On the distances between entangled pseudoscalar mesons states.

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Abstract. Entangled states of pseudoscalar mesons represent a very interesting tool for studying foundations of quantum mechanics, e.g. for testing Bell inequalities. Recently, they also emerged as a test bench for quantum information protocols. On the other hand, from a quantum information point of view, the characterization of the distance between two quantum states is a topic of the utmost importance. In this letter, with the purpose of providing a useful tool for further investigations, we address the problem of which distance allows a better discrimination between density matrices appearing in pseudoscalar phenomenology.

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1 Introduction

Entangled states of neutral pseudoscalar mesons ($K_0$ and $B_0$) represent a very interesting tool for studying foundations of quantum mechanics [1], e.g. as test of Bell inequalities [2,3,4,5]. Recently, they also emerged as a test bench for quantum information protocols [6]: for example protocols for quantum teleportation [7] and quantum eraser [8] have been proposed. These new proposals represent a very interesting innovative application for $\Phi$ and $B$ factories: indeed not only to realize similar experiments with different physical systems is of large interest, but also the use of mesons discloses new possibilities [1,2,3,4,5,6,7].

On the other hand, from a quantum information point of view, a topic of the utmost importance is the characterization of the distance between two quantum states [10], that, beyond its own intrinsic interest, is needed, for example, for assessing teleportation [11], purification [12], quantum cloning [13], remote state preparation [14] and state estimation [15]. The use of the neutral pseudoscalars in this context would represent a further interesting application to quantum information. Furthermore, the notion of distance between quantum states for kaons is also a useful tool when studying the developing of mixed states from pure ones in presence of (quantum gravity induced) decoherence [16] or when searching for CPT violation [17].

The problem of which distance is more convenient for distinguishing different quantum states is not solved in general, but only specific cases, as single qubits in a noisy channel [10], have been discussed.

In this letter we address this problem in neutral pseudoscalar mesons phenomenology. Several different experimental implementable cases, which could represent an interesting ”arena” for future studies, are investigated. The presented results provide a useful tool for future research addressed to study application of mesons both to quantum information and decoherence, suggesting which distance is better to use in specific cases.

2 Distances between entangled pseudoscalar mesons states

In the last years various distances among quantum states have been defined [9] with the purpose of comparing the states and eventually defining an entanglement measure.

Here we will consider three of the most used distances [9]: the Bures, the Hilbert-Schmidt and the trace distance [10].

The Bures distance,

$$D_B(\rho, \sigma) = \sqrt{1 - \text{Tr}[(\sqrt{\sigma} \rho \sqrt{\sigma})^2]}$$

represents the shortest path connecting two fibers (in the Hilbert Schmidt fiber bundle) lying over the two density matrices. It is both Riemannian and monotone [9].

The Hilbert-Schmidt distance,

$$D_{HS}(\rho, \sigma) = \sqrt{\text{Tr}[(\rho - \sigma)(\rho - \sigma)]}$$

is the Euclidean distance deriving from the definition of scalar product in the Hilbert-Schmidt space. It is Riemannian, but not monotone.

Finally the trace distance,

$$D_{tr}(\rho, \sigma) = \frac{1}{2}\text{Tr}|\rho - \sigma|$$

simply derives by the definition of norm. It is not Riemannian, but it is monotone. It coincides with the Hilbert-Schmidt one for single qubits.

1 In the following we use the normalisations as in Ref. [10].
In the following we will consider the basis \( K_L = \{1, 0\} \), \( K_S = \{0, 1\} \), \( K_L, K_S \) being the long and short living states of \( K_0 \) respectively (a similar discussion, \textit{mutatis mutandis}, for mass and beauty eigenstates can be performed for \( B \) mesons as well \[^3\]). The experimental determination of the meson states can be achieved through decays \[^2\].

The density matrix for the singlet state is:

\[
|\psi^-\rangle = \frac{|K^0\rangle|K^0\rangle - |K^0\rangle|K^0\rangle}{\sqrt{2}} = |K_L\rangle|K_S\rangle - |K_S\rangle|K_L\rangle
\]

as typically produced at \( \Phi \) factories, is therefore:

\[
g_s = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]

As a first example, mathematically almost trivial, let us consider the distances between the singlet and the transformed state after that one of the two components has been regenerated in a slab \[^18\], which is an interesting effect for a simple experimental implementation.

The transformation is \[^4\] (\( f \) complex, with typical values per unit thickness \(|f| \sim 10^{-3} \text{ mm}^{-1} \) \[^19\] and \[^20\] )

\[
U = \frac{1}{\sqrt{(1 + |f|^2)}} \begin{pmatrix} 1 & f & 0 & 0 \\ 0 & 1 & 0 & f \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]

In this case the three distances coincide:

\[
D_{Bures}(gs, U gsU^\dagger) = D_{tr}(gs, U gsU^\dagger) = D_{HS}(gs, U psU^\dagger) = \sqrt{\frac{|f|^2}{1 + |f|^2}}
\]

Let us now consider the more interesting case of the decoherence effect on a singlet \[^19\], \( t \) being the time

\[
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2}e^{-it} & 0 \\ 0 & \frac{1}{2}e^{-it} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]

This density matrix can describe both decoherence effects due to environment interaction \[^19\] and to gravity induced decoherence \[^16\]. Thus, one is interested in distinguishing the decohered state from the unaffected singlet: a point that has a large interest for future experiments.

Let us define \( \tau = 1/\ell \) the characteristic decoherence time scale. Again the trace and Hilbert-Schmidt distances coincide. If one looks to fig.11 one can observe as the Bures distance allows a better discrimination from the unaffected singlet at times smaller than \( \tau_s = 0.69\tau \), since it has a larger gradient. On the other hand, the Hilbert-Schmidt one is more sensitive at larger values of time. \( \tau_s \) shifts toward smaller values if a background is added, \( \zeta_d = x|\sigma|D + (1 - x)1/4 \), being \( \tau_s = 0.67\tau \) for a 1% background and going to 0.51 for a 10% one.

As a further example, one can consider the statistical mixture of the singlet with a singlet regenerated by a slab, \( \varrho = xgs + (1 - x)U gsU^\dagger \), whose distance is measured from the singlet. In this case the Hilbert-Schmidt distance, coinciding with the trace one, is always less sensitive than the Bures one.

The general case of the distance between the regenerated singlet \( U(f1)gsU(f1)^\dagger \) and the mixing of the singlet regenerated by two different slabs, \( xU(f1)gsU(f1)^\dagger + (1 - x)U(f2)gsU(f2)^\dagger \) represents an interesting example where a more complex situation can be studied (and that eventually can also be implemented experimentally). This case must be considered with a full variation of modulus and phase of \( f1, f2 \). An example is reported in fig.2, where one can observe the plot of the difference between Bures and Hilbert-Schmidt distances when \( f1, f2 \) are real and \( x = 0.5 \). For this case one can see that, when varying \( |f2| \) at fixed \( |f1| \), the Hilbert-Schmidt distance is more sensitive than the Bures one when \(|f2| < |f1| \) and vice-versa. When a phase is added to \( f1 \), the situation does not change substantially, except for a decrease of the difference \( D_B - D_{HS} \) at small \( f2 \) (vanishing when the phase goes to \( \pi/2 \)) and a small decrease of the inversion point under \(|f1| \). On the other hand, when we scan the variation with \( x \) at fixed values of \( f1, f2 \), the Bures distance is always more sensitive than the Hilbert-Schmidt one when \( x < 0.75 \) and vice-versa for \( x > 0.75 \) (e.g. see fig.3).

In conclusion, we consider the case where the singlet is mixed with a background, \( \varrho = xgs + (1 - x)1/4 \) [in quantum information terminology a "depolarizing channel"] Here, the Hilbert - Schmidt distance is always more sensitive than the trace one. On the other hand it is less sensitive than the Bures distance for small values of \( x (x < 0.5) \), see fig.2. Finally, Bures distance is more discriminant than the trace one up to \( x = 2/3 \) and less for larger \( x \). The situation remains exactly the same when considering the mixing \( U gsU^\dagger + (1 - x)1/4 \) between regenerated singlet with a background and changing the phase and the modulus of \( f \) (in a reasonable interval corresponding to the experimental accessible zone \[^3\]).

In summary, our calculations show that the choice of which distance to consider for studying the 0\( ^{+} \) mesons entanglement depends on the cases, in particular the Hilbert-Schmidt and the Bures ones look to be the most promising. Our results suggest which would be the best selection for some specific different situations of theoretical and experimental interest.

3 Conclusions

In this letter we have presented a study addressed to estimate which distance between states is more sensitive when \(^3\) i.e. \(|f| < 0.1 \). For \(|f| \) approaching 1 the Hilbert-Schmidt distance becomes more sensitive also at smaller \( x_s \).
Fig. 1. Difference between Bures and Hilbert Schmidt distances between singlet and decohered singlet in function of time (measured in units corresponding to decoherence characteristic time 1/l).

comparing different density matrices that can be met in neutral 0^{−+} mesons phenomenology.

This is a relevant problem since neutral pseudoscalar mesons represent an interesting test bench of quantum information protocols, alternative to more traditional ones based on photons and/or atoms. Many different interesting cases, from a quantum information point of view, can be experimentally achieved by exploiting the regeneration phenomenon. Furthermore, our results can find application when studying the developing of mixing from a pure state due to gravitational induced (or not) decoherence.

Albeit our results are far from being exhaustive, nevertheless they include many interesting examples pointing out how the sensitivity of different distances varies from case to case and must be kept carefully into account when applied to neutral pseudoscalar phenomenology. In general, they suggest that Hilbert-Schmidt and Bures distances are the most promising to be considered and they represent a possible guide for a choice related to specific physical examples, that could be of use in further studies addressed to plan meson experiments both on quantum information and decoherence.

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Fig. 2. 3-dimensional plot of the difference between Bures and Hilbert Schmidt distances between $U(f_1)_{\tilde{g}S}U(f_1)^\dagger$ and $(U(f_1)_{\tilde{g}S}U(f_1)^\dagger + U(f_2)_{\tilde{g}S}U(f_2)^\dagger)/2$ in function of $\log(f_1)$ and $\log(f_2)$.
Fig. 3. Difference between Bures and Hilbert Schmidt distances between $U(f_1)gsU(f_1)\dagger$ and $xU(f_1)gsU(f_1)\dagger + (1-x)U(f_2)gsU(f_2)\dagger$, $f_1 = 0.05, f_2 = 0.001$, in function of $x$.

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Fig. 4. Difference between Bures and Hilbert Schmidt distances (solid) and between Bures and Trace distances (dashed) between singlet and singlet mixed with background, \( \rho = x \rho_s + (1 - x) \rho_b \), in function of \( x \).