A dynamical-scanning inner product based on Euclidean inner product

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Abstract. Euclidean dot product plays an important role in data analysis and relation comparison for its intuitive properties. However, for some complicated structures, for example, time series, it would be insufficient to construct a complete theory via this product. In order to resolve and accommodate these issues, we devise a dynamical-scanning inner product which would take the interaction, in particular in the sense of time shift, between two vectors into consideration and yield a relation that contains much more information. Though this device is based on Euclidean dot product, it is more much suitable and flexible for handling some complicated mathematical objects. In addition, we also construct a dynamical-scanning inner product for function spaces. In the end, we show how to apply our devices on both matrix representations and time series. These exploration might delve into some intrinsic properties of some mathematical concepts or models.

1. Introduction

Inner products basically represent a relationship between two vectors. They capture a huge chunk of mathematical properties, in particular linear ones, and widely used in Fourier analysis [1], Banach space, Hilbert space and spectrum theory [2, 3]. Even its weak versions have various applications [4, 5, 6]. It has also been studied and applied in fuzzy theory [7]. There are many forms of inner product. One of the highly used inner products is the Euclidean inner product. Though Euclidean inner product per se has been used for a century, it is cumbersome in dealing with some complex mathematical objects, for example, time series and signal processing. In this article, we will further expand Euclidean dot product, which is static in essence, into some dynamical-scanning inner product (DSIP), which could further capture some complex structures of mathematical objects. DSIP will scan through every parts of the given two vectors and capture the interaction between them via both directional scanning. In other words, DSIP would capture the interactions between the sub-vectors of each vector via two directions. The main property of DSIP is it is dynamical and comprehensive, rather static. The contribution of our DSIP could be applied to explore the properties of some mathematically-based phenomena, for example, data analysis, signal processing, times series, etc.

2. Notations

For any set $S$, we use $|S|$ to denote the cardinality of $S$. Let $E = (\mathbb{R}^n, \cdot)$ denote the real Euclidean space, where the symbol $\cdot$ denotes the Euclidean dot product. Let $\vec{v}$ denote a vector
(v_1, v_2, ..., v_n) \in \mathbb{R}^n$, and \( \vec{0} \) denote the zero vector. Let vectors \( \vec{a}, \vec{b}, \text{ and } \vec{c} \in \mathbb{R}^n \) be arbitrary. Let \( \langle , \rangle_n \) denote an arbitrary inner product over \( \mathbb{R}^n \). We use \( \vec{a}[i : j] \) (where \( i < j \)) to denote the reduced vector (or sub-vector) \( (a_i, a_{i+1}, ..., a_{j-1}, a_j) \). For example, if \( \vec{a} = (1, -2, 3, 9, 4, 5) \), then the sub-vector \( \vec{a}[2 : 4] = (-2, 3, 9) \). In order to introduce a dynamical-scanning inner product, we define the following product over \( \mathbb{R}^n \) first.

**Definition 2.1.**

\[
\langle \langle \vec{a}, \vec{b} \rangle \rangle \equiv \sum_{k=n}^{1} \langle \vec{a}[1 : k], \vec{b}[n - (k - 1) : n] \rangle_k + \sum_{h=1}^{n} \langle \vec{a}[h : n], \vec{b}[1 : n - (h - 1)] \rangle_{n-(h-1)}.
\]

The first part \( \sum_{k=n}^{1} \langle \vec{a}[1 : k], \vec{b}[n - (k - 1) : n] \rangle_k \) could be viewed as \( \vec{a} \) is motionless while \( \vec{b} \) is set correspondingly as \( \vec{a} \) and then moving (or scanning in this article) toward the head of \( \vec{a} \), in which the inner products of the overlapped sub-vectors are added up. The second part \( \sum_{h=1}^{n} \langle \vec{a}[h : n], \vec{b}[1 : n - (h - 1)] \rangle_{n-(h-1)} \) is the same process, with \( \vec{b} \) moving toward the tail of \( \vec{a} \), in which the inner products of the overlapped sub-vectors are added up. The whole process is named a dynamical scanning in this article.

**Example 1.** Suppose \( n = 4, \vec{a} = (1, 2, -1, 0), \vec{b} = (-1, 0, 1, -2) \) and each \( \langle , \rangle_i \) is an Euclidean dot product for all \( 1 \leq i \leq n \). Then

\[
\sum_{k=n}^{1} \langle \vec{a}[1 : k], \vec{b}[n - (k - 1) : n] \rangle_k = \vec{a}[1 : 4] \cdot \vec{b}[1 : 4] + \vec{a}[1 : 3] \cdot \vec{b}[2 : 4] + \vec{a}[1 : 2] \cdot \vec{b}[3 : 4] + \vec{a}[1 : 1] \cdot \vec{b}[4 : 4] = -2 + 4 - 3 - 2 = -3;
\]

similarly, one has \( \sum_{h=1}^{n} \langle \vec{a}[h : n], \vec{b}[1 : n - (h - 1)] \rangle_{n-(h-1)} = -6 \). Hence \( \langle \langle \vec{a}, \vec{b} \rangle \rangle = -6 \).

**Claim 1.** The defined product \( \langle \langle , \rangle \rangle \) in Definition 2.1 satisfies the properties symmetry and linearity.

**Proof.** By Definition 2.1, and the symmetry of each inner product \( \langle , \rangle_k \), one has the results.

Though satisfying symmetry and linearity properties, this product fails the positive definiteness. In order to complete it as an inner product, we need to further specify each \( \langle , \rangle_k \). The simple solution is to choose each \( \langle , \rangle_k \) as the Euclidean dot product for each \( 1 \leq k \leq n \).

**Claim 2.** (Euclidean-based) If \( \langle , \rangle_k \) is an Euclidean dot product for each \( 1 \leq k \leq n \) over \( \mathbb{R}^k \), then \( \langle \langle \vec{a}, \vec{b} \rangle \rangle_s \equiv \langle \langle \vec{a}, \vec{b} \rangle \rangle = \left( \sum_{i=1}^{n} a_i \right) \cdot \left( \sum_{j=1}^{n} b_j \right) + \sum_{k=1}^{n} a_k \cdot b_k \).
Proof. By Definition 2.1, one has

\[
\langle\langle \vec{a}, \vec{b} \rangle\rangle = \sum_{k=n}^{1} \vec{a}[1 : k] \bullet \vec{b}[n - (k - 1) : n] + \sum_{h=1}^{n} \vec{a}[h : n] \bullet \vec{b}[1 : n - (h - 1)]
\]

\[
= \sum_{k=n}^{1} \sum_{i=1}^{k} a_i \cdot b_{i+n-k} + \sum_{h=1}^{n} \sum_{i=1}^{h} a_{i+n-h} \cdot b_i
\]

\[
= \left( \sum_{i=1}^{n} a_i \right) \cdot \left( \sum_{j=1}^{n} b_j \right) + \sum_{k=1}^{n} a_k \cdot b_k.
\]

From now on, we use \( \langle\langle \rangle\rangle_s \) to denote this Euclidean-based dynamical-scanning inner product.

**Lemma 2.1.** \( \langle\langle \rangle\rangle_s \) is an inner product on \( \mathbb{R}^n \).

**Proof.** By Claim 1, it suffices to show \( \langle\langle \vec{a}, \vec{a} \rangle\rangle_s = 0 \) if and only if \( \vec{a} = \vec{0} \). This, by Claim 2, follows immediately from simple arithmetic.

**Corollary 1.** \( \langle\langle \vec{a}, \vec{b} \rangle\rangle_s = \vec{a} \bullet \vec{b} + \left( \sum_{i=1}^{n} a_i \right) \cdot \left( \sum_{j=1}^{n} b_j \right) \).

**Proof.** It follows immediately from above lemma.

**Example 2.** Let us use this equation to double check the result of Example 1. \( \langle\langle \vec{a}, \vec{b} \rangle\rangle_s = -2 + 2 \cdot (-2) = -6 \).

Let \( ||\vec{a}||_E \) denote the norm defined by Euclidean dot product. Let \( d_E(\vec{a}, \vec{b}) \) denote the distance function defined by the norm \( ||\vec{a} - \vec{b}||_E \). Now let us introduce a dynamical-scanning norm (or \( s \)-norm or \( ||\cdot||_s \)) and a dynamical-scanning metric (or \( s \)-metric or \( d_s \)) based on \( \langle\langle \rangle\rangle_s \).

**Definition 2.2.** (\( s \)-norm) \( ||\vec{a}||_s := \sqrt{\langle\langle \vec{a}, \vec{a} \rangle\rangle_s} \).

**Definition 2.3.** (\( s \)-metric) \( d_s(\vec{a}, \vec{b}) := ||\vec{a} - \vec{b}||_s \).

**Claim 3.** (i) \( d_s(\vec{a}, \vec{b}) \geq d_E(\vec{a}, \vec{b}) \);

(ii) \( d_s(\vec{a}, \vec{b}) = d_E(\vec{a}, \vec{b}) \) if and only if \( \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i \).

**Proof.** By the above definitions and Claim 2,

\[
d_s(\vec{a}, \vec{b}) = ||\vec{a} - \vec{b}||_s = \sqrt{\sum_{i=1}^{n} (a_i - b_i)^2 + ||\vec{a} - \vec{b}||_E^2}
\]

\[
= \sqrt{\sum_{i=1}^{n} (a_i - b_i)^2 + d_E^2(\vec{a}, \vec{b})} \geq d_E(\vec{a}, \vec{b}),
\]

and \( d_s(\vec{a}, \vec{b}) = d_E(\vec{a}, \vec{b}) \) if and only if \( \sum_{i=1}^{n} (a_i - b_i)^2 = 0 \), i.e., \( \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i \).
3. Function spaces
Let $T$ denote a finite totally ordered set. Let $|T| = n$ and $T = \{t_i\}_{i=1}^{n}$. Let us assume $t_1 < t_2 < t_3 < ... < t_n$. Let $[T]$ denote all the function whose domain is $T$ and whose range is $\mathbb{R}^n$, i.e., $[T] = T \rightarrow \mathbb{R}$. In this section, we will define an inner product $\langle \cdot , \cdot \rangle_T$ for $[T]$ based on $\langle \cdot , \cdot \rangle_s$.

Claim 4. $[T]$ is a vector space whose multiplication by scalars, addition and subtraction are all defined by the usual function operations.

Proof. The result follows immediately from the definitions.

Definition 3.1. Let $f(T)$ denote the vector $(f(t_1), f(t_2), ..., f(t_n)) \in \mathbb{R}^n$.

Definition 3.2. $(s)$-induced function product

\[
\langle f, g \rangle_T := \langle \langle f(T), g(T) \rangle \rangle_s.
\]

Indeed this definition could also applied for Euclidean dot product, i.e., $\langle f, g \rangle_T := \langle f(T), g(T) \rangle_E$. Here we focus solely on our dynamical-scanning inner product.

Claim 5. $\langle f, g \rangle_T$ is an inner product on $[T]$.

Proof. The first three properties for being an inner product follow immediately from the properties of $\langle \langle f(T), g(T) \rangle \rangle_s$ deduced in Lemma 2.1. Here we show the linearity property. Let $\alpha_1, \alpha_2 \in \mathbb{R}$ be arbitrary. Let $f_1, f_2 \in [T]$ be arbitrary. By Definition 3.2, the operations of functions and Lemma 2.1

\[
\langle \alpha_1 \cdot f_1 + \alpha_2 \cdot f_2, g \rangle_T = \langle \langle \alpha_1 \cdot f_1 + \alpha_2 \cdot f_2(T), g(T) \rangle \rangle_s = \langle \langle \alpha_1 \cdot f_1(T) + \alpha_2 \cdot f_2(T), g(T) \rangle \rangle_s = \langle \langle \alpha_1 \cdot f_1(T), g(T) \rangle \rangle_s + \langle \langle \alpha_2 \cdot f_2(T), g(T) \rangle \rangle_s = \alpha_1 \cdot \langle f_1, g \rangle_T + \alpha_2 \cdot \langle f_2, g \rangle_T.
\]

Now we extend this result to a product vector space $[T] \times [T] \equiv [T]^2$. Let $\alpha \in \mathbb{R}$ be arbitrary. Let $|T| = n$. Let $f, g, f_1, f_2, g_1, g_2 \in [T]$ be arbitrary. We could then define the values and range of this product function via column vectors and matrices.

Definition 3.3. $f \times g, \alpha \cdot (f \times g) : T \rightarrow \mathbb{R}^{2 \times 1}$ are respectively defined by

\[
(f \times g)(t) := \begin{bmatrix} f(t) \\ g(t) \end{bmatrix}, \alpha \cdot (f \times g)(t) := \begin{bmatrix} \alpha \cdot f(t) \\ \alpha \cdot g(t) \end{bmatrix}.
\]

Definition 3.4. $(f \times g)(T) := \begin{bmatrix} f(T) \\ g(T) \end{bmatrix} \in \mathbb{R}^{2 \times n}$.

Definition 3.5. $\alpha \cdot (f \times g)(T) := \begin{bmatrix} \alpha \cdot f(T) \\ \alpha \cdot g(T) \end{bmatrix} \in \mathbb{R}^{2 \times n}$.

Now we define an inner product on $[T]^2$.

Definition 3.6. $(s)$-induced function product

\[
\langle f_1 \times f_2, g_1 \times g_2 \rangle_T := \langle \langle f_1(T), g_1(T) \rangle \rangle_s + \langle \langle f_2(T), g_2(T) \rangle \rangle_s.
\]
By the properties of function operations, we could easily obtain the following results.

**Claim 6.**

(i) \((f_1 \times f_2 + g_1 \times g_2) = (f_1 + g_1) \times (f_2 + g_2)\);

(ii) \([\alpha \cdot (f \times g)] = (\alpha \cdot f) \times (\alpha \cdot g)\).

**Proof.** By the definition of function operations and Definition 3.3, one has for all \(t \in T\)

\[
(f_1 \times f_2 + g_1 \times g_2)(t) = (f_1 \times f_2)(t) + (g_1 \times g_2)(t)
\]

\[
= \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} + \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix} = \begin{bmatrix} (f_1 + g_1)(t) \\ (f_2 + g_2)(t) \end{bmatrix} = [(f_1 + g_1) \times (f_2 + g_2)](t).
\]

Now we show the other statement.

\[
[\alpha \cdot (f \times g)](t) = \alpha \cdot [(f \times g)(t)] = \alpha \cdot \begin{bmatrix} f(t) \\ g(t) \end{bmatrix}
\]

\[
= \begin{bmatrix} (\alpha \cdot f)(t) \\ (\alpha \cdot g)(t) \end{bmatrix} = [(\alpha \cdot f) \times (\alpha \cdot g)](t).
\]

**Theorem 3.1.** \(\langle f_1 \times f_2, g_1 \times g_2 \rangle_T\) is an inner product on \([T]^2\).

**Proof.** By Definition 3.6 and Lemma 2.1,

\[
\langle f_1 \times f_2, f_1 \times f_2 \rangle_T := \langle \langle f_1(T), f_1(T) \rangle \rangle_s + \langle \langle f_2(T), f_2(T) \rangle \rangle_s = 0
\]

if and only if \(f_1(T) = f_2(T) = 0\), i.e., \(f_1 = f_2 = 0\). Moreover, by the symmetry of \(\langle , \rangle_s\), one could show the symmetry of \(\langle , \rangle_T\). Now we show the linearity property. By Definition 3.3, Claim 6, Definition 3.6, and the properties of dynamical-scanning product, we have the following inferences:

\[
\langle \alpha_1 \cdot (f_1 \times g_1) + \alpha_2 \cdot (f_2 \times g_2), f \times g \rangle_T
\]

\[
= \langle \langle \alpha_1 \cdot f_1 \times \alpha_1 \cdot g_1 \rangle \rangle_s + \langle \langle \alpha_2 \cdot f_2 \times \alpha_2 \cdot g_2 \rangle \rangle_s
\]

\[
= \langle \langle \alpha_1 \cdot f_1 + \alpha_2 \cdot f_2 \rangle \rangle_s + \langle \langle \alpha_1 \cdot g_1 + \alpha_2 \cdot g_2 \rangle \rangle_s
\]

\[
= \alpha_1 \cdot \langle \langle f_1(T), f(T) \rangle \rangle_s + \langle \langle g_1(T), g(T) \rangle \rangle_s
\]

\[
+ \alpha_2 \cdot \langle \langle f_2(T), f(T) \rangle \rangle_s + \langle \langle g_2(T), g(T) \rangle \rangle_s
\]

\[
= \alpha_1 \cdot \langle f_1 \times g_1, f \times g \rangle_T + \alpha_2 \cdot \langle f_2 \times g_2, f \times g \rangle_T.
\]

Now we could extend these definitions and properties to any finite products. Let \([T]^m\) denote the function vector space \(\prod_{i=1}^m [T]\). Let function vector \(\vec{f} \in [T]^m\) be arbitrary, i.e., \(\vec{f}\) consists of an array of real-valued functions whose domain is \(T\).

**Definition 3.7.** (finite function product) \(\vec{f} : T \rightarrow \mathbb{R}^{m \times 1}\) is defined by

\[
\vec{f}(t) := \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_m(t) \end{bmatrix}, \quad \alpha \cdot \vec{f}(t) := \begin{bmatrix} \alpha \cdot f_1(t) \\ \alpha \cdot f_2(t) \\ \vdots \\ \alpha \cdot f_m(t) \end{bmatrix}.
\]
Definition 3.8. \( \vec{f}(T) \equiv \begin{bmatrix} f_1(T) \\ f_2(T) \\ \vdots \\ f_m(T) \end{bmatrix}, \) and \((\alpha \cdot \vec{f})(T) \equiv \begin{bmatrix} (\alpha \cdot f_1)(T) \\ (\alpha \cdot f_2)(T) \\ \vdots \\ (\alpha \cdot f_m)(T) \end{bmatrix}.\)

Definition 3.9. (s-induced function product)

\[
\langle \vec{f}, \vec{g} \rangle_T := \sum_{i=1}^{n} \langle f_i(T), g_i(T) \rangle_s.
\]

Theorem 3.2. \( \langle \vec{f}, \vec{g} \rangle_T \) is an inner product on \([T]^m\).

Proof. This could be proved by similar approaches adopted for the proofs in Theorem 3.1, when \( m = 2 \).

4. Application: dynamical matrix inner product

Suppose \( A, B \in \mathbb{R}^{m \times n} \) are an arbitrary matrices. We would like to apply our theorems on the inner products for matrices. Recall that the usual inner product of matrices, in which the inner product of \( A \) and \( B \) - we name it a static inner product - is defined by \( A \bullet B := \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \cdot b_{ij} \). In order to extend it to a dynamical-scanning inner product, we have the following representation of matrices via function vectors. We identify \( A \) with a function \( f_i : T \equiv \{1, 2, \ldots, n\} \to \mathbb{R} \) defined by \( f_i(j) := a_{ij} \), where \( a_{ij} \) is the element located in \( i \)-th row and \( j \)-th column of matrix \( A \). Hence \( A \) could be represented by a function vector \( \vec{f}^A = (f_1^A, f_2^A, \ldots, f_m^A) \in [T]^m \). Based on this, one could then define a dynamical-scanning inner product \( \langle \cdot \rangle_{MX} \) for matrices:

Definition 4.1. Define the inner product \( \langle \cdot \rangle_{MX} : \mathbb{R}^{m \times n} \times \mathbb{R}^{m \times n} \to \mathbb{R} \) by

\[
\langle A, B \rangle_{MX} := \langle \vec{f}^A, \vec{f}^B \rangle_T = \sum_{i=1}^{m} \langle (f_i^A(T), f_i^B(T)) \rangle_s.
\]

This product is not the usual static inner product, but a dynamical-scanning inner product defined in this article.

Example 3. Suppose \( A = \begin{bmatrix} -1 & 2 & -3 & 0 \\ 1 & -4 & 2 & 1 \end{bmatrix} \) and \( B = \begin{bmatrix} 4 & 1 & -1 & 2 \\ 1 & 0 & -2 & -3 \\ -2 & -2 & 2 & 0 \end{bmatrix} \). Then the usual static inner product \( A \bullet B = \sum_{i=1}^{3} \sum_{j=1}^{4} a_{ij} \cdot b_{ij} = -3 \) and \( T = \{1, 2, 3, 4\} \). Since \( f_i^A(T) = (-1, 2, -3, 0) \) and \( f_i^B(T) = (4, 1, -1, 2) \), by Claim 2, one has \( \langle (f_1^A(T), f_1^B(T)) \rangle_s = (-2) \cdot 6 + [-4 + 2 + 3 + 0] = -11 \); similarly, \( \langle (f_2^A(T), f_2^B(T)) \rangle_s = -6 \) and \( \langle (f_3^A(T), f_3^B(T)) \rangle_s = 2 \). Hence by Definition 4.1, one has \( \langle A, B \rangle_{MX} = \langle \vec{f}^A, \vec{f}^B \rangle_T = \sum_{i=1}^{m} \langle (f_i^A(T), f_i^B(T)) \rangle_s = -15 \).

5. Conclusions

We have shown how to devise a dynamical-scanning inner product based on the typical Euclidean dot product. This new device is much more suitable and flexible in handling some complex mathematical structures. We also devise a dynamical-scanning inner product for function spaces. In addition, we show how to apply our devices via matrix representations.
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