Lower and Upper Bound for Computing the Size of All Second Neighbourhoods

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Abstract

We consider the problem of computing the size of each \( r \)-neighbourhood for every vertex of a graph. Specifically, we ask whether the size of the closed second neighbourhood can be computed in subquadratic time.

Adapting the SETH reductions by Abboud et al. (2016) that exclude subquadratic algorithms to compute the radius of a graph, we find that a subquadratic algorithm would violate the SETH. On the other hand, a linear fpt-time algorithm by Demaine et al. (2014) parameterized by a certain ‘sparseness parameter’ of the graph is known, where the dependence on the parameter is exponential. We show here that a better dependence is unlikely: for any \( \delta < 2 \), no algorithm running in time \( O(2^{\alpha \nu_c(G) \cdot n^\delta}) \), where \( \nu_c(G) \) is the vertex cover number, is possible unless the SETH fails.

We supplement these lower bounds with algorithms that solve the problem in time \( O(2^{\nu_c(G)/2} \cdot \nu_c(G)^2 \cdot n) \) and \( O(2^{\omega \cdot w} \cdot n) \).

1 Introduction

For a vertex \( v \) of a graph \( G \) and an integer \( r \geq 1 \), \( N^r(v) \) (\( N^r[v] \), respectively) denotes the set of vertices of \( G \) of distance exactly (at most, respectively) \( r \) from \( v \). For a graph \( G \), \( |G| \) will denote the number of vertices in \( G \). As usual in graph algorithms literature, unless defined differently, \( n \) and \( m \) will denote the number of vertices and edges in the input graph. In this paper, we consider the following two basic problems on graphs.

\textbf{\( r \)-Neighbourhood Sizes}

\begin{tabular}{|l|}
\hline
\textit{Input:} & A graph \( G \) and an integer \( r \). \\
\textit{Problem:} & Compute for every vertex \( v \in G \) the size of \( N^r(v) \). \\
\hline
\end{tabular}

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A graph $G$ and an integer $r$.

Problem: Compute for every vertex $v \in G$ the size of $N^r[v]$.

Since both problems are easily Turing-reducible to each other, we focus on the closed neighbourhood variant in the following with the understanding that all results transfer to the open neighbourhood variant. Without loss of generality, we will assume in the remainder of the paper that the input graph is connected. Clearly we can solve the above problems in time $O(n(m + n))$ by conducting a (truncated) breadth-first search from every vertex. This means $\Omega(n^2)$ time even for sparse connected graphs. The following question is natural: can we solve Closed $r$-Neighbourhood Sizes in a subquadratic (in $n$) time even for $r = 2$? We will show in Theorem 1 that this is not possible provided the Strong Exponential Time Hypothesis (SETH) holds. SETH has been put forward by Impagliazzo and Paturi [5], stating that, for every positive $\varepsilon < 1$, there exists an integer $r$ such that $r$-CNF SAT cannot be solved in time $O(2^{\varepsilon n})$, where $n$ is the number of variables in the input $r$-CNF formula. More precisely, define $s_r$ to be the infimum over all numbers $\delta$ for which there exists an algorithm that solves $r$-CNF SAT in time $2^{\delta n}(n + m)^{O(1)}$. The exponential time hypothesis (ETH) states that $s_3 > 0$, that is, there is no subexponential algorithm solving 3SAT. SETH asserts that the limit of the sequence $(s_r)_{r \in \mathbb{N}}$ is 1.

Since subquadratic algorithms seem to be out of reach for Closed $2$-Neighbourhood Sizes, we ask whether we can trade-off some of the polynomial complexity in the input size for an exponential dependence on some structural parameter of the input graph. Demaine et al. showed that a running time of $O(2^{\Delta_r(G)}n)$ is indeed possible (for the general Closed $r$-Neighbourhood Sizes problem) where $\Delta_r$ is a certain measure of the sparsity of $G$ [4] which we describe briefly below. Without going into further detail here, we note that $\Delta_r$ satisfies $\Delta_2(G) \leq \text{vc}(G)$, where $\text{vc}(G)$ is the minimum size of a vertex cover of $G$, i.e. a set which contains at least one vertex of every edge of $G$. Can we use the following trade-off in the running time of [4]: replace $n$ by a subquadratic function in $n$ and replace $\Delta_2(G)$ by $o(\text{vc}(G))$? We prove in Theorem 2 that the answer to this question is negative, assuming SETH. Therefore, since the parameters treewidth and tree-depth are smaller than the vertex cover number, the same impossibility result follows also if we replace $\text{vc}(G)$ by any of these parameters, see Corollary 1.

In contrast, we show in Theorem 3 that Closed 2-Neighbourhood Sizes can even be solved in linear time in $n$ if the factor is exponential in $\text{vc}(G)/2$ (where the base of the exponent is 2). In Theorem 4, we prove that the same result is true if we replace $\text{vc}(G)/2$ by treewidth.

## 2 Preliminaries

In the following we will make explicit use of the sparsification lemma by Calabro, Impagliazzo, and Paturi [2]:

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1We define these two parameters in the next section.
Lemma 1 (Sparsification Lemma [2]). For every $r \in \mathbb{N}$ and $\varepsilon > 0$ there exists an algorithm which, given an $r$-CNF formula $\phi$ over $n$ variables, outputs in time $2^{3n}n^{O(1)}$ a list of $r$-CNF formulas $(\psi_i)_{i \leq t}$, where $t \leq 2^{3n}$, such that

- $\phi$ is satisfiable if and only if at least one $\psi_i$ is satisfiable and
- each formula $\psi_i$ has at most $n$ variables, each of which occurring at most $O((\frac{\varepsilon}{r})^{3r})$ times.

We now formally define the notions of a tree decomposition and of a nice tree decomposition, which are key to our analysis below.

Definition 1. Given a graph $G = (V,E)$, a tree decomposition of $G$ is a pair $(T, \beta)$, where $T$ is a tree and $\beta : V(T) \to 2^V$ such that $\bigcup_{x \in V(T)} \beta(x) = V$, for each edge $uv \in E$, there exists a node $x \in V(T)$ such that $u, v \in \beta(x)$, and for each $v \in V$, the set $\beta^{-1}(v)$ of nodes form a connected subgraph (i.e. a subtree) in $T$.

The width of $(T, \beta)$ is $\max_{x \in V(T)}(|\beta(x)| - 1)$. The treewidth of $G$ (denoted by $\text{tw}(G)$) is the minimum width of all tree decompositions of $G$.

A path decomposition of $G$ is defined similar to a tree decomposition of $G$, but the only trees $T$ allowed are paths. This leads to the pathwidth of $G$ denoted $\text{pw}(G)$.

Definition 2. Given an undirected graph $G = (V,E)$, a nice tree decomposition $(T, \beta)$ is a tree decomposition such that $T$ is a rooted tree, and each of the nodes $x \in V(T)$ falls under one of the following classes:

- $x$ is a Leaf node: then $x$ has no children in $T$;
- $x$ is an Introduce node: then $x$ has a single child $y$ in $T$, and there exists a vertex $v \notin \beta(y)$ such that $\beta(x) = \beta(y) \cup \{v\}$;
- $x$ is a Forget node: then $x$ has a single child $y$ in $T$, and there exists a vertex $v \in \beta(y)$ such that $\beta(x) = \beta(y) \setminus \{v\}$;
- $x$ is a Join node: then $x$ has exactly two children $y$ and $z$, and $\beta(x) = \beta(y) = \beta(z)$.

It is well-known [6] that any given tree decomposition of a graph can be transformed into a nice tree decomposition of the same width in polynomial time.

For a rooted tree $T$ and a node $i \in T$ we will write $T_i$ to denote the subtree of $T$ which includes $i$ and all its descendants. We consider $T_i$ to be rooted in $i$.

Besides width-measures like treedepth, pathwidth, and treewidth we will further consider the sparseness parameters $\nabla_1$ and $\nabla_2$ (see Definition [3]). Recall that a graph $H$ is a minor of a graph $G$ if $H$ can be obtained from $G$ by contracting a collection of disjoint connected subgraphs and then taking a (not necessarily induced) subgraph. If we impose the restriction that each contracted subgraph further has radius at most $r$ (that is, there exists a vertex in it from which every other vertex has distance at most $r$ within the subgraph), then we say that $H$ is an $r$-shallow minor of $G$ and we write $H \ll_r G$.

Recall that $H$ is a topological minor of a graph $G$ if we can select $|V(H)|$ vertices in $G$ (the nails) and connect them by $|E(H)|$ internally vertex-disjoint paths $P$ such that $uv \in H$ if and only if the corresponding nails $u'$, $v'$ in $G$ are connected by a path in $P$. If we further impose the restriction that all paths in $P$
have length at most $2r + 1$, then $H$ we say that $H$ is an $r$-shallow topological minor of $G$ and we write $H \preceq_r G$. Note that every $r$-shallow topological minor is in particular an $r$-shallow minor.

With these two containment notions, we can now define the sparseness parameters $\nabla_r$ and $\nabla_r$. For a more in-depth introduction to the topic of shallow minors we refer to the book by Nešetřil and Ossona de Mendez [7].

**Definition 3** (Grad and top-grad). For a graph $G$ and an integer $r \geq 0$, we define the greatest reduced average density (grad) at depth $r$ as

$$\nabla_r(G) = \max_{H \preceq_r G} \frac{|E(H)|}{|V(H)|}$$

and the topologically greatest reduced average density (top-grad) at depth $r$ as

$$\tilde{\nabla}_r(G) = \max_{H \preceq_r G} \frac{|E(H)|}{|V(H)|}.$$

The following is a simple observation relevant to our results below:

**Observation 1.** $\tilde{\nabla}_1(G) \leq \nabla_1(G) \leq \mathbf{vc}(G)$.

**Proof.** The first inequality follows immediately since every 1-shallow topological minor is also a 1-shallow minor. To prove the second inequality, let $X \subseteq V(G)$ be a minimal vertex cover of $G$ and let $H$ be a 1-shallow minor of $G$ (i.e. $H \preceq_1 G$) with $\nabla_0(H) = |E(H)|/|V(H)| = \nabla_1(G)$. Let us choose $H$ among all minors that satisfy this relation such that $|V(G)|$ is minimal.

Contracting an 1-shallow minor is equivalent to contracting a star forest. Let $\{S_x\}_{x \in \mathcal{H}}$ be the stars contracted to obtain $H$, identified by the resulting vertex in $H$. Note that every star $S_x$ with more than one vertex necessarily intersects with $X$, thus the number $*$ of such stars is $|\{S_x \in \mathcal{H} \mid \exists x \in X \subseteq V(H) \mid \nabla_0(H) \leq |X| \} \leq |X|$. Let us call these stars big and all other stars small. Note that a small star is simply a single vertex in $V(G) \setminus X$. It follows that for two small stars $S_x, S_y$ we have that $xy \notin H$.

If $\nabla_0(H) \leq |X|$ we are done, thus assume that $\nabla_0(H) > |X|$. By minimality, it follows that the minimum degree $\delta(H)$ of $H$ satisfies $\delta(H) \geq \nabla_0(H) > |X|$, otherwise we could remove a vertex of minimal degree without decreasing $\nabla_0(H)$, contradicting our choice of $H$. But then there cannot be any small stars since their corresponding vertices have degree at most $|X|$. We arrive at a contradiction since then only $|X|$ vertices remain in $H$, making a density of $|X|$ impossible.

Note that the bound of Observation [1] is asymptotically tight. Indeed, consider the graph $K_{s,t}$; clearly, if we keep $t$ fixed and let $s$ grow, the density approaches $t = \mathbf{vc}(K_{s,t})$ from below. Furthermore note that taking a 1-shallow minor does not affect this argument (for example, adding all edges to the side of size $t$ does not improve asymptotic bound).

Let us now discuss the parameter $\tilde{\Delta}_r(G)$. It is defined as the maximum in-degree of so-called transitive fraternal augmentations of $G$. The first augmentation $\tilde{G}_1$ is simply an acyclic augmentation that minimizes the maximum in-degree; $\tilde{G}_1$ is then computed from $\tilde{G}_{i-1}$ by the following two rules:

1. If $uv, vw \in \tilde{G}_{i-1}$ then $uw \in \tilde{G}_i$; and
2. if $uv, uw \in \bar{G}_{i-1}$ then either $uv \in \bar{G}_i$ or $wu \in \bar{G}_i$.

The orientation in the second case is chosen such that $\bar{G}_i$ has the smallest possible maximum in-degree. The following lemma illucidates the realtionship between dtf-augmentations and vertex covers. We have to phrase it slightly weaker than the bounds on $\nabla_1$ and $\nabla_i$ since the value of $\Delta_r$ depends on how the augmentation was computed.

Observation 2. There exists a dtf-augmentation of $G$ with $\Delta_r(G) \leq \text{vc}(G)$ for all $r \geq 1$.

Proof. Let $X$ be a minimal vertex cover of $G$ and let $Y := V(G) \setminus X$. We construct $\bar{G}_1$ by orienting all edges incident to $Y$ towards $Y$ and choose an arbitrary acyclic orientation for all other edges.

Any augmentation built from $\bar{G}_1$ will not add any out-arcs to $Y$; thus the maximum in-degree of vertices in $Y$ is $|Y|$. The maximum in-degree of a vertex in $X$ is $|X| - 1$ since no arc will point from $Y$ to $X$. This proves the claim. □

3 Lower Bounds

We adapt the construction of Abboud, Williams, and Wang \cite{ABW18} to verify our intuition that computing neighbourhood sizes is probably not possible in sub-quadratic time.

Theorem 1. For any $\varepsilon > 0$, Closed 2-Neighbourhood Sizes on a graph $G$ cannot be solved in time $O(|G|^{2-\varepsilon})$, unless SETH fails.

Proof. Consider a Satisfiability instance $\phi$ with variables $x_1, \ldots, x_n$ and a set $C$ of $m$ clauses. For simplicity, assume that $n$ is even and define $N := 2^{n/2}$. We partition the variables into sets $X_l := \{x_1, \ldots, x_{n/2}\}$ and $X_h := \{x_{n/2+1}, \ldots, x_n\}$.

Now construct a graph $G$ as follows: create one vertex for each of the $N = 2^{n/2}$ possible truth assignments $\alpha_l : X_l \rightarrow \{0, 1\}^{n/2}$ of $X_l$; call the set of these vertices $A := \{\alpha_l\}_{l \in \{N\}}$. Proceed similarly for $X_h$ and create a set $B$ of $N$ vertices, corresponding to all truth assignments $\beta_l : X_h \rightarrow \{0, 1\}^{n/2}$. Furthermore create one vertex $c_i$ for every clause in $\phi$ and call by $C$ the resulting set of $m$ vertices. Finally, create two additional vertices $v_a, v_b$.

Now for every partial assignment $\gamma \in A \cup B$, connect $\gamma$ to each clause $c \in C$ which is not satisfied by $\gamma$ (we consider a clause to be satisfied under a partial assignment if at least one variable of the clause is set to true or at least one negative variable is set to false). Finally, connect vertex $v_a$ to vertex $v_b$ and to all vertices in $A \cup C$, and connect vertex $v_b$ to all vertices in $B \cup C$. This concludes the construction of $G$, which can be executed in $O(Nm)$ time. For an illustration of this construction see Figure 1.

Note that, if there exist partial truth assignments $\alpha \in A$ and $\beta \in B$ such that $N[\alpha] \cap N[\beta] = \emptyset$, then the truth assignment $(\alpha, \beta)$ satisfies $\phi$. By construction, $A \cup C \cup \{v_a, v_b\} \subseteq N^2[\alpha]$ for every $\alpha \in A$. Furthermore, for every $\beta \in B \cap N^2[\alpha]$ we know that the truth assignment $(\alpha, \beta)$ does not satisfy $\phi$. We therefore can reformulate the condition under which a satisfying truth assignment $(\alpha, \beta)$ does exist: if for any $\alpha \in A$ we have that $|N^2[\alpha]| < |A| + |B| + |C| + 2 = 2N + m + 2$, then there must be a
some $\beta \in B \setminus N^2[\alpha]$, and thus $(\alpha, \beta)$ is satisfying. Note that the reverse holds as well: if there is a satisfying assignment for $\phi$, the respective restrictions to $X_l$ and $X_h$ are vertices in $G$ with the aforementioned property.

Assume that we can solve \textit{Closed 2-Neighbourhood Sizes} for $G$ in time $O(|G|^{2-\varepsilon})$ for some $\varepsilon > 0$. Since the output consists of $|G|$ numbers, we can test in time $O(|G| \log |G|)$ whether some vertex in $A$ has strictly less than $2N + m + 2 \leq 2$-neighbours. But then we could find a satisfying assignment for $\phi$ in time $O(Nm + |G|^{2-\varepsilon} + |G| \log |G|) = 2^{n(1-\varepsilon/2)}mO(1)$, contradicting SETH.

\textbf{Theorem 2.} For any $\delta < 2$, \textit{Closed 2-Neighbourhood Sizes} cannot be solved in time $O(2^{o(\text{vc}(G))} n^\delta)$, unless SETH fails.

\textbf{Proof.} Let $\phi$ be an $r$-CNF formula and let $\varepsilon > 0$ be some constant we will fix later. Using the sparsification lemma, we construct $t \leq 2^{n}n$ formulas $(\psi_i)_{i \leq t}$, each on $n_i \leq n$ variables and $m_i = O((\frac{r}{2})^3 n)$ clauses. For each $\psi_i$ in turn, we apply the reduction from Lemma 1. Notice that the resulting graph $G$ has $C \cup \{v_a, v_b\}$ as a vertex cover and thus $\text{vc}(G) \leq |C| + 2 = O((\frac{r}{2})^3 n)$.

Assume towards a contradiction that we can solve \textit{Closed 2-Neighbourhood Sizes} in time $O(2^{o(\text{vc}(G))} N^\delta)$. By inspecting the output of this hypothetical algorithm, we can determine again in time $O(N \log N)$ whether $\psi_i$ is satisfiable. The total running time of this algorithm would therefore be

$$O(2^{o(\text{vc}(G))} N^\delta + N \log N) = O(2^{o((\frac{r}{2})^3 n)}2^{n/2} + n/2)nO(n).$$

Thus deciding whether the original formula $\phi$ is decidable would be possible in total time

$$2^{n}nO(1) + 2^{n} \cdot 2^{o((\frac{r}{2})^3 n) + \delta/2}nO(n) = 2^{o(\text{vc}(G)) + \delta/2 + \varepsilon}nO(1).$$

For appropriate choices of $\varepsilon$, we can ensure that asymptotically $o((\frac{r}{2})^3 n) + \delta/2 + \varepsilon < 1$. But then the resulting algorithm contradicts the SETH and we conclude that the statement of the theorem holds.
Finally, let us note that the vertex cover does not need to be provided as input for the lower bound to hold since we can find it in time $O(1.2738^{\text{vc}(G)} + mN)$ which is contained in $O(2^{0.35\text{vc}(G)}N)$.

The above construction implies several other algorithmic results, following from the fact that $\text{wcol}_2(G) \leq \text{td}(G) \leq \text{vc}(G) \leq m + 2$, $\text{tw}(G) \leq \text{pw}(G) \leq \text{td}(G) - 1 \leq m + 1$ and $\overline{\text{vc}}(G) \leq \text{vc}(G) \leq m + 2$ (cf. Observation 1) and $\overline{\text{vc}}_2(G) \leq \text{vc}(G)$ (Observation 2).

**Corollary 1.** Unless either the SETH fails, Closed 2-Neighbourhood Sizes cannot be solved in time $O(2^{o(f(G))}\cdot n^\delta)$ for any $\delta < 2$ and any structural parameter $f \in \{\text{wcol}_2, \overline{\text{vc}}_2, \text{vc}, \text{pw}, \text{tw}, \text{td}, \overline{\text{vc}}_1, \overline{\text{vc}}_1\}$.

## 4 Upper Bounds

The main results of this section are that, for every graph $G$ with $n$ vertices, Closed 2-Neighbourhood Sizes can be solved in $O(2^{\text{vc}(G)/2}\cdot \text{vc}(G)^2 \cdot n)$ time (see Theorem 3) and in $O(2^w \cdot n)$ time (see Theorem 4), where $\text{vc}(G)$ is the size of the minimum vertex cover and $w$ is the treewidth of $G$. Before we proceed with the proof of Theorem 3, we first introduce now some needed infrastructure.

We will use calligraphic letters like $\mathcal{H}, \mathcal{Q}$ for set families and use $\Delta(\mathcal{H}) := \max_{H \in \mathcal{H}} |H|$ to denote the maximum cardinality of sets in $\mathcal{H}$. A weighted set family over a universe set $U$ is a tuple $(\mathcal{H}, w)$ where $\mathcal{H}$ is a family of sets over $U$ and $w : \mathcal{H} \to \mathbb{R}^+$ assigns a positive rational weight to each member $H \in \mathcal{H}$.

**Definition 4** (Weighted set queries). Let $(\mathcal{H}, w)$ be a weighted set family over the universe $U$. We define the following weighted queries for every $S \subseteq U$:

$$w_\subseteq(S) := \sum_{H \subseteq S \in \mathcal{H}} w(H), \quad w_\supseteq(S) := \sum_{H \supseteq S \in \mathcal{H}} w(H), \quad w_\cap(S) := \sum_{H \in \mathcal{H}, H \cap S \neq \emptyset} w(H).$$

In other words, $w_\subseteq(S)$ (resp. $w_\supseteq(S)$) returns total weight of sets in $\mathcal{H}$ that are supersets (resp. subsets) of $S$ and $w_\cap(S)$ the total weights of sets in $\mathcal{H}$ that intersect $S$.

We will assume in the following that functions with domain in $2^U$ are implemented as data structures which allow constant-time lookup and modification. This can be done either in a randomized way via hash-maps or using the following deterministic implementation on a RAM: assuming that $U = [n]$ for some natural number $n$, we store the value to a key $S = \{s_1, s_2, \ldots, s_p\}$ at address $s_1 + n \cdot s_2 + \cdots + n^p \cdot s_p$. The largest address used in this manner has size polynomial in $n$, however, we only need to initialize as many registers as we store values (which will be linear in the following applications).

**Lemma 2.** Given a weighted set family $(\mathcal{H}, w)$ and a set family $\mathcal{Q}$ over $U$ one can compute all values $w_\subseteq(\mathcal{Q})$, $w_\cap(\mathcal{Q})$, $w_\supseteq(\mathcal{Q})$ for $Q \in \mathcal{Q}$ in time $O(2^\Delta(\mathcal{Q})|\mathcal{Q}| + 2^\Delta(\mathcal{H})|\mathcal{H}|)$.

**Proof.** First, we can easily compute all values for $w_\subseteq(S)$ for $S \subseteq U$ as follows: for every $H \in \mathcal{H}$, we increment a counter for each subset $S \subseteq H$. The resulting data structure gives exactly $w_\subseteq$ and it takes $O(\sum_{H \in \mathcal{H}} 2^{|H|}) = O(2^\Delta(\mathcal{H})|\mathcal{H}|)$
Theorem 3. For every graph $G$ with $n$ vertices, CLOSED 2-NEIGHBOURHOOD SIZES can be solved in $O(2^{\text{vc}(G)/2} \cdot \text{vc}(G)^2 \cdot n)$ time.

Proof. Let $X$ be a vertex cover of $G$ containing $t$ vertices (in particular $t = \text{vc}(G)$ if $X$ is a minimum vertex cover). Let $I := V(G) \setminus X$ be the complement independent set to $X$.

First, we compute the second neighbourhood size for vertices in the vertex cover $X$ in time $O(tn)$. To that end, let $\hat{E}$ contain all pairs $u, v$ with $u \in X$ and $v \in X \cup I$ such that $u, v$ have exactly distance two to each other. Since $|\hat{E}| \leq t^2 + tn = O(tn)$ we can easily compute $|\hat{E}|$ in the claimed time. Consequently, we can also compute in $O(tn)$ time the size of the second neighbourhood for every vertex in $X$.

Let us now partition the independent set $I$ into sets $I_l, I_h$ where $I_l$ contains all vertices from $I$ that have degree at most $t/2$ and $I_h$ the remaining vertices. Note that every pair of vertices $u, v \in I_h$ will share at least one common neighbour in $X$, hence all vertices in $I_h$ have exactly distance two to each other—we can therefore count the contribution of $I_h$ to the closed second neighbourhood of each member in $I_h$ as $|I_h|$. It is therefore left to compute the contributions of vertices in $I_l$ to vertices in $I_l$, of vertices in $I_h$ to vertices in $I_l$, and of vertices in $I_l$ to vertices in $I_h$.

As observed above, we can find a vertex cover of size $O(2^{\Delta(G)}|Q|)$ in time $O(2^{\Delta(G)}|Q|)$ by simply following the definition.

For $Q \subseteq Q$ and $S \subseteq Q$ we define the auxiliary weighted query

$$w_Q(S) := \sum_{H \subseteq S \subseteq Q} w(H),$$

that is, $w_Q(S)$ returns the total weight of sets in $H \subseteq \mathcal{H}$ whose intersection with $Q$ is precisely $S$. We can compute $w_Q$ from $w_{\subseteq}$ via the following inclusion-exclusion formula:

$$w_Q(S) := \sum_{S \subseteq S' \subseteq Q} (-1)^{|S'| \setminus S} w_{\subseteq}(S').$$

For a fixed $Q \subseteq Q$, all values $w_Q(S)$ for $S \subseteq Q$ can be computed in time $O(2^{|Q|})$ using the fast Möbius transform [8]. Given $w_Q$, we can then compute $w_{\gamma}(Q)$ using the identity

$$w_{\gamma}(Q) = \sum_{S \subseteq Q} w_Q(S),$$

hence all values of $w_{\gamma}(Q)$, $Q \subseteq Q$ can be computed in time

$$O(2^{\Delta(G)} \Delta(G)|Q| + \sum_{Q \subseteq Q} 2^{|Q|}) = O(2^{\Delta(G)}|Q|).$$

Summing up the time needed to compute $w_{\subseteq}$, $w_{\supseteq}$ and $w_{\gamma}$ yields the claimed bound. 

As observed above, we can find a vertex cover of size $t$ in time $O^*(1.274^t)$ [3], and thus the following result holds even if the vertex cover $X$ is not provided as input. We will use the Iverson bracket notation $[\phi]$ in the proof of Theorem 3, which evaluates to 1 if $\phi$ is a true statement and to 0 otherwise.

Theorem 3. For every graph $G$ with $n$ vertices, CLOSED 2-NEIGHBOURHOOD SIZES can be solved in $O(2^{|E(G)/2} \cdot \text{vc}(G)^2 \cdot n)$ time.
To that end, let \((\mathcal{X}, w^i)\) denote the weighted set family over \(X\) with \(\mathcal{X} := \{N(v)\}_{v \in I_i}\) and where \(w^i(H)\) simply counts the number of vertices in \(I_i\) that have neighbourhood exactly \(H\). Then for \(v \in I_i\) the value \(w^i_v(N(v))\) provides us exactly with the number of vertices in \(I_i\) whose neighbourhood intersects with \(N(v)\) (including \(v\) itself), thus by Lemma 2 we can compute the contribution of \(I_i\) to vertices in \(I_i\) in time

\[
O(2^{\Delta(\mathcal{X})} \Delta(\mathcal{X}_i)|\mathcal{X}_i|) = O(2^{t/2}tn).
\]

Next, to compute how vertices in \(I_i\) contribute to the second neighbourhood of vertices in \(I_i\), let \(N(u) := X \setminus N(u)\) for every \(u \in I_i\). We define the set family \(\mathcal{X}_h := \{N(u)\}_{u \in I_h}\). A vertex \(v \in I_i\) does not contribute to the second neighbourhood of a vertex \(u \in I_h\) if \(N(v) \subseteq N(u)\). Of course, if we know the number of vertices in \(I_i\) that do not contribute to the second neighbourhood of \(u \in I_h\), we can easily compute the number of vertices in \(I_i\) that contribute.

For a given vertex \(u \in I_h\), we therefore want to compute

\[
\sum_{v \in I_i} \|N(v) \subseteq \hat{N}(u)\| = \sum_{H \in \mathcal{X}_i} \|H \subseteq \hat{N}(u)\| \cdot w^i(H)
= \sum_{H \in \mathcal{X}_i: H \subseteq \hat{N}(u)} w^i(H) = w^i_\geq(\hat{N}(u)),
\]

which, again, by Lemma 2 can be computed for all sets in \(\mathcal{X}_h\) in time

\[
O(2^{\Delta(\mathcal{X})} \Delta(\mathcal{X}_i)|\mathcal{X}_i| + 2^{\Delta(\mathcal{X}_h)}|\mathcal{X}_h|) = O(2^{t/2}tn).
\]

Finally, let us compute how vertices in \(I_h\) contribute to the second neighbourhood of vertices in \(I_i\). Let \((\mathcal{X}_h, w^h)\) denote the weighted set family over \(X\) with \(\mathcal{X}_h\) defined as above, where \(w^h(H)\) counts the number of vertices in \(I_h\) that have neighbourhood exactly \(X \setminus H\). To compute the contribution of \(I_h\) to the second neighbourhood of a vertex \(v \in I_i\), we instead count the number of vertices in \(I_h\) that do not contribute. Since \(N(u) \cap N(v) = \varnothing\) exactly when \(N(v) \subseteq X \setminus N(u)\), this quantity is given by

\[
\sum_{H \in \mathcal{X}_h} \|N(v) \subseteq H\| \cdot w^h(H) = \sum_{H \in \mathcal{X}_h: N(v) \subseteq H} w^h(H) = w^h_\leq(N(v)).
\]

We can compute all necessary values \(w^h_\leq(S)\) for \(S \in \mathcal{X}_i\) using Lemma 2 in time

\[
O(2^{\Delta(\mathcal{X}_i)} \Delta(\mathcal{X}_h)|\mathcal{X}_h| + 2^{\Delta(\mathcal{X}_i)}|\mathcal{X}_i|) = O(2^{t/2}tn).
\]

Having computed all second neighbourhoods of vertices in \(X\) and summed up all possible ways in which vertices in \(I_h\) and \(I_i\) can contribute to each other’s second neighbourhood in the claimed running time, we conclude the statement of the theorem.

In the next theorem we prove the second result of this section, namely an algorithm for Closed 2-Neighbourhood Sizes with exponential dependency on the treewidth of the input graph.
Theorem 4. For every graph \( G \) with \( n \) vertices and with a tree decomposition of width \( w \) given as input, Closed 2-Neighbourhood Sizes can be solved in \( O(2^w n) \) time.

Proof. We can assume without loss of generality that the provided tree decomposition \( (X_i)_{i \in T} \) is nice (see Definition 2). For a bag \( X_i \) in the decomposition, we define the past as \( P_i := \bigcup_{j \in \mathcal{E}_i} X_j \setminus X_i \), and the future as \( F_i := V(G) \setminus (P_i \cup X_i) \).

We now pass over the decomposition in a bottom-up manner to compute a collection of dictionaries \( N_i^P \), \( i \in T \) with the following semantic: for every subset \( Y \subseteq X_i \) we have that \( N_i^P[Y] := |N_G(Y) \cap P_i| \). Using backtracking, we afterwards compute the dictionary \( N_i^F \) with \( N_i^F[Y] := |N_G(Y) \cap F_i| \). Note that for a join-bag \( X_h \) with children \( X_i, X_j \) we have that

\[
N_h^P[Y] = N_i^P[Y] + N_j^P[Y], \quad Y \subseteq X_h,
\]

which follows easily from the assumption that the tree decomposition \( (X_i)_{i \in T} \) is nice.

In our next pass over the decomposition, we keep track of the quantities \( |N(u) \cap P_i| \) and \( |N^2(u) \cap P_i| \) as well as the sets \( N(u) \cap X_i \) and the number of 2-neighbours that \( u \) has in \( G[X_i \cup P_i] \) for every \( v \in X_i \). Maintaining appropriate dynamic programming tables is simple for introduce- and forget operations, and the join-case is again a simple addition of table entries.

Consider a vertex \( u \) and let \( X_i \) be the highest bag in which \( u \) appears, i.e. either \( X_i \) is the root bag and contains vertex \( u \), or the parent bag \( X_j \) of \( X_i \) satisfies \( X_j := X_i \setminus \{u\} \). From the dynamic programming table in \( X_i \) we know the size of \( N^2(u) \cap P_i \). Now it holds that

\[
|N^2(u)| = |N^2(u) \cap P_i| + |N^2(u) \cap X_j| + |N^2(u) \cap F_j|
= |N^2(u) \cap P_i| + |N(N(u) \cap X_i) \cap X_i| + N_i^F[N(u) \cap X_i]
\]

and we have all three quantities readily available. To compute the closed second neighbourhood, we add the degree of \( u \). After passing over the whole tree decomposition we therefore have the size of every closed second neighbourhood of every vertex.

The first pass over the decomposition takes time \( O(2^w n) \), the second one maintains tables of size \( O(w^2) \) and computes the degree inside bags in time \( O(w^2) \), for a total running time of \( O(w^2 n) \). We conclude that the total time taken is \( O(2^w n) \), as claimed.

\[\square\]

Conclusion

We used the SETH reduction toolkit by Abboud, Williams, and Wang to show that computing the 2-neighbourhood sizes is neither possible in subquadratic time nor in fpt-time with subexponential dependence on a range of ‘sparseness parameters’. In that sense, the algorithm by Demaine et al. cannot be improved substantially; although a better exponential dependence of course remains possible. We supplemented these lower bounds with algorithms that solve the problem in time \( O(2^{w\vc(G)/2} \cdot \vc(G)^2 \cdot n) \) and \( O(2^w n) \).
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