POLARIZED DIPOLE RADIATION FROM AZIMUTHALLY ANISOTROPIC ELECTRON SCATTERING

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A technique for calculation of polarized bremsstrahlung probability within the equivalent photon approximation is developed, based on reduction of the electron spin matrix element to a form $V^\alpha \gamma^\alpha + A_\gamma^\alpha$. For the fully differential cross-section in the lab frame it is shown that the radiation polarization is directed along a family of circles passing through a pair of intensity dips. Those circles correspond to meridional circles in the initial electron rest frame, and the correspondence with the lab frame emission is classified as a stereographic projection. Integration over momentum transfers in an intrinsically anisotropic target is performed, all the anisotropy effects within the dipole approximation being accumulated into a special modulus-bound transverse vector, $N$. To exemplify a target with $N^2 \sim 1$, we calculate radiation from electron incident at a small angle on an atomic row in oriented crystal. Radiation intensity and polarization dependence on the emission angle and frequency for fixed $N$ is investigated. A prominent feature is the existence of an angle at which the radiation may be completely polarized, in spite of the target isotropy — that owes to existence of an origin-centered tangential circle for polarization in the fully differential radiation probability kernel. Net polarization for the angle-integral cross-section is evaluated, which appears to be proportional to $N^2/2$, and decreases with the increase of the photon energy fraction.

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I. INTRODUCTION

Relativistic electrons interacting with matter emit plenty of gamma-radiation, which may be used for probing nuclei and hadrons [1], or to deliver information about the medium the electrons move in. The full set of the radiation characteristics includes photon polarization, which correlates with the preferential direction of the radiating particle in the medium, and also with the azimuthal angle of photon emission. Detection techniques sensitive to $\gamma$-quantum polarization have been developed to date at various frequency scales [2, 3, 4].

Calculations of polarization of radiation in atomic collisions (bremsstrahlung) date back to 1950’s [3, 4, 5, 6], but at that time they focused solely on electron interaction with one atom, which possesses spherical symmetry. That entails axial symmetry of the process, making net polarization of the radiation (when integrated over the relativistically small emission angles) vanish. Later on, polarized radiation at electron periodic motion in variously anisotropic fields (representing synchrotron or undulator magnets, or electric field in a crystal) had been studied (see [3, 10] and refs. therein), but given the complexity of the problem of particle motion in a field, it had been analytically accomplished only with fields depending on one spatial coordinate – planar or axial geometry, which probably do not cover all the cases of interest.

A simplifying property of electron scattering in atomic matter is that small (relative to the electron mass) momentum transfers to atoms dominate [36]. That enables one to benefit from the equivalent photon approximation [11, 12, 13, 14, 15] (in application to the bremsstrahlung problem also called dipole approximation), according to which the polarization-dependent part of the bremsstrahlung process amplitude is similar to that of Compton scattering. Technically, it is interesting to note that for Compton scattering there is a possibility to simplify the procedure of summation over electron spins, due to a particular choice of photon gauge (purely spatial in the initial electron’s rest frame [16]), and still more if an appropriate matrix basis for electron spin transition amplitudes is employed. We are going to explicate that procedure here, since it may be of more general applicability. Yet, the initial electron rest frame allows to infer some non-trivial symmetry properties of the polarized radiation angular distribution.

Our next, physical objective in this paper is to study electron radiation at passage through matter, but under conditions of azimuthal anisotropy of the scattering. The value of the dipole approximation is that it makes the radiation differential probability simply a quadratic form in the transferred momentum. That permits averaging over the momentum transfers in matter, basically, in a model-independent way, thus separating out the potentially complicated problem of electron motion in an anisotropic external field from the radiation process (but not necessarily relying on the scattering factorization assumption yet). Averaging of the generic quadratic form leads to accumulation of all the anisotropy effects into a single transverse vector, pointing along the preferential

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direction of momentum transfer, and having the absolute value related to the asymmetry degree. Then, all the radiation characteristics are to be expressed in terms of this vector, and studied as functions of its modulus. Last not least, it is important to discuss how high are the experimentally achievable values of the anisotropy degree and the radiation polarization.

The paper is organized as follows. In Sec. II we define the equivalent photon approximation for the bremsstrahlung process, implying at that stage the scattering factorization. Sec. III describes a method for evaluation of polarized Compton cross-section based on a specific Dirac spin matrix basis choice. In Sec. IV we specify the obtained Lorentz-invariant Compton cross-section, and after the scattering. To remind how it formally manifests itself in different popular frameworks, first refer to the target rest frame, where one observes relativistic length [\( z \sim \gamma / \gamma \)], and appears to the radiating electron as a longer object of typical size \( r_a \). The factorization conditions are thus

\[ q_z r_a \ll 1. \]  

(3)

From another viewpoint, in a frame where the electron is non-ultra-relativistic and evolves together with its electromagnetic proper-field at times \( \sim m^{-1} \), now the target atom becomes longitudinally Lorentz-contracted to the size \( \sim r_a / \gamma \), and appears to the radiating electron as a short kicker, leading again to the same condition \( q_z r_a \ll 1 \). Finally, if working in the momentum representation, say, in terms of Feynman diagrams, the emitted real photon typically changes the electron virtuality (square of its 4-momentum in a virtual state) by amount \( \sim m^2 \). As for momentum exchange with the target, individual longitudinal transfers \( q_z^{(i)} \) which are of the order of \( r_a^{-1} \)

\[ \sum q_z^{(i)} = q_z \]  

are kinematically restricted to be \( \ll r_a^{-1} \), though make denominator of the electron’s propagator relativistically large, but a proper compensation arrives from the energy numerator, typical for vector coupling theory (the same reason as for finiteness of forward cross-sections – see, e. g., [17]). However, if the real photon is emitted in between the momentum exchanges with...
the target, it splits a hard electron propagator into two hard ones, without a numerator compensation. Therefore, largest are contributions from diagrams in which the real photon is the first or the last one in the sequence, and the amplitude of radiation at a single scattering [18].

As for the Dirac matrix structure of the scattering amplitude, for small-angle scattering it is particularly simple. In each contribution to the amplitude from propagation between the scatterings (say, on the initial end) 
\[ e \cdot p \cdot \gamma + \sum q^{(i)} \cdot \gamma + m \]
the spin numerator can always be recast as 
\[ e \cdot \gamma \left( p \cdot \gamma + \sum q^{(i)} \cdot \gamma + m \right) e \cdot \gamma \]
\[ = 2 e \cdot p e \cdot \gamma + \left( -p \cdot \gamma - \sum q^{(i)} \cdot \gamma + m \right) e^2. \] (4)

With \( e \cdot p / m = \gamma \) (Lorentz-factor), \( \bar{u} e \cdot \gamma u / \bar{u} u \sim \gamma \), the second term in (1) is generally \( \mathcal{O} (\gamma^{-2}) \) relative to the first one and can be neglected within the accuracy of the factorization approximation [18]. Proceeding so in all orders, the matrix scattering amplitude can be written as 
\( e \cdot \gamma A_{\text{scat}}^{\text{diff}} (q_\perp) \), where \( A_{\text{scat}}^{\text{diff}} (q_\perp) \) is the spin-independent forward angle scattering amplitude including all orders in perturbation theory. Physically, it can be regarded as diffractive, so that its spin independence is intuitive.

Ultimately, the factorization theorem for the small-angle bremsstrahlung process assumes the form 
\[ T_{\text{fact}} = A_{\text{scat}}^{\text{diff}} (q_\perp) \sqrt{4\pi e M_{\text{rad}}} (q_\perp, q') \{ 1 + \mathcal{O} (q_\perp r_a) \} \] (5)

with 
\[ M_{\text{rad}} = \bar{u} \left( \frac{e^{\ast} \cdot \gamma (p \cdot \gamma + q \cdot \gamma + m) e \cdot \gamma}{2 p \cdot q + q^2} - \frac{e \cdot \gamma (p \cdot \gamma - q' \cdot \gamma + m) e^{\ast} \cdot \gamma}{2 p \cdot q'} \right) u, \] (6)

the tree-level radiation matrix element, and \( A_{\text{scat}}^{\text{diff}} (q_\perp) \) – the exact elastic scattering amplitude abridged of the conserved electron bispinors. If we normalize \( A_{\text{scat}}^{\text{diff}} (q_\perp) \), in accord with its diffractive origin, so that the diffractive scattering differential cross-section expresses as 
\[ d\sigma_{\text{scat}} = |A_{\text{scat}}^{\text{diff}} (q_\perp)|^2 \frac{d^2 q_\perp}{(2\pi)^2}, \] (7)

the factorization theorem for probabilities will read
\[ d\sigma_{\text{rad}} = \frac{1}{2E |T_{\text{fi}}|^2} \frac{d^2 q_\perp}{(2\pi)^2} \frac{d^2 q'}{(2\pi)^2} \frac{d\Gamma_{q'}}{2q_0^2} \]
\[ = d\sigma_{\text{scat}} (q_\perp) dW_{\text{rad}} (q_\perp, q'). \] (8)

Here,
\[ dW_{\text{rad}} = \frac{4\pi \alpha}{4EE'} |M_{\text{rad}}|^2 d\Gamma_{q'} \]
\[ (E, E', q_0' \text{ are the energies correspondingly of the initial and final electron, and of the emitted photon}). \]

It is important to stress that the formulated theorem does not require the softness of the emitted photon in the sense that its energy may be of the order of initial electron energy. That is why in Eq. (3) the spin structure of the radiation matrix amplitude is essential.

**B. Comptonization conditions**

Although \( q^2 \) is not subject to exact mass-shell restriction, but in atomic matter \( q \) is typically soft (and space-like), i. e.,
\[ |q^2| \sim r_a^{-2} \ll m^2. \] (11)

Other kinematic invariants in the problem, \( p \cdot q \) and \( p' \cdot q \), are \( \sim m^2 \) (as will be manifest in Sec. LV). So, everywhere except in the overall factor to be isolated later on, \( q^2 \) can be neglected, thus leading to the equivalent photon approximation:
\[ M_{\text{rad}} \approx M_{\text{Compt}}, \] (12)

\[ M_{\text{Compt}} = \bar{u} \left( \frac{e^{\ast} \cdot \gamma (p \cdot \gamma + q \cdot \gamma + m) e \cdot \gamma}{2 p \cdot q} - \frac{e \cdot \gamma (p \cdot \gamma - q' \cdot \gamma + m) e^{\ast} \cdot \gamma}{2 p \cdot q'} \right) \bigg|_{q^2=0}. \] (13)
The actual accuracy to which approximation (12) holds may be not immediately obvious, given that momentum variables are partially contained in electron spinors. We shall see (Sec. IV and Appendix A) that for electron spin dependent part of radiation intensity, the accuracy of the equivalent photon approximation for the radiation block may be $O(q/m)$, instead of $O(q^2/m^2)$ characteristic for a spinless charged particle. Furthermore, when folded with differential cross-section of scattering in fields with Coulombic cores, the accuracy of the equivalent photon approximation becomes only logarithmic (11, 12, 14, 15), which will be concerned with in Sec. V C.

To simplify expression (13), let us now commit ourselves to the gauge of orthogonality to the initial electron momentum, which is known to be advantageous in Compton scattering (16):

$$e = e_p, \quad e' = e'_p,$$

$$e_p \cdot p \equiv 0 \approx e_p \cdot q, \quad e'_p \cdot p \equiv 0 = e'_p \cdot q'. \quad (14)$$

Then, in the matrix element (13) $p \cdot \gamma$ may be anti-commuted with $e_p \cdot \gamma, \ e'_p \cdot \gamma$, to get in a position neighboring to $u_p$, and by virtue of Dirac equations (13) be canceled with the mass:

$$M_{\text{Compt}} = \bar u' \left( \frac{e_p \cdot \gamma q \cdot \gamma e_p \cdot \gamma}{2p \cdot q} + \frac{e_p \cdot \gamma q' \cdot \gamma e'_p \cdot \gamma}{2p \cdot q'} \right) u. \quad (15)$$

In this paper, we will only be interested in $|M_{\text{Compt}}(q_\perp, q')|^2$ averaged over initial electron’s and summed over final electron’s polarizations. By the trace technology (22, 23, 24, 25), the latter quantity should be evaluated as a spur

$$\left| \langle M_{\text{Compt}} \rangle \right|_{e \cdot \text{spin}}^2 = \frac{1}{2} S p \left\{ (p' \cdot \gamma + m) \times \left( \frac{e_p \cdot \gamma q \cdot \gamma e_p \cdot \gamma}{2p \cdot q} + \frac{e_p \cdot \gamma q' \cdot \gamma e'_p \cdot \gamma}{2p \cdot q'} \right) (p \cdot \gamma + m) \times \left( \frac{e_p \cdot \gamma q' \cdot \gamma e'_p \cdot \gamma}{2p \cdot q'} + \frac{e_p \cdot \gamma q' \cdot \gamma e'_p \cdot \gamma}{2p \cdot q} \right) \right\}. \quad (16)$$

Thereat, disappointing is that the polynomial under the spur is of 8th degree in $\gamma$-matrices, and its algebraic decomposition (16) produces quite a lengthy expression, terms of which are yet subject to strong mutual cancelations, whereas the grouping is not unique.

Generally, a faster and technically less ambiguous approach to calculation of QED processes (yet, giving access to electron polarization observables, which will not be our concern here, though) is evaluation of a complete set of electron spin amplitudes related to some Dirac (or Pauli) matrix basis. Techniques along these lines were developed in diverse fashions — see Refs. (19) and citations therein. But Compton kinematics actually appears to be special. We are going to show that it suggests a spin matrix choice different from any of the Refs. (19), and proving best suited for the differential cross-section calculation with the account of initial and final photon polarizations.

### III. COVARIANT CALCULATION OF POLARIZED COMPTON SCATTERING

**Choice of spin matrix basis** Our particular approach for Compton scattering consists in the following. To start with, make use of the standard $3\gamma$-matrix decomposition rule (38)

$$\gamma^\alpha \gamma^\beta \gamma^\gamma = g^{\alpha\beta} \gamma^\gamma - g^{\alpha\gamma} \gamma^\gamma + g^{\beta\gamma} \gamma^\alpha - i \epsilon^{\alpha\beta\gamma} \gamma^\nu \gamma^\mu. \quad (17)$$

To bring $M_{\text{Compt}}$ to a $1\gamma$-matrix form

$$M_{\text{Compt}} = \bar u' \left( V^\alpha \gamma^\alpha - i e_p^{\epsilon^\alpha} G^0 e^{\epsilon^\gamma} e^{\nu} \gamma^\mu \gamma^\nu \gamma^\mu \gamma^\alpha \right) u, \quad (18)$$

with abbreviations

$$V^\alpha = \frac{e_p^\alpha q' \cdot e^{\epsilon^\alpha} - q^\alpha e_p \cdot e^{\epsilon^\alpha}}{2p \cdot q} + \frac{e_p^\alpha q' \cdot e^{\epsilon^\alpha} - q^\alpha e_p \cdot e^{\epsilon^\alpha}}{2p \cdot q'}. \quad (19)$$

$$G^0 = \frac{q^3}{2p \cdot q} - \frac{q'^3}{2p \cdot q'}. \quad (20)$$

Here, it is fortunate that vector $G$, alongside with $e_p$ and $e'_p$ in (18), happens to be orthogonal to $p$:

$$G \cdot p = 0. \quad (21)$$

Thus, one may explicitly project the axial matrix-vector amplitude in Eq. (18) onto its only allowed direction $\parallel p$:

$$\epsilon^\alpha \epsilon^\beta \gamma^\gamma G = \frac{p^\alpha}{m} \epsilon^\alpha \epsilon^\beta \gamma^\gamma G. \quad (22)$$

The action of $p^\alpha \gamma^\nu \gamma^\mu / m$ on $u_p$ in Eq. (18) is equal to that of $-\gamma^5$, and one is left with a representation

$$M_{\text{Compt}} = \bar u' \left( V^\alpha \gamma^\alpha + A_5 \gamma^5 \right) u \quad (21)$$

with

$$A_5 = i \epsilon^{\alpha\beta\gamma} \frac{p^\mu}{m} e_p^{\epsilon^\alpha} e^{\epsilon^\beta} G^\gamma. \quad (22).$$

Yet, without transcending the generic structure of Eq. (18), we are free to add to vector $V^\alpha$ an arbitrary vector proportional to $(p - p')^\alpha$, since by Dirac equations (19, 16) it gives zero contribution to the amplitude. It is advantageous to tune it so that $V$ becomes orthogonal to momentum $p$:

$$V^\alpha \rightarrow V^\alpha_p = \frac{e_p^\alpha q' \cdot e^{\epsilon^\alpha} - q^\alpha e_p \cdot e^{\epsilon^\alpha}}{2p \cdot q} + \frac{e_p^\alpha q' \cdot e^{\epsilon^\alpha} - q^\alpha e_p \cdot e^{\epsilon^\alpha}}{2p \cdot q' \parallel p \cdot p' m^2}$$

$$= \frac{e_p^\alpha e^{\epsilon^\alpha}}{p \cdot p' m^2} [G - e_p^{\epsilon^\alpha} e_p ^\parallel G + G^0 e_p \cdot e_p \cdot p \cdot q \parallel p \cdot q' - m^2]. \quad (23)$$
(in the last equality, for future convenience, we passed from products of polarization vectors with photon momenta to their products with one and the same vector \(G\)). Now, \(V_p^\alpha\) and \(A_5\) contain in total 4 independent components, exactly as many as is appropriate for parametrization of a matrix describing transition between two spin-\(\frac{1}{2}\) on-shell particle states.

Compton differential cross-section averaged over electron polarizations Upon substitution of Eq. (18) to Eq. (16), the spur is calculated trivially \((p'\mu\text{ has nothing to contract with but } p'\nu)\), yielding \(39\)

\[
\left| M_{\text{Compt}} \right|^2_{\text{el.spin}} = \frac{1}{2} S_p \left( p' \cdot \gamma + m \right) \left( V_p \cdot \gamma + A_5 \gamma^5 \right) \left( V_p^* \cdot \gamma - A_5^* \gamma^5 \right) = 2(p \cdot p' - m^2) \left[ -|V_p|^2 + |A_5|^2 \right].
\]

The square of amplitude \(V_p\) from Eq. (23) obviously is

\[
|V_p|^2 = \left| e_p^\alpha e_p^* \cdot G - e_p^{\alpha*} e_p \cdot G \right|^2 + G^2 \left| e_p^\alpha e_p^* \right|^2 \left( \frac{p \cdot q + p' \cdot q'}{p \cdot p' - m^2} \right)^2.
\]

Ultimately, we observe an identity

\[
- \left| e_p^\alpha e_p^* \cdot G - e_p^{\alpha*} e_p \cdot G \right|^2 + |A_5|^2 = -G^2 \left( |e_p|^2 |e_p'|^2 - |e_p^\alpha e_p^*|^2 \right),
\]

which results because in l.h.s. of Eq. (20) the term \(- \left| e_p^\alpha e_p^* \cdot G - e_p^{\alpha*} e_p \cdot G \right|^2\) is a square of vector product of vectors \(G^\alpha\) and \(\varepsilon^{\alpha\beta\mu\nu} \frac{q}{m} e_p^\beta e_p^\nu\) (better recognizable in terms of spatial components of the vectors in the initial electron rest frame: \([ G \times [ e_p \times e_p^* ] ] = e_p (e_p^* \cdot G) - e_p^\alpha (e_p \cdot G)\)), whereas the term \(|A_5|^2\) is a square of the scalar product of the same vectors. In total, that is common known to equal to a product of the vector squares (one of which is \(G^2\)) and the other one \(|e_p \times e_p^*|^2 = |e_p|^2 |e_p'|^2 - |e_p^\alpha e_p^*|^2 \). The overall factor \(G^2\) can be evaluated from (20):

\[
G^2 = \frac{q \cdot q'}{2p \cdot q \cdot q'} = \frac{p \cdot p' - m^2}{2p \cdot q \cdot q'} = \frac{p \cdot q - p' \cdot q'}{2p \cdot q \cdot q'}.
\]

Substituting (25) (27) to (24), one attains the result:

\[
\left| M_{\text{Compt}} \right|^2_{\text{el.spin}} = \frac{1}{p \cdot q \cdot q'} \left( |e_p|^2 |e_p'|^2 - |e_p^\alpha e_p^*|^2 \right) (p \cdot q - p' \cdot q')^2 + |e_p^\alpha e_p^*|^2 (p \cdot q + p' \cdot q')^2.
\]
IV. SPECIFICATION FOR BREMSSTRAHLUNG IN THE LAB FRAME

Our main objective in this paper is to investigate bremsstrahlung in the lab frame. Transition from the center of mass to the lab involves standard kinematical relations. To set the notations, we briefly recollect those relations, and extend them to the polarization vectors.

A. Ultra-relativistic and dipole approximations

The laboratory frame is defined as one in which initial pseudo-photon carry zero energy, whereas their 4-vector of polarization, obeying the Lorentz gauge condition $e \cdot q = 0$, aligns with the time direction (independently of $q$):

$$e = (1, \mathbf{0}),$$

$$q = (0, -\mathbf{q}).$$

Such a definition of $q$ components, with the minus sign, is conventional in the bremsstrahlung theory, meant to make the longitudinal component of momentum transfer $q$ positive. In fact, that longitudinal component is not independent — it is determined by $q'$ and the momentum conservation law (see Eqs. (11) below). Components of the other 4-vectors are designated as

$$e' = (0, e'), \quad |e'| = 1, \quad e' \cdot k = 0,$$

$$q' = (\omega, k), \quad |k| = \omega < E,$$

$$p = (E, \mathbf{p}), \quad E = \sqrt{p^2 + m^2},$$

$$p' = (E', \mathbf{p'}), \quad E' = \sqrt{p'^2 + m^2} = E - \omega,$$

$$q = p - p' - k.$$

As Eq. (24) indicates, the typical values of momentum variables must be those at which denominators $p \cdot q, p' \cdot q'$, engaged in $\varepsilon_p, \varepsilon_{p'}$, are of the order of their minimal values. Let us express one of the denominators in terms of the angle between the entering electron’s and photon’s momenta:

$$p \cdot q' = E\omega - p \cdot k \equiv (E - |p|)\omega + |p|\omega \left(1 - \hat{p} \cdot \hat{k}\right)$$

$$= m^2 \frac{\omega}{2E} \left(1 + \gamma^2 \theta^2\right) \left(1 + \mathcal{O}\left(\gamma^{-2}\right)\right),$$

where

$$\hat{p} = \frac{p}{|p|}, \quad \hat{k} = \frac{k}{\omega}, \quad \hat{p}' = \frac{p'}{|p'|}$$

will signify the corresponding vector directions, and small transverse vector

$$\theta \overset{\text{def}}{=} k - \hat{p}$$

has the meaning of the radiation angle. For (34) to be of the order of its minimal value, the radiation angle must have magnitude

$$|\theta| \sim \gamma^{-1}$$

independently of any other variables.

The second denominator, $p' \cdot q' \approx p \cdot q$, expresses similarly in terms of the angle between $p'$ and $k$:

$$\theta' = k - \hat{p}' \quad (|\theta'| \sim \gamma'^{-1}).$$

Under the conditions of the ultra-relativistic dipole approximation, angles (36) and (38) are, in fact, closely related:

$$E'\theta' = \frac{E'}{\omega} k - \frac{E'}{|p'|} p'$$

$$= \frac{E'}{\omega} k - (p - k - q) \left\{1 + \mathcal{O}\left(\gamma'^{-2}\right)\right\}$$

$$= (E\theta + q) \left\{1 + \mathcal{O}\left(\gamma'^{-2}\right)\right\}$$

$$= E\theta \left\{1 + \mathcal{O}\left(\frac{q}{m}, \gamma'^{-2}\right)\right\}$$

(39a)

(39b)

(the last relation employs (37)).

Given (39b), it is convenient to introduce the universal rescaled radiation emission angle variable

$$\Theta \overset{\text{def}}{=} \gamma \theta = \gamma' \theta' \left\{1 + \mathcal{O}\left(\frac{q}{m}\right)\right\},$$

(40)

and use the longitudinal momentum transfer component

$$q_z \overset{\text{def}}{=} \frac{p \cdot q}{E} = \frac{p' \cdot q + mx_\omega \Theta \cdot q_\bot}{E'} \left\{1 + \mathcal{O}\left(\frac{q_\bot}{m^2}, \gamma^{-2}\right)\right\}$$

(41a)

$$= \frac{mx_\omega \left(1 + \Theta^2\right)}{2\gamma(1 - x_\omega)} \left\{1 + \mathcal{O}\left(\frac{q_\bot}{m}, \gamma^{-2}\right)\right\},$$

(41b)

where

$$x_\omega = \frac{\omega}{E}$$

stands for the photon energy fraction, and the direction of $z$ axis may be chosen parallel to $p$, or $k$, or any other direction within the forward radiation cone of angle width $\sim \gamma^{-1}$, provided

$$x_\omega \gg \frac{q}{m}.$$  

Naively, it might seem that the above condition only concerns the $z$ axis choice freedom, while the true condition of the equivalent photon approximation is $x_\omega \gg \frac{q}{m^2}$. 

However, with the account of electron spin, condition \(43\) proves to be physically significant (see below). Besides that, factorization condition \(3\) with \(44\) implies
\[
\frac{1 - x_\omega}{x_\omega} \gg \frac{r_{s\mu}}{\gamma} \geq \frac{1}{\gamma^2},
\]
separating \(x_\omega\) from 1. Restrictions \(43\) and \(44\) are not stringent for \(x_\omega\) if \(\gamma \geq 10^3\), and virtually leave for its variation the whole interval from 0 to 1.

Before we turn to polarizations, we may express the kinematic invariant ratio entering to Eq. \(29\):
\[
\frac{(p \cdot q - p \cdot q')^2}{p \cdot q \cdot q'} = \frac{x_\omega^2}{1 - x_\omega} \left\{1 + O \left(\frac{q_z}{m}, \gamma^{-2}\right)\right\}
\]
(45) (concerning the accuracy – see also Appendix \(3\)).

### B. Polarization vector dot-product

Next, it is desirable to express the components of \(p\)-gauge polarization vectors in the lab frame. At that, it is definitely beneficial to use the frame with the \(z\)-axis directed precisely along \(p\).

\[
Oz \parallel p.
\]
(46)
as long as the gauge had been fixed relative to it. Then, the longitudinal components of \(e_p\) and \(e_p'\) are almost light-like with the accuracy the same as for \(p\), i.e., \(O(\gamma^{-2})\), and at least for \(e_p\), they may be neglected because large transverse components dominate:
\[
\begin{align*}
e_p &= e - q \cdot \frac{e \cdot p}{p \cdot q} = \left(1, q \frac{e \cdot p}{p \cdot q}\right) \\
&= \left(1, q \left\{1 + O(\gamma^{-2})\right\}, q \frac{1}{q_z} \left\{1 + O(\gamma^{-2})\right\}\right).
\end{align*}
\]
The square of \(47\)
\[
- e_p^2 = \frac{q_z^2}{q_z} \left\{1 + O(\gamma^{-2})\right\} \gg 1
\]
(48)
to which \((e_p \cdot e_p')^2\) will actually be proportional as well) thus emerges as an overall factor in the cross-section. In the equivalent photon approach, this factor in product with \(\left|A_{\text{gauge}}(q_\perp)\right|^2\) are interpreted as the equivalent photon inflow \(41\).

As for the final photon polarization
\[
e_p' = e' - q' \frac{e' \cdot p}{p \cdot q'},
\]
the involved factor
\[
\frac{e' \cdot p}{p \cdot q'} \approx \frac{E e' \cdot \theta}{E' q_z}
\]
is of the order of \(m^{-1}\). Hence, \(e_p'\) has finite transverse and large light-like longitudinal components, which yield comparable contributions to \(e_p'^2\). Actually, since the outgoing photon is truly real, we know without calculation in components that
\[
- e_p'^2 \equiv - e'^2 = 1.
\]
(50)
Yet, for the sake of contraction with \(e_p\), exposing components of \(e_p'\) is expedient:
\[
e_p' \approx - \frac{e' \cdot p}{p \cdot q'}\gamma \cdot \frac{e' \cdot q}{q \cdot q'} \left\{1 + O(\gamma^{-2})\right\}, e_p' = e' - \frac{e' \cdot p}{p \cdot q'}\theta\gamma.
\]
(51)
Scalar product of Eq. \(51\) with Eq. \(47\) results in \(42\)
\[
e_p \cdot e_p' = - \frac{q_z}{q_z} \left(\frac{e'}{\theta} - \frac{e \cdot q}{E' q_z}\right) \left\{1 + O(\gamma^{-2})\right\}.
\]
(52)
This may be viewed as a product of transverse components alone, since longitudinal component contributions in the chosen frame \(10\) cancel with sufficient accuracy.

To explicate that both terms in brackets in Eq. \(52\) are of the same order, substitute \(q_z\) from Eq. \(11\) and pass from angle \(\theta\) to \(\Theta\) as given by \(10\):
\[
e_p \cdot e_p' = - \frac{q_z}{q_z} \left(\frac{e'}{\theta} - \frac{2}{1 + \Theta^2} \Theta(\theta \cdot e')\right).
\]
(53)

### C. Interpretation from the initial electron rest frame

Let us now analyze the kinematical origin of Eq. \(3\). If in addition to using the gauge \(\perp p\) we actually pass to the initial electron rest frame, the kinematics in this frame is as follows. Vector \(q\) becomes nearly light-like, with the longitudinal momentum \(\sim - q_z\gamma\), directed opposite to \(z\) axis, and of the absolute value \(\sim m_{\text{ph}} z\gamma\). That results in photon emission with energy \(\Omega \sim m_{\text{ph}} \Omega z\gamma\) and the radiation angle relative to \(z\)-axis \(\Psi \sim 1\); \(z\)-component of the photon’s momentum \(K\) in the initial electron’s rest frame will be denoted as \(K_z = \Omega \cos \Psi\) (the transverse component \(K_\perp = k_\perp\)) – see Fig. \(2\).

We are interested in the scalar product of purely spatial vectors \(e_p, e_p'\) in the given reference frame. In this frame, we can write
\[
e_p = \left(0, O(\gamma^{-2}), q_z \frac{1}{q_z} \left\{1 + O(\gamma^{-2})\right\}\right),
\]
\[
e_p' = (0, e_p') \quad |e_p'| = 1 \quad e_p' \cdot K = 0,
\]
\[
e_p \cdot e_p' \approx - \frac{q_z}{q_z} \cdot e_p'.
\]
(54)
Now, relating vector \(e_p'\) to the polarization vector \(e'\) of the same photon but in a different reference frame and different gauge is rather simple. Their components transverse to the plane \((K, Oz)\) are equal (because they are...
not altered neither by the boost along $Oz$, nor gauge transformation – translation along 4-vector $q$). As for the components of $e'_p$ and $e'$ belonging to the plane ($K$, $Oz$), owing to conditions $|e'_p|^2 = |e'|^2 = 1$, they must have the same norm and thus be related by a pure rotation. Since $e'$ is nearly orthogonal to $Oz$, whereas $e'_p$ is orthogonal to $K$, the angle of this rotation is just the angle $\Psi$ between $Oz$ and $K$:

$$e'_p = R_{\Psi(\Theta)}e'$$  \hspace{1cm} (55)

($R_{\Psi(\Theta)}$ is a product of an operator of gauge transformation and of a boost operator). So, one can view (54) as

$$e_p \cdot e' = -\frac{q_\perp}{q_z} G e'$$,  \hspace{1cm} (56)

where

$$G(\Theta) = P_\perp R_{\Psi(\Theta)} P_\perp,$$  \hspace{1cm} (57)

$P_\perp$ being an operator of projection onto the plane $\perp Oz$. Note that application of $P_\perp$ to $R_\psi$ makes the tensor symmetric.

To finalize the correspondence with Eq. (53), one needs to construct an explicit representation for $R_{\Psi(\Theta)}$ in terms of the kinematical vectors, particularly of vector $\Theta$ related with $K$ by [43]

$$\Theta = \frac{k_\perp}{\omega} = \frac{k_\perp}{\Omega + K_z}.$$  \hspace{1cm} (58)

The construction proceeds by noting an obvious identity

$$P_\perp R_{\Psi(\Theta)} P_\perp = P_\perp - (1 - \cos \Psi) P_{k_\perp},$$  \hspace{1cm} (59)

where $P_{k_\perp}$ is a projector onto direction $k_\perp$. Taking into account the representation

$$1 - \cos \Psi = \frac{\sin^2 \Psi}{1 + \cos \Psi} = \frac{k_\perp^2}{\Omega(\Omega + K_z)},$$  \hspace{1cm} (60)

one may write

$$(1 - \cos \Psi) P_{k_\perp} = \frac{k_\perp \otimes k_\perp}{\Omega(\Omega + K_z)} = (1 + \cos \Psi) \Theta \otimes \Theta.$$  \hspace{1cm} (61)

Finally, from (58) one expresses $1 + \cos \Psi$ through $\Theta^2$:

$$\Theta^2 = \frac{1 - \cos \Psi}{1 + \cos \Psi}, \quad 1 + \cos \Psi = \frac{2}{1 + \Theta^2}.$$  \hspace{1cm} (62)

Thus, Eqs. (53) (51) (52) indeed are equivalent to Eq. (53). Tensor representation for $G$ is

$$G_{\imath m}(\Theta) = \delta_{\imath m} - \frac{2}{1 + \Theta^2} \Theta_\imath \Theta_m.$$  \hspace{1cm} (63)

From Eq. (58), in particular, it is evident that for $\Psi = \pi/2$, corresponding to $|\Theta| = 1$, operator $G = P_\perp - P_{k_\perp}$ acts as a projector on the direction orthogonal to the radiation plane, i.e., to vector $\Theta$ (that can be observed from representation (53) as well) [44]. Speaking more physically, at $K \perp Oz$ the component of the polarization vector belonging to the emission plane must point along $Oz$, and there its projection on the transverse plane containing $\hat{q}_\perp$ vanishes.

![FIG. 2: Correspondence between the photon emission angles and polarizations: in the initial electron rest frame ($\Psi$, $e'_p$) and in the high-energy transverse plane ($\Theta$, $e'$). For other notations see text. Vectors shown in bold have also components transverse to the plane of the figure. The dashed line indicates the stereographic projection implied by the proportion (59).](image)

### D. Differential probability of bremsstrahlung

Inserting all the ingredients [35][32] and [41][a] into Eq. (20), and this latter to Eq. (48), one arrives at the final expression for the bremsstrahlung differential probability:

$$\frac{dW_{\text{dip}}(e')}{d\Gamma q'} = \frac{4\pi\alpha}{4EE} \left(\left(M_{\text{Compt}}\right)^2\right)_{\text{el.spin}} \frac{\gamma^2}{m^2(1 + \Theta^2)^2} \times \left\{1 + O\left(q_\perp/m, \gamma^{-2}\right)\right\},$$  \hspace{1cm} (64)

where

$$\hat{q}_\perp = \frac{q_\perp}{|q_\perp|}.$$  \hspace{1cm} (65)

We remind that the first term (unity) in braces of Eq. (64) has originated from the first (polarization-un correlated) term of the generalized Klein-Nishina equation (29).

The unpolarized probability corresponding to Eq. (64) is obtained by summing it over the independent directions of $e'$:

$$\frac{dW_{\text{unpol}}}{d\Gamma q'} = \sum_{e'} \frac{dW_{\text{dip}}}{d\Gamma q'} = \frac{q_\perp^2}{m^2(1 + \Theta^2)^2} \times \left\{1 + \frac{2(1 - x_\omega)}{x_\omega^2} (D\hat{q}_\perp)^2\right\}$$  \hspace{1cm} (65)

(the accuracy will not be written explicitly anymore).
Two representations for \((Gq_\perp)^2\) are of utility:

\[
(Gq_\perp)^2 = 1 - \frac{4}{(1 + \Theta^2)^2} (\Theta \cdot \hat{q}_\perp)^2 \tag{66a}
\]

\[
= \left( \frac{\Theta + q_\perp}{1 + \Theta^2} \right)^2 \left( \frac{\Theta - q_\perp}{1 + \Theta^2} \right)^2 \tag{66b}
\]

Eq. \((66a)\) shows that \((Gq_\perp)^2\) can reach 1, whereas \((66b)\) proves that it may also fall to zero:

\[0 \leq (Gq_\perp)^2 \leq 1.\]

One may notice that in the limit \(x_{\perp} \to 0\) intensity \((64)\) reduces to that of classical particle dipole radiation in an undulator \([10]\). In fact, vector \(a_1\) of \([10]\) is similar to our vector \(Gq_\perp\). Although the undulator motion is of permanently accelerated type, not scattering, the description in those cases is largely similar, because Fourier transform expands small-angle deflections at scattering in periodic modes, anyway.

On the other hand, notations \((64-65)\) are not in a transparent relation with those commonly used for non-dipole bremsstrahlung. In Appendix \(\text{A}\) we provide the necessary details of the correspondence, along with a discussion of the accuracy of the dipole approximation. Notation \((64)\) has the merit of not involving large cancelations, and manifestly exposing the polarization direction – being set by the vector \(Gq_\perp\).

\section{V. LIST OF PROPERTIES OF THE RADIATION FULLY DIFFERENTIAL PROBABILITY}

The fully differential radiation probability in all variables \(q_\perp, \Theta, e', x_{\omega}\) is rarely subject to observation – usually measurements are more inclusive, corresponding to integration over all variables but one or two. Nonetheless, in order to completely understand behavior of the integrated probability, it is pre-requisite to appreciate the features of the integrand. Below we will collect all the functional properties of kernel \((64)\), which we will need to refer to in the following study.

\subsection{A. Dependence of differential probability on the variables}

\textbf{Incomplete factorization of \(q_\perp\) dependence.}

Inspection of Eqs. \((64) (65)\) shows that besides the overall proportionality to \(q_\perp^2\), both \(dW_{\text{dip}}/d\Gamma'\) and \(dW_{\text{unpol}}/d\Gamma'\), are still dependent on \(q_\perp\), and thus on \(q_\perp\) in general – in spite of the condition \(q_\perp^2 \ll m^2\). Effects of residual azimuthal correlations in equivalent photon-induced reactions are, in principle, known, at least, for some other problems \([26]\). Actually, it should not be understood as genuine factorization failure, since its physical conditions hold well, but rather as a modification due to the polarization carried by the equivalent photon flow \([12]\). It is also true that the present effect disappears when \(dW_{\text{dip}}/d\Gamma'\) is integrated over the azimuthal directions of \(\Theta\) and summed over \(e'\) (as is usually done in application to inclusive peripheral particle production \([27]\), or to energy losses of fast charged particles in matter, being the original concern of Fermi, Weizsäcker, and Williams \([11]\)).

\textbf{Rutherford-like asymptotics in radiation angle.}

At large \(\Theta\) intensity \((64)\) falls off as \(\Theta^{-4}\), i. e. follows essentially the same law as the Rutherford scattering cross-section. This is a general consequence of proportionality of the amplitude to one hard propagator – in the present case of electron, not of a photon. In fact, in Sec. \(\text{VILC}\) we shall yet encounter a kind of ‘transient asymptotics’ at moderate \(\Theta\) (if \(x_{\omega}\) is sufficiently small).

\textbf{A pair of intensity dips at given \(q_\perp\).} As is indicated by Eq. \((66a)\), there exists a pair of \(\Theta\) values, specifically

\[
\Theta_\perp = \pm q_\perp,
\]

at which \((Gq_\perp)^2\) turns to zero. Those directions correspond to minima in the radiation intensity at a given \(q_\perp\) (see Fig. \(3\)). In the initial electron rest frame they correspond to directions along \(q_\perp\) \((K_z = 0\) in Eq. \((64)\), i. e., along the momentum transfer, and it is quite natural that semi-classical dipole radiation under those directions is zero.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{image.png}
\caption{Logarithm of unpolarized radiation intensity \((65)\), at \(x_{\omega} \to 0\), as a function of \(\Theta\) (the radiation angle vector in units of \(\gamma^{-1}\)). The direction of \(q_\perp\) is chosen to be along \(y\)-axis. A pair of dips (black spots) is manifest. With the increase of \(x_{\omega}\) they get filled in.}
\end{figure}
Polarization alignment along circles at given $\hat{q}_\perp$.

It is easy to show by straightforward solution of the ordinary differential equation

$$d\Theta_y = \frac{G_{ym}(\Theta_x, \Theta_y) q_{\perp m}}{G_{zm}(\Theta_x, \Theta_y) q_{\perp m}}, \quad (68)$$

that curves tangential to the vector field of polarization directions $Gq_y$, are a family of circles

$$\Theta^2 + \text{const}[q \times \Theta]_z = 1$$

passing through two knot points (67) (see Fig. 4). Along with $Gq_y \Theta \rightarrow \pm \hat{q}_\perp$, in those points to zero drops the polarization.

The origin of the circular distribution is explainable, again, from the viewpoint of the initial electron rest frame. There, polarization of the dipole radiation distributes on the sphere of photon emission directions along meridional circles, with the poles at $\pm \hat{q}_\perp$. The relativistic boost of this radiation, and description by $\Theta$ variables (see Eq. (28)) is equivalent to a stereographic projection of this sphere from an equatorial point at the rear end of $z$-axis onto the plane tangent to the sphere in the front point of $z$-axis (see Fig. 2). As is known [28], at a stereographic projection any circle on the sphere is mapped into a circle on the plane.

In what follows, of value for us mainly will be the circle centered at the origin. That such a circle must exist is obvious already from the azimuthal symmetry of the stereographic projection for the given tangential curve on the sphere, with respect to the $z$-axis.

Semi-classical limit. Another instructive point of view on the structure $G_{im} \hat{q}_{\perp m} \cdot \hat{e}'$ emerges in the soft-photon limit $\omega \rightarrow 0$ [40], in the lab (target rest) frame, in the purely spatial photon gauge. The differential probability of radiation then assumes semi-classical form [13, 23]

$$\frac{dW_{\text{semi-cl}}}{d\Gamma q'} = 4\pi \alpha |I_{\text{semi-cl}} \cdot \hat{e}'|^2 \quad (69)$$

with the radiation amplitude

$$I_{\text{semi-cl}} \cdot \hat{e}' = \left( \frac{p'}{p' \cdot q'} - \frac{p}{p \cdot q'} \right) \cdot \hat{e}' \quad (70a)$$

$$\approx \frac{1}{\omega} \left( \frac{v'}{1 - k \cdot v'} - \frac{v}{1 - k \cdot v} \right) \cdot \hat{e}' \quad (70b)$$

$$\approx \frac{1}{\omega} \left\{ \frac{\chi}{1 - k \cdot (v + \chi)} + \left( \frac{1}{1 - k \cdot (v + \chi)} - \frac{1}{1 - k \cdot v} \right) v \right\} \cdot \hat{e}', \quad (70c)$$

$\chi = v' - v$ being the scattering angle, $\hat{k} = k/\omega$.

The dipole approximation consists in expanding this expression to first order in $\chi$. In the first term one neglects $\chi$ dependence in the denominator, and lets $1 - \hat{k} \cdot v \approx \gamma^{-2} + \hat{e}'^2$. That will correspond to the Kronecker-delta term in tensor $G_{im}$ [63], and may be regarded as a direct response of the transverse acceleration current. In the remainder of Eq. (70c) the parentheses

$$\frac{1}{1 - \hat{k} \cdot (v + \chi)} - \frac{1}{1 - k \cdot v} \approx \frac{\hat{k} \cdot \chi}{(1 - k \cdot v)^2} \quad (71)$$

are large, however, they are multiplied by the correspondingly small quantity

$$v \cdot \hat{e}' \approx -\hat{e}' \cdot \theta. \quad (72)$$

The product of (71) and (72) is the counterpart of the second term in $G_{im}$ in Eq. (63). One may say that the angle-dependent rotation of the polarization occurs due to a longitudinal current caused by the variation of the characteristic denominator, or the Lorentz-contracted distance from the source to the observer, i. e., the source brightness.

Finally, note that quantum corrections in the angular-spectral distribution [63] consist of a substitution $\frac{1}{r_z} \rightarrow \frac{1}{r_z + r_z^2}$ (emerging from a rule for frequency change due to the radiation recoil $\omega \rightarrow \omega e^{-1}$ in the amplitude [3], and of a factor $E'^{-1}$ in the final electron phase volume for the probability), and of an additive $x_{\omega}$-independent term representing the spin contribution to radiation.

Monotonic damping of polarization with $x_{\omega}$.

Generally, from (64) it is seen that with the
VI. INTEGRATION OVER ATOMIC RECOIL

Let us now proceed to the description of radiation on a solid target. Then, momentum $\mathbf{q}_\perp$ imparted to the target is beyond detection and has to be integrated over, with the weight $(2\pi)^{-2} |A_{\text{diff}}(\mathbf{q}_\perp)|^2 = \frac{d\sigma_{\text{scat}}}{d\Omega q_\perp}$, and appropriate averaging is due over the atomic configurations:

$$\left\langle \frac{d\sigma_{\text{rad}}}{d\Omega q'} \right\rangle = \left\langle \int d^2q_\perp \frac{d\sigma_{\text{scat}}}{d^2q_\perp} dW_{\text{dif}} \right\rangle. \quad (74)$$

For a macroscopic target, differential cross-section (74) must be proportional to the target area if the beam is still wider than the target, or to the area of the beam transverse section, if it is narrower than the target (as is a usual practice) and is transversely uniform (otherwise, we can consider any beam part uniform relative to target inhomogeneities). Dividing (74) by the interaction area $S$, one obtains a quantity independent of $S$ (but proportional to the target matter density and thickness) and having the meaning of differential probability for the given radiative process to occur per one particle passed through the target.

With $\frac{dW_{\text{dif}}}{d\Omega q'}$ given by Eq. (64), in (74) one encounters two basic integrals:

$$\frac{1}{S} \left\langle \int d\sigma_{\text{scat}} q_\perp^2 \right\rangle = \langle q_\perp^2 \rangle, \quad (75a)$$

$$\frac{1}{S} \left\langle \int d\sigma_{\text{scat}} (2q_\perp m q_{\perp n} - q_\perp^2 \delta_{mn}) \right\rangle = \langle 2q_\perp m q_{\perp n} - q_\perp^2 \delta_{mn} \rangle, \quad (75b)$$

having the meaning of average momentum squares. For our analysis to reach beyond the conventional case of isotropic target, it is prerequisite that average (75b) differs from zero. In particular, that allows one to anticipate non-zero polarization of the bremsstrahlung beam as a whole. Physically, this average is related to the azimuthal anisotropy (“ellipticity”) in scattering, even if being not a true measure of the latter due to the radiative character of the averaging (see subsection VII).

A. Vector anisotropy parameter and the anisotropy degree

Instead of (75b), it is convenient to deal with the ratio of (75b) to (75a), which can serve as a direct measure of the scattering asymmetry. This ratio, which is a symmetric traceless tensor in 2 transverse dimensions, can be characterized by the direction of one of its two eigenvectors and the corresponding eigenvalue (another eigenvector will be orthogonal to the first one and correspond to the eigenvalue opposite in sign). Let $\mathbf{N}$ stand for the eigenvector corresponding to the positive eigenvalue, then we express

$$\frac{\langle 2q_\perp m q_{\perp n} - q_\perp^2 \delta_{mn} \rangle}{\langle q_\perp^2 \rangle} \overset{\text{def}}{=} 2N_m N_n - N^2 \delta_{mn}. \quad (76)$$

If tensor $\langle q_\perp m q_{\perp n} \rangle$ diagonalizes in axes $x$, $y$, and, say, $\langle q_2^2 \rangle \geq \langle q_1^2 \rangle$, then

$$N^2 = \frac{\langle q_2^2 \rangle - \langle q_1^2 \rangle}{\langle q_2^2 \rangle + \langle q_1^2 \rangle}. \quad (77)$$

(and $\mathbf{N} \parallel Oy$). That implies a constraint

$$N^2 \leq 1. \quad (78)$$

Covariantly, one can verify (78) by squaring both sides of (76) and taking double trace.

Now, the average differential probability of radiation can be phrased in terms of the introduced vector $\mathbf{N}$:

$$\left\langle \frac{dW_{\text{dif}}}{d\Omega q'} \right\rangle = \frac{4\pi \alpha \langle q_\perp^2 \rangle}{m^4 (1 + \Theta^2)^2} \left\{ 1 + \frac{2(1 - x_\omega)}{x_\omega^2} [(G_{im} \epsilon_i')(2(1 - N^2) + 2(G_{im} N_m \epsilon_i')^2)] \right\}. \quad (79)$$

At $N^2 = 1$, Eq. (79) essentially coincides with Eq. (64). Decomposing also the leftmost unity in braces of Eq. (79)
as $1 \equiv (1 - N^2) + N^2$, we get a representation in form of an incoherent mixture of bremsstrahlung on isotropic target with that on an anisotropic one, in proportion $(1 - N^2) : N^2$ determined by the target anisotropy degree. But for polarization characteristics that superposition is non-trivial, inasmuch as the polarization direction and degree does not express as any simple superposition.

B. Dipole approximation beyond the scattering factorization

The principal concern at application of bremsstrahlung theory to particle passage through matter is the vulnerability of the scattering factorization condition (3) due to significant target thickness. Fortunately, a way for generalization beyond the factorization is known, which preserves the Dirac matrix structure of the radiation matrix element, only trading the transferred momentum $q_\perp$ times $A_{\text{scal}}$ for some overlap of initial and final electron wave functions, now involving integration over longitudinal coordinates (17). To make the text self-contained, we briefly remind the idea behind that generalization.

In the first place, it is convenient to put the factorized matrix element to a form explicitly linear in $q_\perp$:

$$M_{\text{rad}} = \vec{u}' \left\{ \left( \frac{E}{p' \cdot q} - \frac{E'}{p \cdot q'} \right) e^{*s} \cdot \gamma + \frac{e^{*s} \cdot \gamma q \cdot \gamma \gamma^0 + \gamma^0 q \cdot \gamma e^{*s} \cdot \gamma}{2p' \cdot q'} \right\} u + \frac{\vec{u}' \left\{ \frac{m \omega}{EE'} \frac{1}{q_z} \cdot \Theta e^{*s} \cdot \gamma - \frac{e^{*s} \cdot \gamma q_\perp \cdot \gamma \gamma^0 - \gamma^0 q_\perp \cdot \gamma e^{*s} \cdot \gamma}{2E} \right\} u}{\text{dip}} \approx$$

where one has to mind our definition (32) and note the relations (80)

Substituting Eqs. (81a) to (82a), the spin corrections to wave functions give significant relative contributions, because the term without them appears to be energy-suppressed:

$$T_{\text{u.r.}} = \sqrt{4 \pi i e} \int d^3 r e^{i \mathbf{q} \cdot \mathbf{r}} \varphi^s(r) \left\{ e^{*s} \cdot \gamma + \frac{i}{2E} e^{*s} \cdot \gamma \nabla \cdot \gamma \gamma^0 + \frac{i}{2E} \gamma^0 \nabla \cdot \gamma e^{*s} \cdot \gamma \right\} \varphi(r) u \{1 + O (\gamma^{-2}) \}. \quad (85)$$

In spite that $q_z \ll q_\perp$, one can not neglect $q_z$ component in the exponent here, because at the scale $L_{\text{corr}}$ of contributing longitudinal distances $q_z L_{\text{corr}} \sim 1$.

It is straightforward to bring (81a) into correspondence with (81b) (cf. [8, 18]). To this end, one needs transform the first term in braces in (81a) via

$$\int d^3 r e^{i \mathbf{q} \cdot \mathbf{r}} \varphi^s(r) \varphi(r) \approx \frac{i}{q_z} \int e^{i \mathbf{q} \cdot \mathbf{r}} (\hat{p} \cdot \nabla) \{ \varphi^s(r) \varphi(r) \}$$

$$\equiv \frac{i}{q_z} \int e^{i \mathbf{q} \cdot \mathbf{r}} [(\hat{p} \cdot \nabla \varphi^s) \varphi + \varphi^s \hat{p} \cdot \nabla \varphi] + \frac{i}{q_z} \int e^{i \mathbf{q} \cdot \mathbf{r}} \varphi^s(r) (\hat{p}' - \hat{p}) \cdot \nabla \varphi(r). \quad (86)$$

With the use of wave equations (81a), it is found that there is a cancelation in the first integral in (80), making it
\( O\left(\frac{q^2 + V^2}{m^2}\right) \) relative to the last integral:

\[
\int d^3 r e^{i q \cdot r} \varphi^* (r) \varphi (r) = \frac{i}{q_z} \int e^{i q \cdot r} \varphi^* (r) \left( \mathbf{p}' - \mathbf{p} \right) \cdot \nabla \varphi (r) \left\{ 1 + O \left( \frac{q^2 + V^2}{m^2} \right) \right\}.
\] (87)

Thus, the entire overlap (85) can be cast in terms of the dipole overlap

\[
J_\perp = i q_z \int d^3 r e^{i q_\perp \cdot r} \varphi^* (r) (-i \nabla_\perp) \varphi (r),
\] (88)

specifically,

\[
T_{\text{dip}} = \sqrt{4 \pi e} \frac{1}{q_z} \left\{ \frac{m \omega}{E F} \frac{d}{d q_z} J_{\perp} \cdot \Theta e^{i \omega} \cdot \gamma - \frac{1}{2E} e^{i \omega} \cdot \gamma J_{\perp} \cdot \gamma \frac{\omega}{E} - \frac{1}{2E} \gamma J_{\perp} \cdot \gamma \right\} u.
\] (89)

This is observed to have the very same Dirac matrix structure as (80), only with \( J_{\perp} \) entering in place of \( q_{\perp} \).

As we see, the recipe for the generalization beyond the scattering factorization is to make in the factorized matrix element

\[
T_{\text{fact}} = \sqrt{4 \pi e} A_{\text{scat}}^{\text{diff}} (q_{\perp}) M_{\text{rad}},
\]

with \( M_{\text{rad}} \) given by (80), a substitution

\[
q_{\perp} A_{\text{scat}}^{\text{diff}} (q_{\perp}) \rightarrow J_{\perp} (q_z, q_{\perp}) = i q_z \int d^3 r e^{i q_{\perp} \cdot r} \varphi^* (r) (-i \nabla_\perp) \varphi (r), \quad q_z = q_z (\omega, \Theta)
\] (90)

Here, factor \( e^{i q_{\perp} \cdot z} \) represents the effects of longitudinal coherence sensitivity. Correspondence with the scattering factorization is achieved when this exponential can be put to unity (after a preliminary integration over \( z \) by parts, to make the integrand vanish at infinity):

\[
J_{\perp} = - \int d^2 r_{\perp} e^{i q_{\perp} \cdot r_{\perp}} \int dze^{i q_{\perp} \cdot z} \left( \frac{\partial}{\partial z} \varphi^* (r) (-i \nabla_\perp) \varphi (r) \right)
\]

\[
q_z L_{\text{corr}} \ll 1 - \int d^2 r_{\perp} e^{i q_{\perp} \cdot r_{\perp}} 1 (-i \nabla_\perp) \varphi (r_{\perp}, z) \gg L_{\text{corr}} = q_{\perp} \int d^2 r_{\perp} e^{i q_{\perp} \cdot r_{\perp}} \varphi (r_{\perp}, z) \gg L_{\text{corr}}.
\] (91)

By the Huygens principle (see, e.g., (31)), the latter integrand equals to the scattering amplitude, if it is normalized as in Eq. (7). So,

\[
J_{\perp} \xrightarrow{q_z L_{\text{corr}} \ll 1} q_{\perp} A_{\text{scat}}^{\text{diff}} (q_{\perp}),
\] (92)

thereby justifying consistency of the substitution rule (90).

To get the spin-averaged probability corresponding to the generalized matrix element (80), one needs no special calculation. Obviously, it is a bilinear form in \( J_{\perp} \), which can be retrieved from the factorized bilinear form by replacement \( d\sigma_{\text{scat}} g_{\perp m} q_{\perp n} \rightarrow \frac{d\sigma_{\text{scat}}}{2S} (2\pi)^2 J_{\perp m} J_{\perp n} \). Thereat, the basic averages for our quadratic form from (75a) (75b) promote to

\[
\frac{1}{S} \left\langle \int d\sigma_{\text{scat}} q_{\perp}^2 \right\rangle \xrightarrow{q_z L_{\text{corr}} \sim 1} \frac{1}{S} \left\langle \int \frac{d^2 q_{\perp}}{(2\pi)^2} \left| J_{\perp} \right|^2 \right\rangle,
\] (93a)

\[
\frac{1}{S} \left\langle \int d\sigma_{\text{scat}} (2q_{\perp m} q_{\perp n} - q_{\perp}^2 \delta_{mn}) \right\rangle \xrightarrow{q_z L_{\text{corr}} \sim 1} \frac{1}{S} \left\langle \int \frac{d^2 q_{\perp}}{(2\pi)^2} (2J_{\perp m} J_{\perp n} - \delta_{mn} J_{\perp}^2) \right\rangle.
\] (93b)

The factor \( 1/S \) in the right-hand sides may be explicitly canceled if \( \varphi \) (but not \( \varphi^* \)) is substituted by a normalized wave packet in transverse coordinates.

For the modified averages (93a) (93b) we introduce same shorthands as (75a) (75b) but with the subscript “rad”:

\[
\frac{1}{S} \left\langle \int \frac{d^2 q_{\perp}}{(2\pi)^2} |J_{\perp}|^2 \right\rangle \equiv \langle q_{\perp}^2 \rangle_{\text{rad}},
\]

\[
\frac{1}{S} \left\langle \int \frac{d^2 q_{\perp}}{(2\pi)^2} (2J_{\perp m} J_{\perp n} - \delta_{mn} J_{\perp}^2) \right\rangle \equiv \langle 2q_{\perp m} q_{\perp n} - q_{\perp}^2 \delta_{mn} \rangle_{\text{rad}}.
\]

For expression of the anisotropy parameter, equations (76, 77) remain valid, only with the substitution \( \langle q_{\perp}^2 \rangle \rightarrow \)}
\[ \langle q^2 \rangle_{\text{rad}}: \]
\[ \frac{2q_{\perp m}q_{\perp n} - q_{\perp}^2 \delta_{mn}}{\langle q^2 \rangle_{\text{rad}}} = 2N_mN_n - N^2 \delta_{mn}, \quad (94) \]
\[ N^2 = \frac{\langle q^2 \rangle_{\text{rad}} - \langle q_{\perp}^2 \rangle_{\text{rad}}}{\langle q_{\perp}^2 \rangle_{\text{rad}} + \langle q_{\perp}^2 \rangle_{\text{rad}}}, \quad (95) \]

C. Estimates for azimuthal anisotropy at passage through an oriented crystal

Among conceivable applications of the connection between the target intrinsic anisotropy and radiation polarization is the possibility of preparation of a polarized photon beam. Practicality demands the beam polarization to be at least a few tens percent, and thence about as high must be \( N^2 \). It is not, however, obvious, whether sizeable \( N^2 \) can be attained with macroscopic targets. The main obstacle is the hard isotropic contribution in scattering. Treating interaction with individual atom as perturbative, in (74) in the integral over \( q_{\perp} \), or, in equivalent integral over impact parameters, the contribution from the atomic distance scale \( \sim r_a \) is comparable to that from the distances from the nucleus of the order \( \sim m^{-1} \), where the impact area is smaller but the acting force, and the generated radiation, is stronger. That familiarly leads to a logarithmic divergence of the integral over \( d^2 q_{\perp} \) from \( \frac{d\sigma_{\text{rad}}}{d^2 q_{\perp}} = \frac{d\sigma_{\text{rad}}}{d^2 q_{\perp}} dW_{\text{rad}} \), with \( dW_{\text{rad}} \propto q_{\perp}^4 \) and \( \frac{d\sigma_{\text{rad}}}{d^2 q_{\perp}} \sim q_{\perp}^{-3} q_{\perp}^{-1} \) (Rutherford tail). Introducing appropriate cutoffs, the upper one due to the dipole approximation failure at \( q_{\perp} \sim m \), and the lower one due to the atomic form-factor regulation, one gets with the logarithmic accuracy

\[ \int d^2 q_{\perp} \frac{1}{q_{\perp}} = \pi \int \frac{q_{\perp}^{\text{max}}}{q_{\perp}^{\text{min}}} dq_{\perp} \approx \pi \ln m r_a \approx \pi \ln \frac{1}{\alpha}. \quad (96) \]

Since in vicinities of the nuclei the scattering is isotropic, the anisotropy parameter \( N^2 \) gets suppressed at least by the factor of \( \ln \frac{1}{\alpha} \approx 5 \).

A remedy to the encountered suppression could be sought in using oriented crystals. Once one aligns some strong crystallographic axis at a small angle \( \chi_0 \ll 1 \) relative to the electron incidence direction (see Fig. 6), the anisotropy shall only persist up to the distance of transverse separation of atomic nuclei in the string, \( \Delta r_{\perp} \approx d_a \chi_0 \), where \( d_a > 2r_a \) is the distance between atomic nuclei in the row, while from scale \( \Delta r_{\perp} \) up to \( r_a \) the scattering should become anisotropic (stronger in the direction transverse to the beam-string plane). At a scale greater than \( \chi_0 d_a \), the cross-section will no longer be a sum of logarithmic cross-sections of scattering on individual atoms, but, rather, the motion will be governed by the aggregate potential of the atoms. The number of atoms overlapping at a given impact parameter is \( \sim \frac{1}{\chi_0 d_a} \), and this is the factor the cross-section must increase by, whilst Coulombic logarithms do not develop anymore in this region.

To verify the above assumption, and also get an idea of the longitudinal coherence sensitivity, consider first a problem of electron radiation at scattering on a single atomic row under a small angle of incidence. In capacity of initial and final state wave functions in the atomic row potential \( V(r) \), take for simplicity the eikonal approximation (corresponding to the neglect in (83) of the right-hand sides, as well as neglect of the angle between \( \mathbf{p} \) and \( \mathbf{p}' \)).

\[ \varphi(r) \approx e^{-i \int_{-\infty}^{\infty} dz' V(z', r)}, \quad (97a) \]
\[ \varphi^{\prime*}(r) \approx e^{-i \int_{-\infty}^{\infty} dz' V(z', r')} \quad (97b) \]
\[ \varphi^{*}(r) \varphi(r) = e^{i \chi_0(r_\perp)}, \quad \chi_0(r_\perp) = - \int_{-\infty}^{\infty} dz V(z', r_\perp). \quad (98) \]

Here \( \chi_0(r_\perp) \) is commonly called the eikonal phase. Substituting (97) into (88), one obtains (89) and

\[ \mathcal{J}_{\perp} = -i q_z \int d^3 r e^{i q r + i \chi_0(r_\perp)} \nabla \int_{-\infty}^{\infty} dz' V(z', r_\perp) \]
\[ = \int d^2 r \int d^2 q_{\perp} e^{i q r + i \chi_0(r_\perp)} \int_{-\infty}^{\infty} dz e^{i q_r z} V(z, r_\perp) \]

(99)

The second equality is the result of \( z \)-integration by parts. The integrals of \( |\mathcal{J}_x(q_{\perp})|^2, |\mathcal{J}_y(q_{\perp})|^2 \) needed for \( N^2 \), evaluate in a particularly simple way:

\[ \int d^2 q_{\perp} |\mathcal{J}_{x,y}|^2 = (2\pi)^2 \int d^2 r |\nabla x,y| \int_{-\infty}^{\infty} dz e^{i q_r z} V \]
\[ = \int d^2 q_{\perp} d^2 q_{\perp, x,y} |d^2 r e^{i q r V(r)}|^2 \]

(100)

I. e., the eikonal phase actually does not contribute to the given integral, and the result is equivalent to the Born approximation.

To proceed, we have to specify the potential of the atomic row. Let us, for simplicity, model it by a superposition of individual screened Coulombic potentials.
For incidence on a row of \( N \) identical atoms with nucleus charge \( Z \), in the \( xz \) plane at an angle \( \chi_0 \ll 1 \) to \( z \)-axis,

\[
V_{\text{row}}(r) = Z \alpha \sum_{n=0}^{N-1} \frac{e^{-\beta^2((x-x_0d_an)^2+(z-z_0d_an)^2)}}{\sqrt{g^2 + ((x-x_0d_an)^2+(z-z_0d_an)^2)}}.
\]

(101)

Fourier transform thereof results as

\[
\int d^3r e^{iq \cdot r} V_{\text{row}}(r) = \frac{4\pi Z \alpha}{q_\perp^2 + r_a^2} \sum_{n=0}^{N-1} e^{i(q_x + \chi_0q_z) \cdot d_a n} \int_0^{\infty} \frac{d^2 q_{\perp}}{q_{\perp}^2 + q_{\perp}^2 + r_a^2} \sin \left( \frac{q_x + \chi_0q_z}{\chi_0} \cdot d_a N \right) N \sin \left( \frac{q_x + \chi_0q_z}{\alpha} \cdot d_a \right),
\]

where in the denominator terms \( q_{\perp}^2 \) had been neglected compared to \( r_a^{-2} \). When squaring (101), the sine ratio factor in a familiar way may be approximated by an equidistant sequence of \( \delta \)-functions:

\[
\frac{\sin^2 \left( \frac{q_x + \chi_0q_z}{\alpha} \cdot d_a N \right)}{\sin^2 \left( \frac{q_x + \chi_0q_z}{\alpha} \cdot d_a \right)} \approx \pi N \delta \left( \frac{q_x + \chi_0q_z}{\alpha} \cdot d_a \right) - \pi j
\]

(102a)

for \( \int d^2 q_{\perp} \left| T_x \right| ^2 \), and

\[
\frac{\sin^2 \left( \frac{q_y + \chi_0q_y}{\alpha} \cdot d_a \right)}{\sin^2 \left( \frac{q_y + \chi_0q_y}{\alpha} \cdot d_a \right)} \approx \pi \frac{2}{q_{\perp}^2 + q_{\perp}^2 + r_a^{-2}}
\]

(102b)

for \( \int d^2 q_{\perp} \left| T_y \right| ^2 \). In the final result, it is convenient to treat in sum the term \( n = 0 \) separately. It has sense of continuous potential contribution, constant along the string. In the higher terms one may neglect \( r_a^{-2} \) relative to \( \frac{q_{\perp}^2}{\chi_0d_a} \). Thereby one obtains

\[
\int \frac{d^2 q_{\perp}}{q_{\perp}^2 + q_{\perp}^2 + r_a^2} = \frac{\pi}{2} \sqrt{q_{\perp}^2 + r_a^2}
\]

(103a)

\[
\int \frac{d^2 q_{\perp}}{q_{\perp}^2 + q_{\perp}^2 + r_a^2} = \frac{\pi}{2} \sqrt{q_{\perp}^2 + r_a^2}
\]

(103b)

The summation upper limit is determined by the same principle as that of integration in (100) – it is set at \( q_x \sim m \), so

\[
j_{\text{max}} = j_{\text{max}}(\chi_0) \sim \frac{\chi_0d_a}{2\pi} m,
\]

(104)

For scattering on a definite single string at zero temperature, we do not need any statistical ensemble averaging, and substitution of (103a, 103b) to (77) yields the desired equation

\[
N^2 = \frac{1}{1 + 2 \frac{q_{\perp}^2}{\chi_0^2} + \frac{2}{\pi} \left( 1 + \frac{q_{\perp}^2}{\chi_0^2} \right)^{3/2} \frac{\chi_0d_a L_0(\chi_0)}{r_a}}
\]

(105)

Let us now analyze the explicit formula (104) to assess the effects of scattering in vicinities of the nuclei and of the longitudinal coherence sensitivity. In the denominator, terms \( \frac{q_{\perp}^2}{\chi_0^2} \) reflect the effect of coherence sensitivity on the anisotropy of radiation distribution. Apparently, increase of \( q_z \) through \( \omega \) always suppresses the anisotropy. The last term, containing \( L_0 \), accounts for effects of the string discreteness, which are also suppressing the anisotropy. But due to the factor \( \frac{\chi_0d_a}{r_a} \), those weaken as \( \chi_0 \) decreases, and even at angles as large as

\[
\chi_0 \sim 0.2 \text{rad} \sim 10^\circ
\]

one has \( N^2 \sim 0.5 \). In fact, at angles (105) and \( q_z \sim m/\gamma \) the longitudinal coherence effect on the anisotropy, quantified by the ratio \( \frac{q_{\perp}^2}{\chi_0^2} \sim \frac{1}{\gamma^2 \chi_0^2} \), will be small, provided \( \gamma \geq 10^2 \), which we had presumed for the dipole approximation self-consistency (13, 14). Then, \( N^2 \) can be regarded as independent of \( q_z \), and therethrough of photon variables \( \omega, \theta \).

### D. Multiple scattering on atomic strings

To be realistic, at particle passage through a real oriented crystal, interaction with one string is not the whole story but only an elementary act. Multiple interactions can affect the distribution function in scattering angles, and yet, in case of periodical hitting the strings, modify the radiation spectrum through the form-factor of the periodic structure.

Crucial for successive scattering on atomic rows is that under a nearly continuous string potential, the modulus of the angle between the row and the particle motion is approximately conserved (transverse energy conservation). Hence, in multiple scattering on mutually parallel strings the particle momentum will diffuse over a cone with the axis along the string direction (“doughnut” scattering [31]), and for a sufficiently thick target the scattering must isotropize. To keep the scattering anisotropy significant, one should not permit the passage to such late a stage. The allowable target thickness \( L \) is estimated assuming that at this distance the particle passes through \( \frac{L}{\theta} \) strings, scattering on each one through a small angle

\[
\chi_1 \sim \frac{F_0}{E \chi_0} = \frac{2V_0}{E \chi_0},
\]
whence the change of the azimuth $\frac{d\chi}{\chi_0}$ needs be $\ll 1$, too. Then, the mean square of the scattering azimuthal angle on a sequence of (statistically independent) strings, $L \lambda_0^2 \chi_0^2$, is required to be less than unity. So, the condition for the target thickness ensues as

$$L < L_{\text{isotr}} \sim d_0 \chi_0 \lambda_0^2 \sim d_0 \frac{E^2 \chi_0^3}{4V_0^2} \sim \frac{d_0}{\alpha^2} \frac{1}{\alpha^2}, \quad \left( \frac{m}{V_0} \sim 1 \right) \tag{106}$$

where $\frac{d_0}{\chi_0}$ is of order 1-2 cm. Since we assume $\gamma \geq 10^3$ for the dipole approximation to keep valid, with $\chi_0$ of the order of $10^3$, the effect of doughnut isotropization is weak.

On the other hand, if strings are encountered along the particle path periodically (‘string of strings’ radiation \[31\], similar to coherent bremsstrahlung \[32\]), then even at $\frac{d_0}{\chi_0} \ll 1$ one can still have $\frac{d_0}{\chi_0} \sim 1$, with $d_0$ the distance between the strings. Then, coherence effects in radiation may develop on a larger spatial scale. If the period of string sequence is equal to the coherence length $q_L^{-1}$ at some $\omega, \theta$, then the spectrum contains a resonance radiation peak at this frequency. Within the peak, the value of $q_L$, and therewith of $q_L$, may be regarded as certain (the ‘point effect’ in coherent bremsstrahlung \[32\]). Then, the asymmetry degree, again, would approach unity, minus corrections on thermal atom oscillations and lattice defects, etc.

It is not our aim in this paper to advocate any specific model for $N^2$. In what follows, we are going to treat $N$ as some arbitrary but fixed parameter. Therewith, we will investigate influence of $N$ on angular anisotropy of the radiation intensity and on polarization – both in angular distribution and in the aggregate photon beam. We start with the angular distribution.

### VII. ANGULAR DISTRIBUTION OF THE POLARIZED RADIATION

Our goal in this section is to investigate the dependence of the differential intensity \[79\] on the photon emission angles and the polarization. We start with \[79\] in which substitute $\langle q^2 \rangle \rightarrow \langle q^2 \rangle_{\text{rad}}$:

$$\langle \frac{dW_{\text{dip}}}{dT_{q'}} \rangle = \frac{4\pi\alpha \langle q_{\text{rad}}^2 \rangle}{m^4 (1 + \Theta^2)^2} \left\{ 1 + \frac{2(1 - x_\omega)}{x_\omega^2} \left[ (G_{im}e'_i)^2 (1 - N^2) + 2(G_{im}N_me'_i)^2 \right] \right\} \tag{107}$$

(Expression for $dT_{q'}$ through $\Theta$ variables will be given later – see Eq. \[109\]).

To isolate the unpolarized part and the polarization, one needs to split the dependence of Eq. \[79\] on $e'$ into the isotropic and the quadrupole parts, writing

$$(G_{im}e'_i)^2 = 1 - \frac{4(\Theta \cdot e')^2}{(1 + \Theta^2)^2}$$

$$= 1 + 4(\Theta \cdot e')^2 + 2(\Theta^2 \delta_{ij} - 2\Theta_i \Theta_j) e'_i e'_j,$$

and

$$2(G_{im}N_me'_i)^2 \equiv (G_{im}N_m)^2 + 2G_{im}N_mG_{jm}N_n - (G_{im}N_m)^2 \delta_{ij} e'_i e'_j,$$

where

$$(G_{im}N_m)^2 = N^2 - 4(N \cdot \Theta)^2 \frac{(G_{im}N_m)^2}{(1 + \Theta^2)^2}.$$ For the angular distribution of unpolarized intensity

$$\langle \frac{dW_{\text{unpol}}}{dT_{q'}} \rangle = \sum e'_i \langle \frac{dW_{\text{dip}}}{dT_{q'}} \rangle = \frac{8\pi\alpha \langle q_{\text{rad}}^2 \rangle}{m^4 (1 + \Theta^2)^2} \left\{ 1 + \frac{2(1 - x_\omega)}{x_\omega^2} \frac{N^2 \Theta^2 - 2(N \cdot \Theta)^2}{(1 + \Theta^2)^2} \right\}, \tag{111}$$

Inspection of Eq. \[111\] reveals that the azimuthal anisotropy embodied by the quadrupole dependence on the angle $\phi$ between $\Theta$ and $N$,

$$N^2 \Theta^2 - 2(N \cdot \Theta)^2 = -N^2 \Theta^2 \cos 2\phi,$$

is sizeable only when $N^2 \sim 1$, and only at angles $\Theta \sim 1$. At $\Theta \ll 1$, or $\Theta \gg 1$, the unpolarized radiation differential intensity isotropizes and becomes independent
of $N^2$, at all (however, the polarization will not be neither isotropic, nor small there – see Eq. (121) below). For illustration purposes, and in order to keep the study model-independent, let $N$ be independent of $\Theta$ (in small enough). The distribution of unpolarized intensity (111) in the $\Theta$ plane, for constant $N$, at exemplary values $N^2 = \frac{1}{2}$ and $x_\omega = \frac{1}{3}$, is illustrated in Fig. 7. As compared with Fig. 3, no dips are left at $N^2$ that small (and $x_\omega$ that large), but there still remains a noticeable azimuthal anisotropy, the radiation intensity being enhanced in the direction orthogonal to $N$. That is explained naturally: $N$ determines the dominant direction of the charge acceleration caused by the field of the scatterer (the much greater deflection due to the transverse recoil from the photon emission proves to be of no consequence there), but dipole radiation intensity is known to be largest in directions orthogonal to that of the acceleration. In contrast, electron multiple scattering diffusion in the sample will be fastest in the direction parallel to $N$.

If resolution of radiation angles is feasible in the experiment, measurement of the radiation azimuthal anisotropy (say, at $\Theta \approx 1$) may offer a method for parameter $N$ determination for a given target. Other methods are based on polarization measurements. We now proceed with the discussion of the polarization.

### B. Angular distribution of polarization

As long as linear polarization is a vector quantity, to handle it practically, it is best to know its absolute magnitude (degree) and the direction. However, those variables are not in a linear relation to the calculated differential probability, which, after momentum averaging in matter, turns to a generic kind of tensor in $e^i$ – see (107). This may prompt one to deal with polarization asymmetries in some fixed coordinate frame, such as Stokes parameters. But actually, in 2 transverse dimensions expressing the polarization direction explicitly is not difficult at all, involving at the most quadratic equations.

From Eq. (109), the polarization degree is extracted as an asymmetry

$$P(\Theta, x_\omega; N) \overset{\text{def}}{=} \max_{e} \left\{ \frac{dW_{\text{unpol}}}{d\sigma} \right\} - \min_{e'} \left\{ \frac{dW_{\text{unpol}}}{d\sigma} \right\}$$

$$\overset{\text{max}}{\text{max}}_{e} \left\{ \frac{dW_{\text{unpol}}}{d\sigma} \right\} + \min_{e'} \left\{ \frac{dW_{\text{unpol}}}{d\sigma} \right\} = \frac{4\pi\alpha \langle q_\omega^2 \rangle_{\text{rad}}}{m^4 (1 + \Theta^2)^2} \frac{2(1 - x_\omega)}{x_\omega^2} \frac{\lambda_+ [T_{ij}]}{\frac{1}{2} \frac{dT}{d\sigma_{ij}}}$$

where $\lambda_+ [T_{ij}]$ is the positive eigenvalue of tensor $(2T_{ij} - T \delta_{ij})$. In terms of the latter, $\lambda_+$ may be expressed as

$$\lambda_+ = \sqrt{\frac{1}{2} (2T_{ij} - T \delta_{ij}) (2T_{ij} - T \delta_{ij})} \equiv \sqrt{2T_{ij}T_{ij} - T^2}.$$

In our case (109), tensor $T_{ij}$ is formed from two vectors:

$$T_{ij} = a_i a_j - b_i b_j$$

with

$$a_i = G_{im} N_m, \quad b_i = \sqrt{2(1 - N^2)} \frac{1}{1 + \Theta^2} \Theta_i$$

(those vectors are not mutually orthogonal, in general). Substituting Eq. (114) to Eq. (113), one straightforwardly evaluates

$$\lambda_+ = \sqrt{(a^2 - b^2)^2 + 4(a \times b)^2}$$

$$\equiv |a - b| |a + b|$$

Eigenvectors of a tensor of the form (114) can be expressed covariantly in terms of the vectors $a, b$:

$$t_{\pm} \parallel 2(a \cdot b) b - \pm \lambda_+ b$$

$$t_{\pm} \perp t_{\mp}$$

(the coefficients at $a, b$ in (118a) (118b) are found by solving a system of two linear equations).

Substitution of (113) (116) into Eqs. (117a) (118a) leads to representations
Polarization zero positions now can be found from set-
split, admitting the “polarization flow” into the gaps.

A novel feature at richer structure than the unpolarized intensity in Fig. 7.

pendent of \( \Theta \) to the unit circle), whereas the width of the gap

exhibits “threshold behavior” as \( N \) departs from 1. However, now the points of zero polarization do not correspond to any minima in the radiation intensity (cf. Fig. 7).

Measurement of the value of the angular separation [125] between the polarization zeroes may serve as another calibration method for the parameter \( N \). The virtue of this method is that it does not require absolute measurements of intensity, but only of the angles, and it is particularly sensitive when \( N^2 \) is close to 1 (due to the square root dependence).

The analytic form of the polarization tangential curves in general case is complicated, and we do not aim determining it here. At least, at \( \Theta = 1 \) it is apparent that \( t_- \parallel \Theta \), hence \( t_+ \perp \Theta \), i.e. [40], polarization direction is steered along the unit circle, anyway.

To gain more quantitative understanding of the profile of polarization distribution, it is instructive to examine its two principal profiles: \( \Theta \parallel \mathbf{N} \) and \( \Theta \perp \mathbf{N} \). In those cases, \( \mathbf{a} \parallel \mathbf{b} \), or \( \mathbf{a} \perp \mathbf{b} \), owing to which Eqs. (119) 120 substantially simplify.

(i) If \( \Theta \parallel \mathbf{N} \),

\[
\lambda_+ = \left| N^2 - \frac{4(N \cdot \Theta)^2 + 2(1 - N^2)\Theta^2}{(1 + \Theta^2)^2} \right| + \frac{8(1 - N^2)\Theta^2}{(1 + \Theta^2)^2},
\]

\[
t_+ \parallel \mathbf{N} \quad \text{if either } |\Theta| < \Theta_<, \text{ or } |\Theta| > \Theta_>, \quad (127a)
\]

\[
t_+ \perp \mathbf{N} \quad \text{in the interval } \Theta_< < |\Theta| < \Theta_>, \quad (127b)
\]
(Eq. 127a is inferred from Eq. (120) with the upper sign, Eq. (127b) – from Eq. (120) with the lower sign).

The polarization degree (112) through (126) reduces to

$$P = \frac{|N^2 (1 + \Theta^4) - 2\Theta^2|}{1 + \Theta^4 - 2N^2\Theta^2 + \frac{x_ω^2}{2(1-x_ω)}(1 + \Theta^2)^2}.$$  \hspace{1cm} (128)

This function is displayed in Fig. 9b. It drops to zero at $\Theta = \Theta_{+<,\Theta_{+>}}$, has maxima at $\Theta = 0$ and $\Theta = \infty$, where it achieves the same values $P(\Theta = 0) = P(\Theta = \infty) = N^2 P_{\text{max}}(x_ω)$ (with $P_{\text{max}}(x_ω)$ given by Eq. (73)), and at $\Theta^2 = 1$, where

$$P(\Theta = \pm N/|N|) = \frac{1}{1 + \frac{x_ω^2}{(1-x_ω)(1+N^2)}}.$$

(129)

(ii) In the case of orthogonal profile $\Theta \perp N$,

$$t_+ \perp t_- \parallel \Theta$$

(128) turns to

$$\lambda_+ = N^2 + (1 - N^2)\frac{2\Theta^2}{(1 + \Theta^2)^2},$$

and so polarization degree (112) becomes

$$P = \frac{N^2 (1 + \Theta^4) + 2\Theta^2}{1 + \Theta^4 + 2N^2\Theta^2 + \frac{x_ω^2}{2(1-x_ω)}(1 + \Theta^2)^2}.$$  \hspace{1cm} (130)

(As Fig. 8 indicates, and can be proven based on Eqs. (112) (119), this is the absolute maximum for all $\Theta$). At $\Theta = 0$ and $\infty$, polarization is minimal with the value $N^2 P_{\text{max}}(x_ω)$.

The observed feature that at $\Theta = 1$ the polarization is capable of achieving 1 looks odd, considering that summation of portions of completely polarized light, but with different polarization orientations (vector $\vec{G}_q\perp$ with $\vec{G}$ given by Eq. (63), generally, rotates along with $\vec{q}_\perp$, must result in decrease of the overall polarization degree. The explanation requires recalling the pattern of polarization alignment at a given $\vec{q}_\perp$ (Fig. 4). Since at $\Theta = 1$ polarization is oriented along a perfect circle centered at the origin of the plane, and that circle remains self-collinear under the rotations of $\vec{q}_\perp$ corresponding to azimuthal averaging, the absolute polarization at this special angle is unaffected by the scattering isotropization.

To conclude this subsection, note that the spots of high and directionally stable polarization at $(\Theta_x, \Theta_y) \approx (\pm 1, 0)$ are also attractive for extraction of a polarized photon beam by the radiation collimation technology. However, with a collimation facility at disposal, one can obtain a polarized photon beam on an isotropic target as well – see the next subsection (and (33)). One should mind also that at $\Theta \approx 1$ the radiation intensity is by an order of magnitude lower than at $\Theta \approx 0$ (see Fig. 10). On the other hand, the region $\Theta \approx 0$ is polarized, too, though half that strong, which nevertheless might be beneficial given the order-of-magnitude higher intensity. Thus, for extraction of a polarized photon beam one may consider collimation out the angular strip at $\Theta_y \approx 0$.

C. Isotropic target ($N = 0$)

Since most substances of a natural origin are highly isotropic on macro-scales, all early studies of bremsstrahlung presumed the scattering isotropy. The results of classic works [5, 7, 8] are readily reproduced from our generic equations.
FIG. 10: Angular dependence of the rescaled Bethe-Heitler radiation intensity in the semi-classical limit \(
abla \frac{dW_{\text{isotr}}}{d\Gamma q'} |_{\omega \ll 1} \) (solid line) as compared with the cross-section pre-factor \((1 + \Theta^2)^2\), also representing the differential intensity as \(x_\omega \rightarrow 1\) (dotted line). The soft radiation angular distribution exhibits two “knees”, at \(\Theta \approx \frac{1}{2}\) and \(\Theta \approx 2\) whereas the hard radiation only has one, at \(\Theta \approx 1\).

Putting in our Eq. (79) \(N = 0\), one reproduces the equation for the polarization-dependent differential cross-section of bremsstrahlung in an isotropic medium obtained by May and Wick [16, 17, 20]:

\[
\left\langle \frac{dW_{\text{isotr}}}{d\Gamma q'} \right\rangle = \left\langle \frac{dW_{\text{unpol}}}{d\Gamma q'} \right\rangle |_{N=0} = \frac{8\pi\alpha(q_\perp^2)_{\text{rad}}}{m^4(1 + \Theta^2)^2} \left\{ 1 + \frac{2(1 - x_\omega)}{x_\omega^2 + 1 + \Theta^4} \right\},
\]

(132)

\[P_{\text{isotr}}(\Theta, x_\omega) = P(\Theta, x_\omega)|_{N=0} = \frac{2q^2}{1 + \Theta^4 + \frac{x_\omega^2}{2(1-x_\omega)(1+\Theta^2)^2}}, \]

(133)

\[t_+ \perp \Theta. \]

(134)

Eq. (132) is essentially the Bethe-Heitler’s formula in the leading logarithmic approximation (the logarithmic factor being contained in \((q_\perp^2)_{\text{rad}}\)). Relation (134) is the observation of May and Wick [16]. It may be naturally interpreted as being due to domination of the dipole emissivity in directions orthogonal to that of the acceleration. In general, the sample of events containing a photon at an angle \(\Theta\) is biased towards momentum transfers in matter orthogonal to \(\Theta\). These events are then likely to contain photon polarization collinear with \(q\), and thus perpendicular to \(\Theta\), in agreement with Eq. (134).

There are yet two remarks on the radiation angular distribution which seem worth making.

a. “Double-knee” in the angular distribution of soft radiation. The function

\[
\frac{1 + \Theta^4}{(1 + \Theta^2)^2} \equiv 1 - \frac{2\Theta^2}{(1 + \Theta^2)^2},
\]

(135)
determining at small \(x_\omega\) the r.h.s. of Eq. (132) at small \(x_\omega\), has a minimum at \(\Theta = 1\) (precisely where the dips in the non-averaged \(dW_{\text{dip}}/d\Gamma q'\) are located). However, function \((dW_{\text{isotr}}/d\Gamma q')\) in fact does not develop any minimum or shoulder about this point, because of the pre-factor \(1/(1 + \Theta^2)^2\) which decreases more steeply than \(135\), rises after its minimum. Nevertheless, some more subtle imprint of function (135) remains in behavior of the Bethe-Heitler radiation angular distribution, as we will now demonstrate.

Let us plot logarithm of \(dW_{\text{isotr}}/d\Gamma q'\) vs. the logarithm of \(\Theta\) (at \(x_\omega \ll 1\), to enhance the relative contribution of (135)). In such kind of a plot linear dependence corresponds to a power-law falloff. As compared with the behavior of the pre-factor \(1/(1 + \Theta^2)^2\), which has on this plot only one “knee” at \(\Theta \approx 1\), \((dW_{\text{isotr}}/d\Gamma q')\) apparently has two “knees” – at about \(\Theta_1 \approx 0.5\) and \(\Theta_2 \approx 2\). In between of those two “knees” the behavior is close to \(\Theta^{-2}\). Beyond \(\Theta_2\), the falloff power turns to \(\sim \Theta^{-4}\), as is required by the Rutherford-like law mentioned in Sec. [V]. This 2-knee shape of the radiation angular distribution may be worth minding at poor statistics measurements, since at the second knee the differential cross-section is already down by a factor of nearly 10^{-2}. But as \(x_\omega\) grows, the distribution approaches the 1-knee limiting form.

b. Polarization maximum and its non-dipole destruction. For what concerns polarization (shown in Fig. 15), again, the prominent feature is that it reaches 100% at \(x_\omega \ll 1\), \(\Theta = 1\), in spite of the angular averaging. As already explained in the previous subsection, this owes to the circular distribution of polarization at any given \(\hat{q}_\perp\), but the example at \(N = 0\) is just the most spectacular. With the account of non-dipole effects (see Appendix B and [6]), polarization in the region \(\Theta \approx 1\) must deplete, because the centres of the circles drift aside, and none of them coincides with the origin exactly.

VIII. SPECTRUM AND NET POLARIZATION OF RADIATION CONE AS A WHOLE

If angular resolution of the emitted radiation is not pursued in the experiment (which may become impractical at \(\gamma > 10^4\)), and only the natural collimation due to emission from an ultra-relativistic particle is utilized, one must integrate Eq. (108) over the radiation angles (which being small, basically form a \(\Theta\) plane):

\[
d\Gamma_q = \frac{d\omega}{\omega} \frac{\omega^2}{4\pi} \frac{d\theta}{\omega} \frac{d^2\Theta}{4\pi^2} \frac{d\omega}{\omega} \frac{\omega^2}{4\pi} \frac{d\theta}{\omega} \frac{d^2\Theta}{4\pi^2} \]

\[
= \frac{d\omega m^2 x_\omega^2}{\omega^2 (4\pi)^2} d\Theta d\phi \Theta \frac{d\phi}{2\pi}.
\]

(136)
The angular integrations are carried out with the aid of the basic integrals
\[
\int \frac{d\phi \Theta_i \Theta_j}{2\pi} = \frac{1}{2} \delta_{ij} \Theta^2,
\]
\[
\int \frac{d\phi \Theta_i \Theta_j \Theta_m}{2\pi} = \frac{1}{8} (\delta_{ij} \delta_{lm} + \delta_{il} \delta_{jm} + \delta_{im} \delta_{jl}) \Theta^4,
\]
\[
\int_0^\infty \frac{d\Theta (\Theta^2)^m}{(1 + \Theta^2)^{2+n}} = m!(n-m)!/(n+1)!
\]
(137)
The result is
\[
\omega \left\langle \frac{dW_{\text{dip}}}{d\omega} \right\rangle = \int d\Theta^2 \frac{d\phi \Theta m^2 x^2}{2\pi (4\pi)^2} \left\langle \frac{dW_{\text{dip}}}{d\Gamma_q} \right\rangle
\]
\[
= \frac{\alpha \left\langle q_2^2 \right\rangle_{\text{rad}}}{4\pi m^2} \left\{ \frac{4}{3} (1 - x_\omega) + x_\omega^2 \right\}
\]
(139)

The unpolarized spectral intensity (which otherwise might be obtained by integrating Eq. (111)) ensues
\[
\omega \left\langle \frac{dW_{\text{unpol}}}{d\omega} \right\rangle = \sum_{\alpha} \omega \left\langle \frac{dW_{\text{dip}}}{d\omega} \right\rangle
\]
\[
= \frac{\alpha \left\langle q_2^2 \right\rangle_{\text{rad}}}{2\pi m^2} \left\{ \frac{4}{3} (1 - x_\omega) + x_\omega^2 \right\}
\]
\[
= \frac{\alpha \left\langle q_2^2 \right\rangle_{\text{rad}}}{2\pi m^2} \frac{E'}{E} \left( \frac{E'}{E} + \frac{E'}{E'} - \frac{2}{3} \right).
\]
(140a)

(see Fig. 11a). Therein the dependence on \( N \) completely drops out – quite naturally, recalling that \( N \) is representative of the quadrupole dependence on \( q_2 \), while after integration over \( \Theta \) it can only be contracted with the quadrupole tensor dependence on \( e' \) (as Eq. (139) indicates) – and after summation over \( e' \), all that but averages to zero. The integral of the \( x_\omega \)-dependent expression in Eq. (140a) (the area under the curve in Fig. 11b) is unity:
\[
\int_0^1 dx_\omega \left\{ \frac{4}{3} (1 - x_\omega) + x_\omega^2 \right\} = 1.
\]
(141)

Finally, the polarization deduced from Eq. (139) is directed parallel to \( N \)
\[
t_+ \parallel N,
\]
(142a)
and its degree equals
\[
P_{\text{net}} = \frac{N^2}{2} \frac{1}{1 + \frac{3x^2}{4(1-x_\omega)}}.
\]
(142b)

The semi-classical limit (\( x_\omega \to 0 \)) of (142b) at \( N = 1 \) agrees with the polarization \( \frac{1}{2} \) of dipole radiation from a classical charged particle in a planar undulator \[3\]. The \( x_\omega \)-dependent factor describes the polarization suppression due to the photon recoil. The function \( P_{\text{net}}/N^2 \) is shown in Fig. 11b.

The practical value of the non-zero net polarization is that once there is a target with sizeable \( N^2/2 \), a polarized gamma-ray beam may be obtained on it, without the need for narrow collimation and particular target thinness \[34\]. The common known drawback of incoherent bremsstrahlung radiation is its continuous spectrum, but that may be coped with by measurement of the energies of all final products of the induced reactions.

Vice versa, measurement of the net polarization may be used as yet another way for empirical determination of \( N \) for a given target, now not requiring angular resolution. If one’s aim is to obtain a polarized photon beam, this will be equivalent to the source calibration \textit{in situ}.

IX. SUMMARY AND OUTLOOK

The concept of equivalent photons proves capable of giving yet new insight into the polarized bremsstrahlung problem. It offers fair simplifications to the electron spin summation procedure. In general, it also favors a view from the initial electron rest frame, wherein the polarization-correlated part of differential cross-section of the Compton process takes semi-classical, Thomson-like structure (the polarization-uncorrelated remainder being due to electron spin effects of no classical analog). The relation between those two frames is in itself interest-
ing because it is just a stereographic projection, which is known by its symmetry properties, and in particular renders polarization of the radiation from an ultrarelativistic electron a distribution along a family of circles. Furthermore, one of the circles is centered at the origin and is invariant under isotropization of momentum transfers in the medium. That entails existence of a special radiation opening angle, \( \theta = 1 \cdot \gamma^{-1} \), at which the polarization tends to 100%, regardless of the scattering anisotropy degree in the medium (the target may even be isotropic).

From the physical side, it is interesting that there must exist macroscopic targets, on which relativistic electron scattering, and hence the accompanying radiation, possesses high degree of azimuthal anisotropy. Examples are single crystals oriented by one of their strong crystallographic axes at moderately small orientation angles \( 10^{-1} \div 10^{-2} \text{rad} \) with respect to the electron beam direction. The anisotropy of the scattering is partially spoiled by nuclei vicinities and by the radiation recoil (see Eq. (11b)), but nonetheless, values \( N^2 \geq 0.5 \) look realistic.

Although anisotropy \( N^2 \sim 0.5 \) does not manifest itself sharply in the angular distribution of radiation intensity (Fig. 7), it makes distribution of the polarization quite rich (Fig. 8). Results for distribution of the radiation polarization degree and direction obtained in Sec. VII.B indicate that there exist 4 polarization zeroes, 2 spots of enhanced polarization, and regions of polarizations direction parallel to \( N \), as well as orthogonal to it. All those features may serve for diagnostics of regimes of particle passage through matter.

The major practical value of polarized bremsstrahlung is its suitability for preparation of polarized photon beams (of continuous spectrum). Actually, there is a number of options for extracting polarized photons from the bremsstrahlung flow:

(i) If only isotropic targets are at service, there is no alternative to the traditional method of collimating the photon flow around the angle \( \Theta = 1 \), i. e., \( \theta = 1 \cdot \gamma^{-1} \).

(ii) If the use of absorber collimators is prohibitive due to smallness of the radiation angles, as it tends to be at \( \gamma > 10^4 \), one needs an intrinsically anisotropic target, aggregate (naturally narrow) cone of photons emitted on which is polarized. But its polarization degree, according to Eq. (12b), is \( \leq N^2/2 \), with \( N^2 < 1 \). For efficiency of such a polarized beam, it is desirable to have \( N^2 \) at least \( \sim 0.7 \div 0.8 \).

(iii) Finally, if both the collimation tool and an intrinsically anisotropic target are available, one may either look for the highest polarization degree, isolating one of the two spots of enhanced polarization (see Fig. 8). Or, if moderate polarization degree is acceptable provided the beam intensity is high, there is an option of collimating out the strip of angles perpendicular to \( N \), in between of the polarization zeroes.

In conclusion, regardless of the bremsstrahlung problem, we draw attention to the formula (28) for the differential cross-section of Compton scattering with arbitrary (including elliptic) photon polarizations, and to representation (21) for the electron scattering amplitude, which may prove useful for calculations of other QED processes.

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**APPENDIX A: CORRESPONDENCE WITH OTHER NOTATIONS**

In this appendix we will establish correspondence of our notations for radiation intensity in the dipole approximation with some other notations widely used in the literature, in particular those of the non-dipole bremsstrahlung theory.

1. Unpolarized dipole intensity

For unpolarized radiation intensity, the dipole approximation, as obtained via spurring procedure, and often invoked in studies of radiation at multiple scattering in a medium (see, e. g., [35]), is

\[
\frac{dW_{\text{unpol}}}{d\Omega_q} = \frac{2\pi \alpha q^2}{E E' q_z^2} \left( \frac{E'}{E} + \frac{2m^2 \omega^2}{E^2 q_z^2} (\theta \cdot \hat{q}_\perp)^2 \right). \tag{A1}
\]

This form is reproduced from Eq. (65) if Eq. (66a) is substituted for \((Gq)^2\), and identity

\[
1 + \frac{2(1-x_\omega)}{x_\omega^2} = \frac{E E'}{\omega^2} \left( \frac{E}{E'} + \frac{E'}{E} \right) \tag{A2}
\]

along with Eq. (41a) is utilized.

2. Polarization dependence description

For differential cross-section of (non-dipole) polarized bremsstrahlung in a Coulomb field in the Born approximation, Ghickstein, Hull, and Breit [3] quote a formula, which after some regrouping reads (see also [35], Eq. (8.2))

\[
\frac{d\sigma_{\text{rad}}}{d^2q_\perp d\Omega_q} = \frac{d\sigma_{\text{Ruth}}}{d^2q_\perp} \frac{\pi \alpha}{EE'} \frac{1 + 2(1-x_\omega)}{x_\omega^2} \left( \left( \frac{2E p' \cdot e' - 2E p \cdot e}{p' \cdot q} \right)^2 + \frac{q^2 \omega^2}{p' \cdot q' p \cdot q'} \right) \left( \frac{p' \cdot q' - p \cdot q'}{p' \cdot q' - 2} - q^2 \left( \frac{p' \cdot e'}{p' \cdot q'} - \frac{p \cdot e}{p \cdot q} \right)^2 \right). \tag{A3}
\]

The last term here,

\[
q^2 \left( \frac{p' \cdot e'}{p' \cdot q'} - \frac{p \cdot e}{p \cdot q} \right)^2 \approx \frac{q^2}{m^2} \left( \frac{2\Theta \cdot e'}{1 + \Theta^2} \right)^2 \sim \frac{q^2}{m^2},
\]
is $O\left(\gamma^{-2}\right)$ relative to other terms, so all the polarization
dependence is accumulated in the first parentheses term. The
first parentheses, with the use of the second equality in Eq. (A13),

eq \frac{\Theta q}{2} \left\{ \frac{q^2}{m^2}, \gamma^{-2} \right\},
\end{equation}
and by relations $p\cdot e' - p'\cdot e' = q\cdot e', p'\cdot e' = -m\Theta \cdot e'$,
are reduced to
\begin{equation}
2E\frac{p'\cdot e'}{p'\cdot q'} - 2E'\frac{p\cdot e'}{p\cdot q'} = -2q\left\{ q\cdot e' - \frac{m^2 + \Theta \cdot q}{E'q_z} \right\} \left\{ 1 + O\left(\frac{q}{m}\right) \right\}.
\end{equation}

Here one recognizes a connection of $q\perp$ and $e'$ through tensor $\Theta$, i.e. the product $e_p \cdot e'_p$.

Actually, to a better accuracy one can prove that
\begin{equation}
e_p \cdot e'_p = \left( E\frac{p'\cdot e'}{p'\cdot q'} - E'\frac{p\cdot e'}{p\cdot q'} \right) \left\{ 1 + O\left( \frac{q^2}{m^2} \right) \right\}.
\end{equation}
The proof proceeds by substituting from (A1a)
\begin{equation}
E\frac{p'\cdot e'}{p'\cdot q'} - E'\frac{p\cdot e'}{p\cdot q'} = -e \cdot p' \cdot e' + e' \cdot p\cdot e' - 1 / q \cdot q' / q'.
\end{equation}

On the other hand, the l.h.s. of Eq. (A6) in terms of lab frame spatial polarization vectors $e, e'$ takes the form
\begin{equation}
e_p \cdot e'_p = -e' \cdot \frac{e \cdot e}{p \cdot q} - e_p \cdot e'' \cdot \frac{p}{p \cdot q}.
\end{equation}

Recasting further $q\cdot q' = p\cdot q - p\cdot q' + q^2 / 2$, and grouping terms with the like denominators, yields
\begin{equation}
e_p \cdot e'_p = -e' \left( \Theta \cdot q \cdot q' / q' \right) - e \cdot e \cdot \frac{q^2}{q^2}.
\end{equation}

Obviously, to accuracy $O\left( q^2 / m^2 \right)$ the last term in
Eq. (A10) may be dropped, and $p\cdot q \rightarrow p'\cdot q'$, so r.h.s. of Eq. (A10) coincides with that of Eq. (A11).

Returning to Eq. (A3), its second term simplifies to
\begin{equation}
\frac{q^2 \omega^2}{p' \cdot q' \cdot q \cdot q} \approx \frac{x^2 \theta^2}{(1 - x^2)}.
\end{equation}

Finally, the third block in (A3)
\begin{equation}
\frac{p \cdot q'}{p' \cdot q'} + \frac{p' \cdot q'}{p \cdot q'} - 2 \approx \frac{x^2}{1 - x^2}.
\end{equation}

may be neglected, or combined with (A11) to give exactly factor $q^2 / m^2$. Thereby, the equivalence of Eqs. (A3) and (A13) is proven.

However, let us yet assess the accuracy of Eq. (A12). In its l.h.s.,
\begin{equation}
p \cdot q - p \cdot q' = q \cdot q' \approx \left( \omega_q + \omega \frac{\Theta}{\gamma} \cdot q \right) \approx \omega q_z \left\{ 1 + O\left( \frac{q}{m} \right) \right\}.
\end{equation}

So, the accuracy of the equivalent photon approximation for the electron spin dependent part of the radiation differential probability is worse than that for a spinless particle, and is only $O(q^2 / m^2)$.

Block
\begin{equation}
2E\frac{p' \cdot e'}{p' \cdot q'} - 2E'\frac{p \cdot e'}{p \cdot q'}
\end{equation}
in (A3) is often given an interpretation of the exact amplitude of spinless particle bremsstrahlung on a static potential, the rest of the terms of Eq. (A3) being attributed to effects of electron spin. In our framework, instead, term (A13) originates from the photon polarization correlator (A8), which suggests an interpretation in terms of the initial electron rest frame. Those interpretations are essentially the same.

**APPENDIX B: NON-DIPOLE MODIFICATION OF THE RADIATION ANGULAR DISTRIBUTION**

Some qualitative features of the dipole approximation actually survive also in the non-dipole case – the existence of dips, and the circular pattern of polarization orientation. Consider the main polarization-dependent part of the non-dipole expression (A3) – first parentheses. In terms of radiation angles introduced in Sec. IV it may be written as
\begin{equation}
\frac{\gamma^{-1}}{\gamma^{-2} + \theta^2} = \frac{\gamma^{-1}}{\gamma^{-2} + \theta'^2}.
\end{equation}

Substituting there
\begin{equation}
\theta' = \frac{E \Theta + q}{E'}
\end{equation}
from (59a), and $\theta = \gamma^{-1} \Theta$, (51) converts to
\begin{equation}
\frac{\Theta}{1 + \Theta^2} - \frac{\Theta + \frac{q}{m}}{1 + \left( \Theta + \frac{q}{m} \right)^2}.
\end{equation}
This vector expression has zeroes at
\[ \Theta_{*\pm} = \frac{q}{2m} \left( \pm \sqrt{1 + \frac{4m^2}{q^2}} - 1 \right), \tag{B3} \]
in the dipole limit agreeing with Eq. (67):
\[ \Theta_{*\pm} \rightarrow_{q \ll m} \Theta_{\pm} = \pm \frac{q}{|q|}. \tag{B4} \]

As for the circular distribution of the polarization direction, let us point out that for Eq. (B2), at arbitrary \( q/m \), there exists a proportionality relation
\[
\frac{\Theta}{1 + \Theta^2} - \frac{\Theta + \frac{q}{m}}{1 + \left( \Theta + \frac{q}{m} \right)^2} = C(q, \Theta) \left[ \frac{\Theta - \Theta_{*\pm}^}\circ}{(\Theta - \Theta_{*\pm}^\circ)^2} - \frac{\Theta - \Theta_{*\pm}^\circ}{(\Theta - \Theta_{*\pm}^\circ)^2} \right], \tag{B5} \]
with \( C \) a scalar function
\[
C(q, \Theta) = \frac{(\Theta - \Theta_{*\pm}^\circ)^2 (\Theta - \Theta_{*\pm}^\circ)^2}{(1 + \Theta^2) \left[ 1 + \left( \Theta + \frac{q}{m} \right)^2 \right] \sqrt{1 + \frac{4m^2}{q^2}}}. \tag{B6} \]

In r.h.s. of Eq. (B5), the vector function in brackets has the appearance of a 2d finite-size dipole, while for the latter it is common known, e.g., from the theory of functions of a complex variable, that its force lines are a family of circles passing through the source location points; the latter points coincide with the intensity dips (B3). However, the non-dipole circular pattern will not be consequential that much as for the dipole case (see the end of Sec. VI), because centres \( \Theta_{\text{centr}} \) of those circles lay on the line
\[
\left( \Theta_{\text{centr}} - \frac{\Theta_{*\pm}^\circ + \Theta_{*\pm}^\circ}{2} \right) \cdot \left( \Theta_{*\pm}^\circ + \Theta_{*\pm}^\circ \right) = 0,
\]
and at \( q/2m \approx 1 \) displacements of all centres from the origin are not small compared to the circle radii.

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That is true for elastic scattering, but the latter in fact dominates over inelastic when nuclear charges are $Z \gg 1$, as coherent contribution vs. incoherent ($Z^2$ vs. $Z$).

For the time being, we ascribe momentum $q$ to the initial state, due to our intention to treat the process in similarity with Compton scattering. Later on, when addressing bremsstrahlung in matter, we will denote spatial components of this 4-vector as $-q$, so that $q$ is the final atom recoil.

The sign (phase) of the last term of this identity corresponds to convention $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ adopted in [25] and opposite to that of [15].

Note that $p \cdot p' - m^2 = p \cdot q - p' \cdot q'$, and the electron mass does not appear explicitly in further formulas. This is in contrast to the trace technology [14]. The mass explicitly enters to the initial and final electron density matrices, as well as in propagators, unless been excluded intentionally.

Since $q_z \ll q_\perp$, vector $q$ is nearly transverse.

In the standard equivalent photon method formulation [13], $A_{\text{diff}}$ traditionally (since [12]) is treated in the Born approximation, i.e., as a Fourier transform of the scatterer field, implying the equivalent photon to be single on. But in the present treatment, based on the high-energy factorization theorem such a restriction is relaxed.

It may be worth emphasizing that it is due to the gauge $p$ that the photon polarization 4-vector product differs from zero. In the initial gauge, one would have $e \cdot e' \equiv 0$.

The second equality of (53) reflects the Doppler rule for transformation of photon frequency under (ultra-relativistic) boosts (see [11]). It is rather obvious from the viewpoint of light-cone variables, which only rescale under Lorentz boosts, in proportion $\frac{\Omega + K_z}{\Omega}$. In turn, Eq. (53) is basically a known formula for light aberration (transformation of the emission angles) at an ultra-relativistic boost [14].

That will prove to have valuable implications, because the polarization direction becomes independent of $q_\perp$ (see Sec. VII B, Sec. VII C).

The author acknowledges communication with I. F. Ginzburg on this point.

Although at the same time we are interested in the dipole approximation, whose condition (43) forbids $\omega$ to turn arbitrarily small, nonetheless, there is room for both conditions to coexist, when $\frac{q_\perp}{m} \ll x, \omega \ll 1$.

The only essential condition is that scattering angles do not become comparable to $\gamma^{-1}$ due to multiple scattering within the coherence length $q_z^{-1}$.

However, for $\Theta$ large enough $N$ should become $\Theta$ dependent (at least, model [104] suggests so), and then polarization at large angles will vanish.

For the $+$ sign, the r. h. s. of (120) vanishes, giving indeterminancy for $t_+$. Fortunately, for $t_-$ such a problem does not arise.

That may require a commentary. In course of our derivation in Sec. III, when commuting spin matrices, we had neglected the terms containing $e_p \cdot q$, which are $\propto q^2$. But the entire dipole amplitude appears to be $\propto q_\perp$ (see Eq. (30)). So, for spin $\frac{1}{2}$ particles the accuracy of the equivalent photon approximation is only $O(q_\perp)$. 