LENGTE OF LOCAL COHOMOLOGY IN POSITIVE CHARACTERISTIC AND ORDINARITY

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Abstract. Let $D$ be the ring of Grothendieck differential operators of the ring $R$ of polynomials in $d \geq 3$ variables with coefficients in a perfect field of positive characteristic $p$. We compute the $D$-module length of the first local cohomology module $H^1_f(R)$ of $R$ with respect to an irreducible polynomial $f$ with an isolated singularity, for $p$ large enough. The expression we give is in terms of the Frobenius action on the top coherent cohomology of the structure sheaf of the exceptional divisor of a resolution of the singularity. Our proof rests on a tight closure computation due to Hara. Since the above length is quite different from that of the corresponding local cohomology module in characteristic zero, we also consider a characteristic zero $D$-module whose length is expected to equal that above, for ordinary primes.

1. Introduction

In this note, we compute the positive characteristic $D$-module length of the first local cohomology module of the structure sheaf with support in a hypersurface, in a large class of examples. Our main result can also be seen as part of our study of the $b$-function in positive characteristic, see [2]. On the one hand, in [2] using $D$-module (or unit $F$-module) techniques, for $D$ the ring of Grothendieck differential operators, we associate to a non-constant polynomial $f$ with coefficients in a perfect field of positive characteristic a set of $p$-adic integers, called the roots of the $b$-function of $f$. On the other, one may consider the $F$-jumping exponents of the generalised test ideals of $f$, see [12]. These are positive real numbers which are characterised by their intersection with the unit interval $(0, 1]$ and have been shown to be rational numbers in [7]. In [2] we prove that the roots of the $b$-function of $f$ are exactly the opposites of the $F$-jumping exponents of $f$ which are in $\mathbb{Q} \cap \mathbb{Z}_p$. It would thus seem that the information provided by the $F$-jumping exponents of $f$ which are not in $\mathbb{Q} \cap \mathbb{Z}_p$, i.e. whose denominator is divisible by $p$, let us call them irregular, is lost in the theory. A consequence of the results presented here is that not all the information is lost. Namely the absence of irregular $F$-jumping exponents is well-known to be closely related to phenomena of ordinarity, see [17]. We claim that at the very least the $D$-module (or unit $F$-module) length of the module $N_f$ used to define the $b$-function in [2] distinguishes ordinary primes from supersingular ones, for large enough primes. More precisely, using the terminology of [2], one can see that the joint eigenspace of the action of the higher Euler operators on $N_f$ corresponding to the root $-1$ of the $b$-function of $f$ is isomorphic to the first local cohomology module $H^1_f(R)$, where $R$ is the ring of polynomials. For a $d$-dimensional proper variety $Z$ over a field of characteristic $p > 0$, we let the $p$-genus $g_p(Z)$ of $Z$ be the dimension of the stable part of $\overline{k} \otimes H^d(Z, \mathcal{O}_Z)$, that is $\dim_{\overline{k}}(\bigcap_{t \geq 0} F^t(\overline{k} \otimes H^d(Z, \mathcal{O}_Z)))$, where $F$ is the Frobenius action on coherent cohomology and $\overline{k}$ is an algebraic closure of the base field. Our main result is (see Theorem 1 for the precise general formulation):

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Theorem. Suppose that \( f \) is an irreducible complex polynomial in \( n \geq 3 \) variables with an isolated singularity at the origin and let \( Y \xrightarrow{\pi} X \) be a resolution of the singularity. Then for almost all \( p \), the \( \mathcal{D} \)-module length of \( H^1_{f_p}(R) \) is \( 1 + g_p(Z_p) \), where \( Z_p \) is the reduction modulo \( p \) of the exceptional fiber of \( \pi \).

The proof, which mostly belongs to the theory of unit \( F \)-modules, uses Blickle’s intersection homology \( \mathcal{D} \)-module \([3]\) and Lyubeznik’s enhancement of Matlis duality \([10]\) to reduce the main unit \( F \)-module length computation to a geometric description of the tight closure of 0 in the local cohomology of the singularity, due to Hara \([11]\). One then deduces the \( \mathcal{D} \)-module length from Blickle’s length comparison result \([5]\) and an application of Haastert’s positive characteristic Kashiwara’s equivalence \([10]\).

We note that in characteristic zero, the \( \mathcal{D} \)-module length of the first local cohomology module is of a quite different nature. It actually is a topological invariant. For example, let \( f \) be a rational cubic in three variables which is the equation of an elliptic curve \( E \) in \( \mathbb{P}^3_{\mathbb{Q}} \), of genus \( g = 1 \). Let \( R_C \) be the ring of complex polynomials in 3 variables and for all primes \( p \), let \( R_p \) be the ring of \( \mathbb{F}_p \)-polynomials in 3 variables. Then the \( \mathcal{D} \)-module length of the local cohomology module \( H^1_C(R_C) \) is \( 3 = 1 + 2g \), see e.g. \([3\) Remark 1.2\]. But as will be seen in Example \([1]\) for almost all primes \( p \), the \( \mathcal{D}_p \)-module length of \( H^1_C(R_p) \) is \( 2 = 1 + g \) if \( E_p \) is ordinary and 1 if \( E_p \) is supersingular, where \( E_p \) (resp. \( f_p \)) is the reduction of \( E \) (resp. \( f \)) modulo \( p \) and \( D_p = D_{R_{p}} \) is the ring of Grothendieck differential operators on \( \mathbb{A}_{p}^{3} \). Thus for (almost) all primes \( p \), the lengths of the first local cohomology modules in characteristic zero and in characteristic \( p \) are different. We end this note with a section on comparison with characteristic zero, arguing that in great generality, the \( \mathcal{D} \)-submodule \( \mathcal{D}^{1}_f \) of the local cohomology module \( H^1_f(R_C) \) generated by the class of \( \frac{1}{f} \) (which need not be equal to \( H^1_C(R_C) \)) is a better behaved characteristic zero analogue of \( H^1_{f_p}(R_p) \) than the whole local cohomology module \( H^1_f(R_C) \). (Recall that the left \( \mathcal{D}_p \)-module \( H^1_{f_p}(R_p) \) is generated by the class of \( \frac{1}{f_p} \), by \([1\) Theorem 1.1\].) For example, in the case of the elliptic curve above, we have that the \( \mathcal{D} \)-module length of \( \mathcal{D}^{1}_f \) is equal to the \( \mathcal{D}_p \)-module length of \( H^1_{f_p}(R_p) \), for almost all ordinary primes \( p \) of \( E \). The characteristic zero \( \mathcal{D} \)-module \( \mathcal{D}^{1}_f \) is studied in detail in \([3\). (See also \([18]\) for a different approach.\)

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1.2. Notations. Throughout the note we will use the following notations: For an integer \( n \geq 2 \) and all fields \( K \), we let \( R_K \) be the ring of polynomials in \( n + 1 \) variables \( \{x_0, \ldots, x_n\} \) with coefficients in \( K \) and \( D_{R_K} \) be the ring of Grothendieck differential operators on \( \mathbb{A}^{n+1}_K \).

Let \( k \) be a perfect field of positive characteristic \( p \), we set \( D = D_{R_K} \). If \( A \) is a \( k \)-algebra, we denote by \( A[F] \) the twisted polynomial ring over \( A \) whose multiplication is defined by \( Fa = a^p F \), for all \( a \) in \( A \).

2. LENGTH OF THE FIRST LOCAL COHOMOLOGY IN POSITIVE CHARACTERISTIC

We first recall some definitions.
**Definition 1.** Let $f \in R_k$ be a non-constant polynomial in $n+1$ variables. The first local cohomology module $H^1_f(R_k)$ of $R_k$ with respect to $f$ is the left $D$-module cokernel of the natural inclusion $R_k \subset R_k(\frac{1}{f})$.

**Remark 1.** The Frobenius endomorphism of $R_k$ induces a finitely generated unit $F$-module structure on $H^1_f(R_k)$. The associated action of $D$ is the natural one. Hence it follows immediately from [16, Theorem 3.2] that $H^1_f(R_k)$ is of finite length as a unit $F$-module. It is thus of finite length as a left $D$-module by [16, Theorem 5.7].

The purpose of this note is to give an expression for the length of $H^1_f(R_k)$, when $f$ has an isolated singularity. It will be in terms of the quasilength of a certain $F$-module. We now recall the definitions from [16, Section 4].

**Definition 2.** Let $A$ be a local Noetherian $k$-algebra of Frobenius endomorphism $F$ and let $M$ be a left $A[F]$-module.

- $M^\ast := \cap_{n>0} F^n M$, where $F^n M$ is the $A$-submodule of $M$ generated by the image $F^n(M)$
- $M_{nil} := \cup_{n>0} \ker\{M \xrightarrow{F^n} M\}$

**Definition 3.** Let $A$ be a local Noetherian $k$-algebra of Frobenius endomorphism $F$ and let $M$ be a left $A[F]$-module. Suppose that $M$ is Artinian as an $A$-module.

- A finite chain of length $s$ of $A[F]$-submodules $0 = M_0 \subset \cdots \subset M_s = M$ is quasimaximal if $(M_i/M_{i-1})_{nil}^\ast$ is a simple left $A[F]$-module, for all $i \in \{1, \ldots, s\}$.
- If $M_{nil} \subset M$, then $M$ has a quasimaximal chain of submodules and all such chains are of the same length, called the quasilength $\text{ql}(M)$ of $M$. If $M = M_{nil}$, then we set $\text{ql}(M) = 0$. See [16, Theorems 4.5 and 4.6].

To state our main result, we need to introduce the following notations:

Let $L$ be a field of characteristic 0 and let $g$ be a non-constant polynomial in $n+1$ variables $\{x_0, \ldots, x_n\}$, with coefficients in $L$. Let $(A, m)$ be the local ring of the zero-locus of $g$ at a singular point $z$. Let us fix a resolution $X \xrightarrow{\pi_B} Z = \text{Spec}(A)$ of the singularity $z$ and let $Y$ be the fiber of $\pi$ at $z$.

**Definition 4.** Let $B \subset L$ be a finitely generated subring, containing 1. We say that $B$ is a ring of definition of $\pi$ if the coefficients of $g$ are contained in $B$ and there is a resolution of singularities of $B$-schemes $X_B \xrightarrow{\pi_B} Z_B$ whose base-change $L \otimes_{\text{Frac}(B)} \pi_B$ is isomorphic to $\pi$.

For each closed point $u$ of $\text{Spec}(B)$, we let $X_u \xrightarrow{\pi_u} Z_u$ (resp. $g_u$, resp. $Y_u$) be the fiber of $\pi_B$ (resp. $g$, resp. $Y$) over $k(u)$. Finally, we consider the coherent cohomology groups $H^i(X_u, \mathcal{O}_{X_u})$ (resp. $H^i(Y_u, \mathcal{O}_{Y_u})$) as left $A[F]$-modules (resp. $k(u)[F]$-modules) for the action of the Frobenius endomorphism on the cohomology. Here is our main result:

**Theorem 1.** Suppose that $n \geq 2$ and that $g$ is absolutely irreducible with an isolated singularity at the origin. Then there is a ring of definition $B \subset L$ of $\pi$ such that, for all closed points $u$ of $\text{Spec}(B)$:

1. the unit $F$-module length of the first local cohomology group $H^1_{g_u}(R_{k(u)})$ is $1 + \text{ql}(H^{n-1}(X_u, \mathcal{O}_{X_u})) = 1 + \text{ql}(H^{n-1}(Y_u, \mathcal{O}_{Y_u}))$
2. the $D_{k(u)}$-module length of $H^1_{g_u}(R_{k(u)})$ is $1 + \text{ql}(k(u) \otimes H^{n-1}(X_u, \mathcal{O}_{X_u})) = 1 + \dim_{k(u)}((k(u) \otimes H^{n-1}(Y_u, \mathcal{O}_{Y_u}))^\ast)$,
where \( \overline{k(u)} \) is any algebraic closure of \( k(u) \) and \((-)^*\) is the operation on \( \overline{k(u)}[F]\)-modules from Definition 2.

**Proof.** By Oستrowski’s Theorem, see [11] Lemma 11 for a quick proof, there is a definition ring \( B' \) of \( \pi \) such that, for all closed points \( u \) of \( \text{Spec}(B') \), \( g_u \) is absolutely irreducible.

For every closed point \( u \) of \( \text{Spec}(B') \), we will use the following notation: \((\mathcal{A}_u, \mathfrak{m}_u)\) is the local ring of the singularity, \((R_0, \mathfrak{m}) := ((R_k(\mathfrak{a}))_{(x_0, \ldots, x_n)}(x_0, \ldots, x_n)(R_k(\mathfrak{b}))_{(x_0, \ldots, x_n)})\) and \((\widehat{R}_0, \widehat{\mathfrak{m}}) := ((R_k(\mathfrak{a}))_{(x_0, \ldots, x_n)}, (x_0, \ldots, x_n)(\widehat{R_k(\mathfrak{b}))}_{(x_0, \ldots, x_n)})\). We denote their completion with respect to their maximal ideal by \((\widehat{\mathcal{A}_0}, \widehat{\mathfrak{m}}_0), (\widehat{R_0}, \widehat{\mathfrak{m}})\) and \((\widehat{R}_0, \widehat{\mathfrak{m}}_0)\), respectively.

We have a short exact sequence of both \( D_k(u) \) and unit F-modules:

\[
0 \to \mathcal{L} \to \mathcal{H}^1_{g_u}(\mathcal{R}_k(u)) \to \mathcal{K} \to 0
\]

where \( \mathcal{L} \) is the intersection homology module \( \mathcal{L}(\mathcal{K}^{n+1}_{k(u)}) \{ g_u = 0 \} \) of [9] and \( \mathcal{K} \) is supported at the origin. Tensoring with the completion \( \widehat{R}_0 \) of the local ring at the origin, we get a short exact sequence:

\[
0 \to \widehat{R}_0 \otimes \mathcal{R}_k(u) \to \widehat{R}_0 \otimes \mathcal{R}_k(u) \mathcal{H}^1_{g_u}(\mathcal{R}_k(u)) \to \widehat{R}_0 \otimes \mathcal{R}_k(u) \mathcal{K} \to 0
\]

which we can rewrite as \( 0 \to \mathcal{L}' \to \mathcal{H}^1_{g_u}(\widehat{R}_0) \to \mathcal{K}' \to 0 \), where \( \mathcal{L}' = \mathcal{L}(\frac{\widehat{R}_0}{\mathcal{R}_k(u)}, \widehat{R}_0) \) and \( \mathcal{K}' = \widehat{R}_0 \otimes \mathcal{R}_k(u) \mathcal{K} \). Indeed \( \mathcal{L}' \cong \widehat{R}_0 \otimes \mathcal{R}_k(u) \mathcal{L} \) by [9] Theorem 4.6 and it is well-known that local cohomology commutes with base-change by the completion. Clearly the length of \( \mathcal{K}' \) as a \( D_{\mathcal{R}_0} \)-module (resp. unit F-module) equals the length of \( \mathcal{K} \) as a \( D_{\mathcal{R}_0} \)-module (resp. unit F-module). Hence so is the case for \( \mathcal{H}^1_{g_u}(\mathcal{R}_k(u)) \) and \( \mathcal{H}^1_{g_u}(\widehat{R}_0) \), since \( \mathcal{L} \) and \( \mathcal{L}' \) are irreducible.

Let \( D' := D_{\mathcal{R}_0} \). We also let \( \lg_{\mathcal{R}_0}(\mathcal{R}_k(u)) \) be the unit F-module length and \( \lg_{\mathcal{R}_0}(\mathcal{R}_k(u)) \) to be the \( D' \)-module length. The proof of [11] thus reduces to: There exists a ring of definition \( B \supset B' \) of \( \pi \) such that \( \text{Spec}(B) \subset \text{Spec}(B') \) is a dense open subset and, for all closed points \( u \) of \( \text{Spec}(B') \), \( \lg_{\mathcal{R}_0}(\mathcal{R}_k(u)) \) is equal to \( q\mathcal{L}(\mathcal{H}^n_{\mathcal{R}_0}(\mathcal{A}_u)) \) and \( \mathcal{K}' = \mathcal{H}^n_{\mathcal{R}_0}(\mathcal{A}_u) \) is the tight closure of zero.

Since Lyubeznik-Matlis duality exchanges unit F-module length with quasilength by the proof of [11] Theorems 4.5, we have \( \lg_{\mathcal{R}_0}(\mathcal{R}_k(u)) = 1 + q\mathcal{L}(\mathcal{H}^n_{\mathcal{R}_0}(\mathcal{A}_u)) \). Moreover by Lemma [11] applied to \( R = \mathcal{R}_0 \) and \( M = \mathcal{A}_u \), \( q\mathcal{L}(\mathcal{H}^n_{\mathcal{R}_0}(\mathcal{A}_u)) \) is equal to \( q\mathcal{L}(\mathcal{H}^n_{\mathcal{R}_0}(\mathcal{A}_u)) \).

Finally, since \( A \) is an isolated singularity and \( n \geq 2 \), it is normal. Hence by [11] Theorem 4.7, there exists a ring of definition \( B \supset B' \) of \( \pi \) such that \( \text{Spec}(B) \subset \text{Spec}(B') \) is a dense open subset and, for all closed points \( u \) of \( \text{Spec}(B) \), \( 0_{\mathcal{R}_0}^\mathcal{A}(\mathcal{A}_u) = \mathcal{H}^n_{\mathcal{R}_0}(\mathcal{A}_u, \mathcal{O}_{\mathcal{X}_u}) \), as \( \mathcal{A}_u[F] \)-modules. But by Lemma [11], \( q\mathcal{L}(\mathcal{H}^n_{\mathcal{R}_0}(\mathcal{A}_u, \mathcal{O}_{\mathcal{X}_u})) = q\mathcal{L}(\mathcal{H}^n_{\mathcal{R}_0}(\mathcal{A}_u, \mathcal{O}_{\mathcal{X}_u})) = \mathcal{H}^n_{\mathcal{R}_0}(\mathcal{A}_u, \mathcal{O}_{\mathcal{X}_u}) \). This concludes the proof of [11].

We now prove [11]. From (1), we deduce the short exact sequence

\[
0 \to \overline{k(u)} \otimes \mathcal{L} \to \mathcal{H}^1_{g_u}(\mathcal{R}_k(u)) \to \overline{k(u)} \otimes \mathcal{K} \to 0
\]

Therefore, tensoring with the completion \( \widehat{\mathcal{R}}_0 \) of \( \mathcal{R}_0 \), we also have the short exact sequence

\[
0 \to \overline{k(u)} \otimes \mathcal{L}' \to \mathcal{H}^1_{g_u}(\widehat{\mathcal{R}}_0) \to \overline{k(u)} \otimes \mathcal{K}' \to 0,
\]
with \( \mathcal{L}' \) and \( \mathcal{K}' \) as above. Note that \( \overline{k(u)} \otimes \mathcal{L}' = \mathcal{L}(\overline{R}_{\mathfrak{g}_u R_{\mathfrak{m}}}, \overline{R}_{\mathfrak{m}}) \) by [3] Lemma 5.16. Moreover, since the injective hull \( H_{\mathfrak{m}}^{n+1}(\overline{R}_{\mathfrak{g}_u}) \) of \( \overline{k(u)} = \frac{\overline{\mathcal{L}}_{\mathfrak{m} \otimes \overline{k(u)}}}{\overline{\mathcal{L}}_{\mathfrak{m} \otimes \overline{k(u)}}} \) is isomorphic to \( H_{\mathfrak{m}}^{n+1}(\overline{R}_{\mathfrak{g}_u}) \otimes \overline{k(u)} \), it is easy to see that Matlis duality commutes with the field extension \( - \otimes \overline{k(u)} \). Hence Lyubeznik-Matlis duality commutes with \( - \otimes \overline{k(u)} \). We thus have that \( \overline{k(u)} \otimes \mathcal{L}' \cong \overline{k(u)} \otimes \mathcal{D}(H_{\mathfrak{m}}^{n+1}(\overline{R}_{\mathfrak{g}_u \otimes \overline{A}_0})) \cong \mathcal{D}(\overline{H}_{\mathfrak{m}}^{n+1}(\overline{R}_{\mathfrak{g}_u \otimes \overline{A}_0})). \) Therefore the length of the unit \( F \)-module \( H_{\mathfrak{g}_u}^1(\overline{R}_{\mathfrak{g}_u}) \) is equal to \( 1 + \text{ql}(\overline{k(u)} \otimes \mathcal{D}(H_{\mathfrak{m}}^{n+1}(\overline{R}_{\mathfrak{g}_u \otimes \overline{A}_0}))) = 1 + \text{ql}(\overline{k(u)} \otimes H_{\mathfrak{m}}^{n+1}(\overline{R}_{\mathfrak{g}_u \otimes \overline{A}_0})). \) Also, similarly as above, the unit \( F \)-module length of \( H_{\mathfrak{g}_u}^1(\overline{R}_{\mathfrak{g}_u}) \) as a unit \( F \)-module. Thus for all \( \mathcal{L}' \otimes \mathcal{K} \) is isomorphic to \( \mathcal{D}(H_{\mathfrak{m}}^{n+1}(\overline{R}_{\mathfrak{g}_u \otimes \overline{A}_0})) \) for all \( \mathcal{K} \). By [3] Theorem 1.1], the length of \( H_{\mathfrak{g}_u}^1(\overline{R}_{\mathfrak{g}_u}) \) as a unit \( F \)-module is equal to its length as a \( D_{\mathfrak{g}_u}(\overline{R}_{\mathfrak{g}_u}) \)-module. Finally, we claim that the \( D_{\mathfrak{g}_u}(\overline{R}_{\mathfrak{g}_u}) \)-module length of \( H_{\mathfrak{g}_u}^1(\overline{R}_{\mathfrak{g}_u}) \) is equal to the \( D_{\mathfrak{g}_u}(\overline{R}_{\mathfrak{g}_u}) \)-module length of \( H_{\mathfrak{g}_u}^1(\overline{R}_{\mathfrak{g}_u}) \). This implies part (b) of the theorem.

Let us prove this last claim. We let \( \text{lg}_{D_{\mathfrak{g}_u}(\overline{R}_{\mathfrak{g}_u})}(-) \) denote the \( D_{\mathfrak{g}_u}(\overline{R}_{\mathfrak{g}_u}) \)-module length. Localising at the origin, one sees by [4] Lemma 5.16] that \( \overline{k(u)} \otimes \mathcal{L} \) is the intersection homology module. Thus \( \text{lg}_{D_{\mathfrak{g}_u}(\overline{R}_{\mathfrak{g}_u})}(\overline{k(u)} \otimes \mathcal{L}) = 1 = \text{lg}_{D_{\mathfrak{g}_u}(\overline{R}_{\mathfrak{g}_u})}(\overline{k(u)} \otimes \mathcal{L}). \) Hence the claim reduces to the equality \( \text{lg}_{D_{\mathfrak{g}_u}(\overline{R}_{\mathfrak{g}_u})}(\overline{k(u)} \otimes \mathcal{L}) = \text{lg}_{D_{\mathfrak{g}_u}(\overline{R}_{\mathfrak{g}_u})}(\overline{k(u)} \otimes \mathcal{K}) = \text{lg}_{D_{\mathfrak{g}_u}(\overline{R}_{\mathfrak{g}_u})}(\overline{k(u)} \otimes \mathcal{K}). \) But this follows immediately from the compatibility with base-field extension of Kashiwara’s equivalence, see [10] Corollary 8.13] (the proof of which is well-known to be valid over an arbitrary field of positive characteristic). Indeed by Kashiwara’s equivalence, we have \( \text{lg}_{D_{\mathfrak{g}_u}(\overline{R}_{\mathfrak{g}_u})}(\overline{k(u)} \otimes \mathcal{L}) = \dim_{k(u)}(\overline{k(u)} \otimes V) = \dim_{k(u)}(V) = \text{lg}_{D_{\mathfrak{g}_u}(\overline{R}_{\mathfrak{g}_u})}(\overline{k(u)} \otimes \mathcal{K}), \) for a certain finite dimensional \( k(u) \)-vector space \( V). \)

Recall that a ring \( R \) is \( F \)-finite if it is of positive characteristic, Noetherian and if the Frobenius map \( \text{Spec}(R) \xrightarrow{F} \text{Spec}(R) \) is finite.

**Lemma 1.** Let \((R, \mathfrak{m})\) be an \( F \)-finite regular local ring and let \( M \) be a finitely generated \( R \)-module. Then for all \( i \geq 0 \), the canonical isomorphism \( H_{\mathfrak{m}}^i(M) \cong H_{\mathfrak{m}}^i(M) \) induces an isomorphism of tight closures \( 0^*_{H_{\mathfrak{m}}^i(M)} \cong 0^*_{H_{\mathfrak{m}}^i(M)} \), where \((\hat{R}, \hat{\mathfrak{m}}) \) (resp. \( M \)) is the \( \mathfrak{m} \)-adic completion of \((R, \mathfrak{m}) \) (resp. \( M \)).

**Proof.** The isomorphism \( H_{\mathfrak{m}}^i(M) \cong H_{\mathfrak{m}}^i(M) \) is well-known, see [15] Proposition 2.15]. Furthermore the existence of completely stable big test elements for \( R \) ([14] Theorem p.77]) immediately implies the equality of tight closures.

**Lemma 2.** Let \( K \) be an algebraically closed field of positive characteristic \( p \) and let \( V \) be a \( K \)-finite dimensional left \( K[F] \)-module. Then the quasilength of \( V \) is \( \dim_K(V^*) \), where \((-)^* \) is the operation on \( K[F] \)-modules from Definition 2.

**Proof.** By definition of quasilength, we have \( \text{ql}(V) = \text{ql}(V^*) \). Moreover, \( F \) acts surjectively and thus injectively on \( V^* \). Hence, by [5] Proposition 4.6] for example, \( V^* \) has a \( K \)-basis of vectors fixed by \( F \). The lemma easily follows.
Lemma 3. Let \((A, \mathfrak{m})\) be a local Noetherian \(k\)-algebra of Frobenius endomorphism \(F\) and let \(M\) be a left \(A[F]\)-module. Suppose that \(M\) is Artinian and Noetherian as an \(A\)-module. If \(M\) is supported at the maximal ideal \(\mathfrak{m}\), then \(M\) and \(\frac{M}{\mathfrak{m}M}\) have the same quasi-length.

Proof. Let us show that \(\mathfrak{m}M \subset M_{nil}\). This implies the lemma since \(\text{ql}(M_{nil}) = 0\).

Since \(M\) is supported at \(\mathfrak{m}\) and is Noetherian, \(\mathfrak{m}^lM = 0\) for some \(l \geq 0\). Thus \(F^r(\mathfrak{m}M) \subset \mathfrak{m}^rM = 0\) for some \(r \geq 0\). Hence \(\mathfrak{m}M \subset M_{nil}\), as claimed.

If \(g\) is homogeneous, Theorem 1 may be rephrased without mentioning a resolution of the singularity. Let \(Y\) be the hypersurface of the zeros of \(g\) in \(\mathbb{P}^n\). We first fix the notation.

Definition 5. Let \(B \subset L\) be a finitely generated subring, containing 1. We say that \(B\) is a ring of definition of \(Y\) if the coefficients of \(g\) are contained in \(B\) and there is a smooth projective hypersurface \(Y_B\) of \(\mathbb{P}^n_B\) whose base-change \(L \otimes_{\text{Frac}(B)} Y_B\) is isomorphic to \(Y\).

Given such an hypersurface \(Y_B\), for each closed point \(u\) of \(\text{Spec}(B)\), we let \(Y_u\) be the fiber of \(Y_B\) over \(k(u)\). Here is the result:

Corollary 1. Under the same hypotheses as Theorem 1, assume that \(g\) is homogeneous and let \(Y\) be the hypersurface of its zeros in \(\mathbb{P}^n\). Then there is a ring of definition \(B \subset L\) of \(Y\) such that, for all closed points \(u\) of \(\text{Spec}(B)\):

(a) the unit \(F\)-module length of \(H^1_{g_u}(R_{k(u)})\) is \(1 + \text{ql}(H^{n-1}(Y_u, \mathcal{O}_{Y_u}))\)

(b) the \(D_{k(u)}\)-module length of \(H^1_{g_u}(R_{k(u)})\) is \(1 + \dim_{k(u)}(k(u) \otimes_{k(u)} H^{n-1}(Y_u, \mathcal{O}_{Y_u}))^*\), where \(k(u)\) is any algebraic closure of \(k(u)\) and \((-)^*\) is the operation on \(k(u)[F]\)-modules from Definition 2.

Proof. It is well-known that in this case the blow-up of the origin is a resolution \(\pi'\) of the singularity and that the fiber at the origin is isomorphic to \(Y\). The result then immediately follows from Theorem 1 applied to \(\pi'\).

Thus the \(D_{k}\)-module length of the first local cohomology is closely related to ordinarity. Here is a simple example:

Example 1. Let \(g\) be a rational cubic in three variables which is the equation of an elliptic curve \(E\) in \(\mathbb{P}^2_{\mathbb{Q}}\). Then, for almost all primes \(p\), the \(D_{R_{p}}\)-module length of \(H^1_{g_u}(R_{p})\) is 2 if \(E_p\) is ordinary and 1 if \(E_p\) is supersingular, where \(E_p\) (resp. \(g_p\)) is the reduction of \(E\) (resp. \(g\)) modulo \(p\).

3. Comparison with characteristic zero

Here, given a complex polynomial \(g\), we consider a holonomic \(D_{R_{C}}\)-module \(N_g\) whose length compares well to the \(D_{k(u)}\)-module length of \(H^1_{g_u}(R_{k(u)})\). We set \(D = D_{R_{C}}\).

Definition 6. Let \(g \in R_{C}\) be a complex polynomial. Then \(N_g\) is the left \(D\)-submodule of the first local cohomology \(D\)-module \(H^1_{g}(R_{C})\) generated by the class of \(\frac{1}{g}\).

The following is proved in [3, Theorem 1.1].

Theorem 2. Let \(g\) be a non-constant homogeneous complex polynomial in \(n + 1\) variables with an isolated singularity at the origin. Then, using the notations of Corollary 1, the \(D\)-module length of \(N_g\) is \(1 + \dim_{C} H^{n-1}(Y, \mathcal{O}_{Y})\).
Remark 2. There is a ring of definition $B \subset \mathbb{C}$ of $Y$ such that, for all closed points $u$ of $\text{Spec}(B)$, $\dim_{\mathbb{C}} H^{n-1}(Y, \mathcal{O}_Y) = \dim_{k(u)}(k(u) \otimes_{k(u)} H^{n-1}(u, \mathcal{O}_{Y_u}))$. Hence by Corollary 7 and Theorem 2, there is a ring of definition $B' \supset B$ of $Y$ such that for all closed points $u$ of $\text{Spec}(B')$, if the Frobenius $F$ acts bijectively on $k(u) \otimes_{k(u)} H^{n-1}(u, \mathcal{O}_{Y_u})$, then the length of $N_g$ is equal to the $D_{k(u)}$-module length of $H^1_{g_u}(R_{k(u)})$. Indeed in that case, $\dim_{\mathbb{C}} H^{n-1}(Y, \mathcal{O}_Y) = \dim_{k(u)}(k(u) \otimes_{k(u)} H^{n-1}(u, \mathcal{O}_{Y_u}))^*$. This property of the Frobenius is called weak ordinarity and is expected to hold for a dense set of closed points of $\text{Spec}(B')$, see [17] Conjecture 1.1.

We would like to put forward the following questions:

**Question 1.** Let $g$ be a non-constant complex polynomial in $n+1$ variables. Is there a unitary finitely generated subring $B \subset \mathbb{C}$ containing the coefficients of $g$ such that:

1. (1) for all closed points $u \in \text{Spec}(B)$, $\lg_{D_{k(u)}}(H^1_{g_u}(R_{k(u)})) \leq \lg_D(N_g)$?

2. (2) there is a dense set of closed points of $\text{Spec}(B)$ for which $\lg_{D_{k(u)}}(H^1_{g_u}(R_{k(u)})) = \lg_D(N_g)$?

As explained in Remark 2 for $g$ homogeneous with an isolated singularity and $n \geq 2$, the first part of Question 1 has a positive answer. In the same case, the second part has a positive answer as well, if the weak ordinarity conjecture of [17] Conjecture 1.1] is satisfied by $Y$. We finally note that by Theorem 1, 3 Conjecture 1.4] (which is equivalent to [8 Conjecture 3.8]) implies a positive answer to the first part of Question 1 for $g$ (not necessarily homogeneous) with an isolated singularity at the origin and $n \geq 2$.

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