MODULI STACKS $\mathcal{L}_{g,S}$

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Abstract. This paper is a sequel to the paper by A. Losev and Yu. Manin [LoMa1], in which new moduli stacks $\mathcal{L}_{g,S}$ of pointed curves were introduced. They classify curves endowed with a family of smooth points divided into two groups, such that the points of the second group are allowed to coincide. The homology of these stacks form components of the extended modular operad whose combinatorial models are further studied in [LoMa2]. In this paper the basic geometric properties of $\mathcal{L}_{g,S}$ are established using the notion of weighted stable pointed curves introduced recently by B. Hassett. The main result is a generalization of Keel’s and Kontsevich–Manin’s theorems on the structure of $H^*(\overline{M}_{0,S})$.

À Pierre Cartier, en témoignage de respect et d’amitié

§0. Introduction

This paper, together with [LoMa2], is a sequel to [LoMa1] where new moduli stacks $\mathcal{L}_{g,S}$ were first introduced. Briefly, let $S$ be a finite set of labels partitioned into two subsets, white and black ones. The stack $\mathcal{L}_{g,S}$ parametrizes algebraic curves of genus $g$ endowed with a family of smooth points labeled by $S$ and satisfying a certain stability condition. All points endowed with white labels must be pairwise distinct, and distinct from all points with black labels. Black points may cluster in an arbitrary way. If there are no black points, we get the classical Deligne–Mumford stacks $\overline{M}_{g,S}$. The simplest non–trivial family of $\mathcal{L}_{g,S}$ thoroughly studied in [LoMa1] corresponds to the case $g = 0$, two white points, and arbitrary number of black points. In particular, in [LoMa1] we calculated the Chow/homology groups of these stacks, and studied the representation theory of a family of rings constructed from these groups.

One objective of [LoMa1] was the extension of the homology operad $\{H_*(\overline{M}_{0,n+1})\}$ by a missing $n = 1$ term. We argued that the union of (the homology of) all $\mathcal{L}_{0,S}$ with two white points provides such a term. The spaces $\mathcal{L}_{0,S}$ with arbitrary number of white and black points, and more generally, the stacks $\mathcal{L}_{g,S}$, are necessary to make the whole system closed with respect to the clutching (operadic) morphisms.

In this paper I review the general properties of $\mathcal{L}_{g,S}$ and calculate their Chow and (co)homology groups for $g = 0$ using a further generalization of $\mathcal{L}_{g,S}$ introduced in the recent paper by B. Hassett [H]. Hassett considers labeling sets $S$ whose elements are endowed with rational weights, introduces the appropriate notion of
stability, and extends the basic constructions of the theory of the stacks $\mathcal{M}_{g,S}$ to the weighted sets. Our stacks $\mathcal{L}_{g,S}$ are special cases of Hassett’s stacks corresponding to appropriate weight systems. Hassett’s technics provide a very flexible algebraic geometric tool and ideally serve our needs.

The paper [LoMa2] is dedicated to the combinatorial models of the cohomology and homology of $\mathcal{L}_{0,S}$ and operadic aspects of this family of rings/modules. The main objective of this paper is to present a proof that these combinatorial models are actually isomorphic to the respective cohomology/homology.

The paper is organized as follows. After reminding principal definitions in §1, we introduce the basic structure morphisms between the stacks $\mathcal{L}_{g,S}$ in §2. They include clutching morphisms (translating into operadic composition laws), forgetting morphisms, and repainting white to black morphisms. The latter play a central role in §3 where the main results of this paper are presented. They concern the structure of the Chow/(co)homology groups of $\mathcal{L}_{0,S}$.

Recall that the cohomology rings of $\mathcal{M}_{0,S}$ were calculated by S. Keel ([Ke]). Keel proved that $H^*(\mathcal{M}_{0,S})$ is a quadratic algebra generated by explicit generators (classes of codimension one boundary strata indexed by 2–partitions of $S$) which satisfy a complete system of explicit linear and quadratic relations having a transparent geometric origin. This result was completed in [KM] and [KMK] where a system of additive generators (classes of boundary strata of arbitrary codimension) and a complete system of linear relations between them was deduced from Keel’s theorem. The latter description was crucial for identifying algebras over the homology operad with formal Frobenius manifolds.

In §3 of this paper and in [LoMa2] both theorems are generalized to $\mathcal{L}_{0,S}$. Proofs are rather long and use algebraic geometric and heavy combinatorial arguments. The algebraic geometric part takes the original Keel’s theorem as the base of induction and descends to the more general $S$ by repainting white points to black one by one. The combinatorial part consists in the thorough study of the abstract quadratic algebras generated by the (analogs of) Keel’s generators and relations. One intermediate combinatorial result (Theorem 3.6.1) is taken for granted here; its proof is contained to [LoMa2].

§1. Definitions and notation

1.1. Curves, pointed curves, stability. A semi–stable curve $C$ over an algebraically closed field $k$ is a proper reduced one–dimensional algebraical scheme over this field having only ordinary double points as singularities. (Geometric) genus of $C$ is $g := \dim H^1(C, \mathcal{O}_C)$.

Let $S$ be a finite set. An $S$–pointed curve $C$ is a system $(C, x_s \mid s \in S)$ where $x_s$ is a family of closed non–singular $k$–points of $S$, non necessarily pairwise distinct. The element $s$ is called the label of $x_s$. 
The normalization $\tilde{C}$ of $C$ is a disjoint union of smooth proper curves. Each irreducible component of $\tilde{C}$ carries inverse images of some labeled points $x_s$ and of singular points of $C$. Taken together, these points are called special ones. Instead of passing to the normalization, we may consider branches (local irreducible germs) of $C$ passing through labeled or singular points. They are in a natural bijection with special points.

A painted finite set $S$ is $S$ together with its partition into two disjoint subsets $S = W \cup B$. Labels from $W$ (resp. $B$) are called white (resp. black) ones. We may refer to $x_s$ as a white (resp. black) point, if its label $s$ is white (resp. black).

In the remainder of this paper, (sets labeling) distinguished points on curves are usually assumed to be painted in this sense. This refers to the labeled points $x_s$ as well as the branches at singular points. The latter are always painted white.

The following is the main definition of this section.

**1.1.1. Definition.** A semistable $S$–pointed curve is called painted stable, if it is connected and the following conditions are satisfied:

(i) Each genus zero component of the normalization $\tilde{C}$ contains at least 3 special points of which at least 2 are white. Each genus one component of $\tilde{C}$ contains at least one special point.

(ii) $x_s$ satisfy the following condition:

$$x_s \neq x_t \quad \text{for each } s \neq t, \ s \in W, \ t \in S. \quad (1.1)$$

Notice that in [LoMa1], 4.3.1, such curves were called stringy stable.

In the intermediate constructions we will also widely use weights on $S$, in the sense of Hassett ([H]), and the related notions of weighted stability. In fact, our paintings corresponds to special systems of Hassett’s weights, and his general techniques provide an extremely efficient way of working with moduli stacks of painted curves.

**1.2. Weights and weighted stability.** Let $S$ be an (unpainted) finite set of labels. The weight data on $S$ is a function $A: S \to \mathbb{Q}$, $s \mapsto a_s$, $0 < a_s \leq 1$. We call $S$ together with a weight data a weighted set.

**1.2.1. Definition ([H]).** A semistable $S$–pointed curve $(C, x_s \mid s \in S)$ is called weighted stable (with respect to $A$) if the following conditions are satisfied:

(i) $\omega_C(\sum_s a_s x_s)$ is ample where $\omega_C$ is the dualizing sheaf of $C$.

(ii) For any subset $S' \subset S$ such that $x_t$ pairwise coincide for $t \in S'$, we have $\sum_{t \in T} a_t \leq 1$. 

In (i) and below, we use a shorthand notation. If at least one \( a_s \) is \( \neq 1 \), then only \( \omega^d_C(\sum_s d a_s x_s) \) is an actual invertible sheaf where \( d > 1 \) is a common denominator of all \( a_s \). Ampleness refers to any of these sheaves.

Clearly, (i) implies that \( 2g - 2 + \sum_s a_s > 0 \).

**1.2.2. Lemma.** Let \( S = W \cup B \) be a painted set. Consider a weight data \( A \) on \( S \) satisfying the following conditions:

\[ (*) a_s = 1 \text{ for all } s \in W, \text{ and } \sum_{t \in B} a_t \leq 1. \]

Then a semistable \( S \)-pointed curve \((C, x_s | s \in S)\) is painted stable if and only if it is weighted stable with respect to \( A \).

**Proof.** The lift of \( \omega^d_C(\sum_s a_s x_s) \) to any component \( D \) of the normalization of \( C \) embeds into \( \Omega^1_D(\sum t b_t x_t) \) where summation is taken over all special points of \( D \) and the weight \( A \) is extended to singularities by 1. Sections which come from \( C \) are singled out by the local condition: at pairs of points which get identified in \( C \) the sum of residues vanishes. This shows that the stability condition (i) of the definition 1.2.1 is non-empty only on components of genus 0 and 1. When (*) is satisfied, this stability condition is then equivalent to the condition (i) of the Definition 1.1.1.

Similarly, (*) and 1.2.1(ii) taken together say that on a weighted stable curve black points may pairwise coincide, but each white point must be different from all other points. This is precisely the condition (ii) of painted stability.

**1.3. Families of pointed curves.** Let \( T \) be a scheme, \( S \) a finite set, \( g \geq 0 \). An \( S \)-pointed curve (or family of curves) of genus \( g \) over \( T \) consists of the data

\[ (\pi : C \to T; x_i : T \to C, \ i \in S) \]

where

(i) \( \pi \) is a flat proper morphism whose geometric fibres \( C_t \) are semistable curves of genus \( g \).

(ii) \( x_i, i \in S \), are sections of \( \pi \) not containing singular points of geometric fibres.

Various definitions of stability from the subsections 1.2–1.3 are generalized to families by requiring the respective properties to hold on all geometric fibers of \( \pi \). Ampleness condition can be equivalently stated in terms of the relative dualizing sheaf.

**1.3.1. Stacks of weighted stable curves \( \overline{M}_{g,A} \).** The first main result of \([H]\) is a proof of the following fact. Fix a weighted set of labels \( S \) and a value of genus \( g \). Then families of weighted stable \( S \)-pointed curves of genus \( g \) form (schematic points of) a connected smooth proper over \( \mathbb{Z} \) Deligne–Mumford stack \( \overline{M}_{g,A} \). The respective coarse moduli scheme is projective over \( \mathbb{Z} \).
1.3.2. Stacks of painted stable curves $\mathcal{L}_{g,S}$. Let now $S$ be a painted set. Reinterpreting the painted stability condition as in Lemma 1.2.2, we deduce from Hassett’s theorem the existence of the respective stacks $\mathcal{L}_{g,S}$ which were constructed in [LoMa1] by an alternative method which we called there “adjunction of the generic black point.”

1.4. Graphs. A graph $\tau$, by definition, is a quadruple $(V_\tau, F_\tau, \partial_\tau, j_\tau)$ where $V_\tau$, resp. $F_\tau$, are finite sets of vertices, resp. flags; $\partial_\tau : F_\tau \to V_\tau$ is the boundary, or incidence, map; $j_\tau : F_\tau \to F_\tau$ is an involution of the set of flags. The geometric realization of $\tau$ is a topological space which is obtained from $F_\tau$ copies of $[0, 1]$ by gluing together points $0$ in the copies corresponding to each vertex $v \in V_\tau$, and by gluing together points $1$ in each orbit of $j_\tau$. This motivates considering the following auxiliary sets and their geometric realizations: the set $E_\tau$ of edges of $\tau$, formally consisting of cardinality two orbits of $j_\tau$, and the set $T_\tau$ of tails, consisting of those flags, which are $j_\tau$–invariant.

We will mostly think and speak about graphs directly in terms of their geometric realizations. In particular, $\tau$ will be called connected (resp. tree, resp. forest) if its geometric realization is connected (resp. connected and has no loops, resp. is a disjoint union of trees).

The (dual) modular graph of an $S$–pointed semistable curve is defined in the same way as in the usual case. We use the conventions of [Ma], III.2 where the reader can find further details. Briefly, irreducible components become vertices, pairs of special points that are identified become edges, labeled points become tails, so that tails acquire labeling by $S$. Tails now can be of two types, we may refer to them and their marks as “black” and “white” ones as well, and call the graph painted one. Moreover, each vertex is marked by the genus of the corresponding (normalized) component.

In the genus zero case, all relevant graphs are trees. If we delete a vertex $v$ from the geometric realization of the tree $\tau$, it will break into a set of $\geq 3$ connected components which we will call branches of $\tau$ at $v$. Their set is canonically bijective to the set of flags $F_\tau(v)$ incident to $v$: we can say that the branch starts with the respective flag. In the extreme case, a branch can be a single tail.

§2. Basic morphisms

2.1. Clutching morphisms and boundary strata. First, recall a general construction studied in [Kn], §3. Let $C/T$ be a flat family of semistable curves over a scheme $T$ endowed with two non–intersecting sections $x_s, x_t$. Then there is another family of curves $C'/T$ and a morphism $p : C \to C'$ over $T$ such that $p \circ x_s = p \circ x_t$ and $p$ is universal with this property. Knudsen proves that $p$ is a finite morphism, and $C'/T$ is again semistable.
We will apply this construction and its iterations to the (schematic points of) various products of \( \overline{\mathcal{L}}_{g,S} \). In all instances, we will be gluing together only pairs of white points.

First, let \( S_1 = S' \cup \{s\} \), \( S_2 = S'' \cup \{t\} \), with \( s, t \) white. Put \( S := S' \cup S'' \). Then \( \overline{\mathcal{L}}_{g_1,S_1} \times \overline{\mathcal{L}}_{g_2,S_2} \) carries a family of disconnected curves \( \prod \overline{\mathcal{L}}_{g_1,S_1} \subset \prod \overline{\mathcal{L}}_{g_2,S_2} \) endowed with two nonintersecting sections coming from \( x_s, x_t \). Clutching them together, we obtain a morphism of stacks

\[
\overline{\mathcal{L}}_{g_1,S_1} \times \overline{\mathcal{L}}_{g_2,S_2} \to \overline{\mathcal{L}}_{g_1+g_2,S}
\]

which is a closed immersion and defines a boundary divisor of \( \overline{\mathcal{L}}_{g,S} \), \( g = g_1 + g_2 \). One can similarly define boundary divisors obtained by gluing together a pair of white sections \( s, t \in S' \) of the universal curve over \( \overline{\mathcal{L}}_{g',S'} \). Such divisors exist only when \( g > 0 \).

More generally, let \( \tau \) be a graph whose tails are bijectively labeled by a painted set \( S \) and each vertex \( v \) is endowed with a value of “genus” \( g_v \). All these labels form a part of the structure of \( \tau \). We extend the painting to all flags: halves of edges are white. Assume that \( \tau \) is painted stable: vertices of genus 0 are incident to \( \geq 3 \) flags of which at least 2 are white, and vertices of genus 1 are incident to \( \geq 1 \) flags. Then we put

\[
\overline{\mathcal{T}}_{\tau} := \prod_{v \in V_{\tau}} \overline{\mathcal{T}}_{g_v,F_{\tau}(v)}
\]

where \( F_{\tau}(v) \) is the set of flags incident to \( v \). This stack carries the disjoint union of universal curves lifted from the factors. This union is endowed with a family of pairwise disjoint sections corresponding to the halves of all edges of \( \tau \). We can now clutch together pairs of sections corresponding to the halves of one and the same edge. This produces a boundary stratum morphism

\[
\overline{\mathcal{T}}_{\tau} \to \overline{\mathcal{T}}_{g,S}
\]

where \( g = \sum g_v + \text{rk} \, H_1(\tau) \). In the genus zero case, we should consider only trees whose all vertices have genus zero so that one can forget about the latter labels.

2.2. Forgetting morphisms. Let \( S \) be a painted set, \( g \geq 0 \), and \( S' \subset S \) a subset such that \( \overline{\mathcal{L}}_{g,S} \) and \( \overline{\mathcal{L}}_{g,S'} \) are nonempty. Then we have a canonical forgetting morphism

\[
\varphi : \overline{\mathcal{L}}_{g,S} \to \overline{\mathcal{L}}_{g,S'}
\]

which on the level of curves consists in forgetting the sections labeled by \( S - S' \) and consecutively contracting the components that become unstable.

This is a particular case of forgetting morphisms defined by Hassett ([H], Theorem 4.3). Hassett’s theorem is applicable thanks to Lemma 1.2.2.
2.3. Repainting morphisms. Let $S$ be a painted set, $a \in S$ a white label. Denote by $S'$ the painted set with the same elements in which now $a$ is painted black whereas all other labels keep their initial colors. Assuming again that $\mathcal{T}_{g,S}$ and $\mathcal{T}_{g,S'}$ are nonempty we have a repainting morphism

$$\rho : \mathcal{T}_{g,S} \to \mathcal{T}_{g,S'}$$

(2.5)

which on the level of curves consists in repainting black the section $x_a$ and again consecutively contracting the components that become unstable.

This is a particular case of reduction morphisms defined by Hassett [H], Theorem 4.1 which is applicable thanks to Lemma 1.2.2 as well.

§3. Chow groups of $\mathcal{T}_{0,S}$

3.1. Keel’s relations for $A^*(\mathcal{T}_{0,S})$. In the following we will be considering only $S$–pointed painted stable curves of genus zero. Painted stability of various sets, partitions, trees etc. means that the respective stacks are nonempty. For any painted stable 2–partition $\sigma$ of $S$ we denote by $[D(\sigma)] \in A^1(\mathcal{T}_{0,S})$ the class of the respective boundary divisor (2.1). Call an ordered quadruple of pairwise distinct elements $i,j,k,l \in S$ allowed, if both partitions $ij|kl$ and $kj|il$ are painted stable. For a painted stable $\sigma$ put $\epsilon(\sigma; i,j,k,l) = 1$ if $\{i,j,k,l\}$ is allowed and $ij\sigma kl$; $-1$, if $\{i,j,k,l\}$ is allowed and $kj\sigma il$; and $0$ otherwise. The following theorem generalizing Keel’s presentation is the main result of this section.

3.1.1. Theorem. The classes $[D(\sigma)]$ generate the ring $A^*(\mathcal{T}_{0,S})$. They satisfy the following relations (3.1), (3.2) which provide a presentation of this ring. First, for each allowed quadruple $i,j,k,l$,

$$\sum_{\sigma} \epsilon(\sigma; i,j,k,l) [D(\sigma)] = 0.$$  

(3.1)

Second, let $\sigma, \sigma'$ be two stable painted partitions such that there exists an allowed quadruple $i,j,k,l$ with $ij\sigma kl$, $kj\sigma il$. Then

$$[D(\sigma)] [D(\sigma')] = 0.$$  

(3.2)

We break the algebraic–geometric arguments in the proof into a series of Lemmas.

3.2. Lemma. The classes $[D(\sigma)]$ satisfy (3.1) and (3.2).

Proof. In fact, let $\{i,j,k,l\}$ be allowed. Consider the forgetting morphism $\varphi : \mathcal{T}_{0,S} \to \mathcal{T}_{0,\{ijkl\}}$. Partitions $ij|kl$ and $kj|il$ define two boundary points in
Their inverse images are precisely sums of boundary divisors of \( \mathcal{T}_{0,S} \) entering in (3.1) with coefficients 1, resp. -1. This gives (3.1) whereas (3.2) follows from the fact that fibers of \( \varphi \) over different points do not intersect.

3.3. Lemma. The classes \([D(\sigma)]\) additively generate \( A^1(\mathcal{T}_{0,S}) \).

Proof. This was proved by Keel ([Ke]) for the case when all labels are white. Consider a painted set \( S \) with \( \geq 3 \) white labels. Choose a white label \( a \in S \) and repaint it black. Denote the resulting painted set \( S' \). Consider the repainting morphism \( \rho : \mathcal{T}_{0,S} \to \mathcal{T}_{0,S'} \). Since \( \rho \) is birational, \( \rho_*: A^k(\mathcal{T}_{0,S}) \to A^k(\mathcal{T}_{0,S'}) \) is surjective for each \( k \). We will show that \( \rho_*([D(\sigma)]) \) is a linear combination of boundary divisors for each painted stable 2–partition \( \sigma \) of \( S \). This will prove our statement by induction on the number of black labels.

If \( \sigma \) remains painted stable after repainting \( a \), we have simply \( \rho_*([D(\sigma)]) = [D(\sigma')] \) where \( \sigma' \) is the same partition of \( S' \).

If \( \sigma \) becomes unstable, one part of it must be \( \{a\} \cup F \) where \( F \subset S \) consists only of black labels. When \( |F| \geq 2 \), we have \( \rho_*([D(\sigma)]) = 0 \). In fact, according to [H], Prop. 4.5, \( [D(\sigma)] \) is contracted by \( \rho \).

Finally, assume that one part of \( \sigma \) is of the form \( \{a,b\} \) where \( b \) is black. Choose two white labels \( i, j \) in the other part of \( \sigma \). The quadruple \( i, j, a, b \) is allowed in \( S \). Write the relation (3.1) for it as an expression for \( [D(\sigma)] \):

\[
[D(\sigma)] = - \sum_{\tau: ij\tau ab} [D(\tau)] + \sum_{\tau: aj\tau ib} [D(\tau)] \tag{3.3}
\]

where in the first sum the part containing \( a, b \) must contain at least one more label. It is clear now that applying \( \rho_* \) to any summand in the right hand side we get either a boundary divisor, or zero. This completes the proof.

We will now generalize these results to the classes of boundary strata of arbitrary codimension. Consider a painted stable \( S \)–tree \( \tau \), a vertex \( v \) of it and an allowed quadruple of flags \( I, J, K, L \) at \( v \) (recall that all halves of edges are painted white). For any painted stable 2–partition \( \alpha \) of \( I, J, K, L \) define the respective tree \( \tau(\alpha) \) which has one extra edge replacing \( v \) in \( \tau \) and which breaks \( F_\tau(v) \) according to \( \alpha \) (cf. [LoMa2], 1.3–1.4).

3.4. Lemma. For each \((\tau, v; I, J, K, L)\) as above, we have the following relation between boundary strata in \( A^*(\mathcal{T}_{0,S}) \):

\[
\sum_{\alpha} \epsilon(\alpha; I, J, K, L) [D(\tau(\alpha))] = 0 . \tag{3.4}
\]
Proof. Notice that when $\tau$ is the one–vertex $S$–tree, (3.4) reduces to (3.1). Conversely, (3.4) can be deduced from (3.1) in the following way. The closed stratum $D(\tau)$ is (the image of) $\prod_{w \in V_\tau} \mathcal{T}_{0,F_\tau(w)}$ in $\mathcal{T}_{0,S}$. Replacing in (3.1) the label set $S$ by $F_\tau(v)$ and $i, j, k, l$ by $I, J, K, L$, we get a relation in $A^*(\mathcal{T}_{0,F_\tau(w)})$. Tensor multiplying this identity by the fundamental classes of all remaining $\mathcal{T}_{0,F_\tau(w)}$ and taking the direct image in $A^*(\mathcal{T}_{0,S})$ we finally obtain (3.4).

3.5. Lemma. The classes $[D(\tau)]$ for painted stable $S$–trees $\tau$ additively generate $A^*(\mathcal{T}_{0,S})$.

Proof. We extend the proof of Lemma 3.3 to this case, by starting with Keel’s result for the case when all labels are white, and repainting the necessary amount of white points one by one.

Again, we have to check only that $\rho_*$ maps classes of boundary strata to linear combinations of such classes. We may and will assume that $\tau$ has at least two edges. As above, if $\tau$ remains stable after repainting $a$, this is clear. If $\tau$ becomes unstable, the tail $a$ must be incident to an end vertex $v$ of $\tau$, and all other tails at this vertex must be black. Let their set be $F$. We will first check that if the number of these black tails is $|F| \geq 2$, then $\rho_*([D(\tau)]) = 0$. In fact, Hassett’s description of the repainting map shows that $\rho(D(\tau))$ parametrizes curves in which the structure points $x_s, s \in F$, all have to coincide after the component on which they formerly freely moved in $C_{0,S}$ has been collapsed. Hence $\dim \rho(D(\tau)) < \dim D(\tau)$ so that $\rho_*([D(\tau)]) = 0$.

Consider now the case when an end vertex $v$ of $\tau$ carries only two flags $a, b$, and $b$ is black.

Thus $v$ carries three flags of which two are white and one black. Let us generally call such a vertex critical one. There is a unique maximal sequence of vertices $v := v_0, v_1, \ldots, v_{n-1}, v_n, n \geq 1$, in $\tau$, with the following properties:

(a). For each $0 \leq i \leq n - 1$, $v_i$ and $v_{i+1}$ are opposite vertices of an edge $e_i$.

(b). $v_0, \ldots, v_{n-1}$ are critical vertices whereas $v_n$ is not critical.

For $n \geq 2$, if we delete the vertex $v_n$ from $\tau$, the connected component containing $a$ will be called the critical branch of $a$, and $n - 1$ will be called its length.

The vertex $w := v_n$ completing the critical branch can fail to be critical in one of two ways:

(b1). $F_\tau(w)$ contains only two white flags but $\geq 2$ black flags.

(b2). $F_\tau(w)$ contains at least three white flags.

I contend that in the case (b1) we again have $\rho_*[D(\tau)] = 0$. In fact, consider a curve $C$ corresponding to a generic geometric point of $D(\tau)$. Using Hassett’s description of the repainting map, we see that after repainting $a$ black, all components of $C$ corresponding to $v_0, \ldots, v_n$ get collapsed. Collapsing critical components
$v_0, \ldots, v_{n-1}$ does not diminish the number of moduli since they are all projective lines with three special points. However, collapsing the first non-critical component $v_n$ does diminish the number of moduli since it carries $\geq 4$ special points.

It remains to treat the case (b2).

Since $v_n$ carries at least three white flags, we can choose at $v_n$ two white flags $K, L$ which do not coincide with the half of the edge $e_{n-1}$. Since $v_{n-1}$ is critical, besides the other half of $e_{n-1}$ it carries a black tail which we call $I$ and one more white flag which we call $j$. Denote by $\sigma$ the result of collapsing $e_{n-1}$ into a vertex $u$ and write the relation (3.4) for $(\sigma, u; I, J, K, L)$:

$$\sum_{\alpha: IJ\alpha KL} [D(\sigma(\alpha))] - \sum_{\alpha: KJ\alpha IL} [D(\sigma(\alpha))] = 0.$$  \hspace{1cm} (3.5)

We will keep denoting $e_{n-1}$ the edge replacing $u$ in $\sigma(\alpha)$, and $v_{n-1}, v_n$ its respective vertices. In the first sum, there is one term $[D(\tau)]$ corresponding to the partition $\alpha = IJ\ldots$ of $F_\sigma(u)$. For all other terms, $v_{n-1}$ will cease to be a critical vertex since it will carry $\geq 4$ flags. In the second sum, $v_{n-1}$ is never critical, because it carries a white flag $K$ and two white halves of the edges $e_{n-1}$ and $e_{n-2}$ (if $n = 1$, in place of $e_{n-2}$ we have $a$).

Thus, (3.5) allows us to replace the Chow class of $D(\tau)$ by a linear combination of classes whose critical branches (of $a$) are shorter than that of $\tau$. Applying this procedure to each term of this expression we can reduce the length once more if need be. But when the length becomes zero, the vertex $v_0$ ceases to be critical, and the repainting of the respective class was described at the beginning. This completes the proof of Lemma 3.5.

The remaining part of the proof of Theorem 3.1.1 is purely combinatorial. It relies upon a theorem on the structure of the abstract ring $H^*_S$ whose presentation is given by (3.1) and (3.2). This theorem is proved in [LoMa2]. Below we will summarize the necessary information from [LoMa2] and then complete the proof.

3.6. The ring $H^*_S$. Let $S$ be a painted set with $|S| \geq 3$ containing at least two white elements, $k$ a commutative coefficient ring. Consider the family of independent commuting variables $\{l_\sigma\}$ indexed by painted stable unordered 2–partitions $\sigma$ of $S$ and put $R_S := k[l_\sigma]$.

For an allowed quadruple $i, j, k, l \in S$, put

$$R_{ijkl} := \sum_{\sigma} \epsilon(\sigma; i, j, k, l) l_\sigma \in R_S,$$ \hspace{1cm} (3.6)

For two partitions $\sigma, \sigma'$ such that there exists an allowed quadruple $\{i, j, k, l\}$ with $ij\sigma kl, kj\sigma il$, put

$$R_{\sigma\sigma'} := l_\sigma l_{\sigma'}.$$ \hspace{1cm} (3.7)
Denote by $I_S \subset \mathcal{R}_S$ the ideal generated by all elements (3.6) and (3.7) and define the combinatorial cohomology ring by

$$H_S^* := \mathcal{R}_S / I_S.$$ 

If $\tau$ is a painted stable $S$–tree, we put

$$m(\tau) := \prod_{e \in E_\tau} l_{\sigma_e} \in \mathcal{R}_S$$

where $\sigma_e$ for an edge $e$ of $\tau$ denotes the 2–partition of $S$ obtained by cutting $\tau$ in a midpoint of $e$. Monomials $m(\tau)$ are called good. Such a monomial depends only on the $S$–isomorphism class of $\tau$. If $\tau$ is one–vertex tree, we put $m(\tau) = 1$. For any $m \in \mathcal{R}_S$, we put $[m] := m \mod I_S \in H_S^*$. 

In [LoMa2], the following result is proved:

3.6.1. **Theorem.** (i) $H_S^*$ as a $k$–module is spanned by the classes of good monomials $[m(\tau)]$.

(ii) Let $(\tau, v; I, J, K, L)$ run over systems described before Lemma 3.4. Each such system determines the following relation between good monomials (notation being as in (3.4)):

$$R(\tau, v; I, J, K, L) := \sum_{\alpha} \epsilon(\alpha; I, J, K, L) m(\tau(\alpha)) \in I_S.$$ \hspace{1cm} (3.8)

Moreover, (3.8) span all relations between the classes of good monomials in $H_S^*$. 

Combining this result with Lemmas 3.2–3.5, we get the following statement:

3.6.2. **Proposition.** The map $[l_\sigma] \mapsto [D(\sigma)]$ extends to a surjective ring homomorphism $h_S : H_S^* \to A^*(L_{0,S})$, which sends $[m(\tau)]$ to $[D(\tau)]$ for each painted stable $S$–tree $\tau$.

3.6.3. **Relations between relations.** For further reference, I collect here several relations between the elements (3.8):

$$R(\tau, v; I, J, K, L) = R(\tau, v; K, L, I, J) =$$

$$= R(\tau, v; J, I, L, K) = -R(\tau, v; I, L, K, J),$$

$$R(\tau, v; I, J, K, L) = R(\tau, v; I, J, L, K) + R(\tau, v; I, K, J, L), \hspace{1cm} (3.9)$$

$$R(\tau, v; I, J, K, L) = R(\tau, v; M, K, J, I) + R(\tau, v; M, I, L, K). \hspace{1cm} (3.10)$$
The first group of relations is straightforward. In (3.10), we assume that \( M \notin \{I, J, K, L\} \) is an extra flag in \( F_\tau(v) \) such that all involved quadruples are allowed. To check (3.10), it is convenient to use the following shorthand notation: denote e.g. by \( LMK|JI \) the sum of those terms in (3.8) for which \( L, M, K \) get into one part of \( \alpha \), whereas \( J, I \) get into another part. Hence we have, for example, \( IJ|KL = IJM|KL + IJ|KLM \). Then

\[
R(\tau, v; M, K, J, I) = MK|JI - MI|JK =
\]

\[
LMK|JI + MK|JIL - LMI|JK - MI|JKL,
\]

\[
R(\tau, v; M, I, L, K) = MI|LK - MK|LI =
\]

\[
MI|LKJ + MI|LKM - MKJ|LI - MK|LIJ.
\]

After adding up and canceling, stop tracking where \( M \) goes. We get \( IJ|KL - KJ|IL \) which is \( R(\tau, v; I, J, K, L) \).

**Questions.** Are all additive relations between relations generated by (3.9) and (3.10)? Do (3.9) – (3.10) form the beginning of an interesting resolution of \( H^*_S \)? Are \( H^*_S \) Koszul quadratic algebras?

**3.7. End of the proof of Theorem 3.1.1: the strategy.** It remains to establish that \( h_S \) is injective. This was proved by Keel in the case when all labels are white. We will again argue by induction on the number of black points. Consider a painted set \( S \) with \( \geq 3 \) white labels and fix once for all a white label \( a \in S \). Let \( S' \) be obtained from \( S \) by repainting \( a \) to black one. Using a long and convoluted inductive procedure we will construct a surjective graded homomorphism of \( k \)-modules \( \rho^H : H^*_S \to H^*_S' \) such that the following diagram commutes:

\[
\begin{array}{ccc}
H^*_S & \xrightarrow{h_S} & A^*_S \\
\downarrow{\rho^*_S} & & \downarrow{\rho^*_S} \\
H^*_S' & \xrightarrow{h_{S'}} & A^*_S'
\end{array}
\]  

(3.11)

where \( A^*_S := A^*(\mathcal{L}_{0,S}) \) with coefficients in \( k \) (or the cohomology, which is the same), and \( \rho_* \) is induced by the repainting morphism. By induction, we can assume that \( h_S \) is an isomorphism. Hence to show that \( h_{S'} \) is an isomorphism it suffices to check that \( \text{Ker } h_{S'} = 0 \). To this end we will use the inverse image repainting morphism \( \rho^*_H : H^*_S' \to H^*_S \) fitting into another commutative diagram

\[
\begin{array}{ccc}
H^*_S' & \xrightarrow{h_{S'}} & A^*_S' \\
\downarrow{\rho^*_H} & & \downarrow{\rho^*_H} \\
H^*_S & \xrightarrow{h_S} & A^*_S
\end{array}
\]  

(3.12)
and such that $\rho^* \circ \rho^*_H = \text{id}$. Its construction is quite easy: see 3.9.1 below.

When this is achieved, the completion of the proof is straightforward. Namely, let $\eta \in H^*_S$ be such that $h_S(\eta) = 0$. Since $\rho^*$ is injective and $h_S$ is an isomorphism, from (3.12) we get $\rho^*_H(\eta) = 0$, and then $\eta = \rho^*_H \circ \rho^*_H(\eta) = 0$.

3.8. Construction of $\rho^*_H$: the induction parameter and the inductive statements. Our definition of $\rho^*_H(m(\tau))$ is motivated by the calculation of $\rho^*_H$ in the proofs of 3.3 and 3.5. We will do it by induction on the value of the function

$$l(\tau) = l(\tau, a) := \text{the length of the critical branch of } \tau.$$ 

Recall that a critical vertex is a vertex carrying exactly two white flags and one black. We have $l(\tau) = 0$ if and only if the vertex $v_0$ carrying $a$ is not critical. We have $l(\tau) = n \geq 1$ if and only if $v_0$ is critical, and there is a (unique) sequence of pairwise distinct vertices $v_0, \ldots, v_n := w$ such that $v_1, \ldots, v_{n-1}$ are critical, whereas $v_n$ is not, and that $v_i, v_{i+1}$ are neighbors, that is ends of an edge.

We will sometimes call $l(\tau)$ simply length of $\tau$.

We will say that $w$ (and $\tau$) is of type I, if $w$ carries only two white flags, and therefore $\geq 2$ black flags. We will say that $w$ (and $\tau$) is of type II, if $w$ carries $\geq 3$ white flags.

The $n$-th step of induction, $n \geq 0$, will consist of the following constructions and verifications.

(A)$_n$. A definition of $\rho^*_H(m(\tau))$ for all $S$-labeled stable trees $\tau$ with $l(\tau) = n$.

Actually, this definition generally will depend on arbitrary choices and produce directly only an element of $\oplus km(\sigma)$ where $\sigma$ runs over $S'$-labeled painted stable trees. Hence we will have to check that

(B)$_n$. For $l(\tau) = n$, $\rho^*_H(m(\tau))$ are defined unambiguously modulo $I_{S'}$.

Finally, we will have to check that $\rho^*_H(m(\tau))$ depends only on $[m(\tau)] \in H^*_S$, or equivalently

(C)$_n$. $\rho^*_H$ extended by linearity sends to $I_{S'}$ each standard relation in $I_S$ whose all terms have length $\leq n$.

3.9. Construction of $\rho^*_H$: the case $l(\tau) = 0$.

(A)$_0$. If $\tau$ is of type I, we put $\rho^*_H(m(\tau)) = 0$. The same prescription will hold for type I and any length.

If $\tau$ is of type II, repainting the label $a$ produces a stable $S'$-tree, say, $\tau'$. In this case we put $\rho^*_H(m(\tau)) = [m(\tau')]$.

(B)$_0$. Clearly, this prescription is unambiguous.

(C)$_0$. Let $R(\sigma, \alpha; I, J, K, L)$ be a relation (3.8), such that all its terms $\sigma(\alpha)$ are of length 0. Since $\sigma$ is obtained from any $\sigma(\alpha)$ by collapsing an edge, we
have \( l(\sigma) = 0 \). If \( u \neq w = v_0 \), the type of \( \sigma \) is the same as the type of all \( \sigma(\alpha) \), and we get either \( \rho(R(\sigma, u; I, J, K, L)) = 0 \) (type I), or \( \rho(R(\sigma, u; I, J, K, L)) = R(\sigma', u; I, J, K, L) \) (type II). If \( u = w = v_0 \) and \( \sigma \) is of type I, all \( \sigma(\alpha) \) must be of type I, and again \( \rho(R(\sigma, u; I, J, K, L)) = 0 \).

It remains to consider the case when \( u = w = v_0 \) and \( \sigma \) is of type II. If \( a \notin \{I, J, K, L\} \), then for any stable partition \( \alpha \) of \( F_\sigma(u) \), \( a \) gets into the part of \( \alpha \) containing an extra white label from \( \{I, J, K, L\} \). Hence \( l(\sigma(\alpha)) = 0 \) and moreover, \( \sigma(\alpha) \) is of type II. Therefore \( \rho(R(\sigma, u; I, J, K, L)) = R(\sigma', u; I, J, K, L) \). Finally, let \( a \in \{I, J, K, L\} \), say, \( a = I \) so that \( I \) is white. We assumed that all terms of \( R(\sigma, u; I, J, K, L) \) are of length 0. Hence \( J \) and \( L \) must be white as well, and we have again \( \rho(R(\sigma, u; I, J, K, L)) = R(\sigma', u; I, J, K, L)) \).

### 3.9.1. Construction of \( \rho^*_H \)

Let \( \tau' \) be a painted stable \( S' \)-tree. The label \( a \) in it is black, so let us denote by \( \tau \) the result of repainting it white. Then \( l(\tau) = 0 \). We put \( \rho^*_H(m(\tau')) := [m(\tau)] \). Clearly, after such reverse repainting, each standard relation becomes a standard relation, so that we have a well defined map \( \rho^*_H : H^*_S \to H^*_S \). The commutativity of (3.12) is obvious. Since this construction involves only \( S \)-trees of length 0, the discussion above already shows that \( \rho^*_H \circ \rho^*_H = \text{id} \) identically.

We will now return to the direct image. From now on, we assume that \( n \geq 1 \) and that the statements (A)\(_m\), (B)\(_m\), (C)\(_m\) have been already treated for all \( m \leq n - 1 \).

### 3.10. Passage from \( n - 1 \) to \( n \): prescription (A)\(_n\)

Let \( l(\tau) = n \), and \( \tau \) be of type II (for type I, repainting produces zero for any length). Denote by \( \sigma \) the result of collapsing to a vertex \( u \) the last edge \( e_{n-1} \) with vertices \( v_{n-1}, v_n \) of the critical branch of \( \tau \). Since \( w = v_n \) in \( \tau \) carried \( \leq 3 \) white flags, \( u \) carries \( \leq 2 \) white flags, say, \( K, L \), besides the white flag, say, \( I \), which starts the way from \( u \) to \( a \) for \( n \geq 2 \), or coincides with \( a \) for \( n = 1 \). Denote by \( J \in F_u(\sigma) \) the unique black flag that was carried by the critical vertex \( v_{n-1} \).

With this notation, consider the relation \( R(\sigma, u; I, J, K, L) \). It contains exactly one term \( \sigma(\alpha_0) \) of length \( n \), namely for \( \alpha_0 = IJ|KL \). We have \( \sigma(\alpha_0) = \tau \), and \( l(\sigma(\alpha)) = n - 1 \) for \( \alpha \neq \alpha_0 \). By the inductive assumption, (the lifts of) \( \rho^*_H(\sigma(\alpha)) \) are well defined modulo the subspace generated by the standard relations all terms of which have length \( \leq n - 1 \). We put

\[
\rho^*_H(\tau) := - \sum_{\alpha \neq \alpha_0; IJ|KL} \rho^*_H(\sigma(\alpha)) + \sum_{\alpha; KJ|IL} \rho^*_H(\sigma(\alpha)) \mod I_{S'}.
\]

Equivalently, repainting of such \( R(\sigma, u; I, J, K, L) \) lands in \( I_{S'} \). Thus, applying the prescription (A)\(_n\) is the same as postulating some particular cases of the statement (C)\(_n\).

### 3.11. Passage from \( n - 1 \) to \( n \): the statement (B)\(_n\)

We have to check that (3.13) does not depend on the arbitrary choices of \( K, L \). To pass from one couple
of white flags $K, L$ to another one it suffices to replace one flag in turn, so we will consider two cases.

Replacement $K \mapsto K'$. We know that after repainting $R(\sigma, u; I, J, K, L)$ we land in $I_{S'}$ and wish to establish the same for $R(\sigma, u; I, J, K', L)$. A version of (3.10) gives

$$R(\sigma, u; I, J, K, L) - R(\sigma, u; I, J, K', L) = R(\sigma, u; K, J, K', L). \quad (3.14)$$

Two terms of length $n$ cancel in the left hand side of (3.14). Hence $R(\sigma, u; K, J, K', L)$ contains only terms of length $\leq n-1$, and the result follows by application of $(C)_{n-1}$.

Replacement $L \mapsto L'$. A similar calculation shows that the respective difference will be now

$$R(\sigma, u; L', K, L, I) \quad (3.15)$$

again with all terms of length $n - 1$ and the same conclusion.

For further use, notice that in this reasoning $L'$ might have been black as well: all involved quadruples remain allowed. Therefore any relation of the type $R(\sigma, u; I, J, K, L)$ with white $I, K$ and black $J, L$ will be also repainted to an element of $I_{S'}$.

It remains to prove $(C)_n$. Before doing this, it is convenient to review the structure of the standard relations that must be treated at the $n$-th step.

3.12. **Standard relations having terms of length $n$.** Let $R(\sigma, u; I, J, K, L)$ be a relation all terms of which have length $\leq n$, and such that this bound is achieved.

Consider a term $\sigma(\alpha_0)$ of length $n$. Let $v_0, \ldots, v_{n-1}, v_n = w$ be (the sequence of vertices of) the critical branch of $\sigma(\alpha_0)$ so that $a$ is incident to $v_0$, all vertices $v_0, \ldots, v_{n-1}$ are critical, and $w$ is not critical. Let $e_i$ be the edge with vertices $v_i, v_{i+1}$.

Consider the position of the edge $e(\alpha_0)$ in $\sigma(\alpha_0)$ which was created by $\alpha_0$ and gets collapsed into $u$ in $\sigma$. We will have to treat separately the following (exhaustive) list of alternatives (a), (b), (c), (d).

(a) $e(\alpha_0) = e_i$ for some $i \leq n - 2$.

This is, of course, possible only for $n \geq 2$.

In this case $F_\sigma(u)$ consists of four flags: white flag leading from $u$ to $a$ (or $a$ itself for $i = 0$) which we denote $I$; black flag $J$ incident to $v_i$ in $\sigma(\alpha)$, black flag $L$ incident to $v_{i+1}$ in $\sigma(\alpha)$, white flag $K$ leading from $u$ to $w$. With this notation, $\alpha_0 = IJ|KL$. Put $\sigma_0 := IL|JK$. Up to renaming flags and changing signs, we get a relation comprising only two nonvanishing terms, both of length $n$.

$$R(\sigma, u; I, J, K, L) = m(\sigma(\alpha_0)) - m(\sigma(\sigma_0)). \quad (3.16)$$
We can call (3.16) “exchange of two neighboring black tails on the critical branch of $\sigma$. “ Notice that the length of $\sigma$ is $i - 1$.

(b) $e(\alpha_0) = e_{n-1}$ so that $w$ is a vertex of $e(\alpha_0)$.

We may and will define $I, J$ as above. If $\sigma(\alpha_0)$ is of type I, then $\sigma$ and all $\sigma(\alpha)$ are of type I so that $R(\sigma, w; I, J, K, L)$ must be repainted to zero. Type II will be treated below.

(c) None of the above, but $w$ is a vertex of $e(\alpha_0)$.

This will be the most difficult case.

(d) None of the vertices $v_0, \ldots, v_n = w$ is a vertex of $e(\alpha_0)$.

In this case all terms $\sigma(\alpha)$ as well as $\sigma$ have length $n$ and in fact share the common critical branch.

We will now treat these options in the following order: (b), (a), (d), (c).

3.13. Passage from $n - 1$ to $n$: the statement $(C)_n$ in the case (b). We use the notation defined in 3.12 (b) and assume that $\sigma$ is of type II. Since $\{I, J, K, L\}$ is allowable, $I$ is white, and $J$ is black, $K$ must be white.

If $L$ is white as well, then $\sigma(\alpha_0)$ is the only term of length $n$ in $R(\sigma, w; I, J, K, L)$, and the whole relation repaints to an element of $I_S$: cf. the last remark in 3.10.

If $L$ is black, we refer to the remark made after formula (3.15) which proves the same result.

3.14. Passage from $n - 1$ to $n$: the statement $(C)_n$ in the case (a). Again, we may assume that $\sigma(\alpha_0)$ is of type II. In order to repaint (3.16), we have to apply to both terms the prescription spelled out in 3.10, formula (3.13). However, the notation adopted in (3.13) conflicts with the one adopted in (3.16). We will keep (3.16) and rewrite (3.13) as follows. Let $I'$ be the white flag at $w$ in $\sigma(\alpha_0)$ and in $\sigma(\overline{\alpha}_0)$ leading to $a$, $J'$, $K'$, $L'$ three other flags at $w$ such that $K', L'$ are white. Then we can use $R(\sigma(\alpha_0), w; I', J', K', L')$ and the similar relation for $\overline{\alpha}_0$ in order to replace $m(\sigma(\alpha_0))$ and $m(\sigma(\overline{\alpha}_0))$ modulo $I_S$ by linear combinations of good monomials corresponding to trees of length $\leq n - 1$.

If $i + 1 < n - 1$, we can then pair the terms in the resulting difference in such a way that we will get a linear combination of the exchange relations of the type (3.16) written for a new family of trees which are of smaller length.

3.15. Passage from $n - 1$ to $n$: the statement $(C)_n$ in the case (d). In this case there is no interaction between the surgeries made upon $\sigma$ by the partitions $\alpha$ involved in $R(\sigma, w; I, J, K, L)$ and the partitions $\beta$ of $F_{\sigma}(w)$ which occur in a formula of the type (3.13) which can be chosen common for all terms $\sigma(\alpha)$. Hence we can reorder the surgeries and the summations and start with summing over $\beta$. This will show that $R(\sigma, w; I, J, K, L)$ is repaint to a sum of standard relations.
3.16. Passage from \(n-1\) to \(n\): the statement \((C)_n\) in the case (c).

Consider again a term \(\sigma(\alpha_0)\) of length \(n\). The edge created by \(\alpha_0\) in \(\sigma(\alpha_0)\) is incident to the last vertex of the critical branch, and another vertex of this edge does not belong to the critical branch. This edge collapses to the vertex \(u\) of \(\sigma\).

Denote by \(M\) the flag at \(u = w = v_n\) (notation now refers to \(\sigma\)) belonging to the critical branch. Thus \(M\) leads to \(v_{n-1}\). Denote by \(B\) the (single) black flag at \(v_{n-1}\) and by \(A\) the single white flag at \(v_{n-1}\) leading in the direction of \(a\) (if \(n = 1, B := b, A := a\).) Furthermore, we have an allowed quadruple of flags \(\{I, J, K, L\} \in F_\sigma(u)\) defining the relation \(R(\sigma, u; I, J, K, L)\) that we are going to reaint.

3.16.1. Claim. The total statement \((C)_n\) in the case (c) will follow if we prove it assuming the following condition:

\[ (*) \text{ The critical branch of } \sigma \text{ is of length } n, I = M, J, L \text{ are white.} \]

Proof of the Claim. In the notation explained at the beginning of 3.16, we have either \(M \notin \{I, J, K, L\}\), or \(M \in \{I, J, K, L\}\).

If \(M \notin \{I, J, K, L\}\), we may and will assume that \(I, K\) are white. In fact, from (3.9) it follows that this can be achieved by renaming the flags and changing the sign of the relation if need be. Since for any \(\sigma(\alpha)\) in \(R(\sigma, u; I, J, K, L)\) the flag \(M\) gets at the same vertex as either \(I\), or \(K\), its vertex remains non–critical. Therefore \(l(\sigma(\alpha)) = l(\sigma) = n\), and all \(\sigma(\alpha)\) share the common critical branch with \(\sigma\).

Now write the relation (3.10):

\[
R(\sigma; u; I, J, K, L) = R(\sigma; u; M, K, J, I) + R(\sigma; u; M, I, L, K)
\]

In both groups of flags \(M, K, J, I\) and \(M, I, L, K\) appearing at the right hand side the first, second, and fourth terms are white. Since \(I, K\) are white, the vertex of \(M\) in each term of the relations at the right hand side is not critical. This means that both these relations satisfy the condition \((*)\), and if \((C)_n\) holds for them, it holds for \(R(\sigma, u; I, J, K, L)\) as well.

Let now \(M \in \{I, J, K, L\}\). We may and will assume that \(M = I\) so that \(I\) is white. If moreover both \(J\) and \(L\) are white, then \((*)\) holds as above.

Let us show that cases when \(J\) or \(L\) is black fall into another category. In fact, if say \(J\) is black, then the partition \(\alpha_0 := IJ|KL\ldots\) creates a new critical edge in \(\sigma(\alpha_0)\) so that we are in the case 3.12(a) or 3.12(b) treated earlier.

3.16.2. Treatment of the case 3.16.1 (*). Put \(F := F_\sigma(u) - \{I, J, K, L\}\). To see better how \(R(\sigma, u; I, J, K, L)\) repaints to zero modulo \(I_{\sigma'}\) consider first the simplest case in which \(F = \emptyset\). We keep notation explained at the beginning of 3.16. The last edge of the critical branch of \(\sigma\) has two vertices carrying flags \(A, B\), resp. \(J, K, L\), besides halves of the edge itself. We will denote \(\sigma\) symbolically by \(AB|JKL\). There will be two partitions determining terms \(\sigma(\alpha)\) of \(R(\sigma, u; I, J, K, L)\);
the respective trees will be denoted $AB|J|KL$ and $AB|L|KJ$ so that in the current shorthand notation

$$R(\sigma, u; I, J, K, L) = AB|J|KL - AB|L|KJ.$$  \hspace{1cm} (3.17)

Applying the prescription (3.13) for repainting two terms of (3.17), we have a choice: either $A$, or $B$ can be moved to the middle vertex. We decide to move $A$, interchanging $A$ with $J$, resp. with $L$. At this step the assumption 3.16.1 (*) that $J$ and $L$ are white is critically used: otherwise one or both trees in (3.17) would become unstable. We get symbolically

$$\rho^H_*(R(\sigma, u; I, J, K, L)) = \rho^H_*(BJ|A|KL) - \rho^H_*(BL|A|JK) \mod I_{S'}$$  \hspace{1cm} (3.18)

where at the right hand side both trees have length $n - 1$. Now we will in three consecutive steps interchange $A$ with $K$, $B$ with $K$, $B$ with $A$. More formally, add to (3.18) appropriate relations of length $n - 1$ which are repainted into $I_{S'}$ in view of (C)_{n-1}. For example, interchanging $A$ with $K$ in the first term of the right hand of (3.18) means subtracting (the repainting of) the standard relation $BJ|A|KL - BJ|K|AL$.

Thus, omitting $\rho^H_*$ for brevity, we rewrite (3.18) modulo $I_{S'}$ consecutively into

$$BJ|K|AL - BL|K|AJ;$$  \hspace{1cm} (3.19)

then

$$KJ|B|AL - KL|B|AJ;$$  \hspace{1cm} (3.20)

and finally

$$KJ|A|BL - KL|A|BJ.$$  \hspace{1cm} (3.21)

Now a miracle happens: $KJ|A|BL$ as a tree is isomorphic to $BL|A|JK$, and $KL|A|BJ$ to $BJ|A|KL$. Therefore (3.21) differs from (3.18) by the sign, and hence both expressions vanish over any ring where 2 is invertible.

We now take a deep breath preparing for the last stretch of the proof, and consider the case of arbitrary $F := F_\sigma(u) - \{I, J, K, L\}$.

The formula (3.17) must now be replaced by the sum taken over all ordered 2–partitions $F = F(0) \cup F(1)$ (corresponding to former $\alpha'$s):

$$R(\sigma, u; I, J, K, L) = \sum_{F(0), F(1)} [AB|JF(0)|F(1)KL - AB|LF(0)|F(1)JK]$$  \hspace{1cm} (3.22)

Here we write say, $F(1)KL$, in place of $F(1) \cup \{K, L\}$. Moreover, we interpret say, the expression $AB|JF(0)|F(1)KL$ as a notation for the isomorphism class of the
following $S$–tree: take a linear tree with two edges and three consecutive vertices $u_1, u_2, u_3$ and attach to the vertices the following branches of the tree $\sigma$ which are denoted by their initial flags: $A$ and $B$ to $u_1$, $\{J\} \cup F(0)$ to $u_2$, $F(1) \cup \{KL\}$ to $u_3$.

After repainting a black, each term of (3.22) turns into a similar sum, now taken over partitions $F(0) = F(00) \cup F(01)$, (we omit $\rho^*_H$ for brevity):

$$AB|JF(0)|F(1)KL \simeq$$

$$- \sum_{\substack{F(00), F(01) \neq \emptyset \\ F(00) \subset F}} ABF(00)|JF(01)|F(1)KL + \sum_{\substack{F(00), F(01) \neq \emptyset \\ F(00) \subset F}} BJF(00)|AF(01)|F(1)KL,$$

$$\quad \text{(3.23a)}$$

$$-AB|LF(0)|F(1)JK \simeq$$

$$\sum_{\substack{F(00), F(01) \neq \emptyset \\ F(00) \subset F}} ABF(00)|LF(01)|F(1)JK - \sum_{\substack{F(00), F(01) \neq \emptyset \\ F(00) \subset F}} BLF(00)|AF(01)|F(1)JK.$$

$$\quad \text{(3.23b)}$$

The first sums at the right hand sides of (2.23a), (2.23b) did not appear in the degenerate case (3.18) where $F$ was empty, so they must be treated separately. Let us group the respective terms together in the following way.

$$\sum_{\substack{F(00) \subset F \\ F(00) \neq \emptyset}} \sum_{\substack{F(01), F(1) \neq \emptyset \\ F(01) \cup F(1) = F - F(00)}} [ABF(00)|LF(01)|F(1)JK - ABF(00)|JF(01)|F(1)KL].$$

Each inner sum here is (the result of repainting of) a standard relation. Each term in all these relations has length $\leq n - 1$ because $F(00) \neq \emptyset$. Hence all these terms can be disposed off thanks to $(C)_{n-1}$.

Turning now to the last sums in (3.23a), (3.23b), we consecutively transform them on the pattern of (3.19)–(3.21).

We describe the first step corresponding to the interchange of $A$ and $K$ in some detail. Sum second terms in (3.23a), resp. (3.23b) over all $F(0) \subset F$ and then make the summation over $F(00) \subset F$ external. Each inner sum will become “one half” of a standard relation, and we replace it by another half. To write a formula, extending (3.19), we denote in the inner sum, when $F(00)$ is fixed, $G := F - F(00) = F(01) \cup F(1)$. The inner sum will be extended over partitions $G = G(0) \cup G(1)$. The general case of (3.19) takes form

$$\sum_{\substack{F(00) \subset F \\ F(00) \neq \emptyset}} \sum_{\substack{F(01), G(1) \neq \emptyset \\ F(01) \cup G(1) = F - F(00)}} [BJF(00)|KG(0)|ALG(1) - BLF(00)|KG(0)|AJG(1)].$$

(3.24)
Now reshuffle again, make summation over \( G(1) \) external, and with fixed \( G(1) \) put
\[
H = F - G(1) = F(00) \cup G(0).
\]
The general case of (3.20) takes form
\[
\sum_{G(1)} \sum_{H(0), H(1)} \left[ KJH(0)|BH(1)|ALG(1) - K LH(0)|BH(1)|AJG(1) \right]. \quad (3.25)
\]
Finally, at the last step make summation over \( H(0) \) external and put \( E = F - H(0) \) so that (3.21) becomes
\[
\sum_{H(0)} \sum_{E(0), E(1)} \left[ KJH(0)|AE(0)|BLE(1) - K LH(0)|AE(0)|BJE(1) \right]. \quad (3.26)
\]
As above, (3.26) differs by sign from the sum of all last terms in (3.23a) and (3.23b), because the triples \((H(0), E(0), E(1))\) in (3.26) and \((F(1), F(01), F(00))\) in (3.23) run over all ordered partitions of \( F \) into three pairwise disjoint subsets.

This completes the construction of \( \rho^H_* \) and the proof of the Theorem 3.1.1.

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