Multi-field Cuscuton cosmology

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Abstract. In this paper, we first introduce a multi-field setup of Cuscuton gravity in a curved field space manifold. Then, we show that this model allows for a regular bouncing cosmology and it does not lead to ghosts or other instabilities at the level of perturbations. More precisely, by decomposing the scalar fields perturbations into the tangential and normal components with respect to the background field space trajectory, the entropy mode perpendicular to the background trajectory is healthy which directly depends on the signature of the field-space metric, whereas the adiabatic perturbation tangential to the background trajectory is frozen. In analogy with the standard Cuscuton theory equipped with an extra dynamical scalar field, the adiabatic field does not have its own dynamics, but it modifies the dynamics of other dynamical fields like entropy mode in our scenario. Finally, we perform a Hamiltonian analysis of our model in order to count the degrees of freedom propagated by dynamical fields.

Keywords: Cosmological perturbation theory in GR and beyond, gravity, modified gravity, physics of the early universe

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1 Introduction

In recent years, there has been growing interest in modified gravity theories which can be used to explain some of the unsolved problems in cosmology and General Relativity (GR) such as unknown components of universe called the dark matter and dark energy, the singularity problem and so on.

Most modified gravity theories such as Horndeski and Galileon models [1–7] contain extra degrees of freedom governed by dynamical equations. In spite of this fact, recently the Cuscuton gravity [8, 9] has been proposed as an infrared modification of GR, with no additional degree of freedom. Cuscuton gravity can be implemented by adding a non-canonical scalar field to general relativity. In this sense, the equation of motion of this field does not have any second order time derivatives. It means that the Cuscuton field does not have its own dynamics and acts as an auxiliary field. Therefore, in order to produce dynamics, it is required to include other fields [8]. An increasing number of studies have been carried out on Cuscuton gravity and its extended versions. As an example, in [10, 11] the authors proved that Cuscuton model with a quadratic potential can be considered as a low-energy limit of the non-projectable Horava-Lifshitz gravity model [12]. Further, such a theory provides significant new features such as a distinct Cosmic Microwave Background [9], viable power-law solutions of inflation [13], the absence of spherical caustic instabilities in galileon theories [14], and inflationary solutions [15, 16]. Interestingly, in ref. [17] the authors have also shown that all acceptable Cuscuton solutions are always solutions for VCDM theory, which is a kind of Type-IIa Minimally Modified Gravity theories [18].

Remarkably, it was shown recently that a stable regular bouncing cosmology generated by Cuscuton gravity [19–21]. Generally, regular bouncing models violate the Null Energy
Condition (NEC), which results in either instabilities or a superluminal speed of sound [22–26].\(^1\) To be more explicit, due to the non-dynamical nature of the Cuscuton field, an effective violation of the NEC occurs for the background bounce while the actual dynamical degree of freedom does not violate NEC and remains safe. Furthermore, in [19, 33] the authors have shown that a Cuscuton bounce does not suffer from ghost instabilities and the scalar perturbations remain stable throughout the bounce phase. The stability of these perturbations were investigated in further detail for (generalized) Cuscuton setups in refs. [20, 34].

With all this in mind, it is interesting to ask what happens if one considers a multi-field generalization of Cuscuton gravity. In the last few years there has been a growing interest in multi-field models with curved field-space manifold, ranging from inflation, dark energy, primordial non-Gaussianity and related areas [35–39]. Therefore, in this work we first attempt to extend the single field Cuscuton gravity [8] to the multi-field setup with a curved field space manifold. Then, by projecting equations of motion for scalar fields along the tangent vector to the background field trajectory, we show that the adiabatic combination of scalar fields does not have its own dynamics, while it can modify the dynamics of the other dynamical fields in our scenario. In this respect, we can investigate the possibility of a bounce solution in such a model.

At the level of perturbations, after decomposing scalar fields perturbations into the tangential and normal components with respect to the background field space trajectory, finally we will prove that the multi-field Cuscuton bounce does not have any ghost instabilities and the scalar perturbations are stable throughout the bounce phase. In addition, inspired by ref. [40], which the Hamiltonian analysis of the single field Cuscuton theory has been performed, we consider the full non-linear Hamiltonian analysis of the multi-field Cuscuton gravity with the curved field space manifold and calculate the physical degrees of freedom.

The paper is organized as follows. In section 2, we attempt to construct a generalization of the multi-field Cuscuton model with non-canonical kinetic term. Moreover, we derive the background equation of motions, then we obtain equation of motions along orthogonal and tangent unit vectors to the field trajectory. In section 3, we find a Cuscuton bounce solution for our model. Cosmological perturbations are presented in section 4, and quadratic action in the spatially flat gauge and comoving gauge are considered. Section 5 is devoted to a Hamiltonian formalism of the multi-field Cuscuton gravity in non-linear level. Finally, our conclusions are drawn in section 6.

### 2 Building a model for the multi-field Cuscuton gravity

Let us first consider a multi-field system with a generic field space metric \(G_{ab}\) coupled to the Einstein gravity which is given by [41]

\[
S = \int d^4x \sqrt{-g} \left( \frac{M_p^2}{2} R + P(X) - V(\phi^a) \right),
\]

in which \(M_p\) is the reduced Planck mass, \(R\) is the Ricci scalar associated with the spacetime metric \(g_{\mu\nu}\) and \(P\) is an arbitrary function of the kinetic term \(X = -G_{ab} g^{\mu\nu} \partial_\mu \phi^a \partial_\nu \phi^b/2\).\(^2\) The

\(^1\)The stability and superluminal nature of the bouncing models is still a challenging problem in the literature. For an example, in [27–29] authors have argued that a healthy bounce can be achieved using Galilean action, whereas in [30–32] have been shown that when coupling to matter or other regions of phase-space are included, Galilean models have superluminal speed of sound.

\(^2\)Generally, one can take \(P\) as a function of \(X\) and \(\phi^a\). We here assume \(P\) depends only on \(X\), for simplification.
$V$ is a general potential function depending on the scalar fields $\phi^a$ as well. By varying the action (2.1) with respect to the metric $g_{\mu\nu}$, the energy-momentum tensor is obtained to be

$$ T_{\mu\nu}^a = (P - V)\delta_{\nu}^a + P_X G_{ab} \partial_{\nu} \phi^a \partial^b \phi^b, \quad (2.2) $$

in which $P_X$ denotes the partial derivative of $P$ with respect to $X$. In a spatially flat FLRW spacetime,

$$ ds^2 = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j, \quad (2.3) $$

the energy-momentum tensor (2.2) reduces to that of a perfect fluid with the energy density

$$ \rho = 2XP_X - p, \quad (2.4) $$

and pressure $p = P - V$. The background equation of motion for the scalar field can be obtained by the variation of the action (2.1) with respect to $\phi^a$ which yields [41]

$$ D_t \dot{\phi}^a + \left( 3H + \frac{\dot{P}_X}{P_X} \right) \dot{\phi}^a + \frac{1}{P_X} G^{ab} V_b = 0, \quad (2.5) $$

where $_b$ stands for the derivative with respect to the scalar field $\phi^a$ and $D_t$ is the covariant time derivative which is defined as

$$ D_t A^a \equiv \dot{A}^a + \Gamma^a_{bc} A^b \dot{\phi}^c, \quad (2.6) $$

where $\Gamma^a_{bc}$ is the Christoffel symbol constructed by the field space metric $G_{ab}$ and $D_t G_{ab} = 0$. Because $\phi^a$ are coordinates in the field space, we are allowed to choose other convenient basis. One possible choice is the so-called kinematic basis which is a set of orthogonal unit vectors $\{ e^a_n \} (n = 1, \ldots, N)$ on the $N$-dimensional field space, where the first vector is the unit adiabatic vector defined as

$$ e^a_1 \equiv T^a = \frac{\dot{\phi}^a}{\phi_0}, \quad (2.7) $$

in which $\dot{\phi}_0 = G_{ab} \dot{\phi}^a \dot{\phi}^b = 2X$. The $(N - 1)$ remaining vectors of the basis, $N^a_n$, span the entropy subspace which is orthogonal to the adiabatic direction [35, 42]. One can also define useful matrices as $Z_{mn} \equiv e^a_m D_t e^a_n$ which satisfy the antisymmetry property $Z_{nm} = -Z_{mn}$ as a result of $D_t(e^a_m e^a_n) = 0$ [35, 42].

Therefore, by projecting eq. (2.5) along $T^a$, the adiabatic equation of motion is obtained to be

$$ \ddot{\phi}_0 + \left( 3H + \frac{\dot{P}_X}{P_X} \right) \dot{\phi}_0 + \frac{1}{P_X} V_T = 0, \quad (2.8) $$

where $V_T \equiv V_a T^a$. In addition, by using

$$ \dot{P}_X = P_{XX} \dot{X} = P_{XX} \dot{\phi}_0 \ddot{\phi}_0. \quad (2.9) $$

Eq. (2.8) can be written as [41]

$$ (P_X + 2XP_{XX}) \ddot{\phi}_0 + 3HP_X \dot{\phi}_0 + V_T = 0. \quad (2.10) $$

On the other hand, the entropy part of the equation of motion gives us the rate of change of the adiabatic (tangent) vector $T^a$. Thus, by projecting eq. (2.5) along $N^a_n$, we also arrive at

$$ D_t T^a = -\frac{V_n}{P_X \phi_0} N^a_n. \quad (2.11) $$
where \( V_n = N_n^a V_a \). This also implies that the matrix elements \( Z_{n1} = -Z_{1n} = -V_n / (P_X \dot{\phi}_0) \). In analogy with the single field cuscuton gravity \([8, 9, 19, 21, 33]\), we therefore demand that a multi-field generalization of Cuscuton gravity can be achieved by taking the below constraint

\[
P_X + 2XP_{XX} = 0. \tag{2.12}
\]

Obviously, in this limit, the adiabatic equation of motion (2.10) is not second order due to the absence of the time derivative of \( \dot{\phi}_0 \). It means that the Cuscuton adiabatic field does not have its own dynamics, while it may modify the dynamics of the other dynamical fields such as entropy mode in our scenario (we refer readers to section 4 in more detail.). Moreover, by imposing the above constraint on the speed of sound,

\[
c_s^2 \equiv \frac{P_X}{\rho_X} = \frac{P_X}{P_X + 2XP_{XX}}, \tag{2.13}
\]

this generality leads to superluminal speed of sound which is naturally addressing the violation of Null Energy Condition (NEC) in regular bounce models (we refer to this point in the next section.). Note that a superluminal propagation speed may not necessary indicates that causality is violated \([43]\) (we discuss about this fact in appendix A).

3 Background cosmology and bounce solutions

In this section, we consider the background equations of motion for multi-field Cuscuton model and present a model for a multi-field Cuscuton bounce scenario. Similar to the single-field cuscuton gravity, the constraint (2.12) leads to restrict our choice of \( P \) to

\[
P(X) = \pm \mu^2 \sqrt{2X}, \tag{3.1}
\]

where \( \mu \) is constant. It follows that eq. (2.5) reduces to

\[
\pm \mu^2 G_{ab} D_t \left( \frac{a^3}{\sqrt{2X}} \phi^b \right) = a^3 V_a. \tag{3.2}
\]

It seems that scalar fields are dynamical fields, in contrast with the single field Cuscuton model in which the cuscuton field has not its own dynamical degree of freedom \([8, 9, 19, 21, 33]\). Nevertheless, the adiabatic combination of scalar fields is a field with no dynamics. It means that the adiabatic equation of motion (2.8) converts to

\[
\pm 3\mu^2 \text{sign}(\dot{\phi}_0) H + V_T = 0, \tag{3.3}
\]

without second time derivative of the adiabatic field \( \phi_0 \). Moreover, Friedman equations are obtained to be

\[
3M_p^2 H^2 = V, \tag{3.4}
\]

\[
M_p^2 \dot{H} = -XP_X = -\frac{1}{2}(\pm)\mu^2 \sqrt{2X}. \tag{3.5}
\]

Now allow us to investigate the bounce realization in the framework of multi-field Cuscuton cosmology. An important feature of a regular bounce \((H \neq \pm \infty)\) is that universe
moves from a contracting phase \((H < 0)\) into an expanding phase \((H > 0)\) at finite value of the scale factor \(a_b\). It follows that

\[ H_b = 0, \quad \text{and} \quad \dot{H}_b > 0. \quad (3.6) \]

Obviously, the second condition implies the violation of NEC. Therefore, this condition forces us to consider the negative sign for the adiabatic field in eq. (3.5) to get a bounce solution. This in turns, yields eq. (3.3) becomes

\[ -3\mu^2 \text{sign}(\dot{\phi}_0)H + V_T = 0. \quad (3.7) \]

Without loss of generality, from now on, we only consider solutions with \(\dot{\phi}_0 > 0\). Taking a time derivative from both sides of eq. (3.7), we have

\[ 3\mu^2 \dot{H} = V_{TT} \dot{\phi}_0 + \left(\frac{V_n}{\mu}\right)^2, \quad (3.8) \]

where \(V_n = N_a V_a\) and \(V_{TT} = T^a T^b V_{ab}\). Because the last term in the right hand side of the above equation is always positive, one can consider

\[ V_{TT} > 0, \quad (3.9) \]

near to the bounce where the NEC will be violated \((\dot{H} > 0)\). Moreover, by plugging \(H\) from eq. (3.7) back into eq. (3.4), one obtains

\[ \frac{M_p^2}{3\mu^4} V_T^2 = V, \quad (3.10) \]

taking a time derivative of the above equation, we arrive at

\[ \frac{2M_p^2}{3\mu^4} V_{TT} - 1 = -\frac{2M_p^2}{3\mu^4} \left(\frac{V_n}{\mu}\right)^2 < 0, \quad (3.11) \]

thus

\[ V_{TT} < \frac{3\mu^4}{2M_p^2}. \quad (3.12) \]

As a result, while the shape of the potential in NEC violation along tangent basis is convex \((V_{TT} > 0)\), its convexity is in the below range

\[ 0 < V_{TT} < \frac{3\mu^4}{2M_p^2}. \quad (3.13) \]

This allows that universe experiences a regular bouncing cosmologies, where an initially contracting universe, bounces and starts expanding. With the background quantities established, we can study cosmological perturbations and analyze the existence of ghosts and other instabilities in this model.
4 Cosmological perturbations

In this part, we investigate the dynamical stability of the cosmological scalar perturbations in multi-field Cuscuton gravity. The set up we used here is inspired by the one proposed in refs. [35, 42]. At the perturbation level, we assume $\phi_a = \phi_a^0 + \delta \phi_a$ where $\delta \phi_a$ are the scalar perturbations. Similar to the Riemann normal coordinates [44–46], these perturbations can be expressed as a power series of the tangent vector $Q^a$ to the geodesic trajectory in the field space. Therefore, one can consider the below covariant form for scalar perturbations [35, 42].

$$\delta \phi_a = Q_a - \frac{1}{2} \Gamma^b_{ac} Q^b Q^c + \frac{1}{6} \left( \Gamma^b_{ac} \Gamma^e_{bd} - \Gamma^e_{bc} \Gamma^a_{bd} \right) Q^b Q^c Q^d + \cdots,$$

(4.1)

in which $\Gamma^a_{bc}$ represents the Christoffel symbol associated with the metric $G_{ab}$. Clearly, the field fluctuations $\delta \phi_a$ and $Q^a$ are identical at the linear order, but they are different at higher orders. Furthermore, components of the full metric denote via

$$g_{00} = -N^2 + \beta^i \beta_i, g^{0i} = \beta^i, g^{ij} = \gamma^{ij},$$

where at linear order the scalar perturbation parts are given by

$$N = 1 + \alpha, \quad \beta_i = B_i, \quad \gamma^{ij} = a^2 \delta^{ij}.$$

4.1 Quadratic action in the spatially flat gauge

Let us first verify the cosmological perturbations in the spatially flat gauge. According to our results in previous section, the action (2.1) is given by the following form

$$S = \int d^4x \sqrt{-g} \left( \frac{M_p^2}{2} R - \mu^2 \sqrt{2X} - V(\phi_a) \right).$$

(4.3)

Now by expanding the above action up to the second order of scalar perturbations, the quadratic action takes the following form,

$$S^{(2)} = \int d^4x \frac{a^2}{2} \left[ P_a \dot{Q}_b \dot{D}_a \dot{Q}_b - \frac{1}{2X} P_a \dot{D}_a \dot{Q}_b \dot{Q}_c \dot{\phi}^b \dot{\phi}^c - a^{-2} G_{bc} P_a \partial_i \dot{Q}_c \partial^i \dot{Q}^b \right.$$

$$- 2\alpha (3H^2 \alpha + Q^b \dot{V}_b + 2M_p^2 H \partial_i \dot{\phi}^j B) + 2Q^b \dot{\phi}_b P_a \partial_i \dot{\phi}^j B - Q^b \dot{Q}^c (V_{bc} + \mathbb{R}_{bc,df} \dot{\phi}^d \dot{\phi}^f P_c X),$$

where $V_{ab} = V_{ba}$ and

$$\mathbb{R}_{bd, c} = \Gamma^a_{bd, c} - \Gamma^a_{bc, d} + \Gamma^a_{cc} \Gamma^e_{bd} - \Gamma^a_{de} \Gamma^c_{bd},$$

(4.4)

is the Riemann tensor related to the curved field space manifold. Clearly, one can see that the quadratic Lagrangian is linear in terms of the non-dynamical mode $B$, thus its equation of motion yields

$$\alpha = \frac{Q^b \dot{\phi}_b P_a X}{2M_p^2 H}.$$

(4.5)

In ref. [15], the authors proposed a class of multi-scalar effective field theories (EFTs) that can achieve inflationary solutions. Remarkably, this EFT superficially resembles the multi-field Cuscuton model at low energies.
Inserting the \( \alpha \) relation in action (4.4), the reduced action takes the following form

\[
S^{(2)} = \int d^4x \frac{a^3}{2} \left( P_X D_t Q_b D_t Q^b - M_{bc} Q^b Q^c - \frac{P_X D_t Q_b D_t Q_c \dot{\phi}^b \dot{\phi}^c}{2X} - \frac{G_{bc}}{a^3} P_X \partial_t Q^c \dot{\phi}^b \right),
\]

(4.7)

where the effective mass matrix is introduced by

\[
M_{bc} = V_{bc} + \frac{P_X}{2H} (V_c \dot{\phi}_b + V_b \dot{\phi}_c) + \frac{3}{2M^2} P_X^2 \dot{\phi}_b \dot{\phi}_c + P_X R_{bcdf} \dot{\phi}^d \dot{\phi}^f.
\]

(4.8)

Up to now, our analysis was general, valid for any number of fields. We now consider explicitly the case where only two scalar fields are present. In this context, the entropy subspace is one-dimensional and the basis \( \{e_1^a, e_2^a\} = \{T^a, N^a\} \) is completely determined. In this case, one can choose to write the normal one-form as

\[
N_a = (\operatorname{sgn}(\pm 1)G)^{1/2} e_{ab} T^b,
\]

(4.9)

in which \( G \) is determinant of the metric \( G_{ab} \), the signum function \( \operatorname{sgn}(\pm 1) \) determines the signature of \( G_{ab} \), for instance, \( \operatorname{sgn}(-1) \) is for Lorentzian signature, whereas \( \operatorname{sgn}(\pm 1) \) is chosen for Euclidean signature. In addition, \( e_{ab} \) is the two dimensional Levi-Civita symbol with \( \epsilon_{11} = \epsilon_{22} = 0 \) and \( \epsilon_{12} = -\epsilon_{21} = 1 \). These definitions satisfy that \( T_a T^a = 1 \), \( N_a N^a = \operatorname{sgn}(\pm 1) \) and \( T^a N_a = 0 \) [35, 42].

By making use of eq. (3.1) and (2.11), we also conclude the following relations,

\[
D_t T^a = \frac{V_N}{\mu^2} N^a = \dot{\theta} N^a,
\]

(4.10)

\[
D_t N^a = -\dot{\theta} T^a,
\]

(4.11)

when \( \dot{\theta} = 0 \), the vectors \( T^a \) and \( N^a \) remain covariantly constant with respect to \( D_t \) along the trajectory in the field space. If \( \dot{\theta} > 0 \), the path turns to the left, whereas if \( \dot{\theta} < 0 \), the turn is towards the right. Moreover, the parallel and normal perturbations due to the background trajectory are given respectively by

\[
u^T \equiv Q^T \equiv T_a Q^a,
\]

(4.12)

\[
u^N \equiv Q^N \equiv N_a Q^a.
\]

(4.13)

In this sense, \( u^T \) corresponds to the perturbations parallel to the background trajectory which shows the adiabatic perturbation mode and \( u^N \) equals to perturbations normal to the trajectory which indicates the entropy perturbation mode [35, 42, 47]. By replacing \( \dot{\phi}_0 \) and \( D_t T^c \) by the tangent and normal vectors \( T^c \) and \( N^c \) according to eqs. (4.9) and (4.12), and using the above representations, the quadratic action (4.7), finally reads

\[
S = \int d^4x \frac{a^3}{2} \left[ \operatorname{sgn}(\pm 1) P_X \left( (\dot{u}_N)^2 - \frac{1}{a^2} (\partial u_N)^2 \right) + \operatorname{sgn}(\pm 1) P_X \left( \partial^2 u_T^2 - \frac{1}{a^2} (\partial u_T)^2 \right) + 2\operatorname{sgn}(\pm 1) P_X \theta u_N \dot{u}_N - M_{NN} u_N^2 - 2M_{NT} u_N u_T - M_{TT} u_T^2 \right],
\]

(4.14)
where the symmetric matrix $M_{IJ}$ elements are specified by

$$M_{NN}^{2} = N^{a}N^{b}M_{ab}^{2} = V_{NN} + \text{sgn}(\pm 1)\dot{H}\mathbb{R},$$

(4.15)

$$M_{TT}^{2} = T^{a}T^{b}M_{ab}^{2} = V_{TT} - \frac{\mu^{2}}{2M_{p}^{2}H}V_{T} + \frac{3}{2}\left(\frac{\mu^{4}}{M_{p}^{4}}\right),$$

(4.16)

$$M_{NT}^{2} = M_{TN}^{2} = T^{a}N^{b}M_{ab}^{2} = V_{NT} - \frac{\dot{\theta}_{\mu}^{4}}{2M_{p}^{4}H},$$

(4.17)

where $V_{NT} = N^{a}T^{b}V_{ab}$, $V_{NN} = N^{a}N^{b}V_{ab}$, and $V_{TT} = T^{a}T^{b}V_{ab}$. Since we assume a 2D field space here, the Riemann tensor can be written in the terms of the Ricci scalar $\mathbb{R}$ as

$$\mathbb{R}_{abcd} = \frac{1}{2}\mathbb{R}\left(G_{ac}G_{bd} - G_{ad}G_{bc}\right),$$

(4.18)

According to the quadratic action (4.14), it is obvious that $u_{N}$, i.e. the perturbation mode perpendicular to background trajectory is excited while the perturbation mode tangential to background trajectory, $u_{T}$, does not propagate. Particularly, the entropy mode propagates with a non-zero sound speed, although the sound speed for the adiabatic mode is zero. Furthermore, whether the entropy perturbation is free from the gradient as well as ghost instabilities depends on the signature of the metric, i.e. $\text{sgn}(\pm 1)$. The similar result has been reported for the multi-field Mimetic gravity [48]. Since at the bounce scenario

$$\dot{H} = -\frac{XP_{X}}{M_{p}^{2}} > 0 \Rightarrow XP_{X} < 0 \Rightarrow P_{X} < 0,$$

(4.19)

in the case of the field space with an Lornztian signature ($\text{sgn}(-1) = -1$), the entropy mode is healthy, whereas in the case of the Euclidean manifold with $\text{sgn}(+1) = 1$, the entropy perturbation is pathological. Therefore, we find that the Cuscuton bounce does not suffer from any ghost instabilities and the scalar perturbations remain stable throughout the bounce phase. In the next section, we confirm this finding in the comoving gauge as well.

### 4.2 Quadratic action in the comoving gauge

In comoving gauge for the scalar perturbations, $\psi$ is present in (4.2). Thus, the equation of motion for the non-dynamical mode $B$ leads to the following constraint,

$$\alpha = \frac{\dot{\psi}}{H} + \frac{Q_{b}\dot{\phi}_{b}P_{X}}{2M_{p}^{2}H},$$

(4.20)

if we impose the above relation in the corresponding quadratic action, we will have the below quadratic action of the form

$$S^{(2)} = \int d^{4}x \frac{\alpha^{2}}{2} \left( P_{X}D_{i}Q_{b}D_{i}Q_{b} - M_{bc}Q_{b}Q_{c} - \frac{P_{X}D_{i}Q_{b}D_{i}Q_{b}\dot{\phi}_{b}\dot{\phi}_{c}}{2X} - \frac{G_{bc}}{a^{2}}P_{X}\partial_{i}Q_{c}\partial_{i}Q_{b}\dot{\phi}_{b}\dot{\phi}_{c} \right) - \frac{\dot{\psi}}{2H}Q_{b}V_{b} - \frac{3}{2}\psi \left[ D_{i}Q_{b}\dot{\phi}_{b}P_{X} - Q_{b}V_{b} \right] - \frac{P_{X}}{2a^{2}H} \left[ Q_{b}\dot{\phi}_{b}\dot{\phi}_{b}\dot{\phi}_{b} + \frac{X}{H}(\partial\psi)^{2} \right],$$

(4.21)

in which the mass matrix $M_{ab}$ was introduced in eq. (4.8). Now we are ready to decompose the variable $Q^{a}$ into the directions along and orthogonal to time evolution [49] as follow

$$Q^{a} = Q_{\perp}^{a} + \dot{\phi}_{0}^{a},$$

(4.22)
with the orthogonality condition $G_{a b} \dot{\phi}^a_0 Q^b = 0$. We impose $\pi = 0$ in the comoving gauge. It is worth mentioning that, the $\pi$ mode is the fluctuation in the direction of the time translation. Furthermore, the orthogonal modes, $Q^a_\perp$, are gauge invariant quantities and are generally called “isocurvature” modes [47]. The Mukhanov-Sasaki variable can be introduced as [36]

$$\tilde{Q}^a \equiv Q^a - \frac{\dot{\phi}^a_0}{H} \psi = Q^a_\perp - \frac{\dot{\phi}^a_0}{H} (\psi - H\pi) \equiv Q^a_\perp - \frac{\dot{\phi}^a_0}{H} \pi,$$

(4.23)

or equivalently

$$Q^a \equiv Q^a_\perp - \frac{\dot{\phi}^a_0}{H} (\pi - \psi).$$

(4.24)

In the two-field case, with respect to the orthogonality condition, the mode $Q^a_\perp$ is proportional to the normal vector $N^a$, i.e., $Q^a_\perp \propto N^a$, and the amplitude of $Q_\perp$ is the isocurvature field $\tilde{F}$ [36]. On the other hand, one can replace $\pi$ simply with the curvature perturbation $\mathcal{R}$ in comoving gauge ($\tilde{\pi} = 0$) in which $\psi = \mathcal{R}$. Now by plugging eq. (4.24) into the quadratic action (4.21), the final quadratic action reads

$$S^2 = \int d^4 x \frac{a^3}{2} \left[ \text{sgn}(\pm 1) P_X \left( \bar{\mathcal{F}}^2 - \frac{1}{a^2} (\partial \mathcal{F})^2 \right) - M^2 N \mathcal{F}^2 - \frac{2\mu^2 \dot{\theta}}{H} \mathcal{F} \mathcal{R} + \frac{M^2 \dot{H}}{H^2 a^2} (\partial \mathcal{R})^2 \right].$$

(4.25)

Obviously, the curvature perturbation $\mathcal{R}$ does not propagate in this setup. On the other hand, the signature of the metric $G_{a b}$ determines the stability of perturbations. Therefore, the isocurvature mode does not suffer from ghost and gradient instabilities with a Lorentzian signature ($\text{sgn}(-1) = -1$), whereas in the case of the Euclidean manifold with $\text{sgn}(+1) = 1$, the isocurvature perturbation is pathological.

5 Hamiltonian analysis of the multi-field Cuscuton gravity

This section is devoted to the full non-linear Hamiltonian analysis of the system for counting the correct number of degrees of freedom (DOFs). Thus, we use the Arnowitt-Deser-Misner (ADM) decomposition [50] for performing the non-linear Hamiltonian analysis. ADM decomposition is used to characterize the nature of gravity as a constrained system. The metric components of spacetime take the following form in ADM formalism

$$g^0_0 = -N^2 + \beta^i \beta^i, \quad g^0_i = \beta^i, \quad g^i_0 = 1, \quad g^{i j} = \gamma^{i j} - \frac{\beta^i \beta^j}{N^2},$$

(5.1)

where $N$ is the lapse function and $\beta^i$ is the shift vector. The spatial component of metric $\gamma^{i j}$ is defined on the three-dimensional spatial hypersurface embedded in the full spacetime. In the ADM formalism, the action (4.3) can be taken as

$$S = S_G + S_M,$$

(5.2)

where $S_G$ is related to the pure gravity part, i.e.,

$$S_G = \int d^4 x N \sqrt{\gamma} \frac{M^2 \dot{\theta}}{2} \left( R^{(3)} + K_{i j} K^{i j} - K^2 \right),$$

(5.3)
where $R^{(3)}$ is the curvature of three-dimensional spatial hypersurface, associated with $\gamma_{ij}$. In addition, the extrinsic curvature, $K_{ij}$ is given by

$$K_{ij} = \frac{1}{2N} (\partial_t \gamma_{ij} - \beta_{i;j} - \beta_{j;i}), \quad K \equiv K^{ij},$$

(5.4)

with the covariant derivative defined by the spatial metric $\gamma_{ij}$. Additionally, the matter part of the action (4.3) is

$$S_M = \int d^4x \sqrt{\gamma} \left(-\mu^2 \sqrt{2X} - V(\phi^a)\right),$$

(5.5)

where $X$ has the following form in ADM formalism

$$X = \frac{1}{2} \left( G_{ab} \nabla_i \phi^a \nabla_i \phi^b - \gamma^{ij} G_{ab} \partial_i \phi^a \partial_j \phi^b \right),$$

(5.6)

where

$$\nabla_i \phi^a = \frac{1}{N} (\partial_i \phi^a - \beta^i \partial_\nu \phi^a).$$

(5.7)

The total multi-field Cuscuton action in ADM decomposition can be expressed as

$$S = \int d^4x \left[ \Pi^\mu \partial_\mu \gamma_{ij} + \Pi^a \partial_t \phi^a - N\left( H_G + H_M \right) - \beta^i \left( H_{G_i} + H_{M_i} \right) \right].$$

(5.8)

Note that the time derivative of $N$ and $\beta^i$ do not exist in the action. It means that these phase space variables are not dynamical. Therefore the dynamical variables are $\gamma_{ij}$ and $\phi^a$. The momentum conjugate of $\gamma_{ij}$ and $\phi^a$, have the following forms

$$\Pi^a = \frac{\mu^2 \sqrt{\gamma}}{\sqrt{2X}} \nabla_i \phi^a,$$

(5.9)

$$\Pi^{ij} = \frac{\delta S_G}{\delta \partial_t \gamma_{ij}} = \frac{M_P^2}{2} \sqrt{\gamma} (\gamma^{ij} - \gamma_{ij} K),$$

(5.10)

and

$$H_M \equiv \mu^2 \sqrt{\gamma} \sqrt{\left( \frac{\Pi_a \Pi_b}{\gamma \mu^4} + G_{ab} \right)} g^{ij} \partial_i \phi^a \partial_j \phi^b + \sqrt{\gamma} V(\phi^a), \quad H_{M_i} = \Pi_a \partial_i \phi^a,$$

(5.11)

with $\Pi \equiv \Pi_i$. Furthermore, $H_G$ and $H_G^i$ associated to the gravity part. In this situation, we have four primary constraints $(\Pi_N, \Pi_M, \Pi_G, \Pi_M^i) \approx 0$. By regarding such primary constraints and the action (5.8), the total Hamiltonian, from the standard definition in [50, 51], takes the following form

$$H_T = \int d^3x \left[ N(\mathcal{H}_G + \mathcal{H}_M) + \beta^i (\mathcal{H}_{G_i} + \mathcal{H}_{M_i}) + v_N \Pi_N + v^i \Pi_i \right],$$

(5.12)

where $v_N$ and $v^i$ are Lagrange multipliers. Now the consistency of the primary constraints $\Omega_1 \equiv \Pi_N \approx 0$ and $\Gamma^i_1 \equiv \Pi^i \approx 0$ gives the secondary constraints as follows\(^4\)

$$\Omega_2 \equiv \partial_x \Omega_1 = \{ \Omega_1(x), H_T(y) \} = -(\mathcal{H}_G + \mathcal{H}_M) \delta^3(x - y) \approx 0,$$

$$\Gamma^i_2 \equiv \partial_x \Gamma^i_1 = \{ \Gamma^i_1(x), H_T(y) \} = -(\mathcal{H}_{G^i} + \mathcal{H}_{M^i}) \delta^3(x - y) \approx 0.$$

(5.13)

\(^4\)Note that for our case, the Dirac brackets coincide with the Poisson brackets.
We should now investigate the consistency of the secondary constraints

\[ \Omega_3 \equiv \partial_t \Omega_2 = \{ \Omega_2(x), H_T(y) \} = -N' \{ \Omega_2(x), \Omega_2(y) \} - \beta_i \{ \Omega_2(x), \Gamma_2^i(y) \} \approx 0, \]

\[ \Gamma_3 \equiv \partial_t \Gamma_2 = \{ \Gamma_2^i(x), H_T(y) \} = -N' \{ \Gamma_2^i(x), \Omega_2(y) \} - \beta_j \{ \Gamma_2^i(x), \Gamma_2^j(y) \} \approx 0, \]  

(5.14)

where

\[ \{ \Omega_2(x), \Omega_2(y) \} = \Gamma_2^j(y) \partial_x \delta^{(3)}(x - y) - \Gamma_2^j(x) \partial_y \delta^{(3)}(x - y) \approx 0, \]

\[ \{ \Omega_2(x), \Gamma_2^i(y) \} = -\Omega_2 \partial_x \delta^{(3)}(x - y) \approx 0, \]

\[ \{ \Gamma_2^i(x), \Gamma_2^j(y) \} = \Gamma_2^j(y) \partial_x \delta^{(3)}(x - y) - \Gamma_2^j(x) \partial_y \delta^{(3)}(x - y) \approx 0. \]  

(5.15)

It is obvious that the above expressions vanish on the constraint surface. In fact, consistency of the secondary constraints \( \Omega_2 \) and \( \Gamma_2^i \) determines none of the Lagrangian multipliers and do not generate any additional constraints. Moreover, these eight constraints \( \Omega_1, \Omega_2, \Gamma_1^i, \) and \( \Gamma_2^i \) are all first class constraints, which are represented as the generators of diffeomorphism. In summary, in this model, there are twenty phase space variables containing \( \{ N, \beta^i, \gamma_{ij}, \Pi_N, \Pi^i, \Pi_{ij} \} \) and \( 2M \) total number of the conjugate pair \( \{ \phi_a, \Pi^a \} \). Therefore, the number of DOFs \( [51] \) is

\[ \text{DOF} = (20 + 2M) - 16 = 4 + 2M, \]  

(5.16)

which corresponds to \((4 + 2M)/2\) physical degrees of freedom in the configuration space. Here \( M \) indicates the dimension of the field space manifold or the number of scalar fields. Therefore, we have \( M \) extra physical degree of freedom in addition to the two gravitational degrees of freedom of general relativity. This result implies that the theory does not have the so-called Ostrogradsky ghost \([52]\). In appendix A we investigate a Hamiltonian analysis of the multi-field cuscuton model in the homogeneous limit where \( \partial_t \phi^a = 0 \). In single field Cuscuton case, in the homogeneous limit \( \partial_t \phi = 0 \) the extra degree of freedom (Cuscuton scalar field) is non-dynamical leading to a theory of gravity with just two tensor degrees of freedom \([40]\).

6 Summary and conclusions

In this work, we first extended the idea of Cuscuton gravity to multi-field setup with the curved field space manifold. Then, we found a bounce solution that has no pathologies associated the violation of NEC. More precisely, at the background level, we looked for some suitable conditions on the potential function in such a way that the bounce in the contracting or expanding phase can exist in our scenario. After finding such a condition, we used the cosmological perturbation theory to examine the existence of ghosts and other instabilities in our model.

At the level of perturbations, we have used the kinematic basis in which the perturbations are decomposed into the tangential and the perpendicular to the field space trajectory. In this respect, we found that the perturbation mode tangential to background trajectory in the field space manifold, namely the adiabatic mode does not have its own dynamics at both background and perturbation levels. Nonetheless, the entropy mode perpendicular to background trajectory, originated from the extra scalar field in our model, propagates with a non-zero sound speed. Furthermore, we proved that whether or not the entropy perturbation is pathological directly depends on the signature of the field-space metric. To be more explicit, because the would- be-wrong-sign kinetic term of the Cuscuton is not a true kinetic term
owing to the degenerate nature of the phase space, our scenario is still stable. In summary, despite violating NEC, our model provides a healthy regular bounce.

In addition, we confirmed our findings by performing the full non-linear Hamiltonian analysis of the multi-field Cuscuton theory and calculated the correct number of DOFs necessary to avoid the Ostrogradsky-type ghost.

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A Is multi-field Cuscuton gravity casual?

As mentioned, the superluminal propagation does not necessarily lead to a breakdown of causality [43]. To make it clear, following the approach proposed in ref. [8], we attempt to examine whether or not this theory is causal? Due to the underlying Lorentz symmetry, we can always go to a frame in which locally

\[ D_a \Pi^a = 0 \]

Consequently, there is no local dynamics in our model. To realize the collapse of the phase space structure, let us change the phase space from \( D_a \Pi^a \) to \( D_a \dot{\phi}^a \) by the following Jacobian matrix.

\[
\begin{vmatrix}
\frac{\partial(\phi^a(x)\Pi^a(x'))}{\partial(\phi^b(y)\Pi^b(y'))} \\
\delta^a_b \\
\frac{X-1}{2} \gamma_{ij} G_{ab} \partial_i \phi^a \partial_j \phi^b \\
\frac{\mu^2}{X^{\frac{3}{2}}} \gamma_{ij} \gamma^{ij} \Pi^a \Pi^b + \sqrt{\gamma} V(\phi^a) \\
\end{vmatrix}
\]

Because the Lorentz symmetry permits us to locally rotate \( \partial_i \phi^b(x) = 0 \) (the homogeneous limit) for vectors \( \partial_\mu \phi^a \) such that \( X = -G_{ab} \partial_\mu \phi^a \partial_\mu \phi^b / 2 > 0 \), the symplectic structure of the phase space collapses without carrying any local dynamical degrees of freedom. It means that local perturbations do not carry any microscopic information. Therefore, the causality can not be violated even in the presence of the superluminal sound speed.

Let us now examine the above result through performing a Hamiltonian analysis of the multi-field Cuscuton model in the homogeneous limit where \( \partial_i \phi^a = 0 \). With respect to the action (4.3), the multi-field Cuscuton action in ADM decomposition can be expressed as

\[
S = \int d^4x \left[ \Pi^i \partial_i \gamma_{ij} + \Pi_\mu \partial_\mu \phi^a - N(H_G + H_M) - \beta^i (H_Gi + H_Mi) - \lambda_a \partial_i \phi^a \right]
\]

Here we have imposed the gauge fixing term \( \partial_i \phi^a \) to the canonical Hamiltonian with a Lagrange multiplier \( \lambda_a \). The other Hamiltonian functions are given by

\[
H_M = \mu^2 \sqrt{\gamma} \left( \frac{\Pi_a \Pi_b}{\gamma \mu^4} + G_{ab} \right) g^{ij} \partial_i \phi^a \partial_j \phi^b + \sqrt{\gamma} V(\phi^a), \quad H_Mi = \Pi_\mu \partial_\mu \phi^a
\]

---

5The set up we used here is practically similar to the one applied to type-II minimally modified gravity theories in [18].
where $\Pi \equiv \Pi^i$ and $\Pi^a$, $\mathcal{H}_G$, $\mathcal{H}'_G$ introduced in section 5. Now, we have $4 + 3M$ primary constraints $(P_N, P_{i\beta}, \partial_i \phi^a) \approx 0$. Thus, the total Hamiltonian function from the standard definition in [53] is as follows,

$$H_T = \mathcal{N}(\mathcal{H}_M + \mathcal{H}_G) + \beta^i(\mathcal{H}_Mi + \mathcal{H}_G\iota) + \lambda^i_a \partial_i \phi^a + u_N P_N + u_{i\beta} P_{i\beta}, \quad (A.4)$$

where $u_N$, $\lambda^i_a$, and $u_{i\beta}$ are Lagrange multipliers which enforce the primary constraints.

To identify the secondary constraints, we should check the consistency of the primary constraints using the Poisson brackets [51]. Thus, the time evolution of the primary constraints are obtained to be

$$\Upsilon^i \equiv \partial_t P_N = -\mathcal{H}_M - \mathcal{H}_G, \quad (A.5)$$

$$\Xi^i \equiv \partial_t P^i_{\beta} = -\mathcal{H}^i_M - \mathcal{H}^i_G, \quad (A.6)$$

$$\Theta^a_i \equiv \partial_t (\partial_i \phi^a) = M C^a \partial_i \delta(x - y) = -(C^a \partial_i \mathcal{N} + \mathcal{N} \partial_i C^a) \delta(x - y) \approx 0, \quad (A.7)$$

where

$$C^a \equiv \left\{ \partial_i \phi^a, \mu^2 \sqrt{\gamma} \left( \frac{\Pi_{\mu} \Pi_{\nu} G_{\mu \nu}}{\gamma \mu^4} + G_{cd} \right) g^{kl} \partial_k \phi^c \partial_l \phi^d + V(\phi^c) \right\} = \mu^{-2}\gamma^{-1/2} \left( \frac{\Pi_{\mu} \Pi_{\nu} G_{\mu \nu}}{\gamma \mu^4} + G_{cd} \right) g^{kl} \partial_k \phi^c \partial_l \phi^d. \quad (A.8)$$

Now, we should consider consistency of secondary constraints. Hence, $P^i_{\beta}$ and $\Xi^i$ are first class constraints which imply that the model is invariant under spatial diffeomorphism. The consistency of the secondary constraint $\Theta^a_i$ gives

$$\hat{\Theta}^a_i \equiv \partial_t \Theta^a_i = -\{\Theta^a_i, H_T\} = \Theta^a_i (u_N, \gamma_{ij}, \Pi^{ij}, \Pi^a, \phi_a) \delta(x - y) \approx 0, \quad (A.9)$$

which leads to determine the Lagrangian multiplier $u_N$. In addition, $\mathcal{N}$ is given through phase space variables from eq. (A.7). Furthermore, the constraint $\Upsilon$ satisfies the below consistency relation.

$$\vartheta \equiv \Upsilon = -(C^a \partial_i \lambda^i_a + \lambda^i_a \partial_i C^a) \delta(x - y) \approx 0, \quad (A.10)$$

which helps us to find $\lambda^i_a$. As a consequence of the above consistency relations, $P_N$, $\Upsilon$, $\partial_i \phi^a$ and $\Theta^a_i$ are all second class constraints. Thus, let us now substitute these constraints strongly, i.e., $P_N = 0$ and $\partial_i \phi^a = 0$ into the total Hamiltonian (A.4). Additionally, the $\mathcal{N}$ function can be given by making use of eq. (A.7) and then it can be substituted into the total Hamiltonian. In this respect, the new momentum canonically conjugate to $\phi^a$ is obtained form the corresponding reduced Hamiltonian as follows.

$$\tilde{\Pi}^a = \mu^2 \sqrt{\gamma} (\phi^c \phi^d) - \partial_i \phi^a \tilde{\phi}^a. \quad (A.11)$$

Note that $\tilde{\Pi}_a$ is defined to be $\tilde{\Pi}_a = G_{ab} \tilde{\Pi}^b$. Therefore, one can derive a new primary constraint $\mathcal{A}$ as

$$\mathcal{A} \equiv \tilde{\Pi}^a \tilde{\Pi}_a - \mu^4 \gamma \approx 0. \quad (A.12)$$

Consequently, the reduced total Hamiltonian becomes

$$\mathcal{H}_{\mathcal{T}}^R = \mathcal{N}(\mathcal{H}_G + V(\phi^a)) + \beta^i \mathcal{H}_G i + \tau \mathcal{A} + u_{i\beta} P_{i\beta}, \quad (A.13)$$
As a result of the above discussion, the total number of DOFs is which implies the system has two physical degrees of freedom in the configuration space. We are now ready to count degrees of freedom. One can find that the total number of phase space variables is Ξ, which was previously given by eq. (A.7). Allow us now to consider consistency of primary constraints $P_{i\beta}$ and $\mathcal{A}$ which yield

$$\Xi^{iR} \equiv \partial_i P_{i\beta}^{R} = -\mathcal{H}_R \approx 0, \quad (A.14)$$

$$\Theta^R \equiv \partial_i \mathcal{A} = \{\mathcal{A}, \mathcal{N}(\mathcal{H}_G + V(\phi^a))\} = \mathcal{D} \approx 0. \quad (A.15)$$

Clearly, $\Xi^{iR}$ and $P_{i\beta}^{R}$ are first class constraints. Moreover, $\mathcal{D}$ is a secondary constraint which involves phase space variables. As shown in the below identity, the Lagrange multiplier $\tau$ is obtained from the time evolution of the secondary constraint $\mathcal{D}$

$$\partial_i \mathcal{D} = \{\mathcal{D}, \mathcal{N}(\mathcal{H}_G + V(\phi^a))\} + \tau \{\mathcal{D}, \mathcal{A}\} \approx 0, \quad (A.16)$$

therefore, $\Theta^R$ and $\mathcal{A}$ are both second class constraints. With the completion of these steps, we are now ready to count degrees of freedom. One can find that the total number of phase space variables is $N = 20 + 2M$ and the constraints $\Theta^R, \mathcal{A}, P_N, \mathcal{V}, \partial_i \phi^a, \Theta^a_i$ are all second class constraints. About the Lagrange multiplier $\lambda^i_a$, we count only $M$ not $3M$ degrees of freedom, because using integral by parts technique, $\lambda^i_a \partial_i \phi^a$ can be shown to be proportional to $\lambda^i_a \nabla^2 \phi^a$. Note that, without loss of generality, we have decomposed $\lambda^i_a$ to scalar and vector parts, i.e. $\lambda^i_a = \partial^i \lambda^a_S + \lambda^A_i V$ during this calculation. In addition, the divergenceless condition $\partial_i \lambda^i_a = 0$ was assumed [18].

Furthermore, we take into account $\partial_i \phi^a$ and $\Theta^a_i$ as $2M$ second class constraints. The constraint $\Xi^{iR}$ and $P_{i\beta}^{R}$ are also six first class constraints generating spatial diffeomorphism. As a result of the above discussion, the total number of DOFs is

$$\text{DOF} = (20 + 2M) - 12 - 2 - 2 - 2M = 4, \quad (A.17)$$

which implies the system has two physical degrees of freedom in the configuration space. Namely, this finding confirms that the multi-field Cuscuton theory in the homogeneous limit adds no additional degrees of freedom to a gravitational system.

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