Nonasymptotic convergence of stochastic proximal point algorithms for constrained convex optimization

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Abstract

A very popular approach for solving stochastic optimization problems is the stochastic gradient descent method (SGD). Although the SGD iteration is computationally cheap and the practical performance of this method may be satisfactory under certain circumstances, there is recent evidence of its convergence difficulties and instability for inappropriate parameters choice. To avoid these drawbacks naturally introduced by the SGD scheme, the stochastic proximal point algorithms have been recently considered in the literature.

We introduce a new variant of the stochastic proximal point method (SPP) for solving stochastic convex optimization problems subject to (in)finite intersection of constraints satisfying a linear regularity type condition. For the newly introduced SPP scheme we prove new nonasymptotic convergence results. In particular, for convex and Lipschitz continuous objective functions, we prove nonasymptotic estimates for the rate of convergence in terms of the expected value function gap of order $O\left(\frac{1}{k^{1/2}}\right)$, where $k$ is the iteration counter. We also derive better nonasymptotic bounds for the rate of convergence in terms of expected quadratic distance from the iterates to the optimal solution for smooth strongly convex objective functions, which in the best case is of order $O\left(\frac{1}{k}\right)$. Since these convergence rates can be attained by our SPP algorithm only under some natural restrictions on the stepsize, we also introduce a restarting variant of SPP method that overcomes these difficulties and derive the corresponding nonasymptotic convergence rates. Numerical evidence supports the effectiveness of our methods in real-world problems.

Keywords: Stochastic convex optimization, intersection of convex constraints, stochastic proximal point method, nonasymptotic convergence analysis.

1. Introduction

The randomness in most of the practical optimization applications led the stochastic optimization field to become an essential tool for many applied mathematics areas, such as machine learning Moulines and Bach (2011), distributed optimization Necoara et al. (2011), sensor networks problems Blatt and Hero (2006). Since the randomness usually enters the problem through the cost function and/or the constraint set, in this paper we approach both randomness sources and consider stochastic objective functions subject to stochastic con-
In the following subsections, we recall some popular numerical optimization algorithms for solving the previous unconstrained stochastic optimization model and set the context for our contributions.

1.1 Previous work

A very popular approach for solving the unconstrained stochastic problem (1) is the stochastic gradient method Nemirovski et al. (2009); Moulines and Bach (2011); Rosasco et al. (2014). At each iteration $k$, the SGD method independently samples a component function uniformly at random $S_k$ and then takes a step along the gradient of the chosen individual function, i.e.:

$$x^{k+1} = x^k - \mu_k \nabla f(x^k; S_k),$$

where $\mu_k$ is a positive stepsize. A particular case of the continuous stochastic optimization model is the discrete stochastic model, where the random variable $S$ is discrete and thus, usually the objective function is given by the finite sum of functional components. There exists a large amount of work in the literature on deterministic and randomized algorithms for the finite-sum optimization problems. Linear convergence results on a restarted variant of SGD for finite-sum problems is given in Yang and Lin (2016). On the deterministic side, the incremental gradient methods are the deterministic (cyclic) correspondent of the SGD schemes and they were extensively analyzed in Bertsekas (2011). Another efficient class of algorithms for finite-sums are based on the common idea of updating the current iterate along the aggregated gradient step: e.g. incremental aggregated gradient (IAG) Vanli et al. (2016) or SAGA algorithm Defazio et al. (2014). There is a recent nonasymptotic convergence analysis of SGD provided in Moulines and Bach (2011), under various differentiability assumptions on the objective function. While the SGD scheme is the method of choice in practice for many machine learning applications due to its superior empirical performance, the theoretical estimates obtained in Moulines and Bach (2011) highlights several difficulties regarding its practical limitations and robustness. For example, the stepsize is highly constrained to small values by an exponential term from the convergence rate which could be catastrophically increased by uncontrolled variations of the stepsize. More precisely, the convergence rates of SGD with decreasing stepsize $\mu_k = \frac{\mu_0}{k}$, given for the quadratic mean $\mathbb{E}[\|x^k - x^*\|^2]$, where $x^*$ is an optimal solution of (1), contains certain exponential terms in the initial stepsize of the following form Moulines and Bach (2011):

$$\mathbb{E}[\|x^k - x^*\|^2] \leq \frac{C_1 e^{C_2 \mu_0^2 k}}{k^{\alpha \mu_0}} + O\left(\frac{1}{k}\right),$$

for $\mu_0 > 2/\alpha$, where $C_1, C_2$ and $\alpha$ are some positive constants. It is clear from previous convergence rate that $\mathbb{E}[\|x^k - x^*\|^2]$ can grow exponentially until the stepsizes become sufficiently small, a behavior which can be also observed in practical simulations.

Since these drawbacks are naturally introduced by the SGD scheme, other modifications have been considered for avoiding these aspects. One of them is the stochastic proximal

$$\min_{x \in \mathbb{R}^n} F(x) = (\mathbb{E}[f(x; S)]).$$

(1)
Stochastic proximal point algorithms for constrained convex optimization

point (SPP) algorithm for solving the unconstrained stochastic problem (1) having the following iteration Ryu and Boyd (2016), Toulis et al. (2016), Bianchi (2016):

\[ x^{k+1} = \arg \min_{z \in \mathbb{R}^n} \left[ f(z; S_k) + \frac{1}{2\mu_k} \|z - x^k\|^2 \right]. \]

Note that the SGD is the particular SPP method applied to the linearization of \( f(z; S_k) \) in \( x^k \), that is to the linear function \( I_f(z; x^k, S_k) = f(x^k; S_k) + \langle \nabla f(x^k; S_k), z - x^k \rangle \). Of course, when \( f \) has an easily computable proximal operator, it is natural to use \( f \) instead of its linearization \( I_f \). In Ryu and Boyd (2016), the SPP algorithm has been applied to problems with the objective function having Lipschitz continuous gradient and the following restricted strong convexity property:

\[ f(x; S) \geq f(y; S) + \langle \nabla f(y; S), x - y \rangle + \frac{1}{2} \langle M_S(x - y), x - y \rangle \quad \forall x, y \in \mathbb{R}^n, \tag{2} \]

for some matrix \( M_S \succeq 0 \), satisfying \( \lambda := \lambda_{\text{min}}(\mathbb{E}[M_S]) > 0 \). The SPP algorithm has been analyzed in Ryu and Boyd (2016) under the above assumptions and the central results were the asymptotic global convergence estimates of SPP with decreasing stepsize \( \mu_k = \frac{\mu_0}{k} \) and a nonasymptotic analysis for the SPP with constant stepsize. In particular, it has been proved that SPP converges linearly to a noise-dominated region around the optimal solution. Moreover, the following asymptotic convergence rates in the quadratic mean (i.e. for a sufficiently large \( k \)) have been given:

\[ \mathbb{E}[\|x^k - x^*\|^2] \leq \left( \frac{1}{e} \right)^{\mu_0 \lambda \ln (k+1)} C_1 + \begin{cases} \frac{C_2}{\mu_0 \lambda - 1} k \frac{1}{k^{1 \gamma}} & \text{if } \mu_0 \lambda > 1 \\ \frac{C_2}{\lambda \mu_0^{\lambda} k^{\gamma}} & \text{if } \mu_0 \lambda = 1 \\ \frac{C_2}{(1 - \mu_0 \lambda) k^{\mu_0 \lambda}} & \text{if } \mu_0 \lambda < 1, \end{cases} \]

where \( C_1 \) and \( C_2 \) are some positive constants. With the essential difference that no exponential terms in \( \mu_0 \) are encountered, these rates of convergence have similar orders with those for the classical SGD method with variable stepsize. Although in this paper we make similar assumptions on the objective function, we complement and extend the previous results. In particular, we provide a nonasymptotic convergence analysis of the stochastic proximal point method for a more general stepsize \( \mu_k = \frac{\mu_0}{k^\gamma} \), with \( \gamma > 0 \), and for constrained problems. Moreover, the Moreau smoothing framework used in the present paper leads to more elegant and intuitive proofs. Another paper related to the SPP algorithm is Toulis et al. (2016), where the considered stochastic model involves minimization of the expectation of random particular components \( f(x; S) \) defined by the composition of a smooth function and a linear operator, i.e.:

\[ f(x; S) = f(A^T_S x). \]

Moreover, the objective function \( F(x) = \mathbb{E}[f(A^T_S x)] \) needs to satisfy \( \lambda_{\text{min}}(\nabla^2 F(x)) \geq \lambda > 0 \) for all \( x \in \mathbb{R}^n \). The nonasymptotic convergence of the SPP with decreasing stepsize \( \mu_k = \frac{\mu_0}{k^\gamma} \), with \( \gamma \in (1/2, 1] \), has been analyzed in the quadratic mean and the following convergence rate has been derived in Toulis et al. (2016):

\[ \mathbb{E}[\|x^k - x^*\|^2] \leq C \left( \frac{1}{1 + \lambda \mu_0 \alpha} \right)^{k^{1 - \gamma}} + O \left( \frac{1}{k^\gamma} \right), \]
where $C$ and $\alpha$ are some positive constants. However, the analysis in Toulis et al. (2016) cannot be trivially extended to the general convex objective functions and complicated constraints, since for the proofs it is essential that each component of the objective function has the form $f(A_k x)$. In our paper we consider general convex objective functions which lack the previously discussed structure and also (in)finite number of convex constraints. Further, in Bianchi (2016) a general asymptotic convergence analysis of several variants of SPP scheme within operator theory settings has been provided, under mild (strong) convexity assumptions. A particular optimization model analyzed in Bianchi (2016), related to our paper, is:

$$\min_x f(x) \quad \text{s.t.} \quad x \in \cap_{i=1}^n X_i,$$

for which the following SPP type algorithm has been derived:

$$x_{k+1} = \begin{cases} \arg \min_{z \in \mathbb{R}^n} \left[ f(z) + \frac{1}{2\mu_k} \|z - x_k\|^2 \right] & \text{if } S_k = 0 \\ [x_k]_{X_{S_k}} & \text{otherwise} \end{cases},$$

where $S_k$ is a random variable on $\Omega = \{0, 1, \cdots, m\}$ with probability distribution $\mathbb{P}$. Although this scheme is very similar with the SPP algorithm, only the almost sure asymptotic convergence has been provided in Bianchi (2016). Other stochastic proximal (gradient) schemes together with their theoretical guarantees are studied in several recent papers as we further exemplify. In Atchade et al. (2014) a perturbed proximal gradient method is considered for solving composite optimization problems, where the gradient is intractable and approximated by Monte Carlo methods. Conditions on the stepsize and the Monte Carlo batch size are derived under which the convergence is guaranteed. Two classes of stochastic approximation strategies (stochastic iterative Tikhonov regularization and the stochastic iterative proximal point) are analyzed in Koshal et al. (2013) for monotone stochastic variational inequalities and almost sure convergence results are presented. A new stochastic optimization method is analyzed in Yurtsever et al. (2016) for the minimization of the sum of three convex functions, one of which has Lipschitz continuous gradient and satisfies a restricted strong convexity condition. New convergence results are provided in Rosasco et al. (2017) for the stochastic proximal gradient algorithm suitable for solving a large class of convex composite optimization problems. The authors derive $O\left(\frac{1}{k}\right)$ nonasymptotic bounds in expectation in the strongly convex case, as well as almost sure convergence results under weaker assumptions. In Combettes and Pesquet (2016) the asymptotic behavior of a stochastic forward-backward splitting algorithm for finding a zero of the sum of a maximally monotone set-valued operator and a cocoercive operator in Hilbert spaces is investigated. Weak and strong almost sure convergence properties of the iterates are established under mild conditions on the underlying stochastic processes. In Xu (2011) the author presents a finite sample analysis for the averaged SGD, which shows that it usually takes a huge number of samples for averaged SGD to reach its asymptotic region, for improperly chosen learning rate (stepsize). Moreover, he proposes a simple way to properly set the learning rate so that it takes a reasonable amount of data for averaged SGD to reach its asymptotic region. In Niu et al. (2011) the authors show through a novel theoretical analysis that SGD can be implemented in a parallel fashion without any locking. Moreover, for sparse optimization problems (meaning that most gradient updates only modify small parts of the decision variable) the developed scheme achieves a nearly optimal rate of convergence. A
regularized stochastic version of the BFGS method is proposed in Mokhtari and Ribeiro (2014) to solve convex optimization problems. Convergence analysis shows that lower and upper bounds on the Hessian eigenvalues of the sample functions are sufficient to guarantee convergence of order $O\left(\frac{1}{k}\right)$. A comprehensive survey on optimization algorithms for machine learning problems is given recently in Bottou et al. (2016). Based on their experience, the authors present theoretical results on a straightforward, yet versatile SGD algorithm, discuss its practical behavior, and highlight opportunities for designing new algorithms with improved performance.

1.2 Contributions

In this paper we consider both randomness sources (i.e. objective function and constraints) and thus our problem of interest involves stochastic objective functions subject to (in)finite intersection of constraints. As we previously observed, given the clear superior features of SPP algorithm over the classical SGD scheme, we also consider an SPP scheme for solving our problem of interest. The main contributions of this paper are:

**More general stochastic optimization model and a new stochastic proximal point algorithm:** While most of the existing papers from the stochastic optimization literature consider convex models without constraints or simple constraints, that is the projection onto the feasible set is easy, in this paper we consider stochastic convex optimization problems subject to (in)finite intersection of constraints satisfying a linear regularity type condition. It turns out that many practical applications, including those from machine learning, fits into this framework: e.g. regression problems, finite sum problems, portfolio optimization problems, convex feasibility problems, etc. For this general stochastic optimization model we introduce a new stochastic proximal point (SPP) algorithm. It is worth to mention that although the analysis of an SPP method for stochastic models with complicated constraints is non-trivial, our framework allows us to deal with even an infinite number of constraints.

**New nonasymptotic convergence results for the SPP method:** For the newly introduced SPP scheme we prove new nonasymptotic convergence results. In particular, for convex and Lipschitz continuous objective functions, we prove nonasymptotic estimates for the rate of convergence of the SPP scheme in terms of the expected value function gap and feasibility violation of order $O\left(\frac{1}{k^{1/2}}\right)$, where $k$ is the iteration counter. We also derive better nonasymptotic bounds for the rate of convergence of SPP scheme with decreasing stepsize $\mu_k = \frac{\mu_0}{k^\gamma}$, with $\gamma \in (0, 1]$, for smooth $\sigma_{f,S}$-strongly convex objective functions. For this case the convergence rates are given in terms of expected quadratic distance from the iterates to the optimal solution and are of order:

$$
\mathbb{E}[\|x^k - x^*\|^2] \leq C \left( \mathbb{E} \left[ \frac{1}{1 + \sigma_{f,S}\mu_0} \right] \right)^{k^{1-\gamma}} + O\left(\frac{1}{k^{\gamma}}\right).
$$

Note that the derived rates of convergence do not contain any exponential term in $\mu_0$, as is the case of the SGD scheme.

**Restarted variant of SPP algorithm and the corresponding convergence analysis:** Since the best complexity of our basic SPP scheme can be attained only under some natural restrictions on the initial stepsize $\mu_0$, we also introduce a restarting stochastic proximal point algorithm that overcomes these difficulties. The main advantage of this restarted variant of
SPP algorithm is that it is parameter-free and thus it is easily implementable in practice. Under strong convexity and smoothness assumptions, for $\gamma > 0$ and epoch counter $t$, the restarting SPP scheme with the constant stepsize (per epoch) $\frac{1}{t}$ provides a nonasymptotic complexity of order $O\left(\frac{1}{\epsilon^{1+\frac{1}{\gamma}}}\right)$.

**Paper outline.** The paper is organized as follows. In Section 2 the problem of interest is formulated and analyzed. Further in Section 3, a new stochastic proximal point algorithm is introduced and its relations with the previous work are highlighted. We provide in Section 4 the first main result of this paper regarding the nonasymptotic convergence of SPP in the convex case. Further, stronger convergence results are presented in Section 5 for smooth strongly convex objective functions. In order to improve the convergence of the simple SPP scheme, in Section 6 we introduce a restarted variant of SPP algorithm. Lastly, in Section 7 we provide some preliminary numerical simulations to highlight the empirical performance of our schemes.

**Notations.** We consider the space $\mathbb{R}^n$ composed by column vectors. For $x, y \in \mathbb{R}^n$ denote the scalar product $\langle x, y \rangle = x^T y$ and Euclidean norm by $\|x\| = \sqrt{x^T x}$. The projection operator onto the nonempty closed convex set $X$ is denoted by $[\cdot]_X$ and the distance from a given $x$ to set $X$ is denoted by $\text{dist}_X(x)$. We also define the function $\varphi_\alpha : (0, \infty) \rightarrow \mathbb{R}$:

$$\varphi_\alpha(x) = \begin{cases} (x^\alpha - 1)/\alpha, & \text{if } \alpha \neq 0 \\ \log(x), & \text{if } \alpha = 0. \end{cases}$$

**2. Problem formulation**

In many machine learning applications randomness usually enters the problem through the cost function and/or the constraint set. Minimization of problems having complicating constraints can be very challenging. This is usually alleviated by approximating the feasible set by an (in)finite intersection of simple sets Necoara et al. (2017); Nedic (2011). Therefore, in this paper we tackle the following stochastic convex constrained optimization problem:

$$F^* = \min_{x \in \mathbb{R}^n} F(x) \quad (:= \mathbb{E}[f(x; S)])$$

$$\text{s.t. } x \in X \quad (:= \cap_{S \in \Omega} X_S),$$

where $f(\cdot; S) : \mathbb{R}^n \rightarrow \mathbb{R}$ are convex functions with full domain $\text{dom} f = \mathbb{R}^n$, $X_S$ are nonempty closed convex sets, and $S$ is a random variable with its associated probability space $(\Omega, \mathbb{P})$. Notice that this formulation allows us to include (in)finite number of constraints. We denote the set of optimal solutions with $X^*$ and $x^*$ any optimal point for (3). For the optimization problem (3) we make the following assumptions.

**Assumption 1** For any $S \in \Omega$, the function $f(\cdot; S)$ is proper, closed, convex and Lipschitz continuous, that is there exists $L_{f,S} > 0$ such that

$$|f(x; S) - f(y; S)| \leq L_{f,S} \|x - y\| \quad \forall x, y \in \mathbb{R}^n.$$ 

Notice that Assumption 1 implies that any subgradient $g_f(x; S) \in \partial f(x; S)$ is bounded, that is $\|g_f(x; S)\| \leq L_{f,S}$ for all $x \in \mathbb{R}^n$ and $S \in \Omega$. For the sets we assume:
Assumption 2 Given \( S \in \Omega \), the following two properties hold:

(i) \( X_S \) are simple convex sets (i.e. projections onto these sets are easy).

(ii) There exists \( \kappa > 0 \) such that the feasible set \( X \) satisfies linear regularity:

\[
\text{dist}^2_X(x) \leq \kappa \mathbb{E}[\text{dist}^2_{X_S}(x)] \quad \forall x \in \mathbb{R}^n.
\]

Assumption 2 (ii) is known in the literature as the linear regularity property and it is essential for proving linear convergence for (alternating) projection algorithms, see Necoara et al. (2017); Nedic (2011). For example, when \( X_S \) are hyperplanes, halfspaces or convex sets with nonempty interior, then linear regularity property holds. The linear regularity property is related to quadratic functional growth condition introduced for smooth convex functions in Necoara et al. (2016). In Necoara et al. (2016) it has also been proved that several first order methods converge linearly under functional growth condition and smoothness of the objective function. Notice that this general model (3) covers interesting particular cases which we discuss below.

2.1 Convex feasibility problem

Let us consider the following objective function and constraints:

\[
f(x; S) := \frac{\lambda}{2} ||x||^2 \quad \forall S \in \Omega \quad \text{and} \quad X = \bigcap_{S \in \Omega} X_S,
\]

where \( \lambda > 0 \). Then, we obtain the least norm convex feasibility problem:

\[
\min_{x \in \mathbb{R}^n} \frac{\lambda}{2} ||x||^2 \quad \text{s.t.} \quad x \in \bigcap_{S \in \Omega} X_S.
\]

We can also consider another reformulation of the least norm convex feasibility problem:

\[
f(x; S) := \frac{\lambda_S}{2} ||x||^2 + \mathbb{I}_{X_S}(x) \quad \forall S \in \Omega,
\]

where \( \lambda_S \geq 0 \) and \( \mathbb{E}[\lambda_S] = \lambda \). Then, this leads to the stochastic optimization model:

\[
\min_{x \in \mathbb{R}^n} \mathbb{E} \left[ \frac{\lambda_S}{2} ||x||^2 + \mathbb{I}_{X_S}(x) \right].
\]

Finding a point in the intersection of a collection of closed convex sets represents a modeling paradigm for solving important applications such as data compression, neural networks and adaptive filtering, see Censor et al. (2012) for a complete list.

2.2 Regression problem

Let us consider the matrix \( A \in \mathbb{R}^{m \times n} \). For any \( S \in \Omega \subseteq \mathbb{R} \), let us define:

\[
f(x; S) := \ell(A^T_S x),
\]

where \( \ell \) is some loss function. This results in the following constrained optimization model:

\[
\min_{x \in \mathbb{R}^n} \mathbb{E}[\ell(A^T_S x)] \quad \text{s.t.} \quad x \in \bigcap_{S \in \Omega} X_S.
\]

Many learning problems can be modeled into this form, see e.g. Toulis et al. (2016). This type of optimization model has been also considered in Bianchi (2016); Rosasco et al. (2014).
2.3 Finite sum problem

Let $\Omega = \{1, \cdots, m\}$ and $\mathbb{P}$ be the uniform discrete probability distribution on $\Omega$. Further, we consider convex functions $f(x; i) = \ell_i(x)$. Then, the following constrained finite sum problem is recovered:

$$\min_{x \in \mathbb{R}^n} \frac{1}{m} \sum_{i=1}^m \ell_i(x) \quad \text{s.t.} \quad x \in \bigcap_{i=1}^m X_i.$$ 

This constrained optimization model appears often in statistics and machine learning applications, where the functions $\ell_i(\cdot)$ typically represent loss functions associated to a given estimator and the feasible set comes from physical constraints, see e.g. Defazio et al. (2014); Vanli et al. (2016); Yurtsever et al. (2016). It is also a particular problem of a more general optimization model considered in Bianchi (2016).

3. Stochastic Proximal Point algorithm

In this section we propose solving the optimization problem (3) through stochastic proximal point type algorithms. It has been proved in Necoara et al. (2017) that the optimization problem (3) can be equivalently reformulated under Assumption 2 into the following stochastic optimization problem:

$$\min_{x \in \mathbb{R}^n} F(x) := \mathbb{E} \left[ f(x; S) + \mathbb{I}_{X_S}(x) \right]. \quad (4)$$

Since each component of the stochastic objective is nonsmooth, a first possible approach is to apply stochastic subgradient methods Duchi and Singer (2009); Moulines and Bach (2011), which would yield simple algorithms, but having usually a relatively slow sublinear convergence rate. Therefore, for more robustness, one can deal with the nonsmoothness through the Moreau smoothing framework. However, there are multiple potential approaches in this direction. For a given smoothing parameter $\mu > 0$, we can smooth each functional component and the associated indicator function together to obtain the following smooth approximation for the nonsmooth convex function $f(\cdot; S) + \mathbb{I}_{X_S}$:

$$\tilde{f}_\mu(x; S) := \min_{z \in \mathbb{R}^n} f(z; S) + \mathbb{I}_{X_S}(z) + \frac{1}{2\mu} \|z - x\|^2.$$ 

Let us denote the corresponding prox operator by $\tilde{z}_\mu(x; S) = \text{arg min}_{z \in \mathbb{R}^n} f(z; S) + \mathbb{I}_{X_S}(z) + \frac{1}{2\mu} \|z - x\|^2$. It is known that any Moreau approximation $\tilde{f}_\mu(\cdot; S)$ is differentiable having the gradient $\nabla \tilde{f}_\mu(x; S) = \frac{1}{\mu}(x - \tilde{z}_\mu(x; S))$. Moreover, the gradient is Lipschitz continuous with constants bounded by $\frac{1}{\mu}$. Then, instead of solving nonsmooth problem (4) we can consider solving the smooth approximation:

$$\min_{x \in \mathbb{R}^n} \tilde{F}_\mu(x) \quad (:= \mathbb{E}[\tilde{f}_\mu(x; S)]).$$

Notice that we can easily apply the classical SGD strategy to the newly created smooth objective function, which results in the following iteration:

$$x^{k+1} = x^k - \mu_k \nabla \tilde{f}_{\mu_k}(x^k; S_k) = \tilde{z}_{\mu_k}(x^k; S_k)$$

$$= \text{arg min}_{z \in \mathbb{R}^n} f(z; S_k) + \mathbb{I}_{X_{S_k}}(z) + \frac{1}{2\mu_k} \|z - x^k\|^2.$$
However, the nonasymptotic analysis technique considered in our paper encounters difficulties with this variant of the algorithm. The main difficulty consists in proving the bound \( \| \nabla f_\mu(x;S) \| \leq \| g_f(x;S) + I_{X_S}(x) \| \) for all \( x \in \mathbb{R}^n \), where \( g_f(x;S) + I_{X_S}(x) \in \partial f(x;S) + I_{X_S}(x) \). We believe that such a bound is essential in our convergence analysis and we leave for future work the analysis of this iterative scheme. Therefore, we considered a second approach based on a smooth Moreau approximation only for the functional component \( f(x;S) \) and keeping the indicator function \( I_{X_S} \) in its original form, that is:

\[
f_\mu(x;S) := \min_{z \in \mathbb{R}^n} f(z;S) + \frac{1}{2\mu} \| z - x \|^2
\]

for some smoothing parameter \( \mu > 0 \). Then, instead of solving nonsmooth problem (4), we solve the following composite approximation:

\[
\min_{x \in \mathbb{R}^n} F_\mu(x) := \mathbb{E}[f_\mu(x;S) + I_{X_S}(x)] .
\]

Let us denote the corresponding prox operator by:

\[
z_\mu(x;S) = \arg \min_{z \in \mathbb{R}^n} f(z;S) + \frac{1}{2\mu} \| z - x \|^2 .
\]

Further, on the stochastic composite approximation (5) we can apply the stochastic projected gradient method, which leads to a stochastic proximal point like scheme for solving the original problem (3):

\[
\text{Algorithm SPP } (x_0, \{\mu_k\}_{k \geq 0})
\]

For \( k \geq 1 \) compute:

1. Choose randomly \( S_k \in \Omega \) w.r.t. probability distribution \( \mathbb{P} \)
2. Update: \( y^k = z_\mu(x^k;S_k) \) and \( x^{k+1} = [y^k]_{X_{S_k}} \)

where \( x^0 \in \mathbb{R}^n \) is some initial starting point and \( \{\mu_k\}_{k \geq 0} \) is a nonincreasing positive sequence of stepsizes. We consider that the algorithm SPP returns either the last point \( x^k \) or the average point \( \hat{x}^k = \frac{1}{\sum_{i=0}^{k-1} \mu_i} \sum_{i=0}^{k-1} \mu_i x^i \) when it is called as a subroutine. Since the update rule of the positive smoothing (stepsize) sequence \( \{\mu_k\}_{k \geq 0} \) strongly contributes to the convergence of the scheme, we discuss in the following sections the most advantageous choices.

We first prove the following useful auxiliary result:

**Lemma 3** Let \( \mu > 0 \), \( S \in \Omega \). Then, for any \( g_f(x;S) \in \partial f(x;S) \), the following holds:

\[
\| \nabla f_\mu(x;S) \| \leq \| g_f(x;S) \| \quad \forall x \in \mathbb{R}^n .
\]

**Proof** The optimality condition of problem \( \min_{z \in \mathbb{R}^n} f(z;S) + \frac{1}{2\mu} \| z - x \|^2 \) is given by:

\[
\frac{1}{\mu} (x - z_\mu(x;S)) \in \partial f(z_\mu(x;S);S) .
\]
The above inclusion easily implies that there is \( g_f(z_\mu(x; S); S) \in \partial f(z_\mu(x; S); S) \) such that:

\[
\frac{1}{\mu}||z_\mu(x; S) - x||^2 = \langle g_f(z_\mu(x; S); S), x - z_\mu(x; S) \rangle \\
= \langle g_f(x; S), x - z_\mu(x; S) \rangle + \langle g_f(z_\mu(x; S); S) - g_f(x; S), x - z_\mu(x; S) \rangle \\
\leq \langle g_f(x; S), x - z_\mu(x; S) \rangle,
\]

where in the last inequality we used the convexity of \( f \). Lastly, by applying the Cauchy-Schwarz inequality in the right hand side we get the above statement. \[\blacksquare\]

The following two well-known inequalities, which can be found in Bullen (2003), will be also useful in the sequel:

**Lemma 4 (Bernoulli)** Let \( t \in [0, 1] \) and \( x \in [-1, \infty) \), then the following holds:

\[
(1 + x)^t \leq 1 + tx.
\]

**Lemma 5 (Minkowski)** Let \( x \) and \( y \) be two random variables. Then, for any \( 1 \leq p < \infty \), the following inequality holds:

\[
(\mathbb{E}[|x + y|^p])^{1/p} \leq (\mathbb{E}[|x|^p])^{1/p} + (\mathbb{E}[|y|^p])^{1/p}.
\]

4. **Nonasymptotic complexity analysis of SPP: convex objective function**

In this section analyze, under Assumptions 1 and 2, the iteration complexity of SPP scheme with nonincreasing stepsize rule to approximately solve the optimization problem (3). In order to prove this nonasymptotic result, we first define \( \hat{\mu}_{1,k} = \sum_{i=0}^{k-1} \mu_i \) and \( \hat{\mu}_{2,k} = \sum_{i=0}^{k-1} \mu_i^2 \). Moreover, denote by \( F_k \) the history of random choices \( \{S_k\}_{k \geq 0} \), i.e. \( F_k = \{S_0, \cdots, S_k\} \).

**Lemma 6** Let Assumptions 1 and 2 hold and the sequences \( \{x^k, y^k\}_{k \geq 0} \) be generated by SPP scheme with positive stepsize \( \{\mu_k\}_{k \geq 0} \). If we define the average sequences \( \hat{x}^k = \frac{1}{\hat{\mu}_{1,k}} \sum_{i=0}^{k-1} \mu_i x^i \) and \( \hat{y}^k = \frac{1}{\hat{\mu}_{1,k}} \sum_{i=0}^{k-1} \mu_i y^i \), then the following relation holds:

\[
\mathbb{E} \left[ \text{dist}^2_{X, S_k}(\hat{y}^k) \right] \geq \frac{1}{\kappa} \mathbb{E} \left[ \text{dist}^2_{X}(\hat{x}^k) \right] - \frac{\hat{\mu}_{2,k}}{\hat{\mu}_{1,k}} \sqrt{\mathbb{E}[\text{dist}^2_{X}(\hat{x}^k)]} \sqrt{\mathbb{E}[L^2_{f,S}]}.
\]

**Proof** By using the convexity of the function \( I_{\mu,S}(x) = \frac{1}{2\mu} \text{dist}^2_{X,S}(x) \) and taking the conditional expectation w.r.t. \( S_k \) over the history \( F_{k-1} = \{S_0, \cdots, S_{k-1}\} \), we get:

\[
\mathbb{E}[\mathbb{I}_{1,S_k}(\hat{y}^k)|F_{k-1}] \geq \mathbb{E} \left[ \mathbb{I}_{1,S_k}(\hat{x}^k) + \langle \nabla \mathbb{I}_{1,S_k}(\hat{x}^k), \hat{y}^k - \hat{x}^k \rangle |F_{k-1} \right].
\]
Taking further the expectation over $\mathcal{F}_{k-1}$ we obtain:
\[
\mathbb{E}[\|I_{1,S_k}(\hat{y}^k)\|^2] \geq \mathbb{E}\left[\|I_{1,S_k}(\hat{x}^k)\|^2 + \mathbb{E}\left[\left\langle \nabla I_{1,S_k}(\hat{x}^k), \hat{y}^k - \hat{x}^k \right\rangle \right]\right]
\]
\[
\geq \mathbb{E}\left[\|I_{1,S_k}(\hat{x}^k)\|^2 + \frac{\mathbb{E}\left[\left\langle \nabla I_{1,S_k}(\hat{x}^k), \sum_{i=0}^{k-1} \mu_i \nabla f_{\mu_i}(x^i; S_i) \right\rangle \right]}{\hat{\mu}_{1,k}}\right]
\]
\[
\geq \mathbb{E}\left[\|I_{1,S_k}(\hat{x}^k)\|^2 - \hat{\mu}_{2,k}\mathbb{E}\left[\|\nabla I_{1,S_k}(\hat{x}^k)\|=0 /\sum_{i=0}^{k-1} \frac{\mu_i^2}{\hat{\mu}_{2,k}} \|\nabla f_{\mu_i}(x^i; S_i)\|\right]\right]
\]
\[
\geq \mathbb{E}\left[\|I_{1,S_k}(\hat{x}^k)\|^2 - \hat{\mu}_{2,k}\mathbb{E}\left[\|\nabla I_{1,S_k}(\hat{x}^k)\|=0 /\sum_{i=0}^{k-1} \frac{\mu_i^2}{\hat{\mu}_{2,k}} \|\nabla f_{\mu_i}(x^i; S_i)\|\right]\right].
\]

Further, using Lemma 3, Assumption 2 and Cauchy-Schwartz inequality, we have:
\[
\mathbb{E}\left[\frac{1}{2}\text{dist}^2_{X_{S_k}}(\hat{y}^k)\right] \geq \mathbb{E}\left[\frac{1}{2}\text{dist}^2_{X_{S_k}}(\hat{x}^k)\right] - \hat{\mu}_{2,k}\mathbb{E}\left[\text{dist}_{X_{S_k}}(\hat{x}^k) L_{f,S_k}\right]
\]
\[
\geq \frac{1}{2\kappa}\mathbb{E}\left[\text{dist}^2_{X}(\hat{x}^k)\right] - \frac{\hat{\mu}_{2,k}}{2\hat{\mu}_{1,k}}\sqrt{\mathbb{E}[\text{dist}^2_{X}(\hat{x}^k)] / \mathbb{E}[L_{f,S_k}^2]},
\]

which proves the statement of the lemma.

Now, we are ready to derive the convergence rate of SPP in the average sequence $\hat{x}^k$:

**Theorem 7** Under Assumptions 1 and 2, let the sequence $\{x^k\}_{k \geq 0}$ be generated by the algorithm SPP with nonincreasing positive stepsize $\{\mu_k\}_{k \geq 0}$. Define the average sequence $\hat{x}^k = \frac{1}{\hat{\mu}_{1,k}} \sum_{i=0}^{k-1} \mu_i x^i$ and $R_\mu = \mu_0 \kappa (\|x_0 - x^*\|^2 + \mathbb{E}[L_{f,S}^2]|\hat{\mu}_{2,k})$. Then, the following estimates for suboptimality and feasibility violation hold:
\[
\begin{align*}
-\kappa \mathbb{E}[L_{f,S}^2]\left(\frac{\hat{\mu}_{2,k}}{\hat{\mu}_{1,k}} + 2\mu_0\right) - \sqrt{\mathbb{E}[L_{f,S}^2]\left(\frac{R_\mu}{\hat{\mu}_{1,k}}\right)} \leq \mathbb{E}[F(\hat{x}^k)] - F^* \leq \frac{R_\mu}{2\mu_0 \kappa \hat{\mu}_{1,k}} - \kappa \mathbb{E}[\text{dist}^2_{X}(\hat{x}^k)] \leq 2\kappa^2 \mathbb{E}[L_{f,S}^2]\left(\frac{\hat{\mu}_{2,k}}{\hat{\mu}_{1,k}} + 2\mu_0\right)^2 + \frac{2R_\mu}{\hat{\mu}_{1,k}}.
\end{align*}
\]

**Proof** Since the function $z \rightarrow f(z; S) + \frac{1}{2\mu} \|z - x\|^2$ is strongly convex, we have:
\[
f(z; S) + \frac{1}{2\mu} \|z - x\|^2 \geq f(z(\mu(x; S); S) + \frac{1}{2\mu} \|z(\mu(x; S) - x\|^2 + \frac{1}{2\mu} \|\mu(x; S) - z\|^2
\]
\[
= f(\mu(x; S) + \frac{1}{2\mu} \|\mu(x; S) - z\|^2 \forall z \in \mathbb{R}^n.
\]

By taking $x = x^k, S = S_k, z = x^*, \mu = \mu_k$ in (7) and using the strictly nonexpansive property of the projection operator, see e.g. Nedic (2011):
\[
\|x - [x]_{X_{S_k}}\|^2 \leq \|x - z\|^2 - \|z - [x]_{X_{S_k}}\|^2 \forall z \in X_{S_k}, x \in \mathbb{R}^n,
\]
then these lead to:
\[
\begin{align*}
&f(x^*; S_k) + \frac{1}{2\mu_k} \|x^k - x^*\|^2 
\geq f_{\mu_k}(x^k; S_k) + \frac{1}{2\mu_k} \|y^k - x^*\|^2 \\
&\geq f_{\mu_k}(x^k; S_k) + \frac{1}{2\mu_k} \|[y^k]_{X_{S_k}} - x^*\|^2 + \frac{1}{2\mu_k} \|y^k - [y^k]_{X_{S_k}}\|^2 \\
&= f_{\mu_k}(x^k; S_k) + \frac{1}{2\mu_k} \|x^{k+1} - x^*\|^2 + \frac{1}{2\mu_k} \|y^k - x^{k+1}\|^2.
\end{align*}
\]
(9)

By denoting $\bar{I}_{\mu,S}(x) = \frac{1}{2\mu} \|x - [x]_{X_S}\|^2$, from (9), it can be easily seen that:
\[
\begin{align*}
&\mu_k(f(x^k; S_k) - f(x^*; S_k)) + I_{1,S_k}(y^k) - \frac{\mu_k^2}{2} I_{f,S_k}^2 \\
&\leq \mu_k(f(x^k; S_k) - f(x^*; S_k)) + I_{1,S_k}(y^k) - \frac{\mu_k^2}{2} \|\nabla f(x^k; S_k)\|^2 \\
&= \mu_k(f(x^k; S_k) - f(x^*; S_k)) + I_{1,S_k}(y^k) + \min_{z \in \mathbb{R}^n} \left[ \mu_k(\nabla f(x^k; S_k), z - x^k) + \frac{1}{2} \|z - x^k\|^2 \right] \\
&\leq \mu_k(f(x^k; S_k) - f(x^*; S_k)) + I_{1,S_k}(y^k) + \mu_k(\nabla f(x^k; S_k), y^k - x^k) + \frac{1}{2} \|y^k - x^k\|^2 \\
&= \mu_k(f(x^k; S_k) + \langle \nabla f(x^k; S_k), y^k - x^k \rangle + \frac{1}{2\mu_k} \|y^k - x^k\|^2 - f(x^*; S_k)) + I_{1,S_k}(y^k) \\
&\overset{\text{conv. f}}{\leq} \mu_k(f_{\mu_k}(x^k; S_k) - f(x^*; S_k)) + I_{1,S_k}(y^k) \\
&\leq \frac{1}{2} \|x^k - x^*\|^2 - \frac{1}{2} \|x^{k+1} - x^*\|^2.
\end{align*}
\]

Taking now the conditional expectation in $S_k$ w.r.t. the history $\mathcal{F}_{k-1} = \{S_0, \cdots, S_{k-1}\}$ in the last inequality we have:
\[
\mu_k(F(x^k) - F(x^*)) + E[I_{1,S_k}(y^k)|\mathcal{F}_{k-1}] - \frac{\mu_k^2}{2} E[L_{f,S_k}^2] \leq \frac{1}{2} \|x^k - x^*\|^2 - \frac{1}{2} E[\|x^{k+1} - x^*\|^2|\mathcal{F}_{k-1}].
\]

Taking further the expectation over $\mathcal{F}_{k-1}$ and summing over $i = 0, \cdots, k - 1$, results in:
\[
\begin{align*}
\frac{1}{2} \sum_{i=0}^{k-1} \mu_i \|x^0 - x^*\|^2 &\geq \frac{1}{2} \sum_{i=0}^{k-1} \sum_{i=0}^{k-1} E[\mu_i(F(x^i) - F(x^*))] + E[I_{1,S}(y^i)] - \frac{\mu_k^2}{2} E[L_{f,S_k}^2] \\
&= \frac{1}{2} \sum_{i=0}^{k-1} \sum_{i=0}^{k-1} E[\mu_i(F(x^i) - F(x^*))] + \mu_i E[I_{1,S}(y^i)] - \frac{\mu_k^2}{2} E[L_{f,S_k}^2] \\
&\geq \frac{1}{2} \sum_{i=0}^{k-1} \sum_{i=0}^{k-1} E[\mu_i(F(x^i) - F(x^*))] + \mu_i E[I_{1,S}(y^i)] - \frac{\mu_k^2}{2} E[L_{f,S_k}^2] \\
&\overset{\text{Jensen}}{\geq} E[F(x^k) - F(x^*)] + E[I_{1,S}(y^k)] - \frac{E[L_{f,S_k}^2] \mu_{2,k}}{2\mu_{1,k}}
\end{align*}
\]
(10)
The relation (10) implies the following upper bound on the suboptimality gap:

\[
\mathbb{E}[F(\hat{x}^k) - F(x^*)] \leq \frac{||x^0 - x^*||^2 + \mathbb{E}[L^2_{f,S}]\hat{\mu}_{2,k}}{2\hat{\mu}_{1,k}}. \tag{11}
\]

On the other hand, recalling \(\nabla F(x^*) = \mathbb{E}[\nabla f(x^*; S)]\), we use the following fact:

\[
\mathbb{E}[F(\hat{x}^k) - F(x^*)] \geq \mathbb{E}[\langle \nabla F(x^*), \hat{x}^k - x^* \rangle]
\]

\[
\geq \mathbb{E}[\langle \nabla F(x^*), [\hat{x}^k]_X - x^* \rangle] + \mathbb{E}[\langle \nabla F(x^*), x^k - [\hat{x}^k]_X \rangle]
\]

\[
\geq -\mathbb{E}[L_{f,S}]\mathbb{E}[\text{dist}_X(\hat{x}^k)] \quad \forall k \geq 0, \tag{12}
\]

which is derived from the optimality conditions \(\langle \nabla F(x^*), z - x^* \rangle \geq 0\) for all \(z \in X\), the Cauchy-Schwarz and Jensen inequalities. By denoting \(r_0 = ||x^0 - x^*||\) and combining (10) with Lemma 6 and the last inequality (12), we obtain:

\[
\mathbb{E}[\text{dist}_X^2(\hat{x}^k)] - \kappa \sqrt{\mathbb{E}[L^2_{f,S}]} \left(\frac{\hat{\mu}_{2,k}}{\hat{\mu}_{1,k}} + 2\mu_0\right) \sqrt{\mathbb{E}[\text{dist}_X^2(\hat{x}^k)]} \leq \frac{\mu_0\kappa r_0^2 + \mu_0\kappa \mathbb{E}[L^2_{f,S}]\hat{\mu}_{2,k}}{\hat{\mu}_{1,k}}. \tag{13}
\]

This relation clearly implies an upper bound on the feasibility residual:

\[
\sqrt{\mathbb{E}[\text{dist}_X^2(\hat{x}^k)]} \leq \kappa \sqrt{\mathbb{E}[L^2_{f,S}]} \left(\frac{\hat{\mu}_{2,k}}{\hat{\mu}_{1,k}} + 2\mu_0\right) + \sqrt{\frac{\mu_0\kappa r_0^2 + \mu_0\kappa \mathbb{E}[L^2_{f,S}]\hat{\mu}_{2,k}}{\hat{\mu}_{1,k}}}. \tag{13}
\]

Also, combining (12) and (13) we obtain the lower bound on the suboptimality gap:

\[
\mathbb{E}[F(\hat{x}^k)] - F^* \geq -\kappa \mathbb{E}[L^2_{f,S}] \left(\frac{\hat{\mu}_{2,k}}{\hat{\mu}_{1,k}} + 2\mu_0\right) - \sqrt{\mathbb{E}[L^2_{f,S}]} \sqrt{\frac{\mu_0\kappa r_0^2 + \mu_0\kappa \mathbb{E}[L^2_{f,S}]\hat{\mu}_{2,k}}{\hat{\mu}_{1,k}}}. \tag{14}
\]

From the upper and lower suboptimality bounds (11)-(14) and feasibility bound (13), we deduce our convergence rate results.

Note that the suboptimality bound (11) obtained for the SPP algorithm coincides with the one given for the standard subgradient method Nesterov (2004). Further, we provide the convergence estimates for the algorithm SPP with constant stepsize for a desired accuracy \(\varepsilon\). For simplicity, assume that \(r_0 = ||x^0 - x^*|| \geq 1\) and \(\mathbb{E}[L^2_{f,S}] \geq 2\).

**Corollary 8** Under the assumptions of Theorem 7, let \(\{x^k\}_{k \geq 0}\) be the sequence generated by algorithm SPP with constant stepsize \(\mu > 0\). Also let \(\varepsilon > 0\) be the desired accuracy, \(K\) be an integer satisfying:

\[
K \geq \frac{\mathbb{E}[L^2_{f,S}]||x^0 - x^*||^2}{\varepsilon^2} \max \left\{1, (3\kappa + \sqrt{2\kappa})^2\right\},
\]

and the stepsize be chosen as:

\[
\mu = \frac{\varepsilon}{\mathbb{E}[L^2_{f,S}] (3\kappa + \sqrt{2\kappa})}.
\]
Then, after \( K \) iterations, the average point \( \hat{x}^K = \frac{1}{K} \sum_{i=0}^{K-1} x^i \) satisfies:

\[
|E[F(\hat{x}^K)] - F^*| \leq \epsilon \quad \text{and} \quad \sqrt{E[\text{dist}_X^2(\hat{x}^K)]} \leq \epsilon.
\]

**Proof** We consider \( k = K \) in Theorem 7 and, by taking into account that \( \mu_k = \mu \) for all \( k \geq 0 \), we aim to obtain the lowest value of the right hand side of (6) by minimizing over \( \mu > 0 \). Thus, by recalling that \( r_0 = \|x^0 - x^*\| \), we obtain for the optimal smoothing parameter:

\[
\mu = \sqrt{\frac{r_0^2}{K E[L^2_{f,S}]}},
\]

the optimal rate

\[
E[F(\hat{x}^K)] - F^* \leq \sqrt{\frac{E[L^2_{f,S}]r_0^2}{K}}. \tag{15}
\]

Also using the optimal parameter \( \bar{\mu} \) into the other relations of Theorem 7 results:

\[
E[\text{dist}_X^2(\hat{x}^K)] \leq \frac{r_0^2}{K} (18\kappa^2 + 4\kappa) \tag{16}
\]

and

\[
E[F(\hat{x}^K)] - F^* \geq -(3\kappa + \sqrt{2}\kappa) \sqrt{\frac{E[L^2_{f,S}]r_0^2}{K}}. \tag{17}
\]

From the upper and lower suboptimality bounds (15)-(17) and feasibility bound (16), we deduce the following bound:

\[
K \geq \frac{E[L^2_{f,S}]r_0^2}{\epsilon^2} \max \left\{ 1, (3\kappa + \sqrt{2}\kappa)^2 \right\}
\]

which confirms our result.

In conclusion, Corollary 8 states that for a desired accuracy \( \epsilon \), if we choose a constant stepsize \( \mu = \mathcal{O}(\epsilon) \) and perform a number of SPP iterations \( \mathcal{O}\left(\frac{1}{\epsilon^2}\right) \) we obtain an \( \epsilon \)-optimal solution for our original stochastic constrained convex problem (3). Note that for convex problems with objective function having bounded subgradients the previous convergence estimates derived for the SPP algorithm are similar to those corresponding to the classical deterministic proximal point method Guler (1991) and subgradient method Nesterov (2004).

### 5. Nonasymptotic complexity analysis of SPP: strongly convex objective function

In this section we analyze the convergence behavior of the SPP scheme under smoothness and strong convexity assumptions on the objective function of constrained problem (3). Therefore, in this section the Assumption 1 is replaced by the following assumptions:
**Assumption 9** Each function $f(\cdot;S)$ is differentiable and $\sigma_{f,S}$-restricted strongly convex, that is there exists strong convexity constant $\sigma_{f,S} \geq 0$ such that:

$$f(x;S) \geq f(y;S) + \langle \nabla f(y;S), x-y \rangle + \frac{\sigma_{f,S}}{2} \|x-y\|^2 \quad \forall x, y \in \mathbb{R}^n.$$  

Moreover, the strong convexity constants $\sigma_{f,S}$ satisfy $\sigma_F = \mathbb{E}[\sigma_{f,S}] > 0$.

Notice that if for some function $f(\cdot;S)$ the corresponding constant $\sigma_{f,S}$ is equal to 0, then $f(\cdot;S)$ is only convex function. However, relation $\mathbb{E}[\sigma_{f,S}] = \sigma_F > 0$ implies that the objective function $F$ of problem (3) is strongly convex with constant $\sigma_F > 0$. In the sequel we will analyze the SPP scheme also under the following smoothness assumption:

**Assumption 10** Each function $f(\cdot;S)$ has Lipschitz gradient, that is there exists Lipschitz constant $L_{f,S} > 0$ such that:

$$\|\nabla f(x;S) - \nabla f(y;S)\| \leq L_{f,S} \|x-y\| \quad \forall x, y \in \mathbb{R}^n.$$  

Note that Assumptions 9 and 10 are standard for the convergence analysis of SPP like schemes, see e.g. Moulines and Bach (2011); Ryu and Boyd (2016). We first present an auxiliary result on the behavior of the proximal mapping $z_\mu(\cdot;S)$.

**Lemma 11** Let $f(\cdot;S)$ satisfy Assumption 9. Further, for any $S \in \Omega$ and $\mu > 0$, we define $\theta_S(\mu) = \frac{1}{1+\mu \sigma_{f,S}}$. Then, the following contraction inequality holds for the prox operator:

$$\|z_\mu(x;S) - z_\mu(y;S)\| \leq \theta_S(\mu) \|x-y\| \quad \forall x, y \in \mathbb{R}^n.$$  

**Proof** Let $\sigma_{f,S} \geq 0$ be the strong convexity constant of the function $f(\cdot;S)$. Notice that we allow the convex case, that is $\sigma_{f,S} = 0$ for some $S$. Then, it is known that the Moreau approximation $f_\mu(\cdot;S)$ is also a $\hat{\sigma}_{f,S}$-strongly convex function with strong convexity constant, see e.g. Rockafellar and Wets (1998):

$$\hat{\sigma}_{f,S} = \frac{\sigma_{f,S}}{1 + \mu \sigma_{f,S}}.$$  

Clearly, in the simple convex case, that is $\sigma_{f,S} = 0$, we also have $\hat{\sigma}_{f,S} = 0$. By denoting $\hat{L}_{f,S} = \frac{1}{\mu}$ the Lipschitz constant of the gradient of $f_\mu(\cdot;S)$, the following well-known relation holds for the smooth (and strongly) convex function $f_\mu(\cdot;S)$, see e.g. Nesterov (2004):

$$\langle \nabla f_\mu(x;S) - \nabla f_\mu(y;S), x-y \rangle \geq \frac{1}{\hat{\sigma}_{f,S} + \hat{L}_{f,S}} \|\nabla f_\mu(x;S) - \nabla f_\mu(y;S)\|^2 + \frac{\hat{\sigma}_{f,S} \hat{L}_{f,S}}{\hat{L}_{f,S} + \hat{\sigma}_{f,S}} \|x-y\|^2 \quad \forall x, y \in \mathbb{R}^n. \quad (18)$$  

By using Assumption 9, then it can be also obtained that:

$$\|\nabla f_\mu(x;S) - \nabla f_\mu(y;S)\| \geq \hat{\sigma}_{f,S} \|x-y\| \quad \forall x, y \in \mathbb{R}^n. \quad (19)$$
Using this relation, we further derive that:

\[
\|z_\mu(x; S) - z_\mu(y; S)\|^2 = \|x - y + \mu(\nabla f_\mu(y; S) - \nabla f_\mu(x; S))\|^2 \\
= \|x - y\|^2 + 2\mu(\nabla f_\mu(y; S) - \nabla f_\mu(x; S), x - y) + \mu^2\|\nabla f_\mu(x; S) - \nabla f_\mu(y; S)\|^2 \\
\leq \left(1 - \frac{2\mu\hat{\sigma}_{f,S}\hat{L}_{f,S}}{\hat{L}_{f,S} + \hat{\sigma}_{f,S}}\right)\|x - y\|^2 + \mu \left(\mu - \frac{2}{\hat{L}_{f,S} + \hat{\sigma}_{f,S}}\right)\|\nabla f_\mu(x; S) - \nabla f_\mu(y; S)\|^2 \\
\leq \left[1 + \frac{\sigma^2_{f,S}}{\sigma_{f,S} + \hat{L}_{f,S}} - \frac{2\mu}{\hat{L}_{f,S} + \hat{\sigma}_{f,S}}\right]\|x - y\|^2 \\
= (1 - \hat{\sigma}_{f,S}\mu)^2 \|x - y\|^2 \quad \forall x, y \in \mathbb{R}^n,
\]

which implies our result.

Notice that if all the functions \(f(\cdot; S)\) are just convex, that is they satisfy Assumption 9 with \(\sigma_{f,S} = 0\), then Lemma 11 highlights the nonexpansiveness property of the proximal operator \(z_\mu(\cdot; S)\). We will further keep using the notation \(\theta_S(\mu)\) for the contraction factor of the operator \(z_\mu(\cdot; S)\). Moreover, in all our proofs below, regarding the results in expectation, we use the standard technique of taking first expectation with respect to \(S_k\) conditioned on \(F_{k-1}\) and then take the expectation over the entire history \(F_{k-1}\) (see the proof of Theorem 7). For simplicity of the exposition and for saving space, we omit these details below.

### 5.1 Linear convergence to noise dominated region for constant stepsize SPP

Next we analyze the sequence generated by the SPP scheme with constant stepsize \(\mu > 0\) and provide a nonasymptotic bound on the quadratic mean \(\mathbb{E}[\|x^k - x^*\|^2]\).

**Theorem 12** Under Assumption 9, let the sequence \(\{x^k\}_{k \geq 0}\) be generated by the algorithm SPP with constant stepsize \(\mu > 0\). Further, assume \(\sigma_f^{\max} = \sup_{S \in \Omega} \sigma_{f,S} < \infty\). Then, \(\mathbb{E}[\theta^2_S(\mu)] \leq \mathbb{E}[\theta_S(\mu)] < 1\) and the following linear convergence to some region around the optimal point in the quadratic mean holds:

\[
\mathbb{E}[\|x^k - x^*\|^2] \leq 2 \left(\mathbb{E}[\theta^2_S(\mu)]\right)^k \|x^0 - x^*\|^2 + \frac{2\mu^2\mathbb{E}[\|\nabla f(x^*; S)\|^2]}{\left(1 - \sqrt{\mathbb{E}[\theta^2_S(\mu)]}\right)^2}.
\]

**Proof** First, it can be easily seen that for any \(\mu > 0\) and \(S \in \Omega\) we have \(\theta^2_S(\mu) \leq \theta_S(\mu) \leq 1\) and assuming that \(\sigma_f^{\max} < \infty\) we obtain:

\[
0 \leq \mathbb{E}[\theta^2_S(\mu)] \leq \mathbb{E}[\theta_S(\mu)] = \mathbb{E}\left[\frac{1}{1 + \mu\sigma_{f,S}}\right] \leq 1 - \mathbb{E}\left[\frac{\mu\sigma_{f,S}}{1 + \mu\sigma_{f,S}}\right] \leq 1 - \frac{\mu\sigma_f^{\max}}{1 + \mu\sigma_f^{\max}} < 1.
\]

Then, by applying Lemma 11 with \(S = S_k, x = x^k\) and \(z = x^*\), results in:

\[
\|z_\mu(x^k; S_k) - z_\mu(x^*; S_k)\| \leq \theta_{S_k}(\mu)\|x^k - x^*\|,
\]

\[
\mathbb{E}[\|x^k - x^*\|^2] \leq 2 \left(\mathbb{E}[\theta^2_{S_k}(\mu)]\right)^k \|x^0 - x^*\|^2 + \frac{2\mu^2\mathbb{E}[\|\nabla f(x^*; S_k)\|^2]}{\left(1 - \sqrt{\mathbb{E}[\theta^2_{S_k}(\mu)]}\right)^2}.
\]
which, by the triangle inequality, further implies:
\[ \| z_\mu(x^k; S_k) - x^* \| \leq \theta_{S_k}(\mu) \| x^k - x^* \| + \| z_\mu(x^*; S_k) - x^* \|. \]

By using the nonexpansiveness property of the projection operator we get that \( \| x^{k+1} - x^* \| \leq \| y^k - x^* \| \), then the last inequality leads to the recurrent relation:
\[ \| x^{k+1} - x^* \| \leq \| z_\mu(x^k; S_k) - x^* \| \leq \theta_{S_k}(\mu) \| x^k - x^* \| + \| z_\mu(x^*; S_k) - x^* \|. \]

The relation (20), Minkowski inequality and Lemma 3 lead to the following recurrence:
\begin{align*}
\sqrt{\mathbb{E}[\| x^{k+1} - x^* \|^2]} & \leq \sqrt{\mathbb{E}[(\theta_{S_k}(\mu) \| x^k - x^* \| + \| z_\mu(x^*; S_k) - x^* \|)^2]} \\
& \leq \sqrt{\mathbb{E}[\theta_{S_k}^2(\mu) \| x^k - x^* \|^2]} + \sqrt{\mathbb{E}[\| z_\mu(x^*; S_k) - x^* \|^2]} \\
& = \sqrt{\mathbb{E}\left[\theta_{S_k}^2(\mu)\right]} \sqrt{\mathbb{E}[\| x^k - x^* \|^2]} + \mu \sqrt{\mathbb{E}[\| \nabla f(x^*; S) \|^2]} \\
& \leq \sqrt{\mathbb{E}\left[\theta_{S_k}^2(\mu)\right]} \sqrt{\mathbb{E}[\| x^k - x^* \|^2]} + \mu \sqrt{\mathbb{E}[\| \nabla f(x^*; S) \|^2]}.
\end{align*}

This yields the following relation valid for all \( \mu > 0 \) and \( k \geq 0 \):
\[ \sqrt{\mathbb{E}[\| x^{k+1} - x^* \|^2]} \leq \sqrt{\mathbb{E}\left[\theta_{S_k}^2(\mu)\right]} \sqrt{\mathbb{E}[\| x^k - x^* \|^2]} + \mu \sqrt{\mathbb{E}[\| \nabla f(x^*; S) \|^2]}, \]

Denote \( r_k = \sqrt{\mathbb{E}[\| x^k - x^* \|^2]} \), \( \eta = \sqrt{\mathbb{E}[\| \nabla f(x^*; S) \|^2]} \) and \( \theta(\mu) = \sqrt{\mathbb{E}\left[\theta_{S_k}^2(\mu)\right]} \). Then, we get:
\[ r_{k+1} \leq \theta(\mu) r_k + \mu \eta. \]

Finally, a simple inductive argument leads to:
\begin{align*}
r_k & \leq r_0 \theta(\mu)^k + \mu \eta \left[1 + \theta(\mu) + \cdots + \theta(\mu)^{k-1}\right] \\
& = r_0 \theta(\mu)^k + \mu \eta \frac{1 - \theta(\mu)^k}{1 - \theta(\mu)} \\
& \leq r_0 \theta(\mu)^k + \frac{\mu \eta}{1 - \theta(\mu)}.
\end{align*}

By squaring and returning to our basic notations, we recover our statement. \( \square \)

Theorem 12 proves a linear convergence rate in expectation for the sequence generated by the SPP algorithm with a constant stepsize \( \mu > 0 \) when the sequence \( \{x^k\}_{k \geq 0} \) is outside of a noise dominated neighborhood of the optimal set of radius \( \mu \sqrt{\mathbb{E}[\| \nabla f(x^*; S) \|^2]} \). It also establishes the boundedness of the sequence \( \{x^k\}_{k \geq 0} \) when the stepsize is constant. Notice that in Ryu and Boyd (2016) a similar result has been given for an unconstrained optimization model with the difference that the convergence rate was provided for \( \mathbb{E}[\| x^k - x^* \|] \). However, our proof is simpler and more elegant, based on the properties of Moreau approximation, despite the fact that we consider the constrained case.
5.2 Nonasymptotic sublinear convergence rate of variable stepsize SPP

In this section we derive sublinear convergence rate of order $O(1/k)$ for the variable stepsize SPP scheme, in a nonasymptotic fashion. We first prove the boundedness of $\{x^k\}_{k \geq 0}$ when the stepsize is nonincreasing, which will be useful for the subsequent convergence results.

**Lemma 13** Under Assumption 9, let the sequence $\{x^k\}_{k \geq 0}$ be generated by the algorithm SPP with nonincreasing positive stepsizes $\{\mu_k\}_{k \geq 0}$. Then, the following relation holds:

$$
\mathbb{E} \left[ \|x^k - x^*\| \right] \leq \sqrt{\mathbb{E}[\|x^k - x^*\|^2]} \leq \max \left\{ \|x^0 - x^*\|, \frac{\mu_0 \sqrt{\mathbb{E}[\|\nabla f(x^*; S)\|^2]}}{1 - \sqrt{\mathbb{E}[\theta_S^2(\mu_0)]}} \right\}.
$$

**Proof** By taking $\mu = \mu_k$ in relation (21), we obtain:

$$
\sqrt{\mathbb{E}[\|x^{k+1} - x^*\|^2]} \leq \sqrt{\mathbb{E}[\theta_S^2(\mu_k)]} \sqrt{\mathbb{E}[\|x^k - x^*\|^2]} + \mu_k \sqrt{\mathbb{E}[\|\nabla f(x^*; S)\|^2]}.
$$

By using the notations $r_k = \sqrt{\mathbb{E}[\|x^k - x^*\|^2]}$, $\theta_k = \sqrt{\mathbb{E}[\theta_S^2(\mu_k)]}$ and $\eta = \sqrt{\mathbb{E}[\|\nabla f(x^*; S)\|^2]}$, the last inequality leads to:

$$
r_{k+1} \leq \theta_k r_k + (1 - \theta_k) \frac{\mu_k}{1 - \theta_k} \eta
\leq \max \left\{ r_k, \frac{\mu_k}{1 - \theta_k} \eta \right\} \leq \max \left\{ r_0, \frac{\mu_0}{1 - \theta_0} \eta, \ldots, \frac{\mu_k}{1 - \theta_k} \eta \right\}.
$$

(22)

By observing the fact that $t \mapsto \mathbb{E} \left[ \frac{\sigma_{f,S}}{(1+t\sigma_{f,S})^2} + \frac{\sigma_{f,S}}{1+t\sigma_{f,S}} \right]$ is nonincreasing in $t$, and implicitly:

$$
\frac{\mu_{k-1}}{1 - \theta_{k-1}} = \mathbb{E} \left[ \frac{\sigma_{f,S}}{(1+\mu_{k-1}\sigma_{f,S})^2} + \frac{\sigma_{f,S}}{1+\mu_{k-1}\sigma_{f,S}} \right] \geq \mathbb{E} \left[ \frac{\sigma_{f,S}}{(1+\mu_k\sigma_{f,S})^2} + \frac{\sigma_{f,S}}{1+\mu_k\sigma_{f,S}} \right] = \frac{\mu_k}{1 - \theta_k},
$$

then we have $\max_{0 \leq i \leq k} \frac{\mu_i}{1 - \theta_i} = \frac{\mu_0}{1 - \theta_0}$ and the relation (22) becomes:

$$
r_k \leq \max \left\{ r_0, \frac{\mu_0}{1 - \theta_0} \eta \right\} \quad \forall k \geq 0,
$$

(23)

which implies our result.

Furthermore, we need an upper bound on the sequence $\{\mathbb{E}[\|\nabla f(x^k; S)\|]\}_{k \geq 0}$:

**Lemma 14** Under Assumptions 9 and 10, let the sequence $\{x^k\}_{k \geq 0}$ be generated by the algorithm SPP with nonincreasing positive step sizes $\{\mu_k\}_{k \geq 0}$. Then, the following holds:

$$
\mathbb{E}[\|\nabla f(x^k; S)\|^2] \leq 2\mathbb{E}[\|\nabla f(x^*; S)\|^2] + 2\mathbb{E}[L^2_{f,S}] A^2,
$$

where $A = \max \left\{ \|x^0 - x^*\|, \frac{\mu_0 \sqrt{\mathbb{E}[\|\nabla f(x^*; S)\|^2]}}{1 - \sqrt{\mathbb{E}[\theta_S^2(\mu_0)]}} \right\}$.
Lastly, by using Lemma 13 we obtain our statement.

By taking expectation in both sides we get:

\[ \mathbb{E}[\|\nabla f(x^k; S)\|^2] \leq 2\mathbb{E}[\|\nabla f(x^*; S)\|^2] + 2L^2_{f,S}\mathbb{E}[\|x^k - x^*\|^2]. \]

By taking into account that \( E \) generated by the SPP scheme with nonincreasing stepsizes.

Finally, we provide a non-trivial upper bound on the feasibility gap, which automatically leads to a iterative descent in the distance to the feasible set of the sequence \( \{x^k\}_{k \geq 0} \), generated by the SPP scheme with nonincreasing stepsizes.

**Proof** From the Lipschitz continuity of \( \nabla f(\cdot; S) \) we have that \( \|\nabla f(x; S) - \nabla f(x^*; S)\| \leq L_{f,S}\|x - x^*\| \) for all \( x \in \mathbb{R}^n \), which implies:

\[ \|\nabla f(x^k; S)\|^2 \leq (\|\nabla f(x^*; S)\| + L_{f,S}\|x^k - x^*\|)^2 \leq 2\|\nabla f(x^*; S)\|^2 + 2L^2_{f,S}\|x^k - x^*\|^2. \]

We also present the following useful auxiliary result:

**Lemma 15** Let \( \gamma \in (0, 1] \) and the integers \( p, q \in \mathbb{N} \) with \( q \geq p \geq 1 \). Given the sequence of stepsizes \( \mu_k = \frac{\mu_0}{k^{\gamma}} \) for all \( k \geq 1 \), where \( \mu_0 > 0 \), then the following relation holds:

\[ \prod_{i=p}^{q} \mathbb{E}[\theta_S^2(\mu_i)] \leq \left( \mathbb{E} \left[ \theta_S^2(\mu_0) \right] \right)^{\varphi_{1-\gamma}(q+1) - \varphi_{1-\gamma}(p)}. \]

**Proof** From definition of \( \theta_S(\mu) \) for any \( k \geq 1 \) we have:

\[
\mathbb{E}[\theta_S^2(\mu_k)] = \mathbb{E} \left[ \frac{1}{\left( 1 + \mu_k \sigma_{f,S} \right)^2} \right] = \mathbb{E} \left[ \frac{1}{\left( 1 + \frac{\mu_0}{k^{\gamma}} \sigma_{f,S} \right)^2} \right] \\
\leq \mathbb{E} \left[ \frac{1}{\left( 1 + \mu_0 \sigma_{f,S} \right)^2} \right] \leq \mathbb{E} \left[ \frac{1}{\left( 1 + \mu_0 \sigma_{f,S} \right)^2} \right]^{\frac{1}{\gamma}} = \left( \mathbb{E}[\theta_S^2(\mu_0)] \right)^{\varphi_{1-\gamma}(1)}.
\]

By taking into account that \( \mathbb{E}[\theta_S^2(\mu_0)] = \mathbb{E} \left[ \frac{1}{\left( 1 + \mu_0 \sigma_{f,S} \right)^2} \right] \leq 1 \) and that

\[
\sum_{i=p}^{q} \frac{1}{t^{\gamma}} \geq \varphi_{1-\gamma}(q+1) - \varphi_{1-\gamma}(p) = \int_{p}^{q+1} \frac{1}{t^{\gamma}} dt = \begin{cases} \ln \frac{q+1}{p} \frac{1-\gamma}{1-\gamma} & \text{if } \gamma = 1 \\ \frac{\ln \frac{a_{1-\gamma}+1-p^{1-\gamma}}{1-\gamma} \left( a_{1-\gamma} \right) }{\gamma} & \text{if } \gamma < 1, \end{cases}
\]

then the relation (24) implies:

\[
\prod_{i=p}^{q} \mathbb{E}[\theta_S^2(\mu_i)] \leq \left( \mathbb{E} \left[ \theta_S^2(\mu_0) \right] \right)^{\sum_{i=p}^{q} \frac{1}{\gamma}} \leq \left( \mathbb{E} \left[ \theta_S^2(\mu_0) \right] \right)^{\varphi_{1-\gamma}(q+1) - \varphi_{1-\gamma}(p)}
\]

\[
= \begin{cases} 
\left( \mathbb{E} \left[ \theta_S^2(\mu_0) \right] \right)^{\frac{\ln \frac{q+1}{p} \frac{1-\gamma}{1-\gamma}}{\gamma}} & \text{if } \gamma = 1 \\
\left( \mathbb{E} \left[ \theta_S^2(\mu_0) \right] \right)^{\frac{\ln \frac{a_{1-\gamma}+1-p^{1-\gamma}}{1-\gamma} \left( a_{1-\gamma} \right) }{\gamma}} & \text{if } \gamma < 1,
\end{cases}
\]

which immediately implies the above statement.

Finally, we provide a non-trivial upper bound on the feasibility gap, which automatically leads to a iterative descent in the distance to the feasible set of the sequence \( \{x^k\}_{k \geq 0} \), generated by the SPP scheme with nonincreasing stepsizes.
Lemma 16 Under Assumptions 2, 9 and 10, let the sequence \{x^k\}_{k \geq 0} be generated by SPP scheme with nonincreasing stepsizes \{\mu_k\}_{k \geq 0}. Then, the following relation holds:

\[
\sqrt{E[dist_X^2(x^k)]} \leq \left(1 - \frac{1}{\kappa}\right)^{k/2} \left[dist_X(x^0) + 2\mu_0\kappa B\right] + 2\mu_{k-1/2}\kappa B,
\]

where \( B = \sqrt{2E[\|\nabla f(x^*;\mathcal{S})\|^2]} + A\sqrt{2E[L_f^2,\mathcal{S}]} \).

Proof By using the strictly nonexpansive property of the projection operator (8) and the linear regularity assumption, we obtain:

\[
\mathbb{E}[dist_X^2(x^{k+1})] \leq \mathbb{E}[\|x^{k+1} - [y^k]_X\|^2] \leq \mathbb{E}[\|y^k - [y^k]_X\|^2] - \mathbb{E}[\|y^k - x^{k+1}\|^2]
\]

As \( \frac{2}{\kappa} \mathbb{E}[\|y^k - [y^k]_X\|^2] - \frac{1}{\kappa} \mathbb{E}[\|y^k - [y^k]_X\|^2]
\]

\[
= \left(1 - \frac{1}{\kappa}\right) \mathbb{E}[dist_X^2(y^k)].
\]

On the other hand, from triangle inequality and Minkowski inequality, we obtain:

\[
\sqrt{\mathbb{E}[dist_X^2(y^k)]} \leq \sqrt{\mathbb{E}[\|y^k - [x^k]_X\|^2]} \leq \sqrt{\mathbb{E}[(\|y^k - x^k\| + dist_X(x^k))^2]}
\]

\[
\leq \sqrt{\mathbb{E}[\|z_{\mu_k}(x; S_k) - x^k\|^2]} + \sqrt{\mathbb{E}[dist_X^2(x^k)]}
\]

\[
= \sqrt{\mathbb{E}[dist_X^2(x^k)]} + \mu_k \sqrt{\mathbb{E}[\|\nabla f_{\mu_k}(x^k; S_k)\|^2]}
\]

Lemma 3 \( \leq \sqrt{\mathbb{E}[dist_X^2(x^k)]} + \mu_k \sqrt{\mathbb{E}[\|\nabla f(x^k; S_k)\|^2]}
\]

Lemma 14 \( \leq \sqrt{\mathbb{E}[dist_X^2(x^k)]} + \mu_k \left(\sqrt{2\mathbb{E}[\|\nabla f(x^*; \mathcal{S})\|^2]} + D\sqrt{2\mathbb{E}[L_f^2,\mathcal{S}]}\right). \)

(27)

For simplicity we use notations: \( \alpha = \sqrt{1 - \frac{1}{\kappa}}, d_k = \sqrt{\mathbb{E}[dist_X^2(x^k)]} \) and \( B = \sqrt{2\mathbb{E}[\|\nabla f(x^*; \mathcal{S})\|^2]} + A\sqrt{2\mathbb{E}[L_f^2,\mathcal{S}]} \). Combining (26) and (27) yields:

\[
d_{k+1} \leq \alpha d_k + \alpha \mu_k B \leq \alpha^{k+1}d_0 + B \sum_{i=1}^{k+1} \alpha^i \mu_{k-i+1}.
\]

(28)

Define \( m = \lceil \frac{k+1}{2} \rceil \). By dividing the sum from the right side of (28) in two parts and by taking into account that \{\mu_k\}_{k \geq 0} is nonincreasing, then results:

\[
\sum_{i=1}^{k+1} \alpha^i \mu_{k-i+1} = \sum_{i=1}^{m} \alpha^i \mu_{k-i+1} + \sum_{i=m+1}^{k+1} \alpha^i \mu_{k-i+1}
\]

\[
\leq \mu_{k-m+1} \sum_{i=1}^{m} \alpha^i + \alpha^{m+1} \sum_{i=0}^{k-m} \alpha^i \mu_{k-i-m}
\]

\[
\leq \mu_{k-m+1} \frac{\alpha (1 - \alpha^m)}{1 - \alpha} + \mu_0 \alpha^{m+1} \frac{1 - \alpha^{k-m+1}}{1 - \alpha}
\]

\[
\leq \mu_{k-m+1} \frac{\alpha}{1 - \alpha} + \alpha^{m+1} \mu_0 \frac{\alpha^m}{1 - \alpha}.
\]

(20)
By using the last inequality into (28) and using the bound \( \frac{\sigma}{1 - \sigma} \leq 2\kappa \), then these facts imply the statement of the lemma.

Now, we are ready to derive the nonasymptotic convergence rate of the Algorithm SPP with nonincreasing stepsizes. For simplicity, we will use the following exponential approximation:

\[
e^{\tau} \geq 1 + x \quad \forall x \geq 0.
\]

We will also denote \( \eta = \sqrt{\mathbb{E}[\|\nabla f(x^*; S)\|^{2}]} \) and keep the notations for \( A \) from Lemma 13 and for \( B \) from Lemma 16.

**Theorem 17** Under Assumptions 2, 9 and 10, let the sequence \( \{x^k\}_{k \geq 0} \) be generated by the algorithm SPP with the stepsize \( \mu_k = \frac{\mu_0}{k^{\gamma}} \), for all \( k \geq 1 \), with \( \mu_0 > 0 \) and \( \gamma \in (0, 1] \), and denote \( \theta_0 = \mathbb{E} \left[ \theta_S^{2}(\mu_0) \right] = \mathbb{E} \left[ \frac{1}{(1 + \mu_0 \sigma_{f,S})^{r}} \right] \). Then, the following relations hold:

(i) If \( \gamma \in (0, 1) \), then we have the following nonasymptotic convergence rates:

\[
\mathbb{E}[\|x^k - x^*\|^2] \leq \theta_0^{2\gamma - 1} \frac{r_0^2}{\gamma} + \mathcal{D} \theta_0^{2\gamma - 1} \left( k \frac{1 + 1}{2} \mu_0^{2} \varphi_1 \left( \frac{k}{2} + 1 \right) \right) + \frac{\mathcal{D} \mu_0^{2} \varphi_{k}^{2}}{(1 - \theta_0) k^{\gamma}}.
\]

(ii) If \( \gamma = 1 \), then we have the following nonasymptotic convergence rate:

\[
\mathbb{E}[\|x^k - x^*\|^2] \leq \begin{cases} 
\theta_0^{2\gamma - 1} \frac{r_0^2}{\gamma} + \frac{2\mu_0^{2}}{k \ln \left( \frac{\gamma}{\theta_0} \right) - 1} & \text{if } \theta_0 < \frac{1}{e} \\
\theta_0^{2\gamma - 1} \frac{r_0^2}{\gamma} + \frac{2\mu_0^{2}}{k^{2}} \ln \left( \frac{1}{\theta_0} \right) & \text{if } \theta_0 = \frac{1}{e} \\
\theta_0^{2\gamma - 1} \frac{r_0^2}{\gamma} + \frac{2\mu_0^{2} \ln k}{k^{2}} & \text{if } \theta_0 > \frac{1}{e},
\end{cases}
\]

where \( \mathcal{D} = 4\|\nabla f(x^*)\| \left[ \frac{\text{dist}_{\kappa}(x^0) + 2\mu_0 \kappa B}{\mu_0 \ln(\kappa/(\kappa - 1))} \right] + 3\kappa \mathcal{B} \] \( + 2\eta \sqrt{2\gamma^2 + 2\mathbb{E}[L_{f,S}^2]A^2} + 2\eta A \sqrt{\mathbb{E}[L_{f,S}^2]} \).

**Proof** Let \( \mu > 0, x \in \mathbb{R}^n \) and \( S \in \Omega \), then we have:

\[
\frac{1}{2} \|z_\mu(x; S) - x^*\|^2
\]

\[
= \frac{1}{2} \|z_\mu(x; S) - z_\mu(x^*; S)\|^2 + \langle z_\mu(x; S) - z_\mu(x^*; S), z_\mu(x^*; S) - x^* \rangle + \frac{1}{2} \|z_\mu(x^*; S) - x^*\|^2
\]

\[
\leq \frac{\theta_\mu^2(\mu)}{2} \|x - x^*\|^2 - \mu \langle \nabla f(x^*; S), x - x^* \rangle + \langle z_\mu(x^*; S) - x^* + \mu \nabla f(x^*; S), x - x^* \rangle
\]

\[
+ \langle z_\mu(x; S) - x, z_\mu(x^*; S) - x^* \rangle - \frac{\mu^2}{2} \|\nabla f_\mu(x^*; S)\|^2.
\]

Now we take expectation in both sides and consider \( x = x^k \) and \( \mu = \mu_k \). We thus seek a bound for each term from the right hand side in (30). For the second term, by using the
optimality conditions $\langle \nabla F(x^*), z - x^* \rangle \geq 0$ for all $z \in X$, we have:

$$
\mathbb{E}[\langle \nabla f(x^*; S), x^* - k \rangle] = \mathbb{E}[\langle \nabla F(x^*), x^* - [k]_X \rangle] + \mathbb{E}[\langle \nabla F(x^*), [k]_X - k \rangle] \\
\leq \mathbb{E}[\langle \nabla F(x^*), [k]_X - k \rangle] \\
\leq \|\nabla F(x^*)\| \mathbb{E}[\text{dist}_X(x^* - [k])] \leq \|\nabla F(x^*)\| \sqrt{\mathbb{E}[\text{dist}_X^2(x^*)]}
$$

Lemma 16

By using relation (29) and the fact that $\frac{1}{k} \leq \frac{1}{k^\gamma}$ when $k \geq 1$ and $\gamma \in (0, 1]$, then the last inequality implies:

$$
\mathbb{E}[\langle \nabla f(x^*; S), x^* - k \rangle] \leq \|\nabla F(x^*)\| \left[ \frac{2\text{dist}_X(x^*)}{k \ln (\kappa/(\kappa - 1))} + 2\mu_k - \frac{\kappa}{2} B \right]. \tag{31}
$$

For the third term in (30) we observe from the optimality conditions for $z_{k}(x^*; S)$ that:

$$
\left\| \frac{1}{\mu_k} (z_{k}(x^*; S) - x^*) + \nabla f(x^*; S) \right\| = \|\nabla f(z_{k}(x^*; S); S) - \nabla f(x^*; S)\| \\
\leq L_{f,S} \|z_{k}(x^*; S) - x^*\| = \mu_k L_{f,S} \|\nabla f_{k}(x^*; S)\| \\
\leq \mu_k L_{f,S} \|\nabla f(x^*; S)\|,
$$

which yields the following bound:

$$
\langle z_{k}(x^*; S) - x^* + \mu_k \nabla f(x^*; S), x^k - x^* \rangle \leq \|z_{k}(x^*; S) - x^* + \mu_k \nabla f(x^*; S)\| \cdot \|x^k - x^*\| \\
\leq \mu_k^2 L_{f,S} \|\nabla f(x^*; S)\| \cdot \|x^k - x^*\|.
$$

By taking expectation in both sides and using Lemma 13, we obtain the refinement:

$$
\mathbb{E}[\langle z_{k}(x^*; S) - x^* + \mu_k \nabla f(x^*; S), x^k - x^* \rangle] \leq \mu_k^2 \sqrt{\mathbb{E}[L_{f,S}^2]} \sqrt{\mathbb{E}[\|\nabla f(x^*; S)\|^2]} \mathbb{E}[\|x^k - x^*\|] \\
\leq \mu_k^2 \sqrt{\mathbb{E}[L_{f,S}^2]} \eta A. \tag{32}
$$

Finally, for the fourth term in (30) we use Lemma 14:

$$
\mathbb{E}[\langle z_{k}(x^k; S) - x^k, z_{k}(x^*; S) - x^* \rangle] = \mu_k^2 \mathbb{E}[\|\nabla f_{k}(x^k; S)\| \|\nabla f_{k}(x^*; S)\|] \\
\leq \mu_k^2 \mathbb{E}[\|\nabla f(x^k; S)\| \|\nabla f(x^*; S)\|] \\
\leq \mu_k^2 \sqrt{\mathbb{E}[\|\nabla f(x^k; S)\|^2]} \mathbb{E}[\|\nabla f(x^*; S)\|^2] \\
\leq \mu_k^2 \eta \sqrt{2 \eta^2 + 2 \mathbb{E}[L_{f,S}^2] A^2}. \tag{33}
$$
By taking expectation in (30), using the relations (31)-(33) and taking into account that
\[ \frac{\mu_k}{\mu_{k-\frac{1}{2}}} \leq 3\gamma \text{ for all } k \geq 1, \]
we obtain:
\[
\mathbb{E}[\|z_{\mu_k}(x^k; S) - x^*\|^2] \\
\leq \mathbb{E} \left[ \theta_k^2(\mu_k)\|x^k - x^*\|^2 \right] + 4\mu_k^2\|F(x^*)\| \left[ \frac{\text{dist}_X(x^0)}{\mu_0} + 2\mu_0\kappa\beta \right] + 3\gamma \beta \kappa \\
+ 2\mu_k^2\eta\sqrt{2\eta^2} + 2\mathbb{E}[L_{f,S}^2]A^2 + 2\mu_k^2\eta\sqrt{\mathbb{E}[L_{f,S}^2]} \\
= \mathbb{E} \left[ \theta_k^2(\mu_k) \right] \mathbb{E}[\|x^k - x^*\|^2] + \mu_k^2D.
\]

For simplicity, we use further in the proof the following notations: \( r_k = \sqrt{\mathbb{E}[\|x^k - x^*\|^2]} \) and \( \theta_k = \mathbb{E}[\theta_k^2(\mu_k)] \). Then, through the nonexpansiveness property of the projection operator, the previous inequality turns into:
\[
\begin{align*}
\sum_{i=0}^{k} \theta_i + D \sum_{i=0}^{k} \left( \prod_{j=i+1}^{k} \theta_j \right) \mu_i^2 & \leq r_{k+1}^2 \leq \mathbb{E}[\|z_{\mu_k}(x^k; S) - x^*\|^2] \\
& \leq \theta_k r_k^2 + \mu_k^2D.
\end{align*}
\]

To further refine the right hand side in (34), we first notice from Lemma 15 that we have
\[ \prod_{i=0}^{k} \theta_i \leq \theta_0^{\varphi_1 - \gamma(k+1)}. \]
Then, from (34) we can derive different upper bounds for the two cases of the parameter \( \gamma \): \( \gamma < 1 \) and \( \gamma = 1 \).

Case (i) \( \gamma < 1 \). From Lemma 15, we derive an upper approximation for the second term in the right hand side of (34). Therefore, if we let \( m = \left\lceil \frac{k}{2} \right\rceil \) we obtain:
\[
\sum_{i=0}^{k} \mu_i^2 \left( \prod_{j=i+1}^{k} \theta_j \right) \leq \sum_{i=0}^{m} \mu_i^2 \left( \prod_{j=i+1}^{k} \theta_j \right) + \sum_{i=m+1}^{k} \mu_i^2 \left( \prod_{j=i+1}^{k} \theta_j \right) \\
\leq \sum_{i=0}^{m} \mu_i^2 \theta_0^{\varphi_1 - \gamma(k+1) - \varphi_1 - \gamma(i+1)} + \sum_{i=m+1}^{k} \mu_i \left( \prod_{j=i+1}^{k} \theta_j \right) \\
= \theta_0^{\varphi_1 - \gamma(k+1) - \varphi_1 - \gamma(m+1)} \sum_{i=0}^{m} \mu_i^2 + \sum_{i=m+1}^{k} \mu_i \left( \prod_{j=i+1}^{k} \theta_j \right) \\
= \theta_0^{\varphi_1 - \gamma(k+1) - \varphi_1 - \gamma(m+1)} \sum_{i=0}^{m} \mu_i^2 + \mu_{m+1} \sum_{i=m+1}^{k} \frac{\mu_i}{1 - \theta_i} (1 - \theta_i) \left( \prod_{j=i+1}^{k} \theta_j \right). 
\]

We will further refine the right hand side of (35) by noticing the following two facts. First, the constant \( \frac{\mu_i}{1 - \theta_i} \) can be upper bounded by:
\[ \frac{\mu_i}{1 - \theta_i} = \frac{1}{\mathbb{E} \left[ \frac{\sigma_S}{(1 + \mu_i\sigma_S)^2} + \frac{\sigma_S}{1 + \mu_i\sigma_S} \right]} \leq \frac{\mu_i}{1 - \theta_i} \leq \cdots \leq \frac{\mu_0}{1 - \theta_0}. \]
Second, the sum of products is upper bounded as:
\[
\sum_{i=m+1}^{k} (1 - \theta_i) \left( \prod_{j=i+1}^{k} \theta_j \right) = \sum_{i=m+1}^{k} \left( \prod_{j=i+1}^{k} \theta_j - \prod_{j=i}^{k} \theta_j \right) = 1 - \prod_{j=m+1}^{k} \theta_j \leq 1.
\]

By using the last two inequalities into (35), we have:
\[
\sum_{i=0}^{k} \mu_i^2 \left( \prod_{j=i+1}^{k} \theta_j \right) \leq \theta_0^{\varphi_1-\gamma(k+1)-\varphi_1-\gamma(m+1)} \sum_{i=0}^{m} \mu_i^2 + \mu_{m+1} \frac{\mu_0}{1 - \theta_0}.
\]
(36)

Since \(\sum_{i=0}^{m} \mu_i^2 \leq \mu_0^2(\varphi_1-2\gamma(m) + 2) \leq \mu_0^2(\varphi_1-2\gamma(m) + 2) \leq \mu_0^2(\varphi_1-2\gamma(k + 1) + 2)\) and using (36) into (34), we obtain the above result.

Case (ii) \(\gamma = 1\). In this case we have:
\[
\sum_{i=1}^{k} \mu_i^2 \left( \prod_{j=i+1}^{k} \theta_j \right) \leq 15 \sum_{i=1}^{k} \mu_i^2 \theta_0^{\varphi_0(k+1)-\varphi_0(i+1)} = \sum_{i=1}^{k} \mu_i^2 \ln \frac{1}{\theta_0} = \sum_{i=1}^{k} \mu_i^2 \ln \frac{k + 1}{i + 1} \leq \left( \frac{1}{k} \right) \ln \left( \frac{1}{\theta_0} \right) \sum_{i=1}^{k} \frac{\mu_i^2}{2 i - 2 \ln \frac{1}{\theta_0}}.
\]
Therefore, the variation of \(\theta_0\) leads to the following cases:
\[
\sum_{i=1}^{k} \mu_i^2 \left( \prod_{j=i+1}^{k} \theta_j \right) \leq \begin{cases} 
\frac{\mu_0^2}{k \ln \frac{1}{\theta_0} - 1} & \text{if } \theta_0 < \frac{1}{e} \\
\frac{\mu_0^2}{k \ln \frac{1}{\theta_0}} & \text{if } \theta_0 = \frac{1}{e} \\
\frac{\mu_0^2}{2 i \ln \frac{1}{\theta_0} - 1 - \ln \frac{1}{\theta_0}} & \text{if } \theta_0 > \frac{1}{e},
\end{cases}
\]
which leads to the second part of the result.

For more clear estimates of the convergence rates obtained in Theorem 17, we provide in the next corollary a summary given in terms of the dominant terms:

**Corollary 18** Under the assumptions of Theorem 17 the following convergence rates hold:
(i) If \(\gamma \in (0, 1)\), then we have convergence rate of order:
\[
\mathbb{E}[\|x^k - x^*\|^2] \leq \mathcal{O} \left( \frac{1}{k^\gamma} \right)
\]
(ii) If \(\gamma = 1\), then we have convergence rate of order:
\[
\mathbb{E}[\|x^k - x^*\|^2] \leq \begin{cases} 
\mathcal{O} \left( \frac{1}{k} \right) & \text{if } \theta_0 < \frac{1}{e} \\
\mathcal{O} \left( \ln \frac{k}{\theta_0} \right) & \text{if } \theta_0 = \frac{1}{e} \\
\mathcal{O} \left( \frac{1}{\theta_0} \right)^2 \ln \left( \frac{1}{\theta_0} \right) & \text{if } \theta_0 > \frac{1}{e}.
\end{cases}
\]
Proof  First assume that $\gamma \in (0, \frac{1}{2})$. This assumption implies that $1 - 2\gamma > 0$ and that:

$$\varphi_{1-2\gamma}(\frac{k}{2} + 2) = (\frac{k}{2} + 2)^{1-2\gamma} - 1 \leq (\frac{k}{2} + 2)^{1-2\gamma}. \quad (37)$$

On the other hand, by using the inequality $e^{-x} \leq \frac{1}{1+x}$ for all $x \in \mathbb{R}$, we obtain:

$$\theta_0^{\varphi_{1-\gamma}(k+1)-\varphi_{1-\gamma}(\frac{k+1}{2})} \varphi_{1-\gamma}(\frac{k}{2} + 2) = e^{(\varphi_{1-\gamma}(k+1)-\varphi_{1-\gamma}(\frac{k+1}{2})) \ln \theta_0 \varphi_{1-2\gamma}(\frac{k}{2} + 2) \left( \frac{1}{\theta_0} \right)} \leq \frac{(k+1)^{1-2\gamma}}{2^{1-2\gamma}(1-2\gamma)} \left( \ln \frac{1}{\theta_0} \right) \approx O \left( \frac{1}{k^{\gamma}} \right).$$

Therefore, in this case, the overall rate will be given by:

$$r_{k+1}^2 \leq \theta_0^{O(k^{1-\gamma})} r_0^2 + O \left( \frac{1}{k^{\gamma}} \right) \approx O \left( \frac{1}{k^{\gamma}} \right).$$

If $\gamma = \frac{1}{2}$, then the definition of $\varphi_{1-2\gamma}(\frac{k}{2} + 2)$ provides that:

$$r_{k+1}^2 \leq \theta_0^{O(\sqrt{k})} r_0^2 + \theta_0^{O(\sqrt{k})} O(\ln k) + O \left( \frac{1}{\sqrt{k}} \right) \approx O \left( \frac{1}{\sqrt{k}} \right).$$

When $\gamma \in (\frac{1}{2}, 1)$, it is obvious that $\varphi_{1-2\gamma}(\frac{k}{2} + 2) \leq \frac{1}{2\gamma-1}$ and therefore the order of the convergence rate changes into:

$$r_{k+1}^2 \leq \theta_0^{O(k^{1-\gamma})}[r_0^2 + O(1)] + O \left( \frac{1}{k^{\gamma}} \right) \approx O \left( \frac{1}{k^{\gamma}} \right).$$

Lastly, if $\gamma = 1$, by using $\theta_0^{\ln k+1} \leq \left( \frac{1}{k} \right)^{\ln \frac{1}{\theta_0}}$ we obtain the second part of our result. 

Notice that the above results state that our SPP algorithm with variable stepsize $\frac{\theta_0}{k}$ converges with $O \left( \frac{1}{k^\gamma} \right)$ rate. Similar results have been obtained in Toulis et al. (2016) for a particular objective function of the form $f(A^T_S x)$ without any constraints and for $\gamma \in (1/2, 1]$. Moreover, for $\gamma = 1$ similar convergence rate, but in asymptotic fashion and for unconstrained problems, has been derived in Ryu and Boyd (2016). As we have already mentioned in the introduction section, the convergence rate for the SGD scheme contains an exponential term of the form $\frac{c_{2\gamma}^k \theta_0^k}{k^{2\gamma}}$, which for a given iteration counter $k$ grows exponentially in the initial stepsize $\mu_0$, see Moulines and Bach (2011). Thus, although the SGD method achieves a rate $O(\frac{1}{k})$ for a variable stepsize $\frac{\mu_0}{k}$, if $\mu_0$ is chosen too large, then it can induce catastrophic effects in the convergence rate. However, one should notice that for our SPP method, Theorem 17 does not contain this kind of exponential term, therefore SPP is more robust than SGD scheme even in the constrained case. This can be also observed in numerical simulations, see Section 8 below. Clearly, Corollary 18 directly implies the following complexity estimates for attaining a suboptimal point $x^k$ satisfying $\mathbb{E}[\|x^k - x^*\|^2] \leq \epsilon$. 

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Corollary 19 Under the assumptions of Theorem 17 and $\epsilon > 0$ the following estimates hold. For $\gamma \in (0, 1)$, if we perform:

$$\left[ \mathcal{O} \left( \frac{1}{\epsilon^{1/\gamma}} \right) \right]$$

iterations of SPP scheme with variable stepsize, then the sequence $\{x^k\}_{k\geq 0}$ satisfies $\mathbb{E}[\|x^k - x^*\|^2] \leq \epsilon$. Moreover, for $\gamma = 1$ and $\theta_0 < \frac{1}{\sqrt{\epsilon}}$, if we perform:

$$\left[ \mathcal{O} \left( \frac{1}{\epsilon} \right) \right]$$

iterations of SPP scheme with variable stepsize, then we have $\mathbb{E}[\|x^k - x^*\|^2] \leq \epsilon$.

Proof The proof follows immediately from Corollary 18.

6. A restarted variant of Stochastic Proximal Point algorithm

From previous section we easily notice that an $\mathcal{O} \left( \frac{1}{\epsilon} \right)$ convergence rate is obtained for the SPP algorithm with variable stepsize $\mu_k = \frac{\theta_0}{k}$ only when the initial stepsize $\mu_0$ is chosen sufficiently large such that $\theta_0 < \frac{1}{\sqrt{\epsilon}}$. However, this condition is not easy to check. Therefore, if $\mu_0$ is not chosen adequately, we can encounter the case $\theta_0 > \frac{1}{\sqrt{\epsilon}}$, which leads to a convergence rate for the SPP scheme of order $\mathcal{O} \left( \epsilon^{-\frac{1}{2\ln(1/\theta_0)}} \right)$, that is implicitly dependent on the choice of the initial stepsize $\mu_0$. In conclusion, in order to remove this dependence on the initial stepsize of the simple SPP scheme, we develop a restarting variant of it. This variant consists of running the SPP algorithm (as a routine) for multiple times (epochs) and restarting it each time after a certain number of iterations. In each epoch $t$, the SPP scheme runs for an estimated number of iterations $K_t$, which may vary over the epochs, depending on the assumptions made on the objective function. More explicitly, the Restarted Stochastic Proximal Point (RSPP) scheme has the following iteration:

**Algorithm RSPP**

Let $\mu_0 > 0$ and $x^{0,0} \in \mathbb{R}^n$. For $t \geq 1$ do:

1. Compute stepsize $\mu_t$ and number of inner iterations $K_t$
2. Set $x^{K_{t-1}, t}$ the average output of SPP($x^{K_{t-1}, t-1}, \mu_t$) runned for $K_t$ iterations with constant stepsize $\mu_t$
3. If an outer stopping criterion is satisfied, then STOP, otherwise $t := t+1$ and go to step 1.

We analyze below the nonasymptotic convergence rates of the RSPP algorithm under different assumptions on the objective function: first we assume Assumptions 9 and 10 to hold.
for the objective functions, and then we assume that the objective functions are polyhedral and thus satisfy a sharp minima like condition.

### 6.1 Nonasymptotic sublinear convergence of algorithm RSPP

In this section we analyze the convergence rate of the sequence generated by the RSPP scheme, which repeatedly calls the subroutine SPP with a constant stepsize, in multiple epochs. We consider that SPP runs in epoch $t \geq 1$ with the constant stepsize $\mu_t$ for $K_t$ iterations. As in previous sections, we first provide a descent lemma for the feasibility gap. For simplicity, we keep the notations of $\mathcal{A}$ from Lemma 14 and $\mathcal{B}$ from Lemma 16.

**Lemma 20** Let Assumptions 2, 9 and 10 hold. Also let the sequence $\{x^{K_t, t}\}_{t \geq 0}$ be generated by RSPP scheme with nonincreasing stepsizes $\{\mu_t\}_{t \geq 0}$ and nondecreasing epoch lengths $\{K_t\}_{t \geq 1}$ such that $K_t \geq 1$ for all $t \geq 1$. Then, the following relation holds:

$$\sqrt{\mathbb{E}[\text{dist}^2(x^{K_t, t})]} \leq \left(1 - \frac{1}{K_t}\right)\sum_{i=1}^{K_t} \text{dist}_X(x^{0, 0}) + 2 \left(1 - \frac{1}{K_t}\right)\sum_{i=1}^{K_t} \frac{K_t}{i} \mu_0 \kappa^2 \mathcal{B} + 2 \mu_t \frac{k^2 \kappa^2 \mathcal{B}}{K_t}.$$

**Proof** The proof follows similar lines with the one of Lemma 26. Therefore, by using notations: $\alpha = \sqrt{1 - \frac{1}{K_t}}$ and $d_{k, t} = \sqrt{\mathbb{E}[\text{dist}^2(x^{k, t})]}$ results:

$$d_{k+1, t} \leq \alpha d_{k, t} + \alpha \mu_t \mathcal{B} \leq \alpha^{k+1} d_{0, t} + \mu_t \mathcal{B} \sum_{i=1}^{k+1} \alpha^i \leq \alpha^{k+1} d_{0, t} + \mu_t \mathcal{B} \frac{\alpha}{1 - \alpha}.$$

By setting $k = K_t - 1$, then the last inequality implies:

$$d_{K_t, t} \leq \alpha^{K_t} d_{K_t - 1, t - 1} + \mu_t \mathcal{B} \frac{\alpha}{1 - \alpha} \leq \alpha^{\sum_{i=1}^{t} K_i} d_{0, 0} + \mathcal{B} \frac{\alpha}{1 - \alpha} \sum_{j=0}^{t-1} \alpha^{\sum_{i=1}^{j} K_i} \mu_{t-j}.$$

Now set $m = \left\lceil \frac{t}{2} \right\rceil$. By dividing the sum from the right side of (28) in two parts, by taking into account that $\{\mu_t\}_{t \geq 0}$ is nonincreasing and $\{K_t\}_{t \geq 0}$ is nondecreasing, then results:

$$\sum_{j=0}^{t-1} \alpha^{\sum_{i=1}^{j} K_i} \mu_{t-j} \leq \mu_{t-m} \sum_{j=0}^{m} \alpha^{\sum_{i=1}^{j} K_i} + \mu_0 \alpha^{\sum_{i=1}^{t-m+1} K_i} \mu_{t-j} \leq \mu_{t-m} \frac{1 - \alpha^{m+1}}{1 - \alpha} + \mu_0 \alpha^{\sum_{i=1}^{t-m+1} K_i} \frac{1 - \alpha^{t-m+2}}{1 - \alpha} \leq \mu_{t-m} \frac{1 - \alpha^{m+1}}{1 - \alpha} + \mu_0 \alpha^{\sum_{i=1}^{t-m} K_i} \frac{1 - \alpha^{t-m+2}}{1 - \alpha}.$$
By using the last inequality into (28) and using the bound $\frac{\alpha}{1-\alpha} \leq 2\kappa$, then these facts imply the statement of the lemma.

Next, we provide the non-asymptotic bounds on the iteration complexity of RSPP scheme.

**Theorem 21** Let Assumptions 2, 9 and 10 hold and $\epsilon, \mu_0 > 0$. Also let $\gamma > 0$ and $\{x^{K_1,t}\}_{t \geq 0}$ be generated by RSPP scheme with $\mu_t = \frac{\mu_0}{t}$ and $K_t = [t^\gamma]$. If we perform the following number of epochs:

$$T = \left\lceil \frac{1}{1 + \gamma} \max \left\{ \ln \left( \frac{2r_0^2}{\epsilon} \right), \frac{1}{\ln (1/\theta_0)}, \left( \frac{2^{\gamma+1}D_T \epsilon}{\epsilon} \right)^{1/\gamma} \right\} \right\rceil,$$

then after a total number of SPP iterations of $T^{1+\gamma}$, which is bounded by

$$\left\lceil \frac{1}{1 + \gamma} \max \left\{ \ln \left( \frac{2r_0^2}{\epsilon} \right), \frac{1}{\ln (1/\theta_0)}, \left( \frac{2^{\gamma+1}D_T \epsilon}{\epsilon} \right)^{1+1/\gamma} \right\} \right\rceil,$$

where $D_T = 4\|\nabla F(x^*)\| \left[ \frac{\ln (\kappa)}{\mu_0 \ln (\kappa / (\kappa - 1))} + 3\gamma B \kappa^2 \right] + 2\eta \sqrt{2\eta^2 + 2\E[L_{f,S}]A^2 + 2\eta A \sqrt{\E[L_{f,S}]} ^2}$ and $\mathcal{C} = \frac{1}{2(1-\gamma) \ln 1/\theta_0} + \frac{(1-s)^2}{(1-\theta_0)^2}$, we have $\E[\|x^{K_1,T} - x^*\|^2] \leq \epsilon$.

**Proof** First notice that from $e^x \geq 1 + x$ for all $x \geq 0$, we have $(1 - \frac{1}{\kappa}) \sum_{i=1}^{K_1} \frac{K_i}{2} \leq \frac{1}{K_1 \ln (\kappa / (\kappa - 1))}$, and $(1 - \frac{1}{\kappa}) \sum_{i=t-1}^{t-\frac{1}{\gamma}} \frac{K_i}{2} \leq (1 - \frac{1}{\kappa}) \frac{K_i}{2} \leq \frac{2}{K_1 \ln (\kappa / (\kappa - 1))}$, which imply that Lemma 2 becomes

$$\sqrt{\E[\|x_{K_1,t}\|^2]} \leq \mu_t \frac{2\ln (\kappa / (\kappa - 1))}{\mu_0 \ln (\kappa / (\kappa - 1))} + \mu_t \frac{\ln (\kappa / (\kappa - 1))}{\mu_0 \ln (\kappa / (\kappa - 1))} + 2\mu_t^{-1} \frac{K_i}{2} \kappa^2 B.$$

(38)

It can be seen that by combining (38) with a similar argument as in Theorem 17 we obtain a similar descent as (34). Therefore, let $k \geq 0$ and $x^{k,t}$ be the $k$th iterate from the $t$th epoch. Then, by denoting $r_{k,t}^2 = \E[\|x^{k,t} - x^*\|^2]$, results:

$$r_{k,t}^2 \leq \E[\theta_S(\mu_t)^2]r_{k,t}^2 + \mu_t^2 D_T.$$

Now taking $k = K_t$ results in:

$$r_{0,t+1}^2 = r_{K_1,t}^2 + D_T \mu_t^2 \sum_{i=0}^{K_1} \mu_i \leq r_{0,t}^2 + \mu_t^2 D_T.$$

(39)

Recalling that we chose $\mu_t = \frac{\mu_0}{t}$ and $K_t = [t^\gamma]$, then Lemma 4 leads to:

$$\theta_t^{K_t} \leq \left( \frac{1}{(1 + \mu_0 \sigma_f S)^2} \right)^{K_t} \leq \theta_0.$$

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Therefore, (39) leads to:

\[
    r_{0,t+1} \overset{(39)}{\leq} \theta_{0,r_{0,t}^2} + \frac{D_r \mu_2^2}{1-\theta_t} \leq \theta_{0,r_{0,1}^2} + \sum_{i=1}^{t} \frac{\mu_2^2 \theta_{0,-i}^2}{1-\theta_i}. \tag{40}
\]

Note that \(\frac{\mu_2^2}{1-\theta_i}\) is nonincreasing in \(i\). Then, if we fix \(m = \lceil \frac{t}{2} \rceil\), then the sum \(\sum_{i=1}^{t} \frac{\mu_2^2 \theta_{0,-i}^2}{1-\theta_i}\) can be bounded as follows:

\[
    \sum_{i=1}^{t} \frac{\mu_2^2 \theta_{0,-i}^2}{1-\theta_i} \leq \theta_{0,m}^m \sum_{i=1}^{m} \frac{\mu_2^2}{1-\theta_i} + \sum_{i=m+1}^{t} \frac{\mu_2^2 \theta_{0,-i}^2}{1-\theta_i}
    \leq \theta_{0,m}^m \mu_1 \left( \sum_{i=1}^{m} \mu_i \right) + \frac{\mu_2^2}{(1-\theta_m)(1-\theta_0)}
    \leq \frac{\theta_{0,m}^m \mu_1}{1-\theta_0} \left( \sum_{i=1}^{m} \mu_i \right) + \mu_m \frac{\mu_1}{(1-\theta_0)^2}. \tag{41}
\]

Taking into account that \(\sum_{i=1}^{m} \mu_i \leq \int_{1}^{m} \frac{s}{s^2} ds \leq \frac{2^{\gamma-1}}{(1-\gamma)^{\gamma-\gamma}}\) and that \(\theta_{0,m} \leq \frac{1}{1+\frac{1}{\log m}}\), the previous relation (41) implies:

\[
    \sum_{i=1}^{t} \frac{\mu_2^2 \theta_{0,-i}^2}{1-\theta_i} \leq \left( \frac{2}{t} \right)^{\gamma} \left[ \frac{1}{2(1-\gamma)\ln 1/\theta_0} + \frac{\mu_1^2}{(1-\theta_0)^2} \right]. \tag{42}
\]

By using this bound in relation (41), then in order to obtain \(r_{0,t+1}^2 \leq \epsilon\) it is sufficient that the number of epochs \(t\) to satisfy:

\[
    t \geq \max \left\{ \ln \left( \frac{2^{\gamma+1} D_r C}{\epsilon} \right) \frac{1}{\ln (1/\theta_0)}, \left( \frac{2^{\gamma+1} D_r C}{\epsilon} \right)^{1/\gamma} \right\}. \tag{43}
\]

Finally, the total number of SPP iterations performed by RSPP algorithm satisfies:

\[
    \sum_{i=1}^{t} K_t \geq \sum_{i=1}^{t} i^{\gamma} \geq \int_{0}^{t} s^{\gamma} ds = \frac{t^{1+\gamma}}{1+\gamma}
    \geq \frac{1}{1+\gamma} \max \left\{ \ln \left( \frac{2^{\gamma+1} D_r C}{\epsilon} \right)^{1+\gamma} \frac{1}{\ln (1/\theta_0)^{1+\gamma}}, \left( \frac{2^{\gamma+1} D_r C}{\epsilon} \right)^{1+\gamma} \right\},
\]

which proves the statement of the theorem. \(\blacksquare\)

In conclusion Theorem 21 states that the RSPP algorithm with the choices \((\mu_t, K_t) = \left( \frac{\mu_0}{t^{\gamma}}, \frac{t^{\gamma}}{2} \right)\) requires \(O \left( \epsilon^{-\left(1+\frac{1}{\gamma} \right)} \right)\) simple SPP iterations to reach an \(\epsilon\) optimal point. It is
important to observe that this convergence rate is achieved when the stepsize and the epoch length are not dependent on any inaccessible constant, making our restarting scheme easily implementable. Moreover, the parameter $\gamma$ can be chosen in $(0, \infty)$, i.e. our RSPP scheme allows also stepsizes $\mu_k = \mu_0 k^\gamma$, with $\gamma > 1$. By comparison, an $O(\epsilon^{-1})$ complexity is obtained for SPP with stepsize $\mu_k = \frac{\mu_0}{k}$ only when $\mu_0$ is chosen sufficiently large such that $\theta_0 < \frac{1}{\sqrt{\epsilon}}$. However, this condition is not easy to check. Moreover, we may fall in the case when $\theta_0 > \frac{1}{\sqrt{\epsilon}}$, which leads to a complexity of $O\left(\epsilon^{-\frac{\gamma}{2\gamma+1}}\right)$ of the variable stepsize SPP scheme. Observe that the last convergence rate is implicitly dependent on the constant $\mu_0$ and can be arbitrarily bad, while for $\gamma > 1$ sufficiently large the RSPP scheme achieves the optimal convergence rate $O(\epsilon^{-1})$.

7. Contributions in the light of prior work

Notice that the above results (see Theorem 17) state that our SPP algorithm with variable stepsize $\mu_k = \mu_0 / k^\gamma$ and $\gamma \in (0, 1]$ converges with $O(1/k^\gamma)$ rate for strongly convex smooth constrained optimization problems. When the objective function is strongly convex and smooth and with a proper learning rate for $\gamma = 1$, our algorithm converges with $O(1/k)$ rate, which is optimal for the stochastic methods for this problem class. As we have already discussed in the introduction section, the convergence rate for the SGD scheme contains an exponential term of the form $e^{C\mu_0^2 / k^\alpha}$, which for a given iteration counter $k$ grows exponentially in the initial stepsize $\mu_0$, see Moulines and Bach (2011). Thus, although the SGD method achieves a rate $O(1/k)$ for a variable stepsize $\mu_k = \mu_0 / k$, if $\mu_0$ is chosen too large, then it can induce catastrophic effects in the convergence rate. However, one should notice that for our SPP method, Theorem 17 does not contain this kind of exponential term, therefore SPP is more robust than SGD scheme even in the constrained case. This can be also observed in numerical simulations, see Section 8 below.

Similar convergence results for SPP have been obtained in Toulis et al. (2016) for a particular objective function of the form $f(x; S) = \ell(A_S^T x)$ without any constraints and for $\gamma \in (1/2, 1]$. However, the analysis in Toulis et al. (2016) cannot be trivially extended to the general convex objective functions and complicated constraints, since for the proofs it is essential that each component of the objective function has the form $f(A_S^T x)$. In our paper we consider general convex objective functions which lack the previously discussed structure and also (in)finite number of convex constraints. Moreover, for $\gamma = 1$ similar convergence rate, but in asymptotic fashion and for unconstrained problems, has been derived in Ryu and Boyd (2016). Another paper related to our work is Bianchi (2016), where the author also proposes a SPP-like algorithm and asymptotic convergence is established without rates. To the best of our knowledge, our SPP method is the first stochastic proximal point algorithm that can tackle optimization problems with complicated constraints (3). Moreover, the convergence analysis is non-trivial and does not follow from the analysis corresponding to the unconstrained settings.

8. Numerical experiments

We present numerical evidence to assess the theoretical convergence guarantees of the SPP algorithm. We provide two numerical examples: constrained stochastic least-square with
8.1 Stochastic least-square problems using random data

In this section we evaluate the practical performance of the SPP schemes on finite large scale least-squares models. To do so, we follow a simple normal (constrained) linear regression example from Moulines and Bach (2011); Touli et al. (2016). Let \( m = 10^5 \) be the number of observations, and \( n = 20 \) be the number of features. Let \( x^\ast \) be a randomly a priori chosen ground truth. The random variable \( S \) is decomposed as \( S = (a_1, b_1) \), where the feature vectors \( a_1, \ldots, a_m \approx N_\mu(0, H) \) are i.i.d. normal random variables, and \( H \) is a randomly generated symmetric matrix with eigenvalues \( 1/k \), for \( k = 1, \ldots, n \). The outcome \( b_i \) is sampled from a normal distribution as \( b_i | a_i \approx N(a_i^T x^\ast, 1) \), for \( i = 1, \ldots, m \). Since the typical loss function is defined as the elementary squared residual \( (a_i^T x - b_i)^2 \), which is not strongly convex, we consider batches of residuals to form our loss functions, i.e. we consider \( \ell(x, i) \) of two forms:

\[
\ell(x, i) = \|A_{(i):j(i)+n}x - b_{(i):j(i)+n}\|^2 \quad \text{or} \quad \ell(x, i) = (a_i^T x - b_i)^2,
\]

where \( a_i \) is the \( i \)th row of \( A \) and \( A_{(i):j(i)+n} \in \mathbb{R}^{n \times n} \) is a submatrix containing \( n \) rows of \( A \) so that the function \( x \mapsto \|A_{(i):j(i)+n}x - b_{(i):j(i)+n}\|^2 \) is strongly convex. In our tests we used \( \text{round}(m/2n) \) batches of dimension \( n \) and we let the rest as elementary residuals, thus having in total \( p = m/2 + m/n \) loss functions. Additionally, we impose on the estimator \( x \) also \( p \) linear inequality constraints \( \{x \mid Cx \leq d \} \). This constraints can be found in many applications and they come from physical constraints, see e.g. Censor et al. (2012); Rosasco et al. (2014). We choose randomly the matrix \( C \) for the constraints and \( d = C \cdot x^\ast + [0 0 0 v^T]^T \), where \( v \geq 0 \) is a random vector of appropriate dimension, i.e. three inequalities are active at the solution \( x^\ast \). Besides the SPP and RSPP algorithms analyzed in the previous sections of our paper, we also implemented SGD and the averaged variant of SPP algorithm (A-SPP), which has the same SPP iteration, but outputs the average of iterates: \( \hat{x}^k = (1/\sum_{i=1}^k \mu_i) \sum_{i=1}^k \mu_i x_i \).

In Figure 1 we run algorithms SPP, RSPP, A-SPP and SGD for two values of the initial stepsize: \( \mu_0 = 0.5 \) and \( \mu_0 = 1 \). Each scheme runs for two stepsize exponents: \( \gamma_1 = 1 \) (left) and \( \gamma_2 = 1/2 \) (right). From Figure 1 we can asses one conclusion of Theorem 17: that the best performance for SPP is achieved for stepsize exponent \( \gamma = 1 \). Moreover, we can observe that algorithm RSPP has the fastest behavior, while the averaged variant A-SPP is more robust to changes in the initial stepsize \( \mu_0 \). The performance of SGD is much worse as exponent \( \gamma \) decreases and it is also sensitive to the learning rate \( \mu_0 \). Notice that both tests are performed over \( m \) iterations (i.e. one pass through data).

In the second set of experiments, we generate random least-square problems of the form \( \min_{x: Cx \leq d} 1/2 \|Ax - b\|^2 \), where both matrices \( A \) and \( C \) have \( m = 10^3 \) rows and generated randomly. Now, we do not impose the solution \( x^\ast \) to have the form given in the first test. We let SPP and RSPP algorithms to do one pass through data for various stepsize exponents \( \gamma \). From Figure 2 we can assess the empirical evidence of the \( O(1/e^{1/\gamma}) \) convergence rate of Theorem 17 for SPP and \( O(1/e^{1+1/\gamma}) \) convergence rate of Theorem 21 for RSPP, by
presenting squared relative distance to the optimum solution. Moreover, the simulation results match another conclusion of Theorems 17 and 21 regarding the stepsize exponent $\gamma$: which state that the performance of SPP/RSSP deteriorates with the decrease in the value of the stepsize exponent.

8.2 Markowitz portfolio optimization using real data

Markowitz portfolio optimization aims to reduce the risk by minimizing the variance for a given expected return. This can be mathematically formulated as a convex optimization problem Brodie et al. (2009); Yurtsever et al. (2016):

$$\min_{x \in \mathbb{R}^n} \mathbb{E}[(a_S^T x - b)^2] \quad \text{s.t.} \quad x \in X = \{x : x \geq 0, \ e^T x \leq 1, \ a_{av}^T x \geq b\},$$

where $a_{av} = \mathbb{E}[a_S]$ is the average returns for each asset that is assumed to be known (or estimated), and $b$ represents a minimum desired return. Since new data points are arriving
on-line, one cannot access the entire dataset at any moment of time, which makes the stochastic setting more favorable. For simulations, we approximate the expectation with the empirical mean as follows:

$$\min_{x \in \mathbb{R}^n} \frac{1}{m} \sum_{i=1}^{m} (a_i^T x - b)^2 \quad \text{s.t.} \quad x \in X = X_1 \cap X_2 \cap X_3,$$

where $X_1 = \{x : x \geq 0\}$, $X_2 = \{x : e^T x \leq 1\}$ and $X_3 = \{x : a_{av}^T x \geq b\}$. We use 2 different real portfolio datasets: Standard & Poor’s 500 (SP500, with 25 stocks for 1276 days) and one dataset by Fama and French (FF100, with 100 portfolios for 23,647 days) that is commonly used in financial literature, see e.g. Brodie et al. (2009). We split all the datasets into test (10%) and train (90%) partitions randomly. We set the desired return $a_{av}$ as the average return over all assets in the training set and $b = \text{mean}(a_{av})$. The results of this experiment are presented in Figure 3. We plot the value of the objective function over the datapoints in the test partition $F_{test}$ along the iterations. We observe that SGD is very sensitive to both parameters, initial stepsize ($\mu_0$) and stepsize exponent ($\gamma$), while SPP is more robust to changes in both parameters and also performs better over one pass through data in the train partition.

Figure 3: Performance on real data of SPP, A-SPP, RSPP and SGD schemes for several values of the initial stepsize $\mu_0$ and for two values of the exponent $\gamma = 1/2$ (left) and $\gamma = 1$ (right): dataset SP500 (top), dataset FF100 (bottom).
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