Euclidean resonance and a new type of nuclear reactions

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Abstract

The extremely small probability of quantum tunneling through an almost classical potential barrier may become not small under the action of the specially adapted nonstationary field. The tunneling rate has a sharp peak as a function of the particle energy when it is close to the certain resonant value defined by the nonstationary field (Euclidean resonance). Alpha decay of nuclei has a small probability since the alpha particle should tunnel through a very nontransparent nuclear Coulomb barrier. The incident proton, due to the Coulomb interaction with the tunneling alpha particle, plays the role of a nonstationary field which may result in Euclidean resonance in tunneling of the alpha particle. At the resonant proton energy, which is of the order of 0.2 MeV, the alpha particle escapes the nucleus and goes to infinity with no influence of the nuclear Coulomb barrier. The process is inelastic since the alpha particle releases energy and the proton gains it. This stimulation of alpha decay by a proton constitutes a new type of nuclear reaction.

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I. INTRODUCTION

A control of processes of quantum tunneling through potential barriers by external signals is a part of the field called quantum control which is actively developed now, see, for example, Ref. [1] and references therein. Excitation of molecules, when one should excite only particular chemical bonds [2–4], formation of programmable atomic wave packets [5], a control of electron states in heterostructures [6], and a control of photocurrent in semiconductors [7], are typical examples of control by laser pulses. A control of quantum tunneling through potential barriers is also a matter of interest, since tunneling is a part of many processes in nature. The computation of probability for a classically forbidden region has a certain peculiarity from the mathematical standpoint: there necessarily arises here the concept of motion in imaginary time or along a complex trajectory [8–10]. The famous semiclassical approach of Wentzel, Kramers, and Brillouin (WKB) [8] for tunneling probability can be easily reformulated in terms of classical trajectories in complex time. The method of complex trajectories is also applicable to a nonstationary case [11,12]. The method has been further developed in papers [13–17], where singularities of the trajectories in the complex plane were accounted for an arbitrary potential barrier (see also [18]). Recent achievements in the semiclassical theory are presented in Refs. [19–23].

Let us focus on the main aspects of tunneling under nonstationary conditions. When the electric field $\mathcal{E} \cos \Omega t$ acts on a tunneling particle of the initial energy $E$, it can absorb the quantum $\hbar \Omega$ (with the probability proportional to the small parameter $\mathcal{E}^2$) and tunnel after that in a more transparent part of the barrier with the higher energy $E + \hbar \Omega$. The pay in the absorption probability may be compensated by the probability gain in tunneling. In this case the system tends to absorb further quanta to increase the total probability of passing the barrier. This mechanism of barrier penetration is called photon-assisted tunneling. If $\hbar \Omega$ is not big, the process of tunneling, with the simultaneous multiquanta absorption, can be described in a semiclassical way by the method of classical trajectories in the complex time [13–17]. When a tunneling particle of the energy $E$ is acted by a short-time pulse, the tunneling probability is associated with the particle density carrying away in the outgoing wave packet. The particle energy after escape is $E + \delta E$, where the energy gain $\delta E = N\hbar \omega$, should be extremized with respect to the number of absorbed quanta $N$ and the energy $\hbar \omega$ of each quantum [24].

According to the perturbation theory, the both probabilities, of absorption and emission of quantum, are small being proportional to $\mathcal{E}^2$. After absorption, $\delta E$ is positive which enhances the total probability due to increase of the tunneling rate. After emission, $\delta E$ is negative and the particle should tunnel with a smaller energy in a less transparent part of the barrier; in this case there is no gain in the probability due to tunneling as for absorption. At first sight, tunneling in a nonstationary field cannot be assisted by an emission of quanta of this field since one should pay in probability twice. Thus, one can expect the double loss in probability due to the emission of quanta and the reduction of tunneling transparency. This conclusion is correct as soon as the nonstationary field is small and the perturbation theory is applicable.

Under increase of the nonstationary field the process of tunneling with emission of quanta becomes completely different compared to one, predicted on the basis of the perturbation theory [25]. The crucial role here plays the fact, that the non-perturbative wave function is
mainly determined by its big phase (real or imaginary). In this case there is no the double loss in probability. The quantum process is not simply reduced to separate emission and tunneling. However, the phase behavior can be interpreted in the way of enhancement of the total probability due to emission processes which competes now with the reduction of the tunneling transparency. The competition between the enhancement of the total probability due to emission and the reduction of it due to tunneling results in the unexpected effect: the total probability (defined as a particle density, carrying away by the outgoing wave packet) becomes not exponentially small for the certain particle energy $E_R$ in the well. This energy depends on parameters of the nonstationary field and in a vicinity of $E_R$ the probability sharply peaks as a function of energy. This reminds, formally, a resonant behavior and is called Euclidean resonance [25]. The energy $E_R$ can be called the resonant energy. In III this phenomenon is deduced from the analysis of the quantum mechanical phases using only simple physical arguments.

As well known, nuclear Coulomb barriers may be very significant in nuclear physics, playing a role of blockade for particle approach or escape [26,27]. The famous example of such a nuclear process is alpha decay of nuclei which has a small probability since the alpha particle should penetrate through a very non-transparent nuclear Coulomb barrier [26,27]. As any tunneling process, alpha decay can be influenced by a nonstationary field. The duration of artificially generated pulses (see, for example, [28]) is too long compared to the nucleus characteristic time of $10^{-21}$ s which makes impossible their influence on alpha decay. A role of a nonstationary field can be played by a moving charged particle which collides the decaying nucleus. For example, a proton of the energy of 1 Mev sweeps the tunneling region in, approximately, $10^{-21}$ s. An incident proton, moving towards a tunneling alpha particle, reduces its energy due to the Coulomb interaction between them. Since the energy is lost but not gained during tunneling, Euclidean resonance may be expected. As shown below, this happens. At the certain energy of the incident proton the simultaneous tunneling of alpha particle has not an exponentially small probability, according to Euclidean resonance.

In other words, due to interaction with an incident proton of the certain energy, the alpha particle escapes the nucleus and goes to infinity with no influence of the Coulomb barrier. This stimulation of alpha decay by a proton constitutes the new type of nuclear reaction. Normally, nuclear reactions involve strong interaction at nuclear distances. In this case, only the short range start of the alpha particle from the nucleus is due to strong interaction and the main physics occurs further, at the larger distance where strong forces do not act and only Coulomb effects are involved. This new type of nuclear reactions has a very resonant character with respect to an energy of the incident proton. As shown below, the typical resonance energy of a proton is in the range of 0.2 MeV. One should emphasize, that this resonance results from solely Coulomb effects in contrast, for example, to resonances in nuclear physics due to formation of compound nuclei.

II. PHOTON-ASSISTED TUNNELING

A penetration of a particle through a potential barrier is forbidden in classical mechanics. Only due to quantum effects the probability of passing across a barrier becomes finite and it can be calculated on the basis of WKB approach, which is also called the semiclassical
theory. The transition probability through the barrier, shown in Fig. 1, is

\[ W \sim \exp[-A_0(E)] \]  

(1)

where

\[ A_0(E) = \frac{2}{\hbar} \int dx \sqrt{2m[V(x) - E]} \]  

(2)

is the classical action measured in units of \( \hbar \). The integration goes under the barrier between two classical turning points where \( V(x) = E \). One can use the general estimate \( A_0 \sim V/\hbar \omega \), where \( V \) is the barrier height and \( \omega \) is the frequency of classical oscillations in the potential well. A semiclassical barrier relates to a big value \( V/\hbar \omega \gg 1 \).

What happens when the static potential barrier \( V(x) \) is acted by a weak nonstationary electric field \( E(t) \)? In this case there are two possibilities for barrier penetration: (i) the conventional tunneling, which is not affected by \( E(t) \), shown by the dashed line in Fig. 2(a), and (ii) an absorption of the quantum \( \hbar \Omega \) of the field \( E(t) \) and subsequent tunneling with the new energy \( E + \hbar \Omega \). The latter process is called photon-assisted tunneling. The total probability of penetration across the barrier can be schematically written as a sum of two probabilities

\[ W \sim \exp\left(-\frac{V}{\hbar \omega}\right) + \left(\frac{aE_\Omega}{\hbar}\right)^2 \exp\left(-\frac{V - \hbar \Omega}{\hbar \omega}\right) \]  

(3)

where \( E_\Omega \) is the Fourier component of the field \( E(t) \). The second term in Eq. (3) relates to photon-assisted tunneling and it is a product of probabilities of two quantum mechanical processes: absorption of the quantum \( \hbar \Omega \) and tunneling through the reduced barrier \( V - \hbar \Omega \). The length \( a \) is a typical barrier extension in space. When the frequency is high \( \Omega > \omega \), the second term starts to dominate at sufficiently small nonstationary field \( (aE_\Omega/\hbar)^2 > \exp(-\Omega/\omega) \). This is a feature of tunneling processes. Normally a nonstationary field starts to dominate at bigger amplitudes \( (aE_\Omega/\hbar)^2 > 1 \). When the second term in Eq. (3) exceeds the first one, further orders of perturbation theory should be accounted which correspond to the multiple absorption, shown in Fig. 2(b).

Let us specify a shape of a field pulse in the form

\[ E(t) = \frac{\mathcal{E}}{1 + t^2/\theta^2} \]  

(4)

with the Fourier component \( E_\Omega = \pi \mathcal{E} \theta \exp(-\theta|\Omega|) \). In this case, in addition to the steady particle flux from the barrier, an outgoing wave packet is created which carries away the certain particle density. Then the probability \( W \) of the transition through the process of absorption of \( N \) quanta and subsequent tunneling with the higher energy \( E + N\hbar \Omega \), shown in Fig. 2(b), is

\[ W \sim \left(\frac{aE_\Omega}{\hbar}\right)^{2N} \exp[-A_0(E + N\hbar \Omega)] = \left(\frac{\pi}{\hbar} \theta a \mathcal{E}\right)^{2N} \exp(-A) \]  

(5)

where

\[ A = \frac{2\theta}{\hbar} \delta E + A_0(E + \delta E) \]  

(6)
Here the total energy transfer $\delta E = N\hbar \Omega$ is introduced ($\Omega > 0$). The maximum transition probability through the barrier is determined by some finite value of $\delta E$, which provides a minimum of $A$ and is defined by the condition $\partial A(E + \delta E)/\partial \delta E = 0$. An existence of such a minimum is possible if a grow up of small $\delta E$ reduces $A$, that is, under the condition $2\theta < \hbar | \partial A_0(E)/\partial E |$ of sufficiently short pulses. In other words, sufficiently short and not very small pulses (however, still much smaller than the static barrier field) strongly enhance tunneling by photon assistance.

Eq. (5) omits some details and it is rather an illustration of a tunneling mechanism with quanta absorption. For example, the accurate perturbation theory starts with the linear $\mathcal{E}$-term. The exact nonperturbative theory, which is a generalization of the conventional semiclassical approach, results in the same $A$. The semiclassical approach is also sensitive to the sign of $\mathcal{E}$ since the classical energy transfer is determined by $\mathcal{E}(t) \partial x/\partial t$, where $x(t)$ is a classical particle trajectory. The strong photon-assisted tunneling exists only at positive $\mathcal{E}$.

### III. EUCLIDEAN RESONANCE

Besides the absorption of quanta in Fig. 2, also the emission is possible, shown in Fig. 3. This process is provided by negative frequencies $\Omega < 0$ and the energy transfer is also negative $\delta E < 0$. In this case Eq. (5) gives $A = 2\theta |\delta E|/\hbar + A_0(E - |\delta E|)$ which does not correspond to any extreme since $\partial A/\partial |\delta E| > 0$. So, on the basis of perturbation theory, one can conclude that emission processes cannot enhance a barrier penetration at least for the pulse amplitude on the border of applicability of the perturbation theory $a\mathcal{E} \lesssim \hbar/\theta$.

May an emission process enhance a tunneling rate when the signal amplitude is not small $\hbar/\theta < a\mathcal{E}$?

#### A. Phase connection

At the big pulse amplitude $\mathcal{E}$ its contribution to the classical action, generally speaking, is big, compared to Planck’s constant,

$$ a\mathcal{E}\theta \gg \hbar $$

and the wave function $\psi \sim \exp(i\chi)$ is mainly determined by its big phase $\chi$, which can be imaginary as well. Another condition of a big phase is a slow varying pulse

$$ \frac{\hbar}{V} \ll \theta $$

since a big phase should be built up during a long time. Here $V$ is the barrier height and, therefore, $\hbar/V$ is some intrinsic time of the problem. One has to note, that within the conditions (7) and (8) the pulse amplitude can be still less compared to the static field of the barrier $V/a$.

Suppose $x_1$ and $x_2$ to relate to the classical turning points (1) and (2) in Fig. 3. At $t \to \pm \infty$, when $\mathcal{E}(t) = 0$, there is a conventional tunneling through the barrier. For a symmetric in time pulse $\mathcal{E}(t)$ the modulus of the wave function of the outgoing particle
\(|\psi(x_2, t)|\) has an extreme (maximum) value at \(t = 0\) which relates to the maximum amplitude of the outgoing wave packet at \(t > 0\). According to Feynman [29], an extreme wave function corresponds to a classical trajectory of the particle connecting the points \(\{x_1, 0\}\) and \(\{x_2, 0\}\). But in the present situation there are no classical trajectories under a barrier.

However, one can find a connection between two constants \(\psi(x_1, 0)\) and \(\psi(x_2, 0)\) without an exact solution of Schrödinger equation. The main point of this procedure is the possibility to define the wave function mainly by its big phase (real or imaginary) which is true under the conditions (7) and (8). Then one can consider a particle state at any coordinate as in classical mechanics. Since the moment \(t = 0\) corresponds to the extreme situation of maximum output, one can look for an alternative extreme way to connect phases of \(\psi(x_1, 0)\) and \(\psi(x_2, 0)\). Let us find some formal path from (1) to (2) which relates to an extreme total phase with respect to the energy of the final state (2).

Suppose, that the particle state \((i)\) at \(x = x_i\) is chosen with the same energy \(E - |\delta E|\) as at the point (2) in Fig. 4. The particle from the position \((i)\) can tunnel to the position (2) and the acquired phase is imaginary

\[
\psi(x_2, 0) \sim \psi(x_i, 0) \exp \left( -\frac{1}{2} \frac{A_0(E - |\delta E|)}{\Delta_0} \right) \tag{9}
\]

On the other hand, the particle can absorb \(N = |\delta E|/\hbar \Omega\) quanta, as in Fig. 4(a), and go to the state (1) where \(\psi(x_1, 0) \sim \psi(x_i, 0) (E_\Omega)^N\). This also leads to an imaginary acquired phase

\[
\psi(x_1, 0) \sim \psi(x_i, 0) \exp \left( -\frac{\theta}{\hbar} |\delta E| \right) \tag{10}
\]

We use here the expression for \(E_\Omega\). Analogously, the particle can emit \(N\) quanta and go from (1) to \((i)\), as in Fig. 4(b). This leads to another connection of (1) and \((i)\)

\[
\psi(x_i, 0) \sim \psi(x_1, 0) \exp \left( -\frac{\theta}{\hbar} |\delta E| \right) \tag{11}
\]

According to Eqs. (9) - (11), the connection between (1) and (2) can be written in the form

\[
\psi(x_2, 0) \sim \exp \left( -\frac{1}{2} A_0(E - |\delta E|) \right) \left[ a \exp \left( \frac{\theta}{\hbar} |\delta E| \right) + b \exp \left( -\frac{\theta}{\hbar} |\delta E| \right) \right] \psi(x_1, 0) \tag{12}
\]

where \(a\) and \(b\) are constants. Only the first term in Eq. (12) provides an extreme of the total phase and, therefore, one should put \(a = 1\) and \(b = 0\). This is an alternative extreme way of phase connection. The relation (12) corresponds to a formal (no direct analogy with the exact solution) path connecting the phases (1) and (2). The extreme (at the moment \(t = 0\)) transition probability from (1) to (2) is determined by the found extreme phase difference in (12)

\[
W(1 \rightarrow 2) \sim \left| \frac{\psi(x_2, 0)}{\psi(x_1, 0)} \right|^2 \sim \exp [-A(|\delta E|)] \tag{13}
\]
where
\[ A(|\delta E|) = A_0(E - |\delta E|) - \frac{2\theta}{\hbar}|\delta E| \]  
(14)

As also follows from the solution of Schrödinger equation, \( A(|\delta E|) \) weakly depends on the pulse amplitude \( \mathcal{E} \) under the conditions (7) and (8). The optimum energy transfer \(|\delta E_0|\) should be determined from the extreme condition \( \partial A(|\delta E|)/\partial |\delta E| = 0 \), which reads
\[ \frac{2\theta}{\hbar} = -\frac{\partial A_0(E - |\delta E|)}{\partial E} \]  
(15)

and determines the extreme value \( A = A(|\delta E_0|) \). Again, the physical transition probability at \( t = 0 \), \( W \sim \exp(-A) \), can be considered either as an extreme in \( t \) of the exact solution or as an extreme in \( |\delta E| \) in the above phase connection procedure.

The phase connection, illustrated in Fig. 4 is applicable for big pulses \( \hbar/\theta < a \mathcal{E} \) when the wave function is mainly defined by its big phase (real or imaginary) and the phase difference between (1) and (i) is of the opposite sign compared to one between (i) and (1). This phase connection, as follows from the solution of Schrödinger equation, is applicable in our case of decay of a metastable state. But it does not mean, that its applicability is automatically valid in other situations with a big phase. For example, a penetration of an incident particle into the potential well is not described by this method.

**B. Resonance conditions**

An applicability of the phase connection, besides the conditions (7) and (8), is restricted by the inequality
\[ \exp(-A) \ll 1 \]  
(16)

A pulse amplitude \( \mathcal{E} \) should be negative, since the classical energy transfer is determined through the classical trajectory \( x(t) \) as \( \mathcal{E}(t)\partial x/\partial t \). \( A \) is determined by the particle energy \( E \) in the well and by the pulse duration \( \theta \). Suppose \( \theta \) to be fixed and the particle energy \( E \) to vary. Then the optimum energy transfer \(|\delta E_0(E)|\), determined by Eq. (15), is a function of \( E \) and one can define the certain energy \( E_R(\theta) \) as a solution of the equation
\[ A_0 [E - |\delta E_0(E)|] - \frac{2\theta}{\hbar}|\delta E_0(E)| = 0 \]  
(17)

At the particle energy close to \( E_R \), \( A \simeq 2\theta[E_R - E]/\hbar \) and the peak in tunneling probability at \( t = 0 \), related to the particle density in the outgoing wave packet, is
\[ W \sim \exp\left(-\frac{2\theta}{\hbar}[E_R(\theta) - E]\right) \]  
(18)

The formal applicability of these relations is \( W \ll 1 \), nevertheless, the probability peak can dramatically grow up (upon approaching \( E_R \)), for example, from \( 10^{-37} \) to \( 10^{-2} \). The energy \( E_R \) has the order of magnitude of the barrier height \( V \) and, hence, \( 2\theta(E_R - E)/\hbar \sim (\theta V/\hbar)(E_R - E)/E_R \). This means, that the tunneling probability at \( t = 0 \) has a sharp peak,
like a resonance, as a function of the particle energy $E$ near $E_R$. The effect may be called Euclidean resonance when $E_R$ plays a role of the resonant energy. The origin of the word “Euclidean” is explained below.

Euclidean resonance also can be treated in another way. Suppose the energy level in the well to be fixed. Then one can adjust the pulse parameter $\theta$ to meet the condition $E_R = E$ when the barrier becomes almost transparent at $t = 0$ for the energy $E$. Note, that for other particle energies the barrier remains low-transparent.

In contrast to photon-assisted tunneling, which has a connection with the perturbation theory (see Fig. 2), the phenomenon of Euclidean resonance is completely non-perturbative. One can construct some analogy of Euclidean resonance by adding the narrow potential well $-v(b)\delta(x - b)$ to the potential $V(x)$ in Fig. 1. The distance $b$ is within the under-barrier region and the positive coefficient $v(b)$ is chosen to get the same energy level $E$ in the $\delta$-well as in the main well (resonance tunneling). Suppose $b$ to be close to the main well in Fig. 1 and the peak of the wave function at $x = b$ is of the same order as in the main well. Let us move $b$ slowly (compared to the time $\hbar/V$) away from the main well to get the big peak of the wave function at some point under the barrier. After this $v(b)$ is switched off fast and the remaining distribution, which is not exponentially small, goes partly outside the barrier. So, this non-stationary mechanism provides the outgoing wave packet which is not exponentially small. As in Euclidean resonance, two issues are important in this example: (i) the nonstationary potential should be not small and (ii) it should be very precisely chosen to get the condition of resonance tunneling, otherwise, the peak of the wave function inside the barrier would be exponentially small.

**IV. AN EXAMPLE OF EUCLIDEAN RESONANCE**

Let us consider an electron emission from a metal, left in Fig. 5, to the vacuum due to the applied electric field $\mathcal{E}_0 + \mathcal{E}(t)$ where $\mathcal{E}_0$ is a constant. The energy $E$ is supposed not to be above the Fermi level. An electron emission occurs by tunneling through the barrier $V(x) - x\mathcal{E}(t)$ where the pulse acts only at $x > 0$. The static barrier is $V(x) = V - x\mathcal{E}_0$ at $x > 0$ and $V(x) = 0$ at $x < 0$.

The conventional WKB action (2) has the form

$$A_0(E) = \frac{4}{3\hbar} (V - E)\tau_{00}(E)$$

where

$$\tau_{00}(E) = \frac{1}{\mathcal{E}_0} \sqrt{2m(V - E)}$$

The relation (15) turns to $\tau_{00}(E - |\delta E|) = \theta$ and the optimum energy transfer is

$$|\delta E_0| = \frac{\theta^2 \mathcal{E}_0^2}{2m} - V + E$$

Eq. (17) defines the resonance energy

$$E_R(\theta) = V - \frac{\theta^2 \mathcal{E}_0^2}{6m}$$
One can note, that, under the resonance condition, the input energy \( E_R \) is connected to the output one \( E_R - |\delta E| \) as \( 3(V - E_R) = V - (E_R - |\delta E|) \).

These formulae relate to the Lorentzian shape of a pulse (4). What happens for the nonstationary field \( \mathcal{E}(t) = \mathcal{E} \cos \Omega t \) or of some other shape? The above arguments, based on general physical principles, are not sufficient to answer this question and a more sophisticated treatment is required. This is considered in V.

V. TRAJECTORIES IN IMAGINARY TIME

According to Feynman [29], when the phase of a wave function is big, it can be expressed through classical trajectories of the particle. But in our case there are no conventional trajectories since a classical motion is forbidden under a barrier. Suppose a classical particle to move in the region to the right of the classical turning point (2) in Fig. 5 and to reach the point (2) at \( t = 0 \). Then, close to the point (2), \( x(t) = x_2 + ct^2 \) \((c > 0)\) and there is no a barrier penetration as at all times \( x(t) > x_2 \). Nevertheless, if \( t \) is formally imaginary, \( t = i\tau \), the penetration becomes possible since \( x(i\tau) = x_2 - c\tau^2 \) is less than \( x_2 \). Therefore, one can use classical trajectories in imaginary time to apply Feynman’s method to tunneling.

A. Newton’s equation and classical action

A classical trajectory has to satisfy Newton’s equation in imaginary time

\[
m \frac{\partial^2 x}{\partial \tau^2} - \frac{\partial V(x)}{\partial x} = -\mathcal{E}(i\tau)
\]  

(23)

where \( V(x) \) is the static barrier in Fig. 5. The classical turning point (2) in Fig. 5 is reached at \( \tau = 0 \) with the initial condition

\[
\left. \frac{\partial x}{\partial \tau} \right|_{\tau=0} = 0 \quad \text{point (2)}
\]  

(24)

The point (1) is reached at \( t = i\tau_0 \) when

\[
x(i\tau_0) = 0 \quad \text{point (1)}
\]  

(25)

The “time” \( \tau_0 \) is expressed through the particle energy \( E \) before the barrier where the nonstationary field does not act

\[
E = \frac{m}{2} \left( \frac{\partial x}{\partial \tau} \right)^2_{\tau_0} + V(0)
\]  

(26)

The conditions (24), (25), and (26) define the solution \( x(i\tau) \) of Eq. (23). This solution, in turn, defines the extreme (in time) transition probability related to the particle density in the outgoing wave packet

\[
W \sim \exp(-A)
\]  

(27)
where $A$ is the classical action in units of Planck’s constant

$$A = \frac{2}{\hbar} \int_{\tau_0}^{\tau_0} d\tau \left[ \frac{m}{2} \left( \frac{\partial x}{\partial \tau} \right)^2 + V(x) - xE(i\tau) - E \right]$$

(28)

$\tau_0$ can be treated as “time” of motion under a barrier. Eq. (28) holds for any barrier which is zero at $x < 0$. Without a nonstationary pulse, $E = 0$, $\tau_0$ coincides with its static value

$$\tau_{00} = \sqrt{\frac{m}{2}} \int dx \sqrt{V(x) - E}$$

(29)

and the action (28) turns to $A_0(E)$ (2). The integration in Eq. (29) goes between classical turning points where $V(x) = E$. According to classical mechanics,

$$2\tau_{00} = -\frac{\partial A_0(E)}{\partial E}$$

(30)

**B. An example of imaginary trajectories**

Let us consider the particular pulse (4). In imaginary time $E(i\tau) = 1/(1 - \tau^2/\theta^2)$ diverges at $\tau = \theta$ and this sets $\tau_0$ in Eq. (28) close to $\theta$. A difference between $\tau_0$ and $\theta$ is small at small pulse amplitude $aE \ll V$. For this reason, the “time” interval $(\tau_0 - \theta)$ near $\tau_0$, when $E(i\tau)$ is not small, weakly contributes to the action. During the short “time” $(\tau_0 - \theta)$ the particle energy reduces (because $E < 0$) down to $(E - |\delta E|)$, so that

$$\tau_{00}(E - |\delta E|) \simeq \theta$$

(31)

and the particle motion at $0 < \tau < \theta$ can be considered to be free

$$A = \frac{2}{\hbar} \int_0^{\theta} \left[ \frac{m}{2} \left( \frac{\partial x}{\partial \tau} \right)^2 + V(x) - E \right]$$

(32)

Adding and subtracting the energy transfer $|\delta E|$ in the right-hand side of Eq. (32) and using the equation (31), one can arrive at

$$A = A_0(E - |\delta E|) - \frac{2\theta}{\hbar}|\delta E|$$

(33)

With account of Eqs. (30) and (31), it is obvious, that Eqs. (14) and (33) give the same result.

**C. Why a small pulse may enhance tunneling**

It is already been shown in a simple way in II, that the effective nonstationary field in tunneling processes enhances compared to other ones. This enhancement is clearly seen in the method of trajectories in imaginary time where an influence of a pulse becomes substantial under the approximate condition $E(i\tau) \sim E_0$. Here $E_0 \sim V/a$ is the field of a
static barrier. In the case of the pulse (4), \( E(i\tau) \sim E\theta/(\tau - \theta) \) is enhanced compared to \( E \) since one can choose \( \tau \) very close to \( \theta \). But this enhancement is not unlimited due to the semiclassical condition of slow varying in time \( (\tau - \theta) \ll \hbar/V \). Thus, the condition on the pulse amplitude coincides with (7) or it can be written as

\[
\frac{1}{A_0} < \frac{E}{E_0} \tag{34}
\]

Since \( A_0 \sim V\theta/\hbar \) is big, tunneling can be strongly influenced by a pulse amplitude \( E \) which is less than the static field of the barrier. This statement is true for more general forms of non-stationary fields. For example, the monochromatic \( E \cos \Omega t \) and the Gaussian \( E \exp(-\Omega^2t^2) \) pulses become exponentially big in imaginary time and their stimulation of tunneling occurs at small amplitudes \( E < E_0 \) as well.

**D. Euclidean resonance in real time**

The metric in relativity \( x^2 + y^2 + z^2 - c^2t^2 \) is Euclidean in imaginary time \( t = i\tau \), since the all coefficient become positive, and the action (28) or (32) is called Euclidean action. Extending this analogy, one can name the present phenomenon Euclidean resonance.

The above approach is valid when \( \exp(-A) \ll 1 \), but upon reduction of \( A \) this condition may be violated. When \( \exp(-A) \) becomes no small one should account further contributions \( \exp(-2A), \exp(-3A) \), etc. This is equivalent to an account of multi-instanton contributions.

For the particular barrier \( V(x) = V - xe_0 \) in presence of a non-stationary field one can build up a bridge between trajectories in imaginary time and the solution of Schrödinger equation in real time. One can find this solution in the form \( \psi(x,t) = a(x,t) \exp[iS(x,t)/\hbar] \), where \( S(x,t) \) is the classical action and one can obtain any correction in \( \hbar \) in the pre-exponent \( a(x,t) \). The result for the imaginary part of the action is shown schematically in Fig. 6. Without a pulse, the solution to the left of the point “exit”, \( \psi = c_1\psi_1 + c_2\psi_2 \) consists of the dominant, \( \psi_1 \sim \exp(iS_1/\hbar) \), and the sub-dominant, \( \psi_2 \sim \exp(iS_2/\hbar) \), solutions. \( \psi_2(x = 0) \sim \psi_1(x = 0) \exp(-A_0) \) is exponentially small, at the point “exit” \( \psi_1 \sim \psi_2 \sim \exp(-A_0/2) \), and to the right of “exit” there is only an outgoing wave of the amplitude \( \exp(-A_0/2) \). This is a picture of decay of the metastable state, localized near \( x = 0 \), through a static barrier.

When \( E(t) \) is not zero, the third solution, \( \psi_3 \sim \exp(iS_3/\hbar) \), appears which is shown by the curve (3) in Fig. 6. The maximum of this solution

\[
\psi_3[x_{cl}(t),t] \sim c_3 \exp(-A/2) \tag{35}
\]

(\( A \) is defined by Eq. (28)) is reached at the classical trajectory \( x_{cl}(t) \) in real time, when \( x_{cl}(\pm\infty) = \infty \) and \( x_{cl}(0) \) is the minimum value. We omit not strong effects of quantum smearing of the wave packet. At a fixed moment of time \( t < 0 \) one can find \( \psi_3(x,t) \) for all \( x \) using the generalized semiclassical approach which nowhere breaks down upon sweeping over all \( x \). In this case, \( \psi_3 \) is an independent solution and one should put \( c_3 = 0 \) at \( t < 0 \) since there is no incoming wave. Close to the moment \( t = 0 \) the solution \( \psi_2 \) and \( \psi_3 \) get a tendency to merge at some point, circled by the dashed curve in Fig. 6. In this region \( \partial^2S/\partial x^2 \), calculated semiclassically, becomes big and the semiclassical approximation breaks down in a vicinity of the circled point. This means, that within the short (non-semiclassical)
time interval \( t \sim \hbar/V \) the solution \( \psi_3 \) is formed. Then, at \( t > 0 \), the semiclassical solution recovers at all \( x \) again, but now \( c_3 \) is not zero, which relates to an outgoing particle. From a semiclassical point of view, there is a jump (since a semiclassical approximation does not resolve a short time) from zero to one of the coefficient at the third solution at the moment \( t = 0 \).

Despite of that the above solution is obtained analytically for the barrier \( V(x) = V - xE_0 \), the presented scenario of stimulation of tunneling by a nonstationary field holds qualitatively for a general semiclassical barrier. The jump of the coefficient occurs both in photon-assisted tunneling and in Euclidean resonance.

VI. EUCLIDEAN RESONANCE AND NUCLEAR REACTIONS

In VI we apply the developed ideas of stimulation of tunneling to nuclear reactions where tunneling through a Coulomb barrier is a substantial part of the process. The role of nonstationary field is played now by a charged incident particle.

A. Alpha decay of nuclei

According to Gamov [26,27], alpha decay of nuclei, is described by tunneling of alpha particle through the Coulomb barrier. The potential energy, as a function of the distance \( R \) between alpha particle and the nucleus, is shown in Fig. 7, where the Coulomb tail \( \alpha_M/R \) \((\alpha_M = 2(z_0 - 2)e^2)\) sharply drops at the nuclear size \( x_0 \). For the alpha decay

\[
\frac{235}{92}U \rightarrow \frac{231}{90}Th + \alpha
\]

\( z_0 = 92, E = 4.678 \text{ Mev}, \) and, according to the liquid drop model,

\[
x_0 = 1.2 \left[ (231)^{1/3} + 4^{1/3} \right] \times 10^{-13} \text{cm} \simeq 0.92 \times 10^{-12} \text{cm}
\]

The WKB tunneling rate is

\[
W \sim \exp \left[ -A_M(E, L_M) \right]
\]

where the action in units of Planck’s constant is

\[
A_M(E, L_M) = \frac{\sqrt{8M}}{\hbar} \int_{x_0}^{R_e} dR \sqrt{\frac{\alpha_M}{R} + \frac{L_M^2}{2MR^2} - E}
\]

\( M \) is the mass of alpha particle and the classical exit point \( R_e \) is determined by zero of the square root. \( L_M \gg \hbar \) is the angular momentum. Further we consider parameters under the condition

\[
\frac{L_M^2}{2Mx_0^2} \sim \frac{\alpha_M}{x_0} \gg E
\]

The action can be written in the form

\[
A_M(E, L_M) = \frac{\pi \alpha_M}{\hbar} \sqrt{\frac{2M}{E}} \left[ 1 - \frac{4}{\pi} \sqrt{\frac{Ex_0}{\alpha_M}} f(p) \right]
\]
where
\[ f(p) = \sqrt{1 + p} - \frac{\sqrt{p}}{2} \ln \left[ 1 + 2p + 2\sqrt{p(1 + p)} \right] ; \quad p = \frac{L_M^2/2Mx_0^2}{\alpha_M/x_0} \] (42)

\( p \) is the ratio of the centrifugal energy and the Coulomb one at the nucleus radius. The imaginary time of motion under the barrier
\[ \tau_M(E) = \frac{1}{2} \frac{\partial A_M(E, L_M)}{\partial E} \approx \frac{\pi \alpha_M}{2hE} \sqrt{\frac{M}{2E}} \] (43)
weakly depends on the angular momentum \( L_M \) under conditions (40).

According to the semiclassical applicability, the correction to \( A_M \) due to the second term in Eq. (41) should be much bigger then one, that is
\[ \sqrt{\frac{\hbar^2}{32M \alpha_M}} \ll \sqrt{x_0} \] (44)
This coincides with the conventional WKB condition in Coulomb field. In this problem of alpha decay the condition (44) reads as \( 0.01 \ll 1 \). Since the parameter \( p \sim 1 \), the angular momentum is big
\[ \frac{L_M}{\hbar} \sim \sqrt{\frac{M \alpha_M}{\hbar^2 x_0}} \gg 1 \] (45)
As one can see, semiclassical conditions are fulfilled well for alpha decay.

**B. An incident proton**

What happens to alpha decay when a charged particle (proton, for example) is stopped by the Coulomb field of the nucleus which is ready to emit alpha particle? The classical motion of the proton in the Coulomb field of uranium nucleus is described by the equation
\[ t = \sqrt{\frac{m}{2}} \int dr \left( \varepsilon - \frac{L_m^2}{2mr^2} - \frac{\alpha_m}{r} \right)^{-1/2} \] (46)
where \( m \) is proton mass, \( \alpha_m = z_0 e^2 \), \( L_m \) is the proton angular momentum, and \( \varepsilon \) is the proton energy. The classical trajectory is shown in Fig. 8, where the shortest distance \( r_e \) between the proton and the nucleus (the classical turning point) is given by the zero of the square root in Eq. (46). At \( r < r_e \) the time \( t \) in Eq. (46) becomes imaginary and, starting at the point \( r_e \), the proton reaches the nucleus (the dashed line in Fig. 8) at the “moment” \( t = i \tau_m \). Analogously to Eq. (43), the expression for \( \tau_m \) is
\[ \tau_m(\varepsilon) = \frac{\pi \alpha_m}{2h\varepsilon} \sqrt{\frac{m}{2\varepsilon}} \] (47)
C. Alpha particle meets proton

When the uranium nucleus emits the alpha particle, the additional interaction energy

\[ V_{\text{int}} = \frac{\alpha_{\text{int}}}{|\vec{R}(i\tau) - \vec{r}(i\tau)|} \]  

(48)

where \( \alpha_{\text{int}} = 2e^2 \), results in a connection of motions of the alpha particle and the proton. Now there is a cooperative motion of two particles in imaginary time which starts at \( \tau = 0 \), with zero radial velocities, and terminates at the nucleus at the certain under-barrier time \( \tau_0 \). The total energy \( (E + \varepsilon) \) of two particles conserves. \( \varepsilon \) is the energy of the incident proton and \( E \) is the energy of the alpha particle close to the nucleus at \( \tau = \tau_0 \). The interaction energy is always small excepting a narrow vicinity of the moment \( \tau = \tau_0 \) when two particles are close to the nucleus where \( |\vec{R} - \vec{r}| \sim x_0 \). In the small vicinity of \( \tau_0 \) the interaction redistributes energies, so that the alpha particle leaves the interaction region with the energy \( E - |\delta E| \) and the proton energy becomes \( \varepsilon + |\delta E| \). The major part of the interval \( (\tau_0, 0) \), excepting a small vicinity of \( \tau_0 \), the particles move independently with the redistributed energies, reach the point \( \tau = 0 \), and go to infinity in real time having the same energies \( E - |\delta E| \) and \( \varepsilon + |\delta E| \). The interaction (48) contributes weakly to the action since it works during a short time.

The both trajectories are shown in Fig. 9, where the exit points are \( R_e = \alpha M / (E - |\delta E|) \) and \( r_e = \alpha m (\varepsilon + |\delta E|) \). As \( x_0 \) is small, the both curves are about to merge at \( \tau = \tau_0 \), otherwise the interaction at this region is not effective. This requires the condition

\[ \tau_M (E - |\delta E|) = \tau_m (\varepsilon + |\delta E|) \approx \tau_0 \]  

(49)

With Eqs. (43) and (47) the condition (49) reads

\[ \frac{\varepsilon + |\delta E|}{E - |\delta E|} = \left( \frac{m\alpha^2}{M\alpha^2} \right)^{1/3} \]  

(50)

D. Cooperative motion of alpha particle and proton

The cooperative motion of the alpha particle and the proton relates to the Euclidean action

\[ \tilde{A} = \frac{2}{\hbar} \int_0^{\tau_0} d\tau \left[ \frac{M}{2} \left( \frac{\partial \vec{R}}{\partial \tau} \right)^2 + \frac{\alpha M}{R} + \frac{m}{2} \left( \frac{\partial \vec{r}}{\partial \tau} \right)^2 + \frac{\alpha m}{r} + \frac{\alpha_{\text{int}}}{|\vec{R} - \vec{r}|} - E - \varepsilon \right] \]  

(51)

The classical trajectory satisfies Newton’s equation resulting from a minimization of \( \tilde{A} \) with the initial conditions

\[ \frac{\partial R_x}{\partial \tau} \bigg|_0 = \frac{\partial r_x}{\partial \tau} \bigg|_0 = 0 ; \quad R_y(0) = r_y(0) = 0 ; \quad MR_x \frac{\partial R_y}{\partial \tau} \bigg|_0 = L_M ; \quad mr_x \frac{\partial r_y}{\partial \tau} \bigg|_0 = L_m \]  

(52)

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where $L_M$ and $L_m$ are the angular momenta of particles which conserve for all $\tau$ excepting a close vicinity of $\tau_0$. The another condition is

$$R(\tau_0) = r(\tau_0) = x_0$$

(53)

and $\tau_0$ relates to the energy of alpha particle

$$E = \left[-\frac{M}{2} \left(\frac{\partial \vec{R}}{\partial \tau}\right)^2 + \frac{\alpha_M}{R}\right]_{\tau_0}$$

(54)

In Eqs. (51)-(54) the vectors are defined as $\vec{R} = \{R_x, iR_y\}$ and $\vec{r} = \{r_x, ir_y\}$ since in imaginary time $y$-components are imaginary. This can be seen in the proton motion in the Coulomb field of the nucleus (with no alpha particle) under the condition analogous to (40) for proton

$$r_x = \left(r + \frac{L_m^2}{m\alpha_m}\right) \left(1 + \frac{2eL_m^2}{m\alpha_m}\right)^{-1/2}; \quad r_y = \sqrt{\frac{L_m^2}{2m\varepsilon} + \frac{r\alpha_m}{\varepsilon} - r^2 \left(1 + \frac{m\alpha_m}{2\varepsilon L_m^2}\right)^{-1/2}}$$

(55)

The dependence $r(i\tau)$ is given by Eq. (46) with $t = i\tau$. As follows from Eq. (55), $r_x^2 - r_y^2 = \gamma^2$. Fall of the proton to the nucleus means that $r(i\tau) = 0$, but $r_x(i\tau)$ and $r_x(i\tau)$ separately are not zero.

In contrast to the big energy transfer $|\delta E|$, the interaction (48) results in a very small transfer of angular momentum $\delta L_m$ between alpha particle and proton. On the basis of the classical relation

$$\frac{\partial \delta L_m}{\partial \tau} = \frac{\alpha_{int}}{|\vec{R} - \vec{r}|^3} (r_x R_y - r_y R_x)$$

(56)

with the estimate $R_x \sim r_x \sim |\vec{R} - \vec{r}| \sim \alpha_m/\varepsilon$ and Eq. (55) for the $y$-component, one can easily deduce, that $\delta L_m/L_m \sim \alpha_{int}/\alpha_m \ll 1$. As shown below, the energy transfer $|\delta E|$ is not small since it is determined by by small $|\vec{R} - \vec{r}| \sim x_0$. The proton can change its angular momentum solely by interaction with the nucleus. We consider only spherically symmetric nuclei and the proton angular momentum $L_m$ conserves. The angular momentum $L_M$ of the alpha particle is determined by the interaction with the nucleus and conserves for all $\tau$.

Now it is possible to calculate $\hat{A}$ in Eq. (51). A contribution to $\hat{A}$ from the interaction part $V_{int}$ is small, almost all “time” $\tau$ the alpha particle conserves its energy $E - |\delta E|$, and the same is true for the proton with the energy $\varepsilon - |\delta E|$. For this reason, one can write $\hat{A}$ in Eq. (51) as a sum of two actions of free particles

$$\hat{A} = A_M(E - |\delta E|, L_M) + A_m(\varepsilon + |\delta E|, L_m)$$

(57)

where $A_m$ is determined by Eq. (41) with the substitution $M \rightarrow m$. $|\delta E|$ obeys Eq. (50), $L_m$ is given by the angular momentum of the incident proton, and $L_M$ is obtained by the alpha particle from the nucleus. We do not distinguish in $\alpha_m = z_0 e^2$ between $z_0$ and $(z_0 - 2)$ as the correction is of the same order as the small interaction.

Since the energy transfer $|\delta E|$ is determined by small $|\vec{r} - \vec{r}| \sim x_0$, it strongly depends on angular momenta $L_M$ and $L_m$ which set a smallest distance between particles. $L_M$ and $L_m$ should be chosen to get the energy transfer $|\delta E|$ obeyed Eq. (50). We discuss this below.
E. Phase connection

The action $\hat{A}$ relates to the phase connection

$$\psi_2 \sim \psi_1 \exp \left( -\frac{\hat{A}}{2} \right)$$

(58)

of two states, $(i)$ and $(2)$, shown in Fig. 10, where $(2)$ is the physical final state of the reaction. The state $(i)$ consists of the nucleus $^{231}_{90}$Th with the alpha particle and proton close to it. $(1)$ is the physical initial state with the incident proton. The phase connection between the states $(i)$ and $(1)$ reads

$$\psi_1 \sim \psi_1 \exp \left[ -\frac{1}{2}A_m(\varepsilon, L_m) \right]$$

(59)

The dominant contribution to the imaginary phases comes from motion in the Coulomb field. By means of Eqs. (58) and (59) one can obtain the probability $W$ of the reaction

$$^{235}_{92}U + p \rightarrow ^{231}_{90}Th + \alpha + p$$

(60)

in the form

$$W \sim \left| \frac{\psi_2}{\psi_1} \right|^2 \sim \exp -A$$

(61)

where

$$A = A_M(E - |\delta E|, L_M) + A_m(\varepsilon + |\delta E|, L_m) - A_m(\varepsilon, L_m)$$

(62)

This procedure of phase connection is same as in III.

One can use for the action (62) the approximation (41), which is used in the derivation of Eq. (50). In this approximation $A$ does not depend on $L_m$ due to the cancellation in the last two terms of Eq. (62). This means, that $L_m$ can vary to adjust the energy transfer $|\delta E|$ to one given by Eq. (50) with no impact on $A$. In this situation the minimum $A$ is reached at $L_M = 0$. Finally, the action in Eq. (62) takes the form

$$A = \frac{\pi \alpha_M}{\hbar} \sqrt{\frac{2M}{E - |\delta E|}} - 4\sqrt{\frac{x_0}{\hbar^2/2M\alpha_M}} + \frac{\pi \alpha_m \sqrt{2m}}{\hbar} \left( \frac{1}{\sqrt{\varepsilon + |\delta E|}} - \frac{1}{\sqrt{\varepsilon}} \right)$$

(63)

where $|\delta E|$ satisfies the relation (50).

The physical trajectories of particles in the nuclear reaction (60) are shown in Fig. 11. The probability of the channel $(a)$ in Fig. 11 is almost $100\%$ and the probability of the channel $(b)$, which is the reaction (60), is of the order of $\exp(-A) \ll 1$. In Fig. 12 the classical positions of particles are shown at the moment of time when the incident proton reaches its minimum distance to the nucleus.

The angular momentum $L_m$ of the proton should be exactly of the value to provide the energy transfer (50). Otherwise, the proton and the alpha particle do not meet at the nucleus position at the coincident “times” $\tau_M = \tau_m$, their interaction would be small and the action would be not a result of a cooperative motion of two particles but simply conventional $A_M(E, L_M)$. 

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F. Required angular momentum of the incident proton

Let us calculate the energy transfer $|\delta E|$ at zero angular momentum of the incident proton $L_m = 0$ and then find the proton energy $\varepsilon$ which corresponds to this process. Since $L_m = 0$, only $x$-components are involved which are determined by the classical dynamical equations

$$
M \frac{\partial R_x}{\partial \tau^2} = -\frac{\alpha_M}{R_x^2} + \frac{\alpha_{int}}{(r_x - R_x)^2}; \quad m \frac{\partial r_x}{\partial \tau^2} = -\frac{\alpha_m}{r_x^2} - \frac{\alpha_{int}}{(r_x - R_x)^2}
$$

(64)

close to $\tau_0$ ($\tau_0 \simeq \tau_M \simeq \tau_m$) the solutions have the form

$$
\frac{R_x(i\tau)}{R_s} = \frac{r_x(i\tau)}{r_s} = \left(\frac{\tau_m - \tau}{\tau_m}\right)^{2/3}
$$

(65)

where $R_s$ and $r_s$ are some constants. The energy $\delta E$, gained by the alpha particle,

$$
\delta E = \alpha_{int} \int_0^{\tau_0} \frac{d\tau}{(R_x - r_x)^2} \frac{\partial R_x}{\partial \tau}
$$

(66)

diverges close to $\tau_0$ and should be cut off by the condition $R_s(1 - \tau/\tau_m)^{2/3} > x_0$. This gives

$$
|\delta E| = \frac{\alpha_{int}}{x_0} \left(\frac{r_s}{R_s} - 1\right)^{-2}
$$

(67)

The ratio $R_s/r_s$, as one can see after a little algebra, satisfies the relation

$$
\frac{M\alpha_m}{m\alpha_M} \left(\frac{R_s}{r_s}\right)^3 \left[\left(1 - \frac{R_s}{r_s}\right)^2 + \frac{\alpha_{int}}{\alpha_m}\right] = \left(1 - \frac{R_s}{r_s}\right)^2 - \frac{\alpha_{int}}{\alpha_M} \left(\frac{R_s}{r_s}\right)^2
$$

(68)

Substituting parameters for the reaction (60) $M/m = 4$, $\alpha_M/\alpha_m = 2$, and $\alpha_{int}/\alpha_M = 1/90$, one can obtain $R_s/r_s \simeq 0.715$ and the energy transfer $|\delta E| \simeq 1.89$ MeV. We use the estimate (37) for the nuclear size. With this $|\delta E|$ and the alpha particle energy $E = 4.678$ MeV, the relation (50) gives an unphysical (negative) value of $\varepsilon$. This means, that an incident proton with zero angular momentum transfers too big energy and the real process requires a finite angular momentum $L_m$ in order to effectively increase the minimum distance $x_0$ in Eq. (67) to reduce the energy transfer. The minimum proton-nucleus distance increases when the proton centrifugal energy becomes of the order of the Coulomb one at the nucleus radius. This corresponds to the estimate (40) for proton

$$
\frac{L_m^2}{2mx_0^2} \sim \frac{\alpha_m}{x_0}
$$

(69)

and a typical angular momentum of the incident proton should be $L_m \sim 10\hbar$. 

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G. Euclidean resonance

If to insert the energy transfer $|\delta E|$ from Eq. (50) into Eq. (63), one can obtain

$$A = \frac{\pi \alpha_M}{\hbar} \sqrt{\frac{2M}{E + \varepsilon}} \left[1 + \left(\frac{m\alpha^2_m}{M\alpha^2_M}\right)^{1/3}\right]^{3/2} - \frac{4}{\hbar^2} \sqrt{2Mx_0\alpha_M} - \frac{\pi \alpha_m}{\hbar} \sqrt{\frac{2m}{\varepsilon}}$$  \hspace{1cm} (70)

At $\varepsilon = \varepsilon_{\text{max}}$, where

$$\varepsilon_{\text{max}} = E \left(\frac{m\alpha^2_m}{M\alpha^2_M}\right)^{1/3} \simeq 1.85 \text{ MeV}$$ \hspace{1cm} (71)

$(E = 4.678 \text{ MeV})$ the energy transfer $|\delta E| = 0$, the parameter $L_m^2/2mx_0\alpha_m$ has its maximum value, and the action (70) matches the conventional one (41) resulting in the tunneling probability

$$W \sim \exp\left[-AM(4.678 \text{ MeV}, 0)\right] \simeq e^{-80.75} \simeq 10^{-35}$$ \hspace{1cm} (72)

The result (72) reasonably describes experimental data if to multiply (72) by the nuclear attempt frequency $10^{21} \text{ s}^{-1}$.

At $\varepsilon < \varepsilon_{\text{max}}$ the energy transfer $|\delta E|$ becomes finite, $L_m$ decreases, and the action (70) reduces compared to $A_M(E, 0)$. Upon reduction of $\varepsilon$, the action (70) turns to zero at the certain proton energy $\varepsilon_R$, which relates to Euclidean resonance. For the reaction (60) $\varepsilon_R = 0.25 \text{ MeV}$ and the accompanied energy transfer is $|\delta E| = 1.15 \text{ MeV}$. In other words, when in the reaction (60) the energy of the incident proton is $\varepsilon = 0.25 \text{ MeV}$, it converts into the proton with the energy $\varepsilon + |\delta E| = 1.40 \text{ MeV}$ and the energy of the emitted alpha particle (instead of $E = 4.678 \text{ MeV}$) becomes $E - |\delta E| = 3.53 \text{ MeV}$.

The cross-section of the reaction (60) is not exponentially small at $\varepsilon = \varepsilon_R$, it has a sharp peak at this proton energy, and is determined by the angular momentum $L_m$ which provides the energy transfer $|\delta E| = 1.15 \text{ MeV}$. A rough estimate of this angular momentum (69) results in the impact parameter $h \sim 8x_0$, according to the classical relation $L_m = h\sqrt{2m\varepsilon}$. A geometrical estimate of the cross-section at $\varepsilon = \varepsilon_{\text{max}}$ is reduced to the ring area $\sigma \simeq 2\pi h\delta h$ of the width $\delta h$ near the circle of the radius $h$, as shown in Fig. 12. Since $h$ is some optimum value, related to an extreme action, $\delta h$ can be estimated as $\delta h/h \sim 1/\sqrt{A_M}$. By means of the relation (72), the geometrical estimate of the cross-section produces 15 nuclear impact areas $\sigma \sim 15(\pi x_0^2)$.

At $\varepsilon = \varepsilon_{\text{max}}$ the exit point of alpha particle is $7.3x_0$ and one of the proton is $10x_0$. The incident proton is stopped by the nuclear Coulomb field at $54x_0$, as shown in Fig. 11. The interaction of alpha particle with a moving proton is analogous to its interaction with some non-stationary field, which results in Euclidean resonance. As in Fig. 6, the exponentially small part of alpha particle wave function merges at some moment of time the growing part of the wave function. Trajectories in imaginary time is only a convenient language to describe the effect. In real (physical) time alpha particle does not approach the nucleus and interacts with it solely via the Coulomb field. For example, this practically excludes spin interaction between them.

The above calculations hold for spherical nuclei. Alpha emitters may be not spherical, but real parameters of nonsphericality unlikely essentially violate the estimates.
VII. CONCLUSION

When a proton approaches a nucleus, which is an alpha emitter, it creates a nonstationary Coulomb interaction with the tunneling alpha particle. At the certain proton energy there are conditions for Euclidean resonance and the Coulomb barrier becomes transparent for the passage of the alpha particle. Normally, $^{235}_{92}$U emits alpha particle of the energy 4.678 MeV. When the energy of the incident proton is close to its resonant value 0.25 MeV, it reflects with the energy 1.40 MeV and simultaneously the alpha particle is emitted with the energy 3.53 MeV. Beams of 0.2 MeV-scale protons are “cheap” since they can be technically created in a relatively easy way. For this reason, low energy protons can be used for practical applications, for example, in disactivation of the nuclear waste.

The analytical calculation of Euclidean resonance on the semiclassical basis is given in Ref. [25]. In principal, one can solve numerically the initial Schrödinger equation for decay of a metastable state in presence of nonstationary field using the proper algorithm and accounting the boundary conditions [30]. However, there is a serious problem in such numerical calculations. As one can see from Fig. 6, the branch (3), related to Euclidean resonance, starts to form from the exponentially small branch (2) which is of the order of $\exp(-A_{WKB})$ at the well position. In order to get a numerical calculation accounted this effect, one should choose very short steps $\Delta t$ in time (the number of steps is proportional to $(\Delta t)^{-1}$) and $\Delta x$ in coordinate (the number of steps is proportional to $(\Delta x)^{-1}$). They have to satisfy the condition $\Delta t \sim (\Delta x)^2 < \exp(-A_{WKB})$, otherwise the effect would be missed. Suppose the numerical computation with $10^3$ steps in time and $10^3$ steps in coordinate requires one second. One can easily estimate the total computation time as $\exp(1.5A_{WKB}) \times 10^{-10}$ days. Since for alpha decay $A_{WKB} \simeq 80$, the total computation time should be $10^{42}$ days. For this reason, numerical computation should start not with the initial Schrödinger equation but with some semiclassical approach.

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FIG. 1. The path of tunneling is denoted by the dashed line. $E$ is the energy of the metastable state, $V$ is the barrier height, and $a$ is the typical potential length.
FIG. 2. The particle can absorb a quantum and tunnel in a more transparent part of the barrier with the energy \( E + \hbar \Omega \) (a). The process of the multiquanta absorption with the subsequent tunneling (b).
FIG. 3. The multiquanta emission with the subsequent tunneling in a less transparent part of the barrier.
FIG. 4. (1) is the initial particle state with the energy $E$ and (2) is the final state of the energy $E - |\delta E|$ after tunneling. The state (i) has the same energy as (2). There are two ways to connect phases of the states (i) and (1): by quanta absorption (a) and by quanta emission (b).
FIG. 5. The electron emission from the metal (left) to the vacuum by applying the constant electric field which is weakly modulated in time.
FIG. 6. The branches of the wave function $\psi \sim \exp(-\text{Im}S)$. In the absence of a nonstationary field there are the dominant branch (1) and the sub-dominant one (2) which merge at the point “exit” and convert into the outgoing wave, denoted by the dashed line. With a nonstationary field, at $t = 0$ the branch (3) is generated from the circled region at the sub-dominant branch (2) where the semiclassical condition is violated at $t = 0$. The branch (3) is formed during a short (non-semiclassical) time and then it moves semiclassically, keeping its maximum at the classical trajectory point $x_{cl}(t)$. This maximum value of the branch (3) decreases slowly in time due to quantum effects of smearing.
FIG. 7. The potential barrier which is passed by the alpha particle. The nuclear Coulomb field is cut off at the nuclear radius $x_0$. $R_e$ is the classical exit point.
FIG. 8. The classical trajectory of the proton moving in the nucleus Coulomb field. $r_e$ is the classical turning point.

FIG. 9. The classical trajectories in imaginary time of the proton $r$ and of the alpha particle $R$. The both particles occur at the nucleus position $x_0$ at the same "moment" of imaginary time.
FIG. 10. (1) is the initial state of the reaction which includes the uranium nucleus and the incident proton. (2) is the final state, including the outgoing alpha particle and the proton. The state (i) consists of the thorium nucleus with the alpha particle and the proton close to it, but away of nuclear forces. The state (i) serves to connect the phases of the states (1) and (2).
FIG. 11. The big circle is the nucleus. The incident proton (the smallest circle) moves along the thick curve and may reflect elastically with no stimulation of alpha decay \((a)\). The incident proton can initiate the alpha decay and then it appears, together with the alpha particle (the small circle), in the channel \((b)\). The exit points at the \(x\)-axis are given for the energy of the incident proton 0.25 Mev.

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\text{FIG. 12. } h \text{ is the impact parameter of the incident proton. The cross-section of the reaction is determined by the area between two dashed circles, separated by the distance } \delta h.
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