ON THE INJECTIVITY AND ASYMPTOTIC STABILITY AT INFINITY

ROLAND RABANAL

Abstract. Let \( X : U \to \mathbb{R}^2 \) be a differentiable vector field defined on the complement of a compact subset of the plane. Let \( \text{Spc}(X) \) denote the set of all eigenvalues of the differential \( DX_z \), when \( z \) varies in whole the domain \( U \). We prove that the condition \( \text{Spc}(X) \subset \{ z \in \mathbb{C} : \Re(z) < 0 \} \) is enough in order to obtain that the infinity to be either an attractor or a repellor for \( X \). More precisely, we show that (i) there exist an unbounded sequence of circles, pairwise bounded an annulus whose boundary is transversal to \( X \) and, (ii) there is a neighborhood of infinity free of singularities and periodic trajectories of \( X \), where all the trajectories are unbounded. The main result is obtained after to proving the existence of \( \tilde{X} : \mathbb{R}^2 \to \mathbb{R}^2 \), a topological embedding with convex image, which is equal to \( X \) in the complement of some compact subset of \( U \). Therefore, the associated map of \( X \) is injective in a neighborhood of infinity.

1. Introduction

A very basic example of non–discrete dynamics on the Euclidean \( n \)--space is given by a linear vector field on \( \mathbb{R}^n \). This linear system has well know properties, for instance if the real part of its eigenvalues is negative the standard Matrix ExponentFormula gives the global asymptotic stability of the origin. In the nonlinear case, if \( Y \) is a \( C^1 \)--vector field whose singularities have non–degenerate jacobian matrix, the Hartman–Grobman Theorem [26] tells us that in a sufficiently small neighborhood of the singularity \( q \), the system \( Y \) is topologically equivalent to its linear part \( DY_q \) as long as the eigenvalues of \( DY_q \) are not pure imaginary; therefore, any singular point \( p \) where the eigenvalues of \( DY_p \) have negative real part is asymptotically stable. This property is closer related with the global asymptotic stability conjecture which claims that: “the know singular point of \( Y \) will be globally attractor, if all the linear parts of \( Y \) are asymptotically stable”. This fact, in the planar case was obtained in [11, 12, 13] and recently improved in [17]. However, it is already been proved that the global asymptotic stability fails in \( \mathbb{R}^3 \), even for polynomial vector fields [7].

There has been a great interest in the local study of vector fields around their singularities. A sample of this study is the work done in [6, 9, 30, 11, 31]. However, in order to understand the global behavior of a planar vector field it is absolutely necessary to study its behavior around to infinity. This is one of the reasons to research the so–called asymptotic stability at infinity [25, 15, 14, 19, 29, 2]. In [15], the authors work with a \( C^1 \)--vector field \( Y : \mathbb{R}^2 \to \mathbb{R}^2 \) for which (i) \( \det(DY_z) > 0 \)

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and (ii) \( \text{Trace}(DY_z) < 0 \) in an neighborhood of infinity. By using a result of [14] they show that “if such \( Y \) has a singularity then, the infinity is either a repeller or an attractor”. Moreover, the authors in [15] introduce the Index \( I(Y) = \int \text{Trace}(DY) \) and show that if such \( Y \) has a singularity and \( I(Y) < 0 \) (resp. \( I(Y) \geq 0 \)), then \( Y \) is topologically equivalent to \( z \mapsto -z \) that is “the infinity is a repellor” (resp. to \( z \mapsto z \) that is “the infinity is an attractor”). This Index was recently studied in [2], where the authors consider a one-parameter family \( Y_\lambda \) of \( C^1 \)-vector fields; they show the bifurcation given by the change in the sign of that Index.

In the important work [24], C. Olech showed the existence of a strong connection between the asymptotic stability of a vector field and the injectivity of its associated map. This connection was strengthened and broadened in subsequent works (see for instance [11-20, 25] in the plane, [23, 18] in higher dimensions). The present paper proceeds with the study of the planar case, in connection with the basic spirit for instance [11-20, 25] in the plane, [23, 18] in higher dimensions). The present article, we consider a differentiable result was recently improved in [29]. In the present article, we consider differentiable vector fields; they introduce the so-called \( B \)-condition, that is: “there does not exist an unbounded sequence \( \mathbb{R}^2 \ni (x_k,y_k) \to \infty \) such that \( X((x_k,y_k)) \to p \in \mathbb{R}^2 \) and \( DX_{(x_k,y_k)} \) has a real eigenvalue \( \lambda_k \) satisfying \( |x_k| \lambda_k \to 0 \)”. In this paper [29] has been proved that in order to obtain the injective extension \( \tilde{X} \) of \( X \) (like above), it is enough that \( X \) verifies the \( B \)-condition and \( \text{Spec}(X) \cap (0, +\infty) = \emptyset \). In the affirmative case, the image of the global map \( \tilde{X}(\mathbb{R}^2) \) is convex. This improves [16] and so the injectivity result of [14] (see also [20, 8]).

In [14], the authors also studied \( C^1 \)-vector fields \( Y : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2 \). By using the stability result of [15] they prove that “the infinity will be an attracting or a repelling singularity of \( Y \), if \( \text{Spec}(Y) \) does not intersect the union

\[
(1.2) \quad (-\varepsilon, 0] \cup \{ z \in \mathbb{C} : \Re(z) \geq 0 \},
\]

for some \( \varepsilon > 0 \), where \( \Re(z) \) denotes the real part of \( z \in \mathbb{C} \). The differentiable version of this result has been proved in [19]. By using the \( B \)-condition, this differentiable result was recently improved in [29]. In the present article, we consider a differential vector field \( X : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2 \) whose domain induces a neighborhood \( V := (\mathbb{R}^2 \setminus \overline{D}_\sigma) \cup \{ \infty \} \) of \( \infty \) in the Riemann sphere \( \mathbb{R}^2 \cup \{ \infty \} \) in a natural way. We prove that the condition \( \text{Spec}(X) \subset \{ z \in \mathbb{C} : \Re(z) < 0 \} \) is enough in order to obtain that the vector field \( X : (V, \infty) \to (\mathbb{R}^2, 0) \) —which is differentiable in \( V \setminus \{ \infty \} \), but not necessarily continuous at \( \infty \) — has \( \infty \) as an attracting or a repelling singularity according to the natural definitions below.

Throughout this article, we shall denote by \( (\mathbb{R}^2 \setminus \overline{D}_\sigma) \cup \{ \infty \} \) the subspace of the Riemann sphere \( \mathbb{R}^2 \cup \{ \infty \} \) with the induced topology. Moreover, given a topological
circle $C \subset \mathbb{R}^2$, the compact disc (resp. open disc) bounded by $C$, will be denoted by $\overline{D}(C)$ (resp. $D(C)$).

2. Statement of the results

Given $U \subset \mathbb{R}^2$ the complement of a compact set. We will consider the differentiable maps (or vector fields) $X : U \rightarrow \mathbb{R}^2$ whose jacobian determinant at any point of $U$ is different from zero.

2.1. Injectivity at infinity. For these maps defined in a neighborhood of infinity, our injectivity result is the following.

**Theorem 2.1.** Let $X = (f, g) : \mathbb{R}^2 \setminus \overline{D}_\sigma \rightarrow \mathbb{R}^2$ be a differentiable map, where $\sigma > 0$ and $\overline{D}_\sigma = \{z \in \mathbb{R}^2 : ||z|| \leq \sigma\}$. If $\text{Spc}(X) \subset \{z \in \mathbb{C} : \Re(z) < 0\}$. Then, there are $s \geq \sigma$ and a globally injective local homeomorphism $\tilde{X} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with convex image such that $X$ and $\tilde{X}$ coincide on $\mathbb{R}^2 \setminus \overline{D}_s$.

The map $\tilde{X}$ of Theorem 2.1 is a differentiable embedding, the image of which may be properly contained in the plane.

By a change in the sign of the map, it is not difficult to see that Theorem 2.1 is valid for maps $X$ such that $\text{Spc}(X) \subset \{z \in \mathbb{C} : \Re(z) > 0\}$. Furthermore, if $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is an arbitrary invertible linear map, then Theorem 2.1 applies to the map $A \circ X \circ A^{-1}$.

Theorem 2.1 complements the result of [10], where the authors consider the assumption (1.1), here the negative eigenvalues can approach zero.

Let us proceed to give an idea of the proof of this theorem. Notice that, we consider a differentiable map $X = (f, g) : \mathbb{R}^2 \setminus \overline{D}_\sigma \rightarrow \mathbb{R}^2$ whose jacobian determinant at any point is different from zero. In this context the Local Inverse Function Theorem is true [11, 12]. As a consequence, the level curves $\{f = \text{constant}\}$ make up, on the plane, a $C^\infty$-foliation $\mathcal{F}(f)$. Moreover, the leaves of $\mathcal{F}(f)$ are differentiable curves, and the restriction of the other function $g$ to any of these leaves is strictly monotone; in particular $\mathcal{F}(f)$ and $\mathcal{F}(g)$ are (topologically) transversal to each other. Like in [14], we orient $\mathcal{F}(f)$ (resp. $\mathcal{F}(g)$) in agreement that if $L_p$ is an oriented leaf of $\mathcal{F}(f)$ (resp. $\mathcal{F}(g)$) thought the point $p$, then the restriction $g|_{L_p}$ (resp. $f|_{L_p}$) is an increasing function in conformity with the orientation of $L_p$. The leaf $L_p$, sometimes will be called trajectory.

**Remark 2.2.** Since, the leaves of these continuous foliations are differentiable curves, the following properties holds:

(a) Let $\alpha : (-a, a) \rightarrow L_p$ be a parametrization with $\alpha(0) = p$ of an oriented leaf of $\mathcal{F}(f)$. So, it is not difficult obtain that $\frac{d}{dt} g(\alpha(t)) > 0$. Also, by using the local inverse of $X$ at $X(p)$, it is easy to see that for each $t \in (-a, a)$ there exists $\eta_t > 0$ such that $\alpha'(t) = \eta_t X_f(\alpha(t))$ where $X_f(z) := (-f_y(z), f_x(z))$.

(b) In a similar way, if $\beta : (-a, a) \rightarrow L_p(g)$ is a local parametrization of any trajectory of $\mathcal{F}(g)$, for every $t \in (-a, a)$, $\beta'(t) = -\tilde{\eta_t} X_g(\beta(t))$ for some $\tilde{\eta_t} > 0$.

(c) These tangent vector fields $X_f$ and $-X_g$ are globally defined but it can be discontinuous.

The linear parts of any map in Theorem 2.1 have no positive eigenvalues; thus, we say that these maps are free of positive eigenvalues. By the results of [13], it is know that any global map $\tilde{X} = (\tilde{f}, \tilde{g}) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ free of positive eigenvalues is globally
injective, because $\mathcal{F}(\dot{f})$ and $\mathcal{F}(\dot{y})$ does not have any Half–Reeb component (see Definition ??). This property of the foliations is used in Section ?? where we prove the first version of Theorem 2.1 that is: “for maps free of positive eigenvalues whose foliations have no unbounded Half–Reeb components” (Theorem ??). In Section ?? we present some preliminary results about the flux of the vector field $X$ through some leaves of $\mathcal{F}(f)$. In Section ??, we include the proof of Theorem 2.1 which proceed by contradiction, we suppose the existence of a Half–Reeb component and by using its boundary we construct a contour whose $X$–flux is positive.

2.2. Differentiable vector fields. We consider the following system

\begin{equation}
\dot{z} = X(z),
\end{equation}

where $X : \mathbb{R}^2 \setminus D_\sigma \rightarrow \mathbb{R}^2$ is a differentiable vector field for some $\sigma > 0$. Since, each point on the domain can be an initial condition, such point jointly to (2.1) give an autonomous differential equation, which may have many solutions defined on their maximal interval of existence. Nevertheless, for every of those trajectories —through the same point, kept fixed— all their local funnel sections are compact connected sets (see [21]); moreover, each trajectory has its two limit sets, $\alpha$ and $\omega$ respectively, which are well defined in the sense that only depend of such solution. Notice that, we called trajectory to the curve determined by any solution defined on their maximal interval of existence. If $\gamma_q$ denotes a trajectory through a point $q \in U$, then $\gamma_q^+$ (resp. $\gamma_q^-$) will denote the positive (resp. negative) semi-trajectory of $X$, contained in $\gamma_q$ and starting at $q$. In this way $\gamma_q = \gamma_q^- \cup \gamma_q^+$ and $\gamma_q^- \cap \gamma_q^+ = \{q\}$.

A $C^0$–vector field $X : \mathbb{R}^2 \setminus D_\sigma \rightarrow \mathbb{R}^2 \setminus \{0\}$ (without singularities) can be extended to a map

\[ \hat{X} : ((\mathbb{R}^2 \setminus D_\sigma) \cup \{\infty\}, \infty) \rightarrow (\mathbb{R}^2, 0) \]

(which takes $\infty$ to 0). In this manner, all questions concerning the local theory of isolated singularities of planar vector fields can be formulated and studied in the case of the vector field $\hat{X}$. For instance, if $\gamma_p^+$ (resp. $\gamma_p^-$) is an unbounded semi-trajectory of $X : \mathbb{R}^2 \setminus D_\sigma \rightarrow \mathbb{R}^2$ passing through $p \in \mathbb{R}^2 \setminus D_\sigma$ such that, its $\omega$–limit (resp. $\alpha$–limit) set is empty, we will also say that $\gamma_p^+$ goes to infinity (resp. $\gamma_p^-$ comes from infinity), it will be denoted by $\omega(\gamma_p^+) = \infty$ (resp. $\alpha(\gamma_p^-) = \infty$). In this context, we may also talk about the phase portrait of $X$ in a neighborhood of $\infty$.

As in [19], we need the following concepts.

Definition 2.3. We will say that the infinity is an attractor (resp. a repellor) for the differentiable vector field $X : \mathbb{R}^2 \setminus D_\sigma \rightarrow \mathbb{R}^2$ if

1. There is a sequence of transversal circles to $X$ tending to infinity, that is for every $r \geq \sigma$ there exist a circle $C_r$ such that $D_r \subset D(C_r)$ and $C_r$ is transversal to $X$.

2. For some circle $C_s$ with $s \geq \sigma$, all the trajectories $\gamma_p$ through any point $p \in \mathbb{R}^2 \setminus D(C_s)$, satisfy $\omega(\gamma_p^+) = \infty$ that is $\gamma_p^+$ go to infinity (resp. $\alpha(\gamma_p^-) = \infty$ that is $\gamma_p^-$ come from infinity).

Let $A$ be a measurable subset of $\mathbb{R}^n$, and let $F : A \rightarrow \mathbb{R}$ be a measurable function. We define as usual

\[ F^+(z) = \max \{ F(z), 0 \}, \quad F^-(z) = \max \{ -F(z), 0 \}. \]
Accordingly, we say that $F : A \to \mathbb{R}$ is \textbf{Lebesgue almost–integrable} if
\[
\min \left\{ \int_A F^+ d\mu, \int_A F^- d\mu \right\} < \infty,
\]
in which case we define
\[
\int_A F d\mu = \int_A F^+ d\mu - \int_A F^- d\mu,
\]
which is a well–defined value of the extended real line $[-\infty, +\infty]$.

\textbf{Definition 2.4.} Let $X : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ be a differentiable vector field. If there are $s \geq \sigma$, and a global differentiable vector field $\hat{X} : \mathbb{R}^2 \to \mathbb{R}^2$ such that:

1. in the complement of the disk $\overline{D}_s$ both $X$ and $\hat{X}$ coincides, and
2. the map $z \mapsto \text{Trace}(D\hat{X})$ is Lebesgue almost–integrable in whole $\mathbb{R}^2$.

Then, we define the \textbf{index of $X$ at infinity $I(X)$} as the number of $[-\infty, +\infty]$ given by
\[
I(X) = \int_{\mathbb{R}^2} \text{Trace}(D\hat{X}) dx \wedge dy.
\]

This index $I(X)$ is well–defined and it does not depend of the map $\hat{X}$, more precisely:

\textbf{Remark 2.5.} If we consider the pairs $(s_1, \hat{X}_1)$ and $(s_2, \hat{X}_2)$ which satisfy (1) and (2) of Definition 2.4. Thanks to the Green’s formula as presented in [27, 28], and to an unbounded sequence of compact discs $\overline{D}(C)$, with $C \subset \mathbb{R}^2 \setminus \overline{D}_\sigma$ surrounding the origin; the proof of Proposition 2.1 in [2] shows that
\[
\int_{\mathbb{R}^2} \text{Trace}(D\hat{X}_1) dx \wedge dy = \int_{\mathbb{R}^2} \text{Trace}(D\hat{X}_2) dx \wedge dy.
\]

We are now ready to state our result over differentiable vector fields.

\textbf{Theorem 2.6.} Let $X : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ be a differentiable vector field where $\sigma > 0$ and $\overline{D}_\sigma = \{ z \in \mathbb{R}^2 : ||z|| \leq \sigma \}$. If $\text{Spec}(X) \subset \{ z \in \mathbb{C} : \Re(z) < 0 \}$, then the following is satisfied:

1. For all $p \in \mathbb{R}^2 \setminus \overline{D}_\sigma$, there is a unique positive semi-trajectory starting at $p$.
2. There exist the index of $X$ at infinity, $I(X) \in [-\infty, +\infty)$ such that if $I(X) < 0$ (resp $I(X) \geq 0$) the infinity is a repellor (resp. an attractor) of the vector field $X + v : \mathbb{R}^2 \setminus \overline{D}_s \to \mathbb{R}^2$ for some $v \in \mathbb{R}^2$.

This theorem improves the main result of [19], where the authors consider the condition (1.2). In the new assumption, the negative eigenvalues can tend to zero.

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E-mail address: rrabanal@ictp.it