A CONJECTURE OF TRAUTMAN

HOWARD JACOBOWITZ

1.

In 1998 the physicist Andre Trautman conjectured that a three-dimen-

sional CR manifold is locally realizable if and only if its canonical

bundle admits a closed nowhere zero section. First we review the re-

levant definitions and in the next section give the physical context. In

Section 3 we outline the earlier results in [2] which had proved a weak

version of the Conjecture.

A CR structure on a three-dimensional manifold $M$ is a two-plane

distribution $H \subset TM$ and a fiber preserving anti-involution $J : H \to H$.

We denote this structure by $(M, H, J)$. It is often useful to extend $J$

by complex linearity to a map

$$J : \mathbb{C} \otimes H \to \mathbb{C} \otimes H.$$  

Then $J$ is completely determined by the eigenspace corresponding to

the eigenvalue $i$ (or to the eigenvalue $-i$).

An equivalent definition of a CR structure on a three-dimensional

manifold may be given in terms of a complex line bundle: A CR struc-

tures on $M$ is a line bundle $B \subset \mathbb{C} \otimes TM$ with the property that $B \cap \overline{B}$

contains only the zero section. Then

$$H = \{ \Re Z : Z \in B \}$$

is of rank 2 and $J$ is defined on $\mathbb{C} \otimes H = B \oplus \overline{B}$ by setting

$$J(Z) = iZ \text{ if } Z \in B$$

and

$$J(Z) = -iZ \text{ if } Z \in \overline{B}.$$  

So for $X - iY \in B$

$$JX = Y \text{ and } JY = -X.$$
Example  Let $M^3 \subset \mathbb{C}^2$ be a real hypersurface and let $J$ denote the usual operator on $\mathbb{R}^4$ giving the complex structure. Set $H_p = T_p M \cap JT_p M$ for each $p \in M$. Now $J$ acts on $H$ and $(M,H,J)$ is a CR structure. Or, to use the alternative definition, just take

$$B = T^{1,0}(\mathbb{C}^2) \cap \mathbb{C} \otimes TM$$

where $T^{1,0}$ is the linear span of

$$\left\{ \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2} \right\}$$

(and $T^{0,1}$ is the span of the conjugates).

So later we write $B = T^{1,0}(M) = T^{1,0}$ and write $T^{0,1}$ for $\overline{B}$.

The canonical bundle is another complex line bundle associated to a CR structure. It is a subbundle of the second exterior product. For a real hypersurface in $\mathbb{C}^2$ it is generated by the two-form $dz_1 \wedge dz_2$ restricted to $M$. More generally, if the CR structure is given by a complex line bundle $B$ then

$$\Omega = \{ \omega \in \mathbb{C} \otimes \Lambda^2(TM) : i_b \omega = 0 \text{ for all } b \in \overline{B} \}.$$ 

The interior product $i_b \omega$ is given by $i_b \omega(X) = \omega(b \wedge X)$.

Definition. $(M,H,J)$ is realizable in a neighborhood of $p$ if there exist complex functions $f_1$ and $f_2$ such that

$$(X + iJX)f_k = 0$$

for all $X \in H$ and

$$F : M \to \mathbb{C}^2$$

$$x \to (\Re f_1, \Im f_1, \Re f_2, \Im f_2)$$

is an embedding.

It follows upon identifying $M$ with its image $F(M)$ that the original structure $(M,H,J)$ coincides with the CR structure induced as in the Example.

We digress briefly to discuss higher-dimensional CR structures and return to this in Section 3.

Definition. $(M^{2n+1}, B)$ is a CR manifold if $B \subset \mathbb{C} \otimes TM$ is a vector subspace of rank $n$ with $B \cap \overline{B} = \{0\}$ and $[\Gamma B, \Gamma B] \subset \Gamma B$. I.e., the commutator of local sections of $B$ is always in $B$.

More precisely, we have defined a CR manifold of hypersurface type.
**Definition.** \((M^{2n+1}, B)\) is realizable if there is an embedding \(F: M \to \mathbb{C}^{n+1}\) with, after identifying \(M\) with \(F(M)\),

\[
T^{1,0}(\mathbb{C}^{n+1}) \cap \mathbb{C} \otimes TM = B.
\]

The canonical bundle is now a complex line bundle in the exterior product \(\Lambda^{n+1}(\mathbb{C} \otimes TM^{2n+1})\). Namely,

**Definition.** The canonical bundle is

\[
\Omega = \{ \omega \in \mathbb{C} \otimes \Lambda^{n+1}(TM) : i_v \omega = 0, \forall v \in T^{0,1} \}.
\]

**Definition.** A function \(f: M \to \mathbb{C}\) is a CR function if \(Lf = 0\) for all \(L \in T^{0,1}\).

**Lemma 1.1.** \(M^{2n+1}\) is realizable in \(\mathbb{C}^{n+1}\) if there exist CR functions \(f_1, \ldots, f_{n+1}\) such that

\[
(1.1) \quad df_1 \wedge \ldots \wedge df_{n+1} \neq 0.
\]

**Proof.** Let \(L_1, \ldots, L_n\) be a basis for \(T^{0,1}\) and let \(T\) be any nonzero vector transverse to \(H\). From (1.1) and using that the functions are CR, we have

\[
df_1 \wedge \ldots \wedge df_{n+1}(\overline{L_1}, \ldots, \overline{L_n}, T) \neq 0.
\]

So \(df_j T \neq 0\) for some \(j\), say \(j = n + 1\), which now implies

\[
df_1 \wedge \ldots \wedge f_n \wedge df_{n+1} \wedge df_1 \wedge \ldots \wedge df_n \neq 0.
\]

Thus

\[
F = (f_1, \ldots, f_{n+1})
\]

is a local embedding. Indeed perhaps after multiplying \(F\) by \(i\), \(F(M)\) has the form

\[
\mathcal{I} z_{n+1} = f(z_1, \ldots, z_n, \Re z_{n+1})
\]

The realizability problem is quite subtle. For instance, most three-dimensional \(C^\infty\) CR structures are not locally realizable [4], [7].

Most realizability results in higher dimensions concern strictly pseudo-convex CR structures.

**Definition.** A CR structure \((M, B)\) is strictly pseudo-convex if the quadratic form

\[
L \in B \to [L, \overline{L}] \mod\{B \oplus \overline{B}\}
\]

is definite.
Such structures are realizable if \(\dim M \geq 7\). See [1] and [5] for the original proofs and [11] for a variation.

Although, as we said, the general realizability problem is subtle there are two easy results.

**Proposition 1.** Real analytic CR manifolds are locally realizable.

A proof can be found, for instance, in [3, page 22].

**Proposition 2.** A CR manifold admitting a vector field \(v\) transverse to \(H\) and preserving the CR structure is locally realizable.

To preserve the CR structure means that the Lie derivative in the direction of \(v\) satisfies

\[
\mathcal{L}_v T^{1,0} = T^{1,0}
\]

A generalization of this result is important in Section 3 and will be proved there.

2.

We first wish to explain the observation of [8] that a shear-free congruence of null geodesics on a four-dimensional manifold induces a three-dimensional CR structure on a quotient manifold.

Let \(M^4\) be a Lorentz manifold with metric \(g\) and let \(k\) be a null vector field, \(g(k, k) = 0\). Let \(K\) be the real line bundle generated by \(k\). Set

\[
K^+ = \{ v \in T_p M : g(v, k) = 0 \}.
\]

Note that \(K \subset K^+\) and that \(K^+ / K\) is an \(\mathbb{R}^2\) bundle on \(M\). Following the notation in [10], let \(n \in K^+\). Denote the equivalency class of \(n\) in \(K^+ / K\) by \([n]\) and use the same notation for \(n \in \mathbb{C} \otimes (K^+ / K) = \mathbb{C} \otimes K^+ / \mathbb{C} \otimes K\).

**Lemma 2.1.** The metric \(g\) induces a well-defined positive definite inner product on \(K^+ / K\).

**Proof.** Let \([n_1]\) and \([n_2]\) belong to the fiber of \(K^+ / K\) over some point of \(M\). Define \(g([n_1], [n_2])\) to be \(g(n_1, n_2)\). If \(v_1\) and \(v_2\) are different choices then \(v_j = n_j + a_j k\) and so

\[
g(v_1, v_2) = g(n_1 + a_1 k, n_2 + a_2 k) = g(n_1, n_2)
\]

since \(k\) is a null vector and \(n_j \in K^+\). This shows that \(g\) is well-defined.

To see that \(g\) is definite, assume that for some \([n]\) we have

\[
g([n], [n]) = g(n, n) = 0.
\]
By the definitions of $k$ and $K^\perp$ we also have
\[ g(k, k) = 0 \]
and
\[ g(k, n) = 0. \]
So either $n$ is a multiple of $k$ or $g$ vanishes on a two-dimensional plane. The second alternative is not possible for a Lorentz metric. So $n = ak$ and thus $[n] = 0$. Hence $g$ is definite, and since it arises from a Lorentz metric it is positive definite.

\[ \square \]

Fix an orientation for $K^\perp/K$ (this is not a problem, as long as we care only about local results) and then let $J : K^\perp/K \to K^\perp/K$ be the operation of rotation by $\pi/2$ radians with respect to the induced metric and orientation. Finally, set
\[ N = \{ n \in C \otimes K^\perp : J[n] = -i[n] \}. \]
Note that $N$ is a two-dimensional complex vector bundle on $M$. Extend the inner product $g$ to $N$ as a complex linear form. For $n_1 = \xi + iJ\xi$ and $n_2 = \eta + iJ\eta$ in $N$ we have
\[
\begin{align*}
g(n_1, n_2) &= g(\xi, \xi) + ig(J\xi, \eta) + ig(\xi, J\eta) - g(J\xi, J\eta) \\
&= 0
\end{align*}
\]
since $J$ is rotation by $\pi/2$ radians. So $N$ is said to be totally null. On the other hand,
\[
g(n_1, \overline{n_1}) = 2g(\xi, \xi) \neq 0.
\]
We have
\[ N \subset C \otimes K^\perp \subset C \otimes TM \]
and
\[ N \cap \overline{N} = C \otimes K, \quad N + \overline{N} = C \otimes K^\perp. \]

Now consider the flow generated by the vector field $k$. For small values of the time parameter, the orbit space is a three-dimensional manifold (again, for local results this is clear); call it $M'$. Without additional assumptions on $k$ the bundle $N$ does not project to a well-defined subbundle of $C \otimes TM'$. Here is where physics enters.

We temporarily drop the assumption that $k$ is null.

**Definition.** [8, page 1426] The vector field $k$ is said to be conformally geodesic if the associated flow preserves $K^\perp$ and $g(k, k)$ does not change sign.
Note that this definition depends only on the conformal class of $g$ and also that in Riemannian geometry the condition on the flow and $g(k, k) = c$ imply $\nabla_k k = 0$.

The flow condition may be rewritten as

$$\mathcal{L}_k K^\perp \subset K^\perp.$$  

and is equivalent to

$$(2.1) \quad g(k) \wedge \mathcal{L}_k g(k) = 0$$

where $g(k)$ is the one-form defined by $g(k) v = g(k, v)$. To see this equivalence, we first note that if $v$ is a vector field satisfying $g(k) v = 0$ then also $k(g(k) v) = 0$ and so

$$(2.2) \quad (\mathcal{L}_k g(k)) v + g(k) \mathcal{L}_k v = 0.$$  

We want to derive $g(k) \wedge \mathcal{L}_k g(k) = 0$. It is enough to show that

$$g(k) v = 0 \implies \mathcal{L}_k g(k) v = 0.$$  

That is, if $g(k)$ and $\mathcal{L}_k g(k)$ have the same kernel then these one-forms are linearly dependent. So assume $\mathcal{L}_k K^\perp \subset K^\perp$ and $g(k) v = 0$. We now have

$$g(k) v = 0 \Rightarrow v \in K^\perp \Rightarrow \mathcal{L}_k v \in K^\perp \Rightarrow g(k) \mathcal{L}_k v = 0 \Rightarrow \mathcal{L}_k g(k) v = 0$$

where the last implication follows from (2.2).

On the other hand, if $g(k) \wedge \mathcal{L}_k g(k) = 0$, then

$$g(k) v = 0 \Rightarrow (\mathcal{L}_k g(k)) v = 0 \Rightarrow g(k) \mathcal{L}_k v = 0 \Rightarrow \mathcal{L}_k K^\perp \subset K^\perp.$$  

We are interested in the case where $k$ is a null vector, $g(k, k) = 0$. When $k$ is null the foliation of $M$ by the integral curves of $k$ is called a congruence of null geodesics.

The Lorentz metric $g$ induces a degenerate inner product on $K^\perp$ and therefore also a (degenerate) conformal structure.

**Definition.** A conformally geodesic vector field is shear-free if the associated flow preserves the conformal structure of $K^\perp$.

The physical hypothesis that $k$ generates a shear-free congruence of null geodesics also can be formulated in terms of the Lie derivative.

**Theorem 2.1.** [9] A vector field $k$ on a manifold $M^4$ with Lorentz metric $g$ generates a shear-free congruence of null geodesics if and only if

$$g(k, k) = 0 \quad (2.3)$$

$$\mathcal{L}_k g = \lambda g + \phi \otimes g(k) \quad (2.4)$$
**Conjecture 7**

where \( \lambda \) is a function, \( \phi \) is a one-form, \( g(k) \) is as defined above, and \( \phi \otimes g(k) \) signifies the symmetric product constructed from the one-forms.

**Proof.** We first show that (2.4), together with (2.3), implies (2.1) and hence \( k \) is a conformally geodesic vector field. We start with the Leibniz rule:

\[
k(g(u, v)) = \mathcal{L}_k g(u, v) + g(\mathcal{L}_k u, v) + g(u, \mathcal{L}_k v).
\]

Setting \( u = k \) and rearranging this becomes

\[
\mathcal{L}_k g(k, v) = k(g(k, v)) - g(k, \mathcal{L}_k v).
\]

Further

\[
k(g(k, v)) = k(g(k)v) = (\mathcal{L}_k g(k))v + g(k, \mathcal{L}_k v)
\]

and so

\[
(\mathcal{L}_k g)(k, v) = (\mathcal{L}_k g)(v).
\]

Now

\[
(g(k) \wedge \mathcal{L}_k g(k))(u, v) = (g(k)u)(\mathcal{L}_k g)(k)v - (g(k)v)(\mathcal{L}_k g)(k)u
\]

\[
= (g(k)u)(\mathcal{L}_k g)(k, v) - (g(k)v)(\mathcal{L}_k g)(k, u)
\]

\[
= g(k, u)(\lambda g(k, v) + \phi \otimes g(k)(k, v))
\]

\[
- g(k, v)(\lambda g(k, u) + \phi \otimes g(k)(k, u))
\]

\[
= 0.
\]

We know that \( g(k) \wedge \mathcal{L}_k g(k) = 0 \) implies that \( K^\perp \) is preserved and so \( k \) is conformally geodesic.

To show that the conformal class \( g \) induces on \( K^\perp \) is constant along the flow on \( M \) induced by \( k \), we let \( v \in K^\perp \) be constant along the flow. So \( g(k)v = 0 \) and \( \mathcal{L}_k v = 0 \). Thus

\[
k(g(v, v)) = \mathcal{L}g(v, v)
\]

\[
= (\lambda g + \phi \otimes g(k))(v, v)
\]

\[
= \lambda g(v, v).
\]

This gives us an ordinary differential equation. If local coordinates \((t, x)\) are introduced with \( k = \partial_t \) then the equation has the form

\[
\frac{\partial f(t, x)}{\partial t} = \lambda(t, x)f(t, x)
\]

and the solutions are

\[
f(t, x) = \Lambda(t, x)f(0, x)
\]

for some function \( \Lambda \). Thus

\[
g(v, v)(t, x) = \Lambda(t, x)g(v, v)(0, x).
\]
This shows that the conformal class of the metric on $K^\perp$ does not change under the flow.

Conversely, we want to show that if $k$ generates a shear-free congruence of null geodesics then there exist a scalar function $\lambda$ and a one-form $\phi$ satisfying (2.4). To see this, we start with a frame invariant along the orbits, labeled $e_1, e_2, e_3, e_4$ with $\{e_1, e_2, e_3\}$ a basis for $K^\perp$ and $g(e_4, e_4) = 0$. Let $0 \leq i \leq 3$, $0 \leq j \leq 3$. Note that $g(k) e_i = 0$ and $g(k) x_4 \neq 0$. For $p \in M$ parametrize the orbit through $p$ by $t$. Since the conformal class of $g$ on $K^\perp$ is constant

$$g(e_i, e_j)|_t = \Lambda(t) g(e_i, e_j)|_p,$$

Thus

$$(\mathcal{L}_k g)(e_i, e_j)|_p = (\mathcal{L}_k \Lambda) g(e_i, e_j)|_p.$$

Define

$$\lambda = \mathcal{L}_k \Lambda$$
$$\phi(e_i) = (g(k) e_4)^{-1} \left( (\mathcal{L}_k g)(e_i, e_4) - \mathcal{L}_k \lambda g(e_i, e_4) \right), \quad 0 \leq i, j \leq 4.$$

We have for $0 \leq i, j \leq 3$

$$(\lambda g + \phi \otimes g(k))(e_i, e_j) = (\mathcal{L}_k \Lambda) g(e_i, e_j) = (\mathcal{L}_k g)(e_i, e_j),$$

while for $i \leq 4$ we have

$$(\lambda g + \phi \otimes g(k))(e_i, e_4) = (\mathcal{L}_k \Lambda) g(e_i, e_4) + \phi(e_i) g(k) e_4$$
$$= (\mathcal{L}_k \Lambda) g(e_i, e_4) + \mathcal{L}_k (g(e_i, e_4)) - (\mathcal{L}_k \Lambda) g(e_i, e_4)$$
$$= (\mathcal{L}_k g)(e_i, e_4).$$

Thus

$$\lambda g + \phi \otimes g(k) = \mathcal{L}_k g.$$ 

\[\square\]

Let $\pi$ denote the map of $M$ to the orbit space

$$\pi: M \to M'.$$

**Lemma 2.2.** Under the conditions of the Theorem, $\pi^*(N)$ is a complex line bundle $\overline{B} \subset C \otimes TM'$ which satisfies $B \cap \overline{B} = \{0\}$.

**Proof.** Since $K^\perp$ is itself invariant under the flow, $K^\perp/K$ projects to a well-defined two-plane distribution $\mathcal{H}$ on $M'$ and on $\mathcal{H}$ we have a well-defined conformal class of metrics. Thus $C \otimes \mathcal{H}$ splits into the eigenspaces of $J$

$$C \otimes \mathcal{H} = B \oplus \overline{B},$$

with $\pi^* N = \overline{B}$.

\[\square\]
That is, the physical assumptions lead to a CR structure on the orbit space. Further, as we now show, the same conditions provide a two-form $F$ associated to $N$ which itself also passes down to $M'$. The interest in such a two-form comes from considerations of Maxwell’s equations. In classical physics, the components of the magnetic and electrical fields can be used to construct a real two-form $F$, called the Faraday tensor. Then, in the absence of charge, Maxwell’s equations become $dF = 0$. Naturally, in relativistic physics the situation is more complicated.

To define $F$ we first find a basis for $N$. Let $\xi \in K^\perp$ and $\xi \notin K$. Choose any $\eta \in K^\perp$ such that $J[\xi] = [\eta]$. Then $n = \xi + i\eta$ and $k$ form a basis for $N$.

Let $g(k)$, defined above, and $g(n)$, defined in the same way, be one-forms on $M$. Set

$$ F = g(n) \wedge g(k). $$

Note that $F$ is nowhere zero since the one-forms $g(n)$ and $g(k)$ are independent. For example, $g(n)\overline{\eta} \neq 0$ while $g(k)\overline{\eta} = 0$.

The two-form $F$ is associated to $N$ in the following sense:

**Lemma 2.3.** $N = \{v \in C \otimes TM : i_v F = 0\}$.

**Proof.** We have $g(k, k) = 0$ because $k$ is null; $g(k, n) = 0$ because $N \subset C \otimes K^\perp$; and $g(n, n) = 0$ because $N$ is totally null. So for our basis $i_k F = 0$ and $i_n F = 0$. Thus

$$ N \subset \{v \in C \otimes TM : i_v F = 0\}. $$

Now let $t \in \{v \in C \otimes TM : i_v F = 0\}$. So

$$ g(n, t)g(k) - g(k, t)g(n) = 0. $$

The independence of $g(n)$ and $g(k)$ implies $t \in C \otimes K^\perp$ at some point of $M$. Thus

$$ t = \alpha n + \beta \overline{n} + \gamma k $$

for constants $\alpha, \beta, \gamma$. Since $g(n, t) = 0$ and $g(n, \overline{n}) \neq 0$, we see that $\beta = 0$ and thus $t \in N$. \hfill \Box

We may use $F$ to define a two-form on $M'$: Let $t_1$ and $t_2$ be vectors in $C \otimes TM'$. Lift $t_j$ to a vector $t_j + \alpha_j k$ in $C \otimes TM$. Then

$$ F((t_1 + \alpha_1 k) \wedge (t_2 + \alpha_2 k)) = g(n, t_1 + \alpha_1 k)g(k, t_2 + \alpha_2 k) - g(n, t_2 + \alpha_2 k)g(k, t_1 + \alpha_1 k) = g(n) \wedge g(k)(t_1 \wedge t_2). $$
So $F$ evaluated on the lift is independent of choices and gives a well-defined two-form on $M'$. Call this form $F'$. For $t \in \mathcal{B} = C \otimes T^{0,1}(M')$ the natural lift, also called $t$ is in $N$. Thus from the Lemma

$$t \in C \otimes T^{0,1}(M') \implies i_tF' = 0.$$ 

Hence $F'$ is section of the canonical bundle of $M'$ and is nowhere zero.

In summary, the local quotient of a Lorentzian manifold under a shear-free congruence of null geodesics is a CR manifold which has a nowhere zero section of its canonical bundle. This section being closed is related to Maxwell’s equation and so is a reasonable hypothesis for physicists. We now repeat Trautman’s conjecture.

**Conjecture 2.1.** If a CR manifold $M^3$ admits a nowhere zero closed section of its canonical bundle, then the CR structure is locally realizable.

As we have seen, the converse is true even globally.

3.

A weak version of the conjecture is true and holds for all dimensions. Functions satisfying

$$df_1 \wedge \ldots \wedge df_k \neq 0$$

are called independent. Functions satisfying

$$df_1 \wedge \ldots \wedge df_k \wedge \overline{df_1} \wedge \ldots \wedge \overline{df_k} \neq 0$$

are called strongly independent.

**Example.** The hyperquadric $Q^3 \subset C^2$ is defined by $\mathcal{I}z_2 = |z_1|^2$. The bundle $T^{0,1}$ is generated by

$$L = \partial_{\overline{z_1}} - iz\partial_u$$

where $u = \mathcal{R}z_2$. The CR function $f = z$ is strongly independent; The function $f = u + i|z|^2$ is independent, but not strongly independent (at the origin).

The following theorem preceded the formulation of Trautman’s Conjecture and establishes a weak form.

**Theorem 3.1.** [2] If the CR structure $M^{2n+1}$ has $n$ strongly independent CR functions near $p$ and if the canonical bundle has a closed nowhere zero section then $M^{2n+1}$ is realizable in a neighborhood of $p$.

The proof depends on the following complex version of Proposition 2.
**Proposition 3.** $M$ is realizable in a neighborhood of $p$ if and only if there exists a complex vector field $Y$ near $p$ such that

- $Y$ is transverse to $T^{1,0} \oplus T^{0,1}$
- $\mathcal{L}_YT^{1,0} = T^{1,0}$.

Thus the existence of a real vector field such that $\mathcal{L}_YT^{1,0} = T^{1,0}$ is very special (since most realizable CR structures do not have such a vector field) but the existence of such a complex vector field characterizes realizability.

**Proof.** We first prove the necessity. So assume $M$ is realizable near $p$. Without loss of generality we assume $p = 0$ and $M$ is given as

$$M = \{(z_1, \ldots, z_{n+1}) : \mathfrak{I}z_{n+1} = \rho(z_1, \ldots, z_n, \overline{z}_1, \ldots, \overline{z}_{n-1}, \Re z_{n+1})\}.$$

Define $\overline{Y}$ by

$$(3.1) \quad dz_{n+1}(\overline{Y}) = 1, \quad dz_j(\overline{Y}) = dz_j(\overline{Y}) = 0, \quad 1 \leq j \leq n.$$ 

Note that $\overline{Y}$ (and also $Y$) is transverse to $T^{1,0} \oplus T^{0,1}$. Set

$$\omega = dz_1 \wedge \ldots \wedge dz_{n+1}|_M.$$ 

This is a nowhere zero closed section of the canonical bundle. As a consequence of Cartan’s formula

$$\mathcal{L}_v = di_v + i_v d$$

we have

$$\mathcal{L}_{i\overline{\omega}} = d(i\overline{\omega}) + i\overline{\omega} d\omega = 0.$$ 

This implies $\mathcal{L}_{\overline{Y}}T^{0,1} = T^{0,1}$ and so also

$$\mathcal{L}_YT^{1,0} = T^{1,0}.$$ 

Conversely, we will assume that $\mathcal{L}_YT^{1,0} = T^{1,0}$ with $Y$ transverse to $T^{1,0} \oplus T^{0,1}$, and show that $M$ is locally realizable. This is just a slight modification of a standard proof of Proposition[2] Extend $Y$ and each of the vectors in $T^{1,0}$ to $\mathbb{C} \otimes T(M \times \mathbb{R})$ by taking them constant in the $\mathbb{R}$ direction. Let $Y$ still denote this extension and let $V$ denote the extension of the bundle $T^{1,0}$. Set $Z$ to be the complex line bundle spanned by $Y + i\partial/\partial t$ where $t$ is the natural parameter for $\mathbb{R}$. Then

$$W = V \oplus Z$$

satisfies

$$W \cap \overline{W} = \{0\} \quad \text{and} \quad W + \overline{W} = \mathbb{C} \otimes T(M \times \mathbb{R}).$$

Finally, as is easily seen, $W$ is closed under the commutation of vector fields,

$$[\Gamma W, \Gamma W] \subset \Gamma W.$$ 

Thus $W$ satisfies the conditions of the Newlander-Nirenberg Theorem[6] and so defines a complex structure on $M \times \mathbb{R}$. Since $W \cap \mathbb{C} \otimes TM = \{0\}$...
the CR structure induced on $M$ is the one we started with. □

All that is left to do in the proof of Theorem 3.1 is to show that if $f_1, \ldots, f_n$ are CR functions on $M^{n+1}$ with

$$df_1 \wedge \ldots \wedge df_n \neq 0$$

and if $\omega$ is a nowhere zero section of the canonical bundle with

$$d\omega = 0$$

then there is a complex vector field $Y$ with

- $Y$ transverse to $T^{1,0} \oplus T^{0,1}$
- $\mathcal{L}_Y T^{1,0} = T^{1,0}$.

We just use the closed section to find a replacement for $dz_{n+1}$ in (3.1). Because we prefer to work with the canonical bundle and not its conjugate, we start, as in the Proposition, by defining a vector field $\zeta$ and then let $Y = \overline{\zeta}$. Towards this end, let $\theta$ be a nowhere zero one-form annihilating $T^{1,0} \oplus T^{0,1}$. Then

$$\theta \wedge df_1 \wedge \ldots \wedge df_n$$

is a nowhere zero section of the canonical bundle. This bundle is one dimensional, so

$$\omega = f\theta \wedge df_1 \wedge \ldots \wedge df_n.$$ 

Define $\zeta$ by

$$f\theta(\zeta) = 1, \quad df_j(\zeta) = 0, \quad d\overline{f}_j(\zeta) = 0$$

$\zeta$ can be thought of as a complex version of the Reeb vector field. In particular, it is transverse to $T^{1,0} \oplus T^{0,1}$.

We have

$$\mathcal{L}_{\zeta} \omega = d(\iota_{\zeta} \omega) + i_{\zeta} d\omega$$

$$= d(f\theta(\zeta)) df_1 \wedge \ldots \wedge df_n + i_{\zeta} d\omega$$

$$= 0.$$

Lemma 3.1. If $\mathcal{L}_{\zeta} \omega = 0$ then $\mathcal{L}_{\zeta} T^{0,1} = T^{0,1}$.

Proof. We have for all vector fields $\zeta$ and $v$ and all forms $\omega$

$$\mathcal{L}_{\zeta} \iota_v \omega = \iota_v \mathcal{L}_{\zeta} \omega + \iota_v \mathcal{L}_{\zeta} \omega.$$ 

So, if $v \in T^{0,1}$, hence $\iota_v \omega = 0$, and $\mathcal{L}_{\zeta} \omega = 0$, then

$$\iota_{\mathcal{L}_{\zeta} v} \omega = 0$$

and so $\mathcal{L}_{\zeta} v$ is also in $T^{0,1}$. □
CONJECTURE

This Lemma has a partial converse: If $\mathcal{L}_\zeta T^{0,1} = T^{0,1}$ then $\mathcal{L}_\omega = \alpha \omega$ for some function $\alpha$.

Finally, we set $Y = \overline{\zeta}$. Thus, $Y$ is transverse to $T^{1,0} \oplus T^{0,1}$ and

$$\mathcal{L}_Y T^{1,0} = \mathcal{L}_\zeta T^{0,1} = T^{1,0}$$

and we are done.

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Department of Mathematical Sciences, Rutgers University, Camden New Jersey, jacobowi@rutgers.edu