THIN POSITION FOR KNOTS IN A 3-MANIFOLD

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ABSTRACT. We extend the notion of thin multiple Heegaard splittings of a link in a 3-manifold to take into consideration not only compressing disks but also cut-disks for the Heegaard surfaces. We prove that if $H$ is a c-strongly compressible bridge surface for a link $K$ contained in a closed orientable irreducible 3-manifold $M$ then one of the following is satisfied:

- $H$ is stabilized
- $H$ is meridionally stabilized
- $H$ is perturbed
- a component of $K$ is removable
- $M$ contains an essential meridional surface.

1. INTRODUCTION

The notion of thin position for a closed orientable 3-manifold $M$ was introduced by Scharlemann and Thompson in [8]. The idea is to build the 3-manifold by starting with a set of 0-handles, then alternate between attaching collections of 1-handles and 2-handles keeping the boundary at the intermediate steps as simple as possible and finally add 3-handles. Such a decomposition of a manifold is called a generalized Heegaard splitting. The classical Heegaard splitting where all 1-handles are attached at the same time followed by all 2-handles is an example of a generalized Heegaard splitting. Casson and Gordon [2] show that if $A \cup P B$ is a weakly reducible Heegaard splitting for $M$ (i.e. there are meridional disks for $A$ and $B$ with disjoint boundaries), then either $A \cup P B$ is reducible or $M$ contains an essential surface. Scharlemann and Thompson [8] show that such surfaces arise naturally when a Heegaard splitting in put in thin position.

Suppose a closed orientable 3-manifold $M = A \cup_P B$ contains a link $K$, then we can isotope $K$ so that it intersects each handlebody in boundary parallel arcs. In this case we say that $P$ is a bridge surface for $K$ or that $P$ is a Heegaard surface for the pair $(M, K)$. The idea was first introduced by

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Schubert in the case that $M = S^3$ and $P = S^2$ and was extended by Morimoto and Sakuma for other 3-manifolds. In [5] Hayashi and Shimokawa considered multiple Heegaard splittings for $(M, K)$ using the idea of changing the order in which the 1-handles and the 2-handles are attached. They generalized the result of [3] in this context, i.e. they showed that if $P$ is a strongly compressible bridge surface for $K$, then either $A \cup P B$ is stabilized or cancellable or $M - \eta(K)$ contains an essential meridional surface.

In this paper we will generalize this important result one step further by weakening the hypothesis. Suppose $M$ is a compact orientable 3-manifold and $F \subset M$ is a properly embedded surface transverse to a 1-submanifold $T \subset M$. In some contexts it is necessary to consider not only compressing disks for $F$ but also cut-disks, that is, disks whose boundary is essential on $F - T$ and that intersect $T$ in exactly one point, as for example in [1], [9] and [11]. A bridge surface $P$ for a link $K$ is c-strongly compressible if there is a pair of disjoint cut or compressing disks for $P_K$ on opposite sides of $P$. In particular every strongly compressible bridge surface is c-strongly compressible. We will show that if a bridge surface $P$ for $K$ is c-strongly compressible then either it can be simplified in one of four geometrically obvious ways or $(M, K)$ contains an essential meridional surface.

2. Definitions and preliminaries

Let $M$ be a compact orientable irreducible 3-manifold and let $T$ be a 1-manifold properly embedded in $M$. A regular neighborhood of $T$ will be denoted $\eta(T)$. If $X$ is any subset of $M$ we will use $X_T$ to denote $X - T$. We will assume that any sphere in $M$ intersects $T$ in an even number of points. As all the results we will develop are used in the context when $T$ only has closed components, this is a natural assumption. If $K$ is a link in $M$, then any sphere in $M$ intersects $K$ in an even number of points, since the ball in bounds in $M$ contains no endpoints of $K$.

Suppose $F$ is a properly embedded surface in $M$. An essential curve on $F_T$ is a curve that doesn’t bound a disk on $F_T$ and it is not parallel to a puncture of $F_T$. A compressing disk $D$ for $F_T$ is an embedded disk in $M_T$ so that $F \cap D = \partial D$ is an essential curve on $F_T$. A cut-disk is a disk $D^c \subset M$ such that $D^c \cap F = \partial D^c$ is an essential curve on $F_T$ and $|D \cap T| = 1$. A c-disk is a cut or a compressing disk. $F$ will be called incompressible if it has no compressing disks and c-incompressible if it has no c-disks. $F$ will be called essential if it does not have compressing disks (it may have cut disks), it is not boundary parallel in $M - \eta(T)$ and it is not a sphere that bounds a ball disjoint from $T$.

Suppose $C$ is a compression body ($\partial_C$ may have some sphere components). A set of arcs $t_i \subset C$ is trivial if there is a homeomorphism after
which each arc is either vertical, ie, \( t_i = (\text{point}) \times I \subset \partial_- C \times I \) or there is an embedded disk \( D_i \) such that \( \partial D_i = t_i \cup \alpha_i \) where \( \alpha_i \subset \partial_+ C \). In the second case we say that \( t_i \) is \( \partial_+ \)-parallel and the disk \( D_i \) is a bridge disk. If \( C \) is a handlebody, then all trivial arcs are \( \partial_+ \)-parallel and are called bridges. If \( T \) is a 1-manifold properly embedded in a compression body \( C \) so that \( T \) is a collection of trivial arcs then we will denote the pair by \( (C, T) \).

Let \( (C, T) \) be a pair of a compression body and a 1-manifold and let \( D \) be the disjoint union of compressing disks for \( \partial_+ C \) together with one bridge disk for each \( \partial_+ \)-parallel arc. If \( D \) cuts \( (C, T) \) into a manifold homeomorphic to \((\partial_- C \times I, \text{vertical arcs})\) together with some 3-balls, then \( D \) is called a complete disk system for \( (C, T) \). The presence of such a complete disk system can be taken as the definition of \( (C, T) \).

Let \( M \) be a 3-manifold, let \( A \cup_P B \) be a Heegaard splitting (ie \( A \) and \( B \) are compression bodies) for \( M \) and let \( T \) be a 1-manifold in \( M \). We say that \( T \) is in bridge position with respect to \( P \) if \( A \) and \( B \) intersect \( T \) only in trivial arcs. In this case we say that \( P \) is a bridge surface for \( T \) or that \( T \) as a Heegaard surface for the pair \((M, T)\).

Suppose \( M = A \cup_P B \) and \( T \) is in bridge position with respect to \( P \). The Heegaard splitting is c-strongly irreducible if any pair of \( c \)-disks on opposite sides of \( P_T \) intersect, in this case the bridge surface \( P_T \) is c-weakly incompressible. If there are \( c \)-disks \( D_A \subset A \) and \( D_B \subset B \) such that \( D_A \cap D_B = \emptyset \), the Heegaard splitting is c-weakly reducible and the bridge surface \( P_T \) is c-strongly compressible.

Following [5], the bridge surface \( P_T \) will be called stabilized if there is a pair of compressing disks on opposite sides of \( P_T \) that intersect in a single point. The bridge surface is meridionally stabilized if there is a cut disk and a compressing disk on opposite sides of \( P_T \) that intersect in a single point. Finally the bridge surface is called cancellable if there is a pair of canceling disks \( D_i \) for bridges \( t_i \) on opposite sides of \( P \) such that \( \emptyset \neq (\partial D_1 \cap \partial D_2) \subset (Q \cap K) \). If \( |\partial D_1 \cap \partial D_2| = 1 \) we will call the bridge surface perturbed. In [10] the authors show that is \( M = A \cup_P B \) is stabilized, meridionally stabilized or perturbed, then there is a simpler bridge surface \( P' \) for \( T \) such that \( P \) can be obtained from \( P' \) by one of three obvious geometric operations.

If the bridge surface \( P \) for \( T \) is cancellable with canceling disks \( D_1 \) and \( D_2 \) such that \( |\partial D_1 \cap \partial D_2| = 2 \) then using this pair of disks some closed component \( t \) of \( T \) can be isotoped to lie in \( P \). If this component can be isotoped to lie in the core of one of the compression bodies, \( A \) say, and is disjoint from all other bridge disks in \( A \) then \( A \setminus \eta(t) \) is also a compression body and the 1-manifold \( T \setminus t \) intersects it in a collection of trivial arcs. Thus \( (A \setminus \eta(t)) \cup_P B \) is Heegaard splitting for \((M \setminus \eta(t))\) and \( P \) is a bridge surface for \( T \setminus t \). In this case we will say that \( T \) has a removable
component. A detailed discussion of links with removable components is given in [10].

In the absence of a knot, it follows by a theorem of Waldhausen that a Heegaard splitting of an irreducible manifold is stabilized if and only if there is a sphere that intersects the Heegaard surface in a single essential curve (i.e. the Heegaard splitting is reducible), unless the Heegaard splitting is the standard genus 1 Heegaard splitting of $S^3$. We will say that a bridge surface for $T$ is $c$-reducible if there is a sphere or a twice punctured sphere in $M$ that intersects the bridge surface in a single essential closed curve. Then one direction of Waldhausen’s result easily generalizes to bridge surfaces as the next theorem shows.

**Theorem 2.1.** Suppose $P$ is a bridge surface for a 1-manifold $T$ properly embedded in a compact, orientable 3-manifold $M$ where $P$ is not the standard genus 1 Heegaard splitting for $S^3$. If $P$ is stabilized, perturbed or meridionally stabilized then there exists a sphere $S$, possibly punctured by $T$ twice, which intersects $P$ in a single essential curve $\alpha$ and neither component of $S - \alpha$ is parallel to $P$.

**Proof.** If $P$ is stabilized let $S$ be the boundary of a regular neighborhood of the union of the pair of stabilizing disks, Figure 1. In this case $S$ is a sphere disjoint from $T$. If $P$ is meridionally stabilized, let $S$ be the boundary of a
regular neighborhood of the union of the cut and compressing disks. In this case $S$ is a twice punctured sphere with both punctures on the same side of $S \cap P$. Finally if $P$ is perturbed with canceling disks $E_1$ and $E_2$, let $S$ be the boundary of a regular neighborhood of $E_1 \cup E_2$. Then $S$ is a twice punctured sphere and the punctures are separated by $S \cap P$.

\[ \square \]

3. C-COMPRESSION BODIES AND THEIR PROPERTIES

We will need to generalize the notion of a compression body containing trivial arcs as follows.

**Definition 3.1.** A c-compression body $(C, T)^c$ is a pair of a compression body $C$ and a 1-manifold $T$ such that there is a collection of disjoint bridge disks and c-disks $D^c$ so that $D^c$ cuts $(C, T)^c$ into a 3-manifold homeomorphic to $(\partial_- C \times I, \text{vertical arcs})$ together with some 3-balls. In this case $D^c$ is called a complete c-disk system.

One way to construct a compression body is to take a product neighborhood $F \times I$ of a closed, possibly disconnected, surface $F$ so that any arc of $T \cap (F \times I)$ can either be isotoped to be vertical with respect to the product structure or is parallel to an arc in $F \times 0$ and then attach a collection of pairwise disjoint 2-handles $\Delta$ to $F \times 1$. If we allow some of the 2-handles in $\Delta$ to contain an arc $t \subset T$ as their cocore, the resulting 3-manifold is a c-compression body. The complete c-disk system described in the definition above consists of all bridge disks together with the cores of the 2-handles. We will use this construction as an alternative definition of a c-compression body.

**Remark 3.2.** Recall that a spine of a compression body $C$ is the union of $\partial_- C$ together with a 1-dimensional graph $\Gamma$ such that $C$ retracts to $\partial_- C \cup \Gamma$. An equivalent definition of a c-compression body is that $(C, T)^c$ is a compression body $C$ together with a 1-manifold $T$ and there exists a spine $\Sigma$ for $C$ such that all arcs of $T$ that are not trivial in $C$ can be simultaneously isotoped to lie on $\Sigma$ and be pairwise disjoint. We will however not use this definition here.

**Proposition 3.3.** Let $(C, T)^c$ be a c-compression body. Then $(C, T)^c$ is a compression body if and only if there is no arc $t \subset T$ such that $\partial t \subset \partial_- C$. In particular if $\partial_- C = \emptyset$, then $C$ is a handlebody.

**Proof.** Consider the construction above and note that before the two handles are added no arc of $T$ has both of its endpoints on $F \times 1$. If some 2-handle $D$ attached to $F \times 1$ contains an arc $t \subset T$ as its core, this arc will have
both of its endpoints on $\partial - C$. Thus $C$ is a compression body if and only if no 2-handle contains such an arc.

□

Lemma 3.4. Let $(C, T)^c$ be a $c$-compression body and let $F$ be a $c$-incompressible, $\partial$-incompressible properly embedded surface transverse to $T$. Then there is a complete $c$-disk system $D^c$ of $(C, T)^c$ such that $D^c \cap F$ consists of two types of arcs

- An intersection arc $\alpha$ between a bridge disk in $D^c$ and a twice punctured sphere component of $F$ with both endpoints of $\alpha$ lying on $T$.
- An intersection arc $\beta$ between a bridge disk in $D^c$ and a once-punctured disk component of $F$ with one endpoint of $\beta$ lying on $T$ and the other lying on $\partial_+ C$.

Proof. The argument is similar to the the proof of Lemma 2.2 in [5] so we only give an outline here. Let $D^c$ be a complete $c$-disk system for $(C, T)^c$ chosen to minimize $|D^c \cap F|$. Using the fact that $F_T$ is $c$-incompressible, we may assume that $D^c \cap F$ does not contain any simple closed curves. If $\alpha \subset D^c \cap F$ is an arc with both of its endpoints on $\partial C$, then an outermost such arc either gives a $\partial$-compression for $F$ contrary to the hypothesis or can be removed by an outermost arc argument contradicting the minimality of $|D^c \cap F|$. Note that if $\alpha$ lies on some cut-disk $D^c$, we can still choose the arc so that the disk it cuts from $D^c$ does not contain a puncture. This establishes that $F$ is disjoint from all $c$-disks in $D^c$.

Suppose $\alpha$ is an arc of intersection between a bridge disk $E$ for $T$ and a component $F'$ of $F$. Assume that $\alpha$ is an outermost such arc and let $E' \subset E$ be the subdisk $\alpha$ bounds on $E$. By the above argument at least one endpoint of $\alpha$ must lie on $T$. If both endpoints of $\alpha$ lie on $T$, the boundary of a regular neighborhood of $E'$ gives a compressing disk for $F$ contrary to the hypothesis unless $F'$ is a twice punctured sphere. If $\alpha$ has one endpoint on $T$ and one endpoint on $\partial C$, a regular neighborhood of $E'$ is a $\partial$-compressing disk for $F$ unless $F'$ is a once punctured disk.

□

Corollary 3.5. If $(C, T)^c$ is a $c$-compression body, then $\partial_+ C$ is incompressible.

Proof. Suppose $D$ is a compressing disk for some component of $\partial_+ C$. By Lemma 3.4 there exists a complete $c$-disk system $D^c$ for $(C, T)^c$ such that $D \cap D^c = \emptyset$. But this implies that $D$ is a $\partial$-reducing disk for the manifold $(F \times I, \text{vertical arcs})$, a contradiction.

□

If $M$ is a 3-manifold we will denote by $\tilde{M}$ the manifold obtained from $M$ by filling any sphere boundary components of $M$ with 3-balls.
Lemma 3.6 (Lemma 2.4 [5]). If $F$ is an incompressible, $\partial$-incompressible surface in a compression body $(C, T)$, then $F$ is a collection of the following kinds of components:

- Spheres intersecting $T$ in 0 or 2 points,
- Disks intersecting $T$ in 0 or 1 points,
- Vertical annuli disjoint from $T$,
- Closed surfaces parallel to a component of $\partial\tilde{C}$.

Corollary 3.7. If $F$ is a $c$-incompressible, $\partial$-incompressible surface in a $c$-compression body $(C, T)^c$, then $F$ is a collection of the following kinds of components:

- Spheres intersecting $T$ in 0 or 2 points,
- Disks intersecting $T$ in 0 or 1 points,
- Vertical annuli disjoint from $T$,
- Closed surfaces parallel to a component of $\partial\tilde{C}$.

Proof. Delete all component of the first two types and let $F'$ be the new surface. By Lemma 3.4 there exists a complete $c$-disk system $D$ for $(C, T)^c$ such that $D \cap F' = \emptyset$. Thus each component of $F'$ is contained in a compression body with trivial arcs (in fact in a trivial compression body but we don’t need this fact). The result follows by Lemma 3.6. 

\[ \square \]

4. C-thin position for a pair 3-manifold, 1-manifold

The following definition was first introduced in [5].

Definition 4.1. If $T$ is a 1-manifold properly embedded in a compact 3-manifold $M$, we say that the disjoint union of surfaces $\mathcal{H}$ is a multiple Heegaard splitting of $(M, T)$ if

1. The closures of all components of $M - \mathcal{H}$ are compression bodies $(C_1, C_1 \cap T), \ldots, (C_n, C_n \cap T)$,
2. for $i = 1, \ldots, n$, $\partial_+ C_i$ is attached to some $\partial_+ C_j$ where $i \neq j$,
3. a component of $\partial_- C_i$ is attached to some component of $\partial_- C_j$ (possibly $i = j$).

A component $H$ of $\mathcal{H}$ is said to be positive if $H = \partial_+ C_i$ for some $i$ and negative if $H = \partial_- C_j$ for some $j$. The unions of all positive and all negative components of $\mathcal{H}$ are denoted $\mathcal{H}_+$ and $\mathcal{H}_-$ respectively.

Note that if $\mathcal{H}$ has a single surface component $P$, then $P$ is a bridge surface for $T$.

Using $c$-compression bodies instead of compression bodies, we generalize this definition as follows.
Definition 4.2. If $T$ is a 1-manifold properly embedded in a compact 3-manifold $M$, we say that the disjoint union of surfaces $\mathcal{H}$ is a multiple c-Heegaard splitting of $(M, T)$ if

1. The closures of all components of $M - \mathcal{H}$ are c-compression bodies $(C_1, C_1 \cap T)^c, \ldots, (C_n, C_n \cap T)^c$,
2. for $i = 1, \ldots, n$, $\partial_+ C_i$ is attached to some $\partial_+ C_j$ where $i \neq j$,
3. a component of $\partial_- C_i$ is attached to some component of $\partial_- C_j$ (possibly $i = j$)

As in [8] and [5] we will associate to a multiple c-Heegaard splitting a measure of its complexity. The following notion of complexity of a surface is different from the one used in [5].

Definition 4.3. Let $S$ be a closed connected surfaces embedded in $M$ transverse to a properly embedded 1-manifold $T \subset M$. The complexity of $S$ is the ordered pair $c(S) = (2 - \chi(S_T), g(S))$. If $S$ is not connected, $c(S)$ is the multi-set of ordered pairs corresponding to each of the components of $S$.

As in [8] the complexities of two possibly not connected surfaces are compared by first arranging the ordered pairs in each multi-set in non-increasing order and then comparing the two multi-sets lexicographically where the ordered pairs are also compared lexicographically.

Lemma 4.4. Suppose $S_T$ is meridional surface in $(M, T)$ of non-positive euler characteristic. If $S'_T$ is a component of the surface obtained from $S_T$ by compressing along a c-disk, then $c(S_T) > c(S'_T)$.

Proof. Without loss of generality we may assume that $S_T$ is connected.

Case 1: Let $\tilde{S}_T$ be a possibly disconnected surface obtained from $S_T$ via a compression along a disk $D$. In this case $\chi(S_T) < \chi(\tilde{S}_T)$ as $\chi(D) = 1$ so the result follows immediately if $\tilde{S}_T$ is connected. If $\tilde{S}_T$ consists of two components then by the definition of compressing disk, we may assume that neither component is a sphere and thus both components of $\tilde{S}_T$ have non-positive Euler characteristic. By the additivity of Euler characteristic it follows that if $S'_T$ is a component of $\tilde{S}_T$, then $\chi(S'_T) < \chi(\tilde{S}_T)$ so $2 - \chi(S_T) > 2 - \chi(S'_T)$ as desired.

Case 2: Suppose $\tilde{S}_T$ is obtained from $S_T$ via a compression along a cut-disk $D'$. If $D'$ is separating, then each of the two components of $\tilde{S}_T$ has at least one puncture and if a component is a sphere, then it must have at least 3 punctures, ie each component of $\tilde{S}_T$ has a strictly negative Euler characteristic. By the additivity of Euler characteristic, we conclude that for each component $S'_T$ of $\tilde{S}_T$, $\chi(S'_T) < \chi(\tilde{S}_T) = \chi(S_T)$ and so the first component of the complexity tuple is decreased.
If the cut disk is not separating the cut-compression does not affect the first term in the complexity tuple as $\chi(D^c) = 0$. Note that $\partial D^c$ must be essential in the non-punctured surface $S$ so we can consider $D^c$ as a compressing disk for $S$ in $M$. Then $g(\tilde{S}) < g(S)$ so in this case the second component of the complexity tuple is decreased.

The width of a c-Heegaard splitting is the multiset of pairs $w(\mathcal{H}) = c(\mathcal{H}_+)$. In [5] a multiple Heegaard splitting is called thin if it is of minimum width amongst all possible multiple Heegaard splittings for the pair $(M, T)$. Similarly we will call a c-Heegaard splitting c-thin if it is of minimal width amongst all c-Heegaard splittings for $(M, T)$.

5. Thinning using pairs of disjoint c-disks

**Lemma 5.1.** Suppose $M$ is a compact orientable irreducible manifold and $T$ is a properly embedded 1-submanifold. If $P$ is a c-Heegaard splitting for $(M, T)$ which is c-weakly reducible, then there exists a multiple c-Heegaard splitting $\mathcal{H}'$ so that $w(\mathcal{H}') < w(P)$.

Moreover if $M$ is closed then either

- There is a component of $\mathcal{H}'_T$ that is neither an inessential sphere nor boundary parallel in $M_T$, or
- $P$ is stabilized, meridionally stabilized or perturbed, or a closed component of $T$ is removable.

The first part of the proof of this lemma is very similar to the proof of Lemma 2.3 in [5] and uses the idea of untelescoping. However, in Lemma 2.3 the authors only allow untelescoping using disks while we also allow untelescoping using cut-disks.

**Proof.** Let $(A, A \cap T)^c$ and $(B, B \cap T)^c$ be the two c-compression bodies that $P$ cuts $(M, T)$ into. Consider a maximal collection of c-disks $D^*_A \subset A_T$ and $D^*_B \subset B_T$ such that $\partial D^*_A \cap \partial D^*_B = \emptyset$. Let $A'_T = \text{cl}(A_T - N(D^*_A))$ and $B'_T = \text{cl}(B_T - N(D^*_B))$ where $N(D^*)$ is a collar of $D^*$. Then $A'_T$ and $B'_T$ are each the disjoint union of c-compression bodies. Take a small collar $N(\partial_+ A'_T)$ of $\partial_+ A'_T$ and $N(\partial_+ B'_T)$ of $\partial_+ B'_T$. Let $C^1_T = \text{cl}(A'_T - N(\partial_+ A'_T)), C^2_T = N(\partial_+ A'_T) \cup N(D^*_B), C^3_T = N(\partial_+ B'_T) \cup N(D^*_A)$ and $C^4_T = \text{cl}(B'_T - N(\partial_+ B'_T))$. This is a new multiple c-Heegaard splitting of $(M, T)$ with positive surfaces $\partial_+ C_1$ and $\partial_+ C_2$ that can be obtained from $P$ by c-compressing along $D^*_A$ and $D^*_B$ respectively and a negative surface $\partial - C_2 = \partial + C_3$ obtained from $P$ by compressing along both sets of c-disks.

By Lemma 4.4 it follows that $w(\mathcal{H}') < w(P)$.

To show the second part of the lemma, suppose $A \cup_P B$ is not stabilized, meridionally stabilized or perturbed and no component of $T$ is removable
and, by way of contradiction, suppose that every component of \( \partial_- C_2 \) is a sphere bounding a ball that intersects \( T \) in at most one trivial arc or a torus that bounds a solid torus such that \( t \subset T \) is a core curve of it.

Let \( \Lambda_A \) and \( \Lambda_B \) be the arcs that are the cocores of the collections of c-disks \( D^*_A \) and \( D^*_B \) respectively. If \( D^c \) is a cut-disk, we take \( \lambda \subset T \) as its cocore. Let \( \Lambda = \Lambda_A \cup \Lambda_B \) and note that \( P \) can be recovered from \( \partial_- C_3 \) by surgery along \( \Lambda \). As \( P \) is connected, at least one component of \( \partial_- C_3 \) must be adjacent to both \( \Lambda_A \) and \( \Lambda_B \), call this component \( F \). Unless \( F \) is is an inessential sphere or boundary parallel in \( M_T \) we are done. If \( F \) is an inessential sphere, then by Waldhausen’s result the original Heegaard splitting is stabilized. As \( \partial M = \emptyset \) by hypothesis, the remaining possibility is that \( F \) is parallel in \( M_T \) to part of \( T \); since \( F \) is connected it is either a torus bounding a solid torus with a component of \( T \) as its core or \( F \) is an annulus, parallel to a subarc of \( T \). That is \( F \) bounds a ball which \( T \) intersects in a trivial arc.

Let \( B \) be the ball or solid torus \( F \) bounds. We will assume that \( B \) lies on the side of \( F \) that is adjacent to \( \Lambda_A \) and that \( F \) is innermost in the sense that \( B \cap \Lambda_B = \emptyset \).

Let \( H = \partial_- C_3 \cap B \) and let \( A' \) be the c-compression body obtained by adding the 1-handles corresponding to the arcs \( \Lambda_A \cap B \) to a collar of \( H \). (Some of these 1-handles might have subarcs of \( T \) as their core). Let \( B' = B - A' \). Notice that \( B' \) can be obtained from \( B \) by c-compressing along all c-disks whose cocores are adjacent to \( F \) and thus \( B' \) is a c-compression body. In fact \( \partial B' = \emptyset \) so \( B' \) is a handlebody, let \( H' = \partial B' \). Then \( A' \cup_{H'} B' \) is a c-Heegaard splitting for \( B \) decomposing in into a c-compression body \( A' \) and a handlebody \( B' \). There are two cases to consider: \( B \) being a ball intersecting \( T \) is a trivial arc and \( B \) being a torus. We will consider each case separately and prove that \( A' \cup_{H'} B' \) is actually a Heegaard splitting for \( B \) (i.e. \( A' \) is a compression body) so we can apply previously known results.

\[ \text{Figure 2.} \]

**Case 1:** If \( B \) is a ball and \( B \cap T = t \) is a trivial arc, there are three sub-cases to consider. If \( t \cap H' \neq \emptyset \) then the construction above gives a nontrivial Heegaard splitting for the pair \((B, t)\); \( A' \) is a compression body
by Proposition 3.3 as \( \partial_- A' \) adjacent to two subarcs of \( t \) both of which have their second endpoint on \( \partial_+ A' = H' \). By Lemma 2.1 of [4], \( H' \) is either stabilized or perturbed (in this context if \( H' \) is cancellable, it must be perturbed as \( t \) is not closed) so the same is true for \( P \).

If \( t \subset A' \) and \( t = \Lambda \cap B \) (in particular \( H = F \)), Figure 2 shows a pair of \( c \)-disks demonstrating that \( P \) is meridionally stabilized.

If \( t \subset A' \) and \( t \neq \Lambda \cap B \), consider the solid torus \( V = B - \eta(t) \). Let \( A'' \) be the \( c \)-compression body obtained by 1-surgery on \( H \) along the arcs \( \Lambda \cap V \).

As \( t \cap V = \emptyset \), \( A'' \) is in fact a compression body. Note that \( V - A'' = B' \) as \( B' \cap t = \emptyset \). Thus \( A'' \cup B' \) is a non-trivial Heegaard splitting for the solid torus \( V \). By [7] it must be stabilized and thus so is \( P \).

**Case 2:** Suppose \( F \) bounds a solid torus \( B \), which is a regular neighborhood of closed component \( t \) of \( T \). As \( \partial_- A \cap t = \emptyset \), \( A' \cup H' \) is a Heegaard splitting for \( (V, t) \). By [3] it is cancellable or stabilized. This proves the theorem at hand unless \( H' \) is cancellable but not perturbed so assume this is the case. In particular this implies that \( H' \cap T = 2 \). In this case [3] shows that if \( g(H') \geq 2 \) then \( H' \) is stabilized. Thus it remains to consider the case when \( H' \) is a torus intersecting \( t \) in two points. In this case \( H \) must be the union of \( F \) and a sphere \( S \) intersecting \( t \) in two points and \( \Lambda \cap B \) is a single possibly knotted arc with one endpoint on \( F \) and the other on \( S \). As \( t \) is cancellable, we can use the canceling disk in \( A' \) to isotope \( t \) across \( H' \) so it lies entirely in \( B' \). After this isotopy it is clear that \( F \) and \( H' \) cobound a product region. As \( F \) is the boundary of a regular neighborhood of \( t \), it follows that \( t \) is isotopic to the core loop of the solid torus \( B' \) ie, \( B' - \eta(t) \) is a trivial compression body. \( B \) can be recovered from \( B' \) by 1-surgery so \( B - \eta(t) \) is also a compression body. Thus after an isotopy of \( t \) along the pair of canceling disks, \( P \) is a Heegaard splitting for \( (M - \eta(t), T - t) \) so \( t \) is a removable component of \( T \).

\[ \square \]

6. **Intersection between a Boundary Reducing Disk and a Bridge Surface**

As in Jaco [6] a weak hierarchy for a compact orientable 2-manifold \( F \) is a sequence of pairs \( (F_0, \alpha_0), \ldots, (F_n, \alpha_n) \) where \( F_0 = F \), \( \alpha_i \) is an essential curve on \( F_i \) and \( F_{i+1} \) is obtained from \( F_i \) by cutting \( F_i \) along \( \alpha_i \). The final surface in the hierarchy, \( F_{n+1} \), satisfies the following:

1. Each component of \( F_{n+1} \) is a disc or an annulus at least one boundary component of which is a component of \( \partial F \).
2. Each non-annulus component of \( F \) has at least one boundary component which survives in \( \partial F_{n+1} \).
The following lemma was first proven by Jaco and then extended in [5], Lemma 3.1.

**Lemma 6.1.** Let $F$ be a connected planar surface with $b \geq 2$ boundary components. Let $(F_0, \alpha_0), \ldots, (F_n, \alpha_n)$ be a weak hierarchy with each $\alpha_i$ an arc. If $d$ is the number of boundary components of $F_{n+1}$ then,

- If $F_{n+1}$ does not have annulus components then $d \leq b - 1$
- If $F_{n+1}$ contains an annulus component, then $d \leq b$. If $d = b$ and $b \geq 3$, then $F_{n+1}$ contains a disc component.

**Theorem 6.2.** Suppose $M$ is a compact orientable irreducible manifold and $T$ is a properly embedded 1-manifold in $M$. Let $A \cup_P B$ be a c-Heegaard splitting for $(M,T)$. If $D$ is a boundary reducing disk for $M$ then there exists such disk $D'$ so that $D'$ intersects $P_T$ in a unique essential simple closed curve.

**Proof.** Let $D$ be a reducing disk for $\partial M$ chosen amongst all such disks so that $D \cap P$ is minimal. By Corollary 3.5, $D \cap P \neq \emptyset$. Let $D_A = D \cap A$ and $D_B = D \cap B$.

Suppose some component of $D_A$ is $c$-compressible in $A$ with $E$ the $c$-compressing disk. Let $\gamma = \partial E$ and let $D_\gamma$ be the disk $\gamma$ bounds on $D$. Note that the sphere $D_\gamma \cup E$ must be punctured by $T$ either 0 or two times thus $E$ must be a non-punctured disk. Let $D' = (D - D_\gamma) \cup E$. $D'$ is also a reducing disk for $\partial M$ as $\partial D' = \partial D$ and $D' \cap T = \emptyset$. As $\partial E$ is essential on $D_A$, $D_\gamma$ cannot lie entirely in $A$ so $|D_\gamma \cap P| > |E \cap P|$ and thus $|D' \cap P| < |D \cap P|$ contradicting the choice of $D$. Similarly $D_B$ is $c$-incompressible in $B$.

Suppose that $E$ is a $\partial$-compressing disk for $D_A$ and $E$ is adjacent to $\partial_- A$. $\partial$-compressing $D$ along $E$ gives two disks $D_1$ and $D_2$ at least one of which has boundary essential of $\partial M$, say $D_1$. However $|D_1 \cap P| < |D \cap P|$, a contradiction.

Suppose that $E$ is a $\partial$-compressing disk for $D_A$ and $E$ is adjacent to $P$. Let $\alpha = E \cap D_A$. Use $E$ to isotope $D$ so that a neighborhood of $\alpha$ lies in $B$, call this new disk $D^1$ and let $D^1_A = D^1 \cap A$ and $D^1_B = D^1 \cap B$. Note that $D^1_A$ is obtained from $D_A$ by cutting along $\alpha$ and $D^1_A$ is also $c$-incompressible. Repeat the above operation naming each successive disk $D^i$ until the resulting surface $D^n_A = D^n \cap A$ is $\partial$-incompressible. By Corollary 3.7, $D^n_A$ consists of vertical annuli and disks.

Suppose some component of $D_A$ is $\partial$-compressible but not adjacent to $\partial_- A$. In this case the result of maximally $\partial$-compressing this component has to be a collection of disks. By Case 1 of Lemma 6.1, $|D^n_A \cap P| < |D_A \cap P|$ contradicting our choice of $D$. Thus every boundary compressible component of $D_A$ is adjacent to $\partial_- A$, in particular $\partial D \subset \partial_- A$ and $D_A$ has a unique $\partial$-compressible component $F$. By the minimality assumption.
and Case 2 of Lemma 6.1, some component of $D^n_A$ must be a disk. $D^n_B$ is then a planar surface that we have shown must be c-incompressible and has a component that is not a disk. As $\partial D \cap \partial B = \emptyset$, it follows that some component of $D^n_B$ is $\partial$-compressible into $P$ and disjoint from $\partial - B$. The above argument applied to $D^n_B$ leads to an isotopy of the disk $D$ so as to reduce $D \cap P$ contrary to our assumption. Thus $D_A$ and $D_B$ are both collections of vertical annuli and disks so $D$ is a reducing disk for $\partial M$ that intersects $P$ in a single essential simple closed curve.

□

Corollary 6.3. Let $A \cup P B$ be a c-strongly irreducible c-Heegaard splitting of $(M, T)$ and let $F$ be a component of $\partial M$. If $F_T$ is not parallel to $P_T$, then $F_T$ is incompressible.

Proof. Suppose $D$ is a compressing disk for $F_T \subset \partial - A$ say. By Theorem 6.2 we can take $D$ such that $|D \cap P| = 1$, $D_A = D \cap A$ is a compressing disk for $P_T$ lying in $A$ and $D - D_B$ is a vertical annulus disjoint from $T$. As $F_T$ is not parallel to $P_T$, there is a c-disk for $P_T$ lying in $A$, $D_A$. By a standard innermost disk and outermost arc arguments, we can take $D_A$ so that $D_A \cap D = \emptyset$. But then $D_A$ and $D_B$ give a pair of c-weakly reducing disks for $P_T$ contrary to our hypothesis.

□

7. MAIN THEOREM

Following [5] we will call a c-Heegaard splitting $H$ c-slim if each component $W_{ij} = C_i \cup C_j$ obtained by cutting $M$ along $H_-$ is c-strongly irreducible and no proper subset of $H$ is also a multiple c-Heegaard splitting for $M$. Suppose $H$ is a c-thin c-Heegaard splitting of $M$. If some proper subset of $H$ is also a c-Heegaard splitting of $M$, then this c-Heegaard splitting will have lower width than $H$. If some component $W_{ij}$ of $M - H$ is c-weakly reducible, applying the untelescoping operation described in Lemma 5.1 to that component produces a c-Heegaard splitting of lower width. Thus if $H$ is c-thin, then it is also c-slim.

Theorem 7.1. Suppose $M$ is a closed orientable irreducible 3-manifold containing a link $K$. If $P$ is a c-strongly compressible bridge surface for $K$ then one of the following is satisfied:

- $P$ is stabilized
- $P$ is meridionally stabilized
- $P$ is perturbed
- a component of $K$ is removable
- $M$ contains an essential meridional surface $F$ such that $2 - \chi(F_K) \leq 2 - \chi(P_K)$. 
Proof. Let $\mathcal{H}$ be a c-slim Heegaard splitting obtained from $P$ by untelescoping as in Lemma 5.1, possibly in several steps. Let $\mathcal{H}_-$ and $\mathcal{H}_+$ denote the negative and positive surfaces of $\mathcal{H}$ respectively and let $W_{ij}$ be the components of $M - \mathcal{H}_-$ where $W_{ij}$ is the union of c-compression bodies $C_i$ and $C_j$ along $H_{ij} = \partial_+ C_i = \partial_+ C_j$. Suppose some component of $\mathcal{H}_-$ is compressible with compressing disk $D$. By taking an innermost on $D$ circle of $D \cap \mathcal{H}_-$ we may assume that $\partial_- C_i$ is compressible in $W_{ij}$. By Corollary 6.3 this contradicts our assumption that $\mathcal{H}$ is c-slim. We conclude that $\mathcal{H}_-$ is incompressible.

If some component of $F_K$ of $\mathcal{H}_-$ is neither an inessential sphere nor boundary parallel in $M_K$, then it is essential and $2 - \chi(F_K) \leq 2 - \chi(P_K)$. If every component is either an inessential sphere in $M_K$ or boundary parallel, then by Lemma 5.1 the splitting is perturbed, stabilized, meridionally stabilized or there is a removable component.

\[ \square \]

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