On the solvability of a nonlinear functional integral equations via measure of noncompactness in $L^p(\mathbb{R}^N)$

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Abstract

Using the technique of a suitable measure of non-compactness and the Darbo fixed point theorem, we investigate the existence of a nonlinear functional integral equation of Urysohn type in the space of Lebesgue integrable functions $L^p(\mathbb{R}^N)$. In this space, we show that our functional-integral equation has at least one solution. Finally an example is also discussed to indicate the natural realizations of our abstract result.

Keywords: functional integral equation; measure of noncompactness; existence; Darbo’s fixed point theorem; fixed point.

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1 Introduction

Integral equations appear in many applications in describing numerous real world problems (see, for instance, (30, 31, 5, 18), and references therein).

Also many useful applications in describing problems of the real world and numerous events, which appear in physics, engineering, mechanics, biology, etc. See for example [1, 4, 8, 13, 15] can be depicted and demonstrated by methods of non-linear functional integral equations (for example, we refer to [25, 26, 28]). Apart from that, integral equations are often investigated in research papers and monographs (cf. [6, 12, 16, 18, 29, 32]) and the references cited therein.

2 Preliminaries

We will collect in this section some definitions and basic results which will be used further on throughout the paper. First, we denote \( L^p(U) \) \((U \in \mathbb{R}^N)\) as the space of Lebesgue integrable functions on \( U \) with the standard norm \( \|x\|_{L^p(U)} = \left( \int_U |x(t)|^p \, dt \right)^{1/p} \).

**Theorem 2.1** [1, 8, 9]

Let \( F \) be a bounded set in \( L^p(\mathbb{R}^N) \) with \( 1 \leq p < \infty \). The closure of \( F \) in \( L^p(\mathbb{R}^N) \) is compact if and only if \( \lim_{h \to 0} \| \tau_h f - f \|_{L^p(\mathbb{R}^N)} = 0 \) uniformly in \( f \in F \),

where \( \tau_h f(x) = f(x + h) \) for all \( x, h \in \mathbb{R}^N \). Also for \( \epsilon > 0 \) there is a bounded and measurable subset \( \Omega \subset (\mathbb{R}^N) \) such that

\[ \| f \|_{(\mathbb{R}^N \setminus \Omega)} < \epsilon \quad \text{for all} \quad f \in F. \]

Next, we recall the concept of measure of noncompactness, let \( E \) be an infinite dimensional Banach space with norm \( \| \cdot \| \) and zero element \( \theta \). Denote by \( M_E \) the family of all nonempty and bounded subsets of \( E \), \( N_E \) and \( N_E^W \) the family of all nonempty relatively compact and weakly relatively compact sets, respectively. The symbols \( \bar{X} \) and \( \text{Conv}X \) stand for the closure and closed convex hull of a subset \( X \) of \( E \), respectively. We use the standard notation \( X + Y \) and \( \lambda X \) for algebraic operations on sets, while,

we denote \( B_r = B(\theta, r) \) the closed ball centered at \( \theta \) and with radius \( r \).

**Definition 2.1** (Measure of noncompactness)

\[ \| \cdot \| \]

A mapping \( \mu : M_E \to [0, \infty) \) is said to be a measure of noncompactness in \( E \) if it satisfies the following conditions:
(1) the family $\text{ker}\, \mu = \{X \in M_E : \mu(X) = 0\}$ is nonempty and $\text{ker}\, \mu \subset N_E$, where $\text{ker}\, \mu$ is called the kernel of the measure $\mu$.

(2) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.

(3) $\mu(\text{Conv}X) = \mu(X) = \mu(\overline{X})$.

(4) $\mu[\lambda X + (1 - \lambda)Y] \leq \lambda\mu(X) + (1 - \lambda)\mu(Y)$, $\lambda \in [0, 1]$.

(5) If $X_n \in M_E$, $X_n = \overline{X}_n$ and $X_{n+1} \subset X_n$ for $n = 1, 2, \ldots$ and if

$$\lim_{n \to \infty} \mu(X_n) = 0,$$

then

$$X_\infty = \bigcap_{n=1}^\infty X_n \neq \phi.$$

**Theorem 2.2** \[1\]

Suppose $1 \leq p < \infty$ and $X$ is a bounded subset of $\mathbb{R}^N$. For $x \in X$ and $\epsilon > 0$

$$w^T(x, \epsilon) = \sup\{\|h x - x\|_{L^p(B_T)} : \|h\|_{\mathbb{R}^N} < \epsilon\},$$

$$w^T(X, \epsilon) = \sup\{w^T(x, \epsilon) : x \in X\},$$

$$w^T(X) = \lim_{\epsilon \to 0} w^T(X, \epsilon),$$

$$w(X) = \lim_{T \to \infty} w^T(X),$$

$$d(X) = \lim_{T \to \infty} \sup\{\|x\|_{L^p(\mathbb{R}^N \setminus B_T)} : x \in X\},$$

where $B_T = \{a \in \mathbb{R}^N : \|a\|_{\mathbb{R}^N} \leq T\}$. Then

$$\mu(X) = w(X) + d(X)$$

is a measure of non compactness on $L^p(\mathbb{R}^N)$.

At the end of this section, we recall the fixed point theorem due to Darbo which enables us to prove the existence theorem for solutions of a several integral equations considered in nonlinear analysis. To quote this theorem we need the following definitions.
Definition 2.2 \[13\]
The function \( f : I \times \mathbb{R} \rightarrow \mathbb{R} \) satisfies Carathéodory condition if it satisfies the following two conditions:

(1) \( f \) is measurable in \( t \in I \) for any \( x \in \mathbb{R} \).

(2) \( f \) is continuous in \( x \in \mathbb{R} \) for almost all \( t \in I \).

Definition 2.3 (Darbo condition)\[11\]
Let \( \Omega \) be a nonempty subset of a Banach space \( E \) and let \( A : \Omega \rightarrow E \) be a continuous operator which transforms bounded sets onto bounded ones. We say that \( A \) satisfies the Darbo condition (with a constant \( k \geq 0 \)) with respect to a measure of noncompactness \( \mu \) if for any bounded subset \( X \) of \( \Omega \), we have \( \mu(AX) \leq k\mu(X) \).

Note that, if \( A \) satisfies the Darbo condition with \( k < 1 \), then it is called a contraction operator with respect to \( \mu \).

Theorem 2.3 (Darbo fixed point theorem)\[7\]
Let \( \Omega \) be a nonempty, bounded, closed and convex subset of \( E \) and let \( f : \Omega \rightarrow \Omega \) be a continuous transformation which is a contraction with respect to the measure of noncompactness \( \mu \), i.e. there exists a constant \( k \in [0, 1) \) such that

\[ \mu(fX) \leq k\mu(X), \]

for any nonempty subset \( X \) of \( \Omega \). Then \( f \) has at least one fixed point in the set \( \Omega \).

3 Main result

This section is devoted to discuss the solvability of the following nonlinear functional integral equation

\[ u(x) = f(x) + g_1(x, u(x)) + h_1 \left(x, g_2(x, u(x)), \int_{\mathbb{R}^N} h_2(x, y, (Qu)(y))dy\right). \tag{1} \]

Now, we will try to assume some assumptions under which we can prove our existence theorem. Assume the following conditions are satisfied:

(i) \( f \in L^p(\mathbb{R}^N) \);

(ii) \( g_i : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R} \) satisfy Carathéodory condition (i.e. measurable in \( t \) for all \( x \in \mathbb{R}^N \), and continuous in \( x \) for all \( t \in \mathbb{R} \)) and there exists a constant \( l \in [0, 1) \) and \( a_i \in L^p(\mathbb{R}^N) \) such that

\[ |g_i(x, u) - g_i(y, v)| \leq |a_i(x) - a_i(y)| + l|u - v|, \]

for any \( u, v \in \mathbb{R} \) and almost all \( x, y \in \mathbb{R}^N \) where \( i = 1, 2 \).
(iii) $h_1 : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that
\[
|h_1(x, y, z)| \leq a(x, y) + b_1|z|,
\]
for all $x, y \in \mathbb{R}^N, a \in L^q(\mathbb{R}^N)$, where $|a(x, y)| \leq a_3(x) + b_2 |y|$ where $b_1, b_2 \geq 0$ are constant and $a_3 \in L^q(\mathbb{R}^N)$.

(iv) $|h_2(x, y, u)| \leq k(x, y)\{a_4(y) + b \cdot u\}$, where $h_2 : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, $b > 0$, $a_4 \in L^p(\mathbb{R}^N)$ and $k(x, y)$ satisfies Carathéodory condition $k : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ and there exist $f_1, f_2 \in L^p(\mathbb{R}^N)$ and $f^* \in L^q(\mathbb{R}^N)$ ($\frac{1}{p} + \frac{1}{q} = 1$) such that $|k(x, y)| \leq f^*(y)f_1(x)$, for all $x, y \in \mathbb{R}^N$ and
\[
|k(x_1, y) - k(x_2, y)| \leq f^*(y)|f_2(x_1) - f_2(x_2)|.
\]

(v) The operator $Q$ is bounded linear operator and continuously maps the space $L^p(\mathbb{R}^N)$ into itself. Moreover, there exists a nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that
\[
\|Qu\|_{L^p(\mathbb{R}^N)} \leq \psi(\|u\|_{L^p(\mathbb{R}^N)})
\]
for any $u \in L^p(\mathbb{R}^N)$.

(vi) there exists a positive constant $r_0$ such that
\[
\|f\|_{L^p(\mathbb{R}^N)} + tr_0 + \|g_1(x, 0)\|_{L^p(\mathbb{R}^N)} + \|a_3\|_{L^p(\mathbb{R}^N)} + b_2 tr_0 + b_2 \|g_2(x, 0)\|_{L^p(\mathbb{R}^N)} + b_1 \|k\|_1 \|a_4\|_{L^p(\mathbb{R}^N)} + b b_1 \|K\|_1 \psi(r_0)
\]
\[
\leq r_0,\text{where}
\]
\[
(Ku)(t) = \int_{\mathbb{R}^N} k(x, y)u(y)dy
\]
and
\[
\|K\|_1 = \{\text{Sup} \|Ku\|_{L^p(\mathbb{R}^N)} : \|u\| \leq r\}
\]

Remark 3.1 The linear fredholm integral operator $K : L^p(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ is a continuous operator and $\|K\|_1 \leq \infty$.

Theorem 3.1 If the above assumptions (i)-(vi) are satisfied then the functional integral equation [1] has at least one solution in $L^p(\mathbb{R}^N)$.

Proof: First of all, we define the operator $F : L^p(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ by
\[
(Fu)(x) = f(x) + g_1(x, u(x)) + h_1 \left(x, g_2(x, u(x)), \int_{\mathbb{R}^N} h_2(x, y, (Qu)(y))dy\right),
\]
and $(GU)(x) = h_1 \left(x, g_2(x, u(x)), \int_{\mathbb{R}^N} h_2(x, y, (Qu)(y))dy\right)$. Now $Fu$ is measurable for any $u \in L^p(\mathbb{R}^N)$, we will prove that $Fu \in L^p(\mathbb{R}^N)$ for any $u \in L^p(\mathbb{R}^N)$ as $G : L^p(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ using the above conditions, we have the following inequality
\[
|(GU)(x)| = |h_1 \left(x, g_2(x, u(x)), \int_{\mathbb{R}^N} h_2(x, y, (Qu)(y))dy\right)|
\]
\[
\begin{align*}
&\leq a(x, g_2(x, u(x))) + b_1 \int_{\mathbb{R}^N} h_2(x, y, (Qu)(y))\,dy \\
&\leq a_3(x) + b_2 \mid g_2(x, u(x)) \mid + b_1 \int_{\mathbb{R}^N} h_2(x, y, (Qu)(y))\,dy \\
&\leq a_3(x) + b_2 \mid g_2(x, u(x)) - g_2(x, 0) \mid + b_2 \mid g_2(x, 0) \mid \\
&+ b_1 \int_{\mathbb{R}^N} k(x, y)a_4(y) + b \mid (Qu)(y) \mid dy \\
&\leq a_3(x) + b_2 \mid a_2(x) - a_2(x) \mid + b_2l \mid u \mid + b_2 \mid g_2(x, 0) \mid \\
&+ b_1 \int_{\mathbb{R}^N} k(x, y)a_4(y)dy + bb_1 \int_{\mathbb{R}^N} k(x, y) \mid (Qu)(y) \mid dy \\
&\leq a_3(x) + b_2l \mid u \mid + b_2 \mid g_2(x, 0) \mid + b_1 \int_{\mathbb{R}^N} k(x, y)a_4(y)dy \\
&+ b \ b_1 \int_{\mathbb{R}^N} k(x, y) \mid (Qu)(y) \mid dy.
\end{align*}
\]

Therefore from assumptions (i), (ii), \( F(u) \in L^p(\mathbb{R}^N) \) and \( F \) is well defined

\[
\begin{align*}
&\mid (Fu)(x) \mid \leq \mid f(x) \mid + \\
&\mid g_1(x, u(x)) \mid + \mid Gu \mid \\
&\leq \mid f(x) \mid + l \mid u \mid + \mid g_1(x, 0) \mid + \mid Gu \mid \\
&\mid Fu \mid_{L^p(\mathbb{R}^N)} \leq \mid f \mid_{L^p(\mathbb{R}^N)} + l \mid u \mid_{L^p(\mathbb{R}^N)} + \mid g_1(x, 0) \mid_{L^p(\mathbb{R}^N)} + \mid G \mid_{L^p(\mathbb{R}^N)} \\
&\leq \mid f \mid_{L^p(\mathbb{R}^N)} + l \mid u \mid_{L^p(\mathbb{R}^N)} + \mid g_1(x, 0) \mid_{L^p(\mathbb{R}^N)} + \mid a_3 \mid_{L^p(\mathbb{R}^N)} \\
&+ b_2l \mid u \mid_{L^p(\mathbb{R}^N)} + b_2 \mid g_2(x, 0) \mid_{L^p(\mathbb{R}^N)} \\
&+ b_1 \mid K \mid_{1} \mid a_4 \mid_{L^p(\mathbb{R}^N)} + bb_1 \mid K \mid_{1} \mid Qu \mid_{L^p(\mathbb{R}^N)} \\
&< \infty.
\end{align*}
\]

Next, we show that

\( F : B_{r_0} \to B_{r_0} \) where
Now, we show that \( w_0(FX) \leq (b_2 + 1)w_0(X) \) for any nonempty set \( X \subset B_{r_0} \). To do this, we fix arbitrary \( T > 0 \) and \( \epsilon > 0 \), let us choose \( u \in X \) and for \( x, h \in B_T \) with \( \| h \|_{\mathbb{R}^N} \leq \epsilon \), we have

\[
\frac{|(Gu)(x+h) - (Gu)(x)|}{h}
\]

\[
= \left| h_1 (x + h, g_2(x + h, u(x + h))) \int_{\mathbb{R}^N} h_2(x + h, y, (Qu)(y))dy \right|
- \left| h_1 (x, g_2(x, u(x))) \int_{\mathbb{R}^N} h_2(x, y, (Qu)(y))dy \right|
\]

\[
\leq |a_3(x + h) + b_2 | g_2(x + h, u(x + h)) | - a_3(x) - b_2 | g_2(x, u(x)) |
+ b_1 \left( \int_{\mathbb{R}^N} h_2(x + h, y, (Qu)(y))dy \right) - \left| \int_{\mathbb{R}^N} h_2(x, y, (Qu)(y))dy \right|
\]

\[
\leq |a_3(x + h) - a_3(x) | + b_2 | g_2(x + h, u(x + h)) - g_2(x, u(x)) |
+ b_1 \left( \int_{\mathbb{R}^N} k(x + h, y) |a_4(y) + b | (Qu)(y) | |dy \right)
- \left( \int_{\mathbb{R}^N} k(x, y) \right)
\]

\[
\times |a_4(y) + b | (Qu)(y) | |dy \right)
\]

\[
\leq |a_3(x + h) - a_3(x) | + b_2 | g_2(x + h, u(x + h)) - g_2(x, u(x)) |
+ b_1 \left( \int_{\mathbb{R}^N} k(x + h, y) - k(x, y) |a_4(y) + b | (Qu)(y) | |dy \right)
\]

\[
\leq |a_3(x + h) - a_3(x) | + b_2 | g_2(x + h, u(x + h)) - g_2(x + h, u(x)) |
+ b_2 | g_2(x + h, u(x)) - g_2(x, u(x)) |
+ b_1 \left( \int_{\mathbb{R}^N} f^*(y) | |f_2(x + h) - f_2(x) | |dy \right)
\times |a_4(y) + b | (Qu)(y) | |dy \right)
\]

\[
\leq |a_3(x + h) - a_3(x) | + b_2 | u(x + h) - u(x) | + b_2 | a_2(x + h) - a_2(x) |
+ b_1 \left( \int_{\mathbb{R}^N} f^*(y) | f_2(x + h) - f_2(x) | a_4(y)dy \right)
+ b \left( b_1 \left( \int_{\mathbb{R}^N} f^*(y) | f_2(x + h) - f_2(x) | \right) (Qu)(y) | |dy \right).
\]
\[ \|\tau_h Gu - Gu\|_{L^p} = \left( \int_{B_T} |(Gu)(x + h) - (Gu)(x)|^p dx \right)^{\frac{1}{p}} \]

\[ \leq \left( \int_{B_T} |a_3(x + h) - a_3(x)|^p dx \right)^{\frac{1}{p}} + lb_2 \left( \int_{B_T} |u(x + h) - u(x)|^p dx \right)^{\frac{1}{p}} + \left( \int_{B_T} b_2 |a_2(x + h) - a_2(x)|^p dx \right)^{\frac{1}{p}} + b_1 \left( \int_{B_T} \left( \int_{\mathbb{R}^N} |f^*(y)|^q |a_4(y)| |f_2(x + h) - f_2(x)|^q |a_2(y)|^q dy \right)^{\frac{1}{q}} dx \right)^{\frac{1}{p}} \]

\[ \leq \left( \int_{B_T} |a_3(x + h) - a_3(x)|^p dx \right)^{\frac{1}{p}} + lb_2 \left( \int_{B_T} |u(x + h) - u(x)|^p dx \right)^{\frac{1}{p}} + \left( \int_{B_T} b_2 |a_2(x + h) - a_2(x)|^p dx \right)^{\frac{1}{p}} + b_1 \left[ \int_{B_T} \left( \int_{\mathbb{R}^N} |f^*(y)|^q |f_2(x + h) - f_2(x)|^q |Qu(y)|^q dy \right)^{\frac{1}{q}} dx \right)^{\frac{1}{p}} \]

\[ \leq \left( \int_{B_T} |a_3(x + h) - a_3(x)|^p dx \right)^{\frac{1}{p}} + lb_2 \left( \int_{B_T} |u(x + h) - u(x)|^p dx \right)^{\frac{1}{p}} + \left( \int_{B_T} b_2 |a_2(x + h) - a_2(x)|^p dx \right)^{\frac{1}{p}} + b_1 \left( \int_{B_T} (f_2(x + h) - f_2(x)) \psi(\|u\|) dx \right)^{\frac{1}{p}} \]

\[ |(Fu)(x + h) - (Fu)(x)| \]
Thus, we obtain

\[
\begin{align*}
| f(x + h) - f(x) | + | g_1(x + h, u(x + h)) - g_1(x, u(x)) | \\
+ | (Gu)(x + h) - (Gu)(x) | \\
\leq | f(x + h) - f(x) | + | g_1(x + h, u(x + h)) - g_1(x + h, u(x)) | \\
+ | g_1(x + h, u(x)) - g(x, u(x)) | + | (Gu)(x + h) - (Gu)(x) | \\
\leq | f(x + h) - f(x) | + | a_1(x + h) - a_1(x) | + l | u(x + h) - u(x) | \\
+ | (Gu)(x + h) - (Gu)(x) | \\
\| \tau h Fu - Fu \|_{L^p} \leq (\int_{B^T} | f(x + h) - f(x) |^p \, dx)^{\frac{1}{p}} + l(\int_{B^T} | u(x + h) - u(x) |^q \, dx)^{\frac{1}{q}} \\
+ (\int_{B^T} | a_1(x + h) - a_1(x) |^q \, dx)^{\frac{1}{q}} + \| \tau h Gu - Gu \|_{L^p(B^T)} \\
\leq \| \tau h f - f \|_{L^p(B^T)} + l \| \tau h u - u \|_{L^p(B^T)} + | \tau h a_1 - a_1 |_{L^p(B^T)} \\
+ \| \tau h G - G \|_{L^p(B^T)},
\end{align*}
\]

Thus, we obtain

\[
\begin{align*}
w^T(Fx, \epsilon) & \leq w^T(f, \epsilon) + lw^T(u, \epsilon) + w^T(a_1, \epsilon) + w^T(a_3, \epsilon) + lb_2 w^T(u, \epsilon) \\
& \quad + w^T(a_2, \epsilon) + b_1 w^T(f_2, \epsilon) \| f^* \|_{L_q(\mathbb{R}^N)} \| a_4 \|_{L_p(\mathbb{R}^N)} \\
& \quad + bb_1 \| f^* \|_{L_q(\mathbb{R}^N)} \\
w^T(f_2, \epsilon) \psi(\| u \|)_{L_p(\mathbb{R}^N)}.
\end{align*}
\]

Also, we have \( w^T(f_2, \epsilon), w^T(f, \epsilon), \) and \( w^T(a_i, \epsilon) \to 0 \) as \( \epsilon \to \infty \) where \( i = 1, 2, 3 \)

then, we obtain

\[
w(FX) \leq l(b_2 + 1)w(X), \quad \text{where} \quad l(b_2 + 1) \leq 1.
\]
Next, let us fix an arbitrary number $T > 0$, then taking into account our assumptions,
for an arbitrary function $u \in X$. We have

\[
\begin{align*}
\int_{\mathbb{R}^N} B^T |(F u)(x)|^p dx & \leq \left( \int_{\mathbb{R}^N \setminus B^T} |f(x)|^p dx \right)^{\frac{1}{p}} + \left( \int_{\mathbb{R}^N \setminus B^T} |g_1(x, u(x))|^p dx \right)^{\frac{1}{p}} \\
& \quad + \left( \int_{\mathbb{R}^N \setminus B^T} \left| h_1(x, g_2(x, u(x)), f_{\mathbb{R}^N} h_2(x, y, (Qu)(y)) dy \right| dx \right)^{\frac{1}{p}} \\
& \leq \left( \int_{\mathbb{R}^N \setminus B^T} |f(x)|^p dx \right)^{\frac{1}{p}} + \left( \int_{\mathbb{R}^N \setminus B^T} |g_1(x, u(x)) - g_1(x, 0)|^p dx \right)^{\frac{1}{p}} \\
& \quad + \left( \int_{\mathbb{R}^N \setminus B^T} |g_1(x, 0)|^p dx \right)^{\frac{1}{p}} \\
& \quad + \left( \int_{\mathbb{R}^N \setminus B^T} |a_3(x)|^p dx \right)^{\frac{1}{p}} + b_2 \left( \int_{\mathbb{R}^N \setminus B^T} |u(x)|^p dx \right)^{\frac{1}{p}} + b_2 \left( \int_{\mathbb{R}^N \setminus B^T} |g_2(x, 0)|^p dx \right)^{\frac{1}{p}} \\
& \quad + b_1 \left( \int_{\mathbb{R}^N \setminus B^T} |\int_{\mathbb{R}^N} |k(x, y)| \times |a_4(y) + b(Qu)(y)||dy| dx \right)^{\frac{1}{p}}
\end{align*}
\]

\[
\begin{align*}
& \leq \left( \int_{\mathbb{R}^N \setminus B^T} |f(x)|^p dx \right)^{\frac{1}{p}} + \left( \int_{\mathbb{R}^N \setminus B^T} |u(x)|^p dx \right)^{\frac{1}{p}} + \left( \int_{\mathbb{R}^N \setminus B^T} |g_1(x, 0)|^p dx \right)^{\frac{1}{p}} \\
& \quad + \left( \int_{\mathbb{R}^N \setminus B^T} |a_3(x)|^p dx \right)^{\frac{1}{p}} + b_2 \left( \int_{\mathbb{R}^N \setminus B^T} |u(x)|^p dx \right)^{\frac{1}{p}} \\
& \quad + b_2 \left( \int_{\mathbb{R}^N \setminus B^T} |g_2(x, 0)|^p dx \right)^{\frac{1}{p}} + b_1 \left( \int_{\mathbb{R}^N \setminus B^T} \int_{\mathbb{R}^N} |k(x, y)| \times |a_4(y) + b(Qu)(y)||dy| dx \right)^{\frac{1}{p}}
\end{align*}
\]
\[
+ bb_1 \left( \int_{\mathbb{R}^N \setminus B_T} \left( \int_{\mathbb{R}^N} |k(x,y)|^q (Qu)(y) \, dy \right)^\frac{p}{q} \, dx \right)^\frac{1}{p} 
\]

\[
\leq \|f\|_{L^p(\mathbb{R}^N \setminus B_T)} + l \|u\|_{L^p(\mathbb{R}^N \setminus B_T)} + \|g_1(\cdot, 0)\|_{L^p(\mathbb{R}^N \setminus B_T)} 
+ \|a_3\|_{L^p(\mathbb{R}^N \setminus B_T)} + b_2 l \|u\|_{L^p(\mathbb{R}^N \setminus B_T)} + b_2 \|g_2(\cdot, 0)\|_{L^p(\mathbb{R}^N \setminus B_T)} 
+ b_1 \|f^*\|_{L^q(\mathbb{R}^N)} \cdot \|f_1\|_{L^p(\mathbb{R}^N \setminus B_T)} \cdot (\|a_4\|_{L^p(\mathbb{R}^N \setminus B_T)} + b \psi(\|u\|)_{L^p(\mathbb{R}^N)}).
\]

Also we have \(\|f\|_{L^p(\mathbb{R}^N \setminus B_T)}, \|g_i(\cdot, 0)\|_{L^p(\mathbb{R}^N \setminus B_T)},\)
\(\|f_1\|_{L^p(\mathbb{R}^N \setminus B_T)}, \|a_3\|_{L^p(\mathbb{R}^N \setminus B_T)} \to 0\)
as \(T \to \infty\) where \(i = 1, 2\)
and hence we obtain that
\[
d(FX) \leq l(b_2 + 1)d(X). \tag{16}
\]

Consequently we infer from equation 13, 16
\[
w_0(FX) \leq l(b_2 + 1)w_0(X),
\]
so, the operator \(F\) satisfies all conditions of Darbo fixed point theorem, which enables us to deduce that \(F\) has at least one solution in \(L^p(\mathbb{R}^N)\). Thus the proof is finished.

Next, we will need the following theorem that help us in a coming example.

**Theorem 3.2** \([7]\)

Let \(\Omega \subseteq \mathbb{R}^N\) be a measure space and suppose \(k : \Omega \times \Omega \to \mathbb{R}\) is a measurable function for which there is constant \(C > 0\) such that

\[
\int_{T} |k(x, y)| \, dx \leq C \quad \text{a.e. } y \in \Omega
\]

and

\[
\int_{T} |k(x, y)| \, dy \leq C \quad \text{a.e. } x \in \Omega.
\]

If \(K : L^p(\Omega) \to L^p(\Omega)\) is defined by

\[
(Kf)(x) = \int_{\Omega} f(y) \, dy,
\]
then \(K\) is a bounded and continuous operator and \(\|K\|_1 \leq C\).
Example: consider the integral equation

\[(y_2 1 + y_1^2 + 2e^{-|x_1|}u(x))dx,\]

where

\[x = (x_1, x_2) \in \mathbb{R}^2,\]

and \(\|x\|\) is the Euclidean norm. We study the solvability of this integral equation in the space \(L^p(\mathbb{R}^2)\) for \(p, q > 2\).

Let \(f(x) = e^{-x^2}\), \(g_1(x, u(x)) = \frac{\sin u}{\|x\|+4}\),

\[h_2(x, y, (Qu)(y)) = e^{-\left(\frac{|x_1|+|y_1|}{1 + |x_1| + |y_1|}\right)} + 2e^{-|u(x)|}u(x),\]

\[a(x, y) = e^{-x^2} + \frac{\sin u}{\|x\|+4}\]

with \(b_1 = \frac{1}{8}\), \(a_3(x) = e^{-x^2}\) where \(a_3 \in L^p(\mathbb{R}^2)\) such that \(b_2 = 1\), \(g_2(x, u(x)) = \frac{\sin u}{\|x\|+4}\).

Hence the norm

\[\|f\|_{L^p(\mathbb{R}^2)} = \left(\frac{\pi}{p}\right)^\frac{1}{p}.\]

Next the functions \(g_i(x, u(x)), i = 1, 2\) satisfy the assumption(ii) with \(a_i(x) = \frac{1}{\|x\|+4}, l = \frac{1}{8}\), indeed

\[|g_i(x, u) - g_i(y, v)| = \left|\frac{\sin u}{\|x\|+4} - \frac{\sin v}{\|y\|+4}\right|\]

\[\leq \left|\frac{1}{\|x\|+4} - \frac{1}{\|y\|+4}\right| |\sin u| + \frac{1}{\|y\|+4} |\sin u - \sin v|\]

\[\leq \left|\frac{1}{\|x\|+4} - \frac{1}{\|y\|+4}\right| + \frac{1}{4} |u - v|\]

\[= |a_i(x) - a_i(y)| + l |u - v|\]

where \(a_i(x) \in L_p(\mathbb{R}^2)\) with norm

\[\|a_i\|_{L^p(\mathbb{R}^2)} = \left(\frac{4\pi(2)^{1-p}}{(p-1)(p-2)}\right)^\frac{1}{p},\]

where \(a_4 = \frac{y_2}{1 + y_1}\), with \(\|a_4\|_{L^p(\mathbb{R}^2)} = 0\), also

\[k(x, y) = e^{-\left(\frac{|x_1|+|y_1|}{1 + |x_1| + |y_1|}\right)} + 2e^{-|u(x)|}u(x),\]

\(f^*(x) = \frac{e^{-|x_1|}}{(1 + |x_1|)^2}, \ f_1(x) = f_2(x) = \frac{e^{-|x_1|}}{(1 + |x_1|)^2}\) we see that \(f_1, f_2 \in L^p(\mathbb{R}^2), f^* \in L_q(\mathbb{R}^2)\). Also we have

\[\int_{\mathbb{R}^2} |k(x, y)| \ dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\left(\frac{|x_1|+|y_1|}{1 + |x_1| + |y_1|}\right)} dx_1 dx_2 \leq \frac{1}{3},\]

\[\int_{\mathbb{R}^2} |k(x, y)| \ dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\left(\frac{|x_1|+|y_1|}{1 + |x_1| + |y_1|}\right)} dy_1 dy_2 \leq \frac{2}{9},\]
and thus from the theorem \[ \| K \|_1 \leq \frac{1}{3} \]
furthermore \( b = 2 \), \( Q(u)(x) = e^{-|u(x)|} u(x) \) satisfies the assumption with \( \psi(t) = t \).
Finally, the inequality from assumption (vi) has the form
\[
\| f \|_{L^p(\mathbb{R}^2)} + b_0 + \| g_1(x,0) \|_{L_1(\mathbb{R}^2)} + \| a_3 \|_{L_1(\mathbb{R}^2)} + b_1 \| K \|_1 \| a_4 \|_{L_1(\mathbb{R}^2)} + b_1 \| K \|_1 \psi(r_0) \\
\leq r_0,
\]
\[
2(\pi p)^{\frac{1}{p}} + \frac{1}{2} r_0 + \left(\frac{1}{4}\right) \left(\frac{1}{3}\right) r_0 \leq r_0.
\]
Thus, for the number \( r_0 = \left(\frac{24}{5}\pi p\right)^{\frac{3}{5}} \)
\( \times \left(\frac{2}{p}\right)^{\frac{2}{5}} \). Hence all the assumptions are satisfied and so, Eq.(3.4) has at least one solution in \( L^p(\mathbb{R}^2) \).

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