Decay Rate of Coherent Field Oscillation

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ABSTRACT

In recent studies it has become increasingly clear that presence of infinitely many instability bands of the parametric resonance plays crucial roles in the phenomenon of particle production under periodic classical field oscillation. We extend previous works to a general class of models including both the Yukawa and the quartic type of couplings of the classical field to quantum bose fields. Decay rate from the $n$–th band is derived in the small amplitude limit using the functional Schrödinger picture. It is then shown that this analytic result of the decay rate can also be derived as the zero momentum limit of a physical process, $n$ particles that comprise the classical homogeneous field decaying simultaneously into 2 bose particles. The latter approach uses ordinary perturbation theory, hence the former result is a novel resummation of many perturbative amplitudes, which usually becomes complicated for a large $n$ order.

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1 Introduction

Particle production under periodic perturbation is a physical process that
governs many problems in different areas of physics. In cosmology in particular, it
is a dominant process of entropy generation after inflation, since inflation dilutes
away essentially everything in the observable part of our universe and leaves behind
the field (inflaton) oscillation [1]. As another example one may cite presence of
flat field directions in supersymmetric models. It may either give good effects such
as the Affleck-Dine scenario of baryogenesis [2], or bad effects such as the Polonyi
or the modular problem [3] that may potentially destroy the successful result of
nucleosynthesis. In all these problems effects of the parametric resonance are crucial
if the initial oscillation amplitude is large enough. Thus a deep understanding of the
basic process is inevitable in any of these applications.

In our previous study [4], [5] we proposed to formulate the problem of particle
production by taking a short time average of the quantum density matrix over a few
oscillation periods. In our view this time average is a substitute for a more physical
means of the coarse graining when created particles interact among themselves or
with some other particles. A large fluctuation of the quantum system coupled to the
classical periodic oscillation is essential to this viewpoint, and the quantum system
exhibits a nearly classical behavior even before taking the time average.

In this work we bridge a gap by offering in the language of ordinary perturbation
of interacting quantum field theory a simple understanding of the decay formula
previously derived. We do this by generalizing the coupling of oscillating field to
quantum boson fields beyond that previously analyzed. The model Lagrangian for
the oscillator coupling is thus taken as

\[ \mathcal{L}_{\text{int}} = -\frac{1}{2} g_4^2 \xi^2 \varphi^2 + \frac{1}{2} g_Y m_\xi \xi \varphi^2, \]  

with \( \xi \) the classical oscillating field that takes a simple sinusoidal form,

\[ \xi(t) = \xi_0 \sin(m_\xi t). \]  

\( m_\xi \) is the mass of the \( \xi \) field, and \( \varphi \) is a generic boson field treated here as a quantum
field. In the rest of discussion it is not important to assume a particular relation
between the two dimensionless couplings, the quartic coupling \( g_4 \) and the Yukawa
coupling \( g_Y \), but in order to organize results systematically we assume these to be
of the same order of magnitude; \( g_4/g_Y = O[1] \).
The quantum state coupled to the external $\xi$ oscillation evolves according to the functional Schrödinger equation, which is much simplified to an independent set of equations in Fourier modes due to the translational invariance of the background oscillation. Thus the state vector in each $\vec{k}$ mode obeys the Schrödinger equation with variable frequency,

$$\omega_k^2(t) = \vec{k}^2 + g_1^2 \xi^2(t) - g_2^2 m_\xi \xi(t).$$  \hspace{1cm} (3)

Precise relation between the quantum wave function $\psi_k(q_k, t)$ and the classical oscillator equation has been established [4]. It takes the most definitive form if one starts with an initial quantum state of the ground state of some reference frequency $\omega$; $\psi_k(q_k, t) = (\omega/\pi)^{1/4} e^{-i/2 \omega t} \exp[-\frac{1}{2} \omega q_k^2]$, for $t \approx 0$. At any finite time $t$

$$\psi_k(q_k, t) = \frac{1}{|u_k(t)|} \exp\left[ -\frac{1}{2} |u_k(t)|^2 q_k^2 + \frac{i}{4} \frac{d}{dt} \ln |u_k(t)|^2 \cdot q_k^2 \right],$$ \hspace{1cm} (4)

where the time dependent complex function $u_k(t)$ obeys the classical oscillator equation with the definite initial condition,

$$\frac{d^2 u_k}{dt^2} + \omega_k^2 u_k = 0, \quad u_k(0) = (\omega/\pi)^{-1/2}, \quad \dot{u}_k(0) = i\omega u_k(0).$$ \hspace{1cm} (5)

The meaning of the classical oscillator amplitude $u_k(t)$ is thus unambiguous: its modulus $|u_k|$ governs the Gaussian width of the wave function and at the same time its logarithmic derivative gives the fluctuation of the state in terms of the phase factor. This can be seen most clearly in the Fock space base of harmonic oscillator of the reference frequency: the diagonal density matrix element derived from this wave function has a smallest fluctuation and is related to the average particle number $\langle N_\omega \rangle$ according to $\rho_{2n,2n} \rightarrow e^{-n/\langle N_\omega \rangle}/\sqrt{n!} \langle N_\omega \rangle$ [4], while the off-diagonal density matrix elements contain wildly varying signs of $\pm$, giving zero after taking the time average over a few oscillation periods. This phenomenon occurs in infinitely many band regions of the parameter space $(k, \xi_0)$ within which a generic classical $u_k$ exponentially grows; $u_k(t) \rightarrow e^{\lambda \xi t/2} \times$ (periodic function) with $\lambda > 0$.

The exponential growth implies rapid excitation of high harmonic oscillator levels. With the short time average it can be interpreted that particle production takes place with $\langle N_\omega \rangle \propto e^{\lambda \xi t}$.

After coarse graining of the time average one can discuss the decay law of the initially prepared ground state: it follows the exponential form,

$$\rho_{00} \approx e^{-\Gamma V t}, \quad \Gamma = \sum_{n=1}^{\infty} \Gamma_n, \quad \Gamma_n = \frac{m_\xi}{2 V} \sum_{\vec{k} \in n{-}\text{band}} \lambda_{\vec{k}}.$$ \hspace{1cm} (6)
Γ is the total decay rate per unit volume and per unit time. Computation and interpretation of the decay rate Γ_\text{n} of the n-th band is our main task in the rest of discussion.

2 Small Amplitude Analysis Revisited

It is customary to recast the classical equation into a dimensionless form,

\[
\frac{d^2 u}{dz^2} + \left[ h - 2\theta_1 \cos(4z) - 2\theta_Y \sin(2z) \right] u = 0, 
\]

with \( m \) the mass of the boson quantum field \( \varphi \). Following the general theorem for solutions of differential equation with periodic coefficients, one expands the solution in the form \([\text{I}]\),

\[
u(z) = \sum_{k=-\infty}^{\infty} c_k e^{(\lambda+i\eta+2ik)z},
\]

with \( n = 1, 2, 3, \cdots \). The condition of the existence of non-trivial solution is then equivalent to the condition of non-vanishing matrix determinant of infinite dimensions, with the diagonal entry of \( \gamma_k = h + (\lambda + i\eta + 2ik)^2 \), and the next off-diagonal entry of \( \pm i\theta_Y \) and the next-to-next off-diagonal entry of \(-\theta_4 \) and all others \( = 0 \).

In the small amplitude limit of \( |\xi_0| \ll 1 \) the n-th instability band starts at \( h = n^2 \). The boundary curve \( h = h^{(n)}(\xi_0/m_\xi) \) dividing the stable and the unstable bands corresponds to the eigen-modes of the form, \( \cos(nz) \) and \( \sin(nz) \), in the \( \xi_0 \to 0 \) limit. With the assumption that the coefficients, \( c_0 \) and \( c_{-n} \), are dominant and the rest of \( c_k, \lambda \), and \( h - n^2 \) are all small, one may simplify the structure of this matrix such that \( \gamma_k = \gamma_{-n-k} = -4k(n+k) \) \((k \neq 0, -n)\) and \( \gamma_0 = h - n^2 + 2in\lambda, \ \gamma_{-n} = h - n^2 - 2in\lambda \). Dividing the matrix into a few parts, one first solves the top \((k \geq 1)\) and the down \((k \leq -n-1)\) infinite dimensional parts in favor of \( c_1, c_2 \) and \( c_{-n-1}, c_{-n-2} \),

\[
c_2 = D_{21}^{-1}(\epsilon_1 c_0 + \epsilon_2 c_{-1}) + D_{22}^{-1} \epsilon_2 c_0, \quad c_1 = D_{11}^{-1}(\epsilon_1 c_0 + \epsilon_2 c_{-1}) + D_{12}^{-1} \epsilon_2 c_0,
\]

\[
c_{-n-1} = D_{11}^{-1}(\epsilon_1^* c_{-n} + \epsilon_2^* c_{-n+1}) + D_{12}^{-1} \epsilon_2^* c_{-n}, \quad c_{-n-2} = D_{21}^{-1}(\epsilon_1^* c_{-n} + \epsilon_2^* c_{-n+1}) + D_{22}^{-1} \epsilon_2^* c_{-n},
\]

with \( \epsilon_1 = -i\theta_Y \) and \( \epsilon_2 = \theta_4 \) and the matrix inverse \( D^{-1} \) defined as a limit of big matrix in the left-upper and the right-down (identical) corners. One next solves the
central block \((-1 \geq k \geq -n + 1\) for \(c_{-1}, c_{-2}, c_{-n+2}, c_{-n+1}\) in terms of \(c_0, c_{-n}\). Here one needs to invert, ignoring subleading terms,

\[
\begin{pmatrix}
\gamma_{-1} - \epsilon_1 & -\epsilon_2 & 0 & \cdots & \cdots & 0 \\
-\epsilon_1^* & \gamma_{-2} - \epsilon_1 & -\epsilon_2 & 0 & \cdots & 0 \\
-\epsilon_2^* & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & -\epsilon_2 \\
0 & \cdots & 0 & -\epsilon_2^* & -\epsilon_1^* & \gamma_{-n+2} - \epsilon_1 \\
0 & \cdots & 0 & -\epsilon_2^* & -\epsilon_1^* & \gamma_{-n+1}
\end{pmatrix}
\begin{pmatrix}
c_{-1} \\
c_{-2} \\
c_{-n+2} \\
c_{-n+1}
\end{pmatrix}
= \begin{pmatrix}
\epsilon_1^* c_0 \\
\epsilon_2^* c_0 \\
\epsilon_2 c_{-n} \\
\epsilon_1 c_{-n}
\end{pmatrix}.
\]

(11)

One finally solves the determinental condition for the equation \(c_0, c_{-n}\), which reads to the leading order as

\[
\det \begin{pmatrix}
h - n^2 + 2in\lambda - A & -C \\
-C^* & h - n^2 - 2in\lambda - A
\end{pmatrix} = 0,
\]

(12)

\[A = |\epsilon_1|^2(D_{11}^{-1} + E_{11}^{-1}), \quad C = \epsilon_1(\epsilon_1 E_{1,n-1}^{-1} + \epsilon_2 E_{1,n-2}^{-1}) + \epsilon_2(\epsilon_1 E_{2,n-1}^{-1} + \epsilon_2 E_{2,n-2}^{-1}),\]

(13)

where \(E\) is the matrix in the left hand side of eq.(11).

We regard \(\epsilon_1 = O[\epsilon]\) and \(\epsilon_2 = O[\epsilon^2]\) since \(g_4/g_Y = O[1]\), and work out matrix elements to leading orders of \(\epsilon\). The net result may be summarized as the formula for the growth rate \(\lambda\),

\[
\lambda_n = \frac{1}{2n} \sqrt{\Delta_n^2 - (h - n^2 - \frac{\theta_Y^2}{2(n^2 - 1)})^2}, \quad \Delta_n = \frac{|C_n|}{2^{2(n-1)[(n-1)!]^2}},
\]

(14)

\[
C_n = \det \begin{pmatrix}
-i\theta_Y & \theta_4 & 0 & \cdots & \cdots & 0 \\
-\gamma_{-1} - i\theta_Y & \theta_4 & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & 0 & -\gamma_{-n+2} - i\theta_Y & \theta_4 \\
0 & \cdots & 0 & -\gamma_{-n+1} & -i\theta_Y
\end{pmatrix},
\]

(15)

with \(\gamma_{-k} = 4k(n-k)\). This result generalizes the previous one in ref.[4] in which \(\theta_4 = 0\) was assumed, hence \(C_n = (-i\theta_Y)^n\). The decay rate of the \(n\)-th band is
computed by summing modes within the band in the narrow width approximation,
\[ \Gamma_n = \frac{m_\xi^4}{256\pi} \sqrt{1 - \frac{4m^2}{n^2m_\xi^2}} \Delta_n^2. \] (16)

The case of \( n = 1 \) is an exception to this general formula that must be treated separately, resulting in
\[ \lambda_1(k, \xi_0) = \frac{1}{2} \sqrt{\theta_Y^2 - (h - 1 + \frac{\theta_Y^2}{8})^2}, \quad \Gamma_1 = \frac{g_Y^2m_\xi^2\xi_0^2}{64\pi} \sqrt{1 - \frac{4m^2}{m_\xi^2}}. \] (17)

### 3 Physical Interpretation in terms of Familiar Perturbation Theory

We shall now offer interpretation of the decay rate \( \Gamma_n \) of \( n \)-th band in terms of the ordinary perturbation theory. Ordinary perturbation uses the particle picture and for that purpose it is useful to recast the rate formula in terms of reaction rates of indivisual particles by dividing some powers of the particle number density \( n_\xi = \frac{1}{2} m_\xi \xi_0^2 \). Let us first note the dependence of the rate \( \Gamma_n \) on the oscillation amplitude, \( \propto \xi_0^{2n} \), and the momentum of final \( \varphi \) particles in the zero momentum limit of \( E_\xi \to m_\xi \) for the process, \( n \xi \to \varphi \varphi \); \( p_\varphi = \frac{nm_\xi}{2} \sqrt{1 - \frac{4m^2}{m_\xi^2}} \). This factor appears in the rate formula, eq. (10).

The simplest case is the decay rate of the first band \( \Gamma_1 \), the rate per unit volume and per unit time. Since the unit volume contains \( n_\xi \) particles, the one particle rate having the dimension of the inverse time is
\[ \frac{\Gamma_1}{n_\xi} = \frac{g_Y^2m_\xi}{32\pi} \frac{1}{1 - \frac{4m^2}{m_\xi^2}}. \] (18)

This exactly coincides with the decay rate of one \( \xi \) particle computed in the conventional way using the Yukawa coupling to \( \varphi \), \( \frac{1}{2} g_Y m_\xi \xi^2 \).

The next thing to be checked is the decay rate from the 2nd instability band \( \Gamma_2 \propto \xi_0^4 \). This time one divides the quantity \( \Gamma_2 \) by \( n_\xi^2 \) since two \( \xi \) particles are involved,
\[ \frac{\Gamma_2}{n_\xi^2} = \frac{(g_Y^2 - g_Y^2)^2}{16\pi} \frac{1}{m_\xi} \sqrt{1 - \frac{m^2}{m_\xi^2}}. \] (19)

In ordinary perturbation theory the amplitude for the 2-body process, \( \xi \xi \to \varphi \varphi \), consists of 3 distinct Feynman amplitudes, the contact term with a single quartic
coupling, the t-channel and the u-channel exchange diagrams with two vertices of the Yukawa coupling, adding to
\begin{equation}
- 2ig_4^2 - ig_Y^2 m_\xi^2 \left( \frac{1}{t - m^2} + \frac{1}{u - m^2} \right).
\end{equation}
(20)

In the zero momentum limit of $E_\xi \to m_\xi$, $t - m^2 = u - m^2 \to -m_\xi^2$, giving the total amplitude $2i(g_Y^2 - g_4^2)$. Working out the phase space factor, one finds out that the invariant rate $\sigma_{\text{rel}}$ given by flux $\times$ cross section is identical to $\Gamma_2/n_\xi^n$ given above.

To proceed to the general $n$-th order case, one notes that the propagator in the zero momentum limit is given by
\begin{equation}
\frac{i}{(\sum_{i=1}^k p_i - q_i)^2 - m^2} \to \frac{-i}{k(n - k) m_\xi^2},
\end{equation}
(21)
where $p_i$ is an initial $\xi$ momentum and $q_i$ is one of the final $\varphi$ momenta. Hence the $n$-th order invariant amplitude containing the Yukawa couplings alone is
\begin{equation}
\frac{i (g_Y m_\xi)^n}{m_\xi^{2(n-1)}} \prod_{k=1}^{n-1} \frac{1}{k(n - k)} = \frac{ig_Y^n}{m_\xi^{n-2}} \frac{1}{[(n-1)!]^2}.
\end{equation}
(22)

It is not difficult to check that this leads to the invariant rate precisely equal to the corresponding decay rate of the $n$-th band, $\Gamma_n/n_\xi^n$. What is left to be shown is then the relative weight of the Yukawa and the quartic contribution. Subdiagrams of $\xi \xi \to \varphi \varphi$ for the whole $n_\xi \to \varphi \varphi$ process contribute with a factor, $\frac{ig_Y^2}{k(n - k)}$ for the Yukawa coupling case, considering the propagator above, and with $-ig_4^2$ for the contact quartic coupling case. The ratio of these two terms is exactly equal to the ratio in
\begin{equation}
\det \begin{pmatrix}
-2ig_Y \frac{\xi_0}{m_\xi} & g_Y^2 \frac{\xi_0^2}{m_\xi^2} \\
-4k(n - k) & -2ig_Y \frac{\xi_0}{m_\xi}
\end{pmatrix} = -4k(n - k) \left[ \frac{g_Y^2}{k(n - k)} - g_4^2 \right] \left( \frac{\xi_0}{m_\xi} \right)^2,
\end{equation}
(23)
that appears in the decomposition of the submatrix of $C_n$, eq. 15 for the formula $\Gamma_n$. This proves our assertion that the decay rate of the $n$-th band $\Gamma_n/n_\xi^n$ is equal to the zero momentum limit of the invariant rate computed in the ordinary perturbation theory.

Needless to say, this interpretation is valid only in the small amplitude limit. In applications to realistic problems analytic formula in the large amplitude regime is indispensable. In cosmological application the parameter region along $h = 2\theta_4$, or more precisely $h - 2\theta_4 (= 4E_\xi^2 + m^2)/m_\xi^2 \ll \theta_4$ with the large amplitude of $\theta_4 \gg 1$
is most important, whose case has been worked out in ref \cite{4}. In the present note we clarified the physical meaning of the $n-$th band decay formula when it is taken to the $\xi_0 \to 0$ limit. In view of our result here the large amplitude formula in ref \cite{4} may be considered non-perturbative effect which is directly joined to the present perturbative result.
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