On Synchronization: Comments on the paper ”Synchronization in scale-free dynamical networks: robustness and fragility”, IEEE Trans. Circuits Syst.
I 49 (1) (2002) 54-62

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Abstract—Synchronization problem for linear coupled networks is a hot topic in recent decade. However, until now, some confused concepts and results still puzzle people. To avoid further misleading researchers, it is necessary to point out these misunderstandings, correct these mistakes and give precise results.

Index Terms—Dynamical networks, Complex networks, linear coupling, stability, synchronization, consensus.

I. INTRODUCTION

In discussing synchronization of coupled systems, following concepts are most important and should be addressed precisely:

1) What is the synchronization and what is the synchronized state?
2) Can an individual trajectory \( \dot{s}(t) = f(s(t)) \) of the uncoupled system be the synchronized state of the coupled system?
3) What is the relationship between the stability of a trajectory of the uncoupled system and the stability of the synchronized state of the coupled system;
4) synchronization criteria of the coupled system.

In [1], the authors wrongly consider the synchronization of the coupled system as the stability of an individual trajectory of the uncoupled system. Based on this misunderstanding, the authors define the so called synchronized state inappropriately. Two criteria for the exponential stability of the so called synchronized state are given. Unfortunately, these two criteria are incorrect, too.

In this paper, we address this issue in detail, pointing out why the results given in [1] are incorrect. Furthermore, we clarify the differences and relations among the stability of the trajectory of uncoupled system, stability of the trajectory of coupled system and the synchronization of coupled system.

II. COMMENTS ON [1]

In the paper [1], the authors discussed the following coupled networks

\[
\dot{x}^i(t) = f(x^i(t)) + c \sum_{j=1}^{N} a_{ij} x^j(t) \quad i = 1, \ldots, N
\]

and its synchronization. Here, \( A = \{a_{ij}\}_{i,j=1}^{n} \in \mathbb{R}^{N \times N}, a_{ij} \geq 0, i \neq j \), \( a_{ii} = -\sum_{j \neq i} a_{ij} \) and assumed to be strongly connected, \( \Gamma = \text{diag}[\gamma_1, \ldots, \gamma_n] \).

The authors wrote in [1]:

Hereafter, the dynamical network is said to achieve (asymptotical) synchronization if as

\[
x^1(t) = x^2(t) = \cdots = x^N(t) = s(t), \quad t \to \infty
\]

where \( s(t) \in \mathbb{R}^n \) is a solution of an isolate node, namely

\[
\dot{s}(t) = f(s(t))
\]

Here, \( s(t) \) can be an equilibrium point, a periodic orbit, or a chaotic attractor. Clearly, stability of the synchronized states of network [1] is determined by the dynamics of an isolate node (function \( f \) and solution \( s(t) \)), the coupling strength \( c \), the inner linking matrix \( \Gamma \), and the coupling matrix \( A \).

First of all, mathematically, expression [2] is meaningless. It is our understanding that the authors want to say

\[
\lim_{t \to \infty} (x^i(t) - s(t)) = 0, \quad i = 1, \ldots, N
\]

Following lemmas (main results) are given in [1], too. Lemma 1. Consider the dynamical network [1]. Let

\[
0 = \lambda_1 > \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_N
\]

be the eigenvalues of its coupling matrix \( A \). If the following of \( (N-1) \)-dimensional linear time-varying systems

\[
\dot{w}(t) = (Df(s(t)) + c\lambda_2 \Gamma)w(t) \quad k = 2, \ldots, N
\]

are exponentially stable, then the synchronized states [2] are exponentially stable.

If \( s(t) = \bar{s} \) is an equilibrium point, then a necessary and sufficient condition for the synchronization stability is that the real parts of the eigenvalues of the matrix \( [Df(\bar{s}) + c\lambda_2 \Gamma] \) are all negative.

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This work was supported by the National Science Foundation of China under Grant No. 61203149.
Lemma 2. Consider the network (1). Suppose that there exists an $n \times n$ diagonal matrix $D > 0$ and two constants $\tau > 0$ and $\bar{d} < 0$, such that
\[ [Df(s(t)) + dI]T] D + D[Df(s(t)) + dI] \leq -\tau I_n \] (7)
for all $d < \bar{d}$. If
\[ c\lambda_2 \leq \bar{d} \] (8)
then the synchronized states (6) are exponentially stable.

Unfortunately, the claims given in two lemmas are incorrect.

In the following, we give detail explanations.

Lemma 1*. Consider the dynamical network (1). Let $\delta x(k, t) = x(k, t) - s(t)$, and the asymptotical (exponential) stability of the synchronized state $s(t)$ is equivalent to that $S(t)$ is an asymptotically (exponentially) stable solution of (9).

Let $\delta x(t)$ be the variation near $S(t)$, then
\[ \dot{\delta x}(t) = [I_N \otimes DF(s(t))]\delta x(t) + c(A \otimes \Gamma) \delta x(t) \] (10)
Moreover, write the Jordan decomposition as $\bar{A} = \Phi^T A \Phi$, $\Lambda = \text{diag} \{\lambda_1, \ldots, \lambda_N\}$, where $0 = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_N$, and $\delta u(t) = \Phi \delta x(t) = [\delta u^1(t) \top, \ldots, \delta u^N(t) \top] \top$, then
\[ \dot{\delta u}(t) = [I_N \otimes DF(s(t))]\delta u(t) + c(A \otimes \Gamma) \delta u(t) \] (11)
which also can be written as
\[ \dot{\delta u}(t) = [DF(s(t)) + \lambda_k \Gamma] \delta u^k(t), \quad k = 1, \ldots, N \] (12)
Thus, the asymptotical (exponential) stability of the trajectory $s(t)$ with respect to the coupled system (1) is equivalent to the all following “N” (not $N-1$) equations
\[ \dot{\delta u}(t) = (DF(s(t)) + \lambda_k \Gamma) \delta u(t) \quad k = 1, \ldots, N \] (13)
are asymptotically (exponentially) stable.

Therefore, Lemma 1 and Lemma 2 in (1) should be Lemma 1*. Let
\[ 0 = \lambda_1 > \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_N \] (14)
be the eigenvalues of its coupling matrix $A$. If the following of (N)-dimensional linear time-varying systems
\[ \dot{\delta u}(t) = (DF(s(t)) + \lambda_k \Gamma) \delta u(t) \quad k = 1, \ldots, N \] (15)
are exponentially stable, then the synchronized states (2) are exponentially stable.

Lemma 2*. Consider the network (1). Suppose that there exist an $n \times n$ diagonal matrix $D > 0$ and a constant $\tau > 0$, such that
\[ [Df(s(t))]^T D + D[Df(s(t))] \leq -\tau I_n \] (16)
then the synchronized states (6) are exponentially stable.

Furthermore, we can prove Lemma 1**. In case $\Gamma = I_n$, then the synchronized states (2) are asymptotically (exponentially) stable for the coupled system (1), it is necessary and sufficient that the uncoupled system
\[ \dot{\delta u}(t) = (DF(s(t)))\delta u(t) \] (17)
is asymptotically (exponentially) stable itself.

If $s(t) = \bar{s}$ is an equilibrium point, then a necessary and sufficient condition for the synchronization stability is that the real parts of the eigenvalues of the matrix $DF(\bar{s})$ are all negative.

In fact, any solution of
\[ \dot{\delta x}(t) = [I_N \otimes DF(s(t))]\delta x(t) + c(A \otimes I_n) \delta x(t) \] (10)
can be written as $\dot{\delta x}(t) = e^{[A \otimes I_n]t} \delta x(t)$. Here, $\delta x(t)$ satisfies the variational system near $S(t)$
\[ \dot{\delta x}(t) = [I_N \otimes DF(s(t))] \delta x(t) \] (18)
and is asymptotically (exponentially) stable.

From the asymptotical stability of (17), we have
\[ \lim_{t \to \infty} \delta x(t) = 0 \]
combining with $\delta x(t) = e^{[A \otimes I_n]t} \delta x(t)$ gives
\[ \lim_{t \to \infty} \delta x(t) = 0 \]
which implies
\[ \lim_{t \to \infty} (x(t) - S(t)) = 0 \]
and equivalently,
\[ \lim_{t \to \infty} (x^i(t) - s(t)) = 0, \quad i = 1, \ldots, N \]

Remark 1: It should be noted that in Lemma 1**, the condition $\Gamma = I_n$ plays key role in the proof. In case that $\Gamma = \text{diag} \{\gamma_1, \ldots, \gamma_n\}$ with some $\gamma_i \neq \gamma_j$, it is not yet known whether Lemma 1** is still true. The point is $DF(s(t)) \Gamma \neq DF(s(t))$.

Similarly, in case $s(t) = \bar{s}$ is an equilibrium point, even the real parts of the eigenvalues of the matrix $[DF(\bar{s}) + c\lambda_2 \Gamma]$ are all negative, we still can not derive the real parts of the eigenvalues of the matrix $[DF(\bar{s}) + c\lambda_2 \Gamma]$, $k = 3, \ldots, N$, are all negative, which also means that it is not yet known whether linear time-varying systems
\[ \dot{\delta u}(t) = (DF(\bar{s}) + c\lambda_k \Gamma) \delta u(t) \quad k = 3, \ldots, N \] (19)
are exponentially stable. Therefore, the claim made in Lemma 1 of (1): if $s(t) = \bar{s}$ is an equilibrium point, then a necessary and sufficient condition for $\bar{s}$ being stable is that the real parts of the eigenvalues of the matrix $[DF(\bar{s}) + c\lambda_2 \Gamma]$ are all negative is incorrect.

In the following, we will give a precise description of synchronization and correct results.

Definition 1: Synchronization subspace is the set composed of $S = \{(x^1, \ldots, x^m) \top : x^i = x^{j}, i, j = 1, \ldots, m\}$, where $x^i = [x^{i1}, \ldots, x^{in}] \top \in R^n, \quad i = 1, \ldots, m$.

Definition 2: (Local synchronization see (2), (3), (4)) If for some $\delta > 0$, such that in case the distance between $x(t)$ and $S$ at time $0, d(x(0), S) \leq \delta$, we have
\[ \lim_{t \to \infty} d(x(t), S) = 0, \quad i, j = 1, 2, \ldots, m \]
which implies then that the component in the transverse subspace \( \bar{x} \) is the component in the transverse subspace.

From Figure 1, it can be seen that synchronization means that the component in the transverse subspace \( \bar{x} \) is \( x(t) - \bar{X}(t) \to 0 \) as \( t \to \infty \), and \( \bar{x}(t) \) (not \( s(t) \)) is the synchronized state.

Let \( \delta \bar{x}(t) \) be the variation near \( \bar{x}(t) \), and \( \delta \bar{u}(t) = \Phi \delta \bar{x}(t) = [\delta \bar{u}^1(t)^T, \ldots, \delta \bar{u}^N(t)^T]^T \), then we have (see (24))

\[
\delta \bar{u}(t) = [I_N \otimes DF(\bar{u}(t))]\delta \bar{u}(t) + c(\Lambda \otimes \Gamma)\delta \bar{u}(t) \tag{20}
\]

and

\[
\delta \bar{u}^k(t) = [DF(\bar{x}(t)) + \lambda_k \Gamma]\delta \bar{u}^k(t), \quad k = 1, \ldots, N \tag{21}
\]

Different from

\[
\dot{\delta \bar{u}}^1(t) = [DF(s(t)) + \lambda_1 \Gamma]\delta \bar{u}^1(t) \neq 0 \tag{22}
\]

here, due to \( \delta \bar{u}^1(t) = 0 \), we have

\[
\dot{\delta \bar{u}}^1(t) = [DF(\bar{x}(t)) + \lambda_1 \Gamma]\delta \bar{u}^1(t) = 0 \tag{23}
\]

Thus, we can give

**Proposition 1:** Consider the dynamical network (7). Let

\[
\lambda_1 > \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_N \tag{24}
\]

be the eigenvalues of its coupling matrix \( A \). If the following \( N - 1 \)-dimensional linear time-varying systems

\[
\dot{w}(t) = (DF(\bar{x}(t)) + c\lambda_k \Gamma)w(t) \quad k = 2, \ldots, N \tag{25}
\]

are locally exponentially stable, then

\[
\|x(t) - \bar{x}(t)\| \leq Me^{-\alpha t}
\]

which implies \( \bar{x}(t) \) is the synchronized state.

**Remark 2:** It can be seen that the right side of the following equations

\[
\dot{x}^i(t) = f(x^i(t)) + c \sum_{j=1}^{N} a_{ij} \Gamma x^j(t) \quad i = 1, \ldots, N \tag{26}
\]

contains two terms. The coupling term \( c \sum_{j=1}^{N} a_{ij} \Gamma x^j(t) \) controls \( x(t) - \bar{X}(t) \). It is clear that the coupling term \( c \sum_{j=1}^{N} a_{ij} \Gamma x^j(t) \) does not contain any message of the synchronized state \( s(t) \), except the initial values \( x^i(0) \) are near \( s(0) \). Therefore, the coupling term does not play any role to make an unstable \( s(t) \) turn to be stable. Moreover, there are infinite \( s_\alpha(t) \) satisfying \( \dot{s}_\alpha(t) = f(s_\alpha(t)) \) with \( s_\alpha(0) \) being near \( s(0) \). Which one is the stable synchronized state defined in (11) for the coupled system (11)?

**Remark 3:** A basic prerequisite condition using variation near \( s(t) \) is that all \( x^i(t), \quad i = 1, \ldots, N \), must be close to \( s(t) \). However, as stated above, under the condition (6), one can not prove that \( x^i(t) - s(t) \to 0, \quad i = 1, \ldots, N \). Therefore, variational analysis near \( s(t) \) can not be applied. In particular, it can not be used for chaotic oscillators.

**III. Numerical Examples**

In this section, we will give several examples to illustrate our claims.

**Example 1:** Consider the following coupled system:

\[
\begin{aligned}
\dot{x}^1(t) &= \tanh(x^1(t)) + (-x^1(t) + x^2(t)) \\
\dot{x}^2(t) &= \tanh(x^2(t)) + (x^1(t) - x^2(t))
\end{aligned} \tag{27}
\]

where the coupling matrix is \( A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \). Its eigenvalues are \( \lambda_1 = 0 \) and \( \lambda_2 = -2 \). \( f(s) = \tanh(s) \), and \( s = 0 \) is the unique equilibrium for \( \dot{s}(t) = f(s(t)) \).

\[
\dot{w}(t) = [DF(0) + \lambda_2]w(t) = -w(t) \tag{28}
\]

is stable, and

\[
\dot{w}(t) = DF(0)w(t) = w(t) \tag{29}
\]

is unstable.

**Numerical simulation** (Figure 2) shows even initial values \( x^1(0) = 0.01, \quad x^2(0) = 0.02 \) are chosen very close to 0. However, \( x^1(t) \to 0 \) and \( x^2(t) \to 0 \), as \( t \to \infty \). Therefore, only the stability of the system

\[
\dot{w}(t) = [DF(0) + \lambda_2]w(t) \tag{30}
\]

can not make the coupled system (27) synchronize to the equilibrium point “0” of the uncoupled system \( \dot{s}(t) = \tanh(s(t)) \).

On the other side, it is easy to see that \( DF(\bar{x}(t)) + \lambda_2 < -1 \). Thus,

\[
\dot{w}(t) = (DF(\bar{x}(t)) + \lambda_2)w(t) \tag{31}
\]

is stable. By Proposition 1 it can be concluded that \( x^1(t) - x^2(t) \to 0 \).
is stable, equilibrium points. In case that the equilibrium point
\( i = 1 \) can not make the unstable equilibrium point
\( x \) is unstable.

oscillators to illustrate our claims (see [4]). The initial values
\( x(0) = (0) \) are very close to \( \tilde{s} = 0 \), but when \( t \to \infty \), \( x^i(t) \to 0 \)
and \( x^2(t) \to 0 \). Instead,

\[
\dot{w}(t) = Df(2)w(t) = -w(t)
\]

is stable, \( x^1(t) \to 2 \) and \( x^2(t) \to 2 \). It means that only the

Example 2: Consider following coupled system

\[
\begin{aligned}
\dot{x}^1(t) &= f(x^1(t)) + (-x^1(t) + x^2(t)) \\
\dot{x}^2(t) &= f(x^2(t)) + (x^1(t) - x^2(t))
\end{aligned}
\]

(32)

where

\[
\begin{aligned}
f(x) &= x - 2r, \quad x \in [2r - 1, 2r + 1], \quad r \text{ is even} \\
f(x) &= -(x - 2r), \quad x \in [2r - 1, 2r + 1], \quad r \text{ is odd}
\end{aligned}
\]

(33)

and system \( \dot{s}(t) = f(s(t)) \) has multiple equilibria \( \tilde{s} = 2r \).

It can be seen that

\[
\dot{w}(t) = [Df(0) + \lambda_2]w(t) = -w(t)
\]

is stable, while

\[
\dot{w}(t) = Df(0)w(t) = w(t)
\]

is unstable.

Simulation also shows that even \( x^1(0) = 0.05, x^2(0) = 0.15 \) are very close to \( \tilde{s} = 0 \), but when \( t \to \infty \), \( x^1(t) \to 0 \)
and \( x^2(t) \to 0 \). Instead,

\[
\dot{w}(t) = Df(2)w(t) = -w(t)
\]

(36)

is stable, \( x^1(t) \to 2 \) and \( x^2(t) \to 2 \). It means that only the

stability of the system

\[
\dot{w}(t) = [Df(0) + \lambda_2]w(t) = -w(t)
\]

(37)

can not make the unstable equilibrium point "0" of the
uncoupled system turn to be a synchronized state of the
coupled system.

The uncoupled system in the first example has a single

equilibrium point and in the second example has multiple

equilibrium points. In case that the equilibrium point \( \tilde{s} \) is not
locally stable for the uncoupled system, the trajectories \( x^i(t) \),

\( i = 1, \ldots, N \), of the coupled system \( (7) \) will not converge to
the equilibrium point (the synchronized state \( \tilde{s} \) defined in \( (7) \)).

In the following, we give a coupled system of chaotic
oscillators to illustrate our claims (see [4]). The initial values
\( x_i(0), \quad i = 1, \ldots, n \), are assumed near \( s(0) \). Simulations show
that the coupled system can reach synchronization, but the
synchronized state is not the trajectory of the uncoupled system
\( s(t) \).

Example 3: Consider a coupled system with seven
Chua’s chaotic neural networks

\[
\frac{dx^i}{dt} = -Dx^i(t) + Tg(x^i(t)) + \sum_{j=1}^{7} a_{ij}x^j(t), \quad i = 1, \ldots, 7
\]

(38)

here, \( x^i = (x^i_1, x^i_2, x^i_3)^\top \in \mathbb{R}^3 \), \( D = I_3 \),

\[
T = \begin{bmatrix}
1.2500 & -3.200 & -3.200 \\
-3.200 & 1.1000 & -4.4000 \\
-3.200 & 4.4000 & 1.000
\end{bmatrix}
\]

\[
g(x^i) = (g(x^i_1), g(x^i_2), g(x^i_3)), \quad g(s) = (|s+1|-|s-1|)/2, \quad A = (a_{ij}),
\]

where

\[
a_{ij} = \begin{cases}
1 & i \neq j \\
-6 & i = j
\end{cases} \quad \text{for } i = 1, 2, \ldots, 7
\]

(39)

\( s(t) \) is a solution of uncoupled system with initial value \( s(0) = [0.1, 0.1, 0.1]^\top \).

The initial value for the coupled system are assumed to be

\( x^i_j(0) = 0.1 + \delta x^i_j(0) \), where \( \|\delta x^i(0)\| \leq 0.1, \quad i = 1, 2, \ldots, 7 \).

Define \( K \) as

\[
\frac{1}{7} \sum_{i=1}^{7} < \|x^i(t) - \bar{x}(t)\| > \quad \text{and} \quad W = \frac{1}{7} \sum_{i=1}^{7} < \|x^i(t) - s(t)\| > \quad \text{where} \quad < \cdot \cdot \cdot > \quad \text{denotes average with time}.
\]

Figure 4 shows the first component of the different syn-
chronized states with different perturbations. It is clear that
the synchronized states heavily depend on the initial value, small perturbation of initial value leads to serious change of the synchronized states. Figure 5 shows that $K$ converges to 0, which means that the synchronization manifold is stable; instead, Figure 6 shows that $W$ does not converges to zero, which means that $x_i(t) - s(t) \to 0$. Therefore, even $x_i(0)$ are very close to $s(0)$ and the coupled system can synchronize, but $s(t)$ is not the synchronized trajectory defined in 2.

IV. Conclusions

In summary, we conclude

- The authors of [1] misunderstand the synchronization by considering synchronization of linear coupled system as asymptotically stable of some solution of uncoupled system.
- It can be seen (see the Figure 2) that

$$x(t) - S(t) = [x(t) - \bar{X}(t)] + [\bar{X}(t) - S(t)]$$

From previous derivation, the stability of following $N-1$ dimensional linear time-varying systems

$$\frac{dw(t)}{dt} = (Df(\bar{x}(t)) + c\lambda_k \Gamma)w(t) \quad k = 2, \cdots, N$$

leads to $x(t) - \bar{X}(t) \to 0$. i.e., the coupled system 2 can reach synchronization and the synchronized state is $\bar{X}(t)$. That means that the coupling term in 1 or 2 (the eigenvalues $\lambda_2, \cdots, \lambda_N$) is used to control $x(t) - \bar{X}(t)$. And the stability of the following system

$$\frac{dw(t)}{dt} = Df(s(t))w(t) \quad (39)$$

leads to $\bar{X}(t) - S(t) \to 0$.

The condition that $N-1$ systems

$$\dot{w}(t) = (Df(s(t)) + c\lambda_k \Gamma)w(t) \quad k = 2, \cdots, N \quad (40)$$

are stable can not lead to $x(t) - S(t) \to 0$.

- The synchronized state $\bar{X}(t)$ depends on initial value $x(0)$ heavily. Any prescribed state $\dot{s}(t) = f(s(t))$ is never asymptotically stable for the coupled system, unless $\dot{s}(t) = f(s(t))$ is asymptotically stable itself.
- There are three possibilities of the dynamical behaviors for the uncoupled system $\dot{x}(t) = f(x(t))$:

1) $\dot{s}(t) = f(s(t))$ is asymptotically stable, then under very mild condition (for example, $\Gamma = I_n$), for the coupled system 2

$$x_i(t) - s(t) \to 0, \quad i = 1, \cdots, N.$$  

2) $f = 0$, and the system $\dot{s}(t) = 0$ is neutral stable. For any initial value $x_i(0)$, $i = 1, \cdots, N$, $x_i(t)$ converge to a consensus $\frac{1}{N} \sum_{i=1}^{N} x_i(0)$. But this consensus value is also neutral stable. It is not asymptotically stable. Small perturbation of the initial value will make the different consensus value and will never return.

3) $\dot{s}(t) = f(s(t))$ is unstable, in particular, it is chaotic, any prescribed solution $s(t)$ of the uncoupled system $s(t) = f(s(t))$ is not a synchronized state for the coupled system 2.

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