BOUNDARY CONTROLLABILITY AND BOUNDARY TIME-VARYING FEEDBACK STABILIZATION OF THE 1D WAVE EQUATION IN NON-CYLINDRICAL DOMAINS

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Abstract. In this paper, we deal with boundary controllability and boundary stabilizability of the 1D wave equation in non-cylindrical domains. By using the characteristics method, we prove under a natural assumption on the boundary functions that the 1D wave equation is controllable and stabilizable from one side of the boundary. Furthermore, the control function and the decay rate of the solution are given explicitly.

1. Introduction and preliminaries. In this work, we are interested in the boundary controllability and stabilizability of the one dimensional wave equation in non-cylindrical domains. More precisely, let \( \alpha \) and \( \beta \) be two real functions defined on \( \mathbb{R}_+ \) and \( Q \) be the set

\[
Q = \left\{ (t, x) \in \mathbb{R}^2, \; x \in (\alpha(t), \beta(t)), \; \alpha(t) < \beta(t), \; t \in (0, \infty) \right\},
\]

with \( \alpha(0) = 0 \) and \( \beta(0) = 1 \). We consider the following two systems

\[
\begin{cases}
  y_{tt}(t, x) = y_{xx}(t, x), & \text{in } Q, \\
  y(t, \alpha(t)) = \frac{1}{2} u(t), \; y(t, \beta(t)) = 0, & \text{in } (0, \infty), \\
  y(0, x) = y_0(x), \; y_t(0, x) = y_1(x), & \text{in } (0, 1),
\end{cases}
\] (1)

and

\[
\begin{cases}
  y_{tt}(t, x) = y_{xx}(t, x), & \text{in } Q, \\
  y_t(t, \alpha(t)) = f(t) y_x(t, \alpha(t)), \; y(t, \beta(t)) = 0, & \text{in } (0, \infty), \\
  y(0, x) = y_0(x), \; y_t(0, x) = y_1(x), & \text{in } (0, 1).
\end{cases}
\] (2)

The functions \( u \in H^1_{\text{loc}}(0, \infty) \), and \( f \in C([0, \infty)) \) in (1) and (2) represent the control force and the feedback function respectively.

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Controllability of system (1) has been extensively studied in the recent past years; most of the papers dealt with the case of one moving endpoint with boundary conditions of the form

\[ y(t, 0) = 0, \quad y(t, kt + 1) = u(t), \quad k \in (0, 1), \quad t \in (0, \infty). \]

In [6], it has been shown that, with these boundary conditions, exact controllability holds for all times \( T > e^{\frac{2k(k+1)}{2^{k(k+1)}}} \). The same authors came back in [7] and improved the latter result to \( T > e^{\frac{2}{2^{k(k+1)}}} \). Later, in [18], the controllability time has been improved to be \( T > \frac{2}{1-k} \). In these papers, only a sufficient condition is provided for the exact controllability.

Concerning the two moving endpoints case, the boundary functions considered in [16] are of the form

\[ \alpha(t) = -kt, \quad \beta(t) = rt + 1, \quad t \in (0, \infty), \quad k, r \in [0, 1) \text{ with } r + k > 0. \]

It has been shown that exact controllability holds if, and only if \( T \geq \frac{2}{1-k} \).

Another kind of boundary functions has been considered in [2]. An observability inequality has been established for the dual of system (1) with \( \beta \equiv 1 \) for sufficiently large time under the assumption that the boundary function \( \alpha \) must be periodic and satisfies \( \|\alpha'\|_{L^\infty(0, \infty)} < 1 \). More general boundary functions are considered in [11] with boundary conditions

\[ y(t, 0) = 0, \quad y(t, s(t)) = u(t), \quad t \in (0, \infty), \]

where \( s : [0, \infty) \to (0, \infty) \) is assumed to be a \( C^1 \) function satisfying \( \|s'\|_{L^\infty(0, \infty)} < 1 \). Furthermore, it has been assumed that \( s \) must be in some admissible class of curves (see [11] for more details). Under these assumptions, the authors proved that exact controllability holds if, and only if \( T \geq s^+ \circ (s^-)^{-1}(0) \), where \( s^+ (t) = t \pm s(t) \). Also, they provided a controllability result when the control is located on the non-moving part of the boundary. By considering the boundary conditions

\[ y(t, 0) = u(t), \quad y(t, s(t)) = 0, \quad t \in (0, \infty), \]

they proved that exact controllability holds if, and only if \( T \geq (s^-)^{-1}(1) \). The same result has been proved in [9] by using a different approach. In all the cited works,
the proofs rely on the multipliers technique, domain transform, the non-harmonic Fourier analysis or the d’Alembert solution of the wave equation.

Recently, in [17], a new Carleman estimate has been established for the wave equation in non-cylindrical domains in more general settings. As a consequence, it has been shown for a boundary conditions as in (1) where \( \alpha(t) < \beta(t), \ t \in (0, \infty) \), are smooth functions satisfying \( \| \alpha' \|_{L^\infty(0, \infty)} \| \beta' \|_{L^\infty(0, \infty)} < 1 \), that system (1) is exactly controllable at time \( T \) if \( T > T^* \) and not exactly controllable if \( T < T^* \) where \( T^* \) is the required time by the geometric control condition, in other words, it is the time where a characteristic line with slope one emanating from the point \((0, 0)\) hits the curve \((t, \beta(t))_{t \geq 0}\) and reflected to intersect the curve \((t, \alpha(t))_{t \geq 0}\) in the point \((T^*, \alpha(T^*))\). Actually, this time can be computed explicitly in terms of the boundary curves, that is \( T^* = (a^+)^{-1} \circ \beta^+ \circ (\beta^-)^{-1}(0) \) where the functions \( \alpha^\pm, \beta^\pm \) are defined by \( \alpha^\pm(t) = t \pm \alpha(t), \ \beta^\pm(t) = t \pm \beta(t) \). However, the result doesn’t cover the critical case \( T = T^* \).

As for the boundary stability of system (2) with non-autonomous damping, to the best of our knowledge, the only existing result in the literature is in [1] where the authors dealt with the same system but with only one moving endpoint, i.e.

\[
y(t, 0) = 0, \quad y_t(t, a(t)) + f(t)y_x(t, a(t)) = 0, \quad t \in (0, \infty),
\]

where \( a \) is a strictly positive 1-periodic function with \( \| a' \|_{L^\infty(0, \infty)} < 1 \) and \( f \) is the feedback function. The authors proved exponential stability of system (2) for a particular class of feedbacks \( f \). The proof relies on transforming problem (2) which is posed on non-cylindrical domain into a problem posed on cylindrical one, then making use of some known results of boundary stability of the 1D wave equation. If the damping function \( f \) is constant and the boundary function \( a \) is not periodic with derivative \( \| a' \|_{L^\infty(0, \infty)} < 1 \), it has been shown in [10] for \( f = 1 \) that the solution vanishes at time \( T \) for any \( T > a^+ \circ (a^-)^{-1}(0) \).

In this paper, we will improve all the previous results either for the boundary control or the boundary stability of the 1D wave equation by using the characteristics method. We shall build the unique exact solution to both systems (1) and (2) in an appropriate energy space. To do so, we proceed by transforming both of systems to a first order hyperbolic system by introducing the Riemann invariants

\[
\begin{aligned}
p = y_t - y_x, \\
q = y_t + y_x.
\end{aligned}
\]

An elementary computation shows that system (1) transforms into

\[
\begin{cases}
p_t + p_x = 0, & \text{in } Q, \\
q_t - q_x = 0, & \text{in } Q, \\
(p + q)(t, a(t)) = u'(t), & \text{in } (0, \infty), \\
p(0, x) = \bar{p}(x), & q(0, x) = \bar{q}(x). & \text{in } (0, 1).
\end{cases}
\]

In the same way, system (2) becomes

\[
\begin{cases}
p_t + p_x = 0, & \text{in } Q, \\
q_t - q_x = 0, & \text{in } Q, \\
(p + F(t)q)(t, a(t)) = 0, & \text{in } (0, \infty), \\
p(0, x) = \bar{p}(x), & q(0, x) = \bar{q}(x). & \text{in } (0, 1),
\end{cases}
\]

where \( F(t) = \frac{1-f(t)}{1+f(t)} \) with \( 1 + f(t) \neq 0, \ \forall t \geq 0 \).

Henceforth, we use the following notations:
The idea is to use the boundary conditions

\[ w(x,0) = \text{boundary curves} \]

and the reflection of the characteristic lines \( x = t + c \) meet the curve \( (t, \alpha(t))_{t \geq 0} \) (resp. \( (t, \beta(t))_{t \geq 0} \)) in finite time; also, they serve to ensure that the characteristic lines \( x = t + c \) are not gliding on the boundary curves or are not out of \( Q \). In fact, assumption (10) is necessary for the existence of solutions. A straightforward consequence of assumption (10) is that the functions \( \alpha^{\pm} : [0, \infty) \to [0, \infty) \) and \( \beta^{\pm} : [0, \infty) \to [\pm 1, \infty) \) are invertible. In the sequel, we use the standard notations to denote their inverses by \( (\alpha^{\pm})^{-1} \) and \( (\beta^{\pm})^{-1} \).
Figure 2. An example of a boundary curves \((t, \alpha(t))_{t \geq 0}\) and \((t, \beta(t))_{t \geq 0}\) that do not satisfy assumption (10). The values of the solution are not defined on the green part of the characteristic lines lying under or above these curves.

2. Main results. We start by giving the well-posedness result for system (7).

**Theorem 2.1.** Let \((\tilde{p}, \tilde{q}, v, F) \in [L^2(0, 1)]^2 \times L^2_{\text{loc}}(0, \infty) \times C([0, \infty))\). Assume that the boundary curves \((t, \alpha(t))_{t \geq 0}\) and \((t, \beta(t))_{t \geq 0}\) satisfy (10). Then, there exists a unique solution to system (7) satisfying

\[ (p, q) \in C \left( 0, t; \left[ L^2(\alpha(t), \beta(t)) \right]^2 \right), \ t \geq 0. \]

The proof of this theorem is a straightforward consequence of the explicit construction of the unique solution that will be done in Section 3.

**Remark 1.** By inverting the transformation given in (4), we obtain

\[ y_t = \frac{p + q}{2}, \quad y_x = \frac{q - p}{2}, \]

hence, for any \((y_0, y_1, u, f) \in H^1_{(1)}(0, 1) \times L^2(0, 1) \times H^1_{\text{loc}}(0, \infty) \times C([0, \infty))\), the solutions to systems (1) and (2) satisfy the regularity

\[ y \in C \left( 0, t; H^1_{(\beta(t))}(\alpha(t), \beta(t)) \right) \cap C^1 \left( 0, t; L^2(\alpha(t), \beta(t)) \right), \ t \geq 0. \]

2.1. Controllability result.

**Definition 2.2.** System (1) is said to be exactly controllable at time \(T > 0\) if for any initial state \((y_0, y_1) \in H^1_{(1)}(0, 1) \times L^2(0, 1)\) and for any target state \((h, k) \in H^1(\beta(T))(\alpha(T), \beta(T)) \times L^2(\alpha(T), \beta(T))\), there exists a control \(u \in H^1_{\text{loc}}(0, \infty)\) such that \((y(T), y_t(T)) = (h, k)\).

The following result shows that the minimal time \(T^*\) where exact controllability is possible depends on the movement of the boundaries and can be represented explicitly in terms of the functions \(\alpha^\pm\) and \(\beta^\pm\). Moreover, also the unique exact control for \(T^*\) can be represented explicitly using these functions.

**Theorem 2.3.** Let \((y_0, y_1) \in H^1_{(1)}(0, 1) \times L^2(0, 1)\). Assume that the boundary curves \((t, \alpha(t))_{t \geq 0}\) and \((t, \beta(t))_{t \geq 0}\) satisfy (10). System (1) is exactly controllable at time
T > 0 if, and only if T ≥ T* = (α+)−1 o β+ o (β−)−1 (0). Further, if T = T*, there exists a unique control u ∈ H1(0, T*) steering the solution (y, yt) to system (1) to the equilibrium point (0, 0) given by

\[ u(t) = \begin{cases} 
  \int_0^t y_1 (\alpha^+(s)) \, ds + y_0 (\alpha^+(t)), & \text{if } t \in [0, (\alpha^+)^{-1}(1)], \\
  y_0 (\beta^- o (\beta^+)^{-1} o \alpha^+(t)) + \int_0^{(\alpha^+)^{-1}(1)} y_1 (\alpha^+(s)) \, ds & \text{if } t \in [(\alpha^+)^{-1}(1), T^*), \\
  -\int_{(\alpha^+)^{-1}(1)}^t y_1 (\beta^- o (\beta^+)^{-1} o \alpha^+(s)) \, ds, & \text{if } t \in [T^*, (\alpha^+)^{-1}(1)]. 
\end{cases} \]

(12)

**Remark 2.** The controllability result still makes sense even if the boundary curves \((t, \alpha(t))_{t \geq 0}\) and \((t, \beta(t))_{t \geq 0}\) are allowed to intersect in time larger than \(T^*\).

**Remark 3.** Let us consider the particular case \(\alpha(t) = kt, \beta(t) = rt + 1, k, r \in (-1, 1)\), with \(\frac{2(k-r)}{(1-r)(1+k)} < \frac{1}{2}\) (The last assumption guarantees that the boundary curves do not intersect before \(T^*\)). In this case, it can be checked that \(T^*\) is given by \(T^* = \frac{2}{(1-r)(k+1)}\) which is the same time found in [16]. In particular, if \(\alpha \equiv 0\) and \(\beta \equiv 1\), we obtain the classical result \(T^* = 2\).

**Remark 4.** The minimal time \(T^*\) is precisely the necessary time for the main characteristic line issued from the point \((0, 0)\) to touch again the curve \((t, \alpha(t))_{t \geq 0}\) in the point \((T^*, \alpha(T^*))\) after having been reflected from the curve \((t, \beta(t))_{t \geq 0}\). More precisely, the characteristic line \(x = t\) hits the curve \((t, \beta(t))_{t \geq 0}\) in the point \((\beta^-)^{-1}(0), \beta(\beta^-)^{-1}(0)\). The reflected characteristic line passing through the last point, i.e. \(x = -t + \beta^+ o (\beta^-)^{-1}(0)\) hits the curve \((t, \alpha(t))_{t \geq 0}\) in the point \((T^*, \alpha(T^*))\). If the control \(u\) is located on the curve \((t, \beta(t))_{t \geq 0}\) instead of \((t, \alpha(t))_{t \geq 0}\), then \(T^{**}\) is the analogous time for the main characteristic line issued from the point \((0, 1)\) with negative slope. In this case \(T^{**} = (\beta^-)^{-1} o \alpha^- o (\alpha^+)^{-1}(1)\).

**2.2. Stability result.** For the sake of lighting notations, we introduce the function \(\phi := \phi(\alpha, \beta)\) defined by

\[ \phi := \alpha^- o (\alpha^+)^{-1} o \beta^+ o (\beta^-)^{-1}. \]

(13)
By assumption (10), the function $\phi : [-1, \infty) \to [\alpha^- \circ (\alpha^+)^{-1} (1), \infty)$ is well defined and increasing function as composition of increasing functions, and hence invertible with inverse

$$\phi^{-1} := \beta^- \circ (\beta^+)^{-1} \circ \alpha^+ \circ (\alpha^-)^{-1}.$$  

Let $(\psi_n)_{n \geq 0}$ be a sequence of functions such that

$$\psi_n : [0, \phi(0)) \to [0, \infty)$$

$$\tau \mapsto \psi_n(\tau) = \prod_{i=0}^{n} \left| F \left( (\alpha^-)^{-1} \circ \phi^{[i]}(\tau) \right) \right|.$$  

The notation $\phi^{[n]}$ refers to

$$\phi^{[n]} = \underbrace{\phi \circ \phi \circ \phi \circ \cdots \circ \phi}_{n \text{ times}}$$

with the convention $\phi^{[0]} = I$. The following result shows that the asymptotic behavior of the solution to system (2) relies heavily on the behavior of the sequence of functions $(\psi_n(\tau))_{n \geq 0}$ defined in (14) when $n \to \infty$.

**Theorem 2.4.** Let $(y_0, y_1) \in H^1_{(1)}(0,1) \times L^2(0,1)$. Assume that the boundary curves $(t, \alpha(t))_{t \geq 0}$ and $(t, \beta(t))_{t \geq 0}$ satisfy (10). In addition, assume that

$$\phi(\tau) < \cdots < \phi^{[n]}(\tau) \to \infty, \quad \forall \tau \in [0, \phi(0)),$$

then,

$$\| (y(t), y_t(t)) \|_{H^1_{(\alpha(t), \beta(t))}(0,1) \times L^2(\alpha(t), \beta(t))} \to 0,$$

if, and only if

$$\psi_n(\tau) \to 0, \quad \forall \tau \in [0, \phi(0)).$$  

(16)

If there exists $g \in C(\mathbb{R}, (0, \infty))$ such that

$$\psi_n(\tau) \sim C g \left( \phi^{[n]}(\tau) \right), \quad \forall \tau \in [0, \phi(0)),$$

(17)

for some positive constant $C > 0$, then the solution to system (2) decays like $g(t)$, i.e.

$$\| (y(t), y_t(t)) \|_{H^1_{(\alpha(t), \beta(t))}(0,1) \times L^2(\alpha(t), \beta(t))} \leq C g(t) \| (y_0, y_1) \|_{H^1_{(1)}(0,1) \times L^2(0,1)}.$$  

(18)

In particular, the solution to system (2) $(y(t), y_t(t))$ decays exponentially to zero with growth bound $-\omega < 0$, i.e. there exists $M \geq 1$ such that

$$\| (y(t), y_t(t)) \|_{H^1_{(\alpha(t), \beta(t))}(0,1) \times L^2(\alpha(t), \beta(t))} \leq M e^{-t\omega} \| (y_0, y_1) \|_{H^1_{(1)}(0,1) \times L^2(0,1)}, \quad \forall t \geq 0,$$

(19)

* if, and only if

$$\sup_{\tau \in [0, \phi(0))} \lim_{n \to \infty} \frac{\ln \psi_n(\tau)}{\phi^{[n]}(\tau)} = -\omega.$$  

If $f \equiv 1$, the solution to system (2) vanishes in finite time $T$ if, and only if $T \geq T^* = (\alpha^+)^{-1} \circ \beta^+ \circ (\beta^-)^{-1} (0)$, i.e.

$$y(T) \equiv y_t(T) \equiv 0, \quad \forall T \geq T^* = (\alpha^+)^{-1} \circ \beta^+ \circ (\beta^-)^{-1} (0).$$

Let us illustrate the previous theorem by some examples.
Example 2.5 (Cylindrical domain). If \( Q \) is cylindrical domain, i.e. \( \alpha = 0 \) and \( \beta = 1 \), the function \( \phi \) defined in (13) is given by \( \phi(\tau) = \tau + 2 \), then, \( \phi^{\beta}(\tau) = \tau + 2n \).

Therefore, the sequence of functions \((\psi_n)_{n \geq 0}\) defined in (14) takes the form
\[
\psi_n : [0, 2) \rightarrow [0, \infty) \quad (20)
\]
\[
\tau \mapsto \psi_n(\tau) = \prod_{i=0}^{n} |F(\tau + 2i)|.
\]

In this case, the assumptions of Theorem 2.4 can be checked easily. Note that since the feedback law in system (2) is non-autonomous (\( f \) is time dependent), we can achieve any decay rate we want (even faster than exponential) with a suitable choice of \( f \). Below, we illustrate this fact by several examples:

- **Exponential decay:**
  Let \( f(t) = \frac{2 - \sin(\pi t)}{2 + \sin(\pi t)} \), therefore, \( F(t) = \frac{\sin(\pi t)}{2} \), thus,
  \[
  \psi_n(\tau) = \prod_{i=0}^{n} |F(\tau + 2i)| = \left( \frac{\sin(\pi \tau)}{2} \right)^{n+1}.
  \]
  By (19), we have
  \[
  \sup_{\tau \in (0,1) \cup (1,2)} \lim_{n \to \infty} \psi_n(\tau) \leq \sup_{\tau \in (0,1) \cup (1,2)} \lim_{n \to \infty} \frac{(n+1) \ln \left( \frac{\sin(\pi \tau)}{2} \right)}{\tau + 2n} = \sup_{\tau \in (0,1) \cup (1,2)} \frac{1}{2} \ln \left| \frac{\sin(\pi \tau)}{2} \right| = -\frac{\ln 2}{2}.
  \]
  therefore, exponential decay occurs with growth bound \(-\omega = -\frac{\ln 2}{2}\).

- **Polynomial decay:**
  Let \( f(t) = \frac{(t+1)^{-s} - (t+3)^{-s}}{(t+1)^{-s} + (t+3)^{-s}} \), \( s > 0 \), then \( F(t) = \left( \frac{t+3}{t+1} \right)^{-s} \), consequently, the sequence of functions \((\psi_n)_{n \geq 0}\) defined in (20) takes the form
  \[
  \psi_n(\tau) = \prod_{i=0}^{n} |F(\tau + 2i)| = \prod_{i=0}^{n} \left| \frac{\tau + 2i + 3}{\tau + 2i + 1} \right| = \left( \frac{\tau + 2n + 3}{\tau + 1} \right)^{-s}.
  \]
  Set \( g(t) = (t+1)^{-s} \), \( s > 0 \). A simple computation shows that
  \[
  \lim_{n \to \infty} \frac{\psi_n(\tau)}{g(\phi^n)} = \lim_{n \to \infty} \frac{\psi_n(\tau)}{g(\tau + 2n)} = \frac{1}{\tau + 1}, \ \tau \in [0, 2),
  \]
  thus, by (18), the solution to system (2) decays like \((t+1)^{-s}, s > 0\).

- **Logarithmic decay:**
  Let \( f(t) = \frac{\log^{-s}(t+1) - \log^{-s}(t+3)}{\log^{-s}(t+1) + \log^{-s}(t+3)} \), \( s > 0 \), then \( F(t) = \left( \frac{\log(t+3)}{\log(t+1)} \right)^{-s} \), consequently, we obtain
  \[
  \psi_n(\tau) = \prod_{i=0}^{n} |F(\tau + 2i)| = \prod_{i=0}^{n} \left| \frac{\log(\tau + 2i + 3)}{\log(\tau + 2i + 1)} \right| = \left( \frac{\log(\tau + 2n + 3)}{\log(\tau + 1)} \right)^{-s}.
  \]
  By letting \( g(t) = \log^{-s}(t+1), s > 0 \), we get
  \[
  \lim_{n \to \infty} \frac{\psi_n(\tau)}{g(\phi^n)} = \log^s(\tau + 1) \lim_{n \to \infty} \left( \frac{\log(\tau + 2n + 3)}{\log(\tau + 2n + 1)} \right)^{-s} = \log^s(\tau + 1), \ \tau \in [0, 2),
  \]
  hence, (18) is satisfied with \( g(t) = \log^{-s}(t+1), s > 0 \).
• Super-stability:
  Let \( f(t) = \frac{t^4}{2t^7} \); therefore, \( F(t) = \frac{1}{1+t^4} \), consequently, we obtain
  \[
  \psi_n(\tau) = \prod_{i=0}^{n} \frac{1}{\tau + 2i + 1} = \frac{1}{(\tau+1)^2n!} \prod_{i=1}^{n} \left( \frac{\tau+1}{2i} + 1 \right)^{-1}.
  \]
  A simple computation shows that
  \[
  \log \prod_{i=1}^{n} \left( \frac{\tau+1}{2i} + 1 \right)^{-1} \sim C(\tau) \log n - \tau + 1,
  \]
  where \( C(\tau) \) is a positive constant depending on \( \tau \). So, we get
  \[
  \psi_n(\tau) \sim n \rightarrow \infty C(\tau) \frac{\tau+1}{2n} n^{\tau+1},
  \]
  which by (16) implies that the solution to system (2) decays to zero. To check whether the stability is exponential it suffices to use (19) to compute the growth bound \( -\omega \). By using (21) we obtain
  \[
  \lim_{n \rightarrow \infty} \frac{\ln \psi_n(\tau)}{\phi[n](\tau)} = - \lim_{n \rightarrow \infty} \frac{\ln n^{\tau+1} + \ln 2 + \ln n!}{2n + \tau} = -\infty = -\omega.
  \]
  Therefore, the decay rate for this choice of \( f \) is faster than any exponential function. Actually, this phenomena is called super-stability. For more of details, we refer the reader to [3].

Example 2.6 (Non cylindrical domain). Things are more delicate in the non-cylindrical case. Consider a boundary functions of the form \( \alpha(t) = rt, \beta(t) = kt^{1/2} \), \( r, k \in (-1, 1) \). To guarantee that \( \alpha(t) \neq \beta(t), \forall t \geq 0 \), we assume that \( k \geq r \).

The function \( \phi \) defined in (13) will be given by
  \[
  \phi(\tau) = \frac{(1 + k)(1 - r)}{(1 - k)(1 + r)} + \frac{2(1 - r)}{(1 - k)(1 + r)} = a\tau + b,
  \]
  therefore, we obtain
  \[
  \phi[n](\tau) = \begin{cases} 
  a^n \left( \frac{\tau - \frac{b}{1-a}}{1-a} \right) + \frac{b}{1-a}, & \text{if } r < k, \\
  \tau + \frac{2n}{1+r}, & \text{if } r = k.
  \end{cases}
  \]

Consequently,
  \[
  (\alpha^-)^{-1} \circ \phi[n](\tau) = \begin{cases} 
  a^n \left( \frac{\tau - \frac{b}{1-a}}{1-a} \right) + \frac{b}{1-a}, & \text{if } r < k, \\
  \frac{\tau}{1-r} + \frac{2n}{1+r}, & \text{if } r = k.
  \end{cases}
  \]

For simplicity, let us take \( f \) as in the previous example, \( f(t) = \frac{t^4}{2t^7} \) which implies that \( F(t) = \frac{1}{1+t^4} \). So, we have:

• If \( r < k \):
  From (22), we can check that (15) is satisfied if, and only if \( a > 1 \). To verify (16), it is enough to estimate its asymptotic behaviour for a large \( n \). So, we have
  \[
  \psi_n(\tau) = \prod_{i=0}^{n} \left| a^i \left( \frac{\tau}{1-r} - \frac{b}{(1-a)(1-r)} \right) + \frac{b}{(1-a)(1+r)} \right| = \prod_{i=0}^{n} \frac{1}{|a^i s(\tau) + z|}.
  \]
\[ \psi_n(\tau) \sim C(r, k, \tau) a^{-\frac{n(n+1)}{2}} s^{n-1}(\tau), \quad \forall \tau \in [0, b), \quad (23) \]

where \( C(r, k, \tau) \) is a positive constant depending on \( r, k \) and \( \tau \).

Nevertheless, we still be able to get an idea about the decay rate. From (23), we observe that the term that really matters is \( a^{-\frac{n^2}{2}} \), so, for \( g(t) = e^{-\frac{1}{2} \log_2(t)} \), we obtain

\[ a^{-\frac{n^2}{2}} \sim n \to \infty Cg(a^n s(\tau) + z), \quad \forall \tau \in [0, b). \]

Note that we did not lose too much since \( g \) decays to zero faster than any polynomial function. This loss can be justified by the fact that the characteristic lines will need a larger time to reflect on the two boundary lines when \( t \) becomes larger.

- If \( k = r \):

  In this case, the lines \( x = rt \) and \( x = kt + 1 \) are parallel, therefore, the characteristic speeds are the same for all time, so we might expect stability in finite time with this choice of \( f \). Let us first check that whether the solution to system (2) decays exponentially or not. By (21), the sequence of functions \( (\psi_n)_{n \geq 0} \) behaves like

\[ \psi_n(\tau) = \prod_{i=0}^{n} \left| \frac{1}{1 - \frac{1}{2} (1+r)(1-r)} \right| \]

\[ = \frac{(1+r)^n (1-r)^n}{2^n n!} \prod_{i=1}^{n} \left( \frac{1}{1 - \frac{1}{2i} (1+r)(1-r)} \left( 1 + \frac{1}{1 - \frac{1}{2i} (1+r)(1-r)} \right) \right). \]

\[ \sim n \to \infty C(r, \tau) \frac{(1+r)^n (1-r)^{n+1}}{2^n n! (1 - r) \frac{1}{2i} (1+r)(1-r)} , \]

where \( C(r, \tau) \) is a positive constant depending on \( r \) and \( \tau \). By using (19), we get

\[ \lim_{n \to \infty} \frac{\ln \psi_n(\tau)}{\phi^{[n]}(\tau)} = \lim_{n \to \infty} \frac{\ln n!}{2n} = -\infty = -\omega, \]

therefore, the solution to system (2) is super-stable.

**Example 2.7 (Constant feedback).** Consider the case when \( f \) is a constant such that \( f \neq 1 \) with keeping \( \alpha \) and \( \beta \) as in the previous example. A simple computation yields

\[ \psi_n = F^{n+1} = \left| \frac{f - 1}{f + 1} \right|^{n+1}. \]

Therefore, by using the formula in (19), we arrive at:
• If \( r < k \):
We can check that the decay is not exponential. Indeed,
\[
\lim_{n \to \infty} \ln \frac{\psi_n(\tau)}{\phi_n(\tau)} = \lim_{n \to \infty} \frac{(n+1) \ln |\frac{f-1}{f+1}|}{a^n} = 0, \quad \forall \tau \in [0, b).
\]
Nonetheless, by (18), we can determine the decay rate for a particular values of \( f \). Let \( g(t) = t^{-s} \).
It is easy to check that if \( a^{-s} = \frac{f-1}{f+1} \) for some \( s > 0 \) then
\[
\lim_{n \to \infty} \frac{\psi_n(\tau)}{g(\phi_n(\tau))} = \lim_{n \to \infty} \frac{|\frac{f-1}{f+1}|^{n+1}}{(a^n s(\tau) + z)^{-s}} = C(\tau, r, k), \quad \forall \tau \in [0, b),
\]
where \( C(r, k, \tau) \) is a positive constant depending on \( r, k \) and \( \tau \). Hence, the solution decays like \( t^{-s}, s > 0 \).

• If \( r = k \):
In this case, we have
\[
\lim_{n \to \infty} \ln \frac{\psi_n(\tau)}{\phi_n(\tau)} = \lim_{n \to \infty} \frac{(n+1) \ln |\frac{f-1}{f+1}|}{(1+r)(1-r) \ln |\frac{f-1}{f+1}|} = -\omega,
\]
hence, exponential decay occurs with growth bound \(-\omega\). In particular, if \( Q \) is a cylindrical domain \((r = 0)\), the solution to system (2) is exponentially stable if, and only if
\[
\frac{1}{2} \ln |\frac{f-1}{f+1}| = -\omega < 0,
\]
which is a known result from [15].

**Remark 5.** We have seen in the previous examples that the decay rate is determined in a crucial way by the boundary curves and the damping function. Actually, we can do the converse for system (2). Namely, by setting
\[
F(t) = \frac{g(\phi \circ \alpha^{-}(t))}{g(\alpha^{-}(t))}, \quad \forall t \geq 0,
\]
with \( g(t) \neq 0 \), for all \( t \geq 0 \), we obtain
\[
\psi_n(\tau) = \prod_{i=0}^{n} \left| F \left( (\alpha^{-})^{-1} \circ \phi^i(\tau) \right) \right| = \prod_{i=0}^{n} \left| \frac{g(\phi^{i+1}(\tau))}{g(\phi^i(\tau))} \right| = \left| \frac{g(\phi^{n+1}(\tau))}{g(\phi^n(\tau))} \right| .
\]
In this case, (17) is automatically satisfied, and since \( F = \frac{1-t}{1+t} \), we obtain
\[
\frac{g(\alpha^{-}(t)) - g(\phi \circ \alpha^{-}(t))}{g(\alpha^{-}(t)) + g(\phi \circ \alpha^{-}(t))} = f_3(t), \quad \forall t \geq 0.
\]
The last expression provides an explicit relation between the decay rate and the feedback function \( f \). This means that \( f \) can be determined based on the desired decay rate. Formula (24) has been used to construct \( f \) in the second and the third points in example (2.5).

**Remark 6.** Examples 2.6 and 2.7 illustrate the big influence of the boundary curves nature on the decay rate of the solution to system (2).
Remark 7. Observe that the time of extinction of the solution to system (2) for \( f \equiv 1 \) is the time of exact controllability in Theorem 2.3. This can be explained by the fact that exponential stability implies exact controllability for time reversible systems (see for instance [12, Remark 1.5] or [14]). Even though our system is not time reversible (because of the boundary functions), we have seen that this implication remains true.

3. Construction of the exact solution. The aim now is to find the solution \((p,q)\) to system (7) in all \(Q\). To this end, let us start by splitting \(Q\) into an infinite number of parts. Namely,

\[
Q = \cup_{n \geq 0} \Sigma_n^p = \cup_{n \geq 0} \Sigma_n^q,
\]

where \(\Sigma_n^p, \Sigma_n^q\) are given for \(n = 0, 1\), by

\[
\begin{align*}
\Sigma_n^p &= \{(t, x) \in Q, \ t \in [0, x]\}, \\
\Sigma_1^p &= \{(t, x) \in Q, \ t - x \in [0, \alpha^{-}(\alpha^{+})^{-1}(1)]\}, \\
\Sigma_n^q &= \{(t, x) \in Q, \ t \in [0, 1 - x]\}, \\
\Sigma_1^q &= \{(t, x) \in Q, \ t + x \in [1, \beta^{+}(\beta^{-})^{-1}(0)]\},
\end{align*}
\]

and for all \(n \geq 1\)

\[
\begin{align*}
\Sigma_{2n}^p &= \{(t, x) \in Q, \ t - x \in \left[\phi^{[n-1]} \circ \alpha^{-} \circ (\alpha^{+})^{-1}(1), \phi^{[n]}(0)\right]\}, \\
\Sigma_{2n+1}^p &= \{(t, x) \in Q, \ t - x \in \left[\phi^{[n]}(0), \phi^{[n]} \circ \alpha^{-} \circ (\alpha^{+})^{-1}(1)\right]\}, \\
\Sigma_{2n}^q &= \{(t, x) \in Q, \ t + x \in \left[\xi^{[n-1]} \circ \beta^{+} \circ (\beta^{-})^{-1}(0), \xi^{[n]}(1)\right]\}, \\
\Sigma_{2n+1}^q &= \{(t, x) \in Q, \ t + x \in \left[\xi^{[n]}(1), \xi^{[n]} \circ \beta^{+} \circ (\beta^{-})^{-1}(0)\right]\},
\end{align*}
\]

where \(\xi\) is defined by

\[
\xi := \beta^{+} \circ (\beta^{-})^{-1} \circ \alpha^{-} \circ (\alpha^{+})^{-1}.
\]

The construction of these regions relies on the reflection of the principal characteristic lines with positive and negative slopes emerging from the points \((0, 0)\) and \((0, 1)\) and reflected along the boundary curves. More precisely, the lines \(x = t\) and \(x = -t + 1\) emerging respectively from \((0, 0)\) and \((0, 1)\) meet the curves \((t, \beta(t))_{t \geq 0}\) and \((t, \alpha(t))_{t \geq 0}\) in the points \((\beta^{-}(0))^{-1}, \beta((\beta^{-}(0))^{-1})\) and \((\alpha^{+})^{-1}(1), \alpha((\alpha^{+})^{-1})\) respectively. The regions \(\Sigma_n^p\) and \(\Sigma_n^q\) are those located between \(t = 0\) and these lines. We can do similarly to construct the regions \(\Sigma_n^p, \Sigma_n^q, n \geq 1\), given above. In the sequel, we denote by \(p_n\) and \(q_n\) the restriction of \(p\) and \(q\) solutions to system (7) on \(\Sigma_n^p\) and \(\Sigma_n^q\), \(n \geq 0\).
The proof readily follows from (8).

**Proof.** Let Lemma 3.1.

In particular, if $\alpha \equiv 0$ and $\beta \equiv 1$, the regions $\Sigma_n^p, \Sigma_n^q, n \geq 0$, are simply given by

\[
\Sigma_n^p = \{(t, x) \in \mathbb{R} \times [0, 1], \ t - x \in [n - 1, n]\}, \\
\Sigma_n^q = \{(t, x) \in \mathbb{R} \times [0, 1], \ x + t \in [n, n + 1]\}.
\]

During the construction below, we use the standard density argument by assuming first that the initial states are sufficiently regular then passing to the limit. So, the constructed solutions must be understood in the weak sense. Let us start by finding $p_0$ and $q_0$:

**Lemma 3.1.** Let $(\tilde{p}, \tilde{q}) \in \left[L^2(0,1)\right]^2$. The solution $(p_0, q_0)$ to system (7) is given by

\[
p_0(t,x) = \tilde{p}(x-t), \quad q_0(t,x) = \tilde{q}(x+t).
\]  

**Proof.** The proof readily follows from (8). \hfill \Box

Now, let us find the solution in the regions $\Sigma^p_1, \Sigma^q_1$:

**Lemma 3.2.** Let $(\tilde{p}, \tilde{q}) \in \left[L^2(0,1)\right]^2$. The solution $(p_1, q_1)$ to system (7) is given by

\[
p_1(t,x) = v \left((\alpha^-)^{-1}(t-x)\right) - F \left((\alpha^-)^{-1}(t-x)\right) \tilde{q} \left(\alpha^+ \circ (\alpha^-)^{-1}(t-x)\right),
\]

\[
q_1(t,x) = -\tilde{p} \left(-\beta^- \circ (\beta^+)^{-1}(x+t)\right).
\]

**Proof.** By using (34), we have at the boundary curves

\[
p_0(\tau, \beta(\tau)) = \tilde{p} \left(-\beta^- (\tau)\right), \quad \tau \in \left[0, (\beta^-)^{-1}(0)\right),
\]

\[
q_0(\chi, \alpha(\chi)) = \tilde{q} \left(\alpha^+ (\chi)\right), \quad \chi \in \left[0, (\alpha^+)^{-1}(1)\right).
\]

By using the boundary conditions given in (9), we get

\[
p_1(\tau, \alpha(\tau)) = v(\tau) - F(\tau)q_0(\tau, \alpha(\tau))
\]

\[
= v(\tau) - F(\tau)\tilde{q}(\alpha^+ (\tau)), \quad \tau \in \left[0, (\alpha^+)^{-1}(1)\right),
\]

\[
q_1(\chi, \beta(\chi)) = -p_0(\chi, \beta(\chi)) = -\tilde{p}(-\beta^- (\chi)), \quad \chi \in \left[0, (\beta^-)^{-1}(0)\right).
\]

Consider the latter values as initial states on both regions $\Sigma^p_1, \Sigma^q_1$ and use (8), we write

\[
p_1(t,c-t) = p_1(\tau, \tau-s), \quad q_1(\chi,c + \chi) = q_1(\chi, c + \chi).
\]

**Figure 3.** The regions $\Sigma^p_i$ are those between the red lines and $\Sigma^q_i$ are those between the blue lines.

**Remark 8.** In particular, if $\alpha \equiv 0$ and $\beta \equiv 1$, the regions $\Sigma_n^p, \Sigma_n^q, n \geq 0$, are simply given by

\[
\Sigma_n^p = \{(t, x) \in \mathbb{R} \times [0, 1], \ t - x \in [n - 1, n]\}, \\
\Sigma_n^q = \{(t, x) \in \mathbb{R} \times [0, 1], \ x + t \in [n, n + 1]\}.
\]
By using the fact that \( p \) and \( q \) are constant along the characteristic lines \( x = t - \alpha^- (\tau) \) and \( x = -t + \beta^+ (\chi) \) respectively, we obtain
\[
p_1(t, t - \alpha^- (\tau)) = p_1(\tau, \alpha(\tau)) = v(\tau) - F(\tau)\tilde{q}(\alpha^+(\tau)), \tag{40}
\]
and
\[
q_1(t, -t + \beta^+ (\chi)) = q_1^-(\chi, \beta(\chi)) = -\tilde{p}(-\beta^-(\chi)). \tag{41}
\]
Now, letting \((\alpha^-)^{-1}(t - x) = \tau\) in \eqref{40} and \(\chi = (\beta^+)^{-1}(x + t)\) in \eqref{41} yields the desired result.

\textbf{Remark 9.} Note that \(\alpha^+ \circ (\alpha^-)^{-1}(t - x), (t, x) \in \Sigma^p \) and \(-\beta^- \circ (\beta^+)^{-1}(x + t), (t, x) \in \Sigma^q \) belong to \((0, 1]\) and the above expressions make perfectly sense. To clarify more things, let \((t, x) \in \Sigma^p \) and let \(\tilde{x}(s) = s - t + x\) the line passing through the point \((t, x)\). By moving backwards, this line meets the curve \((s, \alpha(s))_{s \geq 0}\) at the point \(\left((\alpha^-)^{-1}(t - x), \alpha(\alpha^-)^{-1}(t - x)\right)\) where \((\alpha^-)^{-1}(t - x) \in [0, \alpha^-)^{-1}(1)]\). We use again the reflection of the characteristic line with negative slope passing through the latter point, i.e. \(\tilde{x}(s) = -s + \alpha^+ \circ (\alpha^-)^{-1}(t - x)\) lying in \(\Sigma^q\), for \(s = 0\), we obtain \(\tilde{x}(0) = \alpha^+ \circ (\alpha^-)^{-1}(t - x) \in (0, 1]\). We can do similarly for \(-\beta^- \circ (\beta^+)^{-1}(x + t), (t, x) \in \Sigma^q\).

\textbf{Lemma 3.3.} Let \((\tilde{p}, \tilde{q}) \in \left[L^2(0, 1]\right]^2 \). The solution \((p_2, q_2)\) to system \eqref{7} is given by
\[
p_2(t, x) = v \left((\alpha^-)^{-1}(t - x)\right) + F \left((\alpha^-)^{-1}(t - x)\right) \tilde{p}(-\phi^{-1}(t - x)), \tag{42}
\]
\[
q_2(t, x) = -v \left((\alpha^-)^{-1} \circ \beta^- \circ (\beta^+)^{-1}(x + t)\right) + F \left((\alpha^-)^{-1} \circ \beta^- \circ (\beta^+)^{-1}(x + t)\right) \tilde{q}(\xi^{-1}(x + t)), \tag{43}
\]
where \(\phi\) and \(\xi\) are defined in \eqref{13} and \eqref{33}.

\textbf{Proof.} From \eqref{35} and \eqref{36}, we have at the boundary curves
\[
p_1(\tau, \beta(\tau)) = v \left((\alpha^-)^{-1} \circ \beta^- (\tau)\right) - F \left((\alpha^-)^{-1} \circ \beta^- (\tau)\right) \tilde{q} \left((\alpha^+ \circ (\alpha^-)^{-1} \circ \beta^- (\tau)\right), \tag{44}
\]
and
\[
q_1(\chi, \alpha(\chi)) = -\tilde{p} \left(-\beta^- \circ (\beta^+)^{-1} \circ \alpha^+(\chi)\right), \tag{45}
\]
\(\tau \in \left[(\beta^-)^{-1}(0), (\beta^-)^{-1} \circ \phi^{-1} (\alpha^+)^{-1}(1)\right]\) and \(\chi \in \left[(\alpha^+)^{-1}(1), (\alpha^+)^{-1} \circ \beta^+ \circ (\beta^-)^{-1}(0)\right]\).

In order to find \(p_2\) and \(q_2\), we use the boundary conditions \eqref{9} and the values of \(p_1\) and \(q_1\) at the boundary curves given in \eqref{44} and \eqref{45} as initial states. Namely, for any \(\tau \in \left[(\beta^-)^{-1}(0), (\beta^-)^{-1} \circ \phi^{-1} (\alpha^+)^{-1}(1)\right]\) and \(\chi \in \left[(\alpha^+)^{-1}(1), (\alpha^+)^{-1} \circ \beta^+ \circ (\beta^-)^{-1}(0)\right]\), we have along the lines \(x = t - \alpha^- (\tau)\) and \(x = -t + \beta^+ (\chi)\) respectively
\[
p_2(t, t - \alpha^- (\tau)) = p_2(\tau, \alpha(\tau)) = v(\tau) - F(\tau)q_1(\tau, \alpha(\tau)), \tag{46}
\]
\[
q_2(t, \beta^+(\chi) - t) = q_2(\chi, \beta(\chi)) = -p_1(\chi, \beta(\chi)). \tag{47}
\]
Plugging \eqref{44} and \eqref{45} in \eqref{46} and \eqref{47}, we get
\[
p_2(t, t - \alpha^- (\tau)) = v(\tau) + F(\tau)\tilde{p} \left(-\beta^- \circ (\beta^+)^{-1} \circ \alpha^+(\tau)\right), \tag{48}
\]
and
\[ q_2(t, \beta^+(\chi) - t) = -v\left((\alpha^-)^{-1} \circ \beta^-(\chi)\right) + F\left((\alpha^-)^{-1} \circ \beta^-(\chi)\right)\tilde{q}\left(\alpha^+ \circ (\alpha^-)^{-1} \circ \beta^-(\chi)\right). \]

The proof follows immediately for \( \tau = (\alpha^-)^{-1}(t - x) \) and \((\beta^+)^{-1}(x + t) = \chi.\)

**Remark 10.** In the same spirit of Remark 9, the expressions (42) and (43) make perfectly sense. We can use the same reasoning to show that
\[ -\beta^- \circ (\beta^+)^{-1} \circ \alpha^+ \circ (\alpha^-)^{-1}(t - x) \in (0, 1), \forall (t, x) \in \Sigma^n_p, \]
\[ \alpha^+ \circ (\alpha^-)^{-1} \circ \beta^- \circ (\beta^+)^{-1}(x + t) \in (0, 1), \forall (t, x) \in \Sigma^n_q. \]

More generally, we have:

**Lemma 3.4.** Let \((\tilde{p}, \tilde{q}) \in \left[L^2(0, 1)\right]^2\). The solutions \(p_{2n+1}, p_{2n+2}, q_{2n+1}, q_{2n+2}, n \geq 1,\) to system (7) are given by
\begin{align*}
p_{2n+1}(t, x) &= \sum_{k=0}^{n} v\left((\alpha^-)^{-1} \circ (\phi^{-1})^{|k|}(t - x)\right) \prod_{i=0}^{k-1} F\left((\alpha^-)^{-1} \circ (\phi^{-1})^{|i|}(t - x)\right) \\
-\tilde{q}\left((\xi^{-1})^{|n|} \circ \alpha^+ \circ (\alpha^-)^{-1}(t - x)\right) \prod_{k=0}^{n} F\left((\alpha^-)^{-1} \circ (\phi^{-1})^{|k|}(t - x)\right),
\end{align*}

\begin{align*}
p_{2n+2}(t, x) &= \sum_{k=0}^{n} v\left((\alpha^-)^{-1} \circ (\phi^{-1})^{|k|}(t - x)\right) \prod_{i=0}^{k-1} F\left((\alpha^-)^{-1} \circ (\phi^{-1})^{|i|}(t - x)\right) \\
+\tilde{p}\left(- (\phi^{-1})^{|n+1|} \circ (t - x)\right) \prod_{k=0}^{n} F\left((\alpha^-)^{-1} \circ (\phi^{-1})^{|k|}(t - x)\right),
\end{align*}

\begin{align*}
q_{2n+1}(t, x) &= -\sum_{k=0}^{n-1} v\left((\alpha^-)^{-1} \circ (\phi^{-1})^{|k|} \circ \beta^- \circ (\beta^+)^{-1}(x + t)\right) \times \\
\prod_{i=0}^{k-1} F\left((\alpha^-)^{-1} \circ (\phi^{-1})^{|i|} \circ \beta^- \circ (\beta^+)^{-1}(x + t)\right) \\
-\tilde{p}\left(- (\phi^{-1})^{|n|} \circ \beta^- \circ (\beta^+)^{-1}(x + t)\right) \times \\
\prod_{k=0}^{n-1} F\left((\alpha^-)^{-1} \circ (\phi^{-1})^{|k|} \circ \beta^- \circ (\beta^+)^{-1}(x + t)\right),
\end{align*}

\begin{align*}
q_{2n+2}(t, x) &= -\sum_{k=0}^{n} v\left((\alpha^-)^{-1} \circ (\phi^{-1})^{|k|} \circ \beta^- \circ (\beta^+)^{-1}(x + t)\right) \times \\
\prod_{i=0}^{k-1} F\left((\alpha^-)^{-1} \circ (\phi^{-1})^{|i|} \circ \beta^- \circ (\beta^+)^{-1}(x + t)\right) \\
+\tilde{q}\left((\xi^{-1})^{|n+1|}(x + t)\right) \times \\
\prod_{k=0}^{n} F\left((\alpha^-)^{-1} \circ (\phi^{-1})^{|k|} \circ \beta^- \circ (\beta^+)^{-1}(x + t)\right),
\end{align*}
with the convention $\prod_{k=0}^{n-1} = 1$. The functions $\phi$ and $x$ are defined in (13) and (33).

**Proof.** The above expressions can be proved by induction. Let us start by proving (51). At the boundary $x = \beta(t)$, (48) becomes

$$p_{2n+1}(t, \beta(t)) = \sum_{k=0}^{n} v \left( (\alpha^{-1} \circ (\phi^{-1})^k \circ \beta^{-1}(t) \right) \prod_{i=0}^{k-1} F \left( (\alpha^{-1} \circ (\phi^{-1})^i \circ \beta^{-1}(t) \right)$$

$$-q \left( (\xi^{-1})^n \circ \alpha^+ \circ (\alpha^{-1} \circ \beta^{-1}(t) \right) \prod_{k=0}^{n} F \left( (\alpha^{-1} \circ (\phi^{-1})^k \circ \beta^{-1}(t) \right).$$

Now, we use the boundary condition given in (9), i.e.

$$q_{2n+2}(\chi, \beta(\chi)) = -\sum_{k=0}^{n} v \left( (\alpha^{-1} \circ (\phi^{-1})^k \circ \beta^{-1}(\chi) \right) \prod_{i=0}^{k-1} F \left( (\alpha^{-1} \circ (\phi^{-1})^i \circ \beta^{-1}(\chi) \right)$$

$$+ q \left( (\xi^{-1})^n \circ \alpha^+ \circ (\alpha^{-1} \circ \beta^{-1}(\chi) \right) \prod_{k=0}^{n} F \left( (\alpha^{-1} \circ (\phi^{-1})^k \circ \beta^{-1}(\chi) \right).$$

Since $q$ is constant along the characteristic lines of the form $x = c - t$, in particular, on the line $x = \beta^+(\chi) - t$, we have

$$q_{2n+2}(t, \beta^+(\chi) - t) = q_{2n+2}(\chi, \beta(\chi)).$$

Finally, by letting $\chi = (\beta^+)^{-1}(x + t)$ in (52), we obtain the formula in (51). Let us do similarly for $p_{2n+2}$. By taking (50) for $x = \alpha(t)$, we obtain

$$q_{2n+1}(t, \alpha(t)) = -\sum_{k=0}^{n-1} v \left( (\alpha^{-1} \circ (\phi^{-1})^k \circ \beta^{-1} \circ (\beta^+)^{-1} \circ \alpha^+(t) \right) \times$$

$$\prod_{i=0}^{k-1} F \left( (\alpha^{-1} \circ (\phi^{-1})^i \circ \beta^{-1} \circ (\beta^+)^{-1} \circ \alpha^+(t) \right)$$

$$-p \left( (\phi^{-1})^n \circ \beta^{-1} \circ (\beta^+)^{-1} \circ \alpha^+(t) \right) \times$$

$$\prod_{k=0}^{n-1} F \left( (\alpha^{-1} \circ (\phi^{-1})^k \circ \beta^{-1} \circ (\beta^+)^{-1} \circ \alpha^+(t) \right).$$

Using the boundary condition

$$p_{2n+2}(\tau, \alpha(\tau)) = v(\tau) - F(\tau)q_{2n+1}(\tau, \alpha(\tau)),$$

$\tau \in \left[ (\alpha^+)^{-1} \circ \xi^n(1), (\alpha^+)^{-1} \circ \xi^n \circ \beta^+ \circ (\beta^-)^{-1}(0) \right],$

and the fact that $q$ is constant along the characteristic lines $x = c - t$, in particular, on the line $x = t - \alpha^{-1}(\tau)$, we obtain

$$p_{2n+2}(\tau, t - \alpha^{-1}(\tau)) = v(\tau) - F(\tau)q_{2n+1}(\tau, \alpha(\tau)).$$

(54)

By letting $\tau = (\alpha^{-1})^{-1}(t - x)$ in (53) and plugging the result in (54) then using the definition of $\phi$ given in (13), we get

$$p_{2n+2}(t, x)$$
proof of main results.

Proof of the controllability theorem.

4.1. Proof of the controllability theorem. Let $F \equiv 1$ in (35), (36), (42) and (43). The solution $p_1$ sees the control immediately for $t \geq 0$, on the contrary, the component $q_1$ has to wait one more reflection on the curve $(t, \alpha(t))_{t \geq 0}$ to see it as soon as $t \geq (\beta^-)^{-1}(0)$.

Let us start by proving the necessary part: $T_{\varepsilon}^* = T^* - \varepsilon$ for sufficiently small $\varepsilon > 0$; the solution $q$ at this time is given by

$$q(T_{\varepsilon}^*, x) = \begin{cases} q_1^c(T_{\varepsilon}^*, x) & \text{if } x \in [\alpha(T_{\varepsilon}^*), T_{\varepsilon}^* - \beta^+ \circ (\beta^-)^{-1}(0)], \\ q_2^c(T_{\varepsilon}^*, x) & \text{if } x \in \left[T_{\varepsilon}^* - \beta^+ \circ (\beta^-)^{-1}(0), \beta(T_{\varepsilon}^*)\right]. \end{cases}$$

Thus, system (1) will be never exactly controllable since we have for any initial state $\tilde{p}$ and any target state $k$

$$q(T_{\varepsilon}^*, x) = -\tilde{p} \left(-\beta^\circ (\beta^-)^{-1}(x + T_\varepsilon)\right) = k(x), \quad x \in [\alpha(T_{\varepsilon}^*), T_{\varepsilon}^* - \beta^+ \circ (\beta^-)^{-1}(0)],$$

which is clearly a violating of the initial states. □
Now, we prove the sufficient part:

**Proposition 2.** If $T \geq T^* = (\alpha^+)^{-1} \circ \beta^+ \circ (\beta^-)^{-1} (0)$, then system \((7)\) is exactly controllable at time $T$.

**Proof.** It suffices to prove it for $T = T^*$. Let $(h, k) \in L^2(\alpha(T^*), \beta(T^*))$ be a target state and let $T''^* = (\beta^-)^{-1} \circ \alpha^- \circ (\alpha^+)^{-1} (1)$. We have three possible configurations:

**Case 1: $T''^* = T^*$**

In this case, we have $p(T^*) = p_2(T^*)$ and $q(T^*) = q_2(T^*)$, then by making use of \((42)\) and \((43)\) we obtain

$$h(x) = p_2(T^*, x) = v \left( (\alpha^-)^{-1} (T^* - x) \right) + \tilde{p} \left( -\phi^{-1}(T^* - x) \right), \quad x \in (\alpha(T^*), \beta(T^*))$$

$$k(x) = -v \left( (\alpha^-)^{-1} \circ \beta^- \circ (\beta^+)^{-1} (x + T^*) \right) + \tilde{q} \left( \xi^{-1}(x + T^*) \right), \quad x \in (\alpha(T^*), \beta(T^*))$$

Therefore, the control $v$ is given by

$$v(t) = \begin{cases} 
    h \left( T^* - \alpha^- (t) \right) - \tilde{p} \left( -\phi^{-1} \circ \alpha^- (t) \right), & \text{if } t \in \left( (\alpha^-)^{-1} \circ \beta^- (T^*), T^* \right), \\
    \tilde{q} \left( \alpha^+ (t) \right), & \text{if } t \in \left( 0, (\alpha^-)^{-1} \circ \beta^- (T^*) \right), \\
    -k \left( \beta^+ \circ (\beta^-)^{-1} \circ \alpha^- (t) - T^* \right), & \text{if } t \in \left( (\alpha^-)^{-1} \circ \beta^- (T^*), T^* \right).
\end{cases}$$

**Case 2: $T''^* < T^*$**

In this case, $p(T^*)$ and $q(T^*)$ are defined by

$$p(T^*, x) = \begin{cases} 
    p_1(T^*, x), & \text{if } x \in \left( T^* - \alpha^- \circ (\alpha^+)^{-1} (1), \beta(T^*) \right), \\
    p_2(T^*, x), & \text{if } x \in \left( \alpha(T^*), T^* - \alpha^- \circ (\alpha^+)^{-1} (1) \right),
\end{cases}$$

and $q(T^*) = q_2(T^*)$. Thus, by making use of \((35),(42)\) and \((43)\), then making some variable substitutions, we arrive at

$$v(t) = \begin{cases} 
    h_1 \left( T^* - \alpha^- (t) \right) + \tilde{q} \left( \alpha^+ (t) \right), & \text{if } t \in \left( (\alpha^-)^{-1} \circ \beta^- (T^*), (\alpha^+)^{-1} (1) \right), \\
    h_2 \left( T^* - \alpha^- (t) \right) - \tilde{p} \left( -\phi^{-1} \circ \alpha^- (t) \right), & \text{if } t \in \left( (\alpha^+)^{-1} (1), T^* \right), \\
    \tilde{q} \left( \alpha^+ (t) \right), & \text{if } t \in \left( 0, (\alpha^-)^{-1} \circ \beta^- (T^*) \right), \\
    -k \left( \beta^+ \circ (\beta^-)^{-1} \circ \alpha^- (t) - T^* \right), & \text{if } t \in \left( (\alpha^-)^{-1} \circ \beta^- (T^*), T^* \right),
\end{cases}$$

where $h_1$ and $h_2$ are the restrictions of the target state $h$ on the regions $\Sigma_1^p$ and $\Sigma_2^p$ respectively.
Case 3: $T^* > T^*$
In this case, we have $p(T^*) = p_2(T^*)$, and $q(T^*)$ is defined by
\[
q(T^*, x) = \begin{cases} 
q_2(T^*, x), & \text{if } x \in (\alpha(T^*), \xi(1) - T^*), \\
q_1(T^*, x), & \text{if } x \in (\xi(1) - T^*, \beta(T^*)).
\end{cases}
\]
By using (42), (43) and (50) for $n = 1$ and $t = T^*$, then making some variable substitutions, we obtain
\[
v(t) = \begin{cases} 
\tilde{h} \left( T^* - \alpha(t) - t \right) - \tilde{p} (\alpha(t) \circ \alpha^{-1}(t)), & \text{if } t \in \left( (\alpha^{-1} \circ \beta^{-1}(T^*) - T^*) \right), \\
-k_2 (\beta^{-1} \circ \alpha^{-1}(t) - T^*), & \text{if } t \in \left( (0, \alpha^{-1}(1)) \right), \\
-k_3 (\beta^{-1} \circ \alpha^{-1}(t) - T^*), & \text{if } t \in \left( (\alpha^{-1}(1), (\alpha^{-1} \circ \beta^{-1}(T^*)) \right).
\end{cases}
\]
where $k_2$ and $k_3$ are the restrictions of the target state $k$ on the regions $\Sigma^2_2$ and $\Sigma^2_3$, respectively. The above expressions are well defined and the control $v$ is uniquely determined on $[0, T^*)$. In particular, from (42) and (43), we can see that the control
\[
v(t) = \begin{cases} 
\tilde{q} (\alpha(t)), & \text{if } t \in \left( [0, \alpha^{-1}(1)) \right), \\
-\tilde{p} (\beta^{-1} \circ \alpha^{-1}(t)), & \text{if } t \in \left( (\alpha^{-1}(1), T^*) \right), \\
0, & \text{if } t \geq T^*.
\end{cases}
\]
makes $p_2$ and $q_2$ vanish, then by the boundary conditions given in (9) all the solutions $p_n, q_n, n \geq 2$, will be zero. To get an explicit formula of the control $u$, it suffices to inverse the transformation defined in (4), then using the compatibility condition $y_0(0) = u(0)$ to obtain (12).

**Remark 13.** Since we have an explicit formula of the solution for all $t \geq 0$, we can prove that exact controllability holds at any time $T > T^*$ with loss of uniqueness of the control.

4.2. Proof of the stability theorem. In this subsection, we let $v \equiv 0$. We start by proving the sufficient part.

At time $t \geq 0$, the components $p(t)$ and $q(t)$ might involve at most three values of the restrictive solutions $p_n(t)$ and $q_n(t)$ respectively on the contrary of the cylindrical case where $p(t)$ and $q(t)$ might involve at most two values (see Figure 3), (if $p(t)$ or $q(t)$ are defined on four regions, we obtain $\alpha(t) > \beta(t)$). Let us deal with the worst case that might occur. We have for the component $p$:

**Case 1:** $t \in \left( (\alpha^{-1} \circ \phi^{[n-1]} \circ \alpha^{-1}(1), (\alpha^{-1} \circ \phi^{[n]}(0)) \right).

In this case, $p(t)$ might expressed in function of $p_{2n-1}(t), p_{2n}(t), p_{2n+1}(t)$,
\[
p(t, x) = \begin{cases} 
p_{2n-1}(t, x), & \text{if } x \in I_1(t) := \left[ t - \phi^{[n-1]}(0), t - \phi^{[n]}(0) \right), \\
p_{2n}(t, x), & \text{if } x \in I_2(t) := \left[ t - \phi^{[n]}(0), t - \phi^{[n-1]}(0) \right), \\
p_{2n+1}(t, x), & \text{if } x \in I_3(t) := \left[ \alpha(t), \beta(t) \right).
\end{cases}
\]

By definition of the regions $\Sigma^2_n, n \geq 0$, given in (25)-(32), we have for $k = 1, 2, 3$
\[
\left\{ (t, x) \in \left( (\alpha^{-1} \circ \phi^{[n-1]} \circ \alpha^{-1}(1), (\alpha^{-1} \circ \phi^{[n]}(0)) \times I_k(t) \right)
\right\}
\subset \Sigma^2_{2n+k-2},
\]
consequently,
\[
\|p(t)\|_{L^2(I(a(t), \beta(t)))}^2 = \sum_{k=1}^{3} \|p_{2n+k-2}(t)\|_{L^2(I_k(t))}^2
\]
therefore, there exist a sequences given in (48) and (49), we obtain for $k = 1, 2, 3$

\[
\sum_{k=1}^{3} \int_{(t,x) \in \Sigma^2_{n+k-2}} |p_{2n+k-2}(t, x)|^2 \, dx.
\]

which leads us to estimate the right hand side of (56). By using the exact solution formulas

\[
\sum_{k=1}^{3} \int_{(t,x) \in \Sigma^2_{n+k-2}} |p_{2n+k-2}(t, x)|^2 \, dx
\]

\[
\leq \| (\tilde{p}, \tilde{q}) \|^2_{L^2(0,1)} \sum_{k=1}^{3} \sup_{(t,x) \in \Sigma^2_{n+k-2}} \prod_{i=0}^{n-1+\left\lfloor \frac{k-1}{n} \right\rfloor} |F \left( (\alpha^-)^{-1} \circ (\phi^{-1})^{[i]} (t, x) \right)|.
\]

By definition of the regions $\Sigma^2_n$, $n \geq 0$ given in (25)-(32), we have

\[
(t, x) \in \Sigma^2_{n+k} \Leftrightarrow t-x \in \left[ \phi^{-[n-1]} \circ \alpha^- \circ (\alpha^+)^{-1} (1), \phi^{[n]}(0) \right],
\]

\[
(t, x) \in \Sigma^2_{2n+1} \Leftrightarrow t-x \in \left[ \phi^{[n]}(0), \phi^{[n]} \circ \alpha^- \circ (\alpha^+)^{-1} (1) \right],
\]

therefore, there exist a sequences $\tau_1(t, x) \in \left[ \alpha^- \circ (\alpha^+)^{-1} (1), \phi(0) \right]$ and $\tau_2(t, x) \in \left[ 0, \alpha^- \circ (\alpha^+)^{-1} (1) \right]$ such that

\[
(t, x) \in \Sigma^2_{2n} \Leftrightarrow t-x = \phi^{-[n-1]} (\tau_1(t, x)),
\]

\[
(t, x) \in \Sigma^2_{2n+1} \Leftrightarrow t-x = \phi^{[n]} (\tau_2(t, x)).
\]

Observe that when $(t, x)$ runs $\Sigma^2_n$ (resp. $\Sigma^2_{2n+1}$), the bounded sequence $\tau_1(t, x)$ (resp. $\tau_2(t, x)$) rises $\left[ \alpha^- \circ (\alpha^+)^{-1} (1), \phi(0) \right]$ (resp. $\left[ 0, \alpha^- \circ (\alpha^+)^{-1} (1) \right]$). These sequences will play the role of two parameters $\tau_1 \in \left[ \alpha^- \circ (\alpha^+)^{-1} (1), \phi(0) \right]$ and $\tau_2 \in \left[ 0, \alpha^- \circ (\alpha^+)^{-1} (1) \right]$. With these notations, we have

\[
\sum_{k=1}^{3} \sup_{(t,x) \in \Sigma^2_{n+k-2}} \prod_{i=0}^{n-1+\left\lfloor \frac{k-1}{n} \right\rfloor} |F \left( (\alpha^-)^{-1} \circ (\phi^{-1})^{[i]} (t, x) \right)|
\]

\[
\leq \sup_{\tau \in \left[ 0, \alpha^- \circ (\alpha^+)^{-1} (1) \right]} \psi_n(\tau) + \sup_{\tau \in \left[ 0, \alpha^- \circ (\alpha^+)^{-1} (1) \right]} \psi_n(\tau).
\]

So,

\[
\| p(t) \|^2_{L^2(\alpha(t), \beta(t))} \leq \| (\tilde{p}, \tilde{q}) \|^2_{L^2(0,1)} \sup_{\tau \in \left[ 0, \alpha^- \circ (\alpha^+)^{-1} (1) \right]} \psi_n(\tau) \]

\[
+ \| (\tilde{p}, \tilde{q}) \|^2_{L^2(0,1)} \sup_{\tau \in \left[ 0, \alpha^- \circ (\alpha^+)^{-1} (1) \right]} \psi_n(\tau)
\]

\[
+ \| (\tilde{p}, \tilde{q}) \|^2_{L^2(0,1)} \sup_{\tau \in \left[ 0, \alpha^- \circ (\alpha^+)^{-1} (1) \right]} \psi_n(\tau).
\]
Case 2: $t \in \left[\left(\alpha^-\right)^{-1} \circ \phi[n](0), \left(\alpha^+\right)^{-1} \circ \alpha^- \circ \left(\alpha^+\right)^{-1}(1)\right]$.

In this case, $p(t)$ might be expressed in function of $p_{2n}(t), p_{2n+1}(t), p_{2n+2}(t)$

$$p(t, x) = \begin{cases} p_{2n}(t, x), & \text{if } x \in I_4(t) := \left[t - \phi[n](0), \beta(t)\right], \\
p_{2n+1}(t, x), & \text{if } x \in I_5(t) := \left[t - \phi[n] \circ \alpha^- \circ \left(\alpha^+\right)^{-1}(1), t - \phi[n](0)\right], \\
p_{2n+2}(t, x), & \text{if } x \in I_6(t) := \left[\alpha(t), t - \phi[n] \circ \alpha^- \circ \left(\alpha^+\right)^{-1}(1)\right]. \end{cases}$$

(59)

In the same way, we obtain the estimate

$$\left\|p(t)\right\|_{L^2(\alpha(t), \beta(t))}^2 \leq \left\|\left(p, \tilde{q}\right)\right\|_{L^2(0,1)}^2 \sup_{\tau \in \left[\alpha^- \circ \left(\alpha^+\right)^{-1}(1), \phi(0)\right]} \psi_{n-1}(\tau)$$

$$+ \left\|\left(p, \tilde{q}\right)\right\|_{L^2(0,1)}^2 \sup_{\tau \in \left[0, \alpha^- \circ \left(\alpha^+\right)^{-1}(1)\right]} \psi_n(\tau)$$

$$+ \left\|\left(p, \tilde{q}\right)\right\|_{L^2(0,1)}^2 \sup_{\tau \in \left[\alpha^- \circ \left(\alpha^+\right)^{-1}(1), \phi(0)\right]} \psi_n(\tau).$$

(60)

Analogously, we have for the component $q$:

Case 1: $t \in \left[\left(\beta^+\right)^{-1} \circ \xi[n-1] \circ \beta^+ \circ \left(\beta^-\right)^{-1}(0), \left(\beta^+\right)^{-1} \circ \xi[n](1)\right]$.

The expression of $q(t)$ might involve the expressions of $q_{2n-1}(t), q_{2n}(t), q_{2n+1}(t)$

$$q(t, x) = \begin{cases} q_{2n-1}(t, x), & \text{if } x \in J_1(t) := \left[\alpha(t), \xi[n-1] \circ \beta^+ \circ \left(\beta^-\right)^{-1}(0) - t\right], \\
q_{2n}(t, x), & \text{if } x \in J_2(t) := \left[\xi[n-1] \circ \beta^+ \circ \left(\beta^-\right)^{-1}(0) - t, \xi[n](1) - t\right], \\
q_{2n+1}(t, x), & \text{if } x \in J_3(t) := \left[\xi[n](1) - t, \beta(t)\right]. \end{cases}$$

(61)

So, we have

$$\left\|q(t)\right\|_{L^2(\alpha(t), \beta(t))}^2 \leq \sum_{k=1}^3 \int_{\left(t, x\right) \in \Sigma_{2n+k-2}^n} |q_{2n+k-2}(t, x)|^2 \, dx$$

$$\leq \left\|\left(p, \tilde{q}\right)\right\|_{L^2(0,1)}^2 \times \sum_{k=1}^3 \sup_{t \in \left(t, x\right) \in \Sigma_{2n+k-2}^n} n^{-2+\left|\frac{k-1}{2}\right|} \prod_{i=0}^{k-1} \left|F\left((\alpha^-)^{-1} \circ \phi^{-1}[k] \circ \beta^- \circ \left(\beta^+\right)^{-1}(x + t)\right)\right|.$$
therefore, there exist \( \chi_1 := \chi^n(t, x) \in \left[ 0, \phi \circ \beta^{-1} \circ (\beta^+)^{-1}(1) \right] \) and \( \chi_2 := \chi^2(t, x) \in \left[ \phi \circ \beta^{-1} \circ (\beta^+)^{-1}(1), \phi(0) \right] \) such that

\[
(t, x) \in \Sigma_{2n}^n \Leftrightarrow t + x = \beta^+ \circ (\beta^-)^{-1} \circ \phi^{[n-1]}(\chi_1),
\]

\[
(t, x) \in \Sigma_{2n+1}^n \Leftrightarrow t + x = \beta^+ \circ (\beta^-)^{-1} \circ \phi^{[n]}(\chi_2).
\]

Thus, by combining (62), (65) and (66), we obtain

\[
\sum_{k=1}^{3} \sup_{k \in \mathbb{Z}} \prod_{i=0}^{n-2} F(\alpha^{-1} \circ (\phi^{-1})^k \circ \beta^{-1} \circ (\beta^+)^{-1}(x + t)) \leq \sup_{\chi_2 \in \left[ \phi \circ \beta^{-1} \circ (\beta^+)^{-1}(1), \phi(0) \right]} \prod_{i=0}^{n-2} F(\alpha^{-1} \circ (\phi^{-1})^i \circ \beta^{-1} \circ (\beta^+)^{-1}) \Bigg| \sup_{\chi_1 \in \left[ 0, \phi \circ \beta^{-1} \circ (\beta^+)^{-1}(1) \right]} \prod_{i=0}^{n-2} F(\alpha^{-1} \circ (\phi^{-1})^i \circ \beta^{-1} \circ (\beta^+)^{-1}) \Bigg| \sup_{\chi_2 \in \left[ \phi \circ \beta^{-1} \circ (\beta^+)^{-1}(1), \phi(0) \right]} \prod_{i=0}^{n-2} F(\alpha^{-1} \circ \phi^{[n-1]}(\chi_2)) \Bigg| \sup_{\chi_1 \in \left[ 0, \phi \circ \beta^{-1} \circ (\beta^+)^{-1}(1) \right]} \prod_{i=0}^{n-2} F(\alpha^{-1} \circ \phi^{[n-1]}(\chi_1)) \Bigg| \sup_{\chi_2 \in \left[ \phi \circ \beta^{-1} \circ (\beta^+)^{-1}(1), \phi(0) \right]} \prod_{i=0}^{n-1} F(\alpha^{-1} \circ \phi^{[n]}(\chi_2)) \Bigg|.
\]

Finally, we get

\[
\|q(t)\|_{L^2(\alpha(t), \beta(t))}^2 \leq C \|\tilde{\beta}, \tilde{q}\|_{L^2(0, 1)}^2 \sup_{\chi \in \left[ \phi \circ \beta^{-1} \circ (\beta^+)^{-1}(1), \phi(0) \right]} \psi_{n-1}(\chi) + C \|\tilde{\beta}, \tilde{q}\|_{L^2(0, 1)}^2 \sup_{\chi \in \left[ 0, \phi \circ \beta^{-1} \circ (\beta^+)^{-1}(1) \right]} \psi_{n-1}(\chi) + C \|\tilde{\beta}, \tilde{q}\|_{L^2(0, 1)}^2 \sup_{\chi \in \left[ \phi \circ \beta^{-1} \circ (\beta^+)^{-1}(1), \phi(0) \right]} \psi_{n-1}(\chi).
\]

**Case 2:** \( t \in \left( (\beta^+)^{-1} \circ \xi^{[n]}(1), (\beta^+)^{-1} \circ \xi^{[n]} \circ \beta^+ \circ (\beta^-)^{-1}(0) \right) \).

As previously, \( q(t) \) might involve the values of \( q_{2n}(t), q_{2n+1}(t), q_{2n+2}(t) \)

\[
q(t, x) = \begin{cases} 
q_{2n}(t, x), & \text{if } x \in J_4(t) := \bigg( \alpha(t), \xi^{[n]}(1) - t \bigg), \\
q_{2n+1}(t, x), & \text{if } x \in J_5(t) := \bigg( \xi^{[n]}(1) - t, \xi^{[n]} \circ \beta^+ \circ (\beta^-)^{-1}(0) - t \bigg), \\
q_{2n+2}(t, x), & \text{if } x \in J_6(t) := \bigg( \xi^{[n]} \circ \beta^+ \circ (\beta^-)^{-1}(0) - t, \beta(t) \bigg).
\end{cases}
\]

In the same way, the following estimate holds

\[
\|q(t)\|_{L^2(\alpha(t), \beta(t))}^2 \leq C \|\tilde{\beta}, \tilde{q}\|_{L^2(0, 1)}^2 \sup_{\chi \in \left[ 0, \phi \circ \beta^{-1} \circ (\beta^+)^{-1}(1) \right]} \psi_{n-1}(\chi) + C \|\tilde{\beta}, \tilde{q}\|_{L^2(0, 1)}^2 \sup_{\chi \in \left[ \phi \circ \beta^{-1} \circ (\beta^+)^{-1}(1), \phi(0) \right]} \psi_{n-1}(\chi).
\]
From (58), (60), (67) and (69), we deduce that

\[ \sup_{\tau \in [0, \phi(0))} \psi_n(\tau) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty, \]

which finishes the proof of the first statement of Theorem 2.4. The proof of the second and the third statements are just a consequences of (70). By definition of the regions \( \Sigma^p_n, \Sigma^n_b, \)

\( n \geq 0, \) given in (25)-(32), we can see that letting \( t \rightarrow \infty \) is the same as \( \phi_n(\tau) \rightarrow \infty, \]

\( \forall \tau \in [0, \phi(0)) \), so, if there exists a positive function \( g \) such that

\[ Cg(\phi_n(\tau)) \sim \psi_n(\tau), \forall \tau \in [0, \phi(0)), \]

then obviously (18) holds. In particular, exponential stability follows immediately from

\[ \sup_{\tau \in [0, \phi(0))} \psi_n(\tau) = \sup_{\tau \in [0, \phi(0))} \exp \left[ \frac{\ln \psi_n(\tau)}{\phi_n(\tau)} \right]. \]

The proof of the necessary part is straightforward. From (48), (49) and (57), we have

\[
\int_{(t,x) \in \Sigma^p_{n+1}} |p_{2n+1}(t, x)|^2 \, dx + \int_{(t,x) \in \Sigma^p_{n+2}} |p_{2n+2}(t, x)|^2 \, dx \\
\geq C \left[ \frac{\|\tilde{q}\|_{L^2_q(0,1)}^2}{\inf_{x, (t,x) \in \Sigma^p_{n+1}} \prod_{i=0}^n F\left( (\alpha^{-1}) \circ (\phi^{-1})^i (t - x) \right)} \right] \\
+ C \left[ \frac{\|p\|_{L^2_p(0,1)}^2}{\inf_{x, (t,x) \in \Sigma^p_{n+2}} \prod_{i=0}^n F\left( (\alpha^{-1}) \circ (\phi^{-1})^i (t - x) \right)} \right] \\
\geq C \left( \frac{\|\tilde{q}\|_{L^2_q(0,1)}^2 + \|p\|_{L^2_p(0,1)}^2}{\GA} \right) \times \\
\left[ \inf_{\tau \in [0, \alpha^{-1} \circ (\alpha^+)^{-1}(1)]} \psi_n(\tau) + \inf_{\tau \in [\alpha^{-1} \circ (\alpha^+)^{-1}(1), \phi(0)]} \psi_n(\tau) \right],
\]

therefore, if (16) is not satisfied then clearly stability cannot occur.

Let us prove the second claim of Theorem 2.4. If \( f \equiv 1 \) then \( F \equiv 0 \). In this case, we infer from the exact formula of solutions given in (35), (36) and (43) that we have \( p_1 \equiv 0 \) while \( q_1 \neq 0 \), and since \( q \) is constant along the characteristic lines, \( q \) is identically zero from the time that \( q_2 \) will be zero, that is \( t \geq T^* = \frac{1}{\alpha^+} \circ \beta^+ \circ (\beta^+)^{-1}(0) \) which is the same time for boundary controllability of system (1).

5. Further remarks and open questions. Let us discuss briefly some possible variations and generalization of the obtained results in this work.

- It is our hope that the tools developed in this paper may help in dealing with the distributed control case

\[
\begin{align*}
\dot{y}_1(t, x) &= y_{2x}(t, x) + \chi_{\omega_T} h(t, x), & \text{in} & \quad Q_T, \\
\dot{y}(t, \alpha(t)) &= y(t, \beta(t)) = 0, & \text{in} & \quad (0, T), \\
\dot{y}(0, x) &= y_0(x), & \text{in} & \quad (0, 1),
\end{align*}
\]

(71)

where \( \omega_T \) is a moving subset of \( Q_T := (0, T) \times (0, 1) \) defined by

\[ \omega_T = \{(t, x) \in Q_T, x \in (a(t), b(t))\}, \]

and \( (y_0, y_1, h) \in H_0^1(0, 1) \times L^2((0, 1) \times L^2(\omega_T)). \) Actually, we can determine the minimal time for which the time-dependent geometric control condition introduced in [13, Definition 1.6] is satisfied. The latter condition states that every generalized bicharacteristic must meet the moving control region at some time \( T. \) It is easy to verify this condition in the one dimensional settings. Indeed, under assumption
with $a, b \in C^1(0, T)$ and $\|a'\|_{L^\infty(0, T)}, \|b'\|_{L^\infty(0, T)} < 1$, we find that all the characteristics with positive slope or negative slope emerging from the point $(0, x)$, for any $x \in (0, 1)$ meet $\omega_T$ if, and only if $T > T^*$ where $T^*$ is given by

$$T^* = \max\{T_1, T_2\} = \max \left\{ b^+ \circ \beta^+ \circ (\beta^-)^{-1} \circ b(0), a^- \circ \alpha^- \circ (\alpha^+)^{-1} \circ a(0) \right\}.$$  

In particular, if $\alpha \equiv 0$ and $\beta \equiv 1$, the time $T^*$ is given by $T^* = 2 \max\{a, 1 - b\}$ which is exactly the time given in [19].

Controllability and stabilizability of the multidimensional wave equation in non-cylindrical domains has been investigated by Bardos and Chen in [4]. By assuming that the domain is expending, exact controllability and stability have been established by a control and a frictional damping acting on the entire domain.

Distributed controllability of system (71) has been studied in [5] in a cylindrical domain with moving control support, i.e. $\alpha \equiv 0$ and $\beta \equiv 1$. It has been proved that exact controllability holds if the moving control support $\omega_T$ satisfies the geometric control condition without the restriction $\|a'\|_{L^\infty(0, T)}, \|b'\|_{L^\infty(0, T)} < 1$. It worths to mention that problem (71) has been recently studied in [8] with a very particular boundary curves and moving control support. The critical time of control seems to be far from being optimal.

- Observe that we have not used the regularity on the boundary function anywhere. We can weaken this regularity a little bit by assuming that the boundary curves are Lipschitz continuous functions with Lipschitz constant less than one.

- We have taken the wave speed in both systems (1) and (2) equals to one just for the sake of simplicity. The approach used in this paper also works if the first line in both systems (1) and (2) is replaced by $y_{tt} = (a^2(x)y_x)_x$. In this case, it suffices to replace the Riemann invariants introduced in (4) by $p = y_t - a(x)y_x$ and $q = y_t + a(x)y_x$.

- Note that we have not used the $L^2$ settings in a crucial way. The same results can be proved in $L^p$, $p \in [1, \infty)$, or in the space of continuous functions.

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