Detection of genuine tripartite entanglement by multiple sequential observers

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Due to the difficulties present in experimentally preparing genuine tripartite entanglement, it is important to explore the possibility of multiple usage of a single genuine entangled state. In the present paper, we present one such possibility by considering a scenario consisting of three spin-\(\frac{1}{2}\) particles shared between Alice, Bob and multiple Charlies. Alice performs measurements on the first particle, Bob performs measurements on the second particle and multiple Charlies perform measurements on the third particle sequentially and independently. In this scenario, we investigate whether more than one Charlie can detect genuine tripartite entanglement, and we answer this question affirmatively. In order to probe genuine entanglement, we use correlation inequalities whose violations certify genuine tripartite entanglement in a device-independent way. We extend our investigation by using appropriate genuine tripartite entanglement witness operators. Using each of these different tools for detecting genuine tripartite entanglement, we find out the maximum number of Charlies who can detect genuine entanglement in the above scenario.

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I. I. INTRODUCTION

Entanglement [1] is one of the most fascinating non-classical features of quantum mechanics. The demarcation between separable and entangled states is well understood in bipartite scenario. But the situation becomes complex in multipartite scenarios as one can consider entanglement across many possible bipartitions. Moreover, the concept of genuine entanglement [2] appears in the multipartite context. A multipartite state is called genuinely entangled if it is not separable with respect to any partition. The concept of genuine entanglement is not only important for quantum foundational research, but also finds various information theoretic implications, for example, in extreme spin squeezing [3], high sensitive metrology tasks [4, 5], quantum computation using cluster states [6], measurement-based quantum computation [7] and multiparty quantum networks [8–11].

In spite of various successful attempts for the generation and detection of genuine multipartite entangled states [12–14], the complication of the process is appreciated as the detection or verification of entanglement involves tomography or constructions of entanglement witnesses under precise experimental control over the system subjected to measurements. Due the difficulties present in generating genuine entanglement which is the resource for a vast range of information processing tasks, it is a significant question to ask whether genuine entanglement can be preserved partially even after performing a few cycles of local operations. The motivation of the present paper is to address the above question, and we are able to answer it in the affirmative for the tripartite scenario.

The general question as to what extent quantum correlation of an entangled state can be shared by multiple observers who perform measurements sequentially and independently of each other, was first posed in the case of the bipartite scenario. Silva et al. [15] addressed this question in the context of Bell nonlocality [16, 17] by considering a scenario where an entangled pair of two spin-\(\frac{1}{2}\) particles are shared between Alice in one wing and multiple Bobs in another wing. Alice acts on the first particle and multiple Bobs act on the 2nd particle sequentially, where Alice is spatially separated from the multiple Bobs. In this scenario, using a measurement model to optimize the trade-off between information gain and disturbance it was conjectured [15] that at most two Bobs can violate the Bell-CHSH (Bell-Clauser-Horne-Shimony-Holt) inequality [16, 17] with a single Alice. This result that was subsequently confirmed analytically [18] applying a one-parameter positive operator valued measurement (POVM) [19, 20].

Various experiments have been performed to demonstrate this phenomena [22, 23]. Recently, the notion of shareability of quantum nonlocality has been extended to investigate several other kinds of quantum correlations. These include sharing of EPR steering [24, 25], entanglement [26, 27], steerability of local quantum coherence [28], Bell-nonlocality with respect to quantum violations of various other Bell type inequalities [29], and preparation contextuality [30]. These ideas have been applied in randomness generation [31], their classical communication cost [32], quantum teleportation [33], and random access codes [34].

Most of the previous studies have addressed the issue of sharing quantum correlations by multiple sequential ob-
servers in the bipartite scenario. Very recently, the possibility of sequential sharing of genuine tripartite nonlocality by multiple observers has been studied [35]. Quantum entanglement is the primary ingredient for nonlocal correlations, and in the present paper we focus our attention on the sharing of genuine multipartite entanglement. In particular, we consider the scenario where three spin-$\frac{1}{2}$ particles are spatially separated and shared between, say, Alice, Bob and multiple Charlies. Alice measures on the first particle; Bob measures on the second particle and multiple Charlies measure on the third particle sequentially. In this scenario we investigate how many Charlies can detect genuine tripartite entanglement.

In order to detect entanglement, one may the violation of Bell-type inequalities as a criterion, since entanglement is a necessary resource for generating nonlocal correlations. One can construct inequalities which can certify genuine multipartite entanglement from the statistical data alone. This method of device-independent detection of genuine entanglement was first introduced in [36–39] followed by an extensive formalization by Bancal et al. [40]. Pal [41] and Liang et al. [42] have improved the existing inequalities for detecting genuine multipartite entanglement. The Mermin polynomial [43] which is a useful tool for device-independent entanglement-witness can be used to detect genuine tripartite entanglement [36]. In the present study, we use quantum violations of the Mermin inequality [43] and the Uffink inequality [38], respectively, in order to probe detection of genuine tripartite entanglement by multiple sequential Charlies.

Another well developed tool for detection of entanglement is through the entanglement witness operators [44–49]. For each entangled state, there always exists a witness operator which is a consequence of the Hahn-Banach theorem [50]. A similar concept has been formulated for the genuine tripartite entangled states (W-state and GHZ-state) which distinguishes genuine entanglement from the set of all bi-separable states [51–54]. In the present paper, we further analyse the idea of sequential detection of genuine tripartite entanglement using appropriate witness operators.

All our analyses point out that it is indeed possible to detect genuine entanglement sequentially by more that one Charlies. In particular, we show that at most two Charlies can detect genuine entanglement sequentially using the linear as well as nonlinear device-independent genuine entanglement inequalities. On the other hand, through appropriate genuine entanglement witnesses which are suitable for the W-state and the GHZ-state, at most four Charlies and twelve Charlies can respectively, detect genuine entanglement. Hence, the present paper paves a new direction on the possibilities of multiple usage of genuine multipartite quantum correlations in various information processing tasks.

The paper is organized as follows: in Section II we present the basic tools for detecting genuine tripartite entanglement. The measurement scenario involving multiple sequential observers used in this paper is also described in this Section. In Section III, we present the main results of this paper, namely, sequential detection of genuine tripartite entanglement. Finally, we conclude in Section V.

II. PRELIMINARIES

In this Section we will present some basic tools which will be used in our paper. We will also elaborate on the scenario in which sequential detection of genuine tripartite entanglement is studied.

A. Detection of Genuine Entanglement

In order to certify genuine entanglement in a device-independent way, several inequalities have been proposed. For the purpose of the present paper, we will use some of them. A tripartite state $\rho$ is said to be bi-separable if and only if it can be written in the following form,

$$\rho = \sum_\lambda p_\lambda \rho_\lambda^A \otimes \rho_\lambda^B \otimes \rho_\lambda^C + \sum_\mu p_\mu \rho_\mu^A \otimes \rho_\mu^B \otimes \rho_\mu^C + \sum_\nu p_\nu \rho_\nu^C \otimes \rho_\nu^A \otimes \rho_\nu^B,$$

with $0 \leq p_\lambda, p_\mu, p_\nu \leq 1$ and $\sum_\lambda \rho_\lambda = \sum_\mu \rho_\mu = \sum_\nu \rho_\nu = 1$. A tripartite state is called genuinely entangled if and only if it cannot be written in the bi-separable form (1).

Let us begin with presenting the device-independent entanglement-witness provided by the Mermin polynomial [43] as the simplest example for detecting genuine tripartite entanglement [36]. Consider that three spatially separated parties, say, Alice, Bob and Charlie are sharing some quantum system in the state $\rho$. The choices of measurement settings, performed by Alice, Bob and Charlie on the shared state $\rho$ are denoted by $A_x$, $B_y$ and $C_z$ respectively, where $x, y, z \in \{0,1\}$. The outcomes of Alice, Bob and Charlie’s measurements are denoted by $a, b$ and $c$, respectively, with $a, b, c \in \{+1, -1\}$. By repeating the experiment a number of times, the joint probability distributions $P(a,b,c|x,y,z)$ are produced. In this scenario, the Mermin inequality, whose violation certifies the presence of genuine entanglement in a device-independent way, can be expressed as [36, 55]:

$$M = |\langle A_1 B_0 C_0 \rangle + \langle A_0 B_1 C_0 \rangle + \langle A_0 B_0 C_1 \rangle - \langle A_1 B_1 C_1 \rangle| \leq 2\sqrt{2}$$

(2)

Here $\langle A_x B_y C_z \rangle = \sum_{abc} a b c P(a,b,c|x,y,z)$. Here it may be noted that the violation of the inequality initially proposed by Mermin [43] (which is nothing but $M \leq 2$) in general, does not detect genuine entanglement. Subsequently, it has been shown that if the bound is modified then it can detect genuine entanglement [36, 55]. Since quantum violation of the above inequality (2) can be detected by observing the outcome statistics of the local measurements alone, it enables detecting genuine...
entanglement without considering the dimension of the corresponding Hilbert space, and is hence, device independent.

With the motivation of getting stronger device-independent genuine entanglement witness, Uffink designed another nonlinear Bell-type inequality [38] which may distinguish genuine multipartite entanglement from lesser entangled states:

\[
U = (A_1 B_0 C_0 + A_0 B_1 C_0 + A_0 B_0 C_1 - A_1 B_1 C_1)^2 + (A_1 B_1 C_0 + A_0 B_0 C_1 + A_1 B_0 C_1 - A_0 B_1 C_0)^2 \leq 8.
\]  

(3)

So far we have discussed the detection of genuine entanglement by looking at the measurement statistics in a device-independent way. However, there exist scenarios in which the devices are trusted, and one need not resort to the more resource consuming method of device-independent entanglement verification. We now describe the concept of witness operators which can also be used to detect genuine entanglement. A witness operator \( W \) which detects genuine entanglement of a state \( \rho \) is a hermitian operator that satisfies the conditions,

\[
\text{Tr}(W \rho) \geq 0, \quad \forall \rho \in \mathcal{B}_S \quad \exists \text{ at least one } \rho \notin \mathcal{B}_S, \text{ s.t. } \text{Tr}(W \rho) < 0
\]

(4)

where \( \mathcal{B}_S \) is the set of all bi-separable states. The existence of such a witness operator is a consequence of the Hahn-Banach theorem on normed linear spaces [50]. For every genuinely entangled state, there exists a genuine entanglement witness.

In the present study, we consider two types of witness operators that detect genuine entangled states. The first witness operator that we will use is suitable for detecting genuine entanglement of the three-qubit W-state. Consider the three-qubit W state given by, \(|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)\). The witness operator that detects genuine entanglement in the state \(|W\rangle\) is given by [51–54],

\[
W_W = \frac{2}{3}I_3 - |W\rangle\langle W|.
\]

(5)

Whenever a state \( \rho \) gives \( \text{Tr}(W_W \rho) < 0 \), genuine entanglement in the state \( \rho \) is certified.

Next, we discuss the witness operator which is suitable for detecting genuine entanglement of three-qubit GHZ-state. Consider the three-qubit GHZ-state given by, \(|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)\). The witness operator that detects genuine entanglement in the state \(|GHZ\rangle\) is given by [51, 54]

\[
W_{GHZ} = \frac{1}{2}I_3 - |GHZ\rangle\langle GHZ|.
\]

(6)

If a state \( \rho \) gives \( \text{Tr}(W_{GHZ} \rho) < 0 \), then genuine entanglement in the state \( \rho \) is certified.

The advantage of such kind of witness operators is that they can be implemented in the laboratory by performing a finite number of correlated local measurements. Hence, such witness operators can be realized when the observers sharing the quantum state are spatially separated. The witness operator (5) can be written in the following decomposition into a sum of tensor products of operators:

\[
W_W = \frac{1}{24} \left( 13I \otimes I \otimes I + 3 \sigma_z \otimes \sigma_z \otimes I + 3I \otimes \sigma_z \otimes I + 3I \otimes I \otimes \sigma_z + 5 \sigma_z \otimes \sigma_z \otimes I + 5 \sigma_z \otimes I \otimes \sigma_z + 5 I \otimes \sigma_z \otimes \sigma_z \right.
\]
\[
+ 7 \sigma_x \otimes \sigma_x \otimes \sigma_x - I \otimes I \otimes (\sigma_x + \sigma_x) - I \otimes (\sigma_x + \sigma_x) \otimes I - (\sigma_x + \sigma_x) \otimes I \otimes I - I \otimes (\sigma_x + \sigma_x) \otimes (\sigma_x + \sigma_x)
\]
\[
- (\sigma_x + \sigma_x) \otimes I \otimes (\sigma_x + \sigma_x) - (\sigma_x + \sigma_x) \otimes (\sigma_x + \sigma_x) \otimes I - I \otimes (\sigma_x + \sigma_x) \otimes (\sigma_x + \sigma_x) - I \otimes I \otimes (\sigma_x + \sigma_x)
\]
\[
- (\sigma_x - \sigma_x) \otimes I \otimes (\sigma_x - \sigma_x) - I \otimes (\sigma_x - \sigma_x) \otimes I - (\sigma_x - \sigma_x) \otimes I \otimes I - I \otimes (\sigma_x - \sigma_x) \otimes (\sigma_x - \sigma_x) - (\sigma_x - \sigma_x) \otimes (\sigma_x - \sigma_x)
\]
\[
- (\sigma_x - \sigma_x) \otimes (\sigma_x - \sigma_x) \otimes I - (\sigma_x - \sigma_x) \otimes (\sigma_x - \sigma_x) \otimes I - I \otimes I \otimes (\sigma_x - \sigma_x) - I \otimes (\sigma_x - \sigma_x) \otimes I - (\sigma_x - \sigma_x) \otimes I \otimes I
\]
\[
- (\sigma_x + \sigma_y) \otimes I \otimes (\sigma_x + \sigma_y) - (\sigma_x + \sigma_y) \otimes I \otimes (\sigma_x + \sigma_y) - I \otimes I \otimes (\sigma_x - \sigma_y) - I \otimes I \otimes (\sigma_x - \sigma_y) - I \otimes (\sigma_x - \sigma_y) \otimes I
\]
\[
- I \otimes (\sigma_x - \sigma_y) \otimes (\sigma_x - \sigma_y) - I \otimes (\sigma_x - \sigma_y) \otimes (\sigma_x - \sigma_y) - (\sigma_x - \sigma_y) \otimes I \otimes I
\]
\[
- (\sigma_x - \sigma_y) \otimes (\sigma_x - \sigma_y) \otimes (\sigma_x - \sigma_y) - (\sigma_x - \sigma_y) \otimes (\sigma_x - \sigma_y) \otimes (\sigma_x - \sigma_y).
\]

(7)

Note that all the correlations of measurements like \( \sigma_z \otimes \sigma_z \otimes \sigma_z \), \( \sigma_z \otimes \sigma_z \otimes I \), \( \sigma_z \otimes I \otimes \sigma_z \), \( I \otimes \sigma_z \otimes \sigma_z \), \( I \otimes \sigma_z \otimes I \), \( I \otimes I \otimes I \otimes I \), \( I \otimes I \otimes I \) can be determined from the same data. Hence, the above decomposition requires measurements of five correlations:

- \( \sigma_z \otimes \sigma_z \otimes \sigma_z \),
- \( \frac{\sigma_z + \sigma_z}{\sqrt{2}} \otimes \frac{\sigma_z + \sigma_z}{\sqrt{2}} \otimes \frac{\sigma_z + \sigma_z}{\sqrt{2}} \),
- \( \frac{\sigma_z - \sigma_z}{\sqrt{2}} \otimes \frac{\sigma_z - \sigma_z}{\sqrt{2}} \otimes \frac{\sigma_z - \sigma_z}{\sqrt{2}} \).
Charlie multiple Charlies (i.e., Charlie 1 separated observers say Alice, Bob and a sequence of the present paper. Let us consider that three spatially separated party’s setting) is satisfied between Alice, any Charlie as they are spatially separated and spatially separated particle’s setting) is satisfied between Alice, the no-signaling condition (the probability of obtaining of each Charlie are equally probable. Note also, that related with the choices of measurement settings and outcomes: the previous Charlie.

The above decomposition requires measurements of four correlations:

- \( (\sigma_z + \sigma_y) \otimes (\sigma_x + \sigma_y) \otimes (\sigma_x + \sigma_y) \),
- \( (\sigma_z - \sigma_y) \otimes (\sigma_z - \sigma_y) \otimes (\sigma_z - \sigma_y) \).

Similarly, the witness operator (6) can be written in the following decomposition:

\[
W_{GHZ} = \frac{1}{8} \left( 3 I \otimes I \otimes I - I \otimes \sigma_z \otimes \sigma_z - \sigma_z \otimes I \otimes \sigma_z - \sigma_z \otimes \sigma_z \otimes I - 2 \sigma_z \otimes \sigma_z \otimes \sigma_z \right) + \frac{1}{2} (\sigma_z + \sigma_y) \otimes (\sigma_x + \sigma_y) \otimes (\sigma_x + \sigma_y) + \frac{1}{2} (\sigma_x - \sigma_y) \otimes (\sigma_x - \sigma_y) \otimes (\sigma_x - \sigma_y) \right). \tag{8}
\]

The above decomposition requires measurements of four correla-

- \( (\sigma_z \otimes \sigma_z \otimes \sigma_z) \),
- \( (\sigma_z \otimes \sigma_x \otimes \sigma_x) \),
- \( (\sigma_z + \sigma_y) \otimes (\sigma_x + \sigma_y) \otimes (\sigma_x + \sigma_y) \),
- \( (\sigma_z - \sigma_y) \otimes (\sigma_x - \sigma_y) \otimes (\sigma_x - \sigma_y) \).

B. Setting up the measurement context

In this subsection we describe the scenario adopted in the present paper. Let us consider that three spatially separated observers say Alice, Bob and a sequence of multiple Charlies (i.e., Charlie 1, Charlie 2, Charlie 3, ... Charlie n) share a tripartite state \( \rho \) consisting of three spin-\( \frac{1}{2} \) particles. In our scenario, Alice performs projective measurements on the first particle, Bob performs projective measurements on the second particle and multiple Charlies are allowed perform non-projective or unsharp measurements [19, 20] on the third particle sequentially. Let us now clarify the measurement scenario of multiple Charlies. Initially, Charlie 1 performs an unsharp measurement on the third particle, then she sends that particle to Charlie 2. Charlie 1 subsequently passes the third particle to Charlie 3 after performing another unsharp measurement. Charlie 3 also follows the same procedure and so on. This scenario is depicted in Figure 1.

It may be noted here here that the choice of measurement settings of each Charlie is independent and uncorrelated with the choices of measurement settings and outcomes of the previous Charlies. The unbiased input scenario is another assumption that we have adopted in this paper. It implies that all possible measurement settings of each Charlie are equally probable. Note also, that the no-signaling condition (the probability of obtaining one party’s outcome does not depend on the other spatially separated party’s setting) is satisfied between Alice, Bob and any Charlie as they are spatially separated and they perform measurements on three different particles. However, the no-signalling condition is not satisfied between different Charlies as each subsequent Charlie performs measurements on the same particle accessed earlier by the previous Charlie.

In the above scenario, we ask the question as to how many Charlies can detect genuine tripartite entanglement with Alice and Bob. We will address this issue by investigating how many Charlies can have correlations with Alice and Bob such that they violate the Mermin inequality (2) or the Uffink inequality (3). Furthermore, we will also discuss how many Charlies can demonstrate genuine tripartite entanglement if they use the witness operators given by Eq. (5) and Eq. (6), respectively. Here, if any Charlie performs projective measurements, then the entanglement of the state will be completely lost, and there will be no chance to detect entanglement by the
subsequent Charlies. However, it is natural that no such restriction is required for the measurements performed by the last Charlie in the sequence. Hence, in order to deal with the above problem with \( n \) Charlies, the first \(( n - 1)\) Charlie should perform unsharp measurements.

In the following we will briefly discuss the unsharp measurement formalism used in this paper (For details, see [15, 18, 24]).

Following the standard projective measurement scheme proposed by von Neuman [56], after an interaction with a meter having the state \( \phi(q) \), the state \(|\psi\rangle = a|0\rangle + b|1\rangle\) (|0\rangle and |1\rangle form orthonormal basis in \( \mathbb{C}^2 \), \(|a|^2 + |b|^2 = 1\)) of the system (to be measured) of a spin-\( \frac{1}{2} \) particle becomes

\[
|\psi\rangle \otimes \phi(q) \rightarrow a|0\rangle \otimes \phi(q-1) + b|1\rangle \otimes \phi(q+1). \tag{9}
\]

In a general sharp or projective measurement, one obtains the maximum amount of information at the cost of maximum disturbance to the state of the system. On the other hand, the disturbance to the state can be reduced by performing an unsharp measurement where one obtains less amount of information. An unsharp measurement can be characterised by two real parameters: the quality factor \( F \) and the precision \( G \) of the measurements. The quality factor quantifies the extend to which the initial state of the system (to be measured) remains undisturbed during the measurement process. Mathematically, the quality factor is defined as \( F(\phi) = \int_{-\infty}^{\infty} \phi(q+1) \phi(q-1) dq \) Precision \( G \) quantifies the information gain due to the measurement. Mathematically, it is defined as \( G = \int_{-1}^{1} \phi^2(q) dq \). It is obvious that for sharp measurement \( F = 0 \) and \( G = 1 \). An optimal pointer state is the one for which one obtains the greatest precision for a given quality factor. The information-disturbance trade-off relation for an optimal pointer is given by, \( F^2 + G^2 = \frac{1}{2} \) [15].

The above formalism can be recast in terms of unsharp measurements. Unsharp measurement is one particular class of POVMs [19, 20]. POVM is nothing but set of positive operators that add to identity, i.e., \( E \equiv \{ E_i \} \sum E_i = I, 0 < E_i \leq I \}. \) Here, each of the \( E_i \) is called effect operator which represents a particular outcome of the POVM. If we restrict ourselves to the dichotomic unsharp measurement formalism, the effect operators are given by,

\[
E_{\pm}^\lambda = \lambda P_{\pm} + (1 - \lambda) \frac{I}{2}, \tag{10}
\]

where \( (0 < \lambda \leq 1) \) is the sharpness parameter, \( P_{+} \) (\( P_{-} \)) is the projector for the outcomes +1 (−1) respectively. \( \text{Tr}[\rho E_{\pm}^\lambda] \) and \( \text{Tr}[\rho E_{\mp}^\lambda] \) are the probability of getting the outcomes +1 and −1 respectively. Using the generalized von Neumann-Lüders transformation rule [19], the states after the measurements, when the outcomes +1 and −1 occurs, are given by,

\[
\frac{\sqrt{E_{\pm}^\lambda \rho E_{\pm}^\lambda}}{\text{Tr}[E_{\pm}^\lambda \rho]} \quad \text{and} \quad \frac{\sqrt{E_{\mp}^\lambda \rho E_{\mp}^\lambda}}{\text{Tr}[E_{\mp}^\lambda \rho]} \]

respectively. Using the von Neumann-Lüders transformation rule, it can be shown that the quality factor and the precision associated with the above unsharp measurement formalism are given by, \( F = \sqrt{T - N^2} \) and \( G = \lambda \). Hence, the optimal pointer state condition, \( F^2 + G^2 = 1 \), is automatically satisfied in the unsharp measurement formalism [18, 24]. In other words, the unsharp measurement formalism along with the von Neumann-Lüders transformation rule provides the largest amount of information for a given amount of disturbance created on the state due to the measurement.

In our study we will consider that each Charlie, except the final Charlie in the sequence, performs unsharp measurements.

### III. SEQUENTIAL DETECTION OF GENUINE TRIPARTITE ENTANGLEMENT IN THE DEVICE-INDEPENDENT SCENARIO

In this section we find out the maximum number of Charlies that can independently and sequentially detect genuine entanglement through the violation of Mermin inequality (2) or Uffink inequality (3) in the scenario described in subsection II B. We start with the Mermin inequality (2), which is maximally violated by tripartite GHZ state [21] \( \rho_{GHZ} = |\psi_{GHZ}\rangle \langle \psi_{GHZ}|, \) where

\[
|\psi_{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle). \tag{11}
\]

Suppose a tripartite GHZ state given by Eq.(11) is initially shared among Alice, Bob and multiple Charlies. Alice performs dichotomic sharp measurement of spin component observable on her part in the direction \( \hat{x}_0 \), or \( \hat{x}_1 \). Bob performs dichotomic sharp measurement of spin component observable on his particle in the direction \( \hat{y}_0 \) or \( \hat{y}_1 \). Charlie \( m \) (where \( m \in \{ 1, 2, \ldots, n \} \)) performs dichotomic unsharp measurement of spin component observable in the direction \( \hat{z}^m_0 \) or \( \hat{z}^m_1 \). The outcomes of each measurement are ±1.

The projectors associated with Alice’s sharp measurement of spin component observable in the direction \( \hat{x}_i \) (with \( i \in \{0, 1\} \)) can be written as \( P_{a_i \hat{x}_j} = \frac{I}{2} + a \hat{x}_i \cdot \hat{\sigma} \) (with \( a \) being the outcome of the sharp measurement and \( a \in \{+1, -1\} \)). The directions \( \hat{x}_i \) can be expressed as

\[
\hat{x}_i = \sin \theta_i \cos \phi_i \hat{X} + \sin \theta_i \sin \phi_i \hat{Y} + \cos \theta_i \hat{Z}, \tag{12}
\]

where \( 0 \leq \theta_i \leq \pi \); \( 0 \leq \phi_i \leq 2\pi \). \( \hat{X}, \hat{Y}, \hat{Z} \) are three orthogonal unit vectors in Cartesian coordinates.

Similarly, the projectors associated with Bob’s sharp measurement of spin component observable in the direction \( \hat{y}_j \) (with \( j \in \{0, 1\} \)) are given by, \( P_{b_i \hat{y}_j} = \frac{I}{2} + b \hat{y}_j \cdot \hat{\sigma} \) (with \( b \) being the outcome of the sharp measurement and \( b \in \{+1, -1\} \)) and the direction \( \hat{y}_j \) is given by,

\[
\hat{y}_j = \sin \theta_j \cos \phi_j \hat{X} + \sin \theta_j \sin \phi_j \hat{Y} + \cos \theta_j \hat{Z}, \tag{13}
\]

where \( 0 \leq \theta_j \leq \pi \); \( 0 \leq \phi_j \leq 2\pi \).
The effect operators associated with Charlie’s $(m \in \{1, 2, \ldots, n\})$ unsharp measurement of spin component observable in the direction $\hat{z}^m_k$ (with $k \in \{0, 1\}$) are given by,

$$E^\lambda m_k = \lambda_m \frac{I_2 + c^m z^m_k \cdot \vec{\sigma}}{2} + (1 - \lambda_m) \frac{I_2}{2},$$

where $c^m_k$ is the outcome of the unsharp measurement by Charlie and $c^m \in \{+1, -1\}$; $\lambda_m$ (with $0 < \lambda_m \leq 1$) denotes the sharpness parameter associated with Charlie’s unsharp measurement. When we consider a sequence of $n$ Charlies, then the measurements of Charlie $n$ will be sharp, i.e., $\lambda_n = 1$. The the direction $\hat{z}^m_k$ is expressed as

$$\hat{z}^m_k = \sin \theta^m_k \cos \phi^m_k \hat{X} + \sin \theta^m_k \sin \phi^m_k \hat{Y} + \cos \theta^m_k \hat{Z},$$

where $0 \leq \theta^m_k \leq \pi$; $0 \leq \phi^m_k \leq 2\pi$.

Let us first study whether Charlie $1$ and Charlie $2$ can sequentially detect genuine entanglement through quantum violation of Mermin inequality (2) with single Alice and single Bob in the scenario depicted in Figure 1. Since there are only two Charlies in this case, we consider measurements of Charlie $2$ to be sharp, i.e., $\lambda_2 = 1$.

The joint probability distribution of occurrence of the outcomes $a$, $b$, $c^1$, when Alice, Bob perform projective measurements of spin component observables along the directions $\hat{x}_i$ and $\hat{y}_j$ respectively, and Charlie $1$ performs unsharp measurement of spin component observable along the direction $\hat{z}_i^1$, is given by,

$$P(a, b, c^1 | \hat{x}_i, \hat{y}_j, \hat{z}_i^1) = \text{Tr} \left[ \frac{I_2 + a\hat{x}_i \cdot \vec{\sigma}}{2} \otimes \frac{I_2 + b\hat{y}_j \cdot \vec{\sigma}}{2} \otimes E^{\lambda_1}_{c^1 | \hat{z}_i^1} \cdot \rho_{\text{GHZ}} \right].$$

The correlation function between Alice, Bob and Charlie $1$, when Alice, Bob perform projective measurements of spin component observables along the directions $\hat{x}_i$ and $\hat{y}_j$ respectively and Charlie $1$ performs unsharp measurement of spin component observable along the direction $\hat{z}_i^1$, can be written as

$$C^1_{i,j,k} = \sum_{a=-1}^{+1} \sum_{b=-1}^{+1} \sum_{c^{1}=-1}^{+1} a b c^1 P(a, b, c^1 | \hat{x}_i, \hat{y}_j, \hat{z}_i^1).$$

The left hand side of the Mermin inequality (2) associated with Alice, Bob and Charlie $1$ in terms of the correlation functions is expressed as

$$M_1 = |C^1_{100} + C^1_{010} + C^1_{101} - C^1_{111}|.\tag{18}$$

Now it is observed that Alice, Bob and Charlie $1$ get quantum violation of Mermin inequality (2) (i.e., $M_1 > 2\sqrt{2}$) when $\lambda_1 > \frac{1}{\sqrt{2}}$. This happens for the following choice of measurement settings: $(\theta^0_0, \phi^0_0, \theta^0_1, \phi^0_1, \theta^b_0, \phi^b_0, \theta^b_1, \phi^b_1, \theta^c_0, \phi^c_0, \theta^c_1, \phi^c_1) \equiv (\frac{\pi}{2}, \frac{\pi}{2}, 0, \frac{\pi}{2}, \frac{\pi}{2}, 0, \frac{\pi}{2}, \frac{\pi}{2}, 0, \frac{\pi}{2}, \frac{\pi}{2}, 0)$. Charlie $1$ passes her particle to Charlie $2$ after her measurement. The following expression gives the unnormalized post measurement reduced state at Charlie $2$’s end after Alice, Bob get outcomes $a$, $b$ by performing projective measurements of spin component observables along the directions $\hat{x}_i$ and $\hat{y}_j$ respectively and Charlie $1$ gets outcome $c^1$ by performing unsharp measurement of spin component observable along the direction $\hat{z}_i^1$:

$$\rho^2_{un} = \text{Tr}_{AB} \left[ \frac{I_2 + a\hat{x}_i \cdot \vec{\sigma}}{2} \otimes \frac{I_2 + b\hat{y}_j \cdot \vec{\sigma}}{2} \otimes \sqrt{E^{\lambda_1}_{c^1 | \hat{z}_i^1}} \cdot \rho_{\text{GHZ}} \right],$$

where,

$$\sqrt{E^{\lambda_1}_{c^1 | \hat{z}_i^1}} = \frac{1 + \lambda_1}{2} \left( \frac{I_2 + c^1 \hat{z}_i^1 \cdot \vec{\sigma}}{2} \right)^{\frac{1}{2}} + \frac{1 - \lambda_1}{2} \left( \frac{I_2 - c^1 \hat{z}_i^1 \cdot \vec{\sigma}}{2} \right)^{\frac{1}{2}}.\tag{20}$$

Here $\text{Tr}_{AB}[\ldots]$ denotes partial trace over the subsystems of Alice and Bob. On the above reduced state, Charlie $2$ again performs unsharp measurement (with sharpness parameter being denoted by $\lambda_2$) of spin component observable along the direction $\hat{z}_i^2$ and gets the outcome $c^2$. The joint probability distribution of occurrence of the outcomes $a$, $b$, $c^1$, $c^2$ when Alice, Bob perform projective measurements of spin component observables along the directions $\hat{x}_i$ and $\hat{y}_j$ respectively and Charlie $1$, Charlie $2$ perform unsharp measurement of spin component observable along the direction $\hat{z}_i^1$, $\hat{z}_i^2$ respectively, is given by,

$$P(a, b, c^1, c^2 | \hat{x}_i, \hat{y}_j, \hat{z}_i^1, \hat{z}_i^2) = \text{Tr} \left[ E^{\lambda_2}_{c^2 | \hat{z}_i^2} \cdot \rho^2_{un} \right].$$

From this expression, one can obtain the joint probability of obtaining the outcomes $a$, $b$, $c^2$ when Alice, Bob, Charlie $2$ measures spin component observables in the directions $\hat{x}_i$, $\hat{y}_j$, $\hat{z}_i^2$, respectively and when Charlie $1$ has already measured spin component observables in the directions $\hat{z}_i^1$, $\hat{z}_i^2$:

$$P(a, b, c^2 | \hat{x}_i, \hat{y}_j, \hat{z}_i^1, \hat{z}_i^2) = \sum_{c^{1}=-1}^{+1} P(a, b, c^{1}, c^2 | \hat{x}_i, \hat{y}_j, \hat{z}_i^1, \hat{z}_i^2).$$

$$C^2_{ijkl}$ denote the correlation between Alice, Bob and Charlie $2$ when Alice, Bob, Charlie $1$ and Charlie $2$ measure spin component observables in the directions $\hat{x}_i$, $\hat{y}_j$, $\hat{z}_i^1$, $\hat{z}_i^2$, respectively. The expression for $C^2_{ijkl}$ can be obtained from

$$C^2_{ijkl} = \sum_{a=-1}^{+1} \sum_{b=-1}^{+1} \sum_{c^{1}=-1}^{+1} a b c^2 P(a, b, c^2 | \hat{x}_i, \hat{y}_j, \hat{z}_i^1, \hat{z}_i^2).\tag{23}$$
Since Charlie’s choice of measurement settings is independent of the measurement settings of Charlie’s, the above correlation has to be averaged over the two possible measurement settings of Charlie’s (spin component observables in the directions \{\hat{z}^1, \hat{z}^2\}). This average correlation function between Alice, Bob and Charlie’s is given by,

\[ C_{ijkl} = \sum_{k=0,1} C_{ijkl}^2 P(\hat{z}^k_i), \tag{24} \]

where \(P(\hat{z}^k_i)\) is the probability with which Charlie performs unsharp measurement of spin component observables in the direction \(\hat{z}^k_i\) (\(k \in \{0, 1\}\)). For an unbiased input scenario, we take the two measurement settings for Charlie’s to be equally probable, i.e., \(P(\hat{z}^k_i) = P(\hat{z}^k_i) = \frac{1}{2}\).

The left hand side of the Mermin inequality (2) associated with Alice, Bob and Charlie’s in terms of the average correlation functions is expressed as

\[ M_2 = \left( C_{1100}^2 + C_{1010}^2 + C_{0110}^2 - C_{1111}^2 \right). \tag{25} \]

In a similar way by evaluating the average correlation functions between Alice, Bob and Charlie’s, the Mermin inequality can be written as

\[ M_m = \left( C_{1100}^m + C_{1010}^m + C_{0110}^m - C_{1111}^m \right) \leq 2 \sqrt{2}. \tag{26} \]

Violation of this inequality implies detection of genuine entanglement by Alice, Bob and Charlie’s.

Now, we observe that when Charlie’s gets 5% violation of the Mermin inequality (2) (i.e., when \(M_1 = 2.96\)), Charlie’s gets 18% violation of the Mermin inequality (2) (i.e., \(M_2 = 3.34\)). This happens for the following choice of measurement settings: \((\phi^0_0, \phi^0_2, \phi^1_0, \phi^1_2, \phi^0_0, \phi^0_2, \phi^1_0, \phi^1_2, \phi^0_0, \phi^0_2, \phi^1_0, \phi^1_2) \equiv (\frac{\pi}{4}, \frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, 0)\) and when \(\lambda_1 = 0.74\). Hence, Charlie’s and Charlie’s can detect genuine entanglement sequentially through the quantum violations of the Mermin inequality (2). Charlie’s and Charlie’s both get quantum violations of the Mermin inequality (2) when \(\lambda_1 \in (0.71, 0.91)\).

Next, we investigate whether Charlie, Charlie’s and Charlie’s can sequentially detect genuine entanglement through quantum violation of Mermin inequality (2) with single Alice and single Bob in the scenario depicted in Figure 1. In this case the measurements of Charlie’s will be sharp, i.e., \(\lambda_3 = 1\). On the other hand, Charlie’s and Charlie’s perform unsharp measurements. When Charlie’s gets 5% violation and Charlie’s gets 5% violation of the Mermin inequality (2) (i.e., when \(M_1 = 2.96\) and \(M_2 = 2.96\)), then the maximum magnitude of left hand side of Mermin inequality (2) for Charlie’s becomes \(M_3 = 2.62\). This happens for the following choice of measurement settings: \((\phi^0_0, \phi^0_2, \phi^1_0, \phi^1_2, \phi^0_0, \phi^0_2, \phi^1_0, \phi^1_2, \phi^0_0, \phi^0_2, \phi^1_0, \phi^1_2) \equiv (\frac{\pi}{4}, \frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, 0)\) and when \(\lambda_1 = 0.74\) and \(\lambda_2 = 0.88\). In fact, when \(M_1 = 2\sqrt{2}, M_2 = 2\sqrt{2}\), then the maximum of \(M_3 = 2.78 < 2\sqrt{2}\). Hence, Charlie’s, Charlie’s, Charlie’s cannot detect genuine entanglement sequentially through the quantum violations of the Mermin inequality (2).

One important point to be noted here is that Charlie’s may obtain quantum violation of the Mermin inequality (2) if the sharpness parameter of Charlie’s or that of Charlie’s is too small to get a violation. Hence, at most two Charlies can sequentially detect genuine entanglement through quantum violations of the Mermin inequality (2).

Up to now we have used quantum violation of the Mermin inequality (2) to certify genuine entanglement between Alice, Bob and any Charlie. Now, we will investigate whether the number of Charlies who can sequentially detect genuine entanglement, can be increased by using quantum violation of the Uffink inequality (3). The Uffink inequality in terms of the average correlation functions between Alice, Bob and Charlie’s can be expressed as

\[ U_m = \left( C_{1100}^m + C_{1010}^m + C_{0110}^m - C_{1111}^m \right)^2 + \left( C_{1100}^m + C_{1010}^m + C_{0110}^m - C_{1111}^m \right)^2 \leq 8. \tag{27} \]

The average correlation functions can be evaluated following the aforementioned procedure. Violation of this inequality implies that genuine entangled state is shared between Alice, Bob and Charlie’s. In this case too, we assume that the three qubit GHZ state is initially shared between Alice, Bob and Charlie’s as this state gives maximum quantum violation of the Uffink inequality (3).

Let us try to find out whether Charlie, Charlie’s and Charlie’s can sequentially detect genuine entanglement through quantum violation of Uffink inequality (3) with single Alice and single Bob. Here the measurements of Charlie’s is sharp, i.e., \(\lambda_3 = 1\). When Charlie’s gets 5% violation and Charlie’s gets 5% violation of the Uffink inequality (3) (i.e., when \(U_1 = 8.40\) and \(U_2 = 8.40\)), then the maximum magnitude of left hand side of Uffink inequality (3) for Charlie’s becomes \(U_3 = 7.73\). This happens for the following choice of measurement settings: \((\phi^0_0, \phi^0_2, \phi^1_0, \phi^1_2, \phi^0_0, \phi^0_2, \phi^1_0, \phi^1_2, \phi^0_0, \phi^0_2, \phi^1_0, \phi^1_2, \phi^0_0, \phi^0_2, \phi^1_0, \phi^1_2) \equiv (\frac{\pi}{4}, \frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, 0)\) and when \(\lambda_1 = 0.72\) and \(\lambda_2 = 0.86\). In fact, we observe that when \(U_1 = 8, U_2 = 8\), then the maximum of \(U_3 = 7.76\). Hence, at most two Charlies can detect genuine entanglement sequentially through the quantum violations of the Uffink inequality (3).

IV. SEQUENTIAL SHARING OF TRIPARTITE GENUINE ENTANGLEMENT USING ENTANGLEMENT WITNESSES

In this section we are going to use genuine entanglement witnesses, instead of using device-independent genuine entanglement inequalities, in order to probe sequen-
The decomposition of this witness operator in terms of tensor products of operators is given by Eq. (7). However, in the scenario depicted in Figure 1 the local measurements performed by Charlie in a sequence is unsharp. Since, the decomposition (7) of the witness operator \( \mathcal{W}_W \) can be used when each observer performs sharp projective measurements, we have to modify the decomposition (7) of the above witness operator for unsharp measurements at Charlie’s end. In order to do this, we will follow the prescription described in [26].

The joint probability of obtaining the outcomes \( a, b, c^m \), when Alice, Bob perform projective measurements of spin component observables along the directions \( \hat{x}_i \) and \( \hat{y}_j \) respectively and Charlie’s performs unsharp measurement of spin component observable along the direction \( \hat{z}^m_k \), can be evaluated using the formula,

\[
\text{Tr} \left[ \rho \left( P_{a|\hat{x}_i} \otimes P_{b|\hat{y}_j} \otimes E_{c^m|\hat{z}^m_k}^{\lambda_m} \right) \right],
\]

(29)

where \( \rho \) is the average post-measurement state obtained after the previous stage of the measurement processes: \( P_{a|\hat{x}_i} \) and \( P_{b|\hat{y}_j} \) are projection operators corresponding to the projective measurements by Alice and Bob respectively, and \( E_{c^m|\hat{z}^m_k}^{\lambda_m} \) is the effect operator associated with the POVM performed by Charlie’s.

The expectation value of the state \( \rho \) corresponding to the above joint measurements is given by,

\[
\text{Tr} \left[ \left\{ (P_{a|\hat{x}_i} - P_{-|\hat{x}_i}) \otimes (P_{b|\hat{y}_j} - P_{-|\hat{y}_j}) \otimes (E_{c^m|\hat{z}^m_k}^{\lambda_m} - E_{-|\hat{z}^m_k}^{\lambda_m}) \right\} \rho \right].
\]

(30)

Now, \( P_{a|\hat{x}_i} - P_{-|\hat{x}_i} \) and \( P_{b|\hat{y}_j} - P_{-|\hat{y}_j} \) is nothing but \( \hat{x}_i \cdot \hat{\sigma} \) \((\hat{y}_j \cdot \hat{\sigma})\). Let us denote it by \( \sigma_{x_i} \) (\( \sigma_{y_j} \)). Let us also denote \( E_{c^m|\hat{z}^m_k}^{\lambda_m} - E_{-|\hat{z}^m_k}^{\lambda_m} \) as \( \sigma_{c_i}^{\lambda_m} \). Hence, we can write the following,

Noting the above relation one can use the substitution \( \sigma_{x_i} \otimes \sigma_{y_j} \otimes \sigma_{c_i}^{\lambda_m} \rightarrow \lambda_m \sigma_{x_i} \otimes \sigma_{y_j} \otimes \sigma_{c_i}^{\lambda_m} \) in the case of a general \( \lambda_m \) [26], so that the decomposition (7) of the genuine entanglement witness operator \( \mathcal{W}_W \) in this case becomes

\[
\mathcal{W}_W^{\lambda_m} = \frac{1}{24} \left( 13 \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I} + 3 \sigma_z \otimes \mathbf{I} \otimes \mathbf{I} + 3 \mathbf{I} \otimes \sigma_z \otimes \mathbf{I} + 3 \mathbf{I} \otimes \mathbf{I} \otimes \sigma_z + 5 \sigma_z \otimes \sigma_z \otimes \mathbf{I} + 5 \sigma_z \otimes \mathbf{I} \otimes \sigma_z + 5 \mathbf{I} \otimes \sigma_z \otimes \sigma_z + 5 \mathbf{I} \otimes \sigma_z \otimes \sigma_z \right)
\]

\[
+ 5 \mathbf{I} \otimes \sigma_z \otimes \lambda_m + 7 \sigma_z \otimes \sigma_z \otimes \lambda_m \sigma_z - \mathbf{I} \otimes \mathbf{I} \otimes \lambda_m \left( \sigma_z + \sigma_x \right) - \mathbf{I} \otimes \left( \sigma_z + \sigma_x \right) \otimes \mathbf{I} - \left( \sigma_z + \sigma_x \right) \otimes \mathbf{I} - \left( \sigma_z + \sigma_x \right) \otimes \mathbf{I}
\]

\[
- \mathbf{I} \otimes \sigma_z \otimes \mathbf{I} \otimes \lambda_m \left( \sigma_z + \sigma_x \right) - \mathbf{I} \otimes \left( \sigma_z + \sigma_x \right) \otimes \mathbf{I} - \left( \sigma_z + \sigma_x \right) \otimes \mathbf{I}
\]

\[
- \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I}
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\]

\[
- \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I}
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- \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I}
\]

\[
- \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I}
\]

\[
\right) \).
\]

(31)

Now, since we have \( \text{Tr}[\mathcal{W}_W \rho_{BS}] \geq 0 \forall \rho_{BS} \in \text{BS} \) (where \( \text{BS} \) is the set of all bi-seperable states) and \( 0 < \lambda_m \leq 1 \),
The first inequality in the last line of (33) is obtained by minimizing all the expectation values. Hence, we can conclude that the operator $W_W$ even after introducing unsharpness in Charlie’s measurements ($W_W^{\lambda_m}$) can be used as a valid witness of genuine entanglement.

Now, suppose that the three qubit W state given by, $|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$ is initially shared between Alice, Bob and Charlie. When Alice, Bob perform projective measurements and Charlie performs unsharp measurement with sharpness parameter being denoted by $\lambda_1,$ the entanglement witness $W_W^{\lambda_1}$ acquires the following expectation value

$$\text{Tr}[|W\rangle\langle W|W_W^{\lambda_1}] = \frac{1}{18}(7 - 13\lambda_1)$$

(34)

It is clear from the above equation that Charlie$^1$ can detect genuine entanglement with Alice and Bob when $\lambda_1 > \frac{7}{13} \approx 0.54$.

Let us now explore whether there is any possibility for subsequent Charlies, i.e, Charlie$^2$, Charlie$^3$ .... to detect the residual genuine entanglement in the post measurement average state with single Alice and single Bob at other sides. Since any Charlie is ignorant about the choices of measurement settings and outcomes all previous Charlies, we have to average over the previous Charlie’s inputs and outputs to obtain the state shared between Alice, Bob and the Charlie of the current stage of the experiment. After performance of Charlie$^1$’s unsharp measurement, the average state becomes

$$|W\rangle\langle W| \rightarrow \rho_W^{\lambda_1}$$

$$= \frac{1}{5} \sum_{i, i' k} \left(1 \otimes I \otimes \sqrt{E^M_{i, i' k}}\right)|W\rangle\langle W|\left(1 \otimes I \otimes \sqrt{E^M_{i, i' k}}\right),$$

(35)

where $i \in \{+1, -1\}$, $\tilde{z}_k \in \{\tilde{z}, \tilde{x}, \tilde{x} + \tilde{y}, \tilde{x} - \tilde{y}\}$, $\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}$.

In the next step Charlie$^2$ performs unsharp measurements on his part of $\rho_W^{\lambda_1}$ with sharpness parameter $\lambda_2,$ to check with Alice and Bob whether the state is genuinely entangled, by using the witness parameter $W_W^{\lambda_2}$ which

| Charlie$^m$ | Conditions on $\lambda_1$ | Conditions on $\lambda_2$ | Conditions on $\lambda_3$ | Conditions on $\lambda_4$ | Conditions on $\lambda_5$ |
|------------|------------------------|------------------------|------------------------|------------------------|------------------------|
| Charlie$^1$ | $1 \geq \lambda_1 > 0.54$ | -                      | -                      | -                      | -                      |
| Charlie$^2$ | $\lambda_1 = 0.54 + \epsilon_1$ | $1 \geq \lambda_2 > 0.60$ | -                      | -                      | -                      |
| Charlie$^3$ | $\lambda_1 = 0.54 + \epsilon_1$ | $\lambda_2 = 0.60 + \epsilon_2$ | $1 \geq \lambda_3 > 0.69$ | -                      | -                      |
| Charlie$^4$ | $\lambda_1 = 0.54 + \epsilon_1$ | $\lambda_2 = 0.60 + \epsilon_2$ | $\lambda_3 = 0.69 + \epsilon_3$ | $1 \geq \lambda_4 > 0.84$ | -                      |
| Charlie$^5$ | $\lambda_1 = 0.54 + \epsilon_1$ | $\lambda_2 = 0.60 + \epsilon_2$ | $\lambda_3 = 0.69 + \epsilon_3$ | $\lambda_4 = 0.84 + \epsilon_4$ | No valid region for $\lambda_5$ |
acquires the following expectation value,
\[
\text{Tr} \left[ \rho_W^\lambda W_W^\lambda \right] = \frac{1}{90} \left( 35 - (23 + 42 \sqrt{1 - \lambda_1^2}) \lambda_2 \right).
\] (36)

Hence, Charlie\(^2\) can detect genuine entanglement with Alice and Bob if
\[
\frac{1}{90} \left( 35 - (23 + 42 \sqrt{1 - \lambda_1^2}) \lambda_2 \right) < 0.
\]
On the other hand, from Eq.(34) we know that Charlie\(^1\) can detect genuine entanglement with Alice and Bob when \(\lambda_1 > \frac{7}{10}\), i.e., when \(\lambda_1 = \frac{7}{10} + \epsilon\) with \(\epsilon\) being a positive number such that \(\epsilon \leq \frac{6}{10}\). Hence, in order to detect genuine entanglement, Charlie\(^2\) must choose his sharpness parameter \(\lambda_2\) such that it satisfies
\[
\frac{1}{90} \left( 35 - (23 + 42 \sqrt{1 - \frac{7}{10}^2}) \lambda_2 \right) < 0.
\]
If we take \(\epsilon = 0\) (i.e., \(\lambda_1 = \frac{7}{10}\)), then we obtain that Charlie\(^2\) can detect genuine entanglement with Alice and Bob if \(\lambda_2 > 0.60\).

In this way if we proceed it can be observed that at most four Charlies can detect genuine entanglement through the witness operator \(W_W^\lambda\) when the initial shared state is three qubit pure W-state. Allowed ranges of the sharpness parameters associated with different Charlies’ measurements in order to detect genuine entanglement using the witness operator \(W_W^\lambda\) are presented in Table 1.

Now we are going to investigate the maximum number of Charlies that can detect genuine entanglement in the scenario mentioned in Figure 1 using another type of witness operator (suitable for detecting genuine entanglement of three qubit GHZ state) which is given by [51, 54],
\[
W_{GHZ} = \frac{1}{2} I_3 - |GHZ\rangle \langle GHZ| \quad \text{(37)}
\]

Now, when any Charlie\(^m\) performs unsharp measurements with sharpness parameter \(\lambda_m\), the decomposition (8) of the above witness operator is modified in the following way,
\[
W_{GHZ}^\lambda = \frac{1}{8} \left( 3 \mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} - \mathbb{I} \otimes \sigma_z \otimes \mathbb{I} - \mathbb{I} \otimes \sigma_z - \sigma_z \otimes \mathbb{I} \otimes \sigma_z - \sigma_z - \sigma_z \otimes \mathbb{I} - 2 \sigma_z \otimes \sigma_z - \sigma_z \otimes \sigma_z \right.
\]
\[
\left. + \frac{1}{2} \sigma_x + \sigma_y \right) \otimes \left( \sigma_x + \sigma_y \right) \otimes \mathbb{I} - \frac{1}{2} \sigma_x - \sigma_y \otimes \left( \sigma_x - \sigma_y \right) \otimes \sigma_z - \frac{1}{2} \sigma_x - \sigma_y \right).
\] (38)

Now, since we have \(\text{Tr}[W_{GHZ}^\lambda \rho_{BS}] \geq 0 \forall \rho_{BS} \in \text{BS}\) and \(0 < \lambda_m \leq 1\), we can write the following:
\[
\begin{align*}
\text{Tr}[W_{GHZ}^\lambda \rho_{BS}] &= \lambda_m \text{Tr}[W_{GHZ} \rho_{BS}] + \frac{1}{8} \left( 3 - \langle \sigma_z \otimes \sigma_z \otimes \mathbb{I} \rangle \right) \\
&\geq \lambda_m \text{Tr}[W_{GHZ} \rho_{BS}] + \frac{1}{4} (1 - \lambda_m) \\
&\geq 0 \quad \forall \rho_{BS} \in \text{BS}. \quad \text{(39)}
\end{align*}
\]

Hence, one may conclude that the operator \(W_{GHZ}^\lambda\) after introducing unsharpness in Charlie’s measurements can again be used as a valid witness operator of genuine entanglement.

In this case, consider that the three qubit GHZ state given by, \(|GHZ\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle)\), is initially shared between Alice, Bob and Charlie\(^1\). In a similar fashion described earlier, we now investigate how many Charlies can detect genuine entanglement sequentially with single Alice and single Bob. Since Alice, Bob perform projective measurements and Charlie\(^1\) performs unsharp measurement with sharpness parameter \(\lambda_1\), the expectation value of the genuine entanglement witness \(W_{GHZ}^\lambda\) becomes,
\[
\text{Tr}[|GHZ\rangle \langle GHZ| \ W_{GHZ}^\lambda] = \frac{1}{4} (1 - 3\lambda_1)
\] (40)

Hence, it is clear from the above expectation value that Charlie\(^1\) can detect genuine entanglement using the genuine entanglement witness \(W_{GHZ}^\lambda\) with Alice and Bob when \(\lambda_1 > \frac{1}{3} \equiv 0.33\).

After Charlie\(^1\)’s unsharp measurement, the average state shared between Alice, Bob and Charlie\(^2\) becomes
\[
|GHZ\rangle \langle GHZ| \rightarrow \rho_{GHZ}^\lambda, \quad \text{where} \quad \langle \sigma_z \otimes \sigma_z \otimes I \rangle = \mathbb{I} \otimes \mathbb{I} \otimes \sqrt{E_1^{\lambda_1}},
\]
\[
|GHZ\rangle \langle GHZ| \rightarrow \mathbb{I} \otimes \mathbb{I} \otimes \sqrt{E_1^{\lambda_1}},
\] (41)

where \(i \in \{+1, -1\}\), \(\hat{z}^1 \in \{ \hat{z}, \hat{x} + \hat{y}, \hat{x} - \hat{y} \}\).

Next, Charlie\(^2\) performs unsharp measurements on his part of \(\rho_{GHZ}^\lambda\) with sharpness parameter \(\lambda_2\), to check with Alice and Bob whether the state is genuinely entangled. In this case, the expectation value of the witness operator \(W_{GHZ}^\lambda\) becomes,
\[
\text{Tr}[\rho_{GHZ}^\lambda W_{GHZ}^\lambda] = \frac{1}{4} \left[ 1 - (1 + 2 \sqrt{1 - \lambda_1^2}) \lambda_2 \right].
\] (42)

Hence, Charlie\(^2\) can detect genuine entanglement with Alice and Bob using the above witness if \(\frac{1}{4} \left[ 1 - \left( 1 + 2 \sqrt{1 - \lambda_1^2} \right) \lambda_2 \right] < 0\). Since, Charlie\(^1\) can detect genuine entanglement with Alice and Bob when \(\lambda_1 =
where three spin-tite entanglement sequentially. We consider the scenario where multipartite quantum correlation several times is not elusive. Hence, exploring the possibilities of using spin resources [3–5, 10, 57–66]. However, due to the difficulties present in experimentally producing multipartite quantum correlations, their implementation as powerful resources in various information processing tasks are still elusive. Hence, exploring the possibilities of using single multipartite quantum correlation several times is not only interesting for foundational studies but also for several information theoretic applications.

In the present study we address the question as to whether multiple observers can detect genuine tripartite entanglement sequentially. We consider the scenario where three spin-$\frac{1}{2}$ particles are spatially separated and shared between, say, Alice, Bob and multiple Charlies. Alice measures on the first particle; Bob measures on the second particle and multiple Charlies measure on the third particle sequentially. In the course of our study we have used both linear as well as non-linear correlation inequalities which detect genuine entanglement in the device-independent scenario. In this context, we have shown that at most two Charlies can detect genuine entanglement of the GHZ-state. In this context we should mention that the question of sharing of genuine entanglement of the W-state in the device-independent scenario remains to be addressed due to lack of a suitable inequality.

The number of Charlies may be increased by giving up the requirement of device-independence, as we have shown using two types of appropriate genuine entanglement witness operators. Here, we find that at most four Charlies can detect genuine entanglement sequentially with the single Alice and single Bob using the shared W-state. In case of the shared GHZ-state we find that the number of Charlies can increase up to twelve, which may open up interesting possibilities of detection of genuine tripartite entanglement sharing by multiple observers.

Before concluding, it may be noted that the issue of sharing genuine nonlocality in the above scenario has been studied earlier [35]. Hence, it would be interesting to investigate this issue in the intermediate context between entanglement and Bell-nonlocality, viz., sharing of genuine multipartite quantum steering [67–69] by multiple observers measuring sequentially on the same particle. Finally, exploring information theoretic applications of the present study is another direction for future research.

| Charlie | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ | $\lambda_5$ | $\lambda_{10}$ | $\lambda_{11}$ | $\lambda_{12}$ | $\lambda_{13}$ |
|---------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| Charlie1 | $1 > \lambda_1 > 0.33$ | - | - | - | - | - | - | - | - |
| Charlie2 | $\lambda_1 = 0.33 + \epsilon$ | $1 > \lambda_2 > 0.36$ | - | - | - | - | - | - | - |
| Charlie3 | $\lambda_1 = 0.33 + \epsilon$ | $\lambda_2 = 0.35 + \epsilon$ | $1 > \lambda_3 > 0.36$ | - | - | - | - | - | - |

TABLE II: Here we show the permissible ranges of sharpness parameters $\lambda_m$ of Charlie$^m$ for detecting genuine entanglement through the witness operator $W^\lambda_{GHZ}$ with a single Alice and a single Bob at the other sides. The permissible range of each $\lambda_m$ depends on the values $\lambda_1$, $\lambda_2$, ..., $\lambda_{m-1}$. In the above table we have presented the permissible range of each $\lambda_m$ when the values $\lambda_1$, $\lambda_2$, ..., $\lambda_{m-1}$ are constrained to be in particular ranges (with $\epsilon_i \in (0, 0.01], i = 1, 2, ..., (m - 1)$). In general, $\epsilon_i > 0$ can have any values such that $\lambda_i < 1$. For other values of $\epsilon_i$, the permissible ranges of $\lambda_m$ will be smaller than that presented in the table and can be calculated easily. In this scenario we find that at most twelve Charlies can detect genuine entanglement through the witness operator $W^\lambda_{GHZ}$ (The values of $\lambda_5$ to $\lambda_9$ are not displayed for brevity).
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VI. ACKNOWLEDGEMENTS

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