ARITHMETIC DUALITY THEOREMS FOR 1-MOTIVES
OVER FUNCTION FIELDS

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Abstract. In this paper we obtain a Poitou-Tate exact sequence for finite and flat group schemes over a global function field. In addition, we extend the duality theorems for 1-motives over number fields obtained by D.Harari and T.Szamuely to the function field case.

1. Introduction

Let \( X \) be a smooth projective curve over a finite field of characteristic \( p \) and let \( K \) be the function field of \( X \). In this paper we establish a Poitou-Tate exact sequence for finite and flat group schemes of \( p \)-power order over \( K \), thereby extending the well-known Poitou-Tate exact sequence in Galois cohomology [15, Theorem I.4.10, p.70]. See Theorem 4.12 below for the precise statement. In particular, we obtain the following duality theorem.

**Theorem 1.1.** Let \( N \) be a \( p \)-primary finite and flat group scheme over \( K \). Then there exists a perfect pairing of finite groups

\[
\text{III}^1(K, N) \times \text{III}^2(K, N^d) \to \mathbb{Q}_p/\mathbb{Z}_p.
\]

The Tate-Shafarevich groups appearing in the statement of the theorem are defined in terms of flat cohomology.

In addition, we extend the duality theorems for 1-motives over number fields obtained by D.Harari and T.Szamuely [10] to the function field case. In particular, we obtain the following result.

**Theorem 1.2.** Let \( M \) be a 1-motive over \( K \) with dual 1-motive \( M^* \). Then there exists a canonical pairing

\[
\text{III}^1(K, M)(p) \times \text{III}^1(K, M^*)(p) \to \mathbb{Q}/\mathbb{Z}
\]

whose left and right kernels are the maximal divisible subgroups of each group.

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The paper has 6 Sections. In Section 2 we prove some elementary results needed in the sequel. Section 3 is a brief summary of the facts that we need on 1-motives (readers wishing to learn more about the theory of 1-motives are advised to read [1]). In Section 4, which is independent of Sections 5 and 6, we establish the Poitou-Tate exact sequence for \( p \)-primary finite and flat group schemes. In Section 5 we prove an “integral version” of Theorem 1.2, namely an analogous statement with \( \text{Spec} \, K \) replaced by an open affine subset \( U \) of \( X \). Theorem 1.2 is then deduced from this integral version in Section 6 by passing to the limit as \( U \) shrinks to \( \text{Spec} \, K \).

The methods of this paper yield both a general Poitou-Tate exact sequence and a Cassels-Tate dual exact sequence for 1-motives over global fields (extending the results of [10, §5], [11, §5] and [7]). These sequences require a significant amount of extra work in relation to [op.cit.] and will be established in separate publications.

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**2. Preliminaries**

Let \( K \) be a global function field of characteristic \( p > 0 \). For basic information on these fields, and their completions, the reader is referred to [4, Chapters I and II]. For any prime \( v \) of \( K \), \( K_v \), \( \mathcal{O}_v \) and \( k(v) \) will denote, respectively, the completion of \( K \) at \( v \), its ring of integers and the corresponding residue field. Thus \( \mathcal{O}_v \) is a complete discrete valuation ring. We will write \( X \) for the unique smooth complete curve over the field of constants of \( K \) having \( K \) as its function field. The primes of \( K \) will often be identified with the closed points of \( X \). A direct product extending over all \( v \in U \) for some nonempty open subset \( U \) of \( X \) is to be understood as extending over all closed points of \( U \).
For any abelian group $B$ and positive integer $n$, we will write $B_n$ for the $n$-torsion subgroup of $B$ and $B/n$ for the quotient $B/nB$. Further, we will write $B(p) = \bigcup_{m \geq 1} B_{p^m}$ (the $p$-primary torsion subgroup of $B$), $B(p) = \varprojlim_m B/p^m$ (the $p$-adic completion of $B$) and $T_p B = \varprojlim_m B_{p^m}$ (the $p$-adic Tate module of $B$). Also, we set $B_{p\text{-div}} = \bigcap_m p^m B$ (the $p$-primary torsion subgroup of $B$), $B_p = \varprojlim_m B_{p^m}$ (the $p$-adic completion of $B$) and $T_pB = \varprojlim_m B_{p^m}$ (the $p$-adic Tate module of $B$). Also, we set $B_{p\text{-div}} = \bigcap_m p^m B$ (the $p$-primary torsion subgroup of $B$), $B_p = \varprojlim_m B_{p^m}$ (the $p$-adic completion of $B$) and $T_pB = \varprojlim_m B_{p^m}$ (the $p$-adic Tate module of $B$).

**Lemma 2.1.** Let $B$ be any abelian group. Then the canonical map $B \to B(p)$ induces an injection

$$B(p)/p\text{-div} \hookrightarrow B^{(p)}(p).$$

**Proof.** There exists an exact sequence of inverse systems $0 \to p^m B \to B \to B/p^m \to 0$, where the transition maps are $p^{m+1} B \hookrightarrow p^m B$, $\text{Id}: B \to B$ and $\text{proj}: B/p^{m+1} \to B/p^m$. Taking inverse limits, we obtain an exact sequence

$$0 \to B_{p\text{-div}} \to B \to B^{(p)}$$

and therefore an exact sequence

$$0 \to B_{p\text{-div}}(p) \to B(p) \to B^{(p)}(p).$$

Since $B_{p\text{-div}}(p) = B(p)_{p\text{-div}}$, the lemma follows. \qed

If $B$ is any abelian topological group, we will write $\hat{B}$ (or $B^\wedge$) for $\varprojlim_{I \in \mathcal{S}} B/I$, where $\mathcal{S}$ is the family of open subgroups of $B$ of finite $p$-power index. Clearly, if $B$ is a discrete torsion abelian group of finite cotype, then $\hat{B} = B/p\text{-div}$. There exists a canonical isomorphism $\hat{B} = (B(p))^\wedge$, whence there exists a canonical map $B^{(p)} \to \hat{B}$. We set $B^D = \text{Hom}_{\text{cont.}}(B, \mathbb{Q}_p/\mathbb{Z}_p)$, where $\mathbb{Q}_p/\mathbb{Z}_p$ is endowed with the discrete topology. We endow $B^D$ with the compact-open topology, i.e., the open subsets of $B^D$ are arbitrary unions of finite intersections of sets of the form $\{f \in B^D: f(K) \subset U\}$, where $K \subset B$ is compact and $U \subset \mathbb{Q}_p/\mathbb{Z}_p$ is open (i.e., arbitrary). Note that, if $B$ is discrete and finitely generated, then $B^D = (B^{(p)})^D = \hat{B}^D$ is a discrete $p$-primary torsion group and $BD^D = B^{(p)} = \hat{B}$.

A pairing of discrete abelian groups $A \times B \to \mathbb{Q}_p/\mathbb{Z}_p$ is called non-degenerate on the right (resp. left) if the induced homomorphism $B \to\ldots$
$A^D$ (resp. $A \to B^D$) is injective. It is called non-degenerate if it is non-degenerate both on the right and on the left. The pairing is said to be perfect if the homomorphisms $B \to A^D$ and $A \to B^D$ are isomorphisms. It is not difficult to see that a perfect pairing $A \times B \to \mathbb{Q}_p/\mathbb{Z}_p$ induces pairings $A(p) \times (B/p\text{-div}) \to \mathbb{Q}_p/\mathbb{Z}_p$ and $(A/p\text{-div}) \times B(p) \to \mathbb{Q}_p/\mathbb{Z}_p$ which are non-degenerate on the left and on the right, respectively.

**Lemma 2.2.** Let $p$ be a prime number.

(a) Let $B$ be an abelian group and let $A$ be a torsion subgroup of $B$. If $B_p = 0$, then $(B/A)_p = 0$.

(b) Let $A \to B \to C \to 0$ be an exact sequence of discrete torsion abelian groups. Then the induced sequence $0 \to C(p)^D \to B(p)^D \to A(p)^D$ is exact.

**Proof.** Since every element of $A$ is annihilated by an integer which is prime to $p$, $A$ is $p$-divisible. This implies (a). Assertion (b) follows from the fact that $B \to B^D$ is an exact functor on the category of discrete abelian groups. \hfill $\square$

In this paper we consider only commutative group schemes, and therefore the qualification “com mutative” will often be omitted when discussing group schemes. Further, all cohomology groups below are flat (fppf) cohomology groups.

Now let $N$ be a finite, flat (commutative) group scheme over $\text{Spec} K$ and let $\mathcal{F} = \mathcal{F}(N)$ be the set of all pairs $(U,N)$, where $U$ is a nonempty open affine subscheme of $X$ (i.e., $U \neq X$) and $N$ is a finite and flat group scheme over $U$ which extends $N$, i.e., $N \times_U \text{Spec} K = N$. Then $\mathcal{F}$ is a nonempty $[13, \text{p.294}]$ directed and partially ordered set with the partial ordering $(U,N) \leq (V,N')$ if and only if $V \subset U$ and $N|_V = N'$.

Clearly, $\varprojlim_{(U,N) \in \mathcal{F}} U = \bigcap_{(U,N) \in \mathcal{F}} U = \text{Spec} K$.

**Lemma 2.3.** With the above notations, for every $i \geq 0$ the canonical map

$$\varprojlim_{(U,N) \in \mathcal{F}} H^i(U,N) \to H^i(K,N)$$

is an isomorphism.

**Proof.** The result is clear if $i = 0$. Assume now that $i \geq 1$. If $(U,N) \in \mathcal{F}$, then $N$ admits a canonical resolution

$$0 \to N \to G_0 \to G_1 \to 0,$$

\footnote{Note that, if $(U,N), (V,N') \in \mathcal{F}$, then there exists a nonempty open subset of $U \cap V$ over which $N$ and $N'$ are isomorphic.}
where $G_0$ and $G_1$ are smooth affine group schemes of finite type over $U$. See [2, §2.2.1, p.25]. Now, using the fact that flat and étale cohomology coincide on smooth group schemes [14, Theorem 3.9, p.114], we obtain the following exact sequence which is functorial in $(U, N)$:

$$H^{i-1}_{\text{ét}}(U, G_0) \rightarrow H^{i-1}_{\text{ét}}(U, G_1) \rightarrow H^i(U, N) \rightarrow H^i_{\text{ét}}(U, G_0) \rightarrow H^i_{\text{ét}}(U, G_1)$$

An analogous exact sequence exists over $K$, and these exact sequences form the top and bottom row, respectively, of a natural exact commutative diagram. Since the canonical maps

$$\lim_{(U, N) \in \mathcal{F}} H^j_{\text{ét}}(U, G_l) \rightarrow H^j_{\text{ét}}(K, G_l)$$

are isomorphisms for $j = i - 1$ or $i$ and $l = 0$ or $1$ by [8, Theorem VII.5.7, p.361], the five-lemma applied to the direct limit of the diagram mentioned above yields the desired result. □

**Lemma 2.4.** There exists a canonical isomorphism

$$H^1(K, N)^D = \lim_{(U, N) \in \mathcal{F}} H^1(U, N)^D.$$

**Proof.** Let $(U, N) \in \mathcal{F}$ and let $V$ be a nonempty open subset of $U$. Since $H^1_v(O_v, N) = 0$ for any $v$ [15, beginning of §III.7, p.349], the localization sequence for the pair $V \subset U$ [op.cit., Proposition III.0.3(c), p.270] shows that the canonical map of discrete groups $H^1(U, N) \rightarrow H^1(V, N)$ is injective. The lemma now follows from Lemma 2.3 above and [16, Propositions 3 and 7]. □

Let $S$ be a scheme. An $S$-torus $T$ is a smooth $S$-group scheme which, locally for the étale topology on $S$, is isomorphic to $G_m^r$ for some positive integer $r$.

We will use without explicit mention the fact that the étale and fppf cohomology groups of a smooth group scheme coincide. In particular, $H^i(S, T) = H^i_{\text{ét}}(S, T)$ for an $S$-torus $T$ as above.

Finally, for each prime $v$ of $K$, $\text{inv}_v : \text{Br}(K_v) \rightarrow \mathbb{Q}/\mathbb{Z}$ will denote the usual invariant map of local class field theory.

**3. Generalities on 1-motives**

Let $S$ be a scheme. We will write $S_{\text{fppf}}$ for the small fppf site over $S$, $\mathcal{F}_S$ for the category of abelian sheaves on $S_{\text{fppf}}$, $\mathcal{C}^b(\mathcal{F}_S)$ for the category of bounded complexes of objects in $\mathcal{F}_S$ and $\mathcal{D}^b(\mathcal{F}_S)$ for the associated derived category.

Recall that a (smooth) 1-motive $M = (Y, A, T, G, u)$ over $S$ consists of the following data:
1. An $S$-group scheme $Y$ which, locally for the étale topology on $S$, is isomorphic to $\mathbb{Z}^r$ for some $r \geq 0$.

2. A commutative $S$-group scheme $G$ which is an extension of an abelian $S$-scheme $A$ by an $S$-torus $T$:
   
   $$0 \to T \to G \xrightarrow{\pi} A \to 0.$$ 

3. An $S$-homomorphism $u : Y \to G$.

We will often identify $M$ with the mapping cone of $u$, i.e., $M = C^\bullet(u) = (Y \xrightarrow{u} G)$, with $Y$ placed in degree $-1$ and $G$ placed in degree $0$. Thus there exists a distinguished triangle

$$Y \to G \to M \to Y[1].$$

Note also that $M$ defines, in a canonical way, an object of $D_b(fppf)$. Every 1-motive $M$ comes equipped with a natural increasing 3-term weight filtration:

$$W_i(M) = 0 \text{ for } i \leq -3, \quad W_{-2}(M) = (0 \to T), \quad W_{-1}(M) = (0 \to G) \text{ and } W_i(M) = M \text{ for } i \geq 0.$$ 

The 1-motive

$$M' = M/W_{-2}(M) = (Y \xrightarrow{h} A),$$

where $h = \pi \circ u$, will play an auxiliary role below. It fits into an exact sequence

$$0 \to T \to M \to M' \to 0,$$

where $T$ is regarded as a complex concentrated in degree zero. Now, to each 1-motive $M = (Y,A,T,G,u)$ as above, one can associate its Cartier dual $M^* = (Y^*,A^*,T^*,G^*,u^*)$. Here $Y^*$ is the sheaf of characters of $T$, $A^*$ is the abelian scheme dual to $A$ and $T^*$ is the $S$-torus with group of characters $Y$. The $S$-group scheme $G^*$ associated to $M$ may be constructed as follows. Assume first that $M = M'$ (i.e., $T = 0$). In this case $M^* = (M')^* = (0 \to G^*)$, where $G^*$ is the $S$-group scheme which represents the functor $S' \mapsto \text{Ext}^{1}_{S'}(M',\mathbb{G}_m)$ on $\mathcal{C}^b(\mathcal{F}_S)$ (the representability of this functor follows from the generalized Weil-Barsotti formula\(^2\)). The 1-motive $(M')^*$ is naturally endowed with a biextension (in the sense of [5, 10.2.1, p.60]) $\mathcal{P}'$ of $(M',(M')^*)$ by $\mathbb{G}_m$, namely the pullback of the canonical Poincaré biextension of $(A,A^*)$ by $\mathbb{G}_m$ under the map $f' \times g'$, where $f' = (0,\text{Id}) : M' = (Y \to A) \to A$ and $g'$ is the composite $(M')^* \xrightarrow{(0,\text{Id})} G^* \xrightarrow{\pi^*} A^*$.

\(^2\)The referee calls attention to the fact that the Weil-Barsotti formula over imperfect fields has not been properly established in the literature. However, O.Wittenberg has recently given the necessary missing details in the Appendix to [18].
Now let $M$ be an arbitrary 1-motive. By (2), $M$ represents a class in $\text{Ext}^1_S(M', T)$. Thus any $\chi_{s'} \in Y^*(S') = \text{Hom}_S(T, \mathbb{G}_m)$ induces an element $u^*(\chi_{s'}) = (\chi_{s'})_*(M_{s'}) \in \text{Ext}^1_S(M', \mathbb{G}_m) = G^*(S')$, which defines an $S$-homomorphism $u^*: Y^* \to G^*$. The associated 1-motive $M^* = (Y^* \xrightarrow{u^*} G^*)$ is the Cartier dual of $M$. The corresponding biextension $P$ of $(M, M^*)$ by $\mathbb{G}_m$ is the pullback of $P'$ under the map $f \times g$, where $f = (\text{Id}, \pi): M = (Y \to G) \to M' = (Y \to A)$ and $g = (0, \text{Id})$: $M^* = (Y^* \to G^*) \to (M')^* = (0 \to G^*)$ are the natural maps.

Now, as in [9, VII.3.6.5], (the isomorphism class of) $P$ corresponds to a map $M \otimes^L M^* \to \mathbb{G}_m[1]$ in $D^b(F_S)$. This map in turn induces pairings
\[ \mathbb{H}^i(S, M) \times \mathbb{H}^j(S, M^*) \to \mathbb{H}^{i+j+1}(S, \mathbb{G}_m) \]
for each $i, j \geq -1$.

Next, for any positive integer $n$, let
\[ T_{Z/n}(M) = \mathbb{H}^{-1}(C^*(n)) = \mathbb{H}^0(M[-1] \otimes^L \mathbb{Z}/n), \]
where $C^*(n)$ is the mapping cone of the multiplication-by-$n$ map on $M$ (to verify the second equality in (3), use the fact that $\mathbb{Z}/n$ is quasi-isomorphic to the complex of flat modules $(\mathbb{Z} \xrightarrow{n} \mathbb{Z})$). It is a finite and flat $S$-group scheme which fits into an exact sequence
\[ 0 \to G_n \to T_{Z/n}(M) \to Y/n \to 0. \]
See [1, §2.3, p.12]. It is not difficult to see that $T_{Z/n}(M)$ is the sheaf associated to the presheaf $S' \mapsto F_{Z/n}(M)(S')$, where
\[ F_{Z/n}(M)(S') = \{ (g, y) \in G(S') \times Y(S') : ng = -u(y) \}/ \{ (-u(y), ny) : y \in Y(S') \}. \]
The map $M \otimes^L M^* \to \mathbb{G}_m[1]$ induces a perfect pairing
\[ T_{Z/n}(M) \times T_{Z/n}(M^*) \to \mu_n, \]
where $\mu_n$ is the sheaf of $n$-th roots of unity. The above pairing generalizes the classical Weil pairing of an abelian variety $A$, which may be recovered by choosing $M = (0 \to A)$ and $n$ prime to $p$ above. We will also need the following groups attached to $M$:
\[ T_p(M) = \lim_{\leftarrow} T_{Z/p^n}(M) \]
(the $p$-adic realization of $M$), where the transition maps are induced by the maps $F_{Z/p^{n+1}}(M) \to F_{Z/p^n}(M), [(g, y)] \mapsto [(pg, y)]$ (see (4)), and
\[ T(M)\{p\} = \lim_{\leftarrow} T_{Z/p^n}(M) \]
(the $p$-divisible group attached to $M$) with transition maps induced by
$$F_{\mathbb{Z}/p^m}(M) \to F_{\mathbb{Z}/p^{m+1}}(M), [(g,y)] \mapsto [(g,p^m y)].$$

Now let $M$ be a 1-motive over $K$. For each prime $v$ of $K$, we will write
$M_v$ for the $K_v$-1-motive $M_{K_v}$. Further, for each $i \geq -1$, $H^i(K_v, M)$
will denote $H^i$ endowed with the discrete topology. For $i = -1, 1, 2$, the group $H^i(K_v, M_v)$
will be endowed with the discrete topology. For $i = 0$, $H^i(K_v, M)$ will be
denoted with the topology defined in [10, p.99]. Define
$$H^i = \text{Ker } [H^0(K_v, Y) \rightarrow H^0(K_v, G)].$$
Then there exists a surjective and continuous map of profinite groups
$H^{-1}(K_v, M) \overline{\otimes} \to H^{-1}(K_v, M)$, and therefore an injection
$$H^{-1}(K_v, M)^D \hookrightarrow (\mathbb{H}^{-1}(K_v, M))^D = H^{-1}(K_v, M)^D.
$$

**Theorem 3.1.** There exists a continuous pairing
$$H^i(K_v, M) \times H^{1-i}(K_v, M^*) \to \mathbb{H}^2(K_v, \mathbb{G}_m) = \mathbb{Q}/\mathbb{Z}
$$
which induces perfect dualities between the following profinite and discrete
groups, respectively.

(a) $H^{-1}(K_v, M)$ and $H^2(K_v, M^*)(p)$.
(b) $H^0(K_v, M)$ and $H^1(K_v, M^*)(p)$.

For $i \neq -1, 0, 1, 2$, the pairing is trivial.

**Proof.** See [10, Theorem 2.3 and Lemma 2.1].

4. **The Poitou-Tate exact sequence for $p$-primary finite
and flat group schemes**

Let $v$ be a prime of $K$ and let $N_v$ be a $p$-primary finite and flat group
scheme over $K_v$. The group $H^i(K_v, N) := H^i(K_v, N_v)$ (for $i = 0, 1$ or 2)
is canonically endowed with a locally compact topology (see [15, comments following
III.6.5, p.341]). Both $H^0(K_v, N)$ and $H^2(K_v, N)$ are discrete (in fact, finite),
hut $H^1(K_v, N)$ need not be. See [15, Table III.6.8, p.343]. We let $N_v^d = \mathcal{H}om(N_v, \mathbb{G}_m)$ be the Cartier dual of $N_v$. Assume now that $N_v$ extends to a finite and flat group scheme $N_v$
over $\text{Spec } \mathcal{O}_v$. Set $H^i(\mathcal{O}_v, N) = H^i(\mathcal{O}_v, N_v)$. By [13, p.293] or [15,
beginning of §III.7], for each $i \geq 0$ the canonical map $H^i(\mathcal{O}_v, N) \to
H^i(K_v, N)$ embeds $H^i(\mathcal{O}_v, N)$ as a compact and open subgroup of
$H^i(K_v, N)$. We will identify $H^i(\mathcal{O}_v, N)$ with its image in $H^i(K_v, N)$
under this map. Further, we will write $H^i(\mathcal{O}_v, N)$ for the cohomology
group of $N$ with support on Spec $k(v)$ (see [15, Proposition III.0.3,
p.270]).

**Theorem 4.1.** Let $i = 0, 1$ or 2.
(a) There exists a perfect continuous pairing
\[ H^i(K_v, N) \times H^{2-i}(K_v, N^d) \to \mathbb{Q}_p / \mathbb{Z}_p. \]

(b) In the pairing of (a), \( H^i(\mathcal{O}_v, \mathcal{N}) \) is the exact annihilator of \( H^{2-i}(\mathcal{O}_v, \mathcal{N}^d) \).

**Proof.** For (a), see [15, Theorem III.6.10, p.344]. Statement (b) is [op.cit., Corollary III.7.2, p.349]. □

Now let \( N \) be a \( p \)-primary finite and flat group scheme over \( K \). For any prime \( v \) of \( K \), we will write \( N_v = N \times_{\text{Spec} \ K} \text{Spec} \ K_v \) and \( H^i(K_v, N) = H^i(K_v, N_v) \). Recall the set \( \mathcal{F} \) defined in Section 2. The elements of \( \mathcal{F} \) are pairs \((U, \mathcal{N})\), where \( U \) is a nonempty open affine subscheme of \( X \) such that \( N \) extends to a finite and flat group scheme \( N \) over \( U \). If \((U, \mathcal{N}) \in \mathcal{F} \) and \( v \in U \), we will write \( N_v = N \times_U \text{Spec} \mathcal{O}_v \) and \( H^i(\mathcal{O}_v, \mathcal{N}) = H^i(\mathcal{O}_v, \mathcal{N}_v) \). For \((U, \mathcal{N}) \in \mathcal{F} \), let \( H^i_c(U, \mathcal{N}) \) be defined as in [15, comments preceding Proposition III.0.4, pp.270-271] with \( K_v \) replacing the field of fractions of the henselization of the local ring at \( v \). Then [op.cit., Proposition III.0.4(a), p.271] remains valid, i.e., there exists an exact sequence
\[
\ldots \to H^i_c(U, \mathcal{N}) \to H^i(U, \mathcal{N}) \to \bigoplus_{v \notin U} H^i(K_v, N) \to H^{i+1}_c(U, \mathcal{N}) \to \ldots
\]

See [15, III.0.6(b), p.274]. The map \( H^i(U, \mathcal{N}) \to \bigoplus_{v \notin U} H^i(K_v, N) \) appearing above is the sum over \( v \notin U \) of the maps \( H^i(U, \mathcal{N}) \to H^i(K_v, N) \) induced by the composite morphism \( \text{Spec} \mathcal{O}_v \to \text{Spec} \mathcal{O}_v \to U \). Set
\[
D^i(U, \mathcal{N}) = \text{Ker} \left[ H^i(U, \mathcal{N}) \to \bigoplus_{v \notin U} H^i(K_v, N) \right]
\]
\[
= \text{Im} \left[ H^i_c(U, \mathcal{N}) \to H^i(U, \mathcal{N}) \right].
\]

Since \( H^0(U, \mathcal{N}) = N(K) \) and \( H^0(K_v, N) = N(K_v) \) for every \( v \notin U \), the map \( H^0(U, \mathcal{N}) \to \bigoplus_{v \notin U} H^i(K_v, N) \) is injective, i.e., \( D^0(U, \mathcal{N}) = 0 \).

**Lemma 4.2.** For any \((U, \mathcal{N}) \in \mathcal{F} \), the canonical map \( H^1(U, \mathcal{N}) \to H^1(K, N) \) is injective.

**Proof.** The proof is similar to the proof of [15, Lemma III.1.1, p.286]. □

Using the above lemma, we will regard \( D^1(U, \mathcal{N}) \) as a subgroup of \( H^1(K, N) \) for any \((U, \mathcal{N}) \in \mathcal{F} \).
From now on, we will simplify our notations by writing \((V,\mathcal{N})\) for \((V,\mathcal{N}|_V)\) when \((U,\mathcal{N}) \in \mathcal{F}\) and \(V\) is an open subset of \(U\).

**Lemma 4.3.** Let \((U,\mathcal{N}) \in \mathcal{F}\) be arbitrary. Then there exists a nonempty open subset \(U_0\) of \(U\) such that, for any nonempty open subset \(V \subset U_0\), both \(D^1(V,\mathcal{N})\) and \(D^1(V,\mathcal{N}^d)\) are finite.

**Proof.** By a theorem of M.Raynaud (see [17] or [3, Theorem 3.1.1, p. 110]), there exist a nonempty open subset \(U_0\) of \(U\), abelian \(U_0(\mathcal{N})\)-schemes \(\mathcal{A}\) and \(\mathcal{B}\) and an exact sequence \(0 \to \mathcal{N}_0 \to \mathcal{A} \to \mathcal{B} \to 0\), where \(\mathcal{N}_0 = \mathcal{N}|_{U_0(\mathcal{N})}\) and the first nontrivial map is a closed immersion. Let \(V\) be any nonempty open subset of \(U_0\). Then \(0 \to \mathcal{N}_0|_V \to \mathcal{A}|_V \to \mathcal{B}|_V \to 0\) and \(0 \to \mathcal{N}|_V \to \mathcal{A} \to \mathcal{B} \to 0\), for any prime \(v\) of \(K\), are also exact. Here \(\mathcal{A}\) and \(\mathcal{B}\) denote, respectively, the generic fibers of \(\mathcal{A}\) and \(\mathcal{B}\). Using these exact sequences and the fact that \(\mathcal{B}(V) = \mathcal{B}(K)\), we obtain an exact commutative diagram

\[
\begin{array}{cccccc}
B(K) & \longrightarrow & H^1(V,\mathcal{N}_0) & \longrightarrow & H^1(V,\mathcal{A})_m \\
\bigoplus_{v \notin V} B(K_v) & \longrightarrow & \bigoplus_{v \notin V} H^1(K_v,\mathcal{N}) & \longrightarrow & \bigoplus_{v \notin V} H^1(K_v,\mathcal{A}) \\
\end{array}
\]

where \(m\) is any integer which annihilates \(\mathcal{N}_0\). Since the image of \(B(K)\) in \(H^1(V,\mathcal{N}_0)\) is finite by the Mordell-Weil theorem, the finiteness of \(D^1(V,\mathcal{N}_0)\) follows from that of \(D^1(V,\mathcal{A})_m = \text{III}^1(K,\mathcal{A})_m\), which is the main theorem of [13] (for the last equality, see [15, Lemma II.5.5, p.246]). Now repeat the proof with \(\mathcal{N}^d\) in place of \(\mathcal{N}\) and take \(U_0 = U_0(\mathcal{N}) \cap U_0(\mathcal{N}^d)\). \(\square\)

We now define, for \(i = 1\) or \(2\),

\[
\text{III}^i(K,\mathcal{N}) = \text{Ker} \left[ H^i(K,\mathcal{N}) \to \prod_{v \notin V} H^i(K_v,\mathcal{N}) \right].
\]

**Proposition 4.4.** Let \((U,\mathcal{N}) \in \mathcal{F}\) be arbitrary and let \(U_0\) be as in the statement of the previous lemma. Then there exists a nonempty open subset \(U_1 \subset U_0\) such that, for any nonempty open subset \(V\) of \(U_1\), \(\text{III}^1(K,\mathcal{N}) = D^1(V,\mathcal{N})\). In particular, \(\text{III}^1(K,\mathcal{N})\) is a finite group.

**Proof.** (Cf. [10, proofs of Lemma 4.7 and Theorem 4.8, pp.114-115]). By definition, \(\text{III}^1(K,\mathcal{N}) \supset \bigcap_{\emptyset \neq W \subset U_0} D^1(W,\mathcal{N})\). Since each set
$D^1(W,\mathcal{N})$ is finite, we may choose finitely many nonempty open sub-sets $W_1, W_2, \ldots, W_r$ of $U_0$ such that

$$\text{III}^1(K, \mathcal{N}) \supset \bigcap_{j=1}^r D^1(W_j, \mathcal{N}).$$

Let $U_1 = \bigcap_{j=1}^r W_j$ and let $V$ be any nonempty open subset of $U_1$. By [15, Proposition III.0.4(c), p.271, and Remark III.0.6(b), p.274], for any $j$ there exist natural maps $H^1_c(V, \mathcal{N}) \xrightarrow{f_j} H^1_c(W_j, \mathcal{N}) \xrightarrow{g_j} H^1(K, \mathcal{N})$ such that $\text{Im}(g_j \circ f_j) = D^1(V, \mathcal{N})$ and $\text{Im}(g_j) = D^1(W_j, \mathcal{N})$.

It follows that $D^1(V, \mathcal{N}) \subset D^1(W_j, \mathcal{N})$ for every $j$ and we conclude that $D^1(V, \mathcal{N}) \subset \text{III}^1(K, \mathcal{N})$. To prove the reverse inclusion, let $\xi \in \text{III}^1(K, \mathcal{N})$. Then $\xi$ extends to $H^1(W, \mathcal{N})$ for some nonempty open subset $W$ of $U$, which we may assume to be contained in $V$. Then $\xi \in D^1(W, \mathcal{N}) \subset D^1(V, \mathcal{N})$ (by the same argument as above), and the proof is complete. \hfill \Box

**Lemma 4.5.** Let $(U, \mathcal{N}) \leq (V, \mathcal{N}) \in \mathcal{F}$. Then the natural map $H^2(U, \mathcal{N}) \rightarrow H^2(V, \mathcal{N})$ induces a map $D^2(U, \mathcal{N}) \rightarrow D^2(V, \mathcal{N})$.

**Proof.** For each $v$, the boundary map $H^2(K_v, \mathcal{N}) \rightarrow H^3(\mathcal{O}_v, \mathcal{N})$ appearing in the localization sequence for the pair $\text{Spec } K_v \subset \text{Spec } \mathcal{O}_v$ [15, Proposition III.0.3(c), p.270] is an isomorphism [op.cit., comments preceding Theorem III.7.1, p.349]. Thus the localization sequence for the pair $V \subset U$ induces an exact sequence

$$H^2(U, \mathcal{N}) \rightarrow H^2(V, \mathcal{N}) \rightarrow \bigoplus_{v \in U \setminus V} H^2(K_v, \mathcal{N}).$$

It is not difficult to check that the second map in the above exact sequence is the natural one, from which the lemma follows. \hfill \Box

**Proposition 4.6.** There exists a canonical isomorphism

$$\lim_{(U, \mathcal{N}) \rightarrow \mathcal{F}} D^2(U, \mathcal{N}) = \text{III}^2(K, \mathcal{N}).$$

**Proof.** For any $(U, \mathcal{N}) \in \mathcal{F}$, set

$$D^2(U, \mathcal{N}) = \text{Im}[D^2(U, \mathcal{N}) \rightarrow H^2(K, \mathcal{N})].$$

Let $(V, \mathcal{N}) \in \mathcal{F}$ be such that $(U, \mathcal{N}) \leq (V, \mathcal{N})$. By Lemma 4.5, the map $D^2(U, \mathcal{N}) \rightarrow H^2(K, \mathcal{N})$ factors through $D^2(V, \mathcal{N})$, whence $D^2(U, \mathcal{N}) \subset D^2(V, \mathcal{N})$. Now the identification of $D^2(U, \mathcal{N})$ with $\text{Im}[H^2_c(U, \mathcal{N}) \rightarrow H^2(U, \mathcal{N})]$ and the covariance of $H^2_c(\mathcal{N})$ with respect to open immersions show that $D^2(V, \mathcal{N}) \subset D^2(U, \mathcal{N})$. We
conclude that $D^2 (V, \mathcal{N}) = D^2 (U, \mathcal{N})$ for all $(V, \mathcal{N})$ as above and necessarily $D^2 (U, \mathcal{N}) = \Pi^2 (K, \mathcal{N})$ for any $(U, \mathcal{N}) \in \mathcal{F}$. Thus we have a surjection

$$\lim_{\to} D^2 (U, \mathcal{N}) \to \Pi^2 (K, \mathcal{N}).$$

By Lemma 2.3 this is an injection as well, which completes the proof.

We would like to establish an analogue of the previous proposition for $\Pi^1 (K, \mathcal{N})$. However (cf. Lemma 4.5), the natural map $H^1 (U, \mathcal{N}) \to H^1 (V, \mathcal{N})$ need not map $D^1 (U, \mathcal{N})$ into $D^1 (V, \mathcal{N})$ for $(U, \mathcal{N}) \leq (V, \mathcal{N}) \in \mathcal{F}$. The reason for this is that a class $\xi \in D^1 (U, \mathcal{N})$ need not map to zero in $H^1 (K_v, \mathcal{N})$ for primes $v \in U \setminus V$. Following [10], we will circumvent this difficulty in the next proposition by showing that the groups $D^1 (U, \mathcal{N})$ “eventually become constant with value $\Pi^1 (K, \mathcal{N})$”, by which we mean that there exists an element $(U_1, \mathcal{N}) \in \mathcal{F}$ such that, for every $(V, \mathcal{N}) \in \mathcal{F}$ with $(U_1, \mathcal{N}) \leq (V, \mathcal{N})$, $D^1 (V, \mathcal{N})$ can be identified with $\Pi^1 (K, \mathcal{N})$.

**Proposition 4.7.** Let $(U, \mathcal{N}) \in \mathcal{F}$ be arbitrary and let $U_0 \subset U$ be as in the statement of Lemma 4.3. Then there exists a nonempty open subset $U_1 \subset U_0$ such that, for any nonempty open subset $V$ of $U_1$, $\Pi^1 (K, \mathcal{N}) = D^1 (V, \mathcal{N})$. In particular, $\Pi^1 (K, \mathcal{N})$ is a finite group.

**Proof.** (Cf. [10, proofs of Lemma 4.7 and Theorem 4.8, pp.114-115]). By definition, $\Pi^1 (K, \mathcal{N}) \supset \bigcap_{V \neq W \subset U_0} D^1 (W, \mathcal{N})$. Since each set $D^1 (W, \mathcal{N})$ is finite (see Lemma 4.3), we may choose finitely many nonempty open subsets $W_1, W_2, \ldots, W_r$ of $U_0$ such that

$$\Pi^1 (K, \mathcal{N}) \supset \bigcap_{j=1}^r D^1 (W_j, \mathcal{N}).$$

Let $U_1 = \bigcap_{j=1}^r W_j$ and let $V$ be any nonempty open subset of $U_1$. By [15, Proposition III.0.4(c), p.271, and Remark III.0.6(b), p.274], for any $j$ there exist natural maps $H^1_c (V, \mathcal{N}) \xrightarrow{f_j} H^1_c (W_j, \mathcal{N}) \xrightarrow{g_j} H^1 (K, \mathcal{N})$ such that $\text{Im} (g_j \circ f_j) = D^1 (V, \mathcal{N})$ and $\text{Im} (g_j) = D^1 (W_j, \mathcal{N})$. It follows that $D^1 (V, \mathcal{N}) \subset D^1 (W_j, \mathcal{N})$ for every $j$ and we conclude that $D^1 (V, \mathcal{N}) \subset \Pi^1 (K, \mathcal{N})$. To prove the reverse inclusion, let $\xi \in \Pi^1 (K, \mathcal{N})$. Then $\xi$ extends to $H^1 (W, \mathcal{N})$ for some nonempty open subset $W$ of $U$, which we may assume to be contained in $V$. Then $\xi \in D^1 (W, \mathcal{N}) \subset D^1 (V, \mathcal{N})$ (by the same argument as above), and the proof is complete.
Let \((U, \mathcal{N})\) be arbitrary and let \(U_1\) be as in the statement of the previous proposition. Set \[ \mathcal{F}_1 = \{(V, \mathcal{N}) \in \mathcal{F} : (U_1, \mathcal{N}) \leq (V, \mathcal{N})\}. \]

Then, if \((V, \mathcal{N}) \leq (W, \mathcal{N}) \in \mathcal{F}_1\), there exist natural maps \(D^1(W, \mathcal{N}) \to D^1(V, \mathcal{N})\) (the identity map; see Proposition 4.7) and \(D^2(V, \mathcal{N}^d) \to D^2(W, \mathcal{N}^d)\) (see Lemma 4.5). The respective limits are

\[
\lim_{(V, \mathcal{N}) \in \mathcal{F}_1} D^1(V, \mathcal{N}) = \text{III}_1(K, \mathcal{N})
\]

and

\[
\lim_{(V, \mathcal{N}) \in \mathcal{F}_1} D^2(V, (\mathcal{N})^d) = \text{III}_2(K, \mathcal{N}^d)
\]

(see Proposition 4.6).

**Lemma 4.8.** Let \((U, \mathcal{N}) \in \mathcal{F}\) be arbitrary and let \(U_1 \subset U\) be as in the statement of Proposition 4.7. Then, for any nonempty open subset \(V \subset U_1\) and \(i = 1\) or \(2\), there exists a perfect pairing of finite groups

\[ D^i(V, \mathcal{N}) \times D^{3-i}(V, \mathcal{N}^d) \to \mathbb{Q}_p/\mathbb{Z}_p. \]

**Proof.** By [15, Theorem III.8.2, p.361], for any \(i\) there exists a perfect pairing

\[ [-, -] : H^i(V, \mathcal{N}) \times H^{3-i}_c(V, \mathcal{N}^d) \to \mathbb{Q}_p/\mathbb{Z}_p \]

between the torsion discrete group \(H^i(V, \mathcal{N})\) and the profinite group \(H^{3-i}_c(V, \mathcal{N}^d)\). The above pairing induces the middle vertical map in the following natural commutative diagram:

\[
\begin{array}{ccc}
0 & \longrightarrow & D^i(V, \mathcal{N}) \\
\longrightarrow & H^i(V, \mathcal{N}) & \oplus_{v \notin V} H^i(K_v, \mathcal{N}) \\
\downarrow & & \downarrow \\
H^{3-i}_c(V, \mathcal{N}^d)^D & \longrightarrow & \oplus_{v \notin V} H^{2-i}(K_v, \mathcal{N}^d)^D.
\end{array}
\]

It follows that there exists a well-defined pairing

\[ (-, -) : D^i(V, \mathcal{N}) \times D^{3-i}(V, \mathcal{N}^d) \to \mathbb{Q}_p/\mathbb{Z}_p \]

given by \((a, a') = [a, b']\), where \(a \in D^i(V, \mathcal{N}) \subset H^i(V, \mathcal{N})\) and \(b'\) is a preimage of \(a'\) in \(H^{3-i}_c(V, \mathcal{N}^d)\). The above pairing is non-degenerate on the left. Similarly, interchanging \(i\) and \(3 - i\) and \(\mathcal{N}\) and \(\mathcal{N}^d\) above, we obtain a pairing

\[ D^{3-i}(V, \mathcal{N}^d) \times D^i(V, \mathcal{N}) \to \mathbb{Q}_p/\mathbb{Z}_p \]
with trivial left kernel. Setting $i = 1$ above and using the finiteness of $D^1(V,N)$ (see Lemma 4.3 and recall that $U_1 \subset U_0$), we obtain the case $i = 1$ of the lemma. To obtain the case $i = 2$ of the lemma, one argues similarly, interchanging the roles of $N$ and $N^d$ and using the finiteness of $D^1(V,N^d)$ (see Lemma 4.3). □

**Theorem 4.9.** There exists a perfect pairing of finite groups

$$\Pi^1(K,N) \times \Pi^2(K,N^d) \to \mathbb{Q}_p/\mathbb{Z}_p.$$  

**Proof.** This follows at once from Lemma 4.8, using (6), (7) and the finiteness of $\Pi^1(K,N)$ (see Proposition 4.4). □

We will need the following lemma.

**Lemma 4.10.** Let $U_1$ be as in the statement of Proposition 4.7. Then there exists a canonical isomorphism

$$H^2(K,N^d) \cong \text{lim} \leftarrow V \subset U_1 H^2(V,N^d).$$

**Proof.** Let $V$ be a nonempty open subset of $U_1$. Since $D^2(V,N^d)$ is finite by Lemma 4.8, $H^2(V,N^d)$ is finite as well since it fits into an exact sequence

$$0 \to D^2(V,N^d) \to H^2(V,N^d) \to \bigoplus_{v \in V} H^2(K_v,N^d).$$

Therefore $\text{lim} \leftarrow V \subset U_1 H^2(V,N^d)$ is canonically isomorphic to the dual of $\text{lim} \leftarrow V \subset U_1 H^2(V,N^d)$. Now Lemma 2.3 completes the proof. □

For $(U,N) \in \mathcal{F}$ and $0 \leq i \leq 2$, define

$$P^i(U,N) = \bigoplus_{v \in U} H^i(K_v,N) \times \prod_{v \in U} H^i(\mathcal{O}_v,N) \subset \prod_{all v} H^i(K_v,N)$$

with the product topology. It is a locally compact group. Now, for every $v$, $H^0(\mathcal{O}_v,N) = H^0(K_v,N)$ and $H^2(\mathcal{O}_v,N) = 0$ [15, beginning of III.7, p. 348], whence

$$P^0(U,N) = \prod_{all v} H^0(K_v,N)$$

and

$$P^2(U,N) = \bigoplus_{v \notin U} H^2(K_v,N).$$
Note that $P_0^0(U, \mathcal{N})$ is compact and $P_2^2(U, \mathcal{N})$ is finite. Further, if $(U, \mathcal{N}) \leq (V, \mathcal{N}) \in \mathcal{F}$, then $P_i^i(U, \mathcal{N}) \subset P_i^i(V, \mathcal{N})$. Define

$$P_i^i(K, N) = \lim_{(U, \mathcal{N}) \in \mathcal{F}} P_i^i(U, \mathcal{N})$$

$$= \bigcup_{(U, \mathcal{N}) \in \mathcal{F}} P_i^i(U, \mathcal{N}) \subset \prod_{\text{all } v} H_i^i(K_v, N),$$

where the transition maps in the direct limit are the inclusion maps. Thus $P_i^i(K, N)$ is the restricted topological product over $v$ of the groups $H_i^i(K_v, N)$ with respect to the subgroups $H_i^i(O_v, \mathcal{N})$. Now equip $H_i^i(U, \mathcal{N})$ with the discrete topology. There exists a natural map $H_i^i(U, \mathcal{N}) \rightarrow \prod_{v \in U} H_i^i(O_v, \mathcal{N})$, namely the product over $v \in U$ of the maps $H_i^i(U, \mathcal{N}) \rightarrow H_i^i(O_v, \mathcal{N})$ induced by the canonical morphisms $\text{Spec } O_v \rightarrow U$. The product of the above map with the map $H_i^i(U, \mathcal{N}) \rightarrow \prod_{v \notin U} H_i^i(K_v, N)$ introduced previously is a map

$$\beta_i(U, \mathcal{N}) : H_i^i(U, \mathcal{N}) \rightarrow P_i^i(U, \mathcal{N}).$$

If $(U, \mathcal{N}) \leq (V, \mathcal{N}) \in \mathcal{F}$, then there exists a canonical commutative diagram

$$\begin{CD}
H_i^i(U, \mathcal{N}) @> \beta_i(U, \mathcal{N}) >> P_i^i(U, \mathcal{N}) \\
\downarrow \downarrow \downarrow \downarrow \\
H_i^i(V, \mathcal{N}) @> \beta_i(V, \mathcal{N}) >> P_i^i(V, \mathcal{N}),
\end{CD}$$

where the left-hand vertical map is induced by the inclusion $V \subset U$. Consequently, by Lemma 2.3, the direct limit of the maps $\beta_i(U, \mathcal{N})$ is a map

$$\beta_i(K, N) : H_i^i(K, N) \rightarrow P_i^i(K, N)$$

whose kernel is $\text{III}^i(K, N)$. Note that $\beta_0(K, N)$ is injective since it coincides with the canonical map $N(K) \rightarrow \prod_{\text{all } v} N(K_v)$.

Now Theorem 4.1 shows that $P_i^i(K, N)$ is the algebraic and topological dual of $P_{2-i}^i(K, N^d)$. Indeed, there exists an isomorphism of topological groups

$$\phi_i^i(K, N) : P_i^i(K, N) \rightarrow P_{2-i}^i(K, N^d)$$

defined as follows. If $(\xi_v) \in P_i^i(K, N)$ and $(\zeta_v) \in P_{2-i}^i(K, N^d)$ is arbitrary, then

$$\phi_i^i(K, N)(\xi_v)(\zeta_v) = \sum_{v \in \mathbb{Q}_p/\mathbb{Z}_p},$$
where, for each \( v \), \((-,-)_v\) is the pairing of Theorem 4.1(a) (that the sum is finite follows from the definition of \( P^i(K,N) \) and the fact that, for each \( v \), \( H^i(\mathcal{O}_v,N) \) and \( H^{2-i}(\mathcal{O}_v,N^d) \) annihilate each other under the pairing \((-,-)_v\)). Now, for each \((U,N) \in \mathcal{F}\), let

\[
\phi^i_{(U,N)}: P^i(U,N) \rightarrow P^{2-i}(U,N^d)^D
\]

be the composition of the restriction of \( \phi^i_{(K,N)} \) to \( P^i(U,N) \) and the canonical map \( P^{2-i}(K,N^d)^D \rightarrow P^{2-i}(U,N^d)^D \). Note that, if \((\xi_v) \in P^i(U,N) \) and \((\zeta_v) \in P^{2-i}(U,N^d) \) is arbitrary, then

\[
\phi^i_{(U,N)}(\xi_v)(\zeta_v) = \sum_{v \notin U} (\xi_v,\zeta_v)_v.
\]

We now let

\[
\gamma^i_{(U,N)}: P^i(U,N) \rightarrow H^{2-i}(U,N^d)^D
\]

be the composite \( \beta_{2-i}(U,N^d)^D \circ \phi^i_{(U,N)} \). Further, let

\[
\psi^i_{(U,N)}: \bigoplus_{v \notin U} H^i(K_v,N) \rightarrow H^{2-i}(U,N^d)^D
\]

be given by

\[
\psi^i_{(U,N)}((\xi_v))(\zeta) = \sum_{v \notin U} (\xi_v,\zeta|_{K_v})_v
\]

where, for each \( v \notin U \), \( \zeta|_{K_v} \) denotes the image of \( \zeta \in H^{2-i}(U,N^d) \) in \( H^{2-i}(K_v,N^d) \) under the map

\[
H^{2-i}(U,N^d) \rightarrow H^{2-i}(K,N^d) \rightarrow H^{2-i}(K_v,N^d)
\]

induced by the composite morphism \( \text{Spec} K_v \rightarrow \text{Spec} K \rightarrow U \). Then the following diagrams commute

\[
\begin{align*}
(8) & \\
P^i(U,N) & \xrightarrow{\gamma^i_{(U,N)}} H^{2-i}(U,N^d)^D \\
\bigoplus_{v \notin U} H^i(K_v,N) & \xrightarrow{\psi^i_{(U,N)}} \\
\end{align*}
\]

and

\[
\begin{align*}
(9) & \\
P^i(U,N) & \xrightarrow{\gamma^i_{(U,N)}} H^{2-i}(U,N^d)^D \\
\bigoplus_{v \notin U} H^i(K_v,N) & \xrightarrow{\psi^i_{(U,N)}} \\
\end{align*}
\]
Proof. We show first that \( \ker(\gamma_i(K,N)) \subset \im(\beta_i(K,N)) \). Let \( (\xi_v) \in \ker(\gamma_i(K,N)) \). Then \( (\xi_v) \in P^i(U,N) \) for some nonempty open affine subset of \( X \), which we may assume to be contained in the set \( U_1 \) introduced in the proof of Proposition 4.7. The element \( (\xi_v)_{v\in U} \in \bigoplus_{v\in U} H^i(K_v,N) \) is in the kernel of \( \psi^i_{(U,N)} \) by the commutativity of (8). Therefore \( (\xi_v)_{v\in U} \) is in the kernel of the composite map
\[
\bigoplus_{v\in U} H^i(K_v,N) \xrightarrow{\psi^i_{(U,N)}} H^{2-i}(U,N^d) \xrightarrow{d} H^{i+1}_c(U,N),
\]
where the last isomorphism is induced by the perfect pairing
\[
[-,-]: H^{2-i}(U,N^d) \times H^{i+1}_c(U,N) \to \mathbb{Q}_p/\mathbb{Z}_p
\]
between the discrete torsion group \( H^{2-i}(U,N^d) \) and the profinite group \( H^{i+1}_c(U,N) \) (see [15, Theorem III.8.2, p.361]). Consequently, by the exactness of (5), there exists an element \( \xi_U \in H^i(U,N) \) such that \( \xi_{v|K_v} = \xi_v \) for all \( v \not\in U \). The assignment \( (\xi_v)_{v\in U} \mapsto \xi_U \) is functorial in \( U \), i.e., \( \xi_U = \xi_{V|V} \) if \( (U,N) \leq (V,N) \in \mathcal{F} \), where \( \xi_{U|V} \) denotes the image of \( \xi_U \) under the map \( H^i(U,N) \to H^i(V,N) \) induced by the inclusion \( V \subset U \). For any \( U \) as above, let \( \xi = \xi_{U|K} \) be the image of \( \xi_U \) under the map \( H^i(U,N) \to H^i(K,N) \) induced by the morphism \( \text{Spec} K \to U \). Then \( \xi \) is a well-defined element of \( H^i(K,N) \) whose image under \( \beta_i(K,N) \) is \( (\xi_v) \). This shows that \( \ker(\gamma_i(K,N)) \subset \im(\beta_i(K,N)) \).

Next we will show that \( \gamma_i(K,N) \circ \beta_i(K,N) = 0 \), which will show that \( \im(\beta_i(K,N)) \subset \ker(\gamma_i(K,N)) \). Let \( \xi \in H^i(K,N) \). By Lemma 2.3, there exists a pair \( (U,N) \in \mathcal{F} \) and an element \( \xi_U \in H^i(U,N) \) such that \( \xi = \xi_{U|K} \). For any \( (V,N) \in \mathcal{F} \) such that \( (U,N) \leq (V,N) \), set \( \xi_v = \xi_{U|V} \). The image of \( \xi_v \) under the map \( H^i(V,N) \to \bigoplus_{v\in V} H^i(K_v,N) \) maps to zero under the composite map (11) (with \( U \) replaced by \( V \) there). Consequently, it maps to zero under the
map $\psi^i_{(V, N)}$. Now the commutativity of (9) (with $U$ replaced by $V$ there) shows that $\gamma_i(V, N)(\beta_i(V, N)(\xi_v)) = 0$. It now follows from (10) (with $U$ replaced by $V$ there) that $\gamma_i(K, N)(\beta_i(K, N)(\xi))$ is in the kernel of the canonical map $H^{2-i}(K, N^d)^D \to H^{2-i}(V, N^d)^D$. This holds for any $V$ as above. Since the canonical map $H^{2-i}(K, N^d)^D \to \bigcup_{V} H^{2-i}(V, N^d)^D$ is an isomorphism (see Lemmas 2.4 and 4.10 and note that this statement is trivially true if $i = 2$), we conclude that $\gamma_i(K, N)(\beta_i(K, N)(\xi)) = 0$, as desired.

It remains to show that $\gamma_2(K, N)$ is surjective. In fact, we will show that $\gamma_2(U, N)$ is surjective for any $(U, N) \in \mathcal{F}$. By (10) and the bijectivity of the canonical map $H^0(K, N^d)^D \to H^0(U, N^d)^D$, this will complete the proof. Since “Coker $(f)^D = \ker (f^D)$” if $f$ is a map between finite groups, we have a canonical isomorphism

$$\text{Coker}(\gamma_2(U, N))^D = \ker \left[ H^0(U, N^d) \xrightarrow{\gamma_2(U, N)^D} P^2(U, N^d)^D \right].$$

The map $\gamma_2(U, N)^D$ may be identified with the natural map $N^d(K) \to \bigoplus_{v \in U} N^d(K_v)$, which is clearly injective. Thus $\gamma_2(U, N)$ is indeed surjective.

Now, for $i = 0$ or 1, we define a map

$$\delta_i(K, N) : H^{2-i}(K, N^d)^D \to H^{i+1}(K, N)$$

as the composite

$$H^{2-i}(K, N^d)^D \to \bigoplus H^{2-i}(K, N^d)^D \simeq \bigoplus H^{i+1}(K, N) \hookrightarrow H^{i+1}(K, N),$$

where the isomorphism comes from Theorem 4.9 (applied to $N$ and $N^d$). Clearly

$$\ker(\delta_i(K, N)) = \text{Im} \left[ P^{2-i}(K, N^d)^D \to H^{2-i}(K, N^d)^D \right] = \text{Im} \left[ P^i(K, N) \to H^{2-i}(K, N^d)^D \right] = \text{Im}(\gamma_i(K, N)).$$

Further, $\text{Im}(\delta_i(K, N)) = \bigoplus H^{i+1}(K, N) = \ker(\beta_{i+1}(K, N))$. These facts, together with Proposition 4.11, yield the following Poitou-Tate exact sequence in flat cohomology.
**Theorem 4.12.** There exists an exact sequence of locally compact groups and continuous homomorphisms

\[
0 \rightarrow H^0(K, N) \xrightarrow{\beta_0} P^0(K, N) \xrightarrow{\gamma_0} H^2(K, N^d) \xrightarrow{D} H^1(K, N^d) \xrightarrow{\beta_1} P^1(K, N) \xrightarrow{\gamma_1} H^0(K, N) \xrightarrow{\delta_0} H^1(K, N) \xrightarrow{\delta_1} P^2(K, N) \xrightarrow{\gamma_2} P^0(K, N) \xrightarrow{\delta_0} 0.
\]

\[\square\]

5. 1-Motives over open affine subschemes of \(X\)

In this section all groups will be endowed with the discrete topology, with the exception of the groups \(H^0(K_v, M)\), which will be endowed with the topology defined in [10, p.99]. Thus \(H^0(K_v, M)\) contains a closed subgroup of finite index which is a (possibly non-Hausdorff) quotient of \(G(K_v)\). The theory of Lie groups over a local field shows that \(G(K_v)\) is locally compact, compactly generated and completely disconnected. Therefore, by [12, Theorem II.9.8, p.90], \(G(K_v)\) is topologically isomorphic to a product \(\mathbb{Z}^b \times C\), where \(b\) is a non-negative integer and \(C\) is a compact and completely disconnected, i.e., profinite, abelian group. We conclude that, if \(H^0(K_v, M)'\) denotes the quotient of \(H^0(K_v, M)\) by the closure of \(\{0\}\) and \(n\) is any integer, then \(H^0(K_v, M)'/n\) is a profinite group.

Let \(U\) be any nonempty open affine subscheme of \(X\). For any cohomologically bounded complex \(\mathcal{F}^\bullet\) of fppf sheaves on \(U\), there exist cohomology groups with compact support \(H^i_c(U, \mathcal{F}^\bullet)\) which may be defined as in [15, comments preceding Proposition III.0.4, p.271]. There exists an exact sequence

\[
\ldots \rightarrow H^i_c(U, \mathcal{F}^\bullet) \rightarrow H^i(U, \mathcal{F}^\bullet) \rightarrow \bigoplus_{v \notin U} H^i(K_v, \mathcal{F}^\bullet) \rightarrow H^{i+1}_c(U, \mathcal{F}^\bullet) \rightarrow \ldots,
\]

where we have abused notation in the third term by writing \(\mathcal{F}^\bullet\) for the pullback of \(\mathcal{F}^\bullet\) under the composite map \(\text{Spec } K_v \rightarrow \text{Spec } K \rightarrow U\).

For any pair of cohomologically bounded complexes \(\mathcal{F}^\bullet\) and \(\mathcal{G}^\bullet\) as above, there exists a cup-product pairing

\[
H^i(U, \mathcal{F}^\bullet) \times H^j_c(U, \mathcal{G}^\bullet) \rightarrow H^{i+j}_c(U, \mathcal{F}^\bullet \otimes^L \mathcal{G}^\bullet).
\]
Now let $\mathcal{M}$ be a 1-motive over $U$. Set
\[ H^i(U, T_p(\mathcal{M})) = \lim_{m \to 0} H^i(U, T_{Z/p^m}(\mathcal{M})) \]
and
\[ H^i_c(U, T_p(\mathcal{M})) = \lim_{m \to 0} H^i_c(U, T_{Z/p^m}(\mathcal{M})). \]

**Lemma 5.1.** Let $i = 0, 1$ or $2$.

(a) There exists a pairing
\[ H^{i+1}(U, T_p(\mathcal{M}))(p) \times (H^{2-i}_c(U, T(\mathcal{M}^*)\{p\})/p-\text{div}) \to \mathbb{Q}_p/\mathbb{Z}_p \]
which is non-degenerate on the left.

(b) There exists a pairing
\[ (H^i(U, T(\mathcal{M})\{p\})/p-\text{div}) \times H^{3-i}_c(U, T_p(\mathcal{M}^*))(p) \to \mathbb{Q}_p/\mathbb{Z}_p \]
which is non-degenerate on the right.

**Proof.** By [15, Theorem III.8.2, p.361], for every $r \geq 0$ and any $m \geq 1$ there exists a perfect pairing
\[ H^r(U, T_{Z/p^m}(\mathcal{M})) \times H^{3-r}_c(U, T_{Z/p^m}(\mathcal{M}^*)) \to \mathbb{Q}_p/\mathbb{Z}_p \]
between the discrete torsion group $H^r(U, T_{Z/p^m}(\mathcal{M}))$ and the profinite group $H^{3-r}_c(U, T_{Z/p^m}(\mathcal{M}^*))$. Setting $r = i + 1$ and $r = i$ above, we obtain perfect pairings
\[ H^{i+1}(U, T_p(\mathcal{M})) \times H^{2-i}_c(U, T(\mathcal{M}^*)\{p\}) \to \mathbb{Q}_p/\mathbb{Z}_p \]
and
\[ H^i(U, T(\mathcal{M})\{p\}) \times H^{3-i}_c(U, T_p(\mathcal{M}^*)) \to \mathbb{Q}_p/\mathbb{Z}_p. \]
The lemma now follows easily. \(\square\)

For each $i$ such that $-1 \leq i \leq 3$, there exists a canonical pairing
\[ \langle -, - \rangle : \mathbb{H}^i(U, \mathcal{M}) \times \mathbb{H}^{2-i}_c(U, \mathcal{M}^*) \to \mathbb{Q}/\mathbb{Z}. \]
See [10, p.108]. The above pairing induces a pairing
\[ \mathbb{H}^i(U, \mathcal{M})(p)/p-\text{div} \times \mathbb{H}^{2-i}_c(U, \mathcal{M}^*)(p)/p-\text{div} \to \mathbb{Q}_p/\mathbb{Z}_p \]

**Theorem 5.2.** For any 1-motive $\mathcal{M}$ over $U$ and any $i$ such that $0 \leq i \leq 2$, the pairing (14) is non-degenerate.

**Proof.** For each integer $m \geq 1$, there exists a canonical exact sequence
\[ 0 \to \mathbb{H}^{1-i}_c(U, \mathcal{M}^*)/p^m \to H^{2-i}_c(U, T_{Z/p^m}(\mathcal{M}^*)) \to \mathbb{H}^{2-i}_c(U, \mathcal{M}^*)_{p^m} \to 0. \]
See [10, p.109]. Taking the direct limit as \( m \to \infty \), we obtain an exact sequence

\[
0 \to \mathbb{H}^{1-i}(U, \mathcal{M}^*) \otimes \mathbb{Q}_p/ \mathbb{Z}_p \to H^{2-i}_c(U, T(\mathcal{M}^*)\{p\}) \\
\to \mathbb{H}^{2-i}_c(U, \mathcal{M}^*)(p) \to 0.
\]

Consequently, there exists a canonical isomorphism

\[
\mathbb{H}^{2-i}_c(U, \mathcal{M}^*)(p)/p\text{-div} = H^{2-i}_c(U, T(\mathcal{M}^*)\{p\})/p\text{-div}.
\]

On the other hand, for every integer \( m \geq 1 \) there exists a canonical exact sequence

\[
0 \to \mathbb{H}^i(U, \mathcal{M})/p^m \to H^{i+1}(U, TZ/p^m(\mathcal{M})) \to \mathbb{H}^{i+1}(U, \mathcal{M})_{p^m} \to 0.
\]

Taking the inverse limit as \( m \to \infty \), we obtain an exact sequence

\[
0 \to \mathbb{H}^i(U, \mathcal{M})(p) \to H^{i+1}(U, T_p(\mathcal{M})) \to T_p \mathbb{H}^{i+1}(U, \mathcal{M}).
\]

Therefore, there exists a canonical isomorphism

\[
H^{i+1}(U, T_p(\mathcal{M}))(p) = \mathbb{H}^i(U, \mathcal{M})(p).
\]

Using Lemma 2.1, we conclude that there exists a canonical injection

\[
\mathbb{H}^i(U, \mathcal{M})(p)/p\text{-div} \hookrightarrow H^{i+1}(U, T_p(\mathcal{M}))(p).
\]

Now Lemma 5.1(a) shows that \( H^{i+1}(U, T_p(\mathcal{M}))(p) \) injects into

\[
(H^{2-i}_c(U, T(\mathcal{M}^*)\{p\})/p\text{-div})^D = (\mathbb{H}^{2-i}_c(U, \mathcal{M}^*)(p)/p\text{-div})^D,
\]

which shows that (14) is non-degenerate on the left. To see that (14) is non-degenerate on the right, interchange in the above argument \( \mathcal{M} \) and \( \mathcal{M}^* \), \( i \) and \( 2-i \), \( H \) and \( H_c \) and \( \mathbb{H} \) and \( \mathbb{H}_c \), and use Lemma 5.1(b) instead of Lemma 5.1(a). \( \square \)

**Remark 5.3.** The pairings (12) and (13) are compatible, i.e., if

\[
\partial_c : \mathbb{H}^{1-i}_c(U, \mathcal{M}) \to H^{2-i}_c(U, TZ/p^m(\mathcal{M})), \\
\vartheta = \partial_i : H^{i+1}(U, TZ/p^m(\mathcal{M}^*)) \to \mathbb{H}^{i+1}(U, \mathcal{M}^*)_{p^m}
\]

are the maps arising from sequences (15) and (16) in the proof of the theorem (with the roles of \( \mathcal{M} \) and \( \mathcal{M}^* \) interchanged), then

\[
[\partial_c(\zeta), \xi] = \langle \zeta, \vartheta(\xi) \rangle
\]

for every \( \xi \in H^{i+1}(U, TZ/p^m(\mathcal{M}^*)) \) and \( \zeta \in \mathbb{H}^{1-i}_c(U, \mathcal{M}) \).
Now define, for \( i \geq 0 \),
\[
D^i(U, \mathcal{M}) = \text{Im} \left[ \mathbb{H}^i_c(U, \mathcal{M}) \to \mathbb{H}^i(U, \mathcal{M}) \right]
\]
\[
= \text{Ker} \left[ \mathbb{H}^i(U, \mathcal{M}) \to \bigoplus_{v \notin U} \mathbb{H}^i(K_v, M) \right].
\]

**Lemma 5.4.** \( D^1(U, \mathcal{M})(p) \) is a group of finite cotype.

**Proof.** (Cf. [10, proof of Proposition 3.7, p.111]) We need to show that \( D^1(U, \mathcal{M})_p \) is finite. There exists an exact commutative diagram
\[
\begin{array}{cccc}
H^1(U, \mathcal{Y}) & \longrightarrow & H^1(U, \mathcal{G}) & \longrightarrow & \mathbb{H}^1(U, \mathcal{M}) & \longrightarrow & H^2(U, \mathcal{Y}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\bigoplus_{v \notin U} H^1(Y_v) & \longrightarrow & \bigoplus_{v \notin U} H^1(G_v) & \longrightarrow & \bigoplus_{v \notin U} \mathbb{H}^1(M_v) & \longrightarrow & \bigoplus_{v \notin U} H^2(Y_v),
\end{array}
\]

where, to simplify the notation, on the bottom row we have written \( H^1(Y_v) \) for \( H^1(K_v, Y) \) and similarly for the remaining terms. The groups \( H^1(U, \mathcal{Y}) \) and \( \bigoplus_{v \notin U} H^1(Y_v) \) are finite. See the proof of [10, Lemma 3.2(3), p.108] and [15, Corollary I.2.4, p.35]. Using these facts, the above diagram shows that the finiteness of \( D^1(U, \mathcal{M})_p \) follows from that of \( D^1(U, \mathcal{G})_p \) and \( D^2(U, \mathcal{Y}) \). Since \( U \) is affine, \( H^1(U, T) \) is finite [15, Theorem II.4.6(a), p.234], which implies that \( D^2(U, \mathcal{Y}) \) is finite (see [10, proof of Proposition 3.7, p.111]). Further, \( D^1(U, \mathcal{A})_p = \mathbb{H}^1(K, A)_p \) is finite by [13] (see also [15, Lemma II.5.5, p.247]) and the finiteness of \( D^1(U, \mathcal{G})_p \) now follows from that of \( H^1(U, T) \). \( \square \)

Now, for \( m \geq 1 \), let \( S(U, \mathcal{M}^*)_{p^m} \) denote the kernel of the composite map
\[
H^1(U, T_{\mathbb{Z}/p^m}(\mathcal{M}^*)) \to \mathbb{H}^1(U, \mathcal{M}^*)_{p^m} \to \bigoplus_{v \notin U} \mathbb{H}^1(K_v, M^*)_{p^m},
\]
where the first map is the surjection appearing in the exact sequence (16) for \( i = 0 \) and the second map is induced by the canonical map \( \mathbb{H}^1(U, \mathcal{M}^*) \to \bigoplus_{v \notin U} \mathbb{H}^1(K_v, M^*) \). Now, for each \( v \), recall the quotient \( \mathbb{H}^0(K_v, M)' \) of \( \mathbb{H}^0(K_v, M) \) by the closure of \{0\}. By Theorem 3.1(b), \( \mathbb{H}^1(K_v, M^*)(p) \) is isomorphic to the continuous dual of \( \mathbb{H}^0(K_v, M)' \), which is the same as that of its dense subgroup
\[
\text{Im} \left( \mathbb{H}^0(K_v, M)' \to \mathbb{H}^0(K_v, M)' \right).
\]
Since the latter is a quotient of \( \mathbb{H}^0(K_v, M)' \), we conclude that there exists an injection
\[
\mathbb{H}^1(K_v, M^*)_{p^m} \hookrightarrow \left( \left( \mathbb{H}^0(K_v, M)' \right)^D \right)_{p^m} = \left( \mathbb{H}^0(K_v, M)'/p^m \right)^D.
\]
Consequently, there exists a canonical exact sequence
\[(18) \quad 0 \to S(U, \mathcal{M}^*)_{p^m} \to H^1(U, T_{\mathbb{Z}/p^m}(\mathcal{M}^*)) \to \bigoplus_{v \notin U} (\mathbb{H}^0(K_v, M')/p^m)^D.\]

Now, by the comments at the beginning of this Section, the preceding is an exact sequence of discrete groups. Consequently, its dual is exact and we conclude that there exists an exact sequence
\[(19) \quad \prod_{v \notin U} \mathbb{H}^0(K_v, M) \to H_c^2(U, T_{\mathbb{Z}/p^m}(\mathcal{M})) \to (S(U, \mathcal{M}^*)_{p^m})^D \to 0.\]

Now let
\[(20) \quad \delta : \prod_{v \notin U} H^1(K_v, T_{\mathbb{Z}/p^m}(\mathcal{M})) \to H_c^2(U, T_{\mathbb{Z}/p^m}(\mathcal{M}))\]
and
\[(21) \quad \lambda : H^1(U, T_{\mathbb{Z}/p^m}(\mathcal{M}^*)) \to \bigoplus_{v \notin U} H^1(K_v, T_{\mathbb{Z}/p^m}(\mathcal{M}^*))\]
be the canonical maps. Further, for each \(v\), let \(\varrho_v\) denote the dual of the composite map
\[H^1(K_v, T_{\mathbb{Z}/p^m}(\mathcal{M}^*)) \to H^1(K_v, M^*)_{p^m} \to \mathbb{H}^0(K_v, M)^D,\]
where the first map is the local analogue of the surjection appearing in (16). Let
\[(22) \quad \varrho = \prod_{v \notin U} \varrho_v : \prod_{v \notin U} \mathbb{H}^0(K_v, M) \to \prod_{v \notin U} H^1(K_v, T_{\mathbb{Z}/p^m}(\mathcal{M})).\]

Now consider the pairing
\[(-, -) : \prod_{v \notin U} H^1(K_v, T_{\mathbb{Z}/p^m}(\mathcal{M})) \times \bigoplus_{v \notin U} H^1(K_v, T_{\mathbb{Z}/p^m}(\mathcal{M}^*)) \to \mathbb{Q}_p/\mathbb{Z}_p\]
defined by
\[((c_v), (c'_v)) = \sum_{v \notin U} \text{inv}_v(c_v \cup c'_v).\]

This pairing is compatible with (12), i.e.,
\[(23) \quad [\delta(c), x] = (c, \lambda(x))\]
for all \(c \in \prod_{v \in U} H^1(K_v, T_{\mathbb{Z}/p^m}(\mathcal{M}))\) and \(x \in H^1(U, T_{\mathbb{Z}/p^m}(\mathcal{M}^*))\).

**Lemma 5.5.** Let \(c \in \prod_{v \in U} H^1(K_v, T_{\mathbb{Z}/p^m}(\mathcal{M}))\). Then \((c, \lambda(x)) = 0\) for all \(x \in S(U, \mathcal{M}^*)_{p^m} \subset H^1(U, T_{\mathbb{Z}/p^m}(\mathcal{M}^*))\) if and only if \(c\) can be written in the form \(c = c_1 + c_2\), with \(c_1 \in \text{Im}(\varrho)\) and \(c_2 \in \text{Ker}(\delta)\), respectively.
Proof. There exists a canonical diagram

\[
\begin{array}{c}
\prod_{v \in U} H^1(K_v, T_{Z/p^m}(M)) \\
\downarrow \varrho \\
\prod_{v \notin U} \mathbb{H}^0(K_v, M) \rightarrow H^2_c(U, T_{Z/p^m}(\mathcal{M})) \rightarrow (S(U, \mathcal{M}^*)_{p^m})^D,
\end{array}
\]

where the bottom row is the exact sequence (19) and \(\varrho\) and \(\delta\) are the maps (22) and (20), respectively. Let \(c \in \prod_{v \notin U} \mathbb{H}^0(K_v, M)\) map to zero in \((S(U, \mathcal{M}^*)_{p^m})^D\). Then \(\delta(c)\) is the image of an element \(c' \in \prod_{v \notin U} \mathbb{H}^0(K_v, M)\). Let \(c_1 = \varrho(c')\). Then \(c_2 := c - c_1 \in \text{Ker}(\delta)\), which completes the proof. \(\square\)

There exists a canonical commutative diagram

\[
\begin{array}{ccccccc}
0 & \rightarrow & D^i(U, \mathcal{M}) & \rightarrow & \mathbb{H}^i(U, \mathcal{M}) & \oplus & H^2_c(U, T_{Z/p^m}(\mathcal{M}^*))^D \\
& & \downarrow & & \downarrow & & \downarrow \\
& & \mathbb{H}^{2-i}(U, \mathcal{M}^*)^D & \oplus & \mathbb{H}^{1-i}(K_v, M^*)^D
\end{array}
\]

with exact top row. The middle vertical map is induced by the pairing (13). It follows that there exists a well-defined pairing

\[
\{ -, - \} : D^i(U, \mathcal{M}) \times D^{2-i}(U, \mathcal{M}^*) \rightarrow \mathbb{Q}/\mathbb{Z}
\]

given by \(\{a, a'\} = \langle a, b'\rangle\), where \(a \in D^i(U, \mathcal{M}) \subset \mathbb{H}^i(U, \mathcal{M})\) and \(b'\) is a preimage of \(a'\) in \(\mathbb{H}^{2-i}_c(U, \mathcal{M}^*)\).

Lemma 5.6. Let \(a \in D^1(U, \mathcal{M})\). Assume that \(a\) is divisible by \(p^m\) in \(\mathbb{H}^1(U, \mathcal{M})\) and that \(\{a, a'\} = 0\) for all \(a' \in D^1(U, \mathcal{M}^*)_{p^m}\), where \(\{-,-\}\) is the pairing (24). Then \(a \in p^m D^1(U, \mathcal{M})\).
Proof. (Cf. [10, Errata]) Consider the exact commutative diagram

\[
\begin{array}{ccccccccc}
\prod_{v \notin U} \mathbb{H}^0(K_v, M) & \xrightarrow{\theta} & \mathbb{H}^1_c(U, \mathcal{M}) & \xrightarrow{\psi} & \mathbb{H}^1(U, \mathcal{M}) \\
\downarrow{\varepsilon} & & \downarrow{\partial_c} & & \downarrow{\delta} \\
\prod_{v \notin U} H^1(K_v, T_{\mathbb{Z}/p^m}(M)) & \xrightarrow{\delta} & H^2_c(U, T_{\mathbb{Z}/p^m}(\mathcal{M})) & \rightarrow & H^2(U, T_{\mathbb{Z}/p^m}(\mathcal{M}))
\end{array}
\]

where \( \varepsilon \) and \( \delta \) are the maps (22) and (20), and \( \partial_c \) is the map introduced in Remark 5.3 (with \( i = 0 \) there). Since \( a \in D^1(U, \mathcal{M}) = \text{Im}(\psi) \), there exists \( \tilde{a} \in \mathbb{H}^1_c(U, \mathcal{M}) \) with \( \psi(\tilde{a}) = a \). On the other hand, since \( a \) is divisible by \( p^m \) in \( \mathbb{H}^1(U, \mathcal{M}) \), we have \( \partial(a) = 0 \). Consequently, we have \( \partial_c(\tilde{a}) = \delta(c) \) for some \( c \in \prod_{v \notin U} H^1(K_v, T_{\mathbb{Z}/p^m}(M)) \). Now recall the map \( \vartheta_0 \) from Remark 5.3 (with \( i = 0 \) there) and let \( \lambda \) be the map (21). If \( x \in S(U, \mathcal{M}^*)_{p^m} \subset H^1(U, T_{\mathbb{Z}/p^m}(\mathcal{M}^*)) \) is arbitrary, then \( \vartheta_0(x) \in D^1(U, \mathcal{M}^*)_{p^m} \) and

\[
(c, \lambda(x)) = [\delta(c), x] = [\partial_c(\tilde{a}), x] = (\tilde{a}, \vartheta_0(x)) = \{a, \vartheta_0(x)\} = 0,
\]

by (23) and Remark 5.3. Consequently, by Lemma 5.5, we can write \( c = g(c'_1) + c_2 \) with \( c'_1 \in \bigoplus_{v \notin U} \mathbb{H}^0(K_v, M) \) and \( c_2 \in \text{Ker}(\delta) \). Therefore \( \partial_c(\tilde{a}) = \delta(c) = (\delta \circ g)(c'_1) = \partial_c(\theta(c'_1)) \). It follows that \( \tilde{a} - \theta(c'_1) = p^m b \) for some \( b \in \mathbb{H}^1_c(U, \mathcal{M}) \), whence \( a = \psi(\tilde{a}) = \psi(\tilde{a} - \theta(c'_1)) = p^m \psi(b) \in p^m D^1(U, \mathcal{M}) \).

**Theorem 5.7.** The pairing (24) induces a pairing

\[
D^1(U, \mathcal{M})(p) \times D^1(U, \mathcal{M}^*)(p) \rightarrow \mathbb{Q}_p/\mathbb{Z}_p
\]

whose left and right kernels are the maximal divisible subgroups of each group.

Proof. (Cf. [10, Errata]) There exists a natural commutative diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & D^1(U, \mathcal{M})(p) & \rightarrow & \mathbb{H}^1(U, \mathcal{M})(p) & \rightarrow & \bigoplus_{v \notin U} \mathbb{H}^1(K_v, M)(p) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & D^1(U, \mathcal{M}^*)^D & \rightarrow & \mathbb{H}^1_c(U, \mathcal{M}^*)^D & \rightarrow & \bigoplus_{v \notin U} \mathbb{H}^0(K_v, M^*)^D.
\end{array}
\]
It is not difficult to see that the kernel of the middle vertical map is contained in the kernel of the map $\mathbb{H}^1(U, \mathcal{M})(p) \to (\mathbb{H}^1(U, \mathcal{M}^*)(p)/p\text{-div})^D$, which equals $\mathbb{H}^1(U, \mathcal{M})(p)/p\text{-div}$ by Theorem 5.2. Now Lemma 5.6 shows that the kernel of the left vertical map is equal to the maximal divisible subgroup of $D^1(U, \mathcal{M})(p)$. To complete the proof, exchange the roles of $\mathcal{M}$ and $\mathcal{M}^*$.

Now, for $i = 0, 1$ or 2, define
\[ D^i(U, T_p(\mathcal{M})) = \lim_m D^i(U, T_{\mathbb{Z}/p^m}(\mathcal{M})), \]
\[ D^i(U, T(\mathcal{M})\{p\}) = \lim_m D^i(U, T_{\mathbb{Z}/p^m}(\mathcal{M})), \]
where $D^i(U, T_{\mathbb{Z}/p^m}(\mathcal{M}))$ are the groups introduced in Section 4, and
\[ D^i,p(U, \mathcal{M}) = \text{Ker} \left[ \mathbb{H}^i(U, \mathcal{M})(p) \to \bigoplus_{v \notin U} \mathbb{H}^i(K_v, M)(p) \right]. \]

Lemma 5.8. There exist canonical isomorphisms

(a) $D^2(U, \mathcal{M})(p) = D^2(U, T(\mathcal{M})\{p\})$, and
(b) $D^i,(p)(U, \mathcal{M})(p) = D^{i+1}(U, T_p(\mathcal{M}))(p)$. Further, $D^i,(p)(U, \mathcal{M}) = D^{i+1}(U, T_p(\mathcal{M}))$ if $D^{i+1}(U, \mathcal{M})_{p\text{-div}} = 0$.

Proof. (a) Since $\mathbb{H}^1(U, \mathcal{M})$ is torsion by [10, Lemma 3.2(1)] and $\mathbb{Q}_p/\mathbb{Z}_p$ is divisible, the direct limit over $m$ of the exact sequence (16) yields a canonical isomorphism $\mathbb{H}^2(U, \mathcal{M})(p) = H^2(U, T(\mathcal{M})\{p\})$. Similarly, for every prime $v$ of $K$, $\mathbb{H}^2(K_v, M)(p) = H^2(K_v, T(M)\{p\})$ canonically. Assertion (a) now follows easily.

(b) Using the exact sequence (17) over $U$ and over $K_v$ for each prime $v \notin U$, we obtain an exact sequence
\[ 0 \to D^i,(p)(U, \mathcal{M}) \to D^{i+1}(U, T_p(\mathcal{M})) \to T_p D^{i+1}(U, \mathcal{M}). \]
The first assertion in (b) is now clear since $T_p D^{i+1}(U, \mathcal{M})$ is torsion-free. The second assertion follows from the fact that $T_p B = T_p B_{p\text{-div}}$ for any abelian group $B$.

Lemma 5.9. There exists a nonempty open affine subset $\tilde{U}$ of $U$ such that, for every open subset $V$ contained in $\tilde{U}$, both $D^1(V, T_{\mathbb{Z}/p^m}(\mathcal{M}))$ and $D^1(V, T_{\mathbb{Z}/p^m}(*_{\mathcal{M}}))$ are finite for every $m \geq 1$.

Proof. By Lemma 4.3, there exists a set $\tilde{U}$ as in the statement such that, for every open subset $V \subset \tilde{U}$, $D^1(V, \mathcal{G}_p), D^1(V, \mathcal{Y}/p), D^1(V, \mathcal{G}^*_p)$ and $D^1(V, \mathcal{Y}^*/p)$ are all finite. We will now use the exact sequence of fppf sheaves
\[ 0 \to \mathcal{G}_p \to \mathcal{G}_{p,m+1} \xrightarrow{p} \mathcal{G}_{p,m} \to 0 \]
to show that the finiteness of $D^1(V, G_{p^m+1})$ follows from that of $D^1(V, G_{p^m})$. (A similar argument, using the exact sequence $0 \to Y/p \to Y/p^{m+1} \to Y/p^m \to 0$, will show that the finiteness of $D^1(V, Y/p^{m+1})$ follows from that of $D^1(V, Y/p^m)$. Note that the corresponding facts will hold for the duals of the group schemes involved as well). Indeed, the canonical exact commutative diagram

$$
\begin{array}{ccc}
H^1(V, G_p) & \longrightarrow & H^1(V, G_{p^m+1}) \\
\downarrow & & \downarrow \\
\bigoplus_{v \not\in V} H^1(K_v, G_p) & \longrightarrow & \bigoplus_{v \not\in V} H^1(K_v, G_{p^m+1}) \\
\bigoplus_{v \not\in V} H^1(K_v, G_{p^m}) & \longrightarrow & \bigoplus_{v \not\in V} H^1(K_v, G_{p^m})
\end{array}
$$

yields exact sequences

$$l^{-1}(\ker i) \to D^1(V, G_{p^m+1}) \to D^1(V, G_{p^m})$$

and

$$0 \to D^1(V, G_p) \to l^{-1}(\ker i) \to \ker i.$$

Since $\ker i$ is a quotient of the finite group $\bigoplus_{v \not\in V} G(K_v)_{p^m}$, we obtain the desired conclusion. Thus, $D^1(V, G_{p^m}), D^1(V, Y/p^m), D^1(V, G_{p^m}^*)$ and $D^1(V, Y^*/p^m)$ are all finite for all $m \geq 1$. Finally, a similar argument using the exact sequence

$$0 \to G_{p^m} \to T_{Z/p^m}(\mathcal{M}) \to Y/p^m \to 0$$

and the finiteness of $D^1(V, G_{p^m}), D^1(V, Y/p^m)$ and $\bigoplus_{v \not\in V} Y/p^m$ will yield the finiteness of $D^1(V, T_{Z/p^m}(\mathcal{M}^*))$ for all $m \geq 1$. The finiteness of $D^1(V, T_{Z/p^m}(\mathcal{M}^*))$ for all $m$ is obtained similarly, replacing $G$ and $Y$ above by their duals.

Now define

$$D_{\lambda}^0(U, \mathcal{M}) = \ker \left[ \mathbb{H}^0(U, \mathcal{M}) \to \bigoplus_{v \not\in U} \mathbb{H}^0(K_v, M)\right].$$

**Theorem 5.10.** Let $\tilde{U}$ be as in the previous lemma. Then, for every nonempty open subset $V$ of $\tilde{U}$, the pairing (13) induces a pairing

$$D_{\lambda}^0(V, \mathcal{M})(p) \times D^2(V, \mathcal{M}^*)(p) \to \mathbb{Q}_p/\mathbb{Z}_p$$

whose left kernel is trivial and right kernel is $D^2(V, \mathcal{M}^*)(p)_{\text{div}}$. 

Proof. (Cf. [10, Errata]) There exists a natural exact commutative diagram

\[
\begin{array}{ccc}
0 & \to & D^0_\alpha(V, \mathcal{M})(p) \\
& \uparrow & \downarrow \\
& \to & \mathbb{H}^0(V, \mathcal{M})(p) \\
& & \oplus_{v \not\in V} \mathbb{H}^0(K_v, M)^{\sim} \\
& & \downarrow \\
0 & \to & D^2(V, \mathcal{M}^*)(p)^D \\
& \uparrow & \downarrow \\
& \to & \mathbb{H}^2_\alpha(V, \mathcal{M}^*)(p)^D \\
& & \oplus_{v \not\in V} \mathbb{H}^1(K_v, M^*)(p)^D.
\end{array}
\]

The bottom row is exact by Lemma 2.2(b) since \( \mathbb{H}^2_\alpha(V, \mathcal{M}^*) \) is torsion [10, Lemma 3.2(1), p.107]. Now, by [10, Lemma 3.2(3), p.107], \( \mathbb{H}^0(V, \mathcal{M})(p) \) is a finite group, whence Theorem 5.2 yields an injection \( \mathbb{H}^0(V, \mathcal{M})(p) \hookrightarrow \mathbb{H}^2_\alpha(V, \mathcal{M}^*)(p)^D \). Thus the left-hand vertical map induces an injection

\[ D^0_\alpha(V, \mathcal{M})(p) \hookrightarrow (D^2(V, \mathcal{M}^*)(p)/p\text{-div})^D. \]

Now Lemmas 4.8 and 5.8(a) show that there exists a perfect continuous pairing

\[ D^1(V, T_p(\mathcal{M})) \times D^2(V, \mathcal{M}^*)(p) \to \mathbb{Q}_p/\mathbb{Z}_p. \]

The left-hand group is profinite by Lemma 5.9 and the right-hand one is discrete and torsion. Consequently, by [15, Proposition 0.19(e), p.15], the preceding pairing induces a perfect pairing

\[ D^1(V, T_p(\mathcal{M}))(p) \times D^2(V, \mathcal{M}^*)(p)/p\text{-div} \to \mathbb{Q}_p/\mathbb{Z}_p. \]

Thus, by Lemma 5.8(b),

\[ (D^2(V, \mathcal{M}^*)(p)/p\text{-div})^D \hookrightarrow D^1(V, T_p(\mathcal{M}))(p) = D^{0,(p)}(V, \mathcal{M})(p) \]

Now, since \( \mathbb{H}^0(V, \mathcal{M}) \) is finitely generated [10, Lemma 3.2(3), p.107], we have \( \mathbb{H}^0(V, \mathcal{M})^{(p)}(p) = \mathbb{H}^0(V, \mathcal{M})(p) \). Thus \( D^{0,(p)}(V, \mathcal{M})(p) \subset D^0_\alpha(V, \mathcal{M})(p) \) and we conclude that \( (D^2(V, \mathcal{M}^*)(p)/p\text{-div})^D \) is a finite group of order at most equal to the order of \( D^0_\alpha(V, \mathcal{M})(p) \). But then (25) is an isomorphism, as desired. \( \square \)

Remark 5.11. The above proof shows that

\[ D^0_\alpha(V, \mathcal{M})(p) = D^{0,(p)}(V, \mathcal{M})(p) = D^1(V, T_p(\mathcal{M}))(p) \]

for any open set \( V \subset \tilde{U} \). It follows that an inclusion \( V_1 \subset V_2 \) of open subsets of \( \tilde{U} \) induces an inclusion \( D^0_\alpha(V_1, \mathcal{M})(p) \subset D^0_\alpha(V_2, \mathcal{M})(p) \), because the latter holds for \( D^1(\cdot, T_p(\mathcal{M}))(p) \) by the proof of Proposition 4.7 and the left-exactness of the inverse limit functor.
Let $M$ be a 1-motive over $K$ and let $\mathcal{F}$ denote the set of pairs $(U, \mathcal{M})$ where $U$ is a nonempty open affine subscheme of $X$ and $\mathcal{M}$ is a 1-motive over $U$ which extends $M$. Then $\mathcal{F}$ is nonempty, i.e., any 1-motive over $K$ extends to a 1-motive over some nonempty open affine subscheme of $X$. As in Section 2, we order $\mathcal{F}$ by setting $(U, \mathcal{M}) \leq (V, \mathcal{M}')$ if and only if $V \subset U$ and $\mathcal{M}|_V = \mathcal{M}'$.

**Lemma 6.1.** Let $T$ be a torus over a nonempty open affine subscheme $U$ of $X$. Then there exists a nonempty open subset $U_0$ of $U$ such that, for any nonempty open subset $V$ of $U_0$, the canonical map $\text{H}^1(V, T) \to \text{H}^1(K, T)$ is injective.

**Proof.** Assume first that $T$ is flasque (see [6, §1, p.157] for the definition). By [15, Theorem II.4.6(a), p.234], $\text{H}^1(U, T)$ is finite. Let $\{\xi_1, \ldots, \xi_r\}$ be the kernel of the canonical map $\text{H}^1(U, T) \to \text{H}^1(K, T)$. For each $j$, there exists a nonempty open subset $U_j$ of $U$ such that $\xi_j \in \text{Ker}[\text{H}^1(U, T) \to \text{H}^1(U_j, T)]$. Set $U_0 = \bigcap_{j=1}^r U_j$ and let $V$ be any nonempty open subset of $U_0$. Using the fact that the canonical map $\text{H}^1(U, T) \to \text{H}^1(V, T)$ is surjective [6, Theorem 2.2(i), p.161], it is not difficult to see that the map $\text{H}^1(V, T) \to \text{H}^1(K, T)$ is injective. Since it is surjective as well [op.cit., Proposition 1.4, p.158, and Theorem 2.2(i), p.161], it is in fact an isomorphism.

Now let $T$ be arbitrary and choose a flasque resolution of $T$ [6, Proposition 1.3, p.158]:

$$1 \to T'' \to T' \to T \to 1$$

with $T'$ and $T''$ flasque. Let $U_0$ and $U'_0$ be attached to $T'$ and $T''$ as in the first part of the proof and let $U_0 = U'_0 \cap U''_0$. Let $V$ be any nonempty open subset of $U_0$. Then there exists an exact commutative diagram

$$\begin{array}{cccccc}
\text{H}^1(V, T'') & \longrightarrow & \text{H}^1(V, T') & \longrightarrow & \text{H}^1(V, T) & \longrightarrow & \text{H}^2(V, T'') \\
\downarrow & & \downarrow \cong & & \downarrow & & \downarrow \\
\text{H}^1(K, T'') & \longrightarrow & \text{H}^1(K, T') & \longrightarrow & \text{H}^1(K, T) & \longrightarrow & \text{H}^2(K, T'')
\end{array}$$

The rightmost vertical map is injective by [6, Theorem 2.2(ii), p.161] and now the four-lemma completes the proof. $\square$

---

3 This inelegant notation is chosen so that the set denoted $U_0$ below corresponds to the set so denoted in our main reference [10].
Lemma 6.2. Let \((U, \mathcal{M}) \in \mathcal{F}\) be arbitrary. Then there exists a nonempty open subset \(U_1\) of \(U\) such that, for any nonempty open subset \(V\) of \(U_1\), the canonical map \(\mathbb{H}^1(V, \mathcal{M})(p) \to \mathbb{H}^1(K, \mathcal{M})(p)\) is injective.

Proof. Let \(T\) be the toric part of \(\mathcal{M}\) and let \(U_1\) be associated to \(T\) as in the previous lemma. Let \(V\) be any nonempty open subset of \(U_1\). There exists a natural exact commutative diagram

\[
\begin{array}{cccccc}
A(V) & \longrightarrow & H^1(V, T) & \longrightarrow & H^1(V, G) & \longrightarrow & H^1(V, A) \\
\downarrow \simeq & & \downarrow & & \downarrow & & \downarrow \\
A(K) & \longrightarrow & H^1(K, T) & \longrightarrow & H^1(K, G) & \longrightarrow & H^1(K, A).
\end{array}
\]

The first vertical map in the above diagram is an isomorphism by the properness of \(A\), the second one is an injection by the previous lemma and the rightmost one is an injection by [15, proof of Lemma II.5.5, p.247]. The four-lemma now shows that the third vertical map is an injection. Consider now the exact commutative diagram

\[
\begin{array}{cccccc}
H^1(V, \mathcal{Y}) & \longrightarrow & H^1(V, G) & \longrightarrow & \mathbb{H}^1(V, \mathcal{M}) & \longrightarrow & H^2(V, \mathcal{Y}) \\
\downarrow \simeq & & \downarrow & & \downarrow & & \downarrow \\
H^1(K, \mathcal{Y}) & \longrightarrow & H^1(K, G) & \longrightarrow & \mathbb{H}^1(K, \mathcal{M}) & \longrightarrow & H^2(K, \mathcal{Y})
\end{array}
\]

whose top and bottom rows come from the distinguished triangle (1) over \(V\) and over \(K\). The second vertical map was shown to be injective above. The first vertical map is an isomorphism by the proof of [15, Proposition II.2.9, p.209] and the fact that \(H^1(G_S, Y(K_S)) = H^1(K, Y)\) (see [10, p.112, lines 11-17]). The rightmost vertical map is injective when restricted to \(p\)-primary components by [15, Proposition II.2.9, p.209] and [loc.cit.]. The lemma now follows from these facts and the commutativity of the last diagram. \(\square\)

Remarks 6.3. (a) As noted in the proof of the above lemma, the canonical map \(H^2(U, \mathcal{Y})(p) \to H^2(K, \mathcal{Y})(p)\) is injective for any nonempty open affine subset \(U\) of \(X\). We may therefore regard \(D^2(U, \mathcal{Y})(p)\) as a subgroup of \(H^2(K, \mathcal{Y})(p)\) for any such \(U\). Recall also that \(D^2(U, \mathcal{Y})\) is finite, as noted in the proof of Lemma 5.4.

(b) Lemma 6.2 is valid if \(K\) is any global field and \(p\) is any prime number (the proof is essentially the same). In the number field case, D.Harari and T.Szamuely have obtained an alternative proof of Lemma 6.2 using a well-known theorem of T.Ono. See [10, Errata].

\footnote{The notation is as in [15, comments preceding Proposition II.2.9, pp.208-209].}
Lemma 6.4. Let \((U, \mathcal{M}) \in \mathcal{F}\) be arbitrary.

(a) For any prime \(v\) of \(K\), the canonical map \(H^2(\mathcal{O}_v, \mathcal{Y})(p) \to H^2(K_v, \mathcal{Y})(p)\) is injective.

(b) There exists a nonempty open subset \(U_0 \subset U\) such that, for any nonempty open subset \(V \subset U_0\), the group \(D^2(V, \mathcal{Y})(p)\) is contained in \(\mathbf{III}^2(K, \mathcal{Y})(p)\).

Proof. (a) By the localization sequence for the pair \(\text{Spec } K_v \subset \text{Spec } \mathcal{O}_v\), it suffices to show that the quotient of \(H^2_v(\mathcal{O}_v, \mathcal{Y})\) by the image of \(H^1(K_v, \mathcal{Y})\) contains no nontrivial \(p\)-torsion elements. By Lemma 2.2(a), this follows from the triviality of \(H^2_v(\mathcal{O}_v, \mathcal{Y})_p\), which in turn follows from that of \(H^1_v(\mathcal{O}_v, \mathcal{Y}/p)\) [15, beginning of §III.7, p.349, line 3].

(b) Using Remark 6.3(a) above, the proof is formally the same as that of [10, Lemma 4.7, p.114].

We now define, for \(i = 0, 1\) or 2,

\[
\mathbf{III}^i(K, \mathcal{M}) = \text{Ker} \left[ \bigoplus_v H^i(K_v, \mathcal{M}) \to \prod_v H^i(K_v, \mathcal{M}) \right].
\]

Lemma 6.5. Let \((U, \mathcal{M}) \in \mathcal{F}\) be arbitrary and let \(U_-\) and \(U_0\) be as in Lemmas 6.2 and 6.4(b), respectively. Let \(U_1 = U_- \cap U_0\). Then, for any nonempty open subset \(V \subset U_1\), the canonical map \(H^1(V, \mathcal{M})(p) \to H^1(K, \mathcal{M})(p)\) induces an isomorphism \(D^1(V, \mathcal{M})(p) = \mathbf{III}^1(K, \mathcal{M})(p)\). In particular, \(\mathbf{III}^1(K, \mathcal{M})(p)\) is a group of finite cotype.

Proof. The proof is analogous to the proof of [10, Proposition 4.5, p.114], using Lemma 6.4 and an argument similar to that used at the end of the proof of Proposition 4.7 (cf. [10, proof of Theorem 4.8, p.115]). The last assertion of the lemma follows from Lemma 5.4.

The following result is an immediate consequence of the previous lemma and Theorem 5.7.

Theorem 6.6. Let \(M\) be a 1-motive over \(K\). Then there exists a canonical pairing

\[
\mathbf{III}^1(K, \mathcal{M})(p) \times \mathbf{III}^1(K, \mathcal{M}^*)(p) \to \mathbb{Q}_p/\mathbb{Z}_p
\]

whose left and right kernels are the maximal divisible subgroups of each group.

Corollary 6.7. Let \(M\) be a 1-motive over \(K\). Assume that \(\mathbf{III}^1(K, \mathcal{M})(p)\) and \(\mathbf{III}^1(K, \mathcal{M}^*)(p)\) contain no nonzero infinitely divisible elements. Then there exists a perfect pairing of finite groups

\[
\mathbf{III}^1(K, \mathcal{M})(p) \times \mathbf{III}^1(K, \mathcal{M}^*)(p) \to \mathbb{Q}_p/\mathbb{Z}_p.
\]
Proof. This is immediate from the theorem, noting that $\mathcal{I}^1_1(K,M)(p) = \mathcal{I}^1(K,M)(p)/p$-div and $\mathcal{I}^1(K,M^*)(p) = \mathcal{I}^1(K,M^*)(p)/p$-div are both finite. □

We now fix an element $(U,\mathcal{M}) \in \mathcal{F}$ and let $\tilde{U} \subset U$ be the set introduced in Lemma 5.9. Further, we write $\mathcal{S}(\tilde{U})$ for the family of all nonempty open subsets of $\tilde{U}$.

Lemma 6.8. There exists a canonical isomorphism
\[
\lim_{\overset{\longrightarrow}{V \in \mathcal{S}(\tilde{U})}} D^2(V,\mathcal{M})(p) = \mathcal{I}^2_2(K,M)(p).
\]
Proof. This follows by combining Remark 5.3(b) and Proposition 4.6. □

Now define
\[
\mathcal{I}^0_\Lambda(K,M) = \text{Ker} \left[ \mathbb{H}^0(K,M) \to \prod_{v} \mathbb{H}^0(K_v,M) \right].
\]

Theorem 6.9. Let $M$ be a 1-motive over $K$. Then there exists a canonical pairing
\[
\mathcal{I}^0_\Lambda(K,M)(p) \times \mathcal{I}^2(K,M^*)(p) \to \mathbb{Q}_p/\mathbb{Z}_p
\]
whose left kernel is trivial and right kernel is the maximal divisible subgroup of $\mathcal{I}^2(K,M^*)(p)$.

Proof. The proof is similar to the proof of [10, Proposition 4.12, p.116], using Theorem 5.10, Lemma 6.8 and Remark 5.11. □

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