Complex Variable Theorems for Finding Zeroes and Poles of Transcendental Functions

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Abstract. The principle of the argument or the winding number is useful in finding the number of zeros of an analytic function in a given contour. A simple extension of this theorem yields relationships involving the locations of these zeros! The resulting equations can be solved very accurately for the zero locations, thus avoiding initial, guess values, which are required by many other techniques. Examples such as a 20th order polynomial, natural frequencies of a thin wire will be discussed.

1. Introduction

This paper discusses a procedure for finding the zeroes of an analytic function. Cauchy’s theorem is used in finding the number of zeroes in a contour. A simple extension of this theorem yields relationships between the locations of the zeroes. Furthermore, a simple procedure involving integration by parts simplifies the equations involved. Numerical procedures for accurately determining the location of the zeroes have been developed. Numerical examples consisting of: a) 20th order polynomial, b) natural frequencies of a thin wire and c) inverse conformal transformation are also presented.

2. Principle of Argument

Consider a function \( f(z) \) which is meromorphic in a simply connected domain \( D \) containing a Jordan contour \( C \), there are well – established methods to determine the difference between the number of zeroes \( N_o \) and the number of poles \( N_p \) in this contour \( C \). Let us also stipulate that there are no zeroes or poles on the contour \( C \).

\[
\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} \, dz = N_o - N_p
\]

In equation (1), the prime superscript denotes a derivative with respect to \( z \). The integration is around the Contour \( C \) in a counter-clockwise direction. Equation (1) can be written as

\[
\frac{1}{2\pi i} \oint_C \frac{d}{dz} \ln[f(z)] \, dz = \frac{1}{2\pi} \arg \{f(z)\} \text{ (around } C) = N_o - N_p = N_a
\]
3. Location of Zeroes

Let us consider a function \( g(z) \) that is analytic in domain \( D \). If the zeroes of \( f(z) \) occur at \( z_{o_1}, \ldots, z_{o_N} \), using the residue theorem, we can write

\[
\frac{1}{2\pi i} \oint_{C} \frac{f'(z)}{f(z)} g(z) \, dz = \sum_{i=1}^{N_o} g(z_{o_i})
\]

(3)

Here the zeroes are counted according to their multiplicity. In other words, a second order zero will be counted as two zeroes occurring at the same location in the complex plane. We are now free to choose \( g(z) \) as long as it is analytic in and on contour \( C \) [1]. We choose \( g(z) = z^k \) with \( k = 0, 1, 2, \ldots, N_o \).

Representing the left-hand side of equation (3) as \( I_k \), we then have

\[
\sum_{i=1}^{N_o} z_{o_1}^{k} + z_{o_2}^{k} + \cdots + z_{o_{N_o}}^{k} = I_k
\]

(4)

where

\[
\sum_{i=1}^{N_o} z_{o_1}^{2} + z_{o_2}^{2} + \cdots + z_{o_{N_o}}^{2} = I_2
\]

\[
\vdots
\]

\[
\sum_{i=1}^{N_o} z_{o_1}^{N_o} + z_{o_2}^{N_o} + \cdots + z_{o_{N_o}}^{N_o} = I_{N_o}
\]

(5)

It is observed that the case of \( k = 0 \) corresponds to the principle of argument leading to the winding number. In other words, the integral \( I_0 = N_o \). The set of non-linear equations can then be solved for the locations of the zeroes. It is possible to simplify the evaluation of \( I_k \) of equation (5) as follows, using integration by parts, as follows.

\[
I_k = \frac{1}{2\pi i} \oint_{C} z^k \frac{d}{dz} \left( \ln|f(z)| \right) \, dz
\]

\[
\frac{d}{dz} \ln|f(z)| = \frac{d}{dz} \left[ \ln|f(z)| + i \arg(f(z)) \right]
\]

(6)

where the adjusted argument is a continuous function except at the end point of the contour \( C \).

\[
I_k = \frac{k}{2\pi i} \oint_{C_{\text{ini}}} z^k \left[ \frac{d}{dz} \ln|f(z)| + i \frac{d}{dz} \left( \arg(f(z)) \right) \right] \, dz
\]

(7)

We can numerically evaluate the above integral and generate a system of equations.
What is remarkable about this process is that no guess values of the location of the zeroes are needed, as in many other iterative methods such as Newton’s method, Newton-Raphson method and Mueller’s method. We have computerized this process and find that it is extremely efficient in finding zeroes of analytical functions. In many physical problems, we encounter transcendental functions for which the locations of zeroes are sought. We provide some illustrative examples in the following sections.

4. Roots of a Polynomial

A twentieth order polynomial as shown below is constructed as an example.

\[
f(s) = (-5 + j1 \cdot 0)(s + (4 + j1 \cdot 4))(s + (-1 + j2 \cdot 6))
\]

\[
(s + (-5 + j2 \cdot 7))(s + (1 + j1 \cdot 8))(s + (1 + j3 \cdot 95))
\]

\[
(s + (-2 + 4j \cdot 1))(s + (-2 + j1 \cdot 2))(s + (-2 + j6 \cdot 3))
\]

\[
(s + (-2 + 1j2 \cdot 1))(s + (-2 + 1j2 \cdot 1))(s + (-2 + 2j4 \cdot 2))
\]

\[
(s + (-2 + 7j4 \cdot 8))(s + (-3 + j1 \cdot 6))(s + (-3 + j7 \cdot 8))
\]

\[
(s + (4 + 25j \cdot 35))(s + (-4 + 3j2 \cdot 3))(s + (-4 + 75j + 72 \cdot 75))
\]

\[
(s + (-4 + 3j4 \cdot 5))(s + (-4 + 95 + j4 \cdot 95))
\]

We then use our numerical procedure outlined earlier to find the zeroes of \(f(s)\) in the complex s-plane. The scan area for the numerical procedure is shown in Figure 1.

![Figure 1. Scan area showing the locations of zeroes of the polynomial show by dots and the order of the zero is shown in parenthesis](image)

The results of our computation of the location of the zeroes compared by three different routines, to the exact known locations are shown in Table 1. The accuracy of our computations is seen to be excellent.
5. Natural frequencies of a thin wire

In this section, we consider an example of finding the complex natural frequencies [2] of a thin wire whose diameter \( d \) to length \( L \) ratio of \( (d/L) = 0.01 \), shown in Figure 2. Hallen form of the integral equation for the current distribution \( \tilde{i}(z) \) is given by

\[
\int_0^{L_f} \tilde{i}(z') \tilde{R}(z-z') dz' = A \sinh(\gamma z) + B \cosh(\gamma z)
\]

(10)

where the Kernel is given by

\[
\tilde{R}(z-z') = \frac{1}{2\pi a} \int_0^{2\pi} e^{-\frac{\gamma R}{4\pi R}} a d\phi'
\]

(11)

Equation (10) can be cast in a matrix form as follows

\[
\begin{bmatrix} Z_{p,q} \end{bmatrix} \begin{bmatrix} I_p \end{bmatrix} = \begin{bmatrix} f_p(z) \end{bmatrix}
\]

(12)

One can set the forcing function on the right-hand side of equation (12) to zero, the complex zeroes of the determinant \([Z_{p,q}]\) in the s-plane are the natural frequencies of the thin wire. It is noted that the determinant is a complex analytic function of the complex \( s \)-plane. Put in these terms, the problem becomes one of finding zeroes of a complex analytic function. The complex natural frequencies appear in layers and they are shown in Figure 2.

### TABLE 1. Exact and computed locations of zeroes of the twentieth order polynomial

| No. | Exact Location | Location Found by SEARCH | Location Found by NOMEN | Location Found by SEARCH |
|-----|----------------|--------------------------|--------------------------|--------------------------|
| 1   | -5+1.0         | -5.00000000210+0.40000000190 | -5.00000000210+0.40000000190 | -5.00000000210+0.40000000190 |
| 2   | -3+1.4         | -3.00000000858+0.54000000187 | -3.00000000858+0.54000000187 | -3.00000000858+0.54000000187 |
| 3   | -1+2.6         | -1.00000001613+0.61000000161 | -1.00000001613+0.61000000161 | -1.00000001613+0.61000000161 |
| 4   | -5+2.7         | -5.00000001537+0.70000000327 | -5.00000001537+0.70000000327 | -5.00000001537+0.70000000327 |
| 5   | -1+3+1.8       | -1.00000000224+0.80000000127 | -1.00000000224+0.80000000127 | -1.00000000224+0.80000000127 |
| 6   | -1+3+0.96      | -1.00000000278+0.85000000176 | -1.00000000278+0.85000000176 | -1.00000000278+0.85000000176 |
| 7   | -2+4+1.1       | -2.00000002807+0.92000000109 | -2.00000002807+0.92000000109 | -2.00000002807+0.92000000109 |
| 8   | -2+5+2.2       | -2.00000004210+0.97000000192 | -2.00000004210+0.97000000192 | -2.00000004210+0.97000000192 |
| 9   | -2+6+3         | -2.00000004778+0.99000000129 | -2.00000004778+0.99000000129 | -2.00000004778+0.99000000129 |
| 10  | -2+7+4.1       | -2.00000006227+1.000000033096 | -2.00000006227+1.000000033096 | -2.00000006227+1.000000033096 |
| 11  | -2+8+5.2       | -2.00000008262+1.00000005465 | -2.00000008262+1.00000005465 | -2.00000008262+1.00000005465 |
| 12  | -2+9+6.3       | -2.00000010358+1.000000076243 | -2.00000010358+1.000000076243 | -2.00000010358+1.000000076243 |
| 13  | -3+3+1.6       | -3.00000000309+1.000000118+ | -3.00000000309+1.000000118+ | -3.00000000309+1.000000118+ |
| 14  | -3+4+2.7       | -3.00000000347+1.00000002373 | -3.00000000347+1.00000002373 | -3.00000000347+1.00000002373 |
| 15  | -3+5+3.8       | -3.000000003153+1.00000005135 | -3.000000003153+1.00000005135 | -3.000000003153+1.00000005135 |
| 16  | -4+3+1.5       | -4.000000003153+1.00000005135 | -4.000000003153+1.00000005135 | -4.000000003153+1.00000005135 |
| 17  | -4+4+2.6       | -4.000000003153+1.00000005135 | -4.000000003153+1.00000005135 | -4.000000003153+1.00000005135 |
| 18  | -4+5+3.7       | -4.000000003153+1.00000005135 | -4.000000003153+1.00000005135 | -4.000000003153+1.00000005135 |
| 19  | -4+6+4.8       | -4.000000003153+1.00000005135 | -4.000000003153+1.00000005135 | -4.000000003153+1.00000005135 |
| 20  | -4+7+5.9       | -4.000000003153+1.00000005135 | -4.000000003153+1.00000005135 | -4.000000003153+1.00000005135 |
6. Conformal transformation
Conformal (-angle preserving) transformations map a problem in an inconvenient geometry (z plane) into a new plane with convenient geometry. The problem is solved in the mapped plane (w) and the solution is transformed back to the original plane. One of the most celebrated conformal transformations is the Joukowski mapping which transforms a circular cylinder into a family of airfoil shapes. This is a problem in fluid mechanics where one needs to solve for the flow field. Knowing the velocity and pressure of air flow in the circular cylinder case, it can be transformed into the solution around the airfoil. If we know the flow field (velocity and pressure) around the airfoil of interest, the “lift” can then be calculated to complete the problem. The computations are generally performed using numerical methods. Excellent treatments of conformal transformations can be found in [3,4].

6.1 Inverting the conformal transformation
Let the original plane be denoted by \( z = x + i \, y \) and the transformed plane by \( w = u + i \, v \) and the transformation is given by

\[
    z = f(w) \tag{13}
\]

In aerodynamics, \( w \) can be the complex flow field (velocity and pressure), and in electromagnetics, it can be the complex potential (scalar potential and stream function). Equation (13) permits the estimation of \( z \), for a given \( w \). However, we need to invert the transformation

\[
    w = f^{-1}(z) \tag{14}
\]

so that one can compute \( w \) (ex: complex potential) at a given point \( z \) in the physical plane. Knowing the complex potential, the electric field can be written as its gradient.
In general, \( f(w) \) is a complicated transcendental function and the inversion indicated in equation (14) is not straightforward even in simple problems.

6.2 Formulating the Inversion

Consider a new function

\[
F(w) = f(w) - z
\]  

(16)

Finding the complex potential \( w \) for a given \( z \) becomes that of finding the zero of an analytic function \( F(w) \) or

\[
F(w_o) = f(w) - z = 0
\]

(17)

The conformal map ensures a one-to-one correspondence which means, for a given value of \( z \), there is only one complex value of \( w_o \). The problem space in the \( z \)-plane maps into a certain area of the \( w \)-plane. The boundary of this area is denoted by a contour \( C \). In that contour, the analytic function \( F(w) \) has one zero \( w_o \). The principle of argument or the winding number in complex variable theory, as discussed above, gives

\[
\frac{1}{2\pi i} \oint_C \frac{F'(w')}{F(w')} \, dw' = \frac{1}{2\pi i} \oint_C \frac{d}{dw} \ln F(w') \, dw' = 1
\]

(18)

Extending the above principle [6], by introducing an extra \( g(w') \) in the integrand, with \( g(w') \) being analytic in and on contour \( C \)

\[
\frac{1}{2\pi i} \oint_C g(w') \frac{F'(w')}{F(w')} \, dw' = \frac{1}{2\pi i} \oint_C g(w') \frac{d}{dw} \ln F(w') \, dw'
\]

\[= \sum_{m=0}^{n} g(w_m)
\]

(19)

where \( w_m \) is the location of \( m \)-th zero in contour \( C \) and there are \( n \) zeros in contour \( C \). By using \( g(w) = w^k \), we have the \( k \)-th moment as

\[
I_k = \frac{1}{2\pi i} \oint_C (w')^k \frac{d}{dw} \ln F(w') \, dw'
\]

(20)

In the present context, we observe that there is only zero in the Contour \( C \) and hence \( I_0 = 1 \) and \( I_1 = w_o \) and the first moment \( I_1 \) leads to the inversion of the conformal transformation as follows

\[
w_o = \frac{1}{2\pi i} \oint_C \frac{F'(w)}{F(w)} \, dw
\]

\[= \frac{1}{2\pi i} \oint_C \frac{f'(w)}{f(w) - z} \, dw
\]

(21)
6.3 Illustrative Example

We consider a two-dimensional problem of two parallel plates of finite width. The geometry of the problem in the \( z \) plane is shown in Figure 3.

The conformal transformation that maps the \( z \) plane \((x + iy)\) into the complex potential \( w = u + iv \) plane is known \([3, 4]\) to be

\[
\phi = \frac{2i}{\pi} \left[ K(m) Z(m|\phi) + \frac{w \pi}{2 K(m)} \right] \tag{22}
\]

The parameter \( m \) is obtained by solving

\[
\frac{a}{b} = \frac{2}{\pi} \left[ K(m) E(m|m) - E(m) F(m|m) \right] \tag{23}
\]

where

\[
\sin(\phi) = \left( \frac{1}{m} \left( 1 - \frac{E(m)}{K(m)} \right) \right)^{1/2} \tag{24}
\]

\( K, E \) and \( F \) are elliptic functions and \( Z \) is the Jacobi Zeta function.

In accordance with equation (16), we form a function

\[
F(w) = f(w) - \phi = \frac{2i}{\pi} \left[ K(m) Z(m|\phi) + \frac{w \pi}{2 K(m)} \right] - \phi \tag{25}
\]

For a given value of \( \phi \), \( F(w) = 0 \), and

\[
w_{\phi} = \frac{1}{2\pi i} \int w^* \frac{d}{dw} [\ln F(w')] dw' \tag{26}
\]

We can rewrite equation (26) as

\[
w_{\phi}(\phi) = \frac{1}{2\pi i} \int w^* \left[ \frac{d}{dw'} (\ln F(w')) + i \frac{d}{dw'} \text{adj}[\arg(F(w'))] \right] \tag{27}
\]

where \( \text{adj} (\arg F(w')) \) is a continuous function except at the end points of the contour. Integrating by parts, while noting that log magnitude is a continuous function, results in the inversion of the conformal transformation as:
The entire first quadrant of the $z$ plane maps into a small rectangular area in the $w$ plane, which becomes the contour $C$, with an initial value of $C_{ini}$ as the starting point on the contour. Note that no differentiation is required. All we need are the values of the $F(w)$ on the contour and a good integration scheme. An example calculation of the complex potential is shown in Figures 4.

Figure 4. Complex potential contours $w = u + i v$ for the case of $(b/a) = 0.40679$, corresponding to a 100 Ohm line

A parametric study has been performed and the complex potential and the electromagnetic field plots are available [5].

References

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