I. EXACT SITE PERCOLATION ON \( \mathbb{Z}^d \)

Vertices (sites), open with the smallest probability \( p_H \), percolate when they form an infinite open path from graph’s origin \( v_0 \). Usually, \( p_H \) values are approximated, but there are a few instances of special lattices with the exact results, [1, 3].

In finite graph \( \mathbb{Z}^d_k \), \( d \) pairs of opposite arcs are \( k \) edges away from \( v_0 \), Fig. 1 (left). Basis \( \mathcal{B} ( \mathbb{Z}^d ) \) and the integers \( a_i (v) \) assign the place to vertex \( v \in \mathbb{Z}^d_k \):

\[
\begin{align*}
v_0 = 0 & \text{ vertex at origin of } \mathbb{Z}^d_k \subset \mathbb{Z}^d \\
\mathcal{B} ( \mathbb{Z}^d ) & = \{ \downarrow_1, \uparrow_1, \downarrow_2, \uparrow_2, \ldots, \downarrow_d, \uparrow_d \} \\
v & = a_1 \downarrow_1 + a_2 \downarrow_2 + a_3 \uparrow_3 + \ldots + a_d \uparrow_d \\
\|v\| & = \sum_{i=1}^{d} |a_i(v)| \quad \|v\| \leq k \forall v \in \mathbb{Z}^d_k
\end{align*}
\]

\( v \)'s neighbors can be partition into \( d \) up-step neighbors traversed via \( \uparrow_i \) and \( d \) down-step neighbors traversed via \( \downarrow_i \), so that the shortest traversal from \( v_0 \) to arc \( A_k ( \mathbb{Z}^d ) \) is a traversal via up-step neighbors. Arcs in \( \mathbb{Z}^d_k \) look the same, and by rotation of \( \mathbb{Z}^d \), any arc can be \( A_k ( \mathbb{Z}^d ) \):

\[
\begin{align*}
\mathcal{N}_u (v, \mathbb{Z}^d) & = \text{up-step neighbors of } v = \{ v+i \downarrow_1, \ldots, v+i \uparrow_d \} \\
\mathcal{A}_k (\mathbb{Z}^d) & = \bigcup_{v \in \mathcal{A}_{k-1} (\mathbb{Z}^d)} \mathcal{N}_u (v, \mathbb{Z}^d) : \mathcal{A}_1 (\mathbb{Z}^d) = \mathcal{B} (\mathbb{Z}^d) \\
v & \in \mathcal{A}_k (\mathbb{Z}^d) \Rightarrow \|v\| = k
\end{align*}
\]

If there is an open path from \( v_0 \) to \( v \in \mathcal{A}_k (\mathbb{Z}^d) : k \to \infty \), \( \mathbb{Z}^d \) percolates. The shortest paths from open \( v_0 \) to \( \mathcal{A}_k (\mathbb{Z}^d) \) are \( k \)-paths built by the up-step traversal. After the first up-step to \( \mathcal{A}_1 (\mathbb{Z}^d) \), the number of paths \( d \)-tuples, so there are \( d \) 1-paths and the expected number of percolating paths in graph induced by the up-step traversal is \( \psi_1 = dp \). After second up-step, there are \( d^2 \) 2-paths and \( \psi_2 = (\psi_1) d p = (dp)^2 \). Inductively, after \( k \) up-steps, there are \( d^k \) \( k \)-paths and \( \psi_k = (dp)^k \):

\[
\begin{align*}
k\text{-path} & = \text{path from } v_0 \text{ to } v : v \in \mathcal{A}_k (\mathbb{Z}^d) \quad \& \quad \|v\| = k \\
(k+m)\text{-path} & = \text{\( k \)-path extended by } m \text{ edges} \\
n_k (\mathcal{A}_k (\mathbb{Z}^d)) & = \text{number of } k \text{-paths} = d^k \\
\psi (\mathbb{Z}^d, p) & = \text{number of percolating paths in } \mathbb{Z}^d_k \text{ for } k \to \infty \geq \psi_k = (dp)^k \\
\Rightarrow p_H (\mathbb{Z}^d) & \leq \frac{1}{d}
\end{align*}
\]

A down-step, avoiding vertex repetition, extends \( (k+2) \)-path to \( (k+4) \)-path. Two down-steps, avoiding vertex repetition, extend \( k \)-path to \( (k+4) \)-path. There are no more than \( 2d^k \) \( (k+4) \)-paths from \( v_0 \) to \( \mathcal{A}_k (\mathbb{Z}^d) \), because other \( (k+4) \)-paths would have to come from the extensions of non \( (k+2) \)-paths and non \( k \)-paths, which is not possible. Inductively, one down-step,..., and \( m \) down-steps, extend \( (k+2m-2) \)-paths,..., and \( k \)-paths to \( (k+2m) \)-paths. There are no more than \( md^k \) \( (k+2m) \)-paths from \( v_0 \) to \( \mathcal{A}_k (\mathbb{Z}^d) \) or \( n_{k+2m} (\mathcal{A}_k (\mathbb{Z}^d)) \leq md^k \), so

\[
\psi (\mathbb{Z}^d, p) = \lim_{k \to \infty} \sum_{i=0}^{n_{k+2i} (\mathcal{A}_k (\mathbb{Z}^d))} p^{k+2i} \leq \lim_{k \to \infty} (dp)^k \left( 1 + \sum_{i=1}^{\infty} i \cdot p^{2i} \right) = \frac{1}{d} \Rightarrow p_H (\mathbb{Z}^d) \geq \frac{1}{d}
\]

From inequalities (1) and (2),

\[
p_H (\mathbb{Z}^d) = \frac{1}{d}
\]
B. Modifications of $Z^d$

Graph $G$, as result of modification of $Z^d$, embeds $Z^d$ or modified $Z^d$. Vertices in arc $A_k(G)$ are $k$ edges away from the origin $v_0 \in G$ and $A_k(G)$ contains the vertices of $Z^d$ that belong to arcs $A_{k+i}(Z^d)$, which are visited in the shortest up-step traversal of $G$. The shortest traversal to $A_k(G)$ is traversal via neighborhoods $N_u(v, G)$, which builds $k$-paths ending in arcs $A_{k+i}(Z^d)$:

$$G = \text{graph after modification of } Z^d$$

$$A_k(G) = \bigcup_{v \in A_{k-1}(G)} N_u(v, G) : A_1(G) = B(G)$$

Modification of $Z^d$ either removes or adds the edges from $B(Z^d)$ to some vertices in $Z^d$, so $B(G)$ is composed of combinations of edges in $B(Z^d)$:

$$B(G) = \text{basis of } G = \bigcup_{v \in G} (v - v_x) : v_x \in N(v, G)$$

For all the rotations of $G$, there is the greatest number of $k$-paths to some $A_{k+i}(Z^d)$, which minimizes percolating probability:

$$n_k \left( A_{k+i}(Z^d), G \right) = \frac{\text{number of } k\text{-paths ending in } A_{k+i}(Z^d)}{\text{number of } k\text{-paths to } A_k(G)}$$

$$n_k \left( A_k(G) \right) = \sum_i n_k \left( A_{k+i}(Z^d), G \right)$$

$n_k \left( A_{k+i}(Z^d), G \right)$ values are maximized by choosing $N_u(v, G)$, so that in the up-step traversal the greatest number of vertices get closer to $A_k(G)$ in all modifications of $Z^d$ embedded in $G$. Extensions of $k$-paths in $G$ are extensions of $k$-paths in $Z^d$. Thus, the extensions do not lower $p'$:

$$n_k \left( A_{k+i}(Z^d), G \right) \cdot p' \geq 1, \text{ so } p' = p_H(G).$$

1st Example: Triangular Lattice

Triangular lattice $T$ embeds $Z^2$, so it has two pairs of opposite sides, Fig. 2. $n_k \left( A_{k+i}(Z^d), T \right)$ values are maximized when each $v \in T$ has neighbor $\uparrow_1 + \uparrow_2$. $A_k(T)$, $k$ edges away from $v_0 \in T$, contains the vertices in arc $A_{k+i}(Z^d)$ traversed via $N_u(v, T)$:

$$T = \text{triangular lattice with embedded } Z^2$$

$$B(T) = \{ \uparrow_1, \uparrow_2, \uparrow_1 + \uparrow_2 \}$$

$$N_u(v, T) = v + \{ \uparrow_1, \uparrow_2, \uparrow_1 + \uparrow_2 \}$$

$$A_k(T) = \bigcup_{v \in A_{k-1}(T)} N_u(v, T) : A_1(T) = \{ \uparrow_1, \uparrow_2, \uparrow_1 + \uparrow_2 \}$$

After 1st up-step in the traversal of $T$, there are 2 paths that end in $A_1(Z^2)$ and 1 path that ends in $A_2(Z^2)$:

$$n_1 \left( A_1(Z^2), T \right) = n_1(1) = \text{number of } 1\text{-paths to } A_1(Z^2) \text{ traversing } T = 1 \cdot 2^1$$

$$n_1 \left( A_2(Z^2), T \right) = n_1(2) = \text{number of } 1\text{-paths to } A_2(Z^2) \text{ traversing } T = 1 \cdot 2^0$$

$$n_1 \left( A_1(T) \right) = n_1 \left( A_1(Z^2), T \right) + n_1 \left( A_2(Z^2), T \right)$$

After 2nd up-step in the traversal of $T$, the number of paths doubles for the up-steps from $A_1(Z^2)$ to $A_2(Z^2)$ and from $A_2(Z^2)$ to $A_3(Z^2)$. The number of paths does not change for the up-steps from $A_1(Z^2)$ to $A_3(Z^2)$ and from $A_2(Z^2)$ to $A_4(Z^2)$:

$$n_2(2) = 2 \cdot n_1(1) = 1 \cdot 2^2$$

$$n_2(3) = n_1(1) + 2 \cdot n_1(2) = 2 \cdot 2^1$$

$$n_2(4) = n_1(2) = 1 \cdot 2^0$$

After $k$th up-step in the traversal of $T$, the paths from $v_0$ end in $A_k(Z^2)$, $A_{k+1}(Z^2)$, .., $A_{2k}(Z^2)$:

$$n_k(k) = 2 \cdot n_{k-1}(k-1) = \binom{k}{0} \cdot 2^k$$

$$n_k(k+1) = n_{k-1}(k-1) + 2 \cdot n_{k-1}(k) = \binom{k}{1} \cdot 2^{k-1}$$

$$n_k(k+2) = n_{k-1}(k) + 2 \cdot n_{k-1}(k+1) = \binom{k}{2} \cdot 2^{k-2}$$

.. $n_k(2k-1) = n_{k-1}(2k-3) + 2 \cdot n_{k-1}(2k-2) = \binom{k}{k-1} \cdot 2^1$

$$n_k(2k) = n_{k-1}(2k-2) = \binom{k}{k} \cdot 2^0$$
Inductively, after \((k+1)\)th step
\[
\begin{align*}
n_{k+1}(k+1) &= 2 \cdot n_k(k) \\
n_{k+1}(k+2) &= n_k(k) + 2 \cdot n_k(k+1) \\
\vdots \\
n_{k+1}(2k+2) &= n_k(2k)
\end{align*}
\]
\[
\begin{align*}
n_{k+1}(k+1) &= 2 \binom{k}{0} 2^k = \binom{k+1}{0} 2^{k+1} \\
n_{k+1}(k+2) &= \binom{k}{0} 2^k + 2 \binom{k}{1} 2^{k-1} = \binom{k+1}{1} 2^k \\
\vdots \\
n_{k+1}(2k+2) &= \binom{k}{k} 2^0 = \binom{k+1}{k+1} 2^0
\end{align*}
\]
In the up-step traversal and for odd or even \(k \to \infty\), the greatest number of paths end in \(A_{k+\frac{k}{2}+1}(Z^2)\) or \(A_{k+\frac{k}{2}-1}(Z^2)\), respectively, and \(p_H(T) \leq \min p\) for which \(\binom{k}{i} 2^{k-i} \cdot p^k \geq 1:\)
\[
\begin{align*}
n_k\left(A_{k+\frac{k}{2}+1}(Z^2)\right) &= \binom{k}{k-1} 2^{k-(\frac{k-1}{2})} \\
n_k\left(A_{k+\frac{k}{2}-1}(Z^2)\right) &= \binom{k}{\frac{k}{2}-1} 2^{k-(\frac{k}{2}-1)}
\end{align*}
\]
\[
\begin{align*}
\psi(T,p) &= \lim_{k \to \infty} \begin{cases} 
2^{\frac{k}{2}} \left( \binom{k}{\frac{k}{2}-1} \frac{1}{2} \right) \cdot 2^{\frac{k}{2}} \cdot p^k \\
\frac{1}{2^{\frac{k}{2}}} \left( \binom{k}{\frac{k}{2}-1} \frac{1}{2} \right) \cdot 2^{\frac{k}{2}} \cdot p^k
\end{cases}
\end{align*}
\]
\[
\begin{align*}
p_H(T) &= \frac{1}{2^{\frac{k}{2}}} \cdot \lim_{k \to \infty} \begin{cases} 
\frac{1}{2^{\frac{k}{2}}} \left( \binom{k}{\frac{k}{2}-1} \frac{1}{2} \right) \\
\frac{1}{2^{\frac{k}{2}}} \left( \binom{k}{\frac{k}{2}-1} \frac{1}{2} \right)
\end{cases}
\end{align*}
\]
\[
\lim_{k \to \infty} \binom{k}{\frac{k-1}{2}} \approx \lim_{k \to \infty} \binom{k}{\frac{k}{2}-1} \approx 2, \quad [3].
\]

2\textsuperscript{nd} Example: Hexagonal Lattice

Hexagonal lattice \(H\) embeds modified \(Z^2\), since it is obtained by removing from every other vertex \(v \in Z^2\) its neighbor \(v^{\uparrow 1}\). For any rotation of \(H\), the neighborhood of \(v \in H\) is the same:
\[
H = \text{hexagonal lattice with modified } Z^2
\]
\[
\mathcal{N}_u(v, H) = v + \begin{cases} 
\uparrow_2 & \text{if } v \in A_k(Z^2) : k \text{ even} \\
\uparrow_1 & \text{if } v \in A_k(Z^2) : k \text{ odd}
\end{cases}
\]
After \(k\) odd up-steps, \(2^{\frac{k-1}{2}}\) paths end in \(A_k(Z^2)\).
\[
\begin{align*}
n_1(A_1(Z^2), H) &= 2^1 \\
n_2(A_2(Z^2), H) &= 1 \cdot n_1(A_1(Z^2), H) = 2^1 \\
n_3(A_3(Z^2), H) &= 2 \cdot n_2(A_2(Z^2), H) = 2^2 \\
&\vdots \\
\Rightarrow n_k(A_k(Z^2), H) &= \begin{cases} 
2^{\frac{k+1}{2}} : k \text{ odd} \\
2^{\frac{k}{2}} : k \text{ even}
\end{cases}
\end{align*}
\]
When \(k \to \infty\),
\[
\psi(H, p) = \lim_{k \to \infty} \begin{cases} 
2^{\frac{k+1}{2}} \cdot p^k = 2^{\frac{k}{2}} \left( 2^{\frac{k}{2}} \cdot p^k \right) \\
2^{\frac{k}{2}} \cdot p^k = \left( 2^{\frac{k}{2}} \cdot p^k \right)
\end{cases}
\]
\[
\Rightarrow p_H(H) = \frac{1}{2^{\frac{k}{2}}} \approx 0.7071
\]

II. Bond percolation on \(Z^d\)

Bond percolation on \(Z^d\) is equivalent to site percolation if the edges of \(Z^d\) are translated into vertices, which are connected if the bonds in \(Z^d\) shared a vertex, \([3]\):
\[
B_d = \text{translation of edges of } Z^d \text{ to new vertices}
\]
A vertex in \(B_d\) is connected to its neighbors via \(2 \cdot (2d-1)\) edges: each end of edge in \(Z^d\) meets with \(2d-1\) edges, Fig. \([3]\). An edge \(v_i v_j\) and its adjacent edge along the same dimension in \(Z^d\) are translated into vertices \(v_i\) and \(v_j\) in \(B_d\), so the edges \(v_i v_j \in Z^d\) and \(v_i v_j \in B_d\) are along the same dimension. \(v_i v_j \approx \uparrow_1\) is in basis of \(Z^d\) and \(\uparrow_1\) connects \(v_i\) to its up-step and down-step neighbors in \(B_d\). Other \(2 \cdot (2d-2)\) edges connect \(v_i\) to its \((2d-2)\) up-step and \((2d-2)\) down-step neighbors.

After the translation of \(Z^d, Z^d\) and \(Z^{2d-2}\) are embedded in \(B_d\). Each \(v \in B_d\) has neighbors \(v + u:\)
The shortest up-step traversal to arc \( u \in B \) connects \( u \) to its 2 neighbors: \( u \) and \( u \). Edges \( + u \) and \( + u \) connect \( u \) to its 4 other neighbors \( u \), \( u \), \( u \), and \( u \).

\[
\begin{align*}
\mathbb{Z}^{2d-2} &= \text{lattice embedded in } B_d \text{ after translation of } \mathbb{Z}^d \\
B(\mathbb{Z}^d) &= \text{basis of } \mathbb{Z}^d = \{ \uparrow_1, \ldots, \uparrow_{d} \} \\
B(\mathbb{Z}^{2d-2}) &= \text{basis of } \mathbb{Z}^{2d-2} = \{ \uparrow_1, \ldots, \uparrow_{2d-2} \} \\
B(B_d) &= \text{basis of } B_d = B(\mathbb{Z}^d) \cup B(\mathbb{Z}^{2d-2})
\end{align*}
\]

\( B_d \) embeds \( \mathbb{Z}^{2d-2} \), so it has \((2d-2)\) pairs of opposite arcs, which in any rotation of \( B_d \) look the same, Fig. 4.

**A. Up-step Traversal of \( B_d \)**

The shortest up-step traversal to arc \( A_k(B_d) \) is a traversal along edges \( \uparrow_i \) and arcs of \( \mathbb{Z}^{2d-2} \) embedded in \( B_d \):

\[
\begin{align*}
A_k(B_d) &= \text{vertices } k \text{ up-steps away from } v_0 \in B_d \\
A_k(\mathbb{Z}^{2d-2}) &= \text{vertices } k \text{ up-steps away from } v_0 \in \mathbb{Z}^{2d-2} \\
v_0 = B_d &= \text{basis of } \mathbb{Z}^{2d-2} \\
\uparrow_i &= \text{edge toward } A_k(B_d) : k \uparrow_i \in A_k(B_d)
\end{align*}
\]

A vertex \( v \) at the end of edge \( \uparrow_i \) has 1 up-step neighbor \( v \uparrow_i \) and \((2d-2)\) up-step neighbors \( v \uparrow_j \). For \( v \in V \):

\[
\begin{align*}
\left( v \uparrow_i \right) &\in A_{k+1}(B_d) & (u \uparrow_i) &\in A_{k+2}(\mathbb{Z}^{2d-2}) \\
\left( v \uparrow_j \right) &\in A_{k+1}(B_d) & (u \uparrow_j) &\in A_{k+1}(\mathbb{Z}^{2d-2})
\end{align*}
\]

Vertex \( v = v_x \uparrow_i \) is at \( \uparrow_i \) edge and it has 1 up-step neighbor \( v \uparrow_i \) at edge \( \uparrow_i \) and it has \((2d-2)\) up-step neighbors \( v \uparrow_j \), which are not at edge \( \uparrow_i \). Since \( v \uparrow_j \) is not at edge \( \uparrow_i \), it connects to two \( \uparrow_j \) edges in the up-step traversal (and two \( \uparrow_j \) edges in the down-step traversal), Fig. 4. All the other connections from \( v \uparrow_i \) are to the non-\( \uparrow_i \) edges, which are either not in up-step traversal or already traversed via \( v \).
B. Critical Probability for Bond Percolation

For any rotation of \( B_d \), the definition of up-step neighborhoods is the same:

\[ N_u(v, B_d) = v + \left\{ \uparrow_j, \uparrow_i ; \forall \uparrow_j \text{ if } v \text{ is on } \uparrow_i \uparrow_j \right\} \]

\( \uparrow_j \in B(\mathbb{Z}^{d-2}) \)

\( v + \uparrow_j \) \& \( v + \uparrow_j \) are on \( \uparrow_i \) edges

Vertices of \( B_d \) traversed at \( k \)th up-step are in arc \( A_k(B_d) \).

For \( v_0 \) at \( \uparrow_i \),

\[ A_1(B_d) = \left\{ \uparrow_i \right\} \bigcup \bigcup_j \uparrow_j : \uparrow_i, \uparrow_j \in B(\mathbb{Z}^d), \uparrow_j \in B(\mathbb{Z}^{d-2}) \]

\[ A_k(B_d) = \bigcup_{v \in A_{k-1}(B_d)} N_u(v, B_d) \]

From \( v_0 \), there is one up-step via edges in \( B(\mathbb{Z}^{d-2}) \) toward \( A_1(\mathbb{Z}^{d-2}) \subset A_1(B_d) \) and there is one up-step via edges in \( B(\mathbb{Z}^d) \) toward \( A_2(\mathbb{Z}^{d-2}) \subset A_1(B_d) \):

\[ n_1(A_1(\mathbb{Z}^{d-2}), B_d) = n_1(1) = \text{number of 1-paths to } A_1(\mathbb{Z}^{d-2}) \text{ traversing } B_d \]

\[ n_1(A_2(\mathbb{Z}^{d-2}), B_d) = n_1(2) = \text{number of 1-paths to } A_2(\mathbb{Z}^{d-2}) \text{ traversing } B_d \]

\[ n_1(A_1(B_d)) = n_1(1) + n_1(2) = \sum_{i=1}^{2} n_1(i) = (2d - 2) + 1 \]

For each vertex in \( A_1(\mathbb{Z}^{d-2}) \), there are 2 up-step neighbors in \( A_2(\mathbb{Z}^{d-2}) \). For each vertex in \( A_2(\mathbb{Z}^{d-2}) \), there are \((2d-2)\) up-step neighbors in \( A_3(\mathbb{Z}^{d-2}) \) and 1 up-step neighbor in \( A_4(\mathbb{Z}^{d-2}) \):

\[ n_2(2) = 2 \cdot n_1(1) = 2 \cdot (2d - 2) \]

\[ n_2(3) = (2d - 2) \cdot n_1(2) = (2d - 2) \]

\[ n_2(4) = 1 \cdot n_1(2) = 1 \]

\[ n_2(A_2(B_d)) = \sum_{i=2}^{4} n_2(i) \]

Vertices in \( A_3(B_d) \) are vertices of paths ending in arcs \( A_3(\mathbb{Z}^{d-2}), A_4(\mathbb{Z}^{d-2}), A_5(\mathbb{Z}^{d-2}) \), and \( A_6(\mathbb{Z}^{d-2}) \):

\[ n_3(3) = (2d - 2) \cdot n_2(2) = 2 \cdot (2d - 2)^2 \]

\[ n_3(4) = n_2(2) + 2 \cdot n_2(3) = 4 \cdot (2d - 2) \]

\[ n_3(5) = (2d - 2) \cdot n_2(4) = (2d - 2) \]

\[ n_3(6) = n_2(4) = 1 \]

\[ n_3(A_3(B_d)) = \sum_{i=3}^{6} n_3(i) \]

In the traversal from a vertex in \( A_k(\mathbb{Z}^{d-2}) \) to \( A_{k+1}(\mathbb{Z}^{d-2}) \) the number of paths doubles, if \( k \) is odd.

If \( k \) is even, an up-step from a vertex in \( A_k(\mathbb{Z}^{d-2}) \) to \( A_{k+1}(\mathbb{Z}^{d-2}) \) or \( B_{2d-2}(\mathbb{Z}^{d-2}) \) the number of paths, and an up-step from a vertex in \( A_k(\mathbb{Z}^{d-2}) \) to \( A_{k+2}(\mathbb{Z}^{d-2}) \) does not change the number of paths. After \( k \)th up-step, the paths end in \( A_k(\mathbb{Z}^{d-2}), A_{k+1}(\mathbb{Z}^{d-2}), ..., A_{2k}(\mathbb{Z}^{d-2}) \).

For \( D = (2d - 2) \),

\[ \sum_{i=4}^{8} n_4(i) = 4D^2 + 4D^2 + 6D^1 + D^1 + 1 \]

\[ \sum_{i=5}^{10} n_5(i) = 4D^3 + 12D^2 + 6D^2 + 8D^1 + D^1 + 1 \]

\[ \sum_{i=6}^{12} n_6(i) = 8D^3 + 12D^2 + 24D^2 + 8D^2 + 10D^1 + D^1 + 1 \]

\[ \sum_{i=7}^{14} n_7(i) = 8D^4 + 32D^3 + 24D^3 + 40D^2 + 10D^2 + ... \]

\[ \sum_{i=8}^{16} n_8(i) = 16D^4 + 32D^4 + 80D^3 + 40D^3 + 60D^2 + ... \]

\[ \sum_{i=9}^{18} n_9(i) = 16D^5 + 80D^4 + 80D^4 + 160D^3 + 60D^3 + ... \]

\[ \sum_{i=10}^{20} n_{10}(i) = 32D^5 + 80D^5 + 240D^4 + 160D^4 + 280D^3 + ... \]

For even \( k \),

\[ n_{k-1}(A_{k-1}(B_d)) = \sum_{i=k-1}^{2k-2} n_{k-1}(i) = \left( \frac{k}{3} \right) 2^{k-\frac{3}{2}} D^{k-\frac{5}{2}} + \left( \frac{k+1}{3} \right) 2^{k-\frac{7}{2}} D^{k-\frac{9}{2}} + \left( \frac{k+2}{3} \right) 2^{k-\frac{9}{2}} D^{k-\frac{11}{2}} + ... \]

\[ n_k(A_k(B_d)) = \sum_{i=k}^{2k} n_k(i) = \left( \frac{k}{3} \right) 2^{\frac{k}{2}} D^{\frac{k}{2}} + \left( \frac{k+1}{3} \right) 2^{\frac{k}{2}} D^{\frac{k}{2}} + \left( \frac{k+2}{3} \right) 2^{\frac{k}{2}} D^{\frac{k}{2}} + ... \]
Inductively, for \((k + 1)\) and \((k + 2)\),

\[
\sum_{i=k+1}^{2k+2} n_{k+1}(i) = \frac{D \cdot n_k(k) + 1 \cdot n_k(k) + 2 \cdot n_k(k + 1) + D \cdot n_k(k + 2) + 1 \cdot n_k(k + 2) + 2 \cdot n_k(k + 3) + \cdots}{2k+2}
\]

\[
n_{k+1}(A_{k+1}(B_d)) = \left(\frac{k+2}{2}\right)^{2k+2} D^{\frac{k+2}{2}} + \left(\frac{k+2}{3}\right)^{2k+2} D^{\frac{k+2}{3}} + \cdots
\]

\[
\sum_{i=k+2}^{2k+4} n_{k+2}(i) = 2 \cdot n_{k+1}(k + 1) + D \cdot n_{k+1}(k + 2) + D \cdot n_{k+1}(k + 4) + \cdots
\]

\[
n_{k+2}(A_{k+2}(B_d)) = \left(\frac{k+2}{1}\right)^{2k+2} D^{\frac{k+2}{1}} + \left(\frac{k+2}{2}\right)^{2k+2} D^{\frac{k+2}{2}} + \left(\frac{k+2}{3}\right)^{2k+2} D^{\frac{k+2}{3}} + \cdots
\]

When even \(k \to \infty\),

\[
\psi(B_d, p) = \lim_{k \to \infty} n_k(A_k(B_d)) \cdot p^k
\]

\[
\begin{align*}
\psi(B_d, p) &= \lim_{k \to \infty} \left(\frac{D}{2}\right)^{\frac{1}{2k+1}} (2D)^{\frac{k}{2}} p^{\frac{k-1}{2k+1}} + \\
&\quad \left(\frac{1}{2D}\right)^{\frac{1}{2k+1}} \left(\frac{1}{2}\right)^{\frac{k}{2}} (2D)^{\frac{k}{2}} p^{\frac{k-1}{2k+1}} + \\
&\quad \left(\frac{1}{2D}\right)^{\frac{1}{2k+1}} \left(\frac{k+2}{2}\right)^{\frac{k}{2}} (2D)^{\frac{k}{2}} p^{\frac{k-1}{2k+1}} + \\
&\quad \left(\frac{1}{2D}\right)^{\frac{1}{2k+1}} \left(\frac{k+2}{3}\right)^{\frac{k}{2}} (2D)^{\frac{k}{2}} p^{\frac{k-1}{2k+1}} + \\
&\quad \cdots
\end{align*}
\]

For \(D = 2d - 2\) and \(d \geq 2\), \(p_H(B_d)\) is the smallest \(p\) for which \(\psi(B_d, p) \geq 1\):

\[
p_H(B_d) = \frac{1}{2(d-1)^{\frac{1}{2}}} \cdot \min \left\{ \lim_{k \to \infty} \left(\frac{2d-1}{d}\right)^{\frac{1}{2k+1}} \right\}
\]
or

\[
p_H(B_d) = \frac{1}{2(d - 1)^{\frac{1}{2}}} \cdot \min \left\{ \lim_{k \to \infty} \left( \frac{2^{(d - 1)}}{\binom{k+2}{2}} \right)^{\frac{1}{k}} \right\}
\]

\[
= 1
\]

For 0 ≤ m ≤ n and m, n ≤ k and k → ∞,

\[
\begin{align*}
    &\min \left\{ \lim_{k \to \infty} \left( \frac{2^n(d - 1)^m}{\binom{k+n}{n}} \right)^{\frac{1}{k+n+2}} \right\} = 1 \\
p_H(B_1) &= 1 \text{ and for } 2 ≤ d ≤ k \text{ and } k \to \infty,
\end{align*}
\]

\[
p_H(B_d) = \frac{1}{2(d - 1)^{\frac{1}{2}}}
\]

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[1] John M. Hammersley. The number of polygons on a lattice. In *Proceedings Cambridge Philosophical Society*, volume 57, pages 516–523. Cambridge Univ Press, 1961.

[2] Harry Kesten. The critical probability of bond percolation on the square lattice equals. *Communications in Mathematical Physics*, 74, 1980.

[3] Bella Bollobas and Oliver Riordan. *Percolation*, chapter 1, pages 7-8. Cambridge University Press, 2006.

[4] Path is a walk via edges visiting each vertex only once.

[5] Limits at http://www.wolframalpha.com.