ON IRREDUCIBILITY OF MODULES OF WHITTAKER TYPE FOR CYCLIC ORBIFOLD VERTEX ALGEBRAS

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Abstract. We extend the Dong-Mason theorem on irreducibility of modules for orbifold vertex algebras (cf. [18]) to the category of weak modules. Let \( V \) be a vertex operator algebra, \( g \) an automorphism of order \( p \). Let \( W \) be an irreducible weak \( V \)-module such that \( W, W \circ g, \ldots, W \circ g^{p-1} \) are inequivalent irreducible modules. We prove that \( W \) is an irreducible weak \( V^{(g)} \)-module. This result can be applied on irreducible modules of certain Lie algebra \( L \) such that \( W, W \circ g, \ldots, W \circ g^{p-1} \) are Whittaker modules having different Whittaker functions. We present certain applications in the cases of the Heisenberg and Weyl vertex operator algebras.

1. Introduction

The study of quantum Galois theory for vertex operator algebras has been initiated by C. Dong and G. Mason in [18], and this theory has many applications in the vertex operator algebra theory. Let us discuss one such application. Let \( g \) be an automorphism of finite order, and \( V^{(g)} \) be its subalgebra of fixed points under \( g \). Let \( M \) be a module for \( V \). In [18], the authors discuss the structure of \( M \) as a \( V^{(g)} \)-module. In particular, they prove that if \( M \) is an irreducible ordinary \( V \)-module such that \( M \) is not isomorphic to \( M \circ g^i \), for all \( i \), then \( M \) is also an irreducible module for the subalgebra \( V^{(g)} \). This result is important for the construction of irreducible modules for vertex algebras \( M(1)^+ \) and \( V^+_p \) (cf. [14] [15] [19] [20] [1] [21] [2] [14] [5] [16]). Recently, it has also been applied for the realization of irreducible modules for subalgebras of the triplet vertex algebra \( V(p) \) (cf. [7]). The proof in [18] is based on certain applications of Zhu’s algebra theory.

The question then arises as to whether the statement of Dong–Mason theorem is true for any weak \( V \)-module. It is clear that the original Dong-Mason proof cannot be applied to arbitrary weak modules, since we cannot apply Zhu’s algebra theory. In this paper, we present an extension of Dong-Mason theorem for weak modules of Whittaker type.

Recently, Whittaker modules have been investigated in the framework of vertex operator algebra theory in [8] [25] [39]. The latter two present a Lie theoretic proof of irreducibility for certain Whittaker modules for the fixed point subalgebra of Heisenberg vertex algebra. Further motivation for this work is to give a vertex algebraic proof of these statements, which can be applied more generally.

In this paper, we emphasise a new method for proving the irreducibility of orbifold modules and the role of Whittaker modules in the proof. In order to avoid technical details, we study only non-twisted modules and basic examples of Heisenberg and Weyl vertex algebras. In our forthcoming papers, we shall study twisted modules and more examples of Whittaker modules for \( V \)-algebras.
1.1. Irreducibility of orbifolds. Let $W$ be an irreducible weak $V$-module and $g$ an automorphism of $V$ with order $p$. Let $Y_W(v, z)$ be the vertex operator of $v \in V$ operating on $W$. Recall that $W \circ g$ is defined in [18] to be the space $W$ with the vertex operator given by

$$Y_{W \circ g}(v, z) = Y_W(gv, z), \forall v \in V.$$ 

It is clear that $W \circ g$ is also a $V$-module. The following is our first main result (see Theorem 5.3 for part (1) and Theorem 6.3 for part (2)).

**Main Theorem 1.** Let $W$ be an irreducible weak $V$–module and $g$ an automorphism of finite order.

1. Assume that $W \circ g_i \not\cong W$ for all $i$. Then $W$ is an irreducible $V(g)$–module.
2. Assume that $W \cong W \circ g$. Then $W$ is a direct sum of $p$ irreducible $V(g)$–modules.

Let us explain the main new ideas of our proof. For (1), we construct a graded module

$$\mathcal{M} = \bigoplus_{i=0}^{p-1} W \circ g^i = \bigoplus_{i=0}^{p-1} \Delta^{p,i}(W),$$

compatible with the action of the automorphism $g$, such that each component is isomorphic to $W$ as $V(g)$–module. Then we take any non-trivial submodule $S$ of $W$ and identify it with a submodule of $\Delta^{p,0}(W)$. It is then sufficient to prove the following claim:

1.1. For each $w \neq 0$, a vector of the form $(w, \ldots, w) \in \mathcal{M}$ is cyclic in $\mathcal{M}$.

The advantages of our approach are the fact that we do not need Zhu’s algebra and the fact that this approach can be applied for non-weight modules. In Lemma 3.3, we prove relation (1.1) for arbitrary weak module by using the Lie algebra $\mathfrak{g}(V)$ associated to $V$ and its universal enveloping associative algebra. It turns out that (1.1) is just a consequence of a similar statement for associative algebras (cf. Lemma 3.1).

For proof of the part (2) (cf. Theorem 6.3), we slightly modify the methods of [18] and [22] by applying a general version of Schur’s Lemma on the action of the group $G = \mathbb{Z}_p$ on $W$.

1.2. Role of the Whittaker modules in the paper. Although Theorem 11 holds for arbitrary weak $V$–modules, it is not easy to construct examples of modules satisfying the conditions of the theorem. It turns out that these conditions can be checked for a large class of Whittaker modules for certain infinite-dimensional Lie algebras. We use concepts of Whittaker categories which appear in the paper [11] (see also [35]). Since any weak module for a vertex algebra is automatically a module for an infinite-dimensional Lie algebra, such an approach gives a framework for studying many examples. We just need to assume that each module $W \circ g^i$ belongs to a different Whittaker block. This means that each module $W \circ g^i$ has a different Whittaker function. The following is our second main result (see Theorem 7.8) which gives most new applications of our construction.

**Main Theorem 2.** Let $W$ be an irreducible weak $V$–module such that all $W_i = W \circ g^i$ are Whittaker modules whose Whittaker functions $\lambda^{(i)} = n \to \mathbb{C}$ are mutually distinct. Then $W$ is an irreducible weak $V^{(g)}$–module.
1.3. **Examples.** We construct a family of Whittaker modules for Heisenberg and Weyl vertex algebras, and apply our new result to prove irreducibility of orbifold subalgebras. In particular, we show that in these cases, standard (= universal) Whittaker modules are irreducible.

In the case of Heisenberg vertex algebra, we use the new method and present an alternative proof of the $\mathbb{Z}_2$–orbifolds of Heisenberg vertex algebra [28].

In the case of Weyl vertex algebra $M$, we construct a family of Whittaker modules $M_1(\lambda, \mu)$ where $(\lambda, \mu) \in \mathbb{C}^n \times \mathbb{C}^n$. We prove:

**Main Theorem 3 (see Theorem 9.3).** Assume that $\Lambda = (\lambda, \mu) \neq 0$. Then $M_1(\lambda, \mu)$ is an irreducible weak module for the orbifold subalgebra $M^{Z_p}$, for each $p \geq 1$.

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2. **Preliminaries**

**Definition 2.1.** A vertex operator algebra $(V, Y, 1, \omega)$ is a $\mathbb{Z}$-graded vector space $V = \bigoplus_{n \in \mathbb{Z}} V(n)$ such that $\text{wt}(v) = n$ for $v \in V(n)$,

$$\dim V(n) < \infty, \text{ for } n \in \mathbb{Z},$$

and $V(n) = 0$ for $n$ sufficiently small, equipped with a linear map $V \otimes V \to V[[z, z^{-1}]]$, or equivalently,

$$V \to (\text{End}V)[[z, z^{-1}]]$$

$$v \mapsto Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1} \text{ (where } v_n \in \text{End}V),$$

$Y(v, z)$ denoting the vertex operator associated with $v$, and equipped also with two distinguished homogenous vectors $1 \in V(0)$ (the vacuum) and $\omega \in V(2)$. The following conditions are assumed for $u, v \in V$:

- $u_n v = 0$ for $n$ sufficiently large (the lower truncation condition),
- $Y(1, z) = Id$,
- $Y(v, z)1 \in V[[z]]$ and $\lim_{z \to 0} Y(v, z)1 = v$ (creation property),

and the Jacobi identity holds

$$z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y(u, z_1)Y(v, z_2) - z_0^{-1} \delta \left( \frac{z_2 - z_1}{z_0} \right) Y(v, z_2)Y(u, z_1)$$

$$= z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) Y(Y(u, z_0)v, z_2).$$
Also, the Virasoro algebra relations hold (acting on $V$):
\[
[L(m), L(n)] = (m - n)L(m + n) + \frac{1}{12}(m^3 - m)\delta_{n+m,0}\text{rk } V\mathbf{1},
\]
for $m, n \in \mathbb{Z}$, where
\[
L(n) = \omega_{n+1} \text{ for } n \in \mathbb{Z} \text{ i.e., } Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n)z^{-n-2}
\]
and
\[
\text{rk } V \in \mathbb{C},
\]
\[
L(0)v = nv = (\omega v)v \text{ for } n \in \mathbb{Z} \text{ and } v \in V(n),
\]
\[
\frac{d}{dz} Y(v, z) = Y(L(-1)v, z).
\]

We say that $g \in \text{Aut}_\mathbb{C}(V)$ is an automorphism of a vertex operator algebra $V$ if
- $g(a_n b) = g(a)_n g(b)$ for all $a, b \in V$, $n \in \mathbb{Z}$.
- $g(\omega) = \omega$.

For any group $G$ of automorphisms of $V$, we have the orbifold vertex algebra
\[
V^G = \{ v \in V \mid g(v) = v, \ g \in G \},
\]
which is a vertex subalgebra of $V$. If $G = \langle g \rangle$ is cyclic, we write $V^{(g)}$ for $V^G$.

We shall now recall the notions of weak modules and ordinary modules for $V$.

**Definition 2.2.** A weak $V$–module is a pair $(W, Y_W)$ where $W$ is a complex vector space, and $Y_W(\cdot, z)$ is a linear map
\[
Y_W : \ V \to \text{End}(W)[[z, z^{-1}]],
\]
\[
a \mapsto Y_W(a, z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1},
\]
which satisfies the following conditions for $a, b \in V$ and $v \in W$:
- $a_n v = 0$ for $n$ sufficiently large.
- $Y_W(1, z) = I_W$.
- The following Jacobi identity holds:
\[
z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)Y_W(a, z_1)Y_W(b, z_2) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{z_0}\right)Y_W(b, z_2)Y_W(a, z_1)
\]
\[
= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)Y_W(Y(a, z_0)b, z_2).
\]

Let $L(z) = Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n)z^{-n-2}$. Note that every weak $V$–module is a module for the Virasoro algebra generated by $L(n)$, $n \in \mathbb{Z}$.

**Definition 2.3.** A weak $V$–module $(W, Y_W)$ is called an ordinary $V$–module if the following conditions hold:
- The $L(-1)$-derivative property: for any $a \in V$,
\[
Y_W(L(-1)a, z) = \frac{d}{dz} Y_W(a, z).
\]
- The grading property:
\[
W = \oplus_{\alpha \in \mathbb{C}} W(\alpha), \quad W(\alpha) = \{ v \in W \mid L(0)v = \alpha v \}.
\]
such that for every $\alpha$, $\dim W(\alpha) < \infty$ and $W(\alpha + n) = 0$ for sufficiently negative $n \in \mathbb{Z}$.

The following result was proved in [18].

**Theorem 2.4.** [18, Theorem 6.1] Assume that $(W, Y_W)$ is an irreducible ordinary module for the vertex operator algebra $V$. Assume that $g$ is an automorphism of $V$ of prime order $p$ such that $W \circ g \notin W$. Then $W$ is an irreducible module for the orbifold subalgebra $V^g$.

The goal of this paper is to extend this result for irreducible weak modules for vertex operator algebras.

3. On cyclic vectors in a direct sum of irreducible weak modules

In this section, we prove one basic, but important technical result on cyclic vectors in a direct sum of non-isomorphic weak modules for a vertex operator algebra. It turns out that the result can be proved much more easily in the context of associative algebras.

First we include the following result for associative algebras:

**Lemma 3.1.** Let $A$ be an associative algebra with unity. Assume that $L_i$, $i = 1, \ldots, t$, are non-isomorphic irreducible $A$-modules and $\mathcal{L} = \bigoplus_{i=1}^t L_i$. Then for each $w_i \neq 0$, $w_i \in L_i$, a vector of the form $(w_1, w_2, \ldots, w_t)$ is cyclic in $L$.

**Proof.** Let $U = A.(w_1, w_2, \ldots, w_t)$ be the $A$-module generated by $(w_1, w_2, \ldots, w_t)$. Let $J_i = \text{Ann}(w_i) = \{a \in A \mid a.w_i = 0\}$ for $1 \leq i \leq t$. Then $J_i$ is a left ideal in $A$ and $A/J_i \cong L_i$. Since $L_i$'s are irreducible $A$-modules which are mutually non-isomorphic, we conclude:

- ideals $J_i$, $i = 1, \ldots, t$, are maximal left ideals,
- $J_i \neq J_j$ for $i \neq j$,
- $A/\bigcap_{i=1}^t J_i \cong \mathcal{L}$.

Note that in the last conclusion we use the fact that $J_i$'s are maximal left ideals of $A$ and apply Chinese Remainder Theorem. This implies that there is an element

$$u_i \in \bigcap_{1 \leq j \leq t, j \neq i} J_j, \ u_i \notin J_i.$$

Then one can construct the vector

$$u_i(w_1, \ldots, w_i, \ldots, w_t) = (0, \ldots, 0, u_iw_i, 0, \ldots, 0),$$

which belongs to $L_i$, so $L_i \subset U$ for all $i$. Therefore $\mathcal{L} \subset U$, which implies that $\mathcal{L} = U$. \hfill \Box

We want to show the analogous result for weak modules for a vertex operator algebra $V$. For this purpose, we use the Lie algebra $\mathfrak{g}(V)$ associated to the vertex operator algebra $V$ (cf. [12], [17]).

The Lie algebra $\mathfrak{g}(V)$ is realized on the vector space

$$\mathfrak{g}(V) = \frac{V \otimes \mathbb{C}[t, t^{-1}]}{(L(-1) \otimes 1 + 1 \otimes \frac{d}{dt}).V \otimes \mathbb{C}[t, t^{-1}]}.$$
where the commutator is given by
\[ [a \otimes t^n, b \otimes t^m] = \sum_{i=0}^{\infty} \binom{n}{i} (a_i b) \otimes t^{n+m-i}. \]

Then by [17, Lemma 5.1] we have:

**Lemma 3.2.** Let \( V \) be a vertex operator algebra. We have:

- Every weak \( V \)-module \( W \) is a \( g(V) \)-module with the action 
  \[ v \otimes t^n \mapsto v_n \quad (v \in V, n \in \mathbb{Z}). \]

- If \( W \) is an irreducible weak \( V \)-module, then \( W \) is also an irreducible \( g(V) \)-module.

**Lemma 3.3.** Assume that \( L_i, i = 1, \ldots, t \), are non-isomorphic irreducible weak \( V \)-modules and \( L = \bigoplus_{i=1}^{t} L_i \). Then for each \( w_i \neq 0 \), \( w_i \in L_i \), a vector of the form \((w_1, w_2, \ldots, w_t)\) is cyclic in \( L \).

**Proof.** By Lemma 3.2, we have that \( L_i, i = 1, \ldots, t \) are irreducible modules for the associative algebra \( A = U(g(V)) \). Then the assertion follows by applying Lemma 3.1. \( \square \)

4. Main result: order 2 case

We shall first consider the case of automorphisms of order two.

Let \( \theta \) be an order two automorphism of \( V \). Let
\[ V^+ = \{ v \in V \mid \theta(v) = v \}, \quad V^- = \{ v \in V \mid \theta(v) = -v \}. \]
Then \( V^+ \) is a vertex subalgebra of \( V \) and \( V^- \) is a \( V^+ \)-module.

**Theorem 4.1.** Let \( V \) be a vertex operator algebra and \( W \) be an irreducible weak \( V \)-module such that \( W_\theta \neq W \otimes \theta \neq W \). Then \( W \) is an irreducible weak \( V^+ \)-module.

**Proof.** Consider a \( V \)-module \( M = W \oplus W_\theta \). Define now the map
\[ \Delta^\pm : W \to M, \quad w \mapsto (w, \pm w). \]
Let
\[ \Delta^\pm(W) = \{(w, \pm w) \mid w \in W\}. \]
Then we have
\[ M = \Delta^+(W) \bigoplus \Delta^-(W). \]
Moreover, \( \Delta^\pm \) are \( V^+ \)-homomorphisms. Next we notice that
\[
V^+.\Delta^+(W) \\
= \text{Span}_\mathbb{C} \{(v_n w, \theta(v)_n w) \mid v \in V^+, w \in W\} \\
= \text{Span}_\mathbb{C} \{(v_n w, v_n w) \mid v \in V^+, w \in W\} \\
= \Delta^+(W)
\]
and
\[ V^- \Delta^+ (W) = \text{Span}_\mathbb{C} \left\{ (v_n w, \theta (v_n w), v \in V^-, w \in W) \right\} = \text{Span}_\mathbb{C} \left\{ ((v_n w, -v_n w), v \in V^-, w \in W) \right\} = \Delta^- (W) \]

Assume that \( W \) is not irreducible \( V^+ \)-module. Then there is a \( V^+ \)-submodule \( 0 \neq S \subseteq W \). In particular, \( 0 \neq \Delta^+(S) \subseteq \Delta^+(W) \). But Lemma 3.3 implies that \( V.\Delta^+(S) = M \). Since \( V.\Delta^+ \) \( \Delta^+(S) \subset \Delta^+(W) \), we must have that \( V.\Delta^+(S) = \Delta^+(W) \) which is a contradiction. The proof follows. \( \Box \)

5. General case

Assume that \( g \) is an automorphism of arbitrary (not necessarily prime) order \( p \). Then
\[
V = V^0 \oplus V^1 \oplus \cdots \oplus V^{p-1}
\]
where \( V^i = \{ v \in V \mid g v = \zeta^i v \} \) and \( \zeta \) is a primitive \( p \)-th root of unity.

Let \( W \) be a weak \( V \)-module. Let
\[
\mathcal{M} = W_0 \oplus W_1 \oplus \cdots \oplus W_{p-1},
\]
where \( W_i = W \circ g^i, i = 0, 1, \ldots, p-1 \). Let \( \Delta^{(p,i)} \) be the \( V^0 \)-homomorphism defined by
\[
w \mapsto (w, (\zeta^i) w, \cdots, (\zeta^{p-1}) w).
\]

Lemma 5.1. We have:

1. \( \mathcal{M} = \bigoplus_{i=0}^{p-1} \Delta^{(p,i)} (W) \).
2. \( V^j, \Delta^{(p,i)} (W) \subset \Delta^{(p,i+j)} (W) \).

Proof. The proof of (1) is easy. Let us prove (2).
Take \( v \in V^j \). For \( w' \in W \), we have
\[
v_n w' = \zeta^{rj} v_n w',
\]
which implies
\[
v_n (w, (\zeta^i) w, \cdots, (\zeta^{p-1}) w) = (v_n w, \zeta^{i+j} v_n w, \cdots, (\zeta^{i+p-1}) v_n w) \in \Delta^{(p,i+j)} (W).
\]
The Lemma holds. \( \Box \)

Lemma 5.2. Let \( W \) be an irreducible weak \( V \)-module and \( \mathcal{M} \) be as above. Assume that for every \( w \in W, w \neq 0, \)
\[
(w, \cdots, w) \text{ is cyclic in } \mathcal{M}
\]
Then \( W \) is an irreducible weak \( V^0 \)-module.
Proof. Assume that $W$ is not a simple $V^0$–module. Then there is a $V^0$–submodule $0 \neq S \subseteq W$. In particular,

$$0 \neq \Delta^{(p,0)}(S) \subseteq \Delta^{(p,0)}(W).$$

Since each $(w, \cdots, w)$, with $w \neq 0$, is a cyclic vector in $\mathcal{M}$, we get

$$\mathcal{M} = V.\Delta^{(p,0)}(S).$$

This implies that $V^0.\Delta^{(p,0)}(S) = \Delta^{(p,0)}(W)$. A contradiction. The proof follows. □

Lemmas 3.3 and 5.2 imply our main result.

**Theorem 5.3.** Let $W$ be an irreducible weak $V$–module, and $g$ an automorphism of finite order such that $W \circ g^i \not\cong W$ for all $i$. Then $W$ is an irreducible weak $V^0$–module.

6. ON COMPLETE REDUCIBILITY OF CERTAIN $V^{(g)}$–MODULES

In this section we shall use the following very general version of Schur’s Lemma. Proof can be found in [27, Section 4.1.2].

**Lemma 6.1.** Assume that $W_1$ and $W_2$ are irreducible modules for an associative algebra $A$. Assume that $W_1$ and $W_2$ have countable dimensions over $\mathbb{C}$. Then $\dim \text{Hom}_A(W_1, W_2) \leq 1$ and $\dim \text{Hom}_A(W_1, W_2) = 1$ if and only if $W_1 \cong W_2$.

**Lemma 6.2.** Assume that $V$ is a vertex operator algebra. Then every irreducible weak $V$–module $W$ is countably dimensional.

Proof. Note that the vertex operator algebra $V$ is countably dimensional. Take any $w \in W$. Then by [31, Proposition 6.1] (see also [18, Proposition 4.1]),

$$W = V.w = \text{Span}_{\mathbb{C}}\{u_n w \mid u \in V, n \in \mathbb{Z}\},$$

which implies that $W$ is also countably dimensional. □

Assume that $g$ is an automorphism of arbitrary order $p$. Then we have the decomposition (6.1).

**Theorem 6.3.** Assume that $g$ is an automorphism of $V$ of finite order $p$ as above. Assume that $W$ is an irreducible weak $V$–module such that $W \circ g \cong W$. Then $W$ is completely reducible weak $V^0$–module such that

1. $W = \bigoplus_{i=0}^p W^i$, $V^j.W^i \subset W^{i+j \mod(p)}$, where $W^j$, $j = 1, \ldots, p$ are eigenspaces of certain linear isomorphism $\Phi(g) : W \to W$.
2. Each $W^i$ is an irreducible weak $V^0$–module.
3. The modules $W^i$, $i = 0, \ldots, p-1$, are non-isomorphic as weak $V^0$–modules.

Proof. By Lemma 6.2, $W$ is countably dimensional. Let $f : W \to W \circ g$ be a $V$–isomorphism. Then we have

$$f(u_n w) = (gu)_{n} f(w) \quad \forall u \in V, \ w \in W.$$

Applying $f$ $p$-times, we get

$$f^p(u_n w) = (g^p u)_{n} f^p(w) = u_n f^p(w).$$
Therefore \( f^p \) is a \( V \)-endomorphism. Applying Schur’s Lemma for the associative algebra \( A = U(\mathfrak{g}(V)) \), we get that \( f^p = a\text{Id}_W \), where \( a \) is a non-zero constant. By rescaling \( f \) one gets an isomorphism \( \Phi(g) : W \to W \circ g \) such that \( \Phi(g)^p = \text{Id} \). Next we consider \( \Phi(g) \) as a linear operator on \( W \) with the property \( \Phi(g)^p = \text{Id}_W \).

That means \( W \) is a \((g)\)-module and there is \( 0 \leq j \leq p - 1 \) and a vector \( 0 \neq w^j \in W \) such that \( \Phi(g)(w^j) = \zeta^j w^j \). Clearly, for any \( x \in V^i, w^j = \text{Span}_C\{v_nw^j | v^i \in V^j, n \in \mathbb{Z} \} \) and \( 0 \leq i \leq p - 1 \), we have \( \Phi(g)(x) = \zeta^{i+j}x \). Without loss, we may assume there is a \( 0 \neq w \in W \) such that \( \Phi(g)(w) = w \).

Now define \( W^j = V^j.w = \text{Span}_C\{v_nw^j | v^i \in V^j, n \in \mathbb{Z} \} \). Then

- \( \Phi(g)(w^j) = \zeta^j w^j \) for each \( w^j \in W^j \).
- \( \Phi(g)(w^j) = \zeta^j g(u)_n w^j = \zeta^{i+j} u_n w^j \) for \( u \in V^i, w^j \in W^j \).

This implies that \( W = \bigoplus_{i=0}^p W^i, V^i.W^j \subset W^{i+j} \text{ mod}(p) \) and (1) holds.

Assertion (2) follows from (1).

Let \( 0 \neq U \neq W^j \) be a proper \( V^0 \)-submodule of \( W^j \) and consider the \( V \)-submodule \( X = V.U \). Then

\[
X = V^0.U + V^1.U + \ldots + V^{p-1}.U
\]

Since \( U \) is a proper \( V^0 \)-submodule of \( W^j \), then \( V^i.U \subset W^{i+j} \) implies that \( X \) is a proper \( V \)-submodule of \( W \). This is impossible since \( W \) is a simple \( V \)-module. Hence \( W^j \) is irreducible \( V^0 \)-module for each \( j \).

Proof of assertion (3) is completely analogous to that of [18, Theorem 5.1] and it uses Lemma 6.1. For completeness, we shall include it.

Suppose we have a \( V^0 \)-isomorphism \( p : V^i \to W^j, i \neq j \). Take a nonzero \( w \in V^i \) and consider the following \( V \)-submodule \( U \) of \( W \oplus W \)

\[
U = V.(w, p(w)) = \text{Span}_C\{(v_n w, v_n p(w)) | v \in V\}.
\]

Then \( W^i \oplus W^i \) is not in \( U \) and hence \( U \) is a proper submodule of \( W \oplus W \). Since \( W \oplus W \) is a \( (g(V)) \)-module of finite length, the Jordan H"older theorem can be applied. Comparing filtrations

\[
(0) \to W \to W \oplus W, \quad (0) \to U \to W \oplus W,
\]

and using simplicity of \( W \), we get that \( U \cong W \) as \( (g(V)) \)-modules. This implies that \( U \cong W \) as \( V \)-modules.

Then both projection maps from \( U \to W \oplus (0) \) and \( U \to (0) \oplus W \) are \( V \)-isomorphisms. Hence the map

\[
\Phi : u_n w \mapsto u_n p(w), \quad (u \in V)
\]

is also an isomorphism. Using Schur’s lemma we get \( \Phi = a\text{Id} \) for \( a \in \mathbb{C} \), which implies that \( p(w) = aw \in W^i \). This implies \( i = j \). A contradiction

\[\Box\]

7. Whittaker modules: some structural results

First we recall some basic notions from [11].
Definition 7.1. For a Lie algebra \( n \), define ideals \( n_0 := n \) and \( n_i := [n_{i-1}, n], i > 0 \). Then we have a sequence of ideals 
\[
\mathfrak{n} = n_0 \supset n_1 \supset n_2 \supset \cdots .
\]
We say that \( n \) is quasi-nilpotent if \( \cap_{i=0}^{\infty} n_i = 0 \). Obviously, any nilpotent Lie algebra is quasi-nilpotent.

Definition 7.2. Let \( \mathfrak{g} \) be a nonzero complex Lie algebra and let \( n \) be a subalgebra of \( \mathfrak{g} \). If \( M \) is a \( \mathfrak{g} \)–module, then we say that the action of \( n \) on \( M \) is locally finite provided that \( U(n)v \) is finite dimensional for all \( v \in M \). Let \( \text{Wh}(\mathfrak{g}, n) \) denote the full subcategory of the category \( \mathfrak{g} \)--Mod of all \( \mathfrak{g} \)--modules, which consists of all \( \mathfrak{g} \)--modules, the action of \( n \) on which is locally finite.

Let \( V \) be a vertex algebra. Assume that the Lie algebra \( \mathfrak{L} \) is one of the following two types:

1. \( \mathfrak{L} = \mathfrak{g}(V) \), or
2. \( \mathfrak{L} \) is the Lie algebra of modes of generating fields of the vertex algebra \( V \).

Remark 1. In many cases it is possible to replace \( \mathfrak{g}(V) \) with much smaller Lie algebra. For example, this happens in the following cases:

- If \( V \) is the universal affine vertex algebra \( V_k(\mathfrak{g}) \), then we can take \( \mathfrak{L} = \hat{\mathfrak{g}} \), where \( \hat{\mathfrak{g}} \) is the affine Lie algebra associated to the simple Lie algebra \( \mathfrak{g} \) (cf. [8], [6] for studying Whittaker modules in this case).
- If \( V \) is the Heisenberg vertex algebra \( M(1) \), we can take \( \mathfrak{L} = \hat{\mathfrak{h}} \) (cf. Section 8 below).
- If \( V \) is the Weyl vertex algebra, we can take Lie algebra \( \mathfrak{L} \) such that the Weyl algebra \( \hat{\mathfrak{A}} = U(\mathfrak{L}) \) (cf. Section 9 below).

Note that by our assumptions on the vertex algebra, every weak \( V \)--module is a module for the Lie algebra \( \mathfrak{L} \). We also assume the following:

- Let \( n \) be a nilpotent subalgebra of \( \mathfrak{L} \).
- Let \( \text{Wh}(\mathfrak{L}, n) \) denote the full category of \( \mathfrak{L} \)--modules \( W \) such that \( n \) acts locally finitely on \( W \) (cf. [11]).

Definition 7.3. Let \( W \in \text{Wh}(\mathfrak{L}, n) \). A vector \( v \in W \) is called a Whittaker vector provided that \( \langle v \rangle \) is an \( n \)--submodule of \( W \).

Let \( \lambda : n \to \mathbb{C} \) be a Lie algebra homomorphism which will be called a Whittaker function. Let \( U_\lambda = \mathbb{C} u_\lambda \) be the 1--dimensional \( n \)--module such that 
\[
 xu_\lambda = \lambda(x)u_\lambda \quad (x \in n).
\]
Consider the standard (universal) Whittaker \( \mathfrak{L} \)--module 
\[
 M_\lambda = U(\mathfrak{L}) \otimes_{U(n)} U_\lambda.
\]

Definition 7.4. We say that an irreducible \( V \)--module \( W \) is of Whittaker type \( \lambda \) if \( W \) is an irreducible quotient of the standard Whittaker module \( M_\lambda \).

Lemma 7.5 (cf. [11]). Assume that \( W \) is an irreducible \( V \)--module of Whittaker type \( \lambda \). Then 
\[
 W = \{ w \in W \mid \forall x \in n, \quad (x - \lambda(x))^kw = 0 \text{ for } k \gg 0 \}. 
\]
Proof. Let us first prove that
\[ M_\lambda = \{ w \in M_\lambda \mid \forall x \in \mathfrak{n}, (x - \lambda(x))^k w = 0 \text{ for } k > 0 \}. \]
Take an arbitrary element \( w_1 \in U(\mathfrak{g}) \). Since \( \mathfrak{n} \) is a nilpotent subalgebra of \( \mathfrak{g} \), for \( x \in \mathfrak{n}, \) we have
\[ \text{ad}_x^k(w_1) = 0 \quad \text{for } k > 0. \]
Next we notice that
\[ (x - \lambda(x))w_1 \otimes u_\lambda = [x, w_1] \otimes u_\lambda \]
which implies that
\[ (x - \lambda(x))^k w_1 \otimes u_\lambda = \text{ad}_x^k(w_1) \otimes u_\lambda = 0 \quad \text{for } k > 0. \]
The claim now follows from the fact that \( W \) is a quotient of the universal Whittaker module \( M_\lambda \).

Lemma 7.6. Assume that \( \lambda, \mu : \mathfrak{n} \to \mathbb{C} \) are Whittaker functions such that \( \lambda \neq \mu \). Assume that \( W_\lambda \) and \( W_\mu \) are irreducible Whittaker modules of types \( \lambda \) and \( \mu \) respectively. Then
1. \( W_\lambda \) and \( W_\mu \) are inequivalent as \( \mathcal{V} \)-modules.
2. Let \( (w_1, w_2) \in W_\lambda \oplus W_\mu, w_1 \neq 0, w_2 \neq 0 \). Then
\[ V_\lambda(w_1, w_2) = W_\lambda \oplus W_\mu. \]

Proof. (1) Assume that \( f : W_\lambda \to W_\mu \) is an isomorphism. Then \( f(w_\lambda) \) is a non-trivial Whittaker vector in \( W_\mu \) such that
\[ (x - \lambda(x))f(w_\lambda) = 0, \quad \forall x \in \mathfrak{n}. \]
Take \( x \in \mathfrak{n} \) such that \( \lambda(x) \neq \mu(x) \). The for every \( k > 0 \) we have
\[ (x - \mu(x))^k f(w_\lambda) = (x - \lambda(x) + \lambda(x) - \mu(x))^k f(w_\lambda) = (\lambda(x) - \mu(x))^k f(w_\lambda) \neq 0. \]
This contradicts with Lemma 7.5. So (1) holds.

Now let us prove (2).

Claim: There exist \( k > 0 \) and \( x \in \mathfrak{n} \) such that
\[ (x - \lambda(x))^k w_1 = 0 \quad \text{and} \quad (x - \lambda(x))^k w_2 = z' \neq 0. \]
Indeed, take \( x \in \mathfrak{n} \) such that \( \mu(x) - \lambda(x) = A \neq 0 \). Let \( k_1, k_2 \) be the smallest positive integer such that
\[ (x - \lambda(x))^{k_1} w_1 = 0 \quad \text{and} \quad (x - \mu(x))^{k_2} w_2 = 0. \]
Let \( k = \max\{k_1, k_2\} \). We have
\[
(x - \lambda(x))^k w_2 \\
= (x - \mu(x) + A)^k w_2 \\
= \sum_{p=0}^{k-1} \binom{k}{p} (x - \mu(x))^p A^{k-p} w_2 \\
= \sum_{p=0}^{k_2-1} \binom{k}{p} (x - \mu(x))^p A^{k-p} w_2 \\
= A^k w_2 + k A^{k-1} (x - \mu(x)) w_2 + \cdots + \binom{k_{k_2-1}}{k_2-1} A^{k-k_2+1} (x - \mu(x))^{k_2-1} w_2.
\]
Note that \( w_2, (x - \mu(x)) w_2, \ldots, (x - \mu(x))^{k_2-1} w_2 \) are linearly independent. Otherwise, there exist \( p_0, p_1, \ldots, p_{k_2-1} \) such that
\[
p_0 w_2 + p_1 (x - \mu(x)) w_2 + \cdots + p_{k_2-1} (x - \mu(x))^{k_2-1} w_2 = 0.
\]
Let \( I = \{ i = 0, 1, \ldots, k_2 - 1 | p_i \neq 0 \} \) and \( s = \max I \). Then
\[
(x - \mu(x))^s w_2 = \sum_{i=0}^{s-1} p_i (x - \mu(x))^i w_2.
\]
If \( (x - \mu(x))^{k_2-s} w_2, (x - \mu(x))^{k_2-s+1} w_2, \ldots, (x - \mu(x))^{k_2-1} w_2 \) are linearly independent, then \( p_i = 0, i = 0, 1, \ldots, s - 1 \). By (7.1), we obtain \( (x - \mu(x))^s w_2 = 0 \) where \( s < k_2 \), which is a contradiction. So \( (x - \mu(x))^{k_2-s} w_2, \ldots, (x - \mu(x))^{k_2-1} w_2 \) (at most \( k_2 - 1 \) terms) are linearly dependent. Repeating similar argument, we can prove that there exists \( q < k_2 \) such that \( (x - \mu(x))^q w_2 = 0 \), which is a contradiction. Thus we proved \( w_2, (x - \mu(x)) w_2, \ldots, (x - \mu(x))^{k_2-1} w_2 \) are linearly independent and hence
\[
A^k w_2 + k A^{k-1} (x - \mu(x)) w_2 + \cdots + (k_{k_2-1}) A^{k-k_2+1} (x - \mu(x))^{k_2-1} w_2 \neq 0.
\]
Now we have
\[
(x - \lambda(x))^s(w_1, w_2) = (0, z'), \quad z' \in W_\mu, \ z' \neq 0.
\]
Irreducibility of \( W_\mu \) now gives that \( W_\mu \subset V(w_1, w_2) \). Similarly we prove that \( W_\lambda \subset V(w_1, w_2) \). So (2) holds.

\textbf{Remark 2.} The assertion (2) is a consequence of (1) and Lemmas \ref{lem:1} and \ref{lem:3}. So we could omit its proof. But since Whittaker modules are the main new examples on which we can apply Theorem \ref{thm:3}, we think that it is important to keep an independent proof which contains explicit construction of elements in maximal left ideals \( J_i = \text{Ann}(w_i), i = 1, 2 \). In particular, we have elements \( (x - \lambda(x))^k \in J_1 \setminus J_2 \) which correspond to element \( u_i \) (for \( i = 2 \)) constructed in Lemma \ref{lem:1} by using abstract arguments.

We can easily generalize the previous lemma:

\textbf{Lemma 7.7.} Assume that \( \lambda_1, \cdots, \lambda_n : n \to \mathbb{C} \) are Whittaker functions such that \( \lambda_i \neq \lambda_j \) for \( i \neq j \). Assume that \( W_{\lambda_i}, i = 1, \ldots, n \) are irreducible Whittaker modules of types \( \lambda_i \). Then
\begin{enumerate}
\item All \( W_{\lambda_i} \) are inequivalent as \( V \)-modules.
\item Let \( w = (w_1, w_2, \cdots, w_n) \in W_{\lambda_1} \oplus \cdots \oplus W_{\lambda_n}, \) where \( 0 \neq w_i \in W_{\lambda_i} \). Then
\[
V.w = W_{\lambda_1} \oplus \cdots \oplus W_{\lambda_n}.
\]
\end{enumerate}

\textbf{Theorem 7.8.} Assume that \( W \) is a \( V \)-module in the Whittaker category \( \mathcal{W} h(L, n) \) as before. Assume also that \( W_i = W \circ g^i \) has the Whittaker function \( \lambda^{(i)} = n \to \mathbb{C} \) and that all \( \lambda^{(i)} \) are distinct. Then \( W \) is an irreducible \( V^0 \)-module.

\textbf{Proof.} The proof follows immediately from Lemma \ref{lem:7} and Theorem \ref{thm:5}. \qed
8. Example: Heisenberg vertex algebra

First we recall the definition of the Heisenberg Lie algebra $\hat{\mathfrak{h}}$. Let $\mathfrak{h}$ be complex $\ell$-dimensional vector spaces with respect to the non-degenerate bilinear form $(\cdot, \cdot)$. Fix an orthonormal basis $\{h_1, h_2, \ldots, h_{\ell}\}$ with respect to form $(\cdot, \cdot)$. The Heisenberg Lie algebra $\hat{\mathfrak{h}}$ is defined as

$$\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C} [t, t^{-1}] \oplus \mathbb{C} K$$

with the commutator relations

$$[K, \hat{\mathfrak{h}}] = 0, \quad \text{and} \quad [a(m), b(n)] = m\delta_{m+n,0}(a, b)K$$

for $a, b \in \mathfrak{h}$, $m, n \in \mathbb{Z}$ and $a(n) = a \otimes t^n$. We identify $\mathfrak{h}$ with its dual space $\mathfrak{h}^*$ by the form $(\cdot, \cdot)$. Let $\mathbb{C} e^0$ be the one-dimensional module over the Lie algebra $\mathfrak{h}$ with action given by

$$h(n)e^0 = 0, \quad \forall h \in \mathfrak{h}, \quad n \geq 0; \quad Ke^0 = e^0.$$

Define the vector space $M(1)$ by

$$(8.1) \quad M(1) = U(\hat{\mathfrak{h}}) \otimes_{U(\mathfrak{h} \otimes \mathbb{C}[t] \oplus \mathbb{C} K)} \mathbb{C} e^0.$$

On $M(1)$ define the state-field correspondence by

$$(8.2) \quad Y(a^{(1)}(n_1) \cdots a^{(r)}(n_r)e^0, z) = a^{(1)}(z)_{n_1} \cdots a^{(r)}(z)_{n_r} \text{Id}_{M(1)}$$

for $a^{(i)} \in \mathfrak{h}$ and $n_i \in \mathbb{Z}$. The vacuum vector is $1 = e^0$ and the conformal vector is given by

$$\omega = \frac{1}{2} \sum_{i=1}^{\ell} h_i(-1)^2 1.$$

In particular,

$$Y(\omega, z) = L(z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}, \quad L(n) = \frac{1}{2} \sum_{m \in \mathbb{Z}} \L h_i(-m) h_i(m+n)^2.$$

Then $(M(1), Y, 1, \omega)$ is a vertex operator algebra (see [33] for details). Now consider the order two automorphism $\theta$ of the vector space $M(1)$ given by

$$\theta(h_i(-n_1)h_i(-n_2)\cdots h_i(-n_k)1) = (-1)^k h_i(-n_1)h_i(-n_2)\cdots h_i(-n_k)1$$

for $i_j \in \{1, 2, \ldots, \ell\}$ for all $j$ and $n_1 \geq n_2 \geq \cdots \geq n_k > 0$. Let $M(1)^\perp$ be the corresponding subspace of fixed-points with respect to $\theta$:

$$(8.3) \quad M(1)^\perp = \{ v \in M(1) \mid \theta(v) = v \}.$$

Then $M(1)^\perp$ is a vertex operator algebra and its structure and representation are well studied (cf. [19] [11] [21] [5]).

Let $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_r, 0, \cdots)$ be a sequence of elements of $\mathfrak{h}$ with at least one nonzero entry, and $\lambda_n = 0$ for $n \gg 0$. Let $\mathbb{C} e^\lambda$ be the one-dimensional module over the Lie algebra $\mathfrak{h} \otimes \mathbb{C}[t] \oplus \mathbb{C} K$ with action given by

$$h(n)e^\lambda = (h, \lambda_n)e^\lambda, \quad h \in \mathfrak{h}, \quad n \geq 0; \quad Ke^\lambda = e^\lambda.$$

Consider the corresponding induced $U(\hat{\mathfrak{h}})$-module

$$M(1, \lambda) = U(\hat{\mathfrak{h}}) \otimes_{U(\mathfrak{h} \otimes \mathbb{C}[t] \oplus \mathbb{C} K)} \mathbb{C} e^\lambda.$$
Then we see that $C(U x ∈ d_{\text{detailed example of orbifold } M})$ the action of $SL_n$ algebra of $M$ were presented in a recent paper by A. Milas, M. Penn and H. Shao [34].

$M_{\text{Theorem 7.8, Proposition 8.1.}}$

(1) Assume that $f or a particularly interesting subgroup of $O_t$ from (1).

Proof. The proof of assertion (1) follows easily by using Theorem 5.2. Since for a 2-cycle $\sigma$, we have

$$\lambda \circ \sigma \neq \lambda, \forall \sigma \iff \forall i, j, 1 < i, j \leq \ell, \lambda_i \neq \lambda_j$$

$$\iff (\lambda^{h(1)}, \ldots, \lambda^{h(\ell)}) \neq (\lambda^1, \ldots, \lambda^\ell), \forall h \in S_\ell.$$

Therefore for any $g \in S_\ell$ we have $\lambda \circ g \neq \lambda$. Now assertion (2) follows directly from (1).
We also have the following conjecture.

**Conjecture 8.2.** Assume that \( \lambda \circ \sigma \neq \lambda \) for any 2–cycle \( \sigma \in S_2 \). Then \( M(1, \lambda) \) is an irreducible \( M(1)^{S_2} \)-module.

The proof of the conjecture requires certain extension of methods used in the paper. We plan to study the proof of this conjecture in our forthcoming papers.

9. **Example: Weyl vertex algebra**

The Weyl algebra \( \hat{A} \) is the associative algebra with generators \( a(n), a^*(n), n \in \mathbb{Z} \) and relations
\[
[a(n), a^*(m)] = \delta_{n+m,0}, \quad [a(n), a(m)] = [a^*(m), a^*(n)] = 0, \quad n, m \in \mathbb{Z}.
\]

Let \( M \) be the simple Weyl module generated by the cyclic vector \( 1 \) such that
\[
a(n)1 = a^*(n+1)1 = 0 \quad (n \geq 0).
\]
As a vector space,
\[
M \cong \mathbb{C}[a(-n), a^*(-m) \mid n > 0, m \geq 0].
\]

There is a unique vertex algebra \((M, Y, 1)\) (cf. [23, 24, 29]) where the vertex operator is given by
\[
Y : M \to \text{End}(M)[[z, z^{-1}]]
\]
such that
\[
Y(a(-1)1, z) = a(z), \quad Y(a^*(0)1, z) = a^*(z),
\]
\[
a(z) = \sum_{n \in \mathbb{Z}} a(n)z^{-n-1}, \quad a^*(z) = \sum_{n \in \mathbb{Z}} a^*(n)z^{-n}.
\]

We choose the following conformal vector of central charge \( c = -1 \) (cf. [29]):
\[
\omega = \frac{1}{2}(a(-1)a^*(-1) - a(-2)a^*(0))1.
\]

Then \((M, Y, 1, \omega)\) has the structure of a \( \frac{1}{2}\mathbb{Z}_{\geq 0} \)-graded vertex operator algebra. We can define weak and ordinary modules for \((M, Y, 1, \omega)\) as in the case of \( \mathbb{Z} \)-graded vertex operator algebras.

We define the Whittaker module for \( \hat{A} \) to be the quotient
\[
M_1(\lambda, \mu) = \hat{A}/I,
\]
where \( \lambda = (\lambda_0, \ldots, \lambda_n), \mu = (\mu_1, \ldots, \mu_n) \) and \( I \) is the left ideal
\[
I = \langle a(0)-\lambda_0, \ldots, a(n)-\lambda_n, a^*(1)-\mu_1, \ldots, a^*(n)-\mu_n, a(n+1), \ldots, a^*(n+1), \ldots \rangle.
\]

**Proposition 9.1.** We have:

(1) \( M_1(\lambda, \mu) \) is an irreducible \( \hat{A} \)-module.

(2) \( M_1(\lambda, \mu) \) is an irreducible weak module for the Weyl vertex operator algebra \( M \).

**Proof.** It is straightforward to check that the ideal \( I \) defined above is a maximal left ideal in \( \hat{A} \) (see also [13, 20]) and therefore the quotient \( M_1(\lambda, \mu) = \hat{A}/I \) is an simple \( \hat{A} \)-module. Since by construction, \( M_1(\lambda, \mu) \) is a restricted \( \hat{A} \)-module, it is an irreducible \( M \)-module. \( \square \)
Let \( w = 1 + 1 \in M_1(\lambda, \mu) \). Then \( w \) is a cyclic vector and
\[
a(0)w = \lambda_0 w, \ldots, a(n)w = \lambda_n w, a^*(1)w = \mu_1 w, \ldots, a^*(n)w = \mu_n w
\]
and \( a^*(k)w = a(k)w = 0 \) for \( k > n \).

Now we want to identify \( M_1(\lambda, \mu) \) as a Whittaker module for certain Whittaker pair. Let \( \mathfrak{L} \) be the Lie algebra with generators \( a(n), a^*(n), K, n \in \mathbb{Z} \) such that \( K \) is central and
\[
[a(n), a^*(m)] = \delta_{n+m,0} K, \quad [a(n), a(m)] = [a^*(m), a^*(n)] = 0, \quad n, m \in \mathbb{Z}.
\]
Then \( M_1(\lambda, \mu) \) is an irreducible \( \mathfrak{L} \)-module of level 1 (i.e., \( K \) acts as the multiplication with 1).

Let \( \mathfrak{n} \) be the subalgebra of \( \mathfrak{L} \) generated by \( a(n), a^*(n+1) \) for \( n \geq 0 \). Then \( \mathfrak{n} \) is commutative, and therefore a nilpotent subalgebra of \( \mathfrak{L} \).

Define the Whittaker function \( \Lambda = (\lambda, \mu) : \mathfrak{n} \to \mathbb{C} \) by
\[
\Lambda(a(0)) = \lambda_0, \ldots, \Lambda(a(n)) = \lambda_n, \Lambda(a(k)) = 0 \quad (k > n),
\]
\[
\Lambda(a^*(1)) = \mu_1, \ldots, \Lambda(a^*(n)) = \mu_n, \Lambda(a^*(k)) = 0 \quad (k > n).
\]

**Proposition 9.2.** \( M_1(\lambda, \mu) \) is a standard Whittaker module of level 1 for the Whittaker pair \( (\mathfrak{L}, \mathfrak{n}) \) with Whittaker function \( \Lambda = (\lambda, \mu) \).

Let \( \zeta_p = e^{2\pi i/p} \) be \( p \)-th root of unity. Let \( g_p \) be the automorphism of the vertex operator algebra \( M \) which is uniquely determined by the following automorphism of the Weyl algebra \( \mathcal{A} \):
\[
a(n) \mapsto \zeta_p a(n), \quad a^*(n) \mapsto \zeta_p^{-1} a^*(n) \quad (n \in \mathbb{Z}).
\]
Then \( g_p \) is the automorphism of \( M \) of order \( p \).

**Theorem 9.3.** Assume that \( \Lambda = (\lambda, \mu) \neq 0 \). Then \( M_1(\lambda, \mu) \) is an irreducible weak module for the orbifold subalgebra \( M_{\mathcal{O}_p} = M^{(g_p)} \) for each \( p \geq 1 \).

**Proof.** First we notice that \( M_1(\lambda, \mu) \circ g^i = M_1(\zeta_p^i \lambda, \zeta_p^{-i} \mu) \) and therefore modules \( M_1(\lambda, \mu) \circ g^i \) have different Whittaker functions for \( i = 0, \ldots, p-1 \). Now assertion follows from Theorem 7.8. \( \square \)

9.1. **An application to affine VOA.** Let \( \mathfrak{g} \) be a simple Lie algebra and let \( V^k(\mathfrak{g}) \) be its universal affine vertex algebra of level \( k \). Let \( V_k(\mathfrak{g}) \) be its simple quotient. The following result is well-known:

**Lemma 9.4.** If \( W \) is an irreducible weak \( V_k(\mathfrak{g}) \)-module, then \( M \) is an irreducible module for the affine Lie algebra \( \mathfrak{g} \)-module of level \( k \).

Next we show how the Theorem 9.3 gives a construction of new irreducible modules for affine Lie algebra \( \widehat{sl}(2) \) associated to \( sl(2) \). In the case \( p = 2 \), \( \mathbb{Z}_2 \)-orbifold \( M^{\mathcal{O}_2} \) is isomorphic to a simple affine VOA \( V_{-1/2}(sl(2)) \) (cf. \cite{spin} and also \cite{spin} Section 6) associated to affine Lie algebra \( \widehat{sl}(2) \) at level \(-1/2\). The previous theorem gives a realization of large family irreducible modules for VOA \( V_{-1/2}(sl(2)) \).

**Corollary 9.5.** Assume that \( \Lambda = (\lambda, \mu) \neq 0 \). Then \( M_1(\lambda, \mu) \) is an irreducible module for the affine Lie algebra \( \widehat{sl}(2) \) at the level \( k = -\frac{1}{2} \).
Proof. For $p=2$, $M^{Z_2}$ is isomorphic to the affine VOA $V_{-\frac{1}{2}}(sl(2))$. Therefore, module $M_1(\lambda, \mu)$ is irreducible for $V_{-\frac{1}{2}}(sl(2))$. Now Lemma 9.4 implies that $M_1(\lambda, \mu)$ is an irreducible module for affine Lie algebra $\hat{sl}(2)$. □

Remark 3. The irreducible weight modules for the Weyl vertex algebra were analysed in [10]. One can easily show that weight modules, denoted by $\tilde{U}(\lambda)$, have the property $U(\lambda) \circ g_p \cong U(\lambda)$. Then Theorem 6.3 implies that they are direct sum of two irreducible relaxed weight modules for the affine vertex algebra $V_{-\frac{1}{2}}(sl(2))$ (see also [4], [30]).

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