Instability of Black Holes with a Gauss-Bonnet Term

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Abstract

We investigate the instability of hairy black holes in the theory with a nonminimally coupled Gauss-Bonnet (GB) term in asymptotically flat spacetime. Our approach corresponds to the fragmentation instability in a non-perturbative way. Under this approach, we investigate whether the initial black hole can be broken into two black holes. We compare the entropy of the initial black hole with the sum of those of two fragmented black holes. The relation between black hole instability and GB coupling with dilaton hair are presented. We describe the phase diagrams with respect to the mass of the black hole solutions and coupling constants. We find that perturbatively stable black hole is unstable under fragmentation.
1 Introduction

The no-hair theorem states that the black holes in Einstein-Maxwell theory are characterized by only its mass, electric charge and angular momentum [1]. All other observable parameters about a black hole are hidden in the event horizon, i.e. the contributions from other parameters cannot be accessible to outside observer. Various gravity theories motivated by string theory and cosmology have received more and more attention. In this perspective, various kinds of black holes with different hairs have been investigated [2]. The nonminimally complex dilaton coupled with the Maxwell field presents the first hairy black hole [3]. Black hole hair was categorized into two types, primary and secondary. Primary hair gives a new quantum number to a black hole, so the black hole states are expanded [4]. On the other hand, secondary hair grows on the primary hair [5]. Black hole dilaton hair is classified as secondary hair, because the dilaton field appears only when it is coupled with a Maxwell field. Recently, dilaton hairs have been discovered in various theories of gravitation. One theory is motivated to show the next-leading order effect of inverse string tension $\alpha'$ (16$\alpha\kappa$ in the present paper) and includes higher-order curvature called the GB term [6]. The GB term is the simplest one in the lowest effective supergravity action. In four dimensions, the presence of a minimally coupled GB term does not have any ghost particles or any problem of unitarity. In addition, the GB term does not change the second-order equation of motion [6, 7].

In the cosmological model, the dilaton-Gauss-Bonnet (DGB) theory with a nonminimally coupled GB term can provide the possibility of avoiding the initial singularity of the universe [8]. It may violate the energy condition in the singularity theorem thanks to the presence of that term. Recently, the specific inflationary model with the GB term was studied [9]. In the DGB theory, the nontrivial real dilaton field appears in the black hole solution [10, 11, 12, 13, 14] as a scalar hair. Black hole hair is secondary [11] in the DGB theory, because scalar hair is included in the mass of the black hole. There exists the minimum black hole mass below which black hole solutions do not exist. Above that minimum mass there exist upper and lower branch solutions. The upper branch solutions are stable under linear perturbation and approach the Schwarzschild black holes in the large mass limit. The lower branch solutions are unstable under linear perturbation and end at a singular solution [13, 14]. Our goal is to investigate black hole instability under non-perturbative method on the upper branch.

In higher-dimensional spacetime, there exist various rotating black holes for given angular momentum. The Myers-Perry (MP) black hole is a generalized Kerr black hole to higher dimensions [15]. The black ring is another type of solutions, which becomes more stable than MP black hole in higher angular momentum [16, 17, 18, 19]. For large angular momentum, a black hole can undergo fragmentation [17]. Fragmentation is based on the entropy [20] preference between the solutions. Fragmentation allows the upper or lower bound of black hole charges [21]. Unstable black holes are important and related to the non-equilibrium states in the anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence [22].

In this paper, we compute and describe the fragmentation instabilities of the black hole with GB term arising in asymptotically flat 4-dimensional spacetime in which fragmentation instability have not been done before. We found unstable phase which is known as stable solutions under a perturbation. We show black hole instability depending on GB couplings. We also present the phase diagrams on parameter space.

The outline of this paper is as follows: In Sec. 2, we introduce our basic framework and numerical
construction of the black holes for the theory with the GB term where the dilaton field is coupled with the GB term. We obtain the equations of motions and numerically solve to construct hairy black holes. We explain black hole properties in the theory with the GB term. In Sec. 3, we describe instabilities of black holes in several limits. In Sec. 4, we numerically investigate black hole instabilities through fragmentation. Black hole phase diagrams are presented in parameter space. In Sec. 5, we summarize and discuss our results.

2 Hairy black holes in DGB theory

We are interested in the instability of a black hole due to a fragmentation. The fragmentation phenomena of a black hole may occur by a large quantum fluctuation. The Einstein gravity does not allow this phenomena. In this perspective, one could introduce the Einstein theory of gravity with a nonminimally coupled GB term as the effective theory including a quantum correction.

2.1 Action and black hole solutions

To explore the fragmentation phenomena, we consider the action as follows:

\[ I = \int_{\mathcal{M}} \sqrt{-g} d^4x \left[ \frac{R}{2\kappa} - \frac{1}{2} \nabla_\alpha \Phi \nabla^\alpha \Phi + \alpha e^{-\gamma \Phi} R^2_{GB} \right] + \int_{\partial \mathcal{M}} \sqrt{-h} d^3x \frac{K - K_o}{\kappa}, \]  

(1)

where \( g = \text{det} g_{\mu\nu}, \kappa \equiv 8\pi G, \) and \( R \) denotes the scalar curvature of the spacetime \( \mathcal{M} \). The higher curvature GB term is given \( R^2_{GB} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \). The action has a dilaton field \( \Phi \) coupled with the GB term \( \alpha e^{-\gamma \Phi} \) where \( \alpha \) and \( \gamma \) are constants, which we will call as dilaton-Gauss-Bonnet (DGB) theory. The second term on the right-hand side is the boundary term \([23, 24]\) in which \( h \) is the determinant of the first fundamental form, \( K \) and \( K_o \) are the traces of the second fundamental form of the boundary \( \partial \mathcal{M} \) for the metric \( g_{\mu\nu} \) and \( \eta_{\mu\nu} \), respectively. The gravitational field equations can be obtained properly from a variational principle with this boundary term. We adopt the sign conventions in Ref. \([25]\).

From the action (1), we obtain the Einstein equations and the scalar field equation

\[ R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa \left( \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2}g_{\mu\nu} \partial_\rho \Phi \partial^\rho \Phi + T^\text{GB}_{\mu\nu} \right), \]  

(2)

\[ \frac{1}{\sqrt{-g}} \partial_\mu [\sqrt{-g} g^{\mu\nu} \partial_\nu \Phi] - \alpha \gamma e^{-\gamma \Phi} R^2_{GB} = 0, \]  

(3)

where the GB term contribute to the energy-momentum tensor

\[ T^\text{GB}_{\mu\nu} = -8\alpha (\nabla^\rho \nabla^\sigma e^{-\gamma \Phi} R_{\rho\mu\nu\sigma} - \Box e^{-\gamma \Phi} R_{\mu\nu} + 2\nabla_\rho \nabla_\sigma (e^{-\gamma \Phi} R^\rho_{\mu\nu} - \frac{1}{2} \nabla_\mu \nabla_\nu e^{-\gamma \Phi} R) \) 
\[ + 4\alpha (2\nabla_\rho \nabla_\sigma e^{-\gamma \Phi} R^{\rho\sigma} - \Box e^{-\gamma \Phi} R) g_{\mu\nu}, \]  

(4)

and \( \Box \equiv \nabla_\mu \nabla^\mu \) is the d’Alembertian.

In this section, we follow the procedure of Ref. \([11]\). We consider a spherically symmetric static spacetime with the metric

\[ ds^2 = -e^{X(r)} dt^2 + e^{Y(r)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \]  

(5)
where the metric functions depend only on \( r \). Then the dilaton field equation turns out to be

\[
\Phi'' + \Phi' \left( \frac{X' - Y'}{2} + \frac{2}{r} \right) = -\frac{4 \alpha \gamma e^{-\gamma \Phi}}{r^2} \left[ X'Y'e^{-Y} + (1 - e^{-Y}) \left( X'' + \frac{X'}{2}(X' - Y') \right) \right],
\]

(6)

Also, there are three Einstein equations for \((tt)\), \((rr)\), and \((\theta \theta)\) components as follows:

\[
Y' \left( 1 - \frac{4 \alpha \gamma \kappa e^{-\gamma \Phi}}{r} \Phi' \right) (1 - 3e^{-Y}) = \frac{\kappa \Phi'^2}{2} + \frac{1 - e^Y}{r} - \frac{8 \alpha \gamma \kappa e^{-\gamma \Phi}}{r} (\Phi'' - \gamma \Phi'^2) (1 - e^{-Y}),
\]

(7)

\[
X' \left( 1 - \frac{4 \alpha \gamma \kappa e^{-\gamma \Phi}}{r} \Phi' \right) (1 - 3e^{-Y}) = \frac{\kappa \Phi'^2}{2} + \frac{(e^Y - 1)}{r},
\]

(8)

\[
X'' + \left( \frac{X'}{2} + \frac{1}{r} \right) (X' - Y') = -\kappa \Phi'^2 - \frac{8 \alpha \gamma \kappa e^{-\gamma \Phi} - Y}{r} \left( \Phi'X'' + (\Phi'' - \gamma \Phi'^2)X' + \frac{\Phi'X'}{2} (X' - 3Y') \right),
\]

(9)

where only two of three are independent. In other words, one can choose three Eqs. among (6) - (9) as dynamical equations depending on one’s convenience. In the present work we choose three Eqs. (7) - (9) as dynamical equations and the remaining one Eq. (6) as the constraint equation.

Next, we eliminate \( Y' \) in Eqs. (7) and (9) using differentiation of Eq. (8) with respect to \( r \) and rewrite two equations after solving simultaneous equations. Then the equations of motion for \( \Phi'' \) and \( X'' \) are obtained as follows:

\[
\Phi'' = \frac{W_1}{W} \quad \text{and} \quad X'' = \frac{W_2}{W},
\]

(10)

where \( W_1, W_2, \) and \( W \) are functions of only \( X', Y', \Phi, \) and \( \Phi' \) whose detail expressions are shown in Appendix A.

We first examine the existence of a black hole solution with an event horizon. The event horizon is simply the hypersurface at which \( g^{rr}(r_h) = 0 \) or \( g_{rr}(r_h) = \infty \). We check the divergence of the metric function \( e^Y(r) \) at the event horizon \( r_h \). We rearrange the terms in Eq. (8) to get

\[
e^Y = \frac{1}{2} \left[ A \pm \sqrt{A^2 + B} \right],
\]

(11)

where \( A = (r - 4 \alpha \gamma \kappa e^{-\gamma \Phi})X' - \kappa r^2 \Phi'^2 + 1 \) and \( B = 48 \alpha \gamma \kappa e^{-\gamma \Phi} \Phi' X' \). We take the plus sign in Eq. (11).

Assuming \( \Phi_h \) and \( \Phi'_h \) to be finite makes \( X' \to \infty \) at the horizon, as can be seen from Eq. (8). We expand the right-hand-side of Eq. (11) near the event horizon as follows:

\[
e^Y = (r - 4 \alpha \gamma \kappa e^{-\gamma \Phi})X' + \frac{[4r + 32 \alpha \gamma \kappa e^{-\gamma \Phi} - 2r^3 \kappa \Phi'^2 + 8 \alpha \gamma \kappa^2 r^2 e^{-\gamma \Phi} \Phi'^3]}{4(r - 4 \alpha \gamma \kappa e^{-\gamma \Phi} \Phi')} + \mathcal{O} \left( \frac{1}{X'} \right),
\]

(12)

where the quantity \( (r - 4 \alpha \gamma \kappa e^{-\gamma \Phi} \Phi') \) is finite.
After substituting Eq. (12) in Eq. (10), we obtain the following:

\[
\Phi'' = \frac{DH}{\kappa E} X' + O(1),
\]

where

\[
D = r - 4\alpha\gamma\kappa e^{-\gamma\Phi'},
\]

\[
H = 4\alpha\gamma\kappa^2 r^2 e^{-\gamma\Phi} \Phi' - \kappa r^3 \Phi' + 12\alpha\gamma\kappa e^{-\gamma\Phi},
\]

\[
E = r^4 - 4\alpha\gamma\kappa r^3 e^{-\gamma\Phi} \Phi' - 96\alpha^2\gamma^2 \kappa e^{-2\gamma\Phi},
\]

\[
K = r^4 + 16\alpha^2\gamma^2 r^2 e^{-2\gamma\Phi} \Phi'^2 - 8\alpha\gamma\kappa r^3 e^{-\gamma\Phi} \Phi' - 48\alpha^2\gamma^2 \kappa e^{-2\gamma\Phi}.
\]

We check the behaviors of the metric functions and the scalar field at the event horizon \(r_h\). To keep \(\Phi''_{h}\) finite, we choose \(H = 0\). Then we can estimate \(\Phi''_{h} = O(1)\) from Eq. (13) and \(X' = \frac{1}{r-r_h} + O(1)\) from Eq. (14). Under \(H = 0\), \(\Phi'_{h}\) is related to \(\Phi_{h}\) as follows:

\[
\Phi'_{h} = \frac{r_h e^{\gamma\Phi_h}}{8\alpha\gamma\kappa} \left(1 \pm \sqrt{1 - 192 e^{-2\gamma\Phi_h} \alpha^2\gamma^2 \kappa/r_h^4}\right).
\]

From the condition that \(\Phi'_{h}\) has real values we obtain the following condition

\[
e^{-\gamma\Phi_h} < \frac{r_h^2}{\alpha \gamma \sqrt{192\kappa}}.
\]

This is the condition for the existence of a black hole solution with appropriate boundary values \(r_h\) and \(\Phi_h\) in given parameter values.

The asymptotic form of the solutions takes

\[
e^X \simeq 1 - \frac{2M}{r} + O(1/r^3),
\]

\[
\Phi \simeq \Phi_\infty + \frac{Q}{r} + O(1/r^2),
\]

where \(M\) denotes the ADM mass, \(Q\) the scalar charge and \(\Phi_\infty\) the asymptotic value of the scalar field, which will be used to rescale the scalar field and radial coordinate in this work. The mass of a hairy black hole is represented as follows [26]:

\[
M(r) = M(r_h) + 4\pi \int_{r_h}^r r'^2 \left[\frac{1}{2} e^{-Y(\Phi')}^2\right] dr.
\]

where \(M(r_h) = \frac{1}{2} r_h\) is the mass of a black hole subtracting the contribution from the scalar hair. The second term in the right-hand side represents the contribution from the scalar hair, so we will use this term as a amount of hair. The \(M(r)\) increases up to some constant as the distance from the horizon increases, if \(\Phi'\) rapidly decreases to be zero.

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For given non-zero couplings $\alpha$ and $\gamma$, we obtain DGB black holes. However, there is no black hole solutions without a hair in DGB theory. If $\Phi = 0$ exists in DGB theory, dilaton equation of motion in Eq. (3) has only $R^2_{\text{GB}}$ term. However, GB term should be non-zero, so there do not satisfy the equations of motion.

For the $\gamma = 0$ case, the DGB theory becomes the Einstein-Gauss-Bonnet (EGB) theory. The EGB black hole solution with a single coupling $\alpha$ is the same as the Schwarzschild one. This is because the GB term does not contribute to equations of motion as a surface term. However, the GB term contributes to the black hole entropy and influence stability.

### 2.2 Numerical Construction of Black Holes

We obtain the DGB black hole by solving Eqs. (8) and (10). We impose the initial conditions as follows: We first fix the couplings $\alpha$ and $\gamma$ in DGB theory. For a hairy black hole having $r_h$, the maximum value of $\Phi_h$ as one of initial values is satisfied the Eq. (17) with the inequality, and all hairy black holes are solutions for the less than that of $\Phi$ field. And $\Phi'_h$ is obtained from Eq. (16). The initial value of $X'$ is obtained from the relation $X' = 1/r - r_h$. The initial value of $Y$ is obtained from Eq. (8). The initial value of $X$ is obtained for the relation $Y = -X$ to be satisfied in the asymptotic region. The equations are integrated through the 4th-order Runge-Kutta-Fehlberg method from $r_h$ to $r \to \infty$. Our calculations are the precisions of the relative tolerance of $10^{-8}$ and the absolute tolerance of $10^{-8}$. We set $\epsilon = 10^{-10}$. The ADM mass $2M$ is obtained from Eq. (18).

Figure 1: (color online). (a) Scalar field profiles for initial couplings and black hole horizons in $\kappa = 1$, $\gamma = 1/6$, and $\alpha = 1/16$. The five solid lines are different DGB black hole solutions. (b) the numerical solutions represent the metric components $g_{tt}$ and $g_{rr}$ for $r_h = 1$.

The scalar field $\Phi$ should be asymptotically flat, so the values of scalar fields are set to $\Phi_\infty = 0$ at asymptotic region. Under this condition, we redefine $\Phi$ to $\tilde{\Phi} = \Phi - \Phi_\infty$. As the equations of motion have to be invariant under this field shift, a radial coordinate is rescaled to $r \to \tilde{r} = re^{\gamma\Phi_\infty/2}$. In the rescaled system, the mass $M$ and charge $Q$ are also rewritten to $M \to \tilde{M} = Me^{\gamma\Phi_\infty/2}$ and $Q \to \tilde{Q} = Qe^{\gamma\Phi_\infty/2}$, respectively. The other parameters are not changed on rescaled system. Under this rescaling, whole solution space $(r_h, \Phi_h)$ shrinks to unique solution line $(\tilde{r}_h, \tilde{\Phi}_h)$. Therefore, even if
we obtain black hole solutions in specific horizon $r_h$ with couplings $\alpha$ and $\gamma$, the solutions are unique in rescaled system. Once again, there are no dependencies on $r_h$ choice. The detailed discussions are in Appendix B. This rescaled system still satisfies the equations of motions in Eq. (6), (7), (8) and (9) as well as the boundary condition in Eq. (17). We choose the unscaled horizon radius $r_h = 1$ and parameter $\kappa = 1$ for convenience without the loss of generality. Now, we use rescaled variables.

Scalar field $\tilde{\Phi}$ are obtained for given couplings and black hole horizons as shown in Fig. 1(a). The scalar field profiles start at negative values $\tilde{\Phi}_h$ at the black hole horizon $\tilde{r}_h$ for given couplings and monotonically approach to zero. In right-hand side of Fig. 1(a), several profiles are plotted. The bottom profile of blue solid line shows the possible minimum horizon radius and maximum magnitude scalar field $|\tilde{\Phi}_h|$ which satisfies inequality in Eq. (17) for given coupling $\alpha$ and $\gamma$. Upper lines satisfy the inequality in Eq. (17). If DGB black hole horizon becomes larger and larger, the magnitude of scalar field becomes smaller and smaller. In the large horizon radius limit, the scalar field approaches to zero, and then black hole becomes a Schwarzschild black hole. The metric components in Fig. 1(b) shows that the metric component $g_{rr}$ is infinity at the horizon while $g_{tt}$ is approaching zero, whereas both metrics are asymptotically approaching the value 1 along the radial direction.

Figure 2: (color online). Coupling $\gamma$ dependency of upper and lower branches for fixed $\alpha$. For coupling $\alpha = 1/16$, (a) Upper and lower branches exist above $\gamma = \sqrt{2}$ (green). (b) Lower branch disappears between $\gamma = 1.29$ (blue) and $1.30$ (cyan). There is no lower branch below $\gamma = 1.29$ (c) In smaller coupling $\gamma$, the solution approaches the Schwarzschild cases such as $\gamma = 1/2$ (red) and $\gamma = 1/6$ (black).

For fixed $\alpha$, upper and lower branches exist in larger $\gamma$ as shown in Fig. 2(a). The minimum mass $\tilde{M}_{\text{min}}$ is at extremal points as a critical point $C$ as shown in Fig. 2(a) [11, 12, 13, 14]. The upper branch is perturbatively stable and approaches to the Schwarzschild black hole. The lower branch is unstable and ends at singular point $S$ which saturates equality in Eq. (17). We obtain that these branches depend on couplings $\gamma$ and $\alpha$ as shown in Fig. 2(a). In smaller $\gamma$, the singular point $S$ becomes closer to the critical point $C$ as shown in Fig. 2(b). Finally, the point $S$ corresponds to the critical point $C$ between $\gamma = 1.29$ and $1.30$, and the extremum is disappeared as shown in Fig. 2(b). As a result, there is no lower branch below $\gamma = 1.29$. The black hole solution has only upper branch as shown in Fig. 2(c). The solutions are closer to Schwarzschild case as $\gamma$ decreases in Fig. 2(c). We focus on below $\gamma = 1.29$ of which cases are stable under perturbation. We will investigate this upper branch under fragmentation.

The black hole mass and hair depend on couplings $\alpha$ and $\gamma$ as shown in Fig. 3. Each point describes the black hole mass $\tilde{M}$ (black solid line) of which initial conditions is $\tilde{r}_h$ (red solid line) and $\tilde{\Phi}_h$. The
Figure 3: (color online). Rescaled black hole mass $\tilde{M}$ (black) and $\tilde{r}_h$ (red) with respect to $\tilde{\Phi}_h$. The hairy black hole solutions depend on couplings $\alpha$ and $\gamma$.

black hole mass and horizon monotonically increase to infinity with respect to $\tilde{\Phi}_h$. Difference of two lines shows the contribution of black hole hair to the black hole mass from Eq. (20). Black hole hair contributes the largest value in minimum black hole mass and zero at large mass limit. When the coupling $\gamma$ decreases in Fig. 3(a) to (b) or (c) to (d), the lines move down and right-hand side, and slope increases. In this change, the black hole minimum mass becomes smaller, and magnitude of scalar fields approaches faster to zero than larger $\gamma$. In other words, the solutions become those of EGB theory in the limit of $\gamma \to 0$, so the mass profile with respect to $\tilde{\Phi}_h$ should approach to the line of $\tilde{\Phi}_h = 0$. The behaviors changed on smaller $\alpha$ are similar to the change of $\gamma$ as in Fig. 3(a) to (c) or (b) to (d).

3 Instability from Fragmentation

Black holes may undergo instability at some couplings and break apart into black holes. The initial phase is a single black hole having mass $\tilde{M}$, which is the function of an initial horizon $\tilde{r}_h$. The final phase is two black holes far from each other. One of these black holes has a mass $\tilde{m}$ and linear momentum $P_1$, and the other has $\tilde{M} - \tilde{m}$ and $P_2$ under mass and momentum conservation. The
total linear momentum is zero in initial and final phases. This final phase is specified by a mass ratio \\
\delta = \frac{\tilde{m}}{\tilde{M}}. We denote final phase as (\delta, 1 - \delta). The possible minimum mass ratio is given as \delta = \frac{\tilde{M}_{\text{min}}}{\tilde{M}}. The minimum mass ratio has a finite value in DGB black hole, because the black hole has minimum mass \tilde{M}_{\text{min}}. The maximum value of \delta is \frac{1}{2} for half fragmentation. The black holes can be fragmented only when it exceeds twice of minimum mass. With a black hole mass below twice of minimum mass, there are no fragmented black hole solutions, so these black holes are absolutely stable. The mass and momenta of the black hole are related
\\
\tilde{M} = \sqrt{\delta \tilde{M}^2 + \tilde{P}_1^2 + \sqrt{(1 - \delta)^2 \tilde{M}^2 + \tilde{P}_2^2}}. \quad (21)
\\
The linear momenta are arbitrary, so we set \tilde{P}_1 = \tilde{P}_2 = 0 to maximize the total entropy of the final phase. In this condition, the black hole slightly breaks into two black holes with negligible momenta. The initial phase decays to the final phase if the entropy is larger than that of the initial phase. The initial phase is unstable, so that the black hole decays into final phase.

First, as the simplest case, we study possible fragmentation of a Schwarzschild black hole. The fragmentation instability depends on the mass ratio \delta between initial and final phase of a black hole mass. For the case of (\delta, 1 - \delta), the entropy ratio is given as
\\
\frac{S_f}{S_i} = \left( \frac{\delta \tilde{r}_h^2 + (1 - \delta) \tilde{r}_h^2}{\tilde{r}_h^2} \right) = \delta^2 + (1 - \delta)^2, \quad (22)
\\
where we denote initial and final phase entropy \( S_i \) and \( S_f \). The entropy ratio is always smaller than 1, so the entropy of a initial phase is larger than that of a final phase. Therefore, a Schwarzschild black hole is always stable under fragmentation. The entropy ratio marginally approaches 1 in \delta \to 0, because entropy is proportional to the square of a horizon radius, but the mass is proportional to the horizon radius.

These phenomena become different in the theory with the higher order of curvature term. The black hole entropy with the polynomial of the Ricci scalar is given as
\\
S = -2\pi \int_{\Sigma} E^{\mu\nu\rho\sigma} \epsilon_{\mu\nu} \epsilon_{\rho\sigma}, \quad E^{\mu\nu\rho\sigma}_R = \frac{\partial L}{\partial R^{\mu\nu\rho\sigma}}, \quad (23)
\\
where \Sigma, \epsilon_{\mu\nu}, and L, are the bifurcation horizon 2-surface, volume element binormal, and Lagrangian density \cite{27, 28}. In the EGB theory of \gamma = 0, the static black hole metric is the same as that of a Schwarzschild solution and exists for arbitrary mass. However, the entropy has a quantum correction coming from GB term of the higher order of curvature. The initial black hole entropy is
\\
S_i = \frac{A_H}{4G} \left( 1 + \frac{8\alpha\kappa}{\tilde{r}_h^2} \right) = \frac{\pi}{G} \left( \tilde{r}_h^2 + 8\alpha\kappa \right). \quad (24)
\\
Unlike Schwarzschild black holes, the fragmentation instability occurs depending on the fragmentation ratio \delta. For the case of (\delta, 1 - \delta) fragmentation, the final phase entropy is given
\\
S_f = \frac{\pi}{G} \left( (\tilde{r}_h^2)^2 + 8\alpha\kappa \right) + \frac{\pi}{G} \left( ((1 - \delta)\tilde{r}_h^2)^2 + 8\alpha\kappa \right). \quad (25)
\\
The EGB black hole is unstable if,
\\
\frac{S_f}{S_i} = \left( \frac{((\delta \tilde{r}_h^2)^2 + 8\alpha\kappa) + ((1 - \delta)\tilde{r}_h^2)^2 + 8\alpha\kappa)}{(\tilde{r}_h^2 + 8\alpha\kappa)} \right) > 1. \quad (26)
In a small mass limit, the ratio becomes 2 from the dominant correction term. On the other hand, in a large mass limit, the entropy ratio becomes $\delta^2 + (1 - \delta)^2$ as same as that of a Schwarzschild case. Therefore, the EGB black hole with a small mass is unstable, while a massive EGB black hole is stable. There exists a crossing point between initial and final phase entropy. The crossing points are obtained from $S_f/S_i = 1$,

$$\tilde{r}_{cross} = 2\sqrt{\frac{\alpha \kappa}{\delta(1 - \delta)}}.$$  \tag{27}

For given parameter, EGB black holes are unstable below $\tilde{r}_{cross}$. There is no minimum mass of the EGB black hole, so mass ratio $\delta$ has a range of $0 < \delta < \frac{1}{2}$. Several initial and final phase entropies for mass ratios are shown in Fig. 4(a). The smaller mass ratio covers larger mass range as shown in Fig. 4(a). Overall behaviors of entropy are independent on mass ratio as same as MP black hole cases[17, 21]. Each mass ratio $\delta$ makes each line in the phase diagram for given $\alpha$ as shown in 4(b).

The mass ratio can have continuous values, and the black hole has stable and unstable phases. The minimum unstable region is at $\delta = \frac{1}{2}$. For the limit of $\delta \to 0$, all of EGB black holes become unstable for fragmentation as shown in 4(b).

However, the DGB black hole has a GB term coupled with a scalar field, so additional entropy correction comes from the higher curvature term. The DGB black hole entropy \cite{14} is

$$S = \frac{\pi \tilde{r}_h^2}{G} \left( 1 + \frac{8\alpha \kappa}{\tilde{r}_h^2} e^{-\gamma \Phi_h} \right),$$  \tag{28}

where a EGB black hole case corresponds to $\gamma = 0$ case. The DGB black hole entropy ratio between the initial and the final entropy including the higher-curvature corrections is under approximation
\[ \tilde{r}_h \approx 2 \tilde{M}, \]
\[
\frac{S_f}{S_i} = \frac{\left( (\delta \tilde{r}_h)^2 + 8\alpha \kappa \epsilon - \gamma \tilde{\Phi}_\delta \right) + \left( ((1 - \delta) \tilde{r}_h)^2 + 8\alpha \kappa \epsilon - \gamma \tilde{\Phi}_{1-\delta} \right)}{\tilde{r}_h^2 + 8\alpha \kappa \epsilon - \gamma \tilde{\Phi}_h}, \tag{29}
\]

where \( \tilde{\Phi}_h, \tilde{\Phi}_\delta, \) and \( \tilde{\Phi}_{1-\delta} \) are the scalar field values at the initial and final black hole horizon. In the large mass limit \( \tilde{r}_h \gg 1 \), the entropy ratio becomes that of Schwarzschild case,
\[
\frac{S_f}{S_i} = \delta^2 + (1 - \delta)^2 < 1. \tag{30}
\]

Thus, massive DGB black holes are stable under fragmentation. The small mass limits are bounded to \( \tilde{M}_{\text{min}} \). DGB black holes of mass \( \tilde{M}_{\text{min}} \) are absolutely stable, because there are no fragmented black hole solutions. Larger than \( \tilde{M}_{\text{min}} \), the black hole stability is dependent on an entropy correction term.

The entropy ratio is given
\[
\frac{S_f}{S_i} = \frac{\delta^2 + (\delta - 1)^2 + 8\alpha \kappa \epsilon \sin^2 \tilde{\Phi}_\delta + 8\alpha \kappa \epsilon \sin^2 \tilde{\Phi}_{1-\delta}}{1 + 8\alpha \kappa \epsilon \tilde{\Phi}_h} \approx 2, \tag{31}
\]

where the approximation is valid for \( 8\alpha \kappa \epsilon \tilde{\Phi}_\delta \gg \tilde{r}_h^2 \). Thus, DGB black holes are unstable for small mass. By the way, the DGB black hole solutions have a minimum mass for given couplings, so the minimum mass bounds the fragmentation mass ratio. It is not seen in the Schwarzschild black hole or EGB black hole. The DGB black holes have more variety properties and behaviors. We will obtain detailed behaviors through the numerical calculation.

4 Numerical Analysis for Fragmentation Instability

We investigate the fragmentation instability using a numerical analysis. We consider fragmentation cases of \( \delta \leq \delta \leq \frac{1}{2} \) as shown in Figure 5.

![Fragmentation Diagram](image-url)
The initial and final phase entropies with respect to \( r_{h,i} \) for given couplings \( \gamma \) and \( \alpha \). The blue solid line and blue dashed-dot line are initial and final phase entropies in EGB theory as a reference for \((\frac{1}{2}, \frac{1}{2})\). The red solid line and red dashed-dot line are initial and final phase entropies in DGB theory for \((\frac{1}{2}, \frac{1}{2})\). Initial phase exists above red circle for minimum mass. Final phase exists above red box for \((\frac{1}{2}, \frac{1}{2})\). The green solid line represents fragmentation for marginal mass ratio \( \bar{\delta} \).

The DGB black hole entropies in Eq. \((24)\) are shown with respect to the horizon radius in Fig. \(6\) for \( \gamma = \frac{1}{2} \) and \( \gamma = \frac{1}{6} \) with \( \alpha = \frac{1}{16} \). Unlike EGB black holes for a blue line, DGB black holes have a minimum mass \( \tilde{M}_{\text{min}} \) for given parameters. Red circle corresponds to initial black hole having a minimum mass. The red box corresponds to a fragmented black hole having a minimum mass. Overall behaviors are similar to those of EGB black holes. The DGB black hole entropy is slightly larger than that of EGB theory, because of hair contribution. Possible fragmentation of DGB black holes occurs at the twice of the minimum mass such as \((\frac{1}{2}, \frac{1}{2})\) mass ratio. Below half fragmentation, DGB black holes are absolutely stable between red circle and box. In this range, these black holes have no final phase solutions to decay. In Fig. \(6(a)\), initial phases are stable, because all final phase entropies are smaller than that of initial phase above red box. However, in Fig. \(6(b)\), red box is located under crossing point, so DGB black holes are unstable between red box and crossing point. Also, above crossing point, the initial phases are stable. For the limit of \( \gamma \to 0 \), the red box must approach to \( \tilde{r}_h = 0 \). DGB black holes are more unstable in smaller mass ratio \( \bar{\delta} \) as the same to EGB black hole cases. The largest unstable region is given at \( \bar{\delta} \). This fragmentation always starts at \((\frac{1}{2}, \frac{1}{2})\) mass ratio and then appears above as shown in Fig. \(6(b)\). The crossing point from \((\bar{\delta}, 1 - \bar{\delta})\) fragmentation appears larger initial black hole mass than that of EGB black hole. As a result, DGB black holes are stable in larger mass.

The black hole solutions also depend on a coupling \( \alpha \) as shown in Fig. \(7\) for initial mass \( 2\tilde{M} = 1 \) and \( 2\tilde{M} = \frac{1}{2} \) with \( \gamma = \frac{1}{6} \). Compared with EGB black holes, DGB black holes depend on coupling \( \alpha \). DGB black hole entropies behave similar to those of EGB black holes, but DGB black hole solutions cover a finite region as shown in Fig. \(7(a)\) and \(7(b)\). DGB black hole entropies are slightly larger than those of EGB black holes, and the crossing point appears smaller coupling \( \alpha \) than that of EGB black holes. Initial phases are stable for small \( \alpha \) as shown in magnified figures of Fig. \(7\). The initial black hole mass has no effect to the overall behaviors of DGB black holes in Fig. \(7(a)\) and \(7(b)\). For given mass, possible coupling \( \alpha \) has a maximum value represented with a red circle. Between red circle and box, DGB black holes are absolutely stable. There are no final phase solutions. In the twice
of minimum mass at red box, \((\frac{1}{2}, \frac{1}{2})\) fragmentation is possible. Below red box, DGB black holes are stable in small \(\alpha\) and unstable in large \(\alpha\).

For fixed coupling \(\gamma\), the DGB black hole phases are represented with respect to mass \(\tilde{M}\) and coupling \(\alpha\) as shown in Fig. 8 for \(\gamma = \frac{1}{2}\) and \(\gamma = \frac{1}{6}\). In the large \(\gamma\), as we see in Fig. 6(a), the DGB black holes have three phases such as no solution, absolutely stable, and stable regions for given \(\alpha\) as shown in Fig. 8(a). In small coupling \(\gamma\) as shown Fig. 6(b), the DGB black holes have four phases such as no solution, absolutely stable, unstable, and stable regions for given \(\alpha\) as shown in Fig. 8(b). In the limit of \(\gamma \to 0\), the DGB theory approaches the EGB theory, so final phases are dominant on small mass, and the black holes are unstable as shown in Fig. 8(b). The unstable region from fragmentation appears between stable and absolutely stable region. Initial phase are still stable for a large mass. The largest unstable region comes from \((\bar{\delta}, 1 - \bar{\delta})\) fragmentation. These unstable regions start at the origin of Fig. 8(b).
Now, we fix a coupling $\alpha$. DGB black hole phase diagrams are represented with respect to mass $\tilde{M}$ and $\gamma$ in Fig. 9. In the large $\gamma$, DGB black holes have stable, absolutely stable, and no solution phases. In the small $\gamma$ coupling, DGB black holes have four phases such as stable, unstable, absolutely stable, and no solution phases as shown in Figs. 9(a) and (c). The absolutely stable region is bounded by the minimum mass of the black hole. The $\delta$, $1 - \delta$) fragmentation gives the largest unstable region of DGB black holes and meets the absolutely stable region at twice of minimum mass or $(\frac{1}{2}, \frac{1}{2})$ fragmentation. The $(\delta, 1 - \delta)$ fragmentation is a marginal boundary of arbitrary mass ratio fragmentation, so all of fragmentation are ended at the $(\delta, 1 - \delta)$ fragmentation line like $(\frac{1}{2}, \frac{1}{2})$ fragmentation as shown in Fig. 9(b) and (d). For example, the fragmentations for $(\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{4}, \frac{3}{4})$, and $(\frac{1}{10}, \frac{9}{10})$ are shown in Fig. 9(b). In the limit of $\gamma \rightarrow 0$, these behaviors correspond to EGB black hole cases, so black hole solutions exist for all values of mass and have a crossing point in $\gamma = 0$ slice. In this limit, black holes only have two phases, unstable and stable.
5 Summary and Discussion

We have investigated the fragmentation instability black holes with a GB term. To explore these phenomena, we have numerically constructed the static DGB hairy black hole in asymptotically flat spacetime. Two couplings $\alpha$ and $\gamma$ affect the scalar hair and mass of the black hole. The profiles of the scalar fields monotonically go to zero at the asymptotic region. The initial magnitudes of dilaton fields are almost inversely proportional to black hole horizons. When the scalar field on the horizon is the maximum, the DGB black hole solution has the minimum horizon size as shown in Fig. 1. The black hole solutions with respect to couplings $\alpha$ and $\gamma$ are shown in Fig. 3. The DGB black hole has a minimum mass for given couplings. The amount of black hole hair decreases as the DGB black hole mass increases. DGB black hole configurations go to EGB black hole cases for small $\alpha$ and $\gamma$. In EGB black hole cases, the black hole solution exists for all values of black hole mass. In other words, the minimum mass becomes zero.

We have investigated DGB black hole instability with fragmentation where the entropy has an additional contribution in Eq. (24) from the higher-curvature GB term. The fragmentation is a non-perturbative method to test stability. We found that the DGB black hole phase is unstable under fragmentation, even if these phases are stable under perturbation. These instabilities have been numerically investigated with respect to couplings.

In the limit of $\gamma \to 0$, the DGB black hole approaches the EGB black hole. The EGB black hole simply has only two phases, stable and unstable. The small EGB black hole is unstable and is fragmented to final phase. The relatively massive EGB black hole is stable. The mass ratio $\delta = \frac{1}{2}$ gives the smallest unstable region. In the limit of $\delta \to 0$, all of EGB black holes are unstable. For the finite values of $\gamma$, the DGB black hole has a minimum mass, so more phases appear. Since the DGB black hole has a minimum mass, the mass ratio $\delta$ is bounded below $\tilde{\delta}$. The mass ratio has a range between $\tilde{\delta}$ and $\delta = \frac{1}{2}$. The phase diagram for given coupling are shown in Fig. 8. In small $\gamma$, the DGB black hole has four phases such as no solution, absolutely stable, unstable, and stable phases. Below the minimum mass, the DGB theory has no black hole solution. there is no fragmented black hole solution between this minimum mass and the twice of the minimum mass, so the initial black hole is absolutely stable in such range. Above the twice of minimum mass, the black hole can be fragmented with the mass ratio $(\delta, 1 - \delta)$. The fragmentation is bounded to $\tilde{\delta}$ which is the minimum fragmentation for given couplings. Above $\tilde{\delta}$ fragmentation with respect to black hole mass, the DGB black hole becomes stable under fragmentation. These phases reduce to three phases in large $\gamma$. The unstable region under fragmentation approaches absolutely stable region and then disappears. Above the minimum mass, the DGB black hole is stable.

For given coupling $\alpha$, the DGB black hole phases are also shown in Fig. 9. Through these diagrams, we can show that the $\delta$ fragmentation plays a role of the marginal fragmentation. For given $\alpha$, the DGB black hole has four phases such as no solution, absolutely stable, unstable, and stable phases in small $\gamma$. For large $\gamma$, the DGB black hole has three phases such as no solution, absolutely stable, and stable phases. These behaviors are not changed with respect to $\alpha$. The smallest unstable region comes from $\frac{1}{2}$ fragmentation which meets at $\tilde{\delta}$ fragmentation and the absolutely stable region. The mass ratio $\tilde{\delta}$ fragmentation gives the largest unstable region. The $\delta$ fragmentation is the marginal fragmentation for any mass ratio. We have found the phase diagram of the fragmentation instability for a black hole mass and two couplings.
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Appendix A

The detailed forms of functions in Eq. (10) are as follows.

\[
W = 4 \left( 8 \left( -1 + e^Y \right) \alpha \gamma X' \left\{ -3e^{2(Y+\gamma\Phi)} r^2 + 4e^{Y+\gamma\Phi} (-13 + 3e^Y) r \alpha \gamma k \Phi' \
+16 \left( -15 + 7e^Y \right) \alpha^2 \gamma^2 k^2 \Phi'^2 \right\} + e^{Y+\gamma\Phi} \left\{ 24e^{Y+\gamma\Phi} (-1 + e^Y) r \alpha \gamma \
+ \left( -e^{2(Y+\gamma\Phi)} r^4 + 224e^{2Y} \alpha^2 \gamma^2 \kappa - 448e^Y \alpha^2 \gamma^2 \kappa + 224e^{2Y} \alpha^2 \gamma^2 \kappa \right) \Phi' \right\} + 4e^{Y+\gamma\Phi} \left( -7 + 3e^Y \right) r^3 \alpha \gamma k \left( \Phi'^2 \right) + 32 \left( -5 + 3e^Y \right) r^2 \alpha^2 \gamma^2 k^2 \left( \Phi' \right)^3 \right\} \right),
\]

\[
W_1 = 2 \left( X' \right)^2 \left( e^{Y+\gamma\Phi} r + 8 \alpha \gamma \kappa \Phi \right) \left\{ e^{Y+\gamma\Phi} r - 4 \left( -3 + e^Y \right) \alpha \gamma k \Phi' \right\}^2 + X' \left\{ 10e^{3(Y+\gamma\Phi)} (-1 + e^Y) r^2 - 4e^{2(Y+\gamma\Phi)} r \gamma \left( e^{Y+\gamma\Phi} r^2 + 42 \alpha \kappa - 52e^Y \alpha k \right) + 10e^{2Y} \alpha k \right\} \Phi' - e^{Y+\gamma\Phi} \left( 5e^{2(Y+\gamma\Phi)} r^4 + 32e^{Y+\gamma\Phi} r^2 \alpha \gamma^2 + 64e^{2Y+\gamma\Phi} r^2 \alpha^2 \gamma^2 + 768 \alpha^2 \gamma^2 \kappa - 1152 \alpha^2 \gamma^2 \kappa + 384e^{2Y} \alpha^2 \gamma^2 \kappa \right) \left( \Phi' \right)^2 + 4e^{Y+\gamma\Phi} \alpha \gamma \left( -23e^{Y+\gamma\Phi} r^2 + 5e^{2Y+\gamma\Phi} r^2 + 80 \alpha \gamma^2 - 352 \alpha \gamma^2 + 80 \alpha \gamma^2 \kappa \right) \kappa^2 \left( \Phi' \right)^3 + 32 \alpha \gamma^2 \left( -15e^{Y+\gamma\Phi} r^2 + 7e^{2Y+\gamma\Phi} r^2 + 96 \alpha \gamma^2 \kappa - 256 \alpha \gamma^2 \kappa + 96 \alpha \gamma^2 \kappa \right)^3 \left( \Phi' \right)^4 \right\} + 2e^{Y+\gamma\Phi} \left\{ -6e^{3(Y+\gamma\Phi)} (-1 + e^Y) r^2 + 2e^{Y+\gamma\Phi} (-1 + e^Y) \gamma \left( e^{Y+\gamma\Phi} r^2 \right) + 28 \alpha \kappa - 28e^Y \alpha k \right\} \Phi' + e^{Y+\gamma\Phi} \left( 3e^{Y+\gamma\Phi} r^2 + e^{2Y+\gamma\Phi} r^2 + 8 \alpha \gamma^2 - 48e^Y \alpha^2 \gamma \right) + 40e^{2Y} \alpha^2 \gamma^2 \left( \Phi' \right)^2 + \gamma \kappa \left( e^{2(Y+\gamma\Phi)} r^4 + 76e^{Y+\gamma\Phi} r^2 \alpha \gamma^2 - 12e^{2Y+\gamma\Phi} r^2 \alpha \kappa \right) + 256 \alpha^2 \gamma^2 \kappa - 640e^{Y} \alpha^2 \gamma^2 \kappa + 384e^{2Y} \alpha^2 \gamma^2 \kappa \right) \left( \Phi' \right)^3 + 32 \alpha \gamma \gamma \kappa \left( \Phi'^4 \right) + 8 \alpha \gamma \left( e^{Y+\gamma\Phi} r^2 - 4 \alpha \gamma^2 + 12e^Y \alpha^2 \gamma \right)^3 \left( \Phi' \right)^5 \right\},
\]

\[
W_2 = -8 \alpha \gamma \left( X' \right)^3 \left\{ -e^{2(Y+\gamma\Phi)} (-7 + 5e^Y) r^2 + 4e^{Y+\gamma\Phi} \left( 29 - 26e^Y + 5e^2Y \right) r \alpha \gamma k \Phi' \right\} + 32 \left\{ 15 - 16e^Y + 5e^2Y \right\} \alpha^2 \gamma^2 k^2 \left( \Phi' \right)^2 \right\} + 2e^{2(Y+\gamma\Phi)} \Phi' \left\{ 2 \left( -1 + e^Y \right) \left( e^{Y+\gamma\Phi} r^2 + 8 \alpha \gamma^2 - 8 \alpha e^Y \alpha^2 \right) \right\} - 16 \left\{ 1 - 4e^Y + 3e^2Y \right\} r \alpha \gamma k \Phi' + r^2 \left( e^{Y+\gamma\Phi} r^2 + 24 \alpha \gamma^2 \right) \left( \Phi' \right)^2 - 8 \left( -5 + 3e^Y \right) r^3 \alpha \gamma k^2 \left( \Phi' \right)^3 \left\{ 2e^Y \left( X' \right) \right\} + 2e^Y \left( X' \right)^2 \left\{ -20e^{Y+2\gamma\Phi} \left( -1 + e^Y \right) r \alpha \gamma + e^{Y+2\gamma\Phi} \left( e^{2(Y+\gamma\Phi)} r^4 + 8e^{Y+\gamma\Phi} r^2 \alpha \gamma^2 + 8e^{2Y+\gamma\Phi} r^2 \alpha \gamma^2 - 128 \alpha \gamma^2 \kappa \right) + 256e^{Y} \alpha^2 \gamma^2 \kappa - 128e^{2Y} \alpha^2 \gamma^2 \kappa \right\} \Phi' - 2e^{\gamma\Phi} r \alpha \gamma \left( -25e^{Y+\gamma\Phi} r^2 + 7e^{2Y+\gamma\Phi} r^2 - 48 \alpha \gamma^2 - 96e^{Y} \alpha^2 \gamma^2 + 16e^{2Y} \alpha^2 \gamma^2 \kappa \right) \Phi' \left( \Phi' \right)^2 - 16 \alpha \gamma^2 \left( -25e^{Y+\gamma\Phi} r^2 + 13e^{2Y+\gamma\Phi} r^2 - 96 \alpha \gamma^2 \right) + 32 \alpha \gamma \gamma \kappa^2 \left( \Phi' \right)^3 \right\} + e^{Y+\gamma\Phi} \left\{ 16 \alpha \gamma \gamma \kappa^2 \left( -1 + e^Y \right) r \alpha \gamma + 2e^{Y+\gamma\Phi} \left( e^{2(Y+\gamma\Phi)} r^4 + 24e^{Y+\gamma\Phi} r^2 \alpha \gamma^2 \right) \left( \Phi' \right)^2 - r \kappa \left( e^{2(Y+\gamma\Phi)} r^4 + 24e^{Y+\gamma\Phi} r^2 \alpha \gamma^2 + 24e^{2Y+\gamma\Phi} r^2 \alpha \gamma^2 + 960 \alpha \gamma^2 \kappa - 768e^{Y} \alpha^2 \gamma^2 \kappa + 320e^{2Y} \alpha^2 \gamma^2 \kappa \right) \left( \Phi' \right)^3 \right\} - 8e^Y r^2 \alpha \gamma \left( 5e^{\gamma\Phi} r^2 + 48 \alpha \gamma^2 \right) \kappa^2 \left( \Phi' \right)^4 \right\}. \]
Appendix B

The initial conditions for $\Phi_h$ and $r_h$ satisfy inequality in Eq. (17) for given couplings $\alpha$ and $\gamma$. The different choices of $r_h$ provide different values of $\Phi_h$ as shown in Fig. 10(a). The minimum values of $\Phi_h$ in each solid line satisfy the inequality in Eq. (17). Each solid line gives different profiles $\Phi(r)$. As a result, the values of scalar fields at infinity $\Phi_\infty$ are different for each solid line as shown in Fig. 10(b). Using scalar field values $\Phi_\infty$, we obtain rescaled system as shown in Fig. 11. For each $r_h$ choice, the rescaled scalar fields $\tilde{\Phi}_h$ are rearranged in Fig. 11(a), and the rescaled scalar field values $\tilde{\Phi}_h$ are all same for different choices of $r_h$. Next, the horizon radius are also rescaled to $\tilde{r}_h$. Eventually, the different choices of $\Phi_h$ and $r_h$ converge to a unique black solid line in Fig. 11(b). In other words, 2-dimensional solution space reduces to actually 1-dimensional line. Therefore, whatever we choose for any value of $r_h$, there is no loss of generality.

Figure 10: (color online). (a) The allowed values of $\Phi_h$ for given $r_h = 1/2$, 1, and 2 with given couplings. (b) The values $\Phi(r)$ of $r \to \infty$ for given $r_h = 1/2$, 1, and 2 with given couplings.

Figure 11: (color online). (a) The initial conditions $\tilde{\Phi}_h$ for given $r_h = 1/2$, 1, and 2 with given couplings. (b) The different initial conditions of given $r_h = 1/2$, 1, and 2 converge to a line with respect to black hole mass $\tilde{M}$ or horizon $\tilde{r}_h$. 
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