On the convergence of weighted-average consensus

Francisco Pedroche*  Miguel Rebollo†  Carlos Carrascosa†  Alberto Palomares†

Abstract

In this note we give sufficient conditions for the convergence of the iterative algorithm called weighted-average consensus in directed graphs. We study the discrete-time form of this algorithm. We use standard techniques from matrix theory to prove the main result. As a particular case one can obtain well-known results for non-weighted average consensus. We also give a corollary for undirected graphs.

Keywords: Consensus algorithms, iterative methods, distributed consensus, multi-agent consensus, Perron-Frobenius.

1 Introduction

Let $G = (V, E)$ be a graph, with $V = \{v_1, v_2, \ldots, v_n\}$ a non-empty set of $n$ vertices (or nodes) and $E$ a set of $m$ edges. Each edge is defined by the pair $(v_i, v_j)$, where $v_i, v_j \in V$. The adjacency matrix of the graph $G$ is $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ such that $a_{ij} = 1$ if there is a directed edge connecting node $v_i$ to $v_j$, and 0, otherwise. We consider directed graphs (digraphs). The out-degree $d_i$ of a node $i$ is the number of its out-links, i.e., $d_i = \sum_{j=1}^{n} a_{ij}$.

We define the Laplacian matrix of the graph as $L = D - A$ where $D$ is the diagonal matrix with the out-degrees. $D = \text{diag}(d_1, d_2, \ldots, d_n)$.

We recall that a permutation matrix $F$ is just the identity matrix with its rows reordered. Permutation matrices are orthogonal, i.e., $F^T = F^{-1}$. A matrix $A \in \mathbb{R}^{n \times n}$, with $n \geq 2$, is said to be reducible if there is a permutation matrix $F$ of order $n$ and there is some integer $r$ with $1 \leq r \leq n - 1$ such that $F^T A F = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$, where $B \in \mathbb{R}^{r \times r}$, $C \in \mathbb{R}^{r \times (n-r)}$, and $0 \in \mathbb{R}^{(n-r) \times r}$ is a zero matrix. A matrix is said to be irreducible if it is not reducible. It is known (see, e.g., [1]) that the adjacency matrix $A$ of a directed graph is irreducible if and only if the associated graph $G$ is strongly connected. For an undirected graph irreducibility implies connectivity. Note that the Laplacian $L = D - A$ is irreducible if and only if $A$ is irreducible.

*Institut de Matemàtica Multidisciplinària, Universitat Politècnica de València. Camí de Vera s/n. 46022 València. Spain. pedroche@imm.upv.es.
†Departament de Sistemes Informàtics i Computació, Universitat Politècnica de València. Camí de Vera s/n. 46022 València. Spain.
irreducible. Note that \( Le = 0 \), with \( e \) the vector of all ones. \( L \) has 0 as an eigenvalue, and therefore is a singular matrix. Note also that \( L \) is irreducible if and only if \( G \) is connected.

We use the sum norm (or \( l_1 \) norm) as the vector norm: \( \|v\|_1 = |v_1| + |v_2| + \ldots + |v_n| \). We denote \( N = \{1, 2, \ldots, n\} \).

## 2 Weighted-average consensus

Let \( G \) be a directed graph. Let \( x^0 \) be a (column) vector with the initial state of each node. Let \( w = [w_1, w_2, \ldots, w_n]^T \) a vector with the weight associated to each node. The following algorithm (see [2], p. 225) can be used to obtain the value of the weighted-average consensus (that is, a common value for all the nodes, reached by consensus)

\[
W \dot{x} = -Lx
\]  

(1)

with \( W = \text{diag}(w_1, w_2, \ldots, w_n) \), and \( L = D - A \), where \( D \) is a diagonal matrix with the out-degrees. A discretized version of (1) is

\[
x_{i}^{k+1} = x_{i}^{k} + \frac{\epsilon}{w_i} \sum_{j \in N_i} a_{ij} (x_{j}^{k} - x_{i}^{k}), \quad \forall i \in N
\]

(2)

where \( N_i \) denote the set of neighbors of node \( i \), that is: \( j \in N_i \leftrightarrow (v_i, v_j) \in E \). The matrix form of (2) is

\[
x^{k+1} = P_w x^k \quad k = 0, 1, 2, \ldots
\]

(3)

where

\[
P_w = I - \epsilon L_w
\]

(4)

where we have denoted \( L_w = W^{-1}L \).

From (2) it follows that

\[
x^k = P_w^k x^0, \quad k = 1, 2, \ldots
\]

(5)

In this note we prove the following

**Theorem** Let \( G \) be a strongly connected digraph. If \( \epsilon < \min_{i \in N}(w_i/d_i) \) then the scheme (5) converges to the weighted-average consensus given by

\[
x_w = \alpha e
\]

with \( e \) the vector of all ones, and \( \alpha = \frac{1}{\|v\|_1} v^T x^0 \), where \( v \) is a positive eigenvector of \( L_w^T \) associated with the zero eigenvalue, that is

\[
L_w^T v = 0
\]

## 3 Known results

A matrix \( P \) is said to be nonnegative if \( P_{ij} \geq 0, \forall (i, j) \in N \). A matrix is said to be primitive if it is irreducible and has only one eigenvalue of maximum modulus [1].

The following theorem is known as Perron-Frobenius Theorem.
Theorem 3.1 ([1], p. 508). If $P \in M_n$ is nonnegative and irreducible, then
a) $\rho(P) > 0$
b) $\rho(P)$ is an eigenvalue of $P$
c) There is a positive vector $x$ such that $Px = \rho(P)x$
d) $\rho(P)$ is an algebraically (and geometrically) simple eigenvalue of $P$

Theorem 3.2 ([1], p. 516). If $P \in M_n$ is nonnegative and primitive, then
\[
\lim_{k \to \infty} [\rho(P)^{-1}P]^k = T > 0
\]
where $T = xy^T$, $Px = \rho(P)x$, $P^Ty = \rho(P)y$, $x > 0$, $y > 0$, and $x^Ty = 1$.

Remark 3.1. It is known (see [3], p. 48) that given $P$ an irreducible nonnegative matrix then if $P$ has at least a diagonal entry positive, then $P$ is primitive (in fact, the index of primitivity is related with the number of diagonal entries positive). Therefore, we can apply theorem 3.2 to matrices that are irreducible nonnegative with at least a diagonal entry positive.

4 Main Result

In this section we prove the main theorem and a corollary.

Theorem 4.1. Let $G$ be a strongly connected digraph. If $\epsilon < \min_{i \in N} (w_i/d_i)$ then the scheme (5) converges to the weighted average consensus given by
\[
x_w = \alpha e
\]
with $e$ the vector of all ones, and $\alpha = \frac{1}{||v||_1}v^Tx^0$, where $v$ is a positive eigenvector of $L_w^T$ associated with the zero eigenvalue, that is:
\[
L_w^Tv = 0
\]

Proof. Let us begin with the existence of the positive vector $v$. Since $G$ is strongly connected we have that $L_w^T$ is irreducible and therefore $P_w^T$ is irreducible nonnegative. Therefore by Theorem 3.1 there exists a positive vector $v$ such that $P_w^Tv = \rho(P_w^T)v$. And since $P_w^T$ is column stochastic we have $\rho(P_w^T) = 1$. Then $P_w^Tv = v$ and it follows that $(I - \epsilon L_w^T)v = v$ and therefore $L_w^Tv = 0$.

Since $\epsilon < w_i/d_i$ for some $i \in N$ we have that $P_w$ is an irreducible nonnegative matrix with at least a diagonal entry positive. Therefore from remark 3.1 we have that $P_w$ is primitive and we can apply theorem 3.2 to conclude that
\[
\lim_{k \to \infty} [P_w]^k = T > 0
\]
where $T = xv^T$, $P_wx = x$, $P_w^Tv = v$, $x > 0$, $v > 0$, and $x^Tv = 1$. 
Since $P_w$ is row stochastic we have that $P_w e = e$, therefore we can write
\[
 x = \alpha e
\]
for some $\alpha > 0$. But the condition $x^T v = 1$ leads to $\alpha e^T v = 1$, that is
\[
 \alpha = \frac{1}{||v||_1}
\]
And therefore
\[
 T = x v^T = \alpha e v^T
\]
And then the weighted average consensus is given by
\[
 x_w = \lim_{k \to \infty} [P_w]^k x_0 = \alpha e v^T x_0 = \alpha v^T x_0 e
\]
and the proof follows.

**Remark** Note that, as a particular case, taking the weights $w = e$ this theorem gives theorem 2 of [2].

In the symmetric case (undirected graph) we have the following

**Corollary 4.1.** Let $G$ be a strongly connected undirected graph. If $\epsilon < \min_{i \in N} (w_i/d_i)$ then the scheme (5) converges to the weighted-average consensus given by $x_w = \alpha e$ with
\[
 \alpha = \frac{1}{||w||} w^T x_0 = \frac{\sum_i w_i x_i^0}{\sum_i w_i}
\]

**Proof.** From theorem 4.1 we have that
\[
 L^T_w v = 0
\]
that is $(W^{-1} L)^T v = 0$ and therefore
\[
 L^T W^{-1} v = 0
\]
since $W^{-1}$ is a diagonal matrix. Now, since $L$ is a symmetric matrix we have
\[
 L W^{-1} v = 0
\]
which means
\[
 L \begin{bmatrix} v_1/w_1 \\ v_2/w_2 \\ \vdots \\ v_n/w_n \end{bmatrix} = 0
\]
and since $\beta e$ for $\beta \in \mathbb{R}$ is the eigenspace associated to $\lambda = 0$ we have that
\[
\begin{bmatrix}
v_1/w_1 \\
v_2/w_2 \\
\vdots \\
v_n/w_n \\
\end{bmatrix} = \beta \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \forall \beta \in \mathbb{R}
\]

which means

\[v = \beta w\]

Therefore from theorem 4.1 we have that the value of the weighted-average consensus is

\[x_w = \alpha e\]

with

\[\alpha = \frac{1}{\|v\|_1} v^T x^0 = \frac{1}{\|\beta w\|_1} \beta w^T x^0 = \frac{1}{\|w\|_1} w^T x^0.\]

\[
\Box
\]

5 Conclusion

In this note we provide sufficient conditions for the convergence of the so-called weighted-average consensus in discrete form for directed graphs. This algorithm is described in [2], but to our knowledge no sufficient conditions for the convergence of this algorithm has been already published in the literature. As a particular case our result gives known-results on non-weighted consensus. We also provide a corollary for undirected graphs. These algorithms are commonly used in multi-agent systems.

Acknowledgments

This work is supported by Spanish DGI grant MTM2010-18674, Consolider Ingenio CSD2007-00022, PROMETEO 2008/051, OVAMAH TIN2009-13839-C03-01, and PAID-06-11-2084.

References

[1] Horn, R. A. & Johnson, C. H. (1999). Matrix Analysis, Cambridge Univ. Press.

[2] Olfati-Saber, R., Fax, J. A., and Murray, R. M. (2007). Consensus and Cooperation in Networked Multi-Agent Systems, Proceedings of the IEEE, vol. 95, no. 1, pp. 215-233.

[3] Varga, R. S. (2000, 2nd ed.). Matrix Iterative Analysis, Springer.