THE SAITO MODULE AND THE MODULI OF A GERM OF CURVE IN \((\mathbb{C}^2,0)\).

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Abstract. This article proposes to study the moduli space of a germ of curve \(S\) in the complex plane, that is to say the equisingularity class of \(S\) up to analytical equivalence relation. The first part is devoted to proving that this last quotient can be endowed with a reasonable complex structure, yet not canonical. The second part deals with the computation of its generic dimension in terms of topological invariants of \(S\). It can be obtained from the study of the valuations of the Saito module of \(S\), \(\text{Der} (\log S)\), i.e. the module of vector fields tangent to \(S\).

Introduction.

The number of moduli of a germ of curve \(S\) in \((\mathbb{C}^2,0)\) is basically the number of parameters on which depend a topologically miniversal family for \(S\). It is also the generic dimension of the quotient of the topological class of \(S\) up to analytical equivalence relation, provided that this quotient admits a structure from which a notion of dimension can be derived. Indeed, this moduli space defined by the quotient of the topological class of \(S\)

\[ \{ S' | S' \sim_{\text{top}} S \} \]

by the following action of \(\text{Diff} (\mathbb{C}^2,0)\)

\[ \phi \cdot S' = \phi (S'), \quad \phi \in \text{Diff} (\mathbb{C}^2,0) \]

a priori has no particular structure beyond being a set.

The first determination of such a number of moduli goes back to the work of Sherwood Ebey in 1965 [7] who dealt with the irreducible curves - those having only one irreducible component. Ebey proved that the moduli space of \(S\) carries a complex structure compatible with a non separated topology and computed the number of moduli for a particular topological class of curve, namely, that given by the equation \(y^5 = x^9\). In 1973, in [23], Oscar Zariski proposed various approaches to get the number of moduli for irreducible curves beyond the case treated previously by Ebey. He introduced most of the concepts on which future work will be based. In 1978, Delorme [5] studied extensively the case of an irreducible curve with one Puiseux pair. In 1979, Granger [13] and later, in 1988, Briançon, Granger and Maisonobe [1] produced an algorithm to compute the number of the moduli of a non irreducible quasi-homogeneous curve. In 1988, Laudal, Martin and Pfister in [19], improved the work of Delorme and gave an explicit description of a miniversal family. From 2009, in a series of papers [13, 15, 16], Abramo Hefez and Marcelo
Hernandes greatly improved the previous studies and achieved the analytical classification of irreducible curves. Their algorithmic approach provided in particular the number of moduli.

In 2010 and 2011, in [10, 11], Emmanuel Paul and the author described the moduli space of a topologically quasi-homogeneous curve $S$ as the spaces of leaves of an algebraic foliation defined on the moduli space of a foliation whose analytic invariant curve is precisely $S$. This work initiated an approach based on the theory of foliations. In 2019, in [8], the author gave an explicit formula for the number of moduli for a curve $S$, generic in its topological class: this formula involves only very elementary topological invariants of $S$, such as, the topological class of its desingularization.

The aim of this article is to investigate the full general case, that is the number of moduli of a germ of curve in the complex plane. We emphasize that our objective is far from being as ambitious as a complete analytical classification, which would require at least some deep algorithmic procedures, but is rather to obtain a geometric interpretation of these moduli and a procedure to calculate their number from primitive topological invariants.

This work follows the ideas introduced in [8] and illustrated in [9], which focus on the irreducible case.

Section 1 establishes an extension of the result of Ebey [7, Theorem 4] to the non irreducible curve: it concerns the structure of the moduli space. As noticed by Ebey himself at the end of its article, its machinery derives from the theory of algebraic groups and depends on the groups being solvable and connected. Therefore, it cannot be directly carried over to several component curves. Here, we overcome this issue by considering not only curves but curves enriched with a marking which allows us to recover the necessary connexity. As Ebey, we use an adapted complete topological invariant - the semi ring of values - introduced by R. Waldi [24] and some of its properties identified by M. Hernandes and E. de Carvalho in [17]. Finally, we obtain the following result

**Theorem.** The marked moduli space $M^\bullet(S)$ of a germ of curve $S$ in $\mathbb{C}^2$, that is its marked topological class up to analytical marked equivalence relation, can be identified with the quotient of a complex constructible set by an action of a connected solvable algebraic group. In particular, it is endowed with a non separated complex structure.

Notice that passing from the moduli space to the marking moduli space has no effect on the generic dimension.

Section 1 can be read independently from the rest of the article.

Section 2 and 3 aim to develop the study of the module $\text{Der}(\log S)$ of vector fields tangent to $S$, on which depends the computation of the number of moduli of $S$. The starting point is a remark of K. Saito in [22], that, highlighted the freeness of this module - which is specific to the curves embedded in the complex plane. An immediate consequence of the work of Saito is that, the smallest valuation of
the vector fields in $\text{Der}(\log S)$ cannot be too big compared to the valuation of $S$, namely, the following upper bound holds

$$\min_{X \in \text{Der}(\log S)} \nu(X) \leq \frac{\nu(S)}{2}.$$ 

Our purpose is to prove that, generically, this bound is essentially reached. In section 2, the existence of a flat basis of $\text{Der}(\log S)$ is shown in the generic situation, that is, a basis admitting an analytic extension as a basis for the modules $\text{Der}(\log C)$ where $C$ are in a neighborhood of $S$ in $M^*(S)$. As a consequence, using the theory of infinitesimal deformations of foliations of $X$, Gómez-Mont [12], we obtain the following theorem

**Theorem.** For $S$ generic in its moduli space $M^*(S)$, one has

$$\min_{X \in \text{Der}(\log S)} \nu(X) \geq \begin{cases} \left\lfloor \frac{\nu(S)}{2} \right\rfloor & \text{if } S \text{ is not of radial type} \\ \left\lfloor \frac{\nu(S)}{2} \right\rfloor - 1 & \text{else} \end{cases}.$$ 

The definition of $S$ being of radial type will be given in the article. Note that if $S$ is not generic in its moduli space, the above lower bound is false, as it will be illustrated by some examples in the article. Moreover, in section 3, we proceed with the precise description of the various possibilities for the flat basis of $\text{Der}(\log S)$.

Finally, Section 4 illustrates our approach for the computation of the generic dimension of $M^*(S)$. As a consequence of section 2 and 3, we recover the classical dimension of the moduli space of the singularity $x^n + y^n = 0$ with $n \geq 1$.

**Corollary 1** ([13]). The generic dimension of $M^*(S)$ where $S = \{x^n + y^n = 0\}$ is equal to

$$\begin{cases} \frac{(n-2)^2}{4} & \text{if } n \text{ is even} \\ \frac{(n-1)(n-3)}{4} & \text{if } n \text{ is odd} \end{cases}.$$ 

In an upcoming article, we will build an algorithm based upon the results presented here, that computes the generic dimension of the moduli space for more general curves. We implemented, among other procedures, this algorithm on Sage 9.*. See the routine Courbes.Planes following the link

https://perso.math.univ-toulouse.fr/genzmer/

1. Moduli space of marked curve.

Throughout this article, $S$ stands for a germ of singular curve in the complex plane $(\mathbb{C}^2, 0)$. In particular, its algebraic valuation is at least 2. From now on, we fix a decomposition of $S$ in irreducible components

$$S = S^1 \cup \cdots \cup S^r$$
where \( r \) is the number of irreducible components. Here and subsequently, \( \text{Comp} (S) \) stands for the set of the irreducible components of \( S \).

Let \( C \) be a germ of curve topologically equivalent to \( S \) by a germ of homeomorphism of the ambient space \((\mathbb{C}^2, 0)\) denoted by \( h \) and such that
\[
h (S) = C.
\]
The application \( h \) induces a bijective map
\[
\sigma_h : \text{Comp} (S) \to \text{Comp} (C)
\]
Two such homeomorphisms \( h \) and \( h' \) are said to be equivalent if and only if
\[
(1.1) \quad \sigma_h = \sigma_{h'}.
\]
**Definition 1.** A curve marked by \( S \) is a couple \((C, \overline{h})\) where \( C \) is curve topologically equivalent to \( S \) and \( \overline{h} \) a class of homeomorphism between \( C \) and \( S \) for the equivalence relation defined above. We will denote by \( \text{Top}^* (S) \) the set of curves marked by \( S \).

The group \( \text{Diff} (\mathbb{C}^2, 0) \) of germs of automorphisms of the ambient space \((\mathbb{C}^2, 0)\) acts on the set \( \text{Top}^* (S) \) by
\[
\phi \cdot (C, \overline{h}) = (\phi (C), \overline{\phi \circ h})
\]
In what follows, the quotient of \( \text{Top}^* (S) \) by \( \text{Diff} (\mathbb{C}^2, 0) \) will be denoted by
\[
\mathcal{M}^* (S)
\]
and will be refered to as the marked moduli space of \( S \). Although \( \mathcal{M}^* (S) \) cannot be endowed with a complex structure by some general statements about group actions, the result below provides such a structure. Indeed, generalizing a result of Ebey [7], we obtain the

**Theorem 1.** The quotient \( \mathcal{M}^* (S) \) can be identified with the quotient of a complex constructible set by an action of a connected solvable algebraic group.

This result still holds if we drop the assumption of \( S \) being a plane curve, once we replace the topological equivalence by the equisingularity which corresponds to the equality of the semirings of valuations as definded in [17]. Since the general proof consists at most in increasing the complexity of the notations, we state Theorem 1 and prove it only for a curve embedded in the complex plane. We follow Theorem 5 in [7] observing that a connected solvable algebraic action on a complex constructible set admits a complete transversal, that is a constructible subset in correspondance one to one with the orbits of the action. Thus, from Theorem 1 \( \mathcal{M}^* (S) \) inherits of the complex structure of this transversal. Its compatible topology is just the quotient topology : in most case, it is not separated (see for instance [14, 15]).

The goal of the current section is to prove Theorem 1.
1.1. **The ring of of functions of** $((C, \overline{h}))$. Let $(C, \overline{h})$ be in $\text{Top}^\bullet (S)$ and 

$$\gamma_C = \{ \gamma_c : t \in (C, 0) \rightarrow (\mathbb{C}^2, 0) \}_{c \in \text{Comp}(C)}$$

be any system of parametrizations of the irreducible components of $C$. We denote by $C_i$ the component of $C$ defined by the marking $h$

$$C_i = \sigma_h (S^i).$$

From the marking $\overline{h}$ of $C$, we derived a morphism of rings defined by

$$\mathcal{E}_{(C, \overline{h})} : \mathbb{C}[[x, y]] \rightarrow ((\mathbb{C}[[t]])^r \ni \gamma_c^*, u_{i=1,...,r},$$

which factorizes in an monomorphism

$$\mathcal{E}_{(C, \overline{h})} : \hat{\mathcal{O}}_C = \mathbb{C}[[x, y]] \hookrightarrow ((\mathbb{C}[[t]])^r$$

where $f$ is any reduced equation of $C$ and $\hat{\mathcal{O}}_C$ is the completion of $\mathcal{O}_C = \frac{\mathbb{C}[[x, y]]}{(f)}$.

The following result is classic - see [7] for the irreducible case.

**Lemma 1.** Let $(C, \overline{h})$ and $(C', \overline{h}')$ be two marked curves in $\text{Top}^\bullet (S)$. The following properties are equivalent

1. The curves $(C, \overline{h})$ and $(C', \overline{h}')$ are analytically equivalent by a conjugacy preserving the markings.
2. The images of the monomorphisms (1.2) associated to both curves are conjugated by a diagonal formal automorphism of $((\mathbb{C}[[t]])^r$.

1.2. **The tropical semiring of values of** $(C, \overline{h})$. Following [17], we consider $\Gamma_C$ the set defined by

$$\Gamma_C = \{ \nu (G) | G \in \mathcal{O}_C \} \subset (\mathbb{N})^r$$

where $\mathbb{N} = \mathbb{N} \cup \{ \infty \}$. The valuation $\nu$ is defined by

$$\nu (G) = (\nu_0 (\gamma_c^*, G))_{i=1,...,r}$$

where $\nu_0$ is the standard valuation $\mathbb{C} \{ t \}$. Notice that this set depends not only on the curve $C$ but also on its marking.

The set $\Gamma_C$ inherits of a semiring structure defined by

$$\alpha \oplus \beta = (\min \{ \alpha_i, \beta_i \})_{i=1,...,r} \quad \alpha \odot \beta = (\alpha_i + \beta_i)_{i=1,...,r}$$

where we set $k + \infty = \infty$. $\Gamma_C$ is also partially ordered by the product order $\leq$. The quadruplet $(\Gamma_C, \oplus, \odot, \leq)$ is called **the tropical semiring of values of** $(C, \overline{h})$.

**Definition 2.** An element $\alpha \in \Gamma_C$ is said **irreducible** if and only if

$$\alpha = a + b \text{ with } a, b \in \Gamma_C \implies \alpha = a \text{ or } \alpha = b.$$

It is said to be **absolute** if for any non empty proper subset $J$ of the set

$$\mathcal{I}_\alpha = \{ i \in \{ 1, \ldots, r \} | \alpha_i \neq \infty \},$$

the following set

$$F_J (\alpha) = \{ a \in \Gamma_C | \forall i \in \mathcal{I}_\alpha \setminus J, a_i > \alpha_i \text{ and } \forall i \notin \mathcal{I}_\alpha \setminus J, a_i = \alpha_i \}$$
is empty.

The following result gathers some known properties of the semiring of values.

**Theorem 2 ([17][24]).** Two germ of plane curves are topologically equivalent if and only if they share the same semiring of values [24]. More precisely, if $C_1 \cup C_2 \cup \cdots \cup C_r$ and $C'_1 \cup C'_2 \cup \cdots \cup C'_r$ are two curves with same semiring, then there exists an homeomorphism $\phi$ of the ambient space $(\mathbb{C}^2, 0)$ such that for any $i$ $\phi(C_i) = C'_i$.

Moreover,

1. $\Gamma_C$ has a conductor, i.e, there exists a minimal $\sigma \in \Gamma_C$ such that $\sigma + \mathbb{N}^r \subset \Gamma_C$ [14].
2. The set $g$ of irreducible absolute points of $\Gamma_C$ is finite and minimally generates $\Gamma_C$ as semiring [17].
3. Any family $G$ of $O_C$ such that $\nu(G) = g$ is a minimal standard basis of $O_C$ as defined in [17].

In particular, for any element $C \in \text{Top}^\bullet(S)$, one has $\Gamma_C = \Gamma_S$.

For now on, we will denote the mutual semiring for curves in $\text{Top}^\bullet(S)$ simply by $\Gamma$.

1.3. **Truncation and conductor.** The following lemma allows us to truncate elements in the ring $O_C$ (resp. $\hat{O}_C$).

**Lemma 2.** Suppose that $G = (\sum_{k=0}^{\infty} a_{lk}t^k)_{l=1, \ldots, r}$ is an element of $O_C$ (resp. of its completion $\hat{O}_C$). Then, for any $p = (p_1, \ldots, p_r) \in \mathbb{N}^r$ with $p_l \geq \sigma_l - 1$ for $l = 1, \ldots, r$, one has

$$\left(\sum_{k=0}^{p_l} a_{lk}t^k\right)_{l=1, \ldots, r} \in O_C,$$ (resp. $\hat{O}_C$)

**Proof.** By definition of $\sigma$, for any $l = 1, \ldots, r$ and for any $k \geq p_l + 1 \geq \sigma_l$, the $r$–uple

$$\left(\infty, \cdots, \infty, k_{l_{th}}, \infty, \cdots, \infty\right)$$

belongs to $\Gamma$. Thus, an inductive argument on the rank $k \geq p_l + 1$ shows that there exists a formal series $\hat{F}_l \in \mathbb{C}[[x, y]]$ such that

$$\gamma^* \hat{F}_l = \left(0, \cdots, 0, \sum_{k=p_l+1}^{\infty} a_{lk}t^k, 0, \cdots, 0\right).$$
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Now, following [21, Theorem 1, p.493], if \(G\) is convergent, so is \(\hat{F}_i\) and, in any case, evaluating

\[ G = \gamma^* \left( \sum_{i=1}^{r} \hat{F}_i \right) \]

yields the lemma. \(\square\)

1.4. \(\Gamma\)-reduction. The notion of \(\Gamma\)-reduction will allow us to construct normal forms for systems of generators of \(\hat{\mathcal{O}}_C\).

Let \(\overline{P} = (P_i)_{i=1,\ldots,r}\) be a family of \(r\) finite subsets of \(\overline{\mathbb{N}}\) such that for any \(i, \infty \in P_i\).

**Definition 3.** The family \(\overline{P}\) is said to be \(\Gamma\)-reduced if and only if

\[ \Gamma \cap \prod_{i=1,\ldots,r} P_i = \{\infty\} \]

where \(\infty = (\infty, \infty, \ldots, \infty)\)

A \(\Gamma\)-reduction of \(\overline{P}\) is an elementary transformation of \(\overline{P}\) of the following form: suppose that there exists \(n = (n_1, \ldots, n_r) \neq \infty\) such that

\[ n \in \Gamma \cap \prod_{i=1,\ldots,r} P_i. \]

Consider an integer \(i\) such that \(n_i \neq \infty\). Then the family \(\overline{P}^{(1)} = \left(P_i^{(1)}\right)_{i=1,\ldots,r}\) defined by

\[
\begin{cases}
P_j^{(1)} = P_j & \text{for } j \neq i \\
P_i^{(1)} = P_i \setminus \{n_i\}
\end{cases}
\]

is called a \(\Gamma\)-reduction of \(\overline{P}\). To keep track of a \(\Gamma\)-reduction, we denote it by

\[ \overline{P} = \overline{P}^{(0)} \xrightarrow{\mu_i} \overline{P}^{(1)} \]

The following lemma is obvious

**Lemma 3.** For any \(\overline{P}\), there exists a finite sequence of \(\Gamma\)-reductions

\[ \overline{P} = \overline{P}^{(0)} \xrightarrow{\mu_i} \overline{P}^{(1)} \xrightarrow{\mu_i} \cdots \xrightarrow{\mu_i} \overline{P}^{(q)} \]

such that \(\overline{P}^{(q)}\) is \(\Gamma\)-reduced.

Notice that this sequence is not unique.
1.5. **Parametrization of the set** \( \text{Top}^* (S) \). Let \( g = \{ g^1, \cdots , g^q \} \) be the set of irreducible absolute points of \( \Gamma \) and \( G = \{ G^1, \cdots , G^q \} \subset \mathcal{O}_C \) such that for all \( i \),
\[
\nu (G^i) = g^i.
\]

**Lemma 4.** Among the family \( G \) and in the identification
\[
\mathcal{O}_C = \frac{\mathbb{C} [[x,y]]}{(f)},
\]
there are exactly two components \( G^i \) whose linear parts are independant.

**Proof.** Assume that \( C \) contains an irreducible singular component, say \( C_1 \), and consider some coordinates \((x,y)\) such that it is parametrized by
\[
t \to (t^n, t^m + \cdots), \quad n \nmid m
\]
Evaluating the valuation of the coordinate functions \( x \) and \( y \), we obtain that \( \Gamma \) contains two elements of the form
\[
(1.5) \quad \nu (x) = (n, \cdots) \in \Gamma \quad \text{and} \quad \nu (y) = (m, \cdots) \in \Gamma.
\]
If the linear parts of the functions \( G^i \) are dependant two by two, then the set of valuations of the complete ring generated by the family \( G \) can contains either \((n, \cdots)\) or \((m, \cdots)\) or none of them, but certainly not both. However, according to Theorem 2, the complete ring generated by \( G \) is the whole ring \( \hat{\mathcal{O}}_C \), which contradicts (1.5). If \( C \) contains two smooth components, transversal or not, a contradiction can be obtained in much the same way by considering coordinates in which these components are written
\[
t \to ((t,0), (0,t), \cdots) \quad \text{or} \quad t \to ((t,0), (t,t^n), \cdots), \quad n \geq 2.
\]
Finally, the existence of a third function \( G^i \) with a non trivial linear part would contradict the minimality of the family \( g \).

Changing the numbering of the elements in \( g \), we may assume that the two elements identified by the above lemma are \( G^1 \) and \( G^2 \) with \( g^1 < g^2 \) for the lexicographic order. Let us denote \( G^i, i = 1, 2 \) by
\[
(1.6) \quad G^i = \left( \sum_{k=g^l_i}^{\infty} a^i_{lk} t^k \right)_{l=1,\ldots,r}
\]
Notice that in the above expression, \( g^l_i \) may be equal to \( \infty \) and the corresponding component \((G^i)_l\) be equal to \( 0 \). However, one has the following

**Lemma 5.** Assume that \( C \) is not the union of two smooth curves. If \( g^l_i \neq \infty \), then \( g^l_i \leq \sigma_l - 1 \).

**Proof.** The proof is by contradiction. Suppose that for some \( l \), \( g^l_i \neq \infty \) and \( g^l_i \geq \sigma_l \).
Applying Lemma 2 to \( G^i \) with
\[
(p_l)_{i=1,\ldots,r} = (\infty, \cdots, \infty, \sigma_l, \infty, \cdots, \infty)
\]
yields an element \( \overline{g} \in \Gamma \) such that \( \overline{g}_l = \infty \) and \( \overline{g}_k = g^i_k \) for \( k \neq l \). Consider the proper subset of \( I_{g^i} \) defined by 

\[
J = I_{g^i} \setminus \{l\},
\]

and suppose it is non empty. Definition 2 of absolute point ensures that \( F_J (g^i) \) is empty. However, by construction, \( \overline{g} \) belongs to \( F_J (g^i) \) which is a contradiction. Thus, \( J \) is empty and \( I_{g^i} = \{l\} \). Therefore, \( g^i \) is written 

\[
g^i = (\infty, \ldots, \infty, g^i_l, \infty, \ldots, \infty). \]

- If \( r \geq 3 \), we are lead to a contradiction noticing that \( G^i \) would be a function with non trivial linear part vanishing along two distinct components of \( C \).

- Assume \( r = 2 \). Since \( G^i \) is a regular function and \( g^i = (\infty, g^i_2) \) or \((g^i_1, \infty)\), one of the component of \( C \), say \( C_1 \), is smooth. One can choose some coordinates \((x, y)\) such that 

\[
C_1 = \{ \alpha y + \beta x = 0 \}, \quad \alpha, \beta \in \mathbb{C} \\
C_2 = \{ y^p + x^q + \cdots = 0 \}
\]

with \( p < q \). The hypothesis of the lemma ensures that the case \( p = 1 \) is excluded. According to [17], the conductor \( \sigma \) of \( C \) is written 

\[
\sigma = (0, c_2) + \begin{cases} 
(p, p) & \text{if } \beta \neq 0 \\
(q, q) & \text{if } \beta = 0
\end{cases},
\]

where \( c_2 \geq 1 \) is the conductor of the component \( C_2 \). By construction, the function \( G^1 \) is equal to \( \alpha y + \beta x \). Therefore,

\[
g^1 = (\infty, \begin{cases} p & \text{if } \beta \neq 0 \\
q & \text{if } \beta = 0 \end{cases}),
\]

thus \( g^1_2 < \sigma_2 \).

\[\square\]

If \( C \) is a union of two smooth curves then one has 

\[
\sigma = (n, n), \quad g^1 = (\infty, n) \quad \text{and} \quad g^2 = (n, \infty)
\]

where \( n \) is the order of tangency between \( C_1 \) and \( C_2 \). Actually, \( C \) is analytically equivalent to the curve 

\[
y(y + x^n) = 0.
\]

Thus the moduli space of \( C \) reduces to a point and the problem of the analytic classification is trivial. For now on, we will assume that \( C \) is not a union of two smooth curves.

Lemma 2 yields truncations of \( G^i \), \( i = 1, 2 \) that we keep on denoting it by 

\[
(1.7) \quad G^i = \left( \max_{j \neq i} (\sigma_j - 1, g^j) \right) \sum_{k=g^i_j}^{\max(\sigma_l-1, g^i_l)} a_{ik}^i t^k
\]

Notice that some components of (1.7) - but not all - may vanish.
We are going to normalize the expressions of $G^i$ in order to make it unique and depending only on the marked curve $(C, \overline{n})$. The first normalization consists in the following: for $i = 1, 2$ let us consider the smallest $l_i$ such that $g_{l_i}^i \neq \infty$, we impose that

$$a_{l_i, g_{l_i}^i}^i = 1.$$ 

To go further in the normalization, we will use $\Gamma$-reductions. For $i = 1, 2$ let us consider $P_i = (P_i^1, \ldots, P_i^r)$ defined by

$$\begin{cases} P_i^l = \left[ g_{l_i}^i, \max (\sigma_i - 1, g_{l_i}^i) \right] \cap \mathbb{N} \cup \{\infty\} & \text{if } l \neq l_i \\ P_i^l = \left[ g_{l_i}^i + 1, \max (\sigma_i - 1, g_{l_i}^i) \right] \cap \mathbb{N} \cup \{\infty\} & \text{if } l = l_i \end{cases}$$

Notice that if $g_i^i = \infty$ then $P_i^i = \{\infty\}$. In the same way, if $g_i^i = \sigma_i - 1$ then $P_i^i = \{\infty\}$.

For any $n \in \mathbb{N}$, we denote by $\text{Init}^i(n)$ the integer defined by

$$\min \{ k | (n)_k \neq \infty \text{ and } (n)_k \neq g_k^i \}.$$ 

If $n \neq \infty \in \Gamma \cap \prod_{l=1}^r P_i^l$ then $\text{Init}^i(n)$ is well defined since the set of which it is the minimum is non-empty: indeed, if for any $l$, one has $(n)_l = g_l^i$ or $\infty$. In particular, $(n)_l = \infty$. Moreover, $n$ belongs to $F_J(g^i)$ where $J$ is defined by

$$J = \{ k | (n)_k \neq \infty \}.$$ 

The set $J$ is non-empty since $n \neq \infty$ and is proper since $l_i \notin J$. That is impossible because by definition of absolute point, $F_J(g^i)$ is empty.

We choose a sequence of $\Gamma$-reductions of $P_i$

$$P_i = P_i^{(0)} \xrightarrow{n_0} P_i^{(1)} \xrightarrow{n_1} \cdots \xrightarrow{n_{q_i-1}} P_i^{(q_i)}$$

such that for any $t \in \{0, \cdots, q_i - 1\}$,

1. $(P_1)$ Init$^i(n) = \min \{ \text{Init}^i(n) | n \neq \infty \in \Gamma \cap \prod_{l=1}^r P_i^{(t)} \}$ and
2. among the $n_l$'s that satisfy the previous equality, we choose the one for which the integer

$$(n)_{\text{Init}^i(n)}$$

is the smallest possible.

Let us show how the $\Gamma$-reduction

$$P_i^{(t)} \xrightarrow{n_t} P_i^{(t+1)}$$

allows us to normalize $G^i$. The $r$-uple $n_t$ being an element of $\Gamma$, by definition, there exists a sum of the form

$$W^{i,(t)} = \sum_{\beta \in \mathbb{N}^2} \nu^{i,(t)}(\beta_1) G^1)^{\beta_1} (G^2)^{\beta_2}$$

such that $\nu(W^{i,(t)}) = n_t$ and the coefficient of $i_{\nu}^{(t)}(n_t)$ in the $k_t^{i}$th component is equal to 1. The difference

$$G^i - a_{k_t(n_t)}^{i} W^{i,(t)}$$

(1.8)
belongs to \( \mathcal{O}_C \) and the coefficient of \( t^{(nt)} \) in the \( k^\text{th} \) component vanishes. By construction, after a \( \Gamma \)–reduction, the new couple of functions defined by \( (1.8) \) still generates \( \mathcal{O}_C \). Doing the whole process of \( \Gamma \)–reductions for both \( G^i, i = 1, 2 \) and a final truncation at \( \sigma \), we obtain a normalized family of generators that we denote \( (\mathcal{O}^i (G^i))_{i=1,2} \). By construction, following the properties \( (P_1) \) and \( (P_2) \), a normalized family of generators \( (\mathcal{O}^i (G^i))_{i=1,2} \) is written

\[
\mathcal{O}^i (G^i) = \left( \sum_{k \in P_i (q_i)} a_{i,k} t^k, \sum_{k \in P_i (q_i)} a_{i,k} t^k \right).
\]

The main characteristic of this normalized basis is that its parameters are unique: indeed, \( G \) and \( G' \) being two couples of normalized generators as in \( (1.9) \), we consider the valuation \( \gamma = \nu (G^i - (G')^i) \).

By definition, \( \gamma \) is an element of \( \Gamma \). By construction of the normalized family, it is also an element of \( \prod_{i=1,\ldots,r} P_i (q_i) \). Since \( \sum_i (q_i) \) is \( \Gamma \)–reduced, \( \gamma \) is equal to \( \infty \) and \( G^i \) and \( (G')^i \) are equal. Therefore the normalized basis is unique and we can consider the following well defined map

\[
M_S : \begin{cases} \text{Top}^\bullet (S) & \longrightarrow \ \prod_{i=1}^r C_{P_i (q_i)} \\ (C, \tilde{h}) & \longmapsto (a_{i,k}) \end{cases}
\]

that associates to a marked curve in \( \text{Top}^\bullet (S) \), the ordered coefficients of a normalized family of generators of \( \mathcal{O}_C \).

### 1.6. \( \text{Top}^\bullet (S) \) as a constructible set.

In this section, we are going to prove the

**Proposition 1.** The image of \( M_S \) is a constructible analytic set, i.e., a finite union of finite intersections of algebraic subsets and complements of algebraic subsets of the affine set \( \prod_{i=1}^r C_{P_i (q_i)} \).

**Proof.** Consider an element of \( \prod_{i=1}^r C_{P_i (q_i)} \) and the associated couple \( (G^1, G^2) \) as in \( (1.9) \). The complete ring generated by \( G \) is the completion of the ring of a plane curve \( C \) with \( r \) components \( C^1, \ldots, C^r \) given by the coordinates of \( G \). Fix some \( i \) in \( \{1, \ldots, r\} \). We begin by proving that the condition

\[
g^i \in \Gamma_C
\]

is a constructible condition. Choose any reduced equation \( h_i (x,y) \) of the curve

\[
\bigcup_{j \neq i} C^j.
\]
If $\mathcal{I}_i$ is empty, choose simply $h_i = 1$. Consider $\mathcal{N}$ the finite set of couples of integers $(u, v) \in \mathbb{N}^2$ such that

$$
\nu \left( h_i \left( G^1, G^2 \right) (G^1)^u (G^2)^v \right) \not\geq \sigma.
$$

and the set of expressions of the form

$$
(1.10) \quad h_i \left( G^1, G^2 \right) \times \sum_{(u,v)\in\mathcal{N}} \beta_{uv} (G^1)^u (G^2)^v, \quad \beta_{uv} \in \mathbb{C}
$$

where the $\beta_{uv}$’s are coefficients. It follows that $g^i \in \Gamma_C$ is equivalent to the existence of a family $\{\beta_{uv}\}_{uv}$ so that the expression (1.10) has a valuation equal to $g^i$. Let $L^i_{l,k}$ be the coefficient of $t^k$ in the $l^{th}$ component of (1.10). The functions $L^i_{l,k}$ are linear forms in the variables $\beta_{uv}$ whose coefficients are algebraic expressions in the coefficients of the generators $G^i$. The condition $g^i \in \Gamma_C$ is equivalent to require that for each $l = 1, \ldots, r$, the linear form $L^i_{l,g^i_l}$ is linearly independent of the linear forms $L^i_{l,k}$ for $l = 1, \ldots, r$ and $k < g^i_l$. The latter condition is constructible one in the coefficients of the generators $G^i$ since it can be expressed using the ranks of the minors of the matrix of these linear forms. It follows that $g^i \subset \Gamma_C$ and thus $\Gamma \subset \Gamma_C$ is a constructible condition. We can now proceed analogously to prove that $\Gamma = \Gamma_C$ is also a constructible condition: indeed, according to [17], providing that $\Gamma \subset \Gamma_C$, the equality $\Gamma = \Gamma_C$ is equivalent to the equality

$$
\Gamma \cap \prod_{i=1}^r [0, \sigma_i] = \Gamma_C \cap \prod_{i=1}^r [0, \sigma_i]
$$

which induces a finite number of conditions, that can be proven to be constructible with similar arguments. □

1.7. Action on $\text{Top}^\star(S)$. The group $(\text{Diff}(\mathbb{C},0))^r$ acts on the image of $M_S$ the following way: given a point $A$ in the image, consider its corresponding couple of generators $(G^1, G^2)$. Take an element $\phi \in (\text{Diff}(\mathbb{C},0))^r$ and right compose $G^i$, $i = 1, 2$, by $\phi$; apply the process of normalization following a sequence of $\Gamma$–reductions initially fixed and truncate the final expressions. In the end, the coefficients of the new normalized couple of generators

$$
(\mathfrak{N}^i \left( G^i \circ \phi \right))_{i=1,2}
$$

corresponds to some expressions $\phi \cdot A$ which depends on $A$ and $\phi$.

**Lemma 6.** The application $(\phi, A) \rightarrow \phi \cdot A$ is an action. More precisely, for any $\phi, \psi$ in $(\text{Diff}(\mathbb{C},0))^r$

$$
\phi \cdot (\psi \cdot A) = (\psi \circ \phi) \cdot A.
$$

**Proof.** For $i = 1, 2$, consider a normalized basis $(G^1, G^2)$ and the two following normalizations

$$
(\mathfrak{N}^i \left( G^i \circ \psi \circ \phi \right))_{i=1,2} \text{ and } (\mathfrak{N}^i \left( \mathfrak{N}^i \left( G^i \circ \psi \circ \phi \right) \bigcirc \phi \right))_{i=1,2}.
$$
Both are normalized bases of the ring 
\[(\psi \circ \phi)^* \mathcal{O}_C = \{\gamma \circ \psi \circ \phi | \gamma \in \mathcal{O}_C\}.\]
The rings \(\mathcal{O}_C\) and \((\psi \circ \phi)^* \mathcal{O}_C\) share the same semiring of valuations \(\Gamma\). Thus, for \(i = 1, 2\), the valuation 
\[\nu \left( \mathfrak{N}^i \left( G^i \circ \psi \circ \phi \right) - \mathfrak{N}^i \left( \mathfrak{N}^i \left( G^i \circ \psi \circ \phi \right) \circ \phi \right) \right)\]
is an element of \(\Gamma \cap \prod_{l,i} P_i^{l_i(q_i)}\). Since \(P_i^{l_i(q_i)}\) is \(\Gamma\)-reduced, this valuation is \(\infty\) and one has 
\[\mathfrak{N}^i \left( G^i \circ \psi \circ \phi \right) = \mathfrak{N}^i \left( \mathfrak{N}^i \left( G^i \circ \psi \circ \phi \right) \circ \phi \right),\]
which is the lemma. \(\square\)

Let us denote by \(\text{Diff}^c (\mathbb{C}, 0)\) the quotient of \(\text{Diff} (\mathbb{C}, 0)\) by the normal subgroup of elements of the form 
\[t \to t + ut^c + \cdots.\]
The truncation at \(\sigma\) being part of the normalization process, it follows that the previous action factorizes through 
\[(1.11) \quad \prod_{i=1}^r \text{Diff}^{c_i} (\mathbb{C}, 0).\]
Since the group \((1.11)\) is a connected solvable algebraic group, Theorem \(\|\) follows from Lemma \(\|\) and the previous constructions.

1.8. An example. Let \(S\) be the curve \(y \left( y^2 - x^3 \right) = 0\). Figure \(1.1\) shows the semiring \(\Gamma_S\). In this rather simple situation, it can computed by hand. In the general case, there exist algorithms to compute the semiring of a curve with several components - see [2].

![Figure 1.1: Semiring of values the curve \(\{y \left( y^2 - x^3 \right) = 0\}\).](image)

Let \((\mathcal{C}, \mathcal{T})\) be in \(\text{Top}^* (S)\). The conductor \(\sigma\) of \(\Gamma\) is \((3, 5)\). The set of absolute points of \(\Gamma\) is 
\[\{(1, 2), (2, 4), (3, \infty), (\infty, 3)\}.\]
Since \((2, 4) = 2 \times (1, 2)\), following [17], the set that minimaly generates \(\Gamma\) as semiring is 
\[g = \{(1, 2), (3, \infty), (\infty, 3)\}.\]
Since \( \nu(x) = (1, 2) \) and \( \nu(y) = (\infty, 3) \), applying Lemma 2 leads to a couple of generators that are written
\[
\{G^1 = (a_1 t + a_2 t^2, a_2 t^2 + a_3 t^3 + a_4 t^4), G^2 = (0, a_3 t^3 + a_4 t^4)\}
\]
with \( a_1 \neq 0, a_2 \neq 0, a_3 \neq 0 \). Normalizing some initial non vanishing coefficients provides the following couple of generators
\[
\{G^1 = (t + a_1 t^2, a_2 t^2 + a_3 t^3 + a_4 t^4), G^2 = (0, t^3 + a_4 t^4)\}. 
\]
To reduce \( G^1 \) we consider the following data
\[
P^1_{1,0} = \{\infty, 2\} \quad P^1_{1,0} = \{\infty, 2, 3, 4\},
\]
and the two successive \( \Gamma_S \)-reductions defined as follows
\[
P^1_{1,0} \xrightarrow{(2,3),1} P^1_{1,1} = \{\{\infty\}, \{\infty, 2, 3, 4\}\}.
\]
Observe \( P^1_{1,2} \) is \( \Gamma_S \)-reduced. Since \( \nu\left((G^1)^2 + G^2\right) = (2, 3) \), the transformation associated to the first \( \Gamma_S \)-reduction as in (1.3) is written
\[
G^1 - a_1^2 \left((G^1)^2 + G^2\right) = (t + t^3 (\cdots), (\ast) t^2 + (\ast)^3 t^3 + (\ast)^3 t^4 + t^5 (\cdots)),
\]
which leads to a new generator that we still denote by \( G^1 \). Noticing that \( \nu\left(G^2\right) = (\infty, 3) \) yields the transformation
\[
G^1 - (\ast) G^2.
\]
The truncation at \( \sigma = (3, 5) \) finishes the normalization of \( G^1 \). The generator \( G^2 \) is already normalized since \((\infty, 4) \notin \Gamma\).

Therefore, the normalized family \( G \) has for final form
\[
G^1 = (t, at^2 + bt^4), G^2 = (0, t^3 + ct^4)
\]
and its elements depends only on \( E_{(C, \Gamma)} \). The map \( M_S \) is defined by
\[
M_S : \left\{\begin{array}{cc}
\text{Top}^0 (S) & \rightarrow C^3 \\
(C, \Gamma) & \rightarrow (a, b, c)
\end{array}\right.
\]
following (1.12). By construction if \( a \neq 0 \), any curve \( C \) associated to a ring generated by such a family admits a semiring of values \( \Gamma_C \) that contains \( (1, 2) \) and \((\infty, 3) \). It can be checked that \( a \neq 0 \) is the sole condition to ensure that actually, \( \Gamma_C = \Gamma_S \). Thus the image of \( M_S \) is the constructible set \( C \setminus \{0\} \times C^2 \).

Let us compute the action of \( \phi \in \text{Diff}^3 (C, 0) \times \text{Diff}^5 (C, 0) \) on \( A \in C \setminus \{0\} \times C^2 \) induced by the present construction where
\[
\phi = (ut + vt^2, \alpha t + \beta t^2 + \gamma t^3 + \Delta t^4) \quad \text{and} \quad A = (a, b, c).
\]
The action of \( \phi \) on \( A \) is written
\[
\phi \cdot A = \left( \frac{\alpha^2}{u}, -a^2 \alpha^4 v - 2a^2 \alpha^2 \beta cu^2 + \alpha^4 bu^2 + 2a \alpha \gamma u^2 - 5a \beta^2 u^2 - 5a^2 \beta^2 u^2, \alpha c + 3 \frac{\beta}{\alpha} \right)
\]
and the quotient reduces to the class of the point \((1, 0, 0)\). As a matter of fact, the curve \( S \) has no moduli (11).
2. Optimal vector field for a germ of curve $S$.

The space $M^* (S)$ is now endowed with a complex structure. The remainder of the article is interested in the generic dimension of $M^* (S)$. In order to reach this purpose, subsequently, we proceed to the study of the module of vector fields tangent to $S$.

Let $S$ be a germ of curve in $(\mathbb{C}^2, 0)$ and $f$ a reduced equation of $S$. Throughout this article, $\text{Der} (\log S)$ will stand for the module of vector fields tangent to $S$, that is such that the set of vector fields $X$ such that $X \cdot f \in (f)$.

It will be called the Saito module of $S$ in reference to [22]. Associated to the latter, we consider the following analytical invariant

**Definition 4.** The Saito number of $S$ is the integer

$$s (S) = \min_{X \in \text{Der} (\log S)} \nu (X),$$

where $\nu$ is the valuation defined by

$$\nu (a \partial_x + b \partial_y) = \min (\nu (a), \nu (b)).$$

According to [22], the Saito module of $S$ is a free $\mathcal{O}_{(\mathbb{C}^2, 0)}$--module of rank 2. If $\{X_1, X_2\}$ is one of its basis, said to be a Saito basis for $S$, it is easily seen that the number of Saito of $S$ satisfies

$$s (S) = \min (\nu (X_1), \nu (X_2)).$$

Following again [22], $\{X_1, X_2\}$ is a Saito basis for $S$ if and only if the following property holds.

**Criterion** (Criterion of Saito). $\{X_1, X_2\}$ is a Saito basis for $S$ if and only if there exists a germ of unit $u$ such that

$$X_1 \wedge X_2 = uf,$$

where $\cdot \wedge \cdot$ stands for determinant of the vector fields in any coordinates.

The property (2.1) will be referred to as the criterion of Saito. Evaluating the valuation of (2.1) gives the inequality

$$\nu (X_1) + \nu (X_2) \leq \nu (X_1 \wedge X_2) = \nu (f) = \nu (S).$$

In particular, one has

$$s (S) \leq \frac{\nu (S)}{2}.$$

**Definition 5.** A vector field $X \in \text{Der} (\log S)$ is said to be optimal for $S$ if $\nu (X) = s (S)$.
Example 1. Let $S$ be the double cusp given by
\[ S = \{(x^2 - y^3)(y^2 - x^3) = 0\}. \]
Then an optimal vector field can be given by
\[ X = \left(2x^2 + \frac{5}{2}y^3 - \frac{9}{2}x^3y\right)\partial_x + (3xy - 3x^2y^2)\partial_y. \]
In particular
\[ s(S) = 2. \]

Proposition 2. If $X$ is optimal for $S$, then there exists a vector field $Y$ such that $\{X, Y\}$ is a Saito basis for $S$.

Proof. Let $\{X_1, X_2\}$ be any Saito basis for $S$. According to the criterion of Saito, there exists a unit $u$ such that
\[ X_1 \wedge X_2 = uf. \]
Since $\{X_1, X_2\}$ is a basis, there exist functions $u_i$, $i = 1, 2$ such that
\[ X = u_1X_1 + u_2X_2. \]
Since $\nu(X) = s(S) = \min(\nu(X_1), \nu(X_2))$, for some $i$, say $i = 1$, $u_i$ is a unit. Then, using (2.4) yields
\[ X \wedge X_2 = u_1uf, \]
and thus, $\{X, X_2\}$ is a Saito basis for $S$. \hfill \Box

2.1. Curve of radial type. Let $E$ be the single blowing-up at $0$. The total space of the blowing-up will be denoted by $M$,
\[ E : (M, D) \rightarrow (\mathbb{C}^2, 0). \]
For any curve $S$, $S^E$ will stand for the strict transform of $S$ by $E$, that is the closure in $M$ of $E^{-1}(S \setminus \{0\})$. Moreover, for any vector field $Y$, $Y^E$ will be the blown-up vector field divided by the maximal power of a local equation of $D$.

Definition 6. Let $Y$ be a germ of vector field in $(\mathbb{C}^2, 0)$. It is is said dicritical if $Y^E$ is generically transverse to the exceptional divisor $D$.

A vector field $X$ being dicritical, it can be written in some coordinates $(x, y)$
\[ X = R(x, y)(x\partial_x + y\partial_y) + (\cdots) \]
where $R$ is an homogeneous polynomial function and $(\cdots)$ stands for higher order terms. Suppose that $X$ is optimal for $S$ and let $Y$ be such that $\{X, Y\}$ is a basis of $\text{Der} (\log S)$. For any couple of non-vanishing functions $(a, b)$, the initial part of $aX + bY$, that is, its homogeneous part of smallest degree, is written
\[ a(0)R(x, y)(x\partial_x + y\partial_y) + b(0)Y^{(s(S))}, \]
where $Y^{(*)}$ stands for the homogeneous part of degree $*$ of $Y$. If $Y$ is optimal and not dicritical then for $a$ and $b$ generic, $aX + bY$ is not dicritical. Which is why, we consider the following definition

Definition 7. $S$ is said to be of radial type if all optimal vector fields for $S$ are dicritical.
2.2. Flat Saito basis. In this section, we are going to identify an open dense set $U \subset M^\bullet(S)$ for which, the Saito basis of $C \in U$, can be extended locally around $C$ in $M^\bullet(S)$ into a family of Saito bases. Further on, an example will illustrate that this property holds only generically.

**Theorem 3.** There exist an open dense set $U \subset M^\bullet(S)$ on which the Saito number is constant. More precisely, for any $(C, h) \in U$, there exists a germ of analytical family of vector fields

$$c \in (M^\bullet(S), (C, h)) \mapsto X_i(c), \ i = 1, 2$$

such that for any $c$, the family $\{X_1(c), X_2(c)\}$ is a Saito basis for $c$ with

$$\forall c \in (M^\bullet(S), (C, h)), \ \nu(X_1(c)) = s(c).$$

**Proof.** Let $(C, h) \in M^\bullet(S)$ be a regular point for the complex structure of $M^\bullet(S)$.

Consider a miniversal deformation of $C$

$$(\Sigma, C) \subset (\mathbb{C}^{2+N}, 0) \xrightarrow{\pi} (\mathbb{C}^N, 0), \quad \pi(x, t) = t \in \mathbb{C}^N$$

versal for topologically trivial deformations of $C$ and for which the singular locus of $\Sigma$ is $\{0\} \times \mathbb{C}^N$: it is enough to consider the miniversal deformation of any reduced equation of $C$ and to restrict it to the associated smooth $\mu-$constant stratum. We fix an open neighborhood $\mathbb{C}^{2+N} \supset U \ni 0$ on which $\Sigma$ and $C$ are well defined. By shrinking $U$ if necessary, we can also suppose that, out of its singular locus, $\Sigma$ is transverse to the fiber of $\pi$, that is for any $p \in U \setminus \{0\} \times \mathbb{C}^N$,

$$\pi^{-1}(\pi(p)) \not\subset T_p \Sigma$$

The deformation (2.5) is topologically trivial: more precisely, there exists an homeomorphism $H:(\mathbb{C}^{2+N}, 0) \rightarrow (\mathbb{C}^{2+N}, 0)$ such that

1. $\pi H = \pi$
2. $H|_{\pi^{-1}(0)} = h$
3. The following diagram commutes

$$\begin{array}{ccc}
(S \times (\mathbb{C}^N, 0), S) & \xrightarrow{H} & (\Sigma, C) \\
\pi \downarrow & & \pi \\
(\mathbb{C}^N, 0) & & (\mathbb{C}^N, 0)
\end{array}$$

By construction, the map $C$ defined by

$$(2.7) \quad t \in (\mathbb{C}^N, 0) \xrightarrow{\xi} \left(\Sigma|_{\pi^{-1}(t)}, \overline{\mathcal{H}}|_{\pi^{-1}(t)}\right) \in M^\bullet(S)$$

is a local diffeomorphism.

For technical reason, we add to $\Sigma$ an hyperplane $H$ not contained in $\Sigma$ and transverse to $\pi$. Consider $\Sigma^\circ = \Sigma \cup H$. In what follows, $f_{\Sigma^\circ}$ stands for a reduced equation of $\Sigma$. The kernel of the evaluation map

$$\text{Der}(\log \Sigma^\circ) \xrightarrow{d\pi(\cdot)} (\mathcal{O}_{N+2})^N$$
is the sheaf $\text{Der}^\dagger (\log \Sigma^o)$ of *vertical* vector fields tangent to $\Sigma^o$. In the initial coordinates $(x, y, t)$ a section of $\text{Der}^\dagger (\log \Sigma^o)$ is written

$$a (x, y, t) \partial_x + b (x, y, t) \partial_y$$

where $a$ and $b$ are analytic functions. The sheaf $\text{Der} (\log \Sigma^o)$ is coherent, so is $\text{Der}^\dagger (\log \Sigma^o)$. Note that if $X$ is a section of $\text{Der}^\dagger (\log \Sigma^o)$, then for any $t \in \pi (U)$, $X|_{\pi^{-1}(t)}$ is tangent to $\Sigma^o|_{\pi^{-1}(t)}$. Fix a system of generators

$$\{X_1, \cdots, X_n\}$$

of $\text{Der}^\dagger (\log \Sigma^o) (U)$. We are going to use the following remarks which are consequences of the coherence property: for any open set $V \subset U$, the vector fields $X_1|_V, \cdots, X_n|_V$ generate $\text{Der}^\dagger (\log \Sigma^o) (V)$. Moreover,

1. If $V$ does not meet $\Sigma^o$ then $\text{Der}^\dagger (\log \Sigma^o) (V)$ is the set of all holomorphic vertical vector fields on $V$.
2. If $V$ meets the smooth part of $\Sigma^o$, then $\text{Der}^\dagger (\log \Sigma^o) (V)$ is freely generated by the vertical vector fields $x\partial_x$ and $\partial_y$ where $(x, y, t)$ is a local system of coordinates preserving the fibration $\pi$ for which $x = 0$ is an equation of the trace of $\Sigma^o$ on $V$: such a local system of coordinates exists under the transversality property (2.6). In particular, the product

$$x\partial_x \wedge \partial_y$$

vanishes at order 1 along $\Sigma^o$.

All the $X'_i$s cannot vanish identically on a given component of $\Sigma^o$ because for instance the section of $\text{Der}^\dagger (\log \Sigma^o)$ defined by

$$\partial_x (f_{\Sigma^o}) \partial_y - \partial_y (f_{\Sigma^o}) \partial_x$$

does not vanish on any component of $\Sigma^o$. Considering if necessary a combination of the $X'_i$s, we can suppose that $X_1$ does not vanish identically on any component of $\Sigma^o$. We can also suppose that $X_1$ is singular in codimension 2: indeed, if not, there exists $\hat{X}_1$ such that $X_1 = h\hat{X}_1$ where $h$ is an holomorphic map with $h(0) = 0$. Since $h$ cannot vanish identically on any component of $\Sigma^o$, $\hat{X}_1$ is tangent to $\Sigma^o$ and the family

$$\{\hat{X}_1, \cdots, X_n\}$$

still generates the sheaf $\text{Der}^\dagger (\log \Sigma^o)$. Now, if there exists $j \neq 1$ such that

$$X_1 \wedge X_j \equiv 0$$

then, by division, there exists $\phi$ such that $X_j = \phi X_1$, which contradicts the minimality of the system of generators (2.8). Thus, for any $j \neq 1$, there exists a function $g_j \neq 0$ such that

$$X_1 \wedge X_j = f_{\Sigma^o} g_j = x f_{\Sigma} g_j$$

(2.9) where the system of coordinates $(x, y, t)$ is chosen so that $x$ is an equation of the added hyperplan $H$. Consider a point $p$ in the zero set $Z(g_2, \ldots, g_n)$ of the ideal $(g_2, \ldots, g_n)$. If $p$ is not in $\Sigma^o$ then all the generators of $\text{Der}^\dagger (\log \Sigma^o)$ are tangent two by two at $p$, which is impossible in view of the above remark (1). Therefore, one has

$$Z(g_2, \ldots, g_n) \subset \Sigma^o.$$
We are going to improve the above inclusion, showing that one can suppose that
\[ Z(g_2, \ldots, g_n) \subset \{0\} \times C^N. \]

Consider the following set
\[ \Delta = \{ t \in (C^N, 0) \mid \pi^{-1}(t) \subset Z(g_2, \ldots, g_n) \}. \]

It is a closed analytic subset of \((C^N, 0)\) and we remove \(\pi^{-1}(\Delta)\) of \(U\). Now, fixed some \(t\) and denote by \(I\) the canonical injection \(I: (C^2, 0) \to \pi^{-1}(t)\), \(I(x) = (x, t)\). If the intersection
\[ \Delta_t = \Sigma^0|_{\pi^{-1}(t)} \cap Z(g_2, \cdots, g_n) \]
contains \(0 \times \{t\}\) has a non isolated point of \(\Delta_t\), it contains also an analytic curve which is a component of \(I^*f_{\Sigma^0} = 0\). Therefore there is a factor \(h\) of \(I^*f_{\Sigma^0}\) that divides \(I^*g_i\) for any \(i\). Thus, for \(i \geq 2\), one has
\[ X_1|_{\pi^{-1}(t)} \wedge X_i|_{\pi^{-1}(t)} = h^2(\cdots). \]

Since the vector fields \(X_i|_{\pi^{-1}(t)}\) are tangent to \(h = 0\), any couple of element in \(\text{Der}^\dagger(\log \Sigma^0)|_{\pi^{-1}(t)}\) has a contact of order 2 locally around the zero locus of \(h\), which is impossible according to the above remark (2). As a consequence, for any \(t \in (C^N, 0)\), if \(\Delta_t\) contains \(0 \times \{t\}\), it is as an isolated point in \(\Delta_t\). So, \(\Delta_t\) is a finite set.

**Lemma 7.** Let \(W \subset (C^{2+N}, 0)\) be an analytic set such that for any \(t\), \(W \cap \pi^{-1}(t)\) is finite. Then
\[ \{0\} \times C^N \not\subset \overline{W \setminus \{0\} \times C^N}. \]

**Proof.** The hypothesis ensures that \(\text{codim}W \geq 2\). If \(\text{codim}W \geq 3\), the lemma is clear since \(\text{codim}(\{0\} \times C^N \subseteq U) = 2\). Suppose \(\text{codim}W = 2\). Let us write
\[ V = (W \cap V) \cup \left( \overline{V \setminus \overline{W}} \right). \]

Since \(V\) is irreducible, either \(W \cap V = V\), and \(V\) is an irreducible component of \(W\), or \(V \setminus \overline{W} = V\), and \(V \cap W\) is an analytic subset of \(V\) of codimension at least 1 in \(V\), and thus of codimension at least 3 in \(C^{2+N}\). In any case, the lemma is proved. \(\square\)

Following the lemma, the analytic set \(K\) defined by
\[ K = \{0\} \times C^N \cap \overline{\Sigma^0 \cap Z(g_2, \cdots, g_n) \setminus \{0\} \times C^N} \]
is a strict analytic subset of \(\{0\} \times C^N\) such that \(\{0\} \times C^N \setminus K\) admits a neighborhood \(U\) in \((C^{2+N}, 0)\) satisfying
\[ Z(g_2, \cdots, g_n) \cap U \subset \{0\} \times C^N \]

At the level of the ideals, the inclusion above ensures that there exists \(M \in \mathbb{N}\) such that
\[ (x, y)^M \subset (g_2, \ldots, g_n) \]
As a consequence, there exists a relation of the following form

\[ x^M = \sum_{i=2}^{n} h_i g_i \]

and considering \( Y = \sum_{i=2}^{n} h_i X_i \) and the relation (2.3) yields a vector field \( Y \) in \( \text{Der}^\uparrow (\log \Sigma^0) (\mathcal{U}) \) such that

\[ X_1 \wedge Y = f_{\Sigma} x^{M+1}. \]

Notice that \( X_1 \) and \( Y \) are both tangent to \( x = 0 \). Let us write in coordinates

\[
X_1 = x a^1 (x, y, t) \frac{\partial}{\partial x} + (b_1^0 (y, t) + x b_1^1 (x, y, t)) \frac{\partial}{\partial y},
\]

\[
Y = x a^2 (x, y, t) \frac{\partial}{\partial x} + (b_2^0 (y, t) + x b_2^1 (x, y, t)) \frac{\partial}{\partial y}.
\]

Replacing if necessary \( X_1 \) by \( X_1 + Y \), we can suppose that

\[
\nu_y (b_1^0) \leq \nu_y (b_2^0)
\]

where \( \nu_y \) is the valuation in the ring \( \mathbb{C} \{ t \} \{ y \} \). Consider the vertical vector field

\[ \tilde{Y} = \frac{1}{x} (Y - \frac{b_2^0}{b_1^0} X_1) . \]

It is holomorphic removing if necessary, some fibers \( \pi^{-1} (t) \) for \( t \) in some closed analytic set of \( \mathbb{C}^N \) related to the zeros of \( b_1^0 (0, t) \). Moreover, one has

\[ X_1 \wedge \tilde{Y} = f_{\Sigma} x^M. \]

Since \( X_1 \) is tangent to \( x = 0 \) and since its singular locus has codimension 2, \( \tilde{Y} \) is also tangent to \( x = 0 \). The process can be repeated and finally, one obtains two vertical vector fields \( X_1 \) and \( X_2 \) tangent to \( \Sigma \) such that

\[ X_1 \wedge X_2 = f_{\Sigma}. \]

The functions

\[ t \in \pi (\mathcal{U}) \mapsto \nu \left( X_i|_{\pi^{-1} (t)} \right), \quad i = 1, 2 \]

are lower semi-continuous. Replacing if necessary \( X_1 \) by \( X_1 + X_2 \), we consider an open set \( \mathcal{U}' \subset (\mathbb{C}^N, 0) \), whose closure is a neighborhood of 0, on which

\[ \forall t \in \mathcal{U}', \quad \nu \left( X_1|_{\pi^{-1} (t)} \right) \leq \nu \left( X_2|_{\pi^{-1} (t)} \right). \]

According to the criterion of Saito, for any \( t \),

\[ \left\{ X_1|_{\pi^{-1} (t)} , X_2|_{\pi^{-1} (t)} \right\} \]

consists in basis of Saito for the curve \( \Sigma|_{\pi^{-1} (t)} \). Therefore, for any \( t \in \mathcal{U}' \), one has

\[ \nu \left( X_1|_{\pi^{-1} (t)} \right) = \nu \left( \Sigma|_{\pi^{-1} (t)} \right). \]

From (2.7), one can consider the open set \( \mathcal{C} (\mathcal{U}') \subset \mathbb{M}^\bullet (S) \) and the union of such open sets while the above construction is done in the neighborhood of any regular point \( (C, \mathcal{N}) \) in \( \mathbb{M}^\bullet (S) \). By construction, the resulting open set has the desired properties. \( \square \)
From now on, a curve $C$ in $\mathbb{M}^\bullet (S)$ will be said *generic* if it belongs to the open set identified in the theorem above: in that sense, for a generic curve $C$ in its moduli space, we will be allowed to consider a analytical family of Saito bases following any topologically trivial deformation of $C$.

**Example 2.** Consider the union of four regular transversal curves. Up to some change of coordinates, it can be written

$$ S = \{ xy (y + x) (y + t_1 x) = 0 \} $$

where $t_1 \in \mathbb{M}^\bullet (S) = \mathbb{C} \setminus \{0, 1\}$. It can be seen [10] that it admits a miniversal deformation for the topologically trivial deformations of the form

$$ \Sigma = \{ F (x, y, t) = xy (y + x) (y + tx) = 0 \} \in (\mathbb{C}^2 \times \mathbb{C}, (0, 0, t_1)) \, . $$

The basis highlighted in Theorem 3 can be explicited in the above coordinates as

$$ X_1 = x \partial_x + y \partial_y, \quad X_2 = \partial_x F \partial_y - \partial_y F \partial_x. $$

In this case, $X_1$ and $X_2$ is a Saito basis in a whole neighborhood of $t_1 \in \mathbb{M}^\bullet (S)$.

In general, the situation is not so favourable.

**Example 3.** Consider for instance the union of five regular transversal curves, which is written

$$ S = \{ xy (y + x) (y + \alpha x) (y + \beta x) = 0 \} \, . $$

with $\alpha \neq 0, 1$ and $\beta \neq 0, 1, \alpha$. A miniversal deformation of $S$ is written

$$ \Sigma = \{ F = xy (y + x) (y + t_1 x + t_2 x + t_3 x^2) = 0 \} \in (\mathbb{C}^2 \times \mathbb{C}^3, (0, 0, \alpha, \beta, 0)) \, . $$

For the curve $S$, which corresponds to the parameter $(\alpha, \beta, 0)$, a basis of Saito is given by

$$ X_1 = x \partial_x + y \partial_y, \quad X_2 = \partial_x F \partial_y - \partial_y F \partial_x. $$

However, this basis cannot be *extended*, in a whole neighborhood of $(\alpha, \beta, 0)$. Since $X_1$ has a valuation equal to one, the Saito number of $S$ is equal to 1. It can be seen that for any $t_3 \neq 0$, the number of Saito of $\Sigma|_{t=(\alpha, \beta, t_3)}$ is bigger than 2. Indeed, consider a vector field $X$ tangent to $\Sigma|_{t=(\alpha, \beta, t_3)}$. If its valuation is smaller than 1, then it is dicritical. Thus it is written

$$ X = k (x \partial_x + y \partial_y) + (\cdots) $$

where $k$ is a non vanishing constant. Following [6], $X$ is linearizable and in some coordinates in which $X$ is linear, the curve $\Sigma|_{t=(\alpha, \beta, t_3)}$ becomes exactly the union of five germs of straight lines, which is impossible if $t_3 \neq 0$. Finally, it can be seen that if $t_3 \neq 0$ then

$$ \delta \left( \Sigma|_{t=(\alpha, \beta, t_3)} \right) = 2 $$

and an optimal vector field for $\Sigma|_{t=(\alpha, \beta, v)}$ is written

$$ X = (x + \epsilon y) (x \partial_x + y \partial_y) + (\cdots) $$

where $\epsilon \neq 0, 1$. 
2.3. **Saito basis for** \( S \) **and** \( S \cup l \). The process described below allows us to obtain a Saito basis for \( S \) from a Saito basis for \( S \cup l \) where the curve \( l \) is a regular curve. This trick has been already introduced in the proof of Theorem 3. Throughout this article, it will be often a key argument.

Let \( S \) be a germ of curve and \( l \) be a germ of smooth curve that is not a component of \( S \). Let \( \{X_1, X_2\} \) be a Saito basis for \( S \cup l \). The Saito criterion is written

\[
X_1 \wedge X_2 = ufL
\]

where \( u \) is a unity, \( f \) a reduced equation of \( S \) and \( L \) a reduced equation of \( l \). Let us consider a local system of coordinates \((x, y)\) in which \( L = x \). Then, for \( i = 1, 2 \), the vector fields \( X_i \) can be written

\[
X_i = x a_i \partial_x + (b_0^i + x b_1^i) \partial_y, \quad a_i, b_1^i \in \mathbb{C}\{x, y\}, \quad b_0^i \in \mathbb{C}\{y\}.
\]

Considering if necessary a generic change of basis

\[
\{\alpha X_1 + \beta X_2, uX_1 + vX_2\}
\]

where \( \begin{vmatrix} \alpha & \beta \\ u & v \end{vmatrix} \neq 0 \), one can suppose that

\[
\nu(X_i) = s(S \cup l) \quad \text{and} \quad \nu_y(b_1^i(y)) = \nu_y(b_0^i(y))
\]

where \( \nu_y \) is the valuation in the ring \( \mathbb{C}\{y\} \). In particular, the quotient \( \frac{\nu_y(b_1^i)}{\nu_y(b_2^i)} \) extends holomorphically at \((x, y) = (0, 0)\) as a unit. The relation (2.10) leads to

\[
\left( X_1 - \frac{\nu_y}{\nu_y(b_2^i)} X_2 \right) \wedge X_2 = uf,
\]

where \( X'_1 \) extends holomorphically at \((0, 0)\). Since \( L = 0 \) is not a component of \( S \), the vector field \( X'_1 \) leaves invariant \( S \). The Saito criterion ensures that \( \{X'_1, X_2\} \) is a Saito basis for \( S \).

Now, it is clear that

\[
\nu(X'_1) \geq \nu(X_1) - 1 = s(S \cup l) - 1.
\]

Since, \( \nu(X_2) = s(S \cup l) \), one has

\[
s(S) = s(S \cup l) - 1 \quad \text{or} \quad s(S \cup l).
\]

Assume moreover, that \( S \) is not of radial type but \( S \cup l \) is. By definition, \( X_1 \) and \( X_2 \) are dicritical. Thus, the homogeneous part of degree \( s(S \cup l) \) of \( X_i \) is written

\[
X_i^{(s(S \cup l))} = R_i (x \partial_x + y \partial_y)
\]

Therefore the homogeneous part of degree \( s(S \cup l) - 1 \) of \( X'_1 \) is

\[
\frac{1}{x} \left( R_1 - \frac{b_0^1}{b_2^1} (0) R_2 \right) (x \partial_x + y \partial_y).
\]

If the above expression does not identically vanish, then \( X'_1 \) would be dicritical. Since \( X_2 \) is dicritical too, \( S \) would be of radial type, which is impossible. Thus, the
expression \(2.12\) vanishes and \(\nu(X'_1) \geq s(S \cup l)\). Since \(\nu(X_2) = s(S \cup l)\) one has finally

\[ s(S) = s(S \cup l). \]

Gathering the remarks above, we obtain the

**Proposition 3.** Let \(l\) be a germ of smooth curve that is not a component of \(S\). Then

1. In any case, \(s(S) = s(S \cup l) - 1\) or \(s(S \cup l)\).
2. If \(S\) is not of radial type but \(S \cup l\) is then
   \[ s(S) = s(S \cup l). \]

The process described above can be reversed. Consider a Saito basis \(\{X_1, X_2\}\) for \(S\). Changing of basis if necessary, one can consider that

\[ \nu(X_1) = \nu(X_2). \]

Let \(l\) be a generic smooth curve and \(L\) a reduced equation of \(l\). Fix some coordinates \((x, y)\) in which \(l\) has a parametrization of the form

\[ \gamma(t) = (t, \epsilon(t)), \ t \in (\mathbb{C}, 0). \]

The product

\[ X_1(\gamma) \land \gamma' \in \mathbb{C}\{t\} \]

has a valuation in \(\mathbb{C}\{t\}\) equal to \(\nu(X_1)\) or \(\nu(X_1) + 1\) depending on whether or not \(X_1\) is dicritical. Therefore, the quotient

\[ \frac{X_1(\gamma) \land \gamma'}{X_2(\gamma) \land \gamma'} \]

extends holomorphically at \(t = 0\) as a unit \(\phi(t)\). By construction, the vector field

\[ X_1 - \phi(x)X_2 \]

is tangent to the curve \(l\). Finally, in the coordinates \((x, y)\), according to the criterion of Saito, the family

\[ (2.13) \]

\[ \{X_1 - \phi(x)X_2, LX_2\} \]

is a Saito basis for \(S \cup l\).

### 3. Generic element in \(\text{Der}(\log S)\) and adapted Saito Bases.

In \(2.3\), we remark that

\[ s(S) \leq \frac{\nu(S)}{2}. \]

In this section, we will prove that for a curve \(S\) generic in its moduli space the latter inequality is essentially reached, as it will be stated in Theorem [1].
3.1. **Generic value of** $s(S)$. Let $S$ be a curve generic in its topological class and \{X_1, X_2\} be a Saito basis for $S$.

The exceptional divisor $D$ of the blowing-up $E : (\mathcal{M}, D) \to (\mathbb{C}^2, 0)$ can be covered by two open sets $U_1$ and $U_2$ and two charts $(x_1, y_1)$ and $(x_2, y_2)$ defined respectively in some neighborhoods of $U_1$ and $U_2$ such that

$$E(x_1, y_1) = (x_1, y_1 x_1) \quad \text{and} \quad E(x_2, y_2) = (x_2 y_2, y_2).$$

Let $\Theta_S$ be the sheaf on $\mathcal{M}$ of vector fields tangent to $E^{(-1)}(S) = SE \cup D$. Let $\omega$ be a 1--form with an isolated singularity tangent to the vector field $X_1$: if $X_1$ is written

$$X_1 = a \partial_x + b \partial_y,$$

one can choose

$$\omega = ady - bdx.$$

Let us consider the global 1--form on $\mathcal{M}$ defined by the pull-back

$$\Omega = E^* \omega.$$

We denote by $\mathfrak{B}$ the basic operator: this is a morphism of sheaves

$$\mathfrak{B} : \Theta_S \to \Omega^2(\mathcal{M})$$

that is written

$$\mathfrak{B}(T) = L_T \Omega \wedge \Omega = \Omega(T) d\Omega - d(\Omega(T)) \wedge \Omega.$$

Here, $\Omega^2(\mathcal{M})$ is the sheaf on $\mathcal{M}$ of holomorphic 2-forms and $L_T$ is the Lie derivative with respect to the vector field $T$. Following [3], the kernel of $\mathfrak{B}$ consists in the infinitesimal generators of the sheaf of automorphisms of the foliation induced by $X_1^E$, that is,

$$L_T \Omega \wedge \Omega \equiv 0 \implies \forall t \in (\mathbb{C}, 0), \left( (e^{tT})^* X_1^E \right) \wedge X_1^E \equiv 0.$$

The lemma below describes partially the image of $\mathfrak{B}$.

**Lemma 8.** $\mathfrak{B}(\Theta_S) \subset \Omega^2(-\pi D - SE)$ where

- $\pi = 2 \nu(X_1) + \begin{cases} 2 & \text{if } X_1 \text{ is dicritical} \\ 1 & \text{if not} \end{cases}$
- $\Omega^2(-\pi D - SE)$ is the sheaf of 2--forms that vanish along $D$ and $SE$ with at least respective orders $\pi$ and 1.

**Proof.** It is a computation which can be performed in local coordinates. If $X_1$ is dicritical, then out of $\text{Sing}(X_1^E)$, one can write

$$\Omega = u x_1^{\nu(X_1)+1} dy_1,$$

where $u$ is a local unit. A section $T$ of $\Theta_S$ is written

$$T = \alpha x_1 \partial_{x_1} + \beta \partial_{y_1}, \quad \alpha, \beta \in \mathbb{C} \{x_1, y_1\}$$

Thus, applying the morphism $\mathfrak{B}$ yields

$$\mathfrak{B}(T) = \left( u^2 x_1^{2 \nu(X_1)+1} \partial_{x_1} \beta \right) dx_1 \wedge dy_1.$$
If $X_1$ is not dicritical, then out of the locus of tangency between $X_1^E$ and $D$, one can write in some coordinate
\[ \Omega = u x_1^{\nu(X_1)} dx_1, \]
and
\[ \mathfrak{B} (T) = \left( u^2 x_1^{2\nu(X_1)+1} \partial_{y_1} \right) dx_1 \wedge dy_1. \]
Finally, along a regular point of $S^E$, one can write
\[ \Omega = u dy_1, \]
where $y_1 = 0$ is a local equation of $S^E$. A local section of $T$ of $\Theta_S$ is written
\[ T = \alpha \partial_{x_1} + \beta y_1 \partial_{y_1}, \quad \alpha, \beta \in \mathbb{C} \{x_1, y_1\} \]
and
\[ \mathfrak{B} (T) = \left( u^2 y_1 \partial_{x_1} \right) dx_1 \wedge dy_1. \]

Notice that if $c$ is not a tangency point between $X_1^E$ and $D$, then at the level of the stack, one has
\[ (\mathfrak{B} (\Theta_S))_c = \left( \Omega^2 \left( -\pi D - S^E \right) \right)_c, \]
thus the two sheaves $\mathfrak{B} (\Theta_S)$ and $\Omega^2 \left( -\pi D - S^E \right)$ are essentially equal.

The proof of the next lemma is a corollary of an adaptation of the theory of infinitesimal deformations of foliations developed in [12] by Gómez-Mont.

**Lemma 9.** The map in cohomology induced by the inclusion of Lemma 8
\[ H^1 (\mathcal{M}, \Theta_S) \xrightarrow{\mathfrak{D}} H^1 (\mathcal{M}, \Omega^2 \left( -\pi D - S^E \right)) \]
is the zero map.

**Proof.** Let us denote by $\Theta_{X_1}$ the sheaf of tangent vector fields to the foliation induced on $\mathcal{M}$ by $X_1^E$. Let us consider the morphism of sheaves
\[ \Theta_S \xrightarrow{\mathfrak{D}} \text{Hom}(\Theta_{X_1}, \Theta_S/\Theta_{X_1}) \]
defined by $\mathfrak{D} (T) = (X \mapsto \pi [X, T])$ where $[\cdot]$ stands for the Lie bracket and $\pi$ the quotient map $\pi : \Theta_{X_1} \to \Theta_S/\Theta_{X_1}$. Following [12] (Theorem 1.6), one has the following exact sequence
\[ H^1 (\mathcal{M}, \Theta_{X_1}) \to H^1 (\mathcal{M}, \Theta_S) \xrightarrow{\mathfrak{D}} H^1 (\mathcal{M}, \text{Hom}(\Theta_{X_1}, \Theta_S/\Theta_{X_1})). \]
In this sequence, $H^1 (\mathcal{M}, \Theta_{X_1})$ is the first hypercohomology group of the leaf complex associated to the morphism $\mathfrak{D}$ as defined in [12]. It is identified with the space of infinitesimal deformations of the foliation induced by $X_1^E$. The cohomological group $H^1 (\mathcal{M}, \Theta_S)$ is the standard Čech cohomology of sheaves. The first cohomology group $H^1 (\mathcal{M}, \Theta_S)$ is identified with the space of infinitesimal deformations of $S^E$. Finally, $\mathfrak{D}$ is the map induced in cohomology by $\mathfrak{D}$.

Assume that $S$ is generic in its moduli space $\mathcal{M} \cdot (S)$. Theorem 3 ensures that any small deformation of $S^E$ can be followed by a deformation of $X_1^E$. As a consequence,
any infinitesimal deformation of \( S^E \) can be followed by an infinitesimal deformation of \( X^E_i \). In other words, in (3.2) the map \[ H^1 (\mathcal{M}, \Theta_{X_i}) \rightarrow H^1 (\mathcal{M}, \Theta_S) \] is onto. Since the sequence (3.2) is exact, the map \( \overline{\mathcal{D}} \) is the zero map.

Now, let us consider a covering \( \{ U_i \}_{i \in I} \) of \( \mathcal{M} \) and a cocycle \( \{ T_{ij} \}_{ij} \)

\[ \{ T_{ij} \}_{ij} \in Z^1 (\mathcal{M}, \{ U_i \}_{i \in I}, \Theta_S). \]

The map \( \overline{\mathcal{D}} \) being the zero map, the cocycle \( \overline{\mathcal{D}} (\{ T_{ij} \}_{ij}) \) is trivial, that is,

\[ \overline{\mathcal{D}} (\{ T_{ij} \}_{ij}) \equiv 0 \text{ in } H^1 (\mathcal{M}, \text{Hom} (\Theta_{X_i}, \Theta_S/\Theta_{X_i})). \]

By definition, there exists \( \{ \tau_i \}_i \in Z^0 (\mathcal{M}, \{ U_i \}_{i \in I}, \text{Hom} (\Theta_{X_i}, \Theta_S/\Theta_{X_i})) \) such that

\[ [T_{ij}, \cdot] = T_j - T_i. \]

At the level of the stack, the map \( \mathcal{D} \) is onto at any regular point for \( X^E_i \). Thus we can consider a covering of \( U_i \setminus \text{Sing} (X^E_i) = \bigcup_{k \in K} U_{ik} \) by open sets \( U_{ik} \) such that \( \mathcal{D} \) is onto on \( U_{ik} \). By construction, on any \( U_{ik} \) there exists a section \( \tau_{ik} \) of \( \Theta_S \) such that

\[ T_i \equiv [\tau_{ik}, \cdot]. \]

Therefore, on \( U_{ik} \cap U_{ik'} \), \( [\tau_{ik}, \cdot] = [\tau_{ik'}, \cdot] \). Thus, applying \( \mathcal{B} \) yields

\[ L_{\tau_{ik}} \Omega \wedge \Omega = \mathcal{B} (\tau_{ik}) = \mathcal{B} (\tau_{ik'}) = L_{\tau_{ik'}} \Omega \wedge \Omega. \]

Therefore, the 2–forms \( \{ L_{\tau_{ik}} \Omega \wedge \Omega \}_{k \in K} \) paste in a global 2–forms \( \Omega_i \) defined on \( U_i \setminus \text{Sing} (X^E_i) \) which can be extended to \( U_i \) since \( \text{Sing} (X^E_i) \) is of codimension 2. By construction,

\[ \overline{\mathcal{B}} (\{ T_{ij} \}) \equiv \{ \Omega_j - \Omega_i \}, \]

which is the lemma.

The open sets \( U_1 \) and \( U_2 \) defined at the beginning of this section are Stein as open set in \( \mathbb{C} \). Thus, following [23], they admit a system of Stein neighborhoods. Since \( \Omega^2 (-\pi D - S^E) \) is coherent, we deduce that there is a covering \( \{ U_1, U_2 \} \) of \( \mathcal{M} \) that is acyclic for \( \Omega^2 (-\pi D - S^E) \). Therefore, one can compute the cohomology using this covering and thus

\[ H^1 (\mathcal{M}, \Omega^2 (-\pi D - S^E)) = H^1 (\{ U_1, U_2 \}, \Omega^2 (-\pi D - S^E)) \]

which is the quotient

\[ \frac{H^0 (U_1 \cap U_2, \Omega^2 (-\pi D - S^E))}{H^0 (U_1, \Omega^2 (-\pi D - S^E)) \oplus H^0 (U_2, \Omega^2 (-\pi D - S^E))}. \]

The lemma below is the key to get a lower bound for the Saito number \( s(S) \) of the curve \( S \).
Lemma 10. Let $f_1$ be the quotient $\frac{f_0 E}{x_1^{\nu}}$ where $f$ is a reduced equation of $S$. If there exists a Laurent series $A = \sum_{i,j} a_{i,j}^{-1} y_1^i x_1^j$ holomorphic on $U_1 \cap U_2$ with a non-vanishing residue $a_{0,-1}$, such that, in the identification (3.3), one has

\[ [A \cdot f_1 x_1^k dx_1 \land dy_1] = 0 \in H^1 (\mathcal{M}, \Omega^2 (-kD - S^E)) \]

then

\[ k \geq \nu (S). \]

Proof. The global sections of $\Omega^2 (-kD - S^E)$ on each open sets $U_1$, $U_2$ and their intersection are written

\begin{align*}
\Omega^2 (-kD - S^E) (U_1) &= \{ f(x,y) \mid f \in \mathcal{O}(U_1) \} \\
\Omega^2 (-kD - S^E) (U_2) &= \{ g(x,y) \mid g \in \mathcal{O}(U_2) \} \\
\Omega^2 (-kD - S^E) (U_1 \cap U_2) &= \{ h(x,y) \mid h \in \mathcal{O}(U_1 \cap U_2) \}
\end{align*}

where $f_2 = \frac{f_0 E}{y_2}$. Therefore, the cohomological equation induced by the equality (3.3) is written

\[ h(x_1, y_1) f_1 x_1^k dx_1 \land dy_1 = g(x_2, y_2) f_2 y_2^k dx_2 \land dy_2 \\
- f(x_1, y_1) f_1 x_1^k dx_1 \land dy_1 \]

which is equivalent to

\begin{equation}
(3.4) \quad h(x_1, y_1) = y_1^{k-\nu(S)-1} g \left( \frac{1}{y_1} x_1 \right) - f(x_1, y_1)
\end{equation}

The hypothesis of Lemma 10 induces that if we set $h$ to be the series $\sum a_{i,j} x_1^i y_1^j$ then the equation above has a solution. In particular, the monomial $\frac{a_{i,j}}{y_1}$ has to appear in the Laurent expansion of one of the two terms of the expression at the right of (3.4). This is equivalent to require that the following system and

\[ \begin{cases} 0 = j \\ -1 = j - i + k - \nu(S) - 1 \end{cases} \iff \begin{cases} j = 0 \\ i = k - \nu(S) \end{cases} \]

has a solution in $\mathbb{N}^2$. Thus, $k \geq \nu(S)$.

Theorem 4. For $S$ generic in its moduli space $\mathbb{M}^* (S)$, one has

\[ a(S) \geq \begin{cases} \lfloor \frac{\nu(S)}{2} \rfloor & \text{if } S \text{ is not of radial type} \\ \lceil \frac{\nu(S)}{2} \rceil - 1 & \text{else} \end{cases} \]

where $\lfloor \ast \rfloor$ and $\lceil \ast \rceil$ stands respectively for the integer part and the least integer of $\ast$.

In the moduli space $\mathbb{M}^* (S)$, the lower bound above holds only for the generic point. For instance, the Saito number of a union of any number of germs of straight lines is 1, since the radial vector field $x \partial x + y \partial y$ is in the Saito module, whereas the algebraic multiplicity $\nu(S)$ goes to infinity with the number of components. Even if the curve $S$ is irreducible, one cannot drop the assumption of $S$ being generic in
its moduli space, as it can be seen in the following example due to M. Hernandes known as deformation by socle: let $S$ be the irreducible curve
$$\{ y^p - x^q + x^q - 2y^p - 2 = 0 \}$$
with $p \land q = 1$ and $4 = p < q$. Its algebraic multiplicity is equal to $p$ whereas its Saito number $s(S)$ is equal to $2$ regardless the value of $p$. Indeed, the vector field $X_1$ written
$$X_1 = \left( y + \frac{(p-2)(q-2)}{pq}x^{q-4}y^{p-3} \right) (px \partial_x + qu \partial_y)$$
$$+ \left( \frac{(p-2)}{q}x^{q-2} \partial_y - \frac{(p-2)}{pq}x^{q-3}y^3 \partial_x \right)$$
is optimal for $S$.

**Proof of Theorem 4.** Let $X_1$ be a generic optimal vector field for $S$. Since we assume $S$ generic in its moduli space, the operator $B$ associated to $X_1$ and defined in Lemma 9 is trivial.

Suppose, first that $X_1$ is dicritical. Let us suppose that in the coordinates $(x_1, y_1)$, the vector field $X_1^E$ is transverse to $D$ at $(0,0)$ and that $f_1 = \frac{f_{\xi \nu}}{x_1^{\nu}}$, does not vanish at $(0,0)$. We can suppose that, in these coordinates, $\Omega$ is written
$$\Omega = ux_1^{\nu(X_1)+1} dy_1, \quad u(0) \neq 0.$$
The image of the vector field
$$T = \frac{x_1}{y_1} \partial_{y_1}$$
by $\frac{1}{f_1} B$ is written
$$\frac{1}{f_1} B (T) = \frac{1}{f_1} L_T \Omega \land \Omega = \frac{u^2(0,0)}{x_1^{\nu}} \frac{1}{y_1} d x_1 \land d y_1 + x_1^{\nu(S)+1} (\cdots).$$
This meromorphic 2–form considered as a cocycle in $Z^1 (\{ U_1, U_2 \}, \Omega^2 (-\pi D - S^E))$ has to be trivial in cohomology according to Lemma 9. Thus, Lemma 10 ensures that $\nu(S) = \nu(X_1) + 2 \geq \nu(S)$, which is also written
$$s(S) = \nu(X_1) \geq \frac{\nu(S)}{2} - 1.$$ 
Therefore, if $X_1$ is dicritical the theorem is proved.

Suppose now that $X_1$ is not dicritical. Let us suppose that $(0,0)$ is a singular point of $X_1^E$. Locally around $(0,0)$, $\Omega$ can be written
$$\Omega = x_1^{\nu(X_1)} y_1^a dx_1 + x_1^{\nu(X_1)+1} (\cdots)$$
where $a$ is some positive integer. Let us write
$$f_1 = y_1^b v (y_1) + x_1 (\cdots), \quad v(0) \neq 0$$
where $b$ is some positive integer. Considering the meromorphic vector field
$$T = \frac{x_1}{y_1^{2a-b}} \partial_{x_1},$$
we apply the operator $\frac{1}{f_1} 3 S$ and obtain
\[
\frac{1}{f_1} 3 S (T) = \frac{(2a - b)}{v (0)} \frac{x \bar{r}}{y_1} dx_1 \wedge dy_1 + x_1^{p+1} (\cdots).
\]
Suppose that there exists a singular point of $X^E_1$ such that $2a \neq b$. Then, Lemma 11 ensures that $\pi = 2\nu (X_1) + 1 \geq \nu (S)$, which is written
\[
(3.5) \quad s (S) = \nu (X_1) \geq \frac{\nu (S) - 1}{2}.
\]
If the equality $2a = b$ is true for any singular points, then $\nu (S)$ is even. Thus, the theorem is proved when
\begin{itemize}
  \item $\nu (S)$ is odd
  \item or $\nu (S)$ is even and for some singular points of $X^E_1$, one has $b \neq 2a.$
  \item or if $S$ is radial.
\end{itemize}

Finally, suppose that $\nu (S)$ is even and $S$ is not radial. Consider a Saito basis \{X_1, X_2\} for $S$ with $\nu (X_1) = \nu (X_2)$. If $\nu (X_1) = \frac{\nu (S)}{2}$ then the property is proved.

Therefore, assume that $\nu (X_1) \leq \frac{\nu (S)}{2} - 1$. Let $l_1$ be a generic smooth curve. Using the construction introduced at (2.13), we obtain a Saito basis for $S \cup l_1$ of the form
\[
\{X_1 + \phi_1 X_2, L_1 X_2\}, \quad \phi_1 (0) \neq 0 \text{ and } l_1 = \{L_1 = 0\}
\]

If $\nu (X_1 + \phi_1 X_2) = \nu (X_1)$ then
\[
\nu (X_1 + \phi_1 X_2) \leq \frac{\nu (S)}{2} - 1 < \frac{\nu (S \cup l_1) - 1}{2}
\]
which contradicts Theorem 11 applied to $S \cup l_1$, the valuation $\nu (S \cup l_1)$ being odd.

Therefore, $\nu (X_1 + \phi_1 X_2) \geq \frac{\nu (S)}{2}$ and since $\nu (L_1 X_2) \geq \nu (S) / 2$ and $\nu (X_2) \leq \nu (S) / 2 - 1$,

considering if necessary $X_1 + \phi_1 X_2 + L_1 X_2$, we obtain a basis of Saito for $S \cup l_1$ written
\[
(3.6) \quad \{X_1 + \phi_1 X_2, L_1 X_2\}, \quad \phi_1 (0) \neq 0
\]
both of these vector fields being non dicritical and of multiplicity $\nu (S) / 2$. Using again the construction (2.13), we add one more generic curve $l_2$ and obtain a basis of Saito of the form
\[
\left\{L_2 (X_1 + \phi_1 X_2), L_1 X_2 + \phi_2 (X_1 + \phi_1 X_2)\right\}, \quad \phi_2 (0) \neq 0 \text{ and } l_2 = \{L_2 = 0\}.
\]

We can apply Theorem 11 to $S \cup l_1 \cup l_2$ since the latter curve has a smooth component for which $b = 1$ is not even. Therefore, the two above vector fields are of multiplicity $\nu (S) / 2 + 1$ and not dicritical. According to the Saito criterion applied to (3.6), one has
\[(X_1 + \phi_1 X_2) \wedge L_1 X_2 = u L_1 f, \quad u (0) \neq 0.
\]

Therefore, $L_1$ cannot divide $X_1 + \phi_1 X_2$. Now consider any couple of non vanishing functions $\alpha$ and $\beta$. Writing
\[
\alpha Y_1 + \beta Y_2 = (\beta \phi_2 + \alpha L_2) (X_1 + \phi_1 X_2) + \beta L_1 X_2
\]
ensures that $\alpha Y_1 + \beta Y_2$ cannot be divided by $L_1L_2$. Fix some coordinates $(x, y)$ such that $L_1 = x$ and $L_2 = y$. Taking a suitable linear combination of $Y_1$ and $Y_2$ we can suppose that they are written

$$Y_1 = a(x) x^p \partial_x + b(y) y^q \partial_y + xy (\cdots)$$

$$Y_2 = c(x) x^p \partial_x + d(y) y^q \partial_y + xy (\cdots)$$

where $a, b, c$ and $d$ are non-vanishing germs of functions and $p$ and $q$ some integers bigger than $\frac{\nu_2}{2} + 1$. Dividing $Y_1$ and $Y_2$ respectively by $b$ and $d$, and making a suitable change of coordinates of the form $(x, y) \mapsto (u(x), y)$, we can suppose that $Y_1$ and $Y_2$ are written

$$Y_1 = ax^p \partial_x + by^q \partial_y + xy (\cdots)$$

$$Y_2 = cx^p \partial_x + dy^q \partial_y + xy (\cdots)$$

where $a, b$ and $d$ belongs to $C \setminus \{0\}$. Finally, considering the vector field

$$Y_2 - \frac{(c(x) - c(0))}{a} Y_1,$$

we can write

$$Y_1 = ax^p \partial_x + by^q \partial_y + xy (\cdots)$$

$$Y_2 = cx^p \partial_x + dy^q \partial_y + xy (\cdots)$$

where $a, b, c$ and $d$ are non vanishing complex numbers. Now, the Saito criterion written

$$Y_1 \wedge Y_2 = uxyf$$

where $u$ is a unit ensures that

$$(cY_1 - aY_2) \wedge (dY_1 - bY_2) = (ad - bc) Y_1 \wedge Y_2 = u(ad - bc) xyf.$$ 

If $ad - bc = 0$ then considering $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ in the kernel of the matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ yields a linear combination written

$$\begin{pmatrix} cY_1 - aY_2 \\ Y_1 \end{pmatrix} \wedge \begin{pmatrix} dY_1 - bY_2 \\ x \end{pmatrix} = u(ad - bc) f$$

is the Saito criterion for the curve $S$. However, both vector fields in the product above have mutiplicities bigger than $\frac{\nu(S)}{2}$ which is a contradiction with the initial assumption.

3.2. Generic Saito basis. The generic lower bound of Theorem 4 induces some properties for a Saito basis of a generic curve. In this section, we explore some of them.

To do so, we are going to use frequently the following lemma that is a direct consequence of the criterion of Saito.

Lemma 11. Let $\{X_1, X_2\}$ be a Saito basis $S$. Then
(1) If \( \nu(X_1) + \nu(X_2) < \nu(S) \) and \( X_1 \) is dicritical then \( X_2 \) is dicritical.

(2) If \( \nu(X_1) + \nu(X_2) = \nu(S) \) and \( X_1 \) is dicritical then \( X_2 \) is not dicritical.

(3) If \( S \) is generic

(4) in its moduli space, then one can suppose that

\[
\nu(S) - 1 \leq \nu(X_1) + \nu(X_2) \leq \nu(S).
\]

Proof. Properties (1) and (2) are consequences of the following remark

\[
X_1^{(\nu(X_1))} \land X_2^{(\nu(X_2))} \equiv 0 \iff \nu(X_1) + \nu(X_2) < \nu(S).
\]

For (3), first, we recall that the sum \( \nu(X_1) + \nu(X_2) \) cannot exceed \( \nu(S) \) as noticed in (2.2). If \( \nu(S) \) is odd, Theorem 4 gives the inequalities

\[
\nu(X_i) \geq \frac{\nu(S) - 1}{2}, \quad i = 1, 2
\]

and thus \( \nu(X_1) + \nu(X_2) \geq \nu(S) - 1 \), which is the lemma.

If \( \nu(S) \) is even, adding a generic line \( l \) to \( S \) yields a Saito basis of \( S \cup l \) for which, in view of the previous arguments - \( \nu(S \cup l) \) is odd - , one has

\[
\nu(X_1) + \nu(X_2) \geq \nu(S \cup l) - 1 = \nu(S).
\]

By the process described in Proposition 3, the induced Saito basis \( \{X'_1, X_2\} \) of \( S \) satisfies

\[
\nu(X'_1) + \nu(X_2) \geq \nu(X_1) + \nu(X_2) - 1 \geq \nu(S) - 1,
\]

which ends the proof of the lemma.

The next lemma ensures somehow that both inequalities identified in Theorem 4 cannot be reached at the same time.

Lemma 12. Let \( S \) be a generic curve of radial type. Then there is no non dicritical vector field \( X \) in \( \text{Der}(\log S) \) with \( \nu(X) = \left\lceil \frac{\nu(S)}{2} \right\rceil \).

Proof. Consider an optimal dicritical vector field \( X_1 \) and \( X_2 \) a vector field such that \( \{X_1, X_2\} \) is a Saito basis of \( S \). If \( \nu(S) \) is even, then Theorem 4.2 ensures that \( \nu(X_1) \geq \frac{\nu(S)}{2} - 1 \). If \( \nu(X_1) = \frac{\nu(S)}{2} \) then the lemma follows from the definition of \( S \) being radial. If \( \nu(X_1) = \frac{\nu(S)}{2} - 1 \) then either \( X_1^{(\nu(X_1))} \land X_2^{(\nu(X_2))} = 0 \) or \( \nu(X_2) \geq \frac{\nu(S)}{2} + 1 \). In any case, the lemma follows. Finally, if \( \nu(S) \) is odd and \( S \) radial, by definition, every vector field of multiplicity \( \left\lceil \frac{\nu(S)}{2} \right\rceil = \frac{\nu(S) - 1}{2} \) is dicritical.

In the proposition below, we are going to identify precisely the type of Saito basis that may occur for a generic curve. In the statement of the theorem, we introduce some notations for the identified classes.

Theorem 5. Let \( S \) be a curve generic in its moduli space. Then there exists a Saito basis \( \{X_1, X_2\} \) for \( S \) with one of the following forms

\[\begin{align*}
&\text{• if } \nu(S) \text{ is even} \\
&\quad (\mathcal{E}) : \nu(X_1) = \nu(X_2) = \frac{\nu(S)}{2}, \text{ } X_1 \text{ and } X_2 \text{ are non dicritical.} \\
&\quad (\mathcal{E}_d) : \nu(X_1) = \nu(X_2) - 1 = \frac{\nu(S)}{2} - 1, \text{ } X_1 \text{ and } X_2 \text{ are dicritical.}
\end{align*}\]
Remark the basis is adapted.

For any function $L$ if the Saito basis of $X$ has one of the above type.

Let us consider a Saito basis for $S$ and suppose that $\nu(X_1) = \nu(X_2) = \frac{\nu(S)}{2}$. Considering if necessary $X_2 + \alpha X_1$ for a generic value $\alpha \in \mathbb{C}$, one has

$$\nu(X_1) = \nu(X_2) = \frac{\nu(S)}{2}.$$  

Then $X_1$ and $X_2$ are non-dicritical.

Assume $X_1$ is dicritical. If $\nu(X_1) = \frac{\nu(S)}{2}$ then Lemma [11] ensures that $X_2$ is not dicritical and the Saito basis $\{X_1, X_2\}$ is of type $(\mathcal{E})$. If

$$\nu(X_1) = \frac{\nu(S)}{2} - 1,$$

one can suppose that

$$\nu(X_2) = \frac{\nu(S)}{2} - 1 = \frac{\nu(S)}{2} + 1.$$ 

Moreover, if $\{X_1, X_2\}$ is a generic Saito basis for $S$ then there exists an holomorphic function $h$ such that

$$\{X_1, X_2 - hX_1\}$$

has one of the above type.

If the Saito basis of $S$ has one of the forms given by Theorem [5], we will say that the basis is adapted.

**Remark 1.** Notice that if the Saito basis $\{X_1, X_2\}$ of $S$ is of type $(\mathcal{E}_d)$ or $(\mathcal{D}_d)$ then for any function $L$ with a non trivial linear part, the family

$$\{X_1, X_2 + L X_1\}, \quad \begin{cases} L(0) \neq 0 & \text{if } S \text{ is of type } (\mathcal{D}_d) \\ L(0) = 0 & \text{if } S \text{ is of type } (\mathcal{E}_d) \end{cases}$$

is a Saito basis for $S$ of type $(\mathcal{E}_d)$ or $(\mathcal{D}_d)$. In some sense, the bases of type $(\mathcal{E}_d)$ or $(\mathcal{D}_d)$ are exceptional among the one of type $(\mathcal{E})$ or $(\mathcal{D})$.

**Remark 2.** The curves of type $(\mathcal{E}_d)$ are the only curves for which there exists a Saito basis $\{X_1, X_2\}$ with

$$|\nu(X_1) - \nu(X_2)| \geq 2$$

**Remark 3.** One of the interest of adapted Saito bases is their behaviour with respect to the blowing-up. For instance, suppose that $S$ has an adapted Saito basis $\{X_1, X_2\}$ of type $(\mathcal{E}_d)$. Then, blowing-up the Saito criterion (2.1) yields the relation

$$X_1^E \wedge X_2^E = u \circ E \circ E \frac{f}{x_1^{\nu(S)}}.$$ 

Therefore, according to the Saito criterion, the family $\{(X_1^E)_c, (X_2^E)_c\}$ is a Saito basis for $(S^E)_{c}$ for any $c \in D$ but not necessarily adapted. It is a simple matter to check that the latter property holds for any above type of Saito bases.

**Proof of Theorem [2].** Let us consider a Saito basis $\{X_1, X_2\}$ of $S$ and suppose that $\nu(X_1) \leq \nu(X_2)$.

**Case 1.** Suppose first $\nu(S)$ even. If $X_1$ is not dicritical then according to Theorem [4] and (2.2), $\nu(X_1) = \nu(X_2) = \frac{\nu(S)}{2}$. Considering if necessary $X_2 + \alpha X_1$ for a generic value $\alpha \in \mathbb{C}$, one has

$$\nu(X_1) = \nu(X_2) = \frac{\nu(S)}{2}.$$  

Then $X_1$ and $X_2$ are non-dicritical.

Assume $X_1$ is dicritical. If $\nu(X_1) = \frac{\nu(S)}{2}$ then Lemma [11] ensures that $X_2$ is not dicritical and the Saito basis $\{X_1 + X_2, X_2\}$ is of type $(\mathcal{E})$. If

$$\nu(X_1) = \frac{\nu(S)}{2} - 1,$$

one can suppose that

$$\nu(X_2) = \frac{\nu(S)}{2} - 1 = \frac{\nu(S)}{2} + 1.$$ 

If the Saito basis of $S$ has one of the forms given by Theorem [5] we will say that the basis is adapted.
If $\nu(X_2) = \frac{\nu(S)}{2} + 1$ then $X_2$ is not dicritical. Thus, one has a basis of the form

$$(\mathcal{E}_d) \quad \nu(X_1) = \nu(X_2) - 2 = \frac{\nu(S)}{2} - 1$$

$X_1$ is dicritical but not $X_2$.

If $\nu(X_2) = \frac{\nu(S)}{2}$, then $X_2$ is dicritical, and thus

$$(\mathcal{E}_d) \quad \nu(X_1) = \nu(X_2) - 1 = \frac{\nu(S)}{2} - 1$$

$X_1$ and $X_2$ are both dicritical.

Case 2. Suppose now $\nu(S)$ odd. In any case, $\nu(X_2) = \frac{\nu(S)+1}{2}$. Suppose $X_1$ dicritical. If $\nu(X_2) = \frac{\nu(S)+1}{2}$ then $X_2$ is dicritical, and thus

$$(\mathcal{D}_d) \quad \nu(X_1) = \nu(X_2) = \frac{\nu(S)+1}{2}$$

$X_1$ and $X_2$ are dicritical.

If $\nu(X_2) = \frac{\nu(S)+1}{2}$ then $X_2$ is not dicritical, and therefore the basis satisfies

$$(\mathcal{D}_d') \quad \nu(X_1) = \nu(X_2) - 1 = \frac{\nu(S)-1}{2}$$

$X_1$ is dicritical and $X_2$ is non-dicritical.

Finally, suppose that $X_1$ is not dicritical. If $\nu(X_2) = \frac{\nu(S)+1}{2}$ then the basis satisfies

$$(\mathcal{D}) \quad \nu(X_1) = \nu(X_2) - 1 = \frac{\nu(S)-1}{2}$$

$X_1$ and $X_2$ are non-dicritical.

It remains to study the case in which $X_1$ is not dicritical and $\nu(X_2) = \frac{\nu(S)}{2} - 1$. To do so, consider a generic line $l$. The multiplicity of $S \cup l$ is even, thus we can apply the results above to reach the description of the possible bases for $S$.

(1) Suppose first that the Saito basis $\{X_1^l, X_2^l\}$ of $S \cup l$ has the form $(\mathcal{E})$;

$$\nu(X_1^l) = \nu(X_2^l) = \frac{\nu(S)+1}{2}$$

none of these vector fields being dicritical. Let us consider some coordinates in which $l = \{x = 0\}$ and let us write

$$X_1^l = xA_i \partial_x + (y^{\alpha_i}b_i(y) + xB_i) \partial_y$$

with $b_i(0) \neq 0$. By symmetry, one can suppose $\alpha_1 \leq \alpha_2$. Thus, the family

$$\left\{ X_1^l, X_2^l = \frac{1}{x} \left( X_2 - y^{\alpha_2-\alpha_1} \frac{b_2}{b_1} X_1 \right) \right\}$$

is a Saito basis for $S$ such that

$$\nu(X_1^l) = \frac{\nu(S)+1}{2} \quad \text{and} \quad \nu(X_2^l) = \frac{\nu(S)-1}{2}.$$
(2) Suppose that the Saito basis \( \{ X^l_1, X^l_2 \} \) of \( S \cup l \) has the form (\( \mathcal{C}_d \))

\[
\nu(X^l_1) = \nu(X^l_2) - 1 = \frac{\nu(S)}{2} + 1 - 1
\]

both vector fields being dicritical. As before, let us consider some coordinates in which \( l = \{ x = 0 \} \) and let us written

\[
X^l_i = x A_i \partial_x + (y^\alpha_1 b_1(y) + x B_i) \partial_y
\]

with \( b_1(0) \neq 0 \). If \( \alpha_1 \leq \alpha_2 \) then the induced Saito basis \( \{ X^l_1, X^l_2 \} \) for \( S \) satisfies \( \nu(X^l_1) = \frac{\nu(S) - 1}{2} \). Therefore,

\[
\nu(X^l_2) = \frac{\nu(S) - 1}{2} \text{ or } \frac{\nu(S)}{2} + 1.
\]

In any case of the alternative above, there is no non dicritical vector fields of multiplicity \( \frac{\nu(S) - 1}{2} \) in the Saito module of \( S \), which is a contradiction with the property of \( X_1 \). Therefore, \( \alpha_1 > \alpha_2 \). In the induced basis \( \{ X^l_1, X^l_2 \} \), the vector field \( X^l_1 \) is write

\[
X^l_1 = \frac{1}{x} \left( X^l_1 - y^\alpha_1 \alpha_2 b_1(y) b_2(y) X^l_2 \right).
\]

Therefore, \( X^l_1 \) is dicritical since \( \nu(X^l_1) > \nu(X^l_2) \). But since \( X_1 \) is not dicritical, it is a contradiction.

(3) Finally, suppose that the Saito basis of \( S \cup l \) has the form (\( \mathcal{C}_d' \))

\[
\nu(X^l_1) = \nu(X^l_2) - 2 = \frac{\nu(S)}{2} + 1 - 1
\]

with \( X^l_1 \) dicritical and \( X^l_2 \) not dicritical. Then for any linear function \( L \) the Saito basis for \( S \cup l \)

\[
\{ X^l_1, X^l_2 + LX^l_1 \}
\]

is of type (\( \mathcal{C}_d \)) and we are reduced to the previous case.

\[\square\]

3.3. **Base of type (\( \mathcal{C}_d' \)) and (\( \mathcal{D}_d' \)).** Beyond Example \( \Box \) the curve \( S \) defined by

\[
S = \{ y^5 + x^5 + x^6 = 0 \}
\]

belongs to the generic component of the moduli space of five smooth and transversal curves. An optimal vector field \( X_1 \) for \( S \) can be written

\[
X_1 = \left( \frac{1}{5} xy - \frac{1}{25} x^2 y + \frac{6}{125} x^3 y + \frac{36}{125} x^4 y \right) \partial_x + \left( \frac{1}{5} y^2 + \frac{216}{625} x^3 y^2 \right) \partial_y
\]

whose initial part is

\[
y \left( x \partial_x + y \partial_y \right).
\]

Thus \( X_1 \) is dicritical and of multiplicity 2. However, after one blowing-up, \( X^E_1 \) is not transverse to \( D \) at every point : indeed, following (3.9), it is tangent to \( D \) at the point corresponding to the direction \( y = 0 \). To formalize these remarks, let us recall the following definition
THE SAITO MODULE AND THE MODULI OF A GERM OF CURVE IN \((\mathbb{C}^2, 0)\).

\[
\begin{array}{cccccc}
S & f = x & f = xy & f = xy(x + y) & f = xy(x^2 - y^2) \\
\nu(S) & 1 & 2 & 3 & 4 \\
\end{array}
\]

\[
\begin{array}{cccccc}
X_1, X_2 & \partial_x, x\partial_y & x\partial_x, y\partial_y & x\partial_x + y\partial_y, \partial f & x\partial_x + y\partial_y, \partial f \\
\nu(X_1), \nu(X_2) & 0, 1 & 1, 1 & 1, 2 & 1, 3 \\
Type & (\mathcal{O}_d) & (\mathcal{C}_d) & (\mathcal{O}_{d}^*) & (\mathcal{C}_{d}^*) \\
\end{array}
\]

\[
\begin{array}{cccccc}
S & f = xy(x^3 - y^3 + \cdots) & f = xy(x^2 - y^2)(x + 2y + \cdots)(x + 3y + \cdots) \\
\nu(S) & 5 & 6 & 5 \\
X_1, X_2 & x(x\partial_x + y\partial_y) + \cdots & (x + 2y)(x\partial_x + y\partial_y) + \cdots & x(x\partial_x + y\partial_y) + \cdots \\
\nu(X_1), \nu(X_2) & 2, 2 & 2, 3 \\
Type & (\mathcal{O}_d) - 1 \text{ free point} & (\mathcal{C}_d) - 1 \text{ free point} \\
\end{array}
\]

Table 1. Examples of different types of Saito bases.

**Definition 8.** Let \(D\) be a divisor and \(X\) a vector field defined in a neighborhood of \(D\) that does not leave invariant \(D\). The locus of tangency between \(X\) and \(D\) is the common zeros of \(F\) and \(X \cdot F\) where \(F\) is any local equation of \(D\). It is denoted by

\[\Tan(X, D)\].

By definition, the locus of tangency between \(D\) and \(X\) contains the singular points of \(X\) which are on \(D\). In the example (3.3), we have

\[\Tan(X_E, D) = \{(x_1 = 0, y_1 = 0)\} \neq \Tan(S_E, D) = \emptyset\].

This leads us to introduce the following notion.

**Definition 9.** A curve \(S\) of radial type is said to be of pure radial type if for any optimal vector field \(X_1\) the following equality holds

\[\Tan(X_E, D) = \Tan(S_E, D)\].

If \(S\) is not pure radial, then the non empty set

\[\Tan(X_E, D) \setminus \Tan(S_E, D)\]

is called the set of free points of \(X_1\)

Notice that by construction of \(X_1\), in any case, the inclusion

\[\Tan(S_E, D) \subset \Tan(X_E, D)\]

holds. The main feature of this definition relies on the fact that it allows to state a characterization of the curves admitting a basis of type \((\mathcal{C}_d)\) or \((\mathcal{O}_d)\).

**Theorem 6.** The following properties are equivalent:

1. \(S\) is of pure radial type.
(2) $S$ admits a Saito basis of type $(\mathcal{E}_q')$ or $(\mathcal{D}_q')$.

Proof. We begin by proving $(2) \implies (1)$. Assume that $S$ admits an adapted Saito basis of type $(\mathcal{E}_q')$ or $(\mathcal{D}_q')$. According to Remark 8, for any point $c \in D$, the family 
\[
\{(X_1^E)_{\cdot c}, (X_2^E)_{\cdot c}\}
\]

is a Saito basis of the germ of curve $(S^E)_c$. Let $c \in D \setminus \text{Tan}(S^E, D)$. Suppose first that $c \notin \text{Sing}(E^{-1}(S))$. Then following Remark 3, the product $X_1^E \wedge X_2^E$ is a unity at $c$. Now, $X_1$ is dicritical and $X_2$ is not. Thus in local coordinates $(x, y)$ at $c$ in which $x = 0$ is a local equation of $D$, we can write
\[
X_1^E \wedge X_2^E = (u\partial_x + v\partial_y) \wedge (ax\partial_x + by\partial_y), \quad u, v, a, b \in \mathbb{C}\{x, y\}
\]

Therefore $u$ is a unity and $X_1^E$ is transverse to $D$. Suppose now that $c \in \text{Sing}(E^{-1}(S))$. Since $c \in D \setminus \text{Tan}(S^E, D)$ then $S^E$ is regular and transverse to $D$. Now, considering local coordinates $(x, y)$ in which $xy = 0$ is a local equation of $E^{-1}(S)$, we obtain
\[
X_1^E \wedge X_2^E = (u\partial_x + v\partial_y) \wedge (ax\partial_x + by\partial_y) = avx - bu
\]

which has to be of the form $\text{(unity)} \times y$ according to the criterion of Saito. Therefore, $u$ is a unity and $X_1^E$ is still transverse to $D$, which completes the proof of the equality
\[
\text{Tan}(X_1^E, D) = \text{Tan}(S^E, D).
\]

We now proceed to the proof of $(1) \implies (2)$. Let $\{X_1, X_2\}$ be an adapted Saito basis for $S$. The curve $S$ being radial, let us write
\[(3.10)\]

$X_1 = h_1(x\partial_x + y\partial_y) + \cdots .$

The hypothesis is equivalent to assume that the tangent cone of $h_1$ coincide with the locus of tangency $\text{Tan}(S^E, D)$ for any optimal vector field $X_1$. Assume first that $\nu(S)$ is odd. According to Proposition 3, the valuation of $X_1$ is
\[\nu(X_1) = \frac{\nu - 1}{2}.\]

If $X_2$ is not dicritical, then $\nu(X_2) = \frac{\nu - 1}{2}$. Therefore, the basis $\{X_1, X_2\}$ is of type $(\mathcal{D}_q')$ and the proposition is proved. Assume $X_2$ is dicritical and $\nu(X_2) = \frac{\nu - 1}{2}$. As in (3.10), we write
\[X_2 = h_2(x\partial_x + y\partial_y) + \cdots \]

and
\[h_2 = q_2 \cdot \overline{h_2}\]

where the tangent cone of $q_2$ does not meet $\text{Tan}(S^E, D)$. For any value of $\alpha$ and $\beta$, the initial part of $\alpha X_1 + \beta X_2$ is written
\[(\alpha h_1 + \beta q_2\overline{h_2})(x\partial_x + y\partial_y).\]

The hypothesis ensures that the tangent cone of $\alpha h_1 + \beta q_2\overline{h_2}$ is in $\text{Tan}(S^E, D)$. Since the tangent cone of $h_1$ is in $\text{Tan}(S^E, D)$, it can be seen that the function $q_2$ is constant and that there exists a constant $u$ such that
\[h_2 = uh_1\]
Then the basis \( \{ X_1, X_2 - uX_1 \} \) is of type \( (\mathcal{D}') \).

Assume finally that \( \nu (S) \) is even and consider a smooth curve \( l \) which is attached to a point in \( \text{Tan} (S^E, D) \) after on blowing-up. Let \( \{ X_1, X_2' \} \) be an adapted Saito basis for \( S \cup l \). Consider some coordinates in which \( l = \{ x = 0 \} \) and write

\[
X_1' = xa_1 \partial_x + (y^{\alpha_1} b_0 (y) + xb_1') \partial_y
\]

with \( b_1 (0) \neq 0 \). Since \( \nu (S \cup l) \) is odd, a few cases may occur:

1. Assume the basis is of type \( (\mathcal{D}) \). Then, we can suppose that \( \nu (X_1') = \nu (X_2') = \frac{\nu (S)}{2} \) and \( \alpha_1 = \alpha_2 \). The family

\[
\left\{ X_1', X_2' = \frac{1}{x} \left( X_2' - \frac{b_0}{b_1} X_1' \right) \right\}
\]

is a Saito basis for \( S \) with

\[
\nu (X_1') = \frac{\nu (S)}{2} \quad \text{and} \quad \nu (X_2') \geq \frac{\nu (S)}{2} - 1.
\]

If \( \nu (X_2') \geq \frac{\nu (S)}{2} \) then \( \nu (S) \geq \frac{\nu (S)}{2} \) which is impossible. Thus \( \nu (X_2') = \frac{\nu (S)}{2} - 1 \).

But then, following Lemma 11, \( X_2' \) cannot be dicritical which contradicts the radially of \( S \). Finally, the Saito basis of \( S \cup l \) cannot be of type \( (\mathcal{D}) \).

2. Assume it is of type \( (\mathcal{D}_d) \) but not of type \( (\mathcal{D}_d') \). Applying Theorem 6 to this case for which \( \nu (S \cup l) \) is odd ensures that \( S \cup l \) is not pure radial. Thus up to some changes of basis, \( \text{Tan} \left( (X_1')^E, S \cup l \right) \) and \( \text{Tan} \left( (X_2')^E, S \cup l \right) \) contains some points out of \( \text{Tan} \left( (S \cup l)^E, D \right) = \text{Tan} (S^E, D) \). Therefore, we obtain a Saito basis for \( S \) of the form

\[
\left\{ X_1', X_2' \right\}
\]

where the tangent cone of \( X_2' \) is not contained in \( \text{Tan} (S^E, D) \), which contradicts the assumption of \( S \) being pure radial. Therefore, the Saito basis of \( S \cup l \) cannot be of type \( (\mathcal{D}_d) \) but not of type \( (\mathcal{D}_d') \).

3. Finally, \( S \cup l \) admits a Saito basis \( \{ X_1', X_2' \} \) of type \( (\mathcal{D}_d') \) with \( \nu (X_1') = \nu (X_2') - 1 = \frac{\nu (S)}{2} - 1 \). If \( \alpha_2 > \alpha_1 \) then

\[
\left\{ X_1', X_2' = \frac{1}{x} \left( X_2' - y^{\alpha_2 - \alpha_1} b_0^0 b_1' X_1' \right) \right\}
\]

is a Saito basis for \( S \) with

\[
\nu (X_1') = \frac{\nu (S)}{2} \quad \text{and} \quad \nu (X_2') \geq \frac{\nu (S)}{2} - 1
\]

which is impossible. Thus \( \alpha_2 \leq \alpha_1 \) and

\[
\left\{ X_1' = \frac{1}{x} \left( X_1' - \frac{y^{\alpha_1 - \alpha_2} b_0^0 b_1' X_2'}{b_1^2} X_2' \right), X_2' \right\}
\]

is a Saito basis for \( S \) of type \( (\mathcal{E}_d') \).

\( \square \)

Finally, from the proof above we deduce the following
Lemma 13. If $S$ is of type $(\mathcal{O})$ then $S \cup l$ is of type $(\mathcal{E})$. If $S$ is of type $(\mathcal{E}_d)$ or $(\mathcal{E}_d')$ then $S \cup l$ is of type $(\mathcal{O}_d)$ or $(\mathcal{O}'_d)$.

3.4. Cohomology of $\Theta_S$. As we will explain in the next section, the cohomology of the sheaf $\Theta_S$ computes the generic dimension of $M^*(S)$. The associated formula depends on the type of Saito basis of $S$.

Proposition 4. The dimension of the cohomology group $H^1(D, \Theta_S)$ can be obtained from the multiplicities of an adapted Saito basis of $S$ the following way

1. If $\nu(X_1) + \nu(X_2) = \nu(S)$ then
   \[
   \dim H^1(D, \Theta_S) = \frac{(\nu_1 - 1)(\nu_1 - 2)}{2} + \frac{(\nu_2 - 1)(\nu_2 - 2)}{2}
   \]
2. If $\nu(X_1) + \nu(X_2) = \nu(S) - 1$ then
   \[
   \dim H^1(D, \Theta_S) = \frac{(\nu_1 - 1)(\nu_1 - 2)}{2} + \frac{(\nu_2 - 1)(\nu_2 - 2)}{2} + \nu(S) - 2 - \nu_0
   \]
   where $\nu_i = \nu(X_i)$, $i = 1, 2$ and $\nu_0 = \nu(\gcd(X_1^{(\nu(X_1))}, X_2^{(\nu(X_2))}))$.

Proof. The proof of the first equality is in \[3\]. Below, we only give a proof of the second equality. Let us consider the standard system of coordinates defined in a neighborhood of $D$ and introduced in section 3.1.

One can compute the cohomology using the associated covering and thus

\[
(3.11) \quad H^1(D, \Theta_S) = H^1([U_1, U_2], \Theta_S) = \frac{H^0(U_1 \cap U_2, \Theta_S)}{H^0(U_1, \Theta_S) \oplus H^0(U_2, \Theta_S)}.
\]

The task is now to describe in coordinates each $H^0$ involved in the quotient above. To deal with $H^0(U_1, \Theta_S)$, we start with the criterion of Saito

(3.12) \quad X_1 \wedge X_2 = uf.

As $\nu_1 + \nu_2 = \nu(S) - 1$, blowing-up the criterion of Saito in the first chart $(x_1, y_1)$ yields

\[
\frac{E^*X_1}{x_1^{\nu_1} - 1} \wedge \frac{E^*X_2}{x_2^{\nu_2} - 1} = u \circ E \cdot x_1^2 f \circ E \cdot x_2^{\nu(S)}.
\]

Let $Y$ be a section of $\Theta_S$ on $U_1$. By definition, there exists $g_1 \in \mathcal{O}(U_1)$ such that

\[
Y \wedge X_1 = g_1 x_1 \frac{f \circ E}{x_1^{\nu(S)}}.
\]

Hence, one has

\[
\left( x_1 Y - g_1 \frac{1}{u \circ E} X_2 \right) \wedge X_1 = 0.
\]

Assume $X_1$ is not dicritical. Then, $X_1$ has only isolated singularities and there exists $h_1 \in \mathcal{O}(U_1)$ such that

\[
x_1 Y = g_1 \frac{1}{u \circ E} X_2 + h_1 X_1.
\]
If now $X_1$ is dicritical, then $x_1^1$ extends analytically along $D$ and has only isolated singularities. Therefore, there still exists $h_1 \in \mathcal{O}(U_1)$ such that

$$x_1 Y = k_1 \frac{1}{u \circ E} x_2^1 + \frac{h_1}{x_1} X_1^1.$$ 

Since, $x_1 Y$ and $X_2^1$ are tangent to $D$, $x_1$ divides $h_1$. Thus we get

$$H^0(U_1, \Theta_S) = \left\{ Y = \frac{1}{x_1} \left( \phi_1^1 X_1^1 + \phi_2^1 X_2^1 \right) \mid \begin{array}{l} (1) \phi_1^1 \in \mathcal{O}(U_1) \\
(2) Y \text{ extends analytically along } D \end{array} \right\}.$$ 

We now proceed to analyse the second condition highlighted above: let us write the expansion of $X_i$ in homogeneous components

$$X_i = X_i^{(\nu_i)} + X_i^{(\nu_i+1)} + \cdots$$

The relation $\nu_1 + \nu_2 = \nu - 1$ implies that

$$X_1^{(\nu_i)} \wedge X_2^{(\nu_2)} = 0$$

and we can write

$$X_i^{(\nu_i)} = \delta_i X_0$$

where $X_0 = \gcd \left( X_1^{(\nu_1)}, X_2^{(\nu_2)} \right)$ and $\delta_i, i = 1, 2$ are homogeneous functions such that

$$\delta_1 \wedge \delta_2 = 1.$$ 

The expression of $Y$ can be expanded with respect to $x_1$ in

$$Y = \frac{1}{x_1} \sum_{i=1,2} \left( \phi_i^{1,0} (y_1) + x_1 (\cdots) \right) \left( \delta_i^1 X_0^1 + x_1 (\cdots) \right), \quad \delta_i^1 = \frac{\delta_i \circ E}{x_1^{(\delta_i)}}.$$ 

Thus the condition $Y$ being extendable along $D$ reduces to

$$\sum_{i=1,2} \phi_i^{1,0} \delta_i^1 = 0.$$ 

We proceed analogously for the open sets $U_2$ and $U_1 \cap U_2$ and obtain the following description where the exponent 2 refers to the second chart $(x_2, y_2)$

$$H^0(U_1, \Theta_S) = \left\{ Y^1 = \frac{1}{x_1} \sum_{i=1,2} \phi_i^1 X_i^1 \mid \phi_i^1 \in \mathcal{O}(U_1) \right\},$$

(3.13) $$H^0(U_2, \Theta_S) = \left\{ Y^2 = \frac{1}{y_2} \sum_{i=1,2} \phi_i^2 X_i^2 \mid \phi_i^2 \in \mathcal{O}(U_2) \right\},$$

$$H^0(U_1 \cap U_2, \Theta_S) = \left\{ Y^{12} = \frac{1}{x_1} \sum_{i=1,2} \phi_i^{12} X_i^1 \mid \phi_i^{12} \in \mathcal{O}(U_1 \cap U_2) \right\}.$$ 

We may now compute the number of obstructions involved in the cohomological equation describing the quotient (3.11), namely,

$$Y^{12} = Y^2 - Y^1.$$ 

In view of the description above, the cohomological equation splits into the system

$$\phi_i^{12} = \frac{\phi_i^2}{y_1^{(\nu_2)}} - \phi_i^1, \quad i = 1, 2.$$
which we filter with respect to $x_1$ obtaining

\begin{align}
\phi^{12,0}_i &= \frac{\phi^{2,0}_i}{y_1} - \phi^{1,0}_i, \quad i = 1, 2 \\
\phi^{12,1}_i &= \frac{\phi^{2,1}_i}{y_1} - \phi^{1,1}_i, \quad i = 1, 2
\end{align}

where

$$
\phi^{*}_i = \phi^{*,0}_i + x_1 \phi^{*,1}_i,
$$

with $* = 1, 2, 12$. Let us analyse the system (3.14). Since the functions $\delta_i$ are relatively prime, the conditions involved in the description of the cohomological spaces (3.13) ensures that there exist analytical functions $\dot{\phi}^{*,0}_i$ such that

$$
\dot{\phi}^{*,0}_1 = \dot{\phi}^{*,0}_2 \delta_2^* \quad \text{and} \quad \dot{\phi}^{*,0}_2 = -\dot{\phi}^{*,0}_1 \delta_1^*.
$$

for $* = 1, 2, 12$. Thus, the system (3.14) reduces to the sole equation

$$
\dot{\phi}^{12,0}_i = \frac{\dot{\phi}^{12,0}_i}{y_1^{1-\nu_1^{(\delta_2)}}} - \dot{\phi}^{1,0}_i.
$$

Writing the Laurent expansions of the above functions yields the relation

$$
\sum_{k \in \mathbb{Z}} \phi^{12,0}_k y_1^k = \sum_{k \in \mathbb{N}} \phi^{12,0}_k y_1^{k-\nu_1^{(\delta_2)}} - \sum_{k \in \mathbb{N}} \phi^{1,0}_k y_1^k
$$

which implies $\dot{\phi}^{12}_k = 0$ for $-\nu_1 - \nu (\delta_2) + 1 \leq k \leq -1$. These conditions provide the sole $-\nu_1 - \nu (\delta_2) + 1$ obstructions to the cohomological equation (3.11). The system (3.15) involves two independent cohomological equations. We can proceed analogously to identify $(\nu_i-1)(\nu_i-2)$ obstructions $i = 1, 2$, respectively, for the equation $i = 1, 2$. Finally, the formula (2) of the proposition follows from the relation

$$
\nu (\delta_2) = \nu_2 - \nu_0.
$$

Notice that the second case of Proposition 3 may occur when $S$ is of type $(\Omega_d)$ or $(\Xi_d)$. In that case, it can be seen that $\nu (X_1) - \nu_0$ is the number of free points of $X_1$.

4. Dimension of the moduli space of a singular regular point.

In this section, we intend to apply the previous results to compute the generic dimension of $M^* (S)$ for

$$
S = \{ x^n + y^n = 0 \},
$$

and thus, to recover a classical result due to Granger [13].

To achieve this, we have to identify precisely the topology of the generic optimal vector field for $S$. First, we are going to improve somehow the optimality of the generic optimal vector field studying when this optimality is preserved after one blowing-up.
4.1. Optimality after one blow-up.

**Proposition 5.** Let $S$ be a generic curve in its moduli space. Let $c \in D$ a point in the exceptional divisor of the single blowing-up $E$. Assume that

\[ (*) \text{ there exists a germ of regular curve } l \text{ such that } (S^E)_c \cup (l^E)_c \text{ has no Saito basis of type } (E_d). \]

Then there exists a vector field $X$ optimal for $S$ such that $(X^E)_c$ is optimal for $(S^E)_c$.

**Proof.** Let $\{X_1, X_2\}$ be an adapted Saito basis for $S$. If $\nu(X_1) = \nu(X_2)$ which is satisfy when the basis is of type $(\mathcal{E}, (\mathcal{D}_d)$ then for $\alpha$ and $\beta$ generic one has $\alpha X_1^E + \beta X_2^E = (\alpha X_1 + \beta X_2)^E$.

According to Remark [3], $\{ (X_1^E)_c, (X_2^E)_c \}$ is a Saito basis for $(S^E)_c$, therefore at $c$ one has $\nu_c((\alpha X_1 + \beta X_2)^E) = \nu_c((S^E)_c)$.

Thus, in that case, choosing $X = \alpha Y_1 + \beta Y_2$ yields the lemma.

Now, assume that $\nu(X_1) < \nu(X_2)$. Suppose first that $\nu(S)$ is odd then $S$ is of type $(\mathcal{D})$. Let us consider a curve $l$ satisfying the hypothesis of the lemma. According to Lemma [13] an adapted Saito basis $\{X_1^l, X_2^l\}$ is of type $(\mathcal{E})$ with $\nu(X_1^l) = \nu(X_2^l) = \frac{\nu(S) + 1}{2}$.

Applying the process of division, we are lead to an adapted Saito basis $\{\overline{X}_1^l, X_2^l\}$ for $S$ with

\[ \nu(\overline{X}_1^l) = \frac{\nu(S) - 1}{2} < \nu(X_2^l) = \frac{\nu(S) + 1}{2} \]

The blow-up family $\left\{ (\overline{X}_1^l)^E_c, (X_2^l)^E_c \right\}$ is a basis for $(S^E)_c$. Now, suppose that $\nu_c((\overline{X}_1^l)^E) \geq \nu_c((X_2^l)^E) + 1$.

therefore,

$\nu_c(\overline{L}(\overline{X}_1^l)^E) \geq \nu_c((X_2^l)^E) + 2$

where $L$ is a local equation of $l^E$. The family $\left\{ \overline{L}(\overline{X}_1^l)^E, (X_2^l)^E \right\}$ is a Saito basis for $S^E \cup l^E$ at $c$ and following Remark [2] it is of type $(\mathcal{E}_d)$. That is impossible. Hence,

\[ \nu_c((\overline{X}_1^l)^E) \leq \nu_c((X_2^l)^E) \]

and, according to [14] and [12], $X = \overline{X}_1^l$ satisfies the conclusion of the lemma. Finally, if $\nu(S)$ is even then $S$ is of type $(\mathcal{E}_d)$ and it is of type $(\mathcal{D}_d)$ and the arguments are similar. \(\square\)
Corollary 2. If any component of $S^E$ satisfies the hypothesis $(\ast)$ of Proposition 4, then there exists a vector field $X$ optimal for $S$ such that, for any $c$, $(X^E)_c$ is optimal for $(S^E)_c$.

Proof. Indeed, for any point $c$ in the tangent cone of $S$, consider $X_c$ given by Proposition 5 for the curve $(S^E)_c$. Then for a generic family of complex numbers $\{\alpha_c\}$, the vector field $X = \sum \alpha_c X_c$ satisfies the property. \hfill \square

4.2. Dimension of $M^\bullet(S)$ where $S = \{x^n + y^a = 0\}$. The curve $S$ is desingularized by a single blowing-up. From [20], the generic dimension of $M^\bullet(S)$ is equal to

$$\dim H^1(D, \Theta_S).$$

Following Proposition 4, it can be computed from some topological data associated to an adapted basis of Saito for $S$. Below, we are going to describe these bases according to the value of $n$.

If $n = 3$ then there are coordinates $(x, y)$ in which $S = \{f = xy(x + y) = 0\}$. The family

$$\{X_1 = x \partial_x + y \partial_y, \ X_2 = \partial f = \partial_x f \partial_y - \partial_y f \partial_x\}$$

is a Saito basis for $S$. Since $\nu(X_1) = \nu(X_2) - 1 = 1$, $X_1$ is dicritical but not $X_2$, $S$ is of type $(\mathcal{O}_d')$. If $n = 4$ then there are coordinates $(x, y)$ in which $S = \{f = xy(x + y)(x + ay) = 0\}$ for some $a \notin \{0, 1\}$. Hence, the family

$$\{X_1 = x \partial_x + y \partial_y, \ X_2 = \partial f\}$$

is a Saito basis for $S$. Since $\nu(X_1) = \nu(X_2) - 2 = 1$, $X_1$ is dicritical but not $X_2$, $S$ is of type $(\mathcal{E}_d')$. In the latter case, the dimension of $M^\bullet(S)$ is 1.

Now suppose $n \geq 5$.

Proposition 6. The curve $S$ is of type $(\mathcal{O}_d)$ or $(\mathcal{E}_d)$. Moreover, the generic optimal vector $X_1$ is completely regular after a single blowing-up and has $\left[\frac{n}{2}\right] - 2$ free points.

Proof. Following notations introduced in [18, p. 657] for a germ at $p$ of vector field $X$ and a germ of curve $S$ given in coordinates by

$$X = a(x, y) \partial_x + b(x, y) \partial_y \quad S = \{x = 0\}$$

we recall the following definitions:

1. if $S$ is invariant by $X$, the integer $\nu_y(b(0, y))$ is called the index of $X$ at $p$ with respect to $S$ and it is denoted by $\text{ind}(X, S, p)$.

2. if $S$ is not invariant by $X$, the integer $\nu_y(a(0, y))$ is called the tangency order of $X$ at $p$ with respect to $S$ and it is denoted by $\text{tan}(X, S, p)$. 
Suppose \( X_1 \) non dicritical, then according to [18, Lemma 1], one has
\[
\nu(X_1) + 1 = \sum_{c \in D} \text{ind} (X^E_1, D, c).
\]
For any point \( c \) in the tangent cone of \( S \), the curve \((S^E \cup D)_c\) is a union of two transversal smooth curves. Therefore, the index \( \text{ind} (X^E_1, D, c) \) is at least 1 since \((X^E_1)_c\) is singular. Therefore, one has
\[
\sum_{c \in D} \text{ind} (X^E_1, D, c) \geq \# \text{tangent cone} = n.
\]
On the other hand, the optimality of \( X_1 \) ensures that
\[
\nu(X_1) \leq \frac{n}{2}.
\]
The equality [LS] and the inequalities (4.4) and (4.5) are incompatible with \( n \geq 5 \), and thus \( X_1 \) is dicritical. Any component of \( S^E \) is a regular curve. Since the union of two curves is not of type \((E_d')\), any component of \( S^E \) satisfies the hypothesis \((\ast)\) of Proposition 5. As a consequence, we can consider \( X_1 \) to be not only optimal for \( S \) but also optimal after one blowing-up. Since any component of \( S^E \) are regular curve, whose Saito number are equal to 0, the vector field \( X^E_1 \) is regular at the tangent cone of \( S \). Moreover, there exists \( Y \) such that \( \{X_1, Y\} \) is an adapted Saito basis. Thus, after one blowing-up, one can write
\[
X^E_1 \wedge Y^E = u \circ E \frac{f \circ E}{x^1}.
\]
where \( f = x^n + y^n \). Since out of the tangent cone of \( S \), the function \( \frac{f \circ E}{x^1} \) is a unit, \( X^E_1 \) is finally regular at any point of \( D \).

Following again [LS] Lemma 1], one has
\[
\nu(X_1) + 1 = \left\lceil \frac{n}{2} \right\rceil = 2 + \sum_{c \in D} \tan (X^E_1, D, c).
\]
The above relation concludes the proof of the proposition : \( \text{Tan} (S^E, D) \) being empty, any tangency point between \( X^E_1 \) and \( D \) is a free point. \( \square \)

As a consequence of Proposition 6, we recover a classical result of Granger concerning the generic dimension of the moduli space of \( S \) [13]. According to Theorem 6, the Saito basis of \( S \) satisfies
\[
\nu(X_1) = \begin{cases} \frac{n}{2} - 1 & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{else} \end{cases} \quad \text{and} \quad \nu(X_2) = \begin{cases} \frac{n}{2} - 1 & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{else} \end{cases}.
\]
Moreover, by construction, the integer \( \nu_0 \) identified in Proposition 4 satisfies
\[
\nu_0 = \nu(X_1) - (\text{number of free points}) = \begin{cases} \frac{n}{2} - 1 & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{else} \end{cases} - \left( \left\lceil \frac{n}{2} \right\rceil - 2 \right) = 1.
\]
Now, following Proposition 6, the dimension of \( \mathcal{M}^* (S) \) is equal to
\[
\begin{cases} \frac{1}{2} \left( \frac{n}{2} - 2 \right) \left( \frac{n}{2} - 3 \right) + \frac{1}{2} \left( \frac{n}{2} - 1 \right) \left( \frac{n}{2} - 2 \right) + n - 3 = \frac{(n-2)^2}{4} & \text{if } n \text{ is even} \\ \left( \frac{n-1}{2} - 1 \right) \left( \frac{n-1}{2} - 2 \right) + n - 3 = \frac{(n-1)(n-3)}{4} & \text{if } n \text{ is odd} \end{cases}
\]
which coincides with the results in [13].

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