On Integrability of the Camassa–Holm Equation and Its Invariants
A Geometrical Approach

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Abstract Using geometrical approach exposed in (Kersten et al. in J. Geom. Phys. 50:273–302, 2004 and Acta Appl. Math. 90:143–178, 2005), we explore the Camassa–Holm equation (both in its initial scalar form, and in the form of 2 × 2-system). We describe Hamiltonian and symplectic structures, recursion operators and infinite series of symmetries and conservation laws (local and nonlocal).

Keywords Camassa–Holm equation · Integrability · Hamiltonian structures · Symplectic structures · Recursion operators · Symmetries · Conservation laws · Geometrical approach

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1 Introduction

The Camassa–Holm equation was introduced in [4] in the form

\[ u_t + \mu u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \quad \mu \in \mathbb{R}, \quad (1) \]

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and was intensively explored afterwards (see, for example, Refs. [5–7, 16]). Its superizations were also constructed, see [1, 18]. Since (1) is not an evolution equation, its integrability properties (existence and even definition of Hamiltonian structures, conservation laws, etc.) are not standard to establish.

One of the ways widely used to overcome this difficulty is to introduce a new unknown \( m = u - u_{xx} \) and transform (1) to the system

\[
\begin{align*}
    m_t &= -um_x - (2m + \mu)u_x, \\
    m &= u - u_{xx},
\end{align*}
\]

which has almost evolutionary form. We stress this “almost”, because the second equation in (2) (that can be considered as a constrain to the first one) disrupts the picture and, at best, necessitates to invert the operator \( 1 - D_x^2 \). At worst, dealing with (2) as with an evolution equation may lead to fallacious results.

In our approach based on the geometrical framework exposed in Ref. [3], we treat the equation at hand as a submanifold in the manifold of infinite jets and consider two natural extensions of this equation, cf. with Ref. [15]. The first one is called the \( \ell \)-covering and serves the role of the tangent bundle. The second extension, \( \ell^* \)-covering, is the counterpart to the cotangent bundle. The key property of these extensions is that the spaces of their nonlocal (in the sense of [14]) symmetries and cosymmetries contain all essential integrability invariants of the initial equation. The efficiency of the method was tested for a number of problems (see Refs. [11–13]) and we apply it to the Camassa–Holm equation here.

In Sect. 2 we briefly expose the necessary definitions and facts. Section 3 contains computations for the Camassa–Holm equation in its matrix version (computations and results are more compact in this representation), while in Sect. 4 we reformulate them for the original form (1) and compare later the results obtained for the two alternative presentations. Finally, Sect. 5 contains discussion of the results obtained. Throughout our exposition we use a very stimulating conceptual parallel between categories of smooth manifolds and differential equations proposed initially by A.M. Vinogradov and in its modern form presented in Table 1.

This table is not just a toy dictionary but a quite helpful tool to formulate important definitions and results. For example, a bivector on a smooth manifold \( M \) may be understood as a derivation of the ring \( C^\infty(M) \) with values in \( C^\infty(T^*M) \). Translating this statement to the language of differential equations we come to the definition of variational bivectors and their description as shadows of symmetries in the \( \ell^* \)-covering (see Theorem 2 below). Another

| Table 1 | Conceptual parallel between two categories |
| --- | --- |
| **Manifolds** | **Equations** |
| Smooth manifold | Infinitely prolonged equation |
| Point | Formal solution |
| Smooth function | Conservation law |
| Vector field | Higher symmetry |
| Differential 1-form | Cosymmetry |
| de Rham complex | \((n-1)\)st line of Vinogradov’s \( \ell \)-spectral sequence |
| Tangent bundle | \( \ell \)-covering |
| Cotangent bundle | \( \ell^* \)-covering |