Decay widths and scattering processes in massive QED2

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(MIT-CTP-2661, September 1997)

Using mass perturbation theory, we infer the bound-state spectrum of massive QED2 and compute some decay widths of unstable bound states. Further, we discuss scattering processes, where all the resonances and particle production thresholds are properly taken into account by our methods.

1. Introduction

QED2 with one massive fermion,

\[ L = \bar{\Psi}(i\not{D} - e\not{A} + m)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \]  \tag{1}

is an interesting toy model for the study of QCD-like properties \cite{1}-\cite{7} (see \cite{7} for a more extensive list of references). It shares with QCD properties like the occurrence of instantons and a nontrivial θ vacuum, the presence of a fermion condensate and confinement of the fundamental fermion of the theory – the spectrum solely consists of mesons. More precisely, we will find two stable particles in the model, a lightest "Schwinger boson", corresponding to the η′ of QCD, and a second particle that may be thought of as a bound state of two Schwinger bosons. Further, there exist unstable higher bound states that may decay into the two stable particles and behave like resonances in scattering cross sections. Using mass perturbation theory, we will perform a resummation of the n-point functions that will prove essential for our results. Further, we will describe the general bound state structure of the model and show how to compute decay widths of the unstable bound states and scattering cross sections of the stable particles.

The massless model, which is the starting point for mass perturbation theory, may be solved exactly \cite{8} \cite{9}. It is equivalent to the theory of a free, massive boson field φ, where φ is related to the vector current, \[ J_\mu = (1/\pi)^{1/2}e_{\mu\nu}\partial^\nu\Phi. \] The two-point function of φ is just the massive scalar boson propagator \[ D_{\mu0}(x - y), \] \[ \mu_0 = e/\sqrt{\pi}. \] Further VEVs that may be computed are VEVs of chiral densities \[ S_\pm = \bar{\Psi}(1/2)(1 \pm \gamma_5)\Psi, \] \[ \langle S_{H_1}(x_1) \cdots S_{H_n}(x_n) \rangle_0 = \]

\[ e^{ik\theta}(\frac{\Sigma}{2})^n \exp \left[ \sum_{i<j} \sigma_i \sigma_j \pi D_{\mu0}(x_i - x_j) \right] \] \tag{2}

where \( \sigma_i = \pm 1 \) for \( H_i = \pm \), \( \theta \) is the vacuum angle and \( k = n_+ - n_- = \sum \sigma_i \) is the instanton number of the contributing instanton sector. Further \( \Sigma \) is the fermion condensate for \( \theta = 0 \), \( \Sigma = \langle \bar{\Psi}\Psi \rangle_\theta = 0 \).

Now the mass perturbation theory may be traced back to space-time integrations of these chiral VEVs, e.g. \( \langle S = S_+ + S_- \rangle \)

\[ Z(m, \theta) = \langle \sum_{n=0}^{\infty} \frac{m^n}{n!} \prod_{i=1}^{n} \int dx_i \bar{\Psi}(x_i)\Psi(x_i) \rangle_0, \] \tag{3}

\[ \langle \hat{O} \rangle_m = \frac{1}{Z(m, \theta)} \langle \hat{O} \sum_{n=0}^{\infty} \frac{m^n}{n!} \prod_{i=1}^{n} \int dx_i S(x_i) \rangle_0. \] \tag{4}

However, as \( \exp(\pm 4\pi D_{\mu0}(x)) \xrightarrow{x \to \infty} 1 \), the perturbation expansion, as it stands, is IR divergent. Therefore, one has to expand the exponentials into the functions \( E_\pm(x) = e^{\pm 4\pi D_{\mu0}(x)} - 1 \).
When all expressions are reexpressed in $E_{\pm}$, it may be shown that the vacuum energy is extensive, $Z(m, \theta) = \exp(V(e(m, \theta)))$, $(V \ldots$ space-time volume, $e(m, \theta) \ldots$ vacuum energy density), and, therefore, all VEVs are finite for $V \to \infty$.

Further, because $S = S_+ + S_-$ occurs in the perturbation expansion (3,4) and $S_+$ and $S_-$ have slightly different VEVs (2), the Feynman rules of the mass perturbation theory acquire a matrix structure. E.g. the propagator connecting two “interaction vertices” $mS$ is given by the matrix

$$E(p) = \begin{pmatrix} E_+(p) & E_-(p) \\ E_-(-p) & E_+(-p) \end{pmatrix}$$

where the individual $++, +-, \ldots$ entries connect the $S_+, S_+, S_-, \ldots$ chiral densities, see (2,3). Because of (2) an arbitrary number of propagators (5) may meet at one vertex, and each vertex is an $n$-th rank tensor $G$ when $n$ propagators meet there [7]. Only two components of this tensor are nonzero,

$$G_{++ \ldots} = g, \quad G_{-\ldots -} = g^*, \quad g = m \Sigma \frac{e^{i\theta}}{2} \quad (6)$$

(corresponding to $S = S_+ + S_-$). The graphical Feynman rules are given in Fig. 1.

The expression corresponding to Fig. 3 reads

$$\mathcal{G}\Pi(p) := \mathcal{G} + \mathcal{G}E(p)\mathcal{G} + \mathcal{G}E(p)\mathcal{G}E(p)\mathcal{G} + \ldots \quad (7)$$

In (7) there occur two types of terms: terms that factorize in momentum space (their graphs fall into two pieces when they are cut at a vertex); the other type terms we call non-factorizable (n.f.) (they are analogous to 1PI graphs of other theories). Each term in $\Pi$ is a n.f. part times an arbitrary part of $\Pi$, therefore it holds that $\Pi(p) = 1 + \Pi^{n.f.}(p)\Pi(p)$, which may be solved,

$$\Pi(p) = \frac{1}{\text{det}(1 - \Pi^{n.f.}(p))} \begin{pmatrix} 1 - \Pi_{n.f.}^{n.f.}(p) & \Pi_{n.f.}^{n.f.} \\ \Pi_{-}^{n.f.} & 1 - \Pi_{+}^{n.f.} \end{pmatrix} \quad (8)$$

where, in leading order, $\Pi^{n.f.}(p)$ is just $\mathcal{E}(p)\mathcal{G}$, $\Pi_{+}^{n.f.}(p) = gE_+(p) + o(g^2)$, etc. We will be especially interested in the determinant in the denominator of (8), because the zeros of its real part will give us the mass poles of the theory, whereas the imaginary parts at the mass poles lead to the decay widths of the unstable bound states.

In an analogous fashion higher $n$-point functions may be reexpressed in terms of exact propagators and n.f. higher $n$-point functions. We show the graphs for the 3- and 4-point functions in Figs. 4, 5 (the triangle and quadrangle denote the n.f. 3- and 4-point functions, respectively).

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2. $n$-point functions, bound-state structure, decay widths

Using the Feynman graphs of Fig. 1, we find for the bosonic two-point function, after amputation of the external boson lines, Fig. 2, where we introduced the exact propagator $\Pi$ that is defined in Fig. 3.
As stated, the determinant in (8) will lead to the bound states and decay widths, therefore let us investigate it more closely.

\[ N(p) := \det(1 - \Pi^{n.f.}(p)) = 
1 - m\Sigma \cos\theta \tilde{E}_+(p) + o(m^2) \quad (9) \]

\[ \tilde{E}_\pm(p) =: \sum_{n=1}^\infty d_n(p), \quad d_n(p) := \frac{(4\pi)^n}{n!} D_{n}^\pm(p) \quad (10) \]

The \( d_n(p) \) are just \( n \)-boson blobs (up to a factor) and behave like follows. They are singular at \( p^2 = (n\mu)^2 \) (real particle production threshold), therefore they are large enough slightly below to balance the small \( m\Sigma \cos\theta \) in (9) and make the whole expression (9) vanish. Further they acquire an imaginary part above the threshold. As a consequence we find mass poles slightly below all \( n \)-boson thresholds in (8) \((n\)-boson bound states). At the lowest bound-state mass \( M_2^2 = (2\mu)^2 - \Delta_2 \), \( N(p) \) has no imaginary part, therefore the lowest bound state \( M_2 \) is stable, like the fundamental boson \( \mu \). For higher mass poles \( M_n^2 = (n\mu)^2 - \Delta_n \) the \( d_i \) have imaginary parts for \( i = 2, \ldots, n-1 \), therefore the \( n \)-boson bound state may decay into \( 2, \ldots, n-1 \) fundamental bosons.

However, there is still something missing. As just stated, the \( M_2 \) is a stable particle, therefore it should be possible in a final state, so where is it? For an answer we need a further resummation. The \( M_2 \) mass pole and propagator is found, in lowest order, by the graphs of Fig. 6. So let us include into \( N(p) \) all the graphs of Fig. 6.

Each graph in Fig. 7 is n.f., therefore they all contribute to \( N(p) \). Further, Fig. 6 is a perfectly legitimate propagator of a stable particle, therefore Fig. 7 is a perfectly legitimate two-boson blob consisting of one \( \mu \) and one \( M_2 \). We call this blob \( d_{1,1}(p) \) and may find by analogous reasoning e.g. a blob of two \( M_2 \), \( d_{2,0} \), etc. Therefore we write for the further resummed \( N(p) \)

\[ N(p) = 1 - m\Sigma \cos\theta \sum_{m,n} d_{m,n}(p) \quad (11) \]

and the additional \( d_{m,n} \) have properties completely analogous to those discussed for the \( d_n \) above. As a consequence we find bound states \( M_{m,n}^2 = (mM_2 + n\mu)^2 - \Delta_{m,n} \) slightly below all thresholds and possible decays into all combinations of \( \mu \) and \( M_2 \) that are allowed kinematically.

Now the bound-state masses may be computed explicitly by solving the equation \( \text{Re} \, N(p) = 0 \) near the individual thresholds (see e.g. [1, 3]). In the vicinity of the mass poles \( \text{Re} \, N \) may be approximated by the first Taylor coefficient,

\[ N(p) \sim M_{m,n}^2 \]

\[ c_{m,n}(p^2 - M_{m,n}^2) + i \text{Im} \, N(M_{m,n}^2) \quad (12) \]
$c_{m,n}$ may be computed from the respective binding energy of the bound state $M_{m,n}$ (see \[\text{[8]}\]), and is, of course, related to the residue of the exact propagator $\Pi$ at the mass pole $M_{m,n}^2$. Im $N(M_{m,n}^2)$ consists of all Im $d_n$ with thresholds below $M_{m,n}^2$, and these Im $d_n$ are wellknown kinematical functions. E.g. for the $M_3$ three-boson bound state we find ($s := p^2$)

$$N(s \sim M_3^2) = c_3(s - M_3^2) - \text{const.}$$

$$i m \Sigma \cos \theta \frac{\sin^2 \theta}{\cos^2 \theta} \text{Im} d_2(M_3^2) + \text{Im} d_{1,1}(M_3^2)$$

(13)

corresponding to the two partial decay channels $M_3 \rightarrow 2\mu$ and $M_3 \rightarrow M_2 + \mu$. The Im $d_2$ term has an additional $\theta$ factor, because the decay $M_3 \rightarrow 2\mu$ is parity forbidden. Comparing with the general expression for the total decay width $\Gamma$ of a bound state $M$,

$$\frac{1}{N(s \sim M^2)} \sim \frac{\text{const.}}{s - M^2 - i \Gamma M}$$

(14)

we find after inserting all the numbers

$$\Gamma_{M_3 \rightarrow 2\mu} \simeq 3.6 \mu \frac{\sin^2 \theta}{\cos^2 \theta} \exp\left(-0.93 \frac{\mu}{m \cos \theta}\right)$$

(15)

$$\Gamma_{M_3 \rightarrow M_2 + \mu} \simeq 44 \mu \exp\left(-0.93 \frac{\mu}{m \cos \theta}\right)$$

(16)

Analogously we may find for the lightest unstable bound state $M_{1,1}$

$$\Gamma_{M_{1,1} \rightarrow 2\mu} \simeq 21340 \mu \left(\frac{m \cos \theta}{\mu}\right)^5 \frac{\sin^2 \theta}{\cos^2 \theta}$$

(17)

Here only one decay channel exists, and the decay is parity forbidden (see \[\text{[8]}\] for details).

3. Scattering processes

A general elastic scattering cross section is

$$\sigma_{ab \rightarrow ab} = C_{\text{sym}} |\mathcal{M}(s)|^2$$

(18)

where the final state symmetry factor $C_{\text{sym}} = 1/(n!)$ for each $n$ identical particles in the final state, $\mathcal{M}$ is the transition matrix element and $w^2(x, y, z) = (x^2 + y^2 + z^2 - 2xy - 2xz - 2yz)$ is a wellknown kinematical function.

For lowest order two-particle elastic scattering we need the first graph on the r.h.s. of Fig. 5. There 4 exact propagators meet at one vertex, and each exact propagator may describe a $\mu$ or $M_2$, because these are the two stable particle poles of $\Pi$. Choosing e.g. $2\mu \rightarrow 2\mu$ elastic scattering for definiteness we obtain in lowest order

$$\sigma_{2\mu \rightarrow 2\mu}(s) = \frac{r_1^2 (m \Sigma \cos \theta)^2}{2w^2(s, \mu^2, \mu^2)}$$

(19)

where $r_1 = 4\pi = \text{Res} \Pi(\mu^2)$ is the residue of the exact propagator at the first mass pole $\mu$.

The first and resummed second order $s$-channel contribution is given by Fig. 8, see Fig. 5.

We again fix the initial state to be $2\mu$, and we allow for an arbitrary final state, because we want to use the optical theorem in the sequel,

$$\sigma_{2\mu \rightarrow f}(s) = \frac{r_1^2}{w(s, \mu^2, \mu^2)} \text{Im} \mathcal{M}_{2\mu \rightarrow 2\mu}(s)$$

(20)

Observe that only initial state factors occur explicitly in (20), therefore Im $\mathcal{M}$ must produce all the final state factors. Further we will restrict to the case $\theta = 0$ in the sequel, because it is easier. In this case $\mathcal{M}$ reads

$$\mathcal{M}^{\theta=0}_{2\mu \rightarrow 2\mu} = \frac{m \Sigma}{1 - m \Sigma(E_+ - E_-)}$$

$$= \frac{m \Sigma}{1 - m \Sigma(d_2 + d_{2,0} + d_4 + \ldots)}$$

(21)
i.e. only parity even blobs occur. Inserting this into (20) we find

$$\sigma_{2\mu \rightarrow f}^{\text{tot}, \theta=0}(s) = \frac{r^2 m^2 \Sigma^2}{w(s, \mu^2, \mu^2)}$$

We indeed recover the required final state factors. Remember that Im \(d_n = (r^n_1/n!)\text{Im} \hat{D}^n_\mu\). Here \(\text{Im} \hat{D}^n_\mu\) is the final state with the phase space integrations, \(r^n_1\) are the residues of the final state \(\mu\), and \(1/n!\) is the final state symmetry factor. Further, it is obvious from Fig. 8 that \(M_2\) may be present in the final state. Therefore, it must be present in the intermediate state, too, in order to saturate the optical theorem, i.e. our inclusion of the general \(d_{m,n}\) into \(N(\mu)\) in the last section is absolutely crucial for unitarity.

Finally we want to evaluate \(\sigma_{2\mu \rightarrow f}^{\text{tot}, \theta=0}(s)\) for some specific values of \(s\). For \((2\mu)^2 < s < M_{2,0}^2\) the \(m\Sigma\) terms in the denominator of (22) are small compared to 1, and further only \(\text{Im} d_2(s) \neq 0\).

As a consequence, (22) reduces to the first order result (19). Next we investigate (22) at the first bound-state mass, \(s = M_{2,0}^2\). There the term \(1 - m\Sigma \text{Re}(\ldots)\) in the denominator vanishes by definition, and we find a \(m^2 \Sigma^2 (\text{Im} d_2)^2\) in the denominator, a \(m^2 \Sigma^2 \text{Re} d_2\) in the numerator, and a further term proportional to \(\text{Im} d_2\) in the numerator from the initial state factor \(w(M_{2,0}^2, \mu^2, \mu^2)\). I.e. everything cancels and we are left with \(\sigma_{2\mu \rightarrow f}^{\text{tot}, \theta=0}(M_{2,0}^2) = 4\). This is much larger than the first order result, i.e. a resonance occurs. At the first higher production threshold \(s = (2M_2)^2\) the term \(\text{Im} d_{2,0}\) is singular. It occurs linearly in the numerator and quadratically in the denominator, therefore the scattering cross section (22) vanishes. In addition, the \(2\mu \rightarrow 2M_2\) inelastic scattering channel opens at \(s = (2M_2)^2\).

For higher \(s\) the above pattern repeats. Therefore we find the following general behaviour. Far away from all thresholds and bound states the scattering cross section is well described by the lowest order result (19). At the position of a bound state a resonance occurs, and the resonance widths are related to the binding energies, because the cross section goes down at the corresponding particle production threshold (a more detailed discussion may be found in \([6, 7]\)).

4. Summary

We have uncovered quite a rich physical structure in the course of our investigation. We found two stable particles in the theory, namely the Schwinger boson \(\mu\) and the two-boson bound state \(M_2\). Higher (unstable) bound states may be formed out of an arbitrary number of \(\mu\) and \(M_2\). Further, these unstable bound states may decay into all combinations of \(\mu\) and \(M_2\) that are possible kinematically.

For scattering processes we found that far from all resonances and particle production thresholds the scattering cross section is well described by a lowest order computation. Whenever it is near a bound-state mass, the scattering cross section has a local maximum, i.e. a resonance occurs. Moreover, for all energies where a new final state becomes possible kinematically, the corresponding real particle production threshold indeed occurs.

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