Behavior Of Very High Energy Hadronic Cross Sections

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1 Introduction

A classic chapter of elementary particle physics appears to be drawing to a close, and it seems appropriate at this point to describe, in general terms, some of the issues and how they have been resolved.

The question is the scattering of elementary hadrons—strongly interacting particles, notably the proton—at very high energy. The question of the very high energy or asymptotic behavior of the cross section for strongly interacting particles—“hadrons”—is one of simplest to pose, but took the longest to answer. Indeed, it has taken a surprisingly long time to answer, and one of the points we would like to address is why.

We say ‘surprisingly’ because very high energy data for protons or antiprotons have been available for decades, at the big accelerators or from cosmic rays. The energies available, expressed in terms of the center-of-mass energy \( W \) have long been well above any evident scale one might associate with strong interactions. For such a scale, one might consider the proton mass itself, about 1 GeV. Or perhaps the \( \Lambda \) parameter of QCD, the field theory presumed to underly strong interactions. But this is even less, only around 0.2 GeV. Otherwise it’s hard to think of any obvious energy scales. On the other hand, the Fermilab TeVatron surpassed \( W = 1 \, TeV = 1,000 \, GeV \) years ago, and CERN’s LHC recently reached \( W = 13 \, TeV = 13,000 \, GeV \). By any measure, one would have thought, a simple pattern or picture should certainly have emerged long ago. Even given that the quantities in question vary only slowly, logarithmically, this great discrepancy is a real conundrum. Is there perhaps a new, higher mass scale waiting to be discovered in the wings? Those patient enough to read to the end will find there is an answer to this puzzle, but from an unexpected direction.

Our main source of information on very high energy comes from proton-proton (p-p) or antiproton-proton (\( \bar{p} \)-p) studies at accelerators, or from cosmic rays. While many studies of hadron interactions exist, we will concentrate on
these channels, but it is probable that our main points would apply equally well to others, such as (π-p) or (K-p), and with suitable account of ‘vector dominance’, to (γ-p) reactions.

In all these channels there is a low energy region, where the cross sections are characterized by various resonances and particularities of the individual channel. But for center-of-mass energies \( W \) above the tens of GeV, a smooth behavior for the cross section sets in, with probably a universal behavior in all channels.

What is this general behavior and how is it to be described...and perhaps to be explained? Fig 1 shows a plot of very high energy p-p and \( \bar{p} \)-p data, as a function of center-of-mass energy \( W \), for the total (upper curve), the inelastic (middle curve) cross sections and (lower curve) their difference, the elastic cross section. Aside from the evident fact that the cross sections are increasing with energy, the plot is rather featureless and there seems to be no particular relationship between the three curves. However, as we shall see, there is nevertheless a hidden simplicity which describes the data.

![Figure 1: Behavior of proton (blue, purple), antiproton (red, black)-proton scattering up to very high energy, shown as a function of the center-of-mass energy \( W \). The curves are for the total (upper), inelastic (middle), and elastic (lower) cross sections. From ref [1].](image)

2 Rise of the Cross Sections

The increase of the cross sections with energy is in itself not a trivial point, and until the early 70’s, when the rise was first seen at CERN’s ISR, the opinion
was often heard that the cross sections would approach a constant limit at very high energy. One might think, after all, that an incoming proton, regardless of its energy, is just hitting a fixed, unchanging target, an object of constant size. However this turned out not to be the case, as one sees. The question is certainly an old one, with proposals going back at least to Heisenberg in the 50’s [2], who suggested that the total cross section $\sigma$ should increase as the square of a logarithm,

$$\sigma \sim \ln^2(W/W_o),$$

where $W_o$ is some energy scale parameter.

The idea underlying Eq 1 is a field theoretic one, and its validity can be interpreted as yet another manifestation of field theory in fundamental interactions. The argument is essentially that with greater energy, a proton can excite a target, (e.g. produce a meson off another proton) from ever greater distances. Since the radius of interaction $R$ is thus growing, so is the cross section, which goes as $R^2$. Thus Eq 1 corresponds to a logarithmic increase in the range of interaction.

This behavior is actually not surprising and has been known for a long time in the more ordinary process of the ionization produced by a charged particle passing through matter. There is a phenomenon called “the relativistic rise” [3] where due to the relativistic boost of the electric/magnetic fields around a very fast charged particle, it can, as its energy goes up, eject electrons or excite atoms further and further away from the charge.

Analogously, with increasing energy one can expect an increasing intensity or energy density in the fields surrounding the highly relativistic proton and an increasing probability of “ejecting” something or exciting a target at ever greater distances. Thus there is an increasing radius of interaction.

However, in hadronic interactions like proton-proton scattering there is an important difference vis-a-vis the “relativistic rise” in ionization. There, one is concerned with the relativistic ‘boost’ of the long–one could say infinite–range coulomb field. Here, the field around the proton is of short range or Yukawa-like: $\sim (1/r)e^{-\mu r}$. This introduces $\mu$, a mass or inverse length parameter (We use natural units $\hbar = c = 1$ where a mass is also an inverse length).

This exponential cutoff means that even if the fields boost as some power $p$ of the energy, it will take a high energy for them to obtain a significant value at large distances. If some threshold value is required to have significant particle production at a distance $r_{\max}$ we will have the leading condition

$$W^p \times e^{-\mu r_{\max}} \geq threshold,$$

a relationship connecting the energy and the effective maximum range of interaction $r_{\max}$. Taking the log we obtain $r_{\max} = constant \times \ln W$. Since the cross section $\sigma$ goes as $r_{\max}^2$, one has Eq 1.

A detailed concretization of this general argument is provided by the work of the Apsen group [1], where the field density around the proton is taken from electromagnetic form factor measurements and an eikonal methods is used [4]. With a power increase for the interaction, the asymptotic $\ln^2 W$ behavior is
indeed obtained. Furthermore, good fits are found to the other quantities such as the shape and energy dependence of the elastic diffraction peak induced by the absorption \[1\].

Interestingly, more formal arguments, based on the analytic properties of scattering amplitudes in the Mandelstam representation, led to the conclusion that Eq.1 represents in fact the fastest growth possible \[5\] consistent with analyticity. Further steps along these lines even led to an upper bound on the coefficient of the \(\ln^{2}W\), with a dimensional coefficient characterized by the pion mass \(\pi/m_{\pi}^{2}\approx 60\,\text{mb}\). It should be stressed that these mathematical results represent upper bounds and that there is nothing \textit{a priori} wrong with a smaller coefficient or a slower behavior.

For many years, the question of the experimental validity or not of Eq.1 was unclear. This is due to the fact that the logarithm is a very slowly changing function, so that even the great efforts in achieving higher \(W\)'s over the decades led to only relatively modest changes in the log's. Furthermore, whatever the final very high energy behavior, there are non-leading terms which will only slowly disappear, making the extraction of the leading behavior non-trivial. Thus despite the enormous developments in accelerator technologies, it was not easy to extract and distinguish one model for the asymptotic cross sections from another.

3 The Growing ”Black Disc”

Over the years, different groups developed fits to the high energy data \[6\]. Finally some clarity in the situation began to emerge in the last decade when M.M. Block and Francis Halzen of the Aspen group noticed \[7\] something very interesting about their fits. Not only could one get good fits with a leading \(\ln^{2}W\) term, but also the coefficients of the elastic and total cross sections for these terms were accurately in the ratio 1:2.

The fact that the ratio \(\sigma(\text{elastic})/\sigma(\text{total})\) approaches one-half

\[
\frac{\sigma(\text{elastic})}{\sigma(\text{total})} \to \frac{1}{2} \quad W \to \infty
\]

(3)

corresponds to the standard ‘black disc’ limit, enshrined in classical optics as “Babinet’s principle”. One has a disc which is totally ‘black’, that is, everything hitting it is completely absorbed, giving the inelastic cross section. At the same time this ‘absorption’ creates a ‘hole’ in the incoming wave front, leading to an elastic scattering which has the same cross section \[8\]. Thus one has the situation \(\sigma(\text{total}) = \sigma(\text{elastic}) + \sigma(\text{inelastic}) = 2\sigma(\text{elastic})\), and so Eq.3

From the fits it thus appears that in the very high energy limit of proton-proton scattering one approaches a text-book ‘black disc’ and with an energy dependence in accordance with Eq.1 and the Froissart bound.
4 Multiplicity and Cross Sections

It would be good to have some further physical support of the general picture. If, as argued above, the growth of the cross section with energy is connected to the possibility of producing particles at ever-increasing distances or impact parameters, then there should be some connection between the growth of the cross section and the number of particles produced in a collision. This suggests looking at the multiplicity $N(W)$, the average number of particles produced in a collision, which is also increasing with energy. A proposal along these lines [9] was that at very high energy the two quantities should grow in parallel:

$$\sigma \propto N \quad W \to \infty.$$  \hspace{1cm} (4)

![Graph](image)

Figure 2: Ratios of the total (red +), inelastic (green x) and elastic (blue *) cross sections to the total pion multiplicity (which is approximately the total multiplicity), in mb vs. the center-of-mass energy $W$ in GeV. From [10].

For this to be true, in view of Eq. [4] the leading behavior for $N$ should also be as $\ln^2 W$ at very high energy. In fact, a plausible fit where this is the case is possible using LHC and lower energy particle production data [10]. Dividing $\sigma$ by $N$ one then has a certain constant cross section per produced particle at very high energy. The fit leads to $0.31 \text{ mb}$ of total cross section per pion. Like the cross section, the fit for $N$ has the feature that along with the $\ln^2 W$ there is non-
leading $\ln W$ term with a large negative coefficient. Since at presently available energies “Asymptopia” is still far, the $\sigma/N$ are not all constant and we do not yet have the limiting $\sigma(\text{elastic})/N = \frac{1}{2}\sigma(\text{total})/N$. Interestingly, however, $\sigma(\text{elastic})/N$ has reached its limiting value and appears to be constant. These features are shown in Fig. 2. There seems to be experimental support for Eq (4) and for the idea of a simple connection between the rise in the cross sections and rise of the multiplicity.

5 The ‘Edge’ of the Proton

We thus arrive at a simple picture: At very high energy the proton looks like a simple ‘black disc’ where the elastic cross section $\sigma(\text{elastic})$ is half that of the total cross section $\sigma(\text{total})$, as somebody who had scattering theory in their quantum mechanics course might have guessed. And, as he or she might not have guessed, if they didn’t know about Eq (1) the radius of this disc is growing logarithmically.

But along with this pleasant picture come two questions:

A) Why did it take so long for this picture to emerge, until energies in the many TeV range? As said above, these energies are a thousand or ten thousand times greater than any obvious mass scale.

B) A simple “black disc” with an abrupt hard edge seems rather a mathematical idealization. Wouldn’t it be more physical and realistic to have some kind of a soft edge, with a gradual transition from total opacity for central collisions to complete transparency at large distances?

It turns out the answer to A) comes from examining B).

To examine B) we need some quantity which will isolate the possible ‘edge’ hidden in the experimental information. This can be done as follows. Both of the cross sections $\sigma(\text{total})$ and $\sigma(\text{elastic})$ may be written as a sum of contributions over impact parameter $b$. We consider the quantity $\left(\sigma(\text{total}) - 2\sigma(\text{elastic})\right)$, which would be zero at all $b$ for the idealized “black disc”. In this impact parameter representation, this difference peaks in the vicinity of the radius. This occurs because, with only a small real part to the amplitude, both $\sigma(\text{total})$ and $\sigma(\text{elastic})$ are given by the same amplitude, the first linearly via the optical theorem, and the second quadratically via squaring the amplitude. In terms of a transparency $\eta(b)$ the difference $\sigma(\text{total}) - 2\sigma(\text{elastic})$ is given by an integral over impact parameter $b$, namely $4\pi \int_0^\infty \eta(b)(1 - \eta(b))b\,db$. It will be seen that as $\eta(b)$ varies from near zero at $b = 0$ to one at large $b$, the integrand goes from zero to zero with a peak in the middle, near where the idealized edge would be. This makes it a suitable quantity for isolating an “edge”.

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1 Technical note: Because of the identity $\ln^2(W/W_o) = \left(\ln(W/W_o') - \ln(W/W_o'')\right)^2 = \ln^2(W/W_o') - 2\ln(W/W_o')\ln(W/W_o'') + \ln^2(W/W_o'')$, one can trade a linear $\ln$ term for a change in the scale in the argument of the $\ln^2$ term. Our statements about a large non-leading term are relative to the use of a moderate scale in the $\ln^2$ terms, which is $1\, GeV$ in the fits we quote. Note this ambiguity of representation has no effects on the coefficient of the $\ln^2$ term.
Figure 3: The “edge” and the “disc”. The dashed (blue) line is a plot of the ratio Eq. 5, representing \( t \), the effective thickness of the edge. Its constancy exhibits the energy independence of the edge. For comparison the dashed-dotted line (red) represents the black-disc radius \( R \) inferred from the total cross section, \( R = \sqrt{\sigma_{\text{total}}/2\pi} \). The units are in fermi=fm = \( 10^{-13} \)cm. From ref [11].

Furthermore if we normalize to the radius defined by the total cross section, \( R = \sqrt{\sigma_{\text{total}}/2\pi} \), one finds that the quantity

\[
t = \frac{\sigma_{\text{total}} - 2\sigma_{\text{elastic}}}{\sqrt{\pi/2}\sigma_{\text{total}}}
\]

represents the ‘thickness’ of the ‘edge’[11].

One notes that this quantity is nicely constructed from experimental quantities only. Thus it is independent of the fitting procedure, as long as the fits go through the data and for energies where data exists, and is independent of any theoretical prejudices.

One may use the fits of Fig 1 to evaluate Eq 5. The result is shown in Fig 3. The blue dashed line is the value of \( t \). For comparison the radius corresponding to the total cross section \( R = \sqrt{\sigma_{\text{total}}/2\pi} \) is also shown as the red dashed-dot line.

One observes that \( t \) is constant with energy, and has the very reasonable value of \( t \approx 1.1 \) fm. This is the hidden simplicity behind the seemingly featureless Fig 1. What the data actually represent is just a logarithmically expanding
black disc, with a constant edge.

To see what the “edge” looks like, one may also use the further information provided by elastic scattering data at non-zero angles to reconstruct the elastic amplitude as a function of $b$. Fig. 4 shows the resulting edge integrand, the quantity whose integral gives the ratio Eq. 5. One sees how the “edge” remains roughly constant and moves out as the “disc” expands. (The quantity $C_R$ is not exactly 1 to take account of the small real part of the amplitude.)

6 Remote Asymptopia

We seem to have arrived at a simple and satisfying picture. There is a black disc, logarithmically expanding, in analogy with the “relativistic rise” of ionization, and this disc has a constant, smooth edge.

However, a puzzle remains. These simple features still do not stand out very clearly, even at the present very high LHC energies; the approach to “Asymptopia” is very slow. In Fig. 1 we are still far from the limit $\sigma(\text{elastic}) = \frac{1}{2}\sigma(\text{total})$
and the same holds for the $\sigma/N$ ratios in Fig. 2. The limits will finally be reached, according to the fits, but why is it that it takes energies $W$ a thousand or ten thousand times greater than any evident energy or mass scale for this to begin to happen? (I also remind the non-specialist that $W$ is a center-of-mass energy, so that it takes even longer in terms of the laboratory energy, which is the relevant energy for cosmic rays or fixed-target experiments.)

An explanation seems to be provided by Fig. 3. “Asymptopia” will be reached when the leading “disc” is distinctly larger than the subdominant “edge”, and Fig. 3 shows this will not happen until $W$ is at least in the multi-TeV regime. Indeed the complete dominance of the “disc” is delayed to energies that probably will never be reached in accelerator or cosmic ray experiments.

7 The “Edge” versus the “Disc”

But perhaps should we say “description” instead of “explanation”? Fig. 3 is certainly very interesting, but one may rightly say it just shifts the question. We would now like to know why it is that the “disc” is so small or, alternatively, the “edge” so big? In terms of cross section parameters the 1.1 $fm$ thickness $t$ of the edge corresponds to $\pi(1.1)^2 = 37$ $mb$ while the coefficient of the $\ln^2$ term for $\sigma_{total}$ is only the relatively tiny 1.1 $mb$.

This brings us to the intriguing question as to the nature of the “edge”. One might entertain various speculations as to its origin or makeup. One suggestion is that $t$ is associated with the length that the color string of QCD can be stretched before it breaks.

Another very interesting possibility is that the ‘edge’ has to do with the exchange of a pion, the lightest hadron. Firstly, the $t = 1.1$ $fm$ corresponds well with the compton wavelength of the pion, $1/m_\pi \approx 1.4$ $fm$. More significantly, this would put the slow approach to “Asymptopia” in a new light and offers an amusing resolution to the puzzle of the mass scales. The pion is quite light compared to other hadrons. Indeed, according to chiral symmetry, which plays an important role in low energy hadron physics, the pion should be thought of as having initially zero mass, before acquiring a finite mass via small corrections.

From this point of view, we would say that the big “edge” comes from the “almost zero” mass of the pion. Hence the slow approach to “Asymptopia” originates not from some hidden high mass scale, but on the contrary, in the existence of a very low scale, which covers up the ultimately leading behavior until very high energy. Thus while the factor $\pi/m_\pi^2$, long thought to characterize the high energy cross section, is certainly there, it gives the contribution of

2 These considerations also provide an interesting view on the question of the Froissart bound in the chiral limit $m_\pi \to 0$. If the asymptotic $\ln^4 W$ really has as its dimensional prefactor something involving $1/m_\pi^2$ as in [5] this would blow up as $m_\pi \to 0$. On the other hand, our discussion suggests that reducing $m_\pi$ gives an increase in the size of the “edge”. Thus a very small $m_\pi$, results not necessarily in a change of the asymptotic term, but rather in the removal of “Asymptopia” to a very high energy.
the non-leading edge, and at high enough energy is finally overtaken by the $(1.1 \text{mb}) \ln^2 W$ of the “disc”.

This leaves us with the problem of explaining the $(1.1 \text{mb})$. It seems to have no connection with $m_\pi$, but it does closely resemble the scale which one gets with the “usual” hadrons with masses around a GeV: $\pi/\text{GeV}^2 = 1.2 \text{mb}$. It is a challenge to theory to provide a calculation of the $1.1 \text{mb}$, which ultimately gives the cross section at highest energies and so represents a fundamental parameter of hadron physics.

This article is dedicated to the memory of Marty Block, who after a very long and productive career, passed away in July 2016.

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