A FEFFERMAN-STEIN INEQUALITY FOR THE CARLESON OPERATOR

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Abstract. We provide a Fefferman-Stein type weighted inequality for maximally modulated Calderón-Zygmund operators that satisfy an a priori weak type unweighted estimate. This inequality corresponds to a maximally modulated version of a result of Pérez. Applying it to the Hilbert transform we obtain the corresponding Fefferman-Stein inequality for the Carleson operator $C$, that is $C : L^p(M^{p+1}w) \to L^p(w)$ for any $1 < p < \infty$ and any weight function $w$, with bound independent of $w$. Our proof builds on a recent work of Di Plinio and Lerner combined with some results on Orlicz spaces developed by Pérez.

1. Introduction

Let $M$ denote the Hardy-Littlewood maximal operator. In 1971, Fefferman and Stein [10] proved that there is a constant $C_n < \infty$ \footnote{Here and throughout the paper we use the letter $C$ to denote a constant that may change from line to line that is, in particular, independent of the function $f$ and the weight $w$. We also use the notation $A \lesssim B$ to denote that there is a constant $C$ such that $A \leq CB$.} such that for any $1 < p < \infty$

\begin{equation}
\int_{\mathbb{R}^n} |Mf(x)|^pw(x)dx \leq C_n^p \int_{\mathbb{R}^n} |f(x)|^pMw(x)dx
\end{equation}

holds for all weight functions $w$. By weight we mean a non-negative locally integrable function.

Inequalities like (1) have been of considerable interest in recent years. On an informal level, given an operator $U$ and an exponent $p \in [1, \infty)$, one studies if there is a constant $C < \infty$ such that

\begin{equation}
\int_{\mathbb{R}^n} |Uf(x)|^pw(x)dx \leq C \int_{\mathbb{R}^n} |f(x)|^pMw(x)dx
\end{equation}

holds for all admissible functions $f$ and weights $w$, where $M$ is a suitable maximal operator. By an elementary duality argument, one can use inequality (2) to transfer bounds from $M$ to $U$, that is,

\begin{equation}
\|U\|_{L^q \to L^{\hat{q}}} \lesssim \|M\|_{L^{\hat{q}(p') \to L^q(p')}}^{1/p}
\end{equation}

for $q, \hat{q} \geq p$. Then, for any fixed exponent $p$, it is of particular interest to identify a maximal operator $M$ satisfying (2) which is optimal, in the sense that Lebesgue space bounds for $M$ allow one to obtain optimal Lebesgue space bounds for $U$ via (3). See Bennett and Harrison [2] and Bennett [1] for recent examples of such optimal control by maximal operators.

We consider the case of a Calderón-Zygmund operator on $\mathbb{R}^n$, that is, an $L^2$ bounded operator represented as

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y)f(y)dy, \quad x \notin \text{supp } f,$$

where the kernel $K$ satisfies

\begin{enumerate}
\item $|K(x, y)| \leq \frac{C}{|x-y|^n}$ for all $x \neq y$;
\item $|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \leq C\frac{|x-x'|^\delta}{|x-y|^{n+\epsilon}}$ for some $0 < \delta \leq 1$ when $|x-x'| < |x-y|/2$.
\end{enumerate}
Córdoba and Fefferman [5] showed that for \( s > 1, 1 < p < \infty \), there is a constant \( C_{p,s} < \infty \) such that
\[
\int_{\mathbb{R}^n} |Tf(x)|^p w(x)\,dx \leq C_{p,s} \int_{\mathbb{R}^n} |f(x)|^p M_s w(x)\,dx
\] (4)
holds for any weight \( w \), where \( M_s w(x) = (M w^s(x))^{1/s} \). Observe that for each \( s > 1 \), the operator \( M_s \) fails to be bounded on \( L^q \) for \( 1 < q \leq s \). Thus, for a fixed \( 1 < p < \infty \), \( M_s w \) is not an optimal maximal operator, since via the inequality (3) we can only recover \( L^q \) bounds for \( T \) in the restricted range \( p \leq q < ps' \), missing the exponents in \([ps', \infty)\); we recall that \( T \) is an \( L^q \) bounded operator for \( 1 < q < \infty \). This problem was resolved by Wilson [25] in the range \( 1 < p \leq 2 \) and by Pérez [21] in the whole range \( 1 < p < \infty \), who showed that for \( 1 < p < \infty \), there is a constant \( C < \infty \) such that
\[
\int_{\mathbb{R}^n} |Tf(x)|^p w(x)\,dx \leq C \int_{\mathbb{R}^n} |f(x)|^p M^{[p]+1} w(x)\,dx
\] holds for any weight \( w \). Here \([p]\) denotes the integer part of \( p \) and \( M^{[p]+1} \) denotes the \((|p|+1)\)-fold composition of \( M \). The operator \( M^{[p]+1} \) is bounded on \( L^q \), \( 1 < q < \infty \), for any \( p \). Thus, given \( 1 < p < \infty \), it is an optimal maximal operator since we can recover via (3) the \( L^q \) boundedness of \( T \) for the whole range \( p \leq q < \infty \). Furthermore, their result is best possible in the sense that it fails if \( M^{[p]+1} \) is replaced by \( M^p \). It should be noted that for each \( r > 1 \) and \( k \geq 1 \), the pointwise estimate \( M^k w(x) \leq CM_r w(x) \) holds for some constant \( C \) independent of \( w \).

The goal of this paper is to extend (5) to a broad class of maximally modulated Calderón-Zygmund operators studied previously by Grafakos, Martell and Soria [11] and Di Plinio and Lerner [7]. Let \( \Phi = \{\phi_\alpha\}_{\alpha \in A} \) be a family of real-valued measurable functions indexed by an arbitrary set \( A \). Then the maximally modulated Calderón-Zygmund operator \( T^\Phi \) is defined by
\[
T^\Phi f(x) = \sup_{\alpha \in A} |T(\mathcal{M}^{\phi_\alpha} f)(x)|,
\] (6)
where \( \mathcal{M}^{\phi_\alpha} f(x) = e^{2\pi i \phi_\alpha(x)} f(x) \). We will consider operators \( T^\Phi \) that satisfy the \textit{a priori} weak-type inequalities
\[
\|T^\Phi f\|_{L^p,w} \leq \psi(p') \|f\|_p
\] (7)
for \( 1 < p \leq 2 \), where \( \psi \) is a non-decreasing function on \([1, \infty)\). This definition is motivated by the Carleson operator,
\[
C f(x) = \sup_{\alpha \in \mathbb{R}} \left| F \right|_{L^p,w} \left| f \right|_{L^p} = \int_{\mathbb{R}} \frac{e^{2\pi iy}}{x - y} f(y) \, dy,
\] (8)
since it can be recovered from (6) by setting \( T = H \) and \( \Phi \) to be the family of functions given by \( \phi_\alpha(x) = \alpha x \) for \( \alpha \in \mathbb{R} \). Expressing \( C f \) in terms of \( \hat{f} \) allows it to be reconciled with the classical expression for the Carleson maximal operator in terms of partial Fourier integrals.

Implicit in the work of Di Plinio and Lerner [7] there is the following analogue of the estimate (4) for maximally modulated Calderón-Zygmund operators.

**Theorem 1.1.** Let \( T^\Phi \) be a maximally modulated Calderón-Zygmund operator satisfying (7). Then for \( 1 < s < 2 \) and \( 1 < p < \infty \) there is a constant \( C < \infty \) such that for any weight \( w \)
\[
\int_{\mathbb{R}^n} |T^\Phi f(x)|^p w(x)\,dx \leq C \int_{\mathbb{R}^n} |f(x)|^p M_s w(x)\,dx.
\] (9)

Note that the inequality (4) can be recovered from (9) simply by taking \( \phi_\alpha \equiv 0 \) for all \( \alpha \). As in the case of (4), for any fixed \( 1 < p < \infty \) and \( 1 < s < 2 \), Theorem 1.1 does not allow one to recover the full range of Lebesgue space bounds for \( T^\Phi \) from those for \( M_s \) via (3).

One can address this question and obtain optimal control of \( T^\Phi \) by combining the ideas developed by Pérez in [21] and [22] with Di Plinio and Lerner’s argument [7]. The main result of this paper is the following.
Theorem 1.2. Let $T^\Phi$ be a maximally modulated Calderón-Zygmund operator satisfying (7). Then for any $1 < p < \infty$ there is a constant $C < \infty$ such that for any weight $w$

\begin{equation}
\int_{\mathbb{R}^n} |T^\Phi f(x)|^p w(x) dx \leq C \int_{\mathbb{R}^n} |f(x)|^p M|^p|+1 w(x) dx.
\end{equation}

This is best possible in the sense that $|p| + 1$ cannot be replaced by $|p|$. Of course, one can recover the estimate (5) from Theorem 1.2. As observed for (5), given $1 < p < \infty$, the control given by the maximal operator $M|^p|+1$ is optimal here.

Indeed Theorem 1.2 can be viewed as a corollary of a more precise statement, that allows one to replace $M|^p|+1$ by a sharper class of maximal operators. This strategy is the same as the one in Pérez [21] for the case of unmodulated Calderón-Zygmund operators.

Let $A$ be a Young function, that is, $A : [0, \infty) \to [0, \infty)$ is a continuous, convex, increasing function with $A(0) = 0$ and such that $A(t) \to \infty$ as $t \to \infty$. For each cube $Q \subset \mathbb{R}^n$, we define the Luxemburg norm of $f$ over $Q$ by

$$\|f\|_{A,Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q A \left( \frac{|f(y)|}{\lambda} \right) dy \leq 1 \right\}$$

and the maximal operator $M_A$ by

\begin{equation}
M_A f(x) = \sup_{Q \ni x} \|f\|_{A,Q},
\end{equation}

where $f$ is a locally integrable function and the supremum is taken over all cubes $Q$ in $\mathbb{R}^n$ containing $x$.

In this context we are able to characterize the class of Young functions for which a Fefferman-Stein inequality holds with controlling maximal operator $M = M_A$.

Theorem 1.3. Let $T^\Phi$ be a maximally modulated Calderón-Zygmund operator satisfying (7). Suppose that $A$ is a Young function satisfying

\begin{equation}
\int_c^\infty \left( \frac{t}{A(t)} \right)^{p'-1} \frac{dt}{t} < \infty
\end{equation}

for some $c > 0$. Then for any $1 < p < \infty$ there is a constant $C < \infty$ such that for any weight $w$

\begin{equation}
\int_{\mathbb{R}^n} |T^\Phi f(x)|^p w(x) dx \leq C \int_{\mathbb{R}^n} |f(x)|^p M_A w(x) dx.
\end{equation}

In the unmodulated setting, Pérez [21] pointed out that condition (12) is necessary for (13) to hold for the Riesz transforms. Hence it also becomes a necessary condition for Theorem 1.3 to be stated in such a generality, characterizing the class of Young functions for which (13) holds.

As observed in [7], the Carleson operator $C$ satisfies condition (7) and one can thus deduce the corresponding Fefferman-Stein inequality.

Corollary 1.4. Let $C$ be the Carleson operator. Then for any $1 < p < \infty$ there is a constant $C < \infty$ such that for every weight $w$

\begin{equation}
\int_\mathbb{R} |Cf(x)|^p w(x) dx \leq C \int_\mathbb{R} |f(x)|^p M^{|p|+1} w(x) dx.
\end{equation}

Weighted inequalities for the Carleson operator have been previously studied by many authors. Hunt and Young [14] established the $L^p(w)$ boundedness of $C$ for $1 < p < \infty$ and $w \in A_p$. Later Grafakos, Martell and Soria [11] gave new weighted inequalities for weights in $A_q$, as well as vector-valued inequalities for $C$. More recently, Do and Lacey [8] gave weighted estimates for a variation norm version of $C$ in the context of $A_p$ theory that strengthened the results in [14]. Finally, Di Plinio and Lerner [7] obtained $L^p(w)$ bounds for $C$ in terms of the $[w]_{A_q}$ constants for $1 \leq q \leq p$. Note that inequality (14) does not fall within the scope of the classical $A_p$ theory.
This paper is organised as follows. In Section 2 we present some powerful results due to Lerner [19, 18, 7] that allow one to bound in norm $T^\Phi$ by the so-called dyadic sparse operators. In Section 3 we present the results obtained by Pérez in [21] and [22] concerning the maximal operator associated to a Young function. In Section 4 we present the proof of Theorem 1.3 and how to apply it to deduce Theorem 1.2. Finally, Section 5 contains some applications that can be deduced from our main result.

Acknowledgements. The author would like to thank his supervisor Jon Bennett for his continuous support and for many valuable comments on the exposition of this paper.

2. A norm estimate by dyadic sparse operators

Here we present a result in [7] that allows one to reduce the proof of (13) to a Fefferman-Stein inequality for a dyadic sparse operator. This reduction rests on a certain local mean oscillation estimate. Such estimates have been developed by Lerner and other authors and have become a powerful technique over the last few years. See, for instance, [16, 17, 18, 7, 12, 15].

Let $D$ be a general dyadic grid, that is a collection of cubes such that

(i) any $Q \in D$ has sidelength $2^k$, $k \in \mathbb{Z}$;
(ii) for any $Q, R \in D$, we have $Q \cap R = \{Q, R, \emptyset\}$;
(iii) the cubes of a fixed sidelength $2^k$ form a partition of $\mathbb{R}^n$.

We say that $S$ is a sparse family of cubes if for any cube $Q \in S$ there is a measurable subset $E(Q) \subset Q$ such that $|Q| \leq 2|E(Q)|$ and the sets $(E(Q))_{Q \in S}$ are pairwise disjoint.

Given a sparse family $S$ consider the dyadic sparse operator

$$A_{r,S}f(x) = \sum_{Q \in S} \left( \frac{1}{|Q|} \int_Q |f|^r \right)^{1/r} \chi_Q(x),$$

where $\bar{Q} = 2\sqrt{n}Q$. Via a local mean oscillation estimate for $T^\Phi$, Di Plinio and Lerner [7] obtained the following estimate for $\|T^\Phi f\|_{L^p(w)}$ in terms of $\|A_{r,S}\|_{L^p(w)}$.

**Proposition 2.1** ([7]). Let $T^\Phi$ be a maximally modulated Calderón-Zygmund operator satisfying (7). Let $1 < p < \infty$ and let $w$ be an arbitrary weight. Then

$$\|T^\Phi f\|_{L^p(w)} \lesssim \inf_{1 < r \leq 2} \left\{ \psi(r') \sup_{D, S} \|A_{r,S}\|_{L^p(w)} \right\}$$

where the supremum is taken over all dyadic grids $D$ and all sparse families $S \subset D$.

3. Bounds for the maximal operator

Let $B$ be a Young function. We define the complementary Young function $\bar{B}$ to be the Legendre transform of $B$, that is

$$\bar{B}(t) = \sup_{s > 0} \{st - B(s)\}, \quad t > 0.$$ 

We have that $\bar{B}$ is also a Young function, and it satisfies

$$t \leq B^{-1}(t) \bar{B}^{-1}(t) \leq 2t$$

for $t > 0$. Pérez [22] characterised the Young functions $B$ such that $M_B$ is of strong type $(p, p)$ for $p > 1$ and established that the $L^p$ boundedness is equivalent to certain weighted inequalities for $M_B$ and related maximal operators.

**Theorem 3.1** ([22]). Let $1 < p < \infty$. Let $A$ and $B$ be Young functions satisfying $\bar{B}(t) = A(t^p)$. Then the following are equivalent:
(i) There is a constant $c > 0$ such that
\[
\int_{c}^{\infty} \left( \frac{t}{A(t)} \right)^{p-1} \frac{dt}{t} < \infty.
\]

(ii) There is a constant $C < \infty$ such that
\[
\int_{\mathbb{R}^n} M_B f(x)^p \, dx \leq C \int_{\mathbb{R}^n} f(x)^p \, dx
\]
for all non-negative functions $f$.

(iii) There is a constant $C < \infty$ such that
\[
\int_{\mathbb{R}^n} M_B f(x)^p u(x) \, dx \leq C \int_{\mathbb{R}^n} f(x)^p u(x) \, dx
\]
for all non-negative functions $f$ and any weight $u$.

(iv) There is a constant $C < \infty$ such that
\[
\int_{\mathbb{R}^n} M f(x)^p \frac{u(x)}{(M_A w(x))^{p-1}} \, dx \leq C \int_{\mathbb{R}^n} f(x)^p \frac{M u(x)}{w(x)^{p-1}} \, dx
\]
for all non-negative functions $f$ and any weights $u$, $w$.

A classical result from Coifman and Rochberg [4] asserts that for any locally integrable function $w$ such that $Mw(x) < \infty$ a.e. and $0 < \delta < 1$, the function $(Mw)^\delta(x)$ is an $A_1$ weight with constant independent of $w$. More precisely,
\[
M \left( (Mw)^\delta \right)(x) \leq C_n \frac{1}{1-\delta} (Mw)^\delta(x)
\]
for almost all $x \in \mathbb{R}^n$. As Pérez [21] remarks, this result still holds when one replaces the Hardy-Littlewood maximal function by the maximal operator $M_A$.

**Proposition 3.2.** Let $A$ be a Young function. If $0 < \delta < 1$, then $(M_A w)^\delta \in A_1$ with $A_1$ constant independent of $w$. In particular,
\[
M \left( (M_A w)^\delta \right)(x) \leq C_n \frac{1}{1-\delta} (M_A w)^\delta(x)
\]
for almost all $x \in \mathbb{R}^n$.

One can find a proof of this result in [6] (Proposition 5.32). For the sake of completeness, we give in the Appendix an alternative proof following the classical approach from [4].

### 4. Proof of Theorem 1.3

In this section we give a proof of Theorem 1.3 and we use it, thanks to an observation due to Pérez [21, 22], to deduce Theorem 1.2. Our proof follows the same pattern of a proof of Di Plinio and Lerner in [7].

As seen in Section 2, the boundedness of $T^\Phi$ can be essentially reduced to the uniform boundedness of the dyadic sparse operators $A_{r,S}$. In particular, we have the following Fefferman-Stein inequality for $A_{r,S}$.

**Lemma 4.1.** Let $D$ be a dyadic grid and $S \subset D$ a sparse family of cubes. Suppose that $A$ is a Young function satisfying (12). Then for $1 < p < \infty$, there is a constant $C_{n,p,A} < \infty$ independent of $S$, $D$ and the weight $w$ such that
\[
\|A_{r,S}\|_{L^p(w)} \leq C_{n,p,A} \left( \frac{p + 1}{2r} \right)^{1/r} \|f\|_{L^p(M_A w)}
\]
holds for any $1 < r < \frac{p+1}{2}$ and any non-negative function $f$. 
Proof. We first linearize the operator $A_{r,S}$. For any $Q$, by $L^p$ duality, there exists $g_Q$ supported in $\tilde{Q}$ such that $\frac{1}{|Q|} \int_Q g_Q = 1$ and
\[
\left( \frac{1}{|Q|} \int_Q f^r \right)^{1/r} = \frac{1}{|Q|} \int_Q f g_Q.
\]
We can thus define a linear operator $L$ by
\[
Lh(x) = \sum_{Q \in S} \left( \frac{1}{|Q|} \int_Q h g_Q \right) \chi_Q(x).
\]
Note that $L(f) = A_{r,S}f$. Then, in order to obtain an estimate for $\|A_{r,S}\|_{L^p(w)}$, independent of $S$ and $D$, it is enough to obtain the corresponding estimate for $\|Lh\|_{L^p(w)}$ uniformly in the functions $g_Q$. By duality, the estimate
\[
\|Lh\|_{L^p(w)} \leq C_{n,p,A} \left( \left( \frac{p+1}{2r} \right)^{r/p} \right)^{1/r} \|h\|_{L^p(M_A w)}
\]
is equivalent to
\[
\|L^* h\|_{L^{p'}((M_A w)^{1-p'})} \leq C_{n,p,A} \left( \left( \frac{p+1}{2r} \right)^{r/p} \right)^{1/r} \|Mh\|_{L^{p'}((M_A w)^{1-p'})}
\]
where $L^*$ denotes the $L^2(\mathbb{R}^n)$-adjoint operator of $L$. Since $A$ satisfies (12), one can apply Theorem 3.1 with $p$ replaced by $p'$. Then using (15) with $u = 1$, the estimate (16) follows from
\[
\|L^* h\|_{L^{p'}((M_A w)^{1-p'})} \leq C_n \left( \left( \frac{p+1}{2r} \right)^{r/p} \right)^{1/r} \|Mh\|_{L^{p'}((M_A w)^{1-p'})}.
\]
We focus then on obtaining (17). By duality, there exists $\eta \geq 0$ such that $\|\eta\|_{L^p(M_A w)} = 1$ and
\[
\|L^* h\|_{L^{p'}((M_A w)^{1-p'})} = \int_{\mathbb{R}^n} L^* (h) \eta dx = \int_{\mathbb{R}^n} h L\eta dx.
\]
By Hölder’s inequality and the $L^{p'}$ boundedness of $g_Q$,
\[
\int_{\mathbb{R}^n} h L\eta dx = \sum_{Q \in S} \left( \frac{1}{|Q|} \int_Q \eta g_Q \right) \int_Q h \leq \sum_{Q \in S} \left( \frac{1}{|Q|} \int_Q \eta \right)^{1/r} \left( \frac{1}{|Q|} \int_Q h \right)^{1/r} \|\eta\|_{L^p(M_A w)}
\]
\[
\leq \sum_{Q \in S} \left( \frac{1}{|Q|} \int_Q \eta \right)^{1/r} \left( \frac{1}{|Q|} \int_Q h \right) (2\sqrt{n}) |Q|
\]
\[
= (2\sqrt{n})^n \sum_{Q \in S} \left( \frac{1}{|Q|} \int_Q \eta \right)^{1/r} \left( \frac{1}{|Q|} \int_Q h \right)^{1/r} \left( \frac{1}{|Q|} \int_Q h \right)^{\frac{p}{p'} |Q|}.
\]
Recall that by definition of the Hardy-Littlewood maximal operator
\[
\frac{1}{|Q|} \int_Q h(x) dx \leq Mh(y)
\]
holds for every $y \in Q$. Combining this and the sparseness of $S$
\[
\int_{\mathbb{R}^n} h L\eta dx \leq 2(2\sqrt{n})^n \sum_{Q \in S} \left( \frac{1}{|Q|} \int_Q \left( (Mh)^{\frac{p}{p'+r}} \eta \right)^{1/r} \left( \frac{1}{|Q|} \int_Q h \right)^{\frac{p}{p'+r}} |E(Q)|
\]
\[
\leq 2(2\sqrt{n})^n \sum_{Q \in S} \int_{E(Q)} M_r((Mh)^{\frac{p}{p'+r}} \eta)(Mh)^{\frac{p}{p'+r}} dx
\]
\[
\leq 2(2\sqrt{n})^n \int_{\mathbb{R}^n} M_r((Mh)^{\frac{p}{p'+r}} \eta)(Mh)^{\frac{p}{p'+r}} dx,
\]
where we have used that \((E(Q))_{Q \in \mathcal{S}}\) are pairwise disjoint and that (19) also holds for \(y \in E(Q) \subset Q \subset \tilde{Q}\).

By Hölder’s inequality with exponents \(\rho = \frac{p + 1}{2}r\) and \(\rho' = \frac{p + 1}{p + 2}r',\)

\[
(20) \quad 2(2 \sqrt{n})^n \int_{\mathbb{R}^n} M_r((Mh)_{p+1} \eta)(M_Aw)^{p+1} (Mh)^{p+1} \frac{1}{p+1} \frac{1}{r+1} dx
\]

\[
\leq 2(2 \sqrt{n})^n \| M_r((Mh)_{p+1} \eta) \|_{L^{p+1}} ((M_Aw)^{p+1}) \| Mh \|_{L^{p+1}}((M_Aw)^{-p'})^{\frac{1}{p+1}} dx.
\]

For \(r < \frac{p + 1}{2},\) we can apply the classical Fefferman-Stein inequality (1) to the first term in (21)

\[
\| M_r((Mh)_{p+1} \eta) \|_{L^{p+1}} ((M_Aw)^{p+1}) \leq C_n \left( \left( \frac{p + 1}{2r} \right) \right)^{\frac{1}{r+1}} \| Mh \|_{L^{p+1}}((M_Aw)^{p+1})^{\frac{1}{r+1}} dx.
\]

and by Proposition 3.2

\[
\| (Mh)_{p+1} \eta \|_{L^{p+1}}((M_Aw)^{p+1}) \leq C_n \| (Mh)_{p+1} \eta \|_{L^{p+1}}((M_Aw)^{p+1}).
\]

Finally, by an application of Hölder’s inequality with \(\rho = 2p'\) and \(\rho' = \frac{2p}{p + 1}\)

\[
\| (Mh)_{p+1} \eta \|_{L^{p+1}}((M_Aw)^{p+1}) = \left( \int_{\mathbb{R}^n} \left( (Mh)_{\frac{1}{2}} (M_Aw)^{-\frac{1}{2}} \right) \left( \eta_{\frac{p + 1}{2}} (M_Aw)^{\frac{p + 1}{2}} \right) dx \right)^{\frac{1}{p+1}}
\]

\[
\leq \| Mh \|_{L^{p+1}}((M_Aw)^{p+1}) \| \eta \|_{L^{p+1}}((M_Aw)^{p+1}) = \| Mh \|_{L^{p+1}}((M_Aw)^{p+1})^{\frac{1}{p+1}} dx.
\]

where the last equality holds since \(\| \eta \|_{L^{p+1}}((M_Aw)^{p+1}) = 1.\) So altogether,

\[
\| L^p h \|_{L^{p+1}((M_Aw)^{p+1})} \leq 2(2 \sqrt{n})^n C_n \left( \left( \frac{p + 1}{2r} \right) \right)^{\frac{1}{r+1}} \| Mh \|_{L^{p+1}((M_Aw)^{p+1})}.\]

This concludes the proof. \(\square\)

We are now able to prove Theorem 1.3.

**Proof of Theorem 1.3.** By Lemma 2.1, it is enough to show that for any \(1 < p < \infty,\)

\[
(22) \quad \inf_{1 < r \leq 2} \left\{ \psi(r') \sup_{\mathcal{D}, \mathcal{S}} \| \mathcal{A}_{r,S} f \|_{L^p} \right\} \leq \| f \|_{L^p((M_Aw)^{p+1})}.
\]

By Lemma 4.1,

\[
(23) \quad \sup_{\mathcal{D}, \mathcal{S}} \| \mathcal{A}_{r,S} \|_{L^p} \leq C_{n,p,A} \left( \left( \frac{p + 1}{2r} \right) \right)^{\frac{1}{r}} \| f \|_{L^p((M_Aw)^{p+1})}
\]

for any \(1 < r < \frac{p + 1}{2},\) since the bound was independent of \(\mathcal{D}, \mathcal{S}.\)

To see (22), we argue differently depending on whether \(p \geq 3\) or not. If \(p \geq 3,\) the inequality (23) is valid for all \(1 < r < 2.\) Then (22) follows from the estimate

\[
\inf_{1 < r \leq 2} \left\{ \psi(r') \left( \left( \frac{p + 1}{2r} \right) \right)^{\frac{1}{r}} \right\} \leq \psi(3) \left( \left( \frac{p + 1}{3} \right) \right)^{\frac{2}{3}} \leq 4 \psi(3) \leq 4p' \psi(3p'),
\]

where we have bounded the infimum by its value in \(r = 3/2\) and we have used that \(\psi\) is an increasing function.
For $p < 3$, we bound the infimum by its value in $r = \frac{p + 3}{4(p - 1)} < \frac{p + 1}{2}$. Then, using (23),

$$\inf_{1 < r < 2} \left\{ \psi(r') \sup_{D,S} \| A_{r,S} f \|_{L^p(w)} \right\} \leq \psi \left( \frac{p + 3}{4(p - 1)} \right) \sup_{D,S} \| A_{r,S} f \|_{L^p(w)}$$

$$\leq C_{n,p,A} (p') \left( \frac{2(p + 1)}{p + 3} \right)^1 \| f \|_{L^p(M_A w)}$$

$$\leq C_{n,p,A} 4p' \psi(3p') \| f \|_{L^p(M_A w)}.$$

This concludes the proof. \[\square\]

Observe that this proof of Theorem 1.3 could be extended to other operators whose bounds depend in a suitable way on those of $A_{r,S}$.

Now one can deduce Theorem 1.2 from Theorem 1.3 via the following observation due to Pérez [21, 22].

**Proof of Theorem 1.2.** Using Theorem 1.3, it is enough to prove that there exists a Young function $A$ satisfying (12) such that

$$M_A w(x) \leq CM^{[p]+1} w(x)$$

with $C$ independent of $w$. Let $A(t) = t \log^{[p]}(1 + t)$. It is an elementary computation to show that $A$ satisfies (12) for any $c > 0$. Then it suffices to prove that there is a constant $C < \infty$ such that for every cube $Q$

$$\|w\|_{A,Q} \leq C \frac{1}{|Q|} \int_Q M^{[p]} w(x) dx =: \lambda_Q.$$

This is equivalent to showing that

$$\left\| \frac{w}{\lambda_Q} \right\|_{A,Q} \leq 1,$$

which by definition of the Luxemburg norm will follow from

$$\frac{1}{|Q|} \int_Q A \left( \frac{w(x)}{\lambda_Q} \right) dx = \frac{1}{|Q|} \int_Q \frac{w(x)}{\lambda_Q} \log^{[p]} \left( 1 + \frac{w(x)}{\lambda_Q} \right) dx \leq 1.$$

Iterating $[p]$ times the inequality

$$\int_Q f(x) \log^{[p]}(1 + f(x)) dx \leq \tilde{C} \int_Q M f(x) \log^{[p] - 1}(1 + M f(x)) dx$$

from [23], with $f = w/\lambda_Q$, we obtain

$$\frac{1}{|Q|} \int_Q \frac{w(x)}{\lambda_Q} \log^{[p]} \left( 1 + \frac{w(x)}{\lambda_Q} \right) dx \leq \frac{\tilde{C}[p]}{|Q|} \int_Q M^{[p]} \left( \frac{w}{\lambda_Q} \right)(x) dx.$$

By choosing $C = \tilde{C}[p] < \infty$, we have

$$\frac{1}{|Q|} \int_Q A \left( \frac{w(x)}{\lambda_Q} \right) dx \leq 1.$$

Thus $M_A w(x) \leq M^{[p]+1} w(x)$, as required.

Finally, one can’t replace $[p] + 1$ by $[p]$ in the statement of Theorem 1.2, since the resulting inequality is shown to be false for the (unmodulated) Hilbert transform [21]. \[\square\]

### 5. Applications

In this section we apply Theorem 1.2 to some well-known operators. First one needs to check that the specific operator satisfies (7). Fortunately, there is a convenient sufficient condition for (7) to hold.
Proposition 5.1. Let \( T^\Phi \) be a maximally modulated Calderón-Zygmund operator satisfying
\[
\|T^\Phi(\chi_E)\|_{L^p,\infty} \lesssim (p')^m |E|^{1/p},
\]
for \( 1 < p \leq 2 \) for some \( m \geq 1 \). Then (7) holds with
\[
\psi(p') = \sup_{t \geq 1} \frac{\Gamma_m(t)}{tp'},
\]
where \( \Gamma_m(t) = t(\log(e + t))^m \log \log (e^{e^t} + t) \).

Note that \( \psi \) is obviously non-decreasing as a function of \( p' \). Proposition 5.1 follows from combining results in [11] with a result in [7]. We refer to Section 6 in [7] for further comments.

5.1. The Carleson operator. Recall that the Carleson operator can be defined as
\[
Cf(x) = \sup_{\xi \in \mathbb{R}} |H(e^{2\pi i \xi x} f)(x)|.
\]
It is well known [13] that \( C \) satisfies the following restricted weak type inequality
\[
|\{x : C(\chi_E)(x) > \lambda\}|^\frac{1}{p} \leq C \left( \frac{p^2}{p - 1} \right)^{\frac{1}{p}} |E|^{\frac{1}{p}}
\]
for every \( \lambda > 0 \), \( 1 < p \leq 2 \). Thus applying Proposition 5.1 and Theorem 1.2 to the Carleson operator, one obtains Corollary 1.4, that is
\[
\int_{\mathbb{R}} |Cf(x)|^p w(x) dx \leq C \int_{\mathbb{R}} |f(x)|^p M^{[p]+1} w(x) dx
\]
for any weight \( w \).

5.2. Maximal multiplier of bounded variation. The essence of the classical Marcinkiewicz multiplier theorem is the observation that a multiplier of bounded variation on the line often satisfies the same norm inequalities as the Hilbert transform. In particular, if \( m \) is a bounded variation multiplier and \( T_m \) is its associated operator, one can deduce
\[
\int_{\mathbb{R}} |T_m f(x)|^p w(x) dx \leq C \int_{\mathbb{R}} |f(x)|^p M^{[p]+1} w(x) dx
\]
for any weight \( w \). Using Corollary 1.4, one can deduce the analogous maximal-multiplier inequality in the sense of Oberlin [20]. Consider the maximal-multiplier operator
\[
M_{BV} f(x) = \sup_{m : \|m\|_{BV} \leq 1} |\langle m \hat{f}\rangle|^\gamma(x)
\]
where the supremum is taken over all functions whose variation norm is less or equal than 1. Recall that the variation norm is defined by
\[
\|m\|_{BV} := \|m\|_x + \sup_{N, \xi_0 < \cdots < \xi_N} \left( \sum_{i=1}^{N} |f(\xi_i) - f(\xi_{i-1})| \right),
\]
where the supremum is over all strictly increasing finite length sequences of real numbers. The second term in the right hand side of (24) is the total variation of \( m \).

Theorem 5.2. For \( 1 < p < \infty \), there is a constant \( C < \infty \) such that
\[
\int_{\mathbb{R}} |M_{BV} f(x)|^p w(x) dx \leq C \int_{\mathbb{R}} |f(x)|^p M^{[p]+1} w(x) dx
\]
holds for any weight \( w \).

Proof. Since \( m \) is of global bounded variation,
\[
T_m f(x) = cf(x) + \int_{\mathbb{R}} S_{(t, x)} f(x) dm(t) \leq cf(x) + \int_{\mathbb{R}} C f(x) dm(t),
\]
where \( dm \) denotes the Lebesgue-Stieltjes measure associated to \( m \). Then
\[
\sup_{m:|m|_{BV} \leq 1} |T_m f(x)| \leq c |f(x)| + |C f(x)| \quad \sup_{m:|m|_{BV} \leq 1} \int \lambda \, dm(t) \leq c |f(x)| + |C f(x)|,
\]
where the last inequality follows since the integral of \( |dm| \) corresponds to the total variation of \( m \). The proof concludes by taking \( L^p(w) \) norms and using Corollary 1.4. \( \square \)

**Remark 5.3.** Let \( m \) be a multiplier of bounded variation and let \( m(t, \xi) = m(t \xi) \). Consider the maximal operator associated to these multipliers, that is,
\[
T^*_m f(x) = \sup_{t > 0} |(m(t \xi))^{-1}(x)|.
\]
Since \( m \) and \( m_t \) have the same variation norm, we have \( T^*_m f(x) \leq \|m\|_{BV} M_{BV} f(x) \), so the inequality (25) also holds for \( T^*_m \) in place of \( M_{BV} \).

5.3. **Carleson-type operators in higher dimensions.** Concerning higher dimensional Carleson operators, the Fefferman-Stein inequality also holds for the operator
\[
C_P f(x) := \sup_{t > 0} \left| \int_{P(t)} \hat{f} \left( e^{i \omega} \xi \right) dx \right|,
\]
where \( P \) is a polyhedron with finitely many faces and the origin in its interior. Indeed, Fefferman deduced in [9] that the norm of this operator is bounded by the norm of the one-dimensional Carleson operator \( C \) in any Banach space.

6. **Further remarks**

Let \( \Lambda = \{ \lambda_j \}_j \) be a lacunary sequence of integers, that is, \( \lambda_{j+1} \geq \theta \lambda_j \) for all \( j \) and for some \( \theta > 1 \) and consider the lacunary Carleson maximal operator
\[
C_\Lambda f(x) = \sup_{j \in \mathbb{N}} \left| \int_{\mathbb{R}} e^{2 \pi i \lambda_j y} f(y) dy \right|.
\]
Of course one has the pointwise estimate \( C_\Lambda f(x) \leq C f(x) \), so the Fefferman-Stein inequality (10) trivially holds for \( C_\Lambda \). This can be reconciled with a Fefferman-Stein inequality for \( C_\Lambda \) obtained by more classical techniques. Consider the more classical version of the lacunary Carleson operator in terms of the lacunary partial Fourier integrals. Following the lines of [3],
\[
S^*_\Lambda f(x) = \sup_k |S_{\lambda_k} f(x)| \leq c M f(x) + \left( \sum_k |S_{\lambda_k} f \ast \psi_k(x)|^2 \right)^{1/2},
\]
where \( \psi \) is a suitable Schwartz function and \( S_{\lambda_k} \hat{f}(\xi) = \chi_{[-\lambda_k, \lambda_k]}(\xi) \hat{f}(\xi) \). Since \( S_{\lambda_k} \) satisfies the same Lebesgue space inequalities as the Hilbert transform, from the estimate (5) and weighted Littlewood-Paley theory (which can be obtained via a standard Rademacher function argument and the results from Pérez [21] and Wilson [26]), one can deduce the inequality (10) for \( C_\Lambda \) with a higher number of compositions of the Hardy-Littlewood maximal operator \( M \).

Following the lines of [7], one can obtain the corresponding Fefferman-Stein inequality for the Walsh-Carleson maximal operator \( W \). This operator is defined as
\[
W f(x) = \sup_{n \in \mathbb{N}} |W_n f(x)|,
\]
for \( x \in \mathbb{T} = [0, 1] \), where \( W_n \) denotes the \( n \)-th partial Walsh-Fourier sum, often considered as a discrete model of the Fourier case. We refer to [24] for definitions and elementary results on Walsh-Fourier series. It is proven in [7] that for \( 1 < p < \infty \) and any weight \( w \),
\[
\|W f\|_{L^p(w)} \leq \inf_{1 < r \leq 2} \left\{ r' \sup_S \left\| \mathcal{A}_r \mathcal{S} \|_{L^p(w)} \right\} \right.,
\]

where
\[
\tilde{A}_{r,S} f(x) = \sum_{Q \in S} \left( \frac{1}{|Q|} \int_Q |f|^r \right)^{1/r} \chi_Q(x)
\]
and \(S \subset D(\mathbb{T})\) is a sparse family of dyadic cubes. Thus,
\[
\int_{\mathbb{T}} |Wf(x)|^p w(x) dx \leq C \int_{\mathbb{T}} |f(x)|^p M^{[p]+1} w(x) dx
\]
follows by adapting the proof of Lemma 4.1 to the operators \(\tilde{A}_{r,S}\) and to functions defined in \(\mathbb{T}\).

7. Appendix

In this appendix we provide a proof of Proposition 3.2. We give an alternative proof to the one given in Proposition 5.32 in [6]. We follow the same method that Coifman and Rochberg [4] used to prove the classical result \((Mw)^\delta \in A_1\) for any \(0 < \delta < 1\). In contrast to the Hardy-Littlewood maximal operator \(M\), the maximal operator \(M_A\) is not in general of weak-type \((1,1)\). However it will be enough to use the following weaker estimate.

**Proposition 7.1** ([6]). *Let \(A\) be a Young function. For all function \(f\) satisfying \(\|f\|_{A,R} \to 0\) as \(|Q| \to \infty\), and all \(t > 0\),
\[
\left| \left\{ x \in \mathbb{R}^n : M_A f(x) > t \right\} \right| \leq 3^n \int_{\left\{ x \in \mathbb{R}^n : |f(x)| > t/2 \right\}} A \left( \frac{2 \cdot 4^n |f(x)|}{t} \right) dx.
\]

**Proof of Proposition 3.2.** Following the ideas in [4], we need to show that
\[
\frac{1}{|Q|} \int_Q (M_A w)^\delta \leq C(M_A w)^\delta(x)
\]
holds for every \(x \in Q\) and \(C\) independent of \(Q\) and \(w\). Let \(2Q\) denote the cube whose center is the same as \(Q\) and whose side length is twice the side length of \(Q\). Write \(w = w_1 + w_2\), where \(w_1 = w\chi_{2Q}\). By definition of \(M_A\), \(M_A w(x) \leq M_A w_1(x) + M_A w_2(x)\), so for \(0 < \delta < 1\),
\[
(M_A w)^\delta(x) \leq (M_A w_1)^\delta(x) + (M_A w_2)^\delta(x).
\]

Hence, it suffices to show
\[
\frac{1}{|Q|} \int_Q (M_A w_i)^\delta(y) dy = \frac{\delta}{|Q|} \int_0^\infty t^{\delta-1} \left| \left\{ x \in Q : M_A w_1(y) > t \right\} \right| dt
\]
holds for every \(x \in Q\) and \(C\) independent of \(Q\) and \(w\). Since \(w_1\) has compact support, it satisfies \(\|w_1\|_{A,R} \to 0\) as \(|R| \to \infty\). Using Proposition 7.1,
\[
\frac{1}{|Q|} \int_Q (M_A w_i)^\delta(y) dy \leq \frac{\delta}{|Q|} \int_0^\infty t^{\delta-1} \min \left( |Q|, 3^n \int_{\left\{ y \in 2Q : M_A w_1(y) > t/2 \right\}} A \left( \frac{2 \cdot 4^n w_1(y)}{t} \right) dy \right) dt
\]
for every \(x \in Q\) and \(C\) independent of \(Q\) and \(w\). Since \(A\) is a convex and increasing function, by definition of the Luxemburg norm we have that for \(t > 2 \cdot 4^n \cdot 3^n \cdot 2^n \|w\|_{A,2Q}\),
\[
3^n \int_{\left\{ y \in 2Q : M_A w_1(y) > t/2 \right\}} A \left( \frac{2 \cdot 4^n w_1(y)}{t} \right) dy \leq \frac{1}{2^n} \int_{\left\{ y \in 2Q : M_A w_1(y) > t/2 \right\}} A \left( \frac{2 \cdot 4^n \cdot 3^n \cdot 2^n w_1(y)}{t} \right) dy
\]
so we can bound (27) by
\[
(27) \leq \frac{\delta}{|Q|} \int_0^\infty t^{\delta-1} |Q| dt + \frac{3^n \delta}{|Q|} \int_0^\infty \int_{\left\{ y \in 2Q : M_A w_1(y) > t/2 \right\}} A \left( \frac{2 \cdot 4^n w_1(y)}{t} \right) dy dt.
\]
The first term in the right hand side is equal to \((2 \cdot 4^n \cdot 3^n \cdot 2^n) \delta \|w\|^\delta_{A,2Q}\). For the second term, by convexity of \(A\), we have

\[
\frac{3^n \delta}{|Q|} \int_{2 \cdot 4^n \cdot 3^n \cdot 2^n \|w\|_{A,2Q}}^{\infty} t^{\delta-1} \int_{\{y \in 2^Q: w_1(y) > t/2\}} A \left( \frac{2 \cdot 4^n \cdot 3^n \cdot 2^n w_1(y)}{t} \right) dy \, dt \\
\leq \frac{\delta}{2^n |Q|} \int_{2 \cdot 4^n \cdot 3^n \cdot 2^n \|w\|_{A,2Q}}^{\infty} t^{\delta-1} \int_{\{y \in 2^Q: w_1(y) > t/2\}} \frac{2 \cdot 4^n \cdot 3^n \cdot 2^n \|w\|_{A,2Q}}{t} A \left( \frac{w_1(y)}{\|w\|_{A,2Q}} \right) dy \, dt \\
= \frac{\delta}{2^n |Q|} \int_{2 \cdot 4^n \cdot 3^n \cdot 2^n \|w\|_{A,2Q}}^{\infty} t^{\delta-2} \cdot 4^n \cdot 3^n \cdot 2^n \|w\|_{A,2Q} \int_{\{y \in 2^Q: w_1(y) > t/2\}} A \left( \frac{w(y)}{\|w\|_{A,2Q}} \right) dy \, dt \\
\leq \delta \int_{2 \cdot 4^n \cdot 3^n \cdot 2^n \|w\|_{A,2Q}}^{\infty} t^{\delta-2} \cdot 4^n \cdot 3^n \cdot 2^n \|w\|_{A,2Q} dt \\
= \frac{\delta}{1-\delta} (2 \cdot 4^n \cdot 3^n \cdot 2^n) \delta \|w\|^\delta_{A,2Q}.
\]

Thus,

\[
\frac{1}{|Q|} \int_{Q} (M_A w_1)^\delta \leq (2 \cdot 24^n)^\delta \frac{1}{1-\delta} \|w\|^\delta_{A,2Q} \leq (2 \cdot 24^n)^\delta (M_A w)^\delta(x)
\]

for all \(x \in Q\).

To prove (26) for \(i = 2\), we can assume \(M_A w_2(x) > 0\). Let \(y \in Q\) and \(R\) be another cube such that \(y \in R\) and \(\|w_2\|_{A,R} > 0\). Then \(R \not\supset 2Q\) and \(\ell(R) > \frac{1}{2} \ell(Q)\), where \(\ell(\cdot)\) denotes the sidelength of a cube. Thus \(Q \subset 3R\). We claim

\[
\|w_2\|_{A,R} \leq 3^n \|w\|_{A,3R} \leq 3^n M_A w(x)
\]

for every \(y\). Then for every \(y \in Q\), we have

\[
M_A w_2(y) \leq 3^n M_A w(x),
\]

so \((M_A w_2)^\delta(y) \leq 3^n \delta (M_A w)^\delta(x)\) for every \(y \in Q\). Thus

\[
\frac{1}{|Q|} \int_{Q} (M_A w_2)^\delta(y) dy \leq 3^n \delta (M_A w)^\delta(x),
\]

as required. To conclude the proof we still need to show (28). But this follows by convexity and monotonicity of \(A\), since

\[
\frac{1}{|R|} \int_{R} A \left( \frac{w_2(z)}{3^n \|w\|_{A,3R}} \right) dz \leq \frac{1}{3^n |R|} \int_{3R} A \left( \frac{w(z)}{\|w\|_{A,3R}} \right) dz \leq 1,
\]

and this implies, by definition of Luxemburg norm, \(\|w_2\|_{A,R} \leq 3^n \|w\|_{A,3R}\). This completes the proof. \(\square\)

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