On Rainbow $k$-Connection Number of Special Graphs and It’s Sharp Lower Bound

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Abstract. Let $G = (V,E)$ be a simple, nontrivial, finite, connected and undirected graph. Let $c$ be a coloring $c : E(G) \to \{1,2,\ldots,s\}$, $s \in \mathbb{N}$. A path of edge colored graph is said to be a rainbow path if no two edges on the path have the same color. An edge colored graph $G$ is said to be a rainbow connected graph if there exists a rainbow $u - v$ path for every two vertices $u$ and $v$ of $G$. The rainbow connection number of a graph $G$, denoted by $rc(G)$, is the smallest number of $k$ colors required to edge color the graph such that the graph is rainbow connected. Furthermore, for an $l$-connected graph $G$ and an integer $k$ with $1 \leq k \leq l$, the rainbow $k$-connection number $r^k_c(G)$ of $G$ is defined to be the minimum number of colors required to color the edges of $G$ such that every two distinct vertices of $G$ are connected by at least $k$ internally disjoint rainbow paths. In this paper, we determine the exact values of rainbow connection number of some special graphs and obtain a sharp lower bound.

Keywords: Rainbow $k$-Connection Number, Special Graphs, Sharp Lower Bound

1. Introduction

Suppose $G$ is a simple connected graph with a set of vertices $V(G)$ and edges $E(G)$. For a further reference please see Gross, et. al. [6]. Let $G$ be a nontrivial connected graph on which it is defined a coloring $c : E(G) \to \{1,2,\ldots,s\}$, $s \in \mathbb{N}$, of the edges of $G$, where adjacent edges may be colored the same. A $u - v$ path $P$ in $G$ is a rainbow path if no two edges of $P$ are colored the same. The graph $G$ is rainbow-connected (with respect to $c$) if $G$ contains a rainbow $u - v$ path for every two vertices $u$ and $v$ of $G$. In this case, the coloring $c$ is called a rainbow coloring of $G$. If $k$ colors are used, then $c$ is a rainbow $k$-coloring. The minimum $k$ for which there exists a rainbow $k$-coloring of the edges of $G$ is the rainbow connection number $rc(G)$. The completes concept can be found in Chartrand in [3].
A simple observation can be proposed that if $G$ has $n$ vertices then \( rc(G) \leq n - 1 \) but is not sharp. Since a given spanning tree can be assigned with distinct colors, and color the remaining edges with one of the already used colors then the upper bound of \( rc(G) \leq n - 1 \), see Caro [1] for detail. It is also easy to understand that \( rc(G) \geq diam(G) \), where \( diam(G) \) denotes the diameter of $G$, Caro in [1]. Thus, it gives the following \[
 diam(G) \leq rc(G) \leq n - 1 
\]

There have been some results regarded to rainbow connection numbers. Chandran, et. al. in [2] determined rainbow connection number and connected dominating sets, Chakraborty, et. al. in [3] considered hardness and algorithms for rainbow connectivity. Furthermore, Li et. al. in [7] stated Rainbow connections of graphs - A survey. Also Li et. al. in [5] characterized graphs with rainbow connection number and rainbow connection numbers of some graph operations. Schiermeyer in [10] studied rainbow connection in graphs with minimum degree three.

A well-known result shows that in every $l$-connected graph $G$ with $l \geq 1$, there are $k$ internally disjoint $u - v$ paths connecting any two distinct vertices $u$ and $v$ for every integer $k$ with $1 \leq k \leq l$. Chartrand et al. [5] defined the rainbow $k$-connectivity \( rc_k(G) \) of $G$ to be the minimum integer $j$ for which there exists a $j$-edge-coloring of $G$ such that for every two distinct vertices $u$ and $v$ of $G$, there exist at least $k$ internally disjoint $u - v$ rainbow paths.

By the definition of rainbow $k$-connectivity \( rc_k(G) \), we realize that it is almost impossible to derive the exact value or a nice bound of the rainbow $k$-connectivity for a general graph $G$. To answer the problem: given that any connected graph $G$, determine the rainbow connection number \( rc_k(G) \) of any graph $G$? It tends to be NP-hard problem. Thus, the study of rainbow $k$-connectivity of some classes of special graphs is still needed. In this paper we will study the rainbow connection number \( rc_k(G) \) of Triangular Ladder, Wheel graphs, and edge comb of graph $G = C_n \nabla TL_m$ and $G = C_n \nabla K_m$. The edge comb between $L$ and $H$, denoted by $L \nabla H$, is a graph obtained by taking one copy of $L$ and $|E(L)|$ copies of $H$ and grafting the $i$-th copy of $H$ at the $i$-th edges of $L$. The result show that all the rainbow $k$-connection number $rc_k(G)$ of the graph studied in this paper achieve the minimum value.

2. The Results
Before presenting the main results we need to establish the lower bound of $rc_k(G)$ of any graph $G$ such that the graph $G$ is considered to be a $k$-connected graph. Note that the length of the shortest graph cycle (if any) in a given graph is known as a girth, and the length of a longest cycle is known as the graph circumference.

**Theorem 1.** Let $d(u, v)$ be a distance between $u$ and $v$, $C(u, v)$ is a shortest cycle that contains the vertices $u$ and $v$. If $G$ is 2-connected graph then $rc_2(G) \geq \max \{|C(u, v)| - d(u, v), \forall u, v \in V(G)\}$, where $C(u, v)$ and $d(u, v)$ are in one cycle.

**Proof.** Let $G$ be a connected cyclical graph. Thus, the length of second alternative internally disjoint rainbow path for any two vertices $u$ and $v$ is $|C(u, v)| - d(u, v)$ where $C(u, v)$ is a girth that contain vertices $u$ and $v$. The greatest lower bound of
\[ rc_2(G) \geq \max \{|C(u,v)| - d(u,v)|\}. \] By contradiction, if we color the edges of \( G \) by any value less than \( \max \{|C(u,v)| - d(u,v)|\} \) then there exist two vertices \( u \) and \( v \) that do not present two internally disjoint paths.

We can extend the theorem for \( l \)-connected graph.

**Lemma 1.** If \( G \) is \( l \)-connected graph, \( l \geq 2 \), then for every two vertices \( u, v \in V(G) \), there exist at least \( l - 1 \) cycles of \( G \) containing the vertices \( u \) and \( v \).

**Proof.** We can prove this theorem by contradiction. Suppose that there exist two vertices \( u, v \in V(G) \) that contain one less than \( l - 1 \) cycles of \( G \). Suppose that the number of cycles that contain \( u, v \in V(G) \) is \( l - k \) where \( k \geq 2 \). The set \( \{C_i\}_{1 \leq i \leq l - k} \) is \( l - k \) cycles that contain any two vertices in \( V(G) \). One cycle is used to make two internally disjoint paths between \( u \) and \( v \). Two cycles are used to make three internally disjoint paths between \( u \) and \( v \). Since \( u \) and \( v \) are on \( l - k \) cycles then the number of disjoint paths between \( u \) and \( v \) is \( l - k + 1 \). Since \( k \geq 2 \) and we have two vertices with \( l - k + 1 \) disjoint paths connecting \( u \) and \( v \), then \( G \) is \((l - k + 1 < l)\)-connected graph. It is a contradiction.

**Theorem 2.** Let \( d(u,v) \) be a distance between \( u \) and \( v \), \( C_i(u,v) \) be a shortest cycles that contain vertices \( u \) and \( v \). Let \( C \) be cycles whose their common edge is \( uv \). If \( G \) is \( l \)-connected graph then \( rc_2(G) \geq \max\{\max\{|C_i(u,v)| - d(u,v)|\}, 1 \leq i \leq l - 1\}, \forall u, v \in V(G) \}, \) where \( C(u,v) \) and \( d(u,v) \) are in one cycle.

**Proof.** If \( G \) is \( l \)-connected graph, then by Lemma 1 every vertex in \( V(G) \) lays on at least \( l - 1 \) cycles. Suppose the element of \( \{C_i(u,v)|1 \leq i \leq l - 1\} \) \( u, v \in V(G) \} \) have \( l - 1 \) cycles containing \( u, v \in V(G) \), the \( l - 1 \) cycles that contain \( u \) and \( v \) has to be minimum of size \(|C_i(u,v)|\). The number of \( rc_2(G) \) is at least \( \max\{|C_i(u,v)| - d(u,v)|\}, 1 \leq i \leq l - 1\}. \) Otherwise there exist two vertices \( u, v \) that do not give \( k \) internally disjoint rainbow path.

Now we will present some classes of graphs which can be determined their rainbow \( k \)-connection number.

**Theorem 3.** Let \( G \) be a triangular ladder graph, the rainbow 2-connection number of \( G \) is \( rc_2(G) = n \).

**Proof.** Suppose \( G = TL_n \). The graph \( G \) has vertex set \( V(G) = \{x_i, y_i; 1 \leq i \leq n\} \) and edge set \( E(G) = \{x_i, x_{i+1}, y_i, y_{i+1}, x_i, y_{i+1}; 1 \leq i \leq n - 1\} \cup \{x_iy_i; 1 \leq i \leq n\} \). Define a color \( c \) of the edges \( c : E(G) \rightarrow \{1, 2, \ldots, s\}, s \in N \):

\[
c(e) = \begin{cases} 
    n - i, & e \in \{x_i, x_{i+1}; 1 \leq i \leq n - 1\} \\
    i, & e \in \{y_i, y_{i+1}; 1 \leq i \leq n - 1\} \\
    1, & e \in \{x_i, y_{i+1}; 1 \leq i \leq n - 1\} \cup \{x_1y_1\} \\
    n, & e \in \{x_iy_i; 2 \leq i \leq n\}
\end{cases}
\]

It is easy to see that the color \( c(e) \) reach a maximum value when \( e = x_iy_i \) and \( c(e) = n \). Thus, \( rc_2(G) \leq n \). Now we will show that \( rc_2(G) \geq n \). Consider the vertex \( u = y_1 \) and \( v = x_n \). The vertex \( u \) and \( v \) lay on the cycle of size \( 2n \). Since distance, \( d(u,v) = n \), then by Theorem 1 we have \( rc_2(G) \geq 2n - n = n \). It concludes that \( rc_2(G) = n \).
Theorem 4. Let $G$ be a wheel graph of order $n+1$, the rainbow $3$-connection number $G$ is $rc_3(W_n) = n$.

Proof. Given that $G = W_n$. The graph $G$ has vertex set $V(G) = \{x_1; 1 \leq i \leq n\} \cup \{A\}$ and edge set $E(G) = \{Ax_i; 1 \leq i \leq n\} \cup \{x_i x_{i+1}; 1 \leq i \leq n-1\} \cup \{x_1 x_n\}$. Define a color $c$ of the edges $c : E(G) \rightarrow \{1, 2, \ldots, s\}, s \in N$:

$$c(e) = \begin{cases} i, & e \in \{x_i x_{i+1}; 1 \leq i \leq n-1\} \cup \{Ax_i; 1 \leq i \leq n\} \\
, & e \in \{x_1 x_n\} \end{cases}$$

It is easy to see that the color $c(e)$ reach a maximum value when $e = x_1 x_n$, thus $rc_3(W_n) \leq n$. No we will show that $rc_3(W_n) \geq n$. We will use a contradiction. Suppose that $rc_3(W_n) \leq n-1$, take $rc_3(W_n) = n - 1$. Consider edge set $E' = \{x_i x_{i+1}; 1 \leq i \leq n\} \cup \{x_1 x_n\}$ and $|E'| = n + 1$. If we color $n + 1$ edges of $E'$ by $n - 1$ colors, then there exist $e_1, e_2 \in E'$ such that $c(e_1) = c(e_2)$, without loss of generality we can choose $e_1 = x_1 x_2$ and $e_2 = x_i x_{i+1}$. Since $W_n$ is 3-connected graph and $rc_3(W_n) = n - 1$ then there must exist three disjoint paths between any two vertices. Consider vertex $x_1$ and vertex $x_{i+1}$ which give three disjoint paths between $x_1$ and $x_{i+1}$. The first possible rainbow path is $x_1 Ax_{i+1}$, the second is $x_1 x_n x_{n-1} \ldots x_j$, however the third path $x_1 x_2 \ldots x_i x_{i+1}$ for $x_1$ and $x_{i+1}$ is not rainbow path as $c(x_1 x_2) = c(x_i x_{i+1})$. It is a contradiction, thus $rc_3(W_n) \geq n$. It concludes $rc_3(W_n) = n$. \[\square\]

Theorem 5. If $G = C_n \uplus TL_m$ then $rc(G) = \frac{n}{2} + 2m - 2$ for $n$ even and $rc_2(G) = 2m + 1$ for $n = 4$.

Proof. The graph $G = C_n \uplus TL_m$ is a connected graph with vertex set $V(G) = \{x_i; 1 \leq i \leq n\} \cup \{y_{i,j}; 1 \leq i \leq n, 1 \leq j \leq m - 1\} \cup \{z_{i,j}; 1 \leq i \leq n, 1 \leq j \leq m - 1\}$ and edge set $E(G) = \{x_i x_{i+1}; 1 \leq i \leq n-1\} \cup \{x_i y_{i,j}; 1 \leq i \leq n\} \cup \{x_{i+1} z_{i,j}; 1 \leq i \leq n-1\} \cup \{y_{i,j} y_{i,j+1}; 1 \leq i \leq n; 1 \leq j \leq m - 2\} \cup \{y_{i,j} z_{i,j+1}; 1 \leq i \leq n; 1 \leq j \leq m - 2\} \cup \{y_{i,j} z_{i,j+1}; 1 \leq i \leq n; 1 \leq j \leq m - 1\} \cup \{x_{i+1} z_{i,j}; 1 \leq i \leq n\} \cup \{y_{i,j} z_{i,j+1}; 1 \leq i \leq n; 1 \leq j \leq m - 2\}$. The value $|V(G)| = n(2m - 1)$ and $|E(G)| = 3n + 2n(m - 2) + 2n(m - 1)$.

The number $rc_2(G)$ is given by the following...
The maximum value of function is $c - 1$ and $m_2$ and vertex $m_1$.

To prove $rc_2(G) \geq 2m + 1$, consider vertex $y_{2m-1}$ and vertex $z_{1m-1}$, the vertex $y_{2m-1}$ and vertex $z_{1m-1}$ lay on cycle of size of at least $4m - 1$. The distance between $y_{2m-1}$ and $z_{1m-1}$ is $2(m - 1)$ so the length of remaining shortest path between $y_{2m-1}$ and $z_{1m-1}$ is $2m + 1$. This path is the shortest alternative path from $y_{2m-1}$ to $z_{1m-1}$ to get the second internally disjoint rainbow path.

Figure 2. Graph $G = W_6$ with $rc_3(W_6) = 6$
Figure 3. Graph edge comb $G = C_{4} \boxtimes TL_{3}$ with $rc_{2}(G) = 7$.

**Theorem 6.** If $G = C_{n} \boxtimes K_{m}$, then the number $rc(G) = \frac{n}{2} + 1$ for $n$ even and $rc_{2}(G) = 4$, for $n = 4$.

**Proof.** The graph $G = C_{n} \boxtimes K_{m}$ is a connected graph with vertex set $V(G) = \{x_{i}|1 \leq i \leq n\}\cup\{y_{i,j}|1 \leq i \leq n, 1 \leq j \leq m-2\}$ and edge set $E(G) = \{x_{i}x_{i+1}|1 \leq i \leq n-1\}\cup\{x_{n}x_{1}\}\cup\{x_{i}y_{i,j}|1 \leq i \leq n, 1 \leq j \leq m-2\}\cup\{x_{i+1}y_{i,j}|1 \leq i \leq n-1, 1 \leq j \leq m-2\}\cup\{x_{1}y_{n,j}|1 \leq j \leq m-2\} \cup (\bigcup_{i=1}^{m-3}\{y_{i,j,y_{i,j+1}}|1 \leq i \leq n, 1 \leq j \leq m-2-i\})$. The number of vertices and edges of $G$ is $|V(G)| = n + n(m-2)$ and $|E(G)| = n(1 + 2(m-2) + \frac{(m-2)(m-3)}{2})$.

The Diameter of $G$, $diam(G) = \frac{n}{2} + 1$.

The value $rc(G) = \frac{n}{2} + 1$ obtained by the following edge mapping function:

$$c(e) = \begin{cases} 
  i \mod \frac{n}{2}, & e \in \{x_{i}x_{i+1}|1 \leq i \leq n-1\}\cup\{x_{1}x_{i+1}|1 \leq i \leq n-1\}\cup\{x_{i}y_{i,j}|1 \leq i \leq n, 1 \leq j \leq m-2\}\cup\{x_{i+1}y_{i,j}|1 \leq i \leq n-1, 1 \leq j \leq m-2\}\cup\{x_{1}y_{n,j}|1 \leq j \leq m-2\} \\
  n \mod \frac{n}{2}, & e \in \{x_{n}x_{1}\} \\
  \frac{n}{2} + 1, & e \in \{x_{1}y_{n,j}|1 \leq j \leq m-2\}\cup\{x_{i+1}y_{i,j}|1 \leq i \leq n-1, 1 \leq j \leq m-2\} 
\end{cases}$$

The maximum value of $c(e)$ is $\frac{n}{2} + 1$ so $rc(G) \leq \frac{n}{2} + 1$, by applying Inequality 1.

$rc(G) \geq \frac{n}{2} + 1$ and finally we get $rc(G) = \frac{n}{2} + 1$. 


The value $rc_2(G) \geq 4$ for $n = 4$ and any $m$, is obtained by the following:

$$c(e) = \begin{cases} 
  i \mod 2, & e \in \{x_ix_{i+1} | 1 \leq i \leq 3\} \cup \{x_jy_{j+1} | 1 \leq i \leq 4, \\
  1 \leq j \leq m-2\} \cup \{y_{i,j}y_{i,j+1} | 1 \leq i \leq 4, \\
  1 \leq j \leq m-3\} \\
  4 \mod 2, & e \in \{x_4x_1\} \\
  3, & e \in \{x_jy_{i,j} | 1 \leq j \leq m-2\} \cup \\
  \{x_{i+1}y_{i,j} | 1 \leq i \leq 3, 1 \leq j \leq m-2\} \\
  4, & e \in \{x_jy_{i,j} | 1 \leq i \leq 4, 1 \leq j \leq m-2\} \\
  \cup (\bigcup_{i=1}^{m-3} \{(y_{i,j}y_{i,j+1} | 1 \leq i \leq 4, \\
  1 \leq j \leq m-2-i\}) - \{y_{i,j}y_{i,j+1} | 1 \leq i \leq 4, \\
  1 \leq j \leq m-3\}) 
\end{cases}$$

To prove $rc_2(G) \leq 4$ consider vertex $y_{1,j}$ and $y_{2,k}$ for $1 \leq j, k \leq m - 2$. This vertices is contained on cycle with size at least 6. The distance between $y_{1,j}$ and $y_{2,k}$ is 2 so the length of remaining shortest path between $y_{1,j}$ and $y_{2,k}$ is 4. This path is the shortest alternative path from $y_{1,j}$ to $y_{2,k}$ to make second internally disjoint rainbow path.

**Concluding Remarks**

We have studied the rainbow $k$-connection number of $G$. The result show that all the rainbow $k$-connection number $rc_k(G)$ of the graph studied in this paper achieve the minimum value. We have also characterized any graph to have a minimum $k$-connection number, through the following theorem: If $G$ is $l$-connected graph then
rc_l(G) \geq \max \{|C_i(u, v)| - d(u, v), 1 \leq i \leq l - 1\}, \text{ where } |C_i(u, v)| \text{ is a girth that contains the vertices } u \text{ and } v. \text{ However, it is just lower bound, we have not found the sharper upper bound of } rc_k(G) \text{ of any graph. Thus we propose the following open problem.}

**Open Problem 1.** Given that any connected graph \( G \), determine a sharp upper bound of the rainbow \( k \)-connection number \( rc_k(G) \) of \( G \).

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