Multi-fusion categories of Harish-Chandra bimodules

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Abstract. We survey some results on tensor products of irreducible Harish-Chandra bimodules. It turns out that such tensor products are semisimple in suitable Serre quotient categories. We explain how to identify the resulting semisimple tensor categories and describe some applications to representation theory.

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1. Introduction

The notion of tensor category is ubiquitous in representation theory. A classical example is the theory of Tannakian categories (see [58, 17]) which shows that a linear algebraic group can be recovered from the tensor category of its finite dimensional representations. The tensor product in a Tannakian category is commutative in a very strong sense. In this note we will be interested in tensor categories for which the tensor product is not assumed to be commutative. One reason for the relevance of such categories to representation theory is very simple: the category of bimodules over an arbitrary algebra $A$ is a tensor category with tensor product given by tensoring over $A$ and this tensor product is non-commutative in general.

A classical notion in the representation theory of a complex semisimple Lie algebra $g$ is that of Harish-Chandra bimodules. These objects were introduced by Harish-Chandra [24] in order to reduce some questions of continuous representation theory of complex semisimple groups (considered as real Lie groups) to pure algebra. A number of deep results on Harish-Chandra bimodules are known, see e.g. [7, 26, 61]. In this note we will be interested in just one aspect of the theory, namely, in the structure of the tensor category of Harish-Chandra bimodules. Some significant steps towards a complete description of this category were made in [61], however due to non-semisimplicity this description is necessarily quite complicated. It turns out that the classical notion of associated variety (see e.g. [27, 65]) provides us with a kind of filtration on this category; moreover one can define an “associated graded” category with respect to this filtration which is much simpler than the original category but still carries an important information about the
category of Harish-Chandra bimodules. Thus one defines certain interesting semi-simple subquotients of the tensor category Harish-Chandra bimodules associated with various nilpotent orbits in \( \mathfrak{g} \) which we call cell categories, see Section 3.4. The idea of this definition can be traced back to the work of Joseph [29] and the name is justified by the connection with the theory of Kazhdan-Lusztig cells [31]. A nice property of the cell categories is that they are multi-fusion in a sense of [21]. The known results from the theory of multi-fusion categories turned out to be powerful enough to identify these categories with some categories constructed from some finite groups. This is interesting in its own right but also gives a better understanding of some notions of representation theory such as Lusztig’s quotients and Lusztig’s subgroups.

One can hope to apply the ideas above in the following way. The Harish-Chandra bimodules act on various categories of \( \mathfrak{g} \)-modules via tensoring over the universal enveloping algebra of \( \mathfrak{g} \). We can exploit this action restricted to the semisimple subquotients as above in order to obtain interesting information about such categories of \( \mathfrak{g} \)-modules. One example of such application is the theory of finite \( W \)-algebras where this strategy allowed to obtain the information on the number of finite dimensional simple modules, see Section 4. It was suggested by Bezrukavnikov that similar approach might work for Harish-Chandra modules, see Section 4.3.

The cell categories described above can be realized via truncated convolution of some perverse sheaves on the flag variety associated with \( \mathfrak{g} \). One advantage of this description is a greater flexibility. For example we can replace the complex semisimple Lie algebra \( \mathfrak{g} \) by a semisimple algebraic group \( G \) defined over a field of arbitrary characteristic. Moreover, using the tensor categorical construction of the Drinfeld center one connects the cell categories with the theory of character sheaves on \( G \), see [6, 11, 45]. We describe briefly these developments in Section 5.

This paper is organized as follows. In Section 2 we review briefly some notions of the theory of tensor categories. In Section 3 we introduce the Harish-Chandra bimodules and define the cell categories. In Section 4 we explain how to use Whittaker modules and Premet’s \( W \)-algebras in order to establish some basic properties of the cell categories. Conversely we show that the actions of the cell categories can be used in order to get an information about finite dimensional representations of \( W \)-algebras. Finally in Section 5 we describe the interaction of the cell categories and some classes of sheaves on algebraic varieties associated with \( \mathfrak{g} \).

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2. Multi-fusion categories

2.1. Monoidal categories. For the purposes of this note a *monoidal category* is a quadruple \((\mathcal{C}, \otimes, a, 1)\) where \(\mathcal{C}\) is a category, \(\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}\) is a bifunctor (called *tensor product*), \(a\) is a natural isomorphism \(a_{X,Y,Z} : (X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)\) (so \(a\) is called *associativity constraint*), \(1 \in \mathcal{C}\) is an object (called *unit object*) such that the following axioms hold:

1. Pentagon axiom: the following diagram commutes for all \(W, X, Y, Z \in \mathcal{C}\):

   ![Diagram](image)

2. Unit axiom: both functors \(1 \otimes ?\) and \(? \otimes 1\) are isomorphic to the identity functor.

It is well known that this definition of monoidal category reduces to the traditional one (see e.g. [46]) if we fix an isomorphism \(1 \otimes 1 \cong 1\).

Further one defines natural notions of tensor functors and tensor equivalences, see e.g. [46]. From a practical point of view tensor equivalent monoidal categories are indistinguishable.

A basic example of monoidal category is category of \(R\)–bimodules over a ring (with unity) \(R\). In this case the tensor product is tensor product \(\otimes_R\) over \(R\), the unit object is \(R\) considered as a bimodule, and the associativity constraint is the obvious one. A closely related example is a category of endofunctors of a category with tensor product given by the composition. Also modules over a commutative ring form a monoidal category, most familiar example being the category of vector spaces over a field \(k\).

Here is another more abstract example.

**Example 2.1** ([59]). Let \(A\) be a group and let \(S\) be an abelian group, both written multiplicatively. We consider the category where the objects are elements of \(A\) and the morphisms are given by \(\text{Hom}(g, h) = \emptyset\) if \(g \neq h\) and \(\text{Hom}(g, g) = S\) for any \(g\). We have a bifunctor \(g \otimes h = gh\) and \(\alpha \otimes \beta = \alpha \beta\) for objects \(g, h \in A\) and morphisms \(\alpha, \beta \in S\). The associativity constraint amounts to a morphism \(\omega_{g,h,k} \in \text{Hom}(ghk, ghk) = S\) for any three elements \(g, h, k \in A\). One verifies that the pentagon axiom reduces to the equation \(\partial \omega = 1\) which says that \(\omega\) is a 3-cocycle on \(A\) with values in \(S\). Moreover, any 2-cochain \(\psi\) determines a tensor structure on the identity functor between tensor categories with associativity constraints given by 3-cocycles \(\omega\) and \(\omega \cdot \partial \psi\). We see that monoidal structures on our category are parameterized by the cohomology group \(H^3(A, S)\).

We explain now that nontrivial associativity constraints do appear in tensor categories of bimodules.
Example 2.2. Let $R$ be an algebra over a field $k$ with trivial center. Recall that $R$–bimodule $M$ is invertible if there exists a $R$–bimodule $N$ such that $M \otimes_R N \simeq N \otimes_R M \simeq R$. The invertible bimodules form a tensor category with respect to $\otimes_R$ (morphisms being the isomorphisms of bimodules). This category is equivalent to the category of type described in Example 2.1. The group of automorphisms of any object is $k^\times$, so the associativity constraint determines a class $\omega \in H^3(\text{Pic}(R), k^\times)$ where $\text{Pic}(R)$ is the group of isomorphism classes of invertible $R$–bimodules (this is the non-commutative Picard group of $R$). This class is often nontrivial. Indeed for any automorphism $\phi$ of $R$ we can define invertible bimodule $R_\phi$ as follows: $R_\phi = R$ as a vector space and the action is given by $(a,b) \cdot c := ac\phi(b)$. The bimodule $R_\phi$ is isomorphic to $R$ if and only if $\phi$ is inner, so we get a well known embedding $\text{Out}(R) \subset \text{Pic}(R)$. Now assume that $\phi$ is outer automorphism such that $\phi^2$ is inner, that is $\phi^2(x) = gxg^{-1}$ for some invertible element $g \in R$ and all $x \in R$; thus $\phi$ generates subgroup $\mathbb{Z}/2\mathbb{Z} \subset \text{Out}(R) \subset \text{Pic}(R)$. It is easy to see that then $\phi(g) = \pm g$; we leave it to the reader to check that the restriction of the class $\omega$ to $\mathbb{Z}/2\mathbb{Z}$ is nontrivial if and only if $\phi(g) = -g$. Here is an example when this is the case:

$$R = \mathbb{C}(g,x,y)/(xy - yx - 1, g^2 - 1, gx + xg, gy + yg), \quad \phi(g) = -g, \phi(x) = -y, \phi(y) = x.$$ 

Remark 2.3. For a commutative algebra $R$ one defines the category of invertible $R$–modules similarly to Example 2.2. Since we have canonically $M \otimes_R N \simeq N \otimes_R M$, this category has an additional structure of symmetric tensor category. This implies that the cohomology class representing the associativity constraint (see Example 2.1) is always trivial, see [19, Section 7].

A crucially important technical assumption on a monoidal category $\mathcal{C}$ is that of rigidity. We recall that for an object $X \in \mathcal{C}$ its right dual is an object $X^* \in \mathcal{C}$ together with evaluation and coevaluation morphisms $\text{ev}_X : X^* \otimes X \to \mathbf{1}$ and $\text{coev}_X : \mathbf{1} \to X \otimes X^*$ such that the composition

$$X \xrightarrow{\text{coev}_X \otimes \text{id}_X} (X \otimes X^*) \otimes X \xrightarrow{\alpha_{X,X^*,X}} X \otimes (X^* \otimes X) \xrightarrow{\text{id}_X \otimes \text{ev}_X} X$$

equals the identity morphism and the composition

$$X^* \xrightarrow{\text{id}_{X^*} \otimes \text{coev}_X} X^* \otimes (X \otimes X^*) \xrightarrow{\alpha_{X^*,X,X}^{-1}} (X^* \otimes X) \otimes X^* \xrightarrow{\text{ev}_{X^*} \otimes \text{id}_{X^*}} X^*$$

equals the identity morphism. Similarly, a left dual of $X$ is $^*X \in \mathcal{C}$ such that $X$ is right dual of $^*X$. A monoidal category $\mathcal{C}$ is rigid if any object of $\mathcal{C}$ has both left and right duals.

Example 2.4. A vector space considered as an object of the monoidal category of vector spaces has left or right dual if and only if it is finite dimensional. A bimodule over an algebra $A$ has a right dual if and only if it is finitely generated projective when considered as a left $A$–module. The category considered in Example 2.1 is always rigid.
2.2. Semisimplicity. We will fix an algebraically closed field $k$ of characteristic zero. We recall that a $k$–linear category $\mathcal{C}$ is called semisimple if there is a collection $\{L_i\}_{i \in J}$ of objects of $\mathcal{C}$ such that

(i) $\dim_k \text{Hom}(L_i, L_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases}$

(ii) any object of $\mathcal{C}$ is isomorphic to a finite direct sum of the objects $L_i$ and any direct sum (including the empty one) is contained in $\mathcal{C}$.

For example if $\mathcal{A}$ as a $k$–linear abelian category with finite dimensional spaces of morphisms the category of semisimple objects in $\mathcal{A}$ (that is the full subcategory consisting of direct sums of simple objects) is semisimple.

Clearly the isomorphism classes of the objects $L_i$ are uniquely determined by the category $\mathcal{C}$. These objects are simple objects of $\mathcal{C}$ (note that zero object is not simple).

For a semisimple category $\mathcal{C}$ let $K(\mathcal{C})$ be its Grothendieck group; this is a free abelian group with basis $[L_i]_{i \in J}$. For any $M \in \mathcal{C}$ we have its class:

$$[M] := \sum_{i \in J} \dim_k \text{Hom}(L_i, M)[L_i] \in K(\mathcal{C}).$$

We say that a semisimple category is finite if the isomorphism classes of simple objects form a finite set.

**Definition 2.5.** A multi-fusion category over $k$ is a $k$–linear rigid monoidal category which is finite semisimple. A fusion category is a multi-fusion category such that the unit object is simple.

Let $1 = \bigoplus_{i \in I} 1_i$ be the decomposition of the unit object of a multi-fusion category into the sum of simple objects. One shows that the objects $1_i$ are “orthogonal idempotents”, that is $1_i \otimes 1_j \simeq \delta_{ij} 1_i$. For any simple object $L \in \mathcal{C}$ there are unique $i, j \in I$ such that $1_i \otimes L = L = L \otimes 1_j$. We say that a multi-fusion category $\mathcal{C}$ is indecomposable if for any pair $i, j \in I$ there exists a simple $L$ with $L = 1_i \otimes L \otimes 1_j$. One shows that any multi-fusion category naturally decomposes into a unique direct sum of indecomposable ones.

The Grothendieck group of a multi-fusion category has a natural structure of a ring via $[M] \cdot [N] = [M \otimes N]$. The Grothendieck ring $K(\mathcal{C})$ of a multi-fusion category $\mathcal{C}$ together with its basis consisting of the classes of simple objects is a based ring in the sense of [39].

**Remark 2.6.** The Grothendieck ring $K(\mathcal{C})$ of a multi-fusion category $\mathcal{C}$ together with its basis determines the tensor product and the unit object in $\mathcal{C}$ uniquely up to isomorphism. Thus the only part of information describing $\mathcal{C}$ and not contained in $K(\mathcal{C})$ is the associativity constraint.

**Example 2.7.** (i) Let $A$ be a finite group. Consider a fusion category where simple objects $L_g$ are labeled by $g \in A$ and $L_g \otimes L_h \simeq L_{gh}$. Similarly to Example 2.1 the possible associativity constraints in this category are classified by $H^3(A, k^\times)$. We will denote such a category with associativity constraint given by $\omega \in H^3(A, k^\times)$.
by $\text{Vec}_A^\omega$; we set $\text{Vec}_A = \text{Vec}_A^{0\omega}$ where $\omega_0$ is the neutral element of $H^3(A,k^\times)$. The Grothendieck ring $K(\text{Vec}_A^\omega)$ is the group ring $\mathbb{Z}[A]$ with a basis $\{g\}_{g \in A}$.

(ii) Let $R$ be a finite dimensional semisimple $k$–algebra, e.g. $R = k \oplus k$. Then the category of finite dimensional $R$–bimodules with tensor product $\otimes_R$ is a multi-fusion category. Its Grothendieck ring is the ring of matrices over $\mathbb{Z}$ with a basis consisting of matrix units.

(iii) Let $A$ be a finite group and $Y$ is a finite set on which $A$ acts. Consider the category $\text{Coh}_A(Y \times Y)$ of finite dimensional $A$–equivariant vector bundles on the set $Y \times Y$. This category has a natural convolution tensor product defined as follows. Let $p_{ij} : Y \times Y \times Y \to Y \times Y, i, j \in \{1, 2, 3\}$ be the various projections; then for $F_1, F_2 \in \text{Coh}_A(Y \times Y)$ we set $F_1 \ast F_2 = p_{13*}(p_{12}^*(F_1) \otimes p_{23}^*(F_2))$ (here $p_{ij}$, and $p_{ij}^*$ are the functors of direct and inverse image and $\otimes$ is the pointwise tensor product). Then the bifunctor $\ast$ has a natural associativity constraint and thus $\text{Coh}_A(Y \times Y)$ is a multi-fusion category. In the special case of trivial $A$ we get example (ii) above; in the case when $Y$ consists of one point we get the category $\text{Rep}(A)$ of representation of $A$; if the action of $A$ on $Y$ is free and transitive we get category $\text{Vec}_A$ from (i).

2.3. Module categories and dual categories. Let $\mathcal{C}$ be a monoidal category and let $\mathcal{M}$ be a category. We say that $\mathcal{C}$ acts on $\mathcal{M}$ if we are give a tensor functor from $\mathcal{C}$ to the category of endofunctors of $\mathcal{M}$. Equivalently, we have a bifunctor $\mathcal{C} \times \mathcal{M} \to \mathcal{M}, (X,M) \mapsto X \otimes M$ endowed with the natural associativity isomorphism $(X \otimes Y) \otimes M \simeq X \otimes (Y \otimes M)$ such that the counterpart of the pentagon axiom holds and the functor $1 \otimes ? : \mathcal{M} \to \mathcal{M}$ is isomorphic to the identity functor. Thus in such situation we often say that $\mathcal{M}$ is a module category over $\mathcal{C}$. Further one defines module functors between module categories and, in particular, the equivalences of module categories, see [52].

Convention. In the case when both categories $\mathcal{C}$ and $\mathcal{M}$ are $k$–linear we will consider only $k$–linear actions. If $\mathcal{C}$ is a multi-fusion category all module categories over $\mathcal{C}$ are assumed to be finite semisimple and non-zero.

Example 2.8. (i) Let $Y$ be a finite set with an action of a finite group $A$. Then the category $\mathcal{M} = \text{Coh}(Y)$ of finite dimensional vector bundles on $Y$ has an obvious structure of module category over $\text{Vec}_A$. It is easy to recover the $A$–set $Y$ from $\mathcal{M}$; the set $Y$ is just the set of isomorphism classes of simple objects in $\mathcal{M}$ and the action of $A$ is recovered by considering the action of simple objects of $\text{Vec}_A$ on simple objects of $\mathcal{M}$.

(ii) We can generalize the example above to the category $\text{Vec}_A^\omega$ from Example 2.7(i). Pick a cocycle $\hat{\omega}$ representing $\omega$. Let $B \subset A$ be a subgroup and let $\psi$ be a 2-cocahn on $B$ such that $\partial \psi = \hat{\omega}|_B$. Then $\psi$ determines an multiplication morphism $R_B \otimes R_B \to R_B$ where $R_B = \bigoplus_{g \in B} L_g$; moreover this morphism makes $R_B$ into associative algebra in the category $\text{Vec}_A^\omega$. Let $\mathcal{M} = \mathcal{M}(B, \psi)$ be the category of right $R_B$–modules in the category $\text{Vec}_A^\omega$; then the left tensoring with object of $\text{Vec}_A^\omega$ makes $\mathcal{M}$ into module category over $\text{Vec}_A^\omega$. Note that the simple objects of $\mathcal{M}$ are naturally labeled by the cosets $A/B$; moreover the action of simple objects of $\text{Vec}_A^\omega$ on simple objects of $\mathcal{M}$ is the same as the action of $A$ on $A/B$. Thus
we consider the module category $\mathcal{M}(B, \psi)$ as a cohomologically twisted version of the action of $A$ on $A/B$. More generally one can consider a direct sum of module categories of the form $\mathcal{M}(B, \psi)$; this is a twisted version of the action of $A$ on a finite set. We will use for such module categories the notation $\mathcal{M} = \text{Coh}(Y)$ where it is understood that the “set” $Y$ carries the cohomological information describing the module category $\mathcal{M}$ (thus $Y$ is completely determined when a finite collection of pairs $(B, \psi)$ as above is specified). We refer the reader to [13, 4.2] for the notion of “$A$-set of centrally extended points” which is a formalization of the cohomological data above in the special case when $\tilde{\omega}$ is trivial. Such notions are important since it is known that any module category over $\text{Vec}_A^\omega$ is equivalent as a module category to $\text{Coh}(Y)$ where $Y$ is such cohomologically twisted $A-$set, see [53, Example 2.1].

Let $\mathcal{M}$ be a module category over an indecomposable multi-fusion category $\mathcal{C}$. Then one defines the dual category $\mathcal{C}^*\mathcal{M}$ to be the category of all endofunctors of $\mathcal{M}$ which commute with the action of $\mathcal{C}$, see [52, 4.2]. The category $\mathcal{C}^*\mathcal{M}$ has a natural monoidal structure where the tensor product is given by the composition of functors. It is known that the category $\mathcal{C}^*\mathcal{M}$ is again an indecomposable multi-fusion category, see [21, Theorem 2.18] (this result fails if $k$ is allowed to have positive characteristic).

Example 2.9. Let $\mathcal{C} = \text{Vec}_A$ and let $\mathcal{M}$ be as in Example 2.8 (i). In this case the category $\mathcal{C}^*\mathcal{M}$ is precisely the category $\text{Coh}_A(Y \times Y)$ from Example 2.7 (iii). Thus using module categories from Example 2.8 (ii) we get a cohomologically twisted version of the category $\text{Coh}_A(Y \times Y)$. We will use similar notation $(\text{Vec}_A)^*\mathcal{M} = \text{Coh}_A,\omega(Y \times Y)$ where it is understood that the $A-$set $Y$ is cohomologically twisted as in Example 2.8 (ii). We note that the that direct summands $\{1_i\}_{i \in I}$ in the decomposition of the unit object $1 \in \text{Coh}_A(Y \times Y)$ are precisely the projection functors from the module category $\mathcal{M}$ to its indecomposable direct summands; in particular the set $I$ is in natural bijection with the set of such summands.

The category $\mathcal{C}^*\mathcal{M}$ consists of endofunctors of $\mathcal{M}$; thus it acts in an obvious way on $\mathcal{M}$. The following result justifies the terminology:

**Theorem 2.10** (see Remark 2.19 in [21]). Let $\mathcal{C}$ be an indecomposable multi-fusion category and let $\mathcal{M}$ be a module category over $\mathcal{C}$. Then $\mathcal{C}^*\mathcal{M}$ is also indecomposable multi-fusion and the natural functor $\mathcal{C} \rightarrow (\mathcal{C}^*\mathcal{M})^*\mathcal{M}$ is an equivalence of tensor categories.

Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a tensor functor between indecomposable multi-fusion categories. We say that $F$ is injective if it is fully faithful and surjective if it is dominant, that is any object of $\mathcal{D}$ is contained in $F(X)$ for suitable $X \in \mathcal{C}$. Now let $\mathcal{M}$ be a module category over $\mathcal{D}$. Then $\mathcal{M}$ can be considered as a module category over $\mathcal{C}$ and we have a natural dual tensor functor $F^* : \mathcal{D}^*\mathcal{M} \rightarrow \mathcal{C}^*\mathcal{M}$. It is shown in [21, 5.7] that this duality interchanges injective and surjective functors.

Example 2.11. Let $A \xrightarrow{f} \bar{A}$ be a surjective homomorphism of finite groups and let $\tilde{\omega}$ be a 3-cocycle representing class $\omega \in H^3(\bar{A}, k^\times)$ such that $f^*(\omega)$ is zero element of
$H^3(A, k^\times)$. Then any 2-cochain $\psi$ such that $\partial \psi = f^*(\tilde{\omega})$ defines a tensor structure on the functor $F : \text{Vec}_A \to \text{Vec}_A^\omega$ sending $L_g$ to $L_{f(g)}$. The functor $F$ is surjective. Conversely, it is easy to see that any surjective tensor functor $\text{Vec}_A \to \mathcal{C}$ where $\mathcal{C}$ is a fusion category is isomorphic to the one of this form.

It is easy to see that the category $\mathcal{C}_M^*$ is fusion if and only if the module category $\mathcal{M}$ is not a nontrivial direct sum of module categories over $\mathcal{C}$, that is $\mathcal{M}$ is indecomposable over $\mathcal{C}$. We have the following consequence of the discussion above:

**Corollary 2.12** (see Lemma 3.1 in [37]). Let $\mathcal{M}$ be a module category over $\text{Vec}_A$ and let $\mathcal{C}$ be a full multi-fusion subcategory of $(\text{Vec}_A)_M^*$ such that $\mathcal{M}$ is indecomposable over $\mathcal{C}$. Then there exists a surjective functor $F : \text{Vec}_A \to \text{Vec}_A^\omega$ such that the action of $\text{Vec}_A$ on $\mathcal{M}$ factors through $F$ and such that $\mathcal{C} = (\text{Vec}_A)_M^* \subset (\text{Vec}_A)_M^*$.

**Proof.** Let $G : \mathcal{C} \to (\text{Vec}_A)_M^*$ be the embedding functor; clearly it is injective. Then the dual functor $G^* : (\text{Vec}_A)_M^* \to \mathcal{C}_M^*$ is surjective. By Theorem 2.10 we have $(\text{Vec}_A)_M^* \subset \mathcal{V}_A$ and the category $\mathcal{C}_M^*$ is fusion. By Example 2.11 the result follows.

The module categories over a fixed indecomposable multi-fusion category $\mathcal{C}$ form a 2-category, where the morphisms are the module functors and 2-morphisms are the natural transformations of the module functors. This 2-category is semisimple in the following sense: for any module categories $\mathcal{M}_1$ and $\mathcal{M}_2$ the category of module functors $\text{Func}_\mathcal{C}(\mathcal{M}_1, \mathcal{M}_2)$ from $\mathcal{M}_1$ to $\mathcal{M}_2$ is finite semisimple (see [21, Theorem 2.18]). It is clear that the composition of functors makes $\text{Func}_\mathcal{C}(\mathcal{M}_1, \mathcal{M}_2)$ into a module category over $\text{Func}_\mathcal{C}(\mathcal{M}_1, \mathcal{M}_1) = \mathcal{C}_M^*$. One shows that the 2-functor $\text{Func}_\mathcal{C}(\mathcal{M}, ?)$ is a 2-equivalence of 2-categories of module categories over $\mathcal{C}$ and over $\mathcal{C}_M^*$, see [33, Proposition 2.3] or [50]. For example this means that there is one to one correspondence between the module categories over $\text{Coh}_{A,\omega}(Y \times Y)$ and over $\text{Vec}_A^\omega$: moreover to compute the module functors between the corresponding module categories over $\text{Vec}_A^\omega$.

**Example 2.13.** Let $\mathcal{M}_1 = \mathcal{M}(B_1, \psi_1)$ and $\mathcal{M}_2 = \mathcal{M}(B_2, \psi_2)$ be the module categories over $\mathcal{C} = \text{Vec}_A$ as in Example 2.8 (ii). Assume that $\psi_1$ and $\psi_2$ are both trivial. Then the category $\text{Func}_\mathcal{C}(\mathcal{M}_1, \mathcal{M}_2)$ identifies with the category $\text{Coh}_{B_1}(A/B_2)$ of $B_1$-equivariant vector bundles on the $B_1$-set $A/B_2$, see e.g. [33, Proposition 3.2].

In general it is difficult to find a number of simple objects in the category $\text{Func}_\mathcal{C}(\mathcal{M}_1, \mathcal{M}_2)$. Here is a special case when this is possible to do. Let $\mathbf{1} = \oplus_{i \in I} \mathbf{1}_i$ be the decomposition of the unit object of $\mathcal{C}$ into simple summands. Let $\mathcal{C} \otimes \mathbf{1}_i$ be the full subcategory of $\mathcal{C}$ consisting of objects $X$ such that $X \otimes \mathbf{1}_i \simeq X$. It is clear that $\mathcal{C} \otimes \mathbf{1}_i$ is stable under the left multiplications by objects from $\mathcal{C}$. Thus $\mathcal{C} \otimes \mathbf{1}_i$ is a module category over $\mathcal{C}$. Note that for any module category $\mathcal{M}$ over $\mathcal{C}$ the Grothendieck group $K(\mathcal{M})$ is naturally a module over the Grothendieck ring $K(\mathcal{C})$. 
Lemma 2.14 (Lemma 3.4 in [37]). Let \( \mathcal{M} \) be a module category over a multi-fusion category \( \mathcal{C} \). Then the number of simple objects in the category \( \text{Fun}_\mathcal{C}(\mathcal{C} \otimes 1_i, \mathcal{M}) \) equals the dimension of \( \text{Hom}_{K(\mathcal{C})}(K(\mathcal{C} \otimes 1_i), K(\mathcal{M})) \).

2.4. Drinfeld center. One of the most important constructions in the theory of tensor categories is the construction of Drinfeld center, see [30, 47]. One definition in the spirit of Section 2.3 is as follows. A monoidal category \( \mathcal{C} \) acts on itself by left and right multiplications, so \( \mathcal{C} \) is a bimodule category over itself. Then the Drinfeld center \( Z(\mathcal{C}) \) of \( \mathcal{C} \) is the category of endofunctors of \( \mathcal{C} \) commuting with these actions. The composition makes \( Z(\mathcal{C}) \) into a monoidal category, but we have more structure here: \( Z(\mathcal{C}) \) is naturally a braided tensor category, see [30, 47]. It is easy to see ([53, 2.3]) that our definition is equivalent to the classical one: the objects of \( Z(\mathcal{C}) \) are pairs \((X, \phi)\) where \( X \) is an object of \( \mathcal{C} \) and \( \phi \) is an isomorphism of functors \( X \otimes \phi \simeq \phi \otimes X \) satisfying some natural conditions, see [30, 47, 51]. We have a natural forgetful functor \( Z(\mathcal{C}) \rightarrow \mathcal{C} \) sending \((X, \phi)\) to \( X \). The right adjoint of this functor (if it exists) is called the induction functor.

It is known that the Drinfeld center of an indecomposable multi-fusion category is a fusion category, see [21, Theorem 2.15] or [51]; in particular the induction functor exists in this case. Another important property is the Morita invariance: for a module category \( \mathcal{M} \) we have a natural tensor equivalence \( Z(\text{Coh}_{\mathcal{A}, \omega}(Y \times Y)) \simeq Z(\mathcal{C}) \), see [53, Corollary 2.6] or [51].

Example 2.15. Recall that \( \text{Coh}_{\mathcal{A}, \omega}(Y \times Y) \) is \( (\text{Vec}_{\omega}^\mathcal{A})^\mathcal{M} \) for suitable \( \mathcal{M} \). Thus we get a somewhat surprising result: \( Z(\text{Coh}_{\mathcal{A}, \omega}(Y \times Y)) \) does not depend on \( Y \) and is equivalent to \( Z(\text{Vec}_{\omega}^\mathcal{A}) \).

3. Harish-Chandra bimodules

3.1. Basic definitions. Let \( \mathfrak{g} \) be a complex semisimple Lie algebra. Let \( U(\mathfrak{g}) \) be the universal enveloping algebra of \( \mathfrak{g} \) and let \( Z(\mathfrak{g}) \subset U(\mathfrak{g}) \) be the center of \( U(\mathfrak{g}) \). Recall that a central character is a homomorphism \( \chi : Z(\mathfrak{g}) \rightarrow \mathbb{C} \). For a central character \( \chi \) we have two sided ideal \( U(\mathfrak{g})\text{Ker}(\chi) \subset U(\mathfrak{g}) \) and we will set \( U_\chi := U(\mathfrak{g})/U(\mathfrak{g})\text{Ker}(\chi) \).

Recall that for a \( U(\mathfrak{g}) \)-bimodule \( M \) one defines an adjoint \( \mathfrak{g} \)-action by the formula \( \text{ad}(x)m := xm - mx \); we will denote by \( M_{\text{ad}} \) the space \( M \) with this action of \( \mathfrak{g} \).

Definition 3.1. A \( U(\mathfrak{g}) \)-bimodule \( M \) is called \( \text{ad}(\mathfrak{g}) \)-algebraic if \( M_{\text{ad}} \) can be decomposed into a direct sum of finite dimensional \( \mathfrak{g} \)-modules. We say that an \( \text{ad}(\mathfrak{g}) \)-algebraic \( U(\mathfrak{g}) \)-bimodule \( M \) is Harish-Chandra bimodule if it is finitely generated as \( U(\mathfrak{g}) \)-bimodule.

Remark 3.2. The definitions of Harish-Chandra bimodules in the literature (see e.g. [20, 23, 61]) differ slightly from each other with \( \text{ad}(\mathfrak{g}) \)-algebraicity being the crucial part.
Example 3.3. Consider \( U(g) \) as \( U(g) \)–bimodule. Then the Poincaré-Birkhoff-Witt (or PBW) filtration on \( U(g) \) is stable under the adjoint action. Hence \( U(g) \) is a Harish-Chandra bimodule. Since \( U(g) \otimes U(g) \) is Noetherian any subquotient of Harish-Chandra bimodule is again Harish-Chandra bimodule. Hence \( I \) and \( U(g)/I \) are Harish-Chandra bimodules for any two sided ideal \( I \subset U(g) \). In particular \( U_\chi \) is a Harish-Chandra bimodule.

Remark 3.4. Assume that \( g \) is a complexification of the Lie algebra of a real semisimple Lie group \( G_R \) with a maximal compact subgroup \( K \). It was shown by Harish-Chandra that the study of continuous representations of \( G_R \) to a large extent reduces to the study of so-called \((g,K)\)–modules (or Harish-Chandra modules). We recall that a \((g,K)\)–module is a finitely generated \( U(g) \)–module endowed with compatible locally finite action of \( K \), see e.g. [65, 2.1(a)]; thus this is a purely algebraic object. The notion of Harish-Chandra bimodule is a special case of this when we take in the place of \( G_R \) a complex simply connected Lie group with Lie algebra \( g \) considered as a real Lie group (so the complexified Lie algebra is isomorphic to \( g \oplus g \)).

The following well known result is of crucial importance for this note:

Lemma 3.5. If \( M \) and \( N \) are Harish-Chandra bimodules, then so is \( M \otimes U(g) N \).

Proof. It is immediate from definitions that the canonical surjection \( M \otimes N \to M \otimes U(g) N \) commutes with the adjoint action. Hence \( M \otimes U(g) N \) is\( \text{ad}(g) \)–algebraic.

Let \( M_0 \subset M \) be a finite dimensional \( \text{ad}(g) \)–invariant subspace of \( M \) generating \( M \) as \( U(g) \)–bimodule. It is easy to see that \( M_0 \) generates \( M \) as left \( U(g) \)–module and as right \( U(g) \)–module. Let \( N_0 \subset N \) be a similar subspace of \( N \). Then the image of \( M_0 \otimes N_0 \) clearly generates \( M \otimes U(g) N \).

Let \( \mathcal{H} \) denote the category of Harish-Chandra bimodules (where the morphisms are homomorphisms of bimodules). The tensor product over \( U(g) \) with the obvious associativity isomorphisms makes \( \mathcal{H} \) a tensor category with the unit object \( U(g) \), see Example 3.3. However this category has some unpleasant properties: the endomorphism algebra of \( U(g) \) identifies with \( Z(g) \), so the Hom–spaces are infinite dimensional in general.

Remark 3.6. It is easy to see that for any \( K \) as in Remark 3.4 the tensor product \( M \otimes U(g) N \) of a Harish-Chandra bimodule \( M \) and \((g,K)\)–module \( N \) is again \((g,K)\)–module. In other words, the category of Harish-Chandra bimodules acts naturally on the category of \((g,K)\)–modules.

3.2. Irreducible Harish-Chandra bimodules. For a central character \( \chi \) let \( \mathcal{H}^\chi \) be the full subcategory of \( \mathcal{H} \) consisting of bimodules \( M \) such that \( M\text{Ker}(\chi) = 0 \) (in other words, the right action of \( Z(g) \) on \( M \) factorizes through \( \chi \)). A very precise description of category \( \mathcal{H}^\chi \) was given by Bernstein and Gelfand in [7]. This description is based on the category \( \mathcal{O} \) of \( g \)–modules introduced by Bernstein, Gelfand and Gelfand in [8]. We refer the reader to [25] for the basic definitions and results on the category \( \mathcal{O} \).
Recall that for any weight $\lambda$ one defines the Verma module $M(\lambda) \in \mathcal{O}$. The center $Z(\mathfrak{g})$ acts on $M(\lambda)$ via central character $\chi_\lambda$. It follows from Harish-Chandra’s theorem (see e.g. [25, 1.10]) that any central character arises in this way; moreover $\chi_\lambda = \chi_\mu$ if and only if there exists an element $w$ of the Weyl group $W$ such that $w(\lambda + \rho) - \rho = \mu$ where $\rho$ is the sum of the fundamental weights. Thus for any central character $\chi$ there exists a dominant (see [25, 3.5]) weight $\lambda$ such that $\chi = \chi_\lambda$. From now on we will restrict ourselves to the case of regular integral central characters $\chi$ (this means that $\chi = \chi_\lambda$ where $\lambda$ is a highest weight of a finite dimensional representation of $\mathfrak{g}$). For example $Z(\mathfrak{g})$ acts on the trivial $\mathfrak{g}$–module via the regular integral character $\chi_0$.

**Theorem 3.7** (Theorem 5.9 in [7]). Assume that the weight $\lambda$ is regular, integral, and dominant. The functor $M \mapsto M \otimes_{U(\mathfrak{g})} M(\lambda)$ is an equivalence of the category $\mathcal{H}^\chi$ and the subcategory of $\mathcal{O}$ consisting of modules with integral weights.

As a consequence we see that any object of the category $\mathcal{H}^\chi$ has finite length (this holds with no restrictions on $\chi$, see e.g. [28, Satz 6.30]). Also the simple objects in the category $\mathcal{H}^\chi$ are labeled by the integral weights (see [7, Proposition 5.4] for the general case).

Now we consider the left action of $Z(\mathfrak{g})$. For two central characters $\chi_1$ and $\chi_2$ let $\chi_1 \mathcal{H}^\chi_2$ be the category of Harish-Chandra bimodules $M$ such that $\text{Ker}(\chi_1) M = M \text{Ker}(\chi_2) = 0$. We also set $\mathcal{H}(\chi) := \chi \mathcal{H}^\chi$. It is clear that for $M \in \chi_1 \mathcal{H}^\chi_2$ and $N \in \chi_2 \mathcal{H}^\chi_1$ we have $M \otimes_{U(\mathfrak{g})} N \in \chi_1 \mathcal{H}^\chi_1$ and $M \otimes_{U(\mathfrak{g})} N = 0$ unless $\chi_2 = \chi_3$. In particular, the category $\mathcal{H}(\chi)$ is a tensor category with unit object $U_\chi$.

For a regular integral $\chi_2$ Theorem 3.7 implies that the category $\chi_1 \mathcal{H}^\chi_2$ is nonzero if and only if $\chi_1$ is integral. Moreover one shows using Theorem 3.7 that the categories $\mathcal{H}(\chi)$ are tensor equivalent for various regular integral $\chi$. Furthermore, we have

**Corollary 3.8.** Let $\chi = \chi_\lambda$ where $\lambda$ is regular, integral, and dominant. The simple objects of $\mathcal{H}(\chi)$ are naturally labeled by the elements of $W$: for any $w \in W$ there exists $M_w \in \mathcal{H}(\chi)$ such that $M_w \otimes_{U(\mathfrak{g})} M(\lambda)$ is the irreducible $\mathfrak{g}$–module with highest weight $w(\lambda + \rho) - \rho$.

### 3.3. Associated varieties

In this section we identify $\mathfrak{g}^*$ and $\mathfrak{g}$ via the Killing form. Let $G$ be the complex connected adjoint algebraic group with the Lie algebra $\mathfrak{g}$. An element $x \in \mathfrak{g}$ is nilpotent if $\text{ad}(x) : \mathfrak{g} \to \mathfrak{g}$ is nilpotent. Let $\mathcal{N} \subset \mathfrak{g}$ be the nilpotent cone, that is the set of all nilpotent elements. Clearly $\mathcal{N}$ is a closed $G$–invariant subvariety of $\mathfrak{g}$. It is a classical fact (see [32]) that $\mathcal{N}$ is a finite union of $G$–orbits, $\mathcal{N} = \sqcup \mathcal{O}$. The $G$–orbits appearing in $\mathcal{N}$ are called nilpotent orbits. For a nilpotent orbit $\mathcal{O}$ let $\overline{\mathcal{O}} \subset \mathcal{N}$ be its Zariski closure; clearly $\overline{\mathcal{O}}$ is a union of nilpotent orbits.

The associated varieties (see e.g. [69]) provide a convenient measure of “size” of a Harish-Chandra bimodule. Let $M$ be a Harish-Chandra bimodule. Then there exists a finite dimensional $\text{ad}(\mathfrak{g})$–invariant subspace $M_0 \subset M$ generating $M$ as a left $U(\mathfrak{g})$–module. Then the PBW filtration on $U(\mathfrak{g})$ induces a compatible filtration on $M$ (note that this filtration is $\text{ad}(\mathfrak{g})$–invariant, so it is compatible
with both left and right $U(\mathfrak{g})$–actions). The associated graded $gr\, M$ with respect to this filtration is a left module over $gr\, U(\mathfrak{g}) = S(\mathfrak{g})$. Let $V(M)$ be the support of this module in $\mathfrak{g} \cong \mathfrak{g}^* = Spec(S(\mathfrak{g}))$. The following properties of $V(M)$ are easy to verify (see e.g. [64]):

1. $V(M)$ is independent of the choice of $M_0$;
2. $V(M)$ is invariant under the adjoint action of $G$;
3. for an exact sequence $0 \to M_1 \to N \to M_2 \to 0$ we have $V(N) = V(M_1) \cup V(M_2)$;
4. $V(M_1 \otimes_{U(\mathfrak{g})} M_2) \subset V(M_1) \cap V(M_2)$;
5. for $M \in \mathcal{H}^X$ we have $V(M) \subset \mathcal{N}$.

The following result of Joseph is fundamental:

**Theorem 3.9** ([27], see also [14] [65]). Assume that $M \in \mathcal{H}$ is irreducible. Then $V(M) = \emptyset$ for some nilpotent orbit $\mathcal{O}$.

Assume that $M \in \mathcal{H}(\chi)$ where $\chi$ is regular integral. It follows from the results of [11] [2] [27] that $V(M) = \emptyset$ where $\emptyset$ is special nilpotent orbit in the sense of Lusztig, (all the nilpotent orbits are special in type $A$; however this is not the case in other types); and all special nilpotent orbits can be obtained in this way.

The theory of associated varieties is closely related with the theory of Kazhdan-Lusztig cells, see [31]. Namely let us introduce the following equivalence relation on the Weyl group $W$: $u \sim v$ if $V(M_u) = V(M_v)$. Then $W$ is partitioned into equivalence classes labeled by the special nilpotent orbits. It follows from the results of [11] [2] [27] that this partition coincides with the partition of $W$ into two sided cells as defined in [31]. In particular the set of two sided cells is in natural bijection with the set of special nilpotent orbits.

### 3.4. Cell categories.

Let $\chi$ be a regular integral central character and let $\emptyset$ be a nilpotent orbit. We define full subcategories $\mathcal{H}(\chi)_{\subset \emptyset}$ and $\mathcal{H}(\chi)_{\lt \emptyset}$ as follows: $M \in \mathcal{H}(\chi)_{\subset \emptyset}$ (respectively, $M \in \mathcal{H}(\chi)_{\lt \emptyset}$) if and only if $V(M) \subset \emptyset$ (respectively, $V(M) \subset \emptyset - \emptyset$). It follows easily from the properties of associated varieties that $\mathcal{H}(\chi)_{\subset \emptyset}$ and $\mathcal{H}(\chi)_{\lt \emptyset}$ are Serre subcategories of $\mathcal{H}(\chi)$: also $\mathcal{H}(\chi)_{\subset \emptyset}$ is closed under the tensor product and the tensor product of bimodules from $\mathcal{H}(\chi)_{\subset \emptyset}$ and $\mathcal{H}(\chi)_{\lt \emptyset}$ is contained in $\mathcal{H}(\chi)_{\lt \emptyset}$.

We define $\mathcal{H}(\chi)_{\emptyset}$ to be the Serre quotient category $\mathcal{H}(\chi)_{\lt \emptyset}/\mathcal{H}(\chi)_{\subset \emptyset}$ (note that the category $\mathcal{H}(\chi)_{\emptyset}$ is nonzero if and only if the nilpotent orbit $\emptyset$ is special). Let $\mathcal{H}(\chi)_{\emptyset} \subset \mathcal{H}(\chi)_{\emptyset}$ be the full subcategory consisting of the semisimple objects in $\mathcal{H}(\chi)_{\emptyset}$. The tensor product $\otimes_{U(\mathfrak{g})}$ descends to a well defined tensor product functor $\otimes$ on the category $\mathcal{H}(\chi)_{\emptyset}$ endowed with the associativity constraint.

**Theorem 3.10** (see [11] [54] [37]). The restriction of $\otimes$ to $\mathcal{H}(\chi)_{\emptyset} \subset \mathcal{H}(\chi)_{\emptyset}$ takes values in the subcategory $\mathcal{H}(\chi)_{\emptyset}$. Moreover, the category $\mathcal{H}(\chi)_{\emptyset}$ is an indecomposable multi-fusion category.

We will explain some ideas of the proof of Theorem 3.10 in Section 4. In the same time we will give a precise description of the cell category $\mathcal{H}(\chi)_{\emptyset}$ as a multi-fusion category. For now we will explain that the category $\mathcal{H}(\chi)_{\emptyset}$ does contain
the unit object. We recall (see e.g. [20, 1.9]) that a two sided ideal \( I \subset U(\mathfrak{g}) \) is primitive if it is the annihilator of an irreducible \( \mathfrak{g} \)-module. It follows from Schur’s lemma that for a primitive ideal \( I \) the intersection \( I \cap Z(\mathfrak{g}) = \text{Ker}(\chi) \) for some central character \( \chi \); let \( \text{Pr}_\chi \) be the set of all such primitive ideals (this set is finite; it is explicitly known in all cases thanks to the deep work of Joseph [28]).

Let \( I \in \text{Pr}_\chi \). It was proved by Joseph [27] that \( V(U(\mathfrak{g})/I) = \bar{0} \) for some special nilpotent orbit \( \bar{0} \) (this result is closely related with Theorem 3.9 see [63, Corollary 4.7]). It is also known that for any ideal \( I' \supseteq I, I' \neq I \) the dimension of \( V(U(\mathfrak{g})/I') \) is strictly smaller than the dimension of \( V(U(\mathfrak{g})/I) \), see [13, 3.6]. Therefore \( U(\mathfrak{g})/I \) contains a unique simple sub-bimodule \( M_I \); moreover \( V(U(\mathfrak{g})/I) = V(M_I) = \bar{0} \) and \( V(U(\mathfrak{g})/I)/M_I \subset \bar{0} - \bar{0} \). In other words \( U(\mathfrak{g})/I \simeq M_I \) in the category \( \bar{H}(\chi)_0 \); in particular \( U(\mathfrak{g})/I \in \bar{H}(\chi)_0 \). Also for two distinct \( I, J \in \text{Pr}_\chi \) with \( V(U(\mathfrak{g})/I) = V(U(\mathfrak{g})/J) = \bar{0} \) we have \( U(\mathfrak{g})/I \otimes_{U(\mathfrak{g})} U(\mathfrak{g})/J = U(\mathfrak{g})/(I + J) \) whence \( V(U(\mathfrak{g})/I \otimes_{U(\mathfrak{g})} U(\mathfrak{g})/J) \subset \bar{0} - \bar{0} \). Equivalently \( U(\mathfrak{g})/I \otimes_{U(\mathfrak{g})} U(\mathfrak{g})/J = M_I \otimes M_J = 0 \) in the category \( \bar{H}(\chi)_0 \).

It is well known that for a simple Harish-Chandra bimodule \( M \in \chi_1 \bar{H}_\chi \) there exist \( I \in \text{Pr}_\chi \), and \( J \in \text{Pr}_\chi \) such that \( I \) is the annihilator of \( M \) considered as a left \( U(\mathfrak{g}) \)-module and \( J \) is the annihilator of \( M \) considered as a right \( U(\mathfrak{g}) \)-module; moreover \( V(M) = V(U(\mathfrak{g})/I) = V(U(\mathfrak{g})/J) \), see e.g. [20, 7.7, 17.8]. Clearly \( U(\mathfrak{g})/I \otimes_{U(\mathfrak{g})} U(\mathfrak{g})/J = M \otimes_{U(\mathfrak{g})} U(\mathfrak{g})/J = M \). Let \( \text{Pr}_\chi(\bar{0}) \subset \text{Pr}_\chi \) consists of \( I \) with \( V(U(\mathfrak{g})/I) = \bar{0} \). It follows from the above that

\[
1 = \bigoplus_{I \in \text{Pr}_\chi(\bar{0})} U(\mathfrak{g})/I = \bigoplus_{I \in \text{Pr}_\chi(\bar{0})} M_I
\]

is the unit object of \( \bar{H}(\chi)_0 \). Again there is an important connection with the theory of Kazhdan-Lusztig cells [31]: it follows from the results of [11, 2, 38] that two elements \( u, v \in W \) are in the same left cell if and only if there exists \( 1_i \) such that \( M_u \otimes 1_i \simeq M_u \) and \( M_v \otimes 1_i \simeq M_v \). In particular the set \( \text{Pr}_\chi \) is in bijection with the set of left cells in \( W \).

4. Actions of cell categories

4.1. Whittaker modules. Let \( e \in \mathfrak{g} \) be a nilpotent element. By the Jacobson-Morozov theorem we can pick \( h, f \in \mathfrak{g} \) such that \( e, f, h \) is an \( sl_2 \)-triple, that is \( [h, e] = 2e, [h, f] = -2f, [e, f] = h \). Then \( \mathfrak{g} \) decomposes into eigenspaces for \( ad(h) \):

\[
\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i), \quad \mathfrak{g}(i) = \{ x \in \mathfrak{g} | [h, x] = ix \}.
\]

In particular \( e \in \mathfrak{g}(2) \) and \( f \in \mathfrak{g}(-2) \). Using the Killing form \( (\ , \) \) on \( \mathfrak{g} \) one defines a skew-symmetric bilinear form \( x, y \mapsto (e, [x, y]) \) on the space \( \mathfrak{g}(-1) \); it turns out that this form is non-degenerate. Pick a lagrangian subspace \( l \subset \mathfrak{g}(-1) \) and set \( m = m_l = l \oplus \bigoplus_{i \leq -2} \mathfrak{g}(i) \). Then \( \xi(x) = (x, e) \) is a Lie algebra homomorphism \( m \to \mathbb{C} \). Let \( m_\xi \) be the Lie subalgebra of \( U(\mathfrak{g}) \) spanned by \( x - \xi(x) \), \( x \in m \).
Definition 4.1 (39). We say that $\mathfrak{g}$–module is Whittaker if the action of $\mathfrak{m}_\chi$ on it is locally nilpotent.

Let $\mathcal{W}$ be the category of Whittaker modules (this is a full Serre subcategory of category of $\mathfrak{g}$–modules). We have a functor from $\mathcal{W}$ to vector spaces

$$M \mapsto \{m \in M | xm = \xi(x)m, \forall x \in \mathfrak{m}\}.$$ 

Let $U(\mathfrak{g}, e)$ be the algebra of endomorphisms of this functor; thus the functor above upgrades to a functor $\text{Sk} : \mathcal{W} \to U(\mathfrak{g}, e) \mod$. An important result proved by Skryabin [60] (see also [22] and [33]) is that this functor is an equivalence of categories. Thus we call $\text{Sk}$ the Skryabin equivalence.

Remark 4.2. The algebras $U(\mathfrak{g}, e)$ are finite $W$–algebras introduced by Premet [35]. We refer the reader to [35] for a nice survey of their properties.

A particularly important property of algebras $U(\mathfrak{g}, e)$ is that they do not depend on the choice of lagrangian subspace $I$ (more precisely the algebras defined using different choices of $I$ are canonically isomorphic), see [22]. In particular, the centralizer $Q$ of $e, h, f$ in $G$ acts naturally on $U(\mathfrak{g}, e)$, see [35, 2.6].

4.2. Irreducible finite dimensional representations of finite $W$–algebras. Let $M \in \mathcal{W}$ and $N \in \mathcal{W}$. For $x \in \mathfrak{g}$ and $m \otimes n \in M \otimes_{U(\mathfrak{g})} N$ we have $x(m \otimes n) = ad(x)m \otimes n + m \otimes xn$. The subalgebra $\mathfrak{m}$ consists of nilpotent elements, so $ad(x)$ is locally nilpotent for any $x \in \mathfrak{m}$. Hence $M \otimes_{U(\mathfrak{g})} N \in \mathcal{W}$, in other words the tensor category $\mathcal{W}$ acts on the category $\mathcal{W}$. Let $\mathfrak{g} \mathcal{W}$ be the full subcategory of $M \in \mathcal{W}$ such that $Z(\mathfrak{g})$ acts on $M$ through the central character $\chi$. Clearly the action above restrict to an action of $H(\chi)$ on $\mathfrak{g} \mathcal{W}$.

We will be interested in the set $Y = Y(\chi)$ of isomorphism classes of irreducible modules $M$ in $\mathfrak{g} \mathcal{W}$ such that $\text{Sk}(M)$ is finite dimensional. Since $\text{Sk}$ is an equivalence, $Y$ is also the set of irreducible finite dimensional representations of $W$–algebra $U(\mathfrak{g}, e)$. For any $M \in Y$ its annihilator is a primitive ideal of $U(\mathfrak{g})$; thus we get a map $\text{Ann}_\chi : Y \to \text{Pr}_\chi$. It was proved by Premet [50] that any ideal $I$ in the image of this map is contained in $\text{Pr}_\chi(\emptyset)$ where $\emptyset = Ge$ is the nilpotent orbit containing $e$. Moreover, it was conjectured by Premet and proved by Losev [34] (see also [50, 57] [23]) that any $I \in \text{Pr}_\chi(\emptyset)$ is in the image of this map. Recall that the group $Q$ acts on the algebra $U(\mathfrak{g}, e)$. Thus we get an action of $Q$ on the set $Y$. One shows that the unit component $Q^0 \subset Q$ acts trivially, so we get an action of the component group $C(e) = Q/Q^0$ on $Y$ (it is well known that the group $C(e)$ is isomorphic to the component group of the centralizer of $e$ in $G$ or, equivalently, $C(e)$ is equivariant fundamental group of the orbit $\emptyset$). It was proved by Losev [34] that each fiber of the map $\text{Ann}_\chi$ is exactly one $C(e)$–orbit in $Y$. We will seek for a precise description of these orbits. Actually there is a little bit more information here. Let $M \in Y$ and let $Q_M \subset Q$ be its stabilizer in the group $Q$. Then $Q_M$ acts projectively on $\text{Sk}(M)$, so we have a cohomology class in $H^2(Q_M, \mathbb{C}^\times)$ describing this action. The data of the set $Y$ together with $Q$–action and 2-cocycles above (which should be compatible in an obvious way) can be described as the data of
“$Q$—set of centrally extended points”, see Example 2.8 (ii). Recall that in the special case when the group $Q$ is finite, precisely the same data describe a structure of module category over $\text{Vec}_Q$ on the category $\text{Coh}(Y)$, see Example 2.8 (ii). In general, $\text{Coh}(Y)$ acquires the structure of a module category over $\text{Vec}_A$ for any finite subgroup $A \subset Q$.

Let $\chi^\text{Wh}_f \subset \chi^\text{Wh}$ be the full subcategory consisting of semisimple $N$ such that $\text{Sk}(N)$ is finite dimensional. It was proved by Losev [34] that for $M \in \mathcal{H}(\chi)_{\leq 0}$ and $N \in \chi^\text{Wh}_f$ we have $M \otimes_{U(g)} N \in \chi^\text{Wh}_f$. Moreover, $M \otimes_{U(g)} N = 0$ for $M \in \mathcal{H}(\chi)_{<0}$. Thus the category $\mathcal{H}(\chi)_{\leq 0}$ acts on $\chi^\text{Wh}_f$. On the other hand the group $Q$ acts on the category $\chi^\text{Wh}_f$ (or rather on the equivalent category of $U(g,c)$—modules) via twisting: an element $g \in Q$ sends a $U(g,c)$—module to itself with the action of $U(g,c)$ twisted by an automorphism $g$. One shows that these two actions commute. We pick a finite subgroup $A \subset Q$ which surjects on $C(e) = Q/Q^0$ and restrict the above action of $Q$ to $A$. Then the category $\chi^\text{Wh}_f$ is a module category over fusion category $\text{Vec}_A$ (note that $\chi^\text{Wh}_f \simeq \text{Coh}(Y)$, and this is the same structure of the module category as in the previous paragraph).

Since any $M \in \mathcal{H}(\chi)_{\leq 0}$ produces a functor $M \otimes_{U(g)} ? : \chi^\text{Wh}_f \rightarrow \chi^\text{Wh}_f$ commuting with the action of $\text{Vec}_A$ we get a canonical tensor functor

$$\mathcal{H}(\chi)_{\leq 0} \rightarrow \text{Fun}_{\text{Vec}_A}(\chi^\text{Wh}_f, \chi^\text{Wh}_f) = (\text{Vec}_A)^*_{\chi^\text{Wh}_f} = \text{Coh}_A(Y \times Y),$$

see Example 2.9.

**Theorem 4.3.** ([34] [37]) The functor $\mathcal{H}(\chi)_{\leq 0} \rightarrow \text{Coh}_A(Y \times Y)$ is fully faithful.

**Remark 4.4.** An important tool in the proof of Theorem 4.3 is the notion of Harish-Chandra bimodules for $W$—algebras introduced by Ginzburg [23] and Losev [34]. It is possible to replace the group $Q$ by the finite group $A$ since the action of $Q$ on $U(g,c)$ has the following property: there is embedding of the Lie algebra $q$ of $Q$ to $U(g,c)$ (considered as a Lie algebra) such that the differential of $Q$—action on $U(g,c)$ coincides with the adjoint action of $q$, see [34] 1.1(1)].

One consequence of Theorem 4.3 is the fact that the category $\mathcal{H}(\chi)_{\leq 0}$ closed under the tensor product, see [34] Corollary 1.3.2. This is a crucial step in the proof of Theorem 3.10. Moreover one shows that the module category $\chi^\text{Wh}_f$ over $\mathcal{H}(\chi)_{\leq 0}$ is indecomposable, see [37] Theorem 5.1. Thus we can apply Corollary 2.12 and get the following

**Corollary 4.5.** There is a quotient $\tilde{A}$ of $A$ and $\omega \in H^3(\tilde{A}, \mathbb{C}^*)$ such that the action of $\text{Vec}_A$ on $\chi^\text{Wh}_f$ factors through tensor functor $\text{Vec}_A \rightarrow \text{Vec}_{\tilde{A}}$ and the action on $\chi^\text{Wh}_f$ induces tensor equivalence $\mathcal{H}(\chi)_{\leq 0} \simeq \text{Coh}_{A,\omega}(Y \times Y)$.

It turns out that the quotient map $A \rightarrow \tilde{A}$ always factorizes through $A \rightarrow Q \rightarrow Q/Q^0 = C(e)$. Thus $\tilde{A}$ is naturally a quotient of the group $C(e)$. It was shown in [37] that $\tilde{A}$ coincides with the Lusztig’s quotient of $C(e)$ which was introduced by Lusztig [38] Section 13]. Also it was shown in [37] (see also [10]) that the class $\omega$ is trivial in almost all cases. However it is not trivial in the case case of nilpotent orbits corresponding to so called exceptional two sided cells, see [54].
It follows from the results in Section 5.2 below that the rational Grothendieck ring \( K(H(\chi) \otimes \mathbb{Q}) \otimes \mathbb{Q} \) is naturally a quotient of the group algebra \( \mathbb{Q}[W] \). Thus for any module category \( \mathcal{M} \) over \( H(\chi) \) the rational Grothendieck group \( K(\mathcal{M}) \otimes \mathbb{Q} \) is naturally \( W \)-module. In the special case \( \mathcal{M} = H(\chi) \otimes 1_i \) we obtain the constructive representations of \( W \), see [38, 5.29]; these representations are explicitly known. On the other hand let Spr be the Springer representation of \( W \times C(e) \) (this is top rational cohomology of the Springer fiber associated with \( e \in \mathcal{O} \) with the natural action of \( C(e) \) and the action of \( W \) defined by Springer [63]). It was proved by Dodd [18] that there is \( W \times C(e) \)-equivariant embedding \( K(\text{Coh}(\mathcal{Y})) \otimes \mathbb{Q} \subset \text{Spr} \). Using this result together with some results by Lusztig on Springer representation [43] and Lemma 2.14 the module category \( \text{Coh}(\mathcal{Y}) \) over \( \text{Vec}^e_A \) was explicitly determined in all cases in [37]. We recall that the indecomposable summands of \( \text{Coh}(\mathcal{Y}) \) are naturally labeled by the simple summands \( 1_i \) of \( 1 \in H(\chi)_0 \), see Example 2.9. Moreover, each such summand is of the form \( \mathcal{M}(B_i, \psi_i) \), see Example 4.6(ii). It was shown in [37] that we have \( C(e) \)-equivariant isomorphism \( \mathbb{Q}[A/B_i] \cong \text{Hom}_W(1) \mathbb{Q}) \otimes \text{Spr}(\mathcal{O}) \); moreover this determines the subgroups \( B_i \subset A \) uniquely up to conjugacy. It turned out that the subgroups \( B_i \) precisely coincide with Lusztig’s subgroups [39, Proposition 3.8] associated to various left cells in \( W \) (recall that the summands \( 1_i \) are labeled by the left cells contained in the two sided cell corresponding to \( \mathcal{O} \), see Section 5.4). Note that the map \( \text{Ann}_\chi : \mathcal{Y} \rightarrow \text{Pr}_\chi(\mathcal{O}) \) has the following interpretation: for any \( M \in \mathcal{Y} \) there is a unique \( 1_i \) such that \( 1_i \otimes M \simeq M \) and \( \text{Ann}_\chi(M) \) is precisely the primitive ideal \( I \) such that \( 1_i = M_I \), see Section 5.4. This implies that the fiber of the map \( \text{Ann}_\chi \) over \( I \in \text{Pr}_\chi(\mathcal{O}) \) such that \( 1_i = M_I \) is precisely \( C(e) \)-set \( A/B_i \).

Now let us assume that the two sided cell corresponding to the orbit \( \mathcal{O} \) is not exceptional (so the class \( \omega \) is trivial). It follows from the computations described above that there is one class \( \psi \in H^2(A, \mathbb{C}^*) \) such that the classes \( \psi_i \) are just inverse images of \( \psi \) under the embeddings \( B_i \subset A \), see [37, Theorem 7.4]. Equivalently, the class describing the projective action of \( Q_M \) on \( M \in \mathcal{Y} \) is the inverse image of \( \psi \) under the map \( Q_M \subset Q \rightarrow C(e) \rightarrow A \). The recent results of Losev imply that the class \( \psi \) is always trivial. To prove this we can assume that \( g \) is simple. The result certainly holds if \( H^2(A, \mathbb{C}^*) = 0 \). It follows from the classification of the nilpotent orbits that if \( H^2(A, \mathbb{C}^*) \neq 0 \) then either \( g \) is classical or \( g \) is exceptional and \( A \) is the symmetric group on four or five letters. In both cases there exists a 1-dimensional \( U(g, e) \)-module fixed by the action of \( Q \); for the classical \( g \) this is [30, Theorem 1.2] and for the exceptional \( g \) one can use the generalized Miura transform (see [35, 2.2]) since \( e \) must be even in this case. Thus we obtain the desired triviality of \( \psi \) since a projective 1-dimensional representation is equivalent to an actual representation.

Remark 4.6. (i) There is a conjectural extension of the picture above to the case of non-integral central characters \( \chi \), see [37, 7.6]. The computations suggest that in this case non-trivial 2-cocycles will arise quite often.

(ii) The results above give a description of the set \( \mathcal{Y} \) (we note that for the Lie algebras of type \( A \) such a description is due to Brundan and Kleshchev [16]). An immediate next question is what are dimensions of the spaces \( \text{Sk}(M), M \in \mathcal{Y} \), or,
equivalently, what are dimensions of the irreducible representations of $W$–algebras. A complete answer to this question is given in a recent paper [30]; remarkably in the same time some old questions about the Goldie ranks of the primitive ideals are resolved in loc. cit.

4.3. Harish-Chandra modules. It would be interesting to investigate whether the ideas above apply to the categories $\mathcal{H}^K$ of $(g, K)$–modules as in Remark 3.4. Recall that we have a Cartan decomposition $g = t + \mathfrak{p}$ where $t$ is the complexified Lie algebra of $K$. For a finitely generated $(g, K)$–module $M$ one defines its associated variety $\mathcal{V}$ consisting of modules $M$ such that $\mathcal{V}$ is a module category over $\mathcal{H}^K$. Let $\mathcal{H}^K$ be the full subcategory of $\mathcal{H}^K$ consisting of modules $M$ such that $Z(g)$ acts on $M$ via central character $\chi$. Then for $M \in \mathcal{H}^K$ we have $\mathcal{V}(M) \subset \mathfrak{p} \cap \mathcal{N}$, see [65, Corollary 5.13].

Clearly the category $\mathcal{H}(\chi)$ acts on $\mathcal{H}^K$ via $\otimes_{U(g)}$. Let us fix a nilpotent orbit $\mathcal{O}$ and consider the Serre subcategories $\mathcal{H}^K_{\mathcal{O}}$ and $\mathcal{H}_{\mathcal{O}}$ consisting of $M \in \mathcal{H}^K$ with $\mathcal{V}(M) \subset \mathfrak{p} \cap \mathcal{O}$ and $\mathcal{V}(M) \subset \mathfrak{p} \cap (\mathcal{O} - \mathcal{O})$. We can form the quotient category $\mathcal{H}^K_{\mathcal{O}} = \mathcal{H}^K_{\mathcal{O}} / \mathcal{H}^K_{\mathcal{O}}$; then $\otimes_{U(g)}$ gives us a bifunctor $\otimes : \mathcal{H}(\chi)_{\mathcal{O}} \times \mathcal{H}^K_{\mathcal{O}} \rightarrow \mathcal{H}^K$. Let $\mathcal{H}^K_{\mathcal{O}} \subset \mathcal{H}^K_{\mathcal{O}}$ be the full subcategory of semisimple objects.

**Conjecture 4.7.** For $M \in \mathcal{H}(\chi)_{\mathcal{O}}$ and $N \in \mathcal{H}^K_{\mathcal{O}}$ we have $M \otimes N \in \mathcal{H}^K_{\mathcal{O}}$.

Conjecture 4.7 would imply that $\mathcal{H}^K_{\mathcal{O}}$ is a module category over $\mathcal{H}(\chi)_{\mathcal{O}}$. By Corollary 4.5 we have a tensor equivalence $\mathcal{H}(\chi)_{\mathcal{O}} = \text{Coh}_{A, \omega}(Y \times Y)$ and by the results of Section 2.3 we have a classification of all indecomposable module categories over $\text{Coh}_{A, \omega}(Y \times Y)$. It would be very interesting to decompose the category $\mathcal{H}^K_{\mathcal{O}}$ and to identify the indecomposable summands in terms of this classification.

5. Sheaves

Let $F$ be an algebraically closed field of arbitrary characteristic. In this Section we will consider consider various classes of sheaves on algebraic varieties over $F$: $D$–modules ($F$ is of characteristic zero), constructible sheaves in the classical topology ($F = \mathbb{C}$), and constructible $l$–adic sheaves ($l$ is invertible in $F$). The corresponding categories of sheaves are $k$–linear where $k = F$, $k$ is arbitrary of characteristic zero, and $k = \mathbb{Q}_l$ respectively. Recall that the theories of such sheaves are parallel up to some extent. Thus we will not specify the kind of sheaves we deal with below unless this is necessary; the results are parallel in all three setups.

5.1. Convolution and Hecke algebra. Let $G$ be a semisimple algebraic group over $F$ of the same Dynkin type as $g$. Let $\mathcal{B}$ be the flag variety of $G$. We recall that $\mathcal{B}$ is a projective variety which is a homogeneous space for $G$; furthermore the Bruhat decomposition gives a canonical bijection between $G$–orbits on $\mathcal{B} \times \mathcal{B}$ and the Weyl group $W$. Let $D^b_G(\mathcal{B} \times \mathcal{B})$ be the suitable $G$–equivariant derived category of sheaves on $\mathcal{B} \times \mathcal{B}$, see e.g. [11, 2.2]. The category $D^b_G(\mathcal{B} \times \mathcal{B})$ contains a natural abelian subcategory $\mathcal{P}$ consisting of $D$–modules or perverse sheaves. The simple objects in the category $\mathcal{P}$ are the intersection cohomology complexes of closures
of \(G\)-orbits on \(B \times B\). This gives a natural bijection \(w \rightarrow I_w\) between \(W\) and the isomorphism classes of simple objects in \(\mathcal{P}\).

The category \(D^\omega(B \times B)\) has a natural monoidal structure with respect to tensor product given by convolution, see e.g. [11, 2.4] (this construction is parallel to Example 2.7 (iii)). It follows from the Decomposition Theorem [4] that the convolution \(I_u \ast I_v\) is isomorphic to a direct sum of shifted \(I_w\):

\[
I_u \ast I_v \cong \bigoplus_{w \in W} n_{u,v}(i)I_w[i]
\]

where the multiplicities \(n_{u,v}(i) \in \mathbb{Z}_{\geq 0}\). Let \(K(\mathcal{P})\) be the algebra over \(\mathbb{Z}[t, t^{-1}]\) which encodes the multiplicities \(n_{u,v}(i)\) above: the algebra has a basis \(c_w\) and \(c_{u} \cdot c_{v} = \sum_{w \in W} n_{u,v}(i)c_{w} t^{i}\). It is a classical result that the algebra \(K(\mathcal{P})\) together with its basis \(\{c_{w}\}\) identifies with the Hecke algebra together with the Kazhdan-Lusztig basis, see e.g. [63, 2.5]. In particular, the multiplicities \(n_{u,v}(i)\) are computable in principle.

Lusztig defined (see [39]) the asymptotic Hecke algebra \(J\) in the following way: for \(w \in W\) let \(a(w) = \max\{i \in \mathbb{Z} | n_{u,v}(i) \neq 0\}\) for some \(u, v \in W\). Let \(J\) be a free \(\mathbb{Z}\)-module with basis \(t_{w}, w \in W\) endowed with multiplication \(t_{u}t_{v} = \sum_{w \in W} n_{u,v}(a(w))t_{w}\). It was shown by Lusztig that this multiplication is associative and has a unit. Moreover there is a canonical isomorphism of associative algebras \(\mathbb{Q}[W] \simeq J \otimes \mathbb{Q}\), see [39, 3.2]. Furthermore, for any subset \(T \subset W\) let \(J_{T}\) be the abelian subgroup of \(J\) spanned by \(t_{u}, u \in T\). It is easy to see that there is a finest partition \(W = \sqcup C\) such that the decomposition \(J = \bigoplus C J_{C}\) is a direct sum of algebras. This partition is known to coincide with partition of \(W\) into two sided Kazhdan-Lusztig cells, see [39, 3.1]. It is also known that the function \(a\) takes a constant value on any two sided cell \(C\); we will denote this value by \(a(C)\).

The constructions above was categorified in [13]. Namely, for any two sided cell \(C\) let \(\mathcal{J}_{C} \subset \mathcal{P}\) be the full subcategory consisting of direct sums \(I_w, w \in C\). The category \(\mathcal{J}_{C}\) has a monoidal structure given by the truncated convolution \(\bullet\), see [13]. For example

\[
I_u \bullet I_v \cong \bigoplus_{w \in C} n_{u,v}(a(C))I_w.
\]

Hence the assignment \(I_w \mapsto t_{w}\) induces isomorphism of based rings \(K(\mathcal{J}_{C}) \simeq J_{C}\). It was shown in [10] (see also [14]) that the category \(\mathcal{J}_{C}\) is rigid. As a consequence \(\mathcal{J}_{C}\) is an indecomposable multi-fusion category.

5.2. \(D\)-modules and Harish-Chandra bimodules. In this section we assume that \(F\) is of characteristic zero. The Beilinson-Bernstein theorem [3] is a fundamental result in representation theory of \(g\). It states that the category of \(U(g)\)-modules with the trivial central character \(\chi = \chi_{0}\) is equivalent to the category of \(D\)-modules on \(B\). As a consequence one deduces that the category of Harish-Chandra bimodules \(H(\chi_{0})\) is equivalent to the category of \(D\)-modules \(\mathcal{P}\). However the Beilinson-Bernstein equivalence is not a tensor equivalence. Luckily it was shown in [3] that a composition of the Beilinson-Bernstein equivalence and a long intertwining functor has a natural structure of tensor functor. Using a
suitable truncation of this functor the following result was shown in [11 Corollary 4.5(b)]:

**Theorem 5.1.** Let $C$ be the two-sided cell corresponding to a special nilpotent orbit $O$ (see Section 3.3). Then there is a natural tensor equivalence $\mathcal{H}(\chi_0)_0 \simeq J_C$.

Theorem 5.1 gives a useful information on the Grothendieck ring $K(\mathcal{H}(\chi_0)_0)$ (we recall that for a regular integral character $\chi$ the category $\mathcal{H}(\chi)_0$ is tensor equivalent to $\mathcal{H}(\chi_0)_0$, see Section 3.2). In particular we can consider $K(J_C \otimes 1) \otimes \mathbb{Q}$ as $W$–modules; these $W$–representations are precisely the constructible representations discussed in *loc. cit.*

Conversely Theorem 5.1 combined with Corollary 4.5 gives an explicit description of $J_C$ as $\text{Coh}_{\bar{A}}(Y \times Y)$ for the $D$–module version of the category $J_C$. This implies similar description of $J_C$ for other categories of sheaves under the assumption that the ground field $F$ has characteristic 0. It was shown in [10] (see also [54] for the case of exceptional two sided cells) that the same description holds over a field $F$ of arbitrary characteristic.

**Remark 5.2.** Theorem 5.1 was inspired by closely related results of Joseph [29]. Also a similar and related connection between Kazhdan-Lusztig cells and Harish-Chandra bimodules is contained in the work of Mazorchuk and Stroppel [48].

5.3. Drinfeld center and character sheaves. Lusztig introduced a very important class of *character sheaves* on the group $G$, see [40]. We recall the definition in the special case of *unipotent character sheaves*. Let

$$X = \{(b, b', g) \in \mathcal{B} \times \mathcal{B} \times G | gb = b'\}$$

we have two projections $f : X \to \mathcal{B} \times \mathcal{B}$, $f(b, b', g) = (b, b')$ and $\pi : X \to G$, $\pi(b, b', g) = g$. Note that group $G$ acts on itself by conjugations and on $X$ via $h \cdot (b, b', g) = (hb, hb', hgh^{-1})$ and both maps $f, \pi$ are $G$–equivariant. Thus we have a functor $\Gamma : D^b_G(\mathcal{B} \times \mathcal{B}) \to D^b_G(G)$, $\Gamma = \pi_! f^s$. It follows from the Decomposition Theorem that $\Gamma(I_w), w \in W$ is isomorphic to a direct sum of shifted simple $G$–equivariant sheaves on $G$; a simple $G$–equivariant sheaf is called a unipotent character sheaf if it appears in such decomposition (possibly with some shift). Let $\mathcal{U}$ be the set of isomorphism classes of unipotent character sheaves on $G$; clearly this is a finite set.

One observes that the functor $\Gamma$ above has formal properties similar to the induction functor from the monoidal category $D^b_G(\mathcal{B} \times \mathcal{B})$ to its Drinfeld center, see Section 2.4. Moreover, it is possible to identify a suitable version of the Drinfeld center of $G$–equivariant sheaves on $\mathcal{B} \times \mathcal{B}$ with suitable category of character sheaves. This was done in [11] using the abelian tensor category of Harish-Chandra bimodules and in [6] using suitable infinity categories. Furthermore applying a suitably truncated version of the same idea to the categories $J_C$, the following result was proved in [11] (for the field $F$ of characteristic zero) and in [45] (for the field $F$ of arbitrary characteristic):
Theorem 5.3. There is a partition $U = \sqcup C \subset U_C$ such that the Drinfeld center of the category $J_C$ is naturally equivalent to the category of sheaves on $G$ which are direct sums of objects from $U_C$. In particular we have a bijection

$$U_C \leftrightarrow \{\text{simple objects of the Drinfeld center of } J_C\}.$$

The sets $U_C$ were defined by Lusztig in [41, Section 16]. Recall that the category $J_C$ is tensor equivalent to $\text{Coh}_{\mathbb{A},\omega}(Y \times Y)$. Hence the Drinfeld center of $J_C$ is equivalent to the Drinfeld center of $\text{Vec}_{\mathbb{A}}^\omega$, see Example 2.15. The resulting bijection between $U_C$ and simple objects of $Z(\text{Vec}_{\mathbb{A}}^\omega)$ conjecturally coincides with Lusztig’s one from [42, 17.8.3] which gives us a new approach to Lusztig’s classification of character sheaves. On the other hand in [54] character sheaves were used in order to determine the associativity constraint in the categories $J_C$ for exceptional cells $C$.

5.4. Some generalizations. Many constructions described in this paper extend to the case when the Weyl group $W$ is replaced by an arbitrary Coxeter group. An important special case of the affine Weyl groups was considered in [9, 13] following conjectures made by Lusztig. In this case the counterparts of the cell categories are in one to one correspondence with all nilpotent orbits of $g$ and are of the form $\text{Coh}_Q(Y \times Y)$ where the reductive group $Q$ is the same as in Section 4.1 (note that the resulting categories are typically not multi-fusion categories since they have infinitely many simple objects). The set $Y$ has a natural interpretation in terms of non-restricted representations of $g$ over fields of positive characteristic, see [12].

Using recent deep results by Elias and Williamson [20] on Soergel bimodules Lusztig defined in [45, Section 10] the counterparts of the cell categories for an arbitrary Coxeter group $W$ (note that these categories sometimes are not even tensor categories since they lack the unit object; however this is not very serious). It would be very interesting to identify the resulting categories. For example in the case of the dihedral group of order 10 one finds a cell category which contains a fusion subcategory with two simple objects $1, X$ and the tensor product $X \otimes X = 1 \oplus X$. This implies that the cell category is not of the form $\text{Coh}_{\mathbb{A},\omega}(Y \times Y)$ in this case.

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