On the intrinsicness of the Newton polygon

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Abstract

A sufficiently generic bivariate Laurent polynomial with given Newton polygon $\Delta$ defines an algebraic curve $C$, many of whose numerical invariants are encoded in the combinatorics of $\Delta$. These include the genus (classical), the gonality, the Clifford index and the Clifford dimension (an enhancement of recent results by Kawaguchi), the scrollar invariants and, for sufficiently nice instances of $\Delta$, certain secondary scrollar invariants that were introduced by Schreyer (new observation). After discussing these invariants, we study to what extent they allow one to reconstruct $\Delta$ from the abstract geometry of $C$.

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1 Introduction

Let $k$ be an algebraically closed field of characteristic zero, let $\mathbb{T}^2 = (k^*)^2$ be the 2-dimensional torus over $k$, and let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be an irreducible Laurent polynomial. Denote by $U(f)$ the curve in $\mathbb{T}^2$ defined by $f$. Let $\Delta(f) \subset \mathbb{R}^2$ be the Newton polygon of $f$, which we always assume to be two-dimensional. We say that $f$ is non-degenerate with respect to its Newton polygon if for every face $\tau \subset \Delta(f)$ (including $\Delta(f)$ itself) the system

$$f_\tau = \frac{\partial f_\tau}{\partial x} = \frac{\partial f_\tau}{\partial y} = 0$$

has no solutions in $\mathbb{T}^2$. (Here $f_\tau$ is obtained from $f$ by only considering those terms that are supported on $\tau$.) For a two-dimensional lattice polygon $\Delta \subset \mathbb{R}^2$, we say that $f$ is $\Delta$-non-degenerate if it is non-degenerate with respect to its Newton polygon and $\Delta(f) = \Delta$. The condition of non-degeneracy is generically satisfied, in the sense that for every $\Delta$ it is characterized by the non-vanishing of

$$\text{Res}_\Delta \left( f, x \frac{\partial f}{\partial x}, y \frac{\partial f}{\partial y} \right)$$

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(where $\text{Res}_\Delta$ is the sparse resultant; see [9, Prop. 1.2] and [18, Thm. 10.1.2] for an according discussion). An algebraic curve $C/k$ is called $\Delta$-non-degenerate if it is birationally equivalent to $U(f)$ for some $\Delta$-non-degenerate Laurent polynomial $f \in k[x^{\pm 1}, y^{\pm 1}]$.

It is well-known that if $C$ is $\Delta$-non-degenerate, then several of its geometric properties are encoded in the combinatorics of $\Delta$. The most famous instance is that the geometric genus of $C$ equals the number of lattice points in the interior of $\Delta$ [27, §4.4.2]. Other known examples are that one can tell from $\Delta$ whether $C$ is hyperelliptic or not [28, Lem. 3.2], and whether it is trigonal or not [8, Lem. 3].

The central question of this paper is:

**Question 1.** How far can this be pushed, i.e. suppose we are given a $\Delta$-non-degenerate curve $C/k$ as an abstract curve, then to what extent can we recover $\Delta$ from the geometry of $C$?

Before we address this question, in a first part of this paper, we extend the list of curve invariants that are known to be encoded in the Newton polygon:

- the genus (brief review, with an addendum on the canonical ideal; see Section 3.1),
- the gonality (see Section 3.2),
- the Clifford index and the Clifford dimension (see Section 3.3),
- the scrollar invariants (associated to a combinatorial gonality pencil, see Section 3.4),
- Schreyer’s tetragonal invariants (see Section 3.5),
- for sufficiently nice Newton polygons: the secondary scrollar invariants (associated to a combinatorial gonality pencil; see Section 3.6).

The material in Sections 3.2 and 3.3 is heavily inspired by a recent preprint of Kawaguchi [26] – our treatment is somewhat less technical, includes the case where $U(f)$ is birationally equivalent to a smooth projective plane curve (which Kawaguchi omits), and makes a more explicit connection with the language of Newton polygons.

Among the consequences we find that one can combinatorially determine whether $U(f)$ is birationally equivalent to a smooth projective plane curve (see Section 3.3) and whether it is birationally equivalent to a smooth projective curve in $\mathbb{P}^1 \times \mathbb{P}^1$ (as soon as $g \neq 4$; see Section 5.10). A more surprising corollary is that any curve (not
necessarily non-degenerate) admits at most one Weierstrass semi-group of embedding dimension two (see Section 3.2).

In Section 4 we mull on other candidate invariants, such as the graded Betti numbers, the minimal degree of a plane model, and the number of vertices (a seemingly new invariant that might be found interesting in its own right).

Then, in Section 5, we return to Question 1. At this point, let us give some preliminary discussion. At most, one can hope to recover \( \Delta \) up to unimodular equivalence, i.e. up to transformations of the form
\[
\mathbb{R}^2 \to \mathbb{R}^2 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad A \in \text{GL}_2(\mathbb{Z}), \; a_1, a_2 \in \mathbb{Z}
\]
(we write \( \Delta \cong \Delta' \) to indicate that \( \Delta' \) is obtained from \( \Delta \) through a unimodular transformation). Indeed, if a Laurent polynomial
\[
f = \sum_{(i,j) \in \Delta \cap \mathbb{Z}^2} c_{i,j}(x, y)^{(i,j)}
\]
is \( \Delta \)-non-degenerate (where \( (x, y)^{(i,j)} \) means \( x^i y^j \)) and \( \eta \) is a unimodular transformation, then
\[
f^{\eta} = \sum_{(i,j) \in \Delta \cap \mathbb{Z}^2} c_{i,j}(x, y)^{\eta(i,j)}
\]
is \( \eta(\Delta) \)-non-degenerate, and \( U(f) \cong U(f^{\eta}) \). (In fact, a unimodular transformation induces an automorphism of \( \mathbb{T}^2 \).) But even this hope is vain in general. E.g., let \( \Delta = \text{conv}\{(0,0), (d,0), (0,d)\} \) for some \( d \geq 2 \), and let \( f \) be \( \Delta \)-non-degenerate. Then \( U(f) \) is the torus part of a smooth plane curve of degree \( d \). Let \( (x_0,y_0) \in U(f) \) be such that the lines \( x = x_0 \) and \( y = y_0 \) are non-tangent to \( U(f) \). Then \( f' = f(x + x_0, y + y_0) \) is \( \Delta' \)-non-degenerate, with
\[
\Delta' = \text{conv}\{(1,0), (d,0), (0,d), (0,1)\}.
\]
Clearly, \( U(f) \) and \( U(f') \) are birationally equivalent, whereas \( \Delta \not\cong \Delta' \). Essentially, \( \Delta' \)-non-degenerate curves are smooth projective plane curves that happen to pass through the origin: it is clear that this property is not intrinsic. More generally, pruning a vertex off a lattice polygon \( \Delta \) without affecting the interior always boils down to forcing the curve through a certain non-singular point of the corresponding toric surface \( X(\Delta) \) — see our discussion on weak non-degeneracy in Section 2 below.

We are naturally led to wondering whether it is possible to recover \( \Delta^{(1)} \), the convex hull of the interior lattice points of \( \Delta \), up to equivalence. This seems to lie closer to
reality. In Sections 5.6 and 5.7 we will show that this may fail for certain polygons in genus 4 and genus 5, but apart from these, we are not aware of any counterexamples.

**Question 1 (more concrete version).** If a curve $C/k$ is both $\triangle$-non-degenerate and $\triangle'$-non-degenerate, where $\triangle, \triangle'$ are two-dimensional lattice polygons not equivalent to $2\Upsilon, \triangle_1^5, \triangle_2^5, \triangle_3^5$, does it automatically follow that $\triangle^{(1)} \cong \triangle'^{(1)}$?

(See right below for a definition of $\Upsilon$, and Section 5.7 for a definition of the $\triangle_i^5$'s.)

In Section 5 we will prove an affirmative answer to Question 1 for curves of gonality $\gamma \leq 3$, for curves of even genus having gonality $\gamma = 4$, and for all curves of genus $g \leq 8$.

### Notations and conventions

It is convenient to introduce a special notation for three recurring polygons:

\[
\Sigma = \text{conv}\{(0,0), (1,0), (0,1)\}, \\
\Upsilon = \text{conv}\{(-1,-1), (1,0), (0,1)\}, \\
\Box = \text{conv}\{(0,0), (1,0), (1,1), (0,1)\}
\]

(where conv denotes the convex hull in $\mathbb{R}^2$). Thus $\Sigma$ is the standard simplex, and $d \Sigma$ (Minkowski multiple) is the Newton polygon of a generic degree $d$ polynomial. If $\triangle$ is a two-dimensional lattice polygon, we denote by $\triangle^{(1)}$ the convex hull of its interior lattice points. If in turn $\triangle^{(1)}$ is two-dimensional, we denote by $\triangle^{(2)}$ the convex hull of the interior lattice points of the latter, and so on. The boundary of $\triangle$ is denoted by $\partial \triangle$.

We use the notation $\mathcal{Z}(\cdot)$ to denote the algebraic set associated to an ideal, and $\mathcal{I}(\cdot)$ to denote the ideal of an algebraic set.

Unless stated otherwise, a curve is always assumed irreducible, but we don’t a priori require it to be complete and/or non-singular. A canonical curve is a non-singular projective curve that arises as the image of the canonical embedding of a non-singular projective curve that is non-hyperelliptic. A canonical model of a curve $C$ is a canonical curve that is birationally equivalent to $C$.

### 2 Divisors on toric surfaces

We review some facts on divisors on toric surfaces, the primary objective being to fix notation and terminology. Our main references on toric geometry are [10, 17].
Toric surfaces

To a (two-dimensional) lattice polygon $\Delta$ we can associate a projective toric surface $\text{Tor}(\Delta)$ over $k$, in two ways:

- One can consider the (inner) normal fan $\Sigma_{\Delta}$, and let $\text{Tor}(\Delta) = \text{Tor}(\Sigma_{\Delta})$ be the toric surface associated to it.

- One can define $\text{Tor}(\Delta)$ as the Zariski closure of the image of $\varphi : \mathbb{T}^2 \hookrightarrow \mathbb{P}^N : (x, y) \mapsto (x^iy^j)_{(i,j) \in \Delta \cap \mathbb{Z}^2}$ (where $N = \sharp(\Delta \cap \mathbb{Z}^2) - 1$). Explicit equations for $\text{Tor}(\Delta)$ can be read from the combinatorics of $\Delta$, as follows. To each $(i, j) \in \Delta \cap \mathbb{Z}^2$ one associates a variable $X_{i,j}$. Then the ideal of $\text{Tor}(\Delta)$ is generated by the binomials $\prod_{\ell=1}^n X_{i_{\ell},j_{\ell}} - \prod_{\ell=1}^n X_{i'_{\ell},j'_{\ell}}$ for which $\sum_{\ell=1}^n (i_{\ell}, j_{\ell}) = \sum_{\ell=1}^n (i'_{\ell}, j'_{\ell})$.

A result of Koelman states that it suffices to restrict to $n \in \{2, 3\}$, and even to $n = 2$ as soon as $\partial \Delta \cap \mathbb{Z}^2 \geq 4$, see [29, 40].

Examples.

- $\text{Tor}(\Upsilon) = \mathbb{Z}(X_3^0X_0^3 - X_{-1,-1}X_{1,0}X_{0,1}) \subset \mathbb{P}^3$.
- $\text{Tor}(\Box) = \mathbb{Z}(X_0^0X_{1,1} - X_{1,0}X_{0,1}) \subset \mathbb{P}^3$.

Both constructions give rise to the same geometric object, but the second construction comes along with an embedding $\psi : \text{Tor}(\Delta) \hookrightarrow \mathbb{P}^N$, i.e. a very ample invertible sheaf $\psi^*\mathcal{O}_{\mathbb{P}^N}(1)$. Note that every complete fan in $\mathbb{R}^2$ arises as some $\Sigma_{\Delta}$; this allows us to switch back and forth between both worlds.

From divisors to polygons

Let $\Delta$ be a (two-dimensional) lattice polygon. The self-action of $\mathbb{T}^2$ yields an action of $\mathbb{T}^2$ on $\varphi(\mathbb{T}^2)$ that naturally extends to an action on all of $\text{Tor}(\Delta)$. The orbits of the latter are in a dimension-preserving one-to-one correspondence with the faces of $\Delta$. Denote the Zariski closures of the one-dimensional orbits (corresponding to the edges of $\Delta$ and to the rays of $\Sigma_{\Delta}$) by $D_1, \ldots, D_r$. A Weil divisor that arises as a $\mathbb{Z}$-linear combination of the $D_\ell$’s is called torus-invariant. An important example is $K = -\sum_{\ell} D_\ell$, which is a canonical divisor. To a torus-invariant Weil divisor $D = \sum_{\ell} a_\ell D_\ell$ one can associate a polygon

$$\Delta_D = \bigcap_{\ell=1}^r H_\ell, \quad \text{with} \quad H_\ell = \{ (i, j) \in \mathbb{R}^2 \mid <(i, j), v_\ell> \geq -a_\ell \}.$$
where $\langle \cdot, \cdot \rangle$ denotes the standard inner product and $v_\ell$ is the primitive generator of the corresponding ray in $\Sigma_\Delta$. It can be proven that

$$H^0(\text{Tor}(\Delta), D) = \{ f \in k(x, y)^* \mid \text{div}(f) + D \geq 0 \} \cup \{0\} = \langle x^i y^j \rangle_{(i,j) \in \Delta_D \cap \mathbb{Z}^2}$$

(here $\langle \cdot \rangle$ denotes the $k$-linear span; we view $x$ and $y$ as functions on $\text{Tor}(\Delta)$ through $\varphi$). One can also show that $D$ is Cartier if and only if the apex of $H_\ell \cap H_m$ is an element of $\mathbb{Z}^2$ for each pair $\ell, m$ corresponding to adjacent edges of $\Delta$. If moreover every such apex is a vertex of $\Delta_D$ then $D$ is called convex (in particular, if $D$ is a convex torus-invariant Cartier divisor then $\Delta_D$ is a lattice polygon). If this gives a bijective apex-vertex correspondence then $D$ is called strictly convex.

A torus-invariant Cartier divisor $D$ is convex iff it is nef iff it is base-point free (i.e. $\mathcal{O}_{\text{Tor}(\Delta)}(D)$ is generated by global sections). It is strictly convex iff it is ample iff it is very ample. If $D$ is convex then all higher cohomology spaces are trivial (Demazure vanishing). If $D_1$ and $D_2$ are convex torus-invariant Cartier divisors, then their intersection number can be interpreted in terms of a mixed volume:

$$D_1 \cdot D_2 = \text{MV}(\Delta_{D_1}, \Delta_{D_2}) = \text{Vol}(\Delta_1 + \Delta_2) - \text{Vol}(\Delta_1) - \text{Vol}(\Delta_2),$$

where $\text{Vol}(\cdot)$ denotes the (Euclidean) area of the corresponding polygon, and the addition of polygons is in Minkowski’s sense. This is a special case of the Bernstein–Khovanskii–Koushnirenko (BKK) theorem.

Every Weil divisor on $\text{Tor}(\Delta)$ is linearly equivalent to a torus-invariant Weil divisor. Two equivalent torus-invariant Weil divisors $D_1$ and $D_2$ differ by some $\text{div}(x^i y^j)$, hence the corresponding polygons $\Delta_{D_1}$ and $\Delta_{D_2}$ are translates of each other, denoted $\Delta_{D_1} \cong_t \Delta_{D_2}$. Therefore, if one is willing to work modulo $\cong_t$, one can associate a polygon $\Delta_D$ to any Weil divisor $D$ (and in particular to any Cartier divisor and any invertible sheaf). All definitions and statements above carry through.

**Example.** With $\psi : \text{Tor}(\Delta) \hookrightarrow \mathbb{P}^N$ as above, $\Delta_{\psi^* \mathcal{O}_{\mathbb{P}^N}(1)} \cong_t \Delta$.

Now consider a $\Delta$-non-degenerate Laurent polynomial

$$f = \sum_{(i,j) \in \Delta \cap \mathbb{Z}^2} c_{i,j} x^i y^j \in k[x^{\pm 1}, y^{\pm 1}].$$

Let $C \subset \text{Tor}(\Delta)$ be the Zariski closure of $\varphi(U(f))$. The $\Delta$-non-degeneracy of $f$ can be characterized as: $C$ is a smooth curve not containing any of the zero-dimensional toric orbits and intersecting the one-dimensional orbits transversally.
Note that \( \psi(C) \) is just the hyperplane section
\[
\sum_{(i,j) \in \Delta \cap \mathbb{Z}^2} c_{i,j} X_{i,j}.
\]
Therefore, if we view \( C \) as a Cartier divisor, then \( C \sim \psi^* \mathcal{O}_{\mathbb{P}^N}(1) \). In particular, \( C \) is strictly convex, and \( \Delta_C \cong \Delta \).

Let \( \Sigma' \) be a subdivision of \( \Sigma_\Delta \). It naturally induces a birational morphism \( \mu : \text{Tor}(\Sigma') \to \text{Tor}(\Sigma_\Delta) \). Let \( C' \) be the strict transform of \( C \) under \( \mu \) (by non-degeneracy, \( C' \) does not meet the exceptional locus of \( \mu \), so \( C' \cong C \)). Then \( C' \) is still Cartier and convex, albeit no longer strictly convex (unless \( \Sigma' = \Sigma_\Delta \)), and \( \Delta_{C'} \cong \Delta_C \).

**Maximal lattice polygons and weak non-degeneracy**

Assume that \( \Delta^{(1)} \) is two-dimensional. Then the set of lattice polygons \( \Gamma \) for which \( \Gamma^{(1)} = \Delta^{(1)} \) admits a maximum (with respect to inclusion) which we denote by \( \Delta_{\text{max}}^{(1)} \) \[19, \text{Lem. 9}\]. Thus \( \Delta \) is obtained from \( \Delta_{\text{max}}^{(1)} \) by subsequently pruning off a number of vertices, without affecting the interior. One verifies that such a vertex is necessarily smooth, i.e. the primitive inner normal vectors of the adjacent edges form a basis of \( \mathbb{Z}^2 \). Hence the corresponding zero-dimensional toric orbit must be a non-singular point of \( \text{Tor}(\Delta_{\text{max}}^{(1)}) \).

Let \( \Delta \) be a (two-dimensional) lattice polygon and let \( f \in k[x^{\pm 1}, y^{\pm 1}] \) be an irreducible Laurent polynomial such that \( \Delta(f) \subset \Delta \). Then we call \( f \) weakly \( \Delta \)-non-degenerate if for every face \( \tau \subset \Delta \)

- the system
  \[
  f_\tau = \frac{\partial f_\tau}{\partial x} = \frac{\partial f_\tau}{\partial y} = 0
  \]
  has no solutions in \( \mathbb{T}^2 \), or
• \( \tau \) is a smooth vertex.

If \( f \) is \( \Delta \)-non-degenerate, then it is weakly \( \Delta \)-non-degenerate. If \( \Delta^{(1)} \) is two-dimensional and \( f \) is \( \Delta \)-non-degenerate, then it is weakly \( \Gamma \)-non-degenerate for every \( \Gamma \supset \Delta \) for which \( \Gamma^{(1)} = \Delta^{(1)} \) (and for \( \Gamma = \Delta^{\text{max}} \) in particular).

If \( f \) is weakly \( \Delta \)-non-degenerate, then \( \varphi(U(f)) \) still completes to a smooth curve \( C \subset \text{Tor}(\Delta) \) (that now possibly contains some of the non-singular zero-dimensional orbits). All other conclusions of the preceding paragraph remain valid, except for the last phrase: if \( \Sigma' \) subdivides a smooth cone of \( \Sigma_{\Delta} \) then this might affect \( \Delta_C \).

We end by remarking that the polygon \( \Delta^{\text{max}} \) can be characterized as follows. Write \( \Delta^{(1)} \) as an intersection of half-planes

\[
\bigcap_{\ell=1}^{r} H_\ell, \quad \text{with } H_\ell = \{ (i,j) \in \mathbb{R}^2 \mid <(i,j), v_\ell > \geq -a_\ell \},
\]

where \( v_1, \ldots, v_r \) are primitive inward pointing normal vectors of the edges of \( \Delta \). Then

\[
\Delta^{\text{max}} = \bigcap_{\ell=1}^{r} H_\ell^{(-1)}, \quad \text{where } H_\ell^{(-1)} = \{ (i,j) \in \mathbb{R}^2 \mid <(i,j), v_\ell > \geq -a_\ell -1 \}.
\]

When applying this construction to an arbitrary two-dimensional lattice polygon \( \Gamma \), one ends up with a polygon \( \Gamma^{(-1)} \) that is a lattice polygon if and only if \( \Gamma = \Delta^{(1)} \) for some lattice polygon \( \Delta \); see [19, Lem. 11] for a proof of this convenient criterion. (We will sometimes call

\[
\{ (i,j) \in \mathbb{R}^2 \mid <(i,j), v_\ell > = -a_\ell -1 \}
\]

the *outward shift* of the edge corresponding to the index \( \ell \); then a necessary, but generally insufficient condition for \( \Gamma^{(-1)} \) to be a lattice polygon is that the outward shifts of any pair of adjacent edges intersect in a lattice point.)

**Remarks.**

- The criterion gives a method for algorithmically enumerating lattice polygons, as elaborated in [7] and [28, Sect. 4.4]. We will use this in the proof of Theorem 4.

- It also shows that, on the level of divisors, the operation \( \Delta \mapsto \Delta^{(-1)} \) can be interpreted as subtraction by \( K \). Similarly, \( \Delta \mapsto \Delta^{(1)} \) is related to adding a canonical divisor (i.e., adjunction); see [12].
Let $\Delta$ be a (two-dimensional) lattice polygon. A curve $C$ is called $\Delta$-toric if it can be realized as a smooth curve on a toric surface, in such a way that $\Delta_C \cong \Delta$. Then we have the following logical implications:

$$\Delta\text{-non-degenerate} \implies \text{weakly } \Delta\text{-non-degenerate} \implies \Delta\text{-toric}.$$ 

These implications might be strict (it could be impossible to avoid passing through a certain zero-dimensional orbit, resp. to avoid tangency to a certain one-dimensional orbit), but in all theorems and lemmata in this paper, the three notions are exchangeable.

3 Invariants encoded in the Newton polygon

3.1 Genus

**Theorem 2** (H. Baker 1893, Khovanskii 1977). Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be non-degenerate with respect to its Newton polygon $\Delta = \Delta(f)$. Then the (geometric) genus of $U(f)$ equals $\#(\Delta^{(1)} \cap \mathbb{Z}^2)$.

**Proof.** Denote the genus of $U(f)$ by $g$ and let $C = \varphi(U(f))^{\text{Zar.cl.}}$. Let $\Sigma'$ be a smooth subdivision of $\Sigma_\Delta$ and let $C' \subset \text{Tor(\Sigma')}$ be the strict transform of $C$ under the corresponding birational morphism. Let $D_1, \ldots, D_r$ be the torus-invariant prime divisors of $\text{Tor(\Sigma')}$ and let $a_\ell \in \mathbb{Z}$ be such that $C' \sim \sum_{\ell=1}^r a_\ell D_\ell$. Recall that $\Delta_{C'} \cong_\ell \Delta$. By the adjunction formula,

$$D = K + \sum_{\ell=1}^r a_\ell D_\ell = \sum_{\ell=1}^r (a_\ell - 1) D_\ell$$

restricts to a canonical divisor $K_{C'} = D|_{C'}$ on $C'$. One finds

$$H^0(C', K_{C'}) = \langle x^i y^j \rangle_{(i,j) \in \Delta^{(1)} \cap \mathbb{Z}^2}$$

(here $x$ and $y$ should be viewed as elements of the fraction field of $k[x^{\pm 1}, y^{\pm 1}]/(f)$). Since $\Delta_D = \Delta_{C'}^{(1)} \cong_\ell \Delta^{(1)}$, the theorem follows using $g = h^0(C', K_{C'})$. 

Assume $g \geq 2$. Then the proof also tells us that the map

$$\varphi_{\text{can}} : U(f) \to \mathbb{P}^{g-1} : (x, y) \mapsto (x^i y^j)_{(i,j) \in \Delta^{(1)} \cap \mathbb{Z}^2}$$

(1)

factors as $\kappa \circ \varphi|_{U(f)}$, where $\kappa : C \to \mathbb{P}^{g-1}$ is a canonical morphism. From this (and using that the canonical image is rational iff $C$ is hyperelliptic) it follows that $C$ is hyperelliptic iff the interior lattice points of $\Delta$ are collinear; see [28, Lem. 3.2.9] or
for more details. If $C$ is non-hyperelliptic (i.e. $\Delta^{(1)}$ is two-dimensional) it follows that the canonical image lies in $\text{Tor}(\Delta^{(1)}) \subset \mathbb{P}^{g-1}$. Of course, unlike $C$ itself, $\kappa(C)$ does not arise as a hyperplane section of its ambient toric surface. But in case $\Delta^{(2)}$ is non-empty, there is still a nice way of describing the equations that cut out $\kappa(C)$ (this observation seems new). Write

$$f = \sum_{(i,j) \in \Delta \cap \mathbb{Z}^2} c_{i,j} x^i y^j \in k[x^{\pm 1}, y^{\pm 1}].$$

Let $w \in \Delta^{(2)} \cap \mathbb{Z}^2$. Then for each $(i, j) \in \Delta \cap \mathbb{Z}^2$ there exist $u_{i,j}, v_{i,j} \in \Delta^{(1)} \cap \mathbb{Z}^2$ such that

$$(i, j) - w = (u_{i,j} - w) + (v_{i,j} - w).$$

(The existence follows, for instance, by translating the polygon so that $w$ becomes the origin, noting that this implies $\Delta \subset \Delta^{\max} = \Delta^{(1)(-1)} \subset 2\Delta^{(1)}$, and using the normality property of two-dimensional lattice polygons.) The quadratic form

$$Q_w = \sum_{(i,j) \in \Delta \cap \mathbb{Z}^2} c_{i,j} X_{u_{i,j}} X_{v_{i,j}}$$

is well-defined modulo the ideal of $\text{Tor}(\Delta^{(1)})$. It clearly vanishes on the image of $\kappa(C)$, hence it is contained in the ideal of $\kappa(C)$. We claim that, along with the ideal of $\text{Tor}(\Delta^{(1)})$, the quadrics $Q_w$ generate $\mathcal{I}(\kappa(C))$.

It suffices to prove this claim under the assumption that both the ideal of $\text{Tor}(\Delta^{(1)})$ and the ideal of $\kappa(C)$ are generated by quadrics. Indeed:

- Recall from Section 2 that a lattice polygon $\Gamma$ is of the form $\Delta^{(1)}$ if and only if $\Gamma^{(-1)}$ is also a lattice polygon. Together with $\Delta^{(2)} \neq \emptyset$, this implies that $\partial \Delta^{(1)} \cap \mathbb{Z}^2 \geq 4$ (so that Koelman’s criterion guarantees that the ideal of $\text{Tor}(\Delta^{(1)})$ is generated by quadrics), unless $\Delta^{(1)} \cong \Upsilon$.

- From Theorem 9 below it follows that $\Delta^{(2)} \neq \emptyset$ implies that the Clifford index of $C$ is at least 2 (so that Petri’s theorem [38] guarantees that $\mathcal{I}(\kappa(C))$ is generated by quadrics), unless $\Delta^{(1)} \cong \Upsilon$.

- So the only situation of concern is where $\Delta^{(1)} \cong \Upsilon$. But here the claim follows by noting that $\text{Tor}(\Upsilon)$ is a cubic in $\mathbb{P}^3$ and that a canonical curve of genus 4 is of degree 6, so that a single (necessarily unique) quadric suffices.

Now let $\mathcal{I}_2(\kappa(C))$ be the $k$-vector space of quadrics in $\mathcal{I}(\kappa(C))$, and similarly define $\mathcal{I}_2(\text{Tor}(\Delta^{(1)}))$ and $\mathcal{I}_2(\mathbb{P}^{g-1})$. We know that

- $\dim \mathcal{I}_2(\kappa(C)) = \binom{g-2}{2}$ by [38] §3.4,
\[ \dim I_2(\Tor(\Delta^{(1)})) = \binom{g+1}{2} - \sharp(2\Delta^{(1)} \cap \mathbb{Z}^2) \] because the \( k \)-vector space morphism
\[ \chi : I_2(\mathbb{P}^{g-1}) \to k[x^{\pm 1}, y^{\pm 1}] : X_{i_1,j_1}, X_{i_2,j_2} \mapsto x^{i_1+i_2}y^{j_1+j_2} \]
has kernel \( I_2(\Tor(\Delta^{(1)})) \) and surjects onto \( \langle x^iy^j \rangle_{(i,j) \in 2\Delta^{(1)} \cap \mathbb{Z}^2} \).

From this one finds
\[ \dim I_2(\kappa(C)) - \dim I_2(\Tor(\Delta^{(1)})) = 3 - 3g + 4\text{Vol}(\Delta^{(1)}) + \sharp(\partial \Delta^{(1)} \cap \mathbb{Z}^2) + 1 \]
\[ = 3 - 3g + 4\sharp(\Delta^{(2)} \cap \mathbb{Z}^2) + 3\sharp(\partial \Delta^{(1)} \cap \mathbb{Z}^2) - 3 = \sharp(\Delta^{(2)} \cap \mathbb{Z}^2), \]
where the first and second equalities make use of Pick’s theorem. Therefore, to prove the claim, it suffices to show that the quadrics
\[ Q_w, \quad w \in \Delta^{(2)} \cap \mathbb{Z}^2 \]
are \( k \)-linearly independent of each other and of the quadrics in \( I_2(\Tor(\Delta^{(1)})) \). But this follows because
\[ \chi(Q_w) = (x,y)^w \cdot f. \]
Thus any linear combination in which the \( Q_w \)’s appear non-trivially is mapped to a non-zero multiple of \( f \), and must therefore be non-zero itself.

Summarizing:

**Theorem 3.** Let \( f \in k[x^{\pm 1}, y^{\pm 1}] \) be non-degenerate with respect to its Newton polygon \( \Delta = \Delta(f) \) and assume that \( \Delta^{(1)} \) is two-dimensional. Then \( U(f) \) is non-hyperelliptic and the Zariski closure of the image of \( U(f) \) is a canonical model of \( U(f) \). If moreover \( \Delta^{(2)} \neq \emptyset \), then the ideal of the latter is generated by \( I(\Tor(\Delta^{(1)})) \) and the quadrics \( \{ Q_w \} \subset \Delta^{(2)} \cap \mathbb{Z}^2 \).

**Remarks.**

- The above proof shows that \( \chi \) maps any element of \( I_2(\kappa(C)) \) to \( gf \), where \( g \in k[x^{\pm 1}, y^{\pm 1}] \) satisfies \( \Delta(g) \subset \Delta^{(2)} \).
- The upper bound \( g \leq \sharp(\Delta^{(1)} \cap \mathbb{Z}^2) \) holds in general (that is, regardless of the non-degeneracy condition) and was discovered by H. Baker in 1893 already [2]; see [4] for an elementary proof of Baker’s bound. Khovanskii was the first to give a sufficient condition for equality [27, §4, Ass. 2].
3.2 Gonality

**Theorem 4.** Let \( f \in k[x^\pm 1, y^\pm 1] \) be non-degenerate with respect to its Newton polygon \( \Delta = \Delta(f) \). Suppose that \( \Delta^{(1)} \) is not equivalent to any of the following:

\[ \emptyset, \ (d-3)\Sigma \text{ (for some integer } d \geq 3), \ \Upsilon, \ \Gamma_1^5, \ \Gamma_2^5, \ \Gamma_3^5 \]  

(see Section 5.7 for a definition of the \( \Gamma_i^5 \)’s). Then every gonality pencil on \( U(f) \) is computed by projecting along some lattice direction.

By a lattice direction we mean a primitive vector \( v \in \mathbb{Z}^2 \); then by projection along \( v \) we mean a map \( U(f) \to k^* : (x, y) \mapsto x^a y^b \) for a primitive vector \((a,b)\) that is perpendicular to \( v \). A gonality pencil that is computed by projecting along some lattice direction will be called combinatorial.

**Remark.** In case \( \Delta^{(1)} \) is among \( \Upsilon, 2\Upsilon, \Gamma_1^5, \Gamma_2^5, \Gamma_3^5 \), there is only a single corresponding \( \Delta \) (namely, \( 2\Upsilon, \Delta_1^5, \Delta_2^5, \Delta_3^5 \) resp. \( \Delta_3^5 \)).

Before we proceed to the proof of Theorem 4, let us discuss some corollaries. From the énoncé it follows that if \( \Delta \) is non-equivalent to any of the polygons listed in (2), then the gonality of \( U(f) \) equals the lattice width of \( \Delta \), i.e. the minimal height of a horizontal strip in which \( \Delta \) can be mapped unimodularly (notation: \( \text{lw}(\Delta) \); it is convenient to define \( \text{lw}(\emptyset) = -1 \)); if a lattice direction is correspondingly mapped to \((0, \pm 1)\) we say that it computes the lattice width. We have the following properties:

**Lemma 5.** Let \( \Delta \) be a two-dimensional lattice polygon. One has:

1. If \( \Delta^{(1)} \neq \emptyset \) and \( \Delta^{(1)} \not\equiv (d-3)\Sigma \) for some \( d \geq 3 \), then \( \text{lw}(\Delta) = \text{lw}(\Delta^{(1)}) + 2 \), and a lattice direction computes the lattice width of \( \Delta \) if and only if it computes the lattice width of \( \Delta^{(1)} \).

2. There are at most 4 pairs \( \pm v \) of lattice directions computing \( \text{lw}(\Delta) \).

3. \( \text{lw}(\Delta)^2 \leq \frac{4}{9} \text{Vol}(\Delta) \).

**Proof.** For 1. see [8, Thm. 4] or [32, Thm. 13], and for 2. see [13] (each of these items allows for a more precise statement). For 3. see [16].

**Remark.** We will sometimes implicitly use the following corollary to Lemma 5. Let \( \Delta \) be a non-empty lattice polygon and assume for ease of exposition that \( \Delta \subset \{ (i,j) \in \mathbb{R}^2 \mid 0 \leq j \leq \text{lw}(\Delta) \} \). Then we claim that every line \( j = \ell \) (where \( \ell \in \{0, \ldots, \text{lw}(\Delta)\} \) intersects \( \Delta \) in at least one lattice point. Indeed, if \( \Delta^{(1)} = \emptyset \) or \( \Delta^{(1)} \cong (d-3)\Sigma \) for some \( d \geq 3 \), then this can be verified exhaustively. If not, the lemma shows that this is true for the lines \( j = 1 \) and \( j = \text{lw}(\Delta) - 1 \). The claim
then follows by recursively applying it to $\Delta^{(1)}$.

Thus if $\Delta^{(1)}$ is not among the polygons listed in (2) then the gonality of $U(f)$ equals $\text{lw}(\Delta^{(1)}) + 2$. The other instances of $\Delta^{(1)}$ can be analyzed case by case:

- If $\Delta^{(1)} = \emptyset$ then $U(f)$ is rational, hence of gonality 1.
- If $\Delta^{(1)} \cong (d-3)\Sigma$ then $U(f)$ is a smooth plane curve of degree $d$, hence of gonality $d - 1$ by a result of Namba [36] (a proof can also be found in [39, Prop. 3.13]).
- If $\Delta^{(1)} \cong \Upsilon$ then $U(f)$ is a non-hyperelliptic curve of genus 4, hence of gonality 3.
- If $\Delta^{(1)} \cong 2\Upsilon$ then $U(f)$ is birationally equivalent to a smooth intersection of two cubics in $\mathbb{P}^3$, hence of gonality 6 by a result of Martens (see [8, Thm. 9] for more details).
- If $\Delta^{(1)} \cong \Gamma^3_i$ ($i = 1, 2, 3$) then $U(f)$ is birationally equivalent to a non-hyperelliptic, non-trigonal curve of genus 5 by [8, Lem. 3], hence of gonality 4.

We conclude:

**Corollary 6.** Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be non-degenerate with respect to its Newton polygon $\Delta = \Delta(f)$. Then the gonality of $U(f)$ equals $\text{lw}(\Delta^{(1)}) + 2$, unless $\Delta^{(1)} \cong \Upsilon$ (i.e. $\Delta \cong 2\Upsilon$), in which case it equals 3.

**Remarks.**

- This implies a conjecture by the current authors [8, Conj. 1]. It does not imply the corresponding conjecture on metric graphs [8, Conj. 3 + err.].
- The corollary also implies that in case $\Delta^{(1)} \cong \Upsilon$, then a combinatorial gonality pencil cannot exist. The same conclusion holds for $\Delta \cong d\Sigma$ for $d \geq 2$. In all other cases, there exists at least one combinatorial gonality pencil.
- Let $k'$ be an arbitrary field of characteristic 0 with algebraic closure $k$. Suppose that $f \in k'[x^{\pm 1}, y^{\pm 1}]$ is non-degenerate with respect to its Newton polygon when considered as a Laurent polynomial over $k$. If $\Delta(f) \not\cong 2\Upsilon, d\Sigma$ then the above implies that $\gamma(U(f)) = \gamma_{k'}(U(f))$, where $\gamma(U(f))$ is the gonality of $U(f)$, and $\gamma_{k'}(U(f))$ is the minimal degree of a $k'$-rational map to $\mathbb{P}^1$. If $\Delta(f) \cong 2\Upsilon$ or $\Delta(f) \cong d\Sigma$ for some $d \geq 2$ then this may not be true. (Example: $x^2 + y^2 + 1 \in \mathbb{R}[x, y]$.)
• By letting $k' = \mathbb{C}(t)$, the preceding remarks lend prudent support in favor of a conjecture by M. Baker, stating that the gonality of a graph equals the gonality of the associated metric graph \cite{1}, Conj. 3.14.

We now give a proof of Theorem 4. We recall that the main ideas are taken from Kawaguchi \cite{26}, but that our proof covers the case where $U(f)$ is birationally equivalent to a smooth projective plane curve (the key ingredient here being the block of text surrounding (3) below).

**Proof of Theorem 4.** Let $g = \sharp(\Delta(1) \cap \mathbb{Z}^2)$ be the geometric genus of $U(f)$. Note that our assumptions imply $g \geq 2$. Then recall that lw($\Delta(1)$) = 0 if and only if $U(f)$ is hyperelliptic. By Lemma 5 this holds if and only if lw($\Delta$) = 2, hence a $g_1^1$ can be computed by projection along some lattice direction. Since the $g_1^1$ of a hyperelliptic curve is unique, Theorem 4 follows in this case. Thus we may assume that $\Delta(1)$ is two-dimensional and that $U(f)$ is of gonality $\gamma \geq 3$.

As explained in Section 2, $f$ is weakly $\Delta_{\text{max}}$-non-degenerate, thus $\varphi(U(f))$ completes to a smooth plane curve $C \subset \text{Tor}(\Sigma_{\Delta_{\text{max}}})$. Let $\Sigma'$ be a minimal smooth subdivision of $\Sigma_{\Delta_{\text{max}}}$ and let $\mu : \text{Tor}(\Sigma') \to \text{Tor}(\Sigma_{\Delta_{\text{max}}})$ be the corresponding birational morphism. Let $C'$ be the strict transform of $C$ under $\mu$. By weak non-degeneracy and because the smooth subdivision was chosen minimal, $C'$ does not meet the exceptional locus of $\mu$. In particular, $\mu|_{C'}$ is an isomorphism of curves and $\Delta_{C'} \cong \Delta_{\text{max}}$. Since $\text{Tor}(\Sigma')$ is smooth, every Weil divisor is Cartier.

By the BKK theorem (recall that $C'$ is a convex divisor),

$$C'^2 = \text{MV}(\Delta_{\text{max}}, \Delta_{\text{max}}) = 2\text{Vol}(\Delta_{\text{max}}) \geq \frac{3}{4}\text{lw}(\Delta_{\text{max}})^2 = \frac{3}{4}\text{lw}(\Delta)^2,$$

where the third and fourth (in)equalities follow from Lemma 5. For small lattice widths the bound $\text{Vol}(\Delta_{\text{max}}) \geq \frac{3}{8}\text{lw}(\Delta_{\text{max}})^2$ can be improved: using the data from 4 one can computationally verify that

$$C'^2 = 2\text{Vol}(\Delta_{\text{max}}) \geq \begin{cases} 
18 & \text{if } \text{lw}(\Delta_{\text{max}}) = 3, \\
20 & \text{if } \text{lw}(\Delta_{\text{max}}) = 4, \\
25 & \text{if } \text{lw}(\Delta_{\text{max}}) = 5, \\
28 & \text{if } \text{lw}(\Delta_{\text{max}}) = 6 
\end{cases}$$

(remark that by Pick’s theorem it suffices to verify these inequalities for small genus only). For these bounds it is essential that $\Delta_{\text{max}}$ is maximal and that $\Delta(1)$ is not among the polygons listed in 2. The patient reader can also do an elaborate analysis by hand, following Kawaguchi \cite{26} Props. 3.10–3.12,4.3].
We now come to the heart of the proof. Let \( p : C' \to \mathbb{P}^1 \) be a morphism of degree \( \gamma \). A theorem by Serrano \[39, \text{Thm. 3.1}\] states that if \( C'^2 > (\gamma + 1)^2 \) then \( p \) can be extended to a morphism \( \text{Tor}(\Sigma') \to \mathbb{P}^1 \). From this it follows that \( p \) is combinatorial (see the last paragraph of the proof). Unfortunately, in general we only have that

\[
C'^2 \geq \frac{3}{4}\text{lw}(\Delta)^2 \geq \frac{3}{4}\gamma^2
\]

(but note that for ‘most’ lattice polygons, the stronger bound \( C'^2 > (\gamma + 1)^2 \) does hold). To bridge this, we follow an approach of Harui \[23\], who dug into Serrano’s proof to extract Theorem 7 below.

We proceed by contradiction: assume that \( p \) cannot be extended to all of \( \text{Tor}(\Sigma') \). Then by Theorem 7 (note that \( C'^2 > 4\gamma \)) there exists an effective divisor \( V \) on \( \text{Tor}(\Sigma') \) satisfying

\[
0 < s < C' \cdot V - s \leq \gamma \quad \text{and} \quad C'^2 \leq \frac{(\gamma + s)^2}{s},
\]

where \( s = V^2 \). Our bounds on \( C'^2 \) imply that \( s = 1 \), except possibly if \( \gamma \in \{6, 7, 8\} \), in which case we can only conclude \( s \in \{1, 2\} \).

We claim that this implies \( h^0(\text{Tor}(\Sigma'), V) \leq s + 1 \). Suppose not, then \( \Delta_V \) contains at least \( s + 2 \) lattice points. Let \( \Gamma \) be the convex hull of these lattice points and let \( D_\Gamma = \sum_\ell a_\ell D_\ell \) be the torus-invariant divisor obtained by taking the \( a_\ell \)'s minimal such that \( \Gamma \subset H_\ell \). (Here, as ever, the \( D_\ell \)'s are the torus-invariant prime divisors of \( \text{Tor}(\Sigma') \) and the \( H_\ell \)'s are the corresponding half-planes.) One verifies that \( D_\Gamma \) is convex, that \( \Delta_{D_\Gamma} = \Gamma \), and that \( V - D_\Gamma \) is effective.

- Suppose that, up to a unimodular transformation, \( \Gamma \) contains a horizontal line segment \( I \) of length \( \geq \frac{3}{2} \). Then \( C' \cdot V = C' \cdot (D_\Gamma + (V - D_\Gamma)) \) is bounded from below by

\[
C' \cdot D_\Gamma = \text{MV}(\Delta_{\text{max}}, \Gamma) \geq \text{MV}(\Delta_{\text{max}}, I) \geq \frac{3}{2}\text{lw}(\Delta_{\text{max}}) = \frac{3}{2}\text{lw}(\Delta) \geq \frac{3}{2}\gamma.
\]

The first inequality follows because \( \text{MV} \) is an increasing function. Since \( \gamma \geq 3 \) and \( C' \cdot D_\Gamma \) is an integer, this contradicts \( C' \cdot V \leq \gamma + s \).

- Now suppose that \( s = 1 \), so that \( \Gamma \) contains at least 3 lattice points, which by the previous item are non-collinear. Then up to a unimodular transformation, we may assume that \( \Sigma \subset \Gamma \). One finds

\[
C' \cdot V \geq \text{MV}(\Delta_{\text{max}}, \Gamma) \geq \text{MV}(\Delta_{\text{max}}, \Sigma) = \deg f
\]

(3)
where \( \text{deg } f \) is the ‘total degree’ of \( f \), i.e. the minimal \( d \) such that \( \Delta^{(1)} \) is contained in a translate of \( d \Sigma \) (indeed, this follows from Bézout’s theorem). Then \( \Delta^{(1)} \subset (d-3) \Sigma \), and by our assumptions this inclusion is strict. It follows that \( \text{lw}(\Delta^{(1)}) \leq d - 4 \), hence by Lemma 5 that \( \text{lw}(\Delta) = \text{lw}(\Delta^{\max}) \leq d - 2 \). From (3) we conclude that \( C' \cdot V \geq \text{lw}(\Delta) + 2 \). This contradicts \( C' \cdot V \leq \gamma + 1 \).

\[ \bullet \text{ Suppose that } s = 2, \text{ so that } \Gamma \text{ contains at least 4 lattice points. Now we may assume that } \square \subset \Gamma, \text{ from which } \]

\[ C' \cdot V \geq \text{MV}(\Delta^{\max}, \Gamma) \geq \text{MV}(\Delta^{\max}, \square) = a + b, \]

where \((a, b)\) is the ‘bidegree’ of \( f \), i.e. the minimal couple of values for which \( \Delta^{\max} \) is contained in a translate of \([0, a] \times [0, b]\). This follows from the BKK theorem applied to \( \mathbb{P}^1 \times \mathbb{P}^1 \). We conclude that \( C' \cdot V \geq 2 \text{lw}(\Delta) \). This contradicts \( C' \cdot V \leq \gamma + 1 \).

The claim follows.

Then, since \( h^0(\text{Tor}(\Sigma'), V) \leq s + 1 \) and because a lattice polygon having at most 3 lattice points cannot have any interior lattice points, we deduce that \( h^0(\text{Tor}(\Sigma'), V + K) = 0 \), with \( K = -\sum \ell \ell \ell \ell \). The Riemann-Roch theorem yields that

\[ \frac{1}{2}(V + K) \cdot V = h^0(\text{Tor}(\Sigma'), V + K) - h^1(\text{Tor}(\Sigma'), V + K) + h^0(\text{Tor}(\Sigma'), -V) - \chi(\mathcal{O}_{\text{Tor}(\Sigma')}) \]

is bounded by \(-\chi(\mathcal{O}_{\text{Tor}(\Sigma')}) = -1\), i.e. \( K \cdot V \leq -s - 2 \). But then Riemann-Roch also tells us that

\[ h^0(\text{Tor}(\Sigma'), V) = h^1(\text{Tor}(\Sigma'), V) - h^0(\text{Tor}(\Sigma'), K - V) + \frac{1}{2} V \cdot (V - K) + 1 \]

is at least \( s + 2 \).

Thus we run into the desired contradiction, and we conclude that \( p \) can be extended to all of \( \text{Tor}(\Sigma') \). Let \( \tilde{p} : \text{Tor}(\Sigma') \to \mathbb{P}^1 \) be such that \( \tilde{p}|_{C'} = p \). Let \( F \) be a fiber of \( \tilde{p} \), so that \( F \cdot C' = \gamma \). Then \( C' \cdot (F - C') \leq \gamma - \frac{3}{4} \gamma^2 < 0 \). Since \( C' \) is nef it follows that \( h^0(\text{Tor}(\Sigma'), F - C') = 0 \). Now by tensoring the short exact sequence

\[ 0 \to \mathcal{O}_{\text{Tor}(\Sigma')}(C') \to \mathcal{O}_{\text{Tor}(\Sigma')}) \to \mathcal{O}_{C'} \to 0 \]

with \( \mathcal{O}_{\text{Tor}(\Sigma')}(F) \) and taking cohomology, we find the exact sequence

\[ 0 \to H^0(\text{Tor}(\Sigma'), F - C') \to H^0(\text{Tor}(\Sigma'), F) \to H^0(C', F|_{C'}) \to \ldots, \]

16
which proves that $h^0(\text{Tor}(\Sigma'), F) \leq 2$. Thus $|F|$ is a linear system of rank 1, i.e. every element of $|F|$ is a fiber of $\tilde{p}$. Let $D$ be a torus-invariant divisor that is equivalent to $F$. By translating if necessary we may assume that $(0,0) \in \Delta_D$, so that $D$ is effective. But then $D \in |F|$ and

$$H^0(\text{Tor}(\Sigma'), D) = \langle 1, x^a y^b \rangle$$

for some primitive $(a, b) \in \mathbb{Z}^2$. We find that $\tilde{p}|_{\mathbb{T}^2} : (x,y) \mapsto x^a y^b$ (up to an automorphism of $\mathbb{P}^1$), i.e. $p$ is combinatorial. ■

**Theorem 7** (Serrano, 1987). Let $C$ be a smooth projective curve on a smooth projective surface $S$, and let $p : C \to \mathbb{P}^1$ be a surjective morphism of degree $d$. Suppose that $C^2 > 4d$ and that $p$ cannot be extended to a morphism $S \to \mathbb{P}^1$. Then there exists an effective divisor $V$ on $S$ for which

$$0 < V^2 < V \cdot (C - V) \leq d \quad \text{and} \quad C^2 \leq \frac{(d + V^2)^2}{V^2}.$$

**Proof.** By contradiction. Suppose that such an effective divisor $V$ does not exist, then one can replace *Claim 6* in Serrano’s proof [39, p. 401] by the following reasoning (the text below does not make sense without Serrano’s paper at hand):

*Claim 6: $a = 0$. Suppose that $a > 0$. Then $V_1$ is an effective divisor such that $0 < V_1^2 < V_1 \cdot V_2 \leq d$ because $a < e$. On the other hand,

$$C^2 = a + 2e + b \leq a + 2e + \frac{e^2}{a} \leq a + 2d + \frac{d^2}{a} = \frac{(a + V_1^2)^2}{V_1^2}.$$"

Since $V_2 = C - V_1$ this contradicts our hypothesis. Hence, $a = 0$. The rest of the proof can be copied word by word. ■

**Application: Weierstrass semi-groups of embedding dimension 2**

A numerical semi-group is said to have *embedding dimension 2* if it is of the form $a\mathbb{N} + b\mathbb{N}$ for coprime integers $a, b \geq 2$. Using Corollary [6] we can prove the following:

**Theorem 8.** If a smooth projective curve $C/k$ carries a point $P$ having a Weierstrass semi-group of embedding dimension 2, then this semi-group does not depend on the choice of $P$.

**Remark.** This is well-known in the case of hyperelliptic curves of genus $g \geq 2$, all of whose Weierstrass points have semi-group $2\mathbb{N} + (2g + 1)\mathbb{N}$. 17
Proof. If \( C \) has a Weierstrass point with semi-group \( aN + bN \) for coprime integers \( a, b \geq 2 \), then it is of genus \((a - 1)(b - 1)/2\) (by Riemann-Roch – this is the number of gaps in the semi-group). We claim that \( C \) has gonality \( \min\{a, b\} \). Together, this implies that \( a \) and \( b \) are indeed uniquely determined (up to order). To prove the claim, we use a result of Miura stating that \( C \) is birationally equivalent to a smooth affine curve of the form

\[
c_{b,0}x^b + c_{0,a}y^a + \sum_{ia+jb<ab} c_{i,j}x^iy^j, \quad c_{b,0}c_{0,a} \neq 0.
\]

See [35, Thm. 5.17, Lem. 5.30] or [33]. From this it is clear that \( C \) is \( \Delta_{a,b} \)-non-degenerate, where

\[
\Delta_{a,b} = \text{conv}\{(b, 0), (0, a), (0, 0)\}
\]

(an affine translation ensures appropriate behavior with respect to the toric boundary). By Corollary 5 we have that the gonality of \( C \) equals \( \text{lw}(\Delta_{a,b}) = \min\{a, b\} \). ■

Remark. Miura studied curves having a Weierstrass semi-group of the form \( aN + bN \) in the context of coding theory; he called them \( C_{a,b} \) curves. (In a recent past, \( C_{a,b} \) curves have enjoyed fair interest from researchers in explicit algebraic geometry [11, 20, 35]). Then another way to state Theorem 8 is that a curve cannot be simultaneously \( C_{a,b} \) and \( C_{a',b'} \) for distinct pairs \( \{a, b\} \) and \( \{a', b'\} \).

3.3 Clifford index and Clifford dimension

To a smooth projective curve \( C/k \) of genus \( g \geq 4 \) one can associate its Clifford index

\[
\text{ci}(C) = \min\{d - 2r \mid C \text{ carries a divisor } D \text{ with } |D| = g_d \text{ and } h^0(C, D), h^0(C, K - D) \geq 2 \}
\]

(where \( K \) is a canonical divisor on \( C \)) and its Clifford dimension

\[
\text{cd}(C) = \min\{r \mid \text{there exists a } g^r \text{ realizing } \text{ci}(C)\};
\]

see [15]. In the case of a singular and/or non-complete curve \( C/k \), we define \( \text{ci}(C) \) and \( \text{cd}(C) \) to be the corresponding quantities associated to its smooth complete model. In this section we give a combinatorial interpretation for the Clifford index and the Clifford dimension. Again the key trick is due to Kawaguchi, but thanks to our more careful analysis of the planar curve case we obtain a complete statement.

Theorem 9. Let \( f \in k[x^\pm 1, y^\pm 1] \) be non-degenerate with respect to its Newton polygon \( \Delta = \Delta(f) \) and suppose that \( \sharp(\Delta^{(1)} \cap \mathbb{Z}^2) \geq 4 \). Then
• if $\Delta^{(1)} \cong (d - 3)\Sigma$ for $d \geq 5$ then $ci(U(f)) = d - 4$ and $cd(U(f)) = 2$,
• if $\Delta^{(1)} \cong \Upsilon$ then $ci(U(f)) = 1$ and $cd(U(f)) = 1$,
• if $\Delta^{(1)} \cong 2\Upsilon$ then $ci(U(f)) = 3$ and $cd(U(f)) = 3$,
• in all other cases $ci(U(f)) = \text{lw}(\Delta^{(1)})$ and $cd(U(f)) = 1$.

Proof. If a curve $C/k$ satisfies $cd(C) \geq 2$ then the number of gonality pencils is infinite. It follows from Theorem 4 and Lemma 5 that $cd(U(f)) = 1$ as soon as $\Delta^{(1)}$ is not among the polygons listed in (2). In that case $ci(U(f)) = \gamma(U(f)) - 2 = \text{lw}(\Delta^{(1)})$. The other cases respectively correspond to smooth projective plane curves of degree $d \geq 5$, non-hyperelliptic curves of genus 4, and smooth intersections of pairs of cubics in $\mathbb{P}^3$, situations in which the Clifford index and the Clifford dimension are well-known [15].

Remark. For curves $C/k$ of genus $1 \leq g \leq 3$ one sometimes defines
• $ci(C) = 1$ if $C$ is a non-hyperelliptic genus 3 curve, and $ci(C) = 0$ if not,
• $cd(C) = 1$.

With these conventions, Theorem 9 remains valid when one replaces the condition $\#(\Delta^{(1)} \cap \mathbb{Z}^2) \geq 4$ with $\#(\Delta^{(1)} \cap \mathbb{Z}^2) \geq 1$.

Corollary 10. Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be non-degenerate with respect to its (two-dimensional) Newton polygon $\Delta = \Delta(f)$. Then $U(f)$ is birationally equivalent to a smooth projective plane curve if and only if $\Delta^{(1)} = \emptyset$ or $\Delta^{(1)} \cong (d - 3)\Sigma$ for some integer $d \geq 3$.

Proof. The ‘if’ part is easily verified. As for the ‘only if’ part, let $g$ be the geometric genus of $U(f)$, which is necessarily of the form $(d - 1)(d - 2)/2$ for some $d \geq 2$. If $d \geq 5$ then $cd(U(f)) = 2$ and the corollary follows from Theorem 9. If $d = 2$ or $d = 3$ then the statement is trivial. If $d = 4$ then the claim follows because $U(f)$ is non-hyperelliptic, and because $\Sigma$ is the only two-dimensional lattice polygon containing $g = 3$ lattice points (up to unimodular equivalence).

3.4 Scrollar invariants

We begin by recalling some facts on rational normal scrolls and on scrollar invariants, for use in Sections 3.4-3.6. Our main references are [14], [21, §8.26-29] and [41, §1-4].

Let $n \in \mathbb{Z}_{\geq 2}$ and let $E = \mathcal{O}(e_1) \oplus \cdots \oplus \mathcal{O}(e_n)$ be a locally free sheaf of rank $n$ on $\mathbb{P}^1$. Denote by $\pi : \mathbb{P}(E) \to \mathbb{P}^1$ the corresponding $\mathbb{P}^{n-1}$-bundle. We assume that
\[ 0 \leq e_1 \leq e_2 \leq \ldots \leq e_n \text{ and that } e_1 + e_2 + \ldots + e_n \geq 2. \]

Set \( N = e_1 + e_2 + \ldots + e_n + n - 1. \) Then the image of the induced morphism
\[ \mu : \mathbb{P}(\mathcal{E}) \to \mathbb{P}H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)), \]
when composed with an isomorphism \( \mathbb{P}H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \to \mathbb{P}^N, \) is called a rational normal scroll of type \((e_1, \ldots, e_n). \) Up to automorphisms of \( \mathbb{P}^N, \) rational normal scrolls are uniquely determined by their type.

The dimension of a rational normal scroll of type \((e_1, \ldots, e_n)\) equals \(n, \) while its degree equals \( e_1 + \ldots + e_n = N - n + 1. \) This means that the classical lower bound \( \deg(X) \geq \text{codim}_{\mathbb{P}^N}(X) + 1 \) for projective varieties \( X \subset \mathbb{P}^N \) that are not contained in any hyperplane is attained. Varieties for which this holds are said to have minimal degree. They have been classified by Del Pezzo (the surface case, 1886) and Bertini (1907): any projective variety of minimal degree is a cone over a smooth such variety, and the smooth such varieties are exactly the rational normal scrolls with \( e_1 > 0, \) the quadratic hypersurfaces, and the Veronese surface in \( \mathbb{P}^5. \) See [14] for a modern proof.

There is an easy geometric way of describing rational normal scrolls. Consider linear subspaces \( \mathbb{P}^{e_1}, \ldots, \mathbb{P}^{e_n} \subset \mathbb{P}^N \) that span \( \mathbb{P}^N. \) In each \( \mathbb{P}^{e_\ell}, \) take a rational normal curve of degree \( e_\ell, \) e.g. parameterized by
\[ \nu_\ell : \mathbb{P}^1 \to \mathbb{P}^{e_\ell} : (X : Z) \mapsto (Z^{e_\ell}, XZ^{e_\ell-1}, \ldots, X^{e_\ell}). \quad (4) \]

Then
\[ S = \bigcup_{P \in \mathbb{P}^1} \langle \nu_1(P), \ldots, \nu_n(P) \rangle \subset \mathbb{P}^N \]
is a rational normal scroll of type \((e_1, \ldots, e_n), \) and conversely every rational normal scroll arises in this way (from this description, it easily follows that rational normal scrolls are toric varieties). The scroll is smooth if and only if \( e_1 > 0. \) In this case \( \mu : \mathbb{P}(\mathcal{E}) \to S \) is an isomorphism. If \( 0 = e_1 = \ldots = e_\ell < e_{\ell+1} \) with \( 1 \leq \ell < n, \) then the scroll is a cone with an \((\ell - 1)\)-dimensional vertex. In this case \( \mu : \mathbb{P}(\mathcal{E}) \to S \) is a rational resolution of singularities. Outside the exceptional locus, our \( \mathbb{P}^{n-1}\)-bundle \( \pi : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^1 \) corresponds to
\[ S \setminus S^{\text{sing}} \to \mathbb{P}^1 : Q \in \langle \nu_1(P), \ldots, \nu_n(P) \rangle \mapsto P. \]
Abusing notation, we denote this map also by \( \pi. \) Abusing terminology, when talking about the fiber of \( \pi \) above a point \( P, \) we mean the whole space \( \langle \nu_1(P), \ldots, \nu_n(P) \rangle. \)
The Picard group of $\mathbb{P}(E)$ is freely generated by the class $H$ of a hyperplane section (more precisely, the class corresponding to $\mu^*\mathcal{O}_{\mathbb{P}^N}(1)$) and the class $R$ of a fiber of $\pi$; i.e.

$$\text{Pic}(\mathbb{P}(E)) = \mathbb{Z}H \oplus \mathbb{Z}R.$$ 

We have the following intersection products:

$$H^n = e_1 + \ldots + e_n, \quad H^{n-1}R = 1 \quad \text{and} \quad R^2 = 0$$

(where $R^2 = 0$ means that any appearance of $R^2$ annihilates the product).

**Example (the class of a subscroll).** Let $\ell \in \{1, \ldots, n\}$ and consider the subsheaf $E_\ell = \mathcal{O}(e_1) \oplus \cdots \oplus \mathcal{O}(e_{\ell-1}) \oplus 0 \oplus \mathcal{O}(e_{\ell+1}) \oplus \cdots \oplus \mathcal{O}(e_n)$ of $E$. Let $S_\ell \subset S$ be the image of $\mathbb{P}(E_\ell)$ under $\mu$. Then $S_\ell$ is of codimension one in $S$. In fact, it is a rational normal scroll itself, of type $(e_1, \ldots, e_{\ell-1}, e_{\ell+1}, \ldots, e_n)$. In our explicit scrollar description, $S_\ell$ is the subvariety defined by the vanishing of the coordinates $Y_{0,\ell}, \ldots, Y_{e_{\ell-1},\ell}$ of $\mathbb{P}^e$. We claim that

$$\mathbb{P}(E_\ell) \sim H - e_\ell R.$$ 

To see this, one can explicitly verify that the divisor of $S$ cut out by $Y_{i,\ell}$ equals

$$i \cdot \pi^{-1}(0 : 1) + (e_\ell - i) \cdot \pi^{-1}(1 : 0) + S_\ell,$$

from which the claim follows. We will use expression (5) in the proof of Lemma 17.

Now let $C \subset \mathbb{P}^{g-1}$ be a canonical curve of genus $g \geq 3$. Let $K$ be a canonical divisor on $C$ and fix a gonality pencil $g_1^\gamma$. Write $g_1^\gamma = |D|$ for an effective divisor $D$. By the Riemann-Roch theorem, $h^0(C, K - D) = g - \gamma + 1$, hence the linear span $\langle D \rangle \subset \mathbb{P}^{g-1}$ of $D$ is a linear subspace of dimension $\gamma - 2$. (If $D$ is the sum of $\gamma$ distinct points, the linear span of $D$ is just the linear span of these points; in general the multiplicities have to be taken into account, see [41, §2.3].) Denote

$$S = \bigcup_{D \in g_1^\gamma} \langle D \rangle \subset \mathbb{P}^{g-1}.$$ 

It is clear that $S$ is a $(\gamma - 1)$-dimensional scroll containing the curve $C$. Now [13 Thm. 2] states that $S$ is actually a rational normal scroll. Let $(e_1, \ldots, e_{\gamma-1})$ be its type, then the numbers $e_1, \ldots, e_{\gamma-1}$ are called the scrollar invariants of $C$ with respect to $g_1^\gamma$. Note that since $g_1^\gamma$ is base-point free, $C$ does not meet the singular locus of $S$. The restriction of $\pi$ to $C$ is a dominant rational map of degree $\gamma$.

The scrollar invariants of an arbitrary non-hyperelliptic curve $C/k$ with respect to a gonality pencil $g_1^\gamma$ are then defined to be the corresponding invariants of a canonical
model. They generalize the Maroni invariants for trigonal curves. As in the case of Maroni invariants, the scrollar invariants determine the dimensions of the multiples of the gonality pencil:

\[ h^0(C, mD) = \begin{cases} 
  h^0(C, (m-1)D) + 1 = m + 1 & \text{if } 0 \leq m \leq e_1 + 1, \\
  h^0(C, (m-1)D) + 2 & \text{if } e_1 + 1 < m \leq e_2 + 1, \\
  \vdots & \\
  h^0(C, (m-1)D) + \gamma - 1 & \text{if } e_{\gamma - 2} + 1 < m \leq e_{\gamma - 1} + 1, \\
  h^0(C, (m-1)D) + \gamma = m\gamma - g + 1 & \text{if } m > e_{\gamma - 1} + 1.
\]

In fact, this could serve as an alternative definition. If \( C \) is a hyperelliptic curve of genus \( g \geq 2 \), it is natural to define the (single) scrollar invariant of \( C \) as \( g - 1 \).

**Remark.** If \( m > \frac{2g-2}{7} \) then \( h^0(C, mD) = m\gamma - g + 1 \) (since \( \deg(mD) = \deg K \)). By the above characterization, the smallest \( m \) for which \( h^0(C, mD) = m\gamma - g + 1 \) is \( m = e_{\gamma - 1} + 1 \). Therefore \( e_{\gamma - 1} \leq \frac{2g-2}{7} \).

We are now ready to state the main result of this section.

**Theorem 11.** Let \( f \in k[x^{\pm 1}, y^{\pm 1}] \) be non-degenerate with respect to its Newton polygon \( \Delta = \Delta(f) \), suppose that \( \sharp(\Delta(1) \cap \mathbb{Z}^2) \geq 2 \), and assume that \( \Delta \not\supseteq 2\gamma, d\Sigma \) (for some \( d \in \mathbb{Z}_{\geq 4} \)). Let \( g_{\gamma} \) be a combinatorial gonality pencil on \( U(f) \). Without loss of generality, we may assume that \( \Delta \) lies in the horizontal strip \( \{(i, j) \in \mathbb{R}^2 \mid 0 \leq j \leq \gamma \} \) and that the \( g_{\gamma} \) corresponds to the projection \( p : U(f) \to k^\ast : (x, y) \mapsto x \).

For \( \ell = 1, \ldots, \gamma - 1 \), let \( E_{\ell} = -1 + \sharp\{(i, j) \in \Delta(1) \cap \mathbb{Z}^2 \mid j = \ell \} \). Then the multiset of scrollar invariants \( \{e_1, \ldots, e_{\gamma - 1}\} \) of \( U(f) \) with respect to \( g_{\gamma} \) equals the multiset \( \{E_1, \ldots, E_{\gamma - 1}\} \).

**Proof.** If \( U(f) \) is hyperelliptic then this immediately follows from Theorem 2. If not, let \( C \) be the canonical model of \( U(f) \) obtained by taking the Zariski closure of the image of \( U \). For \( \ell \in \{1, \ldots, \gamma - 1\} \), let \( \mathbb{P}^{E_{\ell}} \subset \mathbb{P}^{g-1} \) be the linear subspace defined by \( X_{\ell,j} = 0 \) for all \( (i, j) \in \Delta(1) \cap \mathbb{Z}^2 \) for which \( j \neq \ell \). That is, \( \mathbb{P}^{E_{\ell}} \) is the subspace corresponding to the projective coordinates \( (X_{i,j})_{(i,j)\in\Delta(1)\cap\mathbb{Z}^2} \). Also consider the rational normal curves parameterized by \( \nu_{\ell} : \mathbb{P}^1 \to \mathbb{P}^{E_{\ell}} \) as in (4), i.e.

\[ \forall x \in k^\ast : \nu_{\ell}(x) = (1 : x : \ldots : x^{E_{\ell}}). \]

Note that \( \varphi(x, y) \in \langle \nu_1(x), \ldots, \nu_{\gamma - 1}(x) \rangle \) for all \( (x, y) \in \mathbb{T}^2 \). Now for all but finitely many \( x \in k^\ast \), the inverse image divisor \( p^{-1}(x) \) consists of \( \gamma \) distinct points of \( U(f) \). So for these \( x \), the linear span of \( D = \varphi_{can}(p^{-1}(x)) \) equals \( \langle \nu_1(x), \ldots, \nu_{\gamma - 1}(x) \rangle \). We conclude that the scroll \( S \subset \mathbb{P}^{g-1} \) swept out by our \( g_{\gamma} \) is exactly the rational normal scroll parameterized by the \( \nu_{\ell} \)’s. Hence the multiset of scrollar invariants with
respect to $g_\gamma^1$ equals $\{E_1, \ldots, E_{\gamma-1}\}$.

**Example.** Assume that $\Delta = \Delta(f)$ is as below:

Then $U(f)$ is a genus 14 curve carrying two $g_3^1$'s, one with scrollar invariants $\{1,3,3,3\}$ (corresponding to vertical projection) and one with scrollar invariants $\{2,2,3,3\}$ (corresponding to horizontal projection).

**Remarks.**

- Inheriting the notation of the above proof, we have $C \subset \text{Tor}(\Delta^{(1)}) \subset S \subset \mathbb{P}^{g-1}$. One can verify that $\text{Tor}(\Delta^{(1)})$ intersects the fiber of $\pi$ above a point $x \in k^*$ in a rational normal curve of degree $\gamma - 2$. Above $(1 : 0), (0 : 1) \in \mathbb{P}^1$ this fiber may degenerate.

- Through Theorem 11, the upper bound $\frac{2g-2}{\gamma}$ for the scrollar invariants implies the purely combinatorial inequality

$$\text{lw}(\Delta) \cdot E_\ell \leq 2\#(\Delta^{(1)} \cap \mathbb{Z}^2) - 2,$$

where $\Delta$ is understood to be contained in

$$\{(i, j) \in \mathbb{R}^2 \mid 0 \leq j \leq \text{lw}(\Delta)\}$$

and the $E_\ell$'s are defined as in the statement of the theorem. This inequality holds as soon as $\#(\Delta^{(1)} \cap \mathbb{Z}^2) \geq 1$ (including the cases $\Delta = 2\Upsilon$ and $\Delta = d\Sigma$, which can be verified separately). The bound can be attained. For example, consider the lattice polygon $\Delta_{a,b} = \text{conv}\{(b,0), (0,a), (0,0)\}$, where $a \geq 2$ and $b$ is of the form $ak - 1$ with $k \in \mathbb{Z}_{\geq 2}$. In this case, $\gamma = \text{lw}(\Delta_{a,b}) = a$ and $E_1 = ak - k - 2 = \frac{2g-2}{\gamma}$.

### 3.5 Schreyer’s tetragonal invariants

Let $C/k$ be a tetragonal curve of genus $g$ (necessarily $g \geq 5$) and fix a $g_4^1$ on $C$. As explained in the previous section, the $g_4^1$ gives rise to a rational normal scroll threefold $S \subset \mathbb{P}^{g-1}$ in which $C$ canonically embeds. Let $\mu : \mathbb{P}(\mathcal{E}) \to S$ be the corresponding rational resolution and let $C'$ be the strict transform under $\mu$ of our
canonical model. In [41], Schreyer shows that $C'$ is the complete intersection of surfaces $Y$ and $Z$ in $\mathbb{P}(\mathcal{E})$, with $Y \sim 2H - b_1 R$, $Z \sim 2H - b_2 R$, $b_1 + b_2 = g - 5$ and $-1 \leq b_2 \leq b_1 \leq g - 4$. He moreover shows that $b_1, b_2$ are invariants of the curve: they depend neither on the choice of the $g_1^4$, nor on the choice of $Y$ and $Z$. If $b_1 > b_2$ (e.g. if $g$ is even) then the same conclusion holds for the surface $\mu(Y) \subset \mathbb{P}^{g-1}$ (for a fixed canonical embedding).

**Theorem 12.** Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be non-degenerate with respect to its Newton polygon $\Delta = \Delta(f)$. Suppose that $\ellw(\Delta(1)) = 2$ and that $\Delta \not\sim 2Y$. Then $U(f)$ is tetragonal and Schreyer’s invariants $b_1, b_2$ equal

$$\sharp(\partial \Delta(1) \cap \mathbb{Z}^2) - 4 \quad \text{resp.} \quad \sharp(\Delta(2) \cap \mathbb{Z}^2) - 1.$$  

If moreover $b_1 > b_2$ then Schreyer’s surface $\mu(Y)$ equals $\text{Tor}(\Delta(1))$ (assuming that $U(f)$ is canonically embedded using [3]).

**Proof.** By Corollary [1], we know that $U(f)$ is tetragonal. We can assume that there exists a combinatorial $g_1^4$, because the case $\Delta \cong 5\Sigma$ (the single case where a combinatorial $g_1^4$ does not exist) can be reduced to $\Delta \cong \text{conv}\{ (1, 0), (5, 0), (0, 5), (0, 1) \}$ by using a coordinate transformation of the type discussed in the introduction. Let $g = \sharp(\Delta(1) \cap \mathbb{Z}^2)$ be the genus of $U(f)$ and write $g^{(1)} = \sharp(\Delta(2) \cap \mathbb{Z}^2)$. Define $B = g - g^{(1)} - 4$ and $B' = g^{(1)} - 1$, so that $B + B' = g - 5$. We have to show that $b_1 = B$ and $b_2 = B'$.

We first prove the purely combinatorial inequality $B \geq B'$, which is equivalent to $2g^{(1)} \leq g - 3$. Let $e_1 \leq e_2 \leq e_3$ be the scrollar invariants of $U(f)$ with respect to our $g_1^4$. Then using Theorem [11] one can see that $g^{(1)} \leq e_3 - 1$ (indeed, if $E_2 = e_3$ then $g^{(1)} \leq E_2 - 1$, and if $E_2 < e_3$ then $g^{(1)} \leq E_2$). The claim now follows from $e_3 \leq \frac{2g-3}{4}$.

If $U(f)$ is canonically embedded using [11], then we are in the situation from the end of the previous section:

$$\mu(C') \subset \text{Tor}(\Delta(1)) \subset S \subset \mathbb{P}^{g-1}.$$  

Let $T' \subset \mathbb{P}(\mathcal{E})$ be the strict transform of $\text{Tor}(\Delta(1))$ under $\mu$. Write the divisor class of $T'$ as $aH + bR$ with $a, b \in \mathbb{Z}$. Recall that $\text{Tor}(\Delta(1))$ intersects a typical fiber of $\pi$ in a rational normal curve of degree 2, so we have that

$$a = (aH + bR) \cdot H \cdot R = T' \cdot H \cdot R = 2.$$  

If we compute the intersection product $T' \cdot H^2$, we get the degree of $\text{Tor}(\Delta(1))$, which equals $2\text{Vol}(\Delta(1))$ (e.g. because the Hilbert polynomial of $\text{Tor}(\Delta(1))$ equals the Ehrhart polynomial of $\Delta(1)$). On the other hand,

$$T' \cdot H^2 = (2H + bR) \cdot H^2 = 2(g - 3) + b.$$  

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We obtain that \( b = 2\text{Vol}(\Delta^{(1)}) - 2(g - 3) = -B \), where the latter equality follows from Pick’s theorem. In conclusion, \( T' \sim 2H - BR \). Now if

- \( Y = T'' \) then it is immediate that \( b_1 = B \) and, consequently, \( b_2 = B' \).

- \( Y \neq T' \) then if we intersect \( Y \sim 2H - b_1R \) and \( T' \sim 2H - BR \) on \( \mathbb{P}(\mathcal{E}) \), we obtain a (possibly reducible) curve whose image under \( \mu \) has degree

\[
H \cdot (2H - BR) \cdot (2H - b_1R) = 4(g - 3) - 2b_1 - 2B \leq 4(g - 3) - 2(g - 5) = 2g - 2.
\]

Here we used that \( 2b_1 \geq b_1 + b_2 = g - 5 \) and \( 2B \geq B + B' = g - 5 \). If either of the latter inequalities would be strict, then we would run into a contradiction because \( C' \) is contained in this intersection (and \( \mu(C') \), being a canonical curve, has degree \( 2g - 2 \)). We conclude that \( b_1 = b_2 = B = B' = \frac{g - 5}{2} \).

All conclusions follow.

**Remark.** The invariants \( b_1 \) and \( b_2 \) completely determine the graded Betti numbers of \( C \) (and conversely) \cite[§6.2]{41}. See Section \[4.1\] for some related comments.

### 3.6 Secondary scrollar invariants

To some extent, the material of the previous section generalizes to curves of higher gonality. Let \( C/k \) be a curve of genus \( g \) and gonality \( \gamma \geq 4 \), and fix a \( g^1_\gamma \) on \( C \). The latter gives rise to a \( (\gamma - 1) \)-dimensional rational normal scroll \( S \subset \mathbb{P}^{g-1} \) in which \( C \) canonically embeds. Let \( \mu : \mathbb{P}(\mathcal{E}) \to S \) be the corresponding rational resolution and let \( C' \) be the strict transform under \( \mu \) of our canonical model.

**Lemma 13.** There exist effective divisors \( D_1, \ldots, D_{(\gamma^2 - 3\gamma)/2} \) on \( \mathbb{P}(\mathcal{E}) \) along with integers \( b_1, \ldots, b_{(\gamma^2 - 3\gamma)/2} \), such that \( C' \) is the (scheme theoretic) intersection of the \( D_\ell \)'s, and such that for all \( \ell \) we have \( D_\ell \sim 2H - b_\ell R \). Moreover, the multiset \( \{b_1, \ldots, b_{(\gamma^2 - 3\gamma)/2}\} \) does not depend on the choice of the \( D_\ell \)'s, and

\[
\sum_{\ell=1}^{(\gamma^2 - 3\gamma)/2} b_\ell = (\gamma - 3)g - (\gamma^2 - 2\gamma - 3).
\]

**Proof.** Define \( \beta_\ell = \frac{\ell(\gamma^2 - 2\gamma)}{\gamma - 1} \cdot \binom{\gamma}{\ell} \) and note that \( \beta_1 = (\gamma^2 - 3\gamma)/2 \). The existence follows from \cite[Cor. 4.4]{41} and its proof, where the \( D_\ell \)'s give rise to an exact sequence
of \( \mathcal{O}_\mathbb{P}(E) \)-modules

\[
0 \to \mathcal{O}_\mathbb{P}(E)(-\gamma H + (g - \gamma + 1)R) \to \sum_{\ell=1}^{\beta_\gamma} \mathcal{O}_\mathbb{P}(E)(-(\gamma - 2)H + b_\ell^{(\gamma-3)}R) \to \cdots
\]

\[
\to \sum_{\ell=1}^{\beta_2} \mathcal{O}_\mathbb{P}(E)(-3H + b_\ell^{(2)}R) \to \sum_{\ell=1}^{\beta_1} \mathcal{O}_\mathbb{P}(E)(-2H + b_\ell R) \to \mathcal{O}_\mathbb{P}(E) \to \mathcal{O}_{C'} \to 0. \quad (6)
\]

We first show that

\[
\sum_{\ell} b_\ell = (\gamma - 3)g - (\gamma^2 - 2\gamma - 3). \quad (7)
\]

Note that for all integers \( a \geq -\gamma \) and \( b \gg 0 \) we have

\[
h^0(\mathbb{P}(E), aH + bR) = (g - \gamma + 1) \left( \frac{a + \gamma - 2}{\gamma - 1} \right) + (b + 1) \left( \frac{a + \gamma - 2}{\gamma - 2} \right),
\]

\[
h^i(\mathbb{P}(E), aH + bR) = 0 \quad \text{as soon as } i \geq 1.
\]

Here,

- for \( a < 0 \) we used [34, Lem. 3.1(ii)],
- for \( a \geq 0 \) and \( i = 0 \) this follows from [34, Cor. 3.2(ii)] or [41, §1.3],
- for \( i = 1 \) we used [34, Cor. 3.2(iii)] along with the fact that \( b \) is sufficiently large,
- for \( i \geq 2 \) we used [34, Lem. 3.1(i)] and Grothendieck vanishing.

Now tensor (6) with \( \mathcal{O}_\mathbb{P}(E)(2H + bR) \) for a sufficiently large integer \( b \) and compute the Euler characteristics of the terms in the resulting exact sequence. We get

\[
\chi(\mathcal{O}_{C'}(2H + bR)) = h^0(C', 2K + bD) = 3g - 3 + b\gamma, \quad \text{(where } |D| = g_1^1)\]

\[
\chi(\mathcal{O}_\mathbb{P}(E)(2H + bR)) = h^0(\mathbb{P}(E), 2H + bR) = \gamma g + (b - 1) \frac{\gamma^2 - \gamma}{2},
\]

\[
\chi \left( \sum_{\ell=1}^{\beta_1} \mathcal{O}_\mathbb{P}(E)((b_\ell + b)R) \right) = \sum_{\ell=1}^{\beta_1} h^0(\mathbb{P}(E), (b_\ell + b)R) = (b + 1)\beta_1 + \sum_{\ell=1}^{\beta_1} b_\ell,
\]

and 0 for all other terms. So (7) follows since the alternating sum of the Euler characteristics is zero. To conclude the proof, note that the exact sequence

\[
\sum_{\ell=1}^{\beta_1} \mathcal{O}_\mathbb{P}(E)(-D_\ell) \to \mathcal{O}_\mathbb{P}(E) \to \mathcal{O}_{C'} \to 0
\]
can be extended to a sequence of the form (6) by (the proof of) [11, Thm. 3.2], and that such an exact sequence is unique up to isomorphism, again by [11, Thm. 3.2].

We call the invariants $b_1, \ldots, b_{(\gamma^2 - 3\gamma)/2}$ the secondary scrollar invariants of $C$ with respect to $g^1_\gamma$. The main goal of this section is to give a combinatorial interpretation for these secondary scrollar invariants. We will manage to do so under a seemingly mild assumption on the Newton polygon (that will be introduced after Lemma 14 below).

For now, let $\Delta$ be any lattice polygon for which $\text{lw}(\Delta) \geq 4$ and such that $\Delta \not\cong 2\Upsilon$ and $\Delta \not\cong d\Sigma$ for any integer $d$. Let $C$ be a $\Delta$-non-degenerate curve and fix a combinatorial gonality pencil $g^1_\gamma$ on $C$. Without loss of generality we may assume that $\Delta \subset \{(i,j) \in \mathbb{Z}^2 | 0 \leq j \leq \gamma\}$ and that our $g^1_\gamma$ corresponds to vertical projection (to $(x,y) \mapsto x$, that is). We will also suppose that $C$ is canonically embedded using (1), so that we are in the situation from the end of Section 3.4:

$$C \subset \text{Tor}(\Delta^{(1)}) \subset S \subset \mathbb{P}^{g-1}.$$ 

Let $e_1, \ldots, e_{\gamma-1}$ be the scrollar invariants of $C$ with respect to $g^1_\gamma$. Recall that by Theorem 11 these match with $E_\ell = -1 + \sharp\{(i,j) \in \Delta^{(1)} \cap \mathbb{Z}^2 | j = \ell\}$ (up to order). In what follows, we will assume that $e_1 > 0$, so that $S$ is non-singular. This condition is not essential; it just allows us to work directly on $S$ rather than $\mathbb{P}(E)$. If $e_1 = 0$ one can work with strict transforms under the corresponding rational resolution $\mu : \mathbb{P}(E) \to S$, as in the previous section.

Our first aim is to write $\text{Tor}(\Delta^{(1)})$ as an intersection of divisors on $S$. Pick two integers $2 \leq j_1 \leq j_2 \leq \gamma - 2$ and let $Y_{j_1,j_2} \subset S$ be the subvariety defined by the elements of $I_2(\text{Tor}(\Delta^{(1)}))$ having the form

$$X_{i_1,j_1}X_{i_2,j_2} = X_{i_1,j_1-1}X_{i_2,j_2+1}.$$ 

One can see that $Y_{j_1,j_2}$ is a $(\gamma - 2)$-dimensional toric variety $\text{Tor}(\Omega_{j_1,j_2})$, where $\Omega_{j_1,j_2} \subset \mathbb{R}^{\gamma-2}$ is a full-dimensional lattice polytope. The (Euclidean) volume of this polytope equals

$$\frac{1}{(\gamma - 2)!}(2(E_1 + \ldots + E_{\gamma-1}) - (E_{j_1-1} + E_{j_2+1} - \epsilon_{j_1,j_2})), $$

where $\epsilon_{j_1,j_2}$ is defined as follows. For all $j \in \{1, \ldots, \gamma - 1\}$ let

$$i^-(j) = \min\{i \in \mathbb{Z} | (i,j) \in \Delta^{(1)}\} \text{ and } i^+(j) = \max\{i \in \mathbb{Z} | (i,j) \in \Delta^{(1)}\},$$

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then

\[
\epsilon_{j_1,j_2} = \begin{cases} 
0 & \text{if } i^-(j_1) + i^-(j_2) \leq i^-(j_1 - 1) + i^-(j_2 + 1) \\
1 & \text{if } i^-(j_1) + i^-(j_2) > i^-(j_1 - 1) + i^-(j_2 + 1) \\
2 & \text{if } i^-(j_1) + i^-(j_2) > i^-(j_1 - 1) + i^-(j_2 + 1)
\end{cases}
\]

for some \( \gamma \). This results in \( \Omega_{j_1,j_2} \)’s being 1, 0, and 1, respectively.

Example. Assume that \( \Delta = \Delta(f) \) is as below (here \( \gamma = 5 \)):

\[
\begin{align*}
\Delta, & \quad \Delta^{(1)}, \\
\Omega_{2,2}, & \quad \Omega_{2,3}, \\
& \quad \Omega_{3,3}.
\end{align*}
\]

Here the arrow \( \rightsquigarrow \) means that we levitate the bottom component so that it floats at height 1 above this sheet of paper, and then take the convex hull. The corresponding \( \epsilon_{j_1,j_2} \)'s are 1, 0 and 1, respectively.

Since the restriction of \( Y_{j_1,j_2} \) to a typical fiber \( \pi : S \to \mathbb{P}^1 \) is a quadratic hypersurface, we have \( Y_{j_1,j_2} \sim 2H - B_{j_1,j_2} R \) for some \( B_{j_1,j_2} \in \mathbb{Z} \). Taking the intersection product with \( H^{\gamma-2} \), we get

\[
\deg Y_{j_1,j_2} = Y_{j_1,j_2} \cdot H^{\gamma-2} = 2H^{\gamma-1} - B_{j_1,j_2} H^{\gamma-2} R
= 2(e_1 + \ldots + e_{\gamma-1}) - B_{j_1,j_2}
= 2(E_1 + \ldots + E_{\gamma-1}) - B_{j_1,j_2},
\]

but \( \deg Y_{j_1,j_2} = (\gamma - 2)! \cdot \text{Vol}(\Omega_{j_1,j_2}) \), so \( B_{j_1,j_2} \) equals \( E_{j_1-1} + E_{j_2+1} - \epsilon_{j_1,j_2} \). This implies

\[
\sum_{j_1 \leq j_2} B_{j_1,j_2} = (\gamma - 4)(E_1 + \ldots + E_{\gamma-1}) + E_1 + E_{\gamma-1} - \sum_{j_1 \leq j_2} \epsilon_{j_1,j_2}
= (\gamma - 4)(g - \gamma + 1) + E_1 + E_{\gamma-1} - \sum_{j_1 \leq j_2} \epsilon_{j_1,j_2}.
\]
The term \( \sum_{j_1 \leq j_2} \epsilon_{j_1,j_2} \) counts the number of times that \( \partial \Delta^{(1)} \) intersects the horizontal lines on height 2, \ldots, \( \gamma - 2 \) in a non-lattice point, so
\[
\sharp(\partial \Delta^{(1)} \cap \mathbb{Z}^2) = (E_1 + 1) + (E_{\gamma - 1} + 1) + 2(\gamma - 3) - \sum_{j_1 \leq j_2} \epsilon_{j_1,j_2}
\]
and we can rewrite \( \sum_{j_1 \leq j_2} B_{j_1,j_2} \) as
\[
(\gamma - 4)g - (\gamma^2 - 3\gamma) + \sharp(\partial \Delta^{(1)} \cap \mathbb{Z}^2).
\]
The intersection of all the subvarieties \( Y_{j_1,j_2} \subset S \) equals Tor(\( \Delta^{(1)} \)). By summarizing the above, we obtain the following result.

**Lemma 14.** There exist \( (\gamma - 2) \) effective divisors \( D_{j_1,j_2} \) on \( S \) (with \( 2 \leq j_1 \leq j_2 \leq \gamma - 2 \)) such that

- Tor(\( \Delta^{(1)} \)) is the (scheme theoretic) intersection of the divisors \( D_{j_1,j_2} \),
- \( D_{j_1,j_2} \sim 2H - B_{j_1,j_2}R \) for all \( j_1, j_2 \), where \( B_{j_1,j_2} = E_{j_1-1} + E_{j_2+1} - \epsilon_{j_1,j_2} \).

Here
\[
\sum_{2 \leq j_1 \leq j_2 \leq \gamma - 2} B_{j_1,j_2} = (\gamma - 4)g - (\gamma^2 - 3\gamma) + \sharp(\partial \Delta^{(1)} \cap \mathbb{Z}^2).
\]

From now on, we will add the extra assumptions that \( \Delta^{(2)} \neq \emptyset \) (which given our other assumptions is equivalent to \( \Delta^{(1)} \neq 2\Sigma \)), and that \( \Delta \) is well-aligned with respect to \((0,1)\). The latter condition states that we have sufficient freedom in constructing the quadrics \( Q_w \) from Section 3.1 in the following sense.

**Definition 15.** Let \( \Delta \) be a two-dimensional lattice polygon such that \( \Delta^{(2)} \neq \emptyset \), and let \( v \in \mathbb{Z}^2 \) be a lattice direction. Then we say that \( \Delta \) is well-aligned with respect to \( v \) if for each line \( L \) that is perpendicular to \( v \) and for each \( (i,j) \in \Delta \cap \mathbb{Z}^2 \) there exist two (not necessarily distinct) lines \( M_1, M_2 \) that are parallel to \( v \), such that for all \( w \in L \cap \Delta^{(2)} \cap \mathbb{Z}^2 \)
\[
\exists u_{i,j} \in M_1 \cap \Delta^{(1)} \cap \mathbb{Z}^2, \ v_{i,j} \in M_2 \cap \Delta^{(1)} \cap \mathbb{Z}^2 : (i,j) - w = (u_{i,j} - w) + (v_{i,j} - w).
\]

**Remark.** The existence of \( M_1 \) and \( M_2 \) is trivial if \( (i,j) \in \Delta^{(1)} \). So it suffices to verify this condition for \( (i,j) \in \partial \Delta \).

Although at first sight being well-aligned with respect to a lattice width direction might seem a strong condition, it is not so easy to cook up counterexamples. One reason is that, apparently, the criterion for interior polygons stated at the end of Section 2 works in favor of well-alignedness. To see this in action, we included the following lemma (at no point in this article we will rely on it, so the reader willing to believe us here can skip its technical proof).
Lemma 16. If \((i, j) \in L\), then such lines \(M_1\) and \(M_2\) always exist.

Remark. Since this purely combinatorial statement does not involve Laurent polynomials \(f \in k[x^{\pm 1}, y^{\pm 1}]\), we will use \(x\) and \(y\) as coordinates on \(\mathbb{R}^2\) in the proof below (instead of our usual \(i\) and \(j\), which are clearly occupied).

Proof. Using a unimodular transformation, we may assume that \(j = 0\), that \(L\) is the horizontal line at height 0, and that

\[
L \cap \Delta^{(2)} \cap \mathbb{Z}^2 = \{(0, 0), \ldots, (B, 0)\}.
\]

By the remark, we may also assume that \((i, j) \in \partial \Delta\). By flipping horizontally, we can moreover arrange that \(i < 0\).

- If \(\partial \Delta^{(1)}\) contains a lattice point between \((i, j)\) and \((0, 0)\), then this necessarily concerns \((-1, 0)\); then for \(w = (a, 0)\) we take \(u_{i,j} = (-1, 0)\) and \(v_{i,j} = (a - 1, 0)\). In particular, we can take \(M_1 = M_2 = L\).

- If there is no such lattice point then \((i, j) = (-1, 0)\) and there is an edge \(\tau\) of \(\Delta^{(1)}\) passing between \((-1, 0)\) and \((0, 0)\). Let \((c, d)\) be the endpoint of \(\tau\) for which \(d > 0\). By horizontally skewing if needed, we may assume that \(-d \leq c \leq -1\). Then \((-1, 1) \in \Delta^{(1)}\), being a lattice point of the triangle \(\text{conv}\{(-1, 0), (0, 0), (c, d)\}\) that differs from \((-1, 0)\). We claim that also \((0, -1) \in \Delta^{(1)}\). If not, then \(\tau\) passes through a point \((\varepsilon, -1)\) with \(\varepsilon \in ]0, 1[\) (indeed, \(\varepsilon < 1\) because \((0, 0)\) lies to the right of \(\tau\)). But then the horizontal distance \(\varepsilon\) between \(\tau\) and \((0, -1)\) is strictly smaller than the horizontal distance

\[
\varepsilon + \left(1 - \frac{\varepsilon - c}{d + 1}\right)
\]

between \(\tau\) and \((-1, 0)\). This means that \((-1, 0)\) cannot belong to \(\Delta^{(1)(-1)} = \Delta^{\max}\): a contradiction.

This implies that for \(w = (0, 0)\) we can take \(u_{i,j} = (0, -1)\) and \(v_{i,j} = (-1, 1)\). We claim that for the other \(w\)'s we can keep taking \(u_{i,j}\) and \(v_{i,j}\) at respective heights \(-1\) and 1, yielding the requested \(M_1\) and \(M_2\). To see this, it is sufficient to show that the right-most lattice points of \(\Delta^{(1)}\) at these heights, say \((m, -1)\) and \((n, 1)\), satisfy \(m + n \geq B - 1\). We distinguish between two cases:

- \((B + 1, 0)\) is not a vertex of \(\Delta^{(1)}\). This case is easy: depending on the position of the edge of \(\Delta^{(1)}\) passing to the right of \((B, 0)\), we will already have that \(m \geq B\) or \(n \geq B\).
\((B + 1, 0)\) is a vertex of \(\Delta^{(1)}\). We can assume that \(m, n \leq B\), so that the edges \(\tau_m, \tau_n\) of \(\Delta^{(1)}\) that are adjacent to \((B + 1, 0)\) are leaning to the left. By the criterion for interior polygons stated at the end of Section \(\text{2}\) the outward shifts of \(\tau_m, \tau_n\) must intersect in a lattice point. One easily checks that this holds if and only if they are both parallel to a line of the form \(x = ay\) for some integer \(a\). It follows that \((m, -1)\) and \((n, 1)\) are points at the boundary of \(\Delta^{(1)}\). We claim that

\[
m \geq \frac{B}{2} \quad \text{and} \quad n \geq \frac{B}{2} - 1.
\]

We will only prove the second inequality; the first one follows similarly (or by symmetry: flip vertically and skew horizontally). Suppose that \(n < \frac{B}{2} - 1\). The supporting line of \(\tau_n\),

\[
x + (B + 1 - n)y = B + 1,
\]

intersects \(\tau\) below (and to the right of) where it intersects \(x + y = -1\), which is at height

\[
\frac{B + 2}{B - n} < 2.
\]

So \(\Delta^{(1)} \subset \{(x, y) \in \mathbb{R}^2 \mid y \leq 1\}\). It follows that \(c = -d = -1\) and that the outward shift of \(\tau\) must contain a point at height 2, which necessarily equals \((-2, 2)\). But the outward shift of \(\tau_n\) intersects \(y = 2\) at \(x = 2n - B\), which lies strictly to the left of \((-2, 2)\): contradiction. 

However, there do exist instances of lattice polygons and lattice width directions for which the condition of being well-aligned is not satisfied. The smallest example we have found corresponds to curves of genus 46 and gonality 10:

**Example.** Let \(\Delta\) be as follows (the dashed line indicates \(\Delta^{(1)}\)).
We claim that it is not well-aligned with respect to \((0, \pm1)\). Indeed, take for \(L\) the horizontal line at height 6 and let \((i, j)\) be the top vertex of \(\Delta\). Consider the bold-marked lattice points on \(L\). In both cases, there is a unique decomposition of \((i, j) - w\):

\[(1, 4) = (0, 1) + (1, 3) \quad \text{resp.} \quad (-2, 4) = (-1, 2) + (-1, 2)\]

So one sees that it is impossible to take the \(u_{i,j}\)'s and/or the \(v_{i,j}\)'s on the same line, which proves the claim. Note however that \(\Delta\) is well-aligned with respect to its other pair of lattice width directions \((\pm 1, 0)\).

We now prove:

**Lemma 17.** If \(\Delta\) is well-aligned with respect to \((0, 1)\) then there exist \(\gamma - 3\) effective divisors \(D_\ell\) on \(S\) (with \(2 \leq \ell \leq \gamma - 2\)) such that

- \(C\) is the (scheme theoretic) intersection of \(\operatorname{Tor}(\Delta^{(1)})\) and the divisors \(D_\ell\),
- \(D_\ell \sim 2H - B_\ell R\) for all \(\ell\), where \(B_\ell = -1 + \sharp\{(i, j) \in \Delta^{(2)} \cap \mathbb{Z}^2 | j = \ell\}\).

Here

\[\sum_{2 \leq \ell \leq \gamma - 2} B_\ell = \sharp(\Delta^{(2)} \cap \mathbb{Z}^2) - (\gamma - 3).\]

**Proof.** The latter sum is easily verified, so we focus on the other assertions. Write

\[f = \sum_{(i, j) \in \Delta \cap \mathbb{Z}^2} c_{i,j} x^i y^j \in k[x^{\pm1}, y^{\pm1}].\]

Let \(2 \leq \ell \leq \gamma - 2\) be an integer and let \(w_0, \ldots, w_{B_\ell}\) be the lattice points of \(\Delta^{(2)}\) at height \(\ell\), enumerated from left to right. For each \(w \in \{w_0, \ldots, w_{B_\ell}\}\), consider the quadratic form

\[Q_w = \sum_{(i, j) \in \Delta \cap \mathbb{Z}^2} c_{i,j} X_{u_{i,j}} X_{v_{i,j}}\]

that was introduced in Section 3.1. Since \(\Delta\) is well-aligned with respect to \((0, 1)\), we can choose the second coordinates of \(u_{i,j}, v_{i,j}\) independently of \(w\). A consequence of this choice is that for all \(w, w' \in \{w_0, \ldots, w_{B_\ell}\}\), we have

\[X_{w'} Q_w - X_w Q_{w'} \in \mathcal{I}(S)\]

rather than just \(\mathcal{I}(\operatorname{Tor}(\Delta^{(1)}))\).

Now consider the \((\gamma - 2)\)-plane \(R_{(0:1)} = \pi^{-1}(0 : 1) \subset S\). Note that

\[
\frac{X_{w_1}}{X_{w_0}} = \frac{X_{w_2}}{X_{w_1}} = \ldots = \frac{X_{w_{B_\ell}}}{X_{w_{B_\ell - 1}}}
\]

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is a local parameter for \( R_{(0:1)} \). From (9) it therefore follows that the \( R_{(0:1)} \)-orders of 

\[ Z(Q_{w_1}), Z(Q_{w_2}), \ldots, Z(Q_{w_B}) \]  

increase by 1 at each step. For a similar reason, with \( R_{(1:0)} = \pi^{-1}(1:0) \), the \( R_{(1:0)} \)-orders of (10) decrease by 1 at each step. We conclude that there exist positive integers \( m_\ell, n_\ell \) such that for all \( i = 0, \ldots, B \) we have

\[ Z(Q_w) = (m_\ell + i) \cdot R_{(0:1)} + (n_\ell + (B_\ell - i)) \cdot R_{(1:0)} + D_{\ell,i} \]  

(11)
on \( S \). Here \( D_{\ell,i} \) is an effective divisor whose support (as a set of prime divisors) is disjoint from \( \{ R_{(0:1)}, R_{(1:0)} \} \). Again because of (9), together with (5) we in fact find that \( D_{\ell,i} \) is independent of \( i \). So we might as well write \( D_{\ell} \). Note that (11) implies that \( D_{\ell} \sim 2H - (B_\ell + m_\ell + n_\ell)R \).

The scheme theoretic intersection of the \( D_{\ell} \)'s (with \( \ell \) ranging over \( 2, \ldots, \gamma - 2 \)) with Tor(\( \Delta^{(1)} \)) is equal to \( C \), because this already holds for the \( Z(Q_w) \)'s (where \( w \) ranges over \( \Delta^{(2)} \cap \mathbb{Z}^2 \)) by Theorem 3; the absence of the components \( R_{(1:0)}, R_{(0:1)} \) does not affect this. This means that, together with the \( (\gamma - 2) \) divisors \( D_{j_1,j_2} \) from Lemma 14 the \( D_{\ell} \)'s form a set of \( (\gamma^2 - 3\gamma)/2 \) divisors of type 

\[ 2H - b_\ell R \]

that cut out \( C \). By Lemma 13 we therefore conclude that the \( B_{j_1,j_2} \)'s of Lemma 14 and the integers 

\[ \{ B_\ell + m_\ell + n_\ell \}_{2 \leq \ell \leq \gamma - 2} \]

together form the multiset of secondary scrollar invariants of \( C \) with respect to our \( g_1^1 \). Again by Lemma 13 these must sum up to \( (\gamma - 3)g - (\gamma^2 - 2\gamma - 3) \). Since 

\begin{equation*}
\sum_{2 \leq j_1 \leq j_2 \leq \gamma - 2} B_{j_1,j_2} + \sum_{2 \leq \ell \leq \gamma - 2} (B_\ell + m_\ell + n_\ell)
\end{equation*}

\begin{equation*}
= (\gamma - 4)g - (\gamma^2 - 3\gamma) + \sharp(\partial \Delta^{(1)} \cap \mathbb{Z}^2) + \sharp(\Delta^{(2)} \cap \mathbb{Z}^2) - (\gamma - 3) + \sum_{2 \leq \ell \leq \gamma - 2} (m_\ell + n_\ell)
\end{equation*}

\begin{equation*}
= (\gamma - 3)g - (\gamma^2 - 2\gamma - 3) + \sum_{2 \leq \ell \leq \gamma - 2} (m_\ell + n_\ell),
\end{equation*}

this is possible only if \( m_\ell = n_\ell = 0 \) for all \( \ell \). The lemma follows. \( \blacksquare \)

Remark. The above proof also shows how to find explicit equations for the \( D_{\ell} \)'s: if one constructs the \( Q_w \)'s compatibly with (9) then \( D_{\ell} = Z(Q_{w_1}, \ldots, Q_{w_B}) \) (on \( S \)).

As a corollary to Lemmata 13-17 we finally get:
Theorem 18. Let $f, \Delta, g^1_\gamma$ and $E_1, \ldots, E_{\gamma-1}$ be as in the statement of Theorem 11. Assume moreover that $\text{lw}(\Delta) \geq 4$, that $\Delta^{(2)} \neq \emptyset$ and that $\Delta$ is well-aligned with respect to $(0,1)$. Let $B_\ell = -1 + \# \{(i,j) \in \Delta^{(2)} \cap \mathbb{Z}^2 \mid j = \ell\}$ for $\ell = 2, \ldots, \gamma - 2$. Then the multiset of secondary scrollar invariants of $U(f)$ with respect to $g^1_\gamma$ is given by

$$\{E_{j_1-1} + E_{j_2+1} - \epsilon_{j_1,j_2} \mid 2 \leq j_1 \leq j_2 \leq \gamma - 2\} \cup \{B_\ell \mid 2 \leq \ell \leq \gamma - 2\},$$

where the $\epsilon_{j_1,j_2}$'s are defined as in (8).

We end this section with a natural question.

Question 19. Can we drop the condition of being well-aligned from the statement of Theorem 18?

One essential part of the proof of Lemma 17 that fails in this more general setting is the claim that $Z(Q_{w_i})$ has $R_{(0:1)}$-order $i$. One can use the same counterexample as after the proof of Lemma 16: let $w_0, \ldots, w_3$ be the lattice points of $\Delta^{(2)}$ on the line at height 6. Then the term in $Q_{w_3}$ corresponding to the top vertex will have $R_{(0:1)}$-order 2 only.

4 Other invariants?

In this section we briefly discuss a number of other candidate-invariants that we have been considering and that are potentially useful in attacking Question II.

4.1 Graded Betti numbers

We refer to [41, §0] for background and terminology.

Question 20. Do the graded Betti numbers of a $\Delta$-non-degenerate curve $C/k$ of genus $g \geq 3$ depend on $\Delta$ (or even on $\Delta^{(1)}$) only?

The answer is yes at least for

- $\text{lw}(\Delta^{(1)}) = 0$ and $\text{lw}(\Delta^{(1)}) = 1$, because the graded Betti numbers of hyperelliptic curves [37, Ch. 0] and trigonal curves [41, §6.1] depend on the genus only,

- $\text{lw}(\Delta^{(1)}) = 2$, because the graded Betti numbers of a tetragonal curve are completely determined by the invariants $b_1$ and $b_2$ [41, §6.2], and by the material in Section 3.5 these are combinatorially determined,
• $g = \sharp(\Delta^{(1)} \cap \mathbb{Z}^2) \leq 9$, because in this range, the only $\Delta$’s for which $\text{lw}(\Delta^{(1)}) > 2$ correspond to genus 9 curves having Clifford index 3 and admitting a $g^2_7$ (see our explicit list of polygons in Section 5 below), which by a result of Sagraloff [37, Thm. 4.3.2] determines the Betti table,

• $\Delta^{(1)} \cong (d - 3)\Sigma$ for some $d \geq 4$, by previous work of Loose [31] on graded Betti numbers of smooth plane curves.

Another humble argument in favor of an affirmative answer to Question 20 is that $\text{ci}(\Delta)$ is fully determined by $\Delta^{(1)}$ (by Theorem 9). It follows that if Green’s canonical conjecture (recalled below) is true, then at least the position of the first non-zero entry of the cubic strand in the Betti table depends on $\Delta^{(1)}$ only. (Remark that some $\Delta$-non-degenerate cases of the related Green-Lazarsfeld gonality conjecture have been proven by Kawaguchi [24, 25].)

**Question 21.** If the answer to Question 20 is yes, does there exist an explicit combinatorial description of the Betti table?

**Remark.** If one wishes to study Question 21 for a lattice polygon $\Delta$ for which Question 20 is known to have a positive answer, it suffices to analyze the Betti table of a single $\Delta$-non-degenerate curve. In this, computer calculations may prove helpful.

It is known that the Betti table has the form

\[
\begin{array}{cccccccc}
1 & 0 & 0 & \ldots & 0 & 0 \\
0 & \beta_{1,2} & \beta_{2,3} & \ldots & \beta_{g-3,g-2} & \beta_{g-2,g-1} \\
\beta_{0,2} & \beta_{1,3} & \beta_{2,4} & \ldots & \beta_{g-3,g-1} & 0 \\
0 & 0 & 0 & \ldots & 0 & 1.
\end{array}
\]

Then Green’s conjecture states that the first index $i$ for which $\beta_{i,i+2} \neq 0$ equals $\text{ci}(\Delta)$. One can show that $\beta_{ij} = \beta_{g-2-i,g+1-j}$ and

\[
\beta_{i,i+1} - \beta_{i-1,i+1} = \frac{(g - 1 - i)(g - 1 - 2i)}{i + 1} \binom{g - 1}{i - 1}
\]

(see [31, 41]), so that the Betti table is completely determined by the numbers

\[
\zeta_i := \beta_{i,i+2} \quad i = 0, \ldots, \lfloor (g - 3)/2 \rfloor.
\]

Thus Question 21 boils down to: can we describe the numbers $\zeta_i$ explicitly in terms of the combinatorics of $\Delta$ (or even of $\Delta^{(1)}$)? In an attempt to observe such an explicit description, we have computed Betti tables for (more or less) randomly chosen Laurent polynomials $f \in \mathbb{F}_{10007}[x^\pm 1, y^\pm 1]$ for which $\Delta(f) = \Gamma^{(-1)}$, with $\Gamma$ running over all (two-dimensional) interior lattice polygons containing up
to 14 lattice points. The characteristic was chosen finite for computational reasons and is not expected to influence the outcome. The output can be found at http://wis.kuleuven.be/~u0040935/. See [8, §5] for some additional discussion. The case of hyperelliptic curves was omitted because here it is well-known that

$$\zeta_i = (g - 2 - i) \binom{g - 1}{i},$$

see e.g. [37, Ch. 0]. In the trigonal case we observed

$$\zeta_i = (g - 2 - i) \binom{g - 2}{i - 1},$$

while for tetragonal curves our output is consistent with

$$\zeta_i = (g - 2 - i) \binom{g - 2}{i - 1} - \min\{b_2 + 1, i\} \binom{g - 3}{i}$$

(recall that $b_2 + 1 = \#(\Delta(2) \cap \mathbb{Z}^2)$). We have neither been able to prove these formulas, nor to trace them in the existing literature.

We end with two extrapolating questions:

**Question 22.** Let $\Delta$ be a lattice polygon with $\#(\Delta(1) \cap \mathbb{Z}^2) \geq 3$. Can one read off the sequence $\#(\Delta(1) \cap \mathbb{Z}^2), \#(\Delta(2) \cap \mathbb{Z}^2), \#(\Delta(3) \cap \mathbb{Z}^2), \ldots$ from the Betti table of a $\Delta$-non-degenerate curve?

(Here $\Delta^{(\ell)}$ abbreviates $\Delta^{(1(\ell-1))}$.)

**Question 23.** Is there a connection between the process of peeling off a lattice polygon $\Delta$ (i.e. subsequently taking interiors) and the process of taking a minimal free resolution of the canonical coordinate ring of a $\Delta$-non-degenerate curve?

### 4.2 Minimal degree of a plane model

For a curve $C/k$ one denotes by $s_2(C)$ the minimal degree of a (possibly singular) curve $C' \subset \mathbb{P}^2$ that is birationally equivalent to $C$ (i.e. $s_2(C)$ equals the minimal $d$ for which $C$ carries a simple $g^2_d$). For a non-empty lattice polygon $\Delta$, we define $\text{lw}^2(\Delta)$ to be the minimal $d$ such that $\Delta$ can be mapped unimodularly inside $d\Sigma$. It is convenient to define $\text{lw}^2(\emptyset) = -2$.

**Question 24.** Let $f \in k[x^{+1}, y^{+1}]$ be non-degenerate with respect to its Newton polygon $\Delta = \Delta(f)$. Suppose that $\Delta^{(1)} \not\sim \Upsilon$. Is it true that $s_2(U(f)) = \text{lw}^2(\Delta^{(1)}) + 3$?
Remark. It is well-known that every non-hyperelliptic genus 4 curve can be realized as a plane quintic, so in this case \( s_2(U(f)) = 5 \), while \( \text{lw}^2(T) + 3 = 6 \).

Example. Let \( f = y^2 - h(x) \) with \( h(x) \in k[x] \) square-free of degree \( 2g+1 \) (so that \( U(f) \) is a hyperelliptic curve of genus \( g \)). Let \( \Delta = \Delta(f) \). Then \( \Delta^{(1)} \) is a line segment of integral length \( g-1 \), hence \( \text{lw}^2(\Delta^{(1)}) = g-1 \). Here Question 24 has an affirmative answer. (A model of degree \( g+2 \) can be obtained by factoring \( h(x) = h_1(x)h_2(x) \) with \( \deg h_1 = g \) and \( \deg h_2 = g+1 \), and then substituting \( y \leftarrow h_1(x)y \).)

Example. Let \( f \) be \( \Delta \)-non-degenerate where \( \Delta = [0, a] \times [0, b] \) for integers \( a, b \geq 2 \). Then \( \text{lw}^2(\Delta^{(1)}) + 3 = a + b - 1 \). A model of degree \( a + b - 1 \) can be achieved by setting \( f' = f(x + x_0, y + y_0) \) for some \((x_0, y_0) \in U(f)\), and then substituting \( x \leftarrow x^{-1}, y \leftarrow y^{-1} \).

4.3 Number of vertices

Let \( C/k \) be a smooth projective curve of genus \( g \geq 2 \). We define the number of vertices of \( C \) as follows. For any set of functions \( S \subset k(C) \) we define its convex hull \( \text{conv}(S) \) as

\[
\{ g \in k(C) \mid \exists a_1, \ldots, a_r \in \mathbb{Z}_{\geq 1}, f_1, \ldots, f_r \in S : g^{a_1+\cdots+a_r} = f_1^{a_1} \cdots f_r^{a_r} \}.
\]

Let \( K \) be a canonical divisor on \( C \). Then the number of vertices of \( C \) is

\[
n(C) = \min \{ \# S \mid S \subset k(C) \text{ and } H^0(C, K) = \langle \text{conv}(S) \rangle \}.
\]

This does not depend on the choice of \( K \).

For a lattice polygon \( \Delta \), its number of vertices is denoted by \( n(\Delta) \).

**Lemma 25.** Let \( C/k \) be a smooth projective curve of genus \( g \geq 2 \) and gonality \( \gamma \). Then we have:

1. \( 2 \leq n(C) \leq \min\{2\gamma - 2, g\} \), and \( n(C) = 2 \) iff \( C \) is hyperelliptic,
2. if \( C \) is \( \Delta \)-non-degenerate for a lattice polygon \( \Delta \), then \( n(C) \leq n(\Delta^{(1)}) \).

**Proof.** 1. The bounds \( 2 \leq n(C) \leq g \) are immediate. It is not hard to show that \( n(C) = 2 \) if and only if there exists a canonical divisor \( K \) on \( C \) and some \( x \in k(C) \) such that \( H^0(C, K) = \langle 1, x, \ldots, x^{g-1} \rangle \).

In turn, this is true if and only if the canonical image of \( C \) is a rational normal curve in \( \mathbb{P}^{g-1} \), i.e. if and only if \( C \) is hyperelliptic. To prove that \( n(C) \leq 2\gamma - 2 \) in general,
we may assume that $C \subset \mathbb{P}^{g-1}$ is non-hyperelliptic and canonically embedded, so that $\mathcal{O}_C(1)$ is canonical. Then recall from Section 3.4 that $C$ is contained in a rational normal scroll $S = S(e_1, \ldots, e_{\gamma-1})$ with $e_1 + \cdots + e_{\gamma-1} + \gamma - 1 = g$. For a suitable choice of coordinates $X_{\ell,m}$, the scroll is defined by the minors of

$$
\left( \begin{array}{cccc|cccc}
X_{1,0} & X_{1,1} & \cdots & X_{1,e_1-1} & X_{2,0} & \cdots & X_{2,e_2-1} \\
X_{1,1} & X_{1,2} & \cdots & X_{1,e_1} & X_{2,1} & \cdots & X_{2,e_2} \\
& & & & \vdots & & \\
& & & & X_{\gamma-1,0} & \cdots & X_{\gamma-1,e_{\gamma-1}-1} \\
& & & & X_{\gamma-1,1} & \cdots & X_{\gamma-1,e_{\gamma-1}}
\end{array} \right) .
$$

Therefore, modulo the ideal of $S$ (and of $C$ in particular), one finds that $X_{\ell,m} \in \text{conv}\{X_{\ell,0}, X_{\ell,e_1}\}$ for all $\ell, m$. Hence

$$H^0(C, \mathcal{O}_C(1)) = \langle \text{conv}\{X_{\ell,0}, X_{\ell,e_1}\} \rangle,$$

so that indeed $n(C) \leq 2\gamma - 2$.

2. Using the canonical divisor $K$ from Section 3.1 (denoted $K_{C'}$ there), one finds

$$H^0(C, K) = \langle x^iy^j \rangle_{(i,j) \in \Delta^{(1)}} = \langle \text{conv}\{x^iy^j\} \rangle_{(i,j) \text{ vertex of } \Delta^{(1)}} ,$$

from which $n(C) \leq n(\Delta^{(1)})$.

Remark. If $g$ is even then $2\gamma - 2 \leq g$; if $g$ is odd then $2\gamma - 2 \leq g + 1$ [22, Rem. IV.5.5.1].

In order to exploit the number of vertices for our study of Question 1, we would need an affirmative answer to a question of the following kind:

Question 26. If $\Delta^{(1)} \not\cong \square$, does equality hold in Lemma 25.2?

Remarks.

- If $n(\Delta^{(1)}) = 3$ then the answer to Question 26 is yes. Indeed, in this case $\Delta^{(1)}$ is two-dimensional, so $C$ is not hyperelliptic, hence $3 \leq n(C) \leq n(\Delta^{(1)}) = 3$. In particular, the number of vertices of a smooth plane curve of degree $d \geq 4$ is 3.

- In Section 5.6 we will see that there exist $2\Upsilon$-non-degenerate curves $C$ that are also $3\square$-non-degenerate. In this case $n(C) = 3$, whereas $n((3\square)^{(1)}) = n(\square) = 4$.

We end with a number of moduli-theoretic questions. Let $\mathcal{M}_g$ be the (coarse) moduli space of curves of genus $g$, let $\mathcal{M}_{g,\gamma}$ be the locus of $\gamma$-gonal curves, and let $\mathcal{M}_g^n$ be the locus of curves having $n$ vertices.
Question 27.  
1. Is the locus of curves $C$ having $\min\{g, 2\gamma - 2\}$ vertices dense in $\mathcal{M}_{g, \gamma}$?

2. Is the locus of curves $C$ with gonality $\lceil \frac{n}{2} \rceil + 1$ dense in each component of $\mathcal{M}_g^n$ (or at least in each component of maximal dimension)?

3. Is $\dim \mathcal{M}_g^n = 2g - 3 + n$?

5 To what extent does this determine $\Delta^{(1)}$?

In this final section we discuss to what extent the material from Section 3 allows us to answer Question 1. First note that if a curve $C/k$ of genus $g$ and gonality $\gamma$ is both $\Delta$- and $\Delta'$-non-degenerate, then certainly

- $\#(\Delta^{(1)} \cap \mathbb{Z}^2) = \#(\Delta'^{(1)} \cap \mathbb{Z}^2)$, because both equal $g$,
- $\lw(\Delta^{(1)}) = \lw(\Delta'^{(1)})$ if $\Delta, \Delta' \not\sim 2\Upsilon$, because both equal $\gamma - 2$ by Corollary 6.

In Sections 5.1, 5.2 and 5.3 we will study Question 1 for curves of increasing gonality. In Sections 5.4-5.9 we will focus on the genus, for which we will make use of the explicit list of polygons from [7]. Finally in Section 5.10 we will consider curves that admit a smooth projective model in $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$.

5.1 Rational, hyperelliptic and trigonal curves

Let $C/k$ be $\Delta$-non-degenerate of genus $g$. If $C$ is rational or (hyper)elliptic then there is only one possibility for $\Delta^{(1)}$ (up to equivalence), so Question 1 has an affirmative answer here. If $C$ is trigonal then either

- $\Delta^{(1)} \cong \Sigma$, which happens if and only if $C$ is birationally equivalent to a smooth plane curve of degree 4 (see Section 5.5), or
- $\Delta^{(1)} \cong \Upsilon$, which is excluded, or
- $\Delta^{(1)}$ does not appear in (2). If $g = 4$ then $\Delta^{(1)}$ is fully determined by the geometry of $C$, as explained in Section 5.6. If $g \geq 5$ then the $g^1_3$ on $C$ is unique. Denote its scrollar invariants (Maroni invariants) by $e_1, e_2$. By Theorem 4 the $g^1_3$ is combinatorial, so by Theorem 11 the lattice points of $\Delta^{(1)}$ can be arranged in two horizontal lines, one containing $e_1 + 1$ lattice points and one containing $e_2 + 1$ lattice points. It is easily checked that, up to equivalence, this determines $\Delta^{(1)}$ completely.

Therefore, also for trigonal curves, Question 1 has an affirmative answer.
5.2 Tetragonal curves

If $C$ is tetragonal then, in addition to $\sharp(\Delta(1) \cap \mathbb{Z}^2)$ and $\text{lw}(\Delta(1))$, also $\sharp(\Delta(2) \cap \mathbb{Z}^2)$ is determined by the geometry of $C$, because this equals $b_2 + 1$ (see Section 3.5).

We claim that if

$$\sharp(\Delta(2) \cap \mathbb{Z}^2) \neq \frac{g - 3}{2} \quad (12)$$

then we can recover $\Delta(1)$ (up to equivalence) from the geometry of $C$. Indeed, (12) implies that $b_1 = b_2 - g - 5 > b_2$, so that Theorem 12 applies: given a canonical model $C \subset \mathbb{P}^{g-1}$ and a $g^1_4$, we can retrieve $\text{Tor}(\Delta(1))$ as a distinguished surface in the corresponding rational normal scroll. This surface is independent of the choice of the $g^1_4$. It follows that we can recover $\text{Tor}(\Delta(1))$ up to an automorphism of $\mathbb{P}^{g-1}$ (because we do depend on the choice of the canonical model). By Lemma 28 below, we recover $\Delta(1)$ up to equivalence, as claimed.

**Lemma 28.** Let $\Delta_1, \Delta_2$ be two-dimensional lattice polygons with

$$\sharp(\Delta_1 \cap \mathbb{Z}^2) - 1 = \sharp(\Delta_2 \cap \mathbb{Z}^2) - 1 = N,$$

and suppose that $\text{Tor}(\Delta_1), \text{Tor}(\Delta_2) \subset \mathbb{P}^N$ can be obtained from one another using a projective transformation. Then $\Delta_1 \cong \Delta_2$.

**Proof.** The projective transformation induces an isomorphism $\text{Tor}(\Delta_1) \to \text{Tor}(\Delta_2)$ that sends $O_{\text{Tor}(\Delta_1)}(1)$ to $O_{\text{Tor}(\Delta_2)}(1)$. Recall from Section 2 that

$$\Delta_1 \cong_{\text{t}} \Delta_{O_{\text{Tor}(\Delta_1)}(1)} \quad \text{and} \quad \Delta_2 \cong_{\text{t}} \Delta_{O_{\text{Tor}(\Delta_2)}(1)}.$$

Thus it suffices to prove the following general statement: if

$$\iota : \text{Tor}(\Delta_1) \xrightarrow{\cong} \text{Tor}(\Delta_2)$$

is an isomorphism between two toric surfaces, and if $D$ is a Weil divisor on $\text{Tor}(\Delta_1)$, then

$$\Delta_D \cong \Delta_{\iota(D)}.$$

It is known that two isomorphic toric varieties always admit a toric isomorphism between them [5, Thm. 4.1], i.e. an isomorphism that is induced by a unimodular transformation taking $\Sigma_{\Delta_1}$ to $\Sigma_{\Delta_2}$. It is clear that such an isomorphism preserves polygons (up to equivalence). Therefore we may assume that $\Sigma_{\Delta_1} = \Sigma_{\Delta_2}$ and that $\iota$ is an automorphism of $\text{Tor}(\Sigma_{\Delta_1})$. Every such automorphism can be written as the composition of

- a toric automorphism,
• the automorphism induced by the action of an element of $T^2$,

• a number of automorphisms of the form $e^\lambda_v$, where $\lambda \in \mathbb{k}$ and $v \in \mathbb{Z}^2$ is a column vector of $\Delta_1$, i.e. a primitive vector $v$ for which there exists an edge $\tau \subset \Delta_1$ such that $u + v \in \Delta_1$ for all $u \in (\Delta_1 \setminus \tau) \cap \mathbb{Z}^2$. To describe $e^\lambda_v$ explicitly, assume that $v = (0, -1)$ and that $\tau$ lies horizontally (the general case can be reduced to this case using an appropriate unimodular transformation). Then $\text{Tor}(\Delta_1)$ can be viewed as a compactification of $T^2$ or $(x$-axis$)$ rather than just $T^2$. On $T^2$ or $(x$-axis$)$, $e^\lambda_v$ acts as $(x, y) \mapsto (x, y + \lambda)$. The column vector property ensures that this extends nicely to all of $\text{Tor}(\Delta_1)$.

\textit{Example.} Let $\Delta_1 = \square$ and consider the map

$$\varphi : T^2 \cup (x$-axis$) \hookrightarrow \text{Tor}(\square) : (x, y) \mapsto (1, x, y, xy).$$

The point $(x, y + \lambda)$ is mapped to $(1 : x : y + \lambda : xy + \lambda x)$. So here

$$e^\lambda_{(0, -1)} : (X_{0,0} : X_{1,0} : X_{0,1} : X_{1,1}) \mapsto (X_{0,0} : X_{1,0} : X_{0,1} + \lambda X_{0,0} : X_{1,1} + \lambda X_{1,0}).$$

See [6, Thm. 3.2] for a proof of this statement, along with a more elaborate discussion. Now the first type of automorphisms preserves polygons up to equivalence, as before. The second type also preserves polygons because it preserves torus-invariant Weil divisors. As for the third type, let $D_\tau$ be the torus-invariant prime divisor corresponding to the base edge $\tau$ of $v$. Then by adding a divisor of the form $\text{div}(x^i y^j)$ if needed, one can always find a torus-invariant Weil divisor that is equivalent to $D$ and whose support does not contain $D_\tau$. But such a divisor is preserved by $e^\lambda_v$, hence the lemma follows.

Note that (12) is automatically satisfied when $g$ is even. Thus in the tetragonal even genus case, Question 11 has an affirmative answer.

### 5.3 Curves of gonality $\gamma \geq 5$

If $C$ has gonality $\gamma \geq 5$ then either

• $\Delta^{(1)} \cong (\gamma - 2)\Sigma$, which by Corollary 11 happens if and only if $C$ is birationally equivalent to a smooth plane curve of degree $\gamma + 1$, or

• $\Delta^{(1)} \cong 2\Upsilon$, which by Theorem 9 happens if and only if $\text{cd}(C) = 3$ (only for $\gamma = 6$), or

• $\Delta^{(1)}$ does not appear in (2), so that each $g^{1}_\gamma$ is combinatorial.

In the latter case, the only general tools we have currently available in attacking Question 11 are combinatorial interpretations of (besides the genus)
• the number of $g_1^1$'s,
• their respective scollar invariants,
• their respective secondary scollar invariants (if $\Delta$ and $\Delta'$ are well-aligned with respect to the corresponding lattice width directions).

In Section 5.9 we will elaborate this in genera 9 and 10.

### 5.4 Curves of genus 0, 1, 2

For each $g \in \{0, 1, 2\}$ there is only one possibility for $\Delta^{(1)}$ (up to equivalence). Hence Question 1 trivially has an affirmative answer here.

### 5.5 Curves of genus 3

In genus 3 there are two possibilities for $\Delta^{(1)}$:

\[
\begin{array}{c}
\Gamma_3^{\text{hyp}} \\
\Sigma.
\end{array}
\]

If a genus 3 curve $C$ is $\Delta$-non-degenerate for some lattice polygon $\Delta$, then $\Delta^{(1)} \cong \Gamma_3^{\text{hyp}}$ if and only if $C$ is hyperelliptic. Hence Question 1 has an affirmative answer for $g = 3$.

### 5.6 Curves of genus 4

In genus 4, we have four possibilities for $\Delta^{(1)}$:

\[
\begin{array}{c}
\Gamma_4^{\text{hyp}} \\
\square \\
\Delta_{1,2} \\
\Upsilon.
\end{array}
\]

If a genus 4 curve $C$ is $\Delta$-non-degenerate for some lattice polygon $\Delta$, and if we are given that $\Delta^{(1)} \neq \Upsilon$, then $\Delta^{(1)}$ is determined by the geometry of $C$. Indeed, either

• it is hyperelliptic, or

• it canonically embeds into a unique quadric, which necessarily equals Tor($\Delta^{(1)}$). If the quadric is smooth then $\Delta^{(1)} \cong \square$. If it is singular then $\Delta^{(1)} \cong \Delta_{1,2}$.

We leave it to the reader to show that in fact every genus 4 curve is $\Delta$-non-degenerate, with $\Delta$ among the following three polygons:
The only genus 4 polygon having $\Upsilon$ as its interior is $2\Upsilon$. By the foregoing, a $2\Upsilon$-non-degenerate curve is also $3\Box$- or $3\Delta_{1,2}$-non-degenerate (it is certainly non-hyperelliptic). Both situations are possible:

- Let $f = x^2 + y^2 + x^{-2}y^{-2}$, then $U(f)$ canonically embeds as
  $$V(X_{0,0}^3 - X_{1,0}X_{0,1}X_{-1,-1}, X_{1,0}^2 + X_{0,1}^2 + X_{-1,-1}^2).$$
  So $U(f)$ embeds in a singular quadric.

- If $f = 1 + x^2 + y^2 + x^{-2}y^{-2}$, then $U(f)$ canonically embeds as
  $$V(X_{0,0}^3 - X_{1,0}X_{0,1}X_{-1,-1}, X_{0,0}^2 + X_{1,0}^2 + X_{0,1}^2 + X_{-1,-1}^2).$$
  Here the quadric is smooth.

In particular, Question 1 has a positive answer in genus 4. If we drop the condition $\Delta, \Delta' \not\sim 2\Upsilon$ then the answer becomes negative.

### 5.7 Curves of genus 5

In genus 5, we have five possibilities for $\Delta^{(1)}$:

The polygons $\Gamma_{\text{hyp}}^5$ and $\Gamma_{\text{trig}}^5$ are uniquely determined by their lattice width, hence intrinsic. However, as in the case of genus 4, the three other polygons cannot be identified on purely geometric grounds (note that condition (12) fails here). The situation is quite different, though. For $i \in \{1, 2, 3\}$, denote the unique lattice polygon having interior $\Gamma_i^5$ by $\Delta_i^5$. Then we can prove the following:

**Theorem 29.** For each $i$, the set of irreducible Laurent polynomials $f$ that are supported on $\Delta_i^5$ and for which $U(f)$ is not non-degenerate with respect to any other lattice polygon, is a dense subset of the according coefficient space (in the sense of Zariski topology).
Proof. A genus 5 curve $C/k$ canonically embeds as a complete intersection of three quadrics in $\mathbb{P}^4$, say with defining polynomials $Q_1$, $Q_2$ and $Q_3$. The discriminant curve $\mathcal{D}(C)$ associated to $C$ is the (possibly reducible) quintic in $\mathbb{P}^2 = \operatorname{Proj} k[\lambda_1, \lambda_2, \lambda_3]$ defined by
\[
\det (\lambda_1 \mathcal{M}_{Q_1} + \lambda_2 \mathcal{M}_{Q_2} + \lambda_3 \mathcal{M}_{Q_3}) = 0,
\]
where $\mathcal{M}_{Q_i}$ denotes the symmetric matrix corresponding to $Q_i$. That is, $\mathcal{D}(C)$ parameterizes the singular quadrics in the canonical ideal. It is well-defined up to transformations of $\mathbb{P}^2$. One can show that $\mathcal{D}(C)$ has at most ordinary double points as singularities, and that the latter correspond to the rank 3 quadrics in the canonical ideal [11, Ex. VI.F]. In turn, these rank 3 quadrics are in one-to-one correspondence with the half-canonical $g_1^1$’s (vanishing theta-nulls) that are carried by $C$ [30, Lem. 1.2]. A generic genus 5 curve does not carry such half-canonical $g_1^1$’s. But in the non-degenerate case they always exist. We will prove the theorem by analyzing their locus in the discriminant curve. It suffices to prove the following observations.

1. (a) The discriminant curve of a $\Delta_1^5$-non-degenerate curve has at least two ordinary double points, the line through which is not tangent to any of the branches.
   (b) If $f$ is a generic $\Delta_1^5$-non-degenerate Laurent polynomial, then $\mathcal{D}(U(f))$ is an irreducible curve having exactly two double points, and no line through any of these points intersects $\mathcal{D}(U(f))$ in another point with multiplicity three.

2. (a) The discriminant curve of a $\Delta_2^5$-non-degenerate curve has at least one ordinary double point, along with an inflection point whose tangent line passes through that double point.
   (b) If $f$ is a generic $\Delta_2^5$-non-degenerate Laurent polynomial, then $\mathcal{D}(U(f))$ is an irreducible curve having exactly one double point.

3. (a) The discriminant curve of a $\Delta_3^5$-non-degenerate curve has at least two ordinary double points, the line through which is tangent to one of the branches of one of these double points.
   (b) If $f$ is a generic $\Delta_3^5$-non-degenerate Laurent polynomial, then $\mathcal{D}(U(f))$ is an irreducible curve having exactly two double points. Except for the line connecting both, no line through any of these points intersects $\mathcal{D}(U(f))$ in another point with multiplicity three.

We will restrict our attention to $\Delta_1^5$; the other polygons can be treated in a similar way. Identify $\Delta_1^5$ with $\operatorname{conv}\{(2,0), (0,2), (-2,0), (0,-2)\}$. Let
\[
f = \sum_{(i,j) \in \Delta_1^5 \cap \mathbb{Z}^2} c_{i,j} x^i y^j
\]
be a $\Delta^5_1$-non-degenerate Laurent polynomial. Then the canonical model of $U(f)$ is

$$C(f) = \mathcal{Z}(X_{0,0}^2 - X_{1,0}X_{-1,0}, X_{0,1}^2 - X_{0,1}X_{-1,0}, Q)$$

where

$$Q = c_{2,0}X_{1,0}^2 + c_{0,2}X_{0,1}^2 + c_{-2,0}X_{-1,0}^2 + c_{0,-2}X_{0,-1}^2$$
$$+ c_{1,1}X_{1,0}X_{0,1} + c_{1,-1}X_{1,0}X_{0,-1} + c_{-1,1}X_{-1,0}X_{0,1} + c_{-1,-1}X_{-1,0}X_{0,-1}$$
$$+ c_{0,1}X_{1,0}X_{0,0} + c_{0,-1}X_{-1,0}X_{0,0} + c_{1,-1}X_{1,0}X_{0,0} + c_{0,-1}X_{0,1}X_{-1,0}$$
$$+ c_{0,0}X_{0,0}^2.$$ 

So $\mathcal{D}(U(f))$ is defined by

$$\det \left( \lambda_1 \mathcal{M}_{X_{0,0}^2 - X_{1,0}X_{-1,0}} + \lambda_2 \mathcal{M}_{X_{0,1}^2 - X_{0,1}X_{-1,0}} + \lambda_3 \mathcal{M}_{Q} \right) = 0.$$ 

Dehomogenizing with respect to $\lambda_3$, one obtains a polynomial $\delta(f) = \sum a_{i,j} \lambda_1^i \lambda_2^j$ that is supported on

$$\Xi^5_1 = \text{conv}\{(0,0), (3,0), (3,2), (2,3), (0,3)\}.$$ 

Each $a_{i,j}$ is a polynomial expression in the $c_{i,j}$’s. One can verify that $a_{2,3} = a_{3,2} = 1/16$. It follows that $(1 : 0 : 0)$ and $(0 : 1 : 0)$ are double points of $\mathcal{D}(U(f))$, and that the line connecting both intersects $\mathcal{D}(U(f))$ in $(1 : -1 : 0)$. Assertion 1.(a) follows.

Now we claim that, generically, $\delta(f)$ is $\Xi^5_1$-non-degenerate. To see this, it is sufficient to show that

$$\text{Res}_{\Xi^5_1} \left( \delta(f), \lambda_1 \frac{\partial \delta(f)}{\partial \lambda_1}, \lambda_2 \frac{\partial \delta(f)}{\partial \lambda_2} \right) \in k[c_{i,j}]_{(i,j) \in \Delta^5_1 \cap \mathbb{Z}^2}$$

does not vanish identically. This we do by giving an explicit example: let

$$c_{2,0} = 3, \quad c_{0,2} = 2, \quad c_{-2,0} = c_{0,-2} = c_{1,0} = c_{0,1} = c_{-1,0} = c_{0,-1} = 1$$

and let all other $c_{i,j}$’s be zero. Then $16\delta(f)$ equals

$$\lambda_1^3 \lambda_2^3 - 8\lambda_1^3 + \lambda_1^2 \lambda_2^2 - 7\lambda_1^2 \lambda_2 + 3\lambda_1^2 - 11\lambda_1 \lambda_2^2 + 88\lambda_1 - 12\lambda_2^3 + 4\lambda_2^2 + 84\lambda_2 - 68$$

of which the reader can check that it is indeed $\Xi^5_1$-non-degenerate. So in particular, $\mathcal{D}(U(f))$ is generically irreducible and smooth away from $(1 : 0 : 0)$ and $(0 : 1 : 0)$.

Next, we claim that

$$\delta(f) = \frac{\partial \delta(f)}{\partial \lambda_1} = \frac{\partial^2 \delta(f)}{\partial \lambda_1^2} = 0$$

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generically has no solutions in \( \mathbb{A}^2 \), so that no line through \( (1 : 0 : 0) \) intersects the curve elsewhere with multiplicity three. Unfortunately the corresponding multivariate resultant vanishes identically, because it always detects the solution \( (1 : 0 : 0) \in \mathbb{P}^2 \). More precisely,

\[
\text{Res}_{\lambda_1}(\delta(f), \frac{\partial \delta(f)}{\partial \lambda_1}) \quad \text{and} \quad \text{Res}_{\lambda_2}(\delta(f), \frac{\partial^2 \delta(f)}{\partial \lambda_2})
\]

both contain a factor \( \lambda_2^2 - 4c_{0,2}c_{0,-2} \), corresponding to the branch tangents of \( (1 : 0 : 0) \). But these tangents cannot intersect the curve elsewhere with multiplicity three anyway, so the non-vanishing of

\[
\text{Res}_{\lambda_2}\left( \frac{\text{Res}_{\lambda_1}(\delta(f), \frac{\partial \delta(f)}{\partial \lambda_1})}{\lambda_2^2 - 4c_{0,2}c_{0,-2}}, \frac{\text{Res}_{\lambda_1}(\delta(f), \frac{\partial^2 \delta(f)}{\partial \lambda_2})}{\lambda_2^2 - 4c_{0,2}c_{0,-2}} \right) \in k[c_{i,j}]_{(i,j) \in \Delta^5 \cap 2^2}
\]

is sufficient for our needs. To see that it does not vanish identically one can use the same example as above.

A similar consideration for \( (0 : 1 : 0) \) (again using the same example) then proves 1.(b).

\[\blacksquare\]

**Theorem 30.** For each \( (i, j) \in \{1, 2, 3\}^2 \) there exists a curve \( C/k \) that is both \( \Delta^5_i \)-non-degenerate and \( \Delta^5_j \)-non-degenerate.

**Proof.** We denote by \( M \) the invertible matrix

\[
\begin{pmatrix}
4 & 4 & 1 & 0 & 0 \\
2 & 3 & 1 & 0 & 0 \\
1 & 2 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 2 \\
0 & 0 & -1 & 1 & 3
\end{pmatrix} \in k^{5 \times 5}
\]

and use the following identifications:

\[
\Delta^5_1 = \text{conv}\{(2, 0), (0, 2), (-2, 0), (0, -2)\}, \\
\Delta^5_2 = \text{conv}\{(0, 2), (-2, 0), (0, -2), (2, -2)\}, \\
\Delta^5_3 = \text{conv}\{(0, 2), (-2, -2), (2, -2)\}.
\]

We also introduce three polynomials rings:

\[
R_1 = k[X_{-1,0}, X_{0,0}, X_{1,0}, X_{0,1}, X_{0,-1}], \\
R_2 = k[Y_{0,1}, Y_{0,0}, Y_{0,-1}, Y_{-1,0}, Y_{1,-1}], \\
R_3 = k[Z_{0,-1}, Z_{0,0}, Z_{0,1}, Z_{-1,-1}, Z_{1,-1}].
\]
First, we consider the case \((i, j) = (1, 2)\). Let \(\phi : R_1 \to R_2\) be the ring isomorphism defined by
\[
\begin{pmatrix}
X_{-1,0} \\
X_{0,0} \\
X_{1,0} \\
X_{0,1} \\
X_{0,-1}
\end{pmatrix}
\mapsto
\begin{pmatrix}
Y_{0,1} \\
Y_{0,0} \\
Y_{0,-1} \\
Y_{-1,0} \\
Y_{1,-1}
\end{pmatrix}
\]
(that is, \(\phi(X_{-1,0})\) equals the top entry of the right hand side, and so on) and write \(\psi = \phi^{-1} : R_2 \to R_1\). Note that \(M\) is chosen in such a way that
\[
\phi(X_{0,0}^2 - X_{1,0}X_{-1,0}) = Y_{0,0}^2 - Y_{0,1}Y_{0,-1}.
\tag{13}
\]
Let \(Q_1 \in R_1\) be the quadratic form corresponding to the symmetric matrix
\[
\mathcal{M}_{Q_1} = (M^{-1})^T.
\]
and let \(Q_2 \in R_2\) be corresponding to
\[
\mathcal{M}_{Q_2} = M^T \cdot \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix} \cdot M = \begin{pmatrix}
8 & 12 & 5 & -1 & -3 \\
12 & 18 & 6 & 0 & 0 \\
5 & 6 & 4 & 0 & -1 \\
-1 & 0 & 0 & -2 & -5 \\
-3 & 0 & -1 & -5 & -12
\end{pmatrix}.
\]
One sees that \(\psi(Y_{-1,0}Y_{1,-1} - Y_{0,0}Y_{0,-1}) = Q_1\) and \(\phi(X_{0,0}^2 - X_{0,1}X_{0,-1}) = Q_2\). So the image under \(\phi\) of the ideal
\[
I_1 = (X_{0,0}^2 - X_{1,0}X_{-1,0}, Q_1, X_{0,0}^2 - X_{0,1}X_{0,-1}) \subset R_1
\]
equals

\[
I_2 = (Y_{0,0}^2 - Y_{0,1}Y_{0,-1}, Y_{-1,0}Y_{1,-1} - Y_{0,0}Y_{0,-1}, Q_2) \subset R_2.
\]
Now consider the Laurent polynomials
\[
f_1 &= Q_1(x^{-1}, 1, x, y, y^{-1}) \\
&= -23x^{-2} + 151x^{-1} - 378 + 17x^{-1}y - 14x^{-1}y^{-1} + 444x - 56y \\
&\quad + 46y^{-1} - 199x^2 + 50xy - 41xy^{-1} - 3y^2 - 2y^{-2}
\]
and
\[
f_2 &= Q_2(y, 1, y^{-1}, x^{-1}, xy^{-1}) \\
&= 4y^2 + 12y + 14 - x^{-1}y - 3x + y^{-1} + 2y^{-2} - xy^{-2} - x^{-2} - 6x^2y^{-2}.
\]
It is easy to check that $f_1$ and $f_2$ are non-degenerate with respect to their respective Newton polygons $\Delta^5_1$ and $\Delta^5_2$. For instance, if $\tau \subset \Delta^5_1$ is the face conv\{(2, 0), (0, 2)\}, then $(f_1)_\tau = -199x^2 + 50xy - 3y^2$, which has a nonzero discriminant. Since $I_1$ and $I_2$ are the respective canonical ideals of $U(f_1)$ and $U(f_2)$, it follows that the latter curves are birationally equivalent. So we conclude that there are curves that are both $\Delta^5_1$- and $\Delta^5_2$-non-degenerate.

The cases $(i, j) = (1, 3)$ and $(i, j) = (2, 3)$ can be handled similarly. For $(i, j) = (1, 3)$, the ring isomorphism

$$
\phi : R_1 \rightarrow R_3 : \begin{pmatrix}
X_{-1, 0} \\
X_{0, 0} \\
X_{1, 0} \\
X_{0, 1} \\
X_{0, -1}
\end{pmatrix} \mapsto M \cdot \begin{pmatrix}
Z_{0, -1} \\
Z_{0, 0} \\
Z_{1, 0} \\
Z_{1, -1}
\end{pmatrix}
$$

will map the canonical ideal of the $\Delta^5_1$-non-degenerate curve $U(g_1)$ to the canonical ideal of the $\Delta^5_2$-non-degenerate curve $U(g_3)$, where

$$
g_1 = 25x^2 - 162x^{-1} + 402 - 17x^{-1}y + 14x^{-1}y^{-1} - 468x + 56y - 46y^{-1} + 208x^2 - 50xy + 41xy^{-1} + 3y^2 + 2y^{-2}
$$

and

$$
g_3 = -y^{-2} + 12y^{-1} + 14 - x^{-1}y^{-2} - 3xy^{-2} + 6y + 2y^2 - x - x^{-2}y^{-2} - 6x^2y^{-2}.
$$

In case $(i, j) = (2, 3)$, we use the ring isomorphism $\phi : R_2 \rightarrow R_3$ defined by

$$
\begin{pmatrix}
Y_{0, 1} \\
Y_{0, 0} \\
Y_{0, -1} \\
Y_{-1, 0} \\
Y_{1, -1}
\end{pmatrix} \mapsto M \cdot \begin{pmatrix}
Z_{-1, -1} \\
Z_{0, -1} \\
Z_{1, -1} \\
Z_{0, 0} \\
Z_{0, 1}
\end{pmatrix}
$$

to see that the $\Delta^5_2$-non-degenerate Laurent polynomial

$$
h_2 = 67y^2 - 435y + 1089 - 49x^{-1}y + 33x - 1255y^{-1} + 159x^{-1} - 107xy^{-1} + 547y^{-2} - 140x^{-1}y^{-1} + 94xy^{-2} + 9x^{-2} + 4x^2y^{-2}
$$

and the $\Delta^5_3$-non-degenerate Laurent polynomial

$$
h_3 = -2x^{-2}y^{-2} - 7x^{-1}y^{-2} - 10y^{-2} + x^{-1}y^{-1} + 3x^{-1} - 5xy^{-2} - 2x^2y^{-2} + x + 1 + 5y + 6y^2
$$

give rise to isomorphic canonical curves.

[\boxed{}]

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Remark. The above examples are far from unique. E.g. in the case where \((i, j) = (1, 2)\), any sufficiently generic matrix \(M\) satisfying (13) is expected to work. Our guess is that there even exists a single curve that is \(\Delta^5_i\)-non-degenerate for all \(i \in \{1, 2, 3\}\), but we did not push the effort in finding an explicit example.

In conclusion, Question 1 has a positive answer for \(g = 5\). If we drop the condition \(\Delta, \Delta' \not\sim \Delta^5_1, \Delta^5_2, \Delta^5_3\) then the answer becomes negative. Remark however that

- it in fact suffices to exclude any two polygons among \(\Delta^5_1, \Delta^5_2, \Delta^5_3\) for Question 1 to have a positive answer,

- one could alternatively replace “\(\Delta\)-non-degenerate” by “defined by a sufficiently generic Laurent polynomial having Newton polygon \(\Delta\)”, and similarly for \(\Delta'\); then the answer would be yes, even with \(\Delta^5_1, \Delta^5_2, \Delta^5_3\) included.

### 5.8 Curves of genus 6, 7 or 8

In genus 6, we have 6 possibilities for \(\Delta^{(1)}\): 

\[
\begin{align*}
\Gamma^6_{\text{hyp}} & \quad \Gamma^6_{\text{trig,0}} & \quad \Gamma^6_{\text{trig,2}} & \quad \Gamma^6_1 & \quad \Gamma^6_2 & \quad \Gamma^6_3.
\end{align*}
\]

In genus 7, we have 8 possibilities:

\[
\begin{align*}
\Gamma^7_{\text{hyp}} & \quad \Gamma^7_{\text{trig,1}} & \quad \Gamma^7_{\text{trig,3}} & \quad \Gamma^7_1 & \quad \Gamma^7_2 & \quad \Gamma^7_3 & \quad \Gamma^7_4 & \quad \Gamma^7_5.
\end{align*}
\]

In genus 8, there are 12 possibilities:

\[
\begin{align*}
\Gamma^8_{\text{hyp}} & \quad \Gamma^8_{\text{trig,0}} & \quad \Gamma^8_{\text{trig,2}} & \quad \Gamma^8_1 & \quad \Gamma^8_2 & \quad \Gamma^8_3 & \quad \Gamma^8_4 & \quad \Gamma^8_5 & \quad \Gamma^8_6 & \quad \Gamma^8_7 & \quad \Gamma^8_8 & \quad \Gamma^8_9.
\end{align*}
\]

These are completely covered by Sections 5.1 and 5.2 (note that \(\Gamma^7_1\) violates condition (12), but it is the unique polygon that does so) so Question 1 has an affirmative answer here.
5.9 Curves of genus 9 or 10

The first open genera are 9 and 10. Below, we have check-marked (✓) the polygons that we are able to characterize on geometric grounds.

In genus 9, there are 17 possible interior polygons:

\[ \Gamma_{hyp}^9, \Gamma_{trig,1}^9, \Gamma_{trig,3}^9, \Gamma_1^9, \Gamma_2^9, \Gamma_3^9, \Gamma_4^9, \Gamma_5^9, \Gamma_6^9, \Gamma_7^9, \Gamma_8^9, \Gamma_9^9, \Gamma_{10}^9, \Gamma_{11}^9, \Gamma_{12}^9, \Gamma_{13}^9, \Gamma_{14}^9. \]

Here as before, \( \Gamma_{hyp}^9, \Gamma_{trig,1}^9 \) and \( \Gamma_{trig,3}^9 \) are covered by Section 5.1. The polygons \( \Gamma_1^9 \) and \( \Gamma_2^9 \) correspond to pentagonal curves and can be separated by considering the number of \( g_1 \)'s. The remaining polygons correspond to tetragonal curves. By Section 5.2 if \( \sharp(\Delta^{(2)} \cap \mathbb{Z}^2) \neq 3 \) then we recover \( \Delta^{(1)} \) on geometric grounds. This means that we are left with \( \Gamma_3^9, \ldots, \Gamma_{12}^9. \) The polygon \( \Gamma_7^9 \) gives rise to scrollar invariants \( \{1, 2, 3\} \), whereas the other polygons (which we were not able to separate) correspond to \( \{1, 1, 4\} \).

Finally in genus 10, we have 22 possible \( \Delta^{(1)} \)'s:

\[ \Gamma_{hyp}^{10}, \Gamma_{trig,0}^{10}, \Gamma_{trig,2}^{10}, \Gamma_{trig,4}^{10}, \Gamma_1^{10}, \Gamma_2^{10}, \Gamma_3^{10}, \Gamma_4^{10}, \Gamma_5^{10}, \Gamma_6^{10}, \Gamma_7^{10}, \Gamma_8^{10}, \Gamma_{10}^{10}, \Gamma_{11}^{10}, \Gamma_{12}^{10}, \Gamma_{13}^{10}, \Gamma_{14}^{10}. \]
The polygons $\Gamma_{10}^{\text{hyp}}$, $\Gamma_{10}^{\text{trig},0}$, $\Gamma_{10}^{\text{trig},2}$, $\Gamma_{10}^{\text{trig},4}$ and $\Gamma_{8}^{10}$, ..., $\Gamma_{18}^{10}$ are covered by Sections 5.1 and 5.2. The polygons $\Gamma_{1}^{10} = 2\Upsilon$ and $\Gamma_{7}^{10} = 3\Sigma$ are characterized by the Clifford dimension. The five remaining polygons correspond to pentagonal curves and can be split up into two groups:

- $\Gamma_{2}^{10}, \Gamma_{5}^{10}$ and $\Gamma_{6}^{10}$ correspond to curves having a unique $g_5^1$;
- $\Gamma_{3}^{10}, \Gamma_{4}^{10}$ correspond to curves having two $g_5^1$'s.

Out of the first group, the polygon $\Gamma_{2}^{10}$ can be identified by its multiset of secondary scrollar invariants, using Theorem 18. Note that the condition of being well-aligned is satisfied here. Then in the case of $\Gamma_{2}^{10}$ and $\Gamma_{6}^{10}$ the secondary scrollar invariants read

$$(B_{2,3}, B_{2,2}, B_{3,3}, B_2, B_3) = (1, 2, 3, 2, 0) \text{ resp. } (B_{2,3}, B_{2,2}, B_{3,3}, B_2, B_3) = (2, 3, 2, 1, 0),$$

whereas $\Gamma_{5}^{10}$ corresponds to $(B_{2,3}, B_{2,2}, B_{3,3}, B_2, B_3) = (1, 3, 3, 1, 0)$.

**Remark.** Our calculations (see [http://wis.kuleuven.be/~u0040935/](http://wis.kuleuven.be/~u0040935/)) suggest that $\Gamma_{2}^{10}$ and $\Gamma_{6}^{10}$ give rise to different Betti tables. If this is correct, we can also checkmark these two polygons, leaving only $\Gamma_{3}^{10}$ and $\Gamma_{4}^{10}$ undecided.

### 5.10 Curves on $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$

Corollary 10 gave us a combinatorial characterization of smooth projective curves in $\mathbb{P}^2$, implying that Question 1 has a positive answer as soon as either of $\Delta^{(1)}$, $\Delta'^{(1)}$ is proportional to the standard simplex.

We can give a similar characterization for curves in $\mathbb{P}^1 \times \mathbb{P}^1$:

**Theorem 31.** Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be non-degenerate with respect to its (two-dimensional) Newton polygon $\Delta = \Delta(f)$, and assume that $\Delta \not\sim 2\Upsilon$. Then $U(f)$ is birationally equivalent to a smooth projective curve in $\mathbb{P}^1 \times \mathbb{P}^1$ if and only if $\Delta^{(1)} = \emptyset$ or $\Delta^{(1)} \cong [0, a] \times [0, b]$ for some integers $a \geq b \geq 0$.

**Proof.** We may assume that $U(f)$ is neither rational, nor elliptic or hyperelliptic (and hence that $\Delta^{(1)}$ is two-dimensional) because such curves admit smooth complete models in $\mathbb{P}^1 \times \mathbb{P}^1$. So for the ‘if’ part we can assume that $b \geq 1$. But then $\text{Tor}(\Delta^{(1)}) \cong \mathbb{P}^1 \times \mathbb{P}^1$, and the statement follows using the canonical embedding (11).
The real deal is the ‘only if’ part. At least, if a curve \( C/k \) is birationally equivalent to a (non-rational, non-elliptic, non-hyperelliptic) smooth projective curve in \( \mathbb{P}^1 \times \mathbb{P}^1 \), then it is non-degenerate with respect to a rectangular polygon \( \Delta' = [-1, a+1] \times [-1, b+1] \) for \( a \geq b \geq 1 \): this follows from the material in Section 2 (one can use an automorphism of \( \mathbb{P}^1 \times \mathbb{P}^1 \) to ensure appropriate behavior with respect to the toric boundary). Note that \( \Delta^{(1)} = [0, a] \times [0, b] \). From Theorem 2 we see that the geometric genus of \( g \) equals \((a+1)(b+1)\), and from Corollary 6 that its gonality equals \( \gamma = b + 2 \). We observe that

- \( g \) is composite,
- \( cd(C) = 1 \) by Theorem 9,
- the scrollar invariants of \( C \) (with respect to any gonality pencil) are all equal to \( g/(\gamma - 1) - 1 \); indeed, by Theorem 4 every gonality pencil is computed by projecting along some lattice width direction; if \( a > b \) then the only pair of lattice width directions is \( \pm (0, 1) \); from Theorem 11 we find that the corresponding scrollar invariants are \( a, a, \ldots, a \); if \( a = b \) we also have the pair \( \pm (1, 0) \), giving rise to the same scrollar invariants,
- if \( \gamma \geq 4 \) then the secondary scrollar invariants (with respect to any gonality pencil) take exactly two distinct values: \( 2g/(\gamma - 1) - 2 \) and \( g/(\gamma - 1) - 3 \); indeed, since \( \Delta' \) is clearly well-aligned, by Theorem 18 we find that these are \( 2a, 2a, \ldots, 2a, a - 2, a - 2, \ldots, a - 2 \).

A first consequence is that \( U(f) \) admits a combinatorial gonality pencil. Indeed, \( \Delta \) cannot be of the form \( 2\Upsilon \) (excluded in the statement of the theorem), nor of the form \( d\Sigma \) for some \( d \geq 2 \): the cases \( d = 2 \) and \( d = 3 \) correspond to rational and elliptic curves (excluded at the beginning of this proof), the case \( d = 4 \) corresponds to curves of genus 3 (not composite), and the cases where \( d \geq 5 \) correspond to curves of Clifford dimension 2. Without loss of generality we may then assume that \( \Delta \subset \{(i, j) \in \mathbb{R}^2 \mid 0 \leq j \leq \gamma \} \) and that our gonality pencil corresponds to vertical projection. By Theorem 11 the numbers \( E_\ell = -1 + \sharp \{(i, j) \in \Delta^{(1)} \cap \mathbb{Z}^2 \mid j = \ell \} \) (for \( \ell = 1, \ldots, \gamma - 1 \)) are the corresponding scrollar invariants. Hence the \( E_\ell \)'s must all be equal to \( g/(\gamma - 1) - 1 \). We denote this common quantity by \( E \).

This already puts severe restrictions on the possible shapes of \( \Delta^{(1)} \). By horizontally shifting and skewing we may assume that the lattice points at height \( j = 1 \) are \( (0, 1), \ldots, (E, 1) \) and that the lattice points at height \( j = 2 \) are \( (0, 2), \ldots, (E, 2) \). If \( \gamma = 3 \), it follows that \( \Delta^{(1)} \) has the desired rectangular shape, so we may suppose that \( \gamma \geq 4 \). Then by horizontally flipping if needed, we can assume that the lattice points at height \( j = 3 \) are \( (i, 3), \ldots, (E + i, 3) \) for some \( i \geq 0 \). Now \( i \geq 2 \) is impossible, for this would introduce a new lattice point at height \( j = 2 \); thus \( i = 0 \) or...
Continuing this type of reasoning, we obtain that the lattice points of \( \Delta^{(1)} \) can be seen as a pile of \( n \) blocks of respectively \( m_1, \ldots, m_n \) sheets, where each block is shifted to the right over a distance 1 when compared to its predecessor:

If \( n = 1 \) then \( \Delta^{(1)} \) has the desired rectangular shape, so let us assume that \( n \geq 2 \). We claim that there exists an integer \( m \geq 2 \) such that \( m - 1 \leq m_1, m_n \leq m \) and \( m_2 = \cdots = m_{n-1} = m \). This we prove in a number of steps.

- **Step 1:** \( m_1 \leq m_2 \leq \cdots \leq m_{n-1} \).

The non-horizontal edge adjacent to \((E, 1)\) must be supported on a line having an integer slope, because its outward shift must intersect the line \( j = 0 \) (the outward shift of \( j = 1 \)) in a lattice point (by the criterion for interior polygons stated at the end of Section 2). It follows that this edge contains \((E + 1, m_1 + 1)\). Now if \( n \geq 3 \) we conclude

- that \( m_2 \geq m_1 \), because the contrary would contradict the convexity of \( \Delta^{(1)} \),
- that, similarly as before, the edge going up from \((E + 1, m_1 + 1)\) (which is the same edge if \( m_2 = m_1 \)) has an integer slope, implying that it contains \((E + 2, m_1 + m_2 + 1)\).

Continuing in this way solves Step 1.

- **Step 2:** \( m_2 \geq \cdots \geq m_{n-1} \geq m_n \).

This follows from Step 1, essentially by symmetry: if we rotate \( \Delta^{(1)} \) by \( 180^\circ \) we get a similar picture but with the \( m_i \)'s in reversed order. (We write ‘essentially’, because the situation may not be entirely symmetric: we might have \( m_n = 1 \); this does not affect the argument, however.)

- **Step 3:** \( m_1 \geq m_2 - 1 \). Suppose the contrary. As a consequence to the above proof, \( \Delta^{(1)} \) has an edge passing through \((0, m_1)\) and \((1, m_1 + m_2)\). By considering the outward shift of this edge, along with the outward shifts of \( i = 0 \) and \( j = 1 \), one runs into a contradiction.
• Step 4: $m_n \geq m_{n-1} - 1$. This again follows by symmetry, essentially.

The claim follows.

From this explicit description we see that whenever $w \in \Delta^{(2)} \cap \mathbb{Z}^2$, then $w + (0, \pm 1) \in \Delta^{(1)}$. This we can use to show that $\Delta$ is well-aligned with respect to $(0, 1)$. Taking into account the remark following Definition \[\text{(1)}\] pick an $(i, j) \in \partial \Delta$ and let $L$ be a horizontal line, say at height $\ell$.

• Suppose that $2 \leq j \leq \gamma - 2$ and suppose that $(i, j)$ lies to the right of $\Delta^{(1)}$. Then for every $w \in L \cap \Delta^{(2)} \cap \mathbb{Z}^2$ we have

$$(i, j) - w = \underbrace{(i, j + 1) - w}_{\in \Delta^{(1)}} + \underbrace{(w + (0, -1) - w)}_{\in \Delta^{(1)}}.$$ 

So we can take for $M_1$ the line at height $j + 1$, and for $M_2$ the line at height $\ell - 1$. The case where $(i, j)$ lies to the left of $\Delta^{(1)}$ follows similarly.

• Suppose that $j = 1$. If this case occurs then necessarily $(i, j) = (-1, 1)$. If $\ell > m_1$ then for every $w \in L \cap \Delta^{(2)} \cap \mathbb{Z}^2$ we have

$$(-1, 1) - w = \underbrace{((0, m_1) - w)}_{\in \Delta^{(1)}} + \underbrace{(w + (-1, -m_1 + 1) - w)}_{\in \Delta^{(1)}}.$$ 

So we can take for $M_1$ the line at height $m_1$ and for $M_2$ the line at height $\ell - m_1 + 1$. If $\ell \leq m_1$ then we can write

$$(-1, 1) - w = \underbrace{((0, 1) - w)}_{\in \Delta^{(1)}} + \underbrace{(w + (-1, 0) - w)}_{\in \Delta^{(1)}},$$ 

so here $M_1$ is the line at height 1 and $M_2 = L$. The case $j = \gamma - 1$ follows similarly.

• Suppose that $j = 0$. If $(i, j) = (-1, 0)$ then we proceed as in the case $j = 1$, using $(-1, -m_1)$ instead of $(-1, -m_1 + 1)$, and $(-1, -1)$ instead of $(-1, 0)$. If $i \geq 0$ then we use $(0, -1)$. The case $j = \gamma$ follows similarly.

It follows that we can apply Theorem \[\text{(1)}\]. Then we see that if $n \geq 2$, then there is at least one secondary scrollar invariant having value $E - 1 = g/(\gamma - 1) - 2$. This is distinct from both $2g/(\gamma - 1) - 2$ and $g/(\gamma - 1) - 3$: contradiction. Therefore $n = 1$, i.e. $\Delta^{(1)}$ has the requested rectangular shape.

In conclusion, Question \[\text{(1)}\] also has a positive answer as soon as either of $\Delta^{(1)}, \Delta'^{(1)}$ is a standard rectangle.
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