Characterizing Linear Memory-Rate Tradeoff of Coded Caching: The \((N, K) = (3, 3)\) Case

Daming Cao and Yinfei Xu

Abstract

We consider the cache problem introduced by Maddah-ali and Niesen [1] for the \((N, K) = (3, 3)\) case, and use the computer-aided approach to derive the tight linear memory-rate trade-off. Two lower bounds \(10M + 6R \geq 15\) and \(5M + 4R \geq 9\) are proved, which are non-Shannon type. A coded linear scheme of point \((M, R) = (0.6, 1.5)\) is constructed with the help of symmetry reduction and brute-force search.

Index Terms

coded cache, linear memory-rate trade-off, computer-aided analysis.

I. INTRODUCTION

The coded cache problem in information theory is introduced by Maddah-Ali and Niesen [1]. This framework has been extended to many scenarios and numerous results have been derived. Many of these extensions/results focus on the optimal memory-rate trade-off. Although various approaches have been proposed to study the rate-memory trade-off, for the general case where the cached content can be coded, the optimal characterization of the trade-off remains open, except in some special cases, i.e., \(N = K = 2\) [1], \(K = 2\) and arbitrary \(N\) [2], [3], \(K = 3\) and \(N = 2\) [2].

We revisit the framework in [1] for the \((N, K) = (3, 3)\) case. Instead of fully characterizing the optimal memory-rate trade-off, we derive a weaker characterization where the cached content and the delivery messages are linear block codes. We use the computer-aided approach to prove both achievability and converse. For the converse, we combining two techniques, namely the computed-aided approach with symmetry reduction by Tian [2], and the linear rank inequality with common information property by Hammer et al. [4] and Dougherty et al. [5], and derive two lower bounds which are non-Shannon type. On the other hand, for the achievability, we propose a symmetric structure for the cache placement, which significantly reduce the design complexity of both caches and delivery messages. Based on the numerical solutions from the computed-aided converse, an achievable coded cache placement with the symmetric structure is obtained by using the brute-force search.

II. SYSTEM MODEL AND EXISTING RESULTS

A. System Model

We consider the cache problem introduced by Maddah-ali and Niesen [1] for the \((N, K) = (3, 3)\) case. For completeness, we briefly revisit the system model. The problem consider a system with one server connected to \(K = 3\) users through a shared, error-free link. The server has access to a database of \(N = 3\) independent equal-size files, each of size \(F\) bits, denoted by \(W_1, W_2\) and \(W_3\). Each user is equipped with an equal-size local caches with capacities of \(MF\) bits. The system operates in two phases. In the placement phase, the users are given access to the entire database and fill their caches in an error-free manner. The contents of the caches after the placement phase are denoted by \(Z_1, Z_2\) and \(Z_3\), respectively. In the delivery phase, each user requests a single file from the server, where \(d_k\) denotes the index of the file requested by User \(k, k = 1, 2, 3\). After receiving the demand pair \(D \triangleq (d_1, d_2, d_3)\), the server generating a message of size \(RF\) bits, denoted by \(X_D\), and transmits the message over the shared channels to all users to satisfy their demands.

Let \(W_k, k = 1, 2, 3\) be the independent random variables each uniformly distributed over \([2^F]\) for some \(F \in \mathbb{N}\). Then a \((M, R)\) cache scheme for this system consists of:

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1) $K$ caching functions

$$\phi_k : [2^F]^N \rightarrow [2^{MF}], \quad (1)$$

which map the database into cache contents of the users, denoted by $Z_k = \phi_k(W_1, W_2, W_3)$, $k = 1, 2, 3$.

2) $N^2$ encoding functions, one for each demand pair,

$$f^D : [2^F]^N \rightarrow [2^{RF}], \quad (2)$$

3) $KN^2$ decoding functions, one for each demand pair,

$$g^D_k : [2^{MF}] \times [2^{RF}] \rightarrow [2^F], k = 1, 2, 3, \quad (3)$$

which decodes the desired file $W_{dk}$ as $\hat{W}_{dk}$ at User $k$ from the cached content at User $k$, the messages transmitted over the shared link.

The probability of the cache scheme is defined as:

$$P_e = \max_{D \in [N]^k} \Pr\{(\hat{W}_{d_1}, \hat{W}_{d_2}, \hat{W}_{d_3}) \neq (W_{d_1}, W_{d_2}, W_{d_3})\}. \quad (4)$$

For clarity, we adopt the zero-error decoding criterion.

Given above definitions and setting, the memory-rate trade-off is defined as follows.

**Definition 1.** A pair $(M, R)$ is achievable if for large enough file size $F$, there exists a $(M, R)$ cache scheme with zero error probability. The closure of the set of all $(M,R)$-achievable pair is called the memory-rate region and is denoted as $\mathcal{R}$. Then the memory-rate trade-off is defined as

$$R^*(M) \triangleq \inf\{R : (M, R) \in \mathcal{R}\} \quad (5)$$

Since most of the achievable $(M, R)$ cache schemes are composed of the linear block codes, we are interested in the linear cache scheme.

**Definition 2.** A $(M, R)$ cache scheme is linear if all of the cached content and delivery messages are linear block codes.

Similarly, we can define the linear memory-rate trade-off as follow.

**Definition 3.** A pair $(M, R)$ is linear-achievable if for large enough file size $F$, there exists a $(M, R)$ linear cache scheme with zero error probability. The closure of the set of all $(M,R)$-linear-achievable pair is called the linear memory-rate region and is denoted as $\mathcal{R}_L$. Then the linear memory-rate trade-off is defined as

$$R^*_L(M) \triangleq \inf\{R : (M, R) \in \mathcal{R}_L\} \quad (6)$$

**B. Existing Results**

For the converse, by using the computational approach, Tian [2] provides the best lower bound for $R^*(M)$ under the Shannon-type inequalities, i.e.,

$$\begin{align*}
3M + R^*(M) & \geq 3 \\
6M + 3R^*(M) & \geq 8 \\
M + R^*(M) & \geq 2 \\
2M + 3R^*(M) & \geq 5 \\
M + 3R^*(M) & \geq 3
\end{align*} \quad (7)$$

For the achievability, the best known achievable pairs $(M, R)$ are proposed by several papers, i.e., $(1/3, 2)$ in [6], $(1/2, 5/3)$ in [7], and $(1, 1)$ and $(2, 1/3)$ in [1]. Note that all of these schemes are linear. The existing results are shown in Fig.1.
III. MAIN RESULT

**Theorem 1.** For the (3, 3) cache problem, the linear memory-rate trade-off $R^*_L(M)$ is fully characterized as follow

$$R^*_L(M) = \max \left\{ 3 - 3M, \frac{8 - 6M}{3}, \frac{15 - 10M}{6}, \frac{9 - 5M}{4}, \frac{5 - 2M}{3}, \frac{3 - M}{3} \right\}$$

(8)

In other words, compared to the existing result (see Fig. 2), the linear memory-rate trade-off $R^*_L(M)$ must additionally satisfy:

$$10M + 6R^*_L(M) \geq 15$$

(9)

$$5M + 4R^*_L(M) \geq 9.$$  

(10)

and the new corner point $(M, R) = (3/5, 3/2)$ is linear-achievable.

IV. CONVERSE

A. Preliminaries

Before presenting the converse proof of Theorem 1, in this subsection, we briefly review the two techniques that we combine in this work, namely the computed-aided approach with symmetry reduction by Tian [2], and the linear rank inequality with common information property by Hammer et al. [4] and Dougherty et al. [5].

The main idea in the computed-aided approach by Tian [2] is to use the information-theoretic inequality prover (ITIP) (or a linear programming (LP)) to identify the boundary of the memory-rate trade-off. However, the straightforward application can not work since the size of the linear programming is extremely large and is unbearable for the computer resource. Therefore, a critical step is to identify and formalize the symmetric structure and also to show the existence of optimal symmetric solutions. Subsequently, based on this symmetry property and problem setting, an equivalence relation for the entropy-quantity terms is given, which significantly reduces the size of the variables in LP and further make possible to use a symmetry-reduced LP with computer-aid in the cache problem. Furthermore, this equivalence relation can be described as follow:
1) Symmetric rule: if two random variables terms satisfy some permutation constraint, these corresponding entropy quantities are equal, in other words, both quantities can be represent by a same variable in LP.

2) Decoding rule: if one random variable term can be decoded by the other random variable term, which means the corresponding conditional entropy is zero, both corresponding entropy quantities are equal.

On the other hand, the key tool of the linear rank inequality with common information property by Hammer et al. [4] and Dougherty et al. [5] is the common information property. More specific, a random variable $Z$ is a common information of random variables $A$ and $B$ if it satisfies the following conditions:

\begin{align}
H(Z|A) &= 0, \\
H(Z|B) &= 0, \\
H(Z) &= I(A;B).
\end{align}

Furthermore, if the random variables are generated/coming from a vector spaces, then the common information always exists. Subsequently, by introducing some new auxiliary random variables which are common information, the linear rank inequalities in [4], [5] can be proved by Shannon-type inequalities even they are non-Shannon-type inequalities.

B. Sketch of proof

In the rest of this section, we present the converse proof of theorem 1. The proof follows by combining the techniques in [2] and [4], [5] and is separated into three steps: 1) introduce two auxiliary random variables; 2) update the equivalence relation; 3) use the LP with symmetry reduction.

More specific, firstly, we introduce two auxiliary random variables $K_1$ and $K_2$, where $K_1$ is the common information of the random variables $Z_{13}X_{213}$ and $W_1$, and where $K_2$ is the common information of the random variables $W_1X_{123}$ and $W_2$. Since we consider the linear scheme, the variables $K_1$ and $K_2$ always exist. Secondly, based on the existing equivalence relation for the entropy-quantity terms without containing the auxiliary random variables, we only use the decoding rule to update the equivalence relation. Clearly, this updated equivalence relation
does not require any additional requirement for the optimal symmetric solutions. In other words, our steps do not break the optimality of the symmetric solutions. Finally, we use the symmetry-reduced LP with computer-aid to obtain the low bound.

A detail of the equivalence relation is provided in Appendix A and a “checkable” proof can be found in Appendix B.

Remark 1. It is noteworthy that this “checkable” proof may not be classical or standard since the equivalence relation is used in this proof. However, this non-classical part is equivalent or transformable to some standard converse techniques which uses the permutations to average the performance of all possible cases (cf. [8]). Furthermore, in some sense, this “non-classical” part is to use the permutations before giving solutions, while the standard ones are to use the permutations after giving solutions.

V. Achievability

A. Preliminaries

For ease of notation, the three files are also denoted as $A$, $B$ and $C$, each of which is partitioned into ten subfiles of equal size, denoted as $A_i$, $B_i$ and $C_i$, $i = 1, 2, \ldots, 10$, respectively. Refer to the linear block code, we construct the linear scheme in $GF(2)$ and represent the schemes in a manner of information vectors and generated matrices. Specially, the information vectors is denoted by $W = [A_1, A_2, \ldots, A_{10}, B_1, B_2, \ldots, B_{10}, C_1, C_2, \ldots, C_{10}]$, and the generated matrices of the three files, the cached contents and the delivery messages are represented as the bold of the corresponding random variables. For example, the cached content of User 1 is the codeword vector $Z_1 \cdot W^T$, and the delivery message $X^{123}$ is the codeword vector $X^{123} \cdot W^T$. In this section, we use $W_i$, $i = 1, 2, 3$ and $A, B, C$, interchangeably, and also use the random variables and the corresponding codeword vectors interchangeably.

Furthermore, note that these subfiles are independent and identical uniform distribution, therefore, the entropy of this random variable. For example, the type of the random variable $Z_1$ is

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Furthermore, note that these subfiles are independent and identical uniform distribution, therefore, the entropy of the random variable are the same (up to a constant factor $0.1F$) as the rank of the corresponding generated matrix. In the rest of this section, for ease of notation, we drop this constant normalized factor $0.1F$ for the entropies.

Subsequently, we introduce two column operations $f$ and $g$. For the operation $f$, it permutes the indexes of the columns as follow:

$$f : (1, 2, \ldots, 30) \mapsto (21, 22, \ldots, 30, 1, 2, \ldots, 20)$$

or equivalently, it maps the notations of the files in the codeword vector as follow:

$$f : \begin{cases} A \mapsto B \\ B \mapsto C \\ C \mapsto A \end{cases}$$

(15)

On the other hand, for the operation $g$, it permutes the indexes of the columns as follow:

$$g : (1, 2, \ldots, 30) \mapsto (7, 8, 9, 1, 2, \ldots, 6, 10, 17, 18, 19, 11, 12, \ldots, 16, 20, 27, 28, 29, 21, 22, \ldots, 26, 30)$$

(16)

or equivalently, it maps the indexes of the subfiles for each files in the codeword vector as follow:

$$g : \begin{cases} i \mapsto i + 3 \pmod{9} & 1 \leq i \leq 9 \\ i \mapsto i & i = 10 \end{cases}$$

(17)

For example, if the first component in the cache of User 1, i.e., $Z_1[1] \cdot W^T$, is $A_2 \oplus B_3 \oplus C_9 \oplus C_{10}$, then after processing on $Z_1[1]$ by function $f \circ g$, we have $f \circ g(Z_1[1]) \cdot W^T = B_5 \oplus C_6 \oplus A_3 \oplus A_{10}$.

Moreover, we define the following mapping function $h$:

$$h : \begin{cases} Z_1 \mapsto Z_2 \\ Z_2 \mapsto Z_3 \\ Z_3 \mapsto Z_1 \\ W_1 \mapsto W_2 \\ W_2 \mapsto W_3 \\ W_3 \mapsto W_1 \end{cases}$$

&

$$\begin{cases} Z_1 \mapsto Z_2 \\ Z_2 \mapsto Z_3 \\ Z_3 \mapsto Z_1 \\ W_1 \mapsto W_2 \\ W_2 \mapsto W_3 \\ W_3 \mapsto W_1 \end{cases}$$

(18)

Finally, we denote the set of the random variables which do not contain the delivery messages as the set $\mathcal{H}_Z$. Given a specific random variable in the set $\mathcal{H}_Z$, we define the vector which sequentially contains the element numbers of the files and the caches as the type of this random variable. For example, the type of the random variable $Z_1W_1W_2$ is $(2, 1)$, and the type of the random variable $W_2$ is $(1, 0)$.
B. Design the symmetric structure for the generated matrices of the caches

Recall the new achievable pair is \((M, R) = (0.6, 1.5)\), in other words, the entropy of each cache is at most 6 and the entropy of each delivery message is at most 15.

Now let \(Z_i\) be a binary matrix with size \(6 \times 30\), \(i = 1, 2, 3\). Given an arbitrary assignment for the first two rows of \(Z_1\), in other words, \(Z_1 [1 - 2]\) is arbitrarily given, we construct the rest rows in \(Z_i\), \(i = 1, 2, 3\) as follow:

\[
\begin{align*}
Z_1[3 - 4] &= f(Z_1[1 - 2]), \\
Z_1[5 - 6] &= f(Z_1[3 - 4]), \\
Z_2 &= g(Z_1), \\
Z_3 &= g(Z_2).
\end{align*}
\]

Note that the operators \(f\) and \(g\) are commutative and both composite operators \(f^3\) and \(g^3\) are identity operators, thus we have the following relations:

\[
\begin{align*}
f(Z_i) &= Z_i, & i = 1, 2, 3, \\
g(Z_i) &= h(Z_i), & i = 1, 2, 3
\end{align*}
\]

Using the relations in (23) and (24), we have the following proposition.

**Proposition 2** (Symmetry). For any two random variables in the set \(H_{Z}\), if they have the same type, then the corresponding entropies are equal.

**Proof:** Firstly, we consider the type \((1, 1)\). Note that the operators \(f\) and \(g\) are the composition of some column interchange operations and recall the well-known fact that the column interchange operation does not change the rank of the matrix, thus we have the following conversions:

\[
\begin{align*}
H(Z_iW_j) &= \text{rk} \left( \begin{bmatrix} Z_i \\ W_j \end{bmatrix} \right) = \text{rk} \left( \begin{bmatrix} f(Z_i) \\ f(W_j) \end{bmatrix} \right) = \text{rk} \left( \begin{bmatrix} Z_i \\ h(W_j) \end{bmatrix} \right) = H(Z_i h(W_j)), \\
H(Z_iW_j) &= \text{rk} \left( \begin{bmatrix} Z_i \\ W_j \end{bmatrix} \right) = \text{rk} \left( \begin{bmatrix} g(Z_i) \\ g(W_j) \end{bmatrix} \right) = \text{rk} \left( \begin{bmatrix} h(Z_i) \\ W_j \end{bmatrix} \right) = H(h(Z_i)W_j).
\end{align*}
\]

Therefore, by using the conversions (25) and (26), the entropy of each random variable with type \((1, 1)\) are identical.

Similarly, we can derive the identical relationship for the rest types.

**Remark 2.** Proposition 2 provides an equivalence relation for the entropy-quantity terms of the random variables in the set \(H_Z\). And this equivalence relation satisfies the symmetric rule introduced in the converse part (see Section IV-A and Appendix A), in other words, the designed cache structure is symmetric (without considering the delivery messages).

C. Brute-force search for the generated matrices of the caches

Since the outer bound is obtained by ITIP, we have the numerical solutions for all entropy-quantity terms at the corner point \((0.5, 1.6)\). However, even we only focus on the caches, in other words, we interest in the random variables in the set \(H_Z\), the optimal solutions are not unique. Therefore, to create coded multi-casting opportunities in the delivery phase as much as possible, intuitively, we choose an optimal solution which maximizes the sum of the entropies of the random variables in the set \(H_Z\) and provide it in Table I. Then the goal of this subsection is to find some cache constructions which match the table 1.

Recall the construct of \(Z_i\), \(i = 1, 2, 3\) in (19) to (22) and note the observations \(H(Z_1|W_1) = 6\) and \(H(Z_1|W_1W_2) = 4\) from the table 1, we may assume that the subfiles contained in the codeword vector \(Z_1[1 - 2] \cdot W^T\) are only from two files. It is noteworthy that this encoding assumption does not make any obvious contradiction to the entropy values in the table 1 (without entropy testing) and it reduces the design difficult, i.e., designing the linear combinations from two files rather three files. Without loss of generality, the codeword vector \(Z_1[1 - 2] \cdot W^T\) does not contain any subfiles in the file \(C\).

Now we have reduced the size of all possible generated matrices \(Z_i\), \(i = 1, 2, 3\) from \(2^{540}\) to \(2^{40}\) based on the symmetric structure and the encoding assumption above. However, this size may still be too large and is unbearable/inefficient for the brute-force search (also for the manual design). Therefore, we further assume that the
codeword vector $Z_1[1 - 2] \cdot W^T$ only consider some partial subfiles, namely active encoding subfiles. Note the constraint $H(Z_1 Z_2 W_3 W_2) = H(W_1 W_2 W_3)$, which means that each subfile at least appears once in some caches, therefore, we first consider a simple case that the active encoding subfiles of the codeword vector $Z_1[1 - 2] \cdot W^T$ are the first three subfiles and the last subfiles, i.e., $(A_1, A_2, A_3, A_{10}, B_1, B_2, B_3, B_{10})$. Then the size of all possible generated matrices $Z_i$, $i = 1, 2, 3$ is $2^{16}$ and it is acceptable for the brute-force search. Unfortunately, we do not find an achievable linear scheme\(^1\) even there exists some matched cache constructions. Thus, we slightly enlarge the active encoding subfiles of the codeword $Z_1[1] \cdot W^T$ to the first four subfiles instead of the first three subfiles. Fortunately, we construct an achievable linear scheme and the corresponding cache construction is given in table II.

**Remark 3.** In our construction, i.e., Table II, the subfiles used in each cache are overlapped, e.g. $A_4, A_{10}$, which is an uncommon design for the case $K \cdot M \leq N$ and may also contradicts the intuitive design.

D. Design the generated matrices of the delivery messages

We partition all delivery messages into five parts in Table III and prove Lemma 3 that simplifies the design complexity.

**Lemma 3.** Given any generated matrices of the caches which satisfy the symmetric structure in (19) to(22), in each part of Table III, the generated matrices of the delivery messages are inter-transformable.

**Proof:** For Part 1 in Table III, we give the transform mapping in (27) and prove the achievability for the first mapping (right-arrow).

$$X^{AAA} \xrightarrow{gof} X^{BBB} \xrightarrow{gof} X^{CCC} \xrightarrow{gof} X^{AAA}$$

(27)

Assume that the delivery message $X^{AAA}$ is achievable, in other words,

$$H(W_i|X^{AAA} Z_i) = 0, \quad i = 1, 2, 3.$$  

(28)

\(^1\)Due to the limit of coding ability, we can not conclude that there is no achievable linear scheme for this simple case.
Similarly to the proof of Proposition 2, we have the following conversion:

\[
H(Z_iW_1X^{AAA}) = \text{rk} \left( \begin{bmatrix} Z_i \\ W_1 \\ X^{AAA} \end{bmatrix} \right) = \text{rk} \left( g \circ f \left( \begin{bmatrix} Z_i \\ W_1 \\ X^{AAA} \end{bmatrix} \right) \right)
\]

\[
= \text{rk} \left( \begin{bmatrix} g \circ f(Z_i) \\ g \circ f(W_1) \\ g \circ f(X^{AAA}) \end{bmatrix} \right) = \text{rk} \left( \begin{bmatrix} h(Z_i) \\ W_2 \\ X^{BBB} \end{bmatrix} \right) = H(h(Z_i)W_2X^{BBB}) \tag{29}
\]

In a same way, we have

\[
H(Z_iX^{AAA}) = H(h(Z_i)X^{BBB}) \tag{30}
\]

Combining (28), (29) and (30), we obtain that

\[
H(Z_iW_2X^{BBB}) = H(Z_iX^{BBB}), \quad i = 1, 2, 3. \tag{31}
\]

Thus, we prove that the generated matrix \( X^{BBB} = g \circ f(X^{AAA}) \) is achievable. Similarly, the rest transform mapping is achievable.

For the rest parts in Table III, the transform mappings are provided in Appendix C and the achievability can be proved in a similarly way.

Now, by using Lemma 3, we only need to design the generated matrices of the delivery messages \( X^{AAA}, X^{ABC}, X^{ACB}, X^{ABB} \) and \( X^{ACC} \). Obviously, the delivery message \( X^{AAA} = A \) is always achievable since the rate is bigger than 1.

For the case \( X^{ABC} \), motivated by the proof of Lemma 3 and based on the observations that \( g \circ f(Z_i) = h(Z_i) \) and \( g \circ f(W_j) = h(W_j) \), we may hope that the matrix \( X^{ABC} \) satisfies the condition \( X^{ABC} = g \circ f(X^{ABC}) \). If so, we have

\[
H(W_i | X^{ABC}, Z_i) = H(h(W_i) | X^{ABC}, h(Z_i)). \tag{32}
\]

Then we just need to guarantee that the User 1 can decode his required file \( A \) from his cache \( Z_1 \) and the delivery message \( X^{ABC} \). Furthermore, similarly to the symmetric structure of the cache generated matrices in (19)-(20), we may further assume that the matrix \( X^{ABC} \) satisfies the following structure:

\[
X^{ABC}[6-10] = g \circ f(X^{ABC}[1-5]) \tag{33}
\]

\[
X^{ABC}[11-15] = g \circ f(X^{ABC}[6-10]) \tag{34}
\]

in other words, the construction work is reduced to design the first five rows \( X^{ABC}[1-5] \) instead of the whole matrix \( X^{ABC} \). Following the assumptions above, we find an achievable generated matrix \( X^{ABC} \) (similarly for \( X^{ACB} \) in Table IV). A checkable decoding processes are provided in Appendix D.

For the rest cases \( X^{ABB} \) and \( X^{ACC} \), although we do not have a similar symmetric structure for the generated matrices, we can still partially reduce the design complexity in a similar manner, which is benefited from the symmetric structure of the caches. The construction of the delivery messages \( X^{ABB} \) and \( X^{ACC} \) are given in Table V and a checkable decoding processes are provided in Appendix D.

**Remark 4.** The delivery message \( X^{ABB} \) gives a “contradiction” to an intuitive guess that if a file is not required in some demands, the corresponding delivery messages are independent of this file.
A. The symmetry introduced in [9]

As we use it in the subsequent proof, for the completeness, we briefly restate the symmetry by Tian [9].

Let $\hat{\pi}(\cdot)$ and $\hat{\pi}(\cdot)$ be two permutation functions on the index set $\{1, 2, 3\}$, $Z$ be a subset of $\{Z_1, Z_2, Z_3\}$, $W$ be a subset of $\{W_1, W_2, W_3\}$, and $X$ be a subset of $\{X_D : D \in [3] \times [3]\}$. Define the following operations:

$$
\hat{\pi} \circ \hat{\pi}(W) = \{W_{\hat{\pi}(i)} : W_i \in W\}
$$

(35)

$$
\hat{\pi} \circ \hat{\pi}(Z) = \{Z_{\hat{\pi}(i)} : Z_i \in Z\}
$$

(36)

$$
\hat{\pi} \circ \hat{\pi}(X) = \left\{X^{\hat{\pi}^{-1}(\hat{\pi}(d_1)), \hat{\pi}^{-1}(\hat{\pi}(d_2)), \hat{\pi}^{-1}(\hat{\pi}(d_3))} : X^{(d_1, d_2, d_3)} \in X\right\}
$$

(37)

Now, the symmetry can be represented as

$$
H(W, Z, X) = H(\hat{\pi} \circ \hat{\pi}(W), \hat{\pi} \circ \hat{\pi}(Z), \hat{\pi} \circ \hat{\pi}(X)) \quad \forall(\hat{\pi}(\cdot), \hat{\pi}(\cdot))
$$

(38)

B. The “checkable” converse proof of Theorem 1

Firstly, we specify the equivalence relation as follow:

1) Symmetric rule: it is the equation (38) and is only available for the random variables without containing the auxiliary random variables $K_1$ and $K_2$. Moreover, the permutation is represented by one-line notation, and this equivalence relation is represented by the right arrow.
2) Decoding rule: it follows the following forms/cases:

\[
\begin{align*}
H(\cdot | W_1, W_2, W_3, \cdot) &= 0 \\
H(W_d | X^D, Z_1, \cdot) &= 0 \\
\begin{cases}
H(K_1 | Z_1, X^{213}, \cdot) &= 0, \\
H(K_1 | W_1, \cdot) &= 0, \\
H(K_2 | W_1, X^{123}, \cdot) &= 0, \\
H(K_2 | W_2, \cdot) &= 0,
\end{cases}
\end{align*}
\]

Moreover, this equivalence relation is represented by the equal sign.

For example, \( H(W_1 W_2 Z_1) \) is equivalent to \( H(W_1 W_2 Z_3 K_1 K_2) \) through the following relation:

\[ H(W_1 W_2 Z_1) \rightarrow H(W_1 W_2 Z_3) = H(W_1 W_2 Z_3 K_1 K_2) \quad ((1, 2, 3), (3, 2, 1)) \]  

where the brackets is the corresponding permutation pair, i.e., \( (\bar{\pi}(\cdot), \hat{\pi}(\cdot)) = ((1, 2, 3), (3, 2, 1)) \).

Now, we show the proof of the lower bound \( 10M + 6R \geq 15 \). For ease of checking, we present the equivalence relation for the entropy terms and the original Shannon-type inequality under each inequality. Based on the results of the computer-aided LP, we have:

\[
\begin{align*}
H(W_1 W_2 W_3) - H(W_1 W_2 Z_1) - H(W_2 W_3 Z_1 K_1 K_2) + H(W_2 Z_3 K_1 K_2) &\leq 0 \\
H(W_1 W_2 Z_1) &\rightarrow H(W_1 W_2 Z_3) = H(W_1 W_2 Z_3 K_1 K_2) \quad ((1, 2, 3), (3, 2, 1)) \\
\Leftrightarrow I(W_1 : W_3 | W_2 Z_3 K_1 K_2) &\geq 0 \\
3H(W_1 W_2 W_3) - 3H(W_1 W_2 Z_1 Z_2) - 3H(W_1 W_2 Z_1 X^{123}) + 3H(W_1 W_2 Z_1) &\leq 0 \\
H(W_1 W_2 Z_1 Z_2) &\rightarrow H(W_1 W_2 Z_1 Z_3) \quad ((1, 2, 3), (1, 3, 2)) \\
\Leftrightarrow I(Z_3 : X^{123} | W_1 W_2 Z_1) &\geq 0 \\
5H(W_1 W_2 Z_1 Z_2 X^{123}) - 5H(W_1 W_2 Z_1 X^{123}) - 5H(W_1 Z_1 X^{123}) + 5H(W_1 X^{123}) &\leq 0 \\
H(W_1 W_2 Z_1 Z_2 X^{123}) &\rightarrow H(W_1 W_2 Z_2 X^{123}) \quad ((2, 1, 3), (2, 1, 3)) \\
\Leftrightarrow I(Z_1 : Z_2 | W_1 W_2 X^{123}) &\geq 0 \\
4H(W_1 W_2 Z_1 X^{123}) - 4H(W_1 Z_1) - 4H(W_1 X^{123}) &\leq 0 \\
H(W_1 W_2 Z_1 X^{123}) &\rightarrow H(W_1 W_2 Z_2 X^{123}) = H(W_1 Z_2 X^{123}) \quad ((2, 1, 3), (2, 1, 3)) \\
H(W_1 Z_1) &\rightarrow H(W_1 Z_2) \quad ((1, 2, 3), (2, 1, 3)) \\
\Leftrightarrow I(Z_2 : X^{123} | W_1) &\geq 0 \\
H(W_1 W_2 Z_1 X^{123}) - H(W_1 Z_1 X^{123}) - H(W_1 X^{123}) &\leq 0 \\
H(W_1 W_2 Z_1 X^{123}) &\rightarrow H(W_1 W_2 Z_1 X^{213}) = H(W_1 W_2 Z_1 X^{213} K_1) \quad ((2, 1, 3), (1, 2, 3)) \\
H(W_1 Z_1 X^{123}) &\rightarrow H(W_2 Z_1 X^{213}) = H(W_2 Z_1 X^{213} K_1 K_2) \quad ((2, 3, 1), (1, 3, 2)) \\
\Leftrightarrow I(W_2 Z_3 : Z_1 | W_2 X^{213} K_1 K_2) &\geq 0 \\
H(W_1 W_2 Z_1 Z_2 X^{123}) - 5H(W_1 W_2 Z_1 X^{123}) - 5H(W_1 Z_1 X^{123}) + 5H(W_1 X^{123}) &\leq 0 \\
H(W_1 W_2 Z_1 X^{123}) &\rightarrow H(W_1 W_2 Z_2 X^{123}) \quad ((2, 1, 3), (2, 1, 3)) \\
\Leftrightarrow I(Z_1 : W_2 Z_2 | W_1 X^{123}) &\geq 0 \\
H(W_1 W_2 Z_1 X^{123}) &\rightarrow H(W_1 W_2 Z_1 X^{123}) = H(W_1 W_2 Z_1 X^{213} K_1) \quad ((2, 1, 3), (1, 2, 3)) \\
H(W_1 Z_1 X^{123}) &\rightarrow H(W_2 Z_1 X^{213}) = H(W_2 Z_1 X^{213} K_1 K_2) \quad ((2, 3, 1), (1, 3, 2)) \\
\Leftrightarrow I(W_2 Z_3 : Z_1 | W_2 X^{213} K_1 K_2) &\geq 0 \\
H(W_1 W_2 Z_1 X^{123}) - H(W_1 Z_1 X^{123}) &\leq 0 \\
H(W_1 W_2 Z_1 X^{123}) &\rightarrow H(W_1 W_2 Z_1 X^{123}) = H(W_1 W_2 Z_1 X^{213} K_1) \quad ((2, 1, 3), (1, 2, 3)) \\
H(W_1 Z_1 X^{123}) &\rightarrow H(W_2 Z_1 X^{213}) = H(W_2 Z_1 X^{213} K_1 K_2) \quad ((2, 3, 1), (1, 3, 2)) \\
\Leftrightarrow I(W_1 : W_2 Z_1 | X^{213} K_1 K_2) &\geq 0 \\
H(W_1 W_2 Z_1 X^{123}) &\rightarrow H(W_1 W_2 Z_1) - H(W_3 Z_1 X^{213} K_1) + H(W_3 Z_3 K_1) \leq 0
\end{align*}
\]
by combining (43) to (57), we have:

$$H(W_1W_2Z_1X^{123}) \rightarrow H(W_1W_3Z_3X^{213}) = H(W_1W_2Z_2X^{213}K_1) \quad ((3, 1, 2), (3, 2, 1))$$
$$H(W_1W_2Z_1) \rightarrow H(W_1W_3Z_3) = H(W_1W_2Z_3K_1) \quad ((1, 3, 2), (3, 1, 2))$$
$$\iff I(W_1; X^{213}|W_2Z_3K_1) \geq 0$$

$$H(W_1W_2) - H(W_1) - H(W_2K_1K_2) + H(K_1) \leq 0$$

$$H(W_1W_2) = H(W_1W_2K_1K_2)$$
$$H(W_1) = H(W_1K_1)$$
$$\iff I(W_1; W_2K_2|K_1) \geq 0$$

$$7H(W_1Z_1X^{123}) - 7H(Z_1) - 7H(X^{123}) \leq 0$$

$$H(W_1Z_1X^{123}) = H(Z_1X^{123})$$
$$\iff I(Z_1; X^{123}) \geq 0$$

$$3H(W_1Z_1) - 3H(W_1) - 3H(Z_1) \leq 0$$

$$\iff I(W_1; Z_1) \geq 0$$

$$H(W_2W_3Z_3X^{213}K_1K_2) - H(W_2Z_3K_1K_2) - H(W_2X^{213}K_1K_2) + H(W_2K_1K_2) \leq 0$$

$$H(W_2W_3Z_3X^{213}K_1K_2) = H(W_2Z_3X^{213}K_1K_2)$$
$$\iff I(Z_3; X^{213}|W_2K_1K_2) \geq 0$$

$$H(W_3Z_3X^{213}K_1) - H(X^{213}K_1) - H(W_1Z_1X^{123}) + H(X^{123}) \leq 0$$

$$H(W_1Z_1X^{123}) \rightarrow H(W_3Z_3X^{213}) \quad ((3, 2, 1), (3, 1, 2))$$

$$H(X^{123}) \rightarrow H(X^{213}) \quad ((3, 2, 1), (3, 1, 2))$$
$$\iff I(K_1; W_3Z_3|X^{213}) \geq 0$$

$$H(W_2W_3Z_3K_1K_2) - H(W_1W_2Z_1) - H(W_3Z_3K_1) + H(W_1Z_1) \leq 0$$

$$H(W_1W_2Z_1) \rightarrow H(W_2W_3Z_3) = H(W_2W_3Z_3K_2) \quad ((2, 3, 1), (3, 1, 2))$$
$$H(W_1Z_1) \rightarrow H(W_3Z_3) \quad ((3, 1, 2), (3, 1, 2))$$
$$\iff I(W_2K_2; K_1|W_3Z_3) \geq 0$$

$$3H(W_1W_2Z_1Z_2) - 3H(W_1W_2Z_1Z_2X^{123}) \leq 0$$
$$\iff H(X^{123}|W_1W_2Z_1Z_2) \geq 0$$

$$3H(W_1W_2Z_1X^{123}) - 3H(W_1W_2Z_1Z_2X^{123}) \leq 0$$
$$\iff H(Z_2|W_1W_2Z_1X^{123}) \geq 0$$

by combining (43) to (57), we have:

$$4H(W_1W_2W_3) + H(W_1W_2) + H(W_1W_2Z_1X^{123}) - H(W_1Z_1X^{123}) - 10H(Z_1) - 6H(X^{123}) + H(K_1) \leq 0$$  (58)

Note that

$$H(K_1) = I(Z_1X^{213}; W_1) \quad (59)$$
$$= H(Z_1X^{213}) + H(W_1) - H(W_1Z_1X^{213}) \quad (60)$$
$$= H(W_2Z_1X^{213}) + H(W_1) - H(W_1W_2Z_1X^{213}) \quad (61)$$

and

$$H(W_1Z_1X^{123}) \rightarrow H(W_2Z_1X^{213}) \quad ((2, 1, 3), (1, 2, 3)) \quad (62)$$
$$H(W_1W_2Z_1X^{123}) \rightarrow H(W_1W_2Z_1X^{213}) \quad ((2, 1, 3), (1, 2, 3)) \quad (63)$$

Thus, by combining (58) to (63), we conclude that

$$10M + 6R \geq 15.$$  (64)
Similarly, for the lower bound $5M + 4R \geq 9$, we have

\[
H(W_1W_2W_3) - H(W_1W_2) - H(W_1W_3K_1K_2) + H(W_1K_1K_2) \leq 0
\]

\[
H(W_1W_2) = H(W_1W_2K_1K_2)
\]

\[
H(W_1W_2) \Leftrightarrow I(W_2; W_1|W_1K_1K_2) \geq 0
\]

\[
2H(W_1W_2W_3) - 2H(W_1W_2Z_1Z_2X^{123}) - 2H(W_1W_2Z_1X^{123}X^{231}) + 2H(W_1W_2Z_1X^{123}) \leq 0
\]

\[
H(W_1W_2W_3) = H(W_1W_2Z_1Z_2X^{123}X^{231})
\]

\[
H(W_1W_2Z_1Z_2X^{123}) \Leftrightarrow I(Z_2; X^{231}|W_1W_2Z_1X^{123}) \geq 0
\]

\[
H(W_1W_2Z_1Z_2X^{123}) - H(W_3Z_1Z_3K_1) - H(W_3Z_3X^{213}K_1) + H(W_3Z_3K_1) \leq 0
\]

\[
H(W_1W_2Z_1Z_2X^{123}) \Leftrightarrow H(W_2W_3Z_1Z_3X^{213}) = H(W_2Z_1Z_3X^{213}) (3, 2, 1), (3, 1, 2)
\]

\[
H(W_1W_2Z_1Z_2X^{123}) - 2H(W_1Z_1X^{123}) + H(X^{123}K_2) \leq 0
\]

\[
H(W_1W_2Z_1Z_2X^{123}) = H(W_1W_2Z_1Z_2X^{123}K_2)
\]

\[
H(W_1W_2Z_1Z_2X^{123}) = H(W_1W_2Z_1X^{123}K_2)
\]

\[
H(W_1W_2Z_1Z_2X^{123}) \Leftrightarrow H(W_2Z_2X^{123}) = H(W_2Z_2X^{123}K_2) ((2, 3, 1), (3, 1, 2))
\]

\[
H(W_1W_2Z_1Z_2X^{123}X^{231}) - H(W_1W_2Z_1X^{123}) - H(W_3Z_3X^{123}K_2) + H(W_3Z_3K_2) \leq 0
\]

\[
H(W_1W_2Z_1Z_2X^{123}X^{231}) = H(W_1W_2Z_1X^{123}X^{231}K_2)
\]

\[
H(W_1W_2Z_1Z_2X^{123}) \Leftrightarrow H(W_1W_2Z_1X^{123}X^{231}) = H(W_1W_2Z_1X^{231}K_1K_2) ((2, 1, 3), (1, 3, 2))
\]

\[
H(W_1W_2Z_1Z_2X^{123}) = H(W_1Z_1X^{123}K_1K_2)
\]

\[
H(W_1W_2Z_1Z_2X^{123}) \Leftrightarrow I(W_2X^{231}, X^{123}|W_1Z_1K_1K_2) \geq 0
\]

\[
H(W_1W_2Z_1Z_2X^{123}) - H(W_1Z_1X^{123}) - H(W_1Z_1X^{123}) - H(W_1Z_1K_1K_2) + H(Z_1K_1K_2) \leq 0
\]

\[
H(W_1W_2Z_1X^{123}) \Leftrightarrow H(W_1W_2Z_1X^{213}) = H(W_1W_2Z_1X^{213}K_1K_2) ((2, 1, 3), (1, 2, 3))
\]

\[
H(W_1W_2Z_1X^{123}) \Leftrightarrow H(W_1W_2Z_1X^{213}) = H(W_1W_2Z_1X^{213}K_1K_2) ((2, 1, 3), (1, 2, 3))
\]

\[
H(W_1W_2Z_1X^{123}) \Leftrightarrow I(W_1; W_2X^{213}|Z_1K_1K_2) \geq 0
\]

\[
H(W_1W_2Z_1X^{123}) - H(W_1Z_1X^{123}) - H(W_1Z_1X^{123}) + H(X^{213}K_1) \leq 0
\]

\[
H(W_1W_2Z_1X^{123}) \Leftrightarrow H(W_1W_2Z_1X^{213}) = H(W_1W_2Z_1X^{213}K_1K_2) ((2, 1, 3), (1, 2, 3))
\]

\[
H(W_1W_2Z_1X^{123}) \Leftrightarrow H(W_1W_2Z_1X^{213}) = H(W_1W_2Z_1X^{213}K_1K_2) ((2, 1, 3), (1, 2, 3))
\]

\[
H(W_1W_2Z_1X^{123}) \Leftrightarrow I(W_1; W_2Z_1X^{213}|K_1K_2) \geq 0
\]

\[
H(W_1W_2) - H(W_1W_2K_1K_2) + H(K_1) \leq 0
\]

\[
H(W_1W_2) = H(W_1W_2K_1K_2)
\]

\[
H(W_1) = H(W_1K_1)
\]

\[
I(W_1; W_2K_2|K_1) \geq 0
\]

\[
5H(W_1Z_1X^{123}) - 5H(Z_1) - 5H(X^{123}) \leq 0
\]

\[
H(W_1Z_1X^{123}) = H(Z_1X^{123})
\]
\[
\leftrightarrow I(Z_1; X_{123}^{123}) \geq 0
\]

(75) \[
H(W_3Z_3K_1K_2) - H(W_3Z_3K_1) - H(W_3Z_3K_2) + H(W_1Z_1) \leq 0
\]

(76) \[
H(W_1Z_1) \rightarrow H(W_3Z_3) \quad ((3, 1, 2), (3, 1, 2))
\]

(77) \[
\leftrightarrow I(K_1; K_2|W_3Z_3) \geq 0
\]

(78) \[
H(W_3Z_3K_1K_2) - H(W_1Z_1K_1K_2) - H(W_3Z_3K_1K_2) + H(W_3K_1K_2) \leq 0
\]

(79) \[
H(W_3Z_3X_{123}^{123}K_2) - H(Z_3K_2) - H(X_{123}^{123}K_2) + H(K_2) \leq 0
\]

(80) \[
H(Z_3Z_2) - H(W_3Z_3Z_1K_1K_2) \leq 0
\]

(81) \[
H(Z_1) \rightarrow H(Z_3) \quad ((3, 1, 2), (1, 2, 3))
\]

(82) \[
\leftrightarrow I(Z_3; K_2) \geq 0
\]

(83) \[
H(W_2K_1K_2) - H(W_1) - H(K_1K_2) + H(K_2) \leq 0
\]

(84) \[
H(W_1Z_1X_{123}^{123}) - H(W_3Z_3X_{123}^{123}K_1) + H(X_{123}^{123}) \leq 0
\]

(85) \[
H(W_1W_2X_{123}^{123}) - H(W_1W_2Z_1X_{123}^{123}) \leq 0
\]

(86) \[
3H(W_1W_2W_3) + H(W_1W_2Z_1X_{123}^{123}) + H(W_1W_2X_{123}^{123}) - H(W_1Z_1X_{123}^{123}) - H(W_1X_{123}^{123}) - 2H(W_1) - 5H(Z_1) - 4H(X_{123}^{123}) + H(K_1) + H(K_2) \leq 0
\]

by combining (65) to (85), we have:

Note that

\[
H(K_1) = I(Z_1X_{213}^{213}; W_1)
\]

(87) \[
= H(Z_1X_{213}^{213}) + H(W_1) - H(W_1Z_1X_{213}^{213})
\]

(88) \[
= H(W_2Z_1X_{213}^{213}) + H(W_1) - H(W_1W_2Z_1X_{213}^{213})
\]

(89) \[
H(K_2) = I(W_1X_{123}^{123}; W_2)
\]

(90) \[
= H(W_1X_{123}^{123}) + H(W_2) - H(W_1W_2X_{123}^{123})
\]
and
\[ H(W_1 Z_1 X_{123}^{123}) \rightarrow H(W_2 Z_1 X_{213}^{213}) \quad ((2, 1, 3), (1, 2, 3)) \] (92)
\[ H(W_1 W_2 Z_1 X_{123}^{123}) \rightarrow H(W_1 W_2 Z_1 X_{213}^{213}) \quad ((2, 1, 3), (1, 2, 3)) \] (93)

Thus, by combining (86) to (93), we conclude that
\[ 5M + 4R \geq 9 \] (94)

C. The transform mappings in Lemma 3

The transform mapping for Part 2 of Table III is provided as follow:

\[ X^{ABC} \xrightarrow{f} X^{BCA} \xrightarrow{f} X^{CAB} \xrightarrow{f} X^{ABC} \] (95)

The transform mapping for Part 3 of Table III is provided as follow:

\[ X^{ACB} \xrightarrow{f^2} X^{CBA} \xrightarrow{f^2} X^{BAC} \xrightarrow{f^2} X^{ACB} \] (96)

The transform mapping for Part 4 of Table III is provided as follow:

\[ X^{ABB} \xrightarrow{g} X^{BAB} \xrightarrow{g} X^{BBA} \xrightarrow{g} X^{ABB} \]
\[ \downarrow f \quad \downarrow f \quad \downarrow f \]
\[ X^{BCC} \xrightarrow{g} X^{CBC} \xrightarrow{g} X^{CCB} \xrightarrow{g} X^{BCC} \]
\[ \downarrow f \quad \downarrow f \quad \downarrow f \]
\[ X^{CAA} \xrightarrow{g} X^{ACA} \xrightarrow{g} X^{AAC} \xrightarrow{g} X^{CAA} \]
\[ \downarrow f \quad \downarrow f \quad \downarrow f \]
\[ X^{ABB} \xrightarrow{g} X^{BAB} \xrightarrow{g} X^{BBA} \xrightarrow{g} X^{ABB} \]

The transform mapping for Part 5 of Table III is provided as follow:

\[ X^{ACC} \xrightarrow{g} X^{CAC} \xrightarrow{g} X^{CCA} \xrightarrow{g} X^{ACC} \]
\[ \downarrow f^2 \quad \downarrow f^2 \quad \downarrow f^2 \]
\[ X^{CBB} \xrightarrow{g} X^{BCB} \xrightarrow{g} X^{BBC} \xrightarrow{g} X^{CBB} \]
\[ \downarrow f^2 \quad \downarrow f^2 \quad \downarrow f^2 \]
\[ X^{BAA} \xrightarrow{g} X^{ABA} \xrightarrow{g} X^{AAB} \xrightarrow{g} X^{BAA} \]
\[ \downarrow f^2 \quad \downarrow f^2 \quad \downarrow f^2 \]
\[ X^{ACC} \xrightarrow{g} X^{CAC} \xrightarrow{g} X^{CCA} \xrightarrow{g} X^{ACC} \]
D. The decoding processes for the corner point (0.6, 1.5)

Restate the linear scheme as follow:

| Table VI: The linear scheme for the corner point (0.6, 1.5). |
|-----------------|-------------------------------|-------------------------------|
| Z₁   | A₂ ⊕ B₁  | B₁ ⊕ B₃ ⊕ B₄ ⊕ C₂ ⊕ C₁₀ | C₁ ⊕ C₂ ⊕ C₄ ⊕ A₂ ⊕ A₁₀ |
| Z₂   | A₄ ⊕ A₆ ⊕ A₇ ⊕ B₅ ⊕ B₁₀ | B₄ ⊕ B₆ ⊕ B₇ ⊕ C₅ ⊕ C₁₀ | C₃ ⊕ C₆ ⊕ C₇ ⊕ A₅ ⊕ A₁₀ |
| Z₃   | A₇ ⊕ A₉ ⊕ A₁₀ ⊕ B₈ ⊕ B₁₀ | B₇ ⊕ B₉ ⊕ B₁ ⊕ C₈ ⊕ C₁₀ | C₇ ⊕ C₉ ⊕ C₁ ⊕ A₈ ⊕ A₁₀ |

The decoding process is given in Table VII, where [·] means the content is in the caches, (·) means the content is from the delivery messages, and {·} means the content is the previous decoding result.

| Table VII: The decoding process for the corner point (0.6, 1.5). |
|-----------------|-------------------------------|-------------------------------|
| Demand | User | decoding output | decoding input |
|-------|------|-----------------|-----------------|
| ABC   | Z₁   | A₁ ⊕ A₃ ⊕ A₄   | [A₁ ⊕ A₃ ⊕ A₄ ⊕ B₂ ⊕ B₁₀] ⊕ (B₂ ⊕ B₁₀) |
|       |      | A₂ ⊕ A₁₀       | [C₁ ⊕ C₃ ⊕ C₄ ⊕ A₂ ⊕ A₁₀] ⊕ (C₁ ⊕ C₃ ⊕ C₄) |
|       |      | A₁         | [C₂ ⊕ A₁] ⊕ (C₂) |
|       |      | A₂         | [A₂ ⊕ B₁] ⊕ (B₁) |
|       |      | A₁₀        | {A₂ ⊕ A₁₀} ⊕ {A₂} |
|       |      | A₉         | [B₂ ⊕ C₁] ⊕ (A₁ ⊕ A₉ ⊕ B₂ ⊕ C₁) ⊕ {A₁} |
|       |      | A₄         | [B₁ ⊕ B₃ ⊕ B₄ ⊕ C₂ ⊕ C₁₀] ⊕ (B₁ ⊕ (C₂) ⊕ (C₅ ⊕ C₁₀) ⊕ (B₃ ⊕ B₄ ⊕ C₅ ⊕ A₄) |
|       |      | A₃         | {A₁ ⊕ A₃ ⊕ A₄} ⊕ {A₁} ⊕ {A₄} |
|       |      | A₆         | (A₄ ⊕ A₆ ⊕ A₇) ⊕ {A₄} ⊕ {A₇} |
|       |      | A₈         | (A₈ ⊕ A₁₀) ⊕ {A₁₀} |

Continued on next page
| Demand | User | decoding output | decoding input |
|--------|------|-----------------|----------------|
|        | ABC  | $B_4 \oplus B_6 \oplus B_7$ | $B_4 \oplus B_6 \oplus B_7 \oplus C_5 \oplus C_{10} \oplus (C_5 \oplus C_{10})$ |
|        | Z_2  | $B_5 \oplus B_{10}$ | $[A_4 \oplus A_6 \oplus A_7 \oplus B_5 \oplus A_{10} \oplus (A_4 \oplus A_6 \oplus A_7)$ |
|        |      | $B_4$ | $[A_5 \oplus B_4] \oplus (A_5)$ |
|        |      | $B_5$ | $[B_5 \oplus C_4] \oplus (C_4)$ |
|        |      | $B_{10}$ | $\{B_5 \oplus B_{10}\} \oplus \{B_5\}$ |
|        |      | $B_3$ | $[C_5 \oplus A_4] \oplus (B_3 \oplus B_4 \oplus C_5 \oplus A_4) \oplus \{B_4\}$ |
|        |      | $B_7$ | $[C_4 \oplus C_6 \oplus C_7 \oplus A_5 \oplus A_{10} \oplus (C_4 \oplus (A_5)$ |
|        |      | $\oplus (A_8 \oplus A_{10}) \oplus (C_6 \oplus C_7 \oplus A_8 \oplus B_7)$ |
|        |      | $B_6$ | $\{B_4 \oplus B_6 \oplus B_7\} \oplus \{B_4\} \oplus \{B_7\}$ |
|        |      | $B_9$ | $(B_7 \oplus B_9) \oplus \{B_7\} \oplus \{B_1\}$ |
|        |      | $B_2$ | $(B_2 \oplus B_{10}) \oplus \{B_{10}\}$ |
|        | Z_3  | $C_7 \oplus C_9 \oplus C_1$ | $[C_7 \oplus C_9 \oplus C_1 \oplus A_8 \oplus A_{10} \oplus (A_8 \oplus A_{10})$ |
|        |      | $C_8 \oplus C_{10}$ | $[B_7 \oplus B_9 \oplus B_1 \oplus C_8 \oplus C_{10} \oplus (B_7 \oplus B_9 \oplus B_1)$ |
|        |      | $C_7$ | $B_8 \oplus C_7 \oplus (B_8)$ |
|        |      | $C_8$ | $[C_8 \oplus A_7] \oplus (A_7)$ |
|        |      | $C_{10}$ | $\{C_8 \oplus C_{10}\} \oplus \{C_8\}$ |
|        |      | $C_6$ | $[A_8 \oplus B_7 \oplus (C_6 \oplus C_7 \oplus A_8 \oplus B_7) \oplus \{C_7\}$ |
|        |      | $C_1$ | $[A_7 \oplus A_9 \oplus A_1 \oplus B_8 \oplus B_{10} \oplus (A_7 \oplus (B_8)$ |
|        |      | $\oplus (B_2 \oplus B_{10}) \oplus (A_1 \oplus A_9 \oplus B_2 \oplus C_1)$ |
|        |      | $C_9$ | $\{C_7 \oplus C_9 \oplus C_1\} \oplus \{C_7\} \oplus \{C_1\}$ |
|        |      | $C_3$ | $(C_1 \oplus C_3 \oplus C_4) \oplus \{C_1\} \oplus \{C_4\}$ |
|        |      | $C_5$ | $(C_5 \oplus C_{10}) \oplus \{C_{10}\}$ |
|        | Z_1  | $A_1 \oplus A_3 \oplus A_4$ | $[A_1 \oplus A_3 \oplus A_4 \oplus B_2 \oplus B_{10} \oplus (B_2 \oplus B_{10})$ |
|        |      | $A_2 \oplus A_{10}$ | $[C_1 \oplus C_3 \oplus C_4 \oplus A_2 \oplus A_{10} \oplus (C_1 \oplus C_3 \oplus C_4)$ |
|        |      | $A_1$ | $[C_2 \oplus A_1] \oplus (C_2)$ |
|        |      | $A_2$ | $[B_2 \oplus C_1] \oplus (A_6 \oplus A_7 \oplus C_2 \oplus C_1) \oplus \{A_7}$ |
|        |      | $A_{10}$ | $\{A_2 \oplus A_{10}\} \oplus \{A_2\}$ |
|        |      | $A_7$ | $[B_1 \oplus B_3 \oplus A_1 \oplus C_1 \oplus C_{10} \oplus (B_1 \oplus (C_2)$ |
|        |      | $\oplus (C_4 \oplus C_{10}) \oplus (B_3 \oplus B_4 \oplus C_6 \oplus A_7)$ |
|        |      | $A_6$ | $\{A_1 \oplus A_3 \oplus A_4\} \oplus \{A_1\} \oplus \{A_4\}$ |
|        |      | $A_5$ | $(A_7 \oplus A_9 \oplus A_1) \oplus \{A_1\} \oplus \{A_7\}$ |
|        |      | $A_9$ | $(A_7 \oplus A_9 \oplus A_1) \oplus \{A_1\} \oplus \{A_7\}$ |
|        |      | $A_5$ | $(A_5 \oplus A_{10}) \oplus \{A_{10}\}$ |
|        |      | $C_4 \oplus C_6 \oplus C_7$ | $[C_4 \oplus C_6 \oplus C_7 \oplus A_5 \oplus A_{10} \oplus (A_5 \oplus B_{10})$ |
|        |      | $C_5 \oplus C_{10}$ | $[B_4 \oplus B_6 \oplus B_7 \oplus C_5 \oplus A_{10} \oplus (B_4 \oplus B_6 \oplus B_7)$ |
|        |      | $C_4$ | $[B_5 \oplus C_4] \oplus (B_5)$ |
|        |      | $C_5$ | $[C_5 \oplus A_4] \oplus (A_4)$ |
|        |      | $C_{10}$ | $\{C_5 \oplus C_{10}\} \oplus \{C_5\}$ |
|        |      | $C_1$ | $[A_4 \oplus A_6 \oplus A_7 \oplus B_5 \oplus B_{10} \oplus (A_4 \oplus (B_5)$ |
|        |      | $\oplus (B_2 \oplus B_{10}) \oplus (A_6 \oplus A_7 \oplus B_2 \oplus C_1)$ |
|        |      | $C_9$ | $(A_5 \oplus B_4) \oplus (C_9 \oplus C_1 \oplus A_5 \oplus B_4) \oplus \{C_1\}$ |

Continued on next page
| Demand | User | decoding output | decoding input |
|--------|------|-----------------|----------------|
| Z2     |      |                 |                |
|        |      | $C_6$           | $\{C_4 \oplus C_6 \oplus C_7\} \oplus \{C_4\} \oplus \{C_7\}$ |
|        |      | $C_3$           | $(C_1 \oplus C_3 \oplus C_4) \oplus \{C_4\} \oplus \{C_1\}$ |
|        |      | $C_8$           | $(C_8 \oplus C_{10})\oplus \{C_{10}\}$ |
|        |      | $B_7 \oplus B_9 \oplus B_1$ | $[B_7 \oplus B_9 \oplus B_1 \oplus C_4 \oplus A_{10}] \oplus (C_8 \oplus B_{10})$ |
|        |      | $B_8 \oplus B_{10}$ | $[A_7 \oplus A_9 \oplus A_1 \oplus B_8 \oplus B_{10}] \oplus (A_7 \oplus A_9 \oplus A_1)$ |
|        |      | $B_7$           | $[A_8 \oplus B_7] \oplus \{A_8\}$ |
|        |      | $B_8$           | $[B_8 \oplus C_7] \oplus \{C_7\}$ |
|        |      | $B_{10}$        | $\{B_8 \oplus B_{10}\} \oplus \{B_8\}$ |
| Z3     |      |                 |                |
|        |      | $B_4$           | $[C_7 \oplus C_9 \oplus C_1 \oplus A_8 \oplus A_{10}] \oplus (C_7 \oplus \{A_8\}$ |
|        |      |                  | $\oplus (A_5 \oplus A_{10}) \oplus (C_9 \oplus C_1 \oplus A_5 \oplus B_4)$ |
|        |      | $B_3$           | $[C_8 \oplus A_7] \oplus [B_3 \oplus B_4 \oplus C_8 \oplus A_7] \oplus \{B_4\}$ |
|        |      | $B_9$           | $\{B_7 \oplus B_9 \oplus B_1\} \oplus \{B_7\} \oplus \{B_1\}$ |
|        |      | $B_6$           | $[B_4 \oplus B_6 \oplus B_7] \oplus \{B_7\} \oplus \{B_4\}$ |
|        |      | $B_2$           | $(B_2 \oplus B_{10}) \oplus \{B_{10}\}$ |
| ABB    | Z1   |                 |                |
|        |      | $A_1 \oplus A_3 \oplus A_4$ | $[A_1 \oplus A_3 \oplus A_4 \oplus B_2 \oplus B_{10}] \oplus (B_2 \oplus B_{10})$ |
|        |      | $A_2 \oplus C_{10} \oplus A_1$ | $[A_2 \oplus B_1] \oplus [B_3 \oplus B_4 \oplus C_2 \oplus C_{10}]$ |
|        |      |                  | $\oplus \{C_2 \oplus A_1\} \oplus \{B_3 \oplus B_4\}$ |
|        |      | $A_1 \oplus A_2$ | $\{A_2 \oplus C_{10} \oplus A_1\} \oplus \{C_{10}\}$ |
|        |      | $A_4 \oplus A_5$ | $(A_5 \oplus C_{10} \oplus A_1) \oplus \{C_{10}\}$ |
|        |      | $A_7 \oplus A_8$ | $(A_8 \oplus C_{10} \oplus A_7) \oplus \{C_{10}\}$ |
|        |      | $A_2$           | $[A_2 \oplus B_1] \oplus \{B_1\}$ |
|        |      | $A_4$           | $\{A_4 \oplus A_3\} \oplus \{A_3\}$ |
|        |      | $A_7$           | $\{A_7 \oplus A_8\} \oplus \{A_8\}$ |
|        |      | $A_1$           | $\{A_1 \oplus A_2\} \oplus \{A_2\}$ |
|        |      | $A_3$           | $\{A_1 \oplus A_3 \oplus A_4\} \oplus \{A_1\} \oplus \{A_4\}$ |
|        |      | $A_6$           | $(A_4 \oplus A_6 \oplus A_7) \oplus \{A_4\} \oplus \{A_7\}$ |
|        |      | $A_9$           | $(A_7 \oplus A_9 \oplus A_1) \oplus \{A_7\} \oplus \{A_1\}$ |
| Z2     |      | $B_5 \oplus B_{10}$ | $[A_4 \oplus A_6 \oplus A_7 \oplus B_5 \oplus B_{10}] \oplus (A_4 \oplus A_6 \oplus A_7)$ |
|        |      | $B_6 \oplus B_7$ | $[A_5 \oplus B_4] \oplus [B_4 \oplus B_6 \oplus B_7 \oplus C_5 \oplus C_{10}]$ |
|        |      |                  | $\oplus \{C_5 \oplus A_4\} \oplus (A_5 \oplus C_{10} \oplus A_4)$ |
|        |      | $B_2$           | $(B_2 \oplus B_{10}) \oplus \{B_{10}\}$ |
|        |      | $B_5$           | $\{B_5 \oplus B_{10}\} \oplus \{B_{10}\}$ |
|        |      | $B_4$           | $\{B_5 \oplus B_4\} \oplus \{A_5\}$ |
|        |      | $B_3$           | $(B_3 \oplus B_4) \oplus \{B_4\}$ |
|        |      | $B_7$           | $(B_4 \oplus B_7) \oplus \{B_4\}$ |
|        |      | $B_8$           | $(B_5 \oplus B_8) \oplus \{B_5\}$ |
|        |      | $B_6$           | $\{B_6 \oplus B_7\} \oplus \{B_7\}$ |
|        |      | $B_9$           | $\{B_6 \oplus B_9\} \oplus \{B_6\}$ |
| Z3     |      | $B_8 \oplus B_{10}$ | $[A_7 \oplus A_9 \oplus A_1 \oplus B_8 \oplus B_{10}] \oplus (A_7 \oplus A_9 \oplus A_1)$ |
|        |      | $B_9 \oplus B_1$ | $[A_8 \oplus B_7] \oplus [B_7 \oplus B_9 \oplus B_1 \oplus C_5 \oplus C_{10}]$ |
|        |      |                  | $\oplus \{C_5 \oplus A_4\} \oplus (A_5 \oplus C_{10} \oplus A_4)$ |

Continued on next page
### TABLE VII – continued from previous page

| Demand | User | decoding output                        | decoding input                      |
|--------|------|----------------------------------------|--------------------------------------|
| $Z_3$  |      | $B_2$                                  | $(B_2 \oplus B_{10}) \oplus (B_{10})$ |
|        |      | $B_8$                                  | $\{B_8 \oplus B_{10}\} \oplus (B_{10})$ |
|        |      | $B_7$                                  | $[A_8 \oplus B_7] \oplus (A_8)$      |
|        |      | $B_9$                                  | $(B_9 \oplus B_1) \oplus \{B_1\}$    |
|        |      | $B_4$                                  | $(B_4 \oplus B_7) \oplus \{B_7\}$    |
|        |      | $B_5$                                  | $(B_5 \oplus B_8) \oplus \{B_8\}$    |
|        |      | $B_3$                                  | $(B_3 \oplus B_4) \oplus \{B_4\}$    |
|        |      | $B_6$                                  | $(B_6 \oplus B_9) \oplus \{B_9\}$    |
| $Z_1$  |      | $A_2 \oplus A_{10}$                    | $[C_1 \oplus C_3 \oplus C_4 \oplus A_2 \oplus A_{10}] \oplus (C_1 \oplus C_3 \oplus C_4)$ |
|        |      | $A_3 \oplus A_4 \oplus B_{10}$        | $[C_2 \oplus A_1] \oplus [A_1 \oplus A_3 \oplus A_4 \oplus B_2 \oplus B_{10}]$ |
|        |      |                                          | $\oplus (B_2 \oplus C_1) \oplus (C_1 \oplus C_2)$ |
| $Z_2$  |      | $A_3 \oplus A_4$                       | $\{A_3 \oplus A_4 \oplus B_{10}\} \oplus (B_{10})$ |
|        |      | $A_5$                                  | $(A_5 \oplus A_{10}) \oplus (A_{10})$ |
|        |      | $A_8$                                  | $(A_8 \oplus A_{10}) \oplus (A_{10})$ |
|        |      | $A_2$                                  | $\{A_2 \oplus A_{10}\} \oplus (A_{10})$ |
|        |      | $A_1$                                  | $[C_1 \oplus A_1] \oplus (C_2)$      |
|        |      | $A_3$                                  | $\{A_3 \oplus A_4\} \oplus (A_4)$    |
|        |      | $A_6$                                  | $(A_6 \oplus A_7) \oplus (A_7)$      |
|        |      | $A_9$                                  | $(A_1 \oplus A_9) \oplus \{A_1\}$    |
| $ACC$  |      | $C_4 \oplus C_6 \oplus C_7$           | $[C_4 \oplus C_6 \oplus C_7 \oplus A_5 \oplus A_{10}] \oplus (A_5 \oplus A_{10})$ |
|        |      | $C_5 \oplus B_{10} \oplus C_4$        | $[C_5 \oplus A_4] \oplus [A_4 \oplus A_6 \oplus A_7 \oplus B_5 \oplus B_{10}]$ |
|        |      |                                          | $\oplus (B_5 \oplus C_4) \oplus (A_6 \oplus A_7)$ |
|        |      | $C_5 \oplus C_4$                       | $(C_5 \oplus B_{10} \oplus C_4) \oplus (B_{10})$ |
|        |      | $C_1$                                  | $(C_1 \oplus C_2) \oplus (C_2)$      |
|        |      | $C_5$                                  | $(C_5 \oplus A_4) \oplus (A_4)$      |
|        |      | $C_8$                                  | $(C_8 \oplus C_6) \oplus \{C_5\}$    |
|        |      | $C_4$                                  | $\{C_4 \oplus C_1\} \oplus \{C_4\}$ |
|        |      | $C_7$                                  | $(C_4 \oplus C_7) \oplus \{C_4\}$    |
|        |      | $C_6$                                  | $\{C_4 \oplus C_6 \oplus C_7\} \oplus (C_4 \oplus C_7)$ |
|        |      | $C_9$                                  | $(C_6 \oplus C_9) \oplus \{C_6\}$    |
|        |      | $C_3$                                  | $(C_1 \oplus C_3 \oplus C_4) \oplus \{C_1\} \oplus \{C_4\}$ |
| $Z_3$  |      | $C_7 \oplus C_9 \oplus C_1$           | $[C_7 \oplus C_9 \oplus C_1 \oplus A_8 \oplus A_{10}] \oplus (A_5 \oplus A_{10})$ |
|        |      | $C_8 \oplus B_{10} \oplus C_7$        | $[C_8 \oplus A_7] \oplus [A_7 \oplus A_9 \oplus A_1 \oplus B_8 \oplus B_{10}]$ |
|        |      |                                          | $\oplus (B_8 \oplus C_7) \oplus (A_9 \oplus A_1)$ |
|        |      | $C_8 \oplus C_7$                       | $(C_8 \oplus B_{10} \oplus C_7) \oplus (B_{10})$ |
|        |      | $C_1$                                  | $(C_1 \oplus C_2) \oplus (C_2)$      |
|        |      | $C_8$                                  | $(C_8 \oplus A_7) \oplus (A_7)$      |
|        |      | $C_5$                                  | $(C_5 \oplus C_8) \oplus \{C_8\}$    |
|        |      | $C_7$                                  | $\{C_8 \oplus C_7\} \oplus \{C_8\}$ |
|        |      | $C_4$                                  | $(C_4 \oplus C_7) \oplus \{C_7\}$    |
|        |      | $C_9$                                  | $\{C_7 \oplus C_9 \oplus C_1\} \oplus \{C_1\} \oplus \{C_7\}$ |

Continued on next page
| Demand | User | decoding output | decoding input |
|--------|------|-----------------|----------------|
| ACC    | $Z_3$| $C_6$           | $(C_6 \oplus C_9) \oplus \{C_9\}$ |
|        |      | $C_3$           | $(C_1 \oplus C_3 \oplus C_4) \oplus \{C_1\} \oplus \{C_4\}$ |
E. A linear scheme of Point \((M, R) = (0.5, \frac{5}{3})\) with coded content

We may follow the achievability proof of Theorem 1 to construct a linear scheme of Point \((M, R) = (0.5, \frac{5}{3})\) with coded content. The cache construction is given as follows:

| Z₁   | A₁ ⊕ A₂ ⊕ A₃ ⊕ B₃ | B₃ ⊕ B₂ ⊕ B₁ ⊕ C₃ | C₁ ⊕ C₂ ⊕ C₃ ⊕ A₃ |
|------|--------------------|--------------------|---------------------|
| Z₂   | A₃ ⊕ A₁ ⊕ A₅ ⊕ B₅ | B₅ ⊕ B₁ ⊕ B₂ ⊕ C₅ | C₃ ⊕ C₁ ⊕ C₅ ⊕ A₅ |
| Z₃   | A₅ ⊕ A₆ ⊕ A₁ ⊕ B₁ | B₅ ⊕ B₆ ⊕ B₁ ⊕ C₁ | C₅ ⊕ C₆ ⊕ C₁ ⊕ A₁ |

and the corresponding construction of delivery messages are provided in the following table.

| AAA | A  |
|-----|----|
| ABC | B₂ |
| C₁ ⊕ C₂ ⊕ C₃ | A₁ |
| B₃ | C₁ |
| B₁ ⊕ B₂ ⊕ B₃ ⊕ C₄ ⊕ C₅ ⊕ A₃ ⊕ A₆ ⊕ A₁ | A₅ |
| C₄ | B₂ |
| Z₄ | A₆ |
| A₅ |

| ABC   | B₂ |
| C₃ ⊕ C₆ | B₅ |
| B₃ | A₅ |
| B₁ ⊕ B₂ ⊕ B₃ ⊕ A₃ ⊕ A₄ ⊕ A₆ ⊕ C₅ ⊕ C₆ ⊕ C₁ | A₅ |
| C₃ ⊕ C₆ | B₅ |
| Z₂ | A₆ |
| A₅ |

| ABC   | B₃ |
| C₁ ⊕ C₆ | A₅ |
| A₃ ⊕ A₄ ⊕ A₅ | A₅ |
| B₂ | A₅ |
| B₃ | B₄ |
| A₁ | A₃ |

F. A slight improvement for the lower bound of \(R^*(M)\)

For the general case, we may use a variational version of the Ahlswede and Körner Lemma [10], [11]. Similarly to the linear case, we introduce an auxiliary random variable \(G\), which satisfies the following constraint:

\[ H(G|W₁X^{123}) = 0, \quad (99) \]
\[ H(W₁|G) = H(W₁|X^{213}) \quad (100) \]
\[ H(X^{123}|G) = H(X^{123}|X^{213}) \quad (101) \]
\[ H(W₁X^{123}|G) = H(W₁X^{123}|X^{213}) \quad (102) \]

Then, by using an updated symmetry-reduced LP, we have:

\[ 41M + 31R^*(M) \geq 69 \quad (103) \]

which slightly improve the lower bound of \(R^*(M)\) (see Fig 3). Moreover, this idea has been applied for the secret sharing in [12].

**Remark 5.** This lower bound (103) shows that the point \((2/3, 4/3)\) is not achievable and strengthens the result that the point \((2/3, 4/3)\) is not linear achievable in [2].
Fig. 3. Rate-memory trade-off $R^*(M)$ and $R_L^*(M)$ for the (3, 3) cache problem.

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