Monopoles and Dyons in SU(5) Gauge Unification
(with relation to a Dual Standard Model)

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This article presents a fairly complete description of the monopoles and dyons arising from an 
SU(5) gauge unification of the standard model. Topics discussed include: the spectrum of SU(5) 
monopoles; their gauge equivalence structure; their spherical symmetry; the construction of dyons; 
global charge; the intrinsic angular momenta of scalar boson-monopole composites and monopole 
gauge excitations; and the effects of a theta vacuum. The relevance of each of these topics to 
constructing a dual description of the standard model with SU(5) solitons is discussed in detail.

I. INTRODUCTION

The concept of magnetic charge has presumably been 
around since that of magnetic field. However it was Dirac 
who discovered the magnetic monopole. He showed 
that the gauge configuration \( A_\phi \sim m (1 - \cos \theta) \) describes 
an isolated magnetic charge \( m \). Implicit in this description 
is a string singularity, which Dirac famously showed 
to be unobservable when the magnetic charge is a multi-
ple of \( 2\pi/e \).

Whilst this motivated much interest, monopoles didn’t 
achieve their present status until 't Hooft and Polyakov’s 
celebrated work. They demonstrated such monopoles 
occurred as solitons in an SU(2) gauge theory spontaneously 
broken to U(1). As particle physics is based on similar 
theories, monopoles are thus immediately relevant.

At present monopoles are mainly used in two appli-
cations: confinement and gauge unification. This article 
will concentrate on the latter of these, where monopoles 
are an inevitable consequence of unifying the fundamen-
tal gauge interactions.

Since 't Hooft and Polyakov’s work such monopoles 
have taken a wider significance within field theory. In 
particular their behaviour has a quite unparalleled rich-
ness of structure. For instance their effects include: the 
spin from isospin mechanism, leading to spin half config-
urations in a bosonic gauge theory; the problem of global 
charge, whereby some charges are not properly de-
finned around a monopole; and the effects of a theta 
vacuum, which converts pure monopoles into dyons.

Another important property of monopoles is their 
electric-magnetic duality, where a system of electric 
or magnetic charges behave identically. Consequently 
monopoles offer an alternative description of particles. 
That is, a charged particle is usually considered to source 
\( A^0 \sim e/4\pi r \), but instead the dual gauge potential (such 
that \( E = \nabla \wedge A \)) can be used to represent it as a 
monopole \( \tilde{A}_\phi \sim e (1 - \cos \theta)/4\pi \).

Whilst electric-magnetic duality is fairly established 
within Abelian theories, there are unsolved issues in 
the non-Abelian case; although certainly such a duality 
should exist. Recently, however, Chan and Tsou have 
described an exact non-Abelian electric-magnetic duality. 
They conclude the magnetic gauge symmetry is the 
same group as the electric gauge symmetry, although its 
gauge potential has the opposite parity.

The existence of this non-Abelian duality has a wider 
significance for particle physics, since it offers an alterna-
tive method for formulating the standard model. Such a 
theory would represent the properties of elementary par-
ticles by the behaviour of monopoles. This theory, as 
proposed by Liu and Vachaspati, would be a dual stan-
ard model.

It is quite possible that the construction of a dual stan-
ard model may uncover a hidden simplicity and regu-
larity of form that underlies the conventional standard 
model. Such a hidden structure could prove crucial to 
understanding the nature and origin of the elementary 
particles. Also possible is that new physics may have to 
to be included to arrive at a simple and consistent form.

A primary indication of the dual standard model’s 
structure is from a remarkable discovery by Vachaspati. 
He observed that the magnetic charges of the five sta-
bile monopoles within Georgi-Glashow SU(5) unification 
are identical to the electric charges in one generation of 
elementary particles. That is, a unification of the 
standard model’s magnetic gauge sector leads to a spec-
trum of electric monopoles whose charges precisely mimic 
one generation of elementary particles.

This lead Vachaspati to propose that perhaps the dual 
standard model could be based on the properties of SU(5) 
monopoles. In this sense the elementary particles would 
then originate as solitons from magnetic gauge unifica-
tion. If successful such a theory would offer a particu-
larly simple and elegant method for unifying the gauge 
and matter content of the standard model.

However there are more aspects to the standard model

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than just its charge spectrum; for instance, spin, confinement, mixing, particle mass, chirality and parity violation. A successful dual standard model must either incorporate or explain these properties. At present this is an ongoing task, although many of these features are naturally falling into place. For instance spin can be included through the spin from isospin mechanism [12], and confinement has a natural description as a dual Meissner effect [13]. In addition some potential explanations for other features have been proposed by Vachaspati and Steer [14] and by the author [15].

To fully investigate these effects one must have a complete understanding of the monopoles from Georgi-Glashow SU(5) → SU(3) × SU(2) × U(1)/Z6 gauge unification. However there is no review that provides a reasonably complete picture of such SU(5) monopoles. Hopefully this article will address that issue, whilst also providing some background to the dual standard model.

In several ways this article relates to Dokos and Tomaras’s original description of monopoles and dyons arising within SU(5) → SU(3) × U(1)/Z3 [14]. Their work formed the basis for much research into the implications of monopoles within grand unification. As this article relates to a different phenomenological situation it takes a different slant on the problem. Additionally there have been several important results that have occurred since their work was completed.

The composition of this article is as follows. Firstly the monopoles in SU(5) gauge unification are considered. Sec. (II) describes their spectrum and similarities to the elementary particles. Sec. (III) discusses their gauge transformation properties, finding representations compatible with the elementary particles. Sec. (IV) discusses their spatial rotation properties, motivating such monopoles are scalars. Then SU(5) dyons are described. These correspond to two types: scalar boson-monopole composites, discussed in sec. (V); and monopole gauge excitations, discussed in sec. (VI). It appears that both types of dyon may have intrinsic half-integer angular momenta. Finally, sec. (VII) describes the effects of a theta vacuum.

Before starting note that each of the three classic review articles [4] uses a slightly different convention; this article follows Preskill, in line with the work on the dual standard model.

II. SU(5) MONOPOLES

To start it is important to understand the spectrum of stable monopoles from SU(5) gauge unification. The complete spectrum was first obtained by Gardner and Harvey [13], who showed there are precisely five stable monopoles. More recently it has observed that these have magnetic charges in coincidence with the electric charges in one generation of elementary particles [11]. This motivates that perhaps the elementary particles originate as monopoles from a magnetic gauge unification of the standard model.

The Georgi-Glashow SU(5) gauge unification can be described through the following symmetry breaking [49]

$$SU(5) \rightarrow H_{SM} = [SU(3)_C \times SU(2)_L \times U(1)_Y] / Z_6.$$  \hspace{1cm} (1)

This symmetry breaking is achieved through condensation of an adjoint scalar field $\Phi$; then with respect to the vacuum $\Phi_0 = i \epsilon \text{diag}(\frac{1}{2}, \frac{1}{2}, -1, -\frac{1}{2}, -\frac{1}{2})$ the standard model gauge symmetry is contained within SU(5) as

$$\left( \begin{array}{cc} SU(3)_C & 0 \\ 0 & SU(2)_L \end{array} \right) \times U(1)_Y.$$  \hspace{1cm} (2)

with $U(1)_Y$ along the diagonal. In the breaking $\Phi_0$ a feature to bear in mind is the discrete $Z_6$ quotient: this represents an intersection between the colour-isospin and the hypercharge parts of (1). Then, since $Z_6$ is included twice in (1) but only once in SU(5), it divides out in (1).

That monopoles occur is implied by the non-trivial topology of (1), where each distinct monopole corresponds to a second homotopy class

$$\pi_2 \left( \frac{SU(5)}{H_{SM}} \right) \cong \pi_1 (H_{SM}).$$  \hspace{1cm} (3)

Within this topology the $Z_6$ quotient in (1) determines the basic pattern of monopoles.

To find the monopole spectrum it is convenient to associate each monopole with a magnetic generator $M$

$$\Phi \sim \Phi_0, \quad B \sim \frac{1}{2g} \frac{\hat{r}}{r^2} M.$$  \hspace{1cm} (4)

This is in a unitary gauge, so there is an implicit Dirac string in the gauge potential. To have a well-defined solution this Dirac string must be a gauge artifact, which constrains $M$ through a topological quantisation [20]

$$\exp(i2\pi M) = 1.$$  \hspace{1cm} (5)

As such $M$ has integer eigenvalues. Additionally, a finite energy monopole has a massless long range magnetic field; hence $M$ is a generator of $H_{SM}$.

The individual colour, isospin and hypercharge magnetic charges are defined by a gauge choice that the monopole’s magnetic field takes the form

$$B = T_C B_C + T_C T_C B_C + T_l B_l + T_Y B_Y,$$  \hspace{1cm} (6)

with generators to be defined below. Then the magnetic charges are

$$M = m_C T_C + m_C T_C T_C + m_l T_l + m_Y T_Y.$$  \hspace{1cm} (7)

Within this definition care must be taken with the normalisation of each $T$. To ease later sections of this review
a normalisation $\text{tr} \, T^2 = 1$ is taken, which is slightly different from that used in similar papers. In that case a suitable choice of generators is
\begin{align*}
T_C &= \sqrt{3/2} \lambda_8 = \sqrt{3/2} \text{diag}(-\frac{1}{3}, -\frac{1}{3}, \frac{2}{3}, 0, 0), \\
T_C' &= \frac{1}{\sqrt{2}} \lambda_3 = \frac{1}{\sqrt{2}} \text{diag}(1, -1, 0, 0, 0), \\
T_I &= \sqrt{\frac{1}{2}} \sigma_3 = \frac{1}{\sqrt{2}} \text{diag}(0, 0, 0, -1, -1), \\
T_Y &= \frac{\sqrt{3}}{3} \sigma_0 = \frac{\sqrt{3}}{3} \text{diag}(1, 1, 1, -\frac{2}{3}, -\frac{2}{3}),
\end{align*}
which for convenience are taken to be diagonal and consistent with (3).

The calculation of $M$ for each monopole then becomes a determination of which sets $(m_C, m_C', m_I, m_Y)$ solve (4); this leads straightforwardly to the following magnetic charges for the first six homotopy classes of $\pi_1(H_{SM})$ [13]:

| $\pi_1$ | $m_C$ | $m_I$ | $m_Y$ | diag $M$ | $n_C$ | $n_I$ | $n_Y$ |
|---------|-------|-------|-------|----------|-------|-------|-------|
| 1       | $\sqrt{\frac{3}{2}}$ | $\sqrt{\frac{3}{2}}$ | $\sqrt{\frac{3}{2}}$ | $(0, 0, 1, -1, 0)$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| 2       | $\sqrt{\frac{3}{2}}$ | $0$ | $\sqrt{\frac{3}{2}}$ | $(0, 1, 1, -1, -1)$ | $-1$ | $0$ | $\frac{1}{2}$ |
| 3       | $0$ | $-\sqrt{\frac{3}{2}}$ | $\sqrt{\frac{3}{2}}$ | $(1, 1, 1, -2, 1)$ | $0$ | $\frac{1}{2}$ | $1$ |
| 4       | $\sqrt{\frac{3}{2}}$ | $0$ | $\frac{\sqrt{3}}{2}$ | $(1, 1, 2, -2, 1)$ | $1$ | $0$ | $\frac{1}{2}$ |
| 5       | $0$ | $0$ | $\sqrt{\frac{3}{2}}$ | $(2, 2, 2, -3, 3)$ | $0$ | $0$ | $2$ |

A few remarks are in order about these monopoles:
(i) To simplify the comparison with the elementary particles their charges are expressed in a basis with simpler normalisation; having $n_C = \sqrt{3/2} m_C$, $n_I = \sqrt{2} m_C$ and $n_Y = \sqrt{2} m_Y$. These normalisations play a central role in an associated gauge unification [23].
(ii) Gardner and Harvey have shown that only the above five monopoles are stable for a wide parameter range [13]; with the 5 and the $n \geq 7$ monopoles unstable to fragmentation. Further non-topological charge may be added to the above monopoles, for instance taking $m_C' = m_C + 3$ or $m_I = m_I + 2$, by Brandt and Neri’s stability analysis these are unstable to long range gauge perturbations [23].
(iii) Splitting $Z_6$ into colour and isospin factors $Z_3 \times Z_2$ exhibits the connection between the topology and the monopole spectrum: giving a 1, −1, 0, · · · periodicity of $n_C$ and the $\frac{1}{2}, 0, \cdot \cdot \cdot$ periodicity of $n_I$.
(iv) For a monopole with non-zero colour there are in fact a triplet of $(m_C, m_C')$ colour charges $(0, \sqrt{3})$ and $(\pm \sqrt{\frac{3}{2}}, -\sqrt{\frac{3}{2}})$. In addition each monopole with non-zero isospin is a member of a doublet $m_1 = \pm \sqrt{\frac{1}{2}}$. All these values are completely in line with them forming colour and isospin gauge multiplets.

Table 1 is the central result of this section. From it Vachaspati made the following key observation: The magnetic charges of the five stable SU(5) monopoles are in complete accordance with the electric charges of the five quark and lepton multiplets. To make the correspondence explicit each monopole is labeled by a particle multiplet in the lightest generation. The spectrum is completed by identifying the anti-particles with anti-monopoles.

This observation is remarkable. It is difficult to believe that it is just a coincidence; the charges identify exactly and by some miracle all monopoles not in correspondence are unstable. This suggests a deep connection between the non-perturbative features of the grand unified theory and the elementary particle spectrum of the standard model. As Vachaspati conjectured: This correspondence suggests that perhaps grand unification should be based on a magnetic SU(5) symmetry group with only a bosonic sector and the presently observed fermions are really the monopoles of that theory.

III. GAUGE FREEDOM

It has long been believed that there is a duality between non-Abelian electric particles and magnetic monopoles [23]. This is of primary importance to the particle-monopole correspondence described in sec. (I). There the magnetic charges of SU(5) monopoles identify with the electric charges of the elementary particles, which suggests their interactions should also be the same. That is, their interactions appear to be dual.

Recently, Chan and Tsou have discovered an exact non-Abelian electric-magnetic duality transformation [9]. Within this they found that the magnetic gauge symmetry has the same group structure as the electric sector, though the gauge field is of opposite parity. This supports that the SU(5) monopoles in table 1 interact under the standard model gauge symmetry $H_{SM}$ with the associated charges.

This section examines the monopole’s gauge freedom under the $H_{SM}$ gauge symmetry and compares it to the gauge freedom of the elementary particles under the same gauge group. By constructing the relevant orbits, the global gauge freedom is shown to be precisely the same. It should be noted that the following arguments represent an improved version of those in ref. [24].

A. Gauge Freedom of the $(u, d)$ Monopole

A simple method for demonstrating that the $(u, d)$ monopole and particle multiplet have the same $H_{SM}$ gauge freedom is to examine their gauge orbits. These orbits consist of a collection of states that are rigidly gauge equivalent to one another, so their geometry is characteristic of the gauge freedom. The particle-monopole correspondence is demonstrated upon showing their gauge orbits are the same.

To illustrate the concept of a gauge orbit consider firstly the $(u, d)$ gauge multiplet, which is a tensor product of a colour 3 triplet and a weak isospin 2 doublet
with a hypercharge phase. Its gauge orbit is generated by acting $H_{\text{SM}}$ upon a typical value, say $q_{ij} = \delta_{i1}\delta_{j1}$, which gives

$$O_{(u,d)} = H_{\text{SM}} \cdot q \cong \frac{H_{\text{SM}}}{C(q)},$$

where $C(q)$ leaves $q$ invariant

$$C(q) = \text{SU}(2)_C \times \text{U}(1)_{Y-1} \times \text{U}(1)_{1+Y-2C}/\mathbb{Z}_2.$$  \hspace{1cm} \text{(10)}$$

The appropriate embedding of $C(q)$ is indicated by its colour, isospin and hypercharge subscripts.

Now the task is to describe the $(u,d)$ monopole’s gauge freedom. Just as with the $(H)$ freedom is determined by the action of a set of $H_{\text{SM}}$ rigid gauge transformations, which collectively generate the monopole’s gauge orbit.

For describing the $(u,d)$ monopole’s gauge orbit it will be convenient to express the monopole in a gauge free of the Dirac string. Such a gauge is the radial gauge, where

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the asymptotics are simply those of an SU(2) ‘t Hooft-Polyakov Ansatz embedded within SU(5)

$$\Phi(r) \sim \text{Ad}(\Omega(\hat{r}))\Phi_0, \quad B(r) \sim \frac{1}{2g} \hat{r} \cdot \hat{T}.$$ \hspace{1cm} \text{(11)}$$

Here $\Omega(\hat{r}) = e^{-i\varphi T_3/2}e^{i\delta T_2/2}e^{-i\varphi T_3/2}$ describes the angular behaviour, which is specified by a set of su(2) Pauli matrices embedded in su(5)

$$\hat{T} = \begin{pmatrix} 0 & 2 \\ \sigma & 0 \end{pmatrix}.$$ \hspace{1cm} \text{(12)}$$

For consistency with (4) in the unitary gauge the magnetic generator $M$ equals $T_3$.

Because both the scalar and magnetic field are in the adjoint representation the action of $H_{\text{SM}}$ has a rather simple form upon (11): simply taking $\hat{T} \mapsto \text{Ad}(h) \hat{T}$. This can be interpreted as rigidly moving the monopole through a set of gauge equivalent embeddings. The collection of these form the gauge orbit

$$O_1 \cong \frac{H_{\text{SM}}}{C(T)}.$$ \hspace{1cm} \text{(13)}$$

where $C(T)$ is the subgroup that leaves all three generators $T_i$ invariant

$$C(T) = \text{SU}(2)_C \times \text{U}(1)_{Y-1} \times \text{U}(1)_{1+Y-2C}/\mathbb{Z}_2.$$ \hspace{1cm} \text{(14)}$$

Clearly this is the same as (10) for the $(u,d)$ gauge multiplet. Therefore the gauge multiplet and monopole have a compatible gauge freedom under $H_{\text{SM}}$.

B. Gauge Freedom of the Other Monopoles

The analysis of the $(u,d)$ monopole’s $H_{\text{SM}}$ gauge freedom was fairly simple because it is essentially an SU(2) ‘t Hooft-Polyakov monopole embedded in SU(5). Unfortunately this is not the case for the other monopoles, which complicates the analogous calculation.

It is interesting to note, however, that the other SU(5) monopoles are uncharged under either colour, isospin or both. This suggests that each monopole can be approximated within an effective symmetry breaking that includes only those symmetries relevant to their charges. Within these effective theories it then seems reasonable to use the approach of the $(u,d)$ monopole.

In some sense this assumes that there is a subset of the gauge freedom that is relevant to the long range magnetic monopole interactions. Although this approach has not been rigorously justified it does seem reasonable. Also, crucially, it yields the desired correspondence between monopoles and gauge multiplets.

1. Gauge Freedom of the $u$ and $\bar{d}$ Monopoles

Both the $u$ and $\bar{d}$ multiplets form colour 3 triplets, with some hypercharge. Analogously to (10) their gauge orbits are therefore

$$O_u = O_{\bar{d}} \cong \frac{\text{SU}(3)_C \times \text{U}(1)_{Y}/\mathbb{Z}_3}{\text{SU}(2)_C \times \text{U}(1)_{Y-2C}/\mathbb{Z}_2}.$$ \hspace{1cm} \text{(15)}$$

Approximating the $u$ and $\bar{d}$ monopoles by ones with the same magnetic charges in the symmetry breaking $\text{SU}(4) \rightarrow \text{SU}(3)_C \times \text{U}(1)_{Y}/\mathbb{Z}_3$ leads to the gauge orbits

$$O_2 = O_4 \cong \frac{\text{SU}(3)_C \times \text{U}(1)_{Y}/\mathbb{Z}_3}{\text{SU}(2)_C \times \text{U}(1)_{Y-2C}/\mathbb{Z}_2}.$$ \hspace{1cm} \text{(16)}$$

Therefore the gauge orbits of the $u$ and $\bar{d}$ monopoles and multiplets are the same.

2. Gauge Freedom of the $(\bar{\nu}, \bar{\nu})$ Monopole

The $(\bar{\nu}, \bar{\nu})$ multiplet transform as an isospin 2 doublet with some hypercharge. Therefore its gauge orbit is

$$O_{(\bar{\nu}, \bar{\nu})} \cong \frac{\text{SU}(2)_I \times \text{U}(1)_{Y}/\mathbb{Z}_2}{\text{U}(1)_Q},$$ \hspace{1cm} \text{(17)}$$

where $\text{U}(1)_Q$ lies diagonally between the isospin and hypercharge groups. Note an interesting equivalence between this gauge orbit and the electroweak vacuum manifold; this occurs because the associated scalar doublet has the same representation as $(\bar{\nu}, \bar{\nu})$.

Approximating the $(\bar{\nu}, \bar{\nu})$ monopole by one with the same magnetic charges in $\text{SU}(3) \rightarrow \text{SU}(2)_I \times \text{U}(1)_{Y}/\mathbb{Z}_2$ then leads to a gauge orbit
\[ \mathcal{O}_3 \cong \frac{\text{SU}(2)_L \times \text{U}(1)_Y}{\mathbb{Z}_2}, \]

which is the same as the \((\bar{\nu}, \bar{e})\) multiplet.

3. Gauge Freedom of the \(\bar{e}\) Monopole

As the \(\bar{e}\) particle is only charged under hypercharge it has the rather trivial gauge orbit
\[ \mathcal{O}_{\bar{e}} \cong \text{U}(1)_Y. \]

Similarly, the \(\bar{e}\) monopole is approximated by one with the same magnetic charge in \(\text{SU}(2) \rightarrow \text{U}(1)_Y\); giving a gauge orbit
\[ \mathcal{O}_0 \cong \text{U}(1)_Y. \]

This is the same as the \(\bar{e}\) particle.

IV. SPHERICAL SYMMETRY

It is important to understand that all of the monopoles in Georgi-Glashow \(\text{SU}(5)\) gauge unification have no intrinsic angular momentum.

When constructing a dual standard model from \(\text{SU}(5)\) monopoles this appears somewhat problematic because all elementary particles have spin. However, what this is really saying is that \(\text{SU}(5)\) gauge unification on its own is insufficient to produce a consistent dual standard model. Because of this it is desirable to understand the angular momentum properties of \(\text{SU}(5)\) monopoles; with these properties hopefully indicating where to proceed.

When does a monopole have no intrinsic angular momentum? The accepted understanding \(^{[25]}\) is that they are spherically symmetric under
\[ \vec{R} = \vec{L} + \vec{T}, \]  

where \(\vec{L} = -\vec{r} \wedge \vec{\nabla}\) generates spatial rotations and \(\vec{T}\) generates some \(\text{SU}(2)\) subgroup of \(\text{SU}(5)\). This is saying that a spatial rotation of a spherically symmetric monopole can always be undone through an internal gauge rotation of \(\text{SU}(2)\). That is, every spatial rotation \(S\) is equivalent to an internal rotation \(\Omega(S) \in \text{SU}(2)\)
\[ \Phi(r) = \text{Ad}[\Omega(S)] \Phi(S^{-1}r), \]
\[ B_i(r) = \text{Ad}[\Omega(S)] S_{ij}B_j(S^{-1}r). \]

Here \(S(\alpha, \beta, \gamma)\) can be defined through its Euler angles, in which case \(\Omega(S) = e^{-i\alpha T_3/2}e^{-i\beta T_3/2}e^{-i\gamma T_2/2}\).

The spherical symmetry of each \(\text{SU}(5)\) monopole in table \(\text{I}\) is now examined using methods first developed by Wilkinson and Goldhaber \(^{[25]}\). Some of this treatment relates to ref. \(^{[26]}\).

A. Spherically Symmetric \((u, d)\) Monopoles

A simple illustration of spherical symmetry is provided by the \((u, d)\) monopole. The task is to show \(^{[22][23]}\) holds.

Fortunately this monopole has already been expressed in a spherically symmetric gauge: the radial gauge \(^{[11]}\)
\[ \Phi(r) \sim \text{Ad}[\Omega(\hat{r})]\Phi_0, \]
\[ B_i(r) \sim \frac{1}{2g} \frac{\hat{r}}{r^2} \text{Ad}[\Omega(\hat{r})]M, \]
with \(\Omega(\hat{r}) = e^{-i\phi T_3/2}e^{-i\theta T_3/2}e^{-i\gamma T_2/2}\) and \(\vec{T}\) defined in \(^{[12]}\). Then the action of \(e^{iT_3X}\) upon \(^{[24]}\) is equivalent to
\[ \Omega(\hat{r}) \rightarrow e^{-iT_3X} \Omega(\hat{r}) e^{iT_3X}, \]
The point being that this takes \(\varphi \rightarrow \varphi + \chi\), which is a spatial rotation around the \(z\)-axis. The demonstration of spherical symmetry is then completed by a similar calculation about any other axis.

B. Spherical Symmetry of the Other Monopoles

Unlike the \((u, d)\) monopole the other \(\text{SU}(5)\) monopoles in table \(\text{I}\) are generally fairly complicated. Fortunately there are some simple criteria for describing their spherical symmetry.

Wilkinson and Goldhaber have constructed a general set of spherically symmetric monopoles that satisfy \(^{[22][23]}\) for magnetic generators that decompose into \(^{[25]}\)
\[ \frac{1}{2} M = I_3 - T_3. \]
Here \(I_3\) and \(T_3\) are elements of two \(\text{su}(2)\) algebras, whose generators \(\vec{I}\) and \(\vec{T}\) are constrained under
\[ [\vec{I}, \Phi_0] = 0, \quad [\vec{I}, M] = 0. \]
These criteria describe the spherical symmetry of all \(\text{SU}(5)\) monopoles, in a necessary and sufficient way \(^{[24]}\).

Before discussing these monopoles it will be useful to quickly interpret the meaning of these conditions:
(i) That \(\vec{I}\) commutes with \(\Phi_0\) specifies the embedding of the associated \(\text{SU}(2)\) group to be contained within the residual symmetry.
(ii) The second constraint is a little more subtle. Later it will be revealed that this allows the generators \(\vec{I}\) to be globally defined around the monopole.

To see how this spherical symmetry emerges it will be necessary to construct the asymptotic form of the monopoles satisfying \(^{[24]}\) and \(^{[27]}\). The key point is that there is then a gauge transformation that takes the unitary gauge configuration \(^{[11]}\) to a non-singular radial form. In that gauge
\[ \Phi(r) \sim \text{Ad}[\Lambda(\hat{r})]\Phi_0, \]
\[ B_i(r) \sim \frac{1}{2g} \frac{\hat{r}}{r^2} \text{Ad}[\Lambda(\hat{r})]M, \]
with \( \Lambda(\hat{r}) = \Omega(\hat{r}) \omega^{-1}(\hat{r}) \), where \( \Omega(\hat{r}) \) is as \( \Box \) and \( \omega(\hat{r}) \) is similarly defined in \( I \) as \( e^{-i\phi I_3/2} e^{i\theta I_2/2} e^{-i\phi I_3/2} \).

Although, by \( \Box \), \( \omega(\hat{r}) \) acts trivially upon both \( \Phi_0 \) and \( M \) it is useful for constructing features relating to the angular momentum. For the time being note that \( \Lambda \) could be replaced by \( \Omega \) in \( \Box \) if desired.

That \( \Box \) is symmetric can be seen through the relation \( \text{Ad}[\Omega(S)]\Phi(\hat{r}) = \text{Ad}[\Lambda(S\hat{r})]\Phi(\hat{r}) \),\( \Box \), which holds providing \( X \) commutes with \( M \) \( \Box \). Thus \( \Omega(S) \) acts on the asymptotic fields \( \Box \) as

\[
\Phi(\hat{r}) \rightarrow \text{Ad}[\Omega(S)]\Phi(\hat{r}) = \text{Ad}[\Lambda(S\hat{r})]\Phi(\hat{r}),
\]

\( B(\hat{r}) \rightarrow \text{Ad}[\Omega(S)]B(\hat{r}) = \frac{1}{2g \hat{r}^2} \text{Ad}[\Lambda(S\hat{r})]M; \]

satisfying the definition \( \Box \) of spherical symmetry.

It is now a fairly simple task to construct the relevant generators of the spherically symmetric SU(5) monopoles. Using table \( \Box \) and some trial and error convinces one that the generators \( I_3 \) and \( T_3 \) for a monopole with magnetic generator \( M \) are:

**TABLE II. Generators of spherically symmetric monopoles.**

| \( \mathbf{u} \) | \( \mathbf{d} \) | \( \mathbf{\bar{u}} \) | \( \mathbf{\bar{d}} \) |
|---|---|---|---|
| \( (0, 0, \frac{1}{2}, \frac{1}{2}, 0) \) | \( (0, \frac{1}{2}, \frac{1}{2}, 0, -1) \) | \( (1, 1, 1, -1, 1) \) | \( (\frac{3}{2}, \frac{3}{2}, 0, 0, 0) \) |

However, there appears to be a problem with the \( u \) monopole: it’s \( T_3 \) eigenvalues \( (0, 1, 1, 1, 1) \) do not correspond to an SU(2) representation, which violates the condition below \( \Box \). Because of this the \( u \) monopole is not spherically symmetric.

What does this imply about the \( u \) monopole’s structure? Note that the solution almost appears to be spherically symmetric; for instance if the SU(5) group is enlarged to SU(6) then \( T_3 \) becomes an SU(2) generator. The most likely implication is a small magnetic dipole moment on the \( u \) monopole. Certainly the loss of spherical symmetry is qualitatively different to the angular momentum discussed in the latter parts of this article. However further study is required to fully elucidate the \( u \) monopole’s form.

**C. Spherically Symmetric Monopoles with Non-Topological Charge**

In addition to the above monopoles one can also consider their counterparts with additional non-topological charge. These are generally expected to be of higher energy, because they have larger magnetic charges.

Some examples of spherically symmetric monopoles having extra non-topological colour charge are:

**TABLE III. Non-Topological Monopole Charges.**

| \( \mathbf{u}^* \) | \( \mathbf{u}^{**} \) | \( \mathbf{e}^* \) | \( \mathbf{e}^{**} \) |
|---|---|---|---|
| \( (1, 1, 0, -1, 1) \) | \( (0, 0, 2, -1, 1) \) | \( (\frac{1}{2}, 0, 0, -1, \frac{1}{2}) \) | \( (\frac{3}{2}, 0, 0, -1, \frac{1}{2}) \) |

There are in fact an infinite tower of non-topological magnetic charges on each monopole. Many of these will be spherically symmetric.

**V. SU(5) DYONS**

The rest of this article is mainly concerned with SU(5) dyons, which have electric charge in addition to their original magnetic charge. This charge is crucial to their nature and dynamics; for instance many of these dyons have an intrinsic angular momentum. In some cases this angular momentum is half integer.

That these dyons may have angular momenta of one-half is really very encouraging for the construction of a dual standard model from SU(5) solitons. As the SU(5) monopoles stand they cannot be dual to the elementary particles because they have no intrinsic angular momenta; but by combining them with electric charges the required solitons can be constructed. Indeed it has been shown that each SU(5) monopole has a dyonic counterpart with one-half angular momentum \( \Box \); these also have interesting duality properties \( \Box \).

Thus, for constructing a dual standard model, an important consideration is the spectrum and properties of SU(5) dyons. This motivates a detailed discussion of these dyons, which naturally splits into three parts:

(i) **Scalar boson-monopole composites:** this is the subject of sec. \( \Box \) and is conceptually the simplest case. The inclusion of extra scalar fields allows their quanta to bind to the monopoles, giving dyons. Many of these have intrinsic angular momentum. It is these dyons that have been discussed in refs. \( \Box \).

(ii) **Monopole gauge excitations:** these are the subject of sec. \( \Box \) and arise from the monopole’s gauge freedom. Essentially an electric field is produced by internal motion of the monopole through its gauge equivalent states. Again these dyons appear to have angular momentum.

(iii) **Effects of a \( \theta \) vacuum:** this is the subject of sec. \( \Box \). A theta vacuum changes the definition of the electric field from the Noether charges, which effectively induces electric charge on the monopole. Whilst this induced charge does not induce angular momentum it does have a central effect on the nature of the dyon spectrum.
Much of the discussion in these three sections relies on similar background material. This is discussed over the next few subsection and hopefully provides a useful introduction to the properties of dyons.

A. Dyon Configurations

The first topic of concern is the definition of the electric and magnetic fields around a dyon. These may be considered in a unitary gauge

$$\Phi \sim \Phi_0, \quad E \sim g \frac{\vec{r}}{4\pi r^2} Q, \quad B \sim \frac{1}{2g} \frac{\vec{r}}{r^2} M.$$  (31)

Particular attention is paid to the magnetic and electric charge normalisations, which will be important later.

Before starting note the unitary gauge configuration $[31]$ has a Dirac string, which must be unobservable by the electric charge to be well defined. This constrains $Q$ and $M$ through a non-Abelian Dirac condition

$$\text{tr} Q M \in \mathbb{Z},$$  (32)

which follows from projecting the non-Abelian theory into the Abelian $U(1)_M$ subtheory.

1. Magnetic Charge

As discussed in sec. $[30]$, the magnetic generator is defined through the condition $\exp(i2\pi M) = 1$. This specifies the individual magnetic charges

$$M = m_C T_C + m_{C'} T_{C'} + m_1 T_1 + m_Y T_Y,$$  (33)

with respect to properly normalised colour, isospin and hypercharge generators $[30]$. The relevance of these magnetic charges can be seen by extracting the individual magnetic fields from the full magnetic field $B = B_C T_C + B_{C'} T_{C'} + B_1 T_1 + B_Y T_Y$.

Of particular interest are the monopole’s colour and isospin magnetic charges. In the normalisation of $[30]$ these take the non-zero values

$$(m_C, m_{C'}) = \{-\sqrt{\frac{1}{3}}, \pm \sqrt{\frac{2}{3}}\}, \quad (\sqrt{2}, 0), \quad m_1 = \pm \sqrt{\frac{2}{5}}.$$

Note that these values are identical to the eigenvalues of the colour and isospin generators, for example in table $[13]$ below. This is the principal reason for taking this normalisation.

2. Electric Charge

To treat electric and magnetic charge on a similar footing the dyon’s electric generator is defined as

$$Q = q_C T_C + q_{C'} T_{C'} + q_1 T_1 + q_Y T_Y,$$  (34)

with each $q$ an individual electric charge. In the above normalisation to $\text{tr} T^2 = 1$ the electric charges are related to the Noether current in the usual way; with each charge an eigenvalue of the relevant generator.

For example a fundamental 5 scalar field $H$ has electric charges that are eigenvalues of $T_C, T_{C'}, T_1$ and $T_Y$:

|       | $(q_C, q_{C'})$ | $q_1$ | $q_Y$ | diag $Q_i$ |
|-------|-----------------|-------|-------|------------|
| $H_1$ | $(-\sqrt{\frac{1}{3}}, \sqrt{\frac{2}{3}})$ | 0     | $\sqrt{\frac{3}{5}}$ | $(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$ |
| $H_2$ | $(-\sqrt{\frac{1}{3}}, -\sqrt{\frac{2}{3}})$ | 0     | $\sqrt{\frac{3}{5}}$ | $(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$ |
| $H_3$ | $(\sqrt{\frac{2}{3}}, 0)$            | 0     | $\sqrt{\frac{3}{5}}$ | $(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$ |
| $H_4$ | 0                            | $\sqrt{\frac{1}{2}}$ | $\frac{3}{2} \sqrt{\frac{2}{5}}$ | $(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$ |
| $H_5$ | 0                            | $-\sqrt{\frac{1}{2}}$ | $\frac{3}{2} \sqrt{\frac{2}{5}}$ | $(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$ |

The electric generator of a scalar boson associated with the $H_i$ component is labelled $Q_i$. Around such a boson is the electric field

$$E = g \frac{\vec{r}}{4\pi r^2} Q_i.$$  (35)

This relates to the classical limit of the Noether current, which for a single stationary point charge is $j^0_0 = Q \delta^3(r)$.

Note that the numerical values of these electric charges are the same as the magnetic charges of the monopoles. As stated above, this is due to the normalisation in $[30]$.

B. Global Charge

An important, and initially unexpected, feature of the above dyons $[32]$ is that generally their electric $Q$ and magnetic $M$ generators are not independent. This feature is central to determining the SU(5) dyon spectrum.

The reason for this dependence between $Q$ and $M$ is a rather elegant property of gauge theories, where an electric charge may not be globally defined around a non-Abelian monopole $[31]$. It transpires there is a topological obstruction to defining such electric charge.

A way to appreciate this global charge is to consider how the gauge field is patched around a monopole $[28]$. In the unitary gauge the asymptotic magnetic field $\frac{r}{2g} \frac{\vec{r}}{r} M \varphi / r \sin \vartheta$ can be defined by two different gauge potentials,

$$A_N \sim \frac{1}{2g} \frac{1 - \cos \vartheta}{r \sin \vartheta} M \varphi,$$  (36)

with a Dirac string along the negative $z$-axis, or

$$A_S \sim \frac{1}{2g} \frac{1 + \cos \vartheta}{r \sin \vartheta} M \varphi,$$  (37)
with a Dirac string along the positive z-axis. As these potentials define the same monopole they are gauge equivalent under \( h(\varphi) = \exp(i M \varphi) \).

The point is that a monopole can be well defined everywhere, with no string, in the unitary gauge providing the gauge field is patched around the monopole \([29]\). Then asymptotic space is split into two patches, say the north and south hemispheres, with \( A_N^Y \) and \( A_S^Y \) well defined on their respective patches. The constraint that these two patches can be joined together by a well-defined gauge transformation implies that \( h(2\pi) = 1 \); yielding the topological quantisation \([3]\).

What happens if there is an electric charge? Then there is an additional component to the gauge field

\[
A_0^N \sim \frac{g}{4\pi r} Q. \tag{38}
\]

But this is only consistent with the patching if on the southern hemisphere \( A_0^S = A_0^N \), which has a string singularity along the negative z-axis unless

\[
\{Q, M\} = 0. \tag{39}
\]

In other words an electric charge is only allowed on a magnetic monopole if the electric generator is an element of an ‘allowed’ symmetry group. This group acts trivially on \( M \), otherwise the dyon will not have a well defined gauge field everywhere around it \([3]\).

For each of the monopoles these allowed symmetry groups are given in table \( V \) below.

| \( H_A \) | \( (u, d) \) | \( u \) | \( \bar{\nu}, \bar{e} \) |
| --- | --- | --- | --- |
| \( U(1)_C \times U(1)_Y \times SU(2)_C \times \mathbb{Z}_{12} \) | \( U(1)_C \times SU(2)_I \times U(1)_Y \times SU(2)_C \times \mathbb{Z}_{12} \) | \( SU(3)_C \times U(1)_Y \times SU(2)_C \times \mathbb{Z}_{12} \) | \( SU(3)_C \times SU(2)_I \times U(1)_Y \times \mathbb{Z}_{12} \) |

In each of these there is a quotient by a finite intersection whose details are not central here.

It is important to realise that this non-definition of global charge goes deeper. Even the notion of gauge invariance can fail outside \( H_A \), which effectively restricts the charge-monopole gauge interactions to within this group \([3]\).

### C. Angular Momentum

An important property of these electrically and magnetically charged dyons is that many appear to have non-trivial intrinsic angular momenta. This section describes this phenomenon from a classical perspective for dyons composed of separate electric and magnetic charges. Other methods will be used later for evaluating this angular momenta; all of which are consistent with the results below.

One way of understanding a dyon’s intrinsic angular momentum is through its non-Abelian electric-magnetic field. Strictly speaking these methods are only relevant for dyons whose individual components have no angular momenta; indeed care should be taken when applying them to more complicated situations. The intrinsic angular momentum is derived from the non-Abelian generalisation of the Poynting vector, which gives

\[
J = \int d^3r \text{tr}[r \land (E \land B)]. \tag{40}
\]

Evaluating this for a dyon with magnetic charge centred at the origin and electric charge at \( r \) yields

\[
J = \frac{1}{2} \text{tr}QM \hat{r}. \tag{41}
\]

A simple proof of \((41)\) has been presented by Goddard and Olive \([17]\). Considering the components of \((41)\) gives

\[
J_i = \int d^3r \text{tr}[E_j(\delta_{ij} - x_i x_j)\frac{M}{2g^2}] = \int d^3r \text{tr}[\partial_x E_j x_i \frac{M}{2g^2}] = -\int d^3r \text{tr}[\nabla \cdot E x_i \frac{M}{2g^2}], \tag{42}
\]

which results in \((41)\).

### D. Angular Momentum and Statistics

Application of the Dirac condition \((32)\) to the angular momentum \((41)\) implies that \( J \) must either be an integer or half-integer. Thus, providing that the usual relation between spin and statistics holds, one might expect some dyons to be fermionic even though the constituents are bosonic. That this is indeed the case was demonstrated by Goldhaber \([5]\).

To see this consider two dyons \((Q, M)\) at \( \pm x \) moving with velocities \( \pm v \). Then the current-gauge interaction is

\[
\mathcal{H}_{\text{int}} = -\text{tr}[(Qv) \cdot (A(x) - A(-x))], \tag{43}
\]

with a gauge potential as in \((36)\). At first sight this is a very complicated, velocity dependent, interaction. However it can be simplified by noting

\[
\text{tr} Q(A_\varphi(\vartheta, \varphi) - A_\varphi(\vartheta, \varphi + \pi)) = \frac{1}{ig}(\partial_\varphi \Omega)\Omega^{-1}, \tag{44}
\]

where \( \Omega = \exp(i\varphi \text{tr} QM) \).

Thus if one considers the two-dyon wavefunction \( \Psi(x) \), then the complicated interaction \((43)\) can be removed by a gauge transformation.
\[ \Psi(x) \rightarrow \Omega(x)\Psi(x). \] (45)

Then upon interchange of the dyons \( x \rightarrow -x \), this gauge transformation effects the phase of the wavefunction

\[ \Omega(-x) = \Omega(\varphi + \pi) = \exp(i\pi \text{tr} Q M)\Omega(x). \] (46)

Hence the usual connection between spin and statistics is obtained, where for integer/half-integer \( J \) the wavefunction is symmetric/antisymmetric upon dyon interchange.

**VI. DYONIC SCALAR BOSON-MONOPOLE COMPOSITES**

A simple method for constructing dyons from SU(5) monopoles is to form composites with electric scalar bosons. Although their angular momenta can be evaluated from their electric-magnetic fields \( \{\Omega\} \), it will be useful to examine the nature of the resulting solitons from a semi-classical viewpoint.

To be specific the scalar bosons are taken to be quanta of a 5 scalar field \( H \). Such a field is directly relevant to SU(5) grand unification, since it is generally taken to contain the isospin doublet responsible for electroweak symmetry breaking. In this context, Lykken and Strominger demonstrated the \( \{u, d\} \) monopole to have dyonic counterparts with one-half angular momentum \[29\].

Following their analysis, Vachaspati has examined the angular momenta of the scalar boson-monopole composites in the context of a dual standard model. Using the classical methods of sec. \( \{V C\} \), he verified that each SU(5) monopole in table \( \{I\} \) has a dyonic counterpart with one-half angular momentum. This gives strong support for using such dyons to construct a dual formulation of the standard model.

This section discusses these dyons from the semi-classical perspective of 't Hooft and Hasenfratz. Then the properties of the dyons are described by the quantum mechanics of the scalar bosons in the classical SU(5) monopole background. The appropriate Hamiltonian is specified by the monopole’s gauge potential

\[ \hat{H} = (1/2m)D^2 + V(r), \quad D = i \nabla + A(r) \] (47)

and acts upon the five-component scalar field \( H \).

An important feature, which will be discussed again in sec. \( \{V I\} \), it that a binding potential \( V(r) \) is required between the charge and monopole. For simplicity this is taken to be spherically symmetric. It should be noted that such a potential is required to construct the dual standard model, since otherwise the dyons are not stable bound states.

The spherical symmetry of the Hamiltonian \( \hat{H} \) implies there is a conserved angular momentum, of which the \( z \)-component is

\[ \hat{J}_z = [r \wedge D]_z + \frac{1}{2} M. \] (48)

Then the angular momentum of a composite dyon \( (Q, M) \) is simply the appropriate eigenvalue of \( \hat{J}_3 \) for the \( H \) eigenstate. For a spherically symmetric ground state the angular momentum is then determined only by

\[ \hat{J}_3 H_i(r) = \frac{1}{2} MH_i(r) = J_3 H_i(r). \] (49)

By \( \{4\} \) these eigenvalues are either integer or half integer; giving integer or half-integer angular momenta to the resulting dyons.

It is then a simple task to determine the angular momenta of the different scalar boson-monopole composites; with the \( J_3 \) values simply corresponding to the relevant eigenvalue of \( \frac{1}{2} M \). Using table \( \{I\} \) these are:

**TABLE VI. Angular momenta of the \((Q, M)\) scalar boson-monopole composites.**

| \(u, \bar{d}\) | \((-1, 0, -1, 0)\) | 0 | 0 | 0 | 0 |
| \(d\) | \((-1, 1, 1, -1, 0)\) | 0 | 0 | 0 | 0 |
| \(\bar{u}\) | \((-1, 1, 2, -2, -1)\) | 0 | 0 | 0 | 0 |
| \(\bar{d}\) | \((-2, 2, -2, -3, -3)\) | 0 | 0 | 0 | 0 |

These are consistent with the angular momentum \( \{4\} \). Note the \( u \) monopole has some subtlety, as described in the last paragraph of sec. \( \{V B\} \).

Table \( \{V I\} \) does not constitute the complete spectrum of dyons. In addition there can also be quanta of \( H \) bound to anti-monopoles, or anti-particles of \( \bar{H} \) bound to monopoles/anti-monopoles. To simplify their classification note that this reflects an underlying parity and charge-parity symmetry,

\[ \mathcal{P} : (Q, M) \rightarrow (Q, -M), \quad \mathcal{C} \mathcal{P} : (Q, M) \rightarrow (-Q, M), \quad \mathcal{C} \mathcal{P} : J_3 \rightarrow -J_3. \] (50)

Parity takes a monopole to its anti-monopole, whilst charge-parity takes an electric charge to its anti-charge: both reverse spin. This can be conveniently represented in the following figure, which is based on a discussion in ref. \( \{11\} \).

![FIG. 1. \( H_i \) scalar bosons bound to either a monopole or an anti-monopole.](image-url)
Here $\mathcal{P}$ is a reflection about the $Q$ axis, whilst $\mathcal{C}\mathcal{P}$ reflects about the $M$ axis.

However the above methods do not appear to produce an $\tilde{e}^*$ dyon with one-half angular momentum. If such a state does not exist this would seriously jeopardise the construction of a dual standard model from SU(5) solitons. Fortunately, there are a couple of solutions:

(i) It appears that two quanta of $H$ on an $\tilde{e}$ monopole could have one-half angular momentum [12].

(ii) An $\tilde{e}^*$ monopole with extra non-topological colour charge is spherically symmetric and has scalar excitations with angular momentum one-half.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
$\tilde{e}^*$ & $Q_1$ & $Q_2$ & $Q_3$ & $Q_4$ & $Q_5$ \\
\hline
\end{tabular}
\caption{Angular momenta of the $\tilde{e}^*$ dyons.}
\end{table}

VII. DYONS FROM MONOPOLE GAUGE EXCITATIONS

This section discusses dyons whose electric charge originates within the properties of the semi-classical electric field around a monopole. These have a similar nature to the dyons discussed in sec. (VI); in particular it appears that they may also possess intrinsic angular momentum. Therefore when constructing a dual standard model the dyons formed by combining monopoles with scalar bosons or gauge excitations are equally relevant. The plan of this section is to firstly illustrate how a classical electric field can arise around a monopole. This is then quantised through semi-classical methods. Finally the angular momentum of such dyons is discussed.

A. Classical Dyon Charge

To start it will prove useful to examine the nature of the magnetic and electric fields around a dyon within a classical field theory context. The following analysis is essentially a non-Abelian generalisation of the Julia-Zee dyon [30].

The task is to find a long range component $A_0$ that solves the field equations in the background of the monopole. Taking a unitary gauge and assuming the electric field to be purely radial allows the field equations to be written, asymptotically,

$$D_i D_i A_0 - [\Phi_0, [\Phi_0, A_0]] = 0,$$

where $D_i$ is the covariant derivative in the background of a monopole and, to be specific, the spatial components of the gauge potential are as in [30]. This implies that the leading order contribution to $A_0$ satisfies

$$[A_0, \Phi] = [A_0, M] = 0,$$

for which (52) becomes a Laplace equation. Therefore in the unitary gauge the classical dyon configuration is

$$\Phi(r) \sim \Phi_0, \quad A(r) \sim \frac{1}{2g} \frac{1 - \cos \vartheta}{r \sin \vartheta} M \varphi,$$

$$A_0(r) \sim \frac{g}{4\pi r} Q,$$

with $[Q, \Phi_0] = [Q, M] = 0$, in agreement with the constraint (39) on global charge.

Because this analysis was classical the magnitude of the electric charge can take a continuum of values. Clearly this is not consistent quantum mechanically; for instance it violates Dirac’s condition and also the angular momentum (11) is not constrained to be integer or half-integer. Consistent values are obtained only upon proper quantum mechanical treatment.

This charge can also be interpreted in a slightly different way. Upon performing a time dependent gauge transformation

$$\Phi(r, t) = Ad[U(r, t)]\Phi(r), \quad A(r, t) = Ad[U(r, t)]A(r),$$

the profile (54) can be expressed in an $A_0 = 0$ gauge providing

$$\dot{U} U^{-1} = \frac{g}{4\pi r} Q.$$

Hence the dyon can be thought of as a monopole rotating in internal space under the action of $U(r, t)$. This motion is quantised to discretise charge.

However when quantising such internal motion only the non-trivial actions of $U$ are relevant. For each SU(5) monopole the subgroup of $H_A$ that acts non-trivially is:

\begin{table}[h]
\centering
\begin{tabular}{|c|c|}
\hline
$H_A^{\text{eff}}$ & $\text{SU}(5)$ \\
\hline
$(u,d)$ & $U(1)_M$ \\
$\bar{d}$ & $U(1)_C \times SU(2)_I \times U(1)_Y / \mathbb{Z}_6$ \\
$(\bar{\nu}, \bar{e})$ & $SU(3)_C \times U(1)_I \times U(1)_Y / \mathbb{Z}_6$ \\
u & $U(1)_C \times SU(2)_I \times U(1)_Y / \mathbb{Z}_{12}$ \\
$\bar{e}$ & $SU(3)_C \times SU(2)_I \times U(1)_Y / \mathbb{Z}_6$ \\
\hline
\end{tabular}
\caption{Allowed and Effective Gauge Symmetries.}
\end{table}

B. Quantisation of the Gauge Excitations

A semi-classical quantisation of the charge values (54) can be achieved through the method of collective coordinates. Then the quantum state is described by a wavefunction over the monopole’s internal degrees of freedom, with the quantised excitations described by the charge eigenstates.
Much of this section follows the work of Dixon [31], although Guadagnini’s quantisation of skyrmions [32] is also relevant.

The quantisation of the monopole’s internal degrees of freedom can be considered over the symmetry group $H_A^\text{eff}$, which generates the motion. Then the quantum excitations are described by a wavefunction $\psi(U)$, written as a sum over representations

$$\psi(U) = \sum_r \psi^r(U) = \sum_r \sum_{ij} c^r_{ij} D^r_{ij}(U), \quad U \in H_A^\text{eff},$$

where $D^r_{ij}$ are the matrix elements in the $r$-representation of $H_A^\text{eff}$. Each of the terms in (58) corresponds to a particular gauge excitation. The charge of these excitations is then determined by the action of the gauge group, namely

$$Q \mapsto \text{Ad}[h]Q \Rightarrow U \mapsto hU. \quad (59)$$

The charges therefore fall into representations of the $H_A^\text{eff}$.

The specific charges of the gauge excitations are constructed from the relevant charge operators. These are analogues of the momentum operator $\hat{\mathbf{p}} = -i\hat{\partial}_x$, acting instead on the internal motion

$$\hat{\mathcal{Q}} \psi(U) = \psi(U + \delta U) - \psi(U) = \psi(\delta U), \quad (60)$$

where $\delta U$ is an infinitesimal transformation of the symmetry associated with the conserved charge. This gives the charge operators

$$\hat{Q}_T \psi(U) = \psi(TU) = q_T \psi(U). \quad (61)$$

Its eigenfunctions are specified by $c^r_{ij}$ and $U$.

Then the individual charge eigenstates correspond to the representations of $H_A^\text{eff}$ in table IX, which are

$$D^r(U) = D(U_C)D(U_I)U_Y^r, \quad U_Y = e^{i\theta Y_i T_i}, \quad (62)$$

with $r$ an integer. The first few terms of $\psi(U)$ in (58) determines the charge associated with each excitation, which, for suitable eigenfunctions, are the eigenvalues

$$d^r(T_C)U = q_C U, \quad d^r(T_I)U = q_I U. \quad (63)$$

This gives the spectrum:

| $D(h)$ | $q_C$ | $q_I$ | $q_Y$ | diag $Q$ | allowed on |
|--------|-------|-------|-------|----------|-----------|
| $h_C h_N$ | $\sqrt{3}/2$ | $\sqrt{3}/2$ | $\sqrt{3}$ | $\sqrt{3}$ | all |
| $h_C^2 h_N^2$ | $-\sqrt{3}/2$ | $0$ | $\sqrt{3}$ | $\sqrt{3}$ | $\bar{a}, (\bar{\mathbf{r}}, \bar{\mathbf{e}}), u, \bar{e}$ |
| $h_N h_Y^2$ | $0$ | $1$ | $1$ | $1$ | $\bar{a}, (\bar{\mathbf{r}}, \bar{\mathbf{e}}), u, \bar{e}$ |
| $h_Y^4$ | $\sqrt{3}/2$ | $0$ | $2$ | $2$ | $\bar{a}, (\bar{\mathbf{r}}, \bar{\mathbf{e}}), u, \bar{e}$ |
| $h_C h_Y^4$ | $-\sqrt{3}/2$ | $\sqrt{3}/2$ | $\sqrt{3}$ | $\sqrt{3}$ | $\bar{a}, (\bar{\mathbf{r}}, \bar{\mathbf{e}}), u, \bar{e}$ |
| $h_C^2 h_Y^4$ | $0$ | $2 \sqrt{3}/2$ | $2$ | $2$ | $\bar{a}, (\bar{\mathbf{r}}, \bar{\mathbf{e}}), u, \bar{e}$ |

There are charges other than those in table IX. However these are not central to the following discussion; being highly charged and therefore more energetic.

It is interesting that all electric generators satisfy $\exp(i 2\pi Q) = 1$, and therefore take the same values as the magnetic generators of the monopoles. This is because both the monopoles and gauge excitations occur in specific representations of $H_{SM}$.

C. Spherically Symmetric Gauge Excitations

In sec. (62[21]) the monopoles with no intrinsic angular momentum were expressed in spherically symmetric way

$$\Phi(r) \sim \text{Ad}[\Lambda(\hat{\mathbf{r}})]\Phi_0, \quad B(r) \sim \frac{1}{2g^2 r^2} \text{Ad}[\Lambda(\hat{\mathbf{r}})]M. \quad (64)$$

Then a gauge transformation by $\Omega(S)$ simply takes

$$\Phi(r) \mapsto \Phi(Sr), \quad B_i(r) \mapsto S_i^j B_j(Sr), \quad (65)$$

so that a spatial rotation is equivalent to a gauge transformation.

So what happens when there is an electric charge on the monopole? In the gauge of (63) the electric field is

$$E(r) \sim g \frac{\hat{\mathbf{r}}}{4\pi r^2} \text{Ad}[\Lambda(\hat{\mathbf{r}})]Q, \quad (66)$$

which should be non-singular for those charges allowed on the monopole. From this a gauge transformation by $\Omega(S)$ takes

$$E(r) \mapsto g \frac{\hat{\mathbf{r}}}{4\pi r^2} \text{Ad}[\Lambda(S\hat{\mathbf{r}})] \text{Ad}[\omega(S)]Q. \quad (67)$$

Therefore a gauge excitation is spherically symmetric with no angular momentum if

$$[\hat{\mathcal{L}}, Q] = 0. \quad (68)$$

The determination of the spherically symmetric gauge excitations is generally treated on a case by case basis. However there are some gauge excitations, for instance when the electric charge is proportional to the magnetic charge ($\pm M, M$), when condition (68) is always satisfied. This has relevance to the induced charge from a theta vacuum discussed in sec. (62[21]).

D. Monopole Gauge Excitations with Internal Angular Momenta

On examining the spectrum of monopole gauge excitations in table IX, it transpires that not all are spherically symmetric, since many violate condition (68). For illustration some particular examples are given table X below. That these dyons are not spherically symmetric suggests
they have internal angular momenta, just like the dyons in sec. (V).

Unlike the composite scalar boson-monopole dyons the determination of the intrinsic angular momentum of the monopole gauge excitations is complicated by the internal structure of the gauge excitation. This may be seen by naively applying \( \text{diag} \) to the spherically symmetric gauge excitations of sec. (VII C), where non-sensical results are generally obtained.

However it does seem that some of these monopole gauge excitations are fermionic. This is because Goldhaber’s arguments of sec. (V I) appear to apply; then many dyons, for instance those in table \( \text{X} \), do seem to have anti-symmetric statistics. Unless the spin-statistics relation is violated then these should have half-integer angular momenta.

Unfortunately the angular momenta of these gauge excitations are an area that has not been fully examined. At present there is only one treatment, by Dixon [31], that has studied their properties. He used the semi-classical methods of sec. (VII B) to define their angular momenta and found an agreement with the spin-statistics relation.

In this section a classical approach will be used to model these gauge excitations. For the dyons in table \( \text{X} \) below this gives compatible results with Dixon’s approach.

To model these gauge excitations consider a composite of a spherically symmetric gauge excitation and a gauge boson. Then the electric charge generator can be split into two components

\[
Q = Q_0 + Q_s, \quad [\hat{I}, Q_0] = 0, \tag{69}
\]

where \( Q_0 \) contributes no angular momentum. Taking \( Q_s \) to be \( \text{diag}(1, -1) \) embedded along the diagonal gives a single gauge boson charge eigenstate.

The configurations in (69) can have the same electric and magnetic charges \( (Q, M) \) as the gauge excitations. Therefore they should have the same statistics. Their angular momenta can also be checked to be compatible with Dixon’s results for those in table \( \text{X} \) below. This suggests that they model the gauge excitations. Whether this is true in general, or just for specific cases, such monopole-gauge boson composites should be present in the SU(5) monopole theory.

The angular momentum of these composites is then determined from (11) to be

\[
J_3 = \frac{1}{2} \text{tr} Q_s M + s_3, \tag{70}
\]

which appears to be degenerate in the spin value \( s_3 \) of the gauge boson. This degeneracy is lifted by the spin-magnetic field interaction

\[
\mathcal{H}_{\text{int}} = \text{tr} Q_s \cdot B. \tag{71}
\]

Then the dyon states with least energy have angular momenta

\[
J_3 = \begin{cases} \frac{1}{2} \text{tr} Q_s M - 1, & \text{tr} Q_s M \geq 0, \\ \frac{1}{2} \text{tr} Q_s M + 1, & \text{tr} Q_s M \leq 0, \end{cases} \tag{72}
\]

This construction is illustrated with the following electric colour excitations:

**TABLE X. Dyonic Gauge Excitations with one-half angular momentum.**

| dyon      | diag \( M \) | diag \( Q \) | \( J_3 \) |
|-----------|-------------|-------------|---------|
| \( \vec{r}, e \) | (1, 1, 1, -2, -1) | (0, 1, 1, -1, -1) | \(-\frac{3}{2} + 1 = -\frac{1}{2} \) |
| \( u \) | (1, 1, 1, -2, -2) | (0, 1, 1, -1, -1) | \( \frac{1}{2} - 1 = -\frac{1}{2} \) |
| \( \bar{e} \) | (1, 1, 4, -3, -3) | (0, 1, 1, -1, -1) | \( \frac{3}{2} - 1 = \frac{1}{2} \) |

Within this table the decompositions of \( Q \) into \( Q_0 \) and \( Q_s \) are, respectively,

\[
(0, 1, 1, -1, -1) = (1, 1, 1, -2, -1) + (-1, 0, 0, 1, 0),
(0, 1, 1, -1, -1) = (1, 1, 0, -1, -1) + (-1, 0, 1, 0, 0),
(0, 1, 1, -1, -1) = (1, 1, 0, -1, -1) + (-1, 0, 1, 0, 0).
\]

### VIII. IMPLICATIONS OF A THETA VACUUM

The last topic of concern is the effect of a theta vacuum

\[
\mathcal{L}_\theta = \frac{\theta g^2}{8\pi^2} \text{tr} \, E \cdot B \tag{73}
\]

upon the SU(5) monopole and dyon spectrum. Although (73) is a total divergence, and hence just a boundary term in the action, it does have a physical effect. Witten showed that generically an electric charge is induced on a monopole through the asymptotic gauge potential (52).

A central feature of this theta vacuum is that it violates parity and charge-parity maximally. This occurs because \( \mathcal{L}_\theta \) is both \( P \) and \( CP \) odd; whereas the usual Yang-Mills Lagrangian is even. These violations appear explicitly in the dyon spectra (73) below.

In the context of a dual standard model this parity violation strongly suggests that a theta vacuum should play a central role. Indeed, without including parity violation in a very unnatural manner, it is difficult to conceive of another method for incorporating parity violation within the SU(5) dyon spectrum (54).

That a theta vacuum induces electric charge can be seen directly through the interaction of a monopole with a gauge field \( (\phi, a) \). Following an argument of Coleman’s (17) the electric and magnetic fields

\[
E = \nabla \phi, \quad B = \nabla \wedge a + \frac{1}{2g} \frac{\hat{r}}{r^2} M, \tag{74}
\]

are substituted into (73) to give, upon integration by parts,

\[
\mathcal{L}_\theta = \int d^3r \ L_\theta = -\frac{\theta g}{2\pi} \int d^3r \ |\delta^3(r)| \text{tr} \phi M. \tag{75}
\]

\[
J_3 = \begin{cases} \frac{1}{2} \text{tr} Q_s M - 1, & \text{tr} Q_s M \geq 0, \\ \frac{1}{2} \text{tr} Q_s M + 1, & \text{tr} Q_s M \leq 0, \end{cases} \tag{72}
\]
But this is precisely the interaction between the gauge potential and an electric charge $Q_a = -\frac{g}{2\pi} M$. Therefore a theta vacuum induces electric charge. This argument also carries through for a dyon $(Q, M)$, resulting in the electric and magnetic fields

$$E \sim \frac{g}{4\pi r^2} (Q - \frac{\theta}{2\pi} M), \quad B \sim \frac{1}{2g} \frac{\dot{r}}{r^2} M. \quad (76)$$

That parity and charge-parity are violated is explicit within the spectrum of dyons in (76). For instance the monopole/anti-monopole spectrum $(0, \pm M)$ becomes $\pm (-\theta M/2\pi, M)$, which is not symmetric under either (74) or (71). Also parity violation in the $(Q \pm \frac{\theta M}{2\pi}, \pm M)$ dyon spectrum is seen in fig. 2 below.

**FIG. 2.** $(Q, \pm M)$ dyons in a theta vacuum (c.f. fig. [1]).

Associated with the induced charge there will be an extra electric interaction between electric charges and monopoles. An interesting property of this interaction is that it is always associated with an Abelian gauge symmetry, even for non-Abelian generators $M$. Consequently an electric charge $Q$ will interact with a magnetic monopole $M$ through a Coulomb potential

$$V(r) \sim -\frac{\theta g^2 \text{tr} Q M}{2\pi - 4\pi r}. \quad (77)$$

In addition there will also be the usual electric-magnetic interaction between the charge and monopole.

This theta induced interaction is Abelian because of the global properties of charge around a monopole. Sec. (VIII) described how an electric charge can only be consistently placed on a monopole if $[Q, M] = 0$; otherwise there is an infinite energy string singularity. This leads to a group $H_A$ of globally allowed symmetries, which is relevant to charge-monopole interactions. The point is that $M$ defines an Abelian symmetry $U(1)_M$ such that $H_A = U(1)_M \times H'_A / \mathbb{Z}$, which follows from $H_A$ centralising $M$. Consequently the induced theta interaction is always Abelian.

It is interesting that this induced theta interaction can provide a suitable binding potential $V(r)$ between some monopoles and electric charges, as discussed in sec. (VI). Also notable is the spectrum of dyonic composites for which this potential is binding will maximally violate parity.

**IX. CONCLUSION**

This review has aimed to present a coherent picture of the monopoles and dyons in SU(5) gauge unification. Their behaviour is fairly intricate, although it does fit together in a coherent manner. A central point is that these monopoles have properties which naturally apply to the construction of a dual standard model. This supports Vachaspati’s original proposal that the elementary particles might originate as monopoles from a magnetic gauge unification of the standard model.

The behaviour of these monopoles/dyons and their relation to a dual standard model is as follows:

(i) The central concept is that a charge may either be represented as sourcing $A^0 \sim e/4\pi r$ or as a monopole in the gauge dual potential $\tilde{A}_\rho \sim e(1 - \cos \theta)/4\pi$. Both descriptions are expected to be equivalent by electric-magnetic duality. Therefore, in principle, there is an alternative, dual formulation of the standard model. Such a formulation may uncover a hidden simplicity and regularity of form that underlies the conventional description.

(ii) For scalar masses much less than gauge masses the magnetic charges of the stable SU(5) monopoles are in one-to-one identification with the electric charges of one generation of elementary particles. This suggests that a dual formulation of the standard model should be constructed around the properties of SU(5) monopoles [10,11].

(iii) The gauge degeneracy of these five stable monopoles has an analogous structure to the gauge degeneracy of each associated elementary particle [24]. This supports a duality in their gauge interactions.

(iv) All stable SU(5) monopoles appear to have no intrinsic angular momentum. This means that some modification of their spectrum is required to make them spin and hence give a realistic dual standard model.

(v) A crucial point is that if dyons are considered instead of monopoles, then these may have intrinsic angular momentum. This strongly suggests that a realistic dual standard model should be formulated around the properties of SU(5) dyons [10,12].

(vi) The simplest dyons are scalar boson-monopole composites. Many of these have one-half angular momentum [12]. However it is unclear how the dyons with one-half angular momentum are preferentially selected. It is also unclear how these dyonic composites are stabilised (although see (vii)).

(vii) In addition to the dyons in (v) the SU(5) model also contains dyons associated with a semi-classical quantisation of the electric field around a monopole. Such dyons also appear to have one-half angular momentum [31], although there are many issues that have not been fully understood.

(viii) A theta vacuum induces extra electric charge on each dyon/monopole [8]. This may be important for
incorporating parity violation within a dual standard model \cite{10,11}. It also has relevance to the dyon spectrum because it induces a binding force that stabilises some dyons.

The implications of a theta vacuum for constructing a realistic dual standard model are investigated from the two alternative viewpoints of ref. \cite{14} and ref. \cite{15}.

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