Change, Time and Information Geometry*

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Abstract

Dynamics, the study of change, is normally the subject of mechanics. Whether the chosen mechanics is “fundamental” and deterministic or “phenomenological” and stochastic, all changes are described relative to an external time. Here we show that once we define what we are talking about, namely, the system, its states and a criterion to distinguish among them, there is a single, unique, and natural dynamical law for irreversible processes that is compatible with the principle of maximum entropy. In this alternative dynamics changes are described relative to an internal, “intrinsic” time which is a derived, statistical concept defined and measured by change itself. Time is quantified change.

1 Introduction

The notion that the concepts of time, change and motion are intimately connected goes back to antiquity. According to Aristotle, “time numbers change with respect to before and after.” One aspect of this connection is the order of a sequence of changes, their temporal order. Another aspect is the use of selected motions or changes to measure the length of time intervals, their duration. We begin by considering the notion of change.

In order to establish that a system has changed one must be able to distinguish between the system being in one state and its being in another state. This requires, to begin with, a clear idea of what is meant by a state. As long as one is interested in the study of phenomena that can be deliberately reproduced by controlling a few macroscopic variables it is reasonable to expect that the values – or rather, the expected values – of these few variables are all that is needed for the purposes of prediction. This limited information defines what we mean by the state or, equivalently, the macrostate of the system.

Next, to measure the extent to which states can be distinguished, we assign a probability distribution to each state. The requirement that the assignment

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procedure itself do not introduce any information beyond that which defines
the state demands we use the method of maximum entropy (ME) \cite{1,2}. In this
way the problem of distinguishing between states is transformed into another
problem, that of distinguishing between the corresponding distributions. The
solution to the latter problem is well known. There is a uniquely natural way
to quantify the extent to which one distribution can be distinguished from an-
other: it is given by the distance between them as measured by the Fisher-Rao
information metric \cite{3}-\cite{6}.

If we think of each state as a point in a manifold, the net outcome of these
considerations (Sect. 2) is that the method of ME has transformed the manifold
of states into a metric space. Distinguishability and therefore change is measured
by distance.

There is not yet any implication that change will happen from one state to
another; to this we turn next. Temporal order, as well as the notion of time
itself, are the subject of dynamics.

Typically, having decided on the kinematics appropriate to a certain move-
motion, one defines the dynamics by additional postulates about the equations of motion,
perhaps in the form of a variational principle. The dynamics is postulated. The
dynamical law that we adopt here (Sect. 3) is a variational principle too, but
there is something very peculiar about it, there is no need to postulate it. The
principle is the same we had already introduced when discussing the space of
states, namely, when selecting a distribution subject to certain constraints, the
preferred distribution is that of maximum entropy. It is just the same old ME
principle applied in a somewhat different way. (The nature of the constraints is
different. For a brief account of the ME method in a form tailored to suit the
needs of this paper see Ref.\cite{7}.)

We have no freedom in choosing the dynamical law; it follows from the single
piece of new information available: recognizing that changes happen. Nothing
else. Suppose the system is in a certain state and a small change happens;
the system moves a distance \(dL\). We cannot with certainty predict in which
direction motion occurs but, according to the principle of ME, unless there is
some positive evidence to the contrary, of all the states on the surface of the
sphere of radius \(dL\) there is one to be preferred above all others: it is the state
of maximum entropy.

As so often in the past, it seems that once more the method of ME has
allowed us to get something out of nothing; yet another free lunch. But the
dynamics proposed here is different in one important respect. (We refrain from
saying “deficient” rather than “different” because in the end it may turn out
to be an advantage.) In the conventional Hamiltonian or Lagrangian mechanics
the equations of motion describe changes relative to an external time. Here
changes are described relative to an internal, “intrinsic” time which is a derived,
statistical concept defined and measured by the change \(dL\) itself. Intrinsic time
is quantified change. The system provides its own clock. Perhaps this is a
necessary feature of any fundamental form of mechanics that generates its own
notion of time, that *explains* time.

The introduction of a metric in the space of states is not new; this has been
done by many authors in statistical inference, where the subject is known as Information Geometry [5][8], and in physics, to study both equilibrium [4][10] and nonequilibrium thermodynamics [11][12]. What is different here is the recognition that this is all one needs to define a dynamics.

An interesting consequence of these ideas is that reciprocity relations of the Onsager type [13] valid near and far from equilibrium are obtained (Sect. 4) without any hypothesis about microscopic reversibility; in fact, no mention is made of any microscopic dynamics. By analyzing specific models other authors [14] have reached similar conclusions: reciprocal relations are possible even if the underlying microscopic dynamics is not reversible.

It is, of course, possible to incorporate more information, that is, additional constraints into the dynamics. In Sect. 5 we consider a simple illustrative example, the intrinsic dynamics of two coupled systems as they evolve towards equilibrium along a trajectory constrained by conservation laws.

Our subject can be approached from another direction. The Greeks did not draw a sharp distinction between change in general and the more special kind of change we call motion; the falling of an apple was not viewed as being in any sense more fundamental than the ripening of an apple. The modern view does draw such distinctions; deterministic motion in space and time is considered basic while other kinds of change – notably irreversible processes in macroscopic systems – are not. They must be understood in terms of the deterministic motion of microscopic constituents. Of course, this view is not wrong, but for some purposes it may be misguided, inconvenient.

All theories describing irreversible processes have, in the past, invariably turned out to be rather formidable (see e.g., [15]-[19]). One reason is that the phenomena to be described are themselves quite complicated. But there is another reason, which is that these theories are attempting to achieve two conflicting goals. One goal is to reach an understanding in terms of the microscopic Hamiltonian laws of motion and requires keeping track of microscopic details. The other goal is to achieve a description in terms of the few variables that matter, those that codify the crucial information relevant to making predictions. Information about the other variables, the vast majority, is totally irrelevant. Achieving such a description requires forgetting about all microscopic details.

It is remarkable that theories that accomplish these two seemingly contradictory goals are at all possible. They involve a very delicate balancing act between keeping track of details, at least for a little while (Hamiltonian evolution), and then throwing them away (projections, coarse-graining, tracing over unwanted variables, etc.).

Our proposal cuts through this Gordian knot. If microscopic details are truly irrelevant then the Hamiltonian evolution itself should be largely irrelevant. The information about irrelevant details should be discarded before, not after, it is computed. This requires formulating a dynamics without the benefit (or, in this case, the hindrance) of Hamiltonians.

A potentially serious problem here is the loss of predictive power that stems from the possibility of being able to choose among different dynamical laws.
What would make us prefer one law over another? Remarkably the problem does not arise; once we define what we are talking about, namely, the states and the criterion to distinguish among them, there is a single, unique, and natural dynamical law that is compatible with the principle of maximum entropy.

The views expressed here are clearly biased in favor of the information theory approach to statistical mechanics, but they need not contradict other points of view. The basic explanation of the second law of thermodynamics was given by Boltzmann and Gibbs long ago but later contributions by many authors have generated several different versions of it. The question of which particular version is the right one remains controversial. However, provided one adopts a certain spirit of tolerance in reading the various authors (words such as entropy or probability can be used with very different meanings), one sees that the different views are not always incompatible. The point we wish to make is that irrespective of which is one’s own personal favorite reason for preferring change in the direction of entropy increase over decrease, the same reason should lead one to prefer a large increase over a small one.

This applies whether we favor the information theory approach [1] or one of the perhaps more traditional points of view such as ergodic theory [2]. For example, directing the system toward a certain region of phase space is easier and is less sensitive to external perturbations if the region is large than if it is small; hitting a large target is easier than hitting a small target; that’s all. Thus entropy should increase to the maximum extent allowed by whatever constraints are known to hold.

This last statement is widely recognized as the basis for equilibrium thermodynamics. But it shouldn’t just apply to the final equilibrium state; it should apply to every one of all the intermediate states along the irreversible trajectory and not just to the end point. Clarifying in precisely what sense this statement can be extended from statics to dynamics is yet another way of stating our goals.

2 Quantifying change

Let the microstates of a physical system be labelled by \(x\), and let \(n(x)dx\) be the number of microstates in the range \(dx\). We assume that a state of the system – that is, a macrostate – is defined by the known expected values \(A^\alpha\) of some \(n_A\) variables \(a^\alpha(x) (\alpha = 1, 2, \ldots, n_A)\),

\[
\langle a^\alpha \rangle = \int dx p(x)a^\alpha(x) = A^\alpha.
\] (1)

This limited information will certainly not be sufficient to answering all questions that one could conceivably ask about the system. Choosing the right set of variables \(\{a^\alpha\}\) is perhaps the most difficult problem in statistical mechanics [2]. A crucial assumption is that Eq. (1) is not just any random information, that it happens to be the right information for our purposes.

It is convenient to think of each state as a point in an \(n_A\)-dimensional manifold; the numerical values \(A^\alpha\) associated to each point form a convenient set
of coordinates. The principle of ME allows us to associate a probability distribution to each point in the space of states. The probability distribution \( p(x|A) \) that best reflects the prior information contained in \( m(x) \) updated by the information \( A^\alpha \) is obtained by maximizing the entropy

\[
S[p] = -\int dx \, p(x) \log \frac{p(x)}{m(x)},
\]

subject to the constraints (1). The result is

\[
p(x|A) = \frac{1}{Z} m(x) e^{-\lambda_\alpha a^\alpha(x)},
\]

where the partition function \( Z \) and the Lagrange multipliers \( \lambda_\alpha \) are given by

\[
Z(\lambda) = \int dx \, m(x) e^{-\lambda_\alpha a^\alpha(x)} \quad \text{and} \quad -\frac{\partial \log Z}{\partial \lambda_\alpha} = A^\alpha.
\]

The maximized value of the entropy is

\[
S(A) = -\int dx \, p(x|A) \log \frac{p(x|A)}{m(x)} = \log Z(\lambda) + \lambda_\alpha A^\alpha.
\]

The second prerequisite to establishing that a system has changed from one state to another is a criterion allowing us to assert that two states \( A \) and \( A + dA \) are not the same. Can we distinguish between the two? If \( dA \) is small enough the corresponding probability distributions \( p(x|A) \) and \( p(x|A + dA) \) overlap considerably and it is easy to confuse them. We seek a real positive number to provide a quantitative measure of the extent to which these two distributions can be distinguished.

The following argument is intuitively appealing. Consider the relative difference,

\[
\frac{p(x|A + dA) - p(x|A)}{p(x|A)} = \frac{\partial \log p(x|A)}{\partial A^\alpha} dA^\alpha.
\]

The expected value of the relative difference might seem a good candidate, but it does not work because it vanishes identically,

\[
\int dx \, p(x|A) \frac{\partial \log p(x|A)}{\partial A^\alpha} dA^\alpha = A^\alpha \frac{\partial}{\partial A^\alpha} \int dx \, p(x|A) = 0.
\]

However, the variance does not vanish,

\[
d\ell^2 = \int dx \, p(x|A) \frac{\partial \log p(x|A)}{\partial A^\alpha} \frac{\partial \log p(x|A)}{\partial A^\beta} dA^\alpha dA^\beta = g_{\alpha\beta} \, dA^\alpha dA^\beta.
\]

This is the measure of distinguishability we seek; a small value of \( d\ell^2 \) means the points \( A \) and \( A + dA \) are difficult to distinguish. The \( g_{\alpha\beta} \) are recognized as elements of the Fisher information matrix \( g \).

Up to now no notion of distance has been introduced on the space of states. Normally one says that the reason it is difficult to distinguish between two
points in say, the real space we seem to inhabit, is that they happen to be too close together. It is very tempting to invert the logic and assert that the two points $A$ and $A + dA$ must be very close together whenever they happen to be difficult to distinguish. Thus it is natural to interpret $g_{\alpha\beta}$ as a metric tensor \[4\]. It is known as the Fisher-Rao metric, or the information metric. A disadvantage of these heuristic arguments is that they do not make explicit a crucial property of the Fisher-Rao metric, except for an overall multiplicative constant this Riemannian metric is unique \[5\][6].

To summarize: the very act of assigning a probability distribution $p(x|A)$ to each point $A$ in the space of states, automatically provides the space of states with a metric structure.

The coordinates $A$ are quite arbitrary, they need not be the expected values $\langle a^{\alpha}\rangle$. One can freely switch from one set to another. It is then easy to check that $g_{\alpha\beta}$ are the components of a tensor, that the distance $d\ell^2$ is an invariant, a scalar. Incidentally, $d\ell^2$ is also dimensionless. There is, however, one special coordinate system in which the metric takes a form that is particularly simple. These coordinates are the expected values themselves, $A^{\alpha} = \langle a^{\alpha}\rangle$. In these coordinates,

$$g_{\alpha\beta} = -\frac{\partial^2 S(A)}{\partial A^{\alpha}\partial A^{\beta}} \tag{9}$$

with $S(A)$ given in Eq.(5) and the covariance is not manifest.

3 Intrinsic dynamics and time

Our basic dynamical principle is that small changes from one state to another are possible and do, in fact, happen. We do not explain why they happen but, if we are given the valuable piece of information that some change will occur, we can then venture a guess, make a prediction as to what the most likely change will be.

Before giving mathematical expression to this principle we note that large changes are assumed to be the cumulative result of many small changes. As the system moves it follows a continuous trajectory in the space of states. We almost hesitate to call this self-evident fact an assumption, but as the example of quantum theory shows, trajectories need not exist.

Thus in order to go from one state to another the system will have to move through intermediate states; in order to change by a distance $2d\ell$ the system must have first changed by a distance $d\ell$.

Suppose the system was in the state $A^{\alpha}_{old} = A^{\alpha}$ and that it changes by a small amount $d\ell$ to a nearby state. We have to select one new state $A^{\alpha}_{new} = A^{\alpha} + dA^{\alpha}$ from among those that lie on the surface of an $n_A$-dimensional sphere of radius $d\ell$ centered at $A^{\alpha}$. This is precisely what the ME principle was designed to do \[6\], namely, to select a preferred probability distribution from within a specified given set. The only difference with more conventional applications of the ME principle is the geometrical nature of the constraint.
We want to maximize $S(A^\alpha + dA^\alpha)$ under variations of $dA^\alpha$ constrained by $g_{\alpha\beta} dA^\alpha dA^\beta = d\ell^2$. The notation $dA^\alpha = \dot{A}^\alpha d\ell$ is slightly more convenient; we maximize $S(A^\alpha + \dot{A}^\alpha d\ell)$ under variations of $\dot{A}^\alpha$ constrained by

$$g_{\alpha\beta} \dot{A}^\alpha \dot{A}^\beta = 1.$$  

Introducing a Lagrange multiplier $\omega$,

$$\delta \left[ S(A^\alpha + \dot{A}^\alpha d\ell) - \omega \left( g_{\alpha\beta} \dot{A}^\alpha \dot{A}^\beta - 1 \right) \right] = 0,$$

we get

$$\left[ \frac{\partial S}{\partial A^\alpha} d\ell - 2\omega g_{\alpha\beta} \dot{A}^\beta \right] \delta \dot{A}^\alpha = 0.$$  

Therefore, writing $\omega = \sigma d\ell/2$, we get

$$\dot{A}^\alpha = \frac{1}{\sigma} g^{\alpha\beta} \frac{\partial S}{\partial A^\beta},$$

where $g^{\alpha\beta}$ is the inverse of $g_{\alpha\beta}$. This is our main result; it can be rewritten as

$$\dot{A}^\alpha = \frac{1}{\sigma} \lambda^\alpha$$

where the vector $\lambda^\alpha$,

$$\lambda^\alpha = g^{\alpha\beta} \frac{\partial S}{\partial A^\beta},$$

is the entropy gradient. The interpretation is clear, the system moves along the entropy gradient.

This seems such an obvious result that it can hardly be new. Notice, however, the gradient vector refers to the direction in which there is a maximum increase per unit distance; one cannot talk about the gradient vector without having first introduced a metric. The differential form defined by the derivatives $S_{,\beta} = \partial S/\partial A^\beta = \lambda_\beta$, the gradient one-form, does not define a direction; it is not by itself sufficient to define the trajectory.

The physical significance of the Lagrange multiplier $\sigma$ derives from the constraint Eq.(10) which, using Eq.(14), can be written as

$$\lambda_\alpha \lambda^\alpha = \sigma^2 \quad \text{or} \quad \sigma = (\lambda_\alpha \lambda^\alpha)^{1/2},$$

$\sigma$ is the magnitude of the entropy gradient. Furthermore, from this and Eq.(14), we get $dS = \lambda_\alpha \dot{A}^\alpha d\ell = \sigma d\ell$, or

$$\sigma = \frac{dS}{d\ell}.$$

$\sigma$ is the rate of entropy increase along the trajectory.

The main result, Eq.(14), determines the trajectory followed by the system. It determines the tangent vector $\dot{A}^\alpha = dA^\alpha/d\ell$, but not the “velocity” $dA^\alpha/dt$. To fix this something must be said about the universe external to the system, something that relates the distance $\ell$ relative to the external time $t$. This is,
in part, the role normally played by the Hamiltonian, it fixes the evolution of a system relative to external clocks. If we cannot appeal to such information (presumably because we do not have it, but perhaps because we just do not want to), then the only “time” available must be internal to the system, intrinsic to the geometry of the space of states.

One convenient choice of intrinsic time \( \tau \) is the distance \( \ell \) itself, or \( d\tau = d\ell \).

Intrinsic time is change. The equation of motion is very simple: the trajectory, \( A^\alpha = A^\alpha(\tau) \), is along the entropy gradient, and the system moves with unit velocity, \( A^\alpha \dot{A}_\alpha = 1 \), or \( d\ell/d\tau = 1 \).

The absolute speed \( d\ell/dt \) remains unknown. Interestingly, there is no guarantee that \( \tau \) will elapse relative to our own external \( t \), we could have a situation with \( d\tau/dt = 0 \). A pile of sand could, if left alone, just stay at \( A^\alpha(\tau_0) \) forever; its intrinsic time \( \tau \) has stopped at \( \tau_0 \). The pile does not change, because it did not have (intrinsic) time to change. (One can play endless word games here.)

However, should a measurement of one of the variables, for example \( A^1 \), indicate a change from the value \( A^1(\tau_0) \) to the value that one would normally associate with another state along the trajectory, say the value \( A^1(\tau_1) \) at the later time \( \tau_1 \), then one is immediately led to infer that the system has moved along the trajectory. Most probably all the other variables have also changed from \( A^\alpha(\tau_0) \) to \( A^\alpha(\tau_1) \). In this case the variable \( A^1(\tau) \) is playing the role of an internal clock. The variable \( A^1 \) is a good clock provided one can invert \( A^1 = A^1(\tau) \), to get \( \tau = \tau(A^1) \). Then, the changes in all other variables \( A^\alpha = A^\alpha[\tau(A^1)] = A^\alpha(A^1) \) can be referred, correlated to the change in \( A^1 \). We see that the loss of predictive power due to the unknown absolute speed \( d\ell/dt \) is quite minimal, particularly for high dimensionality (large \( n_A \)).

At this point one could agree that the notion of \( \tau \) is useful, perhaps even elegant. But are we justified in calling \( \tau \) time? Perhaps these are mere word games, but if we do call \( \tau \) time, then being a distance it provides us with a model of duration. Furthermore, the very definite ordering of states along the trajectory \( A^\alpha(\tau) \) provides a realization of a temporal order. Finally, the dynamics is intrinsically asymmetric; the trajectory is intrinsically oriented. There is one direction in which entropy increases providing a clear distinction between earlier and later. So this is our answer: we are justified in calling \( \tau \) time, because if we do, then we have a neat model, an explanation for temporal order, for time asymmetry, and for duration. What better reasons do we need?

We close this section with the observation that the system does not follow a geodesic in the space of states. From Eq.(14) we can show that the acceleration vector, given by the absolute derivative (we assume a Riemannian geometry, with the Levi-Civita connection)

\[
\frac{D\dot{A}^\alpha}{d\tau} = \dot{A}^\alpha_{\beta} \dot{A}^\beta = g^{\alpha\beta} f_{\beta\gamma} \dot{A}^\gamma,
\]

does not vanish. The “thermodynamic force” resembles the Lorentz force law in electrodynamics. The “field strength” tensor \( f_{\alpha\beta} \), given by \( f_{\alpha\beta} = A_{\alpha\beta} - A_{\beta\alpha} \), is antisymmetric as needed to preserve the unit magnitude of the velocity \( \dot{A}^\alpha \).
4 Reciprocal relations

The standard theory of irreversible thermodynamics, due to Onsager \[13\], is based on the usual postulates of equilibrium thermostatics supplemented by the additional postulate that the microscopic laws of motion are symmetric under time reversal. A brief outline is the following.

As the system moves along its trajectory entropy increases at a rate
\[
\frac{dS}{dt} = \frac{\partial S}{\partial A^{\alpha}} \frac{dA^\alpha}{dt} = \lambda_\alpha \frac{dA^\alpha}{dt}
\]
relative to the external time \( t \); the variables \( \lambda_\alpha \) are called thermodynamic forces, and \( dA^\alpha / dt \) are called fluxes. In this theory linear relations between fluxes and forces are postulated,
\[
\frac{dA^\alpha}{dt} = L^{\alpha\beta} \lambda_\beta,
\]
for which there is abundant experimental evidence, at least close to thermodynamic equilibrium.

The significance of these relations lies in that they postulate crossed connections between a flux of type \( \alpha \) and a force of type \( \beta \), and vice versa. (Thus, a temperature gradient will not just generate an heat current; it may also generate electric currents, matter flows, and so on.) The strength of these effects is measured by the phenomenological Onsager coefficients \( L^{\alpha\beta} \). The central result of the theory is the reciprocal relation between these crossed effects. The reciprocity theorem, proved by Onsager on the basis of microscopic reversibility, states that the matrix of phenomenological coefficients is symmetric
\[
L^{\alpha\beta} = L^{\beta\alpha}.
\]

The intrinsic dynamics discussed in the previous section also leads to reciprocal relations. The equation of motion, Eq.(14), gives
\[
\frac{dA^\alpha}{dt} = \frac{d\tau}{dt} \frac{dA^\alpha}{d\tau} = \frac{d\tau}{dt} \frac{1}{\sigma} g^{\alpha\beta} \lambda_\beta.
\]
This allows us to identify the Onsager coefficients as
\[
L^{\alpha\beta} = \frac{d\tau}{dt} \frac{1}{\sigma} g^{\alpha\beta}.
\]
These coefficients are not constants, they vary along the trajectory, \( L^{\alpha\beta} = L^{\alpha\beta}(A) \).

What is interesting here is that their symmetry follows from the symmetry of the metric tensor. No hypothesis about microscopic reversibility was needed; in fact, microscopic dynamics was not mentioned at all. In addition, the validity of Eq.(22) is not restricted to the immediate vicinity of equilibrium. To the extent that the variables \( A \) are the right variables to describe phenomena far from equilibrium, the reciprocal relations should still hold.
5 Dynamics constrained by conservation

Beyond the fact that changes happen, perhaps the most common additional information that one can have about an irreversible process is that some quantities are conserved. As an illustrative example we consider two systems that are allowed to exchange some conserved quantities and evolve towards equilibrium. To fix ideas we could think of an ideal gas filling two vessels at different temperatures and chemical potentials. Once the two vessels are connected, for example by a tube, a little hole, or a porous plug, matter and energy will flow until equilibrium is reached.

To keep this as simple as possible we assume the experimental conditions are such that throughout the process the two systems remain homogeneous and independent. The first system is described by variables \( A \), the second is described by primed variables \( A' \), and the entropies, given by Eq.(5), are additive

\[
S_T(A, A') = S(A) + S'(A').
\]

Since the quantities \( A \) are conserved the dynamics is constrained by \( A' = A_T - A \), with \( A_T \) fixed, or \( \dot{A}' = -\dot{A} \). The conservation constraint could be incorporated using Lagrange multipliers; for this simple example it is just as easy to eliminate \( A' \).

In our ideal gas example, the variables could be energy, \( A^1 = E \), and number of molecules, \( A^2 = N \). This crucial part in setting the problem, choosing the description, is the one most likely to go wrong. If the hole coupling the two vessels is too large, the ME predictions below will fail. The failure is not to be blamed in the ME method, but on the choice of variables: the pair \( E, N \) is not enough to codify the relevant information of say, a turbulent flow. The same remark applies if the connecting porous plug is such that heat can be easily exchanged but there is resistance to matter flow. In this case additional variables are needed, perhaps describing the physical state of the plug and the gas in it.

Suppose the system was in the state \( A \) and that it changes by a small amount \( d\tau \) to a nearby state. To select one new state \( A + \dot{A}d\tau \) from among those that lie on the surface of a sphere of radius \( d\tau \) centered at \( A \), we maximize

\[
S_T(A + \dot{A}d\tau) = S(A + \dot{A}d\tau) + S'(A_T - A - \dot{A}d\tau)
\]

under variations of \( \dot{A} \) constrained by

\[
g_{\alpha\beta} \dot{A}^\alpha \dot{A}^\beta = 1.
\]

where the Fisher-Rao metric, Eq.(8), is given by

\[
g_{\alpha\beta} = -\frac{\partial^2}{\partial A^\alpha \partial A^\beta} (S(A) + S'(A_T - A)).
\]

The result is

\[
\frac{dA^\alpha}{d\tau} = \frac{1}{\sigma} g^{\alpha\beta} (\lambda_\beta - \lambda'_\beta),
\]
where $\sigma$ is the rate of entropy production $\sigma = dS_T/d\tau$. The system evolves until the conjugate variables $\lambda$ are equalized.

6 Final remarks

The main conclusion is simple: unless there is positive evidence to the contrary, our best prediction is that the system evolves along the entropy gradient. What is perhaps not so trivial is that, unlike other conventional forms of dynamics, this intrinsic dynamics does not require an additional postulate. It is the unique dynamics that follows from the maximum entropy principle and nothing else. Another nontrivial aspect is that the model supplies its own notion of time. Since the irreversible macroscopic motion is not explained in terms of a reversible microscopic motion there is no need to explain irreversibility, this question never arises. Similarly, there is no need to explain the second law of thermodynamics; it is the second law (in the form of the ME axioms) that explains everything else.

These ideas can be explored further in a number of directions. There is, for example, the relation with other theories of irreversible processes, such as the equations of hydrodynamics. Another possibility is to extend the theory to account for fluctuations and diffusion. The intrinsic dynamics proposed above is deterministic, but to the extent that the ME principle does not completely rule out distributions of lower entropy, fluctuations about equilibrium and about the deterministic motion are possible.

Perhaps the most intriguing question to pursue stems from the possibility of deriving dynamics from purely entropic arguments. This is clearly valuable in areas where the microscopic dynamics may be too far removed from the phenomena of interest, say in biology or ecology, or where it may just be unknown or perhaps even inexistent, as in economics. One could argue that these theories would be phenomenological as opposed to fundamental, that within physics the search for a fundamental mechanics would still be left open. However, in previous work we have shown that entropic arguments do account for a substantial part of the formalism of quantum mechanics, a theory that is presumably fundamental. Perhaps the fundamental theories of physics are not so fundamental; they are just consistent, objective ways to manipulate information.

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