We reconsider the cosmological model discussed by Sung-Won Kim [Phys. Rev D 53, 6889 (1996)] in the context of Friedmann-Robertson-Walker cosmologies with a traversable wormhole, where it is assumed that the matter content is divided into two parts: the cosmic part that depends on time only and the wormhole part that depends on space only. The cosmic part obeys the barotropic equation of state \( p_c = \gamma \rho_c \). The complete analysis requires further care and reveals more interesting results than what was previously shown by the author. They can be readily applied to the evolution of a large class of cosmological models which are more general than FRW models.

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I. INTRODUCTION

Wormhole models, and in particular evolving ones, have attracted attention in the past few decades. In the Ref. [1] the author considers a Friedmann-Robertson-Walker (FRW) cosmological model with a traversable wormhole, by assuming that the matter is divided into two parts: the cosmic part that depends on time only and the wormhole part that depends on space only.

Specifically, the author considers the metric element of the wormhole in a FRW universe in the form

\[ ds^2 = -dt^2 + R(t)^2 \left( \frac{dr^2}{1 - kr^2 - \frac{\kappa r^2}{r}} + r^2 d\Omega^2 \right), \]

where \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \), \( R(t) \) is the scale factor of the universe and \( k \) is the sign of the curvature of spacetime: +1, 0, -1. The Einstein equation are written in the form

\[ \kappa \rho(t, r) = 3H^2 + \frac{3k}{R^2} + \frac{b'}{R^2 r^2} - \Lambda, \]

\[ \kappa \tau(t, r) = \left( \frac{\ddot{R}}{R} + H^2 + \frac{k}{R^2} \right) + \frac{b}{R^2 r^3} - \Lambda, \]

\[ \kappa P(t, r) = -\left( \frac{2\ddot{R}}{R} + H^2 + \frac{k}{R^2} \right) + \frac{b - r b'}{2R^2 r^2} + \Lambda, \]

where \( \kappa = 8\pi G \), \( H = \dot{R}/R \), and a prime and an overdot denote differentiation with respect to \( r \) and \( t \) respectively. We have added the cosmological constant \( \Lambda \) to the Einstein equations. Here \( \rho(t, r), \tau(t, r), P(t, r) \) are the mass energy density, radial tension and lateral pressure. Note that for the tension we have that \( \tau = -P_r \), where \( P_r(t, r) \) is the radial pressure.

Since the the mass energy density, radial and lateral pressures depend on both \( t \) and \( r \), the following ansatz for matter parts is introduced by the author:

\[ R^2(t) \rho(t, r) = R^2(t) \rho_c(t) + \rho_w(r), \]

\[ R^2(t) \tau(t, r) = R^2(t) P_c(t) + P_w(r), \]

\[ R^2(t) P(t, r) = R^2(t) P_c(t) + P_w(r). \]

The subscripts \( c \) and \( w \) indicate the cosmological and wormhole parts respectively. Note that, without any loss of generality, the Eq. (5) has been rewritten by using the radial pressure \( P_r(t, r) \) instead of the radial tension \( \tau(t, r) \), as made in Ref. [1]. The cosmological part is represented by an isotropic pressure \( P_c \).

The Eqs. (5)-(7) allow us to separate the Einstein equations (2)-(4) into two parts

\[ R^2 \left( \kappa \rho_c - \left[ 3H^2 + \frac{3k}{R^2} - \Lambda \right] \right) = \frac{b'}{r^2} - \kappa \rho_w = l, \]

\[ R^2 \left( \kappa \rho_c - \left[ 3H^2 + \frac{3k}{R^2} - \Lambda \right] \right) = -\frac{b}{r^3} - \kappa P_w = m, \]

where \( l \) and \( m \) are arbitrary separation constants. The separation constants of Eqs. (8) and (9) are equal due to the form of the cosmological part. Notice that for the cosmological part from Eqs. (8) and (9) we can construct the equation

\[ \dot{\rho}_c + 3H(\rho_c + P_c) = \frac{l + 3m}{\kappa} \dot{R}R^{-3}. \]
This is a generalized conservation equation for the cosmological part.

**Case** \( l = -3m \): For parameter values satisfying this constraint we obtain the standard conservation equation for FRW metric. In this case Eqs. (8)-(11) allow us to write the cosmological part in the form

\[
3H^2 + \frac{3(3 - m)}{R^2} = \kappa \rho_c + \Lambda, \tag{12}
\]

\[
-2\frac{\dot{R}}{R} H^2 - \frac{k - m}{R^2} = \kappa P_c - \Lambda, \tag{13}
\]

while the wormhole part is given by

\[
\frac{b'}{r^2} - \kappa \rho_w = -3m, \tag{14}
\]

\[
-\frac{b}{r^3} - \kappa P_w' = m, \tag{15}
\]

\[
\frac{b - rb'}{2r^3} - \kappa P_w = m. \tag{16}
\]

From Eqs. (14)-(16) we obtain that \( \rho_w + P_w' + 2P_w = 0 \). This relation implies that if we require that one of the pressures has a barotropic equation of state then the other one also has the same type of equation of state. So, for example, if we require that the radial pressure has the form \( \rho_r = \omega_r \rho_w \), then the lateral pressure is given by the barotropic equation of state \( P_w = - (1 + \omega_r) \rho_w/2 \).

From the above Eqs. (12)-(13), it is clear that the isotropic cosmological matter distribution \( \rho_c \) determines the behavior of the scale factor \( R(t) \), while the relevant metric function \( b(r) \) is determined by the matter distribution \( \rho_w \). Notice that Eqs. (12) and (13) are a generalization of the FRW equations, from which if \( m = 0 \) we obtain the standard FRW equations. In this case is still valid the relation \( \rho_w + P_w' + 2P_w = 0 \).

Notice that in Eqs. (14)-(16), the energy density, the radial and lateral pressures are written through the relevant metric function \( b(r) \). However, equations of the wormhole part may be rewritten through energy density or one of the pressures. We shall write these equations through the radial pressure \( P_w(r) \). Thus Eqs. (14)-(16) may be rewritten in the following differential form:

\[
b(r) = -mr^3 - \kappa r^3 P_w', \tag{17}
\]

\[
\rho_w(r) = -r^{-2} \left( r^3 P_w' \right)', \tag{18}
\]

\[
P_w(r) = P_w' + \frac{\kappa}{2} P_w'^2. \tag{19}
\]

This allows us to write the metric (11) as

\[
d\bar{s}^2 = -dt^2 + R(t)^2 \times \left( \frac{dr^2}{1 - (k - m)r^2 + \kappa r^2 P_w(r)} + r^2 d\Omega^2 \right). \tag{20}
\]

This analysis reveals that a large class of cosmological models, given by the metric (20), are determined by the arbitrary function \( P_w(r) \) and the matter distribution \( \rho_c(t) \) which, with the help of the Friedmann-like Eqs. (12) and (13), determines the scale factor \( R(t) \).

As a simple example let us consider the barotropic case \( \rho_w + P_w' + 2P_w = 0 \). Thus from Eqs. (12) and (13) we have that \( \rho_w = C r^{-3/2 - 1/\omega_r} \) and \( 2P_w = -C(\omega_r + 1) r^{-3/2 - 1/\omega_r} \), respectively. This specific class of solutions, for \( m = 0 \), was studied in Ref [2]. In this case we can consider any isotropic matter distribution \( \rho_c \), in order to define the scale factor \( R(t) \).

**Case** \( l \neq -3m \): For this parameter relation we can consider that Eq. (11) is the conservation equation of a viscous fluid, so the cosmological part will describe viscous cosmologies. In order to construct this interpretation let us make \( l = 3 \). Thus, the cosmological part may be written in the form

\[
3H^2 + \frac{3(3 + \tilde{l})}{R^2} = \kappa \rho_c + \Lambda, \tag{21}
\]

\[
\kappa P_c + 2\frac{\dot{R}}{R} H^2 + \frac{k + \tilde{l}}{R^2} - \Lambda = \frac{m + \tilde{l}}{R^2}, \tag{22}
\]

while the wormhole part is represented in the form

\[
\frac{b'}{r^2} - \kappa \rho_w = 3\tilde{l}, \tag{23}
\]

\[
-\frac{b}{r^3} - \kappa P_w' = m, \tag{24}
\]

\[
\frac{b - rb'}{2r^3} - \kappa P_w = m. \tag{25}
\]

From Eqs. (23)-(25) we have that \( \kappa \rho_w + \kappa P_w' + 2\kappa P_w = -3(\tilde{l} + m) \). If we require a barotropic radial pressure \( \rho_r = \omega_r \rho_w \), the lateral pressure has a generalized equation of state of the form \( \kappa P_w = -3(\tilde{l} + m)/2 - (1 + \omega_r) \kappa \rho_w/2 \).

In order to consider the viscous models we may introduce the cosmological viscous pressure in the form \( P_{\text{visc}} = \rho_c - 3H \xi \), where \( \xi \) is the bulk viscosity. Thus from Eq. (11) we obtain that

\[
\xi(t) = \frac{\tilde{l} + m}{3\kappa RR}. \tag{26}
\]

It is interesting to remark that for \( \tilde{l} = 0 \) we obtain standard open, flat and closed viscous FRW cosmologies.

In this case, from Eqs. (23)-(25), we can write the relevant metric function \( b(r) \), the radial and lateral pressures through the energy density in the following integral form:

\[
b(r) = C_1 + \tilde{l} r^3 + \kappa \int r^2 \rho_w(r) dr, \tag{27}
\]

\[
\kappa P_w(r) = -C_1 \frac{r^3}{3} - (\tilde{l} + m) - \frac{\kappa}{r^3} \int \rho_w(r)r^2 dr \tag{28}
\]

\[
\kappa P_w(r) = C_1 \frac{r^3}{3} - (\tilde{l} + m) - \frac{\kappa}{2} \rho_w(r) + \kappa \frac{2}{r^3} \int r^2 \rho_w(r) dr. \tag{29}
\]

These equations for the constraint \( \tilde{l} = -m \) may be directly rewritten in the form of Eqs. (14)-(19).
This allows us to write the metric (1) in the following form:

\[ ds^2 = -dt^2 + R(t)^2 \left( \frac{dr^2}{1 + \frac{C_2}{r^2} - (k + \tilde{l})r^2 + \kappa \int r^2 \rho_w(r) dr} + r^2 d\Omega^2 \right). \] (30)

In this case the analysis reveals that a large class of viscous cosmological models, given by the metric (30), are determined by the arbitrary function \( \rho_w(r) \) and the isotropic matter distribution \( \rho_c(t) \) which, with the help of the Friedmann-like Eqs. (21) and (22), determines the scale factor \( R(t) \).

As a simple application let us consider the cosmological model studied in Ref. [1], defined by the equation of state \( p_c = \gamma \rho_c \) for the cosmological part. In this case the conservation equation (11) (with \( l = 3l \)) gives for the cosmic energy density

\[ \kappa \rho_c = CR(t)^{-3(\gamma + 1)} + 3 \frac{\tilde{l} + m}{1 + 3\gamma} R^{-2}(t), \] (31)

where \( C \) is an integration constant. By putting this relationship into Eq. (21) we find that the scale factor satisfies the differential equation

\[ 3H^2 + \frac{3\alpha}{R^2} = CR(t)^{-3(\gamma + 1)} + \Lambda, \] (32)

where \( \alpha = k + (3\gamma l - m)/(1 + 3\gamma) \). The particular solution for Eq. (22) considered in Ref. [1] is expressed by the scale factor

\[ R(t) = R_0 t^{2/(3(\gamma + 1))}, \] (33)

for \( \gamma \neq -1 \). This implies that the constraint \( \alpha = \Lambda = 0 \) must be fulfilled. The bulk viscosity and energy density take the form

\[ \kappa \xi = \frac{(\tilde{l} + m)(1 + \gamma)}{2R_0^2} t^{\frac{2\gamma}{3(\gamma + 1)}}, \] (34)

\[ \kappa \rho_c = \rho_{\text{eff}} t^{-2} + 3 \frac{\tilde{l} + m}{(1 + 3\gamma)R_0^2} t^{\frac{8}{3(\gamma + 1)}}, \] (35)

respectively, where \( \rho_{\text{eff}} = 4/(3\kappa(1 + \gamma)^2) \). Note that a careful analysis indicates that the scale factor (33) defines the expansion for a large class of spacetimes expressed by the metric (30). Independently, the spatial part of this metric is defined exclusively by the matter content \( \rho_w(r) \). Let us consider a particular form of the spatial part. If we require, for example, the condition \( P_w = \beta \rho_w \), from Eq. (28) we find that

\[ \kappa \rho_w = CR^{-\frac{3\beta + 1}{3}} - 3 \frac{\tilde{l} + m}{3\beta + 1}, \] (36)

where \( C \) is a constant of integration. By putting this expression back into Eq. (28) we find that \( C_1 = 0 \) in order to have the required barotropic equation of state \( P_w = \beta \rho_w \). From Eqs. (29) and (36) we obtain for lateral pressure

\[ \kappa P_w = -C(\beta + 1) \frac{3}{2r} r^{-\frac{3\beta + 1}{3\beta + 1}} - 3 \frac{\beta (\tilde{l} + m)}{3\beta + 1}. \] (37)

In this case the metric takes the following form:

\[ ds^2 = -dt^2 + R_0^2 e^{\frac{2}{3\beta + 1}} \times \left( \frac{dr^2}{1 - \alpha r^2 - C \beta r^{-\frac{1 + m}{3\beta + 1}}} + r^2 d\Omega^2 \right), \] (38)

where \( \alpha = -(l + k) - \frac{l + m}{3\beta + 1} \). From Eq. (37) we conclude that the lateral pressure has a barotropic equation of state only for a vanishing bulk viscosity, since \( \tilde{l} = -m \) must be required. The specific Kim’s solution is obtained by putting \( \beta_{\text{eff}} = -(1 + 2\beta_{Kim}) \) and \( \tilde{l} = -m \) into Eqs. (30) and (31).

As another example, we can consider a direct generalization of the previously discussed viscous model by including the constant \( C_1 \). As we can see from Eqs. (27)–(29), in this case the radial and lateral pressures do not obey barotropic equations of state with constant state parameters.

In conclusion, the gravitational configuration defined by Eqs. (1)- (15) may be separated, with the help of the ansatz (5)-(7), into two independent gravitational systems: one with Einstein field equations describing a static gravitational field and another describing a time-dependent gravitational field. In general, the static field part corresponds to a large class of spherically symmetric solutions of the Einstein equations, which may represent zero-tidal-force wormholes or naked singularities, among others static configurations, while the time-dependent field part represents Friedmann-like field equations for FRW models with any curvature \( k \). Notice that they do not significantly differ from the standard FRW ones. For parameter values \( l = m = 0 \), the Friedmann-like equations become the standard FRW ones. These equations define the expansion factor \( R(t) \) of a large class of metrics (1), where the spatial part depends on the arbitrary function \( b(r) \). The presence of the matter content \( \rho_w(r) \) is essential to this spatial part, since from Eqs. (12)- (16) we have for \( \rho_w = P_w = 0 \) that the arbitrary function \( b(r) \) is trivially given by \( b(r) = -mr^3 \). Thus, the standard FRW field equations determine the expansion not only of an isotropic and homogeneous FRW metric, but also of more general metrics given by Eq. (1).

II. ACKNOWLEDGEMENTS

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