EIGENVALUE ESTIMATES FOR A THREE-DIMENSIONAL MAGNETIC SCHRODINGER OPERATOR

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Abstract. We consider a magnetic Schrödinger operator

\[ H^h = (-i\hbar \nabla - \vec{A})^2 \]

with the Dirichlet boundary conditions in an open set \( \Omega \subset \mathbb{R}^3 \), where \( h > 0 \) is a small parameter. We suppose that the minimal value \( b_0 \) of the module \( |\vec{B}| \) of the vector magnetic field \( \vec{B} \) is strictly positive, and there exists a unique minimum point of \( |\vec{B}| \), which is non-degenerate. The main result of the paper is upper estimates for the low-lying eigenvalues of the operator \( H^h \) in the semiclassical limit. We also prove the existence of an arbitrary large number of spectral gaps in the semiclassical limit in the corresponding periodic setting.

1. Preliminaries and main results

1.1. Main assumptions. We would like to analyze the asymptotic behavior, in the semiclassical regime, of the low-lying eigenvalues of the Dirichlet realization of the magnetic Schrödinger operator in an open set \( \Omega \) in \( \mathbb{R}^3 \):

\[ H^h = (hD_{X_1} - A_1(X))^2 + (hD_{X_2} - A_2(X))^2 + (hD_{X_3} - A_3(X))^2, \]

where \( \vec{A} = (A_1, A_2, A_3) \in C^\infty(\bar{\Omega}, \mathbb{R}^3) \) is a magnetic potential and \( h > 0 \) is a small parameter. We will denote the coordinates in \( \mathbb{R}^3 \) as \( X = (X_1, X_2, X_3) = (x, y, z) \).

Let \( \vec{B} = \text{rot} \vec{A} = (B_1, B_2, B_3) \) be the corresponding vector magnetic field:

\[ B_1 = \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}, \quad B_2 = \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}, \quad B_3 = \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}. \]

Put

\[ b_0 = \min\{|\vec{B}(X)| : X \in \Omega\}. \]

We assume that there exist a (connected) bounded domain \( \Omega_1 \subset \subset \Omega \) and a constant \( \epsilon_0 > 0 \) such that

\[ |\vec{B}(X)| \geq b_0 + \epsilon_0, \quad x \notin \Omega_1. \]

We also assume that:

\[ b_0 > 0, \]

and that there exists a unique minimum \( X_0 \in \Omega \) such that \( |\vec{B}(X_0)| = b_0 \), which is non-degenerate: in some neighborhood of \( X_0 \)

\[ C^{-1}|X - X_0|^2 \leq |\vec{B}(X)| - b_0 \leq C|X - X_0|^2. \]

2000 Mathematics Subject Classification. 35P20, 35J10, 47F05, 81Q10.

Key words and phrases. Magnetic Schrödinger operator, eigenvalue asymptotics, magnetic wells, semiclassical limit, spectral gaps.

Y.K. is partially supported by the Russian Foundation of Basic Research (grants 09-01-00389 and 12-01-00519).
1.2. Main statement.

Denote
\[ d = \det \text{Hess} |\vec{B}|(X_0), \quad a = \frac{1}{2b_0} \text{Hess} |\vec{B}| \cdot \vec{B})(X_0). \]

Denote by \( \lambda_0(H^h) \leq \lambda_1(H^h) \leq \lambda_2(H^h) \leq \ldots \) the eigenvalues of the operator \( H^h \).

**Theorem 1.1.** Under current assumptions, for any natural \( m \), there exist \( C_m > 0 \) and \( h_m > 0 \) such that, for any \( h \in (0, h_m) \),
\[ \lambda_m(H^h) \leq h b_0 + h^{3/2} a^{1/2} + h^2 \left[ \frac{1}{2b_0} \left( \frac{d}{2a} \right)^{1/2} (2m + 1) + \nu \right] + C_m h^{9/4}, \]
where \( \nu \) is some explicit constant (which means that it is given by a rather complicated explicit formula).

Theorem 1.1 will be based on a construction of quasimodes.

**Theorem 1.2.** Under current assumptions, for any natural \( j, k \) and \( m \), there exist \( \phi_{j,k,m}^h \in C^\infty(\Omega) \), \( C_{j,k,m} > 0 \) and \( h_{j,k,m} > 0 \) such that
\[ (\phi_{j_1,k_1,m_1}^h, \phi_{j_2,k_2,m_2}^h) = \delta_{j_1,j_2} \delta_{k_1,k_2} \delta_{m_1,m_2} + O_{j_1,k_1,m_1,j_2,k_2,m_2}(h), \]
and, for any \( h \in (0, h_{j,k,m}) \),
\[ ||H^h \phi_{j,k,m}^h - \mu_{j,k,m}^h \phi_{j,k,m}^h|| \leq C_{j,k,m} h^\frac{9}{4} ||\phi_{j,k,m}^h||, \]
where
\[ \mu_{j,k,m}^h = \mu_{j,k,m,0} h + \mu_{j,k,m,2} h^{3/2} + \mu_{j,k,m,4} h^2. \]
with
\[ \mu_{j,k,m,0} = (2k + 1)b_0, \quad \mu_{j,k,m,2} = (2j + 1)(2k + 1)^{1/2} a^{1/2}, \]
and
\[ \mu_{j,k,m,4} = \frac{1}{2b_0} \left( \frac{d}{2a} \right)^{1/2} (2m + 1)(2k + 1) + \nu(j,k), \]
where \( \nu(j,k) \) has the form
\[ \nu(j,k) = \nu_{22}(2k + 1)^2 + \nu_{11}(2j + 1)^2 + \nu_0 \]
with some explicit constants \( \nu_0, \nu_{11}, \nu_{22} \).

**Remark 1.**

It is conjectured that
\[ \lambda_m(H^h) \geq b b_0 + h^{3/2} a^{1/2} + h^2 \left[ \frac{1}{2b_0} \left( \frac{d}{2a} \right)^{1/2} (2m + 1) + \nu \right] - C_m h^{9/4}. \]

1.3. History of the problem.

May be at the level of the mathematical problem, the starting reference for the spectral analysis of self-adjoint realizations of the magnetic Schrödinger operator is the paper by Avron-Herbst-Simon [1] where the role of the module of the magnetic field in the three-dimensional case appears for the first time. Further investigations were inspired by R. Montgomery [14], who was asking “Can we hear the locus of the magnetic field” (by analogy with the celebrated question by M. Kac). In [14], this question was studied for the two-dimensional magnetic Schrödinger operator. Motivated by the question of R. Montgomery, the first author and Mohamed in [12] investigated the asymptotic behavior of the low-lying eigenvalues of the Dirichlet realization of the magnetic Schrödinger operator in the case when the magnetic
field vanishes. This study was continued more recently in [13, 7, 9, 3] (see also [8]). The case when the magnetic field never vanishes was analyzed in detail for the Dirichlet realization in the two-dimensional case in [13] and more recently in [10, 11]. Moreover, there is a big literature devoted to the spectral analysis of the Neumann realization because of its connection with problems in superconductivity (see [4] and the references therein). Finally, we do not also give a complete description of the semi-classical results obtained in the case when an electric potential $V$ is creating the main localization and refer to [5] and [2] for a presentation and references therein.

The paper is organized as follows. In Section 2, we make some simplifications of the operator $H^h$, using linear changes of variables and gauge transformations, and present the normal form appearing in the generic situation. Section 3 is devoted to the analysis of the action of the metaplectic group on our models and to the introduction of a suitable rescaling of the problem. In Section 4 we construct approximate eigenfunctions of the operator $H^h$, completing the proof of Theorem 1.2. In Section 5 we consider the case when the magnetic field is periodic. We combine the constructions of approximate eigenfunctions given in Sections 4 with the results of [6] to prove the existence of arbitrary large number of gaps in the spectrum of the periodic operator $H^h$ in the semiclassical limit.

2. Preliminaries for the proof: Normal form

In this section, we make some simplifications of the operator $H^h$, using linear changes of variables and gauge transformations.

Without loss of generality we will assume that $X_0 = (0, 0, 0)$ and the magnetic field $\vec{B}$ at $X_0$ is $(0, 0, b_0)$. Thus, we can write the Taylor expansion up to order 4 at $(0, 0, 0)$ of the magnetic field:

$$
\begin{aligned}
B_1(X) &= \ell_1 \cdot X + Q_1(X) + C_1(X) + R_1(X) + O(|X|^5), \\
B_2(X) &= \ell_2 \cdot X + Q_2(X) + C_2(X) + R_2(X) + O(|X|^5), \\
B_3(X) &= b_0 + Q_3(X) + C_3(X) + R_3(X) + O(|X|^5),
\end{aligned}
$$

where $\ell_j \in \mathbb{R}^3$, $j = 1, 2$, and $Q_j$, $C_j$ and $R_j$, $j = 1, 2, 3$, are the terms of order 2, 3 and 4, respectively.

By assumption (1.3), we have

$$
(\ell_1 \cdot X)^2 + (\ell_2 \cdot X)^2 + 2b_0Q_3(X) > 0, \quad \forall X \in \mathbb{R}^3.
$$

We also have

$$
\text{div} \, \vec{B} = \frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z} = 0,
$$

which implies

$$
\begin{aligned}
\frac{\partial (\ell_1 \cdot X)}{\partial x} + \frac{\partial (\ell_2 \cdot X)}{\partial y} = 0; \\
\frac{\partial Q_1}{\partial x} + \frac{\partial Q_2}{\partial y} + \frac{\partial Q_3}{\partial z} = 0; \\
\frac{\partial C_1}{\partial x} + \frac{\partial C_2}{\partial y} + \frac{\partial C_3}{\partial z} = 0; \\
\frac{\partial R_1}{\partial x} + \frac{\partial R_2}{\partial y} + \frac{\partial R_3}{\partial z} = 0.
\end{aligned}
$$

(2.3)
With the notation,
\[ \ell_1 = (\alpha_1, \alpha_2, \alpha_3), \quad \ell_2 = (\beta_1, \beta_2, \beta_3), \]
the first line of (2.3) gives
\[ (2.4) \quad \alpha_1 + \beta_2 = 0. \]
We can assume without loss of generality that
\[ (2.5) \quad \alpha_3 = 0. \]
Indeed, either it is already the case, either \( \beta_3 = 0 \) and we can exchange the role of \( x \) and \( y \), or \( \alpha_3 \neq 0 \) and \( \beta_3 \neq 0 \) and a rotation around the \( z \)-axis permits to obtain the cancellation of \( \alpha_3 \).

With the notation
\[ (2.6) \quad Q_1(X) = \sum_{i,j=1}^{3} a_{ij} X_i X_j, \quad Q_2(X) = \sum_{i,j=1}^{3} b_{ij} X_i X_j, \quad Q_3(X) = \sum_{i,j=1}^{3} c_{ij} X_i X_j, \]
the second line of (2.3) reads
\[ (2.7) \quad a_{1j} + b_{2j} + c_{3j} = 0, \quad j = 1, 2, 3. \]
With this notation, (1.3) is equivalent to saying that the matrix
\[ (2.8) \quad Q = \frac{1}{2} \text{Hess}(|\vec{B}|^2)(0) = b_0 \text{Hess}|\vec{B}|(0) \]
or equivalently (cf. (2.2))
\[ Q = \begin{pmatrix}
\alpha_1^2 + \beta_1^2 + 2b_0c_{11} & \alpha_1 \alpha_2 + \beta_1 \beta_2 + 2b_0c_{12} & \beta_1 \beta_3 + 2b_0c_{13} \\
\alpha_1 \alpha_2 + \beta_1 \beta_2 + 2b_0c_{21} & \alpha_2^2 + \beta_2^2 + 2b_0c_{22} & \beta_2 \beta_3 + 2b_0c_{23} \\
\beta_1 \beta_3 + 2b_0c_{31} & \beta_2 \beta_3 + 2b_0c_{32} & \beta_3^2 + 2b_0c_{33}
\end{pmatrix} \]
is positive definite. We will denote the elements of \( Q \) by \( \{q_{ij}\}_{i,j=1,2,3} \).

Without loss of generality, we can, after possibly a gauge transformation, assume that:
\[ A_1 = 0. \]
With this assumption, we get an equation for \( A_2 \) by writing:
\[ B_3 = \frac{\partial A_2}{\partial x} = b_0 + Q_3(X) + C_3(X) + R_3(X) + O(|X|^5), \]
and without loss of generality, we can, after possibly a gauge transformation assume that
\[ A_2(0, y, z) = 0. \]
This implies
\[ (2.9) \quad A_2(x, y, z) = b_0 x + \int_0^x Q_3(\hat{x}) \, d\hat{x} + \int_0^x C_3(\hat{x}) \, d\hat{x} + \int_0^x R_3(\hat{x}) \, d\hat{x} + x O(|X|^5), \]
where \( \hat{X} = (\hat{x}, y, z) \). Similarly, we get an equation for \( A_3 \) by writing:
\[ B_2(x, y, z) = -\frac{\partial A_3}{\partial x} = \ell_2 \cdot X + Q_2(X) + C_2(X) + R_2(X) + O(|X|^5). \]
and without loss of generality we can, after possibly a gauge transformation (keeping the previous properties) assume that
\[ A_3(0, 0, z) = 0. \]

This implies
\begin{equation}
A_3(x, y, z) = -\int_0^x \ell_2 \cdot \hat{X} \, d\hat{x} - \int_0^x Q_2(\hat{X}) \, d\hat{x} - \int_0^x C_2(\hat{X}) \, d\hat{x} - \int_0^x R_2(\hat{X}) \, d\hat{x} + a_3(y, z) + x \mathcal{O}(|X|^5),
\end{equation}
where \( \hat{X} = (\hat{x}, y, z) \) and \( a_3(y, z) = A_3(0, 0, z) \) satisfies
\begin{equation}
a_3(0, 0, z) = 0. \tag{2.11}
\end{equation}

From (2.9) and (2.10), we obtain, using the definition of \( B \),
\[ B_1(x, y, z) = \frac{\partial A_3}{\partial y}(X) - \frac{\partial A_2}{\partial z}(X) = -\beta_2x - \int_0^x \frac{\partial Q_3}{\partial y}(\hat{X}) \, d\hat{x} - \int_0^x \frac{\partial C_3}{\partial y}(\hat{X}) \, d\hat{x} - \int_0^x \frac{\partial R_3}{\partial y}(\hat{X}) \, d\hat{x} + \partial a_3(y, z) \, d\hat{x} - \int_0^x \frac{\partial R_3}{\partial z}(\hat{X}) \, d\hat{x} + x \mathcal{O}(|X|^4), \]
and using (2.9), we obtain
\[ B_1(x, y, z) = a_1x + Q_1(X) - Q_1(0, y, z) + C_1(X) - C_1(0, y, z) + R_1(X) - R_1(0, y, z) + \partial a_3(y, z) \, dx + \mathcal{O}(|X|^5). \]

On the other hand, comparing with (2.11), we obtain
\[ \frac{\partial a_3}{\partial y}(y, z) = a_2y + Q_1(0, y, z) + C_1(0, y, z) + R_1(0, y, z) + \mathcal{O}(|y|^5 + |z|^5). \]

Thus, we obtain, having (2.11) in mind,
\begin{equation}
a_3(y, z) = \frac{1}{2}a_2y^2 + \int_0^y Q_1(0, \hat{y}, z) \, d\hat{y} + \int_0^y C_1(0, \hat{y}, z) \, d\hat{y} + \int_0^y R_1(0, \hat{y}, z) \, d\hat{y} + y \mathcal{O}(|y|^5 + |z|^5). \tag{2.12}
\end{equation}

Coming back to (2.9), we have
\[ A_2(x, y, z) = k_0 x + \frac{1}{3}c_{11}x^3 + c_{12}x^2z + c_{13}x^2y + c_{22}xy^2 + 2c_{23}xyz + c_{31}xz^2 + \int_0^x C_3(\hat{X}) \, d\hat{x} + \int_0^x R_3(\hat{X}) \, d\hat{x} + x \mathcal{O}(|X|^5). \]

Similarly, coming back to (2.10) and using (2.12), we obtain
\[ A_3(x, y, z) = -\frac{1}{2}\beta_1x^2 - \beta_2xy - \beta_3xz + \frac{1}{2}a_2y^2 + \int_0^y C_1(0, \hat{y}, z) \, d\hat{y} + \int_0^y R_1(0, \hat{y}, z) \, d\hat{y}. \]
Let us make a change of variables

\[ \begin{aligned}
- \frac{1}{3} b_{11} x^3 - b_{12} x^2 y - b_{13} x^2 z - b_{22} x y^2 - 2 b_{23} x y z - b_{33} x z^2 \\
+ \frac{1}{3} a_{22} y^3 + a_{23} y^2 z + a_{33} y z^2 - \int_0^x C_2(\hat{X}) \, d\hat{x} - \int_0^x R_2(\hat{X}) \, d\hat{x} \\
+ x \mathcal{O}(|X|^5) + y \mathcal{O}(|y|^5 + |z|^5).
\end{aligned} \]

Finally, what we have got is the following normal form:

**Proposition 2.1.** Using only linear change of variables and gauge transformations, we can assume that

\[ A_1(x, y, z) = 0, \quad A_2(0, y, z) = 0, \quad A_3(0, 0, z) = 0. \]

Moreover:

\[ \begin{aligned}
A_2(x, y, z) &= b_0 x + \frac{1}{3} c_{11} x^3 + c_{12} x^2 y + c_{13} x^2 z + c_{22} x y^2 + 2 c_{23} x y z + c_{33} x z^2 \\
&\quad + \int_0^x C_3(\hat{X}) \, d\hat{x} + \int_0^x R_3(\hat{X}) \, d\hat{x} + x \mathcal{O}(|X|^5), \\
A_3(x, y, z) &= -\frac{1}{2} \beta_1 x^2 - \beta_2 x y - \beta_3 x z \\
&\quad + \frac{1}{2} a_2 y^2 + \int_0^y C_1(0, \hat{y}, z) \, d\hat{y} + \int_0^y R_1(0, \hat{y}, z) \, d\hat{y} \\
&\quad - \frac{1}{3} b_{11} x^3 - b_{12} x^2 y - b_{13} x^2 z - b_{22} x y^2 - 2 b_{23} x y z - b_{33} x z^2 \\
&\quad + \frac{1}{3} a_{22} y^3 + a_{23} y^2 z + a_{33} y z^2 - \int_0^x C_2(\hat{X}) \, d\hat{x} - \int_0^x R_2(\hat{X}) \, d\hat{x} \\
&\quad + x \mathcal{O}(|X|^5) + y \mathcal{O}(|y|^5 + |z|^5).
\end{aligned} \]

**3. Preliminaries for the proof: Metaplectic transformations**

In this section, we make further simplifications of the operator $H^h$, using metaplectic transformations. First, we introduce a suitable $h$-dependent rescaling of the problem and expand the resulting operator in fractional powers of $h$.

**3.1. Rescaling.**

Let us make a change of variables

\[ x = h^\frac{1}{4} \hat{x}, \quad y = h^\frac{1}{4} \hat{y}, \quad z = h^\frac{1}{4} \hat{z}. \]

We will only apply our operator $H^h$ on functions which are a product of cut-off functions localized in a neighborhood of $(0, 0, 0) \in \mathbb{R}^3$ with linear combinations of terms like

\[ h^\nu w(h^{-1/2} x, h^{-1/2} y, h^{-1/4} z), \]

with $w$ in $\mathcal{S}(\mathbb{R}^3)$. These functions are consequently $\mathcal{O}(h^\infty)$ outside a fixed neighborhood of $(0, 0, 0)$. We will start by doing the computations formally in the sense that everything is determined modulo $\mathcal{O}(h^\infty)$, and any smooth function will be replaced by its Taylor’s expansion at $(0, 0, 0)$. We introduce

\[ C_1(X) = \sum_{i \leq j \leq k} p_{ijk} X_i X_j X_k, \]
\[ C_2(X) = \sum_{i \leq j \leq k} q_{ijk} X_i X_j X_k, \]
$C_3(X) = \sum_{i \leq j \leq k} r_{ijk} X_i X_j X_k$.

We will also need the coefficient of $z^4$ in $R_j$:

$$R_j(X) = \delta_j z^4 + \ldots, \quad j = 1, 2, 3.$$  

We have

$$H^h = hP^h,$$

where

$$P^h = D_2^2 + \left(D_y - h^{-1/2} A_2(h^\frac{r}{x}, h^\frac{y}{y}, h^\frac{z}{z})\right)^2 + \left(\frac{h^{1/4} D_z - h^{-1/2} A_3(h^\frac{r}{x}, h^\frac{y}{y}, h^\frac{z}{z})\right)^2.$$

Now, using (2.13), we expand the operator in fractional powers of $h$. First, we compute

$$h^{-1/2} A_2(h^\frac{r}{x}, h^\frac{y}{y}, h^\frac{z}{z})$$

$$= b_0 x + h^x c_{33} x \bar{z}^2 + h^x [c_{13} x^2 \bar{z} + 2c_{23} x y \bar{z} + r_{333} x \bar{z}^3]$$

$$+ h \left[\frac{1}{3} c_{11} x^3 + c_{12} x^2 y + c_{22} x y^2 + \frac{1}{2} r_{133} x^2 \bar{z}^2 + r_{233} x y \bar{z}^2 + \delta_y \bar{z}^4\right]$$

$$+ O(h^{5/4}),$$

and

$$h^{-1/2} A_3(h^\frac{r}{x}, h^\frac{y}{y}, h^\frac{z}{z})$$

$$= -h^x \beta_3 x \bar{z} + h^x \left[-\frac{1}{2} \beta_1 x^2 - \beta_2 x y + \frac{1}{2} \gamma x^2 + a_{33} y \bar{z}^2\right]$$

$$+ h^x \left[2b_{13} x \bar{z} - 2b_{23} x y \bar{z} + a_{23} y \bar{z}^2 - q_{333} x \bar{z}^3 + p_{333} y \bar{z}^3\right] + O(h).$$

Using these formulæ and omitting the tilda’s, we obtain (in the dilated coordinates)

$$P^h = P_0 + h^x P_1 + h^y P_2 + h^z P_3 + h P_4 + O(h^{5/4}),$$

where

$$P_0 = D_2^2 + (D_y - b_{0x})^2,$$

$$P_1 = 0,$$

$$P_2 = -2 c_{33} x z^2 (D_y - b_{0x}) + (D_z + \beta_3 z)^2,$$

$$P_3 = -2 c_{13} z (D_y - b_{0x}) - 2c_{23} [(D_y - b_{0x}) x y z + x y z (D_y - b_{0x})]$$

$$- 2 r_{333} x z^3 (D_y - b_{0x}) + (\beta_1 x^2 + 2 \beta_2 x y - \alpha_2 y^2) (D_z + \beta_3 z)$$

$$+ (b_{33} x - a_{33} y) ((D_z + \beta_3 z) x^2 + x z^2 (D_z + \beta_3 z)),$$

$$P_4 = -(D_y - b_{0x}) \left[\frac{1}{3} c_{11} x^3 + c_{12} x^2 y + c_{22} x y^2 + \frac{1}{2} r_{133} x^2 \bar{z}^2 + r_{233} x y \bar{z}^2 + \delta z x^4\right]$$

$$- \left[\frac{1}{3} c_{11} x^3 + c_{12} x^2 y + c_{22} x y^2 + \frac{1}{2} r_{133} x^2 \bar{z}^2 + r_{233} x y \bar{z}^2 + \delta z x^4\right] (D_y - b_{0x})$$

$$+ c_{33} x^4 + (D_z + \beta_3 z) [b_{13} x^2 z + 2b_{23} x y z - a_{23} y^2 z + q_{333} x \bar{z}^3 - p_{333} y \bar{z}^3]$$

$$+ [b_{13} x^2 z + 2b_{23} x y z - a_{23} y^2 z + q_{333} x \bar{z}^3 - p_{333} y \bar{z}^3] (D_z + \beta_3 z)$$
+ \left( \frac{1}{2} \beta_1 x^2 + \beta_2 xy - \frac{1}{2} \alpha_2 y^2 + b_{33} x z^2 - a_{33} y z^2 \right)^2.

3.2. Partial Fourier transform and gauge transform.

Now we are going to apply some metaplectic transformations. We recall that these
transformations are unitary and therefore preserve the spectrum of the operators.

First, we make a partial Fourier transform in $y$: 

\[ \hat{P}_0 = D_x^2 + (\eta - b_0x)^2, \]
\[ \hat{P}_1 = 0, \]
\[ \hat{P}_2 = -2c_{33} x z (\eta - b_0 x) + (D_z + \beta_3 x z)^2, \]
\[ \hat{P}_3 = -2c_{13} x^2 z (\eta - b_0 x) - 2c_{23}\{(\eta - b_0 x) x (-D_\eta) z + x (-D_\eta) z (\eta - b_0 x)\}
- 2r_{333} x z^3 (\eta - b_0 x) + (\beta_1 x^2 + 2\beta_2 x (-D_\eta) - \alpha_2 D_\eta^2) (D_z + \beta_3 x z)
+ (b_{33} x + a_{33} D_\eta) ((D_z + \beta_3 x z)^2 + z^2 (D_z + \beta_3 x z)), \]
\[ \hat{P}_4 = -(\eta - b_0 x) \left[ \frac{1}{3} c_{11} x^3 + c_{12} x^2 (-D_\eta) + c_{22} x D_\eta^2 + \frac{1}{2} r_{133} x^2 z^2
+ r_{233} x (-D_\eta)^2 + \delta_3 x z^4 \right] - \left[ \frac{1}{3} c_{11} x^3 + c_{12} x^2 (-D_\eta) + c_{22} x D_\eta^2 + \frac{1}{2} r_{133} x^2 z^2
+ r_{233} x (-D_\eta)^2 + \delta_3 x z^4 \right] (\eta - b_0 x)^2 + \frac{1}{2} r_{133} x^2 z^4
+ (D_z + \beta_3 x z) \left[ b_{13} x^2 z + 2 b_{23} x (-D_\eta) z - a_{23} D_\eta^2 z + q_{333} x z^3 - p_{333} (-D_\eta)^2 z^3 \right]
+ \left[ b_{13} x^2 z + 2 b_{23} x (-D_\eta) z - a_{23} D_\eta^2 z + q_{333} x z^3 - p_{333} (-D_\eta)^2 z^3 \right] (D_z + \beta_3 x z)
+ \left( \frac{1}{2} \beta_1 x^2 + \beta_2 x (-D_\eta) - \frac{1}{2} \alpha_2 D_\eta^2 + b_{33} x z^2 + a_{33} D_\eta z^2 \right)^2. \]

Next, we make a translation $\tilde{x} = x - \frac{\eta}{b_0}$ (and forget the "check"):

\[ \tilde{P}_0 = D_x^2 + b_0^2 x^2, \]
\[ \tilde{P}_1 = 0, \]
\[ \tilde{P}_2 = 2 c_{33} (x + \frac{\eta}{b_0}) z^2 b_0 x + (D_z + \beta_3 (x + \frac{\eta}{b_0}) z)^2, \]
\[ \tilde{P}_3 = 2 c_{13} (x + \frac{\eta}{b_0}) z^2 (b_0 x) + 2 c_{23} \left[ (b_0 x) (x + \frac{\eta}{b_0}) \left( \frac{1}{b_0} D_x - D_\eta \right) \right]
+ (x + \frac{\eta}{b_0}) \left( \frac{1}{b_0} D_x - D_\eta \right) z (b_0 x)
+ 2 r_{333} (x + \frac{\eta}{b_0}) z^3 (b_0 x)
+ \left( \beta_1 (x + \frac{\eta}{b_0})^2 + 2 \beta_2 (x + \frac{\eta}{b_0}) \left( \frac{1}{b_0} D_x - D_\eta \right) - \alpha_2 \left( \frac{1}{b_0} D_x - D_\eta \right)^2 \right) \times
(D_z + \beta_3 (x + \frac{\eta}{b_0}) z) + \left( b_{33} (x + \frac{\eta}{b_0}) z - a_{33} \left( \frac{1}{b_0} D_x - D_\eta \right) \right) \times
\left( (D_z + \beta_3 (x + \frac{\eta}{b_0}) z)^2 + z^2 (D_z + \beta_3 (x + \frac{\eta}{b_0}) z) \right), \]
\[ \tilde{P}_4 = b_0 x \left[ \frac{1}{3} c_{11} (x + \frac{\eta}{b_0})^3 + c_{12} (x + \frac{\eta}{b_0})^2 \left( \frac{1}{b_0} D_x - D_\eta \right)
+ c_{22} (x + \frac{\eta}{b_0}) \left( \frac{1}{b_0} D_x - D_\eta \right)^2 + \frac{1}{2} r_{133} (x + \frac{\eta}{b_0}) z^2 \right] \]
Finally, we make a gauge transformation (by \(\exp(-i\frac{\beta_i}{b_0} \eta z^2)\)):

\[
\begin{align*}
\tilde{P}_0 &= D_x^2 + b_0^2 x^2, \\
\tilde{P}_1 &= 0, \\
\tilde{P}_2 &= 2b_0 c_{333}(x + \frac{\eta}{b_0}) z^2 + (D_z + \beta_3 x z)^2, \\
\tilde{P}_3 &= 2b_0 c_{133}(x + \frac{\eta}{b_0}) z^2 + 2c_{233}(b_0 x)(x + \frac{\eta}{b_0}) \left( \frac{1}{b_0} D_x + \frac{\beta_3}{2b_0} z^2 - D_\eta \right) z \\
&+ (x + \frac{\eta}{b_0}) \left( \frac{1}{b_0} D_x + \frac{\beta_3}{2b_0} z^2 - D_\eta \right) z(b_0 x) + 2r_{333}(x + \frac{\eta}{b_0}) z^3(b_0 x) \\
&+ \left( \beta_1(x + \frac{\eta}{b_0}) + 2\beta_2(x + \frac{\eta}{b_0}) \frac{1}{b_0} D_x + \frac{\beta_3}{2b_0} z^2 - D_\eta \right) \\
&- \alpha_2 \left( \frac{1}{b_0} D_x + \frac{\beta_3}{2b_0} z^2 - D_\eta \right)^2 (D_z + \beta_3 x z) \\
&+ \left( b_{333}(x + \frac{\eta}{b_0}) - a_{333} \left( \frac{1}{b_0} D_x + \frac{\beta_3}{2b_0} z^2 - D_\eta \right) \right) \times (D_z + \beta_3 x z) z^2 + z^2(D_z + \beta_3 x z)), \\
\tilde{P}_4 &= b_0 x \left( \frac{1}{3} c_{111}(x + \frac{\eta}{b_0})^3 + c_{122}(x + \frac{\eta}{b_0})^2 \left( \frac{1}{b_0} D_x + \frac{\beta_3}{2b_0} z^2 - D_\eta \right) \right)
\end{align*}
\]
expansion of \( P \) construction. We will construct an approximate eigenfunction of the operator \( u \).

The first equations read (here we omit \( \tilde{\lambda} \)):

\[
\begin{align*}
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(4.1) & \quad (b_0 x) \\
(4.2) & \quad (b_0 x)
\end{align*}
\]

We will express the cancelation of the coefficients of \( \alpha \). In this section, we complete the proof of Theorem 1.2. We start with a formal construction. We will construct an approximate eigenfunction of the operator \( P^h \) in the form

\[
u^h(x, \eta, z) \sim \sum_{j \in \mathbb{N}} h^j u_j(x, \eta, z),
\]

corresponding to an approximate eigenvalue

\[
\lambda^h \sim \sum_{j \in \mathbb{N}} h^j \lambda_j.
\]

We will express the cancelation of the coefficients of \( h^\ell/4 \) \( (\ell \in \mathbb{N}) \) in the formal expansion of \( (P^h - \lambda^h) u^h \), starting with \( \ell = 0 \).

4. Construction of approximate eigenfunctions

In this section, we complete the proof of Theorem 1.2. We start with a formal construction. We will construct an approximate eigenfunction of the operator \( P^h \) in the form

\[
u^h(x, \eta, z) \sim \sum_{j \in \mathbb{N}} h^j u_j(x, \eta, z),
\]

corresponding to an approximate eigenvalue

\[
\lambda^h \sim \sum_{j \in \mathbb{N}} h^j \lambda_j.
\]

We will express the cancelation of the coefficients of \( h^\ell/4 \) \( (\ell \in \mathbb{N}) \) in the formal expansion of \( (P^h - \lambda^h) u^h \), starting with \( \ell = 0 \).

4.1. The first equations.

The first equations read (here we omit \( \tilde{\lambda} \)):

\[
\begin{align*}
(4.1) & \quad (P_0 - \lambda_0) u_0 = 0, \\
(4.2) & \quad (P_0 - \lambda_0) u_1 = \lambda_1 u_0,
\end{align*}
\]
(4.3) \((P_0 - \lambda_0)u_2 = -P_2 u_0 + \lambda_1 u_1 + \lambda_2 u_0\),
(4.4) \((P_0 - \lambda_0)u_3 = -P_3 u_0 - P_2 u_1 + \lambda_1 u_2 + \lambda_2 u_1 + \lambda_3 u_0\),
(4.5) \((P_0 - \lambda_0)u_4 = -P_4 u_0 - P_3 u_1 - P_2 u_2 + \lambda_1 u_3 + \lambda_2 u_2 + \lambda_3 u_1 + \lambda_4 u_0\),
and we now show how to solve them successively.

4.2. **Main equation.**

The first equation (4.1) reads
\[
(D_x^2 + b_0^2 x^2 - \lambda_0)u_0 = 0.
\]
We arrive at the eigenvalue problem for the harmonic oscillator
\[
\mathfrak{h}_1 := D_x^2 + b_0^2 x^2.
\]
Recall that its eigenvalues have the form
\[
\text{Sp}(\mathfrak{h}_1) = \{\mu_k = (2k + 1)b_0 : k \in \mathbb{N}\}.
\]
An eigenfunction of \(\mathfrak{h}_1\) associated with the eigenvalue \(\mu_k\) is given by
\[
\phi_k(x) = b_0^{1/2} H_k(b_0 x) e^{-b_0 x^2/2},
\]
where \(H_k\) is the Hermite polynomial:
\[
H_k(t) = (-1)^k e^{t^2} \frac{d^k}{dt^k}(e^{-t^2}).
\]
The norm of \(\phi_k\) in \(L^2(\mathbb{R}, dx)\) equals the norm of \(H_k\) in \(L^2(\mathbb{R}, e^{-t^2} dt)\), which is given by
\[
\|H_k\| = \sqrt{2^k k! \sqrt{\pi}}.
\]
Some well-known formulae, concerning to the Hermite functions, which we need in the paper, are gathered in the appendix.

We will take \(\lambda_0\) as the \(k\)-th eigenvalue of \(\mathfrak{h}_1\)
\[
\lambda_0 = (2k + 1)b_0
\]
and will look for a solution \(u_0\) to (4.1) in the form
\[
u_0(x, \eta, z) = \phi_k(x) \chi_0(\eta) \Psi_0(z),
\]
with \(\chi_0\) and \(\Psi_0\) in \(\mathcal{S}(\mathbb{R})\) to be determined in the next steps.

Remark that what we do at this step corresponds to the so-called reduction to the \(k\)-th Landau-Level.

4.3. **Second equation : coefficient of \(\mathfrak{h}_1^2\).**

We look for a solution \(u_1\) to (4.1) in the form
\[
u_1(x, \eta, z) = \phi_k(x) v_1(\eta, z),
\]
with \(v_1 \in \mathcal{S}(\mathbb{R}^2)\) to be determined in the next steps, and take
\[
\lambda_1 = 0.
\]
4.4. Third equation: coefficient of $h^2$.

In order to solve the equation (4.3), we find as a necessary condition that, for any $z$ we should have,

$$\tag{4.11} - \frac{1}{\|H_k\|^2} (P_2 u_0, \phi_k)_x + \lambda_2 \chi_0(\eta) \Psi_0(z) = 0,$$

with $(\cdot, \cdot)_x$ denoting the scalar product in $L^2(\mathbb{R}_x)$. Using that $\phi_k^2$ is even, we obtain that

$$\tag{4.12} \frac{1}{\|H_k\|^2} (P_2 u_0, \phi_k)_x = \frac{1}{\|H_k\|^2} ([2b_0 \beta_3 x^2 z^2 + D_z^2 + \beta_3 x^2 z^2]u_0, \phi_k)_x = \chi_0(\eta) \mathfrak{b}_3 \Psi_0(z),$$

where

$$\tag{4.13} \mathfrak{b}_3 := D_z^2 + \frac{2k + 1}{2b_0} (\beta_3^2 + 2c_{33} b_0) z^2.$$

Thus, the condition (4.11) can be rewritten as

$$\tag{4.14} \mathfrak{b}_3 \Psi_0 = \lambda_2 \Psi_0.$$

This determines $\lambda_2$ and $\Psi_0$. We take $\lambda_2$ as the $j$-th eigenvalue eigenvalue of $\mathfrak{b}_3$:

$$\tag{4.15} \lambda_2 = (2j + 1)(2k + 1)^{1/2} \sqrt{\frac{\beta_3^2 + 2c_{33} b_0}{2b_0}} = (2j + 1) \Lambda_2,$$

where

$$\Lambda_2 := (2k + 1)^{1/2} \sqrt{\frac{\beta_3^2 + 2c_{33} b_0}{2b_0}} = (2k + 1)^{1/2} \sqrt{\frac{q_{33}}{2b_0}},$$

and $\Psi_0$ as the corresponding eigenfunction:

$$\Psi_0(z) = \psi_j(z) := \Lambda_2^{1/2} H_j(\Lambda_2^{1/2} z) e^{-\Lambda_2 z^2/2}.$$

Observe that

$$q_{33} = \frac{1}{2} \frac{\beta_3^2 |B|^2}{\partial z^2} (X_0) = 2b_0 a.$$

Thus we obtain

$$\lambda_2 = (2j + 1)(2k + 1)^{1/2} a^{1/2}.$$

To find $u_2$, from (4.3) we obtain that

$$\mathfrak{b}_1 u_2 - \lambda_0 u_2 = -\phi_k(x) \chi_0(\eta) D_z^2 \psi_j(z) - x \phi_k(x) \chi_0(\eta) (2c_{33} \beta_3 z^2 + \beta_3 (D_z^2 + z D_z)) \psi_j(z) - (\beta_3^2 + 2b_0 c_{33}) x^2 \phi_k(x) \chi_0(\eta) \psi_j(z) + \lambda_2 \phi_k(x) \chi_0(\eta) \psi_j(z)$$

$$= -\frac{1}{2b_0^{1/2}} (\phi_{k+1}(x) + 2k \phi_{k-1}(x)) \chi_0(\eta) (2c_{33} \beta_3 z^2 + \beta_3 (D_z^2 + z D_z)) \psi_j(z)$$

$$= -\frac{1}{4b_0} (\beta_3^2 + 2b_0 c_{33}) (\phi_{k+2}(x) + 4k(k-1) \phi_{k-2}(x)) \chi_0(\eta) z^2 \psi_j(z).$$

Therefore, using the identity

$$\mathfrak{b}_1 - \lambda_0 \phi_{k+\ell} = 2\ell b_0 \phi_{k+\ell},$$

we get

$$\tag{4.16} \phi_k(x) \chi_0(\eta) u_2(x, \eta, z) = U_2(x, \eta, z) + \phi_k(x) v_2(\eta, z).$$
where

\[ U_2(x, \eta, z) = - \frac{1}{4b_0^{1/2}} (\phi_{k+1}(x) - 2k\phi_{k-1}(x)) \chi_0(\eta)(2c_33\eta z^2 + \beta_3(D_z z + zD_z))\psi_j(z) \]

and \( \psi_j \in S(\mathbb{R}^2) \) has to be determined in the next steps.

4.5. **Fourth equation: coefficient of \( h^{\frac{1}{2}} \).**

In order to solve the equation (4.4), we find as a necessary condition that, for any \((z, \eta)\) we should have

\[(4.17) \quad \frac{1}{||H_k||^2} \langle P_3^+ u_0, \phi_k \rangle_x = \frac{1}{||H_k||^2} \langle P_2 u_1, \phi_k \rangle_x + \lambda_2 v_1(z, \eta) + \lambda_3 \chi_0(\eta)\psi_j(z) = 0. \]

For the first term in the left-hand side of (4.17), since \( \phi_k \) is even, we have

\[ \frac{1}{||H_k||^2} \langle P_3^+ u_0, \phi_k \rangle_x = \frac{1}{||H_k||^2} \langle P_3^+ u_0, \phi_k \rangle_x, \]

where \( P_3^+ \) is the even part of \( P_3 \) as a differential operator in \( x \). As a sum of homogeneous components in \((\eta, D_\eta)\), the operator \( P_3^+ \) is written as

\[ P_3^+ = P_3^{(2)} + P_3^{(1)} + P_3^{(0)}, \]

where

\[ P_3^{(2)} = \left( \beta_1 \frac{\eta^2}{b_0} - 2\beta_2 \frac{\eta}{b_0} D_\eta - \alpha_2 D_\eta^2 \right) D_z, \]

\[ P_3^{(1)} = 2q_33x^2 \frac{\eta}{b_0} + q_{23} \left( \frac{\eta}{b_0} (xD_x + D_x x) - 2x^2 D_\eta \right) + \alpha_2 \beta_3 \frac{1}{b_0} (xD_x + D_x x) D_\eta \]

\[ + \left[ \frac{3\beta_3}{2b_0^2} \eta + \frac{\alpha_2 \beta_3}{2b_0} D_\eta + b_{33} \frac{\eta}{b_0} + \alpha_3 D_\eta \right] (z^2D_z + D_z z^2), \]

\[ P_3^{(0)} = \frac{q_{33}3}{2b_0} x^2 z^3 + 2(\beta_3 q_{33} + b_3P_{333})x^2 z^3 + (\beta_1 x^2 + 2\beta_2 \frac{1}{b_0} xD_x - \alpha_2 \frac{1}{b_0} D_x^2)D_z \]

\[ - \left( \frac{\alpha_2 \beta_3}{2b_0^2} + \frac{\beta_3 a_{33}}{2b_0} \right) (D_z z^4 + z^4 D_z) - \left[ \frac{\alpha_2 \beta_3^2}{2b_0^2} + \frac{\beta_3 \alpha_3}{2b_0} \right] (xD_x + D_x x) z^3. \]

Now we compute:

\[ (4.18) \quad \frac{1}{||H_k||^2} \langle P_3^{(2)} u_0, \phi_k \rangle_x \]

\[ = \frac{1}{||H_k||^2} \langle P_3^{(2)} u_0, \phi_k \rangle_x + \frac{1}{||H_k||^2} \langle P_3^{(1)} u_0, \phi_k \rangle_x + \frac{1}{||H_k||^2} \langle P_3^{(0)} u_0, \phi_k \rangle_x, \]

where

\[ \frac{1}{||H_k||^2} \langle P_3^{(2)} u_0, \phi_k \rangle_x \]

\[ = \left( \beta_1 \frac{\eta^2}{b_0} - 2\beta_2 \frac{\eta}{b_0} D_\eta - \alpha_2 D_\eta^2 \right) \chi_0(\eta)D_z \psi_j(z) \]

\[ = \left( \beta_1 \frac{\eta^2}{b_0} - 2\beta_2 \frac{\eta}{b_0} D_\eta - \alpha_2 D_\eta^2 \right) \chi_0(\eta) \frac{1}{2} \Lambda_{\frac{1}{2}}^{1/2} i (\psi_{j+1}(z) - 2j \psi_{j-1}(z)). \]
\[
\frac{1}{\|H_k\|^2} \langle P_3^{(1)} u_0, \phi_k \rangle_x \\
= (2k + 1) \left[ q_{33} \frac{\eta}{b_0} - q_{23} \frac{1}{b_0} D_\eta \right] \chi_0(\eta) z \psi_j(z) \\
+ \left[ \frac{\beta_2 \beta_3}{2b_0} \eta + \frac{\alpha_2 \beta_3}{2b_0} D_\eta \right] + b_{33} \frac{\eta}{b_0} + a_{33} D_\eta \right] \chi_0(\eta) (z^2 D_z + D_z z^2) \psi_j(z) \\
= (2k + 1) \left[ q_{33} \frac{\eta}{b_0} - q_{23} \frac{1}{b_0} D_\eta \right] \chi_0(\eta) \frac{1}{2\Lambda_z^2} (\psi_{j+1}(z) + 2j \psi_{j-1}(z)) \\
+ \left[ \frac{\beta_2 \beta_3}{2b_0} \eta + \frac{\alpha_2 \beta_3}{2b_0} D_\eta \right] + b_{33} \frac{\eta}{b_0} + a_{33} D_\eta \right] \chi_0(\eta) \frac{1}{4\Lambda_z^2} (\psi_{j+3}(z) + \\
+ (2j + 2) \psi_{j+1}(z) - 4j^2 \psi_{j-1}(z) - 8j(j - 1)(j - 2) \psi_{j-3}(z)),
\]
Therefore, the condition (4.22) can be rewritten as an equation with respect to $\lambda_3 = 0$.

Using (4.18) and the identity

$$ (h_3 - \lambda_2)\psi_{j+\ell} = 2\ell\Lambda_2\psi_{j+\ell}, $$

we find a solution to (4.19) in the form

$$ v_1 = v_1^{(2)} + v_1^{(1)} + v_1^{(0)}, $$

where

$$ v_1^{(2)} = -\left(\frac{\beta_1}{b_0} - 2\beta_2\frac{\eta}{b_0}D\eta - \alpha_2 D^2\eta\right)\chi_0(\eta)\frac{1}{4\Lambda_2}\left[\psi_{j+1}(z) + 2j\psi_{j-1}(z)\right], $$

$$ v_1^{(1)} = -\left[q_{23}\frac{\eta}{b_0} - q_{32}\frac{1}{b_0}D\eta\right]\chi_0(\eta)\frac{2k + 1}{4\Lambda_2}\left[\psi_{j+1}(z) - 2j\psi_{j-1}(z)\right] $$

$$ - \frac{\beta_2}{2b_0}\eta + \frac{\alpha_2\beta_2}{2b_0}D\eta + b_{23}\frac{\eta}{b_0} + a_{33}D\eta\right]\chi_0(\eta)\frac{1}{24\Lambda_2}\left[\psi_{j+3}(z) + (6j + 6)\psi_{j+1}(z) + 12j^2\psi_{j-1}(z) + 8j(j - 1)(j - 2)\psi_{j-3}(z)\right], $$

$$ v_1^{(0)} = -\left[q_{23}\frac{\eta}{b_0} - q_{32}\frac{1}{b_0}D\eta\right]\chi_0(\eta)\frac{2k + 1}{4\Lambda_2}\left[\psi_{j+1}(z) + 2j\psi_{j-1}(z)\right] $$

$$ + (18j + 18)\psi_{j+1}(z) - 36j^2\psi_{j-1}(z) - 8j(j - 1)(j - 2)\psi_{j-3}(z) $$

$$ - (\beta_1\frac{2k}{b_0} + \beta_2\frac{b_{32}}{b_0} - \alpha_2\frac{2k + 1}{b_0})\chi_0(\eta)\frac{1}{4\Lambda_2}\left[\psi_{j+1}(z) + 2j\psi_{j-1}(z)\right] $$

$$ + \frac{\alpha_2\beta_2}{8b_0}\eta + \frac{\beta_3 a_{33}}{2b_0}\chi_0(\eta)\frac{1}{16\Lambda_2}\left[\frac{1}{10}\psi_{j+3}(z) + (j + 2)\psi_{j+3}(z) + 2(2j^2 + 4j + 3)\psi_{j+1}(z) + 4(2j^2 + 1)j\psi_{j-1}(z) $$

$$ + 8j(j - 1)^2(j - 2)\psi_{j-3}(z) + \frac{16j(j - 1)(j - 2)(j - 3)(j - 4)}{5}\psi_{j-5}(z)\right]. $$

With such a choice of $v_1$, the orthogonality condition (4.17) holds, and a solution $u_3$ to (4.2) exists. Substituting (4.21) in (4.9), we find $u_1$.

4.6. **Fifth equation: coefficient of $h$.**

The necessary condition for solvability of (4.5) reads

$$ (4.22) \quad -\frac{1}{\|H_k\|^2}\langle P_u u_0, \phi_k \rangle_x - \frac{1}{\|H_k\|^2}\langle P_3 u_3, \phi_k \rangle_x = \frac{1}{\|H_k\|^2}\langle P_2 u_2, \phi_k \rangle_x + \lambda_2 \langle u_2, \phi_k \rangle_x + \lambda_4 \chi_0(\eta)\psi_j(z) = 0. $$

Using (4.16) and (4.12), we get

$$ \frac{1}{\|H_k\|^2}\langle P_2 \phi_k \rangle_x = \frac{1}{\|H_k\|^2}\langle P_2 U_\eta, \phi_k \rangle_x + h_3 v_2(\eta, z), $$

and

$$ \frac{1}{\|H_k\|^2}\langle u_2, \phi_k \rangle_x = v_2(\eta, z). $$

Therefore, the condition (4.22) can be rewritten as an equation with respect to $v_2$ in the form
\[ (4.23) \quad h_3 - \lambda_2 v_2(\eta, z) = -\frac{1}{\|H_k\|^2} (P_4 u_0, \phi_k) x - \frac{1}{\|H_k\|^2} (P_3 u_1, \phi_k) x - \frac{1}{\|H_k\|^2} (P_2 U_2, \phi_k) x + \lambda_4 \chi_0(\eta) \psi_j(z). \]

The necessary condition for solvability of (4.23) is given by
\[ (4.24) \quad -\frac{1}{\|H_k\|^2} (P_4 u_0, \phi_k \psi_j)_{x, z} - \frac{1}{\|H_k\|^2} (P_3 u_1, \phi_k \psi_j)_{x, z} - \frac{1}{\|H_k\|^2} (P_2 U_2, \phi_k \psi_j)_{x, z} + \lambda_4 \chi_0(\eta) = 0. \]

**Lemma 4.1.** The identity (4.24) has the form
\[ (4.25) \quad \frac{2k + 1}{2b_0} q_{22} q_{33} - q_{23}^2 D_\eta^2 \chi_0(\eta) - 2k + 1 \frac{q_{12} q_{33} - q_{13} q_{23}}{q_{33}} (\eta D_\eta + D_\eta^2) \chi_0(\eta) \]
\[ + \frac{2k + 1}{2b_0} q_{11} q_{33} - q_{13}^2 D_\eta^2 \chi_0(\eta) + \alpha D_\eta \chi_0(\eta) + \beta \eta \chi_0(\eta) + \gamma \chi_0(\eta) - \lambda_4 \chi_0(\eta) = 0, \]
where \( \alpha = \alpha(j, k), \beta = \beta(j, k), \gamma = \gamma(j, k) \) are of the form
\[ (4.26) \quad \alpha = \alpha(2k + 1)^1/2(2j + 1), \quad \beta = \beta(2k + 1)^1/2(2j + 1), \]
and
\[ (4.27) \quad \gamma = (2j + 1)^2 + \gamma_2(2k + 1)^2 + \gamma_0, \]
where \( \alpha, \beta, \gamma_0, \gamma_1 \) and \( \gamma_2 \) are some explicit constants.

The proof of Lemma 4.1 is given by a long routine computation and will be omitted.

By Lemma 4.1, \( \lambda_4 \) is an eigenvalue of the second order differential operator
\[ h_5 := \frac{2k + 1}{2b_0} q_{22} q_{33} - q_{23}^2 D_\eta^2 - \frac{2k + 1}{2b_0} q_{12} q_{33} - q_{13} q_{23} \frac{q_{33}^2}{q_{33}} (\eta D_\eta + D_\eta^2) \]
\[ + \frac{2k + 1}{2b_0} q_{11} q_{33} - q_{13}^2 D_\eta^2 + \alpha D_\eta + \beta \eta + \gamma. \]

This operator is a globally elliptic operator in \( \mathbb{R} \). Therefore, it has discrete spectrum in \( L^2(\mathbb{R}) \), which is described by the next lemma.

**Lemma 4.2.** The eigenvalues of the operator
\[ H = AD_\eta^2 + B(\eta D_\eta + D_\eta^2) + C \eta^2 + \alpha D_\eta + \beta \eta + \gamma, \quad AC - B^2 > 0, \]
are given by
\[ \lambda_m = (2m + 1) \sqrt{AC - B^2} + \gamma - \frac{1}{4(AC - B^2)} (C \alpha^2 - 2B \alpha \beta + A \beta^2), \quad m \in \mathbb{N}. \]

**Proof.** The operator \( H \) can be written as
\[ H = XX^+ + \sqrt{AC - B^2} + \gamma - |z|^2, \]
where
\[ X = A^{1/2} D_\eta + \frac{B + i \sqrt{AC - B^2}}{A^{1/2}} \eta + z, \quad X^+ = A^{1/2} D_\eta + \frac{B - i \sqrt{AC - B^2}}{A^{1/2}} \eta + \bar{z}, \]
EIGENVALUE ESTIMATES FOR A THREE-DIMENSIONAL MAGNETIC OPERATOR

\[ z = \frac{1}{2A^{1/2}} \alpha + i \frac{A\beta - B\alpha}{2A^{1/2}\sqrt{AC - B^2}}, \]

and we have

\[ [X^+, X] = 2\sqrt{AC - B^2}. \]

Therefore, the lemma is immediately proved by following a well-known computation of the spectrum of the harmonic oscillator by means of creation and annihilation operators. □

For \( h_5 \), we have

\[ A = \frac{2k + 1}{2b_0} \frac{q_{12}q_{33} - q_{23}^2}{q_{33}}, \quad B = \frac{2k + 1}{2b_0} \frac{q_{11}q_{33} - q_{13}^2}{q_{33}}, \quad C = \frac{2k + 1}{2b_0} \frac{q_{11}q_{33} - q_{13}^2}{q_{33}}, \]

and \( \alpha, \beta \) and \( \gamma \) are given by (4.26) and (4.27). In particular, by (2.8), we have

\[ AC - B^2 = \frac{(2k + 1)^2}{4b_0^4} \left( \frac{d}{2a} \right)^2 = \frac{(2k + 1)^2}{4b_0^2} \]

By Lemma 4.2, the eigenvalues of \( h_5 \) have the form

\[ \lambda_m = \frac{1}{2b_0} \left( \frac{d}{2a} \right)^{1/2} (2k + 1)(2m + 1) + \nu(j, k), \quad m \in \mathbb{N}, \]

where

\[ \nu(j, k) = \gamma - \frac{1}{4(AC - B^2)} (C\alpha^2 - 2B\alpha\beta + A\beta^2) \]

\[ = \nu_{22}(2k + 1)^2 + \nu_{11}(2j + 1)^2 + \nu_0, \]

with

\[ \nu_{22} = \gamma_2, \quad \nu_0 = \gamma_0, \]

and

\[ \nu_{11} = \gamma_1 - \frac{a}{d} \left( \frac{1}{b_0} \frac{q_{11}q_{33} - q_{13}^2}{q_{33}} \alpha^2 + 2 \frac{q_{12}q_{33} - q_{13}q_{23}}{q_{33}} \bar{\alpha}\beta + b_0 \frac{q_{22}q_{33} - q_{23}^2}{q_{33}} \beta^2 \right). \]

Taking \( \lambda_4 \) as the \( m \)-th eigenvalue of \( h_5 \) and \( \lambda_0 \) as the corresponding eigenfunction, we obtain that the condition (4.24) holds. Therefore, we can find a solution \( v_2 \) to (4.26). Then the condition (4.22) holds, that allows us to find a solution \( u_4 \) to (4.5).

Thus, for any \( j, k, m \in \mathbb{N} \), we have proved the existence of a solution \( u_\ell, \ell = 0, 1, 2, 3, 4 \), to the system of equations (4.1)–(4.5) with a suitable choice of \( \lambda_\ell, \ell = 0, 1, 2, 3, 4 \) given by (4.7), (4.10), (4.15), (4.20) and (4.28). Setting

\[ u^h(x, \eta, z) = \sum_{\ell=0}^4 h^{\frac{\ell}{2}} u_\ell(x, \eta, z), \quad \lambda^h = \sum_{\ell=0}^4 h^{\frac{\ell}{2}} \lambda_\ell, \]

we obtain

\[ P^h u^h - \lambda^h u^h = O(h^{\frac{3}{2}}). \]

The functions \( u_\ell \) have sufficient decay properties. Therefore, by changing back to the original coordinates and multiplying by a fixed cut-off function, we obtain the desired functions \( \phi^h_{j,k,m} \), which satisfy (1.3) with

\[ \mu^h_{j,k,m} = \sum_{\ell=0}^4 h^{\frac{\ell}{2}} \mu_{j,k,m,\ell}, \]

where

\[ \nu_{22} = \gamma_2, \quad \nu_0 = \gamma_0, \]

and

\[ \nu_{11} = \gamma_1 - \frac{a}{d} \left( \frac{1}{b_0} \frac{q_{11}q_{33} - q_{13}^2}{q_{33}} \alpha^2 + 2 \frac{q_{12}q_{33} - q_{13}q_{23}}{q_{33}} \bar{\alpha}\beta + b_0 \frac{q_{22}q_{33} - q_{23}^2}{q_{33}} \beta^2 \right). \]
where
\[ \mu_{j,k,m,\ell} = \lambda_{\ell}, \quad \ell = 0, 1, 2, 3, 4. \]

The functions \( u_0 \) for different \( j, k \) and \( m \) are orthogonal. Since each change of variables, which we use, is unitary, this implies the condition (1.4).

5. Periodic case and spectral gaps

In this section, we apply the previous results to the problem of existence of gaps in the spectrum of a periodic magnetic Schrödinger operator. For related results on spectral gaps for periodic magnetic Schrödinger operators, see [8] and references therein.

Consider the Schrödinger operator with magnetic potential in the entire Euclidean space \( \mathbb{R}^3 \)
\[ H^h = (hD_X - A_1(X))^2 + (hD_X - A_2(X))^2 + (hD_X - A_3(X))^2. \]

We assume that the vector magnetic field \( \vec{B} = (B_1, B_2, B_3) \) is \( \Gamma \)-invariant for some cocompact lattice \( \Gamma \subset \mathbb{R}^3 \).

Put \( b_0 = \min \{ |\vec{B}(X)| : X \in \mathbb{R}^3 \} \)
and assume that there exist a (connected) fundamental domain \( Q \) for \( \Gamma \) and a constant \( \epsilon_0 > 0 \) such that
\[ |B(X)| \geq b_0 + \epsilon_0, \quad x \in \partial Q. \]

We will consider the operator \( H^h \) as an unbounded self-adjoint operator in the Hilbert space \( L^2(\mathbb{R}^3) \). Using the results of [6], one can immediately derive from Theorem 1.2 the following result on existence of gaps in the spectrum of \( H^h \) in the semiclassical limit.

**Theorem 5.1.**

Suppose that
\[ b_0 > 0, \]
and that there exists a unique minimum \( X_0 \in Q \) such that \( |B(X_0)| = b_0 \), which is non-degenerate: in some neighborhood of \( X_0 \)
\[ C^{-1} |X - X_0|^2 \leq |\vec{B}(X)| - b_0 \leq C |X - X_0|^2. \]

Then, for any natural \( j, k \) and \( N \), there exist \( h_{j,k,N} > 0, c_{j,k} \) and \( C_{j,k,N} \) such that
\[ C_{j,k,N} > c_{j,k} + \frac{1}{b_0} \left( \frac{d}{2a} \right)^{1/2} (2k + 1)N, \]
and the spectrum of \( H^h \) in the interval \([A_{j,k}, B_{j,k,N}]\) where
\[ A_{j,k} = (2k + 1)b_0 + (2j + 1)(2k + 1)^{1/2}a^{1/2}h^{3/2} + h^2 c_{j,k} \]
and
\[ B_{j,k,N} = (2k + 1)b_0 + (2j + 1)(2k + 1)^{1/2}a^{1/2}h^{3/2} + h^2 C_{j,k,N}, \]
has at least \( N \) gaps for any \( h \in [0, h_{j,k,N}] \).
APPENDIX A. HERMITE POLYNOMIALS

For a fixed $\lambda > 0$, put

$$h_m(t) = \lambda^{1/2} H_m(\lambda^{1/2} t)e^{-\lambda t^2/2}.$$  

Here we gather some well-known formulae, concerning to the Hermite functions $h_m$, which we need in the paper:

$$th_m(t) = \frac{1}{2\lambda^{1/2}} [h_{m+1}(t) + 2mh_{m-1}(t)] .$$

$$t^2 h_m(t) = \frac{1}{4\lambda} [h_{m+2}(t) + (4m + 2)h_{m}(t) + 4(m-1)h_{m-2}(t)] .$$

$$t^3 h_m(t) = \frac{1}{8\lambda^{3/2}} [h_{m+3}(t) + (6m + 6)h_{m+1}(t) + 12m^2 h_{m-1}(t) + 8m(m-1)(m-2)h_{m-3}(t)] .$$

$$D_t h_m(t) = \frac{1}{2} \lambda^{1/2} i [h_{m+1}(t) - 2mh_{m-1}(t)] .$$

$$\langle (Dt + D_t^2)h_m, h_m \rangle = 0 .$$

$$(Dt^2 + t^2 D_t)h_m(t) = \frac{1}{4\lambda^{1/2}} i [h_{m+3}(t) + (2m + 2)h_{m+1}(t) - 4m^2 h_{m-1}(t) - 8m(m-1)(m-2)h_{m-3}(t)] .$$

$$(t^4 D_t + D_t t^4)h_m(t) = \frac{1}{16\lambda^{3/2}} i [h_{m+5}(t) + (6m + 12)h_{m+3}(t) + 4(2m^2 + 4m + 3)h_{m+1}(t) - 8(2m^2 + 1)m h_{m-1}(t) - 48m(m-1)^2(m-2)h_{m-3}(t) - 32m(m-1)(m-2)(m-3)(m-4)h_{m-5}(t)] .$$

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