GLIDER REPRESENTATION RINGS WITH A VIEW ON DISTINGUISHING GROUPS

FREDERIK CAENEPEEL AND GEOFFREY JANSSENS

Abstract. Let $G$ be a finite group. The main aim of this paper is to further develop the youngly introduced glider representation theory and to kick start its connections with classical representation theory (over $\mathbb{C}$). Firstly, we obtain that the symmetric monoidal structure of the category $\text{Glid}_1(G)$ of glider representations of length 1 of $G$ determines $G$ uniquely. More precisely we show that $\text{Glid}_1(G)$ is somehow a concrete model of $(\text{Rep}_C(G), F)$, the $G$-representations together with a fiber functor $F$. Thenceforth we introduce and investigate the (reduced) glider representation ring $R(\tilde{G})$ and its finitely versions $R_d(\tilde{G})$. Hereby we obtain a short exact sequence relating the semisimple part of $Q \otimes Z R_1(\tilde{G})$ in a precise way to the representations of $G$ (and subnormal subgroups in $G$). For instance if $G$ is nilpotent of class 2, the aforementioned sequence yields that $Q \otimes Z R(\tilde{G})$ contains as a direct summand $Q(H^{ab})$, the rational group algebra of the abelianization of $H$, for every subgroup $H$ of $G$. We end with pointing out applications on distinguishing isocategorical groups.

Contents

1. Introduction 2
2. Glider representation rings 4
2.1. Construction and preliminaries 4
2.2. Gliders of length 1 versus $\text{Rep}(G)$ as symmetric tensor category 8
3. Induced morphisms between glider representation rings 11
3.1. From $\text{Rep}(H)$ to $\text{Rep}(G)$: construction 11
3.2. From $\mathcal{P}_1(\tilde{H})$ to $\mathcal{P}_1(\tilde{G})$: the functor 14
4. A short exact sequence 17
5. Precise description semisimple part $Q \otimes \mathcal{P}(\tilde{G})$ under vanishing obstructions 22
6. Interpreting the obstructions with a representation eye 25
6.1. The module $Q$ 25
6.2. The module $P$ 27
6.3. The module $R$ 30
7. A look at concrete classes of groups 31
7.1. Nilpotent groups of class 2 31
7.2. Isocategorical groups 33
References 34

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1. Introduction

Let $G$ be a finite group. Somehow the purpose of representation theory is to reconstruct $G$, or group-theoretical pieces of it, from its representations and certain invariants attached to it. This will also be the guiding principle of this article, although here it will be from the point of view of glider representations. This recently introduced theory has been developed by Caenepeel-Van Oystaeyen in a series of papers [2, 3, 4] and a full exposition of this young theory can already be found in their book [5].

Given any field $K$ and a subgroup $H$ in $G$. The first purpose of glider representation theory is to develop a ’relative representation theory’ for a pair $(H,G)$. Secondly, every $KG$-module $M$ has a filtration with irreducible factors (i.e. a composition series), say $M \supseteq M_1 \supseteq \ldots \supseteq M_d$. In this case the theory also aims to provide the necessary language and tools to work with such a chain as an object (i.e. to work with the full chain at once).

More concretely, given a chain of (potentially equal) subgroups $G_0 \leq \ldots \leq G_d = G$ then a glider representation of this chain consists of a $KG$-module $M$ together with a descending chain of $KG_0$-submodules $M_i$ such that $KG_j \cdot M_j \subseteq M_i$ for all $0 \leq i \leq j$. We denote by $\text{Glid}(KG_0 \subseteq \ldots \subseteq KG_d)$ the category obtained. In Section 2.1 we recall the necessary background.

Suppose now that $\text{char}(K) = 0$. Such as $\text{Rep}(G)$ the category of gliders $\text{Glid}(KG_0 \subseteq \ldots \subseteq KG_d)$ is still a symmetric monoidal additive category (see Proposition 2.5), however it is no longer a fusion category because it is both not abelian and it has an infinite number of irreducible gliders. Despite this essential difference, the category fits in the natural aims above.

The category $\text{Glid}_1(K \subset KG)$

In this article we will focus on the ’most basic’ case where the chain is simply $1 < G$. This case is already surprisingly rich and as a first main result we obtain that $\text{Glid}_1(K \subset KG)$, viewed with his full structure, determines $G$ uniquely.

Theorem A (Theorem 2.14). Let $G$ be a finite group. Then the functor

$$\mathcal{F} : \text{Mod}(KG) \to \text{Vec}_K : M \mapsto \text{Hom}_{\text{glid}}((K \supset 0), (M \supset K))$$

is faithful $K$-linear symmetric monoidal and is monoidal natural isomorphic to the forgetful functor $F$. Consequently, $\text{Glid}_1(K \subset KG)$ as symmetric monoidal additive category determines $G$ uniquely.

The proof in fact indicates that $\text{Glid}_1(K \subset KG)$ is a model to work concretely with $(\text{Rep}_K(G), F)$, the representations together with the forgetful functor.

Subsequently, we parametrize the isomorphism classes of ’irreducible gliders’ of length 1 (i.e. the objects in $\text{Glid}_1(K \subset KG)$ without ’trivial’ subglider representations). For this, let $\text{Gr}(U) = \bigsqcup_{j=1}^{d} \text{Gr}(j, U)$ where $U \in \text{Irr}(G)$ and $\dim U = d$. Further denote by $\mathcal{S}_G$ the set of subsets $B \subseteq \bigsqcup_{U \in \text{Irr}(G)} \{j | \dim(U) > 1 \text{Gr}(U)\}$, such that for all $U$ the intersection $B \cap \text{Gr}(j, U)$ is non-empty for at most one $1 \leq j \leq \dim(U)$ and for this $j$ it is in fact a singleton. Then,

Proposition B (Proposition 2.17). Let $G$ be a finite group. There is a bijection

$$\left\{ \text{irreducible } (K \subset KG) \text{- gliders } \right\} \cong \{(A, B) \in \mathcal{P}(G/G') \times \mathcal{S}_G\}.$$

In the previous result $G'$ is the commutator subgroup of $G$ and $\mathcal{P}(G/G')$ the power set of $G/G'$.

The glider representation ring and its structure.

In the rest of the paper we investigate which information is still present in the ’glider
representation ring'. Hereby one has however to be careful, since \( \text{Glid}(KG_0 \subseteq \ldots \subseteq KG_\Delta) \) is not an abelian category and hence we can not simply form its Grothendieck ring. In Section 2.1 we carefully introduce the notions of glider representation ring \( R_G \) and glider character ring \( ch_G \).

Starting from Section 3 we aim to describe the structure of \( R_1(\tilde{G}) \) or rather a quotient of it, called the reduced glider representation ring, and which will denoted by \( \overline{R}_1(\tilde{G}) \). One may think of this being the additive group generated by the isomorphism classes of glider representations of length exactly 1 (see Definition 2.7 for a precise definition). Using the tensor product of gliders (cf. Definition 2.4) we can furthermore make into a unital ring and hence \( Q(\tilde{G}) := \mathbb{Q} \otimes_{\mathbb{Z}} \overline{R}_1(\tilde{G}) \) is a \( \mathbb{Q} \)-algebra. In general this algebra is infinite-dimensional, but taking the quotient with its Jacobson radical will yield a finite-dimensional algebra over \( \mathbb{Q} \). Our second main theorem provides a short exact sequence which relates the latter to the glider representation rings of certain subgroups \( H \) of \( G \).

**Theorem C** (Theorem 4.13). Let \( G \) be a finite group. We have the following short exact sequence of \( \mathbb{Q}(G^{ab}) \)-modules

\[
0 \longrightarrow P_G + Q_G + \sum_{\mathbb{Q} \leq R \subseteq G} \sum_{\mathbb{Q} / \mathbb{N} \ni \Psi} \overline{R}_G(Q(\tilde{H}) / N) \longrightarrow Q(\tilde{G}) / N \longrightarrow Q(\tilde{G}) / J \longrightarrow 0
\]

for concretely defined \( \mathbb{Q}(G^{ab}) \)-modules \( P, Q, R \) and morphism \( \Psi \).

At first, the modules \( P, Q, R \) have 'only' concrete definitions in the language of gliders. Therefore, in Section 6, we connect these modules to natural questions in classical representation theory (in case of \( P \) and \( Q \)) and in group theory (in case of \( R \)). We call these modules 'obstruction modules' because, as we show in Section 5, when they vanish we obtain a precise description of the semisimple quotient of \( Q(\tilde{G}) = \mathbb{Q} \otimes_{\mathbb{Z}} \overline{R}_1(\tilde{G}) \). More concretely,

**Theorem D** (Theorem 5.3). Let \( G \) be a finite group such that \( P = 0 = Q \) and \( R = \mathbb{Q}(G / G') \). Then

\[ Q(\tilde{G}) / J \cong \bigoplus_H \mathbb{Q}(H^{ab}), \]

where the direct sum runs over all subnormal subgroups \( H \) of \( G \).

We should mention that in general \( \mathbb{Q}(G / G') \subseteq R \) and hence equality may indeed be viewed as vanishing of \( R \).

Using the aforementioned interpretations of \( P, Q, R \), obtained in Section 6, we show that they indeed vanish, among others, in case that \( G \) is nilpotent of class 2.

**Theorem E** (Corollary 7.2, Proposition 7.3, Proposition 7.4 & Theorem 7.5). Let \( G \) be a finite nilpotent group of class 2. Then

\[ Q(\tilde{G}) / J \cong \bigoplus_{H \leq G} \mathbb{Q}(H^{ab}). \]

Finally, in Section 7.2 we shortly consider isocategorical groups in the sense of Etingof-Gelaki [9]. Recall that groups \( G_1 \) and \( G_2 \) are called isocategorical if \( \text{Rep}(G_1) \) and \( \text{Rep}(G_2) \) are equivalent as tensor category (so without consideration of the symmetry of their monoidal structure). In [13, Section 4] an (infinite) family of non-isomorphic but isocategorical groups \( G^m \) and \( G^m_n \), with \( 3 \leq m \in \mathbb{N} \), was constructed. Despite that they have isomorphic representation rings we show that \( R_1(G^m) \neq R_1(G^m_n) \).

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Conventions. Throughout the full paper we will assume the following (except stated explicitly otherwise):

- \( K \) is a field of characteristic 0 and from Section 3 onwards also algebraically closed.
- all groups, denoted with the letters \( G \) or \( H \), will be finite,
- all \( KG \)-modules will be left modules,
- \( \mathbb{N} \) denotes the positive integers (with 0 included).
- \( \subset \) and \( < \) will indicate strictly smaller.

2. Glider representation rings

2.1. Construction and preliminaries. In this section we introduce the construction of glider representation and character rings. This is inherent in [4] but there it was only defined in a particular case.

The category of \( FKG \)-gliders.

Give a finite group \( G \) and a chain of subgroups \( G_0 < G_1 < \ldots < G_d = G \), one obtains in a natural way a filtration, by subalgebras, \( FKG \) of the group algebra \( KG \) by defining

\[
F_{-n}KG = 0, F_0KG = KG_0, F_nKG = KG_n \quad \text{for } n > 0 \text{ and where } G_n = G \text{ if } n \geq d.
\]

Definition 2.1. An \( FKG \)-glider consists of a (left) \( KG \)-module \( \Omega \) together with a descending chain of \( KG_0 \)-submodules

\[
\Omega \supseteq M_0 \supseteq M_1 \supseteq M_2 \supseteq \ldots
\]

such that for any \( 0 \leq i \leq j \), and with the action induced from the \( G \)-action of \( \Omega \), it holds that \( KG_{j-i}M_j \subseteq M_i \). This glider is denoted shortly by \( (\Omega \supseteq M) \).

Given a ring \( R \), one can actually define \( FR \)-glider representations for any filtration \( FR \) of \( R \) via so-called \( FR \)-fragments [2]. However we will only consider the (very natural) algebra filtration coming from a chain of subgroups, as above. Therefore we will often not emphasize the filtration and simply speak about a glider (representation) of \( G \).

Literature remark 2.2. In the original definition of a glider representation the module \( \Omega \) was not included in the data and only its existence was assumed. However, since we will be interested in the generalized character ring, as in [3], we take over the convention of loc.cit. One may opt to call in the future glider representations as defined above, i.e. with \( \Omega \) included, ‘pre-giders’.

Let \( (\Omega_M \supseteq M) = \Omega_M \supseteq M = M_0 \supseteq M_1 \supseteq \ldots \) and \( (\Omega_N \supseteq N) = \Omega_N \supseteq N = N_0 \supseteq N_1 \supseteq \ldots \) be two glider representations of \( G \).

Definition 2.3. A \( K \)-linear map \( f : M \rightarrow N \) is called a morphism of gliders if there exists a \( KG \)-module morphism \( F : \Omega_M \rightarrow \Omega_N \) such that \( F_{|M_i} = f, f(M_i) \subseteq N_i \) for all \( 0 \leq i \) and \( f(r \cdot m) = r \cdot f(m) \) for all \( r \in F_iKG \) and \( m \in M_i \). In particular it holds that \( f : M \rightarrow N \) is a \( F_0KG \)-linear map.

Note that a glider morphism \( f \) gives rise to a sequence of maps \( f_i = f_{|M_i} : M_i \rightarrow N_i \) such that \( f_i(\alpha_{j-i}m_j) = \alpha_{j-i}f_j(m_j) \) for all \( 0 \leq i \leq j \), \( \alpha_{j-i} \in KG_{j-i} \) and \( m_j \in M_j \) (hence it has a flavour of morphisms of quiver representations), which justifies the terminology.
It is important to remark that if Ω_M ⊆ Ω is a submodule of a larger KG-module Ω. Then, since KG is semisimple, (Ω_M ⊇ M) and (Ω ⊇ M) are isomorphic gliders. Thus up to isomorphism one may assume that Ω = KGM.

Given gliders (Ω_M ⊇ M_i), i = 1, 2, 3, then the composition of glider morphisms f : M_1 → M_2, g : M_2 → M_3 is simply the composition as K-linear maps, which will again be a glider morphism.

With all these definitions FKG-gliders form a category denoted

\[ \text{Glid}(FKG) \subseteq KG \subseteq \cdots \subseteq KG_d = KG \]

which furthermore inherits a monoidal structure from Mod(KG).

**Definition 2.4.** Let Ω_M ⊇ M ⊇ M_1 ⊇ \cdots, Ω_N ⊇ N ⊇ N_1 ⊇ \cdots be FKG-gliders. Then the descending chain

\[ \Omega_M \otimes_K \Omega_N \supseteq M \otimes_K N \supseteq M_1 \otimes_K N_1 \supseteq \cdots, \]

where KG acts on Ω_M ⊗ Ω_N via the comultiplication map ∆ of KG, is the tensor product of the gliders (Ω_M ⊇ M) and (Ω_N ⊇ N).

Note that the tensor product above indeed spits out an FKG-glider because the comultiplication ∆ : KG → KG ⊗ KG is given by ∆(g) = g ⊗ g, extended linearly.

Next recall that the sum of gliders (Ω_M ⊇ M) and (Ω_N ⊇ N) is the term-wise sum (Ω_M + Ω_N ⊇ M + N). This sum is called a strong fragment direct sum if M_i ⊇ N_i is direct for all i ≥ 0. One now easily checks the following.

**Proposition 2.5.** The category (Glid(KG) ⊇ KG ⊇ \cdots ⊇ KG_d, ⊗) is a symmetric monoidal additive category.

Unfortunately, in contrast to Mod(KG), the category of FKG-gliders is not abelian. In the forthcoming work [12] Henrard-van Roosmalen will show what is the precise categorical framework of the theory of glider representations.

Now recall that the KG-module B(M) = \cap_{n≥0}M_n is called the body of the glider. If there exists a number t ≥ 0 such that M_t ⊇ B(M), but M_{t+1} = B(M), then one says that the glider (Ω ⊇ M) has finite essential length t and we write el(M) = t. Denote by

\[ \text{Glid}_t(KG) \subseteq KG \subseteq \cdots \subseteq KG_d \]

the full subcategory consisting of the gliders of essential length at most t. One immediately sees that, for any t ≥ 0, this subcategory inherits the symmetric monoidal additive category structure.

Given a sequence G_0 < \cdots < G_d, it is useful and important to mention that, by [2, Page 1480], one can reduce the study of glider representations to those of finite essential length (even length at most d) and zero body. Therefore these will be standing assumptions on all the gliders considered in this paper.

The glider representation ring.

Earlier we saw the notion of strong fragment direct sum of gliders, which is the direct sum in the categorical sense in Glid(KG) ⊇ KG ⊇ \cdots ⊇ KG_d. However when dealing with filtrations this is a too strong notion and in fact the more suitable concept is the one of a fragment direct sum. Recall that the sum of the gliders (Ω_M ⊇ M) and (Ω_N ⊇ N) is called fragment direct if for some i ≤ el(M), el(N) we have that M_i is disjoint from N_i and we write M ⊕ N.

**Definition 2.6.** The glider representation ring of length t over K of G corresponding to the chain G_0 < G_1 < \cdots < G_d = G, denoted R_{t,K}(G_0 < G_1 < \cdots < G_d), is the quotient of the free abelian group generated by the isomorphism classes of
FKG-gliders of essential length at most \( t \in \mathbb{N} \) and zero body with the additive subgroup generated by the elements

\[
(\Omega_M \supseteq M) + (\Omega_N \supseteq N) - ((\Omega_M \supseteq M) \oplus (\Omega_N \supseteq N)).
\]

Furthermore \( R_{t,K}(G_0 < G_1 < \ldots < G_d) \) is equipped with the multiplication coming from the tensor product of gliders:

\[
[(\Omega_M \supseteq M)] \cdot [(\Omega_N \supseteq N)] = [(\Omega_M \otimes \Omega_N \supseteq M \otimes N)].
\]

Clearly due to Proposition 2.5 \( R_{t,K}(G_0 < G_1 < \ldots < G_d) \) is a commutative unital ring with unit element \( T \supseteq T \supseteq \ldots \supseteq T \supseteq 0 \supseteq \ldots \) where \( T \) denotes the trivial \( G \)-representation. Note that the additive subgroup \( G_{t-1} \) generated by the FKG-gliders of essential length at most \( t - 1 \) is an ideal in \( R_{t,K}(G_0 < G_1 < \ldots < G_d) \).

**Definition 2.7.** The ring

\[
\overline{R}_{t,K}(G_0 < G_1 < \ldots < G_d) = R_{t,K}(G_0 < G_1 < \ldots < G_d)/G_{t-1}
\]

is called the **reduced glider representation ring** of length \( t \) over \( K \) of the chain \( G_0 < G_1 < \ldots < G_d \).

We could also have considered in Definition 2.6 the free abelian group generated by all FKG-gliders of any (arbitrary large) length modulo the same additive subgroup. In this case we omit the subscript \( t \) in both definitions.

**Notational conventions.** If the field \( K \) is clear from the context we also omit the subscript \( K \). Also, usually the chain \( G_0 < G_1 < \ldots < G_d = G \) will be clear from the context and therefore we will usually use the abbreviated notations \( R_t(\tilde{G}) \) and \( \overline{R}_t(\tilde{G}) \).

**Remark 2.8.** Over a field of characteristic 0, the classical representation ring has a \( \mathbb{Z} \)-basis consisting of the irreducible representations. For gliders this is however no longer true. In a first instance one needs to be careful with the notion of an 'irreducible object' in \( \text{Gld}(G_0 < G_1 < \ldots < G_d) \) since gliders of length at least 1 will always have subobjects such as \( \Omega \supseteq M \supseteq 0 \supseteq \ldots \). A list of 'trivial subgliders' and the notion of an irreducible glider was introduced in [5, Th. 3.2.14.]. Hence they form a generating set for \( R_t(\tilde{G}) \), however it is still an open question whether they form a basis (i.e. whether the decomposition in irreducible gliders is unique).

Since we don’t need the general definition, we will describe in section Section 2.2 only irreducible gliders of essential length 1, which is the context of this paper.

**Glider character ring.**

In [4] character theory for \( FCG \)-gliders was introduced. We recall the definition of a glider character and glider class function (over a field \( K \) with \( \text{char}(K) = 0 \) as in [5, Section 5.8.]) and then we introduce the (reduced) glider character ring.

Let \( \Omega \supseteq M_0 \supseteq \ldots \supseteq M_d \supseteq 0 \) be a glider representation (with \( \text{el}(M) \leq d \)) of the chain \( 1 \subsetneq G_1 \subsetneq \ldots \subsetneq G_d = G \). So for \( i \leq j \) we have that \( KG_iM_j \), i.e. we have \( KG_i \)-modules \( G_iM_j \) with associated character \( \chi_{i,j} \).

**Definition 2.9.** Let \( (\Omega \supseteq M) \) be an FKG-glider with \( \text{el}(M) \leq d \). Then the associated **glider character** is the map \( \chi_{(\Omega \supseteq M)} : G \to K^n \) with \( n = \frac{(d+1)(d+2)}{2} \).
GLIDER REPRESENTATION RINGS WITH A VIEW ON DISTINGUISHING GROUPS

which sends \( g \in G_1 \setminus G_{i-1} \) to

\[
\chi(\Omega \geq M)(g) = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 \\
\chi_i(g) & \chi_{i+1}(g) & \cdots & \chi_{i+d-1}(g) & \chi_{i,d}(g) \\
\chi_{i+1}(g) & \chi_{i+1,d-1}(g) & \cdots & \chi_{i+1,d}(g) \\
\vdots & \vdots & & \ddots & \vdots \\
\chi_{d-1,d-1}(g) & \chi_{d-1,d}(g) & \cdots & \chi_{d,d}(g)
\end{pmatrix}
\]

The image has been written in matrix form \( \chi(\Omega \geq M)(g)_{i,j} = \chi_{i,j}(g) \), however in fact it truly lives inside \( K^n \). Note that if \( g_1, g_2 \in G_i \setminus G_{i-1} \), then \( \chi(\Omega \geq M)(g_1) = \chi(\Omega \geq M)(g_2) \) if and only if \( h^{-1}g_1h = g_2 \) for some \( h \in G \). Hence it is an example of a glider class function. Recall that these are the maps from \( G \) to \( K^n \) that are constant on \( C(g) \cap G_i \setminus G_{i-1} \) for \( g \in G_i \setminus G_{i-1} \) and all \( 0 \leq i \leq d \). The set of glider class functions, denoted \( \mathcal{A}(\tilde{G}) \), also carries the structure of a \( K \)-vector space via component wise addition and \( \lambda \in K \) acts via point wise multiplication with the function \( c_\lambda(g)_{i,j} = \lambda \) if \( i \leq k \leq l \) and 0 otherwise, where \( g \in G_i \setminus G_{i-1} \) (recall that the elements are tuples in \( K^n \), hence the multiplication is the component wise one in \( K^n \) and not matrix multiplication).

**Definition 2.10.** Let

\[
\operatorname{ch}_{i,K} : \text{Glid}_d(KG_0 \subseteq KG_1 \subseteq \ldots \subseteq KG_d) \to \mathcal{A}(\tilde{G}) : (\Omega \geq M) \mapsto \chi(\Omega \geq M)
\]

be the \( K \)-linear map sending a glider on his character. Then \( \operatorname{Im}(\operatorname{ch}_{i,K}) \) is called the glider character ring of length \( t \) over \( K \) corresponding to the chain \( G_0 < \cdots < G_d \) and is denoted by \( \operatorname{ch}_{i,K}(G_0 < \cdots < G_d) \). Furthermore,

\[
\overline{\operatorname{ch}}_{i,K}(G_0 < \cdots < G_d) = \operatorname{ch}_{i,K}(G_0 < \cdots < G_d)/\chi(M) = \chi(M) \leq t - 1 \}
\]

is called the reduced glider character ring.

Again, when the context is clear we will use the abbreviations \( \chi(\tilde{G}) \) and \( \overline{\chi}(\tilde{G}) \).

A first important difference with classical representation theory is that the map \( \operatorname{ch}_{i,K} \) is not injective. Indeed, slightly reformulated [4, Proposition 3.1] tells us the following.

**Proposition 2.11 ([4]).** Let \( (\Omega_M \geq M) \) and \( (\Omega_N \geq N) \) be two irreducible gliders in \( \text{Glid}_d(K \leq KG_1 \subseteq \ldots \subseteq KG_d) \). Then \( \chi(\Omega_M \geq M) = \chi(\Omega_N \geq N) \) exactly when only \( M_d \) is a non-isomorphic \( K \)-vectorspace.

In other words, let \( 1 < G_1 < \ldots < G_d \) be a chain of finite groups and \( (\Omega \geq M) \) an irreducible \( FKG \)-glider. Then the glider character \( \chi(\Omega \geq M) \) determines uniquely the \( KG_i \)-modules \( G_iM_j \) for all \( 0 \leq i \leq j \) except for \( (i,j) = (0,d) \).

If we now denote the image of a glider \( (\Omega \geq M) \), with essential length \( d \), in \( \overline{\mathcal{R}}(\tilde{G}) \) by \( [[\Omega \geq M]] \) and the image of \( \chi(\Omega \geq M) \) in \( \overline{\mathcal{R}}(\tilde{G}) \) by \( \overline{\chi}(\Omega \geq M) \), then Proposition 2.11 immediately yields the following.

**Corollary 2.12.** Let \( (\Omega_M \geq M) \) and \( (\Omega_N \geq N) \) be two irreducible gliders. Then

\[
\overline{\chi}(\Omega_M \geq M) = \overline{\chi}(\Omega_N \geq N)
\]

if and only if \( [[\Omega_M \geq M]] = [[\Omega_N \geq N]] \).

**Literature remark** 2.13. In [3, 4] the authors introduced 'generalized characters' and a ring which they call the 'generalized character ring' for the first time. In the recent monograph [5, Chapter 5] the new terminology 'glider characters' and 'glider representation ring' are coined for these objects. The latter is furthermore denoted by \( R(G_0 < G_1 < \ldots < G_d) \), or \( R(\tilde{G}) \) in short. However the approach in
loc. cit. is less general and hence differs from ours. Nevertheless, over a field $K$ of characteristic $0$, their ‘glider representation ring $R(\hat{G})$’ is isomorphic to $\underline{\text{TR}}_{d,K}(\hat{G})$, the reduced glider representation ring of length $d$ over $K$, in our sense.

2.2. Gliders of length 1 versus $\text{Rep}(G)$ as symmetric tensor category.

From now on, we will always consider chains of the form $1 < G$, where $G$ is a finite group and $1 = e$ is the unit element of $G$. The associated algebra filtration becomes $K \subset KG$. Also, we assume that $K$ is an algebraically closed field of characteristic 0.

We will start by showing that the category $\text{Glid}_1(K \subset KG)$ is surprisingly rich. In fact, when taking the full symmetric monoidal additive category structure into account, gliders of essential length 1 always determine uniquely the group $G$. Thereafter we will parametrise the irreducible gliders of essential length 1.

Recurrent notation. If $(\Omega_M \supset M)$ is a glider of essential length at most 1, we simply write the glider fully: $(\Omega \supset M \supset M_1)$. In case $M$ is a $KG$-module and $\Omega_M = M$ we leave $\Omega_M$ out of the notation (i.e. $(M \supset M_1)$). In particular when writing the glider $(K \supset 0)$ we view $K$ as the trivial $KG$-module.

Retracing the Fiber functor from $\text{Glid}_1(K \subset KG)$.

The forgetful functor $F: \text{Mod}(KG) \to \text{Vect}_K$ sending a module to its underlying $K$-vector space is a faithful $K$-linear monoidal functor. In particular this allows to do Tannaka-Krein reconstruction, i.e. to reconstruct $G$ via

$$G \cong \text{Aut}^\otimes(F).$$

By a theorem of Deligne, see [8, Th. 3.2. (b)] or [7], the forgetful functor is the unique fiber functor (i.e. unique faithful exact $K$-linear symmetric monoidal functor from $\text{Mod}(KG)$ to $\text{Vect}_K$), which we moreover can recover from $\text{Mod}(KG)$ by taking into account its full symmetric tensor structure. Hence, by the above, when considering all the latter data we can reconstruct $G$ uniquely from $\text{Mod}(KG)$. Intriguingly, there can be different non-symmetric faithful exact $K$-linear monoidal functors which leads to the phenomenon that the monoidal structure of $\text{Mod}(KG)$ may be insufficient to recover $G$. Following Etingof-Gelaki [9] such groups are called isocategorical.

We will now show that we can construct the forgetful functor $F$ purely in terms of the structure of the category $\text{Glid}_1(K \subset KG)$, in particular also $G$.

**Theorem 2.14.** Let $G$ be a finite group. Then the functor

$$F: \text{Mod}(KG) \to \text{Vect}_K: M \mapsto \text{Hom}_{\text{glid}}((K \supset 0), (M \supset K))$$

is faithful $K$-linear symmetric monoidal and is monoidal natural isomorphic to the forgetful functor $F$. Consequently, $\text{Glid}_1(K \subset KG)$ as symmetric monoidal additive category determines $G$ uniquely.

**Proof.** We should first point out what the functor $F$ does at level of morphisms. Let $\varphi : N \to M$ be a $KG$-module morphism and $f \in \text{Hom}_{\text{glid}}((K \supset 0), (N \supset K))$. Then we define $F(\varphi)(f) = f_\varphi$ where $f_\varphi(1) = \varphi(f(1))$ $K$-linearly extended. Clearly, $F$ is $K$-linear.

To start, we check that $F$ is indeed faithful. Hence let $N, M$ be $KG$-modules and $\varphi_1, \varphi_2 \in \text{Hom}_{KG}(N, M)$ such that $F(\varphi_1) = F(\varphi_2)$. For $x \in N$, define $f_x \in \text{Hom}_{\text{glid}}((K \supset 0), (N \supset K))$ by $f_x(1) = x$. In this way we get that $\varphi_1(x) = (f_x)_{\varphi_1}(1) = F(\varphi_1)(f_x)(1) = F(\varphi_2)(f_x)(1) = \varphi_2(x)$ for all $x \in N$, as needed. With similar arguments the other properties of $F$ mentioned, follow.
It is easy to see that \( \text{Hom}_{\text{glid}}((K \supset 0), (M \supset K)) \cong \text{Hom}_K(K, M) \) as \( K \)-vector spaces. Therefore define for every \( K \)-module \( M \) the map
\[
\eta_M : \text{Hom}_{\text{glid}}((K \supset 0), (M \supset K)) \to M : f \mapsto f(1)
\]
which is a \( K \)-linear isomorphism. One now immediately checks that \( \eta = (\eta_M)_M : \mathcal{F} \Rightarrow F \) is a natural isomorphism which moreover is compatible with the monoidal structure.

The last statement follows from the first and the discussion before the theorem, after checking that we solely used the symmetric monoidal additive structure of \( \text{Glid}_d(K \subseteq KG) \) in order to obtain the functor \( \mathcal{F} \) as fiber functor.

The main bulk of this paper is about investigating how much the reduced glider representation ring \( \overline{\mathcal{R}}_d(G) \) still remembers of \( G \). Despite the above result, we will show that the ring-structure of \( \overline{\mathcal{R}}_d(G) \) is much richer than the one of the classical representation ring of \( G \) and is for example able to distinguish between certain isocategorical groups.

**Concrete description of the irreducible gliders and characters.**

Let \( (\Omega \supset M \supset M_1) \) be a glider of essential length \( 1 \) of the chain \( 1 \prec G \). If this glider is irreducible, by \([2, \text{Lemma 2.5.}]\), \( KGM_1 = M \). In particular, \( M \) is a \( KG \)-module and hence by definition of a glider morphism, up to isomorphism, we have that \( \Omega = M \). However, there are more restrictions on \( M \) and \( M_1 \). In \([3]\) (or \([5, \text{Theorem 4.1.12.}]\)) irreducible \((K \subseteq KG)\)-glider representations were classified.

Since we never recalled the exact definition of an irreducible glider, the reader can consider the following theorem as a definition.

**Theorem 2.15** ([3]). Let \( G \) be a finite group, \( K \) an algebraically closed field of characteristic zero and let \( \{V_1, \ldots, V_n\} \) be a full set of irreducible \( G \)-representations of resp. dimension \( n_i \). A \((K < KG)\)-glider representation
\[
(M = \bigoplus_{i=1}^n V_i^{\oplus m_i} \supseteq Ka)
\]
with \( a = v_1^1 + \cdots + v_{m_1}^1 + v_1^2 + \cdots + v_{m_2}^2 + \cdots + v_1^n + \cdots + v_{m_n}^n \in M \) is irreducible if and only if
\[
\begin{align*}
(1) & \quad \forall i \quad m_i \leq n_i \\
(2) & \quad \forall i \quad \dim(<v_1^i, \ldots, v_{m_i}^i>) = m_i
\end{align*}
\]
Furthermore, \((K < 0)\) is the unique irreducible glider of essential length \( 0 \).

Different choices of the point \( a \) may however yield isomorphic irreducible gliders.

In order to parametrize the isomorphism classes we need following generalization of \([4, \text{Lemma 7.1}]\).

**Lemma 2.16.** Let \( G \) be a finite group, \( U \) a \( d \)-dimensional irreducible \( G \)-representation and \( m \leq d \). The irreducible \((K \subseteq KG)\)-glider representations \( U^{\oplus m} \supseteq K(u_1 + \ldots + u_m) \) and \( U^{\oplus m} \supseteq K(v_1 + \ldots + v_m) \) are isomorphic if and only if \( \langle u_1, \ldots, u_m \rangle \) and \( \langle v_1, \ldots, v_m \rangle \) determine the same point in the Grassmanian \( \text{Gr}(m, U) \).

**Proof.** Extend \( \{u_1, \ldots, u_m\} \) and \( \{v_1, \ldots, v_m\} \) to \( K \)-bases for \( U \). Then there exists a base change matrix \( B \) such that \( Bu_i = v_i \) for \( 1 \leq i \leq m \) if and only if \( \langle u_1, \ldots, u_m \rangle \) and \( \langle v_1, \ldots, v_m \rangle \) determine the same point in the Grassmanian \( \text{Gr}(m, U) \). \( \Box \)

For an irreducible \( G \)-representation \( U \) of dimension \( d \) we denote \( \text{Gr}(U) = \sqcup_{j=1}^d \text{Gr}(j, U) \) and we denote a point in \( \text{Gr}(j, U) \) by \( (a_1, \ldots, a_j) \in \mathbb{P}^{d-1} \times \cdots \times \mathbb{P}^{d-1} \) (all \( a_k \) different). For \( j = d \), \( \text{Gr}(d, U) \) is a singleton which we denote by \( \{*_{TV}\} \). We denote
by
\[ S = S_G \]
the set of subsets \( B \subseteq \bigcup_{U \in \text{Irr}(G)} 1 \dim(U) > 1 \text{Gr}(U) \), such that for all \( U \) the intersection \( B \cap \text{Gr}(j, U) \) is non-empty for at most one \( 1 \leq j \leq \dim(U) \) and for this \( j \) it is in fact a singleton. We denote by \( M \in S \) the set \( \{ *_U \mid U \in \text{Irr}(G) \} \).

**Proposition 2.17.** Let \( G \) be a finite group. There is a bijection
\[
\begin{align*}
\{ \text{irreducible } (K \subseteq KG) - \text{gliders} \} &\overset{1-1}{\longleftrightarrow} \{(A, B) \in \mathcal{P}(G/G') \times S_G\}
\end{align*}
\]
where \( G'=[G, G] \) is the commutator subgroup of \( G \) and \( \mathcal{P}(G/G') \) the power set of \( G/G' \).

**Proof.** Recall that the 1-dimensional representations of \( G \) correspond to the character group \( G/G' = \text{Hom}_{\text{grp}}(G/G', K^*) \) and moreover \( G/G' \cong G' \). We fix such an isomorphism and use it to fix a correspondence between the 1-dimensional representations and the elements of \( G/G' \). For \( z \in G/G' \) denote the corresponding \( G \)-representation by \( T_z \).

For every \( z \in G/G' \), take an element \( t_z \in T_z \). Now by generalizing Theorem 2.15 to more summands (for abelian group see [4, Lemma 4.3]) we see that there is a one-to-one correspondence
\[
A \in \mathcal{P}(G/G') \overset{1-1}{\longleftrightarrow} \left( \bigoplus_{z \in A} T_z \supset K(\sum_{z \in z} t_z) \right),
\]
between subsets of \( G/G' \) and isomorphism classes of irreducible gliders of essential length at most 1 of \( G/G' \). If \( A = \emptyset \) then the associated glider is \( (K \subset 0) \). This correspondence does not depend on the chosen elements \( t_z \) because of Lemma 2.16.

From Theorem 2.15 we see that in order to make an irreducible \( (K < KG) \)-glider we need to determine the numbers \( m_i \) and choose elements \( v_i^j \in V_i \) with \( 1 \leq j \leq m_i \). In case \( V_i \) is 1-dimensional, as mentioned earlier, the chosen element \( v_i^j \) does not matter and hence the choice reduces whether to pick \( V_i \) or not. Or in other words, by the above, the 1-dimensional summands part corresponds to subsets of \( G/G' \).

For the \( V_i \) of dimension at least 2, the choice correspond by definition (and due to Lemma 2.16) to a point of \( S_G \). So altogether we obtain the statement. \( \square \)

Let us give an example how the correspondence works.

**Example 2.18.** Let \( G = Q_8 = \langle i, j, k \mid i^2 = j^2 = k^2 = ijk \rangle \) be the quaternion group. The abelianization of \( Q_8 \) is \( C_2 \times C_2 \cong \langle a, b \mid a^2 = b^2 = 1 \rangle \) and denote \( ab = c \). Fix the isomorphism between \( Q_8/Q_8' \) and the group of 1-dimensional \( Q_8 \)-representations
\[
1 \mapsto T_1 \quad a \mapsto T_i \quad b \mapsto T_j \quad c \mapsto T_k.
\]
With fixed basis \( \{ e_1, e_2 \} \) of the 2-dimensional representation \( U \), the point \( [A : \mu] \in \mathbb{P}^1 \) determines the glider \( U \supseteq K(\lambda e_1 + \mu e_2) \). We have the correspondences
\[
\chi((b, c), (1 : 1)) \leftrightarrow T_j \oplus T_k \oplus U \supseteq K(t_j + t_k + e_1 + e_2)
\]
and
\[
\chi((1, \{ u \}) \leftrightarrow T_1 \oplus U^{\oplus 2} \supseteq K(t_1 + u_1 + u_2),
\]
where \( \dim_K(\langle u_1, u_2 \rangle) = 2 \).

Finally, let \( (M \supseteq Km), (N \supseteq Kn) \) be irreducible gliders. Remark that \( KG(m \otimes n) \) is a \( KG \)-submodule of the \( KG \)-module \( M \otimes_K N \) (with diagonal \( G \)-action). Hence there exists some \( KG \)-submodule \( V \) complementing \( KG(m \otimes n) \). Therefore
\[
(M \supseteq Km) \otimes (N \supseteq Kn) = (V \subseteq 0) \oplus (KG(m \otimes n) \subset Km \otimes n).
\]
Consequently,
\[(2) \quad \overline{\chi(M_{\geq Kn})} \cdot \overline{\chi(N_{\geq Kn})} = \overline{\chi(KG(m\otimes n))_{\geq Km\otimes n}}\]
in $\overline{\text{ch}}_1(\tilde{G})$. In the rest of the paper this equality will often be used without further notice.

**Recurrent notation.** Given a tuple $(A, B) \in \mathcal{P}(G/G') \times S_G$ we will write
- $M_{(A,B)}$ for the isomorphism class of the irreducible ($K \subseteq KG$)-glider corresponding to it following Proposition 2.17;
- $\chi(A, B)$ for the image of the glider character $\chi_{M_{(A,B)}}$ in the reduced character ring $\overline{\text{ch}}_1(\tilde{G})$ (recall that isomorphic gliders have equal characters);
- $\chi_A$ instead of $\chi(A, B)$ (in spirit of [4] where the abelian case was handled). However both notations will be in use.

Note that the point $(\emptyset, \emptyset)$ corresponds to the glider $(K \supset 0)$ which is of essential length 0 and hence $\chi_{(\emptyset, \emptyset)}$ is equal to zero in $\overline{\text{ch}}_1(\tilde{G})$.

### 3. Induced Morphisms Between Glider Representation Rings

Let $\varphi : H \to G$ be a group morphism between finite groups $G$ and $H$ and denote the irreducible representations of $H$, resp. $G$ by
\[
\text{Irr}(H) = \{W_1, \ldots, W_n\}, \quad \text{Irr}(G) = \{V_1, \ldots, V_m\}.
\]

We would like to define a morphism between the reduced glider representation rings $\overline{\text{Rep}}_H(1 < H)$ and $\overline{\text{Rep}}_G(1 < G)$ which preserves multiplication. In order to arrive at such a morphism, we have to associate to any $H$-representation $W$ a $G$-representation. The underlying idea is to include all the irreducible $G$-representations that are connected to $W$ through $\varphi$.

#### 3.1. From $\text{Rep}(H)$ to $\text{Rep}(G)$: Construction

We proceed as follows: let $V = V_i \in \text{Irr}(G)$. The group morphism $\varphi$ allows to consider $V = V_\varphi$ as an $H$-representation and by our assumption on the ground field, we have a decomposition into irreducible $H$-representations
\[
V_\varphi = \bigoplus_{i=1}^n W_i^{e(V_i, W_i)},
\]
where $W^f$ denotes a direct sum $W^\otimes f$. We fix bases $\{w_{i1}, \ldots, w_{id_i}\}$ for the irreducible $H$-representations $W_i$, $1 \leq i \leq n$ and for every $V_j$, $1 \leq j \leq n$ we denote and fix bases in $(V_j)_\varphi$
\[
\begin{align*}
& w_{1,1}^{1,1}, \ldots, w_{1,d_1}^{1,1} \quad \text{(first $W_1$)} \\
& \vdots \\
& w_{1}^{j,e(V_j, W_1)}, \ldots, w_{1}^{j,e(V_j, W_1)} \quad \text{($e(V_j, W_1)$th $W_1$)} \\
& w_{2,1}^{1,1}, \ldots, w_{2,d_2}^{1,1} \quad \text{(first $W_2$)} \\
& \vdots \\
& w_{2}^{j,e(V_j, W_2)}, \ldots, w_{2}^{j,e(V_j, W_2)} \quad \text{($e(V_j, W_2)$th $W_2$)} \\
& \vdots \\
& w_{n,1}^{1,1}, \ldots, w_{n,d_n}^{1,1} \quad \text{($e(V_j, W_n)$th $W_n$)}
\end{align*}
\]
which establishes the decomposition of $(V_j)_\varphi$ into $H$-components. A basis element $w_{i,j}^{k,l}$ denotes the associated basis element $w_{i,j}$ from $W_i$ embedded into the $l$-th component $W_i$ of $V_k$. In particular, we have $KHW_i^{k,l} = W_i$ for all $1 \leq i \leq n, 1 \leq l \leq
by 

\[ \text{diag}(\iota_{W_1,V_j}, \ldots, \iota_{W_i,V_j}, \iota_{W_{i+1},V_j}, \ldots, \iota_{W_n,V_j}) \]

where there are \( f_1 \iota_{W_1,V_j}'s, f_2 \iota_{W_2,V_j}'s, \ldots, f_n \iota_{W_n,V_j}'s \).

Finally, for \( W \cong \bigoplus_{i=1}^n W_i^{f_i} \) we define the \( H \)-module morphism

\[ \iota_{W,\bar{\varphi}} : W \rightarrow \varphi(W) = \bigoplus_{j=1}^m V_j \sum_{t=1}^n f_t \iota(V_t,W_t) \]

by

\[ \begin{pmatrix}
\iota_{W_1,V_j} \\
\iota_{W_2,V_j} \\
\vdots \\
\iota_{W_m,V_j}
\end{pmatrix} \]

up to reordering the components in \( \varphi(W) \). To summarize, per component \( W_i \) of \( W \), we add \( V_j \) if \( W_i \) appears in the decomposition of \( (V_j)_\phi \).

Because the multiplication in glider representation rings is based upon tensor products of group representations, we now elucidate the behavior of the morphisms \( \iota_{W,\bar{\varphi}}\) under taking tensor products. To this extent, let \( w = w_{1,1} \in W_1, w' = w_{2,1} \in W_2 \) and suppose that \( W_1 \otimes W_2 \cong \bigoplus_{i=1}^n W_i^{f_i k_i} \). Accordingly, there exist \( c_{i,j} \in K \) such that \( w \otimes w' = \sum_{j=1}^n \sum_{k=1}^{k_i} \sum_{j=1}^{f_j} c_{i,j} w_{i,j,k} \). The extra subindex \( k \) of \( w_{i,j,k} \) denotes the \( k \)-th copy of \( W_i \). By construction we have

\[ \iota_{W_1,V_1}(w) = \sum_{l=1}^{f_1} w_{1,1}^{l}, \quad \iota_{W_2,V_1}(w') = \sum_{r=1}^{f_2} w_{2,1}^{r}, \]
for the decomposition of $W_1 \otimes W_2$. By this we mean that the coefficients of the basis vectors $w_{i,j,k}^{u,v}$ are $\lambda_{l,r} \epsilon_{ikj}$, for some $\lambda_{l,r} \in K$. This observation allows to prove the following.

**Lemma 3.2.** For $W, W'$ $H$-representations such that both $\mathcal{P}(W), \mathcal{P}(W') \neq 0$, there exists a $G$-linear embedding $\mathcal{P}(W \otimes W') \subseteq \mathcal{P}(W) \otimes \mathcal{P}(W')$ that maps

$$\iota_{W \otimes W', V(W \otimes W')} (w \otimes w') \mapsto \iota_{W, V(W)}(w) \otimes \iota_{W', V(W')} (w').$$

**Proof.** It suffices to proof this for $W = W_1, W' = W_k$ irreducible $H$-representations. We use coefficients $b_s^l$ to indicate the decomposition $V_j \otimes V_l \cong \bigoplus_{s=1}^m V_s^{l,j}$ and we use $a_s^{l,k}$ for the decomposition of $W_j \otimes W_k$. Suppose that $W_j \otimes W_l \cong \bigoplus_{s=1}^m W_s^{l,j,k}$. By definition, we have

$$\mathcal{P}(W_i) \otimes \mathcal{P}(W_k) \cong \bigoplus_{j,l=1}^m V_j^{e(V_j,W_i)} \otimes V_l^{e(V_l,W_k)}$$

$$\cong \bigoplus_{j,l=1}^m (V_j \otimes V_l)^{e(V_j,W_i) e(V_l,W_k)}$$

$$\cong \bigoplus_{j,l=1}^m (W_j \otimes W_k)^{e(V_j,W_i) e(V_l,W_k)}.$$  

It follows that the amount of $V_l$-components of $\mathcal{P}(W_i) \otimes \mathcal{P}(W_k)$ is bigger or equal then

$$\sum_{j,l=1}^m \sum_{s=1}^n e(V_j,W_i) e(V_l,W_k) a_{s}^{l,k} e(V_l,W_k)$$

$$\geq \sum_{s=1}^n a_{s}^{l,k} e(V_l,W_k)$$

$$= \mathcal{P}(W_i \otimes W_k)_l,$$

where $\mathcal{P}(W_i \otimes W_k)_l$ denotes the number of $V_l$ components of $\mathcal{P}(W_i \otimes W_k)$. Together with the observations made before the lemma, this shows that it is possible to embed $\mathcal{P}(W \otimes W')$ in $\mathcal{P}(W) \otimes \mathcal{P}(W')$ such that

$$\iota_{W \otimes W', \mathcal{P}(W \otimes W')} (w \otimes w') \mapsto \iota_{W, \mathcal{P}(W)}(w) \otimes \iota_{W', \mathcal{P}(W')} (w')$$

$\square$
Example 3.3. We consider again the example of $Q_8 = \langle i, j, k \mid i^2 = j^2 = k^2 = ijk \rangle$ and let $\varphi : \mathbb{Z}_4 = \langle j \rangle \rightarrow Q_8$. Recall that character table of $Q_8$ is given by:

| $T_1$ | $T_1'$ | $T_2$ | $T_2'$ | $U$ |
|-------|-------|-------|-------|-----|
| 1     | 1     | 1     | 1     | 2   |
| 1     | 1     | 1     | -1    | -2  |
| 1     | 1     | -1    | 1     | 0   |
| 1     | 1     | -1    | -1    | 0   |

For the two-dimensional irreducible representation $U$ we fix a basis $\{e_1, e_2\}$ such that $U$ has the following presentation:

\[
i \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad j \mapsto \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.
\]

With regard to this basis, we obtain a decomposition

\[
U \otimes U = \mathbb{C}(e_1 \otimes e_2 - e_2 \otimes e_1) \oplus \mathbb{C}(e_1 \otimes e_2 + e_2 \otimes e_1) \oplus \mathbb{C}(e_1 \otimes e_1 - e_2 \otimes e_2).
\]

and one sees that $KQ_8 e_1 \otimes e_2 = T_1 \oplus T_2 \subseteq U \otimes U$. The irreducible representations of $\mathbb{Z}_4 = \langle j \rangle$ are denoted by $V_1, V_{-1}, V_i, V_{-i}$. The irreducible $Q_8$-representations decompose as $\langle j \rangle$ in the following way:

\[
(T_1)_\varphi = (T_2)_\varphi = V_1, \quad (T_3)_\varphi = (T_4)_\varphi = V_{-1}, \quad U_\varphi = V_i \oplus V_{-i}.
\]

By definition it follows that

\[
\overline{\varphi}(V_{-i}) = U, \quad \overline{\varphi}(V_1 \oplus V_i) = T_1 \oplus T_2 \oplus U.
\]

So on the one hand, we have

\[
\overline{\varphi}(V_{-i} \otimes (V_1 \oplus V_i)) = \overline{\varphi}(V_1 \oplus V_{-i}) = T_1 \oplus T_2 \oplus U
\]

On the other hand, we have

\[
\overline{\varphi}(V_{-i} \otimes V(V_1 \oplus V_i)) = U \oplus U \oplus T_1 \oplus T_2 \oplus T_3 \oplus T_4 \oplus T_5
\]

and the embedding will map the component $U$ diagonally into $U \oplus U$. Adding in vectors, one checks that

\[
\iota_{V_1 \oplus V_{-i}, T_1 \oplus T_2} (v_1 + v_{-i}) = t_1 + t_2 + e_1
\]

and

\[
\iota_{V_{-i}, U} (v_{-i}) \otimes \iota_{V_1 \oplus V_{-i}, T_1 \oplus T_2} (v_1 + v_i) = e_1 \otimes (t_1 + t_2 + e_2)
\]

where we choose the isomorphism $U \otimes T_1 \cong U$ such that $e_1 \otimes t_j$ is mapped to $e_1$ (in general, an isomorphism $U \otimes T_1 \cong U$ maps $e_1 \otimes t_j$ to $\lambda e_1$).

3.2. From $\overline{R}_1(\tilde{H})$ to $\overline{R}_1(\tilde{G})$: the functor. In the previous section we have shown how to connect a $G$-representation to a given $H$-representation. Furthermore this correspondence was shown to behave well with direct sums and tensor products. Therefore we are now on a firm footing to make a map between the reduced representation rings $\overline{R}_1(\tilde{H})$ and $\overline{R}_1(\tilde{G})$.

In fact we will work with the $\mathbb{Q}$-algebra extension of the representations rings, which we denote by:

\[
\mathbb{Q}(\tilde{H}) := \mathbb{Q} \otimes \overline{R}_1(\tilde{H}).
\]
Let $A \subseteq H/H'$ and consider the associated irreducible glider character $\chi_{(A,B)} \in \mathbb{Q}(\tilde{H})$, as in Proposition 2.17. If the corresponding glider representation is
\[ \bigoplus_{i=1}^{n} W_{i}^{f_i} \supseteq Ka, \]
with $a = \sum_{i=1}^{n} a_i^1 + \ldots + a_i^{f_i}$ and $W = \bigoplus_{i=1}^{n} W_{i}^{f_i}$, then we define
\[ A(\varphi)(\chi_{(A,B)}) = \overline{\varphi}(W) \supseteq Kb, \]
where
\[ b = i_{W,\overline{\varphi}(W)}(a) = \sum_{j=1}^{m} \sum_{i=1}^{n} f_i \iota_{W_{i},V_{j}}(a_i^k). \]

**Remark 3.4.** It seems that everything depends on the choice of basis of $W$. However, we are considering glider representations $W \supseteq Ka$ of essential length 1, whence working with a different basis of $W$ leads to writing $a \in W$ as another sum of vectors, such that $i_{W,\overline{\varphi}(W)}(W \supseteq Ka)$ is indeed independent of the choice of decomposition by Lemma 2.16.

**Proposition 3.5.** Let $\chi_{(A,B)}$ be associated to the irreducible $K \subseteq KH$-glider \( \bigoplus_{i=1}^{n} W_{i}^{f_i} \supseteq Ka \). Then if $A(\varphi)(\chi_{(A,B)}) = (\overline{\varphi}(W) \supseteq Kb)$, we have that $KHb \cong W$ as $H$-representations.

**Proof.** By the irreducibility $\chi_{(A,B)}$ and Theorem 2.15 we have that $KH(\sum_{k=1}^{f_i} a_i^k) = W_{i}^{f_i}$. By construction, we have that
\[ KH_{iW,\overline{\varphi}(W)}(a_i^k) \cong W_i \]
as $H$-representations, whence
\[ KHb = KH \sum_{j=1}^{m} \sum_{i=1}^{n} f_i \iota_{W_{i},V_{j}}(a_i^k) \]
\[ = \sum_{i=1}^{n} KH \sum_{j=1}^{m} \sum_{k=1}^{f_i} \iota_{W_{i},V_{j}}(a_i^k) \]
\[ = \sum_{i=1}^{n} KH \sum_{k=1}^{f_i} (\iota_{W_{i},\overline{\varphi}(W)}(a_i^k)) \]
\[ \cong \sum_{i=1}^{n} KH \sum_{k=1}^{f_i} a_i^k \cong \bigoplus_{i=1}^{n} W_{i}^{f_i}. \]

By construction, $A(\varphi)(\chi_{(A,B)})$ yields the $K \subseteq KG$-glider $V \supseteq Kb$ such that $V$ is the “largest” $G$-representation such that $KHb \cong W$ and $KGb = V$. More precisely,

**Proposition 3.6.** Let $W = \bigoplus_{i=1}^{n} W_{i}^{f_i} \supseteq Ka$ be associated to $\chi_{(A,B)}$. Then
\[ \dim_K(\overline{\varphi}(W)) = \max \{ \dim_K(V) \mid \exists \text{irreducible } KG\text{-glider } V \supseteq Kv \text{ such that } KHv \cong W \text{ as } H\text{-representations} \} \]

**Proof.** Let $V \supseteq Kv$ be an irreducible $KG$-glider with $KHv \cong W$. The $KH$-glider $KHv \supseteq Kv$ is irreducible and $A(\varphi)(KHv \supseteq Kv) = \overline{\varphi}(KHv) \supseteq Kv$ for some $v'$. Since $KGv = V$, we have that $V$ embeds in $\overline{\varphi}(KHv)$, whence
\[ \dim_K(V) \leq \dim_K(\overline{\varphi}(KHv)) = \dim_K(\overline{\varphi}(W)). \]

Denote by $N_\varepsilon(\varphi) = \{ W \in \text{Rep}(H) \mid \overline{\varphi}(W) \neq 0 \}$. 

Proposition 3.7. The map $A(\varphi) : Q(\tilde{H}) \to Q(\tilde{G})$ preserves multiplication on the $\mathbb{Z}$-algebra generated by the irreducible gliders $\chi_{(A,B)} \in Q(\tilde{H})$ for which the associated $H$-representation $W$ is in $N_2(\varphi)$.

Proof. Let $W \supseteq K \omega$ and $W' \supseteq K \omega'$ be two irreducible $K \subseteq KH$-glider representations with both $W, W' \in N_2(\varphi)$. By definition, we have that

$$A(\varphi)\left( (W \supseteq K \omega) \cap (W' \supseteq K \omega') \right) = KG_{W \supseteq W'} \chi_{W \supseteq W'}(w \otimes w'),$$

where $KG_{W \supseteq W'} \chi_{W \supseteq W'}(w \otimes w') \subseteq \varphi(W \otimes W')$ and

$$A(\varphi)(W \supseteq K \omega) \cap A(\varphi)(W' \supseteq K \omega') = KG_{W \supseteq W'} \chi_{W \supseteq W'}(w \otimes w') \supseteq KG_{W \supseteq W'} \chi_{W \supseteq W'}(w) \otimes KG_{W \supseteq W'} \chi_{W \supseteq W'}(w').$$

where $KG_{W \supseteq W'} \chi_{W \supseteq W'}(w) \otimes KG_{W \supseteq W'} \chi_{W \supseteq W'}(w') \subseteq \varphi(W) \otimes \varphi(W')$. The $G$-linear embedding $\varphi(W \otimes W') \hookrightarrow \varphi(W) \otimes \varphi(W')$ from Lemma 3.2 then yields the desired isomorphism as $(K \subseteq KG)$-glider representations. \hfill \Box

Example 3.8. The condition in the previous proposition that $\varphi(W) \neq 0$ is necessary. Indeed, consider for example a group $G$ and the projection $\pi : G \to G/G'$. Then $\varphi(W) = 0$ if and only if $W$ decomposes into irreducible $G$-representations of dimension all bigger than 1. In case $G = Q_8$, we have that the unique 2 dimensional irreducible representation $U$ is such that $U \otimes U = T_1 \oplus T_2 \oplus T_3 \oplus T_4$. With notations from [1] we have that

$$A(\varphi)(\chi_{(\emptyset, ([1:0])}))) = A(\varphi)(\chi_{(\{1, a\}, 0)})) = \chi_{(1, a)},$$

which is not equal to

$$A(\varphi)(\chi_{(\emptyset, ([1:0])})))^2 = 0.$$

For a surjective morphism $\varphi : H \to G$, we have that

$$N_2(\varphi) = \{ W \in \text{Rep}(H) : \ker(\varphi) \text{ acts trivial on } W \}.$$

For monomorphisms $\iota : H \hookrightarrow G$ it is clear that $N_2(\iota) = \text{Rep}(H)$.

Suppose now that we have group homomorphisms

$$H \overset{\varphi}{\longrightarrow} G \overset{\psi}{\longrightarrow} E$$

and let $\{ U_1, \ldots, U_r \}$ be a complete set of irreducible $E$-representations.

Lemma 3.9. Let $U \in \text{Irr}(E), W \in \text{Irr}(H)$, then

$$e(U,W) = \sum_{j=1}^{m} e(U,V_j)e(V_j,W).$$

Proof. On the one hand, we have

$$U_{\psi \circ \varphi} \cong \bigoplus_{i=1}^{n} W_i e(U,W_i).$$

On the other hand, we have

$$U_{\psi} \cong \bigoplus_{j=1}^{m} V_j e(U,V_j)$$

whence

$$(U_{\psi})_{\varphi} \cong \bigoplus_{j=1}^{m} \bigoplus_{i=1}^{n} W_i e(U,V_j)e(V_j,W_i) \cong \bigoplus_{i=1}^{n} \sum_{j=1}^{m} e(U,V_j)e(V_j,W_i).$$

Since $U_{\psi \circ \varphi} \cong (U_{\psi})_{\varphi}$, the desired equality follows. \hfill \Box
Proposition 3.10. Let $W$ be an $H$-representation. Then
\[ \psi \circ \varphi(W) = \overline{\psi(\overline{\varphi(W)})} \]
and
\[ A(\psi \circ \varphi) = A(\psi) \circ A(\varphi). \]

Proof. Easy corollary of the foregoing lemma. \qed

Since it is clear that $A(id_G) = id_{\mathbb{Q}G}$, we have defined a functor
\[ A : \text{Grp}_{\text{fin}} \to \mathbb{Z}\text{-mod}, \]
where $\text{Grp}_{\text{fin}}$ is the category of finite groups. If we restrict to the subcategory of all finite groups with morphisms being the monomorphisms, then we actually have a functor to commutative rings, where the morphisms do not need to preserve the identity.

4. A SHORT EXACT SEQUENCE

From the previous section, we obtain that every subgroup $H \leq G$ yields a monomorphism $\mathbb{Q}(H) \hookrightarrow \mathbb{Q}\overline{G}$. In this section, we use these monomorphisms to construct a short exact sequence of $\mathbb{Q}(G'/G'$)-modules.

As in the proof of Proposition 2.17 we fix a group isomorphism $G/G' \cong \overline{G}/\overline{G'}$ and denote the one-dimensional $G$-representations by $\{T_g \mid g \in G/G'\}$.

The map
\[ \mathbb{Q}(G'/G') \hookrightarrow \mathbb{Q}(\overline{G}), e_g \mapsto \chi_{\{g\},\theta} \]
is a ring morphism and this defines a $\mathbb{Q}(G'/G')$-module structure on the glider representation ring. It turns out that we can also define a $\mathbb{Q}(G'/G')$-module structure on $\mathbb{Q}(H)$. Considered as $H$-representation through $\varphi : H \hookrightarrow G$, we have $(T_g)_{\varphi} \cong S = Ks$ for some $H$-representation $S = Ks$. For an irreducible $(K \subseteq KH)$-glider $W \supseteq Kw$ we define
\[ (T_g \supseteq Kt_g) \cdot (W \supseteq Kw) := (S \supseteq Ks) \otimes (W \supseteq Kw). \]

We define a subgroup $A_{\varphi}(H) \leq G/G'$ by
\[ A_{\varphi}(H) = \{ g \in G/G' \mid (T_g)_{\varphi} \cong T_H \text{ as } H\text{-representations} \}, \]
where $T_H$ denotes the trivial $H$-representation.

Lemma 4.1. Let $W$ be an $H$-representation and $T_g, S$ be such that $(T_g)_{\varphi} \cong S$ as $H$-representations. Then
\[ \overline{\varphi}(S \otimes W) \cong T_g \otimes \overline{\varphi}(W). \]

Proof. It suffices to prove this for $W = W_1$ irreducible. The $G$-representation $\overline{\varphi}(W)$ is determined by decomposing all the $(V_j)_\varphi$ into $H$-components and checking whether $W$ appears as a component. Let $h \in A_{\varphi}(H)$, then $W_1$ appears in $(V_j)_\varphi$ if and only if $W$ appears in $T_h \otimes (V_j)_\varphi$. Hence we have two expressions of $\overline{\varphi}(W)$, namely
\[ \bigoplus_{j=1}^{m'} V_j \quad \text{and} \quad \bigoplus_{j=1}^{m'} T_h \otimes V_j, \]
for some $m' \leq m$ (up to reordering). It follows that there exists a permutation $\sigma$ on $m'$ letters such that $V_j \cong T_h \otimes V_{\sigma(j)}$ as $G$-representations. For every $z \in A_{\varphi}(H)$ we have that $T_{gz} \otimes V_j$ lies over $S \otimes W$. However, $T_g \otimes V_j \cong T_{gh} \otimes V_{\sigma(j)}$ for all $1 \leq j \leq m'$ and it follows that
\[ \bigoplus_{j=1}^{m'} T_g \otimes V_j \cong \bigoplus_{j=1}^{m'} T_{gh} \otimes V_{\sigma(j)}. \]
This shows that
\[ \overline{\tau}(S \otimes W) \cong T_g \otimes \overline{\tau}(W) \cong T_{gz} \otimes \overline{\tau}(W) \quad \forall z \in A_p(H). \]

Recall that for a group morphism \( \varphi : H \to G \) we denoted the associated morphism between glider representation rings by \( A(\varphi) \). If \( \varphi \) is a monomorphism, we also denote \( A(\varphi) \) by \( \Phi_H^G \).

**Proposition 4.2.** The map \( \Phi_H^G : Q(\tilde{H}) \to Q(\tilde{G}) \) is a \( Q(G/G') \)-module morphism.

**Proof.** With notations as before we have
\[
\Phi_H^G((T_g \supset Kt_g) \cdot (W \supset Kw)) = \Phi_H^G(S \otimes W \supseteq Ks \otimes w) \\
= \overline{\tau}(S \otimes W) \supseteq Kt \otimes \overline{\tau}(S \otimes W)(s \otimes w) \\
\cong (T_g \otimes \overline{\tau}(W)) \supseteq Kt_g \otimes \overline{\tau}(W)(w) \\
\cong (T_g \supset Kt_g) \otimes (\overline{\tau}(W) \supseteq Kt \otimes \overline{\tau}(W)(w)) \\
= (T_g \supset Kt_g) \otimes \Phi_H^G(W \supseteq Kw). \]

In order to construct a short exact sequence of \( Q(G/G') \)-modules we are forced to introduce three \( Q(G/G') \)-modules \( P, Q \) and \( R \). In the next section we will on the one hand discuss for which groups one can deduce what these submodules exactly are, which then yields a description of the glider representation ring of \( G \) modulo its Jacobson radical \( J = J(Q(G)) \). On the other hand, we explain how these modules are linked with group representation theoretic properties of the group \( G \) as mentioned in the introduction. The proofs in this section will already make clear some of these connections.

First of all, the glider representation ring of a non-abelian group is an infinite dimensional vector space. In [4] the authors gave a description of the Jacobson radical for \( G = Q_8 \) the quaternion group. We actually defined an ideal \( I \) contained in the nilradical \( N = \text{N}(Q(Q_8)) \) and showed that the quotient \( Q(Q_8)/I \) was semisimple, from which we could conclude that \( I = N = J \). It is not clear that the morphisms \( \Phi_H^G \) factorize over the Jacobson radical. Since they do factorize over the nilradical – indeed, every element in the nilradical is nilpotent and the morphisms preserve multiplication – we consider the induced morphisms
\[
\overline{\Phi_H^G} : Q(\tilde{H})/N \to Q(\tilde{G})/N.
\]

Here, we denoted the nilradicals of \( Q(\tilde{H}) \) and \( G(\tilde{H}) \) both by \( N \). It will always be clear from the context of which ring \( R, N \) is the nilradical of.

Let \( \chi_{(A,B)} \in Q(\tilde{G}) \) and consider the cyclic semigroup \( \langle \chi_{(A,B)} \rangle \). If this semigroup is finite, then it contains a unique idempotent element \( e = \chi_{(C,D)}^{n-1} \). If \( n \) is the smallest integer such that \( \chi_{(A,B)}^n = e \), then the difference \( \chi_{(A,B)} - \chi_{(A,B)}^n \) is nilpotent. Since the glider representation ring is infinite dimensional for non-abelian groups, it could be that the cyclic semigroup \( \langle \chi_{(A,B)} \rangle \cong \mathbb{N} \).

**Definition 4.3.** We define \( P \) to be the \( \mathbb{Q} \)-vector space with basis the elements \( \chi_{(A,B)} \) for which the cyclic semigroup is not finite.

**Proposition 4.4.** The vector space \( P \) is a \( Q(G/G') \)-submodule of \( Q(\tilde{G}) \).

**Proof.** Let \( a = \chi_{(A,B)} \in P \) and \( g \in G/G' \). Denote \( b = \chi_{\{a\},g} \). If \( ba \notin P \), then there exists \( n > 0 \) such that \( b^n a^n = e \) is idempotent. But then \( e = b^{[G/G']^n} a^{[G/G']^n} = a^{[G/G']^n} \), contradicting \( a \in P \). \( \square \)
For $\chi(A, B) \notin P$, denote the associated idempotent element by $e(A, B)$. We have the following lemma.

**Lemma 4.5.** If an element of the form $\chi(A, B) \in \mathbb{Q}(\widetilde{G})$ with $A \neq \emptyset$ is idempotent, then $A \subseteq G/G'$ is a subgroup.

**Proof.** Suppose that $\chi(A, B)$ is idempotent, then $A.A \subseteq A$. Let $a \in A$, then $e = a^{o(a)} \in A^{o(a)} \subseteq A$, which entails that $A.A = A$ or that $A \subseteq G/G'$ is a subgroup. □

**Remark 4.6.** It remains a question whether an element of the form $\chi(\emptyset, B)$ can be idempotent.

The idea is to show that certain idempotent elements $e(A, B)$ are in the image $\Phi^G_H$ for some subgroup $H \leq G$. To this extent, we have

**Lemma 4.7.** Let $C \leq G/G'$ be a subgroup and define $H = \bigcap_{c \in C} \ker(T_c) \leq G$. Then

$$A_i(H) = \{ g \in G/G' \mid (T_g)_H \cong T_H \text{ as } H\text{-representations} \} = C,$$

where $i : H \hookrightarrow G$ denotes the embedding.

**Proof.** By definition it follows that $C \subseteq A_i(H)$, from which we obtain

$$\bigcap_{g \in A_i(H)} \ker(T_g) \subseteq \bigcap_{c \in C} \ker(T_c) = H.$$

Let $g \in A_i(H)$, then $H \subseteq \ker(T_g)$. Since this holds for all $g \in A_i(H)$, we obtain that

$$\bigcap_{g \in A_i(H)} \ker(T_g) \subseteq \bigcap_{c \in C} \ker(T_c) \subseteq \bigcap_{g \in A_i(H)} \ker(T_g)$$

and it follows that $A_i(H) = C$. □

Write $e(A, B) = \chi(C, D)$. If $1 \neq C$ and if there exists an idempotent element $\chi(C', D')$ with $D \subseteq D'$, then we call $\chi(A, B)$ non-maximal.

**Definition 4.8.** We define $Q$ to be the $\mathbb{Q}$-vector space with basis elements exactly the $\chi(A, B)$ which are non-maximal.

By definition of $S_G = S$, see (1), $D \subseteq D'$ means that there exists an irreducible $G$-representation $U$ which appears in the decomposition of $\chi(C', D')$ but not in $\chi(C, D)$. The following example shows that $Q \neq 0$ in general.

**Example 4.9.** Let $G = A_4$. The commutator subgroup equals $A_4 = V_4$ and consider a subgroup $C_2 < V_4$. One shows that under the embedding

$$\Phi^{A_4}_{V_4} : \mathbb{Q}(\widetilde{V}_4) \rightarrow \mathbb{Q}(\widetilde{A}_4),$$

the element $\chi_{C_2} = \chi(C_2, \emptyset)$ is sent to $\chi(A_4/V_4, \{a\})$ where $\{a\} \in \text{Gr}(1, U)$, $U$ being the only three dimensional irreducible $A_4$-representation. Since the former element is idempotent, so is the latter. Since $\chi(A_4/V_4, \{a\})$ is also idempotent, $\chi(A_4/V_4, \{a\})$ is indeed non-maximal. We already mention here that $\chi(A_4/V_4, \{a\}) \in Q \cap \Phi^{A_4}_{V_4}(\mathbb{Q}(\widetilde{V}_4)).$

**Proposition 4.10.** The vector space $Q$ is a $\mathbb{Q}(G/G')$-submodule of $\mathbb{Q}(\widetilde{G})$.

**Proof.** Follows because $\langle \chi(A, B) \rangle$ and $\langle \chi(\emptyset, \emptyset) \chi(A, B) \rangle$ have the same idempotent element (if it exists). □

**Lemma 4.11.** Let $E, H$ be subgroups of $G$. If $H \subseteq E$, then

$$\Phi^G_H(\mathbb{Q}(\widetilde{H})) \subseteq \Phi^G_E(\mathbb{Q}(\widetilde{E})).$$
The map $\Psi$ vector of where 1 denotes the unit in $Q$. Let Theorem 4.13. We define the $Q(G/G')$-module $R$ to be generated by all elements $\chi_{(A,B)}$ for which the associated idempotent element $e(A,B)$ is of the form $\chi(\{e\}, D)$ or $\chi(\emptyset, D)$. The reason for including the elements $\chi(\emptyset, D)$ comes from Remark 4.6. We need one more notion to prove the following theorem: let $V \supseteq K\nu$ be the irreducible glider associated to $\chi_{(A,B)}$, then we call $\alpha = (n_1, \ldots, n_m) \in \mathbb{N}^m$ the dimension vector of $\chi_{(A,B)}$ if

$$KG\nu = V \cong \bigoplus_{j=1}^{m} V^{\oplus n_j}.$$ 

**Theorem 4.13.** Let $G$ be a finite group. We have the following short exact sequence of $Q(G/G')$-modules

$$0 \rightarrow \frac{P}{Q} + \frac{Q}{P} + \sum_{G \triangleright H \trianglelefteq G} \Phi_H^G(\overline{\chi}G/\overline{H}) \rightarrow \overline{Q}G/\overline{H} \rightarrow \frac{Q}{C} \rightarrow 0.$$ 

The map $\Psi$ denotes the embedding. 

**Proof.** We first prove the statement for $G$ abelian. It is clear that $P = 0$ since $Q(G)$ is finite dimensional. Moreover, by definition it is also clear that $Q = 0$ and $R = Q(G)$. Let $H \triangleleft G$ be a subgroup, then we define the subgroup

$$H'' = \bigcap_{g \in H} \text{Ker}(T_g) \triangleleft G.$$ 

It holds that $H'' \leq G$ is a proper subgroup if and only if $H \neq e$. By construction, it follows that $\Phi_{H''}(1) = \overline{\chi} H'' \in \overline{Q}/\overline{H}$. If $gH$ is a left coset, then $(T_g)^{H''} \cong T_{gH}$, for some $h' \in H''$ and it follows that $\Phi_{H''}(\chi_{(h')}) = \overline{\chi}_g H''$. This shows that the generators for the cokernel are exactly the elements of the form $\chi_{(g)}$ with $g \in G$.

Now we treat the general case. Let $\overline{\chi}_{(A,B)}$ be a generator of $\overline{Q}/N$ with $\chi_{(A,B)} \notin P$. By definition of $P$, there exists an $n > 0$ such that $\chi_{(A,B)} = \overline{\chi}_{(C,D)}$ is idempotent. In fact, $\chi_{(A,B)} = e(A,B) = \chi_{(C,D)}$ is also idempotent. Lemma 4.5 entails that $C \subseteq G/G'$ is a subgroup. By Lemma 4.7, the subgroup

$$G' \leq H = \bigcap_{c \in C} \text{Ker}(T_c) \leq G$$

is such that $A_e(H) = C$. This entails that

$$\Phi_{H'}^G(1) = \chi_{(C,\emptyset)};$$

where 1 denotes the unit in $Q(H)$, i.e. $1 = \chi(\{e\}, \emptyset)$. Observe moreover that $G' < H < G$ is proper if and only if $C \neq e$ and $C \neq G/G'$.

Firstly, suppose that $C \neq e$, that is $G' \leq H \leq G$. If $U \supseteq K\nu$ is the $(K \subseteq K\nu)$-glider corresponding to $\chi_{(C,D)}$, then the $H$-module $KH\nu$ contains the trivial $H$-representation $T_H$. Moreover, the associated element of $KH\nu \supseteq U$ is of the form $\chi_{(C'', D'')}$ for some $1 \in C'' \subseteq H/H'$ and $D'' \subseteq S_H$. If $D = \emptyset$, then $D'' = \emptyset$ and $\chi_{(C,\emptyset)} = \Phi_H^G(\chi_{(1), \emptyset})$. Suppose that $D \neq \emptyset$. By construction, the $G$-representation $U$ embeds in $V(KH\nu)$. First of all, if an irreducible $H$-representation $W$ appears in the decomposition of $KH\nu$, then there must be at least one irreducible $G$-representation $V$ lying over $W$ that appears in $U$. We also have by definition
of \( H \) that all one dimensional components of \( U \) are exactly all the one dimensional components that lie over \( T_H \). Therefore, we can write

\[
\Phi^G_H(\chi(C''',D''')) = \left( \bigoplus_{c \in C} T_c \oplus U' \right) \supseteq Ka,
\]

with \( U' \) a \( G \)-representation. Suppose that the irreducible \( G \)-representation \( V \) with \( \dim(V) > 1 \) appears in the decomposition of \( U \), then this implies that there exists an irreducible \( H \)-representation \( W \) which appears in \( \chi(C''',D''') \). The \( G \)-representation \( V(KH_u) \) then contains all other irreducible \( G \)-representations \( V'' \) such that \( V'' \) lies over this \( W \). If \( \dim(V) = \dim(V') \), then by [6, Theorem 5] we know that \( V \) and \( V' \) differ only in the projective representation of \( G^{dec}/H \), where \( G^{dec} \) denotes the decomposition group \( H \leq G^{dec} \leq G \). It holds that \( G^{dec}/H \leq G/H \) and \( G/H \cong (G/G')/(H/G') \), whence \( G^{dec}/H \) is abelian and both representations \( V \) and \( V' \) only differ in a one dimensional representation \( S \) of \( G^{dec}/H \). By definition of \( A_u(H) = C \), it follows that there exists \( c \in C \) such that \( V' \cong V \otimes T_c \) as \( H \)-representations. Because the element \( \chi(C,D) \) is idempotent, the \( G \)-representation \( T_c \otimes V \) already appears in \( U \). It follows that

\[
\chi(C,D) \Phi^G_H(\chi(C''',D''')) = \Phi^G_H(\chi(C''',D''')) = \chi(C,E),
\]

for some \( E \in S \). Hence if \( \chi(A,B) \notin Q \), then \( E = D \), meaning that \( \chi(C,D) = \Phi^G_H(\chi(C''',D''')) \).

We return to the element \( \chi(A,B) \). It follows that \( \chi_m(A,B) \) and \( \chi_{m'}(A,B) \) are congruent modulo the nilradical \( N(\mathbb{Q}(\hat{G})) \) if and only if \( m - m' \in n\mathbb{Z} \). Therefore we can replace \( \chi_m(A,B) \) by \( \chi_{m+1}(A,B) \), since \( \chi_{m+1}(A,B) = \frac{\chi_{m+1}(A,B)}{\mathbb{Q}(\hat{G})/N} \) for \( \chi_{m+1}(A,B) \). We have that \( A = gC \) for some \( g \in G/G' \). Indeed, \( \chi_{m}(A,B) = \chi(C,E) \) for some \( E \in S \) and hence \( \chi_{m+1}(A,B) = \chi(C,E') \) for some \( E' \in S \) and where \( A \) contains at least one full left coset of \( C \) in \( G/G' \). If it would be strictly bigger than one coset, the idempotent element in the cyclic semigroup \( \langle A \rangle \) would be of the form \( \chi(C',D') \) with \( C \leq C' \), contradiction. In fact, by replacing \( \chi(A,B) \) by \( \chi_{m+1}(A,B) \) we obtain an equality \( \chi(A,B) \chi(C,D) = \chi(A,B) \) inside \( \mathbb{Q}(\hat{G}) \). This shows that the dimension vectors of \( \chi(A,B) \) and \( \chi(C,D) \) are the same. Since \( A = gC \), \( \chi(A,B) \) corresponds to the glider

\[
\bigoplus_{c \in C} T_{gc} \oplus V \supseteq K(\sum_{c \in C} t_{cg} + v)
\]

for some \( v \in V \). The equality \( \chi(A,B) \chi(C,D) = \chi(A,B) \) states

\[
KG((\sum_{c \in C} t_{cg} \otimes v) \otimes a) \cong \bigoplus_{c \in C} T_{gc} \oplus V.
\]

However, since \( KG(t_{g} \otimes a) \) has the same dimension vector as \( \chi(A,B) \), this shows that we even have the equality

\[
\chi((g),\emptyset) \chi(C,D) = \chi(A,B).
\]

Let \( S \) denote the one-dimensional \( H \)-representation \( (T_g)_H \), then if \( \chi(A'',B'') \) denotes the associated element to the \( (K \subseteq KH) \)-glider

\[
(S \subseteq Ks) \otimes (KH_u \supseteq Ku),
\]

it follows that

\[
\Phi^G_H(\chi(A'',B'')) = \Phi^G_H(\chi((s),\emptyset)) \Phi^G_H(\chi(C'',D''))
\]

\[
= \chi((g),\emptyset) \chi(C,D)
\]

\[
= \chi((g),\emptyset) \chi(C,D) \chi(C,D)
\]

\[
= \chi((g),\emptyset) \chi(C,D) \chi(A,B)
\]
Hence we have shown that any generator $\chi(A,B)$ of $\mathbb{Q}(\tilde{G})/N$ with $\chi(A,B) \notin P, Q$ and for which the associated idempotent $e(A,B) = \chi(C,D)$ has $C \neq e$ lies in the image $\Phi_H^G(\mathbb{Q}(\tilde{E})/N)$ where $E = \bigcap_{C \leq G} \text{Ker}(T_C) \triangleleft G$. Since $C \neq e$, $E \lhd G$ is proper and Lemma 4.11 entails that $\chi(A,B)$ indeed lies in the image of $\Psi$. By running over all $\chi(A,B)$ not in $P$, not in $Q$ and for which $e(A,B)$ has $C \neq e$, we see that we need all subgroups $\tilde{G}' \leq H \triangleleft_{\text{max}} G$.

The only generators $\chi(A,B)$ not in the image of $\Psi$ are the ones with associated idempotent $e(A,B) = \chi(\{\iota\},D)$ or $\chi(\emptyset,D)$. This shows that the cokernel of $\Psi$ is isomorphic to $R/(R \cap N)$.

\begin{remark}
Observe that the sum on the left is not direct. This was already clear from Example 4.9. In the next section we will address this further.
\end{remark}

5. Precise description semisimple part $\mathbb{Q} \otimes \mathcal{P}(\tilde{G})$ under vanishing obstructions

Let $G$ be a finite group and $P, Q, R$ the $\mathbb{Q}(G/G')$-modules from Theorem 4.13. Suppose for the remainder of this section that $P = 0 = Q$ and $R = \mathbb{Q}(G/G')$.

In other words, suppose that the obstruction modules vanish for $G$.

The aim of this section is to prove Theorem 5.3 which gives a concrete decomposition of $\mathbb{Q}(\tilde{G})$ in terms of the group algebras $\mathbb{Q}(H/H')$ where $H$ runs over the subnormal subgroups of $G$.

The short exact sequence of $\mathbb{Q}(G/G')$-modules from Theorem 4.13 takes the form

\[
0 \longrightarrow \sum_{\tilde{G}' \leq H \triangleleft_{\text{max}} G} \overline{\Phi_H^G}(\mathbb{Q}(\tilde{H})/N) \longrightarrow \mathbb{Q}((\tilde{G})/N) \longrightarrow \mathbb{Q}(G/G') \longrightarrow 0
\]

and is split by the map

\[
f : \mathbb{Q}(G/G') \to \mathbb{Q}((\tilde{G})/N), \; \overline{\varphi} \mapsto \chi(\overline{\varphi}, \emptyset),
\]

from which we deduce the isomorphism as $G/G'$-modules

\[
(5) \quad \mathbb{Q}((\tilde{G})/N) \cong \mathbb{Q}(G^{ab}) \oplus \sum_{\tilde{G}' \leq H \triangleleft_{\text{max}} G} \overline{\Phi_H^G}(\mathbb{Q}(\tilde{H})/N),
\]

where $G^{ab} = G/G'$ denotes the abelianization of $G$. In fact, we even obtain an isomorphism of rings: let $\{H_i \mid i \in I\}$ be the set of all minimal subgroups $e < H_i < G/G'$. The elements $\chi(\iota, \emptyset) = \chi(H_i, \emptyset)$ are idempotent elements and look at the monomorphism

\[
\alpha : \mathbb{Q}(G^{ab}) \to \mathbb{Q}((\tilde{G})/N), \; \overline{\varphi} \mapsto \chi(\overline{\varphi}, \emptyset) \prod_{i \in I} \chi(\iota, \emptyset) - \chi(H_i, \emptyset).
\]

One checks that

\[
\alpha(\mathbb{Q}(G^{ab})) \sum_{\tilde{G}' \leq H \triangleleft_{\text{max}} G} \overline{\Phi_H^G}(\mathbb{Q}(\tilde{H})/N) = 0,
\]

by using that $\overline{\Phi_H^G}(\mathbb{Q}(\tilde{H})/N) = \Phi_H^G(1) \overline{\Phi_H^G}(\mathbb{Q}(\tilde{H})/N)$ and

\[
\Phi_H^G(1)\beta = \chi(A_i, H_i, \emptyset)\beta = 0.
\]

To make the last term in the decomposition \((5)\) into a direct sum, we prove the following.
Lemma 5.1. Let $H, E \trianglelefteq G$ be normal subgroups. Then

$$A_i(H)A_j(E) = A_i(H \cap E).$$

Proof. Write $C = A_i(H), D = A_j(E)$. Then

$$H \cap E = \bigcap_{c \in C} \ker(T_c) \cap \bigcap_{d \in D} \ker(T_d) \subseteq \bigcap_{c \in C d \in D} \ker(T_c \otimes T_d) = \bigcap_{e \in CD} \ker(T_e).$$

Since $C, D \subseteq CD$ we also have

$$\bigcap_{e \in CD} \ker(T_e) \subseteq \bigcap_{e \in C} \ker(T_e) \cap \bigcap_{d \in D} \ker(T_d) = H \cap E.$$

Lemma 4.7 entails that $A_i(H \cap E) = CD = A_i(H)A_j(E).$

Proposition 5.2. Let $H, E \trianglelefteq G$ be normal subgroups of $G$. We have the equality

$$\Phi_H^G(Q(H)/N) \cap \Phi_E^G(Q(E)/N) = \Phi_H^E_Q(Q(H \cap E)/N).$$

Proof. One inclusion follows from Lemma 4.11. For the other, let $\chi_{(A,B)} \in \Phi_H^E_Q(Q(H)/N) \cap \Phi_E^G(Q(E)/N)$ and we denote $C = A_{\phi}(H), \ D = A_{\phi}(E)$. Let $\chi_{(V,B)}$ be the unique minimal idempotent in the semigroup $(\chi_{(A,B)})$. It holds that $C \leq V$ and $D \leq V$, whence $A_{\phi}(H \cap E) = CD \leq V$ (see Lemma 5.1). By the proof of Theorem 4.13, we know that $\chi_{(A,B)}$ and also $\chi_{(V,B)}$ are elements of

$$\Phi_P^G(Q(P)/N),$$

where $P = \bigcap_{e \in V} \ker(T_e)$. Because we have that

$$P \subseteq \bigcap_{c \in CD} \ker(T_c \otimes T_d) = H \cap E,$$

it follows that $\chi_{(A,B)} \in \Phi_H^E_Q(Q(H \cap E)/N)$ by Lemma 4.11.

The next step is to mod out $\sum_{G' \leq H_{\max G}} \Phi_H^G(Q(H)/N)$ by $\Phi_H^E_Q(Q(H \cap E)/N)$. In general, this will not be sufficient to arrive at a direct sum over all subgroups $G' \leq H_{\max G}$. Nevertheless, we will obtain a partition of these subgroups: suppose that the subset of maximal normal subgroups is indexed by $I$, then we arrive at a partition

$$I = I_1 \sqcup I_2 \sqcup \cdots \sqcup I_r$$

with $r \geq 2$ and we have the exact sequence of $Q(G/G')$-modules

$$0 \to \Phi_{\bigcap H}(Q(H)/N) \to \bigoplus_{H \in I} \Phi_H^G(Q(H)/N) \to \bigoplus_{k=1}^r \sum_{H \in I_k} \Phi_H^G(Q(H)/N) \to 0$$

Again, we have that the above sequence is split. Indeed, it suffices to send $[\chi_{(A,B)}]$ to $\chi_{(A,B)}$ and again we arrive at an isomorphism of rings. To see this, let $C = \Pi_H A_i(H)$ and consider the idempotent elements

$$\delta = \Pi_D \chi_{(C,D)} - \chi_{(D,D)},$$

where the product runs over all subgroups $D \leq G$ minimal over $C$ and

$$\epsilon_H = \Pi_E \chi_{(A_i(H),E)} - \chi_{(E,E)},$$
where the product runs over all subgroups $E \leq G/G'$ minimal over $A_i(H)$. So for any $H$ there exists $E$ and $D$ such that

$$A_i(H) \leq E \leq C \leq D,$$

which entails that

$$(\chi(C, \mathfrak{g}) - \chi(D, \mathfrak{g})\chi(A_i(H), \mathfrak{g}) - \chi(E, \mathfrak{g})) = 0.$$

We have monomorphisms

$$f : \Phi^G_{G/H}(\mathbb{Q}(\widetilde{H})/N) \to \mathbb{Q}(\widetilde{G})/N, \quad a \mapsto a\delta$$

and

$$g_H : \Phi^G_H(\mathbb{Q}(\widetilde{H})/N) \to \mathbb{Q}(\widetilde{G})/N, \quad b \mapsto bH.$$

It follows that

$$f\left(\Phi^G_{G/H}(\mathbb{Q}(\widetilde{H})/N)\right) g_H \left(\Phi^G_H(\mathbb{Q}(\widetilde{H})/N)\right) = 0$$

since $f(1) = \chi(C, \mathfrak{g})\delta = \delta$, $g_H(1) = \chi(A_i(H), \mathfrak{g})\epsilon_H = \epsilon_H$ and $\delta \epsilon_H = 0$.

Suppose that $|I_1| > 1$, then we can do the same and arrive at the following split exact sequence

$$0 \to \frac{\Phi^G_{G/H}(\mathbb{Q}(\widetilde{H})/N) \mathbb{Q}(\bigcap_{H} Z)/N}{\Phi^G_{G/H}(\mathbb{Q}(\widetilde{H})/N) \mathbb{Q}(\bigcap_{H} Z)/N} \to \sum_{H \in I_1} \frac{\Phi^G_H(\mathbb{Q}(\widetilde{H})/N)}{\Phi^G_H(\mathbb{Q}(\widetilde{H})/N)} \to \bigoplus_{i=1}^s \frac{\Phi^G_{G/H}(\mathbb{Q}(\widetilde{H})/N)}{\Phi^G_{G/H}(\mathbb{Q}(\widetilde{H})/N)},$$

where

$$I_1 = J_1 \cup J_2 \cup \cdots \cup J_s$$

is a partition. Using the same argument as before, one shows that this again yields an isomorphism of rings.

By continuing this procedure we arrive at a partition such that $|I_1| = 1$ (in the next section we will prove that also for all the subgroups $H$ the obstruction modules vanish). Without loss of generalization, we may assume that this was already the case in the first step. By Theorem 4.13 we know what remains of

$$\frac{\Phi^G_H(\mathbb{Q}(\widetilde{H})/N)}{\Phi^G_{G/H}(\mathbb{Q}(\widetilde{H})/N)},$$

namely

$$\mathbb{Q}(H^{ab}) \oplus \sum_{H < E \leq G} \frac{\Phi^G_H(\mathbb{Q}(\widetilde{E})/N)}{\Phi^G_{G/H}(\mathbb{Q}(\widetilde{E})/N)}.$$

Altogether we obtain the following theorem. Recall that a subgroup $H$ is called subnormal if there exists $H_1 \leq G$ such that $H \triangleleft H_1 \triangleleft \cdots \triangleleft H_l = G$.

**Theorem 5.3.** Let $G$ be a finite group such that $P = 0 = Q$ and $R = \mathbb{Q}(G/G')$. Then

$$\mathbb{Q}(\widetilde{G})/J \cong \bigoplus_H \mathbb{Q}(H^{ab}),$$

where the direct sum runs over all subnormal subgroups $H$ of $G$.

**Proof.** The equality $J = N$ follows now since $J$ is the smallest ideal such that $R/J$ is semisimple. That we indeed obtain all the subnormal subgroups follows from a careful analysis of the proof above. □
Interpreting the obstructions with a representation eye

In Theorem 5.3 we saw that if \( P = Q = 0 \) and \( R = Q(G/G') \) (i.e. the obstruction modules vanish) then the exact sequence in Theorem 4.13 takes a particularly nice form. However the current definitions of the modules \( P, Q, R \) are still a bit exotic, making it non-transparent how to check vanishing. The goal of this section is to adjust this by giving descriptions in more classical languages, namely \( \mathbb{C} \)-representation theory and group theory.

### 6.1. The module \( Q \)

**Interpretation.** Let us look at the submodule \( Q \), which keeps track of elements \( \chi_{(A,B)} \) that yield non-maximal idempotent elements. We recall from Example 4.9 that

\[
Q \bigcap \sum_{G' \leq H \leq G} \Phi_H'(Q(H)/N) \neq 0.
\]

To make a connection with representation theoretic questions, we alter \( Q \) by the quotient

\[
\frac{Q \bigcap \sum_{G' \leq H \leq G} \Phi_H'(Q(H)/N)}{Q(\tilde{H})/N},
\]

but we still denote this quotient by \( Q \). From the proof of Theorem 4.13 we deduce that the only possible obstruction of \( e(A,B) = \chi_{(C,D)} \) being non-maximal lies in the existence of irreducible \( G \)-representations \( V, V' \) of different dimension which lie over the same irreducible \( H \)-representation \( W \) for some normal subgroup \( H \) containing \( G' \). Summarised, we obtain the following down to earth interpretation of the module \( Q \).

**Interpretation obstruction.** If \( Q(G) \neq 0 \). Then there exists \( V, W \in \text{Irr}(G) \) and \( G' \leq H \leq G \) such that

1. \( \dim(V) \neq \dim(W) \),
2. \( V|_H \) and \( W|_H \) contain a common irreducible summand \( U \in \text{Irr}(H) \).

In [3, Corollary 3.17] it is shown that this situation cannot occur for \( G \) nilpotent and \( H \triangleleft G \) a normal maximal subgroup strictly containing the center \( Z(G) \). We will now handle other cases where this cannot happen.

**Lemma 6.1.** Let \( G \) be a group with subgroup \( G' \leq H \leq G \) containing the commutator subgroup and \( V, W \in \text{Irr}(G) \) lie over the irreducible \( H \)-representation \( U \). Then \( \dim(V) = \dim(W) = 1 \).

**Proof.** Suppose that \( \dim(V) = 1 \), then so is \( \dim(U) = 1 \). Up to tensoring with a power of \( V \), we can assume that \( U = T_H \) is the trivial \( H \)-representation. Then \( W \) is an irreducible \( G \)-representation lying over the trivial \( G' \)-representation. Since the only irreducible \( G \)-representations lying over \( T_{G'} \) are one-dimensional the claim follows. \( \square \)

**Proposition 6.2.** Let \( G \) be a finite group with maximal subgroup \( G' \leq H \leq G \) containing the commutator subgroup \( G' \). If \( V, W \in \text{Irr}(G) \) lie over \( U \in \text{Irr}(H) \) then \( \dim(V) = \dim(W) \).

**Proof.** Suppose \( \dim(V) < \dim(W) \). By the previous lemma it follows that \( 1 < \dim(V) < \dim(W) \). Any subgroup containing the commutator subgroup is normal, since \( g^{-1}hg = [g^{-1},h]h \in H \), whence we can use the results from [6]. In loc. cit. the author shows that \( V_H \) either remains irreducible or either decomposes and yields a decomposition group \( G_{V'} \). We treat the former case first. In this situation, the \( H \)-representation \( W_H \) cannot be irreducible (for otherwise \( \dim(V) = \dim(W) \))
and since $H \triangleleft G$ is maximal $G_{W}^{\text{dec}}$ is either $H$ or $G$. If $G_{W}^{\text{dec}} = H$, then for some $g_{2}, \ldots, g_{n} \in G \setminus H$

$$W = U \oplus g_{2}U \oplus \cdots \oplus g_{n}U$$

$$= R_{1} \oplus R_{2} \oplus \cdots \oplus R_{n}$$

with all $R_{i}$ non-isomorphic $H$-representations. From [6] we know that $n = [G : G_{W}^{\text{dec}}] = [G : H]$. We also know by [14, Proposition 20.5] that $n^{2} \leq [G : H]$. This leads to $[G : H] = 1$, contradiction. In case $G_{W}^{\text{dec}} = G$, we can write

$$W = R_{1} = U \oplus g_{2}U \oplus \cdots \oplus g_{n}U,$$

where the $g_{i}U$ are isomorphic as $H$-representations and $U = V$. Hence the same result from [14] now entails the inequality

$$\left(\frac{\dim(W)}{\dim(V)}\right)^{2} = n^{2} \leq [G : H].$$

Since $H \leq G$ is maximal and normal, $G/H$ has no non-trivial subgroups, whence is cyclic of prime order $p$. In other words, $[G : H] = p$. Write

$$|G| = \dim(W)k_{u}$$

$$|H| = \dim(V)k_{v}$$

$$\dim(W) = \dim(V)l$$

Here we used the Frobenius divisibility property, see [10, Theorem 4.16]. It follows that

$$l^{2} \leq [G : H] = p = \frac{k_{u}}{k_{v}}.$$

However, $l = p$ contradicts $p^{2} \leq p$ and $l = 1$ entails $\dim(V) = \dim(W)$, also a contradiction. This covers the case $V_{H}$ irreducible. Suppose now that

$$V_{H} = U \oplus g_{2}U \oplus \cdots \oplus g_{n}U$$

$$V_{G_{W}^{\text{dec}}} = R_{1} \oplus \cdots \oplus R_{m}$$

From [6] we know that the appearing irreducible $H$-components of $W_{H}$ are the same as the ones appearing in $V_{H}$, possibly with different multiplicity. We also know that $\dim(R_{1}) = \ldots = \dim(R_{m})$. It follows that

$$W_{G_{W}^{\text{dec}}} = R_{1}' \oplus \cdots \oplus R_{m}',$$

with $\dim(R_{1}') = \ldots = \dim(R_{m}')$. Hence $m = [G : G_{V}^{\text{dec}}] = [G : G_{W}^{\text{dec}}]$ and since both decomposition groups contain $H$, they must be equal, denote this group by $G_{\text{dec}}$. If $G_{\text{dec}} = H$, then all $g_{i}U$ are non-isomorphic $H$-representations, whence $R_{1}' = U$. Therefore $R_{i}' = U^{\oplus s}$ for some $s \geq 1$, but since $R_{1}'$ is an irreducible $G_{\text{dec}}$-representation, $s$ must equal to 1. It then follows that $\dim(V) = \dim(W)$, contradiction. If $G_{\text{dec}} = G$, then $m = 1$ and $U \cong g_{i}U$ as $H$-representation for all $2 \leq i \leq n$. Hence we can write

$$V_{H} = U^{\oplus t} \Rightarrow t^{2} \leq [G : H] = p$$

$$W_{H} = U^{\oplus s} \Rightarrow s^{2} \leq [G : H] = p$$

We can also write $\dim(V) = t \dim(U), \dim(W) = s \dim(U)$ and $|H| = \dim(U)k_{u}$, whence

$$p = [G : H] = \frac{\dim(V)k_{u}}{\dim(U)k_{u}} = \frac{t}{k_{u}} = \frac{\dim(W)k_{u}}{\dim(U)k_{u}} = \frac{k_{w}}{k_{u}} = s \frac{k_{w}}{k_{u}}$$

Clearly $t < s$, whence $t = 1$ and $s = p$. However, $t = 1$ implies that $V_{H} = U$ is an irreducible $H$-rep, contradiction. □
Applications of the interpretation.

We start by showing that the vanishing of $Q$ is preserved for subgroups.

**Proposition 6.3.** Let $G$ be a finite group such that $Q(G) = 0$, then $Q(H) = 0$ for all subgroups $H \leq G$.

**Proof.** Suppose that $e(A, B) = \chi_{(C,D)}$ is a non-maximal idempotent in $\mathbb{Q}(\hat{H})$ and let $f = \chi_{(C,D')}$ be an idempotent such that $fe(A, B) = f$. The elements $\Phi_{H}^{e}(e(A, B)), \Phi_{H}^{f}(f)$ are of the form $\chi_{(C',M)}$, $\chi_{(C',M')}$ respectively, with $M \subseteq M'$. Since $Q(G) = 0$, $M = M'$, but because $\Phi_{H}^{f}$ is injective, $f = e(A, B)$, contradiction. $\square$

From the interpretation of the obstruction module $Q$ we immediately get the following.

**Proposition 6.4.** If $G$ has all its irreducible representations of degree $\leq 2$ then $Q = 0$.

6.2. The module $P$.

**Interpretation.** For non-abelian groups the number of elements $\chi_{(A,B)}$ is infinite, however by Theorem 2.15 the number of dimension vectors is still finite. This allows us to show that under some condition on $B$ or $A$, the cyclic semigroup $\langle \chi_{(A,B)} \rangle$ contains an idempotent and hence does not contribute to $P$.

We say that a representation $U$ completely linearizes if it decomposes into one-dimensional representations.

**Proposition 6.5.** Let $(A, B) \in \mathcal{P}(G/G') \times S_{G}$. If $A \neq \emptyset$ or there exists a $U \in \text{Irr}(G)$ such that $B \cap \text{Gr}(U) \neq \emptyset$ and $U^\otimes n$ completely linearizes for some $n$, then $\langle \chi_{(A,B)} \rangle$ contains an idempotent.

**Proof.** Suppose first that $A \neq \emptyset$, then there exists some $n$ (e.g. $|G|$) such that $\chi_{(A,B)}^{n} = \chi_{(C,D)}$ with $1 \in C$ (i.e. the trivial representation $T$ appears). Consider now the sequence

$$\chi_{(C,D)}, \chi_{(C,D)}^{2}, \chi_{(C,D)}^{3}, \ldots$$

Suppose that $D \cap \text{Gr}(j, V) = \{a_{1}, \ldots, a_{j}\}$ for $V \in \text{Irr}(G)$. Because $T$ appears we have that at least $\{a_{1}, \ldots, a_{j}\}$ appears in $\chi_{(C,D)}^{n}$. This shows that the dimension vector $\alpha(n)$ of $\chi_{(C,D)}^{n}$ is an increasing function. However, since there are only a finite number of dimension vectors, this sequence must stabilize, and again using the argument involving the appearance of $T$, we arrive at an element $\chi_{(E,F)} \in \langle \chi_{(A,B)} \rangle$ which is idempotent.

Suppose that $A = \emptyset$. Consider now $U \in \text{Irr}(G)$ and $n$ as in the statement. Then $\chi_{(A,B)}^{n} = \chi_{(A',B')}^{n}$ with $A' \neq \emptyset$ and hence we are finished by the first part. $\square$

The proof of the previous statement shows the importance of detecting the presence of the trivial representation. This can be done through the next proposition. First recall that for a given normal subgroup $N \triangleleft G$ there exists irreducible characters $\chi_{i}$ such that $N = \bigcap_{i \in I} \text{Ker}(\chi_{i})$

If the index set $I$ is such that removing one of the $\text{Ker}(\chi_{i})$ yields a strictly bigger normal subgroup, we call the intersection a minimal presentation of $N$.

**Proposition 6.6.** Let $N = \bigcap_{i \in I} \text{Ker}(\chi_{i})$ be a minimal presentation. If $U \in \text{Irr}(G)$ with associated character $\chi$ is such that $N \subseteq \text{Ker}(\chi)$, then $U$ appears as a component of

$$\bigoplus_{i \in I} U_{i} \otimes n$$
for some \( n \geq 1 \) (\( U_i \) denotes the irreducible representation associated to \( \chi_i \)).

Proof. The \( G \)-representation \( \bigoplus_{i \in I} U_i \) induces a \( G/N \)-representation \( \overline{V} \) which is faithful. Because \( N \subseteq \text{Ker}(\chi) \), \( U \) also induces a \( G/N \)-representation \( \overline{U} \). Hence there exists \( n \geq 1 \) such that the inproduct in \( G/N \)
\[ \langle \overline{U}, \overline{V} \rangle_{G/N} \neq 0. \]
The result now follows, because for \( G \)-representations \( W, W' \) that induce \( G/N \)-representations \( \overline{W}, \overline{W}' \) we have the equality
\[ \langle W, W' \rangle_{G/N} = \langle \overline{W}, \overline{W}' \rangle_{G}. \]
\[ \square \]

All this yields the following obstruction.

Interpretation obstruction. Given a \( U \in \text{Irr}(G) \) with \( \text{dim} U > 1 \), there exists by Proposition 6.6 an \( n \in \mathbb{N} \) such that the trivial \( G \)-representation appears in the decomposition of \( U^{\otimes n} \). Working with \((K \subseteq KG)\)-glider representations, however, requires keeping track of a vector \( u \in U \), and by definition
\[ (U \otimes K)^{\otimes n} = (K G(u \otimes \cdots \otimes u) \supseteq K u \otimes \cdots \otimes u). \]
In general, \( KG(u \otimes \cdots \otimes u) \subseteq U^{\otimes n} \). If nevertheless we can ensure that \( T \) appears in the decomposition of \( KG(u \otimes \cdots \otimes u) \), then \( \chi(U \supseteq K u) \notin P \).

Interestingly we were unable to find a group such that \( P \neq 0 \). Note that the above interpretation could as well have been done with \( T \) replaced by another one-dimensional \( G \)-representation \( S \). Since simple groups have only one 1-dimensional representation, they form natural candidates with non-vanishing \( P \).

Corollary 6.7. Let \( G \) be a finite group and \( \chi(A,B) \in \mathbb{Q}(\tilde{G}) \). If there exists \( U \in \text{Irr}(G) \) such that \( B \cap \text{Gr}(U) = \{ \ast_U \} \), then \( \chi(A,B) \notin P \).

Proof. Because \( N = \text{Ker}(\chi_U) \subseteq \text{Ker}(\chi_T) \), the previous proposition shows there exists \( n \) such that \( T \) appears in the decomposition of \( U^{\otimes n} \). Because we have the liberty of choosing vectors \( u_1, \ldots, u_{\text{dim}(U)} \) in \( U^{\otimes \text{dim}(U)} \), we can choose them appropriately such that \( T \) appears in the decomposition of
\[ KG(u_1 + \cdots + u_{\text{dim}(U)})^{\otimes n}. \]
Proposition 6.5 now yields the result. \[ \square \]

Applications of the interpretation.
To start we directly obtain the analogon of Proposition 6.3, since we can embed \( \mathbb{Q}(\tilde{H}) \in \mathbb{Q}(\tilde{G}) \) via \( \Phi_H^G \).

Proposition 6.8. Let \( G \) be a finite group such that \( P(G) = 0 \), then \( P(H) = 0 \) for all subgroups \( H \leq G \).

We will give now a first non-trivial application of the interpretation of the module \( P \) obtained earlier. More concretely,

Proposition 6.9. Let \( G \) be a group with an abelian subgroup \( H \) of index 2. Then \( P(G) = 0 \).

Proof. Since \( [G : H] = 2 \) and \( H \) is abelian we know that all the irreducible representations of \( G \) have degree at most 2. Let \( U \in \text{Irr}(G) \) be 2-dimensional and decompose it in its symmetric and antisymmetric part: \( U \otimes U = S(U \otimes U) \oplus A(U \otimes U) \).
We know that $u \otimes u \in S(U \otimes U)$ so $\text{KG}(u \otimes u) \subseteq S(U \otimes U)$. Since $S(U \otimes U)$ is 3-dimensional, it is either of the form $T_1 \oplus T_2 \oplus T_3$ or either $T_1 \oplus V$. In the former case, Proposition 6.5 yields the desired conclusion.

Therefore suppose that $S(U \otimes U) \cong T_1 \oplus V$. Fix a basis for $U$ such that

$$U_H \cong S \oplus S' = Ks \oplus Kt$$

as $H$-representations. By [14, Proposition 20.5] $S \not\cong S'$, since $[G:H] = 2$.

\textbf{Claim 1:} IF $V,W \in \text{Irr}(G)$ lie over an irreducible representation $S$ of $H$. Then $\dim(V) = 1$ if and only if $\dim(W) = 1$.

\textbf{Proof.} Suppose $\dim(V) = 1$ and $\dim(W) = 2$. By tensoring with a power over $V$ we may assume that $S = T$ is the trivial $H$-representation. The result [14, Proposition 20.5] shows that $\chi_W(g) = 0$ for $g \in G \setminus H$. Decompose $W_H = T \oplus T'$. Because $V_H = T_H$ we have

$$0 = |G|(W,V) = \sum_{h \in H} \chi_W(h) = |H| + \sum_{h \in H} \chi_{S'}(h) = |H| + \langle \chi_{T'} , \chi_{T_H} \rangle_H \geq |H|,$$

contradiction. \hfill \Box

\textbf{Claim 2:} $S \otimes S \not\cong S' \otimes S'$ as $H$-representations.

\textbf{Proof.} Suppose that $S \otimes S \cong S' \otimes S'$ and write $V_H = T' \oplus T''$ as $H$-representation. As before, $T' \not\cong T''$ whence

$$V_H \cong S \otimes S \otimes S \otimes S',$$

up to changing the roles of $S$ and $S'$. Consequently, if $A(U \otimes U) = T''$ and then either $V$ and $T''$ or either $V$ and $T_1$ lie over the $H$-representation $S \otimes S'$, which in both cases contradicts the previous lemma. \hfill \Box

Let $u = \lambda s + \mu t$, then

$$u \otimes u = \lambda^2 s \otimes s + \lambda \mu (s \otimes t + t \otimes s) + \mu^2 t \otimes t.$$

If $\lambda \mu \neq 0$, then $\text{KG}(u \otimes u)$ must be 3 dimensional and it reaches a one-dimensional representation, which is sufficient to show that $\langle \chi_{U \otimes U} \rangle$ contains an idempotent. If $\lambda \mu = 0$, then, say, $KHu \not\cong S$. In this case $\text{KG}(w \otimes u)$ is 2 dimensional so isomorphic to $V$. Decompose $V$ as $H$-representation

$$V_H = W \oplus W' = Kw \oplus Kw'.$$

We remark that this decomposition is unique: for $h \in H$, write $h \cdot w = c(h)w$ and $h \cdot w' = d(h)w'$. Since $W \not\cong W'$, there exists $h \in H$ such that $c(h) \neq d(h)$. If $\alpha \omega + \beta \omega'$ is such that $KH(\alpha \omega + \beta \omega') \cong W$, then on the one hand we have

$$h \cdot (\alpha \omega + \beta \omega') = \alpha c(h)w + \beta d(h)w'.$$

and on the other hand

$$h \cdot (\alpha \omega + \beta \omega') = \gamma \alpha(w) + \gamma \beta w'.$$

Up to rescaling, it follows that $\gamma = c(h) = d(h)$, contradiction. Since $H$ is abelian, we can represent $W$ and $W'$ by elements $h$ and $h'$ of $H$. If $h^2 = h^2$, then by the second claim $S(V \otimes V)$ must be of the form $T_1 \oplus T_2 \oplus T_3$ and we see that $(U \cong Ku)^4 = (V \cong Kv)^2$ reaches a one-dimensional representation, which suffices to conclude the existence of an idempotent element. If $h^2 \neq (h')^2$ then one looks at $S(V \otimes V)$ and restarts the reasoning: if $S(V \otimes V) = T_1 \oplus T_2 \oplus T_3$, one concludes.
In the other case, \( S(V \otimes V) = T' \oplus V' \), we check the dimension of \( KG(v \otimes v) \). If it is 3, we are done, if it is 2, then \( KG(v \otimes v) = V' \) and
\[
V'_H \cong W^{\otimes 2} \oplus (W')^{\otimes 2}.
\]
But these \( H \)-representations correspond to \( h^4, (h')^4 \) respectively. If both elements are equal, one concludes, otherwise one restarts. Since \( H \) is finite abelian, there exists \( n \geq 1 \) such that \( h^{2^n} = (h')^{2^n} \) so the above argument stops and we conclude.

For an arbitrary glider representation \( V \supseteq Kv \) we know that if an irreducible representation \( U \) of dimension 2 appears in the decomposition of the \( G \)-representation \( V \), a certain power reaches a one dimensional representation and we can deduce the existence of an idempotent element in \( \langle \chi_V \rangle_{Kv} \). If all appearing representations in \( V \) are 1 dimensional, we are working in \( \mathbb{Q}(G/G') \), which is finite dimensional. Hence we have shown that \( P = 0 \).

Remark 6.10. Amitsur [1] classified all groups having all irreducible representations of dimension bounded by 2. His classification consists of three subclasses: (1) abelian groups; (2) certain groups of nilpotency class 2 and (3) groups having an abelian subgroup of index 2. In section Section 7.4 we will handle arbitrary groups of nilpotency class 2. Hence the groups in (3) remain and this was one of the original motivations to apply the interpretation to the groups above.

6.3. The module \( R \).

To understand the flavour of \( R \) let us come back to Example 4.9. In this case, \( R \) can be strictly bigger than \( \mathbb{Q}(G/G') \). Indeed, \( A_4 \) has a maximal subgroup \( C_3 < A_4 \) which is non-normal and the element
\[
\Phi_C^{A_4}(1) = \chi(\{(1), (a)\}),
\]
where \( \{a\} \in \text{Gr}(U, 1) \) corresponds to \( u \in U \) with the property that \( KC_3u = Ku \), is an idempotent element. By definition, this element sits in \( R \). More general, maximal subgroups \( H \leq G \) which are not normal, yield idempotent elements of the form \( \Phi_H^{G}(1) = \chi(\{(1), D\}) \). By construction, we know that
\[
\Phi_H^{G}(1) = \left( T \oplus V_1^{\oplus m_1} \oplus \cdots \oplus V_n^{\oplus m_n} \geq Ka \right),
\]
for some irreducible \( G \)-representations \( V_i \) and \( 1 \leq m_i \leq \text{dim}(V_i) \). The following proposition shows that when \( H \) is maximal but non-normal, then at least one \( V_i \) has dimension strictly bigger than one. To prove this we use the Frobenius reciprocity law, which states that for an irreducible \( G \)-representation and \( T_H \) the trivial \( H \)-representations we have the equality
\[
\langle V_H, T_H \rangle_H = \langle V, \text{Ind}^G_{T_H}(T_H) \rangle_G.
\]
To prove our claim, it suffices to show that \( \text{Ind}^G_{T_H}(T_H) \) does not completely linearizes.

**Proposition 6.11.** Let \( G \) be a group having a maximal subgroup \( H \leq G \) which is not normal, then \( R \) is strictly bigger than \( \mathbb{Q}(G/G') \).

**Proof.** Non-normality of \( H \) implies that \( G' \not\leq H \). Furthermore, by using the equalities \( g^{-1}hg = [g^{-1}, h]h \) and \( gh^{-1}g^{-1} = h^{-1}[h, g] \) one shows that both \( G' \) and \( H \) are subgroups of \( G \) and by maximality of \( H \) they are both equal to \( G \). Hence we can write
\[
G = H \sqcup g'_1H \sqcup \cdots \sqcup g'_kH
\]
with \( g_i' \in G' \setminus H, 2 \leq i \leq k \). As observed just before the statement of the proposition, it suffices to show that \( \text{Ind}^G_{T_H}(T_H) \) does not completely linearizes. Hence, suppose it
does. This implies that
\[ \left( \text{Ind}_{H}^{G}(T_{H}) \right)_{G'} = T_{G'}^{\oplus(G:H)}. \]
By definition, \( \text{Ind}_{H}^{G}(T_{H}) = T_{H} \oplus g'H_{T_{H}} \oplus \cdots \oplus g'_{k}T_{H} \) if \( g' \cdot g'_{1}t = g'_{1}ht = g'_{k}t \) if \( g' = g'_{k} \). However, since \( g' \cdot g'_{1}t = g'_{1}t \) we have that \( g' \cdot g'_{1} \in g'_{1}H \). In particular, for \( g' = g'_{1} \) we obtain \( g'g'_{1} \in H \), which implies \( g'_{1} \in H \), contradiction. \( \square \)

We expect that the following question is true and hence the content would form a checkable obstruction for \( R = Q(G/G') \).

**Question 6.12.** If \( Q(G/G') \subseteq R \) then there exists a maximal subgroup \( H \) in \( G \) which is not normal.

**Remark 6.13.** In case \( G \) is finite group having an abelian subgroup of index 2 we were unable to prove that \( R(G) = Q(G/G') \).

### 7. A Look at Concrete Classes of Groups

In this section we will apply the short exact sequence of Theorem 4.13 to groups of nilpotent class 2 and to certain isocategorical groups.

#### 7.1. Nilpotent groups of class 2.

We will prove that if \( G \) has nilpotency class 2 then the obstruction modules vanish (i.e. \( Q = 0 = P \) and \( R = Q(G/G') \)) and hence we are in the context of Theorem 5.3. Instrumental in the proofs of the vanishing results is the following characterization of groups of nilpotency class 2, which might be known to experts however we were unable to find a reference.

**Proposition 7.1.** Let \( G \) be a finite group. Then the following are equivalent

1. \( G \) is nilpotent of class 2
2. For every \( V \in \text{Irr}(G) \), there exists \( n \geq 1 \) such that \( V^{\otimes n} \) completely linearizes.

**Proof.** Suppose that the nilpotency class of \( G \) is larger than 2. Then there exists \( g \in G' \setminus Z(G) \). Since \( Z(G) = \bigcap_{\chi \in \text{Irr}(G)} Z(\chi) \), there exists an irreducible character \( \chi \) such that \( |\chi(g)| < |\chi(e)| \). If there would exist an \( n > 1 \) such that \( \chi^{n} \) is a positive linear combination of linear characters of \( G \), then on the one hand \( \chi^{n}(g) = \chi^{n}(e) \), since \( g \in G' \) (the commutator subgroup \( G' \) is the intersection of the kernels of all linear characters). On the other hand \( |\chi^{n}(g)| < |\chi^{n}(1)| \), which gives a contradiction. Conversely, suppose that \( G \) is of nilpotency class at most 2, i.e. \( G' \subseteq Z(G) \) and let \( U \) be an irreducible \( G' \)-representation. Considered as \( G' \)-representation, \( U^{\otimes n} \cong S^{\otimes \dim(U)} \) for some one-dimensional \( G' \)-representation \( S \), since \( G' \subseteq Z(G) \). There exists \( n \geq 1 \) such that \( S^{\otimes n} \) is the trivial \( G' \)-representation \( T_{G'} \). Hence
\[ (U^{\otimes n} \cong T_{G'}^{\otimes \dim(U)}). \]

Hence, \( U^{\otimes n} \) decomposes into irreducible \( G' \)-representations which all lie over the trivial \( G' \)-representation, i.e. \( U^{\otimes n} \) is a sum of one-dimensional representations. \( \square \)

Consequently we may apply Proposition 6.5 to obtain that \( P \) vanishes.

**Corollary 7.2.** If \( G \) is nilpotent of class 2, then \( P = 0 \).

Proposition 7.1 and the methods of its proof also allows to show that \( Q \) must vanish.

**Proposition 7.3.** Let \( G \) be a nilpotent group of class 2, then \( Q(G) = 0 \).
Proof. Let $U, V \in \text{Irr}(G)$. Recall that $\dim(U \otimes V)^G$ counts the multiplicity of $T$, the trivial $G$-representation, as irreducible component of $U \otimes V$. Consequently, since $(U \otimes V)^G \cong (U^{**} \otimes V)^G \cong \text{Hom}_K(U^{**}, V)^G \cong \text{Hom}_{KG}(U^{**}, V)$, the multiplicity is non-zero if and only if $V \cong U^*$. In the latter case, by Shur’s lemma, it is 1-dimensional. Note that if there exists an $n > 1$ such that $U^{\otimes n}$ completely linearizes and if

$$U^{\otimes n-1} = \bigoplus_{i=1}^{k} W_j,$$

with $W_1, \ldots, W_k$ irreducible $G$-representations, then $\dim(W_1) = \ldots = \dim(W_k) = \dim(U)$. Indeed, $U^{\otimes n} \cong \bigoplus_{i=1}^{\dim(U)} W_i \otimes U \cong \bigoplus_{i=1}^{\dim(U)^n} T_j$. Hence every $W_i \otimes U$ contains a one-dimensional representation $T_j$. But then $T \subseteq T_j^* \otimes W_i$ and hence by the start of the proof $U^* \cong T_j^* \otimes W_i$. In particular $\dim(U) = \dim(W_i)$. Using this, we will now check the interpretation we obtained in Section 6 for the obstruction module $Q$.

Suppose that $V, W$ are irreducible $G$-representations that lie over a same irreducible $H$-representation, $H$ some normal subgroup of $G$ which contains $G'$. Since $G' \subseteq Z(G)$, $V_{G'} \cong S_V^{\dim(V)}$, $W_{G'} \cong S_W^{\dim(W)}$, for some irreducible $G'$-representations $S_V, S_W$. By the assumption on $V, W$ we have $S_V \cong S_W$ as $G'$-representations. By Proposition 7.1 there exists $n$ such that $V^{\otimes n}$ completely linearizes and up to taking a multiple of $n$, we may assume that $T$ appears as a component of $V^{\otimes n}$. It follows that $S_{V'}^{\otimes n} \cong T_{G'}$, where $T_{G'}$ denotes the trivial $G'$-representation. This further entails that $V^{\otimes n-1} \otimes W$ contains at least one 1-dimensional representation, say $S$. Decompose $V^{\otimes n-1} \cong U_1 \oplus \cdots \oplus U_k$ into irreducible $G$-representations, then $\dim(U_i) = \dim(V)$ for $1 \leq i \leq k$ by the observation earlier in the proof. Up to renumbering, it follows that $S \subseteq U_1 \otimes W$. Similarly to earlier in the proof, this implies that $\dim(V) = \dim(U_1) = \dim(W)$.

Finally, let us consider the $\mathbb{Q}(G/G')$-module $R$.

**Proposition 7.4.** Let $G$ be a finite group of nilpotency class 2. If the element $\chi(\{e\}, D)$ is idempotent, then $D = \emptyset$. As a corollary $R = \mathbb{Q}(G/G')$.

**Proof.** Let $V \subseteq K v$ be associated to $\chi(\{e\}, D)$ and suppose that an irreducible $G$-representation $U$ with $\dim(U) > 1$ appears. By decomposing $V$, we may find $u \in U$ that appears in the corresponding decomposition of $v$. There exists an $n > 0$ such that $U^{\otimes n}$ completely linearizes. Because $\chi(\{e\}, D)$ is idempotent and by Theorem 2.15 it follows that $KG(u \otimes \cdots \otimes u) \cong T$ which contradicts $KGu = U$. Hence $D = \emptyset$.

Altogether we can apply now Theorem 5.3. Hereby recall that in a nilpotent group all subgroups are subnormal.

**Theorem 7.5.** Let $G$ be a finite nilpotent group of class 2. Then

$$\mathbb{Q}(\hat{G})/J \cong \bigoplus_{H \leq G} \mathbb{Q}(H^{ab}).$$

**Example 7.6.** There are two non-abelian groups of prime cuber order $p^3$, namely $C_{p^2} \ltimes C_p$ and $H_p$ the Heisenberg group. For instance, if $p = 2$ these are simply $D_8$ and $Q_8$. The groups $C_{p^2} \ltimes C_p$ and $H_p$ have the same character table. However they are nilpotent of class 2 and have $Z(G) = G' = C_p$. It follows that the glider representation rings are non-isomorphic since they have a different number of subgroups.
7.2. Isocategorical groups.

As another application we recall that two groups \( G_1 \) and \( G_2 \) are called isocategorical if \( \text{Rep}(G_1) \) and \( \text{Rep}(G_2) \) are equivalent as tensor category (so without consideration of the symmetry). It was proven by Etingof-Gelaki [9, Lemma 3.1.] that if \( G_1 \) and \( G_2 \) are isocategorical, then there exists a Drinfeld twist \( J \) such that \( \mathbb{C}(H)^J \) is isomorphic to \( \mathbb{C}(G) \). In fact, all groups isocategorical to a given group \( G \) can be explicitly classified in group theoretical terms.

More concretely, let \( A \) be a normal abelian subgroup of \( G \) of order \( 2^{2m} \) for some \( m \in \mathbb{N} \) and write \( Q = G/A \). Let \( R : \tilde{A} \to A \) be a \( G \)-invariant skew-symmetric isomorphism between \( A \) and its character group \( \tilde{A} \). This form induces a \( Q \)-invariant cohomology class \( [\alpha] \) in \( H^2(\tilde{A}, K^*)^R \) (where the action of \( \tilde{A} \) on \( K^* \) is the trivial one). By definition, \( qa/\alpha \) is a trivial 2-cocycle for any \( q \in Q \). Hence there exists a 1-cocycle \( z(q) : \tilde{N} \to K^* \) such that \( \partial(z(q)) = qa/\alpha \). Define the cochain

\[
\beta(p, q) := \frac{z(pq)}{z(p)z(q)}.
\]

One can check that it has trivial coboundary and hence \( \beta(p, q) \in \tilde{A} \cong A \). In other words \( \beta(p, q) \in Z^2(Q, N) \). Define now the group \( G_0 \) to be equal to \( G \) as a set, but with multiplication defined by

\[
g \cdot h = b(\bar{g}, \bar{h})gh.
\]

In [9, Theorem 1.3.] Etingof-Gelaki prove that if \( G_2 \) is isocategorical to \( G_1 \), then \( G_2 \cong (G_1)_b \) for \( b \) some cocycle obtained as in the procedure above. In particular, [9, Corollary 1.4.], if a group \( G \) does not have a normal abelian 2-subgroup equipped with a \( G \)-invariant alternating form then it is categorically rigid, i.e. no other group is isocategorical equivalent to it. This holds for example if 4 does not divide \( |G| \).

In [13, Section 4] an infinite family of pairs of non-isomorphic, yet isocategorical groups \( G^m \) and \( G^m_0 \), for \( 3 \leq m \in \mathbb{N} \), was constructed. As proven by Goyvaerts-Meir [11] the case \( m = 3 \) yield the smallest non-isomorphic, but isocategorical, groups (which are thus of order 64).

**Proposition 7.7.** Let \( 3 \leq m \in \mathbb{N} \) and \( G^m, G^m_0 \) be the isocategorical groups from [13]. Then their representation rings over \( \mathbb{C} \) are isomorphic, however the glider representation rings \( \mathcal{P}(G^m) \) and \( \mathcal{P}(G^m_0) \) are non-isomorphic rings.

More generally, suppose that \( G \) and \( H \) are isocategorical. Thus there exists a monoidal equivalence \( F : \text{Rep}(G) \to \text{Rep}(H) \). Then \( F \) clearly induces an isomorphism between the Grothendieck rings \( K_0(\text{Rep}(G)) \) and \( K_0(\text{Rep}(H)) \). Thus the first part of Proposition 7.7 is a general statement about isocategorical groups and hence follows from [9, Theorem 1.3.] and the construction of the groups.

For the second part of Proposition 7.7 we start by recalling the construction in case \( m = 3 \). Let \( G = N \times H \) with \( N \cong \mathbb{Z}_4 \times \mathbb{Z}_4 \) and \( H = \langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \rangle \times \langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle \). Denote the generators of \( N \) by \( n_1 \) and \( n_2 \). The action of \( H \) on \( N \) is as follows

\[
(n^h)^t = h^{-1}n.
\]

For example \( (n_1^{h_1})^t = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \). In other words \( n_1^{h_1} = n_1 \).

Now we need the cocycle \( b \) to twist \( G \). The action of \( H \) on \( \tilde{N} \) is given by \( (h\omega)(n) := \omega(n^h) \). Define

\[
b(h_1^{l_1}h_2^{l_2}, h_1^{r_1}h_2^{r_2}) = n_1^{l_1}n_2^{l_2}.
\]
with \( l_i = \delta_{1,i} \delta_{1,r_i} \). With easy computations one can now check the following.

**Lemma 7.8.** With notations as above we have that \( G = \langle n_1, n_2, h_1, h_2 \mid \mathcal{R}_1 \rangle \) and \( G_b = \langle n_1, n_2, h_1, h_2 \mid \mathcal{R}_2 \rangle \) with

\[
\mathcal{R}_1 = \{ n_1^4 = 1, h_1^2 = 1, (h_1, h_2) = 1, n_1 n_2 = 1, n_1 h_1 = n_1 n_1^2 n_2, n_2 h_2 = n_2 h_1 n_2 \}
\]

\[
\mathcal{R}_2 = \{ n_1^4 = 1, h_1^2 = n_1^2, (h_1, h_2) = 1, (n_1, n_2) = 1, n_1 h_1 = n_1 h_1 n_2, n_2 h_2 = n_2 n_1 \}.
\]

Note that both \( G \) and \( G_b \) are nilpotent of class 2 and in fact their centers equal their commutator subgroups (e.g. \( G' = \langle n_1^4, n_2^2 \rangle = \mathcal{Z}(G) \)). Furthermore, the subgroup lattices of \( G \) and \( G_b \) are isomorphic. However both groups have a different amount of subgroups. Theorem 5.3 now shows that \( \mathcal{R}_1(G) \not\cong \mathcal{R}_1(G_b) \) (since otherwise the same would hold after extension of scalars to \( \mathbb{Q} \) and taking the quotient by the Jacobson radical). Thus we have proven Proposition 7.7 in the case \( m = 3 \). The case of a general \( m \) is analogue but notational more cumbersome.

In upcoming work we will describe, in a more systematic way, data that is contained in \( \mathcal{R}_1(G) \) but which is not necessarily detected by \( \text{Rep}_K(G) \) viewed as tensor category.

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(FREDERIK CAENEPEEL)

SHANGHAI CENTER FOR MATHEMATICAL STUDIES, FUDAN UNIVERSITY, 2005 SONGHU ROAD, SHANGHAI, CHINA

E-MAIl ADDRESS: frederik.caenepeel@fudan.edu.cn
GLIDER REPRESENTATION RINGS WITH A VIEW ON DISTINGUISHING GROUPS

(Geoffrey Janssens)
Departement Wiskunde, Vrije Universiteit Brussel, Pleinlaan 2, 1050 Elsene, Belgium
E-mail address: geofjans@vub.ac.be