OPTIMAL CONTROL OF SOLUTIONS OF A MULTIPOINT INITIAL-FINAL PROBLEM FOR NON-AUTONOMOUS EVOLUTIONARY SOBOLEV TYPE EQUATION

MINZILIA A. SAGADEEVA*
South Ural State University
Institute of Natural Sciences and Mathematics
454080, Chelyabinsk, Lenin av, 76, Russian Federation

SOPHIYA A. ZAGREBINA AND NATALIA A. MANAKOVA
South Ural State University
Institute of Natural Sciences and Mathematics
454080, Chelyabinsk, Lenin av, 76, Russian Federation

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Abstract. The paper presents sufficient conditions for existence of an optimal control of solutions to a non-autonomous degenerate operator-differential evolution equation. We construct families of operators that solve this equation, as well as classical and strong solutions of the multipoint initial-final problem for the equation. We show that there exists a solution of an optimal control problem for a given operator-differential equation with a multipoint initial-final condition. The paper, in addition to the introduction and the bibliography, contains five sections. The first three parts contain information about the solvability of the multipoint initial-final problem for a non-autonomous equation. The fourth section presents the main result of the article; that is, a theorem on existence of optimal control of solutions to a multipoint initial-final problem. In the fifth part, the optimal control problem for the non-autonomous modified Chen – Gurtin model with the multipoint initial-final condition is investigated on the basis of the obtained abstract results.

Introduction. Let \( \Omega \subset \mathbb{R}^m \) be a bounded domain with boundary \( \partial \Omega \) of \( C^\infty \) class. On the interval \((\tau_0, \tau_n) \subset \mathbb{R}\) consider the modified Chen – Gurtin equation \([6]\)

\[
(\lambda - \Delta) x_t(r, t) = \Delta x(r, t) - id^2 x(r, t) + u(r, t), \quad (r, t) \in \Omega \times (\tau_0, \tau_n),
\]

where \( \lambda, d \in \mathbb{R} \), with the boundary conditions

\[
\Delta x(r, t) = x(r, t) = 0, \quad (r, t) \in \partial \Omega \times (\tau_0, \tau_n).
\]

The required complex-valued function \( x(r, t) \) in (1) describes the dynamics of the process, and the complex-valued function \( u(r, t) \) describes an external effect on the system. In the special case of \( d = 0 \) equation (1) describes the process of heat

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* Corresponding author: sagadeevama@susu.ru.
conduction with "two temperatures" [6], as well as the dynamics of fluid pressure in a cracks-porous medium [5] and the process of moisture transfer in soil [12]. Also, equation (1) is a particular case of the linearized classical Ginzburg – Landau equation [2]. If $d = 0$, then (1) is also the linearization of the Oskolkov equation [21].

In this paper, we consider the non-autonomous modified Chen – Gurtin equation of the form

$$\frac{d}{dt}(\lambda - \Delta)x(t) = a(t)(\Delta x(t) - id\Delta^2 x(t)) + u(t),$$

(3)

which allows for taking into account the change in the parameters of the system over time. Note that (3) is a generalization of equation (1), which can be obtained from (3) for $a \equiv 1$. In the case $d = 0$, the autonomous model (1), (2) was investigated on the finite connected oriented graphs (i.e., $m = 1$), for example, in [28], and the non-autonomous model (2), (3) was investigated in [23]. We also note that if the operator on the left-hand side of (1) is invertible, then (1), (2) can be investigated by methods of evolutionary semigroups (see, for example, [15, 13, 22, 4, 3]).

Problem (2), (3) is reduced to a non-autonomous evolutionary equation

$$L\dot{x}(t) = a(t)Mx(t) + g(t) + Bu(t),$$

(4)

where the operators $L \in \mathcal{L}(X; Y)$ and $B \in \mathcal{L}(U; Y)$ (i.e., linear and continuous), and the operator $M \in \mathcal{C}(X; Y)$ (i.e., linear, closed, densely defined in $X$). Equations of the form (4) with $\ker L \neq \{0\}$ are called degenerate differential equations [8] or not solvable with respect to the highest-order derivative (see the historical review in [7]). Recently, the term Sobolev type equation, which was proposed by R. Showalter [25, 26], has been widely used (see, for example, [29, 1]). This research is in the framework of the theory of degenerate operator semigroups, which was founded by G.A. Sviridyuk [29] and is currently actively developed, for example, in [35, 18, 27]. Moreover, this theory was generalized for spaces of random processes [9, 10].

The main feature of such semigroups is that their operators are degenerated not only on the kernel of the operator $L$, but also on the linear span of $M$-adjoint vectors of height at most $p$ of operator $L$ [11, 29, Capter 2]. Note that the degenerate strongly continuous semigroup of solving operators was considered for the first time in [30]. A. Favini and A. Yagi in their work [8] investigated the solvability of degenerate equations using theory of the multivalued operators and differential inclusions. Their results [8] intersect Sobolev type equations theory [29] only in case $p = 0$.

For equation (4), where $P_j$ are some spectral projections. Note that the general formulation of the multipoint initial-final problem (5)

$$P_0(x(t) - x_0) = 0, \quad P_j(x(\tau_j) - x_j) = 0, \quad j = 1, \ldots, n$$

(5)

for equation (4), where $P_j$ are some spectral projections. Note that the general formulation of the multipoint initial-final problem (4), (5) in the relatively $p$-radial case was first introduced in [34]. If $n = 1$, then (5) is called an initial-final condition. Initial-final problem was considered earlier for autonomous Sobolev type equations of the first (see the review in [33]) and higher [36] orders. Note that the multipoint
initial-final conditions are also a generalization of the Showalter – Sidorov condition
[31], which corresponds to \( n = 0 \) (for more details, see [14]).

In Hilbert spaces \( X \), \( Y \) and \( U \) consider the optimal control problem for (4), (5):
to find a vector-function \( v \in \mathcal{U}_{ad} \) such that the relation

\[
J(v) = \inf_{u \in \mathcal{U}_{ad}} J(u)
\]

holds, where all pairs \((x(u), u)\) satisfy the multipoint initial-final problem (4), (5).
Here \( J(u) \) is a cost functional of a special form, and \( \mathcal{U}_{ad} \) is a closed and convex subset
of admissible controls in \( U \). The optimal control problem for the linear autonomous
Sobolev-type equation with the Cauchy condition was first considered in [32]. Later
this problem was studied in different cases [23, 18, 36, 19, 20]. Separately note the
paper [20], where the optimal control problem for the autonomous Chen – Gurtin
equation (1) with boundary (2) and initial-final condition was investigated.

The paper, in addition to the introduction and the bibliography, contains five
parts. The first part provides the necessary information about the theory of relatively
\( p \)-radial operators. In the second and the third parts, the solvability of the
homogeneous non-autonomous equation (4) and the multipoint initial-final prob-
tively \( p \)-radial operators. In the second and the third parts, the solvability of the

1. Relatively \( p \)-radial operators. Let us recall the standard definitions and no-
tations of the theory of relatively \( p \)-radial operators [29, 30, 11]. The proofs of
the statements of this part can be found in [11, 29, Chapter 2].

Let \( X \) and \( \mathcal{Y} \) be Banach spaces, \( \mathcal{L}(X; \mathcal{Y}) \) be the space of continuous linear
operators, and \( \mathcal{C}(X; \mathcal{Y}) \) be the space of closed densely defined linear operators. The
operators \( L \in \mathcal{L}(X; \mathcal{Y}) \) and \( M \in \mathcal{C}(X; \mathcal{Y}) \) are given. Following [29, 30, 11], the sets
\( \rho^L(M) = \{ \mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathcal{Y}; X) \} \) and \( \rho^M(M) = \mathbb{C} \setminus \rho^L(M) \) are called
\( L \)-resolvent set and \( L \)-spectrum of the operator \( M \), respectively. By the results
of [29, 30, 11], the \( L \)-resolvent set is open, and therefore the \( L \)-spectrum of the
operator \( M \) is always closed. The \( L \)-resolvent set of the operator \( M \) can be empty,
for example, if \( \ker L \cap \ker M \neq \{0\} \).

Assuming that \( \rho^L(M) \neq \emptyset \), introduce operator functions \((\mu L - M)^{-1}, R^L_\mu(M) =
(\mu L - M)^{-1}L \) and \( L^L_\mu(M) = L(\mu L - M)^{-1} \) of a complex variable with the
domain \( \rho^L(M) \), which are called \( L \)-resolvent, right \( L \)-resolvent and left \( L \)-resolvent of
the operator \( M \), respectively. Similarly, operator functions

\[
R^L_{(\mu,q)}(M) = \prod_{k=0}^q R^L_{\mu k}(M), \quad L^L_{(\mu,q)}(M) = \prod_{k=0}^q L^L_{\mu k}(M), \quad \mu_k \in \rho^L(M) \quad (k = 0,1, \ldots)
\]

of \((q + 1)\) complex variables with the domain \([\rho^L(M)]^{q+1}\) are called right \((L,q)\)-resolvent and left \((L,q)\)-resolvent of the operator \( M \), respectively. All considered
operator functions are holomorphic in their domains [29, Chapter 2].

Assume that \( \ker L \neq \emptyset \). The vector \( \varphi_0 \in \ker L \setminus \{0\} \) is called an eigenvector of
operator \( L \). An ordered set of vectors \( \{\varphi_0, \varphi_1, \ldots\} \) is called a chain of \( M \)-adjoint
vectors of \( \varphi_0 \) if \( L\varphi_{k+1} = M\varphi_k, \quad k = 0,1, \ldots \) \( (\varphi_l \notin \ker L, \quad l = 1,2, \ldots) \).
We call the largest index of a vector in a chain (starting from 0) its height, thus the eigenvector is an $M$-adjoint vector of height 0.

**Lemma 1.1.** [29, Chapter 2] The vector $\varphi \neq 0$ is an $M$-adjoint vector of height at most $q$ of operator $L$ if and only if $(R_{\mu,p}^L(M))^{q+1}\varphi = 0$.

Using this lemma, we obtain the statement of the following lemma.

**Lemma 1.2.** [29, Chapter 2] Take $\lambda, \mu \in (\rho^L(M))^+$. 

(i) $\ker R_{(\mu,p)}^L(M)$ is the linear span of the set of $M$-adjoint vectors of height at most $q$ of $L$, and $\ker R_{(\mu,p)}^L(M) = \ker R_{(\lambda,q)}^L(M)$;

(ii) $\ker L_{(\mu,p)}^L(M) = \{M \varphi : \varphi \in \ker R_{(\mu,p)}^L(M) \cap \text{dom}M\}$ and $\ker L_{(\lambda,q)}^L(M) = \ker L_{(\lambda,q)}^L(M)$.

**Definition 1.3.** The operator $M$ is called $p$-radial relatively to the operator $L$ (briefly, $(L,p)$-radial), if

(i) $\exists \alpha \in \mathbb{R} : (\alpha, +\infty) \subset \rho^L(M)$;

(ii) $\exists p \in \mathbb{N}_0 \enspace \exists K > 0 : \forall \mu = (\mu_0, \mu_1, \ldots, \mu_p) \in (\alpha, +\infty)^{p+1} \forall n \in \mathbb{N}$

$$\max\{(R_{(\mu,p)}^L(M))^n \|L(X)\|, (L_{(\mu,p)}^L(M))^n \|L(Y)\|\} \leq \frac{K}{\prod_{k=0}^p (\mu_k - \alpha)^n}.$$  

**Remark 1.4.** The concept of $(L,p)$-radial operator $M$ is a generalization of the conditions of the Hille – Yosida theorem [13, Chapter 12, § 3]. More details can be found in [11].

By virtue of Lemma 1.2, introduce the notation

$$\mathcal{X}^0 = \ker R_{(\mu,p)}^L(M), \quad \mathcal{Y}^0 = \ker L_{(\mu,p)}^L(M), \quad L_0 = L \bigg|_{\mathcal{X}^0}, \quad M_0 = M \bigg|_{\text{dom}M \cap \mathcal{X}^0},$$

where $p \in \mathbb{N}_0$ is taken from Definition 1.3. Let $\mathcal{X} = (\mathcal{Y}^1)$ be the closure of the lineal $\text{im}R_{(\mu,p)}^L(M)$ $(\text{im}L_{(\mu,p)}^L(M))$. Note that if the operator $M$ is $(L,p)$-radial, then there exists an operator $M_0^+ \in \mathcal{L}(\mathcal{Y}^0; \mathcal{X}^0)$ [29, Chapter 2].

**Definition 1.5.** A one-parameter family of operators $X^\bullet : \mathbb{R}_+ \to \mathcal{L}(\mathcal{X})$ is called a strongly continuous semigroup ($C_0$-semigroup) of operators, if

(i) $X^sX^t = X^{s+t} \forall s, t \in \mathbb{R}_+$;

(ii) $X^t$ is strongly continuous for $t > 0$ and there exists $\lim_{t \to 0^+} X^t x = x$ for all $x$ from some lineal, which is dense in $\mathcal{X}$.

**Definition 1.6.** A semigroup $\{X^t \in \mathcal{L}(\mathcal{X}) : t \in \mathbb{R}_+\}$ is called exponentially bounded with constants $C$ and $\alpha$, if the following condition holds:

$$\exists C > 0 \exists \alpha \in \mathbb{R} \forall t \in \mathbb{R}_+ \quad \|X^t\|_{\mathcal{L}(\mathcal{X})} \leq Ce^{\alpha t}.$$  

**Theorem 1.7.** [29, Chapter 2] Let $M$ be an $(L,p)$-radial operator $(p \in \mathbb{N}_0)$. Then there exists a $C_0$-semigroup $\{X^t \in \mathcal{L}(\mathcal{X}) : t \geq 0\}$ $(\{X^t \in \mathcal{L}(\mathcal{Y}) : t \geq 0\})$, which is defined in the subspace $\hat{\mathcal{X}}(\hat{\mathcal{Y}})$, where $\hat{\mathcal{X}}(\hat{\mathcal{Y}})$ denotes the closure of the lineal $\mathcal{X}^0 + \text{im}R_{(\mu,p)}^L(M) \mathcal{Y}^0 + \text{im}L_{(\mu,p)}^L(M)$ in the norm of space $\mathcal{X}(\mathcal{Y})$. Moreover, this semigroup is exponentially bounded with the constants $K, \alpha$ from Definition 1.3.
Remark 1.8. Operators of the semigroup \( \{X^t \in \mathcal{L}(\mathfrak{X}) : t \geq 0\} \) from Theorem 1.7 can be represented in the form

\[
X^t = \lim_{k \to \infty} \left( L - \frac{t}{k} M \right)^{-1} = \lim_{k \to \infty} \left( \frac{k}{t} R^K_{\tau}(M) \right)^k, \quad t > 0
\]  
(6)

\[
Y^t = \lim_{k \to \infty} \left( L \left( L - \frac{t}{k} M \right)^{-1} \right)^k = \lim_{k \to \infty} \left( \frac{k}{t} L^K_{\tau}(M) \right)^k, \quad t > 0
\]

Here and below the notation \( s\text{-lim} \) means the limit in the strong topology of the corresponding space of operators.

Remark 1.9. [29, Chapter 2] A unit of the semigroup \( \{X^t \in \mathcal{L}(\mathfrak{X}) : t \in \mathbb{R}_+\} \) \( \{(Y^t \in \mathcal{L}(\mathfrak{Y}) : t \in \mathbb{R}_+)\} \) is the operator \( P = \lim_{t \to 0^+} X^t \) \( (Q = \lim_{t \to 0^+} Y^t) \), which is a projection along \( \mathfrak{X}^0 (\mathfrak{Y}^0) \) onto the subspace \( \mathfrak{X}^1 (\mathfrak{Y}^1) \).

Example. To illustrate these concepts, we consider a degenerate system of ordinary differential equations

\[
\begin{align*}
\dot{y} &= x, \\
0 &= y, \\
\dot{z} &= z.
\end{align*}
\]

Let \( u(t) = \col(x(t), y(t), z(t)) \). Then we can rewrite this system as \( L\dot{u} = Mu \) with operators

\[
L = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix} \quad \text{and} \quad M = E = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

For such operators \( L \) and \( M \), the \( L \)-resolvent, the right \( L \)-resolvent and the left \( L \)-resolvent of the operator \( M \) have the form

\[
(\mu L - M)^{-1} = \begin{pmatrix}
-1 & -\mu & 0 \\
0 & -1 & 0 \\
0 & 0 & \frac{1}{\mu - 1}
\end{pmatrix}, \quad R^L_{\mu_0}(M) = L^L_{\mu}(M) = \begin{pmatrix}
0 & -1 & 0 \\
0 & 0 & 0 \\
0 & 0 & \frac{1}{\mu - 1}
\end{pmatrix}
\]

and

\[
R^L_{\mu_0}(M)R^L_{\mu_1}(M) = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \frac{1}{(\mu_0 - 1)(\mu_1 - 1)}
\end{pmatrix}.
\]

The \( L \)-spectrum of \( M \) consists of one point \( \sigma^L(M) = \{1\} \) and \( \alpha = 1 \) in condition (i) of Definition 1.3. It is clear that condition (ii) of Definition 1.3 is not fulfilled for \( R^L_{\mu_0}(M) \) if \( n = 1 \), and it holds for \( R^L_{\mu_0}(M)R^L_{\mu_1}(M) \) \( \forall n \in \mathbb{N} \). Thus, \( p = 1 \) and the operator \( M \) is \( (L, 1) \)-radial.

Note that \( \ker L = \span \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \) and \( \mathfrak{X}^0 = \ker R^L_{(\mu, 1)} = \span \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \).

Then the projection \( P \) from Remark 1.9 has the form \( P = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix} \).

Introduce the condition

\[
\mathfrak{X} = \mathfrak{X}^0 \oplus \mathfrak{X}^1, \quad \mathfrak{Y} = \mathfrak{Y}^0 \oplus \mathfrak{Y}^1.
\]  
(7)
Let \( L_1 (M_1) \) be the restriction of the operator \( L (M) \) to \( \mathcal{X}^1 \) \( (\text{dom } M \cap \mathcal{X}^1) \). Also, introduce one more additional condition:

\[
\text{there exists an operator } L_1^{-1} \in \mathcal{L}(\mathcal{Y}^1; \mathcal{X}^1).
\]  

(8)

**Theorem 1.10.** [29, Chapter 2] Let the operator \( M \) be \((L, p)\)-radial \( (p \in \mathbb{N}_0) \) and conditions (7), (8) be satisfied. Then

\[ \text{(i) } L_k = L \bigg|_{\mathcal{X}^k} \in \mathcal{L} (\mathcal{X}^k; \mathcal{Y}^k), \quad M_k = M \bigg|_{\text{dom } M_k} \in \mathcal{C} (\mathcal{X}^k; \mathcal{Y}^k); \]

\[ \text{dom } M_k = \text{dom } M \cap \mathcal{X}^k, \quad k = 0, 1; \]

\[ \text{(ii) the operator } H = M_0^{-1} L_0 \in \mathcal{L}(\mathcal{Y}^0) \text{ is nilpotent of degree at most } p; \]

\[ \text{(iii) the operator } S = L_1^{-1} M_1 \in \mathcal{C}(\mathcal{X}^1). \]

2. \( C_0 \)-semiflows of solving operators for non-autonomous evolution equations. Let \( \mathcal{X}, \mathcal{Y} \) be Banach spaces, \( L \in \mathcal{L}(\mathcal{X}; \mathcal{Y}) \) and \( M \in \mathcal{C}(\mathcal{X}; \mathcal{Y}) \) be given operators.

**Definition 2.1.** A two-parameter family \( X(\cdot, \cdot) : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathcal{L}(\mathcal{X}) \) is called a degenerate strongly continuous semiflow (briefly, a degenerate \( C_0 \)-semiflow) of operators, if the following conditions are satisfied:

\[ \text{(i) } X(t, \tau)X(t, \zeta) = X(t, \zeta) \text{ for all } 0 \leq \zeta < \tau < t; \]

\[ \text{(ii) } X(t, \zeta) \text{ are strongly continuous for all } t, \zeta > 0 \text{ and for all } \zeta \geq 0 \text{ there exists } \lim_{t \to \zeta^+} X(t, \zeta)x = x \text{ for all } x \text{ from a dense in linear subspace } \mathcal{X}. \]

Let the function \( a \in C(\mathbb{R}; \mathbb{R}_+) \). Then from (6) we can get the two-parameter family of operators of \( \zeta, t \in \mathbb{R}: \)

\[
X(t, \zeta) = \lim_{k \to \infty} \left( \left( L - \frac{1}{k} M \int_{\zeta}^{t} a(\zeta) d\zeta \right)^{-1} L \right)^k, \quad \zeta < t.
\]  

(9)

**Theorem 2.2.** Let \( M \) be an \((L, p)\)-radial operator \( (p \in \mathbb{N}_0) \), conditions (7), (8) be satisfied and \( a \in C(\mathbb{R}; \mathbb{R}_+) \). Then the family \( \{X(t, \zeta) \in \mathcal{L}(\mathcal{X}) : t > \zeta \geq 0\} \), given by (9), is a degenerate \( C_0 \)-semiflow of operators.

**Proof.** Show that \( X(t, \zeta) \) is a semiflow of operators.

Let us show that condition (i) of Definition 2.1 holds. Let \( 0 \leq \zeta < \tau < t \), then

\[
X(t, \tau)X(\tau, \zeta) = \lim_{k \to \infty} \left( \left( L - \frac{1}{k} M \int_{\tau}^{t} a(\zeta) d\zeta \right)^{-1} L \right)^k \left( \left( L - \frac{1}{k} M \int_{\zeta}^{\tau} a(\zeta) d\zeta \right)^{-1} L \right)^k = \]

\[
= \lim_{k \to \infty} \left( \left( L - \frac{t_1}{k} M \right)^{-1} L \right)^k \left( \left( L - \frac{t_2}{k} M \right)^{-1} L \right)^k = X^{t_1}X^{t_2},
\]

where the notations \( t_1 = \int_{\tau}^{t} a(\zeta) d\zeta, \quad t_2 = \int_{\zeta}^{\tau} a(\zeta) d\zeta \) and the form of semigroup operators (6) are used. By the conditions of the theorem \( t_1, t_2 > 0 \), and, consequently,
from condition (i) of Definition 1.5, we obtain
\[ X^{t_1}X^{t_2} = X^{t_1+t_2} = s\lim_{k \to \infty} \left( \left( L - \frac{1}{k} M \int_{\varsigma}^{t} a(\zeta) d\zeta \right)^{-1} L \right)^k = X(t, \varsigma). \]

Let us show that condition (ii) holds. For \( \varsigma < t \) we replace \( t_3 = \int_{\varsigma}^{t} a(\zeta) d\zeta > 0 \) in (9) and in view of (6), we obtain
\[ X(t, \varsigma) = s\lim_{k \to \infty} \left( \left( L - \frac{1}{k} M \int_{\varsigma}^{t} a(\zeta) d\zeta \right)^{-1} L \right)^k X^{t_3}, \]
and, consequently, the operator \( X(t, \varsigma) \) is strongly continuous by the condition (ii) of Definition 1.5. From (10) for \( \varsigma \geq 0 \) and \( x \in X \) we obtain
\[ \lim_{t \to \varsigma^+} X(t, \varsigma)x = \lim_{t_3 \to 0^+} \lim_{k \to \infty} \left( \left( L - \frac{t_3}{k} M \right)^{-1} L \right)^k x = \lim_{t_3 \to 0^+} X^{t_3}x = Px \]
in view of Remark 1.9.

**Remark 2.3.** In view of Theorem 1.10, we obtain
\[ X(t, \varsigma) \mid_{X^1} = s\lim_{k \to \infty} \left( \left( L - \frac{1}{k} M \int_{\varsigma}^{t} a(\zeta) \, d\zeta \right)^{-1} L(P + I_X - P) \right)^k \mid_{X^1} = \]
\[ = s\lim_{k \to \infty} \left\{ \left( L - \frac{1}{k} M \int_{\varsigma}^{t} a(\zeta) \, d\zeta \right)^{-1} L \right\} P + \left\{ \left( L - \frac{1}{k} M \int_{\varsigma}^{t} a(\zeta) \, d\zeta \right)^{-1} L \right\} (I_X - P) \mid_{X^1}. \]

By Remark 1.9, from this formula we get
\[ X(t, \varsigma) \mid_{X^1} = \]
\[ = s\lim_{k \to \infty} \left( L^{-1} L_1 - \frac{1}{k} L_1^{-1} M_1 \int_{\varsigma}^{t} a(\zeta) \, d\zeta \right)^{-k} = s\lim_{k \to \infty} \left( I_{X^1} - \frac{1}{k} S \int_{\varsigma}^{t} a(\zeta) \, d\zeta \right)^{-k}, \]
where \( S = L_1^{-1} M_1 \in \mathcal{C}(X^1). \)

**Remark 2.4.** By analogy with (9), the semiflow of operators in the space \( \mathcal{L}(\mathfrak{Y}) \) can be also given by the following formula:
\[ Y(t, \varsigma) = s\lim_{k \to \infty} \left( L \left( L - \frac{1}{k} M \int_{\varsigma}^{t} a(\zeta) \, d\zeta \right)^{-1} \right)^k, \quad \varsigma < t. \]
The proof of the semiflow properties for operators (11) is analogous to the proof of Theorem 2.2.

On the interval \((t_0, T] \subset \mathbb{R}_+\), consider the weakened Cauchy problem (in sense of S.G. Krein [16, 15, Chapter 1, \S 3])

\[
\lim_{t \to t_0^+} x(t) = x_{t_0} \tag{12}
\]

for a homogeneous non-autonomous equation

\[
L\dot{x}(t) = a(t)Mx(t), \tag{13}
\]

where the function \(a : [t_0, T] \to \mathbb{R}_+\) will be defined later.

**Definition 2.5.** A vector-function \(x \in C([t_0, T]; X) \cap C^1((t_0, T]; X)\), satisfying (13) on \((t_0, T]\) is called a solution of this equation. A solution of (13) is called a classical solution of the weakened Cauchy problem (12), (13), if it additionally satisfies condition (12).

**Definition 2.6.** A closed set \(P \subset X\) is called a phase space of (13), if

(i) any solution \(x(t)\) of (13) lies in \(P\), i.e. \(x(t) \in P \forall t \in (t_0, T]\);

(ii) for any \(x_{t_0}\) from \(P\) there exists a unique solution of the weakened Cauchy problem (12) for equation (13).

Together with equation (13) we consider the equivalent equation

\[
L(\nu L - M)^{-1}\dot{y} = a(t)M(\nu L - M)^{-1}y \quad \text{with} \quad \nu \in \rho(L). \tag{14}
\]

**Theorem 2.7.** Let \(M\) be an \((L, p)\)-radial operator \((p \in \mathbb{N}_0)\), conditions (7), (8) be satisfied and \(a \in C(\mathbb{R}, \mathbb{R}_+)\). Then the set \(X^1\) is the phase space of equation (13), and the set \(Y^1\) is the phase space of equation (14).

**Proof.** In view of Theorem 1.10, equation (13) is equivalent to the system

\[
\dot{x}^1 = aL^{-1}M_1x^1, \quad M_0^{-1}L_0\dot{x}^0 = ax^0, \tag{15}
\]

where \(x^1 = Px \in X^1\), \(x^0 = (I_X - P)x \in X^0\) and \(P\) is taken from Remark 1.9.

From the second equation in (15), we obtain

\[
x^0 = \frac{1}{a}H\dot{x}^0. \tag{16}
\]

Differentiate (16) and apply the operator \(\frac{1}{a}H\), in view of (16), we obtain

\[
\frac{1}{a}H\dot{x}^0 = \frac{1}{a}H\frac{d}{dt} \left(\frac{1}{a}Hx^0\right) = H^2\frac{1}{a}d\frac{d}{dt} \left(\frac{1}{a}\dot{x}^0\right) = x^0.
\]

Repeating this procedure \(p\) times we obtain

\[
\frac{1}{a}H\dot{x}^0 = H^{p+1}\frac{1}{a}\left(\frac{d}{dt}\right)^p \left(\frac{1}{a}\dot{x}^0\right) = x^0.
\]

Therefore, since the operator \(H\) is nilpotent, we get that \(x^0 = 0\).

Now consider the first equation of (15). The solution of this equation has the form [22, Theorem 6.1.5]

\[
x^1(t) = \lim_{k \to \infty} \left(\mathbb{I}_{X^1} - \frac{1}{k}S_{t_0}^{t}a(\zeta)d\zeta\right)^{-k}x_{t_0},
\]
where $S = L^{-1}M_1 \in \mathcal{C}(\mathcal{X}_1)$. Hence, for $x_{t_0} \in \mathcal{X}_1$, in view of Remark 2.3, the solution of (13) has the form $x(t) = X(t, t_0)x_{t_0} \in \mathcal{X}_1$.

**Definition 2.8.** A semiflow of operators $X(\cdot, \cdot) : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathcal{L}(\mathcal{X})$ is called the semiflow of solving operators of (13), if for any $x_{t_0} \in \mathcal{X}$ the vector-function $x(t) = X(t, t_0)x_{t_0}$ is a solution of (13) (in the sense of Definition 2.5).

In view of Theorems 2.2 and 2.7, the following theorem is true.

**Theorem 2.9.** Let $M$ be an $(L, p)$-radial operator ($p \in \mathbb{N}_0$), conditions (7), (8) be satisfied and $a \in C(\mathbb{R}, \mathbb{R}_+)$, then the family $\{X(t, \cdot) \in \mathcal{L}(\mathcal{X}) : t > \varsigma \geq 0\}$, given by formula (9), is the semiflow of solving operators for (13), and the family $\{Y(t, \cdot) \in \mathcal{L}(\mathcal{Y}) : t > \varsigma \geq 0\}$, given by formula (11), is the semiflow of solving operators for (14).

**Remark 2.10.** Similar results for the relatively $p$-bounded case were obtained in [24].

### 3. Strong solutions of the multipoint initial-final problem for the non-autonomous evolution equation.

Let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert spaces, $L \in \mathcal{L}(\mathcal{X} ; \mathcal{Y})$, $M \in \mathcal{C}(\mathcal{X} ; \mathcal{Y})$ be given operators, $M$ be an $(L, p)$-radial operator ($p \in \mathbb{N}_0$) and conditions (7), (8) be satisfied. On the interval $(\tau_0, \tau_n) \subset \mathbb{R}_+$ consider the inhomogeneous equation

$$Lx(t) = a(t)Mx(t) + g(t)$$

with functions $g : (\tau_0, \tau_n) \to \mathcal{Y}$ and $a \in C([\tau_0, \tau_n]; \mathbb{R}_+)$. Additionally we assume the conditions:

$$\sigma^L(M) = \bigcup_{j=0}^{n} \sigma_j^L(M), \ n \in \mathbb{N}, \text{ and } \sigma_j^L(M) \neq \emptyset,$$

there exist contours $\gamma_j = \partial D_j \subset \mathbb{C}$, where $D_j \supset \sigma_j^L(M)$, such that $\overline{D_j} \cap \sigma_k^L(M) = \emptyset$ and $\overline{D_k} \cap \overline{D_l} = \emptyset$ for all $j, k, l = 1, n, k \neq l$.

In view of holomorphy of the relative resolvents, there exist projections $P_j \in \mathcal{L}(\mathcal{X})$ and $Q_j \in \mathcal{L}(\mathcal{Y})$ ($j = \overline{0, n}$), which have the form, [34],

$$P_j = \frac{1}{2\pi i} \int_{\gamma_j} R^L_\mu(M) d\mu, \quad Q_j = \frac{1}{2\pi i} \int_{\gamma_j} L^L_\mu(M) d\mu \quad \text{for} \quad j = \overline{1, n}$$

and $P_0 = P - \sum_{j=1}^{n} P_j$, $Q_0 = Q - \sum_{j=1}^{n} Q_j$, where $P$ and $Q$ are defined in Remark 1.9.

Introduce the subspaces $\mathcal{X}_j^1 = \text{im} P_j$, $\mathcal{Y}_j^1 = \text{im} Q_j$, $j = \overline{0, n}$. By construction,

$$\mathcal{X}_1^1 = \bigoplus_{j=0}^{n} \mathcal{X}_j^1 \quad \text{and} \quad \mathcal{Y}_1^1 = \bigoplus_{j=0}^{n} \mathcal{Y}_j^1.$$  

Let $L_{1j}$ be the restriction of the operator $L$ to $\mathcal{X}_1^1$, $j = \overline{0, n}$, and $M_{1j}$ be the restriction of the operator $M$ to $\text{dom} M \cap \mathcal{X}_1^1$, $j = \overline{0, n}$. Since it is easy to show that $P_j \varphi \in \text{dom} M$, if $\varphi \in \text{dom} M$, then $\text{dom} M_{1j} = \text{dom} M \cap \mathcal{X}_1^1$ is dense in $\mathcal{X}_j^1$, $j = \overline{0, n}$.

**Theorem 3.1.** (The generalized splitting theorem) [34]. Let $L \in \mathcal{L}(\mathcal{X} ; \mathcal{Y})$ and $M \in \mathcal{C}(\mathcal{X} ; \mathcal{Y})$, the operator $M$ be $(L, p)$-radial ($p \in \mathbb{N}_0$) and conditions (7), (8) and (18) be satisfied. Then
(i) the operators \( L_{1j} \in \mathcal{L}(\mathcal{X}^1_1; \mathcal{Y}^1_1) \), \( M_{1j} \in \mathcal{C}(\mathcal{X}^1_1; \mathcal{Y}^1_1) \), \( j = 0, n \);
(ii) there exist the operators \( L_{1j}^{-1} \in \mathcal{L}(\mathcal{Y}^1_1; \mathcal{X}^1_1) \), \( j = 0, n \).

Fix \( x_j \in \mathcal{X} \) and \( \tau_j \in \mathbb{R}_+ \) \( (j = 0, n) \) such that \( \tau_0 = 0 \) and \( \tau_{j-1} < \tau_j \) \( (1 \leq j \leq n) \).

For such data, consider the multipoint initial-final problem \([33]\)

\[
\lim_{t \to 0^+} P_0(x(t) - x_0) = 0, \quad P_j(x(\tau_j) - x_j) = 0, \quad j = \overline{1, n},
\]
for equation \((17)\).

**Definition 3.2.** A vector-function \( x \in C([\tau_0, \tau_n]; \mathcal{X}) \cap C^1((\tau_0, \tau_n]; \mathcal{X}) \) is called a classical solution of \((17)\), if this function converts the equation into an identity on \((\tau_0, \tau_n)\). A classical solution \( x = x(t) \) of \((17)\) is called a classical solution of the multipoint initial-final problem \((17), (19)\), if it satisfies \((19)\).

**Theorem 3.3.** Let \( M \) be an \((L, p)\)-radial operator \((p \in \mathbb{N}_0)\), \( a \in C^{p+1}([\tau_0, \tau_n]; \mathbb{R}_+) \) and conditions \((7), (8), (18)\) be satisfied. Then for any vectors \( x_j \in \mathcal{X}^1_1 \) \((j = 0, n)\) and vector-function \( g : (\tau_0, \tau_n) \to \mathcal{Y} \) such that \( Qg \in C((\tau_0, \tau_n), \mathcal{Y}^1) \) and \((\mathbb{I}_0 - Q)g \in C^{p+1}((\tau_0, \tau_n), \mathcal{Y}^1)\), there exists a unique classical solution \( x \in C([\tau_0, \tau_n]; \mathcal{X}) \cap C^1((\tau_0, \tau_n]; \mathcal{X}) \) of \((17), (19)\), given by

\[
x(t) = -\sum_{k=0}^{p} H^k M_0^{-1}(\mathbb{I}_0 - Q) \left( \frac{1}{a(t)} \frac{d}{dt} \right)^k g(t) a(t) + \sum_{j=0}^{n} X(t, \tau_j)P_j x_j - \int_{t}^{\tau_j} X(t, s)L_{1j}^{-1} Q_j g(s)ds.
\]

**Proof.** By virtue of Theorem 3.1, problem \((17), (19)\) is equivalent to the system of problems

\[
\dot{x}_j(t) = a(t)S_{1j}x_j(t) + L_{1j}^{-1}g_j(t), \quad x_j(\tau_j) = P_j x_j, \quad j = \overline{0, n},
\]

\[
H \dot{z}_0(t) = a(t)x_0(t) + M_0^{-1} z_0(t),
\]

on subspaces \( \mathcal{X}_1^1 \) \((j = 0, n)\) and \( \mathcal{X}^0 \), respectively. Here, \( H = M_0^{-1}L_0 \in \mathcal{L}(\mathcal{X}^0) \) is nilpotent of degree \( p \in \mathbb{N}_0 \), the operators \( S_{1j} = L_{1j}^{-1}M_{1j} \in \mathcal{C}(\mathcal{X}_1^1) \), moreover, \( \sigma(S_{1j}) = \sigma_j^1(M); g^0 = (\mathbb{I}_0 - Q)g \), \( g_j^1 = Q_j g \), \( x^0 = (\mathbb{I}_X - P)x \), \( x_j^1 = P_j x \), \( j = 0, n \).

The solution of \((21)\) for \( j = \overline{1, n} \) is found using the constant variation method. Represent \( x_j^1(t) = X(t, \tau_j)z_j(t) \), substitute it into \((21)\) and obtain

\[
a(t)L_{1j}^{-1}MX(t, \tau_j)z_j(t) + X(t, \tau_j)\dot{z}_j(t) = a(t)L_{1j}^{-1}MX(t, \tau_j)z_j(t) + L_{1j}^{-1}g_j(t),
\]

thus \( \dot{z}_j(t) = [X(t, \tau_j)]^{-1}L_{1j}^{-1}g_j(t) = X(\tau_j, t)L_{1j}^{-1}g_j(t) \) and \( z_j(\tau_j) = P_j x_j \).

Integrating in the range from \( \tau_j \) to \( t \), we obtain \( z_j(t) = P_j x_j + \int_{\tau_j}^{t} X(\tau_j, s)L_{1j}^{-1}g_j^1(s)ds \), hence

\[
x_j^1(t) = X(t, \tau_j)P_j x_j + \int_{\tau_j}^{t} X(t, s)L_{1j}^{-1}g_j^1(s)ds = X(t, \tau_j)P_j x_j - \int_{t}^{\tau_j} X(t, s)L_{1j}^{-1}g_j^1(s)ds.
\]
The solution of (21) for \( j = 0 \) has the form

\[
x_0^1(t) = X(t, \tau_0)P_0x_0 + \int_{\tau_0}^t X(t, s)L_{10}^{-1}g_0^1(s)ds,
\]

which is directly verified, similarly to [22, Theorem 6.1.5].

Now consider problem (22). From (22) we obtain

\[
x^0(t) = H \frac{\dot{x}^0(t)}{a(t)} - M_0^{-1}g^0(t) \tag{23}
\]

Using (23) we get the equality

\[
Hx^0(t) = H^2 \frac{d}{dt} \left( \frac{\dot{x}^0(t)}{a(t)} \right) - HM_0^{-1} \frac{d}{dt} \left( \frac{g^0(t)}{a(t)} \right)
\]

and, in view of (22), we obtain

\[
a(t)x^0(t) + M_0^{-1}g^0(t) = H^2 \frac{d}{dt} \left( \frac{\dot{x}^0(t)}{a(t)} \right) - HM_0^{-1} \frac{d}{dt} \left( \frac{g^0(t)}{a(t)} \right)
\]

and

\[
x^0(t) = H^2 \frac{1}{a(t)} \frac{d}{dt} \left( \frac{\dot{x}^0(t)}{a(t)} \right) - HM_0^{-1} \frac{1}{a(t)} \frac{d}{dt} \left( \frac{g^0(t)}{a(t)} \right) - M_0^{-1}g^0(t).
\]

Repeating these transformations and using the nilpotency of the operator \( H \), we obtain

\[
x^0(t) = -\sum_{k=0}^{p} H^k M_0^{-1} \left( \frac{1}{a(t)} \frac{d}{dt} \right)^k \frac{g^0(t)}{a(t)}.
\]

Therefore, the solution of (17), (19) has the form \( x(t) = x^0(t) + \sum_{j=0}^{n} x_j^1(t) \).

Construct the space \( H^{p+1}(\mathcal{G}) = \{ \xi \in L_2(\tau_0, \tau_n; \mathcal{G}) : \xi^{(p+1)} \in L_2(\tau_0, \tau_n; \mathcal{G}), p \in \mathbb{N}_0 \} \), which is a Hilbert space in view of the Hilbert property of \( \mathcal{G} \) with an inner product

\[
[\xi, \eta] = \sum_{q=0}^{p+1} \int_{\tau_0}^{\tau_n} \left\langle \xi^{(q)}, \eta^{(q)} \right\rangle_{\mathcal{G}} dt.
\]

**Definition 3.4.** A vector-function \( x \in H^1(\mathcal{X}) = \{ x \in L_2(\tau_0, \tau_n; \mathcal{X}) : \dot{x} \in L_2(\tau_0, \tau_n; \mathcal{X}) \} \) is called a **strong solution of** (17), if this function converts the equation into an identity almost everywhere on \((\tau_0, \tau_n)\). A strong solution \( x = x(t) \) of (17) is called a **strong solution of the multipoint initial-final problem** (17), (19), if it satisfies conditions (19).

**Theorem 3.5.** Let \( M \) be an \((L, p)\)-radial operator \((p \in \mathbb{N}_0)\), \( a \in C^{p+1}([\tau_0, \tau_n]; \mathbb{R}_+) \) and conditions (7), (8), (18) be satisfied. Then, for any \( x_j \in \mathcal{X}_j^1 \) \((j = 0, n)\) and vector-function \( g : (\tau_0, \tau_n) \rightarrow \mathcal{G} \), such that

\[
(\mathbb{I}_\mathcal{G} - Q)g \in H^{p+1}(\mathcal{G}) \quad \text{and} \quad Q_j g \in L_2(\tau_0, \tau_n; \mathcal{G}_j^1) \quad (j = 0, n),
\]

there exists a unique strong solution \( x \in H^1(\mathcal{X}) \) of the multipoint initial-final problem (17), (19), given by (20).

**Proof.** In view of Theorem 3.3 it is clear that the function, given by (20), satisfies (17), (19). We verify that the function \( x(t) \) belongs to the required class. Suppose that \( g(t) \) satisfies the conditions of the theorem; then the third term of (20) belongs to \( H^1(\mathcal{X}) \), and functions...
Remark 3.6. It is clear that if vector-function \( g \in H^{p+1}(\mathcal{Q}) \), then condition (24) is fulfilled automatically.

4. The optimal control problem. Let \( \mathcal{X}, \mathcal{Q} \) and \( \mathcal{U} \) be Hilbert spaces, \( L \in \mathcal{L}(\mathcal{X}; \mathcal{Q}) \), \( M \in \mathcal{C}(\mathcal{X}; \mathcal{Q}) \) be given operators, \( M \) be an \((L, p)\)-radial operator \( (p \in \mathbb{N}_0) \) and conditions (7), (8), (18) be satisfied. On the interval \((\tau_0, \tau_n)\) (\(0 = \tau_0 < \tau_n < +\infty\)) consider the multipoint initial-final problem (19) for equation

\[
L \dot{x}(t) = a(t)Mx(t) + g(t) + Bu(t),
\]

where operator \( B \in \mathcal{L}(\mathcal{U}; \mathcal{Q}) \) and function \( a \in C([\tau_0, \tau_n]; \mathbb{R}_+) \). If the operator \( M \) is an \((L, p)\)-radial operator \( (p \in \mathbb{N}_0) \), as well as conditions (7), (8), (18) are satisfied for function \( a \in C^{p+1}([\tau_0, \tau_n]; \mathbb{R}_+) \), vector-function \( u \in H^{p+1}(\mathcal{U}) \) and \( g : (\tau_0, \tau_n) \to \mathcal{Q} \) satisfying (24), then, in view of Theorem 3.5, for any \( x_j \in \mathcal{X}_j \) \((j = 0, \ldots, n)\) there exists the solution of the multipoint initial-final problem (19), (25).

Let \( \mathfrak{F} \) be a Hilbert space and the operator \( C \in \mathcal{L}(\mathfrak{F}; \mathfrak{F}) \). Consider the optimal control problem for (19), (25) in the form

\[
J(x, u) = \sum_{q=0}^{\tau_n} \int_0^{\tau_q} \|z^{(q)} - z_d^{(q)}\|^2_\mathfrak{F} dt + \sum_{q=0}^{p+1} \int_0^{\tau_n} \langle N_q u^{(q)}, u^{(q)} \rangle_{\mathcal{U}} dt \to \inf,
\]

where \( z = Cx, N_q \in \mathcal{L}(\mathfrak{F}) \) \((q = 0, 1, \ldots, p+1)\) are self-adjoint and positively defined operators, \( z_d = z_d(t) \in \mathfrak{F} \) is a target observation. Note that if \( x \in H^1(\mathcal{X}) \), then \( z \in H^1(\mathfrak{F}) \). Similarly to \( H^{p+1}(\mathcal{Q}) \), consider the space \( H^{p+1}(\mathcal{U}) \), which is the Hilbert space in view of the Hilbert property of \( \mathcal{U} \). Let \( \mathcal{U}_{ad} \) be a non-empty, closed and convex subset of the space \( H^{p+1}(\mathcal{U}) \), which is the set of admissible controls.

Consider a strong solution of the multipoint initial-final problem (19), (25) as a mapping \( D : H^{p+1}(\mathcal{U}) \to H^1(\mathcal{X}) \), given by

\[
Du = \sum_{j=0}^{n} \left( X(t, \tau_j)P_j x_j - \int_0^{\tau_j} X(t, s) P_j L^{-1}M_0(\mathcal{Q}) (g(s) + Bu(s)) ds \right) - \\
- \sum_{k=0}^{p} H^k M_0^{-1}(\mathcal{Q}) \left( \frac{1}{a(t)} \frac{d}{dt} \right)^k \frac{g(t) + Bu(t)}{a(t)}.
\]
Lemma 4.1. Let $X$, $Y$ and $\Omega$ be Hilbert spaces, $M$ be an $(L, p)$-radial operator ($p \in \mathbb{N}_0$), conditions (7), (8), (18) be satisfied, $a \in C^{p+1}([\tau_0, \tau_n]; \mathbb{R}_+)$ also let elements $x_j \in X_j^+$ ($j = 0, n$) and vector-function $g : (\tau_0, \tau_n) \to Y$ satisfying (24) be given. Then the mapping $D : H^{p+1}(\Omega) \to H^1(X)$, given by (27), is continuous.

Proof. Take $u_1, u_2 \in H^{p+1}(\Omega)$ and consider

$$Du_2 - Du_1 = \sum_{j=0}^n \int_{\tau_j}^{\tau_{j+1}} X(t, s) P_j L_j^{-1} Q_j B(u_1(s) - u_2(s)) \, ds - \sum_{k=0}^p H^k M_k^{-1}(I_2) - Q \left( \frac{1}{a(t)} \frac{d}{dt} \right)^k B(u_2(t) - u_1(t)) \frac{a(t)}{a(t)}.$$ 

By the properties of semiflow $X(t, s)$, continuity of $a(t)$ for $t \in [\tau_0, \tau_n]$ and the fact that the operator $B \in L(L, \Omega))$ we have

$$\|Du_2 - Du_1\|_{H^1(X)} \leq K_1 \|u_2 - u_1\|_{H^{p+1}(\Omega)}.$$ 

Remark 4.2. In view of Lemma 4.1, functional (26) depends only on the control function $u(t)$. Namely,

$$J(u) = \frac{1}{2} \sum_{q=0}^{\tau_n} \|C x^{(q)} - z^{(q)}_d\|_2^2 + \sum_{q=0}^{\tau_n} \int_{\tau_0}^{\tau_n} \left\langle N q(u(q), u^{(q)}) \right\rangle \, dt.$$ 

Definition 4.3. A vector-function $v \in \Phi_{ad}$ is called an optimal control for problem (19), (25), if

$$J(v) = \inf_{u \in \Phi_{ad}} J(u).$$ 

Theorem 4.4. Let $M$ be an $(L, p)$-radial operator ($p \in \mathbb{N}_0$), conditions (7), (8), (18) be satisfied and $a \in C^{p+1}([\tau_0, \tau_n]; \mathbb{R}_+)$ also let elements $x_j \in X_j^+$ ($j = 0, n$) and vector-function $g : (\tau_0, \tau_n) \to Y$ satisfying (24), there exists a unique optimal control $v \in \Phi_{ad}$ for problem (19), (25), (29) with the cost functional (28).

Proof. Using the mapping $D$ from Lemma 4.1, rewrite the cost functional (28) in the form

$$J(u) = \|C x - z_d\|_{H^{1}(3)}^2 + [\eta, u],$$

where $x = Du$, $[\eta, u] = \sum_{q=0}^{\tau_n} \int_{\tau_0}^{\tau_n} \left\langle \eta^{(q)}, u^{(q)} \right\rangle \, dt$ and $\eta^{(q)}(t) = N_q u^{(q)}(t)$ ($q = 0, p+1$). Denote $\hat{x} = D\hat{u}$ for $\hat{u} \equiv 0$. In other words, $\hat{x}$ is given by (20). Hence,

$$J(u) = \beta(u, u) - 2 \gamma(u) + \|z_d - C\hat{x}\|_{H^1(3)}^2,$$

where $\beta(u, u) = \|C(x - \hat{x})\|_{H^1(3)}^2 + [\eta, u]$ is a bilinear continuous coercive form on $H^{p+1}(\Omega)$ and

$$\gamma(u) = \langle z_d - C\hat{x}, C(x - \hat{x}) \rangle_{H^1(3)}$$

is a linear and continuous on $H^{p+1}(\Omega)$ form. Therefore, the statement of the theorem follows from [17, Chapter 1].

\[\square\]
5. Optimal control of solutions of the Chen – Gurtin equation with complex coefficients. Let \( \Omega \subset \mathbb{R}^m \) be a bounded domain with boundary \( \partial \Omega \) of \( C^\infty \) class. Reduce equation

\[
(\lambda - \Delta)x_t(r,t) = a(t)(\Delta - id\Delta^2)x(r,t) + u(r,t), \quad (r, t) \in \Omega \times (\tau_0, \tau_n),
\]
where \( \lambda, d \in \mathbb{R} \), with boundary conditions

\[
\Delta x(r, t) = x(r, t) = 0, \quad (r, t) \in \partial \Omega \times (\tau_0, \tau_n)
\]

to (25). Suppose that \( x \in W^2_\alpha(\Omega) \cap W^1_\alpha(\Omega) \), \( \mu \in \mathbb{R} \), \( r \in \partial \Omega \).

Lemma 5.1. [20] For any \( \lambda \in \mathbb{R} \setminus \{0\} \) and \( d \in \mathbb{R} \) operator \( M \) is \( (L, 0) \)-radial and conditions (7), (8) hold.

The statement of Lemma 5.1 follows from [20, Lemma 3.1].

Let \( \{\lambda_k\} \) be a sequence of eigenvalues of the homogeneous Dirichlet problem for the Laplace operator \( \Delta \) in the domain \( \Omega \). Suppose that the sequence \( \{\lambda_k\} \) is numbered in a non increasing order taking into account the multiplicities. Denote by \( \{\varphi_k\} \) the orthonormal (in the sense of \( L_2(\Omega) \)) sequence of corresponding eigenfunctions \( \varphi_k \in C^\infty(\Omega) \), \( k \in \mathbb{N} \). The \( L \)-spectrum of the operator \( M \) has the form

\[
\sigma^L(M) = \left\{ \mu_k = \frac{\lambda_k - id\lambda_k^2}{\lambda - \lambda_k}, k \in \mathbb{N} \setminus \{l : \lambda_l = \lambda\} \right\}.
\]

Suppose that \( \bigcup_{j=0}^n D_j \supset \sigma^L(M) \) and each of the domains \( D_j \ (j = \overline{0,n}) \) contains a finite number of points from \( \sigma^L(M) \). Denote \( \sigma^L_j(M) = \sigma^L(M) \cap D_j \) and construct the projections

\[
P_j = \sum_{k : \mu_k \in \sigma^L_j(M)} \langle \cdot, \varphi_k \rangle \varphi_k, \quad j = \overline{0,n}.
\]

Take \( x_j \in \mathcal{X} \) and \( \tau_j \in \mathbb{R}^+ \ (j = \overline{0,n}) \) such that \( \tau_0 = 0 \) and \( \tau_{j-1} < \tau_j \ (1 \leq j \leq n) \). In the cylinder \( \Omega \times (\tau_0, \tau_n) \), we find the solution of equation (30), satisfying (31) and multipoint initial-final conditions

\[
P_j(x(\tau_j) - x_j) = \sum_{k : \mu_k \in \sigma^L_j(M)} \langle x(\tau_j) - x_j, \varphi_k \rangle \varphi_k = 0, \quad j = \overline{1,n},
\]

\[
\lim_{t \to 0^+} P_0(x(t) - x_0) = \lim_{t \to 0^+} \sum_{k : \mu_k \in \sigma^L_0(M)} \langle x(t) - x_0, \varphi_k \rangle \varphi_k = 0.
\]

The theorem below follows from Theorem 3.5, Remark 3.6 and Lemma 5.1.

Theorem 5.2. For any \( \lambda \in \mathbb{R} \setminus \{0\} \), \( d \in \mathbb{R} \), \( a \in C^1(\{\tau_0, \tau_n]\; \mathbb{R}^+) \), as well as for any \( x_j \in \mathcal{X}_j \ (j = \overline{0,n}) \) and for any \( u \in H^1(\Omega) \), there exists a unique solution \( x \in H^1(\mathcal{X}) \) of the multipoint initial-final problem (30) – (32), given by

\[
x(r, t) = - \sum_{\lambda_k = \lambda} \frac{\langle u(t), \varphi_k \rangle}{a(t)(\lambda_k - id\lambda_k^2)} \varphi_k(r) +
\]
Finally, due to Theorem 4.4, Lemma 5.1 and Theorem 5.2, the following theorem is true.

**Theorem 5.3.** For any \( \lambda \in \mathbb{R} \setminus \{0\}, \ d \in \mathbb{R}, \ a \in C^1([\tau_0, \tau_n]; \mathbb{R}^+), \) as well as for any \( x_j \in \mathbb{X}_j (j = 0, n), \) there exists a unique optimal control \( v \in \mathcal{U}_{ad} \) for (28) – (32).

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*E-mail address: sagadeevama@susu.ru*

*E-mail address: zagrebinasa@susu.ru*

*E-mail address: manakovana@susu.ru*