Influence of wall slip in dilute suspensions

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Abstract.

Navier’s (1823) slip condition, where the tangential velocity is proportional to the shear stress, may apply at micro-scales for a viscous liquid on walls with modified surfaces, e.g. hydrophobic ones in water. Experiments by Lumma et al (2003) show the difficulties involved in measuring such a slip.

At small scales, particle-wall hydrodynamic interactions are important. They are modeled here for a dilute suspension of spherical solid particles near a slip wall. Consider a translating and rotating sphere (on which the no-slip condition applies) in an ambient parabolic flow. Analytical solutions of Stokes equations for the various elementary flow fields were obtained (Feuillebois et al 2009, 2011) as series in bispherical coordinates. The coefficients in the series are solutions of an infinite linear system, which is solved by an extension of Thomas’ algorithm, allowing to calculate a large number of terms. Accurate results are available for the force, torque and stresslet on a sphere, velocity of a freely moving sphere and diffusion tensor.

The Aris-Taylor dispersion of Brownian particles in a shear flow near a slip wall gives a large bias in the measurement of slip (Vinogradova et al, 2009). It is calculated here from the advection-diffusion equation, using the expressions for the particle velocity and diffusion tensor near a slip wall.

1. Introduction

In this paper, prepared at the occasion of the symposium for my 65th birthday, I shall present a summary of results obtained with various collaborators over the recent years, namely Olga I. Vinogradova and Dirk Lumma for the experimental part [1, 2], Hend Loussaief, Laurentiu Pasol for the theoretical part on creeping flow [3]. The paper also contains recent theoretical results [4] obtained with my present coauthors, Néjiba Ghalya, Antoine Sellier and Lassaad Elasmi. I am grateful to Maria L. Ekiel-Jeżewska and to the local committee for the splendid organisation of this symposium.

The idea of a slip boundary condition for a viscous fluid on a solid wall was originally postulated by Navier [5]. This condition is written in term of a slip velocity \( v_w \) along the wall that is proportional to the tangential stress \( \tau_w \). Alternatively, for a flow along a plane wall, \( v_w \) is proportional to the shear rate on the wall \( \left. \frac{dv}{dz} \right|_w \), where \( v \) is the velocity along the wall at distance \( z \):

\[
\frac{v_w}{\mu} = \frac{b}{\mu} \tau_w = b \left. \frac{dv}{dz} \right|_w
\]
Here, $\mu$ is the fluid viscosity. The constant $b$ is the slip length. It may be interpreted as the distance into the wall for which the extrapolated shear flow velocity (based on the shear rate at the wall) would vanish (Fig. 1. Note that in general we do not extrapolate the velocity profile itself to zero; this only happens for a linear shear flow). Thus, alternative ways of representation of the slip condition for a shear flow are the slip velocity $v_w$ and the slip length $b$.

![Figure 1](image1.png)

**Figure 1.** Definitions of the slip velocity $v_w$ and slip length $b$ on a plane wall.

Even though it is now admitted that the no-slip condition usually applies at ordinary scales, this idea of a slip condition reappears in connection with novel applications in microfluidics. For a gas at the scale of the mean free path, it is known that a slip boundary condition applies for the flow outside the Knudsen layer. But it may also apply to a liquid, e.g. for micro-channels with chemically treated surfaces. Obviously, the slip depends on the liquid and wall materials. Fluids considered here are Newtonian. Non-Newtonian fluids, like polymer solutions, require some particular rheological bulk equations and slip boundary conditions. Even for Newtonian liquids, there is a large body of literature and values of $b$ are a subject of discussions. A review of experiments in various systems is given in [6].

There may also be a slip boundary condition on suspended particles in a liquid, either because of their hydrophobic surface or because they are porous, like for a polymer jumble. Indeed, it has been shown by [7] that a slip applies on a porous wall, the slip length being of the order of the pore size. The scope of this paper is restricted to the case of the no-slip boundary condition on particles.

For applications, the advantage of a slip condition is that pushing a fluid through a micro-channel requires a reduced pressure gradient (which otherwise should be quite large).
incidence of a wall slip on dilute suspensions is considered here. This problem is relevant in microfluidics, for instance for separation techniques in analytical chemistry like field-flow-fractionation (FFF) and capillary hydrodynamic chromatography (CHDC) (see [8, 9, 10]). The influence of fluid slip in CHDC was considered recently in [11].

The outline of the paper is as follows. First, recent experiments for measuring the slip length are presented in §2. Important physical parameters which may be responsible for discrepancies in the results are displayed. The importance of hydrodynamic interactions of particles with a slip wall are emphasised. This is an incentive for studying in detail the hydrodynamic interaction of a spherical particle with a slip wall, when in creeping flow, §3. The general case of a moving sphere in a parabolic shear flow is expressed by linearity of Stokes equations of fluid motion as the superposition of elementary flow problems. The solution for an ambient quadratic flow is presented here in §4. It is searched in terms of series in bispherical coordinates and a fast algorithm is then applied to calculate the coefficients in the series. Results for the example case of a fixed sphere embedded in a quadratic shear flow are presented in §5. Results for a freely moving sphere are provided in §6. Then §7 is concerned with Brownian particles entrained in a shear flow near a slip wall. This problem is relevant for the measurements described in §2 and also for the separation techniques briefly mentioned above. Finally, the conclusion and outlooks are in §8.

2. Measurements of slip length and important parameters
The measurements presented here were performed with the double-focus fluorescence cross-correlation technique (DF-FCS). They are fully documented in [1, 2]. The setup and principle of DF-FCS are sketched in Fig. 2.

A laser beam (543 nm line of a 5 mW Helium-Neon laser) is split as two parallel laser beams separated by a distance of 6 μm. The beams cross the slip wall from below. Their foci $F_1$ and $F_2$ provide two spheroidal test volumes located in the fluid region. Fluorescent tracers are carried along the wall in the $x$ direction by the fluid flow with velocity $v(z)$, where $z$ denotes the coordinate normal to the wall. They cross consecutively the two foci, producing two time-resolved fluorescence intensities $I_1(t)$ and $I_2(t)$. The time cross correlation function calculated as $G(\tau) = \langle I_1(t)I_2(t + \tau) \rangle_t / \langle I_1(t) \rangle_t \langle I_2(t) \rangle_t$ has a maximum at some $\tau = \tau_m$, which is characteristic of the transfer time between foci.

The tracers were either 40 nm diameter spherical latex particles or single dye Alexa 568 dye molecules with a hydrodynamic diameter of ca. 1.2 nm. The liquid was water, with various proportions of added dissolved salt to screen out the repulsive electrical forces between particles. The tracers were observed with a confocal microscope.

Measured velocities are averages over those of particles located in the spheroidal test volumes. The $\zeta$ position refers to the position of the center of each spheroidal test volume relative to the channel midplane. A typical velocity profile is shown in Fig. 3. As expected, across the channel (left) the parabolic Poiseuille profile is obtained. The plots (a) and (b) in Fig. 3 (right) show enlargements of the near-wall regions, the walls being represented as dashed lines. Note that $\zeta$ may exceed the distance between the channel wall and the channel midplane. When the whole spheroidal test volume is in the fluid, all particles are illuminated and contribute to the signal. When the center of the test volume is in the transparent wall (the glass slide) while a part of this volume is still in the fluid, some particles close to the wall are still illuminated. The signal then is weaker but provides a value of the velocity in the vicinity of the wall. The plateau, in either (a) or (b), then corresponds to particles located very close to a wall. The non-zero value at the plateau exhibits a slip velocity on the wall. From these figures, the slip length may then be obtained, see Fig. 1.

Results for the slip length presented in Tab. 1 show wide variations depending upon the used particles, fluid and wall material. This illustrates the difficulties involved in using particle
Figure 3. Measured velocity of particles located in the test volume, versus the position $\zeta$ of the center of the test volume (from [1]). Particles are latex spheres. Left: across channel. Right: near walls, showing the slip velocity on the walls. The experimentally determined positions of the walls are shown as dashed lines. When the center of test volume is in the solid transparent wall, some particles in the fluid close to the wall provide a value of the velocity there. This velocity appears as a plateau in the walls (a) and (b).

tracking techniques (like this technique, and also micro-PIV) to measure a slip length.

|                  | glass slide ⇒ | borosilicate | glass slide | mica |
|------------------|---------------|--------------|-------------|------|
| Latex spheres,   | 1.00          | 0.89         | 0.86        |      |
| salt-free water  |                |              |             |      |
| Alexa 568,       | 0.82          | 0.63         | 0.51        |      |
| salt-free water  |                |              |             |      |
| Alexa 568,       | 0.32          | 0.59         |             |      |
| 0.01 mM salted water |            |              |             |      |
| Alexa 568,       | 0.25          | 0.22         |             |      |
| 1 mM salted water|                |              |             |      |

Table 1. Experimental results for the slip length (in $\mu$m).

The differences in the table may be interpreted in terms of various physical phenomena:

(i) Repulsive electrical forces push particles away from wall. Thus they go faster and this increases the apparent slip length. But some values are still not consistent, comparing latex spheres and the Alexa 568 molecules.

(ii) The Aris-Taylor dispersion provides an apparent slip length.

(iii) The hydrodynamic interactions between particles and slip wall modify the relative velocity between the tracer and fluid.

The important influence of item (ii) was resolved in [2]. Experimental results and the model are compared in Fig. 4. The notation for $\zeta$ is like in Fig. 3 and the plateau also corresponds to particles located very close to the wall that is shown as a dashed line. In these results:

(i) A $10^{-4}$ mol/l NaCl solution was used for small repulsive forces.

(ii) Only particles at a distance from the wall were considered in fitting the model and experiment, so that particle-wall interactions could be neglected.

The calculated Aris-Taylor dispersion with these conditions gave a rather large apparent slip length of 740 nm. But subtracting the apparent slip lengths for slip and no-slip walls could provide a true slip length of around 60 to 70 nm.
Figure 4. Experimental results using latex spheres and model from [2]. The experimentally determined position of the wall is shown as a dashed line. As for the plateau, see the explanation in the caption of Fig. 3 and in the text.

Yet, as observed in Fig. 4 (right), there is a discrepancy between the theoretical and experimental points near the wall which could be due in particular to particle-wall hydrodynamic interactions. Progress in this domain will thus arise from a study of this effect. This paper brings its contribution in this respect.

3. Hydrodynamical problem for a spherical particle near a slip wall

Considering the size of particles involved in the measurement techniques of the slip length and in separation methods, the Reynolds number for the flow field relative to a particle is usually low as compared with unity and Stokes equations for creeping flow apply. Particles considered here are solid and spherical. The suspension of particles is dilute in the sense that the volume fraction is low. Then, hydrodynamic interactions between particles may be neglected. That is, particles may be considered as independent in their motion relative to the fluid. The typical problem at hand for the quoted application is that of a particle embedded in a fluid that is moving with a given velocity \( v(z) \) in the \( x \) direction (following the notation of the preceding section) along and in between parallel walls.

There is a large body of literature for spherical particles in creeping flow near no-slip walls (see e.g. [12, 13] and more recent references in [14]), but less results are available for a slip wall. It is shown in [14] that, in view of modeling the hydrodynamic interactions between a particle and two no-slip walls, the approximation of considering only the hydrodynamic interactions with the nearest no-slip wall has a surprisingly large range of application. Thus, the approximation of considering the nearest slip wall may be sufficient for many purposes. This idea is based on the fact that hydrodynamic interactions with a slip wall are weaker than with a no-slip wall.

The presentation follows this line. Consider (Fig. 5) a spherical particle with radius \( a \) centred at a distance \( \ell \) from a slip wall. The particle is moving as a solid body with translational velocity \( \mathbf{U} \) and rotational velocity \( \mathbf{\Omega} \) in an ambient flow field with velocity \( \mathbf{v} = [k_s (z + b) + k_q z^2] \mathbf{e}_x \) along the wall. Here, \( k_s \) and \( k_q \) denote constant coefficients and \( \mathbf{e}_x \) is the unit vector along \( x \). The perturbed flow velocity \( \mathbf{u} \) and pressure \( p \) satisfy the Stokes equations:

\[
\begin{align*}
\nabla \cdot \mathbf{u} & = 0 \quad (1a) \\
\mu \nabla^2 \mathbf{u} & = \nabla p , \quad (1b)
\end{align*}
\]

with boundary conditions:

\[
\mathbf{u} = b \frac{d\mathbf{u}}{dz} \quad \text{on} \quad z = 0 \quad (2a)
\]
Figure 5. Notation for a solid spherical particle moving in an ambient flow field near a slip wall.

\[ \mathbf{u} \rightarrow \mathbf{v} = [k_s(z + b) + k_qz^2] \mathbf{e}_x \quad \text{for} \quad |\mathbf{r}| \rightarrow \infty \quad (2b) \]
\[ \mathbf{u} = \mathbf{U} + \mathbf{\Omega} \times \mathbf{r} \quad \text{on} \quad |\mathbf{r}| = a, \quad (2c) \]

where \( \mathbf{r} \) denotes a position vector originating from the sphere centre. By linearity of the Stokes equations and boundary conditions, this problem may be solved as the superposition of the cases of:

(i) a sphere translating (\( \mathbf{U} \)) and rotating (\( \mathbf{\Omega} \)) in a fluid at rest,
(ii) a fixed sphere in a linear shear flow (\( k_s(z + b) \mathbf{e}_x \)),
(iii) a fixed sphere in a quadratic shear flow (\( k_qz^2 \mathbf{e}_x \)).

Problem (i) was solved by the method of bispherical coordinates in [15]. The system of equations was rearranged in a simpler way in [3]. Reference [3] also treated problem (ii) along the same line. Problems (i) and (ii) were also considered by [16] using the boundary integral method. Problem (iii) presented in [4] is exposed below; the analytical approach follows that of [3].

Quantities of interest in view of modeling the particle motion are the force \( \mathbf{F} \) and torque \( \mathbf{C} \) on the particle, calculated from:

\[ \mathbf{F} = \int_S \sigma \cdot \mathbf{n} \, dS \quad (3a) \]
\[ \mathbf{C} = \int_S \mathbf{r} \times \sigma \cdot \mathbf{n} \, dS, \quad (3b) \]

where \( \sigma \) is the stress tensor, \( \mathbf{n} \) an unit normal vector and \( S \) the sphere surface. From the particle equations of motion, the velocities \( \mathbf{U} \) and \( \mathbf{\Omega} \) (which are assumed to be given in the preceding step) can then be obtained, e.g. for a freely moving particle (that is for zero external force and torque).

The symmetric first moment of stresses, or “stresslet”:

\[ \mathbf{S} = \int_S \left[ \frac{1}{2}(\mathbf{r} \sigma \cdot \mathbf{n} + \sigma \cdot \mathbf{n} \mathbf{r}) - \frac{1}{3} \mathbf{I}(\mathbf{r} \cdot \sigma \cdot \mathbf{n}) \right] dS \quad (4) \]

where \( \mathbf{I} \) is the identity tensor, is also of interest, in view of calculating the rheology of a dilute suspension.
The force, torque and stresslet for problem (1)(2) are obtained from the preceding decomposition as the sum of those for the elementary problems. These quantities may be written in terms of friction factors as:

| Description          | Formula                                                                 |
|----------------------|-------------------------------------------------------------------------|
| Translation          | $F_x = -6\pi a \mu U f_{x,x}$, $C_{xy} = 8\pi a^2 \mu U c_{xy}$, $S_{xz} = 6\pi a^2 \mu U s_{xz}$ |
| Rotation             | $F_y = 6\pi a^2 \mu \Omega f_{y,y}$, $C_{yy} = -8\pi a^3 \mu \Omega c_{yy}$, $S_{xz} = 6\pi a^3 \mu \Omega s_{xz}$ |
| Linear shear flow    | $F_z = 6\pi a \mu (k_s \ell + b) f_{z,z}$, $C_{zx} = 4\pi a^3 \mu k_s c_{zx}$, $S_{zz} = \frac{10}{3} \pi a^3 \mu k_s s_{zz}$ |
| Quadratic shear flow | $F_{q,x} = 6\pi a \ell^2 \mu k_q f_{q,x}$, $C_{q,y} = 8\pi a^3 \ell \mu k_q c_{q,y}$, $S_{q,z} = \frac{20}{3} \pi a^3 \ell \mu k_q s_{q,z}$ |

The force and torque for translation, rotation and linear shear flow (in green in the table) were calculated by [3]. The stresslet for translation and rotation and the force, torque and stresslet for quadratic shear flow (in red in the table) were obtained more recently [4]. Forces, torques and stresslets are here normalised by their counterpart for a sphere in unbounded fluid ($\ell/a \to \infty$), obtained from Faxen’s formulae. Then, all normalised quantities, or “friction factors” (denoted with lower case letters, in obvious notation) are dimensionless and become unity for $\ell/a \to \infty$, except for the coupling terms $c_{txy}$ (torque due to translation) and $f_{rxy}$ (force due to rotation) which vanish. There is also a symmetry relationship between these two coefficients from Lorentz reciprocal theorem: $f_{rxy} = \frac{4}{3} c_{txy}$.

4. Spherical particle in quadratic flow near a slip wall; solution in bispherical coordinates

Using cylindrical coordinates ($\rho, z, \phi$), see Fig. 6 (the sphere centre being at $\rho = 0, z = \ell$), the perturbation flow field $\{w = v - u, q = p - 2\mu k_q\}$ satisfying the momentum equation (1a) is written following [17, 18] as:

$$
 w_\rho = \frac{1}{2} \{\rho Q_1 + U_0 + U_2\} \cos \phi, \quad w_z = \frac{1}{2} \{zQ_1 + 2U_1\} \cos \phi, \\
 w_\phi = \frac{1}{2} \{U_2 - U_0\} \sin \phi, \quad q = \mu Q_1 \cos \phi,
$$

(6a)

(6b)

where $U_0, U_1, U_2, Q_1$ are harmonic functions to be found.

These functions are searched for as series in bispherical coordinates. These coordinates have been used in a number of problems for nearly 100 years. Definition formulae of the bispherical coordinates $(\eta, \xi, \phi)$ and a sketch are displayed in Fig. 6. More details are in [12], Appendix A-19.
Note that $\xi = 0$ represents the plane $z = 0$ and $\xi = \alpha$ represents the sphere surface, with $\ell = a \cosh \alpha$. The application of boundary conditions on the plane and sphere is then made easier.

The expressions of the harmonic functions $U_1, Q_1, U_0, U_2$ as series in these coordinates are:

\begin{equation}
U_1 = d^2(\cosh \xi - \beta)^{1/2} \sin \eta \sum_{n \geq 1} A_n \sinh(\gamma_n \xi) P_n(\beta), \quad (7a)
\end{equation}

\begin{equation}
Q_1 = d(\cosh \xi - \beta)^{1/2} \sin \eta \sum_{n \geq 1} [B_n \cosh(\gamma_n \xi) + C_n \sinh(\gamma_n \xi)] P_n(\beta), \quad (7b)
\end{equation}

\begin{equation}
U_0 = d^2(\cosh \xi - \beta)^{1/2} \sum_{n \geq 0} [D_n \cosh(\gamma_n \xi) + E_n \sinh(\gamma_n \xi)] P_n(\beta), \quad (7c)
\end{equation}

\begin{equation}
U_2 = d^2(\cosh \xi - \beta)^{1/2} \sin^2 \eta \sum_{n \geq 2} [F_n \cosh(\gamma_n \xi) + G_n \sinh(\gamma_n \xi)] P_n(\beta), \quad (7d)
\end{equation}

where $\beta = \cos \eta$, $\gamma_n = n + 1/2$, $P_n(\beta)$ is the Legendre polynomial of order $n$ and $\partial / \partial \beta$. By construction, the harmonic functions vanish at infinity which is represented by $\xi = \eta = 0$ in bispherical coordinates.

The coefficients $A_n, \ldots, G_n$ in the series (7) have to be found from the boundary conditions and also from the continuity equation (1a). Calculations are made simpler by using the following formula, obtained by differentiating the slip boundary condition (2a) and combining with the continuity equation (1a):

\begin{equation}
\frac{\partial w_z}{\partial z} = b \frac{\partial^2 w_z}{\partial z^2} \quad \text{on} \quad z = 0. \quad (8)
\end{equation}

As a result, the unknown coefficients $B_n, D_n, F_n$ are obtained in terms of the other ones $A_n, C_n, E_n, G_n$. The system to be solved for $A_n, C_n, E_n, G_n$ may then be written in the form:

\begin{equation}
\begin{align*}
M_1 X_1 + M_1^+ X_2 &= B_1, \quad (9a) \\
M_n^{-} X_{n-1} + M_n X_{n} + M_n^+ X_{n+1} &= B_n, \quad (n \geq 2) \quad (9b)
\end{align*}
\end{equation}

where the $M$’s are $4 \times 4$ known matrices, the $B$’s are known vectors and the unknowns are:

\begin{equation}
X_1 = \begin{pmatrix} A_1 \\ C_1 \\ E_1 \\ 0 \end{pmatrix}; \quad X_n = \begin{pmatrix} A_n \\ C_n \\ E_n \\ G_n \end{pmatrix} \quad (n \geq 2)
\end{equation}

The forms of $X_1$ and of the matrices $M_1, M_1^+$ and $B_1$ in (9a) are special because the series in (7) start at different values of $n$. Relationships between coefficients were then grouped for conciseness. Details [19] are too lengthy to be reproduced here. The matrices $M$ are, perforce, the same as for the linear shear flow [3]. Only the $B$’s are different, viz.:

\begin{equation}
B_1 = \begin{pmatrix} 0 \\ K_0 \\ K_1 \\ 0 \end{pmatrix}; \quad B_n = \begin{pmatrix} 0 \\ 0 \\ K_n \\ 0 \end{pmatrix} \quad (n \geq 2)
\end{equation}

with

\begin{equation}
K_n = \frac{4 \sqrt{2} (2n+1)(2n+1+\frac{2}{\tanh \alpha})}{3 [1 + e(2n+1)\alpha]}. \quad (10)
\end{equation}
The infinite linear system (9) is closed by noting that all coefficients $A_n, \cdots, G_n$ should vanish for $n \to \infty$ in order that the series in (7) converge. Then,

$$X_n \to 0 \text{ for } n \to \infty.$$ \hfill (11)

The infinite linear system (9) is solved by truncating for some large $n$, say $N$, that is $X_{N+1} = 0$. This gives from (9b) the equation for $n = N$:

$$M_N^{-1}X_{N-1} + M_NX_N = B_N.$$ \hfill (12)

A classic way to solve the system $L = \{(9a), (9b)\}$ for $2 \leq n \leq N - 1$, (12) is first to rearrange it as a linear system for the $4N$ unknowns $(A_1, C_1, E_0, A_2, C_2, E_2, \cdots, A_N, C_N, E_N, G_N)$. By construction, the system has a banded matrix. The system is solved by LU decomposition. At the next step, $N$ is increased by unity, the novel linear system is solved again, etc., until the results for $A_1, C_1, E_0, G_2$ converge. It is then a simple matter to derive all coefficients by the recurrence relationships (9). The method amounts to solving $4N \times 4N$ linear systems of increasingly larger sizes, which usually requires a large amount of computer calculation time and memory. Moreover, when $\ell/a - 1$ is moderate, obtaining a number of terms in the expansions is necessary in order to have a sufficient precision.

A faster method for solving an infinite linear system like (9), but for scalars, was proposed by [20] and applied thereafter to solve other hydrodynamic problems involving a sphere and a plane in [21, 22]. One could think of extending that method to matrices. However, this is not possible here, since the matrices $M_1^+$ and $M_n^+$ $(n \geq 2$ in (9) have zero determinant and are thus not invertible.

For these reasons, the following faster algorithm is proposed. The algorithm for inverting the truncated block tridiagonal matrix $L$ is basically an extension of Thomas’ algorithm for solving a tridiagonal system. It proceeds as follows:

- The formal solution of (9a) for $X_1$ is $X_1 = U_1X_2 + V_1$ with:
  $$U_1 = -[\hat{M}_1]^{-1}\hat{M}_1^+; \quad V_1 = [\hat{M}_1]^{-1}B_1.$$ \hfill (13)

- The formal solution of (9b) for $X_n$ when $2 \geq n \geq N - 1$ is $X_n = U_nX_{n+1} + V_n$ with:
  $$U_n = -[M_n^-U_{n-1} + M_n]^{-1}M_n^+; \quad V_n = [M_n^-U_{n-1} + M_n]^{-1}[B_n - M_n^-V_{n-1}].$$ \hfill (14)

- The formal solution of (12) for $X_N$ is $X_N = V_N$ with:
  $$V_N = [M_N^-U_{N-1} + M_N]^{-1}[B_N - M_N^-V_{N-1}].$$ \hfill (15)

It appears that $(U_1, V_1)$ may be directly calculated from (13), then in turn $(U_n, V_n)$ from (14) for $n = 2, \cdots, N - 1$, then $V_N$ from (15). Once $V_N$ is known, $X_N = V_N$, then all other $X_n$ for $n = N - 1, \cdots, 1$ are calculated by proceeding downwards.

This procedure has two main advantages:

- At each step, only one $4 \times 4$ matrix is inverted, namely $[M_n^-U_{n-1} + M_n]$.
- When $N$ is increased by unity, only $U_{N-1}$ and $U_N, V_N$ should be calculated, thus only the supplementary $[M_N^-U_{N-1} + M_N]$ matrix has to be inverted.

Then the new $X_N$ and $X_n$’s can be obtained.

This algorithm was implemented in Mathematica® computer algebra language to calculate asymptotic expansions for $a/\ell \ll 1$ (large distance from wall).


5. Results for a spherical particle in quadratic shear flow

By performing the integrations in (3) and (4), the force, torque and stresslet friction factors for a fixed sphere in a quadratic shear flow are obtained in terms of the coefficients of the series (7) as:

\[
f_{xx}^{q} = -\frac{\sqrt{2}}{6} \sinh^3 \frac{\alpha}{2} \sum_{n \geq 0} \left\{ E_n + n(n+1)C_n \right\}, \tag{16a}
\]

\[
c_{yx}^{q} = \frac{\sqrt{2}}{4} \sinh^4 \frac{\alpha}{2} \sum_{n \geq 0} \left\{ \frac{E_n}{\tanh \alpha} + 2n(n+1)A_n \right\}, \tag{16b}
\]

\[
s_{xx}^{q} = \frac{3\sqrt{2}}{20} \sinh^4 \frac{\alpha}{2} \sum_{n \geq 0} \left\{ \frac{E_n}{\tanh \alpha} - \frac{1}{3} n(n+1)(2n+1)(B_n + C_n) \right\}. \tag{16c}
\]

These friction factors were calculated in terms of the normalised sphere centre to wall gap, \( \ell/a - 1 \) and of the normalised slip length \( \lambda = b/a \). Results are displayed in figures 7, 8 and 9. Values of \( f_{xx}^{q} \) and \( c_{yx}^{q} \) for \( \lambda = 0 \) are in good agreement with those obtained for a sphere in a quadratic flow near a no-slip wall by [22]. As found in [22], values of \( s_{xx}^{q} \) have a small minimum around \( \ell/a - 1 \approx 0.18 \), which is not so apparent at this scale. When \( \lambda \) increases, it is observed in Fig. 7 that \( f_{xx}^{q} \) decays. Fig. 8 shows that \( c_{yx}^{q} \) also decays for increasing \( \lambda \) for a dimensionless gap \( \ell/a - 1 \) smaller than around 0.2 but increases otherwise with \( \lambda \). The minimum at \( \lambda = 0, \ell/a - 1 \approx 0.18 \) also disappears and is replaced by a maximum at some larger value of \( \ell/a - 1 \) for values of \( \lambda \) of order unity. The physical interpretation of these behaviours is not straightforward, since it depends on a complicated flow structure, see e.g. Fig. 3 of [22] for the case \( \lambda = 0 \).

Asymptotic expansions were obtained for large sphere to wall distances \( a/\ell \ll 1 \); for example, for the force friction factor:

\[
f_{xx}^{q} = 1 + \frac{9}{16} \left( \frac{a}{\ell} \right) + \left( \frac{499}{768} - \frac{9\lambda}{16} \right) \left( \frac{a}{\ell} \right)^2 + \left( \frac{9\lambda^2}{8} - \frac{81\lambda}{128} - \frac{1575}{4096} \right) \left( \frac{a}{\ell} \right)^3 \\
+ \left( -\frac{81\lambda^3}{16} + \frac{405\lambda^2}{256} - \frac{3701\lambda}{4096} - \frac{7263}{65536} \right) \left( \frac{a}{\ell} \right)^4 \\
+ \left( \frac{261\lambda^4}{8} - \frac{891\lambda^3}{128} + \frac{19551\lambda^2}{4096} + \frac{4959\lambda}{16384} + \frac{1596155}{3145728} \right) \left( \frac{a}{\ell} \right)^5 \\
+ \left( -\frac{4005\lambda^5}{16} + \frac{5589\lambda^4}{128} + \frac{17507\lambda^3}{512} - \frac{45081\lambda^2}{32768} + \frac{2197069\lambda}{1048576} + \frac{3036175}{16777216} \right) \left( \frac{a}{\ell} \right)^6 \\
+ \ldots \tag{17a}
\]

Up to 14 terms could be calculated in 5 mn on a PC using Mathematica computer algebra software. Similar results for the torque and stresslet are provided in [4]. Such asymptotic expansions are by definition valid provided each term is of lower order than the preceding one, giving the supplementary condition for validity: \( \lambda a/\ell \ll 1 \). Values calculated with these expansions are compared to the accurate results in figures 7, 8 and 9.

6. A freely moving sphere near a slip wall

Based upon the decomposition in §3, earlier results and the results of §4, the case of a freely moving sphere in the ambient flow field \( \mathbf{v} = [k_s(z+b)+k_qz^2] \mathbf{e}_x \) can now be considered. Writing that the net force and torque vanish:

\[
F_x^s + F_y^s + F_x^e + F_y^e = 0, \quad C_x^t + C_y^t + C_x^r + C_y^r = 0 \tag{18}
\]

1 Figures in this section are reprinted with permission from [4], ©2011, American Institute of Physics.
Figure 7. Friction factor for the force on a fixed sphere in a quadratic flow. Solid lines: accurate results. Dotted lines: asymptotic results for \( a/\ell \ll 1 \).

provides the translational velocity \( U = U e_x \) and rotational velocity \( \Omega = \Omega e_y \) of the sphere with:

\[
U = U_s + U_q, \quad \Omega = \Omega_s + \Omega_q, \tag{19}
\]

in which reduced values of \( U_s, \Omega_s \) for the linear shear flow were derived in [3] to be:

\[
\frac{u_s}{k_s(\ell + b)} = \frac{U_s}{k_s(\ell + b)} = \frac{\left(\frac{4}{a}\right)^2 c_{yy} f_{xx}^s + \frac{1}{2} f_{xy} c_{yx}^s}{\left(\frac{4}{a} + \lambda\right) \left( c_{yy} f_{xx}^s - f_{xy} c_{yx}^s \right)}, \tag{20a}
\]

and reduced values of \( U_q, \Omega_q \) for the quadratic shear flow are [4]:

\[
\frac{\omega_s}{\Omega_s} = \frac{\Omega_s}{\Omega_s} = \frac{c_{yx} f_{xx}^q + 2 \left(\frac{4}{a}\right)^2 f_{xx} c_{yx}^q}{c_{yy} f_{xx}^q - f_{xy} c_{yx}^q}. \tag{20b}
\]

Values of the reduced translational velocity are plotted in Fig. 10\(^2\). The velocity increase with \( \lambda \) is due to the reduced drag on the wall when the slip increases. The velocity can be even larger than that far from the wall when \( \lambda \) is large. This can be understood since, roughly speaking,
the particle close to the wall is then submitted to viscous stresses only on one side of its surface. Based upon the series expansions for large $\ell/a$, a Padé approximant was constructed for $u_q$. The results it provides (see dashed lines in Fig. 10) are in good agreement with the accurate results, even down to small gaps of around $\ell/a - 1 \approx 0.2$. Values of the reduced rotational velocity are plotted in Fig. 11. It is seen that this quantity has small variations with $\lambda$.

The $xz$ component of the stresslet for a freely moving sphere in the ambient flow field $\mathbf{v} = [k_s(z + b) + k_q z^2] \mathbf{e}_x$ can also by linearity be expanded as:

$$ S_{xz} = S_{xz}^t + S_{xz}^r + S_{xz}^s + S_{xz}^q. $$

(22)

The normalised stresslet for the freely moving sphere in quadratic flow field is then obtained as [4]:

$$ s_{xz}^q = \frac{S_{xz}^q}{\frac{2}{3} \pi a^3 \ell \mu k_q} = \left( \frac{9}{10} \frac{\ell}{a} + \frac{3}{10} \frac{a}{\ell} \right) s_{xz}^t u_q + \frac{9}{10} s_{xz}^r \omega_q + s_{xz}^q, $$

(23)

with the normalised $u_q, \omega_q$, see Eq. (21), and the reduced stresslets:

$$ s_{xz}^t = \frac{S_{xz}^t}{6 \pi a^3 \mu U}, \quad s_{xz}^r = \frac{S_{xz}^r}{6 \pi a^3 \mu \Omega}, \quad s_{xz}^q = \frac{S_{xz}^q}{\frac{20}{3} \pi a^3 \ell \mu k_q}, $$

(24)

which have the limit of unity when $\ell/a \to \infty$.

The reduced stresslet $s_{xz}^q$ is plotted in Fig. 12. Padé approximants are seen to fit quite well the accurate results, even close the wall. The stresslet is relevant for the effective viscosity of a dilute suspension near a slip wall. A lower stresslet for larger slip suggests that the effective viscosity of a suspension would be smaller for slip walls.

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**Figure 8.** Friction factor for the torque on a fixed sphere in a quadratic flow. Solid lines: accurate results. Dotted lines: asymptotic results for $a/\ell \ll 1$. 

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Figure 9. Friction factor for the stresslet on a fixed sphere in a quadratic flow. Values of the reduced slip length $\lambda$ correspond to curves from bottom to top. Solid lines: accurate results. Dashed lines: asymptotic results for $a/\ell \ll 1$.

7. Aris-Taylor dispersion of suspension near slip wall
The motion of Brownian particles entrained in a shear flow near a slip wall is relevant for various applications, the measurement technique mentioned in §2 and also separation techniques in analytical chemistry.

When Brownian motion combines with entrainment by an ambient shear flow, Aris-Taylor dispersion of particles occurs. The principle is that:

- Particles diffuse by Brownian motion in $z$ direction normal to the wall to a position where they are entrained by the shear flow along the wall.
- The larger their distance to the wall, the larger their velocity along it.
- They are then widely dispersed in the $x$ direction.

To model this phenomenon, we consider a dilute suspension of equal spherical particles. We write an advection-diffusion equation for the concentration $c(x, y, z, t)$ in particles (any type of definition of the concentration can be used; here, we use e.g. the number of particles per unit volume):

$$\frac{\partial c}{\partial t} + \frac{\partial [U_x c]}{\partial x} + \frac{\partial [U_z c]}{\partial z} = \nabla \cdot [D \cdot \nabla c],$$

in which $U_x, U_z$ are the components of the particles at some position $(x, y, z)$ and some time $t$. The motion $U_x$ along $x$ is essentially due to advection by the shear flow, whereas $U_z$ contains the effect of migration due to electrostatic repulsive forces from the wall. $D$ denotes the Brownian
Figure 10. Normalised translational velocity of a freely moving sphere in a quadratic flow near a slip wall. Solid lines: accurate results. Dashed lines: Padé approximants for small $a/\ell$.

Figure 11. Normalised rotational velocity of a freely moving sphere in a quadratic flow near a slip wall. Accurate results.
diffusion tensor:

\[ \mathbf{D} = \begin{pmatrix} D_x & 0 & 0 \\ 0 & D_y & 0 \\ 0 & 0 & D_z \end{pmatrix}. \] (26)

The components depend on the distance to the wall \( z \) and on the slip length \( b \). Note that \( D_y = D_x \) by symmetry.

The Brownian diffusion tensor (26) is diagonal for spherical particles. Indeed, using Einstein’s argument, Brownian motion is equivalent to a thermodynamic force, that is the gradient of the chemical potential (Helmholtz free energy per unit mole). Then, following arguments of [23], a spherical particle close to a wall and submitted to an external force that is either normal or parallel to the wall moves in the same direction as this force. This result comes from symmetry of the figure and linearity of Stokes equations of fluid motion.

More precisely, the diffusion coefficient \( D_x \) for freely moving spheres along a slip wall may be obtained from Einstein’s argument, which is repeated as follows. The thermodynamic force for a dilute suspension is: \( \mathbf{F}^B = -kT \nabla \log c \), where \( k \) is Boltzmann constant and \( T \) the absolute temperature. Brownian motion gives zero torque on a sphere. The balances of forces along \( x \) and torques along \( y \) are:

\[ F^t_x + F^r_x - kT \frac{d}{dx} \log c = 0, \quad C^t_y + C^r_y = 0. \] (27)

Expressing the forces and torques in terms of velocities and friction coefficients, we obtain in particular the translation velocity \( U_x \) along \( x \), and then the flux in the form:

\[ U_x c = -D_x \frac{dc}{dx}. \] (28)

This is Fick’s law for diffusion, with diffusion coefficient in \( x \) direction:

\[ D_x = \frac{c_{yy}^r}{f_{xx}^y f_{yy}^r - f_{yx}^r f_{xy}^y} D_0, \] (29)
where $D_0 = \frac{kT}{6\pi a\mu}$ is Einstein’s diffusion coefficient.

Along $z$ normal to wall, there is no rotational velocity by symmetry, so that the balance of forces is simply:

$$F_z^t - kT \frac{d}{dz} \log c = 0,$$

(30)
giving:

$$U_z c = -D_z \frac{dc}{dx}.$$

(31)

This is again Fick’s law, with diffusion coefficient in $z$ direction:

$$D_z = \frac{1}{f_{zz}} D_0.$$

(32)

The friction coefficient $f_{zz}$ for a sphere translating normally to a slip wall may be obtained from the calculations of [20] (after correcting misprints).

The advection-diffusion equation (25) with diffusion tensor given by (26) (29) (32) provides a model for Aris-Taylor dispersion. We apply it now to the DF-FCS experimental technique presented in §2.

First, in a fluid at rest, there is an equilibrium concentration profile $c_0(z)$ of particles submitted on one hand to repulsive electrostatic forces from the wall and on the other hand to Brownian diffusion. Let:

$$i_0(x, y, z) = i_{\text{norm}} \exp \left[ -2(x^2 + y^2)/r_0^2 - 2(z - H)^2/z_0^2 \right]$$

(33)

be the 3D profile of the intensity of light at the upstream focus. Here, $H$ denotes the distance of the centre of either focus to the wall, see Fig. 2. The concentration of illuminated particles at the upstream focus is: $c(0, x, y, z) = c_0(x, y, z) i_0(x, y, z)$. This is the initial condition. The calculation proceeds by integrating (25) to follow in time this cloud of illuminated particles. Particles move to the downstream focus (1) where they are then illuminated by the intensity:

$$i_1(x, y, z) = i_{\text{norm}} \exp \left[ -2((x^2 - s) + y^2)/r_0^2 - 2(z - H)^2/z_0^2 \right].$$

(34)

We calculate the total intensity of illuminated particles in the fluid domain $z > 0$:

$$I(t) = \int_{z>0} c(t, x, y, z) i_1(x, y, z) \, dx \, dy \, dz.$$  

(35)

The time evolution is like in Fig. 2, bottom curve. $I(t)$ goes through a maximum for some transit time.

The numerical integration of (25) was performed with the Comsol© finite elements software. Since the problem is three-dimensional, the mesh could not be made small enough to capture with sufficient precision the maximum of $I(t)$. For this reason, it was found more efficient to use a Fourier-transform of Eq. (25) along the $y$ direction (along the wall and normal to the flow direction). The definition of the Fourier transform used here is:

$$\hat{f}(\eta) = \int_{-\infty}^{+\infty} \exp(-i\eta y) f(y) dy.$$  

(36)

The transformed Eq. (25) reads:

$$\frac{\partial \hat{c}}{\partial t} + \frac{\partial [U_x \hat{c}]}{\partial x} + \frac{\partial [U_z \hat{c}]}{\partial z} + \frac{\partial}{\partial x} \left( D_x \frac{\partial \hat{c}}{\partial x^2} \right) + \frac{\partial}{\partial z} \left( D_z \frac{\partial \hat{c}}{\partial z} \right) = -\eta^2 D_y \hat{c}.$$  

(37)
The required intensity is then obtained from the convolution theorem (and using symmetry in $y$) as:

$$I(t) = 2 \int_{\eta=0}^{+\infty} \left[ \int_{x=-\infty}^{+\infty} \int_{z>0} \tilde{c}(t, x, \eta, z) \tilde{i}_1(x, \eta, z) dz dx \right] d\eta.$$  

Gauss-Legendre algorithm was used for the integration in $\eta$. Thus, the two-dimensional equation (37) had to be integrated only for a few values of $\eta$. A typical view of the cloud evolution is shown in Fig. 13. The particle cloud is elongated in the $x$ direction due to Aris-Taylor dispersion and the concentration decays due to diffusion. The light intensity at the downstream focus is represented as an ellipsoid in the right-hand-side of the figure. Roughly speaking, the intersection of both spots corresponds to the intensity $I(t)$ that we are considering. A typical variation of $I(t)$ (for [NaCl] = $10^{-4}$ mol/l, $H = 0$) is shown in Fig. 14 (Note that at the time this calculation was done [19, 3], we did not have yet results for hydrodynamic interactions in quadratic shear flow; thus only interactions in a pure shear flow are considered here). The maximum corresponds to the transit time $\tau_M$ between foci and the velocity is taken as the distance between foci (6 $\mu$m) divided by this transit time.

**Figure 13.** View of the calculated cloud evolution due to Aris-Taylor dispersion and of the light intensity at the downstream focus (from [2]).

**Figure 14.** Evolution the calculated light intensity $I(t)$ showing the maximum corresponding to the transit time between foci (from [2]).

By recalculating the cloud evolution for various values of $H$, we then obtain the velocity profile represented in Fig. 15.

The earlier theoretical results in [2] (see Fig. 4, right) were obtained by neglecting the hydrodynamic interactions with the wall. From the velocity profile, we then obtained a large apparent slip length of 740 nm due to Aris-Taylor dispersion.

If we now compare the theoretical results without hydrodynamic interactions in Fig. 4 to those with interactions in Fig. 15, we observe that the decay close to the wall observed in the experimental results of Fig. 4 (right) is captured in the later case. More calculations have to be done, but these results are encouraging for applying the modeling of near wall hydrodynamic interactions to experimental techniques.

8. Conclusion and outlooks

Experiments [1] show typical difficulties involved in measuring a slip length by following particles. To model this problem, we consider a solid spherical particle moving in an ambient parabolic flow field near a slip wall.

The Stokes flow problem can be decomposed as the sum of elementary ones (translation, rotation of a particle in a fluid at rest and applied linear and quadratic shear flows onto a fixed particle), each of which has a solution as series in bispherical coordinates. The coefficients in the series are solution of 4 coupled recurrence relationships forming an infinite linear system. This
system is solved by an extension of Thomas’ algorithm for inverting a tridiagonal matrix. This algorithm has a cost proportional to the number of terms in the series. It is thus possible to obtain a large number of terms, which are necessary to resolve the lubrication situation when a sphere is close to the slip wall. Accurate results for the force and torque on a sphere, velocity of a freely moving sphere, diffusion tensor are then obtained. The stresslet, that is the symmetric moment of surface stresses on the sphere, is also derived in view of applications to suspension rheology. Expansions of various quantities for a large distance from the wall and for a low slip length are performed on the basis of the analytical solutions, using computer algebra. Padé approximants calculated therefrom provide good approximations.

The Aris-Taylor dispersion of Brownian particles in a shear flow near a slip wall gives a large bias in the measurement of slip [2]. It is calculated from the advection-diffusion equation, using the expressions for the particle velocity and diffusion tensor. It is shown here how wall hydrodynamic interactions with a slip wall modify these results. There are various applications of these results in measurement techniques involving particle tracking and in separation techniques in analytical chemistry.

On the theoretical side, the algorithm for solving this type of infinite linear system can be applied to other problems using the bispherical coordinates technique (in particular asymmetric flow fields problems) so as to obtain results for small gap between surfaces.

The stresslet on a freely moving sphere can be used for deriving the effective viscosity of a dilute suspension of spheres near a slip wall in a linear and parabolic shear flow. Effective collective slip in suspension may then combine with local fluid slip.

The coupled influence of two parallel walls may be important for separation techniques. An extension of the approach of [14] to slip walls may be useful for applications.

As for non-spherical particles near a slip wall, the hydrodynamic problem may be solved by the semi-analytical boundary integral technique. A paper was presented in this symposium on this topic [24].

A non-uniform slip length may also be of interest for applications, e.g. to mixing. A texture optimisation was obtained for a pure fluid flowing between two parallel walls in [25, 26, 27]. Performing optimisations for suspensions would be of interest for microfluidics.
and mixing processes.

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