Meta-conformal algebras in $d$ spatial dimensions

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Abstract

Meta-conformal transformations are constructed as dynamical symmetries of the linear transport equation in $d$ spatial dimensions. In one and two dimensions, the associated Lie algebras are infinite-dimensional and isomorphic to the direct sum of either two or three Virasoro algebras. Co-variant two-point correlators are derived and possible physical applications are discussed.

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1 Introduction

Conformal invariance has found many brilliant applications, for example to string theory and high-energy physics [52], or to two-dimensional phase transitions [7, 24, 31, 54] or the quantum Hall effect [14, 25]. These applications are based on a geometric definition of conformal transformations, considered as local coordinate transformations $r \mapsto r' = f(r)$, of spatial coordinates $r \in \mathbb{R}^2$ such that angles are kept unchanged.\footnote{See [54] and refs. therein for the considerable recent interest into the case $r \in \mathbb{R}^d$ with $d > 2$.} The Lie algebra of these transformations is naturally called the ‘conformal Lie algebra’.

In view of these successes, it appears natural to ask if at least some ideas of conformal invariance can be brought to bear on dynamical problems. Indeed, the first attempt we are aware of concerns the critical dynamics of a two-dimensional statistical system [15]. In general, the global scaling of time and space coordinates is distinguished by the dynamical exponent $z$, according to $t \mapsto t' = b^z t$ and $r \mapsto r' = b r$. In general, $z$ has a non-trivial value [56]. Starting from the well-established conformal invariance in the spatial coordinates, with a generic space-dependent rescaling factor $b = b(r)$, for generic $z$, generalised conformal transformations are derived for both two-time correlators and two-time response functions.\footnote{At equilibrium, these are related by the fluctuation-dissipation theorem.} However, it seems that the absence of supporting physical examples led to the abandon of this idea. We shall present here the first example where this idea can be implemented, and the associated Lie algebra generators be explicitly constructed, at least for a dynamical exponent $z = 1$.

Concerning symmetry approaches to dynamics in the presence of a dynamic scaling behaviour, it turned out to be more fruitful to rather consider conformal transformations in time, e.g. $t \mapsto \frac{\alpha t + \beta}{\gamma t + \delta}$ with $\alpha \delta - \beta \gamma = 1$, and then to choose the spatial transformation such as to obtain a closed Lie group. The free diffusion equation has such a dynamical symmetry, by now usually called the Schrödinger group [50], with dynamical exponent $z = 2$. In any space dimension, this non-semi-simple Lie group has an infinite-dimensional extension, the Schrödinger-Virasoro group, although one usually analyses its Lie algebra [29, 57]. Analogous ideas can be brought forward for generic values of $z$ [32], lead to explicit prediction for the scaling form of the two-time response function and have been tested in a large variety of non-equilibrium systems which undergo dynamical scaling, see [34] for a review and [40] for a tutorial introduction.

In this work, we shall be interested in systems which undergo dynamical scaling with a dynamical exponent $z = 1$. Trivial examples would be given by conformally invariant critical systems at equilibrium, where one of the spatial direction would be relabeled as ‘time’. For the sake of a clear distinction, we refer to the ‘standard’ conformal transformations, which keep angles unchanged and mentioned so far, as ortho-conformal transformations. Here, we shall be interested in Lie algebras of time-space transformations, which as Lie algebras are still isomorphic to the ortho-conformal Lie algebra or at least contain it as a sub-algebra, but which no need to be angle-preserving. Such transformations will be called meta-conformal transformations [38, 39]. To make this idea explicit, consider the infinitesimal generators, acting on a two-dimensional time-space with points $(t, r) \in \mathbb{R}^2$ [32]

\[
X_n = -t^{n+1} \partial_t - \mu^{-1}[(t + \mu r)^{n+1} - t^{n+1}] \partial_r - (n+1)\frac{\gamma}{\mu}[(t + \mu r)^n - t^n] - (n+1)xt^n
\]

\[
Y_n = -(t + \mu r)^{n+1} \partial_r - (n+1)\gamma(t + \mu r)^n
\]

where $x, \gamma$ are constants and $\mu^{-1}$ is a constant universal velocity (‘speed of sound or speed of
light’). The global dilatations are generated by $X_0$ and it is easy to see that indeed $z = 1$. Clearly, these infinitesimal transformations are not angle-preserving in the time-space $(t, r) \in \mathbb{R}^2$, but their Lie algebra $\langle X_n, Y_n \rangle_{n \in \mathbb{Z}}$ obeys

$$[X_n, X_m] = (n - m)X_{n+m}, \quad [X_n, Y_m] = (n - m)Y_{n+m}, \quad [Y_n, Y_m] = \mu(n - m)Y_{n+m} \quad (1.2)$$

The isomorphism of (1.2) with the ortho-conformal Lie algebra can be seen by writing $X_n = \ell_n + \tilde{\ell}_n$ and $Y_n = \mu^{-1}\tilde{\ell}_n$. Then the generators $\langle \ell_n, \tilde{\ell}_n \rangle_{n \in \mathbb{Z}}$ satisfy $[\ell_n, \ell_m] = (n - m)\ell_{n+m}, [\ell_n, \tilde{\ell}_m] = (n - m)\tilde{\ell}_{n+m}, [\ell_n, \tilde{\ell}_m] = 0$. Provided $\mu \neq 0$, the Lie algebra (1.2) is isomorphic to a pair of Virasoro algebras $\text{vect}(S^1) \oplus \text{vect}(S^1)$ with a vanishing central charge [32, 55].

The meta-conformal generators (1.1) are dynamical symmetries of the equation of motion

$$\hat{S}\phi(t, r) = (-\mu\partial_t + \partial_r)\phi(t, r) = 0. \quad (1.3)$$

Indeed, since (with $n \in \mathbb{Z}$)

$$[\hat{S}, X_n] = -(n + 1)t^n\hat{S} + n(n + 1)\mu \left( x - \frac{\gamma}{\mu} \right) t^{n-1}, \quad [\hat{S}, Y_n] = 0 \quad (1.4)$$

a solution $\phi$ of with scaling dimension $x_\phi = x = \gamma/\mu$ is mapped onto another solution of (1.3). Hence the space of solutions of the equation (1.3) is meta-conformally invariant. This is the analogue of the familiar ortho-conformal invariance of the 2D Laplace equation.

Several different types of physical systems with dynamical exponent $z = 1$ are known. First, the dynamical symmetries of the Jeans-Vlassov equation [43, 59, 41, 49, 58, 12, 13, 23, 53] in one space dimension are given by a representation of (1.2), distinct from (1.1) [55]. Second, the ‘non-relativistic limit’ $\mu \to 0$ in the above generators yields a Lie-algebra contraction of (1.2) whose result is called ‘conformal galilean algebra’ CGA($d$) [28] or ‘BMS-algebra’ BMS$_{2+d}$[9], since the contracted generators are immediately generalised to $d \geq 1$ spatial dimensions, and rotations by arbitrary time-dependent angles appear for $d \geq 2$. Remarkably, the Lie algebra CGA($d$) is not isomorphic to the Schrödinger (or Schrödinger-Virasoro) Lie algebra in $d$ dimensions [33, 22]. Applications arise in hydrodynamics [61] or in gravity, e.g. [5, 1, 2, 48, 6, 3], and the bootstrap approach has been tried [48, 4]. Third, the non-equilibrium dynamics of quantum quenches generically has $z = 1$, related to ballistic spreading of signals, see [10, 11, 21] and this apparently holds both for quenches in the vicinity of the quantum critical point [18] as well as for deep quenches into the two-phase coexistence region [60]. The available examples suggest that the value $z = 1$ should be robust with respect to the change from closed to open quantum systems. Forth, effective equations of motion of the form (1.3) arise in recent studies of the generalised hydrodynamics required for the description of strongly interacting non-equilibrium quantum systems [8, 16, 19, 17, 51, 20].

In table 1 we collect the coordinate transformations of these examples of infinite-dimensional groups of time-space transformations. While the physical interpretation of the co-variant $n$-point functions of ortho-conformal invariance as correlators is long-established, for the other groups a systematic extension of the Cartan sub-algebra must considered [33, 36, 37]. This yields either causality conditions appropriate for a response function or symmetry conditions as expected for a correlator (this construction has not yet been carried out for the meta-conformal transformations in two dimensions). The additional time-dependent rotations of

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3Furthermore, without these extensions, i.e. the naïve co-variant two-point functions become
coordinate changes
\[ z' = f(z) \quad z' = f(\bar{z}) \]
\[ z' = z \quad \bar{z}' = \bar{f}(\bar{z}) \]

Schrödinger-Virasoro
\[ t' = \beta(t) \quad r' = (d\beta(t)/dt)^{1/2} r \]
\[ t' = t \quad r' = r + a(t) \]

conformal galilean
\[ t' = \beta(t) \quad r' = (d\beta(t)/dt) r \]
\[ t' = t \quad r' = r + a(t) \]

meta-conformal 1D
\[ t' = f(t) \quad \rho' = f(\rho) \]
\[ t' = t \quad \rho' = a(\rho) \]

meta-conformal 2D
\[ t' = t \quad z' = f(z) \quad \bar{z}' = \bar{f}(\bar{z}) \]
\[ t' = t \quad z' = z \quad \bar{z}' = \bar{f}(\bar{z}) \]
\[ t' = \theta(t) \quad w' = w \quad \bar{w}' = \bar{w} \]

Table 1: Several examples of infinite-dimensional groups of time-space transformations, with the defining coordinate changes. Herein, \( f, \bar{f}, \theta \) are arbitrary functions, and \( a \) an arbitrary vector-valued function, of their argument. In addition, \( z, \bar{z} \) are (complex) light-cone coordinates, \( \rho = t + \mu r \) and \( w = t + \beta z, \bar{w} = t + \beta \bar{z} \). The physical interpretation of the co-variant \( n \)-point functions as either correlators or responses is based on the extension of the Cartan sub-algebra [35, 36, 37].

the Schrödinger and conformal galilean groups for \( d \geq 2 \) dimensions are not explicitly listed. Only the ortho-conformal transformation include rotations between the ‘time’ and ‘space’ coordinates.

Here, we wish to investigate the existence of meta-conformal dynamical symmetries in more than one spatial dimension. We shall therefore look for dynamical symmetries of higher-dimensional analogues of ballistic transport equations. In trying to find such algebras, explicit constructions will begin with the maximal finite-dimensional subalgebra \( \{X_n, Y_n\}_{n \in \{0, \pm 1\}} \), where \( X_{-1}, Y_{-1} \) are time and space translations, respectively, \( X_0 \) is the dilatation generator, \( Y_0 \) is the generator of generalised Galilei transformations and \( X_1, Y_1 \) are ‘special’ meta-conformal transformations.

In section 2, we begin by adding some further aspects of the 1D meta-conformal case, especially the finite transformations listed in table 1 and also a generalisation of the representation of the Lie algebra (1.2). From this, an ansatz for the \( d \)-dimensional construction is extracted and used in section 3 to find the generic form of the generators. Particular attention will be devoted to construct the terms which will describe how primary scaling operators will transform under meta-conformal transformations. In section 4 we shall concentrate on the special case of \( d = 2 \) dimensions, where stronger results are found. First, we find that there exist two distinct, non-isomorphic symmetric algebras, which are distinguished by the values \( p = -1 \) and \( p = \frac{1}{2} \) of a certain parameter which arises in the construction. Second, in the case \( p = -1 \), we shall show that the symmetry algebra is infinite-dimensional and isomorphic to the direct sum of three

\[
\langle \phi(t, r)\phi(0, 0)\rangle_{CGA} = \Phi^{(0)t^{-\xi}} t^\xi \exp[-\xi r/t] \quad \text{for the conformal galilean case} \\
\langle \phi(t, r)\phi(0, 0)\rangle_{meta} = \Phi^{(0)t^{-\xi}} [1 + \mu r/t]^{-\xi/\mu} \quad \text{for the 1D metaconformal case. These show non-physical singularities. Extending the Cartan algebra modifies the results to the physically acceptable forms} \\
\langle \phi(t, r)\phi(0, 0)\rangle_{CGA} = \Phi^{(0)t^{-\xi}} t^\xi \exp[-\xi r/t] \\
\langle \phi(t, r)\phi(0, 0)\rangle_{meta} = \Phi^{(0)t^{-\xi}} [1 + \mu r/t]^{-\xi/\mu} \quad [37]. \text{Bootstrap approaches should reproduce these singularity-free forms.}
Virasoro algebras (without central charge). As shown in table 1, the corresponding Lie group includes ortho-conformal transformations of the spatial variables as a sub-group. The time-dependent transformations might be used to generate the temporal evolution of the physical system. Indeed, the co-variant two-point function is explicitly seen to describe the relaxation towards an ortho-conformally two-point function, which reflects the meta-conformal aspects in this Lie group. On the other hand, in the case \( p = \frac{1}{3} \), we only succeeded to find a finite-dimensional Lie group of dynamical symmetries. The form of the co-variant two-point function is distinct. We conclude in section 5. An appendix gives the details for finding explicitly the finite transformations from the Lie algebra generators.

2 Meta-conformal algebras: general remarks

2.1 Finite 1D meta-conformal transformation

In order to obtain a better geometric picture of the meta-conformal transformations (1.1), we begin by deriving the corresponding finite 1D meta-conformal transformations. Formally, they are given by the Lie series

\[
F_Y(\varepsilon, t, r) = e^{\varepsilon Y_m} F(0, t, r) \quad \text{and} \quad F_X(\varepsilon, t, r) = e^{\varepsilon X_n} F(0, t, r),
\]

with the generators taken from (1.1). They are given as the solutions of the two initial-value problems

\[
\begin{align}
(\partial_\varepsilon + (t + \mu r)^m \partial_r + (m + 1)\gamma (t + \mu r)^m) F_Y(\varepsilon, t, r) &= 0 \\
(\partial_\varepsilon + t^n \partial_t + \mu^{-1} [(t + \mu r)^n + t^n + 1] \partial_r + (n + 1) \left( xt^n + \frac{\gamma}{\mu} [(t + \mu r)^n - t^n] \right) ) F_X(\varepsilon, t, r) &= 0
\end{align}
\]

subject to the initial conditions \( F_X(0, t, r) = F_Y(0, t, r) = \phi(t, r) \).

Rather than presenting the details of that integration (see the appendix for this), it is more instructive to look immediately at the result, see also table 1. A simple form is obtained by using the variable \( \rho = t + \mu r \) instead of \( r \):

\[
\begin{align}
Y_m : \phi'(t, \rho) &= \left( \frac{d\phi'}{d\rho} \right)^{\gamma/\mu} \phi(t', \rho') \quad ; \quad t' = t, \quad \rho' = a(\rho) \\
X_n : \phi'(t, \rho) &= \left( \frac{d\phi'}{dt} \right)^{\gamma - x} \left( \frac{d\phi'}{d\rho} \right)^{\gamma} \phi(t', \rho') \quad ; \quad t' = \beta(t), \quad \rho' = \beta(\rho)
\end{align}
\]

where \( a = a(\rho) \) and \( \beta = \beta(t) \) are arbitrary functions. By expanding \( \beta(t) = t + \varepsilon t^{n+1} \) and \( a(\rho) = \rho + \varepsilon \rho^{m+1} \) the differential equations for the Lie series as they follow from the explicit generators can be recovered.

Eqs. (2.2) give the global form of the 1D meta-conformal transformations and the transformation of the associated primary scaling operators. Since this work is mainly interested in finding new meta-conformal symmetries, we shall leave the construction of the full conformal field-theory based on (2.2) to future work.

We also note the transformation of \( r \) as generated by \( X_n \)

\[
r' = \frac{1}{\mu} [\beta(t + \mu r) - \beta(t)].
\]
2.2 A generalisation of the one-dimensional case

Considering (1.3), or else (2.17) restricted to \( d = 1 \), and focussing on the finite-dimensional sub-algebra of symmetries, it is known that the generators \( X_1, Y_{0,1} \) can be generalised as follows, with new constants \( \alpha, \beta \) [55]:

\[
X_1 = -(t^2 + \alpha r^2) \partial_t - (2tr + \beta r^2) \partial_r - 2xt - 2\mu xr, \\
Y_0 = -\alpha r \partial_t - (t + \beta r) \partial_r - \gamma \\
Y_1 = -\alpha (2tr + \beta r^2) \partial_t \\
- (t^2 + 2\beta tr + (\alpha + \beta^2)r^2) \partial_r - 2\gamma t - 2(\alpha x + \beta \gamma)r.
\]

while the generators \( X_{-1,0}, Y_{-1} \) are un-modified with respect to (1.1). For \( n, m, \in \{0, \pm 1\} \) they satisfy the following commutation relations

\[
[X_n, X_m] = (n-m)X_{n+m}, \quad [X_n, Y_m] = (n-m)Y_{n+m} \\
[Y_n, Y_m] = (n-m)(\alpha X_{n+m} + \beta Y_{n+m}).
\]

Although these commutators look different, the Lie algebra \( \langle X_n, Y_n \rangle_{n \in \{0, \pm 1\}} \cong \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) \) is isomorphic to the ortho-conformal algebra [55, Prop. 1] and hence also to (1.2). Furthermore, the generators (2.4) are indeed dynamical meta-conformal symmetries of the 1D eq. (2.17), if the parameters are chosen as follows

\[
\alpha = \frac{1 + \beta c}{c^2}, \quad x = -\gamma c, \quad c = -\mu^{-1}
\]

Then the dynamical symmetries follow from the commutators

\[
\begin{align*}
\hat{B}, X_1 &= -2 \left( t + \frac{1 + \beta c}{c^2 - cr} \right) \hat{B} \\
\hat{B}, Y_0 &= -\frac{1 + \beta c}{c^2} \hat{B} \\
\hat{B}, Y_1 &= -2 \frac{1 + \beta c}{c^2} (ct + (1 + \beta c)r) \hat{B}
\end{align*}
\]

such that the solution space of (2.17) is indeed invariant.

In order to write an infinite dimensional representation of the algebra (2.5) we first obtain the generator \( X_2 \) from

\[
[X_2, X_{-1}] = 3X_1, \quad [X_2, X_0] = 2X_2, \quad [X_2, Y_{-1}] = 3Y_1
\]

Starting from a general form

\[
X_2 = -a(t, r)\partial_t - b(t, r)\partial_r - c(t, r),
\]

and satisfying the system (2.8) we obtain

\[
\begin{align*}
a(t, r) &= t^3 + 3\alpha tr^2 + \alpha \beta r^3 \\
b(t, r) &= 3t^2 r + 3\beta tr^2 + (\alpha + \beta^2)r^3 \\
c(t, r) &= 3xt^2 + 6\gamma tr + 3(\alpha x + \beta \gamma)r^2
\end{align*}
\]
Next, the higher members of $X_n$ hierarchy can be obtained from the commutator

$$X_n = \frac{1}{n-2}[X_{n-1}, X_1].$$

(2.9)

We obtain

$$X_2 = -(t^3 + \alpha(3t^2 + \beta r^3)) \partial_t - (3t^2r + 3\beta tr^2 + \beta^2 r^3 + \alpha r^3) \partial_r - 3xt^2 - 3\gamma(2tr + \beta r^2) - 3\alpha xr^2$$

(2.10)

$$X_3 = -(t^4 + \alpha(6t^2r^2 + 4\beta^3r^4 + \alpha^2r^4)) \partial_t - (4t^3r + 6\beta t^2r^2 + 4\beta^2 t^3 + \beta^3 r^4 + 2\alpha(2t^3r + \beta r^4)) \partial_r - 4xt^3 - 4\gamma(3t^2r + 3\beta r^2 + \beta^2 r^3) - 4\alpha x(3t^2r + \beta r^3) - 4\alpha\gamma r^3$$

(2.11)

$$X_4 = -(t^5 + \alpha(10t^3r^2 + 10\beta t^2r^3 + 5\beta^2 tr^4 + \beta^3 r^5) + 2\alpha^2(5t^4r^4 + 2\beta r^5)) \partial_t - \left(\frac{1}{\beta}[(t + \beta r)^5 - t^5] + \alpha(10t^2r^3 + 10\beta t^4 + 3\beta^2 r^5 + \alpha^2 r^5)\right) \partial_r - 5xt^4 - 5\gamma (t + \beta r)^4 - t^4 - 5\alpha x(6t^2r^2 + 4\beta tr^3 + \beta^2 r^4) - 4\alpha\gamma (2t^3r + \beta r^4) - 5\alpha^2 xr^4.$$  

(2.12)

Correspondingly, the higher members of $Y_n$ hierarchy are obtained by the commutator

$$Y_n = n^{-1}[X_n, Y_0].$$

(2.13)

We obtain

$$Y_2 = -\alpha(3t^2r + 3\beta tr^2 + \beta^2 r^3 + \alpha r^3) \partial_t - ((t + \beta r)^3 + \alpha(3t^2 + 2\beta r^3)) \partial_r - 3\gamma (t + \beta r)^2 - 3\alpha(\beta x + \gamma) r^2$$

(2.14)

$$Y_3 = -\alpha(4t^3r + 6\beta t^2r^2 + 4\beta^2 t^3 + \beta^3 r^4 + 2\alpha(2t^3r + \beta r^4)) \partial_t - ((t + \beta r)^4 + \alpha(6t^2r^2 + 8\beta tr^3 + 3\beta^2 r^4) + \alpha^2 r^4) \partial_r - 4\gamma (t + \beta r)^3 - \alpha x \frac{4}{\beta} ((t + \beta r)^3 - t^3) - 4\alpha\gamma(3t^2r + 2\beta r^3) - 4\alpha^2 xr^3$$

(2.15)

$$Y_4 = -\alpha \frac{\beta}{\beta} \left( (t + \beta r)^5 - t^5 + \alpha \left( 15\beta t^2 r^3 + \frac{35}{2} \beta^2 t^4 r^2 + \frac{11}{2} \beta^3 r^5 \right) + \frac{9}{4} \alpha^2 r^5 \right) \partial_t$$

$$- \left( (t + \beta r)^5 - t^5 + \alpha(10t^3r^2 + 20\beta t^2r^3 + 15\beta^2 t^4 r^2 + 4\beta^3 r^5) + \frac{5}{4} \alpha^2 (3t^4r^4 + 2\beta r^5) \right) \partial_r$$

$$- 5\gamma (t + \beta r)^4 - \alpha x \frac{5}{\beta} ((t + \beta r)^4 - t^4) - 5\alpha\gamma(6t^2r^2 + 8\beta tr^3 + 3\beta^2 r^4)$$

$$- 5\alpha^2 x(4t^3r^3 + 2\beta r^4) - 5\alpha^2 \gamma r^4.$$  

(2.16)

One can conclude that an infinite-dimensional structure in the generators $X_n, Y_n$ is possible to exist up to terms linear in $\alpha$. However for $n \geq 2$ the terms with higher power of $\alpha$ appear although the commutation relation of the algebra are steel valid for $n = 3$. In particular the strange coefficients in $Y_4$ alarms that the commutation relations are no more satisfied (see for example $[X_{-1}, Y_4] = -5Y_3$). It follows that an infinite-dimensional representation of metacconformal algebra exists only for $\alpha = 0$ and is given by the generators (1.1).
2.3 Ansatz for the $d$-dimensional case

Higher-dimensional analogues of the meta-conformal algebra (1.2) are sought as dynamical symmetries of a ballistic transport equation, of the form

$$\hat{B}f(t, \mathbf{r}) = (\partial_t + \mathbf{c} \cdot \partial_{\mathbf{r}})f(t, \mathbf{r}) = 0$$

(2.17)

where $\mathbf{c} \in \mathbb{R}^d$ is a constant vector, which naturally generalizes eq. (1.3).

Then the generators of translations and dynamical scaling are trivially generalized to the $d$-dimensional case

$$X_{-1} = -\partial_t$$

(2.18a)

$$Y_{j}^{j} = -\partial_{r_j}, \quad j \in \{1, \ldots, d\}$$

(2.18b)

$$X_0 = -t\partial_t - \mathbf{r} \cdot \partial_{\mathbf{r}} - \delta$$

(2.18c)

where $\delta$ stands for a scaling dimension. Then the form of all generators follows from the one of $X_1$. We make the following ansatz

$$X_1 := -(t^2 + \alpha r^2)\partial_t - 2t\mathbf{r} \cdot \partial_{\mathbf{r}} - pr^2\beta \cdot \partial_{\mathbf{r}} - (1-p)(\beta \cdot \mathbf{r})\mathbf{r} \cdot \partial_{\mathbf{r}} - 2\delta t$$

$$-\mathbf{A}(\mathbf{r}, \beta) \cdot \partial_{\beta} - \mathbf{B}(\mathbf{r}, \beta, \gamma) \cdot \partial_{\gamma} - k\gamma \cdot \mathbf{r}$$

(2.19)

where $\alpha$, $p$ and $k$ are scalars, $\beta$, $\gamma$ are constant vectors and $\mathbf{A}, \mathbf{B}$ are vector functions of their arguments. All these must be found self-consistently from the algebra we are going to construct. The form of the $X_1$ is motivated as follows.

- Taking into account the form of $X_1$, eq. (2.4) for $d = 1$ space dimension, we see that the sum of the prefactors of the terms quadratic in $\mathbf{r}$ must give unity

- $X_1$ should be invariant under spatial rotations $[X_1, R_{ij}] = 0$, $R_{ij} = r_i \partial_{r_j} - r_j \partial_{r_i}$. (2.20)

However, even in the simplest case when

$$\mathbf{A} = \mathbf{B} = \gamma = 0$$

(2.21)

the resulting form of $X_1$ eq. (2.19) is not rotation-invariant under the $R_{ij}$ (2.20). Therefore, it will be necessary to include rotations of the vectors $\beta$ and $\gamma$ such that the rotation generator becomes

$$\bar{R}_{ij} = r_i \partial_{r_j} - r_j \partial_{r_i} + \gamma_i \partial_{\gamma_j} - \gamma_j \partial_{\gamma_i} + \beta_i \partial_{\beta_j} - \beta_j \partial_{\beta_i}$$

(2.22)

Furthermore, rotation-invariance of $X_1$ will require the possibility that $\mathbf{A} \neq 0$ and $\mathbf{B} \neq 0$. In addition, taking into account the commutation relation of the one-dimensional case (2.4), especially $[[X_1, Y_{-1}^j], Y_{-1}^j] \sim Y_{-1}^j$, it follows that $\mathbf{A}, \mathbf{B}$ can at most be linear in $\mathbf{r}$. 

Additional restriction on the forms of $X_1$ come from the requirement that it should act as a dynamical symmetry of eq. (2.17). By ‘dynamical symmetry’ we mean the following required commutator [50]

$$[\hat{B}, X_1] = \lambda(t, r) \hat{B}. \tag{2.23}$$

which implies that the space of solutions of $\hat{B}\phi = 0$ is invariant under the action of $X_1$. As we shall see, this requirement leads to new relations between $\alpha$, $p$ and $\beta$. As an example, consider the case $d = 3$. From (2.23) it follows that $\delta = 0$ and the following conditions

\begin{align*}
1 + \beta_x c_x + \frac{1-p}{2} (\beta_y c_y + \beta_z c_z) &= \alpha c_x^2 \tag{2.24a} \\
p\beta_x c_y + \frac{1-p}{2} \beta_y c_x &= \alpha c_x c_y \tag{2.24b} \\
p\beta_x c_z + \frac{1-p}{2} \beta_z c_x &= \alpha c_x c_z \tag{2.24c} \\
1 + \frac{1-p}{2} \beta_x c_y + \frac{1-p}{2} \beta_y c_x &= \alpha c_y^2 \tag{2.24d} \\
p\beta_y c_x + \frac{1-p}{2} \beta_x c_y &= \alpha c_x c_y \tag{2.24e} \\
p\beta_y c_z + \frac{1-p}{2} \beta_z c_y &= \alpha c_z c_y \tag{2.24f} \\
1 + \frac{1-p}{2} (\beta_x c_x + \beta_y c_y) + \beta_z c_z &= \alpha c_z^2 \tag{2.24g} \\
p\beta_z c_x + \frac{1-p}{2} \beta_x c_z &= \alpha c_x c_z \tag{2.24h} \\
p\beta_z c_y + \frac{1-p}{2} \beta_y c_z &= \alpha c_y c_z. \tag{2.24i}
\end{align*}

We look for a solution of the above system for $\beta \neq 0$. Straightforward calculations show that

- The case $p = 1$ leads to contradictions between some of the equations in the system. This means that in this case the generator $X_1$ cannot be a symmetry.

- For $p \neq 1$ we have following solution of the system (2.24)

\begin{align*}
c_j &= \frac{2}{p-1} \frac{\beta_j}{\beta^2}, \quad j = x, y, z \tag{2.25a} \\
\alpha &= \frac{(p+1)(p-1)}{4} \beta^2 \tag{2.25b}
\end{align*}

and the condition (2.23) is satisfied, with $\lambda(t, r) = -2t - (p+1)(\beta \cdot r)$. In particular, it follows that $\alpha = 0$ is only possible for $p = -1$.

In certain case, where a more general form of $X_1$ with $A \neq 0$ or $B \neq 0$ is needed, the sought symmetries generated by $X_1$ can become conditional symmetries, that is some auxiliary conditions on the field $\Phi(t, r, \beta, \gamma)$ must be imposed. This comes from the fact that $A \neq 0$ and $B \neq 0$ are in general linear functions of $r$. 
3 Meta-conformal algebra in $d > 2$ spatial dimensions

Our preliminary computations suggest a principal difference between the algebras which generalize the meta-conformal algebra $\text{mconf}(1, 1)$ to $\text{mconf}(1, d)$ for $d = 2$ and for $d > 2$. We shall consider this two cases separately beginning with generalization for $d > 2$. In particular we shall give possible representations of the algebra $\text{mconf}(1, 3)$. Starting from the ansatz (2.19), we obtain

$$Y_0^j = \frac{1}{2}[X_1, Y_{-1}^j]$$

$$= -\alpha r_j \partial_t - \left( t + \frac{1}{2} (1 - p)(\beta \cdot r) \right) \partial_{r_j} - pr_j \beta \cdot \partial_r - \frac{1}{2} (1 - p)\beta_j r \cdot \partial_r - (k/2)\gamma_j \quad (3.1)$$

In particular for $d = 3$ case we have

$$Y_0^x = -\alpha x \partial_t - \left( t + \beta_x x + \frac{1 - p}{2}(\beta_y y + \beta_z z) \right) \partial_x$$

$$- \left( p\beta_y x + \frac{1 - p}{2} \beta_x y \right) \partial_y - \left( p\beta_z x + \frac{1 - p}{2} \beta_x z \right) \partial_z - (k/2)\gamma_x$$

$$Y_0^y = -\alpha y \partial_t - \left( p\beta_y x + \frac{1 - p}{2} \beta_y y \right) \partial_x - \left( t + \beta_y y + \frac{1 - p}{2}(\beta_x x + \beta_z z) \right) \partial_y$$

$$- \left( p\beta_z x + \frac{1 - p}{2} \beta_y z \right) \partial_z - (k/2)\gamma_y$$

$$Y_0^z = -\alpha z \partial_t - \left( p\beta_z x + \frac{1 - p}{2} \beta_z x \right) \partial_x - \left( p\beta_y x + \frac{1 - p}{2} \beta_z y \right) \partial_y$$

$$- \left( t + \beta_z z + \frac{1 - p}{2}(\beta_y y + \beta_x x) \right) \partial_z - (k/2)\gamma_z \quad (3.2)$$

When calculating $[Y_0^j, Y_0^i]$ for $i \neq j$ we obtain taking into account the value of $\alpha = \frac{(p-1)(p+1)}{4} \beta^2$

$$[Y_0^x, Y_0^y] = \frac{(3p - 1)(p + 1)(p - 1)}{8} (\beta_y x - \beta_x y) \partial_t$$

$$+ \left( \frac{(3p - 1)(p + 1)}{4} (\beta_x \beta_y x - \beta_y^2 y) + \frac{(1 - p)^2}{4} (\beta_z^2 y - \beta_y \beta_z z) \right) \partial_x$$

$$+ \left( \frac{(3p - 1)(p + 1)}{4} (\beta_y^2 x - \beta_x \beta_y y) - \frac{(1 - p)^2}{4} (\beta_z^2 x - \beta_x \beta_z z) \right) \partial_y$$

$$+ p^2 (\beta_y \beta_z x - \beta_x \beta_y y) \partial_z. \quad (3.3)$$

It follows that for the solutions of the equation

$$p^2 - (1 - p)^2/4 = (p + 1)(3p - 1)/4 = 0, \quad (3.4)$$

that is for $p_1 = -1$ and $p_2 = 1/3$ one can write

$$[Y_0^x, Y_0^y] = -p^2 (\beta_x^2 R_{xy} + \beta_x \beta_y R_{yz} - \beta_y \beta_z R_{xz}) \quad (3.5)$$

\(^4\)In the ansatz (2.19) we first take $A = B = 0$, but if necessary we shall redefine $X_1$.\n
Similar calculations shows that for the same values of $p = -1, 1/3$ it is fulfilled

\[ [Y^x_0, Y^z_0] = p^2(\beta_y^2 R_{xy} + \beta_x^2 R_{yz} - \beta_y R_{xz}) \] (3.6)

\[ [Y^y_0, Y^z_0] = -p^2(\beta_x^2 R_{xy} + \beta_y^2 R_{yz} - \beta_x R_{xz}) \] (3.7)

Looking at the expressions (3.5,3.6,3.7) we see two problems

- On the right-hand side the commutators give the generators $R_{ij}$ instead $\bar{R}_{ij}$. In fact one can add directly $R_{\beta_i \beta_j}$ to $R_{ij}$ on the right-hand side because

\[
0 = \beta_z^2 R_{\beta_x \beta_y} + \beta_x \beta_z R_{\beta_x \beta_y} - \beta_y \beta_z R_{\beta_x \beta_y} \\
= \beta_y \beta_z R_{\beta_x \beta_y} + \beta_x \beta_y R_{\beta_x \beta_y} - \beta_y^2 R_{\beta_x \beta_y} \\
= \beta_x \beta_z R_{\beta_x \beta_y} + \beta_y^2 R_{\beta_x \beta_y} - \beta_x \beta_y R_{\beta_x \beta_y}.
\] (3.8)

On the other hand $R_{\gamma \gamma}$ can not be added directly. It can be obtained working with a more general form of $X_1$ (2.19) with $\mathbf{A} = 0$ and appropriate form of $\mathbf{B}(r, \beta, \gamma) \neq 0$ (we shall do this in the second subsection).

- The second problem is also in the right-hand sides of (3.5,3.6,3.7), namely they contain as pre-factors the components of vector $\beta$, which enter in generator of rotations $\bar{R}_{ij}$ and are considered as variables together with $x, y, z$ and $\gamma_x, \gamma_y, \gamma_z$. In order to obtain a closed algebraic structure, which contains the generators $Y_{x, y, z}$ and $\bar{R}_{ij}$ we must require

\[
R_{\beta_i \beta_j} = \beta_i \partial_{\beta_j} - \beta_j \partial_{\beta_i} = 0
\] (3.9)

So to satisfy above we must fix $\beta$ such that it is characterized by unique scalar non-zero parameter, that is if more than one of the components of $\beta$ are non-zero, they must be equal (up to sign). We shall restrict to the case $\beta_x = \beta, \beta_y = \beta_z = 0$, that is a ballistic transport in $x$ direction.

### 3.1 Meta-conformal algebra in $d = 3$ space dimensions with $\beta = (\beta, 0, 0)$ and $\gamma = 0$

Because of the fact that $\gamma = 0$, the generators of rotations have the usual form $R_{ij} = r_i \partial_{r_j} - r_j \partial_{r_i}$. Then for $\alpha = \frac{1}{4} (p + 1)(p - 1) \beta^2$ we obtain

\[
X_1 = -(t^2 + \alpha(x^2 + y^2 + z^2)) \partial_t - (2tx + \beta x^2 + \beta p(y^2 + z^2)) \partial_x \\
-(2t + \frac{1 - p}{2} \beta x) y \partial_y - (2t + \frac{1 - p}{2} \beta x) z \partial_z - 2\delta t.
\] (3.10)

From the above form of $X_1$ we can generate all other generators. Starting from $Y^t_0 = \frac{1}{2} [X_1, Y^t_1]$ we obtain

\[
Y^x_0 = -\alpha x \partial_t - (t + \beta x) \partial_x - \frac{1 - p}{2} \beta \partial_y - \frac{1 - p}{2} \beta \partial_z
\] (3.11)

\[
Y^y_0 = -\alpha y \partial_t - p \beta y \partial_y - (t + \frac{1 - p}{2} \beta x) \partial_y
\] (3.12)

\[
Y^z_0 = -\alpha z \partial_t - p \beta z \partial_z - (t + \frac{1 - p}{2} \beta x) \partial_z.
\] (3.13)
We calculate that \([Y_0^x, Y_0^y] = [Y_0^x, Y_0^z] = 0\), but
\[
[Y_0^y, Y_0^z] = -p^2\beta^2 R_{yz}. \tag{3.14}
\]
Next we obtain also
\[
[Y_0^y, Y_0^z] = \alpha X_{-1} + \beta Y_{-1}^z, \tag{3.15}
\]
\[
[Y_0^y, Y_0^y] = [Y_0^z, Y_0^z] = \alpha X_{-1} + p\beta Y_{-1}^z, \tag{3.16}
\]
\[
[Y_0^y, Y_0^y] = [Y_0^y, Y_0^z] = \frac{1 - p}{2} \beta Y_{-1}^y, \tag{3.17}
\]
\[
[Y_0^y, Y_{-1}^z] = [Y_0^z, Y_{-1}^z] = \frac{1 - p}{2} \beta Y_{-1}^z. \tag{3.18}
\]
Furthermore, from the commutators \(Y_1^j = [X_1, Y_0^j]\) we calculate
\[
Y_1^x = -\alpha \left( 2tx + \beta x^2 + (1 - 2p)\beta (y^2 + z^2) \right) \partial_t - 2\alpha \delta x \\
- \left( t + 2\beta tx + \frac{1 - p}{2} \beta^2 ([p + 2]x^2 + p[y^2 + z^2]) \right) \partial_x \\
- (1 - p) \beta \left( t + \frac{1 - p}{2} \beta x \right) (y \partial_y + z \partial_z) \tag{3.19}
\]
\[
Y_1^y = -2\alpha (t + p\beta x)y \partial_t - 2p\beta \left( t + \frac{p^2 + 4p - 1}{4p} \beta x \right) y \partial_x - 2\alpha \delta y \\
- (t^2 + (1 - p)\beta tx + p^2 \beta^2 (x^2 - y^2 + z^2)) \partial_y + 2p^2 \beta^2 yz \partial_z \tag{3.20}
\]
\[
Y_1^z = -2\alpha (t + p\beta x)z \partial_t - 2p\beta \left( t + \frac{p^2 + 4p - 1}{4p} \beta x \right) z \partial_x - 2\alpha \delta z \\
+ 2p^2 \beta^2 yz \partial_y - (t^2 + (1 - p)\beta tx + p^2 \beta^2 (x^2 + y^2 - z^2)) \partial_z \tag{3.21}
\]
The nonzero commutators are
\[
[Y_1^x, Y_{-1}^x] = 2(\alpha X_0 + \beta Y_{-1}^x) \tag{3.22}
\]
\[
[Y_1^y, Y_{-1}^y] = [Y_1^z, Y_{-1}^z] = 2(\alpha X_0 + p\beta Y_{-1}^x) \tag{3.23}
\]
\[
[Y_1^x, Y_{-1}^y] = [Y_1^y, Y_{-1}^x] = (1 - p)\beta Y_{-1}^y \tag{3.24}
\]
\[
[Y_1^y, Y_{-1}^z] = [Y_1^z, Y_{-1}^y] = (1 - p)\beta Y_{-1}^y \tag{3.25}
\]
\[
[Y_1^y, Y_{-1}^y] = 2\beta^2 R_{yz} = -[Y_1^z, Y_{-1}^z] \tag{3.26}
\]
\[
[Y_1^x, Y_0^x] = \alpha X_1 + \beta Y_1^x \tag{3.27}
\]
\[
[Y_1^y, Y_0^y] = [Y_1^z, Y_0^z] = \alpha X_1 + p\beta Y_1^x \tag{3.28}
\]
\[
[Y_1^x, Y_0^y] = [Y_1^y, Y_0^x] = \frac{1 - p}{2} \beta Y_1^y \tag{3.29}
\]
\[
[Y_1^x, Y_0^z] = [Y_1^z, Y_0^x] = \frac{1 - p}{2} \beta Y_1^z. \tag{3.30}
\]
in addition to
\[
[X_n, Y_m^j] = (n - m)Y_{n+m}^j, \quad n = 0, \pm 1, m = 0, \pm 1, j = x, y, z \tag{3.31}
\]
\[
[Y_m, R_{yz}] = Y_m^z, \quad [Y_m^z, R_{yz}] = -Y_m^y, \quad m = 0, \pm 1. \tag{3.32}
\]
It follows that the algebra
\[
mconf(1, 3) = \{X_{0, \pm 1}, Y_{0, \pm 1}^x, y, z, R_{yz}\}
\]
is closed.
3.2 Meta-conformal algebra in $d = 3$ space dimensions with $\gamma \neq 0$

We first redefine the generator of rotations

$$R_{yz} \rightarrow \tilde{R}_{yz} =: y \partial_z - z \partial_y + \gamma_y \partial_{\gamma_z} - \gamma_z \partial_{\gamma_y}.$$  \hspace{1cm} (3.33)

Then we modify $X_1(3.10)$ by the following ansatz

$$X_1 \rightarrow X_1 + \tilde{X}_1$$
$$\tilde{X}_1 = -a(\beta \cdot r) \gamma \cdot \partial_y - b(\gamma \cdot r) \beta \cdot \partial_y - c(\beta \cdot \gamma) r \cdot \partial_y - k(\gamma, r)$$
$$= -\beta ((a + b + c)x\gamma_x + a(y\gamma_y + z\gamma_z)) \partial_{\gamma_x} - k(x\gamma_x + y\gamma_y + z\gamma_z)$$
$$- \beta(bx\gamma_y + cy\gamma_x) \partial_{\gamma_y} - \beta(bx\gamma_z + cz\gamma_x) \partial_{\gamma_z}$$  \hspace{1cm} (3.34)

where $a, b, c$ and $k$ are constants to be determined. Next we generate $Y_0^{x,y,z}$ and $Y_1^{x,y,z}$ in usual way. In particular we obtain

$$Y_0^x \rightarrow Y_0^x + \tilde{Y}_0^x, \quad Y_0^y \rightarrow Y_0^y + \tilde{Y}_0^y, \quad Y_0^z \rightarrow Y_0^z + \tilde{Y}_0^z$$
$$\tilde{Y}_0^x = -(\beta/2)((a + b + c)x\gamma_x \partial_{\gamma_x} + b(\gamma_y \partial_{\gamma_y} + \gamma_z \partial_{\gamma_z}) - (k/2)\gamma_x$$  \hspace{1cm} (3.35)
$$\tilde{Y}_0^y = -(\beta/2)(a\gamma_y \partial_{\gamma_y} + c\gamma_x \partial_{\gamma_x}) - (k/2)\gamma_y$$  \hspace{1cm} (3.36)
$$\tilde{Y}_0^z = -(\beta/2)(a\gamma_x \partial_{\gamma_x} + c\gamma_z \partial_{\gamma_z}) - (k/2)\gamma_z$$  \hspace{1cm} (3.37)

In order to satisfy the commutation relations relevant to the previous case ($\beta = (\beta, 0, 0), \gamma = 0$) we first obtain $b = a, \quad c = -a$ (from $[Y_0^x, Y_0^y] = [Y_0^x, Y_0^z] = 0$), and $a = \pm 2p$ (from $[Y_0^y, Y_0^z] = -p^2\beta^2 R_{yz}$) and next the following representations of meta-conformal algebra $\text{mconf}(1, 3)$

$$X_1 = -(t^2 + \alpha(x^2 + y^2 + z^2)) \partial_t - (2tx + \beta x^2 + \beta p(y^2 + z^2)) \partial_x$$
$$- (2t + \frac{1-p}{2} \beta x)y \partial_y - (2t + \frac{1-p}{2} \beta x)z \partial_z$$
$$- a\beta(x\gamma_x + y\gamma_y + z\gamma_z) \partial_{\gamma_x} - a\beta(x\gamma_y - y\gamma_x) \partial_{\gamma_y}$$
$$- a\beta(x\gamma_z - z\gamma_x) \partial_{\gamma_z} - 2dt - k(x\gamma_x + y\gamma_y + z\gamma_z),$$  \hspace{1cm} (3.38)

$$Y_0^x = -\alpha x \partial_t - (t + \beta x) \partial_x - \frac{1-p}{2} \beta \partial_y - \frac{1-p}{2} \beta \partial_z$$
$$- \alpha \beta/2(\gamma_x \partial_{\gamma_x} + \gamma_y \partial_{\gamma_y} + \gamma_z \partial_{\gamma_z}) - (k/2)\gamma_x, $$  \hspace{1cm} (3.39)

$$Y_0^y = -\alpha y \partial_t - p\beta y \partial_y - (t + \frac{1-p}{2} \beta x) \partial_y$$
$$- \alpha \beta/2(\gamma_y \partial_{\gamma_y} - \gamma_x \partial_{\gamma_x}) - (k/2)\gamma_y, $$  \hspace{1cm} (3.40)

$$Y_0^z = -\alpha z \partial_t - p\beta z \partial_z - (t + \frac{1-p}{2} \beta x) \partial_z$$
$$- \alpha \beta/2(\gamma_z \partial_{\gamma_z} - \gamma_x \partial_{\gamma_x}) - (k/2)\gamma_z. $$  \hspace{1cm} (3.41)

---

\(^5\)Modifying $X_1$ and correspondingly $Y_0^{x,y,z}$ and $Y_1^{x,y,z}$ by additive terms obviously does not change the commutation relations.
\[ Y^x_1 = -\alpha \left( 2tx + \beta x^2 + (1 - 2p)\beta (y^2 + z^2) \right) \partial_t - \left( t + 2\beta tx + \frac{1 - p}{2}\beta^2 \left( [p + 2]x^2 + p[y^2 + z^2] \right) \right) \partial_x - (1 - p)\beta \left( t + \frac{1 - p}{2}\beta x \right) (y \partial_y + z \partial_z) - a\beta \left( t + \beta x \right) \gamma_x - \frac{1 - p}{2}\beta (y \gamma_y + z \gamma_z) \partial_{\gamma_x} - a\beta \left( t + \beta x \right) \gamma_y - \frac{1 - p}{2}\beta y \gamma_z \partial_{\gamma_y} - a\beta \left( t + \beta x \right) \gamma_z - k\gamma_x - (2\alpha \delta + \beta \gamma_x) x - k\beta \frac{1 - p}{2}(\gamma_y y + \gamma_z z), \]

(3.42)

\[ Y^y_1 = -2\alpha(t + p\beta x)y \partial_t - 2p\beta \left( t + \frac{p^2 + 4p - 1}{4p}\beta x \right) y \partial_y - \left( t^2 + (1 - p)\beta tx + p^2\beta^2 (x^2 - y^2 + z^2) \right) \partial_y + 2p^2\beta^2 y z \partial_z - a\beta \left( p\beta y \gamma_x + (t + \frac{1 - p}{2}\beta x) \gamma_y \right) \partial_{\gamma_x} + a\beta \left( t + \frac{1 - p}{2}\beta x \right) - p\beta y \gamma_y - \frac{a\beta}{2} \gamma_z \partial_{\gamma_y} - a\beta \left( p\beta y \gamma_z - \frac{a\beta}{2} z \gamma_y \right) \partial_{\gamma_z} - k\gamma_y - k\beta \frac{1 - p}{2} \gamma_y x - (2\alpha \delta + k p\beta \gamma_x) y, \]

(3.43)

\[ Y^z_1 = -2\alpha(t + p\beta x)z \partial_t - 2p\beta \left( t + \frac{p^2 + 4p - 1}{4p}\beta x \right) z \partial_z + 2p^2\beta^2 y z \partial_y - \left( t^2 + (1 - p)\beta tx + p^2\beta^2 (x^2 + y^2 - z^2) \right) \partial_z - a\beta \left( p\beta z \gamma_x + (t + \frac{1 - p}{2}\beta x) \gamma_z \right) \partial_{\gamma_x} - a\beta \left( p\beta z \gamma_y - \frac{a\beta}{2} y \gamma_z \right) \partial_{\gamma_y} + a\beta \left( t + \frac{1 - p}{2}\beta x \right) \gamma_x - \frac{a\beta}{2} y \gamma_y - p\beta z \gamma_z \partial_{\gamma_x} - k\gamma_z - k\beta \frac{1 - p}{2} \gamma_z x - (2\alpha \delta + k p\beta \gamma_x) z. \]

(3.44)

Note that the above expressions for the representation of the meta-conformal algebra give in fact four different representations, parameterized by the admissible values of \( p = -1, a = -2, 2 \) and \( p = 1/3, a = -2/3, 2/3 \), while the value of \( k \) is not fixed in the level of commutation relations.
Example: $p = -1$, $a = 2$, $k = 2$

\[
X_1 = -t^2 \partial_t - (2tx + \beta(x^2 - y^2 - z^2)) \partial_x - 2(t + \beta x) y \partial_y - 2(t + \beta x) z \partial_z - \delta (x \gamma_x - y \gamma_y - z \gamma_z) - 2 \beta (x \gamma_x - y \gamma_y - z \gamma_z) \partial_{\gamma_x} - 2 \beta (x \gamma_y - y \gamma_y - z \gamma_z) \partial_{\gamma_y} - 2 \beta (x \gamma_z - y \gamma_y - z \gamma_z) \partial_{\gamma_z}
\]

\[
Y_0^x = -(t + \beta x) \partial_x - \beta y \partial_y - \beta z \partial_z - \beta \gamma_x \partial_{\gamma_x} - \beta \gamma_y \partial_{\gamma_y} - \beta \gamma_z \partial_{\gamma_z} - \gamma_x
\]

\[
Y_0^y = \beta y \partial_x - (t + \beta x) \partial_y - \beta \gamma_x \partial_{\gamma_x} + \beta \gamma_y \partial_{\gamma_y} - \gamma_y
\]

\[
Y_0^z = \beta z \partial_x - (t + \beta x) \partial_z - \beta \gamma_x \partial_{\gamma_x} + \beta \gamma_z \partial_{\gamma_z} - \gamma_z
\]

\[
Y_1^x = -((t + \beta x)^2 - \beta^2 (y^2 + z^2)) \partial_x - 2 \beta (t + \beta x) y \partial_y - 2 \beta (t + \beta x) z \partial_z - 2 \beta ((t + \beta x) \gamma_x + \beta (y \gamma_y + z \gamma_z)) \partial_{\gamma_x} - 2 \beta (t \gamma_y + \beta (x \gamma_x - y \gamma_z)) \partial_{\gamma_y}
\]

\[
Y_1^y = 2 \beta (t + \beta x) y \partial_x - ((t + \beta x)^2 + \beta^2 (z^2 - y^2)) \partial_y + 2 \beta^2 y z \partial_z
\]

\[
Y_1^z = 2 \beta (t + \beta x) z \partial_x + 2 \beta^2 y z \partial_y - ((t + \beta x)^2 + \beta^2 (y^2 - z^2)) \partial_z
\]

\section{3.3 Symmetries}

We shall now verify whether the representations (2.18a, 2.18b, 2.18c, 3.38, 3.39, 3.40, 3.41, 3.42, 3.43, 3.44, 3.33) act as symmetry algebra of an equation in the form (2.17, 2.25a)

\[
\hat{B}_A \Phi_\gamma(t, \mathbf{r}) = \left( \partial_t + \frac{2}{\beta(p - 1)} \partial_x \right) \Phi_\gamma(t, \mathbf{r}) = 0.
\]

We calculate

\[
[\hat{B}_A, X_1] = -(2t + (p + 1) \beta x) \hat{B}_A - 2 \left( \delta + \frac{k}{\beta(p - 1)} \gamma_x + \frac{a}{p - 1} \gamma \cdot \partial_\gamma \right)
\]

\[
[\hat{B}_A, X_0] = -\hat{B}_A,\quad [\hat{B}_A, Y_0^x] = -\frac{p + 1}{2} \beta \hat{B}_A
\]

\[
[\hat{B}_A, Y_0^y] = -\frac{\beta(p + 1)}{2} (2t + (p + 1) \beta x) \hat{B}_A - (p + 1) \beta \left( \delta + \frac{2 + (p - 1)k}{(p - 1)(p + 1)} \gamma_x + \frac{a}{p - 1} \gamma \cdot \partial_\gamma \right)
\]

\[
[\hat{B}_A, X_{-1}] = [\hat{B}_A, Y_{-1}^{x,y,z}] = [\hat{B}_A, Y_{0}^{y,z}] = [\hat{B}_A, Y_{1}^{y,z}] = [\hat{B}_A, \bar{R}_{yz}] = 0.
\]

We conclude the following

- For $\gamma_x = \gamma_y = \gamma_z = 0$ the meta-conformal algebra left invariant the equations (3.52) under condition that $\delta = 0$

- For $\gamma \neq 0$ and under condition that $\Phi_\gamma(t, \mathbf{r}) = \Phi(t, \mathbf{r})$ that is only representations of the algebra depend on $\gamma$, the meta-conformal algebra left invariant the equations (3.52) for $k = 1$ and $\gamma_z = (1 - p) \beta \delta$.  

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Finally, if the fields depend on $\gamma$, the condition they must satisfy is

$$
\left( \delta + \frac{k}{\beta(p-1)} \gamma_x + \frac{a}{p-1} \gamma \cdot \partial \gamma \right) \Phi_\gamma(t, r) = 0 \tag{3.55}
$$

In addition, $k = 1$ is required. It follows that we have on shell or conditional symmetry algebra of equation (3.52).

## 4 Meta-conformal algebra in $d = 2$ spatial dimensions

Generalisations of the one-dimensional case to $d = 2$ space dimensions (with points $(t, x, y) \in \mathbb{R}^3$) can be proceed as follows. The generators of translations and dynamical scaling read

- $X_{-1} = -\partial_t$
- $Y_{-1}^x = -\partial_x$,
- $Y_{-1}^y = -\partial_y$
- $X_0 = -t\partial_t - x\partial_x - y\partial_y - \delta$.

For this case the form of $X_1$ can be obtained from (2.19) putting $A = B = 0$ and $k = 2$. As we shall show it is the simplest case when one is able to find a closed algebraic structure. We write

$$
X_1 = -\left( t^2 + \alpha(x^2 + y^2) \right) \partial_t - \left( 2tx + \beta_x x^2 + (1-p)\beta_y xy + p\beta_x y^2 \right) \partial_x
- \left( 2ty + p\beta_y y^2 + (1-p)\beta_x xy + \beta_y y^2 \right) \partial_y - 2\delta t - 2\gamma_x x - 2\gamma_y y. \tag{4.2}
$$

and the generator of rotations reads

$$
R_{xy} = x\partial_y - y\partial_x + \gamma_x \partial_{\gamma_x} - \gamma_y \partial_{\gamma_y} + \beta_x \partial_{\beta_x} - \beta_y \partial_{\beta_y}. \tag{4.3}
$$

As seen before, we have $[X_1, R_{xy}] = 0$. Combination with the translations gives the next two generators, namely

- $Y_0^x := \frac{1}{2}[X_1, Y_{-1}^x]$
- $Y_0^y := \frac{1}{2}[X_1, Y_{-1}^y]$

Because of (2.5), it is necessary to verify whether $Y_0^x$ and $Y_0^y$ commute or not. We calculate

$$
[Y_0^x, Y_0^y] = \alpha \left( \frac{3p-1}{2} (\beta_y x - \beta_x y) \partial_t + x \partial_y - y \partial_x \right)
+ \frac{p+1}{2} \left( \frac{3p-1}{2} \beta_x \beta_y x - (p\beta_x^2 + \frac{1-p}{2} \beta_y^2) y \right) \partial_x
+ \frac{p+1}{2} \left( (p\beta_y^2 + \frac{1-p}{2} \beta_x^2) x - \frac{3p-1}{2} \beta_x \beta_y y \right) \partial_y.
\tag{4.5}
$$

where in the second row we have substituted $\alpha = \frac{(p+1)(p-1)}{4} \beta^2$ taken from (2.25b). It follows that $[Y_0^x, Y_0^y] = 0$ for two cases:
1. $p = -1$ and consequently $\alpha = 0$.
2. $p = 1/3$, hence $\alpha = -\frac{2}{9}(\beta_x^2 + \beta_y^2)$.

We shall take up these two distinct cases separately.

### 4.1 The case $p = -1$

As mentioned in this case $\alpha = 0$, so the generators $X_1, Y_0^x$ and $Y_0^y$ reduce to

\[
X_1 = -t^2 \partial_t - (2tx + \beta_xx^2 + 2\beta_xy - \beta_xy^2) \partial_x - (2ty - \beta_yx^2 + 2\beta_xy + \beta_yy^2) \partial_y - 2\delta t - 2\gamma x - 2\gamma y. \tag{4.6a}
\]

\[
Y_0^x = -(t + \beta_xx + \beta_yy) \partial_x - (\beta_xy - \beta_yx) \partial_y - \gamma _x. \tag{4.6b}
\]

\[
Y_0^y = -(\beta_yx - \beta_xy) \partial_x - (t + \beta_yx + \beta_xx) \partial_y - \gamma _y. \tag{4.6c}
\]

The last two generators $Y_1^x := [X_1, Y_0^x]$ and $Y_1^y := [X_1, Y_0^y]$ become

\[
Y_1^x = -(t^2 + 2t\beta_xx + 2t\beta_yy + (\beta_x^2 - \beta_y^2)x^2 + 4\beta_x\beta_xy - (\beta_x^2 - \beta_y^2)y^2) \partial_x - (2t\beta_xy - 2t\beta_yx - 2\beta_x\beta_yx^2 + 2(\beta_x^2 - \beta_y^2)xy + 2\beta_x\beta_yy^2) \\
- 2\gamma_x(t + \beta_xx + \beta_yy) - 2\gamma_y(\beta_yx - \beta_x) \tag{4.7a}
\]

\[
Y_1^y = -(2t\beta_yx - 2t\beta_yx + 2\beta_x\beta_xy^2 - 2(\beta_x^2 - \beta_y^2)xy - 2\beta_x\beta_yy^2) \partial_x - (t^2 + 2t\beta_xx + 2t\beta_yy + (\beta_x^2 - \beta_y^2)x^2 + 4\beta_x\beta_xy - (\beta_x^2 - \beta_y^2)y^2) \partial_y \\
- 2\gamma_y(t + \beta_xx + \beta_yy) - 2\gamma_x(\beta_yx - \beta_x) \tag{4.7b}
\]

It is readily checked that $[Y_1^x, Y_1^y] = [X_1, Y_1^x] = [X_1, Y_1^y] = 0$ and finally, that the generators (4.1, 4.6, 4.7, 4.3) satisfy the following commutation relations, with $n, m \in \{0, \pm 1\}$

\[
[X_n, X_m] = (n - m)X_{n+m}, \\
[X_n, Y_0^x] = (n - m)Y_0^x_{n+m}, \quad [X_n, Y_0^y] = (n - m)Y_0^y_{n+m}, \\
[Y_0^x, Y_0^y] = [Y_0^y, Y_0^x] = (n - m)(\beta_yY_0^y_{n+m} + \beta_xY_0^x_{n+m}), \\
[Y_0^x, Y_m^x] = -[Y_0^y, Y_m^x] = (n - m)(\beta_yY_0^y_{n+m} + \beta_xY_0^x_{n+m}), \\
[Y_m^x, \bar{R}_xy] = Y_m^y, \quad [Y_m^y, \bar{R}_xy] = -Y_m^x. \tag{4.8}
\]

It is turned out that the Lie algebra

\[
m_{conf}^A(1, 2) := \langle X_{0, \pm 1}, Y_{0, \pm 1}^x, Y_{0, \pm 1}^y, R_{xy} \rangle \tag{4.9}
\]

is really closed if $\beta_x = \beta, \beta_y = 0$ (or if $\beta_x = \pm \beta_y$, see previous section), when $\beta$ is just a parameter. However all the generators acts as dynamical symmetries of the linear differential equation (even for general $\beta = (\beta_x, \beta_y)$)

\[
\hat{B}f(t, x, y) = (\partial_t + c_x\partial_x + c_y\partial_y)f(t, x, y) = 0 \tag{4.10}
\]

if

\[
\gamma_xc_x + \gamma_yc_y + \delta = 0 \tag{4.11}
\]

\[
\beta_x = -\frac{c_x}{c_x^2 + c_y^2}, \quad \beta_y = -\frac{c_y}{c_x^2 + c_y^2}. \tag{4.12}
\]
Indeed, one has in general that

\[
[B, X_{-1}] = [B, Y^x_1] = [B, Y^y_1] = 0 \quad \Rightarrow \quad [B, X_0] = -B. \tag{4.13}
\]

In addition, under the conditions (4.11, 4.12), we also have

\[
[B, Y^x_0] = [B, Y^y_0] = [B, Y^x_1] = [B, Y^y_1] = 0 \quad \Rightarrow \quad [B, X_1] = -2tB. \tag{4.14}
\]

which implies the invariance of the solution space.

### 4.1.1 Infinite-dimensional extension

A better understanding of the algebraic structure behind the rather awkward set (4.8) of commutators is obtained by choosing the coordinate axes such that the vector \( \beta = (\beta, 0) \) of the orientation of the ballistic transport is along the \( x \)-axis. This leads to a slight simplification (in what follows, because of (4.12), we simply have \( \beta = \beta_x \) and \( \beta_y = 0 \)), in particular, the generator of rotations now reads

\[
R = R_{xy} = x\partial_y - y\partial_x + \gamma x\partial_{\gamma_y} - \gamma y\partial_{\gamma_x} \tag{4.15}
\]

Considering the commutators between \( Y^x_n \) and \( Y^y_n \), the peculiar signs arising suggest to go over to new generators

\[
Y^\pm_n := \frac{1}{2}(Y^x_n \pm iY^y_n) \tag{4.16}
\]

Then the commutators (4.8) simplify to (with \( n, m \in \{\pm 1, 0\} \))

\[
\begin{align*}
[X_n, Y^\pm_m] &= (n - m)Y^\pm_{n+m} \\
[Y^\pm_n, Y^\pm_m] &= \beta(n - m)Y^\pm_{n+m} - [Y^+_n, Y^-_m] = 0 \tag{4.17}
\end{align*}
\]

In addition, we go over to complex spatial coordinates \( z = x - iy \) and \( \bar{z} = x + iy \). The generators are then re-expressed as follows

\[
\begin{align*}
X_{-1} &= -\partial_t \\
Y^+_1 &= -\partial \\
X_0 &= -t\partial_t - z\partial - \bar{z}\bar{\partial} - \delta \\
Y^+_0 &= -(t + \beta z)\partial - \gamma \\
X_1 &= -t^2\partial_t - (2tz + \beta z^2)\partial - (2tz + \beta z^2)\bar{\partial} - 2\delta t - 2\gamma z - 2\bar{\gamma}\bar{z} \\
Y^+_1 &= -(t + \beta z)^2\partial - 2\gamma(t + \beta z)
\end{align*} \tag{4.18}
\]

where \( \partial = \partial_z, \bar{\partial} = \partial_{\bar{z}} \) and the complex components \( \gamma := \frac{1}{2}(\gamma_x + i\gamma_y) \) and \( \bar{\gamma} := \frac{1}{2}(\gamma_x - i\gamma_y) \). The generators \( Y^-_n \) are obtained from \( Y^+_n \) by the replacements \( z \mapsto \bar{z}, \partial \mapsto \bar{\partial} \) and \( \gamma \mapsto \bar{\gamma} \). Clearly, restricting to points \( (t, z) \) or \( (t, \bar{z}) \), we recover the representation (1.1) of the meta-conformal algebra in \( d = 1 \) dimensions, restricted to \( n = 1, 0, 1 \).

The algebra (4.17) is identical to the non-local meta-conformal algebra found recently for the diffusion-limited erosion process in one spatial dimension [38, 39]. Therefore, we define the new generators

\[
A_n := X_n - \frac{1}{\beta}Y^+_n - \frac{1}{\beta}Y^-_n \tag{4.19}
\]
In the basis \( \langle A_n, Y_n^+, Y_n^- \rangle_{n \in \{\pm 1, 0\}} \), the Lie algebra (4.17) becomes the direct sum \( \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) \). The above definition can now be extended to an infinite-dimensional set of generators, with \( n \in \mathbb{Z} \)

\[
A_n = -t^{n+1} \left( \partial_t - \frac{1}{\beta} \partial - \frac{1}{\beta} \bar{\partial} \right) - (n+1)t^n \left( \delta - \frac{\gamma}{\beta} - \bar{\gamma} \right)
\]

\[
Y_n^+ = -(t + \beta z)^{n+1} \partial - (n+1)\gamma(t + \beta z)^n
\]

\[
Y_n^- = -(t + \beta \bar{z})^{n+1} \bar{\partial} - (n+1)\bar{\gamma}(t + \beta \bar{z})^n
\]  

(4.20)

with the only non-vanishing commutators

\[ [A_n, A_m] = (n-m)A_{n+m} \quad [Y_n^\pm, Y_m^\pm] = \beta(n-m)Y_{n+m}^\pm \]  

(4.21)

such that the Lie algebra (4.21) is isomorphic \( \mathfrak{vect}(S^1) \oplus \mathfrak{vect}(S^1) \oplus \mathfrak{vect}(S^1) \), the direct sum of three Virasoro algebras without central charge. Finally, the ballistic operator (4.10) becomes \( \hat{B} = -\partial_t + \frac{1}{\beta}(\partial + \bar{\partial}) \) and obeys the commutators

\[ [A_n, \hat{B}] = (n+1)t^n \hat{B} - (n+1)nt^{n-1} \bar{\delta} \quad [Y_n^\pm, \hat{B}] = 0 \]  

(4.22)

where \( \bar{\delta} := \delta - \frac{\gamma}{\beta} - \bar{\gamma} \). Summarising, we have proven:

**Proposition:** In two spatial dimensions \( r = (x, y) \), the linear ballistic transport equation (2.17) can be brought to the form \( \hat{B} f(t,x,y) = (-\partial_t + \beta^{-1}\partial_x) f(t,x,y) = 0 \), where \( \beta \) is a constant. Its maximal dynamical symmetry is infinite-dimensional, spanned by the generators (4.20), if only \( \bar{\delta} = \delta - \frac{\gamma}{\beta} - \frac{\bar{\gamma}}{\bar{\beta}} = 0 \). Herein, complex coordinates \( z = x - iy, \bar{z} = x + iy \) and the associated derivatives \( \partial = \partial_z \) and \( \bar{\partial} = \partial_{\bar{z}} \) are used and \( \gamma, \bar{\gamma}, \delta \) are constants. The Lie algebra of dynamical symmetries is given by (4.21) and is isomorphic to the direct sum of three centre-less Virasoro algebras.

Working with the coordinates \( w = t + \beta z \) and \( \bar{w} = t + \beta \bar{z} \), we see that the symmetries generated by \( Y_n^\pm \) are ortho-conformal in the variables \( (w, \bar{w}) \), while the action of the genertors \( A_n \) are meta-conformal. This appears to be the first known example which combines ortho- and meta-conformal transformations into a single symmetry algebra. If \( \bar{\delta} = 0 \), we actually have a spectrum-generating algebra for \( \hat{B} = A_0 \). In spite of the symmetric formulation, the equation of motion (4.10) contains a bias, since the transport goes along the axis \( x = \frac{1}{2}(z + \bar{z}) \), if \( \beta \neq 0 \).

### 4.1.2 Finite transformations

The finite transformations associated with the generators \( A_n, Y_n^+, Y_n^- \) with \( n \in \mathbb{Z} \) are given by the corresponding Lie series, for scaling operators which are scalars under spatial rotations. The final result is simple:

\[
Y_n^+ : \quad \phi'(t, z, \bar{z}) = \left( \frac{dz'}{dz} \right)^{-\gamma/\mu} \phi(t', z', \bar{z}') \quad t' = t , \quad z' = a(z) , \quad \bar{z}' = \bar{z} \quad (4.23a)
\]

\[
Y_n^- : \quad \phi'(t, z, \bar{z}) = \left( \frac{dz'}{dz} \right)^{-\gamma/\mu} \phi(t, z, \bar{z}') \quad t' = t , \quad z' = z , \quad \bar{z}' = \bar{a}(\bar{z}) \quad (4.23b)
\]

\[
A_n : \quad \phi'(t, w, \bar{w}) = \left( \frac{dt'}{dt} \right)^{\delta} \phi(t', w', \bar{w}') \quad t' = k(t) , \quad w' = w , \quad \bar{w}' = \bar{w} \quad (4.23c)
\]
with the coordinates \( w = t + \beta z, \bar{w} = t + \beta \bar{z} \) and \( k = k(t), a = a(z), \bar{a} = \bar{a}(\bar{z}) \) are arbitrary functions. Expanding these according to \( k(t) = t + \epsilon t^{n+1} \), and analogously for \( a(z) \) and \( \bar{a}(\bar{z}) \), the explicit differential equations for the Lie series can be recovered. Their direct integration is detailed in the appendix.

Eqs. (4.23) clearly show that the relaxational behaviour described by the 2D meta-conformal symmetry is governed by three independent conformal transformations, rather than two as it is the case for 2D conformal invariance at the stationary state.

### 4.1.3 Two-point function

A simple application of dynamical symmetries is the computation of covariantly transforming two-point functions. Non-trivial results can be obtained from so-called ‘quasi-primary’ scaling operators \( \phi(t, z, \bar{z}) \), which transform co-variantly under the finite-dimensional sub-algebra \( \{A_{\pm 1,0}, Y_{\pm 1,0}\} \). Because of temporal and spatial translation-invariance, we can directly write

\[
F(t, z, \bar{z}) = \langle \phi_1(t, z, \bar{z})\phi_2(0, 0, 0) \rangle
\]

(4.24)

where the brackets indicate a thermodynamic average which will have to be carried out when such two-point functions are to be computed in the context of a specific statistical mechanics model. Extending the generators (4.20) to two-body operators, the covariance is then expressed through the Ward identities \( X_0^{[2]}F = X_1^{[2]}F = Y_0^{[2]}F = Y_1^{[2]}F = 0 \). Each scaling operator is characterised by three constants \( (\tilde{\delta}, \gamma, \bar{\gamma}) \). Standard calculations (along the well-known lines of ortho- or meta-conformal invariance) then lead to

\[
F(t, z, \bar{z}) = F_0 \delta_{\tilde{\delta}_1, \tilde{\delta}_2} \beta \gamma \delta_{\gamma_1, \gamma_2} t^{-2\delta_1} (t + \beta z)^{-2\gamma_1} (t + \beta \bar{z})^{-2\bar{\gamma}_1}
\]

(4.25)

where \( F_0 \) is a normalisation constant. This shows a cross-over between an ortho-conformal two-point function when \( t \ll z, \bar{z} \) and a non-trivial scaling form in the opposite case \( t \gg z, \bar{z} \). We illustrate this for scalar quasi-primary scaling operators, where \( \gamma_1 = \bar{\gamma}_1 \)

\[
F(t, z, \bar{z}) \sim \begin{cases} 
  t^{-2\delta_1} (z \bar{z})^{-2\gamma_1} & \text{if } t \ll z, \bar{z} \\
  t^{-2\delta_1} \exp \left[ -2\beta \gamma_1 \left| \frac{z + \bar{z}}{t} \right| \right] & \text{if } t \gg z, \bar{z}
\end{cases}
\]

(4.26)

If the time-difference is small compared to the spatial distance, the form of the correlator reduces to the one of standard, ortho-conformal invariance. For increasing time-differences \( t \), the behaviour becomes increasingly close to the known one of effectively 1D meta-conformal invariance.\(^6\)

\(^6\)We did not yet carry out the algebraic procedure which should in the \( t \gg z, \bar{z} \) limit produce the non-diverging behaviour \( F \sim t^{-2\delta_1} \exp \left[ -2\beta \gamma_1 \left| \frac{z + \bar{z}}{t} \right| \right] \), see [37].
4.2 The case $p = 1/3$

We shall write the generators $X_1, Y^x_0, Y^y_0$ and non-zero commutation relations for $\beta = (\beta, 0)$, that is for $\alpha = -(2/9)\beta^2$

\[
X_1 = -\left( t^2 - \frac{2}{9} \beta^2 (x^2 + y^2) \right) \partial_t - \left( 2tx + \beta x^2 + \frac{1}{3} \beta ty^2 \right) \partial_x \\
- \left( 2ty + \frac{2}{3} \beta xy \right) \partial_y - 2\delta t - 2\gamma x x - 2\gamma y y.
\]

(4.27a)

\[
Y^x_0 = \frac{2}{9} \beta^2 x \partial_t - (t + \beta x) \partial_x - \frac{1}{3} \beta y \partial_y - \gamma x
\]

(4.27b)

\[
Y^y_0 = \frac{2}{9} \beta^2 y \partial_t - \frac{1}{3} \beta y \partial_x - \left( t + \frac{1}{3} \beta x \right) \partial_y - \gamma y.
\]

(4.27c)

The last two generators $Y^x_1 := [X_1, Y^x_0]$ and $Y^y_1 := [X_1, Y^y_0]$ become

\[
Y^x_1 = \frac{2}{9} \beta^2 \left( 2tx + \beta x^2 + \frac{1}{3} \beta y^2 \right) \partial_t - \left( t^2 + 2\beta tx + \frac{7}{9} \beta^2 x^2 + \frac{1}{9} \beta^2 y^2 \right) \partial_x \\
- \left( \frac{2}{3} \beta ty + \frac{2}{9} \beta^2 xy \right) \partial_y - 2\gamma x t - 2\left( \beta \gamma x - \frac{2}{9} \beta^2 \delta \right) x - \frac{2}{3} \beta \gamma y y
\]

(4.28a)

\[
Y^y_1 = \frac{2}{9} \beta^2 \left( 2ty + \frac{2}{3} \beta xy \right) \partial_t - \left( \frac{2}{3} \beta ty + \frac{2}{9} \beta^2 xy \right) \partial_x \\
- \left( t^2 + \frac{2}{3} \beta tx + \beta \gamma y x - 2\left( \frac{1}{3} \beta \gamma x - \frac{2}{9} \beta^2 \delta \right) y
\]

(4.28b)

It is readily checked that $[Y^x_1, Y^y_1] = [X_1, Y^x_1] = [X_1, Y^y_1] = 0$ and finally, that the generators (4.1, 4.6, 4.7, 4.3) satisfy the following commutation relations, with $n, m \in \{0, \pm1\}$

\[
[X_n, X_m] = (n - m)X_{n+m}, \\
[X_n, Y^x_m] = (n - m)Y^x_{n+m}, \quad [X_n, Y^y_m] = (n - m)Y^y_{n+m},
\]

\[
[Y^x_n, Y^y_m] = [Y^y_n, Y^x_m] = \frac{n - m}{3} \beta Y^y_{n+m}, \\
[Y^x_n, Y^x_m] = (n - m) \left( -\frac{2}{9} \beta^2 X_{n+m} + \beta Y^x_n \right), \\
[Y^y_n, Y^y_m] = (n - m) \left( -\frac{2}{9} \beta^2 X_{n+m} + \frac{\beta}{3} Y^x_{n+m} \right), \\
[Y^x_m, R_{xy}] = Y^y_m, \quad [Y^y_m, R_{xy}] = -Y^x_m
\]

(4.29)

In addition the Lie algebra

\[
m_{\text{conf}}^B := \langle X_{0, \pm 1}, Y^x_{0, \pm 1}, Y^y_{0, \pm 1}, R_{xy} \rangle
\]

(4.30)

acts as dynamical symmetry algebra of the linear differential equation

\[
\dot{B}_p f (t, x, y) = \left( \partial_t - \frac{2}{3} \beta \partial_x \right) f(t, x, y) = 0
\]

(4.31)

if $\delta = (3/\beta) \gamma x$. 

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Indeed, one has in general that
\[ [\hat{B}_p, X_{-1}] = [\hat{B}_p, Y^x_{-1}] = [\hat{B}_p, Y^y_{-1}] = 0, \quad [\hat{B}, X_0] = -\hat{B}. \] (4.32)

In addition, we obtain
\[ [\hat{B}_p, Y^y_0] = [\hat{B}_p, Y^y_1] = 0 \]
\[ [\hat{B}_p, X_1] = -\left(2t + \frac{4}{3} \beta x\right) \hat{B}_p \]
\[ [\hat{B}_p, Y^y_0] = -\frac{2}{3} \beta \hat{B}_p \]
\[ [\hat{B}_p, Y^y_1] = -\frac{4}{3} \beta \left(2t + \frac{2}{3} \beta x\right) \hat{B}_p, \] (4.33)
which proves the symmetries.

Looking at the structure of the algebra (4.30) it can be shown that
\[ \text{mconf}^B(1, 2) \not\cong \text{sl}(2, \mathbb{R}) \oplus \text{sl}(2, \mathbb{R}) \oplus \text{sl}(2, \mathbb{R}). \]

### 4.2.1 Two-point function

For this case, the two-point function is built by the quasi-primary fields \( \phi(t, x, y, \gamma_x = \gamma, \gamma_y = \lambda) \) which transform covariantly under the algebra \( \text{mconf}^B(1, 2)(4.30) \). Taking into account the covariance of space and time translations we write
\[ F = \langle \phi(t_1, x_1, y_1, \lambda_1) \phi(t_2, x_2, y_2, \lambda_2) \rangle = F(t, x, y, \gamma_1, \gamma_2, \lambda_1, \lambda_2), \]
where \( t = t_1 - t_2, x = x_1 - x_2, y = y_1 - y_2 \). The covariance under the other generators is expressed by the following system, to be satisfied by the \( F \)

\[ X_0 : \quad (t \partial_t + x \partial_x + y \partial_y + \delta_1 + \delta_2) F = 0 \]
\[ Y^x_0 : \quad \left(\frac{2}{9} \beta^2 x \partial_t - \left(t + \beta x\right) \partial_x - \frac{\beta}{3} y \partial_y - \gamma_1 - \gamma_2\right) F = 0 \]
\[ Y^y_0 : \quad \left(\frac{2}{9} \beta^2 y \partial_t - \frac{\beta}{3} y \partial_x - \left(t + \frac{\beta}{3} x\right) \partial_y - \lambda_1 - \lambda_2\right) F = 0 \]
\[ X_1 : \quad \left(- \left(t^2 + \frac{2}{9} \beta^2 x^2 + \frac{2}{9} \beta^2 y^2\right) \partial_t + \beta \left(x^2 + \frac{y^2}{3}\right) \partial_x + \frac{2}{3} \beta xy \partial_y\right) F 
- (2t \delta_2 - 2x \gamma_1 - 2y \lambda_1) F = 0 \]
\[ Y^x_1 : \quad \left(\frac{2}{9} \beta^3 \left(x^2 + \frac{y^2}{3}\right) \partial_t + \left(t^2 - \frac{7}{9} \beta^2 x^2 - \frac{1}{9} \beta^2 y^2\right) \partial_x - \frac{2}{9} \beta^2 xy \partial_y\right) F
+ \left(2t \gamma_2 + \beta \left(\frac{4}{9} \beta \delta_1 - 2 \gamma_1\right) x - \frac{2}{3} \lambda_1 y\right) F = 0 \]
\[ Y^y_1 : \quad \left(\frac{4}{27} xy \partial_t - \frac{2}{9} \beta^2 xy \partial_x + \left(t^2 - \frac{\beta^2}{9} (x^2 - y^2)\right) \partial_y\right) F
+ \left(2t \lambda_2 - \frac{2}{3} \beta \lambda_1 x + \beta \left(\frac{4}{9} \beta \delta_1 - \frac{2}{3} \lambda_1\right) y\right) F = 0 \]
\[ R_{xy} : \quad (x \partial_y - y \partial_x + \gamma_1 \partial_{\lambda_1} - \lambda_1 \partial_{\gamma_1} + \gamma_2 \partial_{\lambda_2} - \lambda_2 \partial_{\gamma_2}) F = 0 \]
Next the above system is reduced as follows: the equation corresponding to $X_1$ is replaced by the one corresponding to $X_1 = X_1 + tX_0 + xY_0^x + Y_0^y$ that is instead of fourth equation we write the sum of the forth equation with the first multiplied by $t$, second multiplied by $x$ and third multiplied by $y$. It becomes trivial and we retain

$$\gamma_1 = \gamma_2 = \gamma, \quad \lambda_1 = \lambda_2 = \lambda, \quad \delta_1 = \delta_2 = \delta,$$

and the result is substituted in the other equations. In the same way we replace the fifth equation by an equation corresponding to $\bar{Y}_1^x = Y_1^x - \beta x Y_0^x - (\beta/3)y Y_0^y$ and the sixth equation by an equation corresponding to $\bar{Y}_1^y = Y_1^y - (\beta/3)y Y_0^y - (\beta/3)x Y_0^y$. We obtain a reduced system acting on $F = F(t, x, y, \gamma, \lambda)$

$$\begin{align*}
(t\partial_t + x\partial_x + y\partial_y + 2\delta)F_1 &= 0 \\
\left(\frac{2}{9}\beta^2 x\partial_t - (t + \beta x)\partial_x - \frac{\beta}{3} y\partial_y - 2\gamma\right)F_1 &= 0 \\
\left(\frac{2}{9}\beta^2 y\partial_t - \frac{\beta}{3} y\partial_x - \left(t + \frac{\beta}{3} x\right)\partial_y - 2\lambda\right)F_1 &= 0 \\
\left(t^2 + \beta tx + \frac{2}{9}\beta^2 x^2\right)\partial_x + \frac{\beta}{3} \left(ty + \frac{2}{3}\beta xy\right)\partial_y + 2t\gamma + \frac{4}{9}\beta^2 \delta x \right)F_1 &= 0 \\
\left(\frac{\beta}{3} \left(ty + \frac{2}{3}\beta xy\right)\partial_x + \left(t^2 + \frac{2}{3}\beta tx + \frac{2}{9}\beta^2 y^2\right)\partial_y + 2t\lambda + \frac{4}{9}\beta^2 y\right)F_1 &= 0 \\
(x\partial_y - y\partial_x + \gamma\partial_\lambda - \lambda\partial_\gamma)F &= 0
\end{align*}$$

(4.34)

We shall solve the first three equations, and then substitute the result in the last three equations, to verify that the system (4.34) is satisfied.

From the first equation we express $y\partial_y F_1 = -(t\partial_t + x\partial_x + 2\delta)F_1$ and put it in the second one. We obtain

$$\left(\left(t + \frac{2}{3}\beta tx\right)\left(\frac{\beta}{3}\partial_t - \partial_x\right) + \frac{2\beta}{3} \left(\delta - \frac{3\gamma}{\beta}\right)\right)F = 0.$$  

(4.35)

Then we perform a change of variables $F_1(t, x, y) \rightarrow G(t, u = t + (2/3)\beta x, y); \partial_t \rightarrow \partial_t + \partial_u, \partial_x \rightarrow (2/3)\beta \partial_u$. The equation becomes

$$\left(\partial_t - \partial_u + \frac{2}{u} \left(\delta - \frac{3\gamma}{\beta}\right)\right)G = 0.$$  

(4.36)

Another change of variables $G(t, u, y) \rightarrow H(t, v = t + u, y); \partial_t \rightarrow \partial_t + \partial_v, \partial_u \rightarrow \partial_v$ allows us to obtain

$$\left(\partial_t + \frac{2}{v - t} \left(\delta - \frac{3\gamma}{\beta}\right)\right)H(t, v, y) = 0$$

(4.37)

and determine the dependence on $t$ in $H(t, v, y)$ (integrating with respect to $v - t$) namely

$$H(t, v, y) = H_0(v, y)(v - t)^{2(\delta - 3\gamma/\beta)}.$$  

(4.38)

Next applying the same chain of change of variables we can bring the third equation in the system (4.34) to the form

$$\begin{align*}
\left(\frac{2}{9}\beta^2 y(\partial_t + \partial_v) - \frac{1}{2} v\partial_y - 2\lambda\right)H(t, v, y) &= (v - t)^{2(\delta - 3\gamma/\beta)} \left(\frac{2}{9}\beta^2 y\partial_v - \frac{1}{2} v\partial_y - 2\lambda\right)H_0(v, y) = 0.
\end{align*}$$

(4.39)
The general solution of the above equation is given by the product of a solution of inhomogeneous equation and general solution of homogeneous one. We obtain

\[ H_0(v, y) = H_{01}(w) \exp \left( -\frac{2\lambda}{3\beta} \arctan \left( \frac{\beta|y|}{3|t + \beta x/3|} \right) \right) \]

\[ w = \left( t + \frac{\beta}{3} x \right)^2 + \frac{\beta^2}{9} y^2. \]  

(4.40)

Now substituting result (4.41) in the first equation of (4.34) we obtain

\[ w H_{01}'(w) + (2\delta - 3\gamma/\beta) H_{01}(w) = 0, \quad H_{01}(w) = H_{00} w^{3\gamma/\beta - 2\delta}. \]  

(4.41)

Taking the expression (4.38, 4.40, 4.41) we conclude that the solution of the system (4.34) and correspondingly the form of two-point function covariant under the algebra \( \text{mconf}^B(1, 2) \) is the following

\[ F(t, x, y) = F_0 \delta_{\delta_1, \delta_2} \delta_{\gamma_1, \gamma_2} \delta_{\lambda_1, \lambda_2} \left( \left( t + \frac{\beta}{3} x \right)^2 + \frac{\beta^2}{9} y^2 \right)^{3\gamma/\beta - 2\delta} \left( t + \frac{2}{3} \beta x \right)^{2(\delta - 3\gamma/\beta)} \times \]

\[ \exp \left( -\frac{2\lambda}{3\beta} \arctan \left( \frac{\beta|y|}{3|t + \beta x/3|} \right) \right). \]  

(4.42)

5 Conclusions

In this work, we have explored possible mathematical symmetries, using the linear ballistic transport equation as a simple starting point. Our main results are collected in table 1: The linear transport equation (1.3) admits in 1D and in 2D infinite-dimensional Lie groups of dynamical symmetries. In 2D, these symmetries contain ortho-conformal transformations of the spatial variables as a sub-group and describe the relaxation of the two-time correlator towards it. It will be possible to adapt constructions from local scale-invariance \([34, 40]\) such that this Lie group can also describe the relaxation of single-time correlators, which is work in progress. Remarkably, the Lie algebra of dynamical symmetries in 2D is isomorphic to the one of the spatially non-local stochastic process of 1D diffusion-limited erosion \([39]\).

Furthermore, the 2D case actually admits two distinct, non-isomorphic symmetries. The meaning of these two distinct symmetries remains to be understood.

Given the considerable variety of strongly interacting systems with dynamical exponent \( z = 1 \), explicit model studies are required in order to see which of the several kinds of conformal invariance will describe one or several of the known physical situations. Here a comparison with the explicit correlators (4.25, 4.42) might be instructive.

The other important question is to see how to set up the analogue of the ortho-conformal bootstrap. Work along these lines is in progress.

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Appendix. Finite meta-conformal transformations

We provide the details for the explicit integration of the Lie series, in order to construct the non-infinitesimal, finite meta-conformal transformations. The results are included in table 1.

A.1 One spatial dimension

The Lie series $F_Y(\varepsilon, t, r) = e^{\varepsilon Y} F(0, t, r)$ and $F_X(\varepsilon, t, r) = e^{\varepsilon X} F(0, t, r)$ are solutions of the two initial-value problems eqs. (2.1a,2.1b), subject to the initial conditions $F_X(0, t, r) = F_Y(0, t, r) = \phi(t, r)$. It turns out, however, that the calculations are simplified with the new coordinate $\rho := t + \mu r$.

We begin by finding $F_Y = F(\varepsilon, t, \rho)$. The initial-value problem (2.1a) simplifies to

$$\left( \partial_\varepsilon + \mu \rho^{m+1} \partial_\rho - (m+1)\gamma \rho^m \right) F(\varepsilon, t, \rho) = 0 \ , \ F(0, t, \rho) = \phi(t, \rho) \quad (A.1)$$

Following [44], the change of variables $F(\varepsilon, t, \rho) = G(\varepsilon, t, v)$, where $v = \varepsilon + 1/(m \mu \rho^m)$, reduces this to

$$\left( \partial_\varepsilon + \frac{(m+1)\gamma}{m} - \frac{1}{\mu v - \varepsilon} \right) G(\varepsilon, t, v) = 0$$

and integration yields

$$G(\varepsilon, t, v) = H(t, v)(v - \varepsilon)^{\frac{n+1}{n}} \mu.$$  \quad (A.2)

Therein, the initial condition $G(0, t, u) = \phi(t, r)$ fixes the last undetermined function. Setting $\varepsilon = 0$, we find $H(t, \bar{v}) = \bar{v}^{-\frac{n+1}{m}} \phi(t, (n \mu v)^{-1/n})$. Finally, (A.2) becomes

$$F(\varepsilon, t, \rho) = \phi'(t, \rho) = \left( \frac{\partial a(\rho)}{\partial \rho} \right)^{-\gamma/\mu} \phi(t', \rho') \quad (A.3a)$$

$$t' = t \ , \ \rho' = a(\rho) = \frac{\rho}{[1 + \varepsilon \mu m \rho^m]^{1/m}} \quad (A.3b)$$

where in the second line we give the corresponding transformation of the coordinates.

The second initial value problem (2.1b) for $F_X = F(\varepsilon, t, \rho)$ becomes

$$\left( \partial_\varepsilon + t^{n+1} \partial_t + \rho^{n+1} \partial_\rho + (n+1) \left( xt^n + \frac{\gamma}{\mu} [\rho^n - t^n] \right) \right) F(\varepsilon, t, \rho) = 0 \ , \ F(0, t, \rho) = \phi(t, \rho) \quad (A.4)$$

With the change of variables $F(\varepsilon, t, \rho) = G(\varepsilon, u, v)$, where $u = \varepsilon + 1/(n \mu t^n)$ and $v = \varepsilon + 1/(n \mu \rho^n)$, this becomes

$$\left( \partial_\varepsilon + \frac{n+1}{n} \frac{(\gamma - x)}{\mu} - \frac{1}{u - \varepsilon} + \frac{n+1}{n} \frac{\gamma}{\mu} \frac{1}{u - \varepsilon} \right) G(\varepsilon, u, v) = 0$$

and integrating with respect to $\varepsilon$, we find

$$G(\varepsilon, u, v) = H(u, v)(u - \varepsilon)^{\frac{n+1}{n}} \frac{\gamma}{\mu} (v - \varepsilon)^{\frac{n+1}{n} \gamma} \mu. \quad (A.5)$$
The finite transformation generated by $A$ where we have defined finite transformations generated by the generators spatial dimension, taking into account the expressions (A.3a, A.3b) we can directly write the $G$

Finally substituting in (A.5) for the solution of the second initial problem we find

$$G(\varepsilon, u, v) = \phi'(t, r) = (1 + \varepsilon nt^n)^{-\frac{n+1}{n}(\frac{\gamma}{\mu} - x)}(1 + \varepsilon np^n)^{-\frac{n+1}{n}(\frac{\gamma}{\mu})} \phi \left( \frac{t}{(1 + \varepsilon nt^n)^{1/n}}, \frac{\rho}{(1 + \varepsilon np^n)^{1/n}} \right)$$

where we have defined $\beta(z) = \frac{z}{(1 + \varepsilon nz^n)^{1/n}}$. The coordinate transformations now simply read

$$t' = \beta(t), \quad \rho' = \beta(\rho).$$

(A.7)

**A.2 Two spatial dimensions, case $p = -1$**

In two spatial dimensions, the meta-conformal algebra with $p = -1$ is spanned by the generators $\langle A_n, Y_+^n, Y_-^n \rangle_{n \in \mathbb{Z}}$. We look for the Lie series $e^{\varepsilon A_n}, e^{\varepsilon Y_+^n}$ and $e^{\varepsilon Y_-^n}$.

Since the generator $Y_+^n(Y_-^n)$ can be viewed as a generator meta-conformal algebra in one spatial dimension, taking into account the expressions (A.3a, A.3b) we can directly write the finite transformations generated by the generators $Y_+^n$ and $Y_-^n$

$$Y_+^n: \phi'(t, z, \bar{z}) = \left( \frac{da(z)}{dz} \right)^{-\gamma/\mu} \phi(t', z', \bar{z}),$$

(A.8a)

$$t' = t, \quad z' = a(z) = z(1 + \varepsilon \mu n p^n)^{-1/n}, \quad \bar{z}' = \bar{z}.$$

(A.8b)

$$Y_-^n: \phi'(t, z, \bar{z}) = \left( \frac{da(\bar{z})}{d\bar{z}} \right)^{-\gamma/\mu} \phi(t', z, \bar{z'}),$$

(A.9a)

$$t' = t, \quad z' = z, \quad \bar{z}' = a(\bar{z}) = \bar{z}(1 + \varepsilon \mu n \bar{z}^n)^{-1/n}.$$  

(A.9b)

The finite transformation generated by $A_n$ are obtained from the following initial value problem

$$\left( \partial_\varepsilon + t^{n+1} \partial_t - \frac{t^{n+1}}{\beta} \partial_z - \frac{t^{n+1}}{\beta} \partial_{\bar{z}} + (n+1) \partial t^n \right) F(\varepsilon, t, z, \bar{z}) = 0, \quad F(0, t, z, \bar{z}) = \phi(t, z, \bar{z})$$

(A.10)

The change of variables $F(\varepsilon, t, z, \bar{z}) = G(\varepsilon, t, w, \bar{w})$, where $w = t + \beta z, \bar{w} = t + \beta \bar{z}$ cast this into the form

$$\left( \partial_\varepsilon + t^{n+1} \partial_t + (n+1) \partial t^n \right) G(\varepsilon, t, w, \bar{w}) = 0, \quad G(0, t, w, \bar{w}) = \phi(t, w, \bar{w}).$$

(A.11)

Another change of variables $G(\varepsilon, t, w, \bar{w}) = H(\varepsilon, u, w, \bar{w})$ with $u = \varepsilon + t^{-n}/n$ allows us to write instead

$$\left( \partial_\varepsilon + \frac{n+1}{u} \frac{\partial}{u - \varepsilon} \right) H(\varepsilon, u, w, \bar{w}) = 0$$

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which is immediately integrated, with the result

\[ H(\varepsilon, u, w, \bar{w}) = H_0(u, w, \bar{w})(u - \varepsilon)^{\frac{n+1}{n}}. \]  

(A.12)

The initial condition \( H(0, u, w, \bar{w}) = \phi(t, w, \bar{w}) \) fixes the last function \( H + 0 \)

\[ H_0(\bar{u}, w, \bar{w}) = \bar{u}^{-\frac{3n+1}{n}} \phi\left((n\bar{u})^{-1/n}, w, \bar{w}\right) \]

and the final result reads

\[ H(\varepsilon, u, w, \bar{w}) = \phi'(t, w, \bar{w}) = (1 + \varepsilon nt^n)^{-\frac{n+1}{n}} \phi\left(t(1 + \varepsilon nt^n)^{-1/n}, w, \bar{w}\right). \]  

(A.13)

Summarising, this can can rewritten as follows

\[ \phi'(t, w, \bar{w}) = \left(\frac{dk(t)}{dt}\right)^{\frac{\delta}{\beta}} \phi\left(t', w', \bar{w}'\right) \]  

(A.14a)

\[ t' = k(t) = t(1 + \varepsilon nt^n)^{-\frac{1}{n}}, \quad w' = w, \quad \bar{w}' = \bar{w}. \]  

(A.14b)

The coordinate transformations in terms of \( t, z, \bar{z} \) are easy obtained from the above and read

\[ t' = k(t), \quad z' = z + \frac{t - k(t)}{\beta}, \quad \bar{z}' = \bar{z} + \frac{t - k(t)}{\beta} \]  

(A.15)

These take the form of conformal transformations in time and time-dependent translations in the spatial coordinates \( z, \bar{z} \).
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