Structures in 3D double-diffusive convection and possible approach to the Saturn’s polar hexagon modeling

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Abstract – Three-dimensional double-diffusive convection in a horizontally infinite layer of an uncompressible fluid interacting with horizontal vorticity field is considered in the neighborhood of Hopf bifurcation points. A family of amplitude equations for variations of convective cells amplitude is derived by multiple-scaled method. Shape of the cells is given as a superposition of a finite number of convective rolls with different wave vectors. For numerical simulation of the obtained systems of amplitude equations a few numerical schemes based on modern ETD (exponential time differencing) pseudo-spectral methods were developed. The software packages were written for simulation of roll-type convection and convection with square and hexagonal type cells. Numerical simulation has showed that the convection takes the form of elongated “clouds”, “spots” or “filaments”. It was noted that in the system quite rapidly a state of diffusive chaos is developed, where the initial symmetric state is destroyed and the convection becomes irregular both in space and time. The obtained results may be the basis for the construction of more advanced models of multi-component convection, for instance, model of Saturn’s polar hexagon.

Introduction. – It is believed that the convection is the most common case of gas and liquid flows in the Universe [1]. Among the various types of convection the so called double-diffusive convection holds a special place. Physical systems with double-diffusive convection have two components with significantly different coefficients of diffusion. It can be heat and salt in the sea water, heat and helium in stellar atmospheres, or two reagents in chemical reactors. As a result of various spatial distribution of these components in a gravitational field the convection arises, which can have various forms and lead to a variety of phenomena [2-5]. In oceanography thermohaline convection plays an important role in heat and mass transfer processes in the ocean and affect different small-scale processes that lead to the formation of vertical fine structure [4].

During last 50 years double-diffusive convection is actively studied by both experimental and theoretical methods, including numerical modeling. One of the classical methods to study the system with convective instability near the bifurcation points is the method of amplitude equations. For the case of Rayleigh-Benard convection this method was used by Newell and Whitehead [5]. It allowed to reduce the original PDE system to a nonlinear evolution equation for one roll mode. Also it made possible to obtain the equations for the case of several roll modes with nonlinear interaction, so that the shape of the convective cells can be an arbitrary. Since then, the method of amplitude equations is frequently used to study various convective phenomena.

In 80-90 years the formation of structures in the neighborhood of Hopf bifurcation points for the horizontally translation-invariant systems was actively studied in some works. The development of oscillations in such systems gives rise to different types of waves (eg, standing, running, modulated, chaotic), which is well described by a complex Ginzburg-Landau equations (CGLE). The equations of this type must be derived from the basic system of partial differential equations for the given physical system by asymptotic methods. However, a full and well-grounded derivation of amplitude equations for systems with double-diffusive convection (especially three-dimensional) is still poorly represented in the literature.

For the 2D roll-type double-diffusive convection the amplitude equations of CGLE type were firstly derived and studied numerically in the work [6]. The amplitude equa-
tions for the case of roll-type Rayleigh-Benard convection were derived in the work [7].

The main idea of the present article consists in combining strict mathematical derivation of amplitude equations by multiple-scaled method (following [3]) and considering arbitrary number of interacting roll-type convective modes over horizontal vorticity field (as it was done for the Rayleigh-Benard convection in [5,7]) for obtaining the amplitude equations for three-dimensional double-diffusive system in the neighborhood of Hopf bifurcation points. This also develops the ideas of previous work [4, 9–11], where a two-dimensional and three-dimensional convection with a square-type and roll-type cells was investigated by alike methods. Then the derived amplitude equations are investigated numerically. Possible forms of 3D double-diffusive convection are described.

Formulation of the problem and basic equations. – Consider 3D double-diffusive convection in a liquid layer of a width \( h \), confined by two plane horizontal boundaries. The liquid layer is heated and salted from below. The governing equations in this case are hydrodynamical equations for a liquid mixture in the gravitational field [12]:

\[
\begin{align*}
\partial_t \mathbf{v} + (\mathbf{v} \nabla)\mathbf{v} &= -\rho^{-1} \nabla p + \nu \Delta \mathbf{v} + \mathbf{g}, \\
\partial_t T + (\mathbf{v} \nabla)T &= \chi \Delta T, \\
\partial_t S + (\mathbf{v} \nabla)S &= D \Delta S, \\
\text{div} \mathbf{v} &= 0.
\end{align*}
\]

Where \( \mathbf{v}(t, x, y, z) \) is the velocity field of liquid, \( T(t, x, y, z) \) is the temperature, \( S(t, x, y, z) \) is the salt concentration, \( p(t, x, y, z) \) is the pressure, \( \rho(t, x, y, z) \) is the density of liquid, \( \mathbf{g} \) is the acceleration of gravity, \( \nu \) is the kinematic viscosity of fluid, \( \chi \) is the thermal diffusivity of the liquid, \( D \) is the salt diffusivity. Cartesian frame with the horizontal \( x \)-axis and \( y \)-axis is used, while the \( z \)-axis is directed upward and \( t \) is the time variable.

Distributed sources of heat and salt are absent. On the upper and lower boundaries of the layer the constant values of temperature and salinity are supported, the higher ones are at the lower boundary.

The governing equations are transformed into dimensionless form with the use of Boussinesq approximation and following units for length, time, velocity, pressure, temperature and salinity are respectively: \( h, h^2/\chi, h/\rho_0 \chi^2/2h, T_\Delta, S_\Delta \), where \( T_\Delta \) and \( S_\Delta \) are temperature and salinity differences across the layer. The dimensionless governing equations for momentum and diffusion of temperature and salt are [9]:

\[
\begin{align*}
u_t + (\nu u_x + \nu u_y + \nu u_z) &= -p_x + \sigma \Delta u, \\
\eta_t + (\eta u_x + \eta u_y + \eta u_z) &= -p_y + \sigma \Delta v, \\
w_t + (w u_x + w u_y + w u_z) &= -p_z + \sigma \Delta w + \sigma R_T \theta - \sigma R_S \xi, \\
\theta_t + (\theta u_x + \theta u_y + \theta u_z) &= -\Delta \theta, \\
\xi_t + (\xi u_x + \xi u_y + \xi u_z) &= -\tau \Delta \xi, \\
u_x + v_y + w_z &= 0.
\end{align*}
\]

Where \( \sigma = \nu_0/\chi \) is the Prandtl number \( (\sigma \approx 7.0) \), \( \tau = D/\chi \) is the Lewis number \( (0 < \tau < 1, \text{usually } \tau = 0.01 \pm 0.1) \). \( R_T = (\alpha' h^3/\chi \nu) T_\Delta \) and \( R_S = (\gamma' h^3/\chi \nu) S_\Delta \) are the temperature and the salinity Rayleigh numbers, \( \alpha' \) and \( \gamma' \) are cubic expansion coefficients. Fluid velocity is represented by the vector \( \mathbf{v}(t, x, y, z) = (u, v, w)^T \) with superscript \( \bar{\mathbf{T}} \) denoting transposition. Variables \( \theta(t, x, y, z) \) and \( \xi(t, x, y, z) \) denote deviations of temperature and salinity from their stationary linear profiles, so

\[
\begin{align*}
T(t, x, y, z) &= T_\Delta + T_+ [\theta(t, x, y, z) - z], \\
S(t, x, y, z) &= S_\Delta + S_+ [\xi(t, x, y, z) - z].
\end{align*}
\]

\( T_+ \) and \( S_+ \) are the temperature and salinity at the lower boundary of the area.

Free-slip boundary conditions are used for the dependent variables (the horizontal velocity component is undefined):

\[
u_z = v_z = w = \theta = \xi = 0 \quad \text{at} \quad z = 0, 1.
\]

It is believed that they are suitable to describe the convection in the inner layers of liquid and do not change significantly the convective instability occurrence criteria for the investigated class of systems [13].

Derivation of amplitude equations - general frame of decomposition. – Consider the equations for double-diffusive convection in the vicinity of a bifurcation point, the temperature and salinity Rayleigh numbers for which are designated as \( R_{Tc} \) and \( R_{Sc} \) respectively. In this case the Rayleigh numbers can be represented as follows:

\[
R_T = R_{Tc}(1 + \varepsilon^2 r_T), \quad R_S = R_{Sc}(1 + \varepsilon^2 r_S).
\]

At least one of the values \( r_T \) or \( r_S \) is of unit order, and the small parameter \( \varepsilon \) shows how far from the bifurcation point the system is. In the case when the system is destabilized by increasing the temperature gradient in the layer we have \( r_T = 1 \) and \( r_S = 0 \). Respectively \( R_T = R_{Tc}(1 + \varepsilon^2) \) and \( R_S = R_{Sc} \). According to these expressions the small parameter can be defined by formula:

\[
\varepsilon = \sqrt{\frac{R_T - R_{Tc}}{R_{Tc}}}, \quad \varepsilon = \sqrt{\frac{R_S - R_{Sc}}{R_{Sc}}}.
\]

To derive the amplitude equations we use the derivative-expansion method [13], which is the case of the multiple-scale method. Introduce the slow variables:

\[
T_1 = \varepsilon t, \quad T_2 = \varepsilon^2 t, \quad X = \varepsilon x, \quad Y = \varepsilon y.
\]
In accordance with the chosen method we assume that the
dependent variables now depend on \( t, T_1, T_2, x, y, z, X \),
which are considered as independent. Also we replace the
derivatives in the equations (1) for the prolonged ones by the
rules:
\[
\partial_t \rightarrow \partial_t + \varepsilon \partial_{T_1} + \varepsilon^2 \partial_{T_2}, \quad \partial_x \rightarrow \partial_x + \varepsilon \partial_X, \quad \partial_y \rightarrow \partial_y + \varepsilon \partial_Y.
\]
Then the equations (1) can be written as:
\[
\dot{\vec{X}} = -\varepsilon \vec{L}_1 \overrightarrow{\varphi} - \varepsilon^2 \vec{L}_2 \overrightarrow{\varphi} - \vec{N}_1(\varphi, \varphi) - \varepsilon \vec{N}_2(\varphi, \varphi).
\] (2)

Where we have introduced vector of the dependent vari-
ablets \( \vec{L}, \vec{L}_1 \) and \( \vec{L}_2 \):
\[
\vec{L} = L_0 \partial_t - L_1 \Delta + L_2 \partial_x + L_3 \partial_y + L_4 \partial_z,
\]
\[
\vec{L}_1 = L_0 \partial_t - L_2 \partial_x + L_3 \partial_y + L_4 \partial_z,
\]
\[
\vec{L}_2 = L_0 \partial_t - L_1 \Delta - \sigma \partial_t - L_1 R_{L1} + \sigma R_{S_L} R_{L2}.
\]

Here \( \Delta = \partial_x^2 + \partial_y^2 \). Matrices \( \vec{L}(6 \times 6) \) have the following nonzero elements:
\[
L_a = \text{diag}(1,1,1,1,1,0), \quad L_b = \text{diag}(\sigma, \sigma, \sigma, 1, \tau, 0),
\]
\[
L_c(6,1) = 1, \quad L_c(6,2) = 1, \quad L_b(2,6) = 1,
\]
\[
L_d(6,2) = 1, \quad L_d(2,6) = 1, \quad L_d(6,3) = 1,
\]
\[
L_e(3,6) = 1, \quad L_e(6,3) = 1, \quad L_d(4,3) = 1,
\]
\[
L_f(5,3) = 1, \quad L_R(3,4) = 1, \quad L_R(2,3,5) = 1.
\]

Also nonlinear operators \( \vec{N}_1 \) and \( \vec{N}_2 \) are introduced as the
following vectors:
\[
\vec{N}_1(\varphi, \varphi_j) = (\vec{M}_1(\varphi, u_j), \vec{M}_1(\varphi, v_j), \vec{M}_1(\varphi, w_j), \vec{M}_1(\varphi, \psi_j), 0)^T,
\]
\[
\vec{M}_1(\varphi, u_j, v_j) = \hat{u}_i(\varphi, u_j) + v_i(\varphi, v_j),
\]
\[
\vec{M}_2(\varphi, u_j, v_j) = \hat{u}_i(\varphi, u_j) + v_i(\varphi, v_j).
\]

We seek solutions of equations (2) in the form of asym-
totic series in powers of small parameter \( \varepsilon \):
\[
\varphi = \sum_{i=1}^{\infty} \varepsilon^i \varphi_i = \varepsilon \varphi_1 + \varepsilon^2 \varphi_2 + \varepsilon^3 \varphi_3 + \cdots.
\] (3)

After their substitution in (2) and collection the terms at
\( \varepsilon^n \) we obtain the systems of equations to determine the
systems at higher powers of the small parameter, for exam-
ple, to include in the final amplitude equations a nonlinear
terms of the fifth order.

However, in this article we restrict ourselves to the equa-
tions obtained at no higher than \( \varepsilon^3 \). Thus linear equations
at \( \varepsilon^1 \) will give us the form of solution as the sum of nor-
mal modes and conditions for the absence of secular terms
in the systems at \( \varepsilon^2 \) and \( \varepsilon^3 \) will lead to equations on the
amplitudes of each of the normal modes.

**The terms of the first order in \( \varepsilon \).** At \( O(\varepsilon^1) \) we obtain the following system:
\[
\dot{\varphi}_1 = 0.
\] (5)

This linear system has a solution in the form of sum of \( n \)
normal modes (convective rolls):
\[
\varphi_1 = \sum_{j=1}^{n} \varphi_{1j} + \tilde{\varphi}_1 + \text{c.c.}
\]
\[
= \sum_{j=1}^{n} A_j(X, Y, T_1, T_2) \varphi_{1j} e^{i \lambda t} e^{i \xi j} \left\{ \sin \pi \frac{x}{2} \cos \pi \frac{y}{2} \right\}
\]
\[
+ \tilde{\varphi}_1(X, Y, T_1, T_2) + \text{c.c.}.
\] (6)

The cosine in the braces is selected for variables \( u_1, v_1, p_1 \),
in another cases the sine is selected. Vectors \( \vec{k}_j \) have com-
ponents \( \vec{k}_j = (k_{1j}, k_{2j}) \). Components of the vectors \( \varphi_{1j} \) and \( \tilde{\varphi}_1 \) are:
\[
\varphi_{1j} = \left( \begin{array}{c}
\frac{ik_{1j} \pi}{k^2}, \frac{ik_{2j} \pi}{k^2}, 1, \frac{1}{\lambda + \tau x^2}, \frac{1}{\lambda + \tau y^2}, -\frac{\pi}{k^2}(\lambda + \sigma x^2) \end{array} \right),
\]
\[
\tilde{\varphi}_1 = (\tilde{u}_1, \tilde{v}_1, 0, 0, 0, \tilde{p}_1).
\]

Without the great loss of generality we omit \( \tilde{u}_1, \tilde{v}_1, \) and \( \xi \), which as the other members with caps have sense of
integration constants on slow horizontal variables. More
detailed analysis shows that these terms are zero or do not
lead to a physically meaningful results. The terms \( \tilde{u}_1 \) and
\( \tilde{v}_1 \) form the velocity field, against which the convec-
tion develops.

Components of \( \varphi_{1j} \) are obtained by substitution of the
anzats (4) into equations (5), and it is true \( L_j \varphi_{1j} = 0 \).
Where \( L_j = \lambda \sigma \lambda_x + \tau x^2 \lambda_y + ik_{1j} \lambda_c + ik_{2j} \lambda_d + \pi \lambda e_1 - L_g - \sigma \partial_t + \sigma R_{S_L} R_{L2} \).

**Dispersion relation.** Parameters of each from \( n \) roll-
mode \( \lambda, k_{1j}, k_{2j}, R_{Tc}, R_{Sc} \) are related by the equation:
\[
(\lambda + \sigma x^2)(\lambda + \tau x^2)(\lambda + \tau y^2)
\]
\[
+ \sigma(k^2/x^2)[R_{S_L}(\lambda + \tau x^2) - R_{Tc}(\lambda + \tau y^2)] = 0.
\]

Here \( k^2 = k_{1j}^2 + k_{2j}^2 \), and \( x^2 = k^2 + \pi^2 \). This equation has
three roots, two of which can be complex conjugates. In
the case of Hopf bifurcation these two roots acquire posi-
tive real part at some \( R_{Tc} \) (\( \omega \) is a frequency of convective
waves):
\[
R_{Tc} = \frac{\sigma + \tau}{1 + \sigma} R_{Sc} + \frac{\sigma^6}{\sigma k^2} (1 + \tau)(\tau + \sigma),
\]
\[
\omega^2 = \frac{1 - \tau}{1 + \sigma} R_{Sc} \frac{k^2}{\sigma^2} - \tau^2 x^4 > 0.
\]

Here \( \omega \) is a frequency of convective waves, and it is assumed to be real. This means that the number \( R_{Sc} \) should not be too small. In this paper we consider double-diffusive convection at Hopf bifurcation points, i.e. in all cases \( \lambda = i \omega \).

**Critical wavenumber.** From the expressions (1) one can see that the minimal Rayleigh number \( R_{Tc} \) is obtained at \( k_c = \pi/\sqrt{2} \), which defines the characteristic size of convective cells, arising with an increase of \( R_T \) above the critical value. Along with the mode having the wavenumber \( k_c \), the adjacent modes different from the central mode by an amount \( \varepsilon \) also are exited, which leads to the result that the total wavepackage looks like one mode with the wavenumber \( k_c \) and variable amplitude described by the amplitude equations.

For the sufficiently large Rayleigh numbers the situation is changing so that the characteristic critical wavenumber is of the order \( 0.23 \sqrt{\omega} \) and may reach values of \( k_c \approx 100 \) [1]. As in the case of small \( R_S \) the first losing stability mode is the mode with \( k_c = \pi/\sqrt{2} \). However, with the growth of \( \omega \) the wavelength of the fastest growing mode increases proportionally \( \sqrt{\omega} \). For some \( \varepsilon \) this growth is stabilized at \( k_c \approx 10 - 100 \), which corresponds to a narrow convective cells. And similarly the adjacent modes are excited, forming a wave packet, which looks like one mode with variable amplitude.

So it makes sense to derive the desired amplitude equations for convective cells of an arbitrary width assuming that the specific value of a small parameter each time defines the value of \( k_c \), which we will further denote as \( k \).

**Shape of the cells.** In the studied system any number of roll modes with different wavevectors can be excited simultaneously, producing convective cells of various forms. Thus, superposition of the two rolls at right angles to each other gives the square-type cells, three rolls at angles of 120 degrees form hexagonal cells. In this paper we do not limit ourselves to any one cell shape, but consider the general situation, when \( n \) roll modes at arbitrary angles to each other are exited. The desired amplitude equations will give the opportunity to find out which of modes given initially in some region of space become dominant and determine the final shape of the cells.

**Resolution conditions.**

**General structure of equations.** The obtained systems have the following general form:

\[
\hat{L}\varphi_i = Q_i.
\]

Functions \( Q_i \) include terms, resonating with the left parts of equations, i.e. \( Q_i = Q^{(1)}_i + Q^{(2)}_i + Q^{(3)}_i \). Here \( Q^{(1)}_i \) and \( Q^{(2)}_i \) generate the secular terms of two types in the solutions, but \( Q^{(3)}_i \) doesn't generate any secular terms and contains only unimportant terms for the explored case. The conditions of the first type secular terms absence reduce to demand of orthogonality functions \( Q^{(1)}_i \) and solutions \( F_j \) of the adjoint homogeneous equation \( \hat{L}^*F_j = 0 \) and usually take form of amplitude equations. Terms \( Q^{(2)}_i \) are the constants with respect to quick variables. Not to brake the regularity of the asymptotic expansions [6] they should be equal to zero \( Q^{(2)}_i = 0 \) [14]. These conditions also take form of amplitude equations.

**Scalar products.** Introduce scalar product of the vectors, composed of the dependent variables:

\[
\langle \varphi_i, \varphi_j \rangle_0 = \lim_{l \to \infty} \frac{2}{l^2} \int_0^1 \int_{l/2}^{l/2} (u_i u_j + v_i v_j + w_i w_j + \theta_i \theta_j + \xi_i \xi_j + p_i p_j) dx dy dt dz.
\]

The actual forms of the functions \( \varphi_i \) and \( \varphi_j \), arising in the explored cases can be the following:

\[
\varphi_i = \overline{\varphi_i} e^{int \omega t} e^{i(\vec{k}_i, \vec{\xi}_i)} \left\{ \frac{\sin \pi m z}{\cos \pi m z} \right\} + c.c.,
\]
\[
\varphi_j = \overline{\varphi_j} e^{int \omega t} e^{i(\vec{k}_j, \vec{\xi}_j)} \left\{ \frac{\sin \pi m z}{\cos \pi m z} \right\} + c.c.. \]

Then we get:

\[
\langle \varphi_i, \varphi_j \rangle_0 = D(\vec{k}_i - \vec{k}_j) \delta_{n, n_j} \delta_{m, m_j} \left( \varphi_i, \varphi_j \right) + D(\vec{k}_i + \vec{k}_j) \delta_{-n, n_j} \delta_{m, m_j} \left( \varphi_i, \varphi_j \right) + c.c. = \langle \varphi_i, \varphi_j \rangle_0 + c.c. \cdot
\]

Here as \( \delta_{ij} \) we denoted the Kronecker delta, and function \( D(x) \) is defined as \( D(0) = 1 \) and \( D(x) = 0 \) at \( x \neq 0 \). Also we have introduced scalar product for the amplitudes of vectors of the dependent variables:

\[
\langle \overline{\varphi_i}, \overline{\varphi_j} \rangle_0 = \overline{u_i u_j} \overline{v_i v_j} \overline{w_i w_j} \overline{\theta_i \theta_j} \overline{\xi_i \xi_j} \overline{p_i p_j}.
\]

Thus for the sake of amplitude equations derivation from the resolution conditions we have introduced the three cases of scalar products: \( \langle \varphi_i, \varphi_j \rangle_0 \), \( \langle \varphi_i, \varphi_j \rangle_0 \) and \( \langle \overline{\varphi_i}, \overline{\varphi_j} \rangle_0 \).

The first one is the initial scalar product in the integral form, the second and third are introduced for getting the nonlinear and linear terms of the amplitude equations respectively.

**The resolution condition.** The functions in the right parts of equations \( Q^{(1)}_i \) have the following general form:

\[
Q^{(1)}_i = \sum_{q=1}^{p} \overline{Q^{(1)}_{iq}} e^{i n_q z} e^{i(\vec{k}_i, \vec{\xi}_i)} \left\{ \frac{\sin \pi m z}{\cos \pi m z} \right\} + c.c. \cdot
\]

Here \( p \) is the number of terms of the considered type in the functions \( Q^{(1)}_i \). The resolution conditions in this case have the form:

\[
(Q^{(1)}_i, F_j)_0 = \sum_{q=1}^{p} \left[ D(\vec{k}_i - \vec{k}_j) \delta_{n_q, n_{iq}} \delta_{m_q, m_{iq}} \langle \overline{Q^{(1)}_{iq}}, \overline{F_j} \rangle + D(\vec{k}_i + \vec{k}_j) \delta_{-n_q, n_{iq}} \delta_{m_q, m_{iq}} \langle \overline{Q^{(1)}_{iq}}, \overline{F_j} \rangle \right] = 0.
\]
Here we have used the explicit expressions for the vectors $F_j$ of homogeneous adjoint equation solutions:

$$F_j = F_j e^{i \omega t} e^{i(k_j \cdot \varphi)} \left\{ \frac{\sin \pi z}{\cos \pi z} \right\} + c.c., \quad (L^*_?)^T F_j = 0.$$  

In many cases we can explicitly resolve Kronecker deltas in equations (3), when remain only nonzero terms with $\vec{k} = \vec{k}_j$ and $n_0 = m_q = 1$. Then the resolution conditions for the considered systems of equations will be [15]:

$$\langle Q_{ij}, F_j \rangle = 0 \quad \text{i.e., for the compatibility of the obtained algebraic systems of equations its right parts should be orthogonal to the solutions of the adjoint homogeneous system. The actual form of the vectors $F_j$ in our case is:}$$

$$F_j = \left( i k_{az} \pi, i k_{bz} \pi, k^2 \sigma R_{TE}, \frac{k^2 \sigma R_{SE}}{\lambda^2 + \pi^2}, -\frac{k^2 \sigma R_{SE}}{\lambda^2 + \tau^2}, \pi(\lambda^2 + \sigma^2) \right)^T.$$

Equations at $\varepsilon^2$. –

General frame of derivation. Find the amplitude equations derived from the system at $\varepsilon^2$. Write vector of the right parts $Q_2$ as a sum of three components mentioned earlier:

$$\tilde{L}\varphi_2 = -\tilde{L}_1 \varphi_1 - \tilde{N}_1 (\varphi_1, \varphi_1) = Q_2^{(1)} + Q_2^{(2)} + Q_2^{(3)}.$$

Note that $\varphi_1 = \varphi_1^{(0)} + \tilde{\varphi}_1$, where $\varphi_1^{(0)}$ is the solution of the homogeneous equation $\tilde{L}_1 \varphi_1^{(0)} = 0$, $\tilde{\varphi}_1$ is the averaged fields depending only on slow variables. Then write the expressions for components $Q_2$, omitting zero terms:

$$Q_2^{(1)} = -\tilde{L}_1 \varphi_1^{(0)} - \tilde{N}_1 (\tilde{\varphi}_1, \varphi_1^{(0)}),$$

$$Q_2^{(2)} = -\tilde{L}_1 \tilde{\varphi}_1,$$

$$Q_2^{(3)} = -\tilde{N}_1 (\varphi_1^{(0)}, \varphi_1^{(0)}).$$

Stream function. To exclude secular terms of the second type one should require fulfillment of equality $Q_2^{(2)} = 0$. Written in components it gives the following system:

$$Q_2^{(2)}(1) = -\tilde{u}_1 T_1 - \tilde{p}_1 X = 0,$$

$$Q_2^{(2)}(2) = -\tilde{v}_1 T_1 - \tilde{p}_1 Y = 0,$$

$$Q_2^{(2)}(6) = -\tilde{u}_1 X - \tilde{v}_1 Y = 0.$$

To satisfy these equalities introduce horizontal stream function $\Psi$ by formulas:

$$\tilde{u}_1 = \Psi_Y, \quad \tilde{v}_1 = -\Psi_X, \quad \Psi_{T_1} = 0. \quad (9)$$

Also it is true $\tilde{p}_1 = 0$ with the accuracy to constants on horizontal variables.

Amplitude equations. Calculations show that for $Q_2^{(1)}$ is true the following expression:

$$Q_2^{(1)} = \sum_{j=1}^{\infty} \{Q_{2ja} (A_j T_1 + (ik_{az} \Psi_Y - ik_{bz} \Psi_X) A_j)$$

$$+ Q_{2jb} A_j X + Q_{2jc} A_j Y \} e^{i \phi_j} \left\{ \frac{\sin \pi z}{\cos \pi z} \right\} + c.c.,$$

Here we have introduced phases $\phi_j = \omega t + k_{az} x + k_{bj} y$ of each mode, and components of the vectors in the expression are:

$$Q_{2ja} = -L_a \varphi_{1j}, \quad Q_{2jb} = (2i k_{az} L_b - L_c) \varphi_{1j},$$

$$Q_{2jc} = (2i k_{az} L_b - L_d) \varphi_{1j}.$$

The condition of there be no secular terms of the first type (3) in the solutions of the equations at $\varepsilon^2$ is written as $\langle Q_{2j}, F_j \rangle = 0$ and, after some calculations, it reduces to requirement $\langle \tilde{Q}_{2j}, F_j \rangle = 0$ for each $j = 1 \ldots n$. Or more explicitly:

$$\langle \tilde{Q}_{2j}, F_j \rangle = (Q_{2ja}, F_j) [A_j T_1 + (ik_{az} \Psi_Y - ik_{bz} \Psi_X) A_j]$$

$$+ \langle \tilde{Q}_{2jb}, F_j \rangle A_j X + \langle \tilde{Q}_{2jc}, F_j \rangle A_j Y = 0.$$

Finally the amplitude equations take the following form:

$$A_j T_1 + 2 \alpha_0 (ik_{az} A_j X + ik_{bz} A_j Y)$$

$$+ (ik_{az} \Psi_Y - ik_{bz} \Psi_X) A_j = 0, \quad j = 1 \ldots n. \quad (10)$$

Where $\alpha_0 = \langle \tilde{Q}_{2jb}, F_j \rangle / (2ik_{bz} \langle \tilde{Q}_{2ja}, F_j \rangle)$, or finally:

$$\alpha_0 = \frac{i \omega}{\varepsilon} \left[ 1 + \left( \frac{\pi^2}{2k^2} - 1 \right) \right]$$

$$\times \left( 1 - \frac{\sigma^2}{\omega^2} \left( \frac{\tau + \sigma + \tau \sigma}{\omega^2 + (1 + \tau + \sigma) \sigma^2} \right) \right) \quad (11)$$

Here we have introduced coefficient $\beta$, which is evidently defined by the above expression. Equations (9) in many important cases can be resolved explicitly and usually imply some kind of transport, so further we don’t discuss their solutions.

Equations (10) and (11) together consist the desired system of amplitude equations obtained as a result of consideration of the members at $\varepsilon^2$ in the multiple-scaled method. If the first one is satisfied by introducing a horizontal stream function $\Psi(X, Y, T_2)$ independent on the slow time $T_1$, then the second one will be used to exclude members alike $A_j T_1$ from the final amplitude equations. Obtained for $A(X, Y, T_2)$ and $\Psi(X, Y, T_2)$ solutions of equations (13) one should substitute into the equations (11) to find the dependence of the amplitudes from $T_1$.

Equations at $\varepsilon^3$. –

General frame of derivation. At last we write the resulting family of amplitude equations for the system at $\varepsilon^3$. For this purpose we need the solutions for $\varphi_1$ and $\varphi_2$, which can be expressed in a general form:

$$\varphi_1 = \varphi_1^{(0)} + \varphi_1^{(1)}, \quad \varphi_2 = \varphi_2^{(0)} + \varphi_2^{(1)} + \varphi_2^{(2)} + \varphi_2^{(3)}.$$  

Here $\varphi_1^{(0)}$ are the general solutions of homogeneous equations $\tilde{L}_1 \varphi_1^{(0)} = 0$, $\varphi_2^{(1)}$ and $\varphi_2^{(2)}$ are linear and nonlinear on
amplitude terms of the particular solution of the inhomogeneous equation \( \hat{L}(\varphi_2^{(1)} + \varphi_2^{(2)}) = Q_2, \) \( \varphi_i \) are the averaged fields on the slow horizontal equations, arising as an integrating constants. For \( \varphi_2^{(1)} \) we have the following expression:

\[
\varphi_2^{(1)} = \sum_{j=1}^{n} (\varphi_{2j} A_j + \varphi_{2j} A_j Y) e^{i\omega_j} \left\{ \begin{array}{l} \sin \pi z \\ \cos \pi z \end{array} \right\} + c.c.,
\]

\[
\varphi_{2j} = \varphi_{2j} + 2a_0 k_{a_0} \varphi_{2j} + \varphi_{2j} = (\varphi_{2j} + 2a_0 k_{a_0} \varphi_{2j}),
\]

where \( L_j \varphi_{2j} = Q_{2j}, \) \( L_j \varphi_{2j} = Q_{2j}, \) \( L_j \varphi_{2j} = Q_{2j}. \)

Write the system at \( \varepsilon^3 \) in a general form:

\[
\hat{L}_3 = -\hat{L}_1 \varphi_2 - \hat{L}_2 \varphi_1 - \hat{N}_1(\varphi_1, \varphi_2) - \hat{N}_1(\varphi_2, \varphi_1)
\]

\[
- \hat{N}_2(\varphi_1, \varphi_1) = Q_3^{(1)} + Q_3^{(2)} + Q_3^{(3)}.
\]

Then the expressions for \( Q_3^{(1)} \) and \( Q_3^{(2)} \), the only needed for the derivation of amplitude equations take form:

\[
Q_3^{(1)} = -[\hat{L}_1 \varphi_2^{(1)} + \hat{L}_2 \varphi_1^{(0)} + \hat{N}_1(\varphi_1, \varphi_2^{(1)}) + \hat{N}_1(\varphi_1, \varphi_1^{(0)})
\]

\[
+ \hat{N}_2(\varphi_1^{(0)}, \varphi_1) + \hat{N}_1(\varphi_1^{(0)}, \varphi_2^{(0)}) + \hat{N}_1(\varphi_2, \varphi_1)] = Q_3^{(1)} + Q_3^{(1p)} + Q_3^{(3p)},
\]

\[
Q_3^{(2)} = -[\hat{L}_1 \varphi_2^{(1)} + \hat{L}_2 \varphi_1^{(0)} + \hat{N}_1(\varphi_1^{(1)}, \varphi_1^{(0)})
\]

\[
+ \hat{N}_2(\varphi_1^{(0)}, \varphi_1^{(0)}) + \hat{N}_2(\varphi_1^{(0)}, \varphi_2^{(0)}) + \hat{N}_2(\varphi_2, \varphi_1)].
\]

Here we have separated linear \( Q_3^{(1)} \) and nonlinear \( Q_3^{(2)} \) on \( A_j \) terms, and also terms \( Q_3^{(1p)} \), \( Q_3^{(3p)} \), containing \( \varphi_1 \). Denote \( \beta_0 = -\langle \varphi_{2ja}, \hat{F} \rangle \), then the desired amplitude equations in a general form are:

\[
\frac{1}{\beta_0} (Q_3^{(1)}, F_j) = \frac{1}{\beta_0} (Q_3^{(1)}, F_j) + \frac{1}{\beta_0} (Q_3^{(1)}, F_j) = 0.
\]

**Linear terms.** The detailed calculations give the following formula for the linear terms of equations:

\[
\frac{1}{\beta_0} (Q_3^{(1)}, F_j) = -A_{jT_2} + \frac{1}{\beta_0} (Q_{3j}, \hat{F}_j) A_j
\]

\[
+ \frac{1}{\beta_0} (Q_{3ja}, \hat{F}_j) A_j + \frac{1}{\beta_0} (Q_{3j}, \hat{F}_j) A_j Y Y
\]

\[
+ \frac{1}{\beta_0} (Q_{3jc}, \hat{F}_j) A_j X Y = -A_{jT_2} + r A_j - \alpha_0 \Delta_1 A_j
\]

\[
+ \frac{1}{k^2} \left( k^2 A_j X X + 2 k a_j b_j A_j X Y + k^2 A_j Y Y \right).
\]

**Terms with stream function.** Similarly one can get a formula for the terms with \( \Psi \):

\[
\frac{1}{\beta_0} (Q_3^{(1p)}, F_j) = -\frac{1}{\beta_0} \left( \frac{1}{k^2} \Psi_{3j} (\hat{F}_j) + \frac{1}{k^2} \right) A_j \left[ (k^2 - k^2) \Psi_{XY} + k a_j b_j (\Psi_{YY} - \Psi_{XX}) - J(A_j, \Psi) = J(A_j, \Psi) + \right.
\]

\[
\left. + \frac{1}{k^2} A_j \left[ (k^2 - k^2) \Psi_{XY} + k a_j b_j (\Psi_{YY} - \Psi_{XX}) \right]. \right\}
\]

Here the Jacobian \( J(f, g) = f_x g_y - f_y g_x \) is introduced. In the formulas the following vectors of the right parts of equations are used:

\[
\begin{align*}
Q_{2ja} = -L_a \varphi_{1j}, & \quad Q_{3ja} = L_a \varphi_{2ja}, \\
Q_{3ja} = -\sigma (r_T R_T L_R) - r_s R_s L_R \varphi_{3j}, & \quad Q_{3ja} = (2a_0 k_{a_0} L_a + 2ik_{a_0} L_b - L_c) \varphi_{2ja} + L_b \varphi_{3j}, \\
Q_{3ja} = (2a_0 k_{a_0} L_a + 2ik_{a_0} L_b - L_c) \varphi_{2ja} + L_b \varphi_{3j}, & \quad Q_{3ja} = (2a_0 k_{a_0} L_a + 2ik_{a_0} L_b - L_c) \varphi_{2ja} + L_b \varphi_{3j}, \\
Q_{3ja} = (2a_0 k_{a_0} L_a + 2ik_{a_0} L_b - L_c) \varphi_{2ja} + L_b \varphi_{3j}. & \quad Q_{3ja} = (2a_0 k_{a_0} L_a + 2ik_{a_0} L_b - L_c) \varphi_{2ja} + L_b \varphi_{3j}.
\end{align*}
\]

**Nonlinear terms.** Now calculate nonlinear terms \( (Q_3^{(1)}, F_j) \), in the amplitude equations. Vector of nonlinear on amplitude members \( Q_3^{(1)} \) in the right part of the equations at \( \varepsilon^3 \) one can represent as the sum:

\[
\frac{1}{\beta_0} (Q_3^{(1)}, F_j) = \alpha_2 A_j \sum_{q=1}^{n} |A_j|^2
\]

\[
\sum_{m=1}^{n} \sum_{q=1}^{n} \sum_{p=1}^{n} \left[ D(\tilde{k}_q - \tilde{k}_p - \tilde{k}_m - \tilde{k}_j) \alpha_1^{(1)} m \right] A_m A_n A_p
\]

\[
+ D(\tilde{k}_q - \tilde{k}_p - \tilde{k}_m - \tilde{k}_j) \alpha_1^{(2)} m \right] A_m A_n A_p
\]

\[
+ D(\tilde{k}_q - \tilde{k}_p - \tilde{k}_m - \tilde{k}_j) \alpha_1^{(3)} m \right] A_m A_n A_p
\]

For the coefficients \( \alpha_1^{(s)}, (s = 1, 2, 3) \), the following expressions are true:

\[
\alpha_1^{(1)} = \frac{c_{qp} \pi}{2k^4} \beta_2 \left\{ \left( 1 - c_{mqpq} \right) \frac{\pi^2}{\varepsilon^2} \right\}
\]

\[
+ (2k^2 - c_{mqpq}) \beta_4 \left( 2k^2 - c_{mqpq} \right) \beta_7 \right\},
\]

\[
\alpha_1^{(2)} = \frac{c_{qp} \pi}{2k^4} \right\}
\]

\[
\alpha_1^{(3)} = \frac{c_{qp} \pi}{2k^4} \left( \beta_2 + \frac{\pi^2 c_{mqpq}}{k^2 c_{qp} \beta_7} \right)
\]

\[
(12)
\]
Here coefficients $\beta_4, \beta_5, \beta_6, \beta_7$, depending from $p$ and $q$ are denoted:

$$\beta_4 = \left[1 - D(\vec{k}_q + \vec{k}_p)\right] \times$$

$$\times \left\{2 + \left\{ \frac{c_{qp2} + 2\pi^2}{(\omega + c_{qp2} + 2\pi^2)(\omega + \tau (c_{qp2} + 2\pi^2))} \left( \kappa_2 + \frac{(\omega + c_{qp2} + 2\pi^2)(\omega + \tau (c_{qp2} + 2\pi^2))}{(\omega + c_{qp2} + 2\pi^2)(\omega + \tau (c_{qp2} + 2\pi^2))} \right) \right\}$$

$$\times \left\{ \frac{(\omega + c_{qp2} + 2\pi^2)(\omega + \tau (c_{qp2} + 2\pi^2))}{(\omega + c_{qp2} + 2\pi^2)(\omega + \tau (c_{qp2} + 2\pi^2))} \right\} +$$

$$\beta_5 = \frac{c_{qp1}\tau \kappa_2}{4k^2(\omega^2 - \kappa_2^2)} \left( \frac{c_{qp1} + 2\pi^2}{(\omega + c_{qp2} + 2\pi^2)(\omega + \tau (c_{qp2} + 2\pi^2))} \right)$$

$$\beta_6 = \frac{c_{qp1}\tau \kappa_2}{2(\omega + c_{qp2} + 2\pi^2)(\omega + \tau (c_{qp2} + 2\pi^2))} \left( \frac{c_{qp1} + 2\pi^2}{(\omega + c_{qp2} + 2\pi^2)(\omega + \tau (c_{qp2} + 2\pi^2))} \right)$$

$$\beta_7 = \frac{c_{qp1}\tau \omega^2}{4k^2(\omega^2 - \kappa_2^2)} \left( \frac{c_{qp1} + 2\pi^2}{(\omega + c_{qp2} + 2\pi^2)(\omega + \tau (c_{qp2} + 2\pi^2))} \right)$$

For a more compact form of the formulas the coefficients $\beta_8 = (1 + \pi^2 \tau \sigma)\omega - \kappa_2^2$ and also $\beta_9 = (1 + \pi^2 \tau \sigma)\omega - \kappa_2^2$ and $\beta_{10} = \omega^2 - (\tau + \pi \sigma)\omega - \kappa_2^2$ are introduced. It is worth to mention that coefficient $\beta_7$ turns to zero each time when $\vec{k}_q = -\vec{k}_p$ true. In addition in formulas for a few more coefficients are used:

$$\beta_{11} = \frac{\pi^2}{2\kappa^2} c_{jm2}\beta_2 - k^2, \quad \beta_{12} = 1 + \frac{\pi^2\beta_8}{2\kappa^2(\omega^2 + 2\pi^2)}$$

$$\beta_{13} = \frac{\pi^2}{2\kappa^2} (c_{mq2} + c_{mp2}) \frac{\kappa_2^2}{2\kappa^2(\omega^2 + 2\pi^2)} \left( \frac{c_{mp1} + 2\pi^2}{(\omega + c_{mp2} + 2\pi^2)(\omega + \tau (c_{mp2} + 2\pi^2))} \right)$$

$$\beta_{14} = \frac{\pi^2}{2\kappa^2} \left( \frac{c_{mq1} + 2\pi^2}{(\omega + c_{mp2} + 2\pi^2)(\omega + \tau (c_{mp2} + 2\pi^2))} \right)$$

$$\beta_{15} = \left(1 + \frac{\pi^2}{2\kappa^2} \frac{c_{mp1} + 2\pi^2}{(\omega + \tau (c_{mp2} + 2\pi^2))} \right)$$

Here $\beta_2$ is defined by the formula in (10). In all represented above formulas the values, composed from the scalar products of the mode wavenumbers are used: $c_{mq1} = (\vec{k}_m, \vec{k}_q)$, $c_{mp2} = (\vec{k}_m, \vec{k}_p)$, $c_{mp2} = (\vec{k}_m, \vec{k}_p) - (\vec{k}_m, \vec{k}_p)$, $c_{mp1} = \kappa - (\vec{k}_m, \vec{k}_p)$, $c_{mp2} = \kappa + (\vec{k}_m, \vec{k}_p)$.

**Equation for the stream function.** From the condition of there be no secular terms of the second type in the equations at $\varepsilon^3$ one should require to be true $Q_3^{(2)} = 0$.

Differentiate the first of these equations with respect to $Y$ and subtract from it the second equation differentiated with respect to $X$. Also assume that $\Delta_{\perp} \Psi \tau T_1 = 0$, as it is true in the case of $\Psi$. As a result we obtain the final equation, relating horizontal vorticity $\Psi$ with convection $A_j$.

$$(\partial T_2 - \sigma \Delta_{\perp}) \Delta_{\perp} \Psi = J(\Psi, \Delta_{\perp}) - \frac{\pi^2}{k^2} \sum_{j=1}^{n} \hat{G}_j(|A_j|^2)$$

Here we have introduced linear operator

$$\hat{G}_j(f) = \frac{1}{k^2} (k_{j\alpha} \partial_X + k_{j\beta} \partial_Y) (k_{j\alpha} \partial_Y - k_{j\beta} \partial_X) f$$

**The $A, \Psi$ family of amplitude equations.** Finally we write the resulting family of amplitude equations for the system at $\varepsilon^3$:

$$\partial T_2 A_j = r A_j + \frac{\alpha_1}{k^2} (k_{j\alpha} \partial_X + k_{j\beta} \partial_Y)^2 A_j - \omega_0 \Delta_{\perp} A_j$$

$$+ i k \omega_0 \tilde{G}_j(\Psi) A_j + J(\Psi, A_j) + N_j(A)$$

$$(\partial T_2 - \sigma \Delta_{\perp}) \Omega = J(\Psi, \Omega) - \frac{\pi^2}{k^2} \sum_{j=1}^{n} \hat{G}_j(|A_j|^2)$$

Where $\Delta_{\perp}$ is Laplacian with respect to the slow variables, $\alpha_i$ are complex coefficients. Index $j = 1 \ldots n$ denotes the mode number. This family of the systems of amplitude equations depends on the set of $n$ wavenumbers which define the shape of convective cells. Operator $\hat{G}$ in the equations describes an interaction between convection and field of horizontal vorticity, generation of vortex due to convection.

The functions $N_j(A)$ are the following combination of cubic nonlinear terms:

$$N_j(A) = \alpha_2 A_j \sum_{p=1}^{n} |A_p|^2$$

$$+ \sum_{m=1}^{n} \sum_{q=p+1}^{n} \left[ D(\vec{k}_q + \vec{k}_p - \vec{k}_m - \vec{k}_j) + D(\vec{k}_q - \vec{k}_p - \vec{k}_m - \vec{k}_j) + D(\vec{k}_q + \vec{k}_p - \vec{k}_m - \vec{k}_j) + D(\vec{k}_q - \vec{k}_p - \vec{k}_m - \vec{k}_j) \right]$$

$$D(\vec{k}_q + \vec{k}_p - \vec{k}_m - \vec{k}_j) A_m^{(1)} A_n^{(1)} A_p +$$

$$D(\vec{k}_q - \vec{k}_p - \vec{k}_m - \vec{k}_j) A_m^{(2)} A_n^{(2)} A_p +$$

$$D(\vec{k}_q + \vec{k}_p - \vec{k}_m - \vec{k}_j) A_m^{(3)} A_n^{(3)} A_p$$

$$Q_3^{(2)}(1) = \sigma \Delta_{\perp} \Psi Y - \Psi Y T_2 - \Psi Y T_1 - \tilde{\beta}_2 X - \Psi Y \Psi X Y$$

$$+ \Psi X \Psi Y Y - \frac{\pi^2}{k^2} \sum_{j=1}^{n} (k_j^2 |A_j|^2)_X + k_{j\alpha} k_{j\beta} (|A_j|^2)_Y = 0$$

$$Q_3^{(2)}(2) = -\sigma \Delta_{\perp} \Psi X + \Psi X T_2 + \Psi X T_1 - \tilde{\beta}_2 Y + \Psi Y \Psi X X - \Psi X \Psi Y Y - \frac{\pi^2}{k^2} \sum_{j=1}^{n} (k_{j\alpha} k_{j\beta} (|A_j|^2)_X + k_{j\alpha} (|A_j|^2)_Y = 0$$

$$r = \beta_2 \frac{(\sigma + \tau)(\omega^2 - \omega_0) \tau T_2 - (\sigma + 1)(\tau \omega^2 - \omega_0) \tau \Omega}{(1 - \tau) \frac{1}{k^2} \left( \frac{2\omega}{\omega_0} + \frac{\pi^2}{k^2} \frac{\beta_2}{2} \left( \frac{\pi^2}{k^2} - 1 \right) \frac{\beta_2}{2} + \frac{4\omega^2}{k^2} \beta_2 \beta_3 \right)}$$

$$\alpha_1 = \beta_2 \frac{\pi^2}{k^2} - 1 \left( \frac{2\omega}{\omega_0} + \frac{\pi^2}{k^2} \frac{\beta_2}{2} \left( \frac{\pi^2}{k^2} - 1 \right) \frac{\beta_2}{2} + \frac{4\omega^2}{k^2} \beta_2 \beta_3 \right)$$

$$\alpha_2 = \frac{\pi^2}{4\omega_0} \beta_2 \beta_3 \left( \frac{\pi^2}{k^2} - 1 \right) \frac{\beta_2}{2}$$
Here for the convenience and compactness of the expressions we introduce functions:

\[\beta_1 = \frac{(\tau + \sigma + \sigma)\omega + \tau \sigma x^2}{\omega + (1 + \tau + \sigma) x^2},\]
\[\beta_2 = \frac{(\omega + \tau x^2) x^2}{2\omega (\omega + (1 + \tau + \sigma) x^2)} = \frac{x^2}{\beta_0},\]
\[\beta_3 = \frac{(1 + \tau + 2\sigma)\omega + (1 + \tau + \sigma + \sigma) x^2}{2(\omega + x^2)(\omega + (1 + \tau + \sigma) x^2)}.
\]

(16)

Coefficients \(\alpha_0\) and \(\beta_0\) are given by the formula (11). Coefficients \(\alpha_6\), \(\alpha_7\), \(\alpha_8\), and \(\alpha_9\) in the equations (13) coincide with the same-named coefficients in the article [10]. Therein one can find the expressions for these coefficients at \(k = \pi/\sqrt{2}\) and graphs of their dependence from frequency \(\omega\).

Coefficients at the nonlinear terms \(\alpha^{(s)}_{jmap}\), \((s = 1, 2, 3)\) are presented by the formulas (12).

**Special cases of amplitude equations for the cells of different forms.** –

**Compatibility with solutions for 2D convection.** If we neglect the interaction with the horizontal stream function and consider the dynamics on the single spatial variable the obtained one-mode system reduces to the well known Ginzburg-Landau equation (CGLE):

\[A_T = r A + \alpha_5 A_{XX} + \alpha_2 A|A|^2.\]

Where \(\alpha_5 = \alpha_1 - \alpha_6\). In the limit of high Hopf frequencies the resulting equation reduces to the nonlinear Schrödinger equation (NSE) and has “dark” solitons solutions [5].

**Roll type one-mode convection.** Consider one-mode convection with convective rolls placed along the \(x\)-axis. The wave vector is: \(\vec{k} = (k, 0)\). In this case the equations (13) after some transformations of dependent and independent variables take the following shape:

\[A_T = A + \alpha_6 A_{XX} - \alpha_7 A_{YY} + \alpha_9 \Psi_{XY} A + J(\Psi, A) - iA|A|^2,\]
\[\Omega_T = \alpha_8 A_{\perp} \Omega + J(\Psi, \Omega) - (|A|^2)_{XY},\]
\[\Omega = A_{\perp}.\]

Here the new coefficients are:

\[\alpha_6 = \alpha_5 k^2 x^2/(4\pi^2 \omega), \quad \alpha_7 = \alpha_5 k^2 x^2/(4\pi^2 \omega), \quad \alpha_8 = k^2 x^2/(4\pi^2 \omega), \quad \alpha_9 = i\alpha_9 i.\]

One should especially note, that in the limit of large \(k\) the coefficients \(\alpha_6\) and \(\alpha_7\) don’t vanish and become equal \(\alpha_6 = 3k^2/(8x^2)\) and \(\alpha_7 = i/\lambda\). In this limit at the different values of \(k\) it is true \(\alpha_6 = 0\ldots3i/\lambda\). The coefficient \(\alpha_8\), describing attenuation of the vortex \(\Omega\), vanishes. Nevertheless the cross members describing in the equations interaction of the vortex and convection don’t vanish, as one could expect. For the coefficient \(\alpha_9\) at large \(\omega\) it is true \(\alpha_9 \approx -3\pi^2/2x^2 + \sigma k^2/\omega\). Thus one can assume that for the physical macro systems with double-diffusive convection, for which the sufficiently large values of \(\omega\) are typical, the effects of interaction of the convection with the field of horizontal vorticity, excitation of the vortex due to convection play an essential role.

If we assume \(\Psi = 0\) in equations (17) and exclude the forcing term, then the derived system reduces to the one equation, which is the case of 2D nonlinear Schrödinger equation (NSE):

\[iA_T = -A_{XX} + A_{YY} + A|A|^2.\]

Possibly this equation can play an essential role in modeling of pattern formation processes in various physical systems and describes the so called dry turbulence [16].

**Hexagonal type three-mode convection.** Consider three-mode convection in the case when the convective rolls are placed at the angles 120 degrees with respect to each other. The wave vectors are: \(k_1 = (k, 0), k_2 = (-k/2, k\sqrt{3}/2), k_3 = (-k/2, -k\sqrt{3}/2)\). The system (17) transforms to the following shape:

\[A_T = A + \alpha_6 A_{XX} - \alpha_7 A_{YY} + J(\Psi, A) + \alpha_9 \Psi_{XY} A - iA|A|^2 + \alpha_{11} |A|^2 + |A|^2,\]
\[B_T = B + \alpha_6 B_{XX} + \alpha_7 B_{YY} - \alpha_{12} B_{XY} + J(\Psi, B) - \frac{1}{2} \alpha_3 B_{YY} + \sqrt{3} \Psi_{XY} - \Psi_{XX}],\]
\[C_T = C + \alpha_6 C_{XX} + \alpha_7 C_{YY} + \alpha_{12} C_{XY} + J(\Psi, C) - \frac{1}{2} \alpha_9 C_{YY} - \sqrt{3} \Psi_{XY} - \Psi_{XX}] - i\alpha_{11} C_{YY} + \alpha_{12} |C|^2 + |C|^2,\]
\[\Omega_T = \frac{\alpha_8 A_{\perp} \Omega + J(\Psi, \Omega) - (|A|^2 - \frac{1}{2} |B|^2 - \frac{1}{2} |C|^2)_{XY}}{4 (|B|^2 - |C|^2)_{XY} - \sqrt{3} (|B|^2 - |C|^2)_{YY}},\]
\[\Omega = A_{\perp}.\]

Here we denoted the coefficients: \(\alpha_{61} = \frac{1}{2} \alpha_6 - \frac{i}{2} \alpha_7, \alpha_{71} = \frac{1}{2} \alpha_7 - \frac{i}{2} \alpha_9, \alpha_{61} = \alpha_{71} = \frac{\sqrt{3}}{4} (\alpha_6 + \alpha_7).\)

**Numerical experiments.** –

The details of calculation methods. For numerical simulation of the equations (13) the software packages based on ETD (exponential time differentiating) pseudo-spectral methods [17] were written to study roll-type convection and convection with square and hexagonal type cells.

In the calculations we used the numerical schemes developed in the frames of two-layers method ET2D and ET2DK method from [17]. The number of nodes on both horizontal variables was usually 256. The size of the area for calculations as a rule was chosen as 15 × 15, and
the calculations were led up to the times about \( T = 50 \). In some cases a square areas of the sizes 10 \( \times \) 10 and 25 \( \times \) 25 were used. In all cases we used the periodic boundary conditions natural for the pseudo-spectral methods.

As an initial conditions for simulation we choose either an arbitrary noise with the amplitude \( 10^{-4} \), or Gauss bell-like function \( A = 2 \exp(-0.5(X^2 + Y^2)) \).

We have performed numerical simulation for the tree cases of convection. Parameters and coefficients for these cases are the following:

- **Case 1**: \( \omega = 2000, k_e = 10, \alpha_0 = 0.098 + 0.263i, \alpha_7 = -0.019 + 0.168i, \alpha_8 = 0.974, \alpha_9 = -0.000197 - 0.277, \alpha_{11} = 0.018 - 0.688i. \)
- **Case 2**: \( \omega = 20000, k_e = 32.3, \alpha_0 = 1.043 - 0.42i, \alpha_7 = -0.185 + 0.564i, \alpha_8 = 9.72, \alpha_9 = 0.123 - 0.293i, \alpha_{11} = 0.00658 - 0.961i. \)
- **Case 3**: \( \omega = 150000, k_e = 62.8, \alpha_0 = 0.656 - 0.875i, \alpha_7 = -0.0993 + 0.596i, \alpha_8 = 18.43, \alpha_9 = 0.0374 - 0.173i, \alpha_{11} = 0.0008 - 0.989i. \)

In all cases \( \sigma = 7, \tau = 1/81. \)

**The results of numerical simulation.** Numerical simulation for the Case 2 (\( \omega = 20000 \)) shows, that convection evolves from the initial arbitrary noise to the some developed structure in a time of about \( T = 15 \). This structure has a form of elongated “clouds”, or “sticks” and “spots” for the convective amplitude (see Fig. 1) and a form of “clouds” for the respective stream function (see Fig. 2).

Numerical simulation for the Case 3 (\( \omega = 150000 \)) shows, that convective patterns become more spot-like (see Fig. 3) for the noise initial conditions. And for the Case 1 (\( \omega = 2000 \)) the patterns have a form of threads or filaments (see Fig. 4).

All obtained patterns slowly evolve with time, and system never reaches any stationary state. This is also true for the initial conditions in the form of bell-like function. It was noticed that in this case the system rather quickly (in a time of \( T = 15–35 \)) develops the condition of diffusion chaos, when the initial state is destroyed and symmetrical convection becomes irregular in both space and time. In this regime in some areas for a certain parameters peak bursts of vorticity are noticed.

In the case of two or more modes the total spatial pattern of convection appears as irregular alternation of convective cells of various shapes. Wherein each mode and stream function of the solution are qualitatively similar to the case of roll convection (Fig. 14). Of course, the solutions for each mode nonlinearly interact with each other. As a result a “curly” structure arises, composed of curling threads pieces. For the regular bell-like initial conditions mode interaction at short times gives beautiful regular patterns. For the three-mode equations these patterns may resemble for convection amplitude (see Fig. 5) and for stream function (see Fig. 6) famous Saturn’s polar hexagon.

There were cases when at some values of parameters in multi-mode convection amplitude of the cells grew up to the formation of a singular solution. For the regularization of such situations it was sufficient to put into the equations minor amendments in the form of terms of the fifth order in amplitude. A more detailed descriptions and analysis of the results of numerical modeling of the equations are beyond the scope of this article.

**An approach to the Saturn’s polar hexagon simulation.** – Saturn’s hexagon is a persisting hexagonal cloud pattern around the north pole of Saturn. The sides
of the hexagon are about 13,800 km long. The hexagon does not shift in longitude like other clouds in the visible atmosphere. Saturn’s polar hexagon discovery was made by the Voyager mission in 1981–82, and it was revisited since 2006 by the Cassini mission.

It is believed that the hexagon is described by some kind of solitonic solution. Also it is stated that the hexagon forms where there is a steep latitudinal gradient in the speed of the atmospheric winds in Saturn’s atmosphere. And the speed differential and viscosity parameters should be within certain margins. If this is not fulfilled the polygons don’t arise, as at other likely places, such as Saturn’s South pole or the poles of Jupiter.

Obviously double-diffusive convection plays in the atmospheres of such planets as Saturn or Jupiter an essential role. Here atmosphere is a mixture of hydrogen with helium, and in the upper atmosphere there exist a vertical negative gradient of temperature due to hot lower layers. Thus we have a diffusive type of double-diffusive convection in a rotation system. As a rule, rotation acts as one more diffusive component, which gives actually a case of triple-diffusive convection and complicates the analysis. Nevertheless preliminary considerations show that at large Rayleigh numbers (as in the case of Hexagon) such system behaves qualitatively as the explored double-diffusive system near the Hopf bifurcation points. So one can expect similar amplitude equations for a slow variations of convective amplitude, but with the different coefficients of such equations.

An exact derivation of amplitude equation for the Hexagon’s case is rather cumbersome task, but the obtained in this article results allow to make some hints on possible steps in solving a task of construction the equations having Hexagon as a solution. As we noted, in the case of Hexagon one can expect amplitude equations similar to (13), but with additional terms. As one can see, the solution (Fig. 5-6) for three-mode equations qualitatively resembles Hexagon, but only on not very large times. On large time the solution spreads over all area, and its shape becomes more whimsical. So one should insert into equations the stabilizing terms possibly taking into account centrifugal forces and the curvature of the surface. Thus solution will stay in the restricted area and have stable hexagonal form. This shape itself is defined by tree interacting modes, and coefficients of the equation should answer the question why three-mode solution dominates over other multi-mode solutions.

Conclusion. – The family of amplitude equations (13) describing three-dimensional double-diffusive convection in an infinite layer of fluid, interacting with horizontal vorticity field is derived. The shape of the convective cells is defined by a finite superposition of roll-type modes.

For numerical simulation of the obtained systems of amplitude equations we developed a few numerical schemes based on modern ETD (exponential time differencing) pseudospectral methods [17]. The software packages were written for simulation of roll-type convection and convection with square and hexagonal type cells.

Numerical simulation has showed that the convection in the system takes the form of convective “spots”, “sticks” or “filaments” and elongated “clouds” as for the respective stream functions. In the system quite rapidly (at time T = 15-35) a state of diffusive chaos is developed, where the initial symmetric state is destroyed and convection becomes irregular both in space and time. At the same
time in some areas there are bursts of vorticity.

The obtained results induce a deeper understanding of heat and mass transfer processes in the ocean and the atmosphere, significantly affecting the environment and migration of various impurities. These results will help to describe more adequately the convective and vortex structures that arise in physical systems with convective instability, and may also be the basis for the construction of more advanced models of systems with multi-component convection.

REFERENCES

[1] GETLING A. V., Rayleigh-Benard Convection: Structures and Dynamics, Advanced Series in Nonlinear Dynamics, Vol. 11 (World Scientific, Singapore-River Edge, New Jersey) 1998, p. 245.
[2] HUPPERT H. E. and TURNER J. S., J. Fluid Mech., 106 (1981) 299
[3] RADKO T., Double-diffusive convection (Cambridge University Press) 2013, p. 344.
[4] KOZITSKII S. B., Phys. Rev. E, 72 (2005) 056309-1
[5] NEWELL A. C. and WHITEHEAD J. A., J. Fluid Mech., 38 (1968) 279
[6] BRETHERTON C. S. and SPIEGEL E. A., Phys. Lett., 96A (1983) 152
[7] ZIPPEL J. A. and SIGGIA E. D., Phys. Fluids, 26 (1983) 2905
[8] KOZITSKII S. B., J. Appl. Mech. and Tech. Phys., 41(3) (2000) 429
[9] KOZITSKII S. B., Vestn. Udmurt. Univ. Mat. Mekh. Komp'yut. Nauki, 3 (2008) 46
[10] KOZITSKII S. B., Vestn. Udmurt. Univ. Mat. Mekh. Komp'yut. Nauki, 4 (2010) 13
[11] KOZITSKII S. B., Vestn. Udmurt. Univ. Mat. Mekh. Komp'yut. Nauki, 4 (2012) 23
[12] LANDAU L. D. and LIFSHITZ E. M., Fluid Mechanics, Course of Theoretical Physics Vol. 6, Vol. 6 (Pergamon Press, Oxford) 1999, p. 539.
[13] WEISS N. O., J. Fluid Mech., 108 (1981) 247
[14] BALMFORTH N. J. and BIETLO J. A., J. Fluid Mech., 375 (1998) 203
[15] NAYFEH A. H., Introduction to Perturbation Techniques. (John Wiley & Sons, New York-Chichester-Brisbane-Toronto) 1993, p. 536.
[16] COOKE K. L., J. Math. Anal. and Appl., 24 (1968) 372
[17] COX S. M. and MATTHEWS P. C., J. Comput. Phys., 176 (2002) 430