Invariants of Lepton Mass Matrices and
CP and T Violation in Neutrino Oscillations

C. Jarlskog

Division of Mathematical Physics
LTH, Lund University
Box 118, S-22100 Lund, Sweden

Abstract

CP and T asymmetries in neutrino oscillations, in vacuum as well as in matter, are expressed in terms of invariant functions of lepton mass matrices.

1 Introduction

Four decades have passed since the unexpected discovery [1] of CP violation, originally seen in two decay modes of $K_L$. For about 25 years the superweak ansats by Wolfenstein [2] accounted for all observed effects, including CP violation in the semileptonic decay modes of $K_L$, and provided a simple and intuitive explanation for why CP violation was not seen elsewhere in particle physics. Nowadays the standard framework for understanding CP violation is the electroweak model [3] (hereafter referred to as the standard model) and the Kobayashi-Maskawa scheme [4] within it. Indeed the remarkable experimental determination of the quantity $\epsilon'/\epsilon$ in $K_L$ decays and the discovery of CP violation in the decays of B-mesons, at SLAC and KEK are in agreement with the predictions of the standard model with three families. Nonetheless, understanding CP violation still remains a great challenge, within a broad area in physics. One faces deep questions such as what is the source of CP violation that goes into generating the baryon asymmetry of the universe and why is the theta parameter of QCD so small that it has not been seen. There are also some perhaps simpler questions, one being: is there CP violation in the leptonic sector? The minimal standard model gives a no as the answer to the latter question because it assumes that the neutrinos are massless. But nowadays there is evidence for neutrino oscillations, a phenomenon that requires massive neutrinos [5]. Therefore the minimal standard model needs to be modified, but we don’t know how. In this article, I shall address the issue of CP violation in neutrino oscillations using the method of invariant functions of mass matrices ([6],[7]). In this article, first a short introduction to this method is presented in the next section followed by application to neutrino oscillations, in vacuum as well as in matter, in the following sections.

2 Invariant functions of mass matrices

Consider first the quark sector of the standard model with three families. The identity of the quarks is encoded in the three-by-three quark mass matrices $M_u$ and $M_d$, for the up-type and down-type quarks respectively. However, these mass matrices are basis dependent. Given any pair $M_u$, $M_d$ one can obtain other pairs through unitary rotations, as will be described below,
without affecting the physics. The measurable quantities must be basis independent and therefore they are "invariant functions" under such rotations. These functions were introduced in \[6\] and studied in detail in \[7\]. Actually, what enters, in the standard model, is the pair

\[
S_u \equiv M_u M_u^\dagger, \quad S_d \equiv M_d M_d^\dagger
\] (1)

For simplicity, we shall refer to these quantities as mass matrices. This should cause no confusion because the underlying mass matrices, \(M_u\) and \(M_d\), do not enter in what follows.

An invariant function \(f(S_u, S_d)\) is a function that does not change under the transformation

\[
S_u \rightarrow XS_uX^\dagger, \quad S_d \rightarrow XS_dX^\dagger
\] (2)

where \(X\) is an arbitrary three-by-three unitary matrix. Evidently, the traces of powers of the above quantities, \(tr(S_u^k S_d^l)\), are such invariant functions. Note that the corresponding determinants are not independent invariant functions because any determinant can be expressed as a function of traces. For a detailed discussion of these invariants see \[7\].

When dealing with CP violation in the standard model with three families, a central role is played by the commutator of the quark mass matrices, \([S_u, S_d]\) (see (\[6\], \[8\])) The determinant of this commutator is an invariant function of mass matrices given by (\[8\], \[6\])

\[
det [S_u, S_d] = 2iJ v(S_u)v(S_d)
\] (3)

where \(J\) is the CP-invariant of the quark mixing matrix \(V\),

\[
Im(V_{\alpha j}V_{\beta k}V_{\gamma j}^* V_{\delta k}^*) = J \sum_{\gamma, \delta} \epsilon_{\alpha \beta \gamma \epsilon_{ijkl}}
\] (4)

\(J\) is equal \[9\] to twice the area of any of the six by now well-known unitarity triangles. The quantities \(v(S_u)\) and \(v(S_d)\) are Vandermonde determinants as follows. Denoting the three eigenvalues of \(S_u\) by \(x_i\), \(x_1 = m_u^2, \ x_2 = m_c^2, \ x_3 = m_t^2\), we have

\[
v(S_u) = \sum_{i,j,k} \epsilon_{ijk} x_j x_k^2 = (m_u^2 - m_c^2)(m_c^2 - m_t^2)(m_t^2 - m_u^2)
\] (5)

and similarly \(v(S_d) = (m_d^2 - m_s^2)(m_s^2 - m_b^2)(m_b^2 - m_d^2)\).

In the standard model with three families, the nonvanishing of the above determinant gives the if and only if condition for CP violation in the quark sector. In fact it manifestly unifies the 14 conditions needed, such as the condition that no two quarks with the same charge are allowed to be degenerate if CP is to be violated, or the conditions that none of the mixing angles nor the phase angle is allowed to assume its maximal or minimal value. In other words, the commutator automatically keeps track of these requirements and is thus useful for checking whether a specific model violates CP or not (for a pedagogical discussion see \[10\]).

The determinant in Eq. (3) appears naturally in computations involving CP violation when all the six quarks enter on equal footing. Examples are the renormalization of the \(\theta\)-parameter of QCD by the electroweak interactions and the calculation of the baryon asymmetry of the universe.
in the standard model. The determinant, in a truncated form, enters in many more computations such as when computing the electric dipole moment of a quark, say the down quark. Since in such a computation, the down quark appears in the external legs, it is tacitly assumed that we know the identity of this quark, i.e., \( m_d \neq m_s \) and \( m_d \neq m_b \). Therefore the factors \((m_d^2 - m_s^2)\) and \((m_b^2 - m_d^2)\) will be missing but all the other factors will be present.

The above commutator is the simplest in a family of commutators of functions of mass matrices

\[ C(f, g) \equiv [f(S_u), g(S_d)] \quad (6) \]

\( f \) and \( g \) being functions that are diagonalised with the same unitary matrices that diagonalise \( S_u \) and \( S_d \) respectively. The determinants of these commutators, which are also invariant functions, are given by

\[ \det [f, g] = 2iJ \cdot v(f) v(g) \quad (7) \]

where \( v(f) \equiv v(f(S_u)) \), and \( v(g) \equiv v(g(S_d)) \). More explicitly

\[ v(f) = \sum_{i,j,k} \epsilon_{ijk} f_j f_k^2 = (f_1 - f_2)(f_2 - f_3)(f_3 - f_1) \quad (8) \]

The \( f_j \) denote the three eigenvalues of the matrix \( f(S_u) \) and the quantities related to the down-type quarks are defined similarly.

An essential point is that Eq.(7) holds irrespectively of whether \( f \) and \( g \) are hermitian or not. This property makes the above formalism applicable to neutrino oscillations, as we shall see here below.

### 3 Neutrino oscillations in vacuum

With massive neutrinos, CP violation in neutrino oscillations could manifest itself by the difference of the rates of reactions \( \nu_\alpha \to \nu_\beta \) and \( \bar{\nu}_\alpha \to \bar{\nu}_\beta \). Here \( \alpha \) and \( \beta \) (\( \alpha \neq \beta \)) stand for \( e, \mu, \tau \). The difference of these rates is found \([11]\) to be proportional to

\[ \sin(2\phi_{12}) + \sin(2\phi_{23}) + \sin(2\phi_{31}) = -4\sin(\phi_{12})\sin(\phi_{23})\sin(\phi_{31}) \quad (9) \]

\[ \phi_{jk} = (m_j^2 - m_k^2)\xi, \quad \xi = \frac{L}{4E} \quad (10) \]

\( m \)'s being the neutrino masses; \( E \) denotes the neutrino energy and \( L \) is the distance from the source (for a recent review see \([12]\)). Because of CPT invariance, the above combination of \( \sin \)'s also appears when testing time-reversal asymmetry by comparing the rates of \( \nu_\alpha \to \nu_\beta \) and \( \nu_\beta \to \nu_\alpha \).

As was noted before \([7]\), in models with quark-lepton universality the results for the quark sector, presented in the previous section, can be extended to the leptonic sector by trivial substitutions, \( M_u \to M_\nu \) and \( M_d \to M_l \) where \( M_\nu \) and \( M_l \) denote the neutrino and the charged lepton mass matrices. An interesting question is then: how is the sum (or product) of the above three \( \sin \)'s related to the commutator of lepton mass matrices? One would expect that there be a relationship because, as far as neutrino oscillations are concerned, the leptonic sector is essentially
a copy of the quark sector. An essential point is that the possible Majorana nature of neutrinos is
known not to be important in the computation of the rates (see [13], [14]). The relevant formulæ
are obtained by assuming that there are three active neutrinos and that their ”effective” mass
matrix is three by three. This pattern emerges in many models, some based on the see-saw mechan-
ism (for a recent review see [15]) as well as in models à la Weinberg [16] where the effective
operator that generates the neutrino masses involves only the three active left-handed neutrinos.

To answer the question concerning the relationship between the above sum or product of the
three sin’s and the lepton mass matrices, we introduce, in analogy with the case of the quarks,

$$S_\nu \equiv M_\nu M_\nu^\dagger, \quad S_l \equiv M_l M_l^\dagger$$  \hspace{1cm} (11)

where $M_\nu$ and $M_l$ are the three-by-three neutrino and charged lepton mass matrices respectively.

We introduce the commutators

$$\Delta^\pm \equiv \left[ e^{\pm 2i\xi S_\nu}, \ S_l \right]$$  \hspace{1cm} (12)

where the unitary matrices $U^\pm \equiv e^{\pm 2i\xi S_\nu}$ are inverses of one another and $\Delta^+ = (\Delta^-)^\dagger$. The
determinant of these commutators are invariant functions of lepton mass matrices. Using Eq.(7)
we have

$$\text{det} \Delta^\pm = 2i J_\nu v(S_l) v(e^{\pm 2i\xi S_\nu})$$  \hspace{1cm} (13)

Here $J_\nu$ is the leptonic analogue of the CP invariant of the quark mixing matrix. Here it is more
convenient to use the index $\nu$ instead of ”lep” (for leptons) because when dealing with oscillations
in matter, in the next section, the notation is easily generalised. $J'_\nu$ and $J'_{\bar{\nu}}$ will then denote the
the corresponding quantities for neutrino and antineutrino oscillations in matter.

Furthermore, just as in the case of the quarks, $J_\nu$ is simply twice the area of any of the six
leptonic unitarity triangles. The two Vandermonde determinants in the above equation are given by

$$v(S_l) = (m_e^2 - m_\mu^2)(m_\mu^2 - m_\tau^2)(m_\tau^2 - m_e^2)$$  \hspace{1cm} (14)

and

$$v(e^{\pm 2i\xi S_\nu}) = \mp 8ie^{\pm 2i\xi tr S_\nu} \sin(\phi_{12}) \sin(\phi_{23}) \sin(\phi_{31})$$  \hspace{1cm} (15)

where $tr S_\nu = (m_1^2 + m_2^2 + m_3^2)$ and the $m$’s are again the neutrino masses. The exponential factors
are the determinants of $U^\pm$. Putting these results together, we find

$$\text{det} \Delta^\pm = \pm 16 v(S_l) [J_\nu \sin(\phi_{12}) \sin(\phi_{23}) \sin(\phi_{31})] e^{\pm 2i\xi tr S_\nu}$$  \hspace{1cm} (16)

Note that the phase of the determinant is determined by the sum of the neutrino masses. Evidently
any of the relations $\text{det} \Delta^\pm \neq 0$ provides the necessary and sufficient condition for having CP
violation in reactions $\nu_\alpha \to \nu_\beta$ and $\bar{\nu}_\alpha \to \bar{\nu}_\beta$. The presence of the factors involving the masses
of charged leptons is essential. These keep track of the identity of neutrinos. For example, for
$m_e = m_\mu$ the electron and muon neutrinos would be indistinguishable and therefore there would
be no CP violation. $\text{det} \Delta^\pm \neq 0$ also provides the if and only if condition for T-violation when
comparing the reactions $\nu_\alpha \to \nu_\beta$ and $\nu_\beta \to \nu_\alpha$. 

Neutrino oscillations in matter

At a first glance, the formalism for CP and time-reversal violation in neutrino oscillations in matter looks deceptively similar to that in vacuum. It would seem that all one needs to do is to distinguish the masses and mixings in matter by simply putting say primes on corresponding quantities in vacuum, as we shall do here below, and by introducing explicit indices $\nu$ and $\bar{\nu}$ to keep track of whether we are dealing with neutrino or antineutrino propagation in matter. However, there are subtle points to be taken into account, as we shall see soon.

Consider first the case of neutrinos. To the leading order, in a frame where the charged lepton mass matrix is diagonal, the neutrino mass matrix, $S_\nu = M_\nu M_\nu^\dagger$, is replaced by an effective mass matrix in matter, $S'_\nu = M'_\nu M'^\dagger_\nu$, given by (17), (18)

$$S'_\nu = S_\nu + \Pi, \quad \Pi \equiv diag(\rho, 0, 0)$$

where $\rho$ is proportional to the neutrino momentum as well as the density of the electrons in matter (for a review see, for example, [19]). Here, we shall treat $\rho$ as a constant. The reality is generally far more complicated. But here we are primarily interested in exploring the structure of matter oscillations rather than making realistic calculations. Thus we construct the matter analogue of any of the two vacuum commutators and take its determinant. We take $\Delta^+$, drop the superscript and define

$$\Delta'_\nu \equiv \left[ e^{2i\xi S'_\nu}, S_\nu \right]$$

From Eq.(13) we have

$$det\Delta'_\nu = 16 \left[ J'_\nu, v(S_\nu) \sin(\phi'_{12}) \sin(\phi'_{23}) \sin(\phi'_{31}) \right] e^{2i\xi trS'_\nu}$$

Here the primed quantities are the matter analogues of the unprimed ones in the previous section and, as before, the phase of this determinant is fixed by the sum of the neutrino masses, now taken in matter. Thus $trS'_\nu = (m'^2_1 + m'^2_2 + m'^2_3)$; $m'^i$'s being the neutrino masses in matter, etc. The subscript $\nu$ is a reminder that we are dealing with neutrinos propagating in matter. Evidently $J'_\nu \rightarrow J_\nu$ as $\rho \rightarrow 0$.

Actually, this determinant is not what enters when testing CP violation in neutrino oscillations in matter through the rates discussed above. It will, however, enter if we were to do the science-fictional experiment of comparing the rates of $\nu_\alpha \rightarrow \nu_\beta$ in matter with that of $\bar{\nu}_\alpha \rightarrow \bar{\nu}_\beta$ in the corresponding antimatter. It is ironic that the very same baryon asymmetry of the universe, that owes its existence to CP violation, forbids us to test CP violation in matter in the above ”straight-forward” fashion.

Turning now to antineutrino oscillations in matter, the effective mass matrix, $S'_\bar{\nu} = M'^\dagger_\bar{\nu} M'^\nu_\bar{\nu}$, to the leading order is given by

$$S'_\bar{\nu} = S_\bar{\nu} - \Pi = S_\bar{\nu} - diag(\rho, 0, 0)$$

Because of the different sign of the added term for neutrinos and antineutrinos, the effective neutrino mass of the $j$-th neutrino, $m'_j$, is in general not equal to that of the corresponding antineutrino, $\bar{m}'_j$. Nor are the mixing angles and thus the corresponding $J$'s in general the same.
This is the reason why we need to introduce not only primes to indicate the presence of matter but also appropriate $\nu$ and $\bar{\nu}$ indices to keep track of whether we are talking about neutrinos or antineutrinos. Defining the commutator relevant to antineutrinos, we have

$$det\Delta'_{\bar{\nu}} = 16v(S_l) \left[J'_\nu sin(\bar{\phi}_{12})sin(\bar{\phi}_{23})sin(\bar{\phi}_{31})\right] e^{2i\xi trS'_\nu}$$  \hspace{1cm} (21)$$

Depending on the density profile of matter, the above expressions could, however, be relevant when testing time-reversal violation. The quantity $det\Delta'_{\nu}$ would enter when comparing the rates of $\nu_\alpha \rightarrow \nu_\beta$ and $\nu_\beta \rightarrow \nu_\alpha$ in matter and $det\Delta'_{\bar{\nu}}$ for corresponding antineutrinos processes again in matter.

Since the matter contribution to the mass matrices, $\Pi$, commutes with the lepton mass matrix $S_l$ (which is diagonal), we have

$$[S_\nu, S_l] = [S'_\nu, S_l] = [S_{\bar{\nu}}, S_l]$$  \hspace{1cm} (22)$$

Taking the determinants of these matrices and removing the common factor involving the charged leptons, gives

$$J_{\nu} v(S_\nu) = J'_{\nu} v(S'_\nu) = J_{\bar{\nu}} v(S_{\bar{\nu}})$$  \hspace{1cm} (23)$$

This result can be found in Refs. [20] and [21]. Again $v(S_\nu) = (m^2_1 - m^2_2)(m^2_2 - m^2_3)(m^2_3 - m^2_1)$, the $m_j$’s being the neutrino masses in vacuum; $v(S'_\nu)$ and $v(S'_{\bar{\nu}})$ are the corresponding expressions with neutrino and antineutrino masses in matter respectively. A more detailed discussion of some further matter effects is given in Refs. [22] and [23].

We shall now consider the next simplest commutator in the series, i.e., $[S'^2_\nu, S_l]$. The determinant of this commutator can be written down immediately, in terms of the matter quantities, using the general formula Eq.(7). We have

$$det \left[ S'^2_\nu, S_l \right] = 2iJ_{\nu} v(S_l)v(S'^2_\nu)$$  \hspace{1cm} (24)$$

where

$$v(S'^2_\nu) = w(S'_\nu)v(S'_\nu)$$  \hspace{1cm} (25)$$

$$w(S'_\nu) = (m'^2_1 + m'^2_2)(m'^2_2 + m'^2_3)(m'^2_3 + m'^2_1) = \frac{1}{3}\{ (trS'_\nu)^3 - trS'_\nu \}$$  \hspace{1cm} (26)$$

The determinant in Eq.(24)= can also be computed directly, in terms of vacuum quantities, by substituting $S'_\nu = S_\nu + \Pi$. A straight-forward calculation gives

$$det \left[ S'^2_\nu, S_l \right] = 2iJ_{\nu} v(S_l)v(S'_\nu) \left[w(S_\nu) + \rho I_1 + \rho^2 I_2 \right]$$  \hspace{1cm} (27)$$

where

$$I_1 = (trS_\nu)^2 - \sum_\alpha |V_{a1}|^2 m^4_\alpha$$

$$I_2 = trS_\nu - \sum_\alpha |V_{a1}|^2 m^2_\alpha$$
Here all the masses and mixings refer to vacuum quantities. Note that the coefficient of $\rho^3$ vanishes because the $\Pi [S^n, S_l] \Pi = 0$. Equating the two expressions for the determinant and using that $J'_\nu v(S'_{\nu}) = J_\nu v(S_{\nu})$ gives

$$w(S'_{\nu}) = w(S_{\nu}) + \rho I_1 + \rho^2 I_2$$

Rewriting this relation in terms of traces gives

$$(tr S'_{\nu})^3 - tr S^3_{\nu} = (tr S_{\nu})^3 - tr S^3_{\nu} + 3\rho(tr S_{\nu})^2 - 3tr(S^2_{\nu}\Pi) + 3\rho^2 tr S_{\nu} - 3tr(S_{\nu}\Pi^2)$$

This relation can easily be checked by computing its left-hand side, in terms of vacuum quantities, using again $S'_{\nu} = S_{\nu} + \Pi$.

For antineutrinos the corresponding results are easily obtained by flipping the sign of $\Pi$ and therefore also that of $\rho$.

5 Oscillations in matter with low density

The results obtained in the previous section hold to all orders in the matter-related parameter $\rho$. Here we would like to examine how the results look like in the low density limit, i.e., when terms of order $\rho^2$ and higher can be neglected. We shall consider the case of neutrinos, extension to antineutrinos being trivial. Beginning with the matter commutator in Eq.(18) we write

$$det \Delta'_{\nu} = det \Delta_{\nu} + \rho R + O(\rho^2)$$

where the first term in the RHS is the vacuum contribution. To compute $R$ we expand the exponential in Eq.(18) using Eq.(17) and find

$$\Delta'_{\nu} = \Delta_{\nu} + \sum_{n=1}^{\infty} \frac{(2i\xi)^n}{n!} \sum_{k=0}^{n-1} [S'^{n-1-k}_{\nu} \Pi S^k_{\nu}, S_l] + O(\rho^2)$$

Taking the determinant of the RHS we find

$$R = -v(S_l) \sum_{\alpha,\beta,\gamma,\sigma} E(\alpha, \beta, \gamma, \sigma) F(\alpha, \beta, \gamma, \sigma)$$

where

$$E(\alpha, \beta, \gamma, \sigma) = \sum_{rst} \epsilon_{rst} (\alpha 1)^*(\alpha r) (\beta r)^*(\beta s) (\gamma s)^*(\gamma t) (\sigma t)^*(\sigma 1)$$

$$F(\alpha, \beta, \gamma, \sigma) = \left( \frac{\epsilon^2 m^2 \alpha - \epsilon^2 m^2 \sigma}{m^2 \alpha - m^2 \sigma} \right) e^{2i\xi (m^2 \beta + m^2 \gamma)}$$

Here, for simplicity, we have introduced the short-hand notation $(ar) \equiv V_{ar}$, $V$ being the vacuum lepton mixing matrix. Note that all the dummy indices in the above sums run from one to three. Note that $E^*(\alpha, \beta, \gamma, \sigma) = -E(\sigma, \gamma, \beta, \alpha)$ and since $F$ is symmetric under $\beta \leftrightarrow \gamma$ as well as under
\[ \alpha \leftrightarrow \sigma \] we only need the corresponding symmetric part of the function \( E \). This quantity is found to be

\[
E(\alpha, \beta, \gamma, \sigma) = -2iJ_\nu \epsilon_{\eta \alpha \beta} \{ \delta_{\alpha \sigma} \delta_{\beta \gamma} + (1 - |V_{1\alpha}|^2)(1 - \delta_{\alpha \sigma}) - (1 - |V_{1\sigma}|^2)\delta_{\beta \gamma} \}
\] (33)

where the dummy index \( \eta \) is summed from one to three. Multiplying the above factors, in Eq.(30), yields

\[
\rho R = -16i \xi \eta v(S) J_\nu X e^{2i \xi tr S} \nu
\] (34)

where

\[
X = \frac{1}{2} \sum_{\eta \alpha \beta} \epsilon_{\eta \alpha \beta} \frac{\sin(\phi_{\alpha \beta})}{\phi_{\alpha \beta}} \{ \phi_{\alpha \beta} \cos(\phi_{\alpha \beta}) + 2|V_{1\alpha}|^2 [\cos(\phi_{\beta \eta}) \sin(\phi_{\eta \alpha}) - \phi_{\alpha \beta} \cos(\phi_{\alpha \beta})] \}
\] (35)

Note that \( X \) vanishes, as it should, if any two neutrinos would be degenerate. Furthermore, the measurable quantities \( |V_{1\alpha}|^2 \) are themselves invariant functions of lepton mass matrices. They can be extracted from the mass matrices with help of projection operators \( P \). In our case we have

\[
|V_{1\alpha}|^2 = \frac{1}{\rho} tr(P_{\alpha}(S_\nu)\Pi) = \frac{1}{\rho} \frac{tr[(S_\nu - m_{\alpha}^2)(S_\nu - m_{\eta}^2)\Pi]}{(m_{\alpha}^2 - m_{\beta}^2)(m_{\alpha}^2 - m_{\eta}^2)}
\] (36)

where, as before, \( \xi(m_{\alpha}^2 - m_{\beta}^2) = \phi_{\alpha \beta} \). The identity

\[
\sin(\phi_{\alpha \beta}) \cos(\phi_{\beta \eta}) \sin(\phi_{\eta \alpha}) = \frac{1}{4} [\cos(2\phi_{\alpha \beta}) + \cos(2\phi_{\beta \eta}) + \cos(2\phi_{\eta \alpha}) - 1]
\] (37)

valid for \( \phi_{\alpha \beta} + \phi_{\beta \eta} + \phi_{\eta \alpha} = 0 \), can be used to rewrite \( X \) noting that the RHS of Eq.(37) has the same value for every term in \( X \).

Returning to Eq.(28), the LHS is a function of matter variables and the RHS contains only vacuum variables. Furthermore, the vacuum and the first order correction terms have the same phase. Removing common factors, we find

\[
J'_\nu \sin(\phi'_{12})\sin(\phi'_{23})\sin(\phi'_{31}) e^{2i \xi \nu} = J_\nu \{ \sin(\phi_{12})\sin(\phi_{23})\sin(\phi_{31}) - i \xi \rho X + O(\rho^2) \}
\] (38)

Thus

\[
J'_\nu \sin(\phi'_{12})\sin(\phi'_{23})\sin(\phi'_{31}) = J_\nu \left[ \sin(\phi_{12})\sin(\phi_{23})\sin(\phi_{31}) + O(\rho^2) \right]
\] (39)

Substituting the relation \( J'_\nu v(S'_\nu) = J_\nu v(S_\nu) \), from Eq.(39), we find

\[
\frac{1}{v(S'_\nu)} \sin(\phi'_{12})\sin(\phi'_{23})\sin(\phi'_{31}) = \frac{1}{v(S_\nu)} \sin(\phi_{12})\sin(\phi_{23})\sin(\phi_{31}) + O(\rho^2)
\] (40)

References

[1] J. H. Christenson, J. W. Cronin, V. L. Fitch and R. Turlay, Phys. Rev. Lett. 13 (1964) 138

[2] L. Wolfenstein, Phys. Rev. Lett. 13 (1964) 569
[3] S. L. Glashow, Nucl. Phys. 22 (1961) 579;  
S. Weinberg, Phys. Rev. Lett. 19 (1967) 1264;  
A. Salam, in Elementary Particle Theory, Ed. N. Svartholm (Almqvist and Wiksell, 1968)  
[4] M. Kobayashi and T. Maskawa, Prog. Theor. Phys. 49 (1973) 652  
[5] S. M. Bilenky and B. Pontecorvo, Phys. Rep. 41 (1978) 225  
[6] C. Jarlskog, Zeit. f. Phys. C29 (1985) 491 C29 (1985) 491  
[7] C. Jarlskog, Phys. Rev. D35 (1987) 1685; ibid D36 2128  
[8] C. Jarlskog, Phys. Rev. Lett. 55 (1985) 1039  
[9] C. Jarlskog and R. Stora, Phys. Lett. B208 (1988) 268  
[10] C. Jarlskog in CP Violation, Ed. C. Jarlskog (World Scientific, 1989) p. 3  
[11] V. Barger, K. Whisnant and R. J. N. Phillips, Phys. Rev. Lett. 45 (1980) 2084  
[12] S. M. Bilenky, hep-ph/0410090  
[13] S. M. Bilenky and S. T. Petcov, Rev. Mod. Phys. 59 (1987) 671  
[14] P. G. Langacker, S. T. Petcov, G. Steigman and S. Toshev, Nucl. Phys. B282 (1987) 589  
[15] P. Ramond, hep-ph/0411010  
[16] S. Weinberg, Phy. Rev. Lett. 43 (1979) 1566  
[17] L. Wolfenstein, Phys. Rev. D17 (1978) 2369; ibid D20 (1979) 2634  
[18] S. P. Mikheyev and A. Yu. Smirnov, Sov. J. Nucl. Phys. 42 (1985) 913; Sov. Phys. JETP 64 (1986) 4  
[19] S. M. Bilenky, C. Giunti and W. Grimus, Prog. Part. Nucl. Phys. 43 (1999) 1  
[20] V. A. Naumov, Intern. J. Mod. Phys. D1 (1992) 379  
[21] P. F. Harrison and W. G. Scott, Phys. Lett. B476 (2000) 349  
[22] P. F. Harrison and W. G. Scott, Phys. Lett. B535 (2002) 229  
[23] P. F. Harrison, W. G. Scott and T. J. Weiler, Phys. Lett. B565 (2003) 159