\( \mathcal{N} = 2 \) MOONSHINE

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Abstract. We construct a model of moonshine phenomenon based on the use of \( \mathcal{N} = 2 \) superconformal algebra. We consider an extremal Jacobi form of weight 0 and index 2, and expand it in terms of \( \mathcal{N} = 2 \) massless and massive representations. We find the multiplicities of massive representations are decomposed into a sum of dimensions of irreducible representations of the group \( L_2(11) \).

1. Introduction

Study of the elliptic genus in string compactification by use of the superconformal algebras (SCA) was introduced in \cite{13}. In this approach we use the representation theory of SCA, and decompose elliptic genus in terms of characters of SCA. In superconformal algebras there appear BPS (massless) and non-BPS (massive) representations, and massless characters are mock theta functions which possess unusual modular transformation laws \cite{7,15,16}. Intrinsic structure of mock theta functions is revealed in \cite{23} (see also \cite{22}).

Recently a phenomenon similar to the famous Monstrous moonshine \cite{5} was discovered in this analysis \cite{12}: it was found that the expansion coefficients of \( K3 \) elliptic genus in terms of characters of \( \mathcal{N} = 4 \) SCA are decomposed into a sum of dimensions of irreducible representations of Mathieu group \( M_{24} \). Analogues of McKay–Thompson series in the Monstrous moonshine were constructed \cite{11,11,18,19}, and the decompositions into \( M_{24} \) representations have been verified up to very high degrees. This phenomenon, sometimes called Mathieu moonshine, combines (mock) modular forms, a sporadic discrete group, and geometry of \( K3 \) surface in a curious manner. Since \( M_{24} \) is the symmetry group of...
an error correcting code (Golay code), such a moonshine phenomenon may be also interesting from the point of view of a possible mechanism of information processing inside black holes.

Quite recently, a generalization of the Mathieu moonshine has been proposed in [2]: authors of [2] consider a sequence of higher dimensional analogues of Mathieu moonshine parametrized by \( m = 2, 3, 4, 5, 7 \) where \((m-1)|24\): \( m = 2 \) case corresponds to \( K3 \) and the original \( M_{24} \) moonshine. \( m = 3 \) corresponds to a 4-dimensional complex manifold. From a general theory of Jacobi forms [17] it is known that the elliptic genera of complex \( D \)-dimensional manifold are given by weak Jacobi forms of weight 0 and index \( D/2 \). When \( D = 4 \), there exist two independent weak Jacobi forms with weight 0, index 2

\[
Z_1(z; \tau) = 48 \left[ \left( \frac{\theta_{10}(z; \tau)}{\theta_{10}(0; \tau)} \right)^4 + \left( \frac{\theta_{00}(z; \tau)}{\theta_{00}(0; \tau)} \right)^4 + \left( \frac{\theta_{01}(z; \tau)}{\theta_{01}(0; \tau)} \right)^4 \right],
\]

(1.1)

\[
Z(z; \tau) = 4 \left[ \left( \frac{\theta_{10}(z; \tau)}{\theta_{10}(0; \tau)} \cdot \frac{\theta_{00}(z; \tau)}{\theta_{00}(0; \tau)} \right)^2 + \left( \frac{\theta_{00}(z; \tau)}{\theta_{00}(0; \tau)} \cdot \frac{\theta_{01}(z; \tau)}{\theta_{01}(0; \tau)} \right)^2 + \left( \frac{\theta_{10}(z; \tau)}{\theta_{10}(0; \tau)} \cdot \frac{\theta_{01}(z; \tau)}{\theta_{01}(0; \tau)} \right)^2 \right].
\]

(1.2)

A suitable linear combination of these Jacobi forms, \( Z_1 + n Z \), will reproduce an elliptic genus of some 4-dimensional complex manifold (\( n = 15 \) gives the elliptic genus of Hilbert scheme of two points on \( K3 \) surface \( K3^{(2)} \)). Coefficient of \( Z_1 \) is fixed since it contains the identity representation in the NS sector.

Somewhat surprisingly authors of [2] chose to drop \( Z_1 \) and studied \( Z \) in isolation. Decomposition of \( Z \) in terms of \( \mathcal{N} = 4 \) characters was known in the literature [8], and it was possible to guess a new moonshine phenomenon which is based on the Mathieu group \( M_{12} \). At higher values of \( m \) they apply a similar construction. Drop the analogue of \( Z_1 \) and consider a linear combination of other Jacobi forms which possesses a polar term only in the massive representations of the smallest isospin (an extremal Jacobi form) [2]. Thus in these examples we seem to lose the connection to geometry and elliptic genus, however, there still appear interesting new examples of moonshine phenomena (\( Z \) may still describe an elliptic genus of a non-compact manifold [14]).

Ordinarily, \( \mathcal{N} = 4 \) (resp. \( \mathcal{N} = 2 \)) SCA describes the geometry of hyper-Kähler (resp. Calabi–Yau, CY for short) manifolds. When one drops \( Z_1 \), however, it is not quite clear whether \( \mathcal{N} = 4 \) or \( \mathcal{N} = 2 \) is the relevant symmetry of the theory. In this article we take up the above example \( Z \) at \( D = 4 \), and decompose it in terms of \( \mathcal{N} = 2 \) SCA characters [10] instead of \( \mathcal{N} = 4 \) [9]. This is to see if it is possible to obtain further examples of moonshine phenomena. We in fact find a moonshine phenomenon with respect to the group \( L_2(11) \) which is closely related to \( M_{12} \).

2. \( \mathcal{N} = 2 \) Superconformal Algebras and Character Decomposition

First let us recall the data of representation theory of \( \mathcal{N} = 2 \) algebra. Representations of the extended \( \mathcal{N} = 2 \) algebra with central charge \( c = 3D \) were studied in [20, 21]. The
characters of the extended algebra are obtained by summing over the spectral flow of irreducible $\mathcal{N} = 2$ characters.

There exist BPS (massless) and non-BPS (massive) representations in the theory, parametrized by the conformal weight $h$ and $U(1)$ charge $Q$. In the Ramond sector $\tilde{R}$ (with $(-1)^F$ insertion) characters are given as follows.

- massive (non-BPS) representations:
  
  $h > \frac{D}{8}; Q = \frac{D}{2}, \frac{D}{2} - 1, \ldots, -(\frac{D}{2} - 1), -\frac{D}{2}$ and $Q \neq 0 (D = \text{even}),$
  
  \[
  \begin{align*}
  c^1_{\tilde{R}, \mathcal{N}=2, D,h,Q>0}(z; \tau) &= (-1)^{Q+2}i^{Q+1}q^{\frac{D}{8}}\theta_{11}(z;\tau) e^{2\pi i(Q+\frac{1}{2})z} \\
  &\quad \times \sum_{n \in \mathbb{Z}} q^\frac{D-1}{2}n^2+(Q-\frac{1}{2})n (-e^{2\pi iz})^{(D-1)n},
  \end{align*}
  \]

- massless (BPS) representations:
  
  $h = \frac{D}{8}; Q = \frac{D}{2} - 1, \frac{D}{2} - 2, \ldots, -(\frac{D}{2} - 1),$
  
  \[
  \begin{align*}
  c^0_{\tilde{R}, \mathcal{N}=2, D,h=\frac{D}{8},Q \geq 0}(z; \tau) &= (-1)^D i^{D+1}q^{\frac{D+1}{2}}\theta_{11}(z;\tau) e^{2\pi i(Q+\frac{1}{2})z} \\
  &\quad \times \sum_{n \in \mathbb{Z}} q^\frac{D-1}{2}n^2+(Q+\frac{1}{2})n (-e^{2\pi iz})^{(D-1)n},
  \end{align*}
  \]

and for $h = \frac{D}{8}; Q = \frac{D}{2}$

\[
\begin{align*}
  c^0_{\tilde{R}, \mathcal{N}=2, D,h=\frac{D}{8},Q=\frac{D}{2}}(z; \tau) &= (-1)^D i^{D+1}q^{\frac{D+1}{2}}\theta_{11}(z;\tau) e^{2\pi i(Q+\frac{1}{2})z} \\
  &\quad \times \sum_{n \in \mathbb{Z}} q^\frac{D-1}{2}n^2+(\frac{D+1}{2})n (1-q) (-e^{2\pi iz})^{(D-1)n} \left(1 - e^{2\pi iz} q^n\right) \left(1 - e^{2\pi iz} q^{n+1}\right).
  \end{align*}
\]

The characters for $Q < 0$ are given by

\[
\begin{align*}
  c^1_{\tilde{R}, \mathcal{N}=2, D,h,Q<0}(z; \tau) &= c^1_{\tilde{R}, \mathcal{N}=2, D,h,Q}(z; -\tau).
  \end{align*}
\]

The Witten index of massless representations are given by

\[
\begin{align*}
  c^0_{\tilde{R}, \mathcal{N}=2, D,h=\frac{D}{8},Q \geq 0}(z = 0; \tau) &= \begin{cases} (-1)^{Q+2}, & \text{for } 0 \leq Q < \frac{D}{2}, \\ 1 + (-1)^D, & \text{for } Q = \frac{D}{2}, \end{cases}
  \end{align*}
\]

while all massive representations having a vanishing index.

At the unitarity bound $h = \frac{D}{8}$, a massive character decomposes into a sum of massless characters as

\[
\begin{align*}
  \lim_{h \searrow \frac{D}{8}} c^1_{\tilde{R}, \mathcal{N}=2, D,h,Q+1}(z; \tau) &= c^1_{\tilde{R}, \mathcal{N}=2, D,h=\frac{D}{8},Q+1}(z; \tau) + c^0_{\tilde{R}, \mathcal{N}=2, D,h=\frac{D}{8},Q}(z; \tau),
  \end{align*}
\]

where $Q \geq 0$, and

\[
\begin{align*}
  \lim_{h \searrow \frac{D}{8}} c^0_{\tilde{R}, \mathcal{N}=2, D,h,Q}(z; \tau) \\
  &= c^0_{\tilde{R}, \mathcal{N}=2, D,h=\frac{D}{8},Q}(z; \tau) + c^0_{\tilde{R}, \mathcal{N}=2, D,h=\frac{D}{8},Q-1}(z; \tau) + c^0_{\tilde{R}, \mathcal{N}=2, D,h=\frac{D}{8},Q-\frac{D}{2}}(z; \tau).
  \end{align*}
\]
In our previous paper [10] we pointed out that when the dimension \( D \) of CY manifold is odd, the decomposition of its elliptic genus into \( \mathcal{N} = 2 \) characters becomes essentially the same as the decomposition of the elliptic genus for a corresponding \((D-3)\)-dimensional hyper-Kähler manifold into \( \mathcal{N} = 4 \) characters. This is due to the uniqueness of Jacobi form of index \( 3/2 \) and weight 0.

In the case of even \( D \), however, the decomposition of CY manifolds becomes somewhat different from that of hyper-Kähler manifolds. For convenience we introduce functions \( B_{D,Q}^{\mathcal{N}=2}(z;\tau) \) and \( C_{D}^{\mathcal{N}=2}(z;\tau) \) by

\[
B_{D,Q}^{\mathcal{N}=2}(z;\tau) = (-1)^{Q+\frac{D}{2}-1} q^{-\frac{D}{2}+\frac{D}{2}+\left(\frac{D-1}{2}\right)^2} \left( \text{ch}_{D,h>\frac{D}{2},Q}(z;\tau) + \text{ch}_{D,h>\frac{D}{2},Q}(z;\tau) \right)
\]

\[
= \begin{cases} 
\frac{i\theta_1(z;\tau)}{[\eta(\tau)]^3} \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{D}{2}n^2} (D-1)(n+\frac{1}{2})^2 e^{2\pi i(D-1)(n+\frac{1}{2})z}, & \text{for } 1 \leq Q < \frac{D}{2}, \\
\frac{i\theta_1(z;\tau)}{[\eta(\tau)]^3} \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{D}{2}n^2} (D-1)(n+\frac{1}{2})^2 e^{2\pi i(D-1)(n+\frac{1}{2})z}, & \text{for } Q = \frac{D}{2},
\end{cases}
\]

(2.8)

\[
C_{D}^{\mathcal{N}=2}(z;\tau) = (-1)^{\frac{D}{2}} q^{\frac{D}{2}} \text{ch}_{D,h=\frac{D}{2},Q=0}(z;\tau)
\]

\[
= \frac{i\theta_1(z;\tau)}{[\eta(\tau)]^3} e^{\pi i z} \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{D}{2}n^2+\frac{1}{2}n} q^{2\pi i(D-1)nz}.
\]

(2.9)

\( B_{D,Q}^{\mathcal{N}=2} \) stands for a charge \( Q \) massive character symmetrized under \( z \leftrightarrow -z \). \( C_{D}^{\mathcal{N}=2} \) is the massless character for the charge \( Q = 0 \). The elliptic genus \( Z_{\text{CYD}}(z;\tau) \) for the Calabi–Yau \( D \)-fold (or any weak Jacobi form of index \( D/2 \) and weight 0) is decomposed as

\[
Z_{\text{CYD}}(z;\tau) = \chi C_{D}^{\mathcal{N}=2}(z;\tau) + \sum_{a=1}^{D/2} \Sigma_{D,a}(\tau) B_{D,a}^{\mathcal{N}=2}(z;\tau).
\]

(2.10)

Here \( \chi \) denotes the Euler number. From a mathematical point of view \( \mathcal{N} = 2 \) decomposition (2.10) gives a theta series expansion of (a real analytic) Jacobi form with a half-odd integral index [10], while \( \mathcal{N} = 4 \) decomposition is that of a Jacobi form of an integral index. See also [6] for recent studies of Jacobi forms. Since the massless character \( C_{D}(z;\tau) \) is a mock theta function, the generating functions \( \Sigma_{D,a}(\tau) \) for the multiplicity of massive representations become also mock theta functions as far as \( \chi \neq 0 \).

In the case of \( D = 2 \), the Calabi–Yau 2-fold is the \( K3 \) surface. The character decomposition above reduces to the Mathieu moonshine considered in [12].
Let us now turn to the case $D = 4$, and study a Jacobi form with weight 0 and index 2, $Z(z; \tau)$

$$Z(z; \tau) = 4 \left[ \left( \frac{\theta_{10}(z; \tau)}{\theta_{10}(0; \tau)} \cdot \frac{\theta_{00}(z; \tau)}{\theta_{00}(0; \tau)} \right)^2 + \left( \frac{\theta_{00}(z; \tau)}{\theta_{00}(0; \tau)} \cdot \frac{\theta_{01}(z; \tau)}{\theta_{01}(0; \tau)} \right)^2 + \left( \frac{\theta_{01}(z; \tau)}{\theta_{01}(0; \tau)} \cdot \frac{\theta_{10}(z; \tau)}{\theta_{10}(0; \tau)} \right)^2 \right]$$

$$= \frac{1}{12} [\phi_{1,1}(z; \tau)]^2 - \frac{1}{12} E_4(\tau) [\phi_{-2,1}(z; \tau)]^2. \quad (3.1)$$

As computed in [10], we have a decomposition

$$Z(z; \tau) = 12 C_4^{N=2}(z; \tau) +$$

$$+ q^{-\frac{3}{24}} \left( -2 + 10 q + 20 q^2 + 42 q^3 + 62 q^4 + 118 q^5 + 170 q^6 + 270 q^7 + \cdots \right) B_{4,1}^{N=2}(z; \tau)$$

$$+ q^{-\frac{7}{24}} \left( 12 q + 36 q^2 + 60 q^3 + 120 q^4 + 180 q^5 + 312 q^6 + 456 q^7 + \cdots \right) B_{4,2}^{N=2}(z; \tau) \quad (3.2)$$

$$= 8 \text{ch}_{D=4,h=\frac{1}{2},Q=0}^{R,N=2}(z; \tau) - 2 \text{ch}_{D=4,h=\frac{1}{2},Q=1}^{R,N=2}(z; \tau) - 2 \text{ch}_{D=4,h=\frac{1}{2},Q=-1}^{R,N=2}(z; \tau)$$

$$- \sum_{Q=\pm 1, \pm 2} (-1)^Q \sum_{n=1}^\infty p_Q(n) \text{ch}_{D=4,h=n+\frac{1}{2},Q}^{R,N=2}(z; \tau). \quad (3.3)$$

Expansion coefficients of the massive representations in (3.2) suggest the group $L_2(11)$ being relevant for a moonshine phenomenon. $L_2(11)$ is the group $PSL_2(\mathbb{F}_{11})$ of $2 \times 2$ matrices of determinant one with matrix elements in the field $\mathbb{F}_{11}$.

See Table 1 for a character table of $SL_2(11) \cong 2.L_2(11)$, which is a double cover of $L_2(11)$ [4]. Therein $n_g$ denotes the number of elements in conjugacy class $g$, and the orthogonality relation reads as

$$\sum_g n_g \chi_R^g \chi_{R'}^{g'} = |G| \delta_{R,R'}. \quad (3.4)$$

$|G|$ denotes the order of $G$, and $|SL_2(11)| = 2^3 \cdot 3 \cdot 5 \cdot 11 = 1320$. It is easy to check that at small values of $n$, the number of massive representations $p_a(n)$ can be written as a sum of dimensions of irreducible representations $R$ of $SL_2(11)$

$$\sum_R \text{mult}_{R,a}(n) \dim R = p_a(n), \quad (3.5)$$

with multiplicities $\text{mult}_{R,a}(n)$

$10 = 5 + 5, \ 20 = 2 \times 10, \ 42 = 10 + 10 + 2 \times 11, \cdots$

$12 = 6 + 6, \ 36 = 6 + 6 + 12 + 12, \ 60 = 3 \times (6 + 6) + 12 + 12, \cdots$

It is known that $L_2(11)$ has a permutation representation on 12 symbols (see, e.g., [3]). Representatives of conjugacy classes $g$ are given in Table 2.

As in the case of Mathieu moonshine, we want to construct twisted elliptic genus $Z_g$ for each conjugacy class $g$. It turns out that due to complication of double covering of the group $L_2(11)$ we can not construct twisted elliptic genera for all classes. However, in the following we obtain those twisted elliptic genera which are just enough to determine the
decomposition of the multiplicities of massive representations into the sum of irreducible representations of $SL_2(11)$. 

Let us call the representations $\{\chi_i\}$, $i = 1, 2, 3, 4, 5, 6, 7, 8$ in Table 1 as even and representations $\{\chi_j\}$, $j = 9, 10, 11, 12, 13, 14, 15$ as odd, respectively. We assume as in [2] that multiplicities of $|Q| = 1$ massive representations are decomposed into a sum of even representations, and that those of $|Q| = 2$ massive representations are decomposed into a sum of odd representations.

Twisted elliptic genus is a Jacobi form with weight 0 and index 2 and has a decomposition analogous to (5.2),

$$Z_g(z; \tau) = \chi_g C_4^{N=2}(z; \tau) + \Sigma_{g,1}(\tau) B_{4,1}^{N=2}(z; \tau) + \Sigma_{g,2}(\tau) B_{4,2}^{N=2}(z; \tau). \tag{3.6}$$

Here $\chi_g$ is the Euler number, $\chi_g = Z_g(0; \tau)$, and $\Sigma_{g,a}(\tau)$ are $q$-series with integral Fourier coefficients

$$\Sigma_{g,a}(\tau) = q^{\frac{(2a-1)^2}{24}} \sum_{n=0}^{\infty} p_{g,a}(n) q^n. \tag{3.7}$$
Structure of the character table suggests that the conjugacy classes 5A and 5B have the same twisted elliptic genus, and we use the notation 5AB. Similarly, we assume the same for classes 10A, B, 11A, B and 22A, B, and use the notations 10AB, 11AB, and 22AB, respectively.

In view of the character table and the permutation representatives of conjugacy classes, we suppose that the Euler number for class $g$ is given by

$$\chi_g = \chi_1^g + \chi_6^g. \quad (3.8)$$

We then find

| $g$   | 1A  | 5AB | 11AB | 4A  | 3A  | 12AB |
|-------|-----|-----|------|-----|-----|------|
| $\chi_g$ | 12  | 2   | 1    | 0   | 0   | 0    |

Thus the classes $\{1A, 5AB, 11AB\}$ belong to type I and $\{4A, 3A, 12AB\}$ belong to type II in the terminology of [11].

The original elliptic genus (3.1) is for the class $g = 1A$. By trial and error, we have constructed the twisted elliptic genera $Z_g(z;\tau)$ for classes $g = 5AB, 11AB, 4A, 3A, 12AB$, which are presented in Table 3.

| $g$   | $Z_g^{N=2}(z;\tau)$ | $\phi_{01}(z;\tau)^2$ | $E_4(\tau)$ | $E_6(\tau)$ | $\phi_{01}(z;\tau)^2$ | $\phi_{01}(z;\tau)^2$ | $\phi_{01}(z;\tau)^2$ |
|-------|----------------------|------------------------|-------------|-------------|------------------------|------------------------|------------------------|
| 1A    | $\frac{1}{12}[\phi_{01}(z;\tau)^2] - \frac{1}{12}E_4(\tau)[\phi_{01}(z;\tau)^2]$ | | | | | | |
| 5AB   | $\frac{1}{72}[\phi_{01}(z;\tau)^2] + \left(-\frac{5}{576}\phi_2(\tau) + \frac{25}{288}\phi_5(\tau) + \frac{35}{576}\phi_2(\tau) + \frac{5}{16}\eta(\tau)\eta(3\tau)\eta(5\tau)\eta(15\tau)\right)\phi_{01}(z;\tau)$ | | | | | | |
|       | $+ \left(-\frac{1}{32}E_4(\tau) + \frac{25}{16}\eta(\tau)\eta(5\tau)^2 - \frac{75}{4}\eta(\tau)\eta(3\tau)\eta(5\tau)\eta(15\tau)^2\right)$ | | | | | | |
|       | $+ \frac{5}{96}\left[\phi_2^{(3)}(\tau)\right]^2 - \frac{25}{576}\phi_5^{(2)}(\tau) - \frac{5}{32}\phi_5^{(2)}(\tau)$ | | | | | | |
|       | $+ \frac{175}{96}\phi_2^{(3)}(\tau)\phi_2^{(5)}(\tau) + \frac{175}{96}\phi_2^{(5)}(\tau)\phi_2^{(15)}(\tau)$ | | | | | | |
| 11AB  | $\frac{1}{144}[\phi_{01}(z;\tau)^2] + \left(\frac{11}{72}\phi_2^{(11)}(z;\tau) + \frac{11}{20}\eta(\tau)\eta(11\tau)^2\right)\phi_{01}(z;\tau)$ | | | | | | |
|       | $+ \left(\frac{1}{120}E_4(\tau) - \frac{121}{720}\phi_2^{(11)}(z;\tau)^2 + \frac{1089}{100}\phi_2^{(11)}(z;\tau)\eta(\tau)\eta(11\tau)^2 - \frac{121}{125}\eta(\tau)\eta(11\tau)^4\right)[\phi_{01}(z;\tau)^2]$ | | | | | | |
| 4A    | $-\frac{\eta(\tau)\eta(2\tau)}{\eta(4\tau)}B_{4,1}^{N=2}(z;\tau)$ | | | | | | |
| 12AB  | $\left(\frac{\eta(\tau)\eta(2\tau)}{\eta(4\tau)} + 3\frac{\eta(3\tau)^2\eta(6\tau)}{\eta(7\tau)} - 6\frac{\eta(4\tau)[\eta(6\tau)^4]}{\eta(2\tau)^2\eta(3\tau)^2[\eta(12\tau)]^2}\right)B_{4,1}^{N=2}(z;\tau)$ | | | | | | |
| 3A    | $-2\frac{\eta(2\tau)^3}{\eta(\tau)}B_{3,1}^{N=2}(z;\tau)$ | | | | | | |

**Table 3.** Twisted elliptic genus $Z_g(z;\tau)$.

In the case of conjugacy class 2A we assume

$$\Sigma_{1A,1}(\tau) = \Sigma_{2A,1}(\tau),$$

$$\Sigma_{1A,2}(\tau) = -\Sigma_{2A,2}(\tau), \quad (3.9)$$
corresponding to the sign change in the odd sector of character table (see Table 1). We suppose a similar pairing as above (sign change in the $\Sigma_{g,2}$ part) between 5AB and 10AB, 11AB and 22AB.

In the case of 4A, on the other hand, we set the odd part to vanish

$$\Sigma_{4A,2}(\tau) = 0$$

(3.10)

since the odd elements in the character table all vanish (see Table 1) for class 4A. We also assume that the conjugacy classes, 12AB, 3A, and 6A, have vanishing odd parts.

In the case of the Mathieu moonshine, all the twisted elliptic genera were Jacobi forms on congruence subgroup $\Gamma_0(\text{ord}(g))$ with a possible character. In the present case only the twisted elliptic genera of conjugacy classes 1A, 5AB, 11AB, 4A, 12AB, 3A = 6A are Jacobi forms (level of congruence subgroup is sometimes higher than $\text{ord}(g)$). Due to the sign flip in odd sector (3.9) twisted elliptic genera of the other classes can not be Jacobi forms. If we insist that twisted elliptic genera must be Jacobi forms, twisted elliptic genera do not exist for classes 2A, 10AB, 22AB. This situation is similar to the $\mathcal{N} = 4$ moonshine in [2].

In Table 4, the Fourier coefficients of $\Sigma_{g,a}(\tau)$, i.e., the number of the massive representations are given. We have omitted from Table the odd sector $p_{g,2}$ for classes $g$ whose generating functions vanish identically $\Sigma_{g,2}(\tau) = 0$.

In order to test the moonshine conjecture, we have computed multiplicities $\text{mult}_{R,a}(n)$ of representations $R$

$$p_{g,a}(n) = \sum_{R} \text{mult}_{R,a}(n) \chi_{R}^{g}.$$  

(3.11)

Here $R$ runs over irreducible representations from $\chi_1$ to $\chi_8$ (resp. from $\chi_9$ to $\chi_{15}$) for $a = 1$ (resp. $a = 2$). From the orthogonality relation (3.4), we have

$$\text{mult}_{R,a}(n) = \sum_{g} \frac{n_{a}}{|G|} \chi_{R}^{g} p_{g,a}(n).$$

(3.12)

See Table 5 for the results of the decomposition into irreducible representations. We find that the multiplicities $\text{mult}_{R,1}(n)$ are the same for $R = \chi_2$ and $R = \chi_3$, and also for $R = \chi_7$ and $R = \chi_8$ in the even sector. In the odd sector we have $\chi_9 = \chi_{10}$, $\chi_{11} = \chi_{12} = \chi_{13}$, and $\chi_{14} = \chi_{15}$.

We have verified up to $n = 100$ the positivity and integrality of the multiplicities $\text{mult}_{R,a}(n)$, and consider this to be a strong evidence for a $\mathcal{N} = 2$ moonshine.

4. Discussions

In this paper we have taken up the suggestion of [2] on the extremal Jacobi form. We studied the decomposition of an extremal form of index 2 into characters of $\mathcal{N} = 2$ SCA. We have found a strong evidence for a moonshine phenomenon with respect to the group $L_2(11)$, which is a subgroup or $M_{12}$ which appeared in [2] from decomposition into $\mathcal{N} = 4$ SCA characters.
\( \mathcal{N} = 2 \) MOONSHINE

| \( n \) \( g \) | \( p_{g,1}(n) \) | \( p_{g,2}(n) \) |
|---|---|---|
| \( n \) \( g \) | \( 1A \) | \( 5AB \) | \( 11AB \) | \( 4A \) | \( 12AB \) | \( 3A \) | \( 1A \) | \( 5AB \) | \( 11AB \) |
| 0 | -2 | -2 | -2 | -2 | -2 | -2 | 0 | 0 | 0 |
| 1 | 10 | 0 | -1 | 2 | 2 | -2 | 12 | 2 | 1 |
| 2 | 20 | 0 | -2 | 4 | -2 | 2 | 36 | 1 | 3 |
| 3 | 42 | 2 | -2 | -2 | -2 | 0 | 60 | 5 | 5 |
| 4 | 62 | 2 | -4 | -2 | 4 | 2 | 120 | 10 | -1 |
| 5 | 118 | -2 | -3 | -2 | 4 | 4 | 180 | 15 | 4 |
| 6 | 170 | 0 | -6 | 2 | 8 | -4 | 312 | 27 | 4 |
| 7 | 270 | 0 | -5 | -2 | 4 | 0 | 456 | 36 | 5 |
| 8 | 400 | 0 | -7 | -4 | 14 | 4 | 720 | 60 | 5 |
| 9 | 600 | 0 | -5 | 4 | 10 | 0 | 1020 | 85 | 8 |
| 10 | 828 | -2 | -8 | 4 | 10 | 0 | 1524 | 129 | 6 |
| 11 | 1220 | 0 | -12 | 0 | 12 | 2 | 2124 | 179 | 12 |
| 12 | 1670 | 0 | -13 | -2 | 22 | -4 | 3036 | 251 | 11 |
| 13 | 2330 | 0 | -13 | 2 | 14 | -4 | 4140 | 345 | 15 |
| 14 | 3162 | 2 | -17 | 2 | 20 | 6 | 5760 | 480 | 18 |
| 15 | 4316 | -4 | -18 | 0 | 30 | -4 | 7740 | 645 | 18 |
| 16 | 5730 | 0 | -23 | -6 | 42 | 0 | 10512 | 877 | 18 |
| 17 | 7710 | 0 | -23 | 2 | 38 | 6 | 13896 | 1156 | 25 |
| 18 | 10102 | 2 | -29 | 6 | 42 | -8 | 18540 | 1545 | 27 |
| 19 | 13312 | 2 | -31 | -4 | 50 | -2 | 24240 | 2020 | 29 |
| 20 | 17298 | -2 | -38 | -6 | 66 | 6 | 31824 | 2654 | 34 |
| 21 | 22500 | 0 | -39 | 0 | 72 | -6 | 41124 | 3429 | 39 |
| 22 | 28860 | 0 | -48 | 4 | 70 | 0 | 53292 | 4437 | 41 |
| 23 | 37162 | 2 | -51 | -2 | 82 | 10 | 68220 | 5685 | 53 |
| 24 | 47262 | 2 | -60 | -6 | 96 | -12 | 87420 | 7285 | 58 |
| 25 | 60128 | -2 | -64 | 4 | 112 | -4 | 110880 | 9240 | 66 |
| 26 | 75900 | 0 | -77 | 8 | 116 | 12 | 140724 | 11729 | 67 |
| 27 | 95740 | 0 | -81 | -4 | 128 | -8 | 177072 | 14752 | 82 |
| 28 | 119860 | 0 | -95 | -4 | 152 | -2 | 222780 | 18565 | 85 |
| 29 | 150062 | 2 | -99 | 2 | 170 | 14 | 278280 | 23190 | 101 |
| 30 | 186576 | -4 | -116 | 8 | 182 | -12 | 347424 | 28954 | 110 |
| 31 | 231800 | 0 | -124 | -4 | 206 | -4 | 431136 | 35931 | 123 |
| 32 | 286530 | 0 | -141 | -10 | 236 | 18 | 534492 | 44537 | 134 |
| 33 | 353094 | 4 | -154 | 6 | 252 | -12 | 659220 | 54935 | 155 |
| 34 | 434524 | 4 | -174 | 12 | 270 | -2 | 812160 | 67680 | 162 |
| 35 | 533334 | -6 | -188 | -2 | 310 | 18 | 996084 | 83009 | 188 |
| 36 | 651790 | 0 | -213 | -10 | 350 | -20 | 1220124 | 101679 | 202 |
| 37 | 795490 | 0 | -228 | 2 | 380 | -8 | 1488612 | 124047 | 224 |
| 38 | 967490 | 0 | -257 | 10 | 400 | 26 | 1813860 | 151155 | 246 |
| 39 | 1174962 | 2 | -278 | -6 | 450 | -18 | 2202420 | 183535 | 275 |
| 40 | 1422264 | -6 | -311 | -12 | 504 | 0 | 2670564 | 222549 | 292 |
| 41 | 1719450 | 0 | -334 | 6 | 546 | 30 | 3228048 | 269008 | 329 |
| 42 | 2072480 | 0 | -371 | 12 | 588 | -28 | 3896568 | 324708 | 357 |
| 43 | 2494542 | 2 | -401 | -6 | 648 | -6 | 4690320 | 390860 | 393 |
| 44 | 2994874 | 4 | -448 | -14 | 718 | 34 | 5637960 | 469830 | 427 |
| 45 | 3590404 | -6 | -480 | 4 | 778 | -26 | 6759744 | 563314 | 475 |
| 46 | 4294020 | 0 | -534 | 12 | 834 | -6 | 8093748 | 674483 | 509 |
| 47 | 5128880 | 0 | -574 | -4 | 908 | 38 | 9668448 | 805698 | 570 |
| 48 | 6112362 | 2 | -635 | -18 | 1002 | -36 | 11534040 | 961170 | 606 |
| 49 | 7274774 | 4 | -681 | 10 | 1090 | -10 | 13730220 | 1144185 | 669 |
| 50 | 8641024 | -6 | -752 | 20 | 1166 | 46 | 16323228 | 1360273 | 724 |

Table 4. The number of massive representations, \( p_{g,1}(n) \) and \( p_{g,2}(n) \)
| \( n \setminus R \) | \( \chi_1 \) | \( \chi_2 = \chi_3 \) | \( \chi_4 \) | \( \chi_5 \) | \( \chi_6 \) | \( \chi_7 = \chi_8 \) | \( \chi_9 = \chi_{10} \) | \( \chi_{11} = \chi_{12} \) | \( \chi_{14} = \chi_{15} \) |
|---|---|---|---|---|---|---|---|---|---|
| 0 | -2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 2 | 0 | 0 | 0 | 2 | 0 | 0 | 1 | 0 | 1 |
| 3 | 0 | 0 | 1 | 1 | 2 | 0 | 3 | 0 | 1 |
| 4 | 1 | 0 | 1 | 3 | 1 | 1 | 0 | 5 | 2 | 0 |
| 5 | 0 | 1 | 4 | 2 | 0 | 2 | 8 | 2 | 1 |
| 6 | 0 | 4 | 4 | 2 | 2 | 2 | 14 | 4 | 1 |
| 7 | 0 | 3 | 6 | 4 | 4 | 4 | 19 | 6 | 2 |
| 8 | 2 | 5 | 11 | 5 | 4 | 6 | 31 | 10 | 2 |
| 9 | 2 | 7 | 11 | 9 | 8 | 10 | 44 | 14 | 3 |
| 10 | 1 | 9 | 15 | 13 | 11 | 14 | 66 | 22 | 3 |
| 11 | 2 | 12 | 23 | 19 | 18 | 20 | 92 | 30 | 5 |
| 12 | 3 | 18 | 31 | 23 | 25 | 28 | 129 | 44 | 7 |
| 13 | 3 | 22 | 39 | 35 | 37 | 40 | 177 | 60 | 9 |
| 14 | 7 | 28 | 55 | 49 | 49 | 54 | 246 | 84 | 12 |
| 15 | 6 | 40 | 73 | 63 | 66 | 76 | 330 | 114 | 15 |
| 16 | 11 | 52 | 99 | 83 | 89 | 100 | 448 | 156 | 19 |
| 17 | 15 | 66 | 128 | 116 | 121 | 136 | 591 | 206 | 26 |
| 18 | 17 | 88 | 163 | 151 | 163 | 178 | 789 | 276 | 33 |
| 19 | 23 | 112 | 216 | 198 | 215 | 236 | 1031 | 362 | 42 |
| 20 | 30 | 144 | 282 | 258 | 276 | 308 | 1354 | 476 | 54 |
| 21 | 38 | 187 | 359 | 335 | 364 | 402 | 1749 | 616 | 69 |
| 22 | 47 | 235 | 457 | 435 | 469 | 516 | 2263 | 800 | 89 |
| 23 | 63 | 298 | 588 | 560 | 605 | 666 | 2899 | 1024 | 113 |
| 24 | 75 | 381 | 742 | 708 | 775 | 848 | 3714 | 1314 | 143 |
| 25 | 97 | 481 | 940 | 904 | 983 | 1082 | 4710 | 1668 | 180 |
| 26 | 123 | 600 | 1184 | 1148 | 1243 | 1366 | 5977 | 2120 | 225 |
| 27 | 150 | 755 | 1486 | 1442 | 1576 | 1726 | 7518 | 2668 | 284 |
| 28 | 189 | 942 | 1859 | 1807 | 1973 | 2162 | 9459 | 3360 | 353 |
| 29 | 241 | 1172 | 2322 | 2266 | 2471 | 2710 | 11815 | 4198 | 440 |
| 30 | 289 | 1457 | 2875 | 2817 | 3079 | 3372 | 14750 | 5244 | 546 |
| 31 | 362 | 1802 | 3569 | 3499 | 3830 | 4192 | 18303 | 6510 | 675 |
| 32 | 450 | 2219 | 4411 | 4329 | 4734 | 5184 | 22686 | 8074 | 835 |
| 33 | 550 | 2738 | 5426 | 5344 | 5856 | 6402 | 27981 | 9960 | 1027 |
| 34 | 674 | 3554 | 6658 | 6572 | 7198 | 7868 | 34470 | 12276 | 1260 |
| 35 | 826 | 4106 | 8170 | 8066 | 8832 | 9664 | 42276 | 15058 | 1543 |
| 36 | 1003 | 5018 | 9971 | 9851 | 10899 | 11812 | 51782 | 18450 | 1885 |
| 37 | 1226 | 6112 | 12156 | 12030 | 13196 | 14422 | 63172 | 22514 | 2297 |
| 38 | 1491 | 7416 | 14775 | 14645 | 16053 | 17544 | 76974 | 27438 | 2793 |
| 39 | 1902 | 9004 | 17926 | 17774 | 19512 | 21312 | 93461 | 33320 | 3387 |
| 40 | 2179 | 10886 | 21692 | 21520 | 23619 | 25804 | 113324 | 40410 | 4099 |
| 41 | 2641 | 13143 | 26208 | 26028 | 28561 | 31202 | 136979 | 48850 | 4950 |
| 42 | 3167 | 15838 | 31560 | 31368 | 34447 | 37614 | 165339 | 58974 | 5970 |
| 43 | 3814 | 19043 | 37977 | 37759 | 41470 | 45282 | 199019 | 70994 | 7178 |
| 44 | 4582 | 22842 | 45586 | 45342 | 49792 | 54370 | 239225 | 85346 | 8620 |
| 45 | 5476 | 27378 | 54612 | 54354 | 59712 | 65194 | 286821 | 102334 | 10328 |
| 46 | 6548 | 32720 | 65294 | 65020 | 71428 | 77976 | 343419 | 122540 | 12355 |
| 47 | 7824 | 39052 | 77973 | 77669 | 85324 | 93148 | 410226 | 146388 | 14754 |
| 48 | 9306 | 46535 | 92889 | 92551 | 101714 | 11018 | 489378 | 174648 | 17586 |
| 49 | 11081 | 55358 | 110526 | 110166 | 121067 | 132144 | 582555 | 207012 | 20925 |
| 50 | 13157 | 65710 | 131260 | 130878 | 143811 | 156974 | 692568 | 247190 | 24863 |

Table 5. Multiplicities \( \text{mult}_{R,a}(n) \) up to \( n = 50 \).
Currently, however, the real origin of the moonshine phenomenon is not very well understood and still remains rather mysterious. It seems that we have to construct and study more examples of moonshine phenomena before we figure out the workings behind them. Especially the $\mathcal{N} = 2$ decomposition of models of $[2]$ for higher values of $m = 4, 5, 7$ may be good candidates of moonshine with the group $L_2(7), L_2(5), L_2(3)$, respectively.

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Appendix

A. Modular Forms

As usual we set $q = e^{2\pi i \tau}$ where $\tau$ is in the upper half-plane. The Dedekind $\eta$-function is

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n). \quad (A.1)$$

The Eisenstein series $E_{2k}(\tau)$ is

$$E_{2k}(\tau) = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \left( \sum_{1 \leq r \mid n} r^{2k-1} \right) q^n, \quad (A.2)$$

where $B_k$ is the Bernoulli number

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

We use modular form of weight 2 on $\Gamma_0(M)$

$$\phi_{(M)}(\tau) = \frac{24}{M-1} q \frac{\partial}{\partial q} \log \eta(M\tau) \eta(\tau). \quad (A.3)$$

The Jacobi theta functions are defined as

$$\theta_{11}(z; \tau) = \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}(n + \frac{1}{2})^2} e^{2\pi i (n + \frac{1}{2})(z + \frac{1}{2})},$$

$$\theta_{10}(z; \tau) = \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}(n + \frac{1}{2})^2} e^{2\pi i (n + \frac{1}{2})z}, \quad (A.4)$$

$$\theta_{00}(z; \tau) = \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}n^2} e^{2\pi inz},$$

$$\theta_{01}(z; \tau) = \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}n^2} e^{2\pi in(z + \frac{1}{2})}.$$ 

Some of the Jacobi forms are given by use of these $q$-series as

$$\phi_{-2,1}(z; \tau) = \left[ \frac{[\theta_{11}(z; \tau)]^2}{[\eta(\tau)]^6} \right], \quad (A.5)$$

$$\phi_{0,1}(z; \tau) = 4 \left[ \left( \frac{\theta_{10}(z; \tau)}{\theta_{10}(0; \tau)} \right)^2 + \left( \frac{\theta_{00}(z; \tau)}{\theta_{00}(0; \tau)} \right)^2 + \left( \frac{\theta_{01}(z; \tau)}{\theta_{01}(0; \tau)} \right)^2 \right]. \quad (A.6)$$

See [17] for general properties of the Jacobi forms.
B. \( \mathcal{N} = 4 \) Moonshine

In order to compare the decomposition in \( \mathcal{N} = 2 \) and \( \mathcal{N} = 4 \) SCA, we reproduce the analysis of the \( m = 3 \) case in \[3\] where the \( M_{12} \) moonshine is observed. See Table 6 for the character table of \( 2M_{12} \), which is a double cover of \( M_{12} \). The order is \( |2M_{12}| = 2^7 \cdot 3^3 \cdot 5 \cdot 11 = 190080 \).

Twisted elliptic genus \( Z_{g}^{N=4}(z; \tau) \) of \( \mathcal{N} = 4 \) theory are summarized in Table 7. As in the case of \( \mathcal{N} = 2 \), not all the conjugacy classes have the elliptic genus. Namely, when \( g \) and \( g' \) is a pair of classes with the same values of characters in the even sector (\( \chi_{i}^{g} = \chi_{i}^{g'} \), for \( i = 1, \cdots, 15 \)) and opposite values in the odd sector (\( \chi_{i}^{g} = -\chi_{i}^{g} \), for \( i = 16, \cdots, 26 \)), only either \( Z_{g} \) or \( Z_{g'} \) is a Jacobi form and becomes a twisted genus.

Note that the \( \mathcal{N} = 2 \) twisted elliptic genus for type-I classes are somewhat similar to those of \( \mathcal{N} = 4 \), and we have \( Z_{g}^{N=2}(z; \tau) = \frac{Z_{g}^{N=4}(z; \tau)}{\eta(5\tau)} \) for \( g = 1A \) and 11AB, and

\[
Z_{5AB}^{N=2}(z; \tau) = Z_{5A}^{N=4}(z; \tau) + 5 \frac{[\eta(15\tau)]^{3}}{\eta(\tau)\eta(5\tau)} B_{4,2}^{N=2}(z; \tau). \tag{B.1}
\]

In \( \mathcal{N} = 4 \) SCA, twisted elliptic genera in Table 7 are decomposed as \[4, 9\]

\[
Z_{g}^{N=4}(z; \tau) = \chi_{g} \text{ch} \tilde{R}_{k=2,h=4,\ell=0}^{N=4}(z; \tau) + \sum_{g}^{1}(\tau) B_{2}^{1}(1, N=4; z; \tau) + \sum_{g}^{2}(\tau) B_{2}^{2}(2, N=4; z; \tau), \tag{B.2}
\]

where \( \chi_{g} \) is the Witten index, \( \chi_{g} = \frac{Z_{g}^{N=4}}{Z_{g}}(z = 0; \tau) \), and is given by \( \chi_{g} = \chi_{1}^{g} + \chi_{2}^{g} \).

| \( g \) | 1A | 2B | 3A | 5A | 6C | 8C | 11AB | others |
|-----|----|----|----|----|----|----|-------|--------|
| \( \chi_{g} \) | 12 | 4 | 3 | 2 | 1 | 2 | 1 | 0 |

\( \mathcal{N} = 4 \) massless characters and bases of massive characters are respectively given as

\[
\text{ch} \tilde{R}_{k=2,h=4,\ell=0}^{N=4}(z; \tau) = \frac{\theta_{11}(z; \tau)^2}{\eta(\tau)^3} \frac{i}{\theta_{11}(2z; \tau)} \sum_{n \in \mathbb{Z}} q^{3n^2} e^{12\pi inz} \frac{1 + q^{n}e^{2\pi inz}}{1 - q^{n}e^{2\pi inz}}. \tag{B.3}
\]

\[
B_{2}^{(a), N=4}(z; \tau) = \frac{\theta_{11}(z; \tau)^2}{\eta(\tau)^3} \chi_{1, \frac{-1}{2}}(z; \tau), \tag{B.4}
\]

where \( \chi_{1,j} \) is an \( SU(2) \) spin \( j \) affine character at level 1.

The \( q \)-series \( \Sigma_{g}^{(a)}(\tau) \) is the generating function of the number of \( \mathcal{N} = 4 \) massive representations, and we have

\[
\Sigma_{g}^{(a)}(\tau) = q^{-\frac{1}{2}} \sum_{n=0}^{\infty} A_{g}^{(a)}(n) q^{n}. \tag{B.5}
\]

For comparison with our \( \mathcal{N} = 2 \) moonshine, values of the Fourier coefficients \( A_{g}^{(a)}(n) \) are given in Tables 8. Note that as in the case of \( \mathcal{N} = 2 \) the sign change of odd part in the character table is reflected in \( e.g. \) \( \Sigma_{2A}^{(2)}(\tau) = -\Sigma_{1A}^{(2)}(\tau) \). It should be remarked that we have \( \Sigma_{4C}^{(1)}(\tau) = \Sigma_{2B}^{(1)}(\tau) \).

Multiplicities of massive representations \( A_{g}^{(a)}(n) \) are given by formula like (3.11) with the character table for \( 2M_{12} \) in Table 6. Multiplicities of irreducible representations are completely determined by Table 8.
Table 6. Character table of $G_{2,M}$.

| $\chi_1$ | $\chi_2$ | $\chi_3$ | $\chi_4$ | $\chi_5$ | $\chi_6$ | $\chi_7$ | $\chi_8$ | $\chi_9$ | $\chi_{10}$ | $\chi_{11}$ |
|---------|---------|---------|---------|---------|---------|---------|---------|---------|-----------|------------|
| 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1         | 1          |
| 2       | 2       | 2       | 2       | 2       | 2       | 2       | 2       | 2       | 2         | 2          |
| 3       | 3       | 3       | 3       | 3       | 3       | 3       | 3       | 3       | 3         | 3          |
| 4       | 4       | 4       | 4       | 4       | 4       | 4       | 4       | 4       | 4         | 4          |
| 5       | 5       | 5       | 5       | 5       | 5       | 5       | 5       | 5       | 5         | 5          |
| 6       | 6       | 6       | 6       | 6       | 6       | 6       | 6       | 6       | 6         | 6          |
| 7       | 7       | 7       | 7       | 7       | 7       | 7       | 7       | 7       | 7         | 7          |
| 8       | 8       | 8       | 8       | 8       | 8       | 8       | 8       | 8       | 8         | 8          |
| 9       | 9       | 9       | 9       | 9       | 9       | 9       | 9       | 9       | 9         | 9          |
| 10      | 10      | 10      | 10      | 10      | 10      | 10      | 10      | 10      | 10        | 10         |
| 11      | 11      | 11      | 11      | 11      | 11      | 11      | 11      | 11      | 11        | 11         |

Note: The table continues, but the image does not provide enough information to display the full table.
\( N = 2 \) MOONSHINE

| Table 7. Twisted elliptic genus for \( N = 4 \) SCA. |

\[
\begin{align*}
1A & & g \cdot Z_m^N(z; \tau) \\
& & \frac{1}{17} \left[ \phi_{01}(z; \tau) \right]^4 - \frac{1}{4} E_2(\tau) \left[ \phi_{-21}(z; \tau) \right]^4 \\
2B & & \frac{1}{36} \left[ \phi_{01}(z; \tau) \right]^2 + \frac{1}{8} \phi_{-2}(\tau) \phi_{02}(z; \tau) \phi_{-21}(z; \tau) + \left( -\frac{5}{36} E_2(\tau) + \frac{128}{3} \left[ \phi_{(2\tau)}^4 \right] - \frac{3}{3} E_2(3\tau) \right) \left[ \phi_{-21}(z; \tau) \right]^2 \\
3A & & \frac{1}{18} \left[ \phi_{01}(z; \tau) \right]^2 + \frac{1}{8} \phi_{-2}(\tau) \phi_{02}(z; \tau) \phi_{-21}(z; \tau) + \left( \frac{11}{48} \left[ \phi_{(2\tau)}^4 \right] - \frac{3}{8} E_2(3\tau) \right) \left[ \phi_{-21}(z; \tau) \right]^2 \\
5A & & \frac{1}{72} \left[ \phi_{01}(z; \tau) \right]^2 + \frac{5}{36} \phi_{-2}(\tau) \phi_{02}(z; \tau) \phi_{-21}(z; \tau) + \left( \frac{1}{48} E_2(\tau) - \frac{25}{144} \left[ \phi_{(2\tau)}^4 \right] + \frac{25}{4} \left[ \phi(\tau) \eta(5\tau) \right] \right) \left[ \phi_{-21}(z; \tau) \right]^2 \\
6C & & \frac{1}{144} \left[ \phi_{01}(z; \tau) \right]^2 + \frac{5}{212} \phi_{-2}(\tau) - \frac{1}{124} \phi_{-2}(\tau) - \frac{1}{72} \phi_{-2}(\tau) \phi_{02}(z; \tau) \phi_{-21}(z; \tau) + \left( \frac{1}{16} \left[ \phi_{(2\tau)}^4 \right] + \frac{5}{16} \phi_{-2}(\tau) \phi_{-2}(\tau) - \frac{35}{36} \left[ \phi_{(2\tau)}^4 \right] \phi_{-2}(\tau) - \frac{7}{6} \phi_{-2}(\tau) \phi_{-2}(\tau) \right) \left[ \phi_{-21}(z; \tau) \right]^2 \\
8C & & \frac{1}{72} \left[ \phi_{01}(z; \tau) \right]^2 + \frac{5}{36} \phi_{-2}(\tau) - \frac{1}{8} \phi_{-2}(\tau) + \frac{7}{36} \phi_{-2}(\tau) \phi_{02}(z; \tau) \phi_{-21}(z; \tau) + \left( \frac{1}{16} \phi_{-2}(\tau) - \frac{5}{16} \phi_{-2}(\tau) \phi_{-2}(\tau) + \frac{35}{36} \phi_{-2}(\tau) \phi_{-2}(\tau) - \frac{7}{6} \phi_{-2}(\tau) \phi_{-2}(\tau) \right) \left[ \phi_{-21}(z; \tau) \right]^2 \\
11AB & & \frac{1}{144} \left[ \phi_{01}(z; \tau) \right]^2 + \left( \frac{11}{72} \phi_{02}(\tau) \phi_{02}(z; \tau) \phi_{02}(\tau) \phi_{02}(\tau) + \frac{27}{4} \left[ \phi_{(2\tau)}^4 \right] - \frac{1}{4} \frac{\eta(\tau) \eta(11\tau)}{\eta(3\tau)} \right) \phi_{-21}(z; \tau) \phi_{01}(z; \tau) + \phi_{01}(z; \tau) \phi_{-21}(z; \tau) \phi_{01}(z; \tau) + \phi_{01}(z; \tau) \phi_{-21}(z; \tau) \phi_{01}(z; \tau) + \phi_{01}(z; \tau) \phi_{-21}(z; \tau) \phi_{01}(z; \tau) \\
4A & & -\frac{1}{2} \frac{\eta(\tau)^4}{\eta(2\tau)^2} B^1_{(2)} \phi_{-21}(z; \tau) \\
3B & & \frac{1}{6} \phi_{02}(\tau) \phi_{02}(z; \tau) \phi_{02}(\tau) \phi_{02}(\tau) + \frac{27}{4} \left[ \phi_{(2\tau)}^4 \right] - \frac{1}{4} \frac{\eta(\tau) \eta(11\tau)}{\eta(3\tau)} \right) \phi_{-21}(z; \tau) \phi_{01}(z; \tau) + \phi_{01}(z; \tau) \phi_{-21}(z; \tau) \phi_{01}(z; \tau) + \phi_{01}(z; \tau) \phi_{-21}(z; \tau) \phi_{01}(z; \tau) + \phi_{01}(z; \tau) \phi_{-21}(z; \tau) \phi_{01}(z; \tau) \\
4B & & -\frac{1}{2} \frac{\eta(2\tau)^2}{\eta(4\tau)^2} B^1_{(2)} \frac{\eta(\tau)^4}{\eta(2\tau)^2} \\
12A & & -\frac{1}{2} \frac{\eta(\tau)^4}{\eta(2\tau)^2} B^1_{(2)} \frac{\eta(\tau)^4}{\eta(2\tau)^2} \\
8A & & -\frac{1}{2} \frac{\eta(4\tau)^4}{\eta(2\tau)^2} B^1_{(2)} \frac{\eta(\tau)^4}{\eta(2\tau)^2} \\
20A & & -\frac{1}{2} \frac{\eta(4\tau)^4}{\eta(2\tau)^2} B^1_{(2)} \frac{\eta(\tau)^4}{\eta(2\tau)^2} \\

\end{align*}
\]
\[ A^{(1)}(n) \]

\[
\begin{array}{cccccccccccc}
& 1A & 2B & 3A & 5A & 6C & 8C & 11AB & 4A & 3B & 4B & 12A & 8A & 20A \\
\hline
n \setminus g & 0 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & 0 \\
& 1 & 32 & 0 & -4 & 2 & 0 & 0 & -1 & 8 & 2 & 0 & 2 & 0 & -2 \\
& 2 & 110 & -2 & 2 & 0 & -2 & 2 & 0 & -10 & 2 & 6 & 2 & -2 & 0 \\
& 3 & 288 & 0 & 0 & -2 & 0 & 0 & 2 & 8 & -6 & 0 & 2 & 0 & -2 \\
& 4 & 660 & 4 & -6 & 0 & -2 & 0 & 0 & -20 & 6 & -4 & -2 & 4 & 0 \\
& 5 & 1408 & 0 & 4 & -2 & 0 & 0 & 0 & 32 & 4 & 0 & -4 & 0 & 2 \\
& 6 & 2794 & -6 & 4 & 4 & 0 & -2 & 0 & -30 & -8 & 2 & 0 & 2 & 0 \\
& 7 & 5280 & 0 & -12 & 0 & 0 & 0 & 0 & 40 & 6 & 0 & -2 & 0 & 0 \\
& 8 & 9638 & 6 & 8 & -2 & 0 & -2 & 2 & -58 & 2 & -10 & 2 & -2 & 2 \\
& 9 & 16960 & 0 & 4 & 0 & 0 & 0 & 0 & -2 & 80 & -14 & 0 & 2 & 0 \\
& 10 & 29018 & -6 & -16 & -2 & 0 & 2 & 0 & -102 & 8 & 10 & 0 & 2 & -2 \\
& 11 & 48576 & 0 & 12 & 6 & 0 & 0 & 0 & 112 & 6 & 0 & -2 & 0 & 2 \\
& 12 & 79530 & 10 & 6 & 0 & -2 & 2 & 0 & -150 & -24 & -6 & 0 & 2 & 0 \\
& 13 & 127776 & 0 & -24 & -4 & 0 & 0 & 0 & 200 & 18 & 0 & 2 & 0 & 0 \\
& 14 & 202050 & -14 & 18 & 0 & -2 & -2 & 2 & -230 & 12 & 10 & 4 & 0 & 2 \\
& 15 & 314688 & 0 & 12 & -2 & 0 & 0 & 0 & 272 & -30 & 0 & 2 & 0 & 2 \\
& 16 & 483516 & 12 & -36 & 6 & 0 & 0 & 0 & -348 & 24 & -12 & 0 & -4 & 2 \\
& 17 & 733920 & 0 & 24 & 0 & 0 & 0 & 0 & 440 & 12 & 0 & -4 & 0 & 0 \\
& 18 & 1101364 & -12 & 16 & -6 & 0 & 4 & 0 & -508 & -44 & 20 & -4 & -4 & 2 \\
\end{array}
\]

\[ A^{(2)}(n) \]

\[
\begin{array}{cccccccccccc}
& 1A & 2B & 3A & 5A & 6C & 8C & 11AB & 3B \\
\hline
n \setminus g & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
& 1 & 20 & -4 & 2 & 0 & 2 & -2 & -2 & -4 \\
& 2 & 88 & 8 & -2 & -2 & 2 & 4 & 0 & 4 \\
& 3 & 220 & -12 & 4 & 0 & 0 & -6 & 0 & 4 \\
& 4 & 560 & 16 & 2 & 0 & -2 & 8 & -1 & -4 \\
& 5 & 1144 & -24 & -8 & 4 & 0 & -12 & 0 & 4 \\
& 6 & 2400 & 32 & 6 & 0 & 2 & 16 & 2 & 0 \\
& 7 & 4488 & -40 & 6 & -2 & 2 & -20 & 0 & -12 \\
& 8 & 8360 & 56 & -10 & 0 & 2 & 28 & 0 & 8 \\
& 9 & 14696 & -72 & 8 & -4 & 0 & -36 & 0 & 8 \\
& 10 & 25544 & 88 & 2 & 4 & -2 & 44 & 2 & -16 \\
& 11 & 42660 & -116 & -18 & 0 & -2 & -58 & 2 & 12 \\
& 12 & 70576 & 144 & 16 & -4 & 0 & 72 & 0 & 4 \\
& 13 & 113520 & -176 & 12 & 0 & 4 & -88 & 0 & -24 \\
& 14 & 180640 & 224 & -26 & 0 & 2 & 112 & -2 & 16 \\
& 15 & 281808 & -272 & 18 & 8 & -2 & -136 & -1 & 12 \\
& 16 & 435160 & 328 & 10 & 0 & -2 & 164 & 0 & -32 \\
& 17 & 661476 & -404 & -42 & -4 & -2 & -202 & 2 & 24 \\
& 18 & 996600 & 488 & 30 & 0 & 2 & 244 & 0 & 12 \\
\end{array}
\]

**Table 8.** The number of massive representations \( A^{(a)}(n) \).
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