SYMMETRIES OF STATISTICS ON LATTICE PATHS BETWEEN TWO BOUNDARIES

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ABSTRACT. We prove that on the set of lattice paths with steps $N = (0,1)$ and $E = (1,0)$ that lie between two fixed boundaries $T$ and $B$ (which are themselves lattice paths), the statistics ‘number of $E$ steps shared with $B$’ and ‘number of $E$ steps shared with $T$’ have a symmetric joint distribution. To do so, we give an involution that switches these statistics, preserves additional parameters, and generalizes to paths that contain steps $S = (0,-1)$ at prescribed $x$-coordinates. We also show that a similar equidistribution result for path statistics follows from the fact that the Tutte polynomial of a matroid is independent of the order of its ground set. We extend the two theorems to $k$-tuples of paths between two boundaries, and we give some applications to Dyck paths, generalizing a result of Deutsch, to watermelon configurations, to pattern-avoiding permutations, and to the generalized Tamari lattice.

Finally, we prove a conjecture of Nicolás about the distribution of degrees of $k$ consecutive vertices in $k$-triangulations of a convex $n$-gon. To achieve this goal, we provide a new statistic-preserving bijection between certain $k$-tuples of non-crossing paths and $k$-flagged semistandard Young tableaux, which is based on local moves reminiscent of jeu de taquin.

1. INTRODUCTION

In the first few sections of this paper we present two general theorems about lattice paths, together with several applications. Both theorems concern the set of lattice paths taking unit north and east steps, starting at the origin and ending at some prescribed point $(x,y)$. Additionally, the paths are required to stay within a fixed region, whose boundaries are also lattice paths from the origin to $(x,y)$.

Informally, the first theorem states that the joint distribution of the number of contacts with the top boundary and the number of contacts with the bottom boundary of the given set of lattice paths is symmetric. We provide a bijective proof of a generalization to paths that also contain south steps in Section 3. The second theorem states that the joint distribution of the number of top and the number of right contacts coincides with the joint distribution of the number of bottom and the number of left contacts. Its proof uses matroid theory, and it is

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1For a slightly more elaborate introduction, we refer the reader to the arXiv preprint [17].
presented in Section 4 and Appendix A. When specialized to a triangular region, both theorems reduce to a result of Emeric Deutsch [12, 13] stating that, over the set of Dyck paths, the joint distribution of the number of returns and the height of the first peak is symmetric.

Some applications of our theorems are described in Section 6, including enumerative results on lattice paths with contacts in various regular regions, and a symmetry property on permutations with occurrences of a certain pattern at prescribed positions. We also establish a link between a map used in our proof and the covering relation in the generalized Tamari lattices recently introduced by François Bergeron [3].

Both theorems generalize naturally to families of non-intersecting paths, as shown in Section 5. Again, a particular case of interest is obtained by confining the paths to a triangular region, in which case the configurations are known in physics as ‘watermelons’. To enumerate watermelons according to the number of contacts, we can use our result to reduce the problem to the enumeration of non-intersecting paths in a slightly modified region (see Section 6.5). This can then be tackled using the classical Lindström–Gessel–Viennot lemma, as shown by Christian Krattenthaler [28].

Section 7 contains our second main contribution, which is a new connection between two seemingly separate topics in algebraic combinatorics. We exhibit a natural weight-preserving bijection between $k$-flagged semistandard Young tableaux of shape $\lambda$, as appearing when studying Schubert polynomials [37], and families of $k$ non-intersecting paths confined to a region of the same shape $\lambda$. We use this bijection to establish a refinement of a conjecture on multi-triangulation of polygons by Carlos Nicolás [32], concerning the distribution of degrees of $k$ consecutive vertices.

To conclude the introduction, we point out the similarity between our results and the fact that the $q,t$-Catalan polynomials are symmetric in $q$ and $t$. Finding a bijective proof of this theorem is an outstanding open problem in combinatorics. The $q,t$-Catalan polynomials enumerate Dyck paths according to two natural statistics on Dyck paths, ‘area’ and ‘bounce’, or equivalently, ‘dinv’ and ‘area’. Jim Haglund [23, p. 50] gives a bijection $\zeta$ on Dyck paths (already used in a different context by George Andrews et al. [1]), sending the pair (area, $\text{dinv}$) to the pair ($\text{bounce}$, area). As pointed out by Christian Stump, the map $\zeta$ also sends the number of contacts with the diagonal to the number of contacts with the left boundary. More generally, Nick Loehr’s extension [29] of this map to $m$-Dyck paths has the analogous property.

2. Statement of main results

Let $T$ and $B$ be two lattice paths in $\mathbb{N}^2$ with north steps ($N = (0,1)$) and east steps ($E = (1,0)$) from the origin to some prescribed point $(x, y) \in \mathbb{N}^2$ such that $T$ is weakly above $B$, i.e., the $n$-th east step of $T$ is weakly above the $n$-th east step of $B$ for $1 \leq n \leq x$. Let $\mathcal{P}(T, B)$ be the set of lattice paths with north and east steps from the origin to $(x, y)$ that lie between $T$ and $B$, i.e., weakly above $B$ and weakly below $T$. Thus, the paths $T$ and $B$ are the upper and lower boundaries of the paths in $\mathcal{P}(T, B)$.

In this paper we show that several natural statistics on lattice paths in $\mathcal{P}(T, B)$ have a symmetric distribution. Formally, a statistic on a set $\mathcal{O}$ of objects is simply a function from $\mathcal{O}$ to $\mathbb{N}$. Two $k$-tuples of statistics $(f_1, f_2, \ldots, f_k)$ and $(g_1, g_2, \ldots, g_k)$
have the same joint distribution over $\mathcal{O}$, denoted
\[(f_1, f_2, \ldots, f_k) \sim (g_1, g_2, \ldots, g_k),\]
if
\[
\sum_{P \in \mathcal{O}} x_1^{f_1(P)} \cdots x_k^{f_k(P)} = \sum_{P \in \mathcal{O}} x_1^{g_1(P)} \cdots x_k^{g_k(P)}.
\]
The distribution of $(f_1, f_2, \ldots, f_k)$ is symmetric over $\mathcal{O}$ if
\[(f_1, f_2, \ldots, f_k) \sim (f_{\pi(1)}, f_{\pi(2)}, \ldots, f_{\pi(k)})\]
for every permutation $\pi$ of $[k] = \{1, 2, \ldots, k\}$.

We consider statistics counting the following special steps of paths in $\mathcal{P}(T, B)$:
- a top contact is an east step that is also a step of $T$,
- a bottom contact is an east step that is also a step of $B$,
- a left contact is a north step that is also a step of $T$,
- a right contact is a north step that is also a step of $B$.

We denote the number of top, bottom, left and right contacts of $P \in \mathcal{P}(T, B)$ by $t(P), b(P), \ell(P)$ and $r(P)$, respectively. An example is given in Figure 1.

In the next two sections we give bijective proofs of the following results.

**Theorem 2.1.** The distribution of the pair $(t, b)$ over $\mathcal{P}(T, B)$ is symmetric.

**Theorem 2.2.** The pairs $(b, \ell)$ and $(t, r)$ have the same joint distribution over $\mathcal{P}(T, B)$.

As an example, if $T = NNENEE$ and $B = ENEENN$, then
\[
\sum_{P \in \mathcal{P}(T, B)} x^{t(P)} y^{b(P)} = x^3 + x^2 y + xy^2 + y^3 + 2x^2 + 2xy + 2y^2 + 2x + 2y + 1
\]
and
\[
\sum_{P \in \mathcal{P}(T, B)} x^{b(P)} y^{\ell(P)} = x^3 + x^2 y + y^3 + 2x^2 + 3xy + 3y^2 + 2x + 2y = \sum_{P \in \mathcal{P}(T, B)} x^{t(P)} y^{r(P)}.
\]
Although the two theorems look very similar, we do not know of a uniform proof for them. Let us point out that it is not true that \((t, b, \ell) \sim (b, t, r)\) in general. In Section 2, we exhibit an involution proving a refined and generalized version of Theorem 2.1. The refinement consists of keeping track of the sequence of \(y\)-coordinates of the east steps that are not contacts, while the generalization consists of allowing the paths to have south steps at prescribed \(x\)-coordinates. The analogous refinement and generalization of Theorem 2.2 does not hold, and in fact our proof of Theorem 2.1 given in Section 4 is very different from that of Theorem 2.2. Namely, we show that both

\[
\sum_{P \in \mathcal{P}(T,B)} x^{b(P)} y^{\ell(P)} \quad \text{and} \quad \sum_{P \in \mathcal{P}(T,B)} x^{\ell(P)} y^{r(P)}
\]

can be interpreted as the Tutte polynomial of the lattice path matroid associated with \(\mathcal{P}(T,B)\), as defined in [7]. To do so, we use the definition of the Tutte polynomial in terms of \emph{activities}, which relies on a linear ordering on the ground set of the matroid. The independence of the Tutte polynomial of this ordering then implies Theorem 2.2. In fact, as pointed out to us by Olivier Bernardi, William Tutte’s proof [36] of the well-definedness of his dichromate (essentially the Tutte polynomial restricted to graphs) is almost bijective. We make this explicit in Appendix A.

In Section 5, Theorems 2.1 and 2.2 are generalized to \(k\)-tuples of non-crossing paths. Let \(\mathcal{P}^k(T,B)\) be the set of \(k\)-tuples \(P = (P_1, P_2, \ldots, P_k)\) such that \(P_i \in \mathcal{P}(T,B)\) for all \(i\), and \(P_i\) is weakly above \(P_{i+1}\) for \(1 \leq i \leq k - 1\). Let \(P_0 = T\) and \(P_{k+1} = B\). For \(0 \leq i \leq k\), denote by \(h_i = h_i(P)\) the number of east steps where \(P_i\) and \(P_{i+1}\) coincide. We provide a bijective proof of a generalization of Theorem 2.1, which can be stated in a simplified form as follows.

**Theorem 2.3.** The distribution of \((h_0, h_1, \ldots, h_k)\) over \(\mathcal{P}^k(T,B)\) is symmetric.

To generalize Theorem 2.2 define the \emph{top contacts} of \(P\) to be the top contacts of \(P_1\), and denote their number by \(t(P)\). Similarly, let \(\ell(P)\) be the number of left contacts of \(P_1\), and let \(b(P)\) (respectively \(r(P)\)) be the number of bottom (respectively right) contacts of \(P_k\). Note that \(t(P) = h_0(P)\) and \(b(P) = h_k(P)\) by definition.

**Theorem 2.4.** The pairs \((b, \ell)\) and \((t, r)\) have the same joint distribution over \(\mathcal{P}^k(T,B)\).

The above theorems have consequences for the enumeration of generalized Dyck paths with a given number of top and bottom contacts, as well as to the distribution of statistics on restricted permutations. These are discussed in Section 6, together with a new proof of a result of Richard Brak and John Essam [3] on watermelon configurations.

In the special case that \(T = N^{n-2k-1}E^{n-2k-1}\) and \(B = (EN)^{n-2k-1}\), the elements of \(\mathcal{P}^k(T,B)\) are \(k\)-fans of Dyck paths of semilength \(n - 2k\). We denote this set by \(\mathcal{D}_n^k\). The number of such \(k\)-fans was shown by Jakob Jonsson [26] to be equal to the number of \(k\)-triangulations of a convex \(n\)-gon, that is, maximal sets of diagonals such that no \(k+1\) of them cross mutually. Nicolás [32] made the following conjecture, based on experimental evidence.

**Conjecture 2.5 (32).** The distribution of degrees of \(k\) consecutive vertices over the set of \(k\)-triangulations of a convex \(n\)-gon equals the distribution of the tuple \((h_0, h_1, \ldots, h_{k-1})\) over \(\mathcal{D}_n^k\).
One of the main results of Section 7 is a proof of this conjecture. In fact, we show that it can be extended to give a description of the distribution of the full degree sequence of the $k$-triangulation. One ingredient in our proof is a bijection of Serrano and Stump [34] between $k$-triangulations of a convex $n$-gon and $D_k$. The other ingredient, whose proof occupies most of Section 7, can be stated in a simplified form as follows.

Theorem 2.6. Let $T = N^y E^x$, and let $B$ be any path from the origin to $(x, y)$, weakly below $T$ and ending with a north step. Let $\lambda$ be the Young diagram bounded by $T$ and $B$. There is an explicit bijection between $k$-tuples of paths $P \in P^k(T, B)$ and $k$-flagged SSYT of shape $\lambda$ such that the number of entries equal to $i + 1$ in the tableau is $\lambda_1 - h_i(P)$, for $0 \leq i \leq k$.

3. The symmetry $(t, b) \sim (b, t)$ for a single path

In this section we construct an involution $\Phi$ that proves a generalized version of Theorem 2.1. It not only applies to a more general set of paths, but it also gives a refined result by preserving the sequence of $y$-coordinates of the east steps that are not contacts.

Let $\tilde{\mathcal{P}}(T, B)$ be the set of lattice paths from the origin to $(x, y)$ with north, east and south ($S = (0, -1)$) steps, lying weakly below $T$ and weakly above $B$. Given such a lattice path $P$, the descent set of $P$ is the set of $x$-coordinates where south steps occur. For a fixed subset $D \subset \mathbb{N}$, denote by $\tilde{\mathcal{P}}(T, B, D)$ the set of paths $P \in \tilde{\mathcal{P}}(T, B)$ having descent set $D$. Note that $\tilde{\mathcal{P}}(T, B, \emptyset) = \mathcal{P}(T, B)$ by definition. The definitions of top and bottom contacts generalize trivially to paths in $\tilde{\mathcal{P}}(T, B)$.

An example is given in Figure 2.

For a given sequence $H$ of integers, let $\tilde{\mathcal{P}}(T, B, D, H)$ (respectively $\mathcal{P}(T, B, H)$) be the subset of $\tilde{\mathcal{P}}(T, B, D)$ (respectively $\mathcal{P}(T, B)$) containing those paths whose sequence of $y$-coordinates of the east steps that are not bottom or top contacts equals $H$. Figure 4 shows all the paths in the set $\tilde{\mathcal{P}}(T, B, D, H)$ with $T = NNNEEENEE$, $B = EENEEENNN$, $D = \{2\}$, and $H = 2 2 3$.

We can now state the announced refinement of Theorem 2.1.

Theorem 3.1. For any set $D$ and any sequence $H$ of integers, the distribution of $(t, b)$ over $\tilde{\mathcal{P}}(T, B, D, H)$ is symmetric.
Remark 3.2. Without the refinement by \( H \), a recursive, non-bijective proof of this result has been found independently by Guo Niu Han \[24\].

In the following we encode a path in \( \tilde{P}(T, B) \) by the sequence of \( y \)-coordinates of its east steps, except that we record top contacts using \( t \)'s and bottom contacts using \( b \)'s. Steps that are simultaneously top and bottom contacts are not recorded in this encoding at all. For example, the first path in Figure 5 is encoded by 2t23t.

To prove Theorem 3.1, we construct an involution \( \Phi \) that essentially turns top contacts into bottom contacts one at a time. The transformation that turns a top contact into a bottom contact, which we denote \( \phi \), relies in turn on a transformation on words, denoted \( \mu \), which turns a sequence of \( e \) \( t \)'s and \( f \) \( b \)'s into a sequence of \( e - 1 \) \( t \)'s and \( f + 1 \) \( b \)'s.

### 3.1. A transformation on words

Let us first describe the map \( \mu \), which is defined on words over the alphabet \( \{t, b\} \). We say that such a word \( w_1 w_2 \ldots w_{2n} \) of even length is a Dyck word if it contains the same number of \( t \)'s and \( b \)'s, and in every prefix \( w_1 w_2 \ldots w_i \) with \( 1 \leq i \leq 2n \), the number of \( b \)'s never exceeds the number of \( t \)'s.

Let \( w = w_1 w_2 \ldots w_{e+f} \) be a word over the alphabet \( \{t, b\} \). Any such \( w \) can be factorized uniquely as

\[
(3.1) \quad w = D_1 b D_2 b \ldots b D_j t D_{j+1} t D_{j+2} t \ldots t D_m,
\]

where each \( D_i \) for \( 1 \leq i \leq m \) is a (possibly empty) Dyck word. In such a factorization, the letters \( t \) and \( b \) which are not part of a Dyck word are called unmatched letters. By construction, all the unmatched \( b \)'s are to the left of the unmatched \( t \)'s. Suppose that there is at least one unmatched \( t \). We define \( \mu(w) \) to be the word obtained from \( w \) by replacing the leftmost unmatched \( t \) with a \( b \), that is,

\[
(3.2) \quad \mu(w) = D_1 b D_2 b \ldots b D_j b D_{j+1} t D_{j+2} t \ldots t D_m.
\]

The above factorization can be visualized by representing a word with a path, drawing an northeast step \((1, 1)\) for each letter \( t \) and a southeast step \((1, -1)\) for each letter \( b \). In Figure 3, the effect of the map \( \mu \) applied to the word \( bttbttbttbtttttt \) is shown. In this example, the Dyck words \( D_2 \), \( D_4 \) and \( D_5 \), indicated by the dotted areas, are non-empty.

![Figure 3. A visual description of the map \( \mu \).](image-url)
Remark 3.3. The map $\mu$, in different contexts, belongs to mathematical folklore. Curtis Greene and Daniel Kleitman \cite{22} used it to build a symmetric chain decomposition of the boolean algebra, giving an injection from $i$-element subsets of $[n]$ to $(i+1)$-element subsets of $[n]$ (where $i < n/2$) with the property that each subset is contained in its image. Their construction proves that the boolean algebra has the Sperner property. It also yields a bijection between $i$-element subsets and $(n-i)$-element subsets of $[n]$ where each subset is contained in its image. On the other hand, a bijective argument based on iterates of $\mu$ proves that the number of ballot paths of length $2n$ is $\binom{2n}{n}$.

Lemma 3.4. Let $e, f, u$ be nonnegative integers with $u \geq \max\{e-f, f-e+2\}$. The map $\mu$ is a bijection between

(i) the set of words with $e$ $t$'s and $f$ $b$'s having $u$ unmatched letters, and
(ii) the set of words with $e-1$ $t$'s and $f+1$ $b$'s having $u$ unmatched letters.

Proof. Let $w$ be a word with $e$ $t$'s and $f$ $b$'s having $u$ unmatched letters. Note that $w$ has some unmatched $t$, since otherwise we would have $u = f - e < f - e + 2$, contradicting the assumption on $u$. If the factorization of $w$ is given by $\mu(w)$, then the factorization of $\mu(w)$ is given by $(\ref{3.2})$. In particular, $\mu(w)$ is a word with $e - 1$ $t$'s and $f + 1$ $b$'s having the same number of unmatched letters as $w$. The map is injective because $w$ can be recovered from $\mu(w)$ by replacing the rightmost unmatched $b$ with a $t$, and it is surjective because any word with $e-1$ $t$'s and $f+1$ $b$'s having $u$ unmatched letters must have some unmatched $b$, since otherwise we would have $u = (e-1) - (f+1) < e - f$, contradicting the assumption on $u$. \hfill $\square$

The next lemma states that the left-most unmatched $t$ in $w$ can only be preceded by a $b$ and followed by a $t$ in both $w$ and $\mu(w)$.

Lemma 3.5. Let $w = w_1w_2 \ldots w_{e+f}$ be a word with $e$ $t$'s and $f$ $b$'s having some unmatched $t$. Let $\mu(w) = w'_1w'_2 \ldots w'_{e+f}$, and let $i$ be such that $w_i = t$ and $w'_i = b$. If $i > 1$, then $w_{i-1} = w'_{i-1} = b$, and if $i < e + f$, then $w_{i+1} = w'_{i+1} = t$.

Proof. This follows trivially from the definition of $\mu$. \hfill $\square$

3.2. The maps $\phi$ and $\Phi$. Our next goal is to translate $\mu$, which is a map on words, into a transformation on paths, denoted $\phi$. Then, $\Phi$ will be constructed by iterating $\phi$.

Definition 3.6. For $P \in \overrightarrow{P}(T, B)$, the sequence of contacts of $P$ is the word $w_P$ over $\{t, b\}$ obtained by recording the top and bottom contacts of $P$ from left to right, except for the steps that are simultaneously top and a bottom contacts, which are not recorded.

Definition 3.7. Let $P \in \overrightarrow{P}(T, B)$ be such that $w_P$ contains some unmatched $t$. The east steps of $P$ can be decomposed uniquely as $P = WXYtYZ$, where

- the selected $t$ is the leftmost unmatched $t$ in $w_P$,
- $X$ is maximal such that there is no descent after any of its steps and no (right) endpoint of any of its steps lies on $B$, and
- $Y$ is maximal such that there is a descent before each of its steps.
Let \( h_X \) (respectively \( h_Y \)) be \(-\infty\) if \( X \) (respectively \( Y \)) is empty, and otherwise the \( y \)-coordinate of its last (respectively first) east step. Define

\[
\phi(P) = \begin{cases} 
WXYbZ & \text{if } h_X \leq h_Y, \\
WbXYZ & \text{if } h_X > h_Y.
\end{cases}
\]

**Remark 3.8.** In the case of paths with no descents, the definition of \( \phi \) is simpler: writing \( P \) as \( P = WXtZ \), where \( X \) is maximal not touching \( B \), we have \( \phi(P) = WbXZ \).

![Figure 4](image-url)

**Figure 4.** Two examples of the map \( \phi \) for paths with one contact: one where \( h_X \leq h_Y \) (top) and one where \( h_X > h_Y \) (bottom).

Examples of the map \( \phi \) for paths with \( t(P) = 1 \) and \( b(P) = 0 \) are given in Figure 4. Before showing that \( \phi \) is a bijection between the appropriate sets, let us remark that, on paths with \( t(P) = 1 \) and \( b(P) = 0 \), its definition is forced by the requirement that the sequence of \( y \)-coordinates of east steps of \( P \) that are not contacts is preserved.

**Proposition 3.9.** There is at most one path \( P_1 \) in \( \tilde{P}(T, B, D, H) \) with \( t(P_1) = 1 \) and \( b(P_1) = 0 \), and at most one path \( P_2 \) with \( t(P_2) = 0 \) and \( b(P_2) = 1 \).

**Proof.** We prove the first part of the statement, the second part then follows by symmetry. Suppose that \( P, P' \in \tilde{P}(T, B, D, H) \) are two paths with \( t(P) = t(P') = 1 \) and \( b(P) = b(P') = 0 \). Since both paths have the same sequence of \( y \)-coordinates of non-contact east steps, we can write the paths as \( P = UVtW \) and \( P' = UtVW \). It suffices to show that \( V \) must be empty.

Suppose that \( V \) is not empty, and that the sequence of \( y \)-coordinates of its east steps is \( v_1, v_2, \ldots, v_m \). There cannot be a descent immediately before a top contact,
in particular not between the last step of $V$ and $t$ in $P$. Since the descent sets of $P$ and $P'$ are the same, there is no descent at this position in $P'$ either, so $v_{m-1} \leq v_m$. But then there is no descent between the last two steps of $V$ in $P$, so the same is true at the corresponding position in $P'$, which implies $v_{m-2} \leq v_{m-1}$. Repeating this argument, we obtain that $v_1 \leq v_2$, i.e., there is no descent between the first two steps of $V$ in $P$, so there is no descent between $t$ and the first step of $V$ in $P'$. However, this is impossible: the top contact in $P'$ coincides with an east step of $T$ whose $y$-coordinate is strictly greater than $v_1$, since the first step of $V$ in $P$ is not a top contact.

\begin{lemma}
In the decomposition $P = WXtYZ$ in Definition 3.7, neither $X$ nor $Y$ contains any top or bottom contacts. In particular, the sequence of contacts of $\phi(P)$ is $\mu(w_P)$.
\end{lemma}

\textbf{Proof.} By Lemma 3.5, the contact preceding the selected $t$ must be a bottom contact. Since $X$ contains no bottom contacts, it contains no top contacts either.

Similarly, the contact following the selected $t$ must be a top contact. Since $Y$ contains no top contacts, it contains no bottom contacts either.

\begin{lemma}
Let $e, f, u$ be nonnegative integers with $u \geq \max\{e - f, f - e + 2\}$. The map $\phi$ is a bijection between

(i) the set of paths in $\overline{P}(T, B, D, H)$ whose sequence of contacts has $e$ $t$’s, $f$ $b$’s, and $u$ unmatched letters, and

(ii) the set of paths in $\overline{P}(T, B, D, H)$ whose sequence of contacts has $e - 1$ $t$’s, $f + 1$ $b$’s, and $u$ unmatched letters.
\end{lemma}

\textbf{Proof.} Let $P$ be a path in the set described in (i). As in the proof of Lemma 3.4, $P$ must have some unmatched $t$, so $\mu$ is applicable. Moreover, the sequence of contacts of $\phi(P)$ is $\mu(w_P)$, which has $e - 1$ $t$’s, $f + 1$ $b$’s, and $u$ unmatched letters. It is clear from the definition that $\phi(P) \in \overline{P}(T, B, H)$.

Let us now check that $P$ and $\phi(P)$ have the same descent set, so $\phi(P) \in \overline{P}(T, B, D, H)$. Suppose that $P = WXtYZ$ is the decomposition in Definition 3.7. Consider first the case $h_X \leq h_Y$, so $\phi(P) = WXYbZ$, that is, the block $tY$ in $P$ becomes $Yb$ in $\phi(P)$. By the choice of $Y$ and because there cannot be a descent just before a top contact, it is clear that $P$ has no descent just before and just after the block $tY$, and there are descents at all positions inside the block. Let us check that this is also the case for the block $Yb$ in $\phi(P)$.

- Just before $Yb$: if $X$ is non-empty, then $h_Y \geq h_X > -\infty$, which implies that $Y$ is non-empty and there is no descent between $X$ and $Y$; if $X$ is empty, then either $W$ is empty or its last east step has its right endpoint on $B$, so there is no descent just before $Yb$ either.

- Just after $Yb$: there cannot be a descent just after a bottom contact.

- Inside $Yb$: by the definition of $Y$, there are descents at all positions inside $Y$; at the position between $Y$ and $b$ (if $Y$ is non-empty), there is a descent because the last east step of $Y$ was not a bottom contact in $P$, by Lemma 3.10 so its $y$-coordinate is strictly larger than that of $b$ in $\phi(P)$.

The arguments for the case that $h_X > h_Y$ and $\phi(P) = WbXYZ$ are very similar and thus omitted.

To show that $\phi$ is invertible we will exhibit its inverse. Informally, the description of $\phi^{-1}$ is obtained from that of $\phi$ by rotating the picture by 180 degrees. Explicitly,
Lemma 3.12. Let \( P' \) be a path in the set described in (ii). The proof of Lemma 3.4 shows that \( w_{P'} \) has some unmatched \( b \). The east steps of \( P' \) can be decomposed uniquely as \( P' = RSBUV \), where

- the selected \( b \) is the rightmost unmatched \( b \) in \( w_{P'} \),
- \( S \) is maximal such that there is a descent after each of its steps, and
- \( U \) is maximal such that there is no descent before any of its steps and no (left) endpoint of any of its east steps lies on \( T \).

Let \( h_S \) (\( h_U \)) be \( +\infty \) if \( S \) (respectively \( U \)) is empty, and otherwise the \( y \)-coordinate of its last (respectively first) east step. Define

\[
\phi^{-1}(P) = \begin{cases} \text{RtSUV} & \text{if } h_S \le h_U, \\ \text{RSUtV} & \text{if } h_S > h_U. \end{cases}
\]

Let us check that \( \phi^{-1} \) is indeed the inverse of \( \phi \). Suppose that \( P = WXtYZ \) and \( h_X \le h_Y \), and let \( P' = \phi(P) = WXbYZ \). By Lemma 3.10, \( w_{P'} = \mu(w_P) \), and so, as in the proof of Lemma 3.4, the leftmost unmatched \( t \) of \( w_P \) becomes the rightmost unmatched \( b \) of \( w_{P'} \). Additionally, when applying \( \phi^{-1} \) to \( P' \), the decomposition \( P' = RSBUV \) has \( S = Y \); by definition of \( S \) and the fact that there is no descent just before \( Y \) but there are descents in all the positions inside \( Yb \). Additionally, \( h_S \le h_U \) because there was no descent just after \( Y \) in \( P \). Thus, \( \phi^{-1}(P') = \text{RtSUV} = P \). The case that \( h_X > h_Y \) is similar. \( \square \)

Figure 5 shows examples of the map \( \phi \).

Lemma 3.12. Let \( e > f \). For \( 0 \le i \le e - f \), let \( \mathcal{P}_i \) be the set of paths in \( \mathcal{P}(T, B, D, H) \) whose sequence of contacts has \( e - i \) \( t \)'s, \( f + i \) \( b \)'s, and at least \( e - f \) unmatched letters. Specifically, \( \mathcal{P}_0 \) is the set of all paths in \( \mathcal{P}(T, B, D, H) \) having \( e \) top contacts and \( f \) bottom contacts, and \( \mathcal{P}_{e-f} \) is the set of all paths in \( \mathcal{P}(T, B, D, H) \) having \( f \) top contacts and \( e \) bottom contacts (steps that are simultaneously top and a bottom contacts are disregarded there). Then the map \( \phi \) produces a sequence of bijections

\[ \mathcal{P}_0 \xrightarrow{\phi} \mathcal{P}_1 \xrightarrow{\phi} \cdots \xrightarrow{\phi} \mathcal{P}_{e-f}. \]

Proof. To prove that \( \mathcal{P}_0 \) and \( \mathcal{P}_{e-f} \) are indeed as claimed, note that every word with \( e \) \( t \)'s and \( f \) \( b \)'s has at least \( e - f \) unmatched \( t \)'s, and every word with \( f \) \( t \)'s and \( e \) \( b \)'s has at least \( e - f \) unmatched \( b \)'s.

By Lemma 3.11, \( \phi : \mathcal{P}_i \rightarrow \mathcal{P}_{i+1} \) is a bijection for \( 0 \le i < e - f \), since in this case \( e - f \ge \max\{e - f - 2i, f - e + 2i + 2\} \). \( \square \)

We can now describe the bijection \( \Phi \) that proves Theorem 3.1, which in turn generalizes Theorem 2.1. Figure 5 gives two examples of the map \( \Phi \).

Definition 3.13. For \( P \in \mathcal{P}(T, B) \), define \( \Phi(P) = \phi^{e-f}(P) \), where \( e = t(P) \) and \( f = b(P) \).

Lemma 3.14. The map \( \Phi \) is an involution on \( \mathcal{P}(T, B) \) that preserves the descent set, as well as the sequence of \( y \)-coordinates of the east steps that are not contacts, and satisfies \( t(\Phi(P)) = b(P) \) and \( b(\Phi(P)) = t(P) \).

Proof. For any fixed \( D, H \), and \( e > f \), Lemma 3.12 states that \( \phi^{e-f} \) is a bijection between \( \{ P \in \mathcal{P}(T, B, D, H) \colon t(P) = e, \ b(P) = f \} \) and \( \{ P \in \mathcal{P}(T, B, D, H) \colon t(P) = f, \ b(P) = e \} \), with inverse \( (\phi^{-1})^{e-f} = \phi^{f-e} \). It follows that \( \Phi \) is a
bijection with the stated properties. To see that it is an involution, note that for a path \( P \in \mathcal{P}(T, B) \) with \( e \) top contacts and \( f \) bottom contacts, \( \Phi(\Phi(P)) = \phi^{f-e}(\phi^{e-f}(P)) = P \).

\[ \begin{align*}
\text{Figure 5.} & \quad \text{The involution } \Phi \text{ applied to two paths with two top contacts and no bottom contacts.}
\end{align*} \]

4. **The Symmetry \((b, \ell) \sim (t, r)\) for a Single Path**

In this section we prove Theorem 2.2. Although this theorem looks superficially similar to Theorem 2.1, we have not found a comparable ‘natural’ bijective proof. Instead, our theorem below is a consequence of work of Anna de Mier, Joseph Bonin and Marc Noy [7], and also Federico Ardila [2].

Again, let \( T \) and \( B \) be lattice paths in \( \mathbb{N}^2 \) with north and east steps from the origin to \((x, y)\) such that \( T \) is weakly above \( B \). The paths in this section have no south steps. We encode a path \( P \in \mathcal{P}(T, B) \) as the subset \( \hat{P} \) of \( \mathbb{N} = \{1, 2, \ldots, x+y\} \) given by the indices of the north steps in \( P \). For example, the path \( P \) in Figure 1 is specified by the subset \( \hat{P} = \{2, 3, 4, 8, 9, 15, 16\} \subseteq [17] \). It is shown in [7] that the set \( \mathcal{B}_{T,B} = \{ \hat{P} : P \in \mathcal{P}(T, B) \} \) is the set of bases of a matroid with ground set \( \mathcal{N} \). This matroid, which we denote by \( \Sigma_{T,B} \), is called a lattice path matroid.

Let \( \prec \) be an arbitrary linear order on \( \mathcal{N} \), and let \( \hat{P} \in \mathcal{B}_{T,B} \). Then an element \( e \notin \hat{P} \) is **externally active** with respect to \((\hat{P}, \prec)\) if

\[ \exists n \in \hat{P} \text{ such that } n \prec e \text{ and } \hat{P} \setminus \{n\} \cup \{e\} \in \mathcal{B}_{T,B}, \]

that is, the east step with index \( e \) cannot be switched with a “smaller” north step to produce another path in \( \mathcal{P}(T, B) \). Similarly, an element \( n \in \hat{P} \) is **internally active** with respect to \((\hat{P}, \prec)\) if

\[ \exists e \in \mathcal{N} \setminus \hat{P} \text{ such that } e \prec n \text{ and } \hat{P} \setminus \{n\} \cup \{e\} \in \mathcal{B}_{T,B}, \]

that is, the north step with index \( n \) cannot be switched with a “smaller” east step to produce another path in \( \mathcal{P}(T, B) \).
For a fixed order $≺$, the internal activity of $\hat{P}$ is the number of internally active elements with respect to $(\hat{P}, ≺)$, and similarly its external activity is the number of externally active elements. The Tutte polynomial of $\Sigma_{T,B}$, introduced by Henry Crapo [11] generalizing Tutte’s dichromate for graphs, is the generating polynomial for the internal and external activities of its bases:

$$\sum_{\hat{P} \in B_{T,B}} x^{\text{internal activity of } \hat{P}} y^{\text{external activity of } \hat{P}}.$$

The main ingredient in the proof of Theorem 2.2, restated below for convenience, is the fact that the Tutte polynomial is well-defined, i.e., independent of the ordering of the ground set. In Appendix A we give an activity-preserving bijection on the bases of a matroid relative to two orderings that differ only in the order of two covering elements. This bijection works for any matroid, and in particular for $\Sigma_{T,B}$, so it gives a bijective proof of Theorem 4.1, but it is iterative and rather tedious to apply. It would be interesting to find a direct bijective proof of this theorem.

**Theorem 4.1.** The pairs $(b, ℓ)$ and $(t, r)$ have the same joint distribution over $P(T, B)$.

**Proof.** Let $≺$ be the usual order $1 ≺ 2 ≺ 3 ≺ \cdots$ of the ground set $N$ of the matroid $\Sigma_{T,B}$ described above. As shown in [7, Theorem 5.4], an internally active element of $\hat{P} \in B_{T,B}$ with respect to this order is a left contact of $P \in P(T,B)$, and an externally active element of $\hat{P}$ is a bottom contact of $P$. By equation (4.1), the Tutte polynomial of $\Sigma_{T,B}$ equals $\sum_{P \in P(T,B)} x^{b(P)} y^{ℓ(P)}$.

Since the Tutte polynomial is independent of the ordering on the ground set, the same polynomial can also be obtained as follows. Let now $≺$ be the order $\cdots ≺ 3 ≺ 2 ≺ 1$. With respect to this order, we claim that an internally active element of $\hat{P} \in B_{T,B}$ is a right contact of $P \in P(T,B)$, and an externally active element of $\hat{P}$ is a top contact of $P$. From the claim it follows that the Tutte polynomial of $\Sigma_{T,B}$ also equals $\sum_{P \in P(T,B)} x^{t(P)} y^{r(P)}$, and thus

$$\sum_{P \in P(T,B)} x^{b(P)} y^{ℓ(P)} = \sum_{P \in P(T,B)} x^{t(P)} y^{r(P)}$$

as desired.

It remains to prove the claim, which can be done with the following argument analogous to [7, Theorem 5.4]. Let $P \in P(T,B)$, and suppose that $n \in \hat{P}$, that is, the $n$-th step of $P$ is a north step. If this step is a right contact, then for any $e$ satisfying that $e \notin \hat{P}$ (i.e., the $e$-th step of $P$ is an east step) and $e ≺ n$ (i.e., $e > n$), the path whose north steps are $\hat{P} \setminus \{n\} \cup \{e\}$ does not lie weakly above $B$, since the $n$-th step of this path goes under $B$, so $\hat{P} \setminus \{n\} \cup \{e\} \notin B_{T,B}$. Thus $n$ is internally active.

Conversely, if the $n$-th step of $P$ is not a right contact, let $e$ be the index of the first east step with $e > n$ that touches $B$ with its right endpoint. Then $e \notin \hat{P}$, $e ≺ n$, and the path whose north steps are $\hat{P} \setminus \{n\} \cup \{e\}$ belongs to $P(T,B)$, so $\hat{P} \setminus \{n\} \cup \{e\} \in B_{T,B}$. Thus $n$ is not internally active.

The argument for externally active edges is very similar and thus omitted. □
5. A \(k\)-tuple of paths between two boundaries

In this section we show how to extend Theorems 2.1 and 2.2 to families of \(k\) non-crossing paths. In both cases, the idea is to repeatedly apply the theorems for single paths.

5.1. The symmetry \((t, b) \sim (b, t)\). In our extension of Theorem 2.1 to \(k\)-tuples of paths, we do not allow paths with south steps, unlike in the more general Theorem 3.1. The reason is that the distribution of bottom and top contacts over \(P(T, B)\) is not symmetric if we allow south steps in the boundary paths \(T\) and \(B\). However, we are able to partially incorporate the refinement keeping track of the \(y\)-coordinates of the non-contact east steps.

Recall from Section 2 that, given \(P = (P_1, P_2, \ldots, P_k) \in P^k(T, B)\), with the convention that \(P_0 = T\) and \(P_{k+1} = B\), we denote by \(h_i = h_i(P)\) the number of east steps where \(P_i\) and \(P_{i+1}\) coincide, for \(0 \leq i \leq k\). Recall that \(T\) and \(B\) are paths from \((0, 0)\) to \((x, y)\). For \(1 \leq s \leq y - 1\), let \(u_s(P)\) be the number of east steps with \(y\)-coordinate \(y - s\) that lie strictly between \(T\) and \(B\) and are not used by any of the paths \(P_1, \ldots, P_k\).

For any sequence \(u\) of nonnegative integers, let \(P_k(T, B, u)\) be the set of \(k\)-tuples \(P \in P^k(T, B)\) with \(u(P) = u\).

**Theorem 5.1.** For any sequence \(u\) of nonnegative integers, the distribution of \((h_0, h_1, \ldots, h_k)\) over \(P^k(T, B, u)\) is symmetric.

**Remark 5.2.** Disregarding the refinement by \(u\), two recursive, non-bijective proofs of this theorem have been given by Nicolas [32, Theorem 3] and Han [24]. To the extent of our knowledge, ours is the first bijective proof.

**Proof.** It suffices to show that \((h_0, \ldots, h_{i-1}, h_i, \ldots, h_k) \sim (h_0, \ldots, h_i, h_{i-1}, \ldots, h_k)\) over \(P^k(T, B, u)\) for any \(i\) with \(1 \leq i \leq k\). Fix such an \(i\), and let \(P = (P_1, P_2, \ldots, P_k) \in P^k(T, B, u)\). Regarding \(P_i\) as a path in \(P(P_{i-1}, P_{i+1})\), we can apply the bijection \(\Phi\) from Definition 3.13 to it, obtaining a path \(\Phi(P_i) \in P(P_{i-1}, P_{i+1})\). Let \(Q = (Q_1, Q_2, \ldots, Q_k)\), where \(Q_j = \Phi(P_j)\) and \(Q_j = P_j\) for \(j \neq i\).

By definition, \(h_j(P) = h_j(Q)\) for \(j \notin \{i - 1, i\}\). By Lemma 3.13, the number of east steps where \(P_i\) and \(P_{i+1}\) (respectively \(P_{i-1}\)) coincide equals the number of
east steps where \( \Phi(P_i) \) and \( P_{i-1} \) (respectively \( P_{i+1} \)) coincide, so \( h_i(P) = h_{i-1}(Q) \) and \( h_{i-1}(Q) = h_i(Q) \).

It remains to show that \( u(P) = u(Q) \). The only east steps that need to be checked are those that lie strictly between \( P_{i-1} = Q_{i-1} \) and \( P_{i+1} = Q_{i+1} \). We know by Lemma 3.14 applied to \( P_i \in P(P_{i-1}, P_{i+1}) \) that the multiset of \( y \)-coordinates of east steps of \( P_i \) that do not coincide with either \( P_{i-1} \) or \( P_{i+1} \) equals the corresponding multiset of \( Q_i \). Thus, the number of unused east steps lying between \( P_{i-1} \) and \( P_{i+1} \) at each fixed \( y \)-coordinate is the same in both \( P \) and \( Q \). \( \square \)

5.2. The symmetry \((b, \ell) \sim (t, r)\). Given \( P = (P_1, P_2, \ldots, P_k) \in P^k(T, B) \), letting \( P_0 = T \) and \( P_{k+1} = B \), we denote by \( v_i = v_i(P) \) the number of north steps where \( P_i \) and \( P_{i+1} \) coincide and, as before, by \( h_i = h_i(P) \) the number of east steps where \( P_i \) and \( P_{i+1} \) coincide, for \( 0 \leq i \leq k \). Note that \( t(P) = h_0(P), b(P) = h_k(P), \ell(P) = v_0(P) \) and \( r(P) = v_k(P) \).

**Theorem 5.3.** The pairs \((b, \ell)\) and \((t, r)\) have the same joint distribution over \( P^k(T, B) \).

**Proof.** As in the proof of Theorem 5.1, each \( P_i \), for \( 1 \leq i \leq k \), can be regarded as a path in \( P(P_{i-1}, P_{i+1}) \). In this setting, the statistics involved in the statement of Theorem 5.1 are \( b(P_i) = h_i(P), \ell(P_i) = v_{i-1}(P), t(P_i) = h_{i-1}(P) \) and \( r(P_i) = v_i(P) \). Applying Theorem 5.1 to \( P_i \in P(P_{i-1}, P_{i+1}) \), it follows that there is a bijection between tuples \( P \in P^k(T, B) \) with \( h_i(P) = e \) and \( v_{i-1}(P) = f \), and tuples \( P \in P^k(T, B) \) with \( h_{i-1}(P) = e \) and \( v_i(P) = f \), which preserves the statistics \( h_j \) and \( v_j \) for all \( j \neq i, i+1 \).

Given \( P \in P^k(T, B) \) with \( b(P) = h_k(P) = e \) and \( \ell(P) = v_0(P) = f \), one can apply this bijection first to \( P_k \), then to \( P_{k-1} \) in the resulting tuple, and successively up to \( P_1 \). This composition gives a bijection between tuples \( P \in P^k(T, B) \) with \( b(P) = e \) and \( \ell(P) = f \), and tuples \( P \in P^k(T, B) \) with \( t(P) = h_0(P) = e \) and \( v_1(P) = f \). On the latter set, one can now apply the bijection to \( P_2 \), then to \( P_3 \), and successively down to \( P_0 \), proving that tuples \( P \in P^k(T, B) \) with \( t(P) = e \) and \( v_1(P) = f \) are in turn in bijection with tuples \( P \in P^k(T, B) \) with \( t(P) = e \) and \( r(P) = v_k(P) = f \).

6. Corollaries and Applications

6.1. Dyck paths and generalizations. In the particular case that \( T = N^n E^n \) and \( B = (EN)^n \), the statistics \( t \) and \( b \) become two familiar Dyck path statistics: the height of the last peak and the number of returns, respectively. A bijective proof of the fact that these statistics are equidistributed on Dyck paths was given by Deutsch [12], who later also exhibited a recursively defined involution [13] proving the symmetry of their joint distribution. The equidistribution result is also a consequence of the more recent bijection \( \zeta \) due to Haglund [23, p. 50] and Andrews et al. [1].

Deutsch’s involution has recently been rediscovered by Mireille Bousquet-Mélou, Éric Fusy and Louis-François Préville-Ratelle [9], who consider the bijection between binary trees and Dyck paths obtained by reading the tree in postorder and recording an \( N \) for each leaf (except the first one) and an \( E \) for each internal node. As they point out, the involution on binary trees produced by reflecting along a vertical axes translates via this bijection into an involution on Dyck paths that switches the statistics \( t \) and \( b \). It is not hard to show that this operation coincides
with Deutsch’s involution (up to a minor modification in order to deal with the height of the last peak rather than the first), and in fact it provides a non-recursive description of it.

The symmetry of the statistics ‘height of the last peak’ and ‘number of returns’ on Dyck paths can also be proved using standard generating function techniques, based on the usual recursive decomposition of Dyck paths. However, neither these techniques nor the above bijections seem to extend to the general setting of Theorem 2.1. Our involution \( \Phi \), when restricted to the case of Dyck paths, is quite different from Deutsch’s involution and Haglund’s bijection. In addition to providing an extension to paths between arbitrary boundaries \( T \) and \( B \), our involution can also be used to prove the following.

**Corollary 6.1.** Let \( T \) and \( B \) be arbitrary paths with \( N \) and \( E \) steps from \((0,0)\) to \((x,y)\) and \( T \) weakly above \( B \). The following statements are equivalent:

(i) the number of paths in \( P(T,B) \) with \( i \) top and \( j \) bottom contacts depends only on \( i + j \);

(ii) for every path in \( P(T,B) \), all its bottom contacts occur before its top contacts;

(iii) the last east step of \( B \) is lower than the first east step of \( T \).

Similarly, the following statements are equivalent:

(i') if \( P(T,B,H) \neq \emptyset \) and \( i + j + |H| = x \), then \( P(T,B,H) \) contains precisely one path with \( i \) top and \( j \) bottom contacts;

(ii') for every path in \( P(T,B,H) \), all its bottom contacts occur before its top contacts.

**Proof.** Let us first show that (ii') implies (i'). When the bottom contacts of a path occur before its top contacts, all the letters in its sequence of contacts are unmatched. By assumption, this is the case for all paths in \( P(T,B,H) \). Thus we can apply Lemma 5.12: let \( D = \emptyset \), \( f = 0 \) and \( e = x - |H| \), the number of contacts of a path in \( P(T,B,H) \). We then obtain a sequence of bijections \( P_0 \xrightarrow{\phi} P_1 \xrightarrow{\phi} \cdots \xrightarrow{\phi} P_e \), where

\[ P_s = \{ P \in P(T,B,H) : t(P) = e - s, \ b(P) = s \} \].

Finally, as in the proof of Proposition 3.9 we can see that there is at most one path in \( P_0 \), because this is just the set of paths in \( P(T,B,H) \) with no bottom contacts.

Conversely assume (i') and \( P(T,B,H) \neq \emptyset \). Then there is precisely one path in \( P_0 \). The sequence of contacts in this path contains only unmatched letters. Since \( \phi \) preserves the number of unmatched letters, all paths in \( P(T,B,H) \) have sequences of contacts with unmatched letters only. This implies that a bottom contact cannot appear after a top contact in any path.

The equivalence of (i') and (ii') entails the equivalence of (i) and (ii), because we can use (i') and (ii') for all choices of \( H \) separately. Condition (ii) implies (iii) because if the last east step of \( B \) is at the same height as the first east step of \( T \) or higher we can choose a path beginning with a top contact and ending with a bottom contact. Finally, (iii) obviously implies (ii). \( \square \)

We remark that the naive generalization of Corollary 6.1 to \( k \)-tuples of paths does not hold. For example, in the set \( P^2(T,B) \) where \( T = NNEE \) and \( B = ENEN \), there is only one pair of paths with \( (h_0,h_1) = (0,1,2) \), but there are two pairs of paths with \( (h_0,h_1,h_2) = (1,1,1) \).
Corollary 6.2. Let $T = N^r E^x$, and let $B$ be any path from the origin to $(x, y)$, weakly below $T$ and ending with a north step. Then the number of paths in $P(T, B)$ with $i$ top and $j$ bottom contacts equals the number of paths with north and east steps from the origin to $(x - i - j, y - 2)$ staying weakly above $B$.

Proof. By Corollary 6.1 it is enough to count paths in $P(T, B)$ with $c := i + j$ top contacts and no bottom contacts. Such paths are in bijection (by removing the terminal $NE^x$) with paths from the origin to $(x - c, y - 1)$ strictly above $B$, which in turn are in bijection (by removing the initial $N$) with paths from the origin to $(x - c, y - 2)$ weakly above $B$, as claimed. \hfill \Box

In particular, Corollary 6.2 allows us to refine a formula due to Vladimir Korolyuk \cite{korolyuk} for the number of lattice paths above a linear boundary whose slope is an integer. For natural numbers $m$, $n$, $r$ and $k$, let $p_{r,k}(m, n)$ be the number of lattice paths with north and east steps from the origin to $(m, n)$ that do not go below the line $y = rx - k$. Korolyuk’s formula is equivalent to

$$
(6.1) \quad p_{r,k}(m, n) = \sum_{i=0}^{\lfloor \frac{m+k+1-rm}{n+k+1-ri} \rfloor} (-1)^i \binom{n+k+1-rm}{m+i} \binom{m+n-k-(r+1)i}{i} \binom{k-ri}{r_i},
$$

for $n \geq rm - k$. When $k = 0$, this equation trivially reduces to the generalized ballot theorem,

$$
(6.2) \quad p_{r,0}(m, n) = \frac{n+1-rm}{n+1} \binom{m+n}{m} - \frac{m+n}{m-1} \binom{m+n}{m-1}.
$$

Amazingly, for $r = 1$ Equation (6.1) also has a closed form,

$$
 p_{1,k}(m, n) = \binom{m+n}{m} - \binom{m+n}{m-1-k},
$$

which is not hard to prove using the reflection principle. For some information on the history of these results, see Marc Renault’s note \cite{renault}.

Using Korolyuk’s formula, Corollary 6.2 gives a formula for the number of paths from $(0, 0)$ to $(m, n)$ not going below the line $y = rx - k$ and having $i$ top and $j$ bottom contacts, assuming that $n > rm - k$. By reflecting the picture about the line $y = -x$, a similar result can be obtained for paths not going below a line of slope $1/r$.

It may be surprising that these are the only known formulas that we can use in applications of Corollary 6.2. Indeed, all the explicit formulas or generating function solutions for the number of paths above other boundaries we are aware of, e.g. \cite{elizalde2009, cr09, cr12, cr14, cr15}, are such that the starting and endpoints must lie on the boundary and thus cannot be used in conjunction with Corollary 6.2.

6.2. A decomposition of the generalized Tamari lattice into chains. The generalized Tamari lattices were introduced by Bergeron in the context of diagonal harmonics \cite{bergeron}. In this section, we exhibit a connection between the covering relation in these lattices and the map $\phi$ introduced in Definition 3.7.

For an integer $r \geq 1$, the elements of the $n$-th $r$-Tamari lattice are the paths in $P(T_n^r, B_n^r)$ where $T_n^r = N^n E^n$ and $B_n^r = (N E)^r$. By Equation (6.2), the generalized Tamari lattice has $p_{r,0}(n, rn) = \frac{1}{rn+1} \binom{(r+1)n}{n}$ elements.
Let \( Q \) be a path in \( \mathcal{P}(T_n^{(r)}, B_n^{(r)}) \) regarded as a sequence of east steps, disregarding the final \( r \) east steps, following our convention to ignore steps which are simultaneously top and bottom contacts. Consider any decomposition of \( Q \) of the form \( Q = Wb_1Xb_2Z \), where \( b_1 \) and \( b_2 \) are single east steps with different \( y \)-coordinates (possibly top or bottom contacts), \( X \) is maximal (possibly empty) such that the right endpoints of all its steps are strictly above the line of slope \( 1/r \) beginning at the right endpoint of \( b_1 \). Then \( Q \) is covered by the path \( P = WXb_2h_3Z \), where \( h_3 \) is a single east step with the same \( y \)-coordinate as \( h_2 \). The partial order in the generalized Tamari lattice, which we denote by \( \preceq \), is the reflexive and transitive closure of this covering relation. Its top element is \( T_n^{(r)} \) and its bottom element is \( B_n^{(r)} \).

The connection to the map \( \phi \) from Definition 3.7 is now immediate:

**Proposition 6.3.** Let \( P \in \mathcal{P}(T_n^{(r)}, B_n^{(r)}) \) such that \( \phi(P) \) is defined. Then \( P \) covers \( \phi(P) \) in the generalized Tamari lattice.

**Proof.** We can write the east steps of \( P \) uniquely as \( P = WXtt\ldots t \), where \( X \) is maximal not containing top or bottom contacts. Then \( \phi(P) = WbXt\ldots t \). Since all steps in \( X \) are above the line of slope \( 1/r \) beginning at the right endpoint of \( b \) in \( \phi(P) \), \( P \) indeed covers \( \phi(P) \). \( \square \)

**Corollary 6.4.** Let \( \mathcal{P}_0 \) be the set of paths in \( \mathcal{P}(T_n^{(r)}, B_n^{(r)}) \) without proper bottom contacts, i.e., which are not simultaneously top contacts. Then

\[
\bigcup_{P \in \mathcal{P}_0} (P, \phi(P), \phi^2(P), \ldots, \Phi(P))
\]

is a decomposition of the generalized Tamari lattice into \( p_{r,0}(n-1, r(n-1)) = \frac{1}{r(n-1)+1} \binom{(r+1)(n-1)}{n-1} \) saturated chains.

It turns out that, for \( r = 1 \), this decomposition is symmetric in the following sense.

**Proposition 6.5.** For \( P, Q \in \mathcal{P}(T_n^{(r)}, B_n^{(r)}) \) with \( Q \preceq P \), let \( d(Q, P) \) be the minimal distance from \( Q \) to \( P \) in the Hasse diagram of the generalized Tamari lattice. Then \( d(P, T_n^{(r)}) = rn - t(P) \). For the classical Tamari lattice we have \( d(B_n^{(1)}, P) = n - b(P) \).

**Proof.** Let us first show that for any path \( Q \) we have \( d(Q, T_n^{(r)}) \geq rn - t(Q) \). By definition of the covering relation, the covering path can have at most one more top contact than the path covered. Applying this to a saturated chain from \( Q \) to \( T_n^{(r)} \) of length \( d(Q, T_n^{(r)}) \), we obtain that \( t(Q) + d(Q, T_n^{(r)}) \geq t(T_n^{(r)}) = rn \).

To show that equality holds, we use induction on \( D(Q) = rn - t(Q) \). We have \( D(Q) = 0 \) only for \( Q = T_n^{(r)} \), in which case the formula for the distance is trivially correct. Otherwise, decompose \( Q \) as \( Q = Wb_1t\ldots t \) where \( b_1 \) is the last east step of \( Q \) which is not a top contact. Thus \( Q \) is covered by the path \( P = Wtt\ldots t \) and \( t(P) = t(Q) + 1 \). By the induction hypothesis the formula is correct for \( P \). Since \( d(Q, T_n^{(r)}) \leq d(P, T_n^{(r)}) + 1 \), we have

\[
rn - t(Q) \leq d(Q, T_n^{(r)}) \leq d(P, T_n^{(r)}) + 1 = rn - t(P) + 1,
\]

and therefore equality.
We prove the formula for the distance of a path to the bottom element of the classical Tamari lattice (i.e., \( r = 1 \)) similarly, with the two main ingredients being the following. First, we use the fact that in any covering relation, the covered path can have at most one more bottom contact than the covering path. Let us remark here that the generalized Tamari lattice does not have this property. Second, if \( P \neq B_n^{(1)} \), we decompose it as \( P = WXh_2h_1Z \) where \( h_3 \) is the last bottom contact preceded by an east step with the same \( y \)-coordinate, and \( X \) is maximal such that the right endpoints of all its (east) steps are strictly above the line of slope 1 passing through the right endpoint of \( h_2 \). Then \( P \) covers \( Q = Wh_1Xh_2Z \), where \( h_1 \) and \( h_2 \) are both bottom contacts. In both \( P \) and \( Q \) the east steps in \( X \) do not contain any bottom contacts.

\[ \square \]

6.3. Two conjectures. Given Theorems 6.1 and 6.2, it is natural to ask whether there are other pairs of lattice paths statistics involving contacts that have the same distribution. As a first tentative answer to this question we offer the following conjecture:

**Conjecture 6.6.** Suppose that \( T \) and \( B \) touch only at the origin and at their common endpoint. Then the following statements are equivalent:

(i) the pairs \((b, \ell)\) and \((b, t)\) have the same joint distribution over \( \mathcal{P}(T, B) \),

(ii) the pairs \((b, \ell)\) and \((\ell, r)\) have the same joint distribution over \( \mathcal{P}(T, B) \),

(iii) the pairs \((t, r)\) and \((b, t)\) have the same joint distribution over \( \mathcal{P}(T, B) \),

(iv) the pairs \((t, r)\) and \((\ell, r)\) have the same joint distribution over \( \mathcal{P}(T, B) \),

(v) either \( T = (NE)^n \) or \( B = (EN)^n \) for some \( n \).

Rotating the region by 180 degrees, or reflecting it about the line \( y = -x \) or \( y = x \) we see that it is sufficient to prove the equivalence of the joint distribution of \((b, \ell)\) and \((b, t)\) with the assertion that either \( T = (NE)^n \) or \( B = (EN)^n \).

Moreover, when \( T = (NE)^n \), each top contact coincides with a left contact. Thus, in this case it is clear that \((b, \ell)\) and \((b, t)\) have the same joint distribution. When \( B = (EN)^n \), each bottom contact coincides with a right contact and we obtain \((b, \ell) \sim (t, r) \sim (t, b) \sim (b, t)\) by applying Theorems 6.1 and 6.2.

We remark that \((b, r) \sim (r, b)\) seems to force symmetry of the bottom boundary \( B \), but symmetry of the top boundary \( T \) is neither sufficient nor necessary, as the example \( T = N^2E^2EN^2E^n \) shows.

Another natural question arises from Corollary 6.6 which deals only with the statistics \((b, t)\): are there regions for which we have an analogous result for the pair \((b, \ell)\)? It appears that the answer is affirmative only when the result holds trivially:

**Conjecture 6.7.** Suppose that \( T \) and \( B \) touch only at the origin and at their common endpoint. Then the number of paths in \( \mathcal{P}(T, B) \) with \( i \) bottom and \( j \) left contacts depends only on \( i + j \) if and only if \( T = N^nE^n \) and \( B = (EN)^n \), or \( T = (NE)^n \) and \( B = E^nN^n \) for some \( n \).

We checked the above two conjectures for all pairs of paths \( T \) and \( B \) from the origin to \((n, n)\) for \( n \leq 6 \).

6.4. Patterns in permutations. We now describe an application of Theorem 3.1 to restricted permutations. Let \( S_n \) denote the set of permutations of \([n]\).

**Definition 6.8.** Let \( \pi \in S_n \). We say that \( \pi(i) \) is a right-to-left minimum (right-to-left maximum) of \( \pi \) if \( \pi(i) < \pi(j) \) (respectively, \( \pi(i) > \pi(j) \)) for all \( j > i \). For
1 < i < n, we say that \( \pi \) has an occurrence of the (dashed) pattern 13-2 at position i if there is a j > i + 1 such that \( \pi(i) < \pi(j) < \pi(i + 1) \).

For example, the permutation 35681742 has occurrences of 13-2 at positions 1, 3, and 5. The right-to-left minima of this permutation are 1, 2, and its right-to-left maxima are 8, 7, 4, 2.

**Proposition 6.9.** The set of permutations in \( S_n \) with \( e \) right-to-left minima, \( f \) right-to-left maxima, and having occurrences of the pattern 13-2 exactly at positions \( D \) is in bijection with the set of paths in \( \tilde{P}(T, B, D) \) with \( e \) top contacts and \( f \) bottom contacts, where \( T = N^n E^n \) and \( B = (EN)^n \).

**Proof.** Given a path \( P \in \tilde{P}(T, B, D) \), let \( y_1, y_2, \ldots, y_n \) be the sequence of y-coordinates of its east steps from left to right, and let \( p_i = n + 1 - y_i \) for \( 1 \leq i \leq n \). We associate to \( P \) a permutation \( \pi \in S_n \) as follows: \( \pi(1) = p_1 \) and, for each \( i \) from 2 to \( n \), let \( \pi(i) \) be the \( p_i \)-th smallest number in \( [n] \setminus \{\pi(1), \pi(2), \ldots, \pi(i - 1)\} \). An example is drawn in Figure 7.

With this definition, the \( i \)-th east step of \( P \) is a top contact if \( p_i = 1 \), which happens if and only if \( \pi(i) \) is the smallest number in \( \{\pi(i), \pi(i + 1), \ldots, \pi(n)\} \), and hence a right-to-left minimum. Similarly, the \( i \)-th east step of \( P \) is a bottom contact if and only if \( \pi(i) \) is a right-to-left maximum.

Next we prove that descents in \( P \) correspond to occurrences of the pattern 13-2 in \( \pi \). Fix \( i \), and suppose that the elements in \( [n] \setminus \{\pi(1), \pi(2), \ldots, \pi(i - 1)\} \), when listed in increasing order, are \( r_1 < r_2 < \cdots < r_{n-i} \). Let \( a = p_i \), \( b = p_{i+1} \), and note that \( \pi(i) = r_a \) by definition. Clearly, \( P \) has a descent in position \( i \) if and only if \( y_i > y_{i+1} \), which is equivalent to \( a < b \). In this case, \( \pi(i + 1) = r_{a+1} \), and there is \( j > i \) such that \( \pi(j) = r_{b} \), so that \( \pi(i)\pi(i + 1)\pi(j) \) is an occurrence of 13-2. On the other hand, if \( P \) has no descent in position \( i \), there are two possibilities. If \( a > b \), then \( \pi(i + 1) = r_{b} < r_{a} = \pi(i) \); if \( a = b \), then \( \pi(i + 1) = r_{a+1} > r_{a} = \pi(i) \), but there is no \( j > i \) with \( \pi(i) < \pi(j) < \pi(i + 1) \). In both cases, \( \pi \) has no occurrence of 13-2 at position \( i \). \( \square \)

**Figure 7.** The path corresponding to the permutation 35681742.

**Corollary 6.10.** Let \( D \subseteq [n - 1] \). In the set of permutations \( \pi \in S_n \) having occurrences of 1-32 exactly at positions \( D \), the joint distribution of the statistics ‘number of right-to-left minima’ and ‘number of right-to-left maxima’ is symmetric.
6.5. Watermelon configurations. As a consequence of Theorem 2.3, we can recover a theorem of Brak and Essam [8, Corollary 1] concerning certain families of \(k\) non-intersecting paths called watermelon configurations.

**Definition 6.11.** A watermelon configuration of length \(x\) and deviation \(y\) is a family of \(k\) non-intersecting lattice paths with northeast \((1, 1)\) and southeast \((1, -1)\) steps, starting at \((0, 2i)\) and terminating at \((x, y + 2i)\) for \(0 \leq i \leq k - 1\), not going below the \(x\)-axis.

Brak and Essam derive the following statement using manipulations of a determinant.

**Theorem 6.12** ([8]). The number of watermelon configurations of length \(x\) and deviation \(y\) whose bottom path has \(e\) returns to the \(x\)-axis is the same as the number of families of \(k\) non-intersecting paths where the lower \(k-1\) paths form a watermelon configuration of length \(x\) and deviation \(y\), and the top path terminates at \((x - e - 1, y + 2k + e - 3)\).

Christian Krattenthaler [28, Proposition 6] gives a bijective proof by transforming the configurations into certain semistandard Young tableaux and applying a variant of *jeu de taquin*. However, a more straightforward proof can be given by interpreting it as a special case of Theorem 2.3.

**Proof.** Any watermelon configuration of length \(x\) and deviation \(y\) can be transformed into a family \(P\) of paths in \(\mathcal{P}^k(T, B)\) with \(T = N^{(x+y)/2}E^{(x-y)/2}\) and \(B = (NE)^{(x-y)/2}N^y\), by letting all paths start at the origin and converting each northeast (southeast) step of a path in the watermelon configuration to a north (respectively east) step.

Doing so, the returns to the \(x\)-axis of the bottom path of the watermelon configuration become the bottom contacts of the lower path in \(P\), which are counted by \(b(P)\). The proof of Theorem 2.3 (or alternatively, Theorem 2.4) gives a bijection between tuples \(P \in \mathcal{P}^k(T, B)\) with \(b(P) = e\) and tuples \(P \in \mathcal{P}^k(T, B)\) with \(t(P) = e\). In the latter tuples, the upper path must have its \(e\) top contacts at the end, so it is a path from the origin to \(((x-y)/2 - e, (x+y)/2 - 1)\), followed by the steps \(NE^e\). Removing these forced steps from the corresponding watermelon configuration, we obtain a family as described in the statement. \(\square\)

We remark that our bijection is different from Krattenthaler’s. One might, however, speculate about a connection between *jeu de taquin* and our theorem.

Even though Corollary 6.2 does not seem to generalize nicely to families of paths, Krattenthaler gives an explicit expression [28, Lemma 7] for the number of watermelon configurations with a given number of bottom contacts, based on the theorem above. Via the Lindström–Gessel–Viennot method for non-intersecting lattice paths, this amounts to a determinant evaluation. It should also be possible to give explicit expressions for the number of families of paths when the lower boundary has arbitrary integer slope.

7. Flagged semistandard Young tableaux and \(k\)-triangulations

In this section we discuss connections of Theorem 2.3 which concerns \(k\)-tuples of non-crossing lattice paths, with two other combinatorial objects: flagged semistandard Young tableaux, and generalized triangulations of a convex polygon.
A \textit{k-triangulation} of a convex \( n \)-gon is a maximal set of diagonals such that no \( k + 1 \) of them mutually cross. In particular, a 1-triangulation is a triangulation in the usual sense. Note that every \( k \)-triangulation contains all the diagonals between vertices at distance \( k \) or less, where distance is the number of sides of the polygon that separate the two vertices. We call these \textit{trivial} diagonals, and we disregard them when computing the degree of a vertex. The \textit{neighbors} of a vertex are then the vertices connected to it by a nontrivial diagonal.

It was shown in \cite{14, 30} that all \( k \)-triangulations of a convex \( n \)-gon have the same number \( k(n - 2k - 1) \) of nontrivial diagonals. Jonsson \cite{20} proved that the number of \( k \)-triangulations of an \( n \)-gon is given by the \( k \times k \) determinant of Catalan numbers \( \det (C_{n-i-j})_{i,j=1}^k \). Denoting by \( D_n^k \) the set \( \mathcal{P}_k(T, B) \) where \( T = \{ \infty \} \), and \( B = \{ (EN)^{n-2k-1} \} \), the above determinant is also known to equal \( |D_n^k| \), by the Lindström–Gessel–Viennot method \cite{20}. The elements of \( D_n^k \) are \( k \)-fans of Dyck paths of semilength \( n - 2k \) (or, in the terminology from Section 6.5, watermelon configurations with deviation 0). There are several well-known bijections between 1-triangulations and Dyck paths. For \( k = 2 \), the first bijection between 2-triangulations and pairs of non-crossing Dyck paths was given in \cite{10}. For general \( k \), a bijection between \( k \)-triangulations of the \( n \)-gon and \( D_n^k \) has been recently found by Serrano and Stump \cite{34}.

Nicolas \cite{32} Conjecture 1] discovered experimentally that the distribution of degrees of \( k \) consecutive vertices over the set of \( k \)-triangulations of a convex \( n \)-gon equals the distribution of \( (h_0, h_1, \ldots, h_{k-1}) \) over \( D_n^k \). This is stated as Conjecture \cite{25} above. Serrano and Stump's bijection \cite{34} Theorem 4.4] proves a special case of this conjecture: the degree of any given vertex in the set of \( k \)-triangulations of a convex \( n \)-gon is equidistributed with the number of top contacts of the upper path in \( D_n^k \).

To prove Conjecture \cite{25} in full generality, we construct a new bijection between tuples of paths and certain semistandard Young tableaux. In the rest of this section, let \( T = \{ \infty \} \) and let \( B \) be a path from the origin to \((x, y)\), weakly below \( T \) and ending with a north step. The region enclosed by \( T \) and \( B \) can then be interpreted in an obvious way as the Young diagram (in English notation) of a partition \( \lambda = \lambda(T, B) \). The parts of \( \lambda \) are the lengths of the rows of the diagram, from top to bottom, so that \( x = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_y \geq 0 \).

Recall that a filling of such a diagram with positive integer entries is called a \textit{Young tableau of shape} \( \lambda \), and that such a tableau is called \textit{semistandard} (a \textit{SSYT} for short) if the entries in its rows are weakly increasing and the entries in its columns are strictly increasing. A Young tableau (or simply tableau, from now on) is called \textit{k-flagged} if the entries of row \( r \) are at most \( k + r \) for every \( r \). Finally, the \textit{weight} of a tableau is \( \mu = (\mu_1, \mu_2, \ldots) \) where \( \mu_i \) is the number of entries equal to \( i \).

As in Section \cite{5} given \( P = (P_1, P_2, \ldots, P_k) \in \mathcal{P}_k(T, B) \), with the convention that \( P_0 = T \) and \( P_{k+1} = B \), let \( h_i(P) \), for \( 0 \leq i \leq k \), be the number of east steps where \( P_i \) and \( P_{i+1} \) coincide. Also, let \( u_s(P) \), for \( 1 \leq s \leq y - 1 \), be the number of east steps with \( y \)-coordinate \( y - s \) that lie strictly between \( T \) and \( B \) and are not used by any of the paths \( P_1, \ldots, P_k \).

We can now state the main result of this section.

**Theorem 7.1.** Let \( T = \{ \infty \} \) and let \( B \) be any path from the origin to \((x, y)\), weakly below \( T \) and ending with a north step. There is an explicit bijection \( \Psi \) between \( k \)-tuples of paths \( P \in \mathcal{P}_k(T, B) \) with \( h_i(P) = h_i \) \((0 \leq i \leq k)\) and \( u_s(P) = u_s \).
(1 ≤ s < y), and k-flagged SSYT of shape λ = λ(T, B) and weight

\[(λ_1 - h_0, λ_1 - h_1, \ldots, λ_1 - h_k, u_1, u_2, \ldots, u_{y-1}).\]

**Figure 8.** An example of the weight-preserving bijection Ψ from Theorem 7.1. The 3-tuple on the left has \((h_0, h_1, h_2, h_3) = (4, 3, 3, 3)\) and \((u_1, u_2, u_3, u_4) = (2, 2, 1, 1)\). The 3-flagged SSYT on the right has \(λ_1 = 6\) and weight \(2, 3, 3, 3, 2, 2, 1, 1\).

Figure 8 shows an example of the bijection Ψ, which will be described in Section 7.2. The importance of Ψ lies in the particular way that the statistics on tuples of paths determine the weight of the corresponding tableau. This property is what allows us in Section 7.1 to use Ψ to prove and generalize Conjecture 2.5. In fact, a much easier bijection between \(k\)-tuples of paths in \(P_k(T, B)\) and \(k\)-flagged SSYT of shape \(λ(T, B)\) can be constructed as follows. Given \(P ∈ P_k(T, B)\), first fill each cell of the Young diagram with the number of paths in \(P\) that pass above the cell (this produces a reverse plane partition where all entries are at most \(k\)). Then, for each row \(r\), add \(r\) to every entry in that row. Figure 9 shows an example of this construction. This bijection, which is used by Serrano and Stump [34], does not have the property described in Theorem 7.1, but it proves the following fact.

**Lemma 7.2 ([34]).** Let \(T = N^y E^x\), and let \(B\) an arbitrary path from the origin to \((x, y)\). Then \(|P_k(T, B)|\) equals the number of \(k\)-flagged SSYT of shape \(λ(T, B)\).

**Figure 9.** An example of the non-weight-preserving bijection that proves Lemma 7.2.

It may be surprising that the bijection Ψ only works for the special top boundary of the region as stated in Theorem 7.1. Indeed, one might be led to believe that paths in an arbitrary region should be in bijection with a suitable set of skew-semistandard Young tableaux. We remark that our proof breaks in a subtle way in this setting. It would be interesting to find such a generalization.
7.1. Consequences. Before we prove Theorem 7.1, let us first show how to infer
Conjecture 2.5 from it. To do so, we use a result of Serrano and Stump [34]. To
simplify the statements of Theorems 7.3 and 7.4, we change the labeling of the
vertices of the polygon, so that our vertex \( i \) corresponds to vertex \( n + 1 - i \) in the
notation of [34]. In the next two theorems, \( \mathcal{T}_n^k \) denotes the set of \( k \)-triangulations
of a convex \( n \)-gon with vertices counterclockwise labeled from 1 to \( n \). In such a
\( k \)-triangulation, \( d_i \) denotes the number of (nontrivial) neighbors of vertex \( i \) among
\( i+1, i+2, \ldots, n \). Note that for \( i \geq n-k \) we have \( d_i = 0 \), and that for \( 1 \leq i \leq k+1, \)
d\( i \) is the degree of vertex \( i \), since all its neighbors are among \( i+1, i+2, \ldots, n \). We
call \((d_1, \ldots, d_{n-k-1})\) the degree sequence of the \( k \)-triangulation. The next theorem
is equivalent to [34 Corollary 3.5]. The description of the weight of the \( k \)-flagged
tableau is not given explicitly in [34], but it is immediate from the Edelman-Greene
insertion process that is used.

**Theorem 7.3 ([34]).** There is an explicit bijection between \( k \)-triangulations in \( \mathcal{T}_n^k \)
with degree sequence \((d_1, \ldots, d_{n-k-1})\) and \( k \)-flagged SSYT of shape \((n-2k-1, n-2k-2, \ldots, 1)\) with weight \((\mu_1, \mu_2, \ldots, \mu_{n-k-1})\), where

\[
\mu_i = \begin{cases} 
    n - 2k - 1 - d_i & \text{if } 1 \leq i \leq k + 1, \\
    n - k - i - d_i & \text{if } k + 1 < i \leq n - k - 1.
\end{cases}
\]

Combining Theorems 7.1 and 7.3, we obtain the following refinement of Conjecture 2.5. Note that this refinement gives a simple description of the distribution of the degrees \( d_1, d_2, \ldots, d_{k+1} \) of \( k+1 \) consecutive vertices, and not just \( k \) as
in the original conjecture. In particular, it implies that the joint distribution of
\((d_1, d_2, \ldots, d_{k+1})\) over \( \mathcal{T}_n^k \) is symmetric.

**Theorem 7.4.** There is an explicit bijection between \( k \)-triangulations in \( \mathcal{T}_n^k \) with
degree sequence \((d_1, \ldots, d_{n-k-1})\), and \( k \)-tuples in \( D_n^k \) with parameters \((h_0, h_1, \ldots, h_k)\) and
\((u_1, u_2, \ldots, u_{n-2k-2})\), where

\[
d_i = \begin{cases} 
    h_{i-1} & \text{if } 1 \leq i \leq k + 1, \\
    n - i - k - u_{i-k-1} & \text{if } k + 1 < i \leq n - k - 1.
\end{cases}
\]

**Proof.** The bijection from Theorem 7.3 sends \( k \)-triangulations of the \( n \)-gon to \( k \)-flagged SSYT of shape \((n-2k-1, n-2k-2, \ldots, 1)\), which in turn become \( k \)-tuples in \( D_n^k \) applying the bijection \( \Psi \) from Theorem 7.1 Comparing the weight of the tableau in both bijections, we see that for \( 1 \leq i \leq k+1, \) we have \( n-2k-1-d_i = \mu_i = \lambda_1 - h_{i-1}, \) and since \( \lambda_1 = n-2k-1, \) it follows that the degree of vertex \( i \) is \( d_i = h_{i-1}, \)
Similarly, for \( k+1 < i \leq n - k - 1, \) we have \( n - k - i - d_i = \mu_i = u_{i-k-1}. \)

Figure 10 shows an example of the composition of bijections in Theorem 7.4
In this example, \((d_1, \ldots, d_5) = (1, 2, 2, 0, 1), (\mu_1, \ldots, \mu_5) = (2, 1, 1, 2, 0), \lambda_1 = 3, (h_0, h_1, h_2) = (1, 2, 2), \) and \((u_1, u_2) = (0, 1). \)

We remark that \( k \)-flagged SSYT appear frequently in the literature because the
generating functions of all such tableaux of a given shape with respect to their
weight, called \( k \)-flagged Schur functions, are Schubert polynomials for certain vex-
illary permutations [34, 37]. In particular, we have the following determinantal
expression due to Michelle Wachs [37 Theorem 3.5]. An alternative proof using the
determinantal formula for the number of non-intersecting lattice paths was
given by Ira Gessel and Xavier Viennot [21 Corollary 4].
Theorem 7.3

\[\rightarrow\]

\[\rightarrow\]

\[\Psi^{-1}\]

Problem 7.4

An example of the bijection from Theorem 7.4.

Proposition 7.5

The generating function for \(k\)-flagged SSYT \(S\) of shape \(\lambda\) is

\[
\sum_S x^S = \det (h_{\lambda_i-i+j}(x_1, \ldots, x_{k+i}))_{1 \leq i,j \leq \ell},
\]

where \(x^S = \prod_i x_i^{\mu_i}\) and \((\mu_1, \mu_2, \ldots)\) is the weight of \(S\), \(\ell\) is the number of parts of \(\lambda\), and \(h_n(x_1, \ldots, x_{k+i}) = \sum_{1 \leq j_1 \leq \cdots \leq j_n \leq k+i} x_{j_1} \cdots x_{j_n}\) is the complete homogeneous symmetric function.

The above proposition implies that \(k\)-flagged Schur functions are symmetric in the variables \(x_1, \ldots, x_{k+1}\). Alternatively, this can be proved combinatorially in the same way that one proves that Schur functions, defined as generating functions of SSYT, are symmetric \([35, \text{Theorem 7.10.2}]\). Using this symmetry, Theorem 7.1 provides an alternative proof of Theorem 2.3 in the case that \(T = N^y E^x\). Indeed, the first \(k+1\) components of the weight have a symmetric joint distribution over \(k\)-flagged SSYT of a given shape \(\lambda\), which, by Theorem 7.1, is the same distribution of \((\lambda_1 - h_0, \ldots, \lambda_1 - h_k)\) over \(P_k(T, B)\), where \(T\) and \(B\) are the boundaries of the shape \(\lambda\). Theorem 2.3 follows now immediately for such \(T\) and \(B\).

7.2. Proof of Theorem 7.1

Throughout this section we assume that \(T = N^y E^x\) and that \(B\) is a path from the origin to \((x, y)\). We have defined \(k\)-flagged tableaux without the requirement of being semistandard because of a technical necessity that will become apparent below. We also need a variation of perforated tableaux as introduced by Georgia Benkart, Frank Sottile and Jeffrey Stroomer \([4]\). We say that an entry \(e\) in a \(k\)-flagged tableau is small (respectively large) if \(e \leq k+1\) (respectively \(e > k+1\)). An entry \(e\) in row \(r\) is called maximal if \(e = k+r\). Note that the entries in the first row of a \(k\)-flagged tableau are necessarily small.

Definition 7.6. A \(k\)-perflagged tableau is a \(k\)-flagged Young tableau where any pair of entries \(e_1, e_2\) with \(e_2\) weakly southeast of \(e_1\) satisfies the following conditions:

- If \(e_1\) and \(e_2\) are in the same row, and both are small or both are large, then \(e_1 \leq e_2\).
- If \(e_1\) and \(e_2\) are in different rows and both are large, then \(e_1 < e_2\).
- If \(e_1\) and \(e_2\) are in different rows and both are small, then \(e_1 \leq e_2\). Furthermore, if \(e_1 = e_2\), the following chain condition must be met:

  Suppose that \(e_1\) is in a cell \(a\) in row \(r\) and \(e_2\) is in a cell \(b\) in row \(r+s\). Then there is a sequence of cells \(c_1, \ldots, c_s\) (called a chain) such that for \(1 \leq i \leq s\), cell \(c_i\) is in row \(r+i\) and weakly east of \(c_{i-1}\) (with the convention...
that \( c_0 = a \), and the entry in \( c_1 \) is no larger than the entry in the cell just northeast of \( c_1 \). Furthermore, \( c_x \) is strictly west of \( b \) (in the same row). In particular, \( a \) and \( b \) cannot be in the same column.

Intuitively, a \( k \)-flagged tableau is \( k \)-perflagged if, when restricting to large (respectively small) entries, the semistandard condition (weakly increasing along rows and strictly increasing along columns) is met, with the caveat that a small entry is allowed to be equal to another one strictly southeast of it in some cases. When this happens, any small entries inside the minimal rectangle containing these two equal entries must also be equal to them. For example, the 2-perflagged tableau

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 \\
3 & 4 & 4 & 4 \\
4 & 5 & 5 & 6 \\
6 & 6 & 6 & 3 \\
\end{array}
\]

has two small entries equal to 3 in different rows, one being southeast of the other. The chain condition is satisfied by taking \( c_1 \) to be the cell containing a 4 in the first column, and \( c_2 \) to be the cell containing a 6 in the third column.

To prove Theorem 7.1, we construct a bijection \( \Psi \) in two parts. First, we associate in a simple way a \( k \)-perflagged tableau to each \( k \)-tuple of paths in \( P_k(T, B) \). This tableau has the weight claimed in Theorem 7.1 but in general it will not be semistandard. Then, we modify the tableau using local moves reminiscent of jeu de taquin to remove any occurring violation of the semistandard property. It will be clear that the weight of the tableau is preserved by these operations. However, proving that the operations are invertible and well defined will require some work.

**Definition 7.7.** To each \( k \)-tuple of paths \( P \in P_k(T, B) \) we associate a Young tableau \( \text{Tab}(P) \) of shape \( \lambda(T, B) \) as follows:

(i) for each cell of the Young diagram whose upper boundary is an east step of a path \( P_i \) (with \( i \geq 0 \), where we again use the convention \( P_0 = T \)), take the largest such \( i \) and fill the cell with the number \( i + 1 \);

(ii) fill the remaining cells in each row \( r \) with \( k + r \).

For an example of this construction, see the first step in Figure 11. Note that step (i) fills the cells with small entries, while step (ii) fills the remaining cells with maximal entries. It is clear that for any \( P \in P_k(T, B) \), \( \text{Tab}(P) \) is a \( k \)-perflagged tableau. Indeed, rows are weakly increasing, and if \( e_1 \) is strictly north and weakly west of \( e_2 \) and entries \( e_1, e_2 \) are both large or both small, then \( e_1 < e_2 \) (in particular, the chain condition is not needed).

**Definition 7.8.** A semistandard violation of a \( k \)-perflagged tableau \( S \) is a cell containing an entry \( e \) such that either

- there is a cell immediately above, whose entry \( e_a \) satisfies \( e_a \geq e \), or
- there is a cell immediately to the left, whose entry \( e_l \) satisfies \( e_l > e \).

The minimal semistandard violation of \( S \) is the one with the smallest entry, and among those with the same smallest entry, the one in the leftmost column.

Note that, since \( S \) is \( k \)-perflagged, a semistandard violation as defined above can only happen if \( e \) is a small entry, and either \( e_a \) or \( e_l \) is a large entry. In the 3-perflagged tableau on the left of Figure 11, the four semistandard violations are marked by diamonds, and the minimal one is the entry 3 in the first column.
**Definition 7.9.** A path violation of a $k$-perflagged tableau $S$ is a cell containing a small entry $e$, such that either

1. there is a cell immediately below containing a large entry which is not maximal, or
2. there is a cell immediately to the right containing a large entry, and all the small entries in this cell’s column up to the top row are strictly less than $e$.

The maximal path violation of $S$ is the one with the largest entry, and among those with the same largest entry, the one in the rightmost column.

In the 3-perflagged tableau on the right of Figure 11, the two path violations are circled, and the maximal one is the entry 4 in the third column.

Clearly, a $k$-perflagged tableau with no semistandard violations is a $k$-flagged SSYT. Now we show that $k$-perflagged tableaux with no path violations correspond to $k$-tuples of paths via the construction in Definition 7.7.

**Lemma 7.10.** The map $P \mapsto \text{Tab}(P)$ is a bijection between $\mathcal{P}^k(T,B)$ and the set of $k$-perflagged tableaux of shape $\lambda(T,B)$ with no path violations. The weight of $\text{Tab}(P)$ in terms of $h_i(P)$ and $u_s(P)$ is as in Theorem 7.1.

*Proof.* First we show that for any $P \in \mathcal{P}^k(T,B)$, the $k$-perflagged tableau $\text{Tab}(P)$ has no path violations. By step (ii) in Definition 7.7, all large entries in row $r$ equal $k + r$, and so $\text{Tab}(P)$ has no path violations of type $\mathbf{1}$. To see that it has no path violations of type $\mathbf{2}$, suppose for contradiction that cell $c$ is such a violation, and let $e$ be the entry in $c$. By construction, the upper boundary of $c$ is an east step of $P_{e-1}$. Since $c$ has a cell immediately to its right, the next east step of $P_{e-1}$ is not a bottom contact, and thus it is the upper boundary of a cell containing a small entry $e' \geq e$, contradicting that $c$ is a path violation.

To show that the map $P \mapsto \text{Tab}(P)$ is a bijection, we describe its inverse. Given a $k$-perflagged tableau $S$ of shape $\lambda(T,B)$ with no path violations, construct an element of $\mathcal{P}^k(T,B)$ as follows. For each $1 \leq j \leq x$, consider the $j$-th column of the tableau, and suppose that the small entries of $S$ in this column are $e_1 < e_2 < \cdots < e_x$. Note that they increase from top to bottom because $S$ is $k$-perflagged. Now let the $j$-th east step of paths $P_1, P_2, \ldots, P_{e_{j-1}}$ be the upper boundary of the cell containing $e_1$, and for each $2 \leq i \leq s$, let the $j$-th east step of $P_{e_{i-1}}, P_{e_{i-1}+1}, \ldots, P_{e_i}$ be the upper boundary of the cell containing $e_i$. Finally, let the $j$-th step of $P_{e_s}, P_{e_{s+1}}, \ldots, P_k$ coincide with the $j$-th east step of $B$.

To see that the heights of the east steps of each path weakly increase from left to right, suppose the $j$-th step of path $P_j$ is the upper boundary of a cell $c$ containing the entry $e$. If there is no cell to the right of $c$, then the $(j + 1)$-st step of $P_j$ is at least as high as its $j$-th step. If there is such a cell, then the fact that $c$ is not a path violation guarantees that there is a small entry of size at least $e$ in column $j + 1$, in the same row as $c$ or above. Thus, the $(j + 1)$-st step of $P_j$ is at least as high as its $j$-th step also in this case. Thus, by adding north steps at the obvious places, this construction produces an element $P \in \mathcal{P}^k(T,B)$.

Now we show that this $k$-tuple satisfies $\text{Tab}(P) = S$. First, it is clear from the construction that the small entries of these tableaux agree. Second, the large entries of $S$ in row $r$ equal $k + r$, and so they agree with $\text{Tab}(P)$ as well. Indeed, if $S$ had a large entry that is not maximal, then the entry immediately above it would be either a path violation of type $\mathbf{1}$, or another non-maximal large entry (since...
large entries in any given row strictly increase from top to bottom). In the second case, considering the entry immediately above and repeating the argument would eventually lead to a path violation of type (I), since all the entries in the top row are small entries. It is also clear that \( P \) is the unique \( k \)-tuple of paths satisfying \( \text{Tab}(P) = S \), because all the east steps of the paths are uniquely determined by the small entries in \( S \).

Finally we prove that the weight is as claimed. Let us first consider large entries. A cell in row \( r \) of \( \text{Tab}(P) \), where \( 2 \leq r \leq y \) (recall that the first row has no large entries), is filled with \( k + r \) if and only if its upper boundary is an unused east step (i.e., it does not belong to any path in \( P \)) with \( y \)-coordinate \( y - r + 1 \). Regarding small entries, an entry \( e \) with \( 1 \leq e \leq k + 1 \) appears in a given column of \( \text{Tab}(P) \) if and only if the east steps of \( P_{e-1} \) and \( P_e \) in this column do not coincide. Since the number of east steps of any path in \( P \) is \( \lambda_1 \), the total number of cells filled with \( e \) is \( \lambda_1 - h_{e-1}(P) \). □

We now define invertible, weight-preserving, local operations on \( k \)-perflagged tableaux that ‘correct’ their violations.

**Definition 7.11.** Let \( S \) be a \( k \)-perflagged tableau having its minimal semistandard violation at cell \( c \). Let \( e, e_a, e_l \) be the entries in, above, and to the left of \( c \), respectively. (If one of these cells is missing, we define the corresponding entry to be 0.) Define \( j(S) \) to be the tableau obtained from \( S \) by swapping \( e \) and \( e_l \) if \( e_l > e_a \), and by swapping \( e \) and \( e_a \) otherwise.

In the rest of this section, we will use \( c \) to denote the minimal semistandard violation of a \( k \)-perflagged tableau \( S \), and we will use \( e, e_a \) and \( e_l \) to denote the entries in, above, and to the left of \( c \), respectively.

**Definition 7.12.** Let \( S \) be a \( k \)-perflagged tableau having its maximal path violation at cell \( d \). Let \( f, f_b, f_r \) be the entries in, below and to the right of \( d \), respectively. (In case one of these cells is missing, define the corresponding entry to be 0.) Define \( j^{-1}(S) \) to be the tableau obtained from \( S \) by swapping \( f \) with whichever of \( f_r \) or \( f_b \) is the smallest large entry, or, in case of a tie, with \( f_b \).

Note that in the above definition, at least one of \( f_r \) (for a path violation of type (I)) or \( f_b \) (for type (II)) is a large entry.

The rest of the proof will proceed as follows. In Lemma 7.13 we show that the operation \( j \) preserves the \( k \)-perflagged property. Lemma 7.14 describes how path and semistandard violations change when \( j \) is applied. This and Lemma 7.15 are needed to prove that the minimal semistandard violation of \( S \) becomes the maximal path violation of \( j(S) \) (Lemma 7.16), which is in turn used to show that \( j^{-1} \) is indeed the inverse of \( j \) (Lemma 7.17).

**Lemma 7.13.** Let \( S \) be a \( k \)-perflagged tableau. Then \( j(S) \) is also a \( k \)-perflagged tableau.

Before proceeding to the proof let us remark that it is indeed possible that in \( j(S) \) there are two equal small entries, one strictly southeast of the other, even if there is no such pair in \( S \). This is demonstrated by the following example, where
the transformation $j$ is applied to a 1-perflagged tableau:

\[
\begin{array}{ccc}
1 & 1 & 2 \\
3 & 3 & 3 \\
2 & 2 & 5 \\
\end{array} \quad \xrightarrow{j} \quad \begin{array}{ccc}
1 & 1 & 2 \\
2 & 3 & 3 \\
3 & 3 & 2 \\
\end{array}
\]

Proof. Let $c$, $e$, $e_a$ and $e_l$ be defined as above. Using that $e$ is a small entry, $\max\{e_l, e_a\}$ is a large entry, and small entries in $S$ are weakly increasing in rows and strictly increasing in columns, there are four possibilities for the relative values of these entries:

(i) $e_l \leq e \leq k + 1 < e_a$ ($e_l$ small),
(ii) $e \leq k + 1 < e_l \leq e_a$,
(iii) $e_a < e \leq k + 1 < e_l$ ($e_a$ small),
(iv) $e \leq k + 1 < e_a < e_l$.

Let us first consider cases (i) and (ii), in which $j$ makes the following local move:

\[
\begin{array}{ccc}
e_a & e_a & e_l \\
e_l & e & e_a \end{array} \quad \xrightarrow{j} \quad \begin{array}{ccc}
e_l & e & e_a \\
e & e_l & e_a \end{array}
\]

Since $S$ is $k$-flagged and $e$ is a small entry, it is clear that $j(S)$ is $k$-flagged. To check that $j(S)$ is $k$-perflagged, let us first consider relations between large entries where one is weakly southeast of the other. Since $e_a$ is the only large entry that is moved, we only need to check pairs involving the entry $e_a$ in cell $c$. There are two nontrivial cases for the other entry in the pair: it is either in the same row as $c$, or it is northwest of $c$ in the row immediately above. For large entries in other places, the relative position with respect to the moved entry $e_a$ is the same in $S$ and $j(S)$.

Large entries in the same row as $c$. Since $S$ is $k$-perflagged, large entries to the right of $c$ must be strictly larger than $e_a$. Now consider entries to the left of $c$. In case (ii), $e_l$ is large, so every large entry $f$ to its left satisfies $f \leq e_l < e_a$. In case (i), if there was some large entry to the left of $e_a$, take the rightmost one. Then its right neighbor would be a large entry $g$ with $g \leq e_l < e$, which would create a semistandard violation in $S$ contradicting the minimality of $c$.

Large entries northwest of $c$ one row above $c$. We have to check that there is no entry equal to $e_a$. Suppose otherwise, and let $d$ be the rightmost cell with entry $e_a$. It cannot have a small entry to its right, since this would be a smaller semistandard violation, so $d$ must be the cell immediately northwest of $c$. Thus, the entries around $c$ in $S$ are as follows:

\[
\begin{array}{ccc}
e_a & e_a \\
e_l & e \end{array}
\]

In case (i), the cell containing $e_l$ would be a semistandard violation, contradicting the minimality of $c$. In case (ii), we have two large entries in the same column with $e_l \leq e_a$, contradicting that $S$ is $k$-perflagged.

Next we consider relations between small entries in $j(S)$ where one is weakly southeast of the other. Let us first check pairs involving the moved entry $e$. Denote by $c_t$ the cell immediately above $c$, which contains $e$ in $j(S)$. There are three cases for the other entry in the pair: it is strictly northwest of $c_t$, it is in the same row as $c_t$, or it is strictly southeast of $c_t$. Recall that a $k$-perflagged tableau has no repeated small entries in the same column.

Small entries in the same row as $c_t$. Since $S$ is $k$-perflagged, small entries to the left of $c_t$ must be less than or equal to $e$. If some entry to the right of $c_t$ was strictly
smaller than \( e \), take the leftmost one. The left neighbor in \( S \) of this entry would then be a large entry (possibly \( e_a \)), and so it would create a semistandard violation smaller than \( c \).

**Small entries strictly northwest of \( c_1 \).** Since \( S \) is \( k \)-perflagged, such an entry is either strictly less than \( e \) or equal to \( e \), in which case there is a chain in \( S \) connecting it to \( e \). The same chain without its last cell connects it to \( c_1 \) in \( j(S) \).

**Small entries strictly southeast of \( c_1 \).** Since \( S \) is \( k \)-perflagged, these entries cannot be smaller than \( e \). We only need to check that if there is an entry equal to \( e \) in a cell \( d \) strictly southeast of \( c_1 \), then there is a chain connecting \( c_1 \) and \( d \). If the cell immediately to the right of \( c_1 \) also contains an entry \( e \), then the chain connecting this entry to \( d \) in \( S \) is also a valid chain connecting \( c_1 \) to \( d \) in \( j(S) \):

![Diagram](image)

Otherwise, the entry \( f \) immediately to the right of \( c_1 \) must be a large entry. Indeed, if it was small, the \( k \)-perflagged condition on \( S \) would force \( f \leq e \) (since cell \( d \) is weakly southeast of it), but then the cell containing \( f \) would be a semistandard violation in \( S \) smaller than \( e \):

![Diagram](image)

Since \( S \) is \( k \)-perflagged, we have that \( e_a \leq f \), and \( S \) contains a chain connecting the two entries \( e \) in cells \( c \) and \( d \) (if \( c \) and \( d \) are in the same row, we consider this chain to be empty). Prepending the cell \( c \) to the beginning of this chain, we obtain a chain in \( j(S) \) connecting the two entries \( e \) in cells \( c_1 \) and \( d \).

Finally, we have to check that for any pair of cells \( d_1, d_2 \) different from \( c_1 \) containing the same small entry \( f \), with \( d_2 \) strictly southeast of \( d_1 \), there is a chain connecting \( d_1 \) and \( d_2 \) in \( j(S) \). If \( c_1, c_2, \ldots, c_s \) is a chain in the notation from Definition [7.1], we say that the chain passes through the cells \( c_i \) and the cells just northeast of each of the \( c_2 \)'s. If \( d_1 \) and \( d_2 \) are connected by a chain in \( S \) not passing through either of the cells \( c \) and \( c_1 \), then the same chain connects them in \( j(S) \). If the minimal rectangle containing \( d_1 \) and \( d_2 \) contains also cell \( c_1 \), then \( f = e \). Thus, concatenating a chain from \( d_1 \) to \( c_1 \) with a chain from \( c_1 \) to \( d_2 \), both of which have been shown to exist (allowing a chain to be empty for two cells in the same row), we get a chain from \( d_1 \) to \( d_2 \). The only situation where the rectangle spanned by \( d_1 \) and \( d_2 \) does not contain \( c_1 \) but a chain connecting \( d_1 \) and \( d_2 \) could pass through \( c \) occurs if \( d_1 \) is in the same row as \( c \), and \( c \) is just northeast of a cell in the chain. In this case, the same chain connecting \( d_1 \) and \( d_2 \) in \( S \) is a valid chain in \( j(S) \), since the entry \( e \) in \( c \) has been replaced with a larger entry \( e_a \).

Let us now turn to cases (iii) and (iv), in which \( j \) makes the following local move:

![Diagram](image)

It is immediate that \( j(S) \) is also \( k \)-flagged. To check that \( j(S) \) is \( k \)-perflagged, let us consider large entries first. The only nontrivial relations that need to be checked are those involving the moved entry \( e_1 \) and another large entry in the column it is
moved to, the one containing cell $c$. Large entries below $c$ in its column must be strictly larger than $e_l$, since $S$ is $k$-perflagged. Now consider large entries above $c$ in its column. In case (iii), there are none, because otherwise, the cell immediately below the bottommost such entry would be a semistandard violation smaller than $c$, using the fact that $e_a < e$. In case (iv), $e_a$ is large, so every large entry $f$ above it satisfies $f < e_a < e_l$.

Next we consider relations between small entries in $j(S)$ where one is weakly southeast of the other. Let us first check pairs involving the moved entry $e$. Let $c_{←}$ be the cell immediately to the left of $c$, which contains $e$ in $j(S)$.

**Small entries in the same column as $c_{←}$**. Since $S$ is $k$-perflagged, small entries above $c_{←}$ must be less than or equal to $e$. To show that they are in fact strictly smaller, suppose that one of these entries is equal to $e_{\uparrow}$. Then $\forall \in S$.

**Small entries strictly northwest of $c_{←}$**. Since $S$ is $k$-perflagged, such an entry is either strictly less than $e$ or equal to $e$, in which case there is a chain in $S$ connecting it to $c$. Since $e_l > e_a$, this chain does not contain $c_{←}$, and so the same chain connects it to $c_{←}$ in $j(S)$.

**Small entries strictly southeast of $c_{←}$**. Since $S$ is $k$-perflagged, these entries cannot be smaller than $e$. If there is an entry equal to $e$, then the same chain in $S$ that connects it to cell $c$ is a chain in $j(S)$ that connects it to cell $c_{←}$.

Finally, we have to check that for any pair of cells $d_1, d_2$ different from $c_{←}$ containing the same small entry $f$, with $d_2$ strictly southeast of $d_1$, there is a chain connecting $d_1$ and $d_2$ in $j(S)$. The only non-trivial case is when the chain connecting $d_1$ and $d_2$ in $S$ passes through $c$ or $c_{←}$. If the minimal rectangle containing $d_1$ and $d_2$ contains also cell $c_{←}$, then $f = e$, and so concatenating a chain from $d_1$ to $c_{←}$ with a chain from $c_{←}$ to $d_2$ (allowing chains connecting cells in the same row to be empty) we get a chain from $d_1$ to $d_2$. If this rectangle does not contain $c_{←}$, the only way for a chain connecting $d_1$ and $d_2$ to pass through $c$ occurs if $d_1$ is above $c$ in the same column, and $d_2$ is weakly southeast of $c$. But in this case $f < e$ (since they are in the same column in $S$) and $e \leq f$ (since $d_2$ is weakly southeast of $c$ in $S$), which is a contradiction.

In the following lemmas, $S$ is a $k$-perflagged tableau with minimal semistandard violation at cell $c$, containing entry $e$. We denote by $j(c)$ the cell that this entry is moved to in $j(S)$.

**Lemma 7.14.** Let $S$, $c$, $e$ and $j(c)$ as above. Denote by $V_{\geq e}^{\text{path}}(S)$ the set of path violations of $S$ with entry at least $e$, and by $V_{\text{std}}(S)$ the set of semistandard violations of $S$. Then
\begin{align*}
(a) & \quad V_{\geq e}^{\text{path}}(j(S)) \setminus V_{\geq e}^{\text{path}}(S) = \{j(c)\}, \\
(b) & \quad V_{\text{std}}(j(S)) \setminus V_{\text{std}}(S) \subseteq \{j(c)\}.
\end{align*}

**Proof.** To prove part (a), suppose first that $e_l \leq e_a$, which corresponds to cases (i) and (ii) in the proof of Lemma 7.13, where $j(c)$ is the cell above $c$. Clearly, $j(c)$ is a path violation of type $[\Pi]$ in $j(S)$, because it contains a small entry $e$, and below it there is a large entry $e_a$ which is not maximal, as it occurs in $S$ in a higher row than in $j(S)$. To see that $j(S)$ has no other path violations aside from the ones in...
$S$, it is enough to check that the cell to the left of $c$, which contains $e_l$, is not a path violation of type (2). For this to happen, $e_l$ would have to be a small entry (case (i)), but then the fact that the next column to its right contains a small entry $e$ satisfying $e_l \leq e$ prevents it from being a path violation.

Suppose now that $e_a < e_l$ (cases (iii) and (iv)), where $j(c)$ is the cell to the left of $c$. Now $j(c)$ is a path violation of type (2) in $j(S)$, because it contains a small entry $e$, there is a large entry $e_l$ to its right, and all the small entries above $c$ in $j(S)$ are strictly less than $e$, since $S$ is $k$-perflagged. The only possible additional path violation of $j(S)$ would be the cell above $c$ containing $e_a$, if this was a small entry. But then $e_a < e$, and so this cell is not in \( V_{\text{path}}^\leq(e)(j(S)) \).

It remains to prove part (b). In both cases, whether the local move is

\[
\begin{array}{cccc}
\begin{array}{ccc}
& b & \\
& a & e_a \\
e & c & e
\end{array}
& \mapsto &
\begin{array}{ccc}
& b & \\
a & e & e_a
\end{array}
\\
or
\begin{array}{ccc}
& b & \\
& a & e_a \\
e & c & e
\end{array}
& \mapsto &
\begin{array}{ccc}
& b & \\
a & e & e_a
\end{array}
\end{array}
\]

a semistandard violation is created at $j(c)$ when $a$ or $b$ are large entries. It is clear that no other semistandard violations are created. \( \Box \)

We remark that the above lemma implies that if $j(c)$ is a semistandard violation of $j(S)$, then it is minimal. Indeed, by part (b), the tableau $j(S)$ has no semistandard violations other than $j(c)$ and those in $S$. Since the cell $j(c)$ is either in the same column as $c$ or to its left, and it contains the same entry as the minimal semistandard violation of $S$, it has to be minimal in $j(S)$.

**Lemma 7.15.** Let $S$, $c$, $e$, and $j(c)$ as above, and let $d$ be the minimal semistandard violation of $j(S)$. Suppose that $d \neq j(c)$ and that $d$ also contains the entry $e$. Then $d$ is strictly to the right of $c$.

**Proof.** The entry $e$ in cell $d$ does not move when $j$ is applied to $S$, since $j$ switches the entry $e$ in cell $c$ (moving it to to $j(e)$) with a strictly larger entry. Now, since $c$ and $d$ contain the same entry $e$ in $S$, which is a $k$-perflagged tableau, they have to be in different columns. But if $d$ was to the left of $c$, then $d$ would be a semistandard violation of $S$ contradicting the minimality of $c$. Thus, $d$ is strictly to the right of $c$. \( \Box \)

In the following lemma, $j^m$ denotes the composition of $j$ with itself $m$ times.

**Lemma 7.16.** Let $R$ be a $k$-perflagged tableau with no path violations, and let $S = j^m(R)$ for some $m \geq 0$. If $c$ is the minimal semistandard violation of $S$, then $j(c)$ is the maximal path violation of $j(S)$.

**Proof.** We proceed by induction on $m$. For $m = 0$, this follows from Lemma 7.14(a), since \( V_{\text{path}}^\leq(j(S)) = \{ j(c) \} \) in this case.

Now suppose that the statement is true for $m$, that is, $c$ is the minimal semistandard violation of $S = j^m(R)$, and $j(c)$ is the maximal path violation of $j(S)$. Let $d$ be the minimal semistandard violation of $j(S)$. To prove that the statement holds for $m + 1$, we have to show that the maximal path violation of $j^2(S)$ is $j(d)$.

Let $e$ be the entry in cell $c$ of $S$, and let $f$ be the entry in cell $d$ of $j(S)$. Note that $f \geq e$, because otherwise, by Lemma 7.14(b), $d$ would have been a semistandard violation of $S$ smaller than $c$. Suppose for contradiction that the
maximal path violation of \( j^2(S) \) is some cell \( a \) other than \( j(d) \), and let \( g \) be its entry. Lemma 7.14(a) applied to \( j(S) \) states that
\[
\gamma_{path}^{\geq f}(j^2(S)) \setminus \gamma_{path}^{\geq f}(j(S)) = \{j(d)\}.
\]

It follows that \( j(d) \) is a path violation of \( j^2(S) \), and that \( a \) is a path violation of \( j(S) \), since \( g \geq f \) (because \( a \) is maximal in \( j^2(S) \)). If \( g > e \), then \( a \) would be a path violation of \( j(S) \) larger than \( j(c) \), a contradiction. If \( g = f = e \), then \( a \) must be strictly to the right of \( j(d) \), since it is maximal. But \( a \) and \( d \) cannot be in the same column, since they are different cells containing the same entry in \( j(S) \), and so \( a \) is strictly to the right of \( d \) as well. If \( d \neq j(c) \), then by Lemma 7.16 \( d \) is strictly to the right of \( c \) and \( j(c) \). In all cases, \( a \) is strictly to the right of \( j(c) \), which contradicts the fact that \( j(c) \) is the maximal path violation of \( j(S) \).

**Lemma 7.17.** Let \( R \) be a \( k \)-perflagged tableau with no path violations, and let \( S = j^m(R) \) for some \( m \geq 0 \). Suppose that \( S \) has some semistandard violation. Then
\[
j^{-1}(j(S)) = S.
\]

**Proof.** Let \( c, e, e_l \) and \( e_a \) as before. Suppose first that \( e_l \leq e_a \). If there is a cell just northeast of \( c \), let \( e_l \) be its entry. Then \( j \) makes the following local move when applied to \( S \) (the cell with \( e_l \) may not be there):
\[
\begin{array}{c}
e_a \ne_l \\
e \end{array} \quad \downarrow \quad \begin{array}{c}
e_a \ne_l \\
e \end{array}
\]

By Lemma 7.16 cell \( j(c) \) is the maximal path violation of \( j(S) \). If \( e_l \) is a large entry, then \( e_a \leq e_l \), since \( S \) is \( k \)-perflagged. In any case, when \( j^{-1} \) is applied to \( j(S) \), it switches the entry \( e \) in cell \( j(e) \) with the entry \( e_a \) in cell \( c \).

Suppose now that \( e_a < e_l \). If there is a cell just southwest of \( c \), let \( e_a \) be its entry. Now \( j \) makes the following local move (the cell with \( e_a \) may not be there):
\[
\begin{array}{c}
e \ne_a \\
e \end{array} \quad \downarrow \quad \begin{array}{c}
e \ne_a \\
e \end{array}
\]

Again by Lemma 7.16 \( j(c) \) is the maximal path violation of \( j(S) \). If \( e_a \) is a large entry, then \( e_l < e_a \), since \( S \) is \( k \)-perflagged. Thus, when \( j^{-1} \) is applied to \( j(S) \), it switches the entry \( e \) in cell \( j(c) \) with the entry \( e_l \) in cell \( c \).

**Proof of Theorem 7.1.** Let \( P \in \mathcal{P}^k(T, B) \) with \( h_i(P) = h_i \) and \( u_s(P) = u_s \). By Lemma 7.10 the map \( P \mapsto \text{Tab}(P) \) is a bijection between such \( k \)-tuples of paths and \( k \)-perflagged tableaux of shape \( \lambda(T, B) \) with no path violations and weight \((\lambda_1 - h_0, \lambda_1 - h_1, \ldots, \lambda_1 - h_k, u_1, u_2, \ldots, u_{y-1})\). To transform such a tableau into a SSYT, we repeatedly apply \( j \), until no semistandard violation occurs. To see that this process ends in a finite number of steps, define a function that associates to each tableau the positive integer \( \sum_{i,j}(i+j)e_{ij} \), where \( e_{ij} \) is the entry in row \( i \) from the top and column \( j \) from the left, and the sum is over all cells in the tableau. This function strictly decreases each time that \( j \) is applied, so the process ends in a tableau with no semistandard violations, which we define as \( \Psi(P) \). Since \( j \) is clearly weight-preserving, \( \Psi(P) \) is a SSYT of shape \( \lambda(T, B) \) with the weight as claimed.

It remains to show that \( \Psi \) is a bijection. By Lemma 7.17 the process that transforms a \( k \)-perflagged tableaux of shape \( \lambda(T, B) \) with no path violations into a
k-flagged SSYT is reversible. Thus, when disregarding the statistics \( h_i \) and \( u_s \), our map \( \Psi \) extends to an injection from the set of all \( k \)-tuples in \( \mathcal{P}^k(T, B) \) to the set of all \( k \)-flagged SSYT of shape \( \lambda(T, B) \). Since these two sets have the same cardinality by Lemma 7.2, the map \( \Psi \) is surjective as well.

The example in Figure 11 illustrates the bijection \( \Psi \), starting from a \( k \)-tuple of paths, constructing a \( k \)-perflagged tableau with no path violations, and then repeatedly applying \( j \) to obtain a \( k \)-perflagged SSYT.

![Figure 11. The sequence of tableaux in the construction of \( \Psi(P) \) for a 3-tuple of paths \( P \). The (minimal) semistandard violations are indicated by (bold) diamonds and the (maximal) path violations by (bold) circles.](image_url)

**Appendix A. A bijective proof of the independence of the Tutte polynomial from the ordering of the ground set.**

In Section 4 we mentioned that we do not know of a ‘natural’ bijective proof of Theorem 2.2. In the language of lattice path matroids, this would be a bijection from \( B_{T,B} \) to itself that turns internal and external activity with respect to the ordering \( 1 < 2 < 3 < \cdots \) of the ground set into internal and external activity with respect the ordering \( \cdots < 3 < 2 < 1 \). However, in the basic case that the ordering of the ground set is modified only by transposing two adjacent elements, we can give a bijection from the set of bases of any matroid to itself that preserves internal and external activity. Since we can go from any ordering of the ground set to any other ordering by successive transpositions of adjacent elements, a bijection proving Theorem 2.2 is obtained by composing \( \binom{|E|}{2} \) iterations of the bijection below, where \( E \) is the ground set of the matroid.

Since this appendix applies to an arbitrary matroid, we discontinue the typographic conventions of the other sections. In this section, \( E \) denotes the ground set, and the letters \( a, b, c, d, e, x \) and \( y \) are used to denote elements of \( E \), while \( B \) denotes the set of bases, and the letters \( B, B', C \) and \( D \) are used to denote bases. We use \( B - x + y \) to denote \( B \setminus \{x\} \cup \{y\} \). Furthermore, if \( B \) contains exactly one element of the pair \( \{x, y\} \), we define \( B^{x+y} \) to be \( B - x + y \) if \( x \in B \), or \( B - y + x \) otherwise (note that \( B^{x+y} = B^{y+x} \) by definition). If \( B \in B \), recall that \( x \) is active with respect to \( (B, \prec) \) if it cannot be switched with a smaller element to produce
another base, that is, there is no \( y < x \) such that \( B^{x+y} \) is defined and belongs to \( B \).

In the proof of the following theorem we use the strong basis exchange property of matroids:

**Lemma A.1.** Let \( C \) and \( D \) be bases of a matroid, and let \( d \in D \setminus C \). Then there exists \( c \in C \setminus D \) such that \( C - c + d \) and \( D - d + c \) are bases of the matroid.

**Theorem A.2.** Let \( \prec \) be a linear order on \( E \), and let \( x \prec y \) be adjacent elements in this order. Let \( \prec' \) be the order obtained from \( \prec \) by reversing the relative order of \( x \) and \( y \) and keeping the rest of order relationships unchanged. Define a map \( \varphi_{xy} : B \to B \) by letting \( \varphi_{xy}(B) = B' \), where

\[
\tag{A.1}
B' = \begin{cases} 
B^{x+y} & \text{if } B \text{ contains exactly one element from } \{x, y\}, B^{x+y} \in B, \text{ and either } x \text{ is active w.r.t. } (B, \prec) \text{ or } y \text{ is active w.r.t. } (B, \prec'); \\
B & \text{otherwise.}
\end{cases}
\]

Then \( \varphi_{xy} \) is a bijection with the property that the internal and external activity of \( (B, \prec) \) equal the internal and external activity of \( (B', \prec') \), respectively.

**Proof.** Let us first show that \( \varphi_{xy} \) is a bijection. We claim that given \( B' \in B \), we can recover \( B \) applying the map \( \varphi_{yx} \), which sends \( B' \) to

\[
\tag{A.2}
B'' = \begin{cases} 
B'^{y+x} & \text{if } B' \text{ contains exactly one element from } \{x, y\}, B'^{y+x} \in B, \text{ and either } y \text{ is active w.r.t. } (B', \prec') \text{ or } x \text{ is active w.r.t. } (B', \prec); \\
B' & \text{otherwise.}
\end{cases}
\]

The fact that \( B'' = B \) is clear if either \( x, y \in B \) or \( x, y \notin B \), or if \( B^{x+y} \notin B \), because in these cases \( B'' = B' = B \). Suppose now that \( B \) contains exactly one element from \( \{x, y\} \), and \( B^{x+y} \in B \). If \( x \) is inactive w.r.t. \( (B, \prec) \) and \( y \) is inactive w.r.t. \( (B, \prec') \), then \( B' = B \) by definition, and in this case, since \( y \) is inactive w.r.t. \( (B', \prec') \), \( x \) is inactive w.r.t. \( (B', \prec) \), we have that \( B'' = B \).

The only remaining case is when the conditions in the first part of \( \text{(A.1)} \) hold, and so \( B' = B^{x+y} \). In this case, the following two statements are true:

(i) \( x \) is active w.r.t. \( (B, \prec) \) if and only if \( y \) is active w.r.t. \( (B', \prec') \),

(ii) \( y \) is active w.r.t. \( (B, \prec') \) if and only if \( x \) is active w.r.t. \( (B', \prec) \).

Statement (i) is clear since \( B^{x+y} = B'^{y+x} \) for every \( e \) for which these are defined, and \( e \prec x \) if and only if \( e \prec' y \). Similarly, statement (ii) holds because \( B'^{y+x} = B^{x+y} \) for every \( e \) for which these are defined, and \( e \prec' y \) if and only if \( e \prec x \).

Now we show that the conditions in the first part of \( \text{(A.2)} \) are satisfied, and thus \( B'' = B'^{y+x} = B \). Indeed, since \( B \) contains exactly one element from \( \{x, y\} \), so does \( B' = B^{x+y} \), and we have \( B'^{y+x} = B \in B \). Additionally, since either \( x \) is active w.r.t. \( (B, \prec) \) or \( y \) is active w.r.t. \( (B', \prec') \), the statements (i) and (ii) imply that either \( y \) is active w.r.t. \( (B', \prec') \) or \( x \) is active w.r.t. \( (B', \prec) \).

Next we show that the internal and external activity of \( (B, \prec) \) equal the internal and external activity of \( (B', \prec') \), respectively. Consider first the case that \( B' = B \), which happens if any of the following hold:

(a) either \( x, y \in B \) or \( x, y \notin B \);
(b) \( B^{x+y} \notin B \);
(c) \( x \) is inactive w.r.t. \( (B, \prec) \) and \( y \) is inactive w.r.t. \( (B, \prec') \).
In all three subcases, it is clear that each $e \notin \{x, y\}$ is active w.r.t. $(B, \prec)$ if and only if it is active w.r.t. $(B, \prec')$, since the relative order of $e$ with the other elements of $E$ does not change. Let us now show that for $e \in \{x, y\}$, $e$ is equally active w.r.t. both orderings. If (a) and (b) hold, this is clear, because $x$ and $y$ cannot be switched with each other to produce a base. In case (c), the reason is that $x$ is inactive w.r.t. $(B, \prec)$, so it is also inactive w.r.t. $(B, \prec')$, since the elements smaller than $x$ in $\prec$ are still smaller than $x$ in $\prec'$; similarly, $y$ is inactive w.r.t. $(B, \prec')$, so it is inactive w.r.t. $(B, \prec)$ as well.

Finally, consider the case in which none of (a), (b) and (c) hold, and so $B' = B^e + y$. In this case, $y$ is always inactive w.r.t. $(B, \prec)$ because $x \prec y$ and $B^e + y \in B$. Similarly, $x$ is always inactive w.r.t. $(B', \prec')$ because $y \prec' x$ and $B^e + x = B \in B$. On the other hand, by statement (i) above, $x$ is active w.r.t. $(B, \prec)$ if and only if $y$ is active w.r.t. $(B', \prec')$. Thus, we have proved that the number of active elements among $\{x, y\}$ is the same w.r.t. both $(B, \prec)$ and $(B', \prec')$.

Next we show that any $e \notin \{x, y\}$ is active w.r.t. $(B, \prec)$ if and only if it is active w.r.t. $(B', \prec')$. It is enough to show that such an $e$ cannot be inactive w.r.t. $(B, \prec)$ but active w.r.t. $(B', \prec')$. The symmetric statement then follows from the fact that $\varphi_{yx}(B') = B$.

Suppose for contradiction that $e \notin \{x, y\}$ is inactive w.r.t. $(B, \prec)$ but active w.r.t. $(B', \prec')$. By definition, there is an element $a$ such that $a \prec e$, and $B^e + a$ is defined and is a base. We can easily discard the case that $a \in \{x, y\}$, because letting $b$ be such that $\{a, b\} = \{x, y\}$, we would have $B^e + a = B^e + b \in B$ and $b \prec e$, so $e$ would be inactive w.r.t. $(B', \prec')$. Thus we assume that $a \notin \{x, y\}$. For simplicity of notation, we suppose in what follows that $x \in B$, and so $B' = B - x + y$ (if $y \in B$ instead, just replace all $x$ with $y$ and all $y$ with $x$).

If $e \notin B$, then $B^e + a = B - a + e$. By Lemma A.1 applied to the bases $C = B'$ and $D = B - a + e$ with $d = e$, there is a $c \in C \setminus D = \{y, a\}$ such that $B' - c + e$ and $D - e + c = B - a + c$ are bases. If $c \prec e$, then the fact that $B' - c + e \in B$ contradicts the assumption that $e$ is active w.r.t. $(B', \prec')$. Thus, we must have $e \prec c$, which implies that $c = y$. But then, $B' - y + e = B - x + e \in B$ and $e \prec x$, so $x$ is inactive w.r.t. $(B, \prec)$. Also, $B - a + y \in B$ and $a \prec e \prec y$, so $y$ is inactive w.r.t. $(B, \prec')$. But then (c) would hold, contradicting our assumption.

If $e \in B$, then $B^e + a = B - e + a$ and the argument is very similar: we apply Lemma A.1 with $C = B - c + a, D = B'$ and $d = e$ to conclude that either $e$ is active w.r.t. $(B', \prec')$, or $y$ is inactive w.r.t. $(B, \prec')$ and $x$ is inactive w.r.t. $(B, \prec)$, reaching a contradiction in both cases. 

\[\square\]

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