Output Spectrum of Single-Atom Lasers

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(April 1, 2022)

We consider a laser composed of a single atom in a microcavity, with a coherent or incoherent pump. We consider both three- and four-level gain schemes, and examine the output spectrum of such lasers. We find that the linewidth generally scales as the inverse of the photon number. For large atom-field coupling, a vacuum-Rabi doublet structure is obtained. In the three-level case, this vacuum-Rabi splitting is apparent only for small intracavity photon numbers, and vanishes for large pumps. In the four-level scheme, the vacuum-Rabi structure appears at a nonzero pump level, and is maintained for large pumps, even when the intracavity photon number is larger than unity. This behavior is explained utilizing the quantum trajectory approach.

I. INTRODUCTION

In recent years it has become possible to explore the dynamics of a single atom as it passes through a microcavity at low velocity. By this we mean that the transit time for the atom across the cavity mode is on the order of a hundred spontaneous emission lifetimes. This slow atomic beam is generated by dumping cold atoms from a magneto-optical trap into a microcavity [1]. There has been much work in the past on the micromaser/microlaser, where two-level atoms in the excited state are flown through a microcavity of high-Q, and interact with the field mode of the cavity [2]. These systems are quite interesting and exhibit a variety of nonclassical effects, but we wish to examine here the case where the atom starts in the ground state, lives in the cavity mode for many lifetimes with an essentially constant atom-field coupling strength. Of course this coupling strength will change with time as the atom moves through the Gaussian profile of the mode, but we will assume here that the atom-field dynamics are such that the atom is essentially stationary. Hence we have the one-atom limit of a laser pointer; we have a fixed gain medium in a cavity, pumped by an external source, but the gain medium is composed of a single three- or four-level atom. Some previous work has been done on this system. Smith and Gardiner [3] were the first to consider such a system, but did not explore large enough atom-field couplings to obtain interesting results. The key result of their paper was a way to treat single atom systems in a Fokker-Planck approach. Mu and Savage [4] considered several classes of single atom lasers and focused on the photon statistics. They showed that lasing was possible for such a system, and that for large atom-field coupling the single atom laser emitted amplitude squeezed, or antibunched light. For parameters for which the output light was amplitude squeezed, the linewidth of the laser was shown to increase with pump strength rather than decrease as in the usual Schawlow-Townes type fashion. Ritsch and Pellazari [5] also examined the photon statistics of a single atom laser, and they too predict regions of parameter space where the output is amplitude squeezed. Ginzel et. al. [6] considered various aspects of single-atom laser systems, including the observation of a vacuum-Rabi doublet in the output spectrum, but in their work the emphasis was on the application of a new computational approach, that of the damping basis. In that approach, the basis states were eigenstates of the dissipative part of the master equation. Loffler et. al. [7] considered the spectrum of a three-level single atom laser, and predicted vacuum-Rabi structures in the output spectrum. In this work we show that the two-peaked vacuum-Rabi structure vanishes quickly as the pump strength is increased. More recently Jones et. al. [8] have examined the photon statistics of single-atom laser systems, with particular emphasis on how the systems behaved as β was changed. The parameter β is the fraction of spontaneous emission into the lasing cavity. It is a parameter of much interest in the microlaser community. As β tends towards unity, the laser output rises linearly with pump strength; this has been referred to as a “thresholdless” laser. For more on this subject, we refer the reader to recent work on macroscopic laser systems and their dependence on β. [9,10]

In section 2, we discuss the output spectrum of a three-level incoherently pumped single-atom laser. Section 3 deals with the four-level incoherently pumped single-atom laser. In section 4 we examine the four-level model, but with coherent pumping, to determine if changing the pumping mechanism alters the results. Finally in section 5 we conclude.

A schematic of the system is shown in Figure 1. We adiabatically eliminate the fast transition from the topmost state to the upper lasing level, and use a two-level model, where the incoherent pump is modeled via the Γ term in the following master equation, [11]

\[ \dot{\rho} = \frac{-i}{\hbar} [H_s, \rho] + \kappa(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger) \]
\[ +\frac{\gamma}{2}(2\sigma_+ \rho \sigma_- - \sigma_+ \sigma_- \rho - \rho \sigma_+ \sigma_-) \]
\[ +\frac{\Gamma}{2}(2\sigma_- \rho \sigma_+ - \sigma_- \sigma_+ \rho - \rho \sigma_- \sigma_+) \] (1)

with the system Hamiltonian given by

\[ H_S = i\hbar g(a^\dagger \sigma_- - a \sigma_+) \] (2)

The only nonzero density matrix elements are the diagonal elements (populations of the various levels) and coherences between states \(|n,1\rangle\) and \(|n-1,2\rangle\). Here \(n\) denotes the photon occupation number. The equations for these density matrix elements are

\[ \frac{d\rho_{n,1;1,1}}{dt} = 2\kappa (n+1)\rho_{n+1,1;1,1} - (2\kappa n + \Gamma) \rho_{n,1;1,1} + \gamma \rho_{n,2;2,2} - 2\sqrt{n+1}g\rho_{n,1;1,2} \] (3)

\[ \frac{d\rho_{n,2;2,2}}{dt} = 2\kappa (n+1)\rho_{n+1,2;2,2} - \{2\kappa n + \gamma\} \rho_{n,2;2,2} + \Gamma \rho_{n,1;1,1} + 2\sqrt{n}g\rho_{n,1;1,2} \] (4)

\[ \frac{d\rho_{n,1;1,2}}{dt} = 2\kappa \sqrt{n(n-1)}\rho_{n+1,2;1,1} - \left\{\kappa (n-1) + \frac{\Gamma + \gamma}{2}\right\} \rho_{n,2;1,1} + \sqrt{n}g(\rho_{n-1,2;1,1} - \rho_{n,1;1,1}) \] (5)

We solve these equations in the steady state, to obtain needed initial conditions for spectrum calculations. These matrix elements can also be used to calculate photon statistics, such as the mean photon number and Fano factor (variance over the mean). Typically we start by truncating the photon basis at some small number (3-10). Calculations are checked at the end to check that the population of the highest photon number states is less than 10\(^{-4}\). If that were not the case, the program that does our calculations repeats the process, but increments the maximal photon number, and rechecks that we are keeping enough photon states. The calculation uses \(\sum_{i,n} \rho_{i,n;i,n} = 1\) to solve for \(\rho_{0,-0,-}\) in terms of the other diagonal density matrix elements. In much of what follows, we will refer to \(\beta\), the fraction of spontaneous emission into the cavity mode.

\[ \beta = \frac{2g^2/(\gamma + \Gamma + 2\kappa)}{2g^2/(\gamma + \Gamma + 2\kappa) + \gamma/2} \] (6)

We are interested in calculating the output spectrum of the laser,

\[ S(\omega) = \int_{-\infty}^{\infty} d\tau e^{i\omega \tau} \langle a^\dagger(0) a(\tau) \rangle = 2\Re \int_{0}^{\infty} d\tau e^{i\omega \tau} \langle a^\dagger(0) a(\tau) \rangle . \] (7)

We calculate this spectrum using the quantum regression theorem,

\[ \langle a^\dagger(0) a(\tau) \rangle = \text{tr} \{a(0) A(\tau) \} = \sum_{i,n} \sqrt{n+1} \langle i,n+1 | A(\tau) | i,n \rangle \] (8)

where \(A(0) = \rho_{SS} a^\dagger\) and \(\dot{A} = \mathcal{L} A\). The resulting equations can be written in the form

\[ \frac{d\dot{A}}{dt} = \hat{M} \dot{A} \] (9)

The relevant equations for the matrix elements of \(\dot{A}\) are

\[ \frac{dA_{n+1,1;1,1}}{dt} = 2\kappa \sqrt{(n+2)(n+1)}A_{n+2,1;1,1} - (\kappa (2n+1) + \Gamma) A_{n+1,1;1,1} + \gamma A_{n+1,2;2,2} \]
After taking the Fourier transform of the above differential equations we have

\[
\frac{dA_{n+1,2;n,2}}{dt} = 2\kappa \sqrt{(n+2)(n+1)}A_{n+2,2;n,1,1} - (\kappa(n+1) + \gamma)A_{n+1,2;n,2} + \Gamma A_{n+1,1;n,1} - g\sqrt{n+2}A_{n+2,1;n,2} - g\sqrt{n+1}A_{n+2,1;n,1,1}
\]

(11)

\[
\frac{dA_{n+2,1;n,2}}{dt} = 2\kappa \sqrt{(n+3)(n+1)}A_{n+3,1;n,1,2} - (\kappa(n+1) + \gamma/2 + \Gamma/2)A_{n+2,1;1,2} + g\sqrt{n+2}A_{n+1,2;n,2} - g\sqrt{n+1}A_{n+1,2;n,1,1}
\]

(12)

\[
\frac{dA_{n,2;n,1}}{dt} = 2\kappa(n+1)A_{n+1,2;n,1,1} - (2\kappa + \gamma/2 + \kappa/2)A_{n+1,2;n,1,1} + g\sqrt{n}A_{n,2;n,1,2} - g\sqrt{n+1}A_{n+1,1;n,1}
\]

(13)

After taking the Fourier transform of the above differential equations we have

\[\vec{\tilde{A}}(\omega) = \left\{ \hat{M} - i\omega\hat{T} \right\}^{-1}\tilde{A}(0)\]

(14)

with \(\vec{\tilde{A}}(\omega)\) composed of the Fourier transform of \(\tilde{A}(\tau)\) and then we can easily form the spectrum

\[S(\omega) = \sum_{i,n} \sqrt{n+1} \langle i, n+1 | n, j \rangle \langle n, j | \tilde{A}\rangle |i, n\rangle.
\]

(15)

In solving these equations, we truncate the photon basis at the same value of \(n\) as in the density matrix element equations.

In Figure 2, we plot the output spectrum of the laser for \(g/\gamma = 0.1\) and \(\kappa/\gamma = 0.1\) as a function of pumping strength, \(\Gamma\). We see that the lineshape is approximately Lorentzian, and the width decreases initially as the pump strength is increased. All spectra in this paper are normalized so that the integrated spectrum is unity. At larger pumps we see that the linewidth begins to broaden. This is apparent most readily from examining the peak of the spectrum. It rises rapidly with pump rate, reaches a maximum, and slowly goes back down. As the area is the same (by our normalization) this behavior is indicative of the initial narrowing, which reaches a minimum before beginning to broaden. In Figure 3, we plot the linewidth, obtained by fitting a Lorentzian curve, versus pump strength. For comparison, we also plot \(\Delta \omega_{ST} = \kappa/2(n)\), the Schawlow-Townes result. Here the mean photon number \(\langle n \rangle\) has been directly calculated from the steady state density matrix elements via \(\langle n \rangle = \sum_{n,j} n\langle n, j | \rho_{SS} | n, j \rangle\) Here \(n\) of course is the photon number index, and \(j = 1, 2\) are the atomic level indices. We see that while the linewidth initially decreases with photon number, it is always broader than the Schawlow-Townes result for small pump rates. We see that as the laser turns off with increasing pump strength, the linewidth increases, and goes below the Schawlow-Townes result, but qualitatively it follows that trend. The mean intracavity photon number is plotted as a function of driving field strength, for these same parameters, in Figure 4. We see that the laser does turn off with pump strength. This is a feature of the incoherently pumped three-level laser, both for single atom and macroscopic systems, as shown by Mu and Savage 3. Jones et. al. 6, and Koganov and Shuker 12. This is due to the incoherent nature of the pumping process, which causes the atom to uncouple from the field for high pump rates. This is due to the incoherent pump process decohering the induced dipole on the lasing transition. This can also be viewed as \(\beta \rightarrow 0\) as \(\Gamma \rightarrow \infty\), that the fraction of spontaneous emission into the cavity mode is pump dependent, and there is no spontaneous emission into the cavity for high pump rates. From the usual arguments of quantum electrodynamics, there is also no stimulated emission. From the viewpoint of quantum trajectory theory, the pump mechanism is a jump process, and the atom becomes trapped in the upper state of the lasing transition. As one increases \(g/\gamma\) to 0.6, we have found that the single atom laser emits amplitude squeezed light 6, and that the linewidth indeed increases with pump strength, even for small pumps, as first predicted by Mu and Savage 3. Further increasing \(g/\gamma\) to 1.414, as in Figure 6, we see that the spectrum exhibits a double-peaked structure. This has been predicted by Löffler et. al. 8, and they identified the source of this structure vacuum-Rabi oscillations on the lasing transition. However we see that this structure only exists for very small pump strengths, when there are not very many photons in the cavity. This is not unexpected, as the vacuum-Rabi oscillations are associated with coherent oscillations of the one-photon states \(|0, 2\rangle\) and \(|1, 1\rangle\). When states with higher photon number are occupied, the vacuum-Rabi structure vanishes, as many transition frequencies between various dressed states start to appear. In Figure 8, we show the spectrum for larger pump values, where one can observe the vacuum-Rabi doublet disappear. Essentially the doublet is coming from spontaneous emission.
of the strongly coupled atom-cavity system, and in no sense from a laser. The mean photon number versus pump is exhibited in Figure 7. There is of course no well defined threshold for such a microscopic case. We see that the vacuum-Rabi oscillations vanish well before the mean photon number nears unity. Returning our attention to Figure 6 for a moment, we notice that again in this case, at high pump rates when the mean intracavity photon number begins to decrease, the linewidth begins to increase. Again this is easily apparent from the drop in the value of the normalized spectrum at line center, and is consistent with the turn off of the laser.

We gain further insight into this behavior by examining results of quantum trajectory simulations.\[13,14\] The conditioned wave function is taken to be

\[
|\psi_c(t)\rangle = \sum_{n=0}^{\infty} C_{1,n}(t) e^{-iE_{1,n}t}|1,n\rangle + C_{2,n}(t) e^{-iE_{2,n}t}|2,n\rangle
\]  

(16)

The coherent evolution of the conditioned wave function obeys the Schroedinger equation with the following non-Hermitian Hamiltonian,

\[
H_D = \hbar(\omega - ik)a^+a + i\hbar g (a^+\sigma_{32} - a\sigma_-)
\]  

\[-i\hbar\frac{\gamma}{2}\sigma_+\sigma_- - i\hbar\frac{\Gamma}{2}\sigma_-\sigma_+\]

(17)

At each time step, the system is subject to collapses of the wavefunction according to the collapse operators

\[
\hat{F}_1 = \sqrt{\gamma}\sigma_-
\]  

(18)

\[
\hat{F}_2 = \sqrt{\Gamma}\sigma_+
\]  

(19)

\[
\hat{F}_3 = \sqrt{2\kappa}a.
\]  

(20)

The probability of a collapse is proportional to the size of the time step multiplied by \(\langle \psi_c | F^\dagger F | \psi_c \rangle\). A separate random number is used to determine at each time step whether a particular collapse occurs. In the unlikely case that two collapses are to occur in a given time step, another random number is used to determine which actually occurs. Of course, we must choose our time step to be small compared to the fastest rates in the problem to minimize such occurrences. In Figures 8-9 we plot the induced dipole for the lasing transition, and population of the upper lasing state for two values of \(\Gamma/\gamma\) and \(g/\gamma = 1.414\). For smaller pump strengths, as in Figure 8, there is an obvious vacuum-Rabi oscillation apparent. For larger pump strengths, as in Figure 9, there are effectively no coherent oscillations in the induced dipole or population, hence there is no vacuum-Rabi structure for large pumps. The incoherent pump process interrupts the coherent oscillations. In the quantum trajectory view, the incoherent pump is modeled as an upward jump. A pump event places the atom in the upper state of the lasing transition, and kills off the coherence between the two lasing states. As the pump rate increases, the atom becomes trapped in the upper level of the lasing transition, and decouples from the field. This eventually results in the mean photon number dropping to zero as the pump rate is increased.

II. INCOHERENTLY PUMPED FOUR-LEVEL LASER

This system is shown schematically in Figure 10. Again, we adiabatically eliminate the level above the upper lasing level. We are left with an effective three-level system, with the incoherent pump modeled as in the above work on the incoherently pumped three-level laser. The master equation for the incoherently pumped four-level laser is then

\[
\dot{\rho} = -\frac{i}{\hbar}[H_s,\rho] + \kappa(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a)
\]  

\[+\frac{\Gamma}{2}(2\sigma_{13}\rho\sigma_{31} - \sigma_{31}\sigma_{13}\rho - \rho\sigma_{31}\sigma_{13})
\]  

\[+\frac{\gamma}{2}(2\sigma_{32}\rho\sigma_{23} - \sigma_{23}\sigma_{32}\rho - \rho\sigma_{23}\sigma_{32})
\]  

\[+\frac{\gamma_f}{2}(2\sigma_{21}\rho\sigma_{12} - \sigma_{12}\sigma_{21}\rho - \rho\sigma_{12}\sigma_{21})
\]  

(21)

with

\[
H_S = i\hbar g(a^\dagger\sigma_{32} - a\sigma_{23})
\]  

(22)
and the following definition for atomic raising and lowering operators

\[ \sigma_{ij} = |j\rangle \langle i| \]  

(23)

The equations for the nonzero density matrix elements are

\[
\frac{d\rho_{n,1;1,n,1}}{dt} = 2\kappa(n+1)\rho_{n+1,1;n+1,1} - \{2\kappa n + \Gamma\} \rho_{n,1;n,1} + \gamma \rho_{n,2;n,2}
\]

(24)

\[
\frac{d\rho_{n,2,n,2}}{dt} = 2\kappa(n+1)\rho_{n+1,2;n+1,2} - \{2\kappa n + \gamma f\} \rho_{n,2;n,2} + \gamma \rho_{n,3,n,3} + 2\sqrt{n}g\rho_{n,2,n-1,3}
\]

(25)

\[
\frac{d\rho_{n,3;3,n,3}}{dt} = 2\kappa(n+1)\rho_{n+1,3;n+1,3} - \{2\kappa n + \gamma\} \rho_{n,3;3,n,3} + \gamma \rho_{n,1;n,1} - 2\sqrt{n+1}\rho_{n,3,n+1,2}
\]

(26)

\[
\frac{d\rho_{n,2,n-1,3}}{dt} = 2\sqrt{n(n-1)}\rho_{n+1,2;n-1,3} - \{\kappa(2n-1) + \frac{\gamma}{2}\} \rho_{n,2;n-1,3} + \sqrt{n} \{\rho_{n-1,3;n-1,3} - \rho_{n,2;n,2}\}.
\]

(27)

We again calculate this spectrum using the quantum regression theorem. The relevant equations for the matrix elements of \( A \) in this case are

\[
\frac{dA_{n+1,1;n,1}}{dt} = 2\kappa\sqrt{(n+2)(n+1)}A_{n+2,1;n+1,1} - (\kappa(2n+1) + \Gamma) A_{n+1,1;n,1} + \gamma f A_{n+1,2;n,2}
\]

(28)

\[
\frac{dA_{n+1,2;n,2}}{dt} = 2\kappa\sqrt{(n+2)(n+1)}A_{n+2,2;n+1,2} - (\kappa(2n+1) + \gamma f) A_{n+1,2;n,2} + \gamma A_{n+1,3;n,3} + g\sqrt{n+1}A_{n,3;n,2} + g\sqrt{n}A_{n+1,2;n-1,3}
\]

(29)

\[
\frac{dA_{n+3,1;n,3}}{dt} = 2\kappa\sqrt{(n+2)(n+1)}A_{n+2,3;n+1,3} - (\kappa(2n+1) + \gamma) A_{n+1,3;n,3} + \gamma A_{n+1,1;n,1} - g\sqrt{n+2}A_{n+2,2;n-1} - g\sqrt{n}A_{n+1,3;n+1,2}
\]

(30)

\[
\frac{dA_{n+2,2;n,3}}{dt} = 2\kappa\sqrt{(n+3)(n+1)}A_{n+3,2;n+1,3} - (\kappa(2n+2) + \gamma/2 + \gamma f/2) A_{n,2;n,3} + g\sqrt{n+2}A_{n+1,3;n,3} - g\sqrt{n+1}A_{n+2,2;n-1,3}
\]

(31)

\[
\frac{dA_{n,3,n,2}}{dt} = 2\kappa(n+1)A_{n+1,3;n+1,2} - (2\kappa n + \gamma/2) A_{n,3;n,2} + g\sqrt{n}A_{n,3;n-1,3} - g\sqrt{n+1}A_{n+1,2;n,3}
\]

(32)

After taking the Fourier transform of the above equations, we have

\[
\tilde{\mathbf{A}}(\omega) = \left\{ \mathbf{M} - i\omega \mathbf{I} \right\}^{-1} \mathbf{A}(0)
\]

(33)

with \( \tilde{\mathbf{A}}(\omega) \) composed of the Fourier transform of \( \mathbf{A}(\tau) \) and then we can easily form the spectrum

\[
S(\omega) = \sum_{i,n} \sqrt{n+1} \left\langle i, n+1 \left| \Re \tilde{\mathbf{A}}(\omega) \right| i, n \right\rangle
\]

(34)

In solving these equations, we truncate the photon basis at the same value of \( n \) as in the density matrix element equations.

In Figure [1], we plot the spectrum of the laser for \( g/\gamma = 1.4, \gamma f/\gamma = 10.0 \) and \( \kappa/\gamma = 0.1 \). We see that the spectrum is a single peaked structure, whose linewidth decreases with pump strength initially, and asymptotically approaching a limiting value. In Figure [2], we plot the linewidth of the laser spectrum versus pump strength, for various values of \( \beta \), for \( \kappa/\gamma = 0.1 \) and \( \gamma f/\gamma = 10.0 \). Recall that the fraction of spontaneous emission into the lasing cavity, \( \beta \), is
then determined by \( g \), with \( \kappa, \gamma_f, \) and \( \gamma \) fixed. This linewidth is again obtained by curve fitting a Lorentzian to the output spectrum. For comparison, we also plot the Schawlow-Townes result, \( \kappa/2(n) \); the mean photon number again calculated from the steady state density matrix. We see that the linewidth is always broader than the Schawlow-Townes limit, but that the linewidth decreases with the inverse of the photon number. The photon number pins for large pump rates, as it does no good to pump the single atom laser faster than the fastest decay rate (usually \( \gamma_f \)), and the linewidth also pins at an asymptotic value. As one increases \( \beta \) to the range 0.6 - 0.9, (or increases \( g \) given that \( \kappa/\gamma \) and \( \gamma_f/\gamma \) are fixed), the asymptotic value of the linewidth for large pumps begins to increase, as shown in Figure 14. For a value of \( \beta = 0.998 \), with \( \kappa/\gamma = 0.1 \), and \( \gamma_f/\gamma = 100.0 \), we see that the linewidth increases with pump strength. As in the case of the three-level system, this is concurrent with the emitted light being amplitude squeezed. In fact for all values of \( \beta \) above 0.5 or so, the light is amplitude squeezed. For the uppermost curve in Figure 13, \( g/\gamma = 100.0 \), but there is no chance of vacuum-Rabi oscillations as \( \gamma_f/\gamma = 100.0 \), and so the decoherence rate \( \gamma_f = g \). We can obtain vacuum-Rabi structure in the output spectrum in many cases, for example \( g/\gamma = 10.0 \), \( \gamma_f/\gamma = 2.0 \), and \( \kappa/\gamma = 0.1 \) as shown in Figure 14. We see that there is a single peaked structure at small pump rate, which then evolves into a double peaked, vacuum-Rabi structure at large pumps, reaching an asymptotic spectrum for large pumps. The average photon number in the cavity reaches \( \langle n \rangle = 2.6 \) for these parameters and large pump, and is larger than one, as shown in Figure 13. The location of the two peaks are not well approximated by the complex part of the single-photon eigenvalues for this system, nor is the width well approximated by the real part. This is due to the fact that more than the one-photon state is involved in this process.

To understand this behavior, it is again instructive to look at quantum trajectory simulations. We take the conditioned wave function to be

\[
|\psi_c(t)\rangle = \sum_{n=0}^{\infty} C_{1,n}(t) e^{-iE_{1,n}t} |1,n\rangle + C_{2,n}(t) e^{-iE_{2,n}t} |2,n\rangle + C_{3,n}(t) e^{-iE_{3,n}t} |3,n\rangle
\]

(35)

where again, the unitary evolution of this wave function is governed by a Schrodinger equation with the following non-Hermitian Hamiltonian,

\[
H_D = \hbar(\omega - i\kappa)a^+a + i\hbar g(a^+\sigma_{32} - a\sigma_{23}) - i\hbar\frac{\gamma_f}{2}\sigma_{32}\sigma_{32} - i\hbar\frac{\gamma_f}{2}\sigma_{32}\sigma_{21} - i\hbar\frac{\Gamma}{2}\sigma_{13}a.
\]

(36)

Here we have four associated collapse processes, that are governed by the following four collapse operators,

\[
\hat{F}_1 = \sqrt{\gamma}\sigma_{32}
\]

(37)

\[
\hat{F}_2 = \sqrt{\gamma}\sigma_{21}
\]

(38)

\[
\hat{F}_3 = \sqrt{\Gamma}\sigma_{13}
\]

(39)

\[
\hat{F}_4 = \sqrt{2\kappa}a.
\]

(40)

These trajectories are generated in the same manner as those of the three-level system in the proceeding section. The derivation of \( \beta \) is particularly transparent in the quantum trajectory formalism, using the equations for the probability amplitudes for the various states,

\[
\hat{C}_{1,n} = -\left(\frac{\Gamma}{2} + n\kappa\right) C_{1,n}
\]

(41)

\[
\hat{C}_{2,n+1} = -\left(\frac{\gamma_f}{2} + (n + 1)\kappa\right) C_{2,n+1} + g\sqrt{n+1} C_{3,n}
\]

(42)

\[
\hat{C}_{3,n} = -\left(\frac{\gamma_f}{2} + n\kappa\right) C_{3,n} - g\sqrt{n+1} C_{2,n+1}.
\]

(43)

If \( \gamma_f \gg \gamma, g, \kappa, \Gamma \), then we have

\[
\hat{C}_{3,n} = -\left(\frac{\gamma_f}{2} + n\kappa\right) C_{3,n} - \frac{g^2}{\kappa(n+1) + \gamma_f/2} (n+1) C_{1,n+1}
\]

(44)

In the case of \( n = 0 \), we may read off

\[
\beta = \frac{2g^2/(\gamma_f + 2\kappa)}{2g^2/(\gamma_f + 2\kappa) + \gamma/2}
\]

(45)
as the fraction of spontaneous emission into the cavity mode.

Why is there no double-peaked structure in the spectrum for small pumps? Examining the temporal evolution of the induced dipole on the lasing transition in Figure 14, we see that for small pump strengths the dipole is usually zero, and an essentially random time occurs before the next oscillation occurs, which then lasts for some variable length of time. We see similar behavior in the population of the upper lasing level. For larger pump strengths, as in Figure 13, the dipole is often interrupted by a jump to the ground state of the system (a \( \gamma_f \) event), which is then swiftly followed by a pump event. This sequence most often occurs when the atom has vacuum-Rabi flopped to the lower lasing level. Then the coherent vacuum-Rabi oscillations are begun again. The interruptions due to the collapses also broaden the vacuum-Rabi peaks, but the dipole is still mainly periodic if not pure sinusoidal. Again, we see similar behavior in the population of the upper lasing level. (We also note that the two peaked structure remains even when the mean intracavity photon number is well above one.)

### III. FOUR-LEVEL COHERENTLY PUMPED LASER

In this section we examine a coherently pumped four-level single atom laser, which is shown schematically in Figure 18. The master equation for this system is given by

\[
\dot{\rho} = -\frac{i}{\hbar} [H_S, \rho] + \kappa (2a\rho a^\dagger - a^\dagger a \rho - \rho a^\dagger a) \\
+ \frac{\gamma}{2} (2\sigma_{32}\rho\sigma_{23} - \sigma_{23}\sigma_{32}\rho - \rho\sigma_{23}\sigma_{32}) \\
+ \frac{\gamma_f}{2} (2\sigma_{21}\rho\sigma_{12} - \sigma_{12}\sigma_{21}\rho - \rho\sigma_{12}\sigma_{21}),
\]

with

\[
H_S = i\hbar (a^\dagger \sigma_{32} - a\sigma_{23}) + i\hbar E_{pump}(\sigma_{41} - \sigma_{14}),
\]

where again \( \sigma_{ij} = |j\rangle \langle i| \).

The nonzero density matrix elements satisfy the following equations,

\[
\frac{d\rho_{n,1;n,1}}{dt} = 2\kappa(n+1)\rho_{n+1,1;n+1,1} \\
- 2\kappa n \rho_{n,1;n,1} + \gamma_21 \rho_{n,2;n,2} + 2\Gamma \rho_{n,4;n-1,1}
\]

(48)

\[
\frac{d\rho_{n,2;n,2}}{dt} = 2\kappa(n+1)\rho_{n+1,2;n+1,2} - \{\kappa n + \gamma_{21}\} \rho_{n,2,n,2} \\
+ \gamma\rho_{3,3;n,3} + 2\sqrt{n}g\rho_{n,2;n-1,3}
\]

(49)

\[
\frac{d\rho_{n,3;n,3}}{dt} = 2\kappa(n+1)\rho_{n+1,3;n+1,3} - \{\kappa n + \gamma\} \rho_{n,3;n,3} \\
+ \gamma\rho_{1,1;n,1} - 2\sqrt{n+1}g\rho_{n,3;n+1,2}
\]

(50)

\[
\frac{d\rho_{n,4;n,4}}{dt} = 2\kappa(n+1)\rho_{n+1,3;n+1,3} \\
- \{2\kappa n + \gamma\} \rho_{n,3;n,3} - 2\Gamma \rho_{n,4;n-1,1}
\]

(51)

\[
\frac{d\rho_{n,2;n-1,3}}{dt} = 2\kappa \sqrt{n(n-1)}\rho_{n+1,2;n-1,3} - \{\kappa(n-1) + \frac{\gamma}{2}\} \rho_{n,2;n-1,3} \\
+ \sqrt{n}g \{\rho_{n-1,3;n-1,3} - \rho_{n,2;n,2}\}
\]

(52)

\[
\frac{d\rho_{n,1;n,4}}{dt} = 2\kappa \sqrt{n(n-1)}\rho_{n+1,1;n+1,4} - \{\kappa(n-1) + \frac{\gamma}{2}\} \rho_{n,1;n,4} \\
+ \Gamma \{\rho_{n,4;n,4} - \rho_{n,1;n,1}\}
\]

(53)

We again calculate this spectrum using the quantum regression theorem. The relevant equations for the matrix elements of \( A \) in this case are

\[
\frac{dA_{n+1,1;n,1}}{dt} = 2\kappa \sqrt{(n+2)(n+1)}A_{n+2,1;n+1,1} - \{\kappa(2n+1)\} A_{n+1,1;n,1} + \gamma f A_{n+1,2;n,2} + \\
E (C_{n+1,4;n,1} + C_{n+1,1;n,4})
\]

(54)
As in the case of the incoherently pumped four-level laser,

\[
\beta = \frac{2g^2/(\gamma_f + 2\kappa)}{2g^2/(\gamma_f + 2\kappa) + \gamma^2/2}.
\]  

Figure 13 presents the linewidth of the output spectrum for various values of \( \beta \), with \( \gamma_4/\gamma = 10.0, \kappa/\gamma = 0.1 \), and \( \gamma_f/\gamma = 10.0 \) The results are in qualitative agreement with those of the incoherently pumped four-level laser, although there is a notable difference in the rate of initial increase/decrease. In Figure 20 we plot the output spectrum in the regime where vacuum-Rabi structures are present. Again, the results are qualitatively the same as the incoherently pumped model, with the persistence of vacuum-Rabi structures for large pumps and mean intracavity photon numbers above unity.

IV. CONCLUSIONS

In this chapter we have examined the output spectrum of several types of single atom laser systems. For the incoherently pumped three-level model, we find that for atom-field couplings at the lower range of that needed to produce photons in the cavity, that the spectrum is approximately a Lorentzian with a width broader than the Schawlow-Townes width. The width initially decreases with increasing pump strength as the photon number increases.

The laser linewidth then expands with further increases in pump strength, and the photon number decreases with increasing pump strength. If the atom-field coupling is increased so that \( \beta = 0.5 \), the laser emits amplitudes squeezed light. Since the output field of a laser is not a minimum uncertainty state due to phase diffusion, it is not necessary that with decreases in amplitude noise the phase noise (linewidth) must increase, but it does so here. If we increase the atom field coupling to a value larger than all the other rates in the system, we find a vacuum-Rabi structure in the output structure as predicted by Loffler et. al. [7] This structure persists only for very small pump rates, as the incoherent pump rapidly decoheres the induced dipole. At moderate to larger pump rates, the spectrum is single-peaked.

The incoherently pumped four-level laser also has a single peaked spectrum for smaller atom-field couplings that is approximately Lorentzian. The linewidth decreases with the inverse of the photon number, but is always broader than the Schawlow-Townes limit. The photon and linewidth both pin at asymptotic values as the pump is increased. This is due to the fact that it does no good to pump the system at a rate faster than the ground state of the atom is replenished by decay from the lower lasing level. As \( \beta \) is increased to 0.5 or so, the system emits amplitude squeezed light, and the linewidth increases with pump strength as in the case of the three-level system. Finally, as \( g \) is made larger than all the other rates in the system, the output spectrum has a vacuum-Rabi structure that persists for large pumps, even when the mean intracavity photon number is greater than unity. In this regime, for small pump rates, the spectrum is single-peaked however, even though \( g \) is the largest rate, which might suggest a double-peaked spectrum. This has been explained using quantum trajectory simulations.
We have further considered a coherently pumped four-level single atom laser; the results are very similar to those of the incoherently pumped four-level laser. It is hoped that with recent advances in experimental techniques [1] that these types of systems will be examined in the laboratory soon.

[1] H. Mabuchi, Q. A. Turchette, M. S. Chapman, and H. J. Kimble, Optics Letters 21, 1393 (1996).
[2] A good place to begin exploring the micromaser is H. Walther, Single atom experiments in cavities and traps. Proc. R. Soc. Lond. A 454 (1998) 431-445.
[3] A. M. Smith and C. W. Gardiner, Phys Rev A 38, 4073, 1988.
[4] Y. Mu and C. Savage, Phys. Rev. A. 46, 5944 (1992).
[5] T. Pellizari and H. Ritsch, J. Mod. Opt. 41, 609 (1994).
[6] C. Ginzel, H-J. Briegel, U. Martini, B. Eengler, and A. Schenzle, Phys. Rev. A. 48, 732 (1993).
[7] M. Loffler, G. M. Meyer, and H. Walther, Phys. Rev. A 53, 1143 (1996), Europhys. Lett. 33, 515 (1996)
[8] B. Jones, S. Ghose, J. Clemens, P. Rice, and L. Pedrotti, accepted for publication, Phys. Rev. A.
[9] P. R. Rice and H. J. Carmichael, Phys. Rev. A. 50, 4318 (1994).
[10] Y. Yamamoto, S. Machida, and O. Nilsson, Chapter 11 in Coherence, Amplification, and Quantum Effects in Semiconductor Lasers (Wiley, New York, 1991), ed. by Y. Yamamoto
[11] H. Haken, in Light and Matter, edited by L. Genzel, Handbuch der Physik Vol. XXV/2c (Springer-Verlag, 1970).
[12] G. Koganov and R. Shuker, Phys. Rev. A 58, 1559 (1998).
[13] H. J. Carmichael, An Open Systems Approach To Quantum Optics, (Springer-Verlag, Berlin, 1993), L. Tian and H. J. Carmichael, Phys. Rev. A. 46, R6801 (1992).
[14] J. Dalibard, Y. Castin, and K. Molmer, Phys. Rev. Lett. 68, 580 (1992); R. Dum, P. Zoller, and H. Ritsch, Phys. Rev. A. 45, 4879 (1992).

FIG. 1. Schematic diagram of single three-level atom in a cavity with incoherent pump, and level 4 adiabatically eliminated. For the 4-level system, $\gamma_{ij}$'s are spontaneous emission rates from level $i$ to $j$, $\Gamma'$ is a pump rate. For the 3-level system $\Gamma$ is an effective pump rate, $\gamma$ is the spontaneous emission rate on the lasing transition. For both systems, $\kappa$ is the cavity decay rate and $g$ is the atom-field coupling strength.

FIG. 2. The output spectrum of the single three-level incoherently pumped laser, for $g/\gamma = \kappa/\gamma = 0.1$ as a function of pumping strength $\Gamma/\gamma$.

FIG. 3. The linewidth of the single three-level incoherently pumped laser, as a function of pumping strength $\Gamma/\gamma$ for $\kappa/\gamma = 0.1$ and $g/\gamma = 0.6$ (solid line). The dashed line is a plot of $\kappa/2\langle n \rangle$ for the same parameters.

FIG. 4. Mean intracavity photon number for the incoherently pumped three-level laser, as a function of pumping strength $\Gamma/\gamma$, for $\kappa/\gamma = 0.1$ and $g/\gamma = 0.6$.

FIG. 5. The output spectrum of the single three-level incoherently pumped laser, for $g/\gamma = 1.414$ and $\kappa/\gamma = 0.1$ as a function of pumping strength $\Gamma/\gamma$. This plot is for small pumping strengths.

FIG. 6. The output spectrum of the single three-level incoherently pumped laser, for $g/\gamma = 1.414$ and $\kappa/\gamma = 0.1$ as a function of pumping strength $\Gamma/\gamma$. Here we show the behavior over a broad range of pump values.

FIG. 7. Mean intracavity photon number for the incoherently pumped three-level laser, as a function of pumping strength $\Gamma/\gamma$, for $\kappa/\gamma = 0.1$ and $g/\gamma = 1.414$. 

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(b) Plot of the conditioned induced dipole on the lasing transition for the three-level incoherently pumped laser with $\Gamma/\gamma = 1.0$, $g/\gamma = 1.414$, and $\kappa/\gamma = 0.1$.

(b) Plot of the conditioned induced dipole on the lasing transition for the three-level incoherently pumped laser with $\Gamma/\gamma = 1.0$, $g/\gamma = 1.414$, and $\kappa/\gamma = 0.1$.

FIG. 8. (a) Plot of the conditioned population of the upper level of the lasing transition for the three-level incoherently pumped laser with $\Gamma/\gamma = 1.0$, $g/\gamma = 1.414$, and $\kappa/\gamma = 0.1$.

FIG. 9. (a) Plot of the conditioned population of the upper level of the lasing transition for $\Gamma/\gamma = 10.0$, $g/\gamma = 0.6$, and $\kappa/\gamma = 0.1$.
(b) Plot of the conditioned induced dipole on the lasing transition for $\Gamma/\gamma = 10.0$, $g/\gamma = 0.6$, and $\kappa/\gamma = 0.1$.

FIG. 10. Schematic diagram of single four-level atom in a cavity with incoherent pump, and level 4 adiabatically eliminated. For the 4-level system, $\gamma_{ij}$'s are spontaneous emission rates from level $i$ to $j$, $\Gamma'$ is a pump rate. For the 3-level system $\Gamma$ is an effective pump rate, $\gamma$ is the spontaneous emission rate on the lasing transition, and $\gamma_f$ is the spontaneous emission rate from the lower lasing level. For both systems, $\kappa$ is the cavity decay rate and $g$ is the atom-field coupling strength.

FIG. 11. The output spectrum of the single four-level incoherently pumped laser, for $g/\gamma = \kappa/\gamma = 0.1$, and $\gamma_f/\gamma = 10.0$ as a function of pumping strength $\Gamma/\gamma$.

FIG. 12. The linewidth of the single four-level incoherently pumped laser, as a function of pumping strength $\Gamma/\gamma$ for $\kappa/\gamma = 0.1$ and $g/\gamma = 0.6$ for (a) $\beta = 0.3$, (b) $\beta = 0.4$, and (c) $\beta = 0.5$. The dashed lines are plots of $\kappa/2\langle n \rangle$ for the same $\gamma_f/\gamma, \kappa/\gamma$ and (d) $\beta = 0.3$, (e) $\beta = 0.4$, and (f) $\beta = 0.5$.

FIG. 13. The linewidth of the single four-level incoherently pumped laser, as a function of pumping strength $\Gamma/\gamma$ for $\kappa/\gamma = 0.1$ and $g/\gamma = 0.6$ for (a) $\beta = 0.6$, (b) $\beta = 0.7$, (c) $\beta = 0.8$, and (d) $\beta = 0.9$. In (e) $\beta = 0.998$, with $\gamma_f/\gamma = 100.0$ and $\kappa/\gamma = 0.1$.

FIG. 14. The output spectrum of the single four-level incoherently pumped laser, for $g/\gamma = 10.0$, $\kappa/\gamma = 0.1$, and $\gamma_f/\gamma = 2.0$ as a function of pumping strength $\Gamma/\gamma$.

FIG. 15. Mean intracavity photon number for the incoherently pumped four-level laser, as a function of pumping strength $\Gamma/\gamma$, for $\kappa/\gamma = 0.1$, $\gamma_f/\gamma = 2.0$, and $g/\gamma = 10.0$.

FIG. 16. (a) Plot of the conditioned population of the upper level of the lasing transition the four-level incoherently pumped laser with $\Gamma/\gamma = 1.0$, $g/\gamma = 10.0$, $\gamma_f/\gamma = 2.0$, and $\kappa/\gamma = 0.1$.
(b) Plot of the conditioned induced dipole on the lasing transition for $\Gamma/\gamma = 1.0$, $g/\gamma = 10.0$, and $\kappa/\gamma = 0.1$.

FIG. 17. (a) Plot of the conditioned population of the upper level of the lasing transition the four-level incoherently pumped laser with $\Gamma/\gamma = 10.0$, $g/\gamma = 10.0$, $\gamma_f/\gamma = 2.0$, and $\kappa/\gamma = 0.1$.
(b) Plot of the conditioned induced dipole on the lasing transition for $\Gamma/\gamma = 10.0$, $g/\gamma = 10.0$, and $\kappa/\gamma = 0.1$.

FIG. 18. Schematic diagram of single four-level atom in a cavity with a coherent pump. $E$ is an effective pump rate, $\gamma$ is the spontaneous emission rate on the lasing transition, $\gamma_4$ is the spontaneous emission rate out of the upper pumping level, and $\gamma_f$ is the spontaneous emission rate from the lower lasing level. Also, $\kappa$ is the cavity decay rate and $g$ is the atom-field coupling strength.

FIG. 19. The linewidth of the singlefour-level coherently pumped laser, as a function of pumping strength $\Gamma/\gamma$ for $\kappa/\gamma = 0.1$ and $g/\gamma = 0.6$ for (a) $\beta = 0.6$, (b) $\beta = 0.7$, (c) $\beta = 0.8$, and (d) $\beta = 0.9$. In (e) $\beta = 0.998$, with $\gamma_f/\gamma = 100.0$ and $\kappa/\gamma = 0.1$.

FIG. 20. The output spectrum of the single four-level coherently pumped laser, for $g/\gamma = 10.0$, $\kappa/\gamma = 0.1$, and $\gamma_f/\gamma = 2.0$ as a function of pumping strength $\Gamma/\gamma$. 
$S(\omega)$
$S(\omega)$
$S(\omega)$

$\omega/\gamma$

$\Gamma/\gamma$