A Variation on Mills-Like Prime-Representing Functions

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Abstract
Mills showed that there exists a constant $A$ such that $\lfloor A^{3^n} \rfloor$ is prime for every positive integer $n$. Kuipers and Ansari generalized this result to $\lfloor A^{c^n} \rfloor$ where $c \in \mathbb{R}$ and $c \geq 2.106$. The main contribution of this paper is a proof that the function $\lceil B^{c^n} \rceil$ is also a prime-representing function, where $\lceil X \rceil$ denotes the ceiling or least integer function. Moreover, the first 10 primes in the sequence generated in the case $c = 3$ are calculated. Lastly, the value of $B$ is approximated to the first 5500 digits and is shown to begin with 1.2405547052\ldots.

1 Introduction
Mills [6] showed in 1947 that there exists a constant $A$ such that $\lfloor A^{3^n} \rfloor$ is prime for all positive integers $n$. Kuipers [5] and Ansari [1] generalized this result to $\lfloor A^{c^n} \rfloor$ where $c \in \mathbb{R}, c \geq 2.106$, i.e., there exist infinitely many $A$'s such that the above expression yields a prime for all positive integers $n$. Caldwell and Cheng [2] calculated the minimum constant $A$ for the case $c = 3$ up to the first 6850 digits ($A051021$), and found it to be approximately equal to 1.3063778838\ldots. This process involved computing the first 10 primes $b_i$ in the sequence generated by the function ($A051254$), with $b_{10}$ having 6854 decimal digits.

The main contribution of this paper is a proof that the function $\lceil B^{c^n} \rceil$ satisfies the same criteria, where $\lceil X \rceil$ denotes the ceiling function (the least integer greater than or equal to $X$). In other words, there exists a constant $B$ such that for all positive integers $n$, the expression $\lceil B^{c^n} \rceil$ yields a prime for $c \geq 3, c \in \mathbb{N}$. Moreover, the sequence of primes generated by such functions is monotonically increasing. Lastly, analogously to [2] the case $c = 3$ is studied in more detail and the value of $B$ is approximated up to the first 5500 decimal digits by calculating the first 10 primes $b_i$ of the sequence.

In contrast to Mills’ formula and given that here the floor function is replaced by a ceiling function, the process of generating the prime number sequence $P_0, P_1, P_2, \ldots$ involves taking the greatest prime smaller than $P_n^c$ at each step instead of smallest prime greater than $P_n^c$, in order to find $P_{n+1}$. As a consequence, the sequence of primes generated by $\lceil B^{c^n} \rceil$ is different.
from the one generated by $|A^c|$ for the same value of $c$ and the same starting prime (apart from the first element of course).

2 The prime-representing function

This paper begins with a proof of the case $c = 3$ and will proceed to a generalization of the function to all $c \geq 3, c \in \mathbb{N}$.

By using Ingham’s result [4] on the difference of consecutive primes:

$$p_{n+1} - p_n < Kp_n^{5/8},$$

and analogously to Mills’ reasoning [6], we construct an infinite sequence of primes $P_0, P_1, P_2, \ldots$ such that $\forall n \in \mathbb{N} : (P_n - 1)^3 + 1 < P_{n+1} < P_n^3$ using the following lemma.

**Lemma 1.** $\forall N > K^8 + 1 \in \mathbb{N} : \exists p \in \mathbb{P} : (N - 1)^3 + 1 < p < N^3$, where $\mathbb{P}$ denotes the set of prime numbers.

**Proof.** Let $p_n$ be the greatest prime smaller than $(N - 1)^3$.

$$(N - 1)^3 < p_{n+1}
< p_n + Kp_n^{5/8}
< (N - 1)^3 + K ((N - 1)^3)^{5/8}$$

(since $p_n < (N - 1)^3$)

$$< (N - 1)^3 + (N - 1)^2$$

(since $N > K^8 + 1$)

$$< N^3 - 2N^2 + N$$

$$< N^3.$$

Note that since $(N - 1)^3 + 1 < p_{n+1}$ since $(N - 1)^3 + 1 = N(N^2 - 3N + 3)$ is not prime.

$\square$

Given the above we can construct an infinite sequence of primes $P_0, P_1, P_2, \ldots$ such that for every positive integer $n$, we have: $(P_n - 1)^3 + 1 < P_{n+1} < P_n^3$.

We now define the following two functions:

$$\forall n \in \mathbb{Z}^+ : u_n = (P_n - 1)^{3-n},$$

$$\forall n \in \mathbb{Z}^+ : v_n = P_n^{3-n}.$$

The following statements can immediately be deduced:

- $u_n < v_n$,
• \( u_{n+1} = (P_{n+1} - 1)^{3^n - 1} > ((P_n - 1)^3 + 1) - 1)^{3^n - 1} = (P_n - 1)^{3^n} = u_n, \)
• \( v_{n+1} = P_{n+1}^{3^n - 1} < (P_n^3)^{3^n - 1} = P_n^{3^n} = v_n. \)

It follows that \( u_n \) forms a bounded and monotone increasing sequence.

**Theorem 2.** There exists a positive real constant \( B \) such that \( \lceil B^{3^n} \rceil \) is a prime-representing function for all positive integers \( n \).

**Proof.** Since \( u_n \) is bounded and strictly monotone, there exists a number \( B \) such that
\[
B := \lim_{n \to \infty} u_n.
\]

From the above deduced properties of \( u_n \) and \( v_n \), we have
\[
 u_n < B < v_n,
(P_n - 1)^{3^n} < B < P_n^{3^n},
P_n - 1 < B^{3^n} < P_n.
\]

**Theorem 3.** There exists a positive real constant \( B \) such that \( \lceil B^{c^n} \rceil \) is a prime-representing function for \( c \geq 3, c \in \mathbb{N} \) and all positive integers \( n \).

**Proof.** We can use the generalizations to Mills’ function as shown by Kuipers [5] and Dudley [3] in order to show that \( \lceil B^{c^n} \rceil \) is also a prime-representing function for \( c \geq 3, c \in \mathbb{N} \). This proof is short as it is essentially identical to the one presented above, with the following modifications.

As shown by Kuipers [5] for Mills’ function, we first define \( a = 3c - 4, b = 3c - 1 \). Therefore \( a/b \geq 5/8 \). This means that in Ingham’s equation there exists a constant \( K' \) such that
\[
p_{n+1} - p_n < K' p_n^{a/b}.
\]

Lemma 1 can then be modified by taking \( N > K'^b + 1 \), defining \( p_n \) as the greatest prime smaller than \( (N - 1)^c \) and noticing that \( ca + 1 = b(c - 1) \). Analogously to the proof in Lemma 1, we quickly obtain the bounds \( (N - 1)^c + 1 < p < N^c \). This means that we can construct a sequence of primes \( P_0, P_1, P_2, \ldots \) such that for every positive integer \( n \), \( (P_n - 1)^c + 1 < P_{n+1} < P_n^c \).

This is then concluded with a similar reasoning as in the proof of Theorem 2. 

\[\square\]
3 Numerical calculation of $B$

In this section, a numerical approximation of $B$ is presented for the case $c = 3$. Mills [6] suggested using the lower bound $K = 8$ for the first prime in the classical Mills function $\lfloor A^3n \rfloor$, where $K$ is the constant defined in Ingham’s paper [4]. Other authors, including Caldwell and Cheng [2], decided to begin with the prime 2 and then choose the least possible prime at each step. In this case, since the ceiling function replaces the floor function, we choose the greatest possible prime smaller than $P_n^3$ as the next element $P_{n+1}$.

If $p_i$ denotes the $i^{th}$ prime in the sequence, we obtain

- $p_1 = 2$
- $p_2 = 7$
- $p_3 = 337$
- $p_4 = 38272739$
- $p_5 = 56062005704198360319209$
- $p_6 = 17619999581432728735667120910458586439705503907211069\times 6028654438846269$
- $p_7 = 54703823381492990628407924713718713957740513297193414\times 21259587335767096542227048457036456872683352033529421007878\times 29141860830768725102385452609882503551811073140339908096068\times 8125590506176016285837338387682469$

The primes $p_8$, $p_9$ and $p_{10}$ are far too large to show in this paper — for instance $p_{10}$ has 5528 decimal digits. The primes $p_i$ for $i \leq 8$ were verified using a deterministic primality test in Wolfram Mathematica 11 with the `ProvablePrimeQ` function in the `PrimalityProving` package, while $p_9$ and $p_{10}$ were certified prime by the Primo software [7]. The certification of $p_{10}$ took 14 hours and 23 minutes on an Intel i7-4770 CPU and 4GB RAM. The prime certificates for $p_9$ and $p_{10}$ as well as the primes themselves can be found alongside this paper as auxiliary files.

The value of $B$ was calculated up to its first 5500 decimal digits. The first 600 are presented below:
1.2405547052  5201424067  4695153379  0034521235  3396725255 
9232034386  1886622104  9111642316  9209174137  7064313608 
3109555650  9480848158  9481662421  8378961303  7426392535 
6658242301  8524802142  1960037621  1464734105  8229918628 
4182439221  9437396337  9442594273  8936874985  9158491115 
7886891108  4262398559  2731605607  5719554304  2915944781 
6278755834  4774142491  8125993063  4590081972  8945860313 
1303247244  0907981721  7119324606  1009855753  6063847008 
6985820925  6038920740  0817313213  1691077511  332609476 
3239264899  5703729933  8452155290  5152647430  8960522935 
3735771869  0936560934  8000430515  4856069064  6309177739 
2832001365  6550953673  1549789328  9032942357  7708168137 

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[7] Primo software, available at [http://www.ellipsa.eu/public/primo/primo.html](http://www.ellipsa.eu/public/primo/primo.html).

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(Concerned with sequences [A051021](https://oeis.org/A051021) and [A051254](https://oeis.org/A051254).)

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