(\{2, 3\}, 6)-SPHERES AND THEIR GENERALIZATIONS

MICHEL DEZA AND MATHIEU DUTOUR SIKIRIĆ

ABSTRACT. We consider here 6-regular plane graphs whose faces have size 1, 2 or 3. In Section 2 a practical enumeration method is given that allowed us to enumerate them up to 53 vertices. Subsequently, in Section 3 we enumerate all possible symmetry groups of the spheres that showed up. In Section 4 we introduce a new Goldberg-Coxeter construction that takes a 6-regular plane graph \( G_0 \), two integers \( k \) and \( l \) and returns two 6-regular plane graphs.

Then in the final section, we consider the notions of zigzags and central circuits for the considered graphs. We introduced the notions of tightness and weak tightness for them and we prove an upper bound on the number of zigzags and central circuits of such tight graphs. We also classify the tight and weakly tight graphs with simple zigzags or central circuits.

1. INTRODUCTION

By a \((S, k)\)-sphere we call a plane \( k \)-regular graph such that any face has size in \( S \).

If \( G \) is a 6-regular plane graph, then by Euler formula it satisfies the equality:

\[
\sum_{k \geq 2} p_k (3 - k) = 6
\]

with \( p_k \) the number of \( k \)-gons, i.e. faces of size \( k \). So, if, moreover, \( G \) has only 2- and 3-gonal faces, then it has exactly six 2-gons.

Note that a \((\{2, 3\}, 6)\)-sphere with \( p_3 \) 3-gons has \( n = 2 + \frac{p_3}{2} \) vertices. In [DeDu08] (Theorem 2.0.1) we proved that for any \( n \geq 2 \) there exist a \((\{2, 3\}, 6)\)-sphere with \( n \) vertices. If 1-gons are permitted, then \( 2p_1 + p_2 \) being 6, all possible pairs \((p_1, p_2)\), besides \((0, 6)\), are \((1, 4)\), \((2, 2)\) and \((3, 0)\).

The only possible \((\{s - 1, s\}, k)\)-spheres have \((s, k) = (6, 3)\) (well-known geometrical fullerenes), \((4, 4)\) (considered in [DeSt03, DeDuSh03, DHL02, DuDe10]) and \((3, 6)\) (the object of this paper). \((2, 3), 6\)-spheres are spherical analog of the 6-regular partition \( \{3^6\} \) of the Euclidean plane by regular triangles, with six 2-gons playing role of “defects”, disclinations needed to increase the curvature zero to the one of...
sphere. The problem of existence of plane graphs with a fixed $p$-vector is an active subject of research, see for example [YHZQ10].

In Section 2 we expose a practical method for generating $((1, 2, 3), 6)$-spheres. The main idea is to use a reduction to 3-regular graphs for which very efficient programs exist [BrHaHe03]. Then in Section 3 we determine the possible symmetry groups of $((1, 2, 3), 6)$-spheres with $i$ 1-gons. The methods are reasonably easy except for the $((1, 3), 6)$-spheres for which the symmetry groups are $C_3$, $C_{3h}$ or $C_{3v}$.

In Section 4 we introduce a new Goldberg-Coxeter construction. It takes a 6-regular sphere $G_0$, two integers $k, l$ and returns two 6-regular spheres $G_1, G_2$ with $GC_{k,l}(G_0) = \{G_1, G_2\}$. The construction satisfies a multiplicativity property based on the ring of Eisenstein integers. In the case $k = l = 1$ we call the construction oriented tripling and we have a more explicit description of it. The Goldberg-Coxeter construction defined here generalizes the one introduced in [Gold37, Cox71, DeSt03] for 3- or 4-regular plane graphs and allows to describe explicitly all $((1, 3), 6)$-spheres. It also allows to describe all $((2, 3), 6)$-spheres of symmetry $D_6, D_{6h}, T, T_h, or T_d$.

In a plane graph $G$, a zigzag is a circuit of edges such that any two but no three consecutive edges are contained in the same face. In an Eulerian (i.e. degree of any vertex is even) plane graph, a central circuit is a circuit of edges such that any edge entering a vertex is followed by the edge opposite to the entering one.

A zigzag is called simple if no two edges occur two times and a central circuit is called simple if no two vertices occur two times. Let $Z$ and $Z'$ be (possibly, $Z = Z'$) zigzags of a plane graph $G$ and let an orientation be selected on them. An edge $e$ of intersection $Z \cap Z'$ is called of type I or type II, if $Z$ and $Z'$ traverse $e$ in opposite or same direction, respectively. Let $C$ and $C'$ be (possibly, $C = C'$) central circuits of a 6-regular plane graph and let an orientation be selected on them. A vertex $v$ of intersection $C \cap C'$ is called of type I or type II if $C$ and $C'$ pass by $v$ with orientation shifted by $60^\circ$, respectively, $120^\circ$. We prove in Theorem 13 that the intersection type is always of type II.

We then introduce the notions of tightness and weak tightness for zigzags and central circuits and we prove upper bound on the maximal number of zigzags and central circuit. The results are summarized in Table 2 and Figure 28. Then we determine completely the weakly tight spheres with simple zigzags or central circuits.
2. Generation method

In any \((\{2,3\},6)\)-sphere, one can collapse its 2-gons into simple edges. By doing so one obtains a graph with vertices of degree at most 6 and with faces of size 3 only. So, the dual will be a 3-regular graph with faces of size at most 6.

**Theorem 1.** With the exception of the following \((\{2,3\},6)\)-spheres

\[
egin{array}{c}
6 \times K_2:
\end{array}
\]

\[
\begin{array}{c}
3 \times K_3:
\end{array}
\]

\[
D_{6h}, n = 2 \quad D_{3h}, n = 3
\]

any \((\{2,3\},6)\)-sphere is obtained from a \((\{3,4,5,6\},3)\)-sphere by adding vertices of degree 2 and taking the dual.

**Proof.** Let \(G\) be a \((\{2,3\},6)\)-sphere and let \(G^*\) be its dual. Then, by removing from \(G^*\) its vertices of degree 2, one gets a 3-regular graph \(G_1\). It can happen that \(G_1\) has no vertices and is reduced to a simple circular edge \(e\). In this case, if one adds six vertices on \(e\) and take the dual, one will get the first exceptional graph with 2 vertices. If \(G_1\) has one face \(F\) which is a 1-gon, then we have to add 5 vertices of degree 2 on the edge \(e\) of \(F\). Necessarily, any face adjacent to \(F\) has to be a 1-gon, but this is, clearly, impossible. If \(F\) is a 2-gon and \(F\) is adjacent to at least one 2-gon, then \(G_1\) is reduced to a graph with two vertices and three edges. The corresponding \((\{2,3\},6)\)-sphere is the second exceptional graph. Assume that \(F\) is adjacent to \(F_1\), \(F_2\) with \(F_i\) being a \(a_i\)-gon and \(a_i \geq 3\). If one of \(a_i\) is 3, then the other is 6 and this gives a 1-gon. Thus, the only possibility is \(a_1 = a_2 = 4\). This implies that we have a graph with 4 vertices, two 4-gonal and two 2-gonal faces. But consideration of all possibilities rules out this option. So, \(G_1\) is a 3-regular plane graph with faces of size within \{3, 4, 5, 6\}. \(\square\)

The method can be generalized (in Theorem 2) to deal with graphs with 1-gons. Note that for most \((\{3,4,5,6\},3)\)-spheres one cannot add those vertices of degree 2, in order to get the required spheres, because whenever we add such a 2-gon, we have two faces of size lower than 6 that are adjacent. Graphs admitting such adjacency are relatively rare in the set of \((\{3,4,5,6\},3)\)-spheres. Some such graphs are the \((\{5,6\},3)\)-spheres with the 5-gons organized in pairs, they are part of the class of face-regular maps [DeDu08].

The above theorem gives a method to enumerate \((\{2,3\},6)\)-spheres. First enumerate the \((\{3,4,5,6\},3)\)-spheres using the program CPF, which
Table 1. Number $N_i$ of ($\{1, 2, 3\}, 6$)-spheres with $n$ vertices and $(p_1, p_2) = (i, 6 - 2i)$

| n  | $N_0$ | $N_1$ | $N_2$ | $N_3$ | n  | $N_0$ | $N_1$ | $N_2$ | $N_3$ | n  | $N_0$ | $N_1$ | $N_2$ | $N_3$ |
|-----|-------|-------|-------|-------|----|-------|-------|-------|-------|----|-------|-------|-------|-------|
| 1   | 0     | 0     | 1     | 1     |   | 19    | 69    | 36    | 13    | 1   | 37    | 436   | 133   | 24    | 1     |
| 2   | 1     | 0     | 1     | 0     |   | 20    | 100   | 34    | 28    | 0   | 38    | 581   | 118   | 37    | 0     |
| 3   | 1     | 1     | 3     | 1     |   | 21    | 86    | 46    | 19    | 1   | 39    | 495   | 159   | 32    | 1     |
| 4   | 3     | 1     | 5     | 1     |   | 22    | 133   | 33    | 23    | 0   | 40    | 677   | 112   | 59    | 0     |
| 5   | 2     | 3     | 5     | 0     |   | 23    | 112   | 62    | 16    | 0   | 41    | 582   | 187   | 26    | 0     |
| 6   | 7     | 2     | 8     | 0     |   | 24    | 165   | 44    | 37    | 0   | 42    | 758   | 133   | 53    | 0     |
| 7   | 5     | 6     | 6     | 1     |   | 25    | 144   | 57    | 20    | 1   | 43    | 679   | 180   | 27    | 1     |
| 8   | 12    | 5     | 12    | 0     |   | 26    | 205   | 54    | 27    | 0   | 44    | 869   | 172   | 53    | 0     |
| 9   | 10    | 8     | 8     | 1     |   | 27    | 176   | 75    | 22    | 1   | 45    | 749   | 199   | 43    | 0     |
| 10  | 19    | 6     | 12    | 0     |   | 28    | 251   | 61    | 36    | 1   | 46    | 1000  | 149   | 44    | 0     |
| 11  | 16    | 14    | 9     | 0     |   | 29    | 214   | 95    | 19    | 0   | 47    | 868   | 250   | 30    | 0     |
| 12  | 29    | 11    | 17    | 1     |   | 30    | 299   | 61    | 40    | 0   | 48    | 1101  | 182   | 72    | 1     |
| 13  | 24    | 17    | 10    | 1     |   | 31    | 265   | 96    | 20    | 1   | 49    | 989   | 235   | 35    | 2     |
| 14  | 42    | 16    | 16    | 0     |   | 32    | 360   | 89    | 43    | 0   | 50    | 1259  | 194   | 57    | 0     |
| 15  | 35    | 23    | 15    | 0     |   | 33    | 305   | 111   | 28    | 0   | 51    | 1076  | 270   | 40    | 0     |
| 16  | 59    | 18    | 22    | 1     |   | 34    | 429   | 80    | 33    | 0   | 52    | 1410  | 210   | 61    | 1     |
| 17  | 48    | 33    | 12    | 0     |   | 35    | 375   | 134   | 31    | 0   | 53    | 1228  | 313   | 33    | 0     |
| 18  | 79    | 22    | 22    | 0     |   | 36    | 488   | 105   | 50    | 1   | 41    | 1000  | 182   | 72    | 1     |

is available from [BDDH97] and whose algorithm has been described in [BrHaHe03]. After such enumeration is done, the trick is to add the six vertices of degree 2 in all possibilities. This is relatively easy to do and thus we have an efficient enumeration method. The numbers of graphs are shown in Table 1 for $2 \leq n \leq 41$. We should point out that this algorithm while reasonable for our purpose is very far from being optimal. A better method would be to adapt the algorithm from [BrHaHe03] although this is not easy to do.

Theorem 2. With the exception of the following graphs $T_1$, $T_2$

\[
\text{Trifolium } T_1 : C_{3v}, \ n = 1 \quad \text{and} \quad T_2 : C_{3h}, \ n = 3
\]

and the spheres of the infinite series depicted in Figures 1, 2, 3 and 4, any ($\{1, 2, 3\}, 6$)-sphere with at least one 1-gon is obtained from a ($\{3, 4, 5, 6\}, 3$)-sphere by taking the dual and then splitting some edges according to following two schemes:
Proof. Let us take a $\{(1, 2, 3), 6\}$-sphere $G$ with at least one 1-gon $F$ in its face-set. Clearly, $F$ cannot be adjacent to another 1-gon. If $F$ is adjacent to a 2-gon, then simple considerations yield that $G$ belongs to the infinite series of Figure 5. So, we can assume in the following that all 1-gons, say $F_1, \ldots, F_s$ are adjacent to 3-gons $G_1, \ldots, G_s$. If one of the $G_i$ is adjacent to two 2-gons, then we get the sphere $B_2$ ($C_{2h}$, $n = 2$) depicted in Figure 6. If one of the $G_i$ is adjacent to exactly one 1-gon, then we get the following partial diagram:

![Diagram]

Clearly, such diagram extends to one of the graphs of the infinite series depicted in Figure 6.

So, we can now assume that the $G_i$ are adjacent to 3-gons only. If one of the 3-gons adjacent to a $G_i$ turns out to be another $G_j$, then we get the map $C_2$ from Figure 6. So, we assume further that those 3-gons are not of the type $G_i$.

The faces $G_i$ contains two vertices $v_i$, $v'_i$ with $v_i$ being contained in $F_i$. If $v_i = v'_i$, then we get the exceptional sphere Trifolium. So, we assume further that $v_i \neq v'_i$. If $v_i = v'_j$ for $i \neq j$, then some easy considerations gives the sphere $T_2$ as the only possibility. So, let us assume now that the vertex $v_i$ is contained in a 2-gon. Then we have the following local configuration:

![Diagram]

From that point, after enumeration of all possibilities we get the infinite series of Figures 1 and 2. So, now we have that all vertices $v_i$ are contained in four 3-gons. This implies that $G$ is obtained from a $\{(3, 4, 5, 6), 3\}$-sphere by taking the dual and then splitting some edges according to mentioned above schemes.

Obviously, the above theorem gives us a method to enumerate the $\{(1, 2, 3), 6\}$-spheres. The enumeration results are shown in Table 1. Like for $\{(1, 2, 3), 6\}$-spheres, it would be interesting to have a faster enumeration method.
3. Symmetry groups

We now give the possible groups of the considered spheres. Note that we are using the terminology of points groups in chemistry as explained, for example, in [Dut04a].

**Theorem 3.** The possible symmetry group of a \((\{2,3\}, 6)\)-sphere are \(C_1, C_2, C_{2h}, C_{2v}, C_3, C_{3h}, C_{3v}, C_1, C_s, D_2, D_{2d}, D_{2h}, D_3, D_{3d}, D_{3h}, D_6, D_{6h}, S_4, S_6, T, T_h\) and \(T_d\). The minimal possible representatives are given in Figure 5.

**Proof.** The method is to consider the possible axes of symmetry; they are passing through faces, edges or vertices. As a consequence, the possibilities for a \(k\)-fold axis of symmetry are 2, 3 or 6. The only
groups that could occur, besides 22 given in the theorem, are $D_{6d}$, $C_6$, $C_{6h}$ or $C_{6v}$.

If a 6-fold axis occurs, then it necessarily passes through two vertices, say, $v_1$ and $v_2$. Around this vertex one can add successive rings of triangles as in the classical structure of the triangular lattice. At some point one gets a 2-gon and thus, by the 6-fold symmetry, six 2-gons. Then, one can continue the structure uniquely and the structure is defined uniquely. This completion is the same as the one around $v$ and it implies the existence of a mapping that inverts $v$ with the transformation inverting $v_1$ and $v_2$ and the group are $D_6$, $D_{6h}$ or $D_{6d}$. The group $D_{6d}$ is ruled out because 2-fold axis passes through the 2-gons. □

**Theorem 4.** The possible symmetry group of a $(\{1,2,3\},6)$-sphere with $p_1 > 0$ are

(i) $C_1$ or $C_s$ if $p_1 = 1$.

(ii) $C_1$, $C_2$, $C_i$, $C_s$, $C_{2v}$ or $C_{2h}$ if $p_1 = 2$.

(iii) $C_3$, $C_{3v}$ or $C_{3h}$ if $p_1 = 3$.

The minimal possible representative are given in Figures 6, 7 and 8.

Proof. For (i), the 1-gon has to be preserved by any symmetry which leaves $C_1$ and $C_s$ as the only possibilities. They are both realized. For (ii), we proceed in the same way. (iii) is proved in Theorem 10 □

An interesting question is to consider whether a $(\{1,2,3\},6)$-sphere can be mapped onto the projective plane $\mathbb{P}^2$. This is clearly equivalent to the map having a central inversion. $(\{1,3\},6)$-maps on the projective plane do not exist since no centrally-symmetric $(\{1,3\},6)$-sphere exist. All $(\{2,3\},6)$-maps on the $\mathbb{P}^2$ are antipodal quotients (i.e. with
Figure 5. Minimal representatives for each possible symmetry group of a \((\{2, 3\}, 6)\)-sphere halved \(p\)-vector and number of vertices) of \((\{2, 3\}, 6)\)-spheres whose groups contain the inversion, i.e. those of symmetry \(C_i, C_{2h}, D_{2h}, D_{3d}, D_{6h}, S_6\) and \(T_h\). In the next section we will describe explicitly the \((\{2, 3\}, 6)\)-sphere of symmetry \(D_{6h}\) and \(T_h\).
Figure 6. Minimal representatives for each possible symmetry group of a \(\{1, 2, 3\}, 6\)-sphere with \((p_1, p_2) = (1, 4)\)

Figure 7. Minimal representatives for each possible symmetry group of a \(\{1, 2, 3\}, 6\)-sphere with \((p_1, p_2) = (2, 2)\)

Figure 8. Minimal representatives for each possible symmetry group of a \(\{1, 3\}, 6\)-sphere

4. The Goldberg-Coxeter construction

In [DuDe04] a construction is given, generalizing Goldberg-Coxeter construction given in [Gold37, Cox71] for 3-regular graphs with 6-gonal and 5-, 4-, 3-gonal faces only. For the particular case when \(G\) is a geometrical fullerene, there is a large body of literature, see bibliography.
Figure 9. The tiling by hexagons, the point $A$ and some points in the other bipartite component

of [DuDe04]. It takes a 3- or 4-regular plane graph $G$ and return a 3- or 4-regular plane graph. There the first step was to take the dual and get a triangulation or a quadrangulation of the sphere. The respective triangles and squares were subdivided, then put together and the dual was taken. An instrument in this operation was that the Eisenstein and Gaussian integers are best represented on the tiling of the plane by equilateral triangles, respectively, squares. We are able to generalize this construction to the 6-regular case but there are differences.

First, if $G$ is a $\{2, 3\}, 6$)-sphere, then the dual $G^*$ is a plane graph with faces of size 6 and thus, bipartite. The tessellation of Euclidean plane by regular hexagons is represented on Figure 9. We use there two vectors $v_1, v_2$ to represent the coordinate of the points. In complex coordinates $v_1 = 1$ and $v_2 = j$ with $j = e^{i\pi/3}$. The lattice $L = \mathbb{Z}v_1 + \mathbb{Z}v_2$ is called the Eisenstein ring. The point $A$ is the origin and the point $B(k, l)$ is the point $k + lj$. The points in the bipartite component of $A$ are $L_A = (1 + j)L$, while the points in the component of $B(1, 0)$ are $L_B = 1 + (1 + j)L$. Both sets $L_A$ and $L_B$ are stable under multiplication. We will first define the Goldberg-Coxeter construction for $k + lj \in L_B$. Then we will extend it to any $(k, l) \neq 0$.

**Theorem 5.** If $z = k + lj \in L_B$ and $G_0$ is a 6-regular plane graph with $|G_0|$ vertices, then it is possible to define a plane graph $G' = GC_z(G_0) = GC_{k,l}(G_0)$ such that the following holds:

(i) $G'$ is a 6-regular plane graph with $|G_0|(k^2 + kl + l^2)$ vertices.

(ii) Every face of $G_0$ corresponds to a face of $G'$ with all new faces of $G'$ being 3-gons.

(iii) $G'$ has all rotational symmetries of $G_0$ and all symmetries as well if $l = 0$ or $k = 0$. 
(iv) \( GC_{1,0}(G_0) = G_0 \) and \( GC_{2}(G_0) = GC_{2z^z}(G_0) \).

(v) \( GC_z(GC_{z'}(G_0)) = GC_{z'z'}(G_0) \).

(vi) \( GC_z(G_0) = GC_{z^z}(G_0) \) where \( G_0 \) is the graph that differs from \( G_0 \) only by a plane symmetry.

Proof. Let \( G_0 \) be a 6-regular graph. The dual \( G^*_0 \) is a plane graph with all faces being 6-gons. If \( z = k+l \in L_B \), then the point \( B(k,l) \) belongs to the same connected component as \( B \).

The point \( c = -j^2z \) is the center of an hexagon and we build around it six points \( P_q \):

\[
P_q = c - j^qc \quad \text{for} \quad 0 \leq q \leq 5.
\]

Those six points form a master hexagon that correspond to the original hexagon. Every hexagon of \( G^*_0 \) can be thus modified and we can arrange them together at the boundary between adjacent hexagons. We can thus obtain another plane graph with 6-regular faces. By taking the dual one more time, we get \( GC_{k,l}(G_0) \). Checking the remaining properties is relatively easy.

The above theorem is similar to Proposition 3.1 in [DuDe04]. But there are some differences. In the 3-regular case, we have \( GC_{k,l}(G_0) \) with all symmetries if \( k = l \), while here the case \( k = l \) is impossible. See in Figure 13 the local structure of the Goldberg-Coxeter construction \( GC_{3,2} \) and in Figure 18 \( GC_{4,0} \).
Theorem 6. The \((\{2,3\}, 6)\)-spheres of symmetry \(D_6\) or \(D_{6h}\) are obtained as \(GC_{k,l}(6 \times K_2)\) with \(k + lj \in L_B\).

Proof. Let us take a \((\{2,3\}, 6)\)-sphere \(G\) of symmetry \(D_6\) or \(D_{6h}\) and let us take the dual \(G^*\). The 6-fold axis passes through a 6-gon \(F\) and the 2-gons of \(G\) correspond to 6-regular vertices. But the position of those 2-gons define a master hexagon around \(F\) and thus we get exactly the structure of a graph \(GC_{k,l}(6 \times K_2)\). 

In Theorem 5 we have defined the Goldberg-Coxeter construction \(GC_{k,l}\) for \(k + lj \in L_B\). Now we want to define it for any \(k, l \neq 0\). For that we first introduce the notion of oriented tripling.

Definition 7. If \(G\) is a 6-regular plane graph, then its dual \(G^*\) is bipartite. For each such bipartite class \(C\) we define a graph \(Or_C(G)\) with the following properties:

(i) \(Or_C(G)\) is a 6-regular plane graph with 3 times as many vertices.
(ii) Each vertex of \(G\) corresponds to 3 vertices of \(Or_C(G)\) and 4 triangular faces.
(iii) Every symmetry of \(G\) preserving \(C\) also occur as symmetry of \(Or_C(G)\).

The local configuration of the operation is shown in Figure 11. For every face \(F\) of \(G\), we orient the edges of \(F\) counter-clockwise. Thus for every bipartite class \(C\) of \(G^*\) we get an orientation of the edges of \(G\). Around a vertex \(v\) and its six adjacent vertices, there are three vertices \(w\) to which the edge \(\{v, w\}\) is oriented from \(v\) to \(w\). They are the vertices 1, 3, 5 in Figure 11.

So, if \(G\) has two inequivalent bipartite components \(C_1\) and \(C_2\), then \(Or_{C_1}(G)\) and \(Or_{C_2}(G)\) are not necessarily isomorphic and the smallest such example is shown on Figure 15. In Figure 12 we give two examples of the action of the oriented tripling when the obtained graph is unique.

For the Trifolium \(T_1\), we can define a sequence \(T_i\) of graphs with \(T_{i+1}\) obtained by applying the oriented tripling to \(T_i\). The first 4 terms are shown in Figure 14.

We now introduce the Goldberg-Coxeter construction in the general case. For a sphere \(G\), denote by \(Tr(G)\) the truncation of \(G\), i.e. the sphere obtained by replacing every vertex of degree \(k\) of \(G\) by a \(k\)-gonal face.

We will also use the following result: if \(G\) is a 3-regular sphere with faces of even size, then it is possible to color the faces of \(G\) so that any two adjacent faces have different colors. Such a coloring is unique up to permutation of the colors. If \(G_0\) is a graph with vertices of even
Figure 11. Local configuration around a vertex of the oriented tripling operation.

Figure 12. Two examples of \( \{2,3\},6\)-spheres with unique oriented tripling.

Figure 13. Local structure of the Goldberg-Coxeter construction \( GC_{3,2} \).

degree, then its dual is bipartite and the three colors in \( Tr(G_0) \) come from the vertices of \( G_0 \) and the two classes of faces in \( G_0 \).

**Theorem 8.** For a 6-regular plane graph \( G_0 \) and two integers \( k, l \) with \( k, l \neq 0 \), we can define two 6-regular spheres \( G_1, G_2 \) with \( GC_{k,l}(G_0) = \{G_1, G_2\} \). This will satisfy to the following properties:
Figure 14. First terms of the infinite sequence of $\left\{ \{1,3\},6 \right\}$-spheres $T_i$

Figure 15. Smallest $\left\{ \{2,3\},6 \right\}$-sphere having two non-isomorphic oriented triplings

\begin{enumerate}
\item $Tr(G_i) = GC_{k,l}(Tr(G_0))$ for $i = 1, 2$ with $GC_{k,l}$ being the Goldberg-Coxeter construction for 3-regular spheres.
\item $G_1$ and $G_2$ are 6-regular plane graphs with $|G_0|(k^2 + kl + l^2)$ vertices.
\item Every face of $G_0$ corresponds to a face of $G_1$ and $G_2$ with all new faces of $G_1$ and $G_2$ being 3-gons.
\item $GC_{1,1}(G_0) = \{Or_C(G_0), Or_{C'}(G_0)\}$.
\item If $k + lj \in L_B$, then $G_1 = G_2$.
\item If $GC_{k,l}(G_0) = \{G_1, G_2\}$, then $GC_{k',l'}(G_1) = GC_{k,p,l_p}(G_0)$ with $k_p + l_pj = (k + lj)(k' + l'j)$.
\end{enumerate}

\textbf{Proof.} Let us take a 3-coloring in white, red, blue of the faces of $Tr(G_0)$ with white corresponding to the faces coming from vertices of $G_0$. The 3-regular sphere $GC_{k,l}(Tr(G_0))$ has the faces of $G_0$ and some 6-gonal faces, thus all its faces are of even size and we can find a 3-coloring of them.

One can see directly that all the white faces of $Tr(G_0)$ have the same color in $GC_{k,l}(Tr(G_0))$; we color them white. If $k \equiv l \pm 1 \pmod{3}$ (this contains the case $k + lj \in L_B$), then the faces of $Tr(G_0)$ coming from faces of $G_0$ will not be white in $GC_{k,l}(Tr(G_0))$. Thus by shrinking the
white faces, we get a graph, which is actually the $GC_{k,l}(G_0)$ defined in
Theorem 5 if $k + lj \in L_B$.

If $k \equiv l \pmod{3}$ then all faces of $Tr(G_0)$ will correspond to white faces in $GC_{k,l}(Tr(G_0))$. The remaining 6-gonal faces have color red and blue. This gives two set of faces that can be shrunk and thus two possible graphs. All properties follow easily. □

Theorem 9. (i) Any $k + lj \neq 0$ can be written as $k + lj = (1 + j)^s(k' + l'j)u$ with $s \geq 0$, $u \in \{0, 1\}$ and $k' + l'j \in L_B$.

(ii) The sphere $GC_{k,l}(G_0)$ is obtained by applying the oriented tripling $s$ times and then the Goldberg-Coxeter construction from Theorem 5.

Proof. (i) The ring of Eisenstein integers is a unique factorization domain. That is every $k + lj \neq 0$ can be factorized by into the relevant primes. The condition $k \equiv l \pmod{3}$ is equivalent to $k + lj$ being divisible by $1 + j$. Thus by repeated application of this we can write

$$k + lj = (1 + j)^s(k_2 + l_2j)$$

with $k_2 \equiv l_2 \pm 1 \pmod{3}$.

If $k_2 \equiv l_2 + 1 \pmod{3}$, then we are done, otherwise we divide by $j$.

(ii) follows from the multiplicativity property (vi) of Theorem 8. □

This idea of using the truncation and resulting 3-regular spheres was, perhaps, used for the first time in [GrZa74]. This idea could in principle be applied to the enumeration of the $(\{2, 3\}, 6)$-spheres, since the $(\{4, 6\}, 3)$-spheres can be obtained from the CPF program. But the truncation multiplies the number of vertices by 6 and this makes this method uncompetitive to the one of Section 2.

We cannot say much in general for the symmetry groups of $GC_{k,l}(G_0)$. This is essentially the same situation as for the oriented tripling. What happens is that for 3-regular graphs, Goldberg-Coxeter construction $GC_{k,l}$ preserve all symmetries if $k = 0$ or $k = l$ and only rotational symmetries otherwise. Thus we get the automorphism group $\Gamma$ of $GC_{k,l}(Tr(G_0))$. If $\Gamma$ preserves the set of faces of color red and blue, then $\Gamma$ is a group of symmetries of $GC_{k,l}(G_0)$, otherwise the stabilizer of the red faces is a group of symmetries of $GC_{k,l}(G_0)$. But some accidental symmetries can occur and we have thus to work on a case-by-case basis.

Theorem 10. Let $G$ be a $(\{1, 3\}, 6)$-sphere. The following hold:

(i) $G = GC_{k,l}(Trifolium)$ with $0 \leq l \leq k$ and has $k^2 + kl + l^2$ vertices.

(ii) $G$ has symmetry $C_{3v}$ if $k = 0$, $C_{3h}$ if $k = l$ and $C_3$ otherwise.

(iii) $G$ is obtained as $GC_{k,l}(T_i)$ with $k + lj \in L_B$, where $(T_i)_{i \geq 1}$ is the infinite series of 6-regular graphs obtained by starting from Trifolium.
Proof. Let us take a \((\{1, 3\}, 6)\)-sphere. Then \(Tr(G)\) is a \((\{2, 6\}, 3)\)-sphere. Either from \[GrZa74\] or \[Thur98\], we know that such spheres are obtained as \(GC_{k,l}(3 \times K_2)\) with \(GC\) denoting here the 3-regular Goldberg-Coxeter construction. Since the faces of \(GC_{k,l}(3 \times K_2)\) are of even size, it is possible to define a 3-coloring of those faces. The 2-gonal faces should not be in all 3 different colorings. This can occur only if \(k \equiv l \pmod{3}\). So, \(k + lj\) can be factorized as \((1 + j)(k' + l'j)\) and we get

\[
Tr(G) = GC_{k,l}(3 \times K_2) = GC_{k',l'}(GC_{1,1}(3 \times K_2)) = GC_{k',l'}(Tr(Trifolium)).
\]

Thus we have proved (i).

The symmetry of \(GC_{k,l}(3 \times K_2)\) is \(D_{3h}\) if \(k = 0\) or \(k = l\) and \(D_3\) otherwise. If \(k \equiv l \pmod{3}\) then all 2-gons of \(GC_{k,l}(3 \times K_2)\) are in the same color, say white. The 3-gonal faces that are not white are of two possible colors red and blue. In order for a symmetry of \(Tr(G) = GC_{k,l}(3 \times K_2)\) to induce a symmetry of \(G\) it is necessary and sufficient that it preserves all 3 colors of the coloring. This reduces by a factor of 2 the symmetry group and we get \(C_3, C_{3h}\) and \(C_{3v}\) as possible groups.

Statement (iii) follows from Theorem \(\text{9 (ii)}\).

Note that the possible number of vertices of \((\{2, 6\}, 3)\)-spheres was already determined in \[GrZa74\].

Denote by \(K_2 \times Tetrahedron\) the Tetrahedron with edges doubled.

**Theorem 11.**

(i) Any \((\{2, 3\}, 6)\)-sphere of symmetry \(T, Th\) or \(Td\) is obtained as \(GC_{k,l}(K_2 \times Tetrahedron)\).

(ii) The \((\{2, 3\}, 6)\)-spheres of symmetry \(Td,\) respectively \(Th,\) are of the form \(GC_{k,0}(K_2 \times Tetrahedron),\) respectively \(GC_{k,k}(K_2 \times Tetrahedron)\).

Proof. Take a \((\{2, 3\}, 6)\)-sphere \(G\) of symmetry \(T, Td\) or \(Th\) and consider their truncation \(Tr(G)\). It is a \((\{4, 6\}, 3)\)-sphere which contains a subgroup \(T\) of symmetry. By Theorem 6.2 of \[DeDu05\], this implies that the symmetry group of \(Tr(G)\) is \(O\) or \(O_h\). By \[DuDe04\] Theorem 5.2, \(Tr(G)\) is described by the Goldberg-Coxeter construction applied to the cube, i.e. \(Tr(G) = GC_{k,l}(Cube)\). We need now to determine which graphs \(GC_{k,l}(Cube)\) are of the form \(Tr(G)\). For that we need to consider the 3-coloring of the faces. It is necessary that the 4-gonal faces are not in all 3 colors of the faces. This implies that \(k \equiv l \pmod{3}\). In turn this give us that \(k + lj = (1 + j)(k' + l'j)\), which then
gives us

\[ Tr(G) = GC_{k,l}(\text{Cube}) = GC_{k',l'}(GC_{1,1}(\text{Cube})) = GC_{k',l'}(Tr(K_2 \times \text{Tetrahedron})). \]

(i) follows from Theorem 8.

If a \{(2,3),6\}-sphere is of symmetry \(T_d\) or \(T_h\), then the symmetry group of the truncation is \(O_h\) and such spheres are described as \(GC_{k,0}(\text{Cube})\) and \(GC_{k,k}(\text{Cube})\).

**Theorem 12.** The number of \((\{1,2,3\},6)\)-spheres with \(i\) 1-gons and less than \(n\) vertices grows like \(O(n^{4-i})\).

**Proof.** Take \(G\) a \((\{1,2,3\},6)\)-sphere with \(n\) vertices and \(i\) 1-gons. Then \(Tr(G)\) is a \((\{2,4,6\},3)\)-sphere with \(i\) 2-gons, \(6 - 2i\) 4-gons and \(6n\) vertices. Thus the number of faces of size 2 or 4 is \(6 - i\). The 3-regular plane graphs whose faces have size at most 6 and the set of faces of size less than 6 is fixed are described by the parametrization theory of Thurston [Thur98]. By using it [Sah94] obtained some upper bound on the number of geometric fullerenes. The proof applies just as well for the other classes of graphs and give us the required upper bound.

Note that in principle, Thurston’s theory allows to say more. First it gives that the \((\{2,4,6\},3)\)-spheres with \(i\) 2-gons. are described by \(4 - i\) Eisenstein integers. Not all such spheres correspond to \((\{1,2,3\},6)\)-spheres with \(i\) 1-gons. For that some congruence have to be satisfied.

5. Zigzags and central circuits

For a plane graph \(G\) and a zigzag or central circuit, if we change the orientation, then the type of intersection does not change. Thus, to a zigzag or central circuit of length \(l\) with \(\alpha_1\) and \(\alpha_2\) intersections of type I and II, we attribute the symbol \(l_{\alpha_1,\alpha_2}\) and we define the \(z\)-vector, respectively, \(c\)-vector \(l_{\alpha_1,1,\alpha_2,2,\ldots,k_{\alpha_1,1,\alpha_2,2}}\) to be the vector enumerating such lengths with multiplicities \(m_i\).

**Theorem 13.** For a 6-regular plane graph, it is possible to find an orientation on the zigzags and central circuits such that any edge, respectively, vertex of intersection is of type II.

**Proof.** Let us take a 6-regular plane graph \(G\). Since every vertex is of even degree and \(G\) is planar, the dual graph \(G^*\) is bipartite. Let us take one color \(c\) of the faces of \(G\) and orient the edges of the face of color \(c\) in such a way that they turn clockwise around the face (see Figure 17). It is apparent that such orientation carries over to the zigzags and
central circuits of $G$ and that with this orientation all the intersection are of type II.

For zigzags, this result is not new, see for example [DeDu05, Sh75].

**Theorem 14.** Let us take a 6-regular plane graph $G$ with $z$-vector $l_{i \alpha, \beta}^a, \ldots, l_{i \alpha, \beta}^b, \ldots$ and $c$-vector $k_{j \alpha', \beta'}^a, \ldots, k_{j \alpha', \beta'}^b, \ldots$. Then the $z$-vector and $c$-vector of $GC_{1+4u,0}(G)$ are

$$
\ldots, \left\{ l_i (1 + 3u) \right\}_{\alpha, \beta}^a, \ldots, \left\{ 2k_j (1 + 3u) \right\}_{\alpha', \beta'}^b, \ldots
$$

and

$$
\ldots, \left\{ \frac{1 + 3u}{2} \right\}_{\alpha, \beta}^a, \ldots, \left\{ k_j (1 + 3u) \right\}_{\alpha', \beta'}^b, \ldots
$$

**Proof.** The proof uses the Goldberg-Coxeter construction previously built. One goes into the dual and subdivides the hexagons. The picture in Figure 18 shows that any central circuit of $G$ corresponds to $1 + 2u$ central circuits (named $B$ in the figure) and that the zigzags in $A$ on one side correspond to zigzags in $G$. The result follows similarly for $z$-vector.

**Theorem 15.** (i) If a $(\{1, 2, 3\}, 6)$-sphere has a 1-gon, then it has at least one self-intersecting central circuit and one self-intersecting zigzag.
Figure 18. The local structure of the Goldberg-Coxeter construction $GC_{4,0}$

Figure 19. Self-intersection induced by a 1-gon

(ii) For a $\{1, 3\}, 6)$-spheres, all central circuits and zigzags are self-intersecting.

Proof.

(i) The self-intersection is evident from Figure 19.

(ii) If a central circuit is simple in a $\{1, 2, 3\}, 6)$-sphere $G$, then it splits $G$ into two domains $D_1$ and $D_2$. If one denotes $n_{i,j}$ the number of faces of size $i$ into the domain $D_j$, then one has obviously $2n_{1,j} + n_{2,j} = 3$. So, if $n_{2,j} = 0$, then there is no solution. The proof for zigzags is the same. □

A $z$-, respectively $c$-railroad is the circuit of 3-gons bounded by two parallel zigzags, respectively central circuits. See Figure 20 for an illustration.

A $\{(1, 2, 3), 6\}$-sphere is called $z$-tight if for any zigzag there is at least one 1-gon or 2-gon on each of its side of the sphere. It is called $z$-weakly tight if for any zigzag there is no zigzag parallel to it. We define the corresponding notions for central circuits. See Figures 22 and 21 for some illustration of those notions.

The notion of tightness was introduced in [DeSt03] for 4-regular plane graphs (see also [DeDuSh03]) and in [DDFo09] [DeDu05] for 3-regular
The central circuit case

The zigzag case

Figure 20. A $c$-railroad and a $z$-railroadss bounded by two central circuits, respectively zigzags

A $c$-tight ($\{2,3\},6$)-sphere
A $c$-weakly tight ($\{2,3\},6$)-sphere
A not $c$-weakly tight ($\{2,3\},6$)-sphere

Figure 21. Illustration of the notions of $c$-tightness

A $z$-tight ($\{2,3\},6$)-sphere
A $z$-weakly tight ($\{2,3\},6$)-sphere
A not $z$-weakly tight ($\{2,3\},6$)-sphere

Figure 22. Illustration of the notions of $z$-tightness

plane graphs. For 4-regular plane graphs, central circuits were used. A central circuit is then called reducible if on one of its side there are only 4-gons. This sequence of 4-gons can be completely eliminated to get a reduced graph. For a 3-regular plane graph, a zigzag is called reducible if on one side there is only 6-gons. We can reduce the graph by eliminating those 6-gons only if the zigzag is simple. Moreover, there are several possibilities for this reduced graph while in the 4-regular case, the reduction is uniquely defined.

For a ($\{1,2,3\},6$)-sphere $G$, let $s(G) = p_1(G) + 2p_2(G)$, where $p_i(G)$ is the number of $i$-gonal faces. If $p \neq 3$ a $p$-gon is called incident to a zigzag or central circuit if it share an edge with it. It is called weakly
incident if it is not incident to it but still prevent the existence of a railroad.

**Theorem 16.** For a \( (\{1, 2, 3\}, 6) \)-sphere \( G \) we have:

(i) If \( G \) is \( z \)-, respectively \( c \)-tight, then it has at most \( \frac{s(G)}{2} \) zigzags, respectively central circuits.

(ii) If \( G \) is \( z \)-, respectively \( c \)-weakly tight, then it has at most \( s(G) \) zigzags, respectively central circuits.

**Proof.** Suppose that \( G \) is \( c \)-tight, then for any central circuit \( C \) there is at least one face of of size different from 3 on each sides. Since the number of sides is \( s(G) \) and there is two sides per central circuit, this gives (i). Also, the zigzag case is identical.

If \( G \) is \( c \)-weakly tight, then a \( p \)-gon for \( p \neq 3 \) is incident, respectively weakly incident, to at most \( p \) central circuits. Since it is weakly tight on each side of central circuits, there should be at least one incident or weakly incident central circuit. Thus the maximal number of central circuits is \( s(G) \) and the proof for zigzags is identical.

For \( (\{1, 2, 3\}, 6) \)-spheres with \( i \) 1-gons \( (i = 0, 1, 2, 3) \), this gives the upper bounds of \( (6, 4, 3, 1) \) for tightness and \( (12, 9, 6, 3) \) for weak tightness.

**Theorem 17.** For a \( (\{1, 3\}, 6) \)-sphere it holds:

(i) It cannot be \( c \)-, or \( z \)-tight.

(ii) Every central circuit correspond in a unique way to a zigzag of doubled length.

(iii) If it is \( c \)- or \( z \)-weakly tight, then the number of central circuits, zigzags is 1 or 3.

**Proof.** By Theorem 4, all \( (\{1, 3\}, 6) \)-spheres have symmetry \( C_3, C_{3v} \) or \( C_{3h} \). Hence they have a 3-fold axis of rotation and hence the 1-gons belong to a single orbit under the group. The faces of a 6-regular plane graph are partitioned in two classes, say \( F_1, F_2 \), since their dual graph is bipartite. Clearly, the 1-gons are all in one partition class, say, \( F_1 \).

A \( c \)-, \( z \)-circuit has two sides, and the faces in those sides all belong to the same partition class. Thus on one side of any \( zc \)-circuit, there is only 3-gons and so, (i) holds.

For a central circuit \( C \), denote by \( t_1, \ldots, t_N \) the triangles on the side of \( F_2 \). Clearly, the set of edges of triangles \( t_i \) not contained in \( C \), define a zigzag and (ii) holds.

If \( C \) is a central circuit in a \( c \)-weakly tight \( (\{1, 3\}, 6) \)-sphere \( G \), then on the side of \( F_1 \) there is a 1-gon and there is at most 3 central circuits. 2 is excluded by the group action. \( \square \)
Table 2. The maximal number of zigzags and central circuits for both notions of tightness and 4 types of spheres. Bold numbers are definite answer, while intervals give the possible range

|                | z-tig. | z-w. tig. | c-tig. | c-w. tig. |
|----------------|--------|-----------|--------|-----------|
| (\{2, 3\}, 6)-spheres | 6      | 9         | 6      | [8, 9]    |
| (\{1, 2, 3\}, 6)-spheres, \(p_1 = 1\) | [3, 4] | [5, 7]    | [3, 4] | [5, 7]    |
| (\{1, 2, 3\}, 6)-spheres, \(p_1 = 2\) | 3      | 5         | 3      | [4, 5]    |
| (\{1, 3\}, 6)-spheres         | 0      | 3         | 0      | 3         |

Theorem 18. Table 2 for the maximal number of zigzags and central circuits and both notions of weak tightness and tightness hold.

Proof. For (\{1, 3\}, 6)-spheres, Theorem 17 resolves the question. The existence of specific graphs in Figure 28 shows the lower bounds that are indicated. Theorem 16 shows the required upper bounds for \(z\)-tightness and \(c\)-tightness.

For the notion of weak tightness, we have to provide something more. Let \(G\) be a \(c\)-weakly tight (\{1, 2, 3\}, 6)-sphere with central circuits \(C_1, \ldots, C_l\). The number of 1-gons and 2-gons is \(p_1 = i\), \(p_2 = 6 - 2i\). We obtain \(2l\) sides since every central circuits has two sides. A side \(S\) is called lonely if it is incident or weakly incident to only one 2-gon.

If a side \(S\) is incident to exactly one 2-gon, then Figure 23.a shows that there is a side of parallel central circuit that is weakly incident two times to this 2-gon. Moreover, if it is incident exactly two times then there is another lonely side, see Figure 23.b. A similar structure show up if a side is weakly incident to a 2-gon.

Call \(n_{1a}\) the number of lonely sides in the first case and \(n_{1b}\) the number of lonely sides in the second case. Call \(n_{1c}\) the number of sides incident or weakly incident to exactly one 1-gon. Also let \(n_2\) be the number of sides incident to exactly two \(i\)-gons (identical or not). Let \(n_3\) be the number of sides incident to at least 3 \(i\)-gons (identical or not). Obviously, \(l = \frac{1}{2}(n_{1a} + n_{1b} + n_{1c} + n_2 + n_3)\).

In case (a), a lonely side \(S\) is incident to at least 3 \(i\)-gons so \(n_{1a} \leq n_3\). Clearly, \(n_{1c} \leq 2i\). Every 2-gon can be incident to 0, 1 or 2 lonely sides so \(n_{1a} + \frac{n_{1b}}{2} \leq 6 - 2i\). By an enumeration of incidence we get

\[ n_{1a} + n_{1b} + n_{1c} + 2n_2 + 3n_3 \leq s(G) = 2i + 4(6 - 2i) = 24 - 6i. \]

Denote by \(P_i\) the 5-dimensional polytope defined by these inequalities and \(n_{1a}, \ldots, n_3 \geq 0\). We optimize the quantity \(l\) over \(P_i\) by using cdd
\begin{enumerate}
    \item One lonely side \(S\)
    \item Two lonely sides \(S\) and \(S'\)
\end{enumerate}

**Figure 23.** Local structure around a side \(S\) incident to a 2-gon

\begin{figure}
\centering
\includegraphics[width=0.8\textwidth]{figure23.png}
\caption{Local structure around a side \(S\) incident to a 2-gon}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=0.8\textwidth]{figure24.png}
\caption{Example of patches and their angles}
\end{figure}

\[\text{Fu},\] which uses exact arithmetic, and found the optimal value to be \(9 - 2i\) for \(i \leq 0 \leq 3\). The proof for zigzags is identical.

In the rest of this section, we give a local Euler formula for central circuits in order to enumerate the \((\{2, 3\}, 6)\)-spheres which are \(c\)-weakly tight and with simple central circuits. The method for zigzags is very similar and thus not indicated.

Let \(G\) be a \((\{2, 3\}, 6)\)-sphere. Consider a patch \(A\) in \(G\), which is bounded by \(t\) arcs (i.e., sections of central circuits) belonging to central circuits (different or coinciding).

We admit also 0-gonal patch \(A\), i.e., just the interior of a simple central circuit. Suppose that the patch \(A\) is regular, i.e., the continuation of any of its bounding arcs (on the central circuit, to which it belongs) lies outside of the patch (see Figure 24). Let \(p'_{2}(A)\) be the number of 2-gonal faces in \(A\).

There are two types of intersections of arcs on the boundary of a regular patch: either intersection in an edge of the boundary, or intersection in a vertex of the boundary. Let us call these types of intersections obtuse and acute, respectively (see Figure 24); denote by \(t_{ob}\) and \(t_{ac}\) the respective number of obtuse and acute intersections. Clearly, \(t_{ob} + t_{ac} = t\), where \(t\) is the number of arcs forming the patch. The
following formula can easily be verified:

(1) \[ 6 - t_{ob} - 2t_{ac} = 2p'_2(A). \]

**Theorem 19.** The intersection of every two simple central circuits, respectively zigzags, of a \((\{2, 3\}, 6)\)-sphere, if non-empty, has one of the following forms (and so, its size is 2, 4 or 6):

\[ \text{ob ac obac} \]
\[ \text{ob ac} \]
\[ \text{ac ob} \]
\[ \text{ob ac} \]
\[ \text{ob ac} \]
\[ \text{ob ac} \]
\[ \text{ob ac} \]
\[ \text{ob ac} \]
\[ \text{ob ac} \]
\[ \text{ob ac} \]

**Proof.** Let us consider the central circuit case, the zigzag case being identical. Define \( H \) to be the graph, whose vertices are edges of intersection between simple central circuits \( C \) and \( C' \), with two vertices being adjacent if they are linked by a path belonging to one of \( C, C' \). The graph \( H \) is a plane 4-regular graph and \( C, C' \) define two central circuits in \( H \). Since \( C \) and \( C' \) are simple, the faces of \( H \) are \( t \)-gons with even \( t \).

Applying Formula (1) to a \( t \)-gonal face \( F \) of \( H \), we obtain that the number \( p'_2(F) \) of 2-gons in \( F \) satisfies \( 6 - t_{ob} - 2t_{ac} = 2p'_2(F) \). So, the numbers \( t_{ob} \) and \( t_{ac} \) are even, since \( t = t_{ob} + t_{ac} \). Also, \( 6 - t_{ob} - 2t_{ac} \geq 0 \). So, \( t \leq 6 \).

We obtain the following five possibilities for the faces of \( H \): 2-gons with two acute angles, 2-gons with two obtuse angles, 4-gons with four obtuse angles, 4-gons with two acute and two obtuse angles, 6-gons with six obtuse angles.

Take an edge \( e \) of a 6-gon in \( H \) and consider the sequence (possibly, empty) of adjacent 4-gons of \( H \) emanating from this edge. This sequence will stop at a 2-gon or a 6-gon; the case-by-case analysis of angles yields that this sequence has to stop at a 2-gon (see Figure 25a)).

Take an edge of a 2-gon in \( H \) and consider the same construction. If the angles are both obtuse, then the construction is identical and the sequence will terminate at a 2-gon or a 6-gon. If the angles are both acute, then cases b), c) of Figure 25 are possible.

In the first case, all 4-gons contain two obtuse angles and two acute angles; so, the sequence of 4-gons finishes with an edge of two obtuse angles. In the second case, there is a 4-gon, whose angles are all obtuse; this 4-gon is unique in the sequence and its position is arbitrary. Every pair of opposite edges of a 4-gon belongs to a sequence of 4-gons considered above. So, all angles of a 4-gon are the same, i.e., obtuse.
This fact restricts the possibilities of intersections to the three cases of
the theorem. □

The proof of this theorem is similar to the one of Theorem 6.4 given in
[DeDu05].

**Theorem 20.** The only weakly tight \( \{2,3\}, 6 \)-spheres, having only
simple zigzags, respectively simple central circuits, are the ones of Fig-
ure 27 and 26.

**Proof.** Let us consider first the central circuit case. By Theorem 19,
every two simple central circuits intersect in at most six vertices. If
a \( \{2,3\}, 6 \)-sphere has \( t \) central circuits, this gives an upper bound of
\( 6 \frac{t(t-1)}{2} \) on the number of vertices of intersection. Since any vertex can
be the intersection of only 3 central circuits we get the upper bound of
\( t(t-1) \) on the number of vertices. If one uses the upper bound of Table
2 on \( t \) for weakly tight \( \{2,3\}, 6 \)-spheres, then one gets \( t \leq 9 \) and the
upper bound 72 on the number of vertices, which is too large for the
enumeration done in Table 1. If one looks at the proof of Theorem 18,
then one sees that a lonely side implies a self-intersection of a parallel
central circuit. So, there is no lonely sides in \( \{2,3\}, 6 \)-spheres with
only simple central circuits. This gives the upper bound \( t \leq 6 \) on the
number of central circuits and then 30 on the number of vertices.

For zigzags we have the upper bound \( 3t(t-1) \) on the number of
edges and this gives the same upper bound of \( t(t-1) \) on the number
of vertices. The enumeration result shown in the Figures follow from
the determination results of Section 2. □

An interesting problem is to determine all \( \{2,3\}, 6 \)-spheres with
simple zigzags and/or central circuits.

A \( \{1,2,3\}, 6 \)-sphere is called \( z \)- or \( c \)-knotted if it has only one zigzag
of central circuit.

**Conjecture 21.** (i) A \( z \)-, \( c \)-knotted \( \{1,2,3\}, 6 \)-sphere, except Tri-
folium \( C_{3v} \), has symmetry \( C_1, C_2, C_3, D_2 \) or \( D_3 \).

(ii) The \( \{2,3\}, 6 \)-spheres with only simple central circuits have sym-
metry \( T_d, T_h, D_{6h}, D_{3d}, D_{2d}, D_{2h}, D_3, C_{2h} \) and \( C_{3v} \).

(iii) A \( \{1,2,3\}, 6 \)-sphere of symmetry \( D_{6h}, T_h, T_d \) have only simple
central circuits and zigzags.
Figure 26. The \(c\)-weakly tight \(\{2, 3\}, 6\)-spheres with simple central circuits

(iv) The \(\{2, 3\}, 6\)-spheres of symmetry \(T_d\) have \(v = 4x^2\) vertices, \(c\)-vector \((3x)^4z\) and \(z\)-vector \((5x)^{4z}\).

(v) The \(\{2, 3\}, 6\)-spheres of symmetry \(T_h\) have \(12x^2\) vertices, \(c\)-vector \((6x)^{6z}\) and \(z\)-vector \((12x)^{6z}\).

Conjecture 22. Let \(f_i(v)\) denote the maximal number of central circuits in a \(\{1, 2, 3\}, 6\)-sphere with \(i\) \(1\)-gons and \(v\) vertices. We conjecture:

(i) \(f_2(v) = v + 1\). It is realized exactly by the series (one for each \(v \geq 1\)) having symmetry \(C_{2h}\) and \(c = 1^v, (2v)_{0,v}\).

(ii) \(f_1(v) = \frac{v-1}{2} + 1, \frac{v-1}{2} + 2\) for \(v \equiv 3, 1 \pmod{4}\). It is realized exactly by the series (one for each odd \(v \geq 3\), all of symmetry \(C_s\)) with \(c = 2^{(v-1)/2}, (2v + 1)_{0,v+2}\) if \(v \equiv 3 \pmod{4}\) and \(2^{(v-1)/2}, (v + 1)_{0,v+1}, (v + 1)_{0,v}\) if \(v \equiv 1 \pmod{4}\).

For even \(v\), \(f_1(v) = \lfloor \frac{v-1}{3} \rfloor + 2\). It is realized for \(v \geq 4\) by series of symmetry \(C_s\) (two spheres for \(v \equiv 2 \pmod{6}\) and unique for other even \(v\)) with \(c = 3^{(v-1)/3}, (\frac{v}{3} + 2 + 3\lfloor \frac{v}{18} \rfloor)_{0,2\lfloor \frac{v}{18} \rfloor + 1}, (v + 1 + 3z(v))_{0,4z(v)+1}\), where \(z(v) = 2\lfloor \frac{v}{18} \rfloor + 1\) if \(v \equiv 6, 8, 10 \pmod{18}\) and \(z(v) = 2\lfloor \frac{v+6}{18} \rfloor\) if \(v\) is other even number.
(iii) $f_0(v) = \frac{v}{2} + 1, \frac{v}{2} + 2$ for $v \equiv 0, 2 \pmod{4}$. It is realized exactly by the series (one for each $v \geq 6$) of symmetry $D_{2d}$ with $c = 2\frac{v}{2}, 2v_{0,v}$ if $v \equiv 0 \pmod{4}$ and of symmetry $D_{2h}$ with $c = 2\frac{v}{2}, (v_0, v_0)^2$ if $v \equiv 2 \pmod{4}$.

(iv) For add $v$, $f_0$ is $\left\lfloor \frac{v}{3} \right\rfloor + 3$ if $v \equiv 2, 4, 6 \pmod{9}$ and $\left\lfloor \frac{v}{3} \right\rfloor + 1$, otherwise. Define $t_v$ by $\frac{v-t_v}{3} = \left\lfloor \frac{v}{3} \right\rfloor$. $f_0(v)$ is realized by the series of symmetry $C_{3v}$ if $v \equiv 1 \pmod{3}$ and $D_{3h}$, otherwise. c-vector is $3\left\lfloor \frac{v}{3} \right\rfloor, (2\left\lfloor \frac{v}{3} \right\rfloor + t_v)^0, (\frac{v-2t_v}{3})^3$ if $v \equiv 2, 4, 6 \pmod{9}$ and $3\left\lfloor \frac{v}{3} \right\rfloor, (2v + t_v)^0, v+2t_v$, otherwise.

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Figure 28. The smallest weakly tight and tight $\{1,2,3\},6$-spheres with the maximal known number of zigzags and central circuits

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M. Deza, École Normale Supérieure, Paris
E-mail address: Michel.Deza@ens.fr

M. Dutour Sikirić, Rudjer Bosković Institute, Bijenicka 54, 10000 Zagreb, Croatia
E-mail address: mdsikir@irb.hr