Constructing conformally invariant equations using
the Weyl geometry

Sofiane Faci

Institute of Cosmology, Relativity and Astrophysics (ICRA-CBPF), Rua Dr Xavier Sigaud, 150,
Urca, CEP 22290-180, Rio de Janeiro, RJ, Brazil

E-mail: sofiane@cbpf.br

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Abstract

We present a simple, systematic and practical method to construct conformally
invariant equations in arbitrary Riemann spaces. This method that we call
‘Weyl-to-Riemann’ is based on two features of the Weyl geometry. (i) Weyl
space is defined by the metric tensor and the Weyl vector \( W \); it is equivalent
to the Riemann space when \( W \) is a gradient. (ii) Any homogeneous differential
equation written in the Weyl space by means of the Weyl connection is
conformally invariant. The Weyl-to-Riemann method selects those equations
whose conformal invariance is preserved when reducing to the Riemann
space. Applications to scalar, vector and spin-2 fields are presented, which
demonstrate the efficiency of this method. In particular, a new conformally
invariant spin-2 field equation is exhibited. This equation extends Grishchuk–
Yudin’s equation and fixes its limitations since it does not require the Lorenz
gauge. Moreover, this equation reduces to the Drew–Gegenberg and Deser–
Nepomechie equations in Minkowski and de Sitter spaces, respectively.

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1. Introduction

Conformal symmetry is a fundamental ingredient for theoretical physics. The underlying
reason is that conformal transformations preserve causal structures and propagate the modes
of conformally invariant equations on light cones [1].

This work deals with the significant problem of how to construct such equations. This
is a non-trivial question and several answers to this are proposed in the literature. They
include Boulanger’s method [2] and the unfolded dynamics approach [3–5], both are highly
sophisticated but not very practical. The same critic applies to the tractor calculus [6] and
the \((D+2)\)-dimensional ambient space techniques [7]. Other methods are designed to deal
exclusively with scalar fields [8, 9]. There is also the so-called Ricci-gauging method which
was developed by Iorio et al [10]. The authors start with an \( SO(2,4) \)-invariant equation
in Minkowski space and extend it to reach a Weyl invariant equation in arbitrary Riemann spaces. However, despite its beauty and simplicity, it is not a systematic procedure and has been applied only to second-order equations. See [11, 12] for the link between Weyl and \textit{SO}(2, 4) transformations.

We present a new method that we call ‘Weyl-to-Riemann’ and which takes the opposite direction to that of Iorio et al. We start from a more general geometry than the Riemann one, namely the Weyl geometry. The latter relies on two objects: the metric tensor and the Weyl vector \( W \). On the one hand, when \( W \) is a gradient, the corresponding Weyl space is equivalent to the Riemann space [13]. On the other hand, using the Weyl-covariant derivative and geometrical tensors, defined by means of the Weyl connection, any equation written in the Weyl space is conformally invariant. The only condition is the homogeneity of this equation, which means that its different terms must have the same conformal weight. In general, this equation depends on \( W \) and the game consists in selecting those equations for which \( W \) decouples, at least when \( W = \text{d}f \). Also, the resulting equation is conformally invariant in a Riemann space.

In practice, we look for the right combination of the different terms that cancels \( W \). Imposing \( W = \text{d}f \) then becomes invisible and thereof the conformal invariance is preserved at the Riemann stage. If such a combination does not exist, then the residual gradient \( \text{d}f \) might break the conformal invariance. In some situations, \( W \) does not decouple in the Weyl space, but cancels when it has a gradient form or when some extra constraints are imposed and then recover conformal invariance. The aim of this method is to provide a simple, practical and systematic algorithm. It translates the problem of constructing conformally invariant equations of any order into algebraic systems whose solutions lead to the desired equations.

The Weyl-to-Riemann method was successfully applied to study the conformal properties of scalar, vector and spin-2 fields. In particular, we exhibit the most general conformally invariant second-order equation for a spin-2 field \( h_{\mu\nu} \) defined in an arbitrary Riemann space with a canonical weight +1. It reads

\[
\Box h_{\mu\nu} - \frac{2}{3} \nabla_\mu \nabla_\nu h_{\rho}^\rho + \frac{1}{3} \nabla_\mu \nabla_\rho h^{\rho\beta} g_{\mu\nu} + \frac{1}{3} (\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box) h + R_{\mu\rho\beta\nu} h^{\rho\beta} - R_{\alpha(\mu} h^\nu_{\alpha)} + \lambda C_{\mu\rho\beta\nu} h^{\rho\beta} = 0,
\]

where \( h = h_{\mu}^\mu \) is the field trace, \( \lambda \) is a free parameter and \( R_{\mu\alpha\beta\nu} \) is the Riemann tensor of which the Weyl tensor \( C_{\mu\rho\beta\nu} \) is the traceless part. This equation is gauge dependent, which indicates the presence of both spin-0 and spin-2 components of \( h_{\mu\nu} \) [14]. In other words, the graviton field can be made conformally invariant—then propagates on light cones—by adding a scalar component. Moreover, and for the sake of consistency, let \( h_{\mu\nu} = \phi g_{\mu\nu} \), where \( \phi \) is a scalar field of weight \(-1\), then the above equation leads as expected to the conformally invariant scalar field equation \( (\Box + \frac{\lambda}{6} \phi = 0) \).

This new equation extends and fixes the limitations of the Grishchuk and Yudin equation since it does not require the Lorenz gauge, \( \nabla_\mu h_{\mu}^\nu = 0 \). Indeed, the latter constraint is conformally invariant for a restricted set of Weyl rescalings verifying \( h^{\mu\nu} \delta_\mu \Omega = 0 \). This was the main argument against Grishchuk–Yudin’s equation, led by Bekenstein and Meisels [15]. Note also that the above equation reduces to Drew–Gegenberg’s equation in Minkowski space and, when the field is traceless, to Deser–Nepomechie’s equation in constant curvature spaces.

This paper is organized as follows. The next section underlines some features of the Weyl geometry and states the Weyl-to-Riemann method. Section 3 is dedicated to applications. Section 3.1 expands first-, second- and fourth-order conformally invariant scalar field equations. Section 3.2 deals with vector fields. In particular, the Eastwood–Singer gauge is recovered. In section 3.3, the method is applied to construct the most general conformally
invariant equation for a spin-2 field. Conclusions and outlooks are presented in section 4. Finally, appendix A derives the Weylian geometrical tensors and appendix B provides some useful formulas.

We work in \( d = 4 \) dimensions, but the Weyl-to-Riemann method is, in principle, valid for any dimension \( d \geq 2 \). Spaces are torsion free and equipped with metrics of signature \((+,−−)\). We use a tilde (~) to indicate objects defined in the Weyl space and distinguish them from their Riemannian counterparts.

2. From Weyl-to-Riemann

2.1. Some features of Weyl geometry

In 1918, Herman Weyl invented a new geometry in order to unify gravitation and electromagnetic forces. This goal was not reached, but Weyl had exhibited a simple generalization of the Riemann geometry and had by the way introduced gauge transformations. These turned out to be fundamental concepts for both physics and mathematics [16].

Weyl space is a real manifold equipped with two independent and equally fundamental objects: the metric tensor \( g \) and the Weyl vector \( W \). In the Weyl geometry, conformal transformations mean gauge–Weyl transformations which transform both \( g \) and \( W \):

\[
g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu},
\]

\[
W_\mu \rightarrow \tilde{W}_\mu = W_\mu + \Omega_\mu,
\]

where \( \Omega \) is a real and smooth function and \( \Omega_\mu = \partial_\mu \ln \Omega \). The transformation (1) alone is called rigid-Weyl (or Weyl) rescaling and defines conformal transformations in the Riemann geometry.

There is a unique symmetric, torsion free and linear connection one can construct in the Weyl geometry [17]. It is the Weyl connection,

\[
\tilde{\Gamma}^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - W^\lambda_{\mu\nu}, \quad \text{with}
\]

\[
\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\tau} (g_{\tau(\mu;\nu)} - g_{\mu\nu,\tau}),
\]

\[
W^\lambda_{\mu\nu} = \delta^\lambda_{\mu} W_{\nu} - g_{\mu\nu} W^\lambda,
\]

where \( W^\lambda = g^{\lambda\tau} W_\tau \). It is straightforward to show that the Weyl connection is conformally invariant:

\[
\tilde{\Gamma} \rightarrow \tilde{\Gamma} = \tilde{\Gamma}.
\]

This is why conformally related spaces represent in fact the same Weyl space, which is a conformal equivalence class \([g,W]\). Another important consequence of the definition (3) is that the associated covariant derivative is not metric compatible,

\[
\tilde{\nabla}_\mu g_{\alpha\beta} = \nabla_\mu g_{\alpha\beta} + W^\lambda_{\mu(\alpha} g_{\beta)\lambda} = 2W_{\mu} g_{\alpha\beta} \neq 0,
\]

where \( \nabla_\mu \) is defined by the Christoffel symbol (4). This means that lengths are not preserved by parallel transport. In the Weyl geometry, only angles (ratios of lengths) are meaningful and lengths can only be compared locally. The Weyl geometry is a ‘truly’ differential geometry and this is in fact what Weyl was looking for [18].

In place of \( \tilde{\nabla}_\mu \), one can use the Weyl covariant derivative defined through

\[
D_\mu T^{\alpha\cdots}_{\beta\cdots} = (\tilde{\nabla}_\mu - w W_\mu) T^{\alpha\cdots}_{\beta\cdots} = (\nabla_\mu - w W_\mu) T^{\alpha\cdots}_{\beta\cdots} + W^\lambda_{\mu\beta} T^{\alpha\cdots}_{\lambda\cdots} + \cdots - W^\alpha_{\mu\lambda} T^{\lambda\cdots}_{\beta\cdots} - \cdots,
\]
where $T$ is a given conformal tensor field of rank $(r, s)$ and conformal weight $w = w(T)$. Under conformal transformations, conformal tensors are mapped like

$$T \rightarrow \Omega^w T.$$ \hfill (9)

The Weyl covariant derivative is metric compatible, $D_{\mu} g_{\alpha\beta} = 0$, and is doubly covariant, according to conformal transformations (thanks to the invariance of the Weyl connection) and diffeomorphisms. Moreover, it is linear and verifies the Leibniz rule.

2.2. The Weyl-to-Riemann method

The purpose of the Weyl-to-Riemann method is to provide a simple and systematic algorithm which translates the problem of constructing conformally invariant equations in arbitrary Riemann spaces into a set of algebraic equations. The method is based on the two following features of the Weyl geometry.

- Using the Weyl covariant derivative (8) and the Weyl geometrical tensors given in appendix A, all homogeneous equations written in the Weyl space are conformally invariant. Homogeneous here means that the different terms have the same conformal weight.
- When the Weyl vector is gradient, $W_\mu = \partial_\mu \ln K$, where $K$ is a real and smooth function, the Weyl space is then called the Weyl integrable space (WIS) and the Weyl connection (3) reduces to $\tilde{\Gamma}_{\mu\nu}^\lambda = \bar{\Gamma}_{\mu\nu}^\lambda$, where $\bar{\Gamma}_{\mu\nu}^\lambda$ is the Christoffel connection of the metric tensor $\bar{g}_{\mu\nu} = K^{-2} g_{\mu\nu}$. The covariant derivative becomes metric compatible, which means that the WIS $[g_{\mu\nu}, \partial_\mu \ln K]$ is in fact a Riemann space equipped with the metric $\bar{g}_{\mu\nu}$. Note that some authors use WISs in a different way [19, 20].

In practice, to construct a Riemannian conformally invariant $n$-order equation for a tensor field $T$ of weight $w$, one begins with writing the most general homogeneous equation in the Weyl space, say

$$a_1 \tilde{D}_1 + a_2 \tilde{D}_2 + \cdots + b_1 \tilde{U}_1 + b_2 \tilde{U}_2 + \cdots = 0,$$ \hfill (10)

where $\tilde{D}_i$ are derivative parts acting on $T$, $\tilde{U}_i$ are geometrical parts and $a_i$ and $b_i$ are free parameters. As stated above, this equation is naturally conformally (gauge–Weyl) invariant in the Weyl space. The process consists of looking for combinations $\{w, a_i, b_i\}$ in order to obtain, when reducing to a Riemann space, by imposing $W_\mu = \partial_\mu \ln K$, an equation

$$a_1 D_1 + a_2 D_2 + \cdots + b_1 U_1 + b_2 U_2 + \cdots = 0,$$ \hfill (11)

which is conformally (rigid-Weyl) invariant. Several situations may occur.

1. There is a combination $\{w, a_i, b_i\}$ which cancels the $W$ dependence of equation (10). The latter is thus rigid-Weyl invariant in the Weyl space. Imposing $W$ to be gradient becomes ‘invisible’ and the resulting Riemannian equation (11) is still rigid-Weyl invariant, i.e. conformally invariant.
2. There is no combination $\{w, a_i, b_i\}$ which cancels the $W$ dependence of equation (10) and then three cases are possible.
   a. $W$ decouples when it has a gradient form. Equation (11) is then conformally invariant.
   b. An extra conformally invariant constraint cancels $W$ in (10). Equation (11), together with the involved constraint, becomes conformally invariant.
   c. There is no way to get rid of $W$. The conformal invariance is broken when going to the Riemann space.
3. Applications

The Weyl-to-Riemann method is applied to construct conformally invariant equations for some integer-spin conformal tensor fields $T$. In the Riemann geometry, these fields have canonical weights which are related to the spin $s$ by the relation

$$w = s - 1.$$  \hfill (12)

Otherwise stated, we are not considering derivatives of the Weyl geometrical tensors.

3.1. Conformally invariant scalar field equations

3.1.1. First-order equation. The conformally invariant first-order equation acting on a scalar field $\phi$ defined in the Weyl space reads

$$D_\mu \phi = (\nabla_\mu - wW_\mu) \phi = 0.$$  \hfill (13)

The Weyl vector decouples when $w = 0$, and the resulting Riemannian equation

$$\nabla_\mu \phi = 0$$  \hfill (14)

is then conformally invariant, but not with the canonical weight (12). This corresponds to the situation (i) of section 2.2.

3.1.2. Second-order equation. The most general second-order scalar field equation in the Weyl space involves the second-rank geometrical tensors; it reads

$$g^{\mu\nu}(D_\mu D_\nu + b_1 \tilde{R}_{\mu\nu} + b_2 \tilde{R}g_{\mu\nu}) \phi = (D^2 + \alpha \tilde{R}) \phi = 0,$$  \hfill (15)

where $\alpha = b_1 + 4b_2$. This equation is expanded as

$$(\Box + \alpha R - 2(1 + w)W \nabla - (6\alpha + w) \nabla W + (w^2 + 2w + 6\alpha) W) \phi = 0,$$  \hfill (16)

where $\Box = g^{\mu\nu} \nabla_\mu \nabla_\nu$ and all other contractions are performed using $g_{\mu\nu}$. We can eliminate $W$ from this equation iff

$$1 + w = 0,$$

$$6\alpha + w = 0,$$

$$w^2 + 2w + 6\alpha = 0.$$  \hfill (17)

This system has a unique solution

$$w = -1, \quad \alpha = \frac{1}{5},$$  \hfill (18)

We are here again in the same situation (section 2.2(i)). Equation (16) then reduces to the usual conformally invariant scalar field equation in Riemann spaces,

$$(\Box + \frac{1}{5} R) \phi = 0.$$  \hfill (19)

3.1.3. Fourth-order equation. Let us now consider a fourth-order conformally invariant scalar equation in the Weyl space given by

$$D_\mu (D^\nu D^\rho + a\tilde{R}^{\mu\nu\rho} + b\tilde{R}g^{\mu\nu\rho}) D_\nu \phi = 0.$$  \hfill (20)

Note that this equation involves derivatives of the geometrical tensors. A straightforward calculation shows that there is a unique combination,
which cancels the Weyl vector provided it verifies the free Maxwell equation
\[ \nabla_\mu \nabla^{[\mu} W_{\nu]} = 0. \tag{22} \]
Equation (20) thus reduces to
\[ \nabla_\mu (\nabla^\mu \nabla^\nu + S^{\mu\nu}) \nabla_\nu \phi = 0, \tag{23} \]
where the Eastwood–Singer tensor reads
\[ S^{\mu\nu} = -2R^{\mu\nu} + \frac{2}{3}R g^{\mu\nu}. \tag{24} \]
When \( W_\mu \) is a gradient, i.e. a pure gauge verifying identically (22), equation (23) becomes conformally invariant in a Riemann space with a vanishing conformal weight. Although in a different way, this equation was derived by Eastwood and Singer in [21].

3.2. Conformally invariant vector field equations

Let us apply the Weyl-to-Riemann method to the spin-1 vector field. We first recover the conformal invariance of the Riemannian–Maxwell equation and then look for some conformally invariant gauge fixing conditions.

3.2.1. Maxwell equation. The Maxwell equation in the Weyl space for a 1-form \( A_\mu \) reads
\[ D^\mu D_\mu A_\nu = 0. \tag{25} \]
It is conformally invariant for any weight \( w = w(A) \) and extends to
\[ \nabla^\mu F_{\mu\nu} - w(W^\mu F_{\mu\nu} + (\nabla^\mu - wW^\mu)(W_\mu A_\nu - W_\nu A_\mu)) = 0, \tag{26} \]
where \( F_{\mu\nu} = \nabla_{[\mu}A_{\nu]} \). For \( w = 0 \), the Weyl vector cancels. Thus one obtain the conformally invariant Maxwell equation, \( \nabla^\mu F_{\mu\nu} = 0 \), with the usual vanishing conformal weight in the Riemann space.

3.2.2. Lorenz gauge. The Weylian Lorenz gauge, \( D A = g^{\mu\nu} D_\mu A_\nu = 0 \), yields
\[ \nabla A - (2 + w)WA = 0. \tag{27} \]
The Weyl vector decouples for a non-canonical weight, \( w = -2 \) or when the constraint \( WA = 0 \) is applied. The latter is conformally invariant for a restricted set of Weyl rescalings verifying
\[ \Omega A = 0. \tag{28} \]
Accordingly the Riemannian Lorenz gauge, \( \nabla A = 0 \), is not completely conformally invariant for a canonical weight, \( w = 0 \).

3.2.3. Eastwood–Singer gauge. The simplest third-order differential equation for the field \( A_\mu \) of a vanishing weight in the Weyl space reads
\[ D^2 D A = 0. \tag{29} \]
This equation contains only derivative terms but is perfectly conformally invariant. It expands as
\[
\begin{align*}
D^2 D A &= \nabla_\mu (\nabla_\mu + 2W_\mu) (\nabla^\nu - 2W^\nu) A_\nu \\
&= \nabla_\mu (\nabla^\mu \nabla^\nu + S^{\mu\nu}) A_\nu - 2W^\nu \nabla^\mu F_{\mu\nu} \\
&= 0, \tag{30}
\end{align*}
\]
where $S^{\mu\nu}$ (24) is the Eastwood–Singer tensor. We have an interesting situation corresponding to the case (ii)b of section 2.2. Indeed, equation (30) contains $W$, but when $A_{\mu}$ verifies the Maxwell equation, the Weyl vector decouples. Equation (30) then reduces to the Eastwood–Singer equation

$$\nabla_{\mu}(\nabla^{\alpha}\nabla_{\nu} + S^{\mu\nu})A_{\nu} = 0,$$  \hspace{1cm} (31)

which is a conformally invariant but only when acting on the Maxwell equation solutions. This gauge was used for instance in [22–25]. Note that for a pure gauge field $A_{\nu} = \nabla_{\nu}\phi$, equation (31) reduces to equation (23).

### 3.3. Conformally invariant spin-2 field equation

Let us turn to the spin-2 field which describes weak gravitational fields and in particular gravitational waves in a given curved background. It is well known that the linearized Einstein equations for a spin-2 field in arbitrary Riemann spaces.

We show how the Weyl-to-Riemann method can be applied to construct conformally invariant equations for a spin-2 field in arbitrary Riemann spaces.

A spin-2 field is usually represented by a symmetric tensor field $h_{\mu\nu} = h_{\nu\mu}$. The trace and divergence are free. In order to exploit the entire capacity of the Weyl-to-Riemann method and increase the chances to find new results, let us write the most general second-order spin-2 field equation in a Weyl space,

$$\tilde{\mathcal{D}}_{\mu\nu} + \tilde{\mathcal{U}}_{\mu\nu} = 0,$$  \hspace{1cm} (32)

where the derivative part is given by

$$\tilde{\mathcal{D}}_{\mu\nu} = D_{\mu}h_{\nu\alpha} + a_{1}D_{\mu}D_{\rho}h_{\nu\rho\alpha} + a_{2}D_{\alpha}D_{\rho}h_{\nu\rho\beta}g_{\mu\nu} + a_{3}D_{\mu}D_{\nu}h + a_{4}D_{\mu}D^{2}g_{\mu\nu}$$  \hspace{1cm} (33)

and the geometrical part reads

$$\tilde{\mathcal{U}}_{\mu\nu} = b_{1}\tilde{R}_{\mu\nu\rho\beta}h^{\rho\beta} + b_{2}\tilde{R}_{\mu\nu\rho\beta}h^{\rho\beta}g_{\mu\nu} + b_{3}\tilde{R}_{\alpha(\mu}h^{\alpha\nu)} + b_{4}\tilde{R}h_{\mu\nu} + b_{5}\tilde{R}_{\nu\mu}h + b_{6}\tilde{R}g_{\mu\nu}h.$$  \hspace{1cm} (34)

The free parameters $\{a_{i}, b_{i}\}$ are real constants. The different terms have the same conformal weight $w - 2$, which ensures conformal homogeneity. The appearance of the non-conformally invariant tensor $\tilde{R}$ in (34) comes from the contraction of the conformally invariant tensor $\tilde{R}_{\mu\nu\rho\beta}$. Note that the covariant derivatives do not commute; thus using $D_{\mu}D_{(\rho}g_{\nu)\beta}$ instead of $D_{\mu}D_{\rho}g_{\nu\beta}$ slightly modifies the final form of the equation. However, this modification amounts to select a different set of the geometrical tensors parameters by means of the identity (B.8).

In the following, we expand equation (32), and then we look for combinations of the different parameters which cancels the Weyl vector dependence.

Considering first the derivative part (33) and trying to cancel the terms containing both $W$ and derivatives of $h_{\mu\nu}$, because these terms cannot be compensated by the geometrical part, we obtain the following:

$$\tilde{\mathcal{D}}_{\mu\nu} = D_{\mu\nu} + 2(w - 1)W^{\rho}\nabla_{\nu}h_{\mu\rho} + (2 + a_{1}(w + 2))W^{\rho}\nabla_{\nu}(h_{\mu\rho}) + (2 - a_{1}(w - 4))W^{\rho}_{(\mu}W_{\nu)_{\rho\nu}\rho} + (a_{1} + a_{3}(3 - w))h_{(\mu}W_{\nu)} + (a_{2} - 2a_{4}(1 - w))h_{\mu\nu}W^{\rho}_{\rho\mu\nu}$$

+ non-derivative terms.

The resulting algebraic system reads

$$w - 1 = 0,$$

$$2 + e(w + 2) = 0.$$
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\[ 2 - a_1 (w - 4) = 0, \]
\[ a_1 + a_2 (w + 1) = 0, \]
\[ a_1 + a_3 (3 - w) = 0, \]
\[ a_2 - a_3 + 2a_4 (1 - w) = 0. \]

This system has an infinite set of solutions, which can be put under the form
\[ w = 1, \quad a_1 = -\frac{2}{3}, \quad a_2 = \frac{1}{3}, \quad a_3 = \frac{1}{3}, \quad a_4 = \tau. \]  

where \( \tau \) is a free parameter. The resulting expression for the derivative part reads
\[
\tilde{D}_{\mu\nu} = D_{\mu\nu} + 2 (W_\alpha h^\alpha_{\mu\nu} + h^\alpha_{\mu\nu} W_\alpha^{-1} - (W_{\alpha\beta} + W_{\alpha\beta}) h^{\alpha\beta} g_{\mu\nu} + (\nabla W - 3W^2) h_{\mu\nu}
+ \left( \tau - \frac{1}{3} \right) (\nabla W - W^2) h g_{\mu\nu} - h \left( \frac{1}{2} W_{\mu\nu} + W_{\nu} W_{\nu} - \frac{2}{3} g_{\mu\nu} W^2 \right). \]

The Weyl vector does not decouple completely. Also, we add the geometrical part (34) to get rid of it. The general equation (32) expands into
\[
\tilde{D}_{\mu\nu} + \tilde{U}_{\mu\nu} = D_{\mu\nu} + U_{\mu\nu} + (1 - 2b_3 - 6b_4) \nabla W h_{\mu\nu} + (-3 + b_1 + 4b_3 + 6b_4) W^2 h_{\mu\nu}
+ (2 - b_1 - 2b_3) (W_\alpha W_{\mu\nu} + W_{\alpha\mu\nu}) h_\alpha^\nu + (b_1 + 2b_3) (W_{\alpha\nu} - W_{\nu\alpha}) h_\alpha^\nu
+ (-1 + b_1 - 2b_3) (W_{\alpha\beta} + W_{\alpha\beta}) h^{\alpha\beta} g_{\mu\nu}
+ (-1 + b_1 - 2b_3) (1/2 W_{\mu\nu}) + W_{\mu\nu} h
+ (1/3 - b_1 - 2b_3 - \tau + 6b_4) \nabla W h g_{\mu\nu}
+ (2/3 - b_1 + 2b_2 + 2b_3 - \tau + 6b_6) W^2 h g_{\mu\nu}
= 0. \]

Except the fourth line which cancels when \( W \) is a gradient, eliminating \( W \) leads to the following algebraic system:
\[
1 - 2b_3 - 6b_4 = 0,
-3 + b_1 + 4b_3 + 6b_4 = 0,
-1 + b_1 - 2b_3 = 0,
-1 + b_1 - 2b_5 = 0,
2 - b_1 - 2b_3 = 0,
1/3 - b_1 - 2b_3 + \tau - 6b_6 = 0,
2/3 - b_1 + 2b_2 + 2b_3 - \tau + 6b_6 = 0. \]

This system has an infinite set of solutions which can take the form
\[ b_1 = \lambda + 1, \quad b_2 = \frac{\lambda}{2}, \quad b_3 = \lambda - 1, \quad b_4 = \frac{\lambda}{6}, \quad b_5 = \frac{\lambda}{2}, \quad 6b_6 = \tau - \lambda + 1/3, \]
where \( \lambda \) and \( \tau \) are free parameters. Equation (38) becomes
\[
D_{\mu\nu} + U_{\mu\nu} + 2h^{\alpha\beta} (\tilde{W}_{\mu\alpha\nu}) = 0, \]
where \( \tilde{W}_{\mu\alpha} \) is given by (A.4) and the purely Riemann parts read
\[
D_{\mu\nu} = \square h_{\mu\nu} - \frac{2}{3} \nabla_{\mu} h_{\nu} h_{\alpha\beta} + \frac{1}{3} \nabla_{\nu} h_{\mu} h^{\alpha\beta} g_{\mu\nu} + \frac{1}{3} \nabla_{\nu} h_{\alpha\beta} + \tau \square h g_{\mu\nu}, \]
\[
U_{\mu\nu} = R_{\mu\alpha\beta\nu} h^{\alpha\beta} - R_{\mu\alpha\nu} h_{\alpha\beta} + \frac{1 + 3\tau}{18} R g_{\mu\nu} h + \lambda C_{\mu\alpha\beta\nu} h^{\alpha\beta}. \]

The appearance of the Weyl tensor
\[
C_{\mu\alpha\beta\nu} = R_{\mu\alpha\beta\nu} + \frac{1}{2} (g_{\mu\nu} R_{\alpha\beta} + g_{\alpha\beta} R_{\mu\nu}) - \frac{R}{6} (g_{\mu\nu} g_{\alpha\beta} - g_{\mu\alpha} g_{\nu\beta}), \]

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which is by itself conformally invariant with weight +2, reflects the presence of free parameters in the geometrical part. The remaining $W$ dependence in equation (39) cancels when going to the Riemann geometry. Moreover, the contraction of (39) leads to

$$(1 + 3\tau) \left( \Box + \frac{R}{6} \right) h = 0,$$

which imposes $\tau = -\frac{1}{3}$ if one wants to leave $h$ unconstrained.

The resulting equation reads

$$\Box h_{\mu\nu} - \frac{2}{3} \nabla_{[\mu} \nabla_{\nu]} h_{\alpha}^{\alpha} + \frac{1}{3} \nabla_{\alpha} \nabla_{\beta} h^{\alpha\beta} g_{\mu\nu} + \frac{1}{3} \left( \nabla_{\mu} \nabla_{\nu} - g_{\mu\nu} \Box \right) h + R_{\alpha\beta\mu\nu} h^{\alpha\beta} - R_{\alpha(\mu} h_{\nu)} + \lambda C_{\mu\alpha\beta\nu} h^{\alpha\beta} = 0.$$  (44)

This equation is conformally invariant in the Riemann geometry, with weight $-1$, for the spin-2 field represented by the symmetric tensor $h_{\mu\nu}$ with the canonical weight $+1$. When $h_{\mu\nu} = \phi g_{\mu\nu}$, where $\phi$ is the conformal scalar field with the canonical weight $-1$, this equation reduces to equation (19). This proves the consistency of our result. Note that this equation is gauge variant, which indicates the presence of both spin-0 and spin-2 components of $h_{\mu\nu}$ [14]. In other words, the $h_{\mu\nu}$ field can be made conformally invariant—to propagate on light cones—by adding a scalar component.

The above equation reduces to Drew–Gegenberg’s equation in Minkowski space [26] and to Deser–Nepomechie’s equation in constant curvature spaces [27]. Actually, Deser and Nepomechie presented a conformally invariant equation in general curved spaces, but this was extended from an equation valid in the de Sitter space. This explains why their equation lacks some terms of (44) for a traceless field.

When the field is traceless, $h = 0$, and divergentless, $\nabla_{\mu} h_{\mu} = 0$, equation (44) takes the simple form

$$\Box h_{\mu\nu} + R_{\mu\alpha\beta\nu} h^{\alpha\beta} - R_{\alpha(\mu} h_{\nu)} + \lambda C_{\mu\alpha\beta\nu} h^{\alpha\beta} = 0.$$  (45)

This equation was given by Grishchuk and Yudin [28] and has been criticized afterward by Bekenstein and Meisels [15]. Indeed, contrary to the traceless constraint which is fully conformally invariant, the Lorenz gauge is invariant only for a restricted set of Weyl rescalings. This is because the conformal invariance of $\nabla_{\mu} h_{\mu} = 0$ requires $h^{\mu\nu} \partial_{\mu} \Omega = 0$. This is equivalent to the constraint (28) necessary to make the Lorenz gauge conformally invariant in the case of the Maxwell field. This was the main argument against the Grishchuk and Yudin equation in particular and conformal spin-2 fields in general. Fortunately, equation (44) does not suffer from such a restriction and is perfectly conformally invariant.

After this work was made available online, Nepomechie drew our attention to the paper [29] in which the operator $S_{ab}$ turned out to be similar to that defining equation (44) in four-dimensional spaces.

4. Concluding remarks

The Weyl-to-Riemann method was presented. This method allows us to construct conformally invariant equations in arbitrary Riemann spaces. It exploits two features of the Weyl geometry. First, all homogeneous equations defined in the Weyl space are conformally invariant. Second, a simple procedure reduces the Weyl space to the Riemann space. The method selects those equations whose conformal invariance is preserved when reducing to the Riemann space.

The purpose of this method is to provide a simple, systematic and practical procedure to construct conformally invariant equations in general Riemann spaces. The major part of this paper gathered practical applications. Scalar, vector and spin-2 fields were considered in order
to demonstrate the efficiency of the Weyl-to-Riemann method. These applications showed that contrary to the Ricci-gauging method our technique is more practical, is not limited to second-order differential equations and is more suitable to find new results.

In particular, a new conformally invariant spin-2 field second-order equation (44) was exhibited. This generalizes all other known equations and allows us to avoid the usual critics against the conformal spin-2 field. The physical content of this equation remains to be explored and experimentally tested using for instance gravitational waves future observations.

The Weyl-to-Riemann method can be applied to half-integer spin fields, higher order and higher rank tensor-field equations. An interesting application would be related to the Fierz (third-rank tensor) representation of the spin-2 field. Indeed, the Fierz representation affords remarkable features in the non-conformally invariant sector [30] and one would expect important results in the conformally invariant one.

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Appendix A. Weyl geometrical tensors

Curvature tensors completely determine the geometry in both Riemann and Weyl spaces. They respectively read [31]

\[ R^\mu_{\nu \beta \gamma} = \partial_\nu \Gamma^\lambda_{\mu \beta} - \partial_\beta \Gamma^\lambda_{\mu \nu} + \Gamma^\tau_{\mu \beta} \Lambda^\lambda_{\tau \nu} - \Gamma^\tau_{\mu \nu} \Lambda^\lambda_{\tau \beta} , \]  

(A.1)

\[ \tilde{R}^\mu_{\nu \beta \gamma} = \partial_\nu \tilde{\Gamma}^\lambda_{\mu \beta} - \partial_\beta \tilde{\Gamma}^\lambda_{\mu \nu} + \tilde{\Gamma}^\tau_{\mu \beta} \tilde{\Lambda}^\lambda_{\tau \nu} - \tilde{\Gamma}^\tau_{\mu \nu} \tilde{\Lambda}^\lambda_{\tau \beta} . \]  

(A.2)

The Weylian curvature tensor is conformally invariant, as is the Weyl connection (6). It expands into

\[ \tilde{\Lambda}^\mu_{\nu \beta} = \tilde{R}^\mu_{\nu \beta} + \nabla W^\mu_{\nu \beta} - \nabla W^\mu_{\beta \nu} + W^\tau_{\mu \beta} W^\lambda_{\tau \nu} - W^\tau_{\mu \nu} W^\lambda_{\tau \beta} , \]

\[ = \tilde{R}^\mu_{\nu \beta} - \left( \delta^\lambda_{\nu} W^\mu_{\beta ; \nu} + W^\lambda_{\beta ; \nu} g_{\mu \beta} - W^\lambda_{\gamma} g_{\mu \beta ; \nu} + W^\lambda_{\beta ; \nu} g_{\mu \gamma} - W^\lambda_{\nu} g_{\beta \mu ; \nu} - W^\lambda_{\beta} g_{\mu \nu ; \nu} \right) \]  

(A.3)

This tensor possesses fewer algebraic symmetries than its Riemannian counterpart. As a consequence, the Weyl curvature tensor has two possible contractions. The first contraction is

\[ W^\mu_{\nu \beta} = \frac{1}{2} \tilde{R}^\mu_{\nu \beta} , \]

which reads

\[ \tilde{W}^\mu_{\nu \beta} = \tilde{W}^\mu_{\nu \beta} - \partial_\beta W^\mu_{\nu} - \partial_\nu W^\mu_{\beta} . \]  

(A.4)

This tensor is antisymmetric, conformally invariant and is independent from the metric tensor. Moreover, it is trace free and vanishes identically in the Riemann space (when \( W \) is gradient). Also, it is not very useful for the Weyl-to-Riemann method.

The second contraction of (A.3) leads to the Weylian Ricci tensor \( \tilde{\Lambda}^\mu_{\nu \beta} = \tilde{\Lambda}^\mu_{\nu \beta} - W^\mu_{\nu} \). It is not symmetric and can be safely redefined under the symmetric form \( \tilde{\Lambda}^\mu_{\nu \beta} = \tilde{\Lambda}^\lambda_{\mu \lambda \nu} - W^\mu_{\nu} \). The latter is related to its Riemannian counterparts as follows:

\[ \tilde{\Lambda}^\mu_{\nu \beta} = R^\mu_{\nu \beta} - (\nabla W - 2W^2) g_{\mu \nu} - (\nabla_{(\mu} W_{\nu)} + 2W_{\mu \nu} \tilde{W}_c) . \]  

(A.5)

The contraction of \( \tilde{\Lambda}^\mu_{\nu \beta} \) produces the Weyl–Ricci scalar

\[ \tilde{\Lambda}^\mu_{\nu \beta} = g^{\mu \nu} \tilde{\Lambda}^\mu_{\nu \beta} = R - 6(\nabla W - W^2) . \]  

(A.6)
This tensor is not conformally invariant, since it has a non-vanishing conformal weight. Indeed, it transforms like $\tilde{R} \rightarrow \Omega^{-2}\tilde{R}$. This is related to the fact that the Ricci scalar, unlike the Riemann and Ricci tensors, is not defined only by a connection, but also needs a metric, i.e. a scale. However, the tensor $\tilde{R}_{\mu\nu}$ is conformally invariant.

Thus, the relevant Weylian geometrical tensors for the Weyl-to-Riemann method are $\tilde{R}_{\mu\nu\alpha\beta}$, $\tilde{R}_{\mu\nu}$ and $\tilde{R}_{\mu\nu}$.

Appendix B. Some useful formulas

Let $h_{\mu\nu}$ be a symmetric second-rank tensor with trace $h = h_{\mu}^{\mu}$. Here are some useful formulas:

$$D^2 h_{\mu\nu} = \nabla h_{\mu\nu} - 2(w - 1) h_{\mu\nu} W^{\alpha} + (w^2 - 2w - 2) W^2 h_{\mu\nu} - (w - 2) \nabla W h_{\mu\nu} - 2 W_{\alpha} h_{\mu(\nu)} + 2 W_{(\mu} h_{\nu)\alpha} + 2 w_{\alpha(\mu} h_{\nu)\beta} W^\alpha W^\beta - 2 g_{\mu\nu} h_{\alpha\beta} W^\alpha W^\beta \frac{w}{2}$$

$$D_{(\mu} D_{\nu)} h_{\alpha\beta} = h_{\alpha\beta}^{(\mu;\nu)} - (w - 4) W_{\alpha(\mu} h_{\nu)\beta} - 2 g_{\mu\nu} h_{\alpha\beta}^{(\mu;\nu)} - (w + 2) h_{\alpha(\mu;\nu)} W_{\beta} + (w^2 - 2w - 8) h_{\alpha(\mu} W_{\nu)} - (w + 2) w_{\alpha(\mu} h_{\nu)\beta} + 2 (2 + w) g_{\mu\nu} h_{\alpha\beta} W^\alpha W^\beta + h_{(\mu W_{\nu)} + 2 h \left( \frac{1}{2} W_{\mu(\nu)} - g_{\mu\nu} W^2 + (4 - w) W_{\mu} W_{\nu} \right) \right) \right) (B.1)$$

$$D_{\mu} D_{\nu} h = h_{\mu\nu} + (3 - w) h_{(\mu W_{\nu)} - g_{\mu\nu} h_{\alpha\beta} W^\alpha W^\beta + (2 - w) h \left( \frac{1}{2} W_{\mu(\nu)} - g_{\mu\nu} W^2 + (4 - w) W_{\mu} W_{\nu} \right) \right) \right) \right) (B.2)$$

Here are some useful expressions involving the geometrical tensors:

$$\tilde{R}_{\alpha\beta\mu\nu} h^{\mu\nu} = R_{\alpha\beta\nu} h^{\mu\nu} + (W_{\alpha} W_{\beta} + \nabla_{\alpha} W_{\beta}) h^{\mu\nu} + W^2 h_{\mu\nu} - h_{\mu(\nu} W_{\alpha)\nu} + \frac{w}{2} W_{(\mu(\nu)} - g_{\mu\nu} W^2 + W_{\mu} W_{\nu} \right) \right) \right) \right) (B.3)$$

$$\tilde{R}_{\alpha\mu} h^{\mu\nu} = R_{\alpha\mu} h^{\mu\nu} - 2 h_{\mu(\alpha} (W_{\nu)} + W_{\alpha} W_{\nu}) - 2 \left( \nabla W - 2 W^2 \right) h_{\mu\nu} \right) \right) \right) \right) (B.4)$$

In the Riemann geometry, the covariant derivatives do not commute. As a consequence, one has the following identity:

$$h_{(\mu;\nu)\beta}^{\alpha} - h_{(\mu;\nu)\alpha}^{\beta} = 2 R_{\alpha\beta\mu\nu} h^{\mu\nu} + R_{\alpha(\mu} h_{\nu)}^{\beta} \right) \right) \right) (B.5)$$

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