How Riemannian Manifolds Converge

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Dedicated to Jeff Cheeger for his 65th birthday

Abstract. This is an intuitive survey of extrinsic and intrinsic notions of convergence of manifolds complete with pictures of key examples and a discussion of the properties associated with each notion. We begin with a description of three extrinsic notions which have been applied to study sequences of submanifolds in Euclidean space: Hausdorff convergence of sets, flat convergence of integral currents, and weak convergence of varifolds. We next describe a variety of intrinsic notions of convergence which have been applied to study sequences of compact Riemannian manifolds: Gromov-Hausdorff convergence of metric spaces, convergence of metric measure spaces, Intrinsic Flat convergence of integral current spaces, and ultralimits of metric spaces. We close with a speculative section addressing possible notions of intrinsic varifold convergence, convergence of Lorentzian manifolds and area convergence.

1. Introduction

The strong notions of smoothly or Lipschitz converging manifolds have proven to be exceptionally useful when studying manifolds with curvature and volume bounds, Einstein manifolds, isospectral manifolds of low dimensions, conformally equivalent manifolds, Ricci flow and the Poincare conjecture, and even some questions in general relativity. However many open questions require weaker forms of convergence that do not produce limit spaces that are manifolds themselves. Weaker notions of convergence and new notions of limits have proven necessary in the study of manifolds with no curvature bounds or only lower bounds on Ricci or scalar curvature, isospectral manifolds of higher dimension, Ricci flow through singularities, and general relativity. Here we survey a variety of weaker notions of convergence and the corresponding limit spaces covering both well established concepts, newly discovered ones and speculations.

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We begin with the convergence of submanifolds of Euclidean space as there is a wealth of different weak kinds of convergence that mathematicians have been applying for almost a century. Section 2.1 covers the Hausdorff convergence of sets, Section 2.2 covers Federer-Fleming’s flat convergence of integral currents, and Section 2.3 covers Almgren’s weak convergence of varifolds. Each has its own kind of limits and preserves different properties. Each is useful for exploring different kinds of problems. While Hausdorff convergence is most well known in the study of convex sets, flat convergence in the study minimal surfaces and varifold convergence in the study of mean curvature flow, each has appeared in diverse applications. Keep in mind that these are extrinsic notions of convergence which depend upon how a manifold is located within the extrinsic space.

It is natural to believe that each should have a corresponding intrinsic notion of convergence which should prove useful for studying corresponding intrinsic questions about Riemannian manifolds which do not lie in a common ambient space. The second section vaguely describes some of these intrinsic notions of convergence with pictures illustrating key examples. Section 3.1 covers Gromov-Hausdorff convergence of metric spaces which is an intrinsic Hausdorff distance, Section 3.2 covers various notions of convergence of metric measure spaces, Section 3.3 covers the intrinsic flat convergence on integral current spaces, and Section 3.4 covers the weakest notion of all: ultralimits of metric spaces. Illustrated definitions and key examples will be given for each kind of convergence. This survey is not meant to provide a thorough rigorous definition of any of these forms of convergence but rather to provide the flavor of each notion and direct the reader to further resources.

The survey closes with speculations on new notions of convergence: Section 4.1 describes difficulties arising when attempting to define an intrinsic varifold convergence. Section 4.2 describes the possible notion of area convergence and area spaces. Section 4.3 discusses the importance of weaker notions of convergence for Lorentzian manifolds. The author attempts to include all key citations of initial work in these directions.

The author apologizes for the necessarily incomplete bibliography and encourages the reader to consult mathscinet and the arxiv for the most recent results in each area. For some of the older forms of convergence entire textbooks have been written focussing on one application alone.

2. Converging Submanifolds of Euclidean Space

There are numerous textbooks written covering the three notions of extrinsic convergence we describe in this section. Textbooks covering all three notions include [48] and [45]. Morgan’s text providing pictures and overviews of proofs with precise references to theorems and proofs in Federer’s classic text [28]. The focus is on minimal surface theory. Another source with many pictures and intuition regarding Plateau’s problem, is Almgren’s classic textbook [6]. Lin-Yang’s textbook [45] covers a wider variety of applications and
provides a more modern perspective incorporating recent work of Ambrosio-Kirchheim. An excellent resource on varifolds is Brakke’s book [11] which is freely available on his webpage. Simon’s classic text [57] is another indispensable resource.

2.1. Hausdorff Convergence of Sets

The notion of Hausdorff distance dates back to the early 20th century. Here we define it on an arbitrary metric space, \((Z, d_Z)\), although initially it was defined on Euclidean space. The Hausdorff distance between subsets \(A_1, A_2 \subseteq Z\) is

\[
d_H^2(A_1, A_2) := \inf \left\{ R : A_1 \subseteq T_R(A_2), \ A_2 \subseteq T_R(A_1) \right\},
\]

where \(T_r(A) := \{ y : \exists x \in A \text{ s.t. } d_Z(x, y) < r \}\). We write \(A_j \xrightarrow{H} A\) iff \(d_H^2(A_j, A) \to 0\).

![Figure 1. Hausdorff Convergence](image)

In Figure 1, we see two famous sequences of submanifolds converging in the Hausdorff sense. The sequence \(A_i\) are tori which converge to a circle \(A\), depicting how a sequence may lose both topology and dimension in the limit. Sequence \(B_i\) of jagged paths in the \(yz\) plane converges to a straight segment, \(B\), in the \(y\) axis. Notice that \(L(B_i) = \sqrt{2}\) while \(L(B) = 1\). In addition to a sudden loss of length under a limit, we lose all information about the derivative (much like a \(C_0\) limit). In fact one may construct a sequence of jagged curves that converge in the Hausdorff sense to the nowhere differentiable Weierstrass function.

One property which is preserved by Hausdorff convergence is convexity. For this reason, Hausdorff convergence has often been applied in the study of convex subsets of Euclidean space.

In 1916, Blaschke proved that if a sequence of nonempty compact sets \(K_i\) lie in a ball in Euclidean space, then a subsequence converges in the Hausdorff sense to a nonempty compact set [9]. If the sets are connected, then their limit is connected. Path connectedness, however, is not preserved as can be seen by a sequence approaching the well known set:

\[
\{(x, \sin(1/x)) : x \in (0, 1]\} \cap \{(0, y) : y \in [-1, 1]\}.
\]
Dimension and measure are also not well controlled. The $k$ dimensional Hausdorff measure of a set $X$ is defined by covering $X$ with countable collections of sets $C_i$ of small diameter:

$$H_k(X) := \lim_{r \to 0} \inf \left\{ \sum_{i=1}^{\infty} \alpha(k) \left( \frac{\text{diam}(C_i)}{2} \right)^k : X \subset \bigcup_{i=1}^{\infty} C_i, \, \text{diam}(C_i) < r \right\}$$

where $\alpha(k)$ is the volume of a unit ball of dimension $k$ in Euclidean space.

For a beautiful exposition of this notion see [48]. When $X$ is a submanifold of dimension $k$, then $H_k(X)$ is just the $k$ dimensional Lebesgue measure: when $k = 1$ it is the length, when $k = 2$ it is the area and so on.

The Hausdorff dimension of a set, $X$, is

$$H_{\text{dim}}(X) := \inf\{k \in (0, \infty) : H_k(X) = 0\}.$$  \hfill (3)

A compact $k$-dimensional submanifold, $M^k$, has $H_{\text{dim}}(M) = k$. Notice that in Figure 1 $H_{\text{dim}}(A_j) = 2$ while $H_{\text{dim}}(A) = 1$. It is also possible for the dimension to go up in the limit. In Figure 2 $H_{\text{dim}}(Y_j) = 2$ while $H_{\text{dim}}(Y) = 3$. It is also possible for a sequence of 1 dimensional submanifolds, to converge to a space of fractional dimension like the von Koch curve. Such a sequence of curves must have length diverging to infinity.

In fact, if $X_j$ have uniformly bounded length, $H_1(X_j) < C$, and $X_j$ converge to $X$ in the Hausdorff sense, then

$$H_1(X) \leq \liminf_{j \to \infty} H_1(X_j).$$  \hfill (4)

Note that the sequence $B_j$ depicted in Figure 1 has $H_1(B_j) = \sqrt{2}$ for all $j$, but $H_1(B) = 1$, so lower semicontinuity is the best one can do. This is not true for higher dimensions as can be seen in Figure 2 where $\lim_{j \to \infty} H_2(Y_j) = 1$ yet $H_2(Y) = \infty$.

Figure 2 depicts a famous example of “a disk with splines” described in Almgren’s text on Plateau’s problem [6]. In Plateau’s problem one is given a closed curve and is asked to find the surface of smallest area spanning that curve. In this example, the sequence of smooth surfaces, $Y_j$, have a common boundary, $\partial Y_j = \{(x, y, 0) : x^2 + y^2 = 1\}$ which is a given closed curve, and the area of the $Y_j$ is approaching the minimal area filling that circle. However the sequence does not converge in the Hausdorff sense to the flat disk, $D^2$, which is the solution to the Plateau problem for a circle. Instead, due to
the many splines, the sequence converges in the Hausdorff sense to the solid cylinder. The disk with many splines example convinced mathematicians that the Hausdorff distance was not useful in the study of minimal surfaces. New notions of convergence had to be defined.

2.2. Flat Convergence of Integral Currents

Federer and Fleming introduced the notion of an integral current and the flat convergence of integral currents to deal with examples like the disk with many splines depicted in Figure 2. When viewed as integral currents, the submanifolds depicted in that figure converge in the flat sense to the disk. All the splines disappear in the limit.

A current, $T$, is a linear functional on smooth $k$ forms. Any compact oriented submanifold, $M$, of dimension $k$ with a smooth compact boundary, may be viewed an a $k$ dimensional current, $T$, defined by

$$T(\omega) := \int_M \omega. \quad (5)$$

Notice that in Figure 2 we have

$$\lim_{j \to \infty} \int_{Y_j} \omega = \int_{D^2} \omega \quad (6)$$

for any smooth differentiable 3 form, $\omega$. So, viewed as currents, $Y_j$ converge weakly to the flat disk $D^2$.

Federer-Fleming proved that any sequence of compact $k$ dimensional oriented submanifolds, $M_j$, in a disk in Euclidean space, with a uniform upper bound on $\mathcal{H}^k(M_j) \leq V_0$ and a uniform upper bound on $\mathcal{H}^{k-1}(\partial M_j) \leq A_0$, has a subsequence which converges when viewed as currents in the weak sense. The limit is an “integral current”.

An integral current, $T$, is a current with a canonical set, $R$, and a multiplicity function $\theta$. The canonical set $R$ is countably $\mathcal{H}^k$ rectifiable, which means it is contained in the image of a countable collection of Lipschitz maps, $\varphi_i : E_i \to R$ from Borel subsets, $E_i$, of $k$ dimensional Euclidean space. The multiplicity function $\theta$ is an integer valued Borel function. We define

$$T(\omega) := \int_R \theta \omega = \sum_{i=1}^{\infty} \int_{E_i} \theta \circ \varphi_i \varphi_i^* \omega. \quad (7)$$

It is further required that an integral current have finite mass

$$M(T) := \int_R \theta \, d\mathcal{H}^k \quad (8)$$

and that the boundary, $\partial T$, defined by $\partial T(\omega) := T(d\omega)$ have finite mass, $M(\partial T) < \infty$. Note that Federer-Fleming and Ambrosio-Kirchheim proved
this implies that $\partial T$ also has a rectifiable canonical set although it is one dimension lower than the dimension of $T$.\footnote{See \cite{48} and \cite{45} for more details. Note there are slight differences in the definition as \cite{48} follows Federer-Fleming \cite{29}, while \cite{45} follows the newer version introduced by Ambrosio-Kirchheim \cite{7}.}

When a submanifold $M$ is viewed as an integral current $T$, then $M$ itself is the canonical set, it has multiplicity 1, the boundary, $\partial T$, is just $\partial M$ viewed as an integral current and the mass, $M(T)$, is just the volume of $M$.

Also included as a $k$ dimensional integral current is the 0 current. In Figure 1 the sequence $A_j$ may be viewed as integral currents. They converge in the weak sense as integral currents to the 0 current. More generally, if $\mathcal{H}^k(M_j)$ decreases to zero, then the weak limit of the $k$ dimensional submanifolds $M_j$ viewed as integral currents is also the 0 current. In fact, whenever $M_j$ have a uniform upper bound on $\mathcal{H}^k(M_j)$ and $M_j = \partial N_j$ where $\mathcal{H}^{k+1}(N_j) \to 0$, then $M_j$ viewed as integral currents also converge weakly to the 0 current. The sequence $A_j$ depicted in Figure 1 are the boundaries of solid tori whose volumes decrease to 0. It was not actually necessary that their areas decrease to 0. This idea of filling in the manifold to assess where it converges makes it much easier to see the limits of integral currents and leads naturally to the following definition.

The flat distance between two $k$ dimensional integral currents, $T_1$ and $T_2$, is defined by

$$d_F(T_1, T_2) = \inf \left\{ M(A) + M(B) : \text{int curr } A, B \text{ s.t. } A + \partial B = T_1 - T_2 \right\}, \tag{9}$$

where the infimum is taken over all $k$ dimensional integral currents, $A$, and all $k + 1$ dimensional integral currents, $B$.\footnote{See prior footnote.}

In the figure to the right we see a choice of $A$ with small area that looks like a catenoid, and then we fill in the space between the flat disk $T_2$ and the disk with many splines, $T_1$, to define $B$. If the surface with the splines actually shares the same boundary with the disk (as it does in Figure 2), then we can take $A$ to be the 0 current and $B$ just to be a filling (the region in between them). It is easy to see that the sequence of manifolds $Y_j$ depicted in Figure 2 converges to the disk $D^2$ using the flat distance.

In Figure 1 the sequence $B_j$ viewed as 1 dimensional integral currents, $T_j$, converges to the Hausdorff limit $B$ viewed as a one dimensional current, $T$ (as long as they are given the same orientation left to right). Here $T_j - T$ is again a cycle and we can find surfaces viewed as 2 dimensional currents, $S_j$, such that $\partial S_j = T_j - T$ whose areas, $M(S_j)$, converge to 0.
Federer-Fleming proved that when a sequence of integral currents has a uniform upper bound on mass and on the mass of their boundaries, then they converge weakly iff they converge with respect to the flat distance. Their compactness theorem can now be restated as follows: if $T_j$ is a sequence of $k$ dimensional integral currents supported in a compact subset with $M(T_j) \leq V_0$ and $M(\partial T_j) \leq A_0$ then a subsequence converges in the flat sense to an integral current space.

One of the beauties of this theorem is that the limit space is rectifiable with finite mass, and in fact the mass is lower semicontinuous. This makes flat convergence an ideal notion when studying Plateau’s problem. The limit integral current has the same boundary as the sequence and minimal area.

In addition integral currents have a notion of an approximate tangent plane, which exists almost everywhere and is a subspace of the same dimension as the current. When $M$ is a submanifold, this approximate tangent plane is the usual tangent plane and exists everywhere inside $M$. In 1966 Almgren was able to apply the strong control on the approximate tangent planes to obtain even stronger regularity results for the limits achieved when working on Plateau’s problem [4]. The limit space ends up having multiplicity 1. In general the limit space may have higher multiplicity or regions with higher multiplicity [Figure 3], which we will discuss in more detail later.

In 1999, Ambrosio-Kirchheim extended the notion of an integral current from Euclidean space to arbitrary metric spaces, proving the existence of a solution to Plateau’s problem on Banach spaces [7]. A key difficulty was that on a metric space, $Z$, there is no notion of a differential form. They applied a notion of DeGiorgi [27], replacing a $k$ dimensional differential form like $\omega = f dx_1 \wedge dx_2 \wedge \cdots \wedge dx_k$ with a $k+1$ tuple, $\omega = (f, x_1, x_2, ... x_k)$, of Lipschitz functions satisfying a few rules, including $d\omega := (1, f, x_1, x_2, ... x_k)$.

An integral current, $T$, is defined as a linear functional on $k+1$ tuples using a rectifiable set, $R$, and a multiplicity function, $\theta$, so that as in (7), we have

$$T(\omega) := \sum_{i=1}^{\infty} \int_{E_i} \theta \circ \varphi_i \varphi_i^* \omega. \quad (10)$$

where $\varphi_i^* \omega := f \circ \varphi d(x_1 \circ \varphi) \wedge d(x_2 \circ \varphi) \wedge \cdots \wedge d(x_k \circ \varphi)$ is defined almost everywhere by Rademacher’s Theorem. They can then define mass exactly as in [8] and boundary exactly as before as well. They require integral currents to have bounded mass and their boundaries to have bounded mass.

Ambrosio-Kirchheim prove that if $T_j$ are integral currents in a compact metric space and $M(T_j) \leq V_0$ and $M(\partial T_j) \leq A_0$ then a subsequence converges in the weak sense to an integral current. Furthermore the mass is lower semicontinuous. There is also a notion of an approximate tangent plane, however, here the approximate tangent plane is a normed space. The norm is defined using the metric differential described in earlier work by Korevaar-Schoen on harmonic maps and also by Kirchheim on rectifiable space [44][42].

\[3\]See [7] or [62] for the precise definition.
Wenger extended the class of metric spaces $Z$ which have a solution to Plateau’s problem and defined a flat distance exactly as in [9]. We will use the notation $d^Z$ for the flat distance in $Z$ to be consistent with $d^H$ denoting the Hausdorff distance in $Z$. He proved that on this larger class of spaces, which includes Banach spaces, when $M(T_j) \leq V_0$ and $M(\partial T_j) \leq A_0$ then $T_j$ converge weakly to $T$ iff $T_j$ converges in the flat sense to $T$ [69][70]. Recall that by Rademacher’s Theorem, any separable metric space, $Z$, isometrically embeds into a Banach space, $W$, so we can always study the convergence of integral currents in $Z$ using the flat distance of the push forwards of the currents in Banach space, $d^W$. If $\varphi : Z \rightarrow W$, then the push forward of a current $T$ in $Z$, is a current $\varphi_* T$ in $W$ defined by $\varphi_* T (f, x_1, \ldots, x_k) := T (f \circ \varphi, x_1 \circ \varphi, \ldots, x_k \circ \varphi)$.

Another extension of the notion of integral currents is that of the $Z_n$ integral currents where the multiplicity function takes values in $Z_n$. This was introduced by Fleming [32] and has been extended to arbitrary metric spaces by Ambrosio and Wenger [8]. One application for this notion is the study of minimal graphs (c.f. [6]). To understand this better we will go over an example using Federer-Fleming’s notation.

In Figure 3, the sequence of embedded curves $C_i$ converges in $C^1$ to a limit curve $C$ which is not embedded. Viewed as 1 dimensional currents, $T_i$, they converge in the flat sense to $T$. It is perhaps easier to understand Figure 3 using weak convergence rather than flat convergence. Note that

$$T_i (f_1(x, y) dx + f_2(x, y) dy) = \int_0^1 (f_1 \circ C_i)(t) d(x \circ C_i) + (f_2 \circ C_i)(t) d(y \circ C_i)$$

converges to

$$T (f_1(x, y) dx + f_2(x, y) dy) = \int_0^1 (f_1 \circ C)(t) d(x \circ C) + (f_2 \circ C)(t) d(y \circ C)$$

Notice that $T$ has multiplicity 1 where $C$ does not overlap itself. It has multiplicity 2 on the segment $C$ covers twice in the same direction. The segment depicted with dashes where $C$ passes in opposite directions is not part of the canonical set of $T$ because the integration cancels there. If one uses $Z_n$ valued coefficients we get the same limit space for $n > 2$, but for $n = 2$ the doubled up segment disappears as well as the cancelled dashed segment.

![Figure 3. Doubling and Cancellation of Flat Limits](image)

The cancellation depicted in Figure 3 is problematic for some applications. Note that the same effect can occur in higher dimensions, by extending the curve to a sheet in $\mathbb{R}^3$. While often it is useful for thin splines to disappear
in the hopes of preserving dimension, canceling sheets which overlap with opposing orientations is not necessary to obtain a rectifiable limit space. This is seen using the notion of a varifold.

2.3. Weak Convergence of Varifolds

Varifolds were introduced by Almgren in 1964 [5]. Significant further work was completed by Allard in 1972 [3]. Almgren’s goal was to define a new notion of convergence for submanifolds which had rectifiable limit spaces and a notion of tangent planes for those limit spaces, but did not have the kind of cancellation that occurs when taking flat limits of integral currents. Under varifold convergence, the sequence of 1 dimensional manifolds depicted in Figure 3 converges to the rectifiable set with weight 1 everywhere including the dashed segment. There is no notion of orientation on the limit, but there are tangent planes almost everywhere.

A $k$ dimensional varifold is a Radon measure on $\mathbb{R}^N \times \Gamma(k,N)$ where $\Gamma(k,N)$ is the space of $k$ subspaces of Euclidean space, $\mathbb{R}^N$. A submanifold $M^k \subset \mathbb{R}^n$, may be viewed as the varifold, $V$, defined on any $W \subset \mathbb{R}^N \times \Gamma(k,N)$ as follows:

$$V(W) := \mathcal{H}^k (W \cap \{(x,T_xM) : x \in M \subset \mathbb{R}^n\}),$$

(11)

where $T_xM$ is the tangent space to $x$ translated to the origin.

A sequence of varifolds, $V_j$, is said to converge weakly to a varifold, $V$,

$$\int f dV_j \to \int f dV \quad \forall f \in C_0(\mathbb{R}^N \times \Gamma(k,N)).$$

(12)

For example, let us examine the sequence of curves $C_j$ which converge $C^1$ to the curve $C$ in Figure 3. They may be viewed as integral currents $V_j$,

$$V_j(W) = \mathcal{H}^1 (W \cap \{(C_j(t), \pm C_j'(t)/|C_j'(t)|) : t \in [0,1]\}),$$

(13)

where we view points in $\Gamma(1,2)$ as $\pm v$ where $v$ in a unit 2 vector. Then

$$\int f(x,v) dV_j = \int_0^1 f(C_j(t), \pm C_j'(t)/|C_j'(t)|) dt$$

(14)

converges to $\int_0^1 f(C(t), \pm C(t)/|C(t)|) dt = \int f(x,v) dV$ for the varifold $V$ defined by

$$V(W) := \int_0^1 \chi_W(C(t), \pm C'(t)/|C'(t)|) dt$$

(15)

where $\chi_W$ is the indicator function of $W$. This varifold corresponds to viewing the limit curve $C$ as having weight 1 on the segments where it doesn’t overlap itself and weight 2 on the segments where it does overlap. There is no cancellation. The dashed segment has weight 2.

Not all limit varifolds end up with tangent planes that align well with the rectifiable set as they do in [15]. If we examine instead the sequence $B_j$
in Figure 1 as a sequence of curves $C_j$ converging to a curve $C$, we see that the limit varifolds, $V$, has the form

$$V(W) := a \int_0^1 \chi_W(C(t), \pm av_1) dt + a \int_0^1 \chi_W(C(t), \pm av_2) dt$$

where $v_1 = (1, 1), v_2 = (-1, 1)$ and $a = \sqrt{2}/2$ because $C_j'(t)$ is always in the direction of $v_1$ or $v_2$. This effect was observed by Young and is discussed at length in [45]. Varifolds like $V$ are of importance but are not considered to be integral varifolds because $C_j(t)$ is unrelated to $v_1$ and $v_2$.

An integral varifold, $V$, is defined as a positive integer weighted countable sum of varifolds, $V_j$, which are defined by embedded submanifolds $M_j$ as in (11). While $V$ is a measure on $\mathbb{R}^N$, when $V$ is defined by a submanifold, $M$, then $||V||(A) = \mathcal{H}^k(A \cap M)$.

A $k$ dimensional varifold $V$ is said to have a tangent plane, $T \in \Gamma(k, N)$ with multiplicity $\theta \in (0, \infty)$ at a point $x \in \mathbb{R}^N$ if a sequence of rescalings of $V$ about the point $x$ converges to $\theta(x) T$. An integral varifold, $V$, has a tangent space $||V||$ almost everywhere on $\mathbb{R}^N$.

Varifolds do not have a notion of boundary. Instead a varifold, $V$, has a notion of first variation, $\delta V$, which is a functional that maps functions $f \in C_0^\infty(\mathbb{R}^N, \mathbb{R}^n)$ to $\mathbb{R}$. There is no room here to set up the background for a precise definition so we just describe $\delta V$ when $V$ is defined by a $C^2$ submanifold $M$ with boundary $\partial M$. In this case, $\delta V(f)$ is the first variation in the area of $M$ as it is flowed through a diffeomorphism defined by the vector field $f$:

$$\delta V(f) = -\int_M f(x) \cdot H(x) d\mathcal{H}^k(x) + \int_{\partial M} f(x) \cdot \eta(x) d\mathcal{H}^{k-1}(x)$$

where $H(x)$ is the mean curvature at $x \in M$ and $\eta(x)$ is the outward normal at $x \in \partial M$.

A varifold, $V$, is said to be stationary when $\delta V = 0$.

A varifold has bounded first variation if

$$||\delta V||(W) := \sup\{\delta V(f) : f \in C_0^\infty(\mathbb{R}^N, \mathbb{R}^n), |f| \leq 1, \text{spt}(f) \subset A\} < \infty.$$  

When $V$ corresponds to a submanifold, $M$, as above, then

$$||\delta V||(W) = \mathcal{H}^{k-1}(W \cap \partial M) + \int_{W \cap M} |H(x)| d\mathcal{H}^n(x).$$

There is an isoperimetric inequality for varifolds.

Allard proved that if a sequence of integral varifolds, $V_j$, with a uniform bound on $||\delta V_j||(W)$ depending only on $W$, converges weakly to $V$, then $V$ is an integral varifold as well. In particular a weakly converging sequence of minimal surfaces, $M_j$, with a uniform upper bound on the length of their

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5See [45] for a more precise definition.
6See [45] 6.2 for the full definition.
boundaries, $\partial M_j$, converges to an integral varifold. For more general submanifolds, one needs only uniformly control the volumes of the boundaries and the $L^1$ norms of the mean curvatures.

One key advantage of varifolds is that they have a notion of mean curvature defined using $\delta V$ and an integral similar to the one in [18]. This notion is used to define the Brakke flow, a mean curvature flow past singularities [11].

Recently White has set up a natural map, $F$, from integral varifolds to $\mathbb{Z}_2$ flat chains, which basically preserves the rectifiable set and takes the integer valued weight to a $\mathbb{Z}_2$ weight. Each $\mathbb{Z}_2$ flat chain corresponds uniquely to a $\mathbb{Z}_2$ integral current. He has proven that if a sequence of submanifolds, $M_j$ viewed as varifolds converge weakly to an integral varifold $V$ and satisfy the conditions of Allard’s compactness theorem, and if further $\partial M_j$ converge as $\mathbb{Z}_2$ integral currents, then $M_j$ viewed as $\mathbb{Z}_2$ integral currents converge to a $\mathbb{Z}_2$ integral current $T$ corresponding to $F(V)$. This is easily seen to be the situation for the sequence depicted in Figure 3 [12].

3. Intrinsic Convergence of Riemannian Manifolds

When studying sequences of Riemannian manifolds, $M_j$, the strongest notions of convergence require that $M_j$ be diffeomorphic to the limit space $M$ with the metrics, $g_j$, converging smoothly:

$$\exists \varphi_j : M \rightarrow M_j \text{ such that } \varphi_j^* g_j \rightarrow g.$$  
(21)

In this survey we are concerned with sequences of Riemannian manifolds which do not converge in such a strong sense. Here we describe a few weaker notions of convergence which allow us to better understand sequences which do not converge strongly. We begin with a pair of motivating examples.

The sequence of flat tori, $M_j = S^1_\pi \times S^1_\pi / j$, has volume converging to 0:

$$\text{vol}(M_j) = 2\pi (2\pi / j) \rightarrow 0.$$  
Sequences with this property are called collapsing sequences. They do not converge in a strong sense to a limit which is also a torus. Intuitively one would hope to define a weaker notion of convergence in which these tori converge to a circle. Examples like this lead to Gromov’s notion of an intrinsic Hausdorff convergence.

If one views Figure 2 as a sequence of Riemannian disks with splines, $M_j$, with the induced Riemannian metrics, they do not converge in a strong sense to a flat Riemannian disk. While they are diffeomorphic to the disk and the volumes are converging, $\text{vol}(M_j) \rightarrow \text{vol}(D^2)$, the metrics do not converge smoothly. Examples like this lead to the notion of intrinsic flat convergence.

In this section we first present Gromov-Hausdorff convergence, then metric measure convergence, then the intrinsic flat convergence and finally, weakest of all, the notion of an ultralimit. We include a few key examples, applications and further resources for each notion.
3.1. Gromov-Hausdorff Convergence of Metric Spaces

In 1981, Gromov introduced an intrinsic Hausdorff convergence for sequences of metric spaces [39]. A few excellent references are Gromov’s book [40], the textbook of Burago-Burago-Ivanov [15], Fukaya’s survey [36] and Bridson-Haefliger’s book [12].

The Gromov-Hausdorff distance is defined between any pair of compact metric spaces,

\[ d_{GH}(M_1, M_2) = \inf \left\{ d_Z(\varphi_1(M_1), \varphi_2(M_2)) : \text{isom } \varphi_i : M_i \to Z \right\} \]  \hspace{1cm} (22)

where the infimum is taken over all metric spaces, \( Z \), and all isometric embeddings, \( \varphi_i : M_i \to Z \). An isometric embedding, \( \varphi : X \to Z \), satisfies

\[ d_Z(\varphi(x_1), \varphi(x_2)) = d_X(x_1, x_2) \quad \forall x_1, x_2 \in X. \]  \hspace{1cm} (23)

We write \( M_i \xrightarrow{GH} X \) iff \( d_{GH}(M_j, X) \to 0 \). See Figure 4.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{gromov_hausdorff_convergence.png}
\caption{Gromov-Hausdorff Convergence}
\end{figure}

The sequences of Riemannian manifolds depicted in Figure 4 reveal a variety of properties that are not conserved under Gromov-Hausdorff convergence. The first sequence \( A_j \) are the flat tori \( S^1_\pi \times S^1_{\pi/j} \) converging to a circle \( A = S^1_\pi \). To see this one takes the common space \( Z_j = A_j \) and isometrically embeds \( A \) into \( Z_j \) so that

\[ d_{GH}(A_j, A) \leq d_Z^j(A_j, \varphi_j(A)) = \pi/(2j) \to 0. \]  \hspace{1cm} (24)
Here we see the topology and Hausdorff dimension may decrease in the limit.

Note that if $M_1$ and $M_2$ are compact then $d_{GH}(M_1, M_2) = 0$ iff $M_1$ and $M_2$ are isometric. The Gromov-Hausdorff distance between $M_1$ and $M_2$ is almost 0 iff there is an almost isometry $f : M_1 \to M_2$ satisfying

$$|d_1(x, y) - d_2(f(x), f(y))| < \epsilon$$

and $M_2 \subset T_\epsilon(f(M_1)).$ (25)

Note that an almost isometry need not be continuous. In Figure 4 it is easy to construct $\epsilon_j$ almost isometries, $\varphi_j : B \to B_j$, such that $\epsilon_j \to 0$ and conclude that $B_j \xrightarrow{GH} B$. This example reveals that the Gromov-Hausdorff limit of a sequence of Riemannian manifolds may not be a Riemannian manifold.

Gromov-Hausdorff limits of Riemannian manifolds are geodesic metric spaces. This means that the distance between any pair of points is equal to the length of the shortest curve between them. The shortest curve exists and is called a minimal geodesic. As in Riemannian geometry, a curve $\gamma$ is called a geodesic if for every $t$, there is an $\epsilon > 0$ such that $\gamma$ restricted to $[t - \epsilon, t + \epsilon]$ is a minimal geodesic.

In the third sequence of Figure 4, $C_j$ are also the boundaries of increasingly thin tubular neighborhoods. The limit, $D$, is the Hawaii Ring, a metric space of infinite topological type that has no universal cover. For more about the topology of Gromov-Hausdorff converging sequences of manifolds see [60] [61] and [54].

In the last sequence of Figure 4, $D_j$ are the smoothed boundaries of increasing thin tubular neighborhoods of increasingly dense grids. They converge to a square $D = [0, 1] \times [0, 1]$. The metric on the square is the taxicab metric (also called the $l_1$ metric):

$$d_C((x_1, y_1), (x_2, y_2)) := |x_1 - x_2| + |y_1 - y_2|. \quad (26)$$

This is easiest to see by showing the $D_j$ are Gromov-Hausdorff close to their grids and that the grids converge to the taxicab square.

Gromov proved that sequences of Riemannian manifolds, $M_j$, with uniform upper bounds on their diameter and on the number, $N(r)$, of disjoint balls of radius $r$ have subsequences which converge in the Gromov-Hausdorff sense to a compact geodesic space, $Y$. Conversely, when $M_j$ converge to a compact $Y$, $N(r)$ is uniformly bounded. Consequently, if one views the sequence of disks with splines depicted in Figure 2 as Riemannian manifolds (with the intrinsic distance), the sequence does not converge in the Gromov-Hausdorff sense: a ball of radius $1/2$ about the tip of a spline does not intersect with the a ball of radius $1/2$ about the tip of another spline. As the number of splines approaches infinity so does the number of disjoint balls of radius $1/2$.

By the Bishop-Gromov volume comparison theorem, sequences of manifolds, $M_j$, with uniform lower bounds on Ricci curvature and upper bounds on diameter satisfy these compactness criteria [39]. This includes, for example, the sequence of flat tori, $A_j$, in Figure 4 as well as any other collapsing

7 Alexandrov space geometers use the term “geodesic” to refer to a “minimal geodesic” and “local geodesic” to refer to a geodesic.
sequence of manifolds with bounded sectional curvature. This lead to a series of papers on the geometric properties of $M_j$ with uniformly bounded sectional curvature and $\text{vol}(M_j) \to 0$ by Cheeger-Gromov, Fukaya, Rong, Shioya-Yamaguchi and others \[35\] \[24\] \[25\] \[55\] \[56\] (c.f. \[53\] \[49\]). Collapsing Riemannian manifolds with boundary have been studied by Alexander-Bishop and Cao-Ge \[17\] \[1\] \[2\]. This collapsing theory is an essential component of Perelman’s proof of the Geometrization Conjecture using Hamilton’s Ricci flow \[49\] \[43\] \[16\].

In 1992, Greene-Petersen applied the notion of Gromov’s filling volume to find a lower bound on the volume of a ball in a Riemannian manifold with a uniform geometric contractibility function. A geometric contractibility function is a function $\rho : (0, r_0] \to (0, \infty)$ with $\lim_{r \to 0} \rho(r) = 0$ such that any ball $B_p(r) \subset M$ is contractible in $B_p(\rho(r)) \subset M$. Applying Gromov’s compactness theorem, one may conclude that a sequence $M_j$ with a uniform geometric contractibility function and a uniform upper bound on volume, has a subsequence that converges in the Gromov-Hausdorff sense to a compact metric space. Note that in this setting, volume is uniformly bounded below so the sequence is not collapsing \[37\]. The limits of such sequences of spaces have been studied by Ferry, Ferry-Okun, Schul-Wenger and Sormani-Wenger \[30\] \[31\] \[63\] and \[62\].

Noncollapsing sequences of Riemannian manifolds with Ricci curvature bounded from below were studied by Colding and Cheeger-Colding in their work on almost rigidity in which they weakened the conditions of well known rigidity theorems. One example of a rigidity theorem is the fact that if $M^m$ has Ricci $\geq (m - 1)$ and $\text{vol}(M^m) = \text{vol}(S^m)$ then $M^m$ is isometric to $S^m$. Colding proved a corresponding almost rigidity theorem which states that if $M^m$ has Ricci $\geq (m - 1)$ then $\forall \epsilon > 0, \exists \delta_{m, \epsilon} > 0$ such that $\text{vol}(M^m) \geq \text{vol}(S^m) - \delta_{m, \epsilon}$ implies $d_{GH}(M^m, S^m) < \epsilon$. For a survey of such almost rigidity theorems see \[26\] \[20\] and Section 8 of \[59\]. The proofs generally involve an explicit construction of an almost isometry using distance functions and solutions to elliptic equations on the Riemannian manifolds.

When a sequence of $M_j$ with uniform lower bounds on Ricci curvature does not collapse Cheeger-Colding proved the volume is continuous and the Laplace spectrum converges. Fukaya observed that when a sequence of Riemannian manifolds $A_i$ and $B_i$ with nonnegative Ricci curvature converging to the same limit space $A = B = [0, 1]$ with the standard metric. The eigenvalues converge to real numbers,

$$\lambda_i^A := \lim_{j \to \infty} \lambda_i(A_j) \text{ and } \lambda_i^B := \lim_{j \to \infty} \lambda_i(B_j) \quad (27)$$

but $\lambda_i^A \neq \lambda_i^B$. While $\lambda_i^B$ made sense as eigenvalues for $[0, 1]$, the other collection of numbers $\lambda_i^A$ did not. The alternating sequence $\{A_1, B_1, A_2, B_2, \ldots\}$ also converges in the Gromov-Hausdorff sense but the eigenvalues do not converge at all. \[33\].
This example disturbed Fukaya in light of the successful isospectral compactness theorems of Osgood-Phillips-Sarnak, Brooks-Perry-Petersen and Chang-Yang [52] [13] [18] all of which imposed stronger conditions on the manifolds and involved smooth convergence or even conformal convergence of the sequences. This lead to Fukaya’s notion of metric measure convergence.

3.2. Metric Measure Convergence of Metric Measure Spaces

In 1987, Fukaya introduced the first notion of metric-measure convergence. A sequence of metric measure spaces $(X_j, d_j, \mu_j)$ converge in the metric measure sense to a metric measure space $(X, d, \mu)$ if there is a sequence of $1/j$ almost isometries, $f_j : X_j \to X$, such that push forwards of the measures, $f_j \ast \mu_j$, converge weakly to $\mu$ on $X$. Recall that $\varphi \# \mu(A) := \mu(\varphi^{-1}(A))$.

In Figure 5, $A_j$ and $B_j$ are given probability measures, $\mu_{A_j}$ and $\mu_{B_j}$ proportional to $H^2$. Then $B_j$ converge in the metric measure sense to $[0, 1]$ with the standard metric and $\mu_B(W) = H^1(W)$. Meanwhile $A_j$ converge in the metric measure sense to $[0, 1]$ with the measure

$$\mu_A(W) := \int_W (2 - |2 - 4x|) \, dx.$$ (28)

Defining the Laplacian with respect to these measures, the spectrum for $(A, \mu_A)$ is $\{\lambda_i^A\}$ and the spectrum for $(B, \mu_B)$ is $\{\lambda_i^B\}$.  

Cheeger-Colding then proved that any sequence of compact Riemannian manifolds with uniform lower bounds on Ricci curvature endowed with a probability measure proportional to the Hausdorff measure, has a subsequence which converges in the metric measure sense. The measures on the limit space satisfy the Bishop-Gromov comparison theorem and are therefore doubling [21]. Sturm and Lott-Villani extended the notion of a Ricci curvature bound to general metric measure spaces using mass transport [64] [46] (c.f. [68]). Recently Topping and others have been developing a notion of Ricci flow on this larger class of spaces in hopes of defining Ricci flow through a singularity [66].

Figure 5. Metric Measure Convergence

8See [34] or [23] for more details.
Cheeger-Colding also prove a Poincare inequality on these limit spaces and deduced that the eigenvalues converge \[22, 23\]. They proved that the limit spaces of a sequence of Riemannian manifolds with uniform lower bounds on Ricci curvature also have a notion of tangent plane almost everywhere. The tangent plane at a point is found by rescaling the space \( Y \) outward and taking a pointed Gromov-Hausdorff limit. At many points, one does not get a unique limit under rescaling and the limits are not necessarily planes or cones. However they do exist at every point and are called tangent cones. Cheeger-Colding proved tangent cones are Euclidean planes almost everywhere. In fact they prove \( Y \) is a countably \( H^m \) rectifiable space and is a \( C^{1,\alpha} \) manifold away from the singularities. Key steps in the proof involve the Splitting Theorem and the Poincare inequality (c.f. [19]). There is an example where the limit space has infinite topological type so it isn’t a \( C^{1,\alpha} \) manifold [47]. However, at least the manifold has a universal cover unlike the Hawaii Ring depicted in Figure 4 [60].

It is also natural to study metric measure convergence without Ricci curvature bounds on the sequence of manifolds. As long as the measure is doubling one can apply Gromov’s compactness theorem. Sometimes the measure on the limit space is supported on a smaller set. This occurs for example for the sequence \( B_j \) of Figure 4. Viewed as a metric measure limit space, the limit space \( B \) has a measure which is supported on the two spheres.

In 1981, Gromov introduced \( \Box_\lambda \) convergence to handle this issue [40]. When \( \lambda = 1 \), the limit of the sequence of \( B_j \) is just the two spheres with the line segment removed. Basically, if \((Y, d, \mu)\) is the metric measure limit of a sequence, and if there is a ball such that \( \mu(B_y(r)) = 0 \) then \( y \) is removed from the space. This leaves us with a new, smaller limit space, \((\hat{Y}, d_{\hat{Y}}, \mu)\), which may no longer be connected. This process may be called “reduction of measure”. The spaces are then no longer close in the Gromov-Hausdorff topology.

In 2005, Sturm introduced a new distance between metric measure spaces which leads to such limits more naturally and also interacts well with mass transport notions mentioned above [65]. He uses the Wasserstein distance, \( W_p \), to measure the distance between measures in a set \( Z \) and defines an intrinsic Wasserstein distance using an infimum over all metric spaces \( Z \) and all isometric embeddings \( \varphi_i : X_i \to Z \):

\[
d_{W_p}((X_1, d_1, \mu_1), (X_2, \mu_2, d_2)) := \inf\{d_{W_p}(\varphi_1 \# \mu_1, \varphi_2 \# \mu_2)\}. \tag{29}
\]

Sturm proved convergence with respect to this distance is equivalent to Gromov’s \( \Box_1 \) convergence [65].

Villani recently defined an intrinsic Prokhorov distance replacing the Wasserstein distance of order \( p \) in [29] with the Prokhorov distance between measures. As convergence with respect to the Wasserstein distance and with respect to the Prokhorov distance agree with weak convergence, the intrinsic Prokhorov and intrinsic Wasserstein limits agree as well. He refers to these kinds of convergence as “measure convergence” Two metric measure spaces
are a zero distance apart with respect to these intrinsic measure distances iff there is a measure preserving isometry between them [68].

Villani also described “metric measure distances”. The Gromov-Hausdorff Wasserstein distance is defined: \( d_{W_p}(X_1, d_1, \mu_1, X_2, d_2, \mu_2) := \) 

\[
\inf \{ d_H(\varphi_1(X_1), \varphi_2(X_2)) + d_{W_p}(\varphi_1#\mu_1, \varphi_2#\mu_2) \},
\]

where the infimum is taken over all metric spaces \( Z \) and all isometric embeddings \( \varphi_i : X_i \rightarrow Z \). The Gromov-Hausdorff Prokhorov distance has the same formula replacing the Wasserstein distance by the Prokhorov distance between measures. Convergence with respect to these metric measure distances agrees with Fukaya’s metric measure convergence. The advantage of having multiple distances is that one may be easier estimate than another [68].

A sequence of metric measure spaces with doubling measures has a converging subsequence and the metric measure limit agrees with the measure limit of a sequence [68]. Any sequence of Riemannian manifolds with lower bounds on their Ricci curvature satisfies the doubling condition by the Bishop-Gromov volume comparison theorem. The sequence of \( B_j \) in Figure 4 does not, as the metric measure and measure limits do not agree.

Despite the immense success in applying these definitions of convergence to study manifolds with Ricci curvature bounds, there has been a need to introduce a weaker form of convergence to study sequences of manifolds which do not satisfy these strong conditions. Mathematicians studying manifolds with scalar curvature bounds and those interested only in sequences with an upper bound on volume and diameter without curvature bounds, need a weaker version of convergence. In particular, geometric analysis related to cosmology and the study of the spacelike universe requires a weaker form of convergence.

An important example introduced by Ilmanen may be called the 3-sphere of many splines. It is a sequence of three dimensional spheres with positive scalar curvature whose volume converges to the volume of the standard three sphere, but has an increasingly dense set of thinner and thinner splines of “length” 1. Cosmologically, one may think of these splines as deep gravity wells. The sequence does not converge in the Gromov-Hausdorff sense because balls of radius \( 1/2 \) centered on the tip of each spline are disjoint and the number of such disjoint balls approaches infinity. This example naturally lead Sormani and Wenger to develop a notion of intrinsic flat convergence.

### 3.3. Intrinsic Flat Convergence of Integral Current Spaces

In 2008 Sormani and Wenger introduced the intrinsic flat distance between compact oriented Riemannian manifolds [63][62]: 

\[
d_F(M_1, M_2) := \inf \left\{ d_Z^F(\varphi_1#T_1, \varphi_2#T_2) : \text{isom } \varphi_i : M_i \rightarrow Z \right\}
\]

where the flat distance in \( Z \) is defined as in [9], the \( \varphi_i : M_i \rightarrow Z_i \) are isometric embeddings as in [23], and where \( T_i \) are defined by integration over \( M_i \) so
that \( \varphi_i \# T_i(f, x_1, \ldots x_k) = \)

\[
= T_i(f \circ \varphi_i, x_1 \circ \varphi_i, \ldots x_k \circ \varphi_i) = \int_{M_i} f \circ \varphi_i \ d(x_1 \circ \varphi_i) \wedge \cdots \wedge d(x_k \circ \varphi_i). \tag{32}
\]

One may immediately note that \( d_T(M_1, M_2) \leq \text{vol}(M_1) + \text{vol}(M_2) < \infty \) as we may always take the integral current \( A \) in (9) to be \( \varphi_1 \# T_1 - \varphi_2 \# T_2 \) and \( B = 0 \). The intrinsic flat distance is a distance between compact oriented Riemannian manifolds in the sense that \( d_T(M_1, M_2) = 0 \) iff there is an orientation preserving isometry between \( M_1 \) and \( M_2 \) \[62\].

Note that in practice it is often possible to estimate the intrinsic flat distance using only notions from Riemannian geometry. That is, if \( M^k \) and \( M^k \) isometrically embed into a Riemannian manifold \( N^{k+1} \) such that \( \partial N^{k+1} = \varphi_1(M^k) \cup \varphi_2(M^k) \) and the manifolds have been given an orientation consistent with Stoke’s theorem on \( N^{k+1} \), then \( d_T(M_1, M_2) \leq \text{vol}(N^{k+1}) \) This viewpoint makes it quite easy to see that Ilmanen’s 3 sphere of many splines example describes a sequence converging to the standard sphere. Note that one cannot just embed the sequence into four dimensional Euclidean space and take \( N^4 \) to be the flat region lying between the two spheres because such \( \varphi_i \) would not be isometric embeddings. Instead, one rotates each spline into half a thin 4 dimensional spline and glues it smoothly to a short \( S^3 \times [0, \epsilon_j] \) where \( \epsilon_j \) is “thinness” of the spline. This produces a four dimensional manifold \( N^4 \) with \( S^3 \) isometrically embedded as one boundary and the 3 sphere of many splines isometrically embedded as the other boundary. So the intrinsic flat distance between these two spaces is \( \leq \text{vol}(N^4) \) which is approximately \( \epsilon_j \text{vol}(S^3) \) plus the sum of the volumes of the four dimensional splines each of which is approximately \( \epsilon_j^3 \). See \[62\] for this example and many others.

The limit spaces are called integral current spaces, and are oriented weighted countably \( H^k \) rectifiable metric spaces. They have the same dimension as the sequence. They have a set of Lipschitz charts as well as a notion of an approximate tangent space. The approximate tangent space is a normed space whose norm is defined by the metric differential. The notion of the metric differential was developed in work of Korevaar-Schoen \[44\] and Kirchheim \[42\]. See \[62\] for a number of examples of limit spaces.

Integral current spaces are said to have a “current structure”, \( T \), defined by integration using the orientation and weight. One writes \((X, d, T)\). The current structure defines a mass measure \( ||T|| \). The points in \( x \) all have positive density with respect to this measure. These spaces have a notion of boundary coming from this current structure. The boundary is also an integral current space. \[7\]

An important integral current space is the 0 current space, and collapsing sequences of manifolds, \( \text{vol}(M_j) \to 0 \), converge to the 0 space. Notice that in Figure 4 the sequences \( A_j \), \( C_j \) and \( D_j \) all converge to 0 in the intrinsic

\[9\] See \[62\] for more details about the relationship between \( X \) and \( T \) and how to find the boundary.
flat sense. The sequence $B_j$ converges to a pair of spheres in the intrinsic flat sense where the limit does not include the line segment.

Most results about integral current spaces and intrinsic flat convergence are proven by finding a way to isometrically embed the entire sequence into a common metric space, $Z$, and apply the theorems for integral currents in $Z$. The rectifiability, dimension and boundary properties of the limits follows from Ambrosio-Kirchheim’s corresponding results on integral currents. So does the slicing theorem and the lower semicontinuity of mass. Combining Gromov’s compactness theorem with Ambrosio-Kirchheim’s compactness theorem, one sees that a sequence of manifolds which converge in the Gromov-Hausdorff sense that have a uniform upper bound on volume, also converge in the intrinsic flat sense [62].

In general the intrinsic flat and Gromov-Hausdorff limits do not agree: the intrinsic flat limit may be a strict subset of the Gromov-Hausdorff limit. If the Gromov Hausdorff limit is lower dimensional than the sequence, then the intrinsic flat limit is the $0$ space. Any regions in the Gromov-Hausdorff limit that are lower dimensional disappear from the intrinsic flat limit [62]. Note that this is in contrast with metric measure limits, which do not lose regions of lower dimension that still have positive limit measures. Such a situation can occur, for example, if a thin region collapsing to lower dimension is very bumpy and has a uniform lower bound on volume.

Sequences of Riemannian manifolds may also disappear due to cancellation. Recall that in Euclidean space, a submanifold which curves in on itself as in Figure 3 will have cancellation in the limit due to the opposing orientation. An example described in [63] is a sequence of three dimensional manifolds with positive scalar curvature created by taking a pair of standard three spheres and joining them by increasingly dense and increasingly thin and short tunnels. The Gromov-Hausdorff limit is a 3-sphere while the intrinsic flat limit is the $0$ current space. If each tunnel is twisted, then the intrinsic flat limit is the standard three sphere with multiplicity two.

In some cases the intrinsic flat limits and Gromov-Hausdorff limits agree giving new insight into the rectifiability of the Gromov-Hausdorff limits. They agree for noncollapsing sequences of manifolds with nonnegative Ricci curvature and also for sequences of manifolds with uniform linear contractibility functions and uniform upper bounds on volume [63]. As a consequence the Gromov-Hausdorff limits are countably $H^m$ rectifiable metric spaces. This was already shown by Cheeger-Colding for the sequences with bounded Ricci curvature but is a new result for the sequences with the linear geometric contractibility hypothesis. With only uniform geometric contractibility functions that are not linear, Schul and Wenger have shown the limits need not be so rectifiable [63] and when there is no upper bound on volume, Ferry has shown the limit spaces need not even be finite dimensional [30]. Recall that Cheeger-Colding have proven limits of manifolds with nonnegative Ricci curvature have Euclidean tangent cones almost everywhere. In contrast, there
is a sequence of Riemannian manifolds with uniform linear contractibility functions that converge to the taxicab space \( [62] \).

The key ingredient in the proofs of these noncancellation results is an estimate on the filling volumes of small spheres in the spaces. The filling volumes of the spheres are continuous with respect to the intrinsic flat distance and can be more easily applied to control sequence than the mass (or volume). It is conjectured that a sequence of three dimensional Riemannian manifolds with positive scalar curvature, no interior minimal surfaces and a uniform upper bound on volume and on the area of the boundary converges in the intrinsic flat sense without cancellation. Such manifolds are important in the study of general relativity. \( [63] \)

In fact, Wenger has proven that a sequence of oriented Riemannian manifolds with a uniform upper bound on diameter and on volume has a subsequence which converges in the intrinsic flat sense to an integral current space \( [71] \). The proof involves an even weaker notion of convergence: the Ultralimit.

3.4. Ultralimits

The notion of an ultralimit was introduced by van den Dries and Wilkie in 1984 \( [67] \) and developed by Gromov in \( [38] \). An ultralimit of a sequence of metric spaces is defined using Cartan’s 1937 notion of a nonprincipal ultrafilter: a finitely additive probability measure \( \omega \) such that all subsets \( S \subset \mathbb{N} \) are \( \omega \)-measurable, \( \omega(S) \in \{0, 1\} \) and \( \omega(S) = 0 \) whenever \( S \) is finite. Given an ultrafilter, \( \omega \), and a bounded sequence of \( a_j \in \mathbb{R} \), there exists an ultralimit, \( L = \omega - \lim_{j \to \infty} a_j \), such that \( \forall \varepsilon > 0, \omega \{ j : |a_j - L| < \varepsilon \} = 1 \).

Given a sequence of compact metric spaces, \( (X_j, d_j) \), with a uniform bound on diameter, and given an ultrafilter, \( \omega \), there is an ultralimit \( (X, d) \) which is a metric space that may no longer be compact, but is at least complete. The space, \( X \), is constructed as equivalence classes of sequences \( \{x_j\} \) where \( x_j \in X_j \) and the metric of \( X \) is defined by taking ultralimits:

\[
d(\{x_j\}, \{y_j\}) := \omega - \lim_{j \to \infty} d_j(x_j, y_j). \tag{33}
\]

Two sequences are equivalent when the distance between them is zero. Notice that one does not need a subsequence to find a limit and that in general the ultralimit depends on \( \omega \).

If \( X_j \) has a Gromov-Hausdorff limit, then the ultralimit is the Gromov-Hausdorff limit and there is no dependence on \( \omega \). If one has two sequences \( X_j \xrightarrow{\text{GH}} X \) and \( Y_j \xrightarrow{\text{GH}} Y \), then the ultralimit of the alternating sequence \( \{X_1, Y_1, X_2, Y_2, X_3, \ldots\} \) is \( Y \) iff \( \exists N \) such that \( \forall n > N \) we have \( \omega \{2n, 2n + 2, 2n + 4, 2n + 6, \ldots\} = 1 \). Otherwise the ultralimit is \( X \).

It is often useful to compute the ultralimits of sequences which do not have Gromov-Hausdorff limits. For example, the three sphere of many splines sequence, \( \{M_j\} \), converges to a standard three sphere with countably many unit line segments attached at various points on the sphere. To see this, one may view each \( M_j \) as a union of regions \( W_j \cup U_{1,j} \cup U_{2,j} \cup \ldots \cup U_{N_j,j} \), where each
$U_{i,j}$ covers a spline and $W_j$ covers the spherical portion. Then $W_j \xrightarrow{GH} S^3$, while $U_{i,j} \xrightarrow{GH} [0,1]$. Thus any ultralimit of these $M_j$ is built by connecting countably many intervals to a three sphere. The locations where the intervals are attached to the sphere may depend on the ultrafilter, $\omega$.

If $X_j$ are geodesic spaces, then the ultralimit is a geodesic space. If the $X_j$ are $CAT(0)$ spaces then the ultralimit is as well. Ultralimits have been applied extensively in the study of $CAT(0)$ spaces and also Lie Groups. See for example [12] and [41] for more details as well as applications.

4. Speculation

While the methods of intrinsic convergence defined above have all proven to be useful in a variety of settings, and should in fact have more applications that have not yet been discovered or fully explored, it is clear that each has its disadvantages. Gromov-Hausdorff convergence, like Hausdorff convergence, provides no rectifiability for its limits. The intrinsic flat convergence, like Federer-Fleming’s flat convergence, has difficulty with cancellation in the limit. Ultraconvergence has the same difficulties as Gromov-Hausdorff convergence and the additional concern that the spaces in the sequence are not particularly close to the limit space in a measurable way using a notion of distance.

There are important problems both related to General Relativity and Ricci flow which have not yet been addressed using the above methods. While intrinsic flat convergence may prove useful as it is further explored, other methods of convergence may be necessary to address all the problems that arise.

4.1. Intrinsic Varifold Convergence

For some time, mathematicians have been exploring possible methods of extending Ricci flow through singularities. Mean curvature flow behaves a lot like Ricci flow and the Brakke flow uses varifolds to extend this notion to the nonsmooth setting. The key missing ingredient that has kept people from directly extending the work on Brakke flow to better understand Ricci flow is that there has never been a notion of an intrinsic varifold space and intrinsic varifold convergence.

In the prior section we saw how mathematicians have repeatedly applied Gromov’s trick of isometrically embedding metric spaces into a common space, measuring the distance between them in that common space, and then taking an infimum. Here, however, there is no metric measuring the distance between varifolds, just the natural notion of weak convergence against test functions. One might try to define a notion of distances between varifolds in Euclidean space which defines a convergence that agrees with the standard varifold convergence and then apply Gromov’s trick.

Alternately, one might venture to say that a sequence of Riemannian manifolds, $M_j$, converges to a space, $M$, as varifolds if and only if there
is a sequence of isometric embeddings from, $M_j$, into a common space, $Z$, such that the images of $M_j$ converge as varifolds in $Z$ to an image of $M$. However, there is as yet no notion of convergence as varifolds in a metric space. Recall that if $Z$ is Euclidean space, convergence as varifolds, means weak convergence as measures on $\mathbb{R}^N \times \Gamma(N,k)$. We would need some sort of notion of a $\Gamma(Z,k)$ perhaps representing all possible “tangent spaces” of $Z$ at a point. Perhaps this might be done by first isometrically embedding $Z$ into a Banach space.

4.2. Area Convergence

One may view Gromov-Hausdorff convergence as a notion of convergence defined by the fact that lengths of minimizing geodesics converge. Measure convergence is defined by the fact that volumes converge. It becomes natural to wonder, particularly in the case of three dimensional manifolds, whether there is a notion of convergence defined by the fact that areas, or areas of minimal surfaces, converge.

Note that Ilmanen’s example of the three sphere of many splines, $\{M_j\}$, converges in some area sense to the standard three sphere. There are radial projections $f_j : M_j \to S^3$ which are almost area preserving, one-to-one and onto. Such a map seems perhaps to extend the notion of an almost isometry without involving a distance. However requiring a diffeomorphism to define area convergence seems too strong for any applications.

Burago, Ivanov and Sormani spent a few years investigating possible notions of area convergence and the area distance between spaces. They observed that two distinct metric spaces could easily have a surjection between them that is area preserving. One space could, for example, be the unit square, $X = [0,1] \times [0,1]$, with the Euclidean metric. The other space could be the unit square with a “pulled thread”: $Y = [0,1] \times [0,1]/\sim$ where $(x_1,x_2) \sim (y_1,y_2)$ iff $x_1 = y_1 = 0$ with the metric:

$$d_Y((x_1,x_2),(y_1,y_2)) = \min\{\sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}, |x_1| + |y_1|\}. \quad (34)$$

This seemed to indicate the need for a notion of “area space” defined as an equivalence class of metric spaces. It was essential before even beginning the project to verify that each equivalence class would only include one Riemannian manifold.

Another difficulty arose in that one would not expect to always find an almost area preserving surjection between spaces which should intuitively be close in the area sense. In an almost isometry, there are two requirements: almost distance preserving and almost onto. Both requirements involve distance and one can prove that if there is an almost isometry from a metric space $X$ to a metric space $Y$ then one can find an almost isometry in the reverse direction (although the errors change slightly in reverse) [10]. To take advantage of Gromov’s ideas, it was decided that one needs to convert the notion of area into a notion of distance.

Given a Riemannian manifold, $M$, one may examine the loop space, $\Omega(M)$. The flat distance between loops is defined using the notion of area.
Recall that the flat distance defined in (9) involves both area and length, however, in the setting of closed loops one may choose $A = 0$ and only take an infimum over all possible fillings by surfaces $B$. Perhaps an area convergence of $M_j$ to some sort of area space $M$ could be defined by taking pointed Gromov-Hausdorff limits or ultralimits of the $\Omega(M_j)$.

Preliminary work in this direction was completed by Burago-Ivanov. They proved that if a pair of three dimensional Riemannian manifolds, $M^3$ and $N^3$, have isometric loop spaces, then $M^3$ and $N^3$ are isometric [14]. This step alone was very difficult because $M^3$ is not isometrically embedded into its loop space, $\Omega(M)$, so one must localize points and planes in $TM$ using sequences of loops without controlling the lengths of the loops.

Now one may move forward and investigate whether there is a natural compactness theorem for some notion of area convergence. It would not be surprising if positive scalar curvature on $M_j$ control areas of minimal surfaces well enough to control $\Omega(M_j)$. Or perhaps one could skip compactness theorems, and apply ultralimits to the $\Omega(M_j)$ and see what properties are conserved under such a limit. Ultimately one does not just want to take a limit of $\Omega(M_j)$ but find an area space $X$, such that $\Omega(X)$ is the limit of the $\Omega(M_j)$. Then one could say $X$ is the area limit of $M_j$.

### 4.3. Convergence of Lorentzian manifolds

Another direction of research that is fundamental to General Relativity is the development of weak forms of convergence for Lorentzian manifolds. While Gromov-Hausdorff convergence has proven useful in the study of the stability of the spacelike universe [58], one needs to extend Gromov-Hausdorff convergence to the Lorentzian setting to study the stability of the space-time universe. The techniques described above immediately fail for these spaces because they do not isometrically embed into metric spaces.

Noldus has developed a Gromov-Hausdorff distance between Lorentzian spaces [51]. Further work in this direction including a notion of the arising limit spaces appears in [50] and compactness theorems and an examination of causality in the limit spaces are proven with Bombelli in [10]. This work has not yet been applied to prove stability questions arising in general relativity.

Perhaps similar methods might be applied to extend the notion of the intrinsic flat distance to Lorentzian spaces.

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