Modular Fuss-Catalan Numbers

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Abstract

The modular Catalan numbers $C_{k,n}$, introduced by Hein and Huang in 2016 count equivalence classes of parenthesizations of $x_0 \ast x_1 \ast \cdots \ast x_n$ where $\ast$ is a binary $k$-associative operation and $k$ is a positive integer. The classical notion of associativity is just 1-associativity, in which case $C_{1,n} = 1$ and the size of the unique class is given by the Catalan number $C_n$.

In this paper we introduce modular Fuss-Catalan numbers $C_{m,k,n}$ which count equivalence classes of parenthesizations of $x_0 \ast x_1 \ast \cdots \ast x_n$ where $\ast$ is an $m$-ary $k$-associative operation for $m \geq 2$. Our main results are a closed formula for $C_{m,k,n}$ and a characterisation of $k$-associativity.

Keywords: Fuss-Catalan numbers, Modular Catalan numbers, $m$-Dyck paths, $m$-ary trees, Tamari lattice, $k$-associativity, $m$-ary operations.

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1 Introduction

Let $X$ be a non-empty set with a binary operation $\ast : X^2 \to X$ and let $n \geq 1$ be a positive integer. If $\ast$ is associative, the general associativity law states that the expression $x_1 \ast x_2 \ast \cdots \ast x_n$ is unambiguous for all $x_1, x_2, \ldots, x_n \in X$, i.e. all possible parenthesizations of the expression result in the same evaluation. We say a non-associative operation $\ast$ is left-associative if the order of operation is understood to be from left to right, in which case we write $x_1 \ast \cdots \ast x_n$ to mean $(x_1 \ast (x_2 \ast \cdots \ast x_n))$ for all $x_1, x_2, \ldots, x_n \in X$. Let $k \geq 1$ be a positive integer. There is a notion of $k$-associativity for binary operations which generalises the notion of associativity. A left-associative binary operation $\ast$ is $k$-associative if

$$(x_1 \ast \cdots \ast x_{k+1}) \ast x_{k+2} = x_1 \ast (x_2 \ast \cdots \ast x_{k+1} \ast x_{k+2})$$

for all $x_1, x_2, \ldots, x_{k+2} \in X$.

By setting $k = 1$, we recover the classical notion of associativity for binary operations. In the case where $k > 1$, the general associativity law no longer holds, that is in general the evaluation of the expression $x_1 \ast x_2 \ast \cdots \ast x_n$ depends on its parenthesization. The $k$-associative binary operations are studied in [6].

Fix a positive integer $m \geq 2$. An $m$-ary operation on $X$ is a map $\ast : X^m \to X$. Another way to generalise associativity of binary operations is to consider associative $m$-ary operations. We
Theorem 1.1. It is easy to show that this operation is:

\[
x_1 \cdot \cdots \cdot x_{j-1} \cdot (x_j \cdot x_{j+1} \cdot \cdots \cdot x_{j+m(m-1)}) \cdot x_{j+(m-1)+1} \cdot x_{j+(m-1)+2} \cdots \cdot x_{m+(m-1)}
\]
\[
= x_1 \cdot \cdots \cdot x_{j-1} \cdot x_j \cdot (x_{j+1} \cdot \cdots \cdot x_{j+(m-1)} \cdot x_{j+(m-1)+1}) \cdot x_{j+(m-1)+2} \cdots \cdot x_{m+(m-1)}.
\]

As in the case for associative binary operations, we have a general associativity law stating that the expression \(x_1 \cdot x_2 \cdots \cdot x_n\) is independent of parenthesization (for example, see [1, Theorem 2.1]). We note that \(n\) is not arbitrary in this case, but is of the form \(n = m + g(m-1)\) for some integer \(g \geq 1\). Associative \(m\)-ary operations are important for the study of \(m\)-semmigroups and polyadic groups. These are generalisations of semigroups and groups where we consider \(m\)-ary operations instead of associative binary operations. The \(m\)-semigroups were introduced in [3] and polyadic groups were introduced extensively in [9] and [10].

In this paper we will study \(m\)-ary \(k\)-associative operations, which are a further generalisation of associative binary operations that combines the two above generalisations. We say \(*\) is left-associative if the order of operation is from left to right, meaning for an integer \(g \geq 1\), we write \(x_1 \cdot x_2 \cdots \cdot x_{m+g(m-1)}\) to mean

\[
\left( \ldots \left(x_1 \cdot \cdots \cdot x_m \right) \cdot x_{m+1} \cdot \cdots \cdot x_{m+(m-1)} \right) \cdots \cdot x_{m+g(m-1)+1} \cdots \cdot x_{m+g(m-1)}.
\]

A left-associative \(m\)-ary operation \(*\) is said to be \(k\)-associative if for \(1 \leq j \leq m-1\), the following equality holds:

\[
x_1 \cdot \cdots \cdot x_{j-1} \cdot (x_j \cdot x_{j+1} \cdot \cdots \cdot x_{j+k(m-1)}) \cdot x_{j+k(m-1)+1} \cdot x_{j+k(m-1)+2} \cdots \cdot x_{m+k(m-1)}
\]
\[
= x_1 \cdot \cdots \cdot x_{j-1} \cdot x_j \cdot (x_{j+1} \cdot \cdots \cdot x_{j+k(m-1)} \cdot x_{j+k(m-1)+1}) \cdot x_{j+k(m-1)+2} \cdots \cdot x_{m+k(m-1)}.
\]

We note that the terminology “\(k\)-associativity” is used by Wardlaw in [12] to mean associativity of \(k\)-ary operations. This is not to be confused with the notion of \(k\)-associativity we consider here, which is a generalisation of associativity for \(m\)-ary operations (and coincides with the \(k\)-associativity for binary operation introduced in [6]). Let \(*\) be a \(k\)-associative \(m\)-ary operation. Let \(g \geq 1\) be a positive integer. Then for \(n = m + g(m-1)\) and \(k > 1\), the expression \(x_1 \cdot \cdots \cdot x_n\) is ambiguous without a parenthesization to clarify the order of operation i.e. the general associativity law no longer holds. Let \(p\) and \(p'\) be two parenthesizations of \(x_1 \cdots \cdot x_n\). If we may obtain \(p'\) from \(p\) by finitely many left-hand-side to right-hand-side applications of the \(k\)-associative property (2) to \(p\), we write \(p \preceq_k p'\). This induces a partial order on the set of parenthesizations of \(x_1 \cdot x_2 \cdots \cdot x_n\), called the \(k\)-associative order. The connected components of the \(k\)-associative order are called \(k\)-components. We say that two parenthesizations of \(x_1 \cdot x_2 \cdots \cdot x_n\) are \(k\)-equivalent if they lie in the same \(k\)-component. When \(k = 1\) and \(m = 2\), we recover the well-known Tamari lattice (see for example [4]). In general, determining whether two parenthesizations are \(k\)-equivalent is a non-trivial problem.

Let \(A = (\mathbb{C} \langle u_1, \ldots, u_n \rangle, +, \cdot)\) be the free associative algebra over \(\mathbb{C}\) in \(n\) indeterminates \(u_1, u_2, \ldots, u_n\). Let \(\omega\) be an element of order \(k(m-1)\) in \(A\). We define an \(m\)-ary operation \(\circ : A^m \rightarrow A\) in the following way for \(u_1, \ldots, u_m\) in \(A\):

\[
a_1 \circ \cdots \circ a_m = \omega^{m-1} \cdot a_1 + \omega^{m-2} \cdot a_2 + \cdots \cdot \omega \cdot a_{m-1} + a_m.
\]

It is easy to show that this operation is \(k\)-associative by direct calculation.

**Theorem 1.1.** Let \(* : X^m \rightarrow X\) be a \(k\)-associative \(m\)-ary operation on a set \(X\) where \(m \geq 2\) and \(k \geq 1\) are integers. Let \(p(x_1 \cdots \cdot x_n)\) and \(p'(x_1 \cdots \cdot x_n)\) be two \(m\)-ary parenthesizations
of the $m$-ary expression $x_1 \ast \cdots \ast x_n$, where $n = m + g(m - 1)$ for some positive integer $g$. Then $p(x_1 \ast \cdots \ast x_n)$ is $k$-equivalent to $p'(x_1 \ast \cdots \ast x_n)$ if and only if

$$p(u_1 \circ \cdots \circ u_n) = p'(u_1 \circ \cdots \circ u_n),$$

where the equality comes from evaluating the parenthesizations under $\circ$ in the algebra $A$.

We define the $(k)$-modular Fuss-Catalan number $C_{k,n}^m$ to be the number of $k$-equivalence classes of parenthesizations of $x_0 \ast x_1 \ast \cdots \ast x_n$. By the general associativity law we have that $C_{1,n}^m = 1$, and the size of this class is given by the Fuss-Catalan number $C_n^m = \frac{1}{(m-1)n+1} \binom{m}{n}$.

The following theorem is a generalisation of [6, Proposition 2.10].

**Theorem 1.2.** The $(k)$-modular Fuss-Catalan number is given by the following closed formula,

$$C_{k,n}^m = \sum_{1 \leq l \leq n} \frac{1}{m-1} \sum_{m_1 + \cdots + m_k = n} \binom{n}{m_1 \ldots m_k}.$$

This paper is organised as follows: in §2 we define right $k$-rotations, left $k$-rotations and $k$-equivalence for $m$-ary trees. In §3 we study $k$-equivalence by appealing to $m$-Dyck paths in order to prove Theorem 2.16; which can be thought of as the $m$-ary tree version of Theorem 1.1. In §4, we prove our first main result Theorem 1.1 using some results obtained in the previous section. Finally in §5 we derived the closed formula in Theorem 1.2 using $m$-Dyck paths; a method adopted from [5, §5].

## 2 $m$-ary Trees

In studying $k$-equivalence, it is often more convenient to do so by appealing to other sequences of combinatorial sets counted by the Fuss-Catalan numbers. In this section we will study $k$-equivalence via $m$-ary trees. In order to do this, we use a known bijection between parenthesizations of $m$-ary expressions and $m$-ary trees outlined in [8, §0]. For the rest of this section, we fix integers $m \geq 2$, $g \geq 0$, $k \geq 1$ and $n = m + g(m - 1)$.

**Definition 2.1. $m$-ary Tree** [11, Section 4, A14(b)]. An $m$-ary tree is a rooted tree with the property that each node either has 0 or $m$ linearly ordered children. A node with no children will be referred to as a leaf and the unique node without a parent is called the root.

**Remark 2.2.** The objects we are calling $m$-ary trees in this paper are commonly referred to as full $m$-ary trees in the wider literature.

It will be our convention in this paper to draw $m$-ary trees with the root at the top and leaves at the bottom. We shall denote the set of $m$-ary trees with $n$ leaves by $B_n^m$. We enumerate the leaves from left to right in a linear order starting from 1 on the leftmost leaf, up to $n$ on the rightmost leaf. We will endow the $m$-ary trees with an additional edge labelling with labels drawn from the set $\{l_1, l_2, \ldots, l_m\}$. An edge will be given the label $l_i$ if it links a node with its $i$th child. See the figure below for an example.
Proposition 2.4. The following bijection is well-known, see for example [8, §5].

Consider the tree $t$ with a root and assign to each leaf $x$ the expression $x = ((t_1 \wedge \cdots \wedge t_m)_{m+1} \wedge \cdots \wedge t_{m+(m-1)}) \wedge \cdots \wedge t_{m+(g-1)(m-1)+1} \cdots t_{m+g(m-1)}$.

The following bijection is well-known, see for example [8, §5].

Definition 2.3. Meet. Let $t_1, t_2, \ldots, t_m$ be $m$-ary trees. We define the meet of $t_1, t_2, \ldots, t_m$ to be the $m$-ary tree $t_1 \wedge t_2 \wedge \cdots \wedge t_m$, which has the tree $t_i$ as the subtree rooted at the $i^{th}$ child of the root for $1 \leq i \leq n$.

We regard the meet as a left-associative $m$-ary operation, meaning that the order of operation is from left to right. That is to say, we will write $t_1 \wedge t_2 \wedge \cdots \wedge t_{m+g(m-1)}$ to mean

$$((\ldots((t_1 \wedge \cdots \wedge t_{m+1}) \wedge \cdots \wedge t_{m+(g-1)(m-1)}) \cdots \wedge t_{m+(g-1)(m-1)+1} \cdots t_{m+g(m-1)}).$$

Example 2.5. Let $t$ be the tree in Figure 1. Thinking of the leaves of $t$ as $3$-ary trees consisting of just a root, assign to each leaf $i$ the label $\varepsilon_i$. Consider the tree $t$ expressed as a bracketed meet of its leaves $\varepsilon_i$, where the $\varepsilon_i$ are thought of as trees consisting of just a root. The bijection maps $t$ to the parenthesization obtained by replacing $\wedge$ with * and replacing $\varepsilon_i$ with $x_i$. The inverse map from the set of $m$-ary parenthesizations of the expression $x_1 * x_2 * \cdots * x_n$ to $m$-ary trees with $n$ leaves acts in the naturally opposite way.

Definition 2.6. Right $k$-rotation. Let $k \geq 1$ be a positive integer. Let $t_1, t_2, \ldots, t_{(m-1)+k(m-1)}$ be $m$-ary trees. Let $1 \leq j \leq m-1$. Suppose that $t \in B_n^m$ has a subtree $s = t_1 \wedge t_2 \wedge \cdots \wedge t_{j-1} \wedge (t_j \wedge t_{j+1} \wedge \cdots \wedge t_{j+k(m-1)}) \wedge t_{j+k(m-1)+1} \wedge t_{j+k(m-1)+2} \wedge \cdots \wedge t_{(m-1)+k(m-1)}$ rooted at some node $v$ in $t$. The right $k$-rotation of $t$ at $v$ is the operation of replacing $s$ with the subtree $s' = t_1 \wedge t_2 \wedge \cdots \wedge t_{j-1} \wedge t_{j+1} \wedge \cdots \wedge t_{j+k(m-1)} \wedge t_{j+k(m-1)+1} \wedge t_{j+k(m-1)+2} \wedge \cdots \wedge t_{(m-1)+k(m-1)}$. 

Figure 1: A labelled 3-ary tree.
**Remark 2.7.** It should be clear that under the bijection in Proposition 2.4, right \( k \)-rotation of \( m \)-ary trees corresponds to a left-hand-side to right-hand-side application of the \( k \)-associative rule in (2). We can also define a left \( k \)-rotation dually by switching the roles of \( s \) and \( s' \) in the above definition. In this case, a left \( k \)-rotation corresponds to a right-hand-side to left-hand-side application of the \( k \)-associative rule in (2).

**Definition 2.8.** Let \( t \) and \( t' \) be \( m \)-ary trees with \( n \) leaves. If we can obtain \( t' \) from \( t \) by applying finitely many right \( k \)-rotations to \( t \), we write \( t \preceq_k t' \). The induced partial order on \( B^m_n \) is called the \( k \)-associative order. The connected components (i.e connected components of the Hasse diagram) of \( B^m_n \) under the \( k \)-associative order are called \( k \)-components. We then say two \( m \)-ary trees with \( n \) leaves are \( k \)-equivalent if they belong to the same \( k \)-component of \( B^m_n \).

**Example 2.9.** The example below shows a right 2-rotation on a 3-ary tree. We apply the right 2-rotation at \( v \).

![Diagram showing a right 2-rotation on a 3-ary tree](image)

Figure 2: The tree on the right is a result of a right 2-rotation of the tree on the left at \( v \).

The subtree rooted at \( v \) is \( s = (t_1 \land t_2 \land t_3 \land t_4 \land t_5) \land t_6 \land t_7 = ((t_1 \land t_2 \land t_3) \land t_4 \land t_5) \land t_6 \land t_7 \). The subtree \( s \) is then replaced by the subtree \( s' = t_1 \land (t_2 \land t_3 \land t_4 \land t_5 \land t_6) \land t_7 = t_1 \land ((t_2 \land t_3 \land t_4) \land t_5 \land t_6) \land t_7 \) at \( v \).

The following proposition is a generalisation of [6, Proposition 2.5].

**Proposition 2.10.** Let \( t \) be an \( m \)-ary tree such that we can perform a right \( k \)-rotation of \( t \) at some node \( v \). Suppose that \( k = pk' \) for some positive integers \( p \) and \( k' \). Then the right \( k \)-rotation at \( v \) may be decomposed into a sequence of \( p \) right \( k' \)-rotations of \( t \). The same holds for left \( k \)-rotations.

**Proof.** We argue by induction on \( p \). The case for \( p = 1 \) is trivial. So assume for induction that the statement is true for some \( p \geq 1 \). Suppose that \( k = (p + 1)k' \) for some positive integer \( k' \).

Suppose we have a tree \( t \) which we can right \( k \)-rotate at some node \( v \). Denote by \( s \) the subtree of \( t \) rooted at \( v \). For some \( 1 \leq j \leq m - 1 \),

\[
\begin{align*}
    s &= t_1 \land t_2 \land \cdots \land t_{j-1} \land t_j \land t_{j+1} \land \cdots \land t_{j+pk'(m-1)} \land t_{j+pk'(m-1)+1} \land \cdots \land t_{j+k(m-1)} \\
    &\land t_{j+k(m-1)+1} \land \cdots \land t_{j+(m-1)+k(m-1)}.
\end{align*}
\]

The right \( k \)-rotation replaces the subtree \( s \) with the subtree \( s' \) where

\[
\begin{align*}
    s' &= t_1 \land t_2 \land \cdots \land t_{j-1} \land t_j \land (t_{j+1} \land \cdots \land t_{j+m-1} \land t_{j+m} \land \cdots \land t_{j+pk'(m-1)} \land t_{j+pk'(m-1)+1} \land t_{j+pk'(m-1)+2} \land \cdots \land t_{j+k(m-1)+1} \land \cdots \land t_{(m-1)+k(m-1)}.
\end{align*}
\]

We will show that the result of this right \( k \)-rotation can also be obtained by performing \((p+1)\) right \( k' \)-rotations.
Let \( r \) be the following subtree of \( s \), which is rooted at the \( j \text{th} \) child of the root of \( s \),

\[
r = (t_j \land t_{j+1} \land \cdots \land t_{j+pk'(m-1)} \land t_{j+pk'(m-1)+1} \land \cdots \land t_{j+k(m-1)}).
\]

We can write

\[
r = ((t_j \land t_{j+1} \land \cdots \land t_{j+m-1} \land t_{j+m} \land \cdots \land t_{j+pk'(m-1)}) \land t_{j+pk'(m-1)+1} \land \cdots \land t_{j+k(m-1)})
\]

since the meet operation is left-associative. Performing a right \( pk' \)-rotation of \( t \) at the \( j \text{th} \) child of the root of \( s \), we replace \( r \) with

\[
r' = (t_j \land (t_{j+1} \land \cdots \land t_{j+m-1} \land t_{j+m} \land \cdots \land t_{j+pk'(m-1)}) \land t_{j+pk'(m-1)+1} \land \cdots \land t_{j+k(m-1)}).
\]

By the inductive hypothesis, this right \( pk' \)-rotation is the result of \( p \) right \( k' \)-rotations.

Set

\[
u := (t_{j+1} \land \cdots \land t_{j+m-1} \land t_{j+m} \land \cdots \land t_{j+pk'(m-1)} \land t_{j+pk'(m-1)+1}).
\]

Then

\[
r' = (t_j \land u \land \cdots \land t_{j+k(m-1)}).
\]

Thus the above right \( pk' \)-rotation of \( t \) at the \( j \text{th} \) child of the root of \( s \), replaces \( s \) with \( q \) at the node \( v \) in \( t \) where,

\[
q = t_1 \land t_2 \land \cdots \land t_{j-1} \land (t_j \land u \land t_{j+pk'(m-1)+2} \land \cdots \land t_{j+k(m-1)+1} \land t_{j+k(m-1)+2} \land \cdots \land t_{j+k(m-1)+k(m-1)}).
\]

We then perform a right \( k' \)-rotation of \( t \) at \( v \). This replaces \( q \) with the subtree,

\[
q' = t_1 \land t_2 \land \cdots \land t_{j-1} \land t_j \land (u \land t_{j+pk'(m-1)+2} \land \cdots \land t_{j+k(m-1)+1} \land t_{j+k(m-1)+2} \land \cdots \land t_{j+k(m-1)+k(m-1)}).
\]

It is easy to see that \( s' = q' \), therefore the result of performing the right \( k = (p+1)k' \)-rotation at \( v \) is precisely the result of performing \( (p+1) \) right-\( k' \) rotations. The proof for left \( k \)-rotations is similar.

**Definition 2.11.** Path. Let \( t \) be an \( m \)-ary tree and \( n \) a positive integer. A path \( p \) in \( t \) of length \( n \) from a node \( v \) to a node \( w \) is a sequence \( p = (v_1, v_2, \ldots, v_n) \) of nodes such that \( v_1 = v, v_n = w \) and \((v_i, v_{i+1})\) is an edge in \( t \) for \( 1 \leq i \leq n - 1 \).

**Definition 2.12.** Depth. Let \( t \) be an \( m \)-ary tree with \( n \) leaves and edges labelled by labels drawn from the set \( \{l_1, l_2, \ldots, l_m\} \). For \( i = 1, 2, \ldots, m \) and \( j = 1, 2, \ldots, n \), let \( \delta^i_j(t) \) be the number of edges labelled \( l_i \) in the unique path from the root to the \( j \text{th} \) leaf. Let \( \delta^i(t) = (\delta^i_1(t), \ldots, \delta^i_n(t)) \) and set \( \delta(t) = (\delta^1(t), \ldots, \delta^n(t)) \). We call \( \delta(t) \) the depth of \( t \).

**Example 2.13.** Let \( t \) be the tree in Figure 1. The depth of tree \( t \) is given by

\[
\delta(t) = ((2, 2, 1, 1, 0, 0, 0), (0, 1, 2, 1, 0, 1, 0), (0, 0, 0, 1, 1, 0, 1)).
\]

The following lemmas are easy to verify.

**Lemma 2.14.** Let \( t \) be an \( m \)-ary tree with \( n \) leaves and depth \( (\delta^1(t), \delta^2(t), \ldots, \delta^m(t)) \). We have that \( \delta^i_n(t) \neq 0 \) and \( \delta^i_1(t) = 0 \) for \( i \neq m \). Dually, \( \delta^i_1(t) \neq 0 \) and \( \delta^i_n(t) = 0 \) for \( i \neq 1 \).

This is because the unique path from the root to the \( n \text{th} \) leaf involves choosing the \( n \text{th} \) child at each stage. Similarly for the dual statement.
Lemma 2.15. Let $t$ be an $m$-ary tree with $n$ leaves and depth $(\delta_1(t), \delta_2(t), \ldots, \delta_m(t))$. We have that $\delta_{n-1}^m(t) = 1$. For $1 \leq i \leq m - 2$, we have that $\delta_{n-1}^i(t) = 0$.

This is because the unique path from the root to the $(n - 1)$th leaf involves choosing the $m$th child at every stage but one, in which case, we choose the $(m - 1)$th child.

We shall prove the following result in the next section.

Theorem 2.16. Let $t, t'$ be a pair of $m$-ary trees with $n$ leaves with depths $(\delta_1(t), \delta_2(t), \ldots, \delta_m(t))$ and $(\delta_1(t'), \delta_2(t'), \ldots, \delta_m(t'))$ respectively. We have that $t$ and $t'$ are $k$-equivalent if and only if

$$\sum_{i=1}^{m-1} (m - i)\delta_i(t) \equiv \sum_{i=1}^{m-1} (m - i)\delta_i(t') \mod k(m - 1)$$

where the addition on the $n$-tuples is componentwise.

The strength of the theorem is that it allows us to determine the $k$-equivalence of $m$-ary trees from simply reading their depths. The case $m = 2$ is known, see [6, Proposition 2.11]). We shall prove the case for general $m \geq 2$. To do this, we appeal to another sequence of combinatorial sets counted by the Fuss-Catalan numbers, the $m$-Dyck paths.

3 m-Dyck Paths

In this section we prove Theorem 2.16. In order to do so, we appeal to a generalisation of Dyck paths known as $m$-Dyck paths to further study $k$-equivalence. We prove the theorem by first proving an $m$-Dyck path version of it. For the rest of this section, we fix integers $m \geq 2$, $g \geq 0$, $k \geq 1$ and $n = m + g(m - 1)$.

Definition 3.1. $m$-Dyck Path. An $m$-Dyck path is a lattice path in $\mathbb{Z}^2$ starting at $(0, 0)$ consisting of up-steps $(m, m)$ and down-steps $(1, -1)$, which remains above the $x$-axis and ends on the $x$-axis. The length of a Dyck path is defined to be the number of down-steps it has.

Definition 3.2. Translated $m$-Dyck Path. Let $a, b$ be non-negative integers both not equal to 0. A translated $m$-Dyck path is a lattice path in $\mathbb{Z}^2$ starting at the point $(a, b)$ consisting of up-steps $(m, m)$ and down-steps $(1, -1)$, which remains above the line $y = b$ and ends on the line $y = b$.

We denote the set of $m$-Dyck paths of length $n$ by $D^m_n$. Where it is convenient, we refer to these paths as Dyck paths instead of $m$-Dyck paths. When referring to a translated $m$-Dyck path that is a sub-path of a larger $m$-Dyck path, we will call it a sub $m$-Dyck path or just sub-Dyck path. The following lemma is straightforward, so we state it without proof.

Lemma 3.3. Let $D$ be an $m$-Dyck path of length $n$. Then we can write

$$D = N^{d_1} S N^{d_2} S \ldots S N^{d_n} S,$$

where $N$ denotes the up-step $(1, 1)$ and $S$ denotes the down-step $(1, -1)$. Note that when $m \neq 1$ the up-steps $(1, 1)$ are not steps on the path $D$ since by definition up-steps of $D$ are of the form $(m, m)$. Here $N^{d_i}$ is taken to mean a sequence of $d_i$ consecutive up-steps $N$. The $d_i$ are non-negative integer multiples of $m$ such that $d_1 + \cdots + d_n = n$. Moreover the $n$-tuple $d(D) = (d_1, d_2, \ldots, d_n)$ is unique to each $m$-Dyck path $D$. 

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Figure 3: a 3-Dyck path with 2 (3,3) up-steps. \(D = N^3SN^3SN^0SN^0SN^0SN^0S\).

Let \(D = N^{d_1}SN^{d_2}S \ldots SN^{d_n}S\) be an \(m\)-Dyck path of length \(n\). When expressing \(D\) in this way, if \(d_i = 0\), we will omit \(N^{d_i}\) from the expression. In this form we will also write \(S^l\) to mean a sequence of \(l\) consecutive \(S\) steps. For example, \(D = N^3SN^3SN^0SN^0SN^0S = N^3SN^3S^5\).

We can express any \(m\)-ary tree as the meet of the \(m\)-ary sub-trees rooted at the children of the root. Therefore for \(t\) an \(m\)-ary tree with \(n\) leaves, we write
\[
t = t_1 \land t_2 \land \cdots \land t_m,
\]
where for \(1 \leq i \leq m\) each \(t_i\) is an \(m\)-ary tree with \(n_i\) leaves and \(n_1 + \cdots + n_m = n\).

Let \(\varepsilon\) be the element of the singleton set \(B_m^0\), so \(\varepsilon\) is the \(m\)-ary tree which consists of just a root. We construct a map \(\sigma_m : B_m^0 \to D^{m-1}\) from the set of \(m\)-ary trees to the set of \((m-1)\)-Dyck paths. We define \(\sigma_m\) inductively in the following way,
\[
\sigma_m(t) = \begin{cases} 
N^0S^0 & \text{if } t = \varepsilon; \\
N^{m-1}\sigma_m(t_1)S\sigma_m(t_2)S \ldots S\sigma_m(t_m) & \text{otherwise}.
\end{cases}
\]

This construction generalises a well known map between binary trees (2-ary trees) and Dyck paths (1-Dyck paths); see for example [2, Page 58, Tamari Lattice, Paragraph 2].

**Example 3.4.** Consider the following 3-ary tree \(t = \varepsilon \land \varepsilon \land (\varepsilon \land \varepsilon \land \varepsilon)\). We calculate \(\sigma_3(t)\),
\[
\sigma_3(t) = N^2\sigma_3(\varepsilon)S\sigma_3(\varepsilon)S\sigma(\varepsilon \land \varepsilon \land \varepsilon) \\
= N^2N^0S^0SN^0S^0SN^2S^0S\sigma(\varepsilon)S\sigma(\varepsilon)S\sigma(\varepsilon) \\
= N^2S^2N^2S^2.
\]

See Figure 4.
Theorem 3.6. The map \( \sigma_m : B^m \rightarrow D^{m-1} \) sends \( m \)-ary trees with \( n \) leaves to \((m-1)\)-Dyck paths of length \( n-1 \).

Proof. We argue by induction. Recall that \( n = m + g(m-1) \) for some integer \( g \geq 0 \). We prove the result by induction on \( g \). When \( g = 0 \) there is only one tree to consider, namely \( t = \epsilon \wedge \epsilon \wedge \cdots \wedge \epsilon \).

Now suppose that the result holds for \( n = m + g'(m-1) \) for all \( g' \leq g \). We consider the \( g+1 \) case. Let \( t \) be an \( m \)-ary tree with \( m + (g+1)(m-1) \) leaves. Then we may write \( t = t_1 \wedge t_2 \wedge \cdots \wedge t_m \) with the \( t_i \in B^m \) and \( n_1 + n_2 + \cdots + n_m = m + (g+1)(m-1) \). Then by definition \( \sigma_m(t) = N^{m-1} \sigma_m(t_1) S \sigma_m(t_2) S \cdots \sigma_m(t_m) \) and by the inductive hypothesis, each \( \sigma(t_i) \) is an \((m-1)\)-Dyck paths of length \( n_i - 1 \). In the expression for \( \sigma_m(t) \) we have \( m-1 \) down-steps \( S \) following the \( N^{m-1} \) inbetween the \( \sigma_m(t_i) \). Therefore the length of \( \sigma_m(t) \) is \((n_1 - 1) + (n_2 - 1) + \cdots + (n_m - 1) + (m-1) = m + (g+1)(m-1) - 1 \) as required. So \( \sigma_m \) is indeed a bijection from \( B^m \) to \( D^{m-1} \).

By the above lemma, \( \sigma_m \) induces a map \( \sigma_{m,n} : B^m_n \rightarrow D^{m-1}_{n-1} \). This map is in fact a bijection between \( B^m_n \) and \( D^{m-1}_{n-1} \). Let \( t = t_1 \wedge t_2 \wedge \cdots \wedge t_m \), where for \( 1 \leq i \leq m \) each \( t_i \) is an \( m \)-ary tree with \( n_i \) leaves and \( n_1 + \cdots + n_m = n \). Then \( \sigma_{m,n} \) is defined as follows:

\[
\sigma_{m,n}(t) = N^{m-1} \sigma_{m,n_1}(t_1) S \sigma_{m,n_2}(t_2) S \cdots S \sigma_{m,n,m}(t_m).
\]

**Proposition 3.6.** The map \( \sigma_{m,n} : B^m_n \rightarrow D^{m-1}_{n-1} \) is a bijection.

Proof. It is well known that both the finite sets \( B^m_n \) and \( D^{m-1}_{n-1} \) have cardinality \( \frac{1}{(m-1)!} \binom{m+n-2}{n} \), see for example [7, Section 3]. To show that \( \sigma_{m,n} \) is a bijection it suffices to show that it is a surjection. We argue by induction on \( n \). When \( n = 0 \), it is trivial.

Let \( D \in D^{m-1}_{n-1} \). Then the first step of \( D \) is an up-step \( N^{k_1(m-1)} \) where \( k_1 \geq 1 \) is an integer. So we can write \( D = N^{m-1} N^{k_1(m-1) - (m-1)} \cdots S \) as in Lemma 3.3. Let \( (x_1', m-2) \) be the first point on \( D \) with \( y \)-coordinate \( m-2 \) after the point \( (m-1, m-1) \). Then the step in \( D \) ending at \( (x_1', m-2) \) must be a down-step \( S \) starting at \( (x_1, y_1) = (x_1' + 1, m-1) \). The part of \( D \) from \((m-1, m-1)\) to \((x_1, y_1)\) is a translated \((m-1)\)-Dyck path \( D_1 \). So we see that the path \( D \) starts as \( N^{m-1} D_1 S \). Let \((x_2', m-3)\) be the first point on \( D \) with \( y \)-coordinate \( m-3 \) after
the point \((x'_1, m - 2)\). As above, the step in \(D\) ending at \((x'_2, m - 3)\) must be a down-step \(S\) starting at \((x_2, y_2) = (x'_2 - 1, m - 2)\). The part of \(D\) from \((x'_1, m - 2)\) to \((x_2, y_2)\) is a translated \((m - 1)\)-Dyck path \(D_2\). Therefore \(D = N^{m-1}D_1SD_2S...S\), and continuing this argument we see that can be write \(D = N^{m-1}D_1SD_2S...D_mS\), where the \(D_i\) are translated \((m - 1)\)-Dyck paths for \(1 \leq i \leq m\).

Regarding the translated \((m - 1)\)-Dyck paths \(D_i\) as \((m - 1)\)-Dyck paths, they each have length \(n_i < n\) for \(1 \leq i \leq m\). So by the inductive hypothesis, for each \(D_i\) there exists an \(m\)-ary tree \(t_i\) such that \(D_i = \sigma_{m,n}(t_i)\). It then follows that \(D = \sigma_{m,n}(t_1 \land t_2 \land \cdots \land t_m)\).

\[ \square \]

Going forward, we drop the subscripts on \(\sigma_{m,n}\) and just write \(\sigma\) when it clear what is meant from the context.

**Proposition 3.7.** Let \(t\) be an \(m\)-ary tree with \(n\) leaves and depth \((\delta^0(t), \delta^1(t), \ldots, \delta^n(t))\). Then \(\sigma(t) = N^{d_1}SN^{d_2}S\cdots SN^{d_{n-1}}SN^d\) where the \(d_i\) are given by

\[
d_1 = (m - 1)\delta^1(t),
\]

\[
d_j = \left(\sum_{i=1}^{m} (m - i)(\delta^i_j(t) - \delta^i_{j-1}(t))\right) + 1, \text{ for } 2 \leq j \leq n.
\]

**Proof.** Recall that \(n\) satisfies the equation \(n + g(m-1)\) for some integer \(g \geq 0\). We prove the result by induction on \(g\). When \(g = 0\) there is only one tree to consider, namely \(t = \varepsilon \land \varepsilon \land \cdots \land \varepsilon\).

For this tree we have that \(\delta^i_j = \delta_j\), where the right hand side is the usual Kronecker delta function. We also have that \(\sigma(t) = N^{m-1}SN^0S\cdots SN^{d_{n-1}}SN^d = N^{m-1}S^{m-1}\). We now need to verify that the exponents of the \(S\)'s satify the above relations. Indeed \(d_1 = m - 1 = (m - 1)\delta^1_1\). Moreover \(\sum_{i=1}^{m} (m - i)(\delta^i_j - \delta^i_{j-1}) + 1 = (m - j) + (j - 1) + 1 = (m - m) + (j - 1) = 0 = d_j\) for \(2 \leq j \leq n\).

Now suppose that the result holds for \(n = m + g'(m - 1)\) for all \(g' \leq g\). We consider the \(g + 1\) case. Let \(t\) be an \(m\)-ary tree with \(n = m + (g + 1)(m - 1)\) leaves. Then we may write \(t = t_1 \land \cdots \land t_m\) where each \(t_i\) is the subtree rooted at the \(i\)th child of the root of \(t\). Each subtree \(t_i\) has \(n_i < n\) leaves and \(n_1 + n_2 + \cdots + n_m = n\). In writing \(t\) as the meet of its sub-trees at the root, we partition the leaves of \(t\). We identify each leaf of \(t\) with a pair \((h, j)\) if it lies in the subtree \(t_h\) and it is the \(j\)th leaf in the linear order on the leaves of \(t_h\) where \(1 \leq j \leq n_h\). Therefore for the leaf \((h, j)\) we have that,

\[
\delta^i_{h,j}(t) = \begin{cases} 
\delta^i_{h,j}(t_h) + 1 & \text{if } i = h; \\
\delta^i_{h,j}(t_h) & \text{otherwise.}
\end{cases}
\]

By the inductive hypothesis \(\sigma(t_h) = N^{d_{h,1}}SN^{d_{h,2}}\cdots SN^{d_{h,n_h}}\), where

\[
d_{h,1}(t) = (m - 1)\delta^1_{h,1}(t_h)
\]
therefore the leftmost leaf in the subtree \( t \) is the rightmost leaf in the subtree \( t \).
Therefore by (3),

\[
\sum_{i=1}^{m} (m-i)(\delta_{h,j}^{i}(t_h) - \delta_{h,j-1}^{i}(t_h)) + 1, \text{ for } 2 \leq j \leq n_h.
\]

By definition, \( \sigma(t) = N^{m-1}\sigma(t_1)\sigma(t_2)\ldots \sigma(t_m) \), so

\[
\sigma(t) = N^{m-1}N^{d_1}\sum SN^{d_2}S\ldots SN^{d_m}S = N(m-1)\sum SN^{d_2}S\ldots SN^{d_m}S
\]

Now we verify that the exponents of the \( N \)s satisfy the required relations. We see that

\[
(m-1) + d_{1,1} = (m-1) + (m-1)\delta_{1,1}^{1}(t_1) = (m-1)(\delta_{1,1}^{1}(t_1) + 1) = (m-1)\delta_{1,1}^{1}(t),
\]

so the first exponent satisfies the required relation. We also see that

\[
d_{h,j} = \sum_{i=1}^{m} (m-i)(\delta_{h,j}^{i}(t_h) - \delta_{h,j-1}^{i}(t_h)) + 1, \text{ for } 2 \leq j \leq n_h.
\]

by (3), therefore the \( d_{h,j} \) also satisfy the required relation for \( t \).

The only exponents left to verify are the \( d_{h,1} \) for \( 2 \leq h \leq m \). In this case, the leaf \((h-1,n_{h-1})\) is the rightmost leaf in the subtree \( t_{h-1} \), so by Lemma 2.14, \( \delta_{h-1,n_{h-1}}^{i}(t_{h-1}) = 0 \) when \( i \neq m \). Therefore by (3), \( \delta_{h-1,n_{h-1}}^{i}(t_{h-1}) = 0 \) when \( i \neq m, h-1 \), so \( \delta_{h-1,n_{h-1}}^{i}(t_{h-1}) = 1 \). The leaf \((h,1)\) is the leftmost leaf in the subtree \( t_h \), so by a dual statement of Lemma 2.14, \( \delta_{h,1}^{i}(t_h) = 0 \) when \( i \neq 1 \). Therefore by (3), \( \delta_{h,1}^{i}(t_h) = 0 \) when \( i \neq 1, h \) and \( \delta_{h,1}^{i}(t_h) = 1 \). So it follows that,

\[
\sum_{i=1}^{m} (m-i)(\delta_{h,1}^{i}(t_h) - \delta_{h-1,n_{h-1}}^{i}(t_h)) + 1 = (m-1)\delta_{h,1}^{i}(t_h) + (m-h) - (m - (h-1))
\]

\[
= (m-1)\delta_{h,1}^{i}(t_h) - (m-m)\delta_{h-1,n_{h-1}}^{i}(t_{h-1}) + 1
\]

\[
= d_{h,1}
\]

therefore the \( d_{h,1} \) also satisfy the required relations for \( 2 \leq h \leq m \). This completes the proof.

\[\Box\]

**Remark 3.8.** It is important to note that \( d_n = 0 \) in Proposition 3.7 since otherwise \( D \) is not a Dyck path. We can observe that \( d_n = 0 \) by referencing Lemma 2.14 and Lemma 2.15. So in the above proposition, \( \sigma(t) \) is indeed a Dyck path. Also note that since \( d_n = 0 \), this form of \( \sigma(t) \) is the same as that given in Lemma 3.3.

The bijection between \( m \)-ary trees with \( n \) leaves and \((m-1)\)-Dyck paths of length \( n-1 \) induces an operation corresponding to a \( k \)-rotation on Dyck paths, which we shall call a \( k \)-compression. Recall in the definition of a right \( k \)-rotation we replace a sub-tree of the form,

\[
s = t_1 \land t_2 \land t_3 \land \cdots \land t_j-1 \land (t_j \land t_{j+1} \land \cdots \land t_{j+k(m-1)+1} \land t_{j+k(m-1)+2} \land \cdots \land t_{(m-1)+k(m-1)})
\]

by a subtree of the form,

\[
s' = t_1 \land t_2 \land \cdots \land t_j \land (t_{j+1} \land t_{j+2} \land \cdots \land t_{j+k(m-1)+1} \land t_{j+k(m-1)+2} \land \cdots \land t_{(m-1)+k(m-1)}).
\]

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It is easy to see that
\[
\sigma(s) = N^{m-1}D_1SD_2S \ldots D_{j-1}SN^{k(m-1)}D_jSD_{j+1}S \ldots SD_{m+k(m-1)},
\]
and
\[
\sigma(s') = N^{m-1}D_1SD_2S \ldots D_{j-1}SD_jSN^{k(m-1)}D_{j+1}S \ldots SD_{m+k(m-1)},
\]
where \(D_i = \sigma(t_i)\).

**Definition 3.9. Right \(k\)-Compression.** Let \(k \geq 1\) and \(1 \leq j \leq m - 1\) be positive integers. Let \(D\) be an \((m-1)\)-Dyck path of length \(n - 1\). Suppose \(D\) contains a sub-Dyck path of the form
\[
X = N^{m-1}D_1SD_2S \ldots D_{j-1}SN^{k(m-1)}D_jSD_{j+1}S \ldots SD_{m+k(m-1)},
\]
where the \(D_i\) are (possibly translated) Dyck paths which may be empty. Then a right \(k\)-compression at \(X\) is the operation of replacing \(X\) with the sub-Dyck path
\[
X' = N^{m-1}D_1SD_2S \ldots D_{j-1}SD_jSN^{k(m-1)}D_{j+1}S \ldots SD_{m+k(m-1)}.
\]

The inverse operation of replacing \(X'\) with \(X\) will be called a left \(k\)-compression. Let \(D, D'\) be \((m-1)\)-Dyck paths of length \(n - 1\). Write \(D \preceq_k D'\) to mean that \(D'\) can be obtained from \(D\) by applying finitely many right \(k\)-compressions. The induced partial order on \(D_{n-1}^{m-1}\) is called the \(k\)-associative order. The connected components of \(D_{n-1}^{m-1}\) under the \(k\)-associative order are called \(k\)-components. We will say two \((m-1)\)-Dyck paths are \(k\)-equivalent if they belong to the same \(k\)-component.

Let \(M \subset \mathbb{N}^n\) be the set of \(n\)-tuples of non-negative integers \((e_1, e_2, \ldots, e_n)\) satisfying the following relations,
\[
e_1 + e_2 + \cdots + e_n = n - 1,
\]
\[
\text{for } 1 \leq i \leq n, \quad e_1 + e_2 + \cdots + e_{j-1} \geq j - 1 \quad \text{for all } 1 \leq j \leq n.
\]
Notice that it follows from the first and last relation that \(e_n = 0\).

**Proposition 3.10.** The map \(d: D_{n-1}^{m-1} \to M\) maps an \((m-1)\)-Dyck path of length \((n-1)\) \(D = N^{d_1}SN^{d_2}S \ldots SN^{d_n}\) to the \(n\)-tuple \(d(D) = (d_1, \ldots, d_n)\). This map is a bijection.

**Proof.** Let \(D = N^{d_1}SN^{d_2}S \ldots SN^{d_n}\) be an \((m-1)\)-Dyck path. Let \(d(D) = (d_1, d_2, \ldots, d_n)\). Note that \(d_n = 0\) by Remark 3.8, so the form of \(D\) is precisely as in Lemma 3.3. By Lemma 3.3 the tuple \((d_1, d_2, \ldots, d_{n-1})\) is unique, so the map \(d\) is well-defined. Since \(D\) is an \((m-1)\)-Dyck path, by definition \((m-1)|d_i\). All Dyck paths start and end on the \(x\)-axis, therefore they must go up the same number of times as they go down. Hence if a path has length \(n - 1\), which is the number of down-steps \(S\), then \(d_1 + \cdots + d_n = n - 1\). By definition, Dyck paths cannot go below the \(x\)-axis, this is to say that \(d_1 + \cdots + d_{j-1} \geq j - 1\) for all \(j \geq 1\).

Let \(f: M \to D_{n-1}^{m-1}\) be the map given by \(f(e_1, e_2, \ldots, e_n) = N^{e_1}SN^{e_2}S \ldots SN^{e_n}\). This is a \((m-1)\)-Dyck path by the arguments similar to those above. It is easy to see that \(f(d(D)) = D\), and \(d(f((e_1, \ldots, e_n))) = (e_1, \ldots, e_n)\). Therefore \(d\) is indeed a bijection. \(\square\)

**Proposition 3.11.** Let \(D, D'\) be \((m-1)\)-Dyck paths of length \(n - 1\) with \(d(D) = (d_1, \ldots, d_n)\) and \(d(D') = (d'_1, \ldots, d'_n)\). Suppose that we can obtain \(D'\) from \(D\) by applying a right \(k\)-compression to \(D\). Then there exist \(1 \leq j < i \leq n\) such that \(d'_i = d_i + k(m-1)\), \(d'_j = d_j - k(m-1)\) and \(d'_h = d_h\) for \(h \neq i, j\).
Proof. Recall that in the definition of right $k$-compression, we replace

$$X = N^{m-1}D_1SD_2S \ldots D_{a-1}SN^{k(m-1)}D_aSD_{a+1}S \ldots SD_{m+k(m-1)}$$

with

$$X' = N^{m-1}D_1SD_2S \ldots D_{a-1}SD_aSN^{k(m-1)}D_{a+1}S \ldots SD_{m+k(m-1)},$$

thus moving the substring $N^{k(m-1)}$ from the immediate left of the (possibly translated) Dyck path $D_a$ to the immediate left of (possibly translated) Dyck path $D_{a+1}$. Let $D = N^{d_1}SN^{d_2}S \ldots SN^{d_n}$. We have the sub-strings $N^{k(m-1)}D_a = N^{d_j}S$ and $D_{a+1} = N^{d_{j+1}}S$ in $D$. So in moving $N^{k(m-1)}$ we get the sub-strings $D_a = N^{d_j+k(m-1)}S$ and $N^{k(m-1)}D_{a+1} = N^{d_{j+1}+k(m-1)}S$ in the Dyck path $D'$. This proves the statement of the proposition.

Remark that if we replace right $k$-compression with left $k$-compression in the above proposition, we have that $j > i$ instead.

Corollary 3.12. Let $D, D'$ be $(m-1)$-Dyck paths of length $n - 1$. If $D$ and $D'$ are $k$-equivalent then $d(D) \equiv d(D') \mod k(m-1)$.

Proof. This is an immediate consequence of Proposition 3.11.

Let $D$ be an $m$-Dyck path of length $n$. We say that $D$ is $k$-minimal if it is minimal in its $k$-equivalence class. That is to say there does not exist a Dyck path $D' \in D_n^m$ such that $D' \preceq_k D$. Let $p = (x, y)$ in $\mathbb{Z}^2$ be a point on the $m$-Dyck path $D$. The level of the point $p$ is the integer $y$, and we say that $p$ is on the $y$th level.

Proposition 3.13. An $(m-1)$-Dyck path $D$ is minimal if and only if for $d(D) = (d_1, \ldots, d_n)$, we have that $d_i < k(m-1)$ for all $i \neq 1$.

Proof. Suppose that we have that $d_i < k(m-1)$ for all $i \neq 1$ and $D$ is not minimal. Then we can left $k$-compress $D$ to obtain another dyck path $D'$. By Proposition 3.11 there is some $j > 1$ such that the $j$-th entry of $d(D')$ is $d_j' = d_j - k(m-1)$. By the assumption that $d_i < k(m-1)$ for $i \neq 1$, we must have that $d_j' < 0$, a contradiction. Therefore $D$ must be minimal.

Recall $D$ is of the form $D = N^{d_1}S \ldots SN^{d_i}S \ldots SN^{d_n}$. Suppose that $D$ is minimal and there exists some $i \neq 0$ such that $d_i \geq k(m-1)$. We will show that $D$ is not minimal by demonstrating that we can left $k$-compress $D$. That is to say we will show that we have a sub-Dyck path $X'$ required to perform a left $k$-compression, where

$$X' = N^{m-1}D_1SD_2S \ldots D_{j-1}SD_jSN^{k(m-1)}D_{j+1}S \ldots SD_{(m-1)+k(m-1)}$$

for $1 \leq j \leq (m-1)$.

Suppose the up-step $N^{d_i}$ starts at some point $(b, l)$ and ends at $(b+d_i, l+d_i)$. The immediately preceding down-step $S$ starts at $(b-1, l+1)$ and ends at $(b, l)$. Let $0 \leq x \leq b - 1$ be maximal such that the point $(x, l)$ is on the Dyck path $D$. By the maximality, the point $(x, l)$ is part of an up-step. Let $U$ be the up-step in $D$ beginning at $(x, l)$ if $(x, l)$ is at the start of an up-step; otherwise let $U$ to be the up-step containing $(x, l)$. Let $(x_1, y_1)$ be the end point of the up-step $U$. Let $(x_0, y_0) = (x - (m-1), y_1 - (m-1))$, this is the start point of the up-step $U$. See the figure below.
Let \((x_2, y_2)\) be the point at which it is the first time the Dyck path goes below the level \(y_1\) after the point \((x_1, y_1)\). That is \(y_2 = y_1 - 1\). Then the subpath \(D_1\) starting from \((x_1, y_1)\) and ending at \((x_2 - 1, y_2 + 1)\) is a translated \((m - 1)\)-Dyck path. Note that it could happen that \((x_1, y_1) = (x_2 - 1, y_2 + 1)\), in this case \(D_1\) is just the empty \((m - 1)\)-Dyck path.

Let \((x_3, y_3)\) be the point at which it is the first time the Dyck path goes below the level \(y_1 - 1\) after the point \((x_2, y_2)\), that is \(y_3 = y_1 - 2\). We define \(D_2\) to be the path starting from \((x_2, y_2)\) to \((x_3 - 1, y_3 + 1)\). As before \(D_2\) is a translated \(m\)-Dyck path which starts and ends on the \((y_1 - 1)\)th level.

Let \(j = y_1 - l\). We can repeat this procedure to define translated \(m\)-Dyck paths \(D_3, D_4, \ldots, D_j\). Here each \(m\)-Dyck path \(D_r\) starts at the point \((x_r, y_r)\) and ends at the point \((x_{r+1} - 1, y_{r+1} + 1)\), where the start and end points are defined as above and \(y_{r+1} = y_r - 1 = y_1 - r\). Note that the translated \(m\)-Dyck path \(D_j\) begins on the level \(y_1 = y_1 - (j - 1) = l + 1\), so the last point of \(D_j\) is \((x_{j+1} - 1, l + 1)\) for some \(x_{j+1} - 1 \leq b\). We claim that \((x_{j+1} - 1, l + 1) = (b - 1, l + 1)\). The point \((b - 1, l + 1)\) is the last time we are on the \((l + 1)\)th level before the \(N^{th}\) up-step. By construction, there is a down-step from \((x_{j+1} - 1, l + 1)\) to \((x_{j+1}, l)\). By the maximality of \(x\) we have that \(x_{j+1} = b\) or \(x_{j+1} = x\). Note that \(x_{j+1} \geq x_1 > x\), so we have that \(x_{j+1} = b\).

So far we have constructed a subpath from \((x_0, y_0)\) to \((b, l + 1)\) given by

\[
X'' = N^{m-1} D_1 S D_2 S \ldots D_j S,
\]

where the \(S\) down-steps are the down steps from \((x_1 - 1, y_1 + 1)\) to \((x_i, y_i)\). Note that \(y_i = y_1 - (i - 1)\) for \(2 \leq i < j\) and the \(S\) after \(D_j\) is the one from \((b - 1, l + 1)\) to \((b, l)\). The \(N^{m-1}\) is the up-step \(U\) from \((x_0, y_0)\) to \((x_1, y_1)\).

Since \(d_i \geq k(m - 1)\), there is an up-step \(N^{k(m-1)}\) from \((b, l)\) to \((x_{j+1}, y_{j+1}) = (b + k(m - 1), l + k(m - 1))\). We define \(D_{j+1}\) to be the path from \((x_{j+1}, y_{j+1})\) to \((x_{j+2} - 1, y_{j+2} + 1)\) where \((x_{j+2}, y_{j+2})\) is the point at which the Dyck path first sits on level \(l + k(m - 1) - 1\) after \((x_{j+1}, y_{j+1})\). In the same fashion we define the \((m - 1) - j + k(m - 1)\) sub paths \(D_{j+2}, D_{j+3} \ldots D_{(m-1)+k(m-1)}\). These are all translated \(m\)-Dyck paths by the same arguments as above. By how we construct the Dyck paths, we see that the path \(D_{(m-1)+k(m-1)}\) ends on level \(y_0 = y_1 - (m - 1)\).

So we have successfully constructed the sub-Dyck path of \(D\),

\[
X' = N^{m-1} D_1 S D_2 S \ldots D_{j-1} S D_j S N^{k(m-1)} D_{j+1} S \ldots S D_{(m-1)+k(m-1)}.
\]

As before the \(S\) are the intermediate down steps between the \(D_i\) and the \(D_i\) may also be empty.

We illustrate the constructive proof above with an example for the case where \(k = 2\) and \(m = 3\).
Therefore since \( d \) obtain

Suppose that

Proof. we see that a

\( k \)

case that Dyck path is trivially minimal. This is a contradiction to our assumption. Thus every

\( 3.13 \)

all but the first entries of \( d \) is order reversing. That is \( d \) \( \prec \) \( \ldots \) \( \prec \) \( \ldots \) \( \prec \) \( D \). Applying \( d \) to the descending chain, we get the ascending chain \( d(D) \prec d(D') \prec \ldots \prec d(D') \prec \ldots \). Since \( D_{n-1} \) is finite, this ascending chain must then be a cycle. Since the lexicographic order is anti-symmetric, this cycle must contain only one element. Therefore if a \( k \)-equivalence class doesn’t have a minimal element, it only contains one Dyck path in which case that Dyck path is trivially minimal. This is a contradiction to our assumption. Thus every \( k \)-equivalence class has a minimal Dyck path.

Suppose we have two minimal Dyck paths \( D \) and \( D' \) in an equivalence class. By Proposition 3.13 all but the first entries of \( d(D) \) and \( d(D') \) are strictly less than \( k(m-1) \). But since \( D \) and \( D' \) are \( k \)-equivalent, \( d(D) \equiv d(D') \mod k(m-1) \). This means all but the first entries of \( d(D) \) and \( d(D') \) are equal. The equality of these entries forces the first entries to also be equal since clearly it cannot be the case otherwise. Therefore \( d(D) = d(D') \) which implies \( D = D' \). So the minimal Dyck paths are unique in their equivalence classes.

Theorem 3.15. Let \( D, D' \in D_{n-1}^m \) be two Dyck paths. Then \( D \) and \( D' \) are \( k \)-equivalent if and only if \( d(D) \equiv d(D') \mod k(m-1) \).

Proof. Suppose that \( D \) and \( D' \) are \( k \)-equivalent. Then suppose without loss of generality that we obtain \( D' \) from \( D \) by application of a finite sequence of \( k \)-compressions. From Proposition 3.11, we see that a \( k \)-compression maps \( d(D) \) to an \( n \)-tuple which is congruent to \( d(D) \) modulo \( k(m-1) \). Therefore since \( d(D') \) an \( n \)-tuple which is a result of a finite sequence of \( k \)-compressions on \( D \), then \( d(D) \equiv d(D') \mod k(m-1) \).
Suppose now that \( d(D) \equiv d(D') \mod k(m - 1) \). Consider their respective minimal representatives in their \( k \)-equivalence classes \( D_{\text{min}} \) and \( D'_{\text{min}} \) respectively. Then \( d(D_{\text{min}}) \equiv d(D) \equiv d(D') \equiv d(D'_{\text{min}}) \mod k(m - 1) \). Therefore \( d(D_{\text{min}}) \equiv d(D'_{\text{min}}) \mod k(m - 1) \), hence by Proposition 3.13 we obtain that \( d(D_{\text{min}}) = d(D'_{\text{min}}) \) which means \( D_{\text{min}} = D'_{\text{min}} \). Therefore \( D \) and \( D' \) belong to the same \( k \)-equivalence class. So we conclude \( D \) and \( D' \) are \( k \)-equivalent. \( \square \)

**Theorem 3.16.** Let \( t, t' \) be a pair of \( m \)-ary trees with \( n \) leaves and depth \((\delta^{l_1}(t), \delta^{l_2}(t), \ldots, \delta^{l_m}(t))\), \((\delta^{l_1}(t'), \delta^{l_2}(t'), \ldots, \delta^{l_m}(t'))\) respectively. Then \( t \) and \( t' \) are \( k \)-equivalent if and only if

\[
\sum_{i=1}^{m-1} (m - i)\delta^{l_i}(t) \equiv \sum_{i=1}^{m-1} (m - i)\delta^{l_i}(t') \mod k(m - 1),
\]

where the addition on the \( n \)-tuples is componentwise.

**Proof.** Suppose that \( t \) and \( t' \) are \( k \)-equivalent, then their corresponding Dyck paths \( D = \sigma(t) \) and \( D' = \sigma(t') \) respectively are also \( k \)-equivalent. Therefore by Theorem 3.15 \( d(D) \equiv d(D') \mod k(m - 1) \). By Proposition 3.7,

\[
d_1 = (m - 1)\delta^{l_1}_1(t),
\]

\[
d_j = \sum_{i=1}^{m} (m - i)(\delta^{l_j}_i(t) - \delta^{l_j}_{j-1}(t)) + 1, \text{ for } j > 1.
\]

Since \( d_1 \equiv d'_1 \mod k(m - 1) \), we have that \((m - 1)\delta^{l_1}_1(t) \equiv (m - 1)\delta^{l_1}_1(t') \mod k(m - 1) \). Furthermore, we observe that from the structure of the of \( m \)-ary trees that \( \delta^{l_i}_1(t) = 0 \) and \( \delta^{l_i}_1(t') = 0 \) for \( i \neq 1 \). So we have that

\[
(m - 1)\delta^{l_i}_1(t) \equiv (m - 1)\delta^{l_i}_1(t') \mod k(m - 1), \text{ for } 1 \leq i \leq m,
\]

hence

\[
\sum_{i=1}^{m} (m - 1)\delta^{l_i}_1(t) \equiv \sum_{i=1}^{m} (m - 1)\delta^{l_i}_1(t') \mod k(m - 1).
\]

From the fact that,

\[
d_2 = \sum_{i=1}^{m} (m - i)(\delta^{l_i}_2(t) - \delta^{l_i}_1(t)) + 1 \equiv d'_2 = \sum_{i=1}^{m} (m - i)(\delta^{l_i}_2(t') - \delta^{l_i}_1(t')) + 1 \mod k(m - 1),
\]

we conclude that,

\[
\sum_{i=1}^{m} (m - i)\delta^{l_i}_2(t) \equiv \sum_{i=1}^{m} (m - i)\delta^{l_i}_2(t') \mod k(m - 1).
\]

From this congruence and the congruence

\[
d_3 = \sum_{i=1}^{m} (m - i)(\delta^{l_i}_3(t) - \delta^{l_i}_2(t)) + 1 \equiv d'_3 = \sum_{i=1}^{m} (m - i)(\delta^{l_i}_3(t') - \delta^{l_i}_2(t')) + 1,
\]

we conclude that,

\[
\sum_{i=1}^{m} (m - i)\delta^{l_i}_3(t) \equiv \sum_{i=1}^{m} (m - i)\delta^{l_i}_3(t') \mod k(m - 1).
\]

Continuing in this manner we obtain the following,

\[
\sum_{i=1}^{m} (m - i)\delta^{l_i}_j(t) \equiv \sum_{i=1}^{m} (m - i)\delta^{l_i}_j(t') \mod k(m - 1), \text{ for } 1 \leq j \leq n.
\]
This is the same as saying,
\[ \sum_{i=1}^{m-1} (m - i)\delta^i(t) \equiv \sum_{i=1}^{m-1} (m - i)\delta^i(t') \mod k(m - 1). \]

Now for the converse, suppose that
\[ \sum_{i=1}^{m-1} (m - i)\delta^i(t) \equiv \sum_{i=1}^{m-1} (m - i)\delta^i(t') \mod k(m - 1). \]

This implies that,
\[ \sum_{i=1}^{m} (m - i)\delta^i_j(t) \equiv \sum_{i=1}^{m} (m - i)\delta^i_j(t') \mod k(m - 1), \text{ for } 1 \leq j \leq n. \]

This further implies that
\[ d_1 = (m - 1)\delta^1_j(t) \equiv d'_1 = (m - 1)\delta^1_j(t') \mod k(m - 1) \]
\[ d_j = \sum_{i=1}^{m} (m - i)(\delta^i_j(t) - \delta^i_{j-1}(t)) + 1 \equiv d'_j = \sum_{i=1}^{m} (m - i)(\delta^i_j(t') - \delta^i_{j-1}(t')) + 1 \mod k(m - 1). \]

Therefore by Theorem 3.15 we have that \( D = \sigma(t) \) and \( D' = \sigma(t') \) are \( k \)-equivalent which implies that \( t \) and \( t' \) are \( k \)-equivalent. \( \square \)

4 An Application to \( m \)-ary operations

In this section we will introduce a particular \( k \)-associative \( m \)-ary operation which will be denoted by \( \circ \). This operation will be used to evaluate \( m \)-ary parenthesizations and we will show that this operation characterises \( k \)-equivalence. This is to say that two parenthesizations will be \( k \)-equivalent (\( k \)-associative) if and only if their evaluations under this operation are equal. For the rest of this section, we fix integers \( m \geq 2, g \geq 0, k \geq 1 \) and \( n = m + g(m - 1). \)

Let \( A = \mathbb{C}(u_1, u_2, \ldots, u_n) \) be the free unital associative algebra over \( \mathbb{C} \) in \( n \) indeterminates \( u_1, u_2, \ldots, u_n \). We define a binary operation \( \circ \) on \( A \) as follows. Let \( \omega \) be an element of \( A \) of order \( k(m - 1) \) e.g. \( \omega = e^{\frac{2\pi i}{m}} \). For \( a, b \) in \( A \), we define \( a \circ b := \omega \cdot a + b \), where \( \cdot \) and \( + \) are the multiplication and addition operations in \( A \) respectively. This is taken to be a left-associative operation. Sometimes we will omit the \( \cdot \) for convenience. The binary operation \( \circ \) on \( A \) induces an \( m \)-ary operation on \( A^m \) defined in the following way,
\[ a_1 \circ a_2 \cdots \circ a_m := \omega^{m-1} \cdot a_1 + \omega^{m-2} \cdot a_2 + \cdots + \omega \cdot a_{m-1} + a_m. \]  

(4)

It is easy to see by direct calculation that the following two lemmas are true.

**Lemma 4.1.** The binary operation \( \circ \) on \( A \) is \( k(m - 1) \)-associative.

**Lemma 4.2.** The \( m \)-ary operation on \( A^m \) induced by the binary operation \( \circ \) on \( A \) is \( k \)-associative.

Let \( X \) be a non-empty set and let \( * : X^m \rightarrow X \) be an \( m \)-ary operation. Take \( x_1, x_2, \ldots, x_n \) in \( X \). Recall that there is a bijection between the set of \( m \)-ary trees on \( n \) leaves and the set of \( m \)-ary parenthesizations of the expression \( x_1 * x_2 * \cdots * x_n \), see Proposition 2.4. We will write \( p_t = p(x_1 * x_2 * \cdots * x_n)_t \) to be the \( m \)-ary parenthesization of the expression \( x_1 * x_2 * \cdots * x_n \) corresponding to the \( m \)-ary tree \( t \). We denote the evaluation of \( p_t \) with respect to \( \circ \) by \( p(u_1 \circ u_2 \cdots \circ u_n)_t \). When there is no risk of confusion, we omit the subscript \( t \).
Lemma 4.3. Let \( p(x_1*x_2*\cdots*x_n) \) be an \( m \)-ary parenthesization of \( x_1*x_2*\cdots*x_n \) corresponding to the \( m \)-ary tree on \( n \) leaves \( t \). Let \( \delta^1(t), \delta^2(t), \ldots, \delta^m(t) \) be the depth of \( t \). Then we have that

\[
p(u_1 \circ u_2 \circ \cdots \circ u_n)_t = \omega_{i=1}^{m} (m-i)\delta^i(t) \cdot u_1 + \omega_{i=1}^{m} (m-i)\delta^i(t) \cdot u_2 + \cdots + \omega_{i=1}^{m} (m-i)\delta^i(t) \cdot u_n.
\]

Proof. Recall that \( n \) satisfies the equation \( n = m + g(m-1) \) for some integer \( g \geq 0 \). We prove the result by induction on \( g \). When \( g = 0 \) there is only one tree to consider, namely \( t = \varepsilon \land \varepsilon \land \cdots \land \varepsilon \).

For this tree we have that \( \delta^m = \delta_j \), where the right hand side is the usual Kronecker delta function. It is easy to see that the statement holds in this case by the definition of \( u_1 \circ u_2 \circ \cdots \circ u_m \) in (4).

Now suppose that the result holds for \( n = m + g'(m-1) \) for all \( g' \leq g \). We consider the \( g + 1 \) case. Let \( t \) be an \( m \)-ary tree with \( n = m + (g+1)(m-1) \) leaves. Then we may write \( t = t_1 \land \cdots \land t_m \) where each \( t_i \) is the subtree rooted at the \( i \)th child of the root of \( t \). Each subtree \( t_i \) has \( n_i < n \) leaves and \( n_1 + n_2 + \cdots + n_m = n \). In writing \( t \) as the meet of its sub-trees at the root, we partition the leaves of \( t \). We identify each leaf of \( t \) with a tuple \( (h,j) \) if it lies in the subtree \( t_h \) and it is the \( j \)th leaf in the linear order on the leaves of \( t_h \), where \( 1 \leq j \leq n_h \). Therefore for the leaf \( (h,j) \) we have that,

\[
\delta^i_{h,j}(t) = \begin{cases} 
\delta^i_{h,j}(t_h) + 1 & \text{if } i = h; \\
\delta^i_{h,j}(t_h) & \text{otherwise.}
\end{cases}
\]

(5)

From the above equation, it follows that,

\[
(m-i)\delta^i_{h,j}(t) = \begin{cases} 
(m-i)\delta^i_{h,j}(t_h) + (m-i) & \text{if } i = h; \\
(m-i)\delta^i_{h,j}(t_h) & \text{otherwise.}
\end{cases}
\]

(6)

The identification of the leaves with the tuples \( (h,j) \) gives another labelling of the variables \( u_s \), where \( 1 \leq s \leq n \). Since the variable \( u_s \) corresponds to the \( s \)th leaf of \( t \), and the \( s \)th leaf is identified with \( (h,j) \), then we write \( u_s(h,j) \) for \( u_s \). So we have that

\[
p(u_1 \circ u_2 \circ \cdots \circ u_n)_t = p(u_{(1,1)} \circ u_{(1,2)} \circ \cdots \circ u_{(m,n_m)})_t.
\]

It is then easy to see that,

\[
p(u_{(1,1)} \circ u_{(1,2)} \circ \cdots \circ u_{(m,n_m)})_t = p(u_{(1,1)} \circ \cdots \circ u_{(1,n_1)})_{t_1} \circ p(u_{(2,1)} \circ \cdots \circ u_{(2,n_2)})_{t_2} \circ \cdots \circ p(u_{(m,1)} \circ \cdots \circ u_{(m,n_m)})_{t_m}
\]

\[= \omega^{m-1} p(u_{(1,1)} \circ \cdots \circ u_{(1,n_1)})_{t_1} + \omega^{m-2} p(u_{(2,1)} \circ \cdots \circ u_{(2,n_2)})_{t_2} + \cdots + p(u_{(m,1)} \circ \cdots \circ u_{(m,n_m)})_{t_m}.
\]

By the inductive assumption we have that,

\[
p(u_{(h,1)} \circ u_{(h,2)} \circ \cdots \circ u_{(h,n_h)})_{t_h} = \omega_{i=1}^{m} (m-i)\delta^i_{(h,1)}(t_h) \cdot u_{(h,1)} + \omega_{i=1}^{m} (m-i)\delta^i_{(h,2)}(t_h) \cdot u_{(h,2)} + \cdots + \omega_{i=1}^{m} (m-i)\delta^i_{(h,n_h)}(t_h) \cdot u_{(h,n_h)}.
\]
from which it follows that,

\[ \omega^{m-h} p(u_{(h,1)} \circ u_{(h,2)} \circ \cdots \circ u_{(h,n_h)}) t_h = \sum_{i=1}^{m} (m-i) \delta_{(h,j)}^i (t_h) \cdot u_{(h,1)} + \sum_{i=1}^{m} (m-i) \delta_{(h,j)}^i (t_h) \cdot u_{(h,2)} + \cdots + \omega^{m-h} \sum_{i=1}^{m} (m-i) \delta_{(h,n_h)}^i (t_h) \cdot u_{(h,n_h)}. \]

By equation (6),

\[ \sum_{i=1}^{m} (m-i) \delta_{(h,j)}^i (t_h) + (m-h) = \sum_{i=1}^{m} (m-i) \delta_{(h,j)}^i (t_h) + (m-h) = \sum_{i=1}^{m} (m-i) \delta_{(h,j)}^i (t) \]

Therefore,

\[ p(u_{(1,1)} \circ u_{(1,2)} \circ \cdots \circ u_{(m,m_m)}) t = \sum_{i=1}^{m} (m-i) \delta_{(1,i)}^i (t) \cdot u_{(1,1)} + \sum_{i=1}^{m} (m-i) \delta_{(1,i)}^i (t) \cdot u_{(1,2)} + \cdots + \omega^{m-i} \sum_{i=1}^{m} (m-i) \delta_{(m,m_m)}^i (t) \cdot u_{(m,m_m)}, \]

as required. This completes the proof.

\[ \Box \]

**Theorem 4.4.** Let \( p = p(x_1 \ast x_2 \cdots x_n)_t \) and \( p' = p'(x_1 \ast x_2 \cdots x_n)_{t'} \) be two \( m \)-ary parenthesizations of \( x_1 \ast x_2 \cdots x_n \) corresponding to the \( m \)-ary trees on \( n \) leaves \( t \) and \( t' \) respectively. Then \( p \) and \( p' \) are \( k \)-equivalent with respect to \( k \)-associativity if and only if,

\[ p(u_1 \circ u_2 \cdots u_n)_t = p'(u_1 \circ u_2 \cdots u_n)_{t'}. \]

**Proof.** Suppose the parenthesizations \( p \) and \( p' \) are \( k \)-equivalent. Then \( t \) and \( t' \) are \( k \)-equivalent also. By Theorem 3.16,

\[ \sum_{i=1}^{m-1} (m-i) \delta^i(t) \equiv \sum_{i=1}^{m-1} (m-i) \delta^i(t') \mod k(m-1). \]

Therefore,

\[ p(u_1 \circ u_2 \cdots u_n)_t = p'(u_1 \circ u_2 \cdots u_n)_{t'} \]

by Lemma 4.3.

Suppose that

\[ p(u_1 \circ u_2 \cdots u_n)_t = p'(u_1 \circ u_2 \cdots u_n)_{t'}, \]

then

\[ \sum_{i=1}^{m} (m-i) \delta_{1}^i (t) \cdot u_1 + \sum_{i=1}^{m} (m-i) \delta_{2}^i (t) \cdot u_2 + \cdots + \omega^{m-i} \sum_{i=1}^{m} (m-i) \delta_{m}^i (t) \cdot u_n. \]
the up-step (1 \{ g \geq k \) corresponds to k unique minimal element. Therefore to count the number of
Recall that we define the (5 Modular Fuss-Catalan Number
classes of parenthesizations of x. Since k,n
The depth of x is
Hence, 2-equivalent to
Example 4.5. In example 2.9 we saw that the 3-ary parenthesization ((x1x2x3)x4x5)x6x7 is
2-equivalent to x1((x2x3x4)x5)x6x7. Let us check the above theorem for this example.
The depth of the first tree is
Therefore the valuation of ((x1x2x3)x4x5)x6x7 with respect to o is
\[ \omega^6x_1 + \omega^5x_2 + \omega^4x_3 + \omega^3x_4 + \omega^2x_5 + \omega x_6 + x_7. \]
The depth of x1((x2x3x4)x5)x6x7 is
\[ (\delta^1 = (1, 2, 1, 1, 0, 0, 0), \delta^2 = (0, 1, 2, 1, 2, 1, 0), \delta^3 = (0, 0, 0, 1, 0, 1, 1)), \]
hence the valuation of x1((x2x3x4)x5)x6x7 with respect to o is,
\[ \omega^2x_1 + \omega^5x_2 + \omega^4x_3 + \omega^3x_4 + \omega^2x_5 + \omega x_6 + x_7. \]
Since \( \omega \) has order 4 the valuations are equal.

5 Modular Fuss-Catalan Number
Recall that we define the (k)-modular Fuss-Catalan number C^{mn}_{k,n} to be the number of k-equivalence
classes of parenthesizations of x0 \ast x1 \ast \cdots \ast x_n. In the previous sections we saw that k-associativity
corresponds to k-rotation and k-compression. Therefore C^{mn}_{k,n} also counts the k-equivalence classes
of (m - 1)-Dyck paths of length n. In this section we follow the strategy of [5, §5] to derive a
closed formula for C^{mn}_{k,n}, see Theorem 1.2. By Proposition 3.14, each k-equivalence class has a
unique minimal element. Therefore to count the number of k-equivalence classes, we just need
to count the number of minimal elements. For the rest of this section, we fix integers m \geq 2,
g \geq 0, k \geq 1 and n = m + g(m - 1).
Assume that l is a positive integer in \{1, 2, \ldots, n\} such that (m - 1) divides l. Let N denote
the up-step (1,1) and S denote the down-step (1,-1) in Z^2. Denote by \( C^{mn}_{k,n,l} \) the set of all
strings (lattice paths) of the form N^i S N^{i_1} S U^{i_2} S \cdots S N^{i_n} such that
\[ i_1 + i_2 + \cdots + i_n = n - l \] where
\[ (m - 1) | i_p \text{ for all } 1 \leq p \leq n \text{ and } 0 \leq i_1, \ldots, i_n < k(m - 1). \]
So C^{mn}_{k,n,l} is a set of lattice paths of
length \( n \) where the first up-step is of size \( l \). For integers \( 1 \leq j \leq k \), denote by \( m_j \) the number of \((j-1)(m-1)s\) appearing among the \( i_1, i_2, \ldots, i_n \in \{0, m-1, 2(m-1), \ldots, (k-1)(m-1)\} \).

It is easy to see that

\[
|C_{m,k,n,l}'| = \sum_{m_1 + \ldots + m_k = n} \binom{n}{m_1, m_2, \ldots, m_k}.
\]

For a string \( w = N^l S N^{i_1} S U^{i_2} \ldots S N^{i_m} \) in \( C_{m,k,n,l}' \) and \( j \) in \( \{0, 1, \ldots, n-1\} \) we define

\[
w_{w,j} := N^l S N^{i_1} S U^{i_2} \ldots S N^{i_j} S U^{i_{j+1}} S \ldots S U^{i_n}.
\]

Let \( C_{m,k,n} \) be the subset of strings in \( C_{m,k,n,l}' \) which are \((m-1)-Dyck\) paths of length \( n \). Since \((m-1)-Dyck\) paths can be thought of as \(1-Dyck\) paths where up-steps come in multiples of \( m-1 \), the following lemmas follow by similar arguments to Lemma 5.5 and Lemma 5.6 from [5], which may be thought of as the \( m=2 \) case. So we will state the lemmas without proof.

**Lemma 5.1.** For a string \( w \) in \( C_{m,k,n,l}' \) the set \( \{0 \leq j \leq n-1 : w_{w,j} \in C_{m,k,n,l}\} \) has cardinality \( l \).

**Lemma 5.2.** For a string \( w \) in \( C_{m,k,n,l}' \), the fibre \( \phi^{-1}(w) \) of \( \phi \) over \( w \) has cardinality \( |\phi^{-1}(w)| = l \).

By Proposition 3.14, the \((k)\)-modular Fuss-Catalan number counts the number of minimal \((m-1)\)-Dyck paths. Moreover by Proposition 3.13, minimal Dyck paths \( D \) satisfy \( d(D) = (d_1, d_2, \ldots, d_n) \) where \( d_i < k(m-1) \) for \( i \neq 1 \). Combining the results of Proposition 3.13, Proposition 3.14 and Lemma 5.2 we have that

\[
|C_{m,k,n,l}'| = \frac{l}{n} |C_{m,k,n,l}|.
\]

Let \( C_{m,k,n} \) be the set of minimal \((m-1)\)-Dyck paths, then we have that

\[
|C_{m,k,n}| = \sum_{1 \leq l \leq n} \binom{n}{m_1, \ldots, m_k}.
\]

Therefore,

\[
C_{m,k,n} = |C_{m,k,n}| = \frac{l}{n} \sum_{1 \leq l \leq n} \binom{n}{m_1, \ldots, m_k}
\]

is the number of minimal \((m-1)\)-Dyck paths of length \( n \), so by Proposition 3.6, it is the number of minimal \( m \)-ary trees of length \( n+1 \). This completes the proof of Theorem 1.2.

**Example 5.3.** In this example, we will count the number of 2-equivalence classes of 3-ary trees with 7 leaves. There are twelve 3-ary trees altogether. See the figure below for the complete list.
Figure 5: The complete list of the 3-ary trees with 7 leaves.

Observe that the trees $T_1$, $T_2$ and $T_3$ in the top row correspond to the following parenthesizations

$$((x_1 \cdot x_2 \cdot x_3) \cdot x_4 \cdot x_5) \cdot x_6 \cdot x_7,$$

$$x_1 \cdot ((x_2 \cdot x_3 \cdot x_4) \cdot x_5 \cdot x_6) \cdot x_7,$$

$$x_1 \cdot x_2 \cdot ((x_3 \cdot x_4 \cdot x_5) \cdot x_6 \cdot x_7)$$

respectively. We can see that we get the tree $T_2$ from the tree $T_1$ by a 2-rotation at the root of $T_1$, and likewise we get the tree $T_3$ from the tree $T_2$ by a 2-rotation at the root of $T_2$. Therefore
$T_1, T_2$ and $T_3$ belong to the same 2-equivalence class. Further observe that the other trees cannot
be 2-rotated because they don’t contain a subtree of form required to perform a 2-rotation. So
we conclude that $C_{2,6}^4 = 10$. Let us check this against the closed formula of Theorem 1.2.

\[
C_{2,6}^4 = \sum_{1 \leq l \leq 6} \frac{l}{6} \sum_{m_1 + m_2 = \frac{l}{2}} \left( \binom{6}{m_1, m_2} \right)
= \frac{2}{6} \binom{6}{4, 2} + \frac{4}{6} \binom{6}{5, 1} + \frac{6}{6} \binom{6}{6, 0}
= \frac{1}{3} (15) + \frac{2}{3} (6) + 1 (1)
= 10.
\]

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