Weyl Geometries and Timelike Geodesics

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Abstract: In view of Ehlers-Pirani-Schild formalism, since 1972 Weyl geometries should be considered to be the most appropriate and complete framework to represent (relativistic) gravitational fields. We shall here show that in any given Lorentzian spacetime \((M, g)\) that admits global timelike vector fields any such vector field \(u\) determines an essentially unique Weyl geometry \(((g), \Gamma)\) such that \(u\) is \(\Gamma\)-geodesic (i.e. parallel with respect to \(\Gamma\)).

1. Introduction

We shall here present a result in Geometry which is motivated and finds important applications in relativistic theories of gravitation. Let us recall first that in the early '70s Ehlers, Pirani and Schild (EPS) proposed (see [1]) a set of axioms to fully determine the geometric structure of a spacetime \(M\) able to describe the fundamental physical properties of any reasonable gravitational field, starting from the worldlines of free falling particles and lightrays; see also [2].

They showed that few reasonable and physically well-founded axioms uniquely determine a conformal structure (i.e. a class of conformally equivalent Lorentzian metrics) and a projective structure (i.e. a class of projectively equivalent torsionless connections) in \(M\). The conformal structure \(\mathcal{G} = [g]\) is uniquely characterized by the distribution of lightcones (which in fact are conformally invariant) while the projective structure \(\mathcal{P} = [\Gamma]\) is uniquely defined by the geodesic trajectories in spacetime; see [3].

The conformal structure divides vectors into \([g]\)-timelike, \([g]\)-lightlike and \([g]\)-spacelike vectors. Let us stress that this characterization just depends on the conformal class and it is independent of the choice of a representative. Analogously, the projective structure \(\mathcal{P}\) is associated to geodesic trajectories which are also called \([\Gamma]\)-geodesics or simply \(\Gamma\)-geodesics. Of course also a metric structure \(g\) determines a connection (and a projective structure) through its Levi-Civita connection; the geodesic trajectories determined by the Levi-Civita connection are also called \(g\)-geodesics.

The conformal and projective structures have to be compatible in the sense that \([g]\)-lightlike \(\Gamma\)-geodesics are also \(g\)-geodesics.

We shall define a pair \(((g), [\Gamma])\) of compatible conformal and projective structures in \(M\) to be an EPS structure on spacetime \(M\). A Weyl geometry is a pair \(((g), \Gamma)\) formed by a conformal structure \([g]\) and a single torsionless connection \(\Gamma\) in the class \(\mathcal{P}\) chosen so that there exists a covector \(A\) such that

\[
\nabla_{\Gamma} g = 2A \otimes g \tag{1.1}
\]

Ehlers, Pirani and Schild showed that an EPS structure uniquely determines a Weyl geometry.

Locally this amounts to fix the connection as follows

\[
\Gamma_{\beta\mu}^{\alpha} = \{g\}_{\beta\mu}^{\alpha} + \left( g^{\alpha\rho} g_{\beta\rho} - 2\delta_{(\beta}^{\alpha} \delta_{\mu)}^{\rho} \right) A_{\rho} \tag{1.2}
\]

where \{\(g\)\}_{\beta\mu}^{\alpha} are the Christoffel symbols of \(g\) in any given chart.
The effects of the gravitational field in this framework are described by two objects. The propagation of light rays is governed by the conformal structure while the motion of (massive) particles is governed by the connection $\Gamma$. In a sense, the gravitational field is a mix of these two objects. One has therefore more kinematical freedom in fitting observational data relying on the extra degrees of freedom contained in the covector $A$.

Since the time of EPS paper [1] and especially since late '90s (when cosmological dynamics has been shown to present an unexpected acceleration) a plethora of modified models of the Universe has been proposed to explain these observations. In some cases modified models resort to adding matter sources to the gravitational field; this approach leads to argue that about 95% of gravitational sources in our Universe are unknown (or better they are known only through their observed gravitational effects – which are the reason that requires their introduction). Moreover, most of these unknown components (about 70%) have quite peculiar properties (namely this part produces a negative pressure and it is known as dark energy).

Another approach leads instead to modify dynamics of the gravitational field in order to account for observations without resorting to dark sources. A class of examples of such modified models is formed by so-called $f(R)$-models where the Lagrangian assumed to describe gravity is any (non–linear) function of the scalar curvature of a spacetime $(M, g, \Gamma)$; see [4], [5], [6], [7], [8], [9]. It has been shown that $f(R)$-models (in their metric-affine formulation) naturally lead to an EPS setting in which the gravitational field is in fact described by a Weyl geometry; see [2] and [10].

After EPS a whole field of research has grown to generalize the physical interpretation of standard General Relativity to Weyl geometries; see [11] and references quoted therein. EPS setting has also proved to be effective in studying models with matter interacting with the connection; [12], [13], [14], [15].

Hereafter, in Section 2 we shall show that given a conformal structure $[g]$ over a spacetime $M$ then a congruence of $[g]$-timelike curves (which can be interpreted as a the flow of all worldlines of a fluid) determines an essentially unique EPS–compatible connection $\Gamma$.

This provides a mechanism to determine a Weyl geometry $([g], \Gamma)$ out of potentially observable quantities (the flow of the fluid). It also provides a tool to fully exploit the new aforementioned degrees of freedom to disentangle Geometry from gravitational effects. The final aim of this study is to describe a generic fluid on a fixed reference background $g$. In this framework the gravitational field is encoded into the covector $A$, still depending on the representative $g$ which one is free to fix in the conformal structure.

In Section 3 we shall review what happens if another representative $\tilde{g}$ for the conformal structure is chosen. We shall determine how the conformal rescaling acts on all objects involved in the formalism.

In Section 4 we shall review the main assumptions made in Section 2 where the Weyl connection is determined by a congruence of timelike $\Gamma$-geodesics. In Section 2 it is assumed that a fluid in a relativistic theory is described by a geodesic $[g]$-timelike field $u$ together with two scalar fields representing the pressure $p$ and the density $\rho$. This issue is relevant since in presence of pressure (i.e. with internal forces) one could expect fluid particles to move along non-geodesic trajectories. We shall show that in some relevant cases the fluid can be nevertheless described in terms of a geodesic field also in presence of pressure.
2. Determining Weyl Geometry

Let us consider a conformal structure \((M,[g])\) and any \([g]\)-timelike vector field \(u\). Of course such a generic vector field has no reason to represent a \(g\)-geodesic field. Hence in standard GR (in which the metric structure \(g\) determines both lightrays and particle worldlines through its Levi-Civita connection \(\Gamma = \{g\} \) that in this case plays the role of the gravitational field) the situation is pretty rigid: once a metric structure has been fixed then relatively few curves are allowed to represent the motion of a fluid. Only a congruence of \(g\)-timelike \(g\)-geodesics is allowed.

We shall show hereafter that Weyl geometries are much less rigid. Once a conformal structure has been fixed then potentially any \([g]\)-timelike vectorfield can be used to represent a fluid. In fact one can use the extra degrees of freedom encoded by the covector \(A\) in order to tune up the Weyl connection \(\Gamma\) so that \(u\) is \(\Gamma\)-geodesic.

For any representative \(g \in [g]\) of the conformal structure one can normalize the vector field \(u\) to be a \(g\)-unit, let it be \(n^\mu\).

A Weyl connection for \(g\) is a connection \(\Gamma\) such that the compatibility condition (1.2) holds true. If \(\Gamma\) is a Weyl connection for \(g\) it is a Weyl connection for any other metric conformal to \(g\).

For later convenience, by using compatibility condition (1.2) one also has

\[ \Gamma \nabla \alpha g_{\mu\nu} = 2 A^\alpha g_{\mu\nu} \quad \Gamma \nabla \alpha g^{\mu\nu} = -2 A^\alpha g^{\mu\nu} \]  

We first want to show that there exists in \(M\) a Weyl connection such that the integral curves of \(n\) are \(\Gamma\)-geodesics. For that to be true one should determine \(\Gamma\) so that

\[ n^\mu \Gamma \nabla \mu n^\nu = \varphi n^\nu \]  

holds for some scalar field \(\varphi\).

The equation (2.2) can be expanded by using (1.2). One finds:

\[ n^\mu \Gamma \nabla \mu n^\nu + n^\mu n^\lambda \left( g^{\nu\lambda} g_{\mu\lambda} - 2 \delta^\nu_{(\mu} \delta^\lambda_{\lambda)} \right) A^\alpha = \varphi n^\nu \]  

which should be solved for \(A\).

Equation (2.3) can be recasted under the following form

\[ A^\nu + 2 n^\nu A, n^\nu = n^\mu \nabla \mu n^\nu - \varphi n^\nu \]  

and contracted with \(n^\nu\) to obtain

\[ A, n^\nu - 2 A, n^\nu = n^\mu n^\nu \nabla \mu n^\nu + \varphi \]

\[ A, n^\nu = - \left( \frac{1}{2} n^\mu \nabla \mu |n|^2 + \varphi \right) \equiv - \varphi \]  

This can be replaced back into equation (2.4) above to find finally

\[ A^\nu = n^\mu \nabla \mu n^\nu + \varphi n^\nu \]
Notice how $A$, and hence $\Gamma$, is uniquely determined by $g$, $n$ and $\varphi$. As a consequence there is a family of connections for which the congruence generated by $n$ is $\Gamma$-geodesic, one for any parametrization of the integral curves of $n$ (which for simplicity and physical reasons we here assume to be complete). Notice also that if $n$ happens to be a geodesic field for $\Gamma = \{g\}$ then $A = 0$, as expected. The covector $A$ in fact measures the failure of $n$ being $g$-geodesic.

3. Conformal Invariance

In Section 2 we started by choosing a representative $g$ of the conformal structure. Here we want to investigate what would have happened if we had chosen another representative $\tilde{g} = \Phi^2 g$.

First of all the Weyl connection can be expressed in terms of $\tilde{g}$ and another covector $\tilde{A}$ satisfying (1.2) for $(\tilde{g}, \Gamma)$ as well. In fact,

$$\Gamma^\alpha_{\mu \nu} = \{\tilde{g}\}^\alpha_{\mu \nu} + \left(\tilde{g}^{\alpha \sigma} \tilde{g}_{\mu \nu} - 2 \delta^\alpha_{\mu} \delta^\sigma_{\nu}\right) \tilde{A}_\nu = \{g\}^\alpha_{\mu \nu} + \left(g^{\alpha \sigma} g_{\mu \nu} - 2 \delta^\alpha_{\mu} \delta^\sigma_{\nu}\right) \left(\tilde{A}_\nu - \partial_\nu \ln \Phi\right)$$

(3.1)

Hence we see that the covector (1.2) in transforms as follows

$$\tilde{A} := A + d \ln \Phi$$

(3.2)

Also the unit vector $n$ is affected by the conformal transformation. If $n$ is the $g$-unit, when another conformal representative $\tilde{g}$ is selected one has

$$\tilde{n}^\lambda = \frac{1}{\Phi} n^\lambda$$

(3.3)

Analogously, for its covariant version one has

$$\tilde{n}_\alpha = \tilde{g}_{\alpha \beta} \frac{1}{\Phi} n^\beta = \Phi n_\alpha$$

(3.4)

The geodesic equation (2.2) is also affected by the conformal transformation

$$\tilde{n}^\mu \tilde{\nabla}_\mu \tilde{n}^\nu + \tilde{\nabla}_\nu = \frac{1}{\Phi} n^\mu \nabla_\mu n^\nu + \frac{1}{\Phi} n^\mu \tilde{\nabla}_\mu \frac{1}{\Phi} n^\nu = \frac{1}{\Phi} \left(\varphi - n^\mu \nabla_\mu \ln \Phi \right) \tilde{n}^\nu = \tilde{\varphi} \tilde{n}^\nu$$

(3.5)

where we set $\tilde{\varphi} := \frac{1}{\Phi} \left(\varphi - n^\mu \nabla_\mu \ln \Phi \right)$. Here we denote by $\tilde{\nabla}_\mu$ the covariant derivative when it happens to be independent of the connection, i.e. $\tilde{\nabla}_\mu = \partial_\mu$.

In other words one recovers the conformal invariance by normalizing the vector field and rescaling the metric and the scalar factor accordingly.

Also equation (2.3) is preserved by the conformal transformation. If one started from a conformally equivalent framework, then the equation

$$\tilde{\nabla}^\mu \tilde{\nabla}_\mu \tilde{n}^\nu + \tilde{\nabla}_\nu \tilde{n}^\lambda \left(\tilde{g}^{\nu \sigma} \tilde{g}_{\mu \lambda} - 2 \delta^\nu_{\mu} \delta^\lambda_{\sigma}\right) \tilde{A}_\nu = \tilde{\varphi} \tilde{n}^\nu$$

(3.6)

would be obtained and shown to be equivalent to (2.3).

Finally, the connection determined by the congruence given by equation (2.6) is independent of the conformal rescaling. In fact, one has

$$\tilde{\nabla}_\nu = \tilde{n}^\mu \tilde{\nabla}_\mu \tilde{n}_\nu + \tilde{\varphi} \tilde{n}_\nu = \tilde{n}^\mu \nabla_\mu n_\nu + \tilde{n}^\mu \tilde{n}_\lambda \left(g^{\nu \sigma} g_{\mu \lambda} - 2 \delta^\nu_{\mu} \delta^\lambda_{\sigma}\right) \partial_\nu \ln \Phi + \tilde{\varphi} n_\nu =$$

$$= n^\mu \nabla_\mu n_\nu + n^\mu n_\nu \nabla_\mu \ln \Phi + (n_\nu n^\nu + \delta^\nu_\nu - n^\nu n_\nu) \partial_\nu \ln \Phi + \left(\varphi - n^\mu \nabla_\mu \ln \Phi \right) n_\nu =$$

$$= n^\nu \nabla_\nu n_\nu + \partial_\nu \ln \Phi + \varphi n_\nu = A_\nu + \partial_\nu \ln \Phi$$

(3.7)
which shows that the covector $\hat{A}$ is exactly the covector needed to define the same connection $\Gamma$ in the conformal framework, as one can easily check by using (3.2).

4. Fluids and Geodesics

In Section 2 we determined the connection $\Gamma$ so that the integral curves of the $[g]$-timelike vector $u$ are $\Gamma$-geodesics. If one assumes $u$ to generate the flow lines of a fluid then the assumption for them being geodesics should be motivated. Let us briefly review what happens in standard GR (where the connection $\Gamma$ is identified with the Levi-Civita connection of the metric $g$).

In the case of pure dust (i.e. when there is no pressure, $p = 0$) then there is no much doubt; the dust particles do not interact with anything else than the gravitational field and hence they are freely falling. In this case, the field can be shown to be geodesics as a consequence of the conservation of the energy-momentum tensor of the fluid; see [16].

In fact, in general the energy-momentum tensor of a fluid can be written as

$$T_{\mu\nu} = pg_{\mu\nu} + (p + \rho)n_\mu n_\nu$$  \hspace{1cm} (4.1)

where $\rho$ represents the density and $p$ represents the pressure of the fluid. The conservation of the energy-momentum tensor is $\nabla_\mu T^{\mu\nu} = 0$. One can take the scalar product along $n$ to get

$$n^\mu g_{\mu\nu} \nabla_\mu T^{\mu\nu} = \nabla_\mu pn^\mu - n^\mu \nabla_\mu (p + \rho) - (p + \rho) \nabla_\mu n^\mu + (p + \rho)n_\mu n_\nu \nabla_\mu n_\nu =$$

$$= -p \nabla_\mu n^\mu - \nabla_\mu (\rho n^\mu) = 0$$  \hspace{1cm} (4.2)

where we used the fact that $n$ is a unit vector. This information can be plugged back into the original conservation law and then

$$\nabla_\mu T^{\mu\nu} = \nabla_\mu p g^{\mu\nu} + (\nabla_\mu pn^\mu - p \nabla_\mu n^\mu + p \nabla_\mu n^\mu) n_\nu + (p + \rho)n_\mu \nabla_\mu n_\nu =$$

$$= \nabla_\mu p (g^{\mu\nu} + n^\mu n_\nu) + (p + \rho)n_\mu \nabla_\mu n_\nu = 0$$  \hspace{1cm} (4.3)

One can easily see that if $p = 0$ (and $\rho > 0$) then the conservation of energy momentum tensor implies

$$n^\mu \nabla_\mu n_\nu = 0$$  \hspace{1cm} (4.4)

i.e. $n$ is a geodesics field. If $p \neq 0$ the same cannot follow in general along the same lines.

However, one can guess that there might exist special combinations of the pressure gradient and the metric tensor such that one still has $\nabla_\mu p (g^{\mu\nu} + n^\mu n_\nu) = 0$; in these cases the vector $n$ is geodesic even if $p \neq 0$. We shall see that Friedman-Robertson-Walker relativistics cosmologies do provide one of these cases.

In Cosmology, the cosmological principle claims that the metric is homogeneous and isotropic; see [17]. For such metrics there exists a coordinate system in which the metric tensor $g$ is in the form

$$g = -dt^2 + a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right)$$  \hspace{1cm} (4.5)

where the function $a(t)$ is called the scale factor, and $k = 0, \pm 1$. One should stress that this ansatz is done before fixing any dynamics. In fact, any metric in this form is a solution of Einstein equations if one defines the energy-momentum tensor of matter sources as

$$G_{\mu\nu} = :\kappa T_{\mu\nu}$$  \hspace{1cm} (4.6)
where $G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is the Einstein tensor of $g$.

One can easily see that the Einstein tensor of a metric in the form (4.5) is in the form (4.1), for a specific choice of pressure $p$ and density $\rho$ as a function of the scale factor $a(t)$ (together with its first and second derivatives) and for $n := \partial_t$. Here the relevant point is that the vector $n$ appearing in the energy momentum tensor (4.1) is fixed uniquely by the metric ansatz (4.5).

Hence for any spacetime in Cosmology the metric is associated to a fluid matter source and the fluid flow lines are generated by $n = \partial_t$. In this case one can check directly that, even when pressure is non-zero, the vector $n$ is geodesic for all metrics in the form (4.5).

Hence, at least for all dust matter models and all models in Cosmology (even with pressure) it is reasonable to assume that the flow lines of a fluid are generated by a timelike geodesic congruence.

5. Conclusions and Perspectives

We have shown that in Weyl geometries one can select the connection $\Gamma$ so that any generic timelike congruence is $\Gamma$-geodesic. In EPS framework (and in particular in $f(R)$ models) one can use any $[g]$-timelike congruence as the flowlines of a fluid.

The connection $\Gamma$ determined by a timelike congruence is conformally invariant.

This research is part of a much larger project aiming to model a generic self-gravitating fluid in EPS formalism. One can specify the conformal structure to be the Minkowskian one by setting $g = \eta$ and still be free to model any congruence of (timelike) worldlines as the flow of a fluid. In this framework the gravitational field is encoded into the covector $A$ which in turn determines the Weyl connection $\Gamma$.

Let us stress that in EPS setting there is no freedom in choosing the connection associated to the free fall. In EPS framework the free fall of particles is by construction described by the connection $\Gamma$ while the Levi-Civita connection of $g$ plays just the kinematical role of reference frame in the affine space of connections (which is moreover conformally covariant since one can start from any representative of the conformal structure).

The extra degrees of freedom to determine $\Gamma$ are thence encoded into the covector $A$ which is kinematically free to be generic. The dynamics of the theory determines then the connection $\Gamma$ fixing the covector $A$ in terms of matter fields and $g$.

Of course Weyl geometries are affected by physical interpretation problems mainly related to the (possibly non-trivial) holonomy of the connection $\Gamma$; see [1]. However, these problems arise only if $\Gamma$ is not metric while metric connections do not generate any physical problem of this kind and have just to be interpreted correctly. We stress that in all $f(R)$ models the connection $\Gamma$ turns out to be automatically metric, unless one introduces a matter Lagrangian in which matter couples directly with the connection $\Gamma$.

In a forthcoming paper we shall investigate in detail the energy-momentum tensor in a Weyl framework together with its conservation. In Weyl geometries one has two connections ($\Gamma$ and $\{g\}$), not just one as in standard GR. All derivations and theorems of standard GR need to be generalized to Weyl geometry at least by specifying which is the relevant connection to be used. This has to do also with determining which is the physical frame in $f(R)$ models; see [18].
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References

[1] J. Ehlers, F. A. E. Pirani, A. Schild, *The Geometry of Free Fall and Light Propagation*, in General Relativity, ed. L. O. Ralfeartagh (Clarendon, Oxford, 1972).

[2] M. Di Mauro, L. Fatibene, M. Ferraris, M. Francaviglia, *Further Extended Theories of Gravitation: Part I*, Int. J. Geom. Methods Mod. Phys. Volume: 7, Issue: 5 (2010), pp. 887-898; gr-qc/0911.2841

[3] J. A. Schouten, *Ricci-Calculus: An Introduction to Tensor Analysis and its Geometrical Applications*, Springer Verlag (1954)

[4] S. Capozziello, M. De Laurentis, V. Faraoni *A bird’s eye view of f(R)-gravity* (2009); arXiv:0909.4672

[5] T. P. Sotiriou, V. Faraoni, *f(R) theories of gravity*, (2008); arXiv: 0805.1726v2

[6] S. Capozziello, M. Francaviglia, *Extended Theories of Gravity and their Cosmological and Astrophysical Applications*, Journal of General Relativity and Gravitation 40 (2-3), (2008) 357-420.

[7] S. Capozziello, M. F. De Laurentis, M. Francaviglia, S. Mercadante, *From Dark Energy and Dark Matter to Dark Metric*, Foundations of Physics 39 (2009) 1161-1176 gr-qc/0805.3612v4

[8] S. Capozziello, M. De Laurentis, M. Francaviglia, S. Mercadante, *First Order Extended Gravity and the Dark Side of the Universe II: Matching Observational Data*, Proceedings of the Conference “Univers Invisible”, Paris June 29 July 3, 2009 to appear in 2010

[9] T. Y. Moon, J. Lee, P. Oh, *Conformal invariance in Einstein-Cartan-Weyl space*, Mod. Phys. Lett. A25, 3129 (2010); arXiv:0912.0432[gr-qc]

[10] L. Fatibene, M. Ferraris, M. Francaviglia, S. Mercadante, *Further Extended Theories of Gravitation: Part II*, Int. J. Geom. Methods Mod. Phys. Volume: 7, Issue: 5 (2010), pp. 899-906; gr-qc/0911.284

[11] V. Perlick, *Characterization of standard clocks by means of light rays and freely falling particles* General Relativity and Gravitation, 19(11) (1987) 1059-1073

[12] L. Fatibene, M. Francaviglia, S. Mercadante, *Matter Lagrangians Coupled with Connections* Int. J. Geom. Methods Mod. Phys. Volume: 7, Issue: 5 (2010), pp. 1185-1189; arXiv: 0911.2981

[13] T. P. Sotiriou, *f(R) gravity, torsion and non-metricity*, Class. Quant. Grav. 26 (2009) 152001; gr-qc/0904.2774

[14] T. P. Sotiriou, *Modified Actions for Gravity: Theory and Phenomenology*, Ph.D. Thesis; gr-qc/0710.4438

[15] Olmo Sigh

[16] M. Francaviglia, *Relativistic theories*, Quaderni del CNR-GNFM (Italy, 1988)

[17] H. Stephani, D. Kramer, M. Mac Callum, *Exact Solutions Of Einstein's Field Equations*, Cambridge University Press (2003)

[18] G. Magnano, L. M. Sokolowski, *On Physical Equivalence between Nonlinear Gravity Theories* Phys.Rev. D50 (1994) 5039-5059; gr-qc/9312008