Stability analysis of nonlinear implicit fractional Langevin equation with noninstantaneous impulses

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Abstract
In this paper, we consider a nonlocal boundary value problem of nonlinear implicit fractional Langevin equation with noninstantaneous impulses. We study the existence, uniqueness and generalized Ulam–Hyers–Rassias stability of the proposed model with the help of fixed point approach, over generalized complete metric space. We give an example which supports our main result.

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1 Introduction
At Wisconsin university, Ulam raised a question about the stability of functional equations in 1940. The question of Ulam was: Under what conditions does there exist an additive mapping near an approximately additive mapping?; see [30]. In 1941, Hyers was the first mathematician who gave a partial answer to Ulam’s question [12] in a Banach space. Since then, stability of such form is known as Ulam–Hyers stability. In 1978, Rassias [23] provided a remarkable generalization of the Ulam–Hyers stability of mappings by considering variables. For more information about the topic, we refer the reader to [3, 14–16, 24, 28, 31, 40, 42].

An equation of the form \( m \frac{d^2 x}{dt^2} = \lambda \frac{dx}{dt} + \eta(t) \) is called Langevin equation, introduced by Paul Langevin in 1908. Langevin equations have been widely used to describe stochastic problems in physics, chemistry and electrical engineering. For example, Brownian motion is well described by the Langevin equation when the random fluctuation force is assumed to be white noise. For the removal of noise, mathematicians used fractional order differential equations, which also perform well in reducing the staircase effects compared to integer order differential equations. Thus it is very important to study Langevin equations with fractional derivatives; see, for instance, [2, 10, 20, 21].

Fractional order differential equations are generalizations of the classical integer order differential equations. Fractional calculus has become a fast developing area, and its applications can be found in diverse fields ranging from physical sciences, porous media, electrochemistry, economics, electromagnetics, medicine and engineering to biological...
sciences. Progressively, fractional differential equations play a very important role in thermodynamics, statistical physics, viscoelasticity, nonlinear oscillation of earthquakes, defence, optics, control, electrical circuits, signal processing, astronomy, etc. There are some outstanding articles which provide the main theoretical tools for the qualitative analysis of this research field, and at the same time, show the interconnection as well as the distinction between integral models, classical and fractional differential equations; see [1, 5, 13, 17, 19, 22, 25–27, 29].

Impulsive fractional differential equations are used to describe both physical and social sciences. Also they describe many practical dynamical systems such as evolutionary processes, characterized by abrupt changes of the state at certain instants. In the last few decades, the theory of impulsive fractional differential equations were well utilized in medicine, mechanical engineering, ecology, biology and astronomy, etc. There are some remarkable monographs [8, 11, 18, 32, 33, 35–37, 39, 41], which consider fractional differential equations with impulses.

Recently, many mathematicians devoted considerable attention to the existence, uniqueness and different types of Hyers–Ulam stability of the solutions of nonlinear implicit fractional differential equations with Caputo fractional derivative, see [4, 6, 7]. Wang et al. [34] studied generalized Ulam–Hyers–Rassias stability of the following fractional differential equation

\[
\begin{aligned}
\mathcal{D}_0^\alpha x(t) &= f(t, x(t)), \quad t \in (t_k, s_k], k = 0, 1, \ldots, m, 0 < \alpha < 1, \\
x(t) &= g_k(t, x(t)), \quad t \in (s_{k-1}, t_k], k = 1, 2, \ldots, m.
\end{aligned}
\]

Zada et al. [38] studied existence and uniqueness of solutions by using Diaz–Margolis’s fixed point theorem and presented Ulam–Hyers stability, Ulam–Hyers–Rassias stability, and generalized Ulam–Hyers stability for a class of nonlinear implicit fractional differential equation with noninstantaneous integral impulses and nonlinear integral boundary condition:

\[
\begin{aligned}
\mathcal{D}_0^\alpha x(t) &= f(t, x(t), \mathcal{D}_0^\alpha x(t)), \quad t \in (t_k, s_k], k = 0, 1, \ldots, m, 0 < \alpha < 1, t \in (0, 1], \\
x(t) &= \int_{s_{k-1}}^{s_k} g_k(t, x(t)), \quad t \in (s_{k-1}, t_k], k = 0, 1, \ldots, m, \\
x(0) &= \frac{1}{\Gamma(1-\eta)} \int_0^T (T-s)^{\alpha-1} \eta(s)x(s) \, ds.
\end{aligned}
\]

Motivated by [34, 38], we consider the following nonlocal boundary value problem of nonlinear implicit fractional Langevin equation with noninstantaneous impulses:

\[
\begin{aligned}
\mathcal{D}_0^\alpha (\mathcal{D}_0^\beta + \lambda) x(t) &= f(t, x(t), \mathcal{D}_0^\beta + \lambda) x(t) \quad t \in (t_k, s_k], k = 0, 1, \ldots, m, \\
&\quad + \int_0^t (T-s)^{\alpha-1} f(s, x(s)) \, ds, \quad t \in (t_k, s_k], k = 1, 2, \ldots, m, \\
x(t) &= g_k(t, x(t)), \quad t \in (s_{k-1}, t_k], k = 1, 2, \ldots, m, \\
x(0) &= x_0, \quad x(T) = \theta \int_0^\eta \frac{1}{T^p} (\eta-s)^{p-1} x(s) \, ds, \quad 0 < \eta < T,
\end{aligned}
\]

where \(\mathcal{D}_0^\alpha\) and \(\mathcal{D}_0^\beta\) represent classical Caputo derivatives [5] of order \(\alpha\) and \(\beta\) with the lower bound zero, \(0 = t_0 < s_0 < t_1 < s_1 < \cdots < t_m < s_m = \tau\), \(\tau\) is the free fixed number and
\( \lambda \in \mathbb{R} \setminus \{0\}, 0 < \alpha, \beta < 1, 0 < \alpha + \beta < 2, \sigma, p > 0, x_0, \theta \) are constants, \( f : [0, \tau] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is continuous and \( g_k : [s_{k-1}, t_k] \times \mathbb{R} \to \mathbb{R} \) is continuous for all \( k = 1, 2, \ldots, m \).

In Sect. 2, we create a uniform framework to originate appropriate formula of solutions for our proposed model. In Sect. 3, we study the concept of generalized Ulam–Hyers–Rassias stability of Eq. (1.1). Finally, we give an example to illustrate our main result.

2 Solution framework of linear impulsive fractional Langevin equation

Let \( J = [0, \tau] \) and \( C(J, \mathbb{R}) \) be the space of all continuous functions from \( J \) to \( \mathbb{R} \), and the piecewise continuous function space \( PC(J, \mathbb{R}) = \{ x : f \to \mathbb{R} : x \in ((t_k, t_{k-1}]], \mathbb{R}), k = 0, \ldots, m \) and there exist \( x(t_k^+) \) and \( x(t_k^-) \), \( k = 1, 2, \ldots, m \) with \( x(t_k^+) = x(t_k^-) \).

In the current section, we create a uniform framework to originate an appropriate formula for the solution of impulsive fractional differential equation of the form:

\[
\begin{cases}
\mathcal{D}_0^\alpha \mathcal{D}_0^\beta + \lambda x(t) = f(t), & t \in (t_k, s_k], k = 0, 1, \ldots, m, 0 < \alpha, \beta < 1, \\
x(t) = g_k(t), & t \in (s_{k-1}, t_k], k = 1, 2, \ldots, m, \\
x(0) = x_0, & x(T) = \theta I_p x(\eta)
\end{cases}
\]

where \( I_p x(\eta) = \int_0^\eta \frac{1}{\Gamma(p)} (\eta - s)^{p-1} x(s) \, ds \), \( 0 < \eta < T \).

We recall some definitions of fractional calculus from [17] as follows.

**Definition 2.1** The fractional integral of order \( \alpha \) from 0 to \( t \) for the function \( f \) is defined by

\[
I_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t f(s)(t - s)^{\alpha-1} \, ds, \quad t > 0, \alpha > 0,
\]

where \( \Gamma(\cdot) \) is the Gamma function.

**Definition 2.2** The Riemann–Liouville fractional derivative of fractional order \( \alpha \) from 0 to \( t \) for a function \( f \) can be written as

\[
L_0^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t - s)^{n+1-\alpha}} \, ds, \quad t > 0, n - 1 < \alpha < n,
\]

where \( \Gamma(\cdot) \) is the Gamma function.

**Definition 2.3** The Caputo derivative of fractional order \( \alpha \) from 0 to \( t \) for a function \( f \) can be defined as

\[
^cD_0^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - s)^{n-\alpha-1} f^n(s) \, ds, \quad n = [\alpha] + 1.
\]

**Definition 2.4** The general form of classical Caputo derivative of order \( \alpha \) of a function \( f \) can be given as

\[
^cD_0^\alpha f(t) = L_0^\alpha \left( f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right), \quad t > 0, n - 1 < \alpha < n.
\]
Remark 2.1

(i) If \( f(\cdot) \in C^m([0, \infty), \mathbb{R}) \), then

\[
LD_0^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^m(s)}{(t-s)^{\alpha+1-m}} ds
\]

\[
= L_{0+}^{m-\alpha} f^{(m)}(t), \quad t > 0, m - 1 < \alpha < m.
\]

(ii) In Definition 2.4, the integrable function \( f \) can be discontinuous. This fact can lead us to consider impulsive fractional problems in the sequel.

Lemma 2.1 ([22]) Let \( \alpha > 0, \beta > 0, \) and \( f \in L^1([a, b]) \). Then

\[
P^\alpha P^\beta f(t) = P^{\alpha+\beta} f(t), \quad \mathcal{D}_0^\alpha \mathcal{D}_0^\beta f(t) = \mathcal{D}_0^{\alpha+\beta} f(t) \quad \text{and} \quad P^\alpha D_0^\beta f(t) = f(t), \quad t \in [a, b].
\]

Lemma 2.2 Function \( x \in PC(J, \mathbb{R}) \) is a solution of (2.1) if and only if

\[
x(t) = \begin{cases}
\frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} x(s) ds \\
- \frac{\Delta_{\alpha+\beta}^\beta P^\beta} {\Gamma(\alpha+1)\Gamma(\beta+1)} \int_0^T (T-s)^{\alpha+\beta-1} x(s) ds \\
+ \frac{\Delta_{\alpha+\beta}^\beta P^\beta} {\Gamma(\alpha+1)\Gamma(\beta+1)} \int_0^T (T-s)^{\alpha+\beta-1} x(s) ds \\
+ \frac{\Delta_{\alpha+\beta}^\beta P^\beta} {\Gamma(\alpha+1)\Gamma(\beta+1)} \int_0^T (T-s)^{\alpha+\beta-1} x(s) ds \\
\end{cases}
\]

\[
t \in (s_{k-1}, t_k], k = 1, 2, \ldots, m.
\]

Proof Let \( x \) be a solution of problem (2.1).

Case 1. For \( t \in [0, s_0] \), we consider

\[
\mathcal{D}_0^\alpha \mathcal{D}_0^\beta + \lambda x(t) = f(t) \quad \text{with} \quad x(0) = x_0 \quad \text{and} \quad x(T) = \theta P^\beta x(\eta).
\]

After using fractional integrals \( P^\alpha \) and \( P^\beta \) for the solution of the above fractional Langevin equation, we get

\[
x(t) = \int_0^T \frac{f^m(s)}{(t-s)^{\alpha+1-m}} ds
\]

\[
= L_{0+}^{m-\alpha} f^{(m)}(t), \quad t > 0, m - 1 < \alpha < m.
\]

(2.2)
Using boundary conditions, we obtain

\[
x(t) = \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t - s)^{\alpha+\beta-1} f(s) \, ds - \frac{\lambda}{\Gamma(\beta)} \int_0^t (t - s)^{\beta-1} x(s) \, ds
\]

\[
- \frac{\Delta t^\beta}{\Gamma(\beta + 1) \Gamma(\alpha + \beta)} \int_0^T (T - s)^{\alpha+\beta-1} f(s) \, ds
\]

\[
+ \frac{\lambda \Delta t^\beta}{\Gamma(\beta) \Gamma(\beta + 1)} \int_0^T (T - s)^{\beta-1} x(s) \, ds
\]

\[
+ \frac{\theta \Delta t^\beta}{\Gamma(\beta + 1) \Gamma(\alpha + \beta + p)} \int_0^\eta (\eta - s)^{\alpha+\beta+p-1} f(s) \, ds
\]

\[
- \frac{\theta \Delta \lambda t^\beta}{\Gamma(\beta + 1) \Gamma(\alpha + \beta + p)} \int_0^\eta (\eta - s)^{\beta+p-1} x(s) \, ds
\]

\[
- \left( \frac{\Delta (\theta \eta^p - \Gamma(p + 1) t^\beta)}{\Gamma(p + 1) \Gamma(\beta + 1)} - 1 \right) x_0, \quad t \in [0, s_0].
\]

For \( t \in (s_0, t_1] \), \( x(t) = g_1(t) \).

**Case 2.** For \( t \in (t_1, s_1] \), we consider

\[
\mathcal{C}D^\alpha_0 (\mathcal{D}^\beta_0 + \lambda) x(t) = f(t) \quad \text{with} \quad x(t_1) = g_1(t_1) \quad \text{and} \quad x(T) = \theta I^\beta_{\eta} x(\eta).
\]

Since \( x(t_1) = g_1(t_1) \), Eq. (2.2) is of the following type:

\[
g_1(t_1) = t^{\alpha+\beta} f(t_1) - \lambda t^\beta x(t_1) - \frac{c_0 t^\beta}{\Gamma(\beta + 1)} - c. \quad (2.3)
\]

Using boundary conditions, we get

\[
x(t) = \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t - s)^{\alpha+\beta-1} f(s) \, ds - \frac{\lambda}{\Gamma(\beta)} \int_0^t (t - s)^{\beta-1} x(s) \, ds
\]

\[
+ \frac{\Delta (t^\beta_1 - t^\beta)}{\Gamma(\beta + 1) \Gamma(\alpha + \beta)} \int_0^T (T - s)^{\alpha+\beta-1} f(s) \, ds
\]

\[
- \frac{\lambda \Delta (t^\beta_1 - t^\beta)}{\Gamma(\beta) \Gamma(\beta + 1)} \int_0^T (T - s)^{\beta-1} x(s) \, ds
\]

\[
- \frac{\theta \Delta \lambda (t^\beta_1 - t^\beta)}{\Gamma(\beta + 1) \Gamma(\alpha + \beta + p)} \int_0^\eta (\eta - s)^{\alpha+\beta+p-1} f(s) \, ds
\]

\[
+ \frac{\theta \Delta (t^\beta_1 - t^\beta)}{\Gamma(\beta + 1) \Gamma(\alpha + \beta + p)} \int_0^\eta (\eta - s)^{\beta+p-1} x(s) \, ds
\]

\[
+ \left( \frac{\Delta (t^\beta_1 - t^\beta)}{\Gamma(\beta + 1) \Gamma(\alpha + \beta + p)} - \lambda \right) \frac{1}{\Gamma(\alpha + \beta)} \int_0^{t_1} (t_1 - s)^{\alpha+\beta-1} f(s) \, ds
\]

\[
- \left( \frac{\Delta (t^\beta_1 - t^\beta)}{\Gamma(\beta + 1) \Gamma(\alpha + \beta + p)} - \lambda \right) \frac{c_0 (t_1 - s)^\beta}{\Gamma(\beta + 1)} \int_0^t (t - s)^{\beta-1} x(s) \, ds
\]

\[
- \left( \frac{\Delta (t^\beta_1 - t^\beta)}{\Gamma(\beta + 1) \Gamma(\alpha + \beta + p)} - \lambda \right) g_1(t_1).
\]

Generally speaking, for \( t \in (s_{k-1}, t_k] \), \( x(t_k) = g_k(t) \).
Case 3. For \( t \in (t_k, s_k] \), we consider

\[ cD_{0,a}^\beta (cD_{0,a}^\beta + \lambda)x(t) = f(t, x(t)), \quad \text{with } x(t_k) = g_k(t_k) \quad \text{and} \quad x(T) = \theta F x(\eta). \]

By repeating again the same process, we have

\[
x(t) = \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t - s)^{\alpha + \beta - 1} f(s) \, ds - \frac{\lambda}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1} x(s) \, ds \\
+ \frac{\Delta(t_0^\beta - t^\beta)}{\Gamma(\beta + 1)\Gamma(\alpha + \beta)} \int_0^T (T - s)^{\alpha + \beta - 1} f(s) \, ds \\
- \frac{\lambda \Delta(t_0^\beta - t^\beta)}{\Gamma(\beta + 1)\Gamma(\beta)} \int_0^T (T - s)^{\beta - 1} x(s) \, ds \\
- \frac{\theta \Delta(t_0^\beta - t^\beta)}{\Gamma(\beta + 1)\Gamma(\alpha + \beta + 1)} \int_0^\eta (\eta - s)^{\alpha + \beta + 1 - 1} f(s) \, ds \\
+ \frac{\theta \Delta \lambda(t_0^\beta - t^\beta)}{\Gamma(\beta + 1)\Gamma(\beta + p)} \int_0^\eta (\eta - s)^{\beta + p - 1} x(s) \, ds \\
+ \left( \frac{\Delta(t_0^\beta - t^\beta)}{\Gamma(\beta + 1)} \left( \frac{\theta \eta^\beta - \Gamma(p + 1)}{\Gamma(p + 1)} \right) - \lambda \right) \frac{1}{\Gamma(\alpha + \beta)} \int_0^{t_k} (t_k - s)^{\alpha + \beta - 1} f(s) \, ds \\
- \left( \frac{\Delta(t_0^\beta - t^\beta)}{\Gamma(\beta + 1)} \left( \frac{\theta \eta^\beta - \Gamma(p + 1)}{\Gamma(p + 1)} \right) - 1 \right) \frac{\lambda}{\Gamma(\beta)} \int_0^{t_k} (t_k - s)^{\beta - 1} x(s) \, ds \\
- \left( \frac{\Delta(t_0^\beta - t^\beta)}{\Gamma(\beta + 1)} \left( \frac{\theta \eta^\beta - \Gamma(p + 1)}{\Gamma(p + 1)} \right) - 1 \right) g_k(t_k),
\]

where

\[ \Delta = \frac{\Gamma(\beta + 1)\Gamma(\beta + p + 1)}{\Gamma(\beta + 1)\eta^\beta - \Gamma(\beta + 1)\theta \eta^{\beta + p} + \Gamma(\beta + 1)\eta^{\beta + 1} - \Gamma(p + 1)} \]

with \( t_0^\beta = 0 \) for \( t \in [0, s_0) \) and \( t_k^\beta \neq 0 \), for \( t \in (t_k, s_k] \), \( k = 2, 3, \ldots \).

Conversely, one can verify the fact by proceeding the standard steps to complete the proof. \( \square \)

3 Generalized Ulam–Hyers–Rassias stability

Using the ideas of stability in [24, 31], we can generate a generalized Ulam–Hyers–Rassias stability concept for Eq. (1.1).

Let \( \varepsilon, \psi \geq 0 \) and for a nondecreasing \( \varphi \in PC(J, \mathbb{R}) \) consider

\[
\begin{cases}
|cD_{0,a}^\beta (cD_{0,a}^\beta + \lambda)x(t) - f(t, x(t), cD_{0,a}^\beta (cD_{0,a}^\beta + \lambda)x(t))| \leq \varphi(t), \\
t \in (t_k, s_k], k = 0, 1, \ldots, m, 0 < \alpha, \beta < 1,
\end{cases}
\]

(3.1)

\[
|x(t) - (\frac{t^\beta - t_0^\beta}{\Gamma(\beta + 1)\eta^\beta - \Gamma(\beta + 1)\theta \eta^{\beta + p} + \Gamma(\beta + 1)\eta^{\beta + 1} - \Gamma(p + 1)}) - 1) g_k(t, x(t))| \leq \psi, \quad t \in (s_{k-1}, t_k], k = 0, 1, \ldots, m.
\]

Remark 3.1 A function \( x \in PC(J, \mathbb{R}) \) is a solution of the inequality (3.1) if and only if there is \( G \in PC(J, \mathbb{R}) \) and a sequence \( G_k, k = 1, 2, \ldots, m \) (which depends on \( x \)) such that

(i) \( |G(t)| \leq \psi(t), t \in J \) and \( |G_k| \leq \psi, k = 1, 2, \ldots, m, \)

(ii) \( \lim_{n \to \infty} \int_{t_{n-1}}^{t_n} G(t) \, dt = \lim_{n \to \infty} \int_{t_{n-1}}^{t_n} \varphi(t) \, dt = \lim_{n \to \infty} \int_{t_{n-1}}^{t_n} \psi(t) \, dt = 0 \) as \( n \to \infty \) for all \( t \in J \).
(ii) \( ^tD_{0,t} x(t) = f(t,x(t)), ^tD_{0,t} x(t) + G(t), t \in (s_k,s_{k+1}), k = 1, 2, \ldots, m, \)

(iii) \( x(t) = g_k(t,x(t)) + G_k, t \in (s_{k-1}, s_k), k = 1, \ldots, m. \)

**Definition 3.1** Equation (1.1) is called generalized Ulam–Hyers–Rassias stable with respect to \((\varphi, \psi)\) if there exists \(c_{f,a,b,c,g,\varphi} > 0\) such that for each solution \(y \in PC(J, \mathbb{R})\) of the inequality (3.1) there is a solution \(x \in PC(J, \mathbb{R})\) of Eq. (1.1) with

\[
|y(t) - x(t)| \leq c_{f,a,b,c,g,\varphi}(\varphi(t) + \psi), \quad t \in J.
\]

**Remark 3.2** If \(x \in PC(J, \mathbb{R})\) is a solution of inequality (3.1), then \(x\) is a solution of the following integral inequality:

\[
\begin{align*}
|y(t) - x(t)| & \leq c_{f,a,b,c,g,\varphi}(\varphi(t) + \psi), \\
& \leq c_{f,a,b,c,g,\varphi}(\varphi(t) + \psi), \\
& \leq c_{f,a,b,c,g,\varphi}(\varphi(t) + \psi), \\
& \leq c_{f,a,b,c,g,\varphi}(\varphi(t) + \psi), \\
& \leq c_{f,a,b,c,g,\varphi}(\varphi(t) + \psi).
\end{align*}
\]
In fact, by Remark 3.1, we get

\[
\begin{aligned}
\left\{ \begin{array}{l}
\cd D^\alpha_{0,t}(cD^\beta_{0,t} + \lambda)x(t) = f(t,x(t),cD^\alpha_{0,t}(cD^\beta_{0,t} + \lambda)x(t)) + G(t), \\
t \in (t_k,t_{k+1}], k = 0,1,\ldots,m, 0 < \alpha, \beta < 1, \\
x(t) = (\Delta(t^\beta_k - t^\beta)/(\Gamma(\beta+1)) - 1)g_k(t,x(t)) + G_k, \\
t \in (s_{k-1},t_k], k = 1,2,\ldots,m.
\end{array} \right.
\end{aligned}
\tag{3.3}
\]

Clearly, the solution of Eq. (3.3) is given by

\[
x(t) = \begin{cases}
\frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t-s)^{\alpha+\beta-1}f(s,x(s),cD^\alpha_{0,s}(cD^\beta_{0,s} + \lambda)x(s)) + G(s) \, ds \\
- \frac{\Delta(t^\beta - t^\beta)}{\Gamma(\beta + 1)\Gamma(\alpha + \beta)} \int_0^T (T-s)^{\alpha+\beta-1}f(s,x(s),cD^\alpha_{0,s}(cD^\beta_{0,s} + \lambda)x(s)) + G(s) \, ds \\
+ \frac{\lambda\Delta(t^\beta - t^\beta)}{\Gamma(\beta + 1)\Gamma(\alpha + \beta)} \int_0^T (T-s)^{\beta-1}x(s) \, ds - \frac{\lambda}{\Gamma(\beta+1)} \int_0^t (t-s)^{\beta-1}x(s) \, ds \\
+ \frac{\theta\Delta(t^\beta - t^\beta)}{\Gamma(\beta + 1)\Gamma(\alpha + \beta)p} \int_0^T (T-s)^{\alpha+\beta-1}f(s,x(s),cD^\alpha_{0,s}(cD^\beta_{0,s} + \lambda)x(s)) + G(s) \, ds \\
- \frac{\theta\Delta(t^\beta - t^\beta)}{\Gamma(\beta + 1)\Gamma(\alpha + \beta)p} \int_0^T (T-s)^{\beta-1}x(s) \, ds - \frac{\theta}{\Gamma(\beta+1)p} \int_0^t (t-s)^{\beta-1}x(s) \, ds \\
\times \frac{1}{\Gamma(\alpha + \beta)} \int_0^T (t_k - s)^{\alpha+\beta-1}f(s,x(s),cD^\alpha_{0,s}(cD^\beta_{0,s} + \lambda)x(s)) + G(s) \, ds \\
- (\Delta(t^\beta_k - t^\beta)/(\Gamma(\beta+1)) - 1)\frac{\lambda}{\Gamma(\beta+1)} \int_0^t (t_k - s)^{\beta-1}x(s) \, ds \\
- (\Delta(t^\beta_k - t^\beta)/(\Gamma(\beta+1)) - 1)g_k(t_k,x(t_k)) + G_k, \\
t \in (t_k,t_{k+1}], k = 0,1,\ldots,m, \\
(\Delta(t^\beta_k - t^\beta)/(\Gamma(\beta+1)) - 1)g_k(t_k,x(t_k)) + G_k, \\
t \in (s_{k-1},t_k], k = 1,2,\ldots,m.
\end{cases}
\]

For \(t \in (t_k,t_{k+1}], k = 0,1,\ldots,m\), we get

\[
x(t) = \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t-s)^{\alpha+\beta-1}f(s,x(s),cD^\alpha_{0,s}(cD^\beta_{0,s} + \lambda)x(s)) \, ds \\
+ \frac{\lambda}{\Gamma(\beta+1)} \int_0^t (t-s)^{\beta-1}x(s) \, ds \\
- \frac{\Delta(t^\beta_k - t^\beta)}{\Gamma(\beta + 1)\Gamma(\alpha + \beta)} \int_0^T (T-s)^{\alpha+\beta-1}f(s,x(s),cD^\alpha_{0,s}(cD^\beta_{0,s} + \lambda)x(s)) \, ds \\
+ \frac{\lambda\Delta(t^\beta_k - t^\beta)}{\Gamma(\beta + 1)\Gamma(\alpha + \beta)} \int_0^T (T-s)^{\beta-1}x(s) \, ds \\
- \frac{\theta\Delta(t^\beta_k - t^\beta)}{\Gamma(\beta + 1)\Gamma(\alpha + \beta)p} \int_0^T (T-s)^{\alpha+\beta-1}f(s,x(s),cD^\alpha_{0,s}(cD^\beta_{0,s} + \lambda)x(s)) \, ds \\
+ \frac{\theta\Delta(t^\beta_k - t^\beta)}{\Gamma(\beta + 1)\Gamma(\alpha + \beta)p} \int_0^T (T-s)^{\beta-1}x(s) \, ds \\
- (\Delta(t^\beta_k - t^\beta)/(\Gamma(\beta+1)) - 1)\frac{\lambda}{\Gamma(\beta+1)} \int_0^t (t_k - s)^{\beta-1}x(s) \, ds \\
- (\Delta(t^\beta_k - t^\beta)/(\Gamma(\beta+1)) - 1)g_k(t_k,x(t_k)) + G_k, \\
t \in (t_k,t_{k+1}], k = 0,1,\ldots,m, \\
(\Delta(t^\beta_k - t^\beta)/(\Gamma(\beta+1)) - 1)g_k(t_k,x(t_k)) + G_k, \\
t \in (s_{k-1},t_k], k = 1,2,\ldots,m.
\]
\[
\begin{align*}
&\times \frac{1}{\Gamma(\alpha + \beta)} \int_0^{t_k} (t_k - s)^{\alpha + \beta - 1} f(s, x(s), D^\mu_{\alpha,2}(D^\nu_{\alpha,2} + \lambda) x(s)) \, ds \\
&+ \left( \Delta (t_k^\rho - t^\rho) \left( \frac{\theta \eta^p - \Gamma(p + 1)}{\Gamma(p + 1)} \right) - \lambda \right) \frac{1}{\Gamma(\alpha + \beta)} \int_0^{t_k} (t_k - s)^{\alpha + \beta - 1} x(s) \, ds \\
&+ \left( \Delta (t_k^\rho - t^\rho) \left( \frac{\theta \eta^p - \Gamma(p + 1)}{\Gamma(p + 1)} \right) - 1 \right) \frac{\lambda}{\Gamma(\beta)} \int_0^{t_k} (t_k - s)^{\beta - 1} x(s) \, ds \\
&\leq \frac{1}{\Gamma(\alpha + \beta)} \int_0^{t} (t - s)^{\alpha + \beta - 1} \psi(s) \, ds + \frac{\lambda}{\Gamma(\beta)} \int_0^{t} (t - s)^{\beta - 1} x(s) \, ds \\
&+ \left( \Delta (t_k^\rho - t^\rho) \left( \frac{\theta \eta^p - \Gamma(p + 1)}{\Gamma(p + 1)} \right) - \lambda \right) \frac{1}{\Gamma(\alpha + \beta)} \int_0^{t_k} (t_k - s)^{\alpha + \beta - 1} G(s) \, ds \\
&+ \left( \Delta (t_k^\rho - t^\rho) \left( \frac{\theta \eta^p - \Gamma(p + 1)}{\Gamma(p + 1)} \right) - 1 \right) \frac{\lambda}{\Gamma(\beta)} \int_0^{t_k} (t_k - s)^{\beta - 1} x(s) \, ds + |G_k| \\
&\leq \frac{1}{\Gamma(\alpha + \beta)} \int_0^{t} (t - s)^{\alpha + \beta - 1} \psi(s) \, ds + \frac{\lambda}{\Gamma(\beta)} \int_0^{t} (t - s)^{\beta - 1} x(s) \, ds \\
&+ \left( \Delta (t_k^\rho - t^\rho) \left( \frac{\theta \eta^p - \Gamma(p + 1)}{\Gamma(p + 1)} \right) - \lambda \right) \frac{1}{\Gamma(\alpha + \beta)} \int_0^{t_k} (t_k - s)^{\alpha + \beta - 1} G(s) \, ds \\
&+ \left( \Delta (t_k^\rho - t^\rho) \left( \frac{\theta \eta^p - \Gamma(p + 1)}{\Gamma(p + 1)} \right) - 1 \right) \frac{\lambda}{\Gamma(\beta)} \int_0^{t_k} (t_k - s)^{\beta - 1} x(s) \, ds + \psi.
\end{align*}
\]

Proceeding as above, we derive
\[
|x(t) - \left( \frac{\Delta (t_k^\rho - t^\rho) \left( \frac{\theta \eta^p - \Gamma(p + 1)}{\Gamma(p + 1)} \right) - \lambda}{\Gamma(\beta)} \right) g_k(t, x(t))| \leq |G_k| \leq \psi,
\]
\[t \in [s_{k-1}, t_k], k = 0, 1, \ldots, m,\]
and

\[
\begin{align*}
|x(t)| &= \left| 1 - \frac{\Delta(t\eta^\beta - \Gamma(p + 1)t^\beta)}{\Gamma(p + 1)\Gamma(\beta + 1)} x_0 - \frac{\lambda \Delta t^\beta}{\Gamma(\beta)\Gamma(\beta + 1)} \int_0^T (T-s)^{\beta-1} x(s) \, ds \\
+ \frac{\lambda}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} x(s) \, ds \\
- \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t-s)^{\alpha + \beta - 1} f(s, x(s), cD_{0+}^\alpha(D_{0+}^\beta + \lambda) x(s)) \, ds \\
+ \frac{\Delta t^\beta}{\Gamma(\beta + 1)\Gamma(\alpha + \beta)} \int_0^T (T-s)^{\alpha + \beta - 1} f(s, x(s), cD_{0+}^\alpha(D_{0+}^\beta + \lambda) x(s)) \, ds \\
- \frac{\theta \Delta t^\beta}{\Gamma(\beta + 1)\Gamma(\alpha + \beta + p)} \int_0^\eta (\eta-s)^{\alpha + \beta + p - 1} f(s, x(s), cD_{0+}^\alpha(D_{0+}^\beta + \lambda) x(s)) \, ds \\
+ \frac{\theta \Delta t^\beta}{\Gamma(\beta + 1)\Gamma(\beta + p)} \int_0^\eta (\eta-s)^{\beta + p - 1} x(s) \, ds \\
\right| \\
\leq & \left| \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t-s)^{\alpha + \beta - 1} \psi(s) \, ds \\
+ \frac{\Delta t^\beta}{\Gamma(\beta + 1)\Gamma(\alpha + \beta)} \int_0^T (T-s)^{\alpha + \beta - 1} \psi(s) \, ds \\
+ \frac{\lambda \Delta t^\beta}{\Gamma(\beta)\Gamma(\beta + 1)} \int_0^T (T-s)^{\beta-1} x(s) \, ds \\
+ \frac{\lambda}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} x(s) \, ds + \frac{\theta \Delta t^\beta}{\Gamma(\beta + 1)\Gamma(\alpha + \beta + p)} \int_0^\eta (\eta-s)^{\alpha + \beta + p - 1} \psi(s) \, ds \\
+ \frac{\theta \Delta t^\beta}{\Gamma(\beta + 1)\Gamma(\beta + p)} \int_0^\eta (\eta-s)^{\beta + p - 1} x(s) \, ds, \quad t \in (0, s_0].
\end{align*}
\]

\section{Main results via fixed point methods}

In order to apply a fixed point theorem of the alternative for contractions on a generalized complete metric space to achieve our main result, we want to collect the following facts.

\begin{definition}
For a nonempty set \(V\), a function \(d : V \times V \to [0, \infty]\) is called a generalized metric on \(V\) if and only if \(d\) satisfies
\begin{itemize}
  \item \(d(v_1, v_2) = 0\) if and only if \(v_1 = v_2\);
  \item \(d(v_1, v_2) = d(v_2, v_1)\) for all \(v_1, v_2 \in V\);
  \item \(d(v_1, v_3) \leq d(v_1, v_2) + d(v_2, v_3)\) for all \(v_1, v_2, v_3 \in V\).
\end{itemize}
\end{definition}
Lemma 4.1 ([9]) Let \((V,d)\) be a generalized complete metric space. Assume that \(T : V \to V\) is a strictly contraction operator with the Lipschitz constant \(L < 1\). If there exists a \(k \geq 0\) such that \(d(T^{k+1}v, T^k v) < \infty\) for some \(v \in V\), then the following statements are true:

\((B_1)\) The sequence \(\{T^n v\}\) converges to a fixed point \(v^*\) of \(T\);
\((B_2)\) The unique fixed point of \(T\) is \(v^* \in V^* = \{u \in V \text{ such that } d(T^k v, u) < \infty\}\);
\((B_3)\) If \(u \in V^*\), then \(d(u, v^*) \leq \frac{1}{1-\lambda} d(Tu, u)\).

We can introduce some assumptions as follows:
\((H_1)\) \(f \in C(U \times \mathbb{R} \times \mathbb{R}, \mathbb{R})\).
\((H_2)\) There exists a positive constant \(L_f\) such that
\[
|f(t, u_1, u_2) - f(t, u_2, u_2)| \leq L_f |u_1 - u_2| + \tilde{L}_2 |u_1 - u_2|, \quad \text{for each } t \in T \text{ and all } u_1, u_2 \in \mathbb{R}.
\]
\((H_3)\) \(g_k \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})\) and there are positive constant \(L_{g_k}\), \(k = 1, 2, \ldots, m\) such that
\[
|g_k(t, u_1) - g_k(t, u_2)| \leq L_{g_k} |u_1 - u_2|, \quad \text{for each } t \in (s_{k-1}, t_{k}], \text{ and all } u_1, u_2 \in \mathbb{R}.
\]
\((H_4)\) Let \(\varphi \in C(J, \mathbb{R})\) be a non-decreasing function. There exists \(c_{\varphi} > 0\) such that
\[
\left( \int_0^t (\varphi(s)^{\frac{1}{p}} ds \right)^p \leq C_{\varphi} \varphi(t) \quad \text{for each } t \in I.
\]

Theorem 4.2 Suppose that \((H_1)\) and \((H_2)\) are satisfied and also a function \(y \in PC(J, \mathbb{R})\) satisfies (3.1). Then there exists a unique solution \(x\) of Eq. (1.1) such that
\[
x(t) = \begin{cases}
\frac{1}{\Gamma(\alpha + \beta)} \int_0^t \left( t - s \right)^{\alpha - 1} f(s, x(s), \zeta D_{0^+}^\alpha (D_{0^+}^\beta + \lambda) x(s) ds \\
- \frac{\lambda}{\Gamma(\alpha + \beta)} \int_0^t \left( t - s \right)^{\alpha - 1} x(s) ds + \frac{\lambda}{\Gamma(\beta + 1)} \int_0^t \left( T - s \right)^{\beta - 1} x(s) ds \\
- \Delta^\beta \int_0^t \left( T - s \right)^{\alpha - 1} f(s, x(s), \zeta D_{0^+}^\alpha (D_{0^+}^\beta + \lambda) x(s) ds \\
+ \Delta^\beta \int_0^t \left( T - s \right)^{\alpha - 1} f(s, x(s), \zeta D_{0^+}^\alpha (D_{0^+}^\beta + \lambda) x(s) ds \\
- \Delta^\beta \int_0^t \left( s - x(s) \right)^{\beta - 1} x(s) ds - \left( \Delta^\beta \int_0^t \left( x(s) \right)^{\beta - 1} x(s) ds - \right)
\end{cases}
\]
\[
\times \frac{1}{\Gamma(\alpha + \beta)} \int_0^t \left( t - s \right)^{\alpha - 1} f(s, x(s), \zeta D_{0^+}^\alpha (D_{0^+}^\beta + \lambda) x(s) ds \\
- \frac{\lambda}{\Gamma(\alpha + \beta)} \int_0^t \left( t - s \right)^{\alpha - 1} x(s) ds + \frac{\lambda}{\Gamma(\beta + 1)} \int_0^t \left( T - s \right)^{\beta - 1} x(s) ds \\
- \Delta^\beta \int_0^t \left( s - x(s) \right)^{\beta - 1} x(s) ds - \left( \Delta^\beta \int_0^t \left( x(s) \right)^{\beta - 1} x(s) ds - \right)
\end{cases}
\]
\[
\times \frac{1}{\Gamma(\alpha + \beta)} \int_0^t \left( t - s \right)^{\alpha - 1} f(s, x(s), \zeta D_{0^+}^\alpha (D_{0^+}^\beta + \lambda) x(s) ds \\
- \frac{\lambda}{\Gamma(\alpha + \beta)} \int_0^t \left( t - s \right)^{\alpha - 1} x(s) ds + \frac{\lambda}{\Gamma(\beta + 1)} \int_0^t \left( T - s \right)^{\beta - 1} x(s) ds \\
- \Delta^\beta \int_0^t \left( s - x(s) \right)^{\beta - 1} x(s) ds - \left( \Delta^\beta \int_0^t \left( x(s) \right)^{\beta - 1} x(s) ds - \right)
\end{cases}
\]
\[
g_k \left( t_k, x(t_k) \right), \quad t \in (s_{k-1}, t_k], k = 1, 2, \ldots, m,
\]
\[
g_{l+1} \left( t_{l+1}, x(t_{l+1}) \right), \quad t \in (t_{l+1}, t_{l+2}], k = 1, 2, \ldots, m,
\]
\[
g_m \left( t_m, x(t_m) \right), \quad t \in (s_{m-1}, t_m] \cup \{ t_m \}.
\]
and

\[
|y(t) - x(t)| \leq \left\{ \frac{C_{\psi}}{\Gamma(\alpha + \beta)} \left( \frac{1 - r}{\alpha + \beta - r} \right)^{1-r} s^{\alpha \beta - r} \right. \\
+ \frac{\lambda C_{\psi}(t_k^\beta - t^\beta)}{\Gamma(\beta + 1)} \left( \frac{1 - r}{\alpha + \beta - r} \right)^{1-r} T^{\alpha \beta - r} \\
+ \frac{\lambda \Delta C_{\psi} t_k^\beta}{\Gamma(\beta + 1)\Gamma(\alpha + \beta)} \left( \frac{1 - r}{\beta - r} \right)^{1-r} T^{\beta - r} \\
+ \frac{\theta \Delta C_{\psi} t_k^\beta}{\Gamma(\beta + 1)\Gamma(\alpha + \beta + p)} \left( \frac{1 - r}{\beta + p - r} \right)^{1-r} \eta^{\alpha \beta p - r} \\
+ \frac{C_{\psi}\theta \Delta \lambda (t_k^\beta - t^\beta)}{\Gamma(\beta + 1)\Gamma(\alpha + \beta + p)} \left( \frac{1 - r}{\beta + p - r} \right)^{1-r} \eta^{\beta - r} \\
+ \left( \frac{\alpha \psi(t) + \psi}{1 - \lambda} \right) \left\{ \frac{\alpha}{\alpha + \beta - r} \right\} \\
+ \left( \frac{\alpha}{\alpha + \beta - r} \right)^{1-r} t_k^{\alpha \beta - 1} \\
\times \left( \frac{\psi(t) + \psi}{1 - \lambda} \right)
\]

(4.3)

for all \( t \in J \) if \( 0 < \alpha < \beta < 1 \), with

\[
M = \max\{M_1, M_2\} < 1,
\]

(4.4)

where

\[
M_1 = \max \left\{ \frac{1}{\Gamma(\alpha + \beta)} \left( \frac{1 - r}{\alpha + \beta - r} \right)^{1-r} (L_{f_1} C_{\psi} + L_{f_2} C_{\psi})_k^{\alpha \beta - r} \\
+ \frac{\lambda C_{\psi}(t_k^\beta - t^\beta)}{\Gamma(\beta + 1)} \left( \frac{1 - r}{\alpha + \beta - r} \right)^{1-r} T^{\alpha \beta - r} \\
+ \frac{\lambda \Delta C_{\psi} t_k^\beta}{\Gamma(\beta + 1)\Gamma(\alpha + \beta)} \left( \frac{1 - r}{\beta - r} \right)^{1-r} T^{\beta - r} \\
+ \frac{\theta \Delta C_{\psi} t_k^\beta}{\Gamma(\beta + 1)\Gamma(\alpha + \beta + p)} \left( \frac{1 - r}{\beta + p - r} \right)^{1-r} \eta^{\alpha \beta p - r} \\
+ \frac{C_{\psi}\theta \Delta \lambda (t_k^\beta - t^\beta)}{\Gamma(\beta + 1)\Gamma(\alpha + \beta + p)} \left( \frac{1 - r}{\beta + p - r} \right)^{1-r} \eta^{\beta - r} \\
+ \left( \frac{\alpha \psi(t) + \psi}{1 - \lambda} \right) \left\{ \frac{\alpha}{\alpha + \beta - r} \right\} \\
+ \left( \frac{\alpha}{\alpha + \beta - r} \right)^{1-r} t_k^{\alpha \beta - 1} \\
\times \left( \frac{\psi(t) + \psi}{1 - \lambda} \right) \right\}
\]
and Proof

Consider the space of piecewise continuous functions

\[ V = \{ g : J \to \mathbb{R} \text{ such that } g \in PC(J, \mathbb{R}) \} \]

endowed with the generalized metric on \( V \), defined by

\[ d(g, h) = \inf \left\{ C_1 + C_2 \in [0, +\infty] \middle| \left| g(t) - h(t) \right| \leq (C_1 + C_2)(\varphi(t) + \psi) \text{ for all } t \in J \right\}, \tag{4.5} \]

where

\[ C_1 \in \{ C \in [0, \infty] \text{ such that } \left| g(t) - h(t) \right| \leq C \varphi(t) \text{ for all } t \in (t_k, s_k], k = 0, 1, \ldots, m \} \]

and

\[ C_2 \in \{ C \in [0, \infty] \text{ such that } \left| g(t) - h(t) \right| \leq C \psi \text{ for all } t \in (s_{k-1}, t_k], k = 1, 2, \ldots, m \}. \]

It is easy to verify that \((V, d)\) is a complete generalized metric space [19].
Define an operator $A : V \rightarrow V$ by

\[
(Ax) (t) = \begin{cases}
\frac{1}{\Gamma (\alpha + \beta)} \int_{0}^{t} (t-s)^{\alpha+\beta-1} f(s, x(s), D_{0+}^{\alpha} (D_{0+}^{\beta} + \lambda)) x(s) \, ds \\
- \frac{1}{\Gamma (\alpha + \beta)} \int_{0}^{t} (t-s)^{\beta-1} x(s) \, ds + \frac{\lambda \Delta t^{\beta}}{\Gamma (\alpha + \beta + 1)} \int_{0}^{T} (T-s)^{\beta-1} x(s) \, ds \\
- \frac{\theta \Delta t^{\beta}}{\Gamma (\alpha + \beta + 1)} \int_{0}^{t} (s) \, ds + \frac{\theta \Delta t^{\beta}}{\Gamma (\alpha + \beta + 1)} \int_{0}^{T} (T-s)^{\beta-1} x(s) \, ds \\
- \frac{\theta \Delta t^{\beta}}{\Gamma (\alpha + \beta + 1)} \int_{0}^{t} (s) \, ds + \frac{\theta \Delta t^{\beta}}{\Gamma (\alpha + \beta + 1)} \int_{0}^{T} (T-s)^{\beta-1} x(s) \, ds \\
+ \frac{\theta \Delta t^{\beta}}{\Gamma (\alpha + \beta + 1)} \int_{0}^{t} (s) \, ds + \frac{\theta \Delta t^{\beta}}{\Gamma (\alpha + \beta + 1)} \int_{0}^{T} (T-s)^{\beta-1} x(s) \, ds \\
- \frac{\lambda \Delta t^{\beta}}{\Gamma (\alpha + \beta + 1)} \int_{0}^{T} (T-s)^{\beta-1} x(s) \, ds \\
\end{cases}
\]

(4.6)

for all $x$ belongs to $V$ and $t \in J$. Obviously, according to $(H_1)$, $A$ is a well-defined operator.

Next we shall verify that $A$ is strictly contractive on $V$. Note that according to definition of $(V, d)$, for any $g, h \in V$, it is possible to find $C_1, C_2, C_3, C_4 \in [0, \infty]$ such that

\[
|g(t) - h(t)| \leq \begin{cases}
C_1 \varphi(t), & t \in (t_k, s_k), k = 0, \ldots, m, \\
C_2 \varphi, & t \in (s_{k-1}, t_k), k = 1, \ldots, m,
\end{cases}
\]

(4.7)

and

\[
|D_{0+}^{\alpha} (D_{0+}^{\beta} + \lambda) g(s) - D_{0+}^{\alpha} (D_{0+}^{\beta} + \lambda) h(s)| \\
\leq \begin{cases}
C_3 \zeta(t) \leq C_1 \varphi(t), & t \in (t_k, s_k), k = 0, \ldots, m, \\
C_4 \zeta(t) \leq C_2 \psi, & t \in (s_{k-1}, t_k), k = 1, \ldots, m.
\end{cases}
\]

From the definition of $A \in$ Eq. (4.6), $(H_2), (H_3)$ and (4.7), we obtain that

Case 1. For $t \in [0, s_0]$,

\[
\left| (Ag)(t) - (Ah)(t) \right| \\
\leq \frac{1}{\Gamma (\alpha + \beta)} \int_{0}^{t} (t-s)^{\alpha+\beta-1} \\
\times \left[ f(s, g(s), D_{0+}^{\alpha} (D_{0+}^{\beta} + \lambda) g(s)) - f(s, h(s), D_{0+}^{\alpha} (D_{0+}^{\beta} + \lambda) h(s)) \right] \, ds \\
+ \frac{\lambda \Delta t^{\beta}}{\Gamma (\beta + 1) \Gamma (\alpha + \beta)} \int_{0}^{T} (T-s)^{\beta-1} \left| g(s) - h(s) \right| \, ds + \frac{\lambda \Delta t^{\beta}}{\Gamma (\beta + 1) \Gamma (\alpha + \beta)} \int_{0}^{T} (T-s)^{\beta-1}
\]

\[
	imes \left[ f(s, g(s), D_{0+}^{\alpha} (D_{0+}^{\beta} + \lambda) g(s)) - f(s, h(s), D_{0+}^{\alpha} (D_{0+}^{\beta} + \lambda) h(s)) \right] \, ds \\
+ \frac{\lambda \Delta t^{\beta}}{\Gamma (\beta + 1) \Gamma (\alpha + \beta)} \int_{0}^{T} (T-s)^{\beta-1} \left| g(s) - h(s) \right| \, ds + \frac{\lambda \Delta t^{\beta}}{\Gamma (\beta + 1) \Gamma (\alpha + \beta)} \int_{0}^{T} (T-s)^{\beta-1}
\]
\[
\times |f(s, g(s), \mathcal{D}^\alpha_{0, \lambda} (\mathcal{D}^\beta_{0, \lambda} + \lambda) g(s)) - f(s, h(s), \mathcal{D}^\alpha_{0, \lambda} (\mathcal{D}^\beta_{0, \lambda} + \lambda) h(s))| \, ds \\
+ \lambda \mathcal{D}^\beta (T - s)^{\beta - 1} |g(s) - h(s)| \, ds \\
+ \frac{\theta \mathcal{D}^\beta}{\Gamma(\beta + 1) \Gamma(\alpha + \beta + p)} \int_0^\eta (\eta - s)^{\alpha + \beta - 1} |g(s) - h(s)| \, ds \\
\times |f(s, g(s), \mathcal{D}^\alpha_{0, \lambda} (\mathcal{D}^\beta_{0, \lambda} + \lambda) g(s)) - f(s, h(s), \mathcal{D}^\alpha_{0, \lambda} (\mathcal{D}^\beta_{0, \lambda} + \lambda) h(s))| \\
+ \frac{\theta \Delta \mathcal{D}^\beta}{\Gamma(\beta + 1) \Gamma(\alpha + \beta + p)} \int_0^\eta (\eta - s)^{\alpha + \beta - 1} |g(s) - h(s)| \, ds \\
\leq \frac{1}{\Gamma(\alpha + \beta)} \int_0^T (T - s)^{\alpha + \beta - 1} |g(s) - h(s)| \, ds \\
+ \frac{\lambda}{\Gamma(\beta)} \int_0^T (T - s)^{\beta - 1} |g(s) - h(s)| \, ds \\
+ \frac{\mathcal{D}^\beta}{\Gamma(\beta + 1) \Gamma(\alpha + \beta + p)} \int_0^\eta (\eta - s)^{\alpha + \beta - 1} |g(s) - h(s)| \, ds \\
\times |f(s, g(s), \mathcal{D}^\alpha_{0, \lambda} (\mathcal{D}^\beta_{0, \lambda} + \lambda) g(s)) - f(s, h(s), \mathcal{D}^\alpha_{0, \lambda} (\mathcal{D}^\beta_{0, \lambda} + \lambda) h(s))| \\
+ \frac{\theta \Delta \mathcal{D}^\beta}{\Gamma(\beta + 1) \Gamma(\alpha + \beta + p)} \int_0^\eta (\eta - s)^{\alpha + \beta - 1} |g(s) - h(s)| \, ds \\
= \frac{L_1}{\Gamma(\alpha + \beta)} \int_0^T (T - s)^{\alpha + \beta - 1} |g(s) - h(s)| \, ds \\
+ \frac{L_1}{\Gamma(\beta)} \int_0^T (T - s)^{\beta - 1} |g(s) - h(s)| \, ds \\
+ \frac{\mathcal{D}^\beta}{\Gamma(\beta + 1) \Gamma(\alpha + \beta + p)} \int_0^\eta (\eta - s)^{\alpha + \beta - 1} |g(s) - h(s)| \, ds \\
\times \int_0^T (T - s)^{\alpha + \beta - 1} |c\mathcal{D}^\alpha_{0, \lambda} (\mathcal{D}^\beta_{0, \lambda} + \lambda) g(s) - c\mathcal{D}^\alpha_{0, \lambda} (\mathcal{D}^\beta_{0, \lambda} + \lambda) h(s)| \, ds \\
+ \frac{\lambda \mathcal{D}^\beta}{\Gamma(\beta + 1) \Gamma(\alpha + \beta + p)} \int_0^T (T - s)^{\beta - 1} |g(s) - h(s)| \, ds \\
+ \frac{L_1 \mathcal{D}^\beta}{\Gamma(\beta + 1) \Gamma(\alpha + \beta + p)} \int_0^\eta (\eta - s)^{\alpha + \beta - 1} |g(s) - h(s)| \, ds \\
\times \int_0^T (T - s)^{\alpha + \beta - 1} |c\mathcal{D}^\alpha_{0, \lambda} (\mathcal{D}^\beta_{0, \lambda} + \lambda) g(s) - c\mathcal{D}^\alpha_{0, \lambda} (\mathcal{D}^\beta_{0, \lambda} + \lambda) h(s)| \, ds \\
+ \frac{\theta \mathcal{D}^\beta}{\Gamma(\beta + 1) \Gamma(\alpha + \beta + p)} \int_0^\eta (\eta - s)^{\alpha + \beta - 1} |g(s) - h(s)| \, ds
\[
\begin{align*}
&\leq \frac{L_{f_1}C_1}{\Gamma(\alpha + \beta)} \int_0^t (t-s)^{\alpha-\beta-1} |\varphi(s)| \, ds + \frac{\lambda C_1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} |\varphi(s)| \, ds \\
&+ \frac{\tilde{L}_{f_2} C_1}{\Gamma(\alpha + \beta)} \int_0^t (t-s)^{\alpha-\beta-1} |\varphi(s)| \, ds \\
&+ \frac{L_{f_1} C_1 \Delta t^\theta}{\Gamma(\beta + 1) \Gamma(\alpha + \beta)} \int_0^T (T-s)^{\alpha+\beta-1} |\varphi(s)| \, ds \\
&+ \frac{\lambda C_1 \Delta t^\theta}{\Gamma(\beta) \Gamma(\beta + 1)} \int_0^T (T-s)^{\beta-1} |\varphi(s)| \, ds \\
&+ \frac{\tilde{L}_{f_2} C_1 \Delta t^\theta}{\Gamma(\beta + 1) \Gamma(\alpha + \beta)} \int_0^T (T-s)^{\alpha+\beta-1} |\varphi(s)| \, ds \\
&+ \frac{L_{f_1} C_1 \theta \Delta t^0}{\Gamma(\beta + 1) \Gamma(\alpha + \beta + p)} \int_0^\eta (\eta-s)^{\alpha+\beta+p-1} |\varphi(s)| \, ds \\
&+ \frac{L_{f_2} C_1 \theta \Delta t^0}{\Gamma(\beta + 1) \Gamma(\alpha + \beta + p)} \int_0^\eta (\eta-s)^{\alpha+\beta+p-1} |\varphi(s)| \, ds \\
&+ \frac{C_1 \theta \Delta t^0}{\Gamma(\beta + 1) \Gamma(\beta + p)} \int_0^\eta (\eta-s)^{\alpha+\beta+p-1} |\varphi(s)| \, ds \\
&\leq \frac{L_{f_1} C_1}{\Gamma(\alpha + \beta)} \left( \int_0^t (t-s)^{\alpha+\beta-1} \, ds \right)^{1-r} \left( \int_0^t (\varphi(s))^{\frac{1}{r}} \, ds \right)^r \\
&+ \frac{\tilde{L}_{f_2} C_1}{\Gamma(\alpha + \beta)} \left( \int_0^t (t-s)^{\alpha+\beta-1} \, ds \right)^{1-r} \left( \int_0^t (\varphi(s))^{\frac{1}{r}} \, ds \right)^r \\
&+ \frac{L_{f_1} C_1 \Delta t^\theta}{\Gamma(\beta + 1) \Gamma(\alpha + \beta)} \left( \int_0^T (T-s)^{\alpha+\beta-1} \, ds \right)^{1-r} \left( \int_0^T (\varphi(s))^{\frac{1}{r}} \, ds \right)^r \\
&+ \frac{\tilde{L}_{f_2} C_1 \Delta t^\theta}{\Gamma(\beta + 1) \Gamma(\alpha + \beta)} \left( \int_0^T (T-s)^{\alpha+\beta-1} \, ds \right)^{1-r} \left( \int_0^T (\varphi(s))^{\frac{1}{r}} \, ds \right)^r \\
&+ \frac{C_1 \theta \Delta t^0}{\Gamma(\beta) \Gamma(\beta + 1)} \left( \int_0^T (T-s)^{\alpha+\beta-1} \, ds \right)^{1-r} \left( \int_0^T (\varphi(s))^{\frac{1}{r}} \, ds \right)^r \\
&+ \frac{L_{f_1} C_1 \theta \Delta t^0}{\Gamma(\beta + 1) \Gamma(\alpha + \beta + p)} \left( \int_0^\eta (\eta-s)^{\alpha+\beta+p-1} \, ds \right)^{1-r} \left( \int_0^\eta (\varphi(s))^{\frac{1}{r}} \, ds \right)^r \\
&+ \frac{L_{f_2} C_1 \theta \Delta t^0}{\Gamma(\beta + 1) \Gamma(\alpha + \beta + p)} \left( \int_0^\eta (\eta-s)^{\alpha+\beta+p-1} \, ds \right)^{1-r} \left( \int_0^\eta (\varphi(s))^{\frac{1}{r}} \, ds \right)^r \\
&+ \frac{\theta \Delta t^0 C_1}{\Gamma(\beta) \Gamma(\beta + 1)} \left( \int_0^\eta (\eta-s)^{\alpha+\beta+p-1} \, ds \right)^{1-r} \left( \int_0^\eta (\varphi(s))^{\frac{1}{r}} \, ds \right)^r \\
&\leq \frac{L_{f_1} C_1 C_\varphi(t)}{\Gamma(\alpha + \beta + p)} \left( \frac{1-r}{\alpha + \beta - r} \right)^{1-r} T^{\alpha+\beta-p} + \frac{\tilde{L}_{f_2} C_1 C_\varphi(t)}{\Gamma(\alpha + \beta + p)} \left( \frac{1-r}{\alpha + \beta - r} \right)^{1-r} T^{\alpha+\beta-p} \\
&+ \frac{\lambda C_1 C_\varphi(t)}{\Gamma(\beta + 1)} \left( \frac{1-r}{\beta - r} \right)^{1-r} T^{\beta-p} + \frac{L_{f_1} C_1 C_\varphi(t) \Delta t^\theta}{\Gamma(\beta + 1) \Gamma(\alpha + \beta)} \left( \frac{1-r}{\alpha + \beta - r} \right)^{1-r} T^{\alpha+\beta-p} \\
&+ \frac{\tilde{L}_{f_2} C_1 C_\varphi(t) \Delta t^\theta}{\Gamma(\beta + 1) \Gamma(\alpha + \beta)} \left( \frac{1-r}{\alpha + \beta - r} \right)^{1-r} T^{\alpha+\beta-p} + \frac{C_1 C_\varphi(t) \lambda \Delta t^\theta}{\Gamma(\beta) \Gamma(\beta + 1)} \left( \frac{1-r}{\beta - r} \right)^{1-r} T^{\beta-p} 
\end{align*}
\]
\[ \begin{align*}
&+ \frac{L_{f_{1}} C_{1} C_{\psi}(t) \theta \Delta t^{\beta}}{\Gamma(\beta + 1) \Gamma(\alpha + \beta + p)} \left( \frac{1 - r}{\alpha + \beta + p - r} \right)^{1-r} \eta^{\alpha + \beta + p - r} \\
&+ \frac{\bar{L}_{f_{2}} C_{1} C_{\psi}(t) \theta \Delta t^{\beta}}{\Gamma(\beta + 1) \Gamma(\alpha + \beta + p)} \left( \frac{1 - r}{\alpha + \beta + p - r} \right)^{1-r} \eta^{\alpha + \beta + p - r} \\
&+ \frac{\theta \Delta t^{\beta}}{\Gamma(\beta + 1) \Gamma(\alpha + \beta + p)} \left( \frac{1 - r}{\beta + p - r} \right)^{1-r} \eta^{\beta + p - r} \\
&+ \frac{\lambda \Delta t^{\beta}}{\Gamma(\beta + 1) \Gamma(\alpha + \beta + p)} \left( \frac{1 - r}{\beta - r} \right)^{1-r} T^{\beta - r} \\
&+ \frac{\lambda}{\Gamma(\beta + 1) \Gamma(\alpha + \beta + p)} \left( \frac{r - 1}{\beta - r} \right)^{1-r} s_{0}^{\beta - r} + \frac{\Delta t^{\beta}}{\Gamma(\beta + 1) \Gamma(\alpha + \beta + p)} \left( \frac{1 - r}{\beta - r} \right)^{1-r} (L_{f_{1}} + \bar{L}_{f_{2}}) T^{\alpha + \beta - p - r} \\
&+ \frac{\theta \Delta t^{\beta}}{\Gamma(\beta + 1) \Gamma(\alpha + \beta + p)} \left( \frac{1 - r}{\beta + p - r} \right)^{1-r} \eta^{\beta + p - r} \right) C_{1} C_{\psi}(t).
\end{align*} \]

**Case 2.** For \( t \in (s_{k-1}, t_{k}] \), we have

\[ |(Ag t - (Ah) t)| = |g_{k}(t, g(t)) - g_{k}(t, h(t))| \leq L_{g_{k}} |g(t) - h(t)| \leq L_{g_{k}} C_{2} \psi. \]

**Case 3.** For \( t \in (t_{k}, s_{k}] \) and \( s \in (t_{k}, s_{k}] \),

\[ |(Ag)(t) - (Ah)(t)| \]

\[ \leq \frac{1}{\Gamma(\alpha + \beta)} \int_{0}^{t} (t - s)^{\alpha + \beta - 1} \]

\[ \times \left| f(s, g(s), D_{0}^{\beta} D_{0}^{\alpha} + \lambda) g(s) - f(s, h(s), D_{0}^{\beta} D_{0}^{\alpha} + \lambda) h(s) \right| ds \]

\[ + \frac{\lambda}{\Gamma(\beta + 1) \Gamma(\alpha + \beta + p)} \int_{0}^{T} (T - s)^{\beta - 1} |g(s) - h(s)| ds + \frac{\Delta (t_{k}^{\beta} - t^{\beta})}{\Gamma(\alpha + \beta + p)} \int_{0}^{T} (T - s)^{\beta - 1} |g(s) - h(s)| ds \]

\[ + \frac{\theta \Delta t^{\beta}}{\Gamma(\beta + 1) \Gamma(\alpha + \beta + p)} \int_{0}^{T} (T - s)^{\beta - 1} |g(s) - h(s)| ds \]

\[ + \frac{\lambda \Delta t^{\beta}}{\Gamma(\alpha + \beta + p)} \int_{0}^{T} (T - s)^{\beta - 1} |g(s) - h(s)| ds \]

\[ + \frac{\theta \Delta t^{\beta}}{\Gamma(\alpha + \beta + p)} \int_{0}^{T} (T - s)^{\beta - 1} |g(s) - h(s)| ds \]

\[ + \left( \frac{\lambda (t_{k}^{\beta} - t^{\beta})}{\Gamma(\alpha + \beta + p)} \left( \frac{\theta \eta^{p} - \Gamma(p + 1)}{\Gamma(p + 1)} \right) - \lambda \right) \frac{1}{\Gamma(\alpha + \beta)} \int_{0}^{t_{k}} (t_{k} - s)^{\alpha + \beta - 1} \]

\[ \times \left| f(s, g(s), D_{0}^{\beta} D_{0}^{\alpha} + \lambda) g(s) - f(s, h(s), D_{0}^{\beta} D_{0}^{\alpha} + \lambda) h(s) \right| ds \]

\[ + \left( \frac{\lambda (t_{k}^{\beta} - t^{\beta})}{\Gamma(\alpha + \beta + p)} \left( \frac{\theta \eta^{p} - \Gamma(p + 1)}{\Gamma(p + 1)} \right) - 1 \right) \frac{\lambda}{\Gamma(\beta + 1) \Gamma(\alpha + \beta + p)} \int_{0}^{t_{k}} (t_{k} - s)^{\beta - 1} |g(s) - h(s)| ds \]
\[
\begin{align*}
&\frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t - s)^{\alpha + \beta - 1} \times \left( L_{t_1} |g(s) - h(s)| + \bar{L}_{t_2} |cD_{0+}^{\alpha} (cD_{0+}^{\beta} + \lambda)g(s) - cD_{0+}^{\alpha} (cD_{0+}^{\beta} + \lambda)h(s)| \right) ds \\
&\quad + \frac{\lambda}{\Gamma(\beta + 1)} \int_0^t (t - s)^{\beta - 1} |g(s) - h(s)| ds + \frac{\Delta(t_k^\beta - t^\beta)}{\Gamma(\beta + 1) \Gamma(\alpha + \beta)} \int_0^T (T - s)^{\alpha + \beta - 1} \times \left( L_{t_1} |g(s) - h(s)| + \bar{L}_{t_2} |cD_{0+}^{\alpha} (cD_{0+}^{\beta} + \lambda)g(s) - cD_{0+}^{\alpha} (cD_{0+}^{\beta} + \lambda)h(s)| \right) ds \\
&\quad + \frac{\lambda \Delta(t_k^\beta - t^\beta)}{\Gamma(\beta + 1) \Gamma(\alpha + \beta)} \int_0^T (T - s)^{\beta - 1} |g(s) - h(s)| ds \\
&\quad + \frac{\theta \Delta(t_k^\beta - t^\beta)}{\Gamma(\beta + 1) \Gamma(\alpha + \beta)} \int_0^\eta (\eta - s)^{\alpha + \beta + p - 1} |g(s) - h(s)| ds \\
&\quad + \frac{\theta \Delta(t_k^\beta - t^\beta)}{\Gamma(\beta + 1) \Gamma(\alpha + \beta + p)} \int_0^\eta (\eta - s)^{\alpha + \beta + p - 1} |g(s) - h(s)| ds \\
&\quad + \frac{\Delta(t_k^\beta - t^\beta)}{\Gamma(\beta + 1) \Gamma(\alpha + \beta)} \int_0^T (T - s)^{\alpha + \beta - 1} \times \left( L_{t_1} |g(s) - h(s)| + \bar{L}_{t_2} |cD_{0+}^{\alpha} (cD_{0+}^{\beta} + \lambda)g(s) - cD_{0+}^{\alpha} (cD_{0+}^{\beta} + \lambda)h(s)| \right) ds \\
&\quad + \frac{\Delta(t_k^\beta - t^\beta)}{\Gamma(\beta + 1) \Gamma(\alpha + \beta)} \int_0^T (T - s)^{\beta - 1} |g(s) - h(s)| ds \\
&\quad + \frac{\lambda \Delta(t_k^\beta - t^\beta)}{\Gamma(\beta + 1) \Gamma(\alpha + \beta)} \int_0^T (T - s)^{\beta - 1} |g(s) - h(s)| ds \\
&\quad + \frac{\theta \Delta(t_k^\beta - t^\beta)}{\Gamma(\beta + 1) \Gamma(\alpha + \beta)} \int_0^\eta (\eta - s)^{\alpha + \beta + p - 1} |g(s) - h(s)| ds \\
&\quad + \frac{\theta \Delta(t_k^\beta - t^\beta)}{\Gamma(\beta + 1) \Gamma(\alpha + \beta)} \int_0^\eta (\eta - s)^{\alpha + \beta + p - 1} |g(s) - h(s)| ds \\
&\quad + \frac{\theta \Delta(t_k^\beta - t^\beta)}{\Gamma(\beta + 1) \Gamma(\alpha + \beta)} \int_0^\eta (\eta - s)^{\alpha + \beta + p - 1} |g(s) - h(s)| ds \\
&\quad + \frac{\theta \Delta(t_k^\beta - t^\beta)}{\Gamma(\beta + 1) \Gamma(\alpha + \beta)} \int_0^\eta (\eta - s)^{\alpha + \beta + p - 1} |g(s) - h(s)| ds
\end{align*}
\]
\[\begin{align*}
&\left(\Delta \left(\frac{t_k^\beta - t^\beta}{\Gamma(\beta + 1)}\right) \left(\frac{\partial \eta p - \Gamma(p + 1)}{\Gamma(p + 1)}\right) - \lambda\right) \\
&\times \frac{L_{f_1}}{\Gamma(\alpha + \beta)} \int_0^{t_k} (t_k - s)^{\alpha + \beta - 1} \left| D_{0,\alpha}^\alpha \left(D_{0,\alpha}^\alpha + \lambda \right) g(s) - D_{0,\alpha}^\alpha \left(D_{0,\alpha}^\alpha + \lambda \right) h(s) \right| ds \\
&+ \left(\Delta \left(\frac{t_k^\beta - t^\beta}{\Gamma(\beta + 1)}\right) \left(\frac{\partial \eta p - \Gamma(p + 1)}{\Gamma(p + 1)}\right) - \lambda\right) \frac{L_{f_1}}{\Gamma(\alpha + \beta)} \int_0^{t_k} (t_k - s)^{\alpha + \beta - 1} \left| g(s) - h(s) \right| ds \\
&+ \left(\Delta \left(\frac{t_k^\beta - t^\beta}{\Gamma(\beta + 1)}\right) \left(\frac{\partial \eta p - \Gamma(p + 1)}{\Gamma(p + 1)}\right) - 1\right) \frac{\lambda}{\Gamma \beta} \int_0^{t_k} (t_k - s)^{\beta - 1} \left| g(s) - h(s) \right| ds \\
&+ \left(\Delta \left(\frac{t_k^\beta - t^\beta}{\Gamma(\beta + 1)}\right) \left(\frac{\partial \eta p - \Gamma(p + 1)}{\Gamma(p + 1)}\right) - 1\right) \left| g(t_k, g(t_k)) - g(t_k, h(t_k)) \right| \\
&\leq \frac{L_{f_1} C_1}{\Gamma(\alpha + \beta)} \int_0^t (t - s)^{\alpha + \beta - 1} \left| \varphi(s) \right| ds + \frac{L_{f_2} C_1}{\Gamma(\alpha + \beta)} \int_0^t (t - s)^{\alpha + \beta - 1} \left| \psi(s) \right| ds \\
&+ \frac{\lambda C_1}{\Gamma \beta} \int_0^t (t - s)^{\alpha + \beta - 1} \left| \varphi(s) \right| ds + \frac{\Delta L_{f_1} C_1 (t_k^\beta - t^\beta)}{\Gamma(\beta + 1) \Gamma(\alpha + \beta)} \int_0^T (T - s)^{\alpha + \beta - 1} \left| \varphi(s) \right| ds \\
&+ \frac{\Delta L_{f_2} C_1 (t_k^\beta - t^\beta)}{\Gamma(\beta + 1) \Gamma(\alpha + \beta)} \int_0^T (T - s)^{\alpha + \beta - 1} \left| \varphi(s) \right| ds \\
&+ \frac{\lambda C_1 \Delta (t_k^\beta - t^\beta)}{\Gamma(\beta + 1) \Gamma(\alpha + \beta)} \int_0^T (T - s)^{\alpha + \beta - 1} \left| \varphi(s) \right| ds \\
&+ \frac{L_{f_1} C_1 \theta \Delta (t_k^\beta - t^\beta)}{\Gamma(\beta + 1) \Gamma(\alpha + \beta + p)} \int_0^\eta (\eta - s)^{\alpha + \beta + p - 1} \left| \varphi(s) \right| ds \\
&+ \frac{L_{f_2} C_1 \theta \Delta (t_k^\beta - t^\beta)}{\Gamma(\beta + 1) \Gamma(\alpha + \beta + p)} \int_0^\eta (\eta - s)^{\alpha + \beta + p - 1} \left| \varphi(s) \right| ds \\
&+ \frac{\theta C_1 \Delta \lambda (t_k^\beta - t^\beta)}{\Gamma(\beta + 1) \Gamma(\alpha + \beta + p)} \int_0^\eta (\eta - s)^{\alpha + \beta + p - 1} \left| \varphi(s) \right| ds \\
&+ \left(\Delta \left(\frac{t_k^\beta - t^\beta}{\Gamma(\beta + 1)}\right) \left(\frac{\partial \eta p - \Gamma(p + 1)}{\Gamma(p + 1)}\right) - \lambda\right) \frac{L_{f_1} C_1}{\Gamma(\alpha + \beta)} \int_0^{t_k} (t_k - s)^{\alpha + \beta - 1} \left| \varphi(s) \right| ds \\
&+ \left(\Delta \left(\frac{t_k^\beta - t^\beta}{\Gamma(\beta + 1)}\right) \left(\frac{\partial \eta p - \Gamma(p + 1)}{\Gamma(p + 1)}\right) - \lambda\right) \frac{L_{f_2} C_1}{\Gamma(\alpha + \beta)} \int_0^{t_k} (t_k - s)^{\alpha + \beta - 1} \left| \varphi(s) \right| ds \\
&+ \left(\Delta \left(\frac{t_k^\beta - t^\beta}{\Gamma(\beta + 1)}\right) \left(\frac{\partial \eta p - \Gamma(p + 1)}{\Gamma(p + 1)}\right) - 1\right) \frac{\lambda C_1}{\Gamma \beta} \int_0^{t_k} (t_k - s)^{\beta - 1} \left| \varphi(s) \right| ds \\
&+ \left(\Delta \left(\frac{t_k^\beta - t^\beta}{\Gamma(\beta + 1)}\right) \left(\frac{\partial \eta p - \Gamma(p + 1)}{\Gamma(p + 1)}\right) - 1\right) \left| g(t_k, g(t_k)) - g(t_k, h(t_k)) \right| \\
&\leq \frac{L_{f_1} C_1}{\Gamma(\alpha + \beta)} \left(\int_0^t (t - s)^{\alpha + \beta - 1} ds\right)^{1-r} \left(\int_0^t \left| \varphi(s) \right|^r ds\right)^{\frac{1}{r}} \\
&+ \frac{L_{f_2} C_1}{\Gamma(\alpha + \beta)} \left(\int_0^t (t - s)^{\alpha + \beta - 1} ds\right)^{1-r} \left(\int_0^t \left| \psi(s) \right|^r ds\right)^{\frac{1}{r}} \\
&+ \frac{\lambda C_1}{\Gamma \beta} \left(\int_0^t (t - s)^{\alpha + \beta - 1} ds\right)^{1-r} \left(\int_0^t \left| \varphi(s) \right|^r ds\right)^{\frac{1}{r}} \\
&+ \frac{\Delta (t_k^\beta - t^\beta) L_{f_1} C_1}{\Gamma(\beta + 1) \Gamma(\alpha + \beta)} \left(\int_0^T (T - s)^{\alpha + \beta - 1} ds\right)^{1-r} \left(\int_0^T \left| \varphi(s) \right|^r ds\right)^{\frac{1}{r}}
\end{align*}\]
\begin{align*}
+ \Delta(t_k^\beta - t^{\beta})L_{j_i} C_1 \left( \int_0^T (T-s)^{\frac{\alpha + \beta - 1}{r}} ds \right)^{1-r} \left( \int_0^T (\psi(s))^{\frac{1}{r}} ds \right)^r \\
+ \frac{\lambda \Delta(t_k^\beta - t^{\beta}) C_1}{\Gamma(\beta + 1) \Gamma(\beta)} \left( \int_0^T (T-s)^{\frac{\alpha + \beta - 1}{r}} ds \right)^{1-r} \left( \int_0^T (\psi(s))^{\frac{1}{r}} ds \right)^r \\
+ \frac{\theta \Delta(t_k^\beta - t^{\beta}) L_{j_i} C_1}{\Gamma(\beta + 1) \Gamma(\beta + p)} \left( \int_0^\eta (\eta-s)^{\frac{\alpha + \beta - 1}{r}} ds \right)^{1-r} \left( \int_0^\eta (\psi(s))^{\frac{1}{r}} ds \right)^r \\
+ \frac{C_1 \theta \Delta \psi(t_k^\beta - t^{\beta})}{\Gamma(\beta + 1) \Gamma(\beta + p)} \left( \int_0^\eta (\eta-s)^{\frac{\alpha + \beta - 1}{r}} ds \right)^{1-r} \left( \int_0^\eta (\psi(s))^{\frac{1}{r}} ds \right)^r \\
+ \left( \frac{\Delta(t_k^\beta - t^{\beta})}{\Gamma(\beta + 1)} \left( \frac{\theta \eta^p - \Gamma(p + 1)}{\Gamma(p + 1)} \right) - \lambda \right) \\
\times L_{j_i} C_1 \left( \int_0^{t_k} (t_k-s)^{\frac{\alpha + \beta - 1}{r}} ds \right)^{1-r} \left( \int_0^{t_k} (\psi(s))^{\frac{1}{r}} ds \right)^r \\
+ \left( \frac{\Delta(t_k^\beta - t^{\beta})}{\Gamma(\beta + 1)} \left( \frac{\theta \eta^p - \Gamma(p + 1)}{\Gamma(p + 1)} \right) - \lambda \right) \\
\times L_{j_i} C_1 \left( \int_0^{t_k} (t_k-s)^{\frac{\alpha + \beta - 1}{r}} ds \right)^{1-r} \left( \int_0^{t_k} (\psi(s))^{\frac{1}{r}} ds \right)^r \\
\times L_{j_i} C_1 \left( \int_0^{t_k} (t_k-s)^{\frac{\alpha + \beta - 1}{r}} ds \right)^{1-r} \left( \int_0^{t_k} (\psi(s))^{\frac{1}{r}} ds \right)^r \\
+ \left( \frac{\Delta(t_k^\beta - t^{\beta})}{\Gamma(\beta + 1)} \left( \frac{\theta \eta^p - \Gamma(p + 1)}{\Gamma(p + 1)} \right) - \lambda \right) \\
\times \frac{L_{j_i} C_1 C_\psi(t)}{\Gamma(\alpha + \beta)} \left( \frac{1 - r}{\alpha + \beta - r} \right)^{1-r} \int_0^\eta \frac{1 - r}{\alpha + \beta - r} \int_0^\eta \left( \frac{1 - r}{\alpha + \beta - r} \right)^{1-r} T_{\alpha + \beta - r} \\
\leq \frac{L_{j_i} C_1 C_\psi(t)}{\Gamma(\alpha + \beta)} \left( \frac{1 - r}{\alpha + \beta - r} \right)^{1-r} \int_0^\eta \frac{1 - r}{\alpha + \beta - r} \int_0^\eta \left( \frac{1 - r}{\alpha + \beta - r} \right)^{1-r} T_{\alpha + \beta - r} \\
+ \frac{\Delta(t_k^\beta - t^{\beta}) L_{j_i} C_1 C_\psi(t)}{\Gamma(\beta + 1) \Gamma(\alpha + \beta)} \left( \frac{1 - r}{\alpha + \beta - r} \right)^{1-r} T_{\alpha + \beta - r} \\
+ \frac{\lambda \Delta \psi(t_k^\beta - t^{\beta}) L_{j_i} C_1 C_\psi(t)}{\Gamma(\beta + 1) \Gamma(\beta + p)} \left( \frac{1 - r}{\alpha + \beta - r} \right)^{1-r} T_{\alpha + \beta - r} \\
+ \frac{\theta \Delta(t_k^\beta - t^{\beta}) C_1 C_\psi(t)}{\Gamma(\beta + 1) \Gamma(\beta + p)} \left( \frac{1 - r}{\alpha + \beta - r} \right)^{1-r} T_{\alpha + \beta - r} \\
+ \frac{C_1 C_\psi(t) \theta \Delta \psi(t_k^\beta - t^{\beta})}{\Gamma(\beta + 1) \Gamma(\beta + p)} \left( \frac{1 - r}{\alpha + \beta - r} \right)^{1-r} T_{\alpha + \beta - r} \\
+ \left( \frac{\Delta(t_k^\beta - t^{\beta})}{\Gamma(\beta + 1)} \left( \frac{\theta \eta^p - \Gamma(p + 1)}{\Gamma(p + 1)} \right) - \lambda \right) \\
\times L_{j_i} C_1 C_\psi(t) \left( \frac{1 - r}{\alpha + \beta - r} \right)^{1-r} T_{\alpha + \beta - r}
\end{align*}
Also, for \( t \in (t_2, s_j) \) and \( s \in (s_{k-1}, t_2) \), we have

\[
|\langle Ag(t) - (Ah)(t)\rangle| \\
\leq \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t - s)^{\alpha + \beta - 1} \times \left| f(s, g(s), D^{\delta}_{t_2}(D^{\delta}_{t_2} + \lambda)g(s)) - f(s, h(s), D^{\delta}_{t_2}(D^{\delta}_{t_2} + \lambda)h(s)) \right| ds \\
+ \frac{\lambda}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1} \left| \int_0^T (T - s)^{\delta - 1} |g(s) - h(s)| ds \right| ds \\
+ \frac{\Delta(t_k^\delta - t^\beta)}{\Gamma(\beta + 1) \Gamma(\alpha + \beta)} \int_0^T (T - s)^{\alpha + \beta - 1} \times \left| f(s, g(s), D^{\delta}_{t_2}(D^{\delta}_{t_2} + \lambda)g(s)) - f(s, h(s), D^{\delta}_{t_2}(D^{\delta}_{t_2} + \lambda)h(s)) \right| ds \\
+ \frac{\lambda \Delta(t_k^\delta - t^\beta)}{\Gamma(\beta + 1) \Gamma(\beta)} \int_0^T (T - s)^{\beta - 1} \left| \int_0^T (T - s)^{\delta - 1} |g(s) - h(s)| ds \right| ds \\
+ \frac{\theta \Delta(t_k^\delta - t^\beta)}{\Gamma(\beta + 1) \Gamma(\alpha + \beta + p)} \int_0^\eta (\eta - s)^{\alpha + \beta + p - 1} \times \left| f(s, g(s), D^{\delta}_{t_2}(D^{\delta}_{t_2} + \lambda)g(s)) - f(s, h(s), D^{\delta}_{t_2}(D^{\delta}_{t_2} + \lambda)h(s)) \right| ds \\
+ \frac{\theta \lambda \Delta(t_k^\delta - t^\beta)}{\Gamma(\beta + 1) \Gamma(\beta)} \int_0^\eta (\eta - s)^{\beta + p - 1} \left| \int_0^T (T - s)^{\delta - 1} |g(s) - h(s)| ds \right| ds.
\]
\[ + \left( \frac{t^\beta - t^\beta}{\Gamma(\beta + 1)} \left( \frac{\theta^\beta - \Gamma(p + 1)}{\Gamma(p + 1)} \right) - \lambda \right) \frac{1}{\Gamma(\alpha + \beta)} \int_0^T (t_k - s)^\alpha s^\beta - 1 \]

\[ \times [f(s, g(s), cD^\beta_0(s^\beta + \lambda) h(s)) - f(s, h(s), cD^\beta_0(s^\beta + \lambda) h(s))] \, ds \]

\[ + \left( \frac{t^\beta - t^\beta}{\Gamma(\beta + 1)} \left( \frac{\theta^\beta - \Gamma(p + 1)}{\Gamma(p + 1)} \right) - 1 \right) \frac{\lambda}{\Gamma(\beta)} \int_0^T (t_k - s)^\beta - 1 |g(s) - h(s)| \, ds \]

\[ + \left( \frac{t^\beta - t^\beta}{\Gamma(\beta + 1)} \left( \frac{\theta^\beta - \Gamma(p + 1)}{\Gamma(p + 1)} \right) - 1 \right) \frac{\lambda}{\Gamma(\beta)} \int_0^T (t_k - s)^\beta - 1 |g(t_k, g(t_k)) - g(t_k, h(t_k))| \]

\[ \leq \frac{1}{\Gamma(\alpha + \beta)} \int_0^T (t - s)^\alpha s^\beta - 1 \]

\[ \times [L_1 |g(s) - h(s)| + L_2 |cD^\beta_0(s^\beta + \lambda) g(s) - cD^\beta_0(s^\beta + \lambda) h(s)|] \, ds \]

\[ + \frac{\lambda}{\Gamma(\beta)} \int_0^T (t - s)^\beta - 1 |g(s) - h(s)| \, ds + \frac{\Delta(t^\beta - t^\beta)}{\Gamma(\beta + 1)\Gamma(\alpha + \beta)} \int_0^T (T - s)^\alpha s^\beta - 1 \]

\[ \times [L_1 |g(s) - h(s)| + L_2 |cD^\beta_0(s^\beta + \lambda) g(s) - cD^\beta_0(s^\beta + \lambda) h(s)|] \, ds \]

\[ + \frac{\lambda}{\Gamma(\beta + 1)\Gamma(\alpha + \beta)} \int_0^T (T - s)^\beta - 1 |g(s) - h(s)| \, ds \]

\[ + \frac{\lambda}{\Gamma(\beta + 1)\Gamma(\alpha + \beta)} \int_0^T (T - s)^\beta - 1 |g(t_k, g(t_k)) - g(t_k, h(t_k))| \]

\[ = \frac{L_1}{\Gamma(\alpha + \beta)} \int_0^T (t - s)^\alpha s^\beta - 1 |g(s) - h(s)| \, ds \]

\[ + \frac{L_2}{\Gamma(\alpha + \beta)} \int_0^T (t - s)^\alpha s^\beta - 1 |cD^\beta_0(s^\beta + \lambda) g(s) - cD^\beta_0(s^\beta + \lambda) h(s)| \, ds \]

\[ + \frac{\lambda}{\Gamma(\beta)} \int_0^T (t - s)^\beta - 1 |g(s) - h(s)| \, ds \]

\[ + \frac{\Delta L_1}{\Gamma(\beta + 1)\Gamma(\alpha + \beta)} \int_0^T (T - s)^\alpha s^\beta - 1 |g(s) - h(s)| \, ds \]

\[ + \frac{\Delta L_2}{\Gamma(\beta + 1)\Gamma(\alpha + \beta)} \int_0^T (T - s)^\alpha s^\beta - 1 |cD^\beta_0(s^\beta + \lambda) g(s) - cD^\beta_0(s^\beta + \lambda) h(s)| \, ds \]

\[ \times \int_0^T (T - s)^\alpha s^\beta - 1 |cD^\beta_0(s^\beta + \lambda) g(s) - cD^\beta_0(s^\beta + \lambda) h(s)| \, ds \]
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\[ \frac{\lambda \Delta(t_k^\beta - t^\beta)}{\Gamma(\beta + 1) \Gamma(\beta)} \int_0^T (T - s)^{\beta - 1} |g(s) - h(s)| \, ds \]

\[ + \frac{L_1 \theta \Delta(t_k^\beta - t^\beta)}{\Gamma(\beta + 1) \Gamma(\alpha + \beta + p)} \int_0^\eta (\eta - s)^{\alpha + \beta + p - 1} |g(s) - h(s)| \, ds \]

\[ + \frac{L_2 \theta \Delta(t_k^\beta - t^\beta)}{\Gamma(\beta + 1) \Gamma(\alpha + \beta + p)} \times \int_0^\eta (\eta - s)^{\alpha + \beta + p - 1} |cD_0^\beta \left( cD_0^\beta + \lambda \right) g(s) - cD_0^\beta \left( cD_0^\beta + \lambda \right) h(s) | \, ds \]

\[ + \frac{\theta \Delta \lambda(t_k^\beta - t^\beta)}{\Gamma(\beta + 1) \Gamma(\beta + p)} \int_0^\eta (\eta - s)^{\beta - 1} |g(s) - h(s)| \, ds \]

\[ + \left( \frac{\Delta(t_k^\beta - t^\beta)}{\Gamma(\beta + 1)} \left( \frac{\theta \eta^p - \Gamma(p + 1)}{\Gamma(p + 1)} \right) - \lambda \right) \int_0^{t_k} (t_k - s)^{\beta - 1} |g(s) - h(s)| \, ds \]

\[ \leq \frac{L_1 \psi C_2 \theta}{\Gamma(\alpha + \beta)} \int_0^\eta (t - s)^{\alpha + \beta - 1} \, ds + \frac{\bar{L}_2 C_2 \psi(t_k^\beta - t^\beta)}{\Gamma(\alpha + \beta)} \int_0^\eta (t - s)^{\alpha + \beta - 1} \, ds \]

\[ + \frac{\lambda C_2 \psi}{\Gamma(\beta)} \int_0^\eta (t - s)^{\beta - 1} \, ds + \Delta \frac{L_1 C_2 \psi(t_k^\beta - t^\beta)}{\Gamma(\beta + 1) \Gamma(\alpha + \beta)} \int_0^T (T - s)^{\alpha + \beta - 1} \, ds \]

\[ + \frac{\Delta \bar{L}_2 C_2 \psi(t_k^\beta - t^\beta)}{\Gamma(\beta + 1) \Gamma(\alpha + \beta)} \int_0^T (T - s)^{\beta - 1} \, ds \]

\[ + \frac{L_1 C_2 \psi \theta \Delta(t_k^\beta - t^\beta)}{\Gamma(\beta + 1) \Gamma(\alpha + \beta + p)} \int_0^\eta (\eta - s)^{\alpha + \beta + p - 1} \, ds \]

\[ + \frac{\bar{L}_2 C_2 \psi \theta \Delta(t_k^\beta - t^\beta)}{\Gamma(\beta + 1) \Gamma(\alpha + \beta + p)} \int_0^\eta (\eta - s)^{\alpha + \beta + p - 1} \, ds \]

\[ + \frac{\theta C_2 \psi \Delta \lambda(t_k^\beta - t^\beta)}{\Gamma(\beta + 1) \Gamma(\beta + p)} \int_0^\eta (\eta - s)^{\beta + p - 1} \, ds \]

\[ + \left( \frac{\Delta(t_k^\beta - t^\beta)}{\Gamma(\beta + 1)} \left( \frac{\theta \eta^p - \Gamma(p + 1)}{\Gamma(p + 1)} \right) - \lambda \right) \frac{L_1 C_2 \psi}{\Gamma(\alpha + \beta)} \int_0^{t_k} (t_k - s)^{\alpha + \beta - 1} \, ds \]

\[ + \left( \frac{\Delta(t_k^\beta - t^\beta)}{\Gamma(\beta + 1)} \left( \frac{\theta \eta^p - \Gamma(p + 1)}{\Gamma(p + 1)} \right) - \lambda \right) \frac{\bar{L}_2 C_2 \psi}{\Gamma(\alpha + \beta)} \int_0^{t_k} (t_k - s)^{\alpha + \beta - 1} \, ds \]
\begin{align*}
&+ \left( \frac{(t^\rho_k - t^\delta)}{\Gamma(\beta + 1)} \left( \frac{\theta \eta^p - \Gamma(p + 1)}{\Gamma(p + 1)} \right) - \lambda \frac{C_2 \psi}{\beta \Gamma^\beta} \int_0^{t_k} (t_k - s)^{\beta - 1} \, ds \right) \frac{L_{f_k} C_2 \psi}{\Gamma(\alpha + \beta)(\alpha + \beta)} T^{\alpha + \beta} + \frac{L_{f_k} C_2 \psi}{\Gamma(\alpha + \beta)(\alpha + \beta)} T^{\alpha + \beta} + \frac{\lambda C_2 \psi}{\beta \Gamma^\beta} T^\delta \\
&+ \Delta(t^\rho_k - t^\delta) \frac{L_{f_k} C_2 \psi}{\Gamma(\beta + 1)(\alpha + \beta)} T^{\alpha + \beta} + \Delta(t^\rho_k - t^\delta) \frac{L_{f_k} C_2 \psi}{\Gamma(\beta + 1)(\alpha + \beta)} T^{\alpha + \beta} + \frac{\lambda \Delta(t^\rho_k - t^\delta) C_2 \psi}{\Gamma(\beta + 1)(\alpha + \beta)} T^{\alpha + \beta} + \frac{\theta \Delta(t^\rho_k - t^\delta) L_{f_k} C_2 \psi}{\Gamma(\beta + 1)(\alpha + \beta)(\alpha + \beta + p)} \eta^{\alpha + \beta + p} \\
&+ \frac{\theta \Delta \lambda(t^\rho_k - t^\delta) C_2 \psi}{\Gamma(\beta + 1)(\alpha + \beta + \eta^{\alpha + \beta + p})} + \frac{\lambda \Delta(t^\rho_k - t^\delta) C_2 \psi}{\Gamma(\beta + 1)(\alpha + \beta + 1)} T^\lambda \\
&+ \frac{(t^\rho_k - t^\delta)}{\Gamma(\beta + 1)(\alpha + \beta + 1)} \left( \frac{\theta \eta^p - \Gamma(p + 1)}{\Gamma(p + 1)} \right) - \lambda \frac{L_{f_k} C_2 \psi}{\Gamma(\alpha + \beta)(\alpha + \beta)} T^{\alpha + \beta} + \frac{L_{f_k} C_2 \psi}{\Gamma(\alpha + \beta)(\alpha + \beta)} T^{\alpha + \beta} + \frac{\lambda C_2 \psi}{\beta \Gamma^\beta} T^\delta \\
&+ \frac{\Delta(t^\rho_k - t^\delta) L_{f_k} C_2 \psi}{\Gamma(\beta + 1)(\alpha + \beta + 1)} T^{\alpha + \beta} + \frac{\lambda \Delta(t^\rho_k - t^\delta) C_2 \psi}{\Gamma(\beta + 1)(\alpha + \beta + 1)} T^{\alpha + \beta} + \frac{\theta \Delta(t^\rho_k - t^\delta) L_{f_k} C_2 \psi}{\Gamma(\beta + 1)(\alpha + \beta + 1)(\alpha + \beta + p)} \eta^{\alpha + \beta + p} \\
&+ \frac{\theta \Delta \lambda(t^\rho_k - t^\delta) C_2 \psi}{\Gamma(\beta + 1)(\alpha + \beta + 1)(\alpha + \beta + 1)} \eta^{\alpha + \beta + p} + \frac{\lambda \Delta(t^\rho_k - t^\delta) C_2 \psi}{\Gamma(\beta + 1)(\alpha + \beta + 1)(\alpha + \beta + p)} \eta^{\alpha + \beta + p} \\
&+ \frac{(t^\rho_k - t^\delta)}{\Gamma(\beta + 1)(\alpha + \beta + 1)} \left( \frac{\theta \eta^p - \Gamma(p + 1)}{\Gamma(p + 1)} \right) - \lambda \frac{L_{f_k} C_2 \psi}{\Gamma(\alpha + \beta)(\alpha + \beta)} T^{\alpha + \beta} + \frac{L_{f_k} C_2 \psi}{\Gamma(\alpha + \beta)(\alpha + \beta)} T^{\alpha + \beta} + \frac{\lambda C_2 \psi}{\beta \Gamma^\beta} T^\delta \\
&\times \left( \frac{L_{f_k} C_2 \psi}{\Gamma(\alpha + \beta)(\alpha + \beta)} T^{\alpha + \beta} + \frac{L_{f_k} C_2 \psi}{\Gamma(\alpha + \beta)(\alpha + \beta)} T^{\alpha + \beta} + \frac{\lambda C_2 \psi}{\beta \Gamma^\beta} T^\delta \right) \\
&\times (C_1 + C_2)(\phi(t) + \psi).
\end{align*}
From above, we have
\[
\|(Ag)(t) - (Ah)(t)\| \leq M(C_1 + C_2)(\varphi(t) + \psi), \quad t \in [0, \tau],
\]
that is,
\[
d(Ag, Ah) \leq M(C_1 + C_2)(\varphi(t) + \psi).
\]
Hence, we conclude that
\[
d(Ag, Ah) \leq Md(g,h), \quad \text{for any } g, h \in V.
\]
Since condition (4.4) is strictly contractive, continuity property is thus shown. Now we take \(g_0 \in V\). From the piecewise continuity property of \(g_0\) and \(Ag_0\), it follows that there exists a constant \(0 < G_1 < \infty\) such that
\[
\|(Ag_0)(t) - g_0(t)\| \\
\leq \left| \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t-s)^{\alpha+\beta-1} f(s, g_0(s), D^\alpha_{0+} D^\beta_{0+} + \lambda) g_0(s) \, ds \right| \\
- \frac{\lambda}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} x(s) \, ds \\
- \frac{\Delta t^\beta}{\Gamma(\beta + 1) \Gamma(\alpha + \beta)} \int_0^T (T-s)^{\alpha+\beta-1} f(s, g_0(s), D^\alpha_{0+} D^\beta_{0+} + \lambda) g_0(s) \, ds \\
+ \frac{\lambda \Delta t^\beta}{\Gamma(\beta)} \frac{\Gamma(\beta + 1)}{\Gamma(\alpha + \beta + p)} \int_0^T (T-s)^{\beta-1} x(s) \, ds + \frac{\theta \Delta t^\beta}{\Gamma(\beta + 1) \Gamma(\alpha + \beta + p)} \\
\times \int_0^T (T-s)^{\alpha+\beta+p-1} f(s, g_0(s), D^\alpha_{0+} D^\beta_{0+} + \lambda) g_0(s) \, ds \\
- \frac{\theta \Delta t^\beta}{\Gamma(\beta + 1) \Gamma(\alpha + \beta + p)} \int_0^T (T-s)^{\beta+p-1} x(s) \, ds \\
+ \frac{(\Delta(\theta \eta^p - \Gamma(p+1)) t^\beta + 1)}{\Gamma(p+1) \Gamma(\beta + 1)} x_0 - g_0(t) \bigg| \\
\leq G_1 \varphi(t) \leq G_1 (\varphi(t) + \psi), \quad t \in (0, s_0).
\]
There exists a constant \(0 < G_2 < \infty\) such that
\[
\|(Ag_0)(t) - g_0(t)\| = \left| g_2 (t, g_0(t)) - g_0(t) \right| \leq G_2 \psi \leq G_2 (\varphi(t) + \psi),
\]
\(t \in (s_k-1, t_k], k = 1, 2, \ldots, m\).

Also we can find a constant \(0 < G_3 < \infty\) such that
\[
\|(Ag_0)(t) - g_0(t)\| \\
\leq \left| \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t-s)^{\alpha+\beta-1} f(s, g_0(s), D^\alpha_{0+} D^\beta_{0+} + \lambda) g_0(s) \, ds \right|
\]
is, functions satisfying Eq. (4.2). Using (3.2) and (4.1),

\[ V(\lambda) \mathcal{g}(t) = \frac{T}{\Gamma(\alpha + \beta)} \int_0^T (T - s)^{\alpha - 1} f(s, g_0(s), \mathcal{D}_0^\beta \mathcal{g}_0(s)) ds, \]

and

\[ V(\lambda) \mathcal{g}(t) = \frac{T}{\Gamma(\alpha + \beta)} \int_0^T (T - s)^{\alpha - 1} \mathcal{g}(s) ds - \frac{\theta T}{\Gamma(\alpha + \beta)} \int_0^T (T - s)^{\alpha - 1} \mathcal{g}(s) ds \]

\[ \times \int_0^T (T - s)^{\alpha - 1} \mathcal{g}(s) ds - \frac{\theta T}{\Gamma(\alpha + \beta)} \int_0^T (T - s)^{\alpha - 1} \mathcal{g}(s) ds \]

By using Banach fixed point theorem, there exists a continuous function \( \mathcal{g} \in \mathcal{G} \) such that

\[ (\mathcal{A}\mathcal{g})(t) = \mathcal{g}(t), \]

for all \( t \in (t_2, s_k), \) and for any \( \eta \in (0, \infty) \).

Since \( f, g_0 \) and \( g_0 \) are bounded on \( J \) and \( \phi > 0 \), Eq. (4.5) implies that \( d(\mathcal{A}\mathcal{g}, g_0) < \infty \).

By using Banach fixed point theorem, there exists a continuous function \( x : J \to \mathbb{R} \) such that

\[ \mathcal{A}^n g_0 \to x \text{ in } (V, d) \] as \( n \to \infty \) and \( \Delta x = y_0 \), that is, \( x \) satisfies Eq. (4.2) for every \( t \in J \).

Now we show that \( \{ g \in V \text{ such that } d(\mathcal{g}_0, g) < \infty \} = V \). For any \( g \in V \), since \( g \) and \( g_0 \) are bounded on \( J \) and \( \min_{a \in J} \phi(t) > 0 \), there exists a constant \( 0 < C_g < \infty \) such that

\[ |g_0(t) - g(t)| \leq C_g (\phi(t) + \psi), \quad t \in (t_k, s_k], k = 1, 2, \ldots, m. \]
Summarizing, we have

\[
d(y, x) \leq d(Ay, y) - \frac{d}{1 - M} + \frac{C_y}{\Gamma(\alpha + \beta)} \left( \frac{1 - r}{\alpha + \beta - r} \right)^1 r^{\alpha + \beta - r} + \lambda C_{\varphi} \left( \frac{1 - r}{\beta - r} \right)^1 r^{\beta - r} + \frac{\Delta C_y(t_0^\beta - t^\beta)}{\Gamma(\beta + 1) \Gamma(\alpha + \beta)} \left( \frac{1 - r}{\alpha + \beta - r} \right)^1 r^{\alpha + \beta - r} + \frac{\lambda \Delta C_y(t_0^\beta - t^\beta)}{\Gamma(\beta + 1) \Gamma(\beta)} \left( \frac{1 - r}{\beta - r} \right)^1 r^{\beta - r} + \frac{\theta \Delta C_y(t_0^\beta - t^\beta)}{\Gamma(\beta + 1) \Gamma(\alpha + \beta + p)} \left( \frac{1 - r}{\alpha + \beta + p - r} \right)^1 r^{\alpha + \beta + p - r} + C_{\varphi} \theta \Delta \lambda (t_k^\beta - t^\beta) \left( \frac{1 - r}{\beta + p} \right)^1 r^{\beta + p - r} + \left( \frac{\Delta (t_k^\beta - t^\beta)}{\Gamma(\beta + 1)} \left( \frac{\theta \eta^p - \Gamma(p + 1)}{\Gamma(p + 1)} \right) + \lambda \right) \frac{C_{\varphi}}{\Gamma(\alpha + \beta)} \left( \frac{1 - r}{\alpha + \beta - r} \right)^1 r^{\alpha + \beta - r} + \left( \frac{\Delta (t_k^\beta - t^\beta)}{\Gamma(\beta + 1)} \left( \frac{\theta \eta^p + \Gamma(p + 1)}{\Gamma(p + 1)} \right) - 1 \right) \lambda C_{\varphi} \left( \frac{1 - r}{\beta - r} \right)^1 r^{\beta - r} + 1 \right].
\]

This shows that (4.3) is true for \( t \in J \).

Here, we give an example to illustrate our main result.

**Example 4.3**

\[
\begin{align*}
&\frac{l^\frac{1}{2}}{D_{0+}^\frac{1}{2}}(\frac{l^\frac{1}{2}}{D_{0+}^\frac{1}{2}} + \frac{3}{20})\eta(t) = \frac{\gamma(t)}{8 + x^2 + \delta^2} \int \frac{\gamma(t)}{8 + x^2 + \delta^2} \gamma(t) \, ds, \quad t \in (0, 1] \cup (2, 3), \\
x(t) &= \frac{\gamma(t)}{8 + x^2 + \delta^2}, \quad t \in (1, 2], \\
x(0) &= \frac{\gamma(t)}{8 + x^2 + \delta^2}, \quad x(1) = \frac{\gamma(t)}{8 + x^2 + \delta^2}, \\
\text{and} \\
&\frac{l^\frac{1}{2}}{D_{0+}^\frac{1}{2}}(\frac{l^\frac{1}{2}}{D_{0+}^\frac{1}{2}} + \frac{3}{20})\eta(t) - \frac{\gamma(t)}{8 + x^2 + \delta^2} \int \frac{\gamma(t)}{8 + x^2 + \delta^2} \gamma(t) \, ds \leq \epsilon', \quad t \in (0, 1] \cup (2, 3), \\
&|\frac{\gamma(t)}{8 + x^2 + \delta^2} - \frac{\gamma(t)}{8 + x^2 + \delta^2}| \leq 1, \quad t \in (1, 2].
\end{align*}
\]

Let \( J = [0, 3] \), \( \alpha = \beta = \frac{1}{2} \), \( r = \frac{1}{3} \), \( \Delta = -2.70 \), \( \theta = \frac{5}{6} \), \( p = \frac{5}{3} \), \( \eta = \frac{1}{3} \) and \( 0 = t_0 < s_0 = 1 < t_1 = 2 < s_1 = 3 \). Denote \( f(t, x(t)) = \frac{\gamma(t)}{8 + x^2 + \delta^2} \) with \( L_f = \frac{1}{3} \) for \( t \in (0, 1] \cup (2, 3) \) and \( g_1(t, x(t)) = \frac{a(t)}{3 + x^2 + \delta^2} \) with \( L_{g_1} = \frac{1}{3} \) for \( t \in (1, 2) \). Putting \( \psi = 1, L_{\psi} = \frac{1}{3}, f(t) = \epsilon' \) and \( c_\psi = 1 \), we have \( (\int_0^t (\epsilon')^3 \, ds)^3 \leq \epsilon' \) and let \( M_1 \approx -0.5900, M_2 \approx 0.9713 \), so \( M = 0.9713 < 1 \).
By Theorem 4.2, there exists a unique solution \( x : [0, 3] \to \mathbb{R} \) such that

\[
\begin{align*}
x(t) = & \int_0^t \left( \frac{|x(t)|^{1/4}D_{\alpha/2}^{1/4}(D_{\beta}^{1/2} + \frac{1}{2})x(t)}{8e^{t^2+2}D_{\alpha}^{1/4}(D_{\beta}^{1/2} + \frac{1}{2})|x(t)|} \right) ds - 0.0846 \int_0^t (t-s)^{\frac{1}{2}} x(s) ds \\
&+ 0.0650t \int_0^t \left( \frac{|x(t)|^{1/4}D_{\alpha/2}^{1/4}(D_{\beta}^{1/2} + \frac{1}{2})x(t)}{8e^{t^2+2}D_{\alpha}^{1/4}(D_{\beta}^{1/2} + \frac{1}{2})|x(t)|} \right) ds \\
&- 0.0901t^2 \int_0^t (1-s)^{\frac{3}{2}} x(s) ds \\
&- 0.7454 \sqrt{t} \int_0^t \left( \frac{|x(t)|^{1/4}D_{\alpha/2}^{1/4}(D_{\beta}^{1/2} + \frac{1}{2})x(t)}{8e^{t^2+2}D_{\alpha}^{1/4}(D_{\beta}^{1/2} + \frac{1}{2})|x(t)|} \right) ds \\
&+ 0.1415 \sqrt{t} \int_0^t \left( \frac{|x(t)|^{1/4}D_{\alpha/2}^{1/4}(D_{\beta}^{1/2} + \frac{1}{2})x(t)}{8e^{t^2+2}D_{\alpha}^{1/4}(D_{\beta}^{1/2} + \frac{1}{2})|x(t)|} \right) ds \\
&+ (0.9476\sqrt{t} + 1)x_0, \quad t \in [0, 1], \\
&\int_0^t \left( \frac{|x(t)|^{1/4}D_{\alpha/2}^{1/4}(D_{\beta}^{1/2} + \frac{1}{2})x(t)}{8e^{t^2+2}D_{\alpha}^{1/4}(D_{\beta}^{1/2} + \frac{1}{2})|x(t)|} \right) ds \\
&- 0.0846 \int_0^t (t-s)^{\frac{1}{2}} x(s) ds \\
&- 1.0650(\sqrt{2} - \sqrt{t}) \int_0^t \left( \frac{|x(t)|^{1/4}D_{\alpha/2}^{1/4}(D_{\beta}^{1/2} + \frac{1}{2})x(t)}{8e^{t^2+2}D_{\alpha}^{1/4}(D_{\beta}^{1/2} + \frac{1}{2})|x(t)|} \right) ds \\
&+ 0.0901(\sqrt{2} - \sqrt{t}) \int_0^t (1-s)^{\frac{3}{2}} x(s) ds \\
&+ 0.7454(\sqrt{2} - \sqrt{t}) \int_0^t \left( \frac{|x(t)|^{1/4}D_{\alpha/2}^{1/4}(D_{\beta}^{1/2} + \frac{1}{2})x(t)}{8e^{t^2+2}D_{\alpha}^{1/4}(D_{\beta}^{1/2} + \frac{1}{2})|x(t)|} \right) ds \\
&+ (0.9476(\sqrt{2} - \sqrt{t}) - \frac{3}{20}) \int_0^t \frac{|x(t)|^{1/4}D_{\alpha/2}^{1/4}(D_{\beta}^{1/2} + \frac{1}{2})x(t)}{8e^{t^2+2}D_{\alpha}^{1/4}(D_{\beta}^{1/2} + \frac{1}{2})|x(t)|} ds \\
&- (0.9476(\sqrt{2} - \sqrt{t}) - 1)0.846 \int_0^t (2-s)^{\frac{1}{2}} x(s) ds \\
&- (0.9476(\sqrt{2} - \sqrt{t}) - 1)\frac{|x(t)|^{1/4}D_{\alpha/2}^{1/4}(D_{\beta}^{1/2} + \frac{1}{2})x(t)}{8e^{t^2+2}D_{\alpha}^{1/4}(D_{\beta}^{1/2} + \frac{1}{2})|x(t)|}, \quad t \in (2, 3]
\end{align*}
\]

Then

\[
|y(t) - x(t)| \leq \left( \frac{C_\varphi}{\Gamma(\alpha + \beta) \left( \frac{1 - r}{\alpha + \beta - r} \right)^{1-r}} \right)^{\epsilon^* x \varphi + 1} + \frac{\lambda C_\varphi}{\Gamma(\alpha + \beta) \left( \frac{1 - r}{\alpha + \beta - r} \right)^{1-r}} t^{\beta - r} \\
+ \frac{\Delta C_\varphi (\epsilon_i^p)^{- \beta}}{\Gamma(\beta + 1) \Gamma(\alpha + \beta) \left( \frac{1 - r}{\alpha + \beta - r} \right)^{1-r}} T^{\alpha + \beta - r} + \frac{\lambda \Delta C_\varphi (\epsilon_i^p)^{- \beta}}{\Gamma(\beta + 1) \Gamma(\alpha + \beta) \left( \frac{1 - r}{\alpha + \beta - r} \right)^{1-r}} T^{\beta - r} \\
+ \frac{\theta \Delta C_\varphi (\epsilon_i^p)^{- \beta}}{\Gamma(\beta + 1) \Gamma(\alpha + \beta + p) \left( \frac{1 - r}{\alpha + \beta + p - r} \right)^{1-r}} \eta^{\alpha + \beta + p - r} \\
+ \frac{C_\varphi \theta \Delta C_\varphi (\epsilon_i^p)^{- \beta}}{\Gamma(\beta + 1) \Gamma(\alpha + \beta + p) \left( \frac{1 - r}{\alpha + \beta + p - r} \right)^{1-r}} \eta^{\alpha + \beta + p - r} \\
+ \left( \frac{(\epsilon_i^p - \epsilon_0)}{\Gamma(\beta + 1) \Gamma(\alpha + \beta + p) \left( \frac{1 - r}{\alpha + \beta + p - r} \right)^{1-r}} - \lambda \right) \frac{C_\varphi}{\Gamma(\alpha + \beta) \left( \frac{1 - r}{\alpha + \beta - r} \right)^{1-r}} t^{\gamma + \beta - r} \\
+ \left( \frac{(\epsilon_i^p - \epsilon_0)}{\Gamma(\beta + 1) \Gamma(\alpha + \beta + p) \left( \frac{1 - r}{\alpha + \beta + p - r} \right)^{1-r}} - 1 \right) \frac{\lambda C_\varphi}{\Gamma(\alpha + \beta) \left( \frac{1 - r}{\alpha + \beta - r} \right)^{1-r}} t^{\gamma + \beta - r} + 1 \left( \frac{\varphi(t) + \psi}{1 - M} \right),
\]
which can further be reduced to

$$\left| y(t) - x(t) \right| \leq \frac{C_p}{\Gamma(\alpha + 1)} \left( \frac{1 - r}{\alpha + \beta - r} \right)^{1-r} T^{a \beta - r} + \frac{\lambda C_p}{\Gamma(\beta)} \left( \frac{1 - r}{\beta - r} \right)^{1-r} T^{\beta - r}$$

$$- \frac{\Delta C_p}{\Gamma(\beta + 1) \Gamma(\alpha + 1)} \left( \frac{1 - r}{\alpha + \beta - r} \right)^{1-r} T^{a \beta - r} - \frac{\lambda \Delta C_p}{\Gamma(\beta + 1) \Gamma(\beta)} \left( \frac{1 - r}{\beta - r} \right)^{1-r} T^{\beta - r}$$

$$- \frac{\theta \Delta C_p}{\Gamma(\beta + 1) \Gamma(\alpha + \beta + p)} \left( \frac{1 - r}{\alpha + \beta + p - r} \right)^{1-r} \eta^{a \beta + p - r}$$

$$- \frac{\Delta t^{\beta}}{\Gamma(\beta + 1) \frac{\Delta (p + 1)}{\Gamma(p + 1)}} \left( \frac{1 - r}{\beta - r} \right)^{1-r} \tau^{a \beta - r} + 1 \left( \frac{\psi(t) + \psi}{1 - M} \right).$$

This implies

$$\left| y(t) - x(t) \right| \leq \frac{1}{\Gamma(\alpha + 1)} \left( \frac{1 - \frac{1}{3}}{\alpha + \beta - r} \right)^{1-r} 3^{a \beta - r} + \frac{0.15}{\Gamma(\beta)} \left( \frac{1 - \frac{1}{3}}{\beta - r} \right)^{1-r} 3^{\beta - r}$$

$$- \frac{(-2.7)3^{\beta}}{\Gamma(\beta + 1) \Gamma(\alpha + 1)} \left( \frac{1 - \frac{1}{3}}{\alpha + \beta + p - r} \right)^{1-r} 3^{a \beta + p - r}$$

$$- \frac{(0.15)(-2.7)3^{\beta}}{\Gamma(\beta + 1) \Gamma(\beta)} \left( \frac{1 - \frac{1}{3}}{\beta - r} \right)^{1-r} 3^{\beta - r}$$

Plugging-in the values, we have

$$\left| y(t) - x(t) \right| \leq \left\{ \frac{1}{\Gamma(\alpha + 1)} \left( \frac{1 - \frac{1}{3}}{\alpha + \beta - r} \right)^{1-r} 3^{a \beta - r} + \frac{0.15}{\Gamma(\beta)} \left( \frac{1 - \frac{1}{3}}{\beta - r} \right)^{1-r} 3^{\beta - r} \right\}$$

$$- \frac{(-2.7)3^{\beta}}{\Gamma(\beta + 1) \Gamma(\alpha + 1)} \left( \frac{1 - \frac{1}{3}}{\alpha + \beta + p - r} \right)^{1-r} 3^{a \beta + p - r}$$

$$- \frac{(0.15)(-2.7)3^{\beta}}{\Gamma(\beta + 1) \Gamma(\beta)} \left( \frac{1 - \frac{1}{3}}{\beta - r} \right)^{1-r} 3^{\beta - r}$$
\[
\begin{align*}
\frac{(0.833)(-2.7)3^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2} + 1\right)\Gamma\left(\frac{1}{2} + \frac{4}{3}\right)}
&\cdot \left(\frac{1 - \frac{1}{3}}{\frac{1}{2} + \frac{4}{3} - \frac{1}{3}}\right)\left(0.25\right)^{\frac{1}{2} + \frac{4}{3} - \frac{1}{3}} \\
\frac{(0.833)(-2.7)(0.15)3^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2} + 1\right)\Gamma\left(\frac{1}{2} + \frac{4}{3}\right)}
&\cdot \left(\frac{1 - \frac{1}{3}}{\frac{1}{2} + \frac{4}{3} - \frac{1}{3}}\right)\left(0.25\right)^{\frac{1}{2} + \frac{4}{3} - \frac{1}{3}} \\
\frac{(-2.7)3^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2} + 1\right)}
&\cdot \left(\frac{(0.833)(0.25)\frac{1}{2} - \Gamma\left(\frac{4}{3} + 1\right)}{\Gamma\left(\frac{4}{3} + 1\right)}\right)\left(0.15\right) \\
\times \frac{1}{\Gamma\left(\frac{1}{2} + \frac{4}{3}\right)}
&\cdot \left(\frac{1 - \frac{1}{3}}{\frac{1}{2} + \frac{4}{3} - \frac{1}{3}}\right)\left(3^{\frac{1}{2} + \frac{4}{3} - \frac{1}{3}}\right) \\
\frac{(-2.7)3^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2} + 1\right)}
&\cdot \left(\frac{(0.833)(0.25)\frac{1}{2} - \Gamma\left(\frac{4}{3} + 1\right)}{\Gamma\left(\frac{4}{3} + 1\right)}\right)\left(0.15\right) \\
\times \frac{e^{t} + 1}{1 - 0.9714} \\
\leq 5.4846 \left(\frac{e^{t} + 1}{0.0286}\right) \\
\leq 191.769(e^{t} + 1),
\end{align*}
\]

thus problem (4.8) is Ulam–Hyers–Rassias stable.

### 5 Conclusions

In this article, we considered a nonlocal boundary value problem of nonlinear implicit fractional Langevin equation with noninstantaneous impulses. After introduction, we built a uniform structure for the solutions of our proposed model. We studied the concept of generalized Ulam–Hyers–Rassias stability to our proposed model. And, finally, we presented a particular example for the applicability of our main result.

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### Competing interests

The authors declare that they have no competing interest regarding this work.

### Authors’ contributions

All the authors contributed equally and significantly in writing this paper. All the authors read and approved the final manuscript.

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