ON REPRESENTATIONS OF SEMIDIRECT PRODUCTS OF A COMPACT QUANTUM GROUP WITH A FINITE GROUP

HUA WANG

Abstract. We study unitary representations of semidirect products of a compact quantum group with a finite group. We give a classification of all irreducible unitary representations, a description of the conjugate representation of irreducible unitary representations in terms of this classification, and the fusion rules for the semidirect product.

Contents

1. Introduction 1
2. Semidirect product of a compact quantum group with a finite group 3
3. A first look at unitary representations of $G \rtimes \Lambda$ 5
4. Principal subgroups of $G \rtimes \Lambda$ 9
5. Induced representations of principal subgroups 11
6. Some character formulae 14
7. Dimension of the intertwiner space of induced representations 17
8. The $C^*$-tensor category $\mathcal{CSR}_{\Lambda_0}$ 19
9. Group actions and projective representations 22
10. Pure, stable, distinguished CSRs and representation parameters 24
11. Distinguished representation parameters and distinguished representations 27
12. Density of matrix coefficients of distinguished representations 29
13. Classification of irreducible representations of $G \rtimes \Lambda$ 31
14. The conjugate representation of distinguished representations 32
15. The incidence numbers 34
16. Fusion rules 37
Acknowledgment 38
References 38

1. Introduction

When studying representations of a group $G$, one often wishes to get significant information using representations of some subgroups of $G$. As a trivial example, the study of representations of a direct product $G \times H$ of groups of $G$ and $H$ is easily reduced to the study of representations of $G$ and $H$ separately. Contrary to the direct product case, when one replaces direct products with semidirect products, the situation quickly becomes much more complicated. To get a taste of this complication, [Ser77, §8.2] treats representations of a semidirect product $G \rtimes H$ in the special case where $G$ is abelian and $G, H$ finite.

In the setting of locally compact groups and their unitary representations, via the theories of system of imprimitivity, induced representations, projective representations (a.k.a. ray representations), etc., George Mackey developed a heavy machinery of techniques, often referred as Mackey’s analysis, Mackey’s machine or the small group method, to attack such kind of problems in the 20th century. Subsequent works based on Mackey’s analysis emerge rapidly, making it one of the most powerful tools to study unitary representations of locally compact groups. For an introduction of this development, we refer the reader to [Mac58; Mac49; FD88; KT13] among large volumes of literature on this subject.

The author’s own motivation of entering this subject comes from the joint work [FW18] with Pierre Fima. In [FW18], we give a systematic study of the permanence of property (RD) and polynomial growth of the dual of a bicrossed product of a matched pair consisting of a second countable compact group and a countable discrete group, which is a noncommutative, noncocommutative compact quantum group. The
question of constructing examples of nontrivial bicrossed products with or without (RD) leads us to study closely the representation theory of semidirect products $G \rtimes \Lambda$ of a compact group $G$ with a finite group $\Lambda$. More precisely, as required by the length functions relevant to these properties, we need a classification of all irreducible unitary representations of $G \rtimes \Lambda$, the conjugate (which is also the contragredient since classical groups are of Kac type) of irreducible representations in terms of this classification, and most importantly, the fusion rules of $G \rtimes \Lambda$, i.e. how the tensor product of two irreducible representations decomposes into a direct sum of irreducible representations. While the first two questions can be settled using Mackey’s machine, the fusion rules, however, to the best of the knowledge, is never calculated in the literature.

This paper treats these questions in the more general setting of semidirect products of the form $G \rtimes \Lambda$, where $G$ is a compact quantum group and $\Lambda$ a finite group. However, instead of using systems of imprimitivity, we introduce the notion of representation parameters, which appears naturally when we try to analyze the rigid $C^*$-tensor category $\text{Rep}(G \rtimes \Lambda)$. A representation parameter is a triple $(u, V, v)$, where $u$ is an irreducible representation of $G$ on some finite dimensional Hilbert space $\mathcal{H}$, $V$ is a unitary projective representation of a certain subgroup $\Lambda_0$ of $\Lambda$ on the same space $\mathcal{H}$ that is covariant with $u$ in a certain sense, and $v$ is a unitary projective representation of the same $\Lambda_0$ on some other finite dimensional Hilbert space, such that $V$ and $v$ have opposing cocycles.

Here is an informal summary of the main results of this paper.

(A) Up to equivalence, every irreducible unitary representation of $G \rtimes \Lambda$ is parameterized by an equivalent class of representation parameters (Theorem 13.1);

(B) The conjugate of irreducible representation of $G \rtimes \Lambda$ parameterized by a representation parameter $(u, V, v)$ is itself parameterized by the conjugate of $(u, V, v)$ (Theorem 14.5);

(C) The fusion rules of $G \rtimes \Lambda$ is expressed as a sum of incidence numbers, all of which can be calculated using unitary projective representations of some suitable subgroup of $\Lambda$ through a reduction procedure (Theorem 16.1).

Thus (A) and (B) maybe viewed as the quantum analogue of the corresponding result of Mackey’s analysis in the classical case of groups, while (C) is new even in the case where $G$ is another finite group.

We now describe the organization of this paper. The semidirect product as a compact quantum group is constructed in §2 using the algebraic method as a Hopf-$*$-algebra with a Haar integral (an algebraic compact quantum group) instead of the usual analytical one as a Woronowicz algebra, which among other benefits, shows clearly what the polynomial ring of $G \rtimes \Lambda$ is before the study of representations of $G \rtimes \Lambda$. We then develop the relevant representation theory, especially the theory of induced representations for $G \rtimes \Lambda$ from $G \rtimes \Lambda_0$, where $\Lambda_0$ is a subgroup of $\Lambda$, in the next five sections. The category $\text{CSR}_{\Lambda_0}$ of covariant systems of representations (CSRs) subordinate to a subgroup $\Lambda_0$ of $\Lambda$ is introduced in §8, as a convenient copy of $\text{Rep}(G \rtimes \Lambda_0)$. The passage from $\text{Rep}(G \rtimes \Lambda_0)$ to $\text{CSR}_{\Lambda_0}$ is more than tautological, as a certain family of CSRs, called stably pure, has a nice structure which can be directly constructed via projective representations of $\Lambda_0$ and the underlying dynamics of the semidirect product. We digress a little in §9 to prepare for such a structural result, and study these structures in §10. Along the way, we will see that representation parameters appear naturally, and a special class of representation parameters, called distinguished, emerges as they all parameterize irreducible unitary representations of $G \rtimes \Lambda$ via induction. Moreover, in §11, we show that one can easily determine which distinguished representation parameters parameterize equivalent representations. To this point, one naturally wonders whether all irreducible representations of $G \rtimes \Lambda$ is parameterized by a distinguished representation parameter. This is answered in the affirmative in §12, yielding our first main theorem, the classification result in §11. Starting from the dual object in $\text{CSR}_{\Lambda_0}$, §14 calculates the “correct” conjugate of a representation parameter and establishes our second main theorem on the conjugation of irreducible unitary representations. Finally, the fusion rules of $G \rtimes \Lambda$ requires a somewhat tedious calculation involving various characters and Haar states for various representations of various subgroups of the form $G \rtimes \Lambda_0$, where $\Lambda_0$ is a subgroup of $\Lambda$. In §15, we present the more structural part of this calculation, and use these results to finishes the calculation in §16.

We conclude the introduction by making some conventions in this paper. All representations, and projective representations are finite dimensional. Most of them are unitary, but the contragredient of a unitary representation may not be unitary when the compact quantum group is not of Kac-type. As a compromise, we assume all representations are over a finite dimensional Hilbert space instead of a mere complex vector space. Terminologies and notations concerning compact quantum groups and $C^*$-tensor categories are largely consistent with those in [NT13]. We also use freely the Peter-Weyl theory for projective representations of finite groups as presented in [Che15]. Finally, throughout this paper,
we fix a compact quantum group \( G = (A, \Delta) \), a finite group \( \Lambda \), and an antihomomorphism of groups \( \alpha^*: \Lambda \to \text{Aut}(G) \).

2. SEMIDIRECT PRODUCT OF A COMPACT QUANTUM GROUP WITH A FINITE GROUP

We describe here the construction of the compact quantum semidirect product \( G \rtimes \Lambda \) to fix the notations.

Let \( G = (A, \Delta) \) be a compact quantum group, \( \Lambda \) a finite group. An action of \( \Lambda \) on \( G \) via quantum automorphisms is an antihomomorphism \( \alpha^*: \Lambda \to \text{Aut}(G) \). One can then form the semi-direct \( G \rtimes \Lambda \), or simply \( G \rtimes \Lambda \) if the action \( \alpha^* \) is clear from the context, which is again a compact quantum group. The underlying \( C^* \)-algebra \( A \) of \( G \rtimes \Lambda \) is \( A \otimes C(\Lambda) \), and the comultiplication \( \tilde{\Delta} \) on \( A \) is determined by

\[
\tilde{\Delta}(a \otimes \delta_r) = \sum_{s \in \Lambda} [\{\text{id} \otimes \alpha^*_s\}(\Delta(a))]_{12} (\delta_s \otimes \delta_{s^{-1}r})_{24} \in A \otimes C(\Lambda) \otimes A \otimes C(\Lambda)
\]

for any \( a \in A \) and \( r \in \Lambda \).

Since \( \text{Pol}(G) \otimes C(\Lambda) \) is dense in \( A \otimes C(\Lambda) \), in order to prove that \( G \rtimes \Lambda \) is indeed a compact quantum group, it suffices to make \( (\text{Pol}(G) \otimes C(\Lambda), \tilde{\Delta}) \) a Hopf \( * \)-algebra with a Haar state, where the comultiplication is still defined by (2.1) (which is easily seen to be well-defined). Let \( \epsilon, S \) be the counit and the antipode for the Hopf \( * \)-algebra \( \text{Pol}(G) \) respectively, define

\[
\tilde{\epsilon}: \text{Pol}(G) \otimes C(\Lambda) \to C
\]

\[
\sum_{r \in \Lambda} x_r \otimes \delta_r \mapsto \epsilon(1)
\]

and

\[
\tilde{S}: \text{Pol}(G) \otimes C(\Lambda) \to \text{Pol}(G) \otimes C(\Lambda)
\]

\[
\sum_{r \in \Lambda} x_r \otimes \delta_r \mapsto \sum_{r \in \Lambda} \alpha^*_s(S(x_r)) \otimes \delta_{r^{-1}} = \sum_{r \in \Lambda} (\alpha^*_s(x_r)^{-1}) \otimes \delta_r.
\]

Since \( \epsilon \) is a \( * \)-morphism of algebras, so is \( \tilde{\epsilon} \). Moreover, let \( e \) be the neutral element of the group \( \Lambda \), for any \( x \in \text{Pol}(G) \) and \( r \in \Lambda \), we have

\[
(\tilde{\epsilon} \otimes \text{id})\tilde{\Delta}(x \otimes \delta_r) = (\tilde{\epsilon} \otimes \text{id}) \sum_{s \in \Lambda} x_{(1)} \otimes \delta_s \otimes \alpha^*_s(x_{(2)}) \otimes \delta_{s^{-1}r} = \sum_{s \in \Lambda} \epsilon(x_{(1)})(\alpha^*_s(x_{(2)}) \otimes \delta_{s^{-1}r}) = \sum_{s \in \Lambda} (\alpha^*_s(x_{(2)}) \otimes \delta_{s^{-1}r}) = (\text{id} \otimes \epsilon) \Delta(x \otimes \delta_r).
\]

Hence \( \tilde{\epsilon} \) is indeed a counit for \( \tilde{\Delta} \). Let \( m: \text{Pol}(G) \otimes \text{Pol}(G) \to \text{Pol}(G) \) be the multiplication map, and \( \tilde{m} \) the multiplication map on \( \text{Pol}(G) \otimes C(\Lambda) \), then

\[
\tilde{m}(\tilde{S} \otimes \text{id})\tilde{\Delta}(x \otimes \delta_r) = \tilde{m}(\tilde{S} \otimes \text{id}) \sum_{s \in \Lambda} x_{(1)} \otimes \delta_s \otimes \alpha^*_s(x_{(2)}) \otimes \delta_{s^{-1}r} = \sum_{s \in \Lambda} \alpha^*_s(S(x_{(1)})) \otimes \delta_{s^{-1}} \otimes \alpha^*_s(x_{(2)}) \otimes \delta_{s^{-1}r} = \sum_{s \in \Lambda} [m(S \otimes \text{id})(\alpha^*_s \otimes \alpha^*_s) \Delta(x)] \otimes \delta_{s^{-1}} \cdot \delta_{s^{-1}r} = \delta_{e,r} \sum_{s \in \Lambda} m(S \otimes \text{id})\Delta(\alpha^*_s(x_{(2)})) \otimes \delta_{s^{-1}} = \delta_{e,r} \sum_{s \in \Lambda} \epsilon(\alpha^*_s(x_{(2)})) 1_A \otimes \delta_{s^{-1}} = \delta_{e,r} \epsilon(x) 1_A \otimes \sum_{s \in \Lambda} \delta_{s^{-1}} = \delta_{e,r} \epsilon(x) 1_A \otimes 1_{C(\Lambda)} = \tilde{\epsilon}(x \otimes \delta_r) 1_A \otimes 1_{C(\Lambda)}.
\]

Similarly, since for any \( s \in \Lambda \),

\[
\alpha^*_s, S, \alpha^*_s = \alpha^*_{s^{-1}}, \alpha^*_s S = \alpha^*_{s^{-1}} S,
\]

3
we have
\[
\tilde{m}(\text{id} \otimes \tilde{S}) \Delta(x \otimes \delta_r) = \tilde{m}(\text{id} \otimes \tilde{S}) \sum_{s \in \Lambda} \sum_{r \in \Lambda} x_{(1)} \otimes \delta_s \otimes \alpha_s^*(x_{(2)}) \otimes \delta_{s^{-1}r} \\
= \tilde{m} \sum_{s \in \Lambda} \sum_{r \in \Lambda} x_{(1)} \otimes \delta_s \otimes (\alpha_{s^{-1}r}^* S \alpha_{s}^*)(x_{(2)}) \otimes \delta_{r^{-1}s} \\
= \tilde{m} \sum_{s \in \Lambda} \sum_{r \in \Lambda} x_{(1)} \otimes \delta_s \otimes (\alpha_s^* \delta_{s^{-1}r}) \otimes \delta_{r^{-1}s} \\
= \delta_{e,r} \sum_{s \in \Lambda} x_{(1)} [S(x_{(2)})] \otimes \delta_s = \delta_{e,r} \sum_{s \in \Lambda} \epsilon(x) 1_A \otimes \delta_s \\
= \delta_{e,r} \epsilon(x) 1_A \otimes 1_{C(\Lambda)} = \tilde{c}(x \otimes \delta_r).
\]

Therefore, \( \tilde{S} \) is indeed an antipode for \( (\text{Pol}(G) \otimes C(\Lambda), \tilde{\Delta}) \).

It remains to establish the Haar state on the Hopf \( * \)-algebra \( \text{Pol}(G) \). Suppose \( h: A \to \mathbb{C} \) is the Haar state on \( G \), define

\[
\tilde{h}: \text{Pol}(G) \otimes C(\Lambda) \to \mathbb{C} \\
\sum_r x_r \otimes \delta_r \mapsto |\Lambda|^{-1} \sum_{r \in \Lambda} h(x_r).
\]

It is obvious that \( \tilde{h} \) is a state. For any \( x \in \text{Pol}(G) \), \( r \in \Lambda \),

\[
(\tilde{h} \otimes \text{id}) \tilde{\Delta}(x \otimes \delta_r) = |\Lambda|^{-1} \sum_{s \in \Lambda} \sum_{r \in \Lambda} h(x_{(1)}) \alpha_s^*(x_{(2)}) \otimes \delta_{s^{-1}r} \\
= |\Lambda|^{-1} \sum_{s \in \Lambda} \alpha_s^*(\sum_{r \in \Lambda} h(x_{(1)}) x_{(2)}) \otimes \delta_{s^{-1}r} \\
= |\Lambda|^{-1} \sum_{s \in \Lambda} \alpha_s^*(h(x) 1_A) \otimes \delta_{s^{-1}r} \\
= |\Lambda|^{-1} h(x) \sum_{s \in \Lambda} 1_A \otimes \delta_{s^{-1}r} \\
= \tilde{h}(x \otimes \delta_r) 1_A \otimes 1_{C(\Lambda)}.
\]

The uniqueness of the Haar state implies that \( h \circ \alpha_s^* = h \) for any \( s \in \Lambda \), hence

\[
(\text{id} \otimes \tilde{h}) \tilde{\Delta}(x \otimes \delta_r) = |\Lambda|^{-1} \sum_{s \in \Lambda} \sum_{r \in \Lambda} x_{(1)} h(\alpha_s^*(x_{(2)})) \otimes \delta_s \\
= |\Lambda|^{-1} \sum_{s \in \Lambda} \sum_{r \in \Lambda} x_{(1)} h(x_{(2)}) \otimes \delta_s \\
= |\Lambda|^{-1} h(x) 1_A \otimes \sum_{s \in \Lambda} \delta_s \\
= \tilde{h}(x \otimes \delta_r) 1_A \otimes 1_{C(\Lambda)}.
\]

Therefore, \( \tilde{h} \) is indeed the Haar state on \( (\text{Pol}(G) \otimes C(\Lambda), \tilde{\Delta}) \). So far, we’ve established that \( (\text{Pol}(G) \otimes C(\Lambda), \tilde{\Delta}) \) is a compact algebraic quantum group (cf. [Tim08, chapter 3]).

Now the density of \( \text{Pol}(G) \otimes C(\Lambda) \) in \( A \otimes C(\Lambda) \) implies that \( (A \otimes C(\Lambda), \tilde{\Delta}) \) is indeed a compact quantum group, with

\[
\text{Pol}(G \rtimes \Lambda) = \text{Pol}(G) \otimes C(\Lambda),
\]

and Haar state (which we still denote by \( \tilde{h} \))

\[
\tilde{h}: A \otimes C(\Lambda) \to \mathbb{C} \\
\sum_r x_r \otimes \delta_r \mapsto |\Lambda|^{-1} \sum_{r \in \Lambda} h(x_r).
\]

Furthermore, the counit \( \tilde{\epsilon} \) and the antipode \( \tilde{S} \) of the Hopf \( * \)-algebra \( \text{Pol}(G \rtimes \Lambda) \) are given by (2.2) and (2.3) respectively (cf. [Tim08, §5.4.2]).
Definition 2.1. Using the above notations, the compact quantum group $(A \otimes C(\Lambda), \Delta)$ is called the semidirect product of $G$ and $\Lambda$ with respect to the action $\alpha^*$, and is denoted by $G \rtimes_{\alpha^*} \Lambda$, or simply $G \rtimes \Lambda$ if the underlying action $\alpha^*$ is clear from context.

Remark 1. There is a faster way of establishing $G \rtimes \Lambda$ as a compact quantum group, which we refer to as the analytic approach. Namely, one might use (2.1) directly to define a comultiplication on the $C^*$-algebra $A \otimes C(\Lambda)$ and show that this comultiplication satisfy the density condition in the definition of a compact quantum group in the sense of Woronowicz (cf. [Wor98]). We prefer the more algebraic approach presented above as it provides more insight and indication for our purpose of studying representations of $G \rtimes \Lambda$. As an illustration, from our treatment, one knows immediately that $\text{Pol}(G \rtimes \Lambda) = \text{Pol}(G) \rtimes \Lambda$, a fact which is not immediately clear from the faster analytic approach.

Remark 2. When $G$ comes from a genuine compact group $G$, it is easy to check via Gelfand theory that the antihomomorphism $\alpha^*: \Lambda \to \text{Aut}(G)$ comes from the pull-back of a group morphism $\alpha: \Lambda \to \text{Aut}(G)$, and $G \rtimes \Lambda$ is exactly the compact group $G \rtimes_{\alpha^*} \Lambda$ viewed as a compact quantum group.

In treating the dual objects of some rigid $C^*$-tensor to be presented later, the following result will be useful.

Proposition 2.2. The compact quantum group $G \rtimes \Lambda$ is of Kac type if and only if $G$ is of Kac type.

Proof. Of the many equivalent characterization for a compact quantum group to be of Kac type (see e.g. [NT13, §1.7]), we use the fact that such a quantum group is of Kac type if and only if the antipode of its polynomial algebra preserves adjoints. The proposition now becomes trivial in view of (2.3).

3. A First Look at Unitary Representations of $G \rtimes \Lambda$

A unitary representation $U$ of a classic compact semidirect product $G \rtimes \Lambda$ is determined by the restrictions $U_G$ and $U_\Lambda$ on the subgroups $G \times 1_\Lambda \simeq G$ and $1_G \times \Lambda \simeq \Lambda$ respectively. It is easy to see that for any

\[(3.1) \quad \forall g \in G, r \in \Lambda, \quad U_G(\alpha_s(g))U_\Lambda(r) = U((1, r)(g, 1)) = U((1, r)((1, r)g)) = U_G(r)U_G(g).\]

Conversely, suppose $U_G, U_\Lambda$ are unitary representations on the same Hilbert space of $G$ and $\Lambda$ respectively, if (3.1) is satisfied, then $U_G(r) = U_G(g)U_\Lambda(r)$ defines a unitary representation of $G \rtimes \Lambda$. When $G$ is replaced by a general compact quantum group $G$, even though the “elements” of $G$ are no longer available, one can still establish a reasonable quantum analogue. We begin with a simple lemma.

Lemma 3.1. Let $\epsilon$ be the counit for $\text{Pol}(G)$, $\epsilon_\Lambda$ the counit for $C(\Lambda)$, then $\epsilon \otimes \text{id}_{C(\Lambda)}$ is a Hopf $*$-algebra morphism from $\text{Pol}(G) \otimes C(\Lambda)$ onto $C(\Lambda)$, and $\text{id}_{\text{Pol}(G)} \otimes \epsilon_\Lambda$ is a Hopf $*$-algebra morphism from $\text{Pol}(G) \otimes C(\Lambda)$ onto $\text{Pol}(G)$.

Proof. Since the antipodes are $*$-morphisms of involutive algebras, it suffices to check that both morphisms preserve comultiplication.

Take any $a \in \text{Pol}(G)$, $r \in \Lambda$, we have

\[
[\epsilon \otimes \text{id}]\Delta_G \Delta_\Lambda(a \otimes \delta_r) = \sum_{s \in \Lambda} \epsilon(\alpha_s^*(a_{\langle 2 \rangle}))\delta_s \otimes \delta_{s^{-1} r},
\]

\[
= \sum_{s \in \Lambda} \epsilon(\alpha_s)\epsilon(\alpha_{\langle 2 \rangle})\delta_{s} \otimes \delta_{s^{-1} r},
\]

\[
= \sum_{s \in \Lambda} \epsilon(a)\delta_{s} \otimes \delta_{s^{-1} r} = \Delta_\Lambda(\epsilon \otimes \text{id})(a \otimes \delta_r),
\]

where $\Delta_\Lambda$ is the comultiplication for $\Lambda$ viewed as a compact quantum group. Thus $\epsilon \otimes \text{id}$ preserves comultiplication. On the other hand, note that $\epsilon_\Lambda(\delta_r) = \delta_{r^{-1} 1_N}$, we have

\[
[(\text{id} \otimes \epsilon_\Lambda) \otimes (\text{id} \otimes \epsilon_\Lambda)]\Delta_G \Delta_\Lambda(a \otimes \delta_r)
\]

\[
= \sum_{s \in \Lambda} \delta_{s, 1_N} \delta_{s^{-1} r, 1_N} \sum_{a_{\langle 1 \rangle} \otimes \alpha_{\langle 2 \rangle}^*(a_{\langle 2 \rangle})}
\]

\[
= \delta_{r, 1_N} \sum_{a_{\langle 1 \rangle} \otimes a_{\langle 2 \rangle}}
\]

\[
= \delta_{r, 1_N} \Delta(a) = \Delta[(\text{id} \otimes \epsilon_\Lambda)(a \otimes \delta_r)].
\]

Thus $\text{id} \otimes \epsilon_\Lambda$ preserves comultiplication too.
Let $U \in \mathcal{B}(\mathcal{H}) \otimes \text{Pol}(G) \otimes C(\Lambda)$ be a finite dimensional unitary representation of $G \times \Lambda$. Define the unitaries

$$\text{Res}_G(U) := (id_{\mathcal{B}(\mathcal{H})} \otimes \text{id}_{\text{Pol}(G)} \otimes \epsilon)(U) \in \mathcal{B}(\mathcal{H}) \otimes \text{Pol}(G),$$

and

$$\text{Res}_\Lambda(U) := (id_{\mathcal{B}(\mathcal{H})} \otimes \epsilon \otimes \text{id}_{C(\Lambda)})(U) \in \mathcal{B}(\mathcal{H}) \otimes C(\Lambda).$$

Then by Lemma 3.1, we see that $\text{Res}_G(U)$ is a finite dimensional unitary representation of $G$ and $\text{Res}_\Lambda(U)$ a finite dimensional unitary representation of $\Lambda$. We call $\text{Res}_G(U)$ (resp. $\text{Res}_\Lambda(U)$) the restriction of $U$ to $G$ (resp. $\Lambda$). For reasons to be explained presently, we also write $U_G$ for $\text{Res}_G(U)$ and $U_\Lambda$ for $\text{Res}_\Lambda(U)$.

**Proposition 3.2.** Using the above notations, we have

$$\forall r_0 \in \Lambda, \quad (U_G(r_0) \otimes 1_A)U_G = [(id_{\mathcal{B}(\mathcal{H})} \otimes \epsilon)(U_G)](U_G(r_0) \otimes 1_A)$$

in $\mathcal{B}(\mathcal{H}) \otimes \text{Pol}(G)$. Moreover,

$$U = (U_G)_{12}(U_\Lambda)_{13} \in \mathcal{B}(\mathcal{H}) \otimes \text{Pol}(G) \otimes C(\Lambda).$$

Conversely, suppose $U_G$ and $U_\Lambda$ are finite dimensional unitary representations of $G$ and $\Lambda$ respectively on the same Hilbert space $\mathcal{H}$, if $U_G$ and $U_\Lambda$ satisfy (3.2), then (3.3) defines a finite dimensional unitary representation $U$ of $G \times \Lambda$ on $\mathcal{H}$. Moreover,

$$(3.4a) \quad U_G = (id_{\mathcal{B}(\mathcal{H})} \otimes \text{id}_{\text{Pol}(G)} \otimes \epsilon)(U) \in \mathcal{B}(\mathcal{H}) \otimes \text{Pol}(G),$$

$$(3.4b) \quad U_\Lambda = (id_{\mathcal{B}(\mathcal{H})} \otimes \epsilon \otimes \text{id}_{C(\Lambda)})(U) \in \mathcal{B}(\mathcal{H}) \otimes C(\Lambda).$$

**Proof.** Let $d = \dim \mathcal{H}$, and fix a Hilbert basis $(e_1, \ldots, e_d)$ for $\mathcal{H}$. Let $(e_{ij}, i, j = 1, \ldots, d)$ be the corresponding matrix units (i.e. $e_{ij} \in \mathcal{B}(\mathcal{H})$) is characterized by $e_{ij}(e_k) = \delta_{jk}e_i$). Then there is a unique $U_{ij} \in \text{Pol}(G) \otimes C(\Lambda)$ for each pair of $i, j$, such that

$$U = \sum_{i,j} e_{ij} \otimes U_{ij},$$

with each $U_{ij}$ decomposed further as $U_{ij} = \sum_{r \in \Lambda} U_{ij,r} \otimes \delta_r$, where each $U_{ij,r} \in \text{Pol}(G)$. Since $U$ is a finite dimensional unitary representation of $G \times \Lambda$, for any $i, j \in \{1, \ldots, d\}$, we have

$$\Delta_{G \times \Lambda}(U_{ij}) = \sum_{k=1}^d U_{ik} \otimes U_{kj},$$

where in $\text{Pol}(G) \otimes C(\Lambda) \otimes \text{Pol}(G) \otimes C(\Lambda)$, we have

$$(3.6) \quad \Delta_{G \times \Lambda}(U_{ij}) = \sum_{r, s, t \in \Lambda, \ r = st} [(id_A \otimes \epsilon)(\Delta(U_{ij,r}))]_{13}(1_A \otimes \delta_s \otimes 1_A \otimes \delta_t)$$

and

$$(3.7) \quad \sum_{k=1}^d U_{ik} \otimes U_{kj} = \sum_{k=1}^d \sum_{s, t \in \Lambda} U_{ik,s} \otimes \delta_s \otimes U_{kj,t} \otimes \delta_t.$$ 

Comparing (3.5), (3.6) and (3.7), we get

$$\text{id}_A \otimes \epsilon \Delta(U_{ij,s,t}) = \sum_{k=1}^d U_{ik,s} \otimes U_{kj,t} \in A \otimes A$$

or equivalently (by applying $(\text{id}_A \otimes \epsilon)$ on both sides)

$$(3.9) \quad \Delta(U_{ij,s,t}) = \sum_{k=1}^d U_{ik,s} \otimes \epsilon(U_{kj,t})$$

for every $s, t \in \Lambda$. Since $(\epsilon \otimes \epsilon)\Delta = \text{id} = (\epsilon \otimes \text{id})\Delta$, we have

$$U_{ij,s,t} = \sum_{k=1}^d \epsilon(U_{ik,s})\epsilon^{-1}(U_{kj,t}) = \sum_{k=1}^d \epsilon(U_{kj,t})U_{ik,s}$$

for any $i, j \in \{1, \ldots, d\}, s, t \in \Lambda$. 

6
We have $\epsilon_{\Lambda}(\delta_r) = \delta_{r,1\Lambda}$, thus by definition

\begin{equation}
U_{\mathcal{G}} = \sum_{i,j=1}^{d} e_{ij} \otimes U_{ij,1\Lambda} \in \mathcal{B}(<\mathcal{H}> \otimes \text{Pol}(\mathcal{G})).
\end{equation}

Similarly,

\begin{equation}
U_{\Lambda} = \sum_{r \in \Lambda} \sum_{i,j=1}^{d} \epsilon(U_{ij,r}) e_{ij} \otimes \delta_r \in \mathcal{B}(<\mathcal{H}> \otimes C(\Lambda)).
\end{equation}

Thus

\begin{equation}
U_{\Lambda}(r_0) = \sum_{i,j=1}^{d} \epsilon(U_{ij,r_0}) e_{ij} \in \mathcal{B}(<\mathcal{H}>).
\end{equation}

Hence,

\begin{equation}
(U_{\Lambda}(r_0) \otimes 1_A)U_{\mathcal{G}} = \sum_{i,j,k,l=1}^{d} \delta_{j,k} \epsilon(U_{ij,r_0}) e_{il} \otimes U_{kl,1\Lambda}
= \sum_{i,l=1}^{d} e_{il} \otimes \sum_{k=1}^{d} \epsilon(U_{ik,r_0}) U_{kl,1\Lambda}
= \sum_{i,l=1}^{d} e_{il} \otimes \alpha^*_{r_0}(U_{il,r_0})
\end{equation}

where the last equality follows from (3.10); and

\begin{equation}
[(\text{id} \otimes \alpha^*_{r_0})U_{\mathcal{G}}](U_{\Lambda}(r_0) \otimes 1_A) = \sum_{i,j,k,l=1}^{d} \delta_{j,k} \epsilon(U_{jk,r_0}) e_{il} \otimes \alpha^*_{r_0}(U_{ik,1\Lambda})
= \sum_{i,l=1}^{d} e_{il} \otimes \sum_{k=1}^{d} \epsilon(U_{kl,r_0}) \alpha^*_{r_0}(U_{ik,1\Lambda})
= \sum_{i,l=1}^{d} e_{il} \otimes \alpha^*_{r_0}(U_{il,r_0})
\end{equation}

where (3.10) is used again in the last equality.

Combining (3.14) and (3.15) finishes the proof of (3.2).

By (3.11), (3.12) and (3.10), one has

\begin{equation}
(U_{\mathcal{G}})_{12}(U_{\Lambda})_{13} = \sum_{i,j,k,l=1}^{d} \sum_{r \in \Lambda} \delta_{j,k} \epsilon(U_{kl,r}) e_{il} \otimes U_{ij,1\Lambda} \otimes \delta_r
= \sum_{i,l=1}^{d} e_{il} \otimes \sum_{r \in \Lambda} \left( \sum_{k=1}^{d} \epsilon(U_{kl,r}) U_{ik,1\Lambda} \right) \otimes \delta_r
= \sum_{i,l=1}^{d} e_{il} \otimes \sum_{r \in \Lambda} U_{il,r} \otimes \delta_r = U
\end{equation}

in $\mathcal{B}(<\mathcal{H}> \otimes \text{Pol}(\mathcal{G}) \otimes C(\Lambda))$. This proves (3.3).

Conversely, suppose $U_{\mathcal{G}}$ and $U_{\Lambda}$ are unitary representations on some finite dimensional Hilbert space $\mathcal{H}$. We still use $(e_1, \ldots, e_d)$ to denote a Hilbert basis for $\mathcal{H}$, where $d = \dim \mathcal{H}$, and $(e_{ij}, i,j = 1, \ldots, d)$ the corresponding matrix unit of $\mathcal{B}(<\mathcal{H}>$). Then for each pair $i,j$, one has a unique $u_{ij} \in \text{Pol}(\mathcal{G})$ and a
unique \( f_{ij} \in C(\Lambda) \), such that \( U_G = \sum_{i,j} e_{ij} \otimes u_{ij}, U_\Lambda = \sum_{i,j} e_{ij} \otimes f_{ij} \). By suitably choosing the basis \((e_1, \ldots, e_d)\), we may and do assume \( \epsilon(u_{ij}) = \delta_{i,j} \). Since these are representations, we have

\[
\Delta(u_{ij}) = \sum_{k=1}^d u_{ik} \otimes u_{kj},
\]

(3.17a)

\[
\Delta_\Lambda(f_{ij}) = \sum_{k=1}^d f_{ik} \otimes f_{kj}.
\]

(3.17b)

By definition,

\[
U = \sum_{i,j,k,l=1}^d \delta_{jk} e_{il} \otimes u_{ij} \otimes f_{kl} = \sum_{i,j=1}^d e_{ij} \otimes U_{ij}
\]

with

\[
U_{ij} = \sum_{k=1}^d u_{ik} \otimes f_{kj} = \sum_{r \in \Lambda} \sum_{k=1}^d f_{kj}(r) u_{ik} \otimes \delta_r.
\]

(3.19)

Since \( U_G \) and \( U_\Lambda \) are unitary, so is \( U \). Using \( \epsilon(u_{ij}) = \delta_{i,j} \), one has

\[
(id_{H(\mathcal{M})} \otimes \epsilon \otimes id_{C(\Lambda)})(U) = \sum_{i,j=1}^d e_{ij} \otimes \sum_{k=1}^d \delta_{i,k} f_{kj} = \sum_{i,j=1}^d e_{ij} \otimes f_{ij} = U_\Lambda.
\]

(3.20)

This proves (3.4b). The proof of (3.4a) is more involved and must resort to condition (3.2), which using the above notations, translates to

\[
\forall r \in \Lambda, \sum_{i,j} e_{ij} \otimes \sum_k f_{ik}(r) u_{kj} = \sum_{i,j} e_{ij} \otimes \sum_k f_{kj}(r) \alpha_r^*(u_{ik}),
\]

(3.21)

or equivalently,

\[
\forall r \in \Lambda, i, j \in \{1, \ldots, d\}, \sum_{k=1}^d f_{ik}(r) u_{kj} = \sum_{k=1}^d f_{kj}(r) \alpha_r^*(u_{ik}).
\]

(3.22)

Since \( U_\Lambda(1_\Lambda) = id_{\mathcal{M}} \), one has \( f_{ij}(1_\Lambda) = \delta_{i,j} \). Taking \( r = 1_\Lambda \) in (3.22) yields

\[
(id_{H(\mathcal{M})} \otimes id_{Pol(G)} \otimes \epsilon_\Lambda)(U) = \sum_{i,j=1}^d e_{ij} \otimes \sum_{k=1}^d f_{kj}(1_\Lambda) u_{ik}
\]

\[
= \sum_{i,j=1}^d e_{ij} \otimes \sum_{k=1}^d \delta_{k,j} u_{ik} = \sum_{i,j=1}^d e_{ij} \otimes u_{ij} = U_G,
\]

(3.23)

which proves (3.4a). To finishes the proof of the proposition, it remains to check that the unitary \( U \) is indeed a representation of \( G \rtimes \Lambda \).
Proposition 3.2 states that for a closed subgroup interchangeably.

C-morphism to a closed quantum subgroup of \( G \) sends representations of \( G \) bijectively to pairs of covariant unitary representations of \( G \). Let \( \psi \) be a Hilbert basis of \( \mathcal{H} \), \( \psi = \{ \psi_1, \ldots, \psi_d \} \), where \( \psi_1, \ldots, \psi_d \) are elements of \( \mathcal{H} \). Thus \( U \) is indeed a (unitary) representation.

\[
\Delta_{G \times \Lambda} (U_{ij}) = \sum_{r \in A} \sum_{s \in A} \left[ (\text{id}_{\mathcal{H}} \otimes \alpha_s^*) \Delta \left( \sum_{k=1}^{d} f_{kj}(r) u_{ik} \right) \right] \otimes \delta_{s-1 r}
\]

Using (3.17a), (3.19) and (3.22), one has

\[
\Delta_{G \times \Lambda} (U_{ij}) = \sum_{r \in A} \sum_{s \in A} f_{kj}(r) u_{il} \otimes \delta_s \otimes \alpha_s^* (u_{ik}) \otimes \delta_{s-1 r}
\]

Thus \( U \) is a unitary representation. \( \square \)

Definition 3.3. Let \( U_G \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{P}(G) \) be a finite dimensional unitary representation of \( G \), \( U_\Lambda \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{P}(\Lambda) \) a finite dimensional unitary representation of \( \Lambda \) on the same space \( \mathcal{H} \), we say \( U_G \) and \( U_\Lambda \) are covariant if they satisfy condition (3.2).

We track here a simple criterion for two representations to be covariant using matrix units and matrix coefficients.

Proposition 3.4. Let \( U_G \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{P}(G) \), \( U_\Lambda \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{P}(\Lambda) \) be finite-dimensional unitary representations of \( G \) and \( \Lambda \) respectively. Let \( e_{1}, \ldots, e_{d} \) be a Hilbert basis of \( \mathcal{H} \), \( e_{ij} \in \mathcal{B}(\mathcal{H}) \) the operator with \( e_{ij}(e_{k}) = \delta_{j,k} e_{i} \), and \( U_G = \sum_{i,j} e_{ij} \otimes u_{ij} \), \( U_\Lambda = \sum_{i,j} e_{ij} \otimes f_{ij} \), then \( U_G \) and \( U_\Lambda \) are covariant if and only if

\[
\forall r \in \Lambda, i, j \in \{1, \ldots, d\}, \quad \sum_{k=1}^{d} f_{ik}(r) u_{kj} = \sum_{k=1}^{d} f_{kj}(r) u_{ik}.
\]

Proof. This is just a restatement of condition (3.2). \( \square \)

By Proposition 3.2, unitary representations of \( G \times \Lambda \), at least the finite dimensional ones, correspond bijectively to pairs of covariant unitary representations of \( G \) and \( \Lambda \).

4. Principal Subgroups of \( G \times \Lambda \)

Definition 4.1. Let \( \mathbb{H} = (B, \Delta_B) \), \( \mathbb{K} = (C, \Delta_C) \) be compact quantum groups, we say \( \mathbb{K} \) is isomorphic to a closed quantum subgroup of \( \mathbb{H} \), or simply \( \mathbb{K} \) is a closed subgroup of \( \mathbb{H} \), if there exists a surjective morphism \( \varphi \) of compact quantum groups from \( \mathbb{H} \) to \( \mathbb{K} \). In other words, \( \varphi : B \to C \) is a unital \( * \)-homomorphism that intertwines the comultiplications, i.e. \( (\varphi \otimes \varphi) \Delta_B = \Delta_C \varphi \) as unital \( * \)-morphisms from \( B \) to \( C \otimes C \).

In the context of compact quantum groups, we will use the terms “quantum closed subgroup” and “closed subgroup” interchangeably.
**Remark 3.** If $\mathbb{H}$ and $\mathbb{K}$ are commutative, i.e. they come from genuine compact groups, then $\mathbb{K}$ being isomorphic to a closed subgroup, according to **Definition 4.1**, says exactly that there exists a continuous injective map $\varphi_*$ from $\text{Spec}(C)$, the underlying space of the compact group $\mathbb{K}$, into $\text{Spec}(B)$, the underlying space of the compact group $\mathbb{H}$, such that $\varphi_*$ preserves multiplication. Thus **Definition 4.1** is consistent with the classic case of compact groups.

Recall that $G = (A, \Delta)$, and $C(G \ltimes \Lambda) = A \otimes C(\Lambda)$.

**Proposition 4.2.** Let $\Lambda_0$ be a subgroup of $\Lambda$, then $\varphi: A \otimes C(\Lambda) \to A \otimes C(\Lambda_0)$, $\sum_{r \in \Lambda} a_r \otimes \delta_r \mapsto \sum_{r \in \Lambda_0} a_r \otimes \delta_r$ is a surjective morphism\(^1\) from $G \ltimes \Lambda$ to $G \ltimes \Lambda_0$. In particular, $G \ltimes \Lambda_0$ is a closed subgroup of $G \ltimes \Lambda$.

**Proof.** Obviously $\varphi$ is a unital surjective morphism of $C^*$-algebras. We need to show that $\varphi$ intertwines the comultiplication $\Delta$ on $G \ltimes \Lambda$ and the comultiplication $\Delta_0$ on $G \ltimes \Lambda_0$. For this, by density, it suffices to prove that the restriction

$$\varphi: \text{Pol}(G) \otimes C(\Lambda) \to \text{Pol}(G) \otimes C(\Lambda_0)$$

(4.1)

intertwines the comultiplications. Indeed, given an arbitrary $a_r \in \text{Pol}(G)$ for any $r \in \Lambda$, note that for any $a \in \text{Pol}(G)$ and $\lambda \in \Lambda$, $\varphi(a \otimes \delta_\lambda) = 0$ whenever $\lambda \notin \Lambda_0$, we have

$$\varphi((\varphi \otimes \varphi)\Delta \left(\sum_{r \in \Lambda} a_r \otimes \delta_r\right)) = (\varphi \otimes \varphi)\Delta \left(\sum_{r \in \Lambda} \sum_{s \in \Lambda} (a_r)_{(1)} \otimes \delta_s \otimes \alpha_s^*((a_r)_{(2)}) \otimes \delta_{s^{-1}r}\right)$$

(4.2)

$$= \sum_{r \in \Lambda_0} \sum_{s \in \Lambda_0} (a_r)_{(1)} \otimes \delta_s \otimes \alpha_s^*((a_r)_{(2)}) \otimes \delta_{s^{-1}r}$$

(Since $s, s^{-1}r \in \Lambda_0$ implies $r = s(s^{-1}r) \in \Lambda_0$)

$$= \tilde{\Delta}_0 \left(\sum_{r \in \Lambda_0} a_r \otimes \delta_r\right) = \tilde{\Delta}_0 \varphi \left(\sum_{r \in \Lambda} a_r \otimes \delta_r\right).$$

This shows that $\varphi$ intertwines comultiplications and finishes the proof. □

**Definition 4.3.** We call closed subgroups of $G \ltimes \Lambda$ of the form $G \ltimes \Lambda_0$ a **principal subgroup** of $G \ltimes \Lambda$, where $\Lambda_0$ is a subgroup of $\Lambda$.

**Remark 4.** If we let $p_0 = \sum_{r \in \Lambda_0} \delta_r \in C(\Lambda)$, then $p_0$ is a projection in $C(\Lambda)$, thus $1 \otimes p_0$ is a central projection in $A \otimes C(\Lambda)$. The morphism $\varphi$ is in fact given by the “compression” map $(1 \otimes p_0)(\cdot)(1 \otimes p_0)$. Essentially, these data says that the principal subgroup $G \ltimes \Lambda_0$ is in fact an open subgroup of $G \ltimes \Lambda$. As we don’t really need the general theory of open subgroups in this work, we won’t recall the relevant notions here and refer the interested reader to the articles [DKSS12; KKS16] for a treatment in the more general setting of locally compact quantum groups.

**Corollary 4.4.** Using the notations in **Proposition 4.2**, if $U \in \mathcal{B}\left(\mathcal{H}\right) \otimes A \otimes C(\Lambda)$ is a (unitary) representation of $G \ltimes \Lambda$, then $(\text{id} \otimes \varphi)(U)$ is a (unitary) representation of $G \ltimes \Lambda_0$.

**Proof.** This follows directly from the fact that the restriction of the mapping $\varphi$ as specified in (4.1) is a morphism of Hopf $*$-algebras. □

**Definition 4.5.** Using the above notations, the representation $(\text{id} \otimes \varphi)(U)$ is called the restriction of $U$ to $G \ltimes \Lambda_0$, and will be denoted by $U|_{G \ltimes \Lambda_0}$.

**Remark 5.** Again, when $G$ is an authentic compact group $G$, we recover the restriction of a representation of $G \ltimes \Lambda$ to the subgroup $G \ltimes \Lambda_0$.

There is a natural “conjugate” relation between principal subgroups of the form $G \ltimes \Lambda_0$ where $\Lambda_0$ is a subgroup of $\Lambda$, which will be used to simplify some calculations in our later treatment of representations. This relation is described in the following proposition.

\(^1\)Note that $\delta_r$ has different meanings when viewed as functions in $C(\Lambda)$ and in $C(\Lambda_0)$.
Proposition 4.6. Let $\Lambda_0$ be a subgroup of $\Lambda$, $r \in \Lambda$, $\Ad_r : \Lambda_0 \to r\Lambda_0 r^{-1}$ the isomorphism $s \mapsto rsr$. Then $\alpha^*_r \otimes \Ad^*_r$ is an isomorphism of compact quantum groups from $G \rtimes r\Lambda_0 r^{-1}$ onto $G \rtimes \Lambda_0$.

Proof. Let $\overline{\mathbb{H}}_0$ (resp. $\mathbb{H}_0$) be the Hopf $*$-algebra structure on $A \otimes C(\Lambda_0)$ as constructed in §2, where $\mathbb{G} = (A, \Delta)$. We denote the comultiplication on $\overline{\mathbb{H}}_0$ (resp. on $\mathbb{H}_0$) by $\Delta_0$ (resp. $\Delta_r$), and the counit by $\epsilon_0$ (resp. $\epsilon_r$). It suffices to prove that the unital $*$-isomorphism

$$
\alpha^*_r \otimes \Ad^*_r : A \otimes C(r\Lambda_0 r^{-1}) \to A \otimes C(\Lambda_0)
$$

of involutive algebras preserves counit and comultiplication.

Let $\epsilon$ be the counit for the Hopf $*$-algebra $\text{Pol}(\mathbb{G})$. For any $x \in \text{Pol}(\mathbb{G})$, $\lambda \in \Lambda_0$, we have

$$
\epsilon_0(\alpha^*_r \otimes \Ad^*_r)(x \otimes \delta_{r\lambda^{-1}}) = \epsilon_0[\alpha^*_r(x) \otimes \delta_{\lambda}]
$$

$$
= \delta_{\lambda,1,\lambda} \epsilon_0[\alpha^*_r(x)] = \delta_{\lambda,1,\lambda} \epsilon_0(x) = \epsilon_r(x \otimes \delta_{r\lambda^{-1}}).
$$

(4.3)

This proves $\alpha^*_r \otimes \Ad^*_r$ preserves counit. Furthermore, using $\Ad_r : \Lambda_0 \to r\Lambda_0 r^{-1}$ is an isomorphism, we have

$$
[(\alpha^*_r \otimes \Ad^*_r) \otimes (\alpha^*_r \otimes \Ad^*_r)](x \otimes \delta_{r\lambda^{-1}})
$$

$$
= \sum_{\mu \in \Lambda_0} \sum_{\mu \in \Lambda_0} \alpha^*_r(x_{(1)}) \otimes \delta_{\mu} \otimes \alpha^*_r(x_{(2)}) \otimes \delta_{\mu^{-1}}
$$

$$
= \sum_{\mu \in \Lambda_0} \sum_{\mu \in \Lambda_0} \alpha^*_r(x_{(1)}) \otimes \delta_{\mu} \otimes \alpha^*_r(x_{(2)}) \otimes \delta_{\mu^{-1}}
$$

$$
= \sum_{\mu \in \Lambda_0} \sum_{\mu \in \Lambda_0} \alpha^*_r(x_{(1)}) \otimes \delta_{\mu} \otimes \alpha^*_r(x_{(2)}) \otimes \delta_{\mu^{-1}}
$$

$$
= \Delta_0[(\alpha^*_r \otimes \Ad^*_r)(x \otimes \delta_{r\lambda^{-1}})].
$$

Thus $\alpha^*_r \otimes \Ad^*_r$ also preserves comultiplication. \qed

5. INDUCED REPRESENTATIONS OF PRINCIPAL SUBGROUPS

Let $\Lambda_0$ be a subgroup of $\Lambda$, $U \in B(\mathcal{H}) \otimes \text{Pol}(G) \otimes C(\Lambda_0)$ a finite dimensional unitary representation of $G \rtimes \Lambda_0$. We want to construct the induced representation $\text{Ind}^{G \rtimes \Lambda_0}_G(U)$ of the larger quantum group $G \rtimes \Lambda_0$. The idea of the construction goes as follows: by the results in §3, we know $U$ is determined by its restrictions $U_0 = \text{Res}_{\mathbb{G}}(U)$ and $U_\Lambda = \text{Res}_{\Lambda}(U)$. While one may not be able to directly extend the representation $U_{\Lambda_0} \otimes \Lambda_0$ of $\Lambda_0$ to a representation of $\Lambda$ on the same space $\mathcal{H}$, we do have the right-regular representation $W_\Lambda$ of $\Lambda$ on $L^2(\Lambda) \otimes \mathcal{H}$ using the group structure of $\Lambda$. On the other hand, the direct sum $W_\Gamma$ of various copies of $U_\Gamma$ placed suitably in $L^2(\Lambda) \otimes \mathcal{H}$ will give a representation of $G$ on $L^2(\Lambda) \otimes \mathcal{H}$. It is then easy to check that $W_\Gamma$ and $W_\Lambda$ are covariant, thus determine a representation $W$ of $G \rtimes \Lambda$ on $L^2(\Lambda) \otimes \mathcal{H}$. To retrieve the information of $U_{\Lambda_0}$, which is implicitly encoded in the $\mathcal{H}$ factor of $L^2(\Lambda) \otimes \mathcal{H}$, we consider the subspace $\mathcal{K}$ of $L^2(\Lambda) \otimes \mathcal{H}$ consisting of vectors which behave in a covariant way with the representation $U_{\Lambda_0}$ on $\mathcal{H}$. More precisely, $\mathcal{K}$ is defined by

$$
\mathcal{K} = \left\{ \sum_{r \in \Lambda} \delta_r \otimes \xi_r : \forall r_0 \in \Lambda_0, \xi_{r_0 r} = U_{\Lambda_0}(r_0)\xi_r \right\}.
$$

(5.1)

One checks that $\mathcal{K}$ is an invariant subspace for both $W_\Lambda$ and $W_\Gamma$, hence $\mathcal{K}$ is a subrepresentation $W$ of $W$, and we define $W$ to be the induced representation $\text{Ind}(U)$. We now proceed to carry out this idea precisely.

Definition 5.1. Let $U$, $\mathcal{H}$, $\Lambda_0$ retain their above meanings. In addition, let $(\epsilon_{r,s} : r, s \in \Lambda)$ be the matrix unit of $B(L^2(\Lambda))$ associated with the standard Hilbert basis $(\delta_r : r \in \Lambda)$ for $L^2(\Lambda)$, i.e., $\epsilon_{r,s} \delta_t = \delta_s \delta_r$ for all $r, s, t \in \Lambda$. The right regular representation $W_\Lambda$ of $\Lambda$ is an operator in $B(L^2(\Lambda)) \otimes B(\mathcal{H}) \otimes C(\Lambda)$ defined by

$$
W_\Lambda = \sum_{r,s \in \Lambda} \epsilon_{r,s^{-1},s} \otimes \text{id}_\mathcal{H} \otimes \delta_s.
$$

(5.2)
It is easy to see that if we regard $\ell^2(\Lambda) \otimes \mathcal{H}$ as $\ell^2(\Lambda, \mathcal{H})$, then for any $s \in \Lambda$, $\overline{W}_s$ is the operator in $B(\ell^2(\Lambda, \mathcal{H}))$ sending each $F: \Lambda \to \mathcal{H}$ to $F \circ R_s$, where $R_s: \Lambda \to \Lambda$ is the right multiplication by $s$. Hence $\overline{W}_s$ is indeed a unitary representation of $\Lambda$ on $\ell^2(\Lambda) \otimes \mathcal{H}$, and has the same formal property as the right regular representation of $\Lambda$. By definition, for any $s \in \Lambda$, the unitary operator $\overline{W}_s \in \mathcal{U}(\ell^2(\Lambda) \otimes \mathcal{H})$ is characterized by

$$\overline{W}_s: \ell^2(\Lambda) \otimes \mathcal{H} \to \ell^2(\Lambda) \otimes \mathcal{H},$$

or equivalently

$$\overline{W}_s: \ell^2(\Lambda) \otimes \mathcal{H} \to \ell^2(\Lambda) \otimes \mathcal{H},$$

$$\sum_{r \in \Lambda} \delta_r \otimes \xi_r \mapsto \sum_{r \in \Lambda} \delta_{rs^{-1}} \otimes \xi_r.$$

**Proposition 5.2.** Using the above notations, the unitary operator $\overline{W}_s$ in $B(\ell^2(\Lambda)) \otimes B(\mathcal{H}) \otimes \text{Pol}(G)$ defined by

$$\overline{W}_s = \sum_{e \in \Lambda} e_{s,s} \otimes [(\text{id} \otimes \alpha_{s}^{*})(U_G)]$$

is a unitary representation of $\mathbb{G}$ on $\ell^2(\Lambda) \otimes \mathcal{H}$. Furthermore, for every $s \in \Lambda$, $\delta_s \otimes \mathcal{H}$ is invariant under $\overline{W}_s$, and the subrepresentation $\delta_s \otimes \mathcal{H}$ of $\overline{W}_s$ is unitarily equivalent to direct sum of the unitary representation $[(\text{id} \otimes \alpha_{s}^{*})(U_G)]$ on $\mathcal{H}$, with $s$ runs through $\Lambda$.

**Proof.** For each $s \in \Lambda$, since $\alpha_{s}^{*} \in \text{Aut}(\mathbb{G})$, the unitary operator $(\text{id} \otimes \alpha_{s}^{*})(U_G)$ is indeed a representation of $\mathbb{G}$ on $\mathcal{H}$. It is easy to see that

$$e_{s,s}(\ell^2(\Lambda)) \otimes \mathcal{H} = C \delta_s \otimes \mathcal{H} = \delta_s \otimes \mathcal{H},$$

and $e_{s,s} \otimes \text{id}_{\mathcal{H}}$ is the orthogonal projection in $B(\ell^2(\Lambda) \otimes \mathcal{H})$ onto the subspace $\delta_s \otimes \mathcal{H}$ of $\ell^2(\Lambda) \otimes \mathcal{H}$. We also have the intertwining relation

$$e_{s,s} \otimes \text{id}_{\mathcal{H}} \circ 1 \overline{W}_s = e_{s,s} \otimes [(\text{id} \otimes \alpha_{s}^{*})(U_G)] = \overline{W}_s(e_{s,s} \otimes \text{id}_{\mathcal{H}} \circ 1 \Lambda).$$

Now the theorem follows from the direct sum decomposition

$$\ell^2(\Lambda) \otimes \mathcal{H} = \bigoplus_{s \in \Lambda} \delta_s \otimes \mathcal{H},$$

and the identification of the subspace $\delta_s \otimes \mathcal{H}$ with the Hilbert space $\mathcal{H}$ via $\delta_s \otimes \xi \mapsto \xi$. \hfill $\square$

**Proposition 5.3.** The representations $\overline{W}_s$ and $\overline{W}_s$ are covariant.

**Proof.** For any $s \in \Lambda$, by definition,

$$\overline{W}_s = \sum_{r \in \Lambda} e_{rs^{-1},r} \otimes \text{id}_{\mathcal{H}} \in B(\ell^2(\Lambda)) \otimes B(\mathcal{H}) = B(\ell^2(\Lambda) \otimes \mathcal{H}).$$

Thus

$$\left(\overline{W}_s \otimes 1\right) \overline{W}_s = \left(\sum_{r \in \Lambda} e_{rs^{-1},r} \otimes \text{id}_{\mathcal{H}} \otimes 1\Lambda\right) \sum_{r \in \Lambda} e_{r,t} \otimes [(\text{id} \otimes \alpha_{s}^{*})(U_G)]

= \sum_{r \in \Lambda} e_{rs^{-1},r} \otimes [(\text{id} \otimes \alpha_{s}^{*})(U_G)] = \sum_{r \in \Lambda} e_{rs^{-1},r} \otimes [(\text{id} \otimes \alpha_{s}^{*})(U_G)]

= (\text{id} \otimes \text{id} \otimes \alpha_{s}^{*}) \left(\sum_{r \in \Lambda} e_{rs^{-1},r} \otimes [(\text{id} \otimes \alpha_{s}^{*})(U_G)]\right)

= (\text{id} \otimes \text{id} \otimes \alpha_{s}^{*}) (\overline{W}_s) (\overline{W}_s \otimes 1).

This proves that $\overline{W}_s$ and $\overline{W}_s$ are indeed covariant. \hfill $\square$
Corollary 5.4. The unitary operator
\[
\tilde{W} = (\tilde{W}_G)_{123}(\tilde{W}_\Lambda)_{124} = \sum_{r,s,t \in \Lambda} e_{t,t} e_{r_s, r_t} \otimes [(id \otimes \alpha^*_r)(U_G)] \otimes \delta_s
\]
(5.11)
\[
= \sum_{r,s \in \Lambda} e_{r_s, r} \otimes [(id \otimes \alpha^*_r)(U_G)] \otimes \delta_s \in \mathcal{B}(\ell^2(\Lambda)) \otimes \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{G}) \otimes C(\Lambda)
\]
is a representation of $G \times \Lambda$ on $\ell^2(\Lambda) \otimes \mathcal{H}$.

Proof. This follows from Proposition 3.2 and Proposition 5.3. \(\square\)

We now proceed to prove the invariance of the subspace $\mathcal{H}$ defined in (5.1) under $\tilde{W}_G$ and $\tilde{W}_\Lambda$.

Lemma 5.5. Using the above notations, the following holds:

(a) the orthogonal projection $\pi \in \mathcal{B}(\ell^2(\Lambda) \otimes \mathcal{H})$ with range $\mathcal{H}$ is given by\(^2\) the following formula:
\[
\pi : \ell^2(\Lambda) \otimes \mathcal{H} \rightarrow \ell^2(\Lambda) \otimes \mathcal{H}
\]
(5.12)
\[
\delta_r \otimes \xi \mapsto |A_0|^{-1} \sum_{r_0 \in A_0} \delta_{r_{r_0}} \otimes U_{A_0}(r_0) \xi.
\]
In other words,
\[
\pi = |A_0|^{-1} \sum_{r_0 \in A_0} \sum_{s \in \Lambda} e_{r_0, s} \otimes U_{A_0}(r_0);
\]
(b) $\mathcal{H}$ is invariant under both $\tilde{W}_G$ and $\tilde{W}_\Lambda$, i.e.
\[
(\pi \otimes 1)\tilde{W}_G = \tilde{W}_G(\pi \otimes 1) = (\pi \otimes 1)\tilde{W}_G(\pi \otimes 1),
\]
(5.14a)
\[
(\pi \otimes 1)\tilde{W}_\Lambda = \tilde{W}_\Lambda(\pi \otimes 1) = (\pi \otimes 1)\tilde{W}_\Lambda(\pi \otimes 1).
\]
(5.14b)
In particular, we have
\[
(\pi \otimes 1 \otimes 1)\tilde{W} = \tilde{W}(\pi \otimes 1 \otimes 1) = (\pi \otimes 1 \otimes 1)\tilde{W}(\pi \otimes 1 \otimes 1).
\]
(5.15)

Proof. It is easy to see that $\pi(\ell^2(\Lambda) \otimes \mathcal{H})$ is precisely $\mathcal{H}$ and $\pi(\sum_{r \in \Lambda} \delta_r \otimes \xi_r) = \sum_{r \in \Lambda} \delta_r \otimes \xi_r$ whenever $\sum_{r \in \Lambda} \delta_r \otimes \xi_r \in \mathcal{H}$. To finish the proof, it suffices to check that $\pi$ is self-adjoint (or even stronger, positive). Since
\[
(\pi(\delta_r \otimes \xi_r), \delta_r \otimes \xi_r) = |A_0|^{-1} \sum_{r_0 \in A_0} (\delta_{r_{r_0}} \otimes U_{A_0}(r_0) \xi_r) \delta_r \otimes \xi_r
\]
(5.16)
\[
= |A_0|^{-1} \|\xi\|^2 \geq 0,
\]
$\pi$ is indeed positive. This finishes the proof of (a).

The invariance of $\mathcal{H}$ under $\tilde{W}_\Lambda$ (equation (5.14b)) follows from (5.1) and (5.4). We now prove the invariance of $\mathcal{H}$ under $\tilde{W}_G$ (equation (5.14a)). By the definitions of $\pi$ and $\tilde{W}_G$, we have
\[
|A_0|(\pi \otimes 1)\tilde{W}_G = \sum_{r_0 \in A_0} \sum_{r,s \in \Lambda} (e_{r_0, s} \otimes U_{A_0}(r_0) \otimes 1) (e_{r, r} \otimes [(id \otimes \alpha^*_r)(U_G)]
\]
(5.17)
\[
= \sum_{r \in \Lambda} (id \otimes id \otimes \alpha^*_r) \left( \sum_{r_0 \in A_0} \sum_{s \in \Lambda} e_{r_0, s} e_{r, r} \otimes [(U_{A_0}(r_0) \otimes 1) U_G] \right)
\]
\[
= \sum_{r \in \Lambda} (id \otimes id \otimes \alpha^*_r) \left( \sum_{r_0 \in A_0} e_{r_0, r} \otimes [(U_{A_0}(r_0) \otimes 1) U_G] \right);
\]

\(^2\)Recall that we’ve identified $\mathcal{B}(\ell^2(\Lambda) \otimes \mathcal{H})$ with $\mathcal{B}(\ell^2(\Lambda)) \otimes \mathcal{B}(\mathcal{H})$.
Proposition 3.2
Proposition 4.6

where the last equality used the covariance of $U_G$ on the set of subgroups of $\Lambda$. Justified by considering the case when $\pi$ is a projection. Now (5.15) follows from (5.14a), (5.14b) and (5.11). This proves (b).

\[ (\pi \otimes 1)\tilde{W}_G = \tilde{W}_G (\pi \otimes 1), \]

from which (5.14a) follows by noting that $\pi$ is a projection. Now (5.15) follows from (5.14a), (5.14b) and (5.11). This proves (b).

Proposition 5.6. Using the above notations, let $c_\pi: B(\ell^2(\Lambda) \otimes \mathcal{H}) \to B(\mathcal{H})$ be the compression by the projection $\pi$ (i.e. the graph of $c_\pi(A)$ is the intersection of the graph of $\pi A \pi$ with $\mathcal{H} \times \mathcal{H}$), then the following holds:

(a) the unitary operator

\[ W = (c_\pi \otimes \text{id}_{\mathcal{H}}) \left( \tilde{W}_G \right) \in B(\mathcal{H}) \otimes \text{Pol}(\mathcal{G}) \otimes C(\Lambda) \]

is a unitary representation of $\mathcal{G} \rtimes \Lambda$ on $\mathcal{H}$;

(b) The subrepresentation $\mathcal{H}$ of $\tilde{W}_G$ (resp. $\tilde{W}_\Lambda$) is given by $W_G = (c_\pi \otimes \text{id}) \left( \tilde{W}_G \right)$ (resp. $W_\Lambda = (c_\pi \otimes \text{id}) \left( \tilde{W}_\Lambda \right)$), and

\[ W_G = \text{Res}_G(W), \quad W_\Lambda = \text{Res}_\Lambda(W). \]

Proof. This follows from Proposition 3.2, Corollary 5.4, Lemma 5.5 and the definition of subrepresentations.

Definition 5.7. Using the above notations, we call $W$ the induced representation of $U$, and denote it by $\text{Ind}_{\mathcal{G} \rtimes \Lambda_0}^{\mathcal{G} \rtimes \Lambda}(U)$, or simply $\text{Ind}(U)$ when the underlying compact quantum groups $\mathcal{G} \rtimes \Lambda_0$ and $\mathcal{G} \rtimes \Lambda$ are clear from context.

6. Some character formulae

Let $\Lambda_0$ be a subgroup of $\Lambda$, $U$ a finite dimensional unitary representation of $\mathcal{G} \rtimes \Lambda_0$, $\text{Ind}_{\mathcal{G} \rtimes \Lambda_0}^{\mathcal{G} \rtimes \Lambda}(U)$ the induced representation of the “global” compact quantum group $\mathcal{G} \rtimes \Lambda$. In this section, we aim to calculate the character of the induced representation $\text{Ind}_{\mathcal{G} \rtimes \Lambda_0}^{\mathcal{G} \rtimes \Lambda}(U)$. The approach adopted here emphasizes the underlying group action of $\Lambda$ on the characters of the class of conjugacy of the open subgroup $\mathcal{G} \rtimes \Lambda_0$ as described in Proposition 4.6.

For any subgroup $\Lambda_1$ and any $f_0 \in C(\Lambda_1)$, we use $E_{\Lambda_1}(f_0)$ to denote the function in $C(\Lambda)$ with $[E_{\Lambda_1}(f_0)](r) = 0$ if $r \notin \Lambda_0$ and $[E_{\Lambda_1}(f_0)](r) = f_0(r)$ if $r \in \Lambda_1$. Then $E_{\Lambda_0}: C(\Lambda_1) \to C(\Lambda)$ is a morphism of $C^*$-algebras, which is not unital unless $\Lambda_1 = \Lambda$, in which case $E_{\Lambda_1} = \text{id}_{C(\Lambda)}$. By Proposition 4.6, we have an action

\[ \Lambda \curvearrowright \{ \mathcal{G} \rtimes r\Lambda r^{-1} : r \in \Lambda \} \]

\[ s \mapsto \{ \mathcal{G} \rtimes s \Lambda r^{-1} : \mathcal{G} \rtimes s \Lambda (sr)^{-1} \} \]

of $\Lambda$ on the set of subgroups of $\mathcal{G} \rtimes \Lambda$ conjugate to $\mathcal{G} \rtimes \Lambda_0$ via elements in $\Lambda$ (the word conjugate is justified by considering the case when $\mathcal{G}$ is a genuine compact group).
Our principal result in this section is the following proposition.

**Proposition 6.1.** Let $\Lambda_0$ be a subgroup of $\Lambda$, $U \in B(\mathcal{H}) \otimes \text{Pol}(\mathbb{G}) \subset C(\Lambda_0)$ a finite dimensional unitary representation of $\mathbb{G} \times \Lambda_0$, $W$ the induced representation $\text{Ind}_{\mathbb{G} \times \Lambda_0}^{\Lambda_0}(U)$. Suppose $\chi$ is the character of a unitary representation $U$ of $\mathbb{G} \times \Lambda_0$, and for each $r$, define

$$ (6.2) \quad r \cdot U = (\text{id}_{\mathcal{H}} \otimes \alpha_r^{-1} \otimes \text{Ad}_{r^{-1}})(U) \in B(\mathcal{H}) \otimes \text{Pol}(\mathbb{G}) \otimes C(\Lambda_0 \cdot r^{-1}). $$

Then $r \cdot U$ is a unitary representation of $\mathbb{G} \times r\Lambda_0 \cdot r^{-1}$ with $1 \cdot U = U$, and $(r \cdot s) \cdot U = r \cdot (s \cdot U)$ for all $r, s \in \Lambda$. Denote the character of $r \cdot U$ by $\chi_r$ (so $\chi_{1 \Lambda} = \chi$), then

$$ (6.3) \quad \chi_W = |\Lambda_0|^{-1} \sum_{r \in \Lambda} (\text{id}_A \otimes \text{E}_{r \Lambda_0 \cdot r^{-1}}) \chi_r, $$

where $\chi_W$ is the character of $W$.

**Proof.** That $r \cdot U$ is a finite dimensional unitary representation of $\mathbb{G} \times r\Lambda_0 \cdot r^{-1}$ follows from the fact (Proposition 4.6) that

$$ \alpha_r^* \otimes \text{Ad}^*_r : A \otimes \Lambda_0 \cdot r \rightarrow A \otimes \Lambda_0 $$

is an isomorphism of compact quantum groups for any $r \in \Lambda$. The identities $1_A \cdot U = U$ and $r \cdot (s \cdot U) = (rs) \cdot U$ follows directly from definitions. We proceed to prove the character formula (6.3).

For any $r \in \Lambda$, let $(r \cdot U)_{\mathcal{G}}$ be the restriction of $r \cdot U$ to $\mathcal{G}$, and $(r \cdot U)_{\Lambda_0 \cdot r^{-1}}$ the restriction of $r \cdot U$ to $r\Lambda_0 \cdot r^{-1}$. We denote the character of $(r \cdot U)_{\mathcal{G}}$ (resp. $(r \cdot U)_{\Lambda_0 \cdot r^{-1}}$) by $\chi_{r, \mathcal{G}}$ (resp. $\chi_{r, \Lambda_0 \cdot r^{-1}}$). One easily checks that $\chi_{r, \mathcal{G}} = \alpha_r^* \cdot (\chi_{1\Lambda, \mathcal{G}})$ and $\chi_{r, \Lambda_0 \cdot r^{-1}} = \text{Ad}^*_r \cdot (\chi_{1\Lambda, \Lambda_0})$. Fix a Hilbert basis $(e_1, \ldots, e_d)$ for $\mathcal{H}$, and let $(c_{ij}, i, j = 1, \ldots, d)$ be the corresponding matrix unit for $B(\mathcal{H})$. Using this matrix unit, we can write

$$ (6.4a) \quad U_{\mathcal{G}} = \sum_{i,j=1}^d e_{ij} \otimes u_{ij}, \quad u_{ij} \in \text{Pol}(\mathbb{G}); $$

$$ (6.4b) \quad U_{\Lambda_0} = \sum_{r_0 \in \Lambda_0} U_{\Lambda_0}(r_0) \otimes \delta_{r_0}. $$

Let $e_{r,s}, \pi, \mathcal{H}, \widetilde{W}_\mathcal{G}, \widetilde{W}_\Lambda, W_G$ and $W_\Lambda$ have the same meaning as in §5, then the construction in §5 tells us that

$$ (6.5) \quad \chi_W = (\text{Tr}_{\mathcal{B}(\Lambda)} \otimes \text{Tr}_\mathcal{H} \otimes \text{id}_A \otimes \text{id}_C(\Lambda)) \left[ \pi_{12} \cdot (\widetilde{W}_\mathcal{G})_{123} \cdot \pi_{12} \cdot (\widetilde{W}_\Lambda)_{124} \cdot \pi_{12} \right]. $$

In the following calculations, we often omit the subscripts of the trace functions $\text{Tr}$ on $\mathcal{B}(\Lambda)$ or on $\mathcal{H}$, and also the subscripts for the multiplicative neutral element 1 of various algebras, whenever it is a trivial task to decipher to which trace and multiplicative neutral element we are referring. The same goes with $\text{id}$ without subscripts.

Note that for any $r, s \in \Lambda$, $\text{Ad}^*_r(\delta_s) = \delta_{r^{-1}sr}$. With these preparations, we now have

$$ (6.6) \quad \chi_r = (\alpha_r^* \otimes \text{Ad}_{r^{-1}})(\chi) $$

$$ = (\alpha_r^* \otimes \text{Ad}_{r^{-1}}) \left( \sum_{i,j=1}^d \sum_{r_0 \in \Lambda_0} \text{Tr} \left( e_{ij} U_{\Lambda_0}(r_0) \right) u_{ij} \otimes \delta_{r_0} \right) $$

$$ = \sum_{i,j=1}^d \sum_{r_0 \in \Lambda_0} \text{Tr} \left( e_{ij} U_{\Lambda_0}(r_0) \right) \alpha_{r^{-1}}(u_{ij}) \otimes \delta_{r_0 r^{-1}}. $$

By (5.4), (5.6) and (5.13), we deduce from (6.5) that

$$ (6.7) \quad |\Lambda_0|^3 \chi_W = \sum_{a_0} \sum_{b_0} \sum_{c_0} \sum_{a,b,c \in \Lambda \cdot \Lambda \cdot \Lambda} \sum_{i,j=1}^d \sum_{s,t,e \in \Lambda} \text{Tr} \left( e_{a_0 a, a e_r r e_s s} e_{b_0 b, b e_{s^{-1}t}} e_{c_0 c, c} \right) $$

$$ \text{Tr} \left( U_{\Lambda_0}^{a_0} e_{i,j} U_{\Lambda_0}^{b_0} U_{\Lambda_0}^{c_0} \right) \alpha_r^*(u_{ij}) \otimes \delta_i. $$

On the right side of the above sum, the first trace doesn’t vanish if and only if it is 1, which happens exactly when

$$ a = r = b_0 b, \quad b = s t^{-1}, \quad s = c_0 c, \quad a_0 a = c $$

$$ \iff b = b_0^{-1} a, \quad e = a_0 a, \quad r = a, \quad s = c_0 a_0 a, \quad t = b^{-1} s = a^{-1} b_0 c_0 a_0 a. $$
Using this condition in (6.7), we get
\[ |\Lambda_0|^3 \chi_W \]

\[ = \sum_{a_0,b_0,c_0 \in \Lambda_0} \sum_{a \in \Lambda} \sum_{i,j=1}^d \text{Tr} \left( U_{\Lambda_0}(a_0) e_{i,j} U_{\Lambda_0}(b_0) U_{\Lambda_0}(c_0) \right) \alpha_a^*(u_{ij}) \otimes \delta_{a^{-1} b_0 c_0 a_0} \]

\[ = \sum_{a_0,b_0,c_0 \in \Lambda_0} \sum_{a \in \Lambda} \sum_{i,j=1}^d \text{Tr} \left( e_{i,j} U_{\Lambda_0}(b_0) U_{\Lambda_0}(c_0) U_{\Lambda_0}(a_0) \right) \alpha_a^*(u_{ij}) \otimes \delta_{a^{-1} b_0 c_0 a_0} \]

\[ = \sum_{a_0,b_0,c_0 \in \Lambda_0} \sum_{a \in \Lambda} \sum_{i,j=1}^d \text{Tr} \left( e_{i,j} U_{\Lambda_0}(b_0 c_0 a_0) \right) \alpha_a^*(u_{ij}) \otimes \delta_{a^{-1} b_0 c_0 a_0} \]

(6.9)

\[ = |\Lambda_0|^2 \sum_{r \in \Lambda} r \sum_{a \in \Lambda} \sum_{i,j=1}^d \text{Tr} \left( e_{i,j} U_{\Lambda_0}(r_0) \right) \alpha_a^*(u_{ij}) \otimes \delta_{a^{-1} r_0 a} \]

\[ = |\Lambda_0|^2 \sum_{r \in \Lambda} (\text{id} \otimes E_{r^{-1} \Lambda_0 r})(\chi_r), \]

where the last line uses (6.6) and the change of variable \( r = a^{-1} \). Dividing \( |\Lambda_0|^3 \) on both sides of (6.9) proves (6.3). \( \square \)

**Corollary 6.2.** Using the notations in Proposition 6.1, \( U \) and \( r \cdot U \) induce equivalent unitary representations of \( \mathbb{G} \rtimes \Lambda \) for all \( r \cdot U \).

**Proof.** By Proposition 6.1, we see that \( \text{Ind}(U) \) and \( \text{Ind}(r \cdot U) \) have the same character. \( \square \)

It is worth pointing out that there are in fact many repetitions in the terms of the right side of formula (6.3), as evidenced by the following lemma.

**Lemma 6.3.** Using the notations of Proposition 6.1, the following holds:
(a) for any \( r \in \Lambda \), we have

\[ (\text{id} \otimes E_{r \Lambda_0 r^{-1}}) \gamma = (\alpha_{r^{-1}}^* \otimes \text{Ad}_{r^{-1}}) \left[ (\text{id} \otimes E_{\Lambda_0}) \gamma \right]; \]

in \( \text{Pol}(\mathbb{G}) \otimes C(\Lambda) \);

(b) for any \( r, s \in \Lambda \), if \( r^{-1} s \in \Lambda_0 \), i.e. \( r \Lambda_0 = s \Lambda_0 \) and \( r \Lambda_0 r^{-1} = s \Lambda_0 s^{-1} \), then

\[ (\text{id} \otimes E_{r \Lambda_0 r^{-1}}) \chi_r = (\text{id} \otimes E_{s \Lambda_0 s^{-1}}) \chi_s \]

in \( \text{Pol}(\mathbb{G}) \otimes C(\Lambda) \). In particular,

\[ \chi_r = \chi_s, \]

or equivalently, \( r \cdot U \) and \( s \cdot U \) are unitarily equivalent unitary representations of the same compact quantum group \( \mathbb{G} \rtimes \Lambda \).

**Proof.** Using the same notations as in the proof of Proposition 6.1, it is clear that

\[ (r \cdot U)_{\mathbb{G}} = \sum_{i,j=1}^d e_{i,j} \otimes \alpha_{r^{-1}}^*(u_{ij}), \]

(6.13a)

\[ (r \cdot U)_{\Lambda_0 r^{-1}} = \sum_{r_0 \in \Lambda_0} U_{\Lambda_0}(r_0) \otimes \delta_{r_0 r^{-1}}. \]

(6.13b)

Calculating in \( \text{Pol}(\mathbb{G}) \otimes C(\Lambda) \), we have

\[ (\text{id} \otimes E_{r \Lambda_0 r^{-1}}) \chi_r = \sum_{i,j=1}^d \sum_{r_0 \in \Lambda_0} \text{Tr} \left( e_{i,j} U_{\Lambda_0}(r_0) \right) \otimes \alpha_{r^{-1}}^*(u_{ij}) \otimes \delta_{r_0 r^{-1}} \]

(6.14)

\[ = (\alpha_{r^{-1}}^* \otimes \text{Ad}_{r^{-1}}) \left[ (\text{id} \otimes E_{\Lambda_0}) \chi_r \right]. \]

This proves (a).
By (a), to establish (b), it suffices to show that
\begin{equation}
\forall s_0 \in \Lambda_0, \quad (\text{id} \otimes E_{\Lambda_0})\chi = (\alpha_{s_0}^* \otimes \text{Ad}_{s_0}^*) [(\text{id} \otimes E_{\Lambda_0})\chi].
\end{equation}
Calculating the right side gives
\begin{equation}
(\alpha_{s_0}^* \otimes \text{Ad}_{s_0}^*) [(\text{id} \otimes E_{\Lambda_0})\chi] = \sum_{i,j=1}^{d} \sum_{r_0 \in \Lambda_0} \text{Tr}(e_{i,j}U_{\Lambda_0}(r_0)) \otimes \alpha_{s_0}^*(u_{ij}) \otimes \delta_{s_0^{-1}r_0s_0}
\end{equation}
\begin{equation}
= \sum_{i,j=1}^{d} \sum_{r_0 \in \Lambda_0} \text{Tr}(e_{i,j}U_{\Lambda_0}(s_0r_0s_0^{-1})) \otimes \alpha_{s_0}^*(u_{ij}) \otimes \delta_{r_0}
\end{equation}
\begin{equation}
= \sum_{i,j=1}^{d} \sum_{r_0 \in \Lambda_0} \text{Tr}\left(U(r_0)U_{\Lambda_0}(s_0^{-1})e_{i,j}U(s_0)\right) \otimes \alpha_{s_0}^*(u_{ij}) \otimes \delta_{r_0}.
\end{equation}
Since $U_{\Lambda_0}$ and $U_G$ are covariant, we have
\begin{equation}
\sum_{i,j}^{d} U(s_0)e_{i,j} \otimes u_{ij} = \sum_{i,j=1}^{d} e_{i,j}U(s_0) \otimes \alpha_{s_0}^*(u_{ij}).
\end{equation}
Combining (6.16) and (6.17), we have
\begin{equation}
(\alpha_{s_0}^* \otimes \text{Ad}_{s_0}^*) [(\text{id} \otimes E_{\Lambda_0})\chi] = \sum_{i,j=1}^{d} \sum_{r_0 \in \Lambda_0} \text{Tr}\left(U(r_0)U_{\Lambda_0}(s_0^{-1})e_{i,j}U(s_0)\right) \otimes \alpha_{s_0}^*(u_{ij}) \otimes \delta_{r_0}
\end{equation}
\begin{equation}
= \sum_{i,j=1}^{d} \sum_{r_0 \in \Lambda_0} \text{Tr}\left(U(r_0)e_{i,j}\right) \otimes u_{ij} \otimes \delta_{r_0}
\end{equation}
\begin{equation}
= \sum_{i,j=1}^{d} \sum_{r_0 \in \Lambda_0} \text{Tr}\left(e_{i,j}U(r_0)\right) \otimes u_{ij} \otimes \delta_{r_0}
\end{equation}
\begin{equation}
= (\text{id} \otimes E_{\Lambda_0})\chi.
\end{equation}
This establishes (6.15) and proves (b).

Remark 6. By Lemma 6.3(b) and Proposition 6.1, one can in fact choose a set $L \subseteq \Lambda$ of representatives of the left coset space $\Lambda/\Lambda_0$, and the character formula (6.3) can now be written as
\begin{equation}
\chi_W = \sum_{r \in L} (\text{id} \otimes E_{r\Lambda_0r^{-1}})\chi_r.
\end{equation}
In the classical case where $G$ is a genuine compact group, one can easily check that the usual character formula for the representation induced by a representation of an open subgroup takes the form (6.19). The reason we prefer (6.3) is that it does not involve a seemingly arbitrary choice of a set of representatives $L$ for $\Lambda/\Lambda_0$, and thus, in the author’s opinion, is more aesthetically pleasing. One might also use this choice of left cosets representatives to fabricate the induced representation. However, in our more symmetric approach (cf. §5), everything seems more natural, and the underlying group action of $\Lambda$ on the various characters $\chi_r, r \in \Lambda$ becomes more transparent in (6.3), and we hope this “hidden symmetry” will keep the reader from losing himself/herself in the details of the tedious calculations to be presented later.

7. Dimension of the intertwiner space of induced representations

Let $\Theta, \Xi$ be subgroups of $\Lambda$, $U \in B(\mathcal{H}) \otimes \text{Pol}(\mathbb{C}) \otimes C(\Theta)$ a finite dimensional unitary representation of $G \rtimes \Theta$, $W \in B(\mathcal{H}) \otimes \text{Pol}(\mathbb{C}) \otimes C(\Xi)$ a finite dimensional unitary representation of $G \rtimes \Xi$. For the sake of brevity, we denote the induced representation $\text{Ind}_{G \rtimes \Theta}(U)$ simply by $\text{Ind}(U)$, and $\text{Ind}(W)$ has the similar obvious meaning. Equipped with the character formula established in §6, one naturally wonders how can we calculate $\dim \text{Mor}_{G \rtimes \Lambda}(\text{Ind}(U), \text{Ind}(W))$ in terms of some simpler data. This section focuses on this calculation, and the result here will play an important role in proving the irreducibility of some induced representations (as it turns out, these are all irreducible representations of $G \rtimes \Lambda$ up to equivalence) as well as our later calculation of the fusion rules.
For any representation $\rho$, we use $\chi(\rho)$ to denote the character of the representation. We denote the Haar state on $G$ by $h$, and the Haar state on $G \rtimes \Lambda_0$ by $h^{\Lambda_0}$ whenever $\Lambda_0$ is a subgroup of $\Lambda$.

By the general theory of representation theory of compact quantum groups, we have

$$\dim \text{Mor}_{G \rtimes \Lambda}(\text{Ind}(U), \text{Ind}(W)) = h^\Lambda(\chi(\text{Ind}(U)))^* \chi(\text{Ind}(W)) \vert_{\Lambda_0}.$$  \hspace{1cm} (7.1)

By Proposition 6.1, we have a representation $r \cdot U$ (resp. $r \cdot W$) of $G \rtimes r\Theta^{-1}$ (resp. $G \rtimes r\Xi^{-1}$), and combined with (7.1), we have

$$\dim \text{Mor}(\text{Ind}(U), \text{Ind}(W)) = \frac{1}{\vert \Theta \vert \cdot \vert \Xi \vert} \sum_{r, s \in \Lambda} h^\Lambda \left( (\text{id} \otimes E_{r \cdot \Theta^{-1}}) \chi(r \cdot U) \right)^* \left( (\text{id} \otimes E_{s \Xi^{-1}}) \chi(s \cdot W) \right).$$  \hspace{1cm} (7.2)

To simplify our notations, let $\Lambda(r, s) = r\Theta^{-1} \cap s\Xi^{-1}$ for any $r, s \in \Lambda$.

**Lemma 7.1.** Using the above notations, for any $r, s \in \Lambda$, we have

$$h^\Lambda \left( (\text{id} \otimes E_{r \cdot \Theta^{-1}}) \chi(r \cdot U) \right)^* \left( (\text{id} \otimes E_{s \Xi^{-1}}) \chi(s \cdot W) \right) = \frac{1}{\vert \Lambda : \Lambda(r, s) \vert} \dim \text{Mor}_{G \rtimes \Lambda(r, s)} \left( (r \cdot U)|_{G \rtimes \Lambda(r, s)}, (s \cdot W)|_{G \rtimes \Lambda(r, s)} \right).$$  \hspace{1cm} (7.3)

**Proof.** For any subgroup $\Lambda_0$ of $\Lambda$, whenever $f \in \text{Pol}(G)$, $r_0 \in \Lambda_0$, by (2.4) in §2, we have

$$h^\Lambda(f \otimes \delta_{r_0}) = \frac{1}{\vert \Lambda : \Lambda_0 \vert} h^{\Lambda_0}(f \otimes \delta_{r_0}).$$  \hspace{1cm} (7.4)

Hence,

$$h^\Lambda \circ (\text{id} \otimes E_{\Lambda_0}) = \frac{1}{\vert \Lambda : \Lambda_0 \vert} h^{\Lambda_0}. $$

By definition and a straightforward calculation, we have

$$\left( \text{id} \otimes E_{r \cdot \Theta^{-1}} \right) \chi(r \cdot U) = \sum_{t \in r \cdot \Theta^{-1}} (\text{Tr} \otimes \text{id}) \left( (r \cdot U)_{G} ((r \cdot U)_{r \cdot \Theta^{-1}}(t) \otimes 1) \right) \otimes \delta_t, $$  \hspace{1cm} (7.6a)

$$\left( \text{id} \otimes E_{s \Xi^{-1}} \right) \chi(s \cdot W) = \sum_{t \in s \Xi^{-1}} (\text{Tr} \otimes \text{id}) \left( (s \cdot W)_{G} ((s \cdot W)_{s \Xi^{-1}}(t) \otimes 1) \right) \otimes \delta_t. $$  \hspace{1cm} (7.6b)

It follows from (7.6a) and (7.6b) that

$$
\left( \text{id} \otimes E_{r \cdot \Theta^{-1}} \right) \chi(r \cdot U) \otimes \delta_t = \sum_{t \in r \cdot \Theta^{-1}} \left[ \left( \text{Tr} \otimes \text{id} \right) \left( (r \cdot U)_{G} ((r \cdot U)_{r \cdot \Theta^{-1}}(t) \otimes 1) \right) \right]^* \otimes \delta_t \\
= \chi \left( \left( \frac{(r \cdot U)_{G} ((s \cdot W)_{s \Xi^{-1}}(t) \otimes 1)}{G \rtimes \Lambda(r, s)} \right) \right) \otimes \delta_t.
$$  \hspace{1cm} (7.7)

Taking $\Lambda_0 = \Lambda(r, s)$ in (7.5) and combining with (7.7) proves (7.3).

** Proposition 7.2.** Using the above notations, we have

$$\dim \text{Mor}_{G \rtimes \Lambda}(\text{Ind}(U), \text{Ind}(W)) = \frac{1}{\vert \Theta \vert \cdot \vert \Xi \vert} \sum_{r, s \in \Lambda} \frac{1}{\vert \Lambda : \Lambda(r, s) \vert} \dim \text{Mor}_{G \rtimes \Lambda(r, s)} \left( (r \cdot U)|_{G \rtimes \Lambda(r, s)}, (s \cdot W)|_{G \rtimes \Lambda(r, s)} \right).$$  \hspace{1cm} (7.8)

**Proof.** This follows directly from (7.2) and Lemma 7.1.

**Corollary 7.3.** Let $\Lambda_0$ be a subgroup of $\Lambda$, $U$ a unitary representation of $G \rtimes \Lambda_0$, then the following are equivalent:

(a) the unitary representation $\text{Ind}(U)$ of $G \rtimes \Lambda$ is irreducible;
(b) for any $r, s \in \Lambda$, posing $\Lambda(r, s) = r\Lambda_0 r^{-1} \cap s\Lambda_0 s^{-1}$, we have

$$\dim \text{Mor}_{G \rtimes \Lambda}(r \cdot U)|_{G \rtimes \Lambda(r, s)}, (s \cdot U)|_{G \rtimes \Lambda(r, s)} = \delta_{r\Lambda_0, s\Lambda_0};$$

(c) $U$ is irreducible, and

$$\forall r, s \in \Lambda, \ r^{-1} s \notin \Lambda_0 \implies \dim \text{Mor}_{G \rtimes \Lambda}(r \cdot U)|_{G \rtimes \Lambda(r, s)}, (s \cdot U)|_{G \rtimes \Lambda(r, s)} = 0.$$  \hspace{1cm} (7.10)
In particular, if the above conditions hold, then \( U \) itself is irreducible.

**Proof.** If \( r^{-1} s \in \Lambda_0 \), then \( r \Lambda_0 r^{-1} = s \Lambda_0 s^{-1} \), so \( \Lambda(r, s) = r \Lambda_0 r^{-1} = s \Lambda_0 s^{-1} \). By Proposition 4.6, we see that

\[
(7.11) \quad \dim \text{Mor}_{\mathbb{G}^r \times \mathbb{G}^s}(r \cdot U, r \cdot U) = \dim \text{Mor}_{\mathbb{G}^r \times \mathbb{G}^s}(U, U).
\]

By Proposition 7.2, Lemma 6.3, and the above, we have

\[
(7.12) \quad \dim \text{Mor}_{\mathbb{G}^r}(\text{Ind}(U), \text{Ind}(U)) = \sum_{r, s \in \Lambda} \frac{1}{|\Lambda|} \dim \text{Mor}_{\mathbb{G}^r \times \mathbb{G}^s}(r \cdot U, r \cdot U)
\]

whenever \( r^{-1} s \in \Lambda_0 \) and

\[
(7.13) \quad \dim \text{Mor}_{\mathbb{G}^r \times \mathbb{G}^s}(r \cdot U, s \cdot U) = \dim \text{Mor}_{\mathbb{G}^r \times \mathbb{G}^s}(r \cdot U, r \cdot U) = \dim \text{Mor}_{\mathbb{G}^r \times \mathbb{G}^s}(U, U)
\]

whenever \( r^{-1} s \in \Lambda_0 \) by Lemma 6.3 and Proposition 4.6, the proposition follows from (7.12) and the fact that a representation is irreducible if and only if the dimension of the space of its self-intertwiners is 1. \( \square \)

**Remark 7.** Corollary 7.3 is a quantum analogue for Mackey’s criterion for irreducibility.

### 8. The C*-tensor category \( \mathcal{CSR}_{\Lambda_0} \)

Recall the notation in Proposition 6.1: for any unitary representation \( U_G \in \mathcal{B}(H) \otimes \text{Pol}(G) \) of \( G \) on some finite dimensional Hilbert space \( H \) and \( r \cdot A \), let \( r \cdot U_G \) be the unitary representation \((\text{id}_H \otimes \alpha_{r^{-1}})(U_G)\) of \( G \) on the same space \( H \). It is easy to see that this defines a left group action of \( \Lambda \) on the proper class of all unitary representations of \( G \), and by passing to quotients, this representation induces an action of \( \Lambda \) on \( \text{Irr}(G) \). From now on, whenever we talk about an element \( r \in \Lambda \) acts on a unitary representation \( U_G \) of \( G \), or on some class \( x \in \text{Irr}(G) \), we always refer to these actions unless stated otherwise.

**Definition 8.1.** A subgroup \( \Lambda_0 \) of \( \Lambda \) is called a general isotropy subgroup if it is an isotropy subgroup (subgroup of stabilizer for some point) for the \( n \)-fold product \([\text{Irr}(G)]^n\) as a \( \Lambda \)-set, in other words, there exists a \( n \)-tuple \((x_1, \ldots, x_n)\) with all \( x_i \in \text{Irr}(G) \), such that

\[
\Lambda_0 = \{ r \in \Lambda : \forall i = 1, \ldots, n, \quad r \cdot x_i = x_i \}.
\]

The finite family of all general isotropy subgroups of \( \Lambda \) is denoted by \( \mathcal{G}_{\Lambda_0}(\Lambda) \).
The following proposition is an easy consequence of properties of $A$-sets and Definition 8.1.

**Proposition 8.2.** The family $\mathcal{F}_{\text{sets}}(A)$ is stable under intersection and conjugation by elements of $\Lambda$. □

**Definition 8.3.** Let $\Lambda_0$ be a general isotropy subgroup of $\Lambda$. A covariant system of representations (or CSR for short) subordinate to $\Lambda_0$ is a triple $(\mathcal{H}, u, w)$, where

- $\mathcal{H}$ is a finite dimensional Hilbert space;
- $u$ is a unitary representation of $G$ on $\mathcal{H}$;
- $w$ is a unitary representation of $\Lambda_0$ on $\mathcal{H}$,

such that $u$ and $w$ are covariant. In this paper, CSRs are often denoted by bold faced uppercase letters like $A, B, C, \ldots$ (mostly $S$) with possible subscripts.

By Proposition 3.2, the covariant system of representations subordinate to a general isotropy subgroup $\Lambda_0$ corresponds bijectively to the class of unitary representations of $G \rtimes \Lambda_0$ via $(\mathcal{H}, u, w) \mapsto u_{12}w_{13}$ in one direction, and $U_{\mathcal{H}} \mapsto (\mathcal{H}, U_{\mathcal{H}}, G, U_{\mathcal{H}, \Lambda_0})$ as its inverse, where $\mathcal{H}$ is the underlying space of the representation $U_{\mathcal{H}}$ of $G \rtimes \Lambda_0$, and $U_{\mathcal{H}}, G, U_{\mathcal{H}, \Lambda_0}$ are the restrictions of $U_{\mathcal{H}}$ to $G$ and $\Lambda_0$ respectively. Using this bijection, we can transport the structure of rigid $C^*$-tensor category on $\text{Rep}(G \rtimes \Lambda_0)$ to the class of covariant systems of representations subordinate to $\Lambda_0$, thereby getting a rigid $C^*$-tensor category $CSR_{\Lambda_0}$ whose objects are CSRs subordinate to $\Lambda_0$.

To make this transport of categorical structure less tautological, we make a more convenient identification of the morphisms in $CSR_{\Lambda_0}$.

**Proposition 8.4.** Fix a general isotropy subgroup $\Lambda_0$ of $\Lambda$. For $i = 1, 2$, let $S_i = (\mathcal{H}_i, u_i, w_i)$ be a CSR subordinate to $\Lambda_0$, $U_i = (u_i)_{12}(w_i)_{13}$ the corresponding unitary representation of $G \rtimes \Lambda_0$, $S \in B(\mathcal{H}_1, \mathcal{H}_2)$. Then $S \in \text{Mor}_{G \rtimes \Lambda_0}(U_1, U_2)$ if and only if $S \in \text{Mor}_G(u_1, u_2) \cap \text{Mor}_{\Lambda_0}(w_1, w_2)$.

**Proof.** The condition is easily seen to be sufficient. Indeed, if $S \in \text{Mor}_G(u_1, u_2) \cap \text{Mor}_{\Lambda_0}(w_1, w_2)$, then

$$\text{(S} \otimes 1\text{)}w_1 = w_2(\text{S} \otimes 1), \text{(S} \otimes 1\text{)}u_1 = u_2(\text{S} \otimes 1).$$

Thus

$$\begin{align*}
(\text{S} \otimes 1 \otimes 1)U_1 &= (\text{S} \otimes 1 \otimes 1)(u_1)_{12}(w_1)_{13} = (u_2)_{12}(\text{S} \otimes 1 \otimes 1)(w_1)_{13} \\
&= (u_2)_{12}(w_1)_{13}(\text{S} \otimes 1 \otimes 1) = U_2(\text{S} \otimes 1 \otimes 1).
\end{align*}$$

This means exactly $S \in \text{Mor}_G(U_1, U_2)$.

To show the necessity of this condition, let $\epsilon_G: \text{Pol}(G) \to \mathbb{C}$ be the counit of the Hopf-$*$-algebra, $\epsilon_{\Lambda_0}: C(\Lambda_0) \to \mathbb{C}$ the counit for the Hopf-$*$-algebra $C(\Lambda_0)$. Since $U_i \in B(\mathcal{H}_i) \otimes \text{Pol}(G) \otimes C(\Lambda_0)$ for $i = 1, 2$ and $S \in \text{Mor}_{G \rtimes \Lambda_0}(U_1, U_2)$, we have

$$\text{(S} \otimes 1 \otimes 1)U_1 = U_2(\text{S} \otimes 1 \otimes 1).$$

Applying $\text{id} \otimes \text{id} \otimes \epsilon_{\Lambda_0}$ on both sides of (8.3) yields

$$\begin{align*}
(\text{S} \otimes 1)w_1 &= u_2(\text{S} \otimes 1),
\end{align*}$$

which means $S \in \text{Mor}_G(u_1, u_2)$. Applying $\text{id} \otimes \epsilon_G \otimes \text{id}$ yields

$$\begin{align*}
(\text{S} \otimes 1)w_1 &= w_1(\text{S} \otimes 1),
\end{align*}$$

which means $S \in \text{Mor}_{\Lambda_0}(w_1, w_2)$. □

We now define a pair of functors, $\mathcal{F}_{\Lambda_0}: CSR_{\Lambda_0} \to \text{Rep}(G \rtimes \Lambda_0)$ and $\mathcal{F}_{\Lambda_0}: \text{Rep}(G \rtimes \Lambda_0) \to CSR_{\Lambda_0}$, between $CSR_{\Lambda_0}$ and $\text{Rep}(G \rtimes \Lambda_0)$ that reflects the transport of categorical structures discussed above. On the object level, for any CSR $(u, w) \in CSR_{\Lambda_0}$, let $\mathcal{F}_{\Lambda_0}(u, w)$ be the representation $u_{12}w_{13}$; for any unitary representation $U \in \text{Rep}(G \rtimes \Lambda_0)$, let $\mathcal{F}_{\Lambda_0}(U)$ be the CSR $(U_G, U_{\Lambda_0})$ where $U_G$ (resp. $U_{\Lambda_0}$) is the restriction of $U$ onto $G$ (resp. $\Lambda_0$). On the morphism level, both $\mathcal{F}_{\Lambda_0}$ and $\mathcal{F}_{\Lambda_0}$ act as identity.

By Proposition 8.4 and Proposition 3.2, $\mathcal{F}_{\Lambda_0}$ and $\mathcal{F}_{\Lambda_0}$ are indeed well-defined functors inverses to each other, and they are fiber functors (exact unitary tensor functors [NT13, §§2.1, 2.2]) simply because the rigid $C^*$-tensor category structure on $CSR_{\Lambda_0}$ is transported from that of $\text{Rep}(G \rtimes \Lambda_0)$ via $\mathcal{F}_{\Lambda_0}$.

**Proposition 8.5.** For $i = 1, 2$, let $S_i = (\mathcal{H}_i, u_i, w_i) \in CSR_{\Lambda_0}, U_i = \mathcal{F}_{\Lambda_0}(S_i) \in \text{Rep}(G \rtimes \Lambda_0)$, then $\mathcal{F}_{\Lambda_0}(U_1 \times U_2) = (u_1 \times u_2, w_1 \times w_2)$. In particular, $S_1 \otimes S_2 = (u_1 \times u_2, w_1 \times w_2)$. 

20
Proposition 8.5

For any \( U \) and \( U' \) in \( \mathcal{H} \), we have
\[
(U \otimes U') (U = (U \otimes U')_{234} \in \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2) \otimes \mathcal{P}(G) \otimes C(\Lambda_0),
\]
where we identified \( \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2) \) with \( \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{H}_2 \) canonically.

The restriction of \( U_1 \times U_2 \) onto \( G \) is
\[
(id \otimes id \otimes id \otimes c)(U_1 \times U_2) = (id \otimes id \otimes id \otimes c)(U_1)(id \otimes id \otimes id \otimes c)(U_2)_{234} = (u_1)_{13}(u_2)_{23} = u_1 \times u_2 \in \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2) \otimes \mathcal{P}(G).
\]

Similarly, the restriction of \( U_1 \times U_2 \) onto \( \Lambda_0 \) is
\[
(id \otimes id \otimes \epsilon \otimes id)(U_1 \times U_2) = (id \otimes id \otimes \epsilon \otimes id)(U_1)(id \otimes id \otimes \epsilon \otimes id)(U_2)_{234} = (u_1)_{13}(u_2)_{23} = w_1 \times w_2 \in \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2) \otimes C(\Lambda_0).
\]

Hence by definition, \( S_1 \otimes S_2 \) is \( \mathcal{H}_{\lambda_0}(U_1 \times U_2) = (u_1 \times u_2, w_1 \times w_2) \).

Proposition 8.6. For \( i = 1, 2 \), let \( S_i = (\mathcal{H}, u_i, w_i) \in \mathcal{H} \), \( U_i = \mathcal{H}_{\lambda_0}(S_i) \in \mathcal{S}(G \times \Lambda_0) \), then
\[
\mathcal{H}_{\lambda_0}(U_1 \otimes U_2) = (u_1 \otimes u_2, w_1 \otimes w_2).
\]

Proposition 8.7. For \( \Lambda_0 \neq \Lambda \), let \( \mathcal{F} = (\mathcal{H}, u, w) \in \mathcal{H} \), \( U = \mathcal{H}_{\lambda_0}(S) \in \mathcal{S}(G \times \Lambda_0) \), \( U^c \) the contragredient representation of \( U \) on the conjugate space \( \mathcal{H}^c \) of \( \mathcal{H} \). If \( \rho_U \), the unique invertible positive operator in \( \mathcal{M}(G \times \Lambda_0, U, U^c) \) such that \( \text{Tr}(\rho_U) = \text{Tr}(\rho_U^{-1}) \) on \( \mathcal{H}_{\lambda_0}(U) \), then \( \mathcal{H}_{\lambda_0}(U^c) \) is the dual of \( \mathcal{H}_{\lambda_0}(U) \) in \( \mathcal{S}(G \times \Lambda_0) \) as presented in Proposition 8.7.

Proof. By definition, \( \mathcal{S} = \mathcal{H}_{\lambda_0}(U) \), thus
\[
u' = (id_{\mathcal{H}} \otimes \epsilon_{\mathcal{P}(G)} \otimes \epsilon_{\Lambda_0})(\mathcal{U}) = (id \otimes \epsilon_{\mathcal{H}})(j(\rho_U)^{1/2} \otimes 1 \otimes 1)(j(\rho_U)^{-1/2} \otimes 1 \otimes 1)
\]
\[
= (j(\rho_U)^{1/2} \otimes 1 \otimes 1)(j(\rho_U)^{-1/2} \otimes 1 \otimes 1)
\]

The expression for \( u' \) is proved analogously by applying \( (id_{\mathcal{H}} \otimes \epsilon_{\mathcal{P}(G)} \otimes \epsilon_{\Lambda_0}) \) on (8.9).
Proposition 8.7. The “modular” operator $\rho_U$ of the representation $U$ is derived from the representation theory of $G \rtimes \Lambda_0$ instead of the representation theory of $G$ and (projective) representation theory of $\Lambda_0$. This makes the description of $\mathcal{T}$ in Proposition 8.7 very unsatisfactory in the non-unimodular case. This being said, we point out that as far as the fusion rules of $G \rtimes \Lambda_0$ is concerned, the duals of a sufficiently large family of CSRs admit a more satisfactory description, as will be demonstrated in the following sections.

When $G$ is of Kac-type, the description of the dual in $CSR_{\Lambda_0}$ is a lot easier, as is seen in the following corollary.

Corollary 8.8. Using the notations in Proposition 8.7, if $G$ is of Kac-type, then $\mathcal{T} = \langle u^e, w^e \rangle$.

Proof. Since $G$ is of Kac-type, the contragredient representation $w^e$ is unitary. By Proposition 2.2, $G \rtimes \Lambda_0$ is also of Kac-type, so $\rho_U = \text{id}_{\mathcal{T}}$. Now the corollary follows directly from Proposition 8.7. \hfill \square

9. Group actions and projective representations

Fix a $\Lambda_0 \in \mathcal{I}_\text{iso}(\Lambda)$. Via the functors $\mathcal{J}_\Lambda$ and $\mathcal{R}_\Lambda$, we see that the problem of irreducible representations of $G \rtimes \Lambda_0$ are essentially the same as simple CSRs in $CSR_{\Lambda_0}$. Thus for the moment, it might be too much to hope there exists a satisfactory description of all simple CSRs in $CSR_{\Lambda_0}$. However, as we will see in §10, if we restrict our attention to the so-called stably pure simple CSRs in $CSR_{\Lambda_0}$, then such a description is indeed achievable via the theory of unitary projective representations of $\Lambda_0$. This section studies how such projective representations arise naturally from the action of $\Lambda$ on irreducible representations of $G$, as well as establishes some basic properties of these projective representations. The results here will be used in §10 to obtain a structure theorem of stably pure CSRs in $CSR_{\Lambda_0}$.

We begin with a simple observation which is a trivial quantum analogue of one of the most basic ingredients of the Mackey analysis. Let $U_G$ be a unitary representation of $G$ on some finite dimensional Hilbert space $\mathcal{H}$. Since $\alpha^*: \Lambda \to \text{Aut}(G)$ is an antihomomorphism of groups, we know that $(\text{id}_{B(\mathcal{H})} \otimes \alpha_{r^{-1}}^*)(U_G)$ is again a unitary representation of $G$ on the same space $\mathcal{H}$, and we denote this new representation by $r \cdot U_G$. One checks that $(rs) \cdot U_G = r \cdot (s \cdot U_G)$. Thus this defines a left action of the group $\Lambda$ on the (proper) class of all unitary representation of $G$, which is easily to preserve irreducibility and pass to a well-defined action of $\Lambda$ on the set $\text{Irr}(G)$ by letting $r \cdot [u] = [r \cdot u]$, where $r$ is in $\Lambda$, $u$ is an irreducible unitary representation of $G$ and $[u]$ is the class of $u$ in $\text{Irr}(G)$. Take another unitary representation $W_G$ of $G$ on some other finite dimensional Hilbert space $\mathcal{K}$. For any $r, s \in \Lambda$ and any $T \in B(\mathcal{H}, \mathcal{K})$, we have

$$T \in \text{Mor}_G(r \cdot U_G, W_G) \iff W_G(T \otimes 1) = (T \otimes 1)(\text{id} \otimes \alpha_{r^{-1}}^*)(U_G) \iff (\text{id} \otimes \alpha_{r^{-1}}^*)([W_G])(T \otimes 1) = (T \otimes 1)(\text{id} \otimes \alpha_{r^{-1}}^*)(U_G) \iff T \in \text{Mor}_G(\sigma \cdot U_G, s \cdot W_G).$$

(9.1)

Now take any irreducible unitary representation $u$ of $G$ on some finite dimensional Hilbert space $\mathcal{H}$. Let $x = [u] \in \text{Irr}(G)$, and

$$\Lambda_x = \{ r \in \Lambda : r \cdot x = x \},$$

i.e. $\Lambda_x$ is the isotropy subgroup of $\Lambda$ fixing $x$. Then for any $r_0 \in \Lambda_x$, $u$ and $r_0 \cdot u$ are equivalent by definition, hence there exists a unitary $V(r_0) \in \mathcal{U}(\mathcal{H})$ intertwining $r_0 \cdot u$ and $u$, in other words,

$$V(r_0) \otimes 1)(\text{id} \otimes \alpha_{r^{-1}_0}^*)(u) = u(V(r_0) \otimes 1),$$

which is clearly equivalent to

$$\forall r_0 \in \Lambda_x, \quad (V(r_0) \otimes 1)u = [(\text{id} \otimes \alpha_{r_0}^*)(u)](V(r_0) \otimes 1).$$

(9.2)

It is remarkable that (9.3) takes exactly the same form as the covariance condition (3.2) when we define covariant representations in §3. Now if we choose a

$$V(r_0) \in \text{Mor}_G(r_0 \cdot u, u) \cap \mathcal{U}(\mathcal{H})$$

(9.4)

for each $r_0 \in \Lambda_x$, then for any $s_0 \in \Lambda_x$, by (9.1), we have

$$V(r_0) \in \text{Mor}_G(s_0r_0 \cdot u, s_0 \cdot u), \quad V(s_0) \in \text{Mor}_G(s_0 \cdot u, u), \quad V(s_0r_0) \in \text{Mor}_G(s_0r_0 \cdot u, u),$$

whence

$$\forall r_0, s_0 \in \Lambda_x, \quad V(s_0r_0)[V(r_0)]^*[V(s_0)]^* \in \text{Mor}_G(u, u) \cap \mathcal{U}(\mathcal{H}) = T \cdot \text{id}_\mathcal{H}.$$
This means that \( V: \Lambda_x \rightarrow \mathcal{U}(\mathcal{H}) \) is a unitary projective representation of \( \Lambda_x \) on \( \mathcal{H} \), which satisfies the covariant condition (9.3) for each \( r_0 \in \Lambda_x \).

To facilitate our discussion, we digress now to give a brief summary of some basic terminologies of the theory of group cohomology which we will use (cf. [Bro94]). We regard \( \mathbb{T} \) as a trivial module over any finite group when considering unitary projective representations of finite groups. For any finite group \( \Gamma \), a n-cochain on \( \Gamma \) with coefficients in \( \mathbb{T} \), or simply a n-cochain (on \( \Gamma \)) since we won’t consider coefficient module other that the trivial module \( \mathbb{T} \), is a mapping from the n-fold product \( \Gamma^n = \Gamma \times \cdots \times \Gamma \) to \( \mathbb{T} \). Let \( C^n(\Gamma, \mathbb{T}) \) be the abelian group of n-cochains on \( \Gamma \), \( Z^n(\Gamma, \mathbb{T}) \) the subgroup of 2-cocycles on \( \Gamma \), i.e. mappings \( \omega: \Gamma \times \Gamma \rightarrow \mathbb{T} \) satisfying the cocycle condition
\[
\forall r, s, t \in \Gamma, \quad \omega(r, st)\omega(s, t) = \omega(r, s)\omega(rs, t).
\]
The mapping
\[
\delta: C^1(\Gamma, \mathbb{T}) \rightarrow Z^2(\Gamma, \mathbb{T})
\]
(9.7)
\[
 b \mapsto \{(r, s) \in \Gamma \times \Gamma \mapsto \frac{b(r)b(s)}{b(rs)} \}
\]
is easily checked to be a well-defined group morphism. We use \( B^2(\Gamma, \mathbb{T}) \) to denote the image of \( \delta \), and the 2-cocycles in \( B^2(\Gamma, \mathbb{T}) \) are called 2-coboundaries of \( \Gamma \). The quotient group \( Z^2(\Gamma, \mathbb{T})/B^2(\Gamma, \mathbb{T}) \) is called the second cohomology group of \( \Gamma \) with coefficients in the trivial \( \Gamma \)-module \( \mathbb{T} \), and is denoted by \( H^2(\Gamma, \mathbb{T}) \). Elements in \( H^2(\Gamma, \mathbb{T}) \) are called cohomology class. Note that ker(\( \delta \)) is exactly the group morphisms on \( \Gamma \), i.e. group morphisms from \( \Gamma \) to \( \mathbb{T} \).

We track here the following easy results for the convenience of the reader.

**Lemma 9.1.** Let \( \Gamma \) be a finite group, \( V: \Gamma \rightarrow \mathcal{U}(\mathcal{H}) \) a finite dimensional unitary projective representation of \( \Gamma \) with cocycle \( \omega \). If \( \omega' \in [\omega] \in H^2(\Gamma, \mathbb{T}) \), then there exists a mapping \( b: \Gamma \rightarrow \mathbb{T} \), such that \( bV: \Gamma \rightarrow \mathcal{U}(\mathcal{H}) \) is a unitary projective representation with cocycle \( \omega' \).

**Proof.** Since \( \omega' \in [\omega] \), there is a mapping \( b: \Gamma \rightarrow \mathbb{T} \) such that \( \omega' = (\delta b)\omega \), and obviously, \( bV \) is a unitary projective representation with \( (\delta b)\omega = \omega' \) as its cocycle. \( \square \)

**Lemma 9.2.** Let \( \Gamma \) be a finite group, \( V: \Gamma \rightarrow \mathcal{U}(\mathcal{H}) \) a finite dimensional unitary projective representation of \( \Gamma \) with cocycle \( \omega \), \( b: \Gamma \rightarrow \mathbb{T} \) an arbitrary mapping. The following hold:

(a) \( bV: \Gamma \rightarrow \mathcal{U}(\mathcal{H}), \gamma \mapsto b(\gamma)V(\gamma) \) is a projective representation with cocycle \( (\delta b)\omega \);

(b) \( bV \) and \( V \) have the same cocycle if and only if \( b \in \text{ker}(\delta) \), i.e. \( b \) is a character of \( \Gamma \);

(c) \( bV \) is irreducible if and only if \( V \) is irreducible.

**Proof.** It is clear that (a) and (b) are direct consequences of definitions. We now prove (c). If we denote the character of \( V \) by \( \chi_V \), then the character of \( bV \) is \( b\chi_V \). Hence
\[
\dim \text{Morr}_\Gamma(bV, bV) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} b(\gamma)\chi_V(\gamma)b(\gamma)\chi_V(\gamma) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \chi_V(\gamma)\chi_V(\gamma) = \dim \text{Morr}_\Gamma(V, V),
\]
and \( bV \) is irreducible if and only if \( V \) is. \( \square \)

**Remark 9.** If \( b \) is a character of \( \Gamma, V: \Gamma \rightarrow \mathcal{U}(\mathcal{H}) \) an irreducible unitary projective representation, then \( bV \) is also an irreducible unitary projective representation with the same cocycle as that of \( V \). Note that \( |b(\gamma)| = 1 \) for all \( \gamma \in \Gamma \), we have
\[
\dim \text{Morr}_\Gamma(bV, V) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} b(\gamma)\chi_V(\gamma)\chi_V(\gamma) \leq \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \chi_V(\gamma)\chi_V(\gamma) = \dim \text{Morr}_\Gamma(V, V) = 1,
\]
with equality holds if and only if \( b(\gamma) = 1 \) whenever \( \chi_V(\gamma) \neq 0 \). If equality doesn’t hold in (9.9), then \( \dim \text{Morr}_\Gamma(bV, V) \) must be 0 since it is a natural number. Therefore, whenever \( \Gamma \) is not trivial, it is possible that \( bV \) and \( V \) are irreducible unitary projective representations with the same cocycle but not equivalent.

Now we resume our discussion before we digress. Using terminologies in the theory of group cohomology, and regarding \( \mathbb{T} \) as the trivial \( \Lambda_x \)-module, we see that the 2-cocycle \( \omega_x \in C^2(\Lambda_x, \mathbb{T}) \) of the unitary projective representation \( V \) of \( \Lambda_x \), determined up to a 2-boundary in \( B^2(\Lambda_x, \mathbb{T}) \), because each unitary operator \( V(r_0), r_0 \in \Lambda_x \) is uniquely determined up to a scalar multiple in \( \mathbb{T} \) (Shur’s lemma plus the unitarity of \( V(r_0) \)). In other words, \( [\omega_x] \in H^2(\Lambda_x, \mathbb{T}) \) is a well-defined cohomology class of \( \Lambda_x \) with coefficients in \( \mathbb{T} \).
Conversely, let \( u \) be an irreducible unitary representation of \( G \) on some finite dimensional Hilbert space \( \mathcal{H} \), and \( x = [u] \in \text{Irr}(G) \). If \( \Lambda_0 \) is a subgroup of \( \Lambda \), \( \Lambda_0 \rightarrow \mathcal{U}(\mathcal{H}) \) a unitary projection representation of \( \Lambda_0 \) such that \( u \) and \( V \) satisfy the covariance condition (9.3), then
\[
\forall r_0 \in \Lambda_0, \quad V(r_0) \in \text{Mor}_G(r_0 \cdot u, u).
\]
In particular, \( \Lambda_0 \) fixes \( x = [u] \) under the action \( \Lambda \acts \text{Irr}(G) \). Repeat the above reasoning shows that (9.5) still holds.

We summarize the above discussion in the following proposition, which proves slightly more.

**Proposition 9.3.** Let \( u \) be an irreducible unitary representation of \( G \) on some finite dimensional Hilbert space \( \mathcal{H} \), \( x = [u] \in \text{Irr}(G) \), \( \Lambda_x \) the isotropy group fixing \( x \) (under the action \( \Lambda \acts \text{Irr}(G) \)). For any \( r_0 \in \Lambda_x \), choose a \( V(r_0) \) according to (9.4). Then

(a) \( V: \Lambda_x \rightarrow \mathcal{U}(\mathcal{H}) \), \( r_0 \mapsto V(r_0) \) is a unitary projective representation satisfying the covariance condition (9.3);

(b) let \( \omega \in C^2(\Lambda_0, \mathbb{T}) \) be the 2-cocycle of \( V \), then the cohomology class \( c_x: = [\omega] \in H^2(\Lambda_x, \mathbb{T}) \) depends only on \( x \), i.e. not on any particular choice of \( u \in x \).

Conversely, if \( V_0 : \Lambda_0 \rightarrow \mathcal{U}(\mathcal{H}) \) is a unitary projective representation of some subgroup \( \Lambda_0 \) of \( \Lambda \) that satisfies the covariance condition (9.3), then

(c) for every \( r_0 \in \Lambda_0 \), the condition (9.4) holds;

(d) \( \Lambda_0 \subseteq \Lambda_x \);

(e) there is a choice of \( V : \Lambda_x \rightarrow \mathcal{U}(\mathcal{H}) \) satisfying (9.3) such that \( V|_{\Lambda_0} = V_0 \);

(f) let \( \omega_0 \in C^2(\Lambda_0, \mathbb{T}) \) be the 2-cocycle of \( V_0 \), then \( [\omega_0] \) is the image of \( c_x \) under the morphism of groups
\[
H^2(\Lambda_0) \hookrightarrow H^2(\Lambda_x, \mathbb{T}) \rightarrow H^2(\Lambda_0, \mathbb{T}).
\]

**Proof.** The above discussion already establishes (a), (c) and (d). Assertion (e) follows from (a) and (c), while (f) follows from (e). Moreover, we've seen that \( [\omega] \in H^2(\Lambda_x, \mathbb{T}) \) does not depend on the choice of \( V \). For any \( w \in x \), there exists a unitary intertwiner \( U \in \text{Mor}_G(u, w) \). It is trivial to check that \( V_w(r_0) = UV(r_0)U^* \) defines a unitary projective representation of \( \Lambda_x \) such that
\[
V_w(r_0) \in \text{Mor}_G(r_0 \cdot w, w).
\]

Since \( V_u \) and \( V \) are unitarily equivalent projective representations of \( \Lambda_x \), the 2-cocycle of \( V_u \) coincides with \( \omega \)—the 2-cocycle of \( V \). This proves that \( c_x = [\omega] \in H^2(\Lambda_x, \mathbb{T}) \) indeed depends only on \( x \) and not on any particular choice of \( u \in x \). This proves (b) and finishes the proof of the proposition. \( \Box \)

**Definition 9.4.** Using the notations in Proposition 9.3, we say the cohomology class \( [\omega] \in H^2(\Lambda_x, \mathbb{T}) \) the cohomology class associated with \( x = [u] \in \text{Irr}(G) \), and we often denote by \( c_x \). If \( \Lambda_0 \) is a subgroup of \( \Lambda_x \), the cohomology class \( [\omega_0] \in H^2(\Lambda_0, \mathbb{T}) \) is called the restriction of the cohomology class \( c_x \) on \( \Lambda_0 \), and is denoted by \( c_x|_{\Lambda_0} \).

Obviously, \( c_x|_{\Lambda_0} \) depends on \( \Lambda_0 \) and \( x \), and \( c_x|_{\Lambda_0} = c_x \) if \( \Lambda_0 = \Lambda_x \). To apply the character theory of projective representations, we need to suitably rescale the projective representations in question so that they share the same cocycle (not merely the same cohomology class for their cocycles). In the case where the representation \( u \in \mathcal{B}(\mathcal{H}) \otimes \text{Pol}(\mathbb{G}) \) of \( G \) is irreducible, and \( V : \Lambda_0 \rightarrow \mathcal{U}(\mathcal{H}) \) is a unitary projective representation satisfying the covariance condition (9.3), such a rescaling is implicit in the choice of \( V(r_0) \in \text{Mor}_G(r_0 \cdot u, u) \) for each \( r_0 \in \Lambda_0 \).

**Proposition 9.5.** Let \( x \in \text{Irr}(G), \ u \in x \), \( \Lambda_0 \) a subgroup of \( \Lambda_x \), \( c_0 \in H^2(\Lambda_0, \mathbb{T}) \) is the image of the cohomology class \( c_x \in H^2(\Lambda_x, \mathbb{T}) \) associated with \( x \) under \( H^2(\Lambda_0) \hookrightarrow H^2(\Lambda_x, \mathbb{T}) \). Then for any 2-cocycle \( \omega_0 \in c_0 \), there exists a unitary projective representation \( V \) of the isotropy subgroup \( \Lambda_x \) with cocycle \( \omega_0 \), such that \( V \) and \( u \) are covariant, and such a \( V \) is unique up to rescaling by a character on \( \Lambda_x \).

**Proof.** This is clear from Proposition 9.3, Lemma 9.1 and Lemma 9.2. \( \Box \)

10. Pure, stable, distinguished CSRs and representation parameters

Recall that for any irreducible representation \( u \) of \( G \), we define the support of \( u \), denoted by \( \text{supp}(u) \), to be the set
\[
\{ x \in \text{Irr}(G) : \dim \text{Mor}_G(x, [u]) \neq 0 \}
\]
where \([u]\) is the class of unitary representations of \( G \) equivalent to \( u \). We call \( u \) pure if \( \text{supp}(u) \) is a singleton.
Definition 10.1. Fix a $\Lambda_0 \in \mathfrak{G}_{\text{iso}}(\Lambda)$, $S = (\mathcal{H}, u, w) \in \text{CSR}_{\Lambda_0}$, we call $S$

- pure, if $u$ is pure;
- stable, if $r \cdot [u] = [r \cdot u]$ for all $r \in \Lambda_0$;
- stably pure, if it is both pure and stable;
- maximally stable, if

$$\Lambda_0 = \{ r \in \Lambda : r \cdot [u] = [u] \};$$

- simple, if $S$ is a simple object in $\text{CSR}_{\Lambda_0}$;
- distinguished, if it is maximally stable, pure and simple.

As remarked earlier, while it is not reasonable for the moment to hope for a satisfactory description of all simple CSRs in $\text{CSR}_{\Lambda_0}$, it is possible to describe simple CSRs that are stably pure using unitary projective representations of $\Lambda_0$. Somewhat surprisingly, one can even describe all stably pure CSRs in this way. To achieve the latter, we introduce the following definitions, which is closely related to the materials in §9.

Definition 10.2. Let $\Lambda_0 \in \mathfrak{G}_{\text{iso}}(\Lambda)$, $u$ a unitary representation of $G$ on some finite dimensional Hilbert space $\mathcal{H}$, $V : \Lambda_0 \to \mathcal{U}(\mathcal{H})$ a unitary projective representation of $\Lambda_0$, we say $u$ and $V$ are covariant if they satisfy the covariance condition (9.3), or equivalently $V(r_0) \in \text{Mor}_G(r_0 \cdot u, u)$ for all $r_0 \in \Lambda_0$.

Definition 10.3. Let $x \in \text{Irr}(G)$, $\Lambda_0 \in \mathfrak{G}_{\text{iso}}(\Lambda)$ with $\Lambda_0 \subseteq \Lambda_x$, $u \in x$, $\omega_0 \in c_x,_{\Lambda_0}$ (see Definition 9.4), then a unitary projective representation $V_0$ of $\Lambda_0$ that is covariant with $u$ is said to be a covariant projective $\Lambda_0$-representation of $u$ (with cocycle $\omega_0$).

Remark 10. In the setting of Definition 10.3, fix any covariant projective $\Lambda_0$-representation $V$ of $u$ with cocycle $\omega_0$, the set of covariant projective $\Lambda_0$-representations of $u$ with multiplier $\omega_0$ is in bijection correspondence with the group of characters of $\Lambda_0$, via $b \mapsto bV$ (see Lemma 9.1 and Lemma 9.2).

Proposition 10.4. Fix a $\Lambda_0 \in \mathfrak{G}_{\text{iso}}(\Lambda)$, let $S = (\mathcal{H}, u, w)$ be a stably pure CSR in $\text{CSR}_{\Lambda_0}$, $x \in \text{Irr}(G)$ is the support point of $u$, $u_0 \in x$ a representation on some finite dimensional Hilbert space $\mathcal{H}_0$, $n$ is the multiplicity of $u_0$ in $u$, $V_0$ a covariant projective $\Lambda_0$-representation of $u_0$, then there exists a unique unitary projective representation $v_0 : \Lambda_0 \to \mathcal{U}(\mathcal{C}^n)$ of $\Lambda_0$ on $\mathcal{C}^n$, such that the following hold:

(a) $V_0$ and $v_0$ have opposing cocycles;

(b) $S_0 = (\mathcal{C}^n \otimes \mathcal{H}_0, \epsilon_n \times u_0, v_0 \otimes V_0)$ is a CSR in $\text{CSR}_{\Lambda_0}$, where $\epsilon_n$ is the trivial representation of $G$ on $\mathcal{C}^n$;

(c) $S_0$ and $S$ are isomorphic in $\text{CSR}_{\Lambda_0}$.

Proof. Let $U$ be a unitary intertwiner from $u$ to $\epsilon_n \otimes u_0$. Replacing $S$ with $USU^*$ if necessary, we may assume $\mathcal{H} = \mathcal{C}^n \otimes \mathcal{H}_0$ and $u = \epsilon_n \times u_0 = (u_0)_{\mathcal{H}_0}$. For any $r_0 \in \Lambda_0$, we claim that there exists a unique $v_0(r_0) \in \mathcal{B}(\mathcal{C}^n)$ such that $w(r_0) = v_0(r_0) \otimes V_0(r_0)$. Admitting the claim for the moment, the unitarity of $v_0(r_0)$ follows from the unitarity of $w(r_0)$ and $V_0(r_0)$, and $w$ being a representation and $V_0$ being a projective representation force $v_0$ to be a unitary projective representation with a cocycle opposing to the cocycle of $V_0$. Thus the proposition follows from the claim, which we now prove. Since $\mathcal{B}(\mathcal{C}^n \otimes \mathcal{H}_0) = \mathcal{B}(\mathcal{C}^n) \otimes \mathcal{B}(\mathcal{H}_0)$ by the usual identification, there exists an $m \in \mathbb{N}$, $A_1, \ldots, A_m \in \mathcal{B}(\mathcal{C}^n)$ and $B_1, \ldots, B_m \in \mathcal{B}(\mathcal{H}_0)$, such that

$$w(r_0) = \sum_{i=1}^m A_i \otimes B_i. \quad (10.1)$$

Furthermore, we can and do choose these operators so that $A_1, \ldots, A_m$ are linearly independent in $\mathcal{B}(\mathcal{C}^n)$. Since $u$ and $w$ are covariant, we have

$$w(r_0) \otimes 1 = [(\text{id}_{\mathcal{H}} \otimes \alpha_{r_0}^* \omega_0)](w(r_0) \otimes 1). \quad (10.2)$$

Substituting $u = (u_0)_{\mathcal{H}_0}$ and (10.1) in (10.2) yields

$$\sum_{i=1}^m A_i \otimes [(B_i \otimes 1)u_0] = \sum_{i=1}^m A_i \otimes \left( [(\text{id}_{\mathcal{H}_0} \otimes \alpha_{r_0}^* \omega_0)u_0] (B_i \otimes 1) \right) \in \mathcal{B}(\mathcal{C}^n) \otimes \mathcal{B}(\mathcal{H}_0) \otimes \text{Pol}(G). \quad (10.3)$$

Since $A_1, \ldots, A_m$ are linearly independent, there exists linear functionals $l_1, \ldots, l_m$ on $\mathcal{B}(\mathcal{C}^n)$ such that $l_i(A_j) = \delta_{i,j}$. Applying $l_i \otimes \text{id}_{\mathcal{H}_0} \otimes \text{id}_{\text{Pol}(G)}$ on (10.3) shows that for each $i = 1, \ldots, m,$

$$(B_i \otimes 1)u_0 = [(\text{id} \otimes \alpha_{r_0}^* \omega_0)u_0] (B_i \otimes 1), \quad (10.4)$$
or equivalently
\begin{equation}
B_i \in \operatorname{Mor}_G(r_0 \cdot u_0, u_0) = \mathbb{C}V_0(r_0).
\end{equation}
Now the claim follows from (10.1) and (10.5).

Conversely, we have

**Proposition 10.5.** Fix a \( \Lambda_0 \in \mathcal{G}_{iso}(\Lambda) \), \( x \in \operatorname{Irr}(G) \) with \( \Lambda_0 \subseteq \Lambda_x \). Take a \( u \in u \) acting on some finite dimensional Hilbert space \( \mathcal{H} \), and a covariant projective \( \Lambda_0 \)-representation \( V \) of \( u \), then for any unitary projective representation \( v : \Lambda_0 \rightarrow \mathcal{U}(\mathcal{H}) \) of \( \Lambda_0 \) with cocycle opposing the cocycle of \( V \), the unitary representation \( v \times V \) of \( \Lambda_0 \) is covariant with the unitary representation \( \operatorname{id}_x \otimes u = \epsilon_x \times u \) of \( G \), where \( \epsilon_x \) is the trivial representation of \( G \) on \( \mathcal{H} \), i.e. \( (\mathcal{H} \otimes \mathcal{H}, \epsilon_x \times u, v \times V) \) is a stably pure CSR in \( \mathcal{CSR}_{\Lambda_0} \).

**Proof.** Since \( V \) and \( u \) are covariant, for any \( r_0 \in \Lambda_0 \), we have
\begin{equation}
(V(r_0) \otimes 1)u = [(\operatorname{id} \otimes \alpha_{r_0}^*)u](V(r_0) \otimes 1).
\end{equation}
The proposition follows by tensoring \( v(r_0) \) on the left in (10.6).

By Proposition 10.4 and Proposition 10.5, we have a satisfactory description of stably pure CSRs in \( \mathcal{CSR}_{\Lambda_0} \)—from any irreducible representation \( u \) of \( G \) on \( \mathcal{H} \) such that \( \Lambda_0 \cdot [u] = [u] \), one can choose a covariant projective \( \Lambda_0 \)-representation \( V \) of \( u \) with some cocycle \( \omega \), then any unitary projective representation \( v \) of \( \Lambda_0 \) gives rise to a stably pure CSR in \( \mathcal{CSR}_{\Lambda_0} \), namely \( S(u, V, v) = (\mathcal{H} \otimes \mathcal{H}, \epsilon_x \times u, v \times V) \); and all stably pure CSRs in \( \mathcal{CSR}_{\Lambda_0} \) arise in this way up to isomorphism.

**Remark 11.** Using the above notations, while it is true that \( V \) is determined by \( u \) to a great extent due to the restriction of Shur’s lemma, it is still not completely determined (see Proposition 9.5), and a choice of this \( V \) is vitally relevant as is demonstrated by Remark 9 applied to \( v \). This is why we choose the notation \( S(u, V, v) \).

**Definition 10.6.** Continuing to use the above notations, the triple \((u, V, v)\) is called a **representation parameter** for \( G \times \Lambda \) associated with \( \Lambda_0 \), and the stably pure CSR \( S(u, V, v) \) in \( \mathcal{CSR}_{\Lambda_0} \) is called the CSR parametrized by the representation parameter \((u, V, v)\). If furthermore the unitary projective representation \( v \) is irreducible, we say the representation parameter \((u, V, v)\) is irreducible.

**Corollary 10.7.** Fix a \( \Lambda_0 \in \mathcal{G}_{iso}(\Lambda) \), then every stably pure CSR in \( \mathcal{CSR}_{\Lambda_0} \) is parametrized by some representation parameter associated with \( \Lambda_0 \).

**Definition 10.8.** Fix a \( \Lambda_0 \in \mathcal{G}_{iso}(\Lambda) \). Let \( u \) be an irreducible unitary representation of \( G \) such that \( \Lambda_0 \cdot [u] = [u] \), \( V_1 \) and \( V_2 \) are two covariant projective \( \Lambda_0 \)-representations of \( u \), the unique mapping \( b : \Lambda_0 \rightarrow T \) such that \( V_2 = b V_1 \) is called the **\( u \)-transitional mapping** from \( V_1 \) to \( V_2 \).

**Proposition 10.9.** Fix a \( \Lambda_0 \in \mathcal{G}_{iso}(\Lambda) \). For \( i = 1, 2 \), let \( (u_i, V_i, v_i) \) be a representation parameter associated with \( \Lambda_0 \), \( U_i \) denote the unitary representation \( \mathcal{A}_{\Lambda_0}(S(u_i, V_i, v_i)) \) of \( G \times \Lambda_0 \), then the following holds:
(a) if \( [u_1] \neq [u_2] \) in \( \operatorname{Irr}(G) \), then \( \dim \operatorname{Mor}_{G \times \Lambda_0}(U_1, U_2) = 0 \);
(b) if \( u_1 = u_2 = u \), and \( b : \Lambda_0 \rightarrow T \) the \( u \)-transitional map from \( V_1 \) to \( V_2 \), then
\begin{equation}
\dim \operatorname{Mor}_{G \times \Lambda_0}(U_1, U_2) = \dim \operatorname{Mor}_{\Lambda_0}(v_1, v_2).
\end{equation}

**Proof.** Let \( h \) be the Haar state of \( G \), by (2.4), the Haar state \( h_{\Lambda_0} \) of \( G \times \Lambda_0 \) is the linear functional on \( A \otimes C(\Lambda_0) \) defined by \( a \otimes \delta_{r_0} \mapsto |\Lambda_0|^{-1} h(a) \), where \( a \in A \), \( r_0 \in \Lambda_0 \) (recall that \( A = C(G) \)).

Suppose \( [u_1] \neq [u_2] \). For any \( i = 1, 2 \), by choosing a Hilbert space basis for the representation of \( u_i \), one can write \( u_i \) as a square matrix \( (u_{ij}^{(i)}) \) over \( \operatorname{Pol}(G) \subseteq A \), and \( V_i \) as a matrix \( (V_{jk}^{(i)}) \) over \( C(\Lambda_0) \) of the same size of \( (u_{ij}^{(i)}) \). Then the character \( \chi_i \) of \( U_i \) is given by
\begin{equation}
\chi_i = \sum_{r_0 \in \Lambda_0} \sum_{j=1}^{n_i} \operatorname{Tr}(v_i) \left( \sum_{k=1}^{n_i} u_{j,k}^{(i)}(r_0) u_{j,k}^{(i)} \right) \otimes \delta_{r_0} \in \operatorname{Pol}(G) \otimes C(\Lambda_0).
\end{equation}
The orthogonality relation for the nonequivalent irreducible representations \( u_1 \) and \( u_2 \) implies that
\begin{equation}
\forall j_1, k_1, j_2, k_2, \quad h \left( (u_{j_1,k_1}^{(1)})^* u_{j_2,k_2}^{(2)} \right) = 0.
\end{equation}
Hence, by \((10.8)\) and \((10.9)\),
\[
\dim \text{Mor}_{\mathbb{G} \times \Lambda_0}(U_1, U_2) = h_{\Lambda_0}(\chi_2)
\]
\[(10.10)\]
\[= |\Lambda_0|^{-1} \sum_{r_0 \in \Lambda_0} \sum_{k_1 = 1}^{n_1} \sum_{j_1 = 1}^{n_2} \text{Tr}(v_1(r_0)) \text{Tr}(v_2(r_0)) V_{k_1j_1}^{(1)}(r_0) V_{k_2j_2}^{(2)}(r_0) h \left( (u_{j_1,k_1})^* u_{j_2,k_2} \right) = 0.
\]
This proves (a).

Under the hypothesis of (b), using the same notations as in the previous paragraph, we have \(n_1 = n_2 = \dim U\). We may assume that \(c_j^{(1)} = c_j^{(2)} = c_j\), hence \(u_{jk} := u_{jk}^{(1)} = u_{jk}^{(2)}\) for all possible \(j, k\). Note that \(V_2 = bv_1\) and \(S(u, V_2, v_2) = S(u, V_1, bv_1)\) because \(bv_1 \times v_2 = V_1 \times v_2\), we may assume that \(V_2 = V_1 = V\) and \(b = 1\), with \(V_{jk} := V_{jk}^{(1)} = V_{jk}^{(2)} \in C(\Lambda_0)\) for all possible \(j, k\). Let \(\rho\) be the unique invertible positive operator in \(\text{Mor}_{\mathbb{G}}(u, u^{-1})\) such that \(\text{Tr}(\cdot \rho) = \text{Tr}(\cdot \rho^{-1})\) on \(\text{End}_G(u)\). With these assumptions, by \((10.10)\), the orthogonality relation takes the form
\[
(10.11) h(u_{j,k}^* u_{k,l}) = \frac{\delta_{j,l} (\rho^{-1})_{kk}}{\dim_q U}
\]
where \(\dim_q U = \text{Tr}(\rho) = \text{Tr}(\rho^{-1})\) is the quantum dimension of \(U\) (see [NT13, §1.4]). Since \(\rho\) is positive, we might choose the basis \(e_1, \ldots, e_n\) to diagonalize \(\rho\), so that \(\rho_{kk} = (\rho_{-1}^{-1})_{kk} = 0\) whenever \(k \neq i\). Using this basis, \((10.11)\) and \((10.10)\), we have
\[
\dim \text{Mor}_{\mathbb{G} \times \Lambda_0}(U_1, U_2)
\]
\[= |\Lambda_0|^{-1} \sum_{r_0 \in \Lambda_0} \sum_{k_1 = 1}^{n_1} \sum_{j_1 = 1}^{n_2} \text{Tr}(v_1(r_0)) \text{Tr}(v_2(r_0)) V_{k_1j_1}^{(1)}(r_0) V_{k_2j_2}^{(2)}(r_0) h \left( (u_{j_1,k_1})^* u_{j_2,k_2} \right)
\]
\[= |\Lambda_0|^{-1} \sum_{r_0 \in \Lambda_0} \sum_{k_1 = 1}^{n_1} \sum_{j_1 = 1}^{n_2} \text{Tr}(v_1(r_0)) \text{Tr}(v_2(r_0)) V_{k_1j_1}^{(1)}(r_0) V_{k_2j_2}^{(2)}(r_0) \cdot \frac{\delta_{j_1,j_2} \delta_{k_1,k_2} (\rho_{-1})_{j_2j_1}}{\dim_q U}
\]
\[(10.12)\]
\[= |\Lambda_0|^{-1} \sum_{r_0 \in \Lambda_0} \text{Tr}(v_1(r_0)) \text{Tr}(v_2(r_0)) \sum_{j_1 = 1}^{n} \frac{\sum_{k_1 = 1}^{n_1} V_{k_1j_1}(r_0)}{\dim_q U} (\rho_{-1})_{j_1j_1}
\]
Note that \(V(r_0)\) is unitary
\[= |\Lambda_0|^{-1} \sum_{r_0 \in \Lambda_0} \frac{\text{Tr}(v_1(r_0)) \text{Tr}(v_2(r_0))}{\dim_q U} \sum_{j = 1}^{n} (\rho_{-1})_{jj}
\]
\[= |\Lambda_0|^{-1} \sum_{r_0 \in \Lambda_0} \frac{\text{Tr}(v_1(r_0)) \text{Tr}(v_2(r_0))}{\dim_q U} = \dim \text{Mor}_{\Lambda_0}(v_1, v_2).
\]
This proves (b).

**Corollary 10.10.** Fix a \(\Lambda_0 \in \mathcal{G}_{\text{iso}}(\Lambda)\). Let \((u, V, v)\) be a representation parameter associated with \(\Lambda_0\), then the representation \(\mathcal{R}_{\Lambda_0}(S(u, V, v))\) of \(\mathbb{G} \times \Lambda_0\) is irreducible if and only if the representation parameter \((u, V, v)\) is irreducible.

**11. Distinguished representation parameters and distinguished representations**

Fix a \(\Lambda_0 \in \mathcal{G}_{\text{iso}}(\Lambda)\). For any unitary projective representation \(V : \Lambda_0 \to U(\mathcal{H})\) of \(\Lambda_0\), and any \(r \in \Lambda\), define \(r \cdot V\) to be the unitary projective representation of \(\Lambda_0\) on \(\mathcal{H}\) sending \(s_0 = r r_0^{-1} \in r \Lambda_0 r^{-1}\) to \((V \circ \text{Ad}_{r^{-1}})(s_0) = V(r_0)\). Then \((r s) \cdot V = r \cdot (s \cdot V)\) for all \(r, s \in \Lambda \) with \(\Lambda \cdot V = V, \) in other words, this defines an action of the group \(\Lambda\) on the class of all unitary projective representations of general isotropy subgroup of \(\Lambda\).

It is easy to see from Proposition 4.6 that whenever \(S = (u, w) \in \mathcal{CSR}_{\Lambda_0}\), the pair \(r \cdot S = (r \cdot u, r \cdot w)\) is a CSR in \(\mathcal{CSR}_{r \Lambda_0 r^{-1}}\). If \(U = \mathcal{R}_{\Lambda_0}(S)\) is the unitary representation of \(\mathbb{G} \times \Lambda_0\), then it is easy to see by restriction that \(\mathcal{R}_{\Lambda_0}(S)\) is the unitary representation of \(r \cdot U = (id \otimes \alpha_{r^{-1}}^{*} \otimes \text{Ad}_{r^{-1}})(U)\) of \(\mathbb{G} \times r \Lambda_0 r^{-1}\), as described in Proposition 6.1. Thus by Corollary 5.4, we see that \(\text{Ind}(U)\) and \(\text{Ind}(r \cdot U)\) are equivalent representations of \(\mathbb{G} \times \Lambda\).

Similarly, for any representation parameter \((u, V, v)\) associated with \(\Lambda_0\) and any \(r \in \Lambda\), the triple \((r \cdot u, r \cdot V, r \cdot v)\) is a representation parameter associated with \(r \Lambda_0 r^{-1}\), which we denoted by \(r \cdot (u, V, v)\).
This clearly defines an $\Lambda$-action on the proper class of all representation parameters associated with some general isotropy subgroup of $\Lambda$. A simple calculation shows that
\begin{equation}
\forall r \in \Lambda, \quad r \cdot S(u, V, v) = S(r \cdot (u, V, v)).
\end{equation}

**Definition 11.1.** Let $(u, V, v)$ be a representation parameter associated with some $\Lambda_0 \in \mathcal{G}_{\text{iso}}(\Lambda)$, the induced representation $\text{Ind}(\mathcal{R}_{\Lambda_0}(S(u, V, v)))$ of $G \rtimes \Lambda$ is called the representation of $G \rtimes \Lambda$ parameterized by $(u, V, v)$.

**Proposition 11.2.** Let $(u, V, v)$ be a representation parameter associated with some $\Lambda_0 \in \mathcal{G}_{\text{iso}}(\Lambda)$. Then for any $r \in \Lambda$, the representation parameters $(u, V, v)$ and $r \cdot (u, V, v)$ parameterize equivalent representations of $G \rtimes \Lambda$.

**Proof.** Since $\mathcal{R}_{\Lambda_0}(S(u, V, v))$ and $\mathcal{R}_{r \Lambda_0 r^{-1}}(r \cdot S(u, V, v))$ induces equivalent representations of $G \rtimes \Lambda$, the proposition follows from (11.1) and Definition 11.1. $\square$

**Proposition 11.3.** Fix a $\Lambda_0 \in \mathcal{G}_{\text{iso}}(\Lambda)$. Let $(u, V, v)$ be an irreducible representation parameter associated with $\Lambda_0$, $U$ denote the representation $\mathcal{R}_{\Lambda_0}(S(u, V, v))$. If $\Lambda_0 = \Lambda[u]$, then the induced representation $\text{Ind}(U)$ of $G \rtimes \Lambda$ is irreducible.

**Proof.** By Corollary 7.3, the proposition amounts to show that
\begin{equation}
\forall r, s \in \Lambda, \quad r^{-1}s \notin \Lambda_0 \implies \dim \text{Mor}_{G \rtimes \Lambda}(r \cdot U, s \cdot U) = 0,
\end{equation}
where $\Lambda(r, s) = r \Lambda_0 r^{-1} \cap \Lambda_0 s^{-1}$. Since $\Lambda_0 = \Lambda[u]$, by definition of $\Lambda[u]$, we have $[r \cdot u] \neq [s \cdot u]$ whenever $r^{-1}s \notin \Lambda_0$. Now condition (11.2) holds by Proposition 10.9. $\square$

**Definition 11.4.** Fix a $\Lambda_0 \in \mathcal{G}_{\text{iso}}(\Lambda)$, an irreducible representation parameter $(u, V, v)$ associated with $\Lambda_0$ is called distinguished if $\Lambda_0 = \Lambda[u]$. When this is the case, the irreducible unitary representation $\text{Ind}(U)$ of $G \rtimes \Lambda$ is called distinguished, where $U$ is the unitary representation $\mathcal{R}_{\Lambda_0}(S(u, V, v))$ of $G \rtimes \Lambda_0$.

**Remark 12.** The associated group of a distinguished representation parameter must be an isotropy subgroup of $\Lambda$ for the action $\Lambda \curvearrowright \text{Irr}(G)$. In other words, if the associated group of a representation parameter $(u, V, v)$ is not an isotropy subgroup, then it can never be distinguished. As we will see presently, in the formulation of our results on the classification of irreducible representations of $G \rtimes \Lambda$ and the conjugation on $\text{Irr}(G)$, only distinguished representation parameters are needed. This makes one wonder why we pose the family of general isotropy subgroup $\mathcal{G}_{\text{iso}}(\Lambda)$ instead of only isotropy subgroups. The main reason we need general isotropy subgroups of $\Lambda$ is that in proving these results, as well as the formulation and the proof of the fusion rules, we need to express the dimensions of various intertwiner spaces. The calculation of the dimensions of these intertwiner spaces will rely on Proposition 7.2, which clearly requires us to consider the intersections of isotropy subgroups, i.e. general isotropy subgroups.

**Definition 11.5.** Let $\Lambda_0$ be an isotropy subgroup of $\Lambda$ for the action $\Lambda \curvearrowright \text{Irr}(G)$, $(u_1, V_1, v_1)$ and $(u_2, V_2, v_2)$ two distinguished representation parameters associated with $\Lambda_0$. If the CSRs $S(u_1, V_1, v_1)$ and $S(u_2, V_2, v_2)$ are isomorphic in $\text{CSR}_{\Lambda_0}$, we say $(u_1, V_1, v_1)$ and $(u_2, V_2, v_2)$ are equivalent.

**Proposition 11.6.** Let $\Lambda_0$ be an isotropy subgroup of $\Lambda$ for the action $\Lambda \curvearrowright \text{Irr}(G)$, $(u_1, V_1, v_1)$ and $(u_2, V_2, v_2)$ two distinguished representation parameters associated with $\Lambda_0$. The following are equivalent:

(a) $(u_1, V_1, v_1)$ and $(u_2, V_2, v_2)$ are equivalent;
(b) $(u_1, V_1, v_1)$ and $(u_2, V_2, v_2)$ parameterize equivalent representations of $G \rtimes \Lambda_0$;
(c) there exists a mapping $b : \Lambda_0 \to T$ such that $bV_1$ and $V_2$ share the same cocycle, and both $\text{Mor}_{\Lambda_0}(u_1, u_2) \cap \text{Mor}_{\Lambda_0}(bV_1, V_2)$ and $\text{Mor}_{\Lambda_0}(V_1, v_2)$ are nonzero;
(d) there exists a mapping $b : \Lambda_0 \to T$ such that $bV_1$ and $V_2$ share the same cocycle, and both $\text{Mor}_{\Lambda_0}(u_1, u_2) \cap \text{Mor}_{\Lambda_0}(bV_1, V_2)$ and $\text{Mor}_{\Lambda_0}(V_1, v_2)$ contain unitary operators.

**Proof.** The equivalence of (a) and (b) follows directly from the definitions. It is also clear that (d) implies (c). If (c) holds, and
\begin{equation}
0 \neq S \in \text{Mor}_G(u_1, u_2) \cap \text{Mor}_{\Lambda_0}(bV_1, V_2), \quad \text{and} \quad 0 \neq T \in \text{Mor}_{\Lambda_0}(v_1, v_2) = \text{Mor}_{\Lambda_0}(b^{-1}v_1, v_2),
\end{equation}
then both $S$ and $T$ are invertible as $u_1, u_2, b^{-1}v_1, v_2$ are all irreducible. Since $u_1, u_2, bV_1, V_2, v_1, v_2$ are all unitary, we have
\begin{equation}
0 \neq Y_S \in \text{Mor}_G(u_1, u_2) \cap \text{Mor}_{\Lambda_0}(bV_1, V_2), \quad \text{and} \quad 0 \neq Y_T \in \text{Mor}_{\Lambda_0}(v_1, v_2),
\end{equation}
where \( S = \mathcal{Y}_S[S] \) is the polar decomposition of \( S \), and \( T = \mathcal{T}_T[T] \) the polar decomposition of \( T \). As \( S, T \) are invertible, \( \mathcal{Y}_S \) and \( \mathcal{Y}_T \) are unitary. This proves that (c) implies (d).

Let \( \mathcal{K}'_i \) be the representation space of \( v_i \) for \( i = 1, 2 \). By definition, \( S(u_i, V_i, v_i) = (\text{id}_{\mathcal{K}_i} \otimes u, v_i \times V_i) \), and \( b^{-1}v_i \times bV_i = v_i \times V_i \) for any mapping \( b : \Lambda_0 \to \mathbb{T} \). If (c) holds, let \( S, T \) be operators as in (11.3), then

\[
T \otimes S \in \text{Mor}_G(\text{id}_{\mathcal{K}_1} \otimes u_1, \text{id}_{\mathcal{K}_2} \otimes u_2) \cap \text{Mor}_{\Lambda_0}(v_1 \times V_1, v_2 \times V_2).
\]

Now (a) follows from (11.5), Proposition 8.4 and the fact that both \( S \) and \( T \) are invertible. Thus (c) implies (a).

We conclude the proof by showing (a) implies (d). By Shur’s lemma, and the irreducibility of \( u_1 \) and \( u_2 \), it is easy to see that

\[
\text{Mor}_G(\text{id}_{\mathcal{K}_1} \otimes u_1, \text{id}_{\mathcal{K}_2} \otimes u_2) = \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2) \otimes \text{Mor}_G(u_1, u_2).
\]

Suppose (a) holds. Then the intertwiner space (11.5) is nonzero, and

\[
\text{Mor}_G(u_1, u_2) = \mathbb{C}W_r
\]

for some unitary operator \( W_r \). By (11.6) and (a), there exists a unitary \( W_l \in \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2) \) such that

\[
W_l \otimes W_r \in \text{Mor}_{\Lambda_0}(v_1 \times V_1, v_2 \times V_2) = \text{Mor}_{\Lambda_0}((b^{-1}v_1) \times (bV_1), v_2 \times V_2).
\]

By (11.7), both \( W_lV_1W_r^* \) and \( V_2 \) are covariant projective \( \Lambda_0 \)-representations of \( u_2 \). Thus we can take a \( u_2 \)-transitional mapping \( b \) from \( W_lV_1W_r^* \) to \( V_2 \) (see Definition 10.8), i.e. a mapping \( b : \Lambda_0 \to \mathbb{T} \) such that

\[
W_r(bV_1)W_r^* = b(W_lV_1W_r^*) = V_2,
\]

which forces the cocycles of \( bV_1 \) and \( V_2 \) coincide, and

\[
W_r \in \text{Mor}_{\Lambda_0}(bV_1, V_2) \cap \text{Mor}_G(u_1, u_2).
\]

Now (11.8) and (11.10) forces

\[
W_l \in \text{Mor}_{\Lambda_0}(b^{-1}v_1, v_2) = \text{Mor}_{\Lambda_0}(v_1, bv_2).
\]

Thus (d) holds by (11.10) and (11.11).

\[\square\]

12. Density of matrix coefficients of distinguished representations

The aim of this section is to show that the linear span of matrix coefficients of distinguished representations of \( \mathbb{G} \times \Lambda \) is exactly \( \text{Pol}(\mathbb{G}) \otimes C(\Lambda) \), hence is dense in \( C(\mathbb{G} \times \Lambda) = A \otimes C(\Lambda) \) in particular. As a consequence, any irreducible unitary representation of \( \mathbb{G} \times \Lambda \) is equivalent to a distinguished one.

The following lemma essentially establishes the density of linear span of matrix coefficients of distinguished representations of \( \mathbb{G} \times \Lambda \) in \( C(\mathbb{G} \times \Lambda) = A \otimes C(\Lambda) \).

Lemma 12.1. Let \( u \) be an irreducible unitary representation of \( \mathbb{G} \) on some finite dimensional Hilbert space \( \mathcal{H} \), \( x = [u] \in \text{Irr}(\mathbb{G}) \), \( V \) the covariant projective \( \Lambda \)-representation of \( u \) with cocycle \( \omega \). Let \( M(u) \) denote the linear subspace of \( \text{Pol}(\mathbb{G}) \otimes C(\Lambda) \) spanned by matrix coefficients of distinguished representations of \( \mathbb{G} \times \Lambda \) parameterized by distinguished representation parameters of the form \( (u, V, v) \), where \( v \) runs through all irreducible unitary projective representations of \( \Lambda \) with cocycle \( \omega^{-1} = \overline{\omega} \). For any \( r \in \Lambda \), suppose \( M_r(r \cdot u) \) is the linear subspace of \( \text{Pol}(\mathbb{G}) \) spanned by matrix coefficients, then

\[
M(u) = \sum_{r \in \Lambda} M_r(r \cdot u) \otimes C(\Lambda) = \left( \sum_{r \in \Lambda} M_r(r \cdot u) \right) \otimes C(\Lambda).
\]

Proof. Take any irreducible unitary projective representation \( v \) of \( \Lambda \) on some finite dimensional Hilbert space \( \mathcal{K} \) with cocycle \( \overline{\omega} \), then \( (u, V, v) \) is a distinguished representation parameter. The distinguishedCSR \( S(u, V, v) \) subordinate to \( \Lambda \) parameterized by \( (u, V, v) \) is given by

\[
S(u, V, v) = (\mathcal{K} \otimes \mathcal{H}, \text{id}_{\mathcal{K}} \otimes u, v \times V)
\]

by definition. Let \( U = \mathcal{B}^{\Lambda_0}(S(u, V, v)) \), then the distinguished representation \( W = \text{Ind}(U) \) of \( \mathbb{G} \times \Lambda \) parameterized by \( (u, V, v) \) is obtained as follows by the construction of induced representations presented in §5. First we define a unitary representation

\[
\overline{W} = \sum_{r \in \Lambda} \epsilon_{r-1, r} \otimes \text{id}_{\mathcal{K}} \otimes (\text{id}_{\mathcal{K}} \otimes \alpha_{r-1}^*) (u) \otimes \delta_u \in \mathcal{B}(L^2(\Lambda)) \otimes \mathcal{B}(\mathcal{K}) \otimes \mathcal{B}(\mathcal{K}) \otimes \text{Pol}(\mathbb{G}) \otimes C(\Lambda).
\]

29
of $G \times \Lambda$ on $\ell^2(\Lambda) \otimes \mathcal{H} \otimes \mathcal{H}$. The subspace

$$\mathcal{H}_w(u,v) = \left\{ \sum_{r \in \Lambda} \delta_r \otimes \zeta_r : \zeta_r \in \mathcal{H} \otimes \mathcal{H}, \text{ and } \zeta_{r_0} = (v(r_0) \otimes V(r_0)) \zeta_r \text{ for all } r_0 \in \Lambda_0, r \in \Lambda \right\}$$

of $\ell^2(\Lambda) \otimes \mathcal{H} \otimes \mathcal{H}$ is invariant under $\overline{W}$ and $W$ is the subrepresentation $\mathcal{H}_u(v,v)$ of $\overline{W}$. Recall (Lemma 5.5) that the projection $\pi \in B(\ell^2(\Lambda) \otimes \mathcal{H} \otimes \mathcal{H})$ with range $\mathcal{H}_u,v,v$ is given by

$$\pi = \frac{1}{|\Lambda_0|} \sum_{r_0 \in \Lambda_0} \sum_{s \in \Lambda} c_{r_0,s} \otimes v(r_0) \otimes V(r_0).$$

Since vectors of the form $\delta_r \otimes \xi \otimes \eta, r \in \Lambda, \xi \in \mathcal{H}, \eta \in \mathcal{H}$ span $\ell^2(\Lambda) \otimes \mathcal{H} \otimes \mathcal{H}$, the matrix coefficients of $W$ are spanned by elements of $\text{Pol}(G) \otimes C(\Lambda)$ of the form

$$c(v; r, s, \xi_1, \xi_2, \eta_1, \eta_2) = (\omega_r(\delta_r \otimes \xi_2 \otimes \eta_1) \otimes \id_{\text{pol}(G)} \otimes \id_{C(\Lambda)})(W)$$

$$= (\omega_r(\delta_r \otimes \xi_2 \otimes \eta_1 \otimes \id \otimes \id)(\pi \otimes 1 \otimes 1) \overline{W}(\pi \otimes 1 \otimes 1)),$$

where the last equality follows from Lemma 5.5.

By (12.3) and (12.5), we see that

$$|\Lambda_0| \cdot \overline{W}(\pi \otimes 1 \otimes 1) = \sum_{s \in \Lambda} \sum_{r_0 \in \Lambda_0} \delta_{r_0,s} \otimes v(r_0) \otimes P(s^{-1} \cdot u) \otimes \delta_{s^{-1} \cdot r_0 r}.$$

(12.7)

(Only terms with $t = r$, and $r' = r_0 t = r_0 r$ can be nonzero)

$$= \sum_{s \in \Lambda} \sum_{r_0 \in \Lambda_0} \delta_{r_0,s} \otimes v(r_0) \otimes \left[ (s^{-1} \cdot u) \otimes \delta_{s^{-1} \cdot r_0} \right] \otimes \delta_{s'^{-1} \cdot r_0 r}.$$

Note that $r_0 s^{-1} = s \iff s' = s^{-1} r_0 r \iff s'^{-1} r_0^{-1} = s^{-1}$, by (12.6) and (12.7), we have

$$c(v; r, s, \xi_1, \xi_2, \eta_1, \eta_2) = \sum_{r_0 \in \Lambda_0} \omega_{s^{-1}}(v(s^{-1} \cdot u) \otimes \delta_{s'^{-1} \cdot r_0 r}) \otimes \delta_{s'^{-1} \cdot r_0 r}.$$

For any $r_0 \in \Lambda_0$, we have

$$\omega_{s^{-1}}(v(s^{-1} \cdot u)) \in \mathbb{C} \quad \text{and} \quad \left[ (\omega_{s^{-1}} \otimes \id)(s^{-1} \cdot u) \right] \in M_\kappa(s^{-1} \cdot u).$$

By (12.8) and (12.9), we have

$$c(v; r, s, \xi_1, \xi_2, \eta_1, \eta_2) \in M_\kappa(s^{-1} \cdot u) \otimes C(\Lambda),$$

which proves that

$$M(u) \subseteq \sum_{r' \in \Lambda} M_\kappa(r' \cdot u) \otimes C(\Lambda) = \left( \sum_{r' \in \Lambda} M_\kappa(r' \cdot u) \right) \otimes C(\Lambda).$$

It remains to establish the converse inclusion, which is easily seen to be equivalent to show that for any $r_1, r_2 \in \Lambda$, we have

$$M(u) \supseteq M_\kappa(r_1 \cdot u) \otimes \delta_{r_2}.$$

By the general theory of projective representations, there exists irreducible unitary projective representations $v_1, \ldots, v_m$ on $\mathcal{H}_1, \ldots, \mathcal{H}_m$ respectively, with all cocycle $\zeta_1, \ldots, \zeta_m \in \mathcal{H}_i$, such that

$$\sum_{i=1}^m \omega_{\zeta(i)}(\zeta(i) \otimes \id)(v_i) = \delta_e \in C(\Lambda_2).$$

By (12.8) and (12.13), we see that for any $r, s \in \Lambda$, and any $\eta_1, \eta_2 \in \mathcal{H}$, $M(u)$ contains

$$\sum_{i=1}^n c(v_i; r, s \zeta(i)_1, s \zeta(i)_2, \eta_1, \eta_2) = \sum_{r_0 \in \Lambda_0} \delta_{r_0}(r_0) \left[ \omega_{s^{-1}}(s^{-1} \cdot u) \otimes \delta_{s'^{-1} \cdot r_0 r} \right] \otimes \delta_{s'^{-1} \cdot r_0 r}.$$

(Only terms with $r_0 = e$ can be nonzero, and $V(e) = \id_{\mathcal{H}_r}$)

$$= \left[ \omega_{s^{-1}}(s^{-1} \cdot u) \otimes \delta_{s'^{-1} \cdot r_0 r} \right] \otimes \delta_{s'^{-1} \cdot r_0 r}.$$

Taking $s = r_1^{-1}$ and $r = sr_2 = r_1^{-1}r_2$ in (12.14) proves (12.12) and finishes the proof of the lemma. \qed
Proposition 12.2. The linear span of matrix coefficients of distinguished representations of $G \times \Lambda$ in $\text{Pol}(G) \otimes C(\Lambda)$ is $\text{Pol}(G) \otimes C(\Lambda)$ itself. In particular, every unitary irreducible representation of $G \times \Lambda$ is unitarily equivalent to a distinguished one.

Proof. The first assertion follows from Lemma 12.1, and the second assertion follows from the first and the orthogonality relations of irreducible representations of $G \times \Lambda$. □

13. Classification of irreducible representations of $G \times \Lambda$

For each isotropy subgroup $\Lambda_0$ of $\Lambda$, let $\mathcal{D}_{\Lambda_0}$ denote the collection of equivalence class of distinguished representation parameters associated with $\Lambda_0$. By Proposition 11.6, the mapping

$$\Psi_{\Lambda_0} : \mathcal{D}_{\Lambda_0} \to \text{Irr}(G \times \Lambda)$$

where $\Psi_{\Lambda_0}([(u, V, v)]) = \text{Ind}_{(u, V, v)}(\mathcal{R}_{\Lambda_0}(S(u, V, v)))$

is well-defined and injective. In particular, $\mathcal{D}_{\Lambda_0}$ is a set (instead of a proper class). Let $\mathcal{D}$ be the collection of equivalence classes of distinguished representation parameters associated with any isotropy subgroup of $\Lambda$. By definition, $\mathcal{D}$ is the disjoint union of $\mathcal{D}_{\Lambda_0}$ as $\Lambda_0$ runs through all isotropy subgroups of $\Lambda$, hence $\mathcal{D}$ is also a set. For any $[(u, V, v)] \in \mathcal{D}_{\Lambda_0}$ and any $r \in \Lambda$, $[(r \cdot u, V, v)] = [r \cdot (u, V, v)]$ is a well-defined class in $\mathcal{D}_{\Lambda_0} = \mathcal{D}_{\Lambda_0} \cdot r$. This defines an action of $\Lambda$ on $\mathcal{D}$. We are now ready to state and prove the classification of irreducible representations of $G \times \Lambda$.

Theorem 13.1 (Classification of irreducible representations of $G \times \Lambda$). The mapping

$$\Psi : \mathcal{D} \to \text{Irr}(G \times \Lambda)$$

where $\Psi([(u, V, v)]) = \text{Ind}(\mathcal{R}_{\Lambda_0}(S(u, V, v)))$

is surjective, and the fibers of $\Psi$ are exactly the $\Lambda$-orbits in $\mathcal{D}$.

Proof. By Proposition 12.2, $\Psi$ is surjective. By Corollary 6.2 and (11.1), each $\Lambda$-orbits in $\mathcal{D}$ maps to the same point under $\Psi$. It remains to show that if $[(u_i, V_i, v_i)]$ is a distinguished representation parameter with associated subgroup $\Lambda_i$ for $i = 1, 2$, and

$$\Psi([(u_1, V_1, v_1)]) = \Psi([(u_2, V_2, v_2)])$$

then there exists an $r_0 \in \Lambda$, such that

$$r_0 \cdot [(u_1, V_1, v_1)] = [(u_2, V_2, v_2)] \in \mathcal{D}_{\Lambda_0}.$$

Let $S_i = S(u_i, V_i, v_i)$, $U_i = \mathcal{R}_{\Lambda_i}(S_i)$ for $i = 1, 2$. If $[u_2] \notin \Lambda \cdot [u_1]$, then by Proposition 10.9, we have

$$\forall r, s \in \Lambda, \dim_{\text{Mor}_{G \times \Lambda(r)}}(r \cdot U_1 \cdot U_2) = 0$$

where $\Lambda(r, s) = r\Lambda_1 r^{-1} \cap s\Lambda_2 s^{-1}$. This is because $(r \cdot U_1)_{\Lambda = \Lambda(r, s)}$ is parameterized by the representation parameter $(u_1, V_1 \Lambda(r, s), v_1 \Lambda(r, s))$ associated with $\Lambda(r, s)$, and a similar assertion holds for $(s \cdot U_2)_{\Lambda = \Lambda(r, s)}$. Thus

$$\dim_{\text{Mor}_{G \times \Lambda}}(\text{Ind}(U_1), \text{Ind}(U_2)) = 0$$

by Proposition 7.2, which contradicts to (13.3).

Thus $[u_2] \in \Lambda \cdot [u_1]$, by replacing $[(u_1, V_1, v_1)]$ with $r_0 \cdot [(u_1, V_1, v_1)]$ for some $r_0 \in \Lambda$ if necessary, we may assume without loss of generality that $[u_1] = [u_2] \in \text{Irr}(G)$, and $\Lambda_1 = \Lambda_2$, which we now denote by $\Lambda_0$. It remains to prove that under this assumption, we have

$$[(u_1, V_1, v_1)] = [(u_2, V_2, v_2)] \in \mathcal{D}_{\Lambda_0}.$$

Since when $r^{-1} \notin \Lambda_0$ if and only if $r \cdot [u_1] \neq [u_2]$, we have

$$\forall r, s \in \Lambda, r^{-1} \notin \Lambda_0 \implies \dim_{\text{Mor}_{G \times \Lambda(r, s)}}(r \cdot U_1 \cdot U_2) = 0.$$ 

Note that when $r^{-1} \in \Lambda_0$, we have $\Lambda(r, s) = r\Lambda_0 r^{-1} = s\Lambda_0 s^{-1}$, and $[\Lambda : \Lambda(r, s)] = [\Lambda : \Lambda_0]$. By (13.3), (13.8) and Proposition 7.2, we have

$$1 = \frac{1}{[\Lambda_0]^2} \sum_{r \cdot s \in \Lambda_0 \forall r^{-1} \in \Lambda_0} \dim_{\text{Mor}_{G \times \Lambda_0}(r \cdot U_1, s \cdot U_2)}.$$

Since $r \cdot U_1, s \cdot U_2$ are both irreducible, we have

$$r^{-1} \in \Lambda_0 \implies \dim_{\text{Mor}_{G \times \Lambda_0}(r \cdot U_1, s \cdot U_2)} = 0$$
Note that there are $|\Lambda_0|^2 |\Lambda_0| = |\Lambda| \cdot |\Lambda_0|$ terms on the right side of (13.9), (13.10) forces
\begin{equation}
(13.11) \quad r^{-1}s \in \Lambda_0 \implies \dim \text{Mor}_{G \rtimes \Lambda_0, r^{-1}}(r \cdot U_1, s \cdot U_2) = 1.
\end{equation}
In particular, taking $r = s = 1_\Lambda$ in (13.11) shows that $U_1$ and $U_2$ are equivalent, hence (13.7) holds by Proposition 11.6. This finishes the proof of the theorem. \qed

14. The Conjugate Representation of Distinguished Representations

We now study the conjugation of irreducible representations of $G \times \Lambda$ in terms of the classification presented in Theorem 13.1. There is a small complication here in the non-Kac type case, where the contragredient of a unitary representation need not be unitary. Resolving this kind of question involves the so-called “modular” operator.

We begin with a simple lemma on linear operators.

**Lemma 14.1.** Let $\mathcal{H}$ be a Hilbert space, $U, P \in B(\mathcal{H})$ such that $U$ is unitary, $P$ is invertible and positive, if $PUP^{-1}$ is unitary, then $PUP^{-1} = U$, i.e. $P$ commutes with $U$.

**Proof.** Let $V = PUP^{-1}$. We have
\begin{equation}
(14.1) \quad PU^*P^{-1} = PU^{-1}P^{-1} = V^{-1} = V^* = P^{-1}U^*P.
\end{equation}
Thus $U^*$ commutes with the positive operator $P^2$. Hence $U^*$ commutes with $(P^2)^{1/2} = P$, i.e. $U^*P = PU^*$. Taking adjoints of this proves $PU = UP$. \qed

**Proposition 14.2.** Let $u$ be an irreducible unitary representation of $G$, $\Lambda_0$ a subgroup of the isotropy subgroup $\Lambda_0(u)$, $V$ a covariant projective $\Lambda_0$-representation of $u$. Then any operator $\rho$ in the $\text{Mor}_G(u, u^{cc})$ commutes with $V$ (i.e. $\rho V(r_0) = V(r_0)\rho$ for all $r_0 \in \Lambda_0$).

**Proof.** Since $u$ is irreducible, $\text{Mor}_G(u, u^{cc})$ is a one dimensional space spanned by an invertible positive operator ($[NT13$, Lemma 1.3.12$]$). By definition (see $[NT13$, Proposition 1.4.4 and Definition 1.4.5$]$), the conjugation $\overline{\rho}$ of $\rho$ is given by
\begin{equation}
(14.2) \quad \overline{\rho} = (j(\rho_u)^{1/2} \otimes 1)u^c(j(\rho_u)^{-1/2} \otimes 1),
\end{equation}
where $\rho_u$ is the unique operator in $\text{Mor}_G(u, u^{cc})$ with $\text{Tr}(\rho_u) = \text{Tr}(\rho_u^{-1})$. Since $\text{Mor}_G(u, u^{cc}) = \mathbb{C}\rho_u$, it suffices to show that $\rho_u$ commutes with $V$.

Since $u, V$ are covariant, we have
\begin{equation}
(14.3) \quad \forall r_0 \in \Lambda_0, \quad (V(r_0) \otimes 1)(r_0 \cdot u) = u(V(r_0) \otimes 1).
\end{equation}
Taking the adjoint of both sides of (14.3) then applying $j \otimes \text{id}$, we get
\begin{equation}
(14.4) \quad \forall r_0 \in \Lambda_0, \quad (V^c(r_0) \otimes 1)(r_0 \cdot u^c) = u^c(V(r_0) \otimes 1),
\end{equation}
where
\begin{equation}
(14.5) \quad V^c = (j \otimes \text{id})(V^{-1}) = (j \otimes \text{id})(U^*U)
\end{equation}
is the contragredient of $V$, and
\begin{equation}
(14.6) \quad u^c = (j \otimes \text{id})(u^{-1}) = (j \otimes \text{id})(u^*)
\end{equation}
the contragredient of $u$. We pose
\begin{equation}
(14.7) \quad \overline{V} = (j(\rho_u)^{1/2} \otimes 1)V^c(j(\rho_u)^{-1/2} \otimes 1),
\end{equation}
then by (14.4) and (14.2), we have
\begin{equation}
(14.8) \quad \forall r_0 \in \Lambda_0, \quad (\overline{V}(r_0) \otimes 1)(r_0 \cdot \overline{\rho}) = \overline{\rho}(V(r_0) \otimes 1).
\end{equation}
Thus for any $r_0 \in \Lambda_0$, $\overline{V}(r_0) \in \text{Mor}_G(r_0 \cdot \overline{\rho}, \overline{\rho})$, which is a one dimensional space spanned by a unitary operator since both $r_0 \overline{\rho}$ and $\overline{\rho}$ are irreducible unitary representations of $G$. Note that $V^c(r_0) = j(V(r_0)^*)$ is unitary, by (14.7), we have
\begin{equation}
(14.9) \quad \det(\overline{V}(r_0)) = \det(j(\rho_u)^{1/2}V^c(r_0)j(\rho_u)^{-1/2}) = \det(V^c(r_0)) \in \mathbb{T}.
\end{equation}
This forces $\overline{V}(r_0)$ to be unitary since it is a scalar multiple of a unitary operator. Applying Lemma 14.1 to (14.7) (evaluated on each $r_0 \in \Lambda_0$), we see that
\begin{equation}
(14.10) \quad V^c = \overline{V} = (j(\rho_u)^{1/2} \otimes 1)V^c(j(\rho_u)^{-1/2} \otimes 1).
\end{equation}
Applying $j \otimes \text{id}$ to the inverse of both sides of (14.10) and note that $V_{cc} = V$, we see that
\begin{equation}
V = V_{cc} = (\rho_{u/2}^1 \otimes 1)V_{cc}(\rho_{u/2}^{-1} \otimes 1) = (\rho_{u/2}^1 \otimes 1)V(\rho_{u/2}^{-1} \otimes 1),
\end{equation}
i.e. $\rho_{u/2}$ (hence $\rho_u$) commutes with $V$.

**Proposition 14.3.** Let $(u, V, v)$ be a representation parameter associated with some $\Lambda_0 \in \mathcal{G}_{iso}(\Lambda_0)$, $U$ is the unitary representation of $\mathcal{G} \times \Lambda_0$ parameterized by $(u, V, v)$, then the following holds
\begin{enumerate}[(a)]
\item $(\pi, V^c, v^*)$ is also a representation parameter;
\item $\rho_U = \text{id} \otimes \rho_u$, where $\rho_U$ (resp. $\rho_u$) is the modular operator for the representation $U$ (resp. $u$);
\item $U$ is parameterized by $(\pi, V^c, v^*)$.
\end{enumerate}

**Proof.** As we’ve seen in Proposition 14.2 and its proof, we have $\overline{V}(r_0) \in \text{Mor}_G(r_0 \cdot u, u)$ for all $r_0 \in \Lambda_0$, thus $V^c = \overline{V}$ is covariant with $\pi$. Since
\begin{equation}
\forall r_0 \in \Lambda_0, \quad (v^c \times V^c)(r_0) = j((v^c(r_0))^{-1} \otimes (V^c(r_0))^{-1}) = j((v \times V)v^c(r_0)),
\end{equation}
v^c \times V^c is the contragredient of the unitary representation $v \times V$ of $\Lambda_0$, hence is a unitary representation itself. Thus $v^c$ and $V^c$ are unitary projective representations with opposing cocycles. This proves (a).

To prove (b), by the characterizing property of $\rho_U$, it suffices to show that the invertible positive operator $\text{id} \otimes \rho_u$ satisfies
\begin{equation}
\text{id} \otimes \rho_u \in \text{Mor}_G(U, U^{cc})
\end{equation}
and (by Proposition 8.4 and Shur’s lemma applied to the irreducible representation $u$)
\begin{equation}
\text{Tr}(\text{id} \otimes \rho_u) = \text{Tr}(\text{id} \otimes \rho_u^{-1}) \in \text{End}_{\mathcal{G} \times \Lambda_0}(U) = \text{End}_{\mathcal{G}}(\text{id} \otimes u) \cap \text{End}_{\Lambda_0}(v \times V) = \mathcal{B}(\mathcal{H}_v) \otimes \text{id}.
\end{equation}
Since $\text{Tr}(\rho_u) = \text{Tr}(\rho_u^{-1})$, (14.14) holds. We now prove (14.13). As is seen in the proof of Proposition 14.2, condition (14.3) holds, and a similar calculation by applying $j \otimes \text{id}$ to the inverse of both sides of (14.3) yields (note that $V^c = V$),
\begin{equation}
\forall r_0 \in \Lambda_0, \quad (V(r_0) \otimes 1)(r_0 \cdot u^{cc}) = u^{cc}(V(r_0) \otimes 1).
\end{equation}
By definition, we have
\begin{equation}
U = (\text{id} \otimes u)_{123}(v \times V)_{124} = (\text{id} \otimes u \otimes 1)v_{14}V_{24} \in \mathcal{B}(\mathcal{H}_v) \otimes \mathcal{B}(\mathcal{H}_u) \otimes \text{Pol}(\mathcal{G}) \otimes C(\Lambda_0).
\end{equation}
Thus
\begin{equation}
U^c = (j \otimes j \otimes \text{id} \otimes \text{id})(U^{-1}) = (\text{id} \otimes u^c \otimes 1)v_{14}^cV_{24}.
\end{equation}
and
\begin{equation}
U^{cc} = (\text{id} \otimes u^{cc} \otimes 1)v_{14}^{cc}V_{24}^{cc} = (\text{id} \otimes u^{cc} \otimes 1)v_{14}^cV_{24}.
\end{equation}
By (14.16), (14.18) and Proposition 14.2, we have
\begin{equation}
(\text{id} \otimes \rho_u \otimes 1 \otimes 1)U = (\text{id} \otimes \rho_u \otimes 1 \otimes 1)(\text{id} \otimes u \otimes 1)v_{14}V_{24} = (\text{id} \otimes u^{cc} \otimes 1)v_{14}V_{24} = (\text{id} \otimes u^{cc} \otimes 1)v_{14}^cV_{24}
\end{equation}
This proves (14.13) and finishes the proof of (b).

By Proposition 8.7 and (b), $\overline{U}$ corresponds to the CSR $(u', w')$ in $\mathcal{CSR}_{\Lambda_0}$, where
\begin{equation}
u' = (\text{id} \otimes j(\rho)^{1/2} \otimes 1)(\text{id} \otimes u'(\rho)^{-1/2} \otimes 1) = \text{id} \otimes \pi,
\end{equation}
and
\begin{equation}
w' = (\text{id} \otimes j(\rho)^{1/2} \otimes 1)(V^c_{13}v_{23})(\text{id} \otimes j(\rho)^{-1/2} \otimes 1)
= v_{13}^c(\text{id} \otimes j(\rho)^{1/2} \otimes 1)V_{23}^c(\text{id} \otimes j(\rho)^{-1/2} \otimes 1)
= v_{13}^cV_{23}^c = v^c \times V^c.
\end{equation}
Thus the CSR $(u', w')$, and consequently $\overline{\pi}$, is indeed parameterized by $(\pi, v^c, V^c)$, which proves (c). □
By Proposition 11.6 and Corollary 10.10, it is clear that the conjugation of an irreducible representation parameter is irreducible, and \([u, V, v] = (\overline{u}, V^c, v^c)\) gives a well-defined mapping \(\mathfrak{D} \to \mathfrak{D}\). The following theorem describes how the conjugate representation of irreducible (unitary) representation of \(G \times \Lambda\) looks like in terms of the classification given in Theorem 13.1.

**Theorem 14.5.** Let \([u, V, v] \in \mathfrak{D}, x = \Psi([[u, V, v]]) \in \text{Irr}(G \times \Lambda)\), then

\[
(14.22) \quad \overline{x} = \Psi([[u, V, v]]) = \Psi([[\overline{u}, V^c, v^c]]).
\]

**Proof.** This follows immediately from Proposition 14.3 and the character formula (6.3) for induced representations from principal subgroups of \(G \times \Lambda\).  

15. The incidence numbers

We now turn our attention to the fusion rules of \(G \times \Lambda\). We define and study incidence numbers in this section, and use these numbers to express the fusion rules in §16.

**Definition 15.1.** For \(i = 1, 2, 3 \in G_{\text{iso}}(\Lambda)\), suppose \(U_i\) is a unitary representation of \(G \times \Lambda_i\), and \(r_i \in \Lambda_i\), then the incidence number of \((r_1, r_2, r_3)\) relative to \((U_1, U_2, U_3)\), denoted by \(m_{U_1, U_2, U_3}(r_1, r_2, r_3)\), is defined by

\[
(15.1) \quad m_{U_1, U_2, U_3}(r_1, r_2, r_3) = \dim \text{Mor}_{G \times \Lambda_0}((r_1 \cdot U_1)|_{G \times \Lambda_0}, (r_2 \cdot U_2)|_{G \times \Lambda_0} \times (r_3 \cdot U_3)|_{G \times \Lambda_0})
\]

where \(\Lambda_0 = \cap_{i=1}^3 r_i \Lambda_i r_i^{-1}\).

We now aim to express the incidence numbers in terms of characters. Let \(\Theta, \Xi \subseteq \mathfrak{E}\). Recall that \(C(\mathfrak{G}) = A\). Suppose \(F = \sum_{\alpha \in \Xi} a_{\alpha} \otimes \delta_r\), \(a_{\alpha} \in \text{A}\) is an element of \(C(\mathfrak{G}) \otimes C(\Xi) = A \otimes C(\Xi)\). We use \(F|_{G \times \Theta}\) to denote the element \(\sum_{r \in \Theta} a_r \otimes \delta_r\) in \(G \times \Theta\), and call it the restriction of \(F\) to \(G \times \Theta\). A simple calculation shows that this restriction operation gives a surjective unital morphism of \(C^*\)-algebras from \(C(\mathfrak{G} \times \Xi) = A \otimes C(\Xi)\) to \(C(\mathfrak{G} \times \Theta) = A \otimes C(\Theta)\) that also preserves comultiplication, thus allows us to view \(G \times \Theta\) as a closed subgroup of \(G \times \Xi\) in the sense of Definition 4.1. Recall that we also have the extension morphism \(E_{\Lambda_0} : (C(\Lambda_0) \to C(\Lambda_0), \delta_{0_1} \to \delta_{0_2}\) for every subgroup \(\Lambda_0\) of \(\Lambda\), which simply sends each function in \(C(\Lambda_0)\) to its unique extension in \(C(\Lambda)\) that vanishes outside \(\Lambda_0\). For the sake of discussion, we use \(h_{\Lambda_0}\) to denote the Haar state on \(G \times \Lambda_0\). For \(i = 1, 2, 3 \in G_{\text{iso}}(\Lambda)\), suppose \(U_i\) is a unitary representation of \(G \times \Lambda_i\), \(\chi_i\) is the character of \(U_i\). Let \(\Lambda_0 = \cap_{i=1}^3 r_i \Lambda_i r_i^{-1}\) then we have the following formula to calculate the incidence numbers in terms of characters.

\[
(15.2) \quad \forall r_1, r_2, r_3 \in \Lambda, \quad m_{U_1, U_2, U_3}(r_1, r_2, r_3) = h_{\Lambda_0}(r_1 \cdot \chi_1)|_{G \times \Lambda_0}, (r_2 \cdot \chi_2)|_{G \times \Lambda_0}, (r_3 \cdot \chi_3)|_{G \times \Lambda_0}.
\]

**Proposition 15.2.** Using the above notations, the incidence number \(m_{U_1, U_2, U_3}(s_1, s_2, s_3)\) depends only on the classes \([U_1], [U_2], [U_3]\) of equivalent unitary representations and the left cosets \(r_1 \Lambda_1, r_2 \Lambda_2, r_3 \Lambda_3\).

**Proof.** Note that for any \(i = 1, 2, 3, s_i \Lambda_i s_i^{-1} = r_i \Lambda_i r_i^{-1}\) whenever \(r_i^{-1} s_i \in \Lambda_i\). The proposition follows from (15.2) and Lemma 6.3(b).

By Proposition 15.2, we see immediately that the following definition is well-defined.

**Definition 15.3.** For \(i = 1, 2, 3 \in G_{\text{iso}}(\Lambda)\), suppose \(x_i\) is a class of equivalent unitary representations of \(G \times \Lambda_i\), and \(z_i \in \Lambda_i / \Lambda_i\) is a left coset of \(\Lambda_i\) in \(\Lambda_i\), then the incidence number of \((z_1, z_2, z_3)\) relative to \((x_1, x_2, x_3)\), denoted by \(m_{x_1, x_2, x_3}(z_1, z_2, z_3)\), is defined by

\[
(15.3) \quad m_{x_1, x_2, x_3}(z_1, z_2, z_3) = m_{U_1, U_2, U_3}(r_1, r_2, r_3)
\]

where \(U_i \in x_i, r_i \in \Lambda_i\) for \(i = 1, 2, 3\).

The rest of this section is devoted to the calculation of the incidence number (15.3) in terms of more basic ingredients when \(x_i = \Phi_{\Lambda_i}(p_i)\) for some \(p_i \in \mathfrak{D}_{\Lambda_i}\) (see §13), as this will be the case we need in the calculation of fusion rules of \(G \times \Lambda\) in §16. We begin with a result on the structure of unitary projective representations of some \(\Lambda_0 \subseteq G_{\text{iso}}(\Lambda)\) that are covariant with some unitary representation of \(G\).

**Lemma 15.4.** Fix a \(\Lambda_0 \subseteq G_{\text{iso}}(\Lambda)\). Let \(u_0\) be an irreducible unitary representation of \(G\), \([u_0] \in \text{Irr}(G)\) the class of \(u_0\), such that \(\Lambda_0 \subseteq \Lambda[u_0]\). Suppose \(u\) is a unitary representation of \(G, V : \Lambda_0 \to U(H_u)\) is a unitary projective representation covariant with \(u, p\) is the minimal central projection in \(\text{End}_G(u)\) corresponding to the maximal pure subrepresentation of \(u\) supported by \([u] \in \text{Irr}(G)\). Let \(q = 1 - p\), then \(V\) is diagonalizable along \(p\) in the sense that

\[
(15.4) \quad (p \otimes 1)V = V(p \otimes 1), \quad (q \otimes 1)V = V(q \otimes 1), \quad (p \otimes 1)V(q \otimes 1) = (q \otimes 1)V(p \otimes 1) = 0.
\]
Proof. Since $V$ and $u$ are covariant, we have
\begin{equation}
\forall r_0 \in \Lambda_0, \quad (V(r_0) \otimes 1)(r_0 \cdot u) = u(V(r_0) \otimes 1).
\end{equation}
Note that $p \in \text{End}_G(u) = \text{End}_G(r_0 \cdot u)$ (see (9.1)), then for every $r_0 \in \Lambda_0$, it follows that
\begin{align}
(p \otimes 1)(V(r_0) \otimes 1)(r_0 \cdot u)(q \otimes 1) &= (p \otimes 1)(V(r_0) \otimes 1)(q \otimes 1) = (p \otimes 1)(V(r_0) \otimes 1)(q \otimes 1)
\end{align}
Let $u_p$ (resp. $u_q$) be the subrepresentation of $u$ corresponding to $p$ (resp. $q$), then $r_0^{-1} \cdot u_p$ is equivalent to $u_p$ for all $r_0 \in \Lambda_0$ since $\Lambda_0 \subseteq \Lambda_{|u|}$, and
\begin{equation}
\forall r_0 \in \Lambda_0, \quad pV(r_0)q = 0.
\end{equation}
By (15.6), the operator $pV(r_0)q$, when viewed as an operator from $p(\mathfrak{H}_u)$ to $q(\mathfrak{H}_u)$, intertwines $r_0 \cdot u_q$ and $u_p$. Thus by (15.7),
\begin{equation}
\forall r_0 \in \Lambda_0, \quad qV(r_0)p = 0.
\end{equation}
Hence
\begin{equation}
pV(r_0) = pV(r_0)(p + q) = pV(r_0)p = (p + q)V(r_0)p = pV(r_0),
\end{equation}
and similarly,
\begin{equation}
qV(r_0) = qV(r_0)(p + q) = qV(r_0)q = (p + q)V(r_0)q = V(r_0)q.
\end{equation}
Now (14.4) follows from equations (15.8)–(15.11).

We also need to generalize the notion of representation parameter a little, as the natural candidate of the “tensor product” of two representation parameters need not be a representation parameter, but it still possesses the same covariant property.

Definition 15.5. Let $\Lambda_0 \in \mathfrak{G}_{\text{iso}}(\Lambda)$, we call a triple $(u, V, v)$ a generalized representation parameter (GRP) associated with $\Lambda_0$, if the following hold:
1. $V$ is a unitary projective representation of $\Lambda_0$ on $\mathfrak{H}_u$, such that
\begin{equation}
\forall r_0 \in \Lambda_0, \quad V(r_0) \in \text{Mor}_G(r_0 \cdot u, u);
\end{equation}
2. $v$ is a unitary projective representation (on some other finite dimensional Hilbert space $\mathfrak{H}_v$) of $\Lambda_0$, such that the cocycles of $v$ and $V$ are opposite to each other.

Proposition 15.6. Let $(u, V, v)$ is a GRP associated with some $\Lambda_0 \in \mathfrak{G}_{\text{iso}}(\Lambda)$, then $(\text{id} \otimes u, v \otimes V) \in \mathcal{CSR}_{\Lambda_0}$.

Proof. The proof of Proposition 10.5 applies almost verbatim here.

Definition 15.7. If $(u, V, v)$ is a GRP associated with $\Lambda_0 \in \mathfrak{G}_{\text{iso}}(\Lambda)$, then the CSR $(\text{id} \otimes u, v \otimes V)$ associated with $\Lambda_0$ and the unitary representation $\mathcal{H}_{\Lambda_0}(\text{id} \otimes u, v \otimes V)$ of $G \times \Lambda_0$ is said to be parameterized by $(u, V, v)$.

We now describe a reduction process for generalized representation parameters, which leads to our desired calculation of the incidence numbers in terms of more basic ingredients—the dimension of a certain intertwiner space of two projective representations of some generalized isotropy subgroup of $\Lambda$.

Proposition 15.8. Fix a $\Lambda_0 \in \mathfrak{G}_{\text{iso}}(\Lambda)$. Let $(u, V, v)$ be a GRP, $x \in \text{Ir}(G)$ such that $\Lambda_0 \subseteq \Lambda_x$, and $u_0 \in x$. Suppose $p$ is the minimal central projection of $\text{End}_G(u)$ corresponding to the maximal pure subrepresentation of $u$ supported by $x$. The following holds:
1. $(u_p, V_p, v)$ is a GRP, where $u_p$ (resp. $V_p$) is the subrepresentation of $u$ (resp. $V$) on $p(\mathfrak{H}_u)$;
2. let $n \in \mathbb{N}$ be the multiplicity of $x$ in $u$, $V_0$ a covariant projective $\Lambda_0$-representation of $u_0$, then up to unitary equivalence, there exists a unique unitary projective representation $V_1$ of $\Lambda_0$ on $\mathbb{C}^n$, such that $V_p$ is unitarily equivalent to $V_1 \times V_0$.

35
Lemma 15.4 and its proof, we may suppose (a)

Now (id_{C^n} \otimes u_0, v \times V_1) and (u_0, V_0, v \times V_1) in the category \( \mathcal{CSR}_{\Lambda_0} \). In particular, \((u_0, V_0, v \times V_1)\) and \((u_0, V_1, v \times V_1)\) parameterize equivalent unitary representations of \( G \times \Lambda_0 \).

Proof. By Lemma 15.4, \( u_0 \) and \( V_0 \) are covariant. Since \( V_0 \) is a subrepresentation of \( V \), it has the same cocycle as \( V \), hence \( V_0 \) and \( V \) have opposing cocycles. This proves (a).

The proof of (b) parallels that of Proposition 10.4. Since \( u_0 \) is equivalent to a direct sum of \( n \) copies of \( u_0 \), thus there exists a unitary operator \( U \in \text{Mor}_{\mathcal{F}}(\text{id}_{C^n} \otimes u_0, u_0) \). Replace \((u_0, V_0, v)\) with \((U^* u_0 U, U^* V_0 U, v)\) if necessary, we may assume \( u_0 = \mathbb{C}^n \otimes u_0 \). Repeat the proof of Proposition 10.4 with the small modification of replacing the unitary representation \( w \) there with the unitary projective representation \( V_0 \), we see that there exists a unique unitary projective representation \( V_1 : \Lambda_0 \to U(\mathbb{C}^n) \), such that \( V_0 = V_1 \times V_0 \). This proves (b).

By (b) and its proof, we may suppose \( u_0 = \text{id}_{C^n} \otimes u_0 \). Note that the CSR parameterized by \((u_0, V_0, v)\) is exactly \((\text{id}_{C^n} \otimes \text{id}_{C^n} \otimes u_0, v \times V_1)\), which coincides exactly with the CSR parameterized by \((\text{id}_{C^n} \otimes \text{id}_{C^n} \otimes u_0, V_0, v \times V_1)\) since \( v \times V_0 = v \times V_1 \times V_0 \). This proves (c).

Definition 15.9. Using the notation of Proposition 15.8, the representation parameter \((u_0, V_0, v \times V_1)\) is called a reduction of the GRP \((u, V, v)\) along \((u_0, V_0)\).

Remark 13. Since \( V_1 \) is determined up to unitary equivalence, so is the reduction \((u_0, V_0, v \times V_1)\).

The following result gives a description of the incidence numbers \( m_{(r_1, |r_2|, |r_3|) \Lambda_0} (z_1, z_2, z_3) \) in terms of the dimension of the intertwiner space of some projective representations of \( \Lambda_0 \).

Proposition 15.10. Suppose we are given the following data for each \( i = 1, 2, 3 \):

- a \( \Lambda_i \in \mathcal{F}_{\Lambda_0}(\Lambda_i) \), a left coset \( z_i \in \Lambda_i / \Lambda_i \) and a \( r_i \in z_i \);
- a representation parameter \((u_i, V_i, v_i)\) associated with \( \Lambda_i \);
- the unitary representation \( U_i \) of \( G \times \Lambda_i \) parameterized by \((u_i, V_i, v_i)\).

Let \( \Lambda_0 = \cap_{i=1}^3 z_i \Lambda_i z_i^{-1} \). Suppose \((r_1 \cdot u_1, (r_1 \cdot V_1)|_{\Lambda_0}, (r_2 \cdot u_2)|_{\Lambda_0}, (r_3 \cdot v_3)|_{\Lambda_0} \times V) \) is the reduction of the GRP \(((r_2 \cdot u_2) \times (r_3 \cdot u_3), (r_2 \cdot V_2)|_{\Lambda_0} \times (r_3 \cdot V_3)|_{\Lambda_0}, (r_2 \cdot u_2)|_{\Lambda_0} \times (r_3 \cdot v_3)|_{\Lambda_0} \times V) \) along \((r_1 \cdot u_1, (r_1 \cdot V_1)|_{\Lambda_0}) \). Then the unitary projective representations \((r_i \cdot u_i)|_{\Lambda_0} \) and \((r_i \cdot v_i)|_{\Lambda_0} \times (r_i \cdot v_3)|_{\Lambda_0} \times V \) of \( \Lambda_0 \) have the same cocycle, and

\[
(15.13) \quad m_{(r_1, |r_2|, |r_3|) \Lambda_0} (z_1, z_2, z_3) = \dim \text{Mor}_{\Lambda_0} (r_1 \cdot u_1)|_{\Lambda_0}, (r_2 \cdot u_2)|_{\Lambda_0} \times (r_3 \cdot v_3)|_{\Lambda_0} \times V).
\]

Proof. It is easy to check that \(((r_2 \cdot u_2) \times (r_3 \cdot u_3), (r_2 \cdot V_2)|_{\Lambda_0} \times (r_3 \cdot V_3)|_{\Lambda_0}, (r_2 \cdot v_2)|_{\Lambda_0} \times (r_3 \cdot v_3)|_{\Lambda_0} \times V) \) is indeed a generalized representation parameter. Take the minimal central projection \( p \in \text{End}_{\mathcal{F}}((r_2 \cdot u_2) \times (r_3 \cdot u_3)) \) corresponding to the maximal pure subrepresentation \( u_p \) of \((r_2 \cdot u_2) \times (r_3 \cdot u_3)\) that is supported by \([r_1 \cdot u_1] \in \text{Irr}(G)\). Suppose \( q = 1 - p \). By Lemma 15.4, \( q \) also corresponds to a subrepresentation \( u_q \) of \((r_2 \cdot u_2) \times (r_3 \cdot u_3)\) on \( \mathcal{F}_{\Lambda_0} \). Similarly, let \( V_p \) (resp. \( V_q \)) be the subrepresentation of the unitary projective representation \((r_2 \cdot v_2)|_{\Lambda_0} \times (r_3 \cdot u_3)|_{\Lambda_0} \) on \( p(\mathcal{F}_{\Lambda_0}) \) (resp. \( q(\mathcal{F}_{\Lambda_0}) \)). Let \( U_p \) (resp. \( U_q \)) be the representation of \( G \times \Lambda_0 \) parameterized by the GRP \(((r_2 \cdot u_2) \times (r_3 \cdot u_3), (r_2 \cdot V_2)|_{\Lambda_0} \times (r_3 \cdot v_3)|_{\Lambda_0}) \) (resp. \( (u_q, V_q, (r_2 \cdot u_2)|_{\Lambda_0} \times (r_3 \cdot v_3)|_{\Lambda_0}) \)). By construction, the unitary representation \( U \) of \( G \times \Lambda_0 \) parameterized by \(((r_2 \cdot u_2) \times (r_3 \cdot u_3), (r_2 \cdot V_2)|_{\Lambda_0} \times (r_3 \cdot v_3)|_{\Lambda_0} \times V) \) is the direct sum of \( U_p \) and \( U_q \). By definition,

\[
(15.14) \quad m_{(r_1, |r_2|, |r_3|) \Lambda_0} (z_1, z_2, z_3) = \dim \text{Mor}_{G \times \Lambda_0} (U_1, U) = \dim \text{Mor}_{G \times \Lambda_0} (U_1, U_p) + \dim \text{Mor}_{G \times \Lambda_0} (U_1, U_q).
\]

From our construction, the matrix coefficients of \( u_p \) and \( u_q \) are orthogonal with respect to the Haar state \( h \) of \( G \). Thus the proof of Proposition 10.9(a) applies almost verbatim, and shows that

\[
(15.15) \quad \dim \text{Mor}_{G \times \Lambda_0} (U_1, U_q) = 0.
\]

On the other hand, the cocycles of both \((r_1 \cdot u_1)|_{\Lambda_0} \) and \((r_2 \cdot v_2)|_{\Lambda_0} \times (r_3 \cdot v_3)|_{\Lambda_0} \times V \) are opposite to that of \((r_1 \cdot V_1)|_{\Lambda_0} \) by the reduction process described above, hence these cocycles coincide. By Proposition 10.9(b), we have

\[
(15.16) \quad \dim \text{Mor}_{G \times \Lambda_0} (U_1, U_p) = \dim \text{Mor}_{G_0} (r_1 \cdot u_1)|_{\Lambda_0}, (r_2 \cdot v_2)|_{\Lambda_0} \times (r_3 \cdot v_3)|_{\Lambda_0} \times V).
\]

Now (15.13) follows from (15.15) and (15.16).
16. Fusion rules

We now calculate the fusion rules of $G \rtimes \Lambda$. From the classification theorem Theorem 13.1, up to unitary equivalence, all unitary irreducible representations of $G \rtimes \Lambda$ are distinguished. Thus the task falls to the calculation of

\[(16.1) \dim \text{Mor}_G(\text{Ind}(U_1), \text{Ind}(U_2) \times \text{Ind}(U_3))\]

where, for $i = 1, 2, 3$, $U_i$ is the irreducible unitary representation of $G \rtimes \Lambda$, parameterized (see Definition 11.1 and Definition 11.4) by some distinguished representation parameter $(u_i, V_i, v_i)$ associated with $\Lambda_i$ (recall that $\Lambda_i = \Lambda_{\vert u_i\vert}$ since $(u_i, V_i, v_i)$ is distinguished). Let $h$ be the Haar state on $C(\mathbb{G}) = A$. For any subgroup $\Lambda_0$ of $\Lambda$, we use $h^{\Lambda_0}$ to denote the Haar state on $C(G \rtimes \Lambda_0) = A \otimes C(\Lambda_0)$, and $E_{\Lambda_0} : C(\Lambda_0) \to C(\Lambda)$ is the linear embedding such that $\delta_{\alpha_0} \in C(\Lambda_0) \mapsto \delta_{\alpha_0} \in C(\Lambda)$ (the extension of functions in $C(\Lambda_0)$ to functions in $C(\Lambda)$ that vanishes outside $\Lambda_0$). In particular, $h^{\Lambda}$ is the Haar state on $C(G \rtimes \Lambda) = A \otimes C(\Lambda)$.

For $i = 1, 2, 3$, let $\chi_i = (\text{Tr} \otimes \text{id})(U_i) \in A \otimes C(\Lambda)$ be the character of $U_i$, and $r \cdot \chi_i$ is defined to be the character of the representation $r \cdot U_i$ of $G \rtimes r\Lambda, r^{-1}$.

Using these notations, by Proposition 6.1, we have the following formula for the character of $\text{Ind}(U_i)$,

\[(16.2) \forall i = 1, 2, 3, \quad \chi(\text{Ind}(U_i)) = |\Lambda_i|^{-1} \sum_{r_i \in \Lambda_i} (\text{id} \otimes E_{r_i, \Lambda_i, r_i^{-1}})(r_i \cdot \chi_i)\]

Thus

\[(16.3) \quad \dim \text{Mor}_{G \rtimes \Lambda}(\text{Ind}(U_1), \text{Ind}(U_2) \times \text{Ind}(U_3)) = h^{\Lambda} \left( \chi(\text{Ind}(U_1)) \chi(\text{Ind}(U_2)) \chi(\text{Ind}(U_3)) \right) = \frac{\sum_{r_1, r_2, r_3} h^{\Lambda}(\text{id} \otimes E_{r_1, \Lambda_1, r_1^{-1}})(r_1 \cdot \chi_1)(\text{id} \otimes E_{r_2, \Lambda_2, r_2^{-1}})(r_2 \cdot \chi_2)(\text{id} \otimes E_{r_3, \Lambda_3, r_3^{-1}})(r_3 \cdot \chi_3)}{|\Lambda_1| \cdot |\Lambda_2| \cdot |\Lambda_3|} \]

If $\Theta, \Xi$ are subgroups of $\Lambda$ with $\Theta \subseteq \Xi$, $\sum_{r \in \Xi} a_r \otimes \delta_r$ an arbitrary element of $A \otimes C(\Xi)$ with all $a_r \in A$, we call the element $\sum_{r \in \Xi} a_r \otimes \delta_r$ of $A \otimes C(\Theta)$ the restriction of $\sum_{r \in \Xi} a_r \otimes \delta_r$ and denote it by $\left( \sum_{r \in \Xi} a_r \otimes \delta_r \right)_{\Theta \subseteq \Xi}$. Recall that

\[(16.4) \quad h^{\Lambda_0} = [\Lambda : \Lambda_0] h^{\Lambda} \circ (\text{id} \otimes E_{\Lambda_0})\]

for any subgroup $\Lambda_0$ of $\Lambda$, posing

\[(16.5) \quad \Lambda(r_1, r_2, r_3) = \prod_{i=1}^3 r_i \Lambda_i r_i^{-1},\]

we have

\[(16.6) \quad h^{\Lambda}(\text{id} \otimes E_{r_1, \Lambda_1, r_1^{-1}})(r_1 \cdot \chi_1)(\text{id} \otimes E_{r_2, \Lambda_2, r_2^{-1}})(r_2 \cdot \chi_2)(\text{id} \otimes E_{r_3, \Lambda_3, r_3^{-1}})(r_3 \cdot \chi_3) = h^{\Lambda}(\Theta)(r_1 \cdot \chi_1)_{\Theta \subseteq \Lambda}(r_2 \cdot \chi_2)_{\Theta \subseteq \Lambda}(r_3 \cdot \chi_3)_{\Theta \subseteq \Lambda} = [\Lambda : \Lambda(r_1, r_2, r_3)]^{-1} m_{U_1, U_2, U_3}(r_1, r_2, r_3),\]

where $m_{U_1, U_2, U_3}(r_1, r_2, r_3)$ is the incidence number of $(r_1, r_2, r_3)$ relative to $(U_1, U_2, U_3)$.

By (16.3) and (16.6), we have

\[(16.7) \quad \dim \text{Mor}_{G \rtimes \Lambda}(\text{Ind}(U_1), \text{Ind}(U_2) \times \text{Ind}(U_3)) = \sum_{r_1, r_2, r_3 \in \Lambda} \frac{m_{U_1, U_2, U_3}(r_1, r_2, r_3)}{|\Lambda_1| \cdot |\Lambda_2| \cdot |\Lambda_3| \cdot [\Lambda : \Lambda(r_1, r_2, r_3)]}.\]

As we’ve seen in §15, we have

\[(16.8) \quad (\forall i = 1, 2, 3, r_i \in z_i \in \Lambda/\Lambda_i) \implies m_{U_1, U_2, U_3}(z_1, z_2, z_3) = m_{U_1, U_2, U_3}(r_1, r_2, r_3),\]

and $\Lambda(z_1, z_2, z_3) := \cap_{i=1}^3 \Lambda_i r_i^{-1}$ does not depend on the choices for $r_i \in z_i, i = 1, 2, 3$. Thus (16.7) can be written more succinctly as

\[(16.9) \quad \dim \text{Mor}_{G \rtimes \Lambda}(\text{Ind}(U_1), \text{Ind}(U_2) \times \text{Ind}(U_3)) = \sum_{z_1 \in \Lambda/\Lambda_1} \sum_{z_2 \in \Lambda/\Lambda_2} \sum_{z_3 \in \Lambda/\Lambda_3} \left[ \frac{m_{U_1, U_2, U_3}(z_1, z_2, z_3)}{[\Lambda : \Lambda(z_1, z_2, z_3)]} \right].\]

We summarize the above calculation more formally as the following theorem, which describes the fusion rules of $G \rtimes \Lambda$ in terms of the more basic ingredients of incidence numbers, which in turn is...
Proposition 15.10

Theorem 16.1. The fusion rules for $\mathbb{G} \times \Lambda$ is given as the following. For $i = 1, 2, 3$, let $W_i$ be an irreducible representation of $\mathbb{G} \times \Lambda$. Suppose $U_i$ is the distinguished representation parameterized by some distinguished representation parameter $(u_i, V_i, v_i)$ associated with some isotropy subgroup $\Lambda_i$ of $\Lambda$, such that $W_i$ is equivalent to $\text{Ind}(U_i)$, then

$$\dim \text{Mor}_{\mathbb{G} \times \Lambda}(W_1, W_2 \times W_3) = \sum_{z_1 \in \Lambda} \sum_{z_2 \in \Lambda} \sum_{z_3 \in \Lambda} \frac{m(U_i, U_j, U_k)(z_1, z_2, z_3)}{[\Lambda : \Lambda(z_1, z_2, z_3)]}.$$ (16.10)

Here the incidence numbers

$$m(U_i, U_j, U_k)(z_1, z_2, z_3) = \dim \text{Mor}_{\Lambda(z_1, z_2, z_3)} \left( (r_1 \cdot v_1)|_{\Lambda_0}, (r_2 \cdot v_2)|_{\Lambda_0} \times (r_3 \cdot v_3)|_{\Lambda_0} \times V \right),$$ (16.11)

where $r_i \in z_i$ for $i = 1, 2, 3$, and the unitary projective representation $V$ of $\Lambda(z_1, z_2, z_3)$ is taken from the reduction $(r_1 \cdot u_1, (r_1 \cdot V_1)|_{\Lambda_0}, (r_2 \cdot v_2)|_{\Lambda_0} \times (r_3 \cdot v_3)|_{\Lambda_0} \times V)$ of the generalized representation parameter $((r_2 \cdot u_2) \times (r_3 \cdot u_3), (r_2 \cdot V_2)|_{\Lambda_0} \times (r_3 \cdot V_3)|_{\Lambda_0}, (r_2 \cdot v_2)|_{\Lambda_0} \times (r_3 \cdot v_3)|_{\Lambda_0})$ along $(r_1 \cdot u_1, (r_1 \cdot V_1)|_{\Lambda_0})$.

Proof. The above calculation proves (16.10), and (16.11) follows from Proposition 15.10. \qed

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E-mail address: hua.wang@imj-prg.fr

Institut de mathématiques de Jussieu; Paris Rive Gauche, Université Paris Diderot