Some Consequences from the Dirac-Kähler Theory:
on Intrinsic Spinor Sub-structure of the Different Boson Wave Functions

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Properties of tensors equivalent to the direct product of two different 4-spinors are investigated. It is shown that the tensors obey additional 8 nonlinear restrictions, those are presented in Lorentz covariant form. In the context of the Dirac–Kähler theory, such a property can be interpreted as follows: if one wishes to consider the Dirac–Kähler field as consisting of two 4-spinor fields, one must impose additional restrictions on tensors of the Dirac–Kähler field, which leads to a non-linear wave equation for a complex boson field (composed on the base of two 4-spinor fields).

Instead, the use of four bi-spinor fields gives possibility to construct the Dirac–Kähler tensor set of 16 independent components. However, the formulas relating the Dirac-Kähler boson to four fermion fields are completely different from those previously used in the literature. In explicit form, restrictions on four 4-spinor corresponding to separation of different simplest bosons with spin 0 or 1 and various intrinsic parities, are constructed.

I. INTRODUCTION

The Dirac–Kähler field can be described whether as a 2-rank bispinor or as a set of tensor fields [1, 2]. Such a double representation gave rise to many investigations with accent on equivalence of the Dirac–Kähler field

\[(i\gamma^a \partial_a - M) U = 0, \quad (i\gamma^a \partial_a - M) \Psi^{(A)} = 0, \quad A = 1, 2, 3, 4\]

to four 4-spinor fields

\[U = \Psi \otimes \Psi = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} = \{\Psi^{(1)}, \Psi^{(2)}, \Psi^{(3)}, \Psi^{(4)}\}\]

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We will obtain relationships between four fermion fields and the Dirac-Kähler boson, they are completely different from these (2), previously used in the literature (see [1, 2] and references therein).

II. BOSON CONSISTED OF 2 FERMIONS

It is known that the Dirac–Kähler field can be described whether as a 2-rank bispinor or as a set of tensor fields (see notation in [2])

\[ U = \Psi \otimes \Psi = \left( -i\Phi + \gamma^b\Phi_b + i\sigma^{ab}\Phi_{ab} + \gamma^5\Phi + i\gamma^b\gamma^5\Phi_b \right) E^{-1} ; \quad (1a) \]

pseudo-quantities are referred by a tilde symbol. Such a double representation gave rise to many investigations (if not speculations) with accent on equivalence of the Dirac–Kähler field to four bispinor fields. In this connection we will consider the following problem: let it will given a direct product of two arbitrary and different 4-spinor fields (thereby we have a system consisting of two fermions)

\[
\begin{array}{c|c|c|c}
\xi^1 & A & \Sigma^1 & M \\
\xi^2 & B & \Sigma^2 & N \\
\eta_1 & C & H_1 & K \\
\eta_2 & D & H_2 & L \\
\end{array}
\]

\[ U = \Phi \otimes \Psi = \left| \begin{array}{cccc}
AM & AN & AK & AL \\
BM & BN & BK & BL \\
CM & CN & CK & CL \\
DM & DN & DK & DL \\
\end{array} \right| = \left| \begin{array}{c}
\xi \\
\Delta \\
W \\
\eta \\
\end{array} \right| . \quad (1b) \]

In 2-spinor form, expansion (1a) looks as follows

\[
\Delta(x) = \left[ \Psi_l(x) + i\tilde{\Psi}_l(x) \right] \tilde{\sigma}^l \epsilon , \quad W(x) = \left[ \Psi_l(x) - i\tilde{\Psi}_l(x) \right] \sigma^l \epsilon^{-1} ,
\]

\[
\xi(x) = \left[ -i\Psi(x) - \tilde{\Psi}(x) + i\Sigma^{mn}\Psi_{mn}(x) \right] \epsilon^{-1} , \quad \eta(x) = \left[ -i\Psi(x) + \tilde{\Psi}(x) + i\Sigma^{mn}\Psi_{mn}(x) \right] \epsilon . \quad (2)
\]

and inverse relations are

\[
\Psi^l(x) + i\tilde{\Psi}^l(x) = \frac{1}{2} \text{Sp} \left[ \epsilon^{-1} \sigma^l \Delta(x) \right] , \quad \Psi^l(x) - i\tilde{\Psi}^l(x) = \frac{1}{2} \text{Sp} \left[ \sigma^l W(x) \right] , \quad (3a)
\]
\[-i \Psi(x) - \bar{\Psi}(x) = \frac{1}{2} \operatorname{Sp} [\epsilon \xi(x)], \quad -i \Psi(x) + \bar{\Psi}(x) = \frac{1}{2} \operatorname{Sp} [\dot{\epsilon}^{-1} \eta(x)], \quad (3b)\]

\[-i \Psi^{kl}(x) + \frac{1}{2} \epsilon^{klmn} \Psi_{mn}(x) = \operatorname{Sp} [\epsilon \Sigma^{kl} \xi(x)],\]

\[-i \Psi^{kl}(x) - \frac{1}{2} \epsilon^{klmn} \Psi_{mn}(x) = \operatorname{Sp} [\dot{\epsilon}^{-1} \Sigma^{kl} \eta(x)]. \quad (3c)\]

Remind the notation:

\[\sigma^a = (I, +\sigma^j), \quad \sigma_a = (I, -\sigma^j),\]

\[\epsilon = +i \sigma^2 = \begin{vmatrix} 0 & +1 \\ -1 & 0 \end{vmatrix}, \quad \epsilon^{-1} = -i \sigma^2 = \begin{vmatrix} 0 & -1 \\ +1 & 0 \end{vmatrix}.\]

\[\dot{\epsilon} = +i \sigma^2 = \begin{vmatrix} 0 & +1 \\ -1 & 0 \end{vmatrix}, \quad (\dot{\epsilon})^{-1} = -i \sigma^2 = \begin{vmatrix} 0 & -1 \\ +1 & 0 \end{vmatrix}.\]

Specifying the formulas (3c)–(3c), we get

\[\Psi = -\frac{1}{4i} (BM - AN + CL - DK), \quad \bar{\Psi} = -\frac{1}{4} (BM - AN - CL + DK). \quad (4)\]

Vector and pseudo-vector are

\[\Phi^0 = \frac{1}{4} (AL - BK + DM - CN), \quad \bar{\Phi}^0 = \frac{1}{4i} (AL - BK - DM + CN),\]

\[\Psi^1 = -\frac{1}{4} (AK - BL + CM - DN), \quad \bar{\Psi}^1 = -\frac{1}{4i} (AK - BL - CM + DN),\]

\[\Phi^2 = -\frac{i}{4} (AK + BL + CM + DN), \quad \bar{\Phi}^2 = -\frac{1}{4} (AK + BL - CM - DN),\]

\[\Phi^3 = \frac{1}{4} (BK + AL + DM + CN), \quad \bar{\Phi}^3 = \frac{1}{4i} (BK + AL - DM - CN). \quad (5)\]

An identity folds

\[\Psi^a \bar{\Psi}_a = \Psi^0 \bar{\Psi}^0 - \Psi^1 \bar{\Psi}^1 - \Psi^2 \bar{\Psi}^2 - \Psi^3 \bar{\Psi}^3 = 0. \quad (6)\]

Now, let us turn to antisymmetric tensor:

\[\Psi^{01} + i\Psi^{23} = -\frac{i}{2} (BN - AM), \quad \Psi^{01} - i\Psi^{23} = -\frac{i}{2} (DL - CK),\]
\[\Psi^{02} + i\Psi^{31} = -\frac{1}{2}(AM + BN), \quad \Psi^{02} - i\Psi^{31} = -\frac{1}{2}(CK + DL),\]

\[\Psi^{03} + i\Psi^{12} = -\frac{i}{2}(AN + BM), \quad \Psi^{03} - i\Psi^{12} = -\frac{i}{2}(CL + DK); \quad (7)\]

which lead to

\[\Psi^{01} = \frac{i}{4}(AM - BN + CK - DL), \quad \Psi^{23} = \frac{1}{4}(AM - BN - CK + DL), \]

\[\Psi^{02} = -\frac{1}{4}(AM + BN + CK + DL), \quad \Psi^{31} = \frac{i}{4}(AM + BN - CK - DL), \]

\[\Psi^{03} = -\frac{i}{4}(AN + BM + CL + DK), \quad \Psi^{12} = -\frac{1}{4}(AN + BM - CL - DK). \quad (8)\]

Let us consider possibilities for vanishing some of these tensor constituents relevant to \(\Phi \otimes \Psi\).

First, let scalar and pseudo-scalar be zero

\[\Psi = -\frac{1}{4i}(BM - AN + CL - DK) = 0, \quad \tilde{\Psi} = -\frac{1}{4i}(BM - AN - CL + DK) = 0; \quad (9a)\]

de identical to more simple ones

\[BM - AN = 0 \quad \text{or} \quad \xi^1\Sigma^2 - \xi^2\Sigma^1 = 0 \quad \text{or} \quad \frac{\xi^1}{\Sigma^1} = \frac{\xi^2}{\Sigma^2},\]

\[CL - DK = 0 \quad \text{or} \quad \eta_1H_2 - \eta_2H_1 = 0 \quad \text{or} \quad \frac{\eta_1}{H_1} = \frac{\eta_2}{H_2}; \quad (9b)\]

It means that 2-spinors must be proportional to each other

\[\Sigma^\alpha(x) = \mu \xi(x), \quad H_\dot{\alpha}(x) = \nu \eta_\dot{\alpha}(x); \quad (10a)\]

Linear restrictions (10a) can be presented as

\[M = \mu A, \quad N = \mu B, \quad K = \nu C, \quad L = \nu D. \quad (10b)\]

At this

\[\Psi(x) = 0, \quad \tilde{\Psi}(x) = 0, \quad (11a)\]

and

\[\Phi^0 = \frac{1}{4} (AD - BC)(\nu + \mu), \quad \tilde{\Phi}^0 = \frac{1}{4i} (AD - BC)(\nu - \mu), \]
$$\Psi^1 = -\frac{1}{4}(AC - BD)(\nu + \mu), \quad \bar{\Psi}^1 = -\frac{1}{4i}(AC - BD)(\nu - \mu),$$

$$\Phi^2 = -\frac{i}{4}(AC + BD)(\nu + \mu), \quad \bar{\Phi}^2 = -\frac{1}{4}(AC + BD)(\nu - \mu),$$

$$\Phi^3 = +\frac{1}{4}(BC + AD)(\nu + \mu), \quad \bar{\Phi}^3 = +\frac{1}{4i}(BC + AD)(\nu - \mu).$$

Note that vector and pseudovector are proportional to each other

$$(\nu - \mu)\Phi^a = i (\nu + \mu)\bar{\Phi}^a. \quad (11b)$$

In turn, 3-vectors related to antisymmetric tensor take the form

$$s_1 = \Psi^{01} + i\Psi^{23} = -\frac{i}{2}(B^2 - A^2) \mu, \quad t_1 = \Psi^{01} - i\Psi^{23} = -\frac{i}{2}(D^2 - C^2) \nu.$$

$$s_2 = \Psi^{02} + i\Psi^{31} = -\frac{1}{2}(A^2 + B^2) \mu, \quad t_2 = \Psi^{02} - i\Psi^{31} = -\frac{1}{2}(C^2 + D^2) \nu.$$

$$s_3 = \Psi^{03} + i\Psi^{12} = -i AB \mu, \quad t_3 = \Psi^{03} - i\Psi^{12} = -i CD \nu; \quad (11c)$$

they are isotropic and non-orthogonal to each other

$$s^2 = 0, \quad t^2 = 0, \quad s \cdot t = \frac{\mu \nu}{2}(AD - BC)^2. \quad (11d)$$

On may impose more weak requirements. For instance, let it be $\Psi \neq 0, \bar{\Psi} = 0$, then

$$BM - AN + CL - DK \neq 0, \quad BM - AN - CL + DK = 0,$$

so that

$$BM - AN = +(CL - DK) \neq 0, \quad \xi^2 \Sigma^1 - \xi^1 \Sigma^2 = + (\eta_1 H_2 - \eta_2 H_1) \neq 0. \quad (12)$$

Let it be $\Psi = 0, \bar{\Psi} \neq 0$, then

$$BM - AN + CL - DK = 0, \quad BM - AN - CL + DK \neq 0,$$

that is

$$BM - AN = -(CL - DK), \quad \xi^2 \Sigma^1 - \xi^1 \Sigma^2 = -(\eta_1 H_2 - \eta_2 H_1) \neq 0. \quad (13)$$

Now, let us consider the case of vanishing two 4-vectors (see (5b)):

$$\Psi^a = 0, \quad \bar{\Psi}^a = 0;$$
\[ AL - BK + DM - CN = 0, \quad AL - BK - DM + CN = 0, \]

\[ AK - BL + CM - DN = 0, \quad AK - BL - CM + DN = 0, \]

\[ AK + BL + CM + DN = 0, \quad AK + BL - CM - DN = 0, \]

\[ BK + AL + DM + CN = 0, \quad BK + AL - DM - CN = 0; \quad (14a) \]

which is equivalent to

\[ AL - BK = 0, \quad DM - CN = 0, \quad AK - BL = 0, \quad CM - DN = 0, \]

\[ AK + BL = 0, \quad CM + DN = 0, \quad BK + AL = 0, \quad DM + CN = 0, \]

these lead to

\[ AK = 0, \quad BL = 0, \quad DM = 0, \quad CN = 0, \]

\[ AL = 0, \quad BK = 0, \quad CM = 0, \quad DN = 0. \quad (14b) \]

There exist two solutions (however, they are Lorentz invariant only with respect to continuous transformations)

1) \[ A = 0, \ B = 0, \ M = 0, \ N = 0, \quad \xi^\alpha = 0, \quad \Sigma^\alpha = 0; \]

2) \[ C = 0, \ D = 0, \ K = 0, \ L = 0, \quad \eta_{\dot{\alpha}} = 0, \quad H_{\dot{\alpha}} = 0. \quad (15a) \]

The lead respectively to

the case 1)

\[ \Psi = \frac{i}{4} (CL - DK), \quad \bar{\Psi} = \frac{1}{4} (CL - DK), \]

\[ \Psi^{01} + i\Psi^{23} = 0, \quad \Psi^{01} - i\Psi^{23} = -\frac{i}{2} (DL - CK), \]

\[ \Psi^{02} + i\Psi^{31} = 0, \quad \Psi^{02} - i\Psi^{31} = \frac{1}{2} (CK + DL), \]
\[\Psi^{03} + i\Psi^{12} = 0, \quad \Psi^{03} - i\Psi^{12} = -\frac{i}{2}(CL + DK) ; \quad (15b)\]

the case 2)

\[\Psi = \frac{i}{4} (BM - AN), \quad \tilde{\Psi} = -\frac{1}{4} (BM - AN),\]

\[\Psi^{01} + i\Psi^{23} = -\frac{i}{2} (BN - AM), \quad \Psi^{01} = i\Psi^{23} = 0,\]

\[\Psi^{02} + i\Psi^{31} = -\frac{1}{2} (AM + BN), \quad \Psi^{02} - i\Psi^{31} = 0,\]

\[\Psi^{03} + i\Psi^{12} = -\frac{i}{2} (AN + BM), \quad \Psi^{03} - i\Psi^{12} = 0. \quad (15c)\]

There is possible to impose more weak restriction. For instance, let it be \(\tilde{\Psi} = 0: \)

\[\Phi^0 = \frac{1}{2} (AL - BK), \quad AL - BK = + (DM - CN),\]

\[\Psi^1 = -\frac{1}{2} (AK - BL), \quad AK - BL = + (CM - DN),\]

\[\Phi^2 = -\frac{i}{2} (AK + BL), \quad AK + BL = + (CM + DN),\]

\[\Phi^3 = \frac{1}{2} (BK + AL), \quad BK + AL = + (DM + CN); \quad (16a)\]

which is equivalent to

\[AK = CM, \quad BL = DN, \quad AL = DM, \quad BK = CN\]

or

\[
\begin{align*}
\frac{A}{M} &= \frac{C}{K} \quad \Rightarrow \quad \frac{\xi^1}{\Sigma^1} = \frac{\eta_1}{H_1}, \quad \frac{B}{N} = \frac{D}{L} \quad \Rightarrow \quad \frac{\xi^2}{\Sigma^2} = \frac{\eta_2}{H_2}, \\
\frac{A}{M} &= \frac{D}{L} \quad \Rightarrow \quad \frac{\xi^1}{\Sigma^1} = \frac{\eta_2}{H_2}, \quad \frac{B}{N} = \frac{C}{K} \quad \Rightarrow \quad \frac{\xi^2}{\Sigma^2} = \frac{\eta_1}{H_1},
\end{align*}
\]

or even shorter

\[
\frac{\xi^1}{\Sigma^1} = \frac{\xi^2}{\Sigma^2} = \frac{\eta_1}{H_1} = \frac{\eta_2}{H_2} \quad \iff \quad \Sigma^\alpha = \mu \xi^\alpha, \quad H_\alpha = \mu \eta_\alpha. \quad (16b)
\]

It means that two 4-spinors are proportional to each other.
Now, let us impose another restriction, $\Psi^a = 0$ (see (5b)):

\[
\begin{align*}
AL - BK &= -(DM - CN) , \quad \tilde{\Phi}^0 = \frac{1}{2i} (AL - BK) , \\
AK - BL &= -(CM - DN) , \quad \tilde{\Psi}^1 = -\frac{1}{2i} (AK - BL) , \\
AK + BL &= -(CM + DN) , \quad \tilde{\Phi}^2 = \frac{1}{2} (AK + BL) , \\
BK + AL &= -(DM + CN) , \quad \tilde{\Phi}^3 = \frac{1}{2i} (BK + AL) ,
\end{align*}
\]

which is equivalent to

\[
\begin{align*}
AL &= -DM , \quad BK = -CN , \quad AK = -CM , \quad BL = -DN ;
\end{align*}
\]

or

\[
\begin{align*}
\frac{A}{M} = -\frac{C}{K} &\quad \Rightarrow \quad \frac{\xi^1}{\Sigma^1} = -\frac{\eta_1}{H_1} , \quad \frac{B}{N} = -\frac{D}{L} &\quad \Rightarrow \quad \frac{\xi^2}{\Sigma^2} = -\frac{\eta_2}{H_2} \\
\frac{A}{M} = -\frac{D}{L} &\quad \Rightarrow \quad \frac{\xi^1}{\Sigma^1} = -\frac{\eta_2}{H_2} , \quad \frac{B}{N} = -\frac{C}{K} &\quad \Rightarrow \quad \frac{\xi^2}{\Sigma^2} = -\frac{\eta_1}{H_1} ,
\end{align*}
\]

and shorter

\[
\begin{align*}
\frac{\xi^1}{\Sigma^1} = \frac{\xi^2}{\Sigma^2} = -\frac{\eta_1}{H_1} = -\frac{\eta_2}{H_2} &\quad \Leftrightarrow \quad \Sigma^\alpha = \mu \xi^\alpha , \quad H_\dot{\alpha} = -\mu \eta_\dot{\alpha} .
\end{align*}
\]

Equations (17b) describe system invariant only with respect to continuous transformations.

Relations (17b may be written in more short form

\[
\begin{align*}
\frac{M}{A} = \mu , \quad \frac{N}{B} = \mu , \quad \frac{K}{C} = -\mu , \quad \frac{L}{D} = -\mu .
\end{align*}
\]

Corresponding scalar and pseudo-scalars are

\[
\Psi = 0 , \quad \tilde{\Psi} = 0 . \quad (17c)
\]

\[
\begin{align*}
\Psi^{01} + i\Psi^{23} &= -\frac{i}{2} \mu (BB - AA) , \quad \Psi^{01} - i\Psi^{23} = +\frac{i}{2} \mu (DD - CC) , \\
\Psi^{02} + i\Psi^{31} &= -\frac{1}{2} \mu (BB + AA) , \quad \Psi^{02} - i\Psi^{31} = +\frac{i}{2} \mu (DD + CC) , \\
\Psi^{03} + i\Psi^{12} &= -iAN = -iBM , \quad \Psi^{03} - i\Psi^{12} = -iCL = -iDK . \quad (17d)
\end{align*}
\]
Finally, let us consider the case of vanishing tensor $\Phi^{ab} = 0$:

$$BN - AM = 0, \quad DL - CK = 0.$$  

$$AM + BN = 0, \quad CK + DL = 0,$$

$$AN + BM = 0, \quad CL + DK = 0; \quad (18a)$$

these conditions are equivalent to

$$BN = 0, \quad AM = 0, \quad DL = 0, \quad CK = 0,$$

$$BM = -AN, \quad CL = -DK; \quad (18b)$$

There exist two different solutions – they both are Lorentz invariant only with respect to continuous transformations

1) $A = 0, \quad B = 0, \quad K = 0, \quad L = 0, \quad \xi^\alpha = 0, \quad H_\dot{\alpha} = 0,$

1) $C = 0, \quad D = 0, \quad M = 0, \quad N = 0, \quad \eta_{\dot{\alpha}} = 0, \quad \Sigma^\alpha = 0. \quad (18c)$

At this, remaining constituents become much simpler (see (4b) and (5b)):

1) $\Psi = 0, \quad \tilde{\Psi} = 0, \quad \tilde{\Psi}^a = +i\Psi,$

$$\Phi^0 = \frac{1}{4} (DM - CN), \quad \tilde{\Phi}^0 = \frac{i}{4} (DM - CN),$$

$$\Psi^1 = \frac{1}{4} (DN - CM), \quad \tilde{\Psi}^1 = \frac{i}{4} (DN - CM),$$

$$\Phi^2 = -\frac{i}{4} (CM + DN), \quad \tilde{\Phi}^2 = \frac{1}{4} (CM + DN),$$

$$\Phi^3 = \frac{1}{4} (DM + CN), \quad \tilde{\Phi}^3 = \frac{i}{4} (DM + CN); \quad (18d)$$

2) $\Psi = 0, \quad \tilde{\Psi} = 0, \quad \tilde{\Psi}^a = -i\Psi,$
\[ \Phi^0 = \frac{1}{4} (AL - BK), \quad \tilde{\Phi}^0 = -\frac{i}{4} (AL - BK), \]

\[ \Psi^1 = \frac{1}{4} (AK - BL), \quad \tilde{\Psi}^1 = \frac{i}{4} (AK - BL), \]

\[ \Phi^2 = -\frac{i}{4} (AK + BL), \quad \tilde{\Phi}^2 = -\frac{1}{4} (AK + BL), \]

\[ \Phi^3 = \frac{1}{4} (BK + AL), \quad \tilde{\Phi}^3 = -\frac{i}{4} (BK + AL). \]

(18e)

Because according to the process consider above from 8 independent complex quantities, constituents of two 4-spinors, there are constructed 16 complex-valued tensor components, we should expect existence of additional relations to which these tensors must obey. By direct calculation, we verify identities

\[ \Psi^{ab} \Psi_b = -\tilde{\Psi} \tilde{\Psi}^a, \quad \Psi^{ab} \tilde{\Psi}_b = +\tilde{\Psi} \Psi^a; \] (19a)

besides, note identity

\[ \Psi^a \tilde{\Psi}_a = 0. \] (19b)

In connection with (19) we can speculate about physical meaning of the Dirac-Kähler theory. Indeed, in tensor form this system is governed by the linear system

\[ \partial_l \Psi + m \Psi_l = 0, \quad \partial_l \tilde{\Psi} + m \tilde{\Psi}_l = 0, \quad \partial_l \Psi + \partial_a \Psi^a_l - m \Psi_l = 0, \]

\[ \partial_l \tilde{\Psi} - \frac{1}{2} \epsilon^{a mn}_l \partial_a \Psi_{mn} - m \tilde{\Psi}_l = 0, \quad \partial_m \Psi_n - \partial_n \Psi_m + \epsilon^{ab}_{mn} \partial_a \tilde{\Psi}_b - m \Psi_{mn} = 0. \] (20a)

In assumption that the Dirac-Kähler tensors are constructed in terms of two 4-spinor according to (1), we must assume constraints (19) on these tensors – at this the system (20a) results in

\[ \partial_l \Psi + m \frac{\Psi_{ln}}{+\Psi} \Psi^n = 0, \quad \partial_l \tilde{\Psi} + m \frac{\Psi_{ln}}{-\tilde{\Psi}} \Psi^n = 0, \quad \partial_l \Psi - \tilde{\Psi} \partial_a \Psi^a_l - m \frac{\Psi_{ln}}{+\Psi} \tilde{\Psi}^n = 0, \]

\[ \partial_l \tilde{\Psi} - \frac{1}{2} \epsilon^{a mn}_l \partial_a \Psi_{mn} - m \frac{\Psi_{ln}}{-\tilde{\Psi}} \Psi^n = 0, \quad \partial_m \Psi_n - \partial_n \Psi_m + \epsilon^{ab}_{mn} \partial_a \tilde{\Psi}_b - m \Psi_{mn} = 0. \] (20b)

In other word, here we have non-linear boson equations referring to a couple of starting fermion fields It may be specially stressed that additional constrains are lorentz invariant.
III. BOSON CONSISTED OF 4 FERMIONS

Let us examine a 2-rank bispinor, consisting of four different 4-spinors according to the rule

\[ U'' = \Phi \otimes \Psi + \Phi' \otimes \Psi' \]

Possible coefficients in this combination can be eliminated by relevant redifinitions. Such an object from the very beginning contains 16 independent components, so one can expect 16 independent tensor components of the Dirac-Kähler field. Let us use the notation

\[
\begin{align*}
\Phi &= \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}, & \Psi &= \begin{bmatrix} M \\ N \\ K \\ L \end{bmatrix}, & \Phi' &= \begin{bmatrix} A' \\ B' \\ C' \\ D' \end{bmatrix}, & \Psi' &= \begin{bmatrix} M' \\ N' \\ K' \\ L' \end{bmatrix}, \\
U &= \Phi \otimes \Psi = \begin{bmatrix} AM & AN & AK & AL \\ BM & BN & BK & BL \\ CM & CN & CK & CL \\ DM & DN & DK & DL \end{bmatrix}, & \xi &= \begin{bmatrix} \Delta \\ W \end{bmatrix}, & \eta &= \begin{bmatrix} \Delta' \\ W' \end{bmatrix}, \\
U' &= \Phi' \otimes \Psi' = \begin{bmatrix} A'M' & A'N' & A'K' & A'L' \\ B'M' & B'N' & B'K' & B'L' \\ C'M' & C'N' & C'K' & C'L' \\ D'M' & D'N' & D'K' & D'L' \end{bmatrix}, \end{align*}
\]

Now, instead of (3) we have

\[
\begin{align}
\Psi^I(x) + i \tilde{\Psi}^I(x) &= \frac{1}{2} \text{Sp} \left[ \epsilon^{-1} \sigma^I (\Delta + \Delta') \right], & \Psi^I(x) - i \tilde{\Psi}^I(x) &= \frac{1}{2} \text{Sp} \left[ \epsilon \sigma^I (W + W') \right], & (22a) \\
-i \Psi(x) - \tilde{\Psi}(x) &= \frac{1}{2} \text{Sp} \left[ \epsilon (\xi + \xi') \right], & -i \Psi(x) + \tilde{\Psi}(x) &= \frac{1}{2} \text{Sp} \left[ \epsilon^{-1} (\eta + \eta') \right], & (22b) \\
-i \Psi^{kl}(x) + \frac{1}{2} \epsilon^{klmn} \Psi_{mn}(x) &= \text{Sp} \left[ \epsilon \Sigma^{kl}(\xi + \xi') \right], \\
-i \Psi^{kl}(x) - \frac{1}{2} \epsilon^{klmn} \Psi_{mn}(x) &= \text{Sp} \left[ \epsilon^{-1} \Sigma^{kl}(\eta + \eta') \right]. & (22c)
\end{align}
\]

Explicitly, equivalent tensor constituents look

\[
\Psi'' = -\frac{1}{4i} \left[ (BM - AN + CL - DK) + (B'M' - A'N' + C'L' - D'K') \right] = \Psi + \Psi',
\]
\[ \Psi'' = -\frac{1}{4} \left[ (BM - AN - CL + DK) + (B'M' - A'N' - C'L' + D'K') \right] = \bar{\Psi} + \bar{\Psi}' . \] (23a)

4-vectors

\[ \Phi''^0 = \frac{1}{4} \left[ (AL - BK + DM - CN) + (A'L' - B'K' + D'M' - C'N') \right] = \Phi^0 + \Phi'^0 , \]

\[ \tilde{\Phi}''^0 = \frac{1}{4i} \left[ (AL - BK - DM + CN) + (A'L' - B'K' - D'M' + C'N') \right] = \tilde{\Phi}^0 + \tilde{\Phi}'^0 , \]

\[ \Psi''^1 = -\frac{1}{4} \left[ (AK - BL + CM - DN) + (A'K' - B'L' + C'M' - D'N') \right] = \Psi^1 + \Psi'^1 , \]

\[ \bar{\Psi}''^1 = -\frac{1}{4i} \left[ (AK - BL - CM + DN) + (A'K' - B'L' - C'M' + D'N') \right] = \bar{\Psi}^1 + \bar{\Psi}'^1 , \]

\[ \Phi''^2 = -\frac{i}{4} \left[ (AK + BL + CM + DN) + (A'K' + B'L' + C'M' + D'N') \right] = \Phi^2 + \Phi'^2 , \]

\[ \tilde{\Phi}''^2 = -\frac{1}{4} \left[ (AK + BL - CM - DN) + (A'K' + B'L' - C'M' - D'N') \right] = \tilde{\Phi}^2 + \tilde{\Phi}'^2 , \]

\[ \Phi''^3 = \frac{1}{4} \left[ (BK + AL + DM + CN) + (B'K' + A'L' + D'M' + C'N') \right] = \Phi^3 + \Phi'^3 , \]

\[ \tilde{\Phi}''^3 = \frac{1}{4i} \left[ (BK + AL - DM - CN) + (B'K' + A'L' - D'M' - C'N') \right] = \tilde{\Phi}^3 + \tilde{\Phi}'^3 . \] (23b)

antisymmetric tensor

\[ \Psi''^{01} = \frac{i}{4} \left[ (AM - BN + CK - DL) + (A'M' - B'N' + C'K' - D'L') \right] = \Psi^{01} + \Psi'^{01} , \]

\[ \Psi''^{02} = -\frac{1}{4} \left[ (AM + BN + CK + DL) + (A'M' + B'N' + C'K' + D'L') \right] = \Psi^{02} + \Psi'^{02} , \]

\[ \Psi''^{03} = -\frac{i}{4} \left[ (AN + BM + CL + DK) + (A'N' + B'M' + C'L' + D'K') \right] = \Psi^{03} + \Psi'^{03} , \]

\[ \Psi''^{23} = \frac{1}{4} \left[ (AM - BN - CK + DL) + (A'M' - B'N' + C'K' + D'L') \right] = \Psi^{23} + \Psi'^{23} , \]

\[ \Psi''^{31} = \frac{i}{4} \left[ (AM + BN - CK - DL) + (A'M' + B'N' - C'K' + D'L') \right] = \Psi^{31} + \Psi'^{31} , \]

\[ \Psi''^{12} = -\frac{1}{4} \left[ (AN + BM - CL - DK) + (A'N' + B'M' - C'L' - D'K') \right] = \Psi^{12} + \Psi'^{12} . \] (23c)
It is easily verified that now we have not any additional constraints on 16 tensor components. Indeed, remembering on relativistic symmetry considerations we might assume existence of the relationships

\[
\Psi''^a\Psi''^b = \alpha \tilde{\Psi}''^a \tilde{\Psi}''^b + \rho \Psi''^a \Psi''^b, \quad \Psi''^a \tilde{\Psi}''^b = \beta \tilde{\Psi}''^a \Psi''^b + \sigma \Psi''^a \Psi''^b. \tag{24a}
\]

When restricting to only two different 4-spinors, eqs. (24a) give

\[4\Psi^a \tilde{\Psi}^b = 4\alpha \tilde{\Psi}^a + 4\rho \Psi^a, \quad 4\Psi^a \tilde{\Psi}^b = 4\beta \tilde{\Psi}^a + 4\sigma \Psi^a;\]

however here an exact solution is known – therefore we have identities

\[\alpha = -1, \quad \beta = +1, \quad \rho = \sigma = 0. \tag{24b}\]

Thus, we are to verify only the following relations

\[
\Psi''^a \Psi''^b = -\tilde{\Psi}''^a \tilde{\Psi}''^b, \quad \Psi''^a \tilde{\Psi}''^b = +\tilde{\Psi}''^a \Psi''^b. \tag{24c}
\]

It is the matter of simple calculation to show that identities (24c) do not hold; therefore, all 16 tensor components of the Dirac–Kähler field are independent.

Now, the task is to describe restrictions on 4-spinors relevant to pure boson particles with fixed spin (0 or 1) and fixed intrinsic parity:

scalar particle

\[
S = 0, \quad \tilde{\Psi} = 0, \quad \Psi^a = 0, \quad \Psi^a \neq 0, \quad \Psi^a \neq 0; \tag{27a}
\]
pseudo-scalar particle

\[
\tilde{S} = 0, \quad \Psi = 0, \quad \Psi^a = 0, \quad \tilde{\Psi}^a \neq 0, \quad \tilde{\Psi}^a \neq 0; \tag{27b}
\]
vector particle

\[
S = 0, \quad \tilde{\Psi} = 0, \quad \Psi = 0, \quad \Psi^a = 0, \quad \Psi^a \neq 0, \quad \Psi^a \neq 0; \tag{27c}
\]
pseudo-vector particle

\[
S = 0, \quad \tilde{\Psi} = 0, \quad \Psi = 0, \quad \Psi^a = 0, \quad \tilde{\Psi}^a \neq 0, \quad \Psi^a \neq 0; \tag{27d}
\]

First, let us find constrains separating **scalar particle** (27a). From

\[
\tilde{\Psi}'' = -\frac{1}{4} [BM - AN - CL + DK + B'M' - A'N' - C'L' + D'K'] = 0.\]
it follows

\[(BM - AN + B'M' - A'N') = +(CL - DK + C'M' - D'K') \] ; \hspace{1cm} (28a)

from

\[\tilde{\Psi}'0 = \frac{1}{4i} \left[ AL - BK - DM + CN + A'L' - B'K' - D'M' + C'N' \right] = 0 ,\]

\[\tilde{\Psi}'3 = \frac{1}{4i} \left[ BK + AL - DM - CN + B'K' + A'L' - D'M' - C'N' \right] = 0\]

we get

\[AL - DM + A'L' - D'M' = 0 , \quad BK - CN + B'K' - C'N' = 0 ; \hspace{1cm} (28b)\]

restrictions

\[\tilde{\Psi}'1 = -\frac{1}{4i} \left[ AK - BL - CM + DN + A'K' - B'L' - C'M' + D'N' \right] = 0 ,\]

\[\tilde{\Psi}'2 = -\frac{1}{4} \left[ AK + BL - CM - DN + A'K' + B'L' - C'M' - D'N' \right] = 0\]

give

\[AK - CM + A'K' - C'M' = 0 , \quad BL - DN + B'L' - D'N' = 0 ; \hspace{1cm} (28b')\]

and

\[\Psi''01 = \frac{i}{4} \left[ AM - BN + CK - DL + A'M' - B'N' + C'K' - D'L' \right] = 0 ,\]

\[\Psi''23 = \frac{1}{4} \left[ AM - BN - CK + DL + A'M' - B'N' - C'K' + D'L' \right] = 0\]

give

\[AM - BN + A'M' - B'N' = 0 , \quad CK - DL + C'K' - D'L' = 0 ; \hspace{1cm} (28c)\]

restrictions

\[\Psi''02 = -\frac{1}{4} \left[ AM + BN + CK + DL + A'M' + B'N' + C'K' + D'L' \right] = 0 ,\]

\[\Psi''31 = \frac{i}{4} \left[ AM + BN - CK - DL + A'M' + B'N' - C'K' - D'L' \right] = 0\]
lead to

\[ AM + BN + A'M' + B'N' = 0, \quad CK + DL + C'K' + D'L' = 0; \]  

\[ (28c') \]

from

\[ \Psi''^{03} = -\frac{i}{4} \left[ AN + BM + CL + DK + A'N' + B'M' + C'L' + D'K' \right] = 0, \]

\[ \Psi''^{12} = -\frac{1}{4} \left[ AN + BM - CL - DK + A'N' + B'M' - C'L' - D'K' \right] = 0 \]

it follows

\[ AN + BM + A'N' + B'M' = 0, \quad CL + DK + C'L' + D'K' = 0. \]  

\[ (28c'') \]

Thus we have found 11 additional constraints on 16 variables of four 4-spinors, separating pseudo-scalar particle.

Similarly, for case of **pseudo-scalar particle** (27b) we have:

\[ (BM - AN + B'M' - A'N') = -(CL - DK + C'L' - D'K') ; \]

\[ AL + DM + A'L' + D'M' - C'N' = 0, \quad BK + CN + B'K' + C'N' = 0; \]  

\[ (29a) \]

\[ AM - BN + A'M' - B'N' = 0, \quad CK - DL + C'K' - D'L' = 0; \]

\[ AM + BN + A'M' + B'N' = 0, \quad CK + DL + C'K' + D'L' = 0; \]

\[ AN + BM + A'N' + B'M'' = 0, \quad CL + DK + C'L' + D'K' = 0. \]  

\[ (29b) \]

Again, we have found 11 additional constraints on 16 variables of four 4-spinors.

In similar manner let us consider the case of the **vector particle** (27c):

\[ \Psi'' = -\frac{1}{4t} \left[ (BM - AN + CL - DK) + (B'M' - A'N' + C'L' - D'K') \right] = 0, \]

\[ \tilde{\Psi}'' = -\frac{1}{4} \left[ (BM - AN - CL + DK) + (B'M' - A'N' - C'L' + D'K') \right] = 0 \]
\[ BM - AN + B'M' - A'N' = 0, \quad CL - DK + C'L' - D'K' = 0; \quad (30a) \]

\[ \tilde{\Phi}''^0 = \frac{1}{4i} \left[ (AL - BK - DM + CN) + (A'L' - B'K' - D'M' + C'N') \right] = 0, \]

\[ \tilde{\Phi}''^3 = \frac{1}{4i} \left[ (BK + AL - DM - CN) + (B'K' + A'L' - D'M' - C'N') \right] = \tilde{\Phi}^3 + \tilde{\Phi}^3 \implies \]

\[ AL - DM + A'L' - D'M' = 0, \quad BK - CN + B'K' - C'N' = 0; \quad (30b) \]

\[ \tilde{\Psi}''^1 = -\frac{1}{4i} \left[ (AK - BL - CM + DN) + (A'K' - B'L' - C'M' + D'N') \right] = 0, \]

\[ \tilde{\Phi}''^2 = -\frac{1}{4} \left[ (AK + BL - CM - DN) + (A'K' + B'L' - C'M' - D'N') \right] = 0 \implies \]

\[ AK - CM + A'K' - C'M' = 0, \quad BL - DN + B'L' - D'N' = 0. \quad (30c) \]

Eqs. (30a,b,c) provide us with 6 additional constraints separating the vector particle.

For the case of pseudo-vector particle we have:

\[ BM - AN + B'M' - A'N' = 0, \quad CL - DK + C'L' - D'K' = 0; \]

\[ AL + DM + A'L' + D'M' = 0, \quad BK + CN + B'K' + C'N' = 0; \]

\[ AK + CM + A'K' + C'M' = 0, \quad BL + DN + B'L' + D'N' = 0. \quad (31) \]

Again we have obtained 6 constraints on 16 components of four 4-spinors.

Thus, restrictions on four 4-spinors corresponding to separation of different simple boson with spin 0 or 1 and various intrinsic parities, are constructed in explicit form.

IV. CONCLUSIONS

Specially note that the use of four 4-spinor fields gives possibility to construct the Dirac–Kähler tensor set of 16 independent components. However, the formulas relating the Dirac-Kähler boson to four fermion fields are completely different from those previously used in the literature.

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