METRICAL PROPERTIES FOR CONTINUED FRACTIONS OF FORMAL LAURENT SERIES

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Abstract. Motivated by recent developments in the metrical theory of continued fractions for real numbers concerning the growth of consecutive partial quotients, we consider its analogue over the field of formal Laurent series. Let \( A_n(x) \) be the \( n \)th partial quotient of the continued fraction expansion of \( x \) in the field of formal Laurent series. We consider the sets of \( x \) such that \( \deg A_{n+1}(x) + \cdots + \deg A_{n+k}(x) \geq \Phi(n) \) holds for infinitely many \( n \) and for all \( n \) respectively, where \( k \geq 1 \) is an integer and \( \Phi(n) \) is a positive function defined on \( \mathbb{N} \). We determine the size of these sets in terms of Haar measure and Hausdorff dimension.

1. Introduction and statement of results

The continued fraction expansion of a real number is an alternative (and efficient) way to the decimal representation of a real number. Every irrational \( x \in (0, 1) \) can be uniquely expressed as a simple continued fraction expansion as follows

\[
x = \frac{1}{a_1(x)} + \frac{1}{a_2(x) + \frac{1}{a_3(x) + \ddots}} := [a_1(x), a_2(x), a_3(x), \ldots]
\]

where \( a_n(x) \) are positive integers and are called the partial quotients of \( x \). The metrical theory of continued fractions concerns the quantity study of growth rate of partial quotients. This expansion can be induced by the Gauss map \( T_G : [0, 1) \to [0, 1) \) defined as

\[
T_G(0) := 0, \quad T_G(x) := \frac{1}{x} \pmod{1}, \quad \text{for } x \in (0, 1),
\]

with \( a_1(x) = \lfloor \frac{1}{x} \rfloor \), where \( \lfloor . \rfloor \) represents the floor function and \( a_n(x) = a_1(T_G^{n-1}(x)) \) for \( n \geq 2 \).

Let \( m \geq 1 \) and \( \Phi : \mathbb{N} \to (1, \infty) \) be a positive function. Define the set

\[
\mathcal{D}_m(\Phi) = \left\{ x \in [0, 1) : \prod_{i=1}^{m} a_{n+i}(x) \geq \Phi(n) \text{ for infinitely many } n \in \mathbb{N} \right\}.
\]

The well-known Borel-Bernstein theorem \([5, 7]\) states that the Lebesgue measure of \( \mathcal{D}_1(\Phi) \) is either zero or full according to the convergence or divergence of the series \( \sum_{n=1}^{\infty} 1/\Phi(n) \) respectively. For rapidly growing function \( \Phi \), the Borel-Bernstein theorem does not give any conclusive information other than Lebesgue measure zero. To distinguish between such sets Hausdorff dimension is an appropriate tool. The Hausdorff dimension of the set \( \mathcal{D}_1(\Phi) \) has been comprehensively determined by Wang-Wu \([20]\).

Motivation for considering the growth of product of consecutive partial quotients arose from the works of Kleinbock-Wadleigh \([16]\) where they considered improvements
to Dirichlet’s theorem. They defined the set of \( \phi \)-Dirichlet improvable numbers for some function \( \phi \) (see Section 2.1) and proved that the Lebesgue measure of the set of \( \phi \)\-Dirichlet non-improvable numbers is equivalent to the Lebesgue measure of the set \( D_2(\Phi) \) with \( \Phi(n) = \frac{1}{1-b^\phi(n)} - 1 \) for some \( b > 1 \). In particular, it was proved that the Lebesgue measure of the set of \( \phi \)-Dirichlet non-improvable numbers is zero or full if the series \( \sum_{n} \frac{\log \Phi(n)}{\Phi(n)} \) converges or diverges respectively, see \[16\] Theorem 3.6 and \[16\] Corollary 3.7. The Hausdorff measure of this set was later established in \[4,14\]. Very recently the Lebesgue measure and the Hausdorff dimension of \( D_m(\Phi) \), for any \( m \geq 2 \) has been determined by Huang-Wu-Xu \[13\]. We refer the reader to \[2,3\] for a comparison between the sizes of the classical well-approximable set with the set of \( \phi \)-Dirichlet non-improvable numbers.

These recent developments on the metrical theory of continued fractions for real numbers motivated the study of the analogous theory for the continued fractions over the field of formal Laurent series. Let \( F_q \) be a finite field with \( q \) elements and \( F_q((z^{-1})) \) denotes the field of all formal Laurent series \( x = \sum_{n=-\infty}^{\infty} c_n z^{-n} \) with coefficients \( c_n \in F_q \). If \( x = \sum_{n=v}^{\infty} c_n z^{-n} \) with \( c_v \neq 0 \) to be the first non-zero coefficient in the expansion of \( x \) then the valuation (or norm) of \( x \) is defined by

\[
|0|_\infty := 0, \quad |x|_\infty := q^{-v}.
\]

It is well known that this valuation is non-Archimedean. The topology induced by this norm make \( F_q((z^{-1})) \) locally compact and the ring of polynomials \( F_q[z] \) discrete in the field. Thus, we can think of \( F_q((z^{-1})) \) analogous to the set of real numbers \( \mathbb{R} \) and \( F_q[z] \) akin to the set of integers \( \mathbb{Z} \). Note that \( F_q((z^{-1})) \) is a complete metric space under the metric \( \rho \) defined by \( \rho(x,y) = |x-y|_\infty \). Let \( I \) be the valuation ideal of \( F_q((z^{-1})) \), that is,

\[
I = \{ x \in F_q((z^{-1})) : |x|_\infty < 1 \} = \left\{ \sum_{n=1}^{\infty} c_n z^{-n} : c_n \in F_q \right\}.
\]

Let \( \nu \) be the normalised (to 1) Haar measure on \( I \). For \( x = \sum_{n=v}^{\infty} c_n z^{-n} \in F_q((z^{-1})) \), we call \([x] = \sum_{n=v}^{0} c_n z^{-n} \) the integer part of \( x \) and \( \{x\} = \sum_{n=1}^{\infty} c_n z^{-n} \) to be the fractional part of \( x \).

As in the real case, consider the Gauss transformation \( T : I \to I \) defined by

\[
T(x) := \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor, \quad T(0) := 0. \tag{1.1}
\]

Then each \( x \in I \) has a finite or infinite continued fraction expansion induced by \( T \),

\[
x = \frac{1}{A_1(x) + \frac{1}{A_2(x) + \frac{1}{A_3(x) + \cdots}}} \quad := [A_1(x), A_2(x), \ldots],
\]

where the partial quotients \( A_i(x) \) are polynomials of strictly positive degree defined by

\[
A_i(x) = \frac{1}{T^{i-1}(x)}, \quad i \geq 1.
\]

This form of continued fraction induced from the Gauss map was first introduced by Artin \[11\], see also Berthé and Nakada \[9\].

As in the case of real numbers, the metrical theory of continued fractions of formal Laurent series can be used to prove many Diophantine approximation results such as
the analogues of the Borel-Bernstein theorem, Khintchine theorem, Jarník theorem and so on. The focus, at a fundamental level, has been on the set

\[ F_1(\Phi) := \{ x \in I : \deg A_n(x) \geq \Phi(n) \text{ for infinitely many } n \in \mathbb{N} \}. \]

The Haar measure, denoted as \( \nu \), of \( F_1(\Phi) \) was obtained by Niederreiter in [18], see also [10, Theorem 2.4], proving that the Haar measure of the set \( F_1(\Phi) \) is zero (respectively full) if the series \( \sum_{n \geq 1} q^{-\Phi(n)} \) converges (respectively diverges). The Hausdorff dimension of this set was completely determined for any function \( \Phi \) in [12]. We refer the reader to [8,11,17,19,21] for more related metrical (or distribution of digits) results over formal Laurent series.

Taking inspirations from the study of the growth of the product of consecutive partial quotients for the real numbers we initiate studying the growth of consecutive partial quotients over the field of formal Laurent series. Let \( k \geq 1 \). Define the set

\[ F_k(\Phi) := \left\{ x \in I : \sum_{i=1}^{k} \deg A_{n+i}(x) \geq \Phi(n) \text{ for infinitely many } n \in \mathbb{N} \right\}. \]

There are several natural justifications for the consideration of this set. For any \( n \geq 1 \), consider the \( n \)th convergents of \( x \)

\[ \frac{P_n(x)}{Q_n(x)} = [A_1(x), A_2(x), \ldots, A_n(x)]. \]

It is well known (see for instance [10]) that

\[ \left| x - \frac{P_n(x)}{Q_n(x)} \right|_\infty = \frac{1}{|Q_n(x)|_\infty |Q_{n+1}(x)|_\infty} \]

and

\[ |Q_n(x)|_\infty = \prod_{i=1}^{n} |A_i(x)|_\infty = q^{\sum_{i=1}^{n} \deg A_i(x)}. \]

Both of these facts give information on the relative error of approximation for \( x \) by consecutive convergents as

\[ \log_q \left| x - \frac{P_{n-1}(x)}{Q_{n-1}(x)} \right|_\infty = \deg A_n(x) + \deg A_{n+1}(x) \]

and

\[ \log_q \prod_{i=n}^{n+k} \left| x - \frac{P_{2i-1}(x)}{Q_{2i-1}(x)} \right|_\infty = \sum_{i=2n}^{2n+2k+1} \deg A_i(x). \]

Thus \( F_k(\Phi) \) describes the set of \( x \) which satisfy certain relative growth speed of consecutive approximations by convergents when \( k \) is even.

In this paper, we calculate the \( \nu \)-measure and Hausdorff dimension of the set \( F_k(\Phi) \).

Without loss of generality, we can assume \( \Phi(n) \geq k \) since \( \sum_{i=1}^{k} \deg A_{n+i}(x) \geq k \) for any irrational \( x \in F_q((z^{-1})) \) and any \( n \geq 0 \).

**Theorem 1.1.** Let \( \Phi : \mathbb{N} \to [k, \infty) \) be a positive function. Then

\[ \nu(F_k(\Phi)) = \begin{cases} 
0, & \text{if } \sum_{n=1}^{\infty} \frac{\Phi^{-1}(n)}{q^{\Phi(n)}} < \infty, \\
1, & \text{if } \sum_{n=1}^{\infty} \frac{\Phi^{-1}(n)}{q^{\Phi(n)}} = \infty.
\end{cases} \]

The Hausdorff dimension of \( F_k(\Phi) \) is completely given by the following result.
Theorem 1.2. Let $\Phi : \mathbb{N} \to [k, \infty)$ be a positive function. Let
\[
B := \liminf_{n \to \infty} \frac{\Phi(n)}{n}, \quad \log b := \liminf_{n \to \infty} \frac{\log \Phi(n)}{n}.
\]
Then
\[
\dim_H \mathcal{F}_k(\Phi) = \begin{cases} 
1 & \text{if } B = 0; \\
\frac{1}{1+b} & \text{if } B = \infty; \\
s_k(B) & \text{if } 0 < B < \infty,
\end{cases}
\]
where $s_k(B)$ is the unique solution of the equation
\[
\sum_{j=1}^{\infty} \frac{(q - 1)q^j}{q^{2js+Bf_k(s)}} = 1
\]
and for any $i \geq 1$, $f_i(s)$ is given by the following recursive formula,
\[
f_1(s) = s, \quad f_{i+1}(s) = \frac{s f_i(s)}{1 - s + f_i(s)} \text{ for } i \geq 1.
\]
It is worth noting that the case $B = \infty$ further leads to three sub-cases.
\[
\dim_H \mathcal{F}_k(\Phi) = \begin{cases} 
\frac{1}{2} & \text{if } b = 1, \\
\frac{1}{b+1} & \text{if } 1 < b < \infty, \\
0 & \text{if } b = \infty.
\end{cases}
\]

Another natural problem that we resolve in this paper is to determine the size of the following set which is obtained by replacing “infinitely many $n$” in the definition of $\mathcal{F}_k(\Phi)$ with “for all $n$”. Let
\[
\mathcal{G}_k(\Phi) = \left\{ x \in I : \sum_{i=1}^{k} \deg A_{n+i}(x) \geq \Phi(n) \text{ for all } n \geq 0 \right\}.
\]
We calculate the Hausdorff dimension of $\mathcal{G}_k(\Phi)$ which turns out to be independent of $k$.

Theorem 1.3. Let $\Phi(n)$ be a positive function such that $\Phi(n) \to \infty$ as $n \to \infty$. Then
\[
\dim_H \mathcal{G}_k(\Phi) = \frac{1}{a+1},
\]
where $1 \leq a \leq \infty$ is defined by $\log a = \limsup_{n \to \infty} \frac{\log \Phi(n)}{n}$.

When $k = 1$, Theorem 1.3 is proved as Theorem 2.3 in [12].

The paper is structured in the following way. In section 2 we group together basic definitions and some auxiliary results that we refer to in proving our results in subsequent sections. In section 3 we prove Theorem 1.1. In section 4 we prove the upper bound and in section 5 the lower bound estimates for Theorem 1.2 for case $\Phi(n) = nB$ with $0 < B < \infty$ only. In section 6 we combine all the dimension estimates for all the cases to complete the proof of Theorem 1.2. Finally, we prove Theorem 1.3 in the last section.

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2. Preliminaries and auxiliary results

We first introduce some fundamental properties of continued fractions in the field of formal Laurent series. Let

\[ \frac{P_n(x)}{Q_n(x)} = [A_1(x), A_2(x), \ldots, A_n(x)] \]

be the \( n \)th convergents of \( x \). The convergents can be obtained from the following recursive formula:

\[ \begin{align*}
    P_{-1}(x) &= 1, & P_0(x) &= 0, & P_n(x) &= A_n(x)P_{n-1}(x) + P_{n-2}(x), & (n \geq 2), \\
    Q_{-1}(x) &= 1, & Q_0(x) &= 1, & Q_n(x) &= A_n(x)Q_{n-1}(x) + Q_{n-2}(x), & (n \geq 2).
\end{align*} \]

We list some useful properties of these convergents, see [10][18] for their proofs.

**Proposition 2.1 ([10][18]).** Let \( x \in \mathbb{F}((z^{-1})) \). Then for all \( n \geq 1 \),

\( \begin{enumerate} \\
    \item \((P_n(x), Q_n(x)) = 1.\)
    \item \(|Q_n(x)P_{n-1}(x) - P_n(x)Q_{n-1}(x) = (-1)^n.\)
    \item \(|Q_n(x)|_\infty = \prod_{i=1}^{n} |A_i(x)|_\infty.\)
    \item \(|x - \frac{P_n(x)}{Q_n(x)}|_\infty = \frac{1}{|Q_n(x)Q_{n+1}(x)|_\infty = |A_{n+1}(x)Q_n^2(x)|_\infty}.\)
\end{enumerate} \)

We will make a frequent use of the following two important properties of the continued fractions over formal power series. The first is that the Haar measure \( \nu \) is preserved by the Gauss map \( T \) [18, Lemma 3]. The second property is that the deg \( A_i(x) \) form a sequence of independent and identically distributed random variables [18, Lemma 4].

**Proposition 2.2 ([18]).** Let \( T \) be defined by \((1.1)\). Then

\( \begin{enumerate} \\
    \item The Gauss map \( T \) is measure preserving with respect to \( \nu \).
    \item \( \text{deg } A_1(x), \text{deg } A_2(x), \ldots \) are independent and identically distributed random variables with respect to \( \nu \).
\end{enumerate} \)

These properties are in contrast to the real numbers case. For instance, there exists an interval \( B \subset [0, 1) \) such that \( T_{G-1}B \) and \( B \) have different Lebesgue measure. However, it should not be confused with the fact that the Gauss measure and Lebesgue measure are equivalent.

For any polynomials \( A_1, A_2, \ldots, A_n \in \mathbb{F}_q[z] \) of positive degree, we call

\[ I(A_1, \ldots, A_n) := \{ x \in I : A_1(x) = A_1, \ldots, A_n(x) = A_n \} \]

an \( n \)th order cylinder. For any subset \( U \subset I \), its diameter \( |U| \) can be defined as

\[ |U| = \sup \{ |x - y|_\infty : x, y \in U \}. \]

**Proposition 2.3 ([18]).** The cylinder \( I(A_1, \ldots, A_n) \) is a closed disc with diameter

\[ |I(A_1, \ldots, A_n)| = q^{-2 \sum_{i=1}^{n} \text{deg } A_i}. \]

and Haar measure

\[ \nu(I(A_1, \ldots, A_n)) = q^{-2 \sum_{i=1}^{n} \text{deg } A_i}. \]
Lemma 2.4. Let $A_1, A_2, \ldots, A_n \in \mathbb{F}_q[z]$ be polynomials of positive degree and $m \geq 1$ be an integer. Let

$$G(A_1, A_2, \ldots, A_n) = \bigcup_{\deg A_{n+1} \geq m} I(A_1, A_2, \ldots, A_{n+1}).$$

Then

$$|G(A_1, A_2, \ldots, A_n)| = q^{-m - 2 \sum_{i=1}^n \deg A_i}.$$

Proof. Let $x, y \in G(A_1, \ldots, A_n)$ with

$$x \in I(A_1, \ldots, A_n, A_{n+1}), \ y \in I(A_1, \ldots, A_n, A_{n+1}^*)$$

for some $A_{n+1}, A_{n+1}^* \in \mathbb{F}_q[z]$. Without loss of generality, we assume that $m \leq \deg A_{n+1} \leq \deg A_{n+1}^*$. Let $x_1 = T^{n+1}(x)$ and $y_1 = T^{n+1}(y)$, where $T$ is defined by (1.1). Then

$$x = \frac{(A_{n+1} + x_1)P_n + P_{n-1}}{(A_{n+1} + x_1)Q_n + Q_{n-1}}, \quad y = \frac{(A_{n+1}^* + y_1)P_n + P_{n-1}}{(A_{n+1}^* + y_1)Q_n + Q_{n-1}}.$$

So

$$|x - y|_{\infty} = \frac{|(A_{n+1} + x_1 - A_{n+1}^* + y_1)(P_nQ_{n-1} - P_{n-1}Q_n)|}{|(A_{n+1} + x_1)Q_n + Q_{n-1}|(A_{n+1}^* + y_1)Q_n + Q_{n-1}|}.$$

By Proposition 2.1 it follows that

$$|x - y|_{\infty} = \left| \frac{A_{n+1} - A_{n+1}^*}{A_{n+1}A_{n+1}^*Q_n^2} \right|_{\infty} \leq q^{-m - 2 \sum_{i=1}^n \deg A_i},$$

since $m \leq \deg A_{n+1} \leq \deg A_{n+1}^*$. The equality holds in the above inequality when $\deg A_{n+1} = m$ and $\deg(A_{n+1}^* - A_{n+1}) = \deg A_{n+1}^*$. □

Lemma 2.5. The number of cylinders $I(A_1, A_2, \ldots, A_k)$ such that $\deg A_1 + \deg A_2 + \cdots + \deg A_k = m$ is $(\binom{m-1}{k-1})(q - 1)^k q^m$.

Proof. The number of integer vectors $(n_1, n_2, \ldots, n_k)$ such that $n_i \geq 1$ for all $1 \leq i \leq k$ and $n_1 + n_2 + \cdots + n_k = m$ is $(\binom{m-1}{k-1})$. Since the number of polynomials in $\mathbb{F}_q[z]$ of degree $n_i$ is $(q - 1)q^m$, the conclusion follows. □

In the next two lemmas we investigate the recursive formula (1.3) that plays a significant role in the Hausdorff dimension estimates. Recall that

$$f_i(s) = s, \quad f_{i+1}(s) = \frac{s f_i(s)}{1 - s + f_i(s)} \text{ for } i \geq 1.$$

Lemma 2.6. For any given $i \geq 1$, $f_i(s)$ is continuous, strictly monotonically increasing on $[0, 1]$, and differentiable in $(0, 1)$.

Proof. Clearly, $f_i(0) = 0$ and $f_i(1) = 1$. Note also that $f_i(s)$ is always a rational function for all $i \geq 1$. We shall prove the conclusion by induction on $i$. When $i = 1$, the conclusion holds for $f_1(s) = s$. Suppose that the conclusion follows for some $i \geq 1$. Then

$$1 - s + f_i(s) \geq 1 - s + f_i(0) = 1 - s \neq 0$$

if $s \in [0, 1)$ and

$$1 - s + f_i(s) = f_i(1) = 1$$

if $s = 1$. Thus $1 - s + f_i(s) \neq 0$ for all $s \in [0, 1]$. Since

$$f_{i+1}(s) = \frac{s f_i(s)}{1 - s + f_i(s)},$$
it follows that the rational function \( f_{i+1}(s) \) is continuous on \([0,1]\) and differential in \((0,1)\). Moreover,
\[
f'_{i+1}(s) = \frac{f_i^2(s) + f_i(s) + s(1-s)f'_i(s)}{(1-s + f_i(s))^2}
\]
for \( s \in (0,1) \). Since \( f_i(s) > f_i(0) = 0 \) and \( f'_i(s) > 0 \) for any \( s \in (0,1) \), it follows that \( f'_{i+1}(s) > 0 \) for any \( s \in (0,1) \). This completes the proof.

**Lemma 2.7.** Let \( f_i(s) \) be defined by (1.3) for \( s \in [0,1] \). Then,
\[
f_i(s) = \frac{s^i}{\sum_{m=0}^{i-1} s^m(1-s)^{i-1-m}} = \begin{cases} \frac{1}{2i} & \text{if } s = \frac{1}{2}, \\ \frac{s(2s-1)}{s^2-(1-s)} & \text{if } s \neq \frac{1}{2}. \end{cases}
\]

**Proof.** By (1.3) and Lemma 2.6, we have \( f_i(0) = 0, f_i(1) = 1 \) and \( 0 < f_i(s) < 1 \) for any \( i \geq 1 \) and \( s \in (0,1) \). Since \( f_{i+1}(s) = \frac{s f_i(s)}{1-s+f_i(s)} \), we have
\[
\frac{1}{f_{i+1}(s)} = \frac{1-s}{s} \frac{1}{f_i(s)} + \frac{1}{s}
\]
for \( s \in (0,1) \). Thus if \( s = 1/2 \), we have
\[
\frac{1}{f_i(1/2)} = 2 + 2(i-1) = 2i \quad \implies \quad f_i(1/2) = \frac{1}{2i}.
\]
If \( s \neq 1/2 \), we have
\[
\frac{1}{f_{i+1}(s)} = \left(\frac{1-s}{s}\right)^i \frac{1}{f_i(s)} + \frac{1}{s} \sum_{j=0}^{i-1} \left(\frac{1-s}{s}\right)^j
\]
\[
= \frac{s^{i+1} - (1-s)^{i+1}}{s^{i+1}(2s-1)}.
\]
So we always have
\[
f_i(s) = \frac{s^i}{\sum_{m=0}^{i-1} s^m(1-s)^{i-1-m}}. 
\]

It follows from Lemma 2.7 that if \( 1/2 < s < 1 \) then
\[
f_i(s) = \frac{2s - 1}{1 - (\frac{1-s}{s})^i}
\]
and
\[
0 < 2s - 1 < f_{i+1}(s) < f_i(s) \leq s
\]
for any \( i \geq 1 \).

**Lemma 2.8.** For any given \( k \geq 1 \), let \( s_k(B) \) be the unique solution in \((1/2,1)\) to the equation
\[
\sum_{j=1}^{\infty} (q-1)q^i \frac{1}{q^{2jB}f_i(s)} = 1.
\]
Then \( s_k(B) \) is continuous with respect to \( B \). Moreover,
\[
\lim_{B \to 0} s_k(B) = 1, \quad \lim_{B \to \infty} s_k(B) = \frac{1}{2}.
\]

The proof of this Lemma is similar to Lemma 7.1 in [12]. For completeness, we include a proof here.
Proof. (i) Let
\[ h_B(s) = \sum_{j=1}^{\infty} (q-1)q^j \frac{1}{q^{2js+Bf_k(s)}}. \]
Then for \( s \in (1/2, 1] \), \( h_B(s) = q^{-Bf_k(s)} \frac{q^{1-s}}{q^s-1} \) which is monotonically decreasing and continuous. Moreover, \( h_B(1) = q^{-B} < 1 \) and \( h_B(s) > 1 \) when \( s \in (1/2, 1/2 + \delta) \) for some \( \delta > 0 \) small enough. Thus there exists a unique \( s_k(B) \in (1/2, 1) \) such that \( h_B(s) = 1 \).

(ii) For any \( \epsilon > 0 \), it suffices to prove that
\[ |s_k(B) - s_k(B')| < \epsilon \]
for any \( |B' - B| < \epsilon \). We first consider the case \( B - \epsilon < B' < B \) and prove that
\[ s_k(B) < s_k(B') < s_k(B) + \epsilon. \]
Since \( s_k(\cdot) \) is monotonically decreasing, the left hand part of the inequality is trivial. Whereas, the estimate
\[
\begin{align*}
    h_B'(s_k(B) + \epsilon) &= \sum_{j=1}^{\infty} (q-1)q^j \frac{1}{q^{2js_k(B)\epsilon+Bf_k(s_k(B)+\epsilon)}} \\
    &\leq q^{-2\epsilon} \sum_{j=1}^{\infty} (q-1)q^j \frac{1}{q^{2js_k(B)+Bf_k(s_k(B)+\epsilon)}} \\
    &\leq q^{-2\epsilon} \sum_{j=1}^{\infty} (q-1)q^j \frac{1}{q^{2js_k(B)+Bf_k(s_k(B))}} \\
    &= q^{-2\epsilon} h_B(s_k(B)) q^{B-B'} f_k(s_k(B)) \\
    &\leq q^{-2\epsilon} q^{B-B'} \leq q^{-\epsilon} < 1,
\end{align*}
\]
implies that \( s_k(B') < s_k(B) + \epsilon \). Similarly, in the case \( B < B' < B + \epsilon \), we also have
\[ s_k(B) - \epsilon < s_k(B') < s_k(B). \]

(iii) Since \( h_B(1) = q^{-B} < 1 \), we always have \( s_k(B) < 1 \) for \( B > 0 \). Take \( s = \frac{2}{2+B} \), where \( 0 < B < 2 \) such that \( s \in (1/2, 1) \). Since \( f_k(s) \leq s \) for \( s \in (1/2, 1) \), we have
\[
\begin{align*}
    h_B(s) &= \sum_{j=1}^{\infty} (q-1)q^j \frac{1}{q^{2js+Bf_k(s)}} \\
    &\geq \sum_{j=1}^{\infty} (q-1)q^j \frac{1}{q^{2js+Bs}} \\
    &\geq \sum_{j=1}^{\infty} (q-1)q^j \frac{1}{q^{2j}} = 1.
\end{align*}
\]
Thus \( s_k(B) \geq \frac{2}{2+B} \) when \( 0 < B < 2 \) and it follows that \( \lim_{B \to 0} s_k(B) = 1 \). The proof of the other assertion is similar.

\[ \square \]
2.1. Remarks on Dirichlet improvability. The theory of uniform Diophantine approximation concerns improvements to Dirichlet’s theorem (1842). In a recent paper, Kleinbock and Wadleigh [10] defined the set of \(\phi\)-Dirichlet improvable numbers to be the set of all \(x \in \mathbb{R}\) such that
\[
|qx - p| < \phi(t), \ 1 \leq |q| < t
\]
has an integer solution \((p, q)\) for all large enough \(t\). Here \(\phi\) is a non-increasing function such that \(\phi(t) \to 0\) as \(t \to \infty\).

We investigate the analogue of Dirichlet improvability over formal Laurent series.

**Proposition 2.9.** For any \(x \in \mathbb{F}_q((z^{-1}))\) and \(t > 1\), there exists nonzero \((P, Q) \in \mathbb{F}_q[z] \times \mathbb{F}_q[z]\) such that
\[
|Qx - P|_{\infty} \leq \frac{1}{t}, \ |Q|_{\infty} < t.
\]

*Proof.* Let \(x\) is irrational. Since \(t > 1\) and \(Q_0(x) = 1\), there exists \(n \geq 0\) such that \(|Q_n(x)|_{\infty} < t \leq |Q_{n+1}(x)|_{\infty}\). Then we have
\[
|Q_n(x)x - P_n(x)|_{\infty} = \frac{1}{|Q_{n+1}(x)|_{\infty}} \leq \frac{1}{t}.
\]
If \(x\) is rational, write \(x = A/B\) with co-prime polynomials \(A\) and \(B\). If \(t \leq |B|_{\infty}\), we can get the conclusion by the same arguments as in the irrational case. If \(t > |B|_{\infty}\), we get the conclusion by taking \(Q = B\) and \(P = A\). \(\square\)

This proposition is an analogue of Dirichlet’s Theorem in the field of formal Laurent series. However, note that it is slightly different from the form of Dirichlet’s Theorem over formal Laurent series in [11] Theorem 1.1.

Now we define the \(\phi\)-Dirichlet improvable set \(\text{Dir}(\phi)\) in the field formal Laurent series field as follows. Let \(\text{Dir}(\phi)\) be the set of all \(x \in \mathbb{F}_q((z^{-1}))\) such that
\[
|Qx - P|_{\infty} \leq \phi(t), \ |Q|_{\infty} < t \tag{2.1}
\]
has a nonzero solution \((P, Q) \in \mathbb{F}_q[z] \times \mathbb{F}_q[z]\) for all large enough \(t\).

For \(x \in \mathbb{F}_q((z^{-1}))\), define
\[
\|x\| = \min_{P \in \mathbb{F}_q[z]} |x - P|_{\infty}.
\]

**Lemma 2.10.** Let \(\phi\) be non-increasing. Then an irrational \(x \in \text{Dir}(\phi)\) if and only if
\[
\|Q_{n-1}(x)x\| \leq \phi(|Q_n(x)|_{\infty}) \tag{2.2}
\]
for all sufficiently large \(n\), where \(P_n(x)/Q_n(x)\) is the \(n\)th convergent of \(x\).

*Proof.* Suppose \(x \in \text{Dir}(\phi)\). Take \(t = |Q_n(x)|_{\infty}\) for large enough \(n\). Then by (2.1), there exists \(0 \neq Q \in \mathbb{F}_q[z]\) such that
\[
\|Qx\| \leq \phi(|Q_n(x)|_{\infty}), \ |Q|_{\infty} < |Q_n(x)|_{\infty}.
\]
By [15] Lemma 1,
\[
\|Qx\| \geq \|Q_{n-1}x\| \text{ whenever } |Q|_{\infty} < |Q_n(x)|_{\infty}.
\]
So we have \(\|Q_{n-1}(x)x\| \leq \phi(|Q_n(x)|_{\infty})\) for all large enough \(n\).

Conversely, suppose \(\|Q_{n-1}(x)x\| \leq \phi(|Q_n(x)|_{\infty})\) for all \(n \geq N\). Then for any \(t \geq |Q_N(x)|_{\infty}\), there exists \(n \geq N\) such that \(|Q_{n-1}(x)|_{\infty} < t \leq |Q_n(x)|_{\infty}\). Since \(\phi\) is non-increasing, we have
\[
\|Q_{n-1}(x)x\| \leq \phi(|Q_n(x)|_{\infty}) \leq \phi(t).
\]
Thus \(x\) is \(\phi\)-Dirichlet. \(\square\)
Since
\[ \|Q_{n-1}(x)x\| = |Q_n(x)|^{-1}_\infty = q^{-\deg Q_n(x)} = q^{-(\deg A_1(x) + \deg A_2(x) + \cdots + \deg A_n(x))}, \]
it follows that \(x\) is \(\phi\)-Dirichlet improvable if and only if
\[ |Q_n(x)|_\infty \phi(|Q_n(x)|_\infty) = q^{\sum_{i=1}^n \deg A_i(x)} \geq 1 \]
for all large enough \(n\) by (2.2). Thus we get the following by Lemma 2.10 and estimate (2.3).

**Lemma 2.11.** Let \(\phi\) be non-increasing. Then an irrational \(x \in D_{ir}(\phi)\) if and only if
\[ \phi(q^{\sum_{i=1}^n \deg A_i(x)}) \geq q^{-\sum_{i=1}^n \deg A_i(x)} \quad (2.4) \]
for all sufficiently large \(n\).

Clearly, it follows from Lemma 2.11 that \(D_{ir}(\phi) = F_q((z^{-1}))\) if \(\phi(t) = 1/t\).

**Corollary 2.12.** Let \(\phi\) be non-increasing. If \(\phi(q^n)q^n < 1\) for infinitely many \(n\), then \(D_{ir}(\phi) \neq F_q((z^{-1}))\).

**Proof.** Take \(x \in F_q((z^{-1}))\) with \(\deg A_i(x) = 1\) for all \(i \geq 1\). Then (2.4) does not hold for infinitely many \(n\). It follows that \(x \notin D_{ir}(\phi)\). \(\Box\)

3. Proof of Theorem 1.1

As we have mentioned, the case \(k = 1\) was proved by Niederreiter [18]. Now we assume \(k \geq 2\). For each \(n \geq 0\), let
\[ E_n = \{x \in I : \deg A_{n+1}(x) + \deg A_{n+2}(x) + \cdots + \deg A_{n+k}(x) \geq \Phi(n)\} \]
and
\[ F_n = \{x \in I : \deg A_1(x) + \deg A_2(x) + \cdots + \deg A_k(x) \geq \Phi(n)\}. \]
Then
\[ F_k(\Phi) = \{x \in I : x \in E_n \text{ for infinitely many } n\}. \]
Since the Haar measure is \(T\)-invariant (Proposition 2.2 (i)), we have \(\nu(E_n) = \nu(F_n)\).

Next we calculate \(\nu(F_n)\). Note that for any \(m \geq k\), we have
\[ \left\{ x \in I : \sum_{i=1}^k \deg A_i(x) = m \right\} = \bigcup_{A_1, A_2, \ldots, A_k : \sum_{i=1}^k \deg A_i = m} I(A_1, A_2, \ldots, A_k). \]
By Lemma 2.5 and Proposition 2.3 it follows that
\[ \nu \left( \left\{ x \in I : \sum_{i=1}^k \deg A_i(x) = m \right\} \right) = \binom{m-1}{k-1} (q-1)^k q^m q^{-2m} = \binom{m-1}{k-1} (q-1)^k q^{-m}. \]
We denote by $\lceil \xi \rceil$ to be the smallest integer no less than $\xi \in \mathbb{R}$. Then

$$
\nu(F_n) = \sum_{m=\lceil \Phi(n) \rceil}^{\infty} \nu \left( \left\{ x \in I : \sum_{i=1}^{k} \deg A_i(x) = m \right\} \right)
= \sum_{m=\lceil \Phi(n) \rceil}^{\infty} \binom{m-1}{k-1} (q-1)^k q^{-m}
\geq \left( \frac{\lceil \Phi(n) \rceil - 1}{k-1} \right) (q-1)^k q^{-\lfloor \Phi(n) \rfloor}
= \frac{\Phi^{k-1}(n)}{(k-1)!} (q-1)^k q^{-\Phi(n)-1} \prod_{i=1}^{k-1} \left( 1 - \frac{i}{\lfloor \Phi(n) \rfloor} \right)
\geq \frac{\Phi^{k-1}(n)}{(k-1)!} (q-1)^k q^{-\Phi(n)-1} \left( 1 - \frac{k-1}{k} \right)^{k-1}
= c_1 \frac{\Phi^{k-1}(n)}{q^{\Phi(n)}}
$$

where $c_1 = k^{-k+1} q^{-1} (q-1)^k / (k-1)!$. Next for the upper bound of $\nu(F_n)$, note that

$$
\nu(F_n) \leq \sum_{m=\lceil \Phi(n) \rceil}^{\infty} m^{k-1} (q-1)^k q^{-m}.
$$

Let

$$
b_m = m^{k-1} (q-1)^k q^{-m}
$$

for $m \geq k$. Since

$$
\lim_{n \to \infty} \frac{b_{m+1}}{b_m} = q^{-1}
$$

and

$$
\frac{b_{m+1}}{b_m} \leq \left( 1 + \frac{1}{k} \right)^{k-1} q^{-1}
$$

for all $m \geq k$, there exists a constant $c_2$ depending on $k$ and $q$ such that

$$
\sum_{m=1}^{\infty} b_m \leq c_2 b_i
$$

for all $i \geq k$. Thus

$$
\nu(F_n) \leq c_2 \Phi(n)^{-1} (q-1)^k q^{-\Phi(n)}
\leq c_2 (q-1)^k 2^{k-1} \frac{\Phi^{k-1}(n)}{q^{\Phi(n)}}.
$$

So there exists a constant $c > 0$ depending on $k$ and $q$ such that

$$
c^{-1} \frac{\Phi^{k-1}(n)}{q^{\Phi(n)}} \leq \nu(F_n) \leq c \frac{\Phi^{k-1}(n)}{q^{\Phi(n)}}
$$

for all $n \geq 1$. From the first Borel-Cantelli Lemma, it follows that the $\nu$ measure of $F_k(\Phi)$ is zero if the series $\sum_n \Phi^{k-1}(n) / q^{\Phi(n)}$ converges. For the divergence case, since

$$
\sum_{n=0}^{\infty} \nu(E_n) = \sum_{j=0}^{k-1} \sum_{i=0}^{\infty} \nu(E_{ik+j}),
$$
there exists an integer $0 \leq j_0 \leq k - 1$ such that $\sum_{i=0}^{\infty} \nu(E_{ik+j_0}) = \infty$. By Proposition 2.2 (ii), $E_{j_0}$, $E_{k+j_0}$, $E_{2k+j_0}$, ... are independent with respect to $\nu$. Thus by the Borel-Cantelli Lemma,

$$\nu(\{x \in I : x \in E_{ik+j_0} \text{ for infinitely many } i\}) = 1.$$ 

It follows that $\nu(\mathcal{F}_k(\Phi)) = 1$.

4. The upper bound of $\dim_{\mathcal{H}} \mathcal{F}_k(\Phi)$ for $\Phi(n) = nB$

In this section we prove the upper bound of Theorem 1.2 for the case $\Phi(n) = nB$ with $0 < B < \infty$. Recall that

$$\mathcal{F}_k(\Phi) = \left\{ x \in I : \sum_{i=1}^{k} \deg A_{n+i}(x) \geq \Phi(n) \text{ for infinitely many } n \in \mathbb{N} \right\}.$$

For any given $0 < B < \infty$, if $\Phi(n) = nB$, we denote $\mathcal{F}_k(\Phi)$ by $\mathcal{F}_k(B)$. We shall prove the following theorem.

**Theorem 4.1.** $\dim_{\mathcal{H}} \mathcal{F}_k(B) = s_k(B)$, where $s_k(B)$ is defined by (1.2).

The proof of the theorem splits into two parts: the upper bound and the lower bound. We prove them separately but before that we state the definition of Hausdorff dimension for completeness. Let $U \subset I$. Then for any $\rho > 0$, any finite or countable collection $\{B_i\}$ of subsets of $I$ with diameters $|B_i| \leq \rho$ such that $U \subset \bigcup_{i} B_i$ is called a $\rho$-cover of $U$. Let

$$\mathcal{H}^t(U) = \lim_{\rho \to 0} \left\{ \inf \sum_{i} |B_i|^t \right\},$$

where the infimum is taken over all possible $\rho$-covers $\{B_i\}$ of $U$. The Hausdorff dimension of $U$ is defined by

$$\dim_{\mathcal{H}} U = \inf \left\{ t \geq 0 : \mathcal{H}^t(U) = 0 \right\}.$$

We first estimate the upper bound of the Hausdorff dimension of $\mathcal{F}_k(B)$.

**Lemma 4.2.** $\dim_{\mathcal{H}} \mathcal{F}_k(B) \leq s_k(B)$.

**Proof.** We prove this result by induction. In the case $k = 1$, the result has already been proven as Theorem 7.2 in [12], that is, $\dim_{\mathcal{H}} \mathcal{F}_1(B) = s_1(B)$. Suppose that the conclusion holds for $k$. Then we show that $\dim_{\mathcal{H}} \mathcal{F}_{k+1}(B) \leq s_{k+1}(B)$. For any $0 < \gamma < B$, let

$$F_{k+1}(\gamma, B) = \left\{ x \in I : \begin{array}{c} \sum_{i=1}^{k} \deg A_{n+i}(x) \leq n\gamma \text{ and } \\
\deg A_{n+k+1}(x) \geq nB - \sum_{i=1}^{k} \deg A_{n+i}(x) \end{array} \text{ for infinitely many } n \in \mathbb{N} \right\}.$$ 

Then

$$\mathcal{F}_{k+1}(B) \subseteq \mathcal{F}_k(\gamma) \cup F_{k+1}(\gamma, B).$$

So we have

$$\dim_{\mathcal{H}} \mathcal{F}_{k+1}(B) \leq \inf_{0 < \gamma < B} \max \{ \dim_{\mathcal{H}} \mathcal{F}_k(\gamma), \dim_{\mathcal{H}} F_{k+1}(\gamma, B) \}.$$
By the induction hypothesis, we have \( \dim H F_k(\gamma) \leq s_k(\gamma) \), where \( s_k(\gamma) \) is the unique solution of the equation

\[
\sum_{j=1}^{\infty} (q - 1) q^j \frac{1}{q^{2j + \gamma f_k(x)}} = 1. \tag{4.1}
\]

Next we shall give an upper bound of \( \dim H F_{k+1}(\gamma, B) \). For every \( n \geq 1 \), let

\[
F_{k+1}(\gamma, B, n) = \left\{ x \in I : \sum_{i=1}^{k} \text{deg } A_{n+i}(x) \leq n \gamma \quad \text{and} \quad \text{deg } A_{n+k+1}(x) \geq n B - \sum_{i=1}^{k} \text{deg } A_{n+i}(x) \right\}.
\]

Then

\[
F_{k+1}(\gamma, B) = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} F_{k+1}(\gamma, B, n).
\]

So for any \( N \geq 1 \), \( \{ F_{k+1}(\gamma, B, n) : n \geq N \} \) is a cover of the set \( F_{k+1}(\gamma, B) \). Note that

\[
F_{k+1}(\gamma, B, n) \subseteq \bigcup_{\text{deg } A_{n+k+1} \geq n B - \sum_{i=1}^{k} \text{deg } A_{n+i}} \bigcup_{i=1, 2, \ldots, n} G(A_1, A_2, \ldots, A_{n+k}),
\]

where

\[
G(A_1, A_2, \ldots, A_{n+k}) = \bigcup_{\text{deg } A_{n+k+1} \geq n B - \sum_{i=1}^{k} \text{deg } A_{n+i}} I(A_1, A_2, \ldots, A_{n+k+1}).
\]

The diameter of \( G(A_1, A_2, \ldots, A_{n+k}) \), by Lemma 2.4, is given by

\[
|G(A_1, A_2, \ldots, A_{n+k})| = q^{-n B t + \sum_{i=1}^{k} \text{deg } A_{n+i} - 2 \sum_{i=1}^{n+k} \text{deg } A_i}.
\]

Thus

\[
\Lambda_N := \bigcup_{n \geq N} \bigcup_{\text{deg } A_i, 1 \leq i \leq n} \bigcup_{\text{deg } A_{n+i} \leq n \gamma} G(A_1, A_2, \ldots, A_{n+k})
\]

is a cover of \( F_{k+1}(\gamma, B) \) for any \( N \geq 1 \). For any \( 1/2 < t < 1 \),

\[
\mathcal{H}^t(F_{k+1}(\gamma, B)) \leq \liminf_{N \to \infty} \sum_{G(A_1, A_2, \ldots, A_{n+k}) \in \Lambda_N} |G(A_1, A_2, \ldots, A_{n+k})|^t
\]

\[
\leq \liminf_{N \to \infty} \sum_{n \geq N} \sum_{\text{deg } A_i \geq 1} \sum_{i=1, 2, \ldots, n} \text{deg } A_{n+i} \leq n \gamma |G(A_1, A_2, \ldots, A_{n+k})|^t
\]

\[
\leq \liminf_{N \to \infty} \sum_{n \geq N} q^{-n B t} \left( \sum_{j=1}^{\infty} (q - 1) q^j \frac{1}{q^{2j t + B t}} \right)^n \sum_{\text{deg } A_{n+i} \leq n \gamma} q^{-t \sum_{i=1}^{k} \text{deg } A_{n+i}}
\]

\[
= \liminf_{N \to \infty} \sum_{n \geq N} \left( \sum_{j=1}^{\infty} (q - 1) q^j \frac{1}{q^{2j t + B t}} \right)^n \sum_{j=k}^{\infty} \left( j - 1 \right) (q - 1)^k q^j \frac{1}{q^t}.
\]

For simplicity, denote \( s_{k+1}(B) \) by \( \tilde{s} \). Since \( \tilde{s} \) is the unique solution of the equation

\[
\sum_{j=1}^{\infty} (q - 1) q^j \frac{1}{q^{2j s + B s^{k+1}(\tilde{s})}} = 1,
\]

(4.2)
we have $1/2 < \tilde{s} = s_{k+1}(B) < 1$. Since

$$\sum_{j=k}^{[n\gamma]} (j-1)(q-1)^k q^j \frac{1}{q^{j+t}} \leq \sum_{j=k}^{[n\gamma]} j^k (q-1)^k q^{(1-t)j} \leq (n\gamma)^k (q-1)^k \sum_{j=k}^{[n\gamma]} q^{(1-t)j} \leq (n\gamma)^k (q-1)^k \frac{q^{(1-t)(n\gamma+1)}}{q^{1-t} - 1} = cn^k q^{(1-t)n\gamma}$$

with $c = \gamma^k (q-1)^k q^{1-t}/(q^{1-t} - 1)$, we have

$$\mathcal{H}^t(F_{k+1}(\gamma, B)) \leq \liminf_{N \to \infty} \sum_{n \geq N} cn^k (\sum_{j=1}^{\infty} (q-1)q^j \frac{1}{q^{2jt+Bt}}) q^{(1-t)n\gamma}$$

$$= \liminf_{N \to \infty} \sum_{n \geq N} cn^k (\sum_{j=1}^{\infty} (q-1)q^j \frac{1}{q^{2jt+Bt-(1-t)\gamma}})^n.$$ 

Now we take

$$\gamma = \frac{B\tilde{s}}{1 - \tilde{s} + f_k(\tilde{s})}.$$ 

Then we have

$$B\tilde{s} - (1 - \tilde{s})\gamma = \gamma f_k(\tilde{s}) = B f_{k+1}(\tilde{s}),$$

where the second equality follows by (4.3). For any small $\epsilon > 0$, take

$$t = \tilde{s} + \epsilon.$$ 

Let

$$g(s) := \sum_{j=1}^{\infty} (q-1)q^j \frac{1}{q^{2js+Bs-(1-s)\gamma}}.$$ 

The function $g(s)$ is strictly monotonically decreasing on $(1/2, \infty)$ and

$$g(\tilde{s}) = \sum_{j=1}^{\infty} (q-1)q^j \frac{1}{q^{2j\tilde{s}+B\tilde{s}-(1-\tilde{s})\gamma}} = \sum_{j=1}^{\infty} (q-1)q^j \frac{1}{q^{2j\tilde{s}+Bf_{k+1}(\tilde{s})}} = 1$$

by (4.2) and (4.3). It follows that

$$\sum_{n \geq 1} n^k (g(\tilde{s}))^n = \sum_{n \geq 1} n^k (g(\tilde{s} + \epsilon))^n < \infty$$

and hence

$$\mathcal{H}^t(F_{k+1}(\gamma, B)) \leq \liminf_{N \to \infty} c \sum_{n \geq N} n^k (g(\tilde{s} + \epsilon))^n = 0$$

since $g(\tilde{s} + \epsilon) < g(\tilde{s}) = 1$. Therefore, from the definition of Hausdorff dimension, it follows that

$$\dim_\mathcal{H} F_{k+1}(\gamma, B) \leq \tilde{s}.$$ 

On the other hand, by (4.2) and (4.3), we have

$$\sum_{j=1}^{\infty} (q-1)q^j \frac{1}{q^{2j\tilde{s}+\gamma f_k(\tilde{s})}} = \sum_{j=1}^{\infty} (q-1)q^j \frac{1}{q^{2j\tilde{s}+Bf_{k+1}(\tilde{s})}} = 1.$$
Then by (4.1), we have $s_k(\gamma) = \tilde{s}$. So
$$\dim_H \mathcal{F}_k(B) \leq \max\{s_k(\gamma), \dim F_{k+1}(\gamma, B)\} = \tilde{s}.$$ □

5. The lower bound of $\dim_H \mathcal{F}_k(\Phi)$ for $\Phi(n) = nB$

In this section we shall prove that $\dim_H \mathcal{F}_k(B) \geq s_k(B)$, where $s_k(B)$ is defined by (1.2). To prove the lower bound of the Hausdorff dimension, we shall use the well-known mass distribution principle [9, Proposition 4.2].

**Lemma 5.1.** Let $F \subset I$ and let $\mu$ be a measure with support contained in $F$. If there are positive constants $c, \delta$ such that
$$\mu(U) < c \cdot |U|^s$$
for all discs with $|U| \leq \delta$, we have
$$\dim H F \geq s.$$

To apply this Lemma we will construct a suitable Cantor like subset of $\mathcal{F}_k(B)$ that supports the $\mu$-measure. We then distribute $\mu$-measure on basic subsets. Finally, we calculate measure for any disc satisfying the hypothesis of Lemma 5.1 to conclude the proof.

For $k = 1$, we already have $\dim_H \mathcal{F}_1(B)$ from [12]. So we assume $k \geq 2$ throughout the rest of the section.

For $M \geq 2$, we denote by $s_{k,M}(B)$ the unique solution of
$$\sum_{j=1}^{M} (q-1)q^j \frac{1}{q^{2js+Bf_k(s)}} = 1.$$ (5.1)

Then
$$\lim_{M \to \infty} s_{k,M}(B) = s_k(B).$$

For the remainder of this section, we write $s = s_{k,M}(B)$ for simplicity. Define parameters $\alpha_i$, for $1 \leq i \leq k$ satisfying
$$s\alpha_i = (1-s)\alpha_{i-1}, \sum_{i=1}^{k} \alpha_i = B.$$ (5.2)

Rewriting this relation, we have
$$\alpha_i = \frac{1-s}{s} \alpha_{i-1},$$
and
$$\alpha_1 \sum_{i=0}^{k-1} \left( \frac{1-s}{s} \right)^i = B \quad \Rightarrow \quad \alpha_1 = \frac{B}{\sum_{i=0}^{k-1} \left( \frac{1-s}{s} \right)^i} = \frac{Bs^k(2s-1)}{s(s^k - (1-s)^k)}.$$

Hence
$$\alpha_i = \frac{s^{k-i}(2s-1)(1-s)^{i-1}}{s^k - (1-s)^k} B, \quad 1 \leq i \leq k.$$ (5.3)

Finally, we identify (following Lemma 2.7) that
$$Bf_k(s) = s\alpha_1.$$
5.1. **A subset of** $\mathcal{F}_k(B)$. Let $M \geq 2$ be an integer and $\epsilon > 0$ a real number small enough. Let $\{n_j\}$ be a subsequence of positive integers satisfying

$$\min_{i \leq j \leq k} \frac{n_j \alpha_i}{n_j \alpha_i + 2} \geq \frac{s - \epsilon}{s}, \quad \frac{n_{j+1} - n_j - k}{n_{j+1}} \geq \frac{s - \epsilon}{s} \quad (5.4)$$

for all $j \geq 1$, where $\alpha_1, \ldots, \alpha_k$ are defined by \(5.2\). Now we shall construct a subset $E(B, M, \epsilon, \{n_j\})$ of $\mathcal{F}_k(B)$ as follows.

$$E(B, M, \epsilon, \{n_j\}) = \left\{ x \in I : \begin{array}{l}
\deg A_{n_j+i}(x) = n_j \alpha_i + 1 \text{ for all } j \geq 1, 1 \leq i \leq k, \\
1 \leq \deg A_n(x) \leq M \text{ for other } n
\end{array} \right\}.$$

By (5.2), we have

$$E(B, M, \epsilon, \{n_j\}) \subseteq \mathcal{F}_k(B).$$

For the remainder of the section, by using the mass distribution principle we prove the following

$$\dim_{\mathcal{H}} E(B, M, \epsilon, \{n_j\}) \geq s - \epsilon.$$

5.2. **Fractal structure of** $E(B, M, \epsilon, \{n_j\})$. For any $n \geq 1$, denote by $D_n$ the set of all $(A_1, \ldots, A_n) \in \mathbb{F}_q[\epsilon]^n$ such that

$$\begin{cases}
\deg A_{n_j+i} = [n_j \alpha_i] + 1 \text{ for any } j \geq 1 \text{ and } 1 \leq i \leq k \text{ with } 1 \leq n_j + i \leq n, \\
1 \leq \deg A_m \leq M \text{ for other } 1 \leq m \leq n
\end{cases}.$$

Let

$$D = \bigcup_{n=1}^{\infty} D_n.$$

For any $n \geq 1$ and $(A_1, \ldots, A_n) \in D_n$, define

$$J(A_1, \ldots, A_n) = \bigcup_{A_{n+1} : (A_1, \ldots, A_{n+1}) \in D_{n+1}} I(A_1, \ldots, A_{n+1})$$

and we call $J(A_1, \ldots, A_n)$ a basic set of order $n$. Note that $J(A_1, \ldots, A_n)$ is a union of finitely many disjoint discs. Then

$$E(B, M, \epsilon, \{n_j\}) = \bigcap_{n \geq 1} \bigcup_{(A_1, \ldots, A_n) \in D_n} J(A_1, \ldots, A_n).$$

**Lemma 5.2.** For any $n \geq 1$ and $(A_1, \ldots, A_n) \in D_n$, we have

$$|J(A_1, \ldots, A_n)| = \begin{cases}
q^{-2 \sum_{m=1}^{n} \deg A_m - [n_j \alpha_i] - 1}, & \text{if } n = n_j + i - 1, j \geq 1, 1 \leq i \leq k, \\
q^{-2 \sum_{m=1}^{n} \deg A_m - 1}, & \text{otherwise}.
\end{cases}$$

**Proof.** Let $x, y \in J(A_1, \ldots, A_n)$ with $x \in I(A_1, \ldots, A_n, A_{n+1})$ and $y \in I(A_1, \ldots, A_n, A^*_n)$. Then as in the proof of Lemma 2.4, we have

$$|x - y|_{\infty} = \frac{|A_{n+1} - A^*_{n+1}|}{A_{n+1} A^*_{n+1}} = \left| \frac{A_{n+1} - A^*_{n+1}}{A_{n+1} A^*_{n+1}} \right|_{\infty}.$$

If $n = n_j + i - 1$ for some $j \geq 1$ and $1 \leq i \leq k$, then $\deg A_{n+1} = \deg A^*_{n+1} = [n_j \alpha_i] + 1$. So

$$|x - y|_{\infty} \leq q^{-[n_j \alpha_i] - 1 - 2 \sum_{i=1}^{n} \deg A_i}.$$
where the equality holds when \( \deg(A_{n+1} - A_n^*) = [n_j \alpha_i] + 1 \). If \( n \not\in \{n_j + i - 1 : j \geq 1, 1 \leq i \leq k \} \), we have

\[
|x - y|_\infty \leq q^{-1 - 2 \sum_{i=1}^{n} \deg A_i}.
\]

Here the equality holds when \( \deg A_{n+1} = 1 \) and \( \deg A_n^* = M \).

\[ \square \]

5.3. **The \( \mu \) measure on \( E(B, M, \epsilon, \{n_j\}) \).** To define a measure \( \mu \) on \( E(B, M, \epsilon, \{n_j\}) \), we first distribute the mass on basic sets.

- If \( n = 1 \), define
  \[
  \mu(J(A_1)) = q^{-2s \deg A_1} q^{-Bf(s)}.
  \]
- If \( 2 \leq n \leq n_1 \),
  \[
  \mu(J(A_1, \ldots, A_n)) = q^{-2s \deg A_n} q^{-Bf(s)} \mu(J(A_1, \ldots, A_{n-1})).
  \]
- If \( n_j + 1 \leq n \leq n_j + k \) for some \( j \geq 1 \), write \( n = n_j + i \) for some \( 1 \leq i \leq k \). Define
  \[
  \mu(J(A_1, \ldots, A_n)) = \frac{1}{(q - 1)q^{[n_j \alpha_i] + 1}} \mu(J(A_1, \ldots, A_{n-1})).
  \]
It means that the measure is uniformly distributed on the basic sets of order \( n \) contained in \( J(A_1, \ldots, A_{n-1}) \) if \( n_j + 1 \leq n \leq n_j + k \).

- If \( n_j + k < n \leq n_j + 1 \), define
  \[
  \mu(J(A_1, \ldots, A_n)) = q^{-2s \deg A_n} q^{-Bf(s)} \mu(J(A_1, \ldots, A_{n-1})).
  \]

By (5.1), we have \( \mu(I) = 1 \) and

\[
\sum_{A_n:(A_1, A_2, \ldots, A_n) \in D_n} \mu(J(A_1, A_2, \ldots, A_n)) = \mu(J(A_1, A_2, \ldots, A_{n-1}))
\]

for all \( n \geq 1 \). So \( \mu \) is well defined on basic sets. Thus it can be extended into a probability measure on \( E(B, M, \epsilon, \{n_j\}) \). From the definition of \( \mu \), we have

\[
\mu(I(A_1, A_2, \ldots, A_n)) = \mu(J(A_1, A_2, \ldots, A_n))
\]

for any \( n \geq 1 \) and \( (A_1, A_2, \ldots, A_n) \in D_n \), since \( I(A_1, A_2, \ldots, A_n) \) contains the basic set \( J(A_1, A_2, \ldots, A_n) \) but does not intersect any other basic sets of order \( n \).

5.4. **The measure on basic sets.** In this section, we prove that

\[
\mu(J(A_1, \ldots, A_n)) \leq |J(A_1, \ldots, A_n)|^{s-\epsilon}
\]

(5.5)

for any \( n \geq n_1 \) and \( (A_1, \ldots, A_n) \in D_n \).

We consider all the cases one by one.

- \( n = n_1 \) case.

\[
\mu(J(A_1, \ldots, A_{n_1})) = \prod_{i=1}^{n_1} (q^{-2s \deg A_i} q^{-Bf(s)})
\]

\[
= q^{-2s \sum_{i=1}^{n_1} \deg A_i - n_1 Bf(s)}
\]

\[
= q^{-s(2 \sum_{i=1}^{n_1} \deg A_i + n_1 \alpha_1)}
\]

\[
\leq q^{-(s-\epsilon)(2 \sum_{i=1}^{n_1} \deg A_i + n_1 \alpha_1 + 1)}
\]

\[
\leq |J(A_1, \ldots, A_{n_1})|^{s-\epsilon},
\]

where we have used (5.3) in the third equality, (5.4) in the first inequality and Lemma 5.2 in the last inequality.
• It suffices to prove that if (5.5) holds for \( n = n_j \), then (5.5) also holds for all \( n_j < n \leq n_{j+1} \).

• If \( n = n_j + p \) for some \( 1 \leq p \leq k - 1 \), we have

\[
\mu(J(A_1, A_2, \ldots, A_{n_j+p})) = \mu(J(A_1, \ldots, A_{n_j})) \prod_{i=1}^{p} \frac{1}{q^{n_j \alpha_i}} = \mu(J(A_1, \ldots, A_{n_j})) \prod_{i=1}^{p} \frac{1}{q^{n_j (\alpha_i + \alpha_{i+1})}}
\]

\[
\leq \mu(J(A_1, \ldots, A_{n_j})) \prod_{i=1}^{p} \frac{1}{q^{n_j \alpha_i}} = \mu(J(A_1, \ldots, A_{n_j})) \prod_{i=1}^{p} \frac{1}{q^{n_j (\alpha_i + \alpha_{i+1})}}
\]

\[
= \mu(J(A_1, \ldots, A_{n_j}))q^{-sn_j(\alpha_1 + 2 \sum_{i=2}^{p} \alpha_i + \alpha_{p+1})}
\]

\[
\leq |J(A_1, \ldots, A_{n_j})|^s - \epsilon q^{-sn_j(\alpha_1 + 2 \sum_{i=2}^{p} \alpha_i + \alpha_{p+1})}
\]

\[
= q^{-(s-\epsilon)(2 \sum_{i=1}^{n_j} \deg A_i + [n_j \alpha_1] + 1)} q^{-sn_j(\alpha_1 + 2 \sum_{i=2}^{p} \alpha_i + \alpha_{p+1})}
\]

\[
\leq q^{-2(s-\epsilon) \sum_{j=1}^{n_j+p} \deg A_i - (s-\epsilon)(n_j \alpha_{p+1} + 1)}
\]

\[
\leq |J(A_1, \ldots, A_{n_j+p})|^s - \epsilon,
\]

where we have used (5.2) in the second equality, the fact \( \deg A_{n_j+l} = [n_j \alpha_l] + 1 \) for \( 1 \leq l \leq k \) and (5.4) in the third inequality, Lemma 5.2 in the last inequality.

• If \( n = n_j + k \), then

\[
\mu(J(A_1, A_2, \ldots, A_{n_j+k})) = \mu(J(A_1, \ldots, A_{n_j+k-1})) \frac{1}{(q-1)q^{[n_j \alpha_k] + 1}}
\]

\[
\leq q^{-n_j \alpha_k} \mu(J(A_1, \ldots, A_{n_j+k-1})) \leq q^{-n_j \alpha_k} |J(A_1, \ldots, A_{n_j+k-1})|^s - \epsilon
\]

\[
= q^{-n_j \alpha_k} q^{-(s-\epsilon)(2 \sum_{i=1}^{n_j+k-1} \deg A_i + \deg A_{n_j+k})}
\]

\[
\leq q^{-2(s-\epsilon)(n_j \alpha_k + 2) \sum_{i=1}^{n_j+k-1} \deg A_i + \deg A_{n_j+k}}
\]

\[
\leq q^{-2(s-\epsilon)(n_j \alpha_k + 2) \sum_{i=1}^{n_j+k-1} \deg A_i + \deg A_{n_j+k}}
\]

\[
\leq |J(A_1, \ldots, A_{n_j+k})|^s - \epsilon,
\]

where we have used Lemma 5.2 in the second equality and the last inequality, and (5.4) in the third inequality.

• If \( n_j + k + 1 \leq n \leq n_{j+1} - 1 \), then

\[
\mu(J(A_1, \ldots, A_n)) = \mu(J(A_1, \ldots, A_{n+j+k})) \prod_{i=n_j+k+1}^{n} q^{-2 \deg A_i - B f(s)}
\]

\[
\leq \mu(J(A_1, \ldots, A_{n_j+k})) \prod_{i=n_j+k+1}^{n} q^{-2 \sum_{i=n_j+k+1}^{n+1} \deg A_i}
\]

\[
\leq |J(A_1, \ldots, A_{n_j+k})|^{s - \epsilon} q^{-2 \sum_{i=n_j+k+1}^{n+1} \deg A_i}
\]

\[
= q^{-(s-\epsilon)(2 \sum_{i=1}^{n_j+k} \deg A_i + 1)} q^{-2 \sum_{i=n_j+k+1}^{n+1} \deg A_i}
\]

\[
\leq q^{-(s-\epsilon)(2 \sum_{i=1}^{n_j+k} \deg A_i + 1)}
\]

\[
= |J(A_1, \ldots, A_n)|^{s - \epsilon},
\]

where we have used Lemma 5.2 in the last two equalities.
If $n = \gamma_{j+1}$, we have

$$
\mu(J(A_1, \ldots, A_{j+1})) = \mu(J(A_1, \ldots, A_{n+j})) \prod_{i=n_j+k+1}^{n_{j+1}} q^{-2s \deg A_i - Bf_k(s)}
$$

$$
= \mu(J(A_1, \ldots, A_{n+j})) q^{-2s \sum_{i=n_j+k+1}^{n_{j+1}} \deg A_i - Bf_k(s)(n_{j+1} - n_j - k)}
$$

$$
\leq |J(A_1, \ldots, A_{n+j})|^{s-\epsilon} q^{-2s \sum_{i=n_j+k+1}^{n_{j+1}} \deg A_i - Bf_k(s)(n_{j+1} - n_j - k)}
$$

$$
= q^{-(s-\epsilon)(2 \sum_{i=1}^{n_{j+1}} \deg A_i + 1)} q^{-2s \sum_{i=n_j+k+1}^{n_{j+1}} \deg A_i - \alpha_1(n_{j+1} - n_j - k)}
$$

$$
\leq q^{-(s-\epsilon)(2 \sum_{i=1}^{n_{j+1}} \deg A_i + \alpha_1 n_{j+1} + 1)}
$$

$$
\leq |J(A_1, \ldots, A_{n+j})|^{s-\epsilon},
$$

where we have used Lemma 5.2 in the last equality and the last inequality, (5.3) in the last equality, and (5.3) in the second inequality.

Thus (5.5) holds for all basic sets with order $n \geq n_1$.

5.5. Measure on any disc. Let $x \in E(B, M, \epsilon, \{n_j\})$ and

$$
r_0 := \min_{(A_1, A_2, \ldots, A_n) \in D_{n_1}} |J(A_1, \ldots, A_n)|.
$$

Then there exist polynomials $A_1, A_2, \ldots$ such that $(A_1, A_2, \ldots, A_i) \in D_i$ and $x \in J(A_1, \ldots, A_i)$ for any $i \geq 1$. Let $B(x, r)$ be the disc with the centre $x$ and radius $r$.

Now we prove that there exists a constant $c$ such that for any $r < r_0$,

$$
\mu(B(x, r)) \leq cr^{s-\epsilon}.
$$

By the definition of $r_0$, there exists a unique $n \geq n_1$ such that

$$
|J(A_1, A_2, \ldots, A_{n+1})| \leq r < |J(A_1, A_2, \ldots, A_n)|.
$$

Since $r < |J(A_1, \ldots, A_n)| \leq |I(A_1, \ldots, A_n)|$ and $I(A_1, \ldots, A_n)$ is also a disc, we have

$$
B(x, r) \subset I(A_1, \ldots, A_n).
$$

We divide the proof into three cases.

Case I. We consider $n_j \leq n \leq n_j + k - 1$ for some $j \geq 1$. Write $n = n_j + i - 1$ for some $1 \leq i \leq k$. If $B(x, r)$ intersects only one basic set of order $n+1$, then

$$
\mu(B(x, r)) \leq \mu(J(A_1, A_2, \ldots, A_{n+1})) \leq |J(A_1, A_2, \ldots, A_{n+1})|^{s-\epsilon} \leq r^{s-\epsilon}.
$$

Now we consider the case that $B(x, r)$ intersects at least two basic sets of order $n+1$. By (5.7), the basic sets of order $n+1$ which intersect $B(x, r)$ are all subsets of $J(A_1, \ldots, A_n)$. So there exists $A_{n+1} \neq A_{n+1}$ such that

$$
B(x, r) \cap J(A_1, \ldots, A_n, A_{n+1}) \neq \emptyset.
$$

Note that

$$
I(A_1, \ldots, A_n, A_{n+1}) \cap J(A_1, \ldots, A_n, A_{n+1}) = \emptyset.
$$

Since $| \cdot |_\infty$ is non-Archimedean, any two discs are either disjoint or one is contained in the other. Both discs $B(x, r)$ and $I(A_1, A_2, \ldots, A_{n+1})$ intersect $J(A_1, A_2, \ldots, A_{n+1})$, so we have

$$
I(A_1, A_2, \ldots, A_{n+1}) \subset B(x, r)
$$

by (5.8) and (5.9). Thus

$$
r \geq |I(A_1, A_2, \ldots, A_{n+1})|.
$$
For any \((A_1, \ldots, A_n, A_{n+1}^*) \in D_{n+1}\), we have
\[
\nu(I(A_1, \ldots, A_{n+1})) = \nu(I(A_1, \ldots, A_n, A_{n+1}^*)) = q^{-2(\sum_{m=1}^{n} \deg A_m + [n_j, n_j]+1)}
\]  
(5.11)
and
\[
|I(A_1, A_2, \ldots, A_{n+1})| = |I(A_1, A_2, \ldots, A_n, A_{n+1}^*)|
\]  
(5.12)
since \(\deg A_{n+1} = \deg A_{n+1}^* = [n_j, n_j] + 1\). Thus if
\[
B(x, r) \cap I(A_1, A_2, \ldots, A_n, A_{n+1}^*) \neq \emptyset,
\]
we have
\[
I(A_1, A_2, \ldots, A_n, A_{n+1}^*) \subset B(x, r)
\]  
(5.13)
by (5.10) and (5.12). Let \(N_r\) be the number of basic sets of order \(n+1\) which intersect \(B(x, r)\). Combining (5.11) and (5.13), we get
\[
N_r q^{-2(\sum_{m=1}^{n} \deg A_m + [n_j, n_j]+1)} \leq \nu(B(x, r)) \leq qr.
\]
In the second inequality, we use the fact that
\[
\nu(B(x, r)) = \nu(B(x, q^{m-1})) = q^{-m}
\]
if \(q^{-m-1} \leq r < q^{-m}\) for some \(m \geq 1\). Thus
\[
N_r \leq q^3 r q^{\sum_{m=1}^{n} \deg A_m + [n_j, n_j]}.
\]
and
\[
\mu(B(x, r)) \leq \sum_{A_{n+1}^* : I(A_1, \ldots, A_n, A_{n+1}^*) \cap B(x, r) \neq \emptyset} \mu(J(A_1, \ldots, A_n, A_{n+1}^*))
\]
\[
\leq N_r \mu(J(A_1, \ldots, A_n, A_{n+1}^*))
\]
\[
= N_r \frac{1}{(q-1)q^{[n_j, n_j]+1}} \mu(J(A_1, \ldots, A_n))
\]
\[
\leq q^3 r q^{\sum_{m=1}^{n} \deg A_m + [n_j, n_j]} \mu(J(A_1, \ldots, A_n)).
\]
From (5.7), it follows that
\[
\mu(B(x, r)) \leq \mu(I(A_1, \ldots, A_n)) = \mu(J(A_1, \ldots, A_n)).
\]
Thus
\[
\mu(B(x, r)) \leq \min\{q^3 r q^{\sum_{m=1}^{n} \deg A_m + [n_j, n_j]}, 1\} \mu(J(A_1, \ldots, A_n))
\]
\[
\leq (q^3 r q^{\sum_{m=1}^{n} \deg A_m + [n_j, n_j]})^{s-\epsilon} \mu(J(A_1, \ldots, A_n))
\]
\[
\leq (q^3 r q^{\sum_{m=1}^{n} \deg A_m + [n_j, n_j]})^{s-\epsilon} |J(A_1, \ldots, A_n)|^{s-\epsilon}
\]
\[
\leq (q^3 r q^{\sum_{m=1}^{n} \deg A_m + [n_j, n_j]})^{s-\epsilon} q^{-(s-\epsilon)(2 \sum_{m=1}^{n} \deg A_m + [n_j, n_j])}
\]
\[
= (q^3 r)^{s-\epsilon} \leq q^{3r^{s-\epsilon}}.
\]
In the second inequality, we use the fact \(\min\{x, y\} \leq x^t y^{1-t}\) for \(x, y > 0\) and \(0 < t < 1\).

**Case II.** We consider \(n_j + k \leq n \leq n_{j+1} - 2\) for some \(j \geq 1\). Since \(B(x, r) \subset I(A_1, \ldots, A_n)\), we have
\[
\mu(B(x, r)) \leq \mu(I(A_1, \ldots, A_n))
\]
\[
= \mu(J(A_1, \ldots, A_n))
\]
\[
\leq |J(A_1, \ldots, A_n)|^{s-\epsilon}.
\]
By Lemma 5.2, we have
\[
|J(A_1, \ldots, A_n)| = q^{-2 \sum_{m=1}^{n} \deg A_i - 1}
\]
and
\[ |J(A_1, \ldots, A_{n+1})| = q^{-2 \sum_{i=1}^{n+1} \deg A_i - 1}. \]

Since \( \deg A_{n+1} \leq M \), we have
\[ |J(A_1, \ldots, A_n)| \leq q^{2M} |J(A_1, \ldots, A_{n+1})| \]
and hence
\[ \mu(B(x, r)) \leq q^{2M} |J(A_1, \ldots, A_{n+1})|^{s-\epsilon} \leq q^{2M} r^{s-\epsilon}. \]

**Case III.** In this last case, we consider \( n = n_{j+1} - 1 \) for some \( j \geq 1 \). If \( B(x, r) \) intersects only one basic set of order \( n+1 \), we have
\[ r \geq |I(A_1, \ldots, A_{n+1})| \]
by the same arguments as in the proof of Case I. Since \( n = n_{j+1} - 1 \), we have \( 1 \leq \deg A_{n+1} = \deg A_{n_{j+1}} \leq M \). It follows that
\[
\begin{align*}
r &\geq |I(A_1, \ldots, A_{n+1})| \\
&= q^{-2 \sum_{i=1}^{n+1} \deg A_i - 1} \\
&\geq q^{-2M} q^{-2 \sum_{i=1}^{n} \deg A_i - 1} \\
&= q^{-2M} |I(A_1, \ldots, A_n)| \\
&\geq q^{-2M} |J(A_1, \ldots, A_n)|.
\end{align*}
\]
Thus by (5.7)
\[ \mu(B(x, r)) \leq \mu(I(A_1, \ldots, A_n)) \]
\[ = \mu(J(A_1, \ldots, A_n)) \]
\[ \leq |J(A_1, \ldots, A_n)|^{s-\epsilon} \]
\[ \leq q^{2M} r^{s-\epsilon}. \]

Thus the inequality (5.6) always holds. By Lemma 5.1 we have
\[ \dim_{H} E(B, M, \epsilon, \{n_j\}) \geq s_{k,M}(B) - \epsilon. \]  \hspace{1cm} (5.14)

**Proof of Theorem 4.1** Since \( E(B, M, \epsilon, \{n_j\}) \subset \mathcal{F}_k(B) \) and \( \dim_{H} E(B, M, \epsilon, \{n_j\}) \geq s_{k,M}(B) - \epsilon \), we have
\[ \dim_{H} \mathcal{F}_k(B) \geq s_{k,M}(B) - \epsilon. \]
Let \( \epsilon \to 0 \) and \( M \to \infty \). It follows that
\[ \dim_{H} \mathcal{F}_k(B) \geq s_k(B) \]
and hence, by combining it with the upper bound estimate (Lemma 4.2), we have
\[ \dim_{H} \mathcal{F}_k(B) = s_k(B). \]
\[ \Box \]
6. Proof of Theorem 1.2

When $k = 1$, the conclusion follows from [12, Theorem 2.4]. Therefore, we assume $k \geq 2$ and split the proof into three cases.

**Case 1.** We first handle the case $B = 0$. Then

$$\mathcal{F}_k(\Phi) \supseteq \mathcal{F}_1(\Phi).$$

The conclusion follows since $\dim_H \mathcal{F}_1(\Phi) = 1$.

**Case 2.** Let $0 < B < \infty$. Then for any $\epsilon > 0$, we have

$$\mathcal{F}_k(\Phi) \subseteq \mathcal{F}_k(B - \epsilon) = \left\{ x \in I : \sum_{i=1}^{k} \deg A_{n+i}(x) \geq (B - \epsilon)n \text{ for infinitely many } n \in \mathbb{N} \right\}.$$

Thus $\dim_H \mathcal{F}_k(\Phi) \leq s_k(B - \epsilon)$. Letting $\epsilon \to 0$, we have $\dim_H \mathcal{F}_k(\Phi) \leq s_k(B)$. Now we prove the reverse inequality. For any $\delta > 0$, $M \geq 2$ and $\epsilon > 0$, there exists a sequence $\{n_j\}$ such that

$$\lim_{j \to \infty} \frac{\Phi(n_j)}{n_j} = \liminf_{n \to \infty} \frac{\Phi(n)}{n} = B$$

and (5.4) holds with $s = s_{k,M}(B + \delta)$, where $\alpha_i$ is defined by

$$s \alpha_i = (1 - s) \alpha_{i-1}, \sum_{i=1}^{k} \alpha_i = B + \delta.$$

We define $E(B + \delta, M, \epsilon, \{n_j\})$ as in Section 5.1 by replacing $B$ with $B + \delta$. Then

$$E(B + \delta, M, \epsilon, \{n_j\}) \subseteq \mathcal{F}_k(\Phi)$$

and

$$\dim_H E(B + \delta, M, \epsilon, \{n_j\}) \geq s_{k,M}(B + \delta) - \epsilon$$

by (5.4). It follows that

$$\dim_H \mathcal{F}_k(\Phi) \geq s_{k,M}(B + \delta) - \epsilon.$$

Letting $\epsilon \to 0$, $M \to \infty$ and $\delta \to 0$, we get $\dim_H \mathcal{F}_k(\Phi) \geq s_k(B)$ by Lemma 2.8. So the conclusion follows in this case.

**Case 3.** For the final case, let $B = \infty$. If $\deg A_{n+1}(x) + \ldots + \deg A_{n+k}(x) \geq \Phi(n)$ then there exists some $1 \leq i \leq k$ such that

$$\deg A_{n+i}(x) \geq \frac{1}{k} \Phi(n).$$

Thus

$$\mathcal{F}_1(\Phi) \subseteq \mathcal{F}_k(\Phi) \subseteq \bigcup_{i=1}^{k} \mathcal{F}_1(\psi_i),$$

where

$$\psi_i(n) = \frac{1}{k} \Phi(n + 1 - i)$$

for all $1 \leq i \leq k$. Since

$$\liminf_{n \to \infty} \frac{\psi_i(n)}{n} = \liminf_{n \to \infty} \frac{\Phi(n)}{n} = \infty$$

and

$$\liminf_{n \to \infty} \frac{\log \psi_i(n)}{n} = \liminf_{n \to \infty} \frac{\log \Phi(n)}{n}$$

for all $1 \leq i \leq k$, we have $\dim_H \mathcal{F}_1(\psi_i) = \dim_H \mathcal{F}_1(\Phi)$. Then the conclusion follows from [12, Theorem 2.4].

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7. Proof of Theorem 1.3

For any integer $M \geq k$, let

$$G_k(M) = \left\{ x \in I : \sum_{i=1}^{k} \deg A_{n+i}(x) \geq M \text{ for all } n \geq 0 \right\}.$$ 

Since $G_k(M + 1) \subset G_k(M)$, $\lim_{M \to \infty} \dim_H G_k(M)$ exists.

**Lemma 7.1.** \(\lim_{M \to \infty} \dim_H G_k(M) \leq 1/2.\)

**Proof.** If $x \in G_k(M)$, we have $\deg A_{nk+1}(x) + \deg A_{nk+2}(x) + \cdots + \deg A_{nk+1}(x) \geq M$ for all $i \geq 0$. So for any $n \geq 1$,

$$\Gamma_n := \{I(A_1, A_2, \ldots, A_{nk}) : \sum_{j=ik+1}^{(i+1)k} \deg A_j \geq M \text{ for all } 0 \leq i \leq n-1\}$$

is a cover of $G_k(M)$. For any $t > 1/2$, we have

$$\mathcal{H}^t(G_k(M)) \leq \liminf_{n \to \infty} \sum_{I(A_1, \ldots, A_{nk}) \in \Gamma_n} |I(A_1, \ldots, A_{nk})|^t \leq \liminf_{n \to \infty} \sum_{I(A_1, \ldots, A_{nk}) \in \Gamma_n} q^{-2t \sum_{i=1}^{nk} \deg A_i} = \liminf_{n \to \infty} \left( \sum_{I(A_1, \ldots, A_{nk}) \in \Gamma_1} q^{-2t \sum_{i=1}^{nk} \deg A_i} \right)^n.$$

By Lemma 2.5 it follows that

$$\mathcal{H}^t(G_k(M)) \leq \liminf_{n \to \infty} \left( \sum_{m=M}^{\infty} \left( \frac{m-1}{k-1} \right)^n \frac{(q-1)^k q^m q^{-2tm}}{n} \right).$$

Since $t > 1/2$ and $\sum_{m=M}^{\infty} m^k(q-1)^k q^{(1-2t)m} < \infty$, we have $\sum_{m=M}^{\infty} m^k(q-1)^k q^{-2tm} < 1$ for all $M$ large enough. It follows that $\mathcal{H}^t(G_k(M)) = 0$ and $\dim_H G_k(M) \leq t$ for all $M$ large enough. Thus

$$\lim_{M \to \infty} \dim_H G_k(M) \leq t.$$

Since $t > 1/2$ is arbitrary, we get the desired result. \(\square\)

**Proof of Theorem 1.3.** When $k = 1$, it is Theorem 2.3 in [12]. We assume $k \geq 2$ and distinguish three cases.

**Case 1.** $a = 1$. For any $M \geq k$, let $N \geq 1$ be the smallest integer such that $\Phi(n) \geq M$ for all $n \geq N$. Then

$$G_k(\Phi) \subseteq \{ x \in I : \deg A_{n+1}(x) + \cdots + \deg A_{n+k}(x) \geq M, n \geq N \}.$$

For any $N$th cylinder $I(A_1, \ldots, A_N)$, let

$$f : G_k(M) \to I(A_1, \ldots, A_N) \cap T^{-N}(G_k(M))$$

be defined by

$$f(x) = [A_1, \ldots, A_{N-1}, A_N + x].$$
Then we have

\[ f(x) = \frac{P_N + xP_{N-1}}{Q_N + xQ_{N-1}} \]

with \( P_N/Q_N = [A_1, A_2, \ldots, A_N] \) and hence

\[ |f(x) - f(y)|_\infty = \frac{|x - y|_\infty}{|Q_N|_\infty}. \]

Demonstrating that \( f \) is a bi-Lipschitz map. Note that

\[ \{ x \in I : \deg A_{n+1}(x) + \cdots + \deg A_{n+k}(x) \geq M, n \geq N \} \]

\[ = \bigcup_{\deg A_1 \geq 1, \ldots, \deg A_N \geq 1} I(A_1, \ldots, A_N) \cap T^{-N} G_k(M). \]

Since Hausdorff dimension is countably stable and invariant under a bi-Lipschitz map, we have

\[ \dim_H \{ x \in I : \deg A_{n+1}(x) + \cdots + \deg A_{n+k}(x) \geq M, n \geq N \} = \dim_H G_k(M). \]

So

\[ \dim_H G_k(\Phi) \leq \dim_H G_k(M) \]

for any \( M \geq k \). By Lemma [7.1] it follows that

\[ \dim_H G_k(\Phi) \leq 1/2. \]

For the lower bound, note that

\[ G_1(\Phi) \subseteq G_k(\Phi). \]

Since \( \dim_H G_1(\Phi) = 1/2 \) in this case, we have \( \dim_H G_k(\Phi) \geq 1/2 \).

**Case 2.** \( 1 < a < \infty \). Since

\[ G_1(\Phi) \subseteq G_k(\Phi), \]

we have

\[ \dim_H G_k(\Phi) \geq \dim_H G_1(\Phi) = \frac{1}{1 + a}. \]

For any \( \epsilon > 0 \),

\[ G_k(\Phi) \subseteq F_k(\Psi) \]

with \( \Psi(n) = (a - \epsilon)^n \). By Theorem [1.2] we have

\[ \dim_H G_k(\Phi) \leq \dim_H F_k(\Psi) = \frac{1}{1 + a - \epsilon}. \]

Letting \( \epsilon \to 0 \), the conclusion follows in this case.

**Case 3.** \( a = \infty \). Then for any \( M > 1 \), we have

\[ G_k(\Phi) \subseteq F_k(\Psi) \]

with \( \Psi(n) = M^n \). By Theorem [1.2] it follows that

\[ \dim_H G_k(\Phi) \leq F_k(\Psi) = \frac{1}{1 + M}. \]

Letting \( M \to \infty \), we get the conclusion. \( \square \)
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