W^{2,\delta} ESTIMATES FOR SOLUTION SETS OF FULLY NONLINEAR ELLIPTIC INEQUALITIES ON C^{1,\alpha} DOMAINS

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Abstract. In this paper, we establish boundary W^{2,\delta} estimates for u \in S(\lambda, \Lambda, f) on C^{1,\alpha} domains with f \in L^p as n < p < \infty and C^{1,\alpha} boundary values. Instead of straightening out the boundary, our main idea is to obtain boundary W^{2,\delta} estimates from interior W^{2,\delta_0} estimates and Whitney decomposition for some \delta \leq \delta_0.

1. Introduction

In this paper, we establish boundary W^{2,\delta} estimates for u \in S(\lambda, \Lambda, f) on C^{1,\alpha} domains. Some notions and notations concerning S(\lambda, \Lambda, f) are listed as follows. We refer to [2] and [3] for more details.

Definition 1.1. Let 0 < \lambda \leq \Lambda < \infty. For M \in S(n) (the space of real n \times n symmetric matrices), the Pucci’s extremal operators \mathcal{M}^- and \mathcal{M}^+ are defined as follows:

\[ \mathcal{M}^-(M, \lambda, \Lambda) = \lambda \sum_{e_i > 0} e_i + \Lambda \sum_{e_i < 0} e_i, \]
\[ \mathcal{M}^+(M, \lambda, \Lambda) = \Lambda \sum_{e_i > 0} e_i + \lambda \sum_{e_i < 0} e_i, \]

where \( e_i = e_i(M) \) are the eigenvalues of \( M \).

Definition 1.2. Let \( p > n/2 \) and \( f \in L^p_{\text{loc}}(\Omega) \). A function \( u \in C(\Omega) \) satisfies \( \mathcal{M}^-(D^2u, \lambda, \Lambda) \leq f(x) \) (resp. \( \mathcal{M}^+(D^2u, \lambda, \Lambda) \geq f(x) \)) in \( \Omega \) in the viscosity sense if \( x_0 \in \Omega, \phi \in W^{2,p}_{\text{loc}}(\Omega) \) and \( u - \phi \) has a local minimum (resp. maximum) at \( x_0 \) imply

\[ \text{ess lim inf}_{x \to x_0} \{ \mathcal{M}^-(D^2\phi, \lambda, \Lambda) - f(x) \} \leq 0, \]
\[ \text{ess lim sup}_{x \to x_0} \{ \mathcal{M}^+(D^2\phi, \lambda, \Lambda) - f(x) \} \geq 0. \]

Definition 1.3. Let \( p > n/2 \) and \( f \in L^p_{\text{loc}}(\Omega) \). We denote by \( S(\lambda, \Lambda, f) \) the space of continuous functions \( u \in \Omega \) such that
\[ \mathcal{M}^-(D^2u, \lambda, \Lambda) \leq f(x) \leq \mathcal{M}^+(D^2u, \lambda, \Lambda) \]
in the viscosity sense in \( \Omega \).

Our main result is as follows.

Key words and phrases. Fully Nonlinear Elliptic Equations, W^{2,\delta} Estimates, Whitney decomposition, C^{1,\alpha} Domains.
Theorem 1.4. Let $0 < \alpha < 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{1,\alpha}$ boundary portion $T \subset \partial \Omega$. Suppose that $u$ satisfies
\[
\begin{cases}
  u \in S(\lambda, \Lambda, f) & \text{in } \Omega; \\
  u = g & \text{on } T,
\end{cases}
\]
where $f \in L^p(\Omega)$ for $n < p < \infty$ and $g \in C^{1,\alpha}(T)$. Then there exists a positive constant $\delta$ depending on $n, \lambda, \Lambda, \alpha$ and $p$ such that for any domain $\Omega' \subset \subset \Omega \cup T$, we have
\[
||u||_{W^{2,\delta}(\Omega')} \leq C \left( ||u||_{L^\infty(\Omega)} + ||f||_{L^p(\Omega)} + ||g||_{C^{1,\alpha}(T)} \right),
\]
where $C$ depends on $n, \lambda, \Lambda, \alpha, \delta, p, T, \Omega'$ and $\Omega$.

Remark 1.5. (i) When $T = \partial \Omega$ in Theorem 1.4, we may obtain a global $W^{2,\delta}(\Omega)$ estimate on the bounded $C^{1,\alpha}$ domain.

(ii) Interior $W^{2,\delta_0}$ estimates and boundary Hölder gradient estimates are used in the proof of Theorem 1.4, where $\delta_0$ depends on $n, \lambda$ and $\Lambda$. Interior $W^{2,\delta_0}$ estimates need $p > n - \epsilon$ with $\epsilon$ depending on $n, \lambda$ and $\Lambda$. (See [2] for more details.) Boundary Hölder gradient estimates need $p > n$. (See the following Corollary 3.4 for more details.)

$W^{2,p}$ regularity for fully nonlinear elliptic equations has been studied extensively during recent decades. For $u \in S(\lambda, \Lambda, f)$ and $f \in L^n$, Caffarelli [2] proved interior $W^{2,\delta_0}$ estimates with $\delta_0 > 0$ depending on $n, \lambda$ and $\Lambda$, which was first discovered by Lin [12] for linear elliptic equations in nondivergence form. Then, interior $W^{2,p}$ estimates for fully nonlinear elliptic equations $F(D^2u, x) = f(x)$ with $f \in L^p$ and $n < p < \infty$ were established under the following two assumptions: The homogeneous equations with constants coefficients $F(D^2w, x_0)$ have interior $C^{1,1}$ estimates for any $x_0 \in B_1$ and the oscillation of $F(M, x)$ in $x$ is small in $L^n$ sense. (See [2].) Escauriaza [5] extended the above estimates to the optimal range of exponents $n - \epsilon < p < \infty$ with $\epsilon$ depending on $n, \lambda$ and $\Lambda$. In [9], Li and Zhang relaxed the Caffarelli’s assumptions of $F$ to: $F(D^2u, x)$ have interior $W^{2,p_1}$ estimates for any $x_0 \in B_1$ and the oscillation of $F(M, x)$ in $x$ is small in $L^{p_2}$ sense, where $n - \epsilon < p_0 < p < p_1 \leq \infty$ and $1/p_1 + 1/p_2 = 1/p_0$. Winter [14] extended Caffarelli’s interior $W^{2,p}$ regularity to boundary and established boundary $W^{2,p}$ estimates on flat domains and then on $C^{1,1}$ domains alongside a flattening argument. For oblique boundary value problem for $F(D^2u, x) = f(x)$, based on pointwise boundary $C^{2,\alpha}$ regularity in [11], Byun and Han [1] derived the boundary $W^{2,p}$ estimates on flat domains and then on $C^{2,\alpha}$ domains by straightening out boundary.

The technique of straightening out boundary is widely applied to deal with boundary regularity. However, it is no longer applicable for the $C^{1,\alpha}$ domains. The reason is as follows: If the technique of straightening out boundary is used, then for any $x_0 \in \partial \Omega$, one should find a neighborhood $\mathcal{N}$ of $x_0$ and a diffeomorphism $\psi$ from $\mathcal{N}$ onto $B_1$ such that $\psi(\mathcal{N} \cap \Omega) = B_1^+$ and $\psi(\mathcal{N} \cap \partial \Omega) = B_1 \cap \{x^n = 0\}$. Writing $y = \psi(x)$, $\tilde{u}(y) = u(x)$ and $\tilde{f}(y) = f(x)$ for $x \in \mathcal{N} \cap \Omega$ and $y \in B_1^+$, we calculate
\[
D^2u = D\psi^T(D^2\tilde{u} \circ \psi)D\psi + ((D\tilde{u} \circ \psi)D\psi)_{1 \leq i,j \leq n}.
\]
Then $\psi$ need to be $C^{1,1}$ in view of the second term.
In this paper, instead of straightening out the boundary, we establish boundary $W^{2,\delta}$ estimates for $u \in S(\lambda, \Lambda, f)$ on $C^{1,\alpha}$ domains from interior $W^{2,\delta_0}$ estimates and Whitney decomposition. Whitney decomposition is an effective tool for obtaining boundary estimates from interior estimates. For instance, Cao, Li and Wang \cite{CaoLiWang} utilized it to prove the optimal weighted $W^{2,p}$ estimates for elliptic equations with non-compatible conditions; Li, Li and Zhang \cite{LiLiZhang} utilized it to prove boundary $W^{2,p}$ estimates for linear elliptic equations on $C^{1,\alpha}$ domains.

We illustrate our idea as follows. Let $\{Q_k\}_{k=1}^\infty$ be Whitney decomposition of $\Omega_1$ (Suppose $0 \in \partial \Omega$ and denote $\Omega_r = \Omega \cap B_r$ and $\tilde{Q}_k = \frac{r}{2}Q_k$ the $\frac{r}{2}$—dilation of $Q_k$ with respect to its center. Let $u$ satisfy \[(\text{I.1})\). Deducing from Hölder’s inequality and interior $W^{2,\delta_0}$ estimates that for any $\delta \leq \delta_0$,

$$||D^2u||_{L^\infty(Q_k)}^\delta \leq C d_k^{n(1-\frac{\delta}{n})} ||D^2(u-l)||_{L^\infty(Q_k)}^\delta$$

$$\leq C \left( d_k^{n-2\delta}||u-l||_{L^\infty(Q_k)}^\delta + d_k^{n-\frac{\alpha n}{p}} ||f||_{L^p(Q_k)}^\delta \right)$$

for some universal constant $C$ and any affine function $l$, where $d_k$ denotes the diameter of $Q_k$. By $C^{1,\alpha_0}$ estimates up to the boundary, we can take $l$ such that

$$|u(x) - l(x)| \leq C \text{dist}(x, \partial \Omega)^{1+\alpha_0}, \forall x \in \Omega_1/2$$

for some positive constants $C$ and proper $0 < \alpha_0 < 1$. It then follows from Whitney decomposition that

$$||u-l||_{L^\infty(Q_k)}^\delta \leq C d_k^{(1+\alpha_0)\delta}. $$

Combining the above estimates, we obtain

$$||D^2u||_{L^\infty(Q_k)}^\delta \leq C \left( d_k^{-\delta+\alpha_0\delta+n} + d_k^{n-\frac{\alpha n}{p}} ||f||_{L^p(Q_k)}^\delta \right).$$

By taking sum on both sides with respect to $k$, we obtain the desired boundary $W^{2,\delta}$ estimate for any $\delta < \min \{1/(1-\alpha_0), p/n\}$ that guarantees $\sum_k (d_k^{-\delta+\alpha_0\delta+n} + d_k^{n-\frac{\alpha n}{p}})$ is convergent.

The paper is organized as follows. In Section 2, Whitney decomposition and its relevant properties are concluded. In Section 3, we demonstrate some basic estimates for $u \in S(\lambda, \Lambda, f)$ concerning interior $W^{2,\delta_0}$ estimates and pointwise boundary Hölder gradient estimates. In Section 4, we prove Theorem \text{[[1.4]]}.

We end this section by introducing some notations.

\textbf{Notation.}

1. $e_i = (0, ..., 0, 1, ..., 0) = i^{th}$ standard coordinate vector.
2. $x' = (x_1, x_2, ..., x_n^1)$ and $x = (x', x^n)$.
3. $\mathbb{R}_+^n = \{ x \in \mathbb{R}^n : x_0 > 0 \}$.
4. $B_r(x_0) = \{ x \in \mathbb{R}^n : |x-x_0| < r \}$ and $B_r^+(x_0) = B_r(x_0) \cap \mathbb{R}_+^n$.
5. $B_r' = \{ x \in \mathbb{R}^{n-1} : |x'| < r \}$ and $T_r = \{(x',0) : x' \in B_r' \}$.
6. $\Omega_r(x_0) = \Omega \cap B_r(x_0)$ and $(\partial \Omega)_r(x_0) = \partial \Omega \cap B_r(x_0)$. We omit $x_0$ when $x_0 = 0$.
7. $\text{diam} E = \sup \{|x-y|, x, y \in E\}, \forall E \subset \mathbb{R}^n$.
8. $\text{dist}(E, F) = \inf \{|x-y|, x \in E, y \in F\}, \forall E, F \subset \mathbb{R}^n$. 
2. Whitney decomposition

In what follows, by a cube we mean a closed cube in $\mathbb{R}^n$, with sides parallel to the axes. We say two such cubes are disjoint if their interiors are disjoint.

**Lemma 2.1.** (Whitney decomposition) Let $\Omega$ be a non-empty open set in $\mathbb{R}^n$. Then there exist two sequences of cubes $Q_k$ (called the Whitney cubes of $\Omega$) and $\tilde{Q}_k = \frac{6}{5}Q_k$ (dilation of $Q_k$ with respect to center of $Q_k$) such that

(i) $\Omega = \bigcup_{k=1}^{\infty} Q_k = \bigcup_{k=1}^{\infty} \tilde{Q}_k$;
(ii) The $Q_k$ are mutually disjoint;
(iii) $d_k \leq \text{dist}(Q_k, \partial \Omega) \leq 4d_k$, where $d_k = \text{diam} Q_k$;
(iv) Each point of $\Omega$ is contained in at most $12^n$ of the cubes $\tilde{Q}_k$.

For the proof of the above lemma, we refer to Theorem 1 and Proposition 3 in Section VI.1 in [13].

From now on, we make the following assumption on $\Omega$:

(A) $0 \in \partial \Omega$ and there exists a continuous function $\varphi : B'_1 \to \mathbb{R}$ such that

$$\Omega_1 = \{x^n > \varphi(x'), |x| < 1\} \quad \text{and} \quad (\partial \Omega)_1 = \{x^n = \varphi(x'), |x| < 1\}.$$ 

Recall that $\Omega_1 = \Omega \cap B_1$ and $(\partial \Omega)_1 = \partial \Omega \cap B_1$.

Let $\{Q_k\}_{k=1}^{\infty}$ be Whitney decomposition of $\Omega_1$, $d_k = \text{diam} Q_k$ and $\tilde{Q}_k = \frac{6}{5}Q_k$.

The following lemmas were first proved in [8] and we give their proofs here for completeness.

**Lemma 2.2.** Suppose that $\Omega$ satisfies Assumption (A). Then

$$\Omega_{1/12} \subset \bigcup_{\tilde{Q}_k \subset \Omega_{1/4}} Q_k.$$  \hfill (2.1)

**Proof.** If not, there exist a point $x \in \Omega_{1/12}$ and a cube $Q_k$ such that $x \in Q_k$ but $\tilde{Q}_k \not\subset \Omega_{1/4}$. It follows that there exists a point $y \in \tilde{Q}_k$ with $|y| \geq 1/4$. Then we deduce from Lemma 2.1 (iii) that

$$\text{dist}(Q_k, \partial \Omega_1) \geq \text{diam} Q_k = 5\text{diam} \tilde{Q}_k/6 \geq 5(|y| - |x|)/6 \geq 5/36 > 1/12.$$ 

On the other hand, since $x \in Q_k \cap \Omega_{1/12}$ and $0 \in \partial \Omega$,

$$\text{dist}(Q_k, \partial \Omega_1) \leq |x| \leq 1/12.$$ 

Thus we get a contradiction. \hfill $\Box$

**Lemma 2.3.** Suppose that $\Omega$ satisfies Assumption (A) with $\varphi \in C^{0,1}(B'_1)$. If $q > n - 1$, then

$$\sum_{\tilde{Q}_k \subset \Omega_{1/4}} d_k^q \leq C,$$ \hfill (2.2)

where $C$ depends on $n$, $q$ and $\|\varphi\|_{C^{0,1}(B'_1)}$. 
Proof. Set
\[ F^s = \bigcup_{k} \{ Q_k : 2^{-s-1} < d_k \leq 2^{-s}, \bar{Q}_k \subset \Omega_{1/4} \}, \ s = 1, 2, \ldots \] (2.3)

For any \( Q_k \in F^s \), since \( F^s \subset \Omega_{1/4} \), there exists \( y_k \in (\partial \Omega)_{1/2} \) such that
\[ \text{dist}(Q_k, y_k) = \text{dist}(Q_k, (\partial \Omega)_{1}) = \text{dist}(Q_k, \partial \Omega) \leq 4d_k \leq 2^{-s+2}, \]
where we have used Lemma 2.1 (iii). (Recall that \( (\partial \Omega)_r = \partial \Omega \cap B_r \) for \( r > 0 \).) It follows that
\[ \text{dist}(x, (\partial \Omega)_{1}) \leq d_k + \text{dist}(Q_k, y_k) \leq 2^{-s} + 2^{-s+2} \leq 2^{-s+3} \]
for any \( x \in Q_k \) and \( Q_k \in F^s \), which implies
\[ F^s \subset \Omega_{1/4} \cap \{ \text{dist}(x, (\partial \Omega)_{1}) \leq 2^{-s+3} \}. \] (2.4)

By Assumption (A), we have
\[ \Omega_{1/4} \cap \{ \text{dist}(x, (\partial \Omega)_{1}) \leq 2^{-s+3} \} \]
\[ \subset \{ |x'| \leq 1/4, \varphi(x') \leq x^n \leq \varphi(x') + (||\varphi||_{C^{0,1}(B'_1)} + 1)2^{-s+3} \}. \]

Since
\[ |\{ |x'| \leq 1/4, \varphi(x') \leq x^n \leq \varphi(x') + (||\varphi||_{C^{0,1}(B'_1)} + 1)2^{-s+3} \}| \leq C2^{-s}, \]
we have
\[ |\Omega_{1/4} \cap \{ \text{dist}(x, (\partial \Omega)_{1}) \leq 2^{-s+3} \}| \leq C2^{-s}, \]
where \( C \) depends on \( n \) and \( ||\varphi||_{C^{0,1}(B'_1)} \). Therefore, by (2.4),
\[ |F^s| \leq C2^{-s}. \] (2.5)

Observe that
\[ \bigcup_{Q_k \subset \Omega_{1/4}} Q_k = \bigcup_{s=1}^{\infty} \bigcup_{Q_k \in F^s} Q_k. \]

If \( q > n - 1 \), we derive from (2.4) and (2.5) that
\[ \sum_{Q_k \subset \Omega_{1/4}} d_k^q \leq \sum_{s=1}^{\infty} \left\{ \sum_{Q_k \in F^s} (d_k^q, d_k^q) \right\} \leq C \sum_{s=1}^{\infty} \left\{ 2^{-s(q-n)} \sum_{Q_k \in F^s} d_k^q \right\} \]
\[ \leq C \sum_{s=1}^{\infty} 2^{-s(q-n)}|F^s| \leq C \sum_{s=1}^{\infty} 2^{-s(q-n+1)} \leq C, \]
where \( C \) depends on \( n, q \) and \( ||\varphi||_{C^{0,1}(B'_1)} \). \( \square \)

Remark 2.4. Lemma 2.3 is obvious as \( q \geq n \) since
\[ \sum_{Q_k \subset \Omega_{1/4}} d_k^q \leq \sum_{\bar{Q}_k \subset \Omega_{1/4}} d_k^q \leq C_n|\Omega_{1/4}| \]
for some dimensional constant \( C_n \).
3. Preliminary estimates

The following lemma concerns interior $W^{2,\delta_0}$ regularity for $u \in S(\lambda,\Lambda, f)$ and $f \in L^n$. We refer to [2], [3] and [12] for its proof.

**Theorem 3.1.** There exists a positive constant $\delta_0$ depending on $n$, $\lambda$ and $\Lambda$ such that if

$$u \in S(\lambda,\Lambda, f) \quad \text{in } B_1$$

and

$$f \in L^n(B_1),$$

then $u \in W^{2,\delta_0}(B_{1/2})$ and

$$\|u\|_{W^{2,\delta_0}(B_{1/2})} \leq C \left( \|u\|_{L^\infty(B_1)} + \|f\|_{L^n(B_1)} \right),$$

where $C$ depends on $n$, $\lambda$ and $\Lambda$.

The following lemma concerns pointwise boundary $C^{1,\bar{\alpha}}$ estimates for $u \in S(\lambda,\Lambda,0)$ on flat boundaries with zero boundary values. We owe it to Krylov [7] and see more details in Lian and Zhang [11].

**Lemma 3.2.** Let $u$ satisfy

$$\begin{cases}
    u \in S(\lambda,\Lambda,0) & \text{in } B_1^+; \\
    u = 0 & \text{on } T_1.
\end{cases}$$

Then there exists a constant $0 < \bar{\alpha} < 1$ depending on $n, \lambda$ and $\Lambda$ such that $u$ is $C^{1,\bar{\alpha}}$ at $0$, i.e., there exists a constant $a$ such that

$$|u(x) - ax^n| \leq C|x|^{1+\bar{\alpha}}\|u\|_{L^\infty(B_1^+), \forall x \in B_{1/2}^+}$$

and

$$|Du(0)| = |a| \leq C\|u\|_{L^\infty(B_1^+)},$$

where $C$ depends on $n$, $\lambda$ and $\Lambda$.

Based on Lemma 3.2 and compactness method, Lian and Zhang [11] proved the following lemma concerning pointwise boundary $C^{1,\alpha}$ estimates for $u \in S(\lambda,\Lambda,f)$ on $C^{1,\alpha}$ domains with $0 < \alpha < \bar{\alpha}$.

**Lemma 3.3.** Let $\bar{\alpha}$ be as in Lemma 3.2 and $0 < \alpha < \bar{\alpha}$. Suppose that $\Omega$ satisfies Assumption (A) with $\varphi \in C^{1,\alpha}(B_1^\alpha)$ and $u$ satisfies

$$\begin{cases}
    u \in S(\lambda,\Lambda,f) & \text{in } \Omega_1; \\
    u = g & \text{on } (\partial\Omega)_1,
\end{cases}$$

where $g \in C^{1,\alpha}((\partial\Omega)_1)$ and $f \in L^n(\Omega_1)$ satisfying for some constant $K_f$,

$$\|f\|_{L^n(\Omega_1)} \leq K_f r^\alpha$$

for any $r \in (0,1]$.

Then $u$ is $C^{1,\alpha}$ at $0$, i.e., there exists an affine function $l$ such that

$$|u(x) - l(x)| \leq C|x|^{1+\alpha}\|u\|_{L^\infty(\Omega_1)} + K_f + \|g\|_{C^{1,\alpha}((\partial\Omega)_1)}, \quad \forall x \in \Omega_{r_0}$$

and

$$|Du(0)| = |Dl| \leq C\|u\|_{L^\infty(\Omega_1)} + K_f + \|g\|_{C^{1,\alpha}((\partial\Omega)_1)},$$

where $C$ and $r_0$ depends on $n$, $\lambda$, $\Lambda$, $\alpha$, $\bar{\alpha}$ and $\|\varphi\|_{C^{1,\alpha}(B_1^\alpha)}$. 

If \( f \in L^p(\Omega_1) \) for \( n < p < \infty \), by Hölder’s inequality, there exists a constant \( C_n \) depending only on \( n \) such that
\[
\|f\|_{L^p(\Omega_1)} \leq C_n r^{1-n/p} \|f\|_{L^p(\Omega_r)}
\]
for any \( r \in (0,1] \), which implies \( f \) satisfies \( C \) with \( \alpha = 1 - n/p \). Then we have the following corollary:

**Corollary 3.4.** Under the hypotheses of Lemma 3.3 with \( 3.3 \) replaced by \( f \in L^p(\Omega_1) \) for \( n < p < \infty \). Set \( \alpha_0 = \min\{\alpha, 1 - n/p\} \). Then \( u \) is \( C^{1,\alpha_0} \) at \( 0 \), i.e., there exists an affine function \( l_{x_0} \) such that
\[
|u(x) - l(x)| \leq C \|x\|^{1+\alpha_0} + \|f\|_{L^p(\Omega_1)} \|g\|_{C^{1,\alpha}((\partial\Omega)_1)}, \quad \forall x \in \Omega_{r_0}
\]
and
\[
|Du(0)| = |Dl| \leq C \|u\|_{L^\infty(\Omega_1)} + \|f\|_{L^p(\Omega_1)} + \|g\|_{C^{1,\alpha}((\partial\Omega)_1)},
\]
where \( C \) and \( r_0 \) depends on \( n, \lambda, \Lambda, \alpha, \bar{\alpha} \) and \( \|\varphi\|_{C^{1,\alpha}(B_1)} \).

4. Proof of Theorem 1.4

Theorem 1.4 follows easily from the following Theorem 4.1 which is proved by interior \( W^{2,\delta_0} \) estimates (Theorem 3.1), boundary \( C^{1,\alpha_0} \) estimates (Corollary 3.4) and Whitney decomposition (Lemma 2.1, 2.2 and 2.3).

**Theorem 4.1.** Let \( \bar{\alpha} \) be as in Lemma 3.3 and \( 0 < \alpha < \bar{\alpha} \). Suppose that \( \Omega \) satisfies Assumption (A) with \( \varphi \in C^{1,\alpha}(B_{\delta}^1) \) and \( u \) satisfies

\[
\begin{aligned}
&u \in S(\lambda, \Lambda, f) \quad \text{in} \quad \Omega_1; \\
&u = g \quad \text{on} \quad (\partial\Omega)_1,
\end{aligned}
\]

where

\[
f \in L^p(\Omega_1) \quad \text{for} \quad n < p < \infty \quad \text{and} \quad g \in C^{1,\alpha}((\partial\Omega)_1).
\]

Let \( \delta_0 \) be as in Theorem 3.4 and \( \alpha_0 = \min\{\alpha, 1 - n/p\} \). Then for any
\[
\delta \leq \delta_0 \quad \text{and} \quad \delta < 1/(1 - \alpha_0),
\]
we have

\[
\|u\|_{W^{2,\delta}(\Omega_{1/2})} \leq C (\|u\|_{L^\infty(\Omega_1)} + \|f\|_{L^p(\Omega_1)} + \|g\|_{C^{1,\alpha}((\partial\Omega)_1)}),
\]

where \( C \) depends on \( n, \lambda, \Lambda, \alpha, \bar{\alpha}, \delta, p \) and \( \|\varphi\|_{C^{1,\alpha}(B_1^1)} \).

**Proof.** In the following, we write for simplicity
\[
\mathcal{H} = \|u\|_{L^\infty(\Omega_1)} + \|f\|_{L^p(\Omega_1)} + \|g\|_{C^{1,\alpha}((\partial\Omega)_1)}.
\]

Let \( \{Q_k\}_{k=1}^\infty \) be Whitney decomposition of \( \Omega_1 \), \( d_k = \text{diam} Q_k \) and \( \bar{Q}_k = \frac{6}{5}Q_k \).

For any \( \bar{Q}_k \subset \Omega_{1/4} \), let \( y_k \in (\partial\Omega)_{1/2} \) and \( \tilde{x}_k \in \partial\bar{Q}_k \) such that
\[
|\tilde{x}_k - y_k| = \text{dist}(\bar{Q}_k, \partial\Omega_1) < \text{dist}(Q_k, \partial\Omega_1) \leq 4d_k,
\]
where Lemma 2.1 (iii) is used in the last inequality. Consequently, we see that
\[
|x - y_k| \leq |x - \tilde{x}_k| + |\tilde{x}_k - y_k| \leq 5d_k, \quad \forall x \in \bar{Q}_k.
\]

By Corollary 3.4 \( u \) is \( C^{1,\alpha_0} \) at \( y_k \) and then there exists an affine function \( l_{y_k} \) (written by \( l \) for simplicity in the following) such that
\[
|u(x) - l(x)| \leq C |x - y_k|^{1+\alpha_0} \mathcal{H} \leq Cd_k^{1+\alpha_0} \mathcal{H}, \quad \forall x \in \bar{Q}_k,
\]

where \( C \) depends on \( n, \lambda, \Lambda, \alpha, \bar{\alpha} \) and \( \alpha_0 \).
where $C$ depends on $n, \lambda, \alpha, \bar{\alpha}$ and $\|\varphi\|_{C^{1,\alpha}(B^\prime_1)}$.

Since $u-l \in S(\lambda, \Lambda, f)$ in $\bar{Q}_k$, we deduce from interior $W^{2,\delta_0}$ estimate, (4.3) and Hölder’s inequality that

$$
\int_{Q_k} |D^2(u-l)|^{\delta_0}dx \leq C \left\{ d_k^{n-2\delta_0} \|u-l\|_{L^{\infty}(\bar{Q}_k)} + d_k^{n-\delta_0} \left( \int_{\bar{Q}_k} |f|^p dx \right)^{\delta_0/p} \right\},
$$

where $C$ depends on $n, \lambda, \alpha, \bar{\alpha}, p$ and $\|\varphi\|_{C^{1,\alpha}(B^\prime_1)}$.

For any $\delta \leq \delta_0$, we deduce from Hölder’s inequality and the above estimate that

$$
\int_{Q_k} |D^2u|^\delta dx \leq C d_k^{n(1-\delta/\delta_0)} \left( \int_{Q_k} |D^2(u-l)|^{\delta_0}dx \right)^{\delta/\delta_0}
\leq C \left\{ d_k^{n-\delta_0} |\mathcal{H}^\delta| + d_k^{n-\delta_0} \left( \int_{\bar{Q}_k} |f|^p dx \right)^{\delta/p} \right\},
$$

Taking sum on both sides with respect to $k$, we obtain

$$
\sum_{\bar{Q}_k \subset \Omega_{1/4}} \int_{Q_k} |D^2u|^\delta dx \leq C \left\{ |\mathcal{H}^\delta| \sum_{\bar{Q}_k \subset \Omega_{1/4}} d_k^{n-\delta_0} + \|f\|_{L^p(\Omega_1)}^\delta \sum_{\bar{Q}_k \subset \Omega_{1/4}} d_k^{n-\delta_0} \right\}.
$$

Since $\alpha_0 = \min\{\alpha, 1-n/p\}$ and $\delta < 1/(1-\alpha_0)$, we have

$$
n - \delta n/p \geq n - (1-\alpha_0)\delta > n - 1.
$$

Consequently, by Lemma 2.2, Lemma 2.3, for any $\delta \leq \delta_0$ and $\delta < 1/(1-\alpha_0)$,

$$
\int_{\Omega_{1/12}} |D^2u|^\delta dx \leq \sum_{\bar{Q}_k \subset \Omega_{1/4}} \int_{Q_k} |D^2u|^\delta dx
\leq C \left( \|u\|_{L^\infty(\Omega_1)}^\delta + \|f\|_{L^p(\Omega_1)}^\delta + \|g\|_{C^{1,\alpha}(\partial \Omega_{1/4})}^\delta \right),
$$

where $C$ depends on $n, \lambda, \alpha, \bar{\alpha}, \delta, p$ and $\|\varphi\|_{C^{1,\alpha}(B^\prime_1)}$. The desired estimate (4.2) then follows easily from the above estimate.

Theorem 1.4 then follows from Theorem 4.1. \hfill \Box

**Proof of Theorem 1.4**

Since $T \subset \partial \Omega$ is of $C^{1,\alpha}$, for any $x \in T$, there exist $r_x > 0$ and $\varphi_x \in C^{1,\alpha}(B^\prime_{r_x}(x))$ such that

$$
\Omega_{r_x}(x) = \{x^n > \varphi_x(x')\} \cap B_{r_x}(x) \text{ and } (\partial \Omega)_{r_x}(x) = \{x^n = \varphi_x(x')\} \cap B_{r_x}(x).
$$

Since $\bigcup_{x \in T} B_{r_x/12}(x)$ cover $T \cap \Omega'$ and $T \cap \Omega'$ is compact, we choose a finite subset $B_{r_x/12}(x_i), i = 1, 2, ..., N$, of $\{B_{r_x/12}(x) : x \in T\}$ that still covers $T \cap \Omega'$.

Let $\delta_0$ be as in Theorem 3.1, $\bar{\alpha}$ be as in Lemma 6.2 and $\alpha_0 = \min\{\bar{\alpha}, \alpha, 1-n/p\}$. Utilizing the scaled version of Theorem 4.1 in $\Omega_{r_x}(x_i) = \Omega \cap B_{r_x}(x_i)$, we obtain that for any $\delta \leq \delta_0$ and $\delta < 1/(1-\alpha_0)$,

$$
r_i^{-n/\delta-2} \|u\|_{L^\infty(\Omega_{r_x/12}(x_i))} + r_i^{-n/\delta-1} \||Du|||_{L^p(\Omega_{r_x/12}(x_i))} + r_i^{-n/\delta} \|D^2u|||_{L^4(\Omega_{r_x/12}(x_i))}
\leq C_i \left( r_i^{-2} \|u\|_{L^\infty(\Omega_{r_x}(x_i))} + r_i^{-n/p} ||f||_{L^p(\Omega_{r_x}(x_i))} + r_i^{-2} \|g\|_{C^{1,\alpha}(\partial \Omega_{r_x}(x_i))} \right),
$$

where $C_i, r_i$ depend on $n, \lambda, \alpha, \bar{\alpha}, p$ and $\|\varphi\|_{C^{1,\alpha}(B^\prime_r)}$. The desired estimate (1.2) then follows from the above estimate.
for $i = 1, 2, ..., N$, where $C_i$ depends on $n, \lambda, \alpha, \bar{\alpha}, \delta, p$ and $\|\varphi_{x_i}\|_{C^{2,\alpha}(B_{r_i}(x_i))}$.

Finally, we sum the above inequalities along with the interior $W^{2,\delta_0}$ estimate, to find $u \in W^{2,\delta}(\Omega')$ for $\Omega' \subset \subset \Omega \cup T$, with the desired estimate \[1.2\].

\[\Box\]

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\section*{References}

[1] Byun, S.; Han, J. $W^{2,p}$-estimates for fully nonlinear elliptic equations with oblique boundary conditions. J. Differential Equations. 268 (2020), 2125-2150.

[2] Caffarelli, L. A.; Cabré, X. Fully nonlinear elliptic equations, Colloquium Publications, 43. American Mathematical Society, Providence, R.I., 1995.

[3] Caffarelli, L. A.; Crandall, M.G.; Kocan, A. \'{S}wiech, M. On viscosity solutions of fully nonlinear equations with measurable ingredients. Commun. Pure Appl. Math. 49 (1996), 365-397.

[4] Cao, Y.; Li, D. S.; Wang, L. H. The optimal weighted $W^{2,p}$ estimates of elliptic equation with non-compatible conditions. Commun. Pure Appl. Anal. 10 (2011), 561-570.

[5] Escauriaza, L. Z. $W^{2,\alpha}$ a priori Estimates for Solutions to Fully Nonlinear Equations, Indiana Univ. Math. J. 42 (1993), 413-423.

[6] Gilbarg, D.; Trudinger, N. S. Elliptic partial differential equations of second order. Reprint of the 1998 edition. Classics in Mathematics, Springer-Verlag, Berlin, 2001.

[7] Krylov, N. V. Boundedly inhomogeneous elliptic and parabolic equations in a domain. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 47 (1983), 75-108.

[8] Li, D. S.; Li X. M.; Zhang, K. $W^{2,p}$ estimates for elliptic equations on $C^{1,\alpha}$ domains. Preprint.

[9] Li, D. S.; Zhang, K. $W^{2,p}$ interior estimates of fully nonlinear elliptic equations. Bull. Lond. Math. Soc. 47 (2015), 301-314.

[10] Li, D. S.; Zhang, K. Regularity for fully nonlinear elliptic equations with oblique boundary conditions. Arch. Ration. Mech. Anal. 228 (2018), 923-967.

[11] Lian, Y. Y.; Zhang, K. Boundary pointwise $C^{1,\alpha}$ and $C^{2,\alpha}$ regularity for fully nonlinear elliptic equations. J. Differ. Eq. 269 (2020), 1172-1191.

[12] Lin, F. H. Second derivative $L^p$-estimates for elliptic equations of nondivergent type. Proc. Amer. Math. Soc. 96 (1986), 447-451.

[13] Stein, E. M. Singular Integrals and Differentiability Properties of Functions. Princeton Math. Ser., vol. 30, Princeton University Press, Princeton, N.J., 1970.

[14] Winter, N. $W^{2,p}$ and $W^{1,p}$-estimates at the boundary for solutions of fully nonlinear, uniformly elliptic equations. Z. Anal. Anwend. 28 (2009), 129-164.