Notes on Cohomology

Luis Arenas-Carmona. ¹

Universidad de Chile,
Facultad de Ciencias.
Casilla 653, Santiago, Chile.
learenas@uchile.cl

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Chapter 1

Introduction.

Galois cohomology is a fundamental tool for the classification of certain algebraic structures. To be precise, let $k$ be a field, $G$ a linear algebraic group acting on a space $V$, both defined over $k$. It is known [4], that if $G$ is defined as the set of automorphisms of a tensor $\tau$ on $V$, e.g., a quadratic form or an algebra structure, the cohomology set $H^1(K/k, G_K)$ classifies the $K/k$-forms of $\tau$, i.e., those tensors of the same type also defined over $k$ that become isomorphic to $\tau$ over the larger field $K$ (§4.1). Results of this type, however, hold in much more general settings. In this notes, we give the general facts about cohomology that allow the use of cohomology sets for classification, and give examples of applications to many parts of field theory and number theory. In particular, we devote a whole chapter to the study of the relation between lattices and cohomology.

Such a theory is already hinted at in [11]. In this reference, two finiteness results are proven. The first one deals with the finiteness of the local cohomology set $H^1(G_w, \Gamma_w)$, for an arithmetically defined group $\Gamma$. Notations are as in [11]. The second one deals with the finiteness of the kernel of the map

$$H^1(G, \Gamma) \rightarrow \prod_{v \text{ place of } k} H(G_{w(v)}, \Gamma_{w(v)}),$$

where we have fixed a place $w(v)$ of $K$ dividing each place $v$ of $k$. It is the proof of the second result which requires expressing the given kernel in terms of the set of double cosets

$$G_k \backslash G_{kk} / \prod_w \Gamma_w$$

(see corollary 3.3 in [11]). These double cosets are the same ones that classify the classes of lattices in a genus. This relation is pursued in chapter (crossreference).

1.0.1 Notations

In all of this notes, $k, K, E$ denote number or local fields of characteristic 0, or algebraic extensions of them. If $k$ is a number field, $\Pi(k)$ denotes the set of
places of $k$.

**Remark 1.0.1.** By an algebraic group, we mean a linear algebraic group. All algebraic groups are assumed to be subgroups of the general linear group of a vector space $V$, of finite dimension, over a sufficiently large algebraically closed field $\Omega$ of characteristic 0. We assume that all localizations of number fields inject into $\Omega$. $G$ denotes an algebraic group over $\Omega$. $GL(V), SL(V)$ denote the general and special linear groups over $\Omega$. When we work over a fixed local or number field $k$, we say that $G$ is defined over $k$ if the equations defining $G$ have coefficients in $k$ (see section 2.1.1 in [10]). This is the case for all groups considered here. For any field $E$, $k \subseteq E \subseteq \Omega$, we write $G_E$ for the set of $E$-points of $G$, e.g., if $G = GL(V)$, the set of $E$ points is denoted $GL_E(V)$. The same conventions apply to spaces and algebras. All spaces and algebras are assumed to be finite dimensional.

Exceptions to this rule are the multiplicative and additive groups. We denote $G_m = \Omega^\times, G_a = \Omega$ when considered as algebraic groups. For the set of $k$-points we write $k^\times, k$. Instead of $(G_m)_k, (G_a)_k$.

The orthogonal group of a quadratic form $Q$ on $V$ is written $O_n(Q)$ or $O_n(Q,V)$, where $n = \dim_{\Omega}(V)$. The set of $E$-points is denoted $O_{n,E}(Q)$.

The field on which a particular lattice is defined is always written as a subindex. If $K/k$ an extension of local or number fields and $\Lambda_k$ is a lattice in $V_k$, $\Lambda_K$ denotes the $O_K$-lattice in $V_K$ generated by $\Lambda_k$.

If $G$ is an algebraic group acting on a space $V$, both defined over $k$, and $\Lambda_k$ is a $O_k$-lattice on $V_k$, the stabilizer of $\Lambda_k$ in $G_k$ is denoted $G^\Lambda_k$. If $G = GL(V)$, this set is denoted $GL^\Lambda_k(V)$. Similar conventions apply to special linear or orthogonal groups.

**Remark 1.0.2.** Whenever $K/k$ is a Galois extension of a number field $k$, and $v$ a place of $k$, $w$ denotes a place of $K$ dividing $v$. We assume that one fixed such $w$ has been chosen for every $v$. This convention is also applied for infinite extension, e.g., $K = \kbar$.

**Remark 1.0.3.** $G_{K/k}$ denotes the Galois group of the extension $K/k$. If there is no risk of confusion, we write simply $G$. If $K$ is not specified, we assume $K = \kbar$. If $k$ is a number field and $v \in \Pi(k)$, we also use the notation $G_v = G_{K_v/k_v}$.

If $\Gamma$ is a group acting on a set $S$, $S/\Gamma$ denotes the set of orbits and $S^\Gamma$ the set of invariant points. The action of $\gamma \in \Gamma$ is denoted $s \mapsto s^\gamma$, for $s \in S$. 


Chapter 2

Cohomology and classification

In this chapter we introduced the basic results that are required to connect cohomology and classification. The results in this section are found in chapter 1 in [5], and p. 13-26 in [10].

Definition 2.0.4. Let $G$ be a finite group, and let $A$ a group provided with a $G$-action. $H^1(G, A)$ is defined as the quotient

$$H^1(G, A) = \{ \alpha : G \to A | \alpha(hg) = \alpha(h)\alpha(g)^h \}/\equiv,$$

where $\alpha \equiv \beta$ if and only if there exists $a \in A$ such that $\alpha(g) = a^{-1}\beta(g)a^g$ for all $g \in G$. If $G$ acts trivially on $A$, then $H^1(G, A) \cong \text{Hom}(G, A)/A$, where $A$ acts on $\text{Hom}(G, A)$ by conjugation. In what follows we write $\alpha_g$ instead of $\alpha(g)$.

In case that $A \subseteq B$ is a subgroup, there is a long exact sequence

$$0 \to A^G \to B^G \to (B/A)^G \to H^1(G, A) \to H^1(G, B),$$

and furthermore, under the natural action of $B^G$ on $(B/A)^G$,

$$(B/A)^G / B^G \cong \ker(H^1(G, A) \to H^1(G, B)). \quad (2.1)$$

To simplify notations, in all that follows we assume that whenever a sequence of pointed sets

$$\ldots \to U \to V \to W \to X \to Y \to Z$$

is written, $X, Y, Z$ denote pointed sets, $W, V, U, \ldots$ denote groups, and $W$ acts on $X$ with

$$X/W \cong \ker(Y \to Z).$$

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1 This result is not found in [5], but can be found in [10] p.22.
If $A$ is normal in $B$, we have in the sense just described

$$
\begin{array}{c}
0 \longrightarrow A^G \longrightarrow B^G \longrightarrow (B/A)^G \longrightarrow 0 \\
\end{array}
$$

(2.2)

In case $A$ is central in $B$, the higher order cohomology groups for $A$ are also defined, and we have a long exact sequence

$$
\begin{array}{c}
0 \longrightarrow A^G \longrightarrow B^G \longrightarrow (B/A)^G \longrightarrow 0 \\
\end{array}
$$

Finally, if $A$ and $B$ are both Abelian this sequence extends to cohomology of all orders [13], [6]. All results of this section can be extended via direct limits to profinite groups acting continuously on discrete groups ([13], p. 9 and 42).

### 2.1 The general classification principle

A $(\mathcal{G}, G)$-space is a set $X$ provided with both, a $\mathcal{G}$-action and a $G$-action (denoted $*$) satisfying

$$
g^\sigma * x^\sigma = (g * x)^\sigma
$$

for $x \in P$, $g \in G$, and $\sigma \in \mathcal{G}$. The space $X$ is said to be free, transitive, etc, if the corresponding $G$-action has any of these properties. The most important application for us of the results in the preceding section is the following:

**Proposition 2.1.1.** Let $X$ be a transitive $(\mathcal{G}, G)$-space for all $x \in X$, $g \in G$, and $\sigma \in \mathcal{G}$. Let $x_0 \in X$, and let $H = \text{Stab}_G(x_0)$. Then, $X^G$ is in one-to-one correspondence with the elements of the cohomology set $\ker (H^1(\mathcal{G}, H) \to H^1(\mathcal{G}, G))$.

**Proof.** Since $X$ is isomorphic to $G/H$ as $\mathcal{G}$-modules, the result follows from [2.1].

A principal homogeneous $G$-space is a $(\mathcal{G}, G)$-space where the $G$ action is transitive and free. Given any principal homogeneous $G$-space $P$ and any base elements $x \in P$, there exists a unique $G$-map (but not a $\mathcal{G}$-map, unless $x$ is invariant) $\phi : G \rightarrow P$ that sends $g$ to $gx$. In particular, we can identify $P$ with $G$ as a set. Classifying principal homogeneous $G$-spaces is, therefore, equivalent
to classifying twisted actions on $G$ on $G$ that turn the set $G$ into a principal homogeneous $G$-space.

Let $X$ be the group of all maps from $G$ to $G$. Then $X$ is a $G$-group with an action $f \mapsto f^\sigma$ satisfying $f^\sigma(\lambda) = f(\lambda \sigma^{-1})$. A twisted semiaction of $G$ on $G$ is a map $\rho : G \times G \to G$ which satisfy $\rho(hg, \sigma) = h^\sigma \rho(g, \sigma)$ and $\rho(1,1) = 1$. A twisted semiaction $\rho$ is a twisted action if it satisfies the relation

$$\rho(\rho(g, \sigma), \lambda) = \rho(g, \sigma\lambda),$$

for all $g$ in $G$ and all $\sigma$ and $\lambda$ in $G$. The group $X$ acts transitively on the set $T$ of twisted semiactions by

$$(\alpha \cdot \rho)(g, \sigma) = \rho(g \alpha(\sigma)^{-1}, \sigma),$$

for all $\alpha$ in $X$, all $\rho$ in $T$, all $g$ in $G$, and all $\sigma$ in $G$. The stabilizer of $\rho_0$, where $\rho_0(g, \sigma) = g^\sigma$, is the set of maps satisfying $\alpha(\sigma)^\sigma = \alpha(1)$. This maps form a subgroup $G'$ of $X$ which is isomorphic to $G$ as a $G$-group. The quotient set $X/G' \cong T$ has a natural $G$ action which is translated to an action on $T$ as $\rho^{\lambda^{-1}}(1, \sigma) = \rho(1, \sigma\lambda)^{\lambda^{-1}} \rho(1, \lambda)^{-\lambda^{-1}}$. With this action, a twisted semiaction $\rho$ is invariant if and only if

$$\rho(1, \sigma\lambda) = \rho(1, \lambda)^{\lambda^{-1}} \rho(1, \lambda) = \rho(\rho(1, \sigma), \lambda),$$

and premultiplying both sides by $g^{\sigma\lambda}$ we see that a $G$-invariant action is the same as a twisted action. The following lemma follows easily from proposition 2.1.1.

**Lemma 2.1.2.** The set of twisted actions on $G$ is in natural correspondence, up to isomorphisms of $(G, G)$-actions, with the kernel of the map $H^1(G, G) \to H^1(G, X)$. \qed

**Proposition 2.1.3** (Shapiro’s Lemma). Let $\mathcal{H}$ be a subgroup of $G$. Let $G$ be a $\mathcal{H}$-group and let $G'$ be the set of maps $\phi : G \to G$ such that $\phi(\sigma\lambda) = \phi(\sigma)^\lambda$ for all $\lambda \in \mathcal{H}$. Then $H^1(\mathcal{H}, G') \cong H^1(\mathcal{H}, G)$. \qed

**Corollary 2.1.3.1.** In the notations of lemma 2.1.2 $H^1(G, X) = \{1\}$. \qed

The following result follows from lemma 2.1.2 and corollary 2.1.3.1.

**Proposition 2.1.4** ([13, p. 44]). The set of principal homogeneous $G$-spaces up to isomorphism is in one-to-one correspondence with the elements of the cohomology set $H^1(G, G)$. \qed

In most applications of the results in this chapter $G$ is the Galois group $G_{K/k}$ of a possibly infinite Galois extension $K/k$, where $k$ is a local or number field. The subgroups $A, B, \ldots$ are usually groups of algebraic or arithmetical nature.
Chapter 3

The generalized Hilbert theorem 90

3.1 Rings, units and cohomology

In this section, $A$ is a ring provided with a $G$ action. A left $(A, G)$-module is an $A$-module $B$ provided with a $G$-action, and satisfying $(ab)^\sigma = a^\sigma b^\sigma$ for all $a \in A$, $b \in B$, and $\sigma \in G$.

An element $b \in B$ is a generator if $A b = B$. It is a regular generator if $a b = 0$ implies $a = 0$. In particular, if $b \in B$ is a regular generator, the map $a \mapsto ab$ is an isomorphism of $A$-modules between $A$ and $B$. Assume that $b$ is a regular generator of $B$. Then an element $a b \in B$ is a regular generator if and only if $a \in A^\ast$. We say that the $(A, G)$-module is principal if it has a regular generator.

Proposition 3.1.1. The cohomology set $H^1(G, A^\ast)$ classifies the set of principal left $(A, G)$-modules. The distinguished point of $H^1(G, A^\ast)$ corresponds to the modules that have an invariant regular generator.

Proof. Let $X$ be the group of all maps from $G$ to $A^\ast$. As in §2.1 we define a twisted semiaction as a map $\rho : G \times A \to A$ satisfying:

a) $\rho(aa', \sigma) = a^\sigma \rho(a', \sigma)$.

b) $\rho(1, \sigma)$ is a unit for all $\sigma \in G$.

A twisted action as a twisted semiaction that is an action. Any principal left $(A, G)$-module is isomorphic to $A$ with a twisted action. The group $X$ acts transitively on the set of twisted semiaction by $(\alpha \cdot \rho)(a, \sigma) = \rho(a a(\sigma), \sigma) \alpha(1)^{-1}$, and the stabilizer of $\rho_0$, where $\rho_0(a, \sigma) = a^\sigma$, is the set of maps satisfying $\alpha(\sigma)^\sigma = \alpha(1)$ for all $\sigma \in G$. Now the proof follows as in §2.1.

Example 3.1.2. Let $A = \mathbb{Z}$ with the trivial $G$-action. Then $H^1(G, \mathbb{Z}^\ast) = \text{Hom}(G, \{1, -1\})$, hence there exists a free left $(G, \mathbb{Z})$-module with no invariant generator if and only if $G$ has a normal subgroup of index 2. The reader can easily check this result independently.
Example 3.1.3. More generally, if $A$ has the trivial $\mathcal{G}$-action. Then $H^1(\mathcal{G}, \mathbb{Z}^*) = \text{Hom}(\mathcal{G}, A^*)/\cong$, where $\cong$ denotes the conjugacy relation. The module associated to the homomorphism $\alpha : \mathcal{G} \to A^*$ has the action $(ab)^\sigma = a\alpha(\sigma)b$.

Example 3.1.4. Let $K/k$ be a finite extension of local or number fields with Galois group $\mathcal{G}$. Let $O_K$ and $O_k$ be the respective rings of integers. Then the exact sequence $O_K^* \to K^* \to P_K$, where $P_K$ is the group of principal fractional ideals of $K$, shows that the kernel of $H^1(\mathcal{G}, O_K^*) \to H^1(\mathcal{G}, K^*)$ corresponds to invariant ideals in $P_K$ modulo ideals in $P_k$ (up to a suitable identification). Since we prove later in this chapter that $H^1(\mathcal{G}, K^*) = 1$, then all free left $(O_K, \mathcal{G})$-module of rank 1 are of this form. If the extension $K/k$ is unramified, one can prove that $P_K^{\mathcal{G}}$ is the set of fractional ideals in $k$ that become principal over $K$. It follows that $H^1(\mathcal{G}, O_K^*)$ is in correspondence with a subgroup of the ideal group of $k$.

Example 3.1.5. Let $R$ be a ring with a $\mathcal{G}$-action. Then there is a natural action of $\mathcal{G}$ on $A = \mathbb{M}_n(R)$ such that the matrices $E_{i,j}$ with a 1 in the intersection of the $i$-th row and the $j$-th column and 0 everywhere else are invariant.

Any $A$-module $B$ has a decomposition of the form $B = \bigoplus_{i=1}^n E_{i,i}B$. Furthermore, we claim that $E_{i,j}E_{k,l}B = \delta_{j,k}E_{i,i}$. In fact, the case $j \neq k$ is trivial and the case $j = k$ follows from the contents $E_{i,j}E_{i,j}B = E_{i,i}(E_{i,j}B) \subseteq E_{i,i}B$ and $E_{i,i}B = E_{i,j}(E_{j,i}B) \subseteq E_{i,j}B$. Conversely, if $B'$ is an $R$-module of rank $n$, then $B = \bigoplus_{i=1}^n B_i$ where $B_i \cong B'$ has a natural $A$-module structure satisfying $E_{i,j}B_k = \delta_{j,k}B_i$. It follows that The cohomology set $H^1(\mathcal{G}, GL_n(R))$ classifies free left $(R, \mathcal{G})$-modules of rank $n$.

### 3.2 Invariant generators of vector spaces

In this section, $K/k$ is a finite Galois filed extension. Also, $V_K$ denotes a finite dimensional vector space. We assume that the Galois group $\mathcal{G}$ acts on $V_K$ in such a way that $(\lambda v)^\sigma = \lambda^\sigma v^\sigma$ for any $\sigma \in \mathcal{G}, v \in V_K$, and $\Lambda \in K$.

**Lemma 3.2.1.** The maps $h \mapsto h^\sigma$, are linearly independent.

**Proof.** Assume $\sum_{\sigma \in \mathcal{G}} \alpha_\sigma h^\sigma = 0$ for all $h \in K$. Since any finite separable extension is simple, we can assume $K = k(\omega)$. In particular, $\sum_{\sigma \in \mathcal{G}} \alpha_\sigma \omega^{k\sigma} = 0$. However, the matrix with entries $\omega^{k\sigma}$ is a Vandermonde matrix with non-zero determinant, whence $\alpha_\sigma = 0$ for all $\sigma$. \hfill \Box

**Proposition 3.2.2.** The space $V_K$ has a basis of $\mathcal{G}$-invariant vectors.

**Proof.** Let $V_k$ be the subspace of invariant vectors. It suffices to prove that $\text{dim}_k(V_k) = \text{dim}_K(V_K)$. Let $b : V_K \to V_k$ be the map $b(v) = \sum_{\sigma \in \mathcal{G}} v^\sigma$. If $\text{dim}_k(V_k) < \text{dim}_K(V_K)$, there exists a non-trivial linear form $u$ such that $u(b(v)) = 0$ for all $v$. For any $h \in K$ we have

$$0 = u(b(hv)) = \sum_{\sigma \in \mathcal{G}} h^\sigma u(v^\sigma).$$
Since the functions $h \mapsto h\sigma$ are linearly independent, it follows that $u(v^\sigma) = 0$ for all $v$ and all $\sigma$. In particular $u = 0$.  

**Corollary 3.2.2.1.** There exists a $k$-subspace $V_k$ of $V_K$, such that $V_K \cong K \otimes_k V_k$.  

**Corollary 3.2.2.2** (Hilbert’s Theorem 90). $H^1(G, GL_n(K)) = \{1\}$.  

### 3.3 Cohomology of the group of units of an algebra

In this section, $K/k$ is a finite Galois extension over an infinite field $k$.

**Proposition 3.3.1.** For any finite dimensional algebra $A$, defined over $k$, and any algebraic extension $K/k$, it holds that $H^1(G_K/k, A^*_K) = \{1\}$.

**Proof.** Assume first that $K/k$ is finite. Let $B_K$ be a free left $(A_K, G)$-module of rank 1. Then $B_K$ is a finite dimensional vector space over $K$ satisfying the hypotheses of proposition 3.2.2. It follows that $B_K$ has a basis of invariant vectors $\{v_1, \ldots, v_n\}$. In particular, $B_K \cong K \otimes_k B_k$ for some $k$-vector space $B_k$. The set of generators of $B_K$ is a Zariski open set and therefore it must contain an element of $B_k$ (see some reference). In the general case, if $\{L\}$ is the set of finite subextensions of $K/k$, then

$$H^1(G, A^*_K) = H^1(G, \lim_{\leftarrow} A^*_L) = \lim_{\leftarrow} H^1(G_L/k, A^*_L) = \{1\}. \quad \blacksquare$$

**Corollary 3.3.1.1.** If $A_k$ is a $k$-algebra which is the direct limit of a family $\{B_k\}$ of finite dimensional algebras then $H^1(G_{K/k}, A^*_K) = \{1\}$.  

**Proof.** In fact,

$$H^1(G, A^*_K) = H^1(G, \lim_{\rightarrow} B^*_K) = \lim_{\rightarrow} H^1(G, B^*_K) = \{1\}. \quad \blacksquare$$

**Corollary 3.3.1.2.** If $A_k$ is a $k$-algebra such that every finite subset of $A_k$ generates a finite dimensional subalgebra then $H^1(G_{K/k}, A^*_K) = \{1\}$.

**Proof.** In this case, $A_k$ is the direct limit of its finite dimensional subalgebras.  

Since $GL_k(V) \cong (\text{End}_k(V))^*$, Hilbert’s theorem 90 is a particular case of proposition 3.3.1. However, proposition 3.3.1 has many other applications, as can be seen in these notes.

Let $K/k$ be any field, then we have an exact sequence:

$$0 \rightarrow SL_K(V) \rightarrow GL_K(V) \rightarrow K^* \rightarrow 0$$

which gives a long exact sequence in cohomology:

$$GL_k(V) \rightarrow k^* \rightarrow H^1(G_{K/k}, SL_K(V)) \rightarrow H^1(G_{K/k}, GL_K(V)) = 1.$$  

as the determinant map is always surjective, this proves:
Proposition 3.3.2. for any field extension \( K/k \) we have:

\[
H^1(G_{K/k}, SL_K(V)) = \{1\}.
\]

More generally, if \( A \) is a finite dimensional central simple algebra split by \( K/k \) and \( N_E : A_E \to E^* \) is the reduced norm, we have a sequence:

\[
A_E^* \xrightarrow{N_E} k^* \to H^1(G, SA_K^*) \to H^1(G, A_k^*) = \{1\},
\]

where \( SA_K^* = \ker(N_K) \), therefore:

\[
H^1(G, SA_K^*) = k^*/N_k(A_k^*).
\]

Example 3.3.3. \( A_E = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, A_R = H, N(A_R) = \mathbb{R}^+, \) and \( A_C = \mathbb{M}_2(C) \), hence \( H^1(G, SH_C^*) = \{[1], [-1]\} \). Same result applies to any matrix algebra over \( H \).

Proposition 3.3.4. The multiplicative and additive groups \( G_m \) and \( G_a \) have trivial \( H^1 \).

**Proof.** It is an immediate application of proposition 3.3.1 that \( H^1(G, G_m) = \{1\} \).

Let \( A_k = k[x]/(x^2) \). \( A \) is a local algebra with maximal ideal \( I = (x) \). Let \( U = \{1 + y | y \in I\} \). Then, \( U \cong G_a \). Therefore, there exists an exact sequence

\[
\{1\} \to G_a \to A^* \to G_m \to \{1\}.
\]

It follows that \( H^1(G, G_m) \cong \ker(A_k^* \to k^*) = \{1\} \).

\begin{proof}
In fact, it holds that \( H^i(G, G_a) = \{1\} \) for all \( i > 0 \). One way to prove this is to see that \( K \) is an induced \( G \)-module so that Shapiro’s lemma applies ([6], p.73). If \( k \) has characteristic 0, an alternative proof follows from the fact that \( H^i(G, A) \) is annihilated by \( |G| \) for all \( G \)-module \( A \), while the map \( \lambda \to n\lambda \) is an isomorphism for all \( n \) ([6], p. 84).
\end{proof}

Proposition 3.3.5. If \( V_K \) is a finite dimensional vector space over \( K \) provided with a Galois action, then \( H^1(G, V_K) = \{1\} \).

**Proof.** Let \( \{v_1, \ldots, v_n\} \) be a basis of invariant vectors of \( V_K \), and let \( W = \text{span} \{v_1, \ldots, v_{n-1}\} \). There exists an exact sequence

\[
\{1\} \to W \to V \to G_a \to \{1\}.
\]

Hence, the result follows by induction.

Proposition 3.3.6. If \( V_k \) is an arbitrary vector space over \( k \), then \( H^1(G, V_K) = \{1\} \).
Proof. This follows from the previous result since $V_k$ is a direct limit of its finite dimensional subspaces $\{W_k\}$ and therefore

$$H^1(\mathcal{G}, V_K)) = H^1(\mathcal{G}, \lim_{\rightarrow} W_K) = \lim_{\rightarrow} H^1(\mathcal{G}, W_K) = \{1\}. \qed$$

Example 3.3.7. Let $G$ be a $\mathcal{G}$-group and let $K$ be a field with a faithfull $\mathcal{G}$-action. Let $F = K^G$. It follows from Galois theory that the extension $K/F$ is Galois and $\text{Gal}(K/F) \cong \mathcal{G}$. Let $A_K = K[G]$ be the group algebra. Let $A_F = A_K^G$. It follows from proposition 3.2.2 that $\dim_F(A_F) = \dim_K(A_K)$. In particular, $A_K$ is actually obtained from $A_F$ by extension of scalars and proposition 3.3.1 applies. A basis $S$ of $A_K$ satisfying both $S = S^\sigma$ for $\sigma \in \mathcal{G}$, and $GS = S$, is a principal homogeneous $G$-space. If a principal homogeneous $G$-space is isomorphic to some basis $S$ as above we say that it is represented in $A_K$. Let $T$ be the set of bases $S$ of $A_K$ satisfying $GS = S$. then $A_K^\ast$ acts on $T$ by $S \mapsto Sa$ for $a \in A_K^\ast$. Invariants elements of $T$ are principal homogeneous $G$-spaces represented in $A_K^\ast$. By propositions 2.1.1 and 3.3.1, it follows that $T^G/A_F^\ast$ is in correspondence with $H^1(\mathcal{G}, G)$. It follows that every principal homogeneous $G$-space is represented in $A_K$ and if two bases $S$ and $S'$ that are principal homogeneous $G$-spaces are isomorphic as such, then there exists $a \in A_F^\ast$ such that $S' = Sa$. 

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Chapter 4

Algebraic applications of cohomology

4.1 Tensors and $K/k$-forms

By a tensor of type $(l, m)$ on $V$, we mean an $\Omega$-linear map $\tau : V^\otimes l \rightarrow V^\otimes m$, where $V^\otimes r = \bigotimes_{i=1}^{r} V$ for $r \geq 1$, $V^\otimes 0 = \Omega$.

$\tau$ is said to be defined over $k$, if $\tau(V^\otimes l_k) \subseteq V^\otimes m_k$. All tensors mentioned in this work are assumed to be defined over $k$. $GL(V)$ acts on the set of tensors of type $(l, m)$ by $g(\tau) = g^\otimes m \circ \tau \circ (g^\otimes l)^{-1}$. It makes sense, therefore, to speak about the stabilizer of a tensor.

Let $I$ be any set. By an $I$-family of tensors, we mean a map that associates, to each element $i \in I$, a tensor $t_i$ of type $(n_i, m_i)$. $GL(V)$ acts on the set of all $I$-families by acting in each coordinate. In all that follows, we say a family instead of an $I$-family unless the set of indices needs to be made explicit. Let $\mathfrak{T}$ be a family of tensors and $H = \text{Stab}_{GL(V)}(\mathfrak{T})$. Then, $H$ is a linear algebraic group.

If $K/k$ is a Galois extension with Galois group $\mathcal{G}$, we get an exact sequence

$$\{1\} \rightarrow H_K \rightarrow GL_K(V) \rightarrow X_K \rightarrow \{1\},$$

where $X_K$ is the $GL_K(V)$-orbit of $\mathfrak{T}$. It follows from (4.1), and example 3.2.2.2, that $X_K^G/GL_k(V) \cong H^1(\mathcal{G}, H_K)$. The elements of $X_K^G/GL_k(V)$ can be thought of as isomorphism classes of pairs $(V'_k, \mathfrak{T}')$ that become isomorphic to $(V_k, \mathfrak{T})$ when extended to $K$. These classes are usually called $K/k$-forms of $(V, \mathfrak{T})$, or just $k$-forms if $K = \bar{k}$. Observe that two vector spaces of the same dimension are isomorphic, so we can always assume that the vector space $V$, in which all tensors are defined, is fixed.
Definition 4.1.1. We call a pair \((V, \mathcal{T})\), where \(\mathcal{T}\) is a family of tensors on \(V\), a space with tensors, or simply a space. By abuse of language, we identify \((V, \mathcal{T})\) and \((V, \mathcal{T}')\) whenever \(\mathcal{T}\), \(\mathcal{T}'\) are in the same \(GL_k(V)\)-orbit, i.e., if they correspond to the same \(K/k\)-form. We say that \((V, \mathcal{T}')\) is a \(K/k\)-form of \((V, \mathcal{T})\), if \(\mathcal{T}\) and \(\mathcal{T}'\) are in the same \(GL_K(V)\) orbit.

Example 4.1.2. Let \(Q\) be a non-singular quadratic form on the space \(V\). Then, \(O_n(Q) = \text{Stab}_{GL(V)}(Q)\). Equivalence classes of non-singular quadratic forms on \(V_k\) are classified by \(H^1(G, O_n, \overline{k}(Q))\). A space, in this case, is what is usually called a quadratic space.

4.2 Semi-simple abelian algebras

An abelian algebra is semisimple if it has no nontrivial nilpotent elements. In this section, let \(L\) be an abelian semi-simple algebra defined over a number field \(k\). Then \(L\) is a \(k\)-form of the trivial semi-simple algebra \(A^{(m)} = k \oplus k \oplus \ldots \oplus k\), \(m\) times, whose group of automorphisms equals \(S_m\), the symmetric group on \(m\) symbols. It follows that the set of isomorphism classes of semi-simple abelian algebras of dimension \(m\) over \(k\) is in one-to-one correspondence with the cohomological set \(H^1(G, S_m) \cong \text{Hom}(G, S_m) / \sim\), where \(\phi \sim \psi\) means that there exists \(\sigma \in S_m\) such that \(\psi(g) = \sigma \phi(g) \sigma^{-1}\) for any \(g \in G\).

Let \(\psi : G \to S_m\) be one map in the conjugacy class corresponding to \(L\). Then some properties of the algebra \(L\) can be translated into properties of the map \(\psi\).

According to the general theory, the algebra \(L\) can be defined as the set of invariant points of the corresponding twisted action, i.e.,

\[ L = \{l \in \overline{k}^m | \psi(\sigma)l^\sigma = l \ \forall \sigma \in G\}. \]

Next we describe some properties of \(L\) in terms of \(\Psi = \text{im}(\psi)\).

Proposition 4.2.1. \(L\) is a field if and only if \(\Psi\) acts transitively on the set \(\{1, \ldots, m\}\).

Proof. The algebra \(L\) is a field if it contains a non trivial projection \(P\). Let \(P_t \in \overline{k}^m\) be the projection in the \(t\)-th factor. If \(S \subseteq \{1, \ldots, m\}\), define \(P_S = \sum_{t \in S} P_t\). Any projection \(P \in \overline{k}^m\) is of the form \(P_S\) for some subset \(S\). Since all projections are fixed by the non-twisted action, \(P_S \in L\) if and only if \(\psi(\sigma)P_S = P_S\) for all \(\sigma \in G\). We have \(\psi(\sigma)P_m = P_{\psi(\sigma)(m)}\). Therefore, a non-trivial projection exists if and only if \(\Psi\) is not transitive.

More precisely, if \(O \subseteq \{1, \ldots, m\}\), then \(P_O \in L\) if and only if \(O\) is invariant under \(\Psi\), i.e., is a union of orbits. Any element \(l\) of \(L\) has the form \(l = (l_1, \ldots, l_m)\) where the elements in \(l_1, \ldots, l_m\) corresponding to elements in the same orbit form a complete set of conjugates under the action of the Galois group, which
acts on them by $l_j^\sigma = l_{\psi^{-1}(j)}$. In particular, if $L$ is a field, $\{l_1, \ldots, l_m\}$ is a complete set of conjugates.

**Proposition 4.2.2.** Let $\mathcal{H}$ be the subset of $\mathcal{G}$ that fixes a subfield $L'$ of $\bar{k}$ isomorphic to $L$. Then

$$\ker(\psi) = \bigcap_{\sigma \in \mathcal{G}} \sigma \mathcal{H} \sigma^{-1}.$$ 

The description of $L$ given earlier, implies that $L'$ can be assumed to be the image of $L$ under the projection on the first factor. the group $\ker(\psi)$ is the subgroup of $G$ that fixes $L$ pointwise (in the non-twisted action). Every conjugate of $\mathcal{H}$ is the stabilizer of the image of $L$ under some projection. 

**Corollary 4.2.2.1.** If $L$ is a Galois extension of $k$, then $\text{Gal}(L/k) \cong \Psi$. 

**Corollary 4.2.2.2.** $L$ is a cyclic extension of $k$ if and only if $\text{im}\psi$ is generated by an $m$-cycle.

**Proof.** If $L/k$ is a cyclic extension, then $\text{im}(\psi)$ is cyclic and transitive. 

**Example 4.2.3.** The exact sequence

$$\{1\} \longrightarrow A_m \longrightarrow S_m \longrightarrow \mu_2 \longrightarrow \{1\}$$

defines a cohomology map $d : H^1(G, S_m) \longrightarrow H^1(G, \mu_2) \cong k^*/(k^*)^2$, called the discriminant, whose kernel is $H^1(G, A_m)$. In other words, $L$ has discriminant 1 if and only if $\Psi \subseteq A_m$. It follows that if $L$ is Galois and $n$ is odd then the discriminant of $L$ is 1. The converse is true for $n = 3$.

### 4.3 finite dimensional abelian algebras with nilpotent elements

Let $L_k$ be an arbitrary finite dimensional abelian algebra over $k$, and let $K$ be an algebraic closure of $k$.

**Proposition 4.3.1.** There are no non-trivial $K/k$-forms of $k[x]/(x^n)$.

**Proof.** Set $A_k = k[x]/(x^n)$. Then $A_K = K \oplus I_K$, where $I_K$ is the ideal generated by $x$. Similarly $A_k = k \oplus I_k$. Observe that $I_K^{n-1} \neq 0$ and $I_K^n = 0$. We use induction on $n$. If $n = 1$ there is nothing to prove. If $n \geq 2$, then $G_n = \text{Aut}(A)$ acts on $A/I^{n-1}$. Let $G'$ be the kernel of this action. Then $G_n/G' \cong G_{n-1}$, so that by induction hypothesis $H^1(G, G/G') = \{1\}$.

Let $g \in G'$, and set $g(u) = u + e(u)$ with $e(u) \in I^n$. Then $g(1) = 1$, so that $e(1) = 0$. Also, for $uv \in I^2$, $g(uv) = (u + e(u))(v + e(v)) = uv$, so $e(uv) = 0$. It follows that $G' \cong \text{Hom}(I/I^2, I^n)$. In particular, it is a vector space, so it is acyclic.

A similar argument shows the following result:
Proposition 4.3.2. There are no non-trivial \( K/k \)-forms of \( k[x_1, \ldots, x_n]/(x_1, \ldots, x_n)^m \).

In general, for an algebra \( A_k \) without projectors on the algebraic closure \( \bar{k} \) it is always true that \( A_K = K \oplus I_K \), where \( I_K \) is the nilradical of \( A_K \). We can also define \( G_n = \text{Aut}(A_k/I_k^{n+1}) \). However, in general \( G_n/G' \) is only a subgroup of \( G_{n-1} \), and it is not always true that \( H^1(G, \text{Aut}(A_K)) = \{1\} \).

The following example will make this clear:

Let \( A_k = k[x,y]/(x,y)^3 + (x^2 - y^2) \). Then if \( I_k \) is the image of the ideal \( (x,y) \), then \( I^3 = 0 \), but \( I^2 \neq 0 \). Let \( G \) be the automorphism group of \( A \) and let \( G' \) be the subgroup of automorphisms of \( A \) that induce the trivial automorphism of \( A/I^2 \). Then \( G/G' \) is contained in the group of automorphisms of \( A/I^2 \), i.e., the group \( \text{GL}(V) \), where \( V \) is the vector space generated by \( x \) and \( y \). An element \( g \in \text{GL}(V) \) is in \( G/G' \) if and only if it fixes the ideal \( (x,y)^3 + (x^2 - y^2) \). It follows that there is a short exact sequence

\[
K^* \rightarrow G_K/G'_K \rightarrow O_K(q),
\]

where \( q \) is the quadratic form \( x^2 - y^2 \) and \( O(q) \) its orthogonal group. Since \( K^* \) is acyclic, it follows that \( H^1(G, G_K) \) equals \( H^1(G, O_K(q)) \). In other words, the \( K/k \)-forms of \( A_k \) are the algebras of the form \( k[x,y]/((x,y)^3 + (q'(x,y))) \), where \( q' \) is a quadratic form.

4.4 Skolem-Noether theorem

Let \( \mathfrak{A} \) be a central simple algebra defined over \( k \). Let \( L \) be a maximal semisimple Abelian subalgebra defined over \( k \). \( \mathfrak{A}^* \) acts on the set of maximal semisimple Abelian subalgebras by conjugation. It is a trivial exercise in linear algebra to prove the transitivity of this action over an algebraically closed field. Let \( G \) be the stabilizer of \( L \). It follows from (2.1), and proposition [3.3.1] that the set of conjugacy classes of maximal Abelian subalgebras that are defined over \( k \) is parametrized by \( H^1(G, G) \). Observe that, over the algebraic closure, any automorphism of \( L \) arises from a conjugation. The short exact sequence

\[
\{1\} \rightarrow L^* \rightarrow G \rightarrow \text{Aut}(L) \rightarrow \{1\},
\]

where \( \text{Aut}(L) \) is the set of automorphisms of \( L \) as an \( \Omega \)-algebra, gives

\[
1 = H^1(G, L^*) \rightarrow H^1(G, G) \rightarrow H^1(G, \text{Aut}(L)).
\]

It follows, since \( H^1(G, \text{Aut}(L)) \) classifies isomorphism classes of semisimple Abelian algebras, that any two isomorphic algebras are conjugate.
4.5 Finite subgroups in proyective groups of algebras

Let $k$ be a field and let $\mathfrak{A}_k$ be a finite dimensional $k$-algebra. Let $\Gamma_0$ be a finite subgroup of $\mathfrak{A}_K^*/k^*$. The group $\Gamma_0$ can be regarded as a subgroup of $\mathfrak{A}_K^*/K^*$ for any field extension $K/k$. In this section we describe a cohomology set that classifies finite subgroups $\Gamma$ of $\mathfrak{A}_K^*/k^*$ that become conjugate to $\Gamma_0$ over some separable algebraic extension $K/k$. Let $C_K$ be the centralizer of $\Gamma_0$ in $\mathfrak{A}_K^*/K^*$ and let $W$ be the group of automorphisms of $\Gamma_0$ that can be realized as conjugations from elements in $\mathfrak{A}_K^*/K^*$. In this section we prove the following result:

**Proposition 4.5.1.** Let $K/k$ be a Galois extension. There exists a natural action of $W$ on the cohomology set $H^1(C_K) = H^1(K/k, C_K)$. The set of conjugacy classes of finite subgroups of $\mathfrak{A}_K^*/k^*$ that become conjugate over $K$ to $\Gamma_0$ is in one-to-one correspondence with the set of orbits $H^1(C_K)/W$.

**Proof.** The group $\mathfrak{A}_K^*$ acts on the set of finite subgroups by conjugation. The stabilizer of $\Gamma_0$ is the preimage $N_K \subseteq \mathfrak{A}_K^*$ of the normalizer of $\Gamma_0$ in $\mathfrak{A}_K/K^*$. It follows from Proposition 2.1.1 that, if $X$ is the set of finite subgroups $\Gamma$ of $\mathfrak{A}_K^*/K^*$ that are conjugate to $\Gamma_0$, i.e., the $\mathfrak{A}_K^*$-orbit of $\Gamma_0$, then $X^\sigma/\mathfrak{A}_k^* \cong \ker[H^1(N_K) \to H^1(\mathfrak{A}_K^*)]$. Since $H^1(\mathfrak{A}_K^*) = \{1\}$ (5, p. 16), it follows that $H^1(N_K)$ is in correspondence with the set of conjugacy classes under $\mathfrak{A}_k^*$ of $G$-invariant finite groups $\Gamma$ that are $\mathfrak{A}_K^*$-conjugate to $\Gamma_0$. The initial group $\Gamma_0$ corresponds to the distinguished element of $H^1(N_K)$. There exists a short exact sequence $C_K \hookrightarrow N_K \twoheadrightarrow W$, where $W$ is a subgroup of $\text{Aut}(\Gamma_0)$. Notice that $G$ acts trivially on $\text{Aut}(\Gamma_0)$, and hence also on $W$. It follows from Propositions 38 and 39 in (13, p. 49), that $H^1(C_K)/W \cong \ker[H^1(N_K) \to H^1(W)]$. Now, since the action of $G$ on $W$ is trivial, the set $H^1(W)$ is identified with the set of conjugacy classes of homomorphisms from $G$ to $W$. Under this identification, the map $\pi$ sends a cocycle $\alpha \in H^1(N_K)$ to a map $\phi_\alpha$, where $\phi_\alpha(\sigma) \in W$ acts on $\Gamma_0$ as conjugation by $\alpha_\sigma$. If $\alpha$ is the cocycle corresponding to a group $\Gamma = a\Gamma_0a^{-1}$, then $\alpha_\sigma = a^{-1}a^{\sigma}$. Now,

$$(a^\gamma a^{-1})^{\alpha} = aa_\gamma a a_\sigma^{-1}a^{-1} \quad \forall \gamma \in \Gamma_0 \subseteq \mathfrak{A}_k.$$

It follows that the kernel of the map $\phi_\alpha$ is the subgroup of $G$ corresponding to the Galois extension $k(\Gamma)/k$ generated by the coordinates of the elements of $\Gamma$. Hence $\phi_\alpha$ is trivial if and only if $k(\Gamma) = k$. The result follows.

We give two applications of this result:

**Corollary 4.5.1.1.** Let $k$ be a field whose characteristic does not divide $n$. If $\mathfrak{A}_k$ is a central division algebra over $k$ of dimension $n^2$, and if $\Gamma_0$ contains a subgroup $\Omega \cong \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ that intersects trivially the center $\mathbb{Z}(\Gamma_0)$ of $\Gamma_0$, then every finite group $\Gamma$ of $\mathfrak{A}_K^*/k^*$ that is conjugate to $\Gamma_0$ over $K$ is conjugate to $\Gamma_0$ over $k$. 

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Comparing eigenvalues of the functions $u = x$. Theorem 4.5.1. \( \rho \) follows that $x \in C \rho$ so that $x^n = a$, $y^n = b$, and $xy = \eta yx$ for some $a, b, \eta \in k$. By Kummer’s Theory, we claim that if $\Omega$ is a subgroup of $\mathbb{Z}/n\mathbb{Z}$, then we must have $m > p$. We conclude that $z \in K(x)^*$ and $\eta \in k$. The condition that $\Omega$ meets trivially the center of $\Gamma_0$ implies that $C_K = K^*$, hence $H^1(C_K) = \{1\}$ ([5], p. 16). The result follows by Theorem 4.5.1.

This result applies to groups $\Gamma_0$ containing a copy of $A_2n$ for $n^2 = \dim k \mathfrak{A}_k > 1$. If $n = 2$, it applies to groups isomorphic to $A_4$, $S_4$, or $A_5$.

**Corollary 4.5.1.2.** Let $k$ be a field of characteristic not equal to $p$, where $p$ is a prime. If $\mathfrak{A}_k$ is a central division algebra over $k$ of dimension $p^2$, and if $\Gamma_0$ is isomorphic to $\mathbb{Z}/m\mathbb{Z}$ with $m \neq p$, then every finite group $\Gamma$ of $\mathfrak{A}_k/k^*$ that is conjugate to $\Gamma_0$ over $K$ is conjugate to $\Gamma_0$ over $k$.

**Proof.** Without loss of generality we assume $K/k$ is a Galois extension. The group $\Gamma_0$ is generated by an element $\rho(x)$, where $x \in \mathfrak{A}_k$ satisfies $x^m = k$ and $\rho$ is as in the proof of Corollary 4.5.1.1. Since $\mathfrak{A}_k$ is a division algebra, for any $d < m$ the subalgebra $k(x^d)$ is maximal abelian in $\mathfrak{A}_k$, i.e., $[k(x^d) : k] = p$. In particular, $k(x^d) = k(x)$. Furthermore, we have $K(x)^* \subseteq C_K$. Then any element $z \in \mathfrak{A}_K$ such that $\rho(z)$ centralizes $\rho(x)$ must satisfy $z^{-1}xz = \tau x$ where $\tau$ is an $m$-th root of unity. Since $\tau x$ and $x$ have the same eigenvalues, multiplication by $\tau$ must permute the eigenvalues of $x$. It follows that the order of $\tau$ must divide $p!$. Assume first that $m$ is not a power of $p$. Replacing $x$ by some power if needed we might assume that $m$ is a prime. Since $[k(x) : k] = p$, we must have $m > p$. We conclude that $z \in K(x)^*$. Assume next $m = p^r$ with $r > 1$. The same argument as above shows that $\tau$ must be a $p$-th root of unity. Hence $z$ commutes with $x^p$. Since $k(x^p) = k(x)$, we still have $z \in K(x)^*$. As $H^1(K(x)^*) = 1$ ([5], p. 16), the result follows. \[\square\]
Chapter 5

Lattices and cohomology

5.1 Basic results

Let $k$ be a local or number field, $K/k$ a Galois extension, $G \subseteq GL(V)$ an algebraic group defined over $k$, $\Lambda_k$ a lattice on $V_k$, $L_K$ a $G$-invariant lattice on $V_K$. Let $G = G_{K/k}$.

**Proposition 5.1.1.** If there is an element $\varphi \in G_K$ such that $\varphi(L_K) = \Lambda_K$, then $a_\sigma = \varphi^\sigma \varphi^{-1}$ is a well defined element of $H^1(G, G^\Lambda_K)$. It is independent of the choice of $\varphi$ and depends only on the orbit of $L_K$ under $G_k$. The correspondence assigning, to every such $G_k$-orbit of $O_K$-lattices, an equivalence class of cocycles, is an injection. The image of this map is

$$\ker(H^1(G, G^\Lambda_K) \to H^1(G, G_K)),$$

where $i$ is the inclusion.

**Proof.** $G_K$ acts on the set of $O_K$-lattices in $V_K$. Let $X$ be the orbit of $\Lambda_K$. We have an exact sequence

$$\{1\} \rightarrow G^\Lambda_K \rightarrow G_K \rightarrow X \rightarrow \{1\}.$$

Hence, by (2.1), we get $X^G / G_k \cong \ker(H^1(G, G^\Lambda_K) \rightarrow H^1(G, G_K))$. □

**Example 5.1.2.** Using the fact that $H^1(G, GL_K(V)) = \{1\}$, we obtain that the set of $GL_k(V)$-orbits of $G$-invariant $O_K$-lattices isomorphic to $\Lambda_K$ is in correspondence with $H^1(G, GL^\Lambda_K(V))$.

If $G$ is defined as the stabilizer of a family of tensors, e.g., the unitary group of a hermitian form or the automorphism group of an algebra, we get a more precise result.

Recall that in section 4.1 we identified $K/k$-forms of $(V, \mathfrak{F})$ with the corresponding $GL_k(V)$-orbits of families of tensors.
Definition 5.1.3. Let \((V, \mathfrak{T})\) be a space. A lattice in \((V, \mathfrak{T})\) is a pair \((\Lambda_K, \mathfrak{T})\), where \(\Lambda_K\) is a lattice in \(V_K\). \(GL_K(V)\) acts on the set of pairs \((\Lambda_K, \mathfrak{T}')\), for all families of tensors \(\mathfrak{T}'\), by acting on each component. Two lattices \((\Lambda_K, \mathfrak{T})\), \((L_K, \mathfrak{T}')\) are said to be in the same space if \(\mathfrak{T}, \mathfrak{T}'\) are in the same \(GL_K(V)\)-orbit.

Proposition 5.1.4. Assume that \(G\) is the stabilizer of a family of tensors \(\mathfrak{T}\) on \(V\). The set \(H^1(\mathcal{G}, G^\Lambda_K)\) is in one-to-one correspondence with the set of \(G_k\)-orbits of \(G\)-invariant \(O_K\)-lattices in the same \(G_K\)-orbit, in all spaces that are \(K/k\)-forms of \((V, \mathfrak{T})\). The kernel of the map

\[ H^1(\mathcal{G}, G^\Lambda_K) \xrightarrow{i_\ast} H^1(\mathcal{G}, G_K), \]

where \(i\) is the inclusion, corresponds to the subset of orbits of lattices that are in the same space as \(\Lambda_K\).

Proof. We have an action of \(GL_K(V)\) on the set of all pairs \((L_K, \mathfrak{T}')\), where \(L_K\) is a lattice and \(\mathfrak{T}'\) a family of tensors with a fixed index set. If \(T\) is the orbit of \((\Lambda_K, \mathfrak{T})\), we have a sequence

\[ \{1\} \to G^\Lambda_K \to GL_K(V) \to T \to \{1\}, \]

and the same argument as before applies. Last statement follows from the fact that spaces \((V_K, \mathfrak{T}')\) are classified by \(H^1(\mathcal{G}_K/k, G_K)\), (see section 4.1 or [5], p. 15). \(\square\)

Remark 5.1.5. Recall that \(\Lambda_K = \Lambda_k \otimes_{O_K} O_K\). If \(L_K\) is in the same \(G_k\)-orbit as \(\Lambda_K\), \(L_K = L_k \otimes_{O_K} O_K\), since \(G_k\) also acts on \(V_k\). Recall that we defined the cocycle corresponding to \(L\) by the formula \(a_{\mathfrak{T}} = \phi^\mathfrak{T} \phi^{-1}\) (see prop. 5.1.1). This definition does not depend on \(G\), as long as \(\phi \in G\). It follows that the set of \(G_k\)-orbits of lattices in \(V_k\) that are isomorphic as \(O_k\)-modules, and whose extensions to \(K\) are in the same \(G_K\) orbit, corresponds to

\[ \ker(H^1(\mathcal{G}, G^\Lambda_K) \to H^1(\mathcal{G}, G_K) \times H^1(\mathcal{G}, GL^\Lambda_K(V))). \quad (5.1) \]

In the case that \(G\) is the stabilizer of a family of tensors,

\[ \ker(H^1(\mathcal{G}, G^\Lambda_K) \to H^1(\mathcal{G}, GL^\Lambda_K(V))) \]

corresponds to the set of \(G_k\)-orbits of such lattices in all spaces that are \(K/k\)-forms of \((V, \mathfrak{T})\).

Example 5.1.6. If \(\Lambda_k\) is free, (5.1) corresponds to the set of \(G_k\)-orbits of free lattices on \(V_k\), whose extensions to \(K\) are in the same \(G_K\)-orbit.

Definition 5.1.7. We say that an \(O_K\)-lattice \(\Lambda_K\) is defined over \(k\), if \(\Lambda_K \cong O_K \otimes_{O_K} \Lambda_k\) for some \(\Lambda_k\). We say that \(\Lambda_K\) is a \(k\)-free lattice, if \(\Lambda_k\) is free.

Assume first that \(G\) is the stabilizer of a family of tensors.
Definition 5.1.8. Let $a \in H^1(\mathcal{G}, G_K^\Lambda)$. We say that $a$ is defined over $k$, $k$-free or in $(V, \Xi)$ if some (hence any), lattice in the class corresponding to $a$ has this property. Define

\begin{align*}
\mathcal{L}_{\text{def}}(G, K/k, \Lambda) &= \{ a \in H^1(\mathcal{G}, G_K^\Lambda) | a \text{ is defined over } k \}, \\
\mathcal{L}_u(G, K/k, \Lambda) &= \{ a \in \mathcal{L}_{\text{def}}(G, K/k, \Lambda) | a \text{ is } k\text{-free} \}, \\
\mathcal{L}^V(G, K/k, \Lambda) &= \{ a \in H^1(\mathcal{G}, G_K^\Lambda) | a \text{ is in } (V_K, \mathcal{X}) \}, \\
\mathcal{L}_{\text{def}}^V(G, K/k, \Lambda) &= \mathcal{L}^V(G, K/k, \Lambda) \cap \mathcal{L}_{\text{def}}(G, K/k, \Lambda), \\
\mathcal{L}_u^V(G, K/k, \Lambda) &= \mathcal{L}^V(G, K/k, \Lambda) \cap \mathcal{L}_u(G, K/k, \Lambda).
\end{align*}

Let

\begin{align*}
F_1 : H^1(\mathcal{G}, G_K^\Lambda) &\to H^1(\mathcal{G}, G_K), \\
F_2 : H^1(\mathcal{G}, G_K^\Lambda) &\to H^1(\mathcal{G}, GL_K^\Lambda(V)),
\end{align*}

be the maps defined by the inclusions. Then, we have the following proposition:

**Proposition 5.1.9.** Assume that $\Lambda_k$ is free. The following identities hold:

\begin{align*}
\mathcal{L}^V(G, K/k, \Lambda) &= \ker F_1, \\
\mathcal{L}_u(G, K/k, \Lambda) &= \ker F_2, \\
\mathcal{L}^V_u(G, K/k, \Lambda) &= \ker F_1 \cap \ker F_2. \hfill \Box
\end{align*}

Later, we give a similar interpretation to $\mathcal{L}_{\text{def}}$.

**Example 5.1.10.**

\[
\mathcal{L}_u(\mathcal{O}_n(Q), \bar{k}/k, \Lambda) = \ker(H^1(\mathcal{G}, \mathcal{O}_n^\Lambda(Q)) \to H^1(\mathcal{G}, GL_K^\Lambda(V)))
\]

is in correspondence with the set of isometry classes of free quadratic lattices that become isometric to $\Lambda_k$ over some extension.

**Remark 5.1.11.** Notice that $\mathcal{L}^V, \mathcal{L}^V_{\text{def}}, \mathcal{L}^V_u$ can be defined, even if $G$ is not the stabilizer of a family of tensors, as follows:

\begin{align*}
\mathcal{L}^V(G, K/k, \Lambda) &= \ker(H^1(\mathcal{G}, G_K^\Lambda) \to H^1(\mathcal{G}, G_K)), \\
\mathcal{L}^V_{\text{def}}(G, K/k, \Lambda) &= \{ a \in \mathcal{L}^V(G, K/k, \Lambda) | a \text{ is defined over } k \}, \\
\mathcal{L}^V_u(G, K/k, \Lambda) &= \{ a \in \mathcal{L}^V(G, K/k, \Lambda) | a \text{ is free} \}.
\end{align*}

In this case, the first and last identities of proposition 5.1.9 still hold. Notice that we can still interpret $\mathcal{L}^V$ as a set of equivalence classes of lattices, because of proposition 5.1.11.

### 5.2 $H^1(\mathcal{G}, U_K)$ and the ideal group

Let $k$ be a local or number field, $K/k$ a finite Galois extension. Let $\mathcal{G} = \mathcal{G}_{K/k}$. $U_K = \mathcal{O}_K^*$ denotes the group of units of $\mathcal{O}_k$.

For any local or number field $E$, let $I_E$ be its group of fractional ideals, $P_E$ the subgroup of principal fractional ideals. There is a natural map $\alpha : I_k \to I_K$.
defined by \( \alpha(A) = A \otimes_{\mathcal{O}_k} \mathcal{O}_K \). Clearly \( \alpha(P_k) \subseteq P_K \), so we get a map \( \alpha' : I_k / P_k \to I_K / P_K \).

Apply the general theory to \( \Lambda_k = \mathcal{O}_k, G = \mathbb{G}_m, G^\Lambda_k = U_K \). Any \( \lambda \in K^* \) acts by \( A \mapsto \lambda A \), for \( A \in I_K \). It follows that,

\[
H^1(G, U_K) \cong (P_K)^{\mathbb{G}} / \alpha(P_k).
\]

**Remark 5.2.1.** When this is done over the function field extension \( L(X)/F(X) \), one obtains \( U_K = L^* \). Since \( H^1(G, L^*) = \{1\} \), this implies that every invariant ideal in \( L[X] \) has a generator in \( F[X] \). In particular, if a polynomial \( P \) of \( F[X] \) is an \( n \)-power in \( L[X] \), by taking the principal ideal generated by the \( n \)-th root, we obtain that \( \lambda P \) is an \( n \)-th power in \( F[X] \) for some constant \( \lambda \). Since powers of monic polynomials are monic we conclude that a monic in \( F[X] \) is an \( n \)-th power in \( L[X] \) if and only if it is an \( n \)-th power in \( F[X] \).

Non-zero prime ideals of \( \mathcal{O}_K \) form a set of free generators for \( I_K \) (see [7], p. 18). Let \( A \in I_K \). We can write

\[
A = \prod_{\wp \in \Pi(k)} \left( \prod_{\mathcal{P} \mid \wp} \mathcal{P}^{\beta(\mathcal{P})} \right).
\]

If \( A \) is \( G \)-invariant, all the powers \( \beta(\mathcal{P}) \) corresponding to prime divisors of the same prime of \( k \) must be equal. In other words:

\[
A = \prod_{\wp \in \Pi(k)} \left( \prod_{\mathcal{P} \mid \wp} \mathcal{P}^{\beta(\wp)} \right), \quad (5.4)
\]

where \( \beta(\wp) \) is the common value of \( \beta(\mathcal{P}) \) for all \( \mathcal{P} \) dividing \( \wp \). This ideal is in \( \alpha(I_k) \) if and only if the ramification degree \( e_\wp \) divides \( \beta(\wp) \) for all \( \wp \). Hence, we have an exact sequence

\[
0 \longrightarrow \ker \alpha' \longrightarrow (P_K)^{\mathbb{G}} / \alpha(P_k) \longrightarrow \prod_{\wp \in \Pi(k)} (\mathbb{Z} / e_\wp),
\]

where the image of the last map corresponds to those ideals of the form \((5.4)\) that are principal in \( K \). The image of \( \ker \alpha' \) is what we call \( \mathcal{L}_{\text{det}}(G, K/k, \Lambda) \).

In particular, since all ideals become principal in some extension, we can take a direct limit, to obtain the long exact sequence:

\[
0 \longrightarrow I_k / P_k \longrightarrow H^1(\mathcal{G}_{k/k}, U_k) \longrightarrow \prod_{\wp \in \Pi(k)} (\mathbb{Q} / \mathbb{Z}) \longrightarrow 0.
\]

A refinement of this argument gives

\[
H^1(\mathcal{G}_{k/k}, U_k) \cong (I_k \otimes_{\mathbb{Z}} \mathbb{Q}) / (P_k \otimes_{\mathbb{Z}} \mathbb{Z}), \quad \mathcal{L}_{\text{det}}(G, K/k, \Lambda) = I_k / P_k.
\]
5.3 Localization

Recall remarks 1.0.2 and 1.0.3. Assume \( k \) is a number field. There exist natural localization maps

\[ F_v : H^1(G, G^\Lambda_K) \to H^1(G_w, G^\Lambda_{K_w}), \]

defined by inclusion and restriction. We define \( G^\Lambda_{K_w} = G_{K_w} \) if \( w \) is Archimedean.

Lemma 5.3.1. Let \( F_1 : H^1(G, G^\Lambda_K) \to H^1(G, G_K) \) be the map induced by the inclusion. If the natural map

\[ \tau : H^1(G, G_K) \to \prod_{v \in \Pi(k)} H^1(G_w, G_{K_w}) \]

is injective, then \( \ker F_1 \supseteq \bigcap_v \ker F_v \).

Proof of lemma. Immediate from the following commutative diagram:

\[
\begin{array}{ccc}
H^1(G, G^\Lambda_K) & \xrightarrow{F_1} & H^1(G, G_K) \\
\downarrow F_v & & \downarrow \tau \\
\prod_v H^1(G_w, G^\Lambda_{K_w}) & \xrightarrow{\prod_v F_v} & \prod_v H^1(G_w, G_{K_w}).
\end{array}
\]

Remark 5.3.2. If the hypothesis of this lemma is satisfied, one says that \( G \) satisfies the Hasse principle over \( k \).

Characterisation of \( L_{\text{det}} \). \( L_{\text{det}}(G/K, \Lambda) \) is the set of equivalence classes of lattices defined over \( k \) that become isomorphic over \( K \). A lattice \( L_K \) is defined over \( k \) if and only if it is generated by its \( k \)-points, i.e.,

\[ L_K = \mathcal{O}_K(L_K \cap V_k). \]

This is a local property. On the other hand, for any local place \( v \), all lattices defined over \( k_v \) are \( k_v \)-free, i.e.,

\[ L_{\text{det}}(GL(V), K_w/k_v, \Lambda) = L_{\text{det}}(GL(V), K_w/k_v, \Lambda). \]

The following result is immediate from this observation.

Proposition 5.3.3.

\[ L_{\text{det}}(G/K, \Lambda) = \ker \left( H^1(G, G^\Lambda_K) \to \prod_v H^1(G_w, GL^\Lambda_{K_w}(V)) \right). \]
5.4 Genus and cohomology

Assume that in all of section 5.4, \( k \) is a number field.

**Definition 5.4.1.** Let \( F_v \) be the localization map. Define

\[
C_{\text{gen}}(G, K/k, \Lambda) = \ker(\prod_v F_v).
\]

We call this set the **cohomological genus** of \( \Lambda \) with respect to \( G \).

**Proposition 5.4.2.** For any linear algebraic group \( G \), it holds that

\[
C_{\text{gen}}(G, K/k, \Lambda) \subseteq L_{\text{det}}(G, K/k, \Lambda).
\]

**Proof.** This follows from proposition 5.3.3 and the commutative diagram

\[
\begin{array}{ccc}
H^1(G, G_K^\Lambda) & \rightarrow & \prod_{v \in \Pi(k)} H^1(G_w, G_{K_w}^\Lambda) \\
\downarrow & & \downarrow \\
\prod_{v \in \Pi(k)} H^1(G_w, G_{K_w}^\Lambda) & \rightarrow & \prod_{v \in \Pi(k)} H^1(G_w, GL_{K_w}^\Lambda(V)).
\end{array}
\]

**Remark 5.4.3.** Assume \( G \) is the stabilizer of a family of tensors. This result tells us that the cohomological genus corresponds to a set of equivalence classes of lattices defined over \( k \). In fact, \( a \in C_{\text{gen}}(G, K/k, \Lambda) \) if and only if \( a \) corresponds to a lattice, in some \( K/k \)-form of \((V, \mathcal{F})\), that is in the same \( G_{K_v} \)-orbit, at every place \( v \).

**Definition 5.4.4.** We define the **VC-genus** of \( \Lambda_k \) by the formula

\[
VC_{\text{gen}}(G, K/k, \Lambda) = C_{\text{gen}}(G, K/k, \Lambda) \cap L^V(G, K/k, \Lambda).
\]

In other words, it is the kernel of the map

\[
H^1(G, G_K^\Lambda) \rightarrow H^1(G, G_K) \times \prod_{v \in \Pi(k)} H^1(G_w, G_{K_w}^\Lambda).
\]

Let \( G \) be an arbitrary linear algebraic group. The VC-genus corresponds to a set of \( G_{K_v} \)-orbits of lattices in \( V_k \). In fact, it corresponds to a subset of the set of double cosets \( G_k \backslash G_{K_k}/G_{\Lambda_k} \), i.e., the **genus** of \( G \) (see [10], p. 440). In particular, the following proposition holds.

**Proposition 5.4.5.** If \( G \) has class number 1 with respect to a lattice \( \Lambda_k \), then \( (5.5) \) has trivial kernel for every Galois extension \( K/k \) (compare with corollary 4 on p. 491 of [10]).\]

This, in particular, applies to a group having absolute strong approximation (see [10]). However, we have a stronger result.

**Proposition 5.4.6.** If \( G \) has absolute strong approximation over \( k \), the map \( (5.5) \) is injective.
Proof. Recall remark 1.0.2.
Let $M_K, L_K$ be two $G$-invariant $O_K$-lattices in $V_K$, that are locally in the same $G_{k_v}$-orbit for all $v$. Then, we can choose elements $\sigma_v \in G_{k_v}$, such that $\sigma_v M_{K_w} = L_{K_w}$ for every place $v$, and $\sigma_v = 1$ at all but finite places. Now, any global element $\sigma$, close enough to $\sigma_v$ at all finite places where $\sigma_v \neq 1$, and stabilizing $M_{K_w} = L_{K_w}$ at the remaining finite places, satisfies $\sigma M_K = L_K$, as claimed. □

The following result is just a restatement of lemma 5.3.1.

**Proposition 5.4.7.** If $G$ satisfies the Hasse principle over $k$, then

$$VC_{gen}(G, K/k, \Lambda) = C_{gen}(G, K/k, \Lambda).$$

□

This result tells us that, in the presence of Hasse principle, the cohomological genus corresponds to a subset of the genus (compare with [11], thm 3.3, p. 198).

## 5.5 Spinor norm and genera

Let $G \subseteq GL(V)$ be a semi-simple group with universal cover $\tilde{G}$ and fundamental group $\mu_n$. Let $K = \bar{k}$.

The short exact sequence

$$\{1\} \rightarrow \mu_n \rightarrow \tilde{G}_K \rightarrow G_K \rightarrow \{1\},$$

defines a map $\theta : G_k \rightarrow H^1(G, F) = k^*/(k^*)^n$.

Let $A_k$ be any lattice in $V_k$. The following proposition holds.

**Proposition 5.5.1.** With the above notations, $VC_{gen}(G, K/k, \Lambda)$ is in one-to-one correspondence with the genus of $G$ (compare with theorem 8.13 in [10], p. 490).

**Proof.** It suffices to show that any two $G_k$-orbits in the same genus are identified over some extension. Without loss of generality, we assume $k$ is non-real. It suffices to check that they are in the same spinor genus (see [2]). Spinor genera are classified by

$$J_k / J_k^n k^* \Theta_k(G_k^\Lambda),$$

where $\Theta_k(G_k^\Lambda)$ is the kernel of the local spinor norm (see [11] or [2]). This is a finite set, and the representing adeles can be chosen to have trivial coordinates at almost all places. Therefore, it suffices to take an extension that contains the $n$-roots of unity, and $n$ roots of a finite set of local elements. □

This result allows us to use cohomology to study the genus of any Semisimple group.

---

1 The case of an orthogonal group is already considered in [3].
5.6 Determinant class of a lattice

Let $[A]$ be the $k^*$-orbit of the $\mathcal{O}_K$-ideal $A$. Assume that

$$\Lambda_k = \mathcal{O}_k \oplus \ldots \oplus \mathcal{O}_k.$$  

The map $\text{det}_*: H^1(\mathcal{G}, GL_k^\Lambda(V)) \to H^1(\mathcal{G}, U_K)$ is the map induced in cohomology by the determinant. It is surjective, since $\text{det}$ has a right inverse. However, in general it is not injective, as the example below shows.

**Definition 5.6.1.** Let $L_K$ be a $G$-invariant lattice in $V_K$, and let $a$ be the cocycle class corresponding to the $GL_k(V)$-orbit of $L_K$. We define the determinant class of $L_K$, which we denote $\text{det}_*(L_K)$, by:

$$\text{det}_*(L_K) = \text{det}_*(a) \in H^1(\mathcal{G}, U_K) \cong \mathcal{P}_K / \alpha(P_k),$$

and we identify it with the corresponding ideal class.

**Example 5.6.2.** Using the standard embedding $GL(V) \times GL(W) \to GL(V \oplus W)$, it is easy to prove that $\text{det}_*(\Lambda_K \oplus L_K) = \text{det}_*(\Lambda_K)\text{det}_*(L_K)$. In particular, we obtain that $\text{det}_*(A_1 \oplus \ldots \oplus A_n) = [A_1 \ldots A_n]$.

Assume $k \subseteq K$ are local fields with maximal ideals $\wp$, $\mathcal{P}$. Assume that $\wp \mathcal{O}_K = \mathcal{P}^e$. Then,

$$\text{det}_*(\mathcal{P} \oplus \ldots \mathcal{P}) = [\mathcal{P}^e] = 1 = \text{det}_*(\mathcal{O}_K \oplus \ldots \oplus \mathcal{O}_K),$$

but the latter lattice is defined over $k$ and the first one is not.

Let $L_{\text{def}} = L_{\text{def}}(GL(V), K/k, \Lambda)$. We have the following result:

**Lemma 5.6.3.** $L_{\text{def}} \cap \ker(\text{det}_*) = \{1\}$.

**Proof of Lemma.** This follows from the fact that all $k$-defined lattices are of the form $A_k \oplus O_k \oplus \ldots \oplus O_k$ (see [8], (81:5)). It can also be proved by a diagram chasing argument.

Now observe that, for any algebraic group $G \subseteq GL(V)$, we have $L_{\text{def}}(G, K/k, \Lambda) = i_*^{-1}(L_{\text{def}})$, where $i_*$ is the cohomology map induced by the inclusion.

**Proposition 5.6.4.** If $G \subseteq SL(V)$, then $i_*^{-1}(L_{\text{def}}) = \ker(i_*)$.

**Proof of Proposition.** It is immediate from the commutative diagram

$$
\begin{array}{ccc}
H^1(\mathcal{G}, G^\Lambda_K) & \xrightarrow{i_*} & H^1(\mathcal{G}, SL^\Lambda_K(V)) \\
\downarrow \text{det}_* & & \downarrow \\
H^1(\mathcal{G}, GL^\Lambda_K(V)) & \xrightarrow{i_*^{-1}} & H^1(\mathcal{G}, U_K)
\end{array}
$$
that \( \text{im}(i_*) \subseteq \ker(\det_*) \). Now recall lemma 5.6.3. □

In particular, such a group cannot identify a free lattice to a non-free \( k \)-defined lattice over any extension, although it can identify a free lattice to a non-\( k \)-defined lattice.

In this case, a description of \( \mathcal{L}_e \) is equivalent to a description of \( \mathcal{L}_{det} \), hence \( \mathcal{L}_{det}(G, K/k, \Lambda) \) can be described without resorting to localization.

### 5.7 Cohomology and representation

Let \( M \) be a sublattice of \( \Lambda \). Let \( G_K^{\Lambda, M} \) be the stabilizer of \( M \) in \( G_K^{\Lambda} \). There exists a short exact sequence:

\[
\{1\} \longrightarrow G_K^{\Lambda, M} \longrightarrow G_K^{\Lambda} \longrightarrow X \longrightarrow \{1\},
\]

where \( X \) is the orbit of \( M \) in the set of sublattices. Then

\[
X^G / G_K^{\Lambda} \cong \ker(H^1(G, G_K^{\Lambda, M}) \longrightarrow H^1(G, G_K^{\Lambda}))
\]

can be identified with the set of \( G_K^{\Lambda} \)-orbits of \( G \)-invariant sublattices in the same \( G_K^{\Lambda} \)-orbit.

Let \( F \) be the stabilizer in \( G \) of the space \( W = \Omega M \), \( \Gamma \) its point-wise stabilizer. \( H = F/\Gamma \). There is a natural map \( G_K^{\Lambda, M} \longrightarrow H_K^M \), which induces a map

\[
H^1(G, G_K^{\Lambda, M}) \longrightarrow H^1(G, H_K^M)
\]

in cohomology. If we are interested in lattices that are in the same \( H_k \)-orbit, they will be classified by the kernel of the map

\[
H^1(G, G_K^{\Lambda, M}) \longrightarrow H^1(G, H_K^M) \times H^1(G, G_K^{\Lambda}).
\]

In the applications, \( G \) is the stabilizer of a tensor \( \tau \) of type \((l, m)\), \( W \) a subspace satisfying \( \tau(W^\otimes l) \subseteq W^\otimes m \), and \( H = \text{Stab}_{GL(W)}(\tau|_W) \), where \( \tau|_W \) is the restriction of \( \tau \) to \( W \). The condition on \( W \) is vacuous if \( m = 0 \).

**Example 5.7.1.** Let \( \tau = q \) is a quadratic form. Inequivalent representations of \( M_k \) by \( \Lambda_k \), that become equivalent over \( K \), are in correspondence with

\[
\ker \left( H^1(G, \mathcal{O}_{K,n}^{\Lambda, M}(q)) \longrightarrow H^1(G, \mathcal{O}_{K,p}^M(q|_W)) \times H^1(G, \mathcal{O}_{K,n}^\Lambda(q)) \right),
\]

where \( n = \dim V, p = \dim W \), and \( q|_W \) is the restriction of \( q \) to \( W \).

**Remark 5.7.2.** All result in this paper apply also to lattices over rings of \( S \)-integers. Absolute strong approximation must be replaced by strong approximation with respect to \( S \).
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