Fuzzy Neutrosophic Soft Matrices of Type I and Type II

R. Uma\textsuperscript{a}, S. Sriram\textsuperscript{a} and P. Murugadas\textsuperscript{b}

\textsuperscript{a}Department of Mathematics, Annamalai University, Annamalainagar, India; \textsuperscript{b}Department of Mathematics, Government Arts and Science College, Veerapandi, India

\textbf{ABSTRACT}

In this paper, two types of fuzzy neutrosophic soft matrix have been introduced and it is shown that the set of all fuzzy neutrosophic soft matrices form a semiring. Further it is explored that the set of all fuzzy neutrosophic soft matrices of both types form a vector space.

\textbf{KEYWORDS}

Fuzzy Neutrosophic Soft Set; Fuzzy Neutrosophic Soft Matrix; Fuzzy Neutrosophic Soft Vector Space; Fuzzy Neutrosophic Soft Algebra

\textbf{AMS SUBJECT CLASSIFICATIONS}

03E72; 15B15

1. Introduction

The concept of fuzzy set was introduced by Zadeh \cite{1} in 1965. The traditional fuzzy sets are characterised by the membership value or the grade of membership value. Sometimes it may be very difficult to assign the membership value for fuzzy sets. Consequently the concept of interval valued fuzzy sets was proposed \cite{2} to capture the uncertainty of grade of membership value. Intuitionistic fuzzy sets introduced by Atanassov \cite{3} is appropriate for such a situation. The intuitionistic fuzzy sets can only handle the incomplete information considering both the truth membership (or simply membership) and falsity-membership (or non membership) values. It does not handle the indeterminate and inconsistent information which exists in belief system. Smarandache \cite{4} introduced the concept of neutrosophic set which is a mathematical tool for handling problems involving imprecise, indeterminacy and inconsistent data.

The concept of soft set theory was introduced by Molodtsov \cite{5} in 1999, it is a new approach for modeling vagueness and uncertainty. Maji \cite{6} et al. introduced the concept of fuzzy soft sets with the operations of union, intersection, complement of fuzzy soft sets. The fuzzy soft set concept extended soft sets into intuitionistic fuzzy soft set and fuzzy neutrosophic soft sets.

Rajarajeswari and Dhanalakshmi \cite{7} introduced the intuitionistic fuzzy soft matrices applied in the application of medical diagnosis. Sumathi and Arockiarani \cite{8} introduced new operations on fuzzy neutrosophic soft matrices.

In this paper, we have defined two types of Fuzzy Neutrosophic Soft Matrix (FNSM). In Section 2, we have recalled the definition of soft sets, neutrosophic sets, etc. Operations on neutrosophic soft set of both types have been defined, discussion about two types of
fuzzy neutrosophic soft algebra and two types of fuzzy neutrosophic soft matrix has been introduced in Section 3. In Section 4, we have shown that the set of all FNSM of type-I is a semiring under the operations componentwise addition $\oplus$, componentwise multiplication and the composition $\circ$. Similar results for FNSM of type II are discussed in Section 5.

### 1.1. Preliminaries

**Definition 1.1 ([4]):** A neutrosophic set $A$ on the universe of discourse $X$ is defined as

$$A = \{ (x, T_A(x), I_A(x), F_A(x)) : x \in X \},$$

where $T, I, F : X \to \mathbb{R}$ and

$$-0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+.$$  \hspace{1cm} (1)

From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $\mathbb{R}$. In real-life application especially in scientific and engineering problems, it is difficult to use neutrosophic set with value from real standard or non-standard subset of $\mathbb{R}$. Hence we consider the neutrosophic set which takes the value from the subset of $[0, 1]$.

Therefore we can rewrite Equation (1) as

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3.$$  \hspace{1cm} (2)

In short an element $\tilde{a}$ in the neutrosophic set $A$, can be written as $\tilde{a} = (a^T, a^I, a^F)$, where $a^T$ denotes degree of truth, $a^I$ denotes degree of indeterminacy, $a^F$ denotes degree of falsity such that $0 \leq a^T + a^I + a^F \leq 3$.

**Example 1.2 ([4]):** Assume that the universe of discourse $X = \{x_1, x_2, x_3\}$, where $x_1, x_2, x_3$ characterise the quality, reliability, and price of the objects. It may be further assumed that the values of $\{x_1, x_2, x_3\}$ are in $[0, 1]$ and they are obtained from some investigations of some experts. The experts may impose their opinion in three components viz. the degree of goodness, the degree of indeterminacy and the degree of poorness to explain the characteristics of the objects. Suppose $A$ is a Neutrosophic Set (NS) of $X$, such that $A = \{ (x_1, 0.4, 0.5, 0.3), (x_2, 0.7, 0.2, 0.4), (x_3, 0.8, 0.3, 0.4) \}$, where for $x_1$ the degree of goodness of quality is 0.4, degree of indeterminacy of quality is 0.5 and degree of falsity of quality is 0.3, etc.

**Definition 1.3 ([8]):** A neutrosophic set $A$ is contained in another neutrosophic set $B$. That is $A \subseteq B$ if $\forall x \in X$, $T_A(x) \leq T_B(x)$, $I_A(x) \leq I_B(x)$, $F_A(x) \geq F_B(x)$.

For $\tilde{a}, \tilde{b} \in A$, $\tilde{a} \leq \tilde{b}$ means $a^T \leq b^T$, $a^I \leq b^I$, $a^F \geq b^F$.

**Definition 1.4 ([5]):** Let $U$ be an initial universe set and $E$ be a set of parameters. Let $P(U)$ denotes the power set of $U$. Consider a nonempty set $A, A \subseteq E$. A pair $(F, A)$ is called a soft set over $U$, where $F$ is a mapping given by $F : A \rightarrow P(U)$.

**Definition 1.5 ([9]):** Let $U$ be an initial universe set and $E$ be a set of parameters. Consider a nonempty set $A, A \subseteq E$. Let $P(U)$ denotes the set of all fuzzy neutrosophic sets of $U$. The collection $(F, A)$ is termed to be the Fuzzy Neutrosophic Soft Set (FNSS) over $U$, where $F$ is a mapping given by $F : A \rightarrow P(U)$. Hereafter we simply consider $A$ as FNSS over $U$ instead of $(F, A)$. 
1.2. Fuzzy Neutrosophic Soft Algebra and Fuzzy Neutrosophic Soft Matrices

Definition 1.6: Let \( \langle a^T, a', a^F \rangle, \langle b^T, b', b^F \rangle \) be any two elements in an FNSS \( A \). We define the addition and multiplication in FNSS of type I as follows:

1. **Idempotence:**
   - (i) \( \langle a^T, a', a^F \rangle + \langle a^T, a', a^F \rangle = \langle a^T, a', a^F \rangle \).
   - (ii) \( \langle a^T, a', a^F \rangle . \langle a^T, a', a^F \rangle = \langle a^T, a', a^F \rangle \).
2. **Commutativity:**
   - (i) \( \langle a^T, a', a^F \rangle + \langle b^T, b', b^F \rangle = \langle b^T, b', b^F \rangle + \langle a^T, a', a^F \rangle \).
   - (ii) \( \langle a^T, a', a^F \rangle . \langle b^T, b', b^F \rangle = \langle b^T, b', b^F \rangle . \langle a^T, a', a^F \rangle \).
3. **Associativity:**
   - (i) \( \langle a^T, a', a^F \rangle + (\langle b^T, b', b^F \rangle + \langle c^T, c', c^F \rangle) = (\langle a^T, a', a^F \rangle + \langle b^T, b', b^F \rangle) + \langle c^T, c', c^F \rangle \).
   - (ii) \( \langle a^T, a', a^F \rangle . (\langle b^T, b', b^F \rangle . \langle c^T, c', c^F \rangle) = (\langle a^T, a', a^F \rangle . \langle b^T, b', b^F \rangle) . \langle c^T, c', c^F \rangle \).
4. **Absorption:**
   - (i) \( \langle a^T, a', a^F \rangle + (\langle a^T, a', a^F \rangle . \langle b^T, b', b^F \rangle) = \langle a^T, a', a^F \rangle \).
   - (ii) \( \langle a^T, a', a^F \rangle . (\langle a^T, a', a^F \rangle + \langle b^T, b', b^F \rangle) = \langle a^T, a', a^F \rangle \).
5. **Distributivity:**
   - (i) \( \langle a^T, a', a^F \rangle . (\langle b^T, b', b^F \rangle + \langle c^T, c', c^F \rangle) = (\langle a^T, a', a^F \rangle . \langle b^T, b', b^F \rangle) + (\langle a^T, a', a^F \rangle . \langle c^T, c', c^F \rangle) \).
   - (ii) \( \langle a^T, a', a^F \rangle + (\langle a^T, a', a^F \rangle . \langle b^T, b', b^F \rangle) = (\langle a^T, a', a^F \rangle + \langle b^T, b', b^F \rangle) . \langle a^T, a', a^F \rangle \).
   - (\langle a^T, a', a^F \rangle + \langle a^T, a', a^F \rangle) . (\langle b^T, b', b^F \rangle + \langle c^T, c', c^F \rangle) = \langle a^T, a', a^F \rangle . (\langle b^T, b', b^F \rangle + \langle c^T, c', c^F \rangle) \).
6. **Universal bounds:**
   - (i) \( \langle a^T, a', a^F \rangle + \langle 0, 0, 1 \rangle = \langle a^T, a', a^F \rangle \).
   - (\langle a^T, a', a^F \rangle + \langle 0, 0, 1 \rangle) + \langle 1, 1, 0 \rangle = \langle 1, 1, 0 \rangle \).
   - (ii) \( \langle a^T, a', a^F \rangle . \langle 0, 0, 1 \rangle = \langle a^T, a', a^F \rangle \).
   - (\langle a^T, a', a^F \rangle . \langle 0, 0, 1 \rangle) . \langle 1, 1, 0 \rangle = \langle a^T, a', a^F \rangle . \langle 1, 1, 0 \rangle \).

If we replace additive identity \( \langle 0, 0, 1 \rangle \) by \( \langle 0, 1, 1 \rangle \) and multiplicative identity \( \langle 1, 1, 0 \rangle \) by \( \langle 1, 0, 0 \rangle \) in the 6th postulates we get fuzzy neutrosophic soft algebra for type II.

By using neutrosophic soft set concept, we define the Fuzzy Neutrosophic Soft Matrix (FNSM).
The matrix form of FNSM can be written as $A = (a_{ij}^N)$; all entries of $A$ is of the form $(a_{ij}^N) = (a_{ij}^T, a_{ij}^L, a_{ij}^F)$. For our convenience, we use $(a_{ij})$ instead of $(a_{ij}^N)$.

**Definition 1.7 ([10]):** Let $U = \{c_1, c_2, \ldots, c_m\}$ be the universal set and $E$ be the set of parameters given by $E = \{e_1, e_2, \ldots, e_n\}$. Let $A \subseteq E$. A pair $(F, A)$ be an FNSS over $U$. Then the subset of $U \times E$ is defined by $R_A = \{(u, e); \ e \in A, \ u \in F_A(e)\}$ which is called a relation form of $(F_A, E)$. The membership function, indeterminacy membership function and non membership function are written by $T_{RA} : U \times E \rightarrow [0, 1]$, $I_{RA} : U \times E \rightarrow [0, 1]$ and $F_{RA} : U \times E \rightarrow [0, 1]$ where $T_{RA}(u, e) \in [0, 1], I_{RA}(u, e) \in [0, 1]$ and $F_{RA}(u, e) \in [0, 1]$ are the membership value, indeterminacy value and non membership value respectively of $u \in U$ for each $e \in E$.

If $[(T_{ij}, l_{ij}, F_{ij})] = [(T_{ij}(u_i, e_j), l_{ij}(u_i, e_j), F_{ij}(u_i, e_j))]$ we define a matrix

$$[[T_{ij}, l_{ij}, F_{ij}]]_{m \times n} = \begin{bmatrix}
\langle T_{11}, l_{11}, F_{11} \rangle & \langle T_{12}, l_{12}, F_{12} \rangle & \cdots & \langle T_{1n}, l_{1n}, F_{1n} \rangle \\
\langle T_{21}, l_{21}, F_{21} \rangle & \langle T_{22}, l_{22}, F_{22} \rangle & \cdots & \langle T_{2n}, l_{2n}, F_{2n} \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle T_{m1}, l_{m1}, F_{m1} \rangle & \langle T_{m2}, l_{m2}, F_{m2} \rangle & \cdots & \langle T_{mn}, l_{mn}, F_{mn} \rangle 
\end{bmatrix}$$

which is called an $m \times n$ FNSM of the FNSS $(F_A, E)$ over $U$.

**Definition 1.8:** Let $U = \{c_1, c_2, \ldots, c_m\}$ be the universal set and $E$ be the set of parameters given by $E = \{e_1, e_2, \ldots, e_n\}$. Let $A \subseteq E$. A pair $(F, A)$ be a fuzzy neutrosophic soft set. Then fuzzy neutrosophic soft set $(F, A)$ in a matrix form as $A_{m \times n} = (a_{ij})_{m \times n}$ or $A = (a_{ij}), i = 1, 2, \ldots m, j = 1, 2, \ldots n$ where

$$(a_{ij}) = \begin{cases}
(T(c_i, e_j), l(c_i, e_j), F(c_i, e_j)) & \text{if } e_j \in A \\
(0, 0, 1) & \text{if } e_j \notin A
\end{cases}$$

where $T_j(c_i)$ represents the membership of $c_i$, $l_j(c_i)$ represents the indeterminacy of $c_i$ and $F_j(c_i)$ represents the non-membership of $c_i$ in the FNSS $(F, A)$.

If we replace the identity element $(0, 0, 1)$ by $(0, 1, 1)$ in the above form, we get FNSM of type II.

### 2. Fuzzy Neutrosophic Soft Matrix of Type I

Let $F_{m \times n}$ denote FNSM of order $m \times n$ and $F_n$ denote FNSM of order $n \times n$.

**Definition 2.1:** Let $A = (a_{ij}^T, a_{ij}^L, a_{ij}^F), B = (b_{ij}^T, b_{ij}^L, b_{ij}^F) \in F_{m \times n}$ the componentwise addition and componentwise multiplication are defined as

$$A \oplus B = (\max(a_{ij}^T, b_{ij}^T), \max(a_{ij}^L, b_{ij}^L), \min(a_{ij}^F, b_{ij}^F)).$$

$$A \odot B = (\min(a_{ij}^T, b_{ij}^T), \min(a_{ij}^L, b_{ij}^L), \max(a_{ij}^F, b_{ij}^F)).$$
Definition 2.2: Let \( A \in \mathcal{F}_{m \times n}, B \in \mathcal{F}_{n \times p} \), the composition of \( A \) and \( B \) is defined as
\[
A \circ B = \left( \sum_{k=1}^{n} (a_{ik}^T \land b_{kj}^T), \sum_{k=1}^{n} (a_{ik}^T \land b_{kj}^T), \prod_{k=1}^{n} (a_{ik}^T \lor b_{kj}^T) \right)
\]
equivalently we can write the same as
\[
= \left( \bigvee_{k=1}^{n} (a_{ik}^T \land b_{kj}^T), \bigvee_{k=1}^{n} (a_{ik}^T \land b_{kj}^T), \bigwedge_{k=1}^{n} (a_{ik}^T \lor b_{kj}^T) \right).
\]

The product \( A \circ B \) is defined if and only if the number of columns of \( A \) is same as the number of rows of \( B \). \( A \) and \( B \) are said to be conformable for multiplication. We shall use \( AB \) instead of \( A \circ B \).

Definition 2.3: The \( n \times m \) Zero matrix \( O_1 \) is the matrix all of whose entries are of the form \( (0, 0, 1) \). The \( n \times n \) identity matrix \( I_1 \) is the matrix
\[
I_1 = \begin{cases} 
(1, 1, 0) & \text{if } i = j \\
(0, 0, 1) & \text{if } i \neq j 
\end{cases}
\]
The \( n \times m \) universal matrix \( J_1 \) is the matrix all of whose entries are of the form \( (1, 1, 0) \).

Definition 2.4: Let \( A = (a_{ij}) \) and \( c \in \mathcal{F} = [0, 1] \). Define the fuzzy neutrosophic scalar multiplication as \( cA = (\{\min(c, a_{ij}^T), \min(c, a_{ij}^T), \max(c, a_{ij}^T)\}) \in \mathcal{F}_{m \times n} \).

For the universal matrix \( J_1 \), by the Definition 4.1,
\[
cJ_1 = \min(c \circ (1, 1, 0)) = (\{\min(c, 1), \min(c, 1), \max(c, 0)\}) = (c, c, c)
\]
is the constant matrix all of whose entries are \( c \) further under componentwise multiplication
\[
cJ_1 \circ A = (c, c, c) \circ (a_{ij}^T, a_{ij}^T, a_{ij}^T)
\]
\[
= (\{\min(c, a_{ij}^T), \min(c, a_{ij}^T), \max(c, a_{ij}^T)\})
\]
\[
= cA \tag{2}
\]

Definition 2.5: If \( A = (a_{ij}) \in \mathcal{F}_{m \times n} \), where \( (a_{ij}) = (a_{ij}^T, a_{ij}^T, a_{ij}^T) \), then \( A^c = (b_{ij})_{m \times n} \) where
\[
b_{ij} = (a_{ij}^T, 1 - a_{ij}^T, a_{ij}^T)
\]
is the complement of \( A \).

Theorem 2.6: The set \( \mathcal{F}_{m \times n} \) is a fuzzy neutrosophic soft algebra under the componentwise addition and componentwise multiplication operations \((\oplus, \odot)\) defined as follows. For \( A = (a_{ij}), B = (b_{ij}) \) in \( \mathcal{F}_{m \times n} \),
\[
A \oplus B = (\max(a_{ij}^T, b_{ij}^T), \max(a_{ij}^T, b_{ij}^T), \min(a_{ij}^T, b_{ij}^T))
\]
\[
A \odot B = (\min(a_{ij}^T, b_{ij}^T), \min(a_{ij}^T, b_{ij}^T), \max(a_{ij}^T, b_{ij}^T))
\]

Proof: The postulates \((p_1)\) to \((p_4)\) of a fuzzy neutrosophic soft algebra are automatically hold. \( A \oplus O_1 = A \) and \( A \odot J_1 = A \) for all \( A \in \mathcal{F}_{m \times n} \). Hence the zero matrix \( O_1 \) is the additive identity and the universal matrix \( J_1 \) is the multiplicative identity. Thus identity element relative to the operations \( + \) and \( \odot \) exist. Further \( A \oplus J_1 = J_1 \) and \( A \odot O_1 = A \).
Theorem 2.8: For any three matrices $A, B, C \in \mathcal{F}_{m \times n}$, $A \oplus (B \oplus C) = (A \oplus B) \oplus C$. That is the left distributive property holds.

Proof: Let $A = (a_{ij}^T, a_{ij}', a_{ij}')_{m \times n}$, $B = (b_{ij}^T, b_{ij}', b_{ij}')_{m \times n}$, $C = (c_{ij}^T, c_{ij}', c_{ij}')_{m \times n}$. Therefore $(p_6)$ holds. For $A = (a_{ij}^T, a_{ij}', a_{ij}')$, $B = (b_{ij}^T, b_{ij}', b_{ij}')$, $C = (c_{ij}^T, c_{ij}', c_{ij}') \in \mathcal{F}_{m \times n}$, since $a_{ij}^T, a_{ij}', a_{ij}'$, $b_{ij}^T, b_{ij}', b_{ij}'$, $c_{ij}^T, c_{ij}', c_{ij}'$ are all real numbers in $[0, 1]$ they are comparable. If $a_{ij} \leq b_{ij}$ (or $c_{ij}$ then in both cases, $(\min\{a_{ij}, \max\{b_{ij}, c_{ij}\}\}) = a_{ij}$ and $\max\{\inf\{a_{ij}, b_{ij}\}, \min\{a_{ij}, c_{ij}\}\} = a_{ij}$.

Therefore $ij^{th}$ entry of $A \odot (B \odot C) = (ij)^{th}$ entry of $(A \odot B) \oplus (A \odot C)$. If $a_{ij} \geq b_{ij}$ and $c_{ij}$ then we have two cases, $a_{ij} \geq b_{ij} \geq c_{ij}$ (or) $a_{ij} \geq c_{ij} \geq b_{ij}$ then,

$$\min\{a_{ij}, \max\{b_{ij}, c_{ij}\}\} = b_{ij} = \max\{\inf\{a_{ij}, b_{ij}\}, \min\{a_{ij}, c_{ij}\}\}$$

Therefore $(ij)^{th}$ entry of $A \odot (B \odot C) = (ij)^{th}$ entry of $(A \odot B) \oplus (A \odot C)$. Hence,

$$A \odot (B \odot C) = (A \odot B) \oplus (A \odot C).$$

Similarly we can prove, $A \oplus (B \odot C) = (A \oplus B) \odot (A \odot C).$ Thus the postulate $(p_5)$ of distributivity holds.

Thus $\mathcal{F}_{m \times n}$ is a fuzzy neutrosophic soft algebra with the operations $(\oplus, \odot)$.}

Theorem 2.7: For any three matrices $A, B, C \in \mathcal{F}_{m \times n}$, $B \in \mathcal{F}_{n \times p}$, $A(B \odot C) = AB \odot AC$. That is the left distributive property holds.

Proof: Let $A = (a_{ij}^T, a_{ij}', a_{ij}')$, $B = (b_{ij}^T, b_{ij}', b_{ij}')$, $C = (c_{ij}^T, c_{ij}', c_{ij}')$ such that the ranges of suffixes $i, j, k$ are 1 to $m, j = 1$ to $n, k = 1$ to $p$ respectively.

Now $(jk)^{th}$ element of $(B \odot C) = (b_{jk}^T, b_{jk}', b_{jk}') \oplus (c_{jk}^T, c_{jk}', c_{jk}')$

$$=(\max\{b_{jk}^T, c_{jk}^T\}, \max\{b_{jk}', c_{jk}'\}, \min\{b_{jk}^T, c_{jk}^T\}).$$

$(ik)^{th}$ element in the product of $A$ and $(B \odot C)$ that is $A(B \odot C)$ is the sum of the product of the corresponding elements in $i$th row of $A$ and $k$th column of $(B \odot C)$.

$$= (\sum_{j=1}^n\langle a_{ij}^T, a_{ij}', a_{ij}'\rangle)\langle b_{jk}^T, b_{jk}', b_{jk}'\rangle \oplus (c_{ij}^T, c_{ij}', c_{ij}').$$

$$= (\sum_{j=1}^n\langle a_{ij}^T, a_{ij}', a_{ij}'\rangle\langle b_{jk}^T, b_{jk}', b_{jk}'\rangle \oplus (\sum_{j=1}^n\langle a_{ij}^T, a_{ij}', a_{ij}'\rangle\langle c_{ij}^T, c_{ij}', c_{ij}'\rangle))$$

$$= (ik)^{th}$$

$$= (ik)^{th}$$ element of $(A \odot (B \odot C))$. Hence $A(B \odot C) = AB \odot AC$.

Similarly we can prove the right distributive property.

In the following theorems, we prove that associative property holds under addition for FNSM of type I.

Theorem 2.8: For any three matrices $A, B, C \in \mathcal{F}_{m \times n}$,

$$A \odot (B \odot C) = (A \odot B) \odot C.$$
\[(A \oplus B) \oplus C = (\max\{a_{ij}^T, b_{ij}^T\}, \max\{a_{ij}^T, b_{ij}^T\}, \min\{a_{ij}^T, b_{ij}^T\}) + (\langle c_{ij}^T, c_{ij}^T, c_{ij}^T \rangle)\]
\[(\max\{a_{ij}^T, b_{ij}^T, c_{ij}^T\}, \max\{a_{ij}^T, b_{ij}^T, c_{ij}^T\}, \min\{a_{ij}^T, b_{ij}^T, c_{ij}^T\})\]

from (3) and (4) we get \(A \oplus (B \oplus C) = (A \oplus B) \oplus C\).

In the following theorem, we prove that the associative property holds under multiplication for FNSM.

**Theorem 2.9:** For \(A \in \mathcal{F}_{m \times n}, B \in \mathcal{F}_{n \times p}, C \in \mathcal{F}_{p \times q}\), then \((AB)C = A(BC)\).

**Proof:** With the given type of matrices \((AB)C\) and \(A(BC)\) are both defined and are of type \(m \times q\). Let \(A = ((a_{ij}^T, a_{ij}^T, a_{ij}^T))_{m \times n}, B = ((b_{jk}^T, b_{jk}^T, b_{jk}^T))_{n \times p}, C = ((c_{kl}^T, c_{kl}^T, c_{kl}^T))_{n \times p}\) such that the ranges of the suffixes \(i, j, k, l\) are 1 to \(m\), 1 to \(n\), 1 to \(p\) and 1 to \(q\) respectively.

Now (ik) th element of the product \(AB = ((\sum_{j=1}^{n}a_{ij}^T b_{jk}^T), \sum_{j=1}^{n}(a_{ij}^T b_{jk}^T), \prod_{j=1}^{n}(a_{ij}^T + b_{jk}^T))\).

The (il)th element in the product \((AB)C\) is the sum of product of the corresponding elements in the \(i\) th row of \(AB\) and \(l\) th column of \(C\) with k-common.

Thus (il) th element of
\[
(AB)C = \left(\sum_{j=1}^{n} (a_{ij}^T b_{jk}^T) \sum_{j=1}^{n} (a_{ij}^T b_{jk}^T) \prod_{j=1}^{n} (a_{ij}^T + b_{jk}^T) \right) (\langle c_{kl}^T, c_{kl}^T, c_{kl}^T \rangle)
\]
\[
\left(\sum_{k=1}^{p} \sum_{j=1}^{n} (a_{ij}^T b_{jk}^T c_{kl}^T), \sum_{k=1}^{p} \sum_{j=1}^{n} (a_{ij}^T b_{jk}^T c_{kl}^T), \prod_{k=1}^{p} (a_{ij}^T + b_{jk}^T + c_{kl}^T) \right)
\]

Now (il) th element of the product \(BC = \left(\sum_{k=1}^{p} (b_{jk}^T c_{kl}^T), \sum_{k=1}^{p} (b_{jk}^T c_{kl}^T), \prod_{k=1}^{p} (b_{jk}^T + c_{kl}^T) \right).\)

Again the (il) th element of the product of \(A(BC)\) is the sum of the product of the corresponding element in the \(i\) th row of \(A\) and \(l\) th column of \(BC\).

(ii) th element of
\[
A(BC) = (a_{ij}^T, a_{ij}^T, a_{ij}^T) \left[\sum_{k=1}^{p} (b_{jk}^T c_{kl}^T), \sum_{k=1}^{p} (b_{jk}^T c_{kl}^T), \prod_{k=1}^{p} (b_{jk}^T + c_{kl}^T) \right]
\]
\[
\left(\sum_{j=1}^{n} (a_{ij}^T b_{jk}^T c_{kl}^T), \sum_{j=1}^{n} (a_{ij}^T b_{jk}^T c_{kl}^T), \prod_{j=1}^{n} \prod_{k=1}^{p} (a_{ij}^T + b_{jk}^T + c_{kl}^T) \right)
\]

from (5) and (6) we get \((AB)C = A(BC)\).
By Theorem 4.7, Theorem 4.8 and Theorem 4.9, \((F_{m \times n}, \oplus, \odot)\) is a semiring.

**Theorem 2.10:** Let \(V_n\) will denote the set of all \(n\)-tuples \(\left( (v^T_1, v'_1, v^F_1), \ldots, (v^T_n, v'_n, v^F_n) \right)\). The following operations are defined for

\[ v = \left( (v^T_1, v'_1, v^F_1), \ldots, (v^T_n, v'_n, v^F_n) \right), \quad s = \left( (s^T_1, s'_1, s^F_1), \ldots, (s^T_n, s'_n, s^F_n) \right) \] in \(V_n\) and \(r \in F = [0, 1], \)

\[ (v^T_1, v'_1, v^F_1), \ldots, (v^T_n, v'_n, v^F_n) + (s^T_1, s'_1, s^F_1), \ldots, (s^T_n, s'_n, s^F_n) = \left( (v^T_1 + s^T_1, v'_1 + s'_1, v^F_1 + s^F_1), \ldots, (v^T_n + s^T_n, v'_n + s'_n, v^F_n + s^F_n) \right) \]

and \(r\left( (v^T_1, v'_1, v^F_1), \ldots, (v^T_n, v'_n, v^F_n) \right) = \left( r(v^T_1, v'_1, v^F_1), \ldots, r(v^T_n, v'_n, v^F_n) \right)\).

The members of \(V_n\) have the properties

1. \(v + w = w + v\),
2. \((v + w) + u = (v + w) + u,\)
3. \((ab)v = a(bv),\)
4. \((a + b)v = av + bv,\)
5. \(a(v + w) = av + bw,\)
6. \((1, 1, 0)v = v,\)
7. \((0, 0, 1) + (0, 0, 1) = (0, 0, 1),\)
8. \(0, 0, 1) + v = v.\)

If we write a member of \(V_n\), \(1 \times n\) matrix which is called row vector. The isometric set of \(n \times 1\) matrices is called column vector, and denoted by \(V^n\). For any result above \(V_n\), there exist a corresponding results above \(V^n\). The system \(V_n\) together with these operations called fuzzy neutrosophic soft vector space of type \(I\).

If we replace equation \((1, 1, 0)\) by \((1, 1, 0)\) in (6) and \((0, 0, 1)\) by \((0, 1, 1)\) in 7 and 8 we get fuzzy neutrosophic soft vector space of type II, \(O'' = (0, 1, \ldots, (0, 1, 1)).\)

**Proposition 2.11:** The set \(F_{m \times n}\) is a fuzzy neutrosophic soft vector space under the operations defined as

\[ A \oplus B = \left( \max\{a^T_{ij}, b^T_{ij}\}, \max\{a'_{ij}, b'_{ij}\}, \min\{a^F_{ij}, b^F_{ij}\} \right) \]

\[ cA = \left( \min\{c, a^T_{ij}\}, \min\{c, a'_{ij}\}, \min\{c, a^F_{ij}\} \right) \]

\[ A = \left( \{a^T_{ij}, a'_{ij}, a^F_{ij}\} \right), B = \left( \{b^T_{ij}, b'_{ij}, b^F_{ij}\} \right) \in F_{m \times n} \text{ and } c \in F \]

**Proof:** For \(A, B, C \in F_{m \times n},\)

\[ A \oplus B = B \oplus A \in F_{m \times n} \]

\[ A \oplus (B \oplus C) = (A \oplus B) \oplus C \]

For all \(A \in F_{m \times n}\) there exist an element \(O_1 \in F_{m \times n}\) such that \(A \oplus O_1 = A.\) For \(c \in F = [0, 1],\)

\[ c(A \oplus B) = c_1 \odot (A \oplus B) \odot (\text{by Equation (2)}) = (c_1 \odot A) \odot c_1 \odot B = cA \oplus cB \text{ (by Equation (2))} \]
For $c_1, c_2 \in \mathcal{F}$

$$(c_1 + c_2)A = (c_1 \oplus c_2)(J_1) \odot A \text{ (by Equation (2))}$$

$$= (c_1 J_1 \oplus c_2 J_1) \odot A$$

$$= (c_1 J_1 \odot A) \oplus (c_2 J_1 \odot A)$$

$$= c_1 A \oplus c_2 A. \text{ (by Equation (2))}$$

From the Definition 4.10, the condition 3, 6, 7, 8 are trivial. Hence $\mathcal{F}_{m \times n}$ is a fuzzy neutrosophic soft vector space over $\mathcal{F}$.

**Definition 2.12:** Let $A = (\langle a_{ij}^T, a_{ij}^l, a_{ij}^r \rangle),$ $B = (\langle b_{ij}^T, b_{ij}^l, b_{ij}^r \rangle)$ be two FNSMs of same dimensions. We write $A \leq B$ if $a_{ij}^T \leq b_{ij}^T$, $a_{ij}^l \leq b_{ij}^l$, $a_{ij}^r \geq b_{ij}^r$ for all $i,j$ and we say that $A$ is dominated by $B$ (or) $B$ dominates $A$. $A$ and $B$ are said to be comparable, if either $A \leq B$ (or) $B \leq A$. $A < B$ if $a_{ij}^T < b_{ij}^T$, $a_{ij}^l < b_{ij}^l$, $a_{ij}^r > b_{ij}^r$.

**Theorem 2.13:** Let $A, B \in \mathcal{F}_{m \times n}$. Then $A \leq B \iff A \oplus B = B$.

**Proof:** If $A \leq B$, then

$$A \oplus B = \left( \left\{ \max \left\{ a_{ij}^T, b_{ij}^T \right\}, \max \left\{ a_{ij}^l, b_{ij}^l \right\}, \min \left\{ a_{ij}^r, b_{ij}^r \right\} \right\} \right)$$

$$= (b_{ij}^T, b_{ij}^l, b_{ij}^r)$$

$$= B$$

Conversely if $A \oplus B = B$, then $\langle a_{ij}^T, a_{ij}^l, a_{ij}^r \rangle \leq \langle b_{ij}^T, b_{ij}^l, b_{ij}^r \rangle$, $\forall i,j,$

that is $A \leq B$. Thus $A \leq B \iff A \oplus B = B$.

**Theorem 2.14:** Let $A, B \in \mathcal{F}_{m \times n}$. If $A \leq B$ then for any $C \in \mathcal{F}_{np}, AC \leq BC$ and for any $D \in \mathcal{F}_{pm}, DA \leq DB$.

**Proof:** Since $A \leq B \Rightarrow a_{ik}^T \leq b_{ik}^T, a_{ik}^l \leq b_{ik}^l, a_{ik}^r \geq b_{ik}^r$ for all $i = 1$ to $m$ and $k = 1$ to $n$. By fuzzy multiplication and addition, $a_{ik}^T c_{kj} \leq b_{ik}^T c_{kj}, a_{ik}^l c_{kj} \leq b_{ik}^l c_{kj}, a_{ik}^r + c_{kj} \geq b_{ik}^r + c_{kj}$ for $j = 1$ to $p$. By fuzzy addition we get

$$\sum_{k=1}^n (a_{ik}^T c_{kj}) \leq \sum_{k=1}^n (b_{ik}^T c_{kj}), \quad \sum_{k=1}^n (a_{ik}^l c_{kj}) \leq \sum_{k=1}^n (b_{ik}^l c_{kj}), \quad \prod_{k=1}^n (a_{ik}^T + c_{kj}) \geq \prod_{k=1}^n (b_{ik}^T + c_{kj}).$$

Thus $AC \leq BC, DA \leq DB$ can be proved in the same method.

### 3. Fuzzy Neutrosophic Soft Matrix of Type II

Here we discuss some operations on FNSM of Type II

**Definition 3.1:** Let $A = (\langle a_{ij}^T, a_{ij}^l, a_{ij}^r \rangle), B = (\langle b_{ij}^T, b_{ij}^l, b_{ij}^r \rangle) \in \mathcal{F}_{m \times n}$, the component wise addition and component wise multiplication are defined as
The product of $A$ and $B$ is defined as
\[
A \ast B = \left( \sum_{k=1}^{n} (a_{ik}^T \land b_{kj}^T), \prod_{k=1}^{n} (a_{ik}^T \lor b_{kj}^T), \prod_{k=1}^{n} (a_{ik}^F \lor b_{kj}^F) \right)
\]
equivalently we can write the same as
\[
= \left( \bigvee_{k=1}^{n} (a_{ik}^T \land b_{kj}^T), \bigwedge_{k=1}^{n} (a_{ik}^T \lor b_{kj}^T), \bigwedge_{k=1}^{n} (a_{ik}^F \lor b_{kj}^F) \right).
\]

the product $A \ast B$ is defined if and only if the number of columns of $A$ is same as the number of rows of $B$. $A$ and $B$ are said to be conformable for multiplication.

**Definition 3.3:** The $n \times m$ zero matrix $O_2$ is the matrix all of whose entries are of the form $(0, 1, 1)$. The $n \times n$ identity matrix $I_2$ is the matrix
\[
I_2 = \begin{cases} 
(1, 0, 0) & \text{if } i = j \\
(0, 1, 1) & \text{if } i \neq j
\end{cases}
\]
The $n \times m$ universal matrix $J_2$ is the matrix all of whose entries are of the form $(1, 0, 0)$.

**Definition 3.4:** Let $A = (a_{ij}^T, a_{ij}^F, a_{ij}^F)$ and $c \in \mathcal{F}$, then the fuzzy neutrosophic scalar multiplication is defined by
\[
cA = (\min\{c, a_{ij}^T\}, \max\{c, a_{ij}^F\}, \max\{c, a_{ij}^F\}).
\]

**Definition 3.5:** If $A = (a_{ij}) \in \mathcal{F}_{m \times n}$, where $(a_{ij}) = (a_{ij}^T, a_{ij}^F, a_{ij}^F)$ then
\[
A^c = (b_{ij})_{m \times n} \text{ where } (b_{ij}) = (a_{ij}^T, a_{ij}^F, a_{ij}^F), \text{ the complement of } A.
\]

**Definition 3.6:** Let $A = (a_{ij}^T, a_{ij}^F, a_{ij}^F), B = (b_{ij}^T, b_{ij}^F, b_{ij}^F) \in \mathcal{F}_{m \times n}$. We write $A \leq B$ if $a_{ij}^T \leq b_{ij}^T, a_{ij}^F \geq b_{ij}^F, a_{ij}^F \geq b_{ij}^F$ for all $i, j$ and we say that $A$ is dominated by $B$ (or $B$ dominates $A$). $A$ and $B$ are said to be comparable, if either $A \leq B$ (or) $B \leq A$. $A < B$ if $a_{ij}^T < b_{ij}^T, a_{ij}^F > b_{ij}^F, a_{ij}^F > b_{ij}^F$.

By the operation defined above in FNSM of type II, one can easily verify that $(\oplus, \odot)$ is an FNS of type II, '$\ast$' is distributive over $\oplus, \odot$ is associative, '$\ast$' is associative and $\mathcal{F}_{m \times n}$ is a fuzzy neutrosophic soft vector space over these operations. As the proofs are similar to the proofs of FNSM type-I, we omit the proofs.

**Theorem 3.7:** Let $A, B \in \mathcal{F}_{m \times n}$. Then $A \leq B \iff A \oplus B = B$. 

\[
A \oplus B = \left( \max\{a_{ij}^T, b_{ij}^T\}, \min\{a_{ij}^T, b_{ij}^T\}, \min\{a_{ij}^F, b_{ij}^F\} \right).
\]
\[
A \odot B = \left( \min\{a_{ij}^T, b_{ij}^T\}, \max\{a_{ij}^T, b_{ij}^T\}, \max\{a_{ij}^F, b_{ij}^F\} \right).
\]
Proof: If \( A \leq B \), then
\[
A + B = \left( \left( \max \left\{ a_{ij}^T, b_{ij}^T \right\}, \min \left\{ a_{ij}^I, b_{ij}^I \right\}, \min \left\{ a_{ij}^F, b_{ij}^F \right\} \right) \right) = \left( \left( b_{ij}^T, b_{ij}^I, b_{ij}^F \right) \right) = B
\]

Conversely if \( A \oplus B = B \), then \( \left( \left\{ a_{ij}^T, a_{ij}^I, a_{ij}^F \right\} \right) \leq \left( \left\{ b_{ij}^T, b_{ij}^I, b_{ij}^F \right\} \right), \forall i,j \) that is \( A \leq B \). Thus \( A \leq B \iff A \oplus B = B \).

Theorem 3.8: Let \( A, B \in \mathcal{F}_{m \times n} \). If \( A \leq B \) then for any \( C \in f_{np} \), \( AC \leq BC \) and for any \( D \in f_{pm} \), \( DA \leq DB \).

Proof: Since \( A \leq B \Rightarrow a_{ik}^T \leq b_{ik}^T, a_{ik}^I \geq b_{ik}^I, a_{ik}^F \geq b_{ik}^F \) for all \( i = 1 \) to \( m \) and \( k = 1 \) to \( n \). By fuzzy multiplication and addition \( a_{ik}^T c_{kj}^I \leq b_{ik}^T c_{kj}^I, a_{ik}^I + c_{kj}^I \geq b_{ik}^I + c_{kj}^I, a_{ik}^F + c_{kj}^F \geq b_{ik}^F + c_{kj}^F \) for \( j = 1 \) to \( p \). By fuzzy addition we get, \( \sum_{k=1}^{n} (a_{ik}^I c_{kj}^I) \leq \sum_{k=1}^{n} (b_{ik}^I c_{kj}^I), \prod_{k=1}^{m} (a_{ik}^F + c_{kj}^F) \geq \prod_{k=1}^{m} (b_{ik}^F + c_{kj}^F) \). Thus \( AC \leq BC, DA \leq DB \) can be proved in the same method.

4. Conclusion

In this paper, fuzzy matrix concept is extended to neutrosophic soft matrix using the operations \( \oplus \), \( \odot \) and \( \circ \), it is shown that the set of all FNSM of type I is a semiring. Further it is noted that the results which are true for type II FNSM. Also certain properties are discussed using the above-defined operations.

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No potential conflict of interest was reported by the author(s).

Notes on contributors

R. Uma received her Ph.D Degree in Mathematics from Annamalai University, Tamilnadu, India. She has published research article in international journals. Also, she attended and presented research articles in International conferences. Her specializations include Theory of Fuzzy Matrix and Neutrosophic Matrix.

Dr. S. Sriram is currently professor and Head, Department of Mathematics, Annamalai University, Tamilnadu, India. He completed his Ph.D in Mathematics from Annamalai University. His area of research include the Theory of fuzzy matrix, generalized fuzzy matrices. He has guided Ph.D scholars and has published research articles in national and international journals.

Dr. P. Murugadas is Assistant Professor, Department of Mathematics, Government Arts College, Veerapandi, Tamilnadu, India. He received his Ph.D from Annamalai University in 2011. He has 18 years of teaching experience and his area of research in the Fuzzy Matrix Theory and Fuzzy Algebra.

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