Commutators close to the identity in unital $C^*$-algebras

K MAHESH KRISHNA and P SAM JOHNSON

Department of Mathematical and Computational Sciences, National Institute of Technology Karnataka (NITK), Surathkal, Mangaluru 575 025, India

*Corresponding Author
E-mail: kmaheshak@gmail.com; sam@nitk.edu.in

MS received 8 April 2021; revised 17 June 2021; accepted 22 July 2021

Abstract. Let $H$ be an infinite dimensional Hilbert space and $B(H)$ be the $C^*$-algebra of all bounded linear operators on $H$, equipped with the operator-norm. By improving the Brown–Pearcy construction, Tao (J. Oper. Theory 82(2) (2019) 369–382) extended the result of Popa (On commutators in properly infinite $W^*$-algebras, in: Invariant subspaces and other topics (1982) (Boston, Mass.: Birkhäuser, Basel)) which reads as: for each $0 < \varepsilon \leq 1/2$, there exist $D, X \in B(H)$ with $\|[D, X] - 1_{B(H)}\| \leq \varepsilon$ such that $\|D\|\|X\| = O \left( \log^5 \frac{1}{\varepsilon} \right)$, where $[D, X] := DX - XD$. In this paper, we show that Tao’s result still holds for certain class of unital $C^*$-algebras which include $B(H)$ as well as the Cuntz algebra $O_2$.

Keywords. Commutator; $C^*$-algebra; Cuntz algebra.

2010 Mathematics Subject Classification. Primary: 47A63, 47B47, 46L05.

1. Introduction

Let $n \in \mathbb{N}$, $\mathbb{K}$ be scalar field and $M_n(\mathbb{K})$ be the ring of $n$ by $n$ matrices over $\mathbb{K}$. Using the property of trace map we easily get that there does not exist $D, X \in M_n(\mathbb{K})$ such that $DX - XD = 1_{M_n(\mathbb{K})}$ [3] (writing matrices as commutator goes as early as 1937 beginning with the work of Shoda [9], see the Introduction in the paper [10]). This argument will not work for bounded linear operators on infinite dimensional Hilbert space since the map trace is not defined on the algebra $B(H)$ of all bounded linear operators on an infinite dimensional Hilbert space $H$ (it is defined for a proper subalgebra of $B(H)$ known as the trace class operators [8]). Using the property of spectrum of bounded linear operator, Winter in 1947 [15] proved that the equation $[D, X] := DX - XD = 1_{B(H)}$ fails to exist in $B(H)$. After two years, Wielandt [14] (also see [7]) gave a simple proof for the failure of this equation. Now it is natural to ask whether we can find $D, X \in B(H)$ such that the commutator $[D, X]$ is close to the identity operator, in operator-norm. This was answered by Brown and Pearcy in 1965 [1] showing that $[D, X]$ can be made close to the identity operator. Brown and Pearcy also characterized the class of commutators of operators. Following the paper [1] of Brown and Pearcy, there is a series of papers devoted to the study of commutators on sequence spaces, $L^p$-spaces, Banach spaces, $C^*$-algebras, von Neumann
algebras, Banach *-algebras, etc. However, a quantitative study of commutators close to the identity operator remains untouched. We start with the following quantitative bound given by Popa in 1981 [6] for product of norm of operators whenever the commutator is close to the identity.

**Theorem 1.1** [6]. Let $\mathcal{H}$ be an infinite dimensional Hilbert space. Let $D, X \in \mathcal{B}(\mathcal{H})$ be such that

$$\| [D, X] - 1_{\mathcal{B}(\mathcal{H})} \| \leq \varepsilon$$

for some $\varepsilon > 0$. Then

$$\| D \| \| X \| \geq \frac{1}{2} \log \frac{1}{\varepsilon}.$$

Now the problem in Theorem 1.1 is the existence of $D, X \in \mathcal{B}(\mathcal{H})$ such that the commutator $[D, X]$ is close to the identity operator. This was again obtained by Popa which is stated in the following result. Given real $r$ and positive $s$, by $r = O(s)$, we mean that there is a positive $\gamma$ such that $|r| \leq \gamma s$.

**Theorem 1.2** [6,12]. Let $\mathcal{H}$ be an infinite dimensional Hilbert space. Then for each $0 < \varepsilon \leq 1$, there exist $D, X \in \mathcal{B}(\mathcal{H})$ with

$$\| [D, X] - 1_{\mathcal{B}(\mathcal{H})} \| \leq \varepsilon$$

and

$$\| D \| \| X \| = O(\varepsilon^{-2}).$$

Recently, Tao improved Theorem 1.2 and obtained the following theorem.

**Theorem 1.3** [12]. Let $\mathcal{H}$ be an infinite dimensional Hilbert space. Then for each $0 < \varepsilon \leq 1/2$, there exist $D, X \in \mathcal{B}(\mathcal{H})$ with

$$\| [D, X] - 1_{\mathcal{B}(\mathcal{H})} \| \leq \varepsilon$$

such that

$$\| D \| \| X \| = O \left( \log^5 \frac{1}{\varepsilon} \right).$$

In this paper, following the arguments in [12], we show that Theorem 1.3 remains valid not only for $\mathcal{B}(\mathcal{H})$ but for certain other classes of unital $C^*$-algebras, such as any algebra containing Cuntz algebra $\mathcal{O}_2$. Throughout the paper, we move along the lines of [12]. In the rest of the Introduction, we recall fundamentals of matrices over unital $C^*$-algebras. For more information, we refer to [4,13].

Let $\mathcal{A}$ be a unital $C^*$-algebra. For $n \in \mathbb{N}$, $M_n(\mathcal{A})$ is defined as the set of all $n$ by $n$ matrices over $\mathcal{A}$. It is clearly an algebra with respect to natural matrix operations. We
define the involution of an element \( A := (a_{i,j})_{1 \leq i,j \leq n} \in M_n(\mathcal{A}) \) as \( A^* := (a_{j,i}^*)_{1 \leq i,j \leq n} \). Then \( M_n(\mathcal{A}) \) is a \(*\)-algebra. From the Gelfand–Naimark–Segal theorem, there exists unique universal representation \((\mathcal{H}, \pi)\), where \( \mathcal{H} \) is a Hilbert space, \( \pi : M_n(\mathcal{A}) \to M_n(B(\mathcal{H})) \) is an isometric \(*\)-homomorphism. This gives a norm on \( M_n(\mathcal{A}) \) defined as

\[
\|A\| := \|\pi(A)\|, \; \forall A \in M_n(\mathcal{A}).
\]

This norm makes \( M_n(\mathcal{A}) \) as a \( C^* \)-algebra.

2. Commutators close to the identity in unital \( C^* \)-algebras

In the sequel, \( \mathcal{A} \) is a unital \( C^* \)-algebra. In [12], the following lemma followed by a corollary is proved for \( B(\mathcal{H}) \), where \( \mathcal{H} \) is an infinite dimensional Hilbert space. Same proofs carry over for unital \( C^* \)-algebras.

**Lemma 2.1 (Commutator calculation).** Let \( u, v, b_1, \ldots, b_n \in \mathcal{A} \) and \( \delta > 0 \). Let

\[
X := \begin{pmatrix}
0 & 0 & 0 & \ldots & 0 & \delta b_1 \\
1_A & 0 & 0 & \ldots & 0 & \delta b_2 \\
0 & 1_A & 0 & \ldots & 0 & \delta b_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & \delta b_{n-1} \\
0 & 0 & 0 & \ldots & 1_A & \delta b_n
\end{pmatrix} \in M_n(\mathcal{A})
\]

and

\[
D := \begin{pmatrix}
\frac{v}{\delta} & 1_A & 0 & \ldots & 0 & \delta b_1 u \\
\frac{v}{\delta} & \frac{v}{\delta} & 2.1_A & \ldots & 0 & \delta b_2 u \\
\frac{v}{\delta} & \frac{v}{\delta} & \frac{v}{\delta} & \ldots & 0 & \delta b_3 u \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & \frac{v}{\delta} & (n-1)1_A + \delta b_{n-1}u \\
0 & 0 & 0 & \ldots & \frac{v}{\delta} & \frac{v}{\delta} + \delta b_nu
\end{pmatrix} \in M_n(\mathcal{A}).
\]

Then

\[
[D, X] = 1_{M_n(\mathcal{A})} + \begin{pmatrix}
0 & 0 & 0 & \ldots & 0 & [v, b_1] + 0 + \delta b_2 + \delta b_1 [u, b_n] \\
0 & 0 & 0 & \ldots & 0 & [v, b_2] + [u, b_1] + 2 \delta b_3 + \delta b_2 [u, b_n] \\
0 & 0 & 0 & \ldots & 0 & [v, b_3] + [u, b_2] + 3 \delta b_4 + \delta b_3 [u, b_n] \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & [v, b_{n-1}] + [u, b_{n-2}] + (n-1)\delta b_n + \delta b_{n-1} [u, b_n] \\
0 & 0 & 0 & \ldots & 0 & [v, b_n] + [u, b_{n-1}] + 0 + \delta b_n [u, b_n] - n.1_A
\end{pmatrix}
\]

**COROLLARY 2.2**

Let \( u, v, b_1, \ldots, b_n \in \mathcal{A} \). Assume that for some \( \delta > 0 \), we have equations

\[
[v, b_i] + [u, b_{i-1}] + i \delta b_{i+1} + \delta b_i [u, b_n] = 0, \; \forall i = 2, \ldots, n-1 \quad (1)
\]
and

\[ [v, b_n] + [u, b_{n-1}] + \delta b_n[u, b_n] = n \cdot 1_{M_n(A)}. \]

(2)

Then for any \( \mu > 0 \), there exist matrices \( D_\mu, X_\mu \in M_n(A) \) such that

\[
\|D_\mu\| \leq \frac{\|u\|}{\mu^2 \delta} + \frac{\|v\|}{\mu \delta} + (n - 1) + \delta \sum_{i=1}^{n} \mu^{n-i-1} \|b_i\| \|u\|,
\]

\[
\|X_\mu\| \leq 1 + \delta \sum_{i=1}^{n} \mu^{n-i+1} \|b_i\|
\]

and

\[
\|[D_\mu, X_\mu] - 1_{M_n(A)}\| \leq \mu^{n-1} \|[v, b_1] + \delta b_2 + \delta b_1[u, b_n]\|.
\]

Let \( A \) be a unital \( C^* \)-algebra. Assume that there are isometries \( u, v \in A \) such that

\[
u^*u = v^*v = uu^* + vv^* = 1_A \quad \text{and} \quad u^*v = v^*u = 0.
\]

(3)

Examples of such unital \( C^* \)-algebras are \( B(H) \) (\( H \) is an infinite dimensional Hilbert space) as well as any unital \( C^* \)-algebra which contains the Cuntz algebra \( O_2 \) (see [2] for Cuntz algebra). Note that whenever a unital \( C^* \)-algebra admits a trace map, there are no isometries satisfying Equation (3). In particular, any finite dimensional unital \( C^* \)-algebra does not have such elements. It is also clear that no commutative unital \( C^* \)-algebra can have isometries satisfying Equation (3).

It is shown in [12] that whenever \( H \) is an infinite dimensional Hilbert space, then the Banach algebras \( B(H) \) and \( M_2(B(H)) \) are isometrically isomorphic. We now do these results for \( C^* \)-algebras whenever they have isometries satisfying Equation (3). To do so, we first need a result from the theory of \( C^* \)-algebras.

**Theorem 2.3** [5, 11].

(i) Every *-homomorphism between \( C^* \)-algebras is norm decreasing.

(ii) If a *-homomorphism between \( C^* \)-algebras is injective, then it is isometric.

Theorem 2.3 gives the following theorem from direct computations (see [12] for the case \( A = B(H) \), where \( H \) is an infinite dimensional Hilbert space).

**Theorem 2.4.** Let \( A \) be a unital \( C^* \)-algebra. If there are isometries \( u, v \in A \) such that equation (3) holds, then the map

\[
\phi : A \ni x \mapsto \begin{pmatrix} u^*xu & u^*xv \\ v^*xu & v^*xv \end{pmatrix} \in M_2(A)
\]

is a \( C^* \)-algebra isomorphism with the inverse map

\[
\psi : M_2(A) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto uau^* + ubv^* + vcu^* + vdv^* \in A.
\]
Along with the lines of Theorem 2.4, we can easily derive the following result.

**Theorem 2.5.** Let $\mathcal{A}$ be a unital C*-algebra and $n \in \mathbb{N}$. If there are isometries $u, v \in \mathcal{A}$ such that equation (3) holds, then the map

$$
\phi : M_{n}(\mathcal{A}) \ni X \mapsto \begin{pmatrix} u^*Xu & u^*Xv \\ v^*Xu & v^*Xv \end{pmatrix} \in M_{2n}(\mathcal{A})
$$

is a C*-algebra isomorphism with the inverse map

$$
\psi : M_{2n}(\mathcal{A}) \ni \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto uAu^* + uBv^* + vCu^* + vDv^* \in M_n(\mathcal{A}),
$$

where if $X := (x_{i,j})_{i,j}$ is a matrix, and $a, b \in \mathcal{A}$, by $aXb$, we mean the matrix $(ax_{i,j}b)_{i,j}$. In particular, the C*-algebras $\mathcal{A}, M_2(\mathcal{A}), M_4(\mathcal{A}), \ldots, M_{2n}(\mathcal{A}), \ldots$ are all *-isometrically isomorphic.

In the rest of the paper, we assume that unital C*-algebra $\mathcal{A}$ has isometries $u, v$ satisfying Equation (3). In the next result, we use the following notation. Given a vector $x \in \mathcal{A}^n$, $x_i$ means its $i$-th coordinate. Next two propositions are proved in [12] for the C*-algebra $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is an infinite dimensional Hilbert space. However, they hold for unital C*-algebras which have isometries satisfying Equation (3).

**PROPOSITION 2.6**

Let $n \geq 2$ and $T : \mathcal{A}^{n} \to \mathcal{A}^{n-1}$ be the bounded linear operator defined by

$$
T(b_i)_{i=1}^{n} := ([v, b_1] + [u, b_{i-1}])_{i=2}^{n}, \quad \forall (b_i)_{i=1}^{n} \in \mathcal{A}^{n}.
$$

Then there exists a bounded linear right-inverse $R : \mathcal{A}^{n-1} \to \mathcal{A}^{n}$ for $T$ such that

$$
\|Rb\| = \sup_{1 \leq i \leq n} \|(Rb)_i\| \leq 8\sqrt{2}n^2 \sup_{2 \leq i \leq n} \|b_i\| \leq 8\sqrt{2}n^2 \sup_{1 \leq i \leq n} \|b_i\| = 8\sqrt{2}n^2 \|b\|, \quad \forall b \in \mathcal{A}^{n}.
$$

As given in [12], we try to shift from the systems of Equations (1) and (2) to the solution of single equation. Let $n \geq 2$. Define $a := (0, \ldots, n) \in \mathcal{A}^{n}$,

$$
F : \mathcal{A}^{n} \ni (b_i)_{i=1}^{n} \mapsto (-2b_3, \ldots, -(n-1)b_n, 0) \in \mathcal{A}^{n-1}
$$

and

$$
G : \mathcal{A}^{n} \times \mathcal{A}^{n} \ni ((b_i)_{i=1}^{n}, (c_i)_{i=1}^{n}) \mapsto (-b_2[u, c_n], \ldots, -b_n[u, b_n]) \in \mathcal{A}^{n-1}.
$$

We then have $\|F\| \leq n - 1$ and $\|G\| \leq 2$. 
PROPOSITION 2.7

Systems (1) and (2) have a solution \( b \) if and only if
\[
Tb = a + \delta F(b) + \delta G(b, b).
\] (4)

The above proposition reduces the work of solving systems (1) and (2) to a single operator equation. To solve equation (4), we need an abstract lemma from [12].

Lemma 2.8 [12]. Let \( \mathcal{X}, \mathcal{Y} \) be Banach spaces, \( T, F : \mathcal{X} \to \mathcal{Y} \) be bounded linear operators, and let \( G : \mathcal{X} \times \mathcal{X} \to \mathcal{Y} \) be a bounded bilinear operator with bound \( r > 0 \) and let \( a \in \mathcal{Y} \). Suppose that \( T \) has a bounded linear right inverse \( R : \mathcal{Y} \to \mathcal{X} \). If \( \delta > 0 \) is such that
\[
\delta(2 \| F \| \| R \| + 4r \| R \|^2 \| a \|) < 1,
\] (5)
then there exists \( b \in \mathcal{X} \) with \( \| b \| \leq 2 \| R \| \| a \| \) that solves the equation
\[
Tb = a + \delta F(b) + \delta G(b, b).
\]

Theorem 2.9. Let \( \mathcal{A} \) be a unital C*-algebra. Suppose there are isometries \( u, v \in \mathcal{A} \) such that equation (3) holds. Then we have the following:

(i) For each \( n \geq 2 \), there exists a solution \( b \) to equation (4) such that \( \| b \| \leq 16\sqrt{2}n^3 \).

(ii) For each \( n \geq 2 \), let \( b \) be an element satisfying equation (4) and \( \| b \| \leq 16\sqrt{2}n^3 \).

Then for \( \mu = \frac{1}{2} \), there exist \( D_\mu, X_\mu \in M_n(\mathcal{A}) \) such that
\[
\| D_\mu \| = O(n^5), \quad \| X_\mu \| = O(1), \quad \|[D_\mu, X_\mu] - 1_{M_n(\mathcal{A})}\| = O(n^3 2^{-n}).
\]

(iii) Let \( 0 < \varepsilon \leq 1/2 \). Then there exist an even integer \( n \) and \( D, X \in M_n(\mathcal{A}) \) with
\[
\|[D, X] - 1_{M_n(\mathcal{A})}\| \leq \varepsilon
\]
such that
\[
\| D \| \| X \| = O \left( \log^5 \frac{1}{\varepsilon} \right).
\]

(iv) For each \( 0 < \varepsilon \leq 1/2 \), there exist \( d, x \in \mathcal{A} \) with
\[
\|[d, x] - 1_{\mathcal{A}}\| \leq \varepsilon
\]
such that
\[
\| d \| \| x \| = O \left( \log^5 \frac{1}{\varepsilon} \right).
\]

Remark 2.10. Proof of Theorem 2.9 can be obtained in a similar manner by following the proof of Lemma 1.4 and its consequences in [12].
Example 2.11. Let $A$ be a finite dimensional unital C*-algebra. From the structure theory, we have

$$A \cong M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_r}(\mathbb{C}),$$

for unique (up to permutation) natural numbers $n_1, \ldots, n_r$. This result says that normalized trace map (a trace map $\text{Tr}$ such that $\text{Tr}(1_A) = 1$) exists on $A$. Using this, we make the following two observations:

(i) $A$ can not have isometries satisfying Equation (3). Suppose that there are such isometries. Then

$$1 = \text{Tr}(uu^* + vv^*) = \text{Tr}(uu^*) + \text{Tr}(vv^*) = \text{Tr}(u^*u) + \text{Tr}(v^*v) = 2$$

which is impossible.

(ii) In [12], Tao observed that if $\mathcal{H}$ is a finite dimensional Hilbert space, then there are no $D, X \in \mathcal{B}(\mathcal{H})$ satisfying $\|[D, X] - 1_{\mathcal{B}(\mathcal{H})}\| < 1$. We elaborate this for any finite dimensional unital C*-algebra $A$, namely, there do not exist $d, x \in A$ satisfying $\|[d, x] - 1_A\| < 1$. In other words, Theorem 2.9 fails for every finite dimensional unital C*-algebra. Let $d, x \in A$ be arbitrary. From the structure theory, we identify that $d$ as $D$ and $x$ as $X$ for some matrices $D, X \in M_n(\mathbb{C})$ and for some $n$. Using the commutativity of trace, we then have $\text{Tr}([D, X]) = 0$. Let $\lambda_1, \ldots, \lambda_n$ be eigenvalues of $[D, X]$. Then

$$\sum_{j=1}^{n} \lambda_j = \text{Tr}([D, X]) = 0.$$ 

This gives

$$n = \left| \sum_{j=1}^{n} (\lambda_j - 1) \right| \leq \sum_{j=1}^{n} |\lambda_j - 1|.$$ 

The previous inequality says that there is at least one $j$ such that $|\lambda_j - 1| \geq 1$. We next see that all the eigenvalues of $[D, X] - 1_{M_n(\mathbb{C})}$ are $\lambda_1 - 1, \ldots, \lambda_n - 1$. Using the property of operator norm, we finally get

$$\|[d, x] - 1_A\| = \|[D, X] - 1_{M_n(\mathbb{C})}\| \geq \sup_{1 \leq j \leq n} |\lambda_j - 1| \geq 1.$$ 

Acknowledgements

We thank Prof. Terence Tao, University of California, Los Angeles, USA for a kind reply which made us to understand his paper [12]. We would like to thank the reviewer for his/her positive and insightful comments towards improving our manuscript. We also thank Sachin M Naik for some discussions.

References

[1] Brown A and Pearcy C, Structure of commutators of operators, Ann. Math. (2) 82 (1965) 112–127
[2] Cuntz J, Simple C*-algebras generated by isometries, Comm. Math. Phys. 57(2) (1977) 173–185
[3] Halmos P R, A Hilbert space problem book, volume 19 of Graduate Texts in Mathematics, second edition (1982) (New York–Berlin: Springer)

[4] Murphy G J, C*-algebras and operator theory (1990) (Boston, MA: Academic Press Inc.)

[5] Pedersen G K, C*-algebras and their automorphism groups, Pure and Applied Mathematics (Amsterdam) (2018) (London: Academic Press)

[6] Popa S, On commutators in properly infinite $W$-algebras, in: Invariant subspaces and other topics (Timisoara/Herculane, 1981), volume 6 of Operator Theory: Adv. Appl. (1982) (Boston, Mass.: Birkhäuser, Basel) pages 195–207

[7] Putnam C R, Commutation properties of Hilbert space operators and related topics, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 36 (1967) (New York: Springer-Verlag)

[8] Schatten R, Norm ideals of completely continuous operators, second printing, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 27 (1970) (Berlin–New York: Springer-Verlag)

[9] Shoda K, Einige Sätze über Matrizen, Japan J. Math. 13(3) (1937) 361–365

[10] Stasinski A, Similarity and commutators of matrices over principal ideal rings, Trans. Amer. Math. Soc. 368(4) (2016) 2333–2354

[11] Takesaki M, Theory of operator algebras. I, volume 124 of Encyclopaedia of Mathematical Sciences (2002) (Berlin: Springer-Verlag)

[12] Tao T, Commutators close to the identity, J. Oper. Theory 82(2) (2019) 369–382

[13] Wegge-Olsen N E, K-theory and C*-algebras: A friendly approach, Oxford Science Publications (1993) (New York: The Clarendon Press, Oxford University Press)

[14] Wielandt H, Über die Unbeschränktheit der Operatoren der Quantenmechanik, Math. Ann. 121 (1949) 21

[15] Wintner A, The unboundedness of quantum-mechanical matrices, Phys. Rev. (2) 71 (1947) 738–739

Communicating Editor: B V Rajarama Bhat