ON AUTOMORPHISMS OF ENVELOPING ALGEBRAS

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ABSTRACT. Given an algebraic Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, we canonically associate to it a Lie algebra $\mathfrak{g}_\infty$ defined over $\mathbb{C}_\infty$ - the reduction of $\mathbb{C}$ mod infinitely large prime, and show that for a class of Lie algebras $\mathfrak{g}_\infty$ is an invariant of the derived category of $\mathfrak{g}$-modules. We give two applications of this construction. First, we show that the bounded derived category of $\mathfrak{g}$-modules determines algebra $\mathfrak{g}$ for a class of Lie algebras. Second, given a semi-simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, we construct a canonical homomorphism from the group of automorphisms of the enveloping algebra $U\mathfrak{g}$ to the group of Lie algebra automorphisms of $\mathfrak{g}$, such that its kernel does not contain a nontrivial semi-simple automorphism. As a corollary we obtain that any finite subgroup of automorphisms of $U\mathfrak{g}$ isomorphic to a subgroup of Lie algebra automorphisms of $\mathfrak{g}$.

INTRODUCTION

This paper is motivated by the question whether a Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ can be recovered from its enveloping algebra $U\mathfrak{g}$. One way to make this question more precise is to state the isomorphism problem for enveloping algebras: given two finite dimensional Lie algebras $\mathfrak{g}_1$, $\mathfrak{g}_2$ over $\mathbb{C}$, such that their enveloping algebras are isomorphic $U\mathfrak{g}_1 \cong U\mathfrak{g}_2$, does it follow that $\mathfrak{g}_1 \cong \mathfrak{g}_2$? This problem is widely open in general, it is known to have the positive answer for the cases of semi-simple Lie algebras (easily follows from [AP]) and low dimensional nilpotent Lie algebras (see [H]).

One is tempted to upgrade this isomorphism problem to the following derived isomorphism problem.

Conjecture 1. Let $\mathfrak{g}_1$, $\mathfrak{g}_2$ be finite dimensional Lie algebras over $\mathbb{C}$. If the derived categories of bounded complexes of $U\mathfrak{g}_1$-modules and $U\mathfrak{g}_2$-modules are equivalent, then $\mathfrak{g}_1 \cong \mathfrak{g}_2$.

We will show that the conjecture holds if Lie algebras $\mathfrak{g}_1$, $\mathfrak{g}_2$ satisfy assumptions below, Theorem 2.

A closely related problem is to understand Aut$(U\mathfrak{g})$-the group of automorphisms of the enveloping algebra. Of particular interest are its finite subgroups. In this regard, the study of finite subgroups of automorphisms of $U\mathfrak{g}$ for semi-simple $\mathfrak{g}$ and the corresponding fixed point rings have been of great interest for some time now, see [AP], [C], [CG], [J2]. For the special case of $\mathfrak{g} = sl_2$, all finite subgroups of Aut$(Usl_2)$ where classified by Fleury [F]. More specifically, she proved that if $\Gamma$ is a finite subgroup of Aut$(Usl_2)$, then $\Gamma$ is conjugate to a subgroup
of Aut(\(\mathfrak{sl}_2\)) \(\subset\) Aut(\(\mathfrak{Usl}_2\)). The proof in [F] relies on the explicit knowledge of generators of automorphism groups of primitive quotients of \(\mathfrak{Usl}_2\) (a result by Dixmier), no such results are known for higher rank Lie algebras.

Following ideas and results of Kontsevich and Belov-Kanel on automorphisms of the Weyl algebra [BK], [K], we will approach these problems by reducing mod large prime \(p\). In this context it will be convenient to use the reduction mod the infinitely large prime construction. Recall that given a commutative ring \(R\), its reduction mod the infinitely large prime \(R_\infty\) is defined as follows (see [K], [BK])

\[
R_\infty = \lim_{\to \rightarrow} \left( \prod_{p \in \mathbb{P}} S/pS \right) / \left( \bigoplus_{p \in \mathbb{P}} S/pS \right),
\]

here the direct limit is taken over all finitely generated subrings \(S \subset R\), and \(\mathbb{P}\) denotes the set of all prime numbers [K]. We have the canonical inclusion \(R \otimes \mathbb{Q} \hookrightarrow R_\infty\). Also, we have the Frobenius map \(F_\infty : R_\infty \to R_\infty\), defined as follows:

\[
F_\infty \left( \prod_p x_p \right) = \prod_p x_p^p.
\]

All results in this paper will be about Lie algebras satisfying the following assumptions. Examples of such Lie algebras besides semi-simple and Frobenius ones are certain \(\mathbb{Z}_2\)-contractions of reductive algebras, for example \(\mathfrak{sl}_n(\mathbb{C}) \ltimes \mathbb{C}^n\) (see [Pa]).

**Assumption 1.** Let \(\mathfrak{g}\) be an algebraic Lie algebra over \(\mathbb{C}\) corresponding to a connected algebraic group \(G\), with the trivial center, such that the following properties hold. The algebra of invariants \(\text{Sym}(\mathfrak{g})^G = \mathbb{C}[f_1, \ldots, f_n]\) is a polynomial algebra with homogeneous generators \(f_1, \ldots, f_n\), such that they form a regular sequence in \(\text{Sym}(\mathfrak{g})\). Moreover, the corresponding algebra of coinvariants \(A = \text{Sym}(\mathfrak{g})/(f_1, \ldots, f_n)\) is a normal domain, such that the coadjoint action of \(G\) on \(\text{Spec} A\) has an open orbit.

Given a perfect Lie algebra \(\mathfrak{g}\); \([\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}\), satisfying assumptions [\(1\)] we will construct a canonical group homomorphism (Section 4)

\[
D : \text{Aut}(\mathfrak{Ug}) \to \text{Aut}(\mathfrak{g}_{\mathbb{C}_\infty}),
\]

where \(\mathfrak{g}_{\mathbb{C}_\infty} = \mathfrak{g} \otimes \mathbb{C} \mathbb{C}_\infty\). We have a base change homomorphism \(\text{Fr}^* : \text{Aut}(\mathfrak{g}) \to \text{Aut}(\mathfrak{g}_{\mathbb{C}_\infty})\) induces by the Frobenius embedding \(\text{Fr}_\infty : \mathbb{C} \to \mathbb{C}_\infty\). The following is the main result of the paper.

**Theorem 1.** Let \(\mathfrak{g}\) be a perfect Lie algebra over \(\mathbb{C}\) satisfying assumptions [\(1\)] Then there are no nontrivial semi-simple elements in \(\ker(D)\). Moreover, \(D\) restricts to \(\text{Fr}^*\) on \(\text{Aut}(\mathfrak{g})\). In particular, if \(\Gamma\) is a finite subgroup of \(\text{Aut}(\mathfrak{Ug})\), then there exists a subgroup \(\Gamma'\) of \(\text{Aut}(\mathfrak{g})\), such that \(\Gamma\) is isomorphic to \(\Gamma'\) as an abstract group.
Remark 1. The most interesting application of the above result is for the case of a simple Lie algebra $g$. In principle, this result provides a full classification of isomorphism classes of finite groups of automorphisms of $\mathfrak{U}g$. However, although the construction of the subgroup $\Gamma' \subset \text{Aut}(g)$ is somewhat canonical, we don not know if $\Gamma'$ is conjugate to $\Gamma$ in $\text{Aut}(\mathfrak{U}g)$. In fact, we do not know if a much stronger statement about linearizability holds: Given a finite subgroup $\Gamma \subset \text{Aut}(\mathfrak{U}g)$, if there exists a subgroup $\Gamma' \subset \text{Aut}(g)$, such that $\Gamma'$ is conjugate to $\Gamma$ in $\text{Aut}(\mathfrak{U}g)$.

Throughout the paper, given an Abelian group $M$, we will denote by $M_p$ its reduction $\mod p : M_p = M/pM$.

1. Reduction $\mod p$ Lemmas

In this section, given any finite dimensional Lie algebra $g$ over $C$, we will define $C_\infty$-Lie algebras $g_\infty, \hat{g}_\infty$, and compute them for Lie algebras satisfying assumptions 1, Lemma 3. This construction will play the crucial role in proving our main results.

At first, recall that given an associative flat $Z$-algebra $R$ and a prime number $p$, then the center $Z(R_p)$ of its reduction $\mod p$ acquires the natural Poisson bracket, which we will refer to as the deformation Poisson bracket, defined as follows. Given $a, b \in Z(R_p)$, let $z, w \in R$ be their lifts respectively. Then the Poisson bracket $\{a, b\}$ is defined to be

$$\frac{1}{p}[z, w] \mod p \in Z(R_p).$$

Let $S$ be a finitely generated subring of $C$, and let $g$ be a Lie algebra over $S$ which is a finite rank free $S$-module. Throughout the paper will denote by $B$ the quotient of $\mathfrak{U}g$ by the augmentation ideal of its center

$$B = \mathfrak{U}g/(Z(\mathfrak{U}g) \cap g)\mathfrak{U}g.$$

Then we will define $S_\infty$-Lie algebras $g_\infty, \hat{g}_\infty$, as follows. Let $p >> 0$ be a large prime number. Then the following augmentation ideals

$$m_p = Z(\mathfrak{U}g_p) \cap g_p(\mathfrak{U}g_p), \quad n_p = Z(B_p) \cap g_pB_p$$

are easily seen to be Poisson ideals in $Z(\mathfrak{U}g_p), Z(B_p)$ respectively. Hence $m_p/m_p^2, n_p/n_p^2$ are Lie algebras, and we will view them as Lie algebras over $S_p$ via the the Frobenius map $\text{Fr}_p : S_p \to S_p$. We will put

$$g_\infty = (\prod_{p \in P} m_p/m_p^2) / \bigoplus_{p \in P} m_p/m_p^2, \quad g_\hat{\infty} = (\prod_{p \in P} n_p/n_p^2) / \bigoplus_{p \in P} n_p/n_p^2.$$

Now, given a Lie algebra over $C$, we will define

$$g_\infty = \lim_{f: g: S \subset C} (g_S)_\infty, \quad g_\hat{\infty} = \lim_{f: g: S \subset C} (\hat{g}_S)_\infty.$$
Here, \( \mathfrak{g}_S \) is a model of \( \mathfrak{g} \) over \( S : \mathfrak{g}_S \otimes_S \mathbb{C} = \mathfrak{g} \). We have the natural homomorphism \( \mathfrak{g}_\infty \to \hat{\mathfrak{g}}_\infty \). If \( \mathfrak{g} \) is an algebraic Lie algebra over \( S \), then we will construct a canonical Lie algebra homomorphism

\[
\mathfrak{g}_\infty = \mathfrak{g} \otimes_S S_\infty \to \mathfrak{g}_\infty.
\]

It will follows from Lemma 3 that if \( \mathfrak{g} \) satisfies assumptions 1, then we have the canonical isomorphism

\[
\mathfrak{g}_\infty = \mathfrak{g} \otimes_S \mathbb{C}_\infty \cong \hat{\mathfrak{g}}_\infty.
\]

Next we will recall a key computation of the Poisson bracket for restricted Lie algebras due to Kac and Radul [KR]. First, we will recall and fix some notations associated with enveloping algebras of restricted Lie algebras. Let \( R \) be a commutative reduced ring of characteristic \( p > 0 \). Let \( \mathfrak{g} \) be a restricted Lie algebra over \( R \) (\( \mathfrak{g} \) is assumed to be a finite free \( R \)-module) with the restricted structure map \( \mathfrak{g} \to \mathfrak{g}[p] \), \( \mathfrak{g} \in \mathfrak{g} \). Then by \( Z_p(\mathfrak{g}) \) we will denote the central \( R \)-subalgebra of the enveloping algebra \( \mathfrak{U}(\mathfrak{g}) \) generated by elements of the form \( \mathfrak{g}^p - \mathfrak{g}[p] \), \( \mathfrak{g} \in \mathfrak{g} \). It is well-known that the map \( \mathfrak{g} \to \mathfrak{g}^p - \mathfrak{g}[p] \) induces homomorphism of \( R \)-algebras

\[
i : \text{Sym}(\mathfrak{g}) \to Z_p(\mathfrak{g}),
\]

where \( Z_p(\mathfrak{g}) \) is viewed as an \( R \)-algebra via the Frobenius map \( F : R \to R \). The homomorphism \( i \) is an isomorphism when \( R \) is perfect. Also, recall that the Lie algebra bracket on \( \mathfrak{g} \) defines the Kirillov-Kostant Poisson bracket on the symmetric algebra \( \text{Sym}(\mathfrak{g}) \).

The following is the key result from [KR]. We will include a proof for the reader’s convenience.

**Lemma 1.** Let \( R \) be a finitely generated integral domain over \( \mathbb{Z} \). Let \( G \) be an affine algebraic group over \( R \), let \( \mathfrak{g} \) be its Lie algebra. Let \( G_p, \mathfrak{g}_p \) be reductions mod \( p \) of \( G, \mathfrak{g} \) respectively. Thus \( Z(\mathfrak{U}\mathfrak{g}_p) \) is equipped with the deformation Poisson bracket. Then \( Z_p(\mathfrak{g}_p) \) is a Poisson subalgebra of \( Z(\mathfrak{U}\mathfrak{g}_p) \), moreover the induced Poisson bracket coincides with the negative of the Kirillov-Kostant bracket:

\[
\{a^p - a[p], b^p - b[p]\} = -([a, b]^p - [a, b][p]), \quad a \in \mathfrak{g}_p, b \in \mathfrak{g}_p.
\]

**Proof.** The proof directly follows from a similar result about Weyl algebras in [?]. More specifically, let \( X \) be a smooth affine variety over \( R \), and let \( D_X \) denote the algebra of differential operators on \( X \). Put \( \overline{X} = X \mod p \). Then the center of \( D_X \mod p = D_{\overline{X}} \) can be identified with (the Frobenius twist) of the functions on the cotangent bundle \( T^*_X \) [BMR]. Then the deformation Poisson bracket of \( Z(D_{\overline{X}}) \) is equal to the negative of the Poisson bracket coming from the symplectic structure of the cotangent bundle of \( \overline{X} \).

Now let \( \theta : \mathfrak{g} \to D_G \) be the realization of \( \mathfrak{g} \) as left-invariant vector fields on \( G \). Then we have the corresponding embedding \( \theta : \mathfrak{U}\mathfrak{g} \to D_G \) and the corresponding reduction mod \( p \) \( \bar{\theta} : \mathfrak{U}\mathfrak{g}_p \to D_{G_p} \), which induces the embedding \( Z_p(\mathfrak{g}_p) \to Z(D_{G_p}) \). In this way \( Z_p(\mathfrak{g}_p) \) is a Poisson subalgebra of \( Z(D_{G_p}) \) and the assertion follows. \( \square \)
It is clear that \( i^{-1}(m_p) = g_p \text{Sym}_p \). So, in view of Lemma 1, we have the canonical Lie algebra homomorphism \( g_p \to m_p/m_p^2 \). Hence, we obtain the desired homomorphism

\[
\mathfrak{g}_S \to \mathfrak{g}_R.
\]

We have the following easy

**Lemma 2.** Let \( S \) be a finitely generated subring of \( \mathbb{C} \), let \( \mathfrak{g} \) be a nilpotent Lie algebra over \( S \), and let \( p >> 0 \) be a prime. Let \( I \) be a Poisson ideal of \( Z(\mathfrak{g}_p) \), such that \( Z(\mathfrak{g}_p)/I = S_p \). Then \( I/I^2 \cong m_p/m_p^2 \) as \( S_p \)-Lie algebras.

**Proof.** We claim that given any ideal \( I \) in \( Z(\mathfrak{g}_p) \), such that \( I \cap Z_p(\mathfrak{g}_p) = (g^p, g \in \mathfrak{g}_p) \) and \( Z(\mathfrak{g}_p)/I = S_p \), then \( I = m_p \). Indeed, since \( \mathfrak{g}_p/(g^p, g \in \mathfrak{g}_p)\mathfrak{g}_p \) is a nilpotent \( S_p \)-algebra, it follows that \( Z(\mathfrak{g}_p)/(g^p, g \in \mathfrak{g}_p)Z(\mathfrak{g}_p) \) is also a nilpotent \( S_p \)-algebra: \( m^p \subset (g^p, g \in \mathfrak{g}_p)Z(\mathfrak{g}_p) \) for large enough \( l \). If \( a + y \in I, a \in S_p, y \in m_p \), then

\[
(a + y)^p \in a^p + (g^p, g \in \mathfrak{g}_p)Z(\mathfrak{g}_p),
\]

so \( a = 0 \) and \( I \subset m_p \). Put \( I' = I \cap Z_p(\mathfrak{g}) \). Then \( I' \) is a Poisson ideal and \( Z_p(\mathfrak{g})/I' = S_p \). Hence, \( I' = (g^p - \chi(g), g \in \mathfrak{g}_p) \), where \( \chi : \mathfrak{g}_p \to S_p \) is a Lie algebra homomorphism. Then \( \phi(g) = g - \chi(g) \) induces an automorphism \( \phi : \mathfrak{g}_p \to \mathfrak{g}_p \), such that \( \phi(m_p) = I \). Moreover \( \phi \) admits a lift over \( S/p^2S \). Indeed, take any character \( \hat{\chi} : g/p^2g \to S/p^2 \) that lifts \( \chi \) and put \( \hat{\phi}(g) = \hat{g} - \hat{\chi}(g) \). Then \( \hat{\phi} : \mathfrak{g}(g/p^2g) \to \mathfrak{g}(g/p^2g) \) is an automorphism such that \( \phi = \hat{\phi} \mod p^2 \). Hence, \( \phi \) is a Poisson automorphism of \( Z(\mathfrak{g}_p) \). Thus we obtain the desired isomorphism of \( S_p \)-Lie algebras \( I/I^2 \cong m_p/m_p^2 \).

\( \square \)

From now on we will fix a Lie algebra \( \mathfrak{g} \) satisfying assumptions 1. Then it follows that we may choose a finitely generated subring \( S \subset \mathbb{C} \), and a Lie algebra \( \mathfrak{g}_S \) over \( S \) which is free \( S \)-module, such that \( \mathfrak{g} = \mathfrak{g}_S \otimes_S \mathbb{C}, f_i \in \text{Sym}_S \) and \( A_S = \text{Sym}_S/(f_1, \ldots, f_n) \) is a normal integral domain. Moreover Spec\( A_S \) has a nonempty open subset \( U_S \) which is symplectic over \( S \) under the Kirillov-Kostant bracket. Denote by \( \overline{g}_i \) the image of \( f_i \) under the symmetrization isomorphism \( \text{Sym}(g)^g \to Z(\mathfrak{g}) \). Hence \( \text{gr}(g_i) = f_i \). We will assume that \( S \) is large enough so that \( \overline{g}_i \in \mathfrak{g}_S, 1 \leq i \leq n \). Just as before, we will put \( B = \mathfrak{g}_S/(g_1, \ldots, g_n) \). Hence \( B = B_S \otimes_S \mathbb{C} \).

Given a commutative \( S \)-algebra \( R \), we will denote by \( g_R, R \) and \( A_R \) the base changes of \( g_S, B_S, A_S \) respectively. In particular, for a base change \( R \to k \), where \( k \) is an algebraically closed field, \( A_k \) is a normal integral domain (for \( p >> 0 \)). We will denote images of \( f_i, g_i \) in \( \text{Sym}_R(g), Z(\mathfrak{g}_R) \) by \( f_i^*, g_i^* \).

Given a commutative \( S_p \)-algebra \( R \), we will denote by \( m_R( \text{respectively } n_R) \), the augmentation ideal \( Z(\mathfrak{g}_R) \cap g_R \mathfrak{g}_R \) (respectively \( Z(B_R) \cap g_R B_R \))

Then we have the following.
Lemma 3. Let $g$ be as above. Then, $Z(\mathfrak{u}_p)$ is a free $Z_p(\mathfrak{u}_p)$-module with the basis
\[ \{ \bar{g}_1^{\alpha_1} \cdots \bar{g}_n^{\alpha_n}, 0 \leq \alpha_i < p, 1 \leq i \leq n \}. \]

In particular,
\[ g_\infty = g_{S_\infty} \oplus \bigoplus_{i=1}^n S_\infty g_i, \quad g_\infty = g_{S_\infty}, \]
where $\bigoplus_{i=1}^n S_\infty g_i$ is an Abelian Lie algebra. Moreover, given a Poisson ideal $I$ in $Z(\mathfrak{u}_p)$, such that $Z(\mathfrak{u}_p)/I = S/pS$, then $I/I^2 \cong g_\infty$ as $S_p$-Lie algebras. If $g$ is perfect, then $\mathfrak{n}_p$ is the unique Poisson ideal of $Z(B_p)$, such that $Z(B_p)/\mathfrak{n}_p = S_p$.

Proof. It is enough to verify above statements after a base change $S/pS \rightarrow k$, where $k$ is an algebraically closed field of characteristic $p$. At first, since $A_k = \text{Sym}(g_k)/(\bar{f}_1, \cdots, \bar{f}_n)$ is a normal domain and the Poisson bracket is symplectic on a nonempty open subset of Spec $A_k$, it follows easily that the Poisson center of $A_k$ is $A_k^p$ (see for example Lemma 2.4[Π]). It will suffice to check that the Poisson center of Sym$_k$ is a free module over $(\text{Sym}_k)^p$ with the basis
\[ \{ \bar{f}_1^{\alpha_1} \cdots \bar{f}_n^{\alpha_n}, 0 \leq \alpha_i < p \}. \]

Indeed, let $g$ be in the Poisson center of Sym$_k$. Denote the ideal $(\bar{f}_1, \cdots, \bar{f}_n) \subset \text{Sym}_k$ by $I$. We will show that $g \in \text{Sym}(g_k)^p[\bar{f}_1, \cdots, \bar{f}_n] + I^m$, for all $m$. We will proceed by induction on $m$. Since $\bar{g} = g \mod I$ belongs to the Poisson center of $A_k$, it follows that $\bar{g} \in A_k^p$. Hence $g \in \text{Sym}(g_k)^p + I$. Assume that $g \in \text{Sym}(g_k)^p[\bar{f}_1, \cdots, \bar{f}_n] + I^m$ for some $m \geq 1$. So, there exists $x \in g + \text{Sym}(g_k)^p[\bar{f}_1, \cdots, \bar{f}_n]$, such that
\[ x = \sum_{\alpha_1 + \cdots + \alpha_n = m} \bar{f}_1^{\alpha_1} \cdots \bar{f}_n^{\alpha_n} x_\alpha, \quad x_\alpha \in \text{Sym}(g_k). \]

Then, for any $y \in g_k$, we have
\[ 0 = \{ y, x \} = \sum_i \bar{f}_1^{\alpha_1} \cdots \bar{f}_n^{\alpha_n} \{ y, x_\alpha \}. \]

Since the sequence $(\bar{f}_1, \cdots, \bar{f}_n)$ is regular, we may conclude that $x_\alpha = x_\alpha \mod I \in A_k^p$. Hence,
\[ g \in \text{Sym}(g_k)^p[\bar{f}_1, \cdots, \bar{f}_n] + I^{m+1}. \]

Therefore,
\[ g \in \bigcap_{m \geq 1} (\text{Sym}(g_k)^p[\bar{f}_1, \cdots, \bar{f}_n] + I^m) = \text{Sym}(g_k)^p[\bar{f}_1, \cdots, \bar{f}_n]. \]

Now, suppose that
\[ 0 = \sum_{\alpha_1, \cdots, \alpha_n} \bar{f}_1^{\alpha_1} \cdots \bar{f}_n^{\alpha_n} x_\alpha^p, x_\alpha \in \text{Sym}(g_k), \]
Such that either $x_\alpha = 0$ or $x_\alpha \notin I$. Let $m$ be such that $x_\alpha = 0$ for all $|\alpha| < m$ in the above sum. Since $I^m/I^{m+1}$ is a free $A_k$-module with basis $\{ \bar{f}_1^{\alpha_1} \cdots \bar{f}_n^{\alpha_n}, \sum \alpha_i = m \}$, we get that $x_\alpha^p$
mod $I = 0$, hence $x_α ∈ I$. So $x_α = 0$. Thus, elements \( \{ f_1^{α_1} \cdots f_n^{α_n} \}, 0 ≤ α_i < p \) are linearly independent over \( (\text{Sym}_k g)_p \) as desired. It follows immediately that

\[
m_k/m_k^2 = g_k \bigoplus \bigoplus_{i=1}^n k\tilde{g}_i.
\]

Denote by $\tilde{f}_i$ the image of $f_i^p$ under the isomorphism $\text{Sym}(g_k) \cong Z_p(g_k)$. Then $\text{gr}(f_i) = f_i^p$. Since $A_k$ is a domain, we have

\[
(\text{Sym}_k g)_p \cap I = \sum_i f_i^p \text{Sym}_k g_k.
\]

Therefore

\[
Z_p(g_k) \cap (g_1, \cdots, g_n) = \sum_i f_i Z_p(g_k).
\]

Hence, we conclude that the map $i : \text{Sym}_k g_k \to Z(B_k)$ induces an isomorphism $A_k \cong Z(B_k)$.

In particular, $n_k/n_k^2 = g_k$. Now let $I \subset Z(\Omega g_p)$ be a Poisson ideal such that $Z(\Omega g_p)/I = S_p$. Proceeding as in the proof of Lemma 2 without loss of generality we may assume that $(g^p - g^{[p]}, g \in g_p) \subset I$. Let $\tilde{g}_i - a_i \in I, a_i \in S_p$. In the above proof, replacing $\tilde{g}_i$ with $\tilde{g}_i - a_i$, we conclude that

\[
I/I^2 = g_p \bigoplus \bigoplus_{i=1}^n S_p(\tilde{g}_i - a_i).
\]

If $g_p$ is perfect and $I$ is Poisson ideal in $B_p$, such that $B_p/I = S/pS$, then if follows that $g^p - g^{[p]} - \chi(g) \in I, g \in g_p$, such that $\chi : g_p \to S_p$ is a character. Hence, $\chi$ must be trivial. Since $g^p - g^{[p]}, g \in g_p$ generate $Z(B_p)$, we get that $I = n_p$.

\[\square\]

2. Derived Isomorphisms

As usual, $S$ is a finitely generated subring of $C$. Throughout, given two algebras $A, B$, we will say that they are derived equivalent if the respective derived categories of bounded complexes of (left) modules are equivalent. We will use the following easy consequence of [R].

Lemma 4. Let $A, B$ be flat $S$-algebras that are derived equivalent. Then $Z(A/pA) \cong Z(B/pB)$ as Poisson algebras for all $p \gg 0$. In particular, if $g_1, g_2$ are Lie algebras over $C$ which are either nilpotent or satisfying assumptions [1] such that $\Omega g_1, \Omega g_1$ are derived equivalent, then $(g_1)_\infty \cong (g_2)_\infty$.

Proof. It follows that algebras $A/p^2A, B/p^2B$ are derived equivalent. We have the following exact sequence of $A/p^2A$-bimodules

\[
0 \to A/pA \to A/p^2A \to A/pA \to 0,
\]

where $A/pA \to A/pA$ is the quotient map and $A/pA \to A/p^2A$ is the multiplication by $p$. It follows from [R] that the connecting map of the Hochschild cohomologies

\[
i_m : \text{HH}^m(A/pA) \to \text{HH}^{m+1}(A/pA)
\]
corresponding to the exact sequence above commutes with isomorphisms of Hochschild cohomologies induced by the derived equivalence
\[ HH^*(A/p^2A) \cong HH^*(B/p^2B), \quad HH^*(A/pA) \cong HH^*(B/p^2B). \]

Now, since the deformation Poisson bracket on \( Z(A/pA) \) is defined as
\[ \{a, -\} = i_0(a)|_{Z(A/pA)}, a \in Z(A/pA), \]
the desired result follows. Now supposed that Lie algebras \( \mathfrak{g}_1, \mathfrak{g}_2 \) are derived equivalent. Then there exists a finitely generated subring of \( C \), and Lie algebras \( (\mathfrak{g}_1)_S, (\mathfrak{g}_2)_S \) over \( S \), such that
\[ \mathfrak{g}_1 = (\mathfrak{g}_1)_S \otimes_S C, \quad \mathfrak{g}_2 = (\mathfrak{g}_2)_S \otimes_S C \]
and \( \mathcal{U}(\mathfrak{g}_1)_S \) is derived equivalent to \( \mathcal{U}(\mathfrak{g}_2)_S \). Hence,
\[ Z(\mathcal{U}(\mathfrak{g}_1)_S/p\mathcal{U}(\mathfrak{g}_1)_S) \cong Z(\mathcal{U}(\mathfrak{g}_2)_S/p\mathcal{U}(\mathfrak{g}_2)_S). \]
as Poisson algebras. Now using Lemmas 2 and 3 we may conclude that \( (\mathfrak{g}_1)_\infty \cong (\mathfrak{g}_2)_\infty \). □

Now we can easily prove the following.

**Theorem 2.** Suppose that Lie algebras \( \mathfrak{g}_1, \mathfrak{g}_2 \) satisfy assumptions 1. If \( \mathcal{U}\mathfrak{g}_1 \) is derived equivalent to \( \mathcal{U}\mathfrak{g}_2 \), then \( \mathfrak{g}_1 \cong \mathfrak{g}_2 \).

**Proof.** It follows from Lemmas 4 and 3 that \( (\mathfrak{g}_1)_\infty \oplus C_\infty^n \cong (\mathfrak{g}_2)_\infty \oplus C_\infty^n \). This implies that \( \mathfrak{g}_1 \cong \mathfrak{g}_2 \). □

**Remark 2.** In view of Lemma 4, it would be very interesting to compute \( \mathfrak{g}_\infty \) for nilpotent Lie algebras. We will come back to this question elsewhere.

### 3. The Homomorphisms \( D, \bar{D} \)

We will assume that \( \mathfrak{g} \) is a perfect Lie algebra over \( C \) satisfying Assumption 1. Let \( S \) be a large enough finitely generated subring of \( C \), and let \( \mathfrak{g}_S \) be a model of \( \mathfrak{g} \) over \( S \) (just as in the paragraph preceding Lemma 3). Then we will construct canonical homomorphisms
\[ D_S : \text{Aut}(\mathcal{U}\mathfrak{g}_S) \to \text{Aut}(\mathfrak{g}_S), \quad \bar{D}_S : \text{Aut}(B_S) \to \text{Aut}(\mathfrak{g}_S), \]
as follows. At first, remark that since \( \mathfrak{g} \) is perfect, any automorphism of \( \mathcal{U}\mathfrak{g} \) must preserve the ideal \( Z(\mathcal{U}\mathfrak{g}) \cap \mathfrak{g}\mathcal{U}\mathfrak{g} \). Therefore we have the restriction homomorphism \( \text{Aut}(\mathcal{U}\mathfrak{g}) \to \text{Aut}(B) \).

Let \( p >> 0 \) be a sufficiently large prime. Let \( \phi \in \text{Aut}(B_S) \). Reducing \( \phi \mod p \), we obtain \( \tilde{\phi} \in \text{Aut}(Z(B_p)) \). Since \( B_p \) is obtained by the reduction \mod p, we have the corresponding deformation Poisson bracket on its center. Hence, \( \tilde{\phi} \) preserves the Poisson bracket on \( Z(B_p) \). Now it follows from Lemma 2 that \( \tilde{\phi} \) preserves \( \mathfrak{n}_p \), thus it induces a Lie algebra automorphism on \( \mathfrak{n}_p/\mathfrak{n}_p^2 \cong \mathfrak{g}_p \), which we will denote by \( (\bar{D}_S)_p(\phi) \). Hence, we obtain a canonical homomorphism \( \bar{D}_S_p : \text{Aut}(B_S) \to \text{Aut}(\mathfrak{g}_p) \). Also, given a base change \( S_p \to k \), we will denote
the corresponding homomorphism $\tilde{D}_S \otimes_{S_p} k : \text{Aut}(B_S) \to \text{Aut}(g_{S_p})$ by $\tilde{D}_k$. Also, denote by $D_k : \text{Aut}(\Omega g_S) \to \text{Aut}(g_k)$ the composition of $\tilde{D}_k$ with the restriction $\text{Aut}(\Omega g_S) \to \text{Aut}(B_S)$. The element $\Pi_p(\tilde{D}_S)(\phi)$ gives rise to an element of $\text{Aut}(g_{S_p})$, which we will denote by $\tilde{D}_S(\phi)$. This way we obtain the desired homomorphisms $\tilde{D}_S$. We will define $D_S : \text{Aut}(\Omega g_S) \to \text{Aut}(g_{S_{\infty}})$ as the composition of $\tilde{D}_S$ with the restriction $\text{Aut}(\Omega g_S) \to \text{Aut}(B_S)$.

Now, taking the direct limit of the above homomorphisms $\tilde{D}_S, D_S$ over finitely generated subrings $S \subset C$, we obtain the sought after homomorphisms

$$\tilde{D} : \text{Aut}(B) \to \text{Aut}(g_{C_{\infty}}), \quad D : \text{Aut}(\Omega g) \to \text{Aut}(g_{C_{\infty}}).$$

It is clear from the construction and Lemma 3 that $D(\text{Aut}(g)) = \text{Fr}_{\infty}^*$, where $\text{Fr}_{\infty} : C \to C_{\infty}$ is the canonical inclusion followed by the Frobenius map.

4. THE PROOF

We will start by the following.

**Proposition 1.** Let $g$ be a perfect Lie algebra satisfying assumptions [1]. Then the kernel of the restriction homomorphism $\text{Aut}(\Omega g) \to \text{Aut}(B)$ contains no nontrivial semi-simple automorphisms.

We remark that in general, the homomorphism $\text{Aut}(\Omega g) \to \text{Aut}(B)$ is not injective. For example, in the case of $g = sl_2$, this follows from existence of a non tame automorphism of $\Omega sl_2$, proved by Joseph [11].

For the proof of Lemma 3 we will need a specific set of linearly independent elements of $\text{HH}^2(B)$. Recall that for a semi-simple $g$, one has $\dim \text{HH}^2(B) = n$, as follows immediately from Soergel’s result [5].

Recall that $Z(\Omega g) = C[g_1, \ldots, g_n], g_i \in \Omega g$ and $(g_1, \ldots, g_n)$ is a regular sequence in $\Omega g$. Put $I = (g_1, \ldots, g_n).$ So $B = \Omega g/I$. Let us put $B' = \Omega g/I^2$. Then we have a short exact sequence

$$0 \to I/I^2 \to B' \to B \to 0.$$ 

We have $I/I^2 = \bigoplus_{i=1}^n B\bar{g}_i$, where $\bar{g}_i$ denotes the image of $g_i$ under the quotient map $\Omega g \to B'$. Denote by $\omega \in \text{HH}^2(B, I/I^2) = \bigoplus_{i=1}^n \text{HH}^2(B)\bar{g}_i$ the cohomology class corresponding to the above short exact sequence. Let us put $\omega = \sum_{i=1}^n \omega_i \bar{g}_i$ where $\omega_i \in \text{HH}^2(B)$. Under these notations we have the following.

**Lemma 5.** Elements $\omega_i, 1 \leq i \leq n$ are linearly independent.

**Proof.** Let $\sum_{i=1}^n c_i \omega_i = 0, c_i \in C$. Let $c_j \neq 0$ for some $j$. Let $B_1$ be the quotient of $B'$ by the ideal $\sum_{i \neq j} B' \bar{g}_i$. Then it follows that the quotient map $B_1 \to B_1/B_1 \bar{g}_j = B$ admits a $C$-algebra splitting.
Let $\psi : B \to B_1$ be such a splitting. Let us write

$$
\psi(x) = x + \theta(x)\bar{g}_j, \quad x \in \mathfrak{g}, \quad \theta(x) \in B.
$$

This implies that $\bar{g}_j \in \bar{g}_j(qB')$. Therefore

$$
g_j \in \sum_{i \neq j} g_i \mathfrak{U} g + g_j \mathfrak{U} g.
$$

Thus, there exists $\alpha \in \mathfrak{U} \mathfrak{g} \setminus \mathfrak{g} \mathfrak{U} \mathfrak{g}$ such that $\alpha g_j \in \sum_{i \neq j} g_i \mathfrak{U} g$. Now the regularity of the sequence $(g_1, \ldots, g_n)$ implies that $\alpha \in I$, which is a contradiction.

\[\square\]

**Proof of Proposition 1.** Let $\phi$ be a semi-simple automorphism of $\mathfrak{U} \mathfrak{g}$ that restricts to the identity on $B$. Denote by $\tilde{\phi}$ the restriction of $\phi$ on $B'$. Therefore $\tilde{\phi}$ fixes $\omega = \sum \omega_i \bar{g}_i \in H^2(B, I/I^2)$. Let $\tilde{\phi} : I/I^2 \to I/I^2$ be represented by the matrix

$$(\phi_{ij}), \phi_{ij} \in \mathbb{C}, \quad \tilde{\phi}(\bar{g}_i) = \sum \phi_{ij} \bar{g}_j.$$ 

Entries $\phi_{ij}$ are scalars since $\phi$ preserves $\mathbb{C}[\bar{g}_1, \ldots, \bar{g}_n]$. Thus $\sum \phi_{ij} \omega_j = \omega_i$. Since $\omega_j$ are linearly independent by Lemma 5 we conclude that $\phi|_{I/I^2} = \text{Id}$. Hence $\phi|_{I^n/I^{n+1}} = \text{Id}$ for all $n$. Since $\phi$ is semi-simple, we get that $\phi = \text{Id}$.

\[\square\]

**Remark 3.** It was proved by Polo that when $\mathfrak{g}$ is semi-simple, the action of an automorphism of $\mathfrak{U} \mathfrak{g}$ on its center is given by a Dynkin diagram automorphism of $\mathfrak{g}$.

**Proof of Theorem 1.** In view of Proposition 1 it suffices to check that $\ker(\bar{D})$ has no nontrivial semi-simple automorphisms. Assume that $\phi \in \text{Aut}(B)$ is a non-trivial semi-simple automorphism. Therefore there exists a finitely generated subring $S \subset \mathbb{C}$, and a finite free $\phi$-invariant $S$-submodule $V \subset B_S$, such that $\phi|_V$ is semi-simple over $S$ and $V$ generates $B_S$ as an $S$-algebra. (we are using notations from the paragraph preceding Lemma 3). We will show that for sufficiently large $S$, and for all $p >> 0$, given any homomorphism $S_p \to k$, where $k$ is a field, then $\bar{D}_k(\phi) \neq \text{Id}_{\mathfrak{g}_k}$. Let $y \in V, 1 \neq \alpha \in S$, such that $\phi(y) = ay$. Let us write $y = \sum a_i e_i, 0 \neq a_i \in S$, where $e_i$ are basis elements of $B_S$ as a free $S$-module. We may assume that $a_i, 1 - \alpha$ are invertible is $S$. Now let $p >> 0$ and $\rho : S \to k$ be a base change to a field such that $\bar{D}_k = \text{Id}_{\mathfrak{g}_k}$. Thus, we have a non-trivial semi-simple automorphism $\tilde{\phi} \in \text{Aut}(B_k)$, such that $\tilde{\phi}|_{m^2_k/m^2_k} = \text{Id}$. Then $\tilde{\phi}$ acts trivially on $m^n_k/m^n_k$ for all $n$. Since the action of $\phi$ on $Z(B_k)$ is semi-simple, it follows that the action of $\phi$ on $Z(B_k)$ is trivial. Then by the Noether-Skolem theorem there exists $a \in B_k$, such that

$$
\phi(x) = axa^{-1}, x \in B_k.
$$

But, since $\tilde{\phi}(\bar{y}) = a\bar{y}$ and $1 \neq a \in k, 0 \neq \bar{y} \in B_k$, we get that $a\bar{y} = a\bar{y}a$. Recall that under the PBW filtration on $B_k$, $\text{gr}(B_k) = A_k$ is a commutative domain. Hence,

$$
0 \neq \text{gr}(a)\text{gr}(\bar{y}) = a\text{gr}(\bar{y})\text{gr}(a),
$$

Thus, there exists $\alpha \in \mathfrak{U} \mathfrak{g} \setminus \mathfrak{g} \mathfrak{U} \mathfrak{g}$ such that $\alpha g_j \in \sum_{i \neq j} g_i \mathfrak{U} g$. Now the regularity of the sequence $(g_1, \ldots, g_n)$ implies that $\alpha \in I$, which is a contradiction.

\[\square\]
which is a contradiction.

Let $\theta : C_\infty \to C$ be a homomorphism. Then we will define a (non-canonical) homomorphism $D^* : \text{Aut}(\mathfrak{U}_g) \to \text{Aut}(\mathfrak{g})$ as the composition of $D$ with the base change homomorphisms $\theta^* : \text{Aut}(\mathfrak{g}_{C_\infty}) \to \text{Aut}(\mathfrak{g})$. We will show that $\ker(D^*)$ contains no nontrivial finite order elements. This will imply that given a finite subgroup $\Gamma \subset \text{Aut}(\mathfrak{U}_g)$, then $\Gamma' = D^*(\Gamma) \subset \text{Aut}(\mathfrak{g})$, $\Gamma \cong \Gamma'$. Let $\phi \in \text{Aut}(\mathfrak{U}_g)$ such that $\phi^m = 1$, $\phi \neq 1$. We may choose a finitely generated subring $S \subset C$, $\frac{1}{m} \in S$ containing all $m$-th roots of unity, such that $\phi \subset \text{Aut}(\mathfrak{U}_g_S)$. As it was shown in the preceding paragraph, by enlarging $S$ if necessary, for all $p >> 0$ and a base change $S_p \to k$ to a field, we have $D_k(\phi) \neq \text{Id}_{k\phi}$. Let $f_p(t) \in S_p[t]$ denote the characteristic polynomial of $D_{S_p}(\phi) \in \text{Aut}(\mathfrak{g}_{S_p})$. Put $l = \dim \mathfrak{g}$. We will show that $\theta(\prod_p f_p(t)) \neq (t - 1)^l$ in $S_\infty[t]$. Indeed, let us write $S_p = \prod_i S_{p,i}$, where each $S_{p,i}$ is a domain (since $p$ is unramified in $S$). Since the image of $D_{S_p}(\phi)$ in $\text{Aut}(\mathfrak{g}_{S_p})$ has order $m$, it follows that the image of $f_p(t)$ in $S_{p,i}[t]$ is not equal to $(t - 1)^l$ and is of the form $\prod_{j=1}^m (t - a_j), a_j^m = 1$. Denote by $\Psi \subset S[t]$ the finite set of all degree $l$ monic polynomials not equal to $(t - 1)^l$, of the form $\prod_{j=1}^m (t - a_j), a_j \in S, a_j^m = 1$. For each $g \in \Psi$, denote by $I_g$ the set of pairs $(p, i)$ for which $f_p(t) = g$ in $S_{p,i}$. Then we have $\prod_p S_p = \prod_{g \in \Psi} (\prod_{(p, i) \in I_g} S_{p,i})$. Now, suppose $g \in \Psi$ is such that $\prod_{(p, i) \in I_g} S_{p,i} \not\subset \ker(\theta)$. Then it follows that $\theta(\prod_p f_p(t)) = \theta(g) \neq (t - 1)^l$. Hence $\phi \notin \ker(D^*)$ as desired.

\[ \square \]

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