Modelling Quintessential Inflation with Branes

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Abstract. I discuss why quintessential inflation model-building is more natural in the context of brane cosmology and study a particular model as an example.

1 Introduction

To minimize its fine-tuning problems, such as initial conditions, there have been attempts to unify quintessence with the inflaton field in a single scalar \cite{3, 9, 12, 13}. This way, introducing yet again another unobserved scalar field is avoided. Also, a common theoretical framework can be used to describe both inflation and quintessence. For quintessential inflation one needs a sterile inflaton, with only minimal gravitational coupling to the standard model, because the field should survive until today. The Universe reheats through gravitational particle production \cite{5}. The minimum of the potential (assumed zero) is typically placed at infinity, because it should not have yet been reached. This results in the so-called quintessential tail of the potential. Candidates for the quintessential–inflation scalar field are moduli fields or the radion field.

2 Dynamics

The Universe is modeled as a collection of perfect fluids; the background fluid with density $\rho_B$ (comprised by matter and radiation) and the scalar field $\phi$ with density $\rho_\phi = \rho_{\text{kin}} + V$ and pressure $p_\phi = p_{\text{kin}} - V$, where $\rho_{\text{kin}} \equiv \frac{1}{2}\dot{\phi}^2$ is the kinetic density of $\phi$ and the dot denotes derivative w.r.t. the cosmic time $t$. For every component one defines the barotropic parameter as $w_i \equiv p_i/\rho_i$. The Universe expansion accelerates when $\rho_B < \rho_\phi$ and $w_\phi < -\frac{1}{3}$. Energy-momentum conservation demands $d(a^3\rho) = -p d(a^3)$, which, for decoupled fluids, gives $\rho_i \propto a^{-3(1+w_i)}$, where $a$ is the scale factor. To study the dynamics of the Universe, one also needs the Friedman equation and the $\phi$ field equation:

$$\dot{\rho}_{\text{kin}} + 6H\rho_{\text{kin}} + \dot{V} = 0,$$

where $H \equiv \dot{a}/a$ is the Hubble parameter. In spatially-flat, FRW cosmology the Friedman equation is $H^2 = \rho/3m_P^2$, with $m_P$ being the reduced Planck mass. This results in the evolution equations:

$$H = \frac{2t^{-1}}{3(1+w)} \quad a \propto t^{\frac{3}{3(1+w)}} \quad \rho = \frac{4}{3(1+w)^2} \left( \frac{m_P}{t} \right)^2$$

(1)

where $w$ corresponds to the dominant fluid component. In brane cosmology the Friedman equation is modified for density above the string tension $\lambda$ \cite{11}.
Assuming a 5th dimension, the Friedman equation becomes \( H \simeq \frac{\rho}{\sqrt{6} \lambda m_P} \), where \( \lambda = \frac{1}{2\pi} \left( M_5^2/m_P^2 \right) \), with \( M_5 \) being the fundamental, 5-dim Planck mass. Then the evolution equations are:

\[
H = \frac{t^{-1}}{3(1 + w)} \quad a \propto t^{\frac{1}{3(1 + w)}} \quad \rho = \frac{\sqrt{6} \lambda}{3(1 + w)} \left( \frac{m_P}{t} \right)
\]

The modified dynamics affect the inflationary era due to excessive friction on the roll of the scalar field, which allows inflation with steep potential. Then, the slow-roll parameters become: \( \epsilon \simeq 2\lambda m_P^2 (V')^2/V^3 \) and \( \eta \simeq 2\lambda m_P^2 V''/V^2 \), where the prime denotes derivative w.r.t. \( \phi \). The modified slow-roll changes the amplitude and spectral index of the generated density perturbations [11]:

\[
\frac{\delta \rho}{\rho} \simeq \frac{1}{2\sqrt{6\pi}} \frac{V^3}{\lambda^{3/2}|V'|m_P^2} \quad \text{and} \quad n_s - 1 \simeq -4m_P^2 \frac{\lambda}{V} \left[ 3 \left( \frac{V'}{V} \right)^2 - \frac{V''}{V} \right]
\]

Gravitational reheating creates a thermal bath of temperature \( T_{\text{reh}} = \frac{\alpha^3}{2\pi^2} H_{\text{end}} \) [5], where \( \alpha \) is the reheating efficiency and the subscript 'end' denotes the end of inflation. Due to the inefficiency of reheating, after the end of inflation the Universe becomes dominated by \( \rho_{\text{kin}} \), i.e. \( \rho \simeq \rho_{\text{kin}} \propto a^{-6} \). As long as \( \rho_{\text{kin}} > 2\lambda \) we have \( a \propto t^{1/6} \) and \( \phi(t) = \phi_{\text{end}} + \frac{2}{\sqrt{3}} \sqrt{\lambda/V_{\text{end}}} \left( \sqrt{t/t_{\text{end}}} - 1 \right) m_P \). However, when \( \rho_{\text{kin}} < 2\lambda \) the brane regime ends and one has \( a \propto t^{1/3} \) and \( \phi(t) = \phi_{\text{end}} + \sqrt{2/3} m_P \), where \( t_{\lambda} = \frac{1}{2\sqrt{6}} m_P/\sqrt{\lambda} \) is the crossover time for which \( \phi_{\lambda} = \phi_{\text{end}} + \sqrt{2/3} m_P \). The thermal bath eventually dominates the density of the Universe and the Hot Big Bang begins at the temperature:

\[
T_* = \frac{\alpha^3}{96\pi^2} \frac{g_*}{15} \left( \frac{m_P}{V_{\text{end}}} \right)^{5/2}
\]

where \( g_* \sim 10^2 \) is the number of relativistic degrees of freedom. In order not to affect Nucleosynthesis (BBN) we require \( T_* > 1 \text{ MeV} \). After the onset of the Hot Big Bang \( \rho_{\text{kin}} \) rapidly decreases and the field freezes at the value:

\[
\phi_F = \phi_{\text{end}} + \frac{m_P}{\sqrt{6}} \left[ 14 + \frac{3}{2} \ln \left( \frac{30\pi^2}{g_* \alpha^4} \right) - 4 \ln \left( \frac{m_P^2}{\lambda} \right) + 5 \ln \left( \frac{m_P^2}{V_{\text{end}}} \right) \right]
\]

3 Why Branes

Standard–cosmology quintessential inflation needs a potential with two flat regions: the inflationary plateau and the quintessential tail [3]. BBN and coincidence demand that the density scale of the flat regions differs by \( \sim 10^{100} \). To prepare for this abysmal dive, \( V \) is strongly curved near the end of inflation. As a result, the slow-roll parameter \( \eta \) is large, leading to spectral index \( n_s \) far from unity. This makes it difficult to construct single-branch models\(^4\),

\(^4\)Multi-branch models such as [9, 12, 13] are disfavoured in our minimalistic approach.
for quintessential inflation, although solutions do exist [3]. In contrast, this \( \eta \)-problem does not appear when considering brane cosmology because the modified Friedman equation allows for steep inflation [2, 8, 10, 11, 14]. Hence, one may obtain successful quintessential inflation with a simple potential.

4 The exponential tail

String theory disfavors eternal acceleration [4, 7]. This immediately rules out quintessential tails milder than exponential and also frozen quintessence (in which \( \phi = \phi_F \) at present). In any case, coincidence is hard to achieve with mild tails (e.g. inverse power-law type [8]) and frozen quintessence is virtually indistinguishable from the cosmological constant alternative. On the other hand steeper-than-exponential tails have disastrous attractors [3]. Thus, the most reasonable approach is the exponential tail, 

\[
V \approx V_0 e^{-b\phi/m_P},
\]

where \( b \) is a positive constant. Then, the \( \phi \)-field equation gives an exact attractor solution:

\[
\phi_{\text{attr}} = \frac{2m_P}{b} \ln \left[ \frac{V_0}{2} \frac{1 + w}{1 - w} \right] \quad \text{and} \quad V_{\text{attr}} = \frac{2}{b^2} \left( \frac{1 - w}{1 + w} \right) \left( \frac{m_P}{T} \right)^2
\]

If \( \rho_\phi < \rho_B \) then \( w = w_B \) and \( \rho_\phi/\rho_B = 3(1 + w_B)/b^2 = \text{const.} \), which means \( w_\phi = w_B \geq 0 \) and \( b^2 > 3(1 + w_B) \). If, however, \( \rho_\phi \geq \rho_B \) then \( w = w_\phi \) and \( \rho = \rho_\phi \propto a^{-3(1 + w_\phi)} \). This time \( b^2 = 3(1 + w_\phi) \). Therefore, dark energy dominates without eternal acceleration when \( 2 < b^2 < 3(1 + w_B) \). Brief acceleration is possible at the time when \( \phi \) unfreezes to follow the attractor, due to superfreezing [3]. In fact, superfreezing enlarges the range to \( 2 < b^2 < 24 \) [1].

5 A toy-model example

Consider the potential (also studied in [6]):

\[
V(\phi) = \frac{M^4}{\cosh(b\phi/m_P) - 1} \Rightarrow \left\{ \begin{array}{ll}
V \approx 2M^4 e^{-b\phi/m_P} & \phi \gg m_P/b \\
V \approx 2M^4 (m_P/b\phi)^2 & \phi \ll m_P/b
\end{array} \right.
\]

The slow-roll parameters are: \( \eta = 2A[\cosh(b\phi/m_P) + 2] = \epsilon + 2A \). Thus, inflation ends when \( \eta(\phi_{\text{end}}) \equiv 1 \), where \( A \equiv b^2(\lambda/M^4) \). The spectral index is

\[
n_s - 1 = -4A \frac{3 + \frac{1}{1-6A}}{1 - \frac{1}{1-6A}} \exp\left(-4N_{\text{dec}}A\right)
\]

where \( N_{\text{dec}} \simeq 69 \) is the number of e-folds of remaining inflation corresponding to the scale of the horizon at decoupling. From the observations \( |n_s - 1| \leq 0.1 \), which demands the constraint \( A < \frac{1}{148} \). Using \( A \ll 1 \) the above becomes
\[ n_s - 1 \simeq -\frac{4}{N_{\text{dec} + 1}}, \] which gives \( n_s \simeq 0.94. \) The density contrast with \( A \ll 1 \) is:

\[ \frac{\delta \rho}{\rho} \simeq \frac{2b^2}{\sqrt{6\pi}} \left( \frac{M}{m_P} \right)^2 \sqrt{A (N_{\text{dec}} + 1)^2} \] (9)

Similarly, for the inflationary scale we find \( V_{\text{end}} \simeq 2AM^4 \) or, equivalently,

\[ V_{\text{end}} \simeq \frac{3\pi^2 b^2}{6\pi} \left( \frac{\delta \rho}{\rho} \right)^2 \left( \frac{m_P^4}{(N_{\text{dec}} + 1)^4} \right) \] (10)

The field freezes at \( \phi_F = \phi_{\text{end}} + \frac{1}{\sqrt{6}}(61.7 - 4 \ln b)m_P. \) Then coincidence demands: \( b \simeq 14.5, \) which is too large for brief acceleration. Further, Eq.(10) gives \( V_{\text{end}}^{1/4} \simeq 2 \times 10^{13}\text{GeV}. \) Using \( V_{\text{end}} \simeq 2b^2 \lambda \) one finds \( \lambda^{1/4} \simeq 6 \times 10^{12}\text{GeV} \) and so \( M_5 \simeq 5 \times 10^{14}\text{GeV}. \) The bound on \( A \) suggests: \( M > 7 \times 10^{13}\text{GeV}, \) which enables the identification \( M = M_5. \) The Hot Big Bang begins at temperature:

\[ T_* = \frac{\alpha^3}{16b(N_{\text{dec}} + 1)^4} \sqrt{\frac{2g_*}{15}} \left( \frac{\delta \rho}{\rho} \right)^2 m_P \] (11)

which satisfies the BBN constraint if \( \alpha \gtrsim 0.1. \) With \( \alpha \sim 0.1, \) the modified Friedman equation gives \( T_{\text{reh}} \sim 10^7\text{GeV}, \) which satisfies the gravitino bound.

### 6 Conclusions

Quintessential-inflation model-building is easier in brane-cosmology because the \( \eta \)-problem is overcome by considering steep inflation [2]. The dynamics of the Universe from inflation through to the present have been analysed and employed in the toy-model presented. This model incorporates two natural mass-scales \( (m_P \text{ and } M_5) \) and leads to a spectral index within observational bounds. However, it fails to provide late-time accelerated expansion.

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