Abstract
In this article, we prove the Eyring–Kramers formula for non-reversible metastable diffusion processes that have a Gibbs invariant measure. Our result indicates that non-reversible processes exhibit faster metastable transitions between neighborhoods of local minima, compared to the reversible process considered in Bovier et al. (J Eur Math Soc 6:399–424, 2004). Therefore, by adding non-reversibility to the model, we can indeed accelerate the metastable transition. Our proof is based on the potential theoretic approach to metastability through accurate estimation of the capacity between metastable valleys. We carry out this estimation by developing a novel method to compute the sharp asymptotics of the capacity without relying on variational principles such as the Dirichlet principle or the Thomson principle.

Mathematics Subject Classification 82C44 · 60K35

1 Introduction

In the study of the metastability of stochastic dynamical systems, one of the most important models is the overdamped Langevin dynamics given by a stochastic differential equation (SDE) of the form

\[
    d y_\epsilon(t) = -\nabla U(y_\epsilon(t)) \, dt + \sqrt{2\epsilon} \, d w_t,
\]

(1.1)
where \((w_t)_{t \geq 0}\) represents the standard \(d\)-dimensional Brownian motion, \(\epsilon > 0\) is a small constant parameter corresponding to the magnitude of the noise, and \(U : \mathbb{R}^d \to \mathbb{R}\) is a smooth Morse function\(^1\) with finite critical points. In addition to its importance in large-deviation theory, mathematical physics, and engineering (cf. [9] and references therein), this process is also well-known for approximating the minibatch gradient descent algorithm widely used in deep learning (cf. [15] and references therein).

The analysis of the metastability of this model has attracted considerable attention in recent decades. Its first successful mathematical treatment was carried out in a sequence of pioneering studies by Freidlin and Wentzell in the 1960s from a large-deviation theoretical perspective, and these achievements have been summarized in [9]. Subsequently, the next breakthrough was achieved in [5] from a potential theoretical perspective. In particular, the so-called Eyring–Kramers formula for (1.1) was established as a refinement of the large-deviation result obtained in [9].

Recently, several alternative approaches have been developed in the study of the metastable behavior of the process \(y_\epsilon(\cdot)\). We refer to [34] written by an author of the current article and Rezakhanlou for the Poisson equation approach, and [13] for the quasi-stationary distribution approach.

**Metastable behavior of overdamped Langevin dynamics**

To heuristically explain the metastable behavior of the process, we first consider the overdamped Langevin dynamics \(y_\epsilon(\cdot)\). We regard this process as a small random perturbation of the dynamical system given by an ordinary differential equation (ODE) of the form

\[
dy(t) = -\nabla U(y(t)) \, dt .
\] (1.2)

Note that the stable equilibria of this dynamical system are given by the local minima of \(U\). Hence, provided that \(\epsilon \approx 0\), the process \(y_\epsilon(\cdot)\) starting from a neighborhood of a local minimum of \(U\) will remain there for a sufficiently long time, as the noise is small compared to the drift term that pushes the process toward the local minimum.

The metastability issue arises for the process \(y_\epsilon(\cdot)\) if \(U\) has multiple local minima. To illustrate the corresponding metastable behavior more clearly, we simply assume that \(U\) has two local minima \(m_1\) and \(m_2\) as shown in Fig. 1, and we suppose that the process \(y_\epsilon(\cdot)\) starts at \(m_1\). If there is no noise, i.e., \(\epsilon = 0\), the process always remains at \(m_1\). However, when \(\epsilon\) is small but positive, random noise accumulates over a sufficiently long time and enables the process \(y_\epsilon(\cdot)\) to make a transition to a neighborhood of another minimum \(m_2\), where it then remains for a long time before making another transition. Such rare transitions between the neighborhoods of local minima constitute the dynamical metastable behavior of the process \(y_\epsilon(\cdot)\). We can expect richer behaviors when \(U\) has a more complex landscape.

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\(^1\) All the critical points of \(U\) are non-degenerate (i.e., the Hessian at each critical point is invertible) and isolated from others.
Eyring–Kramers formula

The Eyring–Kramers formula is the sharp asymptotics, as \( \epsilon \to 0 \), of the mean of the time required to observe the transition described above. It was obtained for the one-dimensional case in classical studies \([8,18]\) conducted in the 1930s on the basis of explicit computation. The generalization of this result to arbitrary dimensions was finally accomplished in \([5]\) a few decades later. We recall the double-well situation illustrated in Fig. 1 to explain the Eyring–Kramers formula in a simple form. Let \( \tau_{D_\epsilon(m_2)} \) denote the hitting time with respect to the process \( y_\epsilon(\cdot) \) of the set \( D_\epsilon(m_2) \), which is a ball of radius \( \epsilon \) centered at \( m_2 \). Then, the Eyring–Kramers formula is the sharp estimate of the mean transition time \( \mathbb{E}[\tau_{D_\epsilon(m_2)} | y_\epsilon(0) = m_1] \). The Freidlin–Wentzell theory gives the large deviation estimate for this quantity as

\[
\lim_{\epsilon \to 0} \epsilon \log \mathbb{E}[\tau_{D_\epsilon(m_2)} | y_\epsilon(0) = m_1] = U(\sigma) - U(m_1),
\]

where \( \sigma \) is the saddle point between the two wells as shown in Fig. 1. The Eyring–Kramers formula is a refinement of this result (cf. Corollary 3.7 of the current article), and it gives the precise asymptotics of the expectation in (1.3).

The mean transition time is related to the quantification of the mixing property of the process \( y_\epsilon(\cdot) \). To explain it more precisely, we remark that the unique invariant measure for the process \( y_\epsilon(\cdot) \) is given by

\[
\mu_\epsilon(dx) = \frac{1}{Z_\epsilon} e^{-U(x)/\epsilon} dx,
\]

where \( Z_\epsilon \) is the constant given by

\[
Z_\epsilon = \int_{\mathbb{R}^d} e^{-U(x)/\epsilon} dx < \infty,
\]

where we will impose suitable growth conditions for \( U \) in Sect. 2 to guarantee the finiteness of \( Z_\epsilon \). The measure \( \mu_\epsilon(\cdot) \) corresponds to the Gibbs measure associated to the energy function \( U \) and inverse temperature \( \epsilon \) and hence the constant \( Z_\epsilon \) denotes the associated partition function. Therefore, we can regard the process \( y_\epsilon(\cdot) \) as a sampler of the Gibbs distribution \( \mu_\epsilon(\cdot) \), which is exponentially concentrated on the global minima of \( U \). There are two representative quantities for measuring this mixing
property of the sampler $y_\epsilon(\cdot)$: the spectral gap [6] and the mean transition time of the process from one local minimum to another [5]. Thus, by estimating the latter using the Eyring–Kramers formula, one can precisely measure the mixing property of $y_\epsilon(\cdot)$.

**Main contribution of this article**

In this article, we consider a variant of the classical overdamped Langevin dynamics $y_\epsilon(\cdot)$, which is obtained by adding a vector field to the drift term of the SDE (1.1). More precisely, we focus on the Eyring–Kramers formula for the diffusion process given by an SDE of the form

$$dx_\epsilon(t) = -(\nabla U + \ell)(x_\epsilon(t)) \, dt + \sqrt{2\epsilon} \, dw_t,$$  

(1.6)

where $U$ is the smooth potential function as described above. Further, $\ell : \mathbb{R}^d \to \mathbb{R}^d$ is a vector field that is orthogonal to the gradient field $\nabla U$, i.e.,

$$\nabla U(x) \cdot \ell(x) = 0 \quad \text{for all } x \in \mathbb{R}^d,$$  

(1.7)

and it is incompressible:

$$(\nabla \cdot \ell)(x) = 0 \quad \text{for all } x \in \mathbb{R}^d.$$  

(1.8)

The condition (1.7) guarantees that the quasi-potential of the process $x_\epsilon(\cdot)$ is $U$ (cf. [9, Theorem 3.3.1]), and the condition (1.8) ensures that the invariant measure of the process $x_\epsilon(\cdot)$ is the Gibbs measure $\mu_\epsilon(\cdot)$ (cf. Theorem 2.3). In this sense, the process $x_\epsilon(\cdot)$ is another sampler of the Gibbs distribution $\mu_\epsilon(\cdot)$. Indeed, we prove in Theorem 2.3 that the conditions (1.7) and (1.8) are the necessary and sufficient conditions for the process $x_\epsilon(\cdot)$ to have as an invariant measure the Gibbs distribution $\mu_\epsilon(\cdot)$ defined in (1.4) for all $\epsilon > 0$. For this reason, this generalized model has been investigated in many studies from different perspectives, e.g., [7,16,17,27,28,32,33].

The main contribution of the current article is the proof of the Eyring–Kramers formula for the process $x_\epsilon(\cdot)$ (Theorem 3.5). We verify in Theorem 2.1 that the stable points of the process $x_\epsilon(\cdot)$ are the local minima of $U$ and hence identical to those of the process $y_\epsilon(\cdot)$. Hence, we can compare the Eyring–Kramers formula of $x_\epsilon(\cdot)$ with that of $y_\epsilon(\cdot)$, and this comparison reveals that the mean transition time of the dynamics $x_\epsilon(\cdot)$ from one local minimum of $U$ to another is always faster than that of the overdamped Langevin dynamics $y_\epsilon(\cdot)$. This implies that we can accelerate the stochastic gradient descent algorithm by adding the incompressible field $\ell$, which is orthogonal to $\nabla U$. We remark that such an acceleration has been observed for the model when the diffusivity $\epsilon$ is kept constant (see [7,16,17,27,32,33] and references therein). In particular, we refer to [11] for the explicit relation with the stochastic gradient descent algorithm.

We also remark that in a recent study [28], the model considered in this article was investigated in view of the low-lying spectra. Sharp estimates were established for the exponentially small eigenvalues of the generator associated with the process $x_\epsilon(\cdot)$.
See Corollary 3.8 to understand how our discovery is related to the result presented in [28].

**General methodology of capacity estimation**

Another main result of our study is the establishment of a straightforward and robust method for estimating a potential theoretic notion known as the capacity. In the proof of Eyring–Kramers formula based on the potential theoretic approach developed in [5], it is crucial to estimate the capacity between metastable valleys. In all the existing results based on this approach, such an estimation is carried out via variational principles such as the Dirichlet principle or the Thomson principle.

For the reversible case, this approach is less complex as the Dirichlet principle is an optimization problem over a space of functions. Hence, by taking a suitable test function that approximates the known optimizer of the variational principle, we can bound the capacity in a precise manner. This strategy is the essence of the potential theoretic approach to metastability. By contrast, for the non-reversible case, the variational expression of the capacity is destined to involve both the function and the so-called flow (cf. [21, Theorems 3.2 and 3.3]). Therefore, one must construct both the test function and the test flow to estimate the capacity precisely. Accordingly, when this approach is adopted for the non-reversible model, the major technical difficulty arises in the construction of the test flow. This problem has been resolved in existing studies such as [19,21,35] based on considerable computations.

In this article, we develop a robust methodology to estimate the capacity without relying on these variational principles. We use only a test function in the estimation of the capacity; *no test flow is used even in the non-reversible case*. Hence, our methodology significantly reduces the complexity of the analysis of metastable non-reversible processes to the level of the reversible models. Therefore, our methodology is expected to present new possibilities for the analysis of non-reversible metastable random processes.

In summary, we develop a new methodology to estimate the capacity and use it to establish the Eyring–Kramers formula for the non-reversible and metastable diffusions $x_\varepsilon(\cdot)$.

**Related question 1: Markov chain description of metastable behavior**

Now, we consider two important questions. The first deals with a more comprehensive description of the metastable behavior of the process $x_\varepsilon(\cdot)$. To view this problem in a concrete form, suppose that $U(m_1) = U(m_2)$ in the double-well situation illustrated in Fig. 1. The Eyring–Kramers formula focuses on a single metastable transition. However, this transition will occur repeatedly between the neighborhoods of two metastable points $m_1$ and $m_2$, and one might be interested in describing these repeated transitions simultaneously. To this end, we can try to prove that a suitably time-rescaled process converges in some sense to a Markov chain whose state space consists of two valleys. By doing so, we can completely describe successive metastable transitions as this Markov chain. We consider this problem for the process $x_\varepsilon(\cdot)$ in our companion paper [26].
Related question 2: metastable behavior of the general model

For a vector field \( b : \mathbb{R}^d \to \mathbb{R}^d \), consider the dynamical system in \( \mathbb{R}^d \) given by an ODE of the form
\[
 dz(t) = -b(z(t)) \, dt \quad ; \; t \geq 0 .
\] (1.9)

Suppose that this dynamics has several stable equilibria. An open problem in the study of metastability is to determine the Eyring–Kramers formula for the following small random perturbation of (1.9):
\[
 dz_\epsilon(t) = -b(z_\epsilon(t)) \, dt + \sqrt{2\epsilon} \, d w_t \quad ; \; t \geq 0 .
\] (1.10)

We refer to [4,9,24] for the study of various aspects of this question. There are two sources of difficulties in this open problem. The first one is the non-reversibility of the dynamics, and the second one is the fact that the invariant measure cannot be written in an explicit form in general. In the present article, we make a significant step toward addressing this problem by completely overcoming the former difficulty. However, since we considered only models with a Gibbs invariant measure, the latter difficulty is not addressed and remains to be resolved.

2 Model

In this section, we introduce the fundamental features of the model. The results stated in this section regarding the process \( x_\epsilon(\cdot) \) constitute the essence of this field. However, we could not find a suitable reference that provides detailed proofs. Hence, we decided to develop the full details.

Potential function \( U \)

To introduce the model rigorously, we must explain the potential function \( U : \mathbb{R}^d \to \mathbb{R} \) in the SDE (1.6). We assume that the potential function \( U \in C^3(\mathbb{R}^d) \) is a Morse function that satisfies the growth conditions
\[
 \lim_{n \to \infty} \inf_{|x| \geq n} \frac{U(x)}{|x|} = \infty , \quad \lim_{|x| \to \infty} \frac{x}{|x|} \cdot \nabla U(x) = \infty , \quad \text{and} \quad \lim_{|x| \to \infty} \left( |\nabla U(x)| - 2\Delta U(x) \right) = \infty ,
\] (2.1)(2.2)(2.3)

where \( |x| \) denotes the Euclidean distance in \( \mathbb{R}^d \). These conditions have been introduced in previous studies such as [5,19,34] to guarantee the positive recurrence of the diffusion process \( y_\epsilon(\cdot) \) given by (1.1) and the finiteness of \( Z_\epsilon \) in (1.5). More precisely,
it is well known (cf. [5]) that these conditions imply the tightness condition

\[ \int_{\{x: U(x) \geq a\}} e^{-U(x)/\epsilon} dx \leq C_a e^{-a/\epsilon} \text{ for all } a \in \mathbb{R}, \tag{2.4} \]

where \( C_a \) is a constant that depends only on \( a \), and hence imply the finiteness of the partition function \( Z_\epsilon \). Finally, we remark that the metastability of the reversible process \( y_\epsilon(\cdot) \) has been analyzed in [5] under the same set of assumptions.

**Deterministic dynamical system \( x(\cdot) \)**

To explain the metastable behavior of the process \( x_\epsilon(\cdot) \), we first consider a deterministic dynamical system given by the ODE

\[ dx(t) = -(\nabla U + \ell)(x(t)) \, dt. \tag{2.5} \]

We can demonstrate that this dynamical system has essentially the same phase portrait as \( y(\cdot) \) defined in (1.2).

**Theorem 2.1** The following hold.

1. We have \( \ell(c) = 0 \) for all critical points \( c \in \mathbb{R}^d \) of \( U \).
2. A point \( c \in \mathbb{R}^d \) is an equilibrium of the dynamical system (2.5) if and only if \( c \in \mathbb{R}^d \) is a critical point of \( U \).
3. An equilibrium \( c \in \mathbb{R}^d \) of the dynamical system (2.5) is stable if and only if \( c \) is a local minimum of \( U \).

The proof is given in Sect. 4. We emphasize that the divergence-free condition (1.8) is not used in the proof of this theorem, whereas the orthogonality condition (1.7) plays a significant role. In view of part (3) of the previous theorem, we can observe that the process \( x_\epsilon(\cdot) \) is expected to exhibit metastable behavior when \( U \) has multiple local minima, and this is the situation that we are going to discuss in the current article.

**Diffusion process \( x_\epsilon(\cdot) \)**

Now, we focus on the diffusion process \( x_\epsilon(\cdot) \). Under the conditions (2.1)–(2.3) and condition (1.7), we can prove the following property of the process \( x_\epsilon(\cdot) \). Note again that the condition (1.8) is not used.

**Theorem 2.2** The following hold.

1. There is no explosion for the diffusion process \( x_\epsilon(\cdot) \).
2. The diffusion process \( x_\epsilon(\cdot) \) is positive recurrent.

The proof of this result is given in Sect. 5.
Invariant measure

Since the process \( x_\epsilon (\cdot) \) is positive recurrent, we know that this process has an invariant measure. Now, we prove that \( \mu_\epsilon \) is the unique invariant measure for the process \( x_\epsilon (\cdot) \).

Before proceeding to the statement of this result, we first explain the role of the conditions (1.7) and (1.8). Recall the general model \( z_\epsilon (\cdot) \) given by the SDE (1.10). It is known from [9, Theorem 3.3.1] that if the quasi-potential \( V \) associated with (1.10) is of class \( C^1 \), we can write \( b = \nabla V + \ell \) where \( \nabla V \cdot \ell = 0 \). Hence, the assumption (1.7) is nothing more than the regularity assumption on the quasi-potential. The special assumption regarding the field \( \ell \) is (1.8), and the role of this assumption is summarized below.

**Theorem 2.3** The following hold.

(1) If \( \ell \) satisfies the conditions (1.7) and (1.8), then the Gibbs measure \( \mu_\epsilon (\cdot) \) is the unique invariant measure for the diffusion process \( x_\epsilon (\cdot) \).

(2) On the other hand, suppose that the Gibbs measure \( \mu_\epsilon (\cdot) \) is the invariant measure for the diffusion process \( z_\epsilon (\cdot) \) defined in (1.10) for all \( \epsilon > 0 \). Then, the vector field \( b \) can be written as \( b = \nabla U + \ell \), where \( U \) and \( \ell \) satisfy (1.7) and (1.8).

The proof of this theorem is given in Sect. 5. Therefore, heuristically, the condition (1.8) can be regarded as a necessary and sufficient condition (up to the regularity of the quasi-potential) for the diffusion process \( z_\epsilon (\cdot) \) has the Gibbs invariant measure.

**Construction of \( \ell \)**

The result obtained in this article might be nearly useless if it is extremely difficult to find a non-trivial \( \ell \) satisfying the conditions (1.7) and (1.8) simultaneously. However, there is a simple way to generate a variety of \( \ell \)'s when the potential \( U \) is given. Let \( \mathcal{M}_{d \times d}(\mathbb{R}) \) be a space of \( d \times d \) real matrices and let \( J : \mathbb{R} \rightarrow \mathcal{M}_{d \times d}(\mathbb{R}) \) be a smooth function such that the range of \( J \) consists of only skew-symmetric matrices. Then, a vector field of the form \( \ell(x) = J(U(x)) \nabla U(x) \) satisfies the conditions (1.7) and (1.8). This has been observed in [28, Sect. 1]. Moreover, unless \( J \) is a constant function, the model considered here is different from the one considered in [19].

**Notations regarding \( x_\epsilon (\cdot) \)**

We conclude this section by defining some notations regarding the process \( x_\epsilon (\cdot) \). Let \( \mathcal{L}_\epsilon \) denote the generator associated with the process \( x_\epsilon (\cdot) \). Then, \( \mathcal{L}_\epsilon \) acts on \( f \in C^2(\mathbb{R}^d) \) such that

\[
\mathcal{L}_\epsilon f = - (\nabla U + \ell) \cdot \nabla f + \epsilon \Delta f .
\]  

(2.6)

Under the conditions (1.7) and (1.8) on \( \ell \), we can rewrite this generator in the divergence form as

\[
\mathcal{L}_\epsilon f = \epsilon e^{U/\epsilon} \nabla \cdot \left[ e^{-U/\epsilon} \left( \nabla f - \frac{1}{\epsilon} f \ell \right) \right] .
\]  

(2.7)
Let $P^x_\epsilon$ denote the law of the process $x_\epsilon(\cdot)$ starting from $x$, and let $E^x_\epsilon$ denote the expectation with respect to $P^x_\epsilon$.

3 Main result

In this section, we explain the Eyring–Kramers formula for the diffusion process $x_\epsilon(\cdot)$. The main result is stated in Theorem 3.5 (and Corollary 3.7 for the simple double-well case).

3.1 Structure of metastable valleys

Let $\mathcal{M}$ denote the set of local minima of $U$. The starting point $m_0 \in \mathcal{M}$ of the process $x_\epsilon(\cdot)$ is fixed throughout the article. Note that $m_0$ is a stable equilibrium of $x(\cdot)$ by Theorem 2.1.

Let us fix $H \in \mathbb{R}$ such that $U(m_0) < H$ and define $\Sigma$ as the set of saddle points of level $H$:

$$\Sigma = \{ \sigma : U(\sigma) = H \text{ and } \sigma \text{ is a saddle point of } U \}.$$  

We take $H$ such that $\Sigma \neq \emptyset$. We define

$$\mathcal{H} = \{ x \in \mathbb{R}^d : U(x) < H \},$$

and we assume that $\mathcal{H}$ has multiple connected components; hence, metastability occurs.

We decompose $\mathcal{H} = \mathcal{H}_0 \cup \mathcal{H}_1$, where $\mathcal{H}_0$ is the connected component of $\mathcal{H}$ containing $m_0$ and $\mathcal{H}_1 = \mathcal{H} \setminus \mathcal{H}_0$. Note that $\mathcal{H}_1$ may not be connected. Let $\mathcal{M}_0$ and $\mathcal{M}_1$ denote the sets of local minima belonging to $\mathcal{H}_0$ and $\mathcal{H}_1$, respectively. Let $D_r(x)$ denote an open ball in $\mathbb{R}^d$ centered at $x$ with radius $r$, and define

$$\mathcal{U}_\epsilon := \bigcup_{m \in \mathcal{M}_1} D_\epsilon(m).$$

In this article, we focus on the sharp asymptotics of the mean of the transition time from $m_0$ to $\mathcal{U}_\epsilon$. Figure 2 illustrates the notations introduced above.

Notation 3.1 Since the sets such as $\Sigma$ and $\mathcal{U}_\epsilon$ depend on $H$, we add the superscript $H$ to these notations, e.g., $\Sigma^H$, when we want to emphasize the dependency on $H$.

3.2 Eyring–Kramers constant for $x_\epsilon(\cdot)$

In the remainder of the article, we use the following notations.

Notation 3.2 For each critical point $c$ of $U$, let $\mathbb{H}_c = (\nabla^2 U)(c)$ denote the Hessian of $U$ at $c$ and let $\mathbb{L}_c = (D\ell)(c)$ denote the Jacobian of $\ell$ at $c$. 

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Fig. 2 Example of landscape of the potential function $U$ with five local minima \( \{m_i : 0 \leq i \leq 4\} \) and four saddle points \( \{\sigma_i : 0 \leq i \leq 3\} \). We assume that $U(m_3) = U(m_4)$ and write $H_1 = U(\sigma_1)$, $0 \leq i \leq 3$. Our objective is to compute the transition time from the local minimum $m_0$ to other local minima. We can select the level $H$ according to our detailed objective. By taking $H = H_1$, we have $M_1 = \{m_1, m_2\}$; hence, we focus on the transition time from $m_0$ to $D(\epsilon)(m_1) \cup D(\epsilon)(m_2)$. This occurs at the level of $H_1$ since the process must pass through $\sigma_1$ to make such a transition. For this case, we have $M_0 = \{m_0, m_3, m_4\}$ and $M_0^* = \{m_3, m_4\}$. On the other hand, by taking $H = H_2$, we have $M_1 = \{m_1, m_2, m_3, m_4\}$. For this case, we compute the escape time from the metastable valley around $m_0$. The selection $H = H_3$ is not available since the condition $U(m_0) < H$ is violated; hence $H$ does not contain $m_0$. This level is meaningful when we start from, e.g., $m_3$. Finally, the selection $H = H_0$ is not appropriate as $\Sigma_0$ becomes an empty set. For this case, we refer to Remark 3.6 (4) for further details.

In this subsection, we fix $\sigma \in \Sigma$ and suppose that $\mathbb{H}^\sigma$ has only one negative eigenvalue $-\lambda^\sigma$. In the Eyring–Kramers formula for the reversible process $y_\epsilon(\cdot)$ obtained in [5], an important constant is the so-called Eyring–Kramers constant defined by

$$\omega_{\text{rev}}^\sigma = \frac{\lambda^\sigma}{2\pi \sqrt{-\det \mathbb{H}^\sigma}}. \quad (3.2)$$

Now, we introduce the corresponding constant for the process $x_\epsilon(\cdot)$. To this end, we first introduce the following lemma.

Lemma 3.3 For $\sigma \in \Sigma$, suppose that $\mathbb{H}^\sigma$ has only one negative eigenvalue. Then, the matrix $\mathbb{H}^\sigma + \mathbb{L}^\sigma$ has only one negative eigenvalue and is invertible.

Although this has been verified already in [28, Lemma 1.8], we provide the proof of this Lemma in Sect. 4.3 for the completeness of the article. Let $-\mu^\sigma$ denote the unique negative eigenvalue obtained in this lemma and define the Eyring–Kramers constant at $\sigma$ by

$$\omega^\sigma = \frac{\mu^\sigma}{2\pi \sqrt{-\det \mathbb{H}^\sigma}}. \quad (3.3)$$

Then, we can prove the following comparison result for the Eyring–Kramers constant.

Lemma 3.4 We have $\mu^\sigma \geq \lambda^\sigma$; therefore, $\omega^\sigma \geq \omega_{\text{rev}}^\sigma$.

The proof is also given in Sect. 4.3. In Corollary 3.9, we prove that the process $x_\epsilon(\cdot)$ is faster than $y_\epsilon(\cdot)$ on the basis of this comparison result.
3.3 Eyring–Kramers formula for \( x_\epsilon (\cdot) \)

For \( A \subset \mathbb{R}^d \), let \( \overline{A} \) denote the closure of \( A \). Define

\[
\Sigma_0 = \overline{H_0} \cap \overline{H_1} \subset \Sigma. \tag{3.4}
\]

We assume that \( \Sigma_0 \neq \emptyset \). For each \( \sigma \in \Sigma_0 \), the Hessian \( H^\sigma \) has only one negative eigenvalue as a consequence of the Morse lemma (cf. [29, Lemma 2.2]); hence, the Eyring–Kramers constant \( \omega^\sigma \) at \( \sigma \in \Sigma_0 \) can be defined as in the previous subsection. Then, define

\[
\omega_0 = \sum_{\sigma \in \Sigma_0} \omega^\sigma. \tag{3.5}
\]

Let \( h_0 \) denote the minimum of \( U \) on \( H_0 \) and let \( M_0^\star \) denote the set of the deepest minima of \( U \) on \( H_0 \):

\[
M_0^\star = \{ m \in M_0 : U(m) = h_0 \}. \tag{3.6}
\]

Define

\[
v_0 = \sum_{m \in M_0^\star} \frac{1}{\sqrt{\det H^m}}. \tag{3.7}
\]

Now, we are ready to state the Eyring–Kramers formula for the non-reversible process \( x_\epsilon (\cdot) \), which is the main result of the current article. For a sequence \( (a_\epsilon)_{\epsilon > 0} \) of real numbers, we write \( a_\epsilon = o_\epsilon (1) \) if \( \lim_{\epsilon \to 0} a_\epsilon = 0 \).

**Theorem 3.5** We have

\[
\mathbb{P}^\epsilon_{m_0} [ \tau_{U_\epsilon} ] = [ 1 + o_\epsilon (1) ] \frac{v_0}{\omega_0} \exp \frac{H - h_0}{\epsilon}. \tag{3.8}
\]

**Remark 3.6** We state the following with regard to Theorem 3.5.

1. Heuristically, the process \( x_\epsilon (\cdot) \) starting at \( m_0 \) first mixes among the neighborhoods of minima of \( M_0^\star \), and then makes a transition to \( U_\epsilon \) by passing through a neighborhood of the saddle in \( \Sigma_0 \) according to the Freidlin-Wentzell theory. This is the reason that the formula (3.8) depends on the local properties of the potential \( U \) at \( M_0^\star \) and \( \Sigma_0 \). A remarkable fact regarding the formula (3.8) is that the sub-exponential prefactor is dominated only by these local properties. This is mainly because the invariant measure is the Gibbs measure \( \mu_\epsilon (\cdot) \). It is observed in [4] that an additional factor called “non-Gibbsianess” of the process should be introduced in the general case (i.e., in the analysis of the metastable behavior of the process \( z_\epsilon (\cdot) \)).

2. Theorem 3.5 is a generalization of [5, Theorem 3.2], as the reversible case is the special \( \ell = 0 \) case of our model. Moreover, a careful reading of our arguments reveals that the error term \( o_\epsilon (1) \) is indeed \( O(\epsilon^{1/2} \log \frac{1}{\epsilon}) \) which is the one that appeared in [5, Theorems 3.1 and 3.2].

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2 The case \( \Sigma_0 = \emptyset \) may occur, for instance, if we take \( H = H_0 \) in Fig. 2. We can deal with this situation using our result by modifying \( H \); see Remark 3.6(4).
The constants \( \omega_0, \nu_0, \) and \( h_0 \) and the set \( \mathcal{U}_e \) are not changed if we take a different starting point \( m_0' \in M_0 \). In view of Theorem 3.5, this implies that all the transition times from a point in \( M_0 \) to \( \mathcal{U}_e \) are asymptotically the same. For instance, if we take \( H = H_1 \) in Fig. 2, the expectation of the hitting time \( \tau_{\mathcal{D}_e(m_1) \cup \mathcal{D}_e(m_2)} \) is asymptotically the same for the starting points \( m_0, m_3 \) and \( m_4 \). This is because the process \( x_\epsilon(\cdot) \) sufficiently mixes in the valley \( \mathcal{H}_0 \) before moving to another valley.

Consider the case \( H = H_2 \), where the potential \( U \) is given as Fig. 2 so that we have \( \mathcal{U}_e^{H_2} = \{ m_1, m_2, m_3, m_4 \} \). However, in time scale \( \exp\left\{ \frac{H_2-h_0}{\epsilon} \right\} \), the diffusion process cannot move to the neighborhoods of \( m_1 \) and \( m_2 \), since \( \sigma_2 \) is the only saddle point in \( \Sigma_0^{H_2} \) and \( m_3 \) and \( m_4 \) are the only minima in the connected components of \( \mathcal{H}_1 \) whose boundary contains \( \sigma_2 \). Our proof verifies this as well.

We can tune \( H \) such that \( m_0 \) is the unique local minimum of \( \mathcal{H}_0 \). For example, in Fig. 2, we can achieve this by selecting \( H = H_2 \). Then, the formula (3.8) becomes the asymptotics of the transition time from \( m_0 \) to one of the other local minima, and this is the classic form of the Eyring–Kramers formula. We remark that all the existing studies \([5,19]\) on the Eyring–Kramers formula for metastable diffusion processes have dealt with only this situation. On the other hand, our result is more comprehensive in that we analyzed all the possible levels by carefully investigating the equilibrium potential in Sect. 9. Such a comprehensive result for a diffusion setting was barely known previously, see \([23]\) where a similar setting along with the possibility of degenerate critical points has been discussed.

By selecting \( \ell \) appropriately, we can make \( \omega_0 \) arbitrarily large.

The proof of Theorem 3.5 is given in Sect. 7.

Double-well case

The Eyring–Kramers formula stated above has a simple form in the double-well case. Recall the double-well situation illustrated in Fig. 1. For this case, the only meaningful selection of \( H \) is \( U(\sigma) \), and \( \Sigma_0 = \{ \sigma \} \) for this choice. With this \( H \), we can interpret Theorem 3.5 as following corollary.

**Corollary 3.7** We have

\[
\mathbb{E}^\epsilon_{m_1}[\tau_{\mathcal{D}_e(m_2)}] = [1 + o_\epsilon(1)] \frac{2\pi}{\mu^\sigma} \sqrt{-\det H^\sigma_{m_1}} \exp \frac{U(\sigma) - U(m_1)}{\epsilon}. \tag{3.9}
\]
This is the classical form of the Eyring–Kramers formula. With this simple case, we explain why this result is a refinement of the Freidlin–Wentzell theory. By [9, Theorem 3.3.1], the quasi-potential \( V(x; m_1) \) of the process \( x_\epsilon(\cdot) \) with respect to the local minimum \( m_1 \) is given by \( V(x; m_1) = U(x) - U(m_1) \) on the domain of attraction of \( m_1 \) with respect to the process \( x(\cdot) \). Hence, we can deduce the following large-deviation type result from the Freidlin–Wentzell theory:

\[
\lim_{\epsilon \to 0} \epsilon \log \mathbb{E}_{m_1}^\epsilon [ \tau_{\mathcal{D}_\epsilon(m_2)} ] = U(\sigma) - U(m_1).
\]

In the formula (3.9), we find the precise sub-exponential pre-factor associated with this large-deviation estimate.

We can also deduce from Corollary 3.7 a precise relation between the mean transition time and a low-lying spectrum of the generator \( \mathcal{L}_\epsilon \) for the double-well case. In [28], the sharp asymptotics for the eigenvalue \( \lambda_\epsilon \) of \( \mathcal{L}_\epsilon \) with the smallest real part was obtained. Note that the generator \( \mathcal{L}_\epsilon \) is not self-adjoint; hence, the eigenvalue might be a complex number.

**Corollary 3.8** For the double-well situation, we suppose that \( U(m_1) \geq U(m_2) \). Let \( \lambda_\epsilon \) denote the one with smallest real part among the non-zero eigenvalues of \( \mathcal{L}_\epsilon \). Then, the following holds:

\[
\mathbb{E}_{m_1}^\epsilon [ \tau_{\mathcal{D}_\epsilon(m_2)} ] = \frac{1 + o_\epsilon(1)}{\lambda_\epsilon}.
\]  

(3.10)

Note that \( \lambda_\epsilon \) as well as the error term \( o_\epsilon(1) \) in (3.10) is in general a non-real complex number. Surprisingly, it is verified in [28, Remark 1.10] that \( \lambda_\epsilon \) is indeed a real number if \( U \) is a double-well potential and \( \epsilon \) is sufficiently small. We remark that the inverse relationship between the low-lying spectrum and the mean transition time as in (3.10) has been rigorously verified in [5,6] for a wide class of reversible models including \( y_\epsilon(\cdot) \).

**Comparison with reversible case**

The Eyring–Kramers formula for the reversible process \( y_\epsilon(\cdot) \) has been shown in [5, Theorem 3.2]. We can also recover\(^3\) this result by inserting \( \ell = 0 \). We now explain this result using our terminology and we provide a comparison between reversible and non-reversible cases. Write

\[
\omega_{0, \text{rev}} = \sum_{\sigma \in \Sigma_0} \omega_{\sigma, \text{rev}},
\]

and let \( \mathbb{E}^\epsilon_{x, \text{rev}} \) denote the expectation with respect to the reversible process \( y_\epsilon(\cdot) \) starting from \( x \in \mathbb{R}^d \). Then, as a consequence of Theorem 3.5 with \( \ell = 0 \), we get the following corollary.

\(^3\) Indeed, our result with \( \ell = 0 \) strictly contains what has been established in [5]. See Remark 3.6-(3).
Corollary 3.9 The following holds:

\[
E^\epsilon_{m_0, \text{rev}}[\tau U] = [1 + o_\epsilon(1)] \frac{v_0}{\omega_{0, \text{rev}}} \exp \frac{H - h_0}{\epsilon}.
\]

Therefore, we have \(E^\epsilon_{m_0, \text{rev}}[\tau U] \leq E^\epsilon_{m_0, \text{rev}}[\tau U]\) for all small enough \(\epsilon\).

**Proof** The first assertion follows immediately from the fact that \(\omega^\sigma_{\text{rev}}\) defined in (3.2) corresponds to \(\omega^\sigma\) with \(\ell = 0\). The second assertion follows from Lemma 3.4 which implies that \(\omega_0 \geq \omega_{0, \text{rev}}\).

\(\square\)

In view of the fact that the dynamics \(y_\epsilon(\cdot)\) plays a crucial role in the stochastic gradient descent algorithm, we might be able to accelerate this algorithm by adding a suitable orthogonal, incompressible vector field to the drift part.

### 4 Dynamical system \(x(\cdot)\)

In this section, we prove the properties of the dynamical systems \(x(\cdot)\) given by the ODE (2.5).

#### 4.1 Preliminary results on matrix computations

In this section, we present few technical lemmas. We remark that all the vectors and matrices in this subsection are real. The first lemma below will be used to investigate the stable equilibria of the dynamical system \(x(\cdot)\).

**Lemma 4.1** Let \(A, B\) be square matrices of the same size and suppose that \(A\) is symmetric positive definite and \(AB\) is skew-symmetric. Then, all the eigenvalues of matrix \(A + B\) are either positive real or complex with a positive real part. In particular, the matrix \(A + B\) is invertible.

**Proof** By a change of basis, we may assume that \(A = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_d)\) for some \(\lambda_1, \ldots, \lambda_d > 0\). Let \(\alpha\) be a real eigenvalue of \(A + B\) and let \(u\) be the corresponding non-zero eigenvector. Then, we have

\[
0 < |Au|^2 = Au \cdot (A + B)u = \alpha(Au \cdot u),
\]

where the first identity holds since \(AB\) is skew-symmetric. This proves that \(\alpha > 0\) since \(A\) is positive definite.

Next, let \(z = a + ib\) be a complex eigenvalue of \(A + B\) and let \(u + iw\) be the corresponding non-zero eigenvector, where \(u\) and \(w\) are real vectors. Since \(A\) and \(B\) are real, we have

\[
(A + B)u = au - bw \quad \text{and} \quad (A + B)w = bu + aw.
\]
Since $\mathbb{A}\mathbb{B}$ is skew-symmetric, we get

$$ |\mathbb{A}\mathbf{u}|^2 = \mathbb{A}\mathbf{u} \cdot (\mathbb{A} + \mathbb{B})\mathbf{u} = \mathbb{A}\mathbf{u} \cdot (a\mathbf{u} - b\mathbf{w}), $$

$$ |\mathbb{A}\mathbf{w}|^2 = \mathbb{A}\mathbf{w} \cdot (\mathbb{A} + \mathbb{B})\mathbf{w} = \mathbb{A}\mathbf{w} \cdot (b\mathbf{u} + a\mathbf{w}). $$

By adding these two identities, we get

$$ |\mathbb{A}\mathbf{u}|^2 + |\mathbb{A}\mathbf{w}|^2 = a(\mathbb{A}\mathbf{u} \cdot \mathbf{u} + \mathbb{A}\mathbf{w} \cdot \mathbf{w}). $$

Therefore, we get $a > 0$ since $\mathbb{A}$ is positive definite.

The next lemma is used to analyze the saddle points of the dynamical system (2.5). For a square matrix $\mathbb{M}$, let $\mathbb{M}^\dagger$ denote its transpose, and we write $\mathbb{M}^s = \frac{1}{2}(\mathbb{M} + \mathbb{M}^\dagger)$.

**Lemma 4.2** Let $\mathbb{A}$, $\mathbb{B}$ be square matrices of the same size and suppose that $\mathbb{A}^s$ is positive definite and $\mathbb{B}$ is a non-singular, symmetric matrix that has only one negative eigenvalue. Then, $\mathbb{A}\mathbb{B}$ is invertible and has only one negative eigenvalue with geometric multiplicity 1.

**Proof** By a change of basis, we may assume that $\mathbb{B} = \text{diag}(-\lambda_1, \lambda_2, \ldots, \lambda_d)$ for some $\lambda_1, \ldots, \lambda_d > 0$. It is well known that a matrix $\mathbb{A}$ such that $\mathbb{A}^s$ is positive definite does not have a negative eigenvalue and $\det \mathbb{A} > 0$. Hence, we have $\det \mathbb{A}\mathbb{B} < 0$ so that $\mathbb{A}\mathbb{B}$ is invertible and has at least one negative eigenvalue.

First, assume that $\mathbb{A}\mathbb{B}$ has two different negative eigenvalues, $-a$, $-b$, and let $\mathbf{u} = (u_1, \ldots, u_d)$, $\mathbf{w} = (w_1, \ldots, w_d)$ be the corresponding eigenvectors. We claim that $u_1, w_1 \neq 0$. By contrast, suppose that $u_1 = 0$. Then, we have

$$ \mathbb{B}\mathbf{u} \cdot \mathbb{A}^s \mathbb{B}\mathbf{u} = \mathbb{B}\mathbf{u} \cdot \mathbb{A}\mathbb{B}\mathbf{u} = -a\mathbb{B}\mathbf{u} \cdot \mathbf{u} = -a \sum_{j=2}^{d} \lambda_j u_j^2 < 0, \quad (4.1) $$

which is a contradiction since $\mathbb{A}^s$ is positive definite. By the same argument, we get $w_1 \neq 0$.

By the definition of $a$, $b$ and by the positive definiteness of $\mathbb{A}^s$, for any $t \in \mathbb{R}$,

$$ (\mathbf{u} + t\mathbf{w})^\dagger \mathbb{B} (a\mathbf{u} + bt\mathbf{w}) = -(\mathbf{u} + t\mathbf{w})^\dagger \mathbb{B}\mathbb{A}\mathbb{B} (\mathbf{u} + t\mathbf{w}) < 0. $$

Let $p = -u_1/(bw_1)$. By substituting $t$ with $ap$ in the previous equation, the first coordinate of $a\mathbf{u} + bt\mathbf{w} = a(\mathbf{u} + bp\mathbf{w})$ is zero; thus, we have

$$ 0 > (\mathbf{u} + ap\mathbf{w})^\dagger \mathbb{B} (a\mathbf{u} + abp\mathbf{w}) = a \sum_{j=2}^{d} \lambda_j (u_j + apw_j)(u_j + bpw_j). \quad (4.2) $$
Similarly, substituting \( t \) with \( bp \) makes the first coordinate of \( u + bpw \) zero, and we get

\[
0 > (u + bpw)^\dagger B (au + b^2pw) = \sum_{j=2}^{d} \lambda_j (au_j + b^2pw_j) (u_j + bpw_j) .
\]

(4.3)

Computing \((b/a \times (4.2) + (4.3))\) gives

\[
0 > \sum_{j=2}^{d} \lambda_j (u_j + bpw_j)(bu_j + abpw_j + au_j + b^2pw_j)
\]

\[
= (a + b) \sum_{j=2}^{d} \lambda_j (u_j + bpw_j)^2
\]

which is a contradiction since we have assumed that \( \lambda_2, \ldots, \lambda_d > 0 \). Therefore, \( AB \) has only one negative eigenvalue \(-a\).

Now, let us assume that there are two eigenvectors \( u \) and \( w \) corresponding to \(-a\), which are linearly independent. Then, we can repeat the same computation as that presented above to get a contradiction, as we did not use the fact that \( a \neq b \) in the computation. Hence, the dimension of the eigenspace corresponding to the eigenvalue \(-a\) is 1.

\( \square \)

**Remark 4.3** Indeed, we can show that the algebraic multiplicity of the unique negative eigenvalue is also 1 by considering the Jordan decomposition.

The following lemma is a direct consequence of the previous one. In the application, we substitute \( A \) and \( B \) as \( H_\sigma \) and \( L_\sigma \), respectively, for some \( \sigma \in \Sigma_0 \).

**Lemma 4.4** Let \( A, B \) be square matrices of the same size and suppose that \( A \) is a symmetric non-singular matrix with exactly one negative eigenvalue and \( AB \) is a skew-symmetric matrix. Then, the matrix \( A + B \) is invertible and has only one negative eigenvalue, and its geometric multiplicity is 1.

**Proof** Since \( A \) is symmetric and \( AB \) is skew-symmetric, we have \(-AB = (AB)^\dagger = B^\dagger A\). Therefore, we get \( BA^{-1} = -A^{-1}B^\dagger = -(BA^{-1})^\dagger \); thus, the matrix \( BA^{-1} \) is skew-symmetric. Let \( I \) be the identity matrix with the same size as \( A \). Then, by substituting \( I + BA^{-1} \) and \( A \) for \( I \) and \( A \), respectively, in Lemma 4.2, we conclude the proof since \( A + B = (I + BA^{-1})A \).

\( \square \)

### 4.2 Equilibria of the dynamical system (2.5)

In this subsection, we analyze the equilibria of the dynamical system (2.5) by proving Theorem 2.1. First, we prove part (1) of the theorem.
Proof of part (1) of Theorem 2.1 Let \( c \in \mathbb{R}^d \) be a critical point of \( U \). Since \( \nabla U \cdot \ell = 0 \) by (1.7), we have
\[
0 \equiv \nabla [\nabla U \cdot \ell] = (\nabla^2 U) \ell + (D \ell) \nabla U .
\]
Thus, we have \((\nabla^2 U)(c) \ell(c) = 0\) as \( \nabla U(c) = 0 \). Since \((\nabla^2 U)(c)\) is invertible as \( U \) is a Morse function, we get \( \ell(c) = 0 \).
\(\square\)

Now, we present a lemma that is a consequence of the condition (1.7) and part (1) of Theorem 2.1 that we have just proved. We recall the notations \( H^c \) and \( L^c \) from Notation 3.2.

**Lemma 4.5** For any critical point \( c \) of \( U \), the matrix \( H^c L^c \) is skew-symmetric.

**Proof** For small \( \varepsilon > 0 \) and \( x \in \mathbb{R}^d \), the Taylor expansion implies that
\[
\nabla U(c + \varepsilon x) = \varepsilon H^c x + O(\varepsilon^2) \quad \text{and} \quad \ell(c + \varepsilon x) = \varepsilon L^c x + O(\varepsilon^2),
\]
since we have \( \nabla U(c) = \ell(c) = 0 \) by part (1) of Theorem 2.1. By (1.7), we have
\[
[\varepsilon H^c x + O(\varepsilon^2)] \cdot [\varepsilon L^c x + O(\varepsilon^2)] = 0 .
\]
Dividing both sides by \( \varepsilon^2 \) and letting \( \varepsilon \to 0 \), we get \( (H^c x) \cdot (L^c x) = 0 \). Since the Hessian \( H^c \) is symmetric, we can deduce that \( x \cdot H^c L^c x = 0 \) for all \( x \in \mathbb{R}^d \). This completes the proof. \(\square\)

Now, we focus on parts (2) and (3) of Theorem 2.1.

**Proof of parts (2) and (3) of Theorem 2.1** First, we focus on part (2). If \( c \) is a critical point of \( U \), we have \( (\nabla U + \ell)(c) = 0 \) by part (1); thus, \( c \) is an equilibrium of the dynamical system (2.5). On the other hand, suppose that \( c \) is an equilibrium, i.e., \( (\nabla U + \ell)(c) = 0 \). Then, by (1.7), we have \( 0 = (\nabla U \cdot \ell)(c) = -|\nabla U(c)|^2 \); thus, \( \nabla U(c) = 0 \).

For part (3), suppose that \( c \) is a local minimum of \( U \) such that the Hessian \( H^c \) is positive definite. Since \( H^c L^c \) is skew-symmetric by Lemma 4.5, we can insert \( A := H^c \) and \( B = L^c \) into Lemma 4.1 to conclude that all the eigenvalues of the matrix \( H^c + L^c \) are either positive real or complex with a positive real part; hence, \( c \) is a stable equilibrium of the dynamical system \( x(\cdot) \) since \( H^c + L^c \) is the Jacobian of the vector field \( \nabla U + \ell \) at \( c \).

For the other direction, suppose that \( c \) is a stable equilibrium of the dynamical system (2.5), i.e., the matrix \( H^c + L^c \) is positive definite in the sense that
\[
x \cdot (H^c + L^c)x > 0 \quad \text{for all } x \neq 0 . \quad (4.4)
\]
Suppose now that the symmetric matrix \( H^c \) is not positive definite so that there is a negative eigenvalue \( -\lambda < 0 \). Let \( v \) be the corresponding unit eigenvector. Since \( H^c L^c \) is...
is skew-symmetric by Lemma 4.5 and $\mathbb{H}^e$ is symmetric, we have

$$2(\mathbb{H}^e)^2 = \mathbb{H}^e[\mathbb{H}^e + \mathbb{L}^e] + [\mathbb{H}^e + (\mathbb{L}^e)^\dagger]\mathbb{H}^e,$$

and thus we get

$$2\lambda^2 = v \cdot 2(\mathbb{H}^e)^2 v = \mathbb{H}^e v \cdot [\mathbb{H}^e + \mathbb{L}^e] v + v \cdot [\mathbb{H}^e + (\mathbb{L}^e)^\dagger]\mathbb{H}^e v$$

$$= -\lambda v \cdot [\mathbb{H}^e + \mathbb{L}^e + \mathbb{H}^e + (\mathbb{L}^e)^\dagger] v = -2\lambda v \cdot [\mathbb{H}^e + \mathbb{L}^e] v.$$ 

This contradicts with (4.4) and therefore $\mathbb{H}^e + \mathbb{L}^e$ must be positive definite. This completes the proof. \(\square\)

### 4.3 Saddle points of dynamical system $x(\cdot)$

Now, we focus on the saddle points. First, we prove that, for $\sigma \in \Sigma^0$, the matrix $\mathbb{H}^\sigma + \mathbb{L}^\sigma$ has only one negative eigenvalue as the matrix $\mathbb{H}^\sigma$ has only one negative eigenvalue.

**Proof of Lemma 3.3** Suppose that $\sigma \in \Sigma^0$ such that $\mathbb{H}^\sigma$ has exactly one negative eigenvalue by the Morse lemma. Then, we can insert $A := \mathbb{H}^\sigma$ and $B := \mathbb{L}^\sigma$ into Lemma 4.4 owing to Lemma 4.5, and we can conclude that the matrix $\mathbb{H}^\sigma + \mathbb{L}^\sigma$ has only one negative eigenvalue and is invertible. \(\square\)

Next, we prove Lemma 3.4, which compares the unique (by Lemma 3.3) negative eigenvalues of $\mathbb{H}^\sigma$ and $\mathbb{H}^\sigma + \mathbb{L}^\sigma$ when $\sigma \in \Sigma^0$.

**Proof of Lemma 3.4** Denote by $-\lambda_1, \lambda_2, \ldots, \lambda_d$ the eigenvalues of the symmetric matrix $\mathbb{H}^\sigma$, where $\lambda_1, \ldots, \lambda_d > 0$. Thus, $\lambda_1 = \lambda_1$. Let $u_1, \ldots, u_d$ denote the normal eigenvectors of $\mathbb{H}^\sigma$ corresponding to the eigenvalues $-\lambda_1, \ldots, \lambda_d$, respectively. Let $v$ denote the unit eigenvector of $\mathbb{H}^\sigma + \mathbb{L}^\sigma$ corresponding to the unique negative eigenvalue $-\mu^\sigma$ and write $v = \sum_{i=1}^d a_i u_i$. Since $\mathbb{H}^\sigma \mathbb{L}^\sigma$ is skew-symmetric by Lemma 4.5, we have

$$|\mathbb{H}^\sigma v|^2 = v \cdot \mathbb{H}^\sigma (\mathbb{H}^\sigma + \mathbb{L}^\sigma) v = -\mu^\sigma v \cdot \mathbb{H}^\sigma v.$$

Using the above-mentioned notations, we can rewrite this identity as

$$\sum_{i=1}^d a_i^2 \lambda_i^2 = -\mu^\sigma \left[ -a_1^2 \lambda_1 + \sum_{i=2}^d a_i^2 \lambda_i \right]. \quad (4.5)$$

First, suppose that $a_1 = 0$. Then, we have $\sum_{i=2}^d a_i^2 \lambda_i^2 = -\mu^\sigma \sum_{i=2}^d a_i^2 \lambda_i$ and hence we get $a_2 = \cdots = a_d = 0$. This implies that $v = 0$, which is a contradiction. Thus, $a_1 \neq 0$. By (4.5), we have

\(\square\) Springer
\[ a_1^2 \lambda_1^2 \leq \sum_{i=1}^{d} a_i^2 \lambda_i^2 = \mu^\sigma a_1^2 \lambda_1 - \mu^\sigma \sum_{i=2}^{d} a_i^2 \lambda_i \leq \mu^\sigma a_1^2 \lambda_1. \]

Since \( a_1 \neq 0 \), we get \( \mu^\sigma \geq \lambda_1 = \lambda^\sigma \). \( \square \)

5 Properties of diffusion Process \( x_\epsilon(\cdot) \)

In this section, we prove the basic properties of the diffusion process \( x_\epsilon(\cdot) \).

5.1 Positive recurrence and non-explosion

First, we establish a technical lemma.

**Lemma 5.1** For all \( \epsilon > 0 \), there exists \( r_0 = r_0(\epsilon) > 0 \) such that \( (L_\epsilon U)(x) \leq -3 \) for all \( x \not\in D_{r_0}(0) \).

**Proof** By (2.2) and (2.3), we can take \( r_0 \) to be sufficiently large such that

\[ |\nabla U(x)| - 2 \Delta U(x) > \frac{\epsilon}{2} \quad \text{and} \quad |\nabla U(x)| > 2 \quad \text{(5.1)} \]

for all \( x \not\in D_{r_0}(0) \). Then, for \( x \not\in D_{r_0}(0) \), we have

\[ \Delta U(x) \leq -\frac{\epsilon}{4} + \frac{1}{2} |\nabla U(x)| \leq \frac{1}{4\epsilon} |\nabla U(x)|^2. \]

Therefore,

\[ (L_\epsilon U)(x) = -|\nabla U(x)|^2 + \epsilon \Delta U(x) \leq -\frac{3}{4} |\nabla U(x)|^2 \leq -3. \]

The last inequality follows from the second condition of (5.1). \( \square \)

Now, we prove Theorem 2.2

**Proof of Theorem 2.2** First, we prove part (1), i.e., the non-explosion property. By [36, Theorem at page 197], it suffices to check that there exists a smooth function \( u: \mathbb{R}^d \rightarrow (0, \infty) \) such that

\[ u(x) \rightarrow \infty \quad \text{as} \quad x \rightarrow \infty \quad \text{and} \quad (L_\epsilon u)(x) \leq u(x) \quad \text{for all} \quad x \in \mathbb{R}^d. \quad \text{(5.2)} \]

We claim that \( u = U + k_\epsilon \) with a sufficiently large constant \( k_\epsilon \) satisfies all these conditions. First, we take \( k_\epsilon \) to be sufficiently large such that \( u > 0 \). The former condition of (5.2) is immediate from (2.1). Now, it suffices to check the second condition. By Lemma 5.1, the function \( L_\epsilon u = L_\epsilon U \) is bounded from above. Denote this bound by...
and then take \( k \) to be sufficiently large such that \( \mu(\epsilon) > M \) for all \( \epsilon \in \mathbb{R}^d \). Then, the second condition of (5.2) follows.

The positive recurrence of \( x_\epsilon (\cdot) \) follows Lemma 5.1 and [31, Theorem 6.1.3] \( \Box \)

### 5.2 Invariant measure

By a slight abuse of notation, we write \( \mu_\epsilon (x) = Z_\epsilon^{-1} e^{-U(x)/\epsilon} \) (cf. (1.4)). Now, we prove Theorem 2.3. We can observe from the expression (2.7) of the generator \( \mathcal{L}_\epsilon \) that the adjoint generator \( \mathcal{L}^a_\epsilon \) of \( \mathcal{L}_\epsilon \) with respect to the Lebesgue measure \( dx \) can be written as

\[
\mathcal{L}^a_\epsilon f = \epsilon \nabla \cdot [e^{-U/\epsilon} \nabla (e^{U/\epsilon} f)] + \ell \cdot \nabla (e^{U/\epsilon} f) .
\]

(5.3)

**Proof of Theorem 2.3** First, we prove part (1). With the expression (5.3) and the explicit form of \( \mu_\epsilon (x) \), we can check that \( \mathcal{L}^a_\epsilon \mu_\epsilon = 0 \). Therefore, by [36, Theorem at page 254] and part (1) of Theorem 2.2, the measure \( \mu_\epsilon (\cdot) \) is the invariant measure for the process \( x_\epsilon (\cdot) \). The uniqueness follows from [36, Theorem at page 259] and [36, Theorem at page 260].

For part (2), let us assume that \( \mu_\epsilon (\cdot) \) is the invariant measure for the dynamics \( z_\epsilon (\cdot) \) given in (1.10) for all \( \epsilon > 0 \). Note that the generator associated with the process \( z_\epsilon (\cdot) \) acts on \( f \in C^2 (\mathbb{R}^d) \) as

\[
\mathcal{L}^a_\epsilon f = -b \cdot \nabla f + \epsilon \Delta f .
\]

Hence, its adjoint generator with respect to the Lebesgue measure is given by

\[
\mathcal{L}^a_\epsilon f = \nabla \cdot [f b] + \epsilon \Delta f .
\]

By [36, Theorem at page 259], we must have \( \mathcal{L}^a_\epsilon \mu_\epsilon = 0 \). By writing \( \ell = b - \nabla U \), this equation can be expressed as \( e^{-U/\epsilon} \left[ \frac{1}{\epsilon} \nabla U \cdot \ell + \nabla \cdot \ell \right] = 0 \). Since this holds for all \( \epsilon > 0 \), the vector field \( \ell \) must satisfy both (1.7) and (1.8). \( \Box \)

### 6 Potential theory

In this section, we introduce the potential theory related to the process \( x_\epsilon (\cdot) \). As in the previous studies, we prove the Eyring–Kramers formula based on the relation between the mean transition time and the potential theoretic notions, and this relation is recalled in Proposition 7.1. The difficulty, especially for the non-reversible process, in using this formula arises from the estimation of the capacity term appeared in the formula. In this article, as explained in the Introduction section, we develop a novel and simple way to estimate the capacity. In this section, we explain a formula given by Proposition 6.2 for the capacity which plays a crucial role in our method. We remark that this formula itself is not new; the method for handling this formula is the innovation of the current study, and will be explained in the remainder of this article. To explain this formula, we start by introducing the adjoint process, equilibrium potential, and capacity.
6.1 Adjoint process

The adjoint operator $\mathcal{L}^*_\epsilon$ of $\mathcal{L}_\epsilon$ with respect to the invariant measure $\mu_\epsilon$ can be written as

$$\mathcal{L}^*_\epsilon f = \epsilon e^{U/\epsilon} \nabla \cdot \left( e^{-U/\epsilon} \left( \nabla f + \frac{1}{\epsilon} f \ell \right) \right) = - (\nabla U - \ell) \cdot \nabla f + \epsilon \Delta f . \quad (6.1)$$

6.2 Equilibrium potentials and capacities

In the remainder of this section, we fix two disjoint non-empty bounded domains $A$ and $B$ of $\mathbb{R}^d$ with $C^2, \alpha$-boundaries for some $\alpha \in (0, 1)$ such that the perimeters $\sigma(A)$ and $\sigma(B)$ are finite, and $d(A, B) > 0$. Now, we introduce the equilibrium potential and capacity between the two sets $A$ and $B$. Write $\Omega = (A \cup B)^c$ so that $\partial \Omega = \partial A \cup \partial B$.

The equilibrium potentials $h^\epsilon_{A, B}, h^{\epsilon, *}_{A, B} : \mathbb{R}^d \to \mathbb{R}$ between $A$ and $B$ with respect to the processes $x_\epsilon(\cdot)$ and $x^*_\epsilon(\cdot)$ are given by

$$h^\epsilon_{A, B}(x) = \mathbb{P}_x [ \tau_A < \tau_B ] \quad \text{and} \quad h^{\epsilon, *}_{A, B}(x) = \mathbb{P}^{\epsilon, *}_x [ \tau_A < \tau_B ]$$

for $x \in \mathbb{R}^d$, respectively.

The capacity between $A$ and $B$ with respect to the processes $x_\epsilon(\cdot)$ and $x^*_\epsilon(\cdot)$ are respectively defined by

$$\text{cap}_\epsilon(A, B) = \epsilon \int_{\partial A} (\nabla h^\epsilon_{A, B} \cdot \mathbf{n}_\Omega) \sigma(d\mu_\epsilon)$$

$$\text{cap}^*_{\epsilon}(A, B) = \epsilon \int_{\partial A} (\nabla h^{\epsilon, *}_{A, B} \cdot \mathbf{n}_\Omega) \sigma(d\mu_\epsilon), \quad (6.2)$$

where $\mathbf{n}_\Omega(x)$ is the outward normal vector to $\Omega$ at $x$; hence, $\mathbf{n}_\Omega(x) = -\mathbf{n}_A(x)$ for $x \in \partial A$. Here, $\int_{\partial A} f \sigma(d\mu_\epsilon)$ is a shorthand of $\int_{\partial A} f(x) \mu_\epsilon(x) \sigma(dx)$. These capacities exhibit the following well-known properties.

Lemma 6.1 The following properties hold.

1. We have

$$\text{cap}_\epsilon(A, B) = \text{cap}^*_{\epsilon}(A, B) = \text{cap}^*_{\epsilon}(B, A) = \text{cap}_\epsilon(B, A) .$$
(2) We have

$$\text{cap}_\epsilon (\mathcal{A}, \mathcal{B}) = \epsilon \int_\Omega |\nabla h^\epsilon_{\mathcal{A}, \mathcal{B}}|^2 d\mu_\epsilon = \epsilon \int_\Omega |\nabla h^\epsilon_{\mathcal{A}, \mathcal{B}}|^2 d\mu_\epsilon .$$

**Proof** We refer to [19, Lemmas 3.2 and 3.1] for the proof of parts (1) and (2), respectively.

6.3 Representation of capacity

We keep the sets $\mathcal{A}, \mathcal{B},$ and $\Omega$ from the previous subsection. Then, for a function $f : \mathbb{R}^d \to \mathbb{R}$ that is differentiable at $x \in \mathbb{R}^d,$ we define a vector field $\Phi_f$ at $x$ as

$$\Phi_f (x) = \nabla f (x) + \frac{1}{\epsilon} f (x) \ell (x) . \quad (6.3)$$

Let $C^\infty_0 (\mathbb{R}^d)$ denote the class of smooth and compactly supported functions on $\mathbb{R}^d.$ Let

$$\mathcal{C}_{\mathcal{A}, \mathcal{B}} = \{ f \in C^\infty_0 (\mathbb{R}^d) : f \equiv 1 \text{ on } \mathcal{A} , \quad f \equiv 0 \text{ on } \mathcal{B} \}. \quad (6.4)$$

Hence, for $f \in \mathcal{C}_{\mathcal{A}, \mathcal{B}},$ the vector field $\Phi_f$ is defined on $\mathbb{R}^d.$ The following expression plays a crucial role in the estimation of the capacity.

**Proposition 6.2** For all $f \in \mathcal{C}_{\mathcal{A}, \mathcal{B}},$ we have

$$\epsilon \int_\Omega [\Phi_f \cdot \nabla h^\epsilon_{\mathcal{A}, \mathcal{B}}] d\mu_\epsilon = \text{cap}_\epsilon (\mathcal{A}, \mathcal{B}) . \quad (6.5)$$

**Proof** Since $f$ is compactly supported, we can apply the divergence theorem to rewrite the left-hand side of (6.5) as

$$\epsilon \int_{\partial \Omega} f [\nabla h^\epsilon_{\mathcal{A}, \mathcal{B}} \cdot \mathbf{n}_\Omega] \sigma (d\mu_\epsilon) - \int_\Omega f (\mathcal{L}_\epsilon h^\epsilon_{\mathcal{A}, \mathcal{B}}) d\mu_\epsilon .$$

Since $f = 1_{\partial \mathcal{A}}$ on $\partial \Omega$ by the condition $f \in \mathcal{C}_{\mathcal{A}, \mathcal{B}},$ the first term of the above-mentioned expression is equal to $\text{cap}_\epsilon (\mathcal{A}, \mathcal{B})$ by (6.2). On the other hand, the second integral is 0 since $\mathcal{L}_\epsilon h^\epsilon_{\mathcal{A}, \mathcal{B}} \equiv 0$ on $\Omega$ by the property of the equilibrium potential. \(\square\)

7 Proof of Eyring–Kramers formula

In this section, we prove the Eyring–Kramers formula stated in Theorem 3.5 up to the construction of a test function and analysis of the equilibrium potential.
7.1 Proof of Theorem 3.5

For convenience of notation, we will use the following abbreviations for the capacity and equilibrium potential between a small ball around the minimum $m_0$ and $\mathcal{U}_\epsilon$:

$$cap_\epsilon = cap_\epsilon (\mathcal{D}_\epsilon (m_0), \mathcal{U}_\epsilon),$$
$$h_\epsilon (\cdot) = h_{\mathcal{D}_\epsilon (m_0), \mathcal{U}_\epsilon}^\epsilon (\cdot) \quad \text{and} \quad h_\epsilon^* (\cdot) = h_{\mathcal{D}_\epsilon (m_0), \mathcal{U}_\epsilon}^{\epsilon, *}(\cdot). \quad (7.1)$$

The proof of the Eyring–Kramers formula relies on the following formula regarding the mean transition time.

**Proposition 7.1** We have

$$\mathbb{E}_{m_0}^{\epsilon} [\tau_{\mathcal{U}_\epsilon}] = \left[1 + o_\epsilon (1)\right] \frac{1}{cap_\epsilon} \int_{\mathbb{R}^d} h_\epsilon^* \, d\mu_\epsilon. \quad (7.2)$$

This remarkable relation between the mean transition time and the potential theoretic notions was first observed in [5, Proposition 6.1] for the reversible case. Then, it was extended to the general non-reversible case in [19, Lemma 9.2]. Our proof is identical to that of the latter case; hence, we omit the details. Now, the proof of Theorem 3.5 is reduced to computing the right-hand side of (7.2). We shall estimate the capacity and integral terms separately. We emphasize here that, even if we rely on the general formula (7.2), the estimation of these two terms is carried out in a novel manner. For simplicity of notation, hereafter, we write

$$\alpha_\epsilon = Z_\epsilon^{-1} e^{-H/\epsilon} (2\pi \epsilon)^{d/2}. \quad (7.3)$$

Our main innovation in the proof of the Eyring–Kramers formula is the new strategy to prove the following proposition.

**Proposition 7.2** For $\omega_0$ defined in (3.5), we have

$$cap_\epsilon = \left[1 + o_\epsilon (1)\right] \alpha_\epsilon \omega_0. \quad (7.4)$$

We present our proof, up to the construction of a test function, in the next subsection. Further, we need to estimate the integral term in (7.2).

**Proposition 7.3** For $\nu_0$ defined in (3.7), we have

$$\int_{\mathbb{R}^d} h_\epsilon^* \, d\mu_\epsilon = \left[1 + o_\epsilon (1)\right] Z_\epsilon^{-1} (2\pi \epsilon)^{d/2} e^{-h_0/\epsilon} \nu_0. \quad (7.5)$$

We heuristically explain that the last proposition holds. Define $\mathcal{G} = \{ x : U(x) < H - \beta \}$ for small $\beta > 0$ and let $\mathcal{G}_i = \mathcal{H}_i \cap \mathcal{G}$ for $i = 0, 1$. Since the process starting from a point in $\mathcal{G}_0$ may touch the set $\mathcal{D}_\epsilon (m_0)$ before climbing to the saddle point at level $H$, we can expect that $h_\epsilon^* \simeq 1$ on $\mathcal{G}_0$. By a similar logic, we have $h_\epsilon^* \simeq 0$ on $\mathcal{G}_1$. Since $\mu_\epsilon (\mathcal{G})$ is negligible by (2.4), we can conclude that the left-hand side of (7.5)
is approximately equal to $\mu_\epsilon (G_0)$, whose asymptotics is given by the right-hand side of (7.5). We turn this into a rigorous argument in Sect. 9.4 on the basis of a delicate analysis of the equilibrium potential.

Now, we formally conclude the proof of Eyring–Kramers formula.

**Proof of Theorem 3.5** The proof is completed by combining Propositions 7.1, 7.2, and 7.3.

### 7.2 Strategy to prove Proposition 7.2

Instead of relying on the traditional approach, which uses the variational expression of the capacity given by the Dirichlet principle or the Thomson principle to estimate the capacity, we develop an alternative strategy in this subsection. This strategy is suitable for non-reversible cases in that neither the flow structure nor the test flow is used.

In Sect. 10, we construct a smooth test function $g_\epsilon \in \mathcal{C}^D(m_0, U_\epsilon)$ (cf. (6.4)) satisfying the following property.

**Theorem 7.4** We have

\[ \epsilon \int_{\Omega_\epsilon} \left[ \Phi_{g_\epsilon} \cdot \nabla h_\epsilon \right] d\mu_\epsilon = [1 + o_\epsilon(1)] \alpha_\epsilon \omega_0 + o_\epsilon(1) \left[ \alpha_\epsilon \text{cap}_\epsilon \right]^{1/2}, \tag{7.6} \]

where $\Omega_\epsilon = (D_\epsilon(m_0) \cup U_\epsilon)^c$.

The left-hand side of (7.6) corresponding to $\text{cap}_\epsilon$ by Proposition 6.2 is believed to be equal to the first term at the right-hand side. Thus, the second error term is somewhat unwanted and appears just because of a technical reason explained in more detail at Remark 7.5. We can however absorb this second error term to the first error term at the right-hand side of (7.6) as illustrated in the proof below of Proposition 7.2. Note that we assume Theorem 7.4 at this moment.

**Proof of Proposition 7.2** By Proposition 6.2 and Theorem 7.4, we get

\[ \text{cap}_\epsilon = [1 + o_\epsilon(1)] \alpha_\epsilon \omega_0 + o_\epsilon(1) \left[ \alpha_\epsilon \text{cap}_\epsilon \right]^{1/2}. \]

By dividing both sides by $\alpha_\epsilon$ and substituting $r_\epsilon = \left[ \text{cap}_\epsilon / \alpha_\epsilon \right]^{1/2}$, we can rewrite the previous identity as

\[ r_\epsilon^2 = [1 + o_\epsilon(1)] \omega_0 + o_\epsilon(1) r_\epsilon. \]

By solving this quadratic equation in $r_\epsilon$, we get $r_\epsilon = [1 + o_\epsilon(1)] (\omega_0)^{1/2}$. Squaring this completes the proof.

Now we turn to Theorem 7.4. The core of our strategy is to find a suitable test function $g_\epsilon$ and to compute the left-hand side of (7.6). Indeed, we construct $g_\epsilon$ as an approximation of the equilibrium potential $h_\epsilon^*(\cdot)$ for the adjoint process (cf. (7.1)). The reason...
is that, by the divergence theorem, we can write the left-hand side of (7.6) as

$$
\epsilon \int_{\Omega_\epsilon} \left[ \Phi g_\epsilon \cdot \nabla h_\epsilon \right] d\mu_\epsilon = - \int_{\Omega_\epsilon} h_\epsilon \mathcal{L}_\epsilon^* g_\epsilon d\mu_\epsilon + \text{(boundary terms)}. \quad (7.7)
$$

To control the integration on the right-hand side, we try to make \( \mathcal{L}_\epsilon^* g_\epsilon \) as small as possible (cf. Proposition 8.5); hence, in view of the fact that \( \mathcal{L}_\epsilon^* h_\epsilon^* \equiv 0 \) on \( \Omega_\epsilon \) by the property of the equilibrium potential, the test function \( g_\epsilon \) should be an approximation of \( h_\epsilon^* \). The main contribution for the computation of the left-hand side of (7.7) comes from the boundary terms, and relevant computations are carried out in Proposition 8.6.

The construction of \( g_\epsilon \) particularly focuses on the neighborhoods of the saddle points of \( \Sigma_0 \) as the equilibrium potential (and hence \( g_\epsilon \), which is an approximation of the equilibrium potential) drastically falls from 1 to 0 there. We carry out this construction around the saddle point in Sect. 8 on the basis of a linearization procedure that is now routine in this field, e.g., [5,19]. Then, we extend these functions around the saddle points of \( \Sigma_0 \) to a continuous function on \( \mathbb{R}^d \) belonging to \( \mathcal{C}_{D_\epsilon}(m_0), U_\epsilon \). This process will be performed in Sect. 10, and we finally obtain \( g_\epsilon \) in (10.2). Then, we prove (7.6) on the basis of our analysis of the equilibrium potential carried out in Sect. 9.

**Remark 7.5** (Comparison with reversible case) Our strategy is relatively simple when the underlying process is reversible. In order to get a continuous test function \( g_\epsilon \), we need a mollification procedure (cf. Proposition 10.2), and we must include an additional term \( o_\epsilon(1) \left[ \alpha_\epsilon \cap \epsilon \right]^{1/2} \) in (7.6) to compensate for this additional procedure. However, for the reversible case, we can get a continuous test function without this mollification procedure (cf. Remark 10.1) and we can prove that

$$
\epsilon \int_{\Omega_\epsilon} \left[ \Phi g_\epsilon \cdot \nabla h_\epsilon \right] d\mu_\epsilon = \left[ 1 + o_\epsilon(1) \right] \alpha_\epsilon \omega_0,
$$

instead of (7.6); hence, the proof of the Eyring–Kramers formula is more straightforward. This is the only technical difference between the reversible and non-reversible models in our methodology.

The remainder of this article is devoted to proving Theorem 7.4, and in the course of the proof, Proposition 7.3 will also be demonstrated in Sect. 9.

### 8 Construction of Test Function Around Saddle Point

We explain how we can construct the test function around a saddle point \( \sigma \in \Sigma_0 \). Section 8.1 presents a preliminary analysis of the geometry around the saddle point. We acknowledge that several statements and proofs given in these sections are similar to those given in [19]; however, we try not to omit the proofs of these results, as the details of the computations are slightly different owing to the differences between the models. Then, we construct the test function \( p_\epsilon^\sigma \) on a neighborhood of \( \sigma \) in Sect. 8.
Finally, we explain several computational properties of this test function in Sects. 8.3–8.5. These properties play crucial role in the proof of Theorem 7.4.

Setting

In this section, we fix a saddle point \( \sigma \in \Sigma_0 \) and simply write \( H = H^\sigma = (\nabla^2 U)(\sigma) \) and \( L = L^\sigma = (D\ell)(\sigma) \). Recall that \( H \) has only one negative eigenvalue because of the Morse lemma. Let \( -\lambda_1, \lambda_2, \cdots, \lambda_d \) denote the eigenvalues of \( H \), where \( -\lambda_1 = -\lambda_1^\sigma \) denotes the unique negative eigenvalue. Let \( e_k = e_k^\sigma \) denote the eigenvector associated with the eigenvalue \( \lambda_k \) (\( -\lambda_k \) if \( k = 1 \)). In addition, we assume the direction of \( e_1 \) to be toward \( H_0 \), i.e., for all sufficiently small \( r > 0 \), \( \sigma + re_1 \in H_0 \).

By Lemma 3.3, the matrix \( H + L \) has a unique negative eigenvalue \( -\mu = -\mu^\sigma \). We can readily observe that the matrix \( H - L^\dagger \) is similar to \( H + L \). To see this, first note that, since \( HL \) is skew-symmetric by Lemma 4.5, we have \( HL = -(HL)^\dagger \). Hence, the matrix \( H - L^\dagger \) also has a unique negative eigenvalue \( -\mu \), and let \( v = v^\sigma \) denote the unit eigenvector of this matrix associated with the eigenvalue \( -\mu \). Finally, we assume without loss of generality that \( v \cdot e_1 \geq 0 \). Indeed, this cannot be 0 because of the following lemma, which implies that \( (v \cdot e_1)^2 > 0 \).

**Lemma 8.1** We have

\[
 v \cdot H^{-1} v = -\frac{(v \cdot e_1)^2}{\lambda_1} + \sum_{k=2}^d \frac{(v \cdot e_k)^2}{\lambda_k} = -\frac{1}{\mu} < 0 .
\]

**Proof** The first equality is obvious if we write \( v = \sum_{i=1}^d a_i e_i \). Now, we focus on the second equality. Note that \( H - L^\dagger \) is invertible byLemma 4.1 and (8.1). Hence, we can compute

\[
 v \cdot H^{-1} v = v \cdot H^{-1}(H - L^\dagger)(H - L^\dagger)^{-1} v = -\frac{1}{\mu} v \cdot H^{-1}(H - L^\dagger) v
\]

\[
 = -\frac{1}{\mu} v \cdot v + \frac{1}{\mu} v \cdot H^{-1} L^\dagger H H^{-1} v .
\]

Since \( |v|^2 = 1 \), the first term in the last line is \(-\frac{1}{\mu}\). On the other hand, since \( L^\dagger H = -(HL)^\dagger \) is skew-symmetric and \( H^{-1} \) is symmetric, the second term in the last line is 0. This completes the proof. \( \square \)

For two vectors \( u, w \in \mathbb{R}^d \), let \( u \otimes w \in \mathbb{R}^{d \times d} \) denote their tensor product, i.e., \( (u \otimes w)_{ij} = u_i w_j \), where \( u_i \) and \( w_j \) are the \( i \)th and \( j \)th elements of \( u \) and \( w \), respectively. The following Lemma is a consequence of the previous lemma and is similar to [21, Lemmas 4.1 and 4.2].

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Lemma 8.2 The following hold.

1. The matrix $\mathbb{H} + 2\mu \mathbf{v} \otimes \mathbf{v}$ is symmetric positive definite and $\det(\mathbb{H} + 2\mu \mathbf{v} \otimes \mathbf{v}) = -\det \mathbb{H}$.

2. The matrix $\mathbb{H} + \mu \mathbf{v} \otimes \mathbf{v}$ is symmetric non-negative definite and $\det(\mathbb{H} + \mu \mathbf{v} \otimes \mathbf{v}) = 0$. The null space of the matrix $\mathbb{H} + \mu \mathbf{v} \otimes \mathbf{v}$ is one-dimensional and spanned by the vector $\mathbb{H}^{-1} \mathbf{v}$.

Proof By a change of coordinate, we can assume that $\mathbf{e}_i$ is the $i$th standard unit vector of $\mathbb{R}^d$ such that $\mathbb{H} = \text{diag}(-\lambda_1, \lambda_2, \ldots, \lambda_d)$. First, we show that $\mathbb{H} + \mu \mathbf{v} \otimes \mathbf{v}$ is non-negative definite. If $\mathbf{v}_2 = \cdots = \mathbf{v}_d = 0$, then, we have $\mathbf{v}_1^2 = \mu/\lambda_1$ by Lemma 8.1; thus, $\mathbb{H} + \mu \mathbf{v} \otimes \mathbf{v} = \text{diag}(0, \lambda_2, \ldots, \lambda_d)$ is non-negative definite. Otherwise, for $\mathbf{x} = \sum_{i=1}^d x_i \mathbf{e}_i \in \mathbb{R}^d$, we can compute

$$\mathbf{x} \cdot [\mathbb{H} + \mu \mathbf{v} \otimes \mathbf{v}] \mathbf{x} = -\lambda_1 x_1^2 + \sum_{k=2}^d \lambda_k x_k^2 + \mu \left( \sum_{i=1}^d x_i \mathbf{v}_i \right)^2.$$

By minimizing the right-hand side over $x_1$ and using Lemma 8.1, we get

$$\sum_{k=2}^d \lambda_k x_k^2 - \frac{(\sum_{k=2}^d x_k \mathbf{v}_k)^2}{\sum_{k=2}^d \mathbf{v}_k^2 / \lambda_k},$$

which is non-negative by Cauchy–Schwarz inequality. This proves that $\mathbb{H} + \mu \mathbf{v} \otimes \mathbf{v}$ is non-negative definite. Then, the matrix $\mathbb{H} + 2\mu \mathbf{v} \otimes \mathbf{v}$ is non-negative definite as well. By the well-known formula

$$\det(\mathbb{A} + \mathbf{x} \otimes \mathbf{y}) = (1 + \mathbf{y}^T \mathbb{A}^{-1} \mathbf{x}) \det \mathbb{A}, \quad (8.2)$$

along with Lemma 8.1, we can check that $\det(\mathbb{H} + 2\mu \mathbf{v} \otimes \mathbf{v}) = -\det \mathbb{H} > 0$, and thus, $\mathbb{H} + 2\mu \mathbf{v} \otimes \mathbf{v}$ is indeed positive definite. Finally, we investigate the null space of $\mathbb{H} + \mu \mathbf{v} \otimes \mathbf{v}$. Suppose that $\mathbf{w} \in \mathbb{R}^d$ satisfies $(\mathbb{H} + \mu \mathbf{v} \otimes \mathbf{v}) \mathbf{w} = 0$. Since $\mathbb{H}$ is invertible, we can rewrite this equation as $\mathbf{w} = -\mu (\mathbf{v} \cdot \mathbf{w}) \mathbb{H}^{-1} \mathbf{v}$. Hence, the null space is a subspace of $\langle \mathbb{H}^{-1} \mathbf{v} \rangle$. On the other hand, if $\mathbf{w} = a \mathbb{H}^{-1} \mathbf{v}$ for some $a \in \mathbb{R}$, we can readily check that $(\mathbb{H} + \mu \mathbf{v} \otimes \mathbf{v}) \mathbf{w} = 0$, and hence, $\langle \mathbb{H}^{-1} \mathbf{v} \rangle$ is indeed the null space. \hfill \Box

8.1 Neighborhood of saddle points

In this subsection, we specify the geometry around each saddle point $\sigma$. Figure 3 illustrates the sets appearing in this section.

We focus on a neighborhood of $\sigma$ with size of order $\delta$, which is defined by

$$\delta = \delta(\epsilon) := \left( \epsilon \log \frac{1}{\epsilon} \right)^{1/2}. \quad (8.3)$$
Let $J$ be a sufficiently large constant that is independent of $\epsilon$. There will be several class, e.g., Lemma 10.4, that require $J$ to be sufficiently large; we suppose that $J$ satisfies all such requirements. Define a box $C_\epsilon^\sigma$ centered at $\sigma$ as

$$C_\epsilon^\sigma = \left\{ \sigma + \sum_{i=1}^d \alpha_i e_i^\sigma \in \mathbb{R}^d : -\frac{J \delta}{\lambda_1^{1/2}} \leq \alpha_1 \leq \frac{J \delta}{\lambda_1^{1/2}} \right. \left. \text{ and } -\frac{2J \delta}{\lambda_j^{1/2}} \leq \alpha_j \leq \frac{2J \delta}{\lambda_j^{1/2}} \right\}.$$

Now, decompose the boundary $\partial C_\epsilon^\sigma$ into $\partial_+ C_\epsilon^\sigma$, $\partial_- C_\epsilon^\sigma$, and $\partial_0 C_\epsilon^\sigma$ such that

$$\partial_\pm C_\epsilon^\sigma = \left\{ \sigma + \sum_{i=1}^d \alpha_i e_i^\sigma \in \mathbb{R}^d : \alpha_1 = \pm \frac{J \delta}{\lambda_1^{1/2}} \right\},$$

$$\partial_0 C_\epsilon^\sigma = \partial C_\epsilon^\sigma \setminus (\partial_+ C_\epsilon^\sigma \cup \partial_- C_\epsilon^\sigma). \quad (8.4)$$

**Lemma 8.3** For $x \in \partial_0 C_\epsilon^\sigma$, we have $U(x) \geq H + \frac{5}{4} J^2 \delta^2$ for all sufficiently small $\epsilon > 0$.

**Proof** For $x \in C_\epsilon^\sigma$, by the Taylor expansion of $U$ at $\sigma$, we have

$$U(x) = H + \frac{1}{2} \left[ -\lambda_1 x_1^2 + \sum_{j=2}^d \lambda_j x_j^2 \right] + O(\delta^3). \quad (8.5)$$
For \( x \in \partial_0 C^\sigma_\epsilon, x_i = \pm 2J \delta / \sqrt{\lambda_i} \) for some \( 2 \leq i \leq d \). Therefore,

\[
-\lambda_1 x_1^2 + \sum_{j=2}^{d} \lambda_j x_j^2 \geq -J^2 \delta^2 + \lambda_i \left( \frac{2J \delta}{\lambda_i^{1/2}} \right)^2 = 3J^2 \delta^2.
\]

Inserting this to (8.5) completes the proof. \( \square \)

Hereafter, we assume that \( \epsilon > 0 \) is sufficiently small such that Lemma 8.3 holds. Define, for \( \epsilon > 0 \),

\[
K_\epsilon = \{ x \in \mathbb{R}^d : U(x) < H + J^2 \delta^2 \} \quad \text{and} \quad K = \{ x \in \mathbb{R}^d : U(x) < H + J^2 \} \quad (8.6)
\]

so that \( \mathcal{H} \subset K_\epsilon \subset K \) holds.

By Lemma 8.3, the boundary \( \partial_0 C^\sigma_\epsilon \) does not belong to \( K_\epsilon \). The neighborhood of \( \sigma \) in which we focus on the construction is the set \( \mathcal{B}^\sigma_\epsilon = C^\sigma_\epsilon \cap K_\epsilon \). Now, we decompose the boundary \( \partial B^\sigma_\epsilon \) into \( \partial_+ B^\sigma_\epsilon, \partial_- B^\sigma_\epsilon \), and \( \partial_0 B^\sigma_\epsilon \) such that

\[
\partial_\pm B^\sigma_\epsilon = \partial_\pm C^\sigma_\epsilon \quad \text{and} \quad \partial_0 B^\sigma_\epsilon = \partial B_\epsilon \setminus ( \partial_+ B^\sigma_\epsilon \cup \partial_- B^\sigma_\epsilon )
\]

so that we have \( U(x) = H + J^2 \delta^2 \) for all \( x \in \partial_0 B^\sigma_\epsilon \) by Lemma 8.3.

Now, the set \( K_\epsilon \setminus \cup_{\sigma \in \Sigma_0} B^\sigma_\epsilon \) consists of several connected components. Let \( \mathcal{H}^\epsilon_0 \) denote one such component containing \( \mathcal{M}_0 \) and let \( \mathcal{H}^\epsilon_1 \) denote the union of the other components such that \( \mathcal{M}_1 \subset \mathcal{H}^\epsilon_1 \). By our convention on the direction of the vector \( e_1 = e^\sigma_1 \) mentioned earlier in the current section, we have

\[
\partial_+ B^\sigma_\epsilon \subset \partial \mathcal{H}^\epsilon_0 \quad \text{and} \quad \partial_- B^\sigma_\epsilon \subset \partial \mathcal{H}^\epsilon_1. \quad (8.7)
\]

This is illustrated in Fig. 3.

### 8.2 Construction of test function around \( \sigma \) via linearization procedure

We construct a function \( p^\sigma_\epsilon : \mathbb{R}^d \to \mathbb{R} \) on \( B^\sigma_\epsilon \), which acts as a building block for the global construction carried out in the following sections. As mentioned in Section 7.2, we would like to build a function approximating the equilibrium potential \( h^\epsilon_\ast \) between \( D_\epsilon (m_0) \) and \( U_\epsilon \). Thus, we expect \( p^\sigma_\epsilon \) to satisfy \( \mathcal{L}^\ast_\epsilon p^\sigma_\epsilon \approx 0 \), where \( \mathcal{L}^\ast_\epsilon \) is defined in (6.1). To find this function, we linearize the generator \( \mathcal{L}^\ast_\epsilon \) around \( \sigma \) by the first-order Taylor expansion such that, for smooth \( f \),

\[
\widetilde{\mathcal{L}}^\ast_\epsilon f = \epsilon \Delta f(x) - \nabla f(x) \cdot (\mathbb{H} - \mathbb{L})(x),
\]

and we solve the linearized equation \( \widetilde{\mathcal{L}}^\ast_\epsilon p^\sigma_\epsilon = 0 \). This equation can be explicitly solved using the separation of variables method. Note that in view of (8.7), we would
like to impose boundary conditions of the form $p_\sigma^\epsilon \simeq 1$ on $\partial_+ B_\sigma^\epsilon$ and $p_\sigma^\epsilon \simeq 0$ on $\partial_- B_\sigma^\epsilon$. A test function satisfying all these requirements is given by

$$p_\sigma^\epsilon(x) = \frac{1}{c_\epsilon} \int_{-\infty}^{(x-\sigma) \cdot v} e^{-\frac{\mu}{2\epsilon} t^2} \, dt \quad ; \quad x \in \overline{B_\sigma^\epsilon}, \quad (8.8)$$

where

$$c_\epsilon = \int_{-\infty}^{\infty} e^{-\frac{\mu}{2\epsilon} t^2} \, dt = \sqrt{\frac{2\pi \epsilon}{\mu}}. \quad (8.9)$$

Note that $v$ and $\mu$ are defined at the beginning of the current section. The crucial technical difficulty arises from the fact that the function $p_\sigma^\epsilon$ is not constant along the boundary $\partial_\pm B_\sigma^\epsilon$ unless the dynamics is reversible since $e_1^\sigma$ and $v$ are linearly independent if $\ell \neq 0$. This makes it difficult to patch these functions together. This issue will be thoroughly investigated in Sect. 10.

Since $p_\sigma^\epsilon$ is smooth on $B_\sigma^\epsilon$, we can define $\Phi p_\sigma^\epsilon$ on $B_\sigma^\epsilon$. Next, we must investigate the properties of $p_\sigma^\epsilon$ and $\Phi p_\sigma^\epsilon$. For the simplicity of notation, we assume that $\sigma = 0$ in the remainder of the current section.

### 8.3 Negligibility of $\mathcal{L}_e^* p_\sigma^\epsilon$ on $B_\sigma^\epsilon$

Our construction of $p_\sigma^\epsilon$ suggests that $\mathcal{L}_e^* p_\sigma^\epsilon$ is small on $B_\sigma^\epsilon$. The next lemma precisely quantifies this heuristic observation.

**Notation 8.4** Let $C > 0$ denote a positive constant independent of $\epsilon$ and $x$. Different appearances of $C$ may express different values.

**Proposition 8.5** We have

$$\int_{B_\sigma^\epsilon} |\mathcal{L}_e^* p_\sigma^\epsilon| \, d\mu_\epsilon = o_\epsilon(1) \alpha_\epsilon.$$

**Proof** By inserting the explicit formula (8.8), we get

$$(\mathcal{L}_e^* p_\sigma^\epsilon)(x) = c_\epsilon^{-1} e^{-\frac{\mu}{2\epsilon} (x \cdot v)^2} \left[ - (\nabla U - \ell)(x) \cdot v - \mu (x \cdot v) \right].$$

Now, by applying the Taylor expansion of $\nabla U$ and $\ell$ around $\sigma$, for $x \in B_\sigma^\epsilon$,

$$(\mathcal{L}_e^* p_\sigma^\epsilon)(x) = -c_\epsilon^{-1} e^{-\frac{\mu}{2\epsilon} (x \cdot v)^2} \left[ \{ (\mathbb{H} - \mathbb{L})x + O(\delta^2) \} \cdot v + \mu (x \cdot v) \right]
= -c_\epsilon^{-1} e^{-\frac{\mu}{2\epsilon} (x \cdot v)^2} \left[ x \cdot (-\mu v) + \mu (x \cdot v) + O(\delta^2) \right],$$

where the last line follows from the fact that $v$ is an eigenvector of $(\mathbb{H} - \mathbb{L})^\dagger = \mathbb{H} - \mathbb{L}^\dagger$ associated with the eigenvalue $-\mu$. Now, recall $c_\epsilon$ from (8.9) to deduce that, for some constant $C > 0$,

$$| (\mathcal{L}_e^* p_\sigma^\epsilon)(x) | \leq \frac{C \delta^2}{\epsilon^{1/2}} e^{-\frac{\mu}{2\epsilon} (x \cdot v)^2}.$$
By the second-order Taylor expansion, we can write

\[ U(x) = H + \frac{1}{2} x \cdot \mathbb{H}x + O(\delta^3) \quad \text{for } x \in B_\epsilon^\sigma. \]

This expansion will be repeatedly used in the subsequent computation. Since
\[ e^{-O(\delta^3)/\epsilon} = 1 + o_\epsilon(1) \] by the definition (8.3) of \( \delta \), we can conclude that

\[ \int_{B_\epsilon^\sigma} |\mathcal{L}_\epsilon^\sigma p_\epsilon^\sigma| \, d\mu_\epsilon \leq C \frac{\delta^2}{\epsilon^{1/2}} e^{-H/\epsilon} \int_{B_\epsilon^\sigma} e^{-\frac{1}{2} x \cdot (\mathbb{H} + \mu v \otimes v)x} \, dx. \] \hspace{1cm} (8.10)

Now, the estimation of the last integral remains. This part is similar to [19, Lemma 8.7]; however, we repeat the argument here for the completeness of the proof. By part (2) of Lemma 8.2, let \( \rho_1 = 0 \) and \( \rho_2, \ldots, \rho_d > 0 \) denote the eigenvalues of \( \mathbb{H} + \mu v \otimes v \) and let \( u_1, \ldots, u_d \) denote the corresponding unit eigenvectors. Let \( (u_2, \ldots, u_d) \) denote the subspace of \( \mathbb{R}^d \) spanned by vectors \( u_2, \ldots, u_d \). Since \( B_\epsilon^\sigma \subset C_\epsilon^\sigma \), there exists \( M > 0 \) such that

\[ B_\epsilon^\sigma \subset \bigcup_{a : |a| \leq M \delta} (a u_1 + (u_2, \ldots, u_d)). \]

Hence, along with the change of variables \( x = \sum y_i u_i \), we can bound the last integral in (8.10) by

\[ \int_{-M \delta}^{M \delta} \left[ \int_{\mathbb{R}^{d-1}} \exp \left\{ -\frac{1}{2\epsilon} \sum_{k=2}^{d} \rho_k y_k^2 \right\} d y_2 \cdots d y_d \right] d y_1 = C \delta \epsilon^{(d-1)/2}. \]

By inserting this into (8.10), we get \( \int_{B_\epsilon^\sigma} |\mathcal{L}_\epsilon^\sigma p_\epsilon^\sigma| \, d\mu_\epsilon \leq C \delta^3 \epsilon^{-1} \alpha_\epsilon. \) Since \( \delta^3 \epsilon^{-1} = o_\epsilon(1) \), the proof is completed. \( \square \)

### 8.4 Property of \( \Phi_{p_\epsilon^\sigma} \) at the boundary of \( B_\epsilon^\sigma \)

Next, we prove the following property of the vector field \( \Phi_{p_\epsilon^\sigma} \). Recall \( \omega^\sigma \) from (3.3).

**Proposition 8.6** We have

\[ \epsilon \int_{\partial B_\epsilon^\sigma} \left[ \left( \Phi_{p_\epsilon^\sigma} - \frac{1}{\epsilon} \ell \right) \cdot e_1 \right] \sigma(d\mu_\epsilon) = [1 + o_\epsilon(1)] \alpha_\epsilon \omega^\sigma. \] \hspace{1cm} (8.11)

This estimate is indeed the key estimate in the proof of Theorem 7.4. The left-hand side of (8.11) corresponds to the boundary term in (7.7). The proof of this proposition is slightly complicated. Hence, we first establish some technical lemmas. For simplicity of notation, we assume in this subsection that \( e_i \) is the \( i \)th standard normal vector of
$\mathbb{R}^d$; hence, we can write
\[
H = \text{diag}(-\lambda_1, \lambda_2, \ldots, \lambda_d) \quad \text{and} \quad v = (v_1, \ldots, v_d).
\]

**Change of coordinate on $\partial_+ B_\epsilon^\sigma$**

First, we introduce a change of coordinate that maps $\partial_+ B_\epsilon^\sigma$ to a subset of $\mathbb{R}^{d-1}$ to simplify the integration in (8.11).

For $A \in \mathbb{R}^{d \times d}$ and $u = (u_1, \ldots, u_d) \in \mathbb{R}^d$, define $\widetilde{A} \in \mathbb{R}^{(d-1) \times (d-1)}$ and $\widetilde{u} \in \mathbb{R}^{d-1}$ as
\[
\widetilde{A} = (A_{i,j})_{2 \leq i, j \leq d} \quad \text{and} \quad \widetilde{u} = (u_2, \ldots, u_d),
\]
respectively. It is important to select a point of $\partial_+ B_\epsilon^\sigma$ corresponding to the origin of $\mathbb{R}^{d-1}$ to simplify our computation. To this end, define $y = (y_2, \ldots, y_d) \in \mathbb{R}^{d-1}$ as
\[
y_k = \frac{\lambda_1^{1/2}}{v_1} \cdot \frac{u_k}{\lambda_k} J \delta ; \quad k = 2, \ldots, d.
\]

Note that $v_1 \neq 0$ by Lemma 8.1. Define a map $\Pi_\epsilon : \partial_+ B_\epsilon^\sigma \to \mathbb{R}^{d-1}$ that represents the change of coordinate as
\[
\Pi_\epsilon(x) = \widetilde{x} + y.
\]

Our careful selection of $y$ ensures that this map simplifies the computation of the crucial quadratic form.

**Lemma 8.7** For all $x \in \partial_+ B_\epsilon^\sigma$, we have
\[
x \cdot (H + \mu v \otimes v)x = \Pi_\epsilon(x) \cdot (\widetilde{H} + \mu \widetilde{v} \otimes \widetilde{v}) \Pi_\epsilon(x).
\]

**Proof** Fix $x = \left( \frac{J \delta}{\lambda_1^{1/2}}, x_2, \ldots, x_d \right) \in \partial_+ B_\epsilon^\sigma$ and write $\Pi_\epsilon(x) = y = (y_2, \ldots, y_d)$. Then, by Lemma 8.1, we can write
\[
x \cdot v = \frac{J \delta}{\lambda_1^{1/2}} v_1 + \sum_{k=2}^d (y_k - y_k) v_k = y \cdot \widetilde{v} + \frac{J \delta \lambda_1^{1/2}}{\mu v_1}.
\]
Thus, we can write $x \cdot (H + \mu v \otimes v)x$ as
\[
-\lambda_1 x_1^2 + \sum_{k=2}^d \lambda_k x_k^2 + \mu \left( y \cdot \widetilde{v} + \frac{J \delta \lambda_1^{1/2}}{\mu v_1} \right)^2 = y \cdot (\widetilde{H} + \mu \widetilde{v} \otimes \widetilde{v}) y.
\]
The correction vector $y$ is designed to clear the linear terms and constant term here. □

We can now show that the image of $\Pi_\epsilon(\partial_+ B_\epsilon^\sigma)$ is comparable with a ball centered at the origin with a radius of order $\delta$. □

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Lemma 8.8 There exist constants $r$, $R > 0$ such that
\[ D_{r\delta}^{d-1}(0) \subset \Pi_\epsilon(\partial_+ B^\sigma_\epsilon) \subset D_{R\delta}^{d-1}(0), \quad (8.15) \]
where $D_a^{d-1}(0)$ denotes a sphere on $\mathbb{R}^{d-1}$ centered at the origin with radius $a$.

Proof Since $\partial_+ B^\sigma_\epsilon \subset D_{C\delta}^{d-1}(0)$ for sufficiently large $C > 0$ and $|\gamma| = O(\delta)$, the existence of $R$ is immediate from the definition of $\Pi_\epsilon$.

Now we focus on the first inclusion of (8.15). For $y \in \mathbb{R}^{d-1}$ defined in (8.13), we write
\[ P_\delta = \left\{ x \in \mathbb{R}^d : x_1 = \frac{J\delta}{\lambda_1^{1/2}} \right\} \subset \mathbb{R}^d \quad \text{and} \quad \gamma = \left( \frac{J\delta}{\lambda_1^{1/2}}, -\gamma_2, \ldots, -\gamma_d \right) \in P_\delta. \]

Then, by the Taylor expansion and Lemma 8.1, we can check that
\[ U(\gamma) = H - \frac{\lambda_1}{2 \mu v_1^2} J^2 \delta^2 + O(\delta^3) < H - c_0 J^2 \delta^2 \quad (8.16) \]
for all sufficiently small $\epsilon > 0$, provided that we take $c_0$ to be sufficiently small. Therefore, there exists $r > 0$ such that $D_{r\delta}^{d-1}(0) \cap P_\delta \subset \partial_+ B^\sigma_\epsilon$. Since $\Pi_\epsilon(\gamma) = 0$, we have $D_{r\delta}^{d-1}(0) = \Pi_\epsilon(D_{r\delta}^{d-1}(\gamma) \cap P_\delta)$. This completes the proof. \(\square\)

Now, we present three auxiliary lemmas (Lemmas 8.9, 8.10, and 8.11) that will be used in several instances including the proof of Proposition 8.6. The proofs of these technical results are deferred to the next subsection.

Lemma 8.9 The matrix $\widetilde{H} + \mu \widetilde{v} \otimes \widetilde{v}$ is positive definite and
\[ \det(\tilde{H} + \mu \tilde{v} \otimes \tilde{v}) = \mu \frac{v_1^2}{\lambda_1} \prod_{k=2}^{d} \lambda_k. \]

Proof By (8.2) and Lemma 8.1,
\[ \det(\tilde{H} + \mu \tilde{v} \otimes \tilde{v}) = (1 + \mu \tilde{v}^T \tilde{H}^{-1} \tilde{v}) \det \tilde{H} = \frac{\mu v_1^2}{\lambda_1} \det \tilde{H}. \]
\(\square\)

Recall $\partial_+ C^\sigma_\epsilon$ from (8.4) and define, for $a > 0$,
\[ \partial_+^1 C^\sigma_\epsilon = \left\{ x \in \partial_+ C^\sigma_\epsilon : x \cdot v \geq a J\delta \right\}, \quad (8.17) \]
\[ \partial_+^2 C^\sigma_\epsilon = \left\{ x \in \partial_+ C^\sigma_\epsilon : U(x) \geq H + a J^2 \delta^2 \right\}. \quad (8.18) \]
Lemma 8.10 There exists $a_0 > 0$ such that, for all $a \in (0, a_0)$,

$$\partial_+^1 a C_\epsilon \cup \partial_+^2 a C_\epsilon = \partial_+ C_\epsilon .$$

Hereafter, the constant $a_0$ always refers to the one in the previous lemma. For $a > 0$, we write

$$\partial_+^1 a B_\epsilon^\sigma = \partial_+ B_\epsilon^\sigma \cap \partial_+^1 a C_\epsilon = \{ x \in \partial_+ B_\epsilon^\sigma : x \cdot v \geq a J \delta \},$$

$$\partial_+^2 a B_\epsilon^\sigma = \partial_+ B_\epsilon^\sigma \cap \partial_+^2 a C_\epsilon = \{ x \in \partial_+ B_\epsilon^\sigma : U(x) \geq H + a J^2 \delta^2 \};$$ (8.19)

hence, we have

$$\partial_+ B_\epsilon^\sigma = \partial_+^1 a B_\epsilon^\sigma \cup \partial_+^2 a B_\epsilon^\sigma$$ (8.21)

for all $a \in (0, a_0)$ by the previous lemma. Now, we introduce the last lemma.

Lemma 8.11 Let $\mathbb{D}$ be a positive-definite $(d - 1) \times (d - 1)$ matrix, Then, for all $u_1, u_2 \in \mathbb{R}^{d-1}$ and $c \in (0, 1)$, we have

$$\int_{\Pi_\epsilon(\partial_+ B_\epsilon^\sigma)} \frac{y \cdot u_2 + \delta}{y \cdot u_1 + \delta} e^{-1/(2\epsilon)} y \cdot \Pi_\epsilon(\partial_+ B_\epsilon^\sigma) d y = \left[ 1 + o_\epsilon(1) \right] \frac{(2\pi\epsilon)^{(d-1)/2}}{\sqrt{\det(\mathbb{D})}} .$$

Now, we are ready to prove Proposition 8.6.

Proof of Proposition 8.6 In view of the definition of $\Phi_{p_\epsilon}$ given in (6.3), we can write

$$\epsilon \int_{\partial_+ B_\epsilon^\sigma} \left[ \Phi_{p_\epsilon} - \frac{1}{\epsilon} \ell \right] \cdot e_1 \sigma (d \mu_\epsilon) = I_1 - I_2,$$ (8.22)

where

$$I_1 = \epsilon \int_{\partial_+ B_\epsilon^\sigma} \nabla p_\epsilon^\sigma(x) \cdot e_1 \sigma (d \mu_\epsilon) \quad \text{and} \quad I_2 = \int_{\partial_+ B_\epsilon^\sigma} (1 - p_\epsilon^\sigma) (\ell \cdot e_1) \sigma (d \mu_\epsilon) .$$

First, we compute $I_1$. By the explicit form of $p_\epsilon^\sigma$ and the Taylor expansion of $U$, we can write

$$I_1 = \left[ 1 + o_\epsilon(1) \right] v_1 \frac{\epsilon}{Z_\epsilon} \frac{\mu}{2\pi \epsilon} \int_{\partial_+ B_\epsilon^\sigma} e^{-\frac{1}{2\epsilon} x \cdot (H + \mu \hat{v} \otimes \hat{v}) x} \sigma (dx) .$$ (8.23)

By the change of variables $y = \Pi_\epsilon(x)$, the last integral can be expressed as

$$\int_{\Pi_\epsilon(\partial_+ B_\epsilon^\sigma)} e^{-\frac{1}{2\epsilon} y \cdot (H + \mu \hat{v} \otimes \hat{v}) y} dy = \left[ 1 + o_\epsilon(1) \right] \frac{(2\pi\epsilon)^{(d-1)/2}}{\sqrt{\det (H + \mu \hat{v} \otimes \hat{v})}} .$$
where the equality follows from the change of variables $z = \epsilon^{-1/2} y$ and Lemma 8.8. Summing up, we get

\[
I_1 = \left[1 + o_\epsilon(1)\right] \frac{v_1 \mu^{1/2} \alpha_\epsilon}{2\pi \sqrt{\det (\mathbb{H} + \mu \tilde{v} \otimes \tilde{v})}}.
\]  

(8.24)

Next, we consider $I_2$. Let us take $a \in (0, a_0)$, where $a_0$ is the constant in Lemma 8.10, and decompose

\[
I_2 = I_{2,1} + I_{2,2}.
\]  

(8.25)

where

\[
I_{2,1} = \int_{\partial_1^1, a B^\sigma_\epsilon} (1 - p^\sigma_\epsilon) (\ell \cdot e_1) \sigma(d\mu_\epsilon), \quad I_{2,2} = \int_{\partial_1^1, a B^\sigma_\epsilon \setminus \partial_1^1, a B^\sigma_\epsilon} (1 - p^\sigma_\epsilon) (\ell \cdot e_1) \sigma(d\mu_\epsilon).
\]

First, we compute $I_{2,1}$. Recall the elementary inequality

\[
\frac{b}{b^2 + 1} e^{-b^2/2} \leq \int_b^\infty e^{-t^2/2} dt \leq \frac{1}{b} e^{-b^2/2} \quad \text{for } b > 0.
\]  

(8.26)

Now, for $x \in \partial_1^1, a B^\sigma_\epsilon$, since we have $\sqrt{\frac{Z}{\epsilon}} (x \cdot v) \to \infty$ as $\epsilon \to 0$, we obtain from the definition of $p^\sigma_\epsilon$ and (8.26) that

\[
1 - p^\sigma_\epsilon(x) = \left[1 + o_\epsilon(1)\right] \frac{e^{1/2}}{(2\pi \mu)^{1/2} (x \cdot v)} \exp\left\{-\frac{\mu}{2\epsilon}(x \cdot v)^2\right\}.
\]  

(8.27)

By the Taylor expansion of $\ell$, we have

\[
\ell(x) \cdot e_1 = \mathbb{L} x \cdot e_1 + O(\delta^2).
\]  

(8.28)

Our plan is to insert (8.27) and (8.28) into $I_{2,1}$ to complete the proof. To this end, we first explain that we can ignore the $O(\delta^2)$ term in (8.28). By (8.27), the Taylor expansion of $U$, and Lemma 8.2, we have

\[
\left|\delta^2 \int_{\partial_1^1, a B^\sigma_\epsilon} (1 - p^\sigma_\epsilon) \sigma(d\mu_\epsilon)\right|
\]

\[
\leq C \delta e^{1/2} Z_\epsilon \int_{\partial_1^1, a B^\sigma_\epsilon} \exp\left\{-\frac{\mu}{2\epsilon} x \cdot (\mathbb{H} + \mu v \otimes v) x\right\} \sigma(dx)
\]

\[
\leq C \delta e^{1/2} Z_\epsilon \sigma(\partial_1^1, a B^\sigma_\epsilon) = C \delta e^{1/2} Z_\epsilon \frac{\mu}{2\epsilon} e^{-H/\epsilon} = o_\epsilon(1) \alpha_\epsilon.
\]  

(8.29)

Hence, by combining (8.27), (8.28), and (8.29), we can write

\[
I_{2,1} = o_\epsilon(1) \alpha_\epsilon + \left[1 + o_\epsilon(1)\right] \alpha_\epsilon \frac{1}{(2\pi \epsilon)^{(d+1)/2} \mu^{1/2}} \int_{\partial_1^1, a B^\sigma_\epsilon} e^{-\frac{1}{2\epsilon} x \cdot (\mathbb{H} + \mu v \otimes v) x} \frac{\mathbb{L} x \cdot e_1}{x \cdot v} \sigma(dx).
\]  

(8.30)
By the change of variables \( y = \Pi_\epsilon(x) \) and Lemma 8.7, we can write the last integral as

\[
\int_{\Pi(\partial_+ B_\epsilon) \cap \{y : y \cdot \tilde{\nu} \geq c' J \delta\}} e^{-\frac{1}{2}y \cdot \left[ \Xi + \mu \tilde{\nu} \otimes \tilde{\nu} \right] y} y \cdot \tilde{v} + \frac{\delta \lambda_1}{\mu v_1} d y \\
= (-\mu \Xi^{-1} \nu) \int_{\Pi(\partial_+ B_\epsilon) \cap \{y : y \cdot \tilde{\nu} \geq c' J \delta\}} e^{-\frac{1}{2}y \cdot \left[ \Xi + \mu \tilde{\nu} \otimes \tilde{\nu} \right] y} y \cdot \tilde{w} + \frac{\delta \lambda_1}{\mu v_1} d y
\]

for some \( w \in \mathbb{R}^{d-1} \) and \( c' = a - \frac{2}{\mu v_1} \). Take \( a \in (0, a_0) \) to be sufficiently small such that \( c' < 0 \) (which is possible by the statement of Lemma 8.10). Evaluating the last integral via Lemmas 8.8 and 8.11 and inserting the result into (8.30), we conclude that

\[
I_{2, 1} = \alpha_\epsilon(1) \alpha_\epsilon + [1 + \alpha_\epsilon(1)] \alpha_\epsilon \frac{\mu^{1/2} (-\Xi^{-1} \nu) \cdot e_1}{2\pi \sqrt{\det(\Xi + \mu \tilde{\nu} \otimes \tilde{\nu})}}.
\]  

Next, we consider \( I_{2, 2} \). By Lemma 8.10, we have \( \partial_+ B_\epsilon^\sigma \setminus \partial_+^{1, a} B_\epsilon^\sigma \subset \partial_+^{2, a} B_\epsilon^\sigma \); hence,

\[
|I_{2, 2}| \leq \frac{C}{Z_\epsilon} \int_{\partial_+^{1, a} B_\epsilon^\sigma} e^{-U(x)/\epsilon} \sigma(d x) \leq \frac{C}{Z_\epsilon} e^{-H/\epsilon} e^{-c J^2 \delta^2/\epsilon} \sigma(\partial_+ B_\epsilon^\sigma),
\]

where we applied trivial bounds\(^4\) for \(|1 - p_\sigma^\epsilon(x)| \) and \( \ell \) in the first inequality, while we used the condition \( U(x) \geq H + a J^2 \delta^2 \) for \( x \in \partial_+^{2, a} B_\epsilon \) in the second one. Since \( \sigma(\partial_+ B_\epsilon) = O(\delta^{d-1}) \), we get

\[
|I_{2, 2}| \leq \frac{C \delta^{d-1}}{Z_\epsilon} e^{c J^2/2} = o_\epsilon(1) \alpha_\epsilon
\]

for sufficiently large \( J \). Hence, \( I_{2, 2} \) is negligible. By combining (8.25), (8.31), and (8.33), we get

\[
I_2 = o_\epsilon(1) \alpha_\epsilon + [1 + o_\epsilon(1)] \alpha_\epsilon \frac{\mu^{1/2} (-\Xi^{-1} \nu) \cdot e_1}{2\pi \sqrt{\det(\Xi + \mu \tilde{\nu} \otimes \tilde{\nu})}}.
\]  

By (8.24) and (8.34), we obtain

\[
I_1 - I_2 = [1 + o_\epsilon(1)] \alpha_\epsilon \frac{\mu^{1/2} (\nu + \Xi^{-1} \nu) \cdot e_1}{2\pi \sqrt{\det(\Xi + \mu \tilde{\nu} \otimes \tilde{\nu})}}.
\]

\(^4\) Since \( \partial_+ B_\epsilon^\sigma \subset K \) where \( K \) is defined in (8.6) we can bound \( \ell \) by the \( L^\infty(K) \) norm of \( \ell \). This argument will be used repeatedly in the remainder of the article without further mention.
Since $H L = - L^\dagger H$ by the skew-symmetry of $H L$, we have $L H^{-1} = - H^{-1} L^\dagger$. Hence,

$$(v + L H^{-1} v) \cdot e_1 = (I - H^{-1} L^\dagger H) v \cdot e_1 = H^{-1} (H - L^\dagger) v \cdot e_1$$

$$= -\mu H^{-1} v \cdot e_1 = \frac{\mu}{\lambda_1} v \cdot e_1 = \frac{\mu v_1}{\lambda_1} \tag{8.36}$$

since $-\mu$ is an eigenvalue of $H - L^\dagger$ associated with the eigenvector $v$ and $H^{-1} = \text{diag}(-1/\lambda_1, 1/\lambda_2, \ldots, 1/\lambda_d)$. Inserting this computation and Lemma 8.9 into (8.35), we get

$$I_1 - I_2 = \left[ 1 + o(\epsilon) \right] \alpha_\epsilon \frac{\mu}{2\pi \sqrt{\prod_{k=1}^d \lambda_k}} = \left[ 1 + o(\epsilon) \right] \alpha_\epsilon \omega^\sigma .$$

This completes the proof. \hfill \Box

### 8.5 Proof of Lemmas 8.10 and 8.11

**Proof of Lemma 8.10** By Lemma 8.1, we have $\lambda_1 \sum_{k=2}^d \frac{v_k^2}{\lambda_k} < v_1^2$. Thus, there exists $\varepsilon_0 \in (0, v_1)$ such that

$$(\lambda_1 + \varepsilon_0) \sum_{k=2}^d \frac{v_k^2}{\lambda_k} < (v_1 - \varepsilon_0)^2 . \tag{8.37}$$

Let $a_0 = \varepsilon_0 \min\{1, \lambda_1^{-1/2}, \lambda_1^{-1}\}$, and we claim that this constant $a_0$ satisfies the requirement of the lemma.

Fix $a \in (0, a_0)$, $x \in \partial_+ C_\epsilon$ and suppose, on the other hand, that

$$x \cdot v < a J \delta \leq \varepsilon_0 \frac{J \delta}{\lambda_1^{1/2}} \quad \text{and} \quad U(x) - H < a J^2 \delta^2 \leq \varepsilon_0 \frac{J^2 \delta^2}{\lambda_1}. \tag{8.38}$$

Since $U(x) - H = \frac{1}{2} x \cdot H x + O(\delta^3)$ by the Taylor expansion, the latter condition implies that $x \cdot H x < \varepsilon_0 \frac{J^2 \delta^2}{\lambda_1}$ for all sufficiently small $\varepsilon > 0$.

Write $x \in \partial_+ C_\epsilon$ as $x = \frac{J \delta}{\lambda_1^{1/2}} \left( e_1 + \sum_{k=2}^d x_k e_k \right)$ such that we can rewrite the two conditions of (8.38) respectively as

$$0 < v_1 - \varepsilon_0 < -\sum_{k=2}^d v_k x_k \quad \text{and} \quad \sum_{k=2}^d \lambda_k x_k^2 < \lambda_1 + \varepsilon_0 .$$
By these two inequalities and (8.37), we have
\[
\sum_{j=2}^{d} \lambda_j x_j^2 \sum_{k=2}^{d} \frac{v_k^2}{\lambda_k} < (\lambda_1 + \varepsilon_0) \sum_{k=2}^{d} \frac{v_k^2}{\lambda_k} < (v_1 - \varepsilon_0)^2 < (\sum_{k=2}^{d} x_k v_k)^2,
\]
which contradicts the Cauchy–Schwarz inequality; hence, the claim is proven. □

**Proof of Lemma 8.11** Write \(\zeta = \zeta(\varepsilon) = \sqrt{\log \frac{1}{\varepsilon}}\) and let \(Q_\bullet = \Pi_{\varepsilon}(\partial_+ B_\varepsilon)\). Then, by the change of variables \(z = e^{-1/2} y\), we can write the integral in the statement of the lemma as
\[
e^{(d-1)/2} \int_{e^{-1/2} Q_\bullet \cap \{z \in \mathbb{R}^{d-1} : z \cdot u_1 \geq -c \zeta\}} \frac{z \cdot u_2 + \xi}{z \cdot u_1 + \xi} e^{-(1/2)z \cdot Dz} dz.
\]
Fix \(0 < \alpha < 1\). Then, since \(\zeta \to \infty\) as \(\varepsilon \to 0\), by Lemma 8.8,
\[
D_{\xi^\alpha}^{(d-1)}(0) \subset e^{-1/2} Q_\varepsilon \cap \{z \in \mathbb{R}^{d-1} : z \cdot u_1 \geq -c \zeta\}
\]
for all sufficiently small \(\varepsilon > 0\). Now we decompose the integral into
\[
\left[ \int_{D_{\xi^\alpha}^{(d-1)}(0)} + \int_{\{e^{-1/2} Q_\varepsilon \setminus D_{\xi^\alpha}^{(d-1)}(0)\} \cap \{z \in \mathbb{R}^{d-1} : z \cdot u_1 \geq -c \zeta\}} \right] \frac{z \cdot u_2 + \xi}{z \cdot u_1 + \xi} e^{-(1/2)z \cdot Dz} dz. \tag{8.39}
\]
Let us consider the first integral. Note that
\[
\sup_{z \in D_{\xi^\alpha}^{(d-1)}(0)} \left| \frac{z \cdot u_2 + \xi}{z \cdot u_1 + \xi} - 1 \right| = o_\varepsilon(1).
\]
Thus, the first integral is
\[
[ 1 + o_\varepsilon(1) ] \int_{D_{\xi^\alpha}^{(d-1)}(0)} e^{-(1/2)z \cdot Dz} dz = [ 1 + o_\varepsilon(1) ] \frac{(2\pi)^{(d-1)/2}}{\sqrt{\det(D)}}. \tag{8.40}
\]
since \(D_{\xi^\alpha}^{(d-1)}(0) \uparrow \mathbb{R}^{d-1}\) as \(\varepsilon \to 0\).

Now, we focus on the second integral. Since \(e^{-1/2} Q_\varepsilon \subset D_{R_\xi}^{(d-1)}(0)\) by Lemma 8.8, and since \(z \cdot u_1 \geq -c \zeta\) for \(c \in (0, 1)\) by the statement of the lemma, there exists \(C > 0\) such that
\[
\sup_{z \in e^{-1/2} Q_\varepsilon} \left| \frac{z \cdot u_2 + \xi}{z \cdot u_1 + \xi} \right| \leq C.
\]
Hence, the absolute value of the second integral in (8.39) is bounded from above by

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\[
C \int_{D_{r_\xi}^{(d-1)}(0) \setminus D_{r_\eta}^{(d-1)}(0)} e^{-(1/2)z \cdot Dz} \, dz = o_\epsilon(1). \tag{8.41}
\]
By combining (8.39), (8.40), and (8.41), we complete the proof. \qed

9 Analysis of equilibrium potential

In this section, we establish a bound on the equilibrium potential \( h_\epsilon \) and \( h^*_\epsilon \) in Proposition 9.1. On the basis of this bound, we prove Proposition 7.3 in Sect. 9.4. Further, we remark that this bound plays an important role in the proof of Theorem 7.4 (cf. Sect. 10.4).

For two disjoint non-empty sets \( A, B \subseteq \mathbb{R}^d \), let \( \Gamma_{A,B} \) be a set of all \( C^1 \)-paths \( \gamma : [0, 1] \to \mathbb{R}^d \) such that \( \gamma(0) \in A \) and \( \gamma(1) \in B \). Then, let \( \mathcal{H}_{A,B} \) denote the height of the saddle points between \( A \) and \( B \):

\[
\mathcal{H}_{A,B} := \inf_{\gamma \in \Gamma_{A,B}} \sup_{t \in [0, 1]} U(\gamma(t)).
\]

9.1 Estimates of equilibrium potentials \( h_\epsilon \) and \( h^*_\epsilon \)

In this subsection, we prove the following proposition regarding the so-called leveling property of the equilibrium potential.

**Proposition 9.1** We can find a constant \( C > 0 \) satisfying the following bounds.

1. For all \( y \in \mathcal{H}_0 \), the following holds:

\[
h_\epsilon(y), \ h^*_\epsilon(y) \geq 1 - C \epsilon^{-d} \exp \left( \frac{\mathcal{H}_{\{y\}, D_\epsilon(m_0) - H}{\epsilon} \right).
\]

2. For all \( y \in \mathcal{H}_1 \), the following holds:

\[
h_\epsilon(y), \ h^*_\epsilon(y) \leq C \epsilon^{-d} \exp \left( \frac{U(y) - H}{\epsilon} \right).
\]

The proof of Proposition 9.1 relies on the following two bounds on the capacity.

**Lemma 9.2** There exists \( C > 0 \) such that for all \( y \in \mathcal{W}_0 \) and \( m \in \mathcal{M}_0 \),

\[
cap_\epsilon(D_\epsilon(y), \ D_\epsilon(m)) \geq C \epsilon^d Z_\epsilon^{-1} e^{-\mathcal{H}_{\{y\}, D_\epsilon(m)}/\epsilon}.
\]

**Lemma 9.3** There exists \( C > 0 \) such that for all \( y \in \mathcal{H}_0 \),

\[
cap_\epsilon(D_\epsilon(y), \ U_\epsilon) \leq C Z_\epsilon^{-1} e^{-H/\epsilon}.
\]

We prove Lemmas 9.2 and 9.3 in Sects. 9.2 and 9.3, respectively. Now, we prove Proposition 9.1.
Proof of Proposition 9.1 Since the proofs for \( h_\epsilon \) and \( h_\epsilon^* \) are identical, we consider only \( h_\epsilon \). In [19, Proposition 7.9], it has been shown that there exists \( C > 0 \) such that

\[
h_{A, B}(x) \leq C \frac{\text{cap}_\epsilon(D_\epsilon(x), A)}{\text{cap}_\epsilon(D_\epsilon(x), B)},
\]

provided that \( A \) and \( B \) are disjoint domains of sufficiently smooth bounds. For part (1), we can use this bound to get

\[
1 - h_\epsilon(y) = h_{\mathcal{U}_\epsilon, D_\epsilon(m_0)}(y) \leq C \frac{\text{cap}_\epsilon(D_\epsilon(y), \mathcal{U}_\epsilon)}{\text{cap}_\epsilon(D_\epsilon(y), D_\epsilon(m_0))}.
\]

Now, by applying Lemmas 9.2 and 9.3, we complete the proof of part (1).

For part (2), we fix \( y \in \mathcal{H}_1 \). Then, again by (9.1),

\[
h_\epsilon(y) = h_{D_\epsilon(m_0), \mathcal{U}_\epsilon}(y) \leq C \frac{\text{cap}_\epsilon(D_\epsilon(y), D_\epsilon(m_0))}{\text{cap}_\epsilon(D_\epsilon(y), \mathcal{U}_\epsilon)}.
\]

By the same logic with the proofs of Lemmas 9.2 and 9.3, we get

\[
\text{cap}_\epsilon(D_\epsilon(y), D_\epsilon(m_0)) \leq \frac{Ce^{H/e}e^{-H/e}}{Z_\epsilon} \quad \text{and} \quad \text{cap}_\epsilon(D_\epsilon(y), \mathcal{U}_\epsilon) \geq \frac{Ce^d}{Z_\epsilon} e^{-\mathcal{H}_1(y), \mathcal{U}_\epsilon}/e.
\]

Since \( \mathcal{U}_\epsilon \) contains all the local minima of \( \mathcal{M}_1 \) and \( \mathcal{H}_1 \) is a subset of the domain of attraction of \( \mathcal{M}_1 \), we have \( \mathcal{H}_1(y), \mathcal{U}_\epsilon = U(y) \) and the proof is completed. \( \square \)

9.2 Proof of Lemma 9.2

For the lower bound case, the proof is a consequence of the existing estimate for the reversible case. Let \( \text{cap}_\epsilon^r(\cdot, \cdot) \) denote the capacity with respect to the reversible process \( z_\epsilon(\cdot) \) given in (1.1), whose generator is \( (1/2)(\mathcal{L}_\epsilon + \mathcal{L}_\epsilon^*) \). Then, it is well known that (cf. [12, Lemma 2.5]) for any two disjoint non-empty domains \( A, B \subset \mathbb{R}^d \) with smooth boundaries, we have the following equation:

\[
\text{cap}_\epsilon(A, B) \geq \text{cap}_\epsilon^r(A, B).
\]

Therefore, it suffices to show the inequality for \( \text{cap}_\epsilon^r(D_\epsilon(y), D_\epsilon(m)) \), instead. The lower bound for this capacity can be obtained by optimizing the integration on the tube connecting \( D_\epsilon(y) \) and \( D_\epsilon(m) \). This is rigorously achieved by a parametrization of this tube. When we parametrize the tube successfully, we can use the idea of [5, Proposition 4.7] to complete the proof.

Let \( \omega : [0, L] \to \mathbb{R}^d \) be a smooth path such that \( |\dot{\omega}(t)| = 1 \) for all \( t \in [0, L] \). For \( r > 0 \), define \( A_r(0), A_r(L) \) by

\[
A_r(0) = \{ x \in \mathbb{R}^d : x \cdot \hat{\omega}(0) < 0, |x - \omega(0)| < r \}
\]

\[
A_r(L) = \{ x \in \mathbb{R}^d : x \cdot \hat{\omega}(L) > 0, |x - \omega(L)| < r \}
\]

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and define the tubular neighborhood of $\omega$ of radius $r$ by

$$\omega_r = \{ x \in \mathbb{R}^d : |x - \omega(t)| < r \text{ for some } t \in [0, L] \} \setminus (A_r(0) \cup A_r(L)).$$

For $\rho > 0$, let $\mathcal{D}_\rho^{(d-1)}$ be a $(d - 1)$-dimensional sphere of radius $\rho$ centered at the origin.

**Lemma 9.4** There exists $r_0 > 0$ such that $[0, L] \times \mathcal{D}_r^{(d-1)}$ is diffeomorphic to $\omega_{r_0}$. Furthermore, we can find a diffeomorphism $\varphi : [0, L] \times \mathcal{D}_r^{(d-1)} \rightarrow \omega_{r_0}$ of the form

$$\varphi(t, z) = \omega(t) + \mathbb{A}(t) z$$

for some smooth $d \times (d - 1)$ matrix-valued function $\mathbb{A}(\cdot)$ of rank $d - 1$, and it satisfies

$$\left| \det \frac{\partial \varphi}{\partial(t, z)} \right| \geq \frac{1}{2} \text{ on } [0, L] \times \mathcal{D}_r^{(d-1)}.$$  

**Proof** The proof needs to recall several notions and results from differential geometry. We refer to [25] for a reference. We regard $\omega = \omega([0, L])$ as a one-dimensional compact manifold. Let $N\omega \subset \mathbb{R}^d \times \mathbb{R}^d$ denote the normal bundle of $\omega$. By the tubular neighborhood theorem (cf. [25, Theorem 6.24]), there exists $r_0 > 0$ such that $\omega_{r_0}$ is diffeomorphic to $N\omega_{r_0} = \{ (p, v) \in N\omega : |v| < r_0 \}$. The diffeomorphism $E : N\omega_{r_0} \rightarrow \omega_{r_0}$ is given by $E(p, v) = p + v$. Since $\omega$ is contractible, the vector bundle of $\omega$ is trivial; thus, $N\omega$ is diffeomorphic to $\omega \times \mathbb{R}^{d-1}$. Let $\varphi : \omega \times \mathbb{R}^{d-1} \rightarrow N\omega$ denote the corresponding diffeomorphism. Since this diffeomorphism preserves the vector space structure, the function $\varphi(p, z)$ is linear in $z$ and satisfies $|\pi_2(\varphi(p, z))| = |z|$, where $\pi_2 : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is the projection function for the second coordinate.

Since $\omega \times \mathbb{R}^{d-1}$ is a trivial bundle of rank $d - 1$, there are $d - 1$ smooth sections $\sigma_j : \omega \rightarrow \mathbb{R}^{d-1}$ which are linearly independent. By the Gram–Schmidt operation, we may assume that they are pointwise orthonormal, i.e., $\sigma_i(p) \cdot \sigma_j(p) = \delta_{i,j}$ for all $i, j$ and $p \in \omega$. Define a $d \times (d - 1)$ matrix $\mathbb{B}(p) = [\mathbb{B}_1(p), \ldots, \mathbb{B}_{d-1}(p)]$ by $\mathbb{B}_j(p) = \pi_2(\varphi(p, \sigma_j(p)))$ for $j = 1, \ldots, d - 1$. By the smoothness of $\varphi$ and $\sigma_j$, we can observe that all the elements of $\mathbb{B}(\cdot)$ are smooth. Then, the diffeomorphism $\varphi : [0, L] \times \mathcal{D}_r^{(d-1)} \rightarrow \omega_{r_0}$ can be written as

$$\varphi(t, z) = \varphi(\omega(t), z) = \omega(t) + \mathbb{B}(\omega(t)) z.$$  

We can now take $\mathbb{A} = \mathbb{B} \circ \omega$ to get (9.3). Now we consider (9.4). We can write

$$\frac{\partial \varphi}{\partial(t, z)}(t, 0) = [\dot{\omega}(t), \dot{\mathbb{A}}(t)].$$

Since all the column vectors in the matrix on the right-hand sides are normal and orthogonal to each other, we have $|\det \frac{\partial \varphi}{\partial(t, z)}(t, 0)| = 1$. Hence, by taking $r_0$ to be sufficiently small, we get (9.4).
Proposition 9.5 Let \( \omega : [0, L] \to \mathbb{R}^d \) be a \( C^1 \)-path connecting \( y \) and \( m \) such that \( U(\omega(t)) \leq M \) and \( |\dot{\omega}(t)| = 1 \) for all \( t \). Moreover, let \( f \) be a smooth function such that \( f \equiv 1 \) on \( D_\epsilon(y) \) and \( f \equiv 0 \) on \( D_\epsilon(m) \). Then, there exists a constant \( C > 0 \) such that

\[
\epsilon \int_{\omega_0} |\nabla f|^2 \, d\mu_\epsilon \geq C L^{-1} \epsilon^d Z_\epsilon^{-1} e^{-M/\epsilon},
\]

where \( r_0 \) is the constant obtained in Lemma 9.4 for the path \( \omega \).

**Proof** By Lemma 9.4, we have

\[
\epsilon \int_{\omega_0} |\nabla f|^2 \, d\mu_\epsilon \geq \frac{\epsilon}{2Z_\epsilon} \int_{D_\epsilon^{(d-1)}} \int_0^L |\nabla f (\omega(t) + \hat{A}(t)z) |^2 e^{-U(\omega(t) + \hat{A}(t)z)/\epsilon} \, dt \, dz
\]

for \( \epsilon \in (0, r_0) \), where the factor of 2 appears because (9.4) is used for bounding the Jacobian of the change of variables from below. For \((t, z) \in [0, L] \times D_\epsilon^{(d-1)}\), we have

\[
\frac{d}{dt} f(\omega(t) + \hat{A}(t)z) = \nabla f(\omega(t) + \hat{A}(t)z) \cdot (\dot{\omega}(t) + \dot{\hat{A}}(t)z) \leq 2|\nabla f(\omega(t) + \hat{A}(t)z)|,
\]

where the last inequality holds for sufficiently small \( \epsilon \) since \( |\dot{\omega}(t)| = 1 \) and \( |z| \leq \epsilon \). Summing up, we can write

\[
\epsilon \int_{\omega_0} |\nabla f|^2 \, d\mu_\epsilon \geq \frac{\epsilon}{4Z_\epsilon} \int_{D_\epsilon^{(d-1)}} \int_0^L \left| \frac{d}{dt} f(\omega(t) + \hat{A}(t)z) \right|^2 e^{-U(\omega(t) + \hat{A}(t)z)/\epsilon} \, dt \, dz.
\]

Now, we can apply the idea of [5, Proposition 4.7]. Indeed, we can fix \( z \in D_\epsilon^{(d-1)} \) and write \( f_z(t) = f(\omega(t) + \hat{A}(t)z) \). Then, we can obtain the minimizer of the integral

\[
\int_0^L \left| \frac{d}{dt} f_z(t) \right|^2 e^{-U(\omega(t) + \hat{A}(t)z)/\epsilon} \, dt
\]

explicitly as

\[
f_z(t) = \int_0^L e^{U(\omega(s) + \hat{A}(s)z)/\epsilon} \, ds \int_0^L e^{U(\omega(s) + \hat{A}(s)z)/\epsilon} \, ds.
\]

Inserting this solution into (9.5) gives

\[
\epsilon \int_{\omega_0} |\nabla f|^2 \, d\mu_\epsilon \geq \frac{\epsilon}{4Z_\epsilon} \int_{D_\epsilon^{(d-1)}} \left[ \int_0^L e^{U(\omega(t) + \hat{A}(t)z)/\epsilon} \, dt \right]^{-1} \, dz.
\]

Since \( |z| \leq \epsilon \), we have \( U(\omega(t) + \hat{A}(t)z) \leq M + C \epsilon \) for some constant \( C > 0 \), and the proof is completed. \( \square \)

Now, we are ready to prove Lemma 9.2.
Proof of Lemma 9.2} Fix \( y \in \mathcal{H}_0 \) and for some \( L = L(y) \) be a \( C^1 \)-path connecting \( y \) to \( \mathcal{D}_\epsilon(m) \) such that \( U(\omega(t)) \leq \mathcal{H}_0[y], \mathcal{D}_\epsilon(m) \) and \( |\dot{\omega}(t)| = 1 \) for all \( t \in [0, L] \). Since \( \mathcal{H}_0 \) is bounded, we can find \( L_0 \) such that \( L(y) < L_0 \) for all \( y \in \mathcal{H}_0 \). Then, recall the diffeomorphism \( \phi: [0,L]\times D_{d-1}\rightarrow \omega_0 \) constructed in Lemma 9.4. Then, \( \cap_{\epsilon} (\mathcal{D}_\epsilon(y), \mathcal{D}_\epsilon(m)) \geq \epsilon \int_{\omega_0} |\nabla h_\epsilon|_{\mathcal{D}_\epsilon(y), \mathcal{D}_\epsilon(m)}^2 d\mu_\epsilon \), where \( h_\epsilon(\cdot) \) is the equilibrium potential between \( \mathcal{D}_\epsilon(y) \) and \( \mathcal{D}_\epsilon(m) \) with respect to the reversible process \( y_\epsilon(\cdot) \). Hence, by Proposition 9.5 and the fact that we can take \( L(y) \) to be uniformly bounded by \( L_0 \), the proof is completed. \( \square \)

9.3 Proof of Lemma 9.3

The upper bound cannot be proven by a comparison with reversible dynamics as in the lower bound case unless the dynamics satisfies the so-called sector condition, and that is exactly what has been used in [19]. However, the dynamics \( x_\epsilon(\cdot) \) does not necessarily satisfy the sector condition; hence, we must develop a new argument. We believe that our argument presented below is sufficiently robust to treat a wide class of models.

Proof of Lemma 9.3} For each set \( A \subset \mathbb{R}^d \) and \( r > 0 \), define
\[
A^{[r]} = \{ x \in \mathbb{R}^d : |x - y| \leq r \text{ for some } y \in A \}. \tag{9.6}
\]
Suppose that \( \epsilon \) is sufficiently small such that \( \mathcal{H}_0^{[2\epsilon]} \) is disjoint from \( \mathcal{U}_\epsilon \) and \( \mathcal{H}_0^{[2\epsilon]} \subset \mathcal{K} \) (cf. (8.6)). Take a smooth function \( q_\epsilon: \mathbb{R}^d \rightarrow \mathbb{R} \) such that, for some constant \( C > 0 \),
\[
q_\epsilon \equiv 1 \text{ on } \mathcal{H}_0^{[\epsilon]}, \quad q_\epsilon \equiv 0 \text{ on } \mathbb{R}^d \setminus \mathcal{H}_0^{[2\epsilon]}, \quad \text{and } |\nabla q_\epsilon| \leq \frac{C}{\epsilon} 1_{\mathcal{H}_0^{[2\epsilon]} \setminus \mathcal{H}_0^{[\epsilon]}}. \tag{9.7}
\]
Since \( q_\epsilon \in C_{\mathcal{D}_\epsilon(y), \mathcal{U}_\epsilon} \) (cf. (6.4)), we can deduce from Proposition 6.2 that
\[
\cap_\epsilon (\mathcal{D}_\epsilon(y), \mathcal{U}_\epsilon) = \epsilon \int_{\Omega_\epsilon} \left[ \nabla q_\epsilon \cdot \nabla h_\epsilon + \frac{1}{\epsilon} q_\epsilon \ell \cdot \nabla h_\epsilon \right] d\mu_\epsilon. \tag{9.8}
\]
By the divergence theorem and (1.8), the second term on the right-hand side can be rewritten as
\[
\int_{\partial \Omega_\epsilon} h_\epsilon q_\epsilon [\ell \cdot n_{\Omega_\epsilon}] \sigma(d\mu_\epsilon) - \int_{\Omega_\epsilon} h_\epsilon [\nabla q_\epsilon \cdot \ell] d\mu_\epsilon. \tag{9.9}
\]
Since $h_\epsilon = 1_{\partial D_\epsilon}$ on $\partial \Omega_\epsilon = \partial U_\epsilon \cup \partial D_\epsilon(y)$, $q_\epsilon \equiv 1$ on $\partial D_\epsilon(y)$, and $n_{\Omega_\epsilon} = -n_{D_\epsilon(y)}$, the first integral of (9.9) becomes

$$- \int_{\partial D_\epsilon(y)} [\ell \cdot n_{D_\epsilon(y)}] \sigma(d\mu_\epsilon) = \int_{D_\epsilon(y)} (\nabla \cdot \ell) d\mu_\epsilon + \int_{D_\epsilon(y)} [\ell \cdot \nabla \mu_\epsilon](x) dx$$

(9.10)

by the divergence theorem again. Note that the last two integrals are 0 by (1.8) and (1.7), respectively. Hence the first integral of (9.9) vanishes. For the second integral of (9.9), by the trivial bound $|h_\epsilon| \leq 1$ and the last condition of (9.7), we have

$$\left| \int_{\Omega_\epsilon} h_\epsilon [\nabla q_\epsilon \cdot \ell] d\mu_\epsilon \right| \leq \frac{C}{\epsilon Z_\epsilon} \int_{H_0^{[2\epsilon]} \setminus H_0^{[\epsilon]}} e^{-U(x)/\epsilon} dx \leq \frac{C}{Z_\epsilon} e^{-H/\epsilon},$$

(9.11)

where the second inequality follows from the fact that $U(x) = H + O(\epsilon)$ on $H_0^{[2\epsilon]} \setminus H_0^{[\epsilon]}$ and that $\text{vol}(H_0^{[2\epsilon]} \setminus H_0^{[\epsilon]}) = O(\epsilon)$. Summing up, we obtain from (9.8) that

$$\text{cap}_\epsilon(D_\epsilon(y), U_\epsilon) \leq \epsilon \int_{\Omega_\epsilon} [\nabla q_\epsilon \cdot \nabla h_\epsilon] d\mu_\epsilon + \frac{C}{Z_\epsilon} e^{-H/\epsilon}.$$  

(9.12)

By the Cauchy–Schwarz inequality and part (2) of Lemma 6.1, the integral on the right-hand side is bounded from above by the square root of

$$\epsilon \int_{\Omega_\epsilon} |\nabla q_\epsilon|^2 d\mu_\epsilon \times \text{cap}_\epsilon(D_\epsilon(y), U_\epsilon).$$

By a computation similar to (9.11), we get

$$\epsilon \int_{\Omega_\epsilon} |\nabla q_\epsilon|^2 d\mu_\epsilon \leq \frac{C}{\epsilon Z_\epsilon} \int_{H_0^{[2\epsilon]} \setminus H_0^{[\epsilon]}} e^{-U(x)/\epsilon} dx \leq \frac{C}{Z_\epsilon} e^{-H/\epsilon}.$$

Therefore, we can bound the integral on the right-hand side of (9.12) by

$$\left[ \frac{C}{Z_\epsilon} e^{-H/\epsilon} \text{cap}_\epsilon(D_\epsilon(y), U_\epsilon) \right]^{1/2} \leq \frac{1}{2} \left[ \frac{C}{Z_\epsilon} e^{-H/\epsilon} + \text{cap}_\epsilon(D_\epsilon(y), U_\epsilon) \right].$$

Inserting this into (9.12) completes the proof.

\[ \square \]

### 9.4 Proof of Proposition 7.3

Now, we are ready to prove Proposition 7.3, which is a crucial step in the proof of the Eyring–Kramers formula.

**Proof of Proposition 7.3** Take $\beta > 0$ to be sufficiently small such that there is no critical point $c$ of $U$ such that $U(c) \in [H - \beta, H)$. Then, we can decompose $G = \{x : U(x) <$
\( H - \beta \) into \( \mathcal{G}_0, \mathcal{G}_1 \), where \( \mathcal{G}_0 \subset \mathcal{H}_0 \) and \( \mathcal{G}_1 \subset \mathcal{H}_1 \). Write

\[
\int_{\mathbb{R}^d} h_\epsilon^* \, d\mu_\epsilon = \left[ \int_{\mathcal{G}_0} + \int_{\mathcal{G}_1} + \int_{\mathcal{G}_c^c} \right] h_\epsilon^* \, d\mu_\epsilon \tag{9.13}
\]

and consider the three integrals separately. First, for \( y \in \mathcal{G}_0 \), we have \( \mathcal{H}\{y\} \mathcal{D}_\epsilon(m_0) < H - \beta \); thus, by part (1) of Proposition 9.1, we have \( |h_\epsilon^*(y) - 1| \leq C e^{-d} e^{-\beta/\epsilon} = o_\epsilon(1) \). This bound ensures that

\[
\int_{\mathcal{G}_0} h_\epsilon^* \, d\mu_\epsilon = [1 + o_\epsilon(1)] \mu_\epsilon(\mathcal{G}_0) = [1 + o_\epsilon(1)] Z_\epsilon^{-1} (2\pi \epsilon)^{d/2} e^{-h_0/\epsilon} v_0, \tag{9.14}
\]

where the second identity follows from the Laplace asymptotics for the function \( e^{-U/\epsilon} \).

For the second integral, by part (2) of Proposition 9.1,

\[
\int_{\mathcal{G}_1} h_\epsilon^* \, d\mu_\epsilon \leq \frac{C}{Z_\epsilon^d} \int_{\mathcal{G}_1} e^{U(x) - H/\epsilon} e^{-U(x)/\epsilon} \, dx = o_\epsilon(1) Z_\epsilon^{-1} (2\pi \epsilon)^{d/2} e^{-h_0/\epsilon} v_0, \tag{9.15}
\]

where the last line follows from \( H > h_0 \). Finally, for the last integral, by the bound \( |h_\epsilon^*| \leq 1 \) and (2.4),

\[
\int_{\mathcal{G}_c^c} h_\epsilon^* \, d\mu_\epsilon \leq \mu_\epsilon(\mathcal{G}_c^c) \leq Z_\epsilon^{-1} e^{-(H-\beta)/\epsilon} = o_\epsilon(1) Z_\epsilon^{-1} (2\pi \epsilon)^{d/2} e^{-h_0/\epsilon} v_0. \tag{9.16}
\]

By inserting (9.14), (9.15), and (9.16) into (9.13), the proof is completed. \( \square \)

### 10 Construction of test function and Proof of Theorem 7.4

In this section, we finally construct the test function \( g_\epsilon \in \mathcal{C}_\mathcal{D}_\epsilon(m_0), \mathcal{U}_\epsilon \) satisfying Theorem 7.4.

#### 10.1 Construction of \( g_\epsilon \) and proof of Theorem 7.4

Recall \( \mathcal{H}_0^\epsilon \) and \( p_\epsilon^\sigma \) from Sect. 8.1 and (8.8), respectively, and define \( f_\epsilon : \mathbb{R}^d \to \mathbb{R} \) as

\[
f_\epsilon(x) = \begin{cases} p_\epsilon^\sigma(x) & x \in \mathcal{B}_\epsilon^\sigma \text{ for some } \sigma \in \Sigma_0, \\ 1_{\mathcal{H}_0^\epsilon}(x) & \text{otherwise}. \end{cases}
\]

The function \( f_\epsilon \) is not continuous on \( \mathcal{K}_\epsilon \) in general; instead, it is discontinuous along the boundaries \( \partial_{\pm} \mathcal{B}_\epsilon^\sigma \) and \( \partial \mathcal{K}_\epsilon \).

**Remark 10.1** It can be readily checked that the function \( f_\epsilon \) is continuous on \( \mathcal{K}_\epsilon \) if we consider the reversible case, i.e., \( \ell \equiv 0 \).
For convenience, we formally define $\nabla f_\epsilon(x)$ as

$$\nabla f_\epsilon(x) = \begin{cases} \nabla p_\sigma^\epsilon(x) & x \in B_\sigma^\epsilon \text{ for some } \sigma \in \Sigma_0, \\ 0 & \text{otherwise}. \end{cases} \quad (10.1)$$

Note that this is not a weak derivative of $f_\epsilon$; hence, elementary theorems such as the divergence theorem cannot be applied to this gradient. With this formal gradient, we can define $\Phi_1 f_\epsilon$ formally as

$$\Phi_1 f_\epsilon(x) = \nabla f_\epsilon(x) + \frac{1}{\epsilon} f_\epsilon(x) \ell(x) = \begin{cases} \epsilon^{-1} \ell(x) & x \in H_0^\epsilon, \\ \Phi_\sigma f_\epsilon(x) & x \in B_\sigma^\epsilon \text{ for some } \sigma \in \Sigma_0, \\ 0 & \text{otherwise}. \end{cases}$$

Note that this is a formal definition, and Proposition 6.2 is not applicable to $\Phi_1 f_\epsilon$.

Now, we mollify the function $f_\epsilon$ as in [19] to get the genuine test function $g_\epsilon$. This end, consider a smooth, positive, and symmetric function $\phi : \mathbb{R}^d \to \mathbb{R}$ that is supported on the unit sphere of $\mathbb{R}^d$ and satisfies $\int_{\mathbb{R}^d} \phi(x) \, dx = 1$. Then, for $r > 0$, define $\phi_r(x) = r^{-d} \phi(r^{-1} x)$. For the function $f : \mathbb{R}^d \to \mathbb{R}$ and vector field $V : \mathbb{R}^d \to \mathbb{R}^d$, we write

$$f^{(r)} = f \ast \phi_r \text{ and } V^{(r)} = V \ast \phi_r,$$

where $\ast$ represents the usual convolution.

**Proposition 10.2** We have

$$\epsilon \int_{\mathbb{R}^d} | \Phi_{f_\epsilon^{(\eta)}} - \Phi_{f_\epsilon} |^2 d\mu_\epsilon = o_\epsilon(1) \alpha_\epsilon.$$ 

Next, we prove the following estimate.

**Proposition 10.3** We have

$$\epsilon \int_{\mathbb{R}^d} [ \Phi_{f_\epsilon} \cdot \nabla h_\epsilon ] d\mu_\epsilon = [ 1 + o_\epsilon(1) ] \alpha_\epsilon \omega_0.$$ 

Before proving these propositions, we explain why Theorem 7.4 is a consequence of these propositions. We define the test function $g_\epsilon$ explicitly as

$$g_\epsilon = f_\epsilon^{(\eta)} \quad \text{where } \eta = \epsilon^2. \quad (10.2)$$

**Proof of Theorem 7.4** By Proposition 10.3, it suffices to prove that

$$\epsilon \int_{\mathbb{R}^d} [ (\Phi_{g_\epsilon} - \Phi_{f_\epsilon} \cdot \nabla h_\epsilon ] d\mu_\epsilon = o_\epsilon(1) [ \alpha_\epsilon \text{ cap}_\epsilon ]^{1/2}.$$
With the selection (10.2), this is immediate from the Cauchy–Schwarz inequality, Lemma 6.1, and Proposition 10.2.

In Sects. 10.2 and 10.3, we shall prove Propositions 10.2 and 10.3, respectively. We remark that the proof of Proposition 10.2 is nearly model-independent and is similar to the proof of [19, Lemma 6.4]. Hence, we explain the structure of the proof and refer to [19] for most of the details. Of course, there are several differences in the proofs, and we present the full details for such parts.

10.2 Proof of Proposition 10.2

By the Cauchy–Schwarz inequality, we can write

$$
\epsilon \int_{\mathbb{R}^d} |\Phi_{f_{\epsilon}^{(n)}} - \Phi_{f_{\epsilon}}|^2 d\mu_{\epsilon} \leq 3 (I_1 + I_2 + I_3),
$$

where

$$
I_1 = \epsilon \int_{\mathbb{R}^d} |\nabla (f_{\epsilon}^{(n)})(x) - (\nabla f_{\epsilon}^{(n)})(x)|^2 d\mu_{\epsilon}, \quad I_2 = \epsilon \int_{\mathbb{R}^d} |(\nabla f_{\epsilon}^{(n)})(x) - \nabla f_{\epsilon}(x)|^2 d\mu_{\epsilon}, \quad I_3 = \frac{1}{\epsilon} \int_{\mathbb{R}^d} (f_{\epsilon}^{(n)} - f_{\epsilon})^2 |\ell|^2 d\mu_{\epsilon}.
$$

To conclude the proof of Proposition 10.2, it suffices to prove that $I_1$, $I_2$, $I_3 = o_\epsilon(1) \alpha_\epsilon$. The proofs of $I_1 = o_\epsilon(1) \alpha_\epsilon$ and $I_2 = o_\epsilon(1) \alpha_\epsilon$ are identical to those of [19, Lemma 8.5] and [19, Assertions 8.C and 8.D], respectively. The term $I_3$ has not been investigated previously. We present the proof of $I_3 = o_\epsilon(1) \alpha_\epsilon$. Note that the functions $f_{\epsilon}^{(n)}$ and $f_{\epsilon}$ are supported on $K$ for sufficiently small $\epsilon > 0$, and since $|\ell|$ is bounded on $K$, it suffices to prove the following lemma.

**Lemma 10.4** We have

$$
\frac{1}{\epsilon} \int_{\mathbb{R}^d} (f_{\epsilon}^{(n)} - f_{\epsilon})^2 d\mu_{\epsilon} = o_\epsilon(1) \alpha_\epsilon. \quad (10.3)
$$

**Proof** Recall the notation $A^{[r]}$ from (9.6) and define

$$
\overline{B}_{\epsilon}^\sigma = B_{\epsilon}^\sigma \setminus (\partial B_{\epsilon}^\sigma)^{[n]} \quad \text{and} \quad \overline{\mathcal{T}}_{i}^\epsilon = \mathcal{T}_{i}^\epsilon \setminus \left[ (\partial K_{\epsilon})^{[n]} \cup \left( \sigma \in \Sigma_0 (\partial B_{\epsilon}^\sigma)^{[n]} \right) \right] ; \ i = 1, 2.
$$

By the Cauchy–Schwarz inequality, we have

$$
\left[ (f_{\epsilon}^{(n)} - f_{\epsilon})(x) \right]^2 \leq \left( \int_{\mathbb{R}^d} (f_{\epsilon}(x) - f_{\epsilon}(x - y)) \phi(y) d y \right)^2 \leq \int_{\mathbb{R}^d} (f_{\epsilon}(x) - f_{\epsilon}(x - y))^2 \phi(y) d y.
$$
Since
\[ f_\epsilon(x) = f_\epsilon(x - y) \quad \text{if } x \not\in K_\epsilon^{[\eta]} \text{ and } |y| \leq \eta, \quad (10.4) \]
the left-hand side of (10.3) is bounded from above by
\[ \int_{K_\epsilon^{[\eta]}} \int_{\mathbb{R}^d} \frac{1}{\epsilon} |f_\epsilon(x) - f_\epsilon(x - y)|^2 \phi_\eta(y) \, dy \, \mu_\epsilon(dx). \]

Now, we divide the integral \( \int_{K_\epsilon^{[\eta]}} \) in the previous case into

\[ \int_{\tilde{H}_0} + \int_{\tilde{H}_1} + \int_{(\partial K_\epsilon)^{[\eta]}} + \sum_{\sigma \in \Sigma_0} \int_{\tilde{B}_\sigma^{[\eta]}} + \sum_{\sigma \in \Sigma_0} \int_{(\partial B_\sigma^{[\eta]} \setminus (\partial K_\epsilon)^{[\eta]})} \quad (10.5) \]
and consider the five integrals separately.

The first two integrals are 0 for the same reason with regard to (10.4). Now, we consider the third one. Since \( |f_\epsilon(x) - f_\epsilon(x - y)| \leq 1 \) for all \( x, y \in \mathbb{R}^d \), the integral is bounded from above by
\[ \int_{(\partial K_\epsilon)^{[\eta]}} \int_{\mathbb{R}^d} \frac{1}{\epsilon} \phi_\eta(y) \, dy \, \mu_\epsilon(dx) = \frac{1}{\epsilon} \mu_\epsilon((\partial K_\epsilon)^{[\eta]}). \quad (10.6) \]

Since \( U(y) = H + J^2 \delta^2 \) for \( y \in \partial K_\epsilon \), there exists \( C > 0 \) such that
\[ U(x) \geq H + J^2 \delta^2 - C \eta \quad \text{for all } x \in (\partial K_\epsilon)^{[\eta]} \].

Hence, the right-hand side of (10.6) is bounded by
\[ \frac{C}{\epsilon Z_\epsilon} e^{-H/\epsilon} \int_{(\partial K_\epsilon)^{[\eta]}} \epsilon^{J^2} e^{C\eta/\epsilon} \, dx \leq C \epsilon^{J^2 - d/2 - 1} \alpha_\epsilon vol((\partial K_\epsilon)^{[\eta]}) = o_\epsilon(1) \alpha_\epsilon \]
for sufficiently large \( J \), since vol \((\partial K_\epsilon)^{[\eta]}\) = \(O(1)\).

Next, we consider the fourth term in (10.5). Fix \( \sigma \in \Sigma_0 \) and assume, for simplicity of notation, that \( \sigma = 0 \). By the mean value theorem, for \( x \in \tilde{B}_\sigma^{[\eta]} \) and \( y \in D_\eta(0) \),
\[ |f_\epsilon(x) - f_\epsilon(x - y)| \leq |y| \sum_{k=1}^d \sup_{z \in D_\eta(x)} |\nabla_k f_\epsilon(z)|. \quad (10.7) \]

First, we remark from the expression (10.1) that, for \( u \in \tilde{B}_\sigma^{[\eta]} \),
\[ \nabla_k f_\epsilon(u) = \frac{1}{c_\epsilon} \exp \left\{ -\frac{\mu}{\epsilon} (u \cdot v_\sigma)^2 \right\} v_k. \quad (10.8) \]

Since \( \eta \ll \delta \) and \( |x| = O(\delta) \), we have
\[ (z \cdot v_\sigma)^2 \geq (x \cdot v_\sigma)^2 - C \eta \delta \quad \text{for } x \in \tilde{B}_\sigma^{[\eta]} \text{ and } z \in D_\eta(x). \quad (10.9) \]
By combining (10.8) and (10.9), we get

$$|\nabla_k f_\epsilon(z)|^2 \leq \frac{C}{\epsilon} \exp \left\{ -\frac{\mu}{\epsilon} (x \cdot \nu^\epsilon)^2 \right\}.$$  

Inserting this into (10.7), we obtain, for $x \in \bar{B}_\epsilon^\sigma$,

$$\int_{\mathbb{R}^d} |f_\epsilon(x) - f_\epsilon(x - y)|^2 \phi_\eta(y) dy \leq \frac{C \eta^2}{\epsilon} \exp \left\{ -\frac{\mu}{\epsilon} (x \cdot \nu^\epsilon)^2 \right\}.$$  

Therefore, the integral in the fourth term of (10.5) is bounded by

$$\frac{1}{\epsilon Z_\epsilon} \frac{C \eta^2}{\epsilon} e^{-H/\epsilon} \int_{\bar{B}_\epsilon^\sigma} \exp \left\{ -\frac{1}{2\epsilon} x \cdot (\overline{v}_\epsilon + 2\mu \nu^\epsilon \otimes \nu^\epsilon) x \right\} dx$$

by the Taylor expansion of $U$ around $\nu$. By Lemma 8.2, the last integral is $O(\epsilon^{d/2})$; hence, the whole expression is $o_\epsilon(1) \alpha_\epsilon$.

Now, we consider the last integral of (10.5). We also fix $\nu$ and assume that $\nu = 0$. Since

$$(\partial B_\epsilon^\nu)^{[\eta]} \setminus (\partial K_\epsilon)^{[\eta]} \subset (\partial_+ B_\epsilon^\nu)^{[\eta]} \cup (\partial_- B_\epsilon^\nu)^{[\eta]},$$

it suffices to prove that the integral over $(\partial_+ B_\epsilon^\nu)^{[\eta]}$ is small, as the argument for $(\partial_- B_\epsilon^\nu)^{[\eta]}$ is identical. Since $\eta \ll \delta$, by Lemma 8.10, there exists a constant $a > 0$ such that

$$U(x) \geq a J^2 \delta^2 \quad \text{or} \quad x \cdot \nu^\epsilon \geq a J \delta \quad (10.10)$$

holds for all $x \in (\partial_+ B_\epsilon^\nu)^{[\eta]}$. Let us first assume that the former holds. Then, since $|f_\epsilon| \leq 1$ and $\text{vol}((\partial_+ B_\epsilon^\nu)^{[\eta]}) = O(1)$, by the first condition of (10.10), the integral over $x \in (\partial_+ B_\epsilon^\nu)^{[\eta]}$ satisfying the former condition of (10.10) is bounded from above by

$$\frac{C}{\epsilon Z_\epsilon} \int_{(\partial_+ B_\epsilon^\nu)^{[\eta]}} e^{-U(x)/\epsilon} dx \leq \frac{C}{\epsilon Z_\epsilon} e^{-H/\epsilon} e^{aJ^2} \text{vol}((\partial_+ B_\epsilon^\nu)^{[\eta]}) = o_\epsilon(1) \alpha_\epsilon \quad (10.11)$$

for sufficiently large $J$.

Now, assume that the second condition of (10.10) holds for $x \in (\partial_+ B_\epsilon^\nu)^{[\eta]}$. As in the proof of Lemma 8.6, we can rewrite $1 - f_\epsilon(x)$ as

$$[1 + o_\epsilon(1)] \frac{\epsilon^{1/2}}{(2\pi \mu)^{1/2} (x \cdot \nu^\epsilon)} \exp \left\{ -\frac{\mu}{2\epsilon} (x \cdot \nu^\epsilon)^2 \right\} \leq \frac{C \epsilon^{1/2}}{\delta} e^{1/2} \exp \left\{ -\frac{\mu}{2\epsilon} (x \cdot \nu^\epsilon)^2 \right\}.$$  

Similarly, we can check that, for $y \in D_\eta(0)$,

$$|1 - f_\epsilon(x - y)| \leq \frac{C \epsilon^{1/2}}{\delta} \exp \left\{ -\frac{\mu}{2\epsilon} (x \cdot \nu^\epsilon)^2 \right\}.$$  

By the two bounds above, we can bound $|f_\varepsilon(x) - f_\varepsilon(x - y)|^2$ from above by

$$
2 \left[ |1 - f_\varepsilon(x)|^2 + |1 - f_\varepsilon(x - y)|^2 \right] \leq \frac{C \varepsilon}{\delta^2} \exp \left\{ - \frac{\mu}{\varepsilon} (x \cdot v)^2 \right\}.
$$

Hence, we can bound the last integral of (10.5) and restrict it to $x \in (\partial + B_\sigma^\varepsilon)^[\eta]$, satisfying the second condition of (10.10), from above by

$$
\frac{C \delta^2}{\varepsilon} \int_{(\partial + B_\sigma^\varepsilon)^[\eta]} \exp \left\{ - \frac{\mu}{\varepsilon} (x \cdot v)^2 \right\} \mu_\varepsilon(dx).
$$

By applying the Taylor expansion of $U$ around $\sigma$, this is bounded by

$$
\frac{1}{\delta^2 Z_\varepsilon} e^{-H/\varepsilon} \int_{(\partial + B_\sigma^\varepsilon)^[\eta]} \exp \left\{ - \frac{\mu}{2\varepsilon} x \cdot [H^\sigma + 2\mu v^\sigma \otimes v^\sigma] x \right\} dx.
$$

By Lemma 8.2, there exists $c > 0$ such that $x \cdot [H^\sigma + 2\mu v^\sigma \otimes v^\sigma] x \geq c |x|^2$. Furthermore, there exists $C > 0$ such that $|x| \geq C\delta$ for all $x \in (\partial + B_\sigma^\varepsilon)^[\eta]$. Therefore, we can bound the last centered display from above by

$$
\frac{1}{Z_\varepsilon} e^{-H/\varepsilon} e^{\varepsilon J^2} \text{vol}((\partial + B_\sigma^\varepsilon)^[\eta]) = O_\varepsilon(1) \alpha_\varepsilon
$$

(10.12)

for sufficiently large $J$ since $\text{vol}((\partial + B_\sigma^\varepsilon)^[\eta]) = O(1)$. By (10.11) and (10.12), we can verify that the last integral of (10.5) is $o_\varepsilon(1) \alpha_\varepsilon$, and this completes the proof. \qed

### 10.3 Proof of Proposition 10.3

First, note that we can write

$$
\varepsilon \int_{\mathbb{R}} [\Phi f_\varepsilon \cdot \nabla h_\varepsilon] d\mu_\varepsilon = A_1 + \sum_{\sigma \in \Sigma_0} A_2(\sigma),
$$

(10.13)

where

$$
A_1 = \int_{\mathcal{H}_0^\varepsilon} [\ell \cdot \nabla h_\varepsilon] d\mu_\varepsilon \quad \text{and} \quad A_2(\sigma) = \varepsilon \int_{B_\sigma^\varepsilon} [\Phi p_\sigma \cdot \nabla h_\varepsilon] d\mu_\varepsilon.
$$

To estimate these integrals, we first mention a technical result.

**Lemma 10.5** There exists $C > 0$ such that

$$
\int_{\partial K^\varepsilon} \sigma (d\mu_\varepsilon) \leq C \varepsilon^{l^2-d/2} \alpha_\varepsilon.
$$
Proof Since \( U(x) = H + J^2 \delta^2 \) on \( \partial K^\epsilon \), we have
\[
\int_{\partial K^\epsilon} \sigma(d\mu_\epsilon) = \int_{\partial K^\epsilon} \mu_\epsilon(x) \sigma(dx) = Z_\epsilon^{-1} e^{-H/\epsilon} \epsilon^J \sigma(\partial K^\epsilon).
\]
Since \( \sigma(\partial K^\epsilon) = O(1) \), the proof is completed by the definition (7.3) of \( \alpha_\epsilon \). \( \square \)

We now consider \( A_1 \).

Lemma 10.6 We can write
\[
A_1 = o_\epsilon(1) \alpha_\epsilon + \sum_{\sigma \in \Sigma_0} A_{1,1}(\sigma),
\]
where
\[
A_{1,1}(\sigma) = \int_{\partial^+ B^\sigma} [\ell \cdot n_{\partial H_0^\epsilon}] h_\epsilon \sigma(d\mu_\epsilon).
\] (10.14)

Proof By the divergence theorem, we have
\[
\int_{\mathcal{H}_0^\epsilon} [\ell \cdot \nabla h_\epsilon] d\mu_\epsilon = \int_{\partial \mathcal{H}_0^\epsilon} [\ell \cdot n_{\partial H_0^\epsilon}] h_\epsilon \sigma(d\mu_\epsilon).
\]
Write
\[
\partial \tilde{\mathcal{H}}_0^\epsilon = \partial \mathcal{H}_0^\epsilon \setminus \bigcup_{\sigma \in \Sigma_0} \partial^+ B^\sigma \subset \partial K^\epsilon.
\]
Then, it suffices to prove that
\[
\int_{\partial \tilde{\mathcal{H}}_0^\epsilon} [\ell \cdot n_{\partial H_0^\epsilon}] h_\epsilon \sigma(d\mu_\epsilon) = o_\epsilon(1) \alpha_\epsilon.
\]
Since \(|h_\epsilon|\) and \(|\ell|\) are bounded on \( \partial \tilde{\mathcal{H}}_0^\epsilon \subset K \), and since \( \partial \tilde{\mathcal{H}}_0^\epsilon \subset \partial K^\epsilon \), the absolute value of the left-hand side of the previous case is bounded by \( \int_{\partial K^\epsilon} \sigma(d\mu_\epsilon) \), which is \( o_\epsilon(1) \alpha_\epsilon \) for sufficiently large \( J \) by Lemma 10.5. This completes the proof. \( \square \)

Now, we focus on \( A_2(\sigma) \).

Lemma 10.7 For \( \sigma \in \Sigma_0 \), we can write
\[
A_2(\sigma) = o_\epsilon(1) \alpha_\epsilon + A_{2,1}(\sigma),
\]
where
\[
A_{2,1}(\sigma) = \epsilon \int_{\partial B^\sigma \cup \partial - B^\sigma} [\Phi_{p^\sigma} \cdot n_{B^\sigma}] h_\epsilon \sigma(d\mu_\epsilon).
\] (10.15)
Proof By the divergence theorem, we can write

\[ A_2(\sigma) = - \int_{B_\epsilon} (\mathcal{L}_{e}^{\sigma} p_{e}^{\sigma} ) h_\epsilon \, d\mu_\epsilon + \epsilon \int_{\partial B_\epsilon} \left[ \Phi_{p_{e}^{\sigma}} \cdot n_{B_\epsilon}^{\sigma} \right] h_\epsilon \sigma (d\mu_\epsilon). \]

By Proposition 8.5, the first integral on the right-hand side is \( o_\epsilon(1) \alpha_\epsilon \). Hence, it suffices to prove that

\[ \epsilon \int_{\partial B_\epsilon} \left[ \Phi_{p_{e}^{\sigma}} \cdot n_{B_\epsilon}^{\sigma} \right] h_\epsilon \sigma (d\mu_\epsilon) = o_\epsilon(1) \alpha_\epsilon. \quad (10.16) \]

By the explicit formula for \( p_{e}^{\sigma} \) and by the boundedness of \( \ell \) on \( K \), we can check that there exists \( C > 0 \) such that \( |\Phi_{p_{e}^{\sigma}}| \leq C \epsilon^{-1} \) on \( \partial_0 B_\epsilon^{\sigma} \). Therefore, the absolute value of the left-hand side of (10.16) is bounded from above by \( C \int_{\partial_0 B_\epsilon^{\sigma}} \sigma (d\mu_\epsilon) \). Since \( \partial_0 B_\epsilon^{\sigma} \subset \partial K_\epsilon \), the proof is completed by Lemma 10.5, provided that we take \( J \) to be sufficiently large.

By (10.13) and Lemmas 10.6 and 10.7, it suffices to check the following Lemma to complete the proof of Proposition 10.3.

Lemma 10.8 For \( \sigma \in \Sigma_0 \), we have

\[ A_{1,1}(\sigma) + A_{2,1}(\sigma) = [1 + o_\epsilon(1)] \alpha_\epsilon \omega^{\sigma}. \]

We defer the proof of Lemma 10.8 to the next subsection and conclude the proof of Proposition 10.3 first.

Proof of Proposition 10.3 The proof is completed by combining 10.13 and Lemmas 10.6, 10.7, and 10.8.

10.4 Proof of Lemma 10.8

As a consequence of Proposition 9.1, we can get the following estimate of the equilibrium potential at the boundaries \( \partial_{+} B_\epsilon^{\sigma} \) and \( \partial_{-} B_\epsilon^{\sigma} \) for \( \sigma \in \Sigma_0 \).

Lemma 10.9 There exists a constant \( C > 0 \) such that, for all \( \sigma \in \Sigma_0 \),

\[ h_\epsilon(x) \geq 1 - C \epsilon^{-d} \exp \frac{U(x) - H}{2\epsilon} \quad \forall x \in \partial_{+} B_\epsilon^{\sigma} \quad \text{and} \]

\[ h_\epsilon(x) \leq C \epsilon^{-d} \exp \frac{U(x) - H}{2\epsilon} \quad \forall x \in \partial_{-} B_\epsilon^{\sigma}. \]

Proof Let us consider the first inequality. If \( x \in \partial_{+} B_\epsilon \) satisfies \( U(x) \geq H \), then the inequality is obvious for all sufficiently small \( \epsilon \). Otherwise, \( x \in \mathcal{H}_0 \); hence, the bound follows from part (1) of Proposition 9.1 since we have \( \mathcal{F}_{\{x\}, \mathcal{D}_0(m_0)} = U(x) \) for all sufficiently small \( \epsilon \). The proof of the second one is similar and left to the reader.

In the next lemma, we provide a consequence of the previous lemma.
Lemma 10.10 For $\sigma = \Sigma_0$, we have

$$
\epsilon \int_{\partial_+ B_{\epsilon}^\sigma} |\nabla p_{\epsilon}^\sigma| \cdot (1 - h_{\epsilon}) \sigma(d\mu_{\epsilon}) = o_\epsilon(1) \alpha_{\epsilon}, \quad (10.17)
$$

$$
\int_{\partial_+ B_{\epsilon}^\sigma} (1 - p_{\epsilon}^\sigma) (1 - h_{\epsilon}) \sigma(d\mu_{\epsilon}) = o_\epsilon(1) \alpha_{\epsilon}, \quad (10.18)
$$

$$
\epsilon \int_{\partial_- B_{\epsilon}^\sigma} |\nabla p_{\epsilon}^\sigma| h_{\epsilon} \sigma(d\mu_{\epsilon}) = o_\epsilon(1) \alpha_{\epsilon}, \quad (10.19)
$$

$$
\int_{\partial_- B_{\epsilon}^\sigma} p_{\epsilon}^\sigma h_{\epsilon} \sigma(d\mu_{\epsilon}) = o_\epsilon(1) \alpha_{\epsilon}. \quad (10.20)
$$

Proof Since the proofs of (10.19) and (10.20) are identical to those of (10.17) and (10.18), respectively, we focus only on (10.17) and (10.18).

Let us first consider (10.17). We use the explicit formula for $p_{\epsilon}^\sigma$ and Lemma 10.9 to bound the left-hand side of (10.17) by

$$
C \epsilon^{-1/2 - 3d/2} \alpha_{\epsilon} \epsilon \int_{\partial_+ B_{\epsilon}^\sigma} \exp \left\{ - \frac{U(x) - H}{2\epsilon} - \frac{\mu}{2\epsilon}(x \cdot v_{\sigma})^2 \right\} \sigma(dx). \quad (10.21)
$$

By the Taylor expansion, the last line can be further bounded by

$$
C \epsilon^{-1/2 - 3d/2} \alpha_{\epsilon} \epsilon \int_{\partial_+ B_{\epsilon}^\sigma} \exp \left\{ - \frac{1}{4\epsilon} x \cdot [H_{\sigma} + 2\mu v_{\sigma} \otimes v_{\sigma}] x \right\} \sigma(dx) \leq C \epsilon^{-1/2 - 3d/2} \alpha_{\epsilon} \epsilon \int_{\partial_+ B_{\sigma}^\sigma} \exp \left\{ - \frac{\gamma}{4\epsilon} |x|^2 \right\} \sigma(dx), \quad (10.22)
$$

where $\gamma > 0$ is the smallest eigenvalue of the positive-definite matrix $H + 2\mu v \otimes v$ (cf. Lemma 8.2). Since there exists $C > 0$ such that $|x| \geq CJ\delta$ for all $x \in \partial_+ B_{\sigma}^\sigma$, and since $\sigma(\partial_+ B_{\sigma}^\sigma) = O(\delta^{d-1})$, we can bound (10.22) from above, for some $c, C > 0$, by

$$
C \epsilon^{-1/2 - 3d/2} \delta^{d-1} e^{cJ^2} \alpha_{\epsilon} = C \left( \log \frac{1}{\delta} \right)^{d-1} e^{cJ^2 - d-1} = o_\epsilon(1) \alpha_{\epsilon}
$$

for sufficiently large $J$. This completes the proof of (10.17).

For (10.18), recall $\partial_{+ a} B_{\epsilon}^\sigma$ and $\partial_{+ a} \sigma^2 B_{\epsilon}^\sigma$ from (8.19) and (8.20), respectively. By Lemma 8.10, it suffices to prove that, for $a \in (0, a_0)$,

$$
\int_{\partial_{+ a} B_{\epsilon}^\sigma} (1 - p_{\epsilon}^\sigma) (1 - h_{\epsilon}) \sigma(d\mu_{\epsilon}) = o_\epsilon(1) \alpha_{\epsilon} \quad ; \quad k = 1, 2. \quad (10.23)
$$

For $k = 1$, by (8.27) and Lemma 10.9, we can bound the integral from above by

$$
\frac{C \epsilon^{-H/2} \epsilon^{1/2}}{Z_\epsilon} \epsilon^d \delta \int_{\partial_+ B_{\epsilon}} \exp \left\{ - \frac{U(x) - H}{2\epsilon} - \frac{\mu}{2\epsilon}(x \cdot v_{\sigma})^2 \right\} \sigma(dx).
$$
Hence, we can proceed as in the computation of (10.21) to prove that this is $o_{\epsilon}(1) \alpha_{\epsilon}$.

Now, we finally consider the $k = 2$ case of (10.23). Since $U(x) \geq H + a J^2 \delta^2$ for $x \in \mathcal{B}_\delta^\sigma$, the left-hand side of (10.23) with $k = 2$ is bounded from above by

$$\frac{1}{Z_\epsilon} e^{-H/\epsilon} e^{a J^2} \sigma(\partial \mathcal{B}_\delta^\sigma) \leq \frac{C}{Z_\epsilon} e^{-H/\epsilon} e^{a J^2} \delta^{d-1} = o_{\epsilon}(1) \alpha_{\epsilon}$$

for sufficiently large $J$. This completes the proof.

Now, we are ready to prove Lemma 10.8.

**Proof of Lemma 10.8** In view of the expressions (10.14) and (10.15) for $A_{1,1}(\sigma)$ and $A_{2,1}(\sigma)$, respectively, it suffices to prove the following estimates:

$$\epsilon \int_{\partial \mathcal{B}_\delta^\sigma} \left[ (\Phi - \frac{1}{\epsilon} \ell) \cdot n_{\mathcal{B}_\delta^\sigma} \right] h_\epsilon \sigma(d\mu_\epsilon) = [1 + o_{\epsilon}(1)] \alpha_{\epsilon} \omega_\sigma,$$

(10.24)

$$\epsilon \int_{\partial \mathcal{B}_\delta^\sigma} \left[ \Phi_{\delta_\epsilon} \cdot n_{\mathcal{B}_\delta^\sigma} \right] h_\epsilon \sigma(d\mu_\epsilon) = o_{\epsilon}(1) \alpha_{\epsilon}.$$

(10.25)

Let us first consider (10.24). By (10.17) and (10.18) of Lemma 10.10, we can replace the $h_\epsilon(x)$ term with 1 with an error term of order $o_{\epsilon}(1) \alpha_{\epsilon}$. Then, we can apply Proposition 8.6 to prove (10.24). On the other hand, the estimate (10.25) is a direct consequence of (10.19) and (10.20) of Lemma 10.10.

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