A description of the target wave-function encoded in the source terms

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We argue that the gauge invariant source terms in the formalism of the Color Glass Condensate (CGC) should properly describe the target wave-function if the microscopic structure is concerned in the regime where the parton distribution is not quite dense. The quantum property of color charge density is incorporated in the quantum weight function defined with the source terms. We sketch that the logarithmic source terms encompass a meaningful picture of the microscopic structure of the target wave-function.

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Introduction

The formalism of the Color Glass Condensate (CGC) is elaborated suitably to describe energetic matter in Quantum Chromodynamics (QCD) \[ \frac{1}{x} \frac{d}{dx} \langle x \rangle \]. When the target material like a nucleus whose dynamics is mainly governed by QCD is highly boosted along the positive z direction, more and more partons with small fraction of the longitudinal momentum \( p^+ = (p^0 + p^3) / \sqrt{2} \) take part in the dynamics. Such partons are commonly called the wee partons. The basic idea of the CGC is as follows; the wee partons with \( p^+ \) smaller than a certain separation scale \( \Lambda \) distribute densely at sufficiently high energy that the wee parton distribution is well coherent and can be described by the classical field \( A^\mu(x) \). The description is analogous to classical electromagnetism then once the saturation is reached. The wee parton distribution is to be approximated by solving the classical equations of motion, i.e. Yang-Mills-Maxwell equations. The color source \( \hat{\rho}^a(x) \) which generates \( A^\mu_a(x) \) is provided by fast partons, namely, partons with \( p^+ \) larger than \( \Lambda \).

The name of the CGC is widely understood: Color is the quantum number relevant to QCD partons. Glass is not crystalline microscopically but static enough as compared with our daily time scale. Likewise, the fast-moving partons are relatively static seen from the wee partons due to the time dilation. Condensate means that the wee partons are dense and coherent represented by a classical field like the field-theoretical description of the Bose-Einstein Condensate (BEC). We note, however, that the physics of the CGC is totally different from that of the BEC. The CGC is not a state of matter, but rather an effective description of matter.

The separation scale \( \Lambda \) is physically specified by Bjorken-x which intuitively signifies the longitudinal momentum fraction between the target’s \( P^+ \) and the parton’s \( p^+ \) in the infinite momentum frame where \( P^+ \sim \infty \). The random distribution of \( \hat{\rho}^a(x) \) depends on \( \Lambda \) or \( x \) through the weight function \( W_\Lambda[\hat{\rho}] \). The differential equation of \( W_\Lambda[\hat{\rho}] \) with respect to \( \Lambda \) is the evolution equation by which one can go toward smaller \( \Lambda \) and thus smaller \( x \). The evolution equation is an essential tool to investigate the small-x physics. In the leading-logarithmic approximation the CGC formalism leads us to the evolution equation known as the Jalilian-Marian-Iancu-McLerran-Weigert-Leonidov-Kovner (JIMWLK) equation \[ \frac{d}{dx} \langle x \rangle \frac{d}{dx} \langle x \rangle \]. Although the JIMWLK equation is successful when the color charge density \( \hat{\rho}^a(x) \) is as large as \( \sim 1/g \) in the saturation region of \( Q^2 \) and \( x \), it has turned out that an important process in lower density regimes is missing in the JIMWLK dynamics. That is the fluctuation of the parton (gluon) number in the target wave-function. The JIMWLK equation correctly grasps the saturation mechanism by the merging process of the wee partons. The basic idea of the CGC is as follows; the wee partons with \( \hat{\rho}^a(x) \) travels through the gauge field \( A^\mu_a(x) \) created by the projectile, then the scattering amplitude earns an eikonal phase \( \exp[-i\hat{\rho}^a(x)A^\mu_a(x)] \), that means there is a sort of uncertainty principle between the target and the projectile states. Thus, if the eigenstate of the target wave-function is chosen represented in terms of \( \hat{\rho}^a(x) \), the projectile wave-function is uncertain represented in terms of \( i\delta/\delta\hat{\rho}^a(x) \), which is analogous to \( 2\pi \delta(x-x) \) in Quantum Mechanics. The JIMWLK equation contains the full order in \( \rho^a(x) \) but only the first and second order in \( i\delta/\delta\hat{\rho}^a(x) \) under the assumption that \( \hat{\rho}^a(x) \) is large (and thus \( i\delta/\delta\hat{\rho}^a(x) \) is small).

The higher order term in \( i\delta/\delta\hat{\rho}^a(x) \) has been added in a way consistent with the color dipole model (DM) in Refs. \[ 12, 13, 14 \] and the general evolution kernel or the Hamiltonian including full order in both \( \hat{\rho}^a(x) \) and \( i\delta/\delta\hat{\rho}^a(x) \) has been achieved in Refs. \[ 15, 16, 17, 18 \]. The state evolving according to the Hamiltonian should be specified to write down an evolution equation. Thus, Refs. \[ 16, 19, 21 \] are seminal works in which they showed that the DM weight function evolves properly according to the general Hamiltonian in the dilute regime (Bremsstrahlung Hamiltonian) in the large \( N_c \) limit.

It is not quite clear, however, how we can push the conventional strategy to derive the JIMWLK equation toward a lower density region. Apparently, for one example, the derivation of the general Hamiltonian in Ref. \[ 17 \] does not rely on any information of the target nor projectile wave-function, while the JIMWLK equation deals with the evolution of the target wave-function. Its clear manifestation is; when one derives the JIMWLK equation, one needs to have
the gauge invariant source terms that induce complicated and non-local vertices. In contrast, the general Hamiltonian shows itself without complicated vertices associated with the source action at all. This seemingly mystical simplicity as compared with the derivation of the JIMWLK equation is to be understood as follows. The source terms should carry the information of the target wave-function. The general Hamiltonian works when it acts either on the target wave-function that is embodied as the weight function with the source terms as we will see, or on the projectile wave-function that is supplied as an operator to be evaluated with the weight function. Even though the Hamiltonian is reduced to the JIMWLK kernel in the dense case, it is not the JIMWLK equation until the weight function is specified. Since the parton distribution is highly dense in the JIMWLK problem, the concrete form of the source terms is not relevant to the final result. In the dilute regime, however, the explicit form is significant as demonstrated in the choice of the weight function as in the DM in Ref. [26].

We will start with the generating functional and briefly review how the CGC picture arises from the stationary-point approximation. Then, we illustrate a simple algebraic trick proposed by the present author in Refs. [21, 22] to treat the quantum nature of color charge density. The benefit of this trick is also that the physical meaning of the source terms is transparent. Actually it becomes clear that the source terms should describe the target wave-function, that is, in the dilute regime the source terms should comply with the microscopic structure of the target wave-function. We will claim that the logarithmic form would be a likely candidate.

**Stationary-point approximation** In the presence of the color source that distributes with the weight function $W_{\tau}[\bar{\rho}]$, the QCD generating functional is modeled as

$$Z[j] = \int D\bar{\rho} W_{\tau}[\bar{\rho}] \int^{\tau} D\bar{A} \delta[A^+] \exp \left\{ i S_{YM}[\bar{A}] + i S_W[\bar{\rho}] - i \int d^4x j_+^a(x)A_\mu^a(x) \right\}$$

(1)

in the light-cone gauge. Here $S_{YM}[\bar{A}]$ is the Yang-Mills action and $S_W[\bar{\rho}]$ denotes the source action which is given in terms of the gauge invariant variable,

$$W[A^-](x^-, \vec{x}) = \mathcal{P}_{x^+} \exp \left\{ ig \int_{-\infty}^{\infty} dx^+ A^- (x^+, x^-, \vec{x}) \right\},$$

(2)

when the background field comes only from the target moving along the $x^+$ direction. It should be noted that $A^-$ in the above expression is a color matrix either in the fundamental or adjoint representation in accord with the constituent of the target. It is the central issue of our interest how we should fix the concrete form of $S_W[A^-]$. In writing Eq. (1) we introduced the rapidity variable, $\tau = \ln[P^+/\Lambda] = \ln[1/x]$, instead of the separation scale $\Lambda$ or Bjorken-$x$. Then, one can compute the expectation value of an operator $\mathcal{O}[\bar{A}]$ as

$$\langle \mathcal{O}[\bar{A}] \rangle_\tau = \mathcal{O}[i \delta/\delta j] Z[j] \big|_{j=0} / Z[0].$$

(3)

The CGC results from the stationary-point approximation on Eq. (1). When $\bar{\rho}(x)$ is as large as $\sim 1/g$ at small $x$, the classical approximation works well, so that the functional integral over $A_\mu^a$ can be estimated at the stationary point determined by

$$\left( \frac{\delta S_{YM}}{\delta A_\mu^a} + \frac{\delta S_W}{\delta A_\mu^a} \right)_{\bar{A}=\bar{A}} = 0.$$  

(4)

Then the expectation value (3) is approximated as

$$\langle \mathcal{O}[\bar{A}] \rangle_\tau = \int D\bar{\rho} W_{\tau}[\bar{\rho}] \mathcal{O}[\bar{A}[\bar{\rho}]]$$

(5)

under the assumption that $\mathcal{O}[\bar{A}]$ would hardly affect the stationary condition. In case of computing the scattering amplitude between a heavy target of the CGC and a simple projectile like the color dipole, the above stationary-point approximation is acceptable, while one needs to solve the two-source problem at the classical level if one intends to compute observables like the gluon or quark production associated with the scattering. The leading term of $S_W[\bar{\rho}]$ should reproduce the eikonal form $- \int d^4x \bar{\rho}(\vec{x})A^-_\mu(x^-)$ from the equations of motion (4) the charge density inferred from the source terms should satisfy the covariant conservation, which is fulfilled whenever the source terms are written in terms of the matrix elements of Eq. (2) as one can easily confirm

$$\partial^- \delta^{ab} + g f^{abc} J^+_c = 0,$$

$$J^+_b \propto \left( W_{\infty, x^+} i g T^b W_{x^+,-\infty} \right)_{ij}.$$  

(6)
Here $T^b$ is an element of the SU($N_c$) algebra. The covariant conservation holds for both the fundamental and the adjoint Wilson lines. In Eq. (4) we denoted the Wilson lines with the $x^+$ integration ranging from $x^+ = \infty$ and from $-\infty$ to $x^+$ as $W_{\infty,x^+}$ and $W_{x^+,-\infty}$ respectively.

In the dense regime there is no further restraint onto the choice of $S_W[\bar{\rho}]$ and, though it is not rigorously proved yet, there is likely a sort of universality and the concrete form of $S_W[\bar{\rho}]$ is not relevant to the evolution equation, i.e., the JIMWLK equation.

A further constraint appears in the dilute regime where the color charge distribution in the target is not quite dense. For that purpose it is convenient to transform the generating functional into the following representation:

$$Z[j] = \int \mathcal{D}\rho \mathcal{N}_\tau[\rho] \int DA \, \delta[A^+] \exp\left\{ iS_{YM}[A] - i \int d^4x \, \rho^a(x) A^a_\mu(x) - i \int d^4x \, j^\mu_\tau(x) A^\mu_\tau(x) \right\},$$

where

$$\mathcal{N}_\tau[\rho] = \int D\rho \mathcal{W}_\tau[\bar{\rho}] \exp\left\{ iS_W[-i\delta/\delta\rho, \bar{\rho}] \right\} \delta[\rho].$$

This is just a simple rewriting, and nevertheless, it can bridge the formalism toward a DM-like picture as we will see. Later we will closely discuss the physical meaning of the expression $S_W[\bar{\rho}]$.

It is a notable feature of this representation that one can see the non-commutative nature of color charge density manifestly. The stationary-point of Eq. (4) is determined by the equations of motion:

$$\left( \frac{\delta S_{YM}}{\delta A^a_\mu} - \delta_{\mu-} \rho^a \right)_{A=A_\tau} = 0,$$

and the expectation value of $O[A]$ is then approximated as

$$\langle O[A] \rangle_{\tau} = \int D\rho \mathcal{N}_\tau[\rho] \, O[A[\rho]].$$

It is important to note that $\rho^a(x)$ is dependent on $x^+$, which actually takes care of the full account of $\delta S_W/\delta A^a_\mu$ in Eq. (4). Therefore, we can no longer assume that the color source $\rho^a(x)$ is static once the non-commutativity plays a role. In the functional integral formalism we have the commutation relation as follows;

$$\langle \rho^a(x^+, \vec{x}) \rho^b(x^++\eta, \vec{y}) \rangle_{\eta \to 0^+} - \rho^b(x^++, \vec{y}) \rho^a(x^++\eta, \vec{x})\rangle_{\eta \to 0^+}$$

\[= - (ig)^2 \int D\rho \mathcal{N}_\tau[\rho] \left\{ \rho^a(x^+, \vec{x}) \rho^b(x^+ - \eta, \vec{y}) - \rho^b(x^+, \vec{y}) \rho^a(x^+ - \eta, \vec{x}) \right\}_{\eta \to 0^+} \]

\[= - ig f^{abc} \cdot i (ig) \int D\rho \mathcal{N}_\tau[\rho] \frac{\delta iS_W}{\delta W_{ij}} (W_{\infty,x^+} [T^a, T^b] W_{x^+, -\infty})_{ij} \delta(3) (\vec{x} - \vec{y}) \]

\[= - i g f^{abc} \rho^c(x^+) \delta(3) (\vec{x} - \vec{y}). \]

Here we utilized the following relation,

$$\int dx \left\{ \frac{-d}{dx} \right\} \delta(x) x = \int dx \left\{ f(0) - i f'(0) \frac{d}{dx} \cdots \right\} \delta(x) x = i f'(0).$$

Strictly speaking, in fact, the right hand side of Eq. (11) is zero after average. One can insert an appropriate operator to make the expectation value nonvanishing and thus Eq. (11) should be understood in such a sense. What we have seen here is essentially equivalent with the argument by Kovner and Lublinsky in Ref. [24], though they just mentioned a similar form encountered in the DM as a motivation. We emphasize that the argument so far is applicable for any $S_W[\bar{\rho}]$ as long as $S_W[\bar{\rho}]$ is a function of the Wilson line $W_\tau[\rho]$. In the previous works [21, 22] the present author called $N_\tau[\rho]$ the density of states. Here it would make sense to call $N_\tau[\rho]$ the quantum weight function since it properly handles the quantization relation.

Let us shortly comment on a symmetric representation of the scattering problem of light materials. In that case, as the target is a right mover, we shall denote the above $\bar{\rho}^a(x)$, $W_\tau[\bar{\rho}]$, and $S_W[\bar{\rho}]$ as $\bar{\rho}_R^a(x)$, $W_R^R[\bar{\rho}_R]$, and $S_W^R[\bar{\rho}_R]$ specifically. In general the wave-function of the left-moving projectile can be expressed as

$$O[A] = \int D\bar{\rho}_L W_{L,-}[\bar{\rho}_L] \exp\left\{ iS_W^L[A^+, \bar{\rho}_L] \right\}$$

\[= \int D\bar{\rho}_L D\bar{\rho}_L W_{L,-}[\bar{\rho}_L] \exp\left\{ iS_W^L[-i\delta/\delta\rho_L, \bar{\rho}_L] - i \int d^4x \, \rho^a_L(x) A^a_\tau(x) \right\} \delta[\rho_L]. \]
with $Y$ being the total relative rapidity of the target and projectile. Then, if the scattering amplitude is concerned and $A^+$ in Eq. (13) is simply approximated as $A^+[\rho]$ like in Eq. (10), the scattering amplitude is evaluated as

$$S_Y = \int D\rho D\bar{\rho} D\bar{\rho}_R W_R^\dagger[\bar{\rho}_R] W_{Y-\tau}[\bar{\rho}_L] \exp\{i S_W^\dagger[-i\delta/\delta\rho,\bar{\rho}_R] + i S_W[\rho],\bar{\rho}_L]\} \delta[\rho].$$

(15)

This expression makes sense in the dilute-dilute or dense-dense (where the dense-dilute duality realizes) scatterings, but not in the dense-dense case in which the classical solution $A^+_N[\rho]$ must be evaluated in the presence of two dense sources of the target and projectile. The color glass representation of the onium-onium scattering like as formulated in Ref. [32] is one manifestation of the above. If one is interested in an expectation value of some other operators, such as in the gluon production and in the quark production and so on, or if one is interested in the dense-dense case, one has to use Eq. (14) and solve the two-source problem with $p_R(x)$ and $p_L(x)$ as analyzed in Refs. [22, 24, 27, 28, 29, 30, 31]. Now let us return to the mainstream of this work and discuss the choice of the source terms.

**Choice of the source terms**
We know that in the large $N_c$ limit the DM weight function captures the correct description of the target wave-function in the dilute regime. The dressed dipole weight function reads $[13, 27, 32]$

$$Z'^{DM}[\rho] = \sum_{N=1}^{\infty} \int d\Gamma_N P_N^{DM}(\{z_i\}; \tau) \prod_{i=1}^{N} D^i(z_{i-1}, z_i) \delta[\rho],$$

(16)

where $N$ is the number of dipoles and $d\Gamma_N$ means the integration over all the phase space, namely, $d\Gamma_N = dz_1 \cdots dz_N$ and $D^i(x, y)$ is the creation operator of a dressed dipole located at $x$ and $y$ in transverse space, that is,

$$D^i(x, y) = \frac{1}{N_c} \text{tr}[W(x)W^+ (y)]$$

(17)
in the fundamental representation. In the DM the longitudinal structure in the $x^-$ direction is neglected and the Wilson lines do not depend on the longitudinal coordinate. Now one may well notice that Eq. (16) is a special case of the quantum weight function $[8]$. In other words, the source terms $S_W[A^-]$ must be chosen such that the quantum weight function can reproduce Eq. (10) somehow in the large $N_c$ limit:

$$N_c^{\tau}[\rho] \rightarrow Z'^{DM}[\rho] \quad \text{as} \quad N_c \rightarrow \infty.$$  

(18)

It is important to mention that we always mean the *dressed* dipoles written in terms of the Wilson line. Actually, in the weak field limit, any source terms which satisfy $S_W[A] = -\int d^3 x \bar{\rho}^a(x)A^-_a(x) + \cdots$ would lead to the results of the (undressed) DM if only an ensemble of the dipole-type color source,

$$\bar{\rho}^a(\vec{x}) = Q^a \delta(x^-)\{\delta^{(2)}(x - x_0) - \delta^{(2)}(x - y_0)\},$$

(19)

where $Q^a$ is a Gaussian random variable $[22]$, is allowed by $W_L[\bar{\rho}]$.

The conventional choice adopted in literatures $[4, 5, 6, 7, 8, 9, 10]$ does not meet the requirement of Eq. (18); whatever $W_L[\bar{\rho}]$ cannot lead to the form of the weight function (10). Therefore, the conventional form of the source terms is no longer a right choice microscopically and is ruled out. It is instructive to look further into why the conventional choice is not good. If the above conventional source terms are substituted into the weight function $[8]$ and the exponential is expanded, an arbitrary number of $W$ is allowed at each point. The higher order $W$ terms cannot be controlled by the distribution $W_L[\bar{\rho}]$ and they are not compatible to the specific form of (10). To put it in a more formal way, this incompatibility seems to stem from that the color charge density belonging to the *algebra* and the Wilson line belonging to the *group* are put on the equal footing, which is quite unnatural.

One possibility would be the eikonal coupling with the Wess-Zumino action $[53, 34, 35]$ that takes care of the non-commutative nature of color charge density just like our $N_c^{\tau}[\rho]$. The Wess-Zumino action $S_{WZ}[\rho]$ is a function of $\rho^a(x)$. In principle $S_{WZ}[\rho]$ is given in terms of $W[-i\delta/\delta\rho]$, but in fact, $W$ is implicit in the expression of $S_{WZ}[\rho]$ given in Refs. [32, 27] and it is non-trivial how to retrieve $W$ out of $S_{WZ}[\rho]$. We have no idea how to derive the microscopic information of the target wave-function, not going back to the representation as a product of $W$ but directly using $S_{WZ}[\rho]$ on its own.
Instead, we shall here focus on another specific possibility of the source terms. We will check whether the following logarithmic form \[21, 22, 36\],

\[
S_W[A^-, \bar{\rho}] = i \frac{2}{g} \int d^3x \text{tr} \left\{ \bar{\rho}(\bar{x}) \ln W[A^-](\bar{x}) \right\},
\]  
(21)
in the fundamental representation, or

\[
S_W[A^-, \bar{\rho}] = i \frac{1}{gN_c} \int d^3x \text{tr} \left\{ \bar{\rho}(\bar{x}) \ln \widetilde{W}[A^-](\bar{x}) \right\},
\]  
(22)
in the adjoint representation \((\widetilde{W} \text{ is defined with the adjoint generators, and so is } \bar{\rho}(\bar{x}) \text{ implicitly), is qualified or not as a microscopically meaningful choice. In this form we note that } \ln W \text{ belongs to the algebra unlike } W, \text{ so that the coupling to } \bar{\rho}^{a}(x) \text{ looks reasonable.}

We shall develop one more argument besides Refs. \[21, 22, 36\] that the logarithmic form \((21)\) is motivated. We know that the solution of the covariant conservation \((6)\) takes a form of

\[
\frac{\partial}{\partial t} \rho(x) + \nabla \cdot J(x) = 0,
\]
(23)
with the initial condition being \(\bar{\rho}_{-\infty}\) at infinite past. Using the relations \(W^{ba} = 2tr[t^{b}Wt^{a}]\) and \(W^{\dagger}_{\infty,-\infty} = W^{\dagger}_{\infty,-\infty}W_{\infty,x^{+}}\), we can equivalently rewrite the above as

\[
J_{b}^{+}(x^{+}, \bar{x}) = \tilde{W}^{ba}_{\infty,-\infty}\bar{\rho}_{-\infty}^{a}
\]
(24)
where we expanded \(W^{\dagger} = \{1 - (1 - W)^{-1}\}^{-1}\) in terms of \(1 - W\). Because \(1 - W\) is at least linear in \(A^{-}\) if expanded, intuitively, \((1 - W)^{n}\) in the above represents the (at least) \(n\) gluon insertion. Now we change the variable \(\bar{\rho}_{-\infty}\) so as to allow for all the orderings of the source among the gluon insertion, namely,

\[
\bar{\rho}_{-\infty}(1 - W)^{n} \rightarrow \frac{1}{n} \left\{ \bar{\rho}(1 - W)^{n} + (1 - W) \bar{\rho}(1 - W)^{n-1} + (1 - W)^{2} \bar{\rho}(1 - W)^{n-2} + \cdots \right\}.
\]  
(25)
Then, Eq. \((24)\) simplifies as

\[
J_{b}^{+} = -i \frac{2}{g} \text{tr} \left[ \bar{\rho} \frac{\delta \ln W}{\delta A_{b}} \right],
\]
(26)
which exactly coincides with the current deduced from the logarithmic source action \((21)\). From this argument, it is also clear that our \(\bar{\rho}(\bar{x})\) does not correspond to the initial condition at infinite past. We did not derive Eq. \((21)\) from Eq. \((6)\), for Eq. \((24)\) is a highly non-trivial variable change, but we can insist that Eq. \((21)\) rather than the conventional one is well motivated.

Returning to the main discussion, let us draw an intuitive picture of the microscopic description of the target wave-function which consists of partons propagating along the \(x^{+}\) direction. The classical source \(\bar{\rho}_{a}(x)\) indicates the parton distribution. It is quite important to notice that the distribution of the color orientation of \(\bar{\rho}_{a}(x)\) is not random at each spatial point. Of course, in the dense regime, one can regard the coarse-grained color distribution as random at each point, but one cannot in the dilute regime. Otherwise there would be no correlation between different two points of a quark and an antiquark and the dipole-type operator like Eq. \((17)\) is not permitted but only \(trW(x)trW^{\dagger}(y)\) would arise. The random distribution is expected to be valid at scale longer than the saturation scale \(\sim Q_{s}^{-1}\) and the dipole size would be smaller. The above mentioned is explicitly observed in Eq. \((19)\) where the random variable is common for the quark at \(x_{0}\) and the antiquark at \(y_{0}\).

In order to see the meaning of the logarithmic form, for the simplest example, let us assume that a quark and an antiquark sit at \(x_{0}\) and \(y_{0}\) respectively in the same way as in Eq. \((19)\). (We will consider the adjoint case later.) As we explained above, the spatial resolution of randomness is coarser than the separation scale of the dipole. Then, the source part becomes

\[
\exp \left[ -\frac{2}{g} \text{tr} \left\{ \bar{\rho} \left( \ln W(x_{0}) + \ln W^{\dagger}(y_{0}) \right) \right\} \right],
\]
(27)
where we used the fundamental one \((21)\) and \(- \ln W = \ln W^{\dagger}\). In general, of course, there could be a color rotation between \(\bar{\rho}^{a}(x_{0})\) and \(\bar{\rho}^{a}(y_{0})\). If \(\bar{\rho}^{a}(y_{0})t^{a} = S\bar{\rho}^{a}(x_{0})t^{a}S^{\dagger}\), however, such a color rotation by \(S\) can be absorbed by the
redefinition $S^\dagger W^\dagger(y_0)S \to W^\dagger(y_0)$. The point here is that the random average over $\bar{\rho}$ is taken simultaneously for the quark and antiquark and Eq. (27) is then a natural extension of Eq. (19) to the non-linear situation.

Here one can make use of the Campbell-Hausdorff formula,

$$\ln W(x_0) + \ln W^\dagger(y_0) = \ln [W(x_0)W^\dagger(y_0)] - \frac{1}{2} [\ln W(x_0), \ln W^\dagger(y_0)] + \cdots .$$

(28)

In the large $N_c$ limit only the first term is dominant; the gauge coupling scales as $g \sim 1/\sqrt{N_c}$ which is compensated by quark loops made up in the double line notation of gluons. The higher order commutators lead to the higher order terms in $g$ and yet the color structure is reduced by the commutators, and thus they are suppressed. At the price of mathematical rigor we sketch the intuitive picture of this in Fig. 1. Recalling that the source terms originate from the QCD vertex $g\bar{\psi}_iA^a_{ij}t^a_{ij}\psi_j$ in case of the fundamental representation, Eq. (27) can be expressed as the one-gluon vertex between a propagating quark and gluons as in Fig. 1. Here we adopted the double line notation for gluons at $x_0$ and $y_0$. Roughly the first term of Eq. (28) corresponds to the contribution with the dotted line in Fig. 1 that makes a loop and thus is the leading order in the $N_c$ counting.

We drop the sub-leading commutators in Eq. (28). Therefore, in the presence of a quark and an antiquark and in the large $N_c$ limit, the quantum weight function reads

$$N_\tau[\rho] \sim \int d\bar{\rho} W_\tau[\bar{\rho}] \exp \left[ -\frac{2}{g} \text{tr} \left\{ \bar{\rho} \ln [W(x_0)W^\dagger(y_0)] \right\} \right] \delta[\rho].$$

(29)

Now the calculation goes in the exactly converse manner as in Ref. [21]. From the property, $W_\tau[S\bar{\rho}S^\dagger] = W_\tau[\bar{\rho}]$, one can diagonalize $W(x_0)W^\dagger(y_0)$ to have a polynomial in terms of its eigenvalues. Using $W_\tau[S\bar{\rho}S^\dagger] = W_\tau[\bar{\rho}]$ again, one can conclude that the polynomial is to be expressed in terms of the trace of $W(x_0)W^\dagger(y_0)$ (see also the discussions in Ref. [21]). Hence, when $W_\tau[\bar{\rho}]$ allows for up to the first order of the trace, we reach

$$N_\tau[\rho] \sim D^\dagger(x_0,y_0) \delta[\rho].$$

(30)

What we showed here is that there is a certain $W_\tau[\bar{\rho}]$ that makes the weight function $\tilde{S}$ with the logarithmic source terms [21] being the DM weight function [16], but there is not for the case with the conventional source terms [20].

The generalization to the case with more quarks and antiquarks is straightforward under the assumption that the randomness of color orientation is clustered at the scale of dipoles. Also, the generalization to the adjoint source [22], i.e. the case with the target composed of gluons, is easy too. The above argument works as it is with the Wilson line replaced by the adjoint one $\tilde{W}$. The formula, $t^a_{ij}t^b_{kl} = \frac{1}{2}\delta_{ab}\delta_{jk} - \frac{1}{2\kappa}\delta_{ij}\delta_{kl}$, follows that the leading order at large $N_c$ is given by all the dipole-type combinations of gluon double lines, as exemplified in Fig. 2. Thus, as an ensemble, these combinations eventually lead to the DM picture with a certain weight $P_{\text{DM}}^N$.

Discussions We demonstrated that the logarithmic form, at least in principle, encompasses the desirable microscopic structure of the target wave-function. It had been confirmed that the logarithmic source terms correctly lead to the JIMWLK equation in the dense regime, and in addition to that, it has been checked here that they are qualified also in the dilute regime. In fact, however, the logarithmic form of the source terms alone does not provide sufficient microscopic information. That means, in order to determine the dynamics in the dilute regime, the microscopic information of $W_\tau[\bar{\rho}]$ is required; especially the randomness is only at the unit of the dipole size scale, or in other words, the color orientation has a strong correlation at short distance of the dipole size. The coarse-grained picture of the color average is valid only in the dense regime.

Let us assume that we had a complete evolution equation applicable for the dense, dilute, and intermediate regimes. It would be an intriguing question what would transpire at intermediate density. When the source is large enough in the dense case, the color distribution is regarded conventionally as random. That is, in two-dimensional transverse
space, it is completely disordered. When the density gets lower and lower, the finer structure of randomness becomes significant, and the two-dimensional system is no longer disordered and a sort of localization is important. We have no idea so far what dynamics would separate these two distinct regimes. Is it a continuous crossover from one to the other, or does it result in a discontinuous transition between two regimes?

It is known that neither of the JIMWLK and the DM equations would be a good starting point to tackle this problem in the intermediate density region. The JIMWLK dynamics is valid after coarse graining and thus it contains less information. The DM might be good, but it is unlikely that the DM is capable of taking care of saturation, for which the interaction among partons is important. We would expect that our logarithmic expression may give a hint for future developments.

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