TOPOLOGICAL AND GEOMETRIC ASPECTS OF ALMOST KÄHLER MANIFOLDS VIA HARMONIC THEORY

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Abstract. This paper extends the Kähler identities to the non-integrable setting and deduces several geometric and topological consequences. Among these are identities of various Laplacians, generalized Hodge and Serre dualities, a generalized hard Lefschetz duality, and a Lefschetz decomposition on certain harmonic forms of compact almost Kähler manifolds. We also prove a generalized Hodge Index Theorem for almost Kähler 4-manifolds. In particular, these provide topological bounds on the dimensions of these spaces of harmonic forms, as well as several new obstructions to the existence of a symplectic form compatible with a given almost complex structure.

1. Introduction

Kähler manifolds play a central role at the intersection of complex, symplectic and Riemannian geometry. Their striking set of properties arise most primitive from a set of purely local commutation relations, known as the Kähler identities. In the compact setting, the theory of elliptic operators allows one to transfer these local statements into a set of surprising cohomological properties, called the Kähler package.

In this paper we establish a set of almost Kähler identities and a corresponding almost Kähler package on spaces of harmonic forms. Topological and geometric implications are deduced in the compact case, since such harmonic forms include into the de Rham cohomology. The package includes relations among various Laplacians, generalized Hodge and Serre dualities, a generalized Hodge Theorem and hard Lefschetz duality, as well as a generalized Hodge Index Theorem for almost Kähler 4-manifolds. Additional results are described in what follows.

Recall that the exterior differential of an almost complex manifold has four components, \( d = \bar{\mu} + \bar{\partial} + \partial + \mu \), where \( \bar{\mu} \) and \( \mu \) arise from the Nijenhuis tensor and vanish if and only if the structure is integrable. In the presence of an almost Hermitian metric, there are formal adjoints for each component, and associated Laplacians. Also, the fundamental \((1,1)\)-form defines a Lefschetz operator \( L \) and its adjoint \( \Lambda \).

The family of almost Kähler identities concerning the operators \( \bar{\partial}, \partial, L \) and their adjoints have been previously noted in the literature (see for instance [Don90], [Kot97]). The first fundamental observation of this paper is a set of relations for the operators \( \bar{\mu} \) and \( \mu \), as well as a set of mixed-type relations involving all components.
of $d$. We remark that similar looking (but very different) relations were found by Verbitsky in [Ver11] for 6-dimensional strictly nearly Kähler manifolds, though these do not apply to the present setting. The newly established identities imply, among other Laplacian relations, that

$$\Delta_{\bar{\partial}} + \Delta_{\partial} = \Delta_{\bar{\mu}} + \Delta_{\mu}.$$ 

Note that in the Kähler case, the $\bar{\mu}$- and $\mu$-Laplacians vanish and one recovers the well-known identity $\Delta_{\bar{\partial}} = \Delta_{\partial}$.

In the compact case, the theory of harmonic forms allows one to translate the above local results into geometric and topological statements for almost Kähler manifolds, as we next explain.

Let $\delta$ denote one of the components $\bar{\mu}, \bar{\partial}, \partial, \mu,$ or $d$. Define the space of $\delta$-harmonic forms in bidegree $(p, q)$ by letting

$$\mathcal{H}^{p,q}_\delta := \text{Ker} (\Delta_{\delta} ) \cap \mathcal{A}^{p,q},$$

where $\mathcal{A}^{p,q}$ denotes the space of $(p, q)$-forms. Note that for compact almost Hermitian manifolds, the spaces $\mathcal{H}^{p,q}_{\bar{\partial}}$ and $\mathcal{H}^{p,q}_\partial$ are finite-dimensional by elliptic operator theory (see [Hir54]) but in general, the spaces $\mathcal{H}^{p,q}_\mu$ and $\mathcal{H}^{p,q}_{\bar{\mu}}$ are infinite-dimensional whenever they are non-zero. The finite-dimensional spaces given by $\mathcal{H}^{p,q}_{\bar{\partial}} \cap \mathcal{H}^{p,q}_\mu$, and their conjugates, are shown to have many interesting properties. First, we have various duality results:

**Theorem 4.1.** For any compact almost Kähler manifold of dimension $2m$, and any $0 \leq k \leq 2m$, there is an orthogonal direct sum decomposition

$$\text{Ker} (\Delta_{\bar{\partial}} + \Delta_{\partial} ) \cap \mathcal{A}^k = \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{\bar{\partial}} \cap \mathcal{H}^{p,q}_\mu,$$

and for all $0 \leq p, q \leq m$ the following dualities hold:

1. (Complex conjugation). We have equalities
   $$\mathcal{H}^{p,q}_{\bar{\partial}} \cap \mathcal{H}^{p,q}_\mu = \mathcal{H}^{q,p}_{\bar{\partial}} \cap \mathcal{H}^{q,p}_\mu.$$

2. (Hodge duality). The Hodge $\star$-operator induces isomorphisms
   $$\star : \mathcal{H}^{p,q}_{\bar{\partial}} \cap \mathcal{H}^{p,q}_\mu \to \mathcal{H}^{m-q,m-p}_{\bar{\partial}} \cap \mathcal{H}^{m-q,m-p}_\mu.$$

3. (Serre duality). There are isomorphisms
   $$\mathcal{H}^{p,q}_{\bar{\partial}} \cap \mathcal{H}^{p,q}_\mu \cong \mathcal{H}^{m-p,m-q}_{\bar{\partial}} \cap \mathcal{H}^{m-p,m-q}_\mu.$$

For any compact almost Hermitian manifold we define the finite-dimensional numbers

$$h^{p,q} := \dim \left( \mathcal{H}^{p,q}_{\bar{\partial}} \cap \mathcal{H}^{p,q}_\mu \right).$$

Note that in the integrable case, these are the Dolbeault numbers of the manifold. Theorem 4.1 generalizes the symmetries of the Hodge diamond to the almost Kähler case, giving

$$h^{p,q} = h^{q,p} = h^{m-q,m-p} = h^{m-p,m-q}.$$ 

An open question of Kodaira and Spencer for compact almost complex manifolds, known as Hirzebruch’s Problem 20 [Hir54], is whether the dimensions of the spaces $\mathcal{H}^{p,q}_\delta$ are metric-independent numbers. An updated account of this problem appears in [Kot13], where Kotschick surveys that there seems to have been no progress at all on this problem, apart from an attempt to develop harmonic theory for almost Kähler manifolds by Donaldson [Don90], and the Hodge theory of strictly nearly...
Kähler manifolds of Verbitsky [Ver11]. In [CW18] we address this problem in general, via the introduction of Dolbeault cohomology for arbitrary almost complex manifolds. One can similarly ask whether the numbers $h^{p,q}$ defined above depend on the metric. The following result shows, among other statements, that for compact almost Kähler manifolds, these a priori metric-dependent numbers are bounded by the topology. This is surprising in light of the fact that for any symplectic structure there is an infinite dimensional space of metrics yielding an almost Kähler structure.

Denote by $b^k := \dim H^k_{\text{dR}}(M, \mathbb{C})$ the Betti numbers of a manifold $M$.

**Theorem 4.3.** For any compact almost Kähler manifold of dimension $2m$, the following is satisfied:

1. For any $(p, q)$, we have
   $\mathcal{H}^p_{\bar{\partial}} \cap \mathcal{H}^q_{\partial} = \mathcal{H}^{p+q}_{\text{d}} \subseteq \mathcal{H}^{p+q}_{\text{d}}$
   and for all $k \geq 0$ we have inequalities
   $$\sum_{p+q=k} h^{p,q} \leq b^k.$$

2. In odd degrees, we have
   $$\sum_{p+q=2k+1} h^{p,q} = 2 \sum_{0 \leq p \leq k} h^{p,2k+1-p} \leq b^{2k+1}.$$

3. In even degrees, we have
   $$\sum_{p+q=2k} h^{p,q} = 2 \sum_{0 \leq p < k} h^{p,2k-p} + h^{k,k} \leq b^{2k},$$
   with $h^{k,k} \geq 1$ for all $k \leq m$.

Theorem 4.3 provides new obstructions for the existence of symplectic structures compatible with a given almost complex structure, complementing the known topological-type obstructions arising from the symplectic form and the associated almost complex structure, as well as the results of Taubes via the theory of Seiberg-Witten invariants [Tau94], [Tau95] (see also Gompf’s review on symplectic obstruction theory [Gom01]). Note that in particular, given a compact symplectic manifold with compatible almost complex structure $J$, as well as an integrable almost complex structure $J'$, we obtain inequalities

$$\sum_{p+q=k} h^{p,q}_{J} \leq b^k \leq \sum_{p+q=k} h^{p,q}_{J'}$$

bounding the Betti numbers of the manifold above and below.

For all $p \geq 0$, denote by

$$\Omega^p_{\bar{\partial}} := \text{Ker} (\bar{\partial}) \cap A^{p,0}$$

the space of holomorphic $p$-forms. The consideration of these spaces together with Theorem 4.3 allows for metric-independent statements. For instance, for any almost complex structure on any compact simply connected almost complex manifold, a necessary condition for it to admit a compatible symplectic structure is that there are no holomorphic 1-forms. This follows after showing that, for compact almost Kähler manifolds, the space of holomorphic 1-forms coincides with the space of $(1,0)$-forms that are $d$-harmonic.
Chen gave the first example of how the Hodge numbers of a compact Kähler manifold affect its fundamental group [Che71, Che72]. Chen’s result is extended here to the non-integrable case, showing that if $M$ is a compact connected almost Kähler manifold satisfying $\dim \Omega^1_\partial > \dim \Omega^2_\partial + 1$, then $\pi_1(M)$ contains a free subgroup of rank $\geq 2$. In particular, $\pi_1(M)$ is not solvable. This is an additional obstruction to finding a symplectic structure compatible with a given almost complex structure.

The almost Kähler identities also yield a generalization of hard Lefschetz duality. It is well known that symplectic manifolds generally do not satisfy hard Lefschetz duality in cohomology. In fact, Mathieu points out that a necessary condition is that all odd Betti numbers must be even [Mat95]. Nevertheless, a generalization of hard Lefschetz duality for compact almost Kähler manifolds occurs on the space of $\bar{\partial}$-µ-harmonic forms. We prove:

**Theorem 5.1** (Generalized Hard Lefschetz Duality). For any compact almost Kähler manifold of dimension $2m$, the operators $\{L, \Lambda, H = [L, \Lambda]\}$ define a finite dimensional representation of $\mathfrak{sl}(2, \mathbb{C})$ on

$$H^p_\partial \cap H^q_\mu = \bigoplus_{p,q \geq 0} H^p_\partial \cap H^q_\mu.$$

Moreover, for every $0 \leq p \leq m$ and all $p \leq k \leq m$ the maps

$$L^{m-k} : H^{p,k-p}_\partial \cap H^{p,k-p}_\mu \xrightarrow{\cong} H^{p+m-k,m-p}_\partial \cap H^{p+m-k,m-p}_\mu,$$

are isomorphisms.

Generalizing the Kähler case, the above hard Lefschetz duality gives an orthogonal direct sum decomposition

$$H^p_\partial \cap H^q_\mu = \bigoplus_{j \geq 0} L^j \left( H^{p-j,q-j}_\partial \cap H^{p-j,q-j}_\mu \right)_{\text{prim}},$$

where

$$(H^r_{\partial} \cap H^s_{\mu})_{\text{prim}} := (H^r_{\partial} \cap H^s_{\mu}) \cap \text{Ker} \Lambda.$$

These results, combined with the Hodge-Riemann pairing on middle degree forms, give:

**Theorem 5.5** (Generalized Hodge Index Theorem). For any compact 4-dimensional almost Kähler manifold $M$, we have

$$h^{1,1} - 1 \leq b_2^-$$

$$2h^{2,0} + 1 \leq b_2^+$$

where the intersection pairing on $H^2(M; \mathbb{C})$ has index $(b_2^+, b_2^-)$. Also, if the almost complex structure is not integrable, then $h^{2,0} = 0$.

Note the numbers $h^{1,1}$ and $h^{2,0}$ depend a priori on the almost Hermitian geometry of the manifold, but are bounded above by the purely topological invariants $b_2^\pm$. The Generalized Hodge Index Theorem gives several direct implications. First, in favorable situations (such as for instance, the Kodaira-Thurston manifold studied in Section 6.1), these bounds actually allow one to determine these numbers from the index of the intersection pairing. Second, note that if a compact 4-manifold satisfies $b_2^+ \leq 2$, then a necessary condition for an almost complex structure to admit a compatible symplectic form is that $h^{2,0} = 0$. Examples include smooth manifolds...
homeomorphic to $\mathbb{CP}^2 \# n\mathbb{CP}^2$, among which there are known to be infinitely many exotic examples for various values of $n$.

Lastly, Theorem 5.5 also implies that for all closed symplectic manifolds with positive definite intersection pairing, the symplectic form “stands alone” in the space of $\bar{\partial}$-harmonic $(1, 1)$-forms, for any compatible metric. Moreover, in this case, we also show that all the numbers $h^{p,q}$, as well as the dimensions of $\mathcal{H}^{p,q}_{\bar{\partial}}$, are all metric-independent for almost Kähler metrics compatible with a given almost complex structure. This yields an affirmative answer to Hirzebruch’s Problem 20 [Hir54] for compact almost Kähler metrics on 4-manifolds with positive definite intersection pairing. This includes $\mathbb{CP}^2$ as well as any exotic $(2n + 1)\mathbb{CP}^2$, though to our knowledge, it is not known yet if any such exist.

The consequences of the almost Kähler package detailed here are perhaps only a sample of its potential. There are several important open problems concerning almost Kähler manifolds, including Donaldson’s tameness question in dimension 4 [Don06], as well as the Goldberg Conjecture [Gol69], which asserts that a compact almost Kähler Einstein manifold is Kähler. One may hope the results here provide useful symplectic invariants for such problems, including interesting properties in the presence of curvature restrictions.

Finally, in the physics literature, the algebraic structure present on the differential forms of a Kähler manifold is often referred to as the $N = 2$ supersymmetry algebra; see [Zum79, AGFS1, HKLR87]. It has been noted that when the integrability condition is dropped, the supersymmetry is partially broken [FGR98]. The almost Kähler identities obtained here show that additional symmetries are indeed present, albeit in a more subtle and interesting way depending on the failure of integrability. We hope the results here may open new possibilities for physicists’ construction and study of supersymmetric theories.

This paper is organized as follows. In Section 2 we collect preliminaries on the differential forms on almost complex and Hermitian manifolds. In Section 3 we prove the almost Kähler identities. In Section 4 we prove Theorems 4.1 and 4.3 and detail several topological and geometric consequences. Section 5 is devoted to hard Lefshetz (Theorem 5.1) and the Hodge index for 4-manifolds (Theorem, 5.5). We also obtain various corollaries of these results. Lastly, in Section 6 we exhibit applications of the theory which show that the theoretical results and new techniques allow us to perform new calculations and reach new theoretical conclusions, even in the case of nilmanifolds. In particular, the kernel of the self-adjoint elliptic operator $\Delta_{\bar{\partial}} + \Delta_{\mu}$ is computed using the topological bounds, and the failure of the symmetry $\Delta_{\bar{\partial}} + \Delta_{\mu} = \Delta_{\bar{\partial}} + \Delta_{\bar{\mu}}$ is shown to detect almost complex structures that do not admit almost Kähler structures.

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2. Differential forms on almost complex manifolds

We collect some main definitions and results on the differential forms for almost complex and almost Hermitian manifolds.

Let $(M, J)$ be an almost complex manifold and let

$$\mathcal{A}^k := \mathcal{A}_{\text{dR}}^k(M) \otimes \mathbb{C} = \bigoplus_{p+q=k} \mathcal{A}^{p,q}$$
be the bigraded algebra of complex valued differential forms on $M$. The exterior differential decomposes as
\[ d = \bar{\mu} + \bar{\partial} + \partial + \mu, \]
with the components $\bar{\mu}$ and $\bar{\partial}$ being complex conjugate to $\mu$ and $\partial$, respectively. Note that each component of $d$ is a derivation, with bidegrees given by
\[ |\bar{\mu}| = (-1, 2), \quad |\bar{\partial}| = (0, 1), \quad |\partial| = (1, 0), \quad \text{and} \quad |\mu| = (2, -1). \]
In particular, $\bar{\mu}$ and $\mu$ are linear over functions.

One can show that $\bar{\mu} + \mu$ is equal, up to a scalar, to the dual of the Nijenhuis tensor. In fact,
\[ \bar{\mu} + \mu = -\frac{1}{4} (N_J \otimes id_C)^*, \]
where the right hand side has been extended over all forms as a derivation. Since both sides are derivations, it suffices to check this on 1-forms, which can be done using Cartan’s formula relating the exterior differential and Lie bracket. In particular, $J$ is integrable if and only if $\bar{\mu} \equiv 0$.

Expanding the equation $d^2 = 0$ we obtain the following set of equations:
\[
\begin{align*}
\mu^2 &= 0 \\
\mu \partial + \partial \mu &= 0 \\
\mu \bar{\partial} + \bar{\partial} \mu + \delta^2 &= 0 \\
(\Delta) \mu \bar{\mu} + \bar{\partial} \partial + \partial \bar{\partial} + \mu \mu &= 0 \\
\bar{\mu} \partial + \partial \bar{\mu} + \bar{\partial}^2 &= 0 \\
\bar{\mu} \bar{\partial} + \bar{\partial} \bar{\mu} &= 0 \\
\bar{\mu}^2 &= 0
\end{align*}
\]

For any almost Hermitian manifold $(M, J, \langle , \rangle)$ of dimension $2m$ there is an associated Hodge-star operator
\[ \star : A^{p,q} \to A^{m-q,m-p} \]
defined by $\alpha \wedge \star \beta = \langle \alpha, \beta \rangle \text{vol}$
where vol is the volume form determined by the metric.

There is an associated fundamental $(1,1)$-form defined by
\[ \omega(X,Y) := \langle JX,Y \rangle \]
and Lefschetz operator
\[ L : A^{p,q} \to A^{p+1,q+1}, \]
defined by $L(\eta) := \omega \wedge \eta$.
It has adjoint $\Lambda = L^* = \star^{-1} L^\star$. It is well known that $\{L, \Lambda, H = [L, \Lambda]\}$ defines a representation of $\mathfrak{sl}(2, \mathbb{C})$, with Lefschetz decomposition on complex $k$-forms
\[ \mathcal{A}^k = \bigoplus_{i \geq 0} L^i P^{k-2i}, \]
where $P^i = \text{Ker}(\Lambda) \cap \mathcal{A}^i$. The map
\[ L^{m-k} : P^k \to \mathcal{A}^{2m-k} \]
is injective for $k \leq m$ (see for instance [Wei58]).

The operators $\delta = \bar{\mu}, \bar{\partial}, \partial, \mu$, and $d$ have $L_2$-adjoint operators $\delta^\star$ when $M$ is closed, and one may check that
\[ \bar{\mu}^\star = -\star \mu \star \quad \text{and} \quad \bar{\partial}^\star = -\star \partial \star. \]
The latter equation is well known. The first equation is checked similarly, by using the definition of $\star$ and the fact that $\mu$ is a derivation.

Define the $\delta$-Laplacian by letting

$$\Delta_\delta := \delta \delta^* + \delta^* \delta.$$ 

It satisfies

$$\star \Delta_\delta = \Delta_\delta \star.$$ 

For all $p, q$, will denote by

$$H^{p,q}_\delta := \text{Ker} (\Delta_\delta) \cap A^{p,q} = \text{Ker} (\delta) \cap \text{Ker} (\delta^*) \cap A^{p,q}$$

the space of $\delta$-harmonic forms in bidegree $(p, q)$.

Note that $H^{p,q}_\delta$ is infinite dimensional whenever it is non-zero, since $\mu$ is linear over functions, but $H^{p,q}_\bar{\partial}$ is finite dimensional on a compact manifold, by elliptic theory. We will refer to

$$H^{p,q}_\delta \cap H^{p,q}_\mu$$

as the space of $\bar{\partial}$-$\mu$-harmonic $(p, q)$-forms, which will later be identified as the kernel of $\Delta_{\bar{\partial}} + \Delta_{\mu}$ in bidegree $(p, q)$.

### 3. Almost Kähler identities

An almost Kähler manifold is by definition an almost Hermitian manifold such that the associated $(1,1)$-form is closed. Equivalently, an almost Kähler manifold is a symplectic manifold with a compatible metric, which subsequently defines an orthogonal almost complex structure.

On an almost Kähler manifold, the so-called Kähler identities, involving the differential operator $\bar{\partial}$, the Lefschetz operator $L$, and their complex conjugates and adjoints, hold just as in the case of Kähler manifolds. These are proven in Weil’s book [Wei58], where indeed, the integrability condition is not used. More recent references often prove the Kähler identities by reducing the proof to a computation in $\mathbb{C}^n$, thus restricting to the integrable setting (see for instance [GH94], [Voi07], see also Remark 3.1.14 of [Huy05]).

In this section, we retake Weil’s approach and prove analogous identities involving the operators $\mu$ and $\bar{\mu}$. From these, we obtain several commutation relations involving the four components $\bar{\mu}, \bar{\partial}, \mu$ and $\partial$ of the differential, as well as various relations between Laplacians.

In what follows we define the graded commutator of operators $A$ and $B$ by

$$[A, B] = AB - (-1)^{\text{deg}(A)\text{deg}(B)} BA$$

where $\text{deg}(A)$ denotes the total degree of $A$.

**Proposition 3.1.** For any almost Kähler manifold the following identities hold:

1. $[\Lambda, \bar{\mu}] = [L, \mu] = 0$ and $[\Lambda, \bar{\mu}^*] = [L, \mu^*] = 0$.
2. $[\Lambda, \bar{\partial}] = [L, \partial] = 0$ and $[\Lambda, \bar{\partial}^*] = [L, \partial^*] = 0$.
3. $[L, \bar{\mu}^*] = i\mu$, $[L, \mu^*] = -i\bar{\mu}$ and $[\bar{\Lambda}, \bar{\mu}] = \bar{\mu}^*$, $[\Lambda, \mu] = -i\bar{\mu}^*$.
4. $[L, \bar{\partial}^*] = -i\partial$, $[L, \partial^*] = i\bar{\partial}$ and $[\bar{\Lambda}, \bar{\partial}] = -i\partial^*$, $[\Lambda, \partial] = i\bar{\partial}^*$.
Proof. Since \( \omega \in \mathcal{A}^{1,1} \) is \( d \)-closed we have \( \bar{\mu} \omega = 0 \), and since \( \bar{\mu} \) is a derivation, \([\bar{\mu}, L] = 0\). The remaining cases in the first statement follow by taking complex conjugates and adjoints, since \( \omega \) is real. The proof for the second statements is identical and well known.

For the third and fourth statements, using the primitive decomposition of the exterior algebra of the manifold, and the fact that \( d\omega = 0 \), it is well known that

\[
[A, d] = \ast \bar{\mu}^{-1} d \ast 
\]

where \( \ast \) is the operator that acts on \((p,q)\)-forms by multiplication by \( i^{p-q} \) (c.f. [Huy05 Proposition 3.1.12, p.121-122]). In bidegree \((p,q)\) we have \( \bar{\mu}^{-1} = (-1)^{p-q} \bar{\mu}_{p,q} \), so conjugating an operator of bidegree \((r,s)\) by \( \ast \) acts by multiplication by \((-i)^{r-s}\).

Then using \( \bar{\delta} = -\ast \delta \ast \) for \( \delta = \bar{\mu}, \bar{\partial}, \partial, \) and \( \mu \), it follows that

\[
\ast \bar{\mu}^{-1} \mu \ast = i\mu^*,
\ast \bar{\partial}^{-1} \partial \ast = -i\partial^*,
\ast \bar{\partial}^{-1} \partial \ast = i\partial^*,
\ast \bar{\partial}^{-1} \partial \ast = -i\bar{\mu}^*.
\]

Then \( d = \bar{\mu} + \bar{\partial} + \partial + \mu \) implies the third and fourth statements involving \( \Lambda \). The statements involving \( L \) follow by taking adjoints. \( \square \)

We deduce the following relations concerning the various component of \( d \) and their adjoints. It is helpful to use the graded Jacobi identity:

\[
[A, [B, C]] = [[A, B], C] + (-1)^{\text{deg}(A)\text{deg}(B)}[B, [A, C]].
\]

**Proposition 3.2.** For any almost Kähler manifold the following identities hold:

1. \([\bar{\mu}, \mu^*] = [\mu, \bar{\mu}^*] = 0\).
2. \([\bar{\partial}, \partial^*] = [\bar{\partial}, \mu^*] \) and \([\mu, \bar{\partial}^*] = [\partial, \bar{\mu}^*] \).
3. \([\partial, \bar{\partial}^*] = [\mu^*, \bar{\partial}] + [\mu, \partial^*] \) and \([\bar{\partial}, \partial^*] = [\mu^*, \partial] + [\bar{\mu}, \partial^*] \).

**Proof.** For the first statement

\[
[\bar{\mu}, \mu^*] = i[\bar{\mu}, [\bar{\mu}, \Lambda]] = 0,
\]

and the second follows by conjugation or adjoint.

Next we have

\[
[\bar{\mu}, \partial^*] = i[\bar{\mu}, [\Lambda, \partial]] = i[[\bar{\mu}, \Lambda], \partial] + i[\Lambda, [\bar{\mu}, \partial]]
\]

by the graded Jacobi identity. Since \([\bar{\mu}, \partial] = 0\), and \([\bar{\mu}, \Lambda] = -i\mu^*\), this becomes

\[
[\bar{\mu}, \partial^*] = [\mu^*, \partial] = [\bar{\partial}, \mu^*] .
\]

Then \([\mu, \partial^*] = [\partial, \mu^*] \) by conjugation or adjoint.

The next two claims are also equivalent by conjugation and adjoint. We’ll prove the first one. First, using \([\Lambda, \partial] = i\partial^*\), we compute

\[
[\partial, \bar{\partial}^*] = -i[\partial, [\Lambda, \partial]] = i[\partial^2, \Lambda] = i[\Lambda, [\partial, \partial]]
\]

where in the last step we used \([\mu, \partial] + \partial^2 = 0\). By the graded Jacobi identity,

\[
[\partial, \bar{\partial}^*] = i[[\partial, \Lambda], \partial] + i[\mu, [\Lambda, \partial]] .
\]

Now using \([\Lambda, \mu] = -i\bar{\mu}^*\) and \([\Lambda, \bar{\partial}] = -i\partial^*\) the result follows. \( \square \)
Note that in the first statement $[\bar{\mu}, \mu^*] = 0$ is a zeroth-order (metric-dependent) condition which obstructs an almost Hermitian manifold from being symplectic, whereas $d\omega = 0$ is a first order (metric-independent) condition. We next deduce several relations concerning various Laplacians.

**Proposition 3.3.** For any almost Kähler manifold the following identities hold:

1. $\Delta_{\bar{\mu} + \mu} = \Delta_{\bar{\mu}} + \Delta_{\mu}$.
2. $\Delta_{\bar{\theta}} + \Delta_{\mu} = \Delta_{\bar{\theta}} + \Delta_{\mu}$.
3. $\Delta_d = 2(\Delta_{\bar{\theta}} + \Delta_{\mu} + [\bar{\mu}, \partial^*] + [\mu, \bar{\partial}^*] + [\partial, \bar{\partial}^*] + [\bar{\partial}, \partial^*])$.

**Proof.** The first claim follows by direct calculation using $[\bar{\mu}, \mu^*] = [\mu, \bar{\mu}^*] = 0$. For the second claim, by Proposition 3.1

$$\Delta_{\bar{\mu}} = \bar{\mu} \bar{\mu}^* + \bar{\mu}^* \bar{\mu} = i (\bar{\mu}[\Lambda, \mu] + [\Lambda, \mu] \bar{\mu}) = i (\bar{\mu} \Lambda \mu - \bar{\mu} \mu \Lambda + \Lambda \bar{\mu} - \mu \Lambda \bar{\mu})$$

and similarly

$$\Delta_{\mu} = \mu \mu^* + \mu^* \mu = -i (\mu[\Lambda, \bar{\mu}] + [\Lambda, \bar{\mu}] \mu) = -i (\mu \Lambda \bar{\mu} - \mu \bar{\mu} \Lambda + \Lambda \mu \bar{\mu} - \bar{\mu} \Lambda \mu)$$

so that

$$\Delta_{\bar{\mu}} - \Delta_{\mu} = i (\Lambda (\mu \bar{\mu} + \bar{\mu} \mu) - (\mu \bar{\mu} + \bar{\mu} \mu) \Lambda) = -i (\Lambda (\partial \bar{\partial} + \bar{\partial} \partial) - (\partial \bar{\partial} + \bar{\partial} \partial) \Lambda) = \Delta_{\bar{\theta}} - \Delta_{\theta}.$$ 

The last equality follows from a similar calculation as is done above for $\Delta_{\bar{\mu}} - \Delta_{\mu}$.

Finally, expanding $\Delta_d = [d, d^*]$ and using $d = \bar{\mu} + \bar{\theta} + \partial + \mu$, we have

$$\Delta_d = \Delta_{\bar{\mu}} + \Delta_{\bar{\theta}} + \Delta_{\theta} + \Delta_{\mu}$$

$$+ [\bar{\mu}, \bar{\partial}^*] + [\bar{\mu}, \partial^*] + [\mu, \bar{\mu}^*]$$

$$+ [\bar{\theta}, \bar{\mu}^*] + [\bar{\theta}, \partial^*] + [\partial, \mu^*]$$

$$+ [\partial, \bar{\mu}^*] + [\partial, \bar{\partial}^*] + [\bar{\partial}, \mu^*]$$

$$+ [\mu, \bar{\mu}^*] + [\mu, \bar{\partial}^*] + [\partial, \mu^*]$$

so the final statement follows using Proposition 3.2 and the previous part. \qed

We have one more set of useful relations, which are related to hard Lefschetz duality.

**Corollary 3.4.** For any almost Kähler manifold the following identities hold:

1. $[L, \Delta_{\bar{\theta}}] = [L, \Delta_{\mu}] = -[L, \Delta_{\bar{\theta}}] = -[L, \Delta_{\mu}] = -i[\bar{\partial}, \partial] = i[\bar{\mu}, \mu]$.
2. $[\Lambda, \Delta_{\bar{\theta}}] = [\Lambda, \Delta_{\mu}] = -[\Lambda, \Delta_{\bar{\theta}}] = -[\Lambda, \Delta_{\mu}] = -i[\partial^*, \partial^*] = i[\bar{\mu}^*, \mu^*]$.

**Proof.** Using Propostion 3.1 we calculate

$$[L, \Delta_{\bar{\theta}}] = [\bar{\partial}, [L, \bar{\partial}^*]] = -i[\bar{\partial}, \partial] = i[\bar{\mu}, \mu] = [\bar{\mu}, [L, \bar{\partial}^*]] = [L, \Delta_{\bar{\mu}}]$$

and all remaining relations follow from taking conjugates or adjoints. \qed
In conclusion, for any almost Kähler manifold, there is a $\mathbb{Z}_2$-graded Lie algebra of operators acting on the $(p, q)$-forms, generated by eight odd operators 
\[ \bar{\partial}, \partial, \bar{\mu}, \mu, \bar{\partial}^*, \partial^*, \bar{\mu}^*, \mu^* \]
and even degree operators $L, \Lambda, H$, from which all relations can be deduced from those given above. In the integrable case, this reduces to the so-called $N = 2$ supersymmetry algebra of a Kähler manifold (see for instance [Zum79], [AGFS81], [HKLR87]), also referred to as the $N = (2, 2)$ supersymmetry algebra in [FGR98].

4. Topological and geometric consequences in the compact case

From the almost Kähler identities and the symmetries of the Laplacians, we deduce in this section several results giving combined geometric/topological restrictions in the compact case. We first have various duality results.

**Theorem 4.1.** For any compact almost Kähler manifold of dimension $2m$, and any $0 \leq k \leq 2m$, there is an orthogonal direct sum decomposition

\[ \text{Ker} (\Delta_{\bar{\partial}} + \Delta_{\mu}) \cap \mathcal{A}^k = \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{\bar{\partial}} \cap \mathcal{H}^{p,q}_{\mu}, \]

and for all $0 \leq p, q \leq m$ the following dualities hold:

1. (Complex conjugation). We have equalities
   \[ \mathcal{H}^{p,q}_{\bar{\partial}} \cap \mathcal{H}^{p,q}_{\mu} = \mathcal{H}^{q,p}_{\bar{\partial}} \cap \mathcal{H}^{q,p}_{\mu}. \]

2. (Hodge duality). The Hodge $\star$-operator induces isomorphisms
   \[ \star : \mathcal{H}^{p,q}_{\bar{\partial}} \cap \mathcal{H}^{p,q}_{\mu} \to \mathcal{H}^{m-q,m-p}_{\bar{\partial}} \cap \mathcal{H}^{m-q,m-p}_{\mu}. \]

3. (Serre duality). There are isomorphisms
   \[ \mathcal{H}^{p,q}_{\bar{\partial}} \cap \mathcal{H}^{p,q}_{\mu} \cong \mathcal{H}^{m-p,m-q}_{\bar{\partial}} \cap \mathcal{H}^{m-p,m-q}_{\mu}. \]

**Proof.** For any compact almost Hermitian manifold we have

\[ \text{Ker} (\Delta_{\bar{\partial}} + \Delta_{\mu}) = \text{Ker} (\Delta_{\bar{\partial}}) \cap \text{Ker} (\Delta_{\mu}). \]

Indeed, if $\alpha \in \text{Ker} (\Delta_{\bar{\partial}} + \Delta_{\mu})$, then

\[ 0 = \langle \Delta_{\bar{\partial}} \alpha, \alpha \rangle + \langle \Delta_{\mu} \alpha, \alpha \rangle = \|\bar{\partial} \alpha\|^2 + \|\partial^* \alpha\|^2 + \|\mu \alpha\|^2 + \|\mu^* \alpha\|^2, \]

so that all the norms vanish. If $\alpha$ is a sum of $(p, q)$-forms, each with fixed total degree $p + q$, then $\bar{\partial} \alpha = 0$ implies each $(p, q)$-component is $\bar{\partial}$-closed, and similarly for $\partial^*$, $\mu$, and $\mu^*$. This proves the orthogonal direct sum decomposition.

Similarly, $\text{Ker} (\Delta_{\bar{\partial}} + \Delta_{\mu}) = \text{Ker} (\Delta_{\partial}) \cap \text{Ker} (\Delta_{\mu})$, so the first duality statement follows from the identity

\[ \Delta_{\bar{\partial}} + \Delta_{\mu} = \Delta_{\bar{\partial}} + \Delta_{\mu}, \]

of Proposition 3.3 by considering conjugation.

The next claim follows from the previous and the relations $\star \Delta_{\partial} = \pm \Delta_{\partial} \star$ for $\delta = \bar{\partial}$ and $\delta = \mu$. The Serre Duality claim, which in fact holds for any almost Hermitian manifold and each of the spaces $\mathcal{H}_{\bar{\partial}}$ and $\mathcal{H}_{\mu}$, follows from these same relations and the conjugation isomorphism

\[ \mathcal{H}^{p,q}_{\bar{\partial}} \cap \mathcal{H}^{p,q}_{\mu} \to \mathcal{H}^{m-q,m-p}_{\bar{\partial}} \cap \mathcal{H}^{m-q,m-p}_{\mu}. \]

□
For any compact almost Hermitian manifold we define
\[ h^{p,q} := \dim \left( H_{\bar{\partial}}^{p,q} \cap H_{\bar{\mu}}^{p,q} \right). \]

If the almost complex structure is integrable, these are just the Dolbeault numbers of the complex manifold. In the non-integrable case, the numbers \( h^{p,q} \) are (a priori) metric-dependent, but Theorem 4.3 below shows in fact these numbers are bounded by the topology. For one of the bounds, we first need the following lemma.

**Lemma 4.2.** Let \( \omega \) be the \((1,1)\)-form of a compact almost Kähler manifold. Then \( \omega^k \in H^{k,k}_\bar{\partial} \cap H^{k,k}_{\bar{\mu}} \) for all \( k \geq 0 \).

**Proof.** Since \( \omega \) is pure of type \((1,1)\) and \( d\omega = 0 \) it follows that \( \delta \omega = 0 \) where \( \delta \) denotes any of the homogeneous components of \( d \). By bidegree reasons, we obtain \( \omega \in H^{1,1}_\mu \). To show \( \omega \in H^{1,1}_{\bar{\partial}} \cap H^{1,1}_{\bar{\mu}} \) it suffices to show that \( \bar{\partial}^* \omega = 0 \). This follows from the identity \( [L, \bar{\partial}^*] = -i\partial \). The result follows now by induction on \( k \), using the identities \( [L, \bar{\partial}^*] = -i\partial \) and \( [L, \mu^*] = -i\bar{\mu} \). \( \square \)

**Theorem 4.3.** For any compact almost Kähler manifold of dimension \( 2m \), the following is satisfied:

1. For any \((p, q)\), we have
   \[ H_{\bar{\partial}}^{p,q} \cap H_{\bar{\mu}}^{p,q} = H_d^{p,q} \subseteq H_d^{p+q} \]
   and for all \( k \geq 0 \) we have inequalities
   \[ \sum_{p+q=k} h^{p,q} \leq b^k. \]

2. In odd degrees, we have
   \[ \sum_{p+q=2k+1} h^{p,q} = 2 \sum_{0 \leq p \leq k} h^{p,2k+1-p} \leq b^{2k+1}. \]

3. In even degrees, we have
   \[ \sum_{p+q=2k} h^{p,q} = 2 \sum_{0 \leq p < k} h^{p,2k-p} + h^{k,k} \leq b^{2k}, \]
   with \( h^{k,k} \geq 1 \) for all \( k \leq m \).

**Proof.** For any almost Hermitian manifold we have
\[ H_{\bar{\partial}}^{p,q} \cap H_{\bar{\mu}}^{p,q} \subseteq \text{Ker} (\Delta_d) \cap A^{p,q}, \]
since the space on the left is in the eightfold intersection of kernels of \( \bar{\partial}, \partial, \bar{\mu}, \mu \) and their adjoints. The space on the left is also
\[ \text{Ker} (\Delta_{\bar{\partial}} + \Delta_{\mu} + \Delta_{\partial} + \Delta_{\bar{\mu}}) = \text{Ker} (\Delta_{\bar{\partial}} + \Delta_{\mu}) \cap \text{Ker} (\Delta_{\partial} + \Delta_{\bar{\mu}}), \]
so by Proposition 3.3
\[ H_{\bar{\partial}}^{p,q} \cap H_{\bar{\mu}}^{p,q} \subseteq \text{Ker} (\Delta_d) \cap A^{p,q} = H_d^{k} \cap A^{p,q}. \]
To prove the converse inclusion, note that if a form \( \alpha \) of type \((p, q)\) satisfies \( d\alpha = 0 \) and \( d^*\alpha = 0 \), then the four components of \( d \) and \( d^* \) also vanish by bidegree reasons. The inequalities in (1) now follow from the isomorphism between \( d \)-harmonic forms and de Rham cohomology, given by the classical Hodge theory for compact manifolds. Assertion (2) follows from (1) and the complex conjugation duality
Lastly, (3) follows from the previous arguments, together with Lemma 4.2 which directly implies that $h_{k,k} \geq 1$. \hfill \Box

**Remark 4.4.** Consider a compact symplectic manifold with compatible almost complex structure $J$, as well as an integrable almost complex structure $J'$. In this case, the Frölicher spectral sequence associated to $J'$, together with the above result, give inequalities

$$
\sum_{p+q=k} h^{p,q}_{J} \leq b^{k} \leq \sum_{p+q=k} h^{p,q}_{J'}
$$

bounding the Betti numbers above and below. For instance, this is the case of the Kodaira-Thurston manifolds studied in Section 6.1.

The theorem shows in particular that there is a combined geometric/topological obstruction to the almost Kähler condition. An immediate corollary is:

**Corollary 4.5.** For any compact almost Kähler manifold, if $h^{p,q} \neq 0$ for some $p \neq q$, then $b^{p+q} \geq 2$ if $p + q > 0$ is even and $b^{p+q} \geq 3$ if $p + q > 0$ is odd.

By restricting to 1-forms, we obtain the following metric-independent statements. Denote by

$$
\Omega^{p}_{\tilde{\partial}} := \text{Ker} (\tilde{\partial}) \cap A^{p,0}
$$

the space of holomorphic $p$-forms. As was previously known for compact Kähler manifolds, the space of holomorphic 1-forms coincides with the space of $d$-harmonic forms of pure type $(1,0)$. Indeed, by bidegree reasons together with Theorem 4.3 we have

$$
\Omega^{1}_{\tilde{\partial}} = H^{1,0}_{\tilde{\partial}} = H^{1,0}_{\tilde{\mu}} \cap H^{1,0}_{\mu} = H^{1,0}_{\tilde{\partial}} \cap H^{1,0}_{\mu} = H^{1,0}_{d}.
$$

This immediately gives:

**Corollary 4.6.** A necessary condition for a compact almost complex manifold to admit a compatible symplectic form is that

$$
2 \dim \Omega^{1}_{\tilde{\partial}} \leq b^{1}.
$$

In particular, compact simply connected almost Kähler manifolds have no holomorphic 1-forms.

**Example 4.7.** Any almost Kähler structure on a 4-dimensional manifold with the same cohomology as $\mathbb{CP}^{2}$ must satisfy $h^{p,p} = 1$ for $p = 0, 1, 2$ and $h^{p,q} = 0$ for $p \neq q$. Likewise, any almost Kähler structure on a 4-dimensional manifold with the same cohomology as $S^{2} \times S^{2}$ must satisfy $h^{p,p} = 1$ for $p = 0, 2$. Also, $h^{1,1} \in \{1, 2\}$ and $h^{p,q} = 0$ for $p \neq q$.

Finally, we extend Chen’s results [Che71], [Che72] on fundamental groups of Kähler manifolds, to the almost Kähler case.

**Lemma 4.8.** If $M$ is a compact connected almost Kähler manifold and if

$$
\dim \Omega^{1}_{\tilde{\partial}} > \dim \Omega^{2}_{\tilde{\partial}} + 1,
$$

then $\pi_{1}(M)$ contains a free subgroup of rank $\geq 2$. In particular, $\pi_{1}(M)$ is not solvable.
Proof: Theorem 3.2 of [Che71] states that if there exist closed 1-forms $\alpha$ and $\beta$ such that $\alpha \wedge \beta = 0$ and $[\alpha]$ and $[\beta]$ are linearly independent classes in $H^1(M, \mathbb{C})$, then $\pi_1(M)$ is not solvable. In Theorem 4.1 of [Che72], it is shown that the same conditions imply that $\pi_1(M)$ contains a free subgroup of rank $\geq 2$. We adapt Chen’s Corollary on Kähler manifolds. Let $\alpha_1, \cdots, \alpha_r$ generate $\Omega_{\bar{\partial}}^1$. Since by assumption we have $\dim \Omega_{\bar{\partial}}^2 < r - 1$, the forms $\alpha_1 \wedge \alpha_i$, for $i > 1$, must be linearly dependent. Hence there is a holomorphic 1-form $\alpha$ such that $\alpha_1 \wedge \alpha = 0$, and $\alpha_1$ and $\alpha$ are linearly independent in $\Omega_{\mu}^1$. Since $\Omega_{\mu}^1 = H_{\bar{\partial}}^{1,0}$ and the map $H_{\bar{\partial}}^{1,0} \to H_{dR}^1(M, \mathbb{C})$ is injective, the classes $[\alpha_1]$ and $[\alpha]$ are linearly independent.□

Take for instance a solvable finitely presented group $G$. Then $G$ can be realized as the fundamental group of a symplectic 4-manifold $M$ by a result of Gompf [Gom95].

The above result tells us that for any compatible almost complex structure, the inequality $\dim \Omega_{\bar{\partial}}^1 \leq \dim \Omega_{\mu}^2 + 1$ is satisfied.

5. Lefschetz duality, decomposition, and Hodge index theorem

We prove Lefschetz duality and decomposition on the spaces of $\bar{\partial}$-$\mu$-harmonic forms. We also give a generalized Hodge index theorem in the four dimensional case.

**Theorem 5.1** (Generalized Hard Lefschetz Duality). For any compact almost Kähler manifold of dimension $2m$, the operators $\{ L, \Lambda, H = [L, \Lambda] \}$ define a finite dimensional representation of $\mathfrak{sl}(2, \mathbb{C})$ on

$$H_{\bar{\partial}} \cap H_{\mu} = \bigoplus_{p,q \geq 0} H_{\bar{\partial}}^{p,q} \cap H_{\mu}^{p,q}.$$ 

Moreover, for every $0 \leq p \leq m$ and all $p \leq k \leq m$ the maps

$$L^{m-k} : H_{\bar{\partial}}^{p,k-p} \cap H_{\mu}^{p,k-p} \to H_{\bar{\partial}}^{p+m-k,m-p} \cap H_{\mu}^{p+m-k,m-p},$$

are isomorphisms.

**Proof.** For any almost Hermitian manifold of dimension $2m$ there are isomorphisms

$$L^{m-k} : \mathcal{A}^{p,k-p} \to \mathcal{A}^{p+m-k,m-p}$$

for every $0 \leq p \leq m$ and all $p \leq k \leq m$. By Corollary [BG88] $[L, \Delta_{\bar{\partial}} + \Delta_{\mu}] = 0$ and $[\Lambda, \Delta_{\bar{\partial}} + \Delta_{\mu}] = 0$, so $L$ and $\Lambda$ preserve $H_{\bar{\partial}} \cap H_{\mu}$. It follows the maps

$$L^{m-k} : H_{\bar{\partial}}^{p,k-p} \cap H_{\mu}^{p,k-p} \to H_{\bar{\partial}}^{p+m-k,m-p} \cap H_{\mu}^{p+m-k,m-p},$$

are well defined, and are injective, since they are isomorphisms before restricting the domain. By Hodge duality of Theorem [BG88] the domain and codomain have the same dimension, so the map is an isomorphism.□

**Remark 5.2.** Benson and Gordon showed that if a symplectic nilmanifold $M$ satisfies that $L : H^1 \to H^{2n-1}$ is an isomorphism, then $M$ is a torus [BG88]. On the other hand, there are many non-toral symplectic nilmanifolds, so the above generalized Lefschetz duality has a large family of non-trivial examples which are computable.
Remark 5.3 (Comparison with symplectic Hodge theory). In [Bry88], Brylinski proposed a Hodge theory for compact symplectic manifolds, by introducing a symplectic Hodge star operator $\star_s$, defined using the symplectic form. The space of symplectically-harmonic $k$-forms is

$$\mathcal{H}_{sym}^k := A^k \cap \text{Ker}(d) \cap \text{Ker}(d_s)$$

where $d^s = \star_s \circ d \circ \star_s$. Brylinski showed that in an almost Kähler manifold, a form of pure type $(p, q)$ is in $\mathcal{H}_{sym}^{p+q}$ if and only if it is in $\mathcal{H}_{d^s}^{p+q}$. This follows from Brylinski’s formula for almost Kähler manifolds: if $\alpha \in A^{p,q}$ then $\star_s(\alpha) = i^{p-q} \star_s(\alpha)$. In particular, by (1) of Theorem 4.3, when restricted to forms of pure type, all notions of harmonics agree:

$$\mathcal{H}_{\bar{\partial}}^{p,q} \cap \mathcal{H}_{\mu}^{p,q} = \mathcal{H}_{d}^{p,q} = \mathcal{H}_{sym}^{p+q} \cap A^{p,q}.$$ 

This gives an inclusion

$$\bigoplus_{p+q=k} (\mathcal{H}_{\bar{\partial}}^{p,q} \cap \mathcal{H}_{\mu}^{p,q}) \hookrightarrow \mathcal{H}_{sym}^{p+q}$$

which is strict in general. Indeed, Yan [Yan96] showed that for $k = 0, 1, 2$, every cohomology class has a symplectically-harmonic representative. This is not true for $\bar{\partial}$-$\mu$-harmonic forms (see the example in 6.1 below).

As a consequence of Theorem 5.1 we have the following generalized Lefschetz decomposition.

Corollary 5.4. For any compact almost Kähler manifold of dimension $2m$, and any $p, q$ we have an orthogonal direct sum decomposition

$$\mathcal{H}_{\bar{\partial}}^{p,q} \cap \mathcal{H}_{\mu}^{p,q} = \bigoplus_{j \geq 0} L^j \left( \mathcal{H}_{\bar{\partial}}^{p-j,q-j} \cap \mathcal{H}_{\mu}^{p-j,q-j} \right)_{\text{prim}}$$

where

$$\left( \mathcal{H}_{\bar{\partial}}^{r,s} \cap \mathcal{H}_{\mu}^{r,s} \right)_{\text{prim}} := \left( \mathcal{H}_{\bar{\partial}}^{r,s} \cap \mathcal{H}_{\mu}^{r,s} \right) \cap \text{Ker } \Lambda.$$ 

This may be proved directly, as a finite dimensional representation of $\mathfrak{sl}(2, \mathbb{C})$. This shows that the numbers $h^{p,q}$ satisfy

$$h^{p,q} \leq h^{p+1,q+1} \leq \cdots \leq h^{p+j,q+j}$$

for all $0 \leq p, q \leq m$ and $p + q + 2j \leq m$. Also, the dimensions of the primitive spaces can be written in terms of successive differences of the numbers $h^{p,q}$.

The Hodge-Riemann pairing yields the analogous bilinear Hodge-Riemann relations on the $\bar{\partial}$-$\mu$-harmonic forms. Namely, for any compact almost Kähler manifold of dimension $2m$, if $\alpha \in \left( \mathcal{H}_{\bar{\partial}}^{p,q} \cap \mathcal{H}_{\mu}^{p,q} \right)_{\text{prim}}$ is a non-zero form, then

$$i^{p-q}Q(\alpha, \bar{\alpha}) > 0,$$

where the Hodge-Riemann pairing $Q$ is given by

$$Q(\alpha, \beta) := (-1)^{\frac{(p+q)(p+q-1)}{2}} \int_M \alpha \wedge \beta \wedge \omega^{m-p-q}.$$ 

This is because, for any primitive $(p,q)$-form $\alpha$ on any almost Hermitian manifold,

$$i^{p-q}(-1)^{\frac{(p+q)(p+q-1)}{2}} \alpha \wedge \bar{\alpha} \wedge \omega^{m-p-q}$$
is an inclusion ⨁ at any point where α is non-vanishing (c.f. [Huy05], p.39 Corollary 1.2.36), so the result follows from the primitive decomposition of Corollary 6.4.

In particular, for any 4k-dimensional almost Kähler manifold, one can consider this pairing in the middle dimension ⨁p+q=2k H_0^{p,q} ∩ H_μ^{p,q}. By Theorem 4.3 there is an inclusion ⨁p+q=2k H_0^{p,q} ∩ H_μ^{p,q} ⊆ H_d^{2k}. The intersection pairing defined on H_d^{2k} by

\((\alpha, \beta) \mapsto \int_M \alpha \wedge \beta\)

may be restricted to the subspace ⨁p+q=2k H_0^{p,q} ∩ H_μ^{p,q} and differs from the Hodge-Riemann pairing by a factor of \((-1)^k\). In particular, for 4-manifolds the Hodge-Riemann pairing and the intersection pairing on \(H^2\) differ by a minus sign, which will be used to prove the following.

**Theorem 5.5 (Generalized Hodge Index Theorem).** For any compact 4-dimensional almost Kähler manifold \(M\), we have

\[ h^{1,1} - 1 \leq b_2^- \]
\[ 2h^{2,0} + 1 \leq b_2^+ \]

where the intersection pairing on \(H^2(M; \mathbb{C})\) has index \((b_2^+, b_2^-)\). Also, if the almost complex structure is not integrable, then \(h^{2,0} = 0\).

**Proof.** The Hodge-Riemann pairing is positive on the primitive forms in \(H_0^{1,1} \cap H_μ^{1,1}\), negative on \(\omega\) since \(\omega^2 > 0\), and negative on primitive forms in \(H_0^{p,q} \cap H_μ^{p,q}\) for \((p, q) = (0, 2)\) and \((p, q) = (2, 0)\). Note that all \((0, 2)\)-forms and \((2, 0)\)-forms are primitive, and we have a decomposition

\[ \bigoplus_{p+q=2} H_0^{p,q} \cap H_μ^{p,q} = P^{1,1} \oplus \omega \mathbb{C} \oplus P^{0,2} \oplus P^{2,0}, \]

where \(P^{r,s} = (H_0^{r,s} \cap H_μ^{r,s})_{prim}\). The result follows by noting that this decomposition is orthogonal with respect to the Hodge-Riemann pairing. This can be checked using the general fact that, for almost Hermitian manifolds, the primitive k-forms for \(k \leq m\) are precisely \(\{\alpha | L^{m-k+1} \alpha = 0\}\), so \(L \alpha = 0\) for \(\alpha \in P^{1,1}\).

If the almost complex structure is not integrable, there is an open set \(U\) on which \(\mu : T^{0,1}M \to (T^{1,0}M \wedge T^{1,0}M)\) is pointwise nonzero, and therefore pointwise surjective on \(U\), since \(M\) is a four manifold. Therefore,

\[ \mu^* : \wedge^2 T^{1,0}M \to T^{0,1}M \]

is pointwise injective on \(U\). If \(\eta \in H_0^{2,0} \cap H_μ^{2,0}\), then \(\mu^* \eta = 0\), which implies \(\eta\) is zero on \(U\). But Theorem 4.3 implies \(\eta\) is a harmonic form, so \(\eta\) is zero on \(M\), [Bär97].

**Remark 5.6.** In the case that the containment \(\bigoplus_{p+q=2k} H_0^{p,q} \cap H_μ^{p,q} \subseteq H_d^{2k}\) is an equality, such as for Kähler manifolds, then both inequalities in Theorem 6.4 become equalities and we obtain an equality of signatures as well. In general, the signature of the Hodge-Riemann pairing on \(\bigoplus_{p+q=2k} H_0^{p,q} \cap H_μ^{p,q}\) need not equal the topological signature, as can be seen in Example 6.1.
Example 5.7. A K3 surface is known to have topological index $(3, 19)$. It follows that for any almost Kähler structure on the underlying 4-manifold, $h^{1,1} \leq 20$ and $h^{2,0} \leq 1$, with equalities achieved by any Kähler structure.

We give another application of the Hodge Index Theorem in Example 6.1 to show that an explicit almost Kähler structure on the Kodaira-Thurston manifold satisfies $h^{1,1} = 3$.

Some consequences of Theorem 5.5 are the following. First, for $b_2^+$ small enough, we immediately obtain:

**Corollary 5.8.** Let $M$ be a compact 4-manifold with whose intersection pairing satisfies $b_2^+ \leq 2$. If an almost complex structure on $M$ admits a compatible symplectic structure then $h^{2,0} = 0$.

Examples include smooth manifolds homeomorphic to $\mathbb{C}P^2 \# n\overline{\mathbb{C}P^2}$, among which there are known to be infinitely many exotic examples, for various values of $n$.

Lastly, in the positive-definite situation, we have:

**Corollary 5.9.** For any compact almost Kähler 4-manifold whose intersection pairing is positive definite, we have that $h^{1,1} = 1$, and all of the numbers $h^{p,q}$ are metric-independent among all almost Kähler metrics that are compatible with the given almost complex structure. Moreover, the dimensions of $\mathcal{H}^{p,q}_\partial$ are similarly metric independent for all $(p, q)$.

**Proof.** We may as well assume the almost complex structure is not integrable, so Theorem 5.5 implies $h^{1,1} = 1$ and $h^{2,0} = h^{0,2} = 0$ for any almost Kähler metric compatible with the given almost complex structure. Then $\mathcal{H}^{1,0}_\partial = \mathcal{H}^{1,0}_\bar{\partial} \cap \mathcal{H}^{1,0}_\mu = \operatorname{Ker} (\bar{\partial}) \cap \mathcal{A}^{1,0}$, which is metric independent, and the claim for the remaining bidegrees $(0, 1)$, $(2, 1)$ and $(1, 2)$ follows by the various dualities of Theorem 4.1. For the last claim, we note that $\mathcal{H}^{2,0}_\partial = \operatorname{Ker} (\bar{\partial}) \cap \mathcal{A}^{2,0}$, which is metric independent, and the other cases follow from the first part since $\mathcal{H}^{p,q}_\partial = \mathcal{H}^{p,q}_\bar{\partial} \cap \mathcal{H}^{p,q}_\mu$ for degrees $(1, 0)$ and $(1, 1)$.

This yields an affirmative answer to Hirzebruch’s Problem 20 [Hir54] for compact almost Kähler metrics on 4-manifolds with positive definite intersection pairing. Examples include $\mathbb{C}P^2$ as well as any exotic $(2n + 1)\mathbb{C}P^2$, though to our knowledge, it is not known yet if any such exist.

6. Applications

In general it is very difficult to compute the dimension of the kernel of a self-adjoint elliptic operator. In this section we show the above theory can be used to compute the spaces $\operatorname{Ker} (\Delta_\partial + \Delta_\mu)$, as well as give several other applications involving the above obstructions to almost Kähler structures. For some examples we use nilmanifolds, which are a quotient of a nilpotent real Lie group $G$ by an integral subgroup $\Gamma$. For this purpose, we first briefly describe how the above theory applies in that setting.

An almost complex structure on the Lie algebra $\mathfrak{g}$ of $G$ defines a bigrading on the Chevalley-Eilenberg dg-algebra $\mathcal{A}^\ast_{\mathfrak{g}C}$ associated to the complexification $\mathfrak{g}C$ of $\mathfrak{g}$ and $M$ inherits an almost complex structure. The algebra $\mathcal{A}^\ast_{\mathfrak{g}C}$ may be regarded as the complex algebra of $\Gamma$-invariant forms on $G$ and it includes into the complex algebra of forms of $M$ via a quasi-isomorphism compatible with bigradings. The theory of
harmonic forms for almost complex nilmanifolds is developed in [CW18]. For our purposes here, it suffices to note that, given a left-invariant Hermitian metric, we have an inclusion
\[ L^p_q \boldsymbol{\bar{\partial}} \cap L^p_q \boldsymbol{\bar{\mu}} := \text{Ker} (\Delta_{\bar{\partial}}) \cap \text{Ker} (\Delta_{\bar{\mu}}) \cap A^*_g \subseteq H^p_q \boldsymbol{\bar{\partial}} \cap H^p_q \boldsymbol{\bar{\mu}} \]
of left-invariant \( \bar{\partial} - \bar{\mu} \)-harmonic forms into all \( \bar{\partial} - \bar{\mu} \)-harmonic forms, and that the almost Kähler package is equally valid on these subspaces. We see below that, in some situations, the dimensions of the left-invariant harmonics, together with the bounds offered by the almost Kähler package, are sufficient in order to compute the dimensions \( h^{p,q} \) of the true \( \bar{\partial} - \bar{\mu} \)-harmonics.

The first example shows how the numbers \( h^{p,q} \) can be computed for an explicit almost Kähler structure on the Kodaira-Thurston manifold. The second example shows that the failure of Hard Lefschetz duality can be used to conclude that an almost complex structure admits no left invariant almost Kähler structure, and the last example shows that the failure of the symmetry condition \( \Delta_{\bar{\partial}} + \Delta_{\bar{\mu}} = \Delta_{\partial} + \Delta_{\mu} \) detects many examples of almost complex structures on symplectic manifolds which admit no compatible symplectic form.

6.1. Calculating the spaces of harmonic forms via Betti-number bounds and the Generalized Hodge Index Theorem. The Kodaira-Thurston manifold was originally studied by Kodaira as a complex manifold [Kod64], and by Thurston as the first example of a symplectic manifold which is non-Kähler, [Thu76]. We follow the presentation in [BMin16] as a nilmanifold, giving here an almost Kähler structure.

The Kodaira-Thurston manifold is the 4-dimensional nilmanifold defined as the quotient
\[ H \times \mathbb{Z} / H \times \mathbb{R} \]
where \( H \) is the 3-dimensional Heisenberg Lie group, and \( H \times \mathbb{Z} \) is the integral subgroup. The Lie algebra is spanned by \( X, Y, Z, W \) where the only non-zero bracket is \([X, Y] = -Z\). On the dual basis \( x, y, z, w \), the only non-zero differential is therefore \( dz = xy \).

Let \( \omega = wx + zy \). This is closed, and can be extended to a left invariant symplectic form. We remark that \( L(y) = -[xyw] = -d(zw) = 0 \), so \( L : H^1 \rightarrow H^3 \) is not an isomorphism, i.e. the hard Lefschetz duality does not hold on cohomology.

Consider the non-integrable \( J \) given by
\[ J(W) = X \quad J(Z) = Y. \]

Observe that \( \langle -,- \rangle := \omega(-,J-) \) is a metric for which \( X, Y, Z, W \) is orthonormal, so that \( \omega, J, \langle -,- \rangle \) is a compatible triple, and extending the structures left invariantly gives an almost Kähler manifold.

Let \( A := X - iJX = X + iW \) and \( B := Y - iJY = Y + iZ \) be a basis for the invariant \((1,0)\)-vectors. The only non-trivial brackets are
\[ [A, B] = [A, \bar{B}] = [\bar{A}, B] = [\bar{A}, \bar{B}] = -Z = -\frac{1}{2i} (B - \bar{B}). \]

Letting \( a, b \) be dual to \( A, B \), the only non-zero components of \( d \) in degree one are
\[ \partial b = \frac{1}{2i} ab, \quad \bar{\partial} b = \frac{1}{2i} (a\bar{b} - b\bar{a}), \quad \bar{\mu} b = \frac{1}{2i} \bar{a}b, \]
and the conjugate equations.
A calculation shows that $a \in H^{1,0}_\bar{\partial} \cap H^{1,0}_\mu$ since $a \in \text{Ker} \bar{\partial}$. By Theorem 4.3, the dimension of global (not-necessarily left invariant) harmonic 1-forms is either zero or two, since it must be even and less than or equal to $b^1 = 3$. We conclude
\[ h^{1,0}_j = h^{0,1}_j = 1, \]
and the representing forms are the left invariant 1-forms $a$ and $\bar{a}$, respectively.

Similarly, $b\bar{b} \in H^{1,2}_\bar{\partial} \cap H^{1,2}_\mu$, and $a\bar{b} \in H^{2,1}_\bar{\partial} \cap H^{2,1}_\mu$, so
\[ h^{1,2}_j = h^{2,1}_j = 1, \]
which also follows by Serre duality, Theorem 4.1, and the previous case.

Considering the middle total degree, where $H^{1,1}_\bar{\partial} = H^{1,1}_\bar{\partial} \cap H^{1,1}_\mu$, by Theorem 4.3 we have
\[ 1 \leq \dim (H^{\cdot,1}_\bar{\partial} \cap A^2) \leq 2. \]
A calculation shows $a\bar{a}$, $\omega = -\frac{i}{2}(a\bar{a} + b\bar{b})$, and $a\bar{b} + b\bar{a}$ generate all of the left invariant forms in $H^{1,1}_\bar{\partial}$, so that $\dim H^{1,1}_\bar{\partial} \geq 3$. Therefore, there are no $\bar{\partial}$-harmonic forms in bidegrees $(2,0)$ and $(0,2)$, by Theorem 4.3. It follows that
\[ 3 \leq \dim (H^{1,1}_\bar{\partial}) \leq 4. \]
Thus, the number $h^{1,1}_j$ is equal to 4 if and only if there is a non-left-invariant $\bar{\partial}$-harmonic $(1,1)$-form, and otherwise it is 3.

Here we can use the topological index to determine that there are no non-left invariant $\bar{\partial}$-harmonic $(1,1)$-forms. The topological intersection pairing of the Kodaira-Thurston manifold can be computed with respect to the ordered basis $xz, xw, yz, yw$ of $H^2$; it is the anti-diagonal matrix with entries $-1, +1, +1, -1$, with eigenvalues $1, 1, -1, -1$, so that $b_2^1 = 2$. Therefore, by Theorem 5.5, we conclude
\[ h^{1,1}_j = 3. \]

Finally, to contrast the numbers $h^{p,q}_j$ associated to the almost Kähler structure above, the integrable left invariant complex structure on the Kodaira-Thurston manifold given by $J'(X) = Y$ and $J'(Z) = W$ has numbers
\[ h^{p,q}_j := \dim (H^{p,q}_\bar{\partial}(J')) = \dim (H^{p,q}_\bar{\partial}(J') \cap H^{p,q}_\mu(J')) \]
given by $h^{0,1}_j = h^{1,1}_j$, $h^{2,1}_j = 2$, and $h^{p,q}_j = 1$ for all other $0 \leq p, q \leq 2$ (see for instance [Cor89]).

We summarize all numbers in the following tables:

\[ h^{s,s}_j = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad h^{s,j}_j = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix}. \]

6.2. **Generalized Hard Lefschetz Theorem** detects non-existence of invariant almost Kähler structures. Consider the real 4-dimensional nilpotent filiform Lie algebra with basis $X_1, X_2, X_3, X_4$ and only non-zero brackets
\[ [X_i, X_j] = X_{i+j} \quad \text{for} \quad i = 2, 3. \]

One can check that the Betti numbers of the compact quotient filiform manifold $\Gamma/G$, where $\Gamma$ is a discrete subgroup of the simply connected Lie group $G$, are $b^1 = b^2 = b^3 = 2$. 
Consider the non-integrable left invariant almost complex structure given by $JX_1 = X_2$ and $JX_3 = X_4$. Letting $A = X_1 - iJX_1$ and $B = X_3 - iJX_3$, the dual elements $a$ and $b$ are a basis for the invariant $(1,0)$-forms. One can check that the only non-zero components of the exterior differential are
\[ \partial b = \frac{1}{2i} (\bar{a} \bar{b} - a b), \quad \partial a = \frac{1}{2i} ab, \]
and their conjugates. We’ll use hard Lefschetz duality to show there is no left invariant metric making this almost complex manifold into an almost Kähler manifold.

First, independent of metric we have $a \in H^{1,0} \cap H^{0,1}$, so if there were a metric making this almost Kähler, then also $\bar{a} \in H^{0,1} \cap H^{0,1}$, and since $b^1 = 2$, it follows that for any almost Kähler metric $h^{0,1} = h^{1,0} = 1$. Also, by duality, $h^{1,1} = h^{2,2} = 1$ as well.

It is straightforward to check that a real basis for the left invariant real $(1,1)$-forms is given by
\[ \{ i \bar{a} \bar{b}, i (\bar{a} \bar{b} + a b), (\bar{a} \bar{b} - a b) \}. \]
and that a basis for the $\partial$-closed left invariant real $(1,1)$-forms is given by
\[ \{ i \bar{a} \bar{a}, i (\bar{a} \bar{b} + a b) \}. \]

Then any compatible invariant symplectic form can be written as $\omega = i \bar{a} \bar{a} + i \beta (\bar{a} \bar{b} + a b)$, for some constants $\alpha, \beta$.

The hard Lefschetz duality theorem implies that
\[ L(a) = i \beta \bar{a} \bar{a} \in H^{2,1} \cap H^{2,1}. \]
But, $\partial(ab) = \frac{1}{4} i \bar{a} \bar{b} a$, so that $L(a) \in Im(\partial)$ and therefore $\partial^* L(a) \neq 0$, unless $\beta = 0$. But then $\omega = i \alpha \bar{a} \bar{a}$ is degenerate, which is a contradiction.

We note that this manifold does admit a symplectic form. One example is $\omega = x_1 x_4 + x_2 x_3$, which is almost Kähler for the metric making $\{X_i\}$ orthonormal, and has $J'X_1 = X_4$ and $J'X_2 = X_3$. This manifold does not admit any integrable almost complex structure, as pointed out to us by Aleksandar Milivojevic.

6.3. Laplacian identity detects symplectic manifolds with almost complex structures that admit no compatible almost Kähler structure. Consider the 6-dimensional nilpotent real Lie algebra with basis $\{X_1, \ldots, X_6\}$ and only non-zero brackets given by
\[ [X_1, X_3] = [X_2, X_4] = X_5 \quad \text{and} \quad [X_1, X_4] = -[X_2, X_3] = X_6 \]
This is considered in [CFGU00], denoted as $\mathfrak{g}_5$, and induces a nilmanifold with nilpotent complex structure. Let $G$ be the simply connected group associated to $G$, and let $\Gamma/G$ be a nilmanifold where $\Gamma$ is a discrete lattice.

Note $\Gamma/G$ is a symplectic manifold, with symplectic form given by
\[ \omega = x_1 x_5 + x_2 x_6 + x_3 x_6, \]
where $x_i$ is dual to $X_i$.

Consider the almost complex structure given by
\[ JX_1 = X_2 \quad JX_3 = -X_4 \quad JX_5 = X_6. \]
Let $a = X_5 - iJX_5$, $b = X_3 - iJX_1$, and $c = X_3 - iJX_3$ be a basis for the left invariant $(1,0)$-forms. A straightforward calculation similar to those in previous examples shows that

$$da = -\frac{1}{4} bc$$

so that $\bar{\partial}a = \bar{\mu}a = 0$ and $\partial a = -\frac{1}{4} bc$. Therefore, $a \in \text{Ker} (\Delta_\bar{\partial} + \Delta_\mu)$ but $a \notin \text{Ker} (\Delta_\partial + \Delta_\bar{\mu})$. Hence,

$$\Delta_\bar{\partial} + \Delta_\mu \neq \Delta_\partial + \Delta_\bar{\mu},$$

so by Theorem 3.3 we can conclude that this almost complex structure does not admit a compatible symplectic structure. Of course, $db = dc = 0$ as well, so that $J$ is integrable. So, one could arrive at this same conclusion since $\Gamma/G$ is not Kähler, since for example $h^{1,0} \geq 3 > \frac{h^2}{2} = 2$.

But, the example shows even more: if $(M, J')$ is any almost complex manifold (integrable or not), then the Cartesian product $\Gamma/G \times M$ with product almost complex structure $J \times J'$, does not admit any metric making the product an almost Kähler manifold. Indeed, the pullback of the form $a$ to the product still exhibits that $\Delta_\bar{\partial} + \Delta_\mu \neq \Delta_\partial + \Delta_\bar{\mu}$ on $\Gamma/G \times M$.

In particular, if $M$ is symplectic then $\Gamma/G \times M$ always admits a symplectic form, but never a symplectic form compatible with this almost complex structure.

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