Polarization mixing in optical lattices with uniaxial anisotropy

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Abstract

We study resonant optical excitations of an ultracold gas in an optical lattice in the Mott insulator case. The particles possess anisotropy via an oriented optical transition dipole moment. The anisotropy, prepared by optical pumping into a certain Zeeman sublevel and/or by external static fields, is of a similar type as in uniaxial crystals. Between two mirrors the electronic excitations and the cavity photons of both TE and TM polarizations are coherently mixed to form polariton branches, which can directly be observed in the linear optical spectra. For example, for an incident field of TE polarization we obtain TE and TM transmitted and reflected fields. Also we calculate the phase shift in the TE and TM transmitted and reflected fields relative to the incident field phase, which leads to polarization rotations. Such polarization mixing can serve as a signature of the formation of oriented molecules from spherically symmetric atoms in optical lattices.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

An optical lattice for ultracold particles is generated by pairs of counterpropagating standing laser waves off-resonance with any internal transitions and possesses a lattice constant of half wavelength, \( a = \lambda/2 \) [1]. A gas of sufficiently cold atoms loaded into such an optical lattice can be well described by a Bose–Hubbard Hamiltonian, which exhibits a quantum phase transition from the superfluid into the Mott insulator phase for strong enough laser intensity [2, 3]. In general, the atoms experience different optical lattice potentials depending on the polarizability of the internal atomic state they are prepared in. The optical lattice potentials are in general different in each direction, as the polarization and frequency for each laser pair can vary. Nevertheless, we can assume that ground- and excited-state optical lattice potentials exhibit minima at the same positions [4, 5]. Throughout the paper we assume the ground-state atoms to occupy the lowest Bloch band of the optical lattice potential and to be in the Mott insulator phase with one atom (or molecule) per site. Resonant excitations couple to a second internal atomic level, which is assumed to be also trapped at the same positions with similar potential depth. Note that experimentally, strong resonant light-matter coupling for a Bose–Einstein condensate in an optical cavity has recently been achieved by several groups [9–11], and the addition of an optical lattice to such a setup is straightforward and can be expected at any time.

In previous work, we already studied collective electronic excitations (excitons) in such a system within a cavity and established cavity polaritons as basic elementary excitations to describe the optical response in such a setup [6–8]. In the present paper the atoms are modeled as multilevel systems, where the optical transition dipole moment has a fixed orientation (no permanent dipoles). This case automatically occurs when all the atoms are prepared in a Zeeman sublevel of fixed angular momentum. This can be achieved by optical pumping as is required to prepare a BEC in a magnetic trap [12] and should survive if one transfers them into an optical lattice with some bias field left on. Similarly the use of designed fixed laser polarizations, which could be accompanied by external static fields should stabilize such an orientation. In special cases even independently controllable lattices for different Zeeman sublevels were prepared and used [13]. Alternatively, for an optical lattice filled with molecules, e.g. homonuclear diatomic molecules, the ‘excited’ state has an orientation so that the transition dipole is preferentially parallel to the molecule axis (for example for the SS molecular ground state and the SP excited state). It is now possible to engineer a situation, where due to the design of the optical lattice polarization, all these dipoles at different sites are oriented in
the same direction [14]. This organization of the molecules is due to the fact that the static molecular polarizability parallel to the molecule axis is larger than the polarizability perpendicular to the molecule axis. Hence the molecules tend to align in the direction of the local optical electric field, if their temperatures are sufficiently low. Starting from paired ultracold atoms in a Mott insulator phase which have isotropic optical properties [7], the formation of on-site diatomic molecules can then be recognized through the appearance of their anisotropic properties. The fact that the transition dipole has a fixed direction makes the system strongly anisotropic.

As a bottom line, both cases, namely optically pumped single atoms or oriented molecules, show an aligned transition dipole direction at each site and thus can be considered like an artificial anisotropic crystal of uniaxial type [15]. Compared to solid-state implementations, the advantage of studying optical lattices as anisotropic crystals lies in the wide range of controllable parameters from lattice depth, anisotropy, to defect numbers, which adds large flexibility to design desired photonic properties and compares predictions with concrete experimental realizations [16].

In the following we study such optical lattices with anisotropy as generic examples for this whole field. Through investigating optical spectra we can measure different physical properties of the system. To achieve our goals in an easily controllable system, the 2D anisotropic optical lattice is taken to be localized between two planar cavity mirrors [6]. For each cavity mode two possible orthogonal degenerate polarizations, TE and TM polarizations, of transverse electric and transverse magnetic fields exist respectively [17]. We consider only a single perpendicular cavity mode close to resonance to the above internal atomic transition. As the atomic transition dipole has a fixed direction, which is in general different from the direction of the cavity electric field, the coupling of the internal atomic excitations and the photons is a function of the angle between the transition dipoles and the photon polarizations. In the strong coupling regime the electronic excitations and the cavity photons of both polarizations are coherently mixed to form the new system diagonal eigenmodes, which are called cavity polaritons [18, 19]. Such photon polarization mixing can be observed via linear optical spectra [6, 18]. For an incident field which is, e.g., TE polarized, we expect, due to the anisotropic optical lattice, to observe transmitted and reflected fields of TE and TM polarizations. Photon polarization mixing can also be observed via the phase shift of the transmitted and reflected fields relative to an incident field of a fixed polarization.

This paper is organized as follows. In section 2 we present an anisotropic optical lattice within a cavity. The polarization mixing in the formation of cavity polaritons appears in section 3. Linear optical spectra are derived in section 4 with the optical light shift, for a given incident field. A conclusion is given in section 5.

2. An anisotropic optical lattice within a cavity

We consider ultracold two-level atoms (or molecules) in an optical lattice with one atom per site. The atoms are prepared in a state where the induced transition dipole moment matrix element has a fixed direction at each site, e.g., as in the case of figure 1. The internal atomic transition between the ground state $|g\rangle$, of energy $\hbar\omega_e$, and the excited state $|e\rangle$, of energy $\hbar\omega_g$, has a transition energy of $\hbar\omega_{eg} = \hbar\omega_e - \hbar\omega_g$. The expectation value of the transition dipole operator $\hat{\mu}$ is $\langle \mu \rangle = \langle e | \hat{\mu} | g \rangle$. The atomic excitation Hamiltonian is given by

$$H_A = \sum_i \hbar \omega_{eg} B_i^\dagger B_i ,$$

where $B_i^\dagger$ and $B_i$ are the creation and annihilation operators of electronic excitation at the lattice site $i$, respectively. Here we neglect electrostatic interactions between atoms at different sites, as for small wave vectors they result only in an energy shift which can be absorbed in $\hbar \omega_A$ (their role in the formation of excitons was discussed widely by us in [6, 7]). In using the lattice symmetry, we can formally transform the excitation Hamiltonian into the momentum space in applying the transformation

$$B_i = \frac{1}{\sqrt{N}} \sum_{\mathbf{q}} e^{i \mathbf{q} \cdot \mathbf{n}_i} B_{\mathbf{q}},$$

where $N = M \times M$ is the number of sites in the optical lattice, $\mathbf{n}_i$ is the position of site $i$, and $\mathbf{q}$ is the in-plane wave vector, which takes the values $\mathbf{q} = (q_x, q_y) = \frac{2\pi}{\sqrt{S}} (n_x, n_y)$, where $n_x, n_y = 0, 1, 2, \ldots, \pm M$, with $S = Na^2$. The Hamiltonian is now written as

$$H_A = \sum_{\mathbf{q}} \hbar \omega_{eg} B_{\mathbf{q}}^\dagger B_{\mathbf{q}} ,$$

The 2D optical lattice is taken to be localized in the middle and parallel to planar cavity mirrors, as presented in figure 2. First we consider the case of perfect cavity mirrors (later in order to get the linear optical spectra we consider the case of non-perfect mirrors). Here the electromagnetic field is free in the cavity plane with the in-plane wave vector $\mathbf{k}$, and is quantized in the perpendicular, $z$, direction with wave numbers $k_z = \frac{2\pi}{L}$, where $L$ is the distance between the cavity mirrors, and $m$ takes integer numbers ($m = 0, 1, 2, 3, \ldots$). For each cavity mode ($\mathbf{k}, m$) we have two possible polarizations TE and
TM, which are denoted by the indices \((s)\) and \((p)\), respectively. The two cavity photon polarizations, \((\nu = s, p)\), are assumed to be degenerate, where \(\omega_{\text{km}} = \omega_{\text{kmp}} = \omega_{\text{km}}\), with the cavity photon dispersion

\[
\omega_{\text{km}} = c \sqrt{k^2 + \left(\frac{m\pi}{L}\right)^2},
\]

where \(k = |\mathbf{k}|\). The cavity photon Hamiltonian is

\[
H_C = \sum_{k,m,\nu} \hbar \omega_{\text{km}} a_{\text{km}}^\dagger a_{\text{km}+\nu},
\]

where \(a_{\text{km}}^\dagger\) and \(a_{\text{km}}\) are the creation and annihilation operators of a cavity photon in the mode \((\mathbf{k}, m, \nu)\), respectively.

The electric field operator is defined by

\[
\mathbf{E}(\mathbf{r}, z) = -i \sum_{k,m,\nu} \frac{\hbar \omega_{\text{km}}}{L S e_0} \left\{ \mathbf{u}_n^m(\mathbf{k}, z) e^{i \mathbf{k} \cdot \mathbf{r}} a_{\text{km}+\nu} \\
- \mathbf{u}_p^m(\mathbf{k}, z) e^{-i \mathbf{k} \cdot \mathbf{r}} a_{\text{km}+\nu}^\dagger \right\},
\]

where \(S\) is the cavity mirror quantization area, \(\mathbf{r}\) is the in-plane position and \(z\) is the perpendicular position. The cavity mirrors are taken to be localized at the positions \(z = \pm L/2\), see figure 2. The field vector functions are defined by [17]

\[
\mathbf{u}_n^m(\mathbf{k}, z) = \sin \left[ \frac{m\pi}{L} \left( z + \frac{L}{2} \right) \right] \hat{n}_k,
\]

where \(m = 0, 1, 2, \ldots, \) and

\[
\mathbf{u}_p^m(\mathbf{k}, z) = -\frac{c m\pi}{L \hbar \omega_{\text{km}}} \left\{ i \sin \left[ \frac{m\pi}{L} \left( z + \frac{L}{2} \right) \right] \hat{\mathbf{e}}_k \\
- \frac{k L}{m\pi} \cos \left[ \frac{m\pi}{L} \left( z + \frac{L}{2} \right) \right] \hat{\mathbf{e}}_z \right\},
\]

where \(m = 1, 2, 3, \ldots, \) and for \(m = 0\) we multiply \(\mathbf{u}_p^0(\mathbf{k}, z)\) by the factor \(1/\sqrt{2}\). The unit vectors are: \(\hat{\mathbf{e}}_z\) is along the \(z\)-axis, \(\hat{\mathbf{e}}_k\) is along \(\mathbf{k}\), that is \(\hat{\mathbf{e}}_k = \mathbf{k}/k\), and \(\hat{n}_k = \hat{\mathbf{e}}_k \times \hat{\mathbf{e}}_z\), as illustrated in figure 1.

The optical lattice is located at the middle and parallel to the cavity mirrors at \(z = 0\). We assume only cavity modes with \(m = 1\) which are close to resonance to the atomic transition. We neglect all the other perpendicular modes. For multilevel atoms the cavity photons, of TE or TM linear polarizations, are close to resonance to the appropriate electronic transition with a fixed angular momentum. For the case of ground and/or excited states with degenerate multi-levels, where several allowed transitions are close to resonance to the cavity photons of TE or TM polarizations, by applying external static fields the degenerate states split, and we can stay with a single electronic transition which is close to resonance to the cavity photons.

The coupling between the atomic transition and the cavity modes in the electric dipole approximation is given by the Hamiltonian

\[
H_{AC} = \frac{i}{\hbar} \sum_{k,v,i} \sqrt{\frac{\hbar \omega_{\text{km}}}{L S e_0}} \left\{ (\hat{\mu}_i \cdot \mathbf{u}_i(\mathbf{k})) e^{i k \cdot r} B_{i\nu}^\dagger a_{\text{km}+\nu} \\
- (\hat{\mu}_i \cdot \mathbf{u}_i(\mathbf{k})) e^{-i k \cdot r} a_{\text{km}+\nu}^\dagger B_{i\nu} \right\}.
\]

By transforming the electronic excitation operators into the momentum space we get

\[
H_{AC} = \frac{i}{\hbar} \sum_{k,q,v} \sqrt{\frac{\hbar \omega_{\text{km}}}{N L S e_0}} \left\{ (\hat{\mu}_i \cdot \mathbf{u}_i(\mathbf{k})) e^{i k \cdot q} B_{q\nu}^\dagger a_{\text{km}+\nu} \\
- (\hat{\mu}_i \cdot \mathbf{u}_i(\mathbf{k})) e^{-i k \cdot q} a_{\text{km}+\nu}^\dagger B_{q\nu} \right\},
\]

where the coupling parameter is

\[
h \mathbf{f}_k^\dagger = i \sqrt{\frac{\hbar \omega_{\text{km}}}{L a^2 e_0}} (\hat{\mu}_i \cdot \mathbf{u}_i(\mathbf{k})).
\]

Due to the lattice translational symmetry, the coupling is between cavity photons and excitations with the same in-plane wave vector.

Explicitly, we have

\[
\begin{aligned}
h \mathbf{f}_k^\dagger &= i \sqrt{\frac{\hbar \omega_{\text{km}}}{L a^2 e_0}} \left( \hat{\mathbf{e}}_k \cdot \hat{n}_k \right), \\
h \mathbf{f}_k^\dagger &= \sqrt{\frac{\hbar \omega_{\text{km}}}{L a^2 e_0}} \left( \frac{\omega_{\text{km}}}{\omega_0} \right) \left( \hat{\mu}_i \cdot \hat{\mathbf{e}}_k \right),
\end{aligned}
\]

where \(\omega_0 = c \pi/k\). We are interested in the case of small wave vectors \(k \approx 0\); hence we neglected the contribution of the \(z\) direction, which is of the order of \(ck/\omega_{\text{km}} \ll 1\) (or we can assume \(\mu_{\nu \nu} \approx 0\)).

We take \(\hat{\mu}_i\) to be real, e.g. for \(\hat{\mu}_i = \mu \hat{\mathbf{x}}\) (see figure 1), and also we have

\[
\hat{\mathbf{e}}_k = \cos \theta \hat{\mathbf{x}} + \sin \theta \hat{\mathbf{y}}, \quad \hat{n}_k = \sin \theta \hat{\mathbf{x}} - \cos \theta \hat{\mathbf{y}},
\]
Polarization mixing in the strong coupling regime

The total Hamiltonian of the coupled electronic excitations and cavity photons is given by

\[ H = \hbar \sum_k \left( \omega_A B_k^\dagger B_k + \sum_v \omega_k a_k^\dagger a_k + \sum_v (f_k^p B_k^\dagger a_{kv} + f_k^p a_k^\dagger B_k) \right) \tag{16} \]

In the strong coupling regime, where the coupling is larger than the atomic transition and the cavity photon linewidth, the real system eigenmodes are cavity polaritons [6, 18], which are obtained by diagonalizing the above Hamiltonian, to get

\[ H = \sum_{k,r} \hbar \Omega_{kr} A_{kr}^\dagger A_{kr} \tag{17} \]

where we obtain three polariton branches, with the eigenfrequencies

\[ \Omega_{k\pm} = \frac{\omega_k + \omega_A}{2} \pm \Delta_k, \quad \Omega_{k0} = \omega_k, \tag{18} \]

where

\[ \Delta_k = \sqrt{\Delta_k^2 + |f_k|^2}, \quad |f_k|^2 = \sum_v |f_k^v|^2 \tag{19} \]

with the excitation-photon detuning

\[ \delta_k = \frac{\omega_k - \omega_A}{2}. \tag{20} \]

For the case of atomic transition frequency of \( \omega_A / 2 \pi = 2.5 \times 10^{14} \) Hz, and for the first cavity mode \( m = 1 \) with a distance between the cavity mirrors of \( L = \pi r / \omega_0 \approx 3.77 \) \( \mu \)m, we get zero detuning between the atomic transition and the cavity photon at \( k = 0 \), we plot in figure 3 the three polariton frequency dispersions, \( \Omega_{kr} / 2 \pi, \) as a function of \( k \), at the angle \( \theta = \pi / 4 \), and for transition dipole of \( \mu = 2 \) e\( \text{A} \), and lattice constant of \( a = 2000 \) \( \AA \) = 0.2 \( \mu \)m.

In the limit of large detuning, where \( \delta_k \gg |f_k| \), we get \( \Omega_{k\pm} \approx \omega_k + |f_k|^2 / \omega_k \), \( \Omega_{k0} \approx \omega_A - |f_k|^2 / \omega_k \) and \( \Omega_{k0} = \omega_k \). The upper branch is a photon with a shift due to the coupling to the excitation, and the lower branch is an excitation with a light shift due to the coupling to the cavity photon. Here the middle branch is still a photon as before. In this limit we get birefringence, as we have two refracted cavity fields: the ordinary field of the (0) branch, and the extraordinary field of the (+) branch.

The polariton operators are, in general, a coherent superposition of atomic excitations and cavity photons of both polarizations, namely we have

\[ A_{kr} = X_{kr} B_k + \sum_v Y_{kr}^v a_{kv}, \tag{21} \]

with the relation \( |X_{kr}|^2 + \sum_v |Y_{kr}^v|^2 = 1 \), where the excitation and photon amplitudes of the upper (+) and the lower (−) polariton branches are

\[ X_{k\pm} = \pm \sqrt{\frac{\Delta_k \pm \delta_k}{2\Delta_k}}, \quad Y_{k\pm}^v = \frac{f_k^v}{|f_k|}, \tag{22} \]

and the excitation and photon amplitudes of the middle (0) polariton branch are

\[ X_{k0} = 0, \quad Y_{k0}^v = \frac{f_k^v}{|f_k|}. \tag{23} \]

Note that the middle polariton branch is purely photonic. There exists a direction, for each \( k \), where the polarization direction of a photon, which is a superposition of cavity photons of both polarizations, is orthogonal to the transition dipole, and hence no interaction between this photon and the material is obtained. In other words, as seen in figure 1, the superposition of the two photon polarizations gives two components: the longitudinal component along the transition dipole, which forms in the strong coupling regime with the excitation the two upper and lower polariton branches; the transverse component which is orthogonal to the transition dipole and decouples to the excitation, to form the photonic middle branch.

At the intersection point between the transition frequency and the cavity photon dispersion, where \( \delta_k = 0 \), the polaritons are half excitation and half photon, namely \( |X_{kr}|^2 = 1 / 2 \) and \( \sum_v |Y_{kr}^v|^2 = 1 / 2 \). At large wave vectors the upper branch became photonic, with \( |X_{kr}|^2 \approx 0 \) and \( \sum_v |Y_{kr}^v|^2 \approx 1 \), and
4. Anisotropic linear optical spectra

To observe the system eigenmodes we need to couple the internal system to the external world. Our observation tool is linear optical spectra, where we apply the quasi-mode formalism which is applicable for high-$Q$ cavities [6, 18]. In assuming non-perfect mirrors, the internal system gets coupled to their environment. For an incident external field with a fixed in-plane wave vector and polarization we calculate the transmission, reflection, and absorption spectra, as seen in figure 2. Now the electromagnetic field is divided into two parts, the cavity field and the environment field, and they are coupled through the non-perfect cavity mirrors. The lifetime of the excited state is included phenomenologically.

The external electromagnetic field is given by the following Hamiltonians, for fields at the upper and lower sides of the cavity,

\[
H_U = \sum_{k, \nu} \int d\omega \ h \omega \ b_{k\nu}^\dagger(\omega_k) b_{k\nu}(\omega_k),
\]

\[
H_L = \sum_{k, \nu} \int d\omega \ h \omega \ c_{k\nu}^\dagger(\omega_k) c_{k\nu}(\omega_k),
\]

where $b_{k\nu}(\omega_k)$ and $b_{k\nu}(\omega_k)$ are the creation and annihilation operators of an external photon in the upper side of the cavity, and $c_{k\nu}(\omega_k)$ and $c_{k\nu}(\omega_k)$ are the creation and annihilation operators of an external photon in the lower side of the cavity. The external photon energy is $h \omega_k$. The coupling of the cavity photons to the external fields is given by

\[
V_U = \sum_{k, \nu} \int d\omega \ i h \omega (\omega_k) \left\{ b_{k\nu}^\dagger(\omega_k) a_{k\nu} - a_{k\nu}^\dagger(\omega_k) b_{k\nu}(\omega_k) \right\},
\]

the lower branch became excitation, with $|X_{k\nu}|^2 \approx 1$ and $\sum_{\nu} |Y_{k\nu}|^2 \approx 0$.

In figures 4–6 we plot the excitation and photon weights in the three polariton branches, $|X_{k\nu}|^2$ and $|Y_{k\nu}|^2$, as a function of $k$, for the angle $\theta = \pi/4$. It is seen that for small $k$ the upper and lower polariton branches, in figures 4–5, are a coherent mix of the excitation and the cavity photon of both polarizations. The upper branch, for large $k$, becomes photonic, and for larger $k$ it becomes photon of $(s)$ polarization, as seen in figure 4. While the lower branch becomes excitation for large $k$, as appears in figure 5. The middle branch is purely photonic, as seen in figure 6, and for large $k$ becomes an $(s)$ polarized photon. In figure 7 we plot the excitation and photon weights as a function of the angle $\theta$ at $k = 5 \times 10^{-7} \text{Å}^{-1}$ in the upper branch. The plot indicates the significant role of the angle controllability in the coupling of matter to photons of both polarizations.

Figure 4. The excitation and photon of both polarization weights in the upper branch, $|X_{k\nu}|^2$, $|Y_{k\nu}^+|^2$ and $|Y_{k\nu}^-|^2$ versus $k$, at $\theta = \pi/4$.

Figure 5. The excitation and photon of both polarization weights in the lower branch, $|X_{k\nu}|^2$, $|Y_{k\nu}^+|^2$ and $|Y_{k\nu}^-|^2$ versus $k$, at $\theta = \pi/4$.

Figure 6. The excitation and photon of both polarization weights in the idle branch, $|X_{k\nu}|^2$, $|Y_{k\nu}^+|^2$ and $|Y_{k\nu}^-|^2$ versus $k$, at $\theta = \pi/4$. 

Figure 7. The excitation and photon of both polarization weights in the upper branch, $|X_{k\nu}|^2$, $|Y_{k\nu}^+|^2$ and $|Y_{k\nu}^-|^2$ versus $\theta$, at $k = 5 \times 10^{-7} \text{Å}^{-1}$.
and
\[ V_L = \sum_{k,v} \int d\omega_k i\hbar v(\omega_k) \{ c_{k,\nu}^\dagger(\omega_k) a_{k,v} - c_{k,v}^\dagger(\omega_k) a_{k,\nu} \}. \]  
(27)
where \( u(\omega_k) \) is the coupling parameter through the upper mirror, and \( v(\omega_k) \) through the lower mirror. The coupling is between cavity photons and external photons with the same in-plane wave vector and polarization. No polarization mixing or wave vector scattering can be obtained due to the cavity mirrors.

The total Hamiltonian of the coupled internal system and external field is separable for each in-plane wave vector \( k \); hence we can treat the whole system for each \( k \) separately. The Hamiltonian for a fixed \( k \), where we can drop \( k \) in the following, is given by
\[ H = \sum_{\nu} \hbar \Omega_{\nu} A_{\nu}^\dagger A_{\nu} + \sum_{\nu} \int d\omega \hbar \omega b_{\nu}(\omega) b_{\nu}(\omega) + \sum_{\nu} \int d\omega \hbar \omega c_{\nu}(\omega) c_{\nu}(\omega) \]  
\[ + \sum_{\nu} \int d\omega \hbar u(\omega) \{ Y_{\nu}^\ast b_{\nu}(\omega) A_{\nu} - Y_{\nu} A_{\nu}^\dagger b_{\nu}(\omega) \} \]  
\[ + \sum_{\nu} \int d\omega \hbar v(\omega) \{ Y_{\nu}^\ast c_{\nu}(\omega) A_{\nu} - Y_{\nu} A_{\nu}^\dagger c_{\nu}(\omega) \}, \]  
(28)
where we used the cavity photon operator in terms of polariton operators, in using the inverse transformation \( a_\nu = \sum Y_{\nu}^\ast A_{\nu} \).

The equations of motion for the external field operators are written as
\[ \frac{d}{dt} b_{\nu}(\omega) = -i\omega b_{\nu}(\omega) + u(\omega) \sum \nu Y_{\nu}^\ast A_{\nu}, \]  
\[ \frac{d}{dt} c_{\nu}(\omega) = -i\omega c_{\nu}(\omega) + v(\omega) \sum \nu Y_{\nu}^\ast A_{\nu}, \]  
(29)
and for the polariton operator we get
\[ \frac{d}{dt} A_{\nu} = -i\Omega_{\nu} A_{\nu} - \sum \nu \int d\omega \{ u(\omega) b_{\nu}(\omega) + v(\omega) c_{\nu}(\omega) \}. \]  
(30)

Using the input–output formalisms [21], the equations for the external field operators are solved formally for the initial and final conditions and substituted back into the polariton equation, and after applying the Markov approximation [22], we get the two equations
\[ \frac{d}{dt} A_{\nu} = -i\Omega_{\nu} A_{\nu} - \sum \nu \int d\omega \{ u(\omega) b_{\nu}^\ast(\omega) + v(\omega) c_{\nu}^\ast(\omega) \}, \]  
and
\[ \frac{d}{dt} A_{\nu} = -i\Omega_{\nu} A_{\nu} + \sum \nu \int d\omega \{ \sqrt{\gamma_U} b_{\nu}^\ast(\omega) + \sqrt{\gamma_L} c_{\nu}^\ast(\omega) \}, \]  
(32)
where we defined the damping rates at the two mirrors as constants, and these are given by \( \gamma_U = 2\pi u^2(\omega) \), and \( \gamma_L = 2\pi v^2(\omega) \), and we also defined \( \gamma = \frac{\gamma_U + \gamma_L}{2} \). Furthermore, we defined the input and output fields at the upper cavity mirror by \( b_{\nu}^\ast \) and \( b_{\nu}^\ast \), and at the lower lower mirror by \( c_{\nu}^\ast \) and \( c_{\nu}^\ast \).

The boundary conditions between the cavity and the external fields at the upper and lower mirrors are given by
\[ \sqrt{\gamma_U} \partial_{\nu} = b_{\nu}^\ast + b_{\nu}^\ast, \]  
\[ \sqrt{\gamma_L} \partial_{\nu} = c_{\nu}^\ast + c_{\nu}^\ast. \]  
(33)
The electronic excitation damping rate is \( \Gamma_{\text{ex}} \). Phenomenologically, the \( r \) polariton branch damping rate is \( \Gamma_r = \Gamma_{\text{ex}} |X_r|^2 \), where \( |X_r|^2 \) is the excitation weight of the \( r \) polariton. We define the polariton complex frequency by \( \Omega_r = \Omega_r - i\Gamma_r \).

In applying the Fourier transform, from time \( t \) to frequency \( \omega \) space, we get the system of equations
\[ i(\Omega_r - \omega) A_{\nu} = \gamma \sum \nu Y_{\nu}^\ast a_{\nu} - \sum \nu Y_{\nu} \{ \sqrt{\gamma_U} b_{\nu}^\ast + \sqrt{\gamma_L} c_{\nu}^\ast \}, \]  
\[ i(\Omega_r - \omega) A_{\nu} = -\gamma \sum \nu Y_{\nu}^\ast a_{\nu} + \sum \nu Y_{\nu} \{ \sqrt{\gamma_U} b_{\nu}^\ast + \sqrt{\gamma_L} c_{\nu}^\ast \} \]  
(34)
and
\[ \sqrt{\gamma_L} a_{\nu} = b_{\nu}^\ast + b_{\nu}^\ast, \]  
\[ \sqrt{\gamma_U} a_{\nu} = c_{\nu}^\ast + c_{\nu}^\ast. \]  
(35)
Two assumptions will be made here, in order to simplify the system of equations. The first is to assume an input external field from only the upper mirror, namely we have \( c_{\nu}^\ast = 0 \). Second, we assume the upper and lower mirrors to be identical, namely we have \( \gamma_U = \gamma_L = \gamma \). Hence, in terms of cavity and external photon operators, we get the system of equations
\[ \tilde{a}_{\nu} = \sum \beta \Lambda_{\nu\beta} \{ \gamma \tilde{a}_{\beta} - \sqrt{\gamma_U} b_{\nu}^\ast - \sqrt{\gamma_L} c_{\nu}^\ast \}, \]  
\[ \tilde{a}_{\nu} = \sum \beta \Lambda_{\nu\beta} \{ -\gamma \tilde{a}_{\beta} + \sqrt{\gamma_U} \tilde{b}_{\nu}^\ast \} \]  
(36)
and
\[ \sqrt{\gamma_U} \tilde{a}_{\nu} = \tilde{b}_{\nu}^\ast + \tilde{b}_{\nu}^\ast, \]  
\[ \sqrt{\gamma_L} \tilde{a}_{\nu} = \tilde{c}_{\nu}^\ast + \tilde{c}_{\nu}^\ast. \]  
(37)
where we defined the matrix
\[ \Lambda_{\nu\beta} = i \sum \frac{Y_{\nu}^\ast Y_{\beta}^\ast}{\omega - \Omega_r}. \]  
(38)

Here we will solve the above system of equations for the following case.

For the incident field of only \( s \) polarization, where \( \tilde{b}_{\nu}^\ast = 0 \), the solution is
\[ \frac{\tilde{c}_{\nu}^\ast}{\tilde{b}_{\nu}^\ast} = \frac{\gamma \Lambda_{ss}(1 + \gamma \Lambda_{ss}) - \gamma^2 \Lambda_{sp} \Lambda_{ps}}{D} = \gamma^{(s)} e^{i\Delta \phi_{ss}}, \]  
\[ \frac{\tilde{c}_{\nu}^\ast}{\tilde{b}_{\nu}^\ast} = \frac{\gamma \Lambda_{ps}}{D} = \gamma^{(s)} e^{i\Delta \phi_{sp}}, \]  
\[ \frac{\tilde{b}_{\nu}^\ast}{\tilde{b}_{\nu}^\ast} = \frac{-1 + \gamma \Lambda_{pp}}{D} = \gamma^{(s)} e^{i\Delta \phi_{pp}}, \]  
(39)
where
\[ D = (1 + \gamma \Lambda_{ss})(1 + \gamma \Lambda_{pp}) - \gamma^2 \Lambda_{sp} \Lambda_{ps}. \]  
(40)
The transmission and reflection of \((s)\) polarized fields are

\[
T_s^{(s)} = (t_s^{(s)})^2 = \frac{\gamma^2|\Lambda_{ss}(1 + \gamma \Lambda_{pp}) - \gamma \Lambda_{sp}\Lambda_{ps}|^2}{|D|^2},
\]

\[
R_s^{(s)} = (r_s^{(s)})^2 = \frac{|1 + \gamma \Lambda_{pp}|^2}{|D|^2}.
\]  

(41)

The transmission and reflection of \((p)\) polarized fields are

\[
T_p^{(s)} = R_p^{(s)} = (t_p^{(s)})^2 = (r_p^{(s)})^2 = \frac{\gamma^2|\Lambda_{ps}|^2}{|D|^2}.
\]  

(42)

Even though the incident field is \((s)\) polarized, due to the anisotropy in the optical lattice we get transmitted and reflected fields which are \((p)\) polarized. Moreover, due to the assumption of identical mirrors, the \((p)\) polarized transmission and reflection fields are equal.

The absorption, \(A\), in the cavity medium is calculated from the identity relation

\[
T_s^{(s)} + T_p^{(s)} + R_s^{(s)} + R_p^{(s)} + A^{(s)} = 1.
\]  

(43)

Also from the system of equations (39) we deduce the phase shift in the \((s)\) and \((p)\) polarized transmitted fields, \(\Delta \phi_s^{(ss)}\) and \(\Delta \phi_p^{(ss)}\), and reflected fields, \(\Delta \phi_s^{(rs)}\) and \(\Delta \phi_p^{(rs)}\), relative to the incident field, respectively.

In figures 8 and 9 we plot the transmission and reflection spectra of the \((s)\) polarized fields, \(T_s^{(s)}\) and \(R_s^{(s)}\), as a function of frequency, \(\omega \rightarrow \omega/2\pi\), respectively, and for different angles \(\theta\), at \(k = 5 \times 10^{-7} \ \text{Å}^{-1}\). The three peaks and three dips correspond to the three polariton branches. At \(\theta = 0\) we get zero transmission and complete reflection. The largest transmission peaks and the deepest reflection dips are obtained at \(\theta = \pi/2\). In figure 10 we plot the transmission and reflection spectra of the \((p)\) polarized fields, \(T_p^{(p)}\) and \(R_p^{(p)}\), which are equal for identical cavity mirrors. Even though the incident field is \((s)\) polarized we get transmission and reflection of \((p)\) polarized fields, with maximum at \(\theta = \pi/4\), and zeros at \(\theta = 0\) and \(\theta = \pi/2\). In figure 11 we plot the absorption spectra \(A^{(p)}\). Only two peaks are obtained, corresponding to the upper and lower branches, as the middle branch is purely photonic and no absorption takes place, where the absorption is only for the polariton excitation part. Also here zero absorption at \(\theta = 0\), and maximum absorption at \(\theta = \pi/2\), are obtained. Here, for the excitation and photon damping rates we used \(\gamma/2\pi = 10^9\ \text{Hz}\) and \(\Gamma_{cs}/2\pi = 10^8\ \text{Hz}\). The cavity mirrors provide us with the two functions: first, the coupling between the internal cavity and the external fields, second, the damping of the cavity photons. Hence, the linewidth is wider for peaks and dips which are more photonic than excitation, where the linewidth for peaks and dips which are more photonic is dominated by the cavity linewidth. While the electronic excitation damping is included phenomenologically, for excitation dips and peaks the linewidth approaches the excitation linewidth.

The phase shift in the \((s)\) polarized transmitted field relative to the incident field, \(\Delta \phi_s^{(ss)}\), as a function of frequency, at \(k = 5 \times 10^{-7} \ \text{Å}^{-1}\) and for \(\theta = \pi/4\), is plotted in figure 12, where below the lower polariton branch the phase shift is \(-\pi/2\), and it jumps to \(+\pi/2\) above the upper polariton branch. The phase shift in the \((s)\) polarized reflected field, \(\Delta \phi_s^{(rs)}\), is plotted in figure 13, where below the lower polariton branch the phase shift is \(-\pi\), and it jumps to \(+\pi\) above the upper polariton branch. In figure 14 we plot the phase shift in the transmitted and reflected \((p)\) polarized fields, \(\Delta \phi_p^{(ss)}\) and \(\Delta \phi_p^{(rs)}\), where below the lower polariton branch the phase shift is zero, and it jumps to \(+\pi\) above the upper polariton branch. The two peaks in the \((s)\) and \((p)\) transmitted and reflected phase-shift spectra are a signature of the two polariton resonances. As the coupling through the mirrors is taken to be a real constant, \(\gamma\), we neglect here phase shifts due to the cavity mirrors, which are in the general case can easily be included. The above phase shifts can be observed by homodyne-type experiments [22], in which the incident field splits into two beams, by using beam splitter. One passed through the optical lattice cavity and the other propagates in the free space, then the interference between the two resulting fields yields the phase shifts.

The \((s)\) polarized cavity mean photon number relative to the incident \((s)\) polarized photon number, in using
0° 0.2 0.4 2.4998 2.4999 2.5 2.5001 2.5002 × 10^14

Figure 10. The \((p)\) polarized field transmission and reflection spectra, \(T_{sp}^{(p)}\) and \(R_{sp}^{(p)}\), for different angles, \(\theta\), at \(k = 5 \times 10^{-7} \text{ Å}^{-1}\) of the \((s)\) polarized incident field.

0° 0.2 0.4 2.4998 2.4999 2.5 2.5001 2.5002 × 10^14

Figure 11. The absorption spectra, \(A^{(s)}\), for different angles, \(\theta\), at \(k = 5 \times 10^{-7} \text{ Å}^{-1}\) of the \((s)\) polarized incident field.

0° 0.2 0.4 2.4998 2.4999 2.5 2.5001 2.5002 × 10^14

Figure 12. The phase shift in the \((s)\) polarized transmitted field, \(\Delta \phi^{(s)}\), at \(k = 5 \times 10^{-7} \text{ Å}^{-1}\) and for \(\theta = \pi/4\) of the \((s)\) polarized incident field.

0° 0.2 0.4 2.4998 2.4999 2.5 2.5001 2.5002 × 10^14

Figure 13. The phase shift in the \((s)\) polarized reflected field, \(\Delta \phi^{(s)}\), at \(k = 5 \times 10^{-7} \text{ Å}^{-1}\) and for \(\theta = \pi/4\) of the \((s)\) polarized incident field.

0° 0.2 0.4 2.4998 2.4999 2.5 2.5001 2.5002 × 10^14

Figure 14. The phase shift in the \((p)\) polarized transmitted and reflected fields, \(\Delta \phi^{(p)}\) and \(\Delta \phi^{(r)}\), at \(k = 5 \times 10^{-7} \text{ Å}^{-1}\) and for \(\theta = \pi/4\) of the \((s)\) polarized incident field.

\[ \sqrt{\gamma} \tilde{a}_s = \tilde{c}^{\text{out},s} \] is given by

\[ I^{(s)} = \frac{\langle \tilde{a}^\dagger_s \tilde{a}_s \rangle}{\langle \tilde{b}_\text{in}^\dagger \tilde{b}_\text{in} \rangle} = \frac{\gamma |\Lambda_{ss}(1 + \gamma \Lambda_{pp}) - \gamma \Lambda_{sp} \Lambda_{ps}|^2}{|D|^2}, \quad (44) \]

and the \((p)\) polarized cavity mean photon number relative to the incident \((s)\) polarized photon number, in using \(\sqrt{\gamma} \tilde{a}_p = \tilde{c}^{\text{out},p}\) is given by

\[ I^{(p)} = \frac{\langle \tilde{a}^\dagger_p \tilde{a}_p \rangle}{\langle \tilde{b}_\text{in}^\dagger \tilde{b}_\text{in} \rangle} = \frac{\gamma |\Lambda_{sp}|^2}{|D|^2}. \quad (45) \]

Note that \(I^{(s)} = T^{(s)}/\gamma\), and \(I^{(p)} = T^{(p)}/\gamma = R^{(p)}/\gamma\). Up to the division by \(\gamma\), the plot of \(I^{(s)}\) is as that of \(T^{(s)}\) in figure 8, and the plot of \(I^{(p)}\) is as that of \(T^{(p)}\) and \(R^{(p)}\) in figure 10. Hence the cavity mean photon number of both polarizations can be observed directly through the optical linear spectra.

Identical results are obtained for the case of \((p)\) polarized incident field only in exchanging \((s)\) and \((p)\) in all the previous equations. The incident field can also be in a superposition of both polarizations.
5. Conclusion

We found that for an optical lattice with aligned transition dipole moments, the cavity photon polarizations, of TE and TM modes, are coherently mixed with the electronic excitations to form two cavity polariton branches in the strong coupling regime. As the superposition of the cavity photons of both polarizations has a component which is orthogonal to the atomic transition dipole, a third photonic branch is obtained which decouples to the electronic transitions. The system eigenmodes are observed by linear optical spectra, which also proved the polarization mixing. For an incident field which is TE polarized, we get TE and TM transmission and reflection spectra, where the TM spectra are induced by the anisotropic optical lattice. The absorption spectrum of the cavity medium is calculated by including the excited atom lifetime phenomenologically. Furthermore, the eigenmodes and the polarization mixing can be observed through the phase shifts of the transmitted and reflected fields relative to the incident field.

Polarization mixing is of major importance for electro-optics devices and quantum information, and can be used as an observation tool of many properties of anisotropic optical lattices. The system can serve as a linear optical switch, as the transmitted and reflected field intensities and phase shifts are a function of the angle between the incident field and the transition dipole, and can be controlled by changing the incident photon polarization direction. Moreover, the reflectivity and the phase shifts of the transmitted and reflected fields relative to the incident field.

Polarization mixing can serve as an important non-destructive tool to recognize the formation of molecules in optical lattices. For a system of ultracold atoms in an optical lattice in the Mott insulator case of two atoms per site with isotropic properties, no polarization mixing appears. Through the formation of on-site molecules, we achieve an optical lattice of one diatomic molecule per site with anisotropic properties, which are revealed via polarization mixing in the optical linear spectra.

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