On the Scaling Violations of Diffractive Structure Functions: Operator Approach

Johannes Blümlein\textsuperscript{a} and Dieter Robaschik\textsuperscript{a,b}

\textsuperscript{a} Deutsches Elektronen–Synchrotron, DESY, Platanenallee 6, D–15738 Zeuthen, Germany

\textsuperscript{b} Brandenburgische Technische Universität Cottbus, Fakultät 1, PF 101344, D–03013 Cottbus, Germany

Abstract

A quantum field theoretic treatment of inclusive deep–inelastic diffractive scattering is given. The process can be described in the general framework of non–forward scattering processes using the light–cone expansion in the generalized Bjorken region. Evolution equations of the diffractive hadronic matrix elements are derived at the level of the twist–2 contributions and are compared to those of inclusive deep–inelastic forward scattering (DIS). The diffractive parton densities are obtained as projections of two–variable parton distributions. We also comment on the higher twist contributions in the light–cone expansion.
1 Introduction

Deep inelastic diffractive lepton–nucleon scattering was observed at the electron–proton collider HERA some years ago \cite{1}. This process is measured in detail by now \cite{2} and the structure function $F_2^D(x, Q^2)$ was extracted. The diffractive events are characterized by a rapidity gap between the final state nucleon and the set of the diffractively produced hadrons due to a color–neutral exchange. The (semi-inclusive) structure functions emerging in the diffractive scattering cross section show the same scaling violations as the structure functions in deep inelastic scattering within the current experimental resolution \cite{4}. This is a remarkable fact, which should also be understood with the help of field–theoretic methods within Quantum Chromodynamics (QCD).

In the description of diffractive $ep$ scattering one has to clearly distinguish \cite{5} the case of hard scattering, i.e. large $Q^2$, from that of softer hadronic interactions, cf. also \cite{6}. A perturbative description of the scaling violations can only be hoped for in the former case, to which we limit our considerations in this paper.

There is a vast amount of varying descriptions of the underlying dynamics of the diffractive scattering process\fn{1,2}, ranging from intuitive phenomenological models \cite{9} to non-perturbative semi-classical descriptions \cite{10}, being able to describe and to parameterize \cite{11} the existing data differentially.

In this paper we describe the process of inclusive deep–inelastic diffractive scattering, being a non–forward process, at large space–like momentum transfer using the general light–cone expansion for the non–forward case, cf. \cite{12–14}. As will be shown, the scaling violations of the diffractive structure functions can be described perturbatively in this region. We show that the diffractive parton densities can be derived as a special projection of two–variable distribution functions $f^A(z_+, z_-)$, where $z_+$ and $z_-$ are light–cone momentum fractions; for other projections in similar cases related to different observables see e.g. \cite{13–15}. In particular we have no need to refer to the specific mechanism of the non–perturbative color–singlet exchange between the proton and the set of diffractively produced hadrons and consider instead the expectation values of operators of a given twist and their anomalous dimensions. In this way we can generalize the analysis to operators of higher twist. This formulation is very appropriate for potential later studies of the corresponding operator matrix elements with the help of lattice techniques similar to the case of deep–inelastic scattering \cite{16} calculating their Mellin–moments.

The paper is organized as follows. In section 2 we consider the Lorentz structure of the differential cross section for inclusive deep–inelastic diffractive scattering. The short–distance structure of the matrix element is derived in section 3 using the (non–local) light–cone expansion in the generalized Bjorken region. In section 4 we derive the anomalous dimensions for the case of twist–2, discuss the implications for higher twist operators and compare the case of diffractive scattering to that of deep–inelastic scattering, and section 5 contains the conclusions.

2 Lorentz Structure

The process of deep–inelastic diffractive scattering is described by the diagram Figure 1. The

1\footnote{The measurement of the longitudinal diffractive structure function $F_L^D(x, Q^2)$ has not yet been possible. For the DIS structure function cf. \cite{3}.}

2\footnote{For surveys and a recent account see \cite{7, 8}.}
differential scattering cross section for single–photon exchange is given by
\[ d^5 \sigma_{\text{diff}} = \frac{1}{2(s - M^2)} \frac{1}{4} dPS^{(3)} \sum_{\text{spins}} \frac{e^4}{Q^2} L_{\mu\nu} W_{\mu\nu}. \]

(1)

Here \( s = (p_1 + l)^2 \) is the cms energy of the process squared and \( M \) denotes the nucleon mass.

\[
l \quad q \quad M_X \quad p_1 \quad p_2
\]

**Figure 1:** The virtual photon-hadron amplitude for diffractive \( ep \) scattering

The phase space \( dPS^{(3)} \) depends on five variables since one final state mass varies. They can be chosen as Bjorken \( x = Q^2/(W^2 + Q^2 - M^2) \), the photon virtuality \( Q^2 = -q^2 \), \( t = (p_1 - p_2)^2 \), a variable describing the non–forwardness w.r.t. the incoming proton direction,

\[
x_F = -\frac{2\eta}{1-\eta} = \frac{Q^2 + M_X^2 - t}{Q^2 + W^2 - M^2} \geq x,
\]

(2)

demanding \( M_X^2 > t \) and where

\[
\eta = \frac{q.(p_2 - p_1)}{q.(p_2 + p_1)} \epsilon \left[ -1, \frac{-x}{2-x} \right],
\]

(3)

and \( \Phi \) the angle between the lepton plane \( p_1 \times l \) and the hadron plane \( p_1 \times p_2 \),

\[
\cos \Phi = \frac{(p_1 \times l) \cdot (p_1 \times p_2)}{|p_1 \times l||p_1 \times p_2|}.
\]

(4)

\( W^2 = (p_1 + q)^2 \) and \( M_X^2 = (p_1 + q - p_2)^2 \) denote the hadronic mass squared and the square of the diffractive mass, respectively.

Since the leptonic tensor \( L^{\mu\nu} \) is symmetric for unpolarized scattering and the electromagnetic current is conserved the hadronic tensor, which is generally composed out of the tensors \( g_{\mu\nu}, (q_\mu, p_{1\mu}, p_{2\mu}) \otimes (q_\nu, p_{1\nu}, p_{2\nu}) \), consist out of four structure functions only

\[
W_{\mu\nu} = \left( -g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) W_1 + \left( p_{1\mu} - q_\mu \frac{p_{1\nu} q}{q^2} \right) \left( p_{1\nu} - q_\nu \frac{p_{1\mu} q}{q^2} \right) \frac{W_3}{M^2} \\
\quad + \left( p_{2\mu} - q_\mu \frac{p_{2\nu} q}{q^2} \right) \left( p_{2\nu} - q_\nu \frac{p_{2\mu} q}{q^2} \right) \frac{W_4}{M^2} \\
\quad + \left[ \left( p_{1\mu} - q_\mu \frac{p_{1\nu} q}{q^2} \right) \left( p_{2\nu} - q_\nu \frac{p_{2\mu} q}{q^2} \right) + \left( p_{2\mu} - q_\mu \frac{p_{2\nu} q}{q^2} \right) \left( p_{1\nu} - q_\nu \frac{p_{1\mu} q}{q^2} \right) \right] \frac{W_5}{M^2},
\]

(5)

\(^3\)For the commonly used notation of four variables see e.g. [17].

\(^4\)For a construction method, see e.g. [18].
with
\[ W_i = W_i(x, Q^2, x_F, t) . \]  (6)

The size of the rapidity–gap \[ \Delta y \sim \ln(1/x_F) \] is of the order \[ \Delta y \sim \ln(1/x_F) \] and large in the kinematic domain of HERA.

If target masses can be neglected, \( M^2 \sim 0 \), and \( t \) is considered to be very small compared to all other invariants the in– and outgoing proton four–momenta become proportional, \( p_2 = z p_1 \). The variable \( z \) is then related to \( x_F \) and \( \eta \) by
\[ z = 1 - x_F = \frac{1 + \eta}{1 - \eta} . \]  (7)

In this limit the hadronic tensor is described by two structure functions
\[
W_{\mu\nu} = \left( -g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) W_1 + \left( p_{1\mu} - q_\mu \frac{p_1 \cdot q}{q^2} \right) \left( p_{1\nu} - q_\nu \frac{p_1 \cdot q}{q^2} \right) \frac{W_2}{M^2} ,
\]  (8)

with
\[
W_2 = W_3 + (1 - x_F) W_5 + (1 - x_F)^2 W_4 . \]  (9)

Due to the dependence on \( x_F \) or \( \eta \), Eq. (2), the process is non–forward although the algebraic structure of the hadronic tensor is the same as in the forward case. Finally the generalized Bjorken–limit is carried out,
\[ 2p_1.q = 2M\nu \to \infty , \quad p_2.q \to \infty , \quad Q^2 \to \infty \] with \( x \) and \( x_F \) is fixed ,
(10)

which leads to \( MW_1(x, Q^2, x_F) \to F_1(x, Q^2, x_F) \) and \( \nu W_2(x, Q^2, x_F) \to F_2(x, Q^2, x_F) \). Note that \( Q^2 \gg t, M^2 \) always holds in this region. In the limit \( p_2 = z p_1 \) and \( M^2, t = 0 \) the \( \Phi \)–integral becomes trivial. The scattering cross section reads
\[
\frac{d^3\sigma_{\text{diff}}}{dx dQ^2 dx_F} = \frac{2\pi \alpha^2}{x Q^4} \left[ y^2 2x F_1^{D(3)}(x, Q^2, x_F) + 2(1 - y) F_2^{D(3)}(x, Q^2, x_F) \right] ,
\]  (11)

with \( y = q.p_1/l.p_1 \).

### 3 The Compton Amplitude

The renormalized and time–ordered product of two electromagnetic currents is given by
\[
\hat{T}_{\mu\nu}(x) = iRT \left[ J_\mu \left( \frac{x}{2} \right) J_\nu \left( -\frac{x}{2} \right) S \right] = -e^2 2\pi^2(x^2 - i\epsilon)^2 RT \left[ \overline{\psi} \left( \frac{x}{2} \right) \gamma^\mu \gamma^\lambda \gamma^\nu \psi \left( -\frac{x}{2} \right) - \overline{\psi} \left( -\frac{x}{2} \right) \gamma^\mu \gamma^\lambda \gamma^\nu \psi \left( \frac{x}{2} \right) \right] S
\]  (12)

\( \tilde{x} \) denotes a light–like vector corresponding to \( x \),
\[
\tilde{x} = x + \frac{\zeta}{\xi^2} \left[ \sqrt{x.\zeta^2 - x^2 \zeta^2} - x.\zeta \right] . \]  (13)
and $\zeta$ is a subsidiary vector. Following Refs. [14, 20] the operator $\hat{T}_{\mu\nu}$ can be expressed in terms of a vector and an axial-vector operator by

$$\hat{T}_{\mu\nu}(x) = -e^2 \frac{\tilde{x}^\lambda}{i\pi^2(x^2 - i\epsilon)^2} \left[ S_{\alpha\mu\lambda\nu}O^\alpha \left( \frac{\tilde{x}}{2}, -\frac{\tilde{x}}{2} \right) + i\varepsilon_{\mu\lambda\sigma}O_5^\alpha \left( \frac{\tilde{x}}{2}, -\frac{\tilde{x}}{2} \right) \right],$$

where

$$S_{\alpha\mu\lambda\nu} = g_{\alpha\mu}g_{\lambda\nu} + g_{\lambda\mu}g_{\alpha\nu} - g_{\mu\nu}g_{\lambda\alpha}.$$  

(15)

The bilocal light-ray operators are

$$O^\alpha \left( \frac{x}{2}, -\frac{x}{2} \right) = \frac{i}{2} RT \left[ \psi \left( \frac{x}{2} \right) \gamma^\alpha \psi \left( -\frac{x}{2} \right) - \bar{\psi} \left( -\frac{x}{2} \right) \gamma^\alpha \bar{\psi} \left( \frac{x}{2} \right) \right] S,$$

(16)

$$O_5^\alpha \left( \frac{x}{2}, -\frac{x}{2} \right) = \frac{i}{2} RT \left[ \bar{\psi} \left( \frac{x}{2} \right) \gamma_5 \gamma^\alpha \psi \left( -\frac{x}{2} \right) + \bar{\psi} \left( -\frac{x}{2} \right) \gamma_5 \gamma^\alpha \bar{\psi} \left( \frac{x}{2} \right) \right] S.$$

(17)

The general operator $\hat{T}_{\mu\nu}$ is now to be related to the diffractive scattering cross section. This is possible upon applying Mueller’s generalized optical theorem [21] (Figure 2), which moves the final state proton into an initial state anti-proton.

![Figure 2: A. Mueller’s optical theorem.](image)

Consequently, the Compton amplitude is obtained as the expectation value

$$T_{\mu\nu}(x) = \langle p_1, -p_2 | \hat{T}_{\mu\nu} | p_1, -p_2 \rangle,$$

which is forward w.r.t. to the direction defined by $\langle p_1, -p_2 \rangle$. The quantity $\langle p_1, -p_2 \rangle$, in the strict sense, is not a single physical state of the Hilbert-space of states, since this would imply $t = (p_2 - p_1)^2 > 0$. The analytic continuation is such, that $t$ is kept space-like, which is necessary for the non-perturbative behavior of the matrix element. The twist–2 contributions to the expectation values of the operators (14,17) are obtained

$$\langle p_1, -p_2 | O_{(5)}^{A,\mu}(\kappa_+, \kappa_-) | p_1, -p_2 \rangle = \int_0^1 d\lambda \partial^\mu_x \langle p_1, -p_2 | O_{(5)}^{A}(\lambda \kappa_+, \lambda \kappa_-) | p_1, -p_2 \rangle \bigg|_{x=\tilde{x}}$$

(19)

as partial derivatives of the expectation values of

$$O^A(\kappa_+, \kappa_-) = x^\alpha O^A_\alpha(\kappa_+, \kappa_-).$$

(20)
the corresponding scalar and pseudo-scalar operators. The index $A = q,G$ labels the quark– or gluon operators, cf.\cite{4}. In the unpolarized case only the vector operators contribute. The scalar twist–2 quark operator matrix element has the representation\footnote{For parameterizing hadronic matrix elements see e.g.\cite{29}.} due to the overall symmetry in $x$

$$\langle p_1, -p_2 | O^q(\kappa_+, \kappa_-) | p_1, -p_2 \rangle = x p_1 \int Dz e^{-i\kappa_- p z} f^q(z_+, z_-) + x \pi_- \int Dz e^{-i\kappa_+ p z} f^q(z_+, z_-) ,$$

and is independent of $\kappa_+$. Here we assume that all the trace–terms have been subtracted, see\cite{4}. The action of $\langle p_1, -p_2 \rangle$ onto the scalar operator yield projections onto two directions $p_-$ and $\pi_- = p_+ - p_- / \eta$. $f^A(z_+, z_-)$ and $f^A(z_+, z_-)$ denote the corresponding scalar two–variable distribution amplitudes and the measure $Dz$ is

$$Dz = dz_+ dz_- \theta(1 + z_+ + z_-) \theta(1 + z_+ - z_-) \theta(1 - z_+ + z_-) \theta(1 - z_+ - z_-) .$$

Here, we decomposed the vector $p_z$ as

$$p_z = p_- z_- + p_+ z_+ = p_- \vartheta + \pi_- z_+ ,$$

with $z_{1,2}$ momentum fractions along $p_{1,2}$ and $p_\pm = p_2 \pm p_1$, $z_\pm = (z_2 \pm z_1) / 2$ and

$$\vartheta = z_- + \frac{1}{\eta} z_+ .$$

Note that, $q, \pi_- \equiv 0$. In the approximation $M^2, t \sim 0$, in which we work from now on, the vector $\pi_- \theta$ vanishes. In this limit only the first term contributes to the matrix element Eq.\cite{21}.

The Fourier–transform of the Compton amplitude is given by\cite{20}

$$T_{\mu\nu}(p_1, p_2, q) = i \int d^4 x e^{i q x} T_{\mu\nu}(x)$$

$$= -2 S_{\mu\nu\sigma\tau} \int Dz \left[ \frac{p^-_\sigma Q^\tau_2}{Q^2 + i\varepsilon} - \frac{1}{2} \frac{p^-_\tau p^-_\sigma}{Q^2 + i\varepsilon} + \frac{Q^2 Q^- p^-_\sigma}{(Q^2 + i\varepsilon)^2 p^-_\sigma Q^\tau_2} \right] F(z_+, z_-) ,$$

with $Q^2 = q - p_z / 2$. The function $F(z_+, z_-)$ is related to the distribution function $f(z_+, z_-)$ by

$$F(z_+, z_-) = \int_0^1 \frac{d\lambda}{\lambda^2} f \left( \frac{z_+}{\lambda}, \frac{z_-}{\lambda} \right) \theta(\lambda - |z_+|) \theta(\lambda - |z_-|) .$$

The denominators take the form

$$\frac{1}{Q^2 + i\varepsilon} = - \frac{1}{q p_- (\vartheta - 2\beta + i\varepsilon)} ,$$

where

$$\beta = \frac{x}{x_p} = \frac{q^2}{2 q p_-} .$$

The conservation of the electromagnetic current is easily seen

$$q^\mu T_{\mu\nu}(p_1, p_2, q) = q^\mu T_{\mu\nu}(p_1, p_2, q) = -2 p_- \nu(\mu) \int Dz F(z_+, z_-) = 0 ,$$

$$\text{[5]}$$
since the symmetry relation, \( [20] \)

\[
\int DzF(z_+,z_-) = 0 ,
\]  

holds. Subsequently we will use the distribution

\[
\hat{F}(\vartheta, \eta) = \int DzF(z_+,z_-)\delta(\vartheta - z_- - z_+/\eta) = \int^{-\text{sign}(\vartheta)/\eta}_0 dz \hat{f}(z, \eta) .
\]  

The distribution function \( \hat{f}(z, \eta) \) is related to \( f(z_+, z_-) \) by

\[
\hat{f}(z, \eta) = \int_{\eta(1-z)}^{\eta(1+z)} d\rho \ \theta(1-\rho)\theta(\rho + 1) f(\rho, z - \rho/\eta) ,
\]  

with \( \rho = z_+/\eta \).

The Compton amplitude may be re-written as

\[
T_{\mu\nu}(p_1, p_2, q) = -2 \int_{-1/\eta}^{1/\eta} d\vartheta \left\{ -q.p_{-\mu\nu} - 2\vartheta p_{-\mu}p_{-\nu} + (p_{-\mu}q_{\nu} + p_{-\nu}q_{\mu}) \right\} \frac{1}{Q_z^2 + i\varepsilon}
\]

\[
\times \left\{ \vartheta(q_{\mu}p_{-\nu} + q_{\nu}p_{-\mu}) - \vartheta q_{\mu}p_{-\nu} - \vartheta^2 p_{-\mu}p_{-\nu} \right\} \frac{1}{(Q_z^2 + i\varepsilon)^2}
\]

\[
\times \int^{-\text{sign}(\vartheta)/\eta}_0 dz \hat{f}(z, \eta) .
\]  

The \( \vartheta \)-integrals in Eq. (33) can be simplified using the identities

\[
\int_{-1/\eta}^{1/\eta} d\vartheta \frac{\vartheta^{k-1}}{(\vartheta - 2\beta + i\varepsilon)^2} \int_{-\text{sign}(\vartheta)/\eta}^{\text{sign}(\vartheta)/\eta} dz \hat{f}(z, \eta) = \int_{-1/\eta}^{1/\eta} d\vartheta \frac{k\vartheta^{k-1}}{(\vartheta - 2\beta + i\varepsilon)} \int_{-\text{sign}(\vartheta)/\eta}^{\text{sign}(\vartheta)/\eta} dz \hat{f}(z, \eta)
\]

\[
- \int_{-1/\eta}^{1/\eta} d\vartheta \frac{\vartheta^{k-1}}{(\vartheta - 2\beta + i\varepsilon)} \hat{f}(\vartheta, \eta) .
\]  

One may further re-write Eq. (33) referring to \( p_1 \) instead of \( p_- \),

\[
T_{\mu\nu}(p_1, p_2, q) = 2 \int_{-1/\eta}^{1/\eta} d\vartheta \left\{ -g_{\mu\nu} + \frac{p_{1\mu}q_{\nu} + p_{1\nu}q_{\mu}}{q.p_1} + \vartheta x_{\mu} \frac{p_{1\mu}p_{1\nu}}{q.p_1} \right\} \hat{f}(\vartheta, \eta) \frac{1}{(\vartheta - 2\beta + i\varepsilon)} .
\]  

Before we consider the absorptive part of the Compton amplitude we exploit the symmetry relation for the unpolarized distribution functions \( F^A(z_1, z_2) \), Ref. [20],

\[
F^A(z_1, z_2) = -F^A(-z_1, -z_2) .
\]  

It translates into

\[
\hat{F}^A(\vartheta, \eta) = -\hat{F}^A(-\vartheta, \eta) ,
\]  

and, cf. Eq. (31),

\[
\hat{f}^A(\vartheta, \eta) = -\hat{f}^A(-\vartheta, \eta) .
\]  

7
Seeking the form of Eq. (8), Eq. (35) transforms to
\[
T_{\mu\nu}(p_1, p_2, q) = \int_{-1/\eta}^{1/\eta} d\vartheta \left[ \left( -g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) + \frac{2x}{q.p_1} \left( p_{1\mu} - q_\mu \frac{p_1.q}{q^2} \right) \left( p_{1\nu} - q_\nu \frac{p_1.q}{q^2} \right) \right]
\times \left[ \frac{\hat{f}(\vartheta, \eta)}{\vartheta - 2\beta + i\varepsilon} - \frac{\hat{f}(-\vartheta, \eta)}{\vartheta - 2\beta + i\varepsilon} \right]
+ 2\frac{p_{1\mu} p_{1\nu}}{q.p_1} \int_{-1/\eta}^{1/\eta} d\vartheta (\vartheta x_p - 2x) \frac{\hat{f}(\vartheta, \eta)}{\vartheta - 2\beta + i\varepsilon}.
\]

Taking the absorptive part one obtains
\[
W_{\mu\nu} = \frac{1}{2\pi} \text{Im} T_{\mu\nu}(p_1, p_2, q)
= \left( -g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) F_1(\beta, \eta, Q^2) + \frac{1}{q.p_1} \left( p_{1\mu} - q_\mu \frac{p_1.q}{q^2} \right) \left( p_{1\nu} - q_\nu \frac{p_1.q}{q^2} \right) F_2(\beta, \eta, Q^2)
\]

where
\[
F_1(\beta, \eta, Q^2) = \sum_{q=1}^{N_f} e_q^2 \left[ f_q^D(\beta, Q^2, x_p) + \bar{f}_q^D(\beta, Q^2, x_p) \right] \equiv F_1^{D(3)}(x, Q^2, x_p),
\]

with \(N_f\) the number of flavors, choosing the factorization scale \(\mu^2 = Q^2\), and
\[
F_2(\beta, \eta, Q^2) = 2x F_1(\beta, \eta, Q^2) \equiv F_2^{D(3)}(x, Q^2, x_p).
\]

If we compare the distribution functions \(f_A^D\) with the parton distributions in the deep inelastic case, the role of \(x\) in the latter case is taken by the variable \(\beta\). Both variables have the range \([0, 1]\), through which Mellin transforms w.r.t. these variables can be evaluated. On the contrary, the support in \(x\) of the diffractive structure functions is limited to \([0, x_p]\). Although \(\beta\) in the diffractive case compares to \(x\) in the deep inelastic case, the Callan–Gross relation, at lowest order in \(\alpha_s\), Eq. (42), is given by the factor \(2x\) between the two structure functions.

The diffractive quark and anti–quark densities are given by
\[
\sum_{q=1}^{N_f} e_q^2 f_q^D(\beta, Q^2, x_p) = -\hat{f}(2\beta, \eta, Q^2)
\]
\[
\sum_{q=1}^{N_f} e_q^2 \bar{f}_q^D(\beta, Q^2, x_p) = \hat{f}(-2\beta, \eta, Q^2).
\]

We finally can express the diffractive parton densities in terms of the distribution function \(f(z_+, z_-)\) directly
\[
f^D(\beta, Q^2, x_p) = -\int_{-x_p}^{x_p (1-2x_p/2)} d\rho f(\rho, 2\beta + \rho(2-x_p)/x_p, Q^2)
\]
\[
\bar{f}^D(\beta, Q^2, x_p) = -\int_{-x_p}^{x_p (1-2x_p/2)} d\rho f(-\rho, -2\beta + \rho(2-x_p)/x_p, Q^2).
\]
4 Evolution Equations

The evolution equation of the scalar twist–2 quark and gluon operator Eq. (19) read, cf. [4]
\[ \mu^2 \frac{d}{d\mu^2} O^A(\kappa_+, \kappa_-, \mu^2) = \int D\kappa' \gamma^{AB}(\kappa_+, \kappa_-, \kappa'_+, \kappa'_-; \mu^2) O_B(\kappa'_+, \kappa'_-, \mu^2), \] (45)
with \( \kappa_\pm = (\kappa_2 - \kappa_1)/2 \) and the measure
\[ D\kappa = d\kappa_+ d\kappa_- (1 + \kappa_+ + \kappa_- \theta(1 - \kappa_+ + \kappa_- \theta(1 - \kappa_+ - \kappa_-). \] (46)

We consider the non–forward evolution equations to trace an eventual \( \eta \)-dependence. Here \( \gamma^{AB}(\kappa_+, \kappa_-, \kappa'_+, \kappa'_-, \mu^2) \) is the general non–forward twist–2 singlet evolution matrix. The same evolution equation applies to the matrix elements \( \langle p_1, -p_2 | O^A(\kappa_+, \kappa_-) | p_1, -p_2 \rangle \), which may be related to the distribution functions \( f^A(\vartheta, \eta) \) by:
\[ f^A(\vartheta, \eta) = \int \frac{dk_- \bar{x}p_-}{2\pi} e^{i\kappa_- \bar{x}p_- \vartheta} \langle p_1, -p_2 | O^A(\kappa_+, \kappa_-) | p_1, -p_2 \rangle (\bar{x}p_-)^{1-d_A}, \] (47)
where \( d_q = 1 \) and \( d_G = 2 \). Eq. (17) and the inverse Fourier transform
\[ \langle p_1, -p_2 | O^A(\kappa_+, \kappa_-) | p_1, -p_2 \rangle (\bar{x}p_-)^{1-d_A} = \int_{1/\eta}^{-1/\eta} d\vartheta' e^{-i\kappa_- \bar{x}p_- \vartheta'} f^A(\vartheta', \eta) \] (48)
are used to obtain evolution equations for \( f^A(\vartheta, \eta; \mu^2) \). To perform the \( \kappa_- \)-integral the dependence on this variable has to be made explicit in the anomalous dimensions \( \gamma^{AB} \) to all orders. The scalar operator matrix elements, Eq. (21), do not depend on the light cone mark \( \kappa_+ \) due to translation invariance on the light–cone, which therefore can be set to zero in Eq. (50). Furthermore, the anomalous dimension \( \gamma^{AB} \) obeys the re-scaling relation, cf. [4],
\[ \gamma^{AB}(\kappa_+, \kappa_-, \kappa'_+, \kappa'_-; \mu^2) = \sigma^{dAB} \gamma^{AB}(\sigma \kappa_+, \sigma \kappa_-, \sigma \kappa'_+, \sigma \k'_-; \mu^2), \] (49)
with \( d_{AB} = 2 + d_A - d_B \). The anomalous dimension transforms to
\[ \int D\kappa' \kappa'^{d_A} K^{\kappa'_-; \k'_+; \mu^2} = \int D\alpha \kappa'^{d_A} \bar{K}^{\alpha; \mu^2} \]
\[ = \text{Def} \int_0^1 du (1 - u) \int_0^1 d\xi \]
\[ \times \bar{K}^{\alpha; \mu^2} \]
\[ = \kappa'^{d_A} \int_0^1 du \bar{K}^{\alpha; \mu^2}, \] (50)
with \( \alpha_1 = [1 - (\kappa'_+ + \k'_-)]/\kappa_-] = (1 - u), \alpha_2 = [1 - (\k'_+ - \k'_-)]/\kappa_-] = (1 - \xi)(1 - u). \) The \( \kappa_- \)-integral yields
\[ \int \frac{dk_- \bar{x}p_-}{2\pi} e^{i\kappa_- \bar{x}p_- (\vartheta - \vartheta')} (\bar{x}p_-)^{d_A} = \bar{O}^{AB}(u\vartheta' - \vartheta) = \begin{cases} \delta(u\vartheta' - \vartheta) & \text{for } A = B = q, G \\
\delta_u (u\vartheta' - \vartheta) & \text{for } A = q, B = G \\
\theta(u\vartheta' - \vartheta)/\vartheta & \text{for } A = G, B = q \end{cases} \]
6The non–singlet evolution equations are structurally the same as that with the anomalous dimension \( \gamma^{AB} \).
7The leading order and next–to–leading order non–singlet and singlet anomalous dimensions were calculated in Refs. [23, 24] and [21].
8Note that we identified here \( \tau = \bar{x}p_-/\bar{x}p_+ \) with \( \eta = q_p_+ / q_p_- \) which is possible after the Bjorken limit is taken.
The following evolution equations for the distribution functions $f^A(\vartheta, \eta)$ are obtained:

$$
\frac{\mu^2}{d\mu^2} f^A(\vartheta, \eta; \mu^2) = \int_0^1 du \int_0^{- \text{sign}(\vartheta) / \eta} d\vartheta' \tilde{O}^{AB}(u\vartheta' - \vartheta) \tilde{K}^{AB}(u; \mu^2) f_B(\vartheta', \eta; \mu^2). \quad (51)
$$

The functions

$$
\vartheta' \int_0^1 du \tilde{O}^{AB}(u\vartheta' - \vartheta) \tilde{K}^{AB}(u; \mu^2) \equiv P^{AB}(\vartheta, \mu^2)
$$

are the forward splitting functions, which are independent of $\eta$ resp. $x_P$, leading to

$$
\frac{\mu^2}{d\mu^2} f^A(\vartheta, \eta; \mu^2) = \int_{- \text{sign}(\vartheta) / \eta} d\vartheta' \frac{\partial}{\partial \vartheta'} P^{AB} \left( \frac{\vartheta}{\vartheta'}, \mu^2 \right) f_B(\vartheta', \eta; \mu^2). \quad (53)
$$

They act on the momentum fraction $\vartheta$ of the diffractive parton densities. Note that the range of $\vartheta'$ is yet different from that of the corresponding quantity for deep inelastic scattering. After identifying $\vartheta = 2\beta$ taking the absorptive part, however, one arrives at the twist–2 evolution equation

$$
\frac{\mu^2}{d\mu^2} f^D_A(\beta, x_P; \mu^2) = \int_{- \text{sign}(\beta) / \beta} d\beta' \frac{\partial}{\partial \beta'} P^B \left( \frac{\beta}{\beta'}, \mu^2 \right) f_B(\beta', x_P; \mu^2). \quad (54)
$$

In the case of deep inelastic scattering the corresponding identification for the momentum fraction is $z = x$. The dependence of the diffractive parton densities w.r.t. $\eta$, or $x_P$, is entirely parametric and not changed under the evolution, which affects $\beta$. In the case of the twist–2 contributions factorization proofs were given \cite{25}, leading to the same evolution equations, (54), which are confirmed by the present derivation. For phenomenological applications to next-to-leading order, see e.g. \cite{28}.

We expressed the Compton amplitude with the help of the light–cone expansion at short distances and applied this representation to the process of deep–inelastic diffractive scattering using Mueller’s generalized optical theorem. This representation is not limited to leading twist operators but can be extended to all higher twist operators synonymously. The corresponding evolution equations for the higher twist hadronic matrix elements, which depend on more momentum fractions $\vartheta_i$ than one, transform analogously to the case of twist–2 being outlined above. A central parameter $\kappa_+ = \frac{1}{n} \sum_{i=1}^n \kappa_i$ may be set to zero, and analogous re-scaling relations apply. By virtue of this also here forward evolution equations are derived. However, the connection of the momentum fractions $\vartheta_i$ to the outer kinematic parameters is less trivial than in the case of twist–2 due to the structure of the corresponding Wilson coefficients. This is the case also for deep inelastic scattering, cf. e.g. \cite{29}.

**5 Conclusions**

The differential cross section of unpolarized deep–inelastic $ep$–diffractive scattering is described by four structure functions for pure photon exchange, which depend on the four kinematic variables, $x, Q^2, x_P$ and $t$. In the limit of vanishing target masses and $t \to 0$ only two structure functions contribute. In the case of hard diffractive scattering the scaling violations of these
structure functions can be described perturbatively. This is possible in transforming the non-forward amplitude of the process via Mueller’s optical theorem, by rotating the final-state proton into an initial-state anti-proton. The Compton-amplitude is calculated for the hadronic two-particle state \( \langle p_1, -p_2 \rangle \) and w.r.t. this state in the forward direction. The diffractive parton densities are associated to the general two-variable distribution functions, which describe the hadronic matrix element. The scaling variable of the diffractive parton densities, which directly compares to the Bjorken variable \( x \) in the deep-inelastic case, is \( \beta = x/x_p \). However, the Callan-Gross relation takes the usual form \( F_2^D(x, Q^2, x_p) = 2xF_1^D(x, Q^2, x_p) \). Due to the transformation of the problem, which is made possible applying Mueller’s optical theorem, the anomalous dimensions are the same as for forward scattering. We demonstrated this by an explicit calculation in the case of the twist-2 operators, however, the same mechanism applies also for higher twist operators using the light cone expansion. In the case of the twist-2 contributions the effective momentum fraction, \( \vartheta = z_- + z_+ / \eta \) may be identified with the variable \( 2\beta \) for diffractive scattering. The dependence of the parton densities on \( x_p \) is not affected by QCD-evolution, which acts on the variable \( \beta \).

Acknowledgment. For discussions we would like to thank J. Bartels, W. Buchmüller, J. Dainton, B. Geyer, P. Kroll, and G. Wolf.

References

[1] M. Derrick et al., ZEUS collaboration, Phys. Lett. B315 (1993) 481; T. Ahmed et. al., H1 collaboration, Nucl. Phys. B429 (1994) 477.

[2] J. Breitweg et al., ZEUS collaboration, Eur. Phys. J. C6 (1999) 43; C. Adloff et al., H1 collaboration, Z. Phys. C76 (1997) 613.

[3] C. Adloff et al., H1 collaboration, Phys. Lett. B393 (1997) 452.

[4] H. Abramowicz and J. Dainton, J. Phys. G22 (1996) 911.

[5] J. Bartels and M. Kowalski, Eur. Phys. J. C19 (2001) 693.

[6] A. Donnachie and P.V. Landshoff, Phys. Lett. B191 (1987) 309; Nucl. Phys. B286 (1987) 704; Phys. Lett. B437 (1998) 408; hep-ph/0105088 and references quoted therein.

[7] M. Wüsthoff and A.D. Martin, J. Phys. G24 (1999) R309; A. Hebecker, Phys. Rep. 331 (2000) 1 and references therein.

[8] H. Navelet and R. Peschanski, hep-ph/0105030 and references therein.

[9] G. Ingelman and P. Schlein, Phys. Lett. B152 (1985) 256; J. Bartels and G. Ingelman, Phys. Lett. B235 (1990) 175; G. Ingelman, hep-ph/9912534.

[10] W. Buchmüller and A. Hebecker, Nucl. Phys. B476 (1996) 293; W. Buchmüller, T. Gehrmann, and A. Hebecker, Nucl. Phys. B537 (1999) 477.

[11] J. Bartels, J. Ellis, M. Kowalski, and M. Wüsthoff, Eur. Phys. J. C7 (1999) 443.

[12] D. Müller, D. Robaschik, B. Geyer, F. Dittes, and J. Hořejší, Fortschr. Phys. 42 (1994) 2; X. Ji, J. Phys. G24 (1998) 1181.
[13] A.V. Radyushkin, Phys. Rev. D56 (1997) 5524; hep-ph/0101225.

[14] J. Blümlein, B. Geyer, and D. Robaschik, Nucl. Phys. B560 (1999) 283.

[15] M. Diehl, T. Feldmann, R. Jakob, and P. Kroll, Phys. Lett. B460 (1999) 204; Eur. Phys. J. C8 (1999) 409;
P. Kroll, Nucl. Phys. A666 (2000) 3.

[16] G. Martinelli and C.T. Sachrajda, Phys. Lett. B190 (1987) 151; 196 (1987) 184;
M. Göckeler et al., Nucl. Phys. (Proc. Suppl.) 53 (1997) 81;
M. Guagnelli, K. Jansen, and R. Petronzio, Phys. Lett. B459 (1999) 594; B493 (2000) 77.

[17] M. Diehl, Nucl. Phys. B (Proc. Suppl. 79 (1999) 723, Proc. DIS 99, eds. J. Blümlein and T. Riemann.

[18] B. Tarrach, Nuov. Cim. 28A (1975) 409.

[19] J.C. Collins, hep-ph/9705393.

[20] J. Blümlein and D. Robaschik, Nucl. Phys. B581 (2000) 449.

[21] A.H. Mueller, Phys. Rev. D2 (1970) 2963; Phys. Rev. D4 (1971) 150;
P.D.P. Collins, An Introduction to Regge Theory and High Energy Physics, (Cambridge University Press, Cambridge, 1977), pp. 331.

[22] S. Weinberg, The Quantum Theory of Fields, Vol. 1, (Cambridge University Press, Cambridge, 1995), pp. 452;
J. Blümlein, J. Eilers, B. Geyer, and D. Robaschik, to appear.

[23] T. Braunschweig, B. Geyer, and D. Robaschik, Ann. Phys. (Leipzig) 44 (1987) 407;
I.I. Balitsky and V.M. Braun, Nucl. Phys. B311 (1988/89) 541;
A.V. Radyushkin, Phys. Lett. B385 (1996) 333; Phys. Rev. D56 (1997) 5524;
J. Blümlein, B. Geyer, and D. Robaschik, Phys. Lett. B406 (1997) 161 and Erratum;
I.I. Balitsky and A.V. Radyushkin, Phys. Lett. B413 (1997) 114;
J. Blümlein, B. Geyer, and D. Robaschik, hep-ph/9711403, in : Proc. of the Workshop Deep Inelastic Scattering Off Polarized Targets: Theory Meets Experiment, eds. J. Blümlein et al., (DESY, Hamburg, 1997) DESY 97–200, p. 196.
X. Ji, Phys. Rev. Lett. 78 (1997) 610; Phys. Rev. D55 (1997) 7114;
L. Mankiewicz, G. Piller, and T. Weigl, Eur. J. Phys. C5 (1998) 119.

[24] F.M. Dittes and A.V. Radyushkin, Phys. Lett. B134 (1984) 359;
S.V. Mikhailov and A.V. Radyushkin, Nucl. Phys. B254 (1985) 89;
F.M. Dittes, D. Müller, D. Robaschik, B. Geyer, and J. Hořejší, Phys. Lett. B209 (1988) 325;
A.V. Belitsky, D. Müller, L. Niedermaier, and A. Schäfer, Nucl. Phys. B546 (1999) 279.

[25] A. Berera and D.E. Soper, Phys. Rev. D50 (1994) 4328; D53 (1996) 6162;
J. Collins, Phys. Rev. D57 (1998) 3051; Erratum: D61 (2000) 019902;
F. Hautmann, Z. Kunszt, and D.E. Soper, Nucl. Phys. B563 (1999) 153.

[26] L. Trentadue and G. Veneziano, Phys. Lett. B323 (1994) 201.
[27] M. Grazzini, Phys. Rev. D57 (1998) 4352.

[28] C. Royon, L. Schoeffel, J. Bartels, H. Jung, and R. Peschanski, Phys. Rev. D63 (2001) 074004.

[29] R.K. Ellis, W. Furmanski, and R. Petronzio, Nucl. Phys. B212 (1983) 29; J. Blümlein, V. Ravindran, J. Ruan, and W. Zhu, Phys. Lett. B594 (2001) 235.