Aspects of the $\mathcal{N} = 4$ SYM amplitude - Wilson polygon duality

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Abstract

We discuss formulation of Wilson polygon - MHV amplitude duality at the perturbative level in various regularizations. For four gluons it is shown that at one loop one can formulate diagrammatic correspondence interpolating between the dimensional regularization and the off-shell one. We suggest new interpretation of all types of box diagrams in terms of the dual simplex in dimensional regularization and describe its degeneration to the Wilson polygon. The interesting nullification phenomena for the low-energy amplitudes in the Higgsed phase has been found.
1 Introduction

One of the most surprising recent developments was the discovery of the correspondence between the Wilson polygons

\[ W(C_n) := \frac{1}{N} Tr P \exp[i g \int d\tau \dot{x}^\mu(\tau) A_\mu(x(\tau))] \]  

and MHV scattering amplitudes in the \( \mathcal{N} = 4 \) SYM

\[ \text{Fin log}\left( \frac{A_{\text{all-loop}}^{\text{MHV}}}{A_{\text{tree}}^{\text{MHV}}} \right) = \text{Fin log}\left( \langle W(p_1, p_2, ..., p_N) \rangle \right). \] 

It was first suggested at strong coupling [1] and later was clarified perturbatively at one-loop [2, 3], two-loop level [4, 5] and at strong coupling [6]. Since the origin of the correspondence was miraculous it is necessary to make steps towards its clarification. In [7] the derivation of one-loop MHV - Wilson loop duality was given from the first principle. The analysis of [7] clearly demonstrates that Feynman parameters in the loop integral parameterize the space where the Wilson polygon is defined in.

In this paper we continue to investigate the explicit mapping between two objects and shall focus on the several topics. First, we make continuation from dimensional to off-shell regularization for one-loop four gluons case and formulate the correspondence. Secondly, we consider Higgsed phase of \( \mathcal{N} = 4 \) SYM recently considered in [8], where the VEV of scalars provide the regularization in the gauge invariant manner and show that “AdS” regularized Wilson loop - Higgsed \( \mathcal{N} = 4 \) SYM amplitude duality, while works at one loop breaks down at two loops. Third, we shall consider one loop box diagrams harder than 2me boxes. These diagrams enter the answers for non-MHV one loop amplitudes in dimensional regularization

\[ A_{n:1} = i (2\pi)^4 \delta^4(p) \sum \left( C^{4m} I^{4m} + C^{3m} I^{3m} + C^{2mh} I^{2mh} + C^{2me} I^{2me} + C^{1m} I^{1m} \right). \] 

We shall demonstrate that in this case there exists dual object, namely, simplex which get reduced to the Wilson polygon for MHV amplitude.

The object dual to boxes harder than two-mass easy which appear in \( N^k \) MHV amplitudes turns out to be not polygon, but a simplex. The origin of the simplex is fixed by claiming the dual edges to be light-like. Namely one has to add the additional vertex connected with the vertices of the Wilson polygon and the gauge field propagator is attached just to this extra vertex of the simplex. We demonstrate that at one loop level this picture degenerates to the known duality when we reduce generic box down to the 2me one. In the case of triangle diagram we show that in the proper limit the dual simplex for the 3m diagram reduces to the Wilson triangle with the additional vertex found in [7].

In the last part we will discuss how one can see AdS geometry from weak coupling point of view from both amplitude and Wilson loop side. In the Higgsed phase we
also provide the recipe of the calculation of the low-energy non-MHV amplitude from the effective actions in the external field and observe the peculiar properties of the amplitudes.

The paper is organized as follows. In Section 2 we derive the duality to the “off-shell” regime at one loop level via the explicit mass dependent change of variables. Section 3 concerns the interpretation of the 2mh and harder boxes in terms of dual simplex. Section 4 is devoted to some two-loop dual calculation in the “AdS” regularization. In section 5 we show how AdS$_3$ geometry emerges from massless amplitudes and Wilson loop calculations and in Section 6 the low-energy amplitudes in the Higgsed phase have been obtained. In the last Section we shall make comments on the open questions. In Appendices we collect some relevant notations and integrals as well as present an interesting geometrical interpretation behind the divergencies of the integrals.

2 Going off-shell from four gluons amplitude - Wilson polygon one-loop duality

Here we discuss the duality for four gluon amplitude at one loop and its possible extension off-shell generalizing the approach of [7] to the case of off-shell external particles. In the following section \( D_{IR} = 4 + 2\epsilon \) and \( D_{UV} = 4 - 2\epsilon \).

2.1 On-shell form of duality

Let us remind the formulation of the on-shell duality. At one loop the vacuum expectation of light-like Wilson loop has the following structure

\[
\mathcal{W} = 1 - a(\pi \mu_{UV}^2)^\epsilon \Gamma(1 - \epsilon) \sum_{1 \leq j,k \leq 4} \tilde{I}_{jk}^W (4 - 2\epsilon) \tag{4}
\]

where we have six \((jk)\) diagrams \((12, 13, 14, 23, 24, 34)\), or three pairs of different diagrams. Notations which we use are given in Appendix A.

Four-gluon amplitude at one loop is given by the following expression

\[
\mathcal{M}_{4}^{MHV} = 1 - \frac{a}{2} s u I_{4}^{0m} (4 + 2\epsilon) = 1 - a(\mu_{IR}^2)^\epsilon \Gamma(1 - \epsilon) \frac{\Gamma(1 + \epsilon)^2}{\Gamma(1 + 2\epsilon)} F_4(4 + 2\epsilon) \tag{5}
\]

where \(I_{4}^{0m}\) is massless box diagram.

In this particular case duality can be stated in several different forms:

Version 1 (stronger one).

\[
\sum_{1 \leq j,k \leq 4} \tilde{I}_{jk}^W (4 - 2\epsilon) = F_4(4 + 2\epsilon) \tag{6}
\]
which is true up to all orders in $\epsilon$. Such form of the duality is known to be violated at higher loops due to different divergent sub-leading parts which are governed by different functions. This version also can be rewritten as the following equality
\[
\frac{\Gamma(1 + \epsilon)^2}{\Gamma(1 + 2\epsilon)} \mathcal{W} = \mathcal{M}_4
\]  
(7)

**Version 2 (weaker one).**
\[
\text{Fin log}[\mathcal{W}] = \text{Fin log}[\mathcal{M}_4].
\]  
(8)

where Fin[...] means equality of finite parts up to arbitrary kinematics independent constant. Up to date the most powerful check of this statement is two-loop six gluons calculation [4, 5]. Here we will show how the duality in the strong form can be pushed off-shell and what terms can be attributed to each Wilson diagram in the dual picture.

### 2.2 Connection between massive box diagrams in different dimensions

Remind that in our derivation [7] of the duality in the dimensional regularization two step were important. First, the peculiar change of variables in the space of the Feynman parameters and secondly the relation between the box diagrams in $D$ and $(D - 2)$ dimensions [11]. To formulate the duality in the "off-shell" case let us use once again the connection between integrals in different dimensions that is given in the appendix A for the box diagram with $D = 6 + 2\epsilon$ and $p_i^2 = m^2$. For this case we get
\[
I_4(6 + 2\epsilon, m) = \frac{1}{(1 + 2\epsilon)z_0} \left( I_4(4 + 2\epsilon, m) - \sum_{i=1}^{4} z_i I_3(4 + 2\epsilon, m; 1 - \delta_{ki}) \right)
\]  
(9)

where
\[
\begin{align*}
  z_0 &= \sum_{i=1}^{4} z_i = 2 \frac{s + u - 4m^2}{su - 4m^4} \\
  z_1 &= z_3 = \frac{u - 2m^2}{su - 4m^4} \\
  z_2 &= z_4 = \frac{s - 2m^2}{su - 4m^4}
\end{align*}
\]  
(10)

Let's rewrite equality between integrals in the following useful way
\[
\frac{a}{2} su I_4(4 + 2\epsilon, m) = a \frac{\Gamma(1+\epsilon)^2}{\Gamma(1+2\epsilon)}
\]
\[
[2w_0 \frac{\Gamma(2+2\epsilon)}{\Gamma(1+\epsilon)^2} I_4(6 + 2\epsilon, m) - \sum_{i=1}^{4} w_i \frac{\Gamma(1+2\epsilon)}{\Gamma(1+\epsilon)} I_3(4 + 2\epsilon, m; 1 - \delta_{ki})]
\]  
(11)
\[
\begin{align*}
    w_0 &= \frac{4m^2 - s - u}{2} = (p_1, p_3) = (p_2, p_4) \\
    w_1 &= \frac{u - 2m^2}{2} = (p_2, p_3) \\
    w_2 &= \frac{s - 2m^2}{2} = (p_3, p_4) \\
    w_3 &= \frac{u - 2m^2}{2} = (p_4, p_1) \\
    w_4 &= \frac{s - 2m^2}{2} = (p_1, p_2)
\end{align*}
\] (12)

On the other hand one-loop correction to the amplitude in \( \mathcal{N} = 4 \) for four gluons is given by

\[
\begin{align*}
    \mathcal{M}_4 &= -\frac{a}{2} suI_4(4 + 2\epsilon, m)|_{m \to 0} \\
    \mathcal{M}_4 &= -\frac{a}{2} suI_4(4 + 2\epsilon, m)|_{\epsilon \to 0}
\end{align*}
\] (13) (14)

in dimensional and mass regularizations. If \( m = 0 \) each summand in the RHS of (11) becomes equal exactly to the corresponding Wilson diagram and we get

\[
\mathcal{M}_4 = \frac{\Gamma(1 + \epsilon)^2}{\Gamma(1 + 2\epsilon)} \mathcal{W}
\] (15)

that is our strong version of the duality mentioned above. Thus, (11) can be considered as the continuation of the stronger version of the duality off-shell.

Now we can send \( \epsilon \to 0 \), and get the analog of each Wilson diagram in this case. We have the following changes (strictly speaking since we have no strict definition of off-shell “Wilson loop”, we are free to multiply and divide by the function \( f(m) \) which is \( f(0) = 1 \) to redefine off-shell diagrams)

\[
\begin{align*}
    \frac{\Gamma(1 + \epsilon)^2}{\Gamma(1 + 2\epsilon)} \mathcal{W} &= \mathcal{M}_4 &\leftrightarrow& \frac{1}{1 - \frac{\Delta x}{su}} \mathcal{W} &= \mathcal{M}_4 \\
    \frac{I^{\text{cusp}}_{ii+1}(4 - 2\epsilon|s_{i,i+1})}{(p_i, p_{i+1})} &= -I_3(4|s_{i,i+1}, m^2, m^2) \\
    \frac{I^W_{ij}(4 - 2\epsilon|s, u)}{(p_i, p_j)} &= I_4(6|s, u, m^2, m^2, m^2, m^2)
\end{align*}
\] (16) (17) (18)

There is no duality with the Wilson polygon in this case since objects at the r.h.s. have no such interpretation. In the next Section we suggest the proper geometrical object which substitutes Wilson polygons.
Figure 1: Basic simplex in dual space.

Note that cusp diagram in this case is equal to

\[ I_{i,i+1}^{\text{cusp}}(4|s_{i,i+1}) = -\frac{1}{2} \log^2 \left( \frac{-m^2}{s_{i,i+1}} \right) \]  

(19)

Here the coefficient is important, because the amplitude in the off-shell regularization satisfies the equation

\[ \left( \frac{\partial}{\partial \ln(m^2)} \right)^2 \ln M_4 = -2\Gamma_{\text{cusp}}(a) \]  

(20)

with extra factor of 2. Up to vanishing in \( m \to 0 \) terms this result is reproduced via cutting prescription discussed in [10].

3 Towards the possible interpretation of arbitrary box in dual terms

In the massless case the amplitude is expected to be dual to the Wilson polygon however the generalization to the massive case is not immediate. In this Section we shall consider the object dual to the arbitrary massive box diagrams and it turns out that the polygon has to be substituted by more generic simplex.

To start with note that the arbitrary box in Feynman parametrization is defined in terms of the following integral

\[ I = \int \prod dx_i \frac{\delta(1 - x_1 - x_2 - x_3 - x_4)}{(-\Delta)^{\frac{n-3}{2} \mu}} \]  

(21)
Figure 2: The region of integration.

\[ \Delta = s x_1 x_3 + u x_2 x_4 + m_1^2 x_1 x_2 + m_2^2 x_2 x_3 + m_3^2 x_3 x_4 + m_4^2 x_4 x_1 \]

where \( s = (p_1 + p_2)^2 = (p_3 + p_4)^2 \) and \( u = (p_2 + p_3)^2 = (p_1 + p_4)^2 \). Let’s introduce dual coordinates \( p_i = q_i - q_{i+1} \) (we use \( q \) instead of \( x \) to avoid mixing with Feynman parameters). Then one can rewrite

\[ \Delta = q_1^2 x_1 + q_2^2 x_2 + q_3^2 x_3 + q_4^2 x_4 - (q_1 x_1 + q_2 x_2 + q_3 x_3 + q_4 x_4)^2 \]

and assuming that dual vectors are light-like (or using dual translational invariance),

\[ \Delta = -(q_1 x_1 + q_2 x_2 + q_3 x_3 + q_4 x_4)^2. \]

This form suggests that it could be interpreted as massless propagator in position space. Note that condition \( q_i^2 = 0 \) can not be imposed in general kinematics. Now the box diagram can be interpreted in terms of the object which we will call following [15] basic simplex (see fig.1).

One should notice that general \( n \)-point diagram with massive propagators will be represented by

\[ \Delta_n = \sum (q_i^2 - m_i^2)x_i - (\sum q_i x_i)^2 \]

where \( m_i \) are propagator masses.

Feynman parameters delta-function bounds the integration region to the momenta tetrahedron for which momenta \( p_i' \), \( s'' = (p_1 + p_2)' \) and \( u'' = (p_2 + p_3)' \) serves as edges (see fig.2). The suggested picture reproduces the known picture for the two-mass easy case \( p_1^2 = 0 = p_3^2 \). To get this situation we can choose \( q_1||q_2 \) and \( q_3||q_4 \) (see fig.3).
Taking into account that \( q_2 = q_1 - p_1 \) and \( q_4 = q_3 - p_3 \) we can obtain
\[
\Delta^{2me} = -(q_1(x_1 + x_2) - p_1 x_2 + q_3(x_3 + x_4) - p_4 x_4)^2
\]
\[
= -((x_1 + x_2)(q_1 - p_1 \frac{x_2}{x_1 + x_2}) + (x_3 + x_4)(q_3 - p_4 \frac{x_4}{x_3 + x_4}))^2
\]

Let us introduce the following variables
\[
\tau_1 = \frac{x_2}{x_1 + x_2} \quad \tau_2 = \frac{x_4}{x_3 + x_4}
\]

It is evident that both \( q_1 + p_1 \frac{x_2}{x_1 + x_2} \) and \( q_3 + p_4 \frac{x_4}{x_3 + x_4} \) are light-like hence we can write
\[
\Delta^{2me} = (x_1 + x_2)(x_3 + x_4) \tilde{\Delta}(\tau_1, \tau_2).
\]

where \( \tilde{\Delta}(\tau_1, \tau_2) \) is nothing but Wilson loop diagram propagator (see fig. 4). It turns out that the measure is also reproduced in the right way.

### 3.1 3-point function case

In the case of 3-point function situation is quite similar. The only difference is that instead of four vectors from the origin of the simplex we have three ones.

To check the suggested picture let us reproduce the known picture for the two-mass hard case \( p_1^2 = 0 \). To get this situation we can choose \( q_1 \parallel q_2 \). Let’s use the fact that \( q_2 = q_1 - p_1 \) to rewrite
\[
\Delta^{2mh} = -(q_1(x_1 + x_2) - p_1 x_2 + q_3 x_3)^2
\]
\[
= -((x_1 + x_2)(q_1 - p_1 \frac{x_2}{x_1 + x_2}) + x_3 q_3)^2
\]
Figure 4: Appearance of Wilson loop diagram from the degenerate basic simplex. By double lines light-like edges are denoted.

Figure 5: Dual to two-mass triangle.

Introduce $\tau = \frac{x_2}{x_1 + x_2}$ and note that both $q_1 + p_1 \frac{x_2}{x_1 + x_2}$ and $q_3$ are light-like. Therefore we can write

$$\Delta^{2mh} = (x_1 + x_2)x_3 \tilde{\Delta}(\tau). \quad (26)$$

where $\tilde{\Delta}(\tau)$ is Wilson loop diagram propagator (see fig. 5).

To conclude this section let us briefly comment on the possible strong coupling counterpart of our weak coupling picture. The natural candidate for the massive gluon amplitude is the correlator of the Wilson polygon with the local operator inserted into the vertex of the simplex $\langle W(C)O \rangle$. Such correlators were investigated in the duality context, for instance in [16] and at strong coupling it reduces to the exchange by some string mode. We have presented only one-loop arguments supporting the correct choice of the dual object for the off-shell external legs. However it is clear that more detailed analysis is desired including two-loop calculations.
4 "AdS" regularized Wilson loop - Higgsed regularized amplitude duality at two loops

In this Section we shall try the different regularization scheme to discuss the Higgsed phase of the theory. Our aim is to suit the proper regularization for the discussion of two-loop off-shell duality. In particular we will show that duality between naively "AdS" regularized Wilson loop and Higgsed amplitude is valid at one loop but breaks down at two loops. We use the fact of exponentiation recently found in [8] at two loops and explicitly show that \( \ln^4 m^2 \) do not cancel in Wilson loop calculation, while such terms cancel on the amplitude side of the correspondence.

The perturbative expansion of Wilson loop

\[
W[C_n] := \frac{1}{N} \text{Tr} \mathcal{P} \exp \left[ ig \oint \tau \left( \bar{x}^\mu (\tau) A_\mu (x(\tau)) \right) \right].
\]  

(27)

reads as follows

\[
\langle W[C_n] \rangle = 1 + \sum_{l=1}^{\infty} a^l W_n^{(l)} = \exp \sum_{l=1}^{\infty} a^l w_n^{(l)},
\]  

(28)

\[
w_2^{(2)} = W_2^{(2)} - \frac{1}{2} (W_1^{(1)})^2,
\]  

(29)

where \( a = \frac{g^2 N}{8 \pi^2} \) ( \( C_F = \frac{N}{2} \) and \( C_A = N \) in the planar limit).

Here we introduce - as a prescription - AdS regularized analogue of Feynman gauge propagator \( \Delta_{\mu \nu}(x) := \eta_{\mu \nu} \Delta(x) \), where

\[
\Delta(x) = \frac{1}{4\pi^2 x^2 + z_{U \bar{V}}^2 - i\epsilon}
\]  

(30)

and in the spirit of duality with amplitudes we will try to identify \( z_{U \bar{V}} \) with regulator mass \( m \) in the Higgsed \( U(N) \times U(n) \) model.

Three-point vertex is given by

\[
- i C_F f^{\alpha_1 \alpha_2 \alpha_3} \times \left[ \eta^{\mu_1 \mu_2} (\partial_1^{\mu_1} - \partial_2^{\mu_1}) + \eta^{\mu_2 \mu_3} (\partial_2^{\mu_2} - \partial_3^{\mu_2}) + \eta^{\mu_3 \mu_1} (\partial_3^{\mu_3} - \partial_1^{\mu_3}) \right] G(x_1, x_2, x_3),
\]  

(31)

where \( G(x_1, x_2, x_3) = \int d^D z \Delta(x_1 - z) \Delta(x_2 - z) \Delta(x_3 - z) \)

\[
G(x_1, x_2, x_3) = \frac{i}{64\pi^4} \int \prod_{i=1}^{3} d\alpha_i \delta(1 - \sum_{i=1}^{3} \alpha_i) \frac{1}{\alpha_1 \alpha_2 x_{12}^2 + \alpha_1 \alpha_3 x_{13}^2 + \alpha_2 \alpha_3 x_{23}^2 + m^2},
\]  

(32)

and we have used \( x_{ij}^2 = (x_i - x_j)^2 \).

To verify or disprove the duality and identification of the regularization scales at one loop we need calculate here only cusp diagrams since finite diagrams do not depend on regularization. For cusp diagram (see fig.6) we get
Figure 6: Cusp diagram.

\[ w_{i,i+1} = \frac{x_{i,i+1}^2}{2} \int_0^1 d\tau_1 d\tau_2 \frac{1}{-x_{i,i+1}^2(1-\tau_1)\tau_2 - m^2} \]  

(33)

and simple Mellin-Barnes technics yields

\[ w_{i,i+1} = -\frac{1}{4} \ln^2\left(\frac{x_{i,i+1}^2}{m^2}\right) \]  

(34)

\[ w_4^{(1)cusps} = -\frac{1}{2} \left(\ln^2\left(\frac{s^2}{m^2}\right) + \ln^2\left(\frac{u^2}{m^2}\right)\right) \]  

(35)

Using analogy with dimensional regularization we can define

\[ \left(\frac{\partial}{\partial \ln(m^2)}\right)^2 \ln W_4 = -\Gamma_{\text{cusp}}(a) \]  

(36)

To first order it properly reproduces the known answer

\[ \Gamma_{\text{cusp}}(a) = 2a \]  

(37)

At two loops we will concentrate at \( \ln^4 m^2 \) terms and will show that they do not cancel contrary to the amplitude side. Thus, using notations of [3] even in \( m_i = m \) regime the naive AdS regularization does not do the job. For the notations and general expressions in dimensional regularization see [14].

The fig.7 diagram is given by

\[ w_d = -\left(\frac{x_{i,i+1}^2}{2}\right)^2 \int_0^1 d\sigma_1 d\tau_1 \int_0^{\sigma_1} d\sigma_2 \int_0^{\tau_2} d\sigma_2 \frac{1}{x_{i,i+1}^2\sigma_1\sigma_2 + m^2 x_{i,i+1}^2\tau_1\tau_2 + m^2} \]  

(38)

We introduce two-fold Mellin-Barnes representation and make all the \( \tau \) and \( \sigma \) integrations to get

\[ w_d = -\frac{1}{4} \int d\zeta_1 d\zeta_2 \left(\frac{m^2}{x_{i,i+1}^2}\right)^{\zeta_1+\zeta_2} \frac{\Gamma(-\zeta_1)\Gamma(-\zeta_2)\Gamma(1+\zeta_1)\Gamma(1+\zeta_2)}{\zeta_1\zeta_2(\zeta_1+\zeta_2)^2} \]  

(39)
The deformation of the contour of integration amounts to $\ln^2(m^2)$ terms which are unnecessary for our consideration

\[
w'^{AdS}_d = -\frac{1}{48} \ln^4\left(\frac{x_{i,i+1}^2}{m^2}\right) \tag{40}
\]

\[
w'^{\text{dim.red.}}_d = -\frac{1}{16} \frac{1}{\epsilon^4} \tag{41}
\]

For diagram depicted on fig. 8 we get\(^*\)

\[
w_Y = \frac{1}{2} \frac{x_{i,i+1}^2}{2} \int_0^1 d\tau_1 \int_0^1 d\tau_2 \int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \int_0^1 d\alpha_3 \frac{\delta(1 - \alpha_1 - \alpha_2 - \alpha_3)}{x_{i,i+1}^2 \tau_1 \tau_2 \alpha_2 \alpha_3 + m^2} \tag{42}
\]

\[
w^{AdS}_Y = \frac{1}{96} \ln^4\left(\frac{x_{i,i+1}^2}{m^2}\right) \tag{43}
\]

\[
w^{\text{dim.red.}}_Y = \frac{1}{16} \frac{1}{\epsilon^4} \tag{44}
\]

Thus, we see that while cancel in dimensional reduction scheme these terms do not cancel in $AdS$ regularization. Hence we have demonstrated that naive $AdS$ inspired regularization of the propagator does not work at two loop level.

\(^*\)Remember that here we get factor of 2 from “upside-down” diagram
5 \textit{AdS geometry from weak coupling}

It was shown in [23] that one can recognize \textit{AdS} geometry starting from the Feynman diagrams. Here we will show how \textit{AdS} geometry appears in the different manner \[15\] from the massless box diagram.

\[ I = \int_0^\infty dx_1dx_2dx_3dx_4 \frac{\delta(1 - x_1 - x_2 - x_3 - x_4)}{(sx_1x_3 + ux_2x_4)^{4 - \frac{D}{2}}} \] (45)

\[ = \frac{1}{(-su)^{2 - \frac{D}{4}}} \int_0^\infty dx_1dx_2dx_3dx_4 \frac{\delta(1 - x_1 - x_2 - x_3 - x_4)}{\left(\sqrt{-\frac{u}{s}}x_1x_3 + \sqrt{-\frac{s}{u}}x_2x_4\right)^{4 - \frac{D}{2}}} \]

\[ \alpha^2 = \frac{8}{u} > 0 \]

Now let’s make change of variables \[15\] \( \sum y_i = y_1 + y_2 + y_3 + y_4 \)

\[ x_1 = \frac{\alpha y_1y_3 + \frac{1}{\alpha} y_2y_4}{\sum y_i} y_1 \] (46)

\[ x_2 = \frac{\alpha y_1y_3 + \frac{1}{\alpha} y_2y_4}{\sum y_i} y_2 \]

\[ x_3 = \frac{\alpha y_1y_3 + \frac{1}{\alpha} y_2y_4}{\sum y_i} y_3 \]

\[ x_4 = \frac{\alpha y_1y_3 + \frac{1}{\alpha} y_2y_4}{\sum y_i} y_4 \]

\[ \left| \frac{\partial(x_i)}{\partial(y_i)} \right| = 2 \left( \frac{\alpha y_1y_3 + \frac{1}{\alpha} y_2y_4}{\sum y_i} \right)^4 \]

\[ I = \frac{2}{(-su)^{2 - \frac{D}{4}}} \int_0^\infty \prod_i dy_i \left( \frac{\alpha y_1y_3 + \frac{1}{\alpha} y_2y_4}{\sum y_i} \right)^4 \frac{\delta(1 - \alpha y_1y_3 - \frac{1}{\alpha} y_2y_4)}{(\alpha x_1x_3 + \frac{1}{\alpha} x_2x_4)^{4 - \frac{D}{2}}} \] (47)

\[ = \frac{2}{(-su)^{2 - \frac{D}{4}}} \int_0^\infty \prod_i dy_i (\sum y_i)^{4 - D} \delta(1 - \alpha y_1y_3 - \frac{1}{\alpha} y_2y_4) \]

and upon the additional change of variables

\[ y_1 = \frac{1}{\sqrt{\alpha}} (\bar{y}_1 - \bar{y}_3) \quad y_3 = \frac{1}{\sqrt{\alpha}} (\bar{y}_1 + \bar{y}_3) \] (48)

\[ y_2 = \sqrt{\alpha} (\bar{y}_2 - \bar{y}_4) \quad y_4 = \sqrt{\alpha} (\bar{y}_2 + \bar{y}_4) \]
we obtain
\[ \delta(1 - \alpha y_1 y_3 - \frac{1}{\alpha} y_2 y_4) \rightarrow \delta(1 - [y_1^2 + y_2^2 - y_3^2 - y_4^2]) \] (49)

It is clear that we find ourselves with the integration of some function over the region in \( AdS_3 \).

6 Low-energy amplitudes in the Higgsed phase

Let us comment on the appearance of the similar \( AdS_3 \) geometry in the Higgsed phase in the low-energy scattering regime. To this aim note that the one loop low-energy amplitude can be derived from the one-loop effective action in the external field \( F_{\mu \nu}^a \). It is assumed that the theory is in the Higgsed phase and the invariants of the external field obey the relation \( F^2/m^4 << 1, F \tilde{F}/m^4 << 1 \), where \( m \) is the mass of the particle in the loop.

The low-energy amplitude can be obtained by expanding the effective action in the proper power of the external field and the type of the amplitude is fixed by the chiral structure of the external field. In particular the effective action in the external self-dual field yields the generating function for the all-plus(or all-minus) amplitudes which are known to vanish in SUSY case. To get the \( N^k MHV \) amplitude one has to expand the effective action in \( k + 2 \) power of \( F_\pm \) and any number of \( F_\pm \).

The explicit expression for the effective action in \( \mathcal{N} = 4 \) \( SU(2) \) SYM in the abelian background \( F_{\mu \nu} = \tilde{F}_{\mu \nu} \frac{\mp \sigma_3}{2} \) with non-zero scalar VEV \( m \) reads as [17]
\[ \mathcal{L}_{\text{eff}} = -\frac{1}{4\pi^2} \int_0^\infty ds e^{-m^2 s} \frac{f_1 f_3 s^2}{\sinh(f_1 s) \sinh(f_2 s)} (\cosh(f_1 s) - \cosh(f_2 s))^2 \] (50)

where \( f_1 \) and \( f_2 \) are Euclidean field invariants; \( m \) which can be written through proper invariants of external field is the scalar VEV. While going to Minkowski space \( F_{0i}^E \rightarrow i F_{0i}^M, F_{E}^2 \rightarrow -F_{M}^2, F_{E} \tilde{F}_{E} \rightarrow i F_{M} \tilde{F}_{M} \). Then
\[ f_1^2 \rightarrow b_1^2 \]
\[ f_2^2 \rightarrow -a_2^2 \] (51) (52)

(here we adopt \((a, b)\) notations used in [18]).

In Minkowski space it is assumed that the external field is the sum of the plane waves

\[ F_{\text{tot}} = \sum_i F_i \quad F_{i,\mu \nu}^\pm = k_{i,\mu} e_{\mu \nu}^\pm - k_{i,\nu} e_{i \mu}^\pm \] (53)
To extract the expression for the low energy amplitudes let us expand similar to \cite{18} the invariants in chiral components

\[
\frac{F_{\text{tot}} F_{\text{tot}}}{4} = \chi_+ + \chi_-, \quad \frac{F_{\text{tot}} \tilde{F}_{\text{tot}}}{4} = -i(\chi_+ - \chi_-)
\]  

(54)

where we have used

\[
\chi_+ = \frac{1}{2} \sum_{1 \leq i < j \leq N} [ij]^2
\]  

(55)

and

\[
\chi_- = \frac{1}{2} \sum_{1 \leq i < j \leq N} \langle ij \rangle^2
\]  

(56)

and the standard spinor helicity notations are implied. In terms of these variables the invariants involved into the effective action reads as

\[
a = \sqrt{\chi_+} + \sqrt{\chi_-}, \quad b = -i(\sqrt{\chi_+} - \sqrt{\chi_-})
\]  

(57)

\[
f_1 = \sqrt{\chi_+} + \sqrt{\chi_-}, \quad f_2 = \sqrt{\chi_+} - \sqrt{\chi_-}
\]  

(58)

\[
\mathcal{L}_{\text{eff}} = -\frac{m^4}{4\pi^2} \sum_{N=4}^{\infty} \frac{1}{(m^2)^N} \sum_{k=0}^{N} c_{N=4}(\frac{k}{2}, \frac{N-k}{2}) \chi_k^2 \chi_{N-k}^2
\]  

(59)

From this expression one can extract $N^{k-2}MHV$ low-energy $N$-particle amplitudes. Explicit expression for the amplitude is given by $c_{N=4}(\frac{k}{2}, \frac{N-k}{2})$ multiplied by some kinematical factor (see \cite{18}).

Here we find special properties of the low-energy amplitudes (for explicit formulas see appendix C):

- the effective action vanishes at the self-dual points $f_1 = f_2$ in agreement with the vanishing of all-plus(minus) amplitudes in $\mathcal{N} = 4$ SYM;
- amplitudes with odd number of particles are zero;
- amplitudes with odd number of positive (negative) helicity particles are zero (particularly NMHV amplitudes);
- the only non-zero MHV amplitude is four-particle one.

During calculation we observed the following identities, which lie in the heart of the nullification of the higher points MHV amplitudes:

\[
\sum_{l=0}^{N} \Gamma_{l} \left( \frac{1 - 2^{2l-1} - 2^{2(N-l)-1}}{\Gamma(2l+1)\Gamma(2(N-l)+1)} B_{2l} B_{2(N-l)} \right) = -\frac{1}{4} \sum_{l=0}^{N} \Gamma_{l} \left( \frac{1 - 2^{2l-1} B_{2l}}{\Gamma(2l+1)\Gamma(2(N-l))} + \frac{1 - 2^{2(N-l)-1} B_{2(N-l)}}{\Gamma(2l)\Gamma(2(N-l)+1)} \right)
\]

(60)
which we found to be true for $N \geq 3$ and $0 \leq \alpha \leq 3$. These convolution identities are in the spirit of the ones that were considered in [19], but we have not found the same ones. It is very natural to expect that supersymmetric effective actions serve as another rich source of convolution identities for Bernoulli numbers.

Note that somewhat similar one-loop nullification phenomena has been discovered for the amplitudes in QED [20] where it was found that only non-vanishing all-plus(minus) amplitudes involves four external legs. It is natural to assume that some hidden symmetry is behind the nullification phenomena of the low-energy amplitude. The candidate in the $\mathcal{N} = 4$ case is the Yangian symmetry and the possible arguments implying the nullification could be similar to ones applied for the nullification of the threshold amplitudes in [21].

Now let us turn to the comment on the underlying $AdS$ geometry following [22]. In that paper it was remarked that the one-loop effective action admits the geometrical interpretation. Namely, one can interpret the integrand in the integral over the proper time as transition amplitude for the particle of some mass depending on the space-time spin in $AdS_3$. The proper time measures the length of the corresponding geodesics. Even more suitable interpretation emerges if we consider the $AdS_2$ geometry and the proper time $s$ measures the length of the “Wilson loop” area. The low-energy effective action [22] acquires the form of the matrix element of some operator in the 2d gravity

$$\mathcal{L}_{\text{eff}} \propto \int ds \Psi_1(s)O\Psi_2(s)$$

(61)

where $\Psi(s)$ is the ”wave function” in 2d gravity and $O$ is some function of $s$. It can be a little bit schematically written in the ”length representation” of the wave functions $|f\rangle$

$$\mathcal{L}_{\text{eff}}(f_1, f_2) \propto \langle f_1 |(\vec{L} - \vec{L})^2|f_2\rangle$$

(62)

where $L$ is the length operator acting on the corresponding state. In this representation the Teichmüller phase space is implied and the momenta of external gluons entering the invariants of the external fields mark the corresponding wave functions.

It is useful to have in mind the first-quantized representation of the effective action as the sum over the paths $C$

$$\mathcal{L}_{\text{eff}}(f_1, f_2) = \sum_C \exp(-imL(C)) \exp(i\Phi(C))\langle W(C) \rangle$$

(63)

where $L(C)$ is the length and $\Phi$ is the spin factor. That is the low-energy limit of the amplitudes in the Higgs phase corresponds to a kind of Fourier transform of the Wilson loops with respect to the length. Remark also that the low-energy effective action in the external field can be used to calculate some BPS invariants in the spirit of [24]. It could be interesting to recognize the corresponding invariants which are related to the low-energy $N^k$MHV amplitudes. We hope to discuss these issues elsewhere.
7 Conclusion

In this paper we have looked for the generalization of the amplitude-Wilson polygon
duality for the cases when the massive particles are at the external or internal legs in
the box diagrams. We have suggested a version of duality for off-shell external legs
and amplitudes at one-loop level. It implies some modification of the dual object and
instead of the Wilson polygon the Wilson tetrahedrons emerge. The duality derivation
is similar to the on-shell case and involves the particular change of variables. Note
that our result is the useful step toward the dual description of the NMHV amplitudes
which involve 2mh,3mh and 4mh boxes. However we do not know how the coefficients
in front of the boxes can be derived in the geometric manner.

One-loop duality appears to be quite transparent however its two-loop general-
ization deserves the formulation of the proper regularization of the "Wilson polygon".
The naive AdS inspired regularization does not work and it would be important to
find the regularization distinct from dimensional one. It it also very interesting to
make the link of our approach with the twistor picture.

We have made also a few comments concerning the appearance of the AdS-
like geometry in the one-loop calculation. It turns out that the box diagram can
be attributed to the integration of particular function over the AdS$_3$. In the deep
Higgsed regime we have found the interesting nullification phenomena for the low-
energy amplitudes. It would be interesting to develop the duality arguments which
would exchange small mass and large mass limits.

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Appendix A Basic notations and integrals

\[ a = \frac{g^2 N}{8\pi^2} \]

\[ I^N(D;\nu_k) = -i\pi^{-\frac{D}{2}}(\mu^2)_{IR}^{-\epsilon} \int d^D l \frac{1}{A_1^\nu_1 A_2^\nu_2 ... A_N^\nu_N} \]

\[ C_F = \frac{N^2 - 1}{2N}, \quad C_T = (\mu^2)_{IR}^{\epsilon \frac{\Gamma(1-\epsilon)\Gamma(1+\epsilon)^2}{\Gamma(1+2\epsilon)}}, \]

\[ C_A = N, \quad (\mu^2)_{IR}^{-\epsilon} = (\pi \mu^2_{IR})^\epsilon. \]

\[ I_3(m^2, 0, 0|4 + 2\epsilon) = \frac{c_T}{\epsilon^2} (-m^2)^{-1+\epsilon} \]

\[ I_3(m_1^2, m_2^2, 0|4 + 2\epsilon) = \frac{c_T}{\epsilon^2} \frac{(-m_1^2)^\epsilon - (-m_2^2)^\epsilon}{(-m_1^2) - (-m_2^2)} \]

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For the Wilson cusp diagram with $m^2 = 2(p_i, p_j)$:

\[
G^F_{\mu\nu}(x - y) = -\eta_{\mu\nu} \frac{(\pi\mu_{UV}^2)^\varepsilon}{4\pi^2} \frac{\Gamma(1 - \varepsilon)}{(-(x - y)^2 + i\epsilon)^{1-\varepsilon}} \tag{67}
\]

\[
\frac{I_{\text{cusp}}(m^2|4 - 2\epsilon)}{(p_i, p_j)} = a(-m^2)^{-1+\varepsilon}(\pi\mu_{UV}^2)^\varepsilon\Gamma(1 - \varepsilon) \tag{68}
\]

**Appendix B  Connection of scalar integrals in different dimensions**

Here we briefly explain the connection between the scalar integrals in different dimensions \[12\]. Suppose, we have the following scalar integral (see fig.9)

\[
I^N(D; \nu_k) \equiv -i\pi^{-\frac{D}{2}}(\mu^2)^\varepsilon \int d^Dl \frac{1}{A_1^\nu_1 A_2^\nu_2 ... A_N^\nu_N} \tag{69}
\]

then it can be shown that

\[
I^N(D - 2; \nu_k) = \sum_{i=1}^{N} z_i I^N(D - 2; \nu_k - \delta_{ki}) \tag{70}
\]

\[
+(D - 1 - \sum_{j=1}^{N} \nu_j) z_0 I^N(D; \nu_k)
\]

where

\[
\sum_{i=1}^{N} (r_i - r_j)^2 z_i = 1 \tag{71}
\]
\[ z_0 = \sum_{i=1}^{N} z_i \]

In the main body of the text we choose \( D = 6 + 2\epsilon, \ N = 4, \ \nu_i = 1 \).

We are interested in the case of \( D = 6 \) two-mass easy boxes and their connection with \( D = 4 \) ones \([12]\) hence

\[ I_{2me}(6 + 2\epsilon) = \frac{1}{(1 + 2\epsilon)z_0}(I_{2me}(4 + 2\epsilon) - \sum_{i=1}^{4} z_i I_{2me}(4 + 2\epsilon; 1 - \delta_{ki})) \]

where

\[ z_0 = \sum_{i=1}^{4} z_i = \frac{2s + u - m_2^2 - m_4^2}{su - m_2^2m_4^2} \]

\[ z_1 = \frac{u - m_2^2}{su - m_2^2m_4^2} \]

\[ z_2 = \frac{s - m_4^2}{su - m_2^2m_4^2} \]

\[ z_3 = \frac{u - m_4^2}{su - m_2^2m_4^2} \]

\[ z_4 = \frac{s - m_2^2}{su - m_2^2m_4^2} \]

It can be easily seen the \( \sum_{i=1}^{4} z_i I^4(4 + 2\epsilon; 1 - \delta_{ki}) \) does precisely the job of taking the finite part.

**Appendix C** Low-energy amplitudes in \( \mathcal{N} = 4 \) SYM

\[ \mathcal{L}_{\text{eff}} = -\frac{1}{4\pi^2} \int_0^{\infty} ds e^{-m^2s} \frac{f_1 f_2 s^2}{\sinh(f_1 s) \sinh(f_2 s)} (\cosh(f_1 s) - \cosh(f_2 s))^2 \]

\[ x = f_1 s \]

\[ y = f_2 s \]

\[ \frac{f_1 f_2 s^2}{\sinh(f_1 s) \sinh(f_2 s)} (\cosh(f_1 s) - \cosh(f_2 s))^2 = \]

\[ 2\left( \frac{x}{\sinh x} - \frac{y}{\sinh y} \right) + x \sinh x - \frac{y}{\sinh y} + \frac{x}{\sinh x} y \sinh y \]

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\[
\frac{x}{\tanh x} = \sum_{n=0}^{\infty} \frac{2^n B_{2n} x^{2n}}{2n!}
\]

(74)

\[
\frac{x}{\sinh x} = \sum_{n=0}^{\infty} \frac{2(1 - 2^{2n-1}) B_{2n} x^{2n}}{2n!}
\]

(75)

\[
x \sinh x = \sum_{n=0}^{\infty} \frac{x^{2(n+1)}}{(2n+1)!}
\]

(76)

\[
\mathcal{L}_{\text{eff}} = -\frac{m^4}{4\pi^2} \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{\Gamma(2(l+n-1))}{(m^2)^{2l(n+l)}} f_1^{2n} f_2^{2l} \mathcal{C}(n,l)
\]

(77)

\[
= -\frac{m^4}{8\pi^2} \sum_{n,l=0}^{\infty} \frac{\Gamma(2(l+n-1))}{(m^2)^{2(l+n)}} (f_1^{2n} f_2^{2l} + f_2^{2n} f_1^{2l}) \mathcal{C}(n,l)
\]

(78)

\[
\mathcal{C}(n,l) = C_1(n,l) + C_2(n,l) + C_2(l,n)
\]

(79)

\[
C_1(n,l) = 8 \frac{B_{2n} B_{2l}}{(2n)!(2l)!} (1 - 2^{2n-1} - 2^{2l-1})
\]

(80)

\[
C_2(n,l) = \frac{2}{(2n-1)!(2l)!} (1 - 2^{2l-1}), \quad n \geq 1
\]

(81)

\[
C_2(0,l) = 0
\]

(82)

Due to symmetries of original expression \( \mathcal{C}(n,l) = \mathcal{C}(l,n) \)

\[
f_1 = \sqrt{\chi_+} + \sqrt{\chi_-} \quad f_2 = \sqrt{\chi_+} - \sqrt{\chi_-}
\]

(83)

\[
f_1^{2n} f_2^{2l} + f_2^{2n} f_1^{2l} = \sum_{\alpha=0}^{2n} \sum_{\beta=0}^{2l} \sqrt{\chi_+^{\alpha+\beta}} \sqrt{\chi_-^{2(n+l)-(\alpha+\beta)}} C_\alpha^{2n} C_\beta^{2l}((-1)^\alpha + (-1)^\beta)
\]

(84)

\[
\alpha + \beta = 2K
\]

\[
f_1^{2n} f_2^{2l} + f_2^{2n} f_1^{2l} = \sum_{K=0}^{n+l} \chi_+^{(n+l)-K} \chi_-^{-K} \Upsilon(n, l, K)
\]

(85)

\[
\Upsilon(n, l, K) = 2 \sum_{\beta=0}^{2l} C_{2K-\beta}^{2n} C_\beta^{2l} (-1)^\beta \Theta(\beta > 2K - 2n) \Theta(\beta < 2K)
\]

(86)
Let’s return to (87) and make change $N = n + l$

$$L_{\text{eff}} = -\frac{m_4}{4\pi^2} \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{\Gamma(2(l+n-1))}{(m^2)^{2(l+n)}} f_1^{2n} f_2^{2l} C(n, l) \quad (87)$$

$$= -\frac{m_4}{4\pi^2} \sum_{N=1}^{\infty} \frac{(2N-3)!}{(m^2)^{2N}} \sum_{K=0}^{N} \chi^K N^{-K} A_{K,N} \quad (88)$$

$$A_{K,N} = \sum_{l=0}^{N} C(N - l, l) \Upsilon(N - l, l, K) \quad (89)$$

All plus and all minus amplitudes are zero, because

$$\Upsilon(N - l, l, 0) = 2 \quad (90)$$

$$\sum_{l=0}^{N} C(N - l, l) = 0 \quad (91)$$

In fact non-zero amplitudes appear when $N \geq 2$ which corresponds to four particles.

- $2K$ - number of particles with positive helicity.
- $2N$ - total number of particles.

Using the notation in the main body of the text

$$c_{N=4}(\frac{k}{2}, \frac{N-k}{2}) = A_{K,N} \quad (92)$$

**Appendix D  Geometry of divergencies**

If we have off-shell regularization then we are free to think about triangle or about the box, since they are defined via the same functions - the only difference is hidden in the ”glued” kinematics.

Let’s introduce standard notations

$$x = \frac{m_1^2 m_3^2}{s u} \quad y = \frac{m_2^2 m_4^2}{s u} \quad (93)$$

$$\lambda(x, y) = \sqrt{(1 - x - y)^2 - 4xy} \quad (94)$$

$$\rho(x, y) = \frac{2}{1 - x - y - \lambda}$$

and

$$\Phi(x, y) = \frac{1}{\lambda} \{2[Li_2(-\rho x) + Li_2(-\rho y)] + log \frac{y}{x} \log \frac{1 + \rho y}{1 + \rho x} + \log(\rho x) \log(\rho y) + \frac{\pi^2}{3}\} (95)$$
The massive box diagram is given by
\[ I_{\text{box}} = \frac{i\pi^2}{su} \Phi(x, y) \] (96)

For the triangle we obtain
\[ x = \frac{p_1^2}{p_3^2}, \quad y = \frac{p_2^2}{p_3^2} \]
and the massive triangle diagram is given by
\[ I_{\Delta} = \frac{i\pi^2}{p_3^2} \Phi(x, y) \] (97)

These diagrams can be interpreted in geometrical terms [15].

Then dihedral angles of ideal tetrahedron are
\[ \cos \psi_{12} = \frac{1 + y - x}{2\sqrt{y}} \] (98)
\[ \cos \psi_{13} = \frac{x + y - 1}{2\sqrt{xy}} \] (99)
\[ \cos \psi_{14} = \frac{1 + x - y}{2\sqrt{x}} \] (100)

\[ \text{Cl}_2(x) = \text{Im}[\text{Li}_2(e^{ix})] = -\int_0^x dy \ln|2\sin y/2| = -\sin \theta \int_0^1 \frac{dz \log z}{1 - 2z \cos \theta + z^2} \] (101)

\[ 2i\Omega^{(4)} = \text{Cl}_2(2\psi_{12}) + \text{Cl}_2(2\psi_{13}) + \text{Cl}_2(2\psi_{23}) = \lambda(x, y, 1)\Phi_{D\text{av}}(x, y) \] (102)

where the Källen function \( \lambda(x, y, z) \) is defined as
\[ \lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx \] (103)

Going on-shell means that \( x \to 0 \) or \( y \to 0 \) and this limit should be understood in terms of the analytical continuation.

**Geometrical regions**

We can start with the kinematical region where geometrical picture is clear and
\[ \psi_{12} + \psi_{13} + \psi_{14} = \pi \] (104)

Generically one can consider simple triangle with these angles instead of the hard ideal hyperbolic tetrahedron. This geometrical picture is valid when \( |\cos \psi| < 1 \) or \( (\sqrt{y} - 1)^2 < x < (\sqrt{y} + 1)^2 \). At the boundary \( x = (\sqrt{y} \pm 1)^2 \) the volume vanishes and equivalently area of the triangle vanishes as well. Thus, to go on-shell we need to make the proper analytic continuation.
Analytic continuation of basic functions

Let us introduce notation

\[ \text{Im}[f(z)] = \frac{f(z) - f(z^*)}{2i} \quad \text{Odd}[f(z)] = \frac{f(z) - f(-z)}{2} \]  \hspace{1cm} (105)

Then

\[ \text{Clh}_2(x) = \text{Odd}[\text{Li}_2(e^x)] = -\int_0^x dy \ln|\text{sinh} y/2| = -\sinh x \int_0^1 \frac{dz \log z}{1 - 2z \cosh x + z^2} \]  \hspace{1cm} (106)

\[ 0 < \text{arccosh} \ x < \infty \]  \hspace{1cm} (107)

- \( x > (\sqrt{y} + 1)^2 \)

Note appearance of the sign in \( \psi_{12} \)

\[ \cosh -\psi_{12} = -\frac{1 + y - x}{2\sqrt{y}} \]  \hspace{1cm} (108)

\[ \cosh \psi_{13} = \frac{x + y - 1}{2\sqrt{xy}} \]  \hspace{1cm} (109)

\[ \cosh \psi_{14} = \frac{1 + x - y}{2\sqrt{x}} \]  \hspace{1cm} (110)

and the sum vanishes

\[ \psi_{12} + \psi_{13} + \psi_{14} = 0 \]  \hspace{1cm} (111)

\[ 2\Omega^{(4)} = \text{Clh}_2(2\psi_{12}) + \text{Clh}_2(2\psi_{13}) + \text{Clh}_2(2\psi_{23}) \]  \hspace{1cm} (112)

- \( x < (\sqrt{y} - 1)^2 \)

\[ \cosh \psi_{12} = \frac{1 + y - x}{2\sqrt{y}} \]  \hspace{1cm} (113)

\[ \cosh \psi_{13} = \frac{x + y - 1}{2\sqrt{xy}} \]  \hspace{1cm} (114)

\[ \cosh -\psi_{14} = -\frac{1 + x - y}{2\sqrt{x}} \]  \hspace{1cm} (115)

\[ \psi_{12} + \psi_{13} + \psi_{14} = 0 \]  \hspace{1cm} (116)

\[ 2\Omega^{(4)} = \text{Clh}_2(2\psi_{12}) + \text{Clh}_2(2\psi_{13}) + \text{Clh}_2(2\psi_{23}) \]  \hspace{1cm} (117)

Massless limit \( x \to 0 \) corresponds to \( \psi_{13} \to \infty \) and \( \psi_{14} \to -\infty \)
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