On the validity of the O(18) coupling scheme in \( N \sim Z \) nuclei

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Abstract

Wave-functions from the newly proposed O(18) coupling scheme in IBM-3 are compared with those from a numerical calculation fitted to some Zn, Ge and Se nuclei.

The interacting boson model [1], in which nucleon pairs are approximated by \( s \)- and \( d \)-bosons has enjoyed some success in describing nuclear systematics and, in lighter nuclei, a version incorporating isospin symmetry, IBM-3 [2], has been used. Each boson in IBM-3 has isospin \( T = 1 \) giving it a total of 18 states so that the states of \( N \) bosons belong, trivially, to the totally symmetric representation \([N]\) of the group U(18). The existence of any chain of subgroups of U(18) which contains the group O(3) \( \times \) SU(2), corresponding to the exact symmetries of angular momentum and isospin, provides a coupling scheme with analytic properties. The most natural chain below U(18) starts with U(6) \( \times \) SU\(_c\)(3) which separates the three charge states \( M_T = 1, 0, -1 \) of the boson from the six “orbital” states of \( s \) and \( d \). The isospin group SU(2) is then a sub-group of SU\(_c\)(3) and, below U(6) are the three alternative chains via U(5), O(6) or SU(3) which are familiar from the earliest days of the IBM [1] and describe vibrational, \( \gamma \)-unstable and rotational nuclei respectively. It is found that the U(6) \( \times \) SU\(_c\)(3) chain has validity, with the low states in numerical IBM-3 calculations [3,4] containing large amounts of the totally symmetric states \([N]\) of each of the two factors SU\(_c\)(3) and U(6). For each value of \( T = N, N - 2, \ldots, 1 \) or 0 allowed from SU\(_c\)(3) the set of orbital states is then identical to those in IBM-1.

Recently two new chains starting with the subgroup O(18) of U(18) have been discovered [5]. One of them proceeds via O(6) \( \times \) SU(2) followed by the familiar reduction from the \( \gamma \)-unstable group O(6). The other goes first to O(15) \( \times \) SU\(_s\)(2), in which O(15) belongs entirely to the \( d \)-boson and SU\(_s\)(2) is the isospin of the \( s \)-bosons. Then O(15) is reduced to O(5) \( \times \) SU\(_d\)(2) and finally SU(2) is reached from the product SU\(_s\)(2) \( \times \) SU\(_d\)(2), a vector-coupling of the separate isospins of \( s \) and \( d \)-bosons.

Since the new O(18) classifications generally break the U(6) symmetry, which is known to be reasonably good, it is important to test their validity in numerical
In this note we report an analysis, in the $O(18) \supset O(15)$ basis, of some recent numerical IBM-3 wave-functions [4]. But first we illustrate the new classifications in the simplest non-trivial example of $N = 3, T = 1$. We argue that one of the new chains, $O(6) \times SU(2)$, seems to be of little interest since it is identical to the old $U(6) \times SU_c(3)$ classification for most of the low states.

In the three sections of table 1 we compare the classifications for the case $N = 3, T = 1$ using the old chain $U(6) \times SU_c(3) \supset O(6) \times SU_c(3)$ and the two new chains $O(18) \supset O(6) \times SU(2)$ and $O(18) \supset O(15)$. (For convenience we refer to these three chains as A, B and C respectively.) All three chains have the $O(5)$ group in common so it is unnecessary to specify the $J$-values which belong to each $O(5)$ representation, according to familiar rules [1]. (For example, $(00) J = 0$, $(10) J = 2$, $(20) J = 2, 4$, $(30) J = 0, 3, 4, 6$.) It is sufficient to use single numbers $\alpha$ and $\delta$, as in ref.[5], for the $O(18)$ and $O(15)$ labels since only the totally symmetric representations occur. In each section of the table there must be exactly the same set of $O(5)$ labels $(\tau_1, \tau_2)$ but they are grouped differently in the different chains.

The states expected to be most prominent at lowest energies are given in the first rows of each section. Notice first that the chains A and B have the group $O(6)$ in common and that, since the representations $(300)$ and $(210)$ occur only once, those states must be identical in the two chains. In other words the $(300)$ states in chain B are identical with those in chain A and have full $U(6)$ symmetry [3] while the $(210)$ states in chain B are identical with the mixed symmetry states [21] of chain A. In general the $(100)$ states in B will be mixtures of [3] and [21] but these states are expected to lie high in energy. We conclude that chain B does not have much new interest.

Chain C is more interesting since it has nothing in common with the familiar chain A beyond the $O(5)$ label. For example, the lowest $0^+$ state, with $\alpha = 3$, $\delta = 0$, $(\tau_1, \tau_2) = (00)$ may be expanded in the chain A basis as

$$-\sqrt{\frac{3}{4}}|s^3[3]\rangle + \sqrt{\frac{5}{36}}|sd^2[3]\rangle + \sqrt{\frac{1}{9}}|sd^2[21]\rangle$$

(1)

which, although containing the expected dominance of the $s$-boson and of the full $U(6)$ symmetry [3], nevertheless contains a significant component of $d$-bosons and of the “mixed” symmetry [21]. The situation for the lowest $2^+$ state is more complex because, as shown in the table, there are two states with the same $O(18)$, $O(15)$ and $O(5)$ labels 3, 1 and (10), corresponding to $T_s = 0$ and 2. Their expansions are

$$|T_s = 0\rangle = \sqrt{\frac{17}{36}}|s^2d[3]\rangle + \sqrt{\frac{17}{45}}|s^2d[21]\rangle - \sqrt{\frac{7}{68}}|d^3[3]\rangle - \sqrt{\frac{4}{85}}|d^3[21]\rangle$$

$$|T_s = 2\rangle = \sqrt{\frac{4}{9}}|s^2d[3]\rangle - \sqrt{\frac{5}{9}}|s^2d[21]\rangle$$

(2)

Each of these states shows very strong mixing of the $U(6)$ labels [3] and [21] which would be contrary to experience for the lowest $2^+$ state but a simple calculation
Table 1
The classification of states for $N = 3$, $T = 1$ using the three group chains A) $U(6) \times SU_c(3) \supset O(6) \times SU_c(3)$, B) $O(18) \supset O(6) \times SU(2)$ and C) $O(18) \supset O(15)$. The superscript 2 which appears in chain C indicates that there are two states with the given labels, corresponding to different values of $T_s$ and $T_d$.

| Chain A | $SU_c(3) \equiv U(6)$ | $O(6)$ | $O(5)$ |
|---------|-----------------------|-------|-------|
| [3]     | (300)                 | (00)  | (10)  | (20)  | (30)  |
|         | (100)                 | (00)  | (10)  |
| [21]    | (210)                 | (10)  | (11)  | (20)  | (21)  |
|         | (100)                 | (00)  | (10)  |

| Chain B | $O(18)$ | $O(6)$ | $O(5)$ |
|---------|---------|-------|-------|
| $\alpha$ | $(300)$ | (00)  | (10)  | (20)  | (30)  |
|          | (210)   | (10)  | (11)  | (20)  | (21)  |
|          | (100)   | (00)  | (10)  |
| $\alpha$ | $(100)$ | (00)  | (10)  |

| Chain C | $O(18)$ | $O(15)$ | $O(5)$ |
|---------|---------|--------|-------|
| $\alpha$ | $\delta$ |
| 3       | 0       | (00)   |
| 1       | (10)$^2$|
| 2       | (00)    | (11)  | (20)$^2$|
| 3       | (10)    | (21)  | (30)  |
| 1       | 0       | (00)   |
| 1       | (10)    |

shows that the percentage of the full symmetry [3] can be increased up to a maximum of 97% by suitable combination of the two states (2). Thus, although we would not expect $T_s$ to be a good quantum number, the $O(18)$ and $O(15)$ labels could still be good.

In a recent paper, ref.[4], we deduced an $NT$-dependent IBM-3 hamiltonian from a shell-model mapping and applied it to the Ni, Zn, Ge and Se isotopes in the first half of the $p_{3/2}$, $f_{5/2}$, $p_{1/2}$ shell. In particular we saw that the low states were dominated by the full-symmetry $U(6)$ label $[N]$ with the mixed symmetry states $[N - 1, 1]$ coming in at about 3 MeV. It is therefore of interest to analyse our wave-functions in the new group chain C since the examples above suggest that this chain is not incompatible with a coupling scheme close to $U(6)$. Table 2 shows the results of such an analysis for the low states of each spin for the nuclei considered. The first
Table 2
A comparison of the wave-function percentages in the favoured representations of U(6), first row, and of the new chain C, O(18) ⊃ O(15), in the second row for each nucleus. The favoured representations in chain C are \( \alpha = N \) and \( \delta = 0, 1, 2 \) (3 if \( T = 0 \)) and 2 for \( J = 0, 2, 3 \) and 4 respectively.

| N  | T | 0^+_1 | 0^+_2 | 2_1^+ | 2_2^+ | 2_3^+ | 3_1^+ | 3_2^+ | 4_1^+ | 4_2^+ | 4_3^+ |
|----|---|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| 62Zn | 3 | 1 | 94 | 91 | 96 | 50 | 27 | 13 | 23 | 88 | 27 | 17 |
|     |   |   | 99 | 0 | 93 | 36 | 61 | 87 | 3 | 90 | 45 | 54 |
| 64Zn | 4 | 2 | 93 | 59 | 94 | 79 | 10 | 44 | 22 | 90 | 70 | 5 |
|     |   |   | 96 | 0 | 97 | 9 | 89 | 48 | 46 | 96 | 2 | 44 |
| 64Ge | 4 | 0 | 94 | 89 | 96 | 83 | 19 | 89 | 9 | 95 | 69 | 18 |
|     |   |   | 95 | 0 | 93 | 3 | 3 | 96 | 99 | 90 | 18 | 65 |
| 66Ge | 5 | 1 | 91 | 73 | 93 | 89 | 3 | 51 | 30 | 94 | 83 | 4 |
|     |   |   | 92 | 0 | 96 | 2 | 92 | 37 | 61 | 97 | 3 | 95 |
| 68Se | 6 | 0 | 91 | 80 | 93 | 91 | 90 | 87 | 10 | 94 | 90 | 86 |
|     |   |   | 84 | 0 | 82 | 9 | 0 | 97 | 99 | 84 | 10 | 2 |

row of numbers for each nucleus gives the percentage lying in the fully symmetric U(6) representation \([N]\) and is copied from table 3 of ref.[4]. The second row gives the percentage in the favoured state of chain C, viz. \( \alpha = N \), with the smallest \( \delta \) compatible with each angular momentum, i.e. \( \delta = 0, 1, 2, \) and 2 for \( J = 0, 2, 3 \) and 4, respectively except for \( J = 3, T = 0 \) when \( \delta = 3 \).

It is immediately clear from the table that both chains give good descriptions of the lowest states for \( J = 0, 2 \) and 4 but it is important to look at the comparison in more detail because in most cases the numbers in the table are a total percentage summed over several independent states with the same representation labels while, in other cases, the numbers refer to unique states. In \(^{62}\text{Zn}\) for example, three of the four \( 0^+ \) states lie within the [3] representation of U(6) whereas the figure of 99% refers to a unique state with \( \alpha = 3, \delta = 0 \) in the O(18) ⊃ O(15) chain. In the same nucleus there is a total of seven \( 2^+ \) states of which three lie within the U(6) label [3] and only two within \( \alpha = 3, \delta = 1 \). The new chain is therefore more specific than U(6) in identifying the lowest states and table 2 shows that it also contains a slightly greater percentage, compared with U(6), in the numerical wave-functions of the calculation [4].

The position is less clear for non-yrast states. The zero entries for the \( 0^+_2 \) states (in fact non-zero but very small) are largely a consequence of there being only one state with \( \alpha = N, \delta = 0 \), with most of it being taken up in the \( 0^+_1 \) state. When we look at the content of the \( 0^+_2 \) state, we find that, in \(^{64}\text{Zn}\) and \(^{66}\text{Ge}\) it is well-described by a single O(18) ⊃ O(15) label, about 98% \( \alpha = N, \delta = 3 \), whereas in \(^{62}\text{Zn}\) and \(^{64}\text{Ge}\) there is strong mixing between \( \delta = 3 \) and \( \delta = 2 \), both with \( \alpha = N \). For \( 2^+ \) states, there are two independent states with the favoured O(18) ⊃ O(15) labels of \( \alpha = N, \delta = 1 \) as shown in table 1, and table 2 shows that the second state of this kind is
generally the $2^+_2$. The nuclei with $T = 0$ are exceptional, as in this case there is only one state with the favoured labels, see ref.[5]. It is notable in table 2 that, in all five nuclei, these favoured states are concentrated more than 90% in the lowest three calculated levels, whereas the total number of $2^+$ states in the model ranges from 7 in $^{62}$Zn to 40 in $^{68}$Se. The $2^+_2$ state is mainly described by $\alpha = N$, $\delta = 2$ for which the percentages are 57, 89, 91, 95 and 85% respectively for the nuclei in the table. As discussed earlier in this letter, the two states with $\alpha = N$, $\delta = 1$ correspond roughly to full U(6) symmetry [N] and mixed symmetry [N - 1, 1] and table 2 shows that the mixed symmetry lies mostly in $2^+_2$ but with a larger component in $2^+_2$ for $^{62}$Zn. In the two $T = 0$ nuclei, both $3^+$ states are extremely well described by $\alpha = N$, $\delta = 3$. This is possible because these states contain a large mixed symmetry component, and it is clear from table 2 that mixed symmetry dominates these two states. For $T > 0$, there is an additional mixed symmetry state corresponding to $\alpha = N$, $\delta = 2$ which, from table 2, accounts for 87% of the $3^+_1$ in $^{62}$Zn, while the $3^+_2$ is 97% $\alpha = N$, $\delta = 3$. The $3^+$ states in both $^{64}$Zn and $^{66}$Ge are more strongly mixed between $\delta = 2$ and $\delta = 3$.

In this analysis of wave-functions from a numerical IBM-3 calculation, based on a mapping from the shell model, the new O(18) ⊃ O(15) coupling scheme provides at least as good a first approximation, using largest O(18) label $\alpha = N$ and smallest O(15) label $\delta$, as the more familiar U(6) scheme. Assuming that this result is not a specific property of the rather simple shell model interaction used in ref.[4], we conclude that it may be possible to use the new group chain to simplify IBM-3 calculations either through a truncation of states or by reducing the number of essential parameters in the hamiltonian. Some selection rules have already been given in ref.[5]. The essential physics of the O(18) ⊃ O(15) chain lies in the two-boson invariants, or pair states, for the two groups, which are $(s^2 - \sqrt{5}d^2) (J = T = 0)$ and $d^2 (J = T = 0)$. This contrasts with the more familiar O(6) and O(5) groups which, in IBM-3, would also contain the corresponding pair states with $T = 2$. Thus, the new chain discriminates between $T = 0$ and $T = 2$. The new chain could be said to favour the $\gamma$-unstable O(6) scheme for $T = 0$ pairs and the U(5) scheme for $T = 2$ pairs. It must be remembered that, as for O(6), high seniority is lowest in energy for O(18), so that the O(18) pair state is unfavoured.

Finally, we comment that the general idea behind the new group chain [5] is not specific to IBM-3 and that a corresponding new group chain O(12) ⊃ O(10) exists in IBM-2 and would again imply mixing of U(6) which in IBM-2 implies mixing of F-spin. In fact, the corresponding table 1 for the IBM-2 example of $N_\nu = 2$, $N_\pi = 1$ would be identical to that given here with the headings SU$_c$(3) replaced by SU$F$(2), the F-spin group, O(18) replaced by O(12) and O(15) by O(10). In this case the new chain has pair states $(s^2 - \sqrt{5}d^2) (J = 0)$ and $d^2 (J = 0)$ for the $\nu-\pi$ system but not for $\nu-\nu$ or $\pi-\pi$, which is analogous with the IBM-3 argument given above.

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