Quantum corrections to the Weyl quantization of the classical time of arrival

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Abstract A time of arrival (TOA) operator that is conjugate with the system Hamiltonian was constructed by Galapon without canonical quantization in Galapon (J. Math. Phys. 45:3180–3215, 2004). The constructed operator was expressed as an infinite series but only the leading term was investigated which was shown to be equal to the Weyl-quantized TOA-operator for entire analytic potentials. In this paper, we give a full account of the said operator by explicitly solving all the terms in the expansion. We interpret the terms beyond the leading term as the quantum corrections to the Weyl quantization of the classical arrival time. These quantum corrections are expressed as some integrals of the interaction potential and their properties are investigated in detail. In particular, the quantum corrections always vanish for linear systems but nonvanishing for nonlinear systems. Finally, we consider the case of an anharmonic oscillator potential as an example.

1 Introduction

Quantum mechanics is one of the most successful quantitative theories ever formulated in describing the physical properties of nature. Its foundations have been tested rigorously and its predictions have been experimentally verified to a high degree of precision and accuracy. However, quantum mechanics is still considered incomplete despite being considered as the triumph of the twentieth-century science. One of the reasons is the lack of formal treatment of time quantities as quantum dynamical observables, a problem known in the literature as the quantum time problem. While there is a rich literature devoted to this problem, a consensus on its resolution has yet to be reached (see, for instance, Refs. [1–45] and all references therein).

One facet of the quantum time problem is the quantum time of arrival problem which requires finding the quantum time of arrival distribution of an elementary particle at a given point in the configuration space [14–45]. In classical mechanics, the time of arrival problem is more of a textbook problem. For a structureless particle of mass \( \mu \) initially located at a point \((q, p)\) at \( t = 0 \) in the phase space, the corresponding time of arrival \( t = T_x(q, p) \) at some point \( q(t = T_x) = x \) in the configuration space is given by

\[
T_x(q, p) = -\text{sgn}(p)\sqrt{\frac{\mu}{2} \int_x^q \frac{dq'}{\sqrt{H(q', p) - V(q')}}},
\]

(1)

where \( V(q) \) is the interaction potential and \( H(q, p) \) is the corresponding Hamiltonian. Equation (1) is simply derived by inverting the classical equations of motion of the particle. As a classical observable, \( T_x(q, p) \) is always finite and real valued in all classically accessible regions in the phase space. It may also have multiple values indicating possible multiple arrivals at the arrival point. Most importantly, it satisfies the classical conjugacy requirement with the Hamiltonian, i.e., it satisfies the canonical Poisson bracket relation \( \{T_x(q, p), H(q, p)\} = -1 \).

The quantum time of arrival problem entails obtaining the quantum image of the classical time of arrival. Following the standard theoretical framework of quantum mechanics, the problem implies finding the appropriate TOA-operator in the underlying Hilbert space of the physical system. The physical contents of the constructed operator can then be extracted by investigating its eigenfunctions, eigenvalues, expectation value and the corresponding probability distribution for arrival time measurements. However, the construction of TOA-operators has been initially ignored due to Pauli who argued that no self-adjoint time operators canonically conjugate to semibounded Hamiltonians exist [46]. Pauli’s “theorem” fundamentally forbids the use of the operator formalism in resolving quantum time of arrival problems, strongly suggesting that time quantities should remain as parameters. Nevertheless, it has already been established by one of us that Pauli’s no-go theorem does not hold in single Hilbert spaces and thus there is no a priori reason to dismiss the existence of self-adjoint time operators canonically conjugate to a semibounded Hamiltonian [30, 47]. In addition, there is a growing consensus that time observables are actually positive operator valued measures (POVMs) in which

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they can be formally represented by maximally symmetric but not necessarily self-adjoint operators [3, 8]. Hence, the construction of time of arrival operators remains a valid and meaningful method in solving quantum time of arrival problems.

One possible way of constructing time of arrival operators is the canonical quantization of the classical time of arrival [1, 22–27]. In Ref. [26], quantization was done on the local time of arrival (LTOA) which is the expansion of the classical time of arrival \( T_c(q, p) \) about the free particle arrival time at the arrival point \( x \). For arrivals at the origin \( (x = 0) \), the LTOA is given by

\[
\tau_0(q, p) = -\sum_{k=0}^{\infty} (-1)^k \frac{(2k - 1)!}{k!} \frac{\mu^{k+1}}{p^{2k+1}} \int_0^q dq' \left( V(q) - V(q') \right)^k.
\]

If the classical time of arrival \( T_0(q, p) \) at the origin \( x = 0 \) holds in the entire region \( \Omega = \Omega_q \times \Omega_p \), then the local time of arrival \( \tau_0(q, p) \) holds only in some local neighborhood of \( \omega_q \times \omega_p \) where it is real-valued. We then have the inclusion \( T_0(q, p) \subset T_0(q, p) \), that is, \( T_0(q, p) \) is the analytic continuation of \( T_0(q, p) \) in \( \Omega/\omega \), where \( \omega = \omega_q \times \omega_p \). Specifically, for \( p \neq 0 \) and \( V \) continuous at \( q \), there exists a neighborhood of \( q \) determined by \( |V(q) - V(q')| < p^2/2\mu \) such that \( \tau_0(q, p) \) converges absolutely and uniformly to \( T_0(q, p) \), i.e., \( T_0(q, p) = \tau_0(q, p) \) in the region \( \omega \subset \Omega \). Outside the region \( \omega_q \), the LTOA \( \tau_0(q, p) \) diverges physically signifying non-arrival at the arrival point. Time of arrival with no classical counterpart, such as the case of a tunneling particle, illustrates a diverging LTOA.

Using the rigged Hilbert space formalism (RHS) of quantum mechanics, the Weyl quantization of the local time of arrival in coordinate representation is given by

\[
\langle \hat{T}_W \psi(q) \rangle = \int_{-\infty}^{\infty} dq' \langle q|\hat{T}_W|q'\rangle \psi(q').
\]

We formally refer to the above integral operator as the Weyl-quantized TOA-operator. Its kernel \( \langle q|\hat{T}_W|q'\rangle \) assumes the form

\[
\langle q|\hat{T}_W|q'\rangle = \frac{\mu}{i\hbar} \text{sgn}(q - q') T_W(q, q').
\]

where \( T_W(q, q') \) is its corresponding time kernel factor given by

\[
T_W(q, q') = \frac{1}{4} \int_0^{q+q'} ds_0 F_1 \left( 1; \frac{\mu}{2\hbar^2} \right)(q - q')^2 \left[ V \left( \frac{q + q'}{2} \right) - V \left( \frac{s}{2} \right) \right].
\]

The factor \( F_1(1; z) \) is a specific hypergeometric function. The Weyl-quantized TOA-operator is considered a legitimate TOA-operator since its eigenfunctions exhibit unitary arrival at the intended arrival point at a time equal to its eigenvalue. This provides a direct connection between the collapse of the particle’s wavefunction and its appearance at the arrival point [29, 36]. It has been applied to specific quantum time of arrival problems including the explicit calculations of the time-of-flight of free neutrons [48], the quantum tunneling time in a single rectangular barrier [49], and the traversal time across a potential well [50]; the investigation of the generalized Hartman effect in a multiple barrier system [51], and the interpretation of the time-energy uncertainty relation [33].

However, canonical quantization suffers from a number of issues such as (i) ordering ambiguities due to the non-commutativity of the position and momentum operators, (ii) obstructions to quantization [52, 53], (iii) and circularity of quantization when invoking the correspondence principle [25]. Hence, the method of supraquantization was introduced in Ref. [26] which does not depend on the canonical quantization of classical observables. The constructed operator is referred to as the supraquantized TOA-operator and is constructed from purely quantum mechanical considerations.

The supraquantized TOA-operator \( \hat{T}_S \) and its kernel \( \langle q|\hat{T}_S|q'\rangle \) also share the same general form as the quantized ones (3–4). The difference, however, lies in the form of the kernel factor \( T_S(q, q') \). In supraquantization, the kernel factor \( T_S(q, q') \) is determined as the solution of a second-order partial differential equation, called the time kernel equation (TKE), with specific boundary conditions (see Sect. 2.2). The time kernel equation is a direct consequence of the conjugacy of the supraquantized TOA-operator with the system Hamiltonian \( \hat{H} \) while the boundary conditions ensure that the operator has the correct classical limit.

An important feature of the time kernel equation is that its solution \( T_S(q, q') \) admits the following expansion for entire analytic potentials

\[
T_S(q, q') = T_0(q, q') + T_1(q, q') + T_2(q, q') + ..., \tag{6}
\]

so that the kernel \( \langle q|\hat{T}_S|q'\rangle \) of the supraquantized TOA-operator appears to be

\[
\langle q|\hat{T}_S|q'\rangle = \langle q|\hat{T}_0|q'\rangle + \langle q|\hat{T}_1|q'\rangle + \langle q|\hat{T}_2|q'\rangle + ... = \left( \frac{\mu}{i\hbar} \right) \text{sgn}(q - q') \sum_{n=0}^{\infty} T_n(q, q'). \tag{7}
\]

Accordingly, the supraquantized TOA-operator itself formally appears as the expansion

\[
\hat{T} = \hat{T}_0 + \hat{T}_1 + \hat{T}_2 + ... \tag{8}
\]
Assuming an analytic solution in powers of \( q + q' \) and \( q - q' \), the leading kernel \( \langle q|\hat{T}_0|q' \rangle \) is found to be the Weyl-quantized time kernel \( \langle q|\hat{T}_W|q' \rangle \) defined by (4) and (5). It satisfies the condition
\[
\langle q|\hat{T}_0|q' \rangle = \frac{1}{2\pi \hbar} \int_{-\infty}^{\infty} dp \ \tau_0 \left( \frac{q + q'}{2}, p \right) e^{i(q-q')p/\hbar},
\]
in accordance with the Weyl quantization of the LTOA. Taking the Weyl-Wigner transform of the full kernel \( \langle q|\hat{T}_S|q' \rangle \), one finds that
\[
\mathcal{T}_{n}(q, p) = \int_{-\infty}^{\infty} dv \ T_n \left( \frac{q + v}{2}, q - \frac{v}{2} \right) e^{-ivp/\hbar}
\]
\[
= \sum_{n=0}^{\infty} \frac{\mu}{\hbar} \int_{-\infty}^{\infty} dv \ T_n \left( \frac{q + v}{2}, q - \frac{v}{2} \right) \text{sgn}(v) e^{-ivp/\hbar}
\]
\[= \tau_0(q, p) + \hbar^2 \tau_1(q, p) + \hbar^4 \tau_2(q, p) + ..., \]
where the \( \tau_n(q, p) \)'s are independent of \( \hbar \). The leading term coincides with the local time of arrival (2) while the succeeding terms vanish in the classical limit \( \hbar \to 0 \).

These findings lead to the conclusion that the supraquantized TOA-operator formally consists of the Weyl-quantized TOA-operator as the leading term, i.e., \( \hat{T}_0 = \hat{T}_W \), plus some additional terms \( \hat{T}_n \) whose Weyl-Wigner transforms are dependent on even powers of \( \hbar \). The additional terms \( T_n(q, q') \), and \( \hat{T}_n \) for \( n \geq 1 \) in (6) are interpreted as quantum corrections to the leading kernel factor \( T_0(q, q') \) and the Weyl-quantized TOA-operator \( \hat{T}_W \), respectively.

One major issue with supraquantization is that no closed-form expression exists for each of the quantum correction \( \hat{T}_n \). It is believed the solution of the time kernel equation needed in the construction of the supraquantized TOA-operator is generally intractable to be useful for the case of nonlinear systems [54]. This makes the usefulness of supraquantization limited only to linear systems and just to highlight the significance of the Weyl quantization of the classical time of arrival over other quantizations in terms of which quantization satisfies the conjugacy requirement for various quantum systems.

In previous works, time of arrival problems involving nonlinear systems were dealt by approximating the corresponding TOA-operator as the Weyl-quantization of the classical time of arrival [33, 49–51] or by numerically constructing the supraquantized TOA-operator by quadrature methods [54]. The former method is problematic since the correction terms are generally non-vanishing and important physical contents may have been missed due to the use of such approximation. On the other hand, the latter has only been done for the case of anharmonic oscillator potential and extending it to other more realistic potentials might not be trivial.

The need for solving the quantum corrections \( \hat{T}_n \) in (8) becomes significant when we consider the quantum tunneling time problem. There is still no consensus on whether the tunneling particle traverses the barrier region instantaneously or not. In Ref. [49], one of us has found that quantum tunneling is instantaneous by taking the expectation value of the Weyl-quantized TOA-operator in the presence of a rectangular potential barrier. However, a potential barrier system is clearly a nonlinear system. Since the Weyl-quantized TOA-operator is just the leading term of the supraquantized TOA-operator, it is compelling to ask whether or not the quantum corrections provide a nonvanishing quantum tunneling time. If it will turn out that they do, it should help us determine in what time scale does quantum tunneling occur. Will it be in the attosecond regime or in the zeptosecond regime?

Another equally important issue that highlights the importance of the quantum corrections is knowing the exact role of the time-energy canonical commutation relation to the dynamics of a legitimate TOA-operator. In Ref. [27], it has been hypothesized that the eigenfunctions of the operator that satisfies the conjugacy relation will have the sharpest arrival at the arrival point at their respective eigenvalues. A complete supraquantized TOA-operator will allow us test such hypothesis since it is the only TOA-operator in the RHS formalism that satisfies the conjugacy relation with the Hamiltonian for both linear and nonlinear systems.

The main objective then of this paper is to undertake the explicit construction of the quantum corrections to the Weyl-quantization of the classical time of arrival, thereby completing the supraquantized TOA-operator initially introduced in Ref. [26]. Specifically, we will solve explicitly the kernel factor corrections \( T_n(q, q') \) in (6) so that the quantum corrections \( \hat{T}_n \) are immediately determined in accordance with (3) and (4). We will show that each kernel factor correction can be derived from the following recurrence relation
\[
T_n(a, v) = \left( \frac{\mu}{2\hbar^2} \right) \sum_{r=1}^{n} \frac{1}{(2r + 1)!} \left( \frac{1}{2} \right)^{2r} \int_{0}^{\infty} ds \ V^{(2r+1)} \left( \frac{s}{2} \right) \int_{0}^{\infty} dw \ U^{2r+1} \ T_{n-r}(s, w)
\]
\[
\times \left[ \text{JL} (\frac{\mu}{2\hbar^2}) (v^2 - w^2) \left( V \left( \frac{u}{2} \right) - V \left( \frac{w}{2} \right) \right) \right].
\]
as an example and solve the first three nonvanishing quantum corrections to the corresponding Weyl-quantized TOA-operator. We calculate the Weyl-Wigner transforms of the correction terms and show that they vanish in the classical limit $\hbar \to 0$. Finally, we conclude in Sect. 5.

2 Construction of time of arrival operators without quantization

2.1 Time of arrival operator in the rigged Hilbert space

In the Hilbert space formulation of quantum mechanics, a quantum mechanical system is represented by an infinite dimensional Hilbert space $\mathcal{H}$ over the complex field. A ray in $\mathcal{H}$ represents a pure state while a generally maximally symmetric densely defined operator in $\mathcal{H}$ corresponds to an observable [26, 55]. The eigenvalues of an operator provide the possible measurement values of the corresponding quantum observable. The spectrum of the operator which is the set of the eigenvalues can be discrete, continuous, or combination of both.

It is well established that observables represented by bounded operators with discrete spectrum are defined on the whole of $\mathcal{H}$ and that their eigenvectors belong to $\mathcal{H}$. However, quantum mechanical observables are generally unbounded and that their spectrum has in general a continuous part. The eigenfunctions corresponding to continuous spectrum are non-normalizable. To deal with these observables, one uses Dirac’s bra-ket formalism which generalizes the linear algebra of Hermitian matrices for observables with discrete spectrum to observables with continuous spectrum. However, the bra-ket formalism does not make sense on Hilbert space alone but is formally and mathematically justified by the rigged Hilbert space (RHS) [56]. This is apparent when we consider non-square integrable eigenfunctions, such as $\delta(q - q_0)$ and $\exp(ipq/\hbar)/\sqrt{2\pi\hbar}$, which are outside of the usual Hilbert space of quantum mechanics.

Given a Hilbert space $\mathcal{H}$, its rigged Hilbert space extension is a triplet, $\Phi^\times \ni \mathcal{H} \ni \Phi$, where $\Phi$ is a dense subspace of $\mathcal{H}$ and $\Phi^\times$ is the space of all continuous linear functionals on $\Phi$. The subspace $\Phi$ is essentially the space of test functions while $\Phi^\times$ is the space of distributions. With the above definitions, a rigged Hilbert space can be referred to as a Hilbert space formally equipped with the theory of distributions so that singular objects become meaningful in the distributional sense [56]. For example, the non-square integrable eigenfunctions, such as $\delta(q - q_0)$ and $\exp(ipq/\hbar)/\sqrt{2\pi\hbar}$, which are outside of the usual Hilbert space of quantum mechanics.

To illustrate how quantum observables appear in the rigged Hilbert space formalism, we consider a structureless particle in the real line whose system Hilbert space is $\mathcal{H} = L^2(\mathbb{R})$, the space of Lebesgue square integrable functions over the real line. We choose the rigged Hilbert space $\Phi^\times \ni L^2(\mathbb{R}) \ni \Phi$, where $\Phi$ is the fundamental space of infinitely differentiable functions in the real line with compact supports and $\Phi^\times$ is the space of functionals on $\Phi$. The standard Hilbert space formulation of quantum mechanics is obtained by taking the closures in $\Phi$ with respect to the metric $L^2(\mathbb{R})$ [26, 27]. In this framework, a quantum observable $O$ appears as the mapping $\hat{O} : \Phi \to \Phi^\times$ and is formally defined by the following integral operator

$$ (\hat{O}\psi)(q) = \int_{-\infty}^{\infty} dq' \langle q|\hat{O}|q'\rangle \psi(q'), \quad (12) $$

in the coordinate representation. The kernel $\langle q|\hat{O}|q'\rangle$ is required to be Hermitian, $\langle q|\hat{O}|q'\rangle = \langle q'|\hat{O}|q\rangle^\ast$ to ensure the real valuedness of the expectation value of $\hat{O}$ in $\Phi$. The corresponding classical observable is derived by taking the limit $\hbar \to 0$ from the inverse Fourier transform

$$ O_h(q, p) = \int_{-\infty}^{\infty} dv q + \frac{v}{2} \hat{O} (q - \frac{v}{2}) e^{-ivp/\hbar}. \quad (13) $$

For example, the usual position $\hat{q}$ and momentum $\hat{p}$ operators appear as

$$ (\hat{q}\psi)(q) = \int_{-\infty}^{\infty} dq' q' \delta(q - q')\psi(q'); \quad (\hat{p}\psi)(q) = \int_{-\infty}^{\infty} dq' i\hbar \delta(q - q')\psi(q'), \quad (14) $$

respectively. The known classical and momentum observables can be derived using (13).

Now we proceed with our current problem which is the construction of TOA-operators. The form of the local time of arrival given by (2) clearly shows its dependence on the classical position and momentum observables. Its corresponding quantum image is then expected to be written in terms of the operators $\hat{q}$ and $\hat{p}$ in free form. But these two operators are generally represented by unbounded operators with continuous spectrum. In addition, we require that our TOA-operator to be canonically conjugate with the system Hamiltonian. But Hamiltonians can also be treated within the rigged Hilbert space formalism and that the rigged Hilbert space can contain all the physically meaningful solutions of the Schrödinger equation [57]. Adding to the fact that Dirac’s bra-ket formalism is mathematically justified in the rigged Hilbert space, it is then meaningful to construct TOA-operators within the rigged Hilbert space formulation of quantum mechanics. We do this exactly by constructing our TOA operator $\hat{T}_S$ in the RHS $\Phi^\times \ni L^2(\mathbb{R}) \ni \Phi$ whose general form is defined by (3).
2.2 The time kernel equation

Supraquantization formally enters when we determine the form of the kernel \( \langle q | \hat{T}_S | q' \rangle \) by imposing four conditions on the supraquantized TOA-operator (3). First, we require that the operator is Hermitian, i.e., \( \hat{T}_S = \hat{T}_S^\dagger \), so that it yields real expectation values. This implies that the kernel must satisfy \( \langle q | \hat{T}_S^\dagger | q' \rangle^* = \langle q | \hat{T}_S | q' \rangle \). Second, the supraquantized TOA-operator must satisfy the time reversal symmetry \( \Theta \hat{T}_S \Theta^{-1} = -\hat{T}_S \), where \( \Theta \) is the time reversal operator, so that the kernel must satisfy \( \langle q | \hat{T}_S | q' \rangle^* = -\langle q | \hat{T}_S | q' \rangle \). Third, the supraquantized TOA-operator must reduce to the classical arrival time in the classical limit, i.e., \( T \langle \Psi_1 | q \rangle \langle q' \rangle \) must be symmetric and that it should lead to the known time kernel factor for the free particle in the limit of vanishing potential, \( V \). The kernel factor \( T_S(q, q') \) is derived from the following canonical commutation relation

\[
\langle \tilde{\varphi} | [\hat{H}, \hat{T}_S] | \varphi \rangle = i \hbar \int dq \, \tilde{\varphi}^*(q) \varphi(q),
\]

and performing two successive integration by parts, we arrive at

\[
\langle \tilde{\varphi} | [\hat{H}, \hat{T}_S] | \varphi \rangle = i \hbar \int dq \, \tilde{\varphi}^*(q) \varphi(q) \left( \frac{dT_S(q, q)}{dq} + \frac{\partial T_S(q, q')}{\partial q} + \frac{\partial T_S(q, q')}{\partial q'} \right) + \frac{\mu}{i \hbar} \int dq \int dq' \, \varphi(q) \varphi^*(q) \text{sgn}(q - q') \left[ -\frac{\hbar^2}{2\mu} \frac{\partial^2 T_S(q, q')}{\partial q^2} + \frac{\hbar^2}{2\mu} \frac{\partial^2 T_S(q, q')}{\partial q'^2} \right] + [V(q) - V(q')] T_S(q, q').
\]

Subject to the following boundary condition

\[
\left. \frac{dT_S(q, q')}{dq} + \frac{\partial T_S(q, q')}{\partial q} \right|_{q=q'} + \left. \frac{\partial T_S(q, q')}{\partial q'} \right|_{q'=q} = 1.
\]

Equation (17) is exactly the time kernel equation we referred to earlier. Of course, the boundary condition given by (18) defines a family of operators satisfying the commutation relation with the extended Hamiltonian. The specific boundary conditions are determined by requiring that the kernel factor \( T_S(q, q') \) should lead us to the correct classical time of arrival. Also, \( T_S(q, q') \) must be symmetric and that it should lead to the known time kernel factor for the free particle in the limit of vanishing potential, \( T_F(q, q') = (q + q')/4 \). All of these conditions require us to set the boundary conditions to

\[
T_S(q, q) = \frac{q}{2}, \quad T_S(q, -q) = 0.
\]

The time kernel equation, coupled with the given boundary conditions, admits a unique solution for entire analytic interaction potentials [54].

The above results can be straightforwardly extended from the arrival point at the origin to any arbitrary arrival point \( x \). For this case, the classical time of arrival is obtained by changing variables in (1) to \( \tilde{q} = q - x \) so that

\[
T_s(\tilde{q}, p) = -\text{sgn}(p) \sqrt{\frac{\mu}{2}} \int_0^{\tilde{q}} dq' \sqrt{H(\tilde{q} + x, p) - V(\tilde{q}' + x)}.
\]

Equation (20) implies that the time of arrival problem at arbitrary arrival point \( x \) is equivalent to the time of arrival problem at the origin under the potential \( V(q) = V(q' + x) \).
Consequently, the time kernel factor of the supraquantized TOA operator is solved from the time kernel equation subject to the potential \( V(q) = V(q' + x) \). For this case, the time kernel equation given by (17) becomes

\[
-\frac{\hbar^2}{2\mu} \frac{\partial^2 T_\alpha(q, q')}{\partial q^2} + \frac{\hbar^2}{2\mu} \frac{\partial^2 T_\alpha(q, q')}{\partial q'^2} + \left[ V(q + x) - V(q' + x) \right] T_\alpha(q, q') = 0,
\]

where the solution \( T_\alpha(q, q') \) is subject to the boundary conditions \( T_\alpha(q, \bar{q}) = \bar{q}/2 \) and \( T_\alpha(\bar{q}, -\bar{q}) = 0 \). The kernel factor \( T_\alpha(q, q') \) is then solved by a simple shift in the potential. Because of this, it is sufficient for us to consider the time of arrival problem at the potential \( V \).

2.3 Solution to the time kernel equation for entire analytic potentials

We derive the solution of the time kernel equation for arbitrary entire analytic potentials and show how the quantum corrections to the classical time of arrival arise.

It will be more convenient to express the time kernel equation in canonical form. Performing a change in variable from \((q, q')\) to \((u = q + q', v = q - q')\) in (17) and (19), the time kernel equation assumes the form

\[
-\frac{2\hbar^2}{\mu} \frac{\partial^2 T_S(u, v)}{\partial u \partial v} + \left( \frac{V(u + v)}{2} - \frac{V(u - v)}{2} \right) T_S(u, v) = 0,
\]

subject to the following boundary conditions

\[
T_S(u, 0) = \frac{u}{4}; \quad T_S(0, v) = 0.
\]

For generality, we consider an arbitrary entire analytic potential of the form given by

\[
V(q) = \sum_{s=1}^{\infty} a_s q^s.
\]

where \( a_s \) are some coefficients. For linear systems, \( a_s = 0 \) for \( s \geq 3 \) while at least \( a_3 \) needs to be nonvanishing for nonlinear systems. The corresponding time kernel equation is found to be

\[
-\frac{2\hbar^2}{\mu} \frac{\partial^2 T_S(u, v)}{\partial u \partial v} + \sum_{s=1}^{\infty} a_s \sum_{k=0}^{[s]} \frac{1}{2^{s-k} k+1} u^{s-2k-1} v^{2k+1} T_S(u, v) = 0,
\]

where \([s] = (s - 1)/2 \) for \( s \) odd and \([s] = s/2 - 1 \) for \( s \) even. We assume an analytic solution in powers of \( u \) and \( v \) given by

\[
T_S(u, v) = \sum_{m,n=0}^{\infty} \alpha_{m,n} u^m v^n.
\]

for some unknown coefficients \( \alpha_{m,n} \) where \( \alpha_{m,n} = 0 \) for negative values of \( m \) or \( n \). The boundary conditions given by (23) imply the initial conditions \( \alpha_{m,0} = \delta_{m,1}/4 \), and \( \alpha_{0,n} = 0 \) for all \( m \) and \( n \). Substituting the assumed solution back into (25) and collecting equal powers of \( u \) and \( v \), we find

\[
\alpha_{m,n} = \left( \frac{\mu}{2\hbar^2} \right)^{1/2} \sum_{s=1}^{\infty} \frac{a_s}{2^{s-1}} \sum_{k=0}^{[s]} \frac{1}{2k+1} \alpha_{m-s+2k,n-2k-2}.
\]

By investigating the first few iterates of (27), it can be shown that the coefficients \( \alpha_{m,n} \) vanish for odd \( n \) but nonvanishing for even \( n \). This implies that odd powers of \( v \) do not contribute in the solution given by (26). This is important as it signifies the symmetry of our solution, i.e., \( T_S(u, v) = T(u, -v) \) or \( T_S(q, q') = T(q', q) \) in the original coordinates, which ensures the real valuedness of the expectation value of our supraquantized TOA-operator.

Using (27) and performing some shifting of indices and reordering of summations, (26) leads to

\[
T_S(u, v) = \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \alpha_m v^{-j} \sum_{s=0}^{\infty} \left( \frac{\mu}{2\hbar^2} \right)^{j-s} \alpha_{m+s,j},
\]

where the new coefficients \( \alpha_{m,j}^{(s)} \) satisfy the recurrence relation

\[
\alpha_{m,j}^{(s)} = \frac{1}{m \cdot 2j} \sum_{r=0}^{s} \sum_{l=2r+1}^{m+2r-1} \frac{\alpha_l}{2^{l-1}} \left( \frac{l}{2r+1} \right) \alpha_{m-l+2r,j-l-1}^{(s-r)},
\]
for all $0 \leq s \leq (j - 1)$. The coefficients $a_{m,j}^{(s)}$ are subjected to the conditions $a_{m,j}^{(0)} = \delta_{m,1/4}$ and $a_{m,j}^{(s)} = 0$ for $m, j \leq 0$. Note that the $j = 0$ terms in (28) are vanishing but we retained them for convenience.

Taking the Weyl-Wigner transform of (28) in accordance with (13), we find

$$T_n(q, p) = \frac{i\hbar}{\mu} \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \sum_{s=0}^{j-1} 2^m q^m \left( \frac{\mu}{2\hbar^2} \right)^{j-s} a_{m,j}^{(s)} \int_{-\infty}^{\infty} dv \, v^{2j} \frac{e^{-ivp\hbar}}{\sin(v)}.$$ (30)

The integral along $v$ is evaluated in distributional sense using the identity

$$\int_{-\infty}^{\infty} v^{m-1} \frac{e^{-ivx}}{\sin(v)} dv = \frac{2(m - 1)!}{i^m x^m},$$ (31)

(the inverse Fourier transform of [58], p. 360, no. 18). Equation (30) then leads to

$$T_n(q, p) = \frac{i\hbar}{\mu} \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} (-1)^j 2^{m+1} (2j)! \frac{q^m}{p^{2j}} \sum_{s=0}^{j-1} \left( \frac{\mu}{2\hbar^2} \right)^{j-s} a_{m,j}^{(s)} \frac{e^{-i\alpha s}}{\hbar}. $$ (32)

It can be seen immediately from (32) that the $s = 0$ terms (with $j > 0$) are independent of $\hbar$ while the succeeding terms are dependent on $\hbar^2$ for $s \geq 1$. This implies that the only contributing terms in the classical limit are the $s = 0$ terms. The $s = 1$ terms provide the leading $\hbar$ correction, which is of the order $O(\hbar^2)$ while the succeeding terms, $s \geq 2$, provide corrections of the order $O(\hbar^{2j})$.

It is then meaningful to isolate the $s = 0$ terms in (28) so that the solution assumes the form

$$T_S(u, v) = T_0(u, v) + T_Q(u, v),$$ (33)

where

$$T_0(u, v) = \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \mu^m q^j \left( \frac{\mu}{2\hbar^2} \right)^j a_{m,j}^{(0)};$$ (34)

$$T_Q(u, v) = \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \mu^m q^j \sum_{s=1}^{j-1} \left( \frac{\mu}{2\hbar^2} \right)^{j-s} a_{m,j}^{(s)}.$$ (35)

It can be shown that the Weyl-Wigner transform of (34) converges to the local time of arrival which is just the expansion of the classical time of arrival at the arrival point. Hence, the leading kernel factor $T_0(u, v)$ can be closed in integral form given by

$$T_0(u, v) = \frac{1}{4} \int_0^u ds \, \int_0^s ds' F_1\left(1; 1; \left( \frac{\mu}{2\hbar^2} \right) v^2 \left[ v\left( \frac{u}{2} - s' \right) - v\left( \frac{s}{2} \right) \right] \right).$$ (36)

Comparing with (5), we see that $T_0(u, v)$ is just equal to the Weyl-quantized time kernel factor in the original $(q, q')$ coordinates given by (5), i.e., $T_0(q, q') = T_W(q, q')$. The term $T_Q(u, v)$ defined by (35) consists all the kernel factor corrections to the leading term $T_0(u, v)$.

To simplify the corrections $T_Q(u, v)$, we rewrite (35) by series rearrangement to get

$$T_Q(u, v) = \sum_{n=1}^{\infty} T_n(u, v),$$ (37)

where

$$T_n(u, v) = \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \mu^m q^j 2^{m+2n+1} \left( \frac{\mu}{2\hbar^2} \right)^{j+n+1} a_{m,j+n+1}^{(n)}.$$ (38)

We formally refer $T_n(u, v)$ as the $n$th kernel factor correction. The convergence of (37) is already guaranteed by the results of Ref. [59].

The appropriate boundary conditions for $T_n(u, v)$ can be easily determined by noting that the leading term $T_0(u, v)$ itself satisfies the full boundary conditions given by (23), i.e., $T_0(u, 0) = u/4$ and $T_0(0, v) = 0$. This means that the correction terms $T_Q(u, v)$ satisfy the condition $T_Q(u, 0) = T_Q(0, v) = 0$. Hence, each kernel factor correction satisfies the condition

$$T_n(u, 0) = T_n(0, v) = 0.$$ (39)
3 The quantum corrections and the complete solution of the time kernel equation

We note that \( T_0(u, v) \) given by (36) was derived in Ref. [26] by comparing its Weyl-Wigner transform to the local time of arrival given by (2). Hence, the coefficients \( \alpha_{m,j}^{(0)} \) in (34) were not explicitly solved. However, the same method cannot be extended to the quantum corrections since they vanish in the classical limit. For example, we want to determine the leading kernel factor correction \( T_1(u, v) \) which consists of the \( s = 1 \) terms in (35), that is,

\[
T_1(u, v) = \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} u^m v^{2j} \left( \frac{\mu}{2\hbar^2} \right)^{j-1} \alpha_{m,j}^{(1)}
\]

Using (29), the coefficients \( \alpha_{m,j}^{(1)} \) are determined by solving the following recurrence relation

\[
\alpha_{m,j}^{(1)} = \frac{1}{m \cdot 2j} \sum_{l=1}^{l=1} \frac{l a_l}{2l-1} \alpha_{m-l,j-1}^{(1)} + \frac{1}{m \cdot 2j} \sum_{l=1}^{l+1} \frac{a_l}{2l-1} \left( \frac{1}{3} \right) \alpha_{m-l+2,j-2}^{(0)}.
\]

The above equation cannot be solved without first determining the coefficients \( \alpha_{m,j}^{(0)} \) of the leading kernel factor \( T_0(u, v) \). But recall that these coefficients are also unknown. One can try solving (41) iteratively by assuming few values of \( m \) and \( j \) and then infer a general form of the coefficients \( \alpha_{m,j}^{(1)} \). However, such method is too cumbersome and impractical as we consider the higher-order correction terms.

What we want then is to determine the \( T_n(u, v) \)’s without explicitly solving for the coefficients \( \alpha_{m,j}^{(m)} \) in (38). This can be done by noting that the kernel corrections \( T_n(u, v) \) are, in fact, generating functions of the coefficients \( \alpha_{m,j}^{(m)} \). Hence, we should be able to derive a closed-form expression for each of the kernel factor \( T_n(u, v) \) using the known techniques of obtaining generating functions, such as the series rearrangement technique [60, 61].

3.1 Alternative derivation of the leading term

Before we proceed with the quantum corrections, we rederive the leading time kernel \( T_0(u, v) \) by series rearrangement technique followed by the method of successive approximations. This is done to show that interpreting the kernel factor corrections \( T_n(u, v) \) as generating functions to obtain their integral forms is more straightforward than using the Frobenius method originally prescribed in Ref. [26].

We go back to (34). Isolating the \( j = 0 \) term and using the initial condition \( \alpha_{m,0}^{(0)} = \delta_{m,1}/4 \), we get

\[
T_0(u, v) = \frac{\mu}{4} + \sum_{m=0}^{\infty} \sum_{j=1}^{\infty} u^m v^{2j} \left( \frac{\mu}{2\hbar^2} \right)^j \alpha_{m,j}^{(0)}
\]

where the coefficients \( \alpha_{m,j}^{(0)} \) satisfy the following recurrence relation

\[
\alpha_{m,j}^{(0)} = \frac{1}{m \cdot 2j} \sum_{s=1}^{s=1} \frac{s a_s}{2s-1} \alpha_{m-s,j-1}^{(0)}.
\]

By direct substitution of (43) into (42), we have

\[
T_0(u, v) = \frac{\mu}{4} + \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \frac{u^m v^{2j}}{m \cdot 2j} \left( \frac{\mu}{2\hbar^2} \right)^j \sum_{s=1}^{s=1} \frac{s a_s}{2s-1} \alpha_{m-s,j-1}^{(0)}
\]

Shifting indices from \( l \) to \( l - 1, j \) to \( j - 1 \), and \( m \) to \( m - 2 \), we arrive at

\[
T_0(u, v) = \frac{\mu}{4} + \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \frac{u^{m+2} v^{2j+2}}{m \cdot 2j + 2} \left( \frac{\mu}{2\hbar^2} \right)^{j+1} \sum_{l=0}^{l=1} \frac{(l+1) a_{l+1}}{2^l} \alpha_{m-l+1,j}^{(0)}
\]

Notice that \( T_0(u, v) \) involves three summations along \( m, j, \) and \( l \). Our immediate task is to close at least one of the three sums. This can be facilitated by using the following elementary integrals,

\[
\int_0^u ds s^{m+1} = \frac{u^{m+2}}{m + 2}, \quad \int_0^v dw w^{2j+1} = \frac{v^{2j+2}}{2j + 2},
\]

so that (45) can be rewritten as

\[
T_0(u, v) = \frac{\mu}{4} + \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \int_0^u ds s^{m+1} \int_0^v dw w^{2j+1} \left( \frac{\mu}{2\hbar^2} \right)^{j+1} \sum_{l=0}^{l=1} \frac{(l+1) a_{l+1}}{2^l} \alpha_{m-l+1,j}^{(0)}
\]
Because of the convergence of the integrals, we can safely interchange the order of summations and integrations leading to

\[ T_0(u, v) = \frac{u}{4} + \int_0^u ds \int_0^v dw \, w \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} s^m w^{2j} \left( \frac{\mu}{2\hbar^2} \right)^j \frac{(l + 1)\alpha_{m+1, j}}{2^j} \alpha_{m-l, j}. \]  

(48)

Using the following summation identity [61],

\[ \sum_{n=0}^{\infty} \sum_{k=0}^{n} B(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} B(k, n + k), \]  

(49)

Equation (48) can be rewritten as

\[ T_0(u, v) = \frac{u}{4} + \int_0^u ds \sum_{l=0}^{\infty} (l + 1)\alpha_{l+1} \left( \frac{s}{2} \right)^l \int_0^v dw \, w \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} s^m w^{2j} \left( \frac{\mu}{2\hbar^2} \right)^j \alpha_{m+1, j}. \]  

(50)

Shifting indices from \( l + 1 \) to \( l \), and \( m + 1 \) to \( m \) leads to

\[ T_0(u, v) = \frac{u}{4} + \int_0^u ds \sum_{l=0}^{\infty} \alpha_l \left( \frac{s}{2} \right)^l \int_0^v dw \, w \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} s^m w^{2j} \left( \frac{\mu}{2\hbar^2} \right)^j \alpha_{m, j}. \]  

(51)

We can now close the infinite series along \( l \) by noting that our potential given by (24) implies the following partial derivative

\[ V'(\frac{s}{2}) = \sum_{l=1}^{\infty} l \alpha_l \left( \frac{s}{2} \right)^{l-1}, \]  

(52)

so that (51) can be simplified as

\[ T_0(u, v) = \frac{u}{4} + \left( \frac{\mu}{2\hbar^2} \right) \int_0^u ds \, V'(\frac{s}{2}) \int_0^v dw \, w \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} s^m w^{2j} \left( \frac{\mu}{2\hbar^2} \right)^j \alpha_{m, j}. \]  

(53)

Notice that the infinite series inside the integral along \( w \) is just \( T_0(u, v) \) defined in (34) but in \((s, w)\) coordinates, i.e., \( T_0(u, v) \rightarrow T_0(s, w) \). Hence, (53) simplifies to the following integral equation,

\[ T_0(u, v) = \frac{u}{4} + \left( \frac{\mu}{2\hbar^2} \right) \int_0^u ds \, V'(\frac{s}{2}) \int_0^v dw \, T_0(s, w). \]  

(54)

We immediately see from (54) that the boundary conditions, \( T_0(u, 0) = u/4 \), and \( T_0(0, v) = 0 \), are satisfied. It is straightforward to convert (54) as a partial differential equation. Taking \( \partial^2 / \partial v \partial u \) and using the Leibniz integral rule given by

\[ \frac{d}{dx} \left( \int_a^x f(x, t) \, dt \right) = f(x, x) + \int_a^x \frac{\partial f(x, t)}{\partial x} \, dt, \]  

(55)

we immediately find the following partial differential equation for \( T_0(u, v) \)

\[ \frac{\partial T_0(u, v)}{\partial v \partial u} = \left( \frac{\mu}{2\hbar} \right) V'(\frac{u}{2}) \, v \, T_0(u, v). \]  

(56)

We will show later that the kernel factor corrections \( T_n(u, v) \) are always vanishing for linear systems so that (56) is already the time kernel equation for linear systems. Likewise, the existence and uniqueness of the solution \( T_0(u, v) \) is already guaranteed for this case [59].

An advantage of our method here, compared to the Frobenius method used in Ref. [26], is that we are able to derive the partial differential equation satisfied by the leading kernel \( T_0(u, v) \) alone, applicable for both linear and nonlinear systems, without solving explicitly for the coefficients \( \alpha_{m, j}^{(0)} \) in (43). This differential equation, coupled with the boundary conditions, can be used to validate any result for \( T_0(u, v) \) given a specific potential. In addition, it will become important later when we show that the full solution \( T_S(u, v) \) satisfies the time kernel equation (22) for arbitrary potentials.

We now go back to the integral equation involving \( T_0(u, v) \) given by (54). A quick look should already allow us to see the dependence of the leading kernel factor \( T_0(u, v) \) to the potential \( V(q) \). The problem, however, is the appearance of \( T_0(s, w) \) on the other side of the equation. We then solve the integral equation in (54) using the method of successive approximations, also known as the Picard iteration. This is done by noting that the first term of (54) is already a solution so that when either \( u = 0 \) or \( v = 0 \), the necessary boundary conditions readily emerged. We then choose

\[ T_{0,0}(u, v) = \frac{u}{4} \]  

(57)
as our zeroth-order approximation of (54). The $n$th-order approximation for $n \geq 1$ is determined from the following recurrence relation,

$$T_{0,n}(u, v) = \frac{u}{4} + \left(\frac{\mu}{2\hbar^2}\right) \int_0^u ds V'(\frac{s}{2}) \int_0^s dw w T_{0,n-1}(s, w).$$  (58)

Equation (54) is retrieved from (58) in the limit $n \to \infty$ so that the solution $T_0(u, v)$ is determined from the limit $T_0(u, v) = \lim_{n \to \infty} T_{0,n}(u, v)$.

Given (58), we find the first few approximations of (54). For $n = 1$, (58) leads to

$$T_{0,1}(u, v) = \frac{u}{4} + \left(\frac{\mu}{2\hbar^2}\right) v^2 \int_0^u ds V'(\frac{s}{2}).$$  (59)

We perform an integration by parts to arrive at the following first-order approximation of $T_{0}(u, v)$

$$T_{0,1}(u, v) = \frac{u}{4} + \left(\frac{\mu}{2\hbar^2}\right) \frac{v^2}{4} \int_0^u ds \left[V(\frac{u}{2}) - V(\frac{s}{2})\right].$$  (60)

On the other hand, the second-order approximation is determined by setting $n = 2$ in (58) leading to

$$T_{0,2}(u, v) = \frac{u}{4} + \left(\frac{\mu}{2\hbar^2}\right) \int_0^u ds V'(\frac{s}{2}) \int_0^s dw w T_{0,1}(s, w) - T_{0,1}(u, v) + \left(\frac{\mu}{2\hbar^2}\right) \frac{v^4}{16} \int_0^u ds V'(\frac{s}{2}) \int_0^s ds' \left[V(\frac{s}{2}) - V(\frac{s'}{2})\right].$$  (61)

We rewrite (61) by exploiting the following equality

$$\frac{\partial}{\partial s}\left[V\left(\frac{s}{2}\right) - V\left(\frac{s'}{2}\right)\right] = V'\left(\frac{s}{2}\right)\left[V\left(\frac{s}{2}\right) - V\left(\frac{s'}{2}\right)\right].$$  (62)

and using the Leibniz integral rule given by (55) so that we arrive at

$$T_{0,2}(u, v) = T_{0,1}(u, v) + \left(\frac{\mu}{2\hbar^2}\right) \frac{v^4}{16} \int_0^u ds \left[V\left(\frac{u}{2}\right) - V\left(\frac{s}{2}\right)\right]^2.$$  (63)

Inserting $T_{0,1}(u, v)$ into (63), we finally find the following second-order approximation of $T_{0}(u, v)$

$$T_{0,2}(u, v) = \frac{u}{4} + \left(\frac{\mu}{2\hbar^2}\right) \frac{v^2}{4} \int_0^u ds \left[V\left(\frac{u}{2}\right) - V\left(\frac{s}{2}\right)\right] + \left(\frac{\mu}{2\hbar^2}\right) \frac{v^4}{16} \int_0^u ds \left[V\left(\frac{u}{2}\right) - V\left(\frac{s}{2}\right)\right]^2.$$  (64)

Following similar steps, the third-order approximation, which is the $n = 3$ case of (58), leads to

$$T_{0,3}(u, v) = \frac{u}{4} + \left(\frac{\mu}{2\hbar^2}\right) \frac{v^2}{4} \int_0^u ds \left[V\left(\frac{u}{2}\right) - V\left(\frac{s}{2}\right)\right] + \left(\frac{\mu}{2\hbar^2}\right) \frac{v^4}{16} \int_0^u ds \left[V\left(\frac{u}{2}\right) - V\left(\frac{s}{2}\right)\right]^2 + \left(\frac{\mu}{2\hbar^2}\right) \frac{v^6}{144} \int_0^u ds \left[V\left(\frac{u}{2}\right) - V\left(\frac{s}{2}\right)\right]^3.$$  (65)

The forms of the first three approximations imply the following general form for arbitrary $n$,

$$T_{0,n}(u, v) = \frac{u}{4} \sum_{k=0}^{n-1} \left(\frac{\mu}{2\hbar^2}\right)^k \frac{v^{2k}}{(1)_k k!} \int_0^u ds \left[V\left(\frac{u}{2}\right) - V\left(\frac{s}{2}\right)\right]^k.$$  (66)

Equation (66) is formally proven via mathematical induction. Assuming that the above equation is valid for some $n = k \geq 0$, the next iterate which is $n = k + 1$ also holds true, validating (66).

Because of the continuity of the potential and convergence of the integral, we can interchange the order of summation and integration. Taking the limit $n \to \infty$, we find

$$T_0(u, v) = \frac{1}{4} \int_0^u ds \sum_{k=0}^{\infty} \left(\frac{\mu}{2\hbar^2}\right)^k \frac{v^{2k}}{(1)_k k!} \left[V\left(\frac{u}{2}\right) - V\left(\frac{s}{2}\right)\right]^k.$$  (67)

The infinite series along $k$ can be simplified by using the definition of the hypergeometric function given by

$$_{0}F_{1}(a; b; z) = \sum_{k=0}^{\infty} \frac{z^k}{(b)_k k!}.$$  (68)
Hence, we finally arrive at the following solution
\[ T_0(u, v) = \frac{1}{4} \int_0^u ds \int_0^v dw \left[ V\left(\frac{u}{2}\right) - V\left(\frac{v}{2}\right)\right], \] (69)
which is exactly the same result with (36).

We now extend our method here into the derivation of the kernel factor corrections \( T_n(u, v) \) for \( n \geq 1 \).

3.2 General expression for the \( n \) th-order quantum correction

Using (29) and (38), the explicit form of \( T_n(u, v) \) is given by
\[ T_n(u, v) = \sum_{r=0}^{n} \sum_{m=0}^{\infty} \sum_{j=0}^{m+2r-1} \sum_{l=2r+1}^{m+2r-1} \frac{u^m v^{2j+2n+2}}{m \cdot 2(j + n + 1)} \left( \frac{\mu}{2\hbar^2} \right)^{j+1} \frac{a_i(\frac{l}{2r+1})}{2l+1} \alpha_{m-l+2r,j+n-r}^{(n-r)}. \] (70)

Taking advantage of the following elementary integrals
\[ \int_0^u ds \, s^{m-1} = \frac{u^m}{m}, \quad \int_0^v dw \, w^{2j+2n+1} = \frac{v^{2j+2n+1}}{2j+2n+2}. \] (71)

Equation (70) can be rewritten as
\[ T_n(u, v) = \sum_{r=0}^{n} \sum_{m=0}^{\infty} \sum_{j=0}^{m+2r-1} \int_0^u ds \, s^{m-1} \int_0^v dw \, w^{2j+2n+1} \left( \frac{\mu}{2\hbar^2} \right)^{j+1} \frac{a_i(\frac{l}{2r+1})}{2l+1} \alpha_{m-l+2r,j+n-r}^{(n-r)}. \] (72)

The convergence of the above equation allows us to safely interchange the order of integrations and summations. Doing so leads to
\[ T_n(u, v) = \sum_{r=0}^{n} \int_0^u ds \, s^{-1} \int_0^v dw \, w^{2j+2n+1} \left( \frac{\mu}{2\hbar^2} \right)^{j+1} \frac{a_i(\frac{l}{2r+1})}{2l+1} \alpha_{m-l+1,j+n-r}^{(n-r)} \times \left( \frac{l + 2r + 1}{2r + 1} \right)^{j+1} \alpha_{m+1,j+n-r}^{(n-r)}. \] (73)

We perform a shift in indices from \( l \) to \( l + 2r + 1 \) and \( m \) to \( m + 2 \) to find
\[ T_n(u, v) = \sum_{r=0}^{n} \int_0^u ds \, s \int_0^v dw \, w \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \sum_{l=0}^{m+2r-1} \frac{u^m v^{2j+2n+2}}{m \cdot 2(j + n + 1)} \left( \frac{\mu}{2\hbar^2} \right)^{j+1} \frac{a_i(\frac{l+2r+1}{2l+1})}{2l+1} \alpha_{m-l+1,j+n-r}^{(n-r)} \times \left( \frac{l + 2r + 1}{2r + 1} \right)^{j+1} \alpha_{m+1,j+n-r}^{(n-r)}. \] (74)

We want to simplify (74) by decoupling the sum along \( l \) from the other infinite series. This is facilitated by using the summation identity given by (49) so that we find
\[ T_n(u, v) = \sum_{r=0}^{n} \int_0^u ds \, s \int_0^v dw \, w \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \frac{u^m v^{2j+2n+2}}{m \cdot 2(j + n + 1)} \left( \frac{\mu}{2\hbar^2} \right)^{j+1} \frac{a_i(\frac{l}{2r+1})}{2l+1} \times \left( \frac{l + 2r + 1}{2r + 1} \right)^{j+1} \alpha_{m+1,j+n-r}^{(n-r)}. \] (75)

Shifting index from \( l \) to \( l - 2r - 1 \) allows us to write
\[ T_n(u, v) = 2 \sum_{r=0}^{n} \int_0^u ds \, s^{-2r} \int_0^v dw \, w \sum_{l=2r+1}^{\infty} \left( \frac{l}{2r + 1} \right)^{j+1} \alpha_{d+2r+1,j+n-r}^{(n-r)} \times \left( \frac{l + 2r + 1}{2r + 1} \right)^{j+1} \alpha_{m+1,j+n-r}^{(n-r)}. \] (76)

The above equation can be further simplified by noting that the \( n \)th derivative of the potential \( V(q) \) given by (24) is
\[ V^{(n)}(q) = n! \sum_{l=n}^{\infty} \left( \frac{l}{n} \right) a_i (\frac{s}{2})^l. \] (77)

This allows us to close the infinite series along \( l \) in (76), that is
\[ \sum_{l=2r+1}^{\infty} \left( \frac{l}{2r + 1} \right)^{j+1} \alpha_{d+2r+1,j+n-r}^{(n-r)} = \frac{1}{(2r + 1)!} \left( \frac{s}{2} \right)^{2r+1} V^{(2r+1)}(\frac{s}{2}). \] (78)
Substituting (78) into (76), we arrive at

$$T_n(u, v) = \left( \frac{\mu}{2\hbar^2} \right)^2 \sum_{r=0}^{n} \frac{1}{(2r + 1)!} \frac{1}{2^{2r}} \int_0^u ds \, V^{(2r+1)} \left( \frac{s}{2} \right) \int_0^v dw \, w^{2r+1}$$

$$\times \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} s^m w^{2j+2n} \left( \frac{\mu}{2\hbar^2} \right)^j \alpha^{(n-r)}_{m+1, j+n-r}. \quad (79)$$

Note that the coefficients $\alpha^{(n-r)}_{m+1, j+n-r}$ are nonvanishing only for $j \geq 1$. Performing a shift in indices from $m$ to $m - 1$ and $j$ to $j + 1$, and adding a factor $l = u^{2r-2r}$, we arrive at

$$T_n(u, v) = \left( \frac{\mu}{2\hbar^2} \right)^2 \sum_{r=0}^{n} \frac{1}{(2r + 1)!} \frac{1}{2^{2r}} \int_0^u ds \, V^{(2r+1)} \left( \frac{s}{2} \right) \int_0^v dw \, w^{2r+1}$$

$$\times \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} s^m w^{2j+2(n-r)+2} \left( \frac{\mu}{2\hbar^2} \right)^{j+1} \alpha^{(n-r)}_{m+1, j+(n-r)-1}. \quad (80)$$

If we compare the factor with double summation in the above equation with that of (38), we find that the former is just the $n \to n-r$ case expressed in $(s, w)$ variables, that is,

$$T_{n-r}(s, w) = \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} s^m w^{2j+2(n-r)+2} \left( \frac{\mu}{2\hbar^2} \right)^{j+1} \alpha^{(n-r)}_{m+1, j+(n-r)-1}. \quad (81)$$

Hence, the $n$th kernel factor correction $T_n(u, v)$ (80) simplifies to

$$T_n(u, v) = \left( \frac{\mu}{2\hbar^2} \right)^2 \sum_{r=0}^{n} \frac{1}{(2r + 1)!} \frac{1}{2^{2r}} \int_0^u ds \, V^{(2r+1)} \left( \frac{s}{2} \right) \int_0^v dw \, w^{2r+1} T_{n-r}(s, w), \quad (82)$$

for $n \geq 1$.

Equation (82) can be easily converted as a partial differential equation for $T_n(u, v)$ and is given by

$$\frac{\partial^2 T_n(u, v)}{\partial v \partial u} = \left( \frac{\mu}{2\hbar^2} \right)^2 \sum_{r=0}^{n} \frac{1}{(2r + 1)!} \frac{1}{2^{2r}} V^{(2r+1)} \left( \frac{u}{2} \right) v^{2r+1} T_{n-r}(u, v). \quad (83)$$

subject to the boundary conditions $T_n(u, 0) = T_n(0, v) = 0$. The above equation will be important later when we prove that the full solution $T_s(u, v)$ indeed satisfies the time kernel equation (22).

The existence of the solution of the time kernel equation guarantees the existence of $T_n(u, v)$ [59]. Similar to the partial differential equation for $T_0(u, v)$, it can be shown that the solution $T_n(u, v)$ is also unique with the boundary conditions specified by (39). The uniqueness will also be followed from the use of the method of successive approximations to solve the integral equation in (82).

To start, we isolate the $r = 0$ term in (82) to get

$$T_n(u, v) = \left( \frac{\mu}{2\hbar^2} \right) \int_0^u ds \, V^{(1)} \left( \frac{s}{2} \right) \int_0^v dw \, w \, T_n(s, w) + \left( \frac{\mu}{2\hbar^2} \right)^2 \sum_{r=1}^{n} \frac{1}{(2r + 1)!} \frac{1}{2^{2r}} \int_0^u ds \, V^{(2r+1)} \left( \frac{s}{2} \right) \int_0^v dw \, w^{2r+1} T_{n-r}(s, w). \quad (84)$$

The second term of (84) can be chosen as the zeroth-order approximation of $T_n(u, v)$, that is,

$$T_{n,0}(u, v) = \left( \frac{\mu}{2\hbar^2} \right) \sum_{r=1}^{n} \frac{1}{(2r + 1)!} \frac{1}{2^{2r}} \int_0^u ds \, V^{(2r+1)} \left( \frac{s}{2} \right) \int_0^v dw \, w^{2r+1} T_{n-r}(s, w). \quad (85)$$

The $m$th-order approximation of the solution $T_n(u, v)$ is obtained from the following recurrence equation,

$$T_{n,m}(u, v) = T_{n,0}(u, v) + \left( \frac{\mu}{2\hbar^2} \right)^2 \sum_{r=1}^{n} \frac{1}{(2r + 1)!} \frac{1}{2^{2r}} \int_0^u ds \, V^{(2r+1)} \left( \frac{s}{2} \right) \int_0^v dw \, w \, T_{n,m-1}(s, w), \quad (86)$$

valid for $m \geq 1$. The kernel factor correction $T_n(u, v)$ can then be taken from the limit $T_n(u, v) = \lim_{m \to \infty} T_{n,m-1}(u, v)$.

The explicit forms of the first two iterations corresponding to $m = 1, 2$ are given by the following equations,

$$T_{n,1}(u, v) = T_{n,0}(u, v) + \left( \frac{\mu}{2\hbar^2} \right)^2 \sum_{r=1}^{n} \frac{1}{(2r + 1)!} \frac{1}{2^{2r}} \int_0^u ds \, V^{(2r+1)} \left( \frac{s}{2} \right) \int_0^v dw \, w^{2r+1} \left[ V \left( \frac{u}{2} \right) - V \left( \frac{s}{2} \right) \right] T_{n-r}(s, w). \quad (87)$$
We get the same result if we apply the same methodology to (40) and (41) as explicitly shown in the Appendix. The second kernel factor correction and in essence the main result of this study. Its convergence is guaranteed by the continuity of the potential and becomes a specific hypergeometric function. Hence, we finally find

\[ T_{n}(u, v) = \left( \frac{\mu}{2 \hbar^2} \right) \sum_{r=1}^{n} \frac{1}{(2r + 1)!} \frac{1}{2^{2r}} \int_{0}^{\mu} d \sigma \langle \hat{\sigma}^{(2r+1)} \rangle \int_{0}^{\mu} dw \langle \hat{w}^{2r+1} \rangle T_{n-r}(s, w). \]  

(88)

From the first few iterates of \( T_{n,m}(u, v) \), we can infer the following general form for arbitrary \( m \)

\[ T_{n,m}(u, v) = \left( \frac{\mu}{2 \hbar^2} \right) \sum_{r=1}^{n} \frac{1}{(2r + 1)!} \frac{1}{2^{2r}} \int_{0}^{\mu} d \sigma \langle \hat{\sigma}^{(2r+1)} \rangle \int_{0}^{\mu} dw \langle \hat{w}^{2r+1} \rangle T_{n-r}(s, w) \times \sum_{k=0}^{m} \frac{1}{(1)_k k!} \left( \frac{\mu}{2 \hbar^2} \right)^k (\nu^2 - \nu^2)^k [V \left( \frac{\mu}{2} \right) - V \left( \frac{s}{2} \right)]. \]

(89)

Equation (89) is formally proven by mathematical induction. Taking the limit \( m \to \infty \), the sum along \( k \) becomes a specific hypergeometric function. Hence, we finally find

\[ T_{n}(u, v) = \left( \frac{\mu}{2 \hbar^2} \right) \sum_{r=1}^{n} \frac{1}{(2r + 1)!} \frac{1}{2^{2r}} \int_{0}^{\mu} d \sigma \langle \hat{\sigma}^{(2r+1)} \rangle \int_{0}^{\mu} dw \langle \hat{w}^{2r+1} \rangle T_{n-r}(s, w) \times \sum_{k=0}^{m} \frac{1}{(1)_k k!} \left( \frac{\mu}{2 \hbar^2} \right)^k (\nu^2 - \nu^2)^k [V \left( \frac{\mu}{2} \right) - V \left( \frac{s}{2} \right)]. \]

(90)

valid for \( n \geq 1 \). The leading kernel \( T_{0}(u, v) \) serves as the initial condition. Equation (90) is our final expression for the \( n \)th kernel factor correction and in essence the main result of this study. Its convergence is guaranteed by the continuity of the potential and the absolute convergence of \( \sum_{k=0}^{m} \frac{1}{(1)_k k!} \left( \frac{\mu}{2 \hbar^2} \right)^k (\nu^2 - \nu^2)^k [V \left( \frac{\mu}{2} \right) - V \left( \frac{s}{2} \right)] \) when \( p < q \).

The kernel factor of the leading quantum correction is obtained by setting \( n = 1 \) and is given by

\[ T_{1}(u, v) = \left( \frac{\mu}{48 \hbar^2} \right) \int_{0}^{\mu} d \sigma \langle \hat{\sigma}^{(3)} \rangle \int_{0}^{\mu} dw \langle \hat{w}^{3} \rangle T_{0}(s, w) \times \sum_{r=0}^{\infty} \frac{1}{(1)_r r!} \left( \frac{\mu}{2 \hbar^2} \right)^r (\nu^2 - \nu^2)^r [V \left( \frac{\mu}{2} \right) - V \left( \frac{s}{2} \right)]. \]

(91)

We get the same result if we apply the same methodology to (40) and (41) as explicitly shown in the Appendix. The second kernel factor correction \( T_{2}(u, v) \) is similarly obtained by letting \( n = 2 \) to get

\[ T_{2}(u, v) = \frac{1}{4 \cdot 3!} \left( \frac{\mu}{2 \hbar^2} \right) \int_{0}^{\mu} d \sigma \langle \hat{\sigma}^{(3)} \rangle \int_{0}^{\mu} dw \langle \hat{w}^{3} \rangle T_{1}(s, w) G(s, w) \]

\[ + \frac{1}{16 \cdot 5!} \left( \frac{\mu}{2 \hbar^2} \right) \int_{0}^{\mu} d \sigma \langle \hat{\sigma}^{(5)} \rangle \int_{0}^{\mu} dw \langle \hat{w}^{5} \rangle T_{0}(s, w) G(s, w). \]

(92)

where

\[ G(s, w) = \sum_{r=0}^{\infty} \frac{1}{(1)_r r!} \left( \frac{\mu}{2 \hbar^2} \right)^r (\nu^2 - \nu^2)^r [V \left( \frac{\mu}{2} \right) - V \left( \frac{s}{2} \right)]. \]

(93)

Both the kernel factors \( T_{0}(u, v) \) and \( T_{1}(u, v) \) are already known at this point so that \( T_{2}(u, v) \) can also be solved analytically or numerically for a given potential \( V(q) \). Likewise, the \( n = 3 \) case leads to the third kernel correction \( T_{3}(u, v) \) given by

\[ T_{3}(u, v) = \frac{1}{4 \cdot 3!} \left( \frac{\mu}{2 \hbar^2} \right) \int_{0}^{\mu} d \sigma \langle \hat{\sigma}^{(3)} \rangle \int_{0}^{\mu} dw \langle \hat{w}^{3} \rangle T_{2}(s, w) G(s, w) \]

\[ + \frac{1}{16 \cdot 5!} \left( \frac{\mu}{2 \hbar^2} \right) \int_{0}^{\mu} d \sigma \langle \hat{\sigma}^{(5)} \rangle \int_{0}^{\mu} dw \langle \hat{w}^{5} \rangle T_{1}(s, w) G(s, w) \]

\[ + \frac{1}{64 \cdot 7!} \left( \frac{\mu}{2 \hbar^2} \right) \int_{0}^{\mu} d \sigma \langle \hat{\sigma}^{(7)} \rangle \int_{0}^{\mu} dw \langle \hat{w}^{7} \rangle T_{0}(s, w) G(s, w). \]

(94)

All the higher-order corrections can similarly be determined by (90) so that in principle, we now have a complete supraquantized TOA-operator in accordance with (3–4).

### 3.3 Properties of the kernel factor corrections

We now determine some important properties of the kernel factor corrections \( T_{n}(u, v) \) given by (90).

1. **Vanishing of** \( T_{n}(u, v) \) **for** \( u = 0 \) **or** \( v = 0 \). We can see immediately that \( T_{n}(u, v) \) satisfies the boundary conditions \( T_{n}(0, 0) = 0 \) and \( T_{n}(0, v) = 0 \) so that the full time kernel factor \( T_{3}(u, v) \) satisfies the original boundary conditions given by (23). The vanishing of the corrections for \( u = 0 \) or \( v = 0 \) guarantees that the supraquantized TOA-operator leads to the correct classical arrival time in the classical limit.

2. **Symmetry of** \( T_{n}(u, v) \) **along** \( v \).
Another important property of the corrections $T_n(u, v)$ is its symmetry along $v$. Changing variables from $v$ to $-v$ in (90) and noting that the initial condition $T_0(u, v)$ is symmetric along $v$, we arrive at the relation $T_n(u, v) = T_n(u, -v)$. This guarantees that the expectation value of the corresponding operator, the quantum correction $\hat{T}_n$, is always real valued.

3. Vanishing and non-vanishing of $T_n(u, v)$ for linear and nonlinear systems, respectively.

Probably the most significant property of the time kernel factor corrections $T_n(u, v)$ is its vanishing for linear systems and non-vanishing for nonlinear systems. For linear systems of the form $V(q) = a + bq + cq^2$, the factor $V^{(2r+1)}(s/2)$ in (90) is always zero for $r \geq 1$. Hence, there are no quantum corrections to the Weyl quantization of the classical time of arrival for this case. This is the exact reason why the Weyl-quantized TOA-operator is sufficient for specific quantum arrival time problems involving a free-particle, a particle in a gravitational field and a particle in the presence of harmonic oscillator, among others. On the other hand, the factor $V^{(2r+1)}(s/2)$ in (90) is always non-zero for the case of nonlinear systems. This explains why there always exist quantum corrections to the Weyl-quantized TOA-operator for nonlinear systems. These observations clearly explain why the Weyl-quantized TOA-operator satisfies the conjugacy requirement with the system Hamiltonian for linear systems but not for nonlinear systems. Equivalently, the existence of obstruction to quantization in the quantum time of arrival problem is justified by the appearance of these quantum corrections to the Weyl-quantization of the classical arrival time.

4. Dependence of the corresponding kernel $\langle q | \hat{T}_n | q' \rangle$ on $h^{2n}$.

We first calculate the general form of the Weyl-Wigner transform of the $n$th kernel factor correction. Using the definition of $T_n(u, v)$ in (70), the corresponding time kernel is given by

$$\langle q | T_n | q' \rangle = \frac{\mu}{\lambda} \text{sgn}(q - q') \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} (q + q')^m (q - q')^{2j+2n+2} \left( \frac{\mu}{2 \lambda} \right)^{j+1} a_{m,j+n+1}^{(n)}, \quad (95)$$

where the coefficients $a_{m,j+n+1}^{(n)}$ satisfy recurrence relation given by (29). Its Weyl-Wigner transform is given by

$$T_n(q, p) = \int_{-\infty}^{\infty} dq' \left( q + \frac{v}{2} \right) | T_n(q - \frac{v}{2}) e^{-ipq/\hbar} \right.$$

$$= 2 \frac{\mu}{p^{2n+2}} h^{2n} \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} (-1)^{j+m} (2q)^m (2j + 2n + 2)! \left( \frac{\mu}{2 p^2} \right)^{j+1} a_{m,j+n+1}^{(n)}. \quad (96)$$

Equation (96) can be converted as an integral equation by performing similar series rearrangement as in the previous subsections. The result is given by

$$T_n(q, p) = \frac{\mu}{p} \sum_{r=0}^{n} h^{2r} \frac{(-1)^r}{2^{2r}(2r+1)!} \int_{0}^{q} dq' V^{(2r+1)}(q') \frac{\partial^{2r+1} T_{n-r}(q', p)}{\partial p^{2r+1}}. \quad (97)$$

Solving (97) by the same method of successive approximation as we did before, we arrive at

$$T_n(q, p) = \mu h^{2n} \sum_{r=1}^{n} \frac{(-1)^r}{2^{2r}(2r+1)!} \int_{0}^{q} dq' \exp \left[ (V(q) - V(q')) \frac{\mu}{p} \frac{\partial}{\partial p} \right] \frac{1}{p} V^{(2r+1)}(q') \frac{\partial^{2r+1} T_{n-r}(q', p)}{\partial p^{2r+1}}. \quad (98)$$

for $n \geq 1$. The Weyl-Wigner transform $T_n(q, p)$ is also vanishing (nonvanishing) for linear (nonlinear) systems due to the vanishing (nonvanishing) of the factor $V^{(2r+1)}(q')$ in (98).

We now clearly see the explicit dependence of $T_n(q, p)$ on $h^{2n}$. This vanishes in the classical limit $\hbar \to 0$ so that the supraquantized TOA-operator leads to the classical arrival time and hence satisfies the quantum-classical correspondence principle.

In essence, the $T_n(q, p)$'s can also be regarded as the quantum corrections to the classical arrival time in phase space so that quantizing them using Weyl prescription leads to the time kernel corrections $\langle q | \hat{T}_n | q' \rangle$.

5. Decreasing contribution of $T_n(q, q')$ with increasing $n$.

Numerical evaluations of $T_n(u, v)$ for some specific values of $u, v$ and $V(q)$ imply a decreasing contribution of $T_n(q, q')$ with increasing $n$. This can already be expected since its Weyl-Wigner transform (98) clearly illustrates the explicit dependence on $h^{2n}$ which has a decreasing classical contribution with $n$. Hence, it does make sense to approximate the full time kernel factor $T_S(q, q')$ as a partial sum $T_S(q, q') \approx T_0(q, q') + T_1(q, q') + \ldots + T_m(q, q')$, with the approximation getting more accurate as more terms are added.

One may argue that the quantum corrections are too small and can be neglected right away in any calculations so that approximating the supraquantized TOA-operator as just the Weyl-quantized TOA-operator suffices. However, the choice to neglect the quantum corrections should depend strictly on the specific quantum time of arrival problem being considered. For example, we consider the case of quantum tunneling. In Ref. [49], the leading term of the supraquantized TOA-operator (8) for the barrier case was only considered. Its expectation value is found to be vanishing for the case of quantum tunneling. However, the zero contribution from the leading term does not immediately mean that the expectation values of the quantum corrections identically vanish. In the same vein, the experimental attosecond time measurements implying zero quantum tunneling time are strictly valid up to the attosecond regime only [62–64]. We do not know if the same result extends to the zeptosecond regime. This is exactly one of our motivations why the quantum corrections need to be explicitly determined and the supraquantized TOA-operator needs to be completed.
3.4 The complete time kernel factor and the supraquantized TOA-operator

To completely and finally validate our results, we now show that the full solution

\[ T_S(u, v) = T_0(u, v) + \sum_{n=1}^{\infty} T_n(u, v), \]  

(99)

where \( T_0(u, v) \) and \( T_n(u, v) \) are defined by (69) and (90), respectively, indeed satisfies the time kernel equation in (22), that is,

\[ \frac{2\hbar^2}{\mu} \frac{\partial^2 T_S(u, v)}{\partial u \partial v} = \left( V\left(\frac{u + v}{2}\right) - V\left(\frac{u - v}{2}\right)\right) T_S(u, v). \]  

(100)

The form of \( T_S(u, v) \) as indicated by (99) implies that the left-hand side of the time kernel equation (100) involves the partial derivatives \( \partial^2 T_0(u, v)/\partial u \partial v \) and \( \partial^2 T_n(u, v)/\partial u \partial v \). But these two derivatives are exactly the partial differential equations uniquely satisfied by the kernel factors \( T_0(u, v) \) and \( T_n(u, v) \) as shown in (56) and (83), respectively. The left-hand side of (100) then evaluates to

\[ \frac{2\hbar^2}{\mu} \frac{\partial^2 T_S(u, v)}{\partial u \partial v} = \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{(2r + 1)!} \frac{1}{2^{2r}} V^{(2r+1)}\left(\frac{u}{2}\right) v^{2r+1} T_{n-r}(u, v). \]  

(101)

Using the summation identity in (49), the two summations can be decoupled leading to

\[ \frac{2\hbar^2}{\mu} \frac{\partial^2 T_S(u, v)}{\partial u \partial v} = \sum_{r=0}^{\infty} \frac{1}{(2r + 1)!} \frac{1}{2^{2r}} V^{(2r+1)}\left(\frac{u}{2}\right) v^{2r+1} T_S(u, v). \]  

(102)

Now, the kernel factor \( T_S(u, v) \) satisfies the time kernel equation if the right hand sides of (100) and (102) are equal, which we will explicitly show. To proceed, we need to evaluate the right-hand side of (100). For entire analytic potentials of the form \( V(q) = \sum_{l=1}^{\infty} a_l q^l \) and noting the binomial expansion

\[ (x + y)^l = \sum_{m=0}^{l} \binom{l}{m} x^{l-m} y^m, \]  

(103)

we find the following equality

\[ \left( V\left(\frac{u + v}{2}\right) - V\left(\frac{u - v}{2}\right)\right) T_S(u, v) = \sum_{l=1}^{\infty} \sum_{m=0}^{l} \frac{a_l}{2^{2l}} \sum_{m=0}^{l} \binom{l}{m} u^{l-2m-1} v^{2m+1} T_S(u, v), \]  

(104)

where the index \([l]\) is defined as \([l] = (l - 1)/2) for odd \( l \) and \([l] = l/2 - 1 \) for even \( l \).

We separate the even and odd parts of the sum along \( l \), and then perform a shift in index from \( l \) to \( l + 1 \). Doing so leads us to

\[ \left( V\left(\frac{u + v}{2}\right) - V\left(\frac{u - v}{2}\right)\right) T_S(u, v) = \sum_{l=0}^{\infty} \sum_{m=0}^{l} \frac{a_{2l+1}}{2^{2l+1}} \binom{2l+1}{2m+1} u^{2l-2m} v^{2m+1} T_S(u, v) + \sum_{l=0}^{\infty} \sum_{m=0}^{l} \frac{a_{2l+2}}{2^{2l+1}} \binom{2l+2}{2m+1} u^{2l-2m} v^{2m+1} T_S(u, v). \]  

(105)

Using again the summation identity in (49) followed by a shift of index from \( l \) to \( l - (2m + 1) \), (105) simplifies to

\[ \left( V\left(\frac{u + v}{2}\right) - V\left(\frac{u - v}{2}\right)\right) T_S(u, v) = \sum_{m=0}^{\infty} \sum_{l=2m+1}^{\infty} \frac{V^{2m+1}}{2^{2m}} \sum_{l=2m+1}^{\infty} \frac{a_l}{2^{2m+1}} \binom{l}{2m+1} \left(\frac{u}{2}\right)^{(l-(2m+1))} T_S(u, v). \]  

(106)

Now, the sum along \( l \) is related to the \((2m+1)\)th derivative of the potential \( V(q) \) evaluated at \( q = u/2 \), that is,

\[ V^{(2m+1)}\left(\frac{u}{2}\right) = (2m+1)! \sum_{l=2m+1}^{\infty} a_l \binom{l}{2m+1} \left(\frac{u}{2}\right)^{(l-(2m+1))}. \]  

(107)

Hence, we finally arrive at the following result

\[ \left( V\left(\frac{u + v}{2}\right) - V\left(\frac{u - v}{2}\right)\right) T_S(u, v) = \sum_{m=0}^{\infty} \frac{1}{(2m+1)!} \frac{1}{2^{2m}} V^{(2m+1)}\left(\frac{u}{2}\right) v^{2m+1} T_S(u, v). \]  

(108)

We immediately see that the right-hand side of (108) is equal to the right-hand side of (102). Equating both equations, we immediately arrive at the time kernel equation given by (100). Hence, the full solution \( T_S(u, v) \) satisfies the time kernel equation. This validates our results for the kernel factor corrections \( T_n(u, v) \) and consequently the quantum corrections \( \hat{T}_n \).
4 The TOA problem for the quartic anharmonic oscillator

We now apply our results to a specific quantum system, in particular, the case of quartic anharmonic oscillator of the form \( V(q) = \lambda q^4 \) for some constant \( \lambda \). This potential clearly yields a nonlinear equation of motion.

4.1 The Weyl-quantized TOA-operator

With the given potential, the leading kernel factor \( T_0(u, v) \) in accordance to (91) is given by

\[
T_0(u, v) = \frac{1}{4} \int_0^u ds \ 0F_1\left(\{1; \eta v^2 (u^4 - s^4)\}\right),
\]

(109)

where \( \eta = \mu \lambda / 32 \hbar^2 \). Expanding the integrand and exchanging the order of integration and summation, we arrive at the following equation

\[
T_0(u, v) = \frac{1}{4} \sum_{k=0}^{\infty} \frac{(\eta u^4 v^2)^k}{(1)_k k!} \int_0^u ds \ (u^4 - s^4)^k.
\]

(110)

The integral along \( s \) in (110) is easily evaluated and is given by

\[
\int_0^u ds \ (u^4 - s^4)^k = u^{4k+1} \frac{\Gamma\left(\frac{5}{4}\right) \Gamma(k+1)}{\Gamma\left(k + \frac{5}{4}\right)},
\]

(111)

so that the leading time kernel factor assumes the form

\[
T_0(u, v) = \frac{u}{4} \sum_{k=0}^{\infty} \frac{(\eta u^4 v^2)^k}{(1)_k} \frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(k + \frac{5}{4}\right)}.
\]

(112)

The above solution converges absolutely in the neighborhood of the origin in the \( uv \)-plane. It can be summed in closed-form in terms of a specific hypergeometric function leading to

\[
T_0(u, v) = \frac{u}{4} 0F_1\left(\left\{\frac{5}{4}; \frac{\mu \lambda}{32 \hbar^2}\right\} u^4 v^2\right).
\]

(113)

The above result is also the Weyl-quantized time kernel factor for the anharmonic potential. One could show that \( T_0(u, v) \) satisfies the partial differential equation for the leading kernel factor (56), that is,

\[
\frac{\partial^2}{\partial u \partial v} T_0(u, v) = 8 \eta u v^3 T_0(u, v),
\]

(114)

and the boundary conditions \( T_0(u, 0) = u/4 \) and \( T_0(0, v) = 0 \). Hence, (113) is validated.

The leading kernel of the supraquantized TOA-operator (3) for the anharmonic oscillator is then given by

\[
\langle q|\hat{t}_0|q' \rangle = \frac{\mu}{i \hbar} \sgn(q - q') \left( q + \frac{q'}{4} \right)^4 \ 0F_1\left(\{\frac{5}{4}; \frac{\mu \lambda}{32 \hbar^2}\} (q + \frac{q'}{4}) \sgn(q - q') \right),
\]

(115)

which also coincides with the time kernel of the Weyl-quantized TOA-operator. To get the local time of arrival, we take the Weyl-Wigner transform of (115) in accordance with (13). We have

\[
\tau_0(q, p) = \frac{\mu q}{2i \hbar} \int_{-\infty}^{\infty} dv \left(q + \frac{v}{2}\right) 0F_1\left(\left\{\frac{5}{4}; \frac{\mu \lambda q^4}{\hbar^2}\right\} (q + \frac{v}{2}) \right) \sgn(v) e^{-ipv/\hbar}
\]

(116)

The above integral can be evaluated by expanding the hypergeometric function as an infinite series, exchanging the order of summation and integration, and then integrating term by term using the integral identity in (31). The LTOA for the quartic oscillator is then given by

\[
\tau_0(q, p) = -\frac{\mu q}{p} \sum_{k=0}^{\infty} \frac{(2k)!}{(5/4)_k k!} \left(-\frac{\mu \lambda q^4}{2p^2}\right)^k.
\]

(117)

The corresponding classical time of arrival at the origin is then

\[
T_0(q, p) = -\frac{\mu q}{p} 2F_1\left(\left\{\frac{1}{2}; 1; \frac{5}{4}; \frac{-2 \mu \lambda q^4}{p^2}\right\} \right).
\]

(118)
provided $2\mu \lambda q^4/p^2 < 1$. When this condition is not satisfied, the LTOA diverges indicating non-arrival at the arrival point. Note that $\tau_0(q, p)$ and $T_0(q, p)$ are strictly positive since the initial position $q$ is located to the left of the origin. It is interesting to note that the classical time of arrival appears as a free-partial arrival time $T_F(q, p) = -\mu q/p$ deformed by some function dependent on the potential $V(q) = \lambda q^4$.

4.2 The first three leading kernel factor corrections

Let us now calculate the first three quantum corrections (90) to the Weyl quantization of the classical arrival time for the case of a quartic anharmonic oscillator potential. This is done by explicitly solving for the first three kernel factor corrections $T_1(u, v)$, $T_2(u, v)$, and $T_3(u, v)$.

In (90), the factor $V^{(2r+1)}(s/2)$ is vanishing for $r \geq 2$ so that only the $r = 1$ term contributes in the sum. The $n$th quantum correction then assumes the form

$$T_n(u, v) = \frac{\mu}{48\hbar^2} \int_0^u ds \: V^{(3)} \left( \frac{s}{2} \right) \int_0^v dw \: V^{(3)} \left( \frac{w}{2} \right) \int_0^{w_0} \eta F_1 \left( \frac{1}{2}; \eta \right),$$

for all $n \geq 1$. Substituting our potential $V(q)$ and the leading kernel factor $T_0(u, v)$ into (119), the leading kernel factor correction is given by

$$T_1(u, v) = 2 \eta \int_0^u ds \: \int_0^v dw \: s^2 \int_0^{w_0} \eta F_1 \left( \frac{1}{2}; \eta \right) V^{(3)} \left( \frac{w}{2} \right),$$

where $\eta = \mu \lambda / 32 \hbar^2$.

To evaluate the double integral, we expand the two hypergeometric functions in the integrand, and then interchange the order of summations and integrations. As a result, we find

$$T_1(u, v) = 2 \eta \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{\eta^{k+l}}{(5/4)_k (1)_l} \frac{1}{k!} \int_0^u ds \: s^{2l+3} \int_0^v dw \: w^{2k+3} \left( u^4 - w^4 \right)^l.$$

The integrals along $s$ and $w$ can be straightforwardly evaluated leading to

$$\int_0^u ds \: s^{2k+3} = u^{2k+4+3} \frac{\Gamma(k + 3/4) \Gamma(k + 1)}{4 \Gamma(k + l + 7/4)},$$

$$\int_0^v dw \: w^{2k+3} = v^{2k+2l+4} \frac{\Gamma(k + 2) \Gamma(k + 1)}{2 \Gamma(k + l + 3)}.$$

Substituting (122) and (123) into (124), our leading kernel factor correction $T_1(u, v)$ becomes

$$T_1(u, v) = \frac{\Gamma(5/4)}{4} - \eta u^3 v^4 \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(\eta v^2 w^4)^l}{\Gamma(k + l + 7/4) \Gamma(k + l + 3)} \frac{(k + 1) \Gamma(k + 3/4)}{\Gamma(k + 5/4)} \Gamma(k + 1) \Gamma(k + 7/4) \Gamma(k + l + 3)}.$$

We rearrange the two summations using the following identity [61],

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A(m, n) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A(m, n - m),$$

so that (124) becomes

$$T_1(u, v) = \frac{\Gamma(5/4)}{4} - \eta u^3 v^4 \sum_{l=0}^{\infty} \frac{(\eta v^2 w^4)^l}{\Gamma(l + 7/4) \Gamma(l + 3)} \sum_{k=0}^{l} \frac{(k + 1) \Gamma(k + 3/4)}{\Gamma(k + 5/4)}.$$

The sum along $k$ results to a specific gamma function which enables us to sum the infinite series along $l$ into a specific hypergeometric function. The leading kernel factor correction is simply given by

$$T_1(u, v) = \frac{\eta u^3 v^4}{24} \left[ 5 F_2 \left( 1, 7 \; \frac{5}{2}, \frac{5}{2}; \eta u^4 v^2 \right) - 1 \right].$$
The convergence of the above results is guaranteed by the absolute everywhere convergence of the hypergeometric function $F_\nu$ for $p < q$. We validate the quantum corrections by showing that they satisfy the partial differential equation for $T_n(u, v)$ (83), that is,

$$ \frac{\partial^2 T_n(u, v)}{\partial v \partial u} = 8\eta u^3 v T_n(u, v) + 8\eta u v^3 T_{n-1}(u, v). $$

and the boundary conditions $T_n(u, 0) = 0$ and $T_n(0, v) = 0$.

Clearly, the kernel factor corrections $T_1(u, v), T_2(u, v),$ and $T_3(u, v)$ are non-vanishing and provide corrections to the Weyl-quantized time kernel factor $T_0(u, v)$. In addition, numerical evaluations of $T_0(u, v), T_1(u, v), T_2(u, v),$ and $T_3(u, v)$ implies that $T_3(u, v) < T_2(u, v) < T_1(u, v) < T_0(u, v).$ This supports our earlier assertion that the kernel factor corrections $T_\nu(u, v)$'s have decreasing contribution with $\nu$. For this specific case, it is then unreasonable to expect the full time kernel factor $T_3(q, q')$ can be approximated by the partial sum $\sum_{\nu=0}^3 T_\nu(q, q')$ so that the supraquantized-TOA operator for the quartic anharmonic oscillator is approximated up to the third-order correction. The approximation gets better as we add more terms in the partial sum.

We are able to solve the time kernel factor $T_n(u, v)$ up to the third-order approximation analytically, unlike before where we have no choice but to approximate the supraquantized-TOA operator using just the leading term. In fact, higher-order corrections can still be determined, albeit tediously. The need to continue approximating our time kernel factor depends on how accurate we want our supraquantized TOA-operator to be when compared to the corresponding experimental TOA observable.

For completeness, we also compute for the Weyl-Wigner transforms of the kernel factor corrections. Given the condition $2\mu \lambda q^4/p^2 < 1$, we arrive at the following closed-form expressions:

$$ T_1(q, p) = -\mu^2 \lambda q^3 p^{-3} h^4 \left[ \frac{5}{2} \, F_1 \left( 1, \frac{5}{2}; \frac{7}{4}; \frac{-2\mu \lambda q^4}{p^2} \right) - \frac{1}{2} \, F_1 \left( 1, \frac{5}{2}; \frac{7}{4}; \frac{-2\mu \lambda q^4}{p^2} \right) \right], $$

$$ T_2(q, p) = -\mu^3 \lambda^2 q^5 p^{-5} h^4 \left[ \frac{14}{3} \, F_3 \left( 2, 2, 2, \frac{9}{2}; 1, 1, \frac{9}{4}; \frac{-2\mu \lambda q^4}{p^2} \right) + \frac{301}{3} \, F_3 \left( 2, \frac{11}{27}; \frac{9}{4}; \frac{9}{4}; \frac{86}{27}; \frac{-2\mu \lambda q^4}{p^2} \right) \right. $$

$$ - \frac{175}{4} \, F_3 \left( 1, \frac{9}{2}; \frac{17}{4}; \frac{15}{4}; \frac{-2\mu \lambda q^4}{p^2} \right) + \frac{91}{4} \, F_3 \left( 1, \frac{9}{2}; \frac{9}{4}; \frac{9}{4}; \frac{-2\mu \lambda q^4}{p^2} \right) \right], $$

$$ T_3(q, p) = -\mu^4 \lambda^3 q^7 p^{-7} h^4 \left[ \frac{1166}{3} \, F_3 \left( 2, 2, 2, \frac{60}{7}; 1, 1, \frac{9}{4}; \frac{53}{7}; \frac{-2\mu \lambda q^4}{p^2} \right) + \frac{154}{5} \, F_1 \left( 1, \frac{13}{2}; \frac{9}{4}; \frac{-2\mu \lambda q^4}{p^2} \right) \right. $$

$$ + \frac{891}{2} \, F_3 \left( 2, 2, 2, \frac{13}{2}; 1, 1, \frac{9}{4}; \frac{-2\mu \lambda q^4}{p^2} \right) - \frac{55}{4} \, F_3 \left( 2, 2, 2, \frac{13}{2}; 1, \frac{11}{4}; \frac{-2\mu \lambda q^4}{p^2} \right) \right. $$

$$ + \frac{541}{12} \, F_3 \left( 2, 2, \frac{13}{2}; \frac{9}{4}; \frac{-2\mu \lambda q^4}{p^2} \right) + \frac{284898}{40} \, F_3 \left( 2, 1, \frac{21277}{12644}; \frac{13}{2}; \frac{8633}{12644}; \frac{-2\mu \lambda q^4}{p^2} \right) $$

$$ - \frac{12265}{2} \, F_3 \left( 2, \frac{515}{69}; \frac{13}{2}; \frac{11}{4}; \frac{446}{69}; \frac{-2\mu \lambda q^4}{p^2} \right) + \frac{75075}{8} \, F_3 \left( 1, \frac{13}{2}; \frac{27}{4}; \frac{9}{2}; \frac{25}{4}; \frac{-2\mu \lambda q^4}{p^2} \right) \right. $$

$$ - \frac{15125}{4} \, F_3 \left( 1, \frac{13}{2}; \frac{11}{4}; \frac{-2\mu \lambda q^4}{p^2} \right) + \frac{418}{15} \, F_1 \left( 2, \frac{13}{2}; \frac{9}{4}; \frac{-2\mu \lambda q^4}{p^2} \right) \right]. $$

(133)
We see the explicit \( h^2, h^4, \) and \( h^6 \) dependence of \( T_1(q, p), T_2(q, p), \) and \( T_3(q, p) \) respectively. In the classical limit \( h \to 0 \), their contributions identically vanish so that the supraquantized TOA-operator leads to the correct classical time of arrival given by (118). Of course, the correction terms are required so that the complementary relation with the Hamiltonian strictly holds.

Having constructed the supraquantized TOA-operator up to the third correction, the corresponding physical contents of the operator for the case of anharmonic oscillator potential may already be investigated in the standard way. However, the mathematical construction of the quantum corrections and their implementation to a specific quantum system already suffice for our current purposes. Elsewhere, the physical contents and implications of the quantum corrections are investigated by considering more realistic potentials that can be compared to experimental time of arrival measurements.

5 Conclusion

In this paper, we have determined explicitly the quantum corrections to the Weyl-quantized time of arrival operator and expressed them as some integrals of the interaction potential. These corrections arise by imposing strict conjugacy of our supraquantized TOA-operator with the system Hamiltonian. They always vanish for linear systems but generally nonvanishing for the case of nonlinear systems. We then considered the case of quartic anharmonic oscillator potential where we have computed the corresponding Weyl-quantized TOA-operator and the three leading kernel factor corrections. We showed that the Weyl-Wigner transform of the quantum corrections identically vanish in the classical limit \( h \to 0 \). At this moment, we now have a complete supraquantized time of arrival operator for arbitrary entire analytic potentials which satisfies all important properties of a time of arrival observable such as the quantum-classical correspondence principle, time reversal symmetry, hermiticity, and conjugacy with the Hamiltonian. Expectation values, eigenvalues, eigenfunctions, and probability distributions are constructed from the supraquantized TOA-operator in the standard way. Elsewhere, we will use the results obtained in this paper to investigate the quantum tunneling time of an elementary particle through piecewise rectangular and smooth potential barriers and to explore the exact role of the time-energy canonical commutation relation to the dynamics of time of arrival operators.

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Data Availability There are no data associated with this manuscript.

Appendix

Derivation of the leading quantum correction

We consider the leading quantum correction \( T_1(u, v) \) given by (40) where the coefficients \( \alpha_{m,j}^{(1)} \) satisfy the recurrence relation given by (41). Direct substitution of (41) into (40) leads to

\[
T_1(u, v) = \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} u^m v^{2j} \left( \frac{\mu}{2h^2} \right)^{j-1} \frac{1}{m \cdot 2^j} \sum_{l=1}^{m-1} l a_l \alpha_{m-l, j-1}^{(1)} \\
+ \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} u^m v^{2j} \left( \frac{\mu}{2h^2} \right)^{j-1} \frac{1}{m \cdot 2^j} \sum_{l=3}^{m+1} a_l \left( \frac{l}{3} \right) \alpha_{m-l+2, j-2}^{(1)}.
\]

Performing series rearrangements, shifting of indices, and using the following relations

\[
\int_0^u ds s^{m+l+1} = \frac{u^{m+l+2}}{m \cdot l + 2}, \quad \int_0^v dw w^{2j+1} = \frac{v^{2j+2}}{2j + 2}.
\]

Equation (134) simplifies to

\[
T_1(u, v) = \left( \frac{\mu}{2h^2} \right) \int_0^u ds V^r \left( \frac{s}{2} \right) \int_0^u dw w T_1(s, w) + \frac{1}{24} \left( \frac{\mu}{2h^2} \right) \int_0^u ds V^{rr} \left( \frac{s}{2} \right) \int_0^u dw w^3 T_0(s, w).
\]

Taking \( \partial^2/\partial u \partial v \) and using again the Leibniz integral rule given by (55), we get the following partial differential equation for \( T_1(u, v) \)

\[
\frac{\partial^2 T_1(u, v)}{\partial v \partial u} = \left( \frac{\mu}{2h^2} \right) V^r \left( \frac{u}{2} \right) v T_1(u, v) + \frac{1}{24} \left( \frac{\mu}{2h^2} \right) v^3 V^{rr} \left( \frac{u}{2} \right) T_0(u, v).
\]

It is straightforward to show the uniqueness of the solution \( T_1(u, v) \) with boundary conditions \( T_1(u, 0) = T_1(0, v) = 0 \). Suppose that \( T_{1,a}(u, v) \) and \( T_{1,b}(u, v) \) both satisfy (137). Since the leading kernel factor \( T_0(u, v) \) is unique, it can be shown using the triangle inequality that \( |T_{1,a}(u, v) - T_{1,b}(u, v)| \to 0 \) so that \( T_{1,a}(u, v) = T_{1,b}(u, v) \). Hence, \( T_1(u, v) \) is also unique. In fact, the uniqueness of \( T_1(u, v) \) is also guaranteed by the use of the method of successive approximations later.
Notice that (137) is dependent on \( T_1(s, w) \) and \( T_0(s, w) \) but the latter is just the leading kernel factor which is already known at this point. To solve for \( T_1(u, v) \), we apply again the method of successive approximations used in Sect. (3.1). Since we are solving for \( T_1(u, v) \), our zeroth-order approximation is the second term of (136), that is,

\[
T_{1,0}(u, v) = \frac{1}{24} \left( \frac{\mu}{2\hbar^2} \right) \int_0^u ds \int_0^v dw \, w^3 T_0(s, w). \tag{138}
\]

The \( n \)th-order approximation can then be determined from the following equation,

\[
T_{1,n}(u, v) = T_{1,0}(u, v) + \left( \frac{\mu}{2\hbar^2} \right) \int_0^u ds \int_0^v dw \, T_{1,n-1}(s, w) \tag{139}
\]

The solution \( T_1(u, v) \) of the integral equation in (136) is derived by taking the limit, \( T_1(u, v) = \lim_{n \to \infty} T_{1,n}(u, v) \).

From (139), we determine the first few iterates and also infer the general form for arbitrary \( n \). For \( n = 1 \), we have

\[
T_{1,1}(u, v) = \frac{1}{24} \left( \frac{\mu}{2\hbar^2} \right) \int_0^u ds \int_0^v dw \, w^3 T_0(s, w) + \frac{1}{24} \left( \frac{\mu}{2\hbar^2} \right)^2 \int_0^u ds \int_0^v dw \, T_0(s, w) \left[ V \left( \frac{s}{2} \right) - V \left( \frac{u}{2} \right) \right] \int_0^v dw \, w^3 (v^2 - w^2) T_0(s, w). \tag{140}
\]

For \( n = 2 \), we have

\[
T_{1,2}(u, v) = \frac{1}{24} \left( \frac{\mu}{2\hbar^2} \right) \int_0^u ds \int_0^v dw \, w^3 T_0(s, w) + \frac{1}{24} \left( \frac{\mu}{2\hbar^2} \right)^2 \int_0^u ds \int_0^v dw \, T_0(s, w) \left[ V \left( \frac{s}{2} \right) - V \left( \frac{u}{2} \right) \right] \int_0^v dw \, w^3 (v^2 - w^2) T_0(s, w) + \frac{1}{96} \left( \frac{\mu}{2\hbar^2} \right)^3 \int_0^u ds \int_0^v dw \, T_0(s, w) \left[ V \left( \frac{s}{2} \right) - V \left( \frac{u}{2} \right) \right]^2 \int_0^v dw \, w^3 (v^2 - w^2)^2 T_0(s, w). \tag{141}
\]

Doing the same calculations for \( n \geq 3 \), we infer the following form of \( T_{1,n}(u, v) \) for arbitrary \( n \)

\[
T_{1,n}(u, v) = \frac{1}{(48\hbar^2)^n} \int_0^u ds \int_0^v dw \, w^3 T_0(s, w) \times \sum_{k=0}^{n} \frac{1}{(1)k!} \left( \frac{\mu}{2\hbar^2} \right)^k (v^2 - w^2)^k \left[ V \left( \frac{s}{2} \right) - V \left( \frac{u}{2} \right) \right]^k. \tag{142}
\]

Equation (142) is also proven formally via mathematical induction. Taking the limit \( n \to \infty \) and using the definition of the hypergeometric function given by (68), we find the leading kernel factor correction to be

\[
T_1(u, v) = \left( \frac{\mu}{48\hbar^2} \right) \int_0^u ds \int_0^v dw \, w^3 T_0(s, w) \, {}_2F_1 \left( 1; 1; \frac{V(s)}{2} \right) \left( v^2 - w^2 \right) \left[ V \left( \frac{s}{2} \right) - V \left( \frac{u}{2} \right) \right]. \tag{143}
\]

in its integral form. Equation (143) clearly shows the dependence of the leading kernel correction \( T_1(u, v) \) on the potential \( V(q) \) and the leading kernel factor \( T_0(u, v) \) which is also dependent on the potential. It is straightforward to show that (91) satisfies the partial differential equation for the leading correction given by (137) subject to the boundary conditions \( T_1(u, 0) = T_1(0, v) = 0 \).

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