Favoring Eagerness for Remaining Items: Achieving Efficient and Fair Assignments

Xiaoxi Guo, Sujoy Sikdar, Lirong Xia, Hanpin Wang, and Yongzhi Cao*

Abstract

In the assignment problem, items must be assigned to agents who have unit demands, based on agents’ ordinal preferences. Often the goal is to design a mechanism that is both fair and efficient. In this paper, we first prove that, unfortunately, the desirable efficiency notions rank-maximality, ex-post favoring-higher-ranks, and ex-ante favoring-higher-ranks, which aim to allocate each item to agents who rank it highest over all the items, are incompatible with the desirable fairness notions strong equal treatment of equals (SETE) and sd-weak-envy-freeness (sd-WEF) simultaneously.

In light of this, we propose novel properties of efficiency based on a subtly different notion to favoring higher ranks, by favoring “eagerness” for remaining items and aiming to guarantee that each item is allocated to agents who rank it highest among remaining items. Specifically, we propose ex-post favoring-eagerness-for-remaining-items (ep-FERI) and ex-ante favoring-eagerness-for-remaining-items (ea-FERI). We prove that the eager Boston mechanism satisfies ep-FERI and sd-WSP, and that the uniform probabilistic respecting eagerness mechanism satisfies ea-FERI. We also prove that both mechanisms satisfy SETE and sd-WEF, and show that no mechanism can satisfy stronger versions of envy-freeness and strategyproofness while simultaneously maintaining SETE, and either ep-FERI or ea-FERI.

1. INTRODUCTION

In the assignment problem (Hylland and Zeckhauser, 1979; Zhou, 1990), $n$ agents have unit demands and strict ordinal preferences for $n$ items, each with unit supply, and the goal is to

X. Guo and Y. Cao are with Key Laboratory of High Confidence Software Technologies (MOE), Department of Computer Science and Technology, Peking University, Beijing 100871, China (e-mail: guoxiaoxi@pku.edu.cn; caoyz@pku.edu.cn).
S. Sikdar is with Department of Computer Science, Binghamton University (email: ssikdar@binghamton.edu).
L. Xia is with Department of Computer Science, Rensselaer Polytechnic Institute (email: xial@cs.rpi.edu).
H. Wang is with School of Computer Science and Cyber Engineering, Guangzhou University, China, and Key Laboratory of High Confidence Software Technologies (MOE), Department of Computer Science and Technology, Peking University, Beijing 100871, China (whpxhy@pku.edu.cn).
*Corresponding author: Yongzhi Cao.
compute an assignment which allocates each agent with a single unit of items. This serves as a useful model for a variety of problems where the items may be either indivisible such as houses (Shapley and Scarf, 1974), dormitory rooms (Chen and Sönmez, 2002), and school choice without priorities (Miralles, 2009); or divisible such as natural resources like land and water (Segal-Halevi, 2016), and computational resources in cloud computing (Ghodsi et al., 2011, 2012; Grandl et al., 2014). Due to the wide applicability of the assignment problem, there is rich literature pursuing the design of assignment mechanisms satisfying desirable properties of fairness, efficiency, and strategyproofness. Unfortunately, many of these properties are incompatible with each other, and trade-offs must be made.

The pursuit of fairness. A natural fairness criterion is equal treatment of equals, i.e., agents with identical preference orderings must receive equal allocations. However, this cannot be satisfied by any deterministic assignment, which allocates each item fully to a single agent. This fundamental impossibility has motivated the development of random mechanisms, which output random assignments representing probability distributions over deterministic assignments, that may provide even stronger fairness guarantees ex-ante, by allocating probability “shares” of items to each agent. For example, the classic random priority (RP) mechanism (Abdulkadiroğlu and Sönmez, 1998) satisfies strong equal treatment of equals (SETE), meaning every pair of agents who have a common prefix ordering over some items receive equal shares of items in the common prefix (Nesterov, 2017). The classic fairness property of envy-freeness (Foley, 1966; Varian, 1973) is extended to random assignments through the notion of stochastic dominance (sd), to compare random allocations (defined in Section 2.1). Two properties extensively studied for random mechanisms are sd-envy-freeness (sd-EF), meaning each agent’s allocation weakly dominates the allocation of every other agent, and sd-weak-envy-freeness (sd-WEF), meaning no agent’s allocation is weakly dominated by another agent’s allocation (Bogomolnaia and Moulin, 2001).

Efficiency vs. fairness. Pareto-efficiency (PE), meaning no other assignment makes some agent better off without making any agent worse off, is a popular efficiency criterion, and is compatible with several fairness and strategyproofness properties (Bogomolnaia and Moulin, 2001; Moulin, 2019; Saban and Sethuraman, 2014). However, PE does not maximize first choices (maximize the number of agents allocated their top items), an important criterion in practice, and is often
considered a weak guarantee of efficiency (Dur et al., 2018). This has motivated the study of stronger properties which, at a high level, both maximize first choices and aim to (approximately) maximize total satisfaction by favoring assignments that allocate each item to an agent who ranks it highest among all the items. Notably, for deterministic assignments, rank-maximality (RM) (Irving et al., 2006; Paluch, 2013), requires that no other assignment improves the number of agents allocated their top ranked item, subject to which, increases the number of agents allocated their second ranked item and so on; while Kojima and Ünver (2014) proposed the related, weaker notion favoring-higher-ranks (FHR), requiring that every item is allocated to an agent that ranks it highest unless every such agent is allocated an item they rank higher. RM and FHR have been studied as desirable properties for assigning schools to students by Abraham (2009) and Kojima and Ünver (2014) respectively, while RM has been studied for assigning papers to referees (Garg et al., 2010) and rental items to customers (Abraham et al., 2006).

There is extensive literature on random mechanisms that satisfy ex-post Pareto-efficiency (ep-PE) or sd-Pareto-efficiency (sd-PE), simultaneously with fairness properties (Bogomolnaia and Moulin, 2001; Guo et al., 2021; Wang et al., 2020). In contrast, FHR has only very recently been extended to random mechanisms by Ramezanian and Feizi (2021) who studied ex-post favoring-higher-ranks (ep-FHR) and Chen et al. (2021) who studied its ex-ante variant, ex-ante favoring-higher-ranks \(^1\) (ea-FHR), while we are unaware of such extensions of RM. However, the compatibility of ep-FHR and ea-FHR with desirable fairness properties remains unexplored.

**Efficiency vs. strategyproofness.** Strategyproofness, meaning no agent can manipulate the outcome to her benefit by misreporting her preference, is an important criterion for many applications such as assigning schools to students (Abdulkadiroglu et al., 2006; Pathak, 2017; Pathak and Sönmez, 2008) and kidney exchanges (Roth et al., 2005). Bogomolnaia and Moulin (2001) showed that RP satisfies ep-PE and sd-strategyproofness (sd-SP), which requires that an agent who misreports preferences is either worse off or unaffected compared to when they report truthfully. Unfortunately, Ramezanian and Feizi (2021) showed that ep-FHR is incompatible simultaneously with SETE and the weaker property of sd-weak-strategyproofness (sd-WSP), which requires that no agent can obtain a better allocation by misreporting.

In the pursuit of fair, efficient, and strategyproof assignment mechanisms, and in light of the

\(^1\)Chen et al. (2021) named this property sd-rank-fairness. We rename it here to emphasize its connection with FHR.
incompatibility between the desirable properties discussed above, we address the following open question in this paper: *What is an appropriate notion of efficiency that is stronger than PE and compatible with fairness (such as SETE, sd-EF, and sd-WEF) and strategyproofness (such as sd-SP and sd-WSP)?*

**TABLE I**

Properties of RP, PS, NBM, EBM, PR and PRE.

| Mechanism | ex-post efficiency | ex-ante efficiency | ex-ante fairness | strategyproofness |
|-----------|---------------------|---------------------|------------------|-------------------|
|           | ep-FERI | ep-FHR | ep-PE | ea-FERI | ea-FHR | sd-PE | sd-EF | sd-WEF | SETE | sd-SP | sd-WSP |
| RP        | N^{C2} | N^{1} | Y^{§} | N^{C2} | N^{1} | N^{5} | N^{§} | Y^{§} | Y^{C2} |
| PS        | N^{P15} | N^{1} | Y^{†} | N^{P15} | N^{1} | Y^{5} | Y^{§} | Y^{§} | Y^{P15} |
| NBM       | N^{P16} | Y^{†} | Y^{†} | N^{P16} | N^{1} | N^{2} | N^{16} | Y^{16} | Y^{P16} |
| EBM       | Y^{T1} | N^{P17} | Y^{T2} | N^{P17} | N^{P17} | N^{P17} | N^{P17} | Y^{T2} | Y^{T2} |
| PR        | N^{P18} | Y^{1} | Y^{1} | N^{P18} | N^{P17} | Y^{1} | N^{P17} | Y^{18} | Y^{18} |
| UPRE      | N^{P19} | N^{P19} | Y^{C1} | Y^{T3} | N^{P19} | Y^{C1} | N^{P19} | Y^{T4} | Y^{T4} |

Note: A “Y” indicates that the mechanism at that row satisfies the property at that column, and an “N” indicates that it does not. Results annotated with †, ‡, and § are from (Ramezanian and Feizi, 2021), (Chen et al., 2021), and (Bogomolnaia and Moulin, 2001), respectively. A result annotated with T, P or C refers to a Theorem, Proposition or Corollary in this paper (or Appendix A.1), respectively.

### 1.1. Our Contributions

We first show that ep-FHR and ea-FHR unfortunately may not be the answer to our question, since they are incompatible with SETE and sd-WEF (Proposition 1). We also prove that RM is strictly stronger than FHR, a connection that has surprisingly not been made earlier. Therefore, the incompatibility extends to ex-post RM.

We answer the question above through our main conceptual contribution, two novel properties of efficiency which favor “eagerness”: Every item ranked on top by some agents is allocated to one such agent; then, depending on the allocation of top ranked items, each remaining item is allocated to an agent who is “eager” for it, i.e., ranks it as their *top remaining item*, if such an agent exists; and so on. Specifically, the efficiency properties we propose are called *ex-post favoring-eagerness-for-remaining-items* (ep-FERI) and *ex-ante favoring-eagerness-for-remaining-items* (ea-FERI).
Our main technical contributions are (i) two mechanisms that satisfy these properties, which as we show, existing mechanisms do not satisfy; and (ii) an extensive analysis of their fairness, efficiency, and strategyproofness guarantees. Specifically, we propose and prove that:

- The *eager Boston mechanism* (EBM), defined in Algorithm 1, satisfies ep-FERI, ep-PE, SETE, sd-WEF, and sd-WSP (Theorem 2).
- The family of *probabilistic respecting eagerness* (PRE) mechanisms, defined in Algorithm 2, satisfy ea-FERI, ep-PE and sd-PE (Theorem 3 and Corollary 1), and one of its members, the *uniform probabilistic respecting eagerness* (UPRE) mechanism satisfies SETE and sd-WEF (Theorem 4).

Table I compares EBM and UPRE to the random priority mechanism (RP) (Abdulkadiroğlu and Sönmez, 1998), probabilistic serial mechanism (PS) (Bogomolnaia and Moulin, 2001), naive Boston mechanism (NBM) (Abdulkadiroğlu and Sönmez, 2003; Kojima and Ünver, 2014), and probabilistic rank (PR) mechanism (Chen et al., 2021; Harless, 2018). The properties are defined in Section 2.1, and all acronyms used in this paper are summarized in Appendix B.

In addition, we prove impossibility results that no mechanism can improve upon the guarantees of EBM and UPRE for certain combinations of efficiency, fairness and strategyproofness:

- ep-FERI, ea-FERI, and SETE (Proposition 10);
- ep-FERI and sd-EF (Proposition 11);
- ea-FERI and sd-EF (Proposition 12);
- ep-FERI, SETE and sd-SP (Proposition 13); and
- ea-FERI, SETE, and sd-WSP (Proposition 14).

![Figure 1](image)

**Figure 1.** Relationship between ep-FERI, ea-FERI, ep-FHR, ea-FHR, ep-PE, and sd-PE. The property A points to another property B means that A implies B. An arrow annotated with †, ‡, or § refers to a result due to (Ramezanian and Feizi, 2021), (Chen et al., 2021), and (Bogomolnaia and Moulin, 2001), respectively, and an edge annotated with P refers to a Proposition in this paper.
Figure 1 shows the relationship between our efficiency notions and PE notions ex-post Pareto-efficiency (ep-PE) and sd-Pareto-efficiency (sd-PE). Notice that ea-FERI does not imply ep-FERI (Proposition 8), unlike the case of the ex-ante and ex-post notions based on favoring higher ranks.

1.2. Related Works and Discussions

This paper is most closely related to the works studying FHR (Kojima and Ünver, 2014), ep-FHR (Ramezanian and Feizi, 2021) and ea-FHR (Chen et al., 2021). Kojima and Ünver (2014) showed that FHR characterizes BM, which is popularly used in practice, perhaps because it maximizes first choices, an important criterion in applications such as school choice as Dur et al. (2018) noted. However, BM is criticized for failing strategyproofness (Abdulkadiroglu et al., 2006), which has motivated the development of variants of BM. Most notably, Mennle and Seuken (2021) showed that the adaptive Boston mechanism (ABM) (Alcalde, 1996; Dur, 2019) satisfies sd-WSP and left the fairness of variants of BM as an open question. Our EBM gives a positive answer to this open question as it satisfies sd-WEF and sd-WSP. While EBM and ABM appear similar, they are indeed different mechanisms as we show in Appendix A.2.

Harless (2018) proposed the immediate division$^+$ mechanism and proved it satisfies sd-WEF. This mechanism appears similar to UPRE, though we are unable to prove or disprove their equivalence. In this paper, we define the larger family of PRE mechanisms (of which UPRE is a member) and prove that it is characterized by ea-FERI which was not considered earlier. In addition, we provide new impossibility results for ea-FERI mechanisms.

2. Preliminaries

An instance of the assignment problem is given by a tuple $(N, M)$ and a preference profile $\mathcal{R}$, where $N = \{1, \ldots, n\}$ is a set of $n$ agents, $M = \{o_1, \ldots, o_n\}$ is a set of $n$ items with a single unit of supply of each item.

Preferences. A preference profile $\mathcal{R} = (\succ_j)_{j \in N}$ specifies the ordinal preference of each agent $j \in [n]$ as a strict linear order over $M$. For any $j \in N$, we use $rk(j, o)$ to denote the rank of item $o$ in $\succ_j$, $\text{top}(j, S)$ to denote the item ranked highest in $S \subseteq M$, and $\succ_{-j}$ to denote the collection of preferences of agents in $N \setminus \{j\}$. Let $\mathcal{R}$ be the set of all the preference profiles. For any linear order $\succ$ over $M$ and item $o$, $U(\succ, o) = \{o' \in M | o' \succ o\} \cup \{o\}$ represents the items weakly preferred to $o$. Besides, let $\succ|_o$ be the part of $\succ$ over $U(\succ, o)$. For any pair of
agents \( j, k \in [n] \), we define the common prefix of their preferences as \( \succ_j,k = \succ_j|_{o'} = \succ_k|_{o'} \) where \( o' \) satisfies that \( U(\succ_j, o') \) is as large as possible.

**Allocations, Assignments, and Mechanisms.** A *random allocation* is a stochastic \( n \)-vector \( p = [p_o]_{o \in M} \), describing the probabilistic share of each item. Let \( \Pi \) be the set of all the possible random allocations. A *random assignment* is a doubly stochastic \( n \times n \) matrix \( P = [p_{j,o}]_{j \in N, o \in M} \).

For each agent \( j \in N \), the \( j \)-th row of \( P \), denoted \( P_j \), is agent \( j \)’s random allocation, and for each item \( o \in M \), \( p_{j,o} \) is \( j \)’s probabilistic share of \( o \). We use \( \mathcal{P} \) to denote the set of all possible random assignments. A *deterministic assignment* \( A : N \rightarrow M \) is a one to one mapping from agents to items, represented by a binary doubly stochastic \( n \times n \) matrix. For each agent \( j \in N \), we use \( A(j) \) to denote the item allocated to \( j \), and for each item \( o \in M \), \( A^{-1}(o) \) to denote the agent allocated \( o \). Let \( \mathcal{A} \) denote the set of all the deterministic assignment matrices. By the Birkhoff-Von Neumann theorem, every random assignment \( P \in \mathcal{P} \) describes a probability distribution over \( \mathcal{A} \).

A *mechanism* \( f : \mathcal{R} \rightarrow \mathcal{P} \) is a mapping from preference profiles to random assignments. For any profile \( R \in \mathcal{R} \), we use \( f(R) \) to refer to the random assignment output by \( f \). For every agent \( j \in N \), we use \( f(R)_j \) to denote agent \( j \)’s random allocation, and for every item \( o \in M \), we use \( f(R)_{j,o} \) to denote \( j \)’s share of \( o \).

### 2.1. Desirable Properties

Given a preference profile \( R \), a deterministic assignment \( A \) satisfies (i) **Pareto-efficiency (PE)** if there does not exist another \( A' \) such that \( A'(j) \succ_j A(j) \) for \( j \in N' \neq \emptyset \) and \( A'(k) = A(k) \) for \( k \in N \setminus N' \), (ii) **rank-maximality (RM)** (Irving et al., 2006) if there is no assignment \( A' \) such that its signature dominates \( A \), where the signature of an assignment \( A \) is an \( n \)-vector \( x = (x_r)_{r \leq n} \) such that for each \( r \in [n] \), the \( r \)-th component is the number of agents who are allocated their \( r \)-th ranked item, and a signature \( x \) dominates signature \( y \) if there exists \( r' \) such that \( x_{r'} > y_{r'} \) and for every \( r'' < r' \), \( x_{r''} \geq y_{r''} \), and (iii) **favoring-higher-ranks (FHR)** if for every \( r \leq n \), and every item \( o \in M \) where there exists an agent \( j \) such that \( rk(j, o) \), it holds that either (a) \( rk(A^{-1}(o), o) \leq r \), or (b) for every agent \( k \) such that \( rk(k, o) \leq r \), \( rk(k, A(k)) < rk(k, o) \).

**Example 1.** Consider the preference profile \( R \) in Figure 2. The deterministic assignment \( A \) indicated by the red circled items satisfying FHR has signature \( x = (3, 1, 1, 0, 0, 1) \), and \( A' \) indicated by the blue squared items satisfying RM has signature \( y = (3, 1, 1, 1, 0, 0) \).
Both FHR and RM require that an item should be assigned to one of the agents who rank it highest if such agents exist. Therefore, $a$ and $b$ go to agents 1 and 2, respectively. As for agents 3 to 6, they all rank $c$ highest.

By FHR, $c$ may be allocated to agent 3, like $A$ does, in which case, agent 6 can only get $f$, since $rk(6, d) > rk(j, d)$ and $rk(6, e) > rk(j, e)$ with $j \in \{4, 5\}$.

By RM, $c$ must be allocated to agent 6, as $A'$ does. Otherwise, it leads to an assignment with signature $x$, which is dominated by $y$.

**Definition 1.** (Bogomolnaia and Moulin, 2001) Given a preference relation $\succ$ over $M$, the stochastic dominance relation associated with $\succ$, denoted by $\succeq^{sd}$, is a partial ordering over $\Pi$ such that for any pair of random allocations $p, q \in \Pi$, $p$ (weakly) stochastically dominates $q$, denoted by $p \succeq^{sd} q$, if for any $o \in M$, $\sum_{o' \in \mathcal{U}(\succ, o)} p_{o'} \geq \sum_{o' \in \mathcal{U}(\succ, o)} q_{o'}$.

In general, given a property $X \in \{\text{PE}, \text{FHR}, \text{RM}\}$, a random assignment satisfies ex-post $X$ if it is a convex combination of deterministic assignments satisfying property $X$, and a mechanism $f$ satisfies a property $Y$, if for every profile $R \in \mathcal{R}$, $f(R)$ satisfies $Y$. Given a preference profile $R$, a random assignment $P$ satisfies

(i) **sd-Pareto-efficiency (sd-PE)** if there is no other random assignment $Q \neq P$ such that $Q_j \succeq^{sd}_j P_j$ for any $j \in N$,

(ii) **ex-ante favoring-higher-ranks (ea-FHR)** if for every agent $j \in N$ and every $o \in M$ such
that $p_{j,o} > 0$, it holds that for every $k \in N$ such that $rk(k, o) < rk(j, o)$, $\sum_{o' \in U(k,o)} p_{k,o'} = 1$, (iii) **strong equal treatment of equals (SETE)** if for any two agents $j$ and $k$, $p_{j,o} = p_{k,o}$ for any item $o$ appearing in the common prefix of their preferences $\succ_{j,k}$, (iv) **sd-envy-freeness (sd-EF)** if $P_j \succeq_{sd} P_k$ for any two agents $j$ and $k \in N$, and (v) **sd-weak-envy-freeness (sd-WEF)** if $P_k \succeq_{sd} P_j \implies P_j = P_k$.

A mechanism $f$ satisfies

(i) **sd-strategyproofness (sd-SP)** if for every $R \in \mathcal{R}$, it holds that $f(R) \succeq_{sd} f(R')$ for any $j \in N$ and any $R' = (\succ'_j, \succ_{-j})$,

(ii) **sd-weak-strategyproofness (sd-WSP)** if for every $R \in \mathcal{R}$, any $j \in N$, and any $R' = (\succ'_j, \succ_{-j})$, it holds that $f(R') \succeq_{sd} f(R) \implies f(R')_j = f(R)_j$.

In addition to the relations between efficiency notions shown in Figure 1, sd-EF implies sd-WEF, sd-SP implies sd-WSP (Bogomolnaia and Moulin, 2001), and sd-EF implies SETE (Nesterov, 2017).

### 3. Incompatibility of ep-FHR, ea-FHR and RM with Fairness and Strategyproofness

Ramezanian and Feizi (2021) proved that no mechanism which satisfies SETE and either one of ep-FHR and ea-FHR can also simultaneously satisfy sd-WSP. However, the question of compatibility with variants of envy-freeness remains open. In Example 2, we demonstrate that requiring ep-FHR together with SETE leads to envy, and formalize this incompatibility with envy-freeness in Proposition 1. Example 2 also illustrates the incompatibility with sd-WSP.

**Example 2.** Consider the preference profile $R$ in Figure 2. Satisfaction of FHR requires $a$ and $b$ to be allocated to agents 1 and 2, respectively, and to satisfy SETE, $c$ must be divided equally among agents in $\{3, 4, 5, 6\}$.

Further, for any FHR assignment where one agent in $\{3, 4, 5\}$ is allocated $c$, item $d$ and $e$ must be allocated to the rest but not to agent 6. This means that agent 6 has 0.75 shares of $f$, resulting in the random assignment below, which satisfies ep-FHR and SETE. It is easy to see that this makes agent 6 envy agent 3.
We also observe that agent 6 can benefit by misreporting her preference, for example, to be identical to that of agents 3 - 5, which results in a better allocation. Therefore, ep-FHR is also incompatible with SETE and sd-WSP.

**Proposition 1.** No mechanisms simultaneously satisfies ep-FHR (or ea-FHR), SETE, and sd-WEF.

**Proof.** We prove it using the preference $R$ in Figure 2. By FHR, one of agents 1 and 2 get $a$ and $b$, respectively, and one of agents 4-6 gets $c$. Besides, if 6 does not gets $c$, then she does not get $e$ or $d$ since $rk(6, d) > rk(j, d)$ and $rk(6, e) > rk(j, e)$ with $j \in \{4, 5\}$. Therefore the assignment in Example 2, denoted $P$, is the only one satisfying SETE and ep-FHR. We see that in $P$, agent 6 envies agent 3, because $\sum_{o' \in U(\succ_6, o)} p_{6,o'} \leq \sum_{o' \in U(\succ_6, a)} p_{1,o'}$ holds for any $o \in M$, and it is strict when $o \in \{e, d\}$, which violates sd-WEF. We have the same result if we replace ep-FHR with ea-FHR because ea-FHR implies ep-FHR (Ramezanian and Feizi, 2021).

We show in Proposition 2 that RM implies FHR, which automatically means that RM suffers the same incompatibility with fairness and strategyproofness as FHR. In fact, RM is incompatible even with sd-WEF alone as we show in Proposition 3.

**Proposition 2.** $RM \Rightarrow FHR$, $FHR \not\Rightarrow RM$.

**Proof.** $(RM \Rightarrow FHR)$ We show that for any preference profile $R$, if a deterministic assignment $A$ does not satisfy FHR, then $A$ is not RM.

Let $A$ be a deterministic assignment which does not satisfy FHR with signature $x$. Then, there exist agents $j$ and $k$ such that $A(j) = o \succ_k o' = A(k)$ and $rk(k, o) < rk(j, o)$. Let $r_{k,o} = rk(k, o), r_{j,o} = rk(j, o), r_{k,o'} = rk(k, o')$ and $r_{j,o'} = rk(j, o')$. Then, we have that $r_{k,o} < r_{j,o}$ and $r_{k,o'} < r_{k,o'}$.

Let $A'$ be the assignment obtained from $A$ by swapping the allocations of agents $j$ and $k$: $A'(j) = A(k), A'(k) = o$, and $A'(j') = A(j')$ for any $j' \in N \setminus \{j, k\}$; and let $y$ be the signature.
of $A'$. Then, we have that $y_{rk,o} = x_{rk,o} + 1$, $y_{rj,o} = x_{rj,o} - 1$, $y_{rk,o'} = x_{rk,o'} - 1$, $y_{rj,o'} = x_{rj,o'} + 1$ and $y_r = x_r$ for $r \notin \{rk,o, rj,o, rk,o', rj,o'\}$.

This means $y$ dominates $x$: $y_{rk,o} > x_{rk,o}$, and $y_{r'} \geq x_{r'}$ for any $r' < r_1$. Therefore $A$ is not RM, which means that that RM implies FHR.

(FHR ̸⇒ RM) This follows from Example 1, where the signature $x$ of an assignment $A$ satisfying FHR is dominated by the signature $y$ of another assignment $A'$ satisfying RM.

Proposition 3. No mechanism satisfies ep-RM and sd-WEF simultaneously.

Proof. Let $R$ be the profile in Figure 2. By RM, $a$ and $b$ must be assigned to agents 1 and 2, respectively, and although agents 3-6 all rank $c$ on top, only the assignments which allocate $c$ to agent 6 and $\{d, e, f\}$ to agents 3-5 can satisfy RM. Thus, every RM assignment has signature $y = (3, 1, 1, 1, 0, 0)$. Otherwise, if $c$ is assigned to agent 6, since RM implies FHR, agent 6 can only get $f$ since $rk(6, d) > rk(j, d)$ and $rk(6, e) > rk(j, e)$, where $j \in \{3, 4, 5\}$. Then, agents 3-5 get $\{c, d, e\}$, which results in assignments with signature $x = (3, 1, 1, 0, 0, 1)$ which is dominated by $y$, a contradiction. Then, since any random assignment $P$ satisfying ep-RM is a convex combination over the set of possible RM assignments, we have that $\sum_{o \in U(\succ 2, b)} p_{3,o} = p_{2,c} = 0 < 1 = p_{6,b} = \sum_{o \in U(\succ 6, c)} p_{6,o}$, a violation of sd-WEF.

4. Ex-post Favoring Eagerness

Motivated by the incompatibility results above for properties based on RM and FHR, and the desire for efficiency in the form of maximizing first choice, we propose (ex-post) favoring-eagerness-for-remaining-items (FERI) as a natural alternative property of efficiency, and prove later in this section that it is compatible with desirable properties of fairness and strategyproofness. Informally, a deterministic assignment that satisfies FERI can be decomposed in a manner that every item ranked highest by some agent is allocated to one such agent, subject to which, every remaining item is allocated to a remaining agent who ranks it highest among remaining items, and so on.

Definition 2 (FERI). Given any deterministic assignment $A$, we define $T_{A,0} = \emptyset$ and $T_{A,r} = \{o \in M : o = \text{top}(j, M \setminus \bigcup_{r' < r} T_{A,r'}) \land A(o) \notin \bigcup_{r' < r} T_{A,r'}\}$.

The assignment $A$ satisfies favoring-eagerness-for-remaining-items (FERI) if it holds that $o = \text{top}(A^{-1}(o), M \setminus \bigcup_{r' < r} T_{A,r'})$ for every item $o \in T_{A,r}$ with $r \geq 1$. 
We say that a random assignment $P$ satisfies {f ex-post favoring-eagerness-for-remaining-items (ep-FERI)} if it can be represented as a convex combination of deterministic assignments satisfying FERI. Intuitively, in Definition 2, $T_{A,r}$ tracks the set of items ranked highest among remaining items by some agent who does not get a higher ranked item in any $T_{A,r'}$ with $r' < r$, and $A$ satisfies FERI if every item $o \in T_{A,r}$ is allocated to one such agent.

**Example 3.** Consider again the profile in Figure 2. By FERI, items $a$ and $b$ must be allocated to agents 1 and 2 respectively, similarly to FHR, since they are the only agents to rank them on top.

However, when $c$ is allocated to one of agents 3 - 5, FERI requires that agent 6 be allocated item $d$, in contrast with FHR which requires that agent 6 be denied from receiving $d$. This eliminates agent 6's incentive to manipulate her preference by recognizing that the positions of $a$ and $b$ in 6's preferences may serve to hurt agent 6.

With FERI and SETE, it leads to the assignment $P$ below, and it is easy to verify that $P$ also satisfies sd-WEF.

| Assignment $P$ |
|----------------|
| $a$ | $b$ | $c$ | $d$ | $e$ | $f$ |
| 1 | 1 | 0 | 0 | 0 | 0 |
| 2 | 0 | 1 | 0 | 0 | 0 |
| 3-5 | 0 | 0 | 1/4 | 1/12 | 1/3 | 1/3 |
| 6 | 0 | 0 | 1/4 | 3/4 | 0 | 0 |

Example 4 shows that FERI and FHR are incomparable, and Proposition 4 shows that ep-FERI is a stronger efficiency property than ep-PE.

**Example 4.** [FHR $\not\Rightarrow$ FERI, FERI $\not\Rightarrow$ FHR] Consider again the profile in Figure 2. Let $A$ be the assignment indicated by the circled items, which satisfies FHR, and let $A^*$ be the following assignment, where $j \leftarrow o$ means agent $j$ is allocated item $o$:

$$A^*: 1 \leftarrow a, 2 \leftarrow b, 3 \leftarrow c, 4 \leftarrow e, 5 \leftarrow f, 6 \leftarrow d.$$ 

We see that $A$ violates FERI since $T_{A,1} = \{a, b, c\}$ and $d \in T_{A,2}$ by $d = \text{top}(6, M \setminus T_{A,1})$, but $A^{-1}(d) = 5$ and $d \neq \text{top}(5, M \setminus T_{A,1}) = e$. Besides, $A^*$ violates FHR because $A^*(6) = d$, $rk(6,d) > rk(5,d)$, and $d \succ_5 A^*(5)$. 

Proposition 4. \( \text{ep-FERI} \Rightarrow \text{ep-PE}, \ \text{ep-PE} \not\Rightarrow \text{ep-FERI} \).

Proof. (\( \text{ep-FERI} \Rightarrow \text{ep-PE} \)) Given a preference profile \( R \), assume for the sake of contradiction that there exists an deterministic assignment \( A \) which satisfies FERI but it is dominated by another assignment \( A' \), i.e., there exists a set of agents \( N' = \{j_1, j_2, \cdots, j_h\} \) such that \( A'(j_i) = A(j_{i+1}) \succ_i A(j_i) \) (we define \( j_{h+1} = j_1 \) here) while \( A'(j) = A(j) \) for \( j \notin N \setminus N' \). For ease of exposition, let \( o_i = A(j_i) \), and \( M' = \{o_1, o_2, \cdots, o_h\} \). W.l.o.g. let \( o_i \) be the item in \( T_{A,r} \) with the smallest \( r \) among \( M' \), and we have that \( o_i = \text{top}(A(j_i), M \setminus \bigcup_{r' < r} T_{A,r'}) \) by FERI. With the fact that \( o_{i+1} \succ_i o_i \), it follows that \( o_{i+1} \in T_{A,r'} \) with \( r' < r \), which is a contradiction to the construction that \( r \) is the smallest. Therefore we have that the assignment \( A \) satisfying FERI is PE, which also means that an random assignment satisfying ep-FERI is also ep-PE.

(\( \text{ep-PE} \not\Rightarrow \text{ep-FERI} \)) For the instance with the following profile \( R \) from (Ramezanian and Feizi, 2021), the deterministic assignment \( A \) is ep-PE since it is an outcome of RP with the priority order that \( 2 \succ 1 \succ 3 \).

| Assignment \( A \) | a | b | c |
|------------------|---|---|---|
| \( \succ_1 \) | a | b | c |
| \( \succ_2 \) | a | c | b |
| \( \succ_3 \) | b | a | c |

We see that \( b \in T_{A,1} = M \) since \( \text{top}(3, M) = b \) and \( A(3) = c \notin T_{A,0} \), but \( A^{-1}(b) = 1 \) and \( b \not\in \text{top}(1, M) = a \), which violates FERI. It also means that the expected outcome of RP, which is ep-PE, is not ep-FERI.

4.1. The Eager Boston Mechanism

The eager Boston mechanism (EBM), defined in Algorithm 1, proceeds in multiple rounds, where in each round, each remaining agent \( j \) applies for the item she is most eager for, i.e., her top remaining item. We use \( N_o \) to refer to the set of agents who apply for \( o \). Every agent in \( N_o \) gets \( o \) with probability \( 1/|N_o| \), with the winner determined by \( G \). Given a set of agents \( S \subseteq N \), \( G(S) \) is a single agent drawn from \( S \) uniformly at random. At the end of each round, for every item \( o \) with \( N_o \neq \emptyset \), both the item \( o \) and the winner \( G(N_o) \) are removed.
Algorithm 1  Eager Boston mechanism (EBM)
1: **Input:** An assignment problem $(N, M)$, a strict linear preference profile $R$, and a lottery winner generator $G$.
2: $M' \leftarrow M$. $N' \leftarrow N$. $A \leftarrow 0^{n \times n}$.
3: while $M' \neq \emptyset$ do
4:   for each $o \in M'$ do
5:      $N_o \leftarrow \{j \in N'|\text{top}(j, M') = o\}$.
6:      Run a lottery over $N_o \neq \emptyset$ to pick an agent $j_o = G(N_o)$, and allocate $o$, $A_{j_o, o} \leftarrow 1$.
7:      $M' \leftarrow M' \setminus \{o \in M'|N_o \neq \emptyset\}$. $N' \leftarrow N' \cup \bigcup_{o \in M'}\{j_o\}$.
8: return $A$

**Example 5.** We execute EBM on the instance in Figure 2. The table below shows for each round, which item each agent applies for, and a ‘/’ represents the fact that an agent does not apply for any item. The circled items represent the allocation of an item to the lottery winner.

| Round | Agent | 1 | 2 | 3 | 4 | 5 | 6 |
|-------|-------|---|---|---|---|---|---|
| 1     |       | a | b | c | c | c | c |
| 2     |       | / | / | / | e | d |   |
| 3     |       | / | / | / | f |   |   |

- At round 1, agents 1 and 2 apply for $a$ and $b$, respectively, and win them since they are the only applicants, while agents 3 - 6 apply for $c$ and enter a lottery with equal chances of winning.
- If agent 3 wins $c$ at round 1, then at round 2, agents 4 and 5 apply for $e$, while agent 6 applies for $d$ alone and gets it.
- If agent 4 wins $e$ at round 2, agent 5 applies for and gets $f$ at round 3.

Then, EBM outputs the assignment $A^*$ in Example 4.

The outcome EBM($R$) is a deterministic assignment which can be computed in polynomial time if $G$ runs in polynomial time, as we prove in Proposition 5. Theorem 1 shows that FERI characterizes EBM.

**Proposition 5.** Given a profile $R$, the deterministic assignment EBM($R$) can be computed in
polynomial time in the number of agents, if \( G \) is a polynomial time algorithm\(^2\).

**Proof.** In Algorithm 1, Line 2 is the initial setting which takes \( O(n^2) \) time, and the **While** loop is executed at most \( n \) times because there is at least one item is allocated in each round. Below, we analyze the time for each line in the main body of the **While** loop.

For Line 5, identifying the top item for each agent among \( M' \subseteq M \), takes \( O(n \cdot n) \) time.

For Line 6, it issues a lottery for each \( o \) over \( N_o \) and the implementation runs in in polynomial time w.r.t. \( n \) by the condition. Here the range is \( |N_o| < n \), which means the implementation of lottery is in polynomial time w.r.t. \( n \).

Line 7 take \( O(n') \) time where \( n' \) is the number of items being allocated at that round, and it takes \( O(n) \) time in total since at most \( n \) items are allocated in one run. Together we have that Algorithm 1 runs in polynomial time. \(\square\)

**Remark 1.** Although it is in polynomial time for EBM to output a deterministic assignment as shown in Proposition 5, there is no guarantee for the time complexity of computing the expected results of EBM. We conjecture that it is \#P-complete to compute the expected output of EBM, just as computing the expected result of RP, where the priority order is generated randomly and uniformly is \#P-complete (Saban and Sethuraman, 2015).

**Theorem 1.** Given a profile \( R \), a deterministic assignment satisfies FERI if and only if it is a possible outcome of \( EBM(R) \).

**Proof.** (Satisfaction) Given the preference profile \( R \), let \( A = EBM(R) \). We show by mathematical induction that, items to be assigned at each round \( r \) of Algorithm 1 are exactly those in \( T_{A,r} \) (defined in Definition 2), and \( A \) satisfies that for any \( o \in T_{A,r} \),

\[
o = \text{top}(A^{-1}(o), M \setminus \bigcup_{r^*<r} T_{A,r^*}).
\] (1)

**Base case.** When \( r = 1 \), any \( o \in T_{A,1} \) satisfies that \( o = \text{top}(j, M) \) for some \( j \). In Algorithm 1, all such agents are in \( N_o \) on Line 5, and \( o \) is assigned to one of them by Line 6 at that round. It means that \( o = \text{top}(A^{-1}(o), M) \), which is equivalent to Eq (1) when \( r = 1 \).

\(^2\)Such algorithm exists, like Xorshift RNG (Thomson, 1958) and linear congruential generator (Marsaglia, 2003)
**Inductive step.** With the condition that items to be assigned at round \( r' \) are exactly in \( T_{A,r} \) and \( o' = \text{top}(A^{-1}(o'),M \setminus \bigcup_{r' < r'} T_{A,r'}) \) for any \( r' < r \), we show that \( o = \text{top}(A^{-1}(o),M \setminus \bigcup_{r' < r} T_{A,r'}) \) for any \( o \in T_{A,r} \). Due to this condition, by Line 3, at the beginning of round \( r' \):

- the set of remaining items is \( M' = M \setminus \bigcup_{r' < r} T_{A,r'} \), since items in \( \bigcup_{r' < r} T_{A,r'} \) are allocated before round \( r \), and
- for any agent \( k \in N' \) who has not received any item yet, it holds that \( A(k) \notin T_{A,r'} \) for any \( r' < r \).

Therefore, if there exists an agent \( j \in N_o \subseteq N' \) (i.e. \( o = \text{top}(j,M') \)) on Line 5, then an agent in \( N_o \) gets \( o \) by Line 6, i.e., \( A^{-1}(o) \in N_o \), which implies Eq (1).

By the induction hypothesis we have that Eq (1) holds for any \( o \in T_{A,r} \) with \( r \geq 1 \), which means that \( A \) satisfies FERI.

**(Uniqueness)** Given that \( A \) satisfies FERI for the preference profile \( R \), we show that it is a possible outcome \( A' \) of EBM. We show by mathematical induction that at each round \( r \) of Algorithm 1, \( A^{-1}(o) \in N_o \) for any \( o \in T_{A,r} \) and it is possible that \( A'(A^{-1}(o)) = o \).

**Base case.** For \( r = 1 \), any \( o \in T_{A,1} \) satisfies that \( o = \text{top}(j,M) \) where \( A(j) = o \) by FERI. We have \( j \in N_o \) by Line 5, and then \( o \) is assigned to \( G(N_o) \) by Line 6. Since \( G(N_o) \) is drawn randomly and evenly from \( N_o \), it is possible that \( G(N_o) = j \).

**Induction step.** Given that \( A^{-1}(o') \in N_o' \) for any \( o' \in T_{A,r'} \) and it is possible that \( A'(A^{-1}(o')) = o' \) for any round \( r' < r \), we show them for \( r \). By the condition, we let any \( o' \in T_{A,r'} \) be assigned to \( A^{-1}(o') \) in \( A' \), and therefore \( M' = M \setminus \bigcup_{r' < r} T_{A,r'} \) and \( N' = N \setminus \{A^{-1}(o') \mid o' \in \bigcup_{r' < r} T_{A,r'} \} \). Then for any \( o \in T_{A,r} \) with \( A^{-1}(o) = j, j \in N' \) and \( \text{top}(j,M') = o \) by FERI, which means that \( j \in N_o \) for round \( r \) by Line 5. By Line 6, \( o \) is assigned to \( G(N_o) \). Since \( G(N_o) \) is drawn from \( N_o \), it is possible that \( G(N_o) = j \), i.e., \( A'(j) = o \).

By the induction, it is possible that \( A'(A^{-1}(o)) = o \) for any \( o \in T_{A,r} \) with \( r \geq 1 \). We note that every item \( o \) satisfies that \( o \in T_{A,r} \) for some \( r \). Otherwise, there exists \( r^* \) such that \( T_{A,r^*} = \emptyset \) while \( M \setminus \bigcup_{r' < r^*} T_{A,r'} \neq \emptyset \), which means that \( A(j) \in \bigcup_{r' < r^*} T_{A,r'} \) for any \( j \in N \), a contradiction.

Together we have that it is possible that \( A'(A^{-1}(o)) = o \) for any \( o \in M \), i.e, \( A \) is a possible outcome of EBM. \( \square \)

The expected outcome of EBM is a random assignment, which we refer to as \( \mathbb{E}(\text{EBM}(R)) \). Therefore, EBM may also be viewed as a random mechanism. We prove its properties in Theorem 2.
Theorem 2. EBM satisfies ep-FERI, ep-PE, sd-WEF, sd-WSP and SETE.

Proof. Given an instance \((N, M)\) with a strict linear preference profile \(R\), let \(P = \mathbb{E}(EBM(R))\).

Part 1: \(\mathbb{E}(EBM(R))\) is ep-FERI, ep-PE.

By Proposition 4 and Theorem 1, we have that EBM satisfies FERI and therefore \(P\) is ep-FERI and ep-PE.

Before the proof we introduce some new notations. For ease of exposition, we use a possible world, denoted \(w\), to represent one possible execution of EBM, and \(EBM^w(R)\) is the corresponding deterministic assignment. It is easy to see that if \(w \neq w'\), then \(EBM^w(R) \neq EBM^{w'}(R)\). Let \(W(R)\) be the set of all possible worlds for the given instance with \(R\), and \(W\) for short when \(R\) is clearly given in the context. The probability of \(w\), denoted \(Pr(w)\), can be computed according to lotteries in each round. We use \(l\) to refer to a lottery and \(N(l)\) be the set of agents who participate in \(l\). Let \(L(w, r)\) denote the set of lotteries in round \(r\) of world \(w\) (\(w\) can be omitted when clear), and \(r(w)\) be the total rounds of \(w\). Specially, \(l_o^w\) refers to the lottery for item \(o\) in \(w\). Then we have that \(Pr(w) = \prod_{l \in L(r, r \leq r(w))} \frac{1}{|N(l)|} = \prod_{o \in M} \frac{1}{|N(l)|}\) since every item can only be allocated once through lottery. Let \(Pr(W') = \sum_{w \in W} Pr(w)\) for \(W' \subseteq W\). If \(W'\) is the set of all the worlds with the same lotteries and winners for the first \(r\) rounds, then \(Pr(W') = \prod_{l \in L(r'), r' \leq r} \frac{1}{|N(l)|}\). For \(P = \mathbb{E}(EBM(R))\), we have that \(p_{j,o} = Pr(\{w \in W|EBM^w(R)(j) = o\})\), i.e., the probability of all the worlds where \(j\) gets items \(o\). Besides, for clear reference to variables in Algorithm 1, we use \(M_r^w, N_r^w, N_o^w\) to indicate \(M', N', N_o^r\) at the beginning of each round \(r\) in world \(w\), and \(w\) can be omitted if clear.

Part 2: \(\mathbb{E}(EBM(R))\) is sd-WEF.

For agent \(j\), let \(k\) be another agent such that \(P_k \succeq_{sd} P_j\). W.l.o.g. let \(\succ_j\) be \(o_1 \succ_j o_2 \succ_j \cdots \succ_j o_n\). We show by mathematical induction, for rank \(i = 1, 2, \ldots\) w.r.t. agent \(j\), that the following conditions hold for any \(w \in W\):

Condition (1): if \(j \in N_{o_i} \cap N_r\), then \(k \in N_{o_{i,r}}\).

Condition (2): if \(j\) gets some \(o \succ_j o_i\) at round \(r'\), then for each \(r > r'\) with \(k \in N_r\) and \(M_r \cup U(\succ_j, o_i) \neq \emptyset\), we have that \(k \in N_{top(j, M_r), r}\), and

Condition (3): \(p_{k, o_i} = p_{j, o_i}\).

Base case. First we prove conditions (1)-(3) for \(i = 1\). Since no \(o \succ_j o_1\), we have condition (2) trivially true. For any possible worlds \(w \in W\), we have that \(j \in N_{o_1, 1}\) and \(k \in N_1 = N\). If \(k \not\in N_{o_1, 1}\), then \(k\) does not participate in the lottery for \(o\), and she does not get \(o\) in any \(w\), which
means that \( p_{k,o_1} = 0 < p_{j,o_1} \), a contradiction to \( P_k \geq_{sd} P_j \). Therefore we have condition (1) for \( i = 1 \). It follows that \( p_{j,o_1} = Pr(\{w \in W | EBM^w(R)(j) = o_1 \}) = Pr(\{w \in W | EBM^w(R)(k) = o_1 \}) = p_{k,o_1} \), i.e., (3) for \( i = 1 \).

**Inductive step.** Assume that conditions (1)-(3) holds for \( i' < i \), we show that they also hold for \( i \). We show that \( p_{k,o_i} \leq p_{j,o_i} \) by comparing the probabilities of worlds where \( j \) gets \( o_i \) with those where \( k \) gets \( o_i \) in the following cases (i)-(iii).

**Case (i):** Given any world \( w' \in W \) satisfying that agents \( j \) and \( k \) does not get items better than \( o_i \), let \( r \) satisfy \( j \in N_{o_i,r} \).

Suppose such an \( r \) does not exist. Then \( EBM^{w'}(R)(j) \neq o_i \), and there does not exists \( r^* \) satisfying \( k \in N_{o_i,r^*} \) either. Otherwise, if such \( r^* \) exists, \( o_i \in M_{r^*} \). Let \( o_i = top(j, M_{r^*}) \), which means that \( i > i^* \). By condition (1) for \( i' < i \), when \( j \) applies for \( o_i \), \( k \) does too. It follows that \( i^* \geq i \) since \( o_i \neq o_i \), a contradiction. Together we see that both agents do not get \( o_i \) in \( w' \), and therefore \( w' \) is out of discussion.

Then we consider the case that such \( r \) exists. Let \( W_1 \) be the set of worlds where lotteries and winners are the same as \( w' \) for any round \( r' < r \), and therefore all the worlds in \( W_1 \) have the same \( M_r, N_r, N_{o_i,r} \) as \( w' \) for \( r \). By selection of \( w' \), we have that \( j, k \in N_r \).

If \( k \in N_{o_i,r} \), then she participates in the lottery for \( o_i \) at round \( r \) and her chance to win is equal to \( j \)’s, which means that

\[
Pr(\{w \in W_1 | EBM^{w'}(R)(j) = o_i \}) = Pr(W_1) \cdot \frac{1}{|N(l_{o_i})|} = Pr(\{w \in W_1 | EBM^{w'}(R)(k) = o_i \}).
\]

If \( k \notin N_{o_i,r} \), then by \( j \in N_{o_i,r} \neq \emptyset \), \( o_i \) is allocated to some agent in \( N_{o_i,r} \) and never appears in later rounds, which means that \( Pr(\{w \in W_1 | EBM^{w'}(R)(j) = o_i \}) > Pr(\{w \in W_1 | EBM^{w'}(R)(k) = o_i \}) = 0. \)

**Case (ii):** For any world \( w_k \in W \) in which agent \( j \) gets item \( o_h \) at round \( r' \) and \( k \) does not get any item better than \( o_i \), let \( r \) satisfy \( o_i = top(j, M_r) \).

Supposing such an \( r \) does not exist, we show that \( k \notin N_{o_i,r^*} \) for any \( r^* \). Otherwise, if \( k \in N_{o_i,r^*} \) for some \( r^* \), then \( k \in N_{r^*} \). Let \( o_i = top(j, M_{r^*}) \). We have that \( o_i \succ_j o_i \), i.e., \( i^* < i \), and the fact that \( o_i \neq o_i \) contradicts condition (1) for \( i^* \) if \( j \in N_{r^*} \), and condition (2) for \( i^* \) if \( j \notin N_{r^*} \). Therefore \( k \notin N_{o_i,r^*} \) for any \( r^* \) if such \( r \) does not exist, which means that \( EBM^{w_k}(R)(k) \neq o_i \) and we do not need to consider \( w_k \).
Having that $o_i = \text{top}(j, M_r)$. By condition (1) for $i' \leq h$, neither agents applies for any $o$ with $o_i \succ_j o$ before round $r'$, and it follows that $r > r'$. Let $W_k$ be the set of worlds where lotteries and winners are the same as $w_k$ for any round $r^* < r$. Correspondingly, we find a set of worlds $W_j$ such that

- for any round in $\{1, \ldots, r' - 1, r' + 1, \ldots, r\}$, lotteries and winners are the same as $w_k$,
- for round $r'$, lotteries are the same as $w_k$, and so do winners except the one for item $o_h$, and
- agent $k$ wins the lottery of $o_h$ at round $r'$.

We have $W_j \neq \emptyset$, because $k$ participates in the lottery for $o_h$ at round $r'$ since $k \in N_{o_h,r'}$ by condition (1) for $h$, which means that $k$ is possible to win $o_h$ instead of $j$, and then $j$ participates in the same lotteries instead of $k$ does in $w_k$ till round $r$ by condition (2) for $h < i' \leq i$. By construction of $W_j$ and $W_k$, $Pr(W_j) = Pr(W_k)$. For any $w \in W_j$ and $w' \in W_k$, $M^w_r = M_r^{w'}$, $j \in N^w_r$, $k \in N^{w'}_r$, and $N^w_r \setminus \{j\} = N^{w'}_r \setminus \{k\}$. By selection of $r$ such that $o_i = \text{top}(j, M_r)$, we have that $j \in N^w_{o_i,r}$.

- If $o_i = \text{top}(k, M_r)$, i.e., $k \in N^{w'}_{o_i,r}$, then by construction of $W_j$ and $W_k$, $N(l^w_{o_i}) \setminus \{j\} = N(l^{w'}_{o_i}) \setminus \{k\}$. It follows that $|N(l^w_{o_i})| = |N(l^{w'}_{o_i})|$ and

$$Pr(\{w \in W_j|\text{EBM}^w(R)(j) = o_i\}) = Pr(W_j) \cdot \frac{1}{|N(l^w_{o_i})|} = Pr(W_k) \cdot \frac{1}{|N(l^{w'}_{o_i})|} = Pr(\{w \in W_k|\text{EBM}^w(R)(k) = o_i\}).$$

- If $o_i \neq \text{top}(k, M_r)$, i.e., $k \notin N^{w'}_{o_i,r}$, we discuss in case of $N^{w'}_{o_i,r}$. When $N^{w'}_{o_i,r} \neq \emptyset$, then $o_i$ is allocated to some agent in $N^{w'}_{o_i,r}$, which means that $Pr(\{w \in W_j|\text{EBM}^w(R)(j) = o_i\}) > Pr(\{w \in W_k|\text{EBM}^w(R)(k) = o_i\}) = 0$. When $N^{w'}_{o_i,r} = \emptyset$, we have that $N^{w'}_{o_i,r} = \{j\}$. It means that $j$ is the only applicant for $o_i$, i.e., $|N(l^w_{o_i})| = 1$, and therefore gets it in any $w \in W_j$. As for agent $k$, she applies for $o' \neq o_i$ at round $r$ in $w' \in W_k$, and $o_i \succ_j o'$ by the selection of $o_i$. It follows that

$$Pr(\{w \in W_j|\text{EBM}^w(R)(j) = o_i\}) = Pr(W_j) \cdot \frac{1}{|N(l^w_{o_i})|} = Pr(W_j)$$

$$> 1 - Pr(\{w \in W_k|\text{EBM}^w(R)(k) = o'\})$$

$$\geq Pr(\{w \in W_k|\text{EBM}^w(R)(k) = o_i\}).$$
**Case (iii):** We do not need to consider worlds where both agents $j$ and $k$ get items better than $o_i$, i.e. do not get $o_i$.

**Concluding the inductive step.** From cases (i)-(iii), we have that $p_{k,o_i} \leq p_{j,o_i}$. With the assumption that $P_k \succ^sd P_j$ and condition (3) holds for $i' < i$, it follows that $p_{k,o_i} \geq p_{j,o_i}$. Therefore we have $p_{k,o_i} = p_{j,o_i}$, i.e., condition (3) holds for $i$. The equality also requires $k \in N_{o_i,r}$ in case (i) and $o_i = \text{top}(j,M_r) = \text{top}(k,M_r)$ in case (ii), i.e., conditions (1) and (2) for $i$.

By mathematical induction, we prove $p_{k,o_i} = p_{j,o_i}$ for any $i$. It follows that if $P_k \succ_j P_j$, $P_k = P_j$.

**Part 3:** $E(\text{EBM}(R))$ is sd-WSP.

W.l.o.g. let $\succ_j$ be $o_1 \succ_j o_2 \succ_j \cdots$. Let profile $R' = (\succ'_j,\succ'_r)$ where $\succ'_j$ is any preference that agent $j$ misreports, and $Q = \text{EBM}(R')$. Assume that $Q_j \succ^sd P_j$.

We show by mathematical induction, for rank $i = 1,2,\ldots$ w.r.t. agent $j$, that the following conditions hold:

**Condition (1):** when $j \in N_{o_i,r}^w$ in a world $w \in W(R)$, for any $w' \in W(R')$ where lotteries and winners are the same as $w$ before round $r$, we have that $j \in N_{o_i,r}^{w'}$, and

**Condition (2):** $p_{j,o_i} = q_{j,o_i}$.

**Base case.** First we show condition (1) for $i = 1$. It is easy to see that in any $w \in W(R)$, $j$ applies for $o_1$ at round 1, i.e., $j \in N_{o_1,1}^w$. We claim that $j \in N_{o_1,1}^{w'}$ for any $w' \in W(R')$. Otherwise, if $j \notin N_{o_1,1}^{w'}$ in some $w'$, we show that both of the possible cases below lead to a contradiction to our assumption that $Q_j \succ^sd P_j$.

- When $N_{o_1,1}^{w'} \neq \emptyset$, $o_1$ is assigned to some agent in $N_{o_1,1}^{w'}$ in $w'$. It follows that $p_{j,o_1} > q_{j,o_1} = 0$, a contradiction to the assumption.
- When $N_{o_1,1}^{w'} = \emptyset$, i.e., $N_{o_1,1}^w = \{j\}$, $o_1$ is assigned to the only applicant $j$ in $w$, while she applies for item $o' \neq o_1$ in $w'$ and $o_1 \succ_j o'$ trivially. It follows that

$$p_{j,o_1} = Pr(\{w \in W(R)|\text{EBM}^w(R)(j) = o_1\}) = 1$$

$$> 1 - Pr(\{w \in W(R')|\text{EBM}^{w'}(R')(j) = o'\})$$

$$\geq Pr(\{w \in W(R')|\text{EBM}^{w'}(R')(j) = o_1\}) = q_{j,o_1},$$

a contradiction to the assumption.
In this way, we have \( j \in N_{o_i,1}^{w'} \) for any \( w' \in W(R') \), i.e., condition (1) for \( i = 1 \), which means that \( |N(l^w_{o_i})| = |N(l^w_{o_i})| \) and

\[
p_{j,o_1} = Pr\{w \in W(R) | \text{EBM}^w(R)(j) = o_1\} = \frac{1}{|N(l^w_{o_1})|} = \frac{1}{|N(l^w_{o_i})|} = Pr\{w \in W(R') | \text{EBM}^w(R')(j) = o_1\} = q_{j,o_1},
\]

i.e., condition (2) for \( i = 1 \).

**Inductive step.** Supposing conditions (1) and (2) hold for \( i' < i \), we show that they also hold for \( i \). First we show condition (1) for \( i \). For an arbitrary world \( w^* \in W(R) \) with \( j \in N_{o_i,r}^{w*} \), let \( W_1 \subseteq W(R) \) and \( W_2 \subseteq W(R') \) be the sets of worlds where lotteries and winners are the same as \( w^* \) before round \( r \) w.r.t. \( R \) and \( R' \), respectively. By construction of \( W_1 \) and \( W_2 \), \( Pr(W_1) = Pr(W_2) \). For any \( w \in W_1 \) and \( w' \in W_2 \), \( M^w_r = M'^w_r \), \( N^w_r = N'^w_r \), and \( N^w_{o,r} = N'^w_{o,r} \) for any \( o \in M^w_r \setminus \{o_i\} \). We have that \( j \in N_{o_i,r}^{w,*} \) by the selection of \( w^* \), and we claim that \( j \in N_{o_i,r}^{w'} \) for any \( w' \in W(R') \). Otherwise, if \( j \notin N_{o_i,r}^{w'} \) in some \( w' \), we show that both of the possible cases below lead to a contradiction to our assumption that \( Q_j \succeq^sd P_j \).

- **When** \( N_{o_i,r}^{w'} \neq \emptyset \), \( o_i \) is assigned to some agent in \( N_{o_i,r}^{w'} \) in \( w' \). It follows that

\[
Pr\{w \in W_1 | \text{EBM}^w(R)(j) = o_i\} = Pr(W_1) > Pr\{w \in W_2 | \text{EBM}^w(R')(j) = o_i\} = 0.
\]

With condition (1) for \( i' < i \), in world \( w' \), agent \( j \) can only apply for \( o_i \) at round \( r' \geq r \) not earlier than she does in \( w \), which means that \( p_{j,o_i} > q_{j,o_i} = 0 \) with Eq (2). Together with condition (2) for \( i' < i \), we have a contradiction to the assumption that \( Q_j \succeq^sd P_j \).

- **When** \( N_{o_i,r}^{w'} = \emptyset \), i.e., \( N_{o_i,r}^{w} = \{j\} \), \( o_i \) is assigned to the only applicant \( j \) in \( w \) while she applies for item \( o' \neq o_i \) in \( w' \) and \( o_i >_j o' \) by the selection. It follows that

\[
Pr\{w \in W_1 | \text{EBM}^w(R)(j) = o_i\} = Pr(W_1) > Pr(W_2) - Pr\{w \in W_2 | \text{EBM}^w(R')(j) = o'\} \geq Pr\{w \in W_2 | \text{EBM}^w(R')(j) = o_i\}.
\]

This means that \( p_{j,o_i} > q_{j,o_i} \), a contradiction to the assumption that condition (2) holds for \( i' < i \).
In this way, we have condition (1) for $i$, which means that $|N(l_{w}^{o})| = |N(l_{o}^{w})|$ and
\[ Pr\{w \in W_{1}|EBM^{w}(R)(j) = o_{i}\} \]
\[ = Pr(W_{1}) \cdot \frac{1}{|N(l_{w}^{o})|} = Pr(W_{2}) \cdot \frac{1}{|N(l_{o}^{w})|} \]
\[ = Pr\{\{w \in W_{2}|EBM^{w}(R')(j) = o_{1}\}\}, \]
which implies $p_{j,o_{i}} = q_{j,o_{i}}$, i.e., condition (2) for $i$.

By mathematical induction, we have that $p_{j,o} = q_{j,o}$ for any $o$, and therefore if $Q_{j} \succeq^{sd} P_{j}$, that $Q_{j} = P_{j}$.

**Part 4:** $\mathbb{E}(EBM(R))$ is SETE.

For agents $j$ and $k$, we prove that $p_{j,o} = p_{k,o}$ for any item $o$ appearing in $\succ_{j,k}$. We compare probability of possible worlds where agent $j$ gets $o \in U(\succ_{j,k},o_{m})$ with those where agent $k$ gets $o$.

First we consider the world $w$ where $j$ gets $o$ at round $r$ and $k$ gets $o' \in U(\succ_{j,k},o_{m})$ at round $r'$. Let $w'$ satisfy $j$ gets $o'$ at round $r$, $k$ gets $o$ at round $r'$, and the result of other lotteries keep the same as $w$. In $w'$, we see that $k$ wins the lottery for $o$ instead of $j$, and $j$ participates in lotteries at rounds $r + 1$ to $r'$ instead of $k$. We also see that for every lottery $l_{o}$, $|N(l_{o})|$ keep the same in worlds $w$ and $w'$. Therefore we have that $Pr(w) = Pr(w')$.

Then we consider the world $w$ where $j$ gets $o$ at round $r$ and $k$ does not get items appearing in $\succ_{j,k}$. Let $o'$ be the last item $k$ applies for in $\succ_{j,k}$ at round $r'$, and $W_{j}$ be the set of worlds which are the same as $w$ from rounds 1 to $r'$. Here the probability of $W_{j}$ can also be computed as $Pr(W_{j}) = \prod_{l \in L(r_{*}),r_{*} \leq r'} \frac{1}{|N(l_{o})|}$. We construct another set $W_{k}$ such that for any $w \in W_{j}$, (i) the winners of lotteries are the same as $w$ at round 1 to $r - 1$, (ii) the winner of $l(o)$ is $k$ at round $r$, and any other $l \in L(r)$ is the same as $w$, (iii) $j$ participates in lotteries at rounds $r + 1$ to $r'$ instead of $k$. Then we see that for every lottery $l \in L(r_{*})$ with $r_{*} \leq r'$, $|N(l)|$ are the same in any world $w \in W_{j}$ and $w' \in W_{k}$. Therefore we have that $Pr(W_{j}) = Pr(W_{k})$.

Together we have that $p_{j,o} = p_{k,o}$ for any $o$ appearing in $\succ_{j,k}$.

5. **Ex-ante Favoring Eagerness**

In this section, we propose **ex-ante favoring-eagerness-for-remaining-items (ea-FERI)**, an ex-ante variant of FERI and stronger efficiency notion than sd-PE, which characterizes the family of PRE mechanisms (Algorithm 2). We show that UPRE, a member of PRE satisfies both SETE and sd-WEF. Intuitively, a random assignment satisfies ea-FERI if shares of every
remaining item is distributed among only the agents who are most eager for it first, i.e., rank it as their top remaining item, until either supply of the item is exhausted, or every eager agents’ demand has been satisfied.

**Definition 3 (ea-FERI).** Given a random assignment \( P \), we define \( M_{P,0} = \emptyset \) and for each item \( o \in M \), \( E_{P,0}(o) = \emptyset \). For each \( r = 1, 2, \ldots \), we define

(i) \[ M_{P,r} = \{ o \in M : \sum_{k \in \bigcup_{r'<r} E_{P,r'}(o)} p_{k,o} < 1 \} \]

(ii) \[ E_{P,r}(o) = \{ j \in N : o = \text{top}(j, M_{P,r}) \} \].

A random assignment \( P \) satisfies **ex-ante favoring-eagerness-for-remaining-items (ea-FERI)** if for any item \( o \in M_{P,r} \), it holds that \( \sum_{o' \in U(\succ_j,o)} p_{j,o'} = 1 \) for any \( j \in E_{P,r'}(o) \) with \( r' < r \).

A natural interpretation is that \( M_{P,r} \) refers to the set of items with remaining supply, and \( E_{P,r}(o) \) refers to the agents who consider \( o \) as their top ranked item in \( M_{P,r} \). Then ea-FERI requires that for each \( o \in M_{P,r} \), and each \( r' < r \), every agent \( j \in E_{P,r'}(o) \) is satisfied by items weakly preferred to \( o \). Proposition 6 shows that for deterministic assignments, ea-FERI is equivalent to FERI, which means that ea-FERI is an ex-ante extension of FERI for random assignments.

**Proposition 6.** Given any preference profile, a deterministic assignment satisfies FERI if and only if it satisfies ea-FERI.

**Proof.** The sets \( T_{A,r} \) and \( M_{A,r} \) in the following are defined in Definition 2 and Definition 3, respectively.

(FERI ⇒ ea-FERI) For any \( A \) satisfying FERI, we show that for any \( r \), the following two conditions hold:

**Condition (i):** \( M_{A,r} = M \setminus \bigcup_{r'<r} T_{A,r'} \), and

**Condition (ii):** for any item \( o \in M_{A,r} \) with \( r^* > r \), \( \sum_{\delta \in U(\prec_j,o)} p_{j,\delta} = 1 \) for any \( j \in E_{A,r}(o) \).

They trivially hold for \( r = 0 \). Supposing that they hold for each of \( r' = 1, \ldots, r-1 \), we prove that they also hold for \( r \).

**Condition (i) holds for \( r \).** For any \( r' < r \), by FERI, \( o' = \text{top}(A^{-1}(o'), M_{A,r'}) \) for every item \( o' \in T_{A,r'} \), which means that \( A^{-1}(o') \in E_{A,r'}(o') \) and \( \sum_{j \in \bigcup_{r'<r} E_{A,r'}(o')} A_{j,o'} = 1 \). Therefore \( o' \notin M_{A,r} \). It is easy to prove the opposite direction that \( o \in T_{A,r} \) for any \( o \notin M_{A,r} \) with the similar analysis. Together they mean that \( M_{A,r} = M \setminus \bigcup_{r'<r} T_{A,r'} \), i.e., (i) for \( r \).
Condition (ii) holds for \( r \). For any item \( o \in M_{A,r} \) with \( r^* > r \), \( \sum_{j \in U_{r',<r}} E_{A,r'}(o) A_{j,o} < 1 \). Since \( A \) is deterministic, it means that \( A_{j',o} = 0 \) for any \( j' \in E_{A,r'}(o) \) with \( r' \leq r \). Besides, \( o \in M_{A,r'} \subseteq M_{A,r} \), and then by FERI and (i) for \( r, o \notin T_{A,r} \), which means that for any \( j \in E_{A,r}(o) \), i.e. \( o = \text{top}(j, M_{A,r}) \), we have that \( A(j) \in T_{A,r'} \) with some \( r' < r \). Again by (i) for \( r' \), we know that \( A(j) = \text{top}(j, M_{A,r'}) \succ_j o \) since \( M_{A,r} \subseteq M_{A,r'} \), and therefore \( \sum_{o' \in \cup_{(j',o)}} p_{j',o'} = 1 \).

(ea-FERI\( \Rightarrow \) FERI) For any \( A \) satisfying ea-FERI, we show that for any \( r \), the following two conditions hold:

(i) \( M_{A,r} = M \setminus \bigcup_{r'<r} T_{A,r'} \), and

(ii) \( o = \text{top}(A^{-1}(o), M_{A,r}) \) for every item \( o \in T_{A,r} \).

They trivially hold for \( r = 0 \). Supposing that they hold for each of \( r^* = 1, \ldots, r-1 \), we prove that they also hold for \( r \).

Condition (i) holds for \( r \). For every \( o \in M_{A,r} \), it means that \( \sum_{j \in \cup_{r'<r}} E_{A,r'}(o) A_{j,o} < 1 \), which implies that \( A_{j,o} = 0 \) for any \( j \in \bigcup_{r'<r} E_{A,r'}(o) \) since \( A \) is deterministic. By the assumption that \( A \) is ea-FERI, \( \sum_{\delta \in U_{(r',o)}} p_{\delta,\delta} = 1 \) for any \( j \in \bigcup_{r'<r} E_{A,r'}(o) \).

- If \( E_{A,r'}(o) = \emptyset \) for any \( r' < r \), then there does not exist \( j \) with \( \text{top}(j, M_{A,r'}) = o \) and therefore \( o \notin T_{A,r'} \).

- If \( E_{A,r'}(o) \neq \emptyset \) for some \( r' < r \), then we have that \( o \notin T_{A,r'} \), since condition (ii) holds for \( r' \) and \( A_{j,o} = 0 \) for any \( j \in E_{A,r'}(o) \).

Then, we have that \( o \notin \bigcup_{r'<r} T_{A,r'} \). It is easy to prove the opposite direction that \( o \in M_{A,r} \) for any \( o \notin \bigcup_{r'<r} T_{A,r'} \) with a similar argument. Together they mean that \( M_{A,r} = M \setminus \bigcup_{r'<r} T_{A,r'} \), i.e., condition (i) holds for \( r \).

Condition (ii) holds for \( r \). For any item \( o \in T_{A,r} \), assume for the sake of contradiction that \( o \neq \text{top}(A^{-1}(o), M_{A,r}) \), which means that \( A(j) \neq o \) for any \( j \) with \( o = \text{top}(j, M_{A,r}) \), i.e., \( j \in E_{A,r}(o) \).

Since \( o \in T_{A,r} \), then \( o \in M_{A,r} \) by condition (i) for \( r \) which we proved immediately above, and therefore \( \sum_{j \in \cup_{r'<r}} E_{A,r'}(o) A_{j,o} = 0 < 1 \) since \( A \) is deterministic. Together, we have that \( \sum_{j \in \cup_{r'<r+1}} E_{A,r'}(o) A_{j,o} = 0 < 1 \), which means that \( o \in M_{A,r+1} \). Again by \( o \in T_{A,r} \), we know that there exists \( j' \in E_{A,r}(o) \), i.e., \( o = \text{top}(j', M_{A,r}) \) with \( A(j') \notin \bigcup_{r'<r} T_{A,r'} \). It also means that \( A(j') \in M_{A,r} \) by condition (i) for \( r \), and therefore \( o \succ_j A(j') \). Then \( \sum_{\delta \in U_{(j',o)}} p_{j',\delta} = 0 \) with \( o \in M_{A,r+1} \) and \( j' \in E_{A,r}(o) \), a contradiction to the assumption that \( A \) satisfies ea-FERI. From the contradiction, we have that \( o = \text{top}(A^{-1}(o), M_{A,r}) \) for every item \( o \in T_{A,r} \), i.e., condition (ii) holds for \( r \). \( \square \)
Proposition 7. \( ea\text{-}FERI \Rightarrow sd\text{-}PE, \ sd\text{-}PE \not\Rightarrow ea\text{-}FERI. \)

Proof. (\( ea\text{-}FERI \Rightarrow sd\text{-}PE \)) Assume for the sake of contradiction that \( P \) is \( ea\text{-}FERI \), but not \( sd\text{-}PE \). By assumption and Lemma 1 (in Appendix A.1), we can find a set of agents \( \{j_1, j_2, \ldots, j_h\} \) and items \( M^* = \{o_1, o_2, \ldots, o_h\} \) such that \( o_{i+1} \succ_j o_i \) with \( p_{j_i, o_i} > 0 \) with \( i \leq h \). Here we define \( j_{h+1} = j_1 \) and \( o_{h+1} = o_1 \) for convenience. By Theorem 3, \( P \) is the outcome of Algorithm 2 for some instance. For each \( o_i \), let \( r_i \) be the round where \( j_i \) consumes it, and let \( o'_{i'} = \arg \min_{o_i \in M^*} r_i \). Then we have that all the items in \( M^* \) are available at round \( r_{i'} \) and \( j_{i'} \in N_{o_{i'}} \), which means that \( top(j_{i'}, M') = o_{i'} \) where \( M' \) is the set of all the available items at that round in Algorithm 2. Since \( M^* \subseteq M' \), we have that \( o_{i'} \succ_{i'} o_{i' + 1} \), a contradiction to the assumption.

(\( sd\text{-}PE \not\Rightarrow ea\text{-}FERI \)) For the following preference profile \( R \), the assignment \( P \) is the outcome of PS which satisfies \( sd\text{-}PE \):

\[
\begin{array}{c}
\succ_1: a \succ_1 b \succ_1 c, \\
\succ_2: a \succ_2 c \succ_2 b, \\
\succ_3: b \succ_3 a \succ_3 c.
\end{array}
\]

Assignment \( P \)

\[
\begin{array}{ccc}
& a & b & c \\
1 & 1/2 & 1/4 & 1/4 \\
2 & 1/2 & 0 & 1/2 \\
3 & 0 & 3/4 & 1/4
\end{array}
\]

We see that \( E_{P,1}(b) = \{3\}, \sum_{o \in U(\succ_3, b)} = 3/4 < 1 \), and \( b \in M_{P,2} \), which violates \( ea\text{-}FERI. \)

Proposition 8. \( ep\text{-}FERI \not\Rightarrow ea\text{-}FERI, \ ea\text{-}FERI \not\Rightarrow ep\text{-}FERI. \)

Proof. (\( ep\text{-}FERI \not\Rightarrow ea\text{-}FERI \)) follows from the fact that EBM satisfies \( ep\text{-}FERI \) (Theorem 1), but it does not satisfy \( ea\text{-}FERI \) (Proposition 17).

(\( ea\text{-}FERI \not\Rightarrow ep\text{-}FERI \)) follows from the fact that UPRE satisfies \( ea\text{-}FERI \) (Theorem 3), but it does not satisfy \( ep\text{-}FERI \) (Proposition 19).

5.1. The probabilistic respecting eagerness mechanism

The probabilistic respecting eagerness mechanism (PRE) (Algorithm 2) is a family of algorithms, where each member \( PRE_\omega \) is defined by a parameter \( \omega = (\omega_j)_{j \in N} \). Each \( \omega_j \) is an eating speed function which maps each time instance \( t \) to a rate of consumption such that \( \int_0^1 \omega_j(t)dt = 1, \ \omega_j(t) \geq 0 \) for \( t \in [0,1] \), and \( \omega_j(t) = 0 \) for \( t > 1 \), i.e., agent \( j \) is not allowed to consume after 1 unit of time. At the beginning of execution, we set \( s(o) = 1 \) to refer to
the supply of item $o$, and set $t_j = 0$ to indicate the elapsed time each agent $j$ has spent on consumption. At each round $r$, each agent $j$ determines $\text{top}(j, M')$, her top item among $M'$ in which every item $o$ is available, i.e., $s(o) > 0$. For each item $o \in M'$, $N_o$ is the set of agents for whom $o$ is the top item. All the agents in $N_o$ consume $o$ together for $\gamma_o(N_o, (t_j)_{j \in N}; s(o))$ units of time. For any $N' \subseteq N$, elapsed consumption times $(t_j)_{j \in N}$, and supply $s$, we define:

$$\gamma_o(N', (t_j)_{j \in N}, s) = \min \left\{ \{ \rho \mid \sum_{k \in N'} \int_{t_k}^{t_k+\rho} \omega_k(t)dt = s \} \cup \{ \rho \in [0, 1] \mid \sum_{k \in N'} \int_{t_k}^{t_k+\rho} \omega_k(t)dt = \sum_{k \in N'} \int_{t_k}^{1} \omega_k(t)dt \} \right\},$$

Notice that for any agent $j$, $\int_{t_j}^{1} \omega_j(t)dt$ refers to her remaining demand. Therefore, agents in $N_o$ consume until either the supply of $o$ is exhausted or all of them are satisfied. Then the amount that agent $j$ consumes at this round is the shares of $o$ she gets in the final outcome, and we update the supply $s(o)$ and the elapsed time $t_j$.

**Algorithm 2** Probabilistic respecting eagerness (PRE)

1: **Input:** An assignment problem $(N, M)$, a strict linear preference profile $R$, a collection of eating functions $\omega = (\omega_j)_{j \in N}$.

2: $M' \leftarrow M$, $P \leftarrow 0^{n \times n}$, $s(o) \leftarrow 1$ for every $o$, and $t_j \leftarrow 0$ for every $j$.

3: **while** $M' \neq \emptyset$ **do**

4: $N_o \leftarrow \{ j \in N \mid \text{top}(j) = o \}$.

5: **for** each item $o \in M'$ **do**

6: Agents in $N_o$ consume $o$.

6.1: $\rho_o \leftarrow \gamma_o(N_o, (t_j)_{j \in N}; s(o))$

6.2: For each $j \in N_o$, $p_{j,o} \leftarrow \int_{t_j}^{t_j+\rho_o} \omega_j(t)dt$.

7: $s(o) \leftarrow s(o) - \sum_{k \in N_o} \int_{t_k}^{t_k+\rho_o} \omega_k(t)dt$.

8: For each $j \in N_o$, $t_j \leftarrow t_j + \rho_o$.

9: $M' \leftarrow M' \setminus \{ o \in M' \mid s(o) = 0 \}$.

10: **return** $P$

**Definition 4** (UPRE). The uniform probabilistic respecting eagerness mechanism (UPRE) is a member of PRE, where every agent eats at a uniform eating speed of one unit of item per
one unit of time, i.e., for each \( j \in N \),

\[
\omega_j(t) = \begin{cases} 
1, & t \in [0, 1], \\
0, & t > 1.
\end{cases}
\]

(3)

UPRE, illustrated in Example 6 below, returns a random assignment in polynomial time, as we show in Proposition 9.

**Proposition 9.** Given a profile \( R \), the random assignment \( \text{UPRE}(R) \) can be computed in polynomial time in the number of agents.

**Proof.** Recall that UPRE is Algorithm 2 using Eq (3) as eating functions, and then \( \int_{t_j}^{t_j+\rho} \omega_j(t) \, dt = \min(\rho, 1-t_j) \) when \( t_j < 1 \), which can be computed in \( O(1) \) time. In Algorithm 2, Line 2 is the initial setting which takes \( O(n^2) \) time. The **While** loop is executed at most \( n \) times because an agent consumes different items in each round. Below, we analyze the time for each line in the main body of the **While** loop.

On Line 4, identifying the top item for each agent among \( M' \subseteq M \), takes \( O(n \cdot n) \) time.

For Line 6.1, for each \( o \), we can compute \( \rho_o \) with the following steps:

1. Sort agents in \( N_o \) by \( 1-t_j \) in increasing order and obtain the sequence \( j_1, j_2, \ldots, j_{n_o} \) where \( n_o = |N_o| \), which takes \( O(n^2) \) time since \( n_o \leq n \);
2. For each \( i \in \{1, \ldots, n_o\} \), test if \( \sum_{k \in N_o} \int_{t_k}^{t_k+\rho} \omega_k(t) \, dt \leq s(o) \) with \( \rho = 1-t_{j_i} \) and stop when it is not. This takes \( O(n) \) time;
3. If \( i' < n_o \) is the maximum value for which the test in step (2) passes, then \( \rho_o \geq 1-t_{j_{i'}} \) and computing \( \rho_o = \max(\{\rho|\rho \cdot (n_o - i') + \sum_{i=1}^{i'} (1-t_i) \leq s(o)\}) = \frac{s(o)-\sum_{i=1}^{i'} (1-t_i)}{n_o} \) takes \( O(1) \) time; if \( i = n_o \), then \( \rho_o = 1-t_{j_{n_o}} \).

In this way, we have that Line 6.1 runs in \( O(n^2) \) time.

Line 6.2 needs us to perform one integration for each agent and can also be computed in polynomial time since each integration can be done in \( O(1) \) time.

Line 7 updates the supply of each \( o \in M' \), Line 8 updates \( t_j \) for each agent \( j \), and Line 9 checks if \( s(o) = 0 \), each of which needs addition/subtraction for no more than \( n \) times.

Together we have that UPRE runs in polynomial time. \( \square \)

**Example 6.** The execution of UPRE with the profile in Figure 2 as input, is described below and in Figure 3.
- At round 1, agents 1 and 2 consume $a$ and $b$ respectively, and other agents consume $c$. After consumption, agents 1 and 2 fully get $a$ and $b$, respectively, and the other four agents each get 1/4 units of $c$. The supply of each consumed item is updated as $s(a) = s(b) = s(c) = 0$.
- At round 2, agents 3 - 5 consume $e$ and each get 1/3 units such that $s(e) = 0$, and agent 6 consumes $d$ till satisfied and gets 3/4 units, leaving $s(d) = 1/4$.
- At round 3, agents 3 - 5 consume $d$ and each get 1/12 units each such that $s(d) = 0$.
- At round 4, agents 3 - 5 consume $f$ and get 1/3 units each.

The final output of UPRE is assignment $P$ in Example 3.

Theorem 3. Given a profile $R$, a random assignment $P$ satisfies ea-FERI if and only if there exists an eating speed function $\omega$ such that $P = \text{PRE}_\omega(R)$.

Proof. (Satisfaction) Let $P = \text{PRE}_\omega(R)$ where $\omega$ is any collection of eating functions. To prove that $P$ is ea-FERI, we show the following claim (where $M_{P,r}$ and $E_{P,r}(o)$ are defined in Definition 3).

Claim 1. Given any $R$ and any member $f$ of PRE, $P = f(R)$ satisfies that

(i) for each $o \in M_{P,r}$ and $E_{P,r}(o) = N_o$, it holds that $M_{P,r} = M'$ at round $r$. 

![Figure 3. An example of the execution of UPRE.](image)
(ii) for any agent \( j \) and item \( o^* \) with \( \text{top}(j, M_{P,r-1}) \succ_j o^* \succ_j \text{top}(j, M_{P,r}) \), it holds that \( p_{j,o^*} = 0 \).

(iii) for any item \( o \in M_{P,r^*} \) with \( r^* > r \), it holds that for any \( j \in E_{P,r}(o) \),

\[
\sum_{o' \in U(\succ_j, o)} p_{j,o'} = 1. \tag{4}
\]

**Proof.** We prove the claim by mathematical induction for every \( r \).

**Base case.** When \( r = 1 \), we see that \( s(o) \) is initially set to 1 w.r.t. the supply of item and \( M_{P,1} = M \) which is also the initial value of \( M' \) on Line 3 of Algorithm 2. Therefore \( E_{P,1}(o) = \{ j \mid \text{top}(j, M') \} = N_o \) by Line 4 for round 1. Together we have Claim 1 (i) for \( r = 1 \).

Besides, since no \( o^* \succ_j o = \text{top}(j, M_{P,1}) \) exists for any agent \( j \), we have that Claim 1 (ii) holds for \( r = 1 \) trivially.

Since \( t_j \) is set to 0 for any \( j \in N \) and \( \sum_{k \in N_o} \int_{t_k}^{t_0} \omega_k(t) dt \geq s(o) = 1 \), \( o \) is consumed to exhaustion by agents in \( N_o \) at round 1. It also means that \( \sum_{k \in E_{P,1}} p_{k,o} = 1 \), \( o \notin M_{P,r^*} \) with \( r^* > 1 \), and therefore Claim 1 (iii) holds trivially.

**Inductive step.** Supposing that Claim 1 holds for any \( r' < r \), we show that it also holds for \( r \). In Algorithm 2, at the beginning of round \( r \), \( M' \) only contains item \( o \) with positive supply, i.e., \( s(o) > 0 \), after consumption of previous rounds by Line 8. Because Claim 1 (i) holds for \( r' < r \), we have that only agents in \( \bigcup_{r' < r} E_{P,r'}(o) \) are able to consume \( o \) before round \( r \), which means that \( s(o) = 1 - \sum_{j \in \bigcup_{r' < r} E_{P,r'}(o)} p_{j,o} \), and therefore \( M' = M_{P,r} \). Then we have that \( N_o = \{ j \mid \text{top}(j, M') \} = E_{P,r}(o) \). Together we have that Claim 1 (i) holds for \( r \).

Then we show that Claim 1 (ii) holds for \( r \). For any agent \( j \) and item \( o^* \) such that \( \text{top}(j, M_{P,r-1}) \succ_j \cdots \succ_j o^* \succ_j \cdots \succ_j \text{top}(j, M_{P,r}) \), it means that \( o^* \notin M_{P,r} \) and agent \( j \) cannot get shares of \( o^* \) at round \( r - 1, r \) or later rounds.

- If \( \text{top}(j, M_{P,r-1}) = \text{top}(j, M_{P,r}) \), we have that Claim 1 (ii) is trivially true.
- If \( \text{top}(j, M_{P,r-1}) \neq \text{top}(j, M_{P,r}) \) and \( o^* = \text{top}(j, M_{P,r'}) \) for some \( r' < r - 1 \), we know that \( j \) consumes \( \text{top}(j, M_{P,r-1}) \) at round \( r - 1 \) and \( o^* \) at round \( r' \) because Claim 1 (i) holds for \( r' \), which means that \( \text{top}(j, M_{P,r-1}) \) and \( o^* \) are available at round \( r' \), i.e., both \( o^*, \text{top}(j, M_{P,r-1}) \in M_{P,r'} \), a contradiction to \( \text{top}(j, M_{P,r-1}) \succ_j o^* \). Therefore \( o^* \neq \text{top}(j, M_{P,r'}) \) for any \( r' < r - 1 \), i.e., agent \( j \) does not get shares of \( o^* \) before round \( r - 1 \).

Together we have that \( p_{j,o^*} = 0 \), i.e., Claim 1 (ii) holds for \( r \).
Next, we show Claim 1 (iii) holds for $r$. For any $o \in M'$, if $o \in M_{P,r^*}$ for some $r^* > r$, then
\[
\sum_{k \in \cup_{t' \leq r, E_{P,r^*}} \omega_k} p_{k,o} \leq \sum_{k \in \cup_{t' \leq r, E_{P,r^*}} \omega_k} p_{k,o} < 1.
\] (5)

Eq (5) implies that $o$ is still available after the consumption at round $r$, and \( \sum_{k \in N_o} \int_{t_k}^{1} \omega_k(t)dt = \sum_{k \in N_o} \int_{t_k}^{1} \omega_k(t)dt < s(o) \) where \( N_o' = \{ k' \in N_o \mid t_{k'} < 1 \} \) by Line 6.1. Then for each agent $j \in N_o'$, $p_{j,o} = \int_{t_j}^{1} \omega_j(t)dt$, and $p_{j',o} = 0$ for each $j' \in N_o \setminus N_o'$ because $t_{j'} \geq 1$ and $\omega_j(t) = 0$ when $t > 1$.

Eq (4) trivially holds for $j' \in N_o \setminus N_o'$, and we show that Eq (4) also for $j \in N_o'$ below. For any $o' = \text{top}(j, M_{P,r})$ such that $j \in N_o'$ and $r' < r$, agent $j \in E_{P,r'}(o')$, which means that $o'$ is consumed by $j$ at round $r'$ because Claim 1 (i) holds for $r'$, and $o' \neq o$ because otherwise $o' \in M_{P,r}$ which means that $j$ is satisfied with $t_j = 1$ at round $r'$, a contradiction to $j \in N_o'$. We also have that $o' \succ_j o$ due to $M_{P,r} \subseteq M_{P,r'}$, and it means that items consumed by $j$ in time period $[0, t_k]$ are better than $o$. With this and the fact that Claim 1 (ii) holds for $r' \leq r$, \( \sum_{\delta \in U(\succ_j,o)} P_{j,\delta} = \int_{0}^{1} \omega_j(t)dt = 1 \). Together we have Eq (4) for current $r$ and any $j \in N_o = E_{P,r}(o)$ if $o \in M_{P,r^*}$.

By Claim 1 (iii), we have that for any item $o \in M_{P,r}$, \( \sum_{o' \in U(\succ_j,o)} P_{j,o'} = 1 \) for any $j \in E_{P,r}(o)$ with $r' < r$, which means that $P$ satisfies ea-FERI.

**Uniqueness** Given $Q$ satisfying ea-FERI, we prove that it coincides with the outcome $P = \text{PRE}_\omega(R)$ where the eating functions in $\omega$ are as defined in Eq (6) for each agent $j$:

\[
\omega_j(t) = \begin{cases} 
n \cdot q_{j,o}, & t \in \left[\frac{r}{n+1}, \frac{r}{n}\right], \\
 r = \min\{\hat{r} \mid j \in E_{Q,r}(o)\}, \\
0, & \text{others}. \end{cases} \]

(6)

We prove by mathematical induction that the following conditions hold for any round $r$:

**Condition (1):** for each $o \in M_{Q,r}$, it holds that $M_{P,r} = M_{Q,r}$ and $E_{P,r}(o) = E_{Q,r}(o)$.

**Condition (2):** for any $j \in E_{Q,r}(o)$, if \( \sum_{o' \in U(\succ_j, \text{top}(j, M_{Q,r-1}))} P_{j,o'} < 1 \), it holds that \( t_j = (r-1)/n \) and \( \rho_o = 1/n \), and

**Condition (3):** for any $j \in E_{Q,r}(o)$ and $o \in M_{Q,r}$, it holds that $p_{j,o} = q_{j,o}$.

**Base case.** When $r = 1$, we trivially have that $M_{Q,1} = M' = M$ and $E_{Q,1}(o) = N_o$ for any $o \in M$ at round 1 in Algorithm 2, and each $j \in N_o$ consumes $o$. With Claim 1 (i), we have that condition (1) holds for $r = 1$. 
Then we show condition (2) holds for \( r = 1 \). By Line 2, \( s(o) \) is set to 1 for any \( o \in M' \), and for any \( j \in E_{Q,r}(o) \), \( t_j \) is set to 0, which is equal to \((r - 1)/n\). Since \( \sum_{k \in N_o} \int_{t_k}^{1} \omega_k(t)dt \geq s(o) = 1 \), \( \rho_o \leftarrow \min\{\rho \mid \sum_{k \in N_o} \int_{0}^{\rho} \omega_k(t)dt = s(o)\} \) by Line 6.1, and \( o \) is consumed to exhaustion. We also have that \( \sum_{j \in N_o} q_{j,o} = 1 \) for \( Q \). Otherwise, \( o \in M_{Q,2} \), and there exists \( j' \in E_{Q,1}(o) \) with \( \sum_{o' \in U(o,j)} q_{j,o'} = q_{j,o} < 1 \), a contradiction to \( Q \) satisfying ea-FERI. Therefore \( \sum_{k \in N_o} \int_{0}^{1} \omega_k(t)dt = 1 \) and \( \rho_o = 1/n \) by Eq (6). Together we have that condition (2) holds for \( r = 1 \).

With \( \rho_o = 1/n \), \( p_{j,o} = \int_{0}^{1/n} \omega_k(t)dt = q_{j,o} \), i.e., condition (3) holds for \( r = 1 \).

**Inductive step.** Supposing that conditions (1)-(3) hold for \( r' < r \), we show they also hold for \( r \). First, since conditions (1) and (3) hold for \( r' < r \), we trivially have that condition (1) holds for \( r \).

Then we show condition (2) holds for \( r \). By Claim 1 (i) and the fact that condition (3) holds for \( r' < r \), it holds that \( M_{Q,r} = M' \) and \( E_{Q,r}(o) = N_o \) for each \( o \in M_{Q,r} \). For any \( j \in E_{Q,r}(o) \) with \( \sum_{o' \in U(o,j)} p_{j,o'} < 1 \), \( t_j = (r - 1)/n \) by Line 8 and that condition (2) holds for \( r - 1 \). We show that \( \rho_o = 1/n \) in both of the possible cases below.

- If \( \sum_{k \in N_o} \int_{t_k}^{1} \omega_k(t)dt \geq s(o) \), then \( o \) is consumed to exhaustion, i.e. \( \sum_{j \in N_o} q_{j,o} = s(o) \). Assume for the sake of contradiction that \( \sum_{j \in N_o} q_{j,o} < s(o) \). Then \( o \in M_{Q,r+1} \), and there exists \( j' \in E_{Q,r}(o) \) with \( q_{j',o} < \int_{t_j}^{1} \omega_j(t)dt \), which means that \( \sum_{o' \in U(o,j)} q_{j,o'} < \int_{0}^{1} \omega_j(t)dt = 1 \), a contradiction to \( Q \) satisfying ea-FERI. Therefore by Eq (6),
  \[
  \rho_o = \min\{\rho \mid \sum_{k \in N_o} \int_{t_k}^{t_k+\rho} \omega_k(t)dt = s(o)\} = 1/n.
  \]

- If \( \sum_{k \in N_o} \int_{t_k}^{1} \omega_k(t)dt < s(o) \), then \( o \in M_{Q,r+1} \) and all the agents in \( N_o \) are satisfied, i.e., \( \sum_{k \in N_o} \int_{t_k}^{1+\rho} \omega_k(t)dt = \sum_{k \in N_o} \int_{t_k}^{1} \omega_k(t)dt \) and \( \sum_{j \in N_o} q_{j,o} = \sum_{k \in N_o} \int_{t_k}^{1} \omega_k(t)dt \). Otherwise, there exists \( j' \in E_{Q,r}(o) \) who is not satisfied with \( \sum_{o' \in U(o,j)} q_{j,o'} < 1 \), a contradiction to \( Q \) satisfying ea-FERI. By Line 6.1 and Eq (6), \( \rho_o = 1/n \).

Together we have that condition (2) holds for \( r \).

Finally we show that condition (3) holds for \( r \). For any \( j \in E_{Q,r}(o) \), if \( j \in E_{Q,r'}(o) \) with some \( r' < r \), then we have the proof trivially by the fact that condition (3) holds for \( r' \). We consider the case that \( j \notin E_{Q,r'}(o) \), i.e., \( o \neq top(j,M_{Q,r}) \) with any \( r' < r \). We show \( q_{j,o} = p_{j,o} \) in both of the possible cases below:

- If \( \sum_{o' \in U(o,j)} p_{j,o'} < 1 \), then by the fact that condition (2) for \( r \) which we just proved above, \( p_{j,o} = \int_{(r-1)/n}^{r/n} \omega_k(t)dt = q_{j,o} \).
• If $\sum_{o^j \in U(\succ j, \text{top}(j, M_{Q,r-1}))} p_{j,o^j} = 1$, then $\sum_{o^j \in U(\succ j, \text{top}(j, M_{P,r-1}))} p_{j,o^j} = 1$ by condition (1) for $r - 1$, which means that $j$ is satisfied before round $r$ and does not consume $o$ since $o \neq \text{top}(j, M_{Q,r'})$ for any $r' < r$. Therefore $p_{j,o} = 0$. As for $Q$, by condition (3) for $r' < r$, $\sum_{o^j \in U(\succ j, \text{top}(j, M_{Q,r-1}))} q_{j,o^j} \geq \sum_{o^j \in U(\succ j, \text{top}(j, M_{P,r-1}))} p_{j,o^j} = 1$. Because $o \neq \text{top}(j, M_{Q,r-1})$ and $o \in M' = M_{Q,r} \subseteq M_{Q,r-1}$ by condition (1) for $r$, we have that $\text{top}(j, M_{Q,r-1}) \succ_j o$, and therefore $q_{j,o} = 0 = p_{j,o}$.

Together we have that condition (3) holds for $r$.

From the induction above, we have that conditions (1) and (3) hold for any $r$, i.e., for any $r$, we have that $M_{P,r} = M_{Q,r}$, $E_{P,r}(o) = E_{Q,r}(o)$ for any $o \in M_{Q,r}$, and $p_{j,o} = q_{j,o}$ for any $j \in E_{Q,r}(o)$. In Algorithm 2, shares of $o$ are only allocated to agents in $E_{P,r}(o)$ in each round, and $o$ is exhausted at the end, which means that $p_{j',o} = 0$ if $j' \notin E_{Q,r}(o) = E_{P,r}(o)$ for any $r$. With the fact the the supply of all the items are fully allocated to agents, it follows that $q_{j',o} = 0$ if $j' \notin E_{Q,r}(o)$ for any $r$ by condition (3). Together we have that $P = Q$. \qed

Every member of PRE also satisfies ep-PE and sd-PE due to Proposition 7 and Theorem 3, which leads to Corollary 1.

**Corollary 1.** For any collection of eating speed functions $\omega$, PRE$_\omega$ satisfies ep-PE and sd-PE.

We now turn our attention to fairness, and show that UPRE satisfies sd-WEF and SETE in Theorem 4.

**Theorem 4.** UPRE satisfies sd-WEF and SETE.

**Proof.** Given an instance with $R$, let $P = \text{UPRE}(R)$.

**Part 1: UPRE($R$) is sd-WEF.**

Assume that there exist agents $j$ and $j'$ such that $P_{j'} \succsd_j P_j$. W.l.o.g., let $o_r$ be the item such that $j \in N_{o_r}$ at round $r$, and we have the following claim:

**Claim 2.** For $r' < r$, if $o_r \neq o_{r'}$, then $o_r \succ_j o_{r'}$.

The claim holds because $o_r = \text{top}(j, M_{P,r})$ and $o_{r'} \in M_{P,r'} \subseteq M_{P,r}$.

We prove by mathematical induction that the following conditions hold for any round $r$ with $t_j < 1$:

**Condition (1):** $t_{j'} = t_j$.

**Condition (2):** $p_{j,o'} = p_{j',o'} = 0$ for any $o'$ with $o_{r-1} \succ_j \cdots \succ_j o' \succ_j \cdots \succ_j o_r$. 

**Condition (3):** $j' \in N_{o_r}$, and

**Condition (4):** $p_{j,o_r} = p_{j',o_r}$.

**Base case.** With $j \in N_{o_1}$ at round 1, we know that $t_k = 0$ for every $k$, i.e., condition (1) holds for $r = 1$, and $o_1 = \text{top}(j,M') = \text{top}(j,M)$ by Claim 1 (i).

Condition (2) is trivially true since no item $o' \succ_j o_1$ exists.

Then, we show that condition (3) holds for $r = 1$. Item $o_1$ is consumed to exhaustion at this round by Line 6.1 because $\sum_{k \in N_{o_1}} \int_{t_k}^{t_k+1} \omega_k(t)dt \geq s(o_1) = 1$, which means that no agent can get $o_1$ at any round $r^* > 1$. Therefore $p_{j,o_1} > 0$ due to $j \in N_{o_1}$. If $j' \notin N_{o_1}$, then $j'$ does no consume $o_1$, which means that $p_{j',o_1} = 0 < p_{j,o_1}$, a contradiction to the assumption that $P_{j'} \succ^d P_j$. Then we have that $j' \in N_{o_1}$, i.e., condition (3) holds for $r = 1$.

Because $t_j = t_j' = 0$, $p_{j,o_1} = \int_0^{o_1} \omega_j(r)dt = \int_0^{o_1} \omega_{j'}(r)dt = p_{j',o_1}$ by Eq (3), i.e., condition (4) for $r = 1$.

**Inductive step.** Supposing that conditions (1)-(4) hold for any $r' < r$, we show that they also hold for $r$ with $t_j < 1$.

We see that condition (1) trivially holds for $r$, i.e., $t_j' = t_j$ due to the fact that by condition (3) for any $r' < r$, both $t_j'$ and $t_j$ increase by the same value $\rho_{o,r}$ on Line 6.1 in each round $r'$.

Then, we prove condition (2) for $r$. Here $o_r \neq o_{r-1}$, because otherwise $o_{r-1} \in M_{P,r}$ and by Theorem 3, $\sum_{o \in U(r_j,o_{r-1})} p_{j,o} = 1$. It means that $t_j \geq 1$ at round $r$ so we do not need to consider it.

For any $o'$ with $o_{r-1} \succ_j \cdots \succ_j o' \succ_j \cdots \succ_j o_r$, $p_{j,o'} = 0$ by Claim 1 (i) and (ii), and we show that $p_{j',o'} = 0$. We have that $o' \notin M'$ at round $r$ because $j \in N_{o_r}$, i.e., $\text{top}(j,M') = o_r$, which means that $o'$ is unavailable for round $r^* \geq r$. With Claim 2, $o'$ is not consumed by $j$ at any round $r' < r$, and therefore it is also not consumed by $j'$ according to condition (3) for $r' < r$. Then $o'$ is never consumed by $j$ or $j'$, which means that $p_{j,o'} = p_{j',o'} = 0$, i.e., condition (2) for $r$.

Next, we prove condition (3) for $r$. We consider the following cases.

- If $o_r \in M_{P,r+1}$, then by Claim 2 and condition (2) for $r' \leq r$, it must hold that $\sum_{o' \succ_j o_r} p_{j,o'} = \int_0^{o_j} \omega_j(r)dt = t_j < 1$. By Claim 1 (i), $j \in N_{o_r} = E_{P,r}(o)$. By Theorem 3 and $o_r \in M_{P,r+1}$, $\sum_{o' \in U(r_j,o_r)} p_{j,o'} = 1$. By condition (1) for $r$ that $t_j' = t_j < 1$, $j'$ is not satisfied at the beginning of round $r$. If $j' \in N_{o'}$ with $o' \neq o_r$, it means that $j'$ consumes $o'$ at round $r$ and $p_{j',o'} > 0$, and $o_r \succ_j o'$ since $o_r = \text{top}(j,M')$. By conditions (2) and (4) for $r' < r$, and condition (2) for $r$
which we just prove, we have that
\[
\sum_{o^* \succ_j o_r} p_{j,o^*} = \sum_{o^* \succ_j o_r} p_{j',o^*}.
\]
(7)

Therefore,
\[
\sum_{o^* \in U(\succ_j, o_r)} p_{j,o^*} < 1 - p_{j',o'} < 1 = \sum_{o^* \in U(\succ_j, o_r)} p_{j,o^*},
\]
a contradiction to the assumption that \( P_{j'} \succeq^sd P_j \).

- If \( o_r \notin M_{P,r+1} \), then \( o_r \) is consumed to exhaustion by agents in \( N_{o_r} \) at round \( r \), and no agent consumes \( o_r \) after round \( r \). Since \( j \in N_{o_r} \) and \( t_j < 1 \), \( p_{j,o_r} > 0 \). If \( j' \notin N_{o_r} \), then \( j' \) does not consume \( o_r \) at round \( r^* \geq r \). Moreover, \( j' \) also does not consume \( o_r \) before round \( r \) by condition (3) for \( r' < r \). With Eq (7), we have that
\[
\sum_{o^* \in U(\succ_j, o_r)} p_{j,o^*} > \sum_{o^* \in U(\succ_j, o_r)} p_{j',o^*},
\]
a contradiction to the assumption that \( P_{j'} \succeq^sd P_j \).

Together we show that \( j \in N_{o_r} \).

Finally we have condition (4) for \( r \) trivially because
\[
p_{j,o_r} = \int_{t_j}^{t_j + \rho_{o_r}} \omega_j(r)dr = \int_{t_j'}^{t_j' + \rho_{o_r}} \omega_j'(r)dr = p_{j',o_r}
\]
by Eq (3), conditions (1) and (3) for \( r \).

By the induction, we have conditions (2) and (4) for any \( r \), i.e., \( p_{j,o} = p_{k,o} \) for any item \( o \), which means that \( P_{j'} = P_j \) if \( P_{j'} \succeq^sd P_j \).

**Part 2: UPRE(\( R \)) is SETE.**

We show that before consuming items not in \( \succ_{j,k} \), \( t_j = t_k \), and \( j \) and \( k \) consume the same item in each round by mathematical induction based on rounds.

**Base case.** At round 1, we know that both \( j \) and \( k \) consume the most preferred item \( o \) in \( \succ_{j,k} \). We have \( p_{j,o} = p_{k,o} \) by Eq (3) and \( t_j = t_k = 0 \) which is set initially at the beginning of Algorithm 2.

**Inductive step.** Supposing that \( j \) and \( k \) consume the same item and get the same shares for each round \( r' < r \), we prove that this is also the true for round \( r \). By the inductive assumption, we trivially have that \( t_j = t_k \) at the beginning of \( r \). Let \( j \) consume \( o \), and \( k \) consume \( o' \). Here we do not need to consider the case that both \( o, o' \) not in \( \succ_{j,k} \). If \( o \neq o' \), we assume that \( o \succ o' \), and therefore \( o \) must be in \( \succ_{j,k} \). It means that \( o \) is available at round \( r \), but \( k \) consumes \( o' \), a contradiction to the selection of top items. Therefore \( o = o' \), and \( p_{j,o} = p_{k,o} \) by \( t_j = t_k \) and Eq (3).

By induction we have that \( p_{j,o} = p_{k,o} \) for every \( o \) in \( \succ_{j,k} \).
6. IMPOSSIBILITY RESULTS

We have established that EBM and UPRE both satisfy some notions of efficiency (ep-FERI for EBM and ea-FERI for UPRE) and fairness (SETE and sd-WEF), and EBM is strategyproof (sd-WSP). In this section, we study the possibility of designing mechanisms with stronger efficiency, fairness or strategyproofness guarantees. Unfortunately, the following propositions show that the answer is negative.

Proposition 10 shows that efficiency cannot be improved (to the simultaneous satisfaction of ep-FERI and ea-FERI) given fairness (SETE).

**Proposition 10.** No mechanism simultaneously satisfies ep-FERI, ea-FERI, and SETE.

**Proof.** Before proceeding with the proof, we present the following claim which illustrates how shares of items must be allocated in order to satisfy ea-FERI.

**Claim 3.** Given \( P \) satisfying ea-FERI, for any \( r \) and \( o \in M_{P,r} \), we define

\[
 s_{P,r}(o) = 1 - \sum_{k \in \bigcup_{r'<r} E_{P,r'}(o)} p_{k,o}, \quad d_{P,r}(j) = 1 - \sum_{o' \in U(\succ_j, \text{top}(j,M_{P,r-1}))} p_{j,o'}.
\]

For any \( j \in E_{P,r}(o) \neq \emptyset \),

(i) for any \( o^* \) with top\((j, M_{P,r-1}) \succ_j o^* \succ_j o \), \( p_{j,o^*} = 0 \).

(ii) if \( \sum_{k \in E_{P,r}(o)} d_{P,r}(k) \geq s_{P,r}(o) \), \( o \notin M_{P,r} \) with \( r^* > r \) and \( \sum_{k \in E_{P,r}(o)} p_{k,o} = s_{P,r}(o) \).

(iii) if \( \sum_{k \in E_{P,r}(o)} d_{P,r}(k) \leq s_{P,r}(o) \), \( p_{j,o} = d_{P,r}(j) \).

**Proof.** We have (i) because \( o^* \notin M_{P,r} \) by the condition, i.e., \( \sum_{k \in \bigcup_{r'<r} E_{P,r'}(o)} p_{k,o} = 1 \), and \( j \notin E_{P,r'}(o^*) \) for any \( r' < r \). We have (ii) because if \( o \in M_{P,r} \), by (i) for \( r \) and \( \sum_{k \in E_{P,r}(o)} d_{P,r}(k) \geq s_{P,r}(o) \), there exists agent \( j' \in E_{P,r}(o) \) such that \( \sum_{o' \in U(\succ_{j'},o)} p_{j',o'} = d_{P,r}(j') + p_{j',o} < 1 \), a contradiction to ea-FERI. We have (iii) because if there exists \( j' \in E_{P,r}(o) \) with \( p_{j',o} < d_{P,r}(j') \), then \( \sum_{o' \succ_{j'} o} p_{j',o'} < 1 \) and \( o \in M_{P,r+1} \) by \( s_{P,r+1}(o) \geq s_{P,r}(o) - \sum_{k \in E_{P,r}(o)} d_{P,r}(k) > 0 \), a contradiction. \( \blacksquare \)
Assume that there exists a mechanism \( f \) satisfying ep-FERI, ea-FERI, and SETE. Let \( P = f(R) \) for the following preference profile \( R \):

\[
\succ_{1,2}: \ a_1 \succ_1 a_2 \succ_1 a_3 \succ_1 \text{ others} \\
\succ_{3}: \ a_1 \succ_3 a_2 \succ_3 a_3 \succ_3 \text{ others} \\
\succ_{4,5}: \ b_1 \succ_4 b_2 \succ_4 b_3 \succ_4 \text{ others} \\
\succ_{6}: \ b_1 \succ_6 b_2 \succ_6 b_4 \succ_6 \text{ others} \\
\succ_{7,17}: \ c_1 \succ_7 c_2 \succ_7 c_3 \succ_7 c_4 \succ_7 c_5 \succ_7 c_6 \succ_7 \text{ others} \\
\succ_{x}: \ c_1 \succ_x c_2 \succ_x c_3 \succ_x a_3 \succ_x b_3 \succ_x c_5 \succ_x c_4 \\
\succ_x c_6 \succ_x \text{ others}
\]

- For \( r = 1 \): \( M_{P,1} = M, E_{P,1}(a_1) = \{1, 2, 3\}, E_{P,1}(b_1) = \{4, 5, 6\} \) and \( E_{P,1}(c_1) = \{7, \ldots, 17, x\} \). By Claim 3 (ii), \( a_1, b_1, c_1 \notin M_{P,2} \) since \( d_{P,1}(j) = 1 \) for any \( j \in N \). Then we have \( p_{j,a_1} = 1/3 \) for \( j \in E_{P,1}(a_1) \), \( p_{j',b_1} = 1/3 \) for \( j' \in E_{P,1}(b_1) \) and \( p_{j',c_1} = 1/12 \) for \( j' \in E_{P,1}(c_1) \) by SETE.

- For \( r = 2 \), \( M_{P,2} = \{a_2, a_3, a_4, b_2, b_3, b_4, c_2, \ldots, c_6, \ldots\} \), \( E_{P,2}(a_2) = \{1, 2, 3\}, E_{P,2}(b_2) = \{4, 5, 6\} \) and \( E_{P,2}(c_2) = \{7, \ldots, 17, x\} \). Similar to \( r = 1 \), by Claim 3 (ii), \( a_2, b_2, c_2 \notin M_{P,3} \). Then we have \( p_{j,a_2} = 1/3 \) for \( j \in E_{P,2}(a_2) \), \( p_{j',b_2} = 1/3 \) for \( j' \in E_{P,2}(b_2) \) and \( p_{j',c_2} = 1/12 \) for \( j' \in E_{P,2}(c_2) \) by SETE.

- For \( r = 3 \), \( M_{P,3} = \{a_3, a_4, b_3, b_4, c_3, \ldots, c_6, \ldots\} \), \( E_{P,3}(a_3) = \{1, 2\} \), \( E_{P,3}(a_4) = \{3\}, E_{P,3}(b_3) = \{4, 5\}, E_{P,3}(b_4) = \{6\} \) and \( E_{P,3}(c_3) = \{7, \ldots, 17, x\} \). By Claim 3 (iii), \( p_{j,a_3} = 1/3 \) for \( j \in E_{P,3}(a_3) \), \( p_{3,a_4} = 1/3 \), \( p_{j',b_3} = 1/3 \) for \( j' \in E_{P,3}(b_3) \) and \( p_{6,b_4} = 1/3 \). Since \( \sum_{j \notin U(\succ_j, top(j, M_{P,3}))} \) = 1 for any agent \( j \in N' = \{1, \ldots, 6\} \), the allocation for \( j \) has been determined and we do not need to consider it in the later rounds.

- For \( r = 4 \), \( M_{P,4} = \{a_3, a_4, b_3, b_4, c_4, c_5, c_6, \ldots\} \). Then \( E_{P,4}(c_4) \setminus N' = \{7, \ldots, 17\} \) and \( E_{P,4}(c_3) \setminus N' = \{x\} \). By Claim 3 (ii), \( a_3, c_4 \notin M_{P,5} \), \( p_{x,a_3} = 1/3 \) and \( p_{j',c_4} = 1/3 \) for \( j' \in E_{P,4}(c_4) \) by SETE.

- For \( r = 5 \), \( M_{P,5} = \{a_4, b_3, b_4, c_5, c_6, \ldots\} \), \( E_{P,5}(c_5) \setminus N' = \{7, \ldots, 17\} \) and \( E_{P,5}(b_3) \setminus N' = \{x\} \). By Claim 3 (ii), \( b_3, c_5 \notin M_{P,6} \), \( p_{x,b_3} = 1/3 \) and \( p_{j',c_5} = 1/3 \) for \( j' \in E_{P,5}(c_5) \) by SETE.

- For \( r = 6 \), \( M_{P,6} = \{a_4, b_4, c_6, \ldots\} \), \( E_{P,6}(c_6) \setminus N' = \{7, \ldots, 17, x\} \). By Claim 3 (ii), \( c_6 \notin M_{P,7} \) and \( p_{j',c_6} = 1/3 \) for \( j' \in E_{P,6}(c_6) \) by SETE.

We show the part of \( P \) which has been determined by \( r \leq 6 \) in the following \( P(i) \) for agents \( \{1, 2, 3, x\} \) over items \( \{a_1, \ldots, a_4\} \), \( P(ii) \) for agents \( \{4, 5, 6, x\} \) over items \( \{b_1, \ldots, b_4\} \), and
$P(iii)$ for agents $\{7, \ldots, 17, x\}$ over items $\{c_1, \ldots, c_6\}$.

| Assignment $P(i)$ | Assignment $P(ii)$ |
|-------------------|-------------------|
| $a_1$ $a_2$ $a_3$ $a_4$ | $b_1$ $b_2$ $b_3$ $b_4$ |
| 1 $1/3$ $1/3$ $1/3$ 0 | 4 $1/3$ $1/3$ $1/3$ 0 |
| 2 $1/3$ $1/3$ $1/3$ 0 | 5 $1/3$ $1/3$ $1/3$ 0 |
| 3 $1/3$ $1/3$ 0 $1/3$ | 6 $1/3$ $1/3$ 0 $1/3$ |
| $x$ 0 0 $1/3$ 0 | $x$ 0 0 $1/3$ 0 |

| Assignment $P(iii)$ |
|---------------------|
| $c_1$ $c_2$ $c_3$ $c_4$ $c_5$ $c_6$ |
| 7-17 $1/12$ $1/12$ $1/12$ $1/11$ $1/11$ $1/12$ |
| $x$ $1/12$ $1/12$ $1/12$ 0 0 $1/12$ |

There exists $A$ with $A(x) = c_6$ among the deterministic assignments which constitute the convex combination for $P$. According to $P$, $a_3$ is assigned to one of $\{1, 2\}$ in $A$ since $A(x) = c_6$. Due to the fact that $\succ_1 = \succ_2$, w.l.o.g. let $A(1) = a_3$. With the fact that only agents in $\{1, 2, 3\}$ can get $\{a_1, a_2\}$, we have that agents $\{2, 3\}$ get $\{a_1, a_2\}$. It follows that $b_3$ is assigned to one of $\{4, 5\}$ and $\{4, 5, 6\}$ get $\{b_1, b_2, b_3\}$ for the same token. Due to the fact that $\succ_4 = \succ_5$, w.l.o.g. let $A(4) = b_3$, and therefore $\{5, 6\}$ get $\{b_1, b_2\}$. Agents in $\{7, \ldots, 17\}$ get the rest items, and for ease of exposition, let agent $j_i$ with $i \in \{1, \ldots, 6\}$ satisfy $j_i \in \{7, \ldots, 17\}$ and $j_i = A^{-1}(c_i)$.

- For $r = 1$, $T_{A,1} = \{a_1, b_1, c_1\}$ because $M_1 = M \triangleleft \{\text{top}(j, M_1) = a_1 \text{ for } j \in \{1, 2, 3\}, \text{top}(j', M_1) = b_1 \text{ for } j' \in \{4, 5, 6\}, \text{and } \text{top}(j^*, M_1) = c_1 \text{ for } j^* \in \{7, \ldots, 11, x\}$.\nomorebreak

- For $r = 2$, no matter which $j \in \{2, 3\}$ gets $a_1$ and which $j' \in \{4, 5\}$ gets $b_1$, $T_{A,2} = \{a_2, b_2, c_2\}$ because for $M_2 = M \setminus T_{A,1}$, $\text{top}(j, M_2) = a_2$ for $j \in \{1, 2, 3\}, \text{top}(j', M_2) = b_2$ for $j' \in \{4, 5, 6\}$, and $\text{top}(j^*, M_2) = c_2$ for $j^* \in \{7, \ldots, 11, x\}$.\nomorebreak

- For $r = 3$, we do not consider $j' \in \{2, 3, 5, 6, j_1, j_2\}$ because $A(j') \in \bigcup_{r' < 3} T_{A, r'}$. Then $T_{A,3} = \{a_3, b_3, c_3\}$ because for $M_3 = M \setminus \bigcup_{r' < 3} T_{A, r'}$, $\text{top}(1, M_3) = a_3$, $\text{top}(4, M_3) = b_3$, and $\text{top}(j, M_3) = c_3$ for $j \in \{7, \ldots, 11\}$.\nomorebreak

- For $r = 4$, we do not consider $j' \in \{1, 6, j_1, j_2, j_3\}$. Then $T_{A,4} = \{c_4, c_5\}$ because for $M_4 = M \setminus \bigcup_{r' < 4} T_{A, r'}$, $\text{top}(j, M_4) = c_4$ for $j \in \{7, \ldots, 11\}$ and $\text{top}(x, M_4) = c_5$. However, we have that $A^{-1}(c_5) = j_5$ and $\text{top}(j_5, M_4) = c_4$, which violates FERI. Therefore $A$ does not satisfy
FERI and $P$ does not satisfy ep-FERI, which means that $f$ does not satisfy ep-FERI, ea-FERI, and SETE simultaneously.

Propositions 11 and 12 show that fairness cannot be improved (from sd-WEF to sd-EF) given efficiency (ep-FERI or ea-FERI).

**Proposition 11.** No mechanism simultaneously satisfies ep-FERI and sd-EF.

**Proof.** We prove it with the following preference profile $R$. By FERI, one of agents in \{1, 2, 3\} gets $a$, and agent 4 must get $b$. If assignment $Q$ satisfies ep-FERI and SETE which is implied by sd-EF, then it is in the following form.

\[
\begin{array}{c|cccc}
\succ_1 & a & c & b & d \\
\succ_2 & a & c & b & d \\
\succ_3 & a & b & c & d \\
\succ_4 & b & a & d & c \\
\end{array}
\]

Assignment $Q$

\[
\begin{array}{c|cccc}
 & a & b & c & d \\
1 & 1 & 0 & ? & ? \\
2 & 1 & 0 & ? & ? \\
3 & 1 & 0 & ? & ? \\
4 & 0 & 1 & 0 & 0 \\
\end{array}
\]

Then we do not have $P_1 \succeq_1 P_4$ since $\sum_{o' \in U(\succ_1, b)} p_{1,o'} < \sum_{o' \in U(\succ_1, b)} p'_{4,o'}$, a contradiction to sd-EF.

Proposition 12. No mechanism simultaneously satisfies ea-FERI and sd-EF.

**Proof.** We prove it with the preference profile $R$ in Proposition 11, and $E_{Q,1}(a) = \{1, 2, 3\}$ and $E_{Q,1}(b) = \{4\}$ for any assignment $Q$ satisfying ea-FERI. Since $\sum_{k \in E_{Q,1}(a)} d_{Q,1}(k) > s_{Q,1}(a)$, $a \notin M_{Q,r}$ with $r > 1$ by Claim 3 (ii), which means that only agents 1, 2 and 3 get shares of $a$. It follows that 4 fully gets $b$ for the same token. Then any assignment satisfying ea-FERI and SETE is in the form of $Q$ in Proposition 11, but $Q$ does not satisfy sd-EF.

As for strategyproofness, Proposition 13 shows that sd-WSP cannot be improved to sd-SP given efficiency (ep-FERI) and fairness (SETE); and Proposition 14 shows that even sd-WSP cannot be satisfied given another combination of efficiency (ea-FERI) and fairness (SETE).

**Proposition 13.** No mechanism simultaneously satisfies ep-FERI, SETE, and sd-SP.
Proof. We prove it with the preference profile \( R \) in Proposition 11. If agent 3 misreports her preference as agent 4, i.e., \( R' = (\succ'_3, \succ'_4) \) with \( \succ'_3 = \succ'_4 \), then one of agents 1 and 2 gets \( a \), and one of agents 3 and 4 get \( b \) by FERI. For the remaining items \( M' = \{c,d\} \), \( \text{top}(1,M') = \text{top}(2,M') = c \) and \( \text{top}(3,M') = \text{top}(4,M') = d \), which means that agent 1 (or 2) gets \( c \) when she does not get \( a \). For the same token, agent 3 (or 4) gets \( d \) when she does not get \( b \). Then the following assignment \( Q' \) is the only one satisfying ep-FERI and SETE for \( R' \).

\[
\begin{array}{cccc}
\text{Assignment } Q' \\
\hline
a & b & c & d \\
1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\
2 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\
3 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\
4 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\
\end{array}
\]

Comparing \( Q' \) with \( Q \) in Proposition 11 which is under the true preference \( R \), we see that \( Q_3 \) does not dominate \( Q'_3 \) on \( U(\succ_3,b) \).

Proposition 14. No mechanism simultaneously satisfies ea-FERI, SETE, and sd-WSP.

Proof. Assume such mechanism \( f \) exists. Let \( R \) be:

\[
\begin{align*}
\succ_1: & \quad a \succ_1 b \succ_1 \cdots \succ_1 h \succ_1 c, \\
\succ_2: & \quad a \succ_2 h \succ_2 \cdots \succ_2 b \succ_2 c, \\
\succ_{3.7}: & \quad c \succ d \succ e \succ f \succ g \succ b \succ h \succ a, \\
\succ_8: & \quad c \succ_8 d \succ_8 b \succ_8 e \succ_8 f \succ_8 g \succ_8 h \succ_8 a.
\end{align*}
\]

Let \( P = f(R) \). We continue to use \( s_{P,r}(o) = 1 - \sum_{k \in \bigcup_{r' < r} E_{P,r'}(o)} p_{k,o} \) and \( d_{P,r}(j) = 1 - \sum_{o' \succ_j o} p_{j,o'} \) for any \( j \in E_{P,r}(o) \).

- For \( r = 1 \), by Claim 3 (ii), since \( E_{P,1}(a) = \{1,2\} \) and \( \sum_{k \in E_{P,1}(a)} d_{P,1}(k) > s_{P,1}(a) \), only agents 1 and 2 get shares of \( a \). It follows that only agents in \( E_{P,1}(c) = \{3,\ldots,8\} \) get \( c \) for the same token. Then we have \( p_{1,a} = p_{2,a} = 1/2 \), \( p_{j,c} = 1/6 \) for \( j \in \{3,\ldots,8\} \) by SETE.

- For \( r = 2 \), \( M_{P,2} = \{b,d,\ldots,h\} \), \( E_{P,2}(b) = \{1\} \), \( E_{P,2}(h) = \{2\} \), and \( E_{P,2}(d) = \{3,\ldots,8\} \). With \( \sum_{k \in E_{P,2}(d)} d_{P,2}(k) > s_{P,2}(d) \), we have that \( p_{j,d} = 1/6 \) for \( j \in \{3,\ldots,8\} \) by Claim 3 (ii) and SETE. With \( \sum_{k \in E_{P,2}(b)} d_{P,2}(k) \leq s_{P,2}(b) \), \( p_{1,b} = d_{P,2}(1) = 1/2 \) by Claim 3 (iii), and it follows that \( p_{2,h} = d_{P,2}(2) = 1/2 \) for the same token.
- For $r = 3$, $M_{P,3} = \{b, e, \ldots, h\}$, $E_{P,3}(b) = \{8\}$, and $E_{P,3}(e) = \{3, \ldots, 7\}$. With Claim 3 (ii) and $\sum_{k \in E_{P,3}(d)} d_{P,3}(k) > s_{P,3}(d)$, we have that $p_{8,b} = 1/2$. With $\sum_{k \in E_{P,3}(e)} d_{P,3}(k) > s_{P,3}(e)$, $p_{j,e} = 1/5$ for $j \in \{3, \ldots, 7\}$ by SETE.

With the analysis above, we have assignment $P$ in the following form.

| Assignment $P$ |
|----------------|
| a  | b  | c  | d  | e  | f  | g  | h  |
|----|----|----|----|----|----|----|----|
| 1  | 1/2| 1/2| 0  | 0  | 0  | 0  | 0  |
| 2  | 1/2| 0  | 0  | 0  | 0  | 0  | 1/2|
| 3-7| 0  | 0  | 1/6| 1/6| 1/5| ?  | ?  |
| 8  | 0  | 1/2| 1/6| 1/6| 0  | ?  | ?  |

If agent 8 misreports her preference as

$\succ'_8: c \succ'_8 d \succ'_8 e \succ'_8 b \succ'_8 f \succ'_8 g \succ'_8 h \succ'_8 a,$

then let $P' = f(R')$ for $R' = (\succ'_8, \succ_{-8}).$

- The analysis for $P'$ with $r = 1$ and 2 is the same as $P$.
- For $r = 3$, $M_{P',3} = \{b, e, \ldots, h\}$ and $E_{P',3}(e) = \{3, \ldots, 8\}$. With $\sum_{k \in E_{P',3}(d)} d_{P',3}(k) > s_{P',3}(d), p'_{j,e} = 1/6$ for $j \in \{3, \ldots, 8\}$ by SETE.
- For $r = 4$, $M_{P',4} = \{b, f, g, h\}$, $E_{P',4}(b) = \{8\}$, and $E_{P',4}(f) = \{3, \ldots, 7\}$. With Claim 3 (iii) and $\sum_{k \in E_{P',4}(d)} d_{P',4}(k) = s_{P',4}(d), p'_{8,b} = d_{P',4}(8) = 1/2$.

Then we obtain the assignment $P'$ in the following form.

| Assignment $P'$ |
|-----------------|
| a  | b  | c  | d  | e  | f  | g  | h  |
|----|----|----|----|----|----|----|----|
| 1  | 1/2| 1/2| 0  | 0  | 0  | 0  | 0  |
| 2  | 1/2| 0  | 0  | 0  | 0  | 0  | 1/2|
| 3-7| 0  | 0  | 1/6| 1/6| 1/6| ?  | ?  |
| 8  | 0  | 1/2| 1/6| 1/6| 1/6| 0  | 0  |

We see that $P'_8$ strictly dominates $P_8$ for $\sum_{o \in U(8,e)} p_{8,o} = 5/6 < 1 = \sum_{o \in U(8,e)} p'_{8,o}$ and $\sum_{o \in U(8,o')} p_{8,o} \leq \sum_{o \in U(8,o')} p'_{8,o}$ for other $o' \in M$, a contradiction to the fact that $f$ is sd-WSP.

$\square$
7. Conclusion

In this paper, we prove that our novel efficiency properties ep-FERI and ea-FERI are stronger than ep-PE and sd-PE respectively, and are compatible with desirable fairness properties. Since EBM and UPRE mechanisms proceed in an iterative manner, only requiring agents to report their top remaining item in each round, they may also address concerns of privacy and cognitive burden on agents. In contrast, mechanisms satisfying properties based on RM or FHR require agents to reveal rankings over all items. Since random mechanisms may be applied to the assignment of divisible items, with probability shares of items interpreted as fractional shares, UPRE’s efficiency and fairness guarantees naturally extend to the assignment of divisible items, while maximizing the allocation of first choice items.

For future work, we are interested in extending the notion of favoring eagerness to generalizations of the assignment problem where agents may have multi-unit demands or partial preferences. Another interesting avenue is the pursuit of ex-ante efficiency properties which “maximize first choices” and are compatible with stronger fairness and strategyproofness properties.

Appendix A.

Additional Results

A.1. Properties that RP, PS, NBM, EBM, PR and UPRE Fail to Satisfy

Corollary 2. RP does not satisfy ep-FERI or ea-FERI, but satisfies SETE.

Proof. (not ep-FERI) This follows from Proposition 13 and the fact that it satisfies sd-SP and SETE.

(not ea-FERI) This follows from Proposition 7 and the fact that it is not sd-PE.

(SETE) It is shown by (Nesterov, 2017).

Proposition 15. PS does not satisfy ep-FERI or ea-FERI, but satisfies SETE.

Proof. (not ep-FERI) We show it with the instance with following profile:

\[ \succ_1: a, b, c; \quad \succ_2: a, c, b; \quad \succ_3: b, a, c. \]

The following \( P \) is the outcome of PS, and \( A \) must be among the deterministic assignments which constitute the convex combination for \( P \).
In assignment $A$, $b \in T_{A,1}$ ($T_{A,r}$ is defined in Definition 2) since $top(3, M) = b$, but we see that $A^{-1}(b) = 1$ and $top(1, M) = a$, which violates FERI and means that $P$ do not satisfy ep-FERI.

(not ea-FERI) This follows from Proposition 12 and the fact that it is sd-EF.

(SETE) It follows from the fact that PS satisfies sd-EF, which implies SETE by (Nesterov, 2017).

**Proposition 16.** $NBM$ does not satisfy ep-FERI, ea-FERI or sd-WEF, but satisfies SETE.

**Proof.** (not ep-FERI) For the instance with profile in Figure 2, the assignment indicated by circled item is one of the possible outcomes of NBM, which does not satisfy FERI as we discuss in Example 4, and therefore it is not ep-FERI.

(not ea-FERI) This follows from the fact that it is not sd-PE (Chen et al., 2021).

(not sd-WEF) This follows from Proposition 1 and the fact that it satisfies SETE (shown below) and ep-FHR (Ramezanian and Feizi, 2021).

(SETE) Let $P = \mathbb{E}(NBM(R))$ for any given profile $R$. For any agents $j, k$ and their common prefix $\succ_{j,k}$, given a priority $\succ$ over agents with $j \succ k$, if $j$ gets an item $o$ appearing in $\succ_{j,k}$, then it is easy to see that $k$ gets $o$ given $\succ'$ which just swaps the positions of $j$ and $k$ in $\succ$. Since any such pair of priorities $\succ$ and $\succ'$ have the equal probability to be drawn in NBM, we have that $p_{j,o} = p_{k,o}$, which means SETE.

**Proposition 17.** $EBM$ does not satisfy ep-FHR, sd-PE, ea-FERI, ea-FHR, sd-EF or sd-SP.

**Proof.** (not ep-FHR) For the profile in Figure 2, one of its possible outcome is the assignment $A^*$ in Example 4 which does not satisfy FHR, and therefore EBM is not ep-FHR.

(not sd-PE) We show it by the instance with following $R$:

\[
\begin{align*}
\succ_1: & \quad a, b, c, \text{others; } \\
\succ_2: & \quad a, b, d, \text{others; } \\
\succ_3: & \quad a, b, e, \text{others; } \\
\succ_4: & \quad a, b, f, \text{others; } \\
\succ_5: & \quad a, b, g, c, d, x, y, \text{others; } \\
\succ_6: & \quad a, b, h, e, f, y, x, \text{others. }
\end{align*}
\]
The following are two possible outcomes of EBM where $j \leftarrow o$ means that agent $j$ gets item $o$:

A: $1 \leftarrow a, 2 \leftarrow b, 3 \leftarrow e, 4 \leftarrow f, 5 \leftarrow g, 6 \leftarrow c, 7 \leftarrow d, 8 \leftarrow h, 9 \leftarrow y, 10 \leftarrow x$.

$A': 1 \leftarrow c, 2 \leftarrow d, 3 \leftarrow a, 4 \leftarrow b, 5 \leftarrow g, 6 \leftarrow x, 7 \leftarrow y, 8 \leftarrow h, 9 \leftarrow e, 10 \leftarrow f$

Let $P = \mathbb{E}(AM(R))$. Then $p_{7,y} > 0$ and $p_{10,x} > 0$. With $x \succ_{7} y$ and $y \succ_{10} x$, by following Lemma 1, we have that $x \tau (P,R)y$ and $y \tau (P,R)x$, which means that $P$ is not sd-PE.

\textbf{(not ea-FERI, not ea-FHR)} It follows from the fact that EBM is not sd-PE.

\textbf{(not sd-EF, not sd-SP)} This follows from Proposition 11 and 13 and the fact that it satisfies ep-FERI and SETE.

We recall Lemma 1 from (Bogomolnaia and Moulin, 2001) used in Proposition 17 and the proof of Proposition 7.

\textbf{Lemma 1. (Bogomolnaia and Moulin, 2001)} Given a preference profile $R$ and a random assignment $P$, let $\tau (P,R)$ be the relation over all the items such that: if there exists an agent $j$ such that $o_a \succ_{j} o_b$ and $p_{j,o_b} > 0$, then $o_a \tau (P,R)o_b$. $P$ is sd-PE if and only if $\tau (P,R)$ is acyclic.

\textbf{Proposition 18.} PR does not satisfy ep-FERI or ea-FERI, but satisfies SETE.

\textbf{Proof. (not ep-FERI)} We show it by the instance with $R$ in Figure 2. Let $A$ be the deterministic assignment indicated by circled items, and $P = PR(R)$ is shown in the following.

| Assignment $P$ | a | b | c | d | e | f |
|---------------|---|---|---|---|---|---|
| 1             | 1 | 0 | 0 | 0 | 0 | 0 |
| 2             | 0 | 1 | 0 | 0 | 0 | 0 |
| 3-5           | 0 | 0 | 1/4| 1/3| 1/3| 1/12|
| 6             | 0 | 0 | 1/4| 0 | 0 | 3/4|

We see that $A$ must be among the deterministic assignments which constitute the convex combination for $P$, and $A$ does not satisfy FERI as shown in Example 4, which means that $P$ does not satisfy ep-FERI.

\textbf{(not ea-FERI)} We continue to use the instance with $R$ in Figure 2. In the assignment $P$ above, it is easy to see that $E_{P,1}(c) = \{3, 4, 5, 6\}$ and $M_{P,2} = \{d, e, f\}$ ($M_{P, r}$ and $E_{P, r}(c)$ are defined
in Definition 3). Then we have that $E_{P,2}(d) = \{6\}$ and $\sum_{j \in \bigcup_{r < 3} E_{P,d}} p_{6,d} = 0 < 1$, which means that $d \in M_{P,3}$ and $6 \in E_{P,3}(d)$ while $\sum_{o \in U(6,d)} p_{6,o} = 1/4 < 1$, which violates ea-FERI.

(SETE) PR satisfies equal-rank envy-freeness by (Chen et al., 2021) which requires that in $P = PR(R)$ for the given preference profile $R$, for any agents $j, k$ and item $o$ with $rk(j, o) = rk(k, o)$, $\sum_{o' \succ_{j,k} o} p_{j,o'} + p_{k,o} \leq \sum_{o' \in U(\succ_{j,k} o)} p_{j,o'}$. Then for any item $o$ appearing in $\succ_{j,k}$, $rk(j, o) = rk(k, o)$, and therefore $\sum_{o' \succ_{j,k} o} p_{j,o'} + p_{k,o} = \sum_{o' \in U(\succ_{j,k} o)} p_{j,o'}$, i.e., $p_{j,o} = p_{k,o}$, which means that PR satisfies SETE.

**Proposition 19.** UPRE does not satisfy ep-FERI, ep-FHR, ea-FHR, sd-EF or sd-SP.

**Proof.** (not ep-FERI) This follows from Proposition 10 and the fact that it satisfies ea-FERI by Theorem 3 and SETE by Theorem 4.

(not ep-FHR) For the profile $R$ in Figure 2, the following assignment $P$ is the outcome of UPRE.

| Assignment $P$ | a | b | c | d | e | f |
|----------------|---|---|---|---|---|---|
| 1              | 1 | 0 | 0 | 0 | 0 | 0 |
| 2              | 0 | 1 | 0 | 0 | 0 | 0 |
| 3-5            | 0 | 0 | 1/4| 1/12| 1/3| 1/3|
| 6              | 0 | 0 | 1/4| 3/4 | 0 | 0 |

The deterministic assignment $A^*$ in the following, where $j \leftarrow o$ means agent $j$ is allocated item $o$, is the one in Example 4 which is not FERI. It is easy to see that $A^*$ must be among those which constitute the convex combination for $P$, which means that $P$ does not satisfy ep-FHR.

$$A^* : 1 \leftarrow a, 2 \leftarrow b, 3 \leftarrow c, 4 \leftarrow e, 5 \leftarrow f, 6 \leftarrow d$$

(not ea-FHR) We continue to use the instance with $R$ in Figure 2. In the assignment $P$ above , where $p_{6,d} > 0$ but $rk(3, d) = 3 < rk(2, d) = 6$ and $\sum_{o' \in U(\succ_{3,d} d)} p_{3,o'} < 1$, which violates ea-FHR.

(not sd-EF, not sd-SP) This follows from proposition 12 and 14 and the fact that it satisfies ea-FERI and SETE.

□
A.2. EBM is Not Equivalent to ABM

**Example 7.** EBM is different from ABM with priority drawn from a uniform distribution. Let \( R \) be:

\[
1-3 : a \succ c \succ \{\text{others}\}, \quad 4-5 : b \succ c \succ \{\text{others}\}.
\]

We compare their shares of \( c \) in the outcomes of EBM and ABM. For ease of exposition, we use agents 2 and 5 to represent the two types of agents with different preferences respectively.

First we consider EBM. Let \( A = EBM(R) \) and \( P = \mathbb{E}(EBM(R)) \). In the first round, EBM issues a lottery for \( a \) among agents 1-3 and one for \( b \) among 4, 5. Then \( \Pr(A(2) \neq a) = 2/3 \) and \( \Pr(A(5) \neq b) = 1/2 \). In the second round, EBM issues a lottery for \( c \), and the number of participants is always 3, which means that \( \Pr(A(2) = c|A(2) \neq a) = \Pr(A(5) = c|A(5) \neq b) = 1/3 \). It follows that

\[
p_{2,c} = \Pr(A(2) = c) = 2/9 \quad \text{and} \quad p_{5,c} = \Pr(A(5) = c) = 1/6.
\]

Then we consider ABM. Let \( Q = \mathbb{E}(ABM(R)) \). There are \( A_5^5 = 120 \) priorities in total. Agent 2 gets \( c \) when in the priority,

(1) one agent \( j \in \{1,3\} \) is ranked above her,
(2) the other agent \( \{1,3\} \setminus \{j\} \) is ranked below her, and
(3) the agent \( k \in \{4,5\} \) with lower rank is ranked below her.

Priorities of such kind with \( j = 1 \) and \( k = 5 \) follow the topological order in Figure 4, and there are 7 as listed in the following:

\[
1 \triangleright 2 \triangleright 3 \triangleright 4 \triangleright 5, \quad 1 \triangleright 2 \triangleright 4 \triangleright 3 \triangleright 5,
\]

\[
1 \triangleright 2 \triangleright 4 \triangleright 5 \triangleright 3, \quad 1 \triangleright 4 \triangleright 2 \triangleright 3 \triangleright 5,
\]

\[
1 \triangleright 4 \triangleright 2 \triangleright 5 \triangleright 3, \quad 4 \triangleright 1 \triangleright 2 \triangleright 3 \triangleright 5,
\]

\[
4 \triangleright 1 \triangleright 2 \triangleright 5 \triangleright 3.
\]

With that \( j \) is chosen in \( \{1,3\} \) and \( k \) is chosen in \( \{4,5\} \), there are \( 7 \cdot C_2^1 \cdot C_2^1 = 28 \) priorities where agent 2 gets \( c \), i.e., \( q_{2,c} = 28/120 = 7/30 \).

Agent 5 gets \( c \) when in the priority,

(1) agent 4 is ranked above her,
(2) at least two agents \( j,k \in \{1,2,3\} \) is ranked below her.
Figure 4. A possible topological order of priorities where agent 2 gets c.

Priorities of such kind with $j = 2$ and $k = 3$ follow one of the topological orders in Figure 5, and there are 6 as listed in the following:

1 $\triangleright$ 4 $\triangleright$ 5 $\triangleright$ 2 $\triangleright$ 3, 1 $\triangleright$ 4 $\triangleright$ 5 $\triangleright$ 3 $\triangleright$ 2,

4 $\triangleright$ 1 $\triangleright$ 5 $\triangleright$ 2 $\triangleright$ 3, 4 $\triangleright$ 1 $\triangleright$ 5 $\triangleright$ 3 $\triangleright$ 2,

4 $\triangleright$ 5 $\triangleright$ 1 $\triangleright$ 2 $\triangleright$ 3, 4 $\triangleright$ 5 $\triangleright$ 1 $\triangleright$ 3 $\triangleright$ 2.

With fact that $j, k$ are chosen in $\{1, 2, 3\}$, there are $6 \cdot C_3^2 = 18$ priorities where agent 5 gets c, i.e., $q_{5,c} = 18/120 = 3/20$.

Together we see that $q_{2,c} \neq p_{2,c}$ and $q_{5,c} \neq p_{5,c}$, which means that $Q \neq P$ and therefore EBM is different from ABM.

Figure 5. Possible topological orders of priorities where agent 5 gets c.
APPENDIX B.
ACRONYMS

TABLE II
ACRONYMS FOR PROPERTIES USED IN THIS PAPER.

| Abbr. | full names                                           | category                  |
|-------|------------------------------------------------------|---------------------------|
| ea-FERI | ex-ante favoring-eagerness-for-remaining-items    | ex-ante efficiency        |
| ea-FHR | ex-ante favoring-higher-ranks                      | ex-ante efficiency        |
| ep-FERI | ex-post favoring-eagerness-for-remaining-items    | ex-post efficiency        |
| ep-FHR | ex-post favoring-higher-ranks                      | ex-post efficiency        |
| ep-PE  | ex-post Pareto-efficiency                          | ex-post efficiency        |
| ep-RM  | ex-post rank-maximality                            | ex-post efficiency        |
| FERI   | favoring-eagerness-for-remaining-items             | efficiency”               |
| FHR    | favoring-higher-ranks                              | efficiency*               |
| RM     | rank-maximality                                    | efficiency”               |
| PE     | Pareto-efficiency                                  | efficiency*               |
| sd-EF  | sd-envy-freeness                                   | ex-ante fairness         |
| sd-PE  | sd-Pareto-efficiency                               | ex-ante efficiency        |
| sd-SP  | sd-strategyproofness                               | strategyproofness         |
| sd-WEF | sd-weak-envy-freeness                             | ex-ante fairness         |
| sd-WSP | sd-weak-strategyproofness                          | strategyproofness         |
| SETE   | strong equal treatment of equals                   | ex-ante fairness         |

Note: Properties annotated with * are for deterministic assignments

TABLE III
ACRONYMS FOR MECHANISMS USED IN THIS PAPER.

| Abbr. | full names                                                                 |
|-------|---------------------------------------------------------------------------|
| ABM   | adaptive Boston mechanism (Alcalde, 1996; Dur, 2019)                       |
| EBM   | eager Boston mechanism                                                   |
| NBM   | naive Boston mechanism (Kojima and Ünver, 2014)                           |
| PR    | probabilistic rank (Chen et al., 2021)                                    |
| PRE   | probabilistic respecting eagerness                                       |
| PS    | probabilistic serial (Bogomolnaia and Moulin, 2001)                       |
| RP    | random priority (Abdulkadiroğlu and Sönmez, 1998)                         |
| UPRE  | uniform probabilistic respecting eagerness                               |
REFERENCES

Atila Abdulkadiroğlu and Tayfun Sönmez. School choice: A mechanism design approach. *American Economic Review*, 93(3):729–747, 2003.

Atila Abdulkadiroğlu, Parag A Pathak, Alvin E Roth, and Tayfun Sönmez. Changing the Boston school choice mechanism: Strategy-proofness as equal access. https://web.stanford.edu/~alroth/papers/bostonMay182006.pdf, 2006. Accessed: 2021-09-08.

Atila Abdulkadiroğlu and Tayfun Sönmez. Random serial dictatorship and the core from random endowments in house allocation problems. *Econometrica*, 66(3):689–702, 1998.

David Abraham, Ning Chen, Vijay Kumar, and Vahab S. Mirrokni. Assignment problems in rental markets. In *Internet and Network Economics*, pages 198–213, Berlin, Heidelberg, 2006. Springer.

David John Abraham. *Matching Markets: Design and Analysis*. PhD thesis, School of Computer Science, Carnegie Mellon University, 2009.

Jose Alcalde. Implementation of stable solutions to marriage problems. *Journal of Economic Theory*, 69(1):240–254, 1996.

Anna Bogomolnaia and Hervé Moulin. A new solution to the random assignment problem. *Journal of Economic Theory*, 100(2):295–328, 2001.

Yajing Chen, Patrick Harless, and Zhenhua Jiao. The probabilistic rank random assignment rule and its axiomatic characterization, 2021. URL http://arxiv.org/abs/2104.09165.

Yan Chen and Tayfun Sönmez. Improving efficiency of on-campus housing: An experimental study. *American Economic Review*, 92(5):1669–1686, 2002.

Umut Dur, Timo Mennle, and Sven Seuken. First-choice maximal and first-choice stable school choice mechanisms. In *Proceedings of the 2018 ACM Conference on Economics and Computation*, pages 251–268, New York, USA, 2018. ACM.

Umut Mert Dur. The modified Boston mechanism. *Mathematical Social Sciences*, 101:31–40, 2019.

Duncan Karl Foley. *Resource Allocation and the Public Sector*. PhD thesis, Yale University, 1966.

Naveen Garg, Telikepalli Kavitha, Amit Kumar, Kurt Mehlhorn, and Julián Mestre. Assigning papers to referees. *Algorithmica*, 58(1):119–136, 2010.

Ali Ghodsi, Matei Zaharia, Benjamin Hindman, Andy Konwinski, Scott Shenker, and Ion Stoica.
Dominant resource fairness: Fair allocation of multiple resource types. In *Proceedings of the 8th USENIX Conference on Networked Systems Design and Implementation*, pages 323–336, Boston, USA, 2011.

Ali Ghodsi, Vyas Sekar, Matei Zaharia, and Ion Stoica. Multi-resource fair queueing for packet processing. In *Proceedings of the ACM SIGCOMM 2012 Conference on Applications, Technologies, Architectures, and Protocols for Computer Communication*, pages 1–12, New York, USA, 2012. ACM.

Robert Grandl, Ganesh Ananthanarayanan, Srikant Kandula, Sriram Rao, and Aditya Akella. Multi-resource packing for cluster schedulers. In *Proceedings of the 2014 ACM Conference on SIGCOMM*, pages 455–466, New York, USA, 2014. ACM.

Xiaoxi Guo, Sujoy Sikdar, Haibin Wang, Lirong Xia, Yongzhi Cao, and Hanpin Wang. Probabilistic serial mechanism for multi-type resource allocation. *Autonomous Agents and Multi-Agent Systems*, 35(1):1–48, 2021.

Patrick Harless. Immediate acceptance without priorities: preserving efficiency while respecting rank. https://sites.google.com/site/pdharless/research/r6, 2018. Accessed: 2021-09-06.

Aanund Hylland and Richard Zeckhauser. The efficient allocation of individuals to positions. *Journal of Political Economy*, 87(2):293–314, 1979.

Robert W. Irving, Telikepalli Kavitha, Kurt Mehlhorn, Dimitrios Michail, and Katarzyna E. Paluch. Rank-maximal matchings. *ACM Transactions on Algorithms*, 2(4):602–610, 2006.

Fuhito Kojima and M. Utku Ünver. The “Boston” school-choice mechanism: an axiomatic approach. *Economic Theory*, 55(3):515–544, 2014.

George Marsaglia. Xorshift RNGs. *Journal of Statistical Software*, 8(14):1–6, 2003.

Timo Mennle and Sven Seuken. Partial strategyproofness: Relaxing strategyproofness for the random assignment problem. *Journal of Economic Theory*, 191:105–144, 2021.

Antonio Miralles. School choice: the case for the boston mechanism. In *Auctions, Market Mechanisms and Their Applications*, pages 58–60, Berlin, Heidelberg, 2009. Springer.

Hervé Moulin. Fair division in the internet age. *Annual Review of Economics*, 11(1):407–441, 2019.

Alexander S Nesterov. Fairness and efficiency in strategy-proof object allocation mechanisms. *Journal of Economic Theory*, 170:145–168, 2017.

Katarzyna Paluch. Capacitated rank-maximal matchings. In *International Conference on Algorithms and Complexity*, pages 324–335, Berlin, Heidelberg, 2013. Springer.
Parag A Pathak. What really matters in designing school choice mechanisms. *Advances in Economics and Econometrics*, 1:176–214, 2017.

Parag A Pathak and Tayfun Sönmez. Leveling the playing field: Sincere and sophisticated players in the boston mechanism. *American Economic Review*, 98(4):1636–52, 2008.

Rasoul Ramezanian and Mehdi Feizi. Ex-post favoring ranks: A fairness notion for the random assignment problem. *Review of Economic Design*, 25:157–176, 2021.

Alvin E Roth, Tayfun Sönmez, and M Utku Ünver. Pairwise kidney exchange. *Journal of Economic Theory*, 125(2):151–188, 2005.

Daniela Saban and Jay Sethuraman. A note on object allocation under lexicographic preferences. *Journal of Mathematical Economics*, 50:283–289, 2014.

Daniela Saban and Jay Sethuraman. The complexity of computing the random priority allocation matrix. *Mathematics of Operations Research*, 40(4):1005–1014, 2015.

Erel Segal-Halevi. *Fair Division of Land*. PhD thesis, Computer Science Department, Bar Ilan University, 2016.

Lloyd Shapley and Herbert Scarf. On cores and indivisibility. *Journal of Mathematical Economics*, 1(1):23–37, 1974.

WE Thomson. A modified congruence method of generating pseudo-random numbers. *The Computer Journal*, 1(2):83–83, 1958.

Hal R Varian. Equity, envy, and efficiency. *Journal of Economic Theory*, 9(1):63–91, 1973.

Haibin Wang, Sujoy Sikdar, Xiaoxi Guo, Lirong Xia, Yongzhi Cao, and Hanpin Wang. Multi-type resource allocation with partial preferences. In *Proceedings of the AAAI Conference on Artificial Intelligence*, pages 2260–2267, New York, 2020. AAAI Press.

Lin Zhou. On a conjecture by gale about one-sided matching problems. *Journal of Economic Theory*, 52(1):123–135, 1990.