Appearance of an infalling star in black holes with multiple photon spheres

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Photon spheres play a pivotal role in the imaging of luminous objects near black holes. In this paper, we examine observational appearances of a star freely falling in hairy black holes, which can possess one or two photon spheres outside the event horizon. When there exists a single photon sphere, the total luminosity measured by distant observers decreases exponentially with time at late times. Due to successive arrivals of photons orbiting around the photon sphere different times, a specific observer would see a series of light flashes with decreasing intensity, which share a similar frequency content. Whereas in the case with two photon spheres, photons temporarily trapped between the photon spheres can cause a peak of the total luminosity, which is followed by a slow exponential decay, at late times. In addition, these photons lead to one more series of light flashes seen by the specific observer.

black hole, gravitational lensing, photon sphere, observational appearance

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1 Introduction

The recent announcements of the images of the supermassive black holes M87* [1-8] and Sgr A* [9-14] by the Event Horizon Telescope (EHT) Collaboration open a new window to test general relativity in the strong field regime. These images display two main features, a central brightness depression, dubbed “shadow”, and a bright ring, which is closely relevant to a class of circular and unstable photon orbits. For static spherically symmetric black holes, these circular and unstable photon orbits would form photon spheres outside the event horizon. Since the shadow and the bright ring originate from light deflections by the strong gravitational field near the photon spheres [15-22], black hole images can encode valuable information of the geometry in the vicinity of the photon spheres. In particular, black hole images have been widely studied in the context of different theories of gravity, e.g., nonlinear electrodynamics [23-29], the Gauss-Bonnet theory [30-33], f(R) gravity [34], the Chern-Simons type theory [35, 36], string inspired black holes [40-43], wormholes [44-50], bosonic stars [51,52], and other theories [53-65]. Furthermore, the EHT observations can also be applied to impose constraints on the cosmological parameters [66-71] and the size of extra dimensions [72-74], test the equivalence principle [75-77], and probe some fundamental physics issues including dark matter [78-83] and dark energy [84-88].
Apart from stationary observational appearances of black holes, photon spheres also play an important role in dynamic observations of luminous objects around black holes. A particularly interesting scenario is luminous matter freely falling onto black holes, which occurs frequently and was reported near the Cyg X-1 black hole [89] and the Sgr A* black hole. Note that there exists a single photon sphere outside the event horizon in a Schwarzschild black hole. It showed that, owing to radiations temporarily trapped just outside the photon sphere, late-time radiations are dominated by blueshifted photons or gravitons with a near-critical impact parameter, and the total luminosity decreases exponentially as $e^{-\lambda t}$, where $\lambda$ is the Lyapunov exponent of circular null geodesics at the photon sphere.

On the other hand, a novel class of Einstein-Maxwell-scalar (EMS) models have recently been constructed to understand the formation of hairy black holes [93-97]. In such models, the scalar field non-minimally couples to the electromagnetic field, which can trigger a tachyonic instability to form spontaneously scalarized hairy black holes from Reissner-Nordström (RN) black holes. Properties of the hairy black holes have been extensively studied in the literature, e.g., different non-minimal coupling functions [98-100], massive and self-interacting scalar fields [101, 102], horizonless reflecting stars [103], stability analysis [104-108], higher dimensional scalar-tensor models [109], quasinormal modes [110, 111], two U(1) fields [112], quasi-topological electromagnetism [113], topology and spacetime structure influences [114], and in dS/AdS spacetime [96, 115-117].

Intriguingly, the scalarized hairy black holes have been shown to possess two photon spheres outside the event horizon in certain parameter regions [118, 119]. The existence of an extra photon sphere can significantly affect observable appearances of surrounding accretion disks (e.g., producing bright rings of different radii in the black hole images [118] and noticeably increasing the flux of the observed images [119]) and luminous celestial spheres (e.g., tripling higher-order images of the celestial spheres [120]). Moreover, the effective potential for a scalar perturbation in the hairy black holes with two photon spheres was found to exhibit a double-peak structure, leading to long-lived quasinormal modes [121] and echo signals [122]. It is worth noting that the existence of two photon spheres outside the event horizon has also been reported for dyonic black holes with a quasi-topological electromagnetic term [123], black holes in massive gravity [124, 125], and wormholes in the black-bounce spacetime [126-128].

In this paper, we aim to study how an extra photon sphere affects observational appearances of a freely-falling star in the context of the hairy black holes. The rest of the paper is organized as follows. In sect. 2, we briefly review hairy black hole solutions in the EMS model and introduce the observational settings. Numerical results are presented in sect. 3. Sect. 4 is devoted to our main conclusions. We set $16\pi G = c = 1$ throughout the paper.

### 2 Set up

We consider a 4-dimensional EMS model with the action [93]

$$S = \int d^4 x \sqrt{-g} \left[ R - 2 \partial_\mu \phi \partial^\mu \phi - e^{\phi} F_{\mu \nu} F^{\mu \nu} \right],$$

where $R$ is the Ricci scalar, and $F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength tensor. In this EMS model, the scalar field $\phi$ is non-minimally coupled to the electromagnetic field $A_\mu$ with the coupling function $e^\phi$. Restricting to static and spherically symmetric black hole solutions, one can have the generic ansatz

$$ds^2 = -N(r)e^{-2\phi(r)}dt^2 + \frac{1}{N(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

$$A_\mu dx^\mu = V(r)dt + \phi = \phi(r).$$

For this ansatz, the equations of motion are

$$N'(r) = \frac{1}{r} - \frac{Q^2}{r^2 e^{2\phi(r)}} - rN(r)\left[\phi'(r)\right]^2,$$

$$\left[ r^2 N(r)\phi''(r) \right] = \frac{\alpha Q^2 \phi(r)}{r^2 e^{2\phi(r)}} - r^3N(r)\left[\phi'(r)\right]^3,$$

$$\phi' = -r\left[\phi'(r)\right]^2,$$

$$V' = \frac{Q}{r^2 e^{2\phi(r)}} e^{-\phi(r)},$$

where the integration constant $Q$ can be interpreted as the electric charge of the black hole, and primes stand for derivative with respect to $r$.

Moreover, we impose proper boundary conditions at the event horizon $r_h$,

$$N(r_h) = 0, \quad \delta(r_h) = \delta_0, \quad \phi(r_h) = \phi_0, \quad V(r_h) = 0, \quad (4)$$

where $\delta_0$ and $\phi_0$ can be used to characterize black hole solutions. Specifically, $\phi_0 = \delta_0 = 0$ corresponds to the scalar-free solutions with $\phi = 0$, i.e., RN black holes. When non-zero values of $\phi_0$ and $\delta_0$ are admitted, hairy black hole solutions with a non-trivial scalar field $\phi$ can be obtained. The asymptotic expansions of the solutions at the spatial infinity take the form

$$N(r) = 1 - \frac{2M}{r} + \frac{Q^2 + Q^2}{r^2} + \cdots, \quad \delta(r) = \frac{Q^2}{2r^2} + \cdots,$$

$$\phi(r) = \frac{Q}{r} + \cdots, \quad V(r) = \Phi - \frac{Q}{r} + \cdots,$$
where $M$ is the ADM mass of the black hole, $\Phi$ is the electrostatic potential, and $Q_s$ is the scalar charge. For a given parameter $\alpha$, we use a shooting method built in the NDSolve function of Wolfram ® Mathematica to match the boundary conditions (4) with the asymptotic expansions (5), which relates $\phi_0$ and $\delta_0$ to $M$, $Q$, $\Phi$, and $Q_s$. Here, we choose the black hole mass $M$ and charge $Q$ as independent parameters to specify black hole solutions. Note that the scalar charge $Q_s$ can be expressed in terms of $M$ and $Q$, indicating that the scalar hair is of “secondary type”. In Figure 1, the domains of existence for hairy and RN black holes are exhibited in the $\alpha$-$Q/M$ plane. RN black holes exist in the region below the horizon dashed line, which corresponds to extremal RN black holes. Hairy black holes exist in the blue region bounded by the bifurcation line, where hairy black holes bifurcate from RN black holes as zero modes, and the critical line, where the horizon radius vanishes with $M$ and $Q$ remaining finite.

Interestingly, there is a coexistence region of hairy and RN black holes between the bifurcation and extremal RN black hole lines.

In this paper, we study a point-like star freely falling along the radial direction at $\theta = \frac{\pi}{2}$ and $\varphi = 0$. The equations governing geodesics on the equatorial plane can be derived from the Lagrangian

$$
\mathcal{L} = \frac{1}{2} \left[ -N(r)e^{-2\delta(r)}\dot{t}^2 + \frac{1}{N(r)}\dot{r}^2 + r^2 \dot{\varphi}^2 \right],
$$

(6)

where dots stand for derivative with respect to some affine parameter $\lambda$. Note that $\mathcal{L} = -1/2$ for time-like geodesics if $\lambda$ is the proper time. Since $t$ and $\varphi$ do not explicitly appear in the Lagrangian, one can characterize geodesics by their energy $E$ and angular momentum $L$, namely

$$
\begin{align*}
\dot{t} &\equiv -N(r)e^{-2\delta(r)}E = -E, \\
\dot{\varphi} &\equiv r^2 \dot{\varphi} = L.
\end{align*}
$$

(7)

For simplicity, we specialize to a radially freely falling star starting at infinity with a zero initial velocity, whose four-velocity is

$$
v^\mu(r) = \left( \frac{e^{-2\delta(r)}}{N(r)} \sqrt{e^{-2\delta(r)} - N(r)}, 0, 0 \right).
$$

(8)

Suppose that the star emits photons when it falls radially onto a black hole. Due to spherical symmetry, we can confine ourselves to emissions on the equatorial plane. The Lagrangian (6) can also be used to describe null geodesics on the equatorial plane. Moreover, the constancy of the Lagrangian $\mathcal{L} = 0$ rewrites the radial component of the null geodesic equations as:

$$
\frac{e^{-2\delta(r)}}{L^2} \dot{r}^2 = \frac{1}{b^2} - V_{\text{eff}}(r),
$$

(9)

where $b \equiv L/E$ is the impact parameter, and $V_{\text{eff}} \equiv e^{-2\delta(r)}N(r)r^{-2}$ is the effective potential. Unstable circular null geodesics at radius $r_{\text{ph}}$, which constitute a photon sphere of radius $r_{\text{ph}}$, are determined by

$$
V_{\text{eff}}(r_{\text{ph}}) = \frac{1}{b_{\text{ph}}^2}, \quad V_{\text{eff}}'(r_{\text{ph}}) = 0, \quad V_{\text{eff}}''(r_{\text{ph}}) < 0,
$$

(10)

where $b_{\text{ph}}$ is the corresponding impact parameter. In short, a local maximum of the effective potential corresponds to a photon sphere. Note that stable circular null geodesics, which correspond to a local minimum of the effective potential, lie on an “anti-photon sphere” [129]. Anti-photon spheres can appear in horizonless ultra-compact objects [130] or on the horizons of extreme static black holes [131]. The appearance of anti-photon spheres could signal the existence of long-live modes, which may render the spacetime unstable [132,133]. Nevertheless, anti-photon spheres barely play an important role in observing black holes by a distant observer since photons at anti-photon spheres cannot escape to infinity. For bald black hole solutions in the EMS model (i.e., RN black holes), there exist one photon sphere outside the event horizon and one anti-photon sphere inside or on the event horizon\(^1\). It is expected that the appearance of matter falling onto a RN black hole is similar to the Schwarzschild black hole case. Remarkably, it showed that, when the black hole charge is large enough, $V_{\text{eff}}(r)$ of the

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\(^1\) For a RN black hole with $g_{tt} = -(1 - r_+/r)(1 - r_-/r)$, the effective potential is $V_{\text{eff}}(r) = r^2(1 - r_+/r)(1 - r_-/r)$, where $r_+$ and $r_-$ are the event and Cauchy horizon radii, respectively, and $r_+ \leq r_-$. Since $V_{\text{eff}}(r) = r^2[-2\dot{r}^2 + 3(r_0 + r_+ - 4r_0r_+) - 4r_0r_+\dot{r}]$, $V_{\text{eff}}(r)$ has no more than two extrema. As $r$ approaches the infinity or $r_+$ from the right, one has $V_{\text{eff}}(r) \to 0^+$, which, together with $V_{\text{eff}}(r) > 0$ for $r_+ < r < \infty$, indicates that a maximum occurs for $r > r_+$. As $r$ approaches $r_-$ from the left or $r_-$ from the right, one has $V_{\text{eff}}(r) \to 0^-$, which, together with $V_{\text{eff}}(r) < 0$ for $r_- < r < r_+$, indicates that a minimum occurs for $r_- < r < r_+$. For an extremal RN black hole with $r_+ = r_-$, the minimum of $V_{\text{eff}}(r)$ is at $r = r_+$. 

Figure 1 (Color online) Domains of existence for hairy and RN black holes in the $\alpha$-$Q/M$ plane. RN black holes exist below the extremal RN black hole line (horizon dashed line), and hairy black holes exist in the blue region, which is bounded by the bifurcation line (dashed blue line) and the critical line (red line).
hairy black hole solutions (2) can have two local maxima, leading to a double-peak structure [118, 119]. Consequently, the hairy black holes with the double-peak effective potential possess two photon spheres outside the event horizon. Indeed, the effective potential of the hairy black hole with \( \alpha = 0.9 \) and \( Q = 1.064M \) exhibits a double-peak structure. Interestingly, with a scalar hair, black hole solutions in the EMS model can be more charged than extremal RN black holes, i.e., \( Q/M > 1 \). In the \( Q/M > 1 \) regime, hairy black holes can have two photon spheres outside the event horizon.

To trace light rays from the star to a far-away observers, one needs to supply initial conditions for eqs. (7) and (9). For a photon of four-momentum \( p_{\mu} \), the momentum measured in the rest frame of the star at \( r = r_{e} \) is

\[
p^f \equiv -\left[N(r_{e})e^{-2\delta(r_{e})}\right]^{-1}p_{t} + \sqrt{e^{2\delta(r_{e})} - N(r_{e})}p_{r},
\]

\[
p^o \equiv -\left[N(r_{e})e^{-2\delta(r_{e})}\right]^{-1}p_{t} + \sqrt{1 - N(r_{e})e^{-2\delta(r_{e})}}p_{r} + \sqrt{e^{2\delta(r_{e})} - N(r_{e})}p_{r}, \quad (11)
\]

\[
p^0 = 0, \quad p^o = \frac{p_{\phi}}{r_{e}}.
\]

The emission angle \( \beta \) is defined as:

\[
\cos \beta = \frac{p^\theta}{p^f}, \quad (12)
\]

which is the angle between the propagation direction of the photon and the radial direction in the rest frame of the star. With eqs. (11) and (12), one has the normalized frequency of the photon measured by the far-away observer,

\[
\omega_{e} = \frac{p^f}{\omega_{o}} = 1 - \cos \beta \sqrt{1 - N(r_{e})e^{-2\delta(r_{e})}}, \quad (13)
\]

where \( \omega_{o} \) and \( \omega_{e} \) are rest-frame and observed photon frequencies, respectively. Furthermore, the luminosity is defined as \( L_{i} \equiv dE_{i}/d\tau_{i} \), where \( E_{i} \) is the total energy of the photon, \( \tau_{i} \) is the proper time, and \( i = e \) and \( o \) denote quantities corresponding to the emitter and observer, respectively. Similar to the normalized frequency, we define the normalized luminosity as:

\[
\frac{L_{o}}{L_{e}} = \frac{dE_{o}}{d\tau_{o}}/\frac{dE_{e}}{d\tau_{e}} \approx \frac{\omega_{e}d\tau_{o}}{\omega_{o}d\tau_{e}} \left( \frac{d\tau_{o}}{d\tau_{e}} \right)^{-1}, \quad (14)
\]

where \( n_{o} \) and \( n_{e} \) are the observed and emitted photon numbers, respectively, and we replaced \( d\tau_{o} \) by \( d\tau_{e} \) since they are almost the same for the far-away observer.

### 3 Numerical results

In this section, we numerically study observational appearances of a star falling radially towards hairy black holes with a single or two photon spheres outside the event horizon. During the free fall of the star, photons are emitted isotropically in its rest frame. Specifically, we assume that the star starts emitting photons at \( t = 0 \) and \( r_{e} = 30.65M \), and emits 3200 photons, which are uniformly distributed in the emission angle \( \beta \), every proper time interval \( \delta \tau = 0.002M \). It is worth emphasizing that observational appearances of the infalling star, especially late-time appearances, are rather insensitive to the initial position \( r_{e} \) where the star starts emitting. Here, we simply choose \( r_{e} = 30.65M \), which is in agreement with that of ref. [92], for better comparison with the Schwarzschild black hole case.

The observational appearances are studied for two kinds of observers in this paper. The first kind is observers distributed on a celestial sphere at the radius \( r_{o} = 100M \), which refers to collecting photons in the whole sky at fixed radial coordinate \( r_{o} = 100M \). The measurement by the observers on the celestial sphere would give the frequency and luminosity distributions of photons that reach the celestial sphere. The second kind is a specific observer, who is located at \( \phi_{o} = 0 \) on the equator of the celestial sphere. Among all photons collected on the celestial sphere, we select photons with \( \cos \phi > 0.99 \) to mimic photons detected by the specific observer. To calculate observed luminosities, the collected photons are grouped into packets of 50 (i.e., \( dn_{o} = 50 \)) according to their arrival time. Without loss of generality, we set \( M = 1 \) in the rest of this section.

### 3.1 Single-peak potential

We first consider the hairy black hole with \( \alpha = 0.9 \) and \( Q = 1.054 \), which has a single-peak potential as shown in Figure 2. The photon sphere is located at \( r_{ph} = 1.610 \), and the

![Figure 2](Link-to-figure)

Figure 2. (Color online) The black line denotes the effective potential of null geodesics in the hairy black hole with \( \alpha = 0.9 \) and \( Q = 1.054 \). There is only one potential maximum at \( r_{ph} = 1.610 \), corresponding to the photon sphere with the critical impact parameter \( b_{ph} = 3.673 \). Colored regions are presented in the \( r-b \) parameter space, for which a photon with the impact parameter \( b \) is emitted at \( r \). A far-away observer cannot receive photons emitted in the gray region. Photons emitted in the pink and purple regions have impact parameters close to \( b_{ph} \), hence can be temporarily trapped around the photon sphere. In particular, after photons in the pink (purple) region are emitted outward inside (inward outside) the photon sphere, they would circle around the photon sphere more than once before reaching an observer.
corresponding critical impact parameter is $b_{\text{ph}} = 3.673$. In-going photons with $b < b_{\text{ph}}$ can overcome the effective potential barrier and get captured by the black hole. Therefore for $b < b_{\text{ph}}$, a distant observer can only receive outward-emitted photons. On the other hand, inward-emitted photons with $b > b_{\text{ph}}$ can be observed since they bounce outward from the potential barrier. Interestingly, strong gravitational lensing near the photon sphere causes extreme bending of light rays with a near-critical impact parameter, which makes them temporarily trapped around the photon sphere and take a long time to reach the observer. As a result, light rays with a near-critical impact parameter play a crucial role in the observational appearance of the infalling star at late times. To illustrate photons with a near-critical impact parameter, we classify photons into four categories according to their $b$.

(1) $b < 3.564$. Yellow region in Figure 2 and yellow dots in Figures 3 and 4.

(2) $3.564 \leq b < b_{\text{ph}}$. Pink region in Figure 2 and pink dots in Figures 3 and 4. Figure 5(a) depicts a light ray of $b = 3.564$, which corresponds to an outward-emitted photon that is emitted at $r_0 = 1.2$ inside the photon sphere and reaches the celestial sphere after the change of angular coordinate $\Delta \varphi = 2\pi$. It is worth emphasizing that near-critical photons of $b < b_{\text{ph}}$ that are emitted outward inside the photon

![Figure 3](image3.png) (Color online) The normalized frequency distribution and total luminosity of photons emitted from a freely-falling star in the hairy black hole with a single-peak potential. The star radially falls along $\phi = 0$ on the equatorial plane from spatial infinity at rest and starts emitting isotropically in its rest frame at $r_e = 31.155$. The emitted photons are collected by observers on a celestial sphere at $r_c = 100$. (a) The observers receive photons with a wide range of frequencies. In the early stage, photons emitted in the green region of Figure 2 dominate the frequency observation. The late-time high-frequency observation is determined by photons emitted in the purple region, which undergo extreme bending near the photon sphere. (b) The luminosity is calculated by grouping received photons into packets of 50. At late times, the luminosity is mainly controlled by photons emitted with a near-critical impact parameter in the purple region. The luminosity decays exponentially as $e^{-\lambda t} = e^{-0.140}$, where $\lambda$ is the Lyapunov exponent of circular null geodesics at the photon sphere. Note that the imaginary part of the associated lowest-lying quasinormal modes is $-\lambda/2$.

![Figure 4](image4.png) (Color online) The normalized frequency and luminosity of the infalling star measured by a far-away observer at $r_c = 100$, $\theta = \pi/2$ and $\phi = 0$ in the hairy black hole with a single-peak potential. The colored dots denote photons emitted in the regions with the same color in Figure 2. (a) The received photons form several frequency lines indexed by the number $n$ of photon orbits around the black hole. From left to right, the frequency lines correspond to $n = 0, 1, 2$ and 3, respectively. In particular, photons with an impact parameter close to $b_{\text{ph}}$ give rise to the self-similar $n \geq 1$ lines, which display blueshifts caused by the Doppler effect and the strong gravitational lensing. The time delay between the $n \geq 1$ lines is roughly the period of circular null geodesics at the photon sphere, $\Delta T = 2\pi r_{\text{ph}} = 23$. (b) At early times, the luminosity is dominated by outward-emitted photons with a small impact parameter and gradually decreases. Subsequently, blueshifted $n = 1$ photons start to reach the observer and then become the most dominant contribution, which results in the growth of the luminosity. Later, the luminosity drops sharply with the decrease in the frequency of the $n = 1$ photons. Consequently, a flash is observed around $t_0 = 215$. When the $n = 2$ photons play out, a similar steady rise followed by a steep fall is also observed, leading to a much fainter flash.
sphere would linger near the photon sphere for some time.

3) \(b_{ph} < b \leq 3.900\). Purple region in Figure 2 and purple dots in Figures 3 and 4. Figure 5(b) shows the trajectory of an inward-emitted photon with \(b = 3.900\), which is emitted at \(r_e = 5\) outside the photon sphere and reaches the celestial sphere after the change of angular coordinate \(\Delta \varphi = 2\pi\). Near-critical photons of \(b > b_{ph}\), that are emitted inward can circle around the photon sphere more than once before being observed, and contribute significantly to late-time observations. In this case, the photon sphere acts as a reflecting wall to scatter photons, which indicates that the near-critical photons can undergo a blueshift.

4) \(b > 3.900\). Green region in Figure 2 and green dots in Figures 3 and 4.

In short, we use the orbit number of light rays emitted at \(r_e = 1.2\) or 5 to determine the threshold impact parameters separating the four categories. To sum up, light rays emitted outward at \(r_e = 1.2\) would circle around the black hole less/more than once before being received in the yellow/pink category; light rays emitted inward at \(r_e = 5\) would circle around the black hole less/more than once before being received in the green/purple category. Note that the orbit number of the light rays depends slightly on the emitting position \(r_e\) with a given impact parameter \(b\). So, light rays connecting the star and the observers circle around the black hole approximately more than once in the pink and purple categories, and less than once in the yellow and green categories. As a result, light rays in the pink and purple categories can be temporarily trapped near the photon sphere.

Considering photons emitted with an impact parameter very close to \(b_{ph}\) in the pink and purple regions of Figure 2, their normalized frequencies measured by observers on the celestial sphere are plotted against the position of the star in Figure 6. Particularly, the purple line represents photons emitted inward in the purple region, and the pink line denotes photons emitted outward inside the photon sphere in the pink region. The normalized frequency of a photon is determined by the gravitational redshift and the Doppler effect, which are controlled by the position and the velocity of the photon when it is emitted, respectively. Due to light-bending near the photon sphere, the Doppler effect can increase the observed frequency of inward-emitted photons. At large \(r_e\), the gravitational redshift is weak, and hence the photons in purple line are blueshifted. The blueshift reaches the maximum \(\omega_0/\omega_e = 1.378\) at \(r_e = 8.970\) and becomes 1 at \(r_e = 3.673\). For \(r_e < 3.673\), as a result of strong gravitational redshift near the black hole, the observed frequency is redshifted and approaches zero as \(r_e\) goes to \(r_h\).

In Figure 3(a), we display the normalized frequency distribution of photons, which are emitted from the infalling

\[
\frac{\omega_0}{\omega_e} = \frac{\omega_e}{\omega_0} = 1.378
\]

at \(r_e = 8.970\). For inward-emitted photons, the Doppler effect competes with the gravitational redshift to determine the observed frequency, leading to a blueshift peak \(\omega_0/\omega_e = 1.378\) at \(r_e = 8.970\).
star and collected by the observers distributed on the celestial sphere. At early times, the received photons are dominated by those emitted in the green region of Figure 2. Later, photons emitted inward in the near-critical purple region start to reach the observers after circling around the photon sphere, and come to dominate the high-frequency observation. On the other hand, photons emitted outward in other color regions are found to be severely redshifted since both the Doppler effect and the gravitational redshift decrease the observed frequency. The corresponding normalized total luminosity is presented in Figure 3(b), where a dot corresponds to a packet of 50 photons, and the color of the dot is that having most photons in the packet. The luminosity gradually increases until reaching a peak at early times, and is dominated by photons emitted in the green region roughly before \( t_0 = 215 \), which is in agreement with the frequency observation. At late times, the luminosity is mostly determined by photons with a near-critical impact parameter emitted in the purple region. Interestingly, we find that the luminosity decays exponentially as \( e^{-\lambda t} \), i.e., \( L \propto e^{-\lambda t} \), where \( \lambda \) is the Lyapunov exponent of circular null geodesics at the photon sphere. Indeed, it was argued that the late-time decay of the Lyapunov exponent of circular null geodesics at the photon sphere is determined by the imaginary part of quasinormal frequencies \( \omega_i \), i.e., \( L \propto e^{2I m(\omega i)} \), in the eikonal limit \([92]\). For the hairy black hole with a single-peak potential, the imaginary part of the lowest-lying quasinormal frequencies is \( -\lambda/2 \) in the eikonal limit \([121]\), which gives \( L \propto e^{-\lambda t} \). Note that the finite number of photons in a packet results in the non-smoothness of the luminosity in Figure 3. The luminosity can become smoother by including more photons in a packet, which, nevertheless, reduces the accuracy of our numerical results with a fixed number of total emitted photons. For a specific observer located at \( \varphi = 0 \) and \( \theta = \pi/2 \) on the celestial sphere, the angular coordinate change \( \Delta \varphi \) of light rays connecting the star with the observer is

\[
\Delta \varphi = 2n\pi, \tag{15}
\]

where \( n = 0, 1, 2, \ldots \) is the number of orbits that the light rays complete around the black hole. To obtain the observational appearance of the star seen by the observer, we focus on the collected photons with \( \cos \varphi > 0.99 \). We present the frequency observation in Figure 4(a), which shows a discrete spectrum separated by the received time. The leftmost line is formed by photons with \( n = 0 \), which radially propagate to the observer. As expected, the observed frequency of the \( n = 0 \) photons decreases with the received time because of the gravitational redshift. The rest three lines correspond to light rays with \( n = 1, 2, \) and 3 from left to right, and have a similar shape since the emission angle is almost the same for the \( n \geq 1 \) light rays. As discussed earlier, the \( n \geq 1 \) photons can be blueshifted due to the Doppler effect and the strong gravitational lensing. With a fixed \( n \), the observed frequencies increase slowly, reach a maximum and then decrease rapidly. In addition, the time delay between the adjacent lines roughly equals the time it takes to orbit the photon sphere, i.e., \( \Delta T = 2\pi b_{ph} \approx 23^3 \). Furthermore, although the received \( n = 1 \) photons mainly come from the purple and pink regions of Figure 2, those emitted in the green and yellow regions are detected at early and late times, respectively. Note that the angular coordinate changes of light rays with a given impact parameter vary with the emitted position \( r_e \). When \( r_e \) is small (large) enough, photons emitted in the yellow (green) regions can circle around the black hole once before being received. For \( n \geq 2 \), the observer only receives photons emitted in the purple and pink regions, which can circle around the black hole more than once due to the near-critical impact parameter.

Figure 4(b) shows the observed normalized luminosity as a function of the observed time, which exhibits a steady decline at early times owing to the gravitational redshift. Around \( t_0 \approx 200, \) blueshifted photons with \( n = 1 \) start to play a dominant role, leading to a gradual rise of luminosity until reaching a peak. Afterwards, a sharp drop of luminosity is observed due to strong redshifts of photons emitted at small \( r_e \). Later, photons with \( n = 2 \) start to dominate the contribution to the luminosity. The luminosity of the \( n = 2 \) photons is about 40 times smaller than that of the \( n = 1 \) photons since the \( n = 2 \) photons are much less than the \( n = 1 \) photons. Despite different magnitudes, the luminosities of the photons with \( n = 1 \) and 2 have a similar shape. In short, successive receptions of photons orbiting around the photon sphere bring about a series of observed flashes labelled by \( n \). Note that our numerical simulation does not have enough emitted photons to produce the luminosity of the \( n > 2 \) photons.

3.2 Double-peak potential

The hairy black hole with \( \alpha = 0.9 \) and \( Q = 1.064 \) has a double-peak effective potential as shown in Figure 7. The inner peak corresponds to the inner photon sphere with the critical impact parameter \( b_{in} = 3.455 \) at \( r_{in} = 0.403 \), and the outer peak to the outer photon sphere with \( b_{out} = 3.584 \) at \( r_{out} = 1.463 \). Similarly, emitted photons are sorted into seven categories by their impact parameter \( b \).

(1) \( b < 2.884 \). Yellow region in Figure 7 and yellow dots in Figures 8 and 9.

(2) \( 2.884 \leq b < b_{in} \). Pink region in Figure 7 and pink dots in Figures 8 and 9. In this category, photons emitted

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2) Eq. (7) leads to \( \frac{dr}{d\psi} = b^{-1}V_{eff}^{-1}(r_{ph}) = b_{ph} \), which gives \( \Delta T = 2\pi b_{ph} \).
The observed luminosity and a luminosity peak at $t \lesssim b_{\text{inner}}$ photon sphere before reaching a distant observer. For outward inside the inner photon sphere can circle around the black hole with $\Delta \varphi \geq 2\pi$. Photons emitted in the brown, blue and orange regions have $b_{\text{in}} < b < b_{\text{out}}$ and can be temporarily trapped in the region between the inner and outer photon spheres by orbiting the black hole with $\Delta \varphi \geq 6\pi$. Hence, these photons play a key role in late-time observations.

Figure 7  (Color online) The effective potential of null geodesics in the hairy black hole with $\sigma = 0.9$ and $Q = 1.064$. (a) The potential has two peaks at $r_{\text{in}} = 0.403$ and $r_{\text{out}} = 1.463$, corresponding to the inner photon sphere with $b_{\text{in}} = 3.455$ and the outer one with $b_{\text{out}} = 3.584$, respectively. (b) highlights the region near the inner peak. When photons are emitted outward (inward) at $r < r_{\text{in}}$ ($r > r_{\text{out}}$) in the pink (purple) region, they usually orbit the black hole with $\Delta \varphi \geq 2\pi$. Photons emitted in the brown, blue and orange regions have $b_{\text{in}} < b < b_{\text{out}}$ and can be temporarily trapped in the region between the inner and outer photon spheres by orbiting the black hole with $\Delta \varphi \geq 6\pi$. Hence, these photons play a key role in late-time observations.

Figure 8  (Color online) The normalized frequency distribution (a) and total luminosity (b) of the infalling star measured by observers on a celestial sphere at $r = 100$ in the hairy black hole with a double-peak potential. Before $t_{\text{in}} = 250$, the frequency and luminosity observations are quite similar to the single-peak case as the outer photon sphere plays the role of the photon sphere in the single-peak case. For $200 \leq t_{\text{in}} \leq 250$, the luminosity scales as a decay $L \propto e^{-0.114t}$, in agreement with the Lyapunov exponent $\lambda_{\text{max}} = 0.11420$ at the outer photon sphere. Interestingly, the inward-emitted photons with $b_{\text{in}} < b < b_{\text{out}}$ (i.e., those emitted in the brown, blue and orange regions) usually linger for a long time between the inner and outer photon spheres before being received and kick in after $t_{\text{in}} = 250$. For $250 \leq t_{\text{in}} \leq 300$, a large number of photons with a wide range of frequencies are detected by the observers, thereby leading to an increase of $\Delta \varphi > 5\pi$, which may be associated with sub-long-lived quasinormal modes residing at the outer photon sphere.

outward inside the inner photon sphere can circle around the inner photon sphere before reaching a distant observer. For example, a light ray with $b = 2.884$, which has $\Delta \varphi = 2\pi$, is displayed in Figure 10(a).

3) $b_{\text{in}} < b \leq 3.458$. Brown region in Figure 7 and brown dots in Figures 8 and 9. In this category, photons emitted inward would circle around the inner photon sphere roughly with $\Delta \varphi \geq 10\pi$ before escaping to the celestial sphere. For example, a light ray with $b = 3.458$, which has $\Delta \varphi = 10\pi$, is displayed in Figure 10(b).

4) $3.458 < b \leq 3.547$. Blue region in Figure 7 and blue dots in Figures 8 and 9. In this category, photons can be temporarily trapped between the inner and outer photon spheres by circling around the black hole approximately between three and five times. For example, a light ray with $b = 3.547$, which has $\Delta \varphi = 10\pi$, is displayed in Figure 10(c).

5) $3.547 < b < b_{\text{out}}$. Orange region in Figure 7 and orange dots in Figures 8 and 9. In this category, if photons are emitted outward outside the outer photon sphere, they would linger for some time around the outer photon sphere by orbiting the black hole approximately with $\Delta \varphi \geq 10\pi$.

6) $b_{\text{out}} < b \leq 3.847$. Purple region in Figure 7 and purple dots in Figures 8 and 9. In this category, photons emitted inward outside the outer photon sphere usually circle around the outer photon sphere more than once. For example, a light ray with $b = 3.847$, which has $\Delta \varphi = 2\pi$, is displayed in Figure 10(d).

7) $b > 3.847$. Green region in Figure 7 and green dots in Figures 8 and 9. Similar to the single-peak case, we use the orbit number of light rays emitted at $r_e = 0.35$ or 5 to determine the threshold impact parameters separating the seven categories. In sum-

$\Delta \varphi \geq 5\pi$.
Figure 9 (Color online) The normalized frequency and luminosity of the infalling star measured by a far-away observer at $r_o = 100$, $\theta = \pi/2$ and $\phi = 0$ in the hairy black hole with a double-peak potential. (a) The frequency lines correspond to the orbit number $n = 0, 1, 2, \ldots$ from left to right, and the $n \geq 1$ frequency lines are separated by $\Delta T = 2\pi b_{\text{in}} = 2\pi b_{\text{out}} = 22$. At early times, the $1 \leq n \leq 4$ frequency lines are primarily determined by photons lingering outside the outer photon sphere. After $t_0 = 250$, photons emitted from the star successively reach the observer after they orbit around the black hole between the inner and outer photon spheres, causing the $n > 4$ frequency lines. (b) Before the photons temporarily trapped between the photon spheres arrive, the observer sees a luminosity drop followed by two flashes, which is quite similar to the single-peak case. After $t_0 = 250$, the received $n = 5$ photons lead to a notable flash. Subsequently, the received $n = 6$ and 7 photons give two more but much less luminous flashes.

Figure 10 (Color online) Photon trajectories in the hairy black hole with a double-peak potential. The blue dashed circles denote the photon spheres. (a) A photon is emitted at $r_e = 0.35$ with $b = 2.884$, and the light ray has $\Delta \varphi = 2\pi$. (b) A photon is emitted at $r_e = 5$ with $b = 3.458$, and the light ray has $\Delta \varphi = 10\pi$. (c) A photon is emitted at $r_e = 5$ with $b = 3.547$, and the light ray has $\Delta \varphi = 10\pi$. (d) A photon is emitted at $r_e = 5$ with $b = 3.847$, and the light ray has $\Delta \varphi = 2\pi$. Mary, light rays emitted outward at $r_e = 0.35$ would circle around the black hole outside the inner photon sphere less/more than once before being received in the yellow/pink category; light rays emitted inward at $r_e = 5$ would circle around the black hole more than 5 times before being received in the brown and orange categories; light rays emitted inward at $r_e = 5$ would circle around the black hole between 3 and 5 times before being received in the blue category; light rays...
emitted inward at \( r_e = 5 \) would circle around the black hole outside the outer photon sphere less/more than once before being received in the green/purple category. It is worth mentioning that light rays emitted inward at \( r_e = 5 \) with an impact parameter between the two potential peaks would at least circle around the black hole 3 times. In other words, light rays connecting the star and the observers approximately circle around the black hole more/less than once in the pink/yellow category, outside the outer photon sphere more/less than once in the purple/green category, between the inner and outer photon spheres more than 5 times in the brown and orange categories, and between the inner and outer photon spheres less than 5 times and more than 3 times in the blue category. Consequently, light rays in the brown, blue and orange categories can be temporarily trapped between the inner and outer photon spheres.

In Figure 11, we plot the normalized frequency \( \omega_o/\omega_k \) as a function of the emitted position \( r_e \) for near-critical photons. Specifically, we focus on inward-emitted photons with \( b \) very close to \( b_{in} \) at \( r_e > r_{in} \) in the brown region, outward-emitted photons with \( b \) very close to \( b_{in} \) at \( r_e < r_{in} \) in the pink region, inward-emitted photons with \( b \) very close to \( b_{out} \) at \( r_e > r_{out} \) in the purple region and photons emitted with \( b \) very close to \( b_{out} \) at \( r_e < r_{out} \) in the orange region. The colors of the lines in Figure 11 match those of the corresponding emitted regions in Figure 7. Moreover, photons with \( b \) very close to \( b_{in} \) and \( b_{out} \) are denoted by solid and dashed lines, respectively. Similar to the single-peak case, strong gravitational lensing around the inner and outer photon spheres can cause blueshifts of the near-critical photons when \( r_e \) is large enough. In particular, the normalized frequency of photons with \( b \) very close to \( b_{in} \) (\( b_{out} \)) reaches the maximum \( \omega_o/\omega_k = 1.387 \) at \( r_e = 8.512 \) (\( r_e = 8.784 \)) and becomes the unit at \( r_e = 3.455 \) (\( r_e = 3.584 \)).

In addition, the inset shows that the dashed line is divided into two branches for \( r_{in} < r_e < r_{out} \), and the upper and lower branches correspond to inward-emitted and outward-emitted photons emitted between the two photon spheres, respectively.

The normalized frequency distribution of photons received by the observers distributed on the celestial sphere is presented in Figure 8(a). Roughly for \( t_o < 250 \), the frequency distribution of the received photons bears a resemblance to the single-peak case. At the early stage, most of the received photons are emitted in the green region. Afterwards, photons that are emitted inward in the purple region come to dominate blueshifted photons received by the observers, and photons that are emitted outward in other colored regions suffer from a severe redshift. Remarkably, photons emitted inward in the blue region start reaching the celestial sphere around \( t_o \approx 240 \) after they orbit the black hole several times between the inner and outer photon spheres. Subsequently, the observers receive photons emitted inward in the orange and brown regions, which would linger between the inner and outer photon spheres for a longer time. As indicated previously, the high-frequency and low-frequency photons come from photons emitted at large and small \( r_e \), respectively. When \( r_e \) is large enough, the received photons are blueshifted.

The normalized total luminosity of the infalling star is displayed in Figure 8(b), where colored dots correspond to packets of 50 photons. Before \( t_o \approx 250 \), the luminosity behaves similarly to the single-peak case, i.e., a gradual rise followed by an exponentially decaying tail as \( L \propto e^{-0.114t_o} \). The tail is determined by photons that are emitted in the purple region and linger outside the outer photon sphere. As expected, the Lyapunov exponent \( \lambda_{out} = 0.11420 \) at the outer photon sphere controls the decay rate of the tail. It should be emphasized that the Lyapunov exponent at the inner photon sphere is \( \lambda_{in} = 0.14399 \), and hence the contribution to the luminosity from photons temporarily trapped near the inner photon sphere is suppressed. After \( t_o \approx 250 \), photons temporarily trapped between the inner and outer photon spheres start to play a dominant role. In particular, photons emitted in the blue region give rise to a noticeable increase of the luminosity and a sharp peak. Moreover, the late-time behavior is dominated by photons emitted in the orange region and can be described by an exponential decay, \( L \propto e^{-0.058t_o} \). In ref. [121], we found that the photons emitted in the orange region are associated with a class of sub-long-lived quasinormal modes, thereby leading to a more slowly decaying tail than \( e^{-0.114t_o} \). However unlike the single-peak case, the imaginary parts of the sub-long-lived modes depend on the angular quantum number, and hence the decay rate of the tail cannot be universally determined.
In Figure 9(a), we exhibit the normalized frequency of photons received by an observer located at $\varphi = 0$ and $\theta = \pi/2$ on the celestial sphere. After photons are emitted from the star, they would orbit around the black hole different times before being captured, which results in a discrete set of the observed frequencies indexed by the number of orbits $n$. The orbit number $n$ of the frequency lines increases from left to right, and the leftmost yellow line corresponds to $n = 0$. As previously stated, the $n \geq 1$ frequency lines are similar and separated by the time delay $\Delta T \approx 2\pi b_{in} \approx 2\pi b_{out} \approx 22$. For $1 \leq n \leq 3$, the frequency lines are mainly contributed by photons that are emitted in the purple region and circle around the outer photon sphere $n$ times. When $n = 4$, photons emitted in the purple region are rarely seen, and some observed low-frequency photons are from emission between the photon spheres in the blue and orange regions. Inward-emitted photons in the blue, orange and brown regions can circle around the black hole 5 times between the inner and outer photon spheres and come back to start arriving after $t_0 = 70 \times 2 + 22 \times 5 \approx 250$, which forms the $n = 5$ frequency line. Later, photons orbiting around the black hole more than 5 times successively reach the observer and give rise to the $n > 5$ frequency lines.

The normalized luminosity of the star measured by the observer is displayed in Figure 9(b). Before $t_0 \approx 250$, the luminosity observation resembles that in the single-peak case, i.e., two flashes following the decrease of the luminosity. Owing to the leaking of photons temporarily trapped between the photon spheres, a noticeable flash appears around $t_0 \approx 300$, and two much fainter flashes are seen later on. These three flashes are caused by the received photons orbiting around the black hole 5, 6 and 7 times, respectively. For $n > 7$, we do not have sufficient photons to compute the luminosity.

4 Conclusions

In this paper, we investigated observational appearances of a point-like freely-falling star, which emits photons isotropically in its rest frame, in hairy black holes. In particular, we considered the frequency and luminosity of received photons measured by all observers on a celestial sphere and a specific observer. Interestingly, it was found that the existence of an extra photon sphere can significantly change the late-time observations, which may provide a new tool to detect black holes with multiple photon spheres.

For hairy black holes with a single-peak potential, emitted photons can be temporarily trapped just outside the photon sphere and subsequently reemitted, resulting in several interesting observational signatures. First, photons with a near-critical impact parameter can be blueshifted since the photon sphere acts as a reflecting wall (see Figures 3 and 6). Second, at late times, the total luminosity measured by observers on the celestial sphere decreases exponentially with time, which is controlled by the Lyapunov exponent of circular null geodesics at the photon sphere (see Figure 3). Last but not least, photons orbiting around the photon sphere different times produce a cascade of flashes observed by the specific observer. The luminosity of the flashes decreases sharply with the orbit number, and the frequency contents of the flashes are similar (see Figure 4). It is worth emphasizing that the observational appearances in the hairy black holes with a single-peak potential are similar to those in Schwarzschild black holes reported in ref. [92].

When an extra photon sphere appears outside the event horizon, we found that light rays can get trapped between the two photon spheres longer than just outside the photon spheres, which appreciably impacts late-time observations. In fact, due to arrivals of photons trapped between the photon spheres, the total luminosity rises to a peak, which is succeeded by a slow exponential decay (see Figure 8). This slow decay may be related to sub-long-lived quasinormal modes living near the outer photon sphere, as found in ref. [121]. The specific observer would see two cascades of flashes, which contain a similar frequency content. While the earlier cascade is mainly determined by photons orbiting outside the outer photon sphere, photons trapped between the photon spheres primarily give rise to the later one (see Figure 9). In summary, black holes with multiple photon spheres will lead to distinctive optical appearances of an infalling star. In the future studies, it will be of great interest if our analysis can be generalized to more astrophysically realistic models.

It has been reported that luminous matter falling onto a black hole seems to occur periodically near the Cyg X-1 black hole [89] and the Sgr A* source [90, 91]. Moreover, an infalling gas cloud was simulated for EHT observations in ref. [134], which provides a new possibility for measuring the spin of Sgr A*. Although the timescale required to observe the whole falling process may be enormous, one can still extract some useful information through late-time observations, such as a luminosity decay and luminous flashes. On the other hand, detecting photons circling around black holes several times at late times could be a challenging task due to the scarcity of these photons. Recently, the possibility of using the complex visibility function to distinguish photon rings, which are formed of photons circling around black holes more than once, has been discussed in refs. [135-137]. It showed that precise measurements of photon rings may be feasible with a very long baseline interferometry. Therefore, interferometric signatures of luminous matter falling onto a black hole with multiple photon spheres deserve further study.
