ASYMPTOTIC LIMIT AND DECAY ESTIMATES FOR A CLASS OF DISSIPATIVE LINEAR HYPERBOLIC SYSTEMS IN SEVERAL DIMENSIONS

THINH TIEN NGUYEN
Gran Sasso Science Institute
Department of Mathematics
Viale Francesco Crispi 7, 67100 - L’Aquila, Italy

(Communicated by Rinaldo M. Colombo)

Abstract. In this paper, we study the large-time behavior of solutions to a class of partially dissipative linear hyperbolic systems with applications in, for instance, the velocity-jump processes in several dimensions. Given integers $n, d \geq 1$, let $A := (A^1, \ldots, A^d) \in (\mathbb{R}^{n \times n})^d$ be a matrix-vector and let $B \in \mathbb{R}^{n \times n}$ be not necessarily symmetric but have one single eigenvalue zero, we consider the Cauchy problem for $n \times n$ linear systems having the form

$$\partial_t u + A \cdot \nabla_x u + Bu = 0, \quad (x, t) \in \mathbb{R}^d \times \mathbb{R}_+.$$ 

Under appropriate assumptions, we show that the solution $u$ is decomposed into $u = u^{(1)} + u^{(2)}$ such that the asymptotic profile of $u^{(1)}$ denoted by $U$ is a solution to a parabolic equation, $u^{(1)} - U$ decays at the rate $t^{-\frac{1}{2} \left(\frac{1}{q} - \frac{1}{p}\right) - \frac{1}{2}}$ as $t \to +\infty$ in any $L^p$-norm and $u^{(2)}$ decays exponentially in $L^2$-norm, provided $u(\cdot, 0) \in L^q(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ for $1 \leq q \leq p \leq \infty$. Moreover, $u^{(1)} - U$ decays at the optimal rate $t^{-\frac{1}{2} \left(\frac{1}{q} - \frac{1}{p}\right) - 1}$ as $t \to +\infty$ if the original system satisfies a symmetry property. The main proofs are based on the asymptotic expansions of the fundamental solution in the frequency space and the Fourier analysis.

1. Introduction. Consider the Cauchy problem for the partially dissipative linear hyperbolic systems

$$\begin{cases}
\partial_t u + A \cdot \nabla_x u + Bu = 0, & (x, t) \in \mathbb{R}^d \times \mathbb{R}_+, \\
u(x, 0) = u_0(x),
\end{cases}$$

(1)

where $A = (A^1, \ldots, A^d) \in (\mathbb{R}^{n \times n})^d$ and $B \in \mathbb{R}^{n \times n}$ are not necessarily symmetric. The system (1) can be regarded as the discrete-velocity models of kinetic equations, where $A$ determines the velocities of particles moving in a gas and $B$ gives the transition rates of the velocities after collisions among the particles. For instance, such type of dissipative linear systems arises in the Goldstein–Kac model [6, 8] and the model of neurofilament transport in axons [5].

The large-time behavior of the solution $u$ to the system (1) has been established in terms of decay estimates for years. It follows from [18] that under appropriate

2010 Mathematics Subject Classification. Primary: 35L45; Secondary: 35C20.
Key words and phrases. Large-time behavior, dissipative linear hyperbolic systems, asymptotic expansions, multi-dimensional space, decay estimates.
assumptions, if $u$ is a solution to the system (1) with the initial data $u_0 \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, then for some positive constants $c$ and $C$, one has

$$\|u\|_{L^2} \leq C(1 + t)^{-\frac{d}{4}}\|u_0\|_{L^1} + Ce^{-ct}\|u_0\|_{L^2}, \quad \forall t > 0. \tag{2}$$

Moreover, the estimate (2) was generalized in [3], where $B$ can be written in the conservative-dissipative form $B = \text{diag}(O, D)$ with positive definite matrix $D$ (not necessarily symmetric). The authors in [3] also showed that the conservative part $u^{(1)}$ of $u$ which is governed by the system (1) reduced to the equilibrium manifold $\ker B$ satisfies the estimate

$$\|u^{(1)} - U\|_{L^p} \leq Ct^{-\frac{d}{2}(1 - \frac{1}{p}) - \frac{1}{2}}\|u_0\|_{L^1}, \quad \forall p \geq \min\{d, 2\}, t \geq 1, \tag{3}$$

where $U$ is a solution to a parabolic system given by applying the Chapman–Enskog expansion method, while the dissipative part $u^{(2)}$ of $u$ which is the remainder decays exponentially in $L^2(\mathbb{R}^d)$. A more exact asymptotic parabolic-limit $U$ of $u$ was established for a class of the generalized Goldstein–Kac system in [12], where the matrix $B$ is symmetric. The solution $u$ to the system (1) then satisfies the estimate

$$\|u - U\|_{L^2} \leq C t^{-\frac{d}{2} - \frac{1}{2}}\|u_0\|_{L^1 \cap L^2}, \quad \forall t \geq 1, \tag{4}$$

where $U$ is obtained based on an exhaustive analysis of the dispersion relation and on the application of a variant of the Kirchhoff’s matrix tree theorem from the graph theory. Recently, [13] has optimized the above decay estimates for the solution $u$ to the system (1) in one dimension $d = 1$ with an explicit parabolic-limit also denoted by $U$ and a corrector denoted by $V$, namely, one has

$$\|u - U - V\|_{L^p} \leq C t^{-\frac{1}{2}(1 - \frac{1}{p}) - \delta}\|u_0\|_{L^3}, \quad \forall 1 \leq q \leq p \leq \infty, t \geq 1, \tag{5}$$

where $\delta \in \{1/2, 1\}, U$ which solves a parabolic system arising in the low-frequency analysis decays diffusively and $V$ which solves a hyperbolic system arising in the high-frequency analysis decays exponentially. The decay estimate (5) is remarkable since it holds for general $p$ and $q$ ranging over $[1, \infty]$. Such kind of decay estimates is very well-known e.g. the $L^p$-$L^q$ decay estimate for the linear damped wave equation in [7, 11, 14, 15].

To obtain (5) in the one-dimensional case $d = 1$, one primarily considers the asymptotic expansions of the fundamental solution to the system (1) in the Fourier space which naturally produce the time-asymptotic profile, where the Fourier space is divided into the low frequency, the intermediate frequency and the high frequency. Then, by an interpolation argument once the $L^\infty$-$L^1$ estimate and the $L^p$-$L^p$ estimate for $1 \leq p \leq \infty$ are accomplished, one obtains the desired $L^p$-$L^q$ estimate for any $1 \leq q \leq p \leq \infty$. The same strategy will be applied to the system (1) in several dimensions $d \geq 2$ in this paper. Nevertheless, difficulties occur as the dimension $d$ increases. For instance, as mentioned in [3], one cannot expect that the estimate

$$\|u\|_{L^1} \leq C\|u_0\|_{L^1} \tag{6}$$

holds in general since for large time, $L_0 u$, where $L_0$ is the left eigenvector associated with the eigenvalue $0$ of $B$, behaves as the solution $\omega$ to the reduced system

$$\partial_t \omega + L_0 A R_0 \cdot \nabla_x \omega = 0, \quad (x, t) \in \mathbb{R}^d \times \mathbb{R}_+,$$

where $R_0$ is the right eigenvector associated with the eigenvalue $0$ of $B$, and thus it is known in [4] that (6) is not true in general. The estimate (6) in fact depends strongly on a uniform parabolic operator. Nonetheless, this obstacle can be defeated if $d = 1$ as in [3, 13] or if $0$ is a simple eigenvalue of $B$ since the scalar kernel associated
with the system (7) allows us to obtain (6) as we will see in this paper. Another difficulty arises in the high-frequency analysis due to the lack of integrability and the fact that one cannot perform a uniform expansion of the fundamental solution to the system (1) as the dimension $d$ increases. Hence, the corrector $V$ as in (5) cannot be obtained trivially.

The aim of this paper is to study the $L^p$-$L^q$ decay estimate for the conservative part $u(1)$ of the solution $u$ to the system (1) in several dimensions $d \geq 2$ for general $p$ and $q$ in $[1, \infty]$ in order to generalize (2), (3) and (4), where $B$ is not necessarily symmetric but has one single eigenvalue zero. The $L^p$-$L^q$ estimate as in (5) for the multi-dimensional case $d \geq 2$ is still a challenge for the author.

For $x = (x_1, \ldots, x_d) \in \mathbb{C}^d$, consider the $n \times n$ operators

$$E(x) := B + A(x), \quad A(x) := A \cdot x = \sum_{j=1}^{d} A^j x_j,$$

where $A = (A^1, \ldots, A^d) \in (\mathbb{R}^{n \times n})^d$ and $B \in \mathbb{R}^{n \times n}$. We start with the following reasonable assumptions.

Condition A. [Hyperbolicity] $A = A(w)$ for $w \in S^{d-1}$ is uniformly diagonalizable with real linear eigenvalues i.e. there is an invertible matrix $R = R(w)$ for $w \in S^{d-1}$ such that

$$\sup_{w \in S^{d-1}} |R(w)||R^{-1}(w)| < +\infty$$

for any matrix norm and $R^{-1}AR$ is a diagonal matrix whose nonzero entries are real linear in $w \in S^{d-1}$.

Condition R. [Diagonalizing matrix] There is a matrix $R$ uniformly diagonalizing $A$ such that $R^{-1}BR$ is a constant matrix.

Condition B. [Partial dissipation] The spectrum of $B$ is decomposed into $\sigma(B) = \{0\} \cup \sigma_0$ where $0$ is simple and $\sigma_0 \subseteq \{z \in \mathbb{C} : \text{Re } z > 0\}$.

Moreover, the requisite condition for the decay of the solution $u$ to the system (1) which is strictly related to the Shizuta–Kawashima condition: the eigenvectors of $A(x)$ do not belong to the kernel of $B$ for any $x \neq 0$ (see [10, 17, 19] and therein) is given by

Condition D. [Uniform dissipation] There is a constant $\theta > 0$ such that for any eigenvalue $\lambda = \lambda(ik)$ of $E = E(ik)$ in (8) for $k \in \mathbb{R}^d$, one has

$$\text{Re } \lambda(ik) \geq \frac{\theta |k|^2}{1 + |k|^2}, \quad \forall k \neq 0 \in \mathbb{R}^d.$$

The requirement of the linearity of the eigenvalues of the matrix $A$ satisfying the condition A and the existence of the matrix $R$ satisfying the condition R can be omitted by considering the dissipative structures proposed in [3, 18]. Nonetheless, the structures in [3, 18] require that the system (1) is Friedrich symmetrizable while in our case, the matrix $A$ is only uniformly diagonalizable. The advantage of the linearity of the eigenvalues of the matrix $A$ and the existence of the matrix $R$ is that one can construct the high-frequency asymptotic expansion of $E$ in (8) by subtracting a suitable Lebesgue measure zero set.
We consider the asymptotic parabolic-limit $U$ of the solution $u$ to the system (1). Let $\Gamma$ be an oriented closed curve in the resolvent set of $B$ such that it encloses zero except for the other eigenvalues of $B$. One sets

$$P_0^{(0)} := -\frac{1}{2\pi i} \int_{\Gamma} (B - zI)^{-1} \, dz, \quad Q_0^{(0)} := \frac{1}{2\pi i} \int_{\Gamma} z^{-1} (B - zI)^{-1} \, dz,$$

which are the eigenprojection and the reduced resolvent coefficient associated with the eigenvalue zero of $B$. We consider the Cauchy problem

$$\begin{cases}
\partial_t U + c \cdot \nabla x U - \text{div} (D \nabla x U) = 0, & (x, t) \in \mathbb{R}^d \times \mathbb{R}_+,
U(x, 0) = P_0^{(0)} u_0(x),
\end{cases}$$

(10)

where $c = (c_h) \in \mathbb{R}^d$ and positive definite $D = (D_{h\ell}) \in \mathbb{R}^{d \times d}$ with scalar entries

$$c_h := \text{tr} (A^h P_0^{(0)}), \quad D_{h\ell} := \frac{1}{2} \text{tr} (A^h P_0^{(0)} A^\ell Q_0^{(0)} + A^h Q_0^{(0)} A^\ell P_0^{(0)}).$$

(11)

**Theorem 1.1** ($L^p$-$L^q$ decay estimates). Let $u$ be a solution to the Cauchy problem (1) with the initial data $u_0 \in L^q(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ for $1 \leq q \leq \infty$. Under the assumptions $A$, $R$, $B$ and $D$, the solution $u$ is decomposed into

$$u(x, t) = u^{(1)}(x, t) + u^{(2)}(x, t),$$

(12)

where

$$u^{(1)}(x, t) := \mathcal{F}^{-1} (e^{-E(i\xi)t} P_0(\xi) \chi(\xi)) * u_0(x)$$

and $u^{(2)}$ is the remainder, where $P_0$ is the eigenprojection which is associated with the eigenvalue of $E$ in (8) converging to 0 as $|\xi| \to 0$ and $\chi$ is a cut-off function valued in $[0, 1]$ with support contained in the ball $B(0, \varepsilon) \subset \mathbb{R}^d$ for small $\varepsilon > 0$. Moreover, for any $1 \leq q \leq p \leq \infty$, one has

$$\|u^{(1)} - U\|_{L^p} \leq Ct^{-\frac{d}{2}(\frac{1}{2} - \frac{1}{q}) - \frac{1}{p}} \|u_0\|_{L^q}, \quad \forall t \geq 1,$$

(13)

where $U$ is a solution to (10) with the initial data $U_0 \in L^q(\mathbb{R}^d)$, and

$$\|u^{(2)}\|_{L^2} \leq Ce^{-ct} \|u_0\|_{L^2}, \quad \forall t > 0,$$

(14)

for some constants $c > 0$ and $C > 0$.

In the case where the solution $u$ to the system (1) has finite speed of propagation, since the fundamental solution associated with $u$ has compact support contained in the wave cone $\{(x, t) \in \mathbb{R}^d \times \mathbb{R} : |x/t| \leq C\}$ for some constant $C > 0$, one can decompose $u$ into $u = u^{(1)} + u^{(2)}$, where

$$u^{(1)}(x, t) := \mathcal{F}^{-1} (e^{-E(i\xi)t} \chi(\xi)) * u_0(x)$$

and $u^{(2)}$ is the remainder, where $\chi$ is a cut-off function valued in $[0, 1]$ with support contained in the ball $B(0, \rho) \subset \mathbb{R}^d$ for any $\rho > 0$, and the estimates (13) and (14) still hold. This fact will be proved in the subsequent sections. For instance, it is the case where the system (1) is Friedrich symmetrizable. Nonetheless, in one-dimensional space where $d = 1$, the case $|x/t| > C$ can be treated since the Cauchy integral theorem holds for the whole complex plane, and thus one can use the estimates for the high-frequency asymptotic expansion of the fundamental solution after changing paths of integrals of holomorphic functions (see [13]).

Moreover, consider the one-dimensional $2 \times 2$ linear Goldstein–Kac system

$$\begin{cases}
\partial_t u_1 - \partial_x u_1 = -\frac{1}{2} u_1 + \frac{1}{2} u_2,
\partial_t u_2 + \partial_x u_2 = \frac{1}{2} u_1 - \frac{1}{2} u_2,
\end{cases}$$

$$(x, t) \in \mathbb{R} \times \mathbb{R}_+.$$
It can be checked easily that $w := u_1 + u_2$ satisfies the linear damped wave equation
\[
\begin{aligned}
\partial_t^2 w - \partial_{xx}^2 w + \partial_t w &= 0, \quad (x, t) \in \mathbb{R} \times \mathbb{R}_+, \\
(w(x, 0) &= w_0(x), \\
\partial_t w(x, 0) &= w_1(x),
\end{aligned}
\]
where $w_0$ and $w_1$ are appropriate initial data. It then follows from \([11]\) that
\[
\left\| w - \phi - e^{-\frac{1}{2} \frac{w_0(x + t) + w_0(x - t)}{2}} \right\|_{L^p} \leq Ct^{-\frac{1}{2} \left(\frac{1}{q} - \frac{1}{p}\right) - 1} \| (w_0, w_1) \|_{L^q}
\]  
(15)
for any $1 \leq q \leq p \leq \infty$ and $t \geq 1$, where $\phi$ is a solution to the heat equation
\[
\begin{aligned}
\partial_t \phi - \partial_{xx} \phi &= 0, \quad (x, t) \in \mathbb{R} \times \mathbb{R}_+, \\
\phi(x, 0) &= w_0(x) + w_1(x).
\end{aligned}
\]

Without regarding the exponentially decaying term in (15), there is a difference of $1/2$ between the decay rates (13) and (15). The difference can be explained by a symmetry property that the one-dimensional $2 \times 2$ linear Goldstein–Kac system possesses i.e. under $x \mapsto -x$, $(u_1, u_2)$ is also a solution to the original system up to a linear change of coordinates. Such kind of symmetry properties is already studied in \([13]\) based on the existence of an invertible matrix $S$ commuting with $B$ and anti-commuting with the matrix $A = A^\dagger$ in the one-dimensional dissipative linear hyperbolic systems. More general, in the multi-dimensional case $d \geq 2$, this property is given by

**Condition 5. [Symmetry]** There is an invertible matrix $S = S(w)$ for $w \in S^{d-1}$ such that
\[
SB = BS, \quad SA = -AS,
\]
where $A = A(w)$ for $w \in S^{d-1}$ is given by (8).

In practice, it is easy to check the condition $S$ if there is a constant invertible matrix $S$ satisfying $SB = BS$ and $SA^j = -A^jS$ for all $j \in \{1, \ldots, d\}$ since $A(w) = \sum_{j=1}^d A^j w_j$ by definition.

We will show that under the conditions $B$, $D$, $S$, the decay rate in the estimate (13) increases. We primarily refine the asymptotic profile $U$.

With the coefficients $P^{(0)}_0$ and $Q^{(0)}_0$ in (9) and $D$ in (11), we consider the Cauchy problem
\[
\begin{aligned}
\partial_t U - \text{div} \left( D \nabla_x U \right) &= 0, \quad (x, t) \in \mathbb{R}^d \times \mathbb{R}_+, \\
U(x, 0) &= P^{(0)}_0 u_0(x) + P^{(1)}_0 \cdot \nabla_x u_0(x),
\end{aligned}
\]  
(16)
where $P^{(1)}_0 = (P^{(1)}_{0k}) \in (\mathbb{R}^{n \times n})^d$ with matrix entries
\[
P^{(1)}_{0k} := -P^{(0)}_0 A^h Q^{(0)}_0 - Q^{(0)}_0 A^h P^{(0)}_0.
\]  
(17)

**Theorem 1.2 (Optimal decay rate).** Under the same hypotheses in Theorem 1.1, if the condition $S$ holds in addition, the solution $u$ is also decomposed into $u = u^{(1)} + u^{(2)}$ as in (12) such that for any $1 \leq q \leq p \leq \infty$ and $t \geq 1$, one has
\[
\| u^{(1)} \|_{L^p} < C t^{-\frac{2}{2} \left(\frac{1}{q} - \frac{1}{p}\right) - 1} \| u_0 \|_{L^q},
\]  
(18)
where $U$ is a solution to (16) with the initial data $U_0$. 

DISSIPATIVE LINEAR HYPERBOLIC SYSTEMS
The paper is organized as follows. Section 2 is devoted to proofs and examples of Theorem 1.1 and Theorem 1.2, where the proofs are based on the estimates obtained in Section 5. In order to prove the estimates in Section 5, we primarily invoke some useful tools from the Fourier analysis and the perturbation analysis in Section 3. With these tools, we construct the asymptotic expansions of the operator $E$ in (8) in Section 4 in order to obtain the asymptotic expansions of the fundamental solution to the system (1) which are useful to prove the estimates in Section 5.

Notations and definitions. We introduce here the notations and definitions which will be used frequently throughout this paper. See [1, 2] for more details.

Definition 1.3. Let $u$ be a function from $\mathbb{R}^d$ to a Banach space equipped with the norm $| \cdot |$, we define the Lebesgue spaces $L^p(\mathbb{R}^d)$ for $1 \leq p \leq \infty$ consisting of functions $u$ which satisfy
\[
\|u\|_{L^p} := \left( \int_{\mathbb{R}^d} |u(x)|^p \, dx \right)^{1/p} < +\infty, \quad 1 \leq p < \infty,
\]
and satisfy
\[
\|u\|_{L^\infty} := \text{ess sup}_{\mathbb{R}^d} |u(x)| < +\infty.
\]

Let $\alpha \in \mathbb{N}^d$ be the multi-index $\alpha := (\alpha_1, \ldots, \alpha_d)$ with $\alpha_j \in \mathbb{N}$. One denotes by
\[
\partial^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}},
\]
where $|\alpha| := \alpha_1 + \cdots + \alpha_d$, the partial derivatives of a smooth function $f$ on $\mathbb{R}^d$. Then, for smooth scalar functions $f$ and $g$ on $\mathbb{R}^d$, we have the Leibniz rule
\[
\partial^\alpha (fg) = \sum_{\nu \leq \alpha} \binom{\alpha}{\nu} \partial^\nu f \partial^{\alpha - \nu} g,
\]
where
\[
\binom{\alpha}{\nu} := \frac{\alpha!}{\nu! (\alpha - \nu)!} = \frac{\alpha_1! \cdots \alpha_d!}{\nu_1! \cdots \nu_d! (\alpha_1 - \nu_1)! \cdots (\alpha_d - \nu_d)!}
\]
is the multi-index binomial coefficient, $\nu \leq \alpha$ means that $\nu_j \leq \alpha_j$ for all $j \in \{1, \ldots, d\}$ and the difference $\alpha - \nu$ is defined by
\[
\alpha - \nu := (\alpha_1 - \nu_1, \ldots, \alpha_d - \nu_d).
\]

Definition 1.4. The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is the set of smooth functions $u$ on $\mathbb{R}^d$ such that for any $k \in \mathbb{N}$, we have
\[
\|u\|_{k, \mathcal{S}} := \sup_{|\alpha| \leq k, x \in \mathbb{R}^d} (1 + |x|)^k |\partial^\alpha u(x)| < +\infty.
\]

One denotes by $\mathcal{S}'(\mathbb{R}^d)$ the dual space of $\mathcal{S}(\mathbb{R}^d)$ and $u \in \mathcal{S}'(\mathbb{R}^d)$ is called a tempered distribution. For $u \in \mathcal{S}$, the Fourier transform $\hat{u}(k) = \mathcal{F}(u(x))$ is defined by the integral
\[
\hat{u}(k) := \int_{\mathbb{R}^d} e^{-ix \cdot k} u(x) \, dx,
\]
where $\cdot$ is the usual scalar product on $\mathbb{R}^d$, and the inverse Fourier transform of $\hat{u}$ denoted by $u(x) = \mathcal{F}^{-1}(\hat{u}(k))$ is given by
\[
u \leq \alpha
\]
\[
\binom{\alpha}{\nu} := \frac{\alpha!}{\nu! (\alpha - \nu)!} = \frac{\alpha_1! \cdots \alpha_d!}{\nu_1! \cdots \nu_d! (\alpha_1 - \nu_1)! \cdots (\alpha_d - \nu_d)!}
\]
is the multi-index binomial coefficient, $\nu \leq \alpha$ means that $\nu_j \leq \alpha_j$ for all $j \in \{1, \ldots, d\}$ and the difference $\alpha - \nu$ is defined by
\[
\alpha - \nu := (\alpha_1 - \nu_1, \ldots, \alpha_d - \nu_d).
\]

Definition 1.4. The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is the set of smooth functions $u$ on $\mathbb{R}^d$ such that for any $k \in \mathbb{N}$, we have
\[
\|u\|_{k, \mathcal{S}} := \sup_{|\alpha| \leq k, x \in \mathbb{R}^d} (1 + |x|)^k |\partial^\alpha u(x)| < +\infty.
\]

One denotes by $\mathcal{S}'(\mathbb{R}^d)$ the dual space of $\mathcal{S}(\mathbb{R}^d)$ and $u \in \mathcal{S}'(\mathbb{R}^d)$ is called a tempered distribution. For $u \in \mathcal{S}$, the Fourier transform $\hat{u}(k) = \mathcal{F}(u(x))$ is defined by the integral
\[
\hat{u}(k) := \int_{\mathbb{R}^d} e^{-ix \cdot k} u(x) \, dx,
\]
where $\cdot$ is the usual scalar product on $\mathbb{R}^d$, and the inverse Fourier transform of $\hat{u}$ denoted by $u(x) = \mathcal{F}^{-1}(\hat{u}(k))$ is given by
\[
u \leq \alpha
\]
\[
\binom{\alpha}{\nu} := \frac{\alpha!}{\nu! (\alpha - \nu)!} = \frac{\alpha_1! \cdots \alpha_d!}{\nu_1! \cdots \nu_d! (\alpha_1 - \nu_1)! \cdots (\alpha_d - \nu_d)!}
\]
On the other hand, we can define the Fourier transform of tempered distributions $u \in \mathcal{S}'(\mathbb{R}^d)$ by the inner product $\langle \cdot, \cdot \rangle_{L^2}$ on $L^2(\mathbb{R}^d)$, namely
\[ \langle \hat{u}, \phi \rangle_{L^2} = \langle u, \hat{\phi} \rangle_{L^2}, \quad \forall \phi \in \mathcal{S}(\mathbb{R}^d). \]

**Definition 1.5.** Let $s \in \mathbb{R}$, the Sobolev space $H^s(\mathbb{R}^d)$ consists of tempered distributions $u$ such that $\hat{u} \in L^2_{\operatorname{loc}}(\mathbb{R}^d)$ and
\[ \|u\|_{H^s} = \left( \int_{\mathbb{R}^d} (1 + |k|^2)^s |\hat{u}(k)|^2 \, dk \right)^{1/2} < +\infty. \]

**Definition 1.6.** Let $\rho \in \mathcal{S}'(\mathbb{R}^d)$, $\rho$ is called a Fourier multiplier on $L^p(\mathbb{R}^d)$ for $1 \leq p \leq \infty$ if the convolution $\mathcal{F}^{-1}(\rho) * \phi \in L^p(\mathbb{R}^d)$ for all $\phi \in \mathcal{S}(\mathbb{R}^d)$ and if
\[ \|\rho\|_{M_p} := \sup_{\|\phi\|_{L^p} = 1} \|\mathcal{F}^{-1}(\rho) * \phi\|_{L^p} < +\infty. \]

The linear space of all Fourier multipliers $\rho$ is denoted by $M_p(\mathbb{R}^d)$ and is equipped with the norm $\| \cdot \|_{M_p}$.

2. Proofs and examples of Theorem 1.1 and Theorem 1.2. For $k \in \mathbb{R}^d$, let $E = E(i\kappa) \in \mathbb{R}^{n \times n}$ be in (8). Let $c \in \mathbb{R}^d$ and $D \in \mathbb{R}^{d \times d}$ be in (11). Let $P_0^{(0)} \in \mathbb{R}^{n \times n}$ be in (9) and let $P_0^{(1)} \in (\mathbb{R}^{n \times n})_d$ be in (17).

For $(x, t) \in \mathbb{R}^d \times \mathbb{R}_+$, consider
\[ \Gamma_t(x) := \Gamma(x, t) = \mathcal{F}^{-1}(e^{-E(i\kappa)t}) \in \mathbb{R}^{n \times n}, \quad (19) \]
the kernel associated with the system (1), and
\[ \tilde{\Phi}_t(x) := \tilde{\Phi}(x, t) = \mathcal{F}^{-1}(e^{-c \cdot i\kappa t - k \cdot Dk}) \in \mathbb{R}, \quad (20) \]
the kernel associated with the system (10). Note that
\[ \tilde{\Phi}_t * U_0(x) = \Phi_t * u_0(x), \quad (21) \]
where
\[ \Phi_t(x) := \Phi(x, t) = \mathcal{F}^{-1}(e^{-c \cdot i\kappa t - k \cdot Dk} P_0^{(0)}) \in \mathbb{R}^{n \times n}. \quad (22) \]
Consider also the kernel
\[ \tilde{\Psi}_t(x) := \tilde{\Psi}(x, t) = \mathcal{F}^{-1}(e^{-k \cdot Dk}) \in \mathbb{R} \quad (23) \]
associated with the system (16). One has
\[ \tilde{\Psi}_t * U_0(x) = \Psi_t * u_0(x), \quad (24) \]
where
\[ \Psi_t(x) := \Psi(x, t) = \mathcal{F}^{-1}(e^{-k \cdot Dk} (P_0^{(0)} + P_0^{(1)} \cdot i\kappa)) \in \mathbb{R}^{n \times n}. \quad (25) \]

We are now able to give proofs of Theorem 1.1 and Theorem 1.2 by using the estimates which will be proved later in Section 5.

**Proof of Theorem 1.1.** Let $u \in \mathbb{R}^n$ be a solution to the system (1) with the initial data $u_0$ and let $U \in \mathbb{R}^n$ be a solution to the system (10) with the initial data $U_0$. One has
\[ u(x, t) = \Gamma_t * u_0(x), \quad U(x, t) = \Phi_t * U_0(x). \]
Moreover, by the relation (21), one has
\[ u(x, t) - U(x, t) = (\Gamma_t - \Phi_t) * u_0(x), \]
where $\Phi_t$ is given by (22).
Moreover, the initial data for (10) are chosen as

Proof of Theorem 1.2. The proof is similar to the proof of Theorem 1.1 where $\tilde{\Phi}_t$ and $\Phi_t$ are substituted by $\Psi_t$ and $\Psi_t$ respectively once considering $U$ as a solution to the system (16). We finish the proof. 

Example 1. Consider the three-dimensional $3 \times 3$ linear Goldstein–Kac system (1) where

$$A^i = \begin{pmatrix} v_1^i & 0 & 0 \\ 0 & v_2^i & 0 \\ 0 & 0 & v_3^i \end{pmatrix}, \quad B = \begin{pmatrix} b + c & -c & -b \\ -c & a + c & -a \\ -b & -a & a + b \end{pmatrix},$$

where $v_j^i \in \mathbb{R}$ for $i, j \in \{1, 2, 3\}$ and $a, b, c > 0$, and the initial data are $u_0 = (u_0, u_0^2, u_0^3) \in \mathbb{R}^3$. Let $A_i := (v_1^i, v_2^i, v_3^i)$ for $i \in \{1, 2, 3\}$, we also consider the system (10) where $\mathbf{c} = (\sum_{i=1}^3 A_i)/3$ and if $\mathbf{c} = 0$, the matrix $\mathbf{D}$ is given by

$$\mathbf{D} = \frac{1}{3(ab + bc + ca)}(aA_1 \otimes A_1 + bA_2 \otimes A_2 + cA_3 \otimes A_3).$$

Moreover, the initial data for (10) are chosen as

$$U_0 = \frac{1}{3}(u_0^4 + u_0^5 + u_0^6, u_0^1 + u_0^2 + u_0^3, u_0^0 + u_0^7 + u_0^8) \in \mathbb{R}^3.$$

Theorem 1.1 then implies that the solution $u$ to the Goldstein–Kac system can be decomposed into $u = u^{(1)} + u^{(2)}$ such that $u^{(1)} - U$, where $U$ is a solution to the above system (10), decays in $L^p(\mathbb{R}^d)$ at the rate $t^{-\frac{d}{2}(\frac{1}{2} - \frac{1}{p})}$ as $t \to +\infty$ for any $1 \leq q \leq p \leq \infty$ and $u_0 \in L^q(\mathbb{R}^d)$. The formulas of $\mathbf{c}$ and $\mathbf{D}$ in fact coincide the formulas obtained by using the graph theory as in Example 3.3 p. 412 in [12]. Furthermore, the decay rate is also the same as in (4).

We give a proof of Theorem 1.2.
Example 2. Consider the two-dimensional linearized isentropic Euler equations with damping
\[
\begin{align*}
\partial_t \rho + \text{div} \, v &= 0, \\
\partial_t v + \nabla_x \rho &= -v,
\end{align*}
\tag{26}
\] which can be written in the vectorial form
\[
\partial_t u + A^1 \partial_{x_1} u + A^2 \partial_{x_2} u + Bu = 0,
\]
where \(u = (\rho, v^1, v^2) \in \mathbb{R}^3\) with the initial data \(u_0 = (\rho_0, v^1_0, v^2_0) \in \mathbb{R}^3\) and
\[
A^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]
Moreover, the matrix \(R\) satisfying the condition \(R\) and the matrix \(S\) satisfying the condition \(S\) are given by
\[
R(w_1, w_2) = \frac{1}{2} \begin{pmatrix} 1 & -w_1 & 2w_2 \\ -w_1 & w_1 & 1 \\ -2w_2 & -w_2 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]
Then, Theorem 1.2 implies that \(u = u^{(1)} + u^{(2)}\) such that \(u^{(1)}\) has the asymptotic profile which is the solution \(U \in \mathbb{R}^3\) to the Cauchy problem
\[
\begin{align*}
\partial_t U - \Delta_x U &= 0, \\
U(x, 0) &= U_0(x),
\end{align*}
\]
where \(U_0 = (\rho_0 - \partial_{x_1} v_1^0 - \partial_{x_2} v_2^0, -\partial_{x_1} \rho_0, -\partial_{x_2} \rho_0) \in \mathbb{R}^3\).
Moreover, \(u^{(1)} - U\) decays in \(L^p(\mathbb{R}^d)\) at the optimal rate \(t^{-(\frac{1}{q} - \frac{1}{p})^{-1}}\) as \(t \to +\infty\) for any \(1 \leq q \leq p \leq \infty\) and \(u_0 \in L^q(\mathbb{R}^d)\). This result is comparable with [7] since \(\rho \in \mathbb{R}\) satisfying (26) also satisfies the linear damped wave equation
\[
\begin{align*}
\partial_{tt}^2 \rho - \Delta_x \rho + \partial_t \rho &= 0, \\
\rho(x, 0) &= \rho_0(x), \\
\partial_t \rho(x, 0) &= -\partial_{x_1} v_1^0(x) - \partial_{x_2} v_2^0(x).
\end{align*}
\]
The proofs of Theorem 2.1 in [7] then implies that \(\rho^{(1)} - \phi\) decays in \(L^p(\mathbb{R}^d)\) at the rate \(t^{-(\frac{1}{q} - \frac{1}{p})^{-1}}\) as \(t \to +\infty\) for any \(1 \leq q \leq p \leq \infty\) and \((\rho_0, \partial_t \rho_0) \in L^q(\mathbb{R}^d)\), where \(\phi\) is a solution to the heat equation
\[
\begin{align*}
\partial_t \phi - \Delta_x \phi &= 0, \\
\phi(x, 0) &= \rho_0(x) - \partial_{x_1} v_1^0(x) - \partial_{x_2} v_2^0(x).
\end{align*}
\]
Remark 1 (Proof of the case of finite speed of propagation). In the case where \(\Gamma_i\) in (19) has compact support contained in the wave cone \(((x, t) \in \mathbb{R}^d \times \mathbb{R} : |x/t| \leq C)\) for some constant \(C > 0\), also by Proposition 4 - Proposition 8, \(u^{(1)}\) can be refined by the following
\[
u^{(1)}(x, t) = F^{-1}(e^{-E_0(k)t} \chi_1(k)) * u_0(x),
\]
where \(\chi_1\) is a cut-off function valued in \([0, 1]\) with support contained in the ball \(B(0, \rho) \subset \mathbb{R}^d\) for any \(\rho > 0\). The proof is then similar to the above proofs. Moreover, this property holds for the two previous examples since they are in fact symmetric hyperbolic systems.
3. **Useful lemmas.** This section is devoted to some useful tools from the Fourier analysis in [1, 2] and the perturbation analysis in [9]. They will be used in Section 4 and Section 5.

3.1. **Fourier analysis.** We introduce here the two well-known inequalities which are the Young inequality and the complex interpolation inequality. On the other hand, we also introduce a powerful Fourier multiplier estimate which is the estimate (27) given by Lemma 3.3. The multiplier estimates are very helpful to study the $L^p-L^p$ estimate for $1 \leq p \leq \infty$.

**Lemma 3.1** (Young’s inequality). For $1 \leq p, q, r \leq \infty$ satisfying $1/p + 1/q = 1 + 1/r$ and any $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$, one has $f \ast g \in L^r(\mathbb{R}^d)$ and the inequality

$$\|f \ast g\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q}.$$  

*Proof.* See the proof of Lemma 1.4 p. 5 in [1].

**Lemma 3.2** (Complex interpolation inequality). Consider a linear operator $T$ which continuously maps $L^{p_j}(\mathbb{R}^d)$ into $L^{q_j}(\mathbb{R}^d)$ for $1 \leq p_j, q_j \leq \infty$ with $j \in \{0, 1\}$. Let $\theta \in [0, 1]$ be such that

$$\left(\frac{1}{p_0}, \frac{1}{q_0}\right) := (1-\theta)\left(\frac{1}{p_1}, \frac{1}{q_1}\right) + \theta\left(\frac{1}{p_1}, \frac{1}{q_1}\right),$$

then $T$ continuously maps $L^{p_0}(\mathbb{R}^d)$ into $L^{q_0}(\mathbb{R}^d)$ and one has

$$\|T\|_{L^{p_0}(\mathbb{R}^d) \to L^{q_0}(\mathbb{R}^d)} \leq \|T\|_{L^{p_1}(\mathbb{R}^d) \to L^{q_1}(\mathbb{R}^d)}.$$  

*Proof.* See the proof of Corollary 1.12 p. 12 in [1].

**Lemma 3.3** (Carlson–Beurling). If $\rho \in H^s(\mathbb{R}^d)$ for $s > d/2$, $\rho \in M_p(\mathbb{R}^d)$ and for some constant $C > 0$, one has the estimate

$$\|\rho\|_{M_p} \leq C\|\rho\|_{L^2}^{1-\frac{d}{s}}\left(\sum_{|\alpha| = s} \|\partial^\alpha \rho\|_{L^2}\right)^{\frac{d}{s}}, \quad 1 \leq p \leq \infty. \quad (27)$$

*Proof.* See the proof of Lemma 6.1.5 p.135 in [2].

3.2. **Perturbation analysis.** We consider the perturbation theory for linear operators in [9] that will be used for studying the asymptotic expansions of the fundamental solution $\Gamma_t$ to the system (1) in (19).

Consider the operator $T(z)$ for $z \in \mathbb{C}$ having the form

$$T(z) = T^{(0)} + zT^{(1)} + z^2T^{(2)} + \ldots, \quad T^{(j)} \in \mathbb{R}^{n \times n}. \quad (28)$$

Exceptional points of the analytic operator $T(z)$ for $z \in \mathbb{C}$ in (28) are defined to be points where the eigenvalues of $T(z)$ intersect. Nonetheless, they are of finite number in the plane. In the domain excluding these points, the operator $T(z)$ has $p$ holomorphic distinct eigenvalues with constant algebraic multiplicities. Moreover, the $p$ eigenprojections and the $p$ eigenimpotents associated with them are also holomorphic. In fact, the eigenvalues of $T(z)$ are solutions to the dispersion polynomial

$$\det(T(z) - \mu I) = 0$$

with holomorphic coefficients. The eigenvalues of $T(z)$ are then branches of one or more than one analytic functions with algebraic singularities of at most order $n$. As a consequence, the number of eigenvalues of $T(z)$ is a constant except for a number of points which is finite in each compact set of the plane. The exceptional
points can be either regular points of the analytic functions or branch-points of some eigenvalues of \(T(z)\). In the former case, the eigenprojections and the eigennilpotents associated with the eigenvalues are bounded while in the latter case, they have poles at the exceptional points even if the eigenvalues are continuous there (see [9]).

We study the behavior of the eigenvalues of \(T(z)\) and the associated eigenprojections and eigennilpotents near an exceptional point. Without loss of generality, we assume that the exceptional point is the point 0 \(\in \mathbb{C}\). Let \(\lambda^{(0)}\) be an eigenvalue of \(T^{(0)}\) with algebraic multiplicity \(m\) \(\geq 1\) and let \(P^{(0)}\) and \(N^{(0)}\) be the associated eigenprojection and eigennilpotent. One has

\[
T^{(0)} P^{(0)} = P^{(0)} T^{(0)} = P^{(0)} T^{(0)} P^{(0)} = \lambda^{(0)} P^{(0)} + N^{(0)},
\]

\[
dim P^{(0)} = m, \quad (N^{(0)})^m = O, \quad P^{(0)} N^{(0)} = N^{(0)} P^{(0)}.
\]

The eigenvalue \(\lambda^{(0)}\) is in general split into several eigenvalues of \(T(z)\) for small \(z \neq 0\). The set of these eigenvalues is called the \(\lambda^{(0)}\)-group. The total projection \(P(z)\) of this group is holomorphic at \(z = 0\) and is approximated by

\[
P(z) = P^{(0)} + z P^{(1)} + z^2 P^{(2)} + O(|z|^3),
\]

where \(P^{(j)}\) can be computed in terms of the coefficients \(T^{(j)}\) in (28) and the coefficients \(N^{(0)}, P^{(0)}\) and \(Q^{(0)}\) given respectively by \(N^{(0)} = (T^{(0)} - \lambda^{(0)} I) P^{(0)}\) and

\[
P^{(0)} = -\frac{1}{2\pi i} \int_\Gamma (T^{(0)} - \mu I)^{-1} d\mu, \quad Q^{(0)} = \frac{1}{2\pi i} \int_\Gamma \mu^{-1} (T^{(0)} - \mu I)^{-1} d\mu,
\]

where \(\Gamma\), in the resolvent set of \(T^{(0)}\), is an oriented closed curve enclosing \(\lambda^{(0)}\) except for the other eigenvalues of \(T^{(0)}\). In fact, from [9] (eq. (2.13) p. 76), one has

\[
P^{(1)} = \sum_{i+j=1} X^{(i)} T^{(1)} X^{(j)},
\]

\[
P^{(2)} = \sum_{i+j=1} X^{(i)} T^{(2)} X^{(j)} - \sum_{i+j+k=2} X^{(i)} T^{(1)} X^{(j)} T^{(1)} X^{(k)},
\]

where

\[
X^{(0)} = P^{(0)}, \quad X^{(i)} = (Q^{(0)})^i, \quad X^{(-i)} = -(N^{(0)})^i, \quad \forall i \geq 1.
\]

Moreover, the subspace \(\text{ran } P(z) := P(z) \mathbb{C}^n\) is \(m\)-dimensional and invariant under \(T(z)\). The \(\lambda^{(0)}\)-group eigenvalues of \(T(z)\) are identical with all the eigenvalues of \(T(z)\) in \(\text{ran } P(z)\). In order to determine the \(\lambda^{(0)}\)-group eigenvalues, therefore, we only have to solve an eigenvalue problem in the subspace \(\text{ran } P(z)\), which is in general smaller than the whole space \(\mathbb{C}^n\).

The eigenvalue problem for \(T(z)\) in \(\text{ran } P(z)\) is equivalent to the eigenvalue problem for

\[
T_r(z) = T(z) P(z) = P(z) T(z) = P(z) T(z) P(z).
\]

Thus, the \(\lambda^{(0)}\)-group eigenvalues of \(T(z)\) are exactly those eigenvalues of \(T_r(z)\) which are different from 0, provided \(|\lambda^{(0)}|\) is large enough to ensure that these eigenvalues do not vanish for the small \(z\) under consideration. The last condition does not restrict the generality, for \(T^{(0)}\) could be replaced by \(T^{(0)} + \alpha\) with a suitable scalar \(\alpha\) without changing the nature of the problem (see [9]).

We also have the following result in [9].

**Lemma 3.4 (A simple case).** If \(T(z) = T^{(0)} + z T^{(1)}\) and \(\lambda^{(0)}\) is a simple eigenvalue of \(T^{(0)}\), the eigenvalue \(\lambda(z)\) of \(T(z)\) converging to \(\lambda^{(0)}\) as \(|z| \to 0\) and its associated
eigenprojection $P(z)$ are holomorphic at $z = 0$. Moreover, for small $z \neq 0$, $P(z)$ is approximated by (29) with the coefficients $P^{(j)}$ for $j = 0, 1, 2, \ldots$ and $\lambda(z)$ is approximated by

$$\lambda(z) = \lambda^{(0)} + z\lambda^{(1)} + z^2\lambda^{(2)} + O(|z|^3), \quad (35)$$

where

$$\lambda^{(j)} = \frac{1}{j} \text{tr} (T^{(1)} P^{(j-1)}), \quad j = 1, 2, 3, \ldots \quad (36)$$

On the other hand, the eigennilpotent associated with $\lambda(z)$ which is $N(z) = (T(z) - \lambda(z)I)P(z)$ vanishes identically.

Proof. For any eigenvalue $\lambda^{(0)}$ of $T^{(0)}$ with algebraic multiplicity $m \geq 1$, one primarily considers the weighted mean of the $\lambda^{(0)}$-group defined by

$$\hat{\lambda}(z) := \frac{1}{m} \text{tr} (T(z)P(z)) = \lambda^{(0)} + \frac{1}{m} \text{tr} ((T(z) - \lambda^{(0)}I)P(z)),$$

where $P(z)$ is the total projection associated with the $\lambda^{(0)}$-group.

We study the asymptotic expansions of $\hat{\lambda}(z)$ and $P(z)$ for small $z \neq 0$. The expansion of $P(z)$ is in fact given by (29), and also following [9] (eq. (2.21) p. 78 and eq. (2.30) p. 79), the coefficient $P^{(j)}$ in (29) is computed by

$$P^{(j)} = -\frac{1}{2\pi i} \sum_{\nu_1 + \cdots + \nu_p = j} \frac{(-1)^p}{\nu_1 i_1 + \cdots + \nu_p i_p} \int_{\Gamma} R^{(0)}(\zeta)T^{(\nu_1)}R^{(0)}(\zeta)T^{(\nu_2)} \cdots T^{(\nu_p)}R^{(0)}(\zeta) d\zeta, \quad (37)$$

where $T^{(\nu_i)}$ for $i \in \{1, \ldots, p\}$ are the coefficients in (28), $R^{(0)}(\zeta) := (T^{(0)} - \zeta I)^{-1}$ is the resolvent of $T^{(0)}$ and $\Gamma$ is a small positively-oriented circle around $\lambda^{(0)}$. On the other hand, following [9] (eq. (2.21) p. 78 and eq. (2.30) p. 79), the weighted mean $\hat{\lambda}(z)$ of the $\lambda^{(0)}$-group is approximated by

$$\hat{\lambda}(z) = \lambda^{(0)} + z\hat{\lambda}^{(1)} + z^2\hat{\lambda}^{(2)} + O(|z|^3), \quad (38)$$

where the coefficient $\hat{\lambda}^{(j)}$ is given by

$$\hat{\lambda}^{(j)} = \frac{1}{2\pi i m} \text{tr} \left( \sum_{\nu_1 + \cdots + \nu_p = j} \frac{(-1)^p}{\nu_1 i_1 + \cdots + \nu_p i_p} \int_{\Gamma} T^{(\nu_1)}R^{(0)}(\zeta) \cdots T^{(\nu_p)}R^{(0)}(\zeta) d\zeta, \quad (39)$$

where the relative coefficients are introduced before.

In the case where $T(z) = T^{(0)} + zT^{(1)}$, one has $T^{(j)} = O$, the null matrix, for $j \geq 2$. Furthermore, since $\nu_i$ in (37) and (39) satisfy $\nu_i \geq 1$, it implies that the relevances are $\nu_i = 1$ for all $i$. Hence, we obtain from (37) and (39) that $p = j$ and

$$\hat{\lambda}^{(j)} = \frac{1}{m j} \text{tr} \left( T^{(1)} \frac{(-1)^j}{2\pi i} \int_{\Gamma} R^{(0)}(\zeta)T^{(1)} \cdots T^{(1)}R^{(0)}(\zeta) d\zeta \right) \quad (40)$$

since there are $j - 1$ matrices $T^{(1)}$ in the integrand in (40).

If $\lambda^{(0)}$ is a simple eigenvalue, one has $m = 1$ and $\lambda^{(0)}$ is not split into many eigenvalues of $T(z)$. Thus, the $\lambda^{(0)}$-group contains only one single eigenvalue $\lambda(z)$ of $T(z)$ converging to $\lambda^{(0)}$ as $|z| \to 0$. Hence, $\lambda(z) = \hat{\lambda}(z)$ and the eigenprojection
On the other hand, the eigennilpotent $N$ is exactly the total projection $P(z)$ of the $\lambda^{(0)}$-group. Therefore, one obtains the expansion (35) from (38) and one obtains the formula (36) from (40) where $m = 1$. The eigenilpotent $N(z)$ associated with $\lambda(z)$ is obviously null since $\lambda(z)$ is simple. The proof is done.

Moreover, one obtains the following result from Lemma 3.4.

**Corollary 1 (Symmetry).** Under the same assumptions in Lemma 3.4, if in addition, there is an invertible matrix $S$ such that $ST^{(1)} = -T^{(1)}S$ and $ST^{(0)} = T^{(0)}S$, then $\lambda^{(j)} = 0$ for all $j$ odd, where $\lambda^{(j)}$ is the $j$-th coefficient in the formulas (35) and (36) for $j = 1, 2, \ldots$

**Proof.** Recall $T(z) = T^{(0)} + zT^{(1)}$, one can study the eigenvalue problem for $T(z)$ by considering the operator

$$T_S(z) := ST(z)S^{-1} = ST^{(0)}S^{-1} + zST^{(1)}S^{-1} = T^{(0)} - zT^{(1)} = T^{(0)} + zT_S^{(1)},$$

(41) where $T_S^{(0)} := T^{(0)}$ and $T_S^{(1)} := -T^{(1)}$. Thus, Lemma 3.4 is applied to $T_S(z)$ since $\lambda^{(0)}$ is also a simple eigenvalue of $T_S^{(0)}$. It implies that the eigenvalue $\lambda_S(z)$ of $T_S(z)$ converging to $\lambda^{(0)}$ as $|z| \to 0$ and the associated eigenprojection $P_S(z)$ are holomorphic at $z = 0$. Moreover, for small $z \neq 0$, the expansion of $P_S(z)$ is given by the expansion (29) with coefficients denoted by $P_S^{(j)}$ for $j = 0, 1, 2, \ldots$ and $\lambda_S(z)$ is approximated by

$$\lambda_S(z) = \lambda^{(0)} + z\lambda_S^{(1)} + z^2\lambda_S^{(2)} + O(|z|^3),$$

where

$$\lambda_S^{(j)} = \frac{1}{j} \text{tr} (T_S^{(1)} P_S^{(j-1)}), \quad j = 1, 2, 3, \ldots$$  

(42)

On the other hand, the eigenilpotent $N_S(z)$ associated with $\lambda_S(z)$ vanishes identically.

Consider the total projection $P_S(z)$ associated with the $\lambda^{(0)}$-group of $T_S(z)$ in (29) with the coefficients $P_S^{(j)}$. We also consider the formula (37) of $P_S^{(j)}$, namely

$$P_S^{(j)} = \frac{-1}{2\pi i} \sum_{\kappa_1 + \cdots + \kappa_p = j} (-1)^p \int R_S^{(0)}(\zeta) T_S^{(\kappa_1)} R_S^{(0)}(\zeta) T_S^{(\kappa_2)} \cdots T_S^{(\kappa_p)} R_S^{(0)}(\zeta) d\zeta,$$

where $T_S^{(\kappa_i)}$ for $i \in \{1, \ldots, p\}$ are the coefficients in the expansion (28) of $T_S(z)$, $R_S^{(0)}(\zeta) := (T_S^{(0)} - \zeta I)^{-1}$ is the resolvent of $T_S^{(0)}$ and $\Gamma$ is a small positively-oriented circle around $\lambda^{(0)}$. Then, since $T_S^{(\nu_i)} = O$ for all $\nu_i \geq 2$ and since $\nu_i \geq 1$ for all $i$, one has $p = j$ and

$$P_S^{(j)} = \frac{1}{2\pi i} (-1)^j \int R_S^{(0)}(\zeta) T_S^{(1)} R_S^{(0)}(\zeta) T_S^{(1)} \cdots T_S^{(1)} R_S^{(0)}(\zeta) d\zeta,$$

where there are $j$ matrices $T_S^{(1)}$ in the integrand.

Since $T_S^{(0)} = T^{(0)}$ and $T_S^{(1)} = -T^{(1)}$, it follows that for all $j$, one has

$$P_S^{(j)} = \begin{cases} P^{(j)} & \text{if } j \text{ is even}, \\ -P^{(j)} & \text{if } j \text{ is odd}, \end{cases}$$

(43)

where $P^{(j)}$ is the $j$-th coefficient in the expansion of the total projection $P(z)$ associated with the $\lambda^{(0)}$-group of $T(z) = T^{(0)} + zT^{(1)}$. 

\text{DIS\textsc{ISSITIVE} LINEAR HYPERBOLIC SYSTEMS 1663}
Hence, from (36), (42) and (43), we have
\[
\lambda_S^{(j)} = \begin{cases} 
\lambda^{(j)} & \text{if } j \text{ is even}, \\
-\lambda^{(j)} & \text{if } j \text{ is odd},
\end{cases} 
\] (44)
where \( \lambda_j \) is the \( j \)-th coefficient in the expansion of the eigenvalue \( \lambda(z) \) of \( T(z) = T^{(0)} + zT^{(1)} \) converging to \( \lambda^{(0)} \) as \( |z| \to 0 \).

Finally, since \( \lambda_S(z) \equiv \lambda(z) \) due to (41) and the fact that they are single eigenvalues, we deduce from (44) that \( \lambda^{(j)} = -\lambda^{(j)} = 0 \) for all \( j \) odd. We finish the proof. \( \square \)

Let \( \sigma(T, D) \) be the spectrum of \( T \) considered in the domain \( D \), we finish this section by introducing the reduction method in [9] which can be applied to the semi-simple-eigenvalue case.

**Lemma 3.5** (Reduction process). Let \( T(z) \) be in (28) with the coefficients \( T^{(i)} \) for \( i = 0, 1, 2, \ldots \) and let \( \lambda^{(0)} \) be a semi-simple eigenvalue of \( T^{(0)} \). Let \( P(z) \) in (29) with the coefficients \( P^{(j)} \) for \( i = 0, 1, 2, \ldots \) be the total projection of the \( \lambda^{(0)} \)-group. The following holds for small \( z \neq 0 \)
\[
T(z)P(z) = \sum_{j=1}^{p}(\lambda^{(0)}I + zT_j(z))P_j(z), \quad (45)
\]
where \( T_j(z) \) commutes with \( P_j(z) \) and \( P_j(z) \) satisfies
\[
P_j(z)P_j(z) = \delta_{jj'}P_j(z), \quad \sum_{j=1}^{p}P_j(z) = P(z). \quad (46)
\]
The expansions of \( T_j(z) \) and \( P_j(z) \) are
\[
T_j(z) = \lambda_j^{(0)}I + N_j^{(0)} + O(|z|) \quad (47)
\]
and
\[
P_j(z) = P_j^{(0)} + O(|z|), \quad (48)
\]
where \( \lambda_j^{(0)} \in \sigma(P^{(0)}T^{(1)}P^{(0)}, \ker(T^{(0)} - \lambda^{(0)}I)) \) with the associated eigenprojection \( P_j^{(0)} \) and eigennilpotent \( N_j^{(0)} \) for \( j \in \{1, \ldots, p\} \) and \( p \) is the cardinality of \( \sigma(P^{(0)}T^{(1)}P^{(0)}, \ker(T^{(0)} - \lambda^{(0)}I)) \).

**Proof.** Recall \( T(z) \) and the coefficients \( T^{(j)} \) in (28). Recall the expansion of the total projection \( P(z) \) of the \( \lambda^{(0)} \)-group of \( T(z) \) and the coefficients \( P^{(j)} \) in (29), where the \( \lambda^{(0)} \)-group is generated by the eigenvalue \( \lambda^{(0)} \) of \( T^{(0)} \). If \( \lambda^{(0)} \) is semi-simple, one obtains from (34) that \( (T(z) - \lambda^{(0)}I)P(z) = z\hat{T}(z) \), where
\[
\hat{T}(z) = \hat{T}^{(0)} + z\hat{T}^{(1)} + O(|z|^2), \quad (49)
\]
where
\[
\hat{T}^{(0)} := P^{(0)}T^{(1)}P^{(0)}, \\
\hat{T}^{(1)} := P^{(1)}T^{(0)}P^{(1)} + P^{(1)}T^{(1)}P^{(0)} + P^{(0)}T^{(1)}P^{(1)}.
\]
Thus, the eigenvalues of \( \hat{T}(z) \) in \( \text{ran } P(z) \) are considered and in general, they converge to the eigenvalues of \( \hat{T}^{(0)} \) in \( \text{ran } P^{(0)} = \ker(T^{(0)} - \lambda^{(0)}I) \) as \( |z| \to 0 \) (see Theorem 2.3 p. 82 in [9]). One denotes the distinct eigenvalues of \( \hat{T}^{(0)} \) considered in \( \ker(T^{(0)} - \lambda^{(0)}I) \) by \( \lambda_j^{(0)} \) for \( j \in \{1, \ldots, p\} \). Then, \( \lambda_j^{(0)} \) generates the \( \lambda_j^{(0)} \)-group of \( \hat{T}(z) \) similarly to the \( \lambda^{(0)} \)-group of \( T(z) \) generated by the eigenvalue \( \lambda^{(0)} \) of \( T^{(0)} \).
Moreover, the total projection $P_j(z)$ of the $\lambda_j^{(0)}$-group commutes with $\hat{T}(z)$, satisfies (46) and is approximated by (48).

Applying (34) where $T(z)$ is substituted by $\hat{T}(z)$ and $P(z)$ is substituted by $P_j(z)$, it follows from (48) and (49) that

$$\hat{T}(z)P_j(z) = \lambda_j^{(0)}I + N_j^{(0)} + O(|z|),$$

where $N_j^{(0)}$ is the eigennilpotent associated with $\lambda_j^{(0)}$. Let $T_j(z) := \hat{T}(z)P_j(z)$ and using (46), (50) and the fact that $T(z)P(z) = z\hat{T}(z)$, one obtains (45) and (47). We finish the proof.

4. Preliminaries to Section 5. In this section, we study the asymptotic expansions of $E(ik) = B + A(ik)$ for $k \in \mathbb{R}^d$ in (8) which will be used in Section 5. One has

$$E(ik) = E(\zeta, w) := B + i\zeta A(w),$$

where $\zeta := |k| \in [0, +\infty)$ and $w := k/|k| \in \mathbb{S}^{d-1}$. Moreover, since $\mathbb{S}^{d-1}$ is compact, $\zeta = 0$ is an isolated exceptional point of $E(\zeta, w)$ uniformly for $w \in \mathbb{S}^{d-1}$ while there is a finite number of exceptional curves of $E(\zeta, w)$ for $0 < \zeta < +\infty$. The exceptional point $\zeta = +\infty$ is not a uniformly exceptional point for $w \in \mathbb{S}^{d-1}$ in general (see [3, 9]). Nonetheless, we can approximate $E(\zeta, w)$ near $\zeta = +\infty$ by subtracting a suitable Lebesgue measure zero set due to the conditions A and R.

In this paper, we are only interested in the asymptotic expansions of $E(\zeta, w)$ near $\zeta = 0$ and $\zeta = +\infty$. As a consequence of Lemma 3.4 - Lemma 3.5, we obtain the followings.

**Proposition 1** (Low-frequency approximation). If the assumptions B and D hold, then for small $k \in \mathbb{R}^d$, $E(ik)$ is approximated by

$$E(ik) = \lambda_0(ik)P_0(ik) + \sum_{j=1}^s E_j(ik)P_j(ik),$$

where

$$\lambda_0(ik) = c \cdot ik + k \cdot Dk + O(|k|^3),$$

where $c = (c_h) \in \mathbb{R}^d$ and positive definite $D = (D_{hl}) \in \mathbb{R}^{d \times d}$ with scalar entries

$$c_h = \text{tr} \left( A^h P_0^{(0)} \right), \quad D_{hl} = \frac{1}{2} \text{tr} \left( A^h P_0^{(0)} A^l Q_0^{(0)} + A^h Q_0^{(0)} A^l P_0^{(0)} \right),$$

and

$$P_0(ik) = P_0^{(0)} + P_0^{(1)} \cdot ik + O(|k|^2),$$

where $P_0^{(1)} (P_0^{(1)}) \in (\mathbb{R}^{n \times n})^d$ with matrix entries

$$P_{0h}^{(1)} = -P_0^{(0)} A^h Q_0^{(0)} - Q_0^{(0)} A^h P_0^{(0)},$$

and $E_j(ik)$ commutes with $P_j(ik)$ and one has

$$E_j(ik) = \lambda_j^{(0)}I + N_j^{(0)} + O(|k|),$$

and

$$P_j(ik) = P_j^{(0)} + O(|k|),$$

where $\lambda_j^{(0)}$ with $\text{Re} \lambda_j^{(0)} > 0$ is the $j$-th nonzero eigenvalue of $B$ with the associated eigenprojection $P_j^{(0)}$ and eigennilpotent $N_j^{(0)}$ for $j \in \{1, \ldots, s\}$ and $s$ is the number of the distinct nonzero eigenvalues of $B$. 

Moreover, if the condition S holds in addition, we have

$$\lambda_0(\mathbf{k}) = k \cdot D\mathbf{k} + \mathcal{O}(|\mathbf{k}|^4).$$

(59)

**Proof.** We primarily consider the 0-group of $E(\zeta, \mathbf{w})$ in (51) for small $\zeta > 0$ and $\mathbf{w} \in S^{d-1}$. Recall the spectrum $\sigma(B)$ of $B$. Since $0 \in \sigma(B)$ is simple if the assumption B holds, the eigennilpotent $N_0^{(0)}$ associated with $0 \in \sigma(B)$ is the null matrix and one obtains from (29) and (31) - (33) that the total projection $P_0(\zeta, \mathbf{w})$ of the 0-group is approximated by

$$P_0(\zeta, \mathbf{w}) = P_0^{(0)} + i\zeta P_0^{(1)}(\mathbf{w}) + \mathcal{O}(\zeta^2).$$

(60)

where $P_0^{(0)}$ is the eigenprojection associated with $0 \in \sigma(B)$ and

$$P_0^{(1)}(\mathbf{w}) = -P_0^{(0)}A(\mathbf{w})Q_0^{(0)} - Q_0^{(0)}A(\mathbf{w})P_0^{(0)}$$

$$= -\sum_{h=1}^{d} (P_0^{(0)}A^h Q_0^{(0)} + Q_0^{(0)}A^h P_0^{(0)}) w_h.$$  

(61)

On the other hand, by (35) and (36) in Lemma 3.4, the 0-group of $E(\zeta, \mathbf{w})$ consists of one single eigenvalue $\lambda_0(\zeta, \mathbf{w})$ approximated by

$$\lambda_0(\zeta, \mathbf{w}) = i\zeta \lambda_0^{(1)}(\mathbf{w}) - \zeta^2 \lambda_0^{(2)}(\mathbf{w}) + \mathcal{O}(\zeta^3),$$

(62)

where

$$\lambda_0^{(1)}(\mathbf{w}) = \text{tr} (A(\mathbf{w})P_0^{(0)}) = \sum_{h=1}^{d} \text{tr} (A^h P_0^{(0)}) w_h$$

(63)

and

$$\lambda_0^{(2)}(\mathbf{w}) = \frac{1}{2} \text{tr} (A(\mathbf{w})P_0^{(1)}(\mathbf{w}))$$

$$= -\frac{1}{2} \sum_{h, \ell=1}^{d} \text{tr} (A^h P_0^{(0)} A^\ell Q_0^{(0)} + A^\ell Q_0^{(0)} A^h P_0^{(0)}) w_h w_{\ell}.$$  

(64)

We consider the other groups of $E(\zeta, \mathbf{w})$ for small $\zeta > 0$ and $\mathbf{w} \in S^{d-1}$. Let $\lambda_j^{(0)} \in \sigma(B) \setminus \{0\}$ be the $j$-th nonzero eigenvalue of $B$ for $j \in \{1, \ldots, s\}$, one deduces directly from (29) that the approximation of the total projection $P_j(\zeta, \mathbf{w})$ of the $\lambda_j^{(0)}$-group is given by

$$P_j(\zeta, \mathbf{w}) = P_j^{(0)} + \mathcal{O}(\zeta),$$

(65)

where $P_j^{(0)}$ is the eigenprojection associated with $\lambda_j^{(0)} \in \sigma(B) \setminus \{0\}$. Moreover, due to the discussion above (34), the study of the $\lambda_j^{(0)}$-group of $E(\zeta, \mathbf{w})$ is equivalent to the study of the eigenvalues of $E_j(\zeta, \mathbf{w}) = E(\zeta, \mathbf{w})P_j(\zeta, \mathbf{w})$ in ran $P_j(\zeta, \mathbf{w})$. Furthermore, one has

$$E_j(\zeta, \mathbf{w}) = (B + i\zeta A(\mathbf{w})) (P_j^{(0)} + \mathcal{O}(\zeta))$$

$$= BP_j^{(0)} + \mathcal{O}(\zeta) = \lambda_j^{(0)} I + N_j^{(0)} + \mathcal{O}(\zeta),$$

(66)

where $N_j^{(0)} = (B - \lambda_j^{(0)} I) P_j^{(0)}$ is the eigennilpotent associated with $\lambda_j^{(0)} \in \sigma(B) \setminus \{0\}$. On the other hand, by definition, one also has $E_j(\zeta, \mathbf{w})$ commutes with $P_j(\zeta, \mathbf{w})$.  


Finally, since \( \sum_{j=0}^s P_j(\zeta, w) = I \), the identity matrix, one has

\[
E(\zeta, w) = \sum_{j=0}^s E(\zeta, w)P_j(\zeta, w)
= \lambda_0(\zeta, w)P_0(\zeta, w) + \sum_{j=1}^s E_j(\zeta, w)P_j(\zeta, w). \tag{67}
\]

We thus obtain (52) - (58) once considering (60) - (67) in the coordinates \( k \in \mathbb{R}^d \) except for the fact that the matrix \( D \) in (54) is positive definite.

We now prove that \( D \) is positive definite. Consider the expansion (53) of the eigenvalue \( \lambda_0(ik) \) of \( E(ik) \) for \( k \in \mathbb{R}^d \) with the coefficients \( c \in \mathbb{R}^d \) and \( D \in \mathbb{R}^{d \times d} \). If the assumption \( D \) holds, then since \( c \cdot k \in \mathbb{R} \), there is a constant \( \theta > 0 \) such that for small \( k \neq 0 \in \mathbb{R}^d \), one has

\[
\frac{\theta |k|^2}{1 + |k|^2} \leq \text{Re} \lambda_0(ik) \leq \text{Re} (k \cdot Dk) + C|k|^3.
\]

As \( |k| \to 0 \), one has \( \text{Re} (k \cdot Dk) \geq \theta > 0 \) for all \( w \in S_{d-1} \). Therefore, for any \( x \neq 0 \in \mathbb{R}^d \), one has \( \text{Re} (x \cdot Dx) = |x|^2 \text{Re} ((x/|x|) \cdot D(x/|x|)) > 0 \).

Finally, since the condition \( S \) implies that for \( w \in \mathbb{R}^d \), there is an invertible matrix \( S = S(w) \) satisfying \( S(w)A(w) = -A(w)S(w) \) and \( S(w)B = BS(w) \), we obtain (59) directly from Corollary 1. The proof is done. \( \Box \)

We study \( E(ik) = B + A(ik) \) for large \( k \in \mathbb{R}^d \). Recall \( E(\zeta, w) = B + i\zeta A(w) \) for \( (\zeta, w) \in [0, +\infty) \times S_{d-1} \) in (51). Note that under the assumption \( A \), there is an invertible matrix \( R = R(w) \) for \( w \in S_{d-1} \) such that \( R^{-1}AR \) is a diagonal matrix whose nonzero entries are real linear eigenvalues of \( A = A(w) \) for \( w \in S_{d-1} \). Hence, one can consider the \( \ell \)-th diagonal element of \( R^{-1}AR \) as the linear function

\[
\nu_\ell(w) := \nu^{(0)}_\ell + \sum_{h=1}^d \nu^{(h)}_\ell w_h, \quad w = (w_1, \ldots, w_d) \in S_{d-1}, \tag{68}
\]

where the coefficients \( \nu^{(h)}_\ell \in \mathbb{R} \) for \( h \in \{0, 1, \ldots, d\} \). Let \( \nu_\ell := (\nu^{(0)}_\ell, \ldots, \nu^{(d)}_\ell) \) be the coefficient vector associated with \( \nu_\ell \) for \( \ell \in \{1, \ldots, n\} \), one sets

\[
S_1 := \{ \ell \in \{1, \ldots, n\} : \nu_\ell = \nu_1 \}.
\]

For \( i_j := \min\{\{1, \ldots, n\} \setminus \cup_{h=1}^{j-1} S_h \} \), one defines

\[
S_j := \{ \ell \in \{1, \ldots, n\} : \nu_\ell = \nu_{i_j} \}, \quad j = 2, 3, \ldots
\]

This procedure will stop at some finite step \( r \leq n \) and \( S := \{S_1, \ldots, S_r\} \) is considered as a partition of \( \{1, \ldots, n\} \). One denotes by \( [j] \) the representation of the elements of \( S_j \) and by \( r_j \) the cardinality of \( S_j \) for \( j \in \{1, \ldots, r\} \).

**Lemma 4.1** (Measure-zero-set subtraction). There is a Lebesgue measure zero set contained in \( S_{d-1} \) such that except for this set, the number of distinct eigenvalues of \( A(w) \) for \( w \in S_{d-1} \) is \( r \) and the algebraic multiplicities associated with them are \( r_j \) for \( j \in \{1, \ldots, r\} \).

**Proof.** Recall the partition \( S = \{S_1, \ldots, S_r\} \) with the cardinality \( r \). Assume that there are \( i, j \in \{1, \ldots, r\} \) such that \( i \neq j \) and \( \nu_{ij}(w_0) = \nu_{[j]}(w_0) \) for some \( w_0 \in
\[ S^{d-1} \]. We prove that \( w_0 \) belongs to a Lebesgue measure zero set in \( \mathbb{R}^{d-1} \). In fact, \( w_0 \) belongs to the intersection of the affine hyperplane

\[
(n^{(0)}_{[i]} - n^{(0)}_{[j]}) + \sum_{h=1}^{d} (n^{(h)}_{[i]} - n^{(h)}_{[j]})x_h = 0, \quad (x_1, \ldots, x_d) \in \mathbb{R}^d,
\]

whose dimension is at most \( d - 1 \) since the coefficient vectors \( n^{(i)}_{[i]} \) and \( n^{(i)}_{[j]} \) satisfy \( n^{(i)}_{[i]} \neq n^{(i)}_{[j]} \) for any \( i \neq j \) by definition, and the unit sphere \( S^{d-1} \). Moreover, the dimension of the intersection is at most \( d - 2 \) and it is therefore a Lebesgue measure zero set in \( \mathbb{R}^{d-1} \). Thus, \( n^{[i]}(w) \neq n^{[j]}(w) \) for any \( i \neq j \) and \( w \in S^{d-1} \) subtracted a Lebesgue measure zero set. Finally, since the repeated eigenvalues of \( A(w) \) are \( n_{\ell}(w) \) determined by the coefficient vectors \( n_{\ell} \) for \( \ell \in \{1, \ldots, n\} \), it follows immediately that the number of distinct eigenvalues of \( A(w) \) for \( w \in S^{d-1} \) is \( r \) and the algebraic multiplicities associated with them are \( r_j \), the cardinality of \( S_j \), for \( j \in \{1, \ldots, r\} \) excluding a Lebesgue measure zero set. We finish the proof.

One sets, for \( j \in \{1, \ldots, r\} \), the projection

\[
(P^{(0)}_j)_{n \ell} := \begin{cases} 1 & \text{if } h = \ell \in S_j, \\ 0 & \text{otherwise}. \end{cases}
\]

(69)

Let \( R = R(w) \) for \( w \in S^{d-1} \) be a matrix satisfying the condition \( R \). One has

**Proposition 2** (High-frequency approximation). If the assumptions A, R and D hold, then for large \( k \in \mathbb{R}^d \), \( E(ik) \) is almost everywhere approximated by

\[
E(ik) = R \sum_{j=1}^{r} \sum_{m=1}^{s_j} \Upsilon_{jm}(ik) \Pi_{jm}(ik) R^{-1},
\]

(70)

where the constant \( s_j \leq r_j \) which is also constant as well as \( r \), \( \Upsilon_{jm}(ik) \) commutes with \( \Pi_{jm}(ik) \) and one has

\[
\Upsilon_{jm}(ik) = (\alpha_j(ik) + \beta_jm)I + \Theta^{(0)}_{jm} + O(|k|^{-1})
\]

(71)

and

\[
\Pi_{jm}(ik) = \Pi^{(0)}_{jm} + O(|k|^{-1}),
\]

(72)

where \( \alpha_j(ik) = i[k]n_{[j]}(ik)/|k| \) where \( n_{[j]} \) is in (68), \( \beta_jm \) with \( \text{Re} \beta_jm > 0 \) is the \( m \)-th nonzero eigenvalue of \( \Pi^{(0)}_{jm} R^{-1} B R \Pi^{(0)}_{jm} \) with the associated eigenprojection \( \Pi^{(0)}_{jm} \) and eigen-nilpotent \( \Theta^{(0)}_{jm} \).

**Proof.** Based on Lemma 4.1, if the condition A holds, the spectrum of \( R^{-1}AR(w) \) for \( w \in S^{d-1} \) is the set \( \{\alpha_1(w), \ldots, \alpha_r(w)\} \) where \( \alpha_j(w) = n_{[j]}(w) \) given by (68) for \( j \in \{1, \ldots, r\} \) with finite constant \( r \), the cardinality of \( S \), and \( [j] \) is the representation of the elements of \( S_j \) for almost everywhere. Thus, from here in this proof, we consider always for almost everywhere and we drop \( w \) in the coefficients written in below if they are in fact constant for almost everywhere.

For \( j \in \{1, \ldots, r\} \), we study the \( \alpha_j(w) \)-group of \( E(\eta, w) \) for \( (\eta, w) \in [0, +\infty) \times S^{d-1} \), where

\[
E(\eta, w) := R^{-1}AR(w) - i\eta R^{-1} B R
\]

for small \( \eta > 0 \). One obtains from (29), (31), (32) and (33) that the total projection \( \Pi_j(\eta, w) \) of the \( \alpha_j(w) \)-group is approximated by

\[
\Pi_j(\eta, w) = \Pi^{(0)}_{jm}(w) + O(\eta),
\]
where $\Pi_j^{(0)}(w)$ is the eigenprojection associated with $\alpha_j(w) \in \sigma(R^{-1}AR(w))$, the spectrum of $R^{-1}AR(w)$. Moreover, by the definition of eigenprojection, if $\Gamma_j$ is an oriented closed curve in the resolvent set of $R^{-1}AR(w)$ enclosing $\alpha_j(w)$ except for the other eigenvalues of $R^{-1}AR(w)$, then
\[
\Pi_j^{(0)}(w) = -\frac{1}{2\pi i} \int_{\Gamma_j} \text{diag}((\nu_1(w) - \mu)^{-1}, \ldots, (\nu_n(w) - \mu)^{-1}) d\mu
\]
\[
= \text{diag}\left(-\frac{1}{2\pi i} \int_{\Gamma_j} (\nu_1(w) - \mu)^{-1} d\mu, \ldots, -\frac{1}{2\pi i} \int_{\Gamma_j} (\nu_n(w) - \mu)^{-1} d\mu\right),
\]
and it coincides (72) since for almost everywhere oriented closed curve in the resolvent set of $R^{-1}AR(w)$.

Thus, $\Pi_j^{(0)}(w)$ is constant for almost everywhere and we can write $\Pi_j^{(0)}$ instead. On the other hand, since $\alpha_j(w)$ is semi-simple, one has
\[
\ker(R^{-1}AR(w) - \alpha_j(w)I) = \text{ran} \Pi_j^{(0)}.
\]
Therefore, by (45) - (48) in Lemma 3.5, the formula of $\tilde{E}(\eta, w)$ and (73), we have
\[
\tilde{E}\Pi_j(\eta, w) = \sum_{m=1}^{s_j} (\alpha_j(w)I - i\eta \tilde{E}_{jm}(\eta, w)) \Pi_j(\eta, w),
\]
where $\tilde{E}_{jm}(\eta, w)$ commutes with $\Pi_{jm}(\eta, w)$ and one has
\[
\tilde{E}_{jm}(\eta, w) = \beta_{jm}I + \Theta_{jm}^{(0)} + O(\eta)
\]
and
\[
\Pi_{jm}(\eta, w) = \Pi_{jm}(0) + O(\eta),
\]
where $\beta_{jm}$ is the $m$-th eigenvalue of $\Pi_j^{(0)}R^{-1}BR\Pi_j^{(0)}$ considered in $\text{ran} \Pi_j^{(0)}$ with the associated eigenprojection $\Pi_{jm}^{(0)}$ and eigenipotent $\Theta_{jm}^{(0)}$ and $s_j$ is the number of such eigenvalues of $\Pi_j^{(0)}R^{-1}BR\Pi_j^{(0)}$. Note that $\beta_{jm}$, $\Pi_{jm}^{(0)}$, $\Theta_{jm}^{(0)}$ and $s_j$ are constant due to the fact that $R^{-1}BR$ is constant under the assumption $R$ and $\Pi_j^{(0)}$ is constant for almost everywhere. Moreover, since one has
\[
\Pi_j^{(0)}R^{-1}BR\Pi_j^{(0)}(I - \Pi_j^{(0)}) = O,
\]
where $O$ is the null matrix, i.e. 0 is an eigenvalue of $\Pi_j^{(0)}R^{-1}BR\Pi_j^{(0)}$ considered in $\text{ran} (I - \Pi_j^{(0)})$ with algebraic multiplicity $\dim(I - \Pi_j^{(0)}) = n - \dim \text{ran} \Pi_j^{(0)}$, it follows immediately that $s_j \leq r_j = \dim \text{ran} \Pi_j^{(0)}$, the cardinality of $\mathcal{S}_j$, by definition.

Therefore, since $E(\zeta, w) = i\zeta R\tilde{E}R^{-1}(\zeta^{-1}, w)$ where $\zeta^{-1} \to 0$ as $\zeta \to +\infty$, one obtains
\[
E(\zeta, w) = R \sum_{j=1}^{r} \sum_{m=1}^{s_j} \Upsilon_{jm}(\zeta, w) \Pi_{jm}(\zeta, w) R^{-1},
\]
where $\Upsilon_{jm}(\zeta, w)$ commutes with $\Pi_{jm}(\zeta, w)$ and
\[
\Upsilon_{jm}(\zeta, w) = (i\zeta \alpha_j(w) + \beta_{jm})I + \Theta_{jm}^{(0)} + O(\zeta^{-1})
\]
and
\[
\Pi_{jm}(\zeta, w) = \Pi_{jm}(0) + O(\zeta^{-1}).
\]
On the other hand, it then follows from (75) - (77) that for large \( \zeta \), the eigenvalues of \( E(\zeta, w) \) are the eigenvalues of \( Y_{jm}(\zeta, w) \) and they are approximated by
\[
\lambda_{jm}(\zeta, w) = i\zeta \alpha_j(w) + \beta_{jm} + o(1)
\]
for \( j \in \{1, \ldots, r\} \) and \( m \in \{1, \ldots, s_j\} \). Thus, if the assumption D holds, then since \( \alpha_j(w) \in \mathbb{R} \), there is a constant \( \theta > 0 \) such that for \( 0 < \zeta^{-1} < \varepsilon \) small, one has
\[
\frac{\theta}{1 + \varepsilon^2} \leq \Re \lambda_{jm}(\zeta, w) \leq \Re \beta_{jm} + \varepsilon.
\]
Let \( \varepsilon \to 0 \), one has \( \Re \beta_{jm} \geq \theta > 0 \). Moreover, one observes from (74) that the nonzero eigenvalues of \( \Pi_{j}^{(0)} R^{-1} BR_{j}^{(0)} \) always belong to the set of the eigenvalues of \( \Pi_{j}^{(0)} R^{-1} BR_{j}^{(0)} \) considered in ran \( \Pi_{j}^{(0)} \). Thus, we can consider that \( \beta_{jm} \) are the nonzero eigenvalues of \( \Pi_{j}^{(0)} R^{-1} BR_{j}^{(0)} \) without specifying that they are considered in ran \( \Pi_{j}^{(0)} \) or not. We finish the proof by writing (75) - (77) in the coordinates \( k \in \mathbb{R}^d \).

**Remark 2** (Intermediate-frequency approximation). In this paper, we will not use any expansions of \( E(ik) = E(\zeta, w) \) in the intermediate frequency but note that there is a finite number of exceptional curves of \( E(\zeta, w) \) for \( 0 < \zeta < +\infty \) in general. In the domain excluding these curves, the number of distinct eigenvalues of \( E(\zeta, w) \) and their algebraic multiplicities are constant (see [3, 9]).

5. **Decay estimates (Core of the paper).** In this section, we prove the estimates used in the proofs of Theorem 1.1 and Theorem 1.2. We primarily give a priori estimates for the principal parabolic part of the fundamental solution \( \Gamma_t \) to the system (1) in (19). Then, we estimate \( \Gamma_t \) by dividing the frequency space into: the low frequency, the intermediate frequency and the high frequency. The main proofs are related to the interpolation between the \( L^\infty - L^1 \) estimate and the \( L^p - L^p \) estimate for \( 1 \leq p \leq \infty \). Moreover, the \( L^\infty - L^1 \) estimate is obtained directly while the \( L^p - L^p \) estimates are obtained based on the Carlson–Beurling inequality (27) in Lemma 3.3. Furthermore, since the Carlson–Beurling inequality (27) depends on the analysis of partial derivatives, one primarily considers the followings.

5.1. **Partial derivatives and parabolic estimates.** Let \( \mathcal{I} \) be an index-set given by \( \mathcal{I} := \{i_1, \ldots, i_s\} \) with possibly repeated indices \( i \in \{1, \ldots, d\} \) i.e. we allows \( i_h = i_t \) for some \( h \neq t \). For any partition \( \{\mathcal{I}_j : j = 1, \ldots, r\} \) of \( \mathcal{I} \) where \( \mathcal{I}_j := \{j_1, \ldots, j_{r_j}\} \) for some \( r \in \{1, \ldots, s\} \), one defines the partial derivative \( \partial_{\mathcal{I}} \) of smooth scalar functions \( q = q(x, t) \) on \( \mathbb{R}^d \times \mathbb{R}_+ \) with respect to \( x \in \mathbb{R}^d \) by
\[
\partial_{\mathcal{I}} q(x, t) := \partial_{x_{i_1} \ldots x_{i_s}} q(x, t),
\]
which is an usual partial derivative of order \( s_j \). Note that, once considering a partition, we do not consider any \( \mathcal{I}_j = \emptyset \), and thus \( s_j \geq 1 \) for all \( j \). On the other hand, for any fixed \( \alpha \in \mathbb{N}^d \), if \( |\alpha| = 0 \) i.e. \( \alpha = 0 \), we set \( \mathcal{I}_\alpha := \emptyset \) and \( |\mathcal{I}_\alpha| := 0 \). If \( |\alpha| = s \in \mathbb{Z}_+ \), \( \alpha \) determines an index-set \( \mathcal{I}_\alpha = \{i_1, \ldots, i_s\} \neq \emptyset \) with possibly repeated indices. In fact, if \( \alpha = (\alpha_1, \ldots, \alpha_d) \), we can define the index-set \( \mathcal{I}_\alpha \) having \( \alpha \ell \) indices \( \ell \in \{1, \ldots, d\} \). We also set \( |\mathcal{I}_\alpha| := s \geq 1 \) and \( |\mathcal{I}_\alpha| := s_j \geq 1 \) for \( j \in \{1, \ldots, r\} \) if \( \mathcal{I}_\alpha \neq \emptyset \).
Lemma 5.1 (Partial derivatives). Let \( \alpha \in \mathbb{N}^d \) with \( |\alpha| \geq 0 \), for any smooth scalar functions \( q = q(x, t) \) on \( \mathbb{R}^d \times \mathbb{R}_+ \), we have
\[
\partial^\alpha e^{q(x,t)} = \sum_{\{I_j : j = 1, \ldots, r\}, r \leq |\alpha|} \partial_{I_1} q(x, t) \ldots \partial_{I_r} q(x, t) e^{q(x,t)}, \tag{78}
\]
where \( \{I_j : j = 1, \ldots, r\} \) is any possible partition of the index-set \( I_\alpha \) determined by \( \alpha \).

Proof. We prove by induction. Let \( \alpha \in \mathbb{N}^d \), if \( |\alpha| = 0 \), then since \( I_\alpha = \emptyset \), there is no partition of \( I_\alpha \) to be considered, and thus \( \partial^\alpha e^{q(x,t)} = e^{q(x,t)} \). If \( |\alpha| = 1 \), by the definition of \( \partial^\alpha \), we have
\[
\partial^\alpha e^{q(x,t)} = \partial^1_{x_1} e^{q(x,t)} = \partial^1_{x_1} q(x, t) e^{q(x,t)} \tag{79}
\]
if \( \alpha_1 = 1 \) and \( \alpha_\ell = 0 \) for all \( \ell \neq 1 \). On the other hand, the index-set determined by \( \alpha \) in this case is \( I_\alpha = \{i\} \) since \( \alpha_1 = 1 \). Thus, \( I_\alpha \) has only one possible partition which is itself and (79) coincides (78).

Given an integer \( s \geq 1 \), assume that (78) holds for any \( \alpha \in \mathbb{N}^d \) satisfying \( |\alpha| = s \). For any \( \beta \in \mathbb{N}^d \) with \( |\beta| = s + 1 \), \( \beta = (\alpha_1, \ldots, \alpha_s + 1, \ldots, \alpha_d) \) for some \( \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d \) and \( i \in \{1, \ldots, d\} \). Hence, we have
\[
\partial^\beta e^{q(x,t)} = \partial^1_{x_i} \partial^\alpha e^{q(x,t)}
\]
\[
= \sum_{\{I_j : j = 1, \ldots, r\}, r \leq s} \sum_{\ell = 1}^r \partial_{I_1} q(x, t) \ldots \partial_{I_\ell} q(x, t) \partial_{I_{\ell+1}} q(x, t) \ldots \partial_{I_r} q(x, t) e^{q(x,t)}
\]
\[
+ \sum_{\{I_j : j = 1, \ldots, r\}, r \leq s} \partial_{I_1} q(x, t) \ldots \partial_{I_r} q(x, t) \partial^1_{x_i} q(x, t) e^{q(x,t)}, \tag{80}
\]
where \( \{I_j : j = 1, \ldots, r\} \) is any possible partition of the index-set \( I_\alpha \) determined by \( \alpha \). We then consider all of possible partitions of \( I_\beta \). The first possibilities are the partitions \( \{\{I_j : j = 1, \ldots, r\}, \{i\}\} \) since \( I_\beta \) has \( \alpha_1 + 1 \) indices \( i \). The last choices are that for each partition \( \{I_j : j = 1, \ldots, r\} \) of \( I_\alpha \), we generate the partition \( \{I'_{j} : j = 1, \ldots, r\} \) of \( I_\beta \) by putting \( i \) into \( I_{\ell} \) and let \( I'_{\ell} = I_j \) for all \( j \neq \ell \) for \( \ell \in \{1, \ldots, r\} \). Thus, since \( r \) varies such that \( r \leq s \), there is no other possible partition of \( I_\beta \) to take part in. Therefore, we obtain from (80) that
\[
\partial^\beta e^{q(x,t)} = \sum_{\{I'_{j} : j = 1, \ldots, r'\}, r' \leq s+1} \partial_{I'_{1}} q(x, t) \ldots \partial_{I'_{r'}} q(x, t) e^{q(x,t)},
\]
where the sum is made on all possible partitions \( \{I'_{j} : j = 1, \ldots, r'\} \) of \( I_\beta \) determined by \( \beta \). We thus proved (78).

Remark 3. Lemma 5.1 is applied only to the case where \( q = q(x,t) \) is scalar for \( (x,t) \in \mathbb{R}^d \times \mathbb{R}_+ \), the matrix case is a challenge as the lack of commutativity of \( q \) and its partial derivatives.

Proposition 3 (Parabolic estimates). If \( D \in \mathbb{R}^{d \times d} \) is positive definite, for \( 1 \leq q \leq p \leq \infty \), there is a constant \( C > 0 \) such that for any \( U_0 \in L^p(\mathbb{R}^d) \), one has
\[
\|\mathcal{F}^{-1}(e^{-k D t}) * U_0\|_{L^q} \leq C(1 + t)^{-\frac{D}{2} \left(\frac{1}{q} - \frac{1}{p}\right)} \|U_0\|_{L^p}, \quad \forall t > 0. \tag{81}
\]

Proof. We primarily study the \( L^\infty-L^1 \) estimate. By the Young inequality and since \( D \) is positive definite, there are constants \( c > 0 \) and \( C > 0 \) such that for \( t > 0 \), we
have
\[ \|F^{-1}(e^{-kDk}) \ast U_0\|_{L^\infty} \leq C\|F^{-1}(e^{-kDk})\|_{L^\infty}\|U_0\|_{L^1} \]
\[ \leq C\|e^{-c|\cdot|^2 t}\|_{L^1}\|U_0\|_{L^1} \leq C(1 + t)^{-\frac{d}{4}}\|U_0\|_{L^1}. \]

(82)

We study the $L^p$-$L^p$ estimate for $1 \leq p \leq \infty$. Let $\alpha \in \mathbb{N}^d$ with $|\alpha| \geq 0$, by the formula (78) in Lemma 5.1, we have
\[ \partial^\alpha (e^{-kDk}) = \sum_{\{I_j : j=1, \ldots, r\}, r \leq |\alpha|} \partial_{I_1}(-k \cdot Dk) \cdots \partial_{I_r}(-k \cdot Dk)e^{-kDk}, \]
where $\{I_j : j=1, \ldots, r\}$ is any possible partition of the index-set $\mathcal{I}_\alpha$ determined by $\alpha$.

On the other hand, by the definition of $\partial_{I_j}$, there is a constant $C > 0$ such that
\[ |\partial_{I_j}(-k \cdot Dk)| \leq C \cdot \begin{cases} 0 & \text{if } |I_j| > 2, \\
t & \text{if } |I_j| = 2, \\
|k|t & \text{if } |I_j| = 1, \end{cases} \]
where $|I_j|$ is the number of elements of $I_j$ with possibly repeated indices for $j \in \{1, \ldots, r\}$. We are then not interested in the cases where $|I_j| > 2$ for some $j \in \{1, \ldots, r\}$. Thus, we can consider only the partitions $\{I_j : j=1, \ldots, r\}$ of $\mathcal{I}_\alpha$ where $1 \leq |I_j| \leq 2$. Hence, we have
\[ |\partial_{I_j}(-k \cdot Dk)| \cdots |\partial_{I_r}(-k \cdot Dk)| \leq C|k|^{m\ell + t}, \]
where $m \geq 0$ is the cardinality of the set $\{j \in \{1, \ldots, r\} : |I_j| = 1\}$ and $\ell \geq 0$ is the cardinality of the set $\{j \in \{1, \ldots, r\} : |I_j| = 2\}$. Moreover, by definition, one has $m + 2\ell = |\mathcal{I}_\alpha| = |\alpha|$, where $|\mathcal{I}_\alpha| = \sum_{j=1}^r |I_j|$ is the number of elements of the index-set $\mathcal{I}_\alpha$ determined by $\alpha$ with possibly repeated indices.

Thus, since $D$ is positive definite, there are constants $c > 0$ and $C > 0$ such that
\[ |\partial^\alpha e^{-kDk}| \leq C \sum_{\{I_j : j=1, \ldots, r\}, r \leq |\alpha|} \sum_{1 \leq |I_j| \leq 2} |k|^{m\ell + t} e^{-c|\cdot|^2 t}. \]

(83)

Hence, since $m + 2\ell = |\alpha|$, we have
\[ \|\partial^\alpha e^{-kDk}\|_{L^2}^2 \leq C \sum_{\{I_j : j=1, \ldots, r\}, r \leq |\alpha|} \int_{\mathbb{R}^d} |k|^{2m\ell + 2\ell t} e^{-2c|\cdot|^2 t} \, dk \]
\[ \leq C \sum_{\{I_j : j=1, \ldots, r\}, r \leq |\alpha|} (1 + t)^{m + 2\ell - \frac{d}{4}} = C(1 + t)^{|\alpha| - \frac{d}{4}}. \]

(84)

By the Carlson–Beurling inequality (27) in Lemma 3.3, one has
\[ \|e^{-kDk}\|_{M_p} \leq C\|e^{-kDk}\|_{L^2}^{-\frac{d}{4}} \left( \sum_{|\alpha| = s} \|\partial^\alpha e^{-kDk}\|_{L^2} \right)^{\frac{d}{4}} \]
\[ \leq C(1 + t)^{-\frac{d}{4}(1 - \frac{d}{4p}) + (\frac{d}{2} - \frac{d}{4})\frac{d}{4p}} \leq C \]
for any integer $s > d/2$, $1 \leq p \leq \infty$ and $t > 0$. Therefore, by the definition of the $M_p$-norm, we have the $L^p$-$L^p$ estimate
\[ \|F^{-1}(e^{-kDk}) \ast U_0\|_{L^p} \leq C\|U_0\|_{L^p}, \quad \forall 1 \leq p \leq \infty. \]

(85)
Finally, by applying the interpolation inequality and the estimates (82) and (85), we obtain (81). The proof is done. \(\square\)

**Remark 4.** Note that the derivative estimate (83) is true for all \(k \in \mathbb{R}^d\).

Let \(\chi_j\) for \(j = 1, 2, 3\) be cut-off functions on \(\mathbb{R}^d\) valued in \([0, 1]\) such that \(\text{supp } \chi_1 \subseteq \{k \in \mathbb{R}^d : |k| < \varepsilon\}\) for small \(\varepsilon > 0\), \(\text{supp } \chi_3 \subseteq \{k \in \mathbb{R}^d : |k| > \rho\}\) for large \(\rho > 0\) and \(\chi_2(k) := 1 - \chi_1(k) - \chi_3(k)\) for \(k \in \mathbb{R}^d\). We are now going to study the large-time behavior of the fundamental solution \(\Gamma_t\) to the system (1) in (19) in each partition of the frequency space.

For \((k, t) \in \mathbb{R}^d \times \mathbb{R}_+\), we recall the Fourier transform of the fundamental solution \(\Gamma_t\), namely
\[
\hat{\Gamma}_t(k) = e^{-E(ik)t},
\]
where \(E\) is given by (8). We also recall
\[
\hat{\psi}_t(k) = e^{-c\langle k - \mathbf{D}k \rangle + P(0)}, \quad \hat{\phi}_t(k) = e^{-c\langle k \mathbf{D}k \rangle + P(0) - ik},
\]
where \(c\) and \(D\) are given by (11), \(P(0)\) is given by (9) and \(P(1)\) is given by (17).

**5.2. Low-frequency analysis.** The aim of this subsection is to study the \(L^p-L^q\) estimate for the low-frequency part of \(\Gamma_t\) for any \(1 \leq q \leq p < \infty\). One thus considers \(\hat{\Gamma}_t\chi_1\).

**Lemma 5.2 (Derivative estimates).** Let \(p = p(x)\) be a scalar polynomial on \(\mathbb{R}^d\) such that the lowest order of \(p\) is \(h \geq 1\) and let \(\alpha \in \mathbb{N}^d\) with \(|\alpha| \geq 0\). There is a constant \(C > 0\) such that for small \(\mathbf{x} \in \mathbb{R}^d\) and \(t \geq 0\), we have
\[
|\partial^\alpha e^{p(x)t}| \leq C \sum_{\{I_j : j = 1, \ldots, r\}, r \leq |\alpha|} |\mathbf{x}|^{\sum_{k=1}^{h-1} k m_k} t^{\ell + \sum_{k=0}^{h-1} m_k} |e^{p(x)t}|,
\]
where the integer \(m_k \geq 0\) is the cardinality of \(\{j \in \{1, \ldots, r\} : |I_j| = h - k\}\) for each \(k \in \{0, \ldots, h-1\}\), the integer \(\ell \geq 0\) satisfies
\[
h\ell < |\alpha| - \sum_{k=0}^{h-1} (h - k) m_k
\]
and \(\{I_j : j = 1, \ldots, r\}\) is any possible partition of the index-set \(I_\alpha\) determined by \(\alpha\).

**Proof.** Let \(\alpha \in \mathbb{N}^d\) with \(|\alpha| \geq 0\) and \(p = p(x)\) be a polynomial on \(\mathbb{R}^d\) such that the lowest order of \(p\) is \(h \geq 1\). For any partition \(\{I_j : j = 1, \ldots, r\}\) of \(I_\alpha\) determined by \(\alpha\), by the definition of \(\partial_{I_j}\), there is a constant \(C(j) > 0\) such that
\[
|\partial_{I_j} p(x)| \leq C(j) \cdot \begin{cases} 1 & \text{if } |I_j| \geq h \\ |x|^k & \text{if } |I_j| = h - k \end{cases}
\]
for any \(k \in \{0, \ldots, h-1\}\) and small \(\mathbf{x} \in \mathbb{R}^d\), where \(|I_j|\) is the number of elements of the index-set \(I_j\) with possibly repeated indices. Note that \(\sum_{j=1}^r |I_j| = |I_{|\alpha|}| = |\alpha|\) by definition. It implies that there is a constant \(C(r) = \max_{j} C(j) > 0\) such that for small \(\mathbf{x} \in \mathbb{R}^d\) and \(t \geq 0\), we have
\[
|\partial_{I_1}(p(x)t) \ldots |\partial_{I_r}(p(x)t)| \leq C(r)|\mathbf{x}|^{\sum_{k=1}^{h-1} k m_k t^{\ell} + \sum_{k=0}^{h-1} m_k},
\]
where \( m_k \geq 0 \) is the cardinality of \( \{ j \in \{1, \ldots, r \} : |Z_j| = h-k \} \) for \( k \in \{0, \ldots, h-1 \} \) and \( \ell \geq 0 \) is the cardinality of \( I := \{ j \in \{1, \ldots, r \} : |Z_j| > h \} \). Moreover, we have

\[
 h\ell < \sum_{j \in I} |Z_j| = |\alpha| - \sum_{k=0}^{h-1}(h-k)m_k. \tag{91}
\]

We thus obtain (88) and (89) from (78), (90) and (91) with \( C = \max_{r} C(r) > 0 \). The proof is done. \( \square \)

Let \( P_0 \) be given by (55), we have the following.

**Proposition 4** (Low-frequency estimates). If the assumptions \( B \) and \( D \) hold, then for \( 1 \leq q \leq p \leq \infty \), there is a constant \( C > 0 \) such that for \( t > 0 \), we have

\[
 \| \mathcal{F}^{-1}(\hat{\Gamma}_t(k)P_0(ik) - \hat{\Phi}_t(k))\chi_1(k) \ast u_0 \|_{L^p} \leq C(1 + t)^{-\frac{q}{2} \left( \frac{d}{2} + \frac{1}{2} \right) - \frac{1}{2}} \| u_0 \|_{L^q}. \tag{92}
\]

If the condition \( S \) holds in addition, then we have

\[
 \| \mathcal{F}^{-1}(\hat{\Gamma}_t(k)P_0(ik) - \hat{\Phi}_t(k))\chi_1(k) \ast u_0 \|_{L^p} \leq C(1 + t)^{-\frac{q}{2} \left( \frac{d}{2} + \frac{1}{2} \right) - 1} \| u_0 \|_{L^q}. \tag{93}
\]

On the other hand, for \( 1 \leq q < 2 \leq p \leq \infty \), there are constants \( c > 0 \) and \( C > 0 \) such that for \( t > 0 \), we have

\[
 \| \mathcal{F}^{-1}(\hat{\Gamma}_t(k)(I - P_0(ik))\chi_1(k)) \ast u_0 \|_{L^p} \leq Ce^{-ct} \| u_0 \|_{L^q}. \tag{94}
\]

Moreover, (94) holds for \( 1 \leq q \leq p \leq \infty \) if \( \Gamma_t \) has compact support contained in \( \{(x, t) \in \mathbb{R}^d \times \mathbb{R} : |x/t| \leq C \} \) for some constant \( C > 0 \).

**Proof.** Under the assumptions \( B \) and \( D \), from (52) - (58) in Proposition 1, for small \( k \in \mathbb{R}^d \), one has

\[
 \hat{\Gamma}_t(k)\chi_1(k) = \hat{\Gamma}_t^{(1)}(k)\chi_1(k) + \hat{\Gamma}_t^{(2)}(k)\chi_1(k), \tag{95}
\]

where

\[
 \hat{\Gamma}_t^{(1)}(k) = e^{-\lambda_0(ik)t}P_0(ik) = e^{-c\cdot ik\cdot t - Dk\cdot t + \mathcal{O}(|k|^2)}(P_0^{(0)} + \mathcal{O}(|k|)). \tag{96}
\]

and

\[
 \hat{\Gamma}_t^{(2)}(k) = \sum_{j=1}^{s}e^{-E_j(ik)t}P_j(ik) = \sum_{j=1}^{s}e^{-\lambda_j^{(0)}t}e^{-N_j^{(0)}t + \mathcal{O}(|k|^2)}(P_j^{(0)} + \mathcal{O}(|k|)), \tag{97}
\]

where \( c \in \mathbb{R}^d \) and \( D \in \mathbb{R}^{d \times d} \) are given by (54), \( D \) is positive definite, \( P_0^{(0)} \) is the eigenprojection associated with \( 0 \in \sigma(B) \), \( \lambda_j^{(0)} \in \sigma(B) \setminus \{0\} \) with \( \Re \lambda_j^{(0)} > 0 \), \( P_j^{(0)} \) and \( N_j^{(0)} \) are the eigenprojection and eigen-nilpotent associated with \( \lambda_j^{(0)} \) respectively for \( j \in \{1, \ldots, s\} \) and \( s \) is the cardinality of \( \sigma(B) \setminus \{0\} \).

We now prove Proposition 4 by establishing the \( L^\infty - L^1 \) estimate firstly. Then, by constructing the \( L^p - L^p \) estimate for \( 1 \leq p \leq \infty \), we apply the interpolation inequality.

**Step 1.** \( L^\infty - L^1 \) estimate.

By changing the coordinates \( (x, t) \mapsto (x - ct, t) \), one can always assume that \( c = 0 \) without loss of generality. We study the \( L^\infty - L^1 \) estimate. Consider
By the Young inequality, we have

$$c > 0 \quad \text{and} \quad C > 0 \quad \text{such that} \quad |\hat{\Gamma}^{(1)}(k) - \hat{\Phi}_t(k)| \leq C e^{-C|k|^2 t} (|k|^3 t + |k|) \leq C(1 + t)^{-\frac{d}{2} - \frac{1}{2}}.$$  

Then, there are constants $c > 0$ and $C > 0$ such that

$$|\hat{\Gamma}^{(1)}(k) - \hat{\Phi}_t(k)| \leq |I| + |J| \leq C e^{-c|k|^2 t} (|k|^3 t + |k|).$$

Thus, we have

$$\|\hat{\Gamma}^{(1)}(k) - \hat{\Phi}_t(k)\|_{L^1} \leq C \int_{\mathbb{R}^d} e^{-c|k|^2 t} (|k|^3 t + |k|) \, dk \leq C (1 + t)^{-\frac{d}{2} - \frac{1}{2}}.$$  

By the Young inequality, we have

$$\|F^{-1}((\hat{\Gamma}^{(1)}(k) - \hat{\Phi}_t(k))\chi_1(k)) * u_0\|_{L^\infty} \leq \|((\hat{\Gamma}^{(1)} - \hat{\Phi}_t) \chi_1) \|_{L^1} \|u_0\|_{L^1} \leq C (1 + t)^{-\frac{d}{2} - \frac{1}{2}} \|u_0\|_{L^1}.$$  

Recall $\hat{\Gamma}^{(2)}_t$ in (97), one has

$$\hat{\Gamma}^{(2)}_t(k) = \sum_{j=1}^{s} e^{-E_j(k)t} P_j(k) = \sum_{j=1}^{s} e^{-\lambda_j^{(0)} t} e^{-N_j^{(0)} t + O(|k|) t} (P_j^{(0)} + O(|k|)),$$

where $\lambda_j^{(0)} \in \sigma(B) \backslash \{0\}$ with $\text{Re} \lambda_j^{(0)} > 0$, $P_j^{(0)}$ and $N_j^{(0)}$ are the eigenprojection and eigennilpotent associated with $\lambda_j^{(0)}$ respectively for $j \in \{1, \ldots, s\}$ and $s$ is the cardinality of $\sigma(B) \backslash \{0\}$. Thus, by the Householder theorem which is Theorem 7.1 p. 133 in [16], for any $\varepsilon > 0$, there is an induced norm such that $|N_j^{(0)}| \leq \varepsilon$ and due to the fact that every norm in a finite-dimensional space are equivalent, one deduces that since $|k|$ small and $\text{Re} \lambda_j^{(0)} > 0$ for all $j \in \{1, \ldots, s\}$, there are constants $c, c' > 0$ and $C > 0$ such that

$$|\hat{\Gamma}^{(2)}_t(k)\chi_1(k)| \leq C \sum_{j=1}^{s} e^{-\Re \lambda_j^{(0)} t} e^{ct + c'|k|t} |\chi_1(k)| \leq C e^{-ct} |\chi_1(k)|.$$  

Hence, we obtain

$$\|\hat{\Gamma}^{(2)}_t\chi_1\|_{L^1} \leq C e^{-ct} \left\|\chi_1\right\|_{L^1} \leq C e^{-ct}.$$  

It implies that

$$\|F^{-1}((\Gamma_t^{(2)}(k)\chi_1(k)) \|_{L^\infty} \leq \|\hat{\Gamma}^{(2)}_t\chi_1\|_{L^1} \|u_0\|_{L^1} \leq C e^{-ct} \|u_0\|_{L^1}.$$  

Therefore, the $L^\infty - L^1$ estimate holds, namely

$$\|F^{-1}((\hat{\Gamma}_t(k) P_0(k) - \hat{\Phi}_t(k))\chi_1(k)) \|_{L^\infty} = \|F^{-1}((\hat{\Gamma}_t^{(1)}(k) - \hat{\Phi}_t(k))\chi_1(k)) \|_{L^\infty} \leq C (1 + t)^{- \frac{d}{2} - \frac{1}{2}} \|u_0\|_{L^1}$$

and

$$\|F^{-1}((\hat{\Gamma}_t(k)(I - P_0(k))\chi_1(k)) \|_{L^\infty} = \|F^{-1}((\hat{\Gamma}_t^{(2)}(k)\chi_1(k)) \|_{L^\infty} \leq C e^{-ct} \|u_0\|_{L^1}.$$
Step 2. $L^p$-$L^p$ estimates.

We study the $L^p$-$L^p$ estimate for $1 \leq p \leq \infty$ by Lemma 3.3. In the spirit of Lemma 3.3, we need to estimate the $L^2$-norm of $\partial^\alpha((\hat{\Gamma}_t - \hat{\Phi}_t)\chi_1)$ for $\alpha \in \mathbb{N}^d$.

Recall the decomposition $\hat{\Gamma}_t\chi_1 = \hat{\Gamma}_t^{(1)}\chi_1 + \hat{\Gamma}_t^{(2)}\chi_1$ in (95), where $\hat{\Gamma}_t^{(1)}$ is given by (96) and $\hat{\Gamma}_t^{(2)}$ is given by (97). We primarily estimate the $L^2$-norm of $\partial^\alpha((\hat{\Gamma}_t^{(1)} - \hat{\Phi}_t)\chi_1)$ for any $\alpha \in \mathbb{N}^d$ by considering the decomposition $(\hat{\Gamma}_t^{(1)} - \hat{\Phi}_t)\chi_1 = I + J$ in (98), where $I$ and $J$ are given by (99) and (100) respectively. By the Leibniz rule, one has

$$\partial^\alpha I = \sum_{\nu \leq \alpha} \sum_{\tau \leq \nu} \binom{\alpha}{\nu} \binom{\nu}{\tau} \partial^\tau e^{-kDk} |\partial^{\nu - \tau}(e^{O(|k|^3)t} - 1)\partial^{\nu - \tau} + \alpha_1(k)|$$

$$= I^{(1)} + I^{(2)}, \quad (104)$$

where

$$I^{(1)} := \sum_{\nu \leq \alpha} \sum_{\tau \leq \nu} \binom{\alpha}{\nu} \binom{\nu}{\tau} \partial^\tau e^{-kDk} |\partial^{\nu - \tau}(e^{O(|k|^3)t} - 1)\partial^{\nu - \tau} + \alpha_1(k)|$$

and

$$I^{(2)} := \sum_{\nu \leq \alpha} \sum_{\tau \leq \nu} \binom{\alpha}{\nu} \binom{\nu}{\tau} \partial^\tau e^{-kDk} |\partial^{\nu - \tau} e^{O(|k|^3)t}\partial^{\nu - \tau} + \alpha_1(k)|. \quad (105)$$

By the estimate (88) in Lemma 5.2, since $\chi_1 \in C_c^\infty(\mathbb{R}^d)$ and $D \in \mathbb{R}^{d \times d}$ is positive definite, there are constants $c > 0$ and $C > 0$ such that

$$|I^{(1)}| \leq C \sum_{\nu \leq \alpha} |\partial^\nu e^{-kDk}||e^{O(|k|^3)t} - 1|$$

$$\leq C \sum_{\nu \leq \alpha} \sum_{\{I_{\nu,j} : 1, \ldots, r \} \in \nu} |k|^{m_1 + 3 \ell + m_0 + m_1 + 1} e^{-|c||k|^2 t},$$

where $m_k \geq 0$ is the cardinality of the set $\{j \in \{1, \ldots, r\} : |I_{\nu,j}| = |\nu| - k\}$ for $k = 0, 1$ and $\ell \geq 0$ satisfies

$$2\ell < |\nu| - 2m_0 - m_1$$

and $\{I_{\nu,j} : j = 1, \ldots, r\}$ is any possible partition of the index-set $I_{\nu}$ determined by $\nu$. Thus, we have

$$\|I^{(1)}\|_{L^2}^2 \leq C \sum_{\nu \leq \alpha} \sum_{\{I_{\nu,j} : 1, \ldots, r \} \in \nu} \int_{\mathbb{R}^d} |k|^{2(m_1 + 3)\ell + m_0 + m_1 + 1} e^{-2c\|k\|^2 t} dk$$

$$\leq C \sum_{\nu \leq \alpha} \sum_{\{I_{\nu,j} : 1, \ldots, r \} \in \nu} (1 + t)^{-\frac{d}{2} - 1 + 2m_0 + m_1 + 2\ell}$$

$$\leq C \sum_{\nu \leq \alpha} (1 + t)^{|\nu| - \frac{d}{2} + 1} \leq C(1 + t)^{|\alpha| - \frac{d}{2} - 1} \quad (107)$$

since $|\nu| \leq |\alpha|$ for all $\nu \leq \alpha$.

Similarly, we can estimate $I^{(2)}$ in (106). Since $\chi_1 \in C_c^\infty(\mathbb{R}^d)$ and $D \in \mathbb{R}^{d \times d}$ is positive definite, from (106) and the estimate (88) in Lemma 5.2, there are constants...
c, c' > 0 and C > 0 such that

\[ |I^{(2)}| \]
\[ \leq C \sum_{\nu \leq \alpha} \sum_{\tau < \nu} |\partial^\nu e^{-kD_\tau t} \chi(k)| \leq C \sum_{\nu \leq \alpha} \sum_{\tau < \nu} |k|^{m_0 + m_1 + m_2} e^{-c|k|^2 t + c'|k|^3 t}, \]

where \( m_k \geq 0 \) is the cardinality of the set \( \{ j \in \{1, \ldots, r \} : |I_j| = |\tau| - k \} \) for \( k = 0, 1 \) and \( m_k' \geq 0 \) is the cardinality of the set \( \{ j \in \{1, \ldots, r' \} : |I'_j| = |\nu - \tau| - k \} \) for \( k = 0, 1, 2 \) and \( \ell, \ell' \geq 0 \) satisfies

\[ 2\ell < |\tau| - 2m_0 - m_1, \quad 3\ell' < |\nu - \tau| - 3m_0' - 2m_1' - m_2', \]

and \( \{ I_j : j = 1, \ldots, r \} \) is any possible partition of the index-set \( I_\tau \) determined by \( \tau \) and \( \{ I'_j : j = 1, \ldots, r' \} \) is any possible partition of the index-set \( I_{\nu - \tau} \) determined by \( \nu - \tau \). Hence, since \( |k| \) small, we have

\[ \| I^{(2)} \|^2_{L^2} \leq C \sum_{\nu \leq \alpha} \sum_{\tau < \nu} \int_{\mathbb{R}^d} |k|^{2(m_1 + m_1' + 2m_2')} (1 + t)^{-\frac{d}{2} + 2m_0 + 2m_1 + m_1' + 2\ell + 2\ell'} e^{-2c|k|^2 t} \mathrm{d}k \]

\[ \leq C \sum_{\nu \leq \alpha} \sum_{\tau < \nu} (1 + t)^{-\frac{d}{2} + 2m_0 + 2m_1 + m_1' + 2\ell + 2\ell'} \]

\[ \leq C \sum_{\nu \leq \alpha} \sum_{\tau < \nu} (1 + t)^{-\frac{d}{2} + |\tau| + |\nu - \tau| - (\ell' + m_0 + m_1')} \]

\[ \leq C \sum_{\nu \leq \alpha} \sum_{\tau < \nu} (1 + t)^{|\nu| - \frac{d}{2} - \tau'} \leq C (1 + t)|\alpha|^{-\frac{d}{2} - 1} \quad (108) \]

since \( |\tau| + |\nu - \tau| = |\nu| \leq |\alpha| \) for any \( \tau \leq \nu \) and \( \ell' + m_0 + m_1' + m_2' = r' \geq 1 \) by definition and \( \tau < \nu \).

We continue to estimate \( J \) in (100). By the Leibniz rule, one has

\[ \partial^\alpha J = \sum_{\nu \leq \alpha} \sum_{\tau < \nu} \left( \begin{array}{c} \alpha \\ \nu \\ \tau \end{array} \right) \partial^\nu e^{-kD_\tau t} O(|k|^3) \partial^{\alpha - \nu} \chi(k) \]

\[ = J^{(1)} + J^{(2)}, \quad (109) \]

where

\[ J^{(1)} := \sum_{\nu \leq \alpha} \left( \begin{array}{c} \alpha \\ \nu \\ \tau \end{array} \right) \partial^\nu e^{-kD_\tau t} O(|k|^3) \partial^{\alpha - \nu} \chi(k) \quad (110) \]

and

\[ J^{(2)} := \sum_{\nu \leq \alpha} \sum_{\tau < \nu} \left( \begin{array}{c} \alpha \\ \nu \\ \tau \end{array} \right) \partial^\nu e^{-kD_\tau t} O(|k|^3) \partial^{\alpha - \nu} \chi(k). \quad (111) \]
We then begin with $J^{(1)}$. Since $\chi_1 \in C_c^\infty(\mathbb{R}^d)$ and $D \in \mathbb{R}^{d \times d}$ is positive definite, from (110) and the estimate (88) in Lemma 5.2, there are constants $c, c' > 0$ and $C > 0$ such that

$$|J^{(1)}| \leq C \sum_{\nu \leq \alpha} |\partial^\nu e^{-k^2 D t + O(|k|^3 t)}|O(|k|)|$$

$$\leq C \sum_{\nu \leq \alpha} \sum_{\nu \in \{\mathcal{I}_j : j = 1, \ldots, r\}} |k|^{m_1 + 1 + \ell + m_0 + m_1 - c|k|^2 + c'|k|^3 t},$$

where $m_k \geq 0$ is the cardinality of the set $\{j \in \{1, \ldots, r\} : |\mathcal{I}_j| = |\nu| - k\}$ for $k = 0, 1$ and $\ell \geq 0$ satisfies

$$2\ell < |\nu| - 2m_0 - m_1$$

and $\{\mathcal{I}_j : j = 1, \ldots, r\}$ is any possible partition of the index-set $\mathcal{I}_\nu$ determined by $\nu$. Since $|k|$ small, it implies that

$$\|J^{(1)}\|_2^2 \leq C \sum_{\nu \leq \alpha} \sum_{\nu \in \{\mathcal{I}_j : j = 1, \ldots, r\}} \int_{\mathbb{R}^d} |k|^{2(m_1 + 1) + 2(\ell + m_0 + m_1)} e^{-2c|k|^2 t} \, dk$$

$$\leq C \sum_{\nu \leq \alpha} \sum_{\nu \in \{\mathcal{I}_j : j = 1, \ldots, r\}} (1 + t)^{-\frac{d}{2} - 1 + 2\ell + 2m_0 + m_1}$$

$$\leq C \sum_{\nu \leq \alpha} (1 + t)^{|\nu| - \frac{d}{2} - 1} \leq C(1 + t)^{|\alpha| - \frac{d}{2} - 1}$$

(112)

since $|\nu| \leq |\alpha|$ for all $\nu \leq \alpha$.

We estimate $J^{(2)}$ in (111). Since $\chi_1 \in C_c^\infty(\mathbb{R}^d)$ and $D \in \mathbb{R}^{d \times d}$ is positive definite, from (111) and the estimate (88) in Lemma 5.2, there are constants $c, c' > 0$ and $C > 0$ such that

$$|J^{(2)}| \leq C \sum_{\nu \leq \alpha} \sum_{\nu \in \{\mathcal{I}_j : j = 1, \ldots, r\}} |\partial^\nu e^{-k^2 D t + O(|k|^3 t)}|O(|k|)|$$

$$\leq C \sum_{\nu \leq \alpha} \sum_{\nu \in \{\mathcal{I}_j : j = 1, \ldots, r\}} |k|^{m_1 + \ell + m_0 + m_1 - c|k|^2 + c'|k|^3 t},$$

where $m_k \geq 0$ is the cardinality of the set $\{j \in \{1, \ldots, r\} : |\mathcal{I}_j| = |\tau| - k\}$ for $k = 0, 1$ and $\ell \geq 0$ satisfies

$$2\ell < |\tau| - 2m_0 - m_1$$

and $\{\mathcal{I}_j : j = 1, \ldots, r\}$ is any possible partition of the index-set $\mathcal{I}_\tau$ determined by $\tau$. Thus, we have

$$\|J^{(2)}\|_2^2 \leq C \sum_{\nu \leq \alpha} \sum_{\nu \in \{\mathcal{I}_j : j = 1, \ldots, r\}} \int_{\mathbb{R}^d} |k|^{2m_1 + 2(\ell + m_0 + m_1)} e^{-2c|k|^2 t + 2c'|k|^3 t} \, dk$$

$$\leq C \sum_{\nu \leq \alpha} \sum_{\nu \in \{\mathcal{I}_j : j = 1, \ldots, r\}} (1 + t)^{-\frac{d}{2} + 2\ell + 2m_0 + m_1}$$

$$\leq C \sum_{\nu \leq \alpha} (1 + t)^{|\tau| - \frac{d}{2}} \leq C \sum_{\nu \leq \alpha} (1 + t)^{|\nu| - \frac{d}{2} - 1} \leq C(1 + t)^{|\alpha| - \frac{d}{2} - 1}$$

(113)

since $|\tau| \leq |\nu| - 1$ for all $\tau < \nu$ and $|\nu| \leq |\alpha|$ for all $\nu \leq \alpha$.

Therefore, from (98) - (100) and (104) - (113), one has

$$\|\partial^\alpha((\hat{\Phi}_t^{(1)} - \hat{\Phi}_t)\chi_1)\|_2 \leq C(1 + t)^{|\alpha| - \frac{d}{2} - \frac{1}{2}}, \quad \forall t > 0, \alpha \in \mathbb{N}^d.$$  

(114)
Let $s \in \mathbb{Z}_+$ satisfying $s > d/2$, then by the Carlson–Beurling inequality (27) in Lemma 3.3 and (114), we obtain
\[
\|(\hat{\Gamma}_t^{(1)} - \Phi_t)\chi_1\|_{M_p} \leq \|(\hat{\Gamma}_t^{(1)} - \Phi_t)\chi_1\|_{L_2}^{1-\frac{d}{2p}} \left( \sum_{|\alpha| = s} \|\partial^\alpha((\hat{\Gamma}_t^{(1)} - \Phi_t)\chi_1)\|_{L^2}^\frac{d}{p} \right)^{\frac{1}{2}} \leq C(1 + t)^{-\frac{d}{2p}} \leq C(1 + t)^{-\frac{d}{2}} (115)
\]
for all $1 \leq p \leq \infty$ and $t > 0$.

It follows from (115) and the definition of the $M_p$-norm that for any $u_0 \in L^p(\mathbb{R}^d)$, one has
\[
\|\mathcal{F}^{-1}((\hat{\Gamma}_t(k) P_0(k) - \hat{\Phi}_t(k))\chi_1(k)) * u_0\|_{L^p} \leq \|(\hat{\Gamma}_t^{(1)} - \Phi_t)\chi_1\|_{M_p}\|u_0\|_{L^p} \leq C(1 + t)^{-\frac{d}{2}} \|u_0\|_{L^p} (116)
\]
for all $1 \leq p \leq \infty$ and $t > 0$.

We estimate the remain parts. Recall $\hat{\Gamma}_t^{(2)}$ in (97) and the estimate (101), there are constants $c > 0$ and $C > 0$ such that
\[
|\hat{\Gamma}_t^{(2)}(k)\chi_1(k)| \leq Ce^{-ct}|\chi_1(k)|. (117)
\]
Thus, by the Parseval identity, one has
\[
\|\mathcal{F}^{-1}(\hat{\Gamma}_t(k)(I - P_0(k))\chi_1(k)) * u_0\|_{L^2} = \|\hat{\Gamma}_t^{(2)} \chi_1 \hat{u}_0\|_{L^2} \leq Ce^{-ct} \|\chi_1\|_{L^\infty}\|\hat{u}_0\|_{L^2} \leq Ce^{-ct}\|u_0\|_{L^2}, \quad \forall t > 0. (118)
\]

Moreover, if $\Gamma_t$ has compact support contained in $\{(x, t) \in \mathbb{R}^d \times \mathbb{R} : |x/t| \leq C\}$ for some constant $C > 0$. We then obtain from (117) and the Young inequality that there is $c', c > 0$ and $C > 0$ such that for $1 \leq p \leq \infty$, one has
\[
\|\mathcal{F}^{-1}(\hat{\Gamma}_t(k)(I - P_0(k))\chi_1(k)) * u_0\|_{L^p} \leq C\|\mathcal{F}^{-1}(\hat{\Gamma}_t(k)(I - P_0(k))\chi_1(k))\|_{L^1}\|u_0\|_{L^p} \leq C \left( \int_{|k| \leq C} \int_{|k| < \epsilon} e^{ct\epsilon k}\hat{\Gamma}_t^{(2)}(k)\chi_1(k) \, dk \, dx \right)\|u_0\|_{L^p} \leq Ce^{-ct} t^d\|u_0\|_{L^p} \leq Ce^{-ct}\|u_0\|_{L^p}, \quad \forall t > 0. (119)
\]

**Step 3. Interpolation.**

Finally, consider the interpolation inequality, from (102) and (116), we obtain (92), namely for $1 \leq q \leq p \leq \infty$ and $t > 0$, one has
\[
\|\mathcal{F}^{-1}((\hat{\Gamma}_t(k)P_0(k) - \hat{\Phi}_t(k))\chi_1(k)) * u_0\|_{L^p} \leq C(1 + t)^{-\frac{d}{2}(q-1)}\frac{1}{q-1}\|u_0\|_{L^q}. (120)
\]

We also obtain (94) from (103) and (118) i.e. for $1 \leq q \leq 2 \leq p \leq \infty$, we have
\[
\|\mathcal{F}^{-1}(\hat{\Gamma}_t(k)(I - P_0(k))\chi_1(k)) * u_0\|_{L^p} \leq Ce^{-ct}\|u_0\|_{L^q}. (120)
\]

Moreover, if $\Gamma_t$ has compact support in $\{(x, t) \in \mathbb{R}^d \times \mathbb{R} : |x/t| \leq C\}$ for some constant $C > 0$, then from (103) and (119), we also have (120) for $1 \leq q \leq p \leq \infty$. 


Under the symmetry property $S$.

Moreover, if the condition $S$ holds in addition, then for small $k$, from (52) - (59) in Proposition 1, one has $\hat{\Gamma}_t\chi_1 = \hat{\Gamma}_t^{(1)}\chi_1 + \hat{\Gamma}_t^{(2)}\chi_1$ where

$$\hat{\Gamma}_t^{(1)}(k) = e^{-\lambda_0(ik)t}P_0(ik)$$

$$= e^{-kDk + \mathcal{O}(|k|^4)t}(P_0^{(0)} + P_0^{(1)} \cdot ik + \mathcal{O}(|k|^2))$$

and

$$\hat{\Gamma}_t^{(2)}(k) = \sum_{j=1}^s e^{-E_j(ik)t}P_j(ik)$$

$$= \sum_{j=1}^s e^{-\lambda_j^{(0)}t}e^{-N_j^{(0)}t+\mathcal{O}(|k|^4)t}(P_j^{(0)} + \mathcal{O}(|k|)),$$

where $D \in \mathbb{R}^{d \times d}$ is given by (54) and is positive definite, $P_0^{(0)}$ is the eigenprojection associated with $0 \in \sigma(B)$, $P_0^{(1)} \in (\mathbb{R}^{n \times n})^d$ is in (56), $\lambda_j^{(0)} \in \sigma(B)\setminus\{0\}$ with $\text{Re}\lambda_j^{(0)} > 0$, $E_j^{(0)}$ and $N_j^{(0)}$ are the eigenprojection and eigen-nilpotent associated with $\lambda_j^{(0)}$ respectively for $j \in \{1, \ldots, s\}$ and $s$ is the cardinality of $\sigma(B)\setminus\{0\}$. Hence, consider

$$\hat{\Psi}_t(k) = e^{-kDk}(P_0^{(0)} + P_0^{(1)} \cdot ik),$$

one has

$$\hat{\Gamma}_t^{(1)}(k) - \hat{\Psi}_t(k)\chi_1(k) = I + J,$$

where

$$I := e^{-kDk}(e^{\mathcal{O}(|k|^4)t} - 1)(P_0^{(0)} + P_0^{(1)} \cdot ik)\chi_1(k),$$

$$J := e^{-kDk + \mathcal{O}(|k|^4)t}\mathcal{O}(|k|^2)\chi_1(k).$$

The estimates are then similar to the previous case. We omit the details. We thus obtain for $1 \leq q \leq p \leq \infty$ and $t > 0$

$$\|\mathcal{F}^{-1}(\hat{\Gamma}_t(k)P_0(k) - \hat{\Psi}_t(k))\chi_1(k)\ast u_0\|_{L^p} \leq C(1 + t)^{-\frac{1}{2}(\frac{1}{q} - \frac{1}{p}) - 1}\|u_0\|_{L^q}.$$

The proof is done since the other computations are also similar to before.

5.3. Intermediate-frequency analysis. We consider the intermediate-frequency part by considering $\hat{\Gamma}_t\chi_2$, $\hat{\Phi}_t\chi_2$ and $\hat{\Psi}_t\chi_2$, where $\hat{\Gamma}_t$, $\hat{\Phi}_t$ and $\hat{\Psi}_t$ are given by (86) and (87) respectively. One has the following.

**Proposition 5** (Intermediate-frequency estimates). If the condition $D$ holds, then for $1 \leq q \leq 2 \leq p \leq \infty$, there are constants $c > 0$ and $C > 0$ such that

$$\|\mathcal{F}^{-1}(\hat{\Gamma}_t(k)\chi_2(k))\ast u_0\|_{L^p} \leq Ce^{-ct}\|u_0\|_{L^q}, \quad \forall t > 0.$$  

Moreover, (123) holds for $1 \leq q \leq p \leq \infty$ if $\Gamma_t$ has compact support contained in $\{(x, t) \in \mathbb{R}^d \times \mathbb{R}: |x/t| \leq C\}$ for some constant $C > 0$.

**Proof.** Recall $E(ik) = B + A(ik)$ for $k \in \mathbb{R}^d$ in (8). We consider $\hat{\Gamma}_t$ where $\hat{\Gamma}_t(k) = e^{-E(ik)t}$. Since the condition $D$ holds, $\text{Re}\lambda(ik) > 0$ for any eigenvalue $\lambda(ik)$ of $E(ik)$ and $k \neq 0 \in \mathbb{R}^d$. Thus, the operator $e^{-E(ik)t}$ has the spectral radius $\text{rad}(e^{-E(ik)t}) < 1$ for almost everywhere. It follows from the Householder theorem in [16] that there is an induced norm such that

$$0 < \varphi := \text{ess sup}_{\mathbb{R}^d} |e^{-E(ik)t}| < 1.$$
Then, for $t > 0$ with integer part $m$, since $\log \varphi < 0$, there are $c, C > 0$ such that one has
\[
\|\hat{\Gamma}(\kappa)\|_2 \leq |e^{-E(i\kappa)\cdot t\cdot m}|e^{-E(i\kappa)(t-m)}\|\chi_2(\kappa)| \leq \varphi^{-m}e^{E(i\kappa)|\chi_2(\kappa)|} \leq \varphi^{-1}e^{(m+1)\log \varphi E(i\kappa)|\chi_2(\kappa)|} \leq Ce^{-ct}e^{E(i\kappa)|\chi_2(\kappa)|},
\]
(124)

We study the $L^\infty-L^1$ estimate. By the Young inequality and from (124), there are constants $c > 0$ and $C > 0$ such that for $t > 0$, we have
\[
\|\mathcal{F}^{-1}(\hat{\Gamma}(\kappa)\chi_2(\kappa)) \ast u_0\|_{L^\infty} \leq C\|\mathcal{F}^{-1}(\hat{\Gamma}(\kappa)\chi_2(\kappa))\|_{L^\infty} \|u_0\|_{L^1},

\leq C\|\hat{\Gamma}\chi_2\|_{L^1} \|u_0\|_{L^1} \leq Ce^{-ct} \|u_0\|_{L^1} .
\]
(125)

We prove the $L^2-L^2$ estimate. It follows from the Parseval identity and the estimate (124) that for $t > 0$, one has
\[
\|\mathcal{F}^{-1}(\hat{\Gamma}(\kappa)\chi_2(\kappa)) \ast u_0\|_{L^2} \leq C\|\hat{\Gamma}\chi_2\|_{L^\infty} \|\hat{u}_0\|_{L^2} \leq C e^{-ct} \|u_0\|_{L^2}
\]
(126)
for some constants $c > 0$ and $C > 0$.

Moreover, if $\Gamma_t$ has compact support contained in $\{(x, t) \in \mathbb{R}^d \times \mathbb{R} : |x/t| \leq C\}$ for some constant $C > 0$. From (124) and the Young inequality, there are $c', c > 0$ and $C > 0$ such that for $1 \leq p \leq \infty$, one has
\[
\|\mathcal{F}^{-1}(\hat{\Gamma}(\kappa)\chi_2(\kappa)) \ast u_0\|_{L^p} \leq C\|\mathcal{F}^{-1}(\hat{\Gamma}(\kappa)\chi_2(\kappa))\|_{L^1} \|u_0\|_{L^p} \leq C e^{-ct} \|u_0\|_{L^p},
\]
(127)

We finish the proof of (123) by applying the interpolation inequality and by using the $L^\infty-L^1$ estimate (125), the $L^2-L^2$ estimate (126) and the $L^p-L^p$ estimates (127).

Moreover, we estimate $\mathcal{F}^{-1}(\hat{\Phi}(\kappa)\chi_2(\kappa)) \ast u_0$ and $\mathcal{F}^{-1}(\hat{\Psi}(\kappa)\chi_2(\kappa)) \ast u_0$.

**Proposition 6.** If the conditions $\mathcal{B}$ and $\mathcal{D}$ hold, then for $1 \leq q \leq p \leq \infty$, there are constants $c > 0$ and $C > 0$ such that for $t > 0$, one has
\[
\|\mathcal{F}^{-1}(\hat{\Phi}(\kappa)\chi_2(\kappa)) \ast u_0\|_{L^p} \leq C e^{-ct} \|u_0\|_{L^q} .
\]
(128)

Similarly, we have
\[
\|\mathcal{F}^{-1}(\hat{\Psi}(\kappa)\chi_2(\kappa)) \ast u_0\|_{L^p} \leq C e^{-ct} \|u_0\|_{L^q} .
\]
(129)

**Proof.** We estimate $\mathcal{F}^{-1}(\hat{\Phi}(\kappa)\chi_2(\kappa)) \ast u_0$ and the other is similar. Recall
\[
\hat{\Phi}_t(\kappa) = e^{-c \cdot \kappa t - k \cdot D \cdot t} \cdot P_0^{(0)},
\]
where $c \in \mathbb{R}^d$ and $D \in \mathbb{R}^{d \times d}$ are given by (54) where $D$ is positive definite under the assumptions $\mathcal{B}$ and $\mathcal{D}$.
Since we can assume that $c=0$ and since $\text{supp } \chi_2 \subseteq \{k \in \mathbb{R}^d : \varepsilon \leq |k| \leq \rho\}$ for some $\varepsilon, \rho > 0$, by the Young inequality, there are constants $c, c' > 0$ and $C > 0$ such that for $t > 0$, we have the $L^\infty - L^1$ estimate
\[
\|\mathcal{F}^{-1}(\hat{\Phi}_t(k)\chi_2(k)) * u_0\|_{L^\infty} \leq C\|\mathcal{F}^{-1}(e^{-kD_{\mathbb{R}^d}}P_0^{(0)}\chi_2(k))\|_{L^\infty}\|u_0\|_{L^1} \leq C e^{-c't}\|\hat{\Phi}_t\|_{L^2}, \quad \forall t > 0.
\]

We study the $L^p - L^p$ estimate for $1 \leq p \leq \infty$. Let $\alpha \in \mathbb{N}^d$, by the Leibniz formula, (83) and Remark 4, one has
\[
|\partial^\alpha(e^{-kD_{\mathbb{R}^d}}P_0^{(0)}\chi_2(k))| \leq C \sum_{\nu \leq \alpha} |\partial^\nu e^{-kD_{\mathbb{R}^d}}| |\partial^{\alpha - \nu} \chi_2(k)|
\leq C \sum_{\nu \leq \alpha} \sum_{\{I_j : j = 1, \ldots, r\}, r \leq |\alpha|} |k|^m \|\partial^\nu| e^{-\varepsilon|k|^2} \hat{\partial}^{\alpha - \nu} \chi_2(k)|,
\]
where $\{I_j : j = 1, \ldots, r\}$ is any possible partition of the index-set $I_\alpha$ determined by $\alpha$ and $m + 2\ell = |\alpha|$. Hence, we have
\[
\|\partial^\alpha(e^{-kD_{\mathbb{R}^d}}P_0^{(0)}\chi_2(k))\|_{L^2} \leq C e^{-ct}, \quad \forall t > 0, \alpha \in \mathbb{N}^d.
\]

Therefore, by the Carlson–Beurling inequality (27) in Lemma 3.3, one obtains
\[
\|e^{-kD_{\mathbb{R}^d}}P_0^{(0)}\chi_2(k)\|_{M_p} \leq C e^{-ct}, \quad \forall 1 \leq p \leq \infty, t > 0.
\]

Finally, by the interpolation inequality, the estimates (130) and (131), we obtain (128). The proof is done. \hfill \Box

### 5.4. High-frequency analysis.

The aim of this part is to give the $L^2 - L^2$ estimate of the high-oscillation part of $\Gamma_\ell$ given by (19), which is $\hat{\Gamma}_\ell \chi_3$ in the Fourier space, where $\Gamma_\ell$ is given by (86).

**Proposition 7** (High-frequency estimates). If the conditions $A$, $R$ and $D$ hold, then there are constants $c > 0$ and $C > 0$ such that one has the estimate
\[
\|\mathcal{F}^{-1}(\hat{\Gamma}_\ell(k)\chi_3(k)) * u_0\|_{L^2} \leq C e^{-ct}\|u_0\|_{L^2}, \quad \forall t > 0.
\]

**Proof.** Under the assumptions $A$, $R$, and $D$, for almost everywhere and large $k \in \mathbb{R}^d$, from (70) - (72), we have
\[
\hat{\Gamma}_\ell(k)\chi_3(k) = R \sum_{j=1}^r \sum_{m=1}^{s_j} e^{-\alpha_j(m)k} e^{-\beta_j m t} e^{\Theta_j^{(0)}(t + O(|k|^{-1}))} R^{-1} \chi_3(k),
\]
where $R$ is an invertible matrix satisfying $A$ and $R$, $\alpha_j(m) = j \nu_j^m \chi_j^m / |k|$ where $\nu_j^m$ is given by (68), $\beta_j m$ with $\text{Re } \beta_j m > 0$ is the $m$-th nonzero eigenvalue of $\Pi_j^{(0)} R^{-1} B \Pi_j^{(0)}$ with the associated eigenprojection $\Pi_j^{(0)}$ and eigennilpotent $\Theta_j^{(0)}$ where $\Pi_j^{(0)}$ is in (69).

Thus, by the Householder theorem in [16], for any $\varepsilon > 0$, there is an induced norm such that $|\Theta_j^{(0)}| \leq \varepsilon$ and due to the fact that every norms in a finite-dimensional
Similarly, we have
\|
\mathcal{F}^{-1}(\hat{\chi}_3(k)) * u_0 \|_{L^2} \leq C e^{-ct} \| u_0 \|_{L^2}
\]
for some constants $c, C > 0$ and for all $t > 0$. We finish the proof.

Moreover, we estimate
\|
\mathcal{F}^{-1}(\hat{\Phi}(k)\chi_3(k)) * u_0 \|_{L^2} \leq C e^{-ct} \| u_0 \|_{L^2}
\]
and
\|
\mathcal{F}^{-1}(\hat{\Psi}(k)\chi_3(k)) * u_0 \|_{L^2} \leq C e^{-ct} \| u_0 \|_{L^2}
\]
for $k > 0, 0 < q < p < \infty,$ and $c, C > 0$ such that for $t > 0$.

Proposition 8. If the conditions $B$ and $D$ hold, then for $1 \leq q \leq p \leq \infty$, there are constants $c > 0$ and $C > 0$ such that for $t > 0$, one has
\[
\| \mathcal{F}^{-1}(\hat{\Phi}(k)\chi_3(k)) * u_0 \|_{L^p} \leq C e^{-ct} \| u_0 \|_{L^p}.
\]
Similarly, we have
\[
\| \mathcal{F}^{-1}(\hat{\Psi}(k)\chi_3(k)) * u_0 \|_{L^p} \leq C e^{-ct} \| u_0 \|_{L^p}.
\]

Proof. Similarly to the proof of Proposition 6 where $\chi_2$ is substituted by $\chi_3$. The proof is done.

Acknowledgments. The author is grateful to Professor Corrado Mascia for his suggestion and useful comments to improve this paper. The author also would like to thank the handling editor and the referees for their helpful remarks.

REFERENCES

[1] H. Bahouri, J.-Y. Chemin and R. Danchin, Fourier Analysis and Nonlinear Partial Differential Equations, Springer-Verlag, Berlin-Heidelberg, 2011.
[2] J. Bergh and J. L"{o}fstr"{o}m, Interpolation Spaces. An Introduction, Springer-Verlag, Berlin-Heidelberg, 1976.
[3] S. Bianchini, B. Hanouzet and R. Natalini, Asymptotic behavior of smooth solutions for partially dissipative hyperbolic systems with a convex entropy, Comm. Pure Appl. Math., 60 (2007), 1559–1622.
[4] A. Bressan, An ill posed Cauchy problem for a hyperbolic system in two space dimensions, Rend. Sem. Mat. Univ. Padova, 110 (2003), 103–117.
[5] G. Craciun, A. Brown and A. Friedman, A dynamical system model of neurofilament transport in axons, J. Theoret. Biol., 237 (2005), 316–322.
[6] S. Goldstein, On diffusion by discontinuous movements, and on the telegraph equation, Quart. J. Mech. Appl. Math., 4 (1951), 129–156.
[7] T. Hosono and T. Ogawa, Large time behavior and $L^p-L^q$ estimate of solutions of 2-dimensional nonlinear damped wave equations, J. Differential Equations, 203 (2004), 82–118.
[8] M. Kac, A stochastic model related to the telegrapher’s equation. Reprinting of an article published in 1956, Papers arising from a Conference on Stochastic Differential Equations (Univ. Alberta, Edmonton, Alta., 1972), Rocky Mountain J. Math., 4 (1974), 497–509.
[9] T. Kato, Perturbation Theory For Linear Operators, Springer-Verlag, Berlin, 1995.
[10] S. Kawashima, Global existence and stability of solutions for discrete velocity models of the Boltzmann equation, North-Holland Math. Stud., 98 (1984), 59–85.
[11] P. Marcati and K. Nishihara, The $L^p-L^q$ estimates of solutions to one-dimensional damped wave equations and their application to the compressible flow through porous media, J. Differential Equations, 191 (2003), 445–469.
C. Mascia, Exact representation of the asymptotic drift speed and diffusion matrix for a class of velocity-jump processes, *J. Differential Equations*, 260 (2016), 401–426.

C. Mascia and T. T. Nguyen, $L^p$-$L^q$ decay estimates for dissipative linear hyperbolic systems in 1D, *J. Differential Equations*, 263 (2017), 6189–6230.

T. Narazaki, $L^p$-$L^q$ estimates for damped wave equations and their applications to semi-linear problem, *J. Math. Soc. Japan*, 56 (2004), 585–626.

K. Nishihara, $L^p$-$L^q$ estimates of solutions to the damped wave equation in 3-dimensional space and their application, *Math. Z.*, 244 (2003), 631–649.

D. Serre, *Matrices. Theory and Applications*, Springer-Verlag, New York, 2002.

Y. Shizuta and S. Kawashima, Systems of equations of hyperbolic-parabolic type with applications to the discrete Boltzmann equation, *Hokkaido Math. J.*, 14 (1985), 249–275.

Y. Ueda, R. Duan and S. Kawashima, Decay structure for symmetric hyperbolic systems with non-symmetric relaxation and its application, *Arch. Ration. Mech. Anal.*, 205 (2012), 239–266.

T. Umeda, S. Kawashima and Y. Shizuta, On the decay of solutions to the linearized equations of electromagnetofluid dynamics, *Japan J. Appl. Math.*, 1 (1984), 435–457.

Received for publication August 2017.

*E-mail address*: nguyen.tienthinh@gssi.infn.it