SCHRÖDINGER OPERATORS WITH COMPLEX SINGULAR POTENTIALS

VLADIMIR MIKHAILETIS AND VOLODYMYR MOLYBOGA

To Myroslav Lvovych Gorbachuk on the occasion of his 75th birthday.

Abstract. We study one-dimensional Schrödinger operators $S(q)$ on the space $L^2(\mathbb{R})$ with potentials $q$ being complex-valued generalized functions from the negative space $H^{-1}_{unif}(\mathbb{R})$. Particularly the class $H^{-1}_{unif}(\mathbb{R})$ contains periodic and almost periodic $H^{-1}_{loc}(\mathbb{R})$-functions. We establish an equivalence of the various definitions of the operators $S(q)$, investigate their approximation by operators with smooth potentials from the space $L^1_{unif}(\mathbb{R})$ and prove that the spectrum of each operator $S(q)$ lies within a certain parabola.

1. Introduction and Main Results

In the complex Hilbert space $L^2(\mathbb{R})$ we consider a Schrödinger operator

$$S(q) = -\frac{d^2}{dx^2} + q(x)$$

with potential $q$ that is a complex-valued distribution from the space $H^{-1}_{unif}(\mathbb{R}) \subset H^{-1}_{loc}(\mathbb{R})$. Recall that $H^{-1}_{loc}(\mathbb{R})$ is a dual to the space $H^1_{comp}(\mathbb{R})$ of functions in $H^1(\mathbb{R})$ with compact support and that every $q \in H^{-1}_{loc}(\mathbb{R})$ can be represented as $Q'$ for $Q \in L^2_{loc}(\mathbb{R})$. Then the operator $S(q)$ can be rigorously defined e.g. by so-called regularization method that was used in [1] in the particular case $q(x) = 1/x$ and then developed for generic distributional potential functions in $H^{-1}_{loc}(\mathbb{R})$ in [21,22]; see also recent extensions to more general differential expressions in [7,8]. Namely, the regularization method suggests to define $S(q)$ via

\begin{equation}
S(q)y = l|y| = -(y' - Qy)' - Qy'
\end{equation}

on the natural maximal domain

\begin{equation}
\text{Dom}(S(q)) = \{y \in L^2(\mathbb{R}) \mid y, y' - Qy \in AC_{loc}(\mathbb{R}), l|y| \in L^2(\mathbb{R})\},
\end{equation}

here $AC_{loc}(\mathbb{R})$ is the space of functions that are locally absolutely continuous. It is easy to see that $S(q)y = -y'' + qy$ in the sense of distributions and the above definition does not depend on the particular choice of the primitive $Q \in L^2_{loc}(\mathbb{R})$.

One can also introduce the minimal operator $S_0(q)$, which is the closure of the restriction $S_{00}(q)$ of $S(q)$ onto the set of functions with compact support, i.e. onto

$$\text{Dom}(S_{00}(q)) := \{y \in L^2_{comp}(\mathbb{R}) \mid y, y' - Qy \in AC_{loc}(\mathbb{R}), l|y| \in L^2(\mathbb{R})\}.$$

In the case when potential function $q$ is real-valued the operator $S_{00}(q)$ (and hence $S_0(q)$) is symmetric; moreover, in a standard manner [18] one can prove that $S(q)$ is adjoint of $S_0(q)$. An important question preceding any further analysis of the operator $S(q)$ is whether it is self-adjoint, i.e. $S(q) = S_0(q)$. The case when the potential belongs to the space $H^{-1}_{unif}(\mathbb{R})$ was investigated in [10]. We recall [10] that any $q \in H^{-1}_{unif}(\mathbb{R})$ can be represented (not uniquely) in the form

\begin{equation}
q = Q' + \tau,
\end{equation}

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where derivative is understood in the sense of distributions and $Q$ and $\tau$ belong to Stepanov spaces $L^2_{\text{unif}}(\mathbb{R})$ and $L^1_{\text{unif}}(\mathbb{R})$ respectively, i.e.

$$\|Q\|_{L^2_{\text{unif}}(\mathbb{R})}^2 := \sup_{t \in \mathbb{R}} \int_0^{t+1} |Q(s)|^2 ds < \infty,$$

$$\|\tau\|_{L^1_{\text{unif}}(\mathbb{R})} := \sup_{t \in \mathbb{R}} \int_0^{t+1} |\tau(s)| ds < \infty.$$ 

Given such a representation, the operator $S$ is defined as

$$(4) \quad S(q)y = -(y' - Qy)' - Qy' + \tau y$$

on the domain $D(S)$. This definition also does not depend on the particular choice of $Q$ and $\tau$ above. Theorem 3.5 of the paper [10] claims that for real-valued $q \in H^{-1}_{\text{unif}}(\mathbb{R})$ the operator $S(q)$ as defined by (4) and (2) is self-adjoint and coincides with the operator $S_{F_0}(q)$ constructed by the form-sum method. However the proof given in [10] is incomplete.

The fact that $S(q)$ is indeed self-adjoint is rigorously justified in the paper [18] for the particular case where $q \in H^{-1}_{\text{unif}}(\mathbb{R})$ is periodic. The authors prove therein that $S_0(q)$, $S(q)$, $S_{F_0}(q)$ and the Friedrichs extension $S_F(q)$ of $S_0(q)$ all coincide. However the arguments heavily use periodicity of $q$ and can not be applied to generic real-valued $q \in H^{-1}_{\text{unif}}(\mathbb{R})$. This gap in the proof of Theorem 3.5 of [10] is filled in by the authors in their recent paper [11], see also [14].

This paper deals with the case when the potential $q \in H^{-1}_{\text{unif}}(\mathbb{R})$ is complex-valued. One can easily see that in this case all operators $S_0(q)$, $S(q)$, $S_{F_0}(q)$ and $S_F(q)$ are well-defined and are related by

$$S_0(q) \subset S_F(q) = S_{F_0}(q) \subset S(q), \quad \text{Dom}(S_F(q)) \subset H^1(\mathbb{R}), \quad \text{Dom}(S(q)) \subset L^1_{\text{loc}}(\mathbb{R}) \cap L^2(\mathbb{R}).$$

The main purpose of this paper is to prove that these operators coincide and to investigate their approximation and spectral properties. Let us state the main results.

**Theorem A.** For every function $q \in H^{-1}_{\text{unif}}(\mathbb{R})$ operators $S_0(q)$, $S(q)$, $S_{F_0}(q)$ and $S_F(q)$ are $m$-sectorial and coincide.

Theorem A allows one to link the known results for the Schrödinger operators in the space $L^2(\mathbb{R})$ which are defined in different ways, see e.g. [2, 5, 13, 24].

In the paper [18] the authors proved that for every real-valued 1-periodic function $q \in H^{-1}_{\text{loc}}(\mathbb{R})$ a sequence of smooth 1-periodic functions $q_n$ exists such that the sequence of operators $S(q_n)$ converges to the operator $S(q)$ in the sense of norm resolvent convergence. It is sufficient to establish

$$\|q - q_n\|_{H^{-1}(0,1)} \to 0, \quad n \to \infty.$$

The following theorem generalizes this result in two directions. The potential $q$ may be complex-valued and non-periodic.

**Theorem B.** Let $q$, $q_n$, $n \geq 1$, belong to the space $H^{-1}_{\text{unif}}(\mathbb{R})$. Then the sequence of operators $S(q_n)$, $n \geq 1$, converges to the operator $S(q)$ in the sense of norm resolvent convergence, $R(\lambda, S) := (S - \lambda I)^{-1}$.

$$(5) \quad \|R(\lambda, S(q)) - R(\lambda, S(q_n))\| \to 0, \quad n \to \infty, \quad \lambda \in \text{Resolv}(S(q)) \neq 0,$$

if

$$(6) \quad q_n \xrightarrow{H^{-1}_{\text{unif}}(\mathbb{R})} q, \quad n \to \infty$$

or, equivalently,

$$(7) \quad q_n \xrightarrow{L^2_{\text{unif}}(\mathbb{R})} Q, \quad \tau_n \xrightarrow{L^1_{\text{unif}}(\mathbb{R})} \tau, \quad n \to \infty.$$

Since the set $C^\infty(\mathbb{R}) \cap L^1_{\text{unif}}(\mathbb{R})$ is dense in the space $H^{-1}_{\text{unif}}(\mathbb{R})$ (see Section 3.2 below), then the following corollary holds.

**Corollary B.1.** For every function $q \in H^{-1}_{\text{unif}}(\mathbb{R})$ there is a sequence of functions $q_n \in C^\infty(\mathbb{R}) \cap L^1_{\text{unif}}(\mathbb{R})$ such that the limit relation (5) is true. If the function $q$ is real-valued, then the functions $q_n$ can be chosen to be real-valued as well.
In particular, if $Q$ and $\tau$ are almost periodic Stepanov functions then $Q_n$ and $\tau_n$ can be chosen to be trigonometrical polynomials [13 Theorem 1.5.7.2]. If $Q$ and $\tau$ are bounded and uniformly continuous on the whole real axis $\mathbb{R}$, then $Q_n$ and $\tau_n$ can be chosen to be entire analytic functions [15 Theorem I.1.10.1, Remark].

The following theorem allows one to describe the localization of the spectrum of the operators $S(q)$.

**Theorem C.** The numerical ranges of operators $S(q)$ (and therefore their spectra) lie within the parabola:

\[ |\text{Im} \lambda| \leq 5K \left( \text{Re} \lambda + 4(2K + 1)^4 \right)^{3/4}, \]
\[ K = 2 \left( \|Q\|_{L^2_{ \text{unif}}(\mathbb{R})} + \|\tau\|_{L^2_{ \text{unif}}(\mathbb{R})} \right). \]

If the potential $q$ is real-valued, then the self-adjoint operator $S(q)$ is bounded below by a number

\[ m(K) = \begin{cases} 
-4K, & \text{if } K \in [0, 1/2), \\
-32K^4, & \text{if } K \geq 1/2. 
\end{cases} \]

Note that if a complex-valued potential $q \in H^1_{ \text{unif}}(\mathbb{R})$ is a periodic generalized function, then the spectrum of the operator $S(q)$ lies within a quadratic parabola [16 Theorem 6]. A similar result holds for certain complex-valued measures, see [24] and Section 3.3, formula (32).

Similar problems are considered in the papers [3, 4, 6, 17, 20, 23].

## 2. Preliminaries

This section contains several statements that are used in the proof of Theorem A.

We begin with introduction the dual operators $S_{00}^+(q) \equiv S^+(q)$. The formally adjoint quasi-differential expression $1^+$ for $1$ is defined by [23]:

\[ v^{(0)} := v, \quad v^{(1)} := v - \overline{Q}v, \quad v^{(2)} := (v^{(1)})' + \overline{Q}v^{(1)} + (Q^2 - \tau)v, \]
\[ 1^+[v] := -v^{(2)}, \quad \text{Dom}(1^+) := \left\{ v : \mathbb{R} \to \mathbb{C} \mid v, v^{(1)} \in \text{AC}_{\text{loc}}(\mathbb{R}) \right\}. \]

By $\overline{\tau}$ we denote a complex conjugation. Then

\[ S^+v \equiv S^+(q)v := 1^+[v], \quad \text{Dom}(S^+) := \left\{ v \in L^2(\mathbb{R}) \mid v, v^{(1)} \in \text{AC}_{\text{loc}}(\mathbb{R}), 1^+[v] \in L^2(\mathbb{R}) \right\}, \]
\[ S_{00}^+v \equiv S_{00}^+(q)v := 1^+[v], \quad \text{Dom}(S_{00}^+) := \left\{ v \in \text{Dom}(S^+) \mid \text{supp} v \in \mathbb{R} \right\}. \]

One can easily see that if $\text{Im} q \equiv 0$ then operators $S_{00}(q) \equiv S_{00}^+(q)$, $S(q)$ and $S^+(q)$ coincide.

**Lemma 1** (Theorem 1, Corollary [25]). For arbitrary functions $u \in \text{Dom}(S)$, $v \in \text{Dom}(S^+)$ and finite interval $[a, b]$ the following equality holds:

\[ \int_a^b \overline{l[u]a} dx - \int_a^b \overline{l[u]}v dx = [u, v]_a^b, \]

where

\[ [u, v](t) := u(t)v^{(1)}(t) - u^{(1)}(t)v(t), \]
\[ [u, v]^b_a := [u, v](b) - [u, v](a). \]

**Lemma 2.** For arbitrary functions $u \in \text{Dom}(S)$ and $v \in \text{Dom}(S^+)$ the following limits exist and are finite:

\[ [u, v](\infty) := \lim_{t \to -\infty} [u, v](t), \quad [u, v](\infty) := \lim_{t \to -\infty} [u, v](t). \]

**Proof.** Let us fix the number $b$ in the equality (9) and then pass to the limit as $a \to -\infty$. Whereas due to the assumptions of the lemma $u, v, l[u], l^+[v] \in L^2(\mathbb{R})$, the limit $[u, v](\infty)$ exists and is finite. Similarly one can prove that the limit $[u, v](\infty)$ exists and is finite.

The Lemma is proved.

**Lemma 3** (Generalized Lagrange identity). For all functions $u \in \text{Dom}(S)$, $v \in \text{Dom}(S^+)$ the equality

\[ \int_{-\infty}^\infty \overline{l[u]a} dx - \int_{-\infty}^\infty \overline{l[u]}v dx = [u, v]_{-\infty}^\infty, \]
\[ [u, v]_{-\infty}^\infty := [u, v](\infty) - [u, v](\infty). \]
holds.

Proof. The identity (10) is true due to Lemma 1 and Lemma 2.

In the following proposition we describe the properties of minimal and maximal operators and their adjoints.

**Proposition 4.** For the operators $S$, $S_{00}$ and $S^+$, $S^+_{00}$ the following statements are fulfilled.

1. Operators $S_{00}$ and $S^+_{00}$ are densely defined in the Hilbert space $L^2(\mathbb{R})$.

2. The following relations hold:

$$(S_{00})^* = S^+, \quad (S^+_{00})^* = S.$$

3. Operators $S$, $S^+$ are closed and operators $S_{00}$, $S^+_{00}$ are closable,

$$S_0 := (S_{00})^*, \quad S^+_0 := (S^+_{00})^*.$$

4. Domains of operators $S_0$, $S^+_0$ may be described in the following way:

$$\text{Dom}(S_0) = \{ u \in \text{Dom}(S) \mid [u, v]_{-\infty}^\infty = 0 \ \forall v \in \text{Dom}(S) \},$$

$$\text{Dom}(S^+_0) = \{ v \in \text{Dom}(S^+) \mid [u, v]_{-\infty}^\infty = 0 \ \forall u \in \text{Dom}(S) \}.$$

5. Domains of operators $S$, $S_0$, $S_{00}$ and $S^+$, $S^+_0$, $S^+_{00}$ satisfy the following relations:

$$u \in \text{Dom}(S) \Leftrightarrow \overline{u} \in \text{Dom}(S^+),$$

$$u \in \text{Dom}(S_0) \Leftrightarrow \overline{u} \in \text{Dom}(S^+_0),$$

$$u \in \text{Dom}(S_{00}) \Leftrightarrow \overline{u} \in \text{Dom}(S^+_{00}).$$

The proof of properties 1–4 in Proposition 4 is similar to the proof of similar statements for symmetric operators on semi-axis [25], see also [19]. The property 5 is proved by direct calculation.

We use the following estimates obtained in [10] Lemma 3.2] to prove the main theorems.

**Lemma 5.** Let the functions $Q \in L^2_{unif}(\mathbb{R})$, $\tau \in L^1_{unif}(\mathbb{R})$ and $u \in H^1(\mathbb{R})$. Then $\forall \varepsilon \in (0, 1)$ and $\forall \eta \in (0, 1)$ the estimates hold:

$$\left| (Q, \overline{\tau} u)_{L^2(\mathbb{R})} \right| \leq ||Q||_{L^2_{unif}(\mathbb{R})} \left( \varepsilon \|u\|^2_{L^2(\mathbb{R})} + 4\varepsilon^{-3} \|u\|^2_{L^2(\mathbb{R})} \right),$$

$$\left| (\tau, |u|^2)_{L^2(\mathbb{R})} \right| \leq ||\tau||_{L^1_{unif}(\mathbb{R})} \left( \eta \|u\|^2_{L^2(\mathbb{R})} + 8\eta^{-1} \|u\|^2_{L^2(\mathbb{R})} \right).$$

3. PROOFS

3.1 **Proof of Theorem A** Consider the sesquilinear forms generated by preminimal operators $S_{00}(q)$:

$$i_{S_{00}}[u, v] := (S_{00}(q)u, v)_{L^2(\mathbb{R})} = (u', v')_{L^2(\mathbb{R})} - (Q, \overline{\tau} v + \overline{\tau} v')_{L^2(\mathbb{R})} + (\tau, \overline{\tau} v)_{L^2(\mathbb{R})},$$

$$\text{Dom}(i_{S_{00}}) := \text{Dom}(S_{00}(q)).$$

To them correspond the quadratic forms

$$i_{S_{00}}[u] = (u', u')_{L^2(\mathbb{R})} - (Q, \overline{\tau} u + \overline{\tau} u')_{L^2(\mathbb{R})} + (\tau, |u|^2)_{L^2(\mathbb{R})}.$$ We introduce the notation:

$$t_{Q, \tau}[u, v] := -(Q, \overline{\tau} v + \overline{\tau} v')_{L^2(\mathbb{R})} + (\tau, \overline{\tau} v)_{L^2(\mathbb{R})}, \quad \text{Dom}(t_{Q, \tau}) := \text{Dom}(S_{00}(q)),

$$i_0[u, v] := (u', v')_{L^2(\mathbb{R})}, \quad \text{Dom}(i_0) := \text{Dom}(S_{00}(q)).$$

Then due to Lemma 5 forms $t_{Q, \tau}$ are 0-bounded with respect to the densely defined positive form $i_0$:

$$||t_{Q, \tau}[u]\| \leq K \varepsilon ||i_0[u]\| + 4K \varepsilon^{-3} ||u||^2_{L^2(\mathbb{R})} \quad \forall \varepsilon \in (0, 1], \ u \in \text{Dom}(i_0),$$

$$K := 2 \left( ||Q||_{L^2_{unif}(\mathbb{R})} + ||\tau||_{L^1_{unif}(\mathbb{R})} \right).$$

Formula (11) implies that sesquilinear forms $i_{S_{00}} = i_0 + t_{Q, \tau}$ are closable, $t_{S_{00}} := (i_{S_{00}})^{\sim}$:

$$i_{S_{00}}[u, v] = (u', v')_{L^2(\mathbb{R})} - (Q, \overline{\tau} v + \overline{\tau} v')_{L^2(\mathbb{R})} + (\tau, \overline{\tau} v)_{L^2(\mathbb{R})}, \quad \text{Dom}(t_{S_{00}}) = H^1(\mathbb{R}).$$

Forms $t_{S_{00}}$ are densely defined, closed and sectorial. Then due to the First Representation Theorem 12, with the sesquilinear forms $t_{S_{00}}$ we associate $m$-sectorial operators $S_F(q)$ that are the Friedrichs extensions of operators $S_{00}(q)$. 


Proposition 6. The m-sectorial operators $S_F(q)$ are described in the following way:

$$S_F u = S_F(q)u = [u], \quad \text{Dom}(S_F) = \left\{ u \in H^1(\mathbb{R}) \mid u, u^{[1]} \in AC_{loc}(\mathbb{R}), [u] \in L^2(\mathbb{R}) \right\}.$$ 

The proof of Proposition 6 is similar to the proof of [10] Theorem 3.5 for real-valued distributions $q \in H^{-1}_{unif}(\mathbb{R})$.

Thus we have established that the following relations hold:

\begin{equation}
S_{00} \subset S_0 \subset S_F \subset S.
\end{equation}

Passing in to the adjoint operators (12) and using property 2 of Proposition 4, we obtain:

\begin{equation}
S_{00}^+ \subset S_0^+ \subset S_F^* \subset S^+.
\end{equation}

One can easily prove that operators $S_F^*$ coincide with Friedrichs extensions $S_F^+ \subset S_F^*$ of operators $S_{00}$.

Let us now define the operators (1) as form-sums.

Consider the sesquilinear forms generated by the distributions $q \in H^{-1}_{unif}(\mathbb{R})$:

$$\langle \cdot, \cdot \rangle_q := (q(x) u, v), \quad \text{Dom}(\langle \cdot, \cdot \rangle_q) := C_0^\infty(\mathbb{R}),$$

where $(\cdot, \cdot)$ is a sesquilinear form pairing the spaces of generalized functions $\mathcal{D}'(\mathbb{R})$ and test functions $C_0^\infty(\mathbb{R})$ with respect to the space $L^2(\mathbb{R})$.

Due to Lemma 5 for the forms

$$i_q[u, v] = \langle q(x) u, v \rangle = -(Q, \nabla u \nabla v')_{L^2(\mathbb{R})} + (\tau, |u|^2)_{L^2(\mathbb{R})}, \quad u \in C_0^\infty(\mathbb{R}),$$

the following estimates hold:

$$|i_q[u]| \leq 2 \left( \|Q\|_{L^2_{unif}(\mathbb{R})} + \|\tau\|_{L^2_{unif}(\mathbb{R})} \right) \left( \|u\|_{L^2(\mathbb{R})}^2 + 4 \|u\|_{L^2(\mathbb{R})}^2 \right), \quad u \in C_0^\infty(\mathbb{R}).$$

Therefore forms $i_q$ allow a continuous extension onto the space $H^1(\mathbb{R})$ [20]. The sesquilinear forms $i_q[u, v]$ on the space $H^1(\mathbb{R})$ are represented as:

\begin{equation}
t_q[u, v] = -(Q, \nabla v \nabla u')_{L^2(\mathbb{R})} + (\tau, \nabla u)_{L^2(\mathbb{R})}, \quad \text{Dom}(t_q) = H^1(\mathbb{R}).
\end{equation}

One may easily see that the following Lemma is true applying the estimates of Lemma [5].

Lemma 7. The sesquilinear forms $t_q$ are 0-bounded with respect to the sesquilinear form

$$t_0[u, v] := (u', v')_{L^2(\mathbb{R})}, \quad \text{Dom}(t_0) := H^1(\mathbb{R}).$$

Thus, the sesquilinear forms

\begin{equation}
t[u, v] := t_0[u, v] + t_q[u, v], \quad \text{Dom}(t) := H^1(\mathbb{R}),
\end{equation}

are densely defined, closed and sectorial. According to the First Representation Theorem [12] with the forms $t$ one can associate m-sectorial operators $S_{fs}(q)$, which are called the form-sums and denoted by:

$$S_{fs} \equiv S_{fs}(q) := -\frac{d^2}{dx^2} + q(x),$$

$$\text{Dom}(S_{fs}(q)) := \left\{ u \in H^1(\mathbb{R}) \mid -u'' + q(x)u \in L^2(\mathbb{R}) \right\}.$$ 

Since the forms $t$ coincide with the forms $t_{S_0}$, the form-sum operators $S_{fs}(q)$ and the Friedrichs extensions $S_F(q)$ of operators $S_{00}(q)$ coincide: $S_F(q) = S_{fs}(q)$.

Thus, relations (12) and (13) take the following form:

\begin{align*}
(16) \quad S_{00} \subset S_0 \subset S_F = S_{fs} \subset S, \quad & \text{Dom}(S_F) \subset H^1(\mathbb{R}), \quad \text{Dom}(S) \subset H^1_{loc}(\mathbb{R}), \\
(17) \quad S_{00}^+ \subset S_0^+ \subset S_F^+ = S_{fs}^* = S_F^* = S^*, \quad & \text{Dom}(S_F^+) \subset H^1(\mathbb{R}), \quad \text{Dom}(S^+) \subset H^1_{loc}(\mathbb{R}).
\end{align*}

Proposition 8. Suppose $\text{Dom}(S) \subset H^1(\mathbb{R})$. Then operators $S_0(q)$ and $S_0^+(q)$ are m-sectorial and

$$S_0 = S_F = S_{fs} = S,$$

$$S_0^+ = S_F^+ = S_{fs}^* = S_F^* = S^+.$$
Therefore, taking into consideration that

Indeed, taking into account the property (21)

Obviously, together with (20) the following is also true:

Taking into account (18) and (19), property 4° of Proposition 4 implies the equalities:

Proposition is proved.

Due to Proposition 14 (see Section 3.3 3.3) operators \( S_0(q) \) are quasiaccretive:

In what follows w.l.a.g. we assume that

Obviously, together with (20) the following is also true:

Indeed, taking into account the property 5° of Proposition 4 we get:

The following lemma is used in the proof of Theorem A. It is proved by direct calculation.

**Lemma 9.** Suppose \( u \in \text{Dom}(S) \). Then \( \forall \varphi \in C_0^\infty(\mathbb{R}) \):

i) \( \| \varphi u \| = \varphi \| u \| - \varphi'' u - 2 \varphi' u' \); ii) \( \varphi u \in \text{Dom}(S_00) \).

Now let us prove Theorem A.

Let us prove that operators \( S_0(q) \) are quasi-\( m \)-accretive. It is sufficient to show that

\( \text{def } S_0(q) := \dim (\text{ran } S_0(q)) + = \dim (\text{ker } S_0^+(q)) = 0. \)

Let \( v(x) \) be a solution of the equation

Let us show that \( v(x) \equiv 0. \)

For any real function \( \varphi \in C_0^\infty(\mathbb{R}) \) due to Lemma 9 and property 5° of Proposition 4 we have \( \varphi v \in \text{Dom}(S_0^0) \). Therefore, taking into consideration that \( 1^+ [v] = 0 \) due to (22), one calculates:

Considering (20) and that

\[
\text{Re} \int_R \varphi \varphi' (v\varphi' - v'\varphi) d x = 0,
\]

from (23) we obtain:

\[
\int_R (\varphi')^2 |v|^2 d x \geq \int_R \varphi^2 |v|^2 d x \quad \forall \varphi \in C_0^\infty(\mathbb{R}), \text{ Im } \varphi = 0.
\]

Let us then take a sequence of functions \( \{ \varphi_n \}_{n \in \mathbb{N}} \) such that:

i) \( \varphi_n \in C_0^\infty(\mathbb{R}) \), \( \text{Im } \varphi_n \equiv 0 \);

ii) \( \supp \varphi_n \subset [-n - 1, n + 1] \);

iii) \( \varphi_n(x) = 1, \ x \in [-n, n] \);

iv) \( |\varphi_n'(x)| \leq C. \)
Substituting functions \( \varphi_n \) into (24) we receive
\[
\int_{-n}^{n} |v|^2 dx \leq \int_{\mathbb{R}} \varphi_n^2 |v|^2 dx \leq \int_{\mathbb{R}} (\varphi_n')^2 |v|^2 dx \leq C^2 \int_{n \leq |x| \leq n+1} |v|^2 dx,
\]
that is
\[
(25) \quad \int_{-n}^{n} |v|^2 dx \leq C^2 \int_{n \leq |x| \leq n+1} |v|^2 dx.
\]

Taking into account that \( v(x) \in L^2(\mathbb{R}) \), passing in (25) to the limit as \( n \to \infty \) we obtain \( v(x) \equiv 0 \).

Thus, operators \( S_0(q) \) are proved to be quasi-\( m \)-accractive. Due to Proposition [14] they are \( m \)-sectorial.

Therefore, by the properties of the Friedrichs extensions [12] we have:
\[
S_0(q) = S_\mathcal{F}(q).
\]

Then taking into account property 2 of Proposition [14] from (26) we derive:
\[
S^+(q) = S_\mathcal{F}^+(q), \quad \text{Dom}(S^+(q)) \subset H^1(\mathbb{R}).
\]

Due to the property 5 of Proposition [14] from Proposition [8] we finally get necessary result
\[
S_0 = S_\mathcal{F} = S_{fs} = S.
\]

Theorem [A] is proved completely.

3.2. Proof of Theorem [B]. Let us suppose that the assumptions of theorem, that is the formula (20) (or equivalently (7)), hold. Consider the sesquilinear forms
\[
\begin{align*}
\tilde{t}_0[u, v] &:= (S(q)u, v)_{L^2(\mathbb{R})}, \quad \text{Dom}(\tilde{t}_0) := \text{Dom}(S(q)), \\
\tilde{t}_n[u, v] &:= (S(q_n)u, v)_{L^2(\mathbb{R})}, \quad \text{Dom}(\tilde{t}_n) := \text{Dom}(S(q_n)), \, n \in \mathbb{N}.
\end{align*}
\]
The forms \( \tilde{t}_0 \) and \( \tilde{t}_n \), \( n \in \mathbb{N} \), are densely defined, closable and sectorial. Their closures may be represented in the following way:
\[
\begin{align*}
t_0[u, v] &= (u', v')_{L^2(\mathbb{R})} - (Q, \partial_\nu v + \partial_\nu v')_{L^2(\mathbb{R})} + (\tau, \partial_\nu v)_{L^2(\mathbb{R})}, \quad \text{Dom}(t_0) = H^1(\mathbb{R}), \\
t_n[u, v] &= (u', v')_{L^2(\mathbb{R})} - (Q_n, \partial_\nu v + \partial_\nu v')_{L^2(\mathbb{R})} + (\tau_n, \partial_\nu v)_{L^2(\mathbb{R})}, \quad \text{Dom}(t_n) = H^1(\mathbb{R}).
\end{align*}
\]

Further, applying the estimates of Lemma [5] we get:
\[
(27) \quad |t_n[u] - t_0[u]| \leq a_n \|u\|^2_{L^2(\mathbb{R})} + 4a_n \|u\|^2_{L^2(\mathbb{R})},
\]
where
\[
a_n := 2 \left( \|Q - Q_n\|^2_{L^2_{n+1}(\mathbb{R})} + \|\tau - \tau_n\|^2_{L^2_{n+1}(\mathbb{R})} \right),
\]
and similarly to the proof of Lemma [11] (see below) we obtain:
\[
(28) \quad 2\Re t_0[u] + 4\|u\|^2_{L^2(\mathbb{R})} \geq \|u'\|^2_{L^2(\mathbb{R})}.
\]

Formulas (27) and (28) together with (6), (7) imply:
\[
|t_n[u] - t_0[u]| \leq 2a_n \Re t_0[u] + 8a_n \|u\|^2_{L^2(\mathbb{R})}, \quad a_n \to 0, \, n \to \infty.
\]

To complete the proof we only need to apply [12, Theorem VI.3.6].

Theorem [B] is proved completely.

To prove Corollary [B.1] we need in an auxiliary result. It has an independent interest also.

**Theorem 10.** The set
\[
C^\infty(\mathbb{R}) \cap L^p_{\text{unif}}(\mathbb{R})
\]
is everywhere dense in the Stepanov space \( L^p_{\text{unif}}(\mathbb{R}) \), \( 1 \leq p < \infty \).

**Proof.** Set for \( f \in L^p_{\text{loc}}(\mathbb{R}) \), \( f_n := \chi_{[n, n+1)} f, \, n \in \mathbb{Z} \). Let \( \varepsilon > 0 \) be given. Since the set \( C^\infty_0(a, b) \) is dense in the space \( L^p(a, b) \), there is a function sequence \( g_n \in C^\infty_0(\mathbb{R}) \), supp \( g_n \subset (n, n+1) \), such that \( \|f_n - g_n\|_{L^p(\mathbb{R})} < \varepsilon 2^{-|n| - 2} \). Set \( g := \sum_{n \in \mathbb{Z}} g_n \). Then \( g \in C^\infty(\mathbb{R}) \) and \( \|f - g\|_{L^p(\mathbb{R})} < \varepsilon \). If \( f \in L^p_{\text{unif}}(\mathbb{R}) \), then the function \( g \in L^p_{\text{unif}}(\mathbb{R}) \) since \( \|f - g\|_{L^p_{\text{unif}}(\mathbb{R})} < \varepsilon \). If the function \( f \) is real-valued, then so are the functions \( f_n \) as well. Therefore, the functions \( g_n \) may be chosen to be real-valued.

Theorem [11] and [10, Theorem 2.1] imply the following important statement.

\( \square \)
Corollary 10.1. The set 
\[ C^\infty(\mathbb{R}) \cap L^1_{\text{unif}}(\mathbb{R}) \]
is everywhere dense in the space \( H_{\text{unif}}^{-1}(\mathbb{R}) \).

Then Corollary 13.1 follows from Theorem 12 and Corollary 10.1

3.3. Proof of Theorem C. Theorem C follows from Theorem 13 below regarding perturbations of a positive quadratic form. It is abstract and can be of independent interest.

Let in an abstract Hilbert space \( H \) a densely defined closed positive sesquilinear form \( \alpha_0[u,v] \) with domain \( \text{Dom}(\alpha_0) \subset H \) be given. Let \( \beta[u,v] \) be a sesquilinear form defined on \( H \) with a domain \( \text{Dom}(\beta) \supset \text{Dom}(\alpha_0) \).

Suppose the form \( \beta \) satisfies the following estimate:
\[ \exists a, b, s > 0 : \quad |\beta[u]| \leq a\varepsilon \alpha_0[u] + b\varepsilon^{-s} \|u\|^2_H \quad \forall \varepsilon > 0, \ u \in \text{Dom}(\alpha_0). \]  

Consider on the Hilbert space \( H \) the sum of forms \( \alpha_0 \) and \( \beta \):
\[ \alpha[u,v] := \alpha_0[u,v] + \beta[u,v], \quad \text{Dom}(\alpha) := \text{Dom}(\alpha_0). \]

A sesquilinear form \( \alpha \) is densely defined closed and sectorial form on the Hilbert space \( H \). Let \( \Theta(\alpha) \) be a numerical range of \( \alpha \):
\[ \Theta(\alpha) := \{ \alpha[u], \ u \in \text{Dom}(\alpha), \ |u| \in H = 1 \}. \]

According to our assumptions \( \Theta(\alpha_0) \subset [0, \infty) \). Let us find the properties of the set \( \Theta(\alpha) \). To do that we require the following two lemmas.

Lemma 11. The following estimates hold:
\[ |\text{Im} \alpha[u]| \leq 2a\varepsilon \text{Re} \alpha[u] + 2b\varepsilon^{-s}|u|^2_H, \quad 0 < \varepsilon \leq (2a + 1)^{-1}. \]

Proof. According to our assumptions we have:
\[ \text{Re} \alpha[u] = \alpha_0[u] + \text{Re} \beta[u], \quad \text{Im} \alpha[u] = \text{Im} \beta[u], \]
and due to (30):
\[ |\text{Im} \alpha[u]| \leq a\varepsilon \alpha_0[u] + b\varepsilon^{-s}|u|^2_H. \]

Furthermore given that \( 0 < \varepsilon \leq (2a + 1)^{-1} \) and therefore \( 1 - a\varepsilon \geq \frac{1}{2} \) we have for \( \text{Re} \alpha[u] \):
\[ \text{Re} \alpha[u] \geq \alpha_0[u] - |\text{Re} \beta[u]| \geq (1 - a\varepsilon)\alpha_0[u] - b\varepsilon^{-s}|u|^2_H \geq \frac{1}{2} \alpha_0[u] - b\varepsilon^{-s}|u|^2_H, \]
and
\[ 2a\varepsilon \text{Re} \alpha[u] \geq a\varepsilon_0[u] \quad \text{and} \quad 2a\varepsilon \text{Re} \alpha[u] + b\varepsilon^{-s}|u|^2_H \geq a\varepsilon_0[u] - b\varepsilon^{-s}|u|^2_H. \]

From (32) and (33) we receive the required estimates:
\[ |\text{Im} \alpha[u]| \leq 2a\varepsilon \text{Re} \alpha[u] + 2b\varepsilon^{-s}|u|^2_H. \]

Lemma is proved. \( \square \)

We introduce the following notation:
\[ S_{a,b,s,\varepsilon} := \{ \lambda \in \mathbb{C} \mid |\text{Im} \lambda| \leq 2a\varepsilon \text{Re} \lambda + 2b\varepsilon^{-s} \}, \]
\[ M_{a,b,s} := \bigcap_{0 < \varepsilon \leq (2a + 1)^{-1}} S_{a,b,s,\varepsilon}. \]

Then due to Lemma 11 we have \( \Theta(\alpha) \subset M_{a,b,s} \).

Lemma 12. The set \( M_{a,b,s} \) can be written as:
\[ M_{a,b,s} = \begin{cases} \{ \lambda \in \mathbb{C} \mid |\text{Im} \lambda| \leq \frac{2a}{2a + 1} \text{Re} \lambda + 2b(2a + 1)^s \}, & \lambda_0 \leq \text{Re} \lambda \leq \lambda_1, \\ \{ \lambda \in \mathbb{C} \mid |\text{Im} \lambda| \leq 2(s + 1)b^{1/(s+1)} \left( \frac{a}{s} \right)^{s/(s+1)} (\text{Re} \lambda)^{s/(s+1)} \}, & \lambda_1 < \text{Re} \lambda, \end{cases} \]

where \( \lambda_0 := -\frac{b}{a}(2a + 1)^{s+1} \) is the vertex of sector

\[
\left\{ \lambda \in \mathbb{C} \bigg| \left| \text{Im} \lambda \right| \leq \frac{2a}{2a + 1} \text{Re} \lambda + 2b(2a + 1)^s \right\},
\]

and \( \lambda_1 := \frac{bs}{a}(2a + 1)^{s+1} \).

**Proof.** For convenience we will find the description of the set \( \mathcal{M}_{a,b,s} \) in \( \mathbb{R}^2 \).

Let

\[
y = 2a \varepsilon x + 2b \varepsilon^{-s}
\]

be the line which bounds the corresponding sector from above. Let us find the locus of points of intersection of these lines when \( 0 < \varepsilon < (2a + 1)^{-1} \):

\[
2a \varepsilon_1 x + 2b \varepsilon_1^{-s} = 2a \varepsilon_2 x + 2b \varepsilon_2^{-s},
\]

\[
2a(\varepsilon_1 - \varepsilon_2)x = 2b(\varepsilon_2^{-s} - \varepsilon_1^{-s}),
\]

\[
x = -\frac{b}{a} \frac{\varepsilon_1^{-s} - \varepsilon_2^{-s}}{\varepsilon_1 - \varepsilon_2},
\]

\[
x \xrightarrow{\varepsilon \to \varepsilon_1} \frac{sb}{a} \varepsilon_1^{-s-1},
\]

and

\[
y = 2a \varepsilon_1, \quad \frac{sb}{a} \varepsilon_1^{-s-1} + 2b \varepsilon_1^{-s} = 2(s + 1)b \varepsilon_1^{-s}.
\]

So, for \( \varepsilon = (2a + 1)^{-1} \) the set \( \mathcal{M}_{a,b,s} \) is bounded from above by the line:

\[
y = \frac{2a}{2a + 1}x + 2b(2a + 1)^s,
\]

and for \( 0 < \varepsilon < (2a + 1)^{-1} \) by the curves:

\[
y = 2(s + 1)b \varepsilon^{-s}.
\]

If we express \( \varepsilon \) through \( x \) in the first equality of (36) and substitute it in the equality for \( y \), we obtain an explicit equation for curves (36):

\[
y = 2(s + 1)b^{1/(s+1)} \left( \frac{a}{s} \right)^{s/(s+1)} x^{s/(s+1)}, \quad x > x_1, \quad x_1 := \frac{bs}{a}(2a + 1)^{s+1}.
\]

The set \( \mathcal{M}_{a,b,s} \) is bounded below by curves of the form (35) and (37) with \(-y\) instead of \(y\).

Thus the set \( \mathcal{M}_{a,b,s} \) in \( \mathbb{R}^2 \) may be represented in the following way:

\[
\mathcal{M}_{a,b,s} = \left\{ (x, y) \in \mathbb{R}^2 \bigg| y \leq \frac{2a}{2a + 1}x + 2b(2a + 1)^s \right\}, \quad x_0 \leq x \leq x_1,
\]

\[
\left\{ (x, y) \in \mathbb{R}^2 \bigg| y \leq 2(s + 1)b^{1/(s+1)} \left( \frac{a}{s} \right)^{s/(s+1)} x^{s/(s+1)} \right\}, \quad x_1 < x,
\]

where \( x_0 := -\frac{b}{a}(2a + 1)^{s+1} \) is the vertex of sector

\[
\left\{ (x, y) \in \mathbb{R}^2 \bigg| |y| \leq \frac{2a}{2a + 1}x + 2b(2a + 1)^s \right\},
\]

and \( x_1 = \frac{bs}{a}(2a + 1)^{s+1} \).

Lemma is proved. \( \square \)

Lemmas \( \text{[11]} \) and \( \text{[12]} \) imply the following theorem.
Theorem 13. The numerical range $\Theta(\alpha)$ of the sesquilinear form $\alpha$ is a subset of the set $\mathcal{M}_{a,b,s}$:

$$
\mathcal{M}_{a,b,s} = \left\{ \lambda \in \mathbb{C} \left| \left| \text{Im} \lambda \right| \leq \frac{2a}{2a + 1} \text{Re} \lambda + 2b(2a + 1)^{s} \right\}, \lambda_{0} \leq \text{Re} \lambda \leq \lambda_{1},
\right.
$$

where $\lambda_{0} = -\frac{b}{a}(2a + 1)^{s+1}$ is the vertex of sector

$$
\left\{ \lambda \in \mathbb{C} \left| \left| \text{Im} \lambda \right| \leq \frac{2a}{2a + 1} \text{Re} \lambda + 2b(2a + 1)^{s} \right\},
$$

and $\lambda_{1} = \frac{bs}{a}(2a + 1)^{s+1}$.

Remark 13.1. Direct calculations show that the following inclusion is valid:

$$
\mathcal{M}_{a,b,s} \subset \left\{ \lambda \in \mathbb{C} \left| \left| \text{Im} \lambda \right| \leq 2(s + 1)b^{1/(s+1)} \left( \frac{a}{s} \right)^{s/(s+1)} (\text{Re} \lambda)^{s/(s+1)} \right\}, \left. \lambda \right| \leq \lambda_{1},
$$

where $\lambda_{0} = -\frac{b}{a}(2a + 1)^{s+1}$ is the vertex of sector

$$
\left\{ \lambda \in \mathbb{C} \left| \left| \text{Im} \lambda \right| \leq \frac{2a}{2a + 1} \text{Re} \lambda + 2b(2a + 1)^{s} \right\},
$$

and $\lambda_{1} = \frac{bs}{a}(2a + 1)^{s+1}$.

Theorem 13 is useful for preliminary localisation of a spectrum of various operators.

For instance, if the potential $q \in H^{-1}_{unif}(\mathbb{R})$ is a complex-valued regular Borel measure such that:

$$
q = Q', \quad Q \in \text{BV}_{\text{loc}}(\mathbb{R}) : \quad |q(I)| = \left| \int_{I} dQ \right| \leq K_{0}, \quad K_{0} > 0,
$$

for any interval $I \subset \mathbb{R}$ of a unit length, then forms satisfy the estimates (30) with $a = b = 4K_{0}$, $s = 1$ (24):

$$
|t_{q}[u]| = \left| \int_{I} |u|^{2}dQ \right| \leq 4K_{0} \epsilon \|u\|_{L_{2}(\mathbb{R})}^{2} + 4K_{0} \epsilon^{-1} \|u\|_{L_{2}(\mathbb{R})}^{2} \quad \forall \epsilon \in (0,1], \quad u \in H^{1}(\mathbb{R}).
$$

Then due to Theorem 13 the spectra $\text{spec}(S(q))$ of operators $S(q)$ belong to a quadratic parabola:

$$
\text{spec}(S(q)) \subset \left\{ \lambda \in \mathbb{C} \left| \left| \text{Im} \lambda \right| \leq 16K_{0} \left( \text{Re} \lambda + (8K_{0} + 1)^{2} \right)^{1/2} \right\}, \quad (38)
$$

compare with (24) Proposition 2.3.

Applying Theorem 13 and estimates (11) we obtain a description of the numerical ranges of preminimal operators $S_{00}(q)$ and $S_{00}^{+}(q)$.

Proposition 14. Operators $S_{00}(q)$ and $S_{00}^{+}(q)$ are sectorial: for arbitrary $\epsilon > 0$ numerical ranges $\Theta(S_{00}(q))$ and $\Theta(S_{00}^{+}(q))$ are located within the sector:

$$
S_{K,\epsilon} := \left\{ \lambda \in \mathbb{C} \left| \left| \text{Im} \lambda \right| \leq 2K \epsilon \text{Re} \lambda + 8K \epsilon^{-3} \right\}, \quad 0 < \epsilon \leq (2K + 1)^{-1}.
$$

Furthermore

$$
\Theta(S_{00}(q)) \subset \mathcal{M}_{K}, \quad \Theta(S_{00}^{+}(q)) \subset \mathcal{M}_{K},
$$

where

$$
\mathcal{M}_{K} := \left\{ \lambda \in \mathbb{C} \left| \left| \text{Im} \lambda \right| \leq \frac{2K}{2K + 1} \text{Re} \lambda + 8K(2K + 1)^{3} \right\}, \lambda_{0} \leq \text{Re} \lambda \leq \lambda_{1},
$$

$$
\mathcal{M}_{K} := \left\{ \lambda \in \mathbb{C} \left| \left| \text{Im} \lambda \right| \leq \frac{32}{125/4}K^{3/4} \text{Re} \lambda^{3/4} \right\}, \lambda_{1} < \text{Re} \lambda,
$$

with $\lambda_{0} := -4(2K + 1)^{4}$ and $\lambda_{1} := 12(2K + 1)^{4}$.

Estimates (8) result from Proposition 14 and Remark 13.1.

Now let $\text{Im} q = 0$. We estimate the lower bound of the operator $S(q)$. From (11) for $K \epsilon \leq 1/2$ we get:

$$
(S(q)u, u)_{L_{2}(\mathbb{R})} = \|u^{'}\|_{L_{2}(\mathbb{R})}^{2} + t_{Q}(\epsilon)u^{2} \geq \|u^{'}\|_{L_{2}(\mathbb{R})}^{2} - K\epsilon\|u^{'}\|_{L_{2}(\mathbb{R})}^{2} - 4K\epsilon^{-3}\|u\|_{L_{2}(\mathbb{R})}^{2} = (1 - K\epsilon)\|u^{'}\|_{L_{2}(\mathbb{R})}^{2} - 4K\epsilon^{-3}\|u\|_{L_{2}(\mathbb{R})}^{2} \geq -4K\epsilon^{-3}\|u\|_{L_{2}(\mathbb{R})}^{2}.
$$

The estimates (39) with $\epsilon := \min\{1, (2K)^{-1}\}$ give us the required result:

$$
(S(q)u, u)_{L_{2}(\mathbb{R})} \geq \begin{cases} -4K\|u\|_{L_{2}(\mathbb{R})}^{2}, & \text{if } K < 1/2, \\ -32K^{4}\|u\|_{L_{2}(\mathbb{R})}^{2}, & \text{if } K \geq 1/2. \end{cases}
$$

Thus Theorem 13 is proved completely. \[\square\]
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Institute of Mathematics, National Academy of Science of Ukraine, 3 Tereshchenkivs’ka Str., 01601 Kyiv-4, Ukraine
E-mail address: mikhailets@imath.kiev.ua

Institute of Mathematics, National Academy of Science of Ukraine, 3 Tereshchenkivs’ka Str., 01601 Kyiv-4, Ukraine
E-mail address: molyboga@imath.kiev.ua