Local hidden–variable models for entangled quantum states

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Abstract

While entanglement and violation of Bell inequalities were initially thought to be equivalent quantum phenomena, we now have different examples of entangled states whose correlations can be described by local hidden-variable models and, therefore, do not violate any of the Bell inequalities. We provide an up-to-date overview of the existing literature regarding local hidden-variable models for entangled quantum states, in both the bipartite and multipartite cases, and discuss some of the most relevant open questions in this context. Our review covers twenty-five years of this line of research, beginning with the seminal work by Werner (1989 \textit{Phys. Rev. A} 40 \textbf{8}), which provided the first example of an entangled state with a local model. Werner’s work, in turn, appeared twenty-five years after the seminal work by Bell (1964 \textit{Physics I} 195), about the impossibility of recovering the predictions of quantum mechanics using a local hidden-variable theory.

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(Some figures may appear in colour only in the online journal)

1. Introduction

In 1935, the concepts of the local–hidden variable model and entanglement were introduced in the works by, respectively, Einstein, Podolsky and Rosen [1], and Schrödinger [2]. After them, very few works (see, e.g., [3, 4]) considered the problem of whether local models could provide a more intuitive, and complete, alternative to the quantum formalism. This continued
until 1964, when Bell showed that local models are in fact in contradiction with quantum predictions. In particular, he proved that local models satisfy some inequalities, known thereafter as Bell inequalities, that are violated by the statistics of local measurements on a singlet state [5].

The work of Bell started the study of quantum nonlocality, that is, of those correlations obtained when performing local measurements on entangled states that do not have a classical analogue. Initially, it was believed that entanglement and quantum nonlocality were equivalent phenomena. And, in fact, entanglement and nonlocality do coincide for pure states, as any pure entangled state violates a Bell inequality [6]. However, in 1989 Werner introduced a family of highly symmetric mixed entangled states, today known as the Werner states, and exploited their symmetries to construct an explicit local model reproducing the correlations for some of them [7]. The work by Werner then implied that the relation between nonlocality and entanglement is subtler than expected and, in particular, these two notions do not coincide in the standard scenario originally introduced by Bell.

After Werner’s work, several other results have appeared providing new local models for other entangled states. The main purpose of this article is to review all these existing models. Our hope is that this review will be useful to have a broad vision of what is known today on the relation between entanglement and quantum nonlocality, in particular in the case when an entangled state satisfies all Bell inequalities. As will become clear below, the explicit construction of local models has turned out to be an extremely difficult problem and, at the moment, we only have a few models beyond Werner’s original construction. In fact, in our view, most, if not all, of the existing models can be interpreted in one way or another as variants of Werner’s model.

The structure of the review is as follows: after introducing the main concepts and definitions used in our work, we go through the existing models, both in the bipartite and multipartite scenarios. We then briefly discuss other possible definitions of nonlocality, such as the concept of hidden nonlocality introduced by Popescu [8]. Finally, we present our conclusions and a list of open questions.

2. Preliminaries and notation

We start by introducing notions and definitions repeatedly used throughout the manuscript: entanglement and genuine multipartite entanglement, measurement in quantum theory, and the Bell experiment setup with the corresponding notions of locality.

In what follows by $B(H)$, $1_d$, $\Omega_d$, and $\omega_d$ we denote, respectively, the set of bounded linear operators acting on a finite-dimensional Hilbert space $H$, the $d \times d$ identity matrix, the set $\Omega_d = \{ |\lambda\rangle \in \mathbb{C}^d | \langle \lambda | \lambda \rangle = 1 \}$, and the unique density over $\Omega_d$ invariant under any unitary operation $U$ acting on $\mathbb{C}^d$.

2.1. Entanglement

Let us consider two parties, traditionally called Alice (A) and Bob (B), sharing a bipartite quantum state $\rho_{AB}$ acting on a product Hilbert space $H = \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$. We say that $\rho_{AB}$ is separable iff it can be written as a convex combination of pure product states [7], that is

$$\rho_{AB} = \sum_i p_i \left| \psi_A^i \right\rangle \left\langle \psi_A^i \right| \otimes \left| \phi_B^i \right\rangle \left\langle \phi_B^i \right|, \quad p_i \geq 0, \quad \sum_i p_i = 1 \quad (1)$$
with some $|\psi_i\rangle \in \mathbb{C}^{d_i}$ and $|\varphi_j\rangle \in \mathbb{C}^{d_j}$. Otherwise, it is called entangled. In a general multiparty scenario with $N$ parties we will denote them $A^{(1)}, \ldots, A^{(N)} =: A$. Let $A_k$ be a $k$-element subset of $A$ and $\overline{A}_k$ its complement in the said set. Then, an $N$-partite state $\rho_A$ is said to be biseparable if it can be written as

$$\rho_A = \sum_{A_k, \overline{A}_k} \rho_{A_k, \overline{A}_k}, \quad \sum_{A_k, \overline{A}_k} \rho_{A_k, \overline{A}_k} = 1,$$

where each $\rho_{A_k, \overline{A}_k}$ is separable (see (1)) across the bipartition $A_k \cup \overline{A}_k$ and the sum goes over all such bipartitions. If a state cannot be written as (2), it is called genuinely multiparty entangled (GME).

### 2.2. Measurement in quantum theory

We call a collection $\{P_a\}_{a=0}^{k-1}$ of $k$ projections acting on $\mathbb{C}^d$ a $k$-outcome projective measurement (PM; also called von Neumann measurement) if the $P_a$ are supported on orthogonal subspaces, i.e., $P_a P_b = P_a \delta_{ab}$, and

$$\sum_{a=0}^{k-1} P_a = 1_d.$$  

With a projective measurement with outcomes $\alpha_a$, $a = 0, 1, \ldots, k - 1$, we associate an operator $A = \sum_{a=0}^{k-1} \alpha_a P_a$, called an observable, which is measured in the measurement process. In what follows, we talk about performing measurement $A$. Performed on a state $\rho$, a measurement will give the $a$th outcome $\alpha_a$ corresponding to $P_a$ with the probability $p(a \mid A)$ given by the Born rule

$$p(a \mid A) = \text{tr} \left( P_a \rho \right).$$

The mean value of an observable in the state $\rho$ is given by

$$\langle A \rangle := \text{tr} \left( A \rho \right) = \sum_{a=0}^{k-1} \alpha_a p(a \mid A).$$

Clearly, when $d = 2$, a projective measurement can have only two outcomes represented by rank-one projections $P_a$ (of course, there is also a trivial single-outcome projective measurement with $P_0 = 1_d$, which we do not need to consider here). In general, however, the projectors $P_a$ do not necessarily have to be rank one, and, in fact, if $k < d$ there must exist at least one projector of rank larger than one.

A collection of $k$ operators $A = \{A_a\}_{a=0}^{k-1}$ acting on $\mathbb{C}^d$ is called a $k$-outcome POVM, where POVM stands for positive-operator valued measure, or $k$-outcome generalized measurement if they are positive, i.e., $A_a \geq 0$, and

$$\sum_{a=0}^{k-1} A_a = 1_d.$$ 

The latter, in particular, means that every $A_a$ is upper bounded by the identity, i.e., $A_a \leq 1_d$. The probability of obtaining the outcome $a$ corresponding to $A_a$ is given again by the Born rule, that is, by equation (4) with $P_a$ replaced by $A_a$.

In what follows, the operators $P_a$ forming a projective measurement as well as $A_a$ forming a generalized measurement will be called measurement operators. Obviously, every projective measurement is also a POVM; the opposite, however, is in general not true.
Equivalently speaking, for a given dimension $d$, the set of all projective measurements forms a proper subset of the set of all generalized measurements\(^3\). It is important to note, however, that whenever we deal with $d$ outcome measurements on $\mathbb{C}^d$ they are necessarily projective. For further benefits, we also notice that in our considerations we can in fact assume that measurement operators, both in the case of projective and generalized measurements, are of rank one, or, in other words, the measurements are nondegenerate. This is because any measurement whose operators are of rank larger than one can always be realized as a measurement with rank-one operators. More precisely, since every element $A_a$ of a generalized measurement is a positive operator, it admits the form $A_a = \sum \eta_i^{(a)} P_i^{(a)}$ with $0 \leq \eta_i^{(a)} \leq 1$ and $P_i^{(a)}$ being, respectively, the eigenvalues and the rank-one eigenprojectors of $A_a$. Moreover, due to equation (6), all the eigenvalues must satisfy

$$\sum \eta_i^{(a)} = d. \quad (7)$$

Now, denoting all these new rank-one operators as $A_{a,i} = \eta_i^{(a)} P_i^{(a)}$ it is clear that they also form a POVM which is a ‘finer-grained’ version of $\{A_a\}$. To reproduce the statistics of the ‘coarse-grained’ original POVM, one simply applies the finer POVM $\{A_{a,i}\}$ and forgets the result $i$. In this way, a local model (see the upcoming section for the relevant notion) for POVMs with rank-one measurement operators will imply a local model for any POVM.

It is also worth mentioning that whenever we work with qubits it is beneficial to exploit the Bloch representation of quantum states and projection operations. A state $\rho$ acting on $\mathbb{C}^2$ can be expressed as

$$\rho = \frac{1}{2} (1 + \rho \cdot \sigma), \quad (8)$$

with $\rho \in \mathbb{R}^3$ being the so-called Bloch vector of length $|\rho| \leq 1$ and $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ standing for a vector consisting of the standard Pauli matrices\(^4\). Bloch vectors of unit length correspond to rank-one projectors (pure states).

Within the Bloch representation the measurement operators $P_a$ associated to $A_a$ in a generalized measurement (see the discussion above) are represented by the Bloch vectors $p_a$ that, due to (6), must satisfy

$$\sum_{a=0}^{k-1} \eta_a p_a = 0. \quad (9)$$

In particular, if $k = 2$, the projections $P_a$ form a projective measurement and the above condition simplifies to

$$p_0 + p_1 = 0. \quad (10)$$

This implies that any two-outcome measurement defined on a qubit Hilbert space is fully represented by a single vector, say $p \equiv p_0$. Also, in the case of two-outcome projective measurements, we will mainly use the more standard notation for the outcomes denoting them

\(^3\) The restriction to the fixed dimensions is very important here since any POVM can be realized formally as a PM in a larger space.

\(^4\) The Pauli matrices are defined as:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
\[ R_z = \frac{1}{2} (1 \pm \mathbf{p} \cdot \mathbf{\sigma}), \]  
and the mean value of an observable \( A \) is just \( \langle A \rangle = p(1 \mid A) - p(-1 \mid A) \).

### 2.3. Bell-type experiment and local models

Let us now consider \( N \) spatially separated parties \( A^{(i)} \), \( i = 1, \ldots, N \) sharing some \( N \)-partite quantum state \( \rho_A \) acting on a product Hilbert space \( \mathcal{H} = \mathbb{C}^{d_1} \otimes \ldots \otimes \mathbb{C}^{d_N} \) with \( d_i < \infty \) \( i = 1, \ldots, N \) denoting the local dimensions. On their share of \( \rho_A \), each party is allowed to perform one of \( m \) (possibly generalized) measurements \( \mathcal{A}^{(i)}_{x_i} \) \( x_i = 1, \ldots, m \), each with \( d \) outcomes, which we label as \( a_i = 0, 1, \ldots, d-1 \). We usually refer to such a scenario as \( (N, m, d) \) scenario (of course, one can consider a more general case of different numbers of measurements and outcomes at each site, but such a scenario is a straightforward generalization of the present one). In the case of a small number of parties \( N = 2, 3 \) we will rather denote them by \( A, B, C \), and the corresponding measurements and outcomes by \( \mathcal{A}, \mathcal{B}, \mathcal{C} \) and \( a, b, c \), respectively.

The correlations generated in such an experiment are described by the set of probabilities

\[ p(a_1, \ldots, a_N \mid x_1, \ldots, x_N) := p(a_1, \ldots, a_N \mid \mathcal{A}^{(1)}_{x_1}, \ldots, \mathcal{A}^{(N)}_{x_N}) \]

\[ = \text{tr} \left[ \left( \mathcal{A}^{(1)}_{x_1} \otimes \ldots \otimes \mathcal{A}^{(N)}_{x_N} \right) \rho_A \right] \]  
(12)

of obtaining results \( a_1, \ldots, a_N \) upon measuring \( \mathcal{A}^{(1)}_{x_1}, \ldots, \mathcal{A}^{(N)}_{x_N} \). In what follows, the set of (12) will also be referred to as quantum correlations or simply correlations, and the probabilities themselves will often be called ‘quantum’. Also, it is useful to think of the set \( \{ p(a_1, \ldots, a_N \mid x_1, \ldots, x_N) \} \), after some ordering, as a vector from \( \mathbb{R}^D \) with \( D = d^N \).

Actually, by using nonsignalling constraints (see, e.g., [9, 10]) one finds that a significantly lower number \( D' = \left( m(d-1) + 1 \right)^N - 1 \) of probabilities is sufficient to fully describe the correlations (12). For further purposes, let us finally notice that in the case \( d = 2 \), one can equivalently describe the correlations by the collection of expectation values

\[ \left\langle \mathcal{A}^{(1)}_{x_1} \otimes \ldots \otimes \mathcal{A}^{(N)}_{x_N} \right\rangle, \]  
(13)

with \( x_{i_1}, \ldots, x_{i_k} = 1, \ldots, m, i_1 < \ldots < i_k = 1, \ldots, N \), and \( k = 1, \ldots, N \). Here, \( \mathcal{A}^{(i)}_{x_i} \) are dichotomic observables with outcomes \( \pm 1 \). There are precisely \( [m+1]^N - 1 \) of such expectation values, which matches the number of independent probabilities \( D' \) in this scenario. Importantly, this equivalence does not hold if \( d > 2 \) simply because the number of independent probabilities exceeds that of the correlators (13) (see, nevertheless, [11, 12] for possible ways to overcome this problem).

It is known that the set of quantum correlations, denoted here by \( Q_{N,m,d} \), that can be produced in the above experiment from all states and measurements is convex (provided one does not constrain the dimension of the underlying Hilbert space). A proper subset of \( Q_{N,m,d} \) is formed by those correlations that the parties can obtain by using local strategies and, as the only resource, some shared classical information, often called shared randomness, \( \lambda \), distributed among them with probability density \( \omega(\lambda) \). Mathematically, this is equivalent to saying that each probability (12) can be expressed as
\[ p(a_1, \ldots, a_N | A_{x_{1}}^{(1)}, \ldots, A_{x_{N}}^{(N)}) = \int_\Omega d\omega(\lambda) p(a_1 | A_{x_{1}}^{(1)}, \lambda) \cdot \ldots \cdot p(a_N | A_{x_{N}}^{(N)}, \lambda). \] (14)

with \( \Omega \) denoting the set over which the random variable \( \lambda \) is distributed and \( p(a_i | A_{x_i}^{(i)}, \lambda) \) being local probability distributions conditioned additionally on \( \lambda \).

It follows that in any \((N, m, d)\) scenario, the set of local correlations, i.e., those admitting the representation (14), is a polytope, \( \mathcal{P}_{N,m,d} \), whose vertices are the local deterministic correlations of the form

\[ p(a_1, \ldots, a_N | A_{x_{1}}^{(1)}, \ldots, A_{x_{N}}^{(N)}) = p(a_1 | A_{x_{1}}^{(1)}) \cdot \ldots \cdot p(a_N | A_{x_{N}}^{(N)}). \] (15)

with the local probabilities being deterministic, that is, \( p(a_i | A_{x_i}^{(i)}) \in \{0, 1\} \) for all outcomes and settings. In passing, let us notice that as \( \mathcal{P}_{N,m,d} \) has a finite number of vertices, the integration in (14) can always be replaced by a finite sum over these vertices; for further purposes it is, however, easier for us to use the integral representation.

As already mentioned, \( \mathcal{Q}_{N,m,d} \) is strictly larger than \( \mathcal{P}_{N,m,d} \) and quantum correlations that fall outside \( \mathcal{P}_{N,m,d} \) are called nonlocal. Natural tools to witness nonlocality in correlations are the so-called Bell inequalities \([5]\) (see also \([13]\) and references therein for examples thereof).

These are linear inequalities that constrain the set of local correlations, and their violation is a signature of nonlocality (see figure 1). For a given \((N, m, d)\) scenario, Bell inequalities can be most generally written as

\[ \beta := \sum_{a_1, \ldots, a_N = 0}^{d-1} \sum_{x_1, \ldots, x_N = 1}^{m} T_{x_1, \ldots, x_N}^{a_1, \ldots, a_N} p(a_1, \ldots, a_N | x_1, \ldots, x_N) \leq \beta_L. \] (16)

where \( T \) is a tensor whose entries can always be taken as non-negative and \( \beta_L \) is the so-called classical bound of the inequality and is given by \( \beta_L = \max_{\mathcal{P}_{N,m,d}} \beta \) (clearly, here it suffices to maximize only over the vertices of \( \mathcal{P}_{N,m,d} \) in order to determine \( \beta_L \)).
An illustrative example of a Bell inequality in the simplest \((2, 2, 2)\) scenario is the famous Clauser–Horne–Shimony–Holt (CHSH) Bell inequality [14]:

\[
\sum_{a,b=0}^{1} \sum_{x,y=1}^{2} p(a \oplus b = (x - 1)(y - 1)|x,y) \leq 3,
\]

which can be restated in terms of the expectation values (13) as

\[
\left| \langle A_1 \otimes B_1 \rangle + \langle A_1 \otimes B_2 \rangle + \langle A_2 \otimes B_1 \rangle - \langle A_2 \otimes B_2 \rangle \right| \leq 2.
\]

Here, \(A_i\) and \(B_i\) \((i = 1, 2)\) are pairs of dichotomic observables with eigenvalues of \(\pm 1\) measured by the parties \(A\) and \(B\), respectively. It is known that in the \((2, 2, 2)\) scenario the CHSH inequality (18) is the only ‘relevant’ Bell inequality as it defines all facets of \(P_{2,2,2}\) [15]. In other words, if a bipartite state \(\rho_{AB}\) does not violate the inequality (18) for any choice of the measurements \(A_i\) and \(B_i\), then it does not violate any other Bell inequality in this scenario. The CHSH inequality is an example of a correlation Bell inequality, in which only the joint expectation values \(\langle A \otimes B \rangle\) appear.

A schematic depiction of all the above concepts is provided in figure 1.

Now, within this framework, we call a state \(\rho\) local in the scenario \((N, m, d)\) if the probability distribution that can be generated from it is local for any choice of the measurements \(A_i^{(n)}\) with \(i = 1, \ldots, m\) and \(i = 1, \ldots, N\). Notice that if \(\rho\) is local in the scenario \((N, m, d)\), then it is local in any scenario \((N, m', d')\) with either \(m' < m\) or \(d' < d\). On the other hand, there are states that are local in some scenarios \((N, m, d)\), but their nonlocality can be revealed only when the number of measurements or outcomes is increased.

Finally, a state \(\rho\) is called local if it is local in the scenario with any number of measurements and outcomes. Equivalently, one says that \(\rho\) has a local–hidden variable (LHV) model or simply a local model. Notice that in such cases, one can drop all the subscripts \(x_i\) in (14) because any probability generated from \(\rho\) must take such a form. As a result, (14) simplifies to

\[
p\left(\{a_1, \ldots, a_N\} | A^{(1)}, \ldots, A^{(N)} \right) = \int_{\Omega} d\lambda \omega(\lambda) p\left(\{a_1, \lambda\} | A^{(1)}, \lambda\right) \cdot \ldots \cdot p\left(\{a_N, \lambda\} | A^{(N)}, \lambda\right).
\]

On the other hand, if there is a scenario \((N, m, d)\) in which for some choice of measurements the resulting probability distribution (12) does not admit the form (14), we call it \(\rho\) nonlocal.

Of course, one can further distinguish the case when \(\rho\) is local only under projective measurements, i.e., (14) holds if the measurements \(A_i^{(n)}\) with \(i = 1, \ldots, N\) are projective. Then, one can say that \(\rho\) has a local model for projective measurements. Moreover, for \(d = 2\), one can also formulate the definition of locality of quantum states in terms of expectation values (13). Then, every expectation value involving more than two parties admits the form (19) with local probabilities replaced by the corresponding one-body mean values ‘conditioned’ on \(\lambda\).

Notice, furthermore, that a model reproducing global probability distribution automatically reproduces local probability distributions (those seen individually by the parties) as the latter are just the marginals of the former. The situation is more subtle when we work with the expectation values (13). Here, a local model realizing expectation values involving some number of parties, say \(k\), does not necessarily describe in a proper way expectation values involving less than \(k\) parties. In particular, in the bipartite case, a model reproducing joint expectation values \(\langle A \otimes B \rangle\) may not reproduce \(\langle A \rangle\) and \(\langle B \rangle\); we refer to such model as a model for joint correlations. This observation will be of particular importance in section 3.3.
From the point of view of the resources shared by the parties, the question of whether a state has a local model is the question of whether shared classical randomness can replace it in the described setup (see figure 2).

It is clear that an unentangled state is trivially local in any scenario as the global probability of local measurements is a sum of factorized components. In the remainder of this paper, we thus focus on providing LHV models for entangled states and for genuinely multipartite entangled states in the multipartite case (cf section 4). In the most general setting, such a model simulating statistics of the measurements $A$ and $B$ with the results $a$ and $b$, respectively, will be given as follows (for brevity we give a bipartite version):

**General local model**

1. Alice and Bob are distributed some classical information $\lambda$ belonging to some $\Omega$ with probability density $\omega(\lambda)$,
2. Alice outputs $a$ according to some probability distribution $p(a | A, \lambda)$ conditioned on $A$ and $\lambda$,
3. and, likewise, Bob outputs $b$ according to the probability distribution $p(b | B, \lambda)$.

In the example above, $p(a | A, \lambda)$ and $p(b | B, \lambda)$, also called response functions, determine the strategies that parties need to employ in order to simulate measurements on a given quantum state (it is not difficult to see that without loss of generality they could be taken as deterministic [15]). Once these mappings are proposed, the main task will be to verify whether (19) holds with lhs arising from measurements on a quantum state of interest.
Notice that in the general model we stated, the nature of $\lambda$ is not an issue. It can be a single variable (discrete or continuous), a set of variables, etc. Also, the case of Alice and Bob receiving different $\lambda$s is covered in this very general formulation.

3. Bipartite quantum states

We start our tour through local models with bipartite scenarios. In such settings, the first and most famous model is due to Werner [7]. Who already in 1989 realized that projective measurements on some entangled states give rise to statistics describable with a local hidden-variable model. These states belong to the class that is usually referred to as the Werner states. Besides presenting the original model, we will also discuss several of its modifications. Many years later, Werner’s result was generalized by Barrett [16], who proved that local models may account for statistics even of generalized measurements on some entangled Werner states. The price one had to pay for such extension was the diminishment of the region of parameters of the applicability of the model. At this moment, it is not known whether Werner states are local for projective and generalized measurements in the same region of the parameter.

An important line of research in the area has been the analysis of the nonlocal properties of noisy states, i.e., the ones arising as the mixture of some state with the white (completely depolarized) noise. This problem was addressed in detail by Acín et al [17] and Almeida et al [18]. In the former work, nonlocal properties of such states were related to the Grothendieck constant [19], establishing, in particular, the region in which a two-qubit Werner state is local for projective measurement. In the latter, building on the previous works by Werner and Barrett about local models and the protocol by Nielsen for the deterministic conversion between pure entangled states by local operations and classical communication (LOCC) [20], lower bounds on the values of the noise thresholds for the locality of noisy states were obtained. Upper bounds on these threshold values, on the other hand, have been obtained by analyzing the violation of the Collins–Gisin–Linden–Massar–Popescu (CGLMP) [21] and CHSH inequalities [22].

We will conclude this section with a recent result by Hirsch et al [23] who found a method of constructing states with local models for POVMs from states with local models only for two-outcome projective measurements.

3.1. Werner’s model

The class of states considered by Werner consists of all two-qudit states that are invariant under a bilateral action of any unitary operation, i.e., that commute with $U \otimes U$ for any unitary $U$ acting on $\mathbb{C}^d$. Such states are of the form

$$\rho_W(d, p) = \frac{1}{d(d - 1)} \left( \frac{d - 1 + p}{d} \mathbb{1}_d - pV_d \right),$$

(20)

where $P_d^{(-)}$ (or $P_d^{(+)}$) stands for the projector onto the antisymmetric (symmetric) subspace of $\mathbb{C}^d \otimes \mathbb{C}^d$ and $V_d$ denotes the so-called swap operator defined through

$$V_d \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} \phi_2 \\ \phi_1 \end{bmatrix}$$

(21)
for any pair \( |\phi_1\rangle, |\phi_2\rangle \in \mathbb{C}^d \). It is useful to recall that \( P_d^{\pm} = (|1\rangle \pm V_d)/2 \). Also, when \( d = 2 \),
\[
\left| \psi_+ \right\rangle = \frac{1}{\sqrt{2}} (|10\rangle - |01\rangle).
\]
(22)

It was shown in [7] that Werner states \( \rho_W(d, p) \) are separable if and only if \( p \leq p_{\text{sep}}^{W,c} \) with
\[
p_{\text{sep}}^{W,c} \equiv \frac{1}{d+1}.
\]
(23)

The ‘only if’ direction is particularly easy to verify as for any \( p > 1/(d + 1) \) the swap operator \( V_d \) is also an entanglement witness [24, 25] detecting them, i.e., \( \text{Tr} [\rho_W(d, p) V_d] < 0 \) in this region.

Let us now show, following [7], that for any \( p \leq p_{\text{PM}}^{W,c} \)
\[
p_{\text{PM}}^{W,c} \equiv \frac{d-1}{d}
\]
(24)
these states, when measured with projective measurements, always give rise to statistics with a local model. In fact, it is enough to show this for the boundary value of the mixing parameter \( p = p_{\text{PM}}^{W,c} \), for which equation (20) reduces to
\[
\rho_W^d \left( d, \frac{d-1}{d} \right) = \frac{1}{d^2} \left( \frac{d+1}{d} V_d - \frac{1}{d} \right).
\]
(25)

For all values of \( p < p_{\text{PM}}^{W,c} \) the result will follow immediately because one can always obtain the corresponding \( \rho_W(d, p) \) by mixing \( \rho_W(d, p_{\text{PM}}^{W,c}) \) with some portion of the white noise \( 1/d^2 \) which itself clearly admits a local model and the state given by the mixture of states with local models also has a local model.

Now, it directly stems from equation (25) that the quantum probability of obtaining outcomes \( a \) and \( b \) when performing the von Neumann measurements \( A \) and \( B \) represented by projections \( \{ P_a \} \) and \( \{ Q_b \} \), respectively, amounts to
\[
p_W^W(a, b | A, B) = \frac{1}{d^2} \left[ \frac{d+1}{d} - \text{Tr} \left( P_a Q_b \right) \right].
\]
(26)
where we have used the well-known property of the swap operator that \( \text{Tr} (V_d X \otimes Y) = \text{Tr} (X Y) \) for any pair of matrices \( X, Y \) and have assumed that the projectors \( P_a \) and \( Q_b \) are of rank one (see section 2.2).

To simulate (26) with local strategies, Werner proposed the following model:

**LHV model for projective measurements on Werner states (Werner’s model) [7]**

(0) Alice and Bob are distributed shared randomness represented by a one-qudit pure state \( \left| \lambda \right\rangle \in \mathbb{C}^d \) (the hidden state space is \( \Omega_d = \{ |\lambda\rangle \in \mathbb{C}^d | \langle \lambda | 1\rangle = 1 \} \) with the unique density invariant under any unitary operation \( \omega_d(\lambda) \).
(1) Alice returns the outcome \( a \) for which the overlap \( \langle \lambda | P_a | \lambda \rangle \) is the smallest one, i.e., her response function reads
p(a | A, λ) = \begin{cases} 1, & \text{if } \langle \hat{\lambda} | P_a | \hat{\lambda} \rangle = \min_a \langle \hat{\lambda} | P_a | \hat{\lambda} \rangle, \\ 0, & \text{otherwise} \end{cases} \quad (27)

(2) Bob returns the outcome b with probability

\[ p(b | B, \lambda) = \langle \hat{\lambda} | Q_b | \hat{\lambda} \rangle. \quad (28) \]

It is worth noting that the response functions (27) and (28) have the following equivariance property

\[ p(a | A, U\lambda) = p\left( a | UAU^\dagger, \lambda \right), \quad p(b | B, U\lambda) = p\left( b | UBU^\dagger, \lambda \right) \quad (29) \]

for any unitary U acting on \( \mathbb{C}^d \). Let us mention in passing that originally in [7] both the response functions were exchanged. This, however, does not affect the final result as the Werner states are permutationally invariant. The latter also means that we can even symmetrize the roles of the parties in the protocol. This can be easily done by augmenting the protocol with an additional two-valued hidden variable determining which type of the function, (27) or (28), each party must choose as a response.

Having all the necessary ingredients, we can now demonstrate that indeed \( p^W_0(a, b | A, B) \) admits the form (19), i.e.,

\[ p^W_0(a, b | A, B) = \int_{\Delta_d} d\lambda \omega_d(\lambda)p(a | A, \lambda)p(b | B, \lambda). \quad (30) \]

To be more instructive and explanatory, we will exploit for this purpose the approach of [26] rather than the original one by Werner. We start by noting that the value of the integral in (30) is, by the very construction of the model, invariant under any unitary rotation applied to \( |\lambda\rangle \). This means that we can decompose \( |\lambda\rangle \) in the eigenbasis \( \{|i\rangle\} \) of Alice’s observable \( A \), i.e.,

\[ |\lambda\rangle = \sum_i \lambda_i |i\rangle, \quad (31) \]

with \( \lambda_i \) being some complex coefficients such that \( |\lambda_i|^2 + \ldots + |\lambda_d|^2 = 1 \). In what follows, we will exploit their polar representation, that is, \( \lambda_i = r_i \exp(i\phi_i) \) with \( 0 \leq r_i \leq 1 \) and \( \phi_i \in [0, 2\pi) \) for any \( i \). Clearly, \( r_1^2 + \ldots + r_d^2 = 1 \). In addition, without any loss of generality, we can assume that the projection corresponding to the particular outcome \( a \) appearing in (30) is simply \( P_a = |1\rangle\langle 1| \). All this allows us to rewrite the probabilities (27) and (28) as

\[ p(a | A, \lambda) = \begin{cases} 1, & r_a^2 = \min_{a=2,\ldots,d} r_a^2 \\ 0, & \text{otherwise} \end{cases} \quad (32) \]

and

\[ p(b | B, \lambda) = \sum_{i,j=1}^d \langle i | Q_b | j \rangle r_{rj} e^{i(\phi_j - \phi_i)} 
\]

\[ = \sum_{i=1}^d \langle i | Q_b | i \rangle r_i^2 + \sum_{i \neq j}^d \langle i | Q_b | j \rangle r_{rj} e^{i(\phi_j - \phi_i)}, \quad (33) \]

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respectively. Inserting these into (30) and performing integrations over all the angles $\phi_i$ (clearly, the second summand in the above formula gives zero), one arrives at

$$\int_{\Omega_d} \frac{d\lambda}{\lambda} \omega_d(\lambda) p(a \mid A, \lambda) p(b \mid B, \lambda) = \frac{1}{N} \sum_{i=1}^{d} \langle i \mid Q_b \mid i \rangle J(u_i),$$

(34)

with the functional

$$J(h) = \int_{0}^{1} du_1 \int_{-1}^{1} du_2 \ldots \int_{-1}^{1} du_d h(u_1, \ldots, u_d) \delta(u_1 + \ldots + u_d - 1),$$

(35)

where we have made the substitution $r_i^2 = u_i$ and we have denoted by $h(u_1, \ldots, u_d)$ an arbitrary function of the variables $u_1, \ldots, u_d$. The Dirac delta in the above is due to the normalization of $|\lambda\rangle$, while the lower limits of the integrations over $u_2, \ldots, u_d$ follow from the condition in (32). Finally,

$$N = \int_{0}^{1} du_1 \int_{-1}^{1} du_d \delta(u_1 + \ldots + u_d - 1).$$

(36)

One can significantly simplify the computation of the rhs of (34) by noting that the integrals in (35) are manifestly symmetric under any permutation of the variables $u_2, \ldots, u_d$ (but not $u_1$!) and, therefore, $J(u_2) = \ldots = J(u_d)$. Moreover, the delta function in (35) allows us to conclude that $J(u_2) + \ldots + J(u_d) = J(1) - J(u_1)$, which in turn means that $J(u_i) = (J(1) - J(u_1))/(d - 1)$ for any $i = 2, \ldots, d$. Putting the pieces together, we have that

$$\int_{\Omega_d} \frac{d\lambda}{\lambda} \omega_d(\lambda) p(a \mid A, \lambda) p(b \mid B, \lambda) = \left[ d \operatorname{Tr}(P_a Q_b) - 1 \right] J(u_1) + \left[ 1 - \operatorname{Tr}(P_a Q_b) \right] J(1),$$

(37)

where we have substituted back $P_a$ for $|1\rangle\langle 1|$ which in particular means that $\langle 1 \mid Q_b \mid 1 \rangle$ is now $\operatorname{Tr}(P_a Q_b)$, and we have also used the fact that $\operatorname{Tr} Q_b = 1$ for any $b$. To complete the proof, one then needs to determine the values of the integrals $J(1)$ and $J(u_1)$. Computation of the latter has been done in the appendix and its value has been found to be

$$J(u_1) = N/d^3.$$  

(38)

The value of $J(1)$ can be inferred directly from the above formula (37); exploiting the fact that $p(a \mid A, \lambda)$ and $p(b \mid B, \lambda)$ are proper probability distributions, by summing both its sides over $a$ and $b$, one obtains that $J(1) = N/d$. Inserting both integrals into the rhs of equation (37), one finally obtains the quantum probability (26).

As a result, the above model allows one to show that the probabilities arising from projective measurements performed on the subsystems of the Werner states can always be simulated with a local model for any $\rho_W(d)$. Since the boundary value $(d - 1)/d$ is larger than the separability threshold $1/(d + 1)$ of $\rho_W(d, p)$, this means that for any $d$ there are entangled states with a local model for projective measurements. Noticeably, the critical value for locality grows with $d$, while the one for separability drops, meaning that the models become more powerful for larger $d$ (cf figure 3).

**Remark 1.** Let us also notice that the above model, initially designed for projective measurements at each site, works also if a generalized measurement with any number of outcomes (recall that a projective measurement can have at most $d$ outcomes) is allowed at Bob’s site. This directly follows from the facts that Bob’s response function is linear in the
measurement operators and that the measurement operators of a POVM $B$ can be taken as $B_b = \xi_b Q_b$ with rank-one operators $Q_b$ and some positive constants $\xi_b$ (see section 2.2). Therefore, the probabilities realized by the model in this case are $p_B(a, b \mid A, B) = \xi_b p_Q(a, b \mid A, B')$, where $p_Q(a, b \mid A, B')$ denote unnormalized ‘probabilities’ obtained by measuring projectors $Q_b$ on Bob’s site. At the same time, one realizes that the quantum probabilities are $p_Q(a, b \mid A, B) = \xi_b p_Q(a, b \mid A, B')$, and, therefore, $p_B(a, b \mid A, B) = p_Q(a, b \mid A, B)$.

Remark 2. Interestingly, as shown by Gisin and Gisin in [27], one can simulate the statistics of the singlet state (or, equivalently, the $2 \otimes 2$ Werner state for $p = 1$) with a local model if Alice or Bob (or both) is allowed not to provide outcomes. To state the model we now switch, for simplicity, to the Bloch representation; however, all that follows can be restated in terms of vectors from $\mathbb{C}^d$.

Assume that Alice and Bob receive unit vectors $\mathbf{a}$ and $\mathbf{b}$ representing local projective measurements $A$ and $B$, respectively, and a unit vector $\lambda$ from the Bloch sphere generated according to the uniform density $\omega(\lambda) = 1/4\pi$. Now, Alice follows the same strategy as in Werner’s model, that is, she always outputs $-\text{sgn}(\mathbf{a} \cdot \lambda)$ with sgn denoting the sign function. Then, with probability $|\mathbf{b} \cdot \lambda|$, Bob ‘accepts’ $\lambda$ and outputs $\text{sgn}(\mathbf{b} \cdot \lambda)$ in this case, while if he does not accept $\lambda$, he does not give any outcome. The local expectation values $\langle A \rangle = \langle B \rangle = 0$, while the correlations produced by this model taken over all the accepted $\lambda$ are given by

$$
\langle \mathbf{A} \otimes \mathbf{B} \rangle = -\frac{1}{2\pi} \int d\lambda \ | \mathbf{b} \cdot \lambda | \ \text{sgn}(\mathbf{a} \cdot \lambda) \ \text{sgn}(\mathbf{b} \cdot \lambda)
$$

$$
= -\frac{1}{2\pi} \int d\lambda \ \text{sgn}(\mathbf{a} \cdot \lambda)(\mathbf{b} \cdot \lambda)
$$

$$
= -\mathbf{a} \cdot \mathbf{b},
$$

(39)

where to obtain the last equation we have followed the calculation from the appendix. The corresponding probabilities are given by
Looking at equation (39), one can realize that this model can also be understood as one with density \((1/2\pi)b \cdot \lambda l\) which depends on the measurement of Bob. In other words, a model with a density that depends on the measurements of one of the parties can reproduce even highly nonlocal correlations. In fact, this observation was used in [28] to construct models for the simulation of entangled states using classical communication or supra-quantum resources.

**Remark 3.** Building on the above result, one can introduce a local model reproducing statistics obtained on a two-qubit state that differs from a two-qubit Werner state [23]. Alice follows the same strategy as above, while Bob, whenever he does not accept \(\lambda\), outputs \(b = \pm 1\) with probability \((1 \pm \langle \eta | b \cdot \sigma | \eta \rangle)/2\). After some calculation, one finds that the class of states whose correlations are simulated by this model is given by

\[
p = p \rho_{\psi_+ \psi_+} \otimes |\eta\rangle \langle \eta|,
\]

with \(p \leq 1/2\).

**Remark 4.** Let us conclude by noting that one can also extend the applicability of Werner’s model by playing with the density \(\omega\). An example of a density different from the uniform one has recently been given in [29] and reads \(\omega(\lambda) = (1 + \lambda_3)/4\pi\), where \(\lambda_3\) is the third coordinate of the Bloch vector \(\lambda\). In that case, the model corresponds to the two-qubit state

\[
\rho(p) = p \rho_{\psi_+ \psi_+} + \frac{1 - p}{5} \left( 2 |0\rangle \langle 0| \otimes \frac{1_2}{2} + 3 \frac{1_2}{2} \otimes |1\rangle \langle 1| \right)
\]

with \(p \leq 1/2\). Notice that this state is a convex combination of two states given in equation (41).

### 3.2. Barrett’s model for the Werner states

Let us now present the local model for general measurements for the Werner states due to Barrett [16]. Now, Alice and Bob want to simulate the probability distribution arising from some generalized measurements, \(A\) and \(B\), with the measurement operators \(A_a = \eta_a P_a\) and \(B_b = \xi_b Q_b\) (cf section 2.2) performed on the Werner states (20). One quickly finds that this distribution is not much different from that for projective measurements (26) and simply reads

\[
p^W_{\eta_a \xi_b, \eta_{a'} \xi_{b'}}(a, b | A, B) = \frac{\eta_a \xi_b}{d(d - 1)} \left[ \frac{d - 1 + p}{d} - p Tr(P_a Q_b) \right]
\]

The model now goes as follows:

**LHV model for POVMs on Werner states (Barrett’s model)** [7]

(0) As in Werner’s model, the set of hidden states and the density are, respectively, \(\Omega_d\) and \(\omega_{d_b}\).
(1) Alice returns the outcome \( a \) according to the response function:

\[
p(a \mid A, \lambda) = \langle \lambda \mid A_a \mid \lambda \rangle \Theta \left( \langle \lambda \mid P_a \mid \lambda \rangle - 1/d \right) + \left( 1 - \sum_a \langle \lambda \mid A_a \mid \lambda \rangle \Theta \left( \langle \lambda \mid P_a \mid \lambda \rangle - 1/d \right) \right) \eta_a/d.
\]

(44)

(2) Bob returns the outcome \( b \) with probability

\[
p(b \mid B, \lambda) = \frac{\xi_b}{d-1} \left( 1 - \langle \lambda \mid Q_b \mid \lambda \rangle \right).
\]

(45)

In the above, \( \Theta \) is the Heaviside step function, i.e., \( \Theta(x) = 1 \) for any \( x > 0 \) and \( \Theta(x) = 0 \) for \( x < 0 \).

To demonstrate that the above model recovers (43) and also to determine the range of \( p \) for which this is the case, let us insert the response functions (44) and (45) into (30). After some algebra, this leads us to

\[
p_L^W (a, b \mid A, B) = \frac{\eta_a \xi_b}{d^2} - \frac{1}{d-1} J_{ab} + \frac{\xi_b}{d-1} \sum_{\beta} J_{\beta a} \eta_{\beta b} - \frac{\eta_{\beta b}}{d(d-1)} \sum_{a} J_{ab} + \frac{\eta_{\beta b}}{d(d-1)} \sum_{a} J_{\alpha a} \eta_{\beta b}.
\]

(46)

with

\[
J_{ab} = \eta_a \xi_b \int_{\Omega_d} d\lambda \omega_d(\lambda) \Theta \left( \langle \lambda \mid P_a \mid \lambda \rangle - 1/d \right) \langle \lambda \mid P_b \mid \lambda \rangle \langle \lambda \mid Q_b \mid \lambda \rangle.
\]

(47)

In order to perform the above integration, let us decompose \( |\lambda\rangle \) as in (31) with \( \lambda_i = r_i \exp (i \phi_i) \) and the basis \( \{|i\rangle\} \) chosen so that it contains the vector corresponding to \( P_a \) (recall that now the projectors \( P_a \) as well as \( Q_b \) do not have to be orthogonal). As determined previously, we can assume that \( P_a = |1\rangle \langle 1| \). With this parametrization, one finds that (47) now becomes

\[
J_{ab} = \frac{\eta_a \xi_b}{N} \sum_{k=1}^d \langle \lambda \mid Q_b \mid \lambda \rangle \tilde{J} [u_1 u_k].
\]

(48)

where the normalization factor \( N \) is given by (36), while the functional \( \tilde{J} [h] \) is defined as

\[
\tilde{J} [h] = \int_{U_d} d u_1 \int_0^1 du_2 \ldots \int_0^1 du_d h(u_1, \ldots, u_d) \delta(u_1 + \ldots + u_d - 1).
\]

(49)

By noting that \( \tilde{J} [u_1 u_2] = \ldots = \tilde{J} [u_1 u_d] \), one easily sees that \( \tilde{J} [u_1 u_k] = \tilde{J} [u_1 u_{k+1}] / (d-1) \), which allows one to further simplify (48) to

\[
J_{ab} = \frac{\eta_a \xi_b}{N(d-1)} \left( \frac{d}{d} \text{Tr} (P_a Q_b) - 1 \right) \tilde{J} [u_1^2] + \left[ 1 - \text{Tr} (P_a Q_b) \right] \tilde{J} [u_1] \right\}.
\]

(50)
After inserting this into (46), one arrives at the following expression

\[ p_L^W(a, b | A, B) = \eta_0 \eta_b \left\{ \frac{1}{d^2} + \frac{d \tilde{J}[u_1^2] - \tilde{J}[u_1]}{N(d-1)^2} \left[ \frac{1}{d} - \text{Tr} \{ P_2 Q_b \} \right] \right\}, \]

(51)

which, when compared with (43), shows that the above model reproduces joint probabilities for the Werner states for any \( p \leq p_{\text{POVM}}^W \) with the critical probability given by

\[ p_{\text{POVM}}^W = \frac{d \tilde{J}[u_1]}{N(d-1)} \]

(52)

To finally determine the explicit value of \( p_{\text{POVM}}^W \), one has to compute the integrals \( \tilde{J}(u_1) \) and \( \tilde{J}(u_1^2) \). This can be done by exploiting basically the same approach as in the case of projective measurements (the detailed calculations are moved to the appendix), which leads us to

\[ \tilde{J}[u_1] = \frac{N}{d} \left( \frac{2d - 1}{d} \right) \left( \frac{d - 1}{d} \right)^{d-1} \]

(53)

and

\[ \tilde{J}[u_1^2] = \frac{N}{d^2} \left( \frac{5d - 3}{d + 1} \right) \left( \frac{d - 1}{d} \right)^{d-1}, \]

(54)

and, in consequence,

\[ p_{\text{POVM}}^W = \frac{3d - 1}{d(d + 1)} \left( \frac{d - 1}{d} \right)^{d-1}. \]

(55)

Interestingly, \( p_{\text{POVM}}^W > p_{\text{sep}}^W \) for any \( d \), and as a result Barrett’s model reproduces statistics of generalized measurements for entangled Werner states for any \( d \). However, unlike in Werner’s model, \( p_{\text{POVM}}^W \) is a (monotonically) decreasing function of \( d \). Moreover, it decreases faster than \( p_{\text{sep}}^W \), meaning that the range of \( p \) for which this is the case, i.e., \( p \in (p_{\text{sep}}^W, p_{\text{POVM}}^W) \), shrinks with \( d \to \infty \). Figure 3 compares the three critical values \( p_{\text{sep}}^W, p_{\text{POVM}}^W, p_{\text{PM}}^W \) for various values of the local dimension \( d \).

**Remark 5.** As noticed by Barrett [16], a bipartite state \( \rho \in B(C^{d_A} \otimes C^{d_B}) \) that has a local model for generalized measurements induces a whole family of states with local models (some of which might naturally be separable). The construction goes as follows: let \( \Omega, \omega(\lambda), p^A_\lambda(a | A, \lambda) \) and \( p^B_\lambda(b | B, \lambda) \) denote, respectively, the space of local variables, the density and the response functions in the local model for \( \rho \) for general measurements \( A = \{ A_\lambda \} \) and \( B = \{ B_\lambda \} \). Consider then two (in general different) quantum channels\(^5\) \( \Lambda_X : B(C^{d_X}) \to B(C^{d_X}) \), \( X = A, B \), and the state \( \sigma \) obtained from \( \rho \) through

\[ \sigma = (\Lambda_A \otimes \Lambda_B)(\rho). \]

(56)

Now, \( \sigma \) has a local model for generalized measurements with the same hidden state space \( \Omega \), density \( \omega \) and response functions defined as

\(^5\) Recall that a linear map \( A : B(H) \to B(K) \) with \( H \) and \( K \) being two finite-dimensional Hilbert spaces is called a quantum channel if it is completely positive and trace-preserving.
with the measurements operators of the generalized measurements $\mathcal{A}'$ and $\mathcal{B}'$ given by
\begin{align}
\mathcal{A}'_a &= A^+_A(A_a), & \mathcal{B}'_b &= A^+_B(B_b),
\end{align}
where $A^+_A$ is the dual\(^6\) map of $A$. As the dual map of a quantum channel is positive and unital (it preserves the identity operator), the operators $\mathcal{A}'_a$ and $\mathcal{B}'_b$ form proper quantum measurements. To see eventually that the functions (57) define a local model for $\sigma$, it is enough to apply the following argument
\begin{align}
\int_{\Omega} d\lambda \omega(\lambda)p^2_{A}(a, \lambda)p^2_{B}(b, \lambda) &= \text{Tr}\left(\mathcal{A}'_a \otimes \mathcal{B}'_b \rho\right) \\
&= \text{Tr}\left[ A^+_A(A_a) \otimes A^+_B(B_b) \rho \right] \\
&= \text{Tr}\left[ A_a \otimes B_b (A \otimes B)(\rho) \right] \\
&= \text{Tr}\left[ A_a \otimes B_b \sigma \right],
\end{align}
where to pass from the second to the third line we have exploited the definition of the dual map to $\Lambda$.

Let us conclude by noting that the above argument can also be applied to multipartite states, provided such states with local models for generalized measurements exist. Actually, it applies to any state that has a ‘mixed’ local model, i.e., one which works for projective measurements at some sites and for generalized measurements at the rest. In such cases, to obtain a new state the local quantum channels discussed above can be applied only to those sites. An example of such a state and the corresponding ‘mixed’ local model will be discussed in section 4.1.

### 3.3. Nonlocality of noisy states: the Grothendieck constant

We now move to the analysis of nonlocal properties of noisy quantum states, that is, states of the form
\begin{align}
\rho(d, p) &= p\rho + (1 - p)\frac{1}{d^2}, & \rho &\in \mathcal{B}\left(\mathbb{C}^d \otimes \mathbb{C}^d\right).
\end{align}

As mentioned at the beginning of this section, this problem can be related to the mathematical constant $K_G$ known as the Grothendieck constant [19]. The connection between the latter and nonlocality was first recognized in 1987 by Tsirelson (a.k.a. Cirel’son) [30] who considered the problem of how large the set of quantum correlations is compared to the set of classical ones. The interest in this surprising relationship had its revival almost thirty years later when it was analyzed in greater detail by Acín et al in [17]. The results of the latter paper, relevant for our review, may be summarized as follows (the terminology and necessary definitions will be introduced in what follows):

1. (1) projective measurements on a two–qubit Werner state, $\rho_w(2, p)$ (see equation (20)), can be simulated with an LHV model if and only if $p \leq 1/K_G(3)$,
2. (2) local models for $\rho_w(2, p)$ exist in the whole range where the state does not violate the CHSH inequality (17), that is, for $p \leq 1/\sqrt{2}$, when at least one of the parties is restricted to perform planar measurements,

\(^6\) The dual map to a linear map $\Lambda: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is a linear map $\Lambda^\dagger: \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{H})$ that satisfies $\text{Tr}(\Lambda^\dagger(X)Y) = \text{Tr}(\Lambda(X)Y)$ for all $X \in \mathcal{B}(\mathcal{K})$ and $Y \in \mathcal{B}(\mathcal{H})$. 


whenever $p \leq 1/K_G(2d^2)$ there is a local model reproducing joint correlations of traceless two-outcome observables for the state (60) with any $\rho$. Moreover, for $p > 1/K_G(2 \log d + 1)$ there exists $\rho(d, p)$ without such a model. In particular, in the limit of $d \to \infty$, both bounds match and every noisy state is local below $1/K_G$, and this number cannot be made larger.

Before we proceed, we need the definition of the Grothendieck constant. Let $n \geq 2$ be an integer, and let $M$ be an arbitrary $m \times m$ real matrix such that for all real numbers $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_m$ from the range $[-1, +1]$, it holds

$$\sum_{i,j=1}^{m} M_{ij} a_i b_j \leq 1.$$  

(61)

The Grothendieck constant of order $n$, $K_G(n)$, is defined to be the smallest number such that

$$\sum_{i,j=1}^{m} M_{ij} a_i \cdot b_j \leq K_G(n),$$  

(62)

for all unit vectors $a_k, b_k \in \mathbb{R}^n$. The Grothendieck constant, $K_G$, is then defined through

$$K_G = \lim_{n \to \infty} K_G(n).$$  

(63)

It is quite remarkable that such constant even exists; it is also interesting that the exact values of the constants, besides $K_G(2) = \sqrt{2}$ [31], are not known. The bounds for the constants appearing in our analysis are as follows [31–34]:

$$1.6770 \leq K_G(2) \leq 1.7822,$$

(64)

$$K_G(8) \leq 1.6641,$$

(65)

$$1.417 \leq K_G(3) \leq \frac{\pi}{2c_3},$$

(66)

where $c_3$ is the unique, in the interval $[0, \pi/2]$, solution of the equation

$$\sqrt{c_3} \int_{0}^{c_3} dx \ x^{-3/2} \sin x = 2.$$  

(67)

Numerically, one then finds that

$$K_G(3) \leq 1.5163.$$  

(68)

Let us now elaborate on each of the items (1)–(3) from the list above.

As to the point (1), let us first notice that the reconstruction of the mean values $\langle A \otimes B \rangle$, $\langle A \rangle$, and $\langle B \rangle$ is enough to retrieve full probability distribution as we deal here with two-outcome measurements, in which case the number of mean values matches the number of independent probabilities necessary for this task (see the discussion in section 2). Further, on states with maximally mixed reductions (for two qubits, these are the Bell diagonal states), including the discussed Werner state $\rho_W(2, p)$, local expectation values vanish (recall that the outcomes of all observables are labeled $\pm 1$), $\langle A \rangle = \langle B \rangle = 0$. The key point of the analysis now is that given a model reproducing correctly only $\langle A \otimes B \rangle$, we can turn it into one for which also local mean values vanish. We augment the old protocol with an extra random bit, and in the new model parties just multiply outputs of the old one by the value of this bit, ensuring that joint predictions are still the same and local ones are vanishing. Thus, nonlocal
properties of $\rho_p(2, p)$ are uniquely determined by the joint correlations solely, and in the following analysis we can restrict ourselves to correlation Bell inequalities. The latter in the most general case can be written as

$$\sum_{i,j=1}^{m} M_{ij} \langle A_i \otimes B_j \rangle \leq \beta_L, \quad M_{ij} \in \mathbb{R},$$

(69)

where $\beta_L$ denotes the local bound of a given Bell inequality, that is,

$$\beta_L = \max_{a_i, b_j = \pm 1} \left| \sum_{i,j=1}^{m} M_{ij} a_i b_j \right|$$

(70)

since deterministic strategies are sufficient to achieve it (we stress again that we assume here that the results of the measurements are $\pm 1$). Clearly, we can normalize our inequality such that $\beta_L = 1$ and we will assume this has been done.

Now, correlations on the maximally mixed state vanish, and thus, the violation of any correlation Bell inequality by Werner states is determined by its violation by the singlet state. Since for the latter we have $\langle A_i \otimes B_j \rangle_{\psi^-} = -a_i \cdot b_j$ with $a_i, b_j \in \mathbb{R}^3$ being Bloch vectors representing the observables $A_i, B_j$, this violation maximized over all Bell inequalities can be written as

$$\beta_{Q,W} := \lim_{m \to \infty} \sup_{M_{ij} a_i b_j} \left| \sum_{i,j=1}^{m} M_{ij} a_i b_j \right|.$$

(71)

From this, it immediately follows that no Werner state $\rho_p(p, 2)$ can violate a correlation Bell inequality in the range $p \leq 1/\beta_{Q,W}$. As one can easily see, $\beta_{Q,W}$ is just the Grothendieck constant $K_G(3)$, and so we have recovered that whenever $p \leq 1/K_G(3)$ a two-qubit Werner...
state is local for projective measurements and no local model can exist for values outside this range. Taking into account (68), we realize that this result is an improvement over Werner’s
\[ p \leq 1/2 \text{, as } 1/K_G(3) \geq 0.6595. \] It is known that there is a gap between the exact value and \( 1/\sqrt{2} \approx 0.7071 \) as the Werner states have been demonstrated to violate some Bell inequality for \( p > 0.7056 \) [33] (cf figure 4). Recall that \( \rho_\psi(2, p) \) is entangled iff \( p > 1/3 \).

As announced in (2), there is no such separation when (at least) one of the parties is restricted to perform measurements on a plane in the Bloch sphere. In this case, vectors \( a_i \) are two-dimensional; moreover, \( b_j \) can be taken to lie in the same plane since only the scalar products of \( a \) and \( b \) are contributing to the value of the Bell operator. In this way, it is clear that the bound now involves \( K_G(2) \), which, as mentioned, is equal to \( \sqrt{2} \). The result, which is an improvement over \( \pi /2 \) from [35], then follows.

The statement of (3) is a result of the attempt to generalize (1) to general noisy states (60) in arbitrary dimensions. However, we encounter two difficulties in such generalization. First, we need to restrict ourselves to two-outcome measurements (for the reason discussed in section 2). Second, even in this case we cannot hope to give full characterization of nonlocal properties of a state with the aid of the correlation Bell inequalities only. This stems from the fact that in the general approach we pursue here we cannot assume anything about reductions of the state (in particular, they might not be maximally mixed) and it might be necessary to use inequalities with local terms [36] for this purpose. Thus, we can only hope here to fully characterize the joint correlations. Further, an additional requirement will have to be met: tracelessness of the observables. This condition ensures that the averages on the maximally mixed state are zero (in other words, outcomes are completely random on such state) and the Grothendieck constant approach in the form put forward above can be employed.

To proceed, it might be useful to rephrase (3) using the threshold probability \( p_d^{\text{thr}} \) for states of the form (60) (this quantity will also be used in the upcoming section 3.4). For a given \( d \), this is the minimum over all states \( \rho \) of the maximal \( p \) for which there exists a local model. With this notion in hand, the claims are that

\[ \frac{1}{K_G(2d^2)} \leq p_d^{\text{thr}} \leq \frac{1}{K_G(2\left\lfloor \log_2 d \right\rfloor + 1)} \]  \tag{72}

and

\[ p_d^{\text{thr}} := \lim_{d \to \infty} p_d^{\text{thr}} = \frac{1}{K_G}. \]  \tag{73}

We start with the lhs of (72) and consider first states \( \rho^{(d)}(d, p) \) of the form (60), with \( \rho \) being an arbitrary pure state \( |\psi_i\rangle \). From [17, 30] we know that for any observables \( A, B \) and a state \( |\psi_i\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d \), one can find vectors \( a \) and \( b \) from \( \mathbb{R}^{2d^2} \) such that

\[ \langle A \otimes B \rangle = a \cdot b. \]  \tag{74}

With the assumption of observables being traceless and the definition of the Grothendieck constant in mind, we conclude that the threshold probability for states \( \rho^{(d)}(d, p) \) is at least equal to \( 1/K_G(2d^2) \). Since any \( \rho \) can be expressed as a convex combination of pure states, \( \rho = \sum_q q_i |\psi_i\rangle \langle \psi_i| \), the bound we have just established for \( \rho^{(d)}(d, p) \) must also hold for general \( \rho(d, p) \). This follows from the fact that our local model for these states might be just taken to be the convex combination of models for \( \rho^{(d)}(d, p) \) with weights \( q_i \). In conclusion, whenever \( p \leq 1/K_G(2d^2) \) the state \( \rho(d, p) \) is local regardless of the form of \( \rho \).

Now, let us move to the rhs of (72). Assume we have a set of unit vectors \( a_i, b_j \in \mathbb{R}^m \) with \( 1 \leq i, j \leq m \) and \( n = 2\left\lfloor \log d \right\rfloor + 1 \), maximizing the lhs of equation (62) that is
achieving $K_G(2 \lfloor \log d \rfloor + 1)$. It is known that there exist traceless observables $A_i, B_j$ such that $\langle A_i \otimes B_j \rangle_{\Phi^+} = a_i \cdot b_j$ for the two-qudit maximally entangled state

$$\phi^+_d = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |ii\rangle$$

with $d = 2^{n/2}$. In equation (60), take now $\rho = |\phi^+_d\rangle\langle \phi^+_d|$, and $p = (1 + \epsilon)/K_G(2 \lfloor \log d \rfloor + 1)$ for some $\epsilon > 0$. From the above, it follows that we will achieve $1 + \epsilon$ for the value of the Bell operator, which means the violation of a Bell inequality and rules out existence of a local model for these states. To sum up, for $p > 1/K_G(2 \lfloor \log d \rfloor + 1)$ there exist states (60) with nonlocal joint correlations.

Taking the limit of both sides clearly results in the threshold value equal to $1/K_G$. This concludes the proof of the claims made in (3).

Let us now comment about possible applications of (3). We list them below and then explain the underlying reasoning:

- (3.1) for the noisy qubit states $\rho(2, p)$ of the form (60) there always exists a local model for joint correlations whenever $p \leq 1/K_G(8)$, and
- (3.2) for the isotropic states (that is, those with $\rho = |\phi^+_d\rangle\langle \phi^+_d|$ in (60); see also section 3.4) there is a local model simulating the full probability distribution for traceless observables whenever $p \leq 1/K_G(d^2 - 1)$.

The statement of (3.1) follows from the already-mentioned facts that in the qubit scenario observables are two-outcome and traceless, just as required. Taking into account (65), this gives the threshold at around 0.6009. On the other hand, (3.2) stems from the following: (i) previously mentioned possibility of adding an extra random bit to the protocol to reproduce local mean values, which are zero for the considered state, and (ii) a refinement of one of the facts mentioned earlier, namely, that when the state is maximally entangled, to reproduce mean values in the form $a \cdot b$ both vectors can be drawn from $\mathbb{R}^{d^2-1}$.

Interestingly, the results by Grothendieck [19] and Krivine [31], or more precisely their proofs concerning upper bounds on the Grothendieck constant, allowed Toner [34] to obtain explicit local models for the above-considered cases. We conclude this section by giving one of these models with a sketch of the proof (details can be found in [34]), namely, the one for projective measurements on the two-qubit Werner states in the range of $p < 0.6595$.

Let $a, b$ be unit vectors from $\mathbb{R}^3$ representing Alice and Bob measurements. Let further $f$ and $g$ be mappings $\mathbb{R}^3 \to \bigoplus_{k = 0}^{2k + 1} \mathbb{R}^{k+1}$ defined by the following set of equations:

$$f(a) = \bigoplus_{k=0}^{2k+1} \bigoplus_{m=-1}^{2k+1} f_{2k+1,m}(a),$$

$$f_{2k+1,m}(a) = (-1)^{k+1} \frac{4\pi^{3/2} J_{2k+3/2} (\xi_3) \left[ \operatorname{Re} \left( Y_{2k+1}^m (a) \right), \operatorname{Im} \left( Y_{2k+1}^m (a) \right) \right]}{\sqrt{2c_3}}$$

and

$$g(b) = \bigoplus_{k=0}^{2k+1} \bigoplus_{m=-1}^{2k+1} g_{2k+1,m}(b),$$

$$g_{2k+1,m}(b) = \frac{4\pi^{3/2} J_{2k+3/2} (\xi_3)}{\sqrt{2c_3}} \left[ \operatorname{Re} \left( Y_{2k+1}^m (b) \right), \operatorname{Im} \left( Y_{2k+1}^m (b) \right) \right].$$
where $c_3$ is defined through (67), $Y_l^m$ are the spherical harmonics, and $J_\nu$ are the Bessel functions of the first kind of order $\nu$. The model reads as follows$^7$:

### LHV model for projective measurements for the two-qubit Werner states [34]

1. Alice and Bob each get an infinite sequence of numbers $\lambda = [\lambda_1, \lambda_2, \ldots] \in \mathbb{R}^\infty$, where each $\lambda_i$ is drawn from a normal distribution with the mean equal to 0 and the standard deviation equal to 1,
2. Alice outputs $a = \text{sgn}(f(a) \cdot \lambda)$ with $f$ defined through equation (76),
3. Bob outputs $b = \text{sgn}(g(b) \cdot \lambda)$ with $g$ defined through equation (77).

Let us now recall some important steps from the proof [34] that the model works in the required range.

First one needs to verify that $f(a)$ is a unit vector (for $g(b)$ the reasoning will be similar so in further parts we omit it). One obtains

$$f(a) \cdot f(a) = \frac{\pi}{2c_3} \sum_{k=0}^{\infty} (4k + 3)J_{2k+3/2}(c_3) = \frac{\sqrt{c_3}}{2} \int_0^\infty dx \; x^{-3/2} \sin x = 1,$$

with the last equality following from the definition of $c_3$. Further:

$$f(a) \cdot g(b) = \frac{\pi}{2c_3} \sum_{k=0}^{\infty} (-)^{k+1}(4k + 3)J_{2k+3/2}(c_3)P_{2k+1}(a \cdot b),$$

where $P_l$ are the Legendre polynomials. One further verifies that

$$-\sin(c_3 x) = \sum_{k=0}^{\infty} (-)^{k+1}(4k + 3)J_{2k+3/2}(c_3)P_{2k+1}(x),$$

which in turn means that

$$f(a) \cdot g(b) = -\sin(c_3 a \cdot b).$$

Now, given the random variable $\lambda$ specified by the conditions given in the frame above and $a = \text{sgn}(x \cdot \lambda)$, $b = \text{sgn}(y \cdot \lambda)$, the average of $ab$ over $\lambda$ is equal $\langle ab \rangle = (2/\pi) \sin^{-1}(x \cdot y)$ for arbitrary $x, y \in \mathbb{R}^\infty$ [5, 19, 34]. We thus find that:

$$\langle A \otimes B \rangle = \frac{-2c_3}{\pi} a \cdot b.$$

Having in mind (66) and (68), we conclude the correctness of the model falls within the claimed range of $p < 0.6595$.

$^7$ Notice that the model fits the general scheme given in section 2 since we can write the corresponding response functions as

$$p(\pm 1 | A, \lambda) = \begin{cases} 1, & \text{sgn}(f(a) \cdot \lambda) = \pm 1 \\ 0, & \text{sgn}(f(a) \cdot \lambda) = \mp 1 \end{cases}$$

and analogously for Bob.
The results of this and previous sections concerning existence of local models for two-qubit Werner states are summarized in figure 4.

Let us conclude by noting that with the aid of the complex Grothendieck constant, it is possible to generalize some of the above statements to Bell inequalities with an arbitrary number of outcomes [12].

3.4. Noise robustness of correlations: Almeida et al’s approach and the CGLMP and CHSH inequalities perspective

The research on the robustness of nonlocality in a general scenario was further pursued by Almeida et al [18]. The starting point of their analysis was to check to what extent the nonlocality of maximally entangled states of arbitrary dimension is affected by the white noise. That is, the goal was to determine when a local model exists for states of the form

$$\rho_{iso}(d, p) = p \left| \psi^d_{iso} \right\rangle \left\langle \psi^d_{iso} \right| + (1 - p) \frac{1d^2}{d^2},$$

where $0 \leq p \leq 1$ and $|\psi^d_{iso}\rangle$ is the maximally entangled state (75). These states are known in the literature as the isotropic states and are the unique states which are invariant under bilateral unitary rotations of the form $U \otimes U^*$ [37]. Note that the isotropic states have a clear physical meaning, as they correspond to noisy versions of maximally entangled states, while this physical interpretation is missing for Werner states of dimension larger than two. In the case of qubits, however, the isotropic and Werner states are equivalent up to local unitary transformations.

Inspired directly by Werner’s construction (see section 3.1) the local model simulating projective measurements $A$ and $B$ with measurement operators $\{P_a\}$ and $\{Q_b\}$, respectively, on the isotropic states is given by the LHV model for projective measurements on the isotropic states [18, 38]

1. Alice and Bob each get $|\lambda\rangle \in \Omega_d = \{|\lambda\rangle \in \mathbb{C}^d \langle \lambda | \lambda \rangle = 1\}$ with the uniform density $\omega_d$.

2. Alice’s response function is:

$$p_{iso}(a | A, \lambda) = \begin{cases} 1, & \text{if } \langle \lambda | P_a | \lambda \rangle = \max_a \langle \lambda | P_a | \lambda \rangle, \\ 0, & \text{otherwise} \end{cases}$$

3. Bob’s response function is:

$$p_{iso}(b | B, \lambda) = \langle \lambda | Q_b^T | \lambda \rangle,$$

where $T$ stands for the transposition.

Direct calculation, which can be carried out with techniques analogous to those already presented in section 3.1, shows that the probabilities realized by this model assume the same form (37) with $Q_b$ replaced by $Q_b^T$ and the integral $J[h]$ replaced by $J[h]$ given by (notice the change of the integration limits)
\[ J[h] = \int_0^1 du_1 \int_0^{u_1} du_2 \ldots \int_0^{u_{d-1}} du_d h(u_1, \ldots, u_d) \delta(u_1 + \ldots + u_d - 1). \]  

(86)

As before, \( J[1] \) can be directly determined from (37) and amounts to \( N/d \), while \( J[u_1] \) is computed in the appendix and is proportional to the so-called harmonic number, i.e.,

\[ J[u_1] = \frac{N}{d^2} \sum_{k=1}^{d} \frac{1}{k}. \]  

(87)

After inserting all this into (37), one arrives at the following critical probability

\[ p = p_{PM}^{iso, c} \equiv \frac{1}{d-1} \left( -1 + \sum_{k=1}^{d} \frac{1}{k} \right) \]  

(88)

for which the above model reproduces the statistics of the isotropic states, meaning that the isotropic states are local at least up to \( p_{PM}^{iso, c} \). Clearly, for \( d = 2 \) this reproduces the critical value 1/2 by Werner as it should since, as noted above, the isotropic and the Werner states are related to each other via local unitary rotations. Noticeably, in the limit of large \( d \), the above critical probability scales as \( \log d/d \). On the other hand, it is known [37] that the isotropic states are separable whenever \( p \leq p_{sep}^{iso, c} \) with

\[ p_{sep}^{iso, c} \equiv \frac{1}{d + 1} \]  

(89)

and, thus, the critical probability for the local model is asymptotically \( \log d \) larger than the corresponding value for entanglement (cf figure 5).

We can now move to the general case of arbitrary \( \rho \). As it was already noted in section 3.3, it is enough to construct a model for a pure state \( \rho = \lvert \psi \rangle \langle \psi \rvert \), as the model for a mixed state can be taken to be a convex combination of models for pure states.

The key point of the general approach is the local model for the following mixture of \( \rho \) with a state-dependent noise

\[ \tilde{\rho} = p_{PM}^{iso, c} \lvert \psi \rangle \langle \psi \rvert + (1 - p_{PM}^{iso, c}) \sigma \otimes \frac{I_d}{d}. \]  

(90)
where $\sigma = \text{tr}_b \psi \psi$. This state can be further transformed into a one of the form (60) by admixing it with the following separable state

$$
\frac{1}{d - 1} \sum_{k=1}^{d-1} \sigma_k \otimes \frac{1_d}{d},
$$

(91)

where $\sigma_k = \sum_j \alpha_{j+k(\text{mod} \, d)} |j\rangle \langle j|$ with $\{\alpha, j\}$ being the eigensystem of $\sigma$. The resulting state then reads

$$
\Theta = q \hat{\psi} + \frac{1 - q}{d - 1} \sum_{k=1}^{d-1} \sigma_k \otimes \frac{1_d}{d}.
$$

(92)

It is easy to see that this state is of the desired form (60) when the weights fulfill the condition $q(1 - p) = (1 - q)/(d - 1)$. Since the state $\hat{\psi}$ has been shown to be local, the state $\Theta$ is also local as it is just a convex combination of local states. The condition on $q$, however, implies that the price we have to pay for this transformation is the diminishment of the value of the critical probability, denoted hereafter by $P_{\text{PM}}$, for which we can construct a model.

Let us now move to the details of the construction of the model for $\hat{\psi}$. The main tool is Nielsen’s protocol for the LOCC conversion between pure states [20]. The idea is that some preprocessing based on such transformations can be performed by the source itself on the hidden state $\lambda$, producing with some probabilities $\lambda \langle \lambda |$ and $\lambda \langle \lambda |$, which are later sent to the parties. The parties then follow the protocol for the isotropic state (see the frame above), taking these hidden states instead of the standard one. Thus, having in hand the model for the isotropic states, we get almost for free the model for the noisy states $\hat{\psi}$.

To understand the details, we need to see how the conversion of $|\phi^d\rangle$ to some $|\psi\rangle$ works. Assume that in the Schmidt form the state $\psi$ reads $|\psi\rangle = \sum_k s_k |k\rangle |k\rangle$. Denoting $S = \text{diag}(s_0, s_1, \ldots, s_{d-1})$ and $U_k = \sum_{j=0}^{d-1} \frac{1}{d} |j\rangle \langle j + k (\text{mod} \, d)|$ with $k = 0, 1, \ldots, d - 1$, we can write

$$
|\psi\rangle = \sqrt[d]{\langle X_k \otimes U_k \rangle} |\phi^d\rangle, \quad X_k = SU_k.
$$

(93)

Observe that $M_k \equiv X_k^* X_k \geq 0$ and $\sum_k M_k = 1_d$, which means that the operators $M_k$ constitute valid elements of a POVM measurement; let us call this measurement $\mathcal{M}$. The conversion protocol appears obvious now: Alice performs $\mathcal{M}$ with probability $\langle \phi^d | M_k | \phi^d \rangle = 1/d$ obtaining the outcome $k$, and then sends the index of the obtained result to Bob, who performs appropriate unitary rotation $U_k$, which in turn results in sharing $|\psi\rangle$ between the parties.

As announced earlier, some preprocessing is simulated at the source before the distribution stage of the protocol. More precisely, the measurement $\mathcal{N}$ with elements $N_k \equiv X_k^* X_k$ on the hidden state $|\lambda\rangle$ is simulated. The outcomes are obtained with probabilities $p_{k,\lambda} (\equiv \langle \lambda | N_k | \lambda \rangle)$ and with these probabilities the following states (more precisely their classical descriptions) are sent to Alice and Bob: $|\lambda^A\rangle \equiv (X_k^* \sqrt[p_{k,\lambda}]{\langle \lambda | N_k | \lambda \rangle} | \lambda \rangle)$ and $|\lambda^B\rangle \equiv U_k | \lambda \rangle$.

Alice and Bob give their outputs according to the response functions $\rho(a | A, \lambda^A)$ and $\tilde{\rho}(b | B, \lambda^B)$, respectively, which are the same as in the protocol for the isotropic states, but with $\lambda_A$ and $\lambda_B$ replacing $\lambda$. The statistics they generate are now $(\Omega_d$ and $\omega_d$ are the same as in Werner’s model):

$$
\tilde{\rho}_L(a, b | A, B) = \int_{\Omega_d} d\lambda \omega_d(\lambda) \sum_{i=0}^{d-1} p_{i,\lambda} \tilde{\rho}(a | A, \lambda^A(\lambda)) \tilde{\rho}(b | B, \lambda^B(\lambda)),
$$

(94)

and one can easily show that they indeed reproduce the quantum prediction for $\hat{\psi}$. 

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To complete the protocol for the general state, one must add noise to Alice’s share, which can be done as mentioned above. The model for an arbitrary noisy state is summarized in the frame below (see the text above for the notation). LHV model for projective measurements on general noisy states (60) [18].

The parties follow the protocols $\mathcal{P}$ and $\mathcal{Q}$ listed below with respective fractions $q$ and $1 - q$ of times.

**Protocol $\mathcal{P}$:**

(\(\mathcal{P}_0\)) $|\lambda\rangle$ are drawn from $\Omega_d$ according to the uniform density $\omega_{\lambda,d}$.

(\(\mathcal{P}_1\)) With probability $p_{k,\lambda} = \langle A | N_k | \lambda\rangle$ Alice gets $|\lambda^A_k\rangle \equiv (X^*_{k} / \sqrt{p_{k,\lambda}})|\lambda\rangle$ and Bob gets $|\lambda^B_k\rangle \equiv U_k |\lambda\rangle$.

(\(\mathcal{P}_2\)) Alice’s response function is

$$\tilde{p}\left(a | A, \lambda^A_k\right) = \begin{cases} 1, & \text{if } \langle \lambda^A_k | P_a | \lambda^A_k \rangle = \max_a \langle \lambda^A_k | P_a | \lambda^A_k \rangle, \\ 0, & \text{otherwise} \end{cases} \quad (95)$$

(\(\mathcal{P}_3\)) Bob’s response is:

$$\tilde{p}\left(b | B, \lambda^B_k\right) = \langle \lambda^B_k | Q_b^T | \lambda^B_k \rangle. \quad (96)$$

**Protocol $\mathcal{Q}$:**

(\(\mathcal{Q}_1\)) Alice simulates the statistics of measurements on $\sum_{j=1}^{d-1} \sigma_j / (d - 1)$.

(\(\mathcal{Q}_2\)) Bob outputs random results simulating in this way the statistics of measurements on the maximally mixed states.

As noted above, the critical value $\tilde{p}_{\text{PM}}^c$ for the model presented above is now reduced in comparison to the one for the isotropic state and is equal to

$$\tilde{p}_{\text{PM}}^c = \frac{p_{\text{PM}}^{\text{iso}, c}}{1 - p_{\text{PM}}^{\text{iso}, c}} = \frac{p_{\text{PM}}^{\text{iso}, c}}{(d - 1) + 1}. \quad (97)$$

In the limit of large $d$ this scales as $\log d / d^2$. In comparison, the threshold value $\tilde{p}_{\text{sep}}^c$ for the separability of states (60) lies in the following interval [39]:

$$\tilde{p}_{\text{sep}}^c \in \left[ \frac{1}{d^2 - 1}, \frac{2}{d^2 + 2} \right]. \quad (98)$$

This means, as previously, that the locality threshold is at least asymptotically $\log d$ larger than the corresponding separability value.

Exploiting Barrett’s model (see section 3.2), the ideas presented above can be adapted to the general case of POVMs. The resulting protocol for the general noisy states is as follows: **LHV model for POVMs on the noisy states (60) [18].**

Fraction $q$ of times, the parties follow the below-described protocol $\mathcal{P}'$, and for fraction $1 - q$, the parties follow the admixing protocol $\mathcal{Q}$ described in the frame above. Protocol $\mathcal{P'}$: 

(\(\mathcal{P}_0'\)) $|\lambda\rangle$ are drawn from $\Omega_d$ according to the density $\omega_d(|\lambda\rangle), \ldots, \ldots
with probability \( p_{i,j} = \langle \lambda_1 | N_i | \lambda \rangle \) Alice gets \( | \lambda_i^A \rangle \equiv (X_i | \sqrt{p_{i,j}} | \lambda \rangle \) and Bob gets \( | \lambda_i^B \rangle \equiv U_i | \lambda \rangle \).

(P2') Alice’s response function is

\[
p' \left( a | A, \lambda_i^A \right) = \langle \lambda_i^A | A_a | \lambda_i^A \rangle \Theta \left( \langle \lambda_i^A | P_a | \lambda_i^A \rangle - \frac{1}{d} \right)
\]

\[
+ \left[ 1 - \sum_j \langle \lambda_i^A | A_j | \lambda_i^A \rangle \Theta \left( \langle \lambda_i^A | P_j | \lambda_i^A \rangle - \frac{1}{d} \right) \right] \eta_{a,j} d,
\]

(99)

(P3') Bob’s response is:

\[
p' \left( b | B, \lambda_i^B \right) = \xi_b \langle \lambda_i^B | Q_b' \lambda_i^B \rangle.
\]

(100)

The steps (P2')–(P3') are the ones required to simulate statistics for the isotropic state and obviously could be performed separately on \( | \lambda \rangle \) if this was the task. The step (P1'), as previously, originates from Nielsen’s protocol. We need to combine the protocol with \( Q \) to ensure that the resulting state has the proper form.

Let us now take a closer look at the range of parameters for which the model works. The critical value \( p_{POVM, c}^{iso} \) for the isotropic state is

\[
p_{POVM}^{iso, c} = \frac{(3d - 1)(d - 1)^{d-1}}{(d + 1)d^d},
\]

(101)

which for large \( d \) scales as \( 3/(ed) \). In the general case (i.e., arbitrary \( \rho \) in (60)) the critical threshold value, in analogy to the projective measurement case, is

\[
p_{POVM}^{c} = \frac{p_{POVM}^{iso, c}}{(1 - p_{POVM}^{iso, c})(d - 1) + 1},
\]

(102)

which asymptotically scales as \( 3/(ed^2) \).

It is important to realize that the results presented above constitute only lower bounds on the noise thresholds. This stems from the fact that they have been obtained for particular local models with no guarantee that a better model could exist in a wider range of \( p \). The question of how much these values might differ from the ultimate ones has been addressed in [21] through the analysis of the violation of the CGLMP inequality and in [22] through the analysis of the CHSH inequality.

In particular, it has been shown in [21] that the isotropic states violate the CGLMP inequality with projective measurements, and consequently also with general measurements, thus being necessarily nonlocal, at least for

\[
p > p_{d, upp}^{iso} := \left[ 4d \left( \sum_{k=0}^{[d/2] - 1} \left( 1 - \frac{2k}{d} \right) q_k \right) q_{-(k+1)} \right]^{-1},
\]

(103)

where \( q_k = 1/(2d^3 \sin^2[\pi(k + \frac{1}{2})]) \) and \([x]\) denotes the integer part of \( x \). It is a decreasing function of the dimension \( d \) and asymptotically it achieves the value of
\[ p_{\infty}^{\text{iso, upp}} := \lim_{d \to \infty} p_d^{\text{iso, upp}} = \frac{\pi^2}{16G} \approx 0.67344, \]  
(104)

where \( G \approx 0.91597 \) is Catalan’s constant.

Summing up, we have that the noise threshold value, \( p_d^{\text{iso, thr}} \), separating the locality and nonlocality of the isotropic states, satisfies

\[ p_X^{\text{iso, c}} \leq p_d^{\text{iso, thr}} \leq p_d^{\text{iso, upp}}, \quad X = \text{PM, POVM}, \]  
(105)

which asymptotically gives

\[ \Theta\left(\frac{\log d}{d}\right) \leq p_d^{\text{iso, thr}} \leq \frac{\pi^2}{16G}, \]  
(106)
\[ \Theta\left(\frac{3}{ed}\right) \leq p_d^{\text{iso, thr}} \leq \frac{\pi^2}{16G}, \]  
(107)

for projective and general measurements, respectively.

In the case of an arbitrary state \( \rho \) in equation (60), one needs a different notion of the noise threshold value. Such a quantity, \( p_d^{\text{thr}} \), has already been introduced in section 3.3. Recall that now one wants to find the value of \( p \) below which all states (60), regardless of the state \( \rho \), are local and above which there certainly exists a nonlocal state.

With the aim of finding good upper bound on this value, one needs to find an example of a noisy state particularly resistant to noise. It turns out [22] that a two-qudit state of the form (60) with \( \rho = |\phi^+\rangle\langle\phi^+| \) is such a candidate, as it violates the CHSH inequality with projective measurements for \( p > p_d^{\text{upp}} \) with

\[ p_d^{\text{upp}} := \frac{1 - \left(\frac{d-2}{d}\right)^2}{\sqrt{2} - \left(\frac{d-2}{d}\right)^2}. \]  
(108)

Thus, we have:

\[ \tilde{p}_X^c \leq p_d^{\text{thr}} \leq p_d^{\text{upp}}, \quad X = \text{PM, POVM}. \]  
(109)

Usefulness of this bound is fully recognized in the limit of large \( d \). Then, denoting \( p_\infty^{\text{thr}} := \lim_{d \to \infty} p_d^{\text{thr}} \), one finds that this results in the following bounds

\[ \Theta\left(\frac{\log d}{d^2}\right) \leq p_\infty^{\text{thr}} \leq \Theta\left(\frac{4}{(\sqrt{2} - 1)d}\right), \]  
(110)
\[ \Theta\left(\frac{3}{ed^2}\right) \leq p_\infty^{\text{thr}} \leq \Theta\left(\frac{4}{(\sqrt{2} - 1)d}\right), \]  
(111)

for projective and general measurements, respectively.

3.5. From projective to generalized measurements—Hirsch et al.’s construction

An interesting approach to the construction of states with local models for POVMs was put forward by Hirsch et al [23]. The innovation of their construction consisted in the somewhat different logic compared to the constructions reported above. As already discussed in section 3.2, it had been known that by applying a local channel to a state with a local model for POVMs one obtains another state with an underlying local model for the same type of
measurements. Hirsch et al proposed a method of constructing states with local models for arbitrary measurements departing from states with models for projective measurements.

Assume $\varrho_0 \in B(C^d \otimes C^d)$ has a local model for any dichotomic projective measurements with measurement operators given by $\{P_a, 1 - P_a\}$ and $\{Q_b, 1 - Q_b\}$ for Alice and Bob, respectively. In what follows, we will show that the state

$$\varrho = \frac{1}{d^2} \left\{ \varrho_0 + (d - 1) \left( \varrho_A \otimes \sigma_B + \sigma_A \otimes \varrho_B \right) + (d - 1)^2 \sigma_A \otimes \sigma_B \right\},$$

(112)

where $\varrho_{A,B}$ are the reductions of the original state $\varrho_0$ and $\sigma_{A,B}$ are arbitrary, is local for arbitrary POVMs with elements (see section 2) $A_a = \eta_a P_a$ and $B_b = \xi_b Q_b$, respectively, for Alice and Bob. The proof of this fact relies on the explicit construction of the corresponding local model, which is the following: LHV model for POVMs for $\varrho$ from equation (112) [23]

1. Alice (Bob) chooses $P_a$ ($Q_b$) with probability $\eta_a$ ($\xi_b$),
2. they simulate the measurement of the dichotomic observables $\overline{A}_a = P_a - P_a^\perp$ and $\overline{B}_b = Q_b - Q_b^\perp$, respectively, on the state $\varrho_0$ with $P_a^\perp = 1 - P_a$ and $Q_b^\perp = 1 - Q_b$,
3. if the result in step (2) is $+1$, Alice (Bob) announces $a$ ($b$) as the result of the simulation of the measurement on $\varrho$,
4. if the result in step (2) is $-1$, Alice (Bob) gives arbitrary $a$ ($b$) as the output with probability $\sigma_{A_a} (\sigma_{B_b})$.

Let us now argue that indeed this model correctly reproduces the quantum probability, which, as can be easily verified, reads

$$p_Q(a, b | A, B) = \frac{\eta_a \xi_b}{d^2} \left\{ \text{tr} \left[ (P_a \otimes Q_b) \varrho_0 \right] + (d - 1)^2 \text{tr} \left( P_a \sigma_A \text{tr} \left( Q_b \sigma_B \right) \right) \right. $$

$$\left. + (d - 1)^2 \left\{ \text{tr} \left( P_a \sigma_A \right) \text{tr} \left( Q_b \sigma_B \right) + \text{tr} \left( P_a \sigma_A \right) \text{tr} \left( Q_b \sigma_B \right) \right\} \right],$$

(113)

Both Alice and Bob, following the above protocol, may possibly output either in step (3) or (4), resulting in four probabilities of outputting the pair of outcomes $(a,b)$. Let us begin with the case of both Alice and Bob outputting in step (3). Since the simulation in this step concerns the original state $\varrho_0$, this will happen with probability $(\eta_a \xi_b / d^2) \text{tr} \left[ (P_a \otimes Q_b) \varrho_0 \right]$, which is the first term in (113). Another possibility is that Alice produces an output in step (3) but Bob fails to do the same and outputs in the next step. As one can easily verify this will occur with probability $[(d - 1)/d]^2 \text{tr} \left( A_a \sigma_A \right) \text{tr} \left( B_b \sigma_B \right)$. On the other hand, the event when Alice produces the output in the last step, but Bob does it in the third one, will occur with the probability $[(d - 1)/d]^2 \text{tr} \left( A_a \sigma_A \right) \text{tr} \left( B_b \sigma_B \right)$; these two probabilities are, respectively, the third and fourth terms in equation (113). The remaining case of both Alice and Bob outputting in the last step will happen with probability $[(d - 1)/d]^2 \text{tr} \left( A_a \sigma_A \right) \text{tr} \left( B_b \sigma_B \right)$, which agrees with the second term of (113). Adding up all the terms we eventually arrive at (113).

One might worry that the resulting state (112) will never be entangled so its locality will always be trivial. To show that this is not the case, it suffices to apply the above method to the state $\varrho_0 = (1/2)\varrho_{11} (\varrho_{11} + (1/4)1 \otimes 1_2).$ Moreover, this state has an interesting property: despite having a local model, it displays hidden nonlocality when subject to sequences of measurements [23]. More details about the concept of hidden nonlocality are given in section 5 below.

8 The corresponding local model for this states is provided in section 3.1
Let us finally comment that, in principle, the construction described above can also be applied to the multiparty scenario. We discuss this possibility in section 4.2.

4. Multipartite quantum states

Let us now move to the multipartite scenario. In this case, we will mostly be interested in genuinely multipartite entangled states for two main reasons. First of all, it is trivial to construct an $N$-party entangled state that is not GME but has a local model: it is enough to take the tensor product of a bipartite state with a local model for two of the parties and a product state for the remaining $N-2$ parties. Second, any entangled state that is not GME has a notion of locality, as it can always be decomposed into a probabilistic mixture of states that are separable with respect to some bipartition (see equation (2)). This, in turn, allows one to directly construct a hybrid local model for such states combining different local models for these bipartitions.

Unfortunately, to our knowledge, the literature on the subject is very limited and boils down to a single local model for a three-qubit GME state, which we discuss below, and its recent extension, which we mention in section 4.2. The question about existence of local models in the general multipartite setup remains open, and at this moment it is far from clear whether there exist local $N$-partite GME states for $N \geq 4$.

4.1. Local model for projective measurements on GME tripartite states

In what follows, we will recall the result of [40] showing that there exist three-qubit GME states with a local model for projective (two-outcome) measurements. Here, we present this result in a slightly different manner than in the original work [40].

Let us consider three parties $A$, $B$, and $C$ and the following simple extension of Werner’s model in which parties $A$ and $B$ behave as in the bipartite case, and the additional party, Charlie ($C$), applies the same strategy as Bob.

Our goal is to show that this local model simulates the outcome probabilities of projective measurements performed on some GME tripartite states. To simplify the problem we assume that all the parties perform two-outcome measurements with measurement operators acting on $\mathbb{C}^2$; in other words, we aim to obtain a three-qubit state. Accordingly, the shared randomness is represented by normalized vectors $|\lambda\rangle \in \mathbb{C}^2$ sampled with the probability density $\omega_\lambda()^2$.

In more precise terms, we ask if there exists a three-qubit state $\rho$ such that the probability $p_\rho(a, b, c \mid A, B, C) = \text{Tr} [(P_a \otimes Q_b \otimes R_c) \rho]$ can be written as

$$p_\rho(a, b, c \mid A, B, C) = p_\lambda(a, b, c \mid A, B, C)$$

$$\equiv \int \lambda \omega_\lambda(\lambda)p(a \mid A, \lambda)p(b \mid B, \lambda)p(c \mid C, \lambda), \quad (114)$$

where the response functions of Alice, Bob, and Charlie are given by (27), (28), and

$$p(c \mid C, \lambda) = \langle \lambda \mid R_c \mid \lambda \rangle, \quad (115)$$

respectively, with local measurement operators $P_a$, $Q_b$, and $R_c$ ($a, b, c = 0, 1$) acting on $\mathbb{C}^2$.

To verify that this is the case, let us first compute the above integral. Then we will argue that there exists a quantum state giving such predictions. It will be particularly useful to
exploit the Bloch representation of one-qubit quantum states (see section 2.2). Then let $p_a$, $q_b$, and $r_c$ denote the Bloch vectors representing, respectively, $P_a$, $Q_b$, and $R_c$ on the Bloch sphere and let $\lambda$ represent $|\lambda\rangle\langle\lambda|$. Then, exploiting the fact that $\langle\lambda| P_a |\lambda\rangle = (1/2)(1 + p_a \cdot \lambda)$ etc, equation (114) can be rewritten as

$$p_L(a, b, c | A, B, C) = \frac{1}{16\pi} \int_{p_a \cdot \lambda < 0} d\lambda \left( 1 + q_b \cdot \lambda \right) \left( 1 + r_c \cdot \lambda \right),$$

(116)

where the fact that the integration is taken over the half-sphere given by $p_a \cdot \lambda < 0$ stems from the condition in (27). After computing all integrals (see the appendix for details), the probabilities can be finally expressed in terms of the Bloch vectors $p_a$, $q_b$, and $r_c$ as

$$p_L(a, b, c | A, B, C) = \frac{1}{8} - \frac{1}{16} (p_a \cdot q_b + p_a \cdot r_c) + \frac{1}{24} q_b \cdot r_c.$$  

(117)

Now, it is fairly easy to construct a three-partite state realizing these probabilities. First, we notice that the scalar product of two normalized vectors $x, y \in \mathbb{R}^3$ can be expressed in the 'quantum-mechanical' form as

$$x \cdot y = \sum_{i=1}^{3} \text{Tr} \left( P_{\sigma_i} \otimes Q_{\sigma_i} \right) = \text{Tr} \left( P \otimes Q \sum_{i=1}^{3} \sigma_i \otimes \sigma_i \right),$$

(118)

where $P$ and $Q$ are projectors corresponding, respectively, to $x$ and $y$ in the Bloch representation. Applying the above rule to each scalar product in (117), one sees that the state that leads to probabilities (117) takes the form

$$\rho_{ABC} = \frac{1}{8} - \frac{1}{16} \sum_{i=1}^{3} (\sigma_i \otimes \sigma_i \otimes 1_2 + \sigma_i \otimes 1_2 \otimes \sigma_i) + \frac{1}{24} \sum_{i=1}^{3} 1_2 \otimes \sigma_i \otimes \sigma_i.$$  

(119)

It remains to be seen that it is genuinely multipartite entangled. The state is manifestly invariant under $U \otimes U \otimes U$, $U \in SU(2)$, since $\sum_{i=1}^{3} \sigma_i \otimes \sigma_i$ is $U \otimes U$ invariant. This allows us to apply the necessary and sufficient criteria developed in [41] which confirm that $\rho_{ABC}$ does indeed contain genuine multipartite entanglement. Interestingly, the state is a symmetric extension of the two-qubit Werner state.

**Remark 6.** A slight modification of the model allows us to extend the states (119). Precisely, let us assume that instead of $\lambda$, the party $A$ uses in its response function a modified Bloch vector given by $\lambda' = [a_1 \lambda_1, a_2 \lambda_2, a_3 \lambda_3]$ with some fixed parameters $-1 \leq a_i \leq 1$ with $i = 1, 2, 3$ (notice that $|\lambda'| \leq 1$).

By repeating the above calculations, one then finds that

$$p_L(a, b, c | A, B, C) = \frac{1}{8} - \frac{1}{16} (p'_a \cdot q_b + p'_a \cdot r_c) + \frac{1}{24} q_b \cdot r_c$$  

(120)

with $p'_a$ being $p_a$ with coordinates multiplied by $a_i$. These probabilities correspond to the following class of states

$$\rho'_{ABC} = \frac{1}{8} - \frac{1}{16} \sum_{i=1}^{3} a_i (\sigma_i \otimes \sigma_i \otimes 1_2 + \sigma_i \otimes 1_2 \otimes \sigma_i) + \frac{1}{24} \sum_{i=1}^{3} 1_2 \otimes \sigma_i \otimes \sigma_i.$$  

(121)

9 Recall from section 2 that we need a single vector to characterize a dichotomic measurement. For notational convenience, however, we do not exploit this fact here.
Although these states are in general no longer $U \otimes U \otimes U$-invariant, the criteria from [41] can still be applied and it follows that whenever $2 < a_1 + a_2 + a_3 \leq 3$ holds $\rho_{ABC}$ are GME states. It is worth mentioning that for $c := a_1 = a_2 = a_3$, (121) reproduces the one-parameter class of states considered in [40] which, as it follows from the example above, is GME if $1 \geq c > 2/3$. This, in turn, provides an improvement over the range $1 \geq c > (\sqrt{13} - 1)/3$, found originally in [40].

Clearly, in the same way one can also modify the Bloch vector $\lambda$ at Bob’s and Charlie’s sites. This would give us some additional parameters in the state (121).

Remark 7. Let us also notice that the above model works also if generalized measurements with any number of outcomes are allowed at Bob’s and Charlie’s sites. The argument in favor of this fact is the same as in the case of the Werner states (see remark 1 in section 3.1). This makes it, in particular, possible to construct more examples of three-party states (not necessarily GME) following the reasoning outlined in remark 5 in section 3.2. That is, any state of the form $\sigma_{ABC} = I_A \otimes A_B \otimes A_C(\rho_{ABC})$ with $A_X$ ($X = A, B$) being some quantum channels has also a local model for projective measurements at Alice’s site and generalized measurements at Bob’s and Charlie’s sites.

4.2. Generalizations and discussion

Let us now discuss how one could apply the results described in previous sections for two parties to a multiparty scenario.

As already mentioned in section 3.5, the method described there for the construction of states with a local model for general measurements from states with a local model for two-outcome projective measurements can be automatically applied to a multiparty scenario. More precisely, assume an $N$-partite state $\rho_A$ acting on $\mathbb{C}^{d^N}$ has a local model for dichotomic projective measurements on each of the sites $A_i$. Then, the following state has a local model for arbitrary measurements:

$$\sigma_A = \frac{1}{d^N} \left( \rho_A + (d - 1) \sum_i \rho_{A \setminus A^0} \otimes \sigma_{A^0} 
+ (d - 1)^2 \sum_{i<j} \rho_{A \setminus A^0} \otimes \sigma_{A^0} \otimes \sigma_{A^0} 
+ \cdots 
+ (d - 1)^N \otimes_{i=1}^N \sigma_{A^0} \right),$$

(122)

where $\rho_{A \setminus A^0} \equiv \text{tr}_{A^0} \rho_A$ etc. are the reduced states of $\rho_A$, and $\sigma_{A^0}$ are arbitrary states acting on $\mathbb{C}^{d^N}$.

The local model for $\sigma_A$ is the same as the one in section 3.5. Assuming that measurement operators for party $A^{(i)}$ are $A^{(i)} = x^{(i)}_k P^{(i)}_k$, $k = 0, 1, \ldots, d - 1$, it goes as follows: with probability $x^{(i)}_k/d$ each party chooses $P^{(i)}_k$ and simulates dichotomic measurement with elements $P^{(i)}_k$ and $1 - P^{(i)}_k$ and the corresponding outcomes $+1$ and $-1$. If the result $+1$ is obtained, the party outputs $x^{(i)}_k$; otherwise, arbitrary $x^{(i)}_k$ is outputted with probability $\text{tr}(\sigma_{A^0} A^{(i)})$. Direct calculation confirms the validity of the model for $\sigma_A$.

At the moment, the question of whether this construction may lead to GME states with a local model remains open. Interestingly, however, a slightly modified approach allows one [42] to obtain a tripartite qutrit-qubit-qubit state with a local model for generalized measurements starting from the qubit state given by Tóth and Acín [40] (see section 4.1).
A priori, another interesting question is whether it is possible to apply the machinery based on the Grothendieck constant (see section 3.3) to the multipartite scenario. Unfortunately, the corresponding universal constant does not exist even for three parties. In particular, it is known that in the multiuser case it is not possible to bound the violation of general correlation Bell inequalities for dichotomic observables [43].

Finally, one is tempted to generalize the approach used in section 4.1 for the construction of a GME tripartite state with a local model to a larger number of parties by appending more parties that behave as in Werner’s original model. It turns out, however, that even for \(N = 4\) the resulting state fails to be \(U^\otimes 4\) invariant so the straightforward generalization of the tripartite case is not possible [40]. Moreover, it has also been proven in [40] that in fact there is no four-partite \(U^\otimes 4\) invariant state for which the model with response functions for \(A\), \(B\), and \(C\) taken as in section 4.1 and for \(D\) the same as \(B\) and \(C\) works. Finally, the tripartite extension of Werner’s model for systems of dimension larger than two does not work either [40].

5. Other locality scenarios

In this work, we have reviewed the state of the art concerning local models for entangled states in the original and more standard Bell scenario where the parties perform local measurements on a single copy of an entangled state. It is, however, possible to consider other scenarios in which the parties have access to a larger set of operations and where different notions of locality appear. In these alternative scenarios, states that have a local model in the standard scenario might display nonlocal correlations. Without going into details, we highlight in what follows some of the known results in these alternative scenarios:

- **Copies of a state**: The first possibility consists of a scenario in which the parties have access to several copies of a state and can perform joint measurements on the copies of their subsystems. In this scenario, it has been shown that nonlocality can be activated: there exist states \(\rho\) that have a local model; nevertheless, \(\rho^\otimes k\) is nonlocal for sufficiently large \(k\). Leaving aside some previous partial results for some specific inequalities (e.g., [44, 45]; see also [46, 47] for other activation scenarios), the proof of this result is due to Palazuelos [48]. His result was later generalized in [49], where it was shown that all bipartite entangled states that are useful for teleportation are nonlocal when considering an arbitrary large number of copies.

- **Network approach**: Similar to the previous case, but a more general scenario was introduced in [50] and consists of a setup in which \(k\) copies of a state, say bipartite, are distributed among \(N\) parties. Now, one has to test whether the resulting \(N\)-partite state, made of \(k\) copies of a bipartite state, is nonlocal. In this scenario, activation effects are also possible in the sense that there exist networks of local states that display nonlocal correlations. Moreover, all states that have one-way distillable entanglement are nonlocal in this scenario [50].

- **LOCC preprocessing**: Given a state, it is possible to perform some LOCC (local operations and classical communication) preprocessing before running the Bell test. While classical communication seems at odds with the concept of nonlocality, this is not the case if it takes place before the measurements to be performed have been decided. That is, under this sequential arrangement classical communication does not create any nonlocality. Bell tests including LOCC preprocessing were introduced by Popescu in [8] (see also [51]), who proved that some local Werner states for \(d \geq 5\) become nonlocal in
this scenario. Popescu coined the term *hidden nonlocality* to describe this phenomenon. While Popescu proved existence of hidden nonlocality for states that have a local model for projective measurements, his result has been generalized to general measurements in [23].

- **Copies together with LOCC preprocessing**: Finally, one can also consider a combination of all the previous possibilities. The resulting scenario is basically the same as that studied for entanglement transformations, and, in particular, entanglement distillation. Here, the goal is to distill with LOCC from copies of a given state a new state that violates a Bell inequality. It is clear that in this more general scenario, all entanglement distillable states are nonlocal. The natural question is whether nondistillable states, such as states with positive partial transposition (PPT) are also nonlocal. This question, known as the Peres conjecture [52], was first proven to be false by Dür in the multipartite case [53] (see also [54–57]). The more challenging bipartite case remained open until recently, when Vértesi and Brunner showed existence of a PPT state violating a Bell inequality [58].

As this brief overview shows, there exist several operationally meaningful notions of nonlocality. What is remarkable is that in all these scenarios it is still open whether it is possible to derive an equivalence between entanglement and nonlocality (while this equivalence is known to fail in the standard scenario).

### 6. Conclusions and outlook

We finish our review with a summary of the main results we have considered and a short discussion about open questions.

As to the first part, we feel that the best way to provide such a summary is to recall in a systematized way the constructions covered in the main body of the paper. We thus have:

- **Werner’s model (section 3.1)**: proof of existence of a local model for projective measurements (PM) on bipartite entangled states. The model was constructed for the Werner states (20) in the region

\[
p \leq \frac{d - 1}{d},
\]

- **Barrett’s model (section 3.2)**: proof of existence of a local model for generalized measurements (POVM) on entangled bipartite states. The model was constructed for the Werner states in the region

\[
p \leq \frac{3d - 1}{d(d + 1)} \left( \frac{d - 1}{d} \right)^{d - 1},
\]

- the Grothendieck constant approach (section 3.3): use of the constant to study the robustness of nonlocal correlations of entangled bipartite states. In particular:
  - there exists a model for PM on a two-qubit Werner state \( \rho_W(2, p) \) iff \( p \leq K_G(3) \), where \( K_G(3) \) is the Grothendieck constant of order 3; in particular, \( \rho_W(2, p) \) is local at least up to 0.6595;
  - \( \rho_W(2, p) \) is local for planar (at least at one site) PM up to the CHSH threshold \( p = 1/\sqrt{2} \).
– for every noisy state
\[ \sigma = pq + (1 - p) \frac{1}{d}, \] (123)
there is a local model for joint correlations of traceless two-outcome observables for
\[ p < 1/K_G(2d^2) \] for any \( \varrho \);
– for \( p \geq 1/K_G(2 \lfloor \log d \rfloor + 1) \) there exists a state of the form (123) whose joint
correlations cannot be reproduced;
– (stems from the previous two facts) in the limit \( d \to \infty \) a state (123) with arbitrary \( \varrho \) and
\( p \leq 1/K_G \) \( (K_G \text{ –the Grothendieck constant}) \) is local and this number gives the ultimate
limit;
– there is a local model simulating full probability distribution for traceless observables for
the isotropic state whenever \( p \leq 1/K_G(d^2 - 1) \) (see also below).

• Almeida et al’s construction (section 3.4): local model for the isotropic states (83) and
general results for the nonlocality of noisy entangled states (123). One has:
– a local model for PM on the isotropic states (83) for
\[ p \leq p_{\text{PM}}^{\text{iso}, c} \equiv \frac{\sum_{k=2}^{d} \frac{1}{d - 1}}{d - 1}, \]
– a model for POVM on the isotropic states (83) for
\[ p \leq p_{\text{POVM}}^{\text{iso}, c} \equiv \frac{3d - 1}{d(d + 1)} \left( \frac{d - 1}{d} \right)^{d-1}, \]
– a local model for PM on arbitrary states of the form (123) for
\[ p \leq \frac{p_{\text{PM}}^{\text{iso}, c}}{(1 - p_{\text{PM}}^{\text{iso}, c})(d - 1) + 1}, \]
– a local model for POVM on arbitrary states of the form (123) for
\[ p \leq \frac{p_{\text{POVM}}^{\text{iso}, c}}{(1 - p_{\text{POVM}}^{\text{iso}, c})(d - 1) + 1}. \]

• CGLMP and CHSH perspective on the noise robustness of nonlocality:
– all isotropic states (83) are nonlocal for
\[ p > 2 \left[ 4d \sum_{k=0}^{[d/2]-1} \left( 1 - \frac{2k}{d - 1} \right) \left( q_k - q_{-(k+1)} \right) \right]^{-1}, \] (124)
where \( q_k = 1/(2d^3 \sin^2[k(k + \frac{1}{4})]) \),
– there always exists a nonlocal noisy state \((60)\) whenever

\[
p > \frac{1 - \left(\frac{d-2}{d}\right)^2}{\sqrt{2} - \left(\frac{d-2}{d}\right)^2},
\]

\((125)\)

- Hirsch et al’s construction (section 3.5): novel examples of local states. The work provides a systematic method for deriving states local for POVMs from states local for PMs of two outcomes. Given a state \(\rho_0\) local for dichotomic PM, one builds the state

\[
\rho = \frac{1}{d^2}(\rho_0 + (d-1)(\sigma_A \otimes \sigma_B + \sigma_A \otimes \rho_B) + (d-1)^2 \sigma_A \otimes \sigma_B)
\]

with \(\rho_A, \rho_B\) arbitrary, which is local for POVM,

- Tóth–Acín model (section 4.1): proof of existence of genuinely entangled three-qubit states with a local model for PMs. The model is based on a symmetric extension of the two-qubit Werner model. Building on this result, existence of three-partite GME states local under general measurements has been proven recently [42].

It is worth mentioning that all the previous construction are, in one way or another, based on Werner’s original model. In this sense, it would be interesting to have a model that does not use, at any point, some of the ingredients of Werner’s model.

Finally, let us conclude with a brief discussion of the main open questions in the relation between entanglement and nonlocality. In the case of two parties, we can think of the following three main questions:

1. It is known that in the standard Bell scenario, entanglement and nonlocality are inequivalent. Is it possible to identify a more general scenario in which these two quantum phenomena become equivalent?

2. For many of the families of states considered above, a gap always appears between the region where a local model exists for projective measurements and the one for general measurements. It is an open question whether this gap is an artifact of the construction or if it actually exists. That is, do general measurements offer any advantage to detect the nonlocality of quantum states?

3. An almost unexplored question is to study local models from a quantitative point of view. Existence of a local model for a quantum state operationally means that the correlations of the state can be reproduced using shared randomness. How much shared randomness is needed? How does it scale with the entanglement in the state?

Moving to the multipartite case, basically all questions remain open. To our knowledge, there is not a single example of a local model for an entangled state of more than three parties. But, perhaps the most interesting questions concern whether gaps exist for any number of parties. Thus, we can identify the following questions:

1. The first question concerns whether for any number of parties it is possible to find GME states that have a fully local model. We do not have any strong intuition about this. In principle, based on all the existing results, one may expect a gap for any number of parties. However, when going to many parties, it could be that GME are so entangled that a fully local model cannot reproduce their statistics. If this was the case, it would be interesting to estimate the minimal number of parties such that local models do not exist. Thus, the question is: Is there a finite number of parties such that all GME violate a standard Bell inequality?
2. Perhaps a more fair comparison between entanglement and nonlocality in the multipartite scenario consists of comparing GME with genuinely multipartite nonlocality. Here, we are interested in understanding whether there exist for any number of parties GME states that can be described by a local model in which some of the parties join, as first considered by Svetlichny [59] (see also [60, 61]). Thus, the question is: Is there a finite number of parties such that all GME violate a Svetlichny–Bell inequality? While finishing writing this manuscript, we have proven that the answer to this question is negative. That is, for any number of parties, we have found GME states that can be described by local models in which subset of the parties join [62].

After all these discussions, the main message of this review becomes clear: fifty years after Bell’s seminal paper [5] we are still very far from understanding the relation between entanglement and nonlocality.

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Appendix

Here we compute analytically the integrals that have been introduced in the main text.

The integral $J[u_1]$ in equation (38). Here, following Mermin [26], we will show that $J[u_1] = N/d^3$. Recall that its explicit form reads

$$
J[u_1] = \int_0^1 du_1 \int_0^1 du_2 \cdots \int_0^1 du_d u_1 \delta(u_1 + \cdots + u_d - 1). \quad (A.1)
$$

To simplify computation of this integral, we can, without changing its value, extend the range of all the variables to infinity, i.e.,

$$
J[u_1] = \int_0^\infty du_1 \int_0^\infty du_2 \cdots \int_0^\infty du_d u_1 \delta(u_1 + \cdots + u_d - 1). \quad (A.2)
$$

By changing the variables $u_i = v_i + u_1$ for $i = 2, \ldots$, and then $u_1 = v_1/d$, it can further be rewritten as

$$
\frac{1}{d^2} \int_0^\infty dv_1 \cdots \int_0^\infty dv_d v_1 \delta(v_1 + \cdots + v_d - 1). \quad (A.3)
$$

One finally notices that the value of the integral is the same if $v_1$ is replaced by any $v_i$, and consequently
\[ \frac{1}{d^2} \int_0^\infty dv_1 \ldots \int_0^\infty dv_d \, v_1 \delta(v_1 + \ldots + v_d - 1) \]
\[ = \frac{1}{d} \int_0^\infty dv_1 \ldots \int_0^\infty dv_d \, (v_1 + \ldots + v_d) \delta(v_1 + \ldots + v_d - 1) \]
\[ = \frac{1}{d^3} N, \tag{A.4} \]

which is what we wanted to prove. Recall that \( N \) is defined by equation (36).

*The integral \( \mathcal{J}[u_1] \) in equation (53).* Now, repeating the same tricks as above, we compute

\[ \mathcal{J}[u_1] = \int_{1/d}^1 du_1 \int_{1/d}^1 du_2 \ldots \int_{1/d}^1 du_d \, u_1 \delta(u_1 + \ldots + u_d - 1). \tag{A.5} \]

As before, we can exploit the fact that the argument of integration contains the Dirac delta and extend upper bounds of all integrals to infinity. Then, by a change of the variable \( u_i \to u_i + 1/d \), one gets

\[ \mathcal{J}[u_1] = \int_0^\infty du_1 \ldots \int_0^\infty du_d \, u_1 \delta\left(u_1 + \ldots + u_d - \frac{d-1}{d}\right) \]
\[ + \frac{1}{d} \int_0^\infty du_1 \ldots \int_0^\infty du_d \, \delta\left(u_1 + \ldots + u_d - \frac{d-1}{d}\right). \tag{A.6} \]

Now, we change all variables \( u_i \to u_i (d-1)/d \) with \( i = 1, \ldots, d \), and then use the property \( \delta(ax) = (1/\alpha) \delta(x) \), which gives

\[ \mathcal{J}[u_1] = \left(\frac{d-1}{d}\right)^d \int_0^\infty du_1 \ldots \int_0^\infty du_d \, u_1 \delta\left(u_1 + \ldots + u_d - 1\right) \]
\[ + \frac{1}{d} \left(\frac{d-1}{d}\right)^{d-1} \int_0^\infty du_1 \ldots \int_0^\infty du_d \, \delta\left(u_1 + \ldots + u_d - 1\right). \tag{A.7} \]

To complete the proof, one notices that the first integral has been already computed and amounts to \( N/d \), while the second one is simply \( N \). As a result,

\[ \mathcal{J}[u_1] = \frac{N}{d} \left(\frac{d-1}{d}\right)^{d-1}. \tag{A.8} \]

*The integral \( \mathcal{J}[u_1^2] \) in equation (54).* Following the same steps as before, it is fairly easy to see that

\[ \mathcal{J}[u_1^2] = \int_{1/d}^1 du_1 \int_{1/d}^1 du_2 \ldots \int_{1/d}^1 du_d \, u_1^2 \delta(u_1 + \ldots + u_d - 1) \tag{A.9} \]

can be expressed as

\[ \mathcal{J}[u_1^2] = \frac{N}{d^2} \left(\frac{d-1}{d}\right)^{d-1} + \frac{2N}{d} \left(\frac{d-1}{d}\right)^d \]
\[ \times \int_0^\infty du_1 \ldots \int_0^\infty du_d \, u_1^2 \delta(u_1 + \ldots + u_d - 1). \tag{A.10} \]

The last integral can be computed directly. From the fact that it contains the Dirac delta, one obtains
\[
\int_0^\infty du_1 \cdots \int_0^\infty du_d \, u_1^2 \delta(u_1 + \cdots + u_d - 1) \\
= \int_0^\infty du_1 u_1^2 \int_0^{1-u_1} du_2 \cdots \int_0^{1-u_1-\cdots-u_{d-2}} du_{d-1}.
\] (A.11)

Straightforward integration over the variables \(u_2, \ldots, u_{d-1}\) gives
\[
\int_0^\infty du_1 \cdots \int_0^\infty du_d \, u_1^2 \delta(u_1 + \cdots + u_d - 1) \\
= \frac{1}{(d-2)!} \int_0^1 du_1 u_1^2 (1-u_1)^{d-2}.
\] (A.12)

which further amounts to
\[
\int_0^\infty du_1 \cdots \int_0^\infty du_d \, u_1^2 \delta(u_1 + \cdots + u_d - 1) = \frac{2N}{d(d+1)}.
\] (A.13)

and consequently,
\[
\mathcal{J}[u_1^2] = \frac{N}{d} \left[ \frac{1}{d} + \frac{2}{d-1} \left( 1 + \frac{d-1}{d} \right)^2 \right] \left( \frac{d-1}{d} \right)^{d-1} \\
= \frac{N}{d} \left( d - 3 \left( 1 - \frac{1}{d} \right) \right)^{d-1}.
\] (A.14)

The integral \(\mathcal{J}[u_1]\) in equation (87) Let us now compute the following integral
\[
\mathcal{J}[u_1] = \int_0^1 du_1 \int_0^{u_1} du_2 \cdots \int_0^{u_{d-1}} du_d \delta(u_1 + \cdots + u_d - 1).
\] (A.15)

For this purpose, we rewrite all integrals over the variables \(u_i (i = 2, \ldots, d)\) as
\[
\int_0^{u_i} du_i = \int_0^\infty du_i - \int_0^{u_i} du_i,
\] (A.16)

which after some algebra leads us to
\[
\mathcal{J}[u_1] = \sum_{k=0}^{d-1} \binom{d-1}{k} (-1)^k \int_0^\infty du_1 u_1 \int_0^{u_1} du_2 \cdots \int_0^{u_{k+1}} du_{k+1} \\
\times \int_0^{u_k+2} \cdots du_d \delta(u_1 + \cdots + u_d - 1).
\] (A.17)

where we have exploited the fact that the integrated function is invariant under any permutation of the variables \(u_2, \ldots, u_d\). Now, by changing the variables \(u_i \rightarrow u_i - 1\) with \(i = 2, \ldots, k + 1\), and then \(u_i \rightarrow u_i/(1+k)\), each integration in the summand rewrites as
\[
\int_0^\infty du_1 u_1 \int_0^{u_1} du_2 \cdots \int_0^{u_{k+1}} du_{k+1} \int_0^\infty du_{k+2} \cdots du_d \delta(u_1 + \cdots + u_d - 1) \\
= \int_0^\infty du_1 u_1 \int_0^\infty du_2 \cdots du_d \delta((k+1)u_1 + \cdots + u_d - 1) \\
= \frac{1}{(1+k)^2} \int_0^\infty du_1 u_1 \int_0^\infty du_2 \cdots du_d \delta(u_1 + \cdots + u_d - 1).
\] (A.18)
The last integral has already been computed and it amounts to \( N/d \), which finally gives
\[
J [ u_1 ] = \frac{N}{d} \sum_{k=0}^{d-1} \left( \frac{d-1}{k} \right) \frac{(-1)^k}{(1+k)^2} = \frac{N}{d^2} \sum_{k=1}^{d-1} \frac{1}{k}, \tag{A.19}
\]

To obtain the last equality, we have utilized one of the representations of the harmonic number \( H_d \equiv \sum_{k=1}^{d} \frac{1}{k} \), namely,
\[
H_d = \sum_{k=0}^{d-1} \left( \frac{d-1}{k} \right) \frac{(-1)^k}{k+1}, \tag{A.20}
\]

**Integrals over the Bloch sphere.** Here we show that
\[
\int_{x \cdot \lambda < 0} d\lambda (y \cdot \lambda) = -\pi x \cdot y, \tag{A.21}
\]
where \( x, y, \) and \( \lambda \) are unit vectors from \( \mathbb{R}^3 \).

Let us first note that both integrals are invariant under any rotation of the Bloch sphere, i.e.,
\[
\int_{x \cdot \lambda < 0} d\lambda (y \cdot \lambda) = \int_{x' \cdot \lambda < 0} d\lambda (y' \cdot \lambda), \tag{A.22}
\]
where \( x' = Ox \) and \( y' = Oy \) with \( O \) being an element of the \( SO(3) \) group. This allows us to choose the reference frame such that \( x \) is aligned with its \( z \)-axis, i.e., we choose such \( O \) that \( Ox = [0, 0, 1] \). Now, with respect to this new frame of reference, we can parametrize \( \lambda \) as \( \lambda = [\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta] \) with \( \phi \in [0, 2\pi] \) and \( \theta \in [0, \pi] \). As a result, the integral (A.21) rewrites as
\[
\int_{\pi/2}^{\pi} d\theta \sin \theta \int_{0}^{2\pi} d\phi \left( y_1' \cos \phi \sin \theta + y_2' \sin \phi \sin \theta + y_3' \cos \theta \right), \tag{A.23}
\]
where the range of the first integral comes from the condition \( x \cdot \lambda < 0 \), which after rotation simplifies to \( \cos \theta < 0 \). Performing all integrations in the above, one obtains that
\[
\int_{x \cdot \lambda < 0} d\lambda (y \cdot \lambda) = -\pi y_3' = -\pi x \cdot y. \tag{A.24}
\]

Along exactly the same lines, one also finds that
\[
\int_{x \cdot \lambda < 0} d\lambda (y \cdot \lambda) (z \cdot \lambda) = \frac{2\pi}{3} y \cdot z. \tag{A.25}
\]

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