The reflexive edge strength on some almost regular graphs

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- $P_n \odot P_k$

Abstract

A function $f$ with domain and range are respectively the edge set of graph $G$ and natural number up to $k$, and a function $f$ with domain and range are respectively the vertex set of graph $G$ and the even natural number up to $2k$, are called a total $k$-labeling where $k = \max\{k_e, 2k_v\}$. The total $k$-labeling of graph $G$ by the condition that every two different edges have different weight is called an edge irregular reflexive $k$-labeling, where for any edge $x_1x_2$, the weight is $w(x_1x_2) = f_e(x_1) + f_e(x_2) + f_s(x_1x_2)$. The reflexive edge strength of the graph $G$, denoted by $res(G)$ is the minimum $k$ for graph $G$ which has an edge irregular reflexive $k$-labeling. In this study, we obtained the $res(G)$ of graphs which their vertex degrees show an almost regularity properties.

1. Introduction

A graph $G$ is an object of nonempty vertex set $V$ and edge set $E$ (possibly empty) of 2-element subsets of $V$. If a graph $G$ has vertex set $V(E)$ and edge set $E(G)$, then we write $G = (V(G), E(G))$. In this paper, we will study simple, nontrivial, nonempty, and connected graphs. For more detail of the notions, see [1].

Meanwhile, labeling is a one-to-one mapping that carries a set of graph elements into a set of non negative integers, called labels. We say an edge labeling if the domain is the set of all edges of graph $G$. One of the concepts of edge labeling can be found in [2]. Ghosh et al. also showed some results of labeling in [3]. A complete dynamic book of graph labeling can be found in the Survey of Graph Labeling written by Joseph A. Gallian [4]. The study of labeling of graphs has gained a lot of benefits in the area of graph theory. The benefits indicate in the applications of graph labeling in some branches of science, for instance in x-ray, coding theory, crystallography, cryptography, astronomy, circuit design, and communication network design, see [1].

Furthermore, the notion of the irregularity strength of graphs was introduced by Chartrand et al. in [5, 6]. An irregular $k$-labeling of a graph $G$ is a function of the edge set of graph $G$ as domain and the natural numbers from 1 up to $k$ as range such that all vertex weights are pairwise distinct. The vertex weight is the sum of labels of edges incident to the vertex. The irregularity strength, denoted by $s(G)$ is the smallest integer $k$ in which $G$ has irregular $k$-labeling. The concept of irregularity strength was extended to irregular total $k$-labeling by Bača et al. in [7]. Some previous results of vertex irregular total $k$-labeling can be found in [8].

Bača et al. define a total $k$-labeling is mapping from union of vertex and edge set of graph $G$ to the natural number up to $k$. A total $k$-labeling is defined to be an edge irregular total $k$-labeling of the graph $G$ if for every two distinct edges have different weights. The edge weight is the sum of labels of its incident vertices and labels of that edge. Total edge irregularity strength of the graph $G$, denoted by $res(G)$ is the minimum $k$ for which the graph $G$ has an edge irregular total $k$-labeling. Some previous results of total edge irregularity strength can be seen in [9, 10, 11].

Bača extended the above notion into an edge irregular reflexive $k$-labeling. The total $k$-labeling defined the function $f_e: E(G) \to \{1, 2, \ldots, k_e\}$ and $f_s: V(G) \to \{0, 1, \ldots, 2k_v\}$, where $k = \max\{k_e, 2k_v\}$. For the graph $G$, the total $k$-labeling is called an edge irregular reflexive $k$-labeling if the condition every two different edges $x_1x_2$ and $y_1y_2$ of $G$ satisfies $w(x_1x_2) \neq w(y_1y_2)$, where $w(x_1x_2) = f_e(x_1) + f_e(x_2) + f_s(x_1x_2)$. The smallest value of $k$ for which such labeling exists is called the reflexive edge strength of the graph $G$, denoted by $res(G)$ [12]. Some Lemmas of the lower bound for $res(G)$ can be seen in [12, 13].
Some relevant results by then are found. Ibrahim et al. in [15] determined the rest(G), where G were star, double star \(S_{2,n} \), and caterpillar graphs. Agustin et al. in [13] determined the rest(G), where G were generalised subdivided star, broom, and double star graph \(S_{2,n} \). Bača et al. in [14] determined the rest(G), where G was cycle, cartesian product of cycles, join of cycle graphs and \(K_1 \). Bača in [12] also determined the rest(G), where G were generalized friendship graphs. Tanna et al. in [16] determined the rest(G), where G were wheels, prisms, basket, and fan graphs. J.L.G. Guirao et al. in [17] determined the rest(G), where G was the disjoint union of Generalized Petersen graphs. In this paper we will study the reflexive edge strength on graphs with a specific property, namely almost regular graphs.

The definition of almost regular graphs appeared for the first time in [15]. Based on this definition, we extend a new definition of almost regular graphs. We also consider a degree sequence described in [1]. A sequence \(d_1, d_2, \ldots, d_n \) of non-negative integers is called a degree sequence of a graph G of order n if the degree of the vertices \(v_1, v_2, \ldots, v_n \) of graph G is \(d_i = d_i \), for \(1 \leq i \leq n \).

**Definition 1.** Given that a graph G. For any natural number \(t \geq 1 \), an almost regular graph is a graph G which the elements of its degree sequence are \(t + r, r \in \{0, 1\} \); or \(t + r, r \in \{0, 1, 2\} \).

Almost any regular graph may be constructed by a cartesian and corona product of two graphs. The definition of cartesian and corona products can be described as follows.

**Definition 2.** [1] The Cartesian product \(H \) of two graphs \(H_1 \) and \(H_2 \), commonly denoted by \(H \times H_2 \), is a graph \(V(H) = V(H_1) \times V(H_2) \), where two distinct vertices \((u, v) \) and \((x, y) \) of \(H \times H_2 \) are adjacent if either \(u = x \) and \(v \in V(H_2) \), or \(v = y \) and \(u \in V(H_1) \).

Meanwhile, Lai [19] defined the corona of two graphs \(H_1 \) and \(H_2 \), denoted by \(H_1 \circ H_2 \). This graph is obtained from one copy of \(H_1 \) and \(|V(H_1)| \) copies of \(H_2 \) where the \(i\)th vertex of \(H_1 \) is adjacent to every vertex in the \(i\)th copy of \(H_2 \). For any integer \(i \geq 2 \), we establish the graph \(H_1 \circ H_2 \) recursively from \(H_1 \circ H_2 \) as \(H_1 \circ H_2 = (H_1 \circ H_2) \circ H_2 \). The graph \(H_1 \circ H_2 \) is also called as \(i \)th corona product of \(H_1 \) and \(H_2 \). The formal definition can be found in the following.

**Definition 3.** [19] Given two simple graphs \(H_1 \) and \(H_2 \), the corona product of \(H_1 \) and \(H_2 \), denoted by \(H_1 \circ H_2 \) is a connected graph obtained by taking a number of vertices \(|V(H_1)| \) copy of \(H_2 \), and making the \(i\)th copy of \(V(H_1) \) adjacent to every vertex of the \(i\)th copy of \(V(H_2) \).

2. Reflexive edge strength on almost regular graphs

In the following theorems, we show the results on edge irregular reflexive k-labeling and reflexive edge strength of some almost regular graphs, namely ladder, triangular ladder, \(P_n \times C_3 \), \(P_n \circ P_2 \), and \(P_n \circ C_3 \). To prove each theorem, we use lemma of the lower bound for \(\text{rest}(G) \) which is appeared in [12] and [13].

**Theorem 1.** Let \(L_n \) be a ladder graph. For every natural number \(n \geq 3 \), \(\text{rest}(L_n) = n \).

**Proof.** Let \(L_n \) be a ladder graph with vertex set \(V(L_n) = \{x_i \in \{1, 2, 3, \ldots, n\} \cup \{y_i \in \{1, 2, 3, \ldots, n - 1\}\} \), \(|V(L_n)| = 2n \) and edge set \(E(L_n) = \{x_i y_{i+1} \in \{1, 2, 3, \ldots, n - 1\}\} \cup \{y_i x_{i+1} \in \{1, 2, 3, \ldots, n - 1\}\}, |E(L_n)| = 2n - 2 \). For every \(n \), it gives \(3n - 2 \equiv 0 \pmod{6} \) for odd \(n \) and \(3n - 2 \equiv 4 \pmod{6} \) for even \(n \). We refer to lemmas in [12] and [13] to determine the lower bound of \(\text{rest}(L_n) \).

\[
\text{rest}(L_n) \geq \frac{3n - 2}{3} = n
\]

For the next step, we will determine the upper bound of \(\text{rest}(L_n) \) by defining the function \(f \) and \(g \) of vertex and edge labeling in the following:

**Case 1.** For odd \(n \), we have

\[
f(x_i) = \begin{cases} 0, & \text{if } i \in \{1, 2\} \\ 2, & \text{if } i = 3 \\ 2\left\lfloor\frac{i-3}{2}\right\rfloor + 2, & \text{if } i \in \{4, 5, 6, \ldots, n\} \end{cases}
\]

\[
g(y_i x_{i+1}) = \begin{cases} 0, & \text{if } i = 1 \\ 2\left\lfloor\frac{i-1}{2}\right\rfloor, & \text{if } i \in \{2, 3, 4, \ldots, n\} \end{cases}
\]

We get the edge weight from the function of vertex and edge labeling, the overall edge weight sets are:

\[
\text{wt}(y_i x_{i+1}) = 3i \quad : \quad i \in \{1, 2, 3, \ldots, n - 1\}
\]

\[
\text{wt}(x_i y_i) = 3i - 2 \quad : \quad i \in \{1, 2, 3, \ldots, n\}
\]

**Case 2.** For even \(n \), we have

\[
f(x_i) = \begin{cases} 0, & \text{if } i = 1 \\ 2, & \text{if } i = 2 \\ 2\left\lfloor\frac{i-2}{2}\right\rfloor + 2, & \text{if } i \in \{3, 4, 5, \ldots, n\} \end{cases}
\]

\[
g(y_i x_{i+1}) = \begin{cases} 0, & \text{if } i = 1 \\ 2\left\lfloor\frac{i-1}{2}\right\rfloor, & \text{if } i \in \{2, 3, 4, \ldots, n - 1\} \\ 2i - 2, & \text{if } i \in \{4, 5, 6, \ldots, n\} \end{cases}
\]

We get the edge weight from the function of vertex and edge labeling, the overall edge weight sets are:

\[
\text{wt}(y_i x_{i+1}) = 3i + 2 \quad : \quad i \in \{1, 2, 3, \ldots, n - 1\}
\]

\[
\text{wt}(x_i y_i) = 3i - 1 \quad : \quad i \in \{1, 2, 3, \ldots, n - 1\}
\]

\[
\text{wt}(x_i y_i) = 3i \quad : \quad i \in \{1, 2, 3, \ldots, n\}
\]

We can see the all edge weights from the weight sets above are different. It concludes the proof.

**Theorem 2.** Let \(T_L \) be a triangular ladder graph. For every natural number \(n \geq 3 \),

\[
\text{rest}(T_L) = \begin{cases} \left\lfloor\frac{n-2}{3}\right\rfloor + 1, & \text{if } n \equiv 0 \pmod{3} \\ \left\lfloor\frac{n-3}{3}\right\rfloor, & \text{otherwise}. \end{cases}
\]

**Proof.** Let \(T_L \), \(n \geq 3 \), be a graph with the vertex set \(V(T_L) = \{x_i \in \{1, 2, 3, \ldots, n\} \cup \{y_i \in \{1, 2, 3, \ldots, n\}\} \), \(|V(T_L)| = 2n \) and the edge set \(E(T_L) = \{x_i y_{i+1} \in \{1, 2, 3, \ldots, n - 1\}\} \cup \{y_i x_{i+1} \in \{1, 2, 3, \ldots, n - 1\}\} \cup \{x_i y_i \in \{1, 2, 3, \ldots, n\}\}, |E(T_L)| = 4n - 3 \). 4n - 3 is odd integer, such that 4n - 3 \( \equiv 2 \pmod{6} \). If \(n \equiv 0, 3 \pmod{6} \),
then $4n - 3 \equiv 3 \pmod{6}$. We refer to lemmas in [12] and [13] to determine the lower bound of $\text{res}(T_{I_p})$.

$$\text{res}(T_{I_p}) \geq \left\{ \begin{array}{ll}
\left\lfloor \frac{4n-1}{3} \right\rfloor + 1, & \text{if } n \equiv 0, 3 \pmod{6} \\
\left\lceil \frac{4n-1}{3} \right\rceil, & \text{otherwise}.
\end{array} \right.$$ 

The next step, we determine the upper bound of $\text{res}(T_{I_p})$ by defining the function $f$ and $g$ of vertex and edge labeling. For $n \in \{3, 4, 5, 6\}$, we have the following function of vertex and edge labeling.

$$f(x_i) = \begin{cases}
0, & \text{if } i \in [1, 2] \\
2(i-2), & \text{if } i \in [3, 4, \ldots, n]
\end{cases}$$

$$f(y_i) = \begin{cases}
2(i-2), & \text{if } i \in [1, 2, 3] \\
2(i-2), & \text{if } i \in [4, 5, \ldots, n]
\end{cases}$$

$$g(x_i, x_{i+1}) = \begin{cases}
2, & \text{if } i = 1 \\
4, & \text{if } i \in [2, 3, \ldots, n-1]
\end{cases}$$

$$g(y_i, y_{i+1}) = \begin{cases}
2, & \text{if } i \in [1, 2] \\
4, & \text{if } i = 3 \\
6, & \text{if } i \in [4, 5, \ldots, n-1]
\end{cases}$$

$$g(y_i, x_{i+1}) = \begin{cases}
3, & \text{if } i \in [1, 2, 3] \\
5, & \text{if } i \in [4, 5, \ldots, n-1]
\end{cases}$$

For $n \geq 7$, we have the following function of vertex and edge labeling. Let,

$$k = \left\{ \begin{array}{ll}
\left\lfloor \frac{4n-1}{3} \right\rfloor + 1, & \text{if } n \equiv 0, 3 \pmod{6} \\
\left\lceil \frac{4n-1}{3} \right\rceil, & \text{otherwise}.
\end{array} \right.$$ 

$$f(x_i) = \begin{cases}
2(i-2), & \text{if } i \in [3, 4, 5, \ldots, \left\lfloor \frac{k}{2} \right\rfloor + 1] \\
\left\lfloor \frac{k}{2} \right\rfloor + 2, & \text{if } i \in [\left\lfloor \frac{k}{2} \right\rfloor + 2, \left\lfloor \frac{k}{2} \right\rfloor + 3, \ldots, n]
\end{cases}$$

$$f(y_i) = \begin{cases}
2(i-2), & \text{if } i \in [4, 5, 6, \ldots, \left\lfloor \frac{k}{2} \right\rfloor + 1] \\
\left\lfloor \frac{k}{2} \right\rfloor + 2, & \text{if } i \in [\left\lfloor \frac{k}{2} \right\rfloor + 2, \left\lfloor \frac{k}{2} \right\rfloor + 3, \ldots, n]
\end{cases}$$

$$g(x_i, x_{i+1}) = \begin{cases}
2, & \text{if } i = 1 \\
4, & \text{if } i \in [2, 3, 4, \ldots, \left\lfloor \frac{k}{2} \right\rfloor + 1] \\
4i - 4\left\lfloor \frac{k}{2} \right\rfloor - 2, & \text{if } i \in [\left\lfloor \frac{k}{2} \right\rfloor + 2, \left\lfloor \frac{k}{2} \right\rfloor + 3, \ldots, n-1]
\end{cases}$$

$$g(y_i, y_{i+1}) = \begin{cases}
2, & \text{if } i \in [1, 2] \\
4, & \text{if } i = 3 \\
6, & \text{if } i \in [4, 5, 6, \ldots, \left\lfloor \frac{k}{2} \right\rfloor + 1] \\
4i - 4\left\lfloor \frac{k}{2} \right\rfloor, & \text{if } i \in [\left\lfloor \frac{k}{2} \right\rfloor + 2, \left\lfloor \frac{k}{2} \right\rfloor + 3, \ldots, n-1]
\end{cases}$$

$$g(y_i, x_{i+1}) = \begin{cases}
3, & \text{if } i \in [1, 2, 3] \\
5, & \text{if } i \in [4, 5, 6, \ldots, \left\lfloor \frac{k}{2} \right\rfloor + 1] \\
4i - 4\left\lfloor \frac{k}{2} \right\rfloor - 1, & \text{if } i \in [\left\lfloor \frac{k}{2} \right\rfloor + 2, \left\lfloor \frac{k}{2} \right\rfloor + 3, \ldots, n-1]
\end{cases}$$

We can see the all edge weights from the weight sets above are different. It completes the proof.

**Theorem 3.** For every natural number $n \geq 2$, $\text{res}(P_n \circ C_3) = 2n$.

**Proof.** Let $P_n \times C_3$ be a graph with vertex set $V(P_n \times C_3) = \{v_{ij} : i \in \{1, 2, \ldots, n\}, \{j_1, j_2, j_3\} \neq \{1, 2, \ldots, n\}\}$, and edge set $E(P_n \times C_3) = \{e_{ij} : i \in \{1, 2\}, j \in \{1, 2, \ldots, n\}\} \cup \{e_{ij} : j \in \{1, 2, \ldots, n\}\}$.

For every $n$, it gives $6n - 3 \equiv 3 (\mod 6)$. We refer to lemmas in [12] and [13] to determine the lower bound of $\text{res}(P_n \times C_3)$.

$$\text{res}(P_n \times C_3) \geq \left\lfloor \frac{6n-3}{3} \right\rfloor + 1$$

$$= \left\lfloor \frac{2(2n-1)}{3} \right\rfloor + 1$$

$$= 2n - 1 + 1$$

$$= 2n$$

For the next step, we determined the upper bound of $\text{res}(P_n \times C_3)$ by defining the function $f$ and $g$ of vertex and edge labeling in the following:

$$f(v_{1,1}) = 0$$

$$f(v_{ij}) = 2j : i \in [1, 3], j \in [1, 2, 3, \ldots, n], v_{ij} \neq v_{1,1}$$

$$g(v_{ij}, v_{1,1}) = \begin{cases}
3, & \text{if } j = 1 \\
2j, & \text{if } j \in [2, 3, 4, \ldots, n]
\end{cases}$$

$$g(v_{ij}, v_{1,1}) = \begin{cases}
4, & \text{if } j = 1 \\
2j, & \text{if } j \in [2, 3, 4, \ldots, n-1]
\end{cases}$$

$$g(v_{ij}, v_{1,2}) = 2j - 1 : j \in [1, 2, 3, \ldots, n-1]$$

$$g(v_{ij}, v_{1,2}) = 2j - 2 : j \in [2, 3, 4, \ldots, n]$$

We get the edge weight from the function of vertex and edge labeling, the overall edge weight sets are:

$$\text{wt}(v_{1,1}v_{1,2}) = 6j - 2 : j \in [1, 2, 3, \ldots, n]$$

$$\text{wt}(v_{1,1}v_{1,3}) = 6j - 1 : j \in [1, 2, 3, \ldots, n]$$

$$\text{wt}(v_{1,2}v_{1,3}) = 6j : j \in [1, 2, 3, \ldots, n]$$

$$\text{wt}(v_{1,2}v_{1,3}) = 6j + 1 : j \in [1, 2, 3, \ldots, n-1]$$

$$\text{wt}(v_{1,1}v_{1,2}) = 6j + 2 : j \in [1, 2, 3, \ldots, n]$$

$$\text{wt}(v_{1,3}v_{1,4}) = 6j + 3 : j \in [1, 2, 3, \ldots, n-1]$$

We can see the all edge weights from the weight sets above are different. It concludes the proof.

**Theorem 4.** For every natural number $n \geq 2$,

$$\text{res}(P_n \circ P_2) = \left\{ \begin{array}{ll}
\left\lfloor \frac{4n}{3} \right\rfloor + 1, & \text{if } n \equiv 1, 4 \pmod{6} \\
\left\lceil \frac{4n}{3} \right\rceil, & \text{otherwise}.
\end{array} \right.$$ 

**Proof.** Let $P_n \circ P_2$ be a graph with vertex set $V(P_n \circ P_2) = \{x_i : i \in [1, 2, 3, \ldots, n]\} \cup \{y_{ij} : i \in [1, 2, 3, \ldots, n]\}$, $f(x_i) = 2i - 1$, $g(x_i, y_{i+1}) = 2$. We refer to lemmas in [12] and [13] to determine the lower bound of $\text{res}(P_n \circ P_2)$.
Further step, we determine the upper bound of \( \text{res}(P_1 \circ P_2) \) by defining the function \( f \) and \( g \) of vertex and edge labeling in the following. Let

\[
\begin{align*}
k &= \begin{cases} 
\lfloor \frac{m+1}{3} \rfloor + 1, & \text{if } n \equiv 1,4 \text{(mod 6)} \\
\lfloor \frac{m-1}{3} \rfloor, & \text{otherwise.}
\end{cases} \\
f(x_i) &= \begin{cases} 
2i - 2, & \text{if } i \in [1,2,3,\ldots,\lfloor \frac{n}{2} \rfloor] \\
2i + 1, & \text{if } i \in [\lfloor \frac{n}{2} \rfloor + 1, \lfloor \frac{n}{2} \rfloor + 2,\ldots,n] 
\end{cases} \\
f(y_{i,j}) &= f(x_i) : i \in [1,2,3,\ldots,n], j \in [1,2].
\end{align*}
\]

For \( n \in \{2,3,4\} \), we have the function of edge labeling as follows,

\[
g(x_i) = \begin{cases} 
2 : i \in [1,2,\ldots,n-1] 
\end{cases} \quad \text{and} \quad \begin{cases} 
g(x_i) = 3 : i \in [1,2,\ldots,n] 
\end{cases}
\]

For \( n \geq 5 \), we have the following function of edge labeling.

\[
g(x_i) = \begin{cases} 
2, & \text{if } i \in [1,2,3,\ldots,\lfloor \frac{n}{2} \rfloor] \\
4i - 4[\frac{i}{2}] + 1, & \text{if } i \in [\lfloor \frac{n}{2} \rfloor + 1, \lfloor \frac{n}{2} \rfloor + 2,\ldots,n-1] 
\end{cases} \\
g(y_{i,j}) = \begin{cases} 
1, & \text{if } i \in [1,2,3,\ldots,\lfloor \frac{n}{2} \rfloor] \\
4i - 4[\frac{i}{2}] - 3, & \text{if } i \in [\lfloor \frac{n}{2} \rfloor + 2, \lfloor \frac{n}{2} \rfloor + 3,\ldots,n] 
\end{cases}
\]

We get the edge weight from the function of vertex and edge labeling, the overall edge weight sets are:

\[
\begin{align*}
\text{aw}(x_{i+1}) &= 4i : i \in [1,2,3,\ldots,n-1] \\
\text{aw}(x_i) &= 4i - 3 : i \in [1,2,3,\ldots,n] \\
\text{aw}(x_{i+1}) &= 4i - 1 : i \in [1,2,3,\ldots,n] \\
\text{aw}(y_{i,j}) &= 4i - 2 : i \in [1,2,3,\ldots,n]
\end{align*}
\]

We can see all the edge weights from the weight sets above are different. It completes the proof. \( \square \)

**Theorem 5.** For every natural number \( n \geq 2 \),

\[
\text{res}(P_1 \circ C_3) = \begin{cases} 
\lfloor \frac{7n}{6} \rfloor + 1, & \text{if } 7n - 1 \equiv 2,3 \text{(mod 6)} \\
\lfloor \frac{7n}{6} \rfloor, & \text{otherwise.}
\end{cases}
\]

**Proof.** Let \( P_1 \circ C_3, n \geq 2 \), be a graph with the vertex set \( V(P_1 \circ C_3) = \{v_{i,j} \mid i \in [1,2,3], j \in [1,2,3,\ldots,n] \} \cup \{u_{j} \mid j \in [1,2,3,\ldots,n] \} \), \( |V(P_1 \circ C_3)| = 4n \) and the edge set \( E(P_1 \circ C_3) = \{v_{i,j+1} \mid i \in [1,2], j \in [1,2,3,\ldots,n] \} \cup \{v_{i+1,j} \mid j \in [1,2,3,\ldots,n] \} \cup \{u_{i,j} \mid j \in [1,2,3,\ldots,n-1] \}, |E(P_1 \circ C_3)| = 7n - 1 \). We refer to lemmas in [12] and [13] to determine the lower bound of \( \text{res}(P_1 \circ C_3) \).

\[
\text{res}(P_1 \circ C_3) \geq \begin{cases} 
\lfloor \frac{7n}{6} \rfloor + 1, & \text{if } 7n - 1 \equiv 2,3 \text{(mod 6)} \\
\lfloor \frac{7n}{6} \rfloor, & \text{otherwise.}
\end{cases}
\]

The next step, we determine the upper bound of \( \text{res}(P_1 \circ C_3) \) by defining the function \( f \) and \( g \) of vertex and edge labeling in the following:

\[
\begin{align*}
\text{aw}(x_{i+1}) &= \begin{cases} 
\lfloor \frac{7n}{6} \rfloor - 1, & \text{if } \equiv 2 \text{(mod 6)} \\
\lfloor \frac{7n}{6} \rfloor + 1, & \text{if } j \equiv 3,4 \text{(mod 6)} \\
\lfloor \frac{7n}{6} \rfloor, & \text{otherwise.}
\end{cases} \\
\text{aw}(y_{i,j}) &= f(u_j) = \begin{cases} 
\lfloor \frac{7n}{6} \rfloor - 1, & \text{if } j \equiv 2 \text{(mod 6)} \\
\lfloor \frac{7n}{6} \rfloor + 1, & \text{if } j \equiv 3,4 \text{(mod 6)} \\
\lfloor \frac{7n}{6} \rfloor, & \text{otherwise.}
\end{cases}
\end{align*}
\]

We can see all the edge weights from the weight sets above are different. It concludes the upper bound of \( \text{res}(G) \leq q \). \( \square \)
Table 2. Label of edges on $v_1,v_2,v_3,v_4,v_1'v_2',v_1'v_3',v_1'v_4',v_2'v_3',v_2'v_4'$.

| $j$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ | $7$ | $8$ | $9$ | $10$ | $11$ | $12$ | $13$ | $14$ | $15$ | $...$
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $e_{1,j}v_{1,j}$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ | $7$ | $8$ | $9$ | $10$ | $11$ | $12$ | $13$ | $14$ | $15$ | $...$
| $e_{2,j}v_{1,j}$ | $2$ | $3$ | $4$ | $5$ | $6$ | $7$ | $8$ | $9$ | $10$ | $11$ | $12$ | $13$ | $14$ | $15$ | $16$ | $...$
| $e_{3,j}v_{1,j}$ | $3$ | $4$ | $5$ | $6$ | $7$ | $8$ | $9$ | $10$ | $11$ | $12$ | $13$ | $14$ | $15$ | $16$ | $17$ | $...$
| $e_{4,j}v_{1,j}$ | $4$ | $5$ | $6$ | $7$ | $8$ | $9$ | $10$ | $11$ | $12$ | $13$ | $14$ | $15$ | $16$ | $17$ | $18$ | $...$
| $e_{1,j}v_{2,j}$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ | $7$ | $8$ | $9$ | $10$ | $11$ | $12$ | $13$ | $14$ | $15$ | $...$
| $e_{2,j}v_{2,j}$ | $2$ | $3$ | $4$ | $5$ | $6$ | $7$ | $8$ | $9$ | $10$ | $11$ | $12$ | $13$ | $14$ | $15$ | $16$ | $...$
| $e_{3,j}v_{2,j}$ | $3$ | $4$ | $5$ | $6$ | $7$ | $8$ | $9$ | $10$ | $11$ | $12$ | $13$ | $14$ | $15$ | $16$ | $17$ | $...$
| $e_{4,j}v_{2,j}$ | $4$ | $5$ | $6$ | $7$ | $8$ | $9$ | $10$ | $11$ | $12$ | $13$ | $14$ | $15$ | $16$ | $17$ | $18$ | $...$

Combining this upper bound lemma together with the lemma of the lower bound mentioned in [12] and [13], we have Corollary 1. By this corollary, we can make sure that all the reflexive edge strength $\text{res}(G)$ obtained in those theorems above lay on this bound.

Corollary 1. Let $G$ be a graph of order $p$ and size $q$. The reflexive edge strength of any graph $G$ is $\left\lceil \frac{p}{2} \right\rceil + 1 \leq \text{res}(G) \leq q$, for $q \equiv 2, 3(\mod 6)$; and $\left\lceil \frac{p}{2} \right\rceil \leq \text{res}(G) \leq q$, for otherwise.

3. Concluding remark

We have obtained the exact values of reflexive edge strength of some almost regular graphs, namely ladder, triangular ladder, $C_1 \times P_n$, $P_2 \odot C_2$, and $P_2 \odot C_2$. We have also analysed the upper bound on the reflexive edge strength of any type of graphs, and finally we can complete the bound of $\text{res}(G)$. However, since obtaining the reflexive edge strength of graph is considered to be NP-complete problem, the characterization of the exact value of $\text{res}(G)$ for any almost regular graph $G$ is still widely open. Therefore, we propose the following open problems. (1) Determine the sharper upper bound of reflexive edge strength of any graphs in order to narrow the gap between lower and upper bound. (2) Determine the exact value of reflexive edge strength of any type of almost regular graphs apart from those families. (3) Develop a theorem together with the proof which state an exact value of the reflexive edge strength $\text{res}(G)$ for any almost regular graph.

Declarations

Author contribution statement

I. H. Agustin: Conceived and designed the experiments.
Dafik: Performed the experiments; Analyzed and interpreted the data.
I. Utoy, Slamian, M. Venkatachalal: Contributed reagents, materials, analysis tools or data; Wrote the paper.

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Additional information

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