Some Geometric Properties of a Non-Strict Eight Dimensional Walker Manifold

Silas Longwap¹, Gukat G. Bitrus² and Chibuisi Chigozie³

¹Department of Mathematics, Faculty of Natural Sciences, University of Jos, P.M.B. 2084, Plateau State, Nigeria.
²Department of Mathematics and Computer Science, Federal University of Kashere, P.M.B. 0182, Gombe State, Nigeria.
³Department of Insurance, University of Jos, P.M.B. 2084, Plateau State, Nigeria.

Authors’ contributions
This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

Article Information
DOI: 10.9734/JAMCS/2021/v36i530367
Editor(s):
(1) Dr. Dariusz Jacek Jakóbczak, Koszalin University of Technology, Poland.
Reviewers:
(1) Seyyed Mohammad Ajdani, Islamic Azad University of Zanjan, Iran.
(2) Ning Zhang, Beijing Union University, China.
(3) Hassan Kamal Jassim, University of Thi-Qar, Iraq.
Complete Peer review History: http://www.sdiarticle4.com/review-history/69478

Received: 24 April 2021
Accepted: 30 June 2021
Published: 02 July 2021

Abstract
An 8–dimensional Walker manifold \((M, g)\) is a strict walker manifold if we can choose a coordinate system \(\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}\) on \((M, g)\) such that any function \(f\) on the manifold \((M, g)\), \(f(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = f(x_5, x_6, x_7, x_8)\). In this work, we define a Non-strict eight dimensional walker manifold as the one that we can choose the coordinate system such that for any \(f\) in \((M, g)\), \(f(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = f(x_1, x_2, x_3, x_4)\). We derive canonical form of the Levi-Civita connection, curvature operator, \((0,4)\)-curvature tensor, the Ricci tensor, Weyl tensor and study some of the properties associated with the class of Non-strict 8–dimensional Walker manifold. We investigate the Einstein property and establish a theorem for the metric to be locally conformally flat.

*Corresponding author: E-mail: longwap4all@yahoo.com;
Keywords: Pseudo-Riemannian manifold; eight dimensional walker manifold; non-strict walker manifold.

2010 Mathematics Subject Classification: 53C25; 83C05; 57N16.

1 Introduction

Walker $2n$-manifold is a pseudo-Riemannian manifold which admit a non-trivial parallel null $r$-plane field with $r \leq n$. Walker $2n$-manifold is applicable in physics. Lorentzian Walker manifolds have been studied extensively in physics since they constitute the background metric of the pp-wave models. A pp-wave spacetime admits a covariantly constant null vector field $U$.

The study of the curvature properties of a given class of Pseudo-Riemannian manifolds is important to our knowledge of these spaces. They are used to exemplify some of the main differences between the geometry of Riemannian manifolds and the geometry of pseudo-Riemannian manifolds and thereby illustrate phenomena in pseudo-Riemannian geometry that are quite different from those which occur in Riemannian geometry. See [2], [3],[4],[5],[6] for more on Walker manifolds. The theory of Walker manifolds is outlined in [7]. The authors treated hypersurfaces with nilpotent shape operators, locally conformally flat metrics with nilpotent Ricci operator, degenerate pseudo-Riemannian homogeneous structures, para-Kaehler structures, and 2-step nilpotent Lie groups with degenerate center. The curvature properties of a large class of 4-dimensional Walker metrics are treated in [8] and Several interesting examples are given. In particular as regards local symmetry, conformal flatness and Einstein-like metrics. There are several and interesting studies in pseudo-Riemannian manifolds. Some examples in this direction may be found in [9], [10], [11],[12], [13] [14] and references therein. Recall that a Walker metric is said to be Einstein Walker metric if its Ricci tensor is a scalar multiple of the metric at each point. 4-dimensional Einstein Walker manifolds form the underlying structure of many geometric and physical models such as; $hh$--space in general relativity, $pp$--wave model and other areas, for example, [14] and references therein. In [15], the geometric properties of some curvature tensors of an 8-dimensional Walker manifold are investigated, theorems for the metric to be Einstein, locally conformally flat and for the 8-dimensional manifold to admit a Kähler structure are given.

We want to extent this study to a canonical form for a Non-strict eight dimensional walker manifold. We derive the $(0,4)$-curvature tensor, the Ricci tensor, Weyl tensor and study some of the properties associated with a class of Non-strict 8-dimensional Walker manifold. We investigate the Einstein property and establish a theorem for the metric to be locally conformally flat.

A $2n$-dimensional pseudo-Riemannian manifold $M$ admitting a parallel field of null $n$-dimensional planes $D$ is given by the metric tensor:

\[
\begin{pmatrix}
0 & Id_n \\
Id_n & B
\end{pmatrix}
\]

where $Id_n$ is the $n \times n$ identity matrix and $B$ is a symmetric $n \times n$ matrix whose entries are functions of the coordinates $(x_1, \ldots, x_{2n})$.

In particular, we want to work on an eight dimensional walker manifold $M$ admitting a parallel field of null 4-dimensional planes $D$ given by the metric tensor:

\[
g_{ij} = \begin{pmatrix} 0 & Id_4 \\ Id_4 & B \end{pmatrix}
\]  

(1.1)
Where 0 is a zero $4 \times 4$ matrix, $I_4$ is a $4 \times 4$ identity matrix, and $B$ is a $4 \times 4$ symmetric matrix whose coefficients are functions of $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$ defined as follows:

$$
0 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix},
I_4 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix},
B = \begin{pmatrix}
a & b & 0 & 0 \\
b & b & 0 & 0 \\
0 & 0 & b & c \\
0 & 0 & c & 0 \\
\end{pmatrix}
$$

for an arbitrary smooth functions $a = a(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$, $b = b(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$, and $c = c(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$, defined on an open subset $U$ of $\mathbb{R}^8$.

We use the following equation:

$$
\Gamma^{i}_{jk} = \sum_{\ell=1}^{n} \frac{1}{2} g^{\ell i} (\partial_{\ell} g_{jk} - \partial_{j} g_{\ell k} - \partial_{k} g_{\ell j}), \ i, k, j = 1 - 8 \tag{1.2}
$$

to obtain the non-vanishing components of the Christoffel symbols $\Gamma^{i}_{jk}$ of the Levi-Civita connection of the Walker metric (1.1) as follows:

$$
\Gamma^{i}_{j5} = \frac{1}{2} (\partial_{i} a) + \frac{1}{2} (\partial_{a} b) + \frac{1}{2} (\partial_{b} a), \ \Gamma^{j}_{k5} = \frac{1}{2} (\partial_{j} a) + \frac{1}{2} (\partial_{a} b) + \frac{1}{2} (\partial_{b} a),
$$

$$
\Gamma^{j}_{5k} = \frac{1}{2} (\partial_{j} b) + \frac{1}{2} (\partial_{b} a) + \frac{1}{2} (\partial_{a} b), \ \Gamma^{k}_{5j} = \frac{1}{2} (\partial_{k} b) + \frac{1}{2} (\partial_{b} a) + \frac{1}{2} (\partial_{a} b),
$$

$$
\Gamma^{i}_{55} = \frac{1}{2} (\partial_{i} a) - \frac{1}{2} (\partial_{a} b), \ \Gamma^{j}_{55} = \frac{1}{2} (\partial_{j} b) - \frac{1}{2} (\partial_{b} a), \ \Gamma^{i}_{55} = \frac{1}{2} (\partial_{i} b) - \frac{1}{2} (\partial_{b} a),
$$

$$
\Gamma^{i}_{5k} = \frac{1}{2} (\partial_{i} a) + \frac{1}{2} (\partial_{a} b) - \frac{1}{2} (\partial_{b} a), \ \Gamma^{k}_{5i} = \frac{1}{2} (\partial_{k} b) + \frac{1}{2} (\partial_{b} a) - \frac{1}{2} (\partial_{a} b),
$$

We denote by $\nabla$ the Levi-Civita connection of a pseudo-Riemannian metric (1.1) and by $R$ its curvature tensor, taken with the sign convention:

$$
\nabla_{\partial_{i}} \partial_{j} = \sum_{k} \Gamma^{i}_{jk} \partial_{k}
$$

$$
R(X,Y) = [\nabla_{X} \nabla_{Y}] - \nabla_{[X,Y]}.
$$

(1.3)
From equations (1.3), after a long but straightforward calculations we obtained the following results:

Lemma 1.1. The Non-zero components of the Levi-Civita connection of a Walker metric (1.1) are given by

$$\nabla_{\partial_i} \partial_j = \left( \frac{1}{2} \partial_i a + b \partial_j a + \frac{1}{2} \partial_j a \right) \partial_i + \left( \frac{b}{2} \partial_i a + \frac{1}{2} \partial_i b - \frac{1}{2} \partial_j a \right) \partial_2 + \left( \frac{b}{2} \partial_i a + \frac{c}{2} \partial_j a \right) \partial_8 + \left( \frac{1}{2} \partial_j a \right) \partial_r + \left( \frac{1}{2} \partial_j a \right) \partial_t$$

$$- \frac{1}{2} \partial_j a \partial_i + \left( \frac{1}{2} \partial_i a \right) \partial_4 + \left( \frac{1}{2} \partial_i a \right) \partial_7 + \left( \frac{1}{2} \partial_i a \right) \partial_5 + \left( \frac{1}{2} \partial_i a \right) \partial_6 + \left( \frac{1}{2} \partial_i a \right) \partial_3$$

$$+ \left( -\frac{1}{2} \partial_j a \right) \partial_8,$$

$$\nabla_{\partial_i} \partial_8 = \left( \frac{1}{2} \partial_i b + \frac{1}{2} \partial_i b + \frac{1}{2} \partial_i a \right) \partial_1 + \left( \frac{b}{2} \partial_i b + \frac{1}{2} \partial_i a \right) \partial_2 + \left( \frac{b}{2} \partial_i b + \frac{c}{2} \partial_i a \right) \partial_3$$

$$+ \left( \frac{c}{2} \partial_i b + \frac{1}{2} \partial_i a \right) \partial_4 + \left( -\frac{1}{2} \partial_i b \right) \partial_5 + \left( -\frac{1}{2} \partial_i a \right) \partial_6 + \left( -\frac{1}{2} \partial_i b \right) \partial_7 + \left( -\frac{1}{2} \partial_i b \right) \partial_8,$$

$$\nabla_{\partial_i} \partial_r = \left( \frac{1}{2} \partial_i a \right) \partial_1 + \left( \frac{1}{2} \partial_i b \right) \partial_2 + \left( \frac{1}{2} \partial_i b \right) \partial_3 + \left( \frac{1}{2} \partial_i c \right) \partial_4$$

$$\nabla_{\partial_i} \partial_t = \left( \frac{1}{2} \partial_i b \right) \partial_1 + \left( \frac{1}{2} \partial_i b \right) \partial_2 + \left( \frac{1}{2} \partial_i b \right) \partial_3 + \left( \frac{1}{2} \partial_i c \right) \partial_4$$

$$\nabla_{\partial_i} \partial_8 = \left( \frac{1}{2} \partial_i b \right) \partial_1 + \left( \frac{1}{2} \partial_i b \right) \partial_2 + \left( \frac{1}{2} \partial_i c \right) \partial_4 + \left( \frac{1}{2} \partial_i c \right) \partial_7 + \left( \frac{1}{2} \partial_i c \right) \partial_6$$

$$\nabla_{\partial_i} \partial_8 = \left( \frac{1}{2} \partial_i b \right) \partial_1 + \left( \frac{1}{2} \partial_i b \right) \partial_2 + \left( \frac{1}{2} \partial_i c \right) \partial_4 + \left( \frac{1}{2} \partial_i c \right) \partial_7 + \left( \frac{1}{2} \partial_i c \right) \partial_6$$

$$+ \left( \frac{c}{2} \partial_i c \right) \partial_5 + \left( -\frac{1}{2} \partial_i c \right) \partial_5 + \left( -\frac{1}{2} \partial_i c \right) \partial_5 + \left( -\frac{1}{2} \partial_i c \right) \partial_5 + \left( -\frac{1}{2} \partial_i c \right) \partial_5$$

$$+ \left( \frac{1}{2} \partial_i c \right) \partial_5.$$
Non-strict walker metric (1.1) are as follows:

\[
\nabla \partial_0 \partial_0 = (a_{10} + \frac{b_{10}}{2}) \partial_1 + \frac{b_{a1}}{2} \partial_2 + \frac{b_{a2}}{2} \partial_3 + (b_{a3} + \frac{c_{a3}}{2}) \partial_3
\]

+ (c_{a3} - \frac{a_1}{2}) \partial_0 - \frac{a_2}{2} \partial_0 - \frac{a_3}{2} \partial_0 - \frac{a_4}{2} \partial_0,

\[
\nabla \partial_0 \partial_2 = (a_{20} + \frac{b_{20}}{2}) \partial_1 + (b_{b1} + \frac{b_{b2}}{2}) \partial_2 + \frac{b_{b3}}{2} \partial_3 + (c_{b4} + \frac{c_{b4}}{2}) \partial_3
\]

+ (c_{b3} - \frac{b_1}{2}) \partial_0 - \frac{b_2}{2} \partial_0 - \frac{b_3}{2} \partial_0 - \frac{b_4}{2} \partial_0,

\[
\nabla \partial_0 \partial_3 = (a_{30} + \frac{b_{30}}{2}) \partial_1 + (b_{c1} + \frac{b_{c2}}{2}) \partial_2 + (b_{c3} + \frac{c_{c3}}{2}) \partial_3
\]

+ (c_{c3} - \frac{c_1}{2}) \partial_0 - \frac{c_2}{2} \partial_0 - \frac{c_3}{2} \partial_0 - \frac{c_4}{2} \partial_0.

Proof. This is obtained from the Lemma (1.1).

Using the proposition (2.1), the following is now immediate:

Lemma 2.1. The Non-zero components of the curvature operator of the eight dimensional Non-strict walker metric (1.1) are given by;

\[
R(\partial_0, \partial_0) = (a_{10} + \frac{b_{10}}{2}) \nabla \partial_0 \partial_0 - \frac{b_{10}}{2} \nabla \partial_0 \partial_0, \quad R(\partial_0, \partial_0) = \frac{b_{10}}{2} \nabla \partial_0 \partial_0,
\]

\[
R(\partial_0, \partial_0) = \frac{a_{20} + \frac{b_{20}}{2}}{2} \nabla \partial_0 \partial_0 + \frac{b_{a1}}{2} \nabla \partial_0 \partial_0, \quad R(\partial_0, \partial_0) = \frac{b_{a2}}{2} \nabla \partial_0 \partial_0 + \frac{b_{a3}}{2} \nabla \partial_0 \partial_0,
\]

\[
R(\partial_0, \partial_0) = -\frac{b_{b1}}{2} \nabla \partial_0 \partial_0 + \frac{b_{b2}}{2} \nabla \partial_0 \partial_0, \quad R(\partial_0, \partial_0) = \frac{b_{b3}}{2} \nabla \partial_0 \partial_0 + \frac{c_{c3}}{2} \nabla \partial_0 \partial_0,
\]

\[
R(\partial_0, \partial_0) = \frac{c_{c3}}{2} \nabla \partial_0 \partial_0 - \frac{c_{c3}}{2} \nabla \partial_0 \partial_0, \quad R(\partial_0, \partial_0) = \frac{c_{c3}}{2} \nabla \partial_0 \partial_0 + \frac{c_{c3}}{2} \nabla \partial_0 \partial_0.
\]

From Lemma (2.1), we can now determine all the (0,4)-curvature tensors $R(X, Y, Z, W) = g(R(X, Y)Z, W)$ with respect to $\partial_0$. From a long but routine calculations, we obtain the following results:

Theorem 2.2. The nonzero components of the (0,4)-curvature tensor of the eight dimensional Non-strict walker metric (1.1) are given by:
\[
R(\partial_5, \partial_6, \partial_7, \partial_1) = \frac{b_1b_2}{4} + \frac{a_1b_1}{4} + \frac{a_2b_2}{4}, \quad R(\partial_5, \partial_6, \partial_7, \partial_2) = \frac{b_2b_3}{4} + \frac{a_1b_2}{4} + \frac{a_2b_1}{4}, \\
R(\partial_5, \partial_6, \partial_7, \partial_3) = \frac{b_3b_4}{4} + \frac{a_1b_3}{4} + \frac{a_2b_4}{4}, \quad R(\partial_5, \partial_6, \partial_7, \partial_4) = \frac{b_4b_5}{4} + \frac{a_1b_4}{4} + \frac{a_2b_5}{4}.
\]
The eight dimensional Non-strict walker metric is defined as
\[ R(\partial_k, \partial_s, \partial_r, \partial_i) = \frac{a_1 b_3}{2} - \frac{a_3 b_1}{2} - \frac{b_1^2}{2} + \frac{a_3 b_3}{2} + \frac{a_4 c_4}{2}, \]
\[ R(\partial_r, \partial_s, \partial_r, \partial_i) = \frac{a_1 b_1}{2} + \frac{c_4 a_1}{2} - \frac{b_3 c_1}{2}, \]
\[ R(\partial_r, \partial_s, \partial_r, \partial_i) = \frac{a_1 b_3}{2} + \frac{a_4 c_4}{2} - \frac{b_1 c_3}{2}, \]
\[ R(\partial_r, \partial_s, \partial_r, \partial_i) = \frac{a_1 b_1}{2} + \frac{c_4 a_1}{2} - \frac{b_3 c_3}{2}. \]

The Ricci tensor is defined as \( \text{Ric}(x, y) = \text{trace}\{z \rightarrow R(X, Z)Y\} \) and so from theorem (2.2) using the equation:

\[ \text{Ric}(X, Y) = \sum_{i,j=1}^{8} g^{ij} R(X, \partial_i, \partial_j, Y), \]

we have the following results:

**Theorem 2.3.** The Non-zero components of the Ricci tensor of the eight dimensional Non-strict walker metric (1.1) are:

\[ \text{Ric}(\partial_r, \partial_i) = \frac{a_1 b_3}{2} - \frac{a_3 b_1}{2} - \frac{b_1^2}{2} + \frac{a_3 b_3}{2} + \frac{a_4 c_4}{2}, \]
\[ \text{Ric}(\partial_r, \partial_i) = \frac{b_2}{2} + \frac{b_4 c_2}{2} + \frac{b_4 c_4}{2}, \]
\[ \text{Ric}(\partial_r, \partial_i) = -\frac{a_3 b_1}{2} - \frac{b_1^2}{2}, \]
\[ \text{Ric}(\partial_r, \partial_i) = -\frac{a_3 b_1}{2} - \frac{b_1^2}{2}. \]

Let \( \tau \) denote the scalar curvature of the Non-strict walker metric (1.1). We define the scalar curvature by the equation

\[ \tau = \sum_{i,j=1}^{8} g^{ij} \text{Ric}(i, j) \]

Thus, we have the following result:

**Theorem 2.4.** The eight dimensional Non-strict walker metric (1.1) has zero scalar curvature.

**Proof.** Observe that the metric (1.1) is of the form

\[ g_{ij} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & a & b & 0 & 0 \\ 0 & 1 & 0 & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & b & c & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & c & 0 \end{pmatrix} \quad \text{and} \quad g^{ij} = \begin{pmatrix} -a & -b & 0 & 0 & 0 & 1 & 0 & 0 \\ -b & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -b & -c & 0 & 0 & 1 \\ 0 & 0 & -c & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \]

where \( g^{ij} \) is the inverse of the metric \( g_{ij} \). From the equation \( \tau = \sum_{i,j=1}^{8} g^{ij} \text{Ric}(i, j) \) and theorem (2.3) we observe that \( \text{Ric}(ij) = 0 \) for \( i, j = 1, 2, 3, 4 \) and \( g^{ij} = 0 \) for \( i, j = 5, 6, 7, 8 \). Thus \( \tau = 0 \) for all \( i, j = 1, 2, ..., 8 \).
Definition 2.1. A Walker metric is said to be Einstein Walker metric if its Ricci tensor is a scalar multiple of the metric at each point.

This definition implies that the eight dimensional nonstrict walker metric (1.1) is an Einstein Walker metric if there is a constant \( \mu \) so that \( \text{Ricc} = \mu g \). The Schouten tensor \( C_{ij} \) is defined by the equation \( C_{ij} = \text{Ricc}(ij) - \frac{\tau}{(m-1)} g_{ij} \). Thus we have the following:

Theorem 2.5. The eight dimensional Non-strict walker metric (1.1) is not Einstein.

Proof. Since Schouten equation is given by \( C_{ij} = \text{Ric}(i,j) - \frac{\tau}{(m-1)} g_{ij} \) and from theorem (2.4) the scalar curvature \( \tau = 0 \). Therefore, \( C_{ij} = \text{Ric}(i,j) \) and the result follows. 

The Weyl conformal curvature tensor \( \mathfrak{W} \) is defined by the equation

\[
\mathfrak{W}(X,Y,Z,W) = R(X,Y,Z,W) + \frac{\tau}{(m-1)(m-2)} \{ g(X,W)g(Y,Z) - g(X,Z)g(Y,W) \} \\
+ \frac{1}{(n-2)} \{ \text{Ricc}(Y,Z)g(X,W) - \text{Ricc}(X,Z)g(Y,W) - \text{Ricc}(Y,W)g(X,Z) \\
- \text{Ricc}(X,W)g(Y,Z) \} \}.
\]

Since the Scalar curvature \( \tau = 0 \), for the eight dimensional Non-strict metric (1.1), the Weyl conformal curvature tensor \( \mathfrak{W} \) becomes

\[
\mathfrak{W}(X,Y,Z,W) = R(X,Y,Z,W) + \frac{1}{n-2} \{ \rho(Y,Z)g(X,W) - \rho(X,Z)g(Y,W) \\
- \rho(Y,W)g(X,Z) + \rho(X,W)g(Y,Z) \}.
\]

Lemma 2.6. The Non-zero components of the Weyl conformal tensor of the eight dimensional Non-strict walker metric (1.1) is given by;
\[ \mathfrak{M}(\partial_5, \partial_6, \partial_7, \partial_8) = \frac{b_1 b_2}{4} + \frac{1}{6} \left( \frac{b_2 b_1}{2} + \frac{b_2^3}{2} + \frac{b_3 c_3}{2} + \frac{b_3 c_4}{2} \right), \]
\[ \mathfrak{M}(\partial_5, \partial_6, \partial_7, \partial_8_2) = \frac{b_2 b_2}{4} - \frac{a_1 b_2}{4} + \frac{1}{6} \left\{ -\left( \frac{a_1 b_2}{2} - \frac{a_2 b_1}{2} - \frac{b_2^3}{2} + \frac{a_3 b_3}{2} + \frac{a_4 c_3}{2} + \frac{a_5 c_4}{2} \right) \right\}, \]
\[ \mathfrak{M}(\partial_5, \partial_6, \partial_7, \partial_8_3) = \frac{b_2 b_3}{4} - \frac{a_1 b_3}{4} + \frac{1}{6} \left\{ -\left( \frac{a_1 b_3}{2} - a_2 b_1 - \frac{b_2^3}{2} + \frac{a_3 b_3}{2} + \frac{a_4 c_3}{2} + \frac{a_5 c_4}{2} \right) \right\}, \]
\[ \mathfrak{M}(\partial_5, \partial_6, \partial_7, \partial_8_4) = \frac{b_2 b_4}{4} - \frac{a_1 b_4}{4} + \frac{1}{6} \left\{ -\left( \frac{a_1 b_4}{2} - \frac{a_2 b_1}{2} - \frac{b_2^3}{2} + \frac{a_3 b_3}{2} + \frac{a_4 c_3}{2} + \frac{a_5 c_4}{2} \right) \right\}, \]
\[ \mathfrak{M}(\partial_5, \partial_6, \partial_7, \partial_8_5) = \frac{b_2 b_5}{4} - \frac{a_1 b_5}{4} + \frac{1}{6} \left\{ -\left( \frac{a_1 b_5}{2} - \frac{a_2 b_1}{2} - \frac{b_2^3}{2} + \frac{a_3 b_3}{2} + \frac{a_4 c_3}{2} + \frac{a_5 c_4}{2} \right) \right\}, \]
\[ \mathfrak{M}(\partial_5, \partial_6, \partial_7, \partial_8_6) = \frac{b_2 b_6}{4} - \frac{a_1 b_6}{4} + \frac{1}{6} \left\{ -\left( \frac{a_1 b_6}{2} - \frac{a_2 b_1}{2} - \frac{b_2^3}{2} + \frac{a_3 b_3}{2} + \frac{a_4 c_3}{2} + \frac{a_5 c_4}{2} \right) \right\}, \]
\[ \mathfrak{M}(\partial_5, \partial_6, \partial_7, \partial_8_7) = \frac{b_2 b_7}{4} - \frac{a_1 b_7}{4} + \frac{1}{6} \left\{ -\left( \frac{a_1 b_7}{2} - \frac{a_2 b_1}{2} - \frac{b_2^3}{2} + \frac{a_3 b_3}{2} + \frac{a_4 c_3}{2} + \frac{a_5 c_4}{2} \right) \right\}, \]
\[ \mathfrak{M}(\partial_5, \partial_6, \partial_7, \partial_8_8) = \frac{b_2 b_8}{4} - \frac{a_1 b_8}{4} + \frac{1}{6} \left\{ -\left( \frac{a_1 b_8}{2} - \frac{a_2 b_1}{2} - \frac{b_2^3}{2} + \frac{a_3 b_3}{2} + \frac{a_4 c_3}{2} + \frac{a_5 c_4}{2} \right) \right\}. \]
A pseudo-Riemannian manifold is locally conformally flat if and only if its Weyl tensor vanishes.

\[ \mathfrak{W}(\partial_5, \partial_6, \partial_7, \partial_8) = \frac{c_1 a_1}{4} + \frac{c_2 b_1}{4} + \frac{1}{6}(a_1 c_1 + b_2 c_2 + b_2 a_1 + c_3 c_4), \]

\[ \mathfrak{W}(\partial_5, \partial_6, \partial_7, \partial_2) = \frac{c_1 a_2}{4} + \frac{c_2 b_2}{4}, \quad \mathfrak{W}(\partial_5, \partial_6, \partial_7, \partial_3) = \frac{c_1 a_3}{4} + \frac{c_2 b_3}{4}, \]

\[ \mathfrak{W}(\partial_5, \partial_6, \partial_7, \partial_4) = \frac{c_1 a_4}{4} + \frac{c_2 b_4}{4} + \frac{1}{6}(a_2 b_1 + a_4 c_1 + b_2 b_3 + b_2 c_2), \]

\[ \mathfrak{W}(\partial_5, \partial_6, \partial_7, \partial_1) = -\frac{b_2 a_1}{4} - \frac{b_2 c_1}{4}, \]

\[ \mathfrak{W}(\partial_5, \partial_6, \partial_7, \partial_2) = \frac{b_2 a_2}{4} - \frac{b_2 c_2}{4} + \frac{1}{6}(a_3 b_1 + a_4 c_1 + b_2 b_3 + b_2 c_2), \]

\[ \mathfrak{W}(\partial_5, \partial_6, \partial_7, \partial_3) = \frac{b_2 a_3}{4} - \frac{b_2 c_3}{4} + \frac{1}{6}(a_2 b_1 + b_2^2 + b_4 c_3 + b_2 c_1), \]

\[ \mathfrak{W}(\partial_5, \partial_6, \partial_7, \partial_4) = \frac{b_2 a_4}{4} - \frac{b_2 c_4}{4}, \quad \mathfrak{W}(\partial_6, \partial_7, \partial_1, \partial_8) = \frac{b_1 b_1}{4}, \]

\[ \mathfrak{W}(\partial_6, \partial_7, \partial_2, \partial_8) = \frac{b_1 b_2}{4} + \frac{1}{6}(b_2 b_1 + b_2 b_1 + b_4 a_1) + \frac{b_3 b_4}{4}, \quad \mathfrak{W}(\partial_6, \partial_7, \partial_2, \partial_3) = \frac{b_1 b_3}{4} - \frac{b_3 c_4}{4}, \quad \mathfrak{W}(\partial_6, \partial_7, \partial_2, \partial_4) = \frac{b_1 b_4}{4}. \]

\[ \mathfrak{W}(\partial_6, \partial_7, \partial_1, \partial_8) = \frac{c_1 b_1}{4} + \frac{1}{6}(a_1 c_1 + b_1 c_2 + b_2 c_1 + c_3 c_4), \]

\[ \mathfrak{W}(\partial_6, \partial_7, \partial_1, \partial_2) = \frac{c_1 b_2}{4} + \frac{1}{6}(a_1 c_1 + b_1 c_2 + b_2 c_1 + c_3 c_4). \]

\[ \mathfrak{W}(\partial_6, \partial_7, \partial_1, \partial_3) = -\frac{b_3 c_1}{4}, \quad \mathfrak{W}(\partial_6, \partial_7, \partial_1, \partial_4) = -\frac{c_3 c_4}{4}. \]

**Definition 2.2.** A pseudo-Riemannian manifold is locally conformally flat if and only if its Weyl tensor vanishes.
Theorem 2.7. The eight dimensional Non-strict walker metric \((1.1)\) is locally conformally flat if and only if the functions \(a, b\) and \(c\) are constants or they satisfy the partial differential equation

\[
\begin{align*}
\frac{b_1 b_2}{3} + \frac{b_2^2}{12} + \frac{b_4 c_1}{12} + \frac{b_1 c_4}{12} &= 0, \\
\frac{b_2 b_3}{4} - \frac{a_1 b_3}{4} + \frac{a_2 b_1}{4} = 0, \\
\frac{a_1 b_3}{4} + \frac{a_2 b_1}{4} = 0, \\
\frac{b_2 b_4}{4} - \frac{a_1 b_4}{4} + \frac{a_4 b_1}{4} = 0, \\
\frac{a_1 b_4}{4} - \frac{a_2 b_1}{4} + \frac{b_2^2}{12} + \frac{a_4 c_1}{12} + \frac{a_3 c_4}{12} &= 0,
\end{align*}
\]

Proof. From Lemma (2.6), if \(a, b\) and \(c\) are constants, then \(\mathbb{W}(\partial_i, \partial_j, \partial_k, \partial_l) = 0\) for all \(i, j, k = 5, 6, 7, 8\) and \(l = 1, 2, 3, 4\) \(\square\)
3 Conclusion

The independency of the Ricci tensor on the variables \(\{x_1, x_2, x_3, x_4\}\) is a common feature of the Non-strict walker metric as seen in theorem (2.3). This results to a zero scalar curvature of the metric as shown in (2.4). The zero scalar curvature also lead to a non Einstein property of the metric. If the associated functions \(a, b, c\) are constants then the Non-strict walker metric is locally conformally flat. There are many more properties associated with the Non-strict walker metric that need to be explored.

Acknowledgement

The first author expresses his deepest gratitude to Dr Abdoul Salam Diallo of Universit Alioune DIOP de Bamby, Senegal, for his encouragement and sharing in his wealth of knowledge, particularly in this aspect of geometry.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Abounasr R, Belljaj A, Rasmussen J, Saidi EH. Superstring theory on pp waves with ADE geometries. J. Phys. A. 2006;39:2797-2841. DOI: 10.1088/0305-4470/39/11/015
[2] Salimov, Arif, Murat Iscan. Some properties of Norden-Walker metrics. Kodai Mathematical Journal. 2010;33(2):283-293.
[3] Iscan, Murat, Gulnur Caglar. Para-Khler-Einstein structures on Walker 4-manifolds. International Journal of Geometric Methods in Modern Physics. 2016;13(02):1650006.
[4] Bejan CL, Dru-Romanici SL. Walker manifolds and Killing magnetic curves. Differential Geometry and its Applications. 2014;35:106-116.
[5] Matsushita, Yasuo, Seiya Haze, Peter R. Law. Almost Khler-Einstein structures on 8-dimensional Walker manifolds. Monatshefte fr Mathematik. 2007;150(1):41.
[6] Davidov, Johann, et al. Almost Khler Walker 4-manifolds. Journal of Geometry and Physics. 2007;57(3):1075-1088.
[7] Brozos-Vzquez M, Garc a Ro E, Gilkey P, Nikevi S, Vzquez-Lorenzo R. The geometry of Walker manifolds, vol. 5 of Synthesis Lectures on Mathematics and Statistics. Morgan and Claypool Publishers, Wills ton, VT; 2009. [Author name on title page: Ramn Vzquez-Lorenzo].
[8] . Wafa Batat, Giovanni Calvaruso, Barbara De Leo. On the geometry of four-dimensional Walker manifolds. Rendiconti di Matematica, Serie VII. 2008;29:163173. Roma.
[9] Batat W, Calvaruso G, De Leo B. Curvature properties of Lorentzian manifolds with large isometry groups. Math. Phys. Anal. Geom. 2009;12:201-217.
[10] Brozos-Vzquez M, Garca-Ro E, Gilkey P, Vzquez-Lorenzo R. Examples of signature (2, 2) manifolds with commuting curvature operators. J. Phys. A: Math. Theor. 2007;40:13149-13159.
[11] Calvaruso G. Einstein-like metrics on three-dimensional homogeneous Lorentzian manifolds. Geom. Dedicata. 2007;127:99-119.
[12] Calvaruso G, De Leo B. Curvature properties of four-dimensional generalized symmetric spaces. J. Geom. 2008;90:30-46.

[13] Chaichi M, Garca-Ro E, Vquez-Abal ME. Three-dimensional Lorentz manifolds admitting a parallel null vector field. J. Phys. A: Math. Gen. 2005;38:841-850.

[14] Chaichi M, Garca-Ro E, Matsushita Y. Curvature properties of four-dimensional Walker metrics. Class. Quantum Grav. 2005;22:559-577.

[15] Diallo Abdoul Salam, Longwap Silas, Massamba Fortun. Almost Kähler eight-dimensional walker manifold. Novi Sad J. Math. 2018;48(1):129-141.

©2021 Longwap et al.; This is an Open Access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0), which permits unrestricted use, distribution and reproduction in any medium, provided the original work is properly cited.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://www.sdiarticle4.com/review-history/69478