SCHWARZ-PICK AND LANDAU TYPE THEOREMS FOR SOLUTIONS TO THE DIRICHLET-NEUMANN PROBLEM IN THE UNIT DISK

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ABSTRACT. The aim of this paper is to establish some properties of solutions to the Dirichlet-Neumann problem: 
\[(\partial_z \overline{\partial_z})^2 w = g \text{ in } \mathbb{D},\]
\[w = \gamma_0 \text{ on } \partial \mathbb{D},\]
\[\partial_\nu \partial_z w = \gamma \text{ on } \partial \mathbb{D} \cap \mathbb{D},\]
where \(\partial_\nu\) denotes differentiation in the outward normal direction. More precisely, we obtain Schwarz-Pick type inequalities and Landau type theorem for solutions to the Dirichlet-Neumann problem.

1. Introduction

Let \(\mathbb{C} \cong \mathbb{R}^2\) denote the complex plane and for \(r > 0\), let \(\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}\). Denote by \(\mathbb{D} := \mathbb{D}_1\), the open unit disk, \(\mathbb{T} = \partial \mathbb{D}\), the boundary of \(\mathbb{D}\), and \(\overline{\mathbb{D}} = \mathbb{D} \cup \mathbb{T}\), the closure of \(\mathbb{D}\). Denote by \(\mathcal{C}(\Omega)\), the set of all continuous functions in a domain \(\Omega \subset \mathbb{C}\). The space of integrable functions in \(\Omega\) is denoted by \(L^1(\Omega)\). Denote by \(\mathcal{H}(\mathbb{D}, \mathbb{D})\) (resp. \(\mathcal{A}(\mathbb{D}, \mathbb{D})\)) the class of all complex-valued harmonic (resp. analytic) self-mappings of the unit disk \(\mathbb{D}\).

The Dirichlet and the Neumann boundary value problems in complex analysis have been very well studied in the literature. See \cite{2, 3} for investigations of basic boundary value problems with different kinds of boundary conditions.

In this paper we investigate some properties of solutions to the following Dirichlet-Neumann problem:

\[
\begin{cases}
(\partial_z \overline{\partial_z})^2 w = g & \text{in } \mathbb{D}, \\
w = \gamma_0 & \text{on } \mathbb{T}, \\
\partial_\nu \partial_z w = \gamma & \text{on } \mathbb{T}
\end{cases}
\]

and

\[
\frac{1}{2\pi i} \int_\mathbb{T} w(\zeta) \overline{\frac{d\zeta}{\zeta}} = c,
\]

where \(\partial_\nu\) denotes differentiation in the outward normal direction, \(g \in L^1(\mathbb{D})\), \(\gamma_0\), \(\gamma \in C(\mathbb{T})\), \(c \in \mathbb{C}\) is a constant, and satisfying the condition

\[
\frac{1}{2\pi} \int_0^{2\pi} \gamma(e^{it}) \, dt = \frac{2}{\pi} \int_\mathbb{D} g(\zeta) \, dA(\zeta).
\]

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Here \( \partial_z := \frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \) and \( \partial_{\bar{z}} := \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \) represent the complex differential operators so that \( \Delta = 4 \partial_z \partial_{\bar{z}} \), is the Laplacian. So, \( \partial_z \partial_{\bar{z}} \) and \( (\partial_z \partial_{\bar{z}})^2 \) are called the harmonic and biharmonic operators, respectively. Consequently, \( \partial_z \partial_{\bar{z}} w = 0 \) in \( D \) is equivalent to the statement that \( w \) is harmonic, while a solution to the equation \( (\partial_z \partial_{\bar{z}})^2 w = 0 \) is called a biharmonic function. See [9, 10] and the references therein for certain properties of biharmonic functions.

For \( z \in D \), let
\[
P(z, e^{it}) = \frac{1 - |z|^2}{|1 - ze^{-it}|^2}
\]
denote the Poisson kernel in \( D \). The function \( z \mapsto P(z, e^{it}) \) is harmonic in \( D \).

In [3], it was shown that the condition (1.3) ensures that all solutions to (1.1) satisfying the condition (1.2) are given by the formula
\[
w(z) = -c(1 - |z|^2) + P_{\gamma_0}(z) + G_1[\gamma](z) - G_2[g](z),
\]
where
\[
P_{\gamma_0}(z) = \frac{1}{2\pi} \int_0^{2\pi} P(z, e^{it}) \gamma_0(e^{it}) \, dt,
\]
\[
G_1[\gamma](z) = \frac{1}{4\pi} \int_0^{2\pi} H_2(z, e^{it}) \gamma(e^{it}) \, dt,
\]
and
\[
G_2[g](z) = \int_D H_2(z, \zeta) g(\zeta) \, dA(\zeta),
\]
with
\[
H_2(z, \zeta) = -|\zeta - z|^2 \log |\zeta - z|^2
\]
\[
- (1 - |z|^2) \left[ 4 + \frac{1}{z\zeta} \log(1 - z\zeta) + \frac{1}{\bar{z}\bar{\zeta}} \log(1 - \bar{z}\bar{\zeta}) \right]
\]
\[
- \frac{(\zeta - z)(1 - \zeta\bar{z})}{z} \log(1 - z\zeta) - \frac{(\bar{\zeta} - \bar{z})(1 - \zeta\bar{z})}{\bar{z}} \log(1 - \bar{z}\bar{\zeta}),
\]
and \( dA(\zeta) = (1/\pi) \, dx \, dy \) denotes the normalized area measure in \( D \).

\section{Main Results}

\subsection{A Schwarz type lemma.}
The classical Schwarz lemma states that if \( f \in A(D, D) \) such that \( f(0) = 0 \), then \( |f(z)| \leq |z| \) for all \( z \in D \). This result has been a crucial result in many branches of research for more than a hundred years. The harmonic analog of this statement states that if \( f \in H(D, D) \) such that \( f(0) = 0 \), then (Heniz [16])
\[
(2.1) \quad |f(z)| \leq \frac{4}{\pi} \arctan |z| \quad \text{for } z \in D.
\]
Later, by removing the assumption $f(0) = 0$, Pavlović [22, Theorem 3.6.1] established the following sharp inequality for $f \in \mathcal{H}(\mathbb{D}, \mathbb{D})$:

$$
(2.2) \quad \left| f(z) - \frac{1 - |z|^2}{1 + |z|^2} f(0) \right| \leq \frac{4}{\pi} \arctan |z| \quad \text{for} \quad z \in \mathbb{D}.
$$

The first purpose of this paper is to consider results of the above type for solutions to (1.1) satisfying the conditions (1.2) and (1.3).

**Theorem 2.1.** Suppose that $g \in \mathcal{C}(\overline{\mathbb{D}})$ and $\gamma \in \mathcal{C}(\mathbb{T})$, and that $w \in \mathcal{C}^4(\mathbb{D}) \cap \mathcal{C}(\overline{\mathbb{D}})$ satisfying the equation (1.1) with the conditions (1.2) and (1.3). Then for $z \in \mathbb{D}$,

$$
(2.3) \quad |w(z) - \frac{1 - |z|^2}{1 + |z|^2} P_{\gamma}(0)| \leq \frac{4}{\pi} \|P_{\gamma}\|_\infty \arctan |z| + |c| + \|\gamma\|_\infty N_1(|z|) + \|g\|_\infty N_2(|z|),
$$

where

$$
N_1(t) = 2 \log 4 + \frac{1 - t^2}{2} \left( \frac{2}{3} \pi^2 - 2 \right) \frac{t}{2} + 4 \left( \frac{\pi^2}{6} \right),
$$

$$
N_2(t) = 4 \log 4 + (1 - t^2) \left( \frac{2}{3} \pi^2 - 2 \right) \frac{t}{2} + \frac{16}{3} \left( \frac{\pi^2}{6} \right).$$

$\|P_{\gamma}\|_\infty = \sup_{z \in \mathbb{D}} \{|P_{\gamma}(z)|\}$, $\|\gamma\|_\infty = \sup_{t \in \mathbb{T}} \{|\gamma(z)|\}$ and $\|g\|_\infty = \sup_{z \in \mathbb{D}} \{|g(z)|\}$.

Clearly, if $c = 0$, $\gamma = g \equiv 0$ and $w$ maps $\mathbb{D}$ into itself, then (2.3) coincides with (2.2).

### 2.2. A Schwarz-Pick type lemma.

For a $2 \times 2$ real matrix $M := M_{2 \times 2}$, the matrix norm and the matrix function are defined by

$$
\|M\| = \sup \{|Mz| : z \in \mathbb{T}\} \quad \text{and} \quad \lambda(M) = \inf \{|Mz| : z \in \mathbb{T}\},
$$

respectively. For a complex-valued function $w = f(z) = u(z) + iv(z)$, the Jacobian matrix $D_f$ and Jacobian (determinant) $J_f$ of $f$ are defined by

$$
D_f = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \quad \text{and} \quad J_f := \det D_f = u_x v_y - v_x u_y = |f_z|^2 - |f_{\overline{z}}|^2,
$$

respectively. Obviously

$$
(2.4) \quad \|D_f(z)\| = \sup \{|D_f(z)c| : |c| = 1\} = |f_z(z)| + |f_{\overline{z}}(z)|
$$

and

$$
\lambda(D_f(z)) = \inf \{|D_f(z)c| : |c| = 1\} = |f_z(z)| - |f_{\overline{z}}(z)|.
$$

Colonna [14] obtained a sharp Schwarz-Pick type lemma for $f \in \mathcal{H}(\mathbb{D}, \mathbb{D})$:

$$
(2.5) \quad \|D_f(z)\| \leq \frac{4}{\pi} \frac{1}{1 - |z|^2} \quad \text{for} \quad z \in \mathbb{D}.
$$

Our second aim in this paper is to prove the following Schwarz-Pick type lemma for solutions to (1.1) satisfying the conditions (1.2) and (1.3).
Theorem 2.2. Suppose that $g \in C(\mathbb{D})$ and $\gamma \in C(\mathbb{T})$, and that $w \in C^4(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ satisfying the equation (1.1) with the conditions (1.2) and (1.3). Then for $z \in \mathbb{D},$

\begin{equation}
\|D_w(z)\| \leq \frac{4}{\pi} \|P_{\gamma}\|_\infty \frac{1}{1 - |z|^2} + 2|c| + \|\gamma\|_\infty N_3(|z|) + \|g\|_\infty N_4(|z|),
\end{equation}

where

\begin{align*}
N_3(t) &= 2 \left(\frac{\pi^2}{3} + 1\right) + t \left(2 \frac{\pi^2}{3} - 2\right)^{\frac{1}{2}} + (1 - t^2) \left(\frac{\pi^2}{6} - 1\right)^{\frac{1}{2}} + \left(\frac{\pi^2}{3} - \frac{1}{2}\right)^{\frac{1}{2}}, \\
N_4(t) &= 2(\log 4 + 1) + t \left(2 \frac{\pi^2}{3} - 2\right)^{\frac{1}{2}} + \frac{2}{3}(1 - t^2) \left(\frac{\pi^2}{6} - 1\right)^{\frac{1}{2}} + \frac{2}{3} \left(\frac{\pi^2}{3} - \frac{1}{2}\right)^{\frac{1}{2}}.
\end{align*}

$\|P_{\gamma}\|_\infty$, $\|\gamma\|_\infty$ and $\|g\|_\infty$ are as in Theorem 2.1.

Remark 2.1. We note that if $c = 0, \gamma = g \equiv 0$ and $w$ maps $\mathbb{D}$ into itself, then (2.6) coincides with (2.5).

2.3. A Landau type theorem. The classical Landau theorem says that there is a $\rho = \frac{1}{M + \sqrt{M^2 - 1}}$ such that every function $f$, analytic in $\mathbb{D}$ with $f(0) = f'(0) - 1 = 0$ and $|f(z)| < M$, is univalent in the disk $\mathbb{D}_\rho$. Moreover, the range $f(\mathbb{D}_\rho)$ contains a disk of radius $M\rho^2$, where $M \geq 1$ is a constant (see [15]). The Landau theorem has become an important tool in geometric function theory of one complex variable (cf. [4, 24]). Unfortunately, for a general class of functions, there is no Landau type theorem (cf. [15, 23]). To establish analogs of the Landau type theorem for more general classes of functions, it is necessary to restrict our focus to certain subclasses (cf. [1, 3, 7, 8, 9, 11, 12, 21, 23]).

As an application of Theorem 2.2, we get the following Landau type theorem for the solutions to (1.1) with the conditions (1.2) and (1.3).

Theorem 2.3. Suppose that $\gamma \in C(\mathbb{T}), g \in C(\overline{\mathbb{D}})$, that $w \in C^4(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ satisfying the equation (1.1) with the conditions (1.2), (1.3) and $w(0) = J_w(0) - 1 = 0$, and that $\|\gamma\|_\infty \leq L_1, \|\gamma\|_\infty \leq L_2$ and $\|g\|_\infty \leq L_3$, where $L_j, j \in \{1, 2, 3\}$, are constants. Then

(1) $w$ is univalent in $\mathbb{D}_{r_0}$, where $r_0$ satisfies the following equation

\begin{equation}
\frac{1}{L_4} - 2r_0 \left(\frac{L_1}{\pi} \frac{2 - r_0}{(1 - r_0)^2} + L_5\right) - 8L_3 \log \frac{1 + r_0}{1 - r_0} = 0;
\end{equation}

(2) $w(\mathbb{D}_{r_0})$ contains a univalent disk $\mathbb{D}_{r_0}$ with

\begin{equation}
R_0 \geq \frac{8L_4}{\pi} \left(\frac{r_0}{1 - r_0}\right)^2 + L_5r_0^2 + \frac{8L_3r_0^2(3 - r_0^2)}{(1 - r_0^2)^2},
\end{equation}

where

$L_4 = 2|c| + \frac{4}{\pi}L_1 + L_2N_3(0) + L_3N_4(0), \quad L_5 = |c| + L_2M_1 + L_3M_2, \quad N_3(0)$ and $N_4(0)$ are defined in Theorem 2.2, whereas $M_1$ and $M_2$ are defined in Lemmas 3.1 and 3.2 respectively.
We would like to remark that this article continues the earlier study on this topic from [19]. Recently, many authors have studied the Schwarz type lemma, Schwarz-Pick type lemma and Landau type theorem for solutions of different equations (cf. [7, 8, 13, 20, 21]).

The rest of this article is organized as follows. Section 3 is devoted to stating and proving several useful lemmas. In Section 4, we present the proofs of Theorems 2.1, 2.2 and 2.3.

3. Several Basic Lemmas

In this section, we shall prove three lemmas which will be used later on.

**Lemma 3.1.** Suppose \( \gamma \in C(\mathbb{T}) \), and \( \mathcal{G}_1[\gamma] \) is defined by (1.5). Then for \( z \in \mathbb{D} \),

\[
\left| \frac{\partial}{\partial z} \mathcal{G}_1[\gamma](z) - \frac{\partial}{\partial z} \mathcal{G}_1[\gamma](0) \right| \leq (\|\gamma\|_\infty M_1)|z|
\]

and

\[
\left| \frac{\partial}{\partial \overline{z}} \mathcal{G}_1[\gamma](z) - \frac{\partial}{\partial \overline{z}} \mathcal{G}_1[\gamma](z) \right| \leq (\|\gamma\|_\infty M_1)|z|,
\]

where

\[
M_1 = \frac{1}{2} \left[ \frac{2\pi}{\sqrt{3}} + 1 + \left( \frac{2}{3} \pi^2 - 2 \right)^\frac{1}{2} + \left( \frac{\pi^2}{6} - \frac{5}{4} \right)^\frac{1}{2} + \left( \frac{\pi^2}{6} - 1 \right)^\frac{1}{2} + \left( \frac{\pi^2}{3} - \frac{11}{4} \right)^\frac{1}{2} \right].
\]

**Proof.** To prove the lemma, we only need to show the first inequality, namely,

\[
\left| \frac{\partial}{\partial z} \mathcal{G}_1[\gamma](z) - \frac{\partial}{\partial z} \mathcal{G}_1[\gamma](0) \right| \leq (\|\gamma\|_\infty M_1)|z|
\]

since the proof of the second inequality is similar. Let

\[
I_1(z) = \frac{1}{2\pi} \int_0^{2\pi} (e^{-it} - \overline{z}) \left( \log |1 - ze^{-it}|^2 + 1 \right) \gamma(e^{it}) \, dt,
\]

\[
I_2(z) = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{G}_1[\gamma](z) \left( 4 + \frac{1 - ze^{-it}}{ze^{-it}} \log(1 - ze^{-it}) \right) \frac{1 - \overline{z}e^{it}}{\overline{z}e^{it}} \log(1 - \overline{z}e^{it}) \gamma(e^{it}) \, dt,
\]

\[
I_3(z) = \frac{1}{2\pi} \int_0^{2\pi} \left( 1 - |z|^2 \right) \left( \frac{\log(1 - ze^{-it})}{z^2e^{-it}} + \frac{1}{z} \right) \gamma(e^{it}) \, dt,
\]

and

\[
I_4(z) = \frac{1}{2\pi} \int_0^{2\pi} \left( e^{it} - z^2e^{-it} \log(1 - ze^{-it}) + \frac{1 - ze^{-it}}{z} \right) \gamma(e^{it}) \, dt.
\]

Now, we need to estimate \( |I_j(z) - I_j(0)| \) for \( j = 1, 2, 3, 4 \), respectively.

**Claim 3.1.** \( |I_1(z) - I_1(0)| \leq \|\gamma\|_\infty |z| \left[ 2 \left( \frac{\pi^2}{3} \right)^\frac{1}{2} + 1 \right] \).

By elementary calculations, we get

\[
|I_1(z) - I_1(0)| \leq \|\gamma\|_\infty (2J_1(z) + |z|),
\]
where
\[
J_1(z) = \frac{1}{2\pi} \int_0^{2\pi} \left| \log |1 - ze^{-it}|^2 \right| dt = \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{n=1}^{\infty} \frac{z^n e^{-int}}{n} + \sum_{n=1}^{\infty} \frac{e^{int}}{n} \right| dt.
\]

By the Hölder inequality and Parseval’s theorem, we obtain
\[
J_1(z) \leq \left( \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{n=1}^{\infty} \frac{z^n e^{-int}}{n} + \sum_{n=1}^{\infty} \frac{e^{int}}{n} \right|^2 dt \right)^{\frac{1}{2}}
= \left( \frac{2}{\pi} \sum_{n=1}^{\infty} \left| \frac{z^n}{n^2} \right| \right)^{\frac{1}{2}} \leq |z| \left( \frac{\pi^2}{3} \right)^{\frac{1}{2}},
\]
which proves Claim 3.1.

**Claim 3.2.** \[|I_2(z) - I_2(0)| \leq \|\gamma\|_{\infty} |z| \left( \frac{2}{3} \pi^2 - 2 \right)^{\frac{1}{2}}.\]

Since \(|I_2(z) - I_2(0)| = |I_2(z)|\), Claim 3.2 follows from [19, Claim 2.4].

**Claim 3.3.** \[|I_3(z) - I_3(0)| \leq \|\gamma\|_{\infty} \left[ \left( \frac{\pi^2}{6} - \frac{5}{4} \right)^{\frac{1}{2}} + \left( \frac{\pi^2}{6} - 1 \right)^{\frac{1}{2}} \right].\]

Since
\[
\log(1 - ze^{-it}) \cdot \frac{1}{z^2 e^{-it}} + \frac{1}{z} = -\sum_{n=1}^{\infty} \frac{z^{n-1} e^{-int}}{n+1},
\]
we deduce that
\[
I_3(0) = \frac{1}{2\pi} \int_0^{2\pi} \left( -\frac{e^{-it}}{2} \right) \gamma(e^{it}) dt.
\]
Then
\[
|I_3(z) - I_3(0)| \leq \|\gamma\|_{\infty} (J_2(z) + J_3(z)),
\]
where
\[
J_2(z) = \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{n=2}^{\infty} \frac{z^{n-1} e^{-int}}{n+1} \right| dt \quad \text{and} \quad J_3(z) = \frac{|z|^2}{2\pi} \int_0^{2\pi} \left| \sum_{n=1}^{\infty} \frac{z^{n-1} e^{-int}}{n+1} \right| dt.
\]

As in the proof of the estimate for \(J_1(z)\), by the Hölder inequality and Parseval’s theorem, we obtain
\[
J_2(z) \leq \left( \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{n=2}^{\infty} \frac{z^{n-1} e^{-int}}{n+1} \right|^2 dt \right)^{\frac{1}{2}}
= \left( \sum_{n=2}^{\infty} \left| \frac{z^{2(n-1)}}{(n+1)^2} \right| \right)^{\frac{1}{2}} \leq |z| \left( \frac{\pi^2}{6} - \frac{5}{4} \right)^{\frac{1}{2}}.
\]
Similarly, we know that
\[ J_3(z) \leq |z|^2 \left( \frac{\pi^2}{6} - 1 \right)^{\frac{1}{2}}. \]

Claim 3.3 follows as \(|z| < 1\).

**Claim 3.4.** \(|I_4(z) - I_4(0)| \leq \|\gamma\|_\infty |z| \left( \frac{\pi^2}{3} - \frac{11}{4} \right)^{\frac{1}{2}}.\)

We rewrite \(I_4(z)\) as
\[ I_4(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{-it} G(ze^{-it}) \gamma(e^{it}) \, dt, \]
where
\[ G(z) = \frac{1 - z^2}{z^2} \log(1 - z) + \frac{1}{z} - 1 = 2 \sum_{n=1}^\infty \frac{z^n}{n(n + 2)} - \frac{3}{2}. \]

Then \(I_4(0) = \frac{1}{2\pi} \int_0^{2\pi} e^{-it} \gamma(e^{it}) \, dt\). As before, we find that
\[
|I_4(z) - I_4(0)| \leq \frac{\|\gamma\|_\infty}{2\pi} \int_0^{2\pi} \left| 2 \sum_{n=1}^\infty \frac{z^n e^{-int}}{n(n + 2)} \right| \, dt \\
\leq \|\gamma\|_\infty \left( \frac{1}{2\pi} \int_0^{2\pi} \left| 2 \sum_{n=1}^\infty \frac{z^n e^{-int}}{n(n + 2)} \right|^2 \, dt \right)^{\frac{1}{2}} \\
= \|\gamma\|_\infty \left( \sum_{n=1}^\infty \frac{4|z|^{2n}}{n^2(n + 2)^2} \right)^{\frac{1}{2}} \leq \|\gamma\|_\infty |z| \left( \frac{\pi^2}{3} - \frac{11}{4} \right)^{\frac{1}{2}},
\]

since
\[
\sum_{n=1}^\infty \frac{4}{n^2(n + 2)^2} = \sum_{n=1}^\infty \frac{1}{(n + 2)^2} + \sum_{n=1}^\infty \frac{1}{n^2} - 2 \sum_{n=1}^\infty \frac{1}{n(n + 2)} = \frac{\pi^2}{3} - \frac{11}{4},
\]
and Claim 3.4 follows.

Therefore, by Claims 3.3, 3.2, 3.3, 3.4 and [17, Proposition 2.4], we conclude that
\[
\left| \frac{\partial}{\partial z} G_1[\gamma](z) - \frac{\partial}{\partial z} G_1[\gamma](0) \right| \leq \frac{1}{2} \sum_{j=1}^4 |I_j(z) - I_j(0)| \leq (\|\gamma\|_\infty M_1)|z|,
\]
as required. \(\square\)

**Lemma 3.2.** Suppose \(g \in C(\overline{D})\) and \(G_2[g]\) is defined in (1.6). Then for \(z \in D_r,\)
\[
\left| \frac{\partial}{\partial z} G_2[g](z) - \frac{\partial}{\partial z} G_2[g](0) \right| \leq \|g\|_\infty |z| M_2 + 4\|g\|_\infty \log \frac{1 + |z|}{1 - |z|},
\]
and
\[
\left| \frac{\partial}{\partial \overline{z}} G_2[g](z) - \frac{\partial}{\partial \overline{z}} G_2[g](0) \right| \leq \|g\|_\infty |z| M_2 + 4\|g\|_\infty \log \frac{1 + |z|}{1 - |z|},
\]
where $0 \leq r^* < 2r_0$ and $r_0$ is determined by Eqn. (2.7), and

$$M_2 = \log 4 + 1 + \left( \frac{2}{3} \pi^2 - 2 \right) \frac{1}{4} + \frac{2}{3} \left[ \left( \frac{\pi^2}{6} - \frac{5}{4} \right) \frac{1}{4} + \left( \frac{\pi^2}{6} - 1 \right) \frac{1}{4} + \left( \frac{\pi^2}{3} - \frac{11}{4} \right) \frac{1}{4} \right].$$

**Proof.** To prove the two inequalities, we only need to prove the first inequality, namely,

$$\left| \frac{\partial}{\partial z} G_2[g](z) - \frac{\partial}{\partial z} G_2[g](0) \right| \leq (\|g\|_\infty M_2) |z|,$$

because the proof of the second inequality is similar. To do this, we let

$$I_5(z) = \int_D \left( \frac{1 - z}{z} \log |z - \zeta| + \frac{1 - \zeta}{z} \log |1 - \zeta| \right) g(\zeta) dA(\zeta),$$

$$I_6(z) = \int_D \left[ \frac{\log(1 - z\zeta)}{z^2} + \frac{1}{z} \right] g(\zeta) dA(\zeta),$$

$$I_7(z) = \int_D \left[ \frac{1}{z^2 \zeta} \log(1 - z\zeta) + \frac{|\zeta|^2 - z\zeta}{z^2} \right] g(\zeta) dA(\zeta),$$

and

$$I_8(z) = \int_D \left( \zeta - \frac{z\zeta}{z^2} \right) \log(1 - z\zeta) + \frac{|\zeta|^2 - z\zeta}{z} g(\zeta) dA(\zeta).$$

In the following, we estimate $|I_j(z) - I_j(0)|$ for $j = 5, 6, 7, 8$, respectively.

**Claim 3.5.** $|I_5(z) - I_5(0)| \leq \|g\|_\infty \left[ 4 \log \frac{1 + |z|}{1 - |z|} + (\log 4 + 1)|z| \right]$.

By calculations, we get

$$I_5(z) - I_5(0) = -J_4(z) + J_5(z),$$

where

$$J_4(z) = \int_D \zeta (\log |\zeta - z|^2 + 1) g(\zeta) dA(\zeta)$$

and

$$J_5(z) = \int_D \zeta (\log |\zeta - z|^2 - \log |\zeta|^2) g(\zeta) dA(\zeta).$$

Obviously,

$$|J_4(z)| \leq (\log 4 + 1) \|g\|_\infty |z|. \quad (3.1)$$

In order to estimate $J_5(z)$, we let

$$h(z, \zeta) = \zeta \log |\zeta - z|^2.$$
Then, for $z \in \mathbb{D}_r$, by Fubini’s Theorem, we get
\[ |J_5(z)| \leq \|g\|_{\infty} \int_{\mathbb{D}} |h(z, \zeta) - h(0, \zeta)| \, dA(\zeta) \]
\[ \leq \|g\|_{\infty} \int_{\mathbb{D}} \left( \int_{[0, z]} \left| h_z(z, \zeta) \, dz + h_{\zeta}(z, \zeta) \, d\zeta \right| \right) \, dA(\zeta) \]
\[ \leq \|g\|_{\infty} \int_{\mathbb{D}} \left( \int_{[0, z]} \left( |h_z(z, \zeta)| + |h_{\zeta}(z, \zeta)| \right) \, dz \right) \, dA(\zeta) \]
\[ = \|g\|_{\infty} \int_{[0, z]} H(z, \zeta) \, d\zeta, \]
where
\[ H(z, \zeta) = \int_{\mathbb{D}} \left( |h_z(z, \zeta)| + |h_{\zeta}(z, \zeta)| \right) \, dA(\zeta) = 2 \int_{\mathbb{D}} \frac{\zeta - z}{|\zeta - z|} \, dA(\zeta). \]

In order to estimate $H(z, \zeta)$, we let
\[ \zeta \mapsto \eta = \phi(\zeta) = \frac{z - \zeta}{1 - \zeta \bar{z}} = re^{i\theta} \]
so that $\phi = \phi^{-1}$,
\[ \zeta = \frac{z - \eta}{1 - \eta \bar{z}}, \quad z - \zeta = \frac{\eta(1 - |z|^2)}{1 - \eta \bar{z}}, \quad \phi'(\zeta) = -\frac{1 - |z|^2}{(1 - \zeta \bar{z})^2}, \]
and thus,
\[ dA(\zeta) = |(\phi^{-1})'(\eta)|^2 dA(\eta) = \frac{(1 - |z|^2)^2}{|1 - \eta \bar{z}|^4} \, dA(\eta). \]

Consequently, switching to polar coordinates yields
\[ H(z, \zeta) = \int_{\mathbb{D}} \frac{|z - \eta|(1 - |\eta|^2)}{|\eta| \cdot |1 - \eta \bar{z}|^4} \, dA(\eta) \leq \frac{2(1 - |z|^2)}{\pi} \int_0^1 \int_0^{2\pi} \frac{1}{|1 - r e^{i\theta}|^4} \, d\theta \, dr. \]

By Parseval’s theorem, we get
\[ \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{|1 - re^{-i\theta}|^4} = \sum_{n=0}^{\infty} (n + 1)^2 |z|^{2n}, \]
and thus,
\[ H(z, \zeta) \leq 4(1 - |z|^2) \sum_{n=0}^{\infty} \frac{(n + 1)^2}{2n + 1} |z|^{2n} \leq 4(1 - |z|^2) \sum_{n=0}^{\infty} (n + 1)|z|^{2n} = \frac{4}{1 - |z|^2}, \]
because $\sum_{n=0}^{\infty} (n + 1)z^n = 1/(1 - z)^2$ for all $|z| < 1$. Hence,
\[ (3.2) \quad |J_5(z)| \leq \|g\|_{\infty} \int_{[0, z]} \frac{8}{1 - |z|^2} \, dz = 4\|g\|_{\infty} \log \frac{1 + |z|}{1 - |z|}. \]

Claim 3.5 follows from $|I_5(z) - I_5(0)| \leq |J_4(z)| + |J_5(z)|$, (3.1) and (3.2).

In the following, we let $\zeta = \rho e^{i\theta}$. By calculations and the Hölder inequality, we get
Claim 3.6. \(|I_6(z) - I_6(0)| \leq \|g\|_\infty |z| \left(\frac{2}{3} \pi^2 - 2\right)^{1/2}\).

Claim 3.6 follows from [19, Claim 2.8], because \(|I_6(z) - I_6(0)| = |I_6(z)|\).

Claim 3.7. \(|I_7(z) - I_7(0)| \leq \frac{2}{3} \|g\|_\infty |z| \left[\left(\frac{\pi^2}{6} - \frac{5}{4}\right)^{1/2} + \left(\frac{\pi^2}{6} - 1\right)^{1/2}\right]\).

As in the proof of Claim 3.3 in Lemma 3.1, we use the representation
\[\log(1 - z\zeta) + \frac{1}{z} = -\zeta \sum_{n=1}^{\infty} \frac{(z\zeta)^{n-1}}{n + 1}\]
and obtain that
\[|I_7(z) - I_7(0)| \leq \|g\|_\infty (J_6(z) + J_7(z)),\]
where
\[J_6(z) = \int_{D} \left| \frac{\sum_{n=2}^{\infty} (z\zeta)^{n-1}}{n + 1} \right| |\zeta| dA(\zeta)\]
and
\[J_7(z) = |z|^2 \int_{D} \left| \frac{\sum_{n=1}^{\infty} (z\zeta)^{n-1}}{n + 1} \right| |\zeta| dA(\zeta)\].

By Hölder’s inequality and Parseval’s theorem, we get
\[J_6(z) \leq 2 \int_0^{1} \left( \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{n=2}^{\infty} \frac{(z\rho e^{-it})^{n-1}}{n + 1} \right|^2 dt \right)^{1/2} \rho^2 d\rho \]
\[= 2 \int_0^{1} \left( \sum_{n=1}^{\infty} \frac{|z|^{2n} \rho^{2n}}{(n + 2)^2} \right)^{1/2} \rho^2 d\rho \]
\[\leq 2|z| \left( \sum_{n=1}^{\infty} \frac{1}{(n + 2)^2} \right)^{1/2} \int_0^{1} \rho^2 d\rho = \frac{2}{3}|z| \left( \frac{\pi^2}{6} - \frac{5}{4}\right)^{1/2}\].

By similar reasoning as above, one obtains that
\[J_7(z) \leq \frac{2}{3}|z| \left( \frac{\pi^2}{6} - 1\right)^{1/2}\].

The Claim 3.7 follows.

Claim 3.8. \(|I_8(z) - I_8(0)| \leq \frac{2}{3} \|g\|_\infty |z| \left(\frac{\pi^2}{3} - \frac{11}{4}\right)^{1/2}\).

It follows from
\[\frac{\zeta - z\zeta}{z^2} \log(1 - z\zeta) + \frac{|\zeta|^2 - z\zeta}{z} = \bar{\zeta} \left( \sum_{n=1}^{\infty} \frac{n(1 - |\zeta|^2) + 2}{n(n + 2)} (z\zeta)^n - 1 - \frac{|\zeta|^2}{2} \right)\]
that
\[|I_8(z) - I_8(0)| = \left| \int_{D} \bar{\zeta} \sum_{n=1}^{\infty} \frac{n(1 - |\zeta|^2) + 2}{n(n + 2)} (z\zeta)^n g(\zeta) dA(\zeta) \right| \leq \|g\|_\infty J_8(z),\]
where
\[ J_8(z) = \int_{\mathbb{D}} \left| \sum_{n=1}^{\infty} \frac{n(1-|\zeta|^2) + 2}{n(n+2)} (z\zeta)^n \right| \cdot |\zeta| \, dA(\zeta). \]

Now it is easy to see that
\[ J_8(z) \leq 2 \int_0^1 \Psi(z, \rho) \rho^2 \, d\rho, \quad \zeta = \rho e^{it}, \]
where
\[
\Psi(z, \rho) = \left( \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{n=1}^{\infty} \frac{n(1-\rho^2) + 2}{n(n+2)} (z\rho e^{-it})^n \right|^2 \, dt \right)^{\frac{1}{2}} \\
= \left( \sum_{n=1}^{\infty} \frac{(n(1-\rho^2) + 2)^2 \rho^{2n}}{n^2(n+2)^2} |z|^{2n} \right)^{\frac{1}{2}} \\
\leq |z| \left( \sum_{n=1}^{\infty} \frac{1}{n^2(n+2)^2} \right)^{\frac{1}{2}} = |z| \left( \frac{\pi^2}{3} - \frac{11}{4} \right)^{\frac{1}{2}}.
\]

Here we have used the fact that for each \( n \geq 1 \), \( \phi(\rho) = (n(1-\rho^2) + 2)^2 \rho^{2n} \) is an increasing function \( \rho \). The Claim 3.8 follows.

Now, Claims 3.5, 3.6, 3.7, 3.8, and [17, Proposition 2.4] guarantee that
\[
\left| \frac{\partial}{\partial z} G_2[g](z) - \frac{\partial}{\partial z} G_2[g](0) \right| \leq \sum_{j=5}^{8} \left| I_j(z) - I_j(0) \right| \leq \|g\|_{\infty} |z| M_2 + 4\|g\|_{\infty} \log \frac{1+|z|}{1-|z|},
\]
as required.

**Lemma 3.3.** For constants \( C_j > 0, j \in \{1, 2, 3, 4\} \), let
\[ \phi(x) = C_1 - C_2 x \left[ \frac{2-x}{(1-x)^2} + C_3 \right] - C_4 \log \frac{1+x}{1-x}, \quad x \in [0, 1). \]

Then we have
(1) \( \phi \) is continuous and strictly decreasing in \((0, 1)\);
(2) there is a unique \( x_0 \in (0, 1) \) such that \( \phi(x_0) = 0 \).

**Proof.** For \( x \in [0, 1) \), we find that
\[ \phi'(x) = -\frac{2C_2}{(1-x)^3} - C_2 C_3 - \frac{2C_4}{1-x^2} < 0 \]
showing that \( \phi(x) \) is strictly decreasing in \([0, 1)\). Moreover,
\[ \phi(0) = C_1 > 0 \quad \text{and} \quad \lim_{x \to 1^-} \phi(x) = -\infty < 0 \]
which implies that there is a unique \( x_0 \in (0, 1) \) such that \( \phi(x_0) = 0 \). The proof of the lemma is complete. \( \square \)
4. The proof of main results

In this section, we supply the proofs of Theorems 2.1, 2.2 and 2.3.

Proof of Theorem 2.1. By (1.4) and (2.2), we can quickly deduce that

\[
|w(z) - \frac{1 - |z|^2}{1 + |z|^2} P_{\gamma_0}(0)| \leq \frac{4}{\pi} \|P_{\gamma_0}\|_{\infty} \arctan |z| + |c| + |G_1[\gamma](z)| + |G_2[g](z)|.
\]

We just need to estimate \(|G_1[\gamma](z)|\) and \(|G_2[g](z)|\).

Claim 4.1. \(|G_1[\gamma](z)| \leq \|\gamma\|_{\infty} N_1(|z|)\).

By elementary calculations, (1.5) and Claim 3.2, we obtain

\[
|G_1[\gamma](z)| \leq \|\gamma\|_{\infty} \left[ 4 \log 4 + (1 - |z|^2) \left( \frac{2}{3} \pi^2 - 2 \right)^{\frac{3}{2}} + 4 \|\gamma\|_{\infty} J_9(z) \right],
\]

where

\[
J_9(z) = \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{\log(1 - ze^{-it})}{z} \right| dt.
\]

Now, in order to estimate \(J_9(z)\), we use Hölder’s inequality and Parseval’s theorem to get

\[
J_9(z) \leq \left( \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{n=1}^{\infty} z^{n-1} e^{-int} \right|^2 dt \right)^{\frac{1}{2}} = \left( \sum_{n=1}^{\infty} \frac{|z|^{2(n-1)}}{n^2} \right)^{\frac{1}{2}} \leq \left( \frac{\pi^2}{6} \right)^{\frac{1}{2}}
\]

which proves Claim 4.1.

Claim 4.2. \(|G_2[g](z)| \leq \|g\|_{\infty} N_2(|z|)\).

By elementary calculations, (1.6) and Claim 3.6, we obtain

\[
|G_2[g](z)| \leq \|g\|_{\infty} \left[ 4 \log 4 + (1 - |z|^2) \left( \frac{2}{3} \pi^2 - 2 \right)^{\frac{3}{2}} + 8 \|g\|_{\infty} J_{10}(z) \right],
\]

where

\[
J_{10}(z) = \int_\mathbb{D} \left| \frac{\log(1 - z\zeta)}{z} \right| dA(\zeta).
\]

In order to estimate \(J_{10}(z)\), we let \(\zeta = pe^{it}\). Switching to polar coordinates and by Hölder’s inequality and Parseval’s theorem, we get

\[
J_{10}(z) \leq 2 \int_0^1 \left( \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{n=1}^{\infty} \frac{(ze^{-it})^{n-1}}{n} \right|^2 dt \right)^{\frac{1}{2}} \rho^2 d\rho
\]

\[
\leq 2 \left( \frac{\pi^2}{6} \right)^{\frac{1}{2}} \int_0^1 \rho^2 d\rho = \frac{2}{3} \left( \frac{\pi^2}{6} \right)^{\frac{1}{2}}
\]

which proves Claim 4.2.

Hence, it follows from Claim 4.1 and Claim 4.2 that (2.3) holds, and the proof of the theorem is complete. \(\square\)
Proof of Theorem 2.2. By (1.4) and (2.4), for each \( z \in \mathbb{D} \), we get
\[
\|D_{w}(z)\| = |w(z)| + |w'(z)| \leq 2|c| + \|D_{\gamma_0}(z)\| + \|D_{\gamma_1}(z)\| + \|D_{\gamma_2}(g)(z)\|.
\]
From (2.5) and [19, Lemmas 2.2 and 2.3], we deduce that
\[
\|D_{w}(z)\| \leq \frac{4}{\pi}\|\mathcal{P}_{\gamma_0}\| + \frac{1}{1 - |z|^2} + 2|c| + \|\gamma\|N_3(|z|) + \|g\|N_4(|z|)
\]
as required, and the proof of the theorem is complete. \( \square \)

Before we prove Theorem 2.3, let us recall the following result.

Theorem A. [11] Lemma 1] Suppose \( f \) is a harmonic mapping of \( \mathbb{D} \) into \( \mathbb{C} \) such that \( |f(z)| \leq M \) and \( f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \overline{z}^n \). Then \( |a_0| \leq M \) and for all \( n \geq 1 \),
\[
|a_n| + |b_n| \leq \frac{4M}{\pi}.
\]
This estimate is sharp, and the extreme function is
\[
f_n(z) = \begin{cases} \frac{2M\alpha_1}{\pi} \arg \left( \frac{1 + \beta_1 z^n}{1 - \beta_1 z^n} \right), & |\alpha_1| = |\beta_1| = 1, \quad \text{if } n \geq 1, \\ M & \text{if } n = 0. \end{cases}
\]

Proof of Theorem 2.3. The function \( \mathcal{P}_{\gamma_0} \) is harmonic in \( \mathbb{D} \) and thus, it can be written in the form
\[
\mathcal{P}_{\gamma_0}(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \overline{z}^n.
\]
Applying Theorem 1, we get
\[
\left| \frac{\partial}{\partial z} \mathcal{P}_{\gamma_0}(z) - \frac{\partial}{\partial z} \mathcal{P}_{\gamma_0}(0) \right| + \left| \frac{\partial}{\partial \overline{z}} \mathcal{P}_{\gamma_0}(z) - \frac{\partial}{\partial \overline{z}} \mathcal{P}_{\gamma_0}(0) \right|
\]
\[
= \left| \sum_{n=2}^{\infty} n a_n z^{n-1} \right| + \left| \sum_{n=2}^{\infty} n b_n \overline{z}^{n-1} \right|
\]
\[
(4.1) \leq \sum_{n=2}^{\infty} n(|a_n| + |b_n|)|z|^{n-1} \leq \frac{4L_1}{\pi} \sum_{n=2}^{\infty} n|z|^{n-1} = \frac{4L_1 |z|(2 - |z|)}{(1 - |z|)^2}.
\]
Then by Lemmas 3.1 and 3.2 and (4.1), for \( z \in \mathbb{D}_r \), we obtain
\[
|w_z(z) - w_z(0)| \leq |c| |z| + \left| \frac{\partial}{\partial z} \mathcal{P}_{\gamma_0}(z) - \frac{\partial}{\partial z} \mathcal{P}_{\gamma_0}(0) \right| + \left| \frac{\partial}{\partial \overline{z}} G_1(\gamma)(z) - \frac{\partial}{\partial \overline{z}} G_1(\gamma)(0) \right|
\]
\[
+ \left| \frac{\partial}{\partial z} G_2(g)(z) - \frac{\partial}{\partial z} G_2(g)(0) \right|
\]
\[
(4.2) \leq |z| \left( \frac{4L_1}{\pi} \frac{2 - |z|}{(1 - |z|)^2} + L_5 \right) + 4L_3 \log \frac{1 + |z|}{1 - |z|},
\]

\[
\mathcal{P}_{\gamma_0} \quad \text{are harmonic mappings of} \quad \mathbb{D} \quad \text{into} \quad \mathbb{D}.
\]
and
\[ |w(z) - w(0)| \leq |c||z| + \left| \frac{\partial}{\partial z} \mathcal{P}_{\gamma_0}(z) - \frac{\partial}{\partial z} \mathcal{P}_{\gamma_0}(0) \right| + \left| \frac{\partial}{\partial z} \mathcal{G}_1[\gamma](z) - \frac{\partial}{\partial z} \mathcal{G}_1[\gamma](0) \right| + \left| \frac{\partial}{\partial z} \mathcal{G}_2[g](z) - \frac{\partial}{\partial z} \mathcal{G}_2[g](0) \right| \]
\[ \leq |z| \left( \frac{4L_1}{\pi} \frac{2 - |z|}{(1 - |z|)^2} + L_5 \right) + 4L_3 \log \frac{1 + |z|}{1 - |z|}, \]  
where \( L_5 = |c| + L_2M_1 + L_3M_2. \)

It follows from Theorem 2.2 that
\[ 1 = J_w(0) = \| D_w(0) \| \lambda(D_w(0)) \leq \lambda(D_w(0))L_4, \]  
which gives
\[ \lambda(D_w(0)) \geq \frac{1}{L_4}. \]  

Now, we are ready to finish the proof of the theorem. First, we demonstrate the univalence of the function \( w \) in \( \mathbb{D}_{r_0} \), where \( r_0 \) is determined by Eqn. (2.7). For this, let \( z_1, z_2 \) be two points in \( \mathbb{D}_{r_0} \) with \( z_1 \neq z_2 \), and denote the segment from \( z_1 \) to \( z_2 \) with the endpoints \( z_1 \) and \( z_2 \) by \([z_1, z_2]\). Since
\[ |w(z_2) - w(z_1)| = \left| \int_{[z_1, z_2]} w(z)dz + w(\bar{z})d\bar{z} \right| \]
\[ \geq \left| \int_{[z_1, z_2]} w(z)(dz + d\bar{z}) \right| - \left| \int_{[z_1, z_2]} [w(z) - w(\bar{z})]dz + [w(\bar{z}) - w(z)]d\bar{z} \right|, \]
we see from (4.2), (4.3), (4.4) and Lemma 3.3 that
\[ |w(z_2) - w(z_1)| \geq \lambda(D_w(0)) \cdot |z_2 - z_1| \]
\[ \geq \left[ \frac{1}{L_4} - 2r_0 \left( \frac{4L_1}{\pi} \frac{2 - r_0}{(1 - r_0)^2} + L_5 \right) - 8L_3 \log \frac{1 + r_0}{1 - r_0} \right] |z_2 - z_1| \]
\[ = 0, \]
which implies the univalence of \( w \) in \( \mathbb{D}_{r_0} \).

Next, we prove Theorem 2.3(2). For any \( \zeta = r_0e^{i\theta} \in \partial \mathbb{D}_{r_0} \), we obtain that
\[ |w(\zeta) - w(0)| = \left| \int_{[0, \zeta]} w_z(z) \, dz + w_\overline{\zeta}(z) \, d\overline{z} \right| \]
\[ \geq \left| \int_{[0, \zeta]} w_z(0) \, dz + w_\overline{\zeta}(0) \, d\overline{z} \right| \]
\[ - \left| \int_{[0, \zeta]} [w_z(z) - w_z(0)] \, dz + [w_\overline{\zeta}(z) - w_\overline{\zeta}(0)] \, d\overline{z} \right| \]
\[ \geq \lambda(Dw(0))r_0 - \frac{8L_1}{\pi} \int_0^{r_0} \frac{|z|(2 - |z|)}{(1 - |z|)^2} \, |dz| - 2L_5 \int_0^{r_0} |z| \, |dz| \]
\[ - 8L_3 \int_0^{r_0} \log \frac{1 + |z|}{1 - |z|} \, |dz| \quad \text{(by (4.2) and (4.3))} \]
\[ \geq \frac{r_0}{L_4} - \frac{8L_1}{\pi} \frac{r_0^2}{1 - r_0} - L_5r_0^2 - 8L_3r_0 \log \frac{1 + r_0}{1 - r_0} + \frac{8L_3r_0^2(3 - r_0^2)}{(1 - r_0^2)^2} \]
\[ = \frac{8L_1}{\pi} \left( \frac{r_0}{1 - r_0} \right)^2 + L_5r_0^2 + \frac{8L_3r_0^2(3 - r_0^2)}{(1 - r_0^2)^2} \quad \text{(by (2.7).} \]

Hence \( f(\mathbb{D}_{r_0}) \) contains a univalent disk \( \mathbb{D}_{R_0} \), where
\[ R_0 \geq \frac{8L_1}{\pi} \left( \frac{r_0}{1 - r_0} \right)^2 + L_5r_0^2 + \frac{8L_3r_0^2(3 - r_0^2)}{(1 - r_0^2)^2}. \]

The proof of this theorem is complete. \( \square \)

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