Charged Black Cosmic String

Nakwoo Kim, Yoonbai Kim* and Kyoungtae Kimm

Department of Physics and Center for Theoretical Physics,
Seoul National University, Seoul 151-742, Korea
nakwoo@phya.snu.ac.kr, dragon@phya.snu.ac.kr

*Department of Physics, Sung Kyun Kwan University, Suwon 440-746, Korea
yoonbai@cosmos.skku.ac.kr

Abstract

Global $U(1)$ strings with cylindrical symmetry are studied in anti-de Sitter spacetime. According as the magnitude of negative cosmological constant, they form regular global cosmic strings, extremal black cosmic strings and charged black cosmic strings, but no curvature singularity is involved. The relationship between the topological charge of a neutral global string and the black hole charge is clarified by duality transformation. Physical relevance as straight string is briefly discussed.

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Cosmic strings are viable extended objects in cosmology [1]. A way to understand basic physical ingredients of cosmic strings is to study a straight string along an axis, which reduces one spatial dimension. Then, the (2+1) dimensional correspondence is the particle-like solitonic excitations so-called vortices in curved spacetime, and the conic space due to massive point source is enough for the description of asymptotic region outside the local vortex core. Recently, black hole solutions have been reported in (2+1) dimensional anti-de Sitter spacetime [2] in addition to known hyperbolic solutions [3], and these Bañados-Teitelboim-Zanelli (BTZ) black hole solutions have been extensively studied in a variety of models [4]. Here we may raise a question that what are the string-like counterpart of these BTZ black holes in cosmology. Specifically, whether the vortices in anti-de Sitter space can constitute black holes in (2+1)D, or straight black strings in (3+1)D. The objects of our interest are global $U(1)$ vortices [5, 6].

It has been shown in Ref. [6] that global $U(1)$ strings coupled to Einstein gravity with zero cosmological constant lead to a physical curvature singularity. Then, how does the constant negative vacuum energy affect the global strings? In this paper, we consider the effect of the negative cosmological constant to the global $U(1)$ vortices in (2+1)D and find three types of regular solutions of which base manifolds form (i) smooth hyperbola, (ii) extremal charged black hole and (iii) charged black hole with two horizons. For all these static solutions, the physical singularity can be avoided, which is different from the zero cosmological constant case. Suppose the magnitude of the negative cosmological constant is extremely small like the lower bound of it in the present universe, $|\Lambda| \leq 10^{-83}$GeV$^2$. Under this perfect toy environment with no fluctuation the global string may be born as a black string with large horizon size $r_H$ in the early universe, i.e., $r_H \sim 10^6$pc for the grand unification scale and $r_H \sim 10^{-2}$A.U. for the electroweak scale.

A cylindrically symmetric metric with boost invariance in the $z$-direction can be written as

$$ds^2 = e^{2N(r)}B(r)(dt^2 - dz^2) - \frac{dr^2}{B(r)} - r^2d\theta^2.$$  

(1)

The physics is reduced to (2+1) dimensional one under this metric. Another well-known
(2+1)D static metric is written under conformal gauge:

\[ ds^2 = \Phi(R)dt^2 - b(R)(dR^2 + R^2d\Theta^2). \] (2)

For a spinless point particle source of mass \( m \) at the origin, the general anti-de Sitter solution is

\[
\begin{align*}
    b &= \frac{4\varepsilon c^2}{|\Lambda|R^2[(R/R_0)^{\sqrt{\varepsilon c}} - (R_0/R)^{\sqrt{\varepsilon c}}]^2} \\
    \Phi &= \sqrt{\varepsilon}rac{(R/R_0)^{\sqrt{\varepsilon c}} + (R_0/R)^{\sqrt{\varepsilon c}}}{(R/R_0)^{\sqrt{\varepsilon c}} - (R_0/R)^{\sqrt{\varepsilon c}}},
\end{align*}
\] (3)

where \( \varepsilon \) is \( \pm1 \) for \( \Lambda < 0 \). When \( \varepsilon = +1 \), a coordinate transformation

\[
r = \frac{2}{|\Lambda|^{1/2}} \frac{1}{R^{(1-4Gm)} - R^{-(1-4Gm)}} \quad \text{and} \quad \theta = (1 - 4Gm)\Theta \quad (c = 1 - 4Gm) \] (5)

leads to

\[ ds^2 = (1 + |\Lambda|r^2)dt^2 - \frac{dr^2}{1 + |\Lambda|r^2} - r^2d\theta^2. \] (6)

It describes a hyperbola with deficit angle \( \delta = 8\pi Gm \) where \( 4Gm < 1 \) [3]. When \( \varepsilon = -1 \), another coordinate transformation

\[
r = \frac{c}{|\Lambda|^{1/2}\sin(2c\ln R)} \quad \text{and} \quad \theta = \Theta \quad (e^{k\pi/4c} < r < e^{(k+1)\pi/4c} \quad \text{and} \quad c^2 = 8GM, \ k \in \mathbb{Z})
\] (7)

results in the exterior region of the Schwarzschild type BTZ black hole [4] with missing information of the point particle mass \( m \) in Eqs. (3) and (4):

\[ ds^2 = (|\Lambda|r^2 - 8GM)dt^2 - \frac{dr^2}{|\Lambda|r^2 - 8GM} - r^2d\theta^2. \] (8)

As expected, the BTZ solution is one of general anti-de Sitter solutions, of which physical meaning was not considered in Ref. [3]. Note that the dimension of \( m \) and \( M \) has the square of mass in (3+1)D because it represents the mass density per unit length along the string direction.
Here we want to solve Einstein equations with both a global string source and constant negative cosmological vacuum energy density. We take a complex scalar field $\phi$ with Lagrange density

$$L = -\frac{1}{16\pi G} (R + 2\Lambda) + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{\lambda}{4} (\bar{\phi} \phi - v^2)^2.$$  \hspace{1cm} (9)

This model admits a string solution of the form

$$\phi = |\phi|(r) e^{i \theta}.$$  \hspace{1cm} (10)

For the cylindrically symmetric configurations, the Euler-Lagrange equations read under the metric in Eq. (1):

$$\frac{1}{r} \frac{dN}{dr} = 8\pi G \left( \frac{d|\phi|}{dr} \right)^2$$ \hspace{1cm} (11)

$$\frac{1}{r} \frac{dB}{dr} = 2|\Lambda| - 8\pi G \left\{ B \left( \frac{d|\phi|}{dr} \right)^2 + \frac{n^2}{r^2}|\phi|^2 + \frac{\lambda}{2} (|\phi|^2 - v^2)^2 \right\}$$ \hspace{1cm} (12)

$$\frac{d^2|\phi|}{dr^2} + \left( \frac{dN}{dr} + \frac{1}{B} \frac{dB}{dr} + \frac{1}{r} \frac{d|\phi|}{dr} \right) = \frac{1}{B} \left( \frac{n^2|\phi|}{r^2} + \lambda (|\phi|^2 - v^2)|\phi| \right).$$  \hspace{1cm} (13)

Though we will consider the spacetime with horizons, we concentrate only on the regular configurations connecting $|\phi|(0) = 0$ and $|\phi|(\infty) = v$ smoothly. By asking the reproduction of Minkowski spacetime in the limit of no matter ($T^\mu_\nu = 0$), and zero vacuum energy ($\Lambda \rightarrow 0$), we can choose an appropriate set of boundary conditions, $B(0) = 1$ and $N(\infty) = 0$. The Einstein equation for the metric function $B(r)$ is then expressed in terms of scalar amplitude

$$B(r) = \exp \left[ -8\pi G \int_r^\infty dr' r' \left( \frac{d|\phi|}{dr'} \right)^2 \right] \left\{ 2|\Lambda| \int_0^r dr' r' \exp \left[ 8\pi G \int_{r'}^\infty dr'' r'' \left( \frac{d|\phi|}{dr''} \right)^2 \right] \right\}$$

$$-8\pi G \int_0^r dr' r' \exp \left[ 8\pi G \int_{r'}^\infty dr'' r'' \left( \frac{d|\phi|}{dr''} \right)^2 \right] \left( \frac{n^2}{r^2} |\phi|^2 + \frac{\lambda}{2} (|\phi|^2 - v^2)^2 \right) + e^{N(0)} \right\}.$$  \hspace{1cm} (14)

Under the approximation that $|\phi| = 0$ for $r < r_c$ and $|\phi| = v$ for $r \geq r_c$, we read the possible form of $B(r)$ with the aid of constant $N(r)$ which is a valid approximation for $r \geq r_c$:

$$B(r) \approx |\Lambda| r^2 - 8\pi G v^2 n^2 \ln r/r_c - 4\pi G v^2 n^2 + 1.$$  \hspace{1cm} (15)

The core radius $r_c = \sqrt{2n}/\sqrt{\lambda} v$ is determined by minimizing the core mass. Another scale, the minimum point of $B(r)$, is $r_m \approx \sqrt{4\pi G v^2/|\Lambda| n}$, and it may coincide with the horizon.
scale $r_H$ when $2\pi G v^2 \gg |\Lambda|/\lambda v^2$. Note that the formation of a black hole is favored as the magnitude of the cosmological constant scaled by the square of Higgs mass, $|\Lambda|/\lambda v^2$, becomes small and the symmetry breaking scale $v$ is large. The latter condition is obvious and the former condition can be understood by the balance between the negative energy due to the negative cosmological constant and positive matter contribution. This energy balance also explains the reason why we can have the regular global $U(1)$ vortex solution in singularity-free curved spacetime. Let us recall no-go theorem that this global $U(1)$ scalar model can not support finite-energy static regular vortex configuration in flat spacetime, so the global $U(1)$ vortex contains a logarithmic divergence in its expression of the energy per unit length. This symptom can not be remedied by inclusion of (2+1)D gravity, namely a higher spin (spin 2) field, since (2+1) dimensional Einstein gravity does not have propagating degree. Therefore, the global $U(1)$ vortex coupled to Einstein gravity leads to an unavoidable physical curvature singularity \[3\]. In the model of our consideration, the negative vacuum energy achieves a kind of energy balance in anti-de Sitter spacetime, and both the scalar field and the curvature are regular everywhere even though we have coordinate singularity at the horizons.

From now on we will solve the field equation and Einstein equations, and will show that there exist regular global $U(1)$ vortex solutions in anti-de Sitter space and the base manifolds of these configurations constitute smooth hyperbola, extremal black hole and charged black hole.

Near the origin, $|\phi|(r) \sim \phi_0 r^n$ and the leading term of the power series solution of $B(r)$ is given by

$$B(r) \sim 1 + \left( \frac{|\Lambda|}{\lambda v^2} - 2\pi G v^2 - \frac{8\pi G \phi_0^2}{\lambda v^2} \delta_{n1} \right) (\sqrt{\lambda v} r)^n.$$  \hspace{1cm} (16)

When the cosmological constant rescaled by the symmetry breaking scale is smaller (larger) than the Planck scale, $B(r)$ starts to decrease (increase) near the origin. Despite of the difficulty in systematic series expansion at the asymptotic region, the leading term provides
for sufficiently large $r$:

$$|\phi|(r) \sim v - \frac{\phi_\infty}{r^2}$$  \hspace{1cm} (17)

$$B(r) \sim |\Lambda| r^2 - 8\pi G v^2 n^2 \ln r/r_c - 8G\mathcal{M} + 1 + \mathcal{O}(1/r^2).$$  \hspace{1cm} (18)

Here $\mathcal{M}$ is the integration constant and will be identified as the core mass of the global vortex. Since the form of Eq. (18) is the same as that in Eq. (15) and we know that the magnitude of two terms proportional to $G$ can be larger than the sum of other two positive terms for some appropriate large $r$, the existence of horizons for the small magnitude of the negative cosmological constant can easily be confirmed.

To elicit the above discussion explicitly, we compute the solitons by use of numerical analysis. Two regular $n = 1$ vortex solutions connecting $|\phi|(0) = 0$ and $|\phi|(\infty) = v$ are illustrated in Figure 1. When $|\Lambda|/\lambda v^2$ is large enough, i.e., the second term in the right hand side of Eq. (16) is positive for small $r$ ($|\Lambda|/\lambda v^2 = 1.0$ and $8\pi G v^2 = 1.15$), $B(r)$ is monotonically increasing (See (i - a) in Fig. 1). When $|\Lambda|/\lambda v^2$ is intermediate, $B(r)$ has a positive minimum (See (i - b) in Fig. 1). Suppose that there exists a horizon $r_H$ such that $B(r, H) = 0$, another boundary condition has to be satisfied, which is obtained from the equation for the scalar field

$$\frac{d|\phi|}{dr}_{r_H} = \frac{|\phi|(r_H) \left[ \frac{n^2}{r_H} + \lambda(|\phi|^2(r_H) - v^2) \right]}{8\pi G r_H \left[ \frac{|\Lambda|}{4\pi G} - \left( \frac{n^2}{r_H} |\phi|^2(r_H) + \frac{1}{2}(|\phi|^2(r_H) - v^2)^2 \right) \right]}.$$  \hspace{1cm} (19)

Therefore, we solve the equations in one region with two boundaries for a regular solution, in two regions with three boundaries for an extremal black hole and in three regions with four boundaries for a charged black hole. Since the value of $B(r)$ also vanishes at the horizon $r_H$ for the extremal black hole, the position of the horizon and the value of scalar field are obtained in explicit forms:

$$r_H = \frac{n}{\sqrt{\lambda v} \left( 1 - \sqrt{1 - \frac{|\Lambda|}{2\pi G \lambda v^4}} \right)^{1/2}}$$ and

$$|\phi|_H = v \left( 1 - \frac{|\Lambda|}{2\pi G \lambda v^4} \right)^{1/4}. \hspace{1cm} (20)$$

When a black hole is formed, an intriguing property should be mentioned: The ratio of horizon scale and the core length ($\sim r_c \sim 1/\sqrt{\lambda v}$) tells us that the horizon lies outside
The string core. We investigate the extremal black hole configuration for various $(|\Lambda|/\lambda v^2, 8\pi G v^2)$ values and find one for $|\Lambda|/\lambda v^2 = 0.1$ and $8\pi G v^2 = 1.338$ (See (ii) in Fig. 1). A Reissner-Nordström type charged black hole with two horizons at $r_{in}^H$ and $r_{out}^H$ is also obtained for $|\Lambda|/\lambda v^2 = 0.1$ and $8\pi G v^2 = 1.4$ (See (iii) in Fig. 1).

Although we have a singularity at each horizon, it is not physical singularity but coordinate artifact. This can be checked by reading the (2+1)D Kretschmann scalar:

$$R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = 4G_{\mu\nu} G^{\mu\nu}$$

$$= 4 \text{Tr} \left[ \text{diag} \left( -\frac{1}{2} dB \frac{dB}{dr}, -\frac{1}{2} dB \frac{dN}{dr}, -\frac{1}{2} dB \frac{dN}{dr} -\frac{3 dB}{2} \frac{dN}{dr} -\frac{B}{2} \frac{d^2 N}{dr^2} -\frac{B}{2} \left( \frac{dN}{dr} \right)^2 \right) \right].$$

Since both $N(r)$ and $B(r)$ are regular everywhere for all configurations of our consideration, no divergent curvature appears at the horizon, and for the regular scalar configurations that behave $|\phi|(r) \sim r^n$ for small $r$, the Kretschmann scalar is also finite at the origin. Therefore, unlike the global $U(1)$ strings which include unavoidable curvature singularity in the spacetime with zero cosmological constant, the string configurations obtained in anti-
de Sitter space have no such curvature singularity. Furthermore, since a possible divergent curvature at the core of global vortex is tamed by the regular behavior of scalar amplitude $|\phi|(r)$, no divergence of curvature appears, which is encountered in the charged black hole formed due to infinite electric self energy by a point charge [2].

Now we look into the planar motions of massive and massless test particles orthogonal to the string direction, which are described by the geodesic equations:

$$\frac{1}{2} \left( \frac{dr}{ds} \right)^2 = -\frac{1}{2} \left( B(r) \left( m^2 + \frac{L^2}{r^2} \right) - \frac{\gamma^2}{e^{2N(r)}} \right)$$

(22)

with two constants of motion, $\gamma$ and $L$, associated with two Killing vectors $\partial/\partial t$ and $\partial/\partial \theta$ respectively. From Eq. (22), the elapsed coordinate time $t$ of a test particle which moves from $r_0$ to $r$ is finite for regular solutions and becomes infinite when it approaches to a horizon ($r \to r_H$). As we expected, the spacetime with horizons depicts that of a black hole. The radial motion of a massless test particle ($m = 0$ in Eq. (22)) is unbounded for $\gamma \neq 0$. On the other hand, any massive particle ($m = 1$ in Eq. (22)) can never escape the black hole irrespective of values of $\gamma$ and $L$, since the asymptotic space is hyperbolic. The size of the boundary is determined as a function of $\gamma$ and $L$. Obviously, for the radial motion, the boundary $r_\infty$ of the massive test particle becomes large as $\gamma$ increases, and then the ratio $r_\infty/r_H$ is much larger than one. Details of all possible planar geodesic motions and related physical implication will be provided in Ref. [7].

Under the metric in Eq. (1) the conserved quasilocal mass per unit length measured by the static observer is [8]:

$$8GM_q = 2e^{N(r)} \left( \sqrt{(|\Lambda| r^2 + 1)B(r)} - B(r) \right).$$

(23)

Though the obtained spacetime is not asymptotically flat and thereby $M_q$ is not identified by Arnowitt-Deser-Misner mass, we compute it for sufficiently large $r$

$$M_q \xrightarrow{r \to \infty} \pi n^2 v^2 \ln r/r_c + \mathcal{M}.$$  

(24)

The first logarithmically divergent term comes from the topological sector of the long range Goldstone degree, and the second finite one is the core mass of the global $U(1)$ vortex, which coincide with those in flat spacetime.
If we compare the obtained spacetime of the global vortices with that of the electric field of a point charge [2], we can easily find a similarity between them except for the core region. The reason why we have this resemblance can be explained by duality transformation [9]. The dual transformed theory equivalent to (2+1)D (or (3+1)D) global vortex model of our consideration is written in terms of a dual vector (or second rank antisymmetric tensor) field $A_\mu$ (or $A_{\mu\nu}$):

$$Z = \int [g^{3/2}d\omega[[|\phi|^{-2}d|\phi|][dA_\mu][d\Theta] \exp\left\{ i \int d^3x \sqrt{g} \left[ -\frac{1}{16\pi G} (R + 2\Lambda) \right.ight.$$ 

$$+ \frac{1}{2} g^{\mu\nu} \partial_\mu |\phi| \partial_\nu |\phi| - V(|\phi|) - \frac{\nu^2}{4|\phi|^2} g^{\mu\rho} g^{\nu\sigma} F_{\mu\rho} F_{\nu\sigma} + \frac{\nu \epsilon^{\mu\nu\rho}}{2\sqrt{g}} F_{\mu\nu} \partial_\rho \Omega \right\} \right\},$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and $\Omega$ is the topological sector of the Goldstone degree. For the cylindrically symmetric strings with $\Theta = n\theta$, the last term of the Lagrangian in Eq. (25) describes the charged point source coupled minimally to a dual gauge field $A_\mu$. The kinetic term of this $A_\mu$ is the Maxwell term for large $r$ ($|\phi| \rightarrow \infty$), but the nonpolynomial interaction ($\sim 1/|\phi|^2 \rightarrow \infty$) plays the role of ultraviolet cutoff to remove the possible curvature divergence at the origin. Through this reformulation the role of the topological charge $n$ in the original formulation is transmuted to the electric charge $n$ in the dual transformed theory.

Brief comments about physical implication of these black cosmic strings are in order. Let us consider a universe with tiny negative cosmological constant, e.g., our present universe with $|\Lambda| \leq 10^{-83}$GeV$^2$. From Eq. (20), the characteristic scale to distinguish regular strings from black strings is about $v \sim 0.3$eV. This means that favorite form of survived global strings in anti-de Sitter space is black hole type where the magnitude of the cosmological constant is about the lower bound of the present universe. Estimation of the horizon size in Eq. (20) gives us $r_H \sim 10^6$pc for the grand unification scale ($v \sim 10^{15}$GeV), $r_H \sim 10^{-2}$A.U. for the electroweak scale ($v \sim 10^2$GeV). Though it is estimated in a perfect presumed toy environment, this property that the horizon of GUT scale black cosmic string is larger than the diameter of our galaxy ($\sim 5 \times 10^4$pc) may imply some incompatibility between the black cosmic string produced in such early universe and the extremely small magnitude of
negative cosmological constant. Again, let us emphasize that the scales obtained above are the outcome of three energy scales of big difference, which are the Planck scale (the largest energy scale), the bound of vacuum energy (the smallest measured scale in cosmology) and an intermediate symmetry breaking scale. In this sense, the obtained black string configurations in Fig. 1 seem to be unphysical since $v \sim (10^{18} \sim 10^{19})$GeV and $\lambda \sim 10^{-122}$ for $|\Lambda| \sim 10^{-83}$GeV$^2$ and $1/\sqrt{G} \sim 10^{19}$GeV. Obtaining black string under the physical situation seems beyond our numerical precision. Even if the global cosmic strings are produced, the lifetime of a typical string loop is very short due to the radiation of gapless Goldstone boson, which is the dominant mechanism for energy loss [10]. The space outside the horizon of black cosmic string is almost flat except for tiny attractive force due to negative cosmological constant as shown in Eq. (22) and Eq. (18), and then the massless Goldstone bosons can be radiated outside the horizon. However, almost all the energy accumulated inside the horizon remains eternally. Let us remind you that the physical radius of black cosmic string is decided by the horizon which is usually much larger than the size of normal cosmic string (the core radius $\sim 1/\sqrt{\lambda v}$), but the energy per unit length of both objects is the same as given in Eq. (24). Therefore, the very existence of this horizon is expected to change drastically the physics related to the evolution of global $U(1)$ strings, e.g., the intercommuting of two strings or the production of wakes by moving long strings [4].

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References

[1] For a review, see A. Vilenkin and E.P.S. Shellard, *Cosmic Strings and Other Topological Defects*, (Cambridge, 1994); M.B. Hindmarsh and T.W.B. Kibble, Rept. Prog. Phys. 58 (1995) 477.
[2] M. Bañados, C. Teitelboim and J. Zanelli, Phys. Rev. Lett. 69 (1992) 1849; M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli, Phys. Rev. D 48 (1993) 1506.

[3] S. Deser and R. Jackiw, Ann. Phys. 153 (1984) 405.

[4] G. Clement, Phys. Rev. D 50 7119 (1994); Phys. Lett. B 367 (1996) 70; K.C.K. Chan and R.B. Mann, Phys. Rev. D 50 (1994) 6385, Erratum-ibid D 52 2600; J.S.F. Chan, K.C.K. Chan and R.B. Mann, Phys. Rev. D 54 (1996) 1535.

[5] A. Vilenkin and A.E. Everett, Phys. Rev. Lett. 48 (1982) 1867; E.P.S. Shellard, Nucl. Phys. B 283 (1987) 624; D. Harari and P. Sikivie, Phys. Rev. D 37 (1988) 3438.

[6] R. Gregory, Phys. Lett. B 215 (1988) 663; A.G. Cohen and D.B. Kaplan, Phys. Lett. B 215 (1988) 67; G.W. Gibbons, M.E. Ortiz and F. Ruiz Ruiz, Phys. Rev. D 39 (1989) 1546.

[7] N. Kim, Y. Kim and K. Kimm, in preparation.

[8] J.D. Brown and J.W. York, Phys. Rev. D 47 (1993) 1407; J.D. Brown, J. Creighton, and R.B. Mann, Phys. Rev. D 50 (1994) 6394.

[9] Y. Kim and K. Lee, Phys. Rev. D 49 (1994) 2041; K. Lee, Phys. Rev. D 49 (1994) 4265; C. Kim and Y. Kim, Phys. Rev. D 50 (1994) 1040.

[10] R.L. Davis, Phys. Rev. D 32 (1985) 3172; A. Vilenkin and T. Vachaspati, Phys. Rev. D 35 (1987) 1138; For a review, see Ref. [1] and the references therein.