Data-Driven Safe Control of Uncertain Linear Systems Under Aleatory Uncertainty

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Abstract—Safe control of constrained uncertain linear systems under aleatory uncertainty is considered. Aleatory uncertainty characterizes random noises and is modeled by a probability distribution function (pdf). Data-based probabilistic safe controllers are designed for the cases where the noise pdf is 1) zero-mean Gaussian with a known covariance, 2) zero-mean Gaussian with an uncertain covariance, and 3) zero-mean non-Gaussian with an unknown distribution. Easy-to-check-model-based conditions for guaranteeing probabilistic safety are provided for the first case by introducing probabilistic $\lambda$-contractive sets. These results are then extended to the second and third cases by leveraging distributionally-robust probabilistic safe control and conditional-value-at-risk-based probabilistic safe control, respectively. Data-based implementations of these probabilistic safe controllers are then considered. Moreover, an upper bound on the minimal risk level, under which the existence of a safe controller is guaranteed, is learned using collected data. A simulation example is provided to show the effectiveness of the proposed approach.

Index Terms—Chance constraints, data-driven control, probabilistic safe control.

I. INTRODUCTION

While many applications can benefit from increased autonomy, the safety of autonomous systems must be guaranteed before their penetration into society. A challenge in assuring safety is accounting for uncertainties. Two common sources of uncertainty are often referred to as aleatory uncertainty and epistemic uncertainty. The former characterizes the inherent randomness, and the latter characterizes the lack of knowledge. In control systems, aleatory uncertainty represents the system and/or measurement noise and is generally modeled by a probability distribution function (pdf). On the other hand, epistemic uncertainty represents the lack of knowledge on the system dynamics. Safe control design methods typically rely on reachability analysis [1], [2] or control barrier functions (CBFs) [3], [4], [5]. To account for aleatory uncertainty, probabilistic-CBF-based safe control design has been considered for discrete-time (DT) systems [6], [7]. CBF-based approaches have also been used to certify the safety of reinforcement learning (RL) algorithms [8], [9], [10], [11], [12], [13]. However, due to their computational complexity, the existing CBF-based results for DT systems are limited to finitely supported noise distributions [6], [7]. Moreover, CBF-based methods typically require complete knowledge of the system dynamics (i.e., they cannot deal with both epistemic and aleatory uncertainties). One way to deal with uncertain dynamics is to identify a model using collected data and leverage it to design model-based safe controllers. However, as shown in this article, the data requirement conditions for identifying the system dynamics are generally more restrictive than the data requirement conditions for directly learning a safe controller.

Even though safe controllers are designed for uncertain dynamics in [14] and [15], these approaches ignore the aleatory uncertainty, and, instead, treat it as a bounded disturbance and provide robust safety guarantees. In the presence of aleatory uncertainty, hedging against the worst case uncertainty to guarantee almost sure invariance of the safe set may not be feasible. In this article, data-based probabilistic safe controllers are directly designed for linear DT systems with unknown dynamics affected by noises that are modeled by 1) a zero-mean Gaussian with known covariance, 2) a zero-mean Gaussian with an uncertain covariance, and 3) a zero-mean non-Gaussian with an unknown distribution. Probabilistic set invariance guarantees and stability guarantees are unified by introducing probabilistic $\lambda$-contractive sets. It is shown that the probabilistic safety amounts to the value-at-risk (VaR), distributionally robust VaR, and conditional VaR (CVaR)-based safe control design for cases 1, 2, and 3, respectively. These probabilistic approaches introduce a risk level that specifies how likely a constraint violation may be. The risk level depends on the uncertainty level as well as the $\lambda$ parameter. Since the minimal risk level is not known a priori, a data-based optimization is provided to learn an upper bound on the minimal risk level. Moreover, data-based optimizations are also provided to solve the resulting VaR-, distributionally-robust VaR-, and CVaR-based safe control designs. The contributions of this article are listed as follows.

1) Both VaR- and CVaR-based safe feedback control design methods are presented for stochastic DT systems. These results extend the results of $\lambda$-contractive methods in [16] to stochastic systems with both Gaussian and non-Gaussian noises.
2) In sharp contrast to CBF-based stochastic safe control design methods in [6] and [7], which are limited to finitely-supported noise distributions and must solve a convex optimization at every step, feedback controllers are learned for infinitely-supported noise distributions by solving linear programming optimizations.
3) Data-based optimization problems are developed to not only solve the resulting VaR-, distributionally-robust VaR-, and CVaR-based safe control designs but also to learn the optimal risk level.

A. Notations and Preliminaries

Throughout the article, $R$ denotes the real numbers. For a matrix $A$, $A_{ij}$ stands for its $i$th row and $A_{ij}$ is the element of its $i$th row and $j$th column. If $A$ and $B$ are matrices (or...
vectors) of the same dimensions, then \( A(\leq, \geq) B \) implies a componentwise inequality, i.e., \( A_{ij}(\leq, \geq) B_{ij} \) for all \( i \) and \( j \). \( Q(\leq, \geq) 0 \) denotes that \( Q \) is (negative, positive) semidefinite.

For a set \( \mathcal{F} \), \( \text{supp}(\mathcal{F}) \) stands for its support. \( I \) denotes the identity matrix of the appropriate dimension. All random variables are assumed to be defined on a probability space \((\Omega, \mathcal{F}, \text{Pr})\), with \( \Omega \) as the sample space, \( \mathcal{F} \) as its associated \( \sigma \)-algebra, and \( \text{Pr} \) as the probability measure. For a random variable \( w : \Omega \to \mathbb{R}^n \) defined on the probability space \((\Omega, \mathcal{F}, \text{Pr})\), with some abuse of notation, the statement \( w \in \mathbb{R}^n \) is used to state the dimension of the random variable. \( \mathbb{M} \) denotes the set of all probability measures defined on \((\mathbb{R}^n, \mathcal{B})\), with \( \mathcal{B} \) the Borel \( \sigma \)-algebra of \( \mathbb{R}^n \). \( \mathbb{E} \) indicates the mathematical expectation, and \( \mathbb{E}[w|z] \) denotes the conditional expectation of \( w \) with respect to \( z \), \( w \sim \mathcal{N}(\mu, \Sigma) \) denotes a multivariate Gaussian random vector with the mean \( \mu \) and the covariance \( \Sigma \).

**Definition 1:** Polyhedral Set [16]: Polyhedral set \( \mathcal{F}(F, g) \) is represented by
\[
\mathcal{F}(F, g) = \{ x \in \mathbb{R}^n : Fx \leq g \}
\]
where \( F \in \mathbb{R}^{m \times n} \) is a matrix with rows \( F_i, i = 1, \ldots, q \), and \( g \) is a vector with elements \( g_i, i = 1, \ldots, q \).

**Lemma 1:** Probabilistic Lyapunov Stability [17]: Consider the stochastic system \( x(t + 1) = h(x(t), w(t)) \), where \( h(\cdot) \) is a nonlinear function and \( w(t) \) is a noise signal. Let \( \mathcal{D} \) be a domain containing the origin. Suppose that there exists a continuous function \( V : \mathcal{D} \to \mathbb{R} \) such that
\[
V(0) = 0
\]
\[
V(x(t)) > 0 \quad \forall x(t) \in \mathcal{D} - 0
\]
\[
\mathbb{E}[V(x(t + 1))] - V(x(t)) \leq -c V(x(t)) \quad \forall x(t) \in \mathcal{D}
\]
for some \( 0 < c < 1 \). Then, the origin of the system is exponentially stable in probability (ESIP).

**II. PROBLEM STATEMENT**

Consider a linear DT control system given by
\[
x(t + 1) = Ax(t) + Bu(t) + w(t)
\]
where \( x(t) \in \mathbb{R}^n \) denotes the system’s state at time \( t \), \( u(t) \in \mathbb{R}^m \) denotes the control input, and \( w(t) \in \mathbb{R}^r \) is additive random noise. Moreover, \( A \) and \( B \) are the system matrices of appropriate dimensions and are not known.

**Assumption 1:** The noise \( w \) is either a zero-mean Gaussian noise or a zero-mean non-Gaussian independent and identically distributed (i.i.d) noise.

**Assumption 2:** The pair \((A, B)\) is stabilizable.

Since the set invariance is used as the key tool for safety guarantee, the following definition is provided.

**Definition 2:** Positive Invariant Set in Probability (ISIP) [18]: A set \( \mathcal{I} \) is an ISIP for the system (5) if \( x(0) \in \mathcal{I} \) implies \( \text{Pr}[x(t) \in \mathcal{I}] \geq (1 - \epsilon) \ \forall t \geq 0 \), where \( \epsilon \) is an acceptable risk level.

For the case where the noise is Gaussian, the following problem is formulated.

**Problem 1:** Consider the system (5) under Assumptions 1 and 2. Let the noise \( w \) be generated by a Gaussian distribution with a known or uncertain covariance. Design a linear feedback controller \( u(t) = Kx(t) \) such that 1) the closed-loop system is ESIP and 2) the polyhedral safe set \( \mathcal{F}(F, g) \) is ISIP.

In the following, we show that Problem 1 imposes chance constraints or VaR-based constraints for safety satisfaction. To this end, note that for the system (5) with the control input \( u(t) = Kx(t) \), its state at time \( t \) can be expressed based on its initial condition \( x(0) \) and the noise sequence up to time \( t \) as
\[
x(t) = (A + BK)^t x(0) + \omega
\]
where
\[
\omega = (A + BK)^{t-1} w(0) + (A + BK)^{t-2} w(1) + \ldots + w(t).
\]

Therefore, assuming that a polyhedral set is ISIP is equivalent to imposing a probabilistic constraint in the form of
\[
\text{Pr}[F(A + BK)^t x + F w \leq \omega] \geq (1 - \epsilon) \ \forall t \geq 0
\]
for every \( x \in \mathcal{F}(F, g) \). This probabilistic constraint, also referred to as a chance constraint, guarantees that the risk level that the future state trajectories fall outside the polyhedral safe set is at most \( \epsilon \in (0, 1) \), which is typically near zero. Chance constraints are closely related to the concept of VaR [19], which has been widely used to make risk-aware decisions in many disciplines. To see this, consider the loss function \( f(x, \omega) \) where \( x \in \mathbb{R}^n \) and \( \omega \in \Omega \) is a random event ranging over the set \( \Omega \) of all random events. The VaR of \( f(x, \omega) \) at level \( \epsilon \) is defined as [19]
\[
\text{VaR}_\epsilon(x, f) = \inf \left\{ \alpha \mid \text{Pr}[f(x, \omega) \geq \alpha] \leq \epsilon \right\}.
\]
Then, one has
\[
\text{Pr}[f(x, \omega) \geq 0] \leq \epsilon \iff \text{VaR}_\epsilon(x, f) \leq 0.
\]

**Lemma 2:** The set \( \mathcal{F}(F, g) \) is ISIP if and only if for every \( x \in \mathcal{F}(F, g) \), the following condition is satisfied:
\[
\text{VaR}_\epsilon(x, f) \leq 0
\]
where
\[
f(x, \omega) = F(A + BK)^t x + F w - g,
\]
and \( \omega \) is defined in (7).
$\forall \epsilon_r(x, f) \leq 0$ is also satisfied and, thus, based on Lemma 2, the set $\mathcal{F}(F, g)$ is ISiP.

III. PROBABILISTIC CONTRACTIVE SETS FOR PROBABILISTIC STABILITY AND SAFETY

To solve Problems 1 and 2, the following definition of probabilistic contractive sets is introduced, as an extension of its deterministic counterpart in [16].

Definition 3: Contractive Sets in Probability: Given a $\lambda \in (0, 1)$, the set $\mathcal{F}$ is $\lambda$-contractive in probability for the system (5) if $x(t) \in \mathcal{P}$ implies that $\Pr[x(t+1) \in \mathcal{P}] \geq (1 - \epsilon)$ where $\epsilon$ is a risk level.

The following lemmas make the connection between probabilistic contractive sets and ISiP. Before proceeding, for a random variable $w$, define the following optimization problem:

$$\tilde{h} = \operatorname{Opt}(H_w, \epsilon) = \arg\min \sum h_j$$

s.t. $\Pr[H_w(w(t) \leq \tilde{h}] = (1 - \epsilon)$

where $H_w$ is a matrix and $\tilde{h}$ is a vector. Then, for any matrix $L$ and decision variable $x$

$$\Pr[Lx(t) + H_ww(t)] \leq \tilde{g}] \geq (1 - \epsilon)$$

is equivalent to [21]

$$Lx(t) \leq \tilde{g} - \tilde{h}, \quad \tilde{h} = \operatorname{Opt}(H_w, \epsilon).$$

Lemma 3: Consider the system (5) with $u(t) = Kx(t)$. If a polyhedral set $\mathcal{F}(F, g)$ is a $\lambda$-contractive set in probability with a risk level $\epsilon_1$, then $x(t) \in \mathcal{F}(F, g)$ implies that $\Pr[x(t+1) \in \mathcal{F}(F, g)] \geq (1 - \epsilon_2)$ for some $\epsilon_2 < \epsilon_1$.

Proof: Let $x(t) \in \mathcal{F}(F, g)$. Since $\mathcal{F}(F, g)$ is a $\lambda$-contractive set in probability with the risk level $\epsilon_1$, by definition, it implies that $\Pr[F((A + BK)x(t) + w(t)] \leq \lambda \tilde{g}] \geq (1 - \epsilon_1)$.

Based on the equivalence of (17) and (18), this is equivalent to $F(A + BK)x(t) \leq \lambda \tilde{g} - \tilde{h} = \tilde{h}_1$ where $\tilde{h}_1 = \operatorname{Opt}(F, \epsilon_1)$. On the other hand, since $\lambda \in (0, 1)$, $\lambda \tilde{g} - \tilde{h} = \tilde{h}_2$ for some $\tilde{h}_2 > \tilde{h}_1$. Therefore, based on the equivalence of (17) and (18), $F((A + BK)x(t) \leq \tilde{h} = \tilde{h}_1$ implies that $\Pr[F((A + BK)x(t) + w(t)] \leq \lambda \tilde{g}] \geq (1 - \epsilon_2)$ for some $\epsilon_2$ satisfying $\tilde{h}_2 = \operatorname{Opt}(F, \epsilon_2)$, which guarantees $\Pr[x(t+1) \in \mathcal{F}(F, g)] \geq (1 - \epsilon_2)$. Moreover, based on (16), since $\tilde{h}_2 > \tilde{h}_1$, it implies that $\epsilon_2 < \epsilon_1$.

Lemma 4: If the set $\mathcal{F}(F, g)$ is a $\lambda$-contractive set in probability, then it is also ISiP.

Proof: Let the set $\mathcal{F}(F, g)$ be a $\lambda$-contractive set in probability with the risk level $\epsilon_1$. Then, based on Lemma 3, $\Pr[x(t) \in \mathcal{F}(F, g)] \geq (1 - \epsilon_2)$ for some $\epsilon_2 < \epsilon_1$. Using this property, and since the noise is i.i.d by Assumption 1, and thus, $x(t)$ has a Markov property, one has

$$\Pr[x(2) \in \mathcal{F}(F, g), x(1) \in \mathcal{F}(F, g)] \geq (1 - \epsilon_2)^2.$$

Using this recursive reasoning, one has $\Pr[x(t) \in \mathcal{F}(F, g)] \geq (1 - \epsilon_2)^t = (1 - \epsilon_3)$ for some $\epsilon_3$, which can be greater than or less than $\epsilon_1$, depending on $\lambda$ and $t$.

Remark 2: Note that for large values of $t$, the risk level can become large and unacceptable. However, since the planning horizon for any control system is generally finite, $\lambda$-contractivity guarantees that a set is ISiP with an acceptable risk level for a time duration longer than the planning horizon.

Lemma 5 (see [22]): Consider a joint chance constraint $\Pr[Hx + M \leq \tilde{g}] \geq (1 - \epsilon)$, where $x \in R^n$ is the decision variable, $d \sim \mathcal{N}(\mu, \Sigma)$, $H \in R^{p \times n}$, $M \in R^{q \times n}$, and $g \in R^q$. If the constraints $Hx + M \leq \tilde{g}$ are satisfied, where $k_i = \sqrt{\frac{L_i}{\beta}}$ with $\sum_i \epsilon_i \leq \epsilon$, then the original joint chance constraint is also satisfied.

To provide conditions for probabilistic $\lambda$-contractiveness, select $\epsilon_i, i = 1, \ldots, q$ such that $\sum_i \epsilon_i \leq \epsilon$ and define $l = [l_1, \ldots, l_q]$ where

$$l_i = \sqrt{\frac{1 - \epsilon_i}{\epsilon_i}} \sqrt{F_i} \Sigma F_i^T. \quad (19)$$

We now define the following operator for the system (5) with the safe set $\mathcal{F}(F, g)$ under the state-feedback control $u(t) = Kx(t)$.

$$\mathcal{F}_F(F, g) = \{x | F(A + BK)x \leq \lambda g - l\} \quad (20)$$

which is the set of all previous states for which it is guaranteed that their current states lie inside $\lambda \mathcal{F}(F, g)$ with a probability of at least $1 - \epsilon$.

Lemma 6: Consider the system (5) under Assumptions 1 and 2 with $w = \mathcal{W}(0, \Sigma)$. Let the control input be $u(t) = Kx(t)$. Then, the polyhedral set $\mathcal{F}(F, g)$ is $\lambda$-contractive in probability with the risk level $\epsilon$ if

$$\mathcal{F}(F, g) \subseteq \mathcal{F}_F(F, g). \quad (21)$$

Proof: By Lemma 5, if $F(A + BK)x \leq \lambda g - l$, the joint chance constraint $\Pr[F(A + BK)x + Fw \leq \lambda g] \geq (1 - \epsilon)$ is satisfied. Therefore, the set (20) is a safe underestimation of the set of all previous states for which it is guaranteed that their current state lies inside $\lambda \mathcal{F}(F, g)$ with a probability of at least $1 - \epsilon$. Therefore, $x(t) \in \mathcal{F}_F(F, g)$ implies that $\Pr[x(t+1) \in \lambda \mathcal{F}(F, g)] \geq (1 - \epsilon)$, which proves that (21) is a sufficient condition for $\lambda$-contractive in probability.

Remark 3: It was shown in [23] that the sufficient condition (21) becomes a necessary and sufficient condition for single-chance constraints, i.e., when $g$ is a scalar.

The noise covariance $\Sigma$ is generally unknown. Assuming that only a certain number $N$ of independent realizations of the random vector $w$ are available, its empirical estimate can be found. In this case, the covariance estimate belongs to an ambiguity set. For an arbitrarily-chosen confidence level $\beta \in (0, 1)$, let

$$N \geq \left(2 + \sqrt{2 \ln \frac{2}{\beta}}\right)^2. \quad (22)$$

Then, the ambiguity set $\mathcal{A}$ can be defined as [24]

$$\mathcal{A} := \left\{\Pr \in \mathbb{M} \left| E \left[(w(w))^T\right] \leq r_\beta(\Sigma_N)\right\} \right\} \quad (23)$$

where

$$\Sigma_N = \frac{1}{N} \sum_{i=1}^N w(i)(w(i))^T \quad (24)$$

is an empirical estimate of the noise covariance $\Sigma$ using $N$ samples and

$$r_\beta(\beta) = \frac{2L^2}{\sqrt{N} \left(2 + \sqrt{2 \ln \frac{2}{\beta}}\right)} \quad (25)$$
where \( L_h = \sup_{w \in \supp(h)} |w| \). Based on this ambiguity set, the controller is then designed such that
\[
\inf_{\psi \in \mathcal{A}} \Pr [x(t) \in \mathcal{S}] \geq (1 - \epsilon) \geq (1 - \beta) \quad \forall t \geq 0. 
\] (26)

These constraints are referred to as distributionally robust chance constraints, as the constraints must hold with a given confidence level for all disturbance distributions that belong to the ambiguity set.

Select \( \epsilon_i, i = 1, \ldots, q \) such that \( \sum_{i=1}^q \epsilon_i \leq \epsilon \) and define \( \hat{l} = [l_1, \ldots, l_q] \)
\[
\hat{l}_i = \sqrt{\frac{1 - \epsilon_i}{\epsilon_i}} \sqrt{F_i(\Sigma_N + r_c(\beta))F_i^T}. 
\] (27)

Now, similar to (20), define
\[
\mathcal{S}_{\mathcal{F}}(F, g) = \{x \mid F(A + BK)x \leq \lambda g - \hat{l} \}. 
\] (28)

**Lemma 7:** Consider the system (5) under Assumptions 1 and 2 with \( w \sim \mathcal{N}(0, \Sigma) \) where \( \Sigma \) is unknown and its sample average \( \Sigma_N \) is calculated using (24). Let the control input be \( u(t) = Kx(t) \). Fix \( \beta \in (0, 1) \). Then, with a probability of at least \( 1 - \beta \) the polyhedral set \( \mathcal{S}(F, g) \) \( \lambda \)-contractive in probability if
\[
\mathcal{S}(F, g) \subseteq \mathcal{S}_{\mathcal{F}}(F, g). 
\] (29)

**Proof:** The proof is based on Lemma 5 and is similar to the proof of Lemma 6.

**Theorem 1:** Consider the system (5) under Assumptions 1 and 2. Let \( w \sim \mathcal{A}(0, \Sigma) \). Then, a controller \( u(t) = Kx(t) \) that makes the set \( \mathcal{S}(F, g) \) \( \lambda \)-contractive in probability by satisfying (21) guarantees that the system is ESiP and the set \( \mathcal{S}(F, g) \) is ISiP. Therefore, it solves Problem 1.

**Proof:** Based on Lemma 6, \( \mathcal{S}(F, g) \) is \( \lambda \)-contractive in probability if \( \mathcal{S}(F, g) \subseteq \{x \mid F(A + BK)x \leq \lambda g - l\} \). This, in turns, implies that \( \{x \mid F(A + BK)x \leq \lambda g - l\} \). Based on the Farkas lemma [25], this is equivalent to the existence of a vector \( p \geq 0 \) such that \( F^T p = (F(A + BK)) \) and \( g^T p \leq \lambda g - l \). Defining \( P = [p_1, \ldots, p_q]^T \), one gets (34).

**Theorem 2:** Gaussian noise with known covariance: Consider the system (5) under Assumptions 1 and 2 with \( w \sim \mathcal{A}(0, \Sigma) \). Let \( u(t) = Kx(t) \). Then, the polyhedral set \( \mathcal{S}(F, g) \) is \( \lambda \)-contractive in probability if there exists a nonnegative matrix \( P \) such that
\[
PF = F(A + BK) 
\]
\[
P g \leq \lambda g - l 
\] (34)

**Proof:** Based on Lemma 6, \( \mathcal{S}(F, g) \) is \( \lambda \)-contractive in probability if \( \mathcal{S}(F, g) \subseteq \{x \mid F(A + BK)x \leq \lambda g - l\} \). This, in turns, implies that \( \{x \mid F(A + BK)x \leq \lambda g - l\} \). Based on the Farkas lemma [25], this is equivalent to the existence of a vector \( p_i \geq 0 \) such that \( F^T p = (F(A + BK)) \) and \( g^T p_1 \leq \lambda g - l \). Defining \( P = [p_1, \ldots, p_q]^T \), one gets (34).

**Theorem 3:** Gaussian noise with uncertain covariance: Consider the system (5) under Assumptions 1 and 2 with \( w \sim \mathcal{A}(0, \Sigma) \) where \( \Sigma \) is unknown. Let \( u(t) = Kx(t) \). Then, with a probability of at least \( 1 - \beta \), the covariance belongs to the ambiguity set (23). Then, with a confidence level of \( 1 - \beta \), the \( \mathcal{S}(F, g) \) is \( \lambda \)-contractive in probability with the risk level \( \epsilon \) if there exists a nonnegative matrix \( P \) such that
\[
PF = F(A + BK) 
\]
\[
P g \leq \lambda g - \hat{l} 
\] (35)

**Theorem 4:** Risk bound: Consider the system (5) under Assumptions 1 and 2 with a control input \( u(t) = Kx(t) \). Let \( N \) samples of a Gaussian noise be collected and its empirical covariance be calculated using (24). Let \( N \) satisfy condition (22) and thus with a probability of at least \( 1 - \beta \), the covariance belongs to the ambiguity set (23). Then, with a probability of at least \( 1 - \beta \), the solution \( \epsilon \) to the following optimization problem represents a bound on the lowest risk level for guaranteeing
that the set \( \mathcal{S}(F, g) \) is \( \lambda \)-contractive in probability

\[
\min_{\nu, \Sigma_{ss}, \epsilon} \epsilon \quad \text{s.t.} \quad \left[ \begin{array}{c} \Sigma_{ss}^b \quad - (\Sigma_N + r_c(\beta)) \\ A \Sigma_{ss} + BV \end{array} \right] \left( A \Sigma_{ss} + BV \right)^T \geq 0
\]

This risk level is achieved by the controller gain \( K = V \Sigma_{ss}^{-1} \).

**Proof:** Since \( A + BK \) is strictly stable, the state trajectories of the closed-loop system \( x(t+1) = (A + BK)x(t) + w(t) \) converge to a stationary distribution for which its covariance \( \Sigma_{ss} = \mathbb{E}[xx^T] \) satisfies [26]

\[
\Sigma_{ss} = (A + BK) \Sigma_{ss} (A + BK)^T + \Sigma
\]

where \( \Sigma \) is the actual covariance of the noise \( w \). Since \( \Sigma \) is not known and it is only known that with a probability of \( 1 - \beta \) it belongs to the ambiguity set (23), then \( \Sigma \leq (\Sigma_N + r_c(\beta)) \) with a probability of at least \( 1 - \beta \). Therefore, the solution to the following Lyapunov equation

\[
\Sigma_{ss}^b = (A + BK) \Sigma_{ss} (A + BK)^T + (\Sigma_N - r_c(\beta))
\]

satisfies \( \Sigma_{ss}^b \geq \Sigma_{ss} \) with a probability of at least \( 1 - \beta \). On the other hand, the solution to the inequality

\[
\Sigma_{ss}^b \geq (A + BK) \Sigma_{ss} (A + BK)^T + (\Sigma_N + r_c(\beta))
\]

is an upper bound of \( \Sigma_{ss}^b \). By defining \( K = V \Sigma_{ss}^{-1} \) and using Schur complement, (39) is equivalent to the linear matrix inequality (LMI)

\[
\left[ \begin{array}{c} \Sigma_{ss}^b - (\Sigma_N + r_c(\beta)) \\ A \Sigma_{ss} + BV \end{array} \right] \left( A \Sigma_{ss} + BV \right)^T \geq 0.
\]

Minimizing over the feasible space of this LMI solutions, i.e., the best upper bound of \( \Sigma_{ss}^b \), is equal to \( \Sigma_{ss}^\star \), and, therefore, any solution to the optimization problem is also an upper bound to \( \Sigma_{ss}^\star \), with a probability of at least \( 1 - \beta \). On the other hand, based on Lemma 5, Pr[\( Fx \leq g \)] \( \geq (1 - \epsilon) \) is satisfied if Pr[\( Fx \leq g_i \)] \( \geq (1 - \epsilon_i) \) and \( \eta \sum_{i=1}^N \epsilon_i \leq \epsilon \). Set \( \epsilon_i = \frac{\epsilon}{N} \). Using Chebyshev’s inequality [27], and when the state reaches the stationary condition, one has Pr[\( Fx > g_i \)] \( \leq \frac{\Sigma_{ss}^\star \epsilon_{i}^2}{\eta} \). Therefore, if \( \frac{\Sigma_{ss}^\star \epsilon_{i}^2}{\eta} \leq \frac{\epsilon}{\eta}, i = 1, \ldots, q \), then the original chance constraint is satisfied in the steady state.

When the noise distribution is completely unknown, Problem 2, for which a CVaR constraint is imposed, must be solved. Since the exact evaluation of the expectation in CVaR is difficult due to the piecewise linearity of the operator \( [\cdot]^+ \), it is typically approximated using sample average approximation methods based on \( N \) available i.i.d scenario data of the noise [28], and thus, the CVaR condition (32) is approximated by

\[
\text{CVaR}_e^N = \min_{\eta, z_i} \eta + \frac{1}{1 - \epsilon} \eta \sum_{i=1}^N z_i
\]

\[
\text{s.t.} \quad z_i \geq f_0(x, w_i) - \eta, \quad z_i \geq 0, \quad i = 1, \ldots, N
\]

where \( f_0 \) is defined in (33) and \( \text{CVaR}_e^N \) is the empirical CVaR.

**Theorem 5: Non-Gaussian noise:** Consider the system (5) under Assumptions 1 and 2 and let \( N \) i.i.d samples of the noise \( w_i, i = 1, \ldots, N \) be available. Let \( u(t) = K x(t) \). Then, the polyhedral set \( \mathcal{S}(F, g) \) with \( g \in R \) is \( \lambda \)-contractive in probability with a confidence level depending on \( N \) if and only if there exists a nonnegative matrix \( P \) such that

\[
\min_{\eta, z_i} \left( \eta + \frac{1}{1 - \epsilon} \eta \sum_{i=1}^N z_i \right) \leq 0
\]

\[
PF = F (A + BK)
\]

\[
P g \leq \lambda g - F w_i + z_i \quad \eta, \quad i = 1, \ldots, N
\]

\[
z_i \geq 0.
\]

**Proof:** Based on Corollary 2 and the approximation (41), \( \mathcal{S}(F, g) \) is \( \lambda \)-contractive in probability if \( \mathcal{S}(F, g) \subseteq \{ x : F (A + BK) x + F w_i \leq \lambda g + z_i + \eta \} \) where \( \eta \) and \( z_i \) satisfy (42). Using the Farkas lemma [25] completes the proof.

**Remark 4:** While Theorem 5 is presented for a single constraint, the joint CVaR constraints can also be handled similar to the joint chance constraints in Lemma 6 by splitting joint constraints into single constraints. Moreover, probably approximately correct [27] data-based confidence levels can be found using the sample average estimation of CVaR [29].

### V. DATA-BASED SAFE RISK ASSESSMENT AND CONTROL DESIGN

Let the collected data samples be arranged as

\[
U_0 = [u(0), \ldots, u(N - 1)] \quad (43a)
\]

\[
W_0 = [w(0), \ldots, w(N - 1)] \quad (43b)
\]

\[
X_0 = [x(0), \ldots, x(N - 1)] \quad (43c)
\]

\[
X_1 = [x(1), \ldots, x(N)] \quad (43d)
\]

**Theorem 6: Data-based versions of Theorems 2 and 3:** Consider the system (5) under Assumptions 1 and 2 with \( w \sim \mathcal{N}(0, \Sigma) \). Let the input/output/noise data be collected by applying an open-loop control sequence to the system and arranged by (43a)–(43d). Let the data matrix \( X_0 \) be full row rank and \( N \geq n + 1 \). Then, there exists a controller \( u(t) = K x(t) \) that makes the set \( \mathcal{S}(F, g) \) a probabilistic \( \lambda \)-contractive set if there exist matrices \( G_K \) and \( P \) satisfying

\[
P g \leq \lambda g - a
\]

\[
PF = F (X_1 - W_0) G_K
\]

\[
X_0 G_K = I
\]

\[
X_1 - W_0 = AX_0 + BU_0
\]

where \( a = l \) with \( l \) being defined in (19) when the noise covariance is known, and \( a = l \) with \( l \) being defined in (27) when the noise covariance is estimated through samples. Moreover, the control gain that solves Problem 1 is \( K = U_0 G_K \).

**Proof:** Since the matrix \( X_0 \) is assumed full rank, a right inverse \( G_K \) exists such that \( X_0 G_K = I \). Since \( N \geq n + 1 \), this right inverse is not unique. Based on the data collected in (43a)–(43d) and the stochastic linear system (5), one has

\[
X_1 - W_0 = AX_0 + BU_0
\]

Multiplying both sides of (45) by \( G_K \) from right yields

\[
(X_1 - W_0) G_K = A + BU_0 G_K
\]
Using the control gain \( K = U_0 G_K \), one obtains \( A + BK = (X_1 - W_0)G_K \). Therefore, \( PF = F(A + BK) \) becomes \( PF = F(X_1 - W_0)G_k \). The rest of the proof follows Theorems 2 and 3 for the first case and second case, respectively.

**Corollary 3:** Data-based version of Theorem 5: Consider the system (5) under Assumptions 1 and 2. Let the noise distribution \( w \) be unknown. Let the input/output/noise data be collected by applying an open-loop control sequence to the system and arranged by (43a)–(43d). Let the data matrix \( X_0 \) be full row rank. Then, there exists a controller \( u(t) = K x(t) \) to make the set \( \mathcal{F}(F, g) \) a probabilistic \( \lambda \)-contractive set if there exist matrices \( G_k \) and \( P \) satisfying

\[
\begin{align*}
&\min_{\eta, z_i} \left( \eta + \frac{1}{N(1 - \epsilon)} \sum_{i=1}^{N} z_i \right) \leq 0, \\
&P g \leq \lambda - F w_i + z_i + \eta, \quad i = 1, \ldots, N, \\
&z_i \geq 0, \\
&P F = F(X_1 - W_0)G_k \\
\end{align*}
\]

Moreover, the control gain that solves Problem 2 is \( K = U_0 G_K \).

**Proof:** The proof is similar to the proof of Theorem 6 and uses the results of Theorem 5.

**Remark 5:** In [15], \( \begin{bmatrix} U_0 & X_0 \end{bmatrix} \) must have full row rank, which requires \( n + m \) independent samples. Theorem 6 shows that the presented approach requires \( X_0 \) to be full row rank and \( N \geq n + 1 \). Therefore, for multi-input systems, a weaker data requirement is needed here. This extends the stability results in [30] to safety.

**Theorem 7:** Consider the system (5) and let conditions of Theorem 4 be satisfied. Let the input/output/noise data be collected by applying an open-loop control sequence to the system and arranged by (43a)–(43d). Then, with a probability of at least \( (1 - \beta) \), the solution \( \hat{\Sigma} \) to the following data-based optimization problem represents a bound on the lowest risk level for guaranteeing \( \lambda \)-contractive in the probability of the set \( \mathcal{F}(F, g) \)

\[
\hat{\Sigma} = \min_{V, \Sigma_{ss}} \epsilon \quad \text{s.t.} \quad \begin{bmatrix} \Sigma_{ss} & \Sigma_{x} \\ \Sigma_{x}^T & \Sigma_{xx} \end{bmatrix} \geq 0, \quad \begin{bmatrix} \Sigma_{ss} & \Sigma_{x} \\ \Sigma_{x}^T & \Sigma_{xx} \end{bmatrix} \geq 0, \quad \begin{bmatrix} \Sigma_{ss} & \Sigma_{x} \\ \Sigma_{x}^T & \Sigma_{xx} \end{bmatrix} \geq 0, \quad \begin{bmatrix} \Sigma_{ss} & \Sigma_{x} \\ \Sigma_{x}^T & \Sigma_{xx} \end{bmatrix} \geq 0.
\]

which is achieved by the controller gain \( K = U_0 V \Sigma_{ss}^{-1} \).

**Proof:** Condition (36b) is equivalent to

\[
\begin{bmatrix} \Sigma_{ss} & \Sigma_{x} \\ \Sigma_{x}^T & \Sigma_{xx} \end{bmatrix} \geq 0, \quad \begin{bmatrix} \Sigma_{ss} & \Sigma_{x} \\ \Sigma_{x}^T & \Sigma_{xx} \end{bmatrix} \geq 0, \quad \begin{bmatrix} \Sigma_{ss} & \Sigma_{x} \\ \Sigma_{x}^T & \Sigma_{xx} \end{bmatrix} \geq 0.
\]

It was shown in Theorem 6 that \( A + BK = (X_1 - W_0)G_K \). Using this fact and defining \( G_K = V \Sigma_{ss}^{-1} \) result in (48b).

Moreover, using \( K = U_0 G_K \) and \( G_K = V \Sigma_{ss}^{-1} \), one has \( K = U_0 V \Sigma_{ss}^{-1} \).

**Remark 6:** Note that Theorem 6 requires the measurements of the noise sequences during learning. The noise measurement is not needed after a safe controller is learned. The requirement of measuring the noise signal during learning can be relaxed as follows. It was shown in Theorem 6 that \( A + BK = (X_1 - W_0)G_K \). Since the noise is zero mean by Assumption 1, \( E(A + BK) = X_1 G_K \). Now, define \( \theta = vec(A + BK) \) and \( \bar{\theta} = vec(X_0 G_K - W_0 G_K) \). Then, one has

\[
\text{Var}(\bar{\theta}) = E[(\theta - \bar{\theta})(\theta - \bar{\theta})^T] = (G_K^T G_K \otimes \Sigma).
\]

Therefore, in Theorem 6, one can ignore the noise, and, instead add a soft constraint \( ||G_K^T G_K|| \leq \gamma \) and optimize over \( \gamma \) to learn a controller that achieves safety with maximum probability. This will result in a minimum-variance certainty-equivalence (CE) solution since the learning is performed as if the noise were zero.

**VI. SIMULATION RESULTS**

**Example 1:** Consider a linear system in the form of (5) with the state vector \( x = [x_1, x_2, x_3, x_4]^T \), the constraint \( x_2 \leq 0.1 \), and dynamics

\[
A = \begin{bmatrix} 0.2 & 0.00 & -0.1 & 0.0 \\ -0.0 & -0.200 & 0.500 & 0.1 \\ -0.1 & -0.5 & 1.0 & 0.0 \\ 0.1 & 0.4 & -0.6 & 0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}
\]

and \( w \sim \mathcal{N}(0, 0.1) \). We assume that the control input that is used for data generation is \( U_0 = [u(1), \ldots, u(5)] \) with \( u(i) = [0.5, 0.3, 0.2]^T, \quad i = 1, \ldots, 5 \). Note that one can identify the system using data first and then use the results of Theorems 2, 3, and 5 to design safe controllers. However, system identification requires \( n + m = 4 + 3 = 7 \) independent samples, which fails in this case. The presented data-based approach in Theorem 6, which bypasses system identification, only requires five independent samples to learn a safe control policy. To assure stability besides safety, we impose a large bound on state \( x_1, x_3, \) and \( x_4 \) (i.e., \( x_1 \leq 5, \quad i = 1, 3, 4 \) for a large value of \( g \)) to make the safe set compact. The learned gain \( \bar{K} \) using Theorem 6 for the case with Gaussian noise with known variance is given by

\[
\bar{K} = \begin{bmatrix} 0 & 0.4157 & -0.5 & -0.1 \\ 0.2494 & -0.3 & -0.06 \\ 0.1663 & -0.2 & -0.04 \end{bmatrix}.
\]

The state trajectories for the first two states (due to the page limitations) of the closed-loop system are shown for 100 different realizations of the noise and starting from \( x = [1, 0.1, 1, 1]^T \).
Example 2: Consider a second-order unstable system in form of (5) with the state vector $x = [x_1, x_2]^T$ and
\[
A = \begin{bmatrix}
\frac{4}{5} & \frac{1}{2} \\
-\frac{2}{5} & \frac{6}{5}
\end{bmatrix}, \quad B = \begin{bmatrix}
0 \\
1
\end{bmatrix}.
\]
Let the set $S(F,g)$ be characterized by
\[
F = \begin{bmatrix}
\frac{1}{5} & \frac{2}{5} \\
-\frac{1}{5} & -\frac{2}{5} \\
-\frac{33}{20} & \frac{1}{5} \\
\frac{3}{20} & -\frac{1}{5}
\end{bmatrix}, \quad g = [1 \ 1 \ 1 \ 1]^T.
\]
First, we assume that the noise is zero-mean Gaussian, i.e., $w \sim \mathcal{N}(0, \Sigma)$, with $\Sigma = 0.1$. The variance $\Sigma$ is assumed to be known for Case 1 and is estimated for Case 2. Theorem 6 is used to learn the control gain for these cases. The performance of learned controllers is compared with that of the CE-model-based safe control. The CE approach is risk neutral and guarantees safety in expectation [i.e., it solves (35) with $l = 0$]. Even if the noise is truncated, the robust optimization can be efficiently solved only if noise can be limited to some special convex sets such as polyhedral sets [31]. The comparison between the robust controller in [15] and the presented CVaR-based control is performed in Case 3 for which the noise is non-Gaussian with a bounded support. For Case 3, the noise $w$ is assumed to be generated from
a uniform distribution and Corollary 3 is used to learn the control gain.

For each experiment, 100 different realizations are taken and plotted. The initial conditions for all 100 cases are set the same and only a different random noise trajectory is sampled. To do a fair comparison between the CE approach and the presented approach in Theorem 6 for Case 1 and Case 2, as well as between the robust control approach and the presented approach in Corollary 3 for Case 3, the noise trajectories in all 100 realizations are generated in the beginning and used by all of them. As can be seen from Figs. 3 to 5, the safety performance of the presented risk-averse control is considerably better than that of the risk-neutral CE approach. Besides, more robustness is provided when the noise covariance is known when using the presented approach in Theorem 6. To compare the robust control approach with the results of Corollary 3 for non-Gaussian noise, the noise is assumed to be generated uniformly on the interval \([-\alpha, \alpha]\). The simulation results confirm that the robust control approach becomes infeasible most of the time for \(\alpha > 0.01\) while Corollary 3 learns a solution for values of \(\alpha\) as large as 0.3.

Figs. 6 and 7 show the results for the CVaR and the robust control approach for \(\alpha = 0.01\) and \(\alpha = 0.2\), respectively. The results of this example show that the robust control approach can lead to no solution for larger values of noises.

VII. CONCLUSION

Data-based safe controllers are presented for stochastic uncertain linear DT systems under aleatory uncertainties. Different assumptions on the noise pdf are considered and the concept of probabilistic \(\lambda\)-contractive sets is leveraged to design probabilistic safe controllers. A bound on the risk level is found using only the collected data and then data-based risk-averse safe controllers are designed.

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