Hawking radiation from acoustic black holes in two space dimensions

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Abstract

We study the Hawking radiation for the acoustic black hole. In the beginning we follow the outline of T.Jacobson but then we use a different $2 + 1$ vacuum state similar to the vacuum state constructed by W.Unruh. We also use a special form of the wave packets. The focus of the paper is to treat the 2 dimensional case, in particular, the case when the radial and angular velocity are variable.

1 Introduction

Consider the acoustic wave equation of the form

\begin{equation}
\Box g u(x_0, x) = \frac{1}{\sqrt{g(x)}} \sum_{j,k=0}^{2} \frac{\partial}{\partial x_j} \left( \sqrt{g} g^{jk}(x) \frac{\partial u(x_0, x)}{\partial x_k} \right) = 0,
\end{equation}

where $x_0 \in \mathbb{R}$ is the time variable, $x = (x_1, x_2) \in \mathbb{R}^2$, $g(x) = (\det[g^{jk}]_{j,k=0}^{2})^{-1}$,

\begin{align}
g^{00} &= 1, & g^{0j} &= g^{j0} = v^j, & 1 \leq j \leq 2, \\
g^{jk} &= (-\delta_{ij} + v^j v^k), & 1 \leq j, k \leq 2,
\end{align}

$v = (v_1, v_2) = \frac{A}{\rho} \hat{x} + \frac{B}{\rho} \hat{\theta}$ is the velocity flow in a vortex, $\rho = |x|, \hat{x} = \frac{x}{|x|}$, 
$\hat{\theta} = \left( -\frac{x_2}{|x|}, \frac{x_1}{|x|} \right)$. We assume, for the simplicity, that the density and the sound speed are equal to 1. We assume also that

\begin{equation}
A < 0, \quad B \neq 0.
\end{equation}
The equation (1.1) describes the wave propagation in the moving fluid with velocity \((v_1, v_2) = \frac{A}{\rho} \hat{x} + \frac{B}{\rho} \hat{\theta}\) (cf. [27]). The metric corresponding to (1.1) has the form

\[
ds^2 = dx_0^2 - (dx - v dx_0)^2,
\]

where \(x = (x_1, x_2), v = (v_1, v_2)\).

It is a Lorentzian metric in \(\mathbb{R}^2 \times \mathbb{R}\) called the acoustic metric. It is one of many examples of analogue gravity metrics (see, for example, the survey [2]) where many others physical examples of analogue gravity are given).

In the general relativity the metric is a solution of the Einstein equation and this sets the analogue metrics apart. From other side analogue metrics exhibit many properties of metrics in general relativity, in particular, they both may have black holes. This fact spurred the interest of physicists in the theoretical and experimental studies of analogue gravity (cf. [16], [22], [26], [27]).

The black hole, by the definition, is a domain in a spacetime where the signals or particles can not escape from. It was a remarkable discovery of S.Hawking that when quantum effects are added, the black hole emits particles ([13]). This phenomenon is called the Hawking radiation and there is a large literature devoted to this subject (cf. [1], [4], [5], [12], [16], [17], [26]). Hawking radiation holds for analogue black holes too and this ignites an additional interest in Analogue Gravity. Note that it is possible to demonstrate the Hawking radiation for analogue black holes while the experimental verification of Hawking radiation in general relativity is not realistic. The pioneering work in this direction was done by W.Unruh in [22]. For more recent experimental results see [3], [10], [20], [21], [29].

Hawking radiation from acoustic black holes was considered in [25], [30].

In all preceding works on Hawking radiation the case of \(1 + 1\) dimensions or spherically symmetric case in \(3 + 1\) dimensions were studied.

The focus of this paper is to investigate the Hawking radiation in \(2 + 1\) dimensions, primary the rotating acoustic black holes with variable radial and angular velocities.

Note that classical acoustic black hole with the variables velocity were treated in [6], [7], [18], [19], [28].

As an introduction to the quantum field theory on curved spacetimes we use the lecture notes of T.Jacobson [15] (see also [14] and references there).

The main departure from [15] in this paper is a new definition of the vacuum state that follows the definition of the Unruh vacuum (cf. [23], [24]).
Another novelty is a special construction of wave packets that is used in the computation of the Hawking radiation. For comparison, K. Fredenhagen and R. Haag [11] in one of few rigorous derivation of Hawking radiation used a limiting procedure when $x_0 \to -\infty$ to obtain the expression for the Hawking radiation. Using special wave packets we avoid taking the time limit. Instead, we use the limit $a \to \infty$ of some parameter $a$ that characterizes the closeness of the wave packet to the black hole.

The plan of the paper is the following:

In §2 we describe needed facts from the quantum field theory on curved spacetimes and define the new vacuum state.

In §3 we study a more simple case of the acoustic black hole when $A < 0$ and $B \neq 0$ are constant. We introduce a wave packets of a special form and compute the Hawking radiation produced by such wave packets.

Finally, in §4 we consider the acoustic black holes with the variable velocity. In §4.1 we study the Hawking radiation in the case when $A < 0$ is constant and $|B(\varphi)| > 0$ is periodic in $\varphi$ and in §4.2 we study the case of general acoustic black hole where $A(\rho, \varphi) < 0, |B(\rho, \varphi)| > 0$.

Some $A(\rho, \varphi)$, $B(\rho, \varphi)$ have direct physical meaning (cf. [27]). In order for $A_\rho \hat x + B_\rho \hat \theta$ to represent a fluid flow there should be a harmonic function $\Psi(\rho, \varphi)$ such that $A = \rho \frac{\partial \Psi}{\partial \rho}$ and $B = \frac{\partial \Psi}{\partial \varphi}$ (see [27]). If we take $\Psi = A_0 \log \rho + B_0 \varphi + C_1 \rho \cos \varphi + C_2 \rho \sin \varphi$, then $A(\rho, \varphi) = A_0 + C_1 \rho \cos \varphi + C_2 \rho \sin \varphi$, $B(\rho, \varphi) = B_0 - C_1 \rho \sin \varphi + C_2 \rho \cos \varphi$ (cf. [7], §4).

Note that one can take any harmonic polynomials instead of $C_1 \rho \cos \varphi + C_2 \rho \sin \varphi$.

2 Elements of the second quantization

For the elements of the quantum field theory on the curved space-time used in this paper see the lectures notes of T. Jacobson [15] (see also [14] and the further references there).

Denote by $f_k^+(x_0, x)$ the solution of $\Box_g u = 0$ in $\mathbb{R}^2 \times \mathbb{R}$ with the initial conditions having the following form in polar coordinates $(\rho, \varphi)$:

$$f_k^+(x_0, x)|_{x_0=0} = \gamma_k e^{ik \cdot x},$$

where $k = (\eta_\rho, \eta_\varphi), x = (\rho, \varphi), k \cdot x = \eta_\rho \rho + \eta_\varphi \varphi,$

$$\frac{\partial f_k^+}{\partial x_0}|_{x_0=0} = i\lambda_\rho(k) \gamma_k e^{ik \cdot x},$$
where

\[(2.3) \quad \lambda_0^-(k) = -\frac{A}{\rho} \eta_\rho - \frac{B}{\rho^2} \eta_\varphi - \sqrt{\frac{\eta_\rho^2}{\rho^2} + \frac{\eta_\varphi^2}{d^2}}, \quad d \text{ is arbitrary.} \]

The normalization factor \(\gamma_k\) will be chosen later. Also denote by \(f_k^-(x_0, x)\) the solution of \(\Box_g u = 0\) with the initial condition

\[(2.4) \quad f_k^-(0, x) = \gamma_k e^{ik \cdot x}, \quad \frac{\partial f_k^-}{\partial x_0} \bigg|_{x_0=0} = i\lambda_0^+(k) \gamma_k e^{ik \cdot x}, \]

where \(\lambda_0^+(k) = -\frac{A}{\rho} \eta_\rho - \frac{B}{\rho^2} \eta_\varphi + \sqrt{\frac{\eta_\rho^2}{\rho^2} + \frac{\eta_\varphi^2}{d^2}}\). Note that

\[(2.5) \quad f_k^+(x_0, x) = f_{-k}(x_0, x), \]

since \(-\lambda^+(\rho, \eta_\rho, \varphi) = \lambda^-(\rho, -\eta_\rho, -\varphi)\) and \(\gamma_k = \gamma_{-k}\) is positive.

Let

\[(2.6) \quad < f, h > = i \int_{x_0=0} \sum_{j=0}^2 g^{ij} \left( \int f \frac{\partial h}{\partial x_j} - \frac{\partial f}{\partial x_j} h \right) dx_1 dx_2. \]

The bracket \((2.6)\) is called the Klein-Gordon (KG) inner product. Since the acoustic metric \((1.4)\) is stationary, the inner product \((2.6)\) is independent of \(t\) when \(f, h\) are solutions of \(\Box_g u = 0\) (cf. [15]). We have, in \((\rho, \varphi)\) coordinates:

\[< f_k^+, f_{k'}^+ > = i \int_{x_0=0} \left[ \left( \frac{\partial f_k^+}{\partial x_0} - \frac{\partial f_{k'}^+}{\partial x_0} \right) + \frac{A}{\rho} \left( \frac{\partial f_k^+}{\partial \rho} - \frac{\partial f_{k'}^+}{\partial \rho} \right) \right] \rho d\rho d\varphi \]

\[+ \frac{B}{\rho^2} \left( \frac{\partial f_k^+}{\partial \varphi} - \frac{\partial f_{k'}^+}{\partial \varphi} \right) \rho d\rho d\varphi \]

\[= i \int_{x_0=0} f_k^+ f_{k'}^+ \left( i\lambda_0^- (k') + i\frac{A}{\rho} \eta_\rho' + i\frac{B}{\rho^2} \eta_\varphi' + i\lambda_0^- (k) + i\frac{A}{\rho} \eta_\rho + i\frac{B}{\rho^2} \eta_\varphi \right) \rho d\rho d\varphi, \]

where \(k = (\eta_\rho, \eta_\varphi), k' = (\eta_\rho', \eta_\varphi').\) Note that \(\gamma_k\) is real-valued. Then derivatives of \(\gamma_k\) do not contribute to KG norm. Also note that \(\lambda_0^- (k) + \frac{A}{\rho} \eta_\rho + \frac{B}{\rho^2} \eta_\varphi = -\sqrt{\eta_\rho^2 + \frac{1}{d^2} \eta_\varphi^2}.\) Therefore

\[(2.7) \quad < f_k^+, f_{k'}^+ > = \int_{x_0=0} \left( \sqrt{\eta_\rho^2 + \frac{\eta_\varphi^2}{d^2}} + \sqrt{(\eta_\rho')^2 + \frac{(\eta_\varphi')^2}{d^2}} \right) \gamma_k \gamma_{k'} e^{-i(k-k') \cdot x} \rho d\rho d\varphi. \]
We choose a normalizing factor
\[ \gamma_k = \frac{1}{\sqrt{\sqrt{\pi r^2 + \frac{\pi}{4}}} \sqrt{2/(2\pi)^2}}. \]
Then \( < f_k^+, f_{k'}^+ > = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{-i(k-k')x} d\rho d\varphi = \delta(k-k') \). Analogously,
\[ < f_k^-, f_{k'}^- > = -\delta(k-k'), \quad < f_k^+, f_{k'}^- > = 0. \]
Note that \( \varphi \in [0, 2\pi) \) and \( \eta_{\varphi} = m \in \mathbb{Z} \). Therefore \( \int_{\mathbb{R}^2} \) is, in fact, a sum \( \sum_{m=-\infty}^{\infty} \) and \( \delta(\eta_{\varphi} - \eta_{\varphi'}) = \delta_{mm'} \), \( \delta_{mm'} = 1 \) when \( m = m' \) and \( \delta_{mm'} = 0 \) when \( m \neq m' \).

We shall continue to use \( \eta_{\varphi} \in \mathbb{R} \) as in (2.1)-(2.4) simply for the shortness of notations.

Let \( \Phi \) be the field operator, i.e.
\[ \Phi = \int_{\mathbb{R}^2} (\alpha^+_k f_k^+(x_0, x) - \alpha^-_k f_{-k}^-(x_0, x)) dk, \]
where
\[ \alpha^+_k = < f_k^+, \Phi >, \quad \alpha^-_k = (\alpha^+_k)^* = < f_{-k}^-, \Phi > \]
are the annihilations and creations operators, respectively.

It follows from (2.7), (2.8) (cf. [15]) that operators \( \alpha^+_k \) and \( \alpha^-_k \) satisfy the following commutation relations
\[ [\alpha^+_k, \alpha^-_{k'}] = \delta(k-k')I, \quad [\alpha^+_k, \alpha^+_k] = 0, \quad [\alpha^-_k, \alpha^-_{k'}] = 0, \quad \alpha^-_k = (\alpha^+_k)^*. \]

Let \( C(x_0, \rho, \varphi) \) be a solution of (1.1) with initial conditions at \( x_0 = 0 \) having a support in the exterior of the black hole \{\( \rho > |A| \}\). We shall call \( C \) a wave packet.

Expanding \( C \) in a basis \( f_k^+(x_0, x), f_{-k}^-(x_0, x) \) we have
\[ C = \int_{\mathbb{R}^2} (C^+(k)f_k^+(x_0, x) - C^-(k)f_{-k}^-(x_0, x)) dk, \]
where \( dk = d\eta_{\rho} d\eta_{\varphi} \). As it was mentioned above the integration in \( \eta_{\varphi} \) is indeed a summation \( \sum_{m=-\infty}^{\infty} \). In (2.12)
\[ C^+(k) = < f_k^+, C >, \quad C^-(k) = < f_{-k}^-, C >. \]
Note that integrals in (2.9) are understood in a distribution sense but $C^+(k), C^-(k)$ are decaying if the initial conditions for $C$ are smooth. It follows from (2.9), (2.12) that

\begin{equation}
< C, \Phi > = \int_{\mathbb{R}^2} (C^+(k)\alpha_+^k - C^-(k)\alpha_-^k)dk.
\end{equation}

It will be convenient to split $f_k^+(x_0, x)$ and $f_k^-(x_0, x)$ onto two parts

\begin{equation}
\begin{aligned}
f_k^{++} &= f_k^+\theta(\eta_\rho), \\
f_k^{+-} &= f_k^+(1 - \theta(\eta_\rho)).
\end{aligned}
\end{equation}

where $\theta(\eta_\rho) = 1$, for $\eta_\rho > 0$, $\theta(\eta_\rho) = 0$ for $\eta_\rho < 0$. Analogously

\begin{equation}
\begin{aligned}
f_k^{-+} &= f_k^-\theta(\eta_\rho), \\
f_k^{--} &= f_k^-(1 - \theta(\eta_\rho)).
\end{aligned}
\end{equation}

Remark 2.1. We shall describe the behavior of null-bicharacteristics starting at $x_0 = 0$ and corresponding to $f_k^{++}, f_k^{+-}, f_k^{-+}, f_k^{--}$ that will show a huge difference between $f_k^{++}, f_k^{--}$ and $f_k^{+-}, f_k^{-+}$.

The Hamiltonian corresponding to $f_k^+$ is $\lambda_0^+ = -\frac{\Lambda}{\rho}\eta_\rho - \frac{B}{\rho^2}\eta_\varphi - \sqrt{\eta_\rho^2 + \eta_\varphi^2}$ (cf. (2.3)).

Denote by $\gamma^{++}$ the null-bicharacteristic with the initial data $(\rho, \varphi, \lambda_0^+, \eta_\rho, \eta_\varphi)$ for $\eta_\rho > 0, \rho > |\Lambda|$, and by $\gamma^{+-}$ the null-bicharacteristic with the same initial data except $\eta_\rho < 0$.

It is not difficult to show (cf. [E8]) that $\gamma^{+-}$ crosses the event horizon $\{\rho = |\Lambda|\}$ when $x_0 \to +\infty$ and $\gamma^{++}$ approaches spiraling the event horizon when $x_0 \to -\infty$. One can say that $\gamma^{++}$ “emerges” from the black hole when $x_0$ increases.

Analogously, the Hamiltonian for $f_k^-$ is $\lambda_0^- = -\frac{\Lambda}{\rho}\eta_\rho - \frac{B}{\rho^2}\eta_\varphi + \sqrt{\eta_\rho^2 + \eta_\varphi^2}$ (cf. (2.4)).

If $\gamma^{+-}$ are the null-bicharacteristics starting at $(\rho, \varphi, \lambda_0^+, \eta_\rho, \eta_\varphi)$ where $\eta_\rho > 0$ for $\gamma^{++}$ and $\eta_\rho < 0$ for $\gamma^{+-}$, then $\gamma^{+}$ crosses the event horizon when $x_0 \to +\infty$ and $\gamma^{-}$ approaches spiraling the event horizon when $x_0 \to -\infty$ (cf. [E8]).

Therefore the null-bicharacteristics corresponding to $f_k^{++}(x)$ and $f_k^{--}(k)$, and the null-characteristics corresponding to $f_k^{+}(x)$ and $f_k^{-}(x)$ have drastically different behavior.
Using (2.15), (2.16) we can rewrite (2.12) in the following form

\[
C = \int_{\mathbb{R}^2} \left( (C^+)(k)f^+_{k} + C^-(k)f^-_{k} \right. \\
\left. - C^+(-k)f^+_{-k} - C^-(-k)f^-_{-k} \right) dk,
\]

where

\[
C^+ = C^+ \theta(\eta_{\rho}), \quad C^- = C^+ (1-\theta(\eta_{\rho})), \quad C^+ = C^+ \theta(\eta_{\rho}), \quad C^- = C^- (1-\theta(\eta_{\rho})).
\]

Denote

\[
C_{+}(\rho, \varphi) = \int_{\eta_{\rho}>0} C^+(k)f^+_{k}(0, x)dk - \int_{\eta_{\rho}<0} C^-(k)f^-_{k}(0, x)dk.
\]

Changing \( \eta_{\rho} \) to \(-\eta_{\rho}\) in the second integral in (2.18) we get

\[
C_{+}(\rho, \varphi) = \int_{0}^{\infty} \sum_{m=-\infty}^{\infty} C^{(1)}(k) \frac{e^{im\eta_{\rho} + im\varphi}}{\sqrt{\rho/\sqrt{\eta_{\rho}^2 + m^2}}} d\eta_{\rho}.
\]

Therefore \( \sqrt{\rho}C_{+}(\rho, \varphi) \) admits an analytic continuation in \( \rho \) to the half-plane \( \Im \rho > 0 \). Analogously

\[
C_{-}(\rho, \varphi) = \int_{\eta_{\rho}<0} C^+(k)f^+_{k}(0, x)dk - \int_{\eta_{\rho}>0} C^-(k)f^-_{k}(0, x)dk
\]

\[
= \int_{-\infty}^{0} \sum_{m=-\infty}^{\infty} C^{(1)}(k) \frac{e^{im\eta_{\rho} + im\varphi}}{\sqrt{\rho/\sqrt{\eta_{\rho}^2 + m^2}}} d\eta_{\rho}.
\]

Thus \( \sqrt{\rho}C_{-}(\rho, \varphi) \) has an analytic continuation in \( \rho \) to the half-plane \( \Im \rho < 0 \). Note that

\[
C(0, \rho, \varphi) = C_{+} + C_{-}.
\]

Hence \( \sqrt{\rho}C = \sqrt{\rho}C_{+} + \sqrt{\rho}C_{-} \). Therefore, by the well-known formula (cf., for example, [9]),

\[
C_{\pm} = \frac{\pm i}{2\pi} \int_{-\infty}^{\infty} \sqrt{\frac{\rho'}{\rho}} C(0, \rho', \varphi) d\rho'
\]
Using (2.15), (2.16) we can rewrite (2.14) in the form

\[(2.22) \quad < C, \Phi > = \int_{\mathbb{R}^2} (C^{++} \alpha_k^{++} + \overline{C}^{--} \alpha_k^{--} - \overline{C}^{--} \alpha_k^{++} - C^{++} \alpha_k^{--}) \, dk,\]

where

\[(2.23) \quad \alpha_k^{++} = < f_k^{++}, \Phi >, \quad \alpha_k^{+-} = < f_k^{+-}, \Phi >, \quad \alpha_k^{-+} = < f_k^{-+}, \Phi >, \quad \alpha_k^{--} = < f_k^{--}, \Phi >, \quad (\alpha_k^{++})^* = \alpha_k^{--}, \quad (\alpha_k^{+-})^* = \alpha_k^{-+}.

We shall define now the vacuum state |\Psi\rangle. In the case of Minkowski metric the vacuum space is defined by the conditions \(\alpha_k^{++}|0\rangle = 0, \alpha_k^{+-}|0\rangle = 0\) for all \(k\). It was emphasized by Unruh [22] and Jacobson [15] the need to modify the definition of the vacuum states in different situations. We shall define, similarly to [23], [24], the vacuum state |\Psi\rangle by the requirements

\[(2.24) \quad \alpha_k^{++} |\Psi\rangle = 0 \quad \text{for all} \quad k = (\eta_\rho, \eta_\phi) \quad \text{such that} \quad \eta_\rho > 0, \quad \alpha_k^{--} |\Psi\rangle = 0 \quad \text{for all} \quad k = (\eta_\rho, \eta_\phi) \quad \text{such that} \quad \eta_\rho < 0.

Note that \((\alpha_k^{+-})^* = \alpha_k^{--}.

It follows from (2.22), (2.24) that

\[(2.25) \quad < C, \Phi > |\Psi\rangle = \int_{\mathbb{R}^2} (C^{--}(k) \alpha_k^{--} - \overline{C}^{++}(k) \alpha_k^{++}) \, dk |\Psi\rangle.

Let

\[(2.26) \quad N(C) = < C, \Phi >^* < C, \Phi >

be the number of particle operator (cf. [14], [15]).

The expectation value of the number operator

\[(2.27) \quad \langle \Psi | N(C) | \Psi \rangle

is the average number of particles created by the wave packet \(C\).

**Theorem 2.1.** The average number of particles created by the wave packet \(C\) is given by the formula

\[(2.28) \quad \langle \Psi | N(C) | \Psi \rangle = - < C_-, C_- >,

where \(C_-\) is given by (2.19).
Proof: We have from (2.24) that

\[ \langle \Psi | (\alpha_k^{++})^* = 0, \quad \langle \Psi | (\alpha_k^{-})^* = \langle \Psi | \alpha_k^{+-} = 0. \]

Therefore

\[ \langle \Psi | < C, \Phi >^* = \langle \Psi | \int_{\mathbb{R}^2} (C^+ - (\alpha_k^{+-})^* - C^+ (\alpha_k^{-})^*) dk. \]

Hence, combining (2.30) and (2.25) we get

\[ \langle \Psi | < C, \Phi >^* < C, \Phi > | \Psi \rangle = \langle \Psi | \int_{\mathbb{R}^2} (C^+ - (\alpha_k^{+-})^* - C^+ (\alpha_k^{-})^*) dk \cdot \int_{\mathbb{R}^2} (C^+ (k') \alpha_k^{+-} - C^+(k) \alpha_k^{+-} ) dk' | \Psi \rangle. \]

Therefore

\[ \langle \Psi | N(C) | \Psi \rangle = \int_{\mathbb{R}^2} ( - |C^+ - (k)|^2 + |C^+ (k)|^2 ) dk. \]

We used that

\[ (\alpha_k^{+-})^* \alpha_k^{+-} = \alpha_k^{+-} (\alpha_k^{+-})^* - I \delta(k - k'), \quad \alpha_k^{+-} (\alpha_k^{+-})^* (\alpha_k^{+-}) = I \delta(k - k'), \quad - \alpha_k^{+-} (\alpha_k^{+-})^*. \]

It follows from (2.19) that

\[ < C_-, C_+ > = \int_{\mathbb{R}^2} ( |C^+ - (k)|^2 - |C^+ (k)|^2 ) dk \]

Therefore

\[ \langle \Psi | N(C) | \Psi \rangle = - < C_-, C_+, >, \]

where \( C_- \) is given by (2.19).
3 Hawking radiation in the case of rotating black hole

The Hamiltonian of (1.1) has the following form in polar coordinates \((\rho, \varphi)\)

\[
H(\rho, \varphi, \xi_0, \xi_\rho, m) = \left(\xi_0 + \frac{A}{\rho} \eta_\rho + \frac{B}{\rho^2} m\right)^2 - \xi_\rho^2 - \frac{m^2}{\rho^2} = 0, \text{ where } (\xi_0, \eta_\rho, m) \text{ are dual variables to } (x_0, \rho, \varphi).
\]

Let \(S = -\eta_0 x_0 + S_1(\rho) + m \varphi\) be the solution of the eikonal equation

\[
(3.1) \quad \left( - \eta_0 + \frac{A}{\rho} S_1 + \frac{B}{\rho^2} m \right)^2 - S_1^2 - \frac{m^2}{\rho^2} = 0.
\]

We are looking for the solution of (3.1) such that \(S_1(\rho) \to +\infty\) when \(\rho \to |A|\).

Solving the quadratic equation (3.1) we get

\[
(3.2) \quad S_1(\rho) = \frac{A}{\rho} (\eta_0 - \frac{B m}{\rho^2}) \pm \sqrt{\frac{A^2}{\rho^2} (\eta_0 - \frac{B m}{\rho^2})^2 - (\frac{A^2}{\rho^2} - 1) \left[ (\eta_0 - \frac{B m}{\rho^2})^2 - \frac{m^2}{\rho^2} \right]}.
\]

Let

\[
(3.3) \quad \xi_0 = \eta_0 - \frac{B}{|A|^2 m}.
\]

Since we are looking for \(S_1(\rho) \to +\infty\) when \(\rho - |A| \to 0\) we have

\[
(3.4) \quad S_1(\rho) = \frac{\xi_0 |A|}{\rho - |A|} + O(\rho - |A|).
\]

Thus \(S_1 = \xi_0 |A| \ln(\rho - |A|) + O(\rho - |A|)\).

We define a wave packet \(C_0(x_0, \rho, \varphi)\) as the exact solution of the wave equation (1.1) with the following initial conditions

\[
(3.5) \quad C_0 \big|_{x_0=0} = \theta(\rho - |A|) \frac{1}{\sqrt{\rho}} (\rho - |A|)^{\frac{\epsilon}{2}} e^{-a(\rho - |A|)} e^{i \xi_0 |A| \ln(\rho - |A|) + i m \varphi},
\]

\[
(3.6) \quad \frac{\partial C_0}{\partial x_0} \big|_{x_0=0} = i \beta (\rho - |A|) \left( \frac{1}{\sqrt{\rho}} (\rho - |A|)^{\frac{\epsilon}{2}} e^{-a(\rho - |A|)} e^{i \xi_0 |A| \ln(\rho - |A|) + i m \varphi},
\]
where $a > 0$, $\varepsilon > 0$,

$$
(3.7) \quad \beta = -\frac{A}{\rho} \frac{\xi_0 |A|}{\rho - |A|} - \frac{B}{\rho^2} m - \frac{\xi_0 |A|}{\rho - |A|} = \left( \frac{|A|}{\rho} - 1 \right) \frac{\xi_0 |A|}{\rho - |A|} - \frac{B}{\rho^2} m
$$

$$
= -\frac{\xi_0 |A|}{\rho} - \frac{Bm}{\rho^2} = -\eta_0 + O(\rho - |A|).
$$

We used in (3.7) that $A < 0$ and $\xi_0 = \eta_0 - \frac{Bm}{|A|^2}$.

The convenience of this choice of the initial conditions (3.5), (3.6) will be clear later.

We shall compute the KG norm of $C_0$.

Let, as in (2.6), $\{v_1, v_2\} = \sum_{j=0}^{2} \left( \frac{\partial v_1}{\partial x_j} - \frac{\partial v_2}{\partial x_j} \right)$. Note that if $C_0 = h(\rho, \varphi)C_01$, where $h(\rho, \varphi) = f(\rho) \sqrt{\rho}$ is real-valued, then $\{C_0, C_0\} = h^2(\rho, \varphi)\{C_01, C_01\}$. Therefore

$$
\{C_0, C_0\} = \frac{\theta(\rho - |A|) f^2(\rho)(-2i\xi_0 |A|)}{\rho(\rho - |A|)}.
$$

Thus

$$
(3.8) \quad < C_0, C_0 > = \int_{|A|}^{\infty} \int_{0}^{2\pi} \frac{2\xi_0 |A|}{\rho - |A|} (\rho - |A|)^{2\varepsilon} e^{-2a(\rho - |A|)} d\rho d\varphi = \frac{4\pi \xi_0 |A| \Gamma(2\varepsilon)}{(2a)^{2\varepsilon}},
$$

since $f(\rho) = (\rho - |A|)^{\varepsilon} e^{-a(\rho - |A|)}$.

We are going to compute $\langle \Psi | N(C_0) | \Psi \rangle$ (cf. (2.26)), i.e. the average number of particles created by the wave packet $C_0$.

It follows from (2.32) that

$$
(3.9) \quad \langle \Psi | N(C_0) | \Psi \rangle = \int_{\mathbb{R}^2} (|C_0^{++}(k)|^2 - |C_0^{+-}(k)|^2) dk,
$$

where

$$
C_0^{++}(k) = < f_{k}^{++}, C_0 >, \quad C_0^{+-}(k) = < f_{-k}^{+-}, C_0 >.
$$

We have (cf. (2.1), (2.2), (3.5), (3.6))

$$
(3.10) \quad C_0^{+-}(k) = C_1^{+-}(k) + C_2^{+-}(k),
$$

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where

\[
C_{1}^{+\pm}(k) = \int_{0}^{\infty} \int_{0}^{\infty} \frac{e^{-i\rho\eta_{\rho} - im'\varphi}}{\sqrt{2} 2\pi \sqrt{\rho}} \frac{\theta(\rho - |A|)(\rho - |A|)^{\varepsilon}}{\sqrt{\rho}} e^{-a(\rho - |A|) + i\xi_{0}|A| \ln(\rho - |A|) + im'\varphi} \rho d\rho d\varphi,
\]

\[
k = (\eta_{\rho}, m'), \eta_{\rho} < 0.
\]

Analogously,

\[
C_{2}^{+\pm}(k) = \int_{0}^{\infty} \int_{0}^{\infty} \frac{e^{-i\rho\eta_{\rho} - im'\varphi}}{\sqrt{2} 2\pi \sqrt{\rho}} \frac{\theta(\rho - |A|)(\rho - |A|)^{\varepsilon}}{\sqrt{\rho}} \cdot \left( \frac{\xi_{0}|A|}{\rho - |A|} - \frac{A}{\rho} \left( \frac{-i\varepsilon}{\rho - |A|} + ia \right) \right) e^{-a(\rho - |A|) + i\xi_{0}|A| \ln(\rho - |A|) + im'\varphi} \rho d\rho d\varphi,
\]

where while computing the KG inner product we used that (cf. (3.9))

\[
\beta_{\rho} = \frac{A \xi_{0}|A|}{\rho - |A|} + \frac{B}{\rho^2} m' = -\frac{\xi_{0}|A|}{\rho - |A|}.
\]

Since \( \int_{0}^{2\pi} e^{-im'\varphi + im'\varphi} d\varphi = 2\pi \delta_{mm'} \) where \( \delta_{mm'} = 1 \) when \( m = m' \) and \( \delta_{mm'} = 0 \) when \( m \neq m' \), we get

\[
C_{1}^{+\pm} = \delta_{mm'} \int_{0}^{\infty} \frac{1}{\sqrt{2}} (\eta_{\rho}^2 + m'^2)^{\frac{1}{4}} \theta(\rho - |A|) e^{-a(\rho - |A|)} e^{(i\xi_{0}|A| + \varepsilon) \ln(\rho - |A|) - i\rho\eta_{\rho}} d\rho.
\]

Change variable \( \rho - |A| = x \) in (3.13) and perform the Fourier transform in \( x \). Using the well-known formula

\[
F(x_{-}^{\lambda} e^{-ax}) = \frac{\Gamma(\lambda + 1)e^{-i(\lambda + 1)\frac{a}{2}}}{(\eta_{1} - ia)^{\lambda + 1}},
\]

(see, for example, formula (11.10) in [E7]), we get, having \( \lambda = i\xi_{0}|A| + \varepsilon \),

\[
C_{1}^{+\pm} = \frac{\delta_{mm'} (\eta_{\rho}^2 + m'^2)^{\frac{1}{4}} e^{-i|A|\eta_{\rho}} \Gamma(i\xi_{0}|A| + \varepsilon + 1) e^{-i(i\xi_{0}|A| + \varepsilon + 1)\frac{a}{2}}}{(\eta_{\rho} - ia)^{i\xi_{0}|A| + \varepsilon + 1}}.
\]
Analogously,

\begin{equation}
C_2^{\pm} = \frac{\delta_{mm'}(\xi_0|A| - i\varepsilon)(\eta^2 + \frac{m^2}{a^2})^{\frac{1}{2}}e^{-i|A|\eta_\rho\Gamma(i\xi_0|A| + \varepsilon)}e^{-i(i\xi_0|A| + \varepsilon)\frac{\pi}{2}}}{(\eta - ia)^{\xi_0|A| + \varepsilon}} + \frac{\delta_{mm'} \cdot O(\frac{1}{|\eta_{\rho} - ia|^{2\varepsilon + 1}})}{\sqrt{2}(\eta^2 + \frac{m^2}{a^2})^\varepsilon}.
\end{equation}

Noting that \(\eta_{\rho} < 0, a > 0\) we get

\[\ln(\eta_{\rho} - ia) = \ln|\eta_{\rho} - ia| + i(-\pi + \sin^{-1}\frac{a}{\sqrt{\eta^2 + a^2}}).\]

Also \(\Gamma(\lambda + 1) = \lambda\Gamma(\lambda)\). Therefore

\begin{equation}
C_1^{\pm}C_2^{\pm} = \delta_{mm'}\frac{\xi_0|A| + i\varepsilon}{\pi}(i\xi_0|A| + \varepsilon)|\Gamma(i\xi_0|A| + \varepsilon)|^2e^{-i\frac{\pi}{2} + \pi|\xi_0|A|}
\end{equation}

\begin{equation}
\cdot e^{-\left(2\pi - 2\sin^{-1}\frac{i\xi_0|A|}{\sqrt{\eta^2 + a^2}}\right)}\frac{\xi_0|A|}{|\eta - ia|^{2\varepsilon}(\eta_{\rho} - ia)} + O\left(\frac{1}{|\eta_{\rho} - ia|^{2\varepsilon + 2}}\right).
\end{equation}

Note that

\begin{equation}
\Gamma(i\xi_0|A| + \varepsilon) = \int_0^\infty e^{(i\xi_0|A| + \varepsilon - 1)\ln x - x}dx.
\end{equation}

Using the Cauchy theorem we can replace the integration over real semiaxis by the integration over the imaginary semiaxis:

\begin{equation}
\Gamma(i\xi_0|A| + \varepsilon) = i\int_0^\infty e^{(i\xi_0|A| + \varepsilon - 1)(\ln y + i\frac{\pi}{2}) - iy}dy = e^{-\frac{\pi}{2}\xi_0|A|}\Gamma_1(\xi_0|A|),
\end{equation}

where

\[\Gamma_1(\xi_0|A|) = i\int_0^\infty e^{(i\xi_0|A| + \varepsilon - 1)\ln y + i(\varepsilon - 1)\frac{\pi}{2} - iy}dy.\]
Therefore

\[(3.20)\]

\[C_1^+ \overline{C}_2^- = \delta_{mm'} \frac{(\xi_0|A|)^2 + \varepsilon^2}{2} |\Gamma_1|^2 e^{-\frac{2\pi - 2 \sin^{-1} a}{\sqrt{\eta^2 + a^2}} \xi_0|A|} \]

\[\cdot \frac{(\eta + ia)}{|\eta - ia|^{2\varepsilon+2}} + O\left(\frac{1}{|\eta - ia|^{2\varepsilon+2}}\right), \quad \eta < 0.\]

Consider now \(C^{-+} = < f^-_k \theta(\eta), C_0 >\). When we change \(\eta\) to \(-\eta\), then the only difference with (3.10), (3.11) is that \((\eta^2 + a^2)^{\frac{1}{4}}\) in (3.11) is replaced by \(- (\eta^2 + a^2)^{\frac{1}{4}}\).

Therefore

\[C^{-+} = -C_1^+ + C_2^+.\]

We have

\[|C^{+-}|^2 = |C_1^+ + C_2^+|^2 = |C_1^+|^2 + C_1^+ \overline{C}_2^+ + C_1^+ \overline{C}_2^+ + |C_2^+|^2,\]

\[|C^{-+}|^2 = |C_1^+|^2 - C_1^+ \overline{C}_2^- - C_2^+ \overline{C}_1^- + |C_2^-|^2.\]

Therefore

\[(3.21)\]

\[\langle \Psi | N(C_0) | \Psi \rangle = \sum_{m' = -\infty}^{\infty} \int_{-\infty}^{0} (|C^{+-}|^2 - |C^{+-}|^2) d\eta = \sum_{m' = -\infty}^{\infty} \int_{-\infty}^{0} 4 \Re(C_1^+ \overline{C}_2^-) d\eta.\]

It follows from (3.20) that

\[(3.22)\]

\[\langle \Psi | N(C_0) | \Psi \rangle = 2e^{-2\pi |\xi_0|A|}((|\xi_0|A|)^2 + \varepsilon^2) |\Gamma_1|^2 \int_{-\infty}^{0} \frac{|\eta| e^{2\xi_0|A| \sin^{-1} a \sqrt{\eta^2 + a^2}}}{(\eta^2 + a^2)^{\varepsilon+1}} d\eta + \int_{-\infty}^{0} O\left(\frac{1}{|\eta - ia|^{2\varepsilon+2}}\right) d\eta.\]
Making the change of variable $\eta_\rho \to a\eta_\rho$, we get

\[
(3.23) \quad \langle \Psi | N(C_0) | \Psi \rangle = 2e^{-2\pi |\xi_0|A|((|\xi_0|A|^2+\varepsilon^2)} a^{-2\varepsilon} |\Gamma_1|^2 \int_{-\infty}^{0} \frac{d\eta_\rho}{(\eta_\rho^2 + 1)^{\varepsilon+1}} \left[ e^{2\xi_0|A|\sin^{-1}\frac{1}{\sqrt{\eta_\rho^2 + 1}}} \right] d\eta_\rho
\]

We now normalize $C_0$ replacing it by $C_n = \frac{C_0}{<C_0,C_0>^{1/2}}$.

We have $N(C_n) = \frac{N(C_0)}{<C_0,C_0>}$. Noting that $<C_0,C_0> = \frac{4\pi |\xi_0|A|\Gamma(\varepsilon)}{(2\pi)^{1/2}}$ (see (3.8)) we get from (3.23) that

\[
(3.24) \quad \lim_{a\to \infty} \langle \Psi | N(C_n) | \Psi \rangle = \frac{2^{2\varepsilon} e^{-2\pi |\xi_0|A|\Gamma|1|^2}}{2\pi \Gamma(\varepsilon)} \int_{-\infty}^{0} \frac{d\eta_\rho}{(\eta_\rho^2 + 1)^{\varepsilon+1}} \left[ e^{2|\xi_0|A|\sin^{-1}\frac{1}{\sqrt{\eta_\rho^2 + 1}}} \right] d\eta_\rho.
\]

**Theorem 3.1.** Let $\xi = \eta_0 - \frac{Bm}{|A|^2}$ and let $C_0(x_0, \rho, \varphi)$ be the solution of (1.1) with the initial conditions (3.5), (3.6). Then the average number of particles created by the normalized wave packet $C_n(x_0, \rho, \varphi)$ is given by (3.24) when $a \to \infty$.

**Remark 3.1** K. Fredenhagen and R. Haag (see [11]) use the limit when the time $T$ tends to $-\infty$ to find the Hawking radiation in a spherically symmetric case. Note that when $T \to -\infty$ the wave packet becomes closer and closer to the black hole. In our approach the time is fixed and in $(\rho - |A|)^\varepsilon e^{-a(\rho - |A|)}$ the parameter $a$ characterizes the closeness to the black hole. Thus, as $a \to +\infty$ we get the value of the Hawking radiation.

### 4 Hawking radiation for analogue black holes with variable $A$ and $B$

In this section we extend the results of §3 to the case when $A$ and $B$ depend on $(\rho, \varphi), \rho > 0, \varphi \in [0, 2\pi]$ (see [6], [7], [18], [19], [23], [27], [28], where physically relevant examples of such acoustic metrics are studied).
4.1 The case of $A < 0$ constant and $|B(\varphi)| > 0$ periodic

Consider first a more simple case when $A < 0$ is a constant and $B(\varphi)$ is a periodic function of $\varphi$, $|B(\varphi)| > 0$. In this case $\rho = \sqrt{A^2 + B^2(\varphi)}$ is an ergosphere and $\{\rho < |A|\}$ is a black hole.

The eikonal $S = -x_0 \eta_0 + S_1(\rho, \varphi)$ is the solution of the equation (4.1)

$$
(\eta_0 + A \frac{\partial S_1}{\partial \rho} + \frac{B(\varphi)}{\rho^2} \frac{\partial S_1}{\partial \varphi})^2 - \left(\frac{\partial S_1}{\partial \rho}\right)^2 - \frac{1}{\rho^2} \left(\frac{\partial S_1}{\partial \varphi}\right)^2 = 0, \ \rho > |A|, \varphi \in [0, 2\pi].
$$

Solving the quadratic equation (4.1) we get (cf. (3.2))

$$
S_{1\rho} = \frac{A (\eta_0 - \frac{B(\varphi)}{\rho^2} \frac{\partial S_1}{\partial \varphi}) \pm \sqrt{(\eta_0 - \frac{B(\varphi)}{\rho^2} \frac{\partial S_1}{\partial \varphi})^2 + \left(\frac{\partial S_1}{\partial \rho}\right)^2}}{A^2 - 1}
$$

We shall find a simple approximation for the eikonal near $\rho = |A|$. Assuming that $S_{1\varphi}(\rho, \varphi)$ is continuous at $\rho = |A|$, we get from (4.2)

$$
S_{1\rho} = \frac{(\eta_0 - \frac{B(\varphi)}{|A|^2} S_{1\varphi}(\rho, \varphi)) |A|}{\rho - |A|} + O(1).
$$

Denote by $S_2(\rho, \varphi)$ the solution of the first order partial differential equation

$$
(\rho - |A|) S_{2\rho}(\rho, \varphi) - \eta_0 |A| + \frac{B(\varphi)}{|A|} S_{2\varphi}(\rho, \varphi) = 0.
$$

There is an alternative way to obtain the approximation of $S_1(\rho, \varphi)$ by $S_2(\rho, \varphi)$: Consider the eikonal equation

$$
-\eta_0 + A \frac{S_1}{\rho} + \frac{B(\varphi)}{\rho^2} S_1 = -\sqrt{S_{1\rho}^2 + \frac{1}{\rho^2} S_{1\varphi}^2}.
$$

We are looking for $S_1$ such that $S_1 \to +\infty$ when $\rho \to |A|$. Then $\sqrt{S_{1\rho}^2 + \frac{1}{\rho^2} S_{1\varphi}^2} = S_{1\rho} \left(1 + \frac{\frac{\partial S_1}{\partial \varphi}}{\rho^2 S_{1\rho}}\right)^{\frac{1}{2}}$, where $\frac{\partial S_1}{\partial \varphi}$ is small. Replacing $\sqrt{S_{1\rho}^2 + \frac{1}{\rho^2} S_{1\varphi}^2}$ by $S_{1\rho}$ we get the linear equation

$$
-\eta_0 + \frac{\rho - |A|}{\rho} S_{1\rho} + \frac{B(\varphi)}{\rho^2} S_{1\varphi} = 0,
$$
which becomes (4.3) when $\rho$ is replaced by $|A|$. 

Equation (4.3) can be solved explicitly and we take $S_2(\rho, \varphi)$ as the approximation of $S_1(\rho, \varphi)$. We have

$$S_2(\rho, \varphi) = \eta_0 |A| \ln(\rho - |A|) + S_3(\rho, \varphi),$$

where $S_3(\rho, \varphi)$ is the solution of the homogeneous equation:

$$(\rho - |A|)S_3' + \frac{B(\varphi)}{|A|} S_3 = 0.$$ 

Consider the characteristic equation

$$\frac{d\rho}{\rho - |A|} - \frac{|A| d\varphi}{B(\varphi)} = 0.$$ 

Then

$$S_3(\rho, \varphi) = g \left( \ln(\rho - |A|) - \int_0^{\varphi} \frac{|A| d\varphi'}{B(\varphi')} \right)$$

is the general solution of a homogeneous equation for arbitrary $g(t)$. We take $g(t) = a_0 t$, where $a_0$ is such that $S_3(\rho, \varphi + 2\pi) = S_3(\rho, \varphi) + 2\pi m$, $m \in \mathbb{Z}$. Thus $a_0$ satisfies the equation $a_0 \int_0^{2\pi} \frac{d\varphi}{B(\varphi)} |A| = 2\pi m$, i.e.

$$(4.4) \quad a_0 = \frac{2\pi m}{|A| \int_0^{2\pi} \frac{1}{B(\varphi)} d\varphi}.$$ 

Therefore

$$S_3(\rho, \varphi) = a_0 \ln |\rho - |A|| + S_4(\varphi),$$

where

$$(4.5) \quad S_4(\varphi) = -a_0 \int_0^{\varphi} \frac{|A| d\varphi'}{B(\varphi')},$$

Finally,

$$S_2(\rho, \varphi) = (\eta_0 |A| + a_0) \ln(\rho - |A|) + S_4(\varphi).$$

Define the wave packet $\hat{C}_0$ as the solution of the wave equation $\Box_g u = 0$ having the initial conditions (cf. (3.5), (3.6))

$$(4.6) \quad \hat{C}_0|_{x_0=0} = \theta(\rho - |A|) f(\rho) e^{i(\eta_0 |A| + a_0) \ln(\rho - |A|) + iS_4(\varphi)},$$

$$(4.7) \quad \frac{\partial \hat{C}_0}{\partial x_0} |_{x_0=0} = i\beta \theta(\rho - |A|) f(\rho) e^{i(\eta_0 |A| + a_0) \ln(\rho - |A|) + iS_4(\varphi)},$$

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where \( a_0 \) and \( S_4(\varphi) \) are the same as in (4.4), (4.5), \( f(\rho) = \frac{1}{\sqrt{\rho}}(\rho - |A|)^{\varepsilon}e^{-a(\rho - |A|)} \), and (cf. (3.7))

\[
\hat{\beta} = -\frac{A(\eta_0|A| + a_0)}{\rho} - \frac{B(\varphi)}{\rho - |A|} \left( \frac{\partial S_4}{\partial \varphi} - \frac{\eta_0|A| + a_0}{\rho - |A|} \right) - \frac{B(\varphi)}{\rho^2}(-a_0|A|B^{-1}(\varphi))
\]

\[
= -\frac{\eta_0|A| + a_0}{\rho} + \frac{a_0|A|}{\rho^2} = -\eta_0 + O(\rho - |A|).
\]

The computation of \( \langle \Psi | N(\hat{C}_0) | \Psi \rangle \) are the same as in \( \S 3 \) with \( \xi_0|A| \) replaced by \( \eta_0|A| + a_0 \) and \( e^{im\varphi} \) replaced by \( e^{iS_4(\varphi)} \). Let

\[(4.9) \hat{C}_0^{+-} = \hat{C}_1^{+-} + \hat{C}_2^{+-},\]

(cf. (3.10)). Then, as in (3.11)

\[(4.10) \hat{C}_1^{+-} = \int_0^\infty \int_0^\infty \frac{\theta(\rho - |A|)(\rho - |A|)^{\varepsilon}}{\sqrt{\rho}} e^{-a(\rho - |A|) + (\eta_0|A| + a_0)ln(\rho - |A|) + iS_4(\varphi)} \rho d\rho d\varphi,\]

Analogously (cf. (3.12))

\[(4.11) \hat{C}_2^{+-} = \int_0^\infty \int_0^\infty \frac{e^{-i\rho_0 - im'\varphi}}{\sqrt{2\pi}} \sqrt{\rho} \left( \eta_0^2 + \frac{m'^2}{\rho} \right)^{\frac{1}{2}} \sqrt{2\pi} \frac{\theta(\rho - |A|)(\rho - |A|)^{\varepsilon}}{\sqrt{\rho}} \left( \frac{(\eta_0|A| + a_0)}{\rho - |A|} - A \left( \frac{-i\varepsilon}{\rho} \right) \right) e^{-a(\rho - |A|) + i(\eta_0|A| + a_0)ln(\rho - |A|) + iS_4(\varphi)} \rho d\rho d\varphi,\]

Let

\[(4.12) \hat{\gamma}_{m'} = \frac{1}{2\pi} \int_0^{2\pi} e^{-im'\varphi + iS_4(\varphi)} d\varphi.\]
Then, integrating in $\rho$ and in $\varphi$ as in (3.13), (3.16), we get

$$
(4.13) \hat{C}_{1+}^{+} = \frac{\hat{\gamma}_{m'}}{\sqrt{2}} e^{-\frac{1}{2} |i\rho|_{\eta_0} \Gamma(i(\eta_0 |A| + a_0) + \varepsilon + 1)} e^{-i(\eta_0 |A| + a_0) + \varepsilon + 1} \frac{(\eta_0 - ia)^{i(\eta_0 |A| + a_0) + \varepsilon + 1}}{(\eta_0 - ia)^{i(\eta_0 |A| + a_0) + \varepsilon + 1}}
$$

Analogously,

$$
(4.14) \hat{C}_{2+}^{+} = \frac{\hat{\gamma}_{m'}}{\sqrt{2}} e^{-\frac{1}{2} |i\rho|_{\eta_0} \Gamma(i(\eta_0 |A| + a_0) + \varepsilon + 1)} e^{-i(\eta_0 |A| + a_0) + \varepsilon + 1} \frac{(\eta_0 - ia)^{i(\eta_0 |A| + a_0) + \varepsilon + 1}}{(\eta_0 - ia)^{i(\eta_0 |A| + a_0) + \varepsilon + 1}}
$$

Therefore, integrating in $\eta_0$ and summing in $m'$, we get (cf. (3.23)):

$$
(4.15) \langle \Psi | N(\hat{C}_0) | \Psi \rangle = 2 e^{-2\pi(\eta_0 |A| + a_0)} \sum_{m' = -\infty}^{\infty} |\hat{\gamma}_{m'}|^2 (|\eta_0 |A| + a_0|^2 + \varepsilon^2) a^{-2\varepsilon} |\Gamma_1(\eta_0 |A| + a_0)|^2 \int_{-\infty}^{\infty} \frac{\eta_0 |e^{2(\eta_0 |A| + a_0) \sin^{-1} \frac{1}{\sqrt{\eta_0^2 + 1}}}}{(\eta_0^2 + 1)^{\varepsilon + 1}} d\eta_0 + O(a^{-2\varepsilon - 1}).
$$

The Parseval’s equality gives

$$
(4.16) \sum_{-m' = -\infty}^{\infty} |\hat{\gamma}_{m'}|^2 = \frac{1}{2\pi} \int_{0}^{2\pi} |e^{iS_4(\varphi)}|^2 d\varphi = 1.
$$

If we replace $\hat{C}_0$ by the normalized wave packet $\hat{C}_n = \sqrt{\frac{\eta_0}{\hat{C}_0, \hat{C}_n}}$ and take the limit as $a \to \infty$, we get

$$
(4.17) \lim_{a \to \infty} \langle \Psi | N(\hat{C}_n) | \Psi \rangle = 2^{2\varepsilon} e^{-2\pi(\eta_0 |A| + a_0)} |\Gamma_1(\eta_0 |A| + a_0)|^2 \frac{(\eta_0 |A| + a_0)^2 + \varepsilon^2}{\eta_0 |A| + a_0} \int_{-\infty}^{0} \frac{\eta_0}{(\eta_0^2 + 1)^{1+\varepsilon}} e^{2(\eta_0 |A| + a_0) \sin^{-1} \frac{1}{\sqrt{\eta_0^2 + 1}} + \varepsilon + 1} d\eta_0.
$$
We proved the following theorem:

**Theorem 4.1.** Let $a_0$ be the same as in (4.4). The average number of particles of created by the normalized wave packet $\hat{C}_n(x_0, \rho, \varphi)$ is given by (4.17) when $a \to \infty$.

### 4.2 General acoustic metric

Now consider acoustic metrics of general form, i.e. when $A(\rho, \varphi), B(\rho, \varphi)$ are functions of $(\rho, \varphi)$, $A(\rho, \varphi) < 0, |B(\rho, \varphi)| > 0$. The equation of the ergosphere is $\frac{A^2 + B^2}{\rho^2} = 1$ and we assume that the ergosphere is a smooth Jordan curve.

It was shown in [6], [7] that there exists a black hole $\{\rho < \rho_0(\varphi)\}$ inside the ergosphere where $\rho = \rho_0(\varphi)$ is a smooth Jordan curve that is a characteristic curve for the spatial part of the wave operator $\Box_g$. Let

\[
\eta_0 + \frac{A(\rho, \varphi)}{\rho} S_\rho + \frac{B(\rho, \varphi)}{\rho^2} S_\varphi \right)^2 - S_\rho^2 - \frac{1}{\rho^2} S_\varphi^2 = 0
\]

be the eikonal equation (cf. (4.1)).

Make change of variable

\[
(4.19) \quad \tilde{\rho} = \rho - \rho_0(\varphi), \quad \varphi = \varphi.
\]

Let $\tilde{S}(\tilde{\rho}, \varphi) = S(\rho, \varphi)$. The eikonal equation (4.18) has the following form in $(\tilde{\rho}, \varphi)$ coordinates

\[
(4.20) \quad \left( - \eta_0 + \left( \frac{A}{\rho} - \frac{B}{\rho^2} \rho_0'(\varphi) \right) \tilde{S}_\rho + \frac{B}{\rho^2} \tilde{S}_\varphi \right)^2 - \tilde{S}_\rho^2 - \frac{1}{\rho^2} (\tilde{S}_\varphi - \rho_0'(\varphi) \tilde{S}_\rho)^2 = 0,
\]

where $\rho = \rho_0(\varphi) + \tilde{\rho}, \rho_0'(\varphi) = \frac{d\rho_0(\varphi)}{d\varphi}$.

We shall rewrite (4.20) as a quadratic equation in $\tilde{S}_\rho$:

\[
(4.21) \quad \bar{A}(\tilde{\rho}, \varphi) \tilde{S}_\rho^2 + 2 \bar{B}(\eta_0, \tilde{\rho}, \varphi, \tilde{S}_\varphi) \tilde{S}_\rho + \bar{C}(\eta_0, \tilde{\rho}, \varphi, \tilde{S}_\varphi) = 0,
\]
where

\[ \tilde{A}(\tilde{\rho}, \varphi) = \left( \frac{A}{\rho} - \frac{B}{\rho^2} \rho'_{0} \right)^2 - 1 - \frac{\rho'^2}{\rho^2}, \]

\[ \tilde{B}(\eta_0, \tilde{\rho}, \varphi, \tilde{S}_\varphi) = \left( \frac{A}{\rho} - \frac{B}{\rho^2} \rho'_{0} \right) \left( - \eta_0 + \frac{B}{\rho^2} \tilde{S}_\varphi \right) + \frac{\rho'_{0}}{\rho^2} \tilde{S}_\varphi, \]

\[ \tilde{C}(\eta_0, \tilde{\rho}, \varphi, \tilde{S}_\varphi) = \left( - \eta_0 + \frac{B}{\rho^2} \tilde{S}_\varphi \right)^2 - \frac{\tilde{S}_\varphi^2}{\rho^2}, \]

\[ \rho = \tilde{\rho} + \rho'_{0}(\varphi). \]

Since \( \tilde{\rho} = 0 \) is a characteristic curve we have

\[ \tilde{A}(\tilde{\rho}, \varphi) = \tilde{A}_0(\tilde{\rho}, \varphi) \tilde{\rho}, \]

where \( \tilde{A}_0(0, \varphi) \neq 0. \)

We have

\[ \tilde{S}_{\tilde{\rho}}(\tilde{\rho}, \varphi) = \frac{-\tilde{B} \pm \sqrt{\tilde{B}^2 - \tilde{A}\tilde{C}}}{\tilde{A}_0(\tilde{\rho}, \varphi)}. \]

The root \( \tilde{S}_{\tilde{\rho}} \) that tends to \( \infty \) when \( \tilde{\rho} \to 0 \) has the form

\[ \tilde{S}_{\tilde{\rho}}(\tilde{\rho}, \varphi) = \frac{-2\tilde{B}(\eta_0, 0, \varphi, \tilde{S}_\varphi)}{\tilde{A}_0(0, \varphi) \tilde{\rho}} + O(1). \]

It follows from (4.20) that

\[ \frac{2B(\eta_0, 0, \varphi, \tilde{S}_\varphi)}{\tilde{A}_0(0, \varphi)} = \tilde{B}_1(\varphi) \eta_0 + \tilde{B}_2(\varphi) \tilde{S}_\varphi. \]

As in (4.3) we approximate \( \tilde{S}(\rho, \varphi) \) by the solution of linear first order partial differential equation

\[ \tilde{\rho}\tilde{S}_1(\tilde{\rho}, \varphi) + \tilde{B}_1(\varphi) \eta_0 + \tilde{B}_2(\varphi) \tilde{S}_1(\varphi) = 0. \]

We shall solve (4.26) explicitly. We look for a particular solution \( \tilde{S}_2(\tilde{\rho}, \varphi) \) of nonhomogeneous equation (4.26) in the form

\[ \tilde{S}_2(\rho, \varphi) = \eta_0 b_0 \ln \tilde{\rho} + \tilde{S}_3(\varphi), \]

where \( b_0 \) will be determined below, \( \tilde{S}_3(\varphi) \) is periodic in \( \varphi \). For \( \tilde{S}_3(\varphi) \) we get an equation

\[ \tilde{B}_2(\varphi) \tilde{S}_3(\varphi) + \tilde{B}_1(\varphi) \eta_0 + b_0 \eta_0 = 0. \]
A necessary and sufficient condition of the existence of a periodic solution of (4.27) is

\begin{equation}
\int_{0}^{2\pi} \left( \frac{\tilde{B}_1(\varphi)}{B_2(\varphi)} + \frac{b_0}{B_2(\varphi)} \right) d\varphi = 0.
\end{equation}

Thus (4.28) determines $b_0$. Note that $\tilde{B}_2(\varphi) \neq 0$ and

\begin{equation}
b_0 = \frac{-\int_{0}^{2\pi} \frac{\tilde{B}_1(\varphi)}{B_2(\varphi)} d\varphi}{\int_{0}^{2\pi} \tilde{B}_2^{-1}(\varphi) d\varphi}.
\end{equation}

Consider the homogeneous equation (4.26):

\begin{equation}
\tilde{\rho} \tilde{S}_4(\tilde{\rho}, \varphi) + \tilde{B}_2(\varphi) \tilde{S}_4(\tilde{\rho}, \varphi) = 0.
\end{equation}

As in (4.3) the general solution of (4.25) has the form

\[ \tilde{S}_4(\tilde{\rho}, \varphi) = g \left( \ln \tilde{\rho} - \int_{0}^{\varphi} \frac{1}{\tilde{B}_2(\varphi')} d\varphi' \right), \]

where $g(t)$ is arbitrary.

We choose $g(t) = b_1 t$, where $b_1$ is such that

\begin{equation}
b_1 \int_{0}^{2\pi} \frac{1}{B_2(\varphi')} d\varphi' = 2\pi m, \quad m \in \mathbb{Z}.
\end{equation}

Therefore finally the solution of (4.26) has the form

\[ \tilde{S}_1(\tilde{\rho}, \varphi) = (\eta b_0 + b_1) \ln \tilde{\rho} + \tilde{S}_5(\varphi), \]

where $b_0, b_1$ are determined in (4.29) and (4.31) and $\tilde{S}_5(\varphi)$ has the form

\begin{equation}
\tilde{S}_5(\varphi) = \tilde{S}_3(\varphi) - b_1 \int_{0}^{\varphi} \frac{1}{\tilde{B}_2(\varphi')} d\varphi'.
\end{equation}
Note that $\tilde{S}_5(\varphi + 2\pi) = \tilde{S}_5(\varphi) + 2\pi m$, $m \in \mathbb{Z}$.

Now we shall define the wave packet $\tilde{C}_0$ as the solution of $\square_g u = 0$ having the following initial conditions in $({\tilde{\rho}}, \varphi)$ coordinates

$$
(4.33) \quad \tilde{C}_0|_{x_0=0} = \theta(\tilde{\rho}) \frac{\tilde{f}(\tilde{\rho})}{\sqrt{\rho_0(\varphi) + \tilde{\rho}}} e^{i(y_0 b_0 + b_1) \ln \tilde{\rho} + i\tilde{S}_5(\varphi)},
$$

$$
\frac{\partial \tilde{C}_0}{\partial x_0}|_{x_0=0} = i \tilde{\beta} \theta(\tilde{\rho}) \frac{\tilde{f}(\tilde{\rho})}{\sqrt{\rho_0(\varphi) + \tilde{\rho}}} e^{i(y_0 b_0 + b_1) \ln \tilde{\rho} + i\tilde{S}_5(\varphi)},
$$

where

$$
(4.34) \quad \tilde{f}(\tilde{\rho}) = \tilde{\rho}^\sigma e^{-a\tilde{\rho}},
$$

$$
\tilde{\beta} = -\left( \frac{A(\rho, \varphi)}{\rho} - \frac{B(\rho, \varphi)}{\rho^2} \rho'_{0}(\varphi) \right) \left( \frac{\eta_0 b_0 + b_1}{\tilde{\rho}} \right) - \frac{B(\rho, \varphi)}{\rho^2} \frac{\partial S_5(\varphi)}{\partial \varphi} - \frac{\eta_0 b_0 + b_1}{\tilde{\rho}}.
$$

Let $\tilde{f}_k^+(x_0, x)$, $\tilde{f}_k^-(x_0, x)$ be the solutions of $\square_g u = 0$ with the initial conditions (cf. (2.1), (2.2))

$$
(4.35) \quad \tilde{f}_k^+(x_0, x)|_{x_0=0} = \tilde{\gamma}_k e^{i\eta_0 \tilde{\rho} + im' \varphi}, \quad \frac{\partial \tilde{f}_k^+(x_0, x)}{\partial x_0}|_{x_0=0} = i \tilde{\lambda}_0^-(k) \tilde{\gamma}_k e^{i\eta_0 \tilde{\rho} + im' \varphi},
$$

where (cf. (2.3) and (4.13))

$$
(4.36) \quad \tilde{\lambda}_0^-(k) = -\left( \frac{A}{\rho} - \frac{B}{\rho^2} \rho'_{0}(\varphi) \right) \eta_0 - \frac{B}{\rho^2} m' - \sqrt{\eta_0^2 + m'^2},
$$

$$
(4.37) \quad \tilde{\gamma}_k = \frac{1}{\sqrt{\rho_0(\varphi) + \tilde{\rho}} \left( \eta_0^2 + \frac{m'^2}{d^2} \right)} \frac{1}{\sqrt{2(2\pi)^2}}, \quad \rho = \rho_0(\varphi) + \tilde{\rho}, \quad d \text{ is arbitrary.}
$$

Analogously (cf. (2.4))

$$
(4.38) \quad \tilde{f}_k^-(x_0, x)|_{x_0=0} = \tilde{\gamma}_k e^{i\eta_0 \tilde{\rho} + im' \varphi},
$$

and

$$
\frac{\partial \tilde{f}_k^-(x_0, x)}{\partial x_0}|_{x_0=0} = i \tilde{\lambda}_0^+(k) \tilde{\gamma}_k e^{i\eta_0 \tilde{\rho} + m' \varphi}.
$$
where $\lambda_0^+(k)$ is similar to $\lambda_0^-(k)$ with a positive square root. Note that $\tilde{f}_k^\pm$ satisfy “orthogonality conditions” of the forms (2.7), (2.8).

Expanding $\tilde{C}_0$ with respect to the basis $\tilde{f}_k^{++}, \tilde{f}_k^{+-}, \tilde{f}_k^{--}, \tilde{f}_k^{--}$ we get, as in (2.17):

\[(4.39) \tilde{C}_0 = \int \tilde{\tilde{C}}^{++}(k) \tilde{f}_k^{++} + \tilde{\tilde{C}}^{+-}(k) \tilde{f}_k^{+-} - \tilde{\tilde{C}}^{-+}(k) \tilde{f}_k^{-+} - \tilde{\tilde{C}}^{--}(k) \tilde{f}_k^{--}) dk,
\]

where $k = (\eta, m')$ and $dk$ means integration in $\eta$ and the summation in $m'$ (cf. (2.12)). Note that

\[(4.40) \tilde{\tilde{C}}^{+-}(\eta, m') = \tilde{\tilde{C}}_1^{+-}(k) + \tilde{\tilde{C}}_2^{+-}(k)
\]

have the same form as (3.12), (3.13) with $\xi_0|A|$ replaced by $(\eta_0 b_0 + b_1)$, $e^{im\varphi}$ replaced by $e^{iS_5(\varphi)}$ and $\rho - |A|$ replaced by $\tilde{\rho}$.

Let, as in (4.12),

\[(4.41) \tilde{\gamma}_{m'} = \frac{1}{2\pi} \int_0^{2\pi} e^{-im'\varphi + iS_5(\varphi)} d\varphi.
\]

If we replace $\tilde{C}_0$ by $\tilde{C}_n = \frac{\tilde{C}_n}{\langle \tilde{C}_0, \tilde{C}_0 \rangle}$ then analogously to (4.15) we have

\[(4.42) \lim_{a \to \infty} \langle \Psi | N(\tilde{C}_n) | \Psi \rangle = \frac{2^{2\varepsilon}}{2\pi \Gamma(\varepsilon)} e^{-2\pi(\eta_0 b_0 + b_1) |\Gamma_1(\eta_0 b_0 + b_1)|^2} \frac{(\eta_0 b_0 + b_1)^2 + \varepsilon^2}{\eta_0 b_0 + b_1}
\cdot \int_0^{\infty} \frac{|\eta_0|}{(\eta_0^2 + 1)^{\varepsilon + 1}} e^{2(\eta_0 b_0 + b_1) \sin^{-1} \frac{1}{\sqrt{\eta_0^2 + 1}}} d\eta_0.
\]

Thus we proved the following theorem:

**Theorem 4.2.** Let $b_0, b_1$ be the same as in (4.29), (4.30). Then $\langle \Psi | N(\tilde{C}_n) | \Psi \rangle$ has the form (4.42) when $a \to \infty$.

### 5 Conclusion

In this paper we study the Hawking radiation for the rotating acoustic black holes with variable radial and angular velocities.
In the general case of rotating black holes, i.e. when \( A(\rho, \varphi) < 0, |B(\rho, \varphi)| > 0 \), there are always black holes (see \([6], [7]\)). Some of them have a direct physical meaning as it was shown in §1.

There are two main steps in the derivation of the Hawking radiation: finding an appropriate vacuum state and finding appropriate vacuum wave packets, more precisely, sequence of wave packets, for the computation of the Hawking radiation.

The choice of vacuum state is not the same as in the Minkowsky space. We used a vacuum state similar to the one found by W.Unruh in \([23], [24]\). As in \([23], [24]\) we split the set of the eigenfunctions of four subsets instead of two.

Regarding the wave packets we are looking for the wave packets having the initial condition of the form \( u = C(\rho)e^{iS(\rho, \varphi)}, \) \( S(\rho, \varphi) \) is a simplified eikonal function and

\[
C(\rho) = \theta(\rho - |A|)\frac{(\rho - |A|)^{\varepsilon}}{\sqrt{\rho}} e^{-a(\rho - |A|)}.
\]

When the parameter \( a \) increases the wave packets becomes closer and closer to the black hole \( \{\rho < |A|\} \).

Assuming that the KG norm of the wave packet is one and taking taking the limit when \( a \to \infty \) we are getting the Hawking radiation. Note that in a rigorous derivation of the Hawking radiation in the spherically symmetric case K.Fredenhagen and R.Haag \([11]\) also used the limiting procedure when the time \( T \to -\infty \) to obtain the Hawking radiation.

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