Thermodynamic phase transition and Joule Thomson adiabatic expansion for dS/AdS Bardeen Black Holes with consistent 4D Gauss-Bonnet gravity

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Abstract

Instead of the work \cite{1} which in according to the Lovelock theorem it could not applicable for all types of 4D curved spacetimes of Einstein Gauss Bonnet (EGB) gravity, authors of the work \cite{2} applied break of diffeomorphism property to present a consistent EGB gravity theory. In this work we use the latter model by adding a nonlinear electromagnetic field lagrangian density to study affects of GB coupling constant into the thermodynamic phase transition and Joule Thomson (JT) adiabatic expansion of a 4D dS/AdS GB Bardeen black hole.

1 Introduction

From the theoretical point view we know the black holes are made from metric solutions of the Einstein’s metric field equation with no temperature and so they are not supposed to show any thermodynamic behavior. For the first time, Hawking presented an important theorem where the event horizon of the black holes should never be decreased because all objects are absorbed by them \cite{3}. This is called now as the Hawking’s area theorem. After this presentation Bekenstein suggested that for the black hole should be assigned an entropy appropriate to the area of its horizon\cite{4}. In analogy with the thermodynamics rules of the ordinary systems, four laws proposed for the black holes thermodynamics. But by considering this analogy, there was obtained a problem for thermodynamics of the black holes.

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as follows. Actually the first law of the thermodynamics of the black holes lacks the pressure and volume components. Because there is no a clear concept for thermodynamic volume and pressure of the black hole. The first idea to solve the pressure problem led to the consideration of a negative cosmological constant \cite{5, 6, 7, 8, 9} which it is called conjugate variable for the thermodynamic volume. There are done a lot of research where the pressure-volume (PV) criticality of thermodynamics of AdS black holes mimic thermodynamic behavior of the well known Van der Waals ordinary gases\cite{10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28}. Recently Glavan and Lin released a paper \cite{29} in which an alternative generally covariant gravity theory is defined which in $4D$ curved space-times can propagates just massless gravitons by bypassing the Lovelock's theorem. This alternative higher order derivatives gravity theory has two correction terms called as Gauss-Bonnet topological invariant and cosmological constant respectively. In $4D$ curved spacetimes the Gauss-Bonnet coupling constant parameter diverges to an infinite value. In this singular limit the Gauss-Bonnet topological invariant term gives rise non-trivial contributions to the gravitational dynamics, while preserving the number of degrees of freedom of graviton and being free from Ostrogradsky instability. They reported some appealing corrections to the dispersion relation of cosmological tensor-scalar modes in the cosmological space times and also singularity resolution in the spherically symmetric space times. As an spherically symmetric static black hole metric solutions of this model, authors of the work \cite{30} obtained a Bardeen type of the black hole solutions and generated its thermodynamic variables via the horizon calculation. They obtained a critical location for the black hole horizon where the corresponding Hawking temperature raises to a maximum value for which a second-order phase transition is happened because the heat capacity diverges to infinity. Existence of these appealing thermodynamic behavior encourages us to study Joule-Thomson adiabatic free expansion phenomena for this black hole given in the AdS background.

If we want to describe this expansion briefly, this is down as follows. In fact this expansion is happened when a gas is allowed to move from a high pressure region to a low pressure one without to change its enthalpy. As it established in the above, the black hole mass would be taken as enthalpy in an extended thermodynamic phase space, so during the Joule Thomson expansion phenomena the mass remains constant (isentropic process). In presence of this expansion, the black hole usually could reach to one of two the heating or cooling phases finally. After to present the pioneer works about the black hole
Joule Thomson expansion given by [31, 32, 33], many other scientists investigated this phenomena for several black holes interacting with many types of the material fields [34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49]. To study this phenomena one can usually investigate the inversion curves which mimics behavior of the van der Waals fluid. In general there are several AdS black holes which behave as different with respect to the van der Waals fluid and they do not mimic completely with the inversion curves (see for instance [30, 31]). In this work, we want to examine the possibility of the emergence of the expansion phenomenon of Joule Thomson for dS/AdS GB Bardeen black hole. In fact we will see importance of parameters of this black hole namely the magnetic charge and the GB coupling constant into its heating-cooling phase transition. The paper is organized as follows.

In the second section we define briefly the consistent 4D GB gravity model given by [2] and as an application we consider the Bardeen black hole nonlinear electromagnetic field lagrangian density. In the third section we derive metric field equations of dS/AdS GB Bardeen black hole and solve them by using some physical arguments. In fourth section we calculate the horizon equation, the Hawking temperature, the equation of state and the Joule Thomson coefficient for different regimes of the 4D GB dS/AdS black hole mass density function. Then by plotting P-V diagrams at constant temperature and also the isenthalpic T-P curves we study possibility of the phase transitions and JT expansion of the black hole under consideration. Fifth section is dedicated to conclusion and outlook of the work.

2 Consistent 4D EGB gravity

By according to the work [2] we define consistent EGB gravity in $D \to 4$ limit with the first term of the lagrangian density in the following action functional in which the second term of the lagrangian density $\mathcal{L}_{\text{matter}}$ denotes to source part.

$$I = \frac{1}{16\pi G} \int dtd^3x N \sqrt{\gamma} \left( \mathcal{L}_{\text{EGB}}^{4D} + \mathcal{L}_{\text{matter}} \right), \quad (2.1)$$

where

$$\mathcal{L}_{\text{EGB}}^{4D} = 2R - 2\Lambda - \mathcal{M} \quad (2.2)$$

$$+ \frac{\tilde{\alpha}}{2} \left[ 8R^2 - 4R\mathcal{M} - \mathcal{M}^2 - \frac{8}{3} \left( 8R_{ij} R^{ij} - 4R_{ij} \mathcal{M}^{ij} - \mathcal{M}_{ij} \mathcal{M}^{ij} \right) \right],$$
and $G$ is the Newton’s gravitational coupling constant. $R$ and $R_{ij}$ are the Ricci scalar and the Ricci tensor of the spatial 3-metric $\gamma_{ij}$ respectively. In the definition (2.2) we have

\[ M_{ij} = R_{ij} + \mathcal{K}_k^i \mathcal{K}_j^k - \mathcal{K}_i^k \mathcal{K}_j^k, \quad M = M_i^i \]  

(2.3)

with

\[ \mathcal{K}_{ij} = \frac{1}{2N} (\dot{\gamma}_{ij} - 2D_i N_j - 2D_j N_i - \gamma_{ij} D_k D^k \lambda_{GF}). \]  

(2.4)

Here, dot $\dot{}$ denotes derivative with respect to the time $t$ and all the effects of the constraint stemming from the gauge fixing (GF) are now encoded in lagrange multiplier $\lambda_{GF}$. $D_i$ is spatial covariant derivative and re-scaled regular EGB coupling constant $\alpha$ is defined versus the irregular GB coupling constant $\alpha_{GB}$ such that $\alpha = (D - 4)\alpha_{GB}$ which in limits of $D \to 4$ dimensions become finite. The above EGB gravity action functional satisfies the following gauge condition for all spherically symmetric and cosmological backgrounds (see [2] and [50]).

\[ \sqrt{\gamma} D_k D^k \left( \pi_{ij} \gamma_{ij} / \sqrt{\gamma} \right) \approx 0. \]  

(2.5)

In fact the above EGB action functional is generated from ADM decomposition of the 4D background metric as $1 + 3$ dimensions such that

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -N^2 dt^2 + \gamma_{ij} (dx^i + N^i dt)(dx^j + N^j dt) \]  

(2.6)

where $N, N_i, \gamma_{ij}$ are the lapse function, the shift vector, and the spatial metric respectively. $\gamma$ factor in the action functional (2.1) is absolute value of determinant of the spatial 3-metric $\gamma_{ij}$. This ADM decomposition is done on the background metric to remove divergent boundary term of the higher order metric derivatives in the GB term of the action functional (2.1) in general 4 dimensional form [2]. First term in the theory defined by (2.1) has the time re-parametrization symmetry $t \to t = t(t')$. We now set the matter source $I_{\text{matter}}$ to be action of a nonlinear electromagnetic antisymmetric Maxwell field $F_{\mu\nu}$ with Ayon Beato Garcia form of the lagrangian density as follows.

\[ \mathcal{L}_{\text{matter}} = \mathcal{L}_{\text{ABG}}(F) = \frac{12M_{\text{ADM}}}{Q^3} \left( \frac{\sqrt{2Q^2 F}}{1 + \sqrt{2Q^2 F}} \right)^{\frac{5}{2}} \]  

(2.7)

where $F = F_{\mu\nu} F^{\mu\nu}$, $M_{\text{ADM}}$ is ADM mass of the black hole and $Q$ is Bardeen magnetic charge (see eq. (8) in ref. [51]).
By comparing the line element with general form of a spherically symmetric static 4D metric field

\[ ds^2 = -e^{2A(r)} \left( 1 - \frac{2M(r)}{r} \right) dt^2 + \frac{dr^2}{1 - \frac{2M(r)}{r}} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \]  

(3.1)

we infer that the lapse function \( N \), and the shift vector \( N_i \) and the spatial metric components \( \gamma_{ij} \) and gauge fixing lagrange multiplier \( \lambda_{GF} \) should be \( r \) dependent so that we can write

\[ N = e^{A(r)} \sqrt{1 - \frac{2M(r)}{r}}, \quad N_{r, \theta, \phi} = 0, \]  

(3.2)

\[ \gamma_{rr} = \frac{1}{1 - \frac{2M(r)}{r}}, \quad \gamma_{\theta\theta} = r^2, \quad \gamma_{\phi\phi} = r^2 \sin^2 \theta, \quad \lambda_{GF} = \lambda_{GF}(r). \]

By substituting (3.1) into (2.2) we obtain

\[ \mathcal{L}^{4D}_{EGB} = R(\gamma_{ij}) - 2\Lambda + 12q^2 + \frac{\tilde{\alpha}}{2} \left[ 3R^2(\gamma) + \frac{88}{3} q^2 R(\gamma) - 272q^4 - 8R_{ij}(\gamma)R^{ij}(\gamma) \right] \]  

(3.3)

in which

\[ q(r) = \frac{e^{-A(r)}}{r^2} \left[ r^2 \lambda_{GF}(r) \right]', \quad K_{ij} = -q \gamma_{ij} \]  

(3.4)

\[ R(\gamma) = -\frac{4M'}{r^2}, \quad R_{ij}(\gamma)R^{ij}(\gamma) = \frac{6}{r^6} (M - rM')^2 \]  

(3.5)

and \( t \) denotes derivative with respect to \( r \). From the Maxwell equations, we can prove that the magnetic field has the form

\[ F_{\theta\phi}(\theta) = Q \sin \theta \]  

(3.6)

which for the metric equation (3.1) the EM lagrangian density become

\[ F = \frac{1}{4} F_{\mu\nu}F^{\mu\nu} = \frac{Q^2}{2r^4} \]  

(3.7)

for which (2.7) leads to the following form

\[ \mathcal{L}_{Bardeen}(r) = \frac{12M_{ADM}}{Q^3} \left( \frac{Q^2}{r^2 + Q^2} \right)^{\frac{3}{2}}. \]  

(3.8)
By adding (3.8) and by substituting (3.3) and (3.5) and by integrating the action functional (2.1) on the 2-sphere $0 \leq \theta \leq \pi$, $0 \leq \varphi \leq 2\pi$ we obtain

$$I = \frac{1}{4G} \int dt \int dr r^2 e^{A(r)} \left\{ -\frac{4M'}{r^2} - 2\Lambda + 12q^2 + \frac{12M_{ADM}}{Q^3} \left( \frac{Q^2}{r^2 + Q^2} \right)^{\frac{5}{2}} (3.9) \right.$$ 

$$- \tilde{\alpha} \left[ \frac{176M'q^2}{3r^2} + 136q^4 + \frac{24M^2}{r^6} - \frac{48MM'}{r^5} \right] \right\}.$$

Euler Lagrange equation for $q$ reads

$$q \left[ 12 - \tilde{\alpha} \left( \frac{176M'}{3r^2} + 136q^2 \right) \right] = 0 \quad (3.10)$$

which has two different solutions as

$$q_1 = 0, \quad q_2 = \frac{\pm 1}{\sqrt{136}} \frac{12}{\tilde{\alpha}} - \frac{176M'}{3r^2}. \quad (3.11)$$

By substituting these two different gauge fixing conditions into the equation (3.4) and the action functional (3.9) we obtain

$$\lambda^{(1)}_{GF}(r) \sim \frac{1}{r} \quad (3.12)$$

$$\lambda^{(2)}_{GF}(r) = \int^r \frac{dr'}{r'^2} \int^{r''} r''q_2(r''')e^{A(r''')} dr'' \quad (3.13)$$

and

$$I_1 = I_2 = \frac{1}{4G} \int dt \int dr r^2 e^{A(r)} \left\{ -\frac{4M'}{r^2} - 2\Lambda + \frac{12M_{ADM}}{Q^3} \left( \frac{Q^2}{r^2 + Q^2} \right)^{\frac{5}{2}} (3.14) \right.$$ 

$$+ \tilde{\alpha} \left[ - \frac{24M^2}{r^6} + \frac{48MM'}{r^5} \right] \right\}.$$

This shows that two different gauge fixing conditions $q_{1,2}$ reach to similar action functional $I_1 = I_2$ and so similar metric solutions. The Euler Lagrange equations for the function $A(r)$ and the mass distribution function $M(r)$ reduce to the following relations respectively.

$$\frac{4M'}{r^2} = \frac{12M_{ADM}}{Q^3} \left( \frac{Q^2}{r^2 + Q^2} \right)^{\frac{5}{2}} - 2\Lambda - \frac{24\tilde{\alpha}M^2(r)}{r^6} \quad (3.15)$$
and

$$A'(r) = \frac{-24\tilde{\alpha}M(r)}{r^3} \left(1 - \frac{12\tilde{\alpha}M(r)}{r^4}\right).$$  \hspace{1cm} (3.16)$$

Now we are in position to solve the above nonlinear differential equations. Here we try to obtain analytical solutions for the metric equations which are applicable in studying of the black hole thermodynamic. To do so we pay attention to the equation (3.15) which up to the Gauss Bonnet term $\tilde{\alpha} = 0$ reads to the following solution.

$$M_0(r) = \frac{M_{ADM}r^3}{(r^2 + Q^2)^{3/2}} - \frac{\Lambda r^3}{6},$$  \hspace{1cm} (3.17)

for which

$$A_0 = \text{constant} = 0. \hspace{1cm} (3.18)$$

By substituting $\Lambda = 0$ into the above zero order solution of the mass function we obtain $\lim_{r \to \infty} M_0(r) = M_{ADM}$ and the metric equation (3.1) reads to the original form of the Bardeen black hole. By applying (3.17) one can show that the metric fields reaches to a vacuum (anti) de Sitter metric solution asymptotically at center of the black hole and Schwarzschild (anti) de Sitter form of a black hole asymptotically at far from its central regions such that

$$\lim_{r < < |Q|} \left(1 - \frac{2M_0(r)}{r}\right) \sim \left(1 - \frac{\tilde{\Lambda}}{3} r^2\right), \quad \tilde{\Lambda} = \frac{6M_{ADM}}{Q^3} - \Lambda,$$  \hspace{1cm} (3.19)

and

$$\lim_{r > > |Q|} \left(1 - \frac{2M_0(r)}{r}\right) \sim \left(1 - \frac{2M_{ADM}}{r} + \frac{\Lambda}{3} r^2\right).$$  \hspace{1cm} (3.20)

By looking at the asymptotically solution (3.19) one can infer that the quantity $\frac{6M_{ADM}}{Q^3}$ behaves as alternative cosmological constant to make as nonsingular the central region of the black hole.

However we try to solve the equation (3.15) to obtain an analytic solution for the mass function $M(r)$ as follows. According to the zero order solution (3.17), it is easy to write the equation (3.15) as follows.

$$\frac{4M'}{r^2} = \frac{4M'}{r^2} - \frac{24\tilde{\alpha}M^2}{r^6} \left(1 - \frac{12\tilde{\alpha}M}{r^4}\right).$$  \hspace{1cm} (3.21)
which is equivalent with
\[ r^2 \frac{d}{dr} (M - M_0) = \frac{6\tilde{\alpha} r}{r^2} (M^2 - M_0^2) = \frac{6\tilde{\alpha} d}{dr} \left( \frac{M^2}{r} \right). \] (3.22)

This equation is a nonlinear differential equation for mass function \( M(r) \) and its nonlinearity is generated from the GB parameter \( \tilde{\alpha} \). To study thermodynamics of this black hole metric we need an analytic solution which for nonlinear differential equations it is possible to obtain via perturbation series method for small \( \tilde{\alpha} \). To do so we define
\[ M(r; \tilde{\alpha}) = M_0(r) + \tilde{\alpha} M_1(r) + \tilde{\alpha}^2 M_2(r) + O(3) \] (3.23)
in which we defined
\[ M_0(r) = M(r; 0), \quad M_1(r) = \frac{\partial M}{\partial \tilde{\alpha}} \bigg|_{\tilde{\alpha}=0}, \quad M_2(r) = \frac{\partial^2 M}{\partial \tilde{\alpha}^2} \bigg|_{\tilde{\alpha}=0} \] (3.24)
and so on. By substituting the above series expansion into the equation (3.22) and by keeping coefficients of the GB parameter as different linear differential equations and solving them as step by step we can obtain for up to third order term
\[ M_1(r) = \int \frac{6}{r^2} d \left( \frac{M_0^2(r)}{r} \right) = \frac{6M_0^2(r)}{r^3} + 12 \int \frac{M_0^2(r)}{r^4} dr \] (3.25)
and
\[ M_2(r) = \int \frac{12}{r^2} d \left( \frac{M_0 M_1}{r} \right) = \frac{12M_0 M_1}{r^3} + \int \frac{24M_0 M_1}{r^4} dr. \] (3.26)

For small value of the GB parameter \( \tilde{\alpha} \) the first order solution of the above mass function is enough to seek how can affects the GB parameter on the thermodynamics properties of this kind of black holes. Linear order solution of the mass function reads
\[ M(r) \approx M_0(r) + \tilde{\alpha} M_1(r) \] (3.27)
for which \( M_0(r) \) should be substituted via the equation (3.17) and \( M_1(r) \) is calculated trivially as follows.
\[ M_1(r) = \frac{6M_0^2(r)}{r^3} + \frac{\Lambda^2 r^3}{9} + \frac{4\Lambda M_{ADM} r}{\sqrt{r^2 + Q^2}} \] (3.28)
For the mass distribution solution (3.27) the exterior horizon equation of the 4D GB dS/AdS Bardeen black hole is obtained by solving the equation $g_{rr}(r) = 0$ given by (3.30) as follows.

\[ 1 = x^2 \left( \frac{2m}{(x^2 + 1)^{\frac{3}{2}}} - \frac{\lambda}{3} \right) + \mu \left\{ \frac{3m^2(x^4 + 4x^2 - 1)}{(x^2 + 1)^3} + \frac{5x^2\lambda^2}{9} + \frac{8m\lambda}{(x^2 + 1)^{\frac{3}{2}}} + 3m^2 \arctan \left( \frac{x}{x} \right) - \frac{8m\lambda}{9} \arcsin \left( \frac{x}{x} \right) \right\} \]

in which we defined dimensionless ADM mass \( m \) and dimensionless cosmological parameter \( \lambda \), dimensionless horizon position \( x = \frac{r_+}{Q} \) and dimensionless GB parameter \( \mu \) as follows.

\[ m = \frac{M_{ADM}}{Q}, \quad \lambda = \Lambda Q^2, \quad x = \frac{r_+}{Q}, \quad \mu = \frac{\tilde{\alpha}}{Q^2}. \]

The Hawking temperature of this kind of the black hole is obtained vs the surface gravity such that

\[ T = \frac{-g_{tt}'}{4\pi} \bigg|_{r_+} = \frac{e^{2A(r_+)} - 2M(r_+)}{4\pi} \left( \frac{2M(r_+)}{r_+^2} - \frac{2M'(r_+)}{r_+} \right) \]
which by substituting $r_+ = 2M(r_+)$, the equation $M'(r)$ given by (3.15) and the solution (3.29) reads to the following dimensionless form.

$$t(x) = 4\pi QT = \frac{x^{1+\lambda\mu}}{(x^2 - 6\mu)} \left[ 1 - \frac{3\mu}{x^2} - \frac{6mx^2}{(x^2 + 1)^2} + \lambda x^2 \right]$$

$$(4.4)$$

$$\times \exp \left\{ 6\mu m \left[ \arctan h\left( \frac{1}{\sqrt{x^2 + 1}} \right) - \frac{1}{\sqrt{x^2 + 1}} \right] \right\}$$

where we substituted the dimensionless quantities (4.2). To study thermodynamic behavior of this kind of black hole it is simpler to use asymptotic behavior of the event horizon equation and the Hawking temperature equation instead of their exact forms. In fact calculations with the exact equations take on complicated forms. To do so we can study two different approaches as $x < 1$ and $x > 1$. In the case $x > 1$ asymptotic series forms of the horizon equation and the Hawking temperature take on divergent forms and so give not suitable physical situations for the p-v curves at constant temperature and other thermodynamic diagrams but for small scale black holes $x < 1$ the behavior of p-v diagram predicts small to large black hole phase transition and vice versa dependent to choose numeric values which we use for the GB parameter $\mu$.

By according to the GB parameter $\mu$ which takes on some small values $\mu < 1$ let us we start to substitute the mass parameter $m(x)$ given by the horizon equation (4.1) into the temperature equation (4.4) which takes on a long length form and so we calculate its series expansion about small values of the parameters $\mu < 1$ and $x < 1$ such that

$$t \approx \frac{(213\mu - 20)}{10x} - \frac{6\mu}{x^3} \left( 4 + \ln \left| \frac{2}{x} \right| \right) + \frac{2\mu(4 - \ln 2)\lambda}{x} + O(x, x \ln x, \mu^2)$$

$$(4.5)$$

and

$$m \approx \frac{5\mu}{2x^4} + \frac{A}{x^2} + B + O(x^2, \mu^2)$$

$$(4.6)$$

where we defined

$$A = \frac{1}{2} + \frac{99}{20}\mu - \lambda\mu, \quad B = \frac{3}{4} + \frac{\lambda}{6} + \frac{1629}{560}\mu - \frac{\lambda^2\mu}{3} - \frac{11\lambda\mu}{10}.$$  

$$(4.7)$$

In the above series expansion forms the $O$ terms are not divergent for small values of $x < 1$ and $\mu < 1$ and so we can remove them when we study thermodynamic behavior of small scale black holes. To obtain equation of state
we set thermodynamic specific volume $v$ and dS/AdS pressures respectively as follows.

$$\text{AdS} : \quad \lambda = -8\pi p, \quad v = -\frac{16\pi \mu (4 - \ln 2)}{x} \quad (4.8)$$

and

$$\text{dS} : \quad \lambda = 8\pi p, \quad v = \frac{16\pi \mu (4 - \ln 2)}{x} \quad (4.9)$$

for which the temperature equation (4.5) reads

$$dS/AdS : \quad t = pv + 6\mu \epsilon \left( \frac{v}{16\pi \mu (4 - \ln 2)} \right)^3 \left( 4 + \ln \left( \frac{v}{8\pi \mu (4 - \ln 2)} \right) \right) \quad (4.10)$$

where $\epsilon = +1(-1)$ for AdS(dS) sector of the background space time. By looking at the positivity condition of the specific volumes (4.8) and (4.9) one can infer that for AdS sector in which $x < 0$ we should choose negative magnetic charge but for dS sector in which $x > 0$ we should choose $Q > 0$.

In order to plot $p - v$ diagrams at constant temperature, the critical points are needed to be obtained from the critical equations of $\frac{\partial t}{\partial v} | _p = 0$ and $\frac{\partial^2 t}{\partial v^2} | _p = 0$ which by substituting the equation of state (4.10) read

$$v_c = 8\pi \mu (4 - \ln 2)e^{-\frac{2p}{\mu}}, \quad p_c = \frac{\epsilon [45e^{-\frac{2p}{\mu}} + 426\mu - 40]}{320\pi \mu (4 - \ln 2)}, \quad t_c = \frac{\epsilon \mu e^{-\frac{2p}{\mu}}}{2}. \quad (4.11)$$

It is useful to rewrite the equation of state (4.10) versus the following re-scaled thermodynamic variables.

$$\bar{v} = \frac{v}{v_c}, \quad \bar{p} = \frac{p}{p_c}, \quad \bar{t} = \frac{t}{t_c} \quad (4.12)$$

such that

$$dS/AdS : \quad \bar{t} = a(\mu)\bar{p}\bar{v} + \frac{3}{2}\bar{v}^3 \left[ \ln \bar{v} - \frac{5}{6} \right] + b(\mu)\bar{v} \quad (4.13)$$

where we defined

$$a(\mu) = \frac{p_cv_c}{t_c} = \frac{45\mu + (426\mu - 40)e^{\frac{2p}{\mu}}}{20\mu}, \quad b(\mu) = \frac{(20 - 213\mu)e^{\frac{2p}{\mu}}}{10\mu}. \quad (4.14)$$
This equation of state has same form for both dS and AdS sector of the space time and we plot $p - v$ diagram at constant temperature for critical temperature $\tilde{t}_c = 1$ and its below values $\tilde{t} < \tilde{t}_c$ and its upper values $\tilde{t} > \tilde{t}_c$ in figure 2 for different values of the GB parameter. To do so we define a critical value for the GB parameter as $\mu_c = 0.09323992798$ which is calculated by substituting $\tilde{v} = \tilde{v}_c = 1$, $\tilde{p} = \tilde{p}_c = 1$ and $\tilde{t} = \tilde{t}_c = 1$ into the equation of state (4.13) and we plot several p-v diagrams at constant temperature for $\mu < \mu_c$, $\mu = \mu_c$ and $\mu > \mu_c$. These diagrams show that a large to small black hole phase transition for $\mu \geq \mu_c$ (see figures 1-a,b,d,f) and small to large phase transition for $\mu < \mu_c$ (see figures 1-b,c,e). In the figures 1-a,b,d,f the diagrams have a local maximum point at higher specific volume and a local minimum point at smaller specific volume. It is known that thermodynamic stability is happened in the minimum point of the p-v diagram while its instability is happened at maximum point of the diagram. By looking at the diagrams given at the figure 1 one can infer that position of the minimum and the maximum points of the diagrams are exchanged for $\mu \geq \mu_c$ and $\mu < \mu_c$ respectively. In the next section we investigate Joule-Thomson expansion of this kind of black hole by plotting isenthalpic $t - p$ diagrams and inversion curves.

### 4.1 JT Expansion

To study the Joule Thomson adiabatic expansion of the 4D GB dS/AdS Bardeen black hole we should investigate isenthalpic curves for all scales of the black holes $x < 1$ and $x > 1$ and so we can not use series solutions of the horizon equation and the Hawking temperature given in the previous section for small values $x < 1$ and so we must be use exact forms of these equations but we can still use series expansions of them for small GB parameter $\mu < 1$. Hence we first solve the horizon equation vs the cosmological parameter $\lambda$ leading to two different solution which one of them reaches to some physical situations where the JT expansion is happened. By substituting it into the temperature equation we can obtain a suitable form for the temperature equation to be independent of the $\lambda$ term but having a long length form. Hence we calculate its Taylor series expansion about small values of $\mu$ which up to the second order term reads

$$
\lambda \simeq \frac{6m}{(x^2 + 1)^{\frac{3}{2}}} - \frac{3}{x^2} - \frac{3\mu}{x^5(x^2 + 1)^{\frac{3}{2}}} \times \frac{1}{x^3} \quad (4.15)
$$
\[
4m(x^2 + 1)^{3/2}(12mx \arcsin hx + 5x^2 + 6) \\
-3m(x^2+1)^3(8 \arcsin hx+mx^2 \arctan x)-5x(x^2+1)^3-3m^2(x^7+\frac{32}{3}x^5+15x^3)
\]

in which \( p = -\frac{\epsilon \lambda}{8\pi} \) with \( (\epsilon = +(-)) \) for AdS(dS) sector of the space time, is pressure of dS/AdS background space. Moreover we approximate the relation of temperature for small \( \mu \) as follow:

\[
t \simeq \left( \frac{1}{x} + \lambda x - \frac{6mx}{(x^2 + 1)^{3/2}} \right) - \mu \left\{ \frac{3}{x^3} - \frac{6m}{(x^2 + 1)^{3/2}} \right\} \tag{4.16}
\]

We plot the isoenthalpic t-p curves for all scales of the 4D GB dS/AdS Bardeen black holes with three chosen numeric values for dimensionless GB parameter as \( \mu = 0.0001, 0.001, \) and \( 0.01. \) To determine inversion curves where the cooling phase and heating phase of the black hole is separated with intersection point of the inversion curve and isenthalpic curves (see figures 2) we should calculate the JT coefficient through \( \mu_{JT} = \frac{\partial t}{\partial p} \bigg|_m \) (see appendix) and solve \( \mu_{JT} = 0 \) for which we obtain a long length relation for the inversion enthalpy (the mass \( m_i \)) and so we do not mention it in this paper. To plot inversion curves we substitute the inversion mass \( m_i \) inside the relation of (4.15) and (4.16) to obtain suitable relations for the inverse temperature and inversion pressure respectively vs \( x. \) Then all isenthalpic curves together with inversion curves are plotted for different values of the horizon parameters \( x. \) These diagrams are collected in the figure 2. One can see that left side of intersection point of the inversion curve with the isenthalpic curves where the slope of the curves or the JT coefficient take on some positive values the black hole system participate in the heating phase but for right side of the intersection point it participates in the cooling phase. Also one can look at the figures 2-a,b,c,d,e to infer that by raising the GB parameter the black hole system at constant enthalpy take on other subsystem which does not participate in the JT adiabatic expansion (please compare 2-a,b,c with 2-d,e). These are shown with straight-lines Particularly for large values of the GB parameter (see figure 2-f) there is not a intersection between the inversion curve (solid blue line) with the isenthalpic curves. In fact to clear a JT adiabatic expansion in a thermodynamic system the inversion curves should
intersect maximum point of the isenthalpic curves which are appeared in all figures 2-a,b,c,d,e but not in the 2-f. Furthermore we should point that the inversion curves have two different branches which one of them intersects with the isenthalpic curves for small values mass of the black holes (see figures 2-a,c,e) while for massive black hole these two branches are approached to each other and in fact two branches of the inversion curves intersect maximum point of the isenthalpic t-p curves (see figures 2-b,d).

5 Conclusion

In this work we used a consistent model of Einstein Gauss Bonnet gravity to study thermodynamics of 4D $dS/AdS$ GB Bardeen black hole. Metric source of this type of black hole is nonlinear electromagnetic fields with a non-vanishing magnetic charge. Physical importance of this type of black holes is nonsingular property which have and they are applicable to study black hole structure of center of galaxies. We obtain analytical solution of the metric field equation for small values of the Gauss Bonnet coupling constant. We calculated exterior horizon of this kind of black holes and the Hawking temperature. At last by calculating the equation of state in presence of both dS and AdS space pressures we investigated the black hole thermodynamics by focusing on the phase transitions and Joule Thomson adiabatic expansion of the black hole in presence of Gauss Bonnet counterterms. In this way we saw that the Gauss Bonnet coupling constant plays important role in these phase transitions. Positivity condition on the thermodynamic volume causes to be valid negative (positive) magnetic charge for AdS (dS) space pressure. Our mathematical calculations predict that 4D AdS GB Bardeen black hole takes on small to large phase transition form small values of the Gauss Bonnet coupling constant and vice versa for larger values of this coupling constant. Also we understand that for small values of the Gauss bonnet term effect the black hole has a one subsystem which participates in the JT adiabatic expansion but for larger values of the Gauss Bonnet effect there is a second subsystem for the black hole which does not participate in the cooling - Heating (JT expansion) phase transition. As an extension of this work one can investigate JT expansion for other types of nonsingular magnetic charged black holes (see for instance [57, 58]). As an future work we like to investigate other thermodynamic properties of the 4D GB dS/AdS Bardeen black hole such that holographic entanglement entropy and complexity growth rate via
complexity=action conjecture.

Appendix

As we mentioned previously the JT expansion occurs at constant enthalpy
\( H = U + PV \) for which we can write

\[
dH = TdS + VdP
\]

(5.1)
where we used

\[
TdS = dQ = dU + PdV.
\]

(5.2)
Applying \( dH = 0 \) the equation (5.1) reduces to the form \( 0 = TdS + VdP \)
which can be rewritten as

\[
\frac{dH}{dP} = 0 = T \left( \frac{\partial S}{\partial P} \right)_H + V.
\]

(5.3)
If we assume that the entropy depends to the temperature \( T \) and the pressure \( P \) as \( S = S(T, P) \) then we can write \( dS = \left( \frac{\partial S}{\partial P} \right)_P dP + \left( \frac{\partial S}{\partial T} \right)_P dT \) for which we will have

\[
\left( \frac{\partial S}{\partial P} \right)_H = \left( \frac{\partial S}{\partial P} \right)_T + \left( \frac{\partial S}{\partial T} \right)_P \left( \frac{\partial T}{\partial P} \right)_H.
\]

(5.4)
Substituting (5.4) into the equation (5.3) we obtain

\[
0 = -T \left( \frac{\partial V}{\partial T} \right)_P + C_P \left( \frac{\partial T}{\partial P} \right)_H + V
\]

(5.5)
where we used \( C_P = T \left( \frac{\partial S}{\partial T} \right)_P \) and the Maxwell relationship \( \left( \frac{\partial S}{\partial T} \right)_P = -\left( \frac{\partial V}{\partial T} \right)_P \)
which by solving \( \left( \frac{\partial T}{\partial P} \right)_H \) one can obtain the JT coefficient \( \mu_{JT} = \left( \frac{\partial T}{\partial P} \right)_H \).

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Figure 1: p-v diagrams for small scale 4D GB dS/AdS Bardeen black holes
Figure 2: Isenthalpic and inversion curves for 4D GB dS/AdS Bardeen black holes with different values for the enthalpy $m$