A new characterization of Auslander algebras

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Abstract

Let $\Lambda$ be a finite dimensional Auslander algebra. For a $\Lambda$-module $M$, we prove that the projective dimension of $M$ is at most one if and only if the projective dimension of its socle $\text{soc} M$ is at most one. As an application, we give a new characterization of Auslander algebra $\Lambda$, and prove that a finite dimensional algebra $\Lambda$ is an Auslander algebra provided its global dimension $\text{gl.d} \Lambda \leq 2$ and an injective $\Lambda$-module is projective if and only if the projective dimension of its socle is at most one.

Key words and phrases: Auslander algebra, projective dimension, global dimension.

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1 Introduction

A finite dimensional algebra $\Lambda$ is called an Auslander algebra if its global dimension is at most two and dominant dimension is at least two, that is, in the minimal injective resolution $0 \rightarrow \Lambda \rightarrow I_0 \rightarrow I_1 \rightarrow I_2 \rightarrow 0$ of $\Lambda$, both $I_0$ and $I_1$ are projective $\Lambda$-modules. It is introduced by Auslander when studying representation-finite algebras in [1] and can be constructed in the following way: let $R$ be a finite dimensional algebra of finite representation type and $M_1, M_2, \ldots, M_n$ be a complete set of representatives of the isomorphism classes of indecomposable $R$-modules, then $\Lambda = \text{End}_R(\oplus_{i=1}^{n} M_i)$ is the Auslander algebra of $R$. Moreover, this construction induces a mutually inverse bijection between Morita equivalence classes of representations-finite algebras and Morita equivalence classes of Auslander algebras. This bijection is called the Auslander correspondence and the correspondence is generalized by Iyama to a higher dimensional version in [2]. It is also known that Auslander algebras have close relations with quasi-hereditary algebras, preprojective algebras and projective quotient algebras, see [3,4,5] for details.

Let $M=\oplus_{i=1}^{n} M_i$ be the additive generator of $R$ and $\Lambda = \text{End}_R M$ be the corresponding Auslander algebra. We denote by $S_i$ the simple top of the indecomposable projective $\Lambda$-module $\text{Hom}_R(M, M_i)$, then the projective dimension $\text{pd}_\Lambda S_i$ of $S_i$ is at most two since the global dimension $\text{gl.d} \Lambda$ of $\Lambda$ is at most two. According to [1,VI, Proposition 5.11], we know that $\text{pd}_\Lambda S_i \leq 1$ if $M_i$ is a projective $R$-module and otherwise $\text{pd}_\Lambda S_i = 2$. Thus the projective dimensions of all simple $\Lambda$-modules are clear. However, for a non-simple $\Lambda$-module $M$, we don’t know whether its projective dimension $\text{pd}_\Lambda M$ is one or two. In this paper, we show that the projective dimension $\text{pd}_\Lambda M$ of $M$ is determined by the projective dimension of its socle $\text{soc}M$. Since $\text{soc}M$ is the direct sum of the simple submodules of $M$, by the above discussion, we can easily calculate the projective dimension of $\text{soc}M$.

Note that the socle of $M$ coincides with the socle of its injective envelope $I(M)$. Thus in order to investigate the relations between the projective dimensions of $M$ and $\text{soc} M$, we should study injective $\Lambda$-modules and their socles at first. Let $C(\Lambda)$ be the full subcategory of mod-$\Lambda$ whose objects are all the projective-injective $\Lambda$-modules, then we prove the following result about injective $\Lambda$-modules.
**Theorem A.** Let $\Lambda$ be a finite dimensional Auslander algebra and $I$ be an indecomposable injective $\Lambda$-module. Then $I$ is projective if and only if the projective dimension of its socle is at most one, that is, $\mathcal{C}(\Lambda) = \{I \in \text{mod-}\Lambda \mid I \text{ is injective and } \text{pd}_\Lambda \text{soc } I \leq 1\}$.

Now we show the relations between the projective dimensions of $\Lambda$-modules and their socles. Let $\mathcal{P}^1(\Lambda)$ be the full subcategory of $\text{mod-}\Lambda$ whose objects are all the $\Lambda$-modules $M$ with $\text{pd}_\Lambda M \leq 1$, we give the second result of this paper.

**Theorem B.** Let $\Lambda$ be a finite dimensional Auslander algebra and $M$ be a $\Lambda$-module. Then the projective dimension of $M$ is at most one if and only if the projective dimension of its socle $\text{soc } M$ is at most one, that is, $\mathcal{P}^1(\Lambda) = \{M \in \text{mod-}\Lambda \mid \text{pd}_\Lambda \text{soc } M \leq 1\}$.

Theorem B also implies that $\text{pd}_\Lambda M = 2$ if and only if $\text{pd}_\Lambda \text{soc } M = 2$. Before we complete this paper, Eiriksson gives a characterization of $\mathcal{P}^1(\Lambda)$ in [7], which states that $\mathcal{P}^1(\Lambda)$ consists of all $\Lambda$-modules cogenerated by projective $\Lambda$-modules. We investigate $\mathcal{P}^1(\Lambda)$ from a different point of view and stress that the projective dimension of a $\Lambda$-module $M$ is completely determined by the projective dimension of its socle.

The above two theorems state the properties of Auslander algebras, then it is natural to ask whether Auslander algebras can be characterized by these properties, that is, whether a finite dimensional algebra of global dimension at most two is an Auslander algebra provided it satisfies the properties in Theorem A or Theorem B. We give a positive answer to this question as following.

**Theorem C.** Let $\Lambda$ be a finite dimensional algebra. Then $\Lambda$ is an Auslander algebra if and only if its global dimension $\text{gl.d } \Lambda \leq 2$ and $\mathcal{C}(\Lambda) = \{I \in \text{mod-}\Lambda \mid I \text{ is injective and } \text{pd}_\Lambda \text{soc } I \leq 1\}$.

We also provide an example to show that a finite dimensional algebra $\Lambda$ with $\text{gl.d } \Lambda \leq 2$ and $\mathcal{P}^1(\Lambda) = \{M \in \text{mod-}\Lambda \mid \text{pd}_\Lambda \text{soc } M \leq 1\}$ is not necessarily an Auslander algebra.

This paper is arranged as follows. In section 2, we fix the notions and recall some necessary facts needed for our research. In section 3, we prove Theorem A and B. Section 4 is devoted to the proof of Theorem C.
2 Preliminaries

Throughout this paper, let \( k \) be an algebraically closed field and we consider basic finite dimensional \( k \)-algebras. For a finite dimensional \( k \)-algebra \( A \), we denote by \( \text{mod-} A \) the category of finitely generated right \( A \)-modules and by \( \text{gl.d} \, A \) the global dimension of \( A \). For a right \( A \)-module \( M \), \( \text{pd}_A M \) (\( \text{Id}_A M \)) is the projective (injective) dimension of \( M \) and \( \text{soc} \, M (\text{rad} \, M) \) is the socle (radical) of \( M \). We denote by \( \text{add} \, M \) the full subcategory of \( \text{mod-} A \) whose objects are direct summands of finite direct sums of copies of \( M \). \( \tau_A \) is the Auslander-Reiten translation of \( A \).

Let \( R \) be a finite dimensional \( k \)-algebra of finite representation type and \( M_1, M_2, \ldots, M_n \) be a complete set of representatives of the isomorphism classes of indecomposable \( R \)-modules. Then \( M = \bigoplus_{i=1}^n M_i \) is the additive generator of \( R \) and \( \Lambda = \text{End}_R M \) is the corresponding Auslander algebra. For a \( R \)-module \( X \), we denote by \( P_X = \text{Hom}_R(M, X) \) the corresponding projective \( \Lambda \)-module. Now we recall some basic properties of Auslander algebras.

**Proposition 2.1.** [6,Proposition 2.3] Let \( S_i (i = 1, 2, \ldots, n) \) be the simple top of the indecomposable projective \( \Lambda \)-module \( P_{M_i} = \text{Hom}_R(M, M_i) \). Then we have

1. \( \text{pd}_\Lambda S_i \leq 1 \) if and only if \( M_i \) is projective. Then \( 0 \to P_{\text{rad} M_i} \to P_{M_i} \to S_i \to 0 \) is a minimal projective resolution of \( S_i \).

2. \( \text{pd}_\Lambda S_i = 2 \) if and only if \( M_i \) is non-projective. Then the almost split sequence \( 0 \to \tau M_i \to E \to M_i \to 0 \) gives a minimal projective resolution \( 0 \to P_{\tau M_i} \to P_E \to P_{M_i} \to S_i \to 0 \) of \( S_i \).

The above proposition shows the relationship between almost split sequences in \( \text{mod-} R \) and projective resolutions of simple \( \Lambda \)-modules.

**Lemma 2.2.** [6,Lemma 2.4] Let \( R \) and \( \Lambda \) be as above and \( M_i \) be a non-projective \( R \)-module. Then we have \( \text{Ext}_\Lambda^2(S_i, \Lambda) \neq 0 \) and \( \text{Ext}_\Lambda^j(S_i, \Lambda) = 0 \) if \( j \neq 2 \).

If \( j = 0 \), we get \( \text{Hom}_\Lambda(S_i, \Lambda) = 0 \) and this implies that the socle of \( \Lambda \) is the direct sum of simple \( \Lambda \)-modules whose projective dimensions are at most one. Now we give a general
result about algebras of global dimension two.

**Lemma 2.3.** Let $A$ be a finite dimensional $k$-algebra with $\text{gl.d} \ A = 2$. Then we have $\text{pd}_A \text{soc} \ A \leq 1$.

**Proof.** Obviously we can get a short exact sequence $0 \to \text{soc} \ A \to A \to A/\text{soc} A \to 0$. If $A/\text{soc} A$ is a projective $A$-module, then this sequence splits and $\text{soc} \ A$ is also projective. Otherwise $\text{pd}_A \text{soc} \ A = \text{pd}_A (A/\text{soc} A) - 1 \leq 1$ since $A$ is projective and $\text{gl.d} \ A = 2$.

We also need the following lemma in this paper.

**Lemma 2.4.** [1,VI, Lemma 5.5] Let $A$ be a finite dimensional $k$-algebra and $\text{gl.d} \ A = n$. Then we have $\text{Id}_A \ A = n$.

The following proposition shows the relations between the projective dimensions of the three modules in a short exact sequence, which is very useful.

**Proposition 2.5.** [8,Appendix, Proposition 4.7] Let $A$ be a finite dimensional $k$-algebra and $0 \to L \to M \to N \to 0$ be a short exact sequence of $A$-modules. Then we have

1. $\text{pd}_A \ N \leq \max(\text{pd}_A \ M, 1 + \text{pd}_A \ L)$, and the equality holds if $\text{pd}_A \ M \neq \text{pd}_A \ L$.
2. $\text{pd}_A \ L \leq \max(\text{pd}_A \ M, -1 + \text{pd}_A \ N)$, and the equality holds if $\text{pd}_A \ M \neq \text{pd}_A \ N$.
3. $\text{pd}_A \ M \leq \max(\text{pd}_A \ L, \text{pd}_A \ N)$, and the equality holds if $\text{pd}_A \ N \neq 1 + \text{pd}_A \ L$.

Throughout this paper, we follow the standard terminologies and notations used in the representation theory of algebras, see [1,8].

3 Projective dimensions of modules over Auslander algebras

Let $A$ be a finite dimensional Auslander $k$-algebra. In this section, we investigate the $A$-modules with projective dimension at most one and show that they are determined by
the projective dimension of their socles.

By Lemma 2.3, we know that the socle of \( \Lambda \) is the direct sum of simple \( \Lambda \)-modules with projective dimension at most one. Furthermore, we prove that all simple \( \Lambda \)-modules with projective dimension at most one are contained in the socle of \( \Lambda \).

**Proposition 3.1.** Let \( \Lambda \) be a finite dimensional Auslander \( k \)-algebra and \( S_i \) be a simple \( \Lambda \)-module. Then we have \( \text{pd}_\Lambda S_i \leq 1 \) if and only if \( S_i \in \text{add soc} \Lambda \).

**Proof.** If \( S_i \in \text{add soc} \Lambda \), then \( \text{pd}_\Lambda S_i \leq 1 \) since \( \text{pd}_\Lambda \text{soc} \Lambda \leq 1 \) by Lemma 2.2.

Now assume \( \text{pd}_\Lambda S_i \leq 1 \), we only need to show \( \text{Hom}_\Lambda (S_i, \Lambda) \neq 0 \). According to Proposition 2.1, there exists an indecomposable projective \( R \)-module \( M_i \) such that

\[
0 \rightarrow \text{Hom}_R(M, \text{rad} M_i) \rightarrow \text{Hom}_R(M, M_i) \rightarrow S_i \rightarrow 0
\]

is a minimal projective resolution of \( S_i \). Applying \( \text{Hom}_\Lambda (\cdot, \Lambda) \) to the above short exact sequence, we get an exact sequence

\[
0 \rightarrow \text{Hom}_\Lambda (S_i, \Lambda) \rightarrow \text{Hom}_\Lambda (\text{Hom}_R(M, M_i), \Lambda) \rightarrow \text{Hom}_\Lambda (\text{Hom}_R(M, \text{rad} M_i), \Lambda)
\]

Note that \( \Lambda = \text{End}_R M \) and \( \text{Hom}_R(M, -) \) induces an equivalence between \( \text{add} M \) and \( \text{add} \Lambda \), then we get the following exact sequence

\[
0 \rightarrow \text{Hom}_\Lambda (S_i, \Lambda) \rightarrow \text{Hom}_R (M_i, M) \xrightarrow{g^*} \text{Hom}_R (\text{rad} M_i, M)
\]

Let \( \varphi \) be the epimorphism from \( M_i \) to \( \text{top} M_i \) and we decompose \( M = \text{top} M_i \oplus M' \), then \( (\varphi, 0)^t \in \text{Hom}_R(M_i, M) \). Obviously \( g^*((\varphi, 0)^t) = g^*(0) = 0 \) and \( g^* \) is not injective. This implies that \( \text{Hom}_\Lambda (S_i, \Lambda) \neq 0 \).

\[\square\]

It is known that the socle of a \( \Lambda \)-module \( M \) coincides with the socle of its injective envelope \( I(M) \), so we investigate the relationship between injective \( \Lambda \)-modules and their socles at first.

**Theorem 3.2.** Let \( \Lambda \) be a finite dimensional Auslander \( k \)-algebra and \( I \) be an indecomposable injective \( \Lambda \)-module. Then \( I \) is projective if and only if the projective dimension of
its socle is at most one, that is, \( C(\Lambda) = \{ I \in \text{mod-}\Lambda \mid I \text{ is injective and } \text{pd}_\Lambda \text{soc} I \leq 1 \} \).

**Proof.** If \( I \) is a projective \( \Lambda \)-module, then \( \text{soc} I \in \text{add} \text{soc} \Lambda \) and by Proposition 3.1, \( \text{pd}_\Lambda \text{soc} I \leq 1 \).

Conversely, assume \( \text{pd}_\Lambda \text{soc} I \leq 1 \). Again by Proposition 3.1, we have \( \text{soc} I \in \text{add} \text{soc} \Lambda \). This implies that \( I \) is a direct summand of the injective envelope \( I(\Lambda) \) of \( \Lambda \). Since the dominant dimension of \( \Lambda \) is at least two, \( I(\Lambda) \) is a projective \( \Lambda \)-module. Thus \( I \) is also projective.

\( \Box \)

Now we are in a position to prove the main result of this section.

**Theorem 3.3.** Let \( \Lambda \) be a finite dimensional Auslander \( k \)-algebra and \( M \) be a \( \Lambda \)-module. Then the projective dimension of \( M \) is at most one if and only if the projective dimension of its socle \( \text{soc} M \) is at most one, that is, \( P^1(\Lambda) = \{ M \in \text{mod-}\Lambda \mid \text{pd}_\Lambda \text{soc} M \leq 1 \} \).

**Proof.** Assume \( \text{pd}_\Lambda M \leq 1 \), then there exists a short exact sequence \( 0 \to P_1 \to P_0 \to M \to 0 \) where \( P_0 \) and \( P_1 \) are projective \( \Lambda \)-modules. If \( \text{pd}_\Lambda \text{soc} M = 2 \), there exists a simple \( \Lambda \)-module \( S \in \text{add} \text{soc} M \) such that \( \text{pd}_\Lambda S = 2 \). Let \( \varphi : S \to M \) be the inclusion, then we have the following commutative diagram with exact rows.

\[
\begin{array}{cccccc}
0 & \longrightarrow & P_1 & \longrightarrow & X & \longrightarrow & S & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & h & & \varphi \\
0 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0
\end{array}
\]

Since \( \varphi \) is an injection, by Snake lemma, \( h \) is also injective. According to Lemma 2.2, we get \( \text{Ext}_1^\Lambda(S, P_1) = 0 \). So the higher short exact sequence splits and \( X = P_1 \oplus S \). It follows that \( S \) is contained in the socle of \( P_0 \), which contradicts the fact that \( \text{pd}_\Lambda \text{soc} P_0 \leq 1 \).

Conversely, assume \( \text{pd}_\Lambda \text{soc} M \leq 1 \). Let \( I(M) \) be the injective envelope of \( M \), then we get a short exact sequence \( 0 \to M \to I(M) \to I(M)/M \to 0 \). Note that
soc \, M = \text{soc} \, I(M), \text{ we get } \text{pd}_\Lambda \text{soc} \, I(M) \leq 1. \text{ By Theorem 3.2, } I(M) \text{ is projective. If } I(M)/M \text{ is a projective } \Lambda\text{-module, } M \text{ is also projective. Otherwise, } \text{pd}_\Lambda \, M = \text{pd}_\Lambda \, I(M)/M - 1 \leq 1.

The above theorem also implies that for a \Lambda\text{-module } M, \text{pd}_\Lambda \, M = 2 \text{ if and only if } \text{pd}_\Lambda \text{soc} \, M = 2. \text{ This shows that the projective dimension of } M \text{ is completely determined by the projective dimension of its socle. By Proposition 2.1, the projective dimensions of all simple } \Lambda\text{-modules are clear, then we can get the projective dimension of any module over the Auslander algebra } \Lambda.

4 Characterizations of Auslander algebras

Auslander algebras are characterized by the properties that global dimension is at most two and dominant dimension is at least two. In [5], Crawley-Boevey and Sauter give a new characterization of the Auslander algebra \Lambda, that is gl.d \Lambda \leq 2 \text{ and there exists a tilting } \Lambda\text{-module which is generated and cogenerated by projective-injective } \Lambda\text{-modules. In this section, we also give a new characterization of Auslander algebras.}

Theorem 3.2 and Theorem 3.3 state two properties of the Auslander algebra \Lambda, that is \mathcal{C}(\Lambda) = \{I \in \text{mod-}\Lambda \mid I \text{ is injective and } \text{pd}_\Lambda \text{soc} \, I \leq 1\} \text{ and } \mathcal{P}^1(\Lambda) = \{M \in \text{mod-}\Lambda \mid \text{pd}_\Lambda \text{soc} \, M \leq 1\}. \text{ Then it is natural to consider whether a finite dimensional algebra of global dimension at most two is an Auslander algebra provided it satisfies these properties. We prove the following result.}

**Theorem 4.1.** Let \Lambda be a finite dimensional \, k\text{-algebra. Then } \Lambda \text{ is an Auslander algebra if and only if its global dimension } \text{gl.d} \, \Lambda \leq 2 \text{ and } \mathcal{C}(\Lambda) = \{I \in \text{mod-}\Lambda \mid I \text{ is injective and } \text{pd}_\Lambda \text{soc} \, I \leq 1\}.

**Proof.** If \Lambda is an Auslander algebra, then \text{gl.d} \, \Lambda \leq 2 \text{ and by Theorem 3.2, we have } \mathcal{C}(\Lambda) = \{I \in \text{mod-}\Lambda \mid I \text{ is injective and } \text{pd}_\Lambda \text{soc} \, I \leq 1\}.\]
Now assume $\Lambda$ is a finite dimensional $k$-algebra with $\text{gl.d} \Lambda \leq 2$ and $\mathcal{C}(\Lambda) = \{ I \in \text{mod-}\Lambda \mid I \text{ is injective and } \text{pd}_\Lambda \text{soc } I \leq 1 \}$. We only need to show that the dominant dimension of $\Lambda$ is at least two.

If $\text{gl.d} \Lambda \leq 1$, then all injective $\Lambda$-modules are projective and $\Lambda$ is a self-injective algebra. For a $\Lambda$-module $M$, there exists a short exact sequence $0 \to P_1 \to P_0 \to M \to 0$ where $P_0$ and $P_1$ are projective $\Lambda$-modules. Since $P_1$ is also injective, this short exact sequence splits and $M$ is projective. It follows that $\Lambda$ is a semi-simple algebra. Thus $\Lambda$ is an Auslander algebra.

If $\text{gl.d} \Lambda = 2$, by Lemma 2.4, we have $\text{Id}_\Lambda \Lambda = 2$. Let $0 \to \Lambda \xrightarrow{f} I_0 \to I_1 \to I_2 \to 0$ be a minimal injective resolution of $\Lambda$. Since $I_0$ is the injective envelope of $\Lambda$, we have $\text{soc } I_0 = \text{soc } \Lambda$. According to Lemma 2.3, $\text{pd}_\Lambda \text{soc } I_0 = \text{pd}_\Lambda \text{soc } \Lambda \leq 1$, then $I_0$ is projective. Consider the short exact sequence $0 \to \Lambda \xrightarrow{f} I_0 \to \text{Im } f \to 0$. We claims that $\text{pd}_\Lambda \text{soc } \text{Im } f \leq 1$.

Otherwise, if $\text{pd}_\Lambda \text{soc } \text{Im } f = 2$, let $g : \text{soc } \text{Im } f \to \text{Im } f$ be the inclusion. Then we get the following commutative diagram with exact rows.

\[
\begin{array}{ccccccc}
0 & \longrightarrow & \Lambda & \longrightarrow & E & \longrightarrow & \text{soc } \text{Im } f & \longrightarrow & 0 \\
& & h & & & & g & \\
0 & \longrightarrow & \Lambda & \longrightarrow & I_0 & \longrightarrow & \text{Im } f & \longrightarrow & 0 \\
\end{array}
\]

Since $g$ is injective, by Snake lemma, $h$ is also an injection. Note that $\text{pd}_\Lambda \text{soc } \text{Im } f = 2 \neq 1 + \text{pd}_\Lambda \Lambda$, by Proposition 2.5, we have that $\text{pd}_\Lambda E = \max(\text{pd}_\Lambda \Lambda, \text{pd}_\Lambda \text{soc } \text{Im } f) = 2$. Consider the short exact sequence $0 \to E \to I_0 \to Y \to 0$. Since $I_0$ is projective, we have $\text{pd}_\Lambda Y = \text{pd}_\Lambda E + 1 = 3$, a contradiction.

Thus $\text{pd}_\Lambda \text{soc } \text{Im } f \leq 1$ and $I_1$ is projective since $I_1$ is the injective envelope of $\text{Im } f$ and $\text{soc } I_1 = \text{soc } \text{Im } f$. Then the dominant dimension of $\Lambda$ is at most two and this completes our proof.

$\square$
We should mention that a finite dimensional $k$-algebra $\Lambda$ with $\text{gl.d} \Lambda \leq 2$ and $\mathcal{P}^1(\Lambda) = \{ M \in \text{mod-}\Lambda \mid \text{pd} \Lambda \text{soc} M \leq 1 \}$ is not necessarily an Auslander algebra. The following is a counter-example.

**Example.** Let $\Lambda = kQ/\langle \beta \alpha \rangle$ be a finite dimensional $k$-algebra where $Q$ is the quiver
\[ \begin{array}{cccc}
1 & \overset{\alpha}{\rightarrow} & 2 & \overset{\beta}{\rightarrow} & 3 & \overset{\gamma}{\leftarrow} & 4.
\end{array} \]
We have $\text{pd}_\Lambda S_1 = 2$ and $\text{pd}_\Lambda S_i \leq 1$ for $i = 2, 3, 4$. Moreover, $S_1$ is the only indecomposable $\Lambda$-module whose projective dimension is two. Then $\text{gl.d} \Lambda = 2$ and it is easy to know $\mathcal{P}^1(\Lambda) = \{ M \in \text{mod-}\Lambda \mid \text{pd}_\Lambda \text{soc} M \leq 1 \}$. However, the indecomposable injective $\Lambda$-module $S_4$ is not projective and $\Lambda$ is not an Auslander algebra since its dominant dimension is zero.

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