Integrable Models of 1+1 Dimensional Dilaton Gravity Coupled to Scalar Matter

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August 6, 2018

Abstract

A class of explicitly integrable models of 1+1 dimensional dilaton gravity coupled to scalar fields is described in some detail. The equations of motion of these models reduce to systems of the Liouville equations endowed with energy and momentum constraints. The general solution of the equations and constraints in terms of chiral moduli fields is explicitly constructed and some extensions of the basic integrable model are briefly discussed. These models may be related to high dimensional supergravity theories but here they are mostly considered independently of such interpretations. A brief review of other integrable models of two-dimensional dilaton gravity is also given.

1 Introduction

It is well known that 1+1 dimensional dilaton gravity coupled to scalar matter fields is a reliable model for some aspects of high dimensional black holes, cosmological models and branes. The connection between high and low dimensions has been demonstrated in different contexts of gravity and string theory and in some cases allowed to find general solution or some special classes of solutions in high dimensional theories\(^1\). For example, spherically symmetric gravity coupled to Abelian gauge fields and massless scalar matter fields exactly reduces to a 1+1 dimensional dilaton gravity and can be explicitly solved if the scalar fields are constants independent of coordinates. Such solutions may describe some interesting physical objects – spherical static black holes, simplest cosmologies, etc. However, when the scalar matter fields, which presumably play a significant cosmological role, are not constant, few exact analytical solutions of high dimensional theories are known. Correspondingly, the two-dimensional models of dilaton gravity nontrivially coupled to scalar matter are usually not integrable.

To obtain integrable models of this sort one has to make some serious approximations, in other words, to deform the original two-dimensional model obtained by a direct dimensional reductions of realistic higher dimensional theories\(^2\). Nevertheless, these models may qualita-

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\(^1\)For a more detailed discussion of this connection and references see e.g. [1], [2] and next section

\(^2\)Note that some important four-dimensional space-times having symmetries defined by two commuting Killing vectors may also be described by two-dimensional dilaton gravity. For example, the cylindrical gravitational waves are described by the 1+1 dimensional dilaton gravity coupled to one scalar field. The stationary axially symmetric pure gravity may be described by a 0+2 dimensional dilaton gravity coupled to one scalar field (this may be related to the previous cylindrical case by the analytic continuation of one space variable to imaginary values). Similar but more general dilaton gravity models were also obtained in string theory. Some of them may be solved by using modern mathematical methods developed in the soliton theory.
tively describe some special physically interesting solutions of higher dimensional gravity or supergravity theories related to the low energy limit of superstring theories.

When preparing and writing this report I often recalled very stimulating talks with Misha Saveliev about the Liouville and other integrable theories. I cannot express my gratitude to him in this world but I wish to mention that he strongly influenced my interest and my approach to matters discussed below.

2 Some integrable models of 1+1 dimensional dilaton gravity coupled to scalar matter

The effective Lagrangian of the 1+1 dimensional dilaton gravity coupled to scalar fields that may be obtained by dimensional reductions of a higher dimensional spherically symmetric (super)gravity can usually be (locally) transformed to the following form

\[ \mathcal{L}^{(2)} = \sqrt{-g} \left[ \varphi R(g) + V(\varphi, \psi) + \sum_n Z_n(\varphi, \psi) g^{ij} \partial_i \psi_n \partial_j \psi_n \right]. \] (1)

Here \( g_{ij}(x^0, x^1) \) is a generic 1+1 metric with signature (-1,1), \( g = \det|g_{ij}| \) and \( R \) is the Ricci curvature of the two-dimensional space-time,

\[ ds^2 = g_{ij} dx^i dx^j, \quad (i, j = 0, 1). \] (2)

The effective potentials \( V \) and \( Z_n \) depend on the dilaton \( \varphi(x^0, x^1) \) and on \( N-2 \) scalar fields \( \psi_n(x^0, x^1) \) (note that \( Z_n < 0 \)). They may depend on other parameters characterizing the parent higher dimensional theory (e.g. on charges introduced by solving the equations for the Abelian fields). Here we mainly consider the simplest kinetic terms, with \( Z_n(\varphi, \psi) = Z_n(\varphi) \), or even with constant \( Z_n \) that are independent of the fields. We also used in (1) a Weyl transformation to exclude the gradient term for the dilaton.

To simplify derivations we will use the equations of motion in the light-cone metric, \( ds^2 = -4f(u, v) du dv \). By first varying this Lagrangian in generic coordinates and then going to the light-cone ones we obtain the equations of motion

\[ \partial_u \partial_v \varphi + f V(\varphi, \psi) = 0, \] (3)

\[ f \partial_i(\partial_i \varphi/f) = \sum_n Z_n(\partial_i \psi_n)^2, \quad (i = u, v). \] (4)

\[ \partial_v(Z_n \partial_u \psi_n) + \partial_u(Z_n \partial_v \psi_n) + f V_{\psi_n}(\varphi, \psi) = \sum_m Z_{m, \psi_n} \partial_u \psi_m \partial_v \psi_m, \] (5)

\[ \partial_u \partial_v \ln |f| + f V_{\psi}(\varphi, \psi) = \sum_n Z_{n, \varphi} \partial_u \psi_n \partial_v \psi_n, \] (6)

where \( V_{\varphi} = \partial_{\varphi} V, V_{\psi_n} = \partial_{\psi_n} V, Z_{n, \varphi} = \partial_{\varphi} Z_n, \) and \( Z_{m, \psi_n} = \partial_{\psi_n} Z_m. \) These equations are not independent. Actually, (6) follows from (3) – (5). Alternatively, if (3), (4), (6) are satisfied, one of the equations (5) is also satisfied.

Consider the solution with constant scalar field \( \psi = \psi_0 \) (the ‘scalar vacuum’ solution). This solution exists if \( V(\varphi, \psi_0) = 0 \), see eq. (5). The constraints (4) can now be solved because their right-hand sides are identically zero. It is a simple exercise to prove that there

\[ \text{[For a detailed motivation and specific examples see \[1\], where references to other related papers may be found. Due to space limitations, only absolutely necessary references are given here.]} \]
exist chiral fields \(a(u)\) and \(b(v)\) such that \(\varphi(u, v) \equiv \varphi(\tau)\) and \(f(u, v) \equiv \varphi'(\tau) a'(u) b'(v)\), where \(\tau \equiv a(u) + b(v)\) (the primes denote derivatives with respect to the corresponding argument). Using this result it is easy to prove that \(\phi\) has the integral \(\varphi' + N(\varphi) = M\), where \(N(\varphi)\) is defined by the equation \(N'(\varphi) = V(\varphi, \psi)\) and \(M\) is the constant (integral) of motion. The horizon, defined as a zero of the metric \(h(\tau) \equiv M - N(\varphi)\), exists because the equation \(M = N(\varphi)\) has at least one solution in some interval of values of \(M\). These solutions are actually one - dimensional (‘automatically’ dimensionally reduced) and can be interpreted as black holes (Schwarzschild, Reissner Nordstrøm, etc.) or as cosmological models. Actually, the scalar vacuum theory is equivalent to the pure dilaton gravity\(^4\), which is a topological theory, and this explains this drastic simplification of the equations of motion. These facts are known for long time and were derived by many authors using different approaches (for references see e.g the recent review \([2]\)).

The properties of the general dilaton gravity theories \([1]\) are much more complex. They do not reduce to one - dimensional models and are in general not integrable. If \(Z_n \neq 0\) the theory may be integrable only with very special potentials \(V(\varphi, \psi)\) and \(Z_n(\varphi, \psi)\). Roughly speaking, the two - dimensional models may be integrable if either the potentials \(Z_n\) can be transformed to constants or the potential \(V\) is zero. In the first case, \(V\) should have very special form that is described below. In the second case, for the theory to be explicitly analytically integrable\(^5\) the potentials \(Z_n\) must be very special functions.

In search for integrable 1+1 dimensional models it may be useful to also consider ‘mixed’ cases. Roughly speaking, this means the following. Suppose we have three different species of scalar matter fields \(- \psi_n, \chi_k, \text{and} \sigma_s\). The characteristic properties of these fields are: the potential \(V\) depends on \(\varphi\) and \(\psi_n\); \(Z_n\) and \(Z_k\) are constants, say, \(Z_n = Z_k = -1\); \(Z_s = Z_s(\varphi)\) do not depend on \(\sigma\) and \(\psi\). Thus the \(Z\)-part of the Lagrangian \([1]\) is

\[
\sqrt{-g} g^{ij} \left[ \sum_n Z_n \partial_i \psi_n \partial_j \psi_n + \sum_k Z_k \partial_i \chi_k \partial_j \chi_k + \sum_s Z_s(\varphi) \partial_i \sigma_s \partial_j \sigma_s \right] \tag{7}
\]

Looking at the equations \([3]\) - \([6]\) we see that \(\chi_k\) are free massless fields (i.e. \(\partial_k \partial_i \chi_k = 0\)), the equations \([3]\), \([5]\) for \(\varphi\), and \(\psi_n\) are independent of the other fields, and, finally, the equations for \(\sigma_s\) are linear. The right - hand sides of the constraints \([4]\),

\[
\sum_n Z_n (\partial_i \psi_n)^2 + \sum_k Z_k (\partial_i \chi_k)^2 + \sum_s Z_s(\varphi) (\partial_i \sigma_s)^2, \quad (i = u, v), \tag{8}
\]

depend on all three species of the matter fields, so that the fields are not decoupled (this is natural because some linear combinations of the constraints give the total energy and momentum of the system that are constrained to zero). The most general system of this sort may be explicitly integrable if its nonlinear sector is explicitly integrable and the constraints can be explicitly solved with arbitrary solutions for \(\sigma\) and \(\chi\) fields. Unfortunately, in such a general setting the constraint cannot be solved. As is shown in the next section, when we have only fields \(f, \varphi\) and \(\psi\) (by taking constant \(\sigma\) fields), each constraint involves only functions of one variable \((u\ or\ v, \text{resp.})\). However, this condition cannot be satisfied for

\(^4\)If \(V = V(\varphi)\) and \(Z_n \equiv 0\) the theory \([1]\) is called the pure dilaton gravity. It is integrable with arbitrary potential \(V(\varphi)\).

\(^5\)We call the theory explicitly integrable if one can explicitly write its general solution analytically, in terms of a sufficient number of arbitrary functions, in our case, of chiral free fields and of their derivatives. For further use in physics problems we even need to have simple enough expressions for the general solution.
the term $Z_s(\varphi)(\partial_t \sigma_s)^2$ in the integrable models discussed here. For this reason, mainly the models with the matter fields $\psi$ and $\chi$ will be considered here. Nevertheless, it is useful to keep $\sigma$-fields in mind for two reasons. First, in some cases these terms may be taken into account as a perturbation. The second reason is the following. In further reductions to dimensions $0+1$ or $1+0$ the $\sigma$ fields may be exactly solved giving additional terms to the potential $V$. The resulting theory may be integrable or approximately integrable. We do not consider such possibilities here and concentrate on the theories with the fields $\psi$ and $\chi$.

Let us first discuss the second ‘pure’ case (including $\chi$-fields is trivial and we omit it). So, suppose that $V \equiv 0$, $\psi \equiv \chi \equiv 0$, and $Z_s = Z(\varphi)$. Then, from the first equation we have (up to the coordinate transformation $u \mapsto a(u)$, $b \mapsto b(v)$): $\varphi = u + v \equiv r$ and the scalar fields satisfy the linear equation ($t \equiv u - v$)

$$(\partial_t^2 - \partial_r^2)\sigma(r, t) + \frac{Z'(r)}{Z(r)} \partial_t \sigma(r, t) = 0. \tag{9}$$

When $Z(\varphi) = -\varphi$ this is the Euler – Darboux equation, for the general solution of which one can write a rather complex integral representation (see e.g. [3]). Moreover, the expression for the metric is very difficult to analyze. In general this does not allow us to find an explicit analytic expression for the metric. Thus, such theories may be called integrable but I would say that they are not explicitly integrable. However, if $Z_n(\varphi)$ are solutions of the one-dimensional Liouville equation, the potentials produced by such $Z_n(r)$, are ‘non reflecting’, the equations for $\sigma$ are explicitly integrable and are simple functions of the chiral free massless fields $a(u)$ and $b(v)$ (see [3]). The metric is then explicitly expressed through these chiral fields and thus the theory is explicitly (and elementary) integrable. These solutions may describe plane waves of scalar matter coupled to gravity or certain inhomogeneous cosmologies.

The qualitative difference between the solutions of [2] for ‘non reflecting’ potentials and the ‘realistic’ ones (corresponding to $Z = -\varphi$) can be made more clear if we use the following beautiful, compact solutions of eq.(3) with $Z = -\varphi^{2\lambda+1}$ ($\lambda > -\frac{1}{2}$). The first solution was essentially found by S.D.Poisson,

$$\sigma(r, t) = \int_0^\pi d\alpha \ q(t - r \cos \alpha) \sin^{2\lambda} \alpha , \tag{10}$$

the second one was derived later by E.W.Hobson\footnote{Known from 19-th century, these solutions and their generalizations may be found in the reprint of a still useful old book [5].},

$$\sigma(r, t) = \int_0^\infty d\alpha \ q(t \pm r \cosh \alpha) \sinh^{2\lambda} \alpha , \tag{11}$$

where $q(x)$ is an arbitrary ‘generating function’. Now, the potential corresponding to $Z = -\varphi^{2\lambda+1}$ becomes ‘non reflecting’ if $\lambda = l + \frac{1}{2}$, with integer $l$. Indeed, then (14) is equivalent to

$$(\partial_r^2 - \partial_t^2)\sigma(r, t) - \frac{l(l+1)}{r^2} \sigma(r, t) = 0 , \tag{12}$$

where $\tilde{\sigma} \equiv r^{l+1} \sigma$. The potential $l(l+1)/r^2$ is a special case of the ‘non reflecting potentials’ considered in [4], with which the solution of this wave equation can be explicitly written as a linear combination of a free massless field $\chi = a(t+r) + b(t-r)$ and of its derivatives (with coefficients depending on $r$). One can easily prove that the solutions (10) and (11) reduce to
such local fields when $\lambda = l + \frac{1}{2}$ with integer $l$. If $l$ is non integer the solution is a continual superposition of waves with different ‘velocities’ $\sec \alpha$ or $\cosh^{-1} \alpha$. In this sense, the solution is generally nonlocal and, in fact, not simpler than one used in [3]. It may be instructive to rewrite Eq.(10), with half - integer $\lambda$, in the form

$$\sigma(r, t) = \frac{1}{r} \int_{t-r}^{t+r} dx q(x) \left[(x - (t - r))(t + r - x)\right]^l. \quad (13)$$

If $l = 0$ we find that

$$\sigma(r, t) = \frac{1}{r} \left[Q(t + r) - Q(t - r)\right],$$

where $Q'(x) \equiv q(x)$. Similarly, one can find the local expressions of $\sigma$ in terms of massless free fields for any integer $l$.

Originally, the equations (10) and (11) were applied to description of simple cylindrical waves (see e.g. [5]). More recently, it has been shown that they can be generalized to describe rather complex cylindrical gravitational waves (see e.g. [6] in which a generalization of these classical solution in the context of the Penrose twistor program is also discussed). In fact, the simplest cylindrical gravitational waves derived by A.Einstein and N.Rosen (see [7]) may be found by solving just the equation (9). To find the most general cylindrical gravitational waves one has to solve the following nonlinear equation

$$\partial_t (rM^{-1}\partial_t M) - \partial_r (rM^{-1}\partial_r M) = 0, \quad (14)$$

where $M$ is a $2 \times 2$ symmetric matrix depending on the cylindrical metric. This equation may be solved by a generalization of the Poisson formula [6] or by more general methods developed in the theory of solitons.

In fact, the equation (14) describing cylindrical gravitational waves is a special case of the field equations of $\sigma$-models coupled to gravity. This equation and its generalizations may be attempted to solve by the inverse scattering methods (ISM) of the soliton theory. Applications of these methods started with papers [8] and [9] which consider integrability of the stationary axial gravity. Further results and references to numerous papers dealing with this subject may be found in [10] and [11]. The methods for solving the integrable two - dimensional theories reducible to solving equations like (14) are not elementary and, for this reason, are not included in our classification of the matter terms (15) that may lead to elementary solvable two - dimensional dilaton gravity. In our notation, the models that may be integrated by ISM have $V = 0$, $Z_n = Z_k = 0$ but, in general, more complex structure of the $\sigma$-terms:

$$\sum_{\alpha,\beta} Z_{\alpha\beta}(\varphi; \sigma) \partial_\alpha \sigma_\alpha \partial_\beta \sigma_\beta \quad (15)$$

In simple cases one may transform these terms to the diagonal form similar to (15) but with the potentials $Z$ depending on $\sigma$-fields. For example, the Lagrangian (2.8) of the two - dimensional dilaton gravity derived in [12] for the stationary, axially symmetric Einstein gravity can easily be transformed to the form of our ‘general’ dilaton gravity [1] but this does not help to solve the equations of motion by elementary methods. On the other hand, we will see that there exist reasonably wide class of physically interesting, integrable two - dimensional theories with simple (constant!) $Z_n$ but rather complex potentials $V$.

The best studied examples of such integrable models (belonging to the first ‘pure’ class, with constant $Z_n$) are Callan - Giddings - Harvey - Strominger (CGHS) model ($V = g_0$)
and the Jackiw-Teitelboim (JT) model \((V = g_1 \varphi)\). In CGHS model \(R = 0\) and in JT model \(R = -g_1\). A strong generalization of these models, in which the two-dimensional curvature is not conformal constant \((V = g_1 e^{q_\varphi} + g_1 e^{-q_\varphi})\) was proposed by the present author in [13]. Like CGHS and JT models, this model belongs to the ‘mixed’ case having free fields \(\chi_k\). It was called the 2-Liouville model because the fields \(\psi_\pm \equiv \ln f \pm \varphi\) satisfy two Liouville equations. The Liouville equations are, of course, explicitly solved in terms of two pairs of the chiral free massless fields \(a_\pm(u), b_\pm(v)\). By using the constraints (4) one of the free scalar fields \(\chi\) can be expressed in terms of \(a, b\) and of other \(\chi\)-fields. So one may be inclined to decide that the two-Liouville model is explicitly solved in [13]. However, there remained one ‘small’ problem – the right-hand sides of the constraints are negative definite while the left-hand sides may change sign. In [13] I did not find an explicit analytic expression for the chiral fields \(a\) and \(b\) that fully satisfy the constraints. The explicit solution of the constraints (4) was constructed only five years later when I have found a much more general N-Liouville model [14], [1]. Here I will show how this class of models may be constructed and give its complete solution in terms of chiral moduli fields.

### 3 N-Liouville 1+1 dimensional model and its solution

Let us suppose that the theory is defined by the Lagrangian,

\[
\mathcal{L}^{(2)} = \sqrt{-g} \left[ \varphi R(g) + V(\varphi, \psi) + g^{ij} \left( \sum_n Z_n \partial_i \psi_n \partial_j \psi_n + \sum_k Z_k \partial_i \chi_k \partial_j \chi_k \right) \right],
\]

with the following potentials:

\[
|f| V = \sum_{n=1}^N 2g_n e^{q_n}, \quad Z_n = Z_k = -1.
\]

Here \(f\) is the light-cone metric, \(ds^2 = -4f(u, v)\, du\, dv\), and

\[
q_n \equiv F + a_n \varphi + \sum_{m=3}^N \psi_m a_{mn} \equiv \sum_{m=1}^N \psi_m a_{mn},
\]

where \(\psi_1 + \psi_2 \equiv \ln |f| \equiv F \left( f \equiv e^{\varepsilon F}, \varepsilon = \pm 1 \right), \psi_1 - \psi_2 \equiv \varphi\) and thus \(a_{1n} = 1 + a_n\), \(a_{2n} = 1 - a_n\). By varying the Lagrangian (16) in \(N - 2\) scalar fields, dilaton, and in \(g_{ij}\) and then passing to the light-cone metric we find \(N\) equations of motion for \(N\) functions \(\psi_n\),

\[
\epsilon_n \partial_u \partial_v \psi_n = \sum_{m=1}^N \varepsilon g_m e^{q_n} a_{mn}; \quad \epsilon_1 = -1, \quad \epsilon_n = +1, \text{ if } n \geq 2,
\]

as well as two constraints,

\[
C_i \equiv f \partial_i (\partial_i \varphi / f) + \sum_{n=3}^N (\partial_i \psi_n)^2 = 4D_i(i), \quad i = (u, v),
\]

where \(4D_i(i) \equiv -\sum (\partial_i \chi_k)^2\) is the contribution of the free fields \(\chi\) (see eq.(8)) that depends on one variable because \(\chi_k = \alpha_k(u) + \beta_k(v)\) (here and below \(k = 1, \ldots, K\)).
With arbitrary coefficients \( a_{mn} \) these equations of motion are not integrable. However, as proposed in [13], the equations (13) are integrable and the constraints (20) can be solved if the \( N \)-component vectors \( v_n \equiv (a_{mn}) \) are pseudo - orthogonal. Then the equations (19) reduce to \( N \) independent, explicitly integrable Liouville equations for \( q_n \),

\[
\partial_u \partial_v q_n - \tilde{g}_n e^{q_n} = 0,
\]

where \( \tilde{g}_n = \varepsilon \lambda_n g_n, \lambda_n = \sum \epsilon_m a_{nm}^2, \) and \( \varepsilon \equiv f/|f| \) (note that the equations for \( q_n \) depend on \( \epsilon_n \) and \( a_{mn} \) only implicitly, through the normalization factor \( \lambda_n \))\(^7\).

The expression for the original fields in terms of the Liouville fields \( q_n \) may be found by using the orthogonality relations for \( a_{mn} (m \neq n) \) and the definition of \( \lambda_n \equiv \gamma_n^{-1} \) (for \( m = n \)) combined in the equation:

\[
\sum_{l=1}^{N} \epsilon_l a_{lm} a_{ln} = \lambda_n \delta_{mn} \equiv \gamma_n^{-1} \delta_{mn}.
\]

This equation may be written in the matrix form, if we define the matrices \( a = (a_{mn}) \), \( \gamma = (\gamma_m \delta_{mn}) \) and \( \epsilon = (\epsilon_m \delta_{mn}) \):

\[
a^T \epsilon a = \gamma^{-1}, \quad a \gamma a^T = \epsilon.
\]

It is important to note that in addition to these relation the matrix \( a \) satisfies two conditions \( a_{1n} = 1 + a_n, a_{2n} = 1 - a_n \).

Now, using (22) or (23) we can invert the definition (18) and get

\[
\psi_m = \epsilon_m \sum_{n=1}^{N} a_{mn} \gamma_n q_n, \quad F = -2 \sum_{n=1}^{N} \gamma_n a_n q_n, \quad \varphi = -2 \sum_{n=1}^{N} \gamma_n q_n.
\]

Writing the first equation for \( n = 1, 2 \), expressing \( a_{1n}, a_{2n} \) in terms of \( a_n \) and using linear independence of the functions \( \psi_n \) we may find very useful identities (‘sum rules’) for \( \gamma_n, a_n \) and \( a_{mn} \) for \( m \geq 3 \). We have

\[
\psi_1 \equiv -\psi_1 \sum_{n=1}^{N} \gamma_n (1 + a_n)^2 - \psi_2 \sum_{n=1}^{N} \gamma_n (1 - a_n^2) - \sum_{m=3}^{N} \psi_m \sum_{n=1}^{N} a_{mn} (1 + a_n) \gamma_n,
\]

\[
\psi_2 \equiv \psi_1 \sum_{n=1}^{N} \gamma_n (1 - a_n^2) + \psi_2 \sum_{n=1}^{N} \gamma_n (1 - a_n)^2 + \sum_{m=3}^{N} \psi_m \sum_{n=1}^{N} a_{mn} (1 - a_n) \gamma_n.
\]

Thus we immediately find that the following identities should be satisfied

\[
\sum_{n=1}^{N} \gamma_n = 0; \quad \sum_{n=1}^{N} \gamma_n a_n = -\frac{1}{2}; \quad \sum_{n=1}^{N} \gamma_n a_n^2 = 0; \quad \sum_{n=1}^{N} a_{mn} \gamma_n = \sum_{n=1}^{N} a_{mn} a_n \gamma_n = 0, m \geq 3.
\]

Also, using the orthogonality relations (22) it can be proven that one and only one norm \( \gamma_n \) is negative. We thus choose \( \gamma_1 < 0 \) while other \( \gamma_n \) are positive. In physically motivated models the parameters \( a_{mn}, \gamma_n \) and \( g_n \) may satisfy some further relations. For example, the

\(^7\)Here we suppose that \( \lambda_n \neq 0 \) and \( g_n \neq 0 \). Otherwise the solution of the constraints should be modified in a fairly obvious way. We also denote \( \gamma_n \equiv \lambda_n^{-1} \).
signs of $\gamma_n$ and $g_n$ may be correlated so that $g_n/\gamma_n < 0$. However, such relations do not follow from the orthogonality conditions and we ignore them in our discussion.\footnote{In fact, the coupling constants have nothing to do with integrability and may be arbitrary numbers.}

The most important fact is that the constraints can be explicitly solved. First, we write the solutions of the Liouville equations \cite{21} in the form suggested by the conformal symmetry properties of the Liouville equation \cite{15},

$$e^{-q_n/2} = a_n(u)b_n(v) - \frac{1}{2} \tilde{g}_n \bar{a}_n(u) \bar{b}_n(v) \equiv X_n(u,v),$$

where the chiral fields $a_n(u)$, $b_n(v)$, $\bar{a}_n(u)$ and $ar{b}_n(v)$ satisfy the equations (do not mix $a_n(u)$ with $a_n$ used above):

$$a_n(u)\bar{a}_n'(u) - a_n'(u)\bar{a}_n(u) = 1, \quad b_n(v)\bar{b}_n'(v) - b_n'(v)\bar{b}_n(v) = 1. \quad (29)$$

Using (29) we can express $\bar{a}$ and $\bar{b}$ in terms of $a$ and $b$ and thus write $X_n$ as

$$X_n(u,v) = a_n(u)b_n(v)\left[1 - \frac{1}{2} \tilde{g}_n \int \frac{du}{a_n^2(u)} \int \frac{dv}{b_n^2(v)}\right]. \quad (30)$$

It is not so straightforward but not very difficult to rewrite the constraints \cite{20} in the form:

$$C_u \equiv 4 \sum_{n=1}^N \gamma_n \frac{a_n''(u)}{a_n(u)} = 4D_u(u), \quad C_v \equiv 4 \sum_{n=1}^N \gamma_n \frac{b_n''(v)}{b_n(v)} = 4D_v(v). \quad (31)$$

We first note that from \cite{18} and \cite{23} it is easy to find the identities

$$\partial_i F \partial_i \varphi + \sum_{n=3}^N (\partial_i \psi_n)^2 = \sum_{n=1}^N \gamma_n (\partial_i q_n)^2. \quad (32)$$

It follows that

$$C_i = \sum_{n=1}^N \gamma_n ((\partial_i q_n)^2 - 2\partial_i^2 q_n) = 4 \sum_{n=1}^N \gamma_n \partial_i^2 X_n/X_n. \quad (33)$$

Now, from the definition of $X_n$ it is not difficult to see that

$$\frac{\partial_i^2 X_n}{X_n} = \frac{a_n''(u)}{a_n(u)}, \quad \frac{\partial_i^2 X_n}{X_n} = \frac{b_n''(v)}{b_n(v)}. \quad (34)$$

Using the relation $\sum \gamma_n = 0$ we can explicitly solve these constraints. We show this for $C_u$ as the derivation for $C_v$ is similar. Let us introduce temporary notation $a_n'(u)/a_n(u) \equiv r_n(u)$, use it in expression for $C_u$ and then shift all $r_n$ by an unknown auxiliary function $R(u)$, i.e. define $\rho_n(u) \equiv r_n(u) + R(u)$. Thus we may express $C_u$ in terms of $\rho_n(u)$ and $R(u)$:

$$\frac{1}{4} C_u = \sum_{n=1}^N \gamma_n \frac{a_n''(u)}{a_n(u)} = \sum_{n=1}^N \gamma_n (r_n' + r_n^2) = \sum_{n=1}^N \gamma_n [\rho_n'(u) + \rho_n^2(u) - 2\rho_n(u)R(u)], \quad (35)$$

where we used the identity $\sum \gamma_n = 0$. Now it is easy to see that the constraint $C_u = 4D_u$ will be solved if we take

$$R(u) = \frac{1}{2} \left[ \sum_{n=1}^N \gamma_n (\rho_n'(u) + \rho_n^2(u)) - D_u(u) \right] \left[ \sum_{n=1}^N \gamma_n \rho_n(u) \right]^{-1}, \quad (36)$$
where \( \rho_n(u) \) are arbitrary functions. Thus, if we choose \( a'_n(u)/a_n(u) = \rho_n - R(u) \), where \( R \) is given by (36), the constraint \( C_u = 4D_u \) will be satisfied\(^9\).

Now we introduce new moduli fields

\[
\mu_n(u) \equiv \rho_n(u) - \frac{1}{2} \left[ \sum_{n=1}^{N} \gamma_n \rho_n^2(u) - D_u(u) \right] \left[ \sum_{n=1}^{N} \gamma_n \rho_n(u) \right]^{-1},
\]

(37)

Using this definition and the identity \( \sum \gamma_n = 0 \), it is easy to check that

\[
\sum \gamma_n \mu_n(u) = \sum \gamma_n \rho_n(u), \quad \sum \gamma_n \mu_n^2(u) = D_u(u) = -\frac{1}{4} \sum (\alpha'_n(u))^2
\]

(here and in what follows the limits of summation are omitted when the summation extends over all possible values of \( n = 1, \ldots, N \) and \( k = 1, \ldots, K \)). Repeating this derivation for the second constraint \( C_v \) and defining the \( v \)-analogue, \( \nu_n(v) \), of the moduli \( \mu_n(u) \), we finally write both constraints in terms of arbitrary chiral fields \( \mu_n(u), \nu_n(v), \alpha_k(u), \beta_k(v) \):

\[
\sum \gamma_n \mu_n^2(u) + \frac{1}{4} \sum (\alpha'_n(u))^2 = 0, \quad \sum \gamma_n \nu_n^2(v) + \frac{1}{4} \sum (\beta'_k(v))^2 = 0.
\]

(39)

These constraints are equivalent to the original ones, (20), if the fields \( f, \varphi, \psi, \chi \) are solutions of the equations of motion.

Returning to the original moduli fields \( \rho_n \) and to their relation to \( a'_n(u)/a_n(u) = r_n(u) \) (and to their \( v \)-analogues), one can see that the constraints are equivalent to the first order differential equations for \( a_n(u) \) and \( b_n(v) \):

\[
\frac{a'_n(u)}{a_n(u)} = \mu_n(u) - \frac{1}{2} \sum \frac{\gamma_n \mu_n'(u)}{\gamma_n \mu_n(u)}, \quad \frac{b'_n(v)}{b_n(v)} = \nu_n(v) - \frac{1}{2} \sum \frac{\gamma_n \nu_n'(v)}{\gamma_n \nu_n(v)},
\]

(40)

where \( \mu_n(u) \) and \( \nu_n(v) \) are arbitrary functions satisfying the constraints (39). This is a small ‘miracle’ allowing us to explicitly solve the equations of motion (including the constraints) in terms of a ‘sufficient’ number of arbitrary functions. The number of independent arbitrary chiral functions in r.h.s. of (40) is \( 2(N - 1 + K) \), where \( N - 2 \) is the number of the scalar matter fields \( \psi \), and \( K \) is the number of the free scalar matter fields \( \chi \). Together with the ‘gravitational’ degrees of freedom \( \varphi, F \) (or \( \psi_1, \psi_2 \)) the theory without constraints would have \( 2(N + K) \) chiral degrees of freedom. We will show in a moment that the residual coordinate transformations allow us to further reduce the number of independent arbitrary chiral functions to \( 2(N - 2 + K) \).

By integrating the first order differential equations (40) for \( a_n(u) \) and \( b_n(v) \) we find the general solution of the \( N \)-Liouville dilaton gravity in terms of the chiral moduli fields \( \mu_n(u) \) and \( \nu_n(v) \) satisfying the constraints (39):

\[
a_n(u) = \left[ \sum \gamma_n \mu_n \right]^{-\frac{1}{2}} \exp \int du \mu_n(u), \quad b_n(v) = \left[ \sum \gamma_n \nu_n \right]^{-\frac{1}{2}} \exp \int dv \nu_n(v).
\]

(41)

Let us choose arbitrary chiral \( \chi \)-fields \( \alpha_k(u) \) and \( \beta_k(v) \). Then the moduli fields \( \mu_n(u) \) and \( \nu_n(v) \) are not independent due to the constraints (39). In addition, we may use residual coordinate transformation \( u \to U(u) \) and \( v \to V(v) \) to choose two gauge (coordinate) conditions. We will show in a moment that writing

\[
\left| \sum \gamma_n \mu_n(u) \right| \equiv U'(u), \quad \left| \sum \gamma_n \nu_n(v) \right| \equiv V'(v)
\]

(42)

\(^9\)Note that the \( \rho_n - R \) are not independent functions. In fact, the chiral fields \( a_n \) depend on \( N - 1 \) arbitrary, independent functions of \( u \). This will be clear in a moment from a somewhat different representation of the solution of the constraint.
is indeed equivalent to choosing \((U,V)\) as a new coordinate system. With this aim, define
new chiral fields,

\[
A_n \equiv \exp \int du \mu_n(u), \quad B_n \equiv \exp \int dv \nu_n(v), \tag{43}
\]
that we consider as the functions of the new coordinates \((U,V)\) (due to implicit transformations \((42)\)). Using \((41)\) and \((42)\) we may rewrite \((30)\) in terms of \(A_n(U), B_n(V)\):

\[
Y_n(U, V) = A_n(U)B_n(V) \left[ 1 - \frac{1}{2} \tilde{g}_n \int \frac{dU}{A_n^2(U)} \int \frac{dV}{B_n^2(V)} \right], \tag{44}
\]
where we have defined

\[
Y_n(U, V) \equiv [U'(u)V'(v)]^\frac{n}{2} X_n(u, v) \tag{45}
\]

Now, with the above definitions we may derive the metric \(f(u, v) = U'(u)V'(v) \prod_{n=1}^{N} [Y_n(U, V)]^{4\gamma_n a_n} \equiv \hat{f}(U, V)U'(u)V'(v), \tag{46}\)

the dilaton \(\varphi\), and the scalar fields \(\psi_m\) \((m \geq 3)\):

\[
e^\varphi = \prod_{n=1}^{N} [Y_n(U, V)]^{4\gamma_n}, \quad e^{\psi_m} = \prod_{n=1}^{N} [Y_n(U, V)]^{-2a_m \gamma_n}. \tag{47}\]

We see that \(ds^2 = -4f(u, v)dudv \equiv -4\hat{f}(U, V)dUdV\) and thus everything is expressed in terms of the new coordinates \((U,V)\).

The constraints \((39)\) for the moduli parameters can easily be solved by expressing one of the moduli in terms of others. However, it may be more convenient and instructive to introduce new moduli that make clear the topological nature of the moduli space and thus may allow interesting topological classification of the solutions of the \(N\)-Liouville theory. To simplify notation we define the new moduli for the case of zero \(\chi\)-fields (the general case is not essentially different). Thus, setting \(\chi \equiv 0\) we introduce the following unit \((N-1)\)-vectors (recall that \(\gamma_1 < 0\) and \(\gamma_k > 0\) for \(k \geq 2\)):

\[
\dot{\xi}_k(u) \equiv \frac{\mu_k(u) \sqrt{\gamma_k}}{\mu_1(u) \sqrt{\gamma_1}}, \quad \hat{\eta}_k(v) \equiv \frac{\nu_k(v) \sqrt{\gamma_k}}{\nu_1(v) \sqrt{\gamma_1}}, \quad k = 2, ..., N. \tag{48}\]

These vectors, moving on the \((N-2)\)-dimensional unit sphere \(S^{(N-2)}\), determine the solution up to a choice of the coordinate system, which can be fixed by the above gauge conditions \((12)\). They now look as follows:

\[
U'(u) = |\gamma_1 \mu_1(u)|(1 - \cos \theta_{\xi}(u)), \quad V'(v) = |\gamma_1 \nu_1(v)|(1 - \cos \theta_{\eta}(v)), \tag{49}\]

\[
\cos \theta_{\xi}(u) = \sum_{k=2}^{N} \hat{\gamma}_k \dot{\xi}_k(u) \equiv \hat{\gamma} \dot{\xi}, \quad \cos \theta_{\eta}(v) = \sum_{k=2}^{N} \hat{\gamma}_k \hat{\eta}_k(v) \equiv \hat{\gamma} \hat{\eta}, \tag{50}\]

where \(\hat{\gamma}\) is the constant unit \((N-1)\)-vector, \(\hat{\gamma}_k = (\gamma_k/\gamma_1)^{\frac{1}{2}}\).

The moduli fields \(\dot{\xi}(u)\) and \(\hat{\eta}(v)\) are very useful because they give very simple representation of the most important solutions and, in particular, visualize relations between solutions of different dimensions. For example, if the vectors \(\dot{\xi}(u)\) and \(\hat{\eta}(v)\) are constant and equal \((\dot{\xi} = \hat{\eta})\), they give a static or a cosmological solution. The static solution has a horizon if
Another interesting static solution, which is flat at infinity, is given by the constant vectors satisfying the condition $\hat{\xi} = \hat{\eta} = \hat{\gamma}$. Two-dimensional solutions may be represented by pairs of curves in $S^{(N-2)}$ that we denote by $(\hat{\xi}(u), \hat{\eta}(v))$. Those that asymptotically interpolate between the above one-dimensional solutions are of special importance and define additional structure on $S^{(N-2)}$, which may be used for a physically motivated classification of the solutions of the $N$-Liouville theory.

Especially interesting are the new solutions that, probably, may be localized in space (‘lumps’ or soliton-like waves of scalar matter). They correspond to constant but not equal vectors $\hat{\xi}$ and $\hat{\eta}$ (obviously, this is possible for $N \geq 3$). In the coordinates $(U, V)$ ($U + V \equiv r$, $U - V \equiv t$) the solutions may be written as

$$Y_n(U, V) = C_n \cosh[R_n^{-1}((r - r_n) - v_n t)],$$

where the parameters $C_n$, $R_n$, $r_n$, $v_n$ rationally depend on the moduli and other parameters characterizing the Lagrangian and the coordinate system (recall that this parameters are not independent, due to the constraints and the gauge conditions). Although the exponentials $e^q = Y_n^{-2}$ are localized in space, this is not necessarily sufficient for the matter fields to be localized. However, using freedom in choosing moduli and other parameters it may be possible to write a truly localized solution. A detailed derivation and investigation of these interesting solutions will be presented in a separate publication.

We thus have the general solution of the 1+1 dimensional dilaton gravity coupled to any number of scalar fields. It is explicitly expressed in terms of a sufficient number of arbitrary chiral fields and thus we may solve the Cauchy problem and study the evolution of cosmological or black hole type solutions, etc. The representation of the general solution in terms of the chiral fields $a_n(u)$ and $b_n(v)$ may give us a good starting point in attempts to quantize our $N$-Liouville dilaton gravity. Even more useful may be the chiral moduli fields $\mu_n(u)$ and $\nu_n(v)$ (or $\hat{\xi}_k(u)$ and $\hat{\eta}_k(v)$). In terms of these moduli fields the dimensional reduction of the solutions becomes very transparent and this may simplify the derivation and physical interpretation of the evolution of one-dimensional solutions and suggest new approaches to quantization based on analogy with the simple 1-dimensional case.

### 4 Discussion and outlook

The explicitly analytically integrable models presented here may be of interest for different applications. Most obviously we may use them to construct first approximations to generally non integrable theories. Realistic theories describing black holes and cosmologies are usually not integrable. However, explicit general solutions of the integrable approximations may allow one to construct different sorts of perturbation theories.

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10 At first sight, the expressions for the fields $\bar{f}$, $\varphi$, $\psi_m$ may look nonlocal. However, denoting the indefinite integrals in (44) by new chiral fields, we see that $Y_n$ are essentially local functions of new fields and of their derivatives.

11 The one-dimensional models obtained by simple dimensional reduction may easily be quantized. Our formulation is as close to the one-dimensional theory as possible and this supports hopes to finally find a quantum version of the $N$-Liouville theory. It should be emphasized that, up to now, the interesting results obtained in quantum Liouville theory did not shed much light on the $N$-Liouville theory. This may be explained by the fact that the main difficulty and the main content of the $N$-Liouville, as distinct from the standard Liouville theory, lies in the constraints.
For example, spherically symmetric static black holes non minimally coupled to scalar fields are described by the integrable 0+1 dimensional N-Liouville model. Similarly, spherically symmetric cosmological models may be described by 1+0 dimensional N-Liouville theory. However, the corresponding 1+1 theory is not integrable because the scalar coupling potentials $Z_n$ are not constant (actually, $Z_n \sim \varphi$). To obtain approximate analytic solutions of the 1+1 theory one then may try to approximate $Z_n$ by properly chosen constants.

This approach may be combined with the recently proposed analytic perturbation theory allowing one to find solutions close to horizons for the most general non integrable 0+1 dilaton gravity theories [16]. Near the horizons we can use the integrable 1+1 dimensional dilaton gravity (with $Z_n = -1$) as a good approximation to a realistic theory (with $Z_n$ depending on the dilaton $\varphi$).

In cosmological applications, the behaviour of the 1+1 dimensional solutions near the singularity at $\varphi = 0$ is of great interest (see e.g. [17]). Integrable 1+1 dimensional N-Liouville theories could give, at best, a rough qualitative approximation of the exact solutions near the singularity. A more quantitative approximation might be obtained by first asymptotically solving the exact theory in the vicinity of $\varphi = 0$ and then sewing the asymptotic solutions with those of the integrable theory. To realize such a program one needs a very simple and explicit analytic solutions of the integrable theory. Our simple model having the solutions represented in terms of the moduli $\hat{\xi}$ and $\hat{\eta}$ may give a good starting point for such a work. Of course, before applications to realistic cosmologies become possible, one should study in detail and completely classify and interpret the behaviour of the 1+1 dimensional solutions and their precise relation to the 1+0 dimensional reduction.

The reduction from dimension 1+1 both to dimension 1+0 (‘cosmological’) and to dimension 0+1 (‘static’ or ‘black hole’) is especially transparent in the moduli representation for the solutions of the 1+1 dimensional N-Liouville model. However, as emphasized in [1], the whole procedure of the dimensional reduction should be reconsidered from a more general point of view, taking into account more general dimensional reductions. A detailed consideration of generalized dimensional reductions will be given elsewhere.

5 Appendix

Here we sketch a simple approach to solving the pseudo orthogonality conditions for $a_{mn}$ with additional restrictions $a_{1n} = 1 + a_n$, $a_{2n} = 1 - a_n$. The equations one has to solve are

$$a_i + a_j = \frac{1}{2} \sum_{n=3}^{N} a_n a_{nj}, \quad i < j.$$  \hspace{1cm} (52)

These $N(N-1)/2$ equations are nonlinear but there exist a simple recursive algorithm reducing their solution to solving linear equations. To find it one has to choose which of the parameters should be regarded as unknown ones. Analyzing the cases $N = 3$ and $N = 4$ one may find that the convenient choice of the division of the parameters into unknown and arbitrarily fixed is the following. Let us fix $a_{33}$ and $a_{mn}$ for $m > n$, $m \geq 3$ and thus all other parameters are unknown. First, consider the equations for $i, j = 1, 2, 3$. Denoting the r.h.s. of (52) by $A_k$ where $(ijk) = (123)_{\text{cyclic}}$ we find

$$2a_i = -A_i + A_j + A_k, \quad (ijk) = (123)_{\text{cyclic}}.$$  \hspace{1cm} (53)
The next step is to consider the equations (52) for $i = 1, 2, 3$ and $j = 4$. These constitute three linear equations for three unknowns $a_4, a_{34}, a_{44}$. They have a unique solution provided that the r.h.s. and the determinant $\Delta$ do not vanish.

Now the general construction should be clear. On the $j$-th step, where $j = 4, \ldots, N$, we have $j - 1$ equations for $j - 1$ unknowns, $a_j$ and $a_{nj}$ for $3 \leq n \leq j$:

$$-2a_j + \sum_{n=3}^{j} a_{ni}a_{nj} = 2a_i - \sum_{n=j+1}^{N} a_{ni}a_{nj}, \quad i = 1, \ldots, j - 1. \quad (54)$$

The r.h.s. of these equations depend both on arbitrary and previously found parameters. The nontrivial part of the procedure is that the determinant, starting from $j = 5$, depend on the previously found parameters and thus the condition $\Delta \neq 0$ is difficult to control. For $j = 4$ the determinant,

$$\Delta = 2[(a_{33} - a_{31})(a_{43} - a_{42}) - (a_{33} - a_{32})(a_{43} - a_{41})],$$

depends on the arbitrary parameters only.

Finally, let us write the parameters of the general model with $N = 3$:

$$2a_i = a_{3i}a_{3j} - a_{3j}a_{3k} + a_{3k}a_{3i}, \quad \gamma_i^{-1} = (a_{3i} - a_{3j})(a_{3i} - a_{3k}), \quad (ijk) = (123)_{\text{cyclic}}.$$

Acknowledgment: The author appreciates financial support of the Dept. of Theoretical Physics of the University of Turin and INFN (Section of Turin), of CERN-TH, of the MPI and the W.Heisenberg Institute (Munich), where some results were obtained. Useful discussions with P. Fré, D. Luest, D. Maison, and G. Veneziano are kindly acknowledged. Useful remarks of G.A. Alekseev concerning Section 2 and references are also appreciated. Author is especially grateful to V. de Alfaro for his support for many years and very fruitful collaboration; in particular, some of the results presented here were obtained in close collaboration with him. This report was completed while the author was visiting CERN-TH; kind hospitality and support of the members of TH is highly appreciated. This work was also partly supported by RFBR grant 03-01-00781-a.
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