Inequivalent quantizations from gradings and $\mathbb{Z}_2 \times \mathbb{Z}_2$ parabosons

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Abstract
This paper introduces the parastatistics induced by $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded algebras. It accommodates four kinds of particles: ordinary bosons and three types of parabosons which mutually anticommute when belonging to different type (so far, in the literature, only parastatistics induced by $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superalgebras and producing parafermions have been considered). It is shown how to detect $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded parabosons in the multi-particle sector of a quantum model. The difference with respect to a system composed by ordinary bosons is spotted by measuring some selected observables on certain given eigenstates. The construction of the multi-particle states is made through the appropriate braided tensor product. The application of $\mathbb{Z}_2$- and $\mathbb{Z}_2 \times \mathbb{Z}_2$-gradings produces 9 inequivalent multi-particle Hilbert spaces of a $4 \times 4$ matrix oscillator. The $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded parabosonic Hilbert space is one of them.

Keywords: gradings, parabosons, parastatistics, multi-particle quantum Hamiltonians, braided tensor products, $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras

1. Introduction

This paper introduces the parastatistics induced by $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras and gives the proof that the associated particles, the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded parabosons, can be detected by performing a measurement in the multi-particle sector of a quantum model.

$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras and Lie superalgebras were introduced by Rittenberg and Wyler in [1, 2]. The term ‘color (super)algebra’ was used (see also [3]) to describe both cases. The particles (bosons and fermions) of an ordinary theory can be associated with 1 bit of information (let us say 0 for bosons and 1 for fermions), while the particles of a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded theory are described by 2 bits of information (00, 10, 11, 01).

The four types of particles in models based on $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebras are (see [4]) the ordinary bosons (00), two types (10 and 01) of parafermions and the exotic bosons

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The four types of particles in models based on $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras are (see [5]) the ordinary bosons (00) and three types (10, 11 and 01) of parabosons; the parabosons of different type anticommute.

The $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded physics, with particles accommodated according to Lie superalgebras, is an obvious extension of ordinary physics. Indeed, ordinary bosons and fermions can be recovered from, respectively, the 00 and 10 sectors, while leaving empty the 11 and 01 sectors. Only recently, however, the open question that was lingering around was solved in [6], by showing that the colored world of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebras produces quantum models which cannot be mimicked by black/white ordinary bosons and fermions alone.

Symbolically, this result can be expressed as

$$\mathbb{Z}_2^1 \cdot \text{LSA} \subset \mathbb{Z}_2^2 \cdot \text{LSA}, \quad (1)$$

meaning that the systems recovered from $\mathbb{Z}_2$-graded Lie superalgebras are a proper subset of those recovered from $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebras.

The approach of [6] emphasizes the role of the braided tensor product, as defined in [7], in the construction of the multi-particle states. This approach is here extended to derive the physics of the parabosons obtained from $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras. A symbolical consequence of the present work can be expressed as

$$\mathbb{Z}_2^0 \cdot \text{LA} \subset \mathbb{Z}_2^2 \cdot \text{LA}, \quad (2)$$

meaning that the bosonic systems recovered from ordinary Lie algebras are a proper subset of those recovered from $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras.

In the first years after the introduction of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebras their physical applications received some limited attention, see [8–11]. More recently, a systematic investigation of their role as symmetries of dynamical systems started. Indeed, they appear [12, 13] as symmetries of the Lévy-Leblond equations for nonrelativistic spinors; furthermore, classical invariant worldline [4] and two-dimensional [14] sigma models have been constructed, invariant quantum mechanical models have been presented in [15, 16] and conformal quantum mechanics in [17]. The parastatistics induced by $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebras was introduced in [18, 19] and further investigated in [6, 20–25].

Despite this activity on graded Lie superalgebras, the possibility of applications of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras has been ignored, probably because they are not extensions of ordinary [26] Lie superalgebras and do not include fermions. Indeed, a consistent number of present works are focusing on even larger (the $\mathbb{Z}_n^2$, for $n > 2$) graded extension of Lie superalgebras, see e.g. [27–29] and references therein for the mathematical literature.

On the other hand, as pointed out very recently in [5], several constructions (invariant models, the graded superspace of [30], etc) which are available for graded superalgebras can be extended to $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras. This is the starting point of the present investigation.

The scheme of the paper is as follows. It is shown at first that the $4 \times 4$ matrix Hamiltonian discussed in [15, 16], besides being supersymmetric and invariant under the one-dimensional $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Poincaré superalgebra, is also invariant under a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebra. This offers the possibility to apply the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded parabosonic statistics to its multi-particle sector.

The Hilbert space is constructed for the special case of the harmonic oscillator potential. It is shown that different statistics can be implemented by using $\mathbb{Z}_2$- and $\mathbb{Z}_2 \times \mathbb{Z}_2$- gradings.
As a consequence, a total number of 9 inequivalent multi-particle Hilbert spaces are encountered. This statement can also be rephrased as ‘9 inequivalent multi-particle quantizations’. The analysis of [6] discussed just three of them (bosonic, supersymmetric and \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-graded parafermionic variants). Among the extra quantizations presented in this paper a case corresponds to the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-graded parabosons, while another case corresponds to a different implementation of the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-graded parafermions.

The multi-particle states are constructed by taking the raising operators as elements of a universal enveloping graded Lie (super)algebra and by applying the associated coproducts. The proof that the 9 variants indeed produce inequivalent multi-particle models is given. In particular, observables discriminating the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-graded parabosonic variant from the bosonic one are constructed.

To better clarify the construction here presented it is worth mentioning that, when applied to single-particle quantum Hamiltonians, the term ‘(para)particles’ associated with the \( \mathbb{Z}_2 \)- and \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-gradings is just a conventional label void of any physical significance. This is no longer the case when multi-particle quantum Hamiltonians are considered. In this setting the statistics of the (para)particles plays a physical role and the labels cease to be conventional. Indeed, the multi-particle wave functions possess mixed symmetry. Their ‘mixed’ properties are encoded in signs entering the braided tensor products. The \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-graded parastatistics approach in this paper follows the lines of [6, 20, 23], which are based on the Hopf algebra framework of [7]. Other works on \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-graded parastatistics, see [18, 24, 25], realize the Green’s trilinear relations [31] in terms of graded orthosymplectic superalgebras (the connection between the Hopf algebras and the trilinear relations approaches to parastatistics are discussed in [32, 33]). Further comments about the construction are given throughout the text.

The obtained results and the future perspectives are discussed in the conclusions.

2. The \( 4 \times 4 \) graded Hamiltonian

The \( 4 \times 4 \) Hermitian matrix Hamiltonian \( H \), given by

\[
H = \frac{1}{2} \begin{pmatrix}
-\partial_x^2 + W^2(x) + W'(x) & 0 & 0 & 0 \\
0 & -\partial_x^2 + W^2(x) + W'(x) & 0 & 0 \\
0 & 0 & -\partial_x^2 + W^2(x) - W'(x) & 0 \\
0 & 0 & 0 & -\partial_x^2 + W^2(x) - W'(x)
\end{pmatrix},
\]

(3)

depends on the prepotential \( W(x) \). We set in the above formula \( W'(x) = \frac{d}{dx} W(x) \). \( H \) is invariant, see [15, 16], under both supersymmetry and the one-dimensional \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-graded Poincaré superalgebra.

We point out here that \( H \) is also invariant under a \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-graded Lie algebra. Let us introduce the Hermitian first-order matrix operators \( Q_{10} \), \( Q_{01} \) and constant matrix \( Z \) as

\[
Q_{10} = \frac{-1}{\sqrt{2}} \begin{pmatrix}
0 & 0 & \partial_x + W(x) & 0 \\
0 & 0 & 0 & \partial_x + W(x) \\
\partial_x - W(x) & 0 & 0 & 0 \\
0 & \partial_x - W(x) & 0 & 0
\end{pmatrix},
\]

\[
Q_{01} = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & 0 & 0 & \partial_x + W(x) \\
0 & 0 & \partial_x + W(x) & 0 \\
0 & -\partial_x + W(x) & 0 & 0 \\
-\partial_x + W(x) & 0 & 0 & 0
\end{pmatrix},
\]

(3)
\[ Z = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \] (4)

The Hamiltonian \( H \) is invariant under the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-graded abelian Lie algebra \( \mathfrak{a} \) defined by the following set of (all vanishing) 6 (anti)commutators:

\[
[H, Q_{10}] = [H, Q_{01}] = [H, Z] = 0, \quad \{Q_{10}, Q_{01}\} = \{Z, Q_{10}\} = \{Z, Q_{01}\} = 0. \tag{5}
\]

The algebra \( \mathfrak{a} \) is listed as ‘A7’ in the table 1 classification of minimal graded algebras presented in [5]. The grading assignment, according to the (A.8) decomposition, of the \( \mathfrak{a} \) generators is

\[
H \in \mathfrak{a}_{00}, \quad Q_{10} \in \mathfrak{a}_{10}, \quad Q_{01} \in \mathfrak{a}_{01}, \quad Z \in \mathfrak{a}_{11}. \tag{6}
\]

The operators \( Q_{10}, Q_{01} \) are square roots of the Hamiltonian \((Q_{10}^2 = Q_{01}^2 = H)\). It follows that, besides (5), \( H \) is invariant under the \( \mathbb{Z}_2 \)-graded Lie superalgebra

\[
\{Q_{10}, Q_{01}\} = \{Q_{01}, Q_{01}\} = 2H, \quad [H, Q_{10}] = [H, Q_{01}] = 0, \tag{7}
\]

which defines \( H \) as a supersymmetric quantum mechanics [34] Hamiltonian.

We anticipate that these two graded invariant structures produce inequivalent quantum models in the multi-particle sectors. Essentially, this is due to the fact that the fermions present in (7) obey the Pauli exclusion principle; this is not the case for the parabosons that, as we will see, are obtained from (5).

If we specialize \( W(x) = -x \), \( H \) becomes the Hamiltonian of the one-dimensional, \( 4 \times 4 \) matrix oscillator. It will be denoted as \( H_{\text{osc}} \); we have

\[
H_{\text{osc}} = \frac{1}{2} \begin{pmatrix} -\partial_x^2 + x^2 - 1 & 0 & 0 & 0 \\ 0 & -\partial_x^2 + x^2 - 1 & 0 & 0 \\ 0 & 0 & -\partial_x^2 + x^2 + 1 & 0 \\ 0 & 0 & 0 & -\partial_x^2 + x^2 + 1 \end{pmatrix}. \tag{8}
\]

The single-particle Hilbert space \( \mathcal{H} \) of the \( H_{\text{osc}} \) Hamiltonian was constructed in [6, 15]. It is obtained by applying raising operators to a lowest weight vector state denoted as \( |0;00\rangle \).

The creation/annihilation oscillators \( a, a^\dagger \), given by

\[
a = \frac{i}{\sqrt{2}}(\partial_x + x), \quad a^\dagger = \frac{i}{\sqrt{2}}(\partial_x - x), \tag{9}
\]

satisfy the commutator

\[
[a, a^\dagger] = 1. \tag{10}
\]

The matrix raising (lowering) operators \( f_{11}^\dagger, f_{10}^\dagger, f_{01}^\dagger (f_{11}, f_{10}, f_{01}) \) can be introduced as

\[
f_{11}^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad f_{10}^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad f_{01}^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{11}
\]
\[ f_{11} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad f_{10} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad f_{01} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \] (11)

The suffix is chosen in order to denote, in the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-gradings, the matrix decompositions expressed in (A.8).

In terms of these operators the Hamiltonian \( H_{\text{osc}} \) can be re-expressed as
\[ H_{\text{osc}} = a^\dagger a \cdot I_4 + f_{10}^\dagger f_{10} + f_{01}^\dagger f_{01} = a^\dagger a \cdot I_4 + \Lambda, \quad \text{with} \quad \Lambda = \text{diag}(0, 0, 1, 1). \] (12)

Here and in the following we denote an \( m \times m \) identity matrix as \( I_m \).

The normalized lowest weight vector \( |0;00\rangle \) satisfies the conditions
\[ a |0;00\rangle = f_{11} |0;00\rangle = f_{10} |0;00\rangle = f_{01} |0;00\rangle = 0. \] (13)

We have
\[ |0;00\rangle = \pi^{-\frac{1}{4}} e^{-\frac{x^2}{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \] (14)

The single-particle Hilbert space \( \mathcal{H} \) is spanned by the orthonormal vectors
\[ |n;00\rangle, |n;11\rangle, |n;10\rangle, |n;01\rangle \] introduced through
\[ |n;00\rangle = (a^\dagger)^n |0;00\rangle, \quad |n;10\rangle = (a^\dagger)^n f_{10}^\dagger |0;00\rangle, \]
\[ |n;11\rangle = (a^\dagger)^n f_{11}^\dagger |0;00\rangle, \quad |n;01\rangle = (a^\dagger)^n f_{01}^\dagger |0;00\rangle. \] (15)

At most a single power of \( f_{11}^\dagger, f_{10}^\dagger, f_{01}^\dagger \) enters the spanning vectors since we have, for any pair of such operators,
\[ f_a^\dagger f_b^\dagger = 0, \quad \text{with} \quad a, b \in \{11, 10, 01\}. \] (16)

Due to the commutators
\[ [H_{\text{osc}}, a^\dagger] = a^\dagger, \quad [H_{\text{osc}}, f_{10}^\dagger] = f_{10}^\dagger, \quad [H_{\text{osc}}, f_{01}^\dagger] = f_{01}^\dagger, \quad [H_{\text{osc}}, f_{11}^\dagger] = 0, \] (17)
the (15) states are energy eigenstates whose eigenvalues are read from
\[ H_{\text{osc}} |n;00\rangle = n |n;00\rangle, \quad H_{\text{osc}} |n;10\rangle = (n + 1) |n;10\rangle, \]
\[ H_{\text{osc}} |n;11\rangle = n |n;11\rangle, \quad H_{\text{osc}} |n;01\rangle = (n + 1) |n;01\rangle. \] (18)

One should note that the vacuum state is doubly degenerate:
\[ H_{\text{osc}} |0;00\rangle = H_{\text{osc}} |0;11\rangle = 0. \] (19)
For later convenience we introduce the exchange matrices $X_{11}, X_{10}, X_{01}$. They are Hermitian operators which mutually interchange the 11, 10 and 01 sectors. Their suffix indicates the $(\mathbb{A}, 8)$ decomposition when a $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading is applied. We have

$$X_{11} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad X_{10} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad X_{01} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (20)$$

The matrices $X_{11}, X_{10}, X_{01}$ are the building blocks in the construction of the observables which are presented in section 5.

3. The construction of multi-particle Hilbert spaces

The $n$-particle Hilbert space $\mathcal{H}^{(n)}$ of the (8) Hamiltonian $H_{\text{osc}}$ is a subset of the tensor products of $n$ single-particle Hilbert spaces $\mathcal{H}$:

$$\mathcal{H}^{(n)} \subset \mathcal{H}^{{\otimes n}}. \quad (21)$$

$\mathcal{H}^{(n)}$ is a lowest weight vector space whose lowest weight vector $|0; 00\rangle^{(n)}$ is a tensor product of the single-particle lowest weight vector $|0; 00\rangle$ given in (14):

$$|0; 00\rangle^{(n)} = |0; 00\rangle \otimes \cdots \otimes |0; 00\rangle. \quad (22)$$

The space coordinates entering the tensor products of the Hilbert spaces $\mathcal{H}^{(n)}$ are denoted as $x_1, x_2, \ldots, x_n$. In the two-particle case we set, for simplicity, $x_1 = x, x_2 = y$. Therefore, the normalized lowest weight vector $|0; 00\rangle^{(2)}$ is

$$|0; 00\rangle^{(2)} = \pi^{-\frac{1}{2}} e^{-\frac{1}{2}(x^2+y^2)} \cdot v_1. \quad (23)$$

Here and in the following we denote as $v_j$, for $j = 1, 2, \ldots, 16$, the 16-component column vector with entry 1 in the $j$th position and 0 otherwise.

The construction of the two-particle Hilbert space assumes the raising operators $a^\dagger, f_{11}^\dagger, f_{10}^\dagger, f_{01}^\dagger$ introduced in (9), (11) to be elements of a graded algebra $g$ (the admissible gradings for $g$ are discussed in section 4). The graded algebra $g$ defines its universal enveloping algebra $U \equiv U(g)$. As recalled in appendix A, $U$ is endowed with a Hopf algebra structure and in particular of an operation, the coproduct $\Delta$, which satisfies (A.10)–(A.12).

The two-particle states are recovered from applying the coproducts

$$\Delta \left((a^\dagger)^r (f_{11}^\dagger)^{r_{11}} (f_{10}^\dagger)^{r_{10}} (f_{01}^\dagger)^{r_{01}}\right) \in U \otimes U \quad (24)$$

to the vector $|0; 00\rangle^{(2)}$ which induces the lowest weight representation. Following the convention of appendix A, a hat denotes the evaluation of the coproduct in the given representation. Therefore

$$\Delta \left((a^\dagger)^r (f_{11}^\dagger)^{r_{11}} (f_{10}^\dagger)^{r_{10}} (f_{01}^\dagger)^{r_{01}}\right) \in \text{End}(\mathcal{H}^{(2)}). \quad (25)$$

The Hilbert space $\mathcal{H}^{(2)}$ is spanned by

$$|n; r_{11}, r_{10}, r_{01}\rangle^{(2)} = \Delta \left((a^\dagger)^r (f_{11}^\dagger)^{r_{11}} (f_{10}^\dagger)^{r_{10}} (f_{01}^\dagger)^{r_{01}}\right) \cdot |0; 00\rangle^{(2)}. \quad (26)$$
The identification $|0; 0, 0, 0\rangle^{(2)} \equiv |0; 00\rangle^{(2)}$ holds.

In (26) $n$ is a non-negative integer $(n \in \mathbb{N}_0)$; the restrictions on $r_{11}, r_{10}, r_{01}$, as discussed in section 4, depend on the grading.

As a useful example, it follows that the formula of the two-particle creation operator $\Delta(a^\dagger)$ is

$$\Delta(a^\dagger) = \frac{i}{\sqrt{2}}(\partial_x - x + \partial_y - y).$$

(27)

For non-interacting multi-particle states the energy is additive. In the two-particle case the additivity is expressed by the Hamiltonian $H^{(2)}_{osc}$, given by

$$H^{(2)}_{osc} = H_{osc} \otimes 1_4 + 1_4 \otimes H_{osc}.$$  

(28)

In terms of the coproduct, within the Hopf algebra framework, the additivity requires the Hamiltonian to be a primitive element (see [35, 36]) of some enlarged graded algebra $g$, so that $H_{osc} \in g$; the Hopf algebra is defined for the universal enveloping algebra $U(g)$. Following [6] one can therefore set $H^{(2)}_{osc} = \Delta(H_{osc}).$ The evaluation of $H_{osc}$ is in the representation specified by formula (8).

The construction of the $n+1$-particle Hilbert spaces, for $n > 1$, is made iteratively by replacing $\Delta \equiv \Delta^{(1)}$ with $\Delta^{(n)}$. Induced by the coassociativity (A.11) of the coproduct, $\Delta^{(n)}$ is defined as

$$\Delta^{(n)} = (id \otimes \Delta^{(1)})\Delta^{(n-1)}, \quad (\Delta^{(1)} \equiv \Delta).$$

(29)

4. The 9 inequivalent two-particle Hilbert spaces

The oscillator Hamiltonian $H_{osc}$ given in (8) possesses nine inequivalent multi-particle quantizations. They are induced by the different gradings assigned to the raising operators $f^1_{11}, f^1_{10}, f^1_{01}$ introduced in (11) and under the assumption that the lowest weight vector is bosonic. Three of the quantizations (bosonic, supersymmetric and a version of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded parafermions) were already discussed in [6]. The extra quantizations are divided into standard and non-standard. Among the standard ones we obtain the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded para-bosons; the non-standard ones include an alternative quantization based on $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded parafermions.

The construction goes as follows: in one case (the bosonic one) $f^1_{11}, f^1_{10}, f^1_{01}$ are assumed to be elements of an ordinary abelian Lie algebra; alternatively, they are assumed to be even/odd elements of a $\mathbb{Z}_2$-graded abelian Lie superalgebra, of a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded abelian Lie superalgebra (parafermions) or of a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded abelian Lie algebra (parabosons).

We proceed at first to discuss the 6 standard gradings.

4.1. The 6 standard gradings

In the $\mathbb{Z}_2$-grading assignment the $4 \times 4$ matrix Hamiltonian $H_{osc}$ corresponds to a block-diagonal supermatrix of $(4 - p|p)$ type, with $p = 0, 1, 2, 3$. The $(4|0)$ case for $p = 0$ coincides with the ordinary bosonic matrix. The $p = 4$ case is excluded if we require the vacuum state to be even (bosonic).
The six assignments are:

1. \( \{ f_{11}^1, f_{10}^1, f_{01}^1 \} \in 0, \{ \emptyset \} \in 1 \) for (4|0);
2. \( \{ f_{11}^1, f_{10}^1 \} \in 0, \{ f_{01}^1 \} \in 1 \) for (3|1);
3. \( \{ f_{11}^1 \} \in 0, \{ f_{10}^1, f_{01}^1 \} \in 1 \) for (2|2);
4. \( \{ \emptyset \} \in 0, \{ f_{11}^1, f_{10}^1, f_{01}^1 \} \in 1 \) for (1|3);
5. \( \{ f_{11}^1, f_{10}^1, f_{01}^1 \} \in 2 \mathbb{Z}_2 \cdot LSA \);
6. \( \{ f_{11}^1, f_{10}^1, f_{01}^1 \} \in 2 \mathbb{Z}_2 \cdot LA \).

The corresponding vanishing (anti)commutators defining the graded abelian algebras \( a_j \), where \( j = 1, 2, \ldots, 6 \), are

\[
\begin{align*}
& a_1 : [f_{11}^1, f_{10}^1] = [f_{10}^1, f_{01}^1] = [f_{01}^1, f_{11}^1] = 0; \\
& a_2 : [f_{11}^1, f_{10}^1] = [f_{10}^1, f_{01}^1] = [f_{01}^1, f_{11}^1] = \{ f_{01}^1, f_{01}^1 \} = 0; \\
& a_3 : [f_{11}^1, f_{10}^1] = \{ f_{10}^1, f_{01}^1 \} = [f_{10}^1, f_{11}^1] = \{ f_{10}^1, f_{10}^1 \} = \{ f_{11}^1, f_{11}^1 \} = 0; \\
& a_4 : [f_{11}^1, f_{10}^1] = [f_{10}^1, f_{01}^1] = [f_{01}^1, f_{11}^1] = \{ f_{01}^1, f_{01}^1 \} = 0; \\
& a_5 : [f_{11}^1, f_{10}^1] = [f_{10}^1, f_{01}^1] = [f_{11}^1, f_{11}^1] = \{ f_{01}^1, f_{01}^1 \} = 0; \\
& a_6 : [f_{11}^1, f_{10}^1] = [f_{10}^1, f_{01}^1] = [f_{01}^1, f_{11}^1] = \{ f_{01}^1, f_{01}^1 \} = 0.
\end{align*}
\]

(30)

The three cases already discussed in [6] correspond to the numbers 1 (the bosonic version of the theory), 3 (the supersymmetric version) and 5 (a \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-graded parafermionic version).

In the above construction we followed the standard block-diagonal matrix format of Lie superalgebras and, for the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) grading, the (A.8) decomposition. Non-standard supermatrix formats are discussed in [37]. The procedure for the non-standard decompositions is presented in the subsection 4.2.

Each algebra \( a_i \) is extended to the graded algebra \( \overline{a}_i = \{ a^i, f_{11}^i, f_{10}^i, f_{01}^i \} \) which contains the creation operator \( a^i \), introduced in (9), as extra generator. Depending on the case, \( a^i \) belongs to either the 0- or the 00-sector. Its commutators are vanishing (\( [a^i, f_j^i] = 0 \) for \( j = 11, 10, 01 \)).

The multi-particle quantizations are recovered, as explained in section 3, from the coproducts defined on the corresponding universal enveloping algebras \( U(\overline{a}_i) \). The multi-particle states are constructed according to formula (26). The signs entering the braided tensor products depend on the different grading assignments of each one of the above cases. They are given by (A.3) and (A.13).

The restrictions on the \( r_{11}, r_{10}, r_{01} \) exponents entering (26) are due to these respective signs. For instance, in the parafermionic quantization \( r_{10} \) takes the values 0, 1; the values taken by \( r_{10} \) in the parabosonic case are 0, 1, 2.

The 6 standard multi-particle quantizations, associated to the respective (30) gradings, are denoted as follows:

\[
\begin{align*}
(4|0) & : a_1, & (2|2) & : a_3, & \mathbb{Z}_2^2 & : PF : a_5, \\
(3|1) & : a_2, & (1|3) & : a_4, & \mathbb{Z}_2^2 & : PB : a_6.
\end{align*}
\]

(31)

In the last column \( PF \) and \( PB \) stand for, respectively, parafermions and parabosons.
4.2. The 3 non-standard gradings

The non-standard cases are obtained by applying decompositions of the supermatrices which do not coincide with the ordinary block-diagonal decompositions; these non-standard formats are discussed in [37]. For the model under consideration these extra cases can be recovered from standard decompositions applied to a different diagonal Hamiltonian whose diagonal entries are permuted with respect to $H_{osc}$.

Before proceeding with the construction of the non-standard quantizations let us recall that the $(11)$ raising operators $f^\dagger_{11}, f^\dagger_{10}, f^\dagger_{01}$ create, see (17), particles of respective energy $0, 1, 1$.

In a $\mathbb{Z}_2$-grading, the standard decomposition of a vector $v^T = (B, B, F, F)$ with $2$ bosons and $2$ fermions can be replaced, for instance, by the decomposition $v^T = (B, F, B, F)$. In these two examples the entries of the fermionic supermatrices are respectively accommodated according to

\[
\text{Standard case: } \begin{pmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ * & * & 0 & 0 \\ * & * & 0 & 0 \end{pmatrix}, \quad \text{non-standard case: } \begin{pmatrix} 0 & * & 0 & * \\ * & 0 & * & 0 \\ 0 & * & 0 & * \\ * & 0 & * & 0 \end{pmatrix}.
\]

(32)

For $3$ bosons and $1$ fermion we can pass from $v^T = (B, B, B, F)$ to, e.g. $v^T = (B, F, B, B)$. In these new examples the entries of the fermionic supermatrices are respectively accommodated according to

\[
\text{Standard case: } \begin{pmatrix} 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ * & * & * & 0 \end{pmatrix}, \quad \text{non-standard case: } \begin{pmatrix} 0 & * & 0 & 0 \\ * & 0 & * & * \\ 0 & * & 0 & 0 \\ 0 & * & 0 & 0 \end{pmatrix}.
\]

(33)

The key issue to notice is that the raising operator $f^\dagger_{11}$ becomes fermionic in the non-standard decompositions above. This implies that the Pauli exclusion principle applies to the $0$-energy particles created by $f^\dagger_{11}$. In the standard cases these particles are bosons. This affects the degeneracy of the energy levels of the multi-particle Hamiltonian producing inequivalent results.

Similarly, a non-standard decomposition of a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra is realized, e.g. by accommodating the $00, 11, 10, 01$ sectors according to

\[
M_{00} = \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}, \quad M_{10} = \begin{pmatrix} 0 & * & 0 & 0 \\ * & 0 & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix},
\]

\[
M_{11} = \begin{pmatrix} 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \\ * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \end{pmatrix}, \quad M_{01} = \begin{pmatrix} 0 & 0 & 0 & * \\ 0 & 0 & * & 0 \\ 0 & * & 0 & 0 \\ * & 0 & 0 & 0 \end{pmatrix}.
\]

(34)

Contrary to the standard decomposition (A.8), in this case the $0$-energy particles created by $f^\dagger_{11}$ are no longer exotic bosons, but parafermions.

On the other hand in the parabosonic case induced by the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebra, the non-standard decomposition above does not produce an inequivalent quantization with respect
Table 1. Spanning vectors of the standard finite dimensional two-particle Hilbert spaces of the $4 \times 4$ matrix oscillator. The first four columns correspond to supermatrices: $(4|0)$, i.e. the bosonic case, $(3|1)$, $(2|2)$, i.e. the supersymmetric case, and $(1|3)$. The last two columns present the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Hilbert spaces for parafermions ($\mathbb{Z}_2^2$-PF) and parabosons ($\mathbb{Z}_2^2$-PB). The ‘X’ denotes the presence of the vector.

|     | $(4|0)$ | $(3|1)$ | $(2|2)$ | $(1|3)$ | $\mathbb{Z}_2^2$-PF | $\mathbb{Z}_2^2$-PB |
|-----|---------|---------|---------|---------|------------------|------------------|
| $V_1$ | $v_1$   | X       | X       | X       | X                | X                |
| $V_2$ | $v_b$   | X       | X       | X       | X                | X                |
| $V_3$ | $v_{11}$| X       | X       | X       | X                | X                |
| $V_4$ | $v_{16}$| X       | X       | X       | X                | X                |
| $V_5$ | $\frac{1}{2} (v_2 + v_3)$ | X       | X       | X       | X                | X                |
| $V_6$ | $\frac{1}{2} (v_3 + v_4)$ | X       | X       | X       | X                | X                |
| $V_7$ | $\frac{1}{2} (v_4 + v_{13})$ | X       | X       | X       | X                | X                |
| $V_8$ | $\frac{1}{2} (v_7 + v_{10})$ | X       | X       | X       | X                | X                |
| $V_9$ | $\frac{1}{2} (v_7 - v_{10})$ | X       | X       | X       | X                | X                |
| $V_{10}$ | $\frac{1}{2} (v_8 + v_{14})$ | X       | X       | X       | X                | X                |
| $V_{11}$ | $\frac{1}{2} (v_8 - v_{14})$ | X       | X       | X       | X                | X                |
| $V_{12}$ | $\frac{1}{2} (v_{12} + v_{15})$ | X       | X       | X       | X                | X                |
| $V_{13}$ | $\frac{1}{2} (v_{12} - v_{15})$ | X       | X       | X       | X                | X                |

A careful inspection shows that in three cases the non-standard decompositions for the Hamiltonian $H_{\text{osc}}$ are not equivalent to the standard ones. Nevertheless, in all three cases these decompositions can be recovered from their corresponding standard ones after changing the Hamiltonian $H_{\text{osc}} = a^\dagger a \cdot I_4 + \Lambda$, with $\Lambda = \text{diag}(0,0,1,1)$, into the permuted Hamiltonian $\overline{H}_{\text{osc}}$ given by

$$
\overline{H}_{\text{osc}} = a^\dagger a \cdot \overline{I}_4 + \overline{\Lambda}, \quad \text{with} \quad \overline{\Lambda} = \text{diag}(0,1,1,0).
$$

These three non-standard multi-particle quantizations are denoted as $(3|1)_{ns}$, $(2|2)_{ns}$, $\mathbb{Z}_2^2$-PF$_{ns}$. Their corresponding graded algebras are

$$(3|1)_{ns} : \alpha_2 \quad \text{for} \quad H_{osc} \mapsto \overline{H}_{osc},$$

$$(2|2)_{ns} : \alpha_3 \quad \text{for} \quad H_{osc} \mapsto \overline{H}_{osc},$$

$$(\mathbb{Z}_2^2)_{PF} : \alpha_5 \quad \text{for} \quad H_{osc} \mapsto \overline{H}_{osc}. \quad (36)$$

4.3. The 2-particle Hilbert spaces

The orthonormal vectors spanning the two-particle Hilbert spaces, from (23), (26), (27), have the form

$$
|m; \ell \rangle = \frac{1}{\sqrt{m!}} \left( \frac{i}{2} (\partial_x + \partial_y - x - y) \right)^m \cdot \left( \pi^{-\frac{1}{2}} e^{-\frac{1}{2}(x^2+y^2)} \right) \cdot V_i,
$$

where $V_i$ are $16$-component constant orthonormal vectors which can be expressed in the $v_j$ basis (we recall that $v_j$ has entry $1$ in the $j$th position and $0$ otherwise).
The two-particle Hilbert spaces induced by the 6 standard gradings will be denoted as $\mathcal{H}_k^{(2)}$; the suffix $k = 1, 2, \ldots, 6$ denotes the respective (30) graded algebras. The finite dimensional Hilbert spaces $\mathcal{H}_k = \mathcal{H}_k^{(2)} \subset \mathcal{H}_k^{(3)}$ are spanned by the $V_I$ vectors by taking $m = 0$ (the Gaussian factor can be dropped for convenience). The spanning vectors $V_I$ entering the six standard quantizations (31) are read from the following table.

The construction of the vectors corresponding to the three columns (40) (bosonic case), (2|2) (supersymmetric case) and $Z_2\text{-PF}$ was presented in [6].

One can observe, in certain cases, the absence of the vectors $\mathcal{H}_k^{(2)}$ recovered from the standard decompositions are therefore spanned by the vectors $|m; I\rangle$, with $m = 0, 1, 2, \ldots$, while $I$ is restricted according to

$$d_1 = d_5 = 10, \quad d_2 = 9, \quad d_3 = d_6 = 8, \quad d_4 = 7.$$  \hspace{1cm} (38)

The six two-particle Hilbert spaces $\mathcal{H}_k^{(2)}$ recovered from the standard decompositions are therefore spanned by the vectors $|m; I\rangle$, with $m = 0, 1, 2, \ldots$, while $I$ is restricted according to

$$\mathcal{H}_1^{(2)} |m; I\rangle \quad \text{for} \quad m \in \mathbb{N}_0 \quad \text{and} \quad I = 1, 2, 3, 4, 5, 6, 7, 8, 10, 12;$$

$$\mathcal{H}_2^{(2)} |m; I\rangle \quad \text{for} \quad m \in \mathbb{N}_0 \quad \text{and} \quad I = 1, 2, 3, 5, 6, 7, 8, 10, 12;$$

$$\mathcal{H}_3^{(2)} |m; I\rangle \quad \text{for} \quad m \in \mathbb{N}_0 \quad \text{and} \quad I = 1, 2, 3, 5, 6, 7, 8, 10, 13;$$

$$\mathcal{H}_4^{(2)} |m; I\rangle \quad \text{for} \quad m \in \mathbb{N}_0 \quad \text{and} \quad I = 1, 5, 6, 7, 9, 11, 13;$$

$$\mathcal{H}_5^{(2)} |m; I\rangle \quad \text{for} \quad m \in \mathbb{N}_0 \quad \text{and} \quad I = 1, 2, 5, 6, 7, 9, 11, 12;$$

$$\mathcal{H}_6^{(2)} |m; I\rangle \quad \text{for} \quad m \in \mathbb{N}_0 \quad \text{and} \quad I = 1, 2, 3, 4, 5, 6, 7, 9, 11, 13.$$  \hspace{1cm} (39)

One should note, see (36), that the three two-particle Hilbert spaces recovered from the non-standard decompositions coincide with the associated standard Hilbert spaces. The difference is encoded in the modified Hamiltonian, $H^{(2)}_{\text{osc}} \mapsto \overline{H}^{(2)}_{\text{osc}}$, with the latter given in (35). Therefore, we have

$$\mathcal{H}_2^{(1)} \text{ for (3|1)$_{\text{ns}}$,} \quad \mathcal{H}_3^{(2)} \text{ for (2|2)$_{\text{ns}}$,} \quad \mathcal{H}_5^{(2)} \text{ for } \overline{Z}_2\text{-PF$_{\text{ns}}$.}$$  \hspace{1cm} (40)

Any vector $|m; I\rangle$ is an energy eigenstate.

For the standard quantizations the energy eigenvalues $E_{m; I}$ are read from

$$H^{(2)}_{\text{osc}} |m; I\rangle = E_{m; I} |m; I\rangle, \quad \text{with} \quad E_{m; I} = m + S_I$$

$$<S_1 = S_2 = S_5 = 0, \quad S_6 = S_7 = S_8 = S_9 = S_{10} = S_{11} = 1, \quad S_{12} = S_{13} = 2>.$$  \hspace{1cm} (41)

For the non-standard quantizations the energy eigenvalues $\overline{E}_{m; I}$ are read from

$$\overline{H}^{(2)}_{\text{osc}} |m; I\rangle = \overline{E}_{m; I} |m; I\rangle, \quad \text{with} \quad \overline{E}_{m; I} = m + \overline{S}_I$$

$$<\overline{S}_1 = \overline{S}_2 = \overline{S}_3 = \overline{S}_6 = \overline{S}_{10} = \overline{S}_{11} = \overline{S}_{12} = \overline{S}_{13} = 1, \overline{S}_2 = \overline{S}_3 = \overline{S}_8 = \overline{S}_9 = 2>.$$  \hspace{1cm} (42)
Table 2. The numbers give the degeneracy of the two-particle energy eigenvalues for each one of the nine quantizations of the $4 \times 4$ quantum oscillator. Different numbers indicate inequivalent quantizations. The inequivalence of the quantizations 1 versus 6, 3 versus 5 and 8 versus 9 cannot be read from this table; it requires a subtler analysis of other observables.

|   | $E = 0$ | $E = 1$ | $E = n \geq 2$ |
|---|---------|---------|----------------|
| 1$^*$—(4|0) | 3 | 7 | 10 |
| 2—(3|1) | 3 | 7 | 9 |
| 3$^*$—(2|2) | 3 | 7 | 8 |
| 4—(1|3) | 2 | 6 | 7 |
| 5$^*$—$Z_2^+$—PF | 3 | 7 | 8 |
| 6$^*$—$Z_2^+$—PB | 3 | 7 | 10 |
| 7—(3|1)$_{ns}$ | 2 | 6 | 9 |
| 8—(2|2)$_{ns}$ | 2 | 6 | 8 |
| 9$^*$—$Z_2^+$—PF$_{ns}$ | 2 | 6 | 8 |

For all quantizations (standard and non-standard) the spectrum of the energy eigenvalues $E_n$ is given by the non-negative integers $0, 1, 2, \ldots$:

$$E_n = n \in \mathbb{N}_0. \quad (43)$$

We now discuss the degeneracy of the energy levels and the inequivalence of the multi-particle quantizations.

4.4. Degeneracy of the energy levels

The degeneracy of an energy level depends on the given quantization and is obtained from (41), (42). The results are summarized in the table 2 below which presents the nine cases (1–6 corresponding to the standard decompositions, 7, 8 and 9 to the non-standard ones). For any given quantization the degeneracy of its energy levels $n = 2, 3, 4, \ldots$ is the same. We have

The final proof of the inequivalence of the nine quantizations is given in section 5 with the construction of the observables discriminating the cases 1 versus 6 and 8 versus 9. The observables discriminating the cases 3 versus 5 are found in [6].

Remark: due to the coassociativity of the coproduct, see (A.11), nine inequivalent $M$-particle graded Hilbert spaces are recovered for any integer number $M > 1$. The formulas are straightforward generalizations of the two-particle construction. In [6] inequivalent three-particle Hilbert spaces were presented for the supersymmetric and (standard) parafermionic gradings.

5. Discriminating two-particle observables

In (39) we presented the two-particle Hilbert spaces $\mathcal{H}_k^{(2)}$ (for $k = 1, 2, \ldots, 6$) which were used to derive the nine (standard and non-standard) quantizations entering table 2. We present here the proof that these nine quantizations are all inequivalent.
Since the construction of the observables which discriminate the parafermionic case \( \mathbb{Z}_2^2\text{-PF} \) from the supersymmetric case \( (2|2) \) was given in [6], what is left here is to present:

(i) Observables which discriminate the parabosonic case \( \mathbb{Z}_2^2\text{-PB} \) from the bosonic case \( (4|0) \), 
(ii) At least one observable which discriminates the non-standard cases \( \mathbb{Z}_2^2\text{-PF}_{m} \) versus \((2|2)_{m}\).

Let us proceed.

5.1. Discriminating \( \mathbb{Z}_2 \times \mathbb{Z}_2\)-graded parabosons from bosons

The two-particle observables discriminating parabosons from bosons should satisfy the following requirements:

(i) They should apply to both bosonic and parabosonic Hilbert spaces,
(ii) They should be Hermitian and 
(iii) They should belong to the 00-graded sector of the parabosonic theory in order to have real (00-graded) eigenvalues.

The following set of two-particle observables, constructed in terms of the exchange operators \( X_{11}, X_{10}, X_{01} \) introduced in (20), satisfy the above three criteria.

We have

\[
X_s = X_{10} \otimes X_{10}, \quad X_t = X_{01} \otimes X_{01}, \quad X_u = X_{11} \otimes X_{11}, \quad X_\ast = X_t + X_s + X_u
\]

and

\[
Y_s = (I_4 \otimes X_{11} + X_{11} \otimes I_4)(I_4 \otimes X_{10} + X_{10} \otimes I_4)(I_4 \otimes X_{01} + X_{01} \otimes I_4),
\]

\[
Y_t = (I_4 \otimes X_{10} + X_{10} \otimes I_4)(I_4 \otimes X_{01} + X_{01} \otimes I_4)(I_4 \otimes X_{11} + X_{11} \otimes I_4),
\]

\[
Y_u = (I_4 \otimes X_{01} + X_{01} \otimes I_4)(I_4 \otimes X_{11} + X_{11} \otimes I_4)(I_4 \otimes X_{10} + X_{10} \otimes I_4),
\]

\[
Y_\ast = X_s + Y_t + Y_u.
\]

Under the \( S_3 \) permutations which interchange the parabosonic sectors 11, 10, 01, the operators \( X_s, X_t, Y_s, Y_t \) are mapped into \( X_u \) \( (Y_u) \), while \( X_u \) and \( Y_\ast \) are \( S_3 \)-invariant. Without loss of generality we can therefore consider the four operators \( X_u, X_s, Y_t, Y_\ast \). Their 16 \( \times \) 16 matrix representations are given in appendix B.

For the purpose of making easier the comparison of the bosonic versus parabosonic Hilbert spaces it is convenient to rename the respective vectors \( V_I \) entering table 1.

They will be expressed in terms of a sign \( \varepsilon \) (\( \varepsilon = +1 \) for bosons, \( \varepsilon = -1 \) for parabosons); the corresponding finite-dimensional Hilbert spaces will be denoted as \( H^{(2)} \). The \( \varepsilon \) sign encodes the property that the bosonic wave functions are totally symmetric, while the parabosonic wave functions have mixed symmetry. We set
Unlike $X$, one can determine whether a system under consideration is composed by ordinary bosons or the presence of the tensor products in (45). This observation explains the presence of the eigenvalues obtained from the $XuU$, $YvW$, $Z\times Z\ast$-graded parabosons.

The eigenvectors of $XuU$ with nonvanishing eigenvalues are $U_\pm$ and $W_{11,\varepsilon}$:

$$X_u U_\pm = \pm U_\pm \quad \text{for} \quad U_\pm = U_{00,C} \pm U_{00,D}, \quad X_u W_{11,\varepsilon} = \varepsilon W_{11,\varepsilon}. \quad (47)$$

The eigenvectors of $X_\varepsilon$ with their respective nonvanishing eigenvalues are

$$X_\varepsilon (U_{00,B} - U_{00,C}) = -(U_{00,B} - U_{00,C}), \quad X_\varepsilon W_{11,\varepsilon} = \varepsilon W_{11,\varepsilon},$$

$$X_\varepsilon (U_{00,C} - U_{00,D}) = -(U_{00,C} - U_{00,D}), \quad X_\varepsilon W_{10,\varepsilon} = \varepsilon W_{10,\varepsilon}, \quad (48)$$

$$X_\varepsilon (U_{00,D} - U_{00,B}) = -(U_{00,D} - U_{00,B}), \quad X_\varepsilon W_{01,\varepsilon} = \varepsilon W_{01,\varepsilon}. \quad (49)$$

The presence of the $\varepsilon$ eigenvalues in (47), (48) proves that, by performing $X_\varepsilon, X_\varepsilon$ measurements, one can determine whether a system under consideration is composed by ordinary bosons or by $Z_2 \times Z_2$-graded parabosons.

A basis of eigenvectors with respective eigenvalues for $Y_\varepsilon$ is given by

$$Y_\varepsilon U_{00,A} = 0,$$

$$Y_\varepsilon (U_{00,B} + U_{00,C} + U_{00,D}) = (12 + 4\varepsilon)(U_{00,B} + U_{00,C} + U_{00,D}),$$

$$Y_\varepsilon (U_{00,B} - U_{00,C}) = -2\varepsilon(U_{00,B} - U_{00,C}),$$

$$Y_\varepsilon (U_{00,C} - U_{00,D}) = -2\varepsilon(U_{00,C} - U_{00,D}),$$

$$Y_\varepsilon U_{11} = 2U_{11},$$

$$Y_\varepsilon U_{10} = 2U_{10},$$

$$Y_\varepsilon U_{01} = 2U_{01},$$

$$Y_\varepsilon W_{11,\varepsilon} = (6 + 4\varepsilon)W_{11,\varepsilon},$$

$$Y_\varepsilon W_{10,\varepsilon} = (6 + 4\varepsilon)W_{10,\varepsilon},$$

$$Y_\varepsilon W_{01,\varepsilon} = (6 + 4\varepsilon)W_{01,\varepsilon}. \quad (49)$$

Unlike $X_\varepsilon$, the operator $Y_\varepsilon$ is $\varepsilon$-dependent, see formula (B.2), due to the braiding properties of the tensor products in (45). This observation explains the presence of the $\varepsilon$ sign in the eigenvalues obtained from the $U_{00,\bullet}$ vectors.
The $Y_u$ eigenvectors and eigenvalues are
\begin{align}
Y_u U_{00,0} &= 0, \\
Y_u U_{11} &= 2 U_{11}, \\
Y_u U_{10} &= 0, \\
Y_u U_{01} &= 0, \\
Y_u W_{11,0} &= 2 W_{11,0}, \\
Y_u W_{10,0} &= (2 + 2 \varepsilon) W_{10,0}, \\
Y_u W_{01,0} &= (2 + 2 \varepsilon) W_{01,0}, \\
Y_u (U_{00,C} - U_{00,D}) &= -2 \varepsilon (U_{00,C} - U_{00,D}) \quad (50)
\end{align}

and, for $\varepsilon = 1$,
\begin{align}
Y_u (U_{00,B} + \frac{1}{2} U_{00,C} + \frac{1}{2} U_{00,D}) &= 6 (U_{00,B} + \frac{1}{2} U_{00,C} + \frac{1}{2} U_{00,D}), \\
Y_u (U_{00,B} - U_{00,C} - U_{00,D}) &= 0, \quad (51)
\end{align}

while, for $\varepsilon = -1$, one has
\begin{align}
Y_u \left( U_{00,B} + \frac{1}{4} (-3 \pm \sqrt{17}) (U_{00,C} + U_{00,D}) \right) \\
= (1 \pm \sqrt{17}) \left( U_{00,B} + \frac{1}{4} (-3 \pm \sqrt{17}) (U_{00,C} + U_{00,D}) \right). \quad (52)
\end{align}

5.2. Discriminating two non-standard quantizations

The matrix operator $X_u$ is an observable for both Hilbert spaces giving the non-standard cases $\mathbb{Z}_2^2 - \text{PF}_{ns}$ and $(2|2)_{ns}$. We rename the vectors entering the finite-dimensional Hilbert spaces as
\begin{align}
V_1 &= v_1, \quad V_2 = v_5, \quad V_3 = \frac{1}{\sqrt{2}} (v_2 + v_8), \quad V_4 = \frac{1}{\sqrt{2}} (v_3 + v_9), \quad V_5 = \frac{1}{\sqrt{2}} (v_4 + v_13), \\
V_{6,\delta} &= \frac{1}{\sqrt{2}} (v_7 + \delta v_{10}), \quad V_{7,\delta} = \frac{1}{\sqrt{2}} (v_8 + \delta v_{14}), \quad V_{8,\delta} = \frac{1}{\sqrt{2}} (v_{12} - \delta v_{15}). \quad (53)
\end{align}

The sign $\delta = \pm 1$ corresponds to the $\mathbb{Z}_2^2 - \text{PF}_{ns}$ case for $\delta = -1$ and to the $(2|2)_{ns}$ case for $\delta = 1$.

The $X_u$ eigenvalues are read from
\begin{align}
X_u V_J &= 0 \quad (\text{for } J = 1, 2, 3, 4, 5), \\
X_u V_{6,\delta} &= 0, \\
X_u V_{7,\delta} &= 0, \\
X_u V_{8,\delta} &= \delta V_{8,\delta}, \quad (54)
\end{align}

Due to the presence of the $\delta$ eigenvalue in the last equation, a measurement of $X_u$ allows to discriminate the two non-standard cases.
This concludes the proof of the inequivalence of the nine quantizations presented in table 2.

6. Conclusions

This paper presents the 9 inequivalent multi-particle quantizations of the $4 \times 4$ matrix oscillator given in (8). Each quantization is recovered from different $\mathbb{Z}_2$- and $\mathbb{Z}_2 \times \mathbb{Z}_2$-gradings (and associated statistics) which are consistently imposed on the component particles. 6 of the quantizations are obtained from the standard block-decompositions of supermatrices, 3 of them from the non-standard ones.

This analysis completes the multi-particle quantizations discussed in [6] for just three cases (bosonic, supersymmetric and the standard version of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded parafermions).

The extra quantizations presented in the paper include, in particular, a non-standard version of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded parafermions and the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded parabosonic statistics induced by $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras; unlike the parafermionic statistics induced by $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebras, see [6, 20–25], this parastatistic has not been previously considered in the literature.

Furthermore, we showed that suitable measurements of observables allow to distinguish if a multi-particle system is composed by $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded parabosons or by ordinary bosons. This result gives to the notion of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded parabosons a legitimate status in physics, proving that it is not just a mathematical artifact void of physically measurable consequences.

A next step, in this line of research, would involve the construction of phenomenological $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded parabosonic models which could be put to experimental test. A possible scenario could apply to emergent particles in condensed matter. Concerning model-building, a general framework to construct $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded parafermionic models was presented in [4] for the classical case and [16] for the quantum case. As pointed out in [5], an extension of the method allows to derive $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded parabosonic models.

On a separate line, the inequivalent multi-particle quantizations induced by gradings shed some light on open issues regarding the quantization, as discussed in [38] both from a historical and an actual perspective.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix A: Relevant formulas for graded (super)algebras

In order to make the paper self-consistent we collect the relevant formulas concerning
(i) $\mathbb{Z}_2$-graded Lie superalgebras,
(ii) $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebras and 
(iii) $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras.  
(A.1)

As recalled in the text they induce inequivalent multi-particle quantizations of the $4 \times 4$ matrix harmonic oscillator (8). Following [5, 6] we use a compact notation to describe, at once, the three cases. Ordinary (bosonic) Lie algebras can be assumed to be $\mathbb{Z}_2$-graded Lie superalgebras with empty odd (fermionic) sector. This allows, e.g. to identify the bosonic case listed in table 1 with $(4|0)$ supermatrices.

Depending on the case under consideration, a graded algebra $g$ is decomposed into

(i) : $g = g_0 \oplus g_1$,
(ii) and (iii) : $g = g_{00} \oplus g_{01} \oplus g_{10} \oplus g_{11}$.  
(A.2)

The even (0) and odd (1) generators in (i) are bosonic (fermionic). The four sectors of (ii) and (iii) are described by 2 bits. The grading of a generator in (i) is given by $\vec{\alpha} = \alpha \in \{0, 1\}$. The grading of a generator in (ii) and (iii) is given by the pair $\vec{\alpha} = (\alpha_1, \alpha_2)$, with $\alpha_{1,2} \in \{0, 1\}$.

Three respective inner products, with addition mod 2, are defined:

(i) : $\vec{\alpha} \cdot \vec{\beta} := \alpha \beta \in \{0, 1\}$,
(ii) : $\vec{\alpha} \cdot \vec{\beta} := \alpha_1 \beta_1 + \alpha_2 \beta_2 \in \{0, 1\}$,
(iii) : $\vec{\alpha} \cdot \vec{\beta} := \alpha_1 \beta_2 - \alpha_2 \beta_1 \in \{0, 1\}$.  
(A.3)

The graded algebra $g$ is endowed with the operation $(\cdot, \cdot) : g \times g \rightarrow g$.

Let $a, b, c \in g$ be three generators whose respective gradings are $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$. The bracket $(\cdot, \cdot)$ is defined as

$$(a, b) := ab - (-1)^{\vec{\alpha} \cdot \vec{\beta}} ba,$$  
(A.4)

resulting in either commutators or anticommutators.

The operation satisfies the graded Jacobi identities

$$( -1 )^{\vec{\alpha} \cdot \vec{\beta}} (a, (b, c)) + ( -1 )^{\vec{\beta} \cdot \vec{\gamma}} (b, (c, a)) + ( -1 )^{\vec{\gamma} \cdot \vec{\alpha}} (c, (a, b)) = 0.$$  
(A.5)

The grading $\text{deg}[(a, b)]$ of the generator $(a, b)$ is the mod 2 sum

$\text{deg}[(a, b)] = \vec{\alpha} + \vec{\beta}$.  
(A.6)

**Remark:** in the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superalgebra case (ii) the only sectors which are on equal footing and can be interchanged are 10 and 01. In the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded algebra case (iii) the three sectors 11, 10, 01 are on equal footing and can be interchanged. In sections 4 and 5 we made use of this observation.

A graded algebra $g$ is represented on a graded vector space $\mathcal{V}$ such that

(i) : $\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_1$;  
(ii) and (iii) : $\mathcal{V} = \mathcal{V}_{00} \oplus \mathcal{V}_{01} \oplus \mathcal{V}_{10} \oplus \mathcal{V}_{11}$.  
(A.7)
The grading of a vector $v \in \mathcal{V}$ is denoted with $\vec{\nu}$. Depending on the case, it is either $\vec{\nu} \equiv \nu \in \{0, 1\}$ or $\vec{\nu} = \nu_1, \nu_2$ with $\nu_{1,2} \in \{0, 1\}$. A generator $a \in \mathfrak{g}$ (of grading $\vec{\alpha}$) is represented by the operator $\hat{a} \in \text{End}(\mathcal{V})$. The compatibility of the gradings requires that the grading $\vec{\nu}'$ of the transformed vector $v' = av \in \mathcal{V}$ is $\vec{\nu}' = \vec{\alpha} + \vec{\nu}$. The sum is taken mod 2.

Without loss of generality, see the discussion in section 4, we can assume the $\mathbb{Z}_2$-graded matrices to be split into block-diagonal even and odd sectors. For the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded (super)algebras, also without loss of generality, the $4 \times 4$ graded matrices can be decomposed according to

$$M_{00} = \begin{pmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & m_4 \end{pmatrix} \in \mathfrak{g}_{00}, \quad M_{11} = \begin{pmatrix} 0 & m_5 & 0 & 0 \\ m_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_7 \\ 0 & 0 & m_8 & 0 \end{pmatrix} \in \mathfrak{g}_{11},$$

$$M_{10} = \begin{pmatrix} 0 & 0 & m_9 & 0 \\ 0 & 0 & 0 & m_{10} \\ m_{11} & 0 & 0 & 0 \\ 0 & m_{12} & 0 & 0 \end{pmatrix} \in \mathfrak{g}_{10}, \quad M_{01} = \begin{pmatrix} 0 & 0 & 0 & m_{13} \\ 0 & 0 & m_{14} & 0 \\ 0 & m_{15} & 0 & 0 \\ m_{16} & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{g}_{01}.\quad (A.8)$$

where the entries $m_1, m_2, \ldots, m_{16}$ are either constant numbers or, as in (4), operators.

By assuming this convention, the graded vector space $\mathcal{V}$ is decomposed according to

$$v_{00} = \begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix} \in \mathcal{V}_{00}, \quad v_{11} = \begin{pmatrix} 0 \\ v \\ 0 \end{pmatrix} \in \mathcal{V}_{11}, \quad v_{10} = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} \in \mathcal{V}_{10}, \quad v_{01} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ v \end{pmatrix} \in \mathcal{V}_{01}.\quad (A.9)$$

The construction of the multi-particle states is based, see [6, 7], on the notions of coproduct and braided tensor product as applied in the context of Hopf algebras. For all three cases (i), (ii), (iii) in (A.1) the universal enveloping algebra $U \equiv U(\mathfrak{g})$ of a graded algebra $\mathfrak{g}$ is endowed with a Hopf algebra structure. Definition and properties of Hopf algebras can be found in [7]. We limit here to recall the properties that we have used in the main text.

The coproduct $\Delta$ is a map

$$\Delta : U \rightarrow U \otimes U \quad (A.10)$$

which satisfies the coassociativity property

$$(\Delta \otimes \text{id})\Delta(U) = (\text{id} \otimes \Delta)\Delta(U). \quad (A.11)$$

The action $\Delta(u)$ of the coproduct on a generic element $u \in U$ can be recovered from the action on the identity $1 \in U(\mathfrak{g})$, the action on a primitive element $g \in \mathfrak{g}$ and from the comultiplication. We have

$$\Delta(1) = 1 \otimes 1,$$

$$\Delta(g) = 1 \otimes g + g \otimes 1,$$

$$\Delta(u_1u_2) = \Delta(u_1) \cdot \Delta(u_2). \quad (A.12)$$
Concerning the braided tensor product, let $a, b, c, d \in g$. We assume, as before, the grading of $b, c$ to be respectively given by $\vec{\beta}, \vec{\gamma}$. The braiding of two tensor spaces is expressed by the formula

$$ (a \otimes b) \cdot (c \otimes d) = (-1)^{\vec{\beta} \cdot \vec{\gamma}} ac \otimes bd. \quad (A.13) $$

For the $\mathbb{Z}_2$- and $\mathbb{Z}_2 \times \mathbb{Z}_2$-gradings the braiding corresponds to the $(-1)^{\vec{\beta} \cdot \vec{\gamma}}$ sign. Its expression, depending on case (i), (ii) or (iii), is given in formula (A.3).

**Appendix B. Representations of two-particle observables**

We present here for completeness the $16 \times 16$ constant Hermitian matrices which realize the two-particle observables $X_u, X_*, Y_u, Y_*$ introduced in section 5, formulas (44) and (45). These observables allow to discriminate whether the system under consideration is composed by ordinary bosons or by $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded parabosons.

We have

$$ X_u = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} $$

(B.1)

$$ X_* = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} $$

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The $\varepsilon$ sign entering the (B.2) matrices takes the value $\varepsilon = +1$ in the bosonic case and $\varepsilon = -1$ in the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded parabosonic case. Unlike $X_\mu$, $X_\ast$, the operators $Y_\mu$, $Y_\ast$ are $\varepsilon$-dependent.

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