump \(t = 0\)). And when \(K/F\) is quadratic wildly ramified extension, the ramification jump is \(t \geq 1\).

3. Epsilon factors of Symplectic characters

Lemma 3.1. Let \(K/F\) be a quadratic wildly ramified extension of \(F/\mathbb{Q}_2\) and let \(\psi_F\) be a fixed nontrivial additive character of \(F\). Let \(\psi\) be a nontrivial character of \(K\) of the form

\[
\psi := c \cdot (\psi_F \circ T_{K/F}), \text{ where } c \in K \text{ with } T_{K/F}(c) = 0.
\]

Then \(\psi\) is trivial on \(F\) and the conductor of \(\psi\) is:

\[
n(\psi) = 2 \cdot n(\psi_F) + \nu_K(c) + d_{K/F},
\]

where \(d_{K/F}\) is the exponent of the different \(D_{K/F}\).

Proof. We know that (cf. Lemma 3.4.3 of [2]) the conductor of \(\psi\) is

\[
n(\psi) = n(c \cdot (\psi_F \circ T_{K/F})) = \nu_K(c) + n(\psi_F \circ T_{K/F})
\]

\[
= \nu_K(c) + e_{K/F} \cdot n(\psi_F) + d_{K/F} = \nu_K(c) + 2 \cdot n(\psi_F) + d_{K/F}.
\]
Again by the given condition, we have $Tr_{K/F}(c) = 0$. Therefore, for any $x \in F$, we can show that:

$$
\psi(x) = \psi_F(Tr_{K/F}(xc)) = \psi_F(0) = 1.
$$

This proves that $\psi$ is trivial on $F$. \hfill \Box

**Remark 3.2. i).** Conversely, if we choose a nontrivial character $\psi$ which is trivial on $F$, then it can be proved that $\psi$ is of the form

$$
\psi = c \cdot (\psi_F \circ Tr_{K/F}),
$$

where $c \in K$ with $Tr_{K/F}(c) = 0$ and $\psi_F$ is some suitable nontrivial additive character of $F$.

Furthermore from Lemma 3.1, we can write:

$$
n(\psi) = n(\psi_F \circ Tr_{K/F}) + \nu_K(c) = 2 \cdot n(\psi_F) + d_{K/F} + \nu_K(c) \equiv d_{K/F} + \nu_K(c) \pmod{2}.
$$

But $[K : F] = 2$ implies that $\ker(Tr_{K/F})$ is a 1-dimensional $F$-space. Thus any other $c'$ has the form $c' = ac$ where $a \in F$, hence

$$
\nu_K(c') = \nu_K(a) + \nu_K(c) = 2\nu_F(a) + \nu_K(c) \equiv \nu_K(c) \pmod{2}.
$$

Therefore $\nu_K(c') \pmod{2}$ does not depend on the choice of $c$.

**ii).** Again by suitable choice of $\psi_F$ and $c$, we can always construct (by using Theorem 2.6 on p. 1999 of [7] and equation (3.1)) such nontrivial additive characters $\psi$ which are taken in Theorem 1.2 and Theorem 1.3.

**Proof of Theorem 1.1.** By local class field theory, we have $\text{Gal}(K/F) \cong F^\times/N_{K/F}(K^\times)$ and $\omega_{K/F}$ is the quadratic character of $F^\times/N_{K/F}(K^\times)$, i.e., $\omega_{K/F}$ trivial on the norm $N_{K/F}(K^\times)$.

We also know that

$$
U_n^K \cap F^\times = \begin{cases} 
U_n^F & \text{when } n \text{ is even}, \\
U_{n+1}^F & \text{when } n \text{ is odd}.
\end{cases}
$$

Let $s$ be the conductor of the quadratic character $\omega_{K/F}$. Then from the definition of conductor we can say $\omega_{K/F}$ is trivial on $U_s^F$, but nontrivial on $U_{s+1}^F$. The character $\omega_{K/F}$ is also trivial on norm $N_{K/F}(K^\times)$. By Corollary 3 on p. 85 of [9], we have

$$
N_{K/F}(U_T^\Phi(n)) = U_T^F \text{ for } n > t \text{ and } N_{K/F}(U_T^\Phi(n+1)) = U_T^{F+1} \text{ for } n \geq t.
$$

This implies $\omega_{K/F}$ trivial on $U_T^{F+1}$ when $n \geq t$ and trivial on $U_T^F$ when $n > t$. Therefore $t+1 \geq s$.

Now we have show that $s = t+1$. We assume the conductor of $\omega_{K/F}$ is greater than or equal to $t+2$, therefore $\omega_{K/F}$ is nontrivial on $U_T^{F+1}$ which is not possible. Hence $s = t+1$. Therefore the conductor of $\omega_{K/F}$ is $t+1$.

Let $\chi$ be a symplectic type character with conductor $m$. Therefore $\chi|_{U_T^F} = 1$ but $\chi|_{U^{-1}_T} \neq 1$.

Again $\chi|_{F^\times} = \omega_{K/F}$, then from relation (3.2) we get $\omega_{K/F}$ is trivial on $U_T^F$ for $m$ even, and trivial on $U_T^F$ for $m$ odd. These gives use $m \geq 2(t+1)$ (when $m$ even) and $m \geq 2t+1$ (when $m$ odd). Now we have to prove that when $m$ is odd, and it is exactly $2t+1$, i.e., $a(\chi) = 2t+1$. Suppose the conductor of $\chi$ is $2t + r$ where $r \geq 3$ and co-prime to 2. Therefore $\chi|_{U_T^{2+1}} = 1$ but $\chi|_{U_T^{2+r}} \neq 1$. Here $r$ can be written as $r = 2b + 1$, where $b \geq 1$. We can write

$$
U_T^{2t+r-1} = U_T^{2t+2b} = U_T^{t+b}U_T^{2t+2b+1} = U_T^{t+b}U_T^{2t+r}.
$$
Now if we restrict $\chi$ to $U_K^{2t+r-1}$ we have

$$\chi|_{U_K^{2t+r-1}} = \chi(U_F^{t+b} U_K^{2t+r}) = \chi|_{U_F^{t+b}} \times \chi|_{U_K^{2t+r}} = \omega_{K/F}|_{U_F^{t+b}} = 1,$$

since $a(\omega_{K/F}) = t + 1$ and $a(\chi) = 2t + r$. But the left hand side of equation (3.3), $\chi|_{U_K^{2t+r-1}} \neq 1$ which is a contradiction. Therefore it is impossible that the conductor of $\chi$ (when conductor is odd) is $2t + r (r \geq 3)$. Thus we conclude that when the conductor of symplectic type character is odd, it is exactly $2t + 1$.

Furthermore, let $\chi, \chi'$ be two symplectic type characters. Then there exists a nontrivial character $\eta : K^\times \to \mathbb{C}^\times$ with $\eta|_{F^\times} \equiv 1$ such that

$$\chi' = \eta \cdot \chi.$$

Since we just notice that $a(\chi'), a(\chi)$ are $\geq \{2t + 1, 2l\}$, where $l \geq (t + 1)$, the conductor of $\eta$ must be a even number greater than $2t + 1$.

Again we notice that there exists a symplectic type character $\chi_{\text{odd}}$ of $K^\times$ with conductor $2t + 1$. It can be proved all symplectic type characters $\chi$ can be represented as follows:

$$\chi = \eta \cdot \chi_{\text{odd}}, \quad \text{where} \quad \eta : K^\times \to \mathbb{C}^\times \quad \text{with} \quad \eta|_{F^\times} \equiv 1.$$

Then

$$a(\chi) = a(\eta \cdot \chi_{\text{odd}}) = \max\{a(\eta), a(\chi_{\text{odd}})\} = a(\eta).$$

This completes the proof.

Proof of Theorem 1.2. By our choice $\pi_K$ is a uniformizer of $K$ and $\pi_F = N_{K/F}(\pi_K)$ is a uniformizer of $F$. This implies $\pi_F = -\pi_K^2$. Let $t$ be the ramification break of the extension $K/F$. We also know from Theorem 1.1 that $2d \geq 2(t + 1)$, where $t$ is odd and $t < 2e$, where $e$ is the absolute ramification index of $F$. Therefore when conductor of symplectic type character is $2d$, then $d \geq 2$.

Now for applying Lamprecht-Tate formula theorem 2.2 we have to choose $c' \in K$ such that the expression $\epsilon(\chi, \psi)$ is good for proving our assertion. The Lamprecht-Tate formula works for any $c'$ such that $\nu_K(c') = a(\chi) + n(\psi) = 2d$. We take $c' = \pi_K^{2d}$ and by the given condition, $a(\chi) = 2d$. Now the conductor of $c'^{-1}\psi$ is:

$$n(c'^{-1}\psi) = \nu_K(c'^{-1}) + n(\psi) = -2d + 0 = -a(\chi).$$

Therefore here we can choose $c' = \pi_K^{2d}$ for applying Lamprecht-Tate formula.
Since $a(\chi)$ is even, here we use the Lamprecht-Tate formula (cf. Theorem 2.2) and we have
\[
e(\chi, \psi) = \chi(c') \times \psi(c'^{-1}) = \chi(\pi_K^{2d}) \times \psi(\pi_K^{-2d}) = \chi(\pi_K^d) \times \psi((\pi_K^d)^{-d}) = \chi(-\pi_F)^d \times \psi((-\pi_F)^{-d}) \quad \text{since } \pi_K^2 = -\pi_F = \chi(\pi(-1)) = 1, \]

Moreover, $\psi|_F = 1$ by Lemma 3.1, so we can write
\[
\psi(c'^{-1}) = \psi(\pi_K^{-2(\omega + 1)}) = \psi((-\pi_F)^{-t+1}) = 1,
\]

and
\[
\chi(c') = \chi(\pi_K^{2(t+1)}) = \chi(\pi_K^{2(t+1)}) = \chi((-\pi_F)^{t+1}) = \chi(-\pi_F)^{t+1} = \chi(-1)^{t+1} \omega_{K/F}(\pi_F)^{t+1} = \chi(-1)^{t+1},
\]

Now from the second part of the Lamprecht-Tate formula 2.4 we can write
\[
\epsilon(\chi, \psi) = \chi(c') \psi_1(c'^{-1}) \cdot G(Q) = \chi(-1)^{t+1} G(Q)
= \begin{cases}
G(Q) & \text{if } \chi(1) = 1 \\
(\chi)^t \cdot G(Q) & \text{if } \chi(-1) = -1.
\end{cases}
\]

where
\[
G(Q) := q_K^{-1/2} \sum_{x \in P_K} Q(x), \quad Q(x) := (1 + x)(c'^{-1})\psi(x).
\]

**Proof of Theorem 1.3.** By our choice $\pi_K$ is a uniformizer of $K$, and $N_{K/F}(\pi_K) = \pi_F$ is a uniformizer of $F$. Since $a(\chi) = 2t + 1$ and $n(\psi) = 2l + 1$, therefore we choose (it can be checked in the same way what we have done in the proof of Theorem 1.2 that $c'$ is suitable for applying Lamprecht-Tate formula)
\[
c' = \pi_K^{a(\chi)+n(\psi)} = \pi_K^{2t+1+2l+1} = \pi_K^{2(t+1)}.
\]

Moreover, $\psi|_F = 1$ by Lemma 3.1, so we can write

**Conjecture 1.** With the above notations, $G(Q)$ is a sign.

**Remark 3.3.** Let $\kappa_K$ be the residue field of $K$. In the above theorem, due to the choice of $c'$, we observe that $Q(x)$ is a function on $P_K$. Such that:

\[
\frac{Q(x+y)}{Q(x)Q(y)} = \chi \left(1 + \frac{xy}{1 + x + y}\right) = (c'^{-1})\psi(x), \quad \text{for } x, y \in P_K.
\]
Since \(a(\chi) = 2t + 1 \neq 1\) and \(p = 2\) hence \(\epsilon(\chi, \psi) = \chi(c') \psi(c^{-1}) \cdot G(Q)\), for \(c' \in K^\times\), which means that \(Q(x) = \chi^{-1}(1 + x)(c^{-1} \psi)(x)\) is a function on \(P_F^\times/P_F^{t+1}\) such that \(\epsilon \in \kappa_K \mapsto \overline{Q}(\epsilon) := Q(\pi_K \epsilon)\) satisfies
\[
\frac{\overline{Q}(\epsilon_1 + \epsilon_2)}{\overline{Q}(\epsilon_1) \overline{Q}(\epsilon_2)} = (\pi_K c^{-1} \psi)(\epsilon_1 \epsilon_2).
\]
Therefore \(\overline{Q}\) is (in the sense of Subsection 1.1.5 of [5]) a quadratic character of \(\kappa_K\) which is related to the additive character \(\tau := \pi_K c^{-1} \psi\), and then by the second part of Lemma 3.1 in Subsection 1.1.9 of [5], we have
\[G(Q) = \gamma(\overline{Q})\] is an 8th root of unity such that:
\[
\gamma(\overline{Q})^2 = \overline{Q}(c'(\tau)), \quad \gamma(\overline{Q})^4 = (-1)^{[\kappa_K:F_2]},
\]
where \(c'(\tau) \in \kappa_K^\times\) is determined by the relation \(\tau(\epsilon) = (-1)^{Tr(\epsilon/c'(\tau)^2)}\) for all \(\epsilon\), and \(Tr := Tr_{\kappa_K:F_2}\).

**Remark 3.4.** When \(K/F\) is a quadratic unramified or tamely ramified, we have the explicit signs of epsilon factor of symplectic type characters due to Dipendra Prasad (see [8], pp. 22-23). He uses Deligne’s twisting formula (cf. Corollary [2,3,3]). One also can determine the explicit sign directly by using Lamprecht-Tate formula. Let \(K/F\) be a quadratic unramified extension of local field \(F\). Let \(\chi\) be a ramified symplectic type character of \(K^\times\), i.e., conductor \(a(\chi) \geq 1\) and \(\psi_b\) be a nontrivial additive character of \(K\) of the form \(\psi_b(x) := \psi_K(bx) = \psi_F(tr_{K/F}(bx))\) for all \(x \in K\) and \(b \in K^\times\) with \(tr_{K/F}(b) = 0\). Then we have (cf. [8], p. 22-23)
\[\epsilon(\chi, \psi_b) = \begin{cases} 1 & \text{when } a(\chi) \text{ even,} \\ -1 & \text{when } a(\chi) \text{ odd.} \end{cases} \]

If \(K/F\) is tamely ramified, then \(t = 0\). Then from Theorem [1,1] we can see that the conductor of symplectic type character is **one or even** and the conductor of \(\omega_{K/F}\) is 1, i.e., \(\omega_{K/F}\) is trivial on \(1 + \pi_F O_F\). In the same case we choose \(\pi_K\) as a uniformizer and then \(\pi_F = N_{K/F}(\pi_K)\) is a uniformizer of \(F\). Now fix a character \(\omega_{\tilde{K}/F}\) of \(K^\times\) which extends the character \(\omega_{K/F}\) of \(F^\times\). We use this isomorphism \(U_K/U_K^\times \cong U_F/U_F^\times\) (since \(K/F\) is a quadratic tamely ramified) to extend \(\omega_{K/F}\) to \(U_K\). And we know \(a(\omega_{K/F}) = 1\), then it can be seen that the conductor of \(\omega_{\tilde{K}/F}\) is one. If \(\chi\) is a character which extends the character \(\omega_{K/F}\) with \(a(\chi) = 1\), then \(\chi\) is either \(\omega_{\tilde{K}/F}\) or \(\omega_{\tilde{K}/F} \mu\) where \(\mu\) is an unramified character of \(K^\times\) taking value \(-1\) at \(\pi_K\). Then we have,
\[
\epsilon(\omega_{\tilde{K}/F} \mu, \psi) = \mu(\pi_K) \epsilon(\omega_{\tilde{K}/F}, \psi) = -\epsilon(\omega_{\tilde{K}/F}, \psi),
\]
where \(\psi\) is an additive character of \(\tilde{K}\) with conductor zero. By using the quadratic classical Gauss sum formula (cf. [6], p. 199, Theorem 5.15) we can give more explicit formula of \(\epsilon(\omega_{\tilde{K}/F}, \psi_{-1})\), where \(\psi_{-1}\) is non trivial additive character of \(K\) with conductors \(-1\). Then
\[\epsilon(\omega_{\tilde{K}/F}, \psi_{-1}) = (-1)^{s-1} \quad \text{when } p \equiv 1 \pmod{4}, \quad \text{and } q_F = p^s.
\]
And when \(p \equiv 3 \pmod{4} \) and \(q_F = p^{2r} (r \geq 1)\), we have
\[\epsilon(\omega_{\tilde{K}/F}, \psi_{-1}) = (-1)^{r-1}.
\]
Now let \(\chi\) be a symplectic type character of \(K^\times\) with conductor \(2d(d \geq 1)\). Let \(\psi\) be a nontrivial additive character of \(K\) which is trivial on \(F\) with conductor zero. Then by using
Lamprecht-Tate formula we have

\[ \epsilon(\chi, \psi) = \begin{cases} 
1 & \text{if } q_F \equiv 1 \pmod{4} \\
(-1)^d & \text{if } q_F \equiv 3 \pmod{4}.
\end{cases} \tag{3.7} \]

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