TOTALLY NON-COISOTROPIC DISPLACEMENT AND ITS APPLICATIONS TO HAMILTONIAN DYNAMICS

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Abstract. In this paper we prove the Conley conjecture and the almost existence theorem in a neighborhood of a closed nowhere coisotropic submanifold under certain natural assumptions on the ambient symplectic manifold. Essential to the proofs is a displacement principle for such submanifolds. Namely, we show that a topologically displaceable nowhere coisotropic submanifold is also displaceable by a Hamiltonian diffeomorphism, partially extending the well-known non-Lagrangian displacement property.

1. Introduction and Main Results

In this paper we study Hamiltonian dynamics in a neighborhood of a nowhere coisotropic submanifold. As a starting point, we establish the following displacement principle: a closed nowhere coisotropic submanifold of a symplectic manifold is (infinitesimally) displaceable provided that there are no topological obstructions to displaceability. (In general, a compact subset \( M \) of a symplectic manifold is said to be displaceable if it can be disjoined from itself by a Hamiltonian diffeomorphism \( \varphi_H \), i.e., \( \varphi_H(M) \cap M = \emptyset \). Thus, by definition, a displaceable set is topologically displaceable.) Then we develop a new version of the theory of action selectors and use it, combined with the displacement principle, to prove two results in Hamiltonian dynamics. Namely, we prove the Conley conjecture (for non-negative Hamiltonians) and the almost existence theorem in a neighborhood of a closed nowhere coisotropic submanifold, under certain natural assumptions on the ambient manifold. The Conley conjecture, \( [Co] \), concerns time-dependent Hamiltonian systems and asserts the existence of infinitely many periodic points for a Hamiltonian diffeomorphism. The almost existence theorem due to Hofer and Zehnder and to Struwe, \( [HZ1, HZ2, St] \), asserts that almost all regular level sets of a proper autonomous Hamiltonian on \( \mathbb{R}^{2n} \) carry periodic orbits. A similar result has also been proved for \( \mathbb{C}P^n \), symplectic vector bundles, subcritical Stein manifolds, and certain other symplectic manifolds; see, e.g., \( [FS, CG, HY, Ke2, Ln2, Sc] \) and also the survey \( [Gi3] \) and references therein. Here, similarly to \( [CGK, FS, GI1, G2, GK1, GK2, Ke1, Ke2, Ln1, Mac, Pol2, Sch] \), we focus on these theorems for Hamiltonians supported in a neighborhood of a closed submanifold.

We also introduce the notion of a wide symplectic manifold, which means that the manifold is open and admits an arbitrarily large compactly supported Hamiltonian without contractible fast periodic orbits. Immediate examples of such manifolds...
include $\mathbb{R}^{2n}$, cotangent bundles, Stein manifolds and twisted cotangent bundles. The essence of this property lies in the fact that on a wide manifold the top degree Floer homology is non-zero for any non-negative compactly supported Hamiltonian which is not identically zero. This allows us to construct an action selector for geometrically bounded wide manifolds and, thus, prove a version of the Conley conjecture.

From now on $W$ will always stand for a wide manifold while $P$ will denote a general symplectic manifold.

Let us now state the main results of the paper.

1.1. **Displacement Principle.** Let $M$ be a closed connected submanifold of a symplectic manifold $(P^{2n}, \omega)$. We say that $M$ is nowhere coisotropic if $T_xM$ is not a coisotropic subspace of $T_xP$ for any $x \in M$. For example, a symplectic submanifold is nowhere coisotropic; a submanifold of middle dimension is nowhere coisotropic if and only if $\omega|_M$ does not vanish at any point, i.e., $T_xM$ is not a Lagrangian subspace for any $x \in M$.

Our first result is the following principle which extends or complements the works of Laudenbach and Sikorav, [LauS], and of Polterovich, [Pol1], and plays a crucial role in the proofs of the Conley conjecture and the almost existence theorem near nowhere coisotropic submanifolds.

**Theorem 1.1 (Displacement Principle).** Let $M$ be a closed, connected submanifold of a symplectic manifold $(P, \omega)$. Assume that $M$ is nowhere coisotropic and the normal bundle to $M$ admits a non-vanishing section. Then $M$ is infinitesimally displaceable, i.e., there exists a non-vanishing Hamiltonian vector field which is nowhere tangent to $M$.

When $M$ is of middle dimension, Theorem 1.1 was proved by Laudenbach and Sikorav, [LauS], under a less restrictive assumption that $M$ is non-Lagrangian and (under the extra assumption that $TM$ has a Lagrangian complement) by Polterovich, [Pol1]. It has also been known for a long time that $M$ is always displaceable when $\dim M < n$. (Note that such a submanifold is automatically nowhere coisotropic.) Thus, Theorem 1.1 can be thought of as an extension of the displacement principle to submanifolds of dimension greater than $n$.

In contrast with the middle-dimensional case, the condition that $M$ is nowhere coisotropic cannot be replaced by the requirement that $M$ is (somewhere) non-coisotropic. For a non-coisotropic submanifold can contain a Lagrangian submanifold and, in this case, $M$ is not displaceable due to the Lagrangian intersection property. However, this assumption can possibly be relaxed. We will examine generalizations of the displacement principle elsewhere.

Theorem 1.1 is proved in Section 4.3. Let us now proceed with the applications.

1.2. **The Conley Conjecture.** In its original form, the Conley conjecture asserts that every Hamiltonian diffeomorphism on $T^{2n}$ has infinitely many simple periodic points, [Co, SZ]. Here “simple” means that the orbits sought are not iterated. A similar conjecture makes sense and is interesting for other symplectic manifolds too. (Observe that the example of an irrational rotation on $S^2$ demonstrates that the conjecture, as stated, fails for manifolds admitting spheres. However, the statement can be suitably modified to be meaningful and non-trivial for such manifolds as well; cf. [FrHa].)
Recently, Ginzburg, [Gi5], proved the Conley conjecture for all closed symplectically aspherical manifolds. Prior to Ginzburg’s work, some particular cases of this conjecture were established. When the manifold is closed, the conjecture was proved by Salamon and Zehnder, [SZ], under the additional assumption that the fixed points are weakly non-degenerate, and Hingston, [Hi], established the conjecture for $\mathbb{T}^{2n}$ without the non-degeneracy assumption.

Other partial results, not necessarily for closed manifolds, were obtained under assumptions on the size of the Hamiltonian. Namely, the conjecture is also known to hold is when the support of the (time-dependent) Hamiltonian is displaceable. For instance, in $\mathbb{R}^{2n}$ every compactly supported Hamiltonian has displaceable support, in which case the conjecture has been proved by Viterbo, [V1], and by Hofer and Zehnder, [HZ2, HZ3]. Admittedly this is a very restrictive assumption, especially for closed manifolds. Yet, this is essentially the only situation in which the conjecture is known to hold for open manifolds. Under the assumption that the support is displaceable, the conjecture was proved by Schwarz, [Sc], for closed symplectically aspherical manifolds and by Frauenfelder and Schlenk, [FS], for manifolds that are convex at infinity. (Recall that a manifold is called convex at infinity if it is isomorphic at infinity to the symplectization of a compact contact manifold.) The question is still open for many “natural” symplectic manifolds such as cotangent bundles.

We establish this conjecture for Hamiltonians supported in a neighborhood of a nowhere coisotropic submanifold under certain assumptions on the ambient manifold. More precisely, we prove

**Theorem 1.2.** Assume that $M$ is a closed, nowhere coisotropic submanifold of a symplectically aspherical manifold $(W, \omega)$. Let $H$ be a non-zero time-dependent Hamiltonian, supported in a sufficiently small neighborhood of $M$.

- If $W$ is geometrically bounded and wide and $H \geq 0$, then $H$ has simple (contractible) periodic orbits with positive action and arbitrarily large period, provided that the time-one map $\varphi_H$ has isolated fixed points with positive action.
- If $W$ is closed, then $H$ has simple (contractible) periodic orbits with non-zero action and arbitrarily large period, provided that the time-one map $\varphi_H$ has isolated fixed points with non-zero action.

We say that $W$ is wide if the manifold admits arbitrarily large, compactly supported, autonomous Hamiltonians $F$ such that all non-trivial contractible periodic orbits of $F$ have periods greater than one; see Section 3.1 for a discussion of this concept. This condition is satisfied for all examples of open geometrically bounded manifolds known to us.

When $W$ is closed or convex, this theorem can be proved using the displacement principle, Theorem 1.1, and the action selectors introduced in [Sc] or [FS] respectively. Moreover, in these cases it suffices to assume that $W$ is weakly-exact rather than symplectically aspherical. However, the constructions of the action selectors for closed or convex manifolds do not extend to open manifolds which are merely geometrically bounded. Hence, the proof of Theorem 1.2 for geometrically bounded manifolds requires developing a new version of the theory of action selectors; see Section 3.2.

An immediate consequence of Theorem 1.2 is the following corollary.
Corollary 1.3. Let \((W, \omega)\) be geometrically bounded, symplectically aspherical and wide, and let \(M\) be a closed and nowhere coisotropic submanifold of \(W\). Then, for every non-zero time-dependent Hamiltonian \(H \geq 0\) supported in a sufficiently small neighborhood of \(M\), the time-one map \(\varphi_H\) has infinitely many simple periodic points corresponding to contractible periodic orbits of \(H\) with positive action. (A similar statement for closed manifolds has been proved in \([Sc]\).)

Remark 1.4. If \(W\) is assumed to be convex, the condition that \(H \geq 0\) can be removed both in Theorem 1.2 and in Corollary 1.3, provided that the orbits are required only to have non-zero, rather than positive, action; see \([FS]\). Moreover, in Theorem 1.2 the assumption that \(W\) is geometrically bounded and wide can be replaced by the assumption that \(W\) admits an exhaustion \(W_1 \subset W_2 \subset \ldots\) by open sets such that each \(W_k\) is symplectomorphic to an open subset of a geometrically bounded and wide manifold, perhaps depending on \(k\).

1.3. The Almost Existence Theorem. Combining the displacement principle and the results from \([Sch]\), we prove the following almost existence theorem for periodic orbits in a neighborhood of a closed nowhere coisotropic submanifold; see Section 4.2.

Theorem 1.5. Assume that \(M\) is a closed, nowhere coisotropic submanifold of a symplectic manifold \((P, \omega)\) which is geometrically bounded and strongly semi-positive. Then the almost existence theorem holds near \(M\): there exists a sufficiently small neighborhood \(U\) of \(M\) in \(P\) such that for any smooth proper Hamiltonian \(H : U \to \mathbb{R}\), the level sets \(H^{-1}(c)\) carry contractible-in-\(P\) periodic orbits of the Hamiltonian flow of \(H\) for almost all \(c\) in the range of \(H\).

Here \((P^{2n}, \omega)\) is said to be strongly semi-positive if \(c_1(A) \geq 0\) for every \(A \in \pi_2(P)\) such that \(\omega(A) > 0\) and \(c_1(A) \geq 2 - n\). The condition that \(P\) is geometrically bounded (e.g. convex) is a way to have sufficient control of the geometry of \(P\) at infinity; see Section 2 for the definition and examples.

Remark 1.6. The displacement results of \([Sch]\) rely heavily on \([LalMc1, McDS]\). In Section II we will give a simple proof of this theorem for symplectically aspherical manifolds \((P, \omega)\) which are either closed or geometrically bounded and wide.

As a particular case, Theorem 1.5 implies the almost existence of periodic orbits in a neighborhood of a closed symplectic submanifold, provided that \(P\) is strongly semi-positive and geometrically bounded. Note in this connection that almost existence in a neighborhood of a symplectic submanifold satisfying certain additional hypotheses was proved by Kerman, \([Ko2]\). On the other hand, almost existence in a neighborhood of a non-Lagrangian submanifold of middle dimension was established by Schlenk, \([Sch]\). Kerman’s theorem holds when the ambient manifold \(P\) is symplectically aspherical while Schlenk’s requires \(P\) to be only strongly semi-positive. Furthermore, G. Lu, \([Lu2]\), has proved the almost existence theorem for neighborhoods of symplectic submanifolds in any symplectic manifold by showing that the contractible Hofer-Zehnder capacity of such a neighborhood is finite using a deep and difficult result due to Liu and Tian, \([LT]\).

The almost existence theorem is closely related to the existence problem for periodic orbits of a charged particle in a magnetic field, also known as the magnetic problem, and to the generalized Moser-Weinstein theorem; see \([Gi1, GG, GK1, GK2, Ko1, Mo]\). To be more precise, let \(M\) be a closed Riemannian manifold and
let \( \eta \) be a closed two-form (magnetic field) on \( M \). Equip \( T^*M \) with the twisted symplectic structure \( \omega = \omega_0 + \pi^*\eta \), where \( \omega_0 \) is the standard symplectic form on \( T^*M \) and \( \pi: T^*M \to M \) is the natural projection. It is known that \( (T^*M, \omega) \), a twisted cotangent bundle, is geometrically bounded for any \( \eta \); see [AL, CGK, Lu1]. Finally, let \( H \) be the standard kinetic energy Hamiltonian on \( T^*M \). The Hamiltonian flow of \( H \) on \( W \), called a twisted geodesic flow, is of interest because it describes, for example, the motion of a charge on \( M \) in the magnetic field \( \eta \).

In this setting, as a particular case of Theorem 1.5, we obtain the existence of contractible twisted geodesics on almost all low energy levels, provided that the magnetic field is nowhere zero – a result complementing numerous other theorems on the existence of twisted geodesics; see, e.g., [CGK, Gil, CGI, GK1, GK2, Kel, Ke2, Lu1, Mac, Pol2, Schl]. Note that the assumption that \( \eta \) is nowhere zero ensures that \( M \) is nowhere coisotropic.

1.4. Organization of the paper. In Section 2 we set the conventions and notation and recall relevant results concerning filtered Floer homology and homotopy maps. The goal of Section 3 is two-fold. We first introduce and discuss the notion of a wide symplectic manifold. Then we construct an action selector for wide manifolds which are geometrically bounded and symplectically aspherical. Here we also state and prove the properties of this selector. In Section 4 we prove the main results of this paper.

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2. Preliminaries

2.1. Floer homology. Let \((P, \omega)\) be an open symplectic manifold. In order for the Floer homology to be defined, we need to impose some additional conditions on the manifold. To this end, we will always assume that \( P \) is geometrically bounded. This assumption gives us sufficient control of the geometry of \( P \) at infinity which is necessary in the case of open manifolds. Examples of such manifolds include symplectic manifolds that are convex at infinity (e.g. compact symplectic manifolds, \( \mathbb{R}^{2n} \), cotangent bundles) as well as twisted cotangent bundles, which, in general, fail to be convex at infinity. For the sake of completeness we recall the definition.

Definition 2.1. A symplectic manifold \((P, \omega)\) is said to be geometrically bounded if \( P \) admits an almost complex structure \( J \) and a complete Riemannian metric \( g \) such that

- \( J \) is uniformly \( \omega \)-tame, i.e., for some positive constants \( c_1 \) and \( c_2 \) we have
  \[
  \omega(X, JX) \geq c_1 \|X\|^2 \quad \text{and} \quad |\omega(X, Y)| \leq c_2 \|X\| \|Y\|
  \]
for all tangent vectors $X$ and $Y$ to $P$;

- the sectional curvature of $(P, g)$ is bounded from above and the injectivity radius of $(P, g)$ is bounded away from zero.

We refer the reader to [AL, CGK, Lu1] for a discussion of geometrically bounded manifolds. In particular, though we have not yet recalled the definition of Floer homology, let us note that the compactness theorem for the moduli spaces of Floer’s trajectories for open geometrically bounded manifolds holds; this is a consequence of Sikorav’s version of the Gromov compactness theorem; see [AL].

Furthermore, assume that $(P, \omega)$ is symplectically aspherical, i.e.,

$$\omega|_{\pi_2(P)} = 0 \quad \text{and} \quad c_1(TP)|_{\pi_2(P)} = 0.$$  

We will indicate when $P$ need not be symplectically aspherical, as is the case in Theorem 1.5.

Among manifolds which are symplectically aspherical and geometrically bounded are $\mathbb{R}^{2n}$, symplectic tori, cotangent bundles and twisted cotangent bundles when the form on the base is weakly exact. Under these hypotheses, the filtered $\mathbb{Z}$-graded Floer homology of a compactly supported Hamiltonian on $P$ is defined as follows.

Recall that for a time-dependent Hamiltonian $H: S^1 \times P \to \mathbb{R}$, the action functional on the space of smooth contractible loops $\Lambda P$ is defined as

$$A_H(x) = - \int_{S^1} \bar{x}^* \omega + \int_{S^1} H(t, x) \, dt,$$

(2.1)

where $x: S^1 \to P$ is a contractible loop and $\bar{x}: D^2 \to P$ is a map of a disk, bounded by $x$, and $S^1 = \mathbb{R}/\mathbb{Z}$. Since $P$ is symplectically aspherical $A_H(x)$ is well-defined. The Hamiltonian vector field $X_H$ is defined by the equation $i_{X_H} \omega = -dH$. Let $\varphi_H^t$ denote the time-dependent flow of $X_H$ and, in particular, $\varphi_H = \varphi_H^1$ denote the time-one flow.

By the least action principle, the critical points of $A_H$ are exactly contractible one-periodic orbits of the Hamiltonian flow of $H$. We denote by $\mathcal{P}_H$ the collection of such orbits and let $\mathcal{P}_H^{(a, b)} \subset \mathcal{P}_H$ stand for the collection of orbits with action in the interval $(a, b)$. The action spectrum $S(H)$ of $H$ is the set of critical values of $A_H$. In other words, $S(H) = \{ A_H(x) \mid x \in \mathcal{P}_H \}$. This is a zero measure set; see, e.g., [HZ3, Sc].

Throughout this paper we will assume that $H$ is compactly supported and set $\supp H = \bigcup_{t \in S^1} \supp H_t$. In this case, $S(H)$ is closed and hence nowhere dense.

Let $J = J_t$ be a time-dependent almost complex structure on $P$. A Floer anti-gradient trajectory $u$ is a map $u: \mathbb{R} \times S^1 \to P$ satisfying the equation

$$\frac{\partial u}{\partial s} + J_t(u) \frac{\partial u}{\partial t} = -\nabla H_t(u).$$

(2.2)

Here the gradient is taken with respect to the time-dependent Riemannian metric $\omega(\cdot, J_t \cdot)$. Denote by $u(s)$ the curve $u(s, \cdot) \in \Lambda P$.

The energy of $u$ is defined as

$$E(u) = \int_{-\infty}^{\infty} \left\| \frac{\partial u}{\partial s} \right\|^2_{L^2(S^1)} \, ds = \int_{-\infty}^{\infty} \int_{S^1} \left\| \frac{\partial u}{\partial t} - J \nabla H_t(u) \right\|^2 \, dt \, ds.$$  

(2.3)

We say that $u$ is asymptotic to $x^\pm \in \mathcal{P}_H$ as $s \to \pm \infty$, or connecting $x^-$ and $x^+$, if $\lim_{s \to \pm \infty} u(s) = x^\pm$ in $\Lambda P$. In this case

$$A_H(x^-) - A_H(x^+) = E(u).$$
We denote the space of Floer trajectories connecting $x^-$ and $x^+$, with the topology of uniform $C^\infty$-convergence on compact sets, by $\mathcal{M}_H(x^-,x^+,J)$. This space carries a natural $\mathbb{R}$-action $(\tau \cdot u)(t,s) = u(t,s + \tau)$ and we denote by $\hat{\mathcal{M}}_H(x^-,x^+,J)$ the quotient $\mathcal{M}_H(x^-,x^+,J)/\mathbb{R}$.

Recall that $x \in \mathcal{P}_H$ is said to be non-degenerate if $d\varphi_H: T_{x(0)}P \to T_{x(0)}P$ does not have one as an eigenvalue. In this case, the so-called Conley–Zehnder index $\mu_{CZ}(x) \in \mathbb{Z}$ is defined; see, e.g., [Sa, SZ]. Here we normalize $\mu_{CZ}$ so that $\mu_{CZ}(x) = n$ when $x$ is a non-degenerate maximum of an autonomous Hamiltonian with a small Hessian. Assume that all periodic orbits with actions in the interval $(a, b)$, including $x^\pm$, are non-degenerate. Then, for a generic $J$, suitable transversality conditions are satisfied and $\mathcal{M}_H(x^-,x^+,J)$ is a smooth manifold of dimension $\mu_{CZ}(x^+) - \mu_{CZ}(x^-)$; see, e.g., [FH, SZ] and references therein.

### 2.2. Filtered Floer homology

In this section we briefly outline the construction of filtered Floer homology following closely [Gi4]; see also [BPS, CGK, FH, GG, Sc].

#### 2.2.1. Filtered Floer homology: definitions

Let $H$ be a compactly supported Hamiltonian on an open symplectic manifold $P$ which is symplectically aspherical and geometrically bounded. Assume that all contractible one-periodic orbits of $H$ with positive action are non-degenerate. This is a generic condition. Consider an interval $(a, b)$, with $a > 0$, such that $a$ and $b$ are outside $S(H)$. Then the collection $\mathcal{P}_H^{(a,b)}$ is finite. Assume furthermore that $J$ is regular, i.e., the necessary transversality conditions are satisfied for moduli spaces of Floer trajectories connecting orbits from $\mathcal{P}_H^{(a,b)}$. This is again a generic property as can be readily seen by applying the argument from [FH, FHS, SZ].

Let $\text{CF}_k^{(a,b)}(H)$ be the vector space over $\mathbb{Z}_2$ generated by $x \in \mathcal{P}_H^{(a,b)}$ with $\mu_{CZ}(x) = k$. Define

$$\partial: \text{CF}_k^{(a,b)}(H) \to \text{CF}_{k-1}^{(a,b)}(H)$$

by

$$\partial x = \sum_y \#(\hat{\mathcal{M}}_H(x,y,J)) \cdot y,$$

where the summation extends over all $y \in \mathcal{P}_H^{(a,b)}$ with $\mu_{CZ}(y) = \mu_{CZ}(x) - 1$ and $\#(\hat{\mathcal{M}}_H(x,y,J))$ is the number of points, modulo 2, in $\hat{\mathcal{M}}_H(x,y,J)$. (Recall that in this case $\hat{\mathcal{M}}_H(x,y,J)$ is a finite set by the compactness theorem.) Then, as is well known, $\partial^2 = 0$. The resulting complex $\text{CF}^{(a,b)}(H)$ is the filtered Floer complex for $(a, b)$. Its homology $\text{HF}^{(a,b)}(H)$ is called the filtered Floer homology. This is essentially the standard definition of Floer homology with critical points having action outside $(a, b)$ being ignored. In general, $\text{HF}^{(a,b)}(H)$ depends on the Hamiltonian $H$, but not on $J$; see, e.g., [Gi4].

**Remark 2.2.** It is clear that the same construction, with suitable modifications, works for closed manifolds. In this case, $\text{HF}(H) = \text{HF}^{(-\infty,\infty)}(H)$ is the ordinary Floer homology. Moreover, $\text{HF}_*(H) = H_{*+n}(P; \mathbb{Z}_2)$. 

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Let $a < b < c$. Assume that all of the above assumptions are satisfied for all three intervals $(a, c)$ and $(a, b)$ and $(b, c)$. Then clearly $\text{CF}^{(a, b)}(H)$ is a subcomplex of $\text{CF}^{(a, c)}(H)$, and $\text{CF}^{(b, c)}(H)$ is naturally isomorphic to the quotient complex $\text{CF}^{(a, c)}(H)/ \text{CF}^{(a, b)}(H)$. As a result, we have the long exact sequence

$$\ldots \rightarrow \text{HF}^{(a, b)}(H) \rightarrow \text{HF}^{(a, c)}(H) \rightarrow \text{HF}^{(b, c)}(H) \rightarrow \ldots . \tag{2.4}$$

In the construction of the action selector for open manifolds given in Section 3.2.1, we will work with filtered Floer homology for the interval $(0, b)$ even though $0$ is necessarily a critical value of the action functional. This homology is defined as

$$\text{HF}^{(0, b)}(H) = \lim_{\epsilon \to 0^+} \text{HF}^{(\epsilon, b)}(H), \tag{2.5}$$

where the inverse limit is taken with respect to the quotient maps and $\epsilon \to 0^+$ in the complement of $S(H)$. (It is clear that this definition is equivalent to the original one when $P$ is closed and $0$ is not in $S(H)$.)

### 2.2.2. Homotopy

Let us now examine the dependence of $\text{HF}^{(a, b)}(H)$ on $H$. Consider a homotopy $H^s$ of Hamiltonians from $H^0$ to $H^1$. By definition, this is a family of Hamiltonians parametrized by $s \in \mathbb{R}$, and such that $H^s = H^0$ when $s$ is large negative and $H^s = H^1$ when $s$ is large positive. Furthermore, let $J^s$ be a family of $t$-dependent almost complex structures such that again $J^s \equiv J^0$ when $s \ll 0$ and $J^s \equiv J^1$ when $s \gg 0$. (We will most of the time suppress $H^s$ as the homotopy.)

For $x \in \mathcal{P}^{(a_0, b_0)}_{H^0}$ and $y \in \mathcal{P}^{(a_1, b_1)}_{H^1}$ denote by $\mathcal{M}_{H^s}(x, y, J^s)$ the space of solutions of (2.2) with $H = H^s$ and $J = J^s$.

The regularity property takes the following form for open manifolds: $(H^s, J^s)$ is said to be regular if the transversality requirements are met along all homotopy trajectories connecting periodic orbits with positive action. This is a generic property as can be seen by arguing as in [FH, FHS, SZ]. (When $P$ is closed, regularity of a homotopy $(H^s, J^s)$ is understood in the standard sense, i.e., the standard transversality requirements are met by the homotopy $(H^s, J^s)$; see [FH, FHS, SZ].)

When the transversality conditions are satisfied, $\mathcal{M}_{H^s}(x, y, J^s)$ is a smooth manifold of dimension $\mu_{CZ}(x) - \mu_{CZ}(y)$. In particular, $\mathcal{M}_{H^0}(x, y, J^0)$ is a finite set when $\mu_{CZ}(x) = \mu_{CZ}(y)$. Define the homotopy map

$$\Psi_{H^0,H^1} : \text{CF}^{(a_0, b_0)}(H^0) \to \text{CF}^{(a_1, b_1)}(H^1)$$

by

$$\Psi_{H^0,H^1}(x) = \sum_y \#(\mathcal{M}_{H^s}(x, y, J^s)) \cdot y.$$

Here the summation is over all orbits $y \in \mathcal{P}^{(a_1, b_1)}_{H^1}$ with $\mu_{CZ}(y) = \mu_{CZ}(x)$ and $\#(\mathcal{M}_{H^s}(x, y, J^s))$ is the number of points, modulo 2, in this moduli space.

The map $\Psi_{H^0,H^1}$ depends on the entire homotopy $(H^s, J^s)$ and in general is not a map of complexes. However, $\Psi_{H^0,H^1}$ becomes a homomorphism of complexes when $(a_0, b_0) = (a_1, b_1)$ and the homotopy is monotone decreasing, i.e., $\partial_s H^s \leq 0$ point-wise. Moreover, the induced map in homology is then independent of the homotopy, within the class of decreasing homotopies, and commutes with the maps from the exact sequence (2.4). (The reader is referred to, e.g., [BPS, CGK, FH, Sa, SZ, Sc, V12], for the proofs of these facts for both open and closed manifolds.)

There are other instances when the same is true. In particular, this is the case when
the location of the intervals \((a_0, b_0)\) and \((a_1, b_1)\) is compatible with the growth of the Hamiltonians in the homotopy, as streamlined by the following theorem from [Gi4]; see also [Sc]. (This theorem holds for both open and closed manifolds.)

**Theorem 2.3 ([Gi4]).** Let \(H^s\) be a homotopy such that 
\[
\int_{-\infty}^{\infty} \int_{S^1} \max_p \partial_s H^s_t dt ds \leq C,
\]
where \(C \in \mathbb{R}\). Then 
\[
\Psi_{H^0 H^1} : \text{CF}^{(a, b)}(H^0) \to \text{CF}^{(a+C, b+C)}(H^1)
\]
is a homomorphism of complexes for any interval \((a, b)\). Hence, \(\Psi_{H^0 H^1}\) induces a map in Floer homology, also denoted by \(\Psi_{H^0 H^1}\). This map sends the exact sequence (2.4) for \(H^0\) and \((a, b, c)\) to the exact sequence (2.4) for \(H^1\) and \((a+C, b+C, c+C)\), i.e., on the level of homology \(\Psi_{H^0 H^1}\) commutes with all maps in the long exact sequence (2.4).

3. Action selectors in wide manifolds

The proof of Theorem 1.2 relies on the theory of action selectors, which is one of the standard approaches to the problem, [FS, HZ3, Sc, Vi1]. This theory is well developed for weakly-exact closed or convex manifolds; see, e.g., [FGS, FS, Sc] and also [Oh] for the theory in a more general setting. The main ingredient in these constructions of an action selector is an identification between Floer homology spaces for different Hamiltonians. For example, for symplectically aspherical closed or convex manifolds, the Floer homology of any Hamiltonian is isomorphic to the homology of the manifold. Then an action selector can be associated to any homology class. However, these constructions do not generalize to the case of open manifolds which are merely geometrically bounded. The main obstacle is that the Floer homology for Hamiltonians on such manifolds is no longer an invariant, i.e., it depends on the Hamiltonian. For the homology can be defined only for action intervals that do not contain zero and hence, in contrast with the case of closed or convex manifolds, there is, in general, no relation between the Floer homology for different Hamiltonians. Nevertheless, to construct an action selector it suffices to have a class in the Floer homology of a Hamiltonian, which is in some sense canonical, even though the homology group it belongs to depends on the Hamiltonian. In this section we construct an action selector for geometrically bounded wide manifolds. The homology class for which the selector is defined is essentially the fundamental class of the manifold modulo infinity. Then this selector, for non-negative Hamiltonians, has properties similar to those of action selectors constructed in [FS, Sc].

3.1. Wide manifolds. Let us now introduce and discuss the notion of a wide symplectic manifold.

**Definition 3.1.** A symplectic manifold \((W, \omega)\) is said to be wide if it is open and if for every constant \(C \geq 0\) and for every compact subset \(X \subset W\), there exists a function \(K : W \to [0, \infty)\) such that
\begin{itemize}
  \item[(W1)] \(K\) is compactly supported;
  \item[(W2)] \(|K|_X \geq C\);
\end{itemize}
(W3) the Hamiltonian flow of $K$ has no non-trivial contractible fast periodic orbits.

We call a non-trivial orbit fast if its period is less than or equal to one. Otherwise an orbit will be called slow.

**Remark 3.2.** The condition (W2) can be replaced by the condition $(W2')$: $\max K \equiv K|_X \equiv C$. For, as explained in [GG, HZ3], one can cut off a function meeting the requirement (W2) without creating new fast periodic orbits and produce another function satisfying the condition $(W2')$.

Examples of wide manifolds include cotangent bundles and Stein manifolds, or more generally symplectic manifolds that are convex at infinity. More importantly, twisted cotangent bundles, which are geometrically bounded but, in general, fail to be convex at infinity, are wide. Non-compact covering spaces of compact manifolds are also examples of wide manifolds, [CH]. Furthermore, the product of two wide manifolds is wide and so is the product of a compact and a wide manifold. On the other hand, a manifold with finite contractible Hofer-Zehnder capacity cannot be wide, as easily follows from Definition 3.1; see, e.g., Section 4.2 for the definition of this capacity.

**Remark 3.3.** Wideness, although indispensable for our construction, appears to be a rather mild assumption: the author is not aware of any example of an open geometrically bounded manifold which is not wide. It is not yet clear, however, whether every open geometrically bounded manifold is wide. The relation between geometrical boundedness and the concept of wideness is interesting but quite complicated, and we will address this question elsewhere.

The property of wideness can also be viewed in terms of the restricted relative Hofer-Zehnder capacity introduced in [GG] or as the property of a manifold to admit a slow proper function. Namely, we have

**Proposition 3.4.** Let $(W, \omega)$ be a symplectic manifold. Then the following statements are equivalent.

1. $(W, \omega)$ is wide.
2. $\overline{c}_{HZ}(W, X) = \infty$ for every compact subset $X \subset W$, where $\overline{c}_{HZ}(W, X)$ is the restricted (or contractible) relative Hofer-Zehnder capacity introduced in [GG].
3. $(W, \omega)$ admits a non-negative proper function without non-trivial contractible fast periodic orbits.

We omit the proof of Proposition 3.4 for it is essentially straightforward and we will mainly be using Definition 3.1.

**Remark 3.5.** One could also replace the condition (W3) in Definition 3.1 by the one that the Hamiltonian flow of $K$ has no non-trivial fast periodic orbits, hence dropping the requirement that the orbits be contractible. However, this would be a more restrictive requirement on $W$; for instance, a cylinder of finite area, which is wide according to Definition 3.1, would then fail to be wide.

Let us now turn to constructing an action selector for geometrically bounded wide manifolds.
3.2. An action selector for wide manifolds. We assume that \((W^{2n}, \omega)\) is a symplectically aspherical, geometrically bounded and wide manifold. We will construct an action selector for non-negative compactly supported Hamiltonians on \(W\).

3.2.1. The definition. Let \(H: S^1 \times W \to \mathbb{R}\) be a compactly supported Hamiltonian such that \(H \geq 0\). It is easy to see that, since \(W\) is wide, there exists a smooth compactly supported function \(F: W \to [0, \infty)\) without non-trivial contractible fast periodic orbits and such that \(F \geq H\) point-wise. (This is essentially the definition of a wide manifold.) Without loss of generality we may assume that \(\text{supp}(F)\) is a smooth connected manifold with boundary and that \(F\) is a Morse function with finitely many critical points when restricted to the interior of the support. (These requirements are generic.) From now on we will call such functions \(wide\) and we will reserve the notation \(F\) for them.

Under these assumptions,
\[
\text{HF}^{(0, \infty)}_*(F) \cong \text{HM}^{(0, \infty)}(F) ,
\]
where \(\text{HM}^{(0, \infty)}(F)\) and \(\text{HF}^{(0, \infty)}_*(F)\) denote, respectively, the (filtered) Morse and Floer homology of \(F\) for the interval \((0, \infty)\).

For the sake of completeness, let us explain first why this isomorphism holds. Intuitively, this is obvious. For \(F\) has no non-trivial contractible one-periodic orbits by definition and, hence, Floer and Morse complexes are the same as vector spaces. Note, however, that the two differentials may be different. We, nevertheless, claim that the resulting homology groups are isomorphic.

To this end, let \(\epsilon > 0\) be small enough so that \(\epsilon F\) is \(C^2\)-small. Recall that for \(C^2\)-small functions, when \(W\) is symplectically aspherical, Morse and Floer homology groups are isomorphic. Then
\[
\text{HF}^{(0, \infty)}_*(\epsilon F) \cong \text{HM}^{(0, \infty)}(\epsilon F).
\]
Consider a monotone decreasing homotopy \(F_s\), for \(s \in [0, 1]\), from \(F\) to \(\epsilon F\). Choose \(\delta > 0\) such that it is below any critical values of \(\epsilon F\). As a result, action values for \(F_s\) for all \(s \in [0, 1]\) are greater than \(\delta > 0\). By the homotopy invariance of Floer homology, we then have the isomorphism
\[
\text{HF}^{(0, \infty)}_*(F_s) \cong \text{HF}^{(0, \infty)}_*(\epsilon F_s) \quad \text{for all } s \in [0, 1].
\]
Also note that \(\text{HF}^{(0, \infty)}_*(F_s) = \text{HF}^{(0, \infty)}_*(\epsilon F_s)\) for all \(s \in [0, 1]\) since zero is an isolated critical value and \(\delta\) is smaller than any critical value of \(F_s\).

Finally, we have
\[
\text{HF}^{(0, \infty)}_*(F) \cong \text{HF}^{(0, \infty)}_*(\epsilon F) \cong \text{HM}^{(0, \infty)}(\epsilon F) \cong \text{HM}^{(0, \infty)}_*(F).
\]

Remark 3.6. Observe that \(\text{HM}^{(0, \infty)}(F)\) is just the ordinary homology of \(\text{supp}(F)\) modulo its boundary, i.e., \(\text{HM}^{(0, \infty)}(F) = H_{*+n}(\text{supp}\ F, \partial(\text{supp}\ F); \mathbb{Z}_2)\).

Let us now turn to the definition of the action selector. Since \(\text{supp}(F)\) is connected, equation (3.1), in particular, implies that
\[
\text{HF}^{(0, \infty)}_*(F) \equiv \text{HM}^{(0, \infty)}_2(F) \equiv \mathbb{Z}_2.
\]
Denote by \([\text{max}_F]\) the generator of \(\text{HF}^{(0, \infty)}_*(F) \equiv \mathbb{Z}_2\), which can be thought of as the fundamental class. Consider the image of this class under the monotonicity map
\[
\Psi_{F_H}: \text{HF}^{(0, \infty)}_*(F) \to \text{HF}^{(0, \infty)}_*(H),
\]
induced by a monotone decreasing homotopy from $F$ to $H$. (This map is independent of the choice of the homotopy, as discussed in Section 2.) Using the homotopy invariance of the Floer homology it is not hard to show that $\Psi_{FH}[\max F]$ is independent of the choice of $F$; denote this class by $[\max]$.  

**Definition 3.7.** The action selector is defined as

$$\sigma(H) = \inf \{ a > 0 \mid j_a^H[\max] = 0 \},$$

where

$$j_a^H : \text{HF}^{(0, \infty)}_n(H) \to \text{HF}^{(a, \infty)}_n(H),$$

is induced by the quotient map.

**Remark 3.8.** Note that this definition makes sense for arbitrary Hamiltonians, i.e., we need not assume that $H$ is non-negative. However, it is not clear whether $\sigma$ would then have the properties (AS0)-(AS6) listed below, which are crucial for applications. Also, when $W$ is convex, this selector is equal to the one constructed in [FS].

3.2.2. Properties of the action selector. Let $S(H)$ denote the action spectrum of $H$ and let $\text{Ham}_c^+(W)$ denote the cone in the group of compactly supported Hamiltonian diffeomorphisms of $W$ generated by non-negative Hamiltonians. The action selector $\sigma : \text{Ham}_c^+(W) \to [0, \infty)$ constructed above has the following properties:

1. **(AS0)** $\sigma(H)$ is a spectral value: $\sigma(H) \in S(H)$;
2. **(AS1)** $\sigma(H)$ is monotone: if $0 \leq H \leq K$ then $0 \leq \sigma(H) \leq \sigma(K)$;
3. **(AS2)** $\sigma(H)$ is non-degenerate: $0 < \sigma(H) < \infty$ if $H \neq 0$;
4. **(AS3)** $\sigma(H)$ is continuous in $H$ with respect to the Hofer norm, and, in particular, it is $C^0$-continuous;
5. **(AS4)** $\sigma(H) \leq \int_0^1 \max H_t \, dt = \|H\|$, where $\| \cdot \|$ denotes the Hofer norm;
6. **(AS5)** if $H$ is autonomous and has no non-trivial contractible fast periodic orbits, then $\sigma(H) = \max H$;
7. **(AS6)** if $H$ and $K$ generate the same time-one flow and $\varphi^t_H$ and $\varphi^t_K$ are homotopic (with fixed end points) in $\text{Ham}_c^+(W)$, then $\sigma(H) = \sigma(K)$;
8. **(AS7)** $\sigma(H) \leq e_V$ for any $H$ supported in $V \subset W$, where $e_V$ denotes the displacement energy of $V$. Thus, $\sigma(H)$ is *a priori* bounded from above by $e_V < \infty$ if $V$ is displaceable.

3.2.3. Proofs of the properties of the selector. We will prove these properties in varying degree of detail; some proofs are very similar to those for the selectors constructed in [FS, Sc], whereas some proofs require modifications. We will mainly focus on the new parts and refer to the literature for the standard ones.

**(AS0)** First recall that $S(H)$ is compact and nowhere dense.

Assume the contrary: $\sigma(H) \notin S(H)$. Then, since $S(H)$ is compact, for a small enough number $\delta > 0$ we have $[\sigma(H) - \delta, \sigma(H) + \delta] \cap S(H) = \emptyset$. Hence, by the definition of the selector, there exists a number $a$ with $\sigma(H) < a \leq \sigma(H) + \delta$ such that $j_a^H[\max] = 0$.

Let $c$ be such that $\sigma(H) - \delta \leq c < \sigma(H)$. Then we have the isomorphism

$$\text{HF}^{(a, \infty)}_n(H) \cong \text{HF}^{(c, \infty)}_n(H),$$
since there are no critical values of $A_H$ in $[c, a]$, and the diagram

$$
\begin{array}{ccc}
\text{HF}_{n_1}^{(0, \infty)}(H) & \xrightarrow{\psi_{KH}} & \text{HF}_{n_2}^{(a, \infty)}(H) \\
0 = j^H_n & \cong & j^H_n \text{HF}_{n_1}^{(0, \infty)}(H) \\
\end{array}
$$

commutes. Thus $j^H_n[\max] = 0$. This contradicts the definition of $\sigma(H)$. We conclude that $\sigma(H) \in \mathcal{S}(H)$.

(AS1) Let $K \geq H \geq 0$ and let $F$ be a wide function such that $F \geq K \geq H$. Monotonicity is a consequence of the commutativity of the following diagram:

$$
\begin{array}{ccc}
\text{HF}_{n_1}^{(0, \infty)}(F) & \xrightarrow{\psi_{FH}} & \text{HF}_{n_2}^{(0, \infty)}(H) \\
\text{HF}_{n_1}^{(0, \infty)}(K) & \xrightarrow{\psi_{FH}} & \text{HF}_{n_2}^{(a, \infty)}(K) \\
\end{array}
$$

(AS2) Finiteness of the selector follows from the compactness of $\mathcal{S}(H)$, for $\text{HF}_{n_1}^{(a, \infty)}(H) = 0$ for any $a > \sup \mathcal{S}(H)$.

Proving the non-degeneracy, i.e., $\sigma(H) > 0$ for any $H \neq 0$, requires more work. First observe that we can find a $C^2$-small space-time bump function $f \neq 0$ such that $0 \leq f \leq H$ for all $t \in S^1$. This is simply because $H \geq 0$ and $H \neq 0$ for some $t$. Hence, by the monotonicity of the selector, it suffices to show that $\sigma(f) > 0$.

More precisely, let $f(t, x) = f_{s1}(t) : f_{w}(x) : S^1 \times W \to [0, \infty)$ be a space-time bump function satisfying $0 \leq f \leq H$ for all $t \in S^1$. Here $f_{w}(x) : W \to [0, \infty)$ and $f_{s1}(t) : S^1 \to [0, \infty)$ are both bump functions in the usual sense, and $f_{w}$ is autonomous and $C^2$-small. The time-one flow of $f$ differs from the time-one flow of $f_{w}$ only by a positive factor equal to the integral of $f_{s1}$ over the circle. Let us assume, for the sake of simplicity, that $\int f_{s1}(t) \, dt = 1$. This can be achieved by choosing $f_{w}$ sufficiently small so that $f$ still fits underneath $H$. Then the action spectra of $f$ and $f_{w}$ are the same.

We claim that $\text{HF}_{n_1}^{(a, b)}(f) \cong \text{HF}_{n_1}^{(a, b)}(f_{w})$ for all positive intervals of action $(a, b) \subseteq (0, \infty)$ such that $a$ and $b$ are not in $\mathcal{S}(f) = \mathcal{S}(f_{w})$. To see this, let $K_s = s f_{w} + (1 - s) f$ be the linear homotopy from $f$ to $f_{w}$ for $s \in [0, 1]$. It is easy to see that all Hamiltonians in this homotopy have the same time-one flow as that of $f_{w}$ and, hence, the only critical points of $A_{K_s}$ are constant one-periodic orbits, i.e., the critical points of $f_{w}$. Thus the action spectrum $\mathcal{S}(K_s)$ for any $s$ consists of two action values: zero and $\max f_{w}$. This implies that for $a, b \notin \mathcal{S}(K_s)$ no periodic orbit with action outside the range $(a, b)$ will enter or exit this interval during the course of the homotopy. In this case the Floer homology groups are isomorphic for all $K_s$; see, e.g., [BPS, GM, VT2]. In particular, $\text{HF}_{n_1}^{(a, b)}(f) \cong \text{HF}_{n_1}^{(a, b)}(f_{w})$. (Note that this isomorphism is not induced by the homotopy map since $K_s$ is not a monotone homotopy.)

We next show that $\sigma(f_{w}) = \max f_{w} > 0$. To this end, let $F$ be a wide function satisfying $F \geq f_{w}$ and recall that $\text{HF}_{n_1}^{(0, \infty)}(F) \cong \text{HM}_{n_1}^{0, \infty}(F)$. Moreover, since $f_{w}$ is a $C^2$-small bump function, Morse and Floer homology groups are isomorphic:
HF_2^{(0,\infty)}(f_w) \cong HM_2^{(0,\infty)}(f_w). Then the diagram
\[
\begin{array}{ccc}
Z_2 \cong HF_n^{(0,\infty)}(F) & \xrightarrow{\cong} & HM_2^{(0,\infty)}(F) \cong Z_2 \\
\Psi_{F/f} & \downarrow \Psi_{F/f_w} & \downarrow \Psi_{F/f}\ \\
HF_n^{(0,\infty)}(f_w) & \xrightarrow{\cong} & HM_2^{(0,\infty)}(f_w) \\
\downarrow j_{f/w} & & \downarrow j_{f/w} \\
HF_n^{(a,\infty)}(f_w) & \xrightarrow{\cong} & HM_2^{(a,\infty)}(f_w)
\end{array}
\]
is commutative. This can easily be seen by factoring the horizontal isomorphisms through isomorphisms induced by monotone homotopies and using the fact that all such diagrams commute when the functions involved are $C^2$-small.

Let us focus on the right-hand side of the diagram. By Morse theory, the map
\[
Z_2 \cong HM_2^{(0,\infty)}(F) \xrightarrow{\Psi_{F/f_w}} HM_2^{(0,\infty)}(f_w) \cong Z_2
\]
is non-zero, and sends $[\max_F]$ to $[\max_{f_w}]$. Moreover, $j_{f/w}[\max_{f_w}] = [\max_{f_w}]$ for any positive $a < \max_{f_w}$ and $j_{f/w}[\max_{f_w}] = 0$ for any $a \geq \max_{f_w}$. Commutativity of the diagram then implies that $\sigma(f_w) = \max_{f_w} > 0$.

As the last step observe that $\sigma(f) = \sigma(f_w) = \max_{f_w} > 0$, which finishes the proof of non-degeneracy. To see this, note that the diagram
\[
\begin{array}{ccc}
HF_n^{(0,\infty)}(F) & \xrightarrow{\Psi_{F/f}} & HF_n^{(0,\infty)}(f) \\
\downarrow \Psi_{F/f_w} & & \downarrow \Psi_{F/f_w} \\
HF_n^{(0,\infty)}(f_w) & \xrightarrow{\cong} & HF_n^{(a,\infty)}(f_w)
\end{array}
\]
is also commutative, where $F$ is a wide function satisfying $F \geq f$ and $F \geq f_w$.

**Remark 3.9.** An important consequence of non-degeneracy of $\sigma$ is that $HF_n^{(0,\infty)}(H) \neq 0$ for any non-negative Hamiltonian $H \neq 0$. To see this, note that the **non-zero** map $HF_n^{(0,\infty)}(F) \to HF_n^{(0,\infty)}(f_w)$ can be factored through $HF_n^{(0,\infty)}(H)$, where $F$ and $f_w$ are as in the proof above. This fact is used in [Gi4].

**(AS3)** Recall that this property asserts the continuity of the selector with respect to the Hofer norm. Namely, we claim that
\[
|\sigma(H) - \sigma(K)| \leq \|H - K\| \text{ for non-negative } H \text{ and } K.
\]

Note first that it suffices to prove continuity for non-degenerate Hamiltonians. Since we are assuming that $W$ is open, this means that all one-periodic orbits with positive action are non-degenerate.

Keeping the notation from Section 2.2.2 let $K = H^0$ and $H = H^1$, and consider a linear homotopy $H^s$ from $H^0$ to $H^1$, i.e., $H^s = (1 - \phi(s))H^0 + \phi(s)H^1$, where $\phi: \mathbb{R} \to [0, 1]$ is a smooth monotone increasing function equal to zero near $-\infty$ and equal to one near $\infty$. Then,
\[
\int_{-\infty}^{\infty} \int_{S^1} \max_W \partial_t H^s_t \, dt \, ds = \int_{S^1} \max_W (H^1_t - H^0_t) \, dt.
\]
Let, as customary, $e^+ = e^+(H^1 - H^0) = \int_{S^1} \max_{H^1} (H^1 - H^0) \, dt$. (Recall that $\|H\| = e^+(H) - e^-(H)$, where $e^-(H) = \int_{S^1} \min_{H} H \, dt$.) Hence, by Theorem 2.3 for $a \notin S(H^*)$, we have the monotonicity maps $\Psi_{H^0; H^1}$ for two intervals: $HF_n^{(a, \infty)}(H^0) \to HF_n^{(a, e^+, \infty)}(H^1)$ and $HF_n^{(0, \infty)}(H^0) \to HF_n^{(e^+, \infty)}(H^1)$. Now the diagram

$$
\begin{array}{ccc}
HF_n^{(0, \infty)}(F) & \xrightarrow{\Psi_{F, H^1}} & HF_n^{(0, \infty)}(H^1) \\
\Psi_{F, H^0} \downarrow & & \downarrow \\
HF_n^{(0, \infty)}(H^0) & \xrightarrow{\Psi_{H^0, H^1}} & HF_n^{(e^+, \infty)}(H^1) \\
\Psi_{F, H^0} \downarrow & & \downarrow j_{n^0} \\
HF_n^{(a, \infty)}(H^0) & \xrightarrow{\Psi_{H^0, H^1}} & HF_n^{(e^+, \infty)}(H^1),
\end{array}
$$

where $F$ is a wide function satisfying $F \geq H^0$ and $F \geq H^1$, is commutative; see [Gi4]. Here the vertical maps on the right-hand side of the diagram are just the maps induced by taking quotient complexes, and their composition is the map $j_{n^0} \cdot H^1$. Consequently, we have

$$
\sigma(H^1) \leq \sigma(H^0) + e^+ = \sigma(H^0) + e^+(H^1 - H^0).
$$

Moreover, exchanging the roles of $H^0$ and $H^1$ we get

$$
\sigma(H^0) \leq \sigma(H^1) + e^+(H^0 - H^1) = \sigma(H^1) - e^-(H^1 - H^0).
$$

Thus, we have

$$
e^-(H^1 - H^0) \leq \sigma(H^1) - \sigma(H^0) \leq e^+(H^1 - H^0).
$$

$C^0$-continuity of the selector follows immediately. In order to prove the continuity in Hofer’s norm, note first that $e^+(H) \geq 0$ and $e^-(H) \leq 0$ for any compactly supported Hamiltonian $H$. (This is also true for any Hamiltonian on a closed manifold.) Consequently, we have

$$
-e^+(H^1 - H^0) + e^-(H^1 - H^0) \leq e^-(H^1 - H^0)
\leq \sigma(H^1) - \sigma(H^0)
\leq e^+(H^1 - H^0)
\leq e^+(H^1 - H^0) - e^-(H^1 - H^0),
$$

and hence

$$
|\sigma(H^1) - \sigma(H^0)| \leq e^+(H^1 - H^0) - e^-(H^1 - H^0) = \|H^1 - H^0\|.
$$

This finishes the proof of continuity.

(A54) The assertion readily follows from (A3).

(A55) We refer the reader to [Gi5, Lemma 3.5] for a proof of the property that $\sigma(H) = \max H$ for an autonomous Hamiltonian $H$ without non-trivial contractible fast periodic orbits. The proof in [Gi5] is set theoretic in nature and works in any setting where the selector has the properties (A0), (A1), (A3) and the claimed property holds for autonomous $C^2$-small functions. (See the proof of (A2) above for the fact that $\sigma(H) = \max H$ when $H$ is a $C^2$-small autonomous function having no non-trivial contractible fast periodic orbits.)

(A56) It is well-known that the action spectrum of a compactly supported Hamiltonian on an open symplectically symplectically aspherical manifold depends only
on the time-one flow; see, e.g., [FS, HZ3]. Thus, if \( H \) and \( K \) generate the same time-one flow and \( \varphi^t_H \) and \( \varphi^t_K \) are homotopic (with fixed end points) in \( \text{Ham}^+_c(W) \), then the action spectrum stays the same throughout the homotopy. On the other hand, due to (AS3), \( \sigma \) varies continuously in the course of the homotopy. As the action spectrum is nowhere dense, \( \sigma \) must be constant.

(AS7) It is a standard fact that an action selector defined on a displaceable domain in a closed or convex manifold is a priori bounded from above. However, the proofs existing in literature rely on the sub-additivity of the action selector and the fact that the selectors are defined for all Hamiltonians (in particular, not necessarily non-negative Hamiltonians). Hence, these arguments do not apply to the action selector introduced here for wide manifolds, and, for the sake of completeness, we provide a proof of (AS7).

Assume that \( V \subset W \) is open and displaceable, and denote by \( e_V \) the displacement energy of \( V \); see, e.g., [HZ3, Sch1, Pol2]. Let \( H : S^1 \times V \to [0, \infty) \) be a compactly supported Hamiltonian whose support is contained in \( V \). Let \( K \) be a compactly supported Hamiltonian such that \( \varphi^K \) displaces \( V \). Moreover, without loss of generality, we may assume that \( K \geq 0 \). For, otherwise, we first shift \( K \) up so that \( \min K = 0 \) and then cut it off away from its original support. The new function is non-negative, still displaces \( V \) and has the same Hofer norm as the original function.

Recall that, in general, for any two Hamiltonians \( H \) and \( K \) generating the time-one flows \( \varphi^t_H \) and \( \varphi^t_K \), the Hamiltonian generating the composition flow \( \varphi^t_H \circ \varphi^t_K \) is given by \( H \circ K = H(t, x) + K(t, (\varphi^t_H)^{-1}(x)) \). Since \( K \geq 0 \) and the selector is monotone, we then have

\[
\sigma(H) \leq \sigma(H \circ K). \tag{3.2}
\]

Since \( \varphi^K \) displaces \( \text{supp} \ H \), one-periodic orbits of \( \varphi^t_H \circ \varphi^t_K \) are exactly the one-periodic orbits of \( \varphi^K \). In fact, we claim that \( S(H \circ K) = S(K) \). Observe that this assertion immediately follows from

\[
S(\varphi^t_H \circ \varphi^t_K) = S(\varphi^t_K * (\varphi^t_H \circ \varphi^t_K)) = S(\varphi^t_K),
\]

where * denotes the concatenation of \( \varphi^t_K \) and \( \varphi^t_H \circ \varphi^t_K \). Here the concatenation \( a(t) * b(t) \) of paths \( a(t) \) and \( b(t) \) with domain \([0, 1]\) is defined by traversing \( a(2t) \) for \( 0 \leq t \leq 1/2 \) and then traversing \( b(2t - 1) \) for \( 1/2 \leq t \leq 1 \).

The first identity above is due to the fact that \( \varphi^t_H \circ \varphi^t_K \) and \( \varphi^K * (\varphi^t_H \circ \varphi^t_K) \) are homotopic with fixed end points. (It is straightforward to write a specific formula for this homotopy.) The second identity is specific to our situation. Namely, observe that one-periodic orbits of the concatenation cannot be in \( \text{supp} \ H \), essentially since \( \varphi^K \) displaces this support. But when a point is outside \( \text{supp} \ H \), the flow \( \varphi^t_H \) is identity and, hence, such a point can correspond to a one-periodic orbit of the concatenation only if it is fixed by \( \varphi^K \). Therefore, one-periodic orbits of the concatenation are just reparametrizations of one-periodic orbits of \( \varphi^K \). As a result, actions acquired in both cases are the same. This proves the second identity.

We now have \( S(H \circ K) = S(K) \). The same, of course, holds when \( H \) is replaced by \( \lambda H \) where \( \lambda \in [0, \infty) \), i.e., \( S(\lambda H \circ K) = S(K) \) for any non-negative \( \lambda \). But \( \sigma \) is continuous and the action spectrum is nowhere dense. Therefore, we conclude from \( \sigma(\lambda H \circ K) \in S(\lambda H \circ K) = S(K) \) that \( \sigma(\lambda H \circ K) \) is independent of \( \lambda \). Setting \( \lambda = 0 \) and \( \lambda = 1 \) yields \( \sigma(K) = \sigma(H \circ K) \). Thus, also using (3.2) and property
(AS4), we have

\[ \sigma(H) \leq \sigma(H \# K) = \sigma(K) \leq \|K\|. \]

Finally, \( e_V = \sup_K \|K\| \). Hence, the selector is \textit{a priori} bounded from above by \( e_V \), which is finite when \( V \) is displaceable.

4. Proofs

In this section we prove the main results of this paper.

4.1. The Conley Conjecture. We will focus on the case \( W \) is open. For closed manifolds, Theorem 1.2 follows from Theorem 1.1 and the results from [Sc].

In what follows we denote by \( S^+(\cdot) \) the positive part of the action spectrum of a Hamiltonian. Theorem 1.2 is a consequence of the following proposition.

**Proposition 4.1.** Let \( V \) be an open displaceable subset of a symplectically aspherical manifold \((W, \omega)\) which is geometrically bounded and wide. Let \( G \geq 0 \) be a non-zero Hamiltonian supported in \( V \). Then, \( \varphi_G \) has infinitely many periodic points with positive action, corresponding to contractible periodic orbits of \( G \). Moreover, assume that \( S^+(G) \) is separated from zero, i.e., \( \inf S^+(G) > 0 \). Then, there exists a sequence of integer periods \( T_k \to \infty \) such that for every \( T_k \), the Hamiltonian \( G \) has such a periodic orbit with minimal period \( T_k \).

Let us first derive the proof of Theorem 1.2 from Proposition 4.1.

**Proof of Theorem 1.2.** Consider \( M \times S^1 \subset W \times T^*S^1 \), where \( T^*S^1 \) is equipped with the standard symplectic structure, also referred to as the “stabilization” of \( M \) [Mad, Pol1]. Note that \( M \times S^1 \) is again nowhere coisotropic and, moreover, the normal bundle to \( M \times S^1 \) admits a non-vanishing section. Theorem 1.1 now implies that a small neighborhood \( V = U \times (S^1 \times (-\epsilon, \epsilon)) \subset W \times T^*S^1 \) of the product manifold is infinitesimally displaceable. Here \( U \subset W \), a neighborhood of \( M \) in \( W \), and \( \epsilon > 0 \) are both sufficiently small.

Let \( H \geq 0 \) be a Hamiltonian as in the statement of Theorem 1.2. Let \( K : T^*S^1 \to [0, 1] \) be an autonomous fiber-wise bump function, depending only on the distance to the zero section, which is supported in \( S^1 \times (-\epsilon, \epsilon) \) and such that \( \max K = K|_{S^1 \times 0} = 1 \). Note that \( K \) has no non-trivial contractible periodic orbits.

Consider the Hamiltonian \( G = H \cdot K \) supported in the displaceable open set \( V \). Then \( G \geq 0 \) and \( G \neq 0 \). Observe that every contractible periodic orbit of the vector field \( X_{G_t} = H_t \cdot X_K + K \cdot X_H \), with positive action must be of the form \( (u(t), v(t)) \in W \times T^*S^1 \), where \( u(t) \) is contractible in \( W \) and \( v(t) \) is constant with \( K(v(t)) = 1 \), i.e., \( v(t) \) is a point on \( S^1 \). To see this, note first that \( v'(t) = H_t(u(t)) \cdot X_K(v(t)) \) where \( X_K \) points in the direction of the angular coordinate. The assumption that \( H \geq 0 \) then implies that \( v(t) \in T^*S^1 \) can be contractible only when \( v(t) \) is constant. Since the pair \( (u(t), v(t)) \) is contractible, \( u(t) \) must also be contractible. Furthermore, the action on the orbit \( (u(t), v(t)) \) can be positive only when \( v(t) \) is a point on the zero-section.

Coming back to the proof of Theorem 1.2, note that by the previous observation we have \( S^+(H) = S^+(G) \). Moreover, since \( \varphi_H \) is assumed to have isolated fixed points with positive action, \( S^+(H) \) is separated from zero. Then, \( S^+(G) \) is also separated from zero and Proposition 4.1 applies. Finally, note that \( T_k \)-periodic orbits of \( \varphi_H \) from the proposition will correspond to infinitely many simple (contractible) periodic orbits of \( \varphi_H \). \( \square \)
Let us now prove Proposition 4.1.

**Proof of Proposition 4.1.** Note first that, by property (AS7) and the assumption that $V$ is displaceable, we have $\sigma(G) \leq e_V < \infty$, where $e_V$ is the displacement energy of $V$.

Let $G^k$ denote the Hamiltonian generating $\varphi_G^k$. Then, since $G \geq 0$, we have $G^l \leq G^k$ whenever $l < k$. (Here we are using the explicit formula for the Hamiltonian generating the composition flow; see, e.g., the proof of (AS7) for this formula.) By the monotonicity and non-degeneracy of and the *a priori* bound on $\sigma$, we then have the following series of inequalities:

$$0 < \sigma(G) \leq \sigma(G^2) \leq \ldots \leq e_V.$$ 

Assume the contrary: $\varphi_G$ has finitely many (simple) periodic points with positive action. Then, for a sufficiently large prime number $p$, fixed points of $\varphi_G^p$ must all be $p$-th iterations of fixed points of $\varphi_G$. Consequently, $S^+(\varphi_G^p) = p S^+(\varphi_G)$ and, in particular, $\sigma(G^p) = p \sigma(G)$.

Now observe that it suffices to prove the “moreover” assertion of the proposition. For when $S^+(G)$ is not separated from zero $\varphi_G$ has infinitely many fixed points and hence has infinitely many periodic points. Thus, assume that $\inf S^+(G) > \delta > 0$ for a sufficiently small $\delta > 0$.

Since $\sigma(G^p) > 0$ and $\sigma(G^p) \in p S^+(G)$, we necessarily have $\sigma(G^p) > p \cdot \delta > 0$. Hence, $\sigma(G^p) \to \infty$ as $p \to \infty$. This contradicts the fact that the selector is *a priori* bounded from above.

**Remark 4.2.** In fact, we have proved that for any prime number $p > e_V/\delta$ there exists a simple contractible $p$-periodic orbit of $G$. Furthermore, the number of simple (contractible) periodic orbits of $\varphi_G$ with period less than or equal to $k \in \mathbb{N}$ is at least a constant, depending on $\varphi_G$, times $k$. This can be seen by applying the argument in [VII, Proposition 4.13].

### 4.2. Almost Existence

We will next prove the almost existence theorem.

**Proof of Theorem 4.3.** The assertion follows from Theorem 1.1 and the results established in [Schl]. Namely, similarly to the proof of Theorem 1.2, a sufficiently small neighborhood of $M \times S^1 \subset P \times T^* S^1$ is infinitesimally displaceable by the displacement principle for nowhere coisotropic submanifolds. Let $V = U \times (S^1 \times (\cdots, \epsilon)) \subset P \times T^* S^1$ be such a neighborhood and let, as before, $e_V$ denote the displacement energy of $V$.

Let us recall the definition of the contractible Hofer-Zehnder capacity $c^0_{HZ}(U)$ of a domain $U \subset P$. Denote by $\mathcal{H}_{HZ}(U)$ the space of compactly supported smooth non-negative Hamiltonians $H: U \to \mathbb{R}$ which are constant near their maxima and have no non-trivial *contractible-in-$P$* fast periodic orbits. The contractible Hofer-Zehnder capacity is then defined to be

$$c^0_{HZ}(U) = \sup \{ \max H \mid H \in \mathcal{H}_{HZ}(U) \}.$$ 

Removing the condition that the orbits are contractible in $P$ yields a finer capacity, the ordinary Hofer-Zehnder capacity $c_{HZ}(U)$. Hence, $c_{HZ}(U) \leq c^0_{HZ}(U)$.

As an immediate consequence of the finiteness of $e_V$ and the energy-capacity inequality established in [Schl, Theorem 1.1], we obtain the estimate

$$c_{HZ}(U) \leq c^0_{HZ}(U) \leq 4 e_V < \infty.$$
In particular, $c_{HZ}(U) < \infty$ and the almost existence theorem follows from the finiteness of $c_{HZ}(U)$ by the standard arguments; see, e.g., [FGS, HZ3, CG]. □

Remark 4.3. As we have already mentioned, the displacement energy-capacity inequality of [Schl] relies heavily on the work of Lalonde–McDuff and McDuff–Slimowitz, [LaMc1, McDSl]. Let us give a simple proof of Theorem 1.1 for symplectically aspherical wide manifolds $(W, \omega)$ using the selector we have constructed in Section 3.2.4.

Recall that using an action selector one can define an invariant, the homological capacity $c_{\text{hom}}(V)$, of a domain $V \subset W \times T^* S^1$ as follows:

$$c_{\text{hom}}(V) = \sup \{ \sigma(H) \mid H : S^1 \times V \to \mathbb{R}, \text{ where } H \geq 0 \text{ is compactly supported} \}.$$ 

By definition $c_{\text{hom}}(V) < \infty$, provided that the selector is a priori bounded from above; for instance, when $V$ is displaceable. In our case, this is guaranteed by property (AS7). Furthermore, property (AS5) implies that $c_{HZ}(V) \leq c_{\text{hom}}(V)$. Hence, we obtain $c_{HZ}(V) < \infty$ whenever $V$ is displaceable, and, consequently, the almost existence theorem holds for $V$. An argument similar to the one in the proof of Theorem 1.2 finishes the proof.

It is clear that this proof also works for closed manifolds, albeit using the selector constructed in [Sc].

4.3. Displacement Principle. The proof of Theorem 1.1 is essentially identical to, if not slightly simpler than, the proof of this statement in the middle-dimensional case due to Laudenbach and Sikorav, [LauSi]. Therefore, we will only outline the proof of this theorem and refer the reader to [LauSi] for the details of the argument.

Note that the bundle $TM^\omega$ is isomorphic to the normal bundle to $M$ and hence admits a non-vanishing section. To prove Theorem 1.1 it suffices to find a non-vanishing section $v$ of $TM^\omega$ such that $dK(v) > 0$ along $M$ for some function $K$ defined near $M$. (The Hamiltonian vector field $X_K$ of $K$ would then be nowhere tangent to $M$.) On the other hand, for a fixed $v$, the existence of such a function is guaranteed by a result due to Sullivan, [Su], whenever $\omega$ is non-recurrent, i.e., no trajectory of $\omega$ is contained entirely in $M$. Then the proof of Theorem 1.1 is similar to the argument in [LauSi], is based on constructing a non-recurrent section of $TM^\omega$.

Let $\xi$ be a non-vanishing section of $TM^\omega$. A priori this section may be somewhere tangent to $M$. The idea is to turn $\xi$ into a non-recurrent vector field by killing the recurrence. To this end, let $R \subset M$ denote the set of all trajectories of $\xi$ contained in $M$. Pick finitely many (mutually) disjoint balls $B_i \subset M$ in such a way that every trajectory of $\xi$ in $M$ intersects the interior of at least one $B_i$. This is possible since $M$ is compact. Denote by $B$ the (disjoint) union of $B_i$’s.

Next let us modify $\xi$ inside $B$ to make $R = \emptyset$. Observe that, if the balls are chosen small enough, inside each $B_i$, the vector field $\xi$ is almost tangent to $M$ by continuity. Thus, let us choose a normal vector $\zeta_i \in TM^\omega$ within each $B_i$ and add these to $\xi$ using cut-off functions supported in $B_i$’s. The resulting vector field $v$ has the desired property. (The real situation is slightly more complicated, for actually $\xi$ need not be tangent to $M$ everywhere in $B_i$ and adding $\zeta_i$ to $\xi$ may force $v$ to vanish in some $B_i$. However, Thom’s jet transversality theorem guarantees that $\zeta_i$’s can be chosen so that they not parallel to $\xi$ in each $B_i$.) This finishes the construction. Let us point out that the essence of assuming $M$ to be nowhere coisotropic is now more transparent: in order for $\zeta_i$’s to exist, some “normal space”
in $TM^\omega$ is needed; for instance, this would be impossible if $TM^\omega \subset TM$ at a point in $R$.

Now it is easy to see that $v$ is non-recurrent, just as in [LauSi].

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