ZETA-LIKE MULTIZETA VALUES FOR $\mathbb{F}_q[t]$

JOSE ALEJANDRO LARA RODRÍGUEZ AND DINESH S. THAKUR

Dedicated to the genius of Srinivasa Ramanujan

Abstract. We prove and conjecture several relations between multizeta values for $\mathbb{F}_q[t]$, focusing on zeta-like values, namely those whose ratio with the zeta value of the same weight is rational (or equivalently algebraic). In particular, we describe them conjecturally fully for $q = 2$, or more generally for any $q$ for ‘even’ weight (‘eulerian’ tuples). We provide some data in support of the guesses.

1. Introduction

Relations between multizeta values defined by Euler have been investigated extensively for the last two decades, and the conjectural forms of these relations have many structural connections with several interesting areas (see [Ct2001, Z2012] and references there) of mathematics. In some sense, the relations have been at least conjecturally understood, though much remains to be proved and relating the general framework to specific instances is often hard.

We will look at the function field analog [T2004, AT2009, T2009, Tbanff, L2011, L2012], where the relations are still not conjecturally understood, though in contrast, there are also some very strong results transcendence and independence results [CY2007, Ch2012, CPY] proved.

While for the Euler multizeta values, the relations come via comparing the two families of shuffle relations, in our function field setting, there is only one shuffle family [T2010]. While the rational number field is the prime field in characteristic zero giving coefficients for the relations, in the function field case the prime field is not the analogous rational function field, but just the finite field, which does not see all the relations. Some such relations were proved in [T2009, L2011].

While second author’s student George Todd is doing extensive numerical study of general relations using an analog of the ‘LLL method’, in this paper we focus on the two term relations of special type, namely zeta-like multizeta, i.e., those whose ratio with the (Carlitz) zeta value of the same weight is rational (or equivalently algebraic). We provide several results, and conjectures, with full conjectural description for $q = 2$, or more generally for any $q$ with ‘even’ weight (‘eulerian’ tuples).

We first fix the notation and give the basic definitions. Next we summarize the known and the new results on zeta-like values, and state the conjectures. Then we give the proofs of the results. Finally we discuss the numerical data, calculated by the first author, giving some evidence for the conjectures made from it.

Date: December 18, 2013.

The authors supported in part by PROMEP grant F-PROMEP-36/Rev-03 SEP-23-006 and by NSA grant H98230-13-1-0244 respectively.
2. Notation and Basic definitions

\[ Z = \{ \text{integers} \}, \]
\[ Z^+ = \{ \text{positive integers} \}, \]
\[ q = \text{a power of a prime } p, \ q = p^s, \]
\[ \mathbb{F}_q = \text{a finite field of } q \text{ elements}, \]
\[ A = \text{the polynomial ring } \mathbb{F}_q[t], \ t \text{ a variable} \]
\[ A^+ = \text{monics in } A, \]
\[ K = \text{the function field } \mathbb{F}_q(t), \]
\[ K_\infty = \mathbb{F}_q((1/t)) = \text{the completion of } K \text{ at } \infty, \]
\[ A_{d^+} = \{ \text{elements of } A^+ \text{ of degree } d \}, \]
\[ [n] = t^n - t, \]
\[ \ell_n = \prod_{i=1}^{n} (t - t^n') = (-1)^n L_n = (-1)^n [n][n-1] \cdots [1], \]
\[ \text{‘even’ multiple of } q - 1, \]

We first recall definitions of power sums, iterated power sums, zeta and multizeta values [T2004, T2009].

For \( s \in Z^+ \) and \( d \geq 0 \), write

\[ S_d(s) := \sum_{a \in A_{d^+}} \frac{1}{a^s} \in K. \]

(This is \( S_d(-s) \) in the notation of [T2004].)

Given integers \( s_i \in Z^+ \) and \( d \geq 0 \) put

\[ S_d(s_1, \ldots, s_r) = S_d(s_1) \sum_{d > d_2 > \cdots > d_r \geq 0} S_{d_2}(s_2) \cdots S_{d_r}(s_r) \in K. \]

For \( s_i \in Z^+ \), we define multizeta values

\[ \zeta(s_1, \ldots, s_r) := \sum_{d_1 > \cdots > d_r \geq 0} S_{d_1}(s_1) \cdots S_{d_r}(s_r) = \sum_{a_{d_1}^{-1} \cdots a_{d_r}^{-1} \in K_\infty} \in K_\infty, \]

where the second sum is over all \( a_{d_i} \in A_{d_i} \) of degree \( d_i \) such that \( d_1 > \cdots > d_r \geq 0 \).

We say that this multizeta value (or rather the tuple \((s_1, \ldots, s_r)\)) has depth \( r \) and weight \( \sum s_i \). In depth one, we recover the Carlitz zeta.

We refer to [C1935, G1996, T2004] for background on this and general function field analogies. Carlitz proved analog of Euler’s result that for ‘even’ \( s \), \( \zeta(s) \) is rational multiple of \( \tilde{\pi}^s \), where the Carlitz period \( \tilde{\pi} \) is analog of \( 2\pi i \).

A multizeta value \( \zeta(s_1, \ldots, s_r) \) of depth \( r \) (or the \( r \)-tuple \((s_1, \ldots, s_r)\)) is zeta-like if the ratio

\[ \zeta(s_1, \ldots, s_r)/\zeta(s_1 + \cdots + s_r) \]

is rational. (We always use depth \( r > 1 \) below, sometimes without mention, because in the \( r = 1 \) case everything is zeta-like by definition). A multizeta value of weight \( w \) is called eulerian, if it is a rational multiple of \( \tilde{\pi}^w \). So eulerian is a special case of zeta-like for ‘even’ weight, by Carlitz result (mentioned above) which says that in depth one, all the zeta values of ‘even’ weight are eulerian.

A strong transcendence result [Ch2012] proved in the function field case shows that if the ratios in the definition are not rational, they are not even algebraic and in fact the multizeta and corresponding zeta values are then algebraically independent.

Another strong transcendence result [CY2007] shows that the Carlitz zeta value of not ‘even’ weight and \( \tilde{\pi} \) are algebraically independent.
Since $\zeta(ps_1, \ldots, ps_r) = \zeta(s_1, \ldots, s_r)^p$, in all the discussion we can restrict to tuples where not all $s_i$’s are divisible by $p$. We call such tuples primitive.

3. Old and New Results on Zeta-like Values

For the Euler multizeta in the number field case, the classical sum shuffle relation immediately implies that $\zeta(2n, 2n)$ are eulerian, and combined with the usual transcendence conjectures, it implies that $\zeta(2n + 1, 2n + 1)$ are not zeta-like. In the function field case, the classical sum shuffle relation does not hold in general, so we only know by this method [T2004, Thm. 5.10.6] that, when $p \neq 2$, that $\zeta(kp^n, kp^n)$ is not zeta-like, if $2k \leq q$ ([CY2007] giving the required transcendence result). (Another instance of different shuffle [L2012, Thm. 6.3] similarly shows that $\zeta(q^n - 1, q^n)$ is not zeta-like, for $q > 2$). In [T2004, T2009, L2011], more examples of zeta-like and non-zeta-like values of ‘even’ and ‘odd’ weights were proved. Combining with general shuffle relations [T2010], some more such results can be proved. But we have now proved much stronger results, which we will recall below.

In [CPY], using the interpretation [AT2009] of multizeta values as periods of iterated extensions of tensor powers of Carlitz-Anderson $t$-motives, it was proved that if $\zeta(s_1, \ldots, s_r)$ is zeta-like (eulerian in the first version), then $\zeta(s_2, \ldots, s_r)$ is eulerian, so that all $\zeta(s_k, \ldots, s_r)$ are eulerian and $s_k$ are ‘even’, for $i \geq 2$. (See [T2009] 5.3).

Remark This implies some, but not all, of the non-zeta-like special results mentioned above. Many can be proved by direct appeal to [CY2007] and shuffle and other results proved, a few (such as $\zeta(2, 1)$ is not zeta-like for $q = 2$) were proved [T2004] Thm. 5.10.12 without using [CY2007].

While [CPY] was being proved for the eulerian case, we had conjectured this (and a few more implications) for zeta-like case, but only in depth 2 and were starting calculations in general depth, which give many interesting conjectural restrictions recalled below.

We now state some families of zeta-like (so eulerian, if the weight is ‘even’) multizeta values of depth two. Proofs will be given in the fifth section.

Theorem 3.1. For any (prime power) $q$, we have

$$\zeta(q^n - \sum_{i=1}^{s} q^{k_i}, (q - 1)q^n) = \frac{(-1)^s}{q^n} \prod_{i=1}^{s} [n - k_i]q^{k_i}\zeta(q^{n+1} - \sum_{i=1}^{s} q^{k_i}),$$

where $n > 0$, $1 \leq s < q$, $0 \leq k_i < n$.

Let $n \geq 0$, $0 \leq k_i \leq n + 1$, $1 \leq s_1 \leq q$, $0 \leq s_2 \leq q - s_1$. Then for $a = s_1q^n$ and $b = s_1(q^{n+1} - q^n) + \sum_{i=1}^{s_2} (q^{n+1} - q^{k_i})$, we have

$$\zeta(a, b) = \frac{1}{\ell_1^{s_1}q^{s_1}}\zeta(a + b).$$

(2)

$$\zeta(q^2 - (q - 1), (q - 1)(q^2 + 1)) = \frac{1 - [2|q]}{\ell_1^{s_1 - 1}q^{s_2}}\zeta(q^3),$$

(3)

$$\zeta(2q - 1, (q - 1)(q^2 + q - 1)) = \frac{1 - [2|q]}{\ell_1^{s_1}q^{s_2}}\zeta(q^3).$$

(4)

$$\zeta(1, q^2 - 1) = \zeta(q^2)(1/\ell_1 + 1/\ell_2).$$

(5)
For \( q > 2, n \geq 0 \) and \(-1 \leq j \leq n,\)

\[
\zeta((-1)^{q^n - 1}, (q - 1)^{q^{n+1}} + q^n - q^{n-j}) = -\frac{[n+1]}{[1][q-1]q^n} \zeta(q^{n+2} - q^{n-j} - 1).
\]

Next we state a theorem (proved in section 5) giving zeta-like family of arbitrary depth.

**Theorem 3.2.** For any \( q, \)

\[
\zeta(1, q-1, (q-1)q, \ldots, (q-1)q^n) = (-1)^n + 1 \prod_{i=0}^{n+1} \zeta(q^{i+1} - 1).
\]

4. **Observations, Guesses and Conjectures**

Now we state some conjectures (based on the numerical data and on consistency with the theorems and the proof methods) with varying degrees of confidence and evidence!

**Conjecture 4.1. Tuple restrictions** If \((s_1, \ldots, s_r)\) is zeta-like, then

1. \( s_i \leq s_{i+1}, i = 1, \ldots, r-1. \) Furthermore, \((q-1)s_i \leq s_{i+1} \leq (q^2 - 1)s_i. \)
2. \((s_2, \ldots, s_r)\) is eulerian and \((s_1, \ldots, s_{r-1})\) is zeta-like.

Note that part 2 can be iterated and implies that \( s_i \) are ‘even’ for \( i \geq 2 \) (already proved together with the first part of (2), in [CPY], as mentioned before).

**Conjecture 4.2. Splicing of tuples** when \( q = 2 \) If \((s_1, \ldots, s_k)\) and \((s_k, \ldots, s_r)\) are zeta like and the total weight \( \sum_{i=1}^{r} s_i \) is a power of 2 or a power of 2 minus one, then \((s_1, \ldots, s_r)\) is zeta-like, except when the two tuples to be spliced are \((1,1)\) and \((1,1)\).

**Remarks** We have not seen any more failures in the limited data we have. We are investigating the situation for general \( q, \) where splicing conditions seem to be much more restrictive and seem to depend (in the limited data we have) on combinatorics of digit expansions. But splicing of eulerian tuples seems to work in weights \( q^n - 1. \)

**Conjecture 4.3. Weight restrictions**

1. Eulerian multizeta value (in depth \( r > 1 \)) can occur only in weights \( p^m(q^k - 1), \) with primitive ones only in weights \( q(q-1) \) or \( q^n - 1 \) for \( q > 2 \) and in weights \( 2^a - 1 \) and \( 2^n, \) if \( q = 2. \)
2. When \( q = p, \) depth \( r > 1, \) the weight of zeta-like but non-eulerian(tuple \( p^m \) times number with no zero digit and at most one 1 digit.
3. In depth \( r, \) the smallest weight of zeta-like value is \( q^{r-1}. \)
4. When \( q > 2, \) the smallest weight of eulerian value is \( q^{r-1} - 1, (q-1)q, \) and \( q - 1 \) according as depth \( r > 2, r = 2 \) and \( r = 1 \) respectively.
5. Weight \( q^k \) is not a zeta-like weight of a primitive tuple, if \( k > r > 3, \) and also when \( k > 3 \) if \( r = 3 \), or \( 2, \) where \( r \) is the depth.
Remarks (0) Let us recall the known results for the Euler multizeta. (We use the standard short-form \( \{X\}_k \) standing for the tuple \( X \) repeated \( k \) times.) Euler proved that \( \zeta(3,1) = \zeta(4)/4 \) and \( \zeta(2,1) = \zeta(3) \), which generalizes to zeta-like \( \zeta_k(2, \{1\}_k) = \zeta(k+2) \) (special case of Hoffman-Zagier duality relation) and \( \zeta(\{3, 1\}_k) = \zeta(\{2\}_{2k})/(2k+1) \) (Broadhurst’s result, conjectured by Zagier) which is known to be eulerian as \( \zeta(\{2\}_k) = \pi^{2k}/(2k+1)! \) (see e.g., [Z2012]). In fact, \( \zeta(\{2n\}_k) \) is also eulerian. (Proof: Let the induction hypothesis \( P(k) \) be that \( A_k := \zeta(\{2n\}_k) \) and \( B_{k,m} := \sum \zeta(X(i)) \) are eulerian for all \( n, m \), where \( X(i) \) runs through all length \( k \) tuples with \( k - 1 \) entries \( 2n \) and one entry \( 2nm \). The sum shuffle gives \( \zeta(2n)A_k = (k+1)A_{k+1} + B_{k,2} \) and \( \zeta(2mn)A_k = B_{k+1,m} + B_{k,m+1} \) proving the result by induction. For a proof using generating functions, see [BBB1997]. We thank J. Zhao for the reference). In our very limited numerical search (weight \( \leq 50 \) for depth 2, and even lower for depths 3, 4), as well as limited search of the vast literature, we did not find any other zeta-like tuples. We do not know whether there are any more examples, or conjectures based on theoretical or numerical evidence.

For Euler’s multizeta, all even weights \( > 2 \) are eulerian in depth more than one, as \( \zeta(2k) \) is eulerian. In our case, for \( q \geq 3 \), even \( \zeta(2,2) \) is not eulerian by [T2004 Thm. 5.10.12], and this conjecture predicts much stringent weight conditions. It is conjectured for the Euler multizeta that the eulerian case occurs only in even weights. In our case, we know by [Ch2012] that the eulerian case occurs only in ‘even’ weights. For the Euler’s multizeta, \( \zeta(2n, 2n) \) are eulerian of weight 4n and depth 2, though weight 4n + 2 does not seem to be eulerian weight in depth 2.

(1) The weight \( p^m(q^k - 1) \), with \( m \geq 0 \) for eulerian value can occur with primitive tuples, e.g., \( q = 2 \) and \( (1,1), (1,3), (3,5) \) or \( q = 3 \) and \( (2,4) \).

(2) The parts 3 and 4 are known for depth 1, and the occurrence in predicted weights is either proved in our main theorem or also follows from the higher depth families conjectures below. So the ‘smallest’ is the real conjectural part. More data may allow to conjecture the depth dependence of possible \( m \) and \( k \) in the first part.

(3) While the Theorem 5.1 shows that weights \( q, q^2, q^3 \) are zeta-like weights for any \( q \), the last part suggests that weight \( q^4 \) is not a weight of a zeta-like tuple (in depth more than one, of course).

Conjecture 4.4. Depth 2, weight at most \( q^2 \) All zeta-like primitive tuples of weight at most \( q^2 \) and depth 2 are exactly \( (i,j(q-1)), i = 1, \ldots, q, j = i, \ldots, [q^2 - i]/(q-1) \) equals \( q + 1 \) or \( q \), depending if \( i = q \) or \( i < q \):

\[
\begin{align*}
(1, q-1) & \quad (1,2(q-1)) & \quad (1,3(q-1)) & \quad (1,4(q-1)) & \ldots & \quad (1,(q+1)(q-1)) \\
(2,2(q-1)) & \quad (2,3(q-1)) & \quad (2,4(q-1)) & \ldots & \quad (2,(q+1)(q-1)) \\
(3,3(q-1)) & \quad (3,4(q-1)) & \ldots & \quad (3,(q+1)(q-1)) \\
& \quad \vdots & \quad \vdots & \quad \vdots & \quad (q,q(q-1))
\end{align*}
\]

Note that our theorems imply that those tuples are zeta-like. The converse is the conjectural part.

Conjecture 4.5. \( q = 2 \), Depth 2 Let \( q = 2 \), the zeta-like (eulerian equivalently) primitive tuples of depth two are exactly \( (1,1), (1,3), (3,5) \) and \( (2^n-1, 2^n), (2^n, 2^{n+1}+2^n-1) \).

Again, our theorems imply all of these are eulerian, the converse is conjectural. (This seems to be true up to weight 128 from the numerical data). Note that
4.1, 4.2, 4.3 part (1) and 4.5 conjecturally completely describe all eulerian tuples, if \( q = 2 \), and remark after 4.2 reduces the general \( q \) eulerian case to depth 2. We are trying to verify the guess that all the depth 2 primitive eulerian tuples are exactly (covered by Theorem 3.1) \( \zeta(q - 1, (q - 1)^2) \), \( \zeta(q^n - 1, (q - 1)q^n) \) and \( \zeta(q^n(q - 1), q^{n+2} - 1 - q^n(q - 1)) \), for \( q > 2 \). (These miss (1, 3), (3, 5) for \( q = 2 \)).

**Conjecture 4.6. Conjectural zeta-like families of arbitrary depth**

1. For any \( q, n \geq 1 \) and \( r \geq 2 \), we have
   \[
   \zeta(q^n - 1, (q - 1)q^n, \ldots, (q - 1)q^{n+r-2}) = \frac{[n + r - 2][n + r - 3] \cdots [n]}{[1][q^{n+r-2}][2][q^{n+r-3}] \cdots [r - 1]q^n} \zeta(q^{n+r-1} - 1).
   \]

2. For any \( q, n \geq 0 \),
   \[
   \zeta(1, q^2 - 1, (q - 1)q^2, \ldots, (q - 1)q^{n+1}) = \frac{[n + 2] - 1}{\ell_1[n + 2]} \frac{1}{\ell_1(q-1)q^n \ell_2(q-1)q^{n-1} \cdots \ell_{n-1}(q-1)q} \zeta(q^{n+2}).
   \]

3. For \( q > 2, n \geq 0 \) and \( r \geq 2 \),
   \[
   \zeta((q - 1)q^n - 1, (q - 1)q^{n+1}, \ldots, (q - 1)q^{n+r-1})
   \]
   equals
   \[
   \frac{(-1)^{r+1}[n + r - 1][n + r - 2] \cdots [n + 1]}{[1][q^{r+1} - 1]q^n [2][q^{r+2} - 1]q^{n-1} \cdots [r - 1](q - 1)q^n} \zeta(q^{n+r} - q^n - 1).
   \]

It seems quite likely that these families can be proved by a proof similar to that of Theorem 5.2 below, but this has not been carried out yet. Note also that in the depth 2 case, all of these are proved in Theorem 3.1. (Here the second one reduces for \( n = 0 \) to \( 5 \) by usual conventions on empty products, sums, patterns and indexing).

5. **Proofs**

The following formulas, which are consequence of Theorems 1 and 3 in [LT], will be used in the proof of the main theorem.

1. For \( 1 \leq s < q \) and \( 0 \leq k_i < k \) with \( 1 \leq i \leq s \), we have

   \[
   S_d(q^k - \sum_{i=1}^{s} q^{k_i}) = \ell_d^{(s-1)q^k} S_d(q^k - q^{k_1}) \cdots S_d(d, q^k - q^{k_n}).
   \]

2. For \( 1 \leq s \leq q \), and any \( 0 \leq k_i \leq k \), with \( 1 \leq i \leq s \), we have

   \[
   S_{<d}(\sum_{i=1}^{s} (q^k - q^{k_i})) = \prod_{i=1}^{s} S_{<d}(q^k - q^{k_i}).
   \]

We also recall Carlitz’ evaluations (see e.g., [T2009 3.3.1, 3.3.2])

1. \( S_d(a) = 1/\ell_d^a, \quad (a \leq q) \)
2. \( S_d(q^j - 1) = \ell_{d+j-1}/\ell_{d+j-1}^q \)
3. \( S_{<d}(q^j - 1) = \ell_{d+j-1}/\ell_{d+j-1}^q \).
Proof of Theorem 3.1. Let \( a = q^n - \sum_{i=1}^s q^k_i \) and \( b = (q - 1)q^n \). By definition, we have

\[
\zeta(a, b) = \sum_{d=1}^\infty S_d(a, b) = \sum_{d=1}^\infty S_d(a)S_{<d}(b).
\]

Using (7), (8), (10) and (11), by straight calculations we get

\[
S_d(a)S_{<d}(b) = \frac{(-1)^s}{\ell^d_1} \prod_{i=1}^s [n - k_i]q^d_i S_{d-1}(a + b).
\]

By summing over \( d \) the claim (11) follows. The proofs of claims (2) and (6) are similar, once we note that for (2) we have

\[
a + b = q^{n+2} - (q - s_1 - s_2)q^{n+1} - \sum_{i=1}^s (q^{n+1} - q^{k_i}),
\]

and for (6), the requirement \( q > 2 \) guarantees that the formula for \( S_d(q^{n+1} - q^n) \) can be applied.

Now, let \( a = q^2 - (q - 1)q^2 + (q^2 - 1) \) and \( b = q^2 - (q - 1) \). Using formulas (7) and (8) again, a straight calculations yields

\[
S_d(a, b) = \frac{1}{\ell^d_1 - \ell^d_2} \frac{t^q - t^{q+2}}{\ell^d_{d-1}}.
\]

Recall that the inverse around origin of the Carlitz exponential \( e(z) \) is the Carlitz logarithm \( \log(z) = \frac{z^q}{\ell^d} \) and it satisfies \( t \log(z) = \log(tz) = \log(z^q) \). Therefore, \( t \log(1) = \log(t) + \log(1) \) or equivalently \( \log(t) = (t - 1) \log(1) \). Since \( \zeta(1) = \log(1) \) and \( \zeta(1)^q = \zeta(q^3) \), by summing over \( d \) we get

\[
\sum_{d=1}^\infty \frac{t^q - t^{q+2}}{\ell^d_{d-1}} = t^q \sum_{d=0}^\infty \frac{1}{\ell^d_1} - \sum_{d=0}^\infty \frac{t^{d+3}}{\ell^d_1} = t^q \log(1)^q - \log(t)^q
\]
\[
= t^q \zeta(q^3) - (t^q - 1) \zeta(q^3)
\]
\[
= (-2)^q + 1) \zeta(q^3),
\]
and claim (3) follows.

Now, for (4), let \( a = 2q - 1 \) and \( b = q^2 - q + (q - 1)(q^2 - 1) \). We have

\[
S_d(a, b) = \frac{1}{\ell^d_1 + \ell^d_2 + 1} \frac{1}{\ell^d_{d-1}} (-[d + 1]^q)
\]

By summing over \( d \geq 1 \), we obtain

\[
\sum_{d=1}^\infty \frac{t^q - t^{q+2}}{\ell^d_{d-1}} = (-2)^q + 1) \zeta(q^3),
\]
and the result follows.

Finally, (5) is proved in [T2009 Thm. 5] \( \square \)

Proof of Theorem 3.2. We claim that

\[
S_d(1, q - 1, q(q - 1), \ldots, q^n(q - 1)) = \frac{1}{\ell_{n+1}^{q-1} \ell_{n}^{q(q-1)} \ldots \ell_{1}^{q^{n-1}(q-1)}} S_{d-(n+1)}(q^{n+1}).
\]
Summing the claimed equality over \(d\) proves the Theorem.

For \(n = 1\) and all \(d\), this is proved in [T2009, 3.4.6]. We prove it by induction by assuming it for \(n\) replaced by \(n - 1\), and considering it for \(n\) as claimed.

For \(d < n + 1\) both sides are zero, and for \(n = d + 1\), it follows using (we use these often below) (9) for \(a = 1\), \(q - 1\) together with the obvious \(\mathcal{S}_d(j) = \mathcal{S}_d(1)\). We write \(S_n(d) := \sum_{j=0}^{d-1} \mathcal{S}_j(q(q-1), \ldots, q^n(q-1))\) and \(f_n(d) := \frac{\ell_d/\ell_{n+1}\ell_{n-1} \cdots \ell_1^{q^n(q-1)} \ell_1^{q^n+1} d}{d-1} \). It is enough to show that \(S_n(d) = f_n(d)\) for all \(d > n + 1\). Now \(S_n(d+1) - S_n(d)\) is

\[
S_d(q-1, \ldots, q^n(q-1)) = S_d(1) \sum_{j=0}^{d-1} \mathcal{S}_j(q(q-1), \ldots, q^n(q-1)) = S_d(1)S_{n-1}(d)^q.
\]

Now \(S_{n-1}(d) = f_{n-1}(d)\) by induction, and a simple manipulation shows that \(f_n(d+1) - f_n(d) = S_d(1)f_{n-1}(d)^q\) thus completing the proof of the claim and the theorem by induction.

Remarks It might be worthwhile to point out a very special low weight case of (2) of Theorem 3.1 that \((n, m(q-1))\) is zeta-like, if \(1 \leq n \leq q\) and \(n \leq m \leq q\).

6. Data

Theory of continued fractions for function fields was first developed by Emil Artin in his thesis. (See [T2004, Chap. 9] for a survey.). We use them to find the zeta-like values as follows. We calculate the multizeta divided by zeta of same weight numerically (i.e., approximation where we use first few degrees rather than all), and calculate its continued fraction. If the ratio of actual values is rational, the continued fraction thus calculated will be same as the continued fraction of this rational for the first few partial quotients and then there will be very large partial quotient indicating small error in approximation. We detect this and then we double check by increasing the precision that we do get the stabilized part, followed by increasing partial quotient (corresponding to reducing error), followed by non-stabilized part.

The calculation was done (in stages, with guesses verified with more data) over several months by programing in SAGE and using laptops and mainframes. In lower depths, and small weights, small \(q\)’s the calculation was exhaustive (i.e., going through all tuples looking for zeta-like values), and sometimes guesses of higher depth, weight, \(q\)’s were checked separately to some extent. For \(q = 2\), depth 2 and 3 and weight up to 128 and 32, respectively, and for \(q = 3\), depth 2 and 3 and weight up to 81, calculation was exhaustive. For \(q = 4, 5\), depths 2 and 3, we went through all weights up to \(q^3\), but assuming that \(s_i\) is ‘even’ (called restrictive search) for \(i \geq 2\), and \(s_i \leq s_{i+1}\). However, we checked to some extent that the tuples not satisfying the increasing condition are not zeta-like. Also, we decreased precision often, otherwise the calculation would have taken much more time.

We only list primitive tuples. The tuples marked with * are covered by the theorems.
6.1. **Data for** $q = 2$. Zeta like tuples of depth 2 and weight at most 128.

| Depth 2, Weight up to 128 | Depth 2, Weight up to 128 |
|--------------------------|--------------------------|
| (1, 1)*                  | (1, 2)*                  |
| (1, 3)*                  | (2, 5)*                  |
| (3, 4)*                  | (3, 5)*                  |
| (4, 11)*                 |                          |
| (7, 8)*                  | (8, 23)*                 |
| (15, 16)*                | (31, 32)*                |
| (16, 47)*                | (32, 95)*                |
| (63, 64)*                |                          |

$q = 2$. Zeta like tuples of depth 3, weight at most $q^5 = 32$, and more.

| Depth 3, Weight up to 32 | Depth 3, Weight up to 32 |
|--------------------------|--------------------------|
| (1, 1, 2)*               | (1, 2, 4)*               |
| (1, 2, 5)*               | (1, 3, 4)                |
| (3, 4, 8)*               | (7, 8, 16)               |
| (15, 16, 32)             | (31, 32, 64)             |

$q = 2$. Some zeta like tuples of depth 4.

| Depth 4, Weight up to 64 | Depth 4, Weight up to 64 |
|--------------------------|--------------------------|
| (1, 1, 2, 4)*            | (1, 2, 4, 8)             |
| (1, 3, 4, 8)             | (7, 8, 16, 32)           |
| (15, 16, 32, 64)         | (31, 32, 64, 128)        |

$q = 2$. Some zeta-like tuples of depth 5.

| Depth 5, Weight up to 128 | Depth 5, Weight up to 128 |
|---------------------------|---------------------------|
| (1, 1, 2, 4, 8)*          | (1, 2, 4, 8, 16)          |
| (1, 3, 4, 8, 16)          | (3, 4, 8, 16, 32)         |
| (7, 8, 16, 32, 64)        | (15, 16, 32, 64, 128)     |
| (31, 32, 64, 128, 256)    |                           |

$q = 2$. Some zeta-like tuples of depth 6.

| Depth 6, Weight up to 256 | Depth 6, Weight up to 256 |
|---------------------------|---------------------------|
| (1, 1, 2, 4, 8, 16)*      | (1, 2, 4, 8, 16, 32)      |
| (1, 3, 4, 8, 16, 32)      | (7, 8, 16, 32, 64, 128)   |
| (15, 16, 32, 64, 128, 256)|                           |
| (31, 32, 64, 128, 256, 512)|                          |

6.2. **Data for** $q = 3$. Zeta-like tuples of depth 2 and weight up to $q^4 = 81$:

| Weight up to 81 | Weight up to 81 |
|-----------------|-----------------|
| (1, 2)*         | (1, 4)*         |
| (1, 6)*         | (1, 8)*         |
| (2, 4)*         | (2, 6)*         |
| (3, 14)*        | (3, 20)*        |
| (3, 22)*        | (5, 12)*        |
| (5, 18)*        | (5, 20)*        |
| (5, 22)*        | (7, 18)*        |
| (7, 20)*        | (8, 18)*        |
| (9, 44)*        |                |
| (15, 62)*       | (17, 36)*       |
| (17, 54)*       | (18, 62)*       |
| (23, 54)*       | (25, 54)*       |
| (26, 54)*       |                |

Zeta-like tuples of depth 3, weight $\leq q^4 = 81$ and more:

| Weight up to 162 | Weight up to 162 |
|-----------------|-----------------|
| (1, 2, 6)*      | (1, 6, 18)      |
| (2, 6, 18)      | (1, 6, 20)      |
| (1, 8, 18)      | (5, 18, 54)     |
| (7, 18, 54)     | (8, 18, 54)     |
| (17, 54, 162)   | (23, 54, 162)   |

| Weight up to 486 | Weight up to 486 |
|-----------------|-----------------|
| (1, 2, 6, 18)*  | (1, 6, 18, 54)  |
| (2, 6, 18, 54)  | (1, 8, 18, 54)  |
| (5, 18, 54, 162)| (7, 18, 54, 162)|
| (8, 18, 54, 162)|(17, 54, 162, 486)|

6.3. **Data for** $q = 4$. Zeta-like tuples (restricted) of depth 2, weight $\leq q^3 = 64$:

| Weight up to 128 | Weight up to 128 |
|-----------------|-----------------|
| (1, 3)*         | (1, 6)*         |
| (1, 9)*         | (1, 12)*        |
| (1, 15)*        | (2, 9)*         |
| (2, 21)*        | (2, 27)*        |
| (3, 9)*         | (3, 12)*        |
| (4, 27)*        | (4, 39)*        |
| (4, 51)*        | (4, 57)*        |
| (5, 18)*        | (5, 24)*        |
| (5, 27)*        | (7, 24)         |
| (7, 36)*        | (7, 48)*        |
| (7, 51)*        | (7, 54)         |
| (7, 57)*        |                |
| (8, 39)*        | (8, 51)*        |
| (10, 51)*       | (11, 36)*       |
| (11, 48)*       | (11, 51)*       |
| (12, 51)*       | (13, 48)*       |
| (13, 51)*       | (15, 48)*       |

Zeta-like tuples (restricted search) of depth 3 up to weight $q^3 = 64$:

| Weight up to 128 | Weight up to 128 |
|-----------------|-----------------|
| (1, 3, 12)*     | (1, 6, 24)      |
| (1, 12, 48)     | (3, 12, 48)     |
| (1, 12, 51)     | (1, 15, 48)     |
6.4. Data for \(q = 5\). Zeta-like tuples (restricted) of depth 2, weights \(\leq 125\):

\[
\begin{align*}
(1, 4)^* & \quad (1, 8)^* & \quad (1, 12)^* & \quad (1, 16)^* & \quad (1, 20)^* & \quad (1, 24)^* \\
(2, 8)^* & \quad (2, 12)^* & \quad (2, 16)^* & \quad (2, 20)^* & \quad (3, 12)^* & \quad (3, 16)^* \\
(3, 20)^* & \quad (4, 16)^* & \quad (4, 20)^* & \quad (5, 44)^* & \quad (5, 64)^* & \quad (5, 68)^* \\
(5, 84)^* & \quad (5, 88)^* & \quad (5, 92)^* & \quad (5, 104)^* & \quad (5, 108)^* & \quad (5, 112)^* \\
(5, 116)^* & \quad (9, 40) & \quad (9, 60) & \quad (9, 64) & \quad (9, 80) & \quad (9, 84) \\
(9, 88) & \quad (9, 100)^* & \quad (9, 104) & \quad (9, 108) & \quad (9, 112) & \quad (9, 116)^* \\
(10, 64)^* & \quad (10, 84)^* & \quad (10, 88)^* & \quad (10, 104)^* & \quad (10, 108)^* & \quad (10, 112)^* \\
(13, 60) & \quad (13, 80) & \quad (13, 84) & \quad (13, 100)^* & \quad (13, 104) & \quad (13, 108) \\
(13, 112) & \quad (14, 60) & \quad (14, 80) & \quad (14, 84) & \quad (14, 100)^* & \quad (14, 104) \\
(14, 108) & \quad (15, 84)^* & \quad (15, 104)^* & \quad (15, 108)^* & \quad (17, 80) & \quad (17, 100)^* \\
(17, 104) & \quad (17, 108) & \quad (18, 80) & \quad (18, 100)^* & \quad (18, 104) & \quad (19, 80)^* \\
(19, 100)^* & \quad (19, 104)^* & \quad (20, 104)^* & \quad (21, 100)^* & \quad (21, 104)^* & \quad (22, 100)^* \\
(23, 100)^* & \quad (24, 100)^* \\
\end{align*}
\]

\(q = 5\). Some zeta-like tuples.

\[
\begin{align*}
(1, 4, 20)^* & \quad (1, 20, 104) & \quad (1, 24, 100) & \quad (2, 20, 100) \\
(4, 20, 100) & \quad (3, 20, 100) & \quad (3, 20, 100, 500) & \quad (19, 100, 500) \\
(19, 100, 500, 2500) & \quad \\
\end{align*}
\]

Summary of depth and weights classified by eulerian and zeta-like.

| \(q\) | depth | Eulerian weights | Zeta-like weights |
|------|-------|-----------------|------------------|
| 2    | 2     | 2, 3, 4, 7, 8, 15, 31, 63 | 3, 5, 7, 9, 17, 23, 25, 27, 53, 57, 71, 77, 79 |
| 2    | 3     | 4, 7, 8, 15, 31, 63, 127 | 9, 25, 27, 77, 79, 233, 239 |
| 2    | 4     | 8, 15, 16, 31, 63, 127, 255 | 27, 79, 81, 239, 241, 719 |
| 2    | 5     | 16, 31, 32, 63, 127, 255, 511 | 4, 7, 10, 11, 13, 16, 23, 29, 31, 32, 43, 46, 47, 55, 58, 59, 61, 62, 64 |
| 2    | 6     | 32, 63, 64, 127, 255, 511, 1023 | 16, 31, 61, 64 |
| 3    | 2     | 6, 8, 26, 80 | 5, 9, 10, 13, 14, 15, 17, 18, 19, 21, 22, 23, 25, 49, 69, 73, 74, 89, 93, 94, 97, 98, 99, 109, 113, 114, 117, 118, 119, 121, 122, 123, 125 |
| 3    | 3     | 26, 80 | 25, 122, 123, 125, 619 |
| 3    | 4     | 80, 242 | 623, 3119 |
Acknowledgments. The first author is grateful to the Centro de Cómputo del Cuerpo Académico Modelado y Simulación Computacional de Sistemas Físicos (UADY-CA-101) de la Universidad Autónoma de Yucatán as well as to the University of Arizona for allowing us to use their computer servers for our computations.

References

[AT2009] G.W. Anderson and D. Thakur. Multizeta values for $\mathbb{F}_q[t]$, their period interpretation and relations between them. International Mathematics Research Notices, 2009(11):2038–2055, May 2009.

[BBB1997] J.M. Borwein, D.M. Bradley and D.J. Broadhurst. Evaluations of k-fold Euler/Zagier sums: a compendium of results for arbitrary k, Electron. J. Combinatorics 4(2) (1997) R5, 1-21.

[C1935] L. Carlitz. On certain functions connected with polynomials in a Galois field. Duke Math. J., 1(2):137–168, 1935.

[CB2001] P. Cartier. Fonctions Polylogarithmes, nombres polyzeta, et groupes pro-unipotents. Sem. Bourbaki no. 881, Mars (2001).

[Ch2012] C.-Y. Chang. Linear independence of monomials of multizeta values in positive characteristic. ArXiv e-prints, July 2012.

[CPY] C.-Y. Chang, M. Papanikolas and J. Yu. A criterion of Eulerian multizeta values in positive characteristic. Preprint 2013.

[CY2007] C.-Y. Chang and J. Yu. Determination of algebraic relations among special zeta values in positive characteristic, Adv. Math. 216 (2007), 321-345.

[GY1996] D. Goss, Basic structures of Function Field Arithmetic, Springer Verlag, NY 1996.

[L2011] J. A. Lara Rodríguez. Relations between multizeta values in characteristic p. J. Number Theory, 131(4):2081–2099, 2011.

[L2012] J. A. Lara Rodríguez. Special relations between function field multizeta values and parity results. Journal of the Ramanujan Mathematical Society, 27(3):275–293, 2012.

[LT] J. A. Lara Rodríguez and D. Thakur. Multiplicative relations between coefficients of logarithmic derivatives of $\mathbb{F}_q$-linear functions and applications. Preprint.

[T2004] D. Thakur, Function Field Arithmetic, World Sci., NJ, 2004.

[Tbanff] D. Thakur. Multizeta in function field arithmetic. To appear in the proceedings of Banff workshop to be published by EMS.

[T2009] D. Thakur, Relations between multizeta values for $\mathbb{F}_q[t]$, International Mathematics Research Notices, 2009(12):2318–2346.

[T2010] D. Thakur. Shuffle Relations for Function Field Multizeta Values. Int. Math. Res. Not. IMRN, 2010(11):1973–1980, 2010.

[Z2012] D. Zagier. Evaluation of the multizeta values $\zeta(2,\cdots,2,3,2,\cdots,2)$. Annals of Math 175 (2012), 977-1000.