Revisiting Calculation of Moments of Number of Comparisons
used by the Randomized Quick Sort Algorithm

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Abstract
We revisit the method of Kirschenhofer, Prodinger and Tichy [KHT87] [KHT] to
calculate the moments of comparisons used by the quick sort algorithm. We reemphasize
that this approach helps in calculating these quantities with less computation. We also
point out that as observed by [KS97] this method also gives moments for total path length
of a binary search tree built over a random set of \( n \) keys.

1 Introduction

Consider the following variant of quick sort algorithm from [SW11]: the quick sort algorithm
recursively sorts numbers in an array by partitioning it into two smaller and independent
subarrays, and thereafter sorting these parts. The partitioning procedure chooses the last
element in the array as \textit{pivot} and puts it in its right place where numbers to the left of it are
smaller than it, and those to its right are larger than it.

For purposes of this analysis assume that the input array to the quick sort algorithm
contains distinct numbers which are randomly ordered. We may assume the input to the
algorithm is simply a permutation of \( \{1, 2, \cdots, n\} \) (if the input array has \( n \) elements).

Let \( S \) be the set of all \( n! \) permutations of \( \{1, 2, \cdots, n\} \). Consider a uniform probability
distribution on the set \( S \), and define for all \( s \in S \), \( C_n(s) \) to be the number of comparisons
used to sort \( s \) by the quick sort algorithm. We wish to calculate mean and variance of \( C_n \)
over the uniform distribution on \( S \).

Our aim here is to obtain following [KHT87] [KHT] to obtain

Theorem 1.1 (Knuth [Knu98]). We have

\[
\text{Mean}(C_n) = 2(n + 1)(H_{n+1} - 1);
\]

and

\[
\text{Var}(C_n) = 7n^2 - 4(n + 1)^2 H_n^{(2)} - (2n + 1)H_n + 13n,
\]

over the uniform probability distribution on \( S \). Here we have used the notation \( H_n = \sum_{k=1}^{n} \frac{1}{k} \)
and \( H_n^{(2)} = \sum_{k=1}^{n} \frac{1}{k^2} \).

Before proceeding we would like to point out that Hennequin [Hen91] has computed the
first five cumulants of the number of comparisons of Quicksort.
2 Calculation of mean and variance

Let $a_{n,s}$ be the number of permutations of $n$ elements requiring a total of $s$ comparisons to sort by the procedure of quicksort.

We start by defining the corresponding probability generating function:

$$G_n(z) = \sum_{k \geq 0} \frac{a_{n,k} z^k}{n!}.$$  

Theorem 2.1. For $n \geq 1$

$$G_n(z) = \frac{z^{n+1}}{n} \sum_{1 \leq j \leq n} G_{n-j}(z) G_{j-1}(z),$$  

and

$$G_0(z) = 1.$$  

Proof. The first partitioning stage requires $n + 1$ comparisons (for some other variants this might be $n - 1$). If the pivot element is $k$th largest, the sub arrays after partitioning are of size $k - 1$ and $n - k$. Thus we can write

$$a_{n,s} = \sum_{1 \leq k \leq n} \binom{n-1}{k-1} \sum_{i+j=s-(n+1)} a_{n-k,i} a_{k-1,j}.$$  

Multiplying equation (3) by $z^s$ and dividing by $n!$ we get

$$\frac{a_{n,s} z^s}{n!} = \sum_{1 \leq k \leq n} \frac{z^s}{n} \sum_{i+j=s-(n+1)} \frac{a_{n-k,i}}{(n-k)!} \cdot \frac{a_{k-1,j}}{(k-1)!}$$

$$= \sum_{1 \leq k \leq n} \frac{z^s}{n} \cdot \left\{ \text{coefficient of } z^{s-(n+1)} \text{ in } G_{n-k}(z) \cdot G_{k-1}(z) \right\}$$

$$= \sum_{1 \leq k \leq n} \frac{z^{n+1}}{n} \cdot z^{s-(n+1)} \left\{ \text{coefficient of } z^{s-(n+1)} \text{ in } G_{n-k}(z) \cdot G_{k-1}(z) \right\}$$

after which summing on $s$ gives us equation (1). \qed

We will now consider the double generating function $H(z,u)$ defined by

$$H(z,u) = \sum_{n \geq 0} G_n(z) u^n.$$  

Corollary 2.2. We have

$$\frac{\partial H(z,u)}{\partial u} = z^2 \cdot H^2(z,zu),$$  

and

$$H(1,u) = (1 - u)^{-1}. $$
Proof. From equation (1) we have
\[
\frac{\partial H(z,u)}{\partial u} = z^2 \sum_{n \geq 1} (uz)^{n-1} \sum_{1 \leq j \leq n} G_{n-j}(z)G_{j-1}(z)
\]
\[
= z^2 \sum_{n \geq 1} (uz)^{n-1} \cdot \{ \text{coefficient of (uz)}^{n-1} \text{ in } H(z,uz) \cdot H(z,uz) \}
\]
\[
= z^2 \cdot H(z,uz) \cdot H(z,uz).
\]

Equation (6) follows from the fact that \( G_n(1) = 1 \).

Now we write the sth factorial moments \( \beta_s(n) \) of the random variable with the aid of the probability generating function \( G_n(z) \):
\[
\beta_s(n) = \left[ \frac{d^s}{dz^s} G_n(z) \right]_{z=1}.
\]

The generating functions \( f_s(u) \) of \( \beta_s(n) \) are
\[
f_s(u) = \sum_{n \geq 0} \beta_s(n)u^n.
\]

By Taylor’s formula and equation (7) we get
\[
H(z,u) = \sum_{s \geq 0} f_s(u) \frac{(z-1)^s}{s!}.
\]

Theorem 2.3. For \( s \geq 1 \) we have
\[
f_s'(u) = s! \cdot \sum_{i+j+k+l+m=s} \frac{a_i \cdot f_j^{(k)}(u) \cdot f_l^{(m)}(u) \cdot u^{k+m}}{j! \cdot k! \cdot l! \cdot m!},
\]
where
\[
a_k = \begin{cases} 
1 & \text{if } k = 0; \\
2 & \text{if } k = 1; \\
1 & \text{if } k = 2; \\
0 & \text{if } k > 2.
\end{cases}
\]

Proof. Using Taylor’s theorem we can write
\[
f_j(x) = \sum_{k \geq 0} \frac{f_j^{(k)}(u)(x-u)^k}{k!}
\]
which on substituting \( x = uz \) gives
\[
f_j(uz) = \sum_{k \geq 0} \frac{f_j^{(k)}(u)(z-1)^k u^k}{k!}.
\]
Now substituting equation (10) in equation (9) gives:

\[ 
\sum_{s \geq 0} f_s(u) \frac{(z-1)^s}{s!} = z^2 \cdot \sum_{p \geq 0} f_p(uz) \frac{(z-1)^p}{p!} \cdot \sum_{r \geq 0} f_r(uz) \frac{(z-1)^r}{r!} 
\]

\[ 
= \sum_{i \geq 0} a_i(z-1)^i \cdot \frac{(z-1)^i}{p!} \cdot \sum_{l \geq 0} u(l)^2 \cdot \sum_{r \geq 0} f_r(uz) \frac{(z-1)^r}{r!} \cdot \sum_{m \geq 0} \frac{f_r^{(m)}(uz)}{m!} (z-1)^{m} u^m 
\]

where in the second last line we replaced \( z^2 \) by \( \sum_{i \geq 0} a_i(z-1)^i \). Now comparing coefficients on both sides of the equation gives

\[ 
\sum_{i \geq 0} a_i(z-1)^i \cdot \frac{(z-1)^i}{p!} \cdot \sum_{l \geq 0} u(l)^2 \cdot \sum_{r \geq 0} f_r(uz) \frac{(z-1)^r}{r!} \cdot \sum_{m \geq 0} \frac{f_r^{(m)}(uz)}{m!} (z-1)^{m} u^m 
\]

\[ 
= \sum_{h \geq 0} (z-1)^h \sum_{i+j+k+l+m=h} a_i \cdot \frac{1}{j!} \cdot \frac{f_j^{(k)}(u) u^k}{k!} \cdot \frac{1}{l!} \cdot \frac{f_l^{(m)}(u) u^m}{m!} 
\]

Remark 2.4. For asymptotic theory of differential equations originating here we recommend reader the paper [CHT02].

Corollary 2.5. We have

\[ 
f_0(u) = (1 - u)^{-1}, \quad (12) \]

\[ 
f_1(u) = \frac{2}{(1-u)^2} \log \frac{1}{1-u}, \quad (13) \]

\[ 
f_2(u) = \frac{8 \log^2(1-u)}{(1-u)^3} - \frac{8 \log(1-u)}{(1-u)^3} - \frac{4 \log^2(1-u)}{(1-u)^2} + \frac{12 \log(1-u)}{(1-u)^2} + \frac{6}{(1-u)^3} - \frac{6}{(1-u)^2}. \quad (14) \]

Proof. The equation (12) follows from the fact that \( \beta_0(u) = 1 \).

Setting \( s = 1 \) in equation (10) gives

\[ 
f_1(u) = a_1 \cdot f_0^{(0)}(u) \cdot f_0^{(0)}(u) + f_0^{(1)}(u) \cdot f_0^{(0)}(u) \cdot u + f_0^{(1)}(u) \cdot f_0^{(0)}(u) \cdot u + f_1^{(0)}(u) \cdot f_0^{(0)}(u) \cdot u 
\]

\[ 
+ f_1^{(0)}(u) \cdot f_0^{(0)}(u) + f_0^{(0)}(u) \cdot f_1^{(0)}(u) 
\]

\[ 
= \frac{2}{(1-u)^2} + \frac{u}{(1-u)^3} + \frac{u}{(1-u)^2} + \frac{f_1(u)}{(1-u)} + \frac{f_1(u)}{(1-u)} 
\]

where we used the fact that \( f_0(u) = (1 - u)^{-1} \). The above equation is

\[ 
f_1(u) - \frac{2 f_1(u)}{1-u} = \frac{2 u}{(1-u)^3} + \frac{2}{(1-u)^3}. \quad (15) \]

Solving the linear differential equation (15) by multiplying with integrating factor \( (1-u)^2 \) gives

\[ 
f_1(u) = \frac{2}{(1-u)^2} \log \frac{1}{1-u} + f_1(0) = \frac{2}{(1-u)^2} \log \frac{1}{1-u}. 
\]
Plugging $s = 2$ in (10) and solving the resultant differential equation gives

$$f_2(u) = \frac{8\log^2(1-u)}{(1-u)^3} - \frac{8\log(1-u)}{(1-u)^2} - \frac{4\log^2(1-u)}{(1-u)^2} + \frac{12\log(1-u)}{(1-u)^2} + \frac{6}{(1-u)^3} - \frac{6}{(1-u)^2}.$$  

\[ \square \]

**Corollary 2.6.** We have

$$\beta_1(n) = 2((n + 1)H_n - n),$$

and

$$\beta_2(n) = 4(n + 1)^2(H_n^2 - H_n^{(2)}) + 4(n + 1)^2H_n - 8(n + 1)H_n + 8nH_n - 4nH_n(5 + 3n) + 11n^2 + 15n.$$  

**Proof.** We use following expansions from [GK07]

$$\frac{1}{(1-u)^{m+1}} \log \left( \frac{1}{1-u} \right) = \sum_{n \geq 0} (H_{n+m} - H_m) \binom{n+m}{n} u^n;$$

$$\frac{1}{(1-u)^{m+1}} \log^2 \left( \frac{1}{1-u} \right) = \sum_{n \geq 0} ((H_{n+m} - H_m)^2 - (H_{n+m}^{(2)} - H_m^{(2)})) \binom{n+m}{n} u^n,$$

to conclude the assertion. \[ \square \]

**Proof of Theorem 1.1** We conclude the results after noting

$$\text{Mean}(C_n) = \beta_1(n),$$

$$\text{Var}(C_n) = \beta_2(n) - (\beta_1(n))^2 + \beta_1(n).$$  

\[ \square \]

### 3 Similar Partial Differential Functional Equations

We point out that following two examples from [KS97] can be analyzed using the method employed here:

1. Moments of total path length $L_n$ of a binary search tree built over a random set of $n$ keys can be extracted from the

$$\frac{\partial L(z,u)}{\partial z} = L^2(zu,u), \quad \frac{\partial L(0,u)}{\partial z} = 1,$$

where $L(z,u) = \sum_{n \geq 0} L_n(u) z^n$ is the bivariate generating function.

2. A *digital* search tree for which the bivariate generating function $L(z,u)$ satisfies

$$\frac{\partial L(z,u)}{\partial z} = L^2 \left( \frac{1}{2} z u, u \right),$$

with $L(z,0) = 1$. 

5
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