Double-logarithmic nonlinear electrodynamics

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Keywords: birefringence, unitarity of NED, nonlinear electrodynamics (NED)), Double-Logarithmic electrodynamics

Abstract

A new model of nonlinear electrodynamics is introduced and investigated. The theory carries one dimensionful parameter \(\beta\) as in Born-Infeld electrodynamics. It is shown that the dual symmetry and dilatation (scale) symmetry are broken in the proposed model. The electric field of a point-like charge is derived for this model, showing that it is non-singular at the origin. Using this electric field, the static electric energy of a point-like charge is calculated. In the presence of an external magnetic field, the theory shows a phenomenon known as vacuum birefringence. The refraction index of two polarizations, parallel and perpendicular to the external magnetic induction field, are calculated. The canonical and symmetrical Belinfante energy-momentum tensors are obtained. Using the causality and unitarity principles, the regions where the theory becomes causal and unitary are found.

1. Introduction

It is known that there is a deep connection between nonlinearity and strong fields. When a classical field becomes strong enough, it invalidates the predictions made by the linear theory. There are examples of classical fields in which the relation between nonlinear effects and strong fields has been applied. A well-known one is Born-Infeld (BI) electrodynamics [1, 2]. To solve the problem of singularities, namely the infinite self-energy of a point-like charge, which comes out in Maxwell’s electrodynamics, Born and Infeld introduced nonlinearity to the classical electrodynamics. BI electrodynamics solved this problem by introducing a dimensionful parameter that gives an upper bound to the electric field of a point-like charge. Moreover, BI type Lagrangians are used to understand the phenomena of meson multiple productions in the strong-field regime [3], and also the propagation of shock waves [4]. Furthermore, one-loop quantum corrections in quantum electrodynamics fixes the Lagrangian of classical electrodynamics by nonlinear terms [5–7]. It is also possible to get nonlinear electrodynamical effects in strong gravitational fields. Besides BI electrodynamics, there are many other types of nonlinear electrodynamics models. Some of them are known as logarithmic electrodynamics [8], exponential electrodynamics [9], and arcsin electrodynamics which is proposed by Kruglov (Kr.) [10]. Also, different types of nonlinear electrodynamics are considered in [11–15]. Among all these theories the introduced ‘double-logarithmic’ electrodynamics has advantages concerning some criteria, discussed in the following subsection, and which make it a convenient NED theory.

1.1. Our motivation for a new model

We assume that a NED model is, in principle, a correction to the classical Maxwell’s linear electrodynamics (LED). Accordingly, a feasible NED model should (i) reduce the LED in the weak-field limit. Furthermore, one expects a simple structure for the NED model such that, for instance, (ii) when it is applied to point charges, results in a bounded function for the fields which can be expressed in terms of elementary functions. Finally, (iii) when the NED model is coupled to gravity, the generalized Reisner-Nordström (RN) black hole solution should also be expressed explicitly in terms of analytic functions. We note that RN black hole solution [16] is an exact solution to Einstein-Maxwell’s equations, which are already non-linear. The solution may also be particle-like or geons, as it was called by Misner and Wheeler in their geometrodynamics theory [17]. In their proposal, Maxwell’s theory in flat spacetime is replaced by a more general covariant formalism due to Rainich [18].
 Accordingly, as it was stated in [17] ‘if classical physics be regarded as comprising gravitation, source free electromagnetism, unquantized charge, and unquantized mass of concentrations of electromagnetic field energy (geons), then classical physics can be described in terms of curved empty space, and nothing more’. Perhaps a combination of the geometrodynamics theory and NED can be of ultimate interest which is, however, out of the scope of this paper. In Table 1 we tabulated some of the proposed models in the literature and their status in terms of the criteria we mentioned above. It is worth recalling that, as it was stated in [19] (see the paragraph after equation (67)), except LED, any other NED model of the form \( \mathcal{L} = \mathcal{L}(F) \) violates the causality conditions. Therefore, from Table 1 it is observed, practically a NED model that satisfies all criteria is still missing. This, however, does not mean that these models should be discarded. On the contrary, every model proposed so far, has been suggested for certain reasons, which have been fulfilled satisfactorily. Inspired by these facts, we are motivated to develop a model of NED that satisfies the best of our criteria. The model presented in this paper achieves our goal in the sense that it satisfies most of the conditions. We also add that, we studied the new NED coupled with gravity in a separate paper [20].

The layout of the paper is as follows. In section 2, we introduce the model and show that Maxwell’s electrodynamics can be regained in the \( \beta \rightarrow 0 \) limit. Also, the equation of motion is calculated, through which Maxwell’s equations are derived. Using the field equations it is shown that the dual symmetry of the theory is broken. The electric field of a point particle is calculated at the point where the particle is located. In section 3, we calculate the speed of the electromagnetic wave in presence of a constant and uniform magnetic current, and energy of a point-like charge are obtained in section 4. Section 5 is devoted to the unitary and causality analysis of the theory. We end up with a conclusion part in section 6.

Hereafter, we take \( \hbar = c = \epsilon_0 = \mu_0 = G = 1 \) and the Minkowski metric with mostly plus signature. The coordinates are defined as \( x^\mu = (t, r, \theta, \phi) \). Greek indices run from 0 to 3 and Latin indices run from 1 to 3.

2. The model

In the Maxwell’s electrodynamics model, the Lagrangian is directly proportional to \( F = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{B^2 - E^2}{2} \), i.e.,

\[
\mathcal{L} = -F.
\]

On the other hand, the most well-known nonlinear theory i.e., BI theory, proposes the following Lagrangian [1, 2]

\[
\mathcal{L} = b^2 \left( 1 - \frac{1}{\sqrt{1 + \frac{F^2}{b^2} - \frac{G^2}{b^4}}} \right)
\]

in which \( G = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -B \cdot E \). The other well-known nonlinear theory is the one proposed by Heisenberg and Euler in 1936, [3, 5] with

| NED | \( \mathcal{L} \) | LED limit | Finite & | Closed form field | Closed form BH |
|-----|-----------------|-----------|---------|------------------|---------------|
| BI [2] | \( b^2 \left( 1 - \sqrt{1 + \frac{F^2}{b^2}} \right) \) | Yes | Yes | Yes | No |
| HE [8] | \( -F + b(F^2 + 7G^2) \) | Yes | No | No | No |
| PM [21] | \( bF^2 \) | Yes | No | Yes | Yes |
| Arc-sin [10] | \( -F - \frac{1}{2} \arcsin(bF) + \frac{7}{2}G^2 \) | Yes | Yes | No | Yes |
| Exp. [22] | \( b^2 \left( \exp \left( -\frac{F^2}{b^2} \right) - 1 \right) \) | Yes | No | No | No |
| Log. [23] | \( -b^2 \ln \left( 1 + \frac{F^2}{b^2} \right) \) | Yes | Yes | Yes | Yes |
| Exp. [24] | \( -F \exp \left( -bF \right) \) | Yes | No | No | No |
| Kr. [25, 26] | \( \frac{-F}{1 - \sqrt{1 - \beta F}} \) | Yes | Yes | Yes | Yes (pure electric) |
| Kr. [26] | \( \frac{-F}{1 + \sqrt{1 - \beta F}} \) | Yes | Yes | Yes | Yes (pure magnetic) |
| Kr. [27] | \( -F + \frac{\beta F^2}{2b^2} \) | Yes | Yes | No | Yes (pure magnetic) |
\( \mathcal{L} = -\mathcal{F} - \beta \mathcal{F}^2 - \gamma \mathcal{G}^2 + \ldots \)  
(3)

Apart from these historical models of nonlinear electrodynamics, there are some recent models which have been intensively studied in the literature. Among them, we find Maxwell’s power law theory with [21, 29, 30]

\[ \mathcal{L} = \alpha (-\mathcal{F})^\beta, \]
(4)

which was proposed by M. Hassaine and C. Martinez. More recently, the arcsin model has been introduced by Kruglov with [10]

\[ \mathcal{L} = -\mathcal{F} - \frac{C}{\beta} \arcsin(\beta \mathcal{F}) + \frac{\gamma}{2} \mathcal{G}^2. \]
(5)

It is not difficult to see that, for a single point charge, the electric field of the theories such as BI and Kruglov [10] is confined. However, although the electric field in BI theory is expressed in terms of an elementary function, in the theories such as arcsin model, such simple expression does not exist. Apart from the model introduced in this manuscript, there are few nonlinear electrodynamics models that admit an exact solution of the electric field [25]. Having the electric field confined and expressed in terms of an elementary function is considered an advantage of a theory on nonlinear electrodynamics. This is because of its further applications.

Keeping in mind the finiteness and closed expression of the electric field of a point charge, here, in this paper we introduce a new nonlinear electrodynamics model given by

\[ \mathcal{L} = \frac{1}{2\beta} \left[ (1 - \mathcal{Y}) \ln(1 - \mathcal{Y}) + (1 + \mathcal{Y}) \ln(1 + \mathcal{Y}) \right], \]
(6)

\( \mathcal{Y} = \sqrt{-2\beta \mathcal{F} + \sigma \beta^3 \mathcal{G}^2}, \) with \( \mathcal{F} \) and \( \mathcal{G} \) the Maxwell invariants. We shall refer to this as the double-logarithmic nonlinear electrodynamics model. Herein, the electromagnetic field and its dual are given by \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) and \( \tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \) in which \( A_\nu \) is the gauge potential and \( \epsilon_{\mu\nu\rho\sigma} \) is a completely antisymmetric tensor with the convention \( \epsilon_{0123} = 1. \) Furthermore, the coupling constant \( \sigma \) can only take values of 1, -1 and 0 and the Maxwell invariants should satisfy \(-2\beta \mathcal{F} + \sigma \beta^3 \mathcal{G}^2 < 1\) in order for \( \mathcal{L} \) to be physically acceptable. Note that, when \(-2\beta \mathcal{F} + \sigma \beta^3 \mathcal{G}^2 < 0\) the first term and the second term in the Lagrangian (6) are the complex conjugate of each other such that their addition becomes real. On the other hand, for \(-2\beta \mathcal{F} + \sigma \beta^3 \mathcal{G}^2 = 0, (6) \) becomes zero too.

The electromagnetic strength tensor and its dual are explicitly given by

\[
\begin{pmatrix}
0 & E^1 & E^2 & E^3 \\
-\mathcal{F} & 0 & B^3 & -B^2 \\
-\mathcal{F} & -B^3 & 0 & B^1 \\
-\mathcal{F} & B^2 & -B^1 & 0 \\
\end{pmatrix}, \quad \tilde{F}^{\mu\nu} = \begin{pmatrix}
0 & B^1 & B^2 & B^3 \\
-B^1 & 0 & -E^3 & E^2 \\
-B^2 & E^3 & 0 & -E^1 \\
-B^3 & -E^2 & E^1 & 0 \\
\end{pmatrix}.
\]

The \( \beta \to 0 \) limit of (6) gives

\[
\lim_{\beta \to 0} \mathcal{L} = -\mathcal{F} - \frac{1}{6} (2\mathcal{F}^2 + 3\mathcal{G}^2) \beta - \frac{1}{15} (4\mathcal{F}^3 + 5\mathcal{F}\mathcal{G}^2) \beta^2 + \mathcal{O}(\beta^3),
\]
(8)

where the zeroth order term is the usual Maxwell’s theory which can be achieved when \( \beta = 0. \) The first order correction to the Maxwell’s theory is \( 2\mathcal{F}^2 + 3\mathcal{G}^2 \) term. For \( \sigma = 0 \) the expansion becomes

\[
-\mathcal{F} + \frac{1}{3} \mathcal{F}^2 \beta = -\frac{4}{15} \mathcal{F}^3 \beta^2 + \mathcal{O}(\beta^3).
\]
(9)

On the other hand, upon considering \(-2\beta \mathcal{F} + \sigma \beta^3 \mathcal{G}^2 < 1, \lim_{\beta \to \infty} \mathcal{L} = 0. \)

2.1. The field equations of the model

With the Lagrangian (6) we can write the following action

\[
I = \int d^4x \sqrt{-\eta} \mathcal{L},
\]
(10)
in which \( \eta \) is the determinant of \( \eta_{\mu\nu} \), the metric of the Minkowski spacetime. Using the Euler–Lagrange equations

\[
\frac{\partial \mathcal{L}}{\partial A_\mu} - \frac{\partial}{\partial \partial A_\mu} \frac{\partial \mathcal{L}}{\partial \partial A_\mu} = 0,
\]
(11)

the field equations can be found as

\[
\partial_\mu \left\{ \frac{(-F^{\mu\nu} + \sigma \beta G^{\mu\nu})}{\sqrt{-2\beta \mathcal{F} + \sigma \beta^3 \mathcal{G}^2}} \ln \left( \frac{1 + \sqrt{-2\beta \mathcal{F} + \sigma \beta^3 \mathcal{G}^2}}{1 - \sqrt{-2\beta \mathcal{F} + \sigma \beta^3 \mathcal{G}^2}} \right) \right\} = 0.
\]
(12)
The electric displacement field can be calculated, using the expression \( \mathbf{D} = \frac{\partial \mathbf{E}}{\partial t} \), and is given by

\[
\mathbf{D} = \frac{1}{2} \left( \mathbf{E} - \sigma \beta \mathbf{G} \right) \ln \left( \frac{1 + \sqrt{-2 \beta \mathbf{F} + \sigma \beta^2 \mathbf{G}^2}}{1 - \sqrt{-2 \beta \mathbf{F} + \sigma \beta^2 \mathbf{G}^2}} \right).
\]

We can write (13) in terms of the electric permittivity tensor \( \varepsilon_{ij} \) which is

\[
\varepsilon_{ij} = \varepsilon (\delta_{ij} + \sigma \beta B_i B_j),
\]

where

\[
\varepsilon \equiv \frac{1}{2} \sqrt{-2 \beta \mathbf{F} + \sigma \beta^2 \mathbf{G}^2} \ln \left( \frac{1 + \sqrt{-2 \beta \mathbf{F} + \sigma \beta^2 \mathbf{G}^2}}{1 - \sqrt{-2 \beta \mathbf{F} + \sigma \beta^2 \mathbf{G}^2}} \right).
\]

Then the electric displacement field (13) takes the following form

\[
\mathbf{D}^i = \varepsilon (\delta_{ij} + \sigma \beta B_i B_j) \mathbf{E}^j = \varepsilon^i \mathbf{E}^j.
\]

Once \( \sigma \) is taken 0, the electric displacement field becomes \( \mathbf{D}^i = \mathbf{G}^i \), where \( \mathbf{G}^i \) is the electric field. Without the Maxwell invariant \( \mathbf{G} \), the magnetic field becomes \( \mathbf{B}^i = \mathbf{G}^i \). In the first two pair of Maxwell's equations (16) and (19) the appearance of the electric permittivity tensor \( \varepsilon_{ij} \) and magnetic permeability tensor \( \mu_{ij} \), respectively, points to a medium with anisotropic nonlinear and inhomogenous properties.

The equations of motion of the electrovacuum (12) can be rewritten in the form of the first pair of Maxwell's equations upon using (13) and (17). For \( \mu = 0 \) in (12), we get

\[
\nabla \cdot \mathbf{D} = 0
\]

and for \( \mu = i \) we get the second Maxwell equation

\[
- \frac{\partial \mathbf{D}}{\partial t} + \nabla \times \mathbf{H} = 0.
\]

The second pair of Maxwell's equations can be obtained by utilizing the Bianchi identity \( \partial_{\mu} F_{\mu\nu} = 0 \). First we set \( \nu = 0 \) in the Bianchi identity, which gives

\[
\nabla \cdot \mathbf{B} = 0,
\]

and then \( \nu = j \), to get

\[
\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0.
\]

In order to see the dual symmetry of the theory we take the dot product of the magnetic field (17) and the electric displacement field (13) i.e.,

\[
\mathbf{D} \cdot \mathbf{H} = \varepsilon (1 - \sigma \beta) \mathbf{E} \cdot \mathbf{B} = \mathbf{E} \cdot \mathbf{B}
\]

which states that the dual symmetry is broken for \( \sigma = 0 \) and \( \sigma \neq 0 \), unlike in the BI theory. When we take the \( \beta \to 0 \) limit we arrive at classical electrodynamics and the dual symmetry is recovered

\[
\mathbf{D} \cdot \mathbf{H} = \mathbf{E} \cdot \mathbf{B}.
\]
2.2. Electrostatics

Let us consider electrostatics for which \( \mathbf{B} = \mathbf{H} = 0 \) and find the electric field at the origin of the point-like charged particle namely at \( r = 0 \). The equation of point-like charge is

\[
\nabla \cdot \mathbf{D}_0 = q \delta(\mathbf{r})
\]

in which \( q \) is the electric charge. The latter admits a solution given by

\[
\mathbf{D}_0 = D_0 \mathbf{r} = -\frac{q}{4\pi r^2} \mathbf{r},
\]

where

\[
D_0 = \frac{1}{2\sqrt{-2\beta \mathcal{F}}} \ln \left( \frac{1 + \sqrt{-2\beta \mathcal{F}}}{1 - \sqrt{-2\beta \mathcal{F}}} \right) E_0.
\]

When there is no magnetic field the Maxwell invariant \( \mathcal{F} \) becomes \( \mathcal{F} = -\frac{1}{2} E_0^2 \) such that

\[
D_0 = \frac{1}{2\sqrt{\beta}} \ln \left( \frac{1 + \sqrt{\beta E_0}}{1 - \sqrt{\beta E_0}} \right).
\]

Furthermore, the equation takes the following form

\[
\frac{1}{2\sqrt{\beta}} \ln \left( \frac{1 + \sqrt{\beta E_0}}{1 - \sqrt{\beta E_0}} \right) = \frac{q}{4\pi r^2},
\]

and finally the solution for the electric field is given by

\[
E_0 = \tanh \left( \frac{2\sqrt{\beta}}{4\pi r^2} \right).
\]

Here we can define unitless variable to check the \( r \to 0 \) limit by defining

\[
x \equiv \frac{2\pi^2}{q \sqrt{\beta}}, \quad y \equiv E_0 \sqrt{\beta},
\]

then the \( r \to 0 \) limit becomes equivalent to \( x \to 0 \) limit. With these unitless variables the electric field equation (32) is rewritten as

\[
y = \tanh \left( \frac{1}{2x} \right),
\]

whose \( x \to 0 \) limit gives \( y \to 1 \). Therefore, the maximum value for the electric field at the origin, where the charged particle is placed, is given by

\[
E_{\text{max}} = \sqrt{\frac{1}{\beta}},
\]

which shows the finiteness of the electric field at the location of the point particle as in BI electrodynamics. This can be also done by considering the particles to be extended in the space, instead of being unphysical points. This is interesting that NED theories allow using point charges instead of extended ones, without producing singularities in the spacetime. It should also be remarked that some of the properties which typically appear in the electrodynamics of extended bodies are missing. Specifically, an interesting phenomenon has been uncovered in a recent paper [31], using Maxwell’s linear electrodynamics, named self-oscillation. Moreover, based on the results of [31], one estimates the possibility of fundamental particles as electromagnetic solitons within the Einstein-Maxwell formalism. If so, we add the following open question. Can NED harbor topological solitary waves in a flat spacetime [31]?.

3. The energy-momentum tensor

In this section, we derive the energy-momentum tensor to find the dilatation current and the energy of a point-like charge. The general expression of energy-momentum tensor is given by Noether’s theorem, which states the correspondence between global symmetries and conserved charges of the field theory,

\[
T^\mu(C) = (\partial_\nu A_\nu) \frac{\partial \mathcal{L}}{\partial (\partial_\nu A_\nu)} - \delta^\mu_\nu \mathcal{L},
\]
which is called canonical energy-momentum tensor. Upon using (6) in (36) one obtains
\[ T^\mu_{(C)} = \varepsilon (\partial_\nu A_\nu)(-F^\mu_{\nu} + \sigma \beta G F^\mu_{\nu}) - \delta^\mu_{\nu} L. \] (37)

While this tensor is conserved \( \partial_\mu T^\mu_{\nu (\nu)} = 0 \) it is not symmetric and gauge-invariant. The conserved quantities arising from Noether’s theorem are usually observable and have clear physical meaning; such as the symmetry under translations refers to energy and momentum or the symmetry under scale transformations refers to conserved dilatation current of the field. However, in general densities of conserved quantities do not possess these properties, for example, the gravitational energy density becomes non-localizable \[ 32 \]. On the other hand, being non-symmetric results in a complicated relation between the canonical energy-momentum tensor and angular momentum tensor \[ 33 \]. Therefore, we have to obtain the symmetric Belinfante tensor \[ 34–36 \] by introducing
\[ T^\mu_{(B)} = T^\mu_{(C)} + \partial_\nu \chi^\nu_{\beta \mu}. \] (38)

Herein,
\[ \chi^\nu_{\beta \mu} = \frac{1}{2}[\Pi^\nu_{\beta \nu}(\Sigma_{\nu \rho})^\mu_{\rho} - \Pi^\mu_{\nu}(\Sigma_{\beta \nu})^\rho_{\rho} - \Pi^\rho_{\nu}(\Sigma^\beta_{\rho \nu})^\mu_{\rho}] A_\rho, \] (39)

where
\[ \Pi^\nu_{\beta \nu} = \frac{\partial L}{\partial (\partial_\nu A_\nu)} = \varepsilon (-F^\nu_{\mu} + \sigma \beta G F^\nu_{\mu}), \] (40)

and the generators of Lorentz transformations \( \Sigma_{\mu \nu} \) have the matrix elements
\[ (\Sigma_{\mu \nu})^\rho_{\rho} = \eta_{\mu \alpha} \eta_{\nu \beta} - \eta_{\mu \beta} \eta_{\nu \alpha}. \] (41)

Using (39) and (41), one achieves
\[ \chi^\nu_{\beta \mu} = \Pi^\nu_{\beta \nu} A_\nu, \] (42)

which shows \( \chi^\nu_{\beta \mu} = - \chi^\nu_{\mu \beta} \) that gives rise to \( \partial^\nu \chi^\beta_{\beta \nu} = 0 \). Therefore, the symmetrical Belinfante tensor is conserved. From (42) and (40), we find
\[ \partial_\alpha \chi^\alpha_{\nu \beta} = \varepsilon (-F^\alpha_{\mu} \partial_\alpha A_\nu + \sigma \beta G F^\alpha_{\mu} \partial_\alpha A_\nu), \] (43)

where \( \partial_\nu \Pi^\nu_{\beta \mu} = 0 \) upon (12). Finally, the Belinfante tensor can be written as follows
\[ T^\mu_{(B)} = -\varepsilon (-F^\mu_{\rho} F_{\rho} + \frac{\sigma \beta G F^\mu_{\rho} F_{\rho}}{\beta}) + \delta^\mu_{\nu} L. \] (44)

The trace of the Belinfante tensor (44) is found to be
\[ T^\mu_{(B)} = \frac{2F}{\sqrt{1 - 2\beta F + \sigma \beta^2 G^2}} \left\{ \ln \frac{1 + \sqrt{1 - 2\beta F + \sigma \beta^2 G^2}}{1 - \sqrt{1 - 2\beta F + \sigma \beta^2 G^2}} \right\} - \frac{2}{\beta} \ln (1 - \beta \Delta), \] (45)

which will be needed to derive the dilatation current. To obtain the linear electrodynamics \( \beta \) must be taken to zero limits and, in this limit, the trace of the energy-momentum tensor becomes zero. However, in this model (6) a non-zero trace of energy-momentum tensor has appeared this contrasts with the electrovacuum of Einstein-Maxwell equations, introducing a non zero Ricci’s scalar of curvature \[ 37 \] in the context of gravity. Rewriting (45) as
\[ T^\mu_{(B)} = \frac{2F}{\sqrt{1 - 2\beta F + \sigma \beta^2 G^2}} \left\{ \ln \left( \frac{1 + \sqrt{\beta \Delta}}{1 - \sqrt{\beta \Delta}} \right) \right\} - \frac{2}{\beta} \ln (1 - \beta \Delta), \] (46)

after defining
\[ \Delta \equiv -2F + \sigma \beta G, \] (47)

it can be easily seen that \( T^\mu_{(B)} = 0 \) for \( \Delta = 0 \) in which case the theory decease. In order to have a unitary and causal theory we have the condition \( 0 < \beta \Delta < 1 \) (see section 4) then the logarithmic terms in front of \( F \) and \( \frac{1}{\beta} \) in (46) becomes positive and negative respectively. Afterwards, (46) can be stated as
\[ T^\mu_{(B)} = aF + \frac{b}{\beta}, \] (48)

where both \( a > 0 \) and \( b > 0 \). Therefore, (48) can take positive and negative values depending on the signs of \( F \) and \( \beta \), meaning that the double-logarithmic NED vacuum represents both positively and negatively curved spacetimes.
3.1. Dilatation current

In this part the dilatation current is calculated to see the scale symmetry of the model. The modified dilatation current is defined as [36]

\[ D^\beta_{\mu} = x^\alpha T^\beta_{\alpha\mu} + V_\mu, \]  

(49)

where the field-virial is given by

\[ V_\mu = \Pi_{\alpha\beta} F^\alpha_{\mu} E^\beta_{\rho} - (\Sigma_{\mu\rho})^\beta_{\beta} A_{\rho}. \]  

(50)

It can be shown easily that \( V_\mu = 0 \), and the modified dilatation current becomes \( D^\beta_{\mu} = x^\alpha T^\beta_{\alpha\mu} \). The divergence of the dilatation current is

\[ \partial^\mu D^\beta_{\mu} = T^{\beta}. \]  

(51)

As a result, the scale symmetry is broken due to the dimensional parameter \( \beta \). Although Maxwell’s theory is conformally symmetric, in BI theory both the scale and conformal symmetries are broken. For nonlinear dynamical systems symmetry breaking, by which the randomness of the system vanishes, is a typical feature [31].

3.2. Energy density of pure electric field

The energy density, \( \rho_E = T^{(B)}_{00} \) for pure electric energy (\( B = 0 \)) is

\[ T^{(B)}_{00} = -\frac{1}{2\beta} \ln \left( 1 - \beta E^2 \right), \]  

(52)

which is positive as it is expected. The total electric energy of a point charge is given by \( E = \int \rho_E dV \) where \( \rho_E \) is the energy density derived above. Hence, one writes

\[ E = -\frac{2\pi}{\beta} \int_0^\infty \ln \left( 1 - \beta E^2 \right) r^2 dr. \]  

(53)

In terms of the dimensionless variables, this equation becomes

\[ E = \frac{2\pi}{\beta} \left( \frac{q\sqrt{\beta}}{4\pi} \right)^2 \int_0^\infty \frac{\ln \left( \cosh x \right)}{x^{5/2}} dx, \]  

(54)

where we have used \( x = \left( \frac{r^2}{r_0^2} \right) \) and \( r_0 = \sqrt{\frac{q\sqrt{\beta}}{4\pi}} \). Our numerical calculation yields

\[ E = 0.101 \frac{q^2}{r_0}. \]  

(55)

Following [38], if we assume the rest mass of the electron is electromagnetic one obtains

\[ mc^2 = 0.101 \frac{q^2}{r_0}. \]  

(56)

which in turn provides estimations for \( \beta \). In [31] it was shown that for the charged extended particles in Maxwell’s theory, the electromagnetic origin of mass is sufficient to drive Newton’s second law of motion, the relativistic energy, and the quantum potential by using the Lienard–Wiechert potentials [39]. This is interesting to observe that in the NED, introduced here in this paper, \( r_0 \) plays the role of the radius of a hypothetical charged extended particle in NED. As \( r_0 \) depends on \( \beta \) and the charge of the particle \( q \) one may employ the formalism introduced in [31, 39] to see the results in terms of \( \beta \). This will be discussed in a separate work.

4. The causality and unitarity

In this part, we will analyze the unitarity and causality of the particle spectrum of the theory. Based on the causality principle, the group velocity of the elementary electromagnetic excitations over a background field is smaller than the speed of light in the vacuum to avoid tachyons in the theory spectrum. The unitarity principle, on the other hand, imposes the positive definiteness of the norm of every elementary excitation of the vacuum such that ghosts are avoided. To obtain the corresponding restrictions on the NED Lagrangian, an electromagnetic wave is considered to be propagating on the background of an electromagnetic field which is constant in time and space. Besides, in [40], a particular case of the background field is assumed so that \( E.B = 0 \). This in turn means that, in a given Lorentz frame, the background field is either purely electric or magnetic. Hence, in this configuration, the propagating electromagnetic wave splits into two orthogonal polarized propagating modes. This is what is called vacuum birefringence. The following restriction conditions which are imposed on each of the polarized modes have been determined in [40].
These inequalities are applicable on any NED theory. Here the subscripts refers to the partial derivatives of the Lagrangian density with respect to the indicated invariant.

In our model the derivatives of the Lagrangian density (6) are

\[ L_F = -\frac{\tanh^{-1}(\sqrt{\beta \Delta})}{\sqrt{\beta \Delta}}, \]

\[ L_{FF} = -\frac{\beta \tanh^{-1}(\sqrt{\beta \Delta})}{(\beta \Delta)^2} + \frac{1}{\Delta - \beta \Delta^2}, \]

\[ L_{\varphi} = \frac{\sigma \beta G \tanh^{-1}(\sqrt{\beta \Delta})}{\sqrt{\beta \Delta}}, \]

and

\[ L_{\varphi \varphi} = -\frac{G^2 \beta \sigma^2 \sqrt{\beta \Delta}}{(\beta \Delta)^{3/2}(1 + \beta \Delta)} + \frac{2\beta \sigma F(1 - \beta \Delta) \tanh^{-1}(\sqrt{\beta \Delta})}{(\beta \Delta)^{3/2}(1 + \beta \Delta)}. \]

The other two inequalities of the principles (57) involve

\[ L_F + 2FL_{FF} = -\frac{\sigma \beta G^2 \tanh^{-1}(\sqrt{\beta \Delta})}{\sqrt{\beta \Delta}} + \frac{2F}{\Delta - \beta \Delta^2}, \]

and

\[ 2FL_{\varphi \varphi} - L_F = -\frac{2FG^2 \beta \sigma^2 \sqrt{\beta \Delta}}{(\beta \Delta)^{3/2}(1 + \beta \Delta)} + \frac{4\beta \sigma F^2(1 - \beta \Delta) \tanh^{-1}(\sqrt{\beta \Delta})}{(\beta \Delta)^{3/2}(1 + \beta \Delta)} + \frac{\tanh^{-1}(\sqrt{\beta \Delta})}{\sqrt{\beta \Delta}}, \]

where we have used (47).

4.1. Causality and unitarity of the model

The first and the second condition in (57) states

\[ \frac{\tanh^{-1}(\sqrt{\beta \Delta})}{\sqrt{\beta \Delta}} \geq 0, \]

and

\[ \frac{1}{\Delta(1 - (\beta \Delta))} \geq \frac{\beta \tanh^{-1}(\sqrt{\beta \Delta})}{\sqrt{\beta \Delta}}, \]

respectively which are both satisfied identically provided for \( \beta > 0 \) and \( \beta < 0 \) we impose \( \beta \Delta < 1 \) and \( 0 < \beta \Delta < 1 \) respectively. In the sequel we consider \( 0 < \beta \Delta < 1 \) which is valid for all values of \( \beta \). Using (61), the third condition of (57) can be expressed as

\[ C_3 := -2\sigma F(1 - \beta \Delta) \tanh^{-1}(\sqrt{\beta \Delta}) + G^2 \beta \sigma^2 \sqrt{\beta \Delta} \geq 0. \]

From the fourth condition of (57) we get

\[ C_4 := -\frac{2\beta F}{1 - \beta \Delta} + \frac{\sigma \beta G^2 \tanh^{-1}(\sqrt{\beta \Delta})}{\sqrt{\beta \Delta}} \geq 0, \]

and finally the last one of (57) represents

\[ C_5 := \left(\frac{4\beta \sigma F^2}{\beta \Delta} - 1 \right) \tanh^{-1}(\sqrt{\beta \Delta}) + \frac{2FG^2 \beta \sigma^2}{\sqrt{\beta \Delta} (\beta \Delta - 1)} \geq 0. \]

For \( \sigma = 0 \), the only condition left to be satisfied is (67) which reduces to

\[ \frac{2\beta F}{1 - \beta \Delta} \leq 0. \]

Upon considering \( 0 < \beta \Delta < 1 \) with \( \Delta = -\frac{2\beta F}{4\beta \sigma F^2} \) is trivially satisfied.

For \( \sigma = \pm 1 \), all three conditions should be checked carefully. In figure 1 we plot \( C_3, C_4 \) and \( C_5 \) in terms of \( t = \beta \Delta \) for some fine tuned \( \beta \) and \( \sigma = 1 \). Figure 1 reveals that for each set of parameters (in this specific case \( \beta = 1 > 0 \) and \( \sigma = 1 \)) one finds regions where the conditions are all satisfied. However, for all values of \( E \) and \( B \)
there may be one or more conditions unsatisfied. In figure 2 we plot the same three conditions i.e., \( C_3, C_4, \) and \( C_5 \) together with \( \mathcal{G}^2 \) in terms of \( t = \beta \Delta \) for \( \beta = \sigma = 1 \) and \( F = 0 \ldots 1 \) with equal steps. In analogy with figure 1, for \( F > 0 \) the conditions are all satisfied provided \( 0 \leq \beta \Delta < (t)_{\text{critical}} < 1 \). At \( (t)_{\text{critical}} \) the condition \( C_4 \) is zero. Note that, in this case i.e., \( \beta > 0 \) and \( \sigma = 1 \), the region with \( F < 0 \) should be excluded in order to have \( \mathcal{G}^2 > 0 \).

5. Vacuum birefringence

In this part, we are going to discuss the effect of vacuum birefringence which relates the phase velocity to the polarization of the electromagnetic wave. Due to the quantum effects, photon-photon interaction is possible. For photons with energy less than the electron’s rest mass, the EH Lagrangian predicts photon splitting and vacuum birefringence in the presence of the classical electromagnetic field. In the PVLAS experiment, both phenomena have been observed \[41\]. Birefringence can be understood as follows: If a beam of photons, initially linearly polarized, enters a strong electromagnetic field, it may partially be scattered into perpendicularly polarized modes. Such phenomena takes place for light coming from the surface of a neutron star, for example, where the strong magnetic field of the neutron star affects its polarization. This can be measured from the surface of Earth \[42–45\]. Hence, let us take an external magnetic induction field \( \mathbf{B}_0 = (B_0, 0, 0) \) which is uniform and constant together with the plane electromagnetic wave \( \mathbf{e} \), \( \mathbf{b} \) as

\[
\mathbf{e} = e_0 \exp \left[ -i(\omega t - k z) \right], \quad \mathbf{b} = b_0 \exp \left[ -i(\omega t - k z) \right]
\]  

(70)

propagating in the \( z \)-direction. As a result, the total electromagnetic field becomes \( \mathbf{E} = \mathbf{e} + \mathbf{b} = \mathbf{B}_0 \) since we are interested in strong magnetic induction field the amplitudes of the electromagnetic wave \( e_0, b_0 \) become small compared to the magnetic induction field, that is \( e_0, b_0 \ll B_0 \). After linearizing the equations (13) (17) the electric permittivity and the magnetic permeability tensors can be found as

\[
\varepsilon_{ij} = \varepsilon (\eta_{ij} + \sigma \beta \eta_{ij} \eta_{ij} B_0^2),
\]  

(71)

and

\[
\mu_{ij} = \mu \eta_{ij}, \quad \mu = \varepsilon^{-1},
\]  

(72)

respectively. From Maxwell’s equations one can obtain the following wave equation

\[
\partial_t^2 E_i - \mu \varepsilon \partial_j \partial_j E_j - \partial_i \partial_j E_j = 0.
\]  

(73)

Choosing the polarization of the electric field parallel to the external magnetic field i.e., \( \mathbf{e}_0 = e_0 (1, 0, 0) \) and solving the wave equation (73) we find the dispersion relation.
\[ \mu \varepsilon_1 \omega^2 = k^2. \]  

The index of refraction \( n_\parallel = \sqrt{\mu \varepsilon_1} \) can be found as

\[ n_\parallel = \sqrt{\frac{1}{\varepsilon_1} \varepsilon (\eta_{11} + \sigma \beta \eta_{12} \eta_{11} B_0^2) + 1 + \sigma \beta B_0^2}. \]  

If the polarization is chosen such that the electromagnetic wave is perpendicular to the external induction field, i.e., \( e_0 = e_0(0, 1, 0) \), then (73) gives

\[ \mu \varepsilon_2 \omega^2 = k^2, \]  

and the index of refraction becomes

\[ n_\parallel = \sqrt{\mu \varepsilon_2} = \sqrt{\frac{1}{\varepsilon_2} \varepsilon (\eta_{12} + \sigma \beta \eta_{12} \eta_{21} B_0^2)} = 1. \]  

Finally, the phase velocity depends on the polarization and takes the value \( v_\parallel = \frac{1}{n_\parallel} \) for parallel polarization i.e., \( e \parallel B_0 \) and \( v_\perp = 1 (c = 1) \) for perpendicular polarization i.e., \( e \perp B_0 \). This shows the effect of vacuum birefringence. On the other side, if we get rid of the Maxwell invariant \( G \) by setting \( \sigma = 0 \) the effect of vacuum birefringence cancel out.

6. Conclusion

We have introduced a new nonlinear electrodynamics model, the double-logarithmic model. The model contains both of Maxwell’s invariants \( F \) and \( G \) and it carries one dimensionful parameter \( \beta \). After finding the field equations of the theory, we calculated the electric field of a point-like charge and we showed that at the origin—the location of the charge—it takes a finite value, that is \( E_{\text{max}} = \frac{1}{\eta_{11}} \), that is not singular. We obtain the canonical and symmetric Belinfante energy-momentum tensors in order to calculate the dilatation current. We showed that the dilatation symmetry is broken due to the dimensional parameter \( \beta \). Moreover, the self-energy of a point-like charge is calculated. We have also discussed the unitarity and causality of the model and conclude that the general model has a unitary region for fine-tuned parameters. We showed also that, in presence of a magnetic field the model admits the effect of vacuum birefringence in which the phase velocities of the electromagnetic wave depend on the polarization. This effect disappears once Maxwell’s invariant \( G \) is disposed of the Lagrangian.

Acknowledgments

We thank to the referees for their constructive and affirmative review of the manuscript.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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References

[1] Born M 1934 Proc. R. Soc. A 143 410
[2] Born M and Infeld L 1934 Proc. R. Soc. A 144 425
[3] Heisenberg W 1939 Z. Phys. 113 61
   Heisenberg W 1949 Z. Phys. 126 519
   Heisenberg W 1952 Z. Phys. 133 79
[4] Taniuti T 1959 Prog. Theor. Phys. Supply. 9 69
[5] Heisenberg W and Euler H 1936 Z. Phys. 98 714
[6] Schwinger J 1951 Phys. Rev. D 82 664
[7] Adler S L 1971 Ann. Phys. 67 599
[8] Gaete P and Helayel-Neto J 2014 Eur. Phys. J. C 74 2816
[9] Hendi S H 2013 Ann. Phys. 333 282
[10] Kruglov S I 2015 Ann. Phys. 527 797–401
[11] Kruglov S I 2007 Phys. Rev. D 75 117301
[12] Kruglov S I 2001 Ann. Phys. 293 328
[13] Kruglov S I 2008 Mod. Phys. Lett. A 23 245
[14] Kruglov S I 2007 Phys. Lett. B 652 146
[15] Kruglov S I 2015 Ann. Phys. 353 299
[16] Reissner H 1916 Annalen der Physik (in German) 50 106
Nordström G 1918 Verhandl. Kominkl. Ned. Akad. Wetenschap., Afdel. Natuurk., Amsterdam 26 1201
[17] Misner C and Wheeler J A 1957 Ann. Phys. 2 525
[18] Rainich G Y 1925 Trans. Amer. Math. Soc. 17 124
[19] Schelistede G O, Perlick V and Lammerzah C 2016 Ann. Phys. 528 738
[20] Gullu I and Mazharimousavi S H Black Holes in Double-Logarithmic Nonlinear Electrodynamics arXiv:2010.04603 [gr-qc]
[21] Hassaine M and Martinez C 2007 Phys. Rev. D 75 027502
[22] Hendi S H 2012 JHEP 03 065
[23] Soleng H H 1995 Phys. Rev. D 52 6178
[24] Kruglov S I 2017 Ann. Phys. 378 59
[25] Mazharimousavi S H and Halilsoy M 2019 Ann. Phys. 531 236
[26] Kruglov S I 2017 Ann. Phys. 529 75
[27] Kruglov S I 2015 Ann. Phys. 353 299
[28] Ejlli A et al 2020 Phys. Reps. 871 1
[29] Gonzalez H A, Hassaine M and Martinez C 2009 Phys. Rev. D 80 104008
[30] Maeda H, Hassaine M and Martinez C 2009 Phys. Rev. D 79 044012
[31] Lopez A G 2020 Non-linear Dyn. 102 621–34
[32] Faddeev L D 1982 Soviet Physics Uspekhi 25 130
[33] Weinberg S 1995 The Quantum Theory of Fields Vol. 1 (Cambridge: Cambridge University Press)
[34] Belinfante F 1939 Physica 6 887
[35] Rosenfeld L 1940 Mem. Acad. Roy. Belg. Sci. XVIII 1536
[36] Coleman S and Jackiw R 1971 Ann. Phys. 67 552
[37] Misner C and Wheeler J A 1957 Ann. Phys. 2 525–603
[38] Born M and Infeld L 1933 Nature 132 970
[39] Lopez A G 2021 Phys. Scr. 96 015006
[40] Shabad A E and Usov V V 2011 Phys. Rev. D 83 105006
[41] Della Valle F, Ejlli A, Gastaldi U, Messineo G, Milotti E, Pengo R, Ruoso G and Zavattini G 2016 Eur. Phys. J. C 76 24
[42] Baier R and Breitenlohner F 1967 Acta Phys. Austriaca 25 212
[43] Baier R and Breitenlohner F 1967 Nuovo Cimento B 47 117
[44] Bialynicka-Birula Z and Bialynicki-Birula I 1970 Phys. Rev. D 2 2341
[45] Adler S L 1971 Ann. Phys. (N.Y.) 67 599