Multiparticle entanglement percolation

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We present percolation strategies based on multipartite measurements to propagate entanglement in quantum networks. We consider networks spanned on regular lattices whose bonds correspond to pure but non-maximally entangled pairs of qubits, with any quantum operation allowed at the nodes. Despite significant effort in the past, improvements over naive (classical) percolation strategies have been found for only few lattices, often with restrictions on the initial amount of entanglement in the bonds. In contrast, multipartite entanglement percolation outperform the classical percolation protocols, as well as all previously known quantum ones, over the entire range of initial entanglement and for every lattice that we considered.

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Quantum networks, which consist of neighboring nodes (or stations) sharing entangled pairs of particles, are the future of quantum communication [1]. Establishment of entanglement between two nodes of the network that are separated by an arbitrarily large distance represents the most important task and challenge in the study of quantum networks. The first solution to this problem was by means of quantum repeaters [2], a (one-dimensional) line of intermediate stations that share multiple noisy entangled pairs. Long-distance entanglement is then achieved via entanglement distillation at the repeater stations and the use of quantum memories.

A new approach for long-distance entanglement distribution was introduced in [3], exploiting the connectivity of multi-dimensional networks. In this setting networks are spanned on regular lattices, where bonds are partially entangled pure states of two qubits and with arbitrary quantum operations allowed at the nodes. Using local operations and classical communication (LOCC) one can probabilistically convert such states into perfect singlets [4]. Therefore, the problem is solved by classical bond percolation: entanglement propagation is possible when the conversion probability exceeds the percolation threshold of the lattice [5, 6]. However, it has been shown in [3, 7, 8] that this classical entanglement percolation (CEP) is not optimal. In fact it is sometimes advantageous to consider quantum entanglement percolation (QEP): one precedes the singlet conversion by a suitably designed set of local quantum operations, and applies CEP afterwards. Despite several studies devoted to these strategies, little is known about entanglement propagation protocols in general lattices.

In this Letter we introduce a powerful class of QEP protocols that exploit multipartite entanglement. In contrast to the strategies studied so far, which solely employ entanglement swapping and conversion into singlets, we make full use of both the classical and quantum aspects of quantum networks: connectivity and multipartite entanglement, respectively. The interplay of geometrical lattice transformations and entanglement manipulations is in fact a key ingredient to surpass CEP. Before explaining our protocols in detail, let us first state the main results of our work: i) multipartite strategies relate for the first time entanglement propagation to classical site percolation in a natural way [9]; ii) they systematically outperform all previously known classical or quantum strategies, regardless of the initial entanglement of the bonds and for every lattice that we considered; iii) they are obviously applicable to any lattice relevant for physical problems.

Let us now be more specific and recall the main idea of CEP. Each bond of the lattice is given by the state $|\varphi\rangle = \sqrt{\varphi_0} |00\rangle + \sqrt{\varphi_1} |11\rangle$, with $\varphi_0 \geq \varphi_1$ and $\varphi_0 + \varphi_1 = 1$, and can be optimally converted into a singlet (or equivalently into the Bell pair $|\Phi^+\rangle \propto |00\rangle + |11\rangle$), with probability $p(\varphi) = 2\varphi_1$. If $p(\varphi)$ is larger than the threshold for the corresponding bond percolation, then a giant cluster of maximally entangled pairs appears and long-distance entanglement between two nodes of this cluster is achieved by performing entanglement swappings.

Concerning multipartite protocols, the main idea is to produce Greenberger-Horne-Zeilinger (GHZ) states, which are the generalization of the Bell pair $|\Phi^+\rangle$ to $n$ qubits: $|\text{GHZ}_n\rangle \propto |00\ldots0\rangle + |11\ldots1\rangle$. For instance, let us show how to build a tripartite GHZ state from two neighboring bonds, that is, from two copies of $|\varphi\rangle$ sharing a common node. Guided by the optimality of Bell measurements in the ZZ basis [10] for one-repeater entanglement swapping [7, Sect. III A], we perform on the two qubits of the common node a measurement $M$ consisting of the two operators $|0\rangle\langle 0| + |1\rangle\langle 1|$ and $|0\rangle\langle 0| + |1\rangle\langle 1|$. The first outcome leads to a perfect GHZ state among the three nodes, but with probability $\varphi_0^2 + \varphi_1^2$ the second outcome yields the (unnormalized) state $\varphi_0 |000\rangle + \varphi_1 |111\rangle$. As for a pair of qubits, the latter state is transformed into a GHZ state with prob-
ability $2\varphi_1/(\varphi_0^2 + \varphi_1^2)$. Summing the two possibilities, we find that two copies of $|\varphi\rangle$ are converted into a GHZ state of three qubits with optimal probability $2\varphi_1$. The multipartite method derives its power from the fact that, while the value of this probability is the same as for the two-qubit swapping used in previous entanglement percolation studies, we now have three entangled qubits rather than two.

Although in this work we use only three-qubit GHZ states as building blocks for multipartite entanglement percolation, we can generalize the above procedure to construct GHZ states of $n+1$ qubits starting from $n$ copies of $|\varphi\rangle$ sharing a common node. In this case we have $2^{n-1}$ measurement operators $E_m$ of the form $|0\rangle\langle m| + |1\rangle\langle \bar{m}|$, where $\bar{m}$ is the complement of $m$ written in base 2. One can check that an $(n+1)$-qubit GHZ state is then created with probability

$$P(\text{GHZ}_{n+1}) = 1 - (\varphi_0 - \varphi_1) \sum_{k=0}^{[x]} \binom{2k}{k} (\varphi_0 \varphi_1)^k,$$  \hspace{1cm} (1)

where $[x]$ denotes the integer part of $x$. This probability, which is only optimal for $n \leq 2$, can be understood as a sequence of $n-1$ entanglement swappings in the ZZ basis on a chain of $n$ links, as calculated in [7, Eq. D3].

The multipartite strategy consists in creating a lattice $\mathcal{L}$, where nodes represent the GHZ states created from a lattice $\mathcal{L}'$. Two vertices in $\mathcal{L}$ are connected by a bond if the corresponding GHZ states share a common node in the original lattice, see Fig. 1. This defines a site percolation process with occupation probability $p = 2\varphi_1$. Above the percolation threshold of the new lattice, entanglement is propagated over a large distance as follows. Consider the situation in which two GHZ states of size $n$ and $m$ sharing one node have been created. One builds a larger GHZ state on $n+m-1$ particles with unit probability by applying the measurement $M$ on the two qubits of the common node. This generalized entanglement swapping can be iterated, and one eventually gets a giant GHZ state spanning the network. Finally, note that, given a GHZ state of any size, a perfect Bell pair is created between any two of its qubits by measuring all other qubits in the $X$ basis.

At this point let us define the quantities characterizing long-distance entanglement. Connectivity in percolation is characterized by the correlation length $\xi(p)$ which approaches 0 as $p \to 0$ and as $p \to 1$, and diverges as $p \to p_c$ and $p \to 1$. In CEP we are interested in the probability $P'(A \leftrightarrow B)$ of creating a Bell pair between two nodes $A$ and $B$ separated by a distance $L$ and chosen from the set of all nodes with the largest coordination number $Z_{\text{max}}$. We shall assume that $A$ and $B$ satisfy $L \gg \xi'(p)$. For $p < p'_c$, $P'(A \leftrightarrow B)$ decays exponentially in $L/\xi'(p)$, while for $p > p'_c$, the two nodes are connected only if they are both in $\mathcal{C}'$. In this limit the events \{A $\in \mathcal{C}'$\} and \{B $\in \mathcal{C}'$\} are independent, which, together with translational invariance, gives $P'(A \leftrightarrow B) = \theta^2(p)$, where $\theta(p) \equiv P'(A \in \mathcal{C}')$. Thus the problem is reduced to studying $\theta(p)$. For QEP, we further restrict the two distant nodes to be chosen only from those at which no measurement $M$ is to be made. In fact only these nodes retain the coordination number $Z_{\text{max}}$. In terms of the probability measure on the site percolation process on $\mathcal{L}$ we have $\theta(p) \equiv P(\bigcup_i \{A_i \in \mathcal{C}'\})$, where the union is over all the $Z_{\text{max}}$ sites $A_i$ possessing a qubit that is also in $A$. It follows that $\hat{P}(A \leftrightarrow B) = \hat{\theta}^2(p)$, and thus a direct comparison of $\theta'(p)$ and $\hat{\theta}(p)$ tells us which from CEP or QEP is favorable. Note that this is the case because in all our examples we have $p = p' = \hat{p} = 2\varphi_1$.

Now that the basic ingredients have been presented, we apply them to several natural examples of lattices. In the following paragraphs we prove for eight lattices that multipartite entanglement percolation protocols beats CEP. First, we examine the thresholds and show that $\hat{p}_c < p_c$ for each lattice. Then, we compute the expansions $\theta'(p)$ and $\hat{\theta}(p)$ in the high-density (i.e. maximally-entangled) limit to prove that $\hat{\theta}(p) > \theta'(p)$ as $p$ tends to unity. Finally, using Monte Carlo techniques, we show that
TABLE I: Improvement of entanglement percolation thresholds using a multiparticle strategy, and relative gain $\Delta \equiv 1 - \hat{p}_c/p_c$. We performed Monte Carlo simulations to calculate $\hat{p}_c$ for lattices 2–8; all other values can be found, with higher precision, in [8, 12, 13].

| #  | Lattice       | $\hat{p}'$ | $\hat{p}_c$ | $\Delta$ [%] |
|----|---------------|------------|-------------|-------------|
| 1  | (4, 8$^2$)    | 0.6768     | 0.6499      | 4.0         |
| 2  | Hexagonal     | 0.6527     | 0.609(0)    | 6.7         |
| 3  | Kagomé        | 0.5244     | 0.427(1)    | 18.6        |
| 4  | Square        | 0.5000     | 0.392(8)    | 21.4        |
| 5  | Dice          | 0.4755     | 0.375(5)    | 21.0        |
| 6  | Triangular    | 0.3441     | 0.294(9)    | 16.8        |
| 7  | Bowtie        | 0.4045     | 0.294(9)    | 27.1        |
| 8  | Triangular    | 0.3472     | 0.273(5)    | 21.2        |

TABLE II: Series expansions of $\theta'(p)$ and $\hat{\theta}(p)$, for $p = 1 - \varepsilon$ and $0 \leq \varepsilon \ll 1$. These formulae have been derived for a fraction $f = d/d'$ of nodes, where $d'$ ($d$) is the density of nodes of higher connectivity in the original (transformed) lattice. Note that $d' = 1$ for the Archimedean lattices, since by definition all their vertices are equivalent, while $d' = 1/3$ for the dice and $d' = 1/2$ for the bowtie lattice.

We provided numerical evidence that multiparticle entanglement percolation yields better thresholds than CEP, and therefore that the connection probability between two widely separated nodes is increased when the bond entanglement $p(\phi)$ is small and lies in the interval $(\hat{p}_c, p_c)$. We now consider the opposite regime in which connections are highly entangled, with $p = 1 - \varepsilon$ and $0 \leq \varepsilon \ll 1$. In this situation an analytical study becomes possible, and we use the perimeter method [8, App. C] to compute high-density series expansions of $\theta'(p)$ and $\hat{\theta}(p)$. The two lowest non-trivial orders of the expansions are easily calculated and are sufficient to show that our strategy leads to an improvement over CEP for all lattices, see Tab. II. Interestingly, this improvement does not appear at the first non-trivial order since there is no way to increase the number of independent connections of a node. In fact generalized entanglement swappings remove such connections, so that only the nodes where no quantum operation is performed share the property $\hat{\theta}(p) > \theta'(p)$.

The previous analyses show that the proposed QEP strategy works well near the classical thresholds and near the ideal situation of perfect connections. We thus expect that our transformations give $\hat{P}(A \leftrightarrow B) > P'(A \leftrightarrow B)$ for all values of bond entanglement. We performed Monte Carlo studies for the eight lattices studied in this work, using importance sampling via the perimeter method in the high-density region, attesting that $\hat{\theta}(p)$ is indeed strictly larger than $\theta'(p)$ for $p \in (\hat{p}_c, 1)$. This is shown for three lattices in Fig. 2.

It is now legitimate to wonder about the optimality of our multiparticle entanglement schemes. It is not difficult to see that they cannot be optimal for every value of the bond entanglement. In fact, let us consider a square lattice and the deterministic “centipede” strategy introduced in [8, Sec. VI, A]. The idea is that part of the links are used to create a spiral centipede whose “legs”
### FIG. 2: Numerical determination of $\hat{\theta} - \theta'$ showing that multipartite percolation yields better long-distance entanglement for any initial entangled state $|\varphi\rangle$, with $p = 2\varphi_1$. We show here the results for the triangle, square and hexagonal lattices (from left to right). High-density expansions are shown as light grey lines.

...have length one. To that end, one performs two entanglement swappings (measurements in the $XYZ$ basis) on three links of a square. The resulting state is combined with the fourth unused link to get a maximally entangled state $|\varphi\rangle$. Clearly, this method can only be applied to lattices whose nodes have four or more neighbors. For square lattices, it leads to $f = 1/2$ and $\theta(p) = 1$ above the point $\tilde{p}_e \approx 0.684$. Therefore, in the case of highly entangled bonds, this method is better than the present protocol, which gives $\theta(p) = 1$ only at the trivial point $p = 1$. However the centipede strategy fails whenever $p$ lies between 0.5 and $\tilde{p}_e$, and therefore does not even reach the classical threshold.

In conclusion, we have proposed a new approach to entanglement percolation in quantum networks that makes significant improvements over CEP. It makes use of multipartite entangled states to improve the probability of long-distance entanglement generation between nodes, regardless of the lattice topology or the degree of entanglement of the bonds. In fact, we have provided both analytical and numerical evidence for several well-studied lattices showing that our protocols yield better results (lower thresholds and higher probabilities) than all previously known strategies. At the moment, we do not know if the thresholds obtained from our multipartite QEP are optimal. This brings us to the important open question of whether there is a minimum amount of initial pure-state entanglement necessary for long-distance entanglement. In one-dimensional chains we know that this is possible only if the connections are maximally entangled, but while any percolation protocol defines a sufficient condition for long-distance entanglement distribution in higher-dimensional systems, a proof of the necessary amount of entanglement is lacking.

Finally, the most important challenge is to generalize the results obtained so far to the realistic case of mixed states. Recently, some results have been presented in [13], where bonds are mixed states of rank two, which may be a first step in this direction. However realistic noisy networks are best described by full-rank mixed states, and even if we do not touch this problem in this Letter, the generality of our results suggests that multipartite strategies may also be of great importance in that case.

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9. Note that bond percolation can always be “artificially” mapped to site percolation.
10. We define a Bell measurement in the $ZZ$ or $XZ$ basis as a Bell measurement in which the local bases are the eigenvectors of the Pauli matrices $\sigma_Z$ and $\sigma_Z$, or $\sigma_X$ and $\sigma_Z$, respectively.
11. Throughout this work, percolation parameters with a circumflex, such as $\hat{\theta}$ and $\hat{p}_e$, refer to multipartite entanglement protocols, while the same parameters with a prime refer to CEP.
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