On freeness of the random fundamental group

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Abstract

Let \( Y(n, p) \) denote the probability space of random 2-dimensional simplicial complexes in the Linial–Meshulam model, and let \( Y \sim Y(n, p) \) denote a random complex chosen according to this distribution. In a paper of Cohen, Costa, Farber, and Kappeler, it is shown that for \( p = o(1/n) \) with high probability \( \pi_1(Y) \) is free. Following that, a paper of Costa and Farber shows that for values of \( p \) which satisfy \( 3/n < p \ll n^{-46/47} \), with high probability \( \pi_1(Y) \) is not free. Here we improve on both of these results to show that there are explicit constants \( \gamma_2 < c_2 < 3 \), so that for \( p < \gamma_2/n \) with high probability \( Y \) has free fundamental group and that for \( p > c_2/n \), with high probability \( Y \) has fundamental group which either is not free or is trivial.

1 Introduction

For positive integers \( n \) and \( d \) and \( p = p(n) \in [0,1] \), the space of Linial–Meshulam random \( d \)-dimensional simplicial complexes, first introduced in [10] and [13] and denoted \( Y_d(n, p) \), is defined to be the probability space of \( d \)-dimensional simplicial complexes on \( n \) vertices with complete \((d-1)\)-skeleton where each of the possible \( \binom{n}{d+1} \) \( d \)-dimensional faces is included independently with probability \( p \). Here we are primarily interested in the \( d = 2 \) case and so we suppress the dimension parameter and write \( Y(n, p) \) for \( Y_2(n, p) \). Now the question of the fundamental group of \( Y \sim Y(n, p) \) is nontrivial and has been studied in [4, 6, 7, 8]. Additionally the series of papers [1, 2, 3, 12], study \( Y_d(n, p) \) in the regime \( p = c/n \). We will describe these results below, but we introduce two constants first introduced in [3] and [1] that are needed to state our main theorem. Let \( \gamma_2 = (2x(1-x))^{-1} \) where \( x \) is the unique nonzero solution to \( \exp(-\frac{1-x}{2x}) = x \) and let \( c_2 = \frac{-\log y}{(1-y)^2} \) where \( y \) is the unique root in \((0,1)\) of \( 3(1-y) + (1+2y)\log y = 0 \). Here we build on the work of [3], [7], and [12] to prove the following result about the fundamental group of a random 2-complex. Note that most of the theorems stated here are asymptotic results and we use the phrase “with high probability”, abbreviated w.h.p., to mean that a property holds for \( Y \sim Y_d(n, p) \) with probability tending to 1 as \( n \) tends to infinity. The following theorem is the main result of this paper.

Theorem 1. If \( c < \gamma_2 \) and \( Y \sim Y(n, c/n) \), then with high probability \( \pi_1(Y) \) is a free group and if \( c > c_2 \) and \( Y \sim Y(n, c/n) \) then with high probability \( \pi_1(Y) \) is not a free group.

Now \( \gamma_2 \) is first defined in [3] and \( c_2 \) is first defined in [1] and approximations are computed as \( \gamma_2 \approx 2.455407 \) and \( c_2 \approx 2.753806 \).

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2 The lower bound

In this section we prove the first part of theorem that for \( c < \gamma_2 \) one has \( \pi_1(Y) \) is a free group with high probability for \( Y \sim Y(n, c/n) \). This result will follow by adapting the argument of [3] used to prove the following result.

**Theorem 2** (2-dimensional case of Theorem 1.4 from [3]). Let \( \gamma_2 \) be as above. If \( c < \gamma_2 \) then w.h.p. \( Y \sim Y(n, c/n) \) is 2-collapsible or contains \( \partial \Delta_3 \) as a subcomplex.

We first define what it means for a simplicial complex to be \( d \)-collapsible. For a \( d \)-dimensional simplicial complex \( Y \), we say that a \((d-1)\)-dimensional face \( \tau \) is free if it is contained in exactly one \( d \)-dimensional face \( \sigma \in Y \). For a free \((d-1)\)-face \( \tau \) an elementary collapse at \( \tau \) is defined to be the simplicial complex \( Y' \) obtained from \( Y \) by removing \( \tau \) and the unique \( d \)-face \( \sigma \) in \( Y \) containing \( \tau \). If there is a sequence of elementary collapses that removes all \( d \)-dimensional faces of \( Y \) we say that \( Y \) is \( d \)-collapsible. Observe that elementary collapses are homotopy equivalences, so if a 2-complex is 2-collapsible (to a graph) then in particular it has free fundamental group. Therefore theorem 2 above almost proves the lower bound except for the problem of tetrahedron boundaries. Note that it is impossible to rule out \( \partial \Delta_3 \) appearing as a subcomplex of \( Y \sim Y(n, c/n) \) for any \( c > 0 \) since the expected number of copies of \( \partial \Delta_3 \) in \( Y \sim Y(n, c/n) \) approaches a Poisson distribution with mean \( c^4/24 \). Additionally, [3] does not state any result about partial collapsibility in the presence of a few copies of \( \partial \Delta_3 \) and indeed it is not clear that any partial collapsibility result would hold. However such a result is not needed to imply that the fundamental group of \( Y \sim Y(n, c/n) \) is free for \( c < \gamma_2 \) as we will see below.

Following the convention of [3] define a core to be a 2-dimensional simplicial complex in which every edge is contained in at least two faces. Also for a 2-complex \( Y \), let \( R(Y) \) denote the simplicial complex obtained by collapsing all the free edges of \( Y \) and let \( R_\infty(Y) \) denote the simplicial complex obtained after repeatedly collapsing at all free edges until no free edges remain. The two key results of [3] that we will use are the following.

**Theorem 3** (2-dimensional case of Theorem 4.1 from [3]). For every \( c > 0 \) there exists a constant \( \delta = \delta(c) > 0 \) such that w.h.p. every core subcomplex \( K \) of \( Y \sim Y(n, c/n) \) with \( f_2(K) \leq \delta n^2 \) must contain the boundary of a tetrahedron.

**Theorem 4** (2-dimensional case of Theorem 5.3 from [3]). Let \( \delta > 0 \) and \( 0 < c < \gamma_2 \) be fixed and suppose \( Y \sim Y(n, c/n) \). Then w.h.p. \( f_2(R_\infty(Y)) \leq \delta n^2 \).

Now to bound the probability that \( \pi_1(Y) \) is not a free group for \( Y \sim Y(n, c/n) \), we will bound the probability that \( Y \sim Y(n, c/n) \) for \( c < \gamma_2 \) has a core which contains no tetrahedron boundary or has a pair of tetrahedron boundaries that are not face disjoint. This will be an upper bound to the probability that \( \pi_1(Y) \) is not free by the following proposition.

**Proposition 5.** Let \( Y \) be a 2-dimensional simplicial complex. If every core of \( Y \) contains a tetrahedron boundary and all the tetrahedron boundaries are face-disjoint then \( \pi_1(Y) \) is free.

**Proof.** Let \( \tilde{Y} \) be the 3-dimensional simplicial complex obtained from \( Y \) by adding a 3-simplex inside all the tetrahedron boundaries of \( Y \). Now \( \tilde{Y}^{(2)} = Y \) so \( \pi_1(Y) = \pi_1(\tilde{Y}) \). Now let \( Z \) be obtained from \( \tilde{Y} \) by collapsing at a free 2-dimensional face at every 3-dimensional face, such a collapse will remove
all the tetrahedra from $\tilde{Y}$ as the tetrahedron boundaries in $Y$ are face disjoint so every tetrahedron in $\tilde{Y}$ has that all of its faces are free. Equivalently, $Z$ is obtained from $Y$ by deleting one face from every tetrahedron boundary of $Y$. Now collapsing at free faces is a homotopy equivalence so $\pi_1(Z) = \pi_1(\tilde{Y})$. Furthermore $Z$ is 2-collapsible. Indeed $Z$ has no cores as a core $K$ of $Z$ would be a core in $Y$ as well since $Z$ is obtained from $Y$ by removing faces. But every core of $Y$ contains a tetrahedron boundary and $Z$ has no tetrahedron boundaries. Since $Z$ does not have a core it must be 2-collapsible, otherwise deleting all the isolated edges of $R_\infty(Z)$ would give us a subcomplex of $Z$ that has no faces of degree zero or one, so such a subcomplex would be a core. Thus $Z$ is 2-collapsible, in particular $Z$ is homotopy equivalent to a graph so $\pi_1(Z)$ is a free group.

Now we are ready to prove the first part of Theorem 1, that is for $c < \gamma_2$ and $Y \sim Y(n, c/n)$, $\pi_1(Y)$ is a free group with high probability.

**Proof of lower bound on Theorem 1.** Let $c < \gamma_2$ be fixed and suppose $Y \sim Y(n, c/n)$, by proposition 5, the probability that $\pi_1(Y)$ is not free is bounded above by the sum of the probability that $Y$ contains tetrahedron boundaries that share a face and the probability that $Y$ has a core with no tetrahedron boundary. First it is easy to bound the probability that $Y$ contains tetrahedron boundaries that share a face. Two tetrahedron boundaries in a simplicial complex sharing a face must meet in exactly one face. Two tetrahedron boundaries meeting at one face is a simplicial complex with 5 vertices and 7 faces, the expected number of such subcomplexes in $Y \sim Y(n, p)$ is $O(n^5p^7)$ which in this case is $O(c^7/n^2) = o(1)$. So by Markov’s inequality the probability that there are tetrahedron boundaries in $Y$ that are not face disjoint is $o(1)$.

We will now use the two theorems from [3] above to show that the probability that $Y$ has a core with no tetrahedron boundary is $o(1)$. Let $\delta = \delta(c)$ be the $\delta$ given by Theorem 3. Let $\mathcal{F}$ denote the collection of 2-complexes on $n$ vertices containing a core with no tetrahedron boundary and let $\mathcal{G}$ denote the collection of 2-complexes on $n$ vertices for which all cores have size at most $\delta n^2$. Note that if $Y \notin \mathcal{G}$, then $f_2(R_\infty(Y)) > \delta n^2$ since cores are unaffected by elementary collapses, so $Pr(Y \notin \mathcal{G}) = o(1)$ by Theorem 4. Now we bound $Pr(Y \in \mathcal{F})$.

$$
Pr(Y \in \mathcal{F}) = Pr(Y \in \mathcal{F} \cap \mathcal{G}) + Pr(Y \in \mathcal{F} \setminus \mathcal{G}) \\
\leq Pr(Y \in \mathcal{F} \cap \mathcal{G}) + Pr(Y \notin \mathcal{G}) \\
\leq Pr(Y \in \mathcal{F} \cap \mathcal{G}) + o(1) \\
\leq Pr(Y \in \{X : X \text{ has a core } K \text{ with at most } \delta n^2 \text{ faces and no } \partial \Delta_3\}) + o(1)
$$

Now by the choice of $\delta$ and Theorem 3, we know that the probability that $Y$ has a core which has at most $\delta n^2$ faces but no tetrahedron boundary is $o(1)$. Thus we have that $Pr(Y \in \mathcal{F}) = o(1)$ which completes the proof. □

### 3 The upper bound

We now turn our attention to proving that when $c > c_2$ and $Y \sim Y(n, c/n)$, with high probability $\pi_1(Y)$ is not a free group. In fact relevant results by Costa and Farber ([7]) will prove that the cohomological dimension is 2. We refer the reader to [5] for background on group cohomology theory. The main result of [7] is the following:
**Theorem 6** (Theorem 2 of [7]). Assume that $p \ll n^{-46/47}$, then w.h.p. a random 2-complex $Y \sim Y(n,p)$ is asphericable. That is the complex $Z$ obtained from $Y$ by removing one face from each tetrahedron of $Y$ is aspherical (i.e. the universal cover of $Z$ is contractible).

From here Costa and Farber prove the following result.

**Theorem 7** (Theorem 3B of [7]). For any constants $c > 3$ and $0 < \epsilon < 1/47$ and $p$ satisfying $c/n < p < n^{-1+\epsilon}$, the cohomological dimension of $Y \sim Y(n,p)$ equals 2 with high probability.

To prove our upper bound from Theorem 1 we will use the following result of Linial and Peled [12] to reduce the constant 3 in Theorem 7 to $c_2$, the argument will follow exactly the argument of Costa and Farber in their proof of Theorem 7, but with the current state-of-the-art (and best-possible, also by [12]) threshold for emergence of homology in degree 2 for a random 2-complex.

**Theorem 8** (Special case of Theorem 1.3 from [12]). Suppose $c > c_2$, then w.h.p. $Y \sim Y(n,c/n)$ has
\[
\dim H_2(Y; \mathbb{R}) = \Theta(n^2).
\]

In [12], the constant implicit in $\Theta(n^2)$ is given explicitly, but we do not need it here. We are now ready to prove the second part of theorem 1.

**Proof of upper bound on Theorem 1.** Fix $c > c_2$ and suppose that $Y$ is a simplicial complex drawn from $Y(n,c/n)$. Now let $Z$ be obtained from $Y$ by removing one face from every tetrahedron boundary. With high probability $\pi_1(Y) = \pi_1(Z)$, and by theorem 6, $Z$ is aspherical. Therefore showing that $\beta_2(Z) \neq 0$ would imply that the cohomological dimension of $\pi_1(Y)$ is at least two. Now by theorem 8 we know that with high probability $\beta_2(Y) = \Theta(n^2)$. Also we have by a first moment argument that the expected number of tetrahedron boundaries is bounded above by $c^4/24$. Therefore by Markov’s inequality with high probability $Y$ has no more than, say, $n$ tetrahedron boundaries. Now given a 2-dimensional simplicial complex, removing a face can drop $\beta_2$ by at most one. Therefore when we remove one face from from each tetrahedron boundary of $Y$ to obtain $Z$ we drop $\beta_2(Y)$ by at most $n$, then w.h.p. $\beta_2(Z) = \Theta(n^2) > 0$. Thus the cohomological dimension of $\pi_1(Y)$ is at least two (actually equality holds by theorem 6), in particular $\pi_1(Y)$ is not a free group.

**4 Concluding Remarks**

The statement of Theorem 1 perhaps implicitly suggests a sharp threshold for the property that a random 2-complex has fundamental group which is not free. However, it is worth mentioning that the property that the fundamental group of a simplicial complex is free is not a monotone property, so it is not obvious at all that a sharp threshold should exist. However, by theorem 1 and theorem 7, for $c_2/n < p < 3 \log n/n$ and $Y \sim Y(n,p)$, with high probability $\pi_1(Y)$ is not free. Combining this with a result from [8] that for $p > (2 \log n + \omega(n))/n$ (with $\omega(n) \rightarrow \infty$ as $n \rightarrow \infty$) and $Y \sim Y(n,p)$, $\pi_1(Y)$ has property (T) with high probability, we have that for $p > c_2/n$, the fundamental group of $Y \sim Y(n,p)$ is with high probability free only if it is trivial as the only free group with property (T) is the trivial group. On the other hand, [6] proves that for $p = o(1/n)$, $Y \sim Y(n,p)$ collapses to a graph with high probability. Thus we do have at least a coarse threshold of $1/n$ for the fundamental group of a random 2-complex to be either not free or trivial.
It remains to discover the fundamental group for $Y(n, p)$ for $\gamma_2 \leq c \leq c_2$ and $p = c/n$. Right now, there does not seem to be enough evidence to establish a conjecture. There are the following three possibilities for what happens to the fundamental group of $Y(n, p)$ in this intermediate regime:

1. $\gamma_2/n$ is the sharp threshold for the fundamental group of $Y(n, p)$ to go from a free group to a non-free group.

2. $c_2/n$ is the sharp threshold for the fundamental group of $Y(n, p)$ to go from a free group to a non-free group.

3. Neither $\gamma_2/n$ nor $c_2/n$ is the sharp threshold for the fundamental group of $Y(n, p)$ to go from a free group to a non-free group.

Any of these three would be interesting in their own way.

If (1) holds, then by [7], $Y(n, c/n)$ in the regime $c \in (\gamma_2, c_2)$ has cohomological dimension equal to 2 and $Y(n, c/n)$ is asphericable. Thus when we remove a face from every tetrahedron we have that $Y(n, c/n)$ is a $K(G, 1)$ for a group $G$ which has cohomological dimension 2. However, the reason for it to have cohomological dimension 2 must be different than the reason for $\pi_1(Y(n, c/n))$ to have cohomological dimension 2 in the regime $c > c_2$. Indeed, in the regime $c > c_2$, $H_2(Y(n, c/n), \mathbb{R}) \neq 0$ is enough to imply that the cohomological dimension of the fundamental group is at least 2. For $c < c_2$ the second homology group of $Y(n, c/n)$ with coefficients in $\mathbb{R}$ is trivial, after the removal of a face from each tetrahedron boundary.

Moreover, there is an apparent lack of torsion in $H_1(Y(n, c/n))$ if $c < c_2$. This has not been proved, but extensive experiments conducted in [9] provide evidence to support this, and [14] state the following conjecture regarding torsion in homology:

**Conjecture** (2-dimensional case of the conjecture from [14]). For every $p = p(n)$ such that $|np - c_2|$ is bounded away from 0, $H_1(Y_d(n, p); \mathbb{Z})$ is torsion-free with high probability.

Torsion in homology is observed experimentally for $p$ close to $c_d/n$. For more about this torsion see [9]. All of this is to say that it is likely that one will not be able to prove that the cohomological dimension of $\pi_1(Y(n, c/n)) \geq 2$ for $\gamma_2 < c < c_2$ by proving $H_2(Y(n, c/n), \mathbb{Z}/q\mathbb{Z}) \neq 0$ for some prime $q$.

On the other hand if (2) holds, then $Y(n, c/n)$ in the regime $\gamma_2 < c < c_2$ is homotopy equivalent to a wedge of circles after the removal of one face from each tetrahedron boundary. This follows from the fact that $Y(n, c/n)$ is asphericable and that an aspherical space which is a CW-complex is unique up to homotopy equivalence. Now for $c < \gamma_2$, as we show above following the results of [3], $Y(n, c/n)$ is homotopy to a wedge of circles after the removal of a face from each tetrahedron boundary. However, this homotopy equivalence is given by a sequence of elementary collapses which reduces the complex to a graph. It is proved in [2] that such a series of elementary collapses is not possible for $c > \gamma_2$. Furthermore, [11] points out that in the regime $\gamma_2 < c < c_2$, $Y(n, c/n)$ is far from being 2-collapsible in the sense that a constant fraction of the faces must be deleted to arrive at a complex which is 2-collapsible. Thus $Y(n, c/n)$ in the regime $\gamma_2 < c < c_2$ if (2) holds would be homotopy equivalent to a wedge of circles, but not via the same type of homotopy equivalence.
which exists for smaller values of $c$.

In summary, regardless of whether the truth is (1) or (2), new techniques will almost certainly be required to prove which is correct. Of course, (3) is a possibility as well. Indeed it is possible that no sharp threshold exists for the property that $\pi_1(Y(n, p))$ as we discuss above. It could also be that there is some $c^* \in (c_2, c_2)$ so that $c^*/n$ is the sharp threshold for $\pi_1(Y(n, p))$ to be non-free, or that within this intermediate regime there is a positive probability $Y(n, p)$ is free and a positive probability that it is not.

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