Universal adic approximation, invariant measures and scaled entropy

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Abstract. We define an infinite graded graph of ordered pairs and a canonical action of the group $\mathbb{Z}$ (the adic action) and of the infinite sum of groups of order two $\mathcal{D} = \sum_{1}^{\infty} \mathbb{Z}/2\mathbb{Z}$ on the path space of the graph. It is proved that these actions are universal for both groups in the following sense: every ergodic action of these groups with invariant measure and binomial generator, multiplied by a special action (the ‘odometer’), is metrically isomorphic to the canonical adic action on the path space of the graph with a central measure. We consider a series of related problems.

Keywords: graph of ordered pairs, universal action, adic transformation, scaled entropy.

§1. Introduction

This paper is devoted to the solution of several interrelated problems:
– we prove the existence of a universal adic realization of an arbitrary ergodic action of the group $\mathbb{Z}$ and the group $\mathcal{D} = \sum_{1}^{\infty} \mathbb{Z}/2\mathbb{Z}$, which is an infinite direct sum of groups of order two, on the path space of the graded graph (Bratteli diagram) of ordered pairs, which is denoted below by OP (Ordered Pairs);
– we list all ergodic central measures on the path space of the graph OP;
– we prove that the scaling sequence for a dyadic filtration does not depend on the choice of the iterated metric;
– we prove that for the adic actions of the groups $\mathbb{Z}$ and $\mathcal{D}$ on the path space of the graph OP there are invariant measures having a given subadditive growth of scaling sequences in the definition of scaled entropy, and here the entropy of the tail filtration with respect to this measure has the same growth.

Let us comment on these problems and their solutions.

1.1. Universal adic realization. In the classical lemma of Rokhlin, a periodic approximation of arbitrary order is constructed for any aperiodic automorphism with invariant measure. It is possible to construct a coherent system of ‘Rokhlin towers’ defining an exhaustive periodic approximation (for example, of orders $2^n$, $n \in \mathbb{N}$): in [1] and [2], the so-called ‘adic realization’ of any ergodic action was

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constructed. Namely, it was proved that for every ergodic automorphism $S$ there is a graded graph $\Gamma$, an adic structure on $\Gamma$, and a central measure $\nu$ on the path space $T(\Gamma)$ of $\Gamma$ such that the adic shift on $T(\Gamma)$ is isomorphic to $S$. The following question arises: is it possible to refine this construction and implement arbitrary ergodic actions on the path space of the same graph by modifying only the invariant measure? One can make the question more specific: consider the so-called symbolic action on the space $2^G$ of a discrete group $G$ by (left) shifts; this is a universal model: any action with a binomial generator can be realized in this way. Then our question, even in the case when $G = \mathbb{Z}$, actually reduces to the following: can we provide the construction of a sequence of approximations of a ‘Rokhlin tower’ with a universal nature, that is, make it into the adic approximation? However, a positive answer to this question suggests a positive answer to another, more modest, question: is it possible to give a universal (if possible, constructive) proof of Rokhlin’s lemma for an arbitrary homeomorphism, for example, for a shift in the space $2^{\mathbb{Z}}$, simultaneously for all invariant Borel probability measures? Approximately this question was posed to the first author by Rokhlin. The point is that an uncountable induction was used in all proofs known so far, and it is desirable to avoid this induction without using any properties of a particular measure. It would then be possible to raise the same question about the adic approximation.\footnote{\textit{added in proof}.} It is shown in this paper that such a universal construction is possible if the question is modified as follows: approximate (or construct an adic realization of) a direct product of an arbitrary action by the ‘odometer’ rather than by an arbitrary action of $\mathbb{Z}$. By the odometer we mean the operation of adding one to any element of the group $\mathbb{Z}_2$ of dyadic numbers; this is an ergodic automorphism with a dyadic spectrum. In the group $\mathcal{D} = \sum_1^{\infty} \mathbb{Z}/2\mathbb{Z}$ considered below, the role of the odometer is played by the action of $\mathcal{D}$ on the group $\left(\mathbb{Z}/2\mathbb{Z}\right)^N = \prod_{j=1}^{\infty} \left(\mathbb{Z}/2\mathbb{Z}\right)$, which is the group of characters of $\mathcal{D}$. Thus, the universality of the approximation is readily achieved by multiplying directly by an action of the simplest kind. As a universal space, the direct product of the space $2^\mathbb{D}$ and the odometer space arises, which, in our case, coincides with the path space of the graph of ordered pairs. Thus, a universal dyadic approximation is constructed. It would be interesting to find out whether or not such a possibility exists for non-Abelian amenable groups.\footnote{\textit{The general problem is as follows: suppose that a hyperfinite equivalence relation (that is, the trajectory partition of a Borel action of the group $\mathbb{Z}$; see [3]) is defined on the standard Borel space or, in particular, on a separable metric space. Then find an effective Borel isomorphism between this space and the path space of some graded locally finite graph that maps the tail equivalence relation on the path space of the graph into a given equivalence relation. It suffices to solve this problem for the space $2^\mathbb{Z}$ and the usual action of the group $\mathbb{Z}$ by shifts on this space. It is not known for what amenable groups this construction is possible in principle.}}

1.2. Invariant central measures. Finding invariant measures for group actions on a given compact or topological space is a traditional problem of the theory of
dynamical systems. In recent papers it has been shown that, in many cases, this problem can be reduced to the description of the so-called central measures on the path spaces of graded graphs (or Bratteli diagrams). In our case, we speak of a specific graph, namely, the graph OP of ordered pairs. Its construction (see §2) is very simple: the set of vertices on the next floor is the set of all ordered pairs of vertices on the previous floor. This graph was studied in [4], as well as a more complicated graph of unordered pairs (a ‘tower of measures’).

There is a natural bijection between the set of central measures on the path space of the graph of ordered pairs and the set of measures on the space $2^\mathbb{Z} \times [0, 1]$ that are invariant under the direct product of the actions of the group $\mathbb{Z}$ (the shift on the first component and the odometer on the segment). Along with the direct products of invariant measures (the invariant measure is unique for the odometer), there is a series of the most interesting ergodic measures with respect to which this action is a skew product. We describe them completely. Moreover, it is possible to describe completely the types of actions of the group $\mathbb{Z}$ with respect to these measures: they are the actions that have a factor isomorphic to the odometer. We note that the necessity of this condition is trivial, while the sufficiency is not so obvious. A similar result holds for the group $\mathcal{D}$.

The case under consideration is part of the following general problem. When there are two continuous actions of a group $G$ on two compact spaces, $X$ and $Y$, describe all measures invariant under the direct product of actions of $G$ on the product $X \times Y$ modulo the invariant measures on $X$ and $Y$ (here we also know in addition that an invariant measure is unique on one of the spaces, for example, on $X$). However, even under this assumption, the problem looks immense since the answer must include descriptions of all quotient actions of $G$ on both spaces. In our case, we used the fact that all quotient actions of the odometer are well known, and therefore the problem of invariant measures has a clear answer.

1.3. Scaled entropy of actions and filtrations. The scaled entropy of dynamical systems was defined and studied in recent papers of the first author ([5], [6]). The theory of filtrations (decreasing sequences of sigma-algebras) was a motivation for the definition: scaled entropy was first defined as an invariant of a filtration (see [7]–[10]). Below we establish the numerical coincidence of these notions in our situation. However, the importance of the notion of scaled entropy of dynamical systems has forced us to give an interpretation in terms of an adic implementation of the action, which is done in this paper. The novelty of the notion is that, although it is a purely metric (rather than topological) invariant of the action, it is defined using the $\varepsilon$-entropy of metric spaces rather than the metric entropy of partitions. The scaling sequence is first defined for an automorphism of a measure space equipped additionally with a metric (that is, an admissible triple: cf., for example, [11]), as a sequence that normalizes the $\varepsilon$-entropy of the space equipped with an invariant measure and an iterated metric. A conjecture of the first author, which was proved in the thesis of the second (see [12] and [13]), is that the asymptotic behaviour of this sequence does not depend on the choice of the original metric. Thus, the scaled entropy is a new metric invariant of an action of groups, which considerably generalizes Kolmogorov’s entropy theory.
As mentioned above, the notion of scaled entropy arose from an analysis of the preceding notion, namely, the \textit{entropy of filtration}, which was suggested as a metric invariant of filtration, that is, of a decreasing sequence of sigma-algebras (or measurable partitions). In this paper, we dwell on the theory of filtrations only within the limits in which it is necessary for the main topic of the paper, the more so because this theory, in its modern version, will be described in detail in a forthcoming paper of the first author. In one of the special cases (homogeneous partitions), the entropy of a filtration was defined without using any metric on a measure space. However, in the general case, it is convenient to define this entropy (as well as the scaled entropy of an action) first in a space with a metric (or semimetric) and then prove the independence of the choice of the original metric. Therefore, in this paper we present an argument about the independence analogous to that used for the scaling sequence of an action. Namely, we show that, in the definition of a scaling sequence of a filtration, it suffices to restrict ourselves to any admissible metric without taking the supremum over all admissible semimetrics.

In various special cases, ideas similar to that of the scaled entropy of an action have already been used by various authors (see [14] for the sequential entropy or the Kirillov–Kushnirenko entropy, [15] and [16] for loosely Bernoulli property, [17] for slow entropy, and [18] for complexity entropy). One of the first applications of the idea of scaled entropy is that the boundedness of a scaling sequence is equivalent to the condition of a purely point spectrum (see [14], [18], [11]). The definition of scaled entropy seems to be the most general and synthetic, uniting the concepts of metric and \(\varepsilon\)-entropy and revealing the auxiliary (non-principal) role of the metric.

A result outlined in the paper [19] of Ferenczi and Park means that the scaling sequence for ergodic automorphisms can have any intermediate asymptotic behaviour, and this was proved rigorously in [13]. In [13], examples of special central measures on the path space of the graph \(\text{OP}\) were constructed for which the adic transformation has an arbitrary prescribed subadditive growth of the scaling sequence, and in [20] it was shown that a subadditive sequence must exist in any non-empty class of scaling sequences. In the above examples with intermediate growth, a free action of the group \(D\) is constructed using a non-free action on \(2^D\) for which a measure is concentrated on the set of functions that are constant on the cosets of some subgroup of \(D\), which, in turn, is chosen for the given scaling sequence. In a more cumbersome way, this effect of intermediate growth can also be explained for the group \(Z\) as an effect of an action on some class of functions (that is, on \(2^Z\)) with hidden symmetries, but a good formulation has yet to be found.

The result on the arbitrariness of the asymptotic behaviour of a scaling sequence for the group \(Z\) was proved in [20] using the adic implementation of the action. Here it is also proved for locally finite Abelian groups. For some special measures this asymptotic behaviour for the adic transformation coincides with the asymptotic behaviour of the entropy of the tail filtration of the path space of the graph of ordered pairs. This coincidence is possibly of a more general nature and not connected with specific features of the graph and the measures considered on the path space.
Certainly, our consideration of the graph of ordered pairs and dyadic groups can readily be generalized, for example to the graph of ordered triples or \(n\)-tuples, with the dyadic group replaced by the triadic, and so on. This introduces no fundamental changes into the results, only the numerical characteristics being modified. Moreover, we can consider the \(r_n\)-adic case in which the number of joined points (instead of pairs) varies from floor to floor, but remains permanent on each floor. This is the case of the so-called \(\{r_n\}\)-adic filtrations (see [9]). The groups corresponding to the group \(D\) are direct sums of the corresponding groups \(\mathbb{Z}/r_n\mathbb{Z}\). The odometer is replaced by an automorphism whose discrete spectrum is the corresponding countable subgroup of the roots of unity. Finally, we can consider the whole project for the odometer whose spectrum is the group of all roots of unity of positive integer degrees, replacing the group \(D\) by the ad`ele group and the graph \(OP\) by a more complicated graph, which is yet to be studied.

1.4. Plan of the paper. In §2 we present the necessary definitions and constructions in the theory of Brattell diagrams, construct the graph \(OP\) of ordered pairs, and define the action of the groups \(\mathbb{Z}\) and \(D = \sum \mathbb{Z}/2\mathbb{Z}\) on the space \(T(OP)\) of infinite paths of the graph \(OP\).

In §3 we give an independent description of the path space of the graph \(OP\) which is convenient for the study of the action of \(D\). We also give a series of examples of central measures on the space \(T(OP)\) which, as will be seen below, show the intermediate growth of the scaling sequences.

In §4 we study the action of an adic transformation on the space \(T(OP)\). By making an appropriate change of variables, we reduce the problem of describing the central measures on the path space \(T(OP)\) of the graph of ordered pairs \(OP\) to the study of measures on \(I^\mathbb{Z} \times I^\mathbb{N}\) invariant under the product of the shift \(S\) and the odometer \(O\).

§5 is devoted to the study of the set of measures on \(I^\mathbb{Z} \times I^\mathbb{N}\) invariant under \(S \times O\). It turns out that such measures are divided into two types, measures of periodic type and aperiodic measures. We give a description of ergodic measures of both types and study their properties.

In §6 we discuss the scaling sequences of actions of \(\mathbb{Z}\) and \(D\) on the space \(T(OP)\) with special measures \(\mu^\sigma\), prove that the scaling sequence of a filtration is independent of the choice of the iterated metric, and also describe the calculation of a scaling sequence of the tail filtration on the space \(T(OP)\) with these special measures.

§2. The graph of ordered pairs and actions on this graph

2.1. Bratteli diagrams, central measures and filtration. Let \(\Gamma\) be a graded graph (Bratteli diagram) whose floors are finite and indexed by non-negative integers. The (oriented) edges in this graph can only connect vertices on adjacent floors, and the beginning of every edge lies on the floor with the smaller index. The set of vertices on floor \(n\) is denoted by the symbol \(\Gamma_n\), and the set of edges going from \(\Gamma_n\) to \(\Gamma_{n+1}\) is denoted by \(E_n\). We denote by \(T(\Gamma)\) the set of infinite (oriented) paths in \(\Gamma\) going from the vertices of the zeroth floor.
The set $\mathcal{T}(\Gamma)$ is equipped with a topology in the standard way. A base of this topology is formed by the so-called elementary cylindrical sets, that is, sets of paths having a fixed origin. In this topology, the elementary cylindrical sets are also closed, and the whole space $\mathcal{T}(\Gamma)$ is compact.

For every $n \geq 0$ we consider the partition $\xi_n$ of $\mathcal{T}(\Gamma)$ into sets of paths that coincide starting at floor $n$. These partitions are in a sense independent complements of the algebras of cylindrical sets. Denote by $\xi$ the tail filtration consisting of the sequence of these partitions, $(\xi_n)_{n \geq 0}$. The equivalence relation generated by the set-theoretic intersection of all partitions $\xi_n$ is called the tail equivalence relation of $\Gamma$: two paths are equivalent if and only if they coincide starting at some place.

An additional (adic) structure on a graded graph is the reverse lexicographic ordering of paths (see [1]). It is defined using introducing a total order on the set of edges entering each vertex from the vertices at the previous level. We say that a path $y \in \mathcal{T}(\Gamma)$ is greater than a path $x \in \mathcal{T}(\Gamma)$ if for some positive integer $n$ they coincide starting at floor $n$ and the edge along which the path $y$ arrives at floor $n$ is greater than the edge along which the path $x$ arrives at the same place. The adic order is a total order on each class of tail equivalence. An adic transformation $A$ is defined on the partially ordered space of paths $\mathcal{T}(\Gamma)$. It assigns to every path the next path in the order. We note that the next path need not exist for a given path, and therefore the adic transformation is defined on a proper subset of $\mathcal{T}(\Gamma)$.

One can consider Borel measures on the topological space $\mathcal{T}(\Gamma)$. We note that the partitions $\xi_n$ are automatically measurable with respect to any Borel measure. An important class of measures is formed by the central measures (measures with maximum entropy). A measure $\mu$ is said to be central if the beginnings of a path are equiprobable for a fixed tail, that is, the conditional measures on the elements of the partition $\xi_n$ are uniform. We note that the centrality of a measure is equivalent to its invariance under the action of the adic transformation. The set of all central measures of $\Gamma$ is denoted by the symbol $\text{Inv}(\Gamma)$. The set of ergodic central measures is the Choquet boundary of $\text{Inv}(\Gamma)$ and is called the absolute of $\Gamma$.

For details concerning the general theory of graded graphs and central measures, see [21]–[26].

2.2. Description of the graph OP of ordered pairs and of the actions on the path space of the graph.

2.2.1. Description of the graph OP of ordered pairs. The graded graph OP of ordered pairs is defined as follows. The set of vertices on the zeroth floor is defined as $\text{OP}_0 = I = \{0, 1\}$. Further, the set of vertices on floor $n+1$ is equal to $\text{OP}_{n+1} = \text{OP}_n \times \text{OP}_n$, $n \geq 0$, that is, the set of all possible ordered pairs of vertices on floor $n$. The set of edges $E_n$, $n \geq 0$, leading from the vertices on floor $n$ to those on floor $n+1$ is constructed as follows. Suppose that $v \in \text{OP}_{n+1}$, $v = (v_0, v_1)$, where $v_0, v_1 \in \text{OP}_n$. We draw edges from $v_0$ and $v_1$ to $v$ and specify the natural order on these edges, namely, the edge from $v_0$ to $v$ is assumed to be the zeroth and the edge from $v_1$ to $v$ the first. If $v_0 = v_1$, then there are two edges simultaneously from $v_0$ to the vertex $V = (v_0, v_0)$. 
By induction, it can readily be seen that the number of vertices on floor $n$ of OP is equal to $2^2n$, and exactly $2^n$ paths starting from floor 0 enter every vertex on floor $n$. Since exactly two edges enter every vertex, except for the vertices on the zeroth floor, it follows that the tail filtration $\xi$ for the graph is dyadic.

An adic automorphism $A$ (an action of the group $\mathbb{Z}$) is defined on the space $\mathcal{T}(OP)$, and one can define the canonical (adic) action of the group

$$\mathcal{D} = \bigoplus_{i=0}^{\infty} \mathbb{Z}/2\mathbb{Z}.$$ 

2.2.2. Description of the adic transformation on the graph OP. The graph of ordered pairs OP is equipped with an order on the edges entering every vertex, and therefore an adic transformation $A$ is defined automatically (see the general definition in §2.1). More specifically, for a path $x \in \mathcal{T}(OP)$ passing through vertices $v_n \in OP_n, n \geq 0$, we can find the first edge of $x$ whose index is 0. Let this edge pass from a vertex $v_n$ to $v_{n+1}$. We define the path $Ax$ in such a way that it coincides with $x$ starting from floor $n+1$, enters $v_{n+1}$ along an edge with index 1, and the previous edges all have index 0. This transformation is defined only for paths having at least one edge with index 0.

2.2.3. Description of the canonical action of the group $\mathcal{D}$ on the graph OP. The simplest way to define the canonical (adic) action $\kappa$ of $\mathcal{D}$ is to define it on the generators. We denote by $\kappa(g)$ the action of an element $g$ of $\mathcal{D}$ on $\mathcal{T}(OP)$. Let the group $\mathcal{D} = \bigoplus_{i=0}^{\infty} \mathbb{Z}/2\mathbb{Z}$ be generated by the elements $g_i, i \geq 0$. We denote by the symbol $\mathcal{D}_n, n \geq 0$, the finite subgroup generated by $g_0, \ldots, g_{n-1}$. Let $g_n$ be a generator of $\mathcal{D}$ and let $x \in \mathcal{T}(OP)$ be a path. We define a path $\kappa(g_n)x$ in such a way that it coincides with $x$, starting at floor $n+1$, the index of its edge entering the vertex on floor $n+1$ differs from the index of the corresponding edge in the path $x$, and the indices of the edges between floors $i$ and $i + 1, 0 \leq i < n$, in the paths $x$ and $\kappa(g_n)x$ coincide. The actions (of the generators) defined in this way satisfy the commutation relations of the group $\mathcal{D}$, and therefore define an action.

We note that the action $\kappa$ of $\mathcal{D}$ is closely related to the tail filtration $\xi$ and to the set of central measures Inv(OP). Namely, the partition $\xi_n$ is precisely the partition into orbits of the action of the subgroup $\mathcal{D}_n$. The partition into orbits of the whole group $\mathcal{D}$ is the tail partition for the graph OP. A Borel measure on the space $\mathcal{T}(OP)$ is central if and only if it is invariant under the action $\kappa$ of $\mathcal{D}$.

§3. Independent description of the path space of the graph OP

3.1. The space $I^\mathcal{D} \times I^\mathbb{N}$, the action of the group $\mathcal{D}$, the filtration and the isomorphism theorem. Consider the space $I^\mathcal{D} \times I^\mathbb{N}$, consisting of all possible pairs $(w, \alpha)$, where $w \in I^\mathcal{D}$ is a configuration on the group $\mathcal{D}$ and $\alpha \in I^\mathbb{N}$ is an infinite sequence of zeros and ones. The cylindrical sets on this space, and also the topology of the product of compact spaces, are defined in the standard way. The cylindrical sets are open and closed in this topology.
The space $I^N$ with coordinatewise addition forms a commutative group, and the group $D$ is isomorphic to the subgroup of $I^N$ consisting of finite sequences (with zeros starting at some place).

**Definition 1.** We denote by $\tau$ the embedding of the group $D$ in the group $I^N$ which takes every element $g \in D$, $g = \sum \alpha_i g_{i-1}$, to the finite sequence $(\alpha_i)_{i \geq 1}$ of the coefficients of its expansion in the generators.

The group $D$ acts on the space $I^D$ by shifting the argument,

$$w(\cdot) \mapsto w(\cdot + g), \quad g \in D, \quad w \in I^D,$$

and on the space $I^N$ by addition,

$$\alpha \mapsto \alpha + \tau(g), \quad \alpha \in I^N, \quad g \in D.$$

The direct product of these actions, which is the action of $D$ on the product $I^D \times I^N$, is given by the formula

$$\text{diag}(g): (w(\cdot, \alpha)) \mapsto (w(\cdot + g, \alpha + \tau(g)), \quad g \in D, \quad w \in I^D, \quad \alpha \in I^N. \quad (1)$$

This action and the dyadic structure (the sequence of embedded subgroups $D_n$) of the group $D$ define a dyadic filtration on the space $I^D \times I^N$, which is the tail filtration for this action. Namely, for $n \geq 0$ we define the partition $\zeta_n$ of the space $I^D \times I^N$ as the partition into orbits of the action of the subgroup $D_n$. It should be noted that this filtration is not the direct product of the tail filtrations of the corresponding actions of $D$ on $I^D$ and $I^N$.

**Theorem 1.** The path space $T(\text{OP})$ of the graph $\text{OP}$ of ordered pairs with the action $\kappa$ and the space $I^D \times I^N$ with the action $\text{diag}$ are isomorphic as topological $D$-spaces.

To prove this theorem, we need some preliminary considerations.

### 3.2. Labels on the vertices of the graph and proof of Theorem 1

The construction of the graph $\text{OP}$ of ordered pairs enables us to parametrize its vertices by configurations on finite subgroups of $D$ in a natural way. We inductively define a map $\Phi$ which assigns configurations on $D_n$, $n \geq 0$, bijectively to vertices on floor $n$. When $v \in \text{OP}_0 = I$ we define $\Phi[v]$ as the configuration that takes the single element of the subgroup $D_0 = \{0\}$ to $v$. Further, suppose that we have already defined $\Phi$ on $\text{OP}_n$. Let $v \in \text{OP}_{n+1}$, $v = (v_0, v_1)$, where $v_0, v_1 \in \text{OP}_n$. We define the configuration $\Phi[v]$ on $D_{n+1}$ by the relation

$$\Phi[v](h) = \begin{cases} 
\Phi[v_0](h), & h \in D_n, \\
\Phi[v_1](g_n + h), & h \in D_{n+1} \setminus D_n. 
\end{cases} \quad (2)$$

**Remark 1.** For each $n \geq 0$ the map $\Phi$ is a bijection between the floor $\text{OP}_n$ of the graph $\text{OP}$ and the set $I^D_n$ of configurations on the subgroup $D_n$. 

We note that for two vertices \( v \in \mathcal{O}_n \) and \( v' \in \mathcal{O}_{n+1} \) joined by an edge with index \( \beta \in \{0, 1\} \), the corresponding configurations on \( \mathcal{D}_n \) are connected by the relation
\[
\Phi[v](h) = \Phi[v'](h + \beta g_n), \quad h \in \mathcal{D}_n.
\] (3)

**Proof of Theorem 1.** To prove the theorem, we shall explicitly construct an isomorphism \( \Psi : \mathcal{T}(\mathcal{O}) \to I^D \times I^N \). To do this, we define two maps, a map \( F \) of \( \mathcal{T}(\mathcal{O}) \) onto \( I^D \) and a map \( A \) of \( \mathcal{T}(\mathcal{O}) \) onto \( I^N \). Then the map \( \Psi = (F, A) \) will be the desired homeomorphism of \( \mathcal{T}(\mathcal{O}) \) onto \( I^D \times I^N \).

Let \( x \in \mathcal{T}(\mathcal{O}) \) be the path passing through the vertices \( v_n \in \mathcal{O}_n, n \geq 0 \), with edges \( e_n \in E_n, n \geq 0 \) (\( e_n \) joins \( v_n \) and \( v_{n+1} \)). We recall that, by the construction of the graph of ordered pairs, an order is given on the two edges leading to a given vertex (the numbers 0 and 1). For \( n \geq 1 \) we set \( \alpha_n \) equal to the ordinal number of the edge \( e_{n-1} \). This defines a sequence \( \alpha = (\alpha_n)_{n \geq 1} \in I^N \), which we call the image of \( x \) under the map \( A \):
\[
A[x] := \alpha.
\]

The map \( F \) is defined in a somewhat more complicated way. We have to define a configuration \( F[x] \in I^D \) on the group \( \mathcal{D} \). Here, every vertex \( v_n, n \geq 0 \), of the path \( x \) is taken by the map \( \Phi \) to the configuration \( \Phi[v_n] \) on the finite subgroup \( \mathcal{D}_n \). We note that the sequence of configurations \( \Phi[v_n], n \geq 0 \), is not consistent in general, that is, the configuration \( \Phi[v_n] \) is not the restriction of the configuration \( \Phi[v_{n+1}] \) to the subgroup \( \mathcal{D}_n \). A direct consequence of the equation (3) and the definition of the number \( \alpha_{n+1} \) is the identity
\[
\Phi[v_n](\cdot) = \Phi[v_{n+1}](\cdot + \alpha_{n+1} g_n) \quad \text{on} \quad \mathcal{D}_n, \quad n \geq 0.
\]

This implies the relation
\[
\Phi[v_n]\left(\cdot + \sum_{i=0}^{n-1} \alpha_{i+1} g_i\right) = \Phi[v_{n+1}]\left(\cdot + \sum_{i=0}^{n} \alpha_{i+1} g_i\right) \quad \text{on} \quad \mathcal{D}_n, \quad n \geq 0,
\]
which means that the sequence of configurations \( \Phi[v_n](\cdot + \sum_{i=0}^{n-1} \alpha_{i+1} g_i), n \geq 0, \) is consistent. Therefore, we can specify a configuration on \( \mathcal{D} \) whose restriction to every subgroup \( \mathcal{D}_n \) coincides with the configuration \( \Phi[v_n](\cdot + \sum_{i=0}^{n-1} \alpha_{i+1} g_i) \). We define this configuration on \( \mathcal{D} \) to be the image \( F[x] \) of the path \( x \) under the map \( F \):
\[
F[x](g) = \Phi[v_n]\left(g + \sum_{i=0}^{n-1} \alpha_{i+1} g_i\right), \quad g \in \mathcal{D}_n, \quad n \geq 1.
\] (4)

As already noted above, we define the map \( \Psi \) as a pair \((F, A)\):
\[
\Psi[x] = (F[x], A[x]), \quad x \in \mathcal{T}(\mathcal{O}).
\]

We claim that \( \Psi \) is a bijection of the path space \( \mathcal{T}(\mathcal{O}) \) of the graph \( \mathcal{OP} \) onto \( I^D \times I^N \). We note that formula (4) enables us to recover uniquely the vertices \( v_n, n \geq 0, \) of \( x \) from the configuration \( F[x] \) and the sequence \( \alpha = A[x] \). Further,
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the edge $e_n, n \geq 0$, can be recovered uniquely from its end $v_{n+1}$ and the ordinal number $\alpha_{n+1}$. Moreover, the configuration can be chosen arbitrarily in $I^D$, and the sequence $\alpha$ can be chosen arbitrarily in $I^N$. This proves that the map $\Psi$ is a bijection.

It is obvious from the definition of $\Psi$ that the image of a cylindrical set in $T(OP)$ under $\Psi$ is a cylindrical set in $I^D \times I^N$. This implies that $\Psi$ is a homeomorphism.

It remains to prove that $\Psi$ is $\mathcal{D}$-equivariant. Let us verify the necessary commutation relation for an arbitrary $m \geq 0$:

$$
\Psi[\kappa(g_m)x] = \text{diag}(g_m)\Psi[x], \quad x \in T(OP).
$$

By the definition of the canonical action $\kappa$, the path $\kappa(g_m)x$ coincides with the path $x$ starting at floor $m + 1$, the ordinal numbers of the edges in these paths coincide up to floor $m$, and the ordinal number of the edge going from floor $m$ to floor $m + 1$ is different. By the definition of $A$, this implies that

$$
A[\kappa(g_m)x]_i = A[x]_i, \quad i \neq m + 1, \quad A[\kappa(g_m)x]_{m+1} \neq A[x]_{m+1}.
$$

In other words, $A[\kappa(g_m)x] = A[x] + \tau(g_m)$. The equation $F[\kappa(g_m)x](\cdot) = F[x](\cdot + g_m)$ follows from formula (4). Thus, we have proved relation (5) and the $\mathcal{D}$-equivariance of $\Psi$. This completes the proof of Theorem 1. □

3.3. Examples of central measures on the path space $T(OP)$ of the graph OP, and the measures $\mu^\sigma$. One of the central problems of the theory of graded graphs is that of describing the central measures on the path space of a graph. §5 below is devoted to a detailed study of the set Inv(OP) of central measures on the path space $T(OP)$ of OP. In this subsection we discuss a class of examples of central measures.

The equivariant homeomorphism $\Psi$ takes the set Inv(OP) of central measures to the set of measures on $I^D \times I^N$ invariant under the action $\text{diag}$ of the group $\mathcal{D}$ given by formula (1). The projections of these measures to $I^D$ and $I^N$ are also invariant under the corresponding actions of $\mathcal{D}$. The only measure on $I^N$ invariant under this action of $\mathcal{D}$ is the Lebesgue measure $m$.

Measures of the form $\mu \times m$, where $\mu$ is a $\mathcal{D}$-invariant measure on $I^D$ and $m$ is the Lebesgue measure on $I^N$, can serve as examples of measures on $I^D \times I^N$ that are invariant under the action diag of $\mathcal{D}$. Among these measures, we single out a special class.

Definition 2. Let $\sigma = (\sigma_i)_{i \geq 0}$ be a sequence of zeros and ones. Let

$$
\mathcal{D}^\sigma = \langle g_i : \sigma_i = 0, i \geq 0 \rangle
$$

be the subgroup of $\mathcal{D}$ generated by those elements $g_i$ for which $\sigma_i = 0$. Consider the quotient map $\mathcal{D} \to \mathcal{D}/\mathcal{D}^\sigma$ and the map of configurations $I^D/\mathcal{D}^\sigma \to I^D$ induced by it. We denote by $m^\sigma$ the measure on $I^D$ which is the pushforward of the Lebesgue measure on $I^D/\mathcal{D}^\sigma$ under the map of configurations, and by $\omega^\sigma$ the product $m^\sigma \times m$ of $m^\sigma$ and the Lebesgue measure on $I^N$. 


The measures $\mu^\sigma$ on the path space $T(\text{OP})$ of OP (the pushforward of the measures $\omega^\sigma$ under the map $\Psi^{-1}$) were studied in [27]. The scaling sequences (see §6) of the adic automorphism on the space $(T(\text{OP}), \mu^\sigma)$ were calculated there, as well as the scaling sequence of the canonical action $\kappa$ of $\mathcal{D}$ on this space.

§ 4. Action of the group $\mathbb{Z}$

4.1. Action of the group $\mathbb{Z}$ on $I^\mathcal{D} \times I^\mathbb{N}$. We proceed with the study of the action of $\mathbb{Z}$ on the path space $T(\text{OP})$ of the graph OP, that is, the adic transformation $\mathcal{A}$ defined in §2.2.

Definition 3. Let $I^\mathbb{N}_0$ denote the set of all sequences in $I^\mathbb{N}$ containing only finitely many zeros or ones and $I^\mathbb{N}_\infty$ the complement of $I^\mathbb{N}_0$, that is, the set of sequences containing infinitely many zeros and ones.

Recall the definition of odometer.

Definition 4. The odometer is a transformation $\mathcal{O}$ defined on the set $I^\mathbb{N}$. Let $\alpha \in I^\mathbb{N}$ be such that $\alpha_1 = \alpha_2 = \cdots = \alpha_{n-1} = 1$ and $\alpha_n = 0$ for some $n \geq 0$. Then the sequence $\mathcal{O}[\alpha]$ is defined in such a way that $\mathcal{O}[\alpha]_i = \alpha_i$ when $i > n$, $\mathcal{O}[\alpha]_n = 1$, and $\mathcal{O}[\alpha]_i = 0$ when $i < n$.

The domain of the odometer in this definition is, generally speaking, not the whole of $I^\mathbb{N}$. We note that $I^\mathbb{N}$ is the group $\mathbb{Z}_2$ of integer 2-adic numbers in the ‘digital’ representation. Here the odometer $\mathcal{O}$ is the addition of one in the group $\mathbb{Z}_2$. The odometer defined on $I^\mathbb{N}$ using the structure of 2-adic numbers is already defined on the whole set. However, we will not use this extension but confine ourselves to Definition 4. This approach is more convenient when working with the graph OP.

Remark 2. All the $\mathcal{O}^n[\alpha]$, $n \in \mathbb{Z}$, are defined for an $\alpha \in I^\mathbb{N}$ if and only if $\alpha \in I^\mathbb{N}_\infty$.

We recall that the adic transformation is also defined only on a subset of the path space $T(\text{OP})$ of the graph OP rather than on the whole space.

Definition 5. Let $T_0(\text{OP}) \subset T(\text{OP})$ be the set of all possible paths $x \in T(\text{OP})$ for which all the $\mathcal{A}^n x$, $n \in \mathbb{Z}$, are defined.

Remark 3. Let $x \in T(\text{OP})$. Then $x \in T_0(\text{OP})$ if and only if $A[x] \in I^\mathbb{N}_\infty$. The map $\Psi$ establishes a bijection between $T_0(\text{OP})$ and $I^\mathcal{D} \times I^\mathbb{N}_\infty$.

We now study the transformation $\mathcal{A}^\Psi = \Psi \circ \mathcal{A} \circ \Psi^{-1}$ on $I^\mathcal{D} \times I^\mathbb{N}_\infty$. It follows from the definition of the adic transformation that for $x \in T_0(\text{OP})$ the sequence $A[Ax]$ is the sequent to $A[x]$ in the reverse lexicographic order. In other words, the transformation $\mathcal{A}^\Psi : I^\mathcal{D} \times I^\mathbb{N}_\infty \to I^\mathcal{D} \times I^\mathbb{N}_\infty$ acts on the second coordinate as the odometer $\mathcal{O}$. On the other hand, the orbits of the actions of the groups $\mathbb{Z}$ and $\mathcal{D}$ on the space $T_0(\text{OP})$ coincide, and both actions are free. Hence, for every $x \in T_0(\text{OP})$ there is a unique element $g \in \mathcal{D}$ such that $Ax = \kappa(g)x$. Then $\mathcal{O}[A[x]] = A[\mathcal{A}x] = A[\kappa(g)x] = A[x] + \tau(g)$, that is, $g = \tau^{-1}(\mathcal{O}[A[x]] - A[x])$. Consequently,

$$F[\mathcal{A}x](\cdot) = F[\kappa(g)x](\cdot) = F[x](\cdot + g).$$
Thus, we have proved the following assertion.

**Proposition 1.** The transformation \( \mathcal{A}^{\Psi} \) on \( I^D \times I_\infty^N \) satisfies the relation
\[
\mathcal{A}^{\Psi}(f, \alpha) = (f \cdot \tau^{-1}(\mathcal{O}([\alpha] - \alpha)), \mathcal{O}([\alpha])), \quad f \in I^D, \ \alpha \in I_\infty^N.
\]  

4.2. Isomorphism of the transformation \( \mathcal{A}^{\Psi} \) and the direct product of the actions on \( I^Z \times I^N \). The map \( \Lambda \). For a more detailed study of the action of the group \( \mathbb{Z} \) it is reasonable to replace the phase space \( I^D \times I_\infty^N \) by \( I^Z \times I^N \).

Here we shall do this in such a way that the transformation \( \mathcal{A}^{\Psi} \) will pass to the transformation \( S \times \mathcal{O} \), where \( S \) stands for the left shift on \( I^Z \).

We want to construct a map \( \Lambda: I^D \times I_\infty^N \rightarrow I^Z \times I_\infty^N \) which closes the commutative diagram
\[
\begin{array}{ccc}
I^D \times I_\infty^N & \xrightarrow{\Lambda} & I^Z \times I_\infty^N \\
\downarrow{\mathcal{A}} & & \downarrow{S \times \mathcal{O}} \\
I^D \times I_\infty^N & \xrightarrow{\Lambda} & I^Z \times I^N \\
\end{array}
\]

If a measure \( \nu \) on \( I^D \times I_\infty^N \) is invariant under the action of \( \mathcal{D} \), then the set \( I^D \times I_\infty^N \) has the full \( \nu \)-measure since the projection of \( \nu \) to \( I^N \) is the Lebesgue measure \( m \). Thus, the map \( \Lambda \) is defined mod 0 with respect to \( \nu \).

**Definition 6.** A Borel measurable map from the product \( I^D \times I_\infty^N \) to the product \( I^Z \times I_\infty^N \) is said to be fibrewise if it preserves the second component, that is, for every \( \alpha \in I_\infty^N \) the image of the fibre \( I^D \times \{\alpha\} \) is contained in the fibre \( I^Z \times \{\alpha\} \).

We restrict ourselves to the class of fibrewise maps \( \Lambda \) whose fibres \( \Lambda(\cdot, \alpha): I^D \rightarrow I^Z \) are induced for \( \alpha \in I_\infty^N \) by some maps \( \lambda_\alpha: \mathbb{Z} \rightarrow \mathcal{D} \), that is, are changes of the argument of the configurations:
\[
\Lambda(f, \alpha) = (f \circ \lambda_\alpha, \alpha), \quad f \in I^D, \ \alpha \in I_\infty^N.
\]

For such maps \( \Lambda \) commutative diagram (7) is obviously equivalent to the following relation on the family of maps \( (\lambda_\alpha)_{\alpha \in I_\infty^N} \):
\[
\lambda_\mathcal{O}([\alpha]) \circ \tau^{-1}(\mathcal{O}([\alpha] - \alpha)) = \lambda_\alpha(k + 1), \quad \alpha \in I_\infty^N, \ k \in \mathbb{Z}.
\]

**Lemma 1.** The map \( \Lambda: I^D \times I_\infty^N \rightarrow I^Z \times I_\infty^N \) defined by equation (8) and satisfying commutation relation (7) is Borel measurable if and only if the map \( \lambda: \alpha \mapsto \lambda_\alpha(0) \) from \( I_\infty^N \) to \( \mathcal{D} \) is Borel measurable. Here the map \( \Lambda \) is completely determined by the map \( \lambda \).

**Proof.** Let \( k \in \mathbb{Z}, \ w^0 \in I \). Let \( W = \{w \in I^Z: w(k) = w^0\} \). Then
\[
\Lambda^{-1}(W \times I_\infty^N) = \bigcup_{\alpha \in I_\infty^N} \{f \in I^D: f(\lambda_\alpha(k)) = w^0\} \times \{\alpha\}
\]
\[
= \bigcup_{g \in \mathcal{D}} \{f \in I^D: f(g) = w^0\} \times \{\alpha \in I_\infty^N: \lambda_\alpha(k) = g\}.
\]

The measurability of \( \Lambda \) is equivalent to the condition that the set in formula (10) is measurable for every \( k \in \mathbb{Z}, \ w^0 \in I \).
If Λ is measurable, then the set in formula (10) is measurable when \( k = 0 \) and \( w^0 = 1 \). Let \( g_0 \in D \) and choose \( f^0 \in I^D \) in such a way that \( f^0(g) = 1 \) holds for \( g \in D \) only when \( g = g_0 \). Then the set
\[
\{ \alpha \in I_\infty^N : (f^0, \alpha) \in \Lambda^{-1}(W \times I_\infty^N) \} = \{ \alpha \in I_\infty^N : f^0 \circ \lambda_\alpha \in W \}
\]
\[
= \{ \alpha \in I_\infty^N : f^0(\lambda_\alpha(0)) = 1 \} = \{ \alpha \in I_\infty^N : \lambda_\alpha(0) = g_0 \}
\]
\[
= \{ \alpha \in I_\infty^N : \lambda(\alpha) = g_0 \}
\]
is measurable. Thus, \( \lambda \) is measurable.

If \( \lambda \) is measurable, then it follows from formula (9) that for every \( k \in \mathbb{Z} \) the map \( \alpha \mapsto \lambda_\alpha(k) \) from \( I_\infty^N \) to \( D \) is also measurable. However, in this case, the set on the right-hand side of formula (10) is a countable union of measurable sets, that is, it is measurable. Hence \( \Lambda \) is measurable.

The last assertion of the lemma is trivial, since the maps \( \lambda_\alpha \) are uniquely defined by \( \lambda \) via formula (9).

Choose a map \( \lambda : I_\infty^N \to D \) which takes everything to the zero element of the group \( D \). By relations (9), this map generates a family of maps \( \lambda_\alpha : \mathbb{Z} \to D, \alpha \in I_\infty^N \), which, by formula (8), in turn defines a fibrewise map \( \Lambda : I^D \times I_\infty^N \to I^Z \times I_\infty^N \). We fix the symbols \( \lambda_\alpha \) and \( \Lambda \) for these objects.

Thus, the action of the adic transformation \( \mathcal{A} \) on the path space \( T(OP) \) of the graph \( OP \) is Borel isomorphic to the action of the transformation \( S \times \emptyset \) on the space \( I^Z \times I^N \).

This implies the following theorem on the universal adic approximation of automorphisms.

**Theorem 2.** Let \( T \) be an automorphism of a Lebesgue space \((X, p)\) with a binomial generator. Then there is a measure \( \mu \in \text{Inv}(OP) \) such that the adic transformation \( \mathcal{A} \) on the path space \( T(OP) \) of the graph \( OP \) with the measure \( \mu \) is isomorphic to the transformation \( T \times \emptyset \) on the direct product \( X \times I^N \) with the measure \( p \times m \).

**Definition 7.** We denote by \( \mathbb{M} \) the set of Borel probability measures on the space \( I^Z \times I^N \) that are invariant under the transformation \( S \times \emptyset \) and by \( \text{Ex}(\mathbb{M}) \) the set of ergodic measures in \( \mathbb{M} \), which is the Choquet boundary consisting of the extreme points.

**Remark 4.** The map \( \Lambda \circ \Psi \) is a bijection between the set of central measures \( \text{Inv}(OP) \) on the path space \( T(OP) \) of the graph \( OP \) and the set \( \mathbb{M} \).

The following remark gives an explicit formula for the fibrewise map \( \Lambda \).

**Remark 5.** For every \( \alpha \in I_\infty^N \) the map \( \lambda_\alpha \) satisfies the relation
\[
\lambda_\alpha \left( \sum_{i=0}^{n} (-1)^{\alpha_{i+1} + \beta_i} 2^i \right) = \sum_{i=0}^{n} \beta_i g_i,
\]
where \( n \geq 0 \) and \( \beta_i \in \{0, 1\} \).

The pre-image of the subgroup \( D_n \) under the map \( \lambda_\alpha \) is the segment of integers \( \{-a_n, \ldots, -a_n + 2^n - 1\} \), where \( a_n = \sum_{i=1}^{n} \alpha_i 2^{i-1} \). Here \( \lambda_\alpha^{-1}(D_n) \) is the left half of \( \lambda_\alpha^{-1}(D_{n+1}) \) if \( \alpha_{n+1} = 0 \) and the right half if \( \alpha_{n+1} = 1 \).
§ 5. Invariant measures on $I^Z \times I^N$

5.1. Direct products and measures of periodic type. In this subsection, we study the measures on $I^Z \times I^N$ that are invariant under the action of $S \times O$. We recall that we have denoted the set of these measures by $M$. This set of measures is taken under the map $\Lambda^{-1}$ to the set of all measures on $D^Z \times I^N$ invariant under the action $\text{diag}$ of the group $D$. Under the map $\Psi^{-1} \circ \Lambda^{-1}$, this set passes into the set of central measures on the path space $T(\text{OP})$ of the graph $\text{OP}$ of ordered pairs.

The simplest examples of measures in $M$ are the direct products. If $\eta$ is a Borel probability measure on $I^Z$ invariant under the shift $S$, then its direct product with the Lebesgue measure $m$ on $I^N$, that is, the measure $\eta \times m$, is invariant under the transformation $S \times O$.

**Definition 8.** Let $\nu$ be a measure on $I^Z \times I^N$. We denote by $P[\nu]$ the projection of $\nu$ to $I^Z$. Let $k \geq 0$ and $0 \leq r \leq 2^k - 1$ and let $A_{r,k} = \{\alpha \in I^N: \sum_{i=1}^{k} \alpha_i 2^{i-1} = r\}$ be a cylinder in $I^N$. Let $P_{r,k}[\nu]$ denote the normalized projection to $I^Z$ of the restriction of the measure $\nu$ to the set $I^Z \times A_{r,k}$, $\theta_k[\nu]$ the measure $P_{0,k}[\nu]$, and $\Theta[\nu]$ the sequence of measures $(\theta_k[\nu])_{k \geq 0}$.

A slightly more complicated example of a measure in $M$ is given by the following construction, and we call it a *measure of periodic type*. **Definition 9.** We denote by $M_k$, $k \geq 0$, the set of all Borel probability measures on $I^Z$ that are invariant under $S^{2^k}$, and by $\text{Ex}(M_k)$ the set of extreme points in $M_k$.

**Remark 6.** For every $k \geq 0$ the set $M_k$ with the topology of weak convergence of measures is a Poulsen simplex, that is, the set $\text{Ex}(M_k)$ of its extreme points is dense in $M_k$.

**Definition 10.** Let $k \geq 0$ and $\eta \in M_k$. We call the measure $D_k[\eta] = \sum_{i=0}^{2^k-1} S_i \eta \times (\chi_{A_{i,k}} m)$ (13) on $I^Z \times I^N$ a *measure of periodic type $k$ with base $\eta$*.

We note that the measures of periodic type 0 are simply direct products.

**Remark 7.** If $k \geq 0$ and $\eta \in M_k$, then $D_k[\eta] \in M$.

**Remark 8.** When $0 \leq k \leq n$ and $\eta \in M_k$ the equation $D_n[\eta] = D_k[\eta]$ holds.

**Definition 11.** We say that a measure $\nu$ on $I^Z \times I^N$ *is a measure of periodic type $k$* if $k$ is the least possible number for which there is a measure $\eta \in M_k$ such that $\nu = D_k[\eta]$. We denote by $M_k$ the set of all measures of periodic type not greater than $k$ on $I^Z \times I^N$ and by $\text{Ex}(M_k)$ the set of extreme points of $M_k$.

We say that a measure $\nu$ is a *measure of a periodic type* if it is a measure of periodic type $k$ for some $k \geq 0$.

A measure $\nu \in M$ which is not a measure of periodic type is said to be *aperiodic*. We denote by $M_\infty$ the set of aperiodic measures.
Remark 9. The set \( \mathbb{M} \) of invariant measures on \( I^Z \times I^N \) can be decomposed into the disjoint union of the set of aperiodic measures and the sets of measures of periodic type \( k, k \geq 0 \):
\[
\mathbb{M} = \mathbb{M}_\infty \cup \mathbb{M}_0 \cup \left( \bigcup_{k \geq 1} (\mathbb{M}_k \setminus \mathbb{M}_{k-1}) \right).
\]

For every \( k \geq 0 \) the map \( D_k \) is an affine homeomorphism between the sets \( \mathcal{M}_k \) and \( \mathbb{M}_k \), and hence a bijection between \( \text{Ex}(\mathcal{M}_k) \) and \( \text{Ex}(\mathbb{M}_k) \).

5.2. Approximation by measures of periodic type. We proceed with the study of measures in \( \mathbb{M} \) of general form.

Remark 10. For every measure \( \nu \in \mathbb{M} \) the family of measures \( \mathcal{P}_{r,k}[\nu], k \geq 0, 0 \leq r < 2^k \), on \( I^Z \) corresponding to \( \nu \) satisfies the following relations:
\[
\mathbb{S} \mathcal{P}[\nu] = \mathcal{P}[\nu]; \quad \mathbb{S}^2 \mathcal{P}_{r,k}[\nu] = \mathcal{P}_{r,k}[\nu]; \quad \mathbb{S} \mathcal{P}_{r,k}[\nu] = \mathcal{P}_{r+1,k}[\nu], \quad 0 \leq r < 2^k - 1; \quad \mathbb{S} \mathcal{P}_{2^k-1,k}[\nu] = \mathcal{P}_{0,k}[\nu]; \quad \mathcal{P}_{r,k}[\nu] = \frac{1}{2}(\mathcal{P}_{r,k+1}[\nu] + \mathcal{P}_{r+2^k,k+1}[\nu]) = \frac{1}{2}(\mathcal{P}_{r,k+1}[\nu] + \mathbb{S}^2 \mathcal{P}_{r,k+1}[\nu]).
\]

Corollary 1. Let \( \nu \in \mathbb{M} \). The sequence of measures \( \theta_k[\nu], k \geq 0, \) uniquely determines the whole family of measures \( \mathcal{P}_{r,k}[\nu], \) and hence also the measure \( \nu \). Moreover, the following relations hold:
\[
\mathbb{S}^{2^k} \theta_k = \theta_k, \quad \theta_k = \frac{1}{2}(\theta_{k+1} + \mathbb{S}^{2^k} \theta_{k+1}), \quad k \geq 0.
\]

Further, if a sequence of measures \( \theta_k, k \geq 0, \) on \( I^Z \) satisfies relations (17), then there is a unique measure \( \nu \in \mathbb{M} \) for which \( \theta_k = \theta_k[\nu], k \geq 0 \).

Definition 12. We denote by \( \text{proj}_k \) the projection of the set \( \mathcal{M}_{k+1} \) to \( \mathcal{M}_k \) given by the formula
\[
\text{proj}_k \theta = \frac{1}{2}(\theta + \mathbb{S}^{2^k} \theta), \quad \theta \in \mathcal{M}_{k+1}.
\]

Lemma 2. The map \( \Theta \) is an affine homeomorphism of the space \( \mathbb{M} \) of measures with the weak topology onto the projective limit \( \lim \mathcal{M}_k, \text{proj}_k \).

Definition 13. We say that a sequence of measures \( (\theta_k)_{k \geq 0} \) on \( I^Z \) stabilizes at a moment \( n, n \geq 0 \), if \( \theta_k = \theta_{k+1} \) for \( k \geq n \) and \( \theta_{n-1} \neq \theta_n \).

Remark 11. For every \( k \geq 0 \) the map \( \Theta \) is a bijection between the set \( \mathbb{M}_k \setminus \mathbb{M}_{k-1} \) of measures of periodic type \( k \) and the sequences stabilizing at the moment \( k \). Here the following identity holds for \( \eta \in \mathcal{M}_k \):
\[
\theta_n[D_k[\eta]] = \eta, \quad n \geq k.
\]

Corollary 2. The measures of periodic type are dense in \( \mathbb{M} \).

Proof. It can readily be seen that every measure \( \nu \in \mathbb{M} \) is approximated in the weak topology by the sequence of measures \( D_k[\theta_k[\nu]], k \to \infty. \)
As noted in Remark 6, the set $\mathbb{M}_k$ is a Poulsen simplex for every $k \geq 0$, that is, its Choquet boundary, the set $\text{Ex}(\mathbb{M}_k)$, is dense in $\mathbb{M}_k$. Corollary 2 means that the set $\bigcup_{k \geq 0} \mathbb{M}_k$ of measures of periodic type is dense in the simplex $\mathbb{M}$ of all invariant measures on the space $I^\mathbb{Z} \times I^\mathbb{N}$. This implies that the set $\bigcup_{k \geq 0} \text{Ex}(\mathbb{M}_k)$ is dense in $\mathbb{M}$. However, some of the measures in $\bigcup_{k \geq 0} \text{Ex}(\mathbb{M}_k)$ are not extreme points of $\mathbb{M}$ (see Lemma 5 below). Thus, a natural question arises: is $\mathbb{M}$ a Poulsen simplex? At the moment, the authors do not know the answer to this question. It seems that the question of whether or not the Poulsen property is preserved under certain operations over simplices of invariant measures has not been studied.

5.3. Ergodic invariant measures on the space $I^\mathbb{Z} \times I^\mathbb{N}$. Let us proceed with the study of the set $\text{Ex}(\mathbb{M})$, which is the set of measures in $\mathbb{M}$ that are ergodic with respect to $S \times \emptyset$.

The following lemma gives a characterization of ergodic measures in $\mathbb{M}$ in terms of their partial projections $\theta_k[\cdot], k \geq 0$.

**Lemma 3.** Let $\nu \in \mathbb{M}$. Then the following conditions are equivalent:

1) $\nu \in \text{Ex}(\mathbb{M})$;
2) $\theta_k[\nu] \in \text{Ex}(\mathbb{M}_k)$ for every $k \geq 0$;
3) $\theta_k[\nu] \in \text{Ex}(\mathbb{M}_k)$ for some $n \geq 0$ and every $k \geq n$.

**Proof.** To show that 1) implies 2), let $k \geq 0$ and recall that the symbol $A_{r,k}$ denotes the coordinate cylinders in $I^\mathbb{N}$ (see Definition 8). Let $B \subset I^\mathbb{Z}$ be an $S^{2k}$-invariant set. Then the set

$$\bigcup_{r=0}^{2^k-1} (S^r B) \times A_{r,k}$$

is $S \times \emptyset$-invariant. Hence, by the ergodicity of the measure $\nu$, the value of $\nu$ on this set is zero or one. This implies that $\theta_k[\nu](B)$ is also zero or one.

2) obviously implies 3). Now suppose that condition 3) holds. To prove 1), suppose the contrary, that $\nu$ is not ergodic. Then there are distinct $\nu_1, \nu_2 \in \mathbb{M}$ for which $\nu_1 + \nu_2 = 2\nu$. Then for every $k \geq n$ we have $\theta_k[\nu_1] + \theta_k[\nu_2] = 2\theta_k[\nu]$. Since the measure $\theta_k[\nu]$ is ergodic, we have the equation $\theta_k[\nu_1] = \theta_k[\nu_2]$ for $k \geq n$. By relation (17), this also holds for all $k \geq 0$. Then the measures $\nu_1$ and $\nu_2$ coincide, which contradicts the procedure of their construction. □

**Corollary 3.** Let $\nu \in \text{Ex}(\mathbb{M})$. Then for every $k \geq 0$ and distinct $r_1$ and $r_2$ in the set $\{0, \ldots, 2^k - 1\}$, the measures $\mathcal{P}_{r_1,k}[\nu]$ and $\mathcal{P}_{r_2,k}[\nu]$ either coincide or are mutually singular.

**Proof.** Each of the measures $\mathcal{P}_{r,k}[\nu], 0 \leq r \leq 2^k - 1$, is ergodic for the transformation $S^{2^k}$. Any two ergodic measures either coincide or are mutually singular. □

5.3.1. Ergodic measures of periodic type. We need the following simple lemma.

**Lemma 4.** Let $T$ be an automorphism of a Lebesgue space $(X, \mu)$. If the transformation $T^2$ is ergodic on $(X, \mu)$, then for every $k \geq 1$ the transformation $T^2^k$ is also ergodic on $(X, \mu)$.
Proof. The ergodicity of the transformation $T^2$ implies that of the automorphism $T$ itself, and also the fact that $-1$ is not an eigenvalue of the unitary operator $U_T$ on $L^2(X, \mu)$ corresponding to the automorphism $T$. Since the eigenvalues of $U_T$ form a group, this implies that the only one which is a root of degree $2^k$ of 1 is the number 1. Therefore, 1 is a multiplicity-free eigenvalue of the operator $T^{2^k}$. □

The following lemma gives a description of the set of measures of periodic type in $\text{Ex}(\mathcal{M})$.

Lemma 5. Let $k \geq 0$ and $\eta \in \mathcal{M}_k$. Then $D_k[\eta] \in \text{Ex}(\mathcal{M})$ if and only if $\eta \in \text{Ex}(\mathcal{M}_{k+1})$.

Proof. It follows from Remark 11 that the equation $\theta_n[D_k[\eta]] = \eta$ holds for $n \geq k$.

If the measure $D_k[\eta]$ is ergodic for $S \times \emptyset$, then, by Lemma 3, the measure $\eta = \theta_{k+1}[D_k[\eta]]$ is ergodic for $S^{2k+1}$.

If the measure $\eta$ is ergodic for $S^{2k+1}$, then, by Lemma 4, it is ergodic for $S^{2n}$, $n > k$. Hence, the measure $D_k[\eta]$ satisfies condition 3) of Lemma 3. This implies that $D_k[\eta]$ is ergodic for $S \times \emptyset$ and completes the proof of the lemma. □

Corollary 4. Let $k \geq 0$. Then (here and below we take $\mathcal{M}_{-1} = \emptyset$)

$$\text{Ex}(\mathcal{M}) \cap \mathcal{M}_k \setminus \mathcal{M}_{k-1} = \text{Ex}(\mathcal{M}_k) \cap \text{Ex}(\mathcal{M}_{k+1}) \setminus \text{Ex}(\mathcal{M}_{k-1}). \quad (18)$$

Proof. We first prove that the left-hand side of equation (18) is contained on the right-hand side. If $\nu \in \text{Ex}(\mathcal{M}) \cap \mathcal{M}_k \setminus \mathcal{M}_{k-1}$, then, obviously, $\nu \in \text{Ex}(\mathcal{M}_k)$. Further, there is a measure $\eta \in \mathcal{M}_k \setminus \mathcal{M}_{k-1}$ for which $\nu = D_k[\eta]$. By Lemma 5, since $\nu \in \text{Ex}(\mathcal{M})$, it follows that $\eta \in \text{Ex}(\mathcal{M}_{k+1})$. Hence, $\nu \in \text{Ex}(\mathcal{M}_{k+1})$. Here $\eta \notin \mathcal{M}_{k-1}$, and therefore $\nu \notin \text{Ex}(\mathcal{M}_{k-1})$.

To prove the reverse, let $\nu \in \text{Ex}(\mathcal{M}_k) \cap \text{Ex}(\mathcal{M}_{k+1}) \setminus \text{Ex}(\mathcal{M}_{k-1})$. Then $\nu = D_k[\eta]$ for some $\eta \in \mathcal{M}_k \cap \text{Ex}(\mathcal{M}_{k+1})$. It follows from Lemma 5 that $\nu \in \text{Ex}(\mathcal{M})$. Moreover, if $\nu \in \text{Ex}(\mathcal{M})$, then, obviously, $\nu \in \text{Ex}(\mathcal{M}_{k-1})$, that is, we arrive at a contradiction. Therefore $\nu \in \text{Ex}(\mathcal{M}) \cap \mathcal{M}_k \setminus \mathcal{M}_{k-1}$. □

Moreover, the following relation is obvious:

$$\text{Ex}(\mathcal{M}) \cap \mathcal{M}_\infty = \text{Ex}(\mathcal{M}_{\infty}).$$

Corollary 5. The set of ergodic measures in $\mathcal{M}$ admits the following grading:

$$\text{Ex}(\mathcal{M}) = \text{Ex}(\mathcal{M}_{\infty}) \cup \left( \bigcup_{k \geq 0} \left( \text{Ex}(\mathcal{M}_k) \cap \text{Ex}(\mathcal{M}_{k+1}) \setminus \text{Ex}(\mathcal{M}_{k-1}) \right) \right).$$

5.3.2. Aperiodic ergodic measures.

Theorem 3. Let $\nu \in \text{Ex}(\mathcal{M}_\infty)$. Then for every $k \geq 0$ and distinct $r_1$ and $r_2$ in the set $\{0, \ldots, 2^k - 1\}$, the measures $\mathcal{P}_{r_1,k}[\nu]$ and $\mathcal{P}_{r_2,k}[\nu]$ are mutually singular.

Proof. Assume that for some $k \geq 0$, $\mathcal{P}_{r_1,k}[\nu]$ and $\mathcal{P}_{r_2,k}[\nu]$ are not mutually singular and let $k$ be minimal among such numbers. We may assume without loss of generality that $r_1 < r_2$. By Corollary 3, the measures $\mathcal{P}_{r_1,k}[\nu]$ and $\mathcal{P}_{r_2,k}[\nu]$ coincide.
Since $k$ is minimal, all the measures $P_{r,k-1}[\nu]$, $0 \leq r < 2^{k-1}$, are pairwise singular. Using this argument and relation (16), we see that $r_2 - r_1 = 2^{k-1}$. Hence, $P_{r_1,k-1}[\nu]$ coincides with $P_{r_1,k}[\nu]$ and is ergodic for the transformation $S^{2^k}$. Thus, the measure $\theta_{k-1}[\nu]$ is also ergodic for $S^{2^k}$. By Lemma 4, $\theta_{k-1}[\nu]$ is ergodic for $S^{2^n}$ when $n \geq k$. On the other hand, when $n > k$, by formula (17), the measure $\theta_{k-1}[\nu]$ is the arithmetic mean of the shifts of the measure $\theta_n[\nu]$ which are invariant under $S^{2^n}$. Hence, $\theta_n[\nu] = \theta_{k-1}[\nu]$ when $n \geq k$. Hence, $\nu = D_k[\theta_{k-1}[\nu]]$, that is, $\nu$ is a measure of periodic type, which contradicts the assumption. □

**Definition 14.** For $n \geq 0$ we denote the number $e^{2^{1-n} \pi i}$ by $\varrho_n$ (it is the $2^n$th root of unity with the least positive argument).

**Corollary 6** (properties of aperiodic ergodic measures). Let $\nu$ be an aperiodic measure in $Ex(\mathbb{M})$. Then

1) the measures $\theta_k[\nu]$ and $S^r \theta_k[\nu]$ are mutually singular for $k \geq 1$ and $1 < r < 2^k - 1$;

2) (delta-type property) for $\theta_0[\nu]$-almost all $w \in I^Z$ the conditional measures in the sections $\{w\} \times I^N$ are delta measures;

3) the projection of $I^Z \times I^N$ to $I^Z$ is an isomorphism of the dynamical systems $(I^Z \times I^N, \nu, S \times \mathbb{O})$ and $(I^Z, \theta_0[\nu], S)$;

4) the numbers $\varrho_n$, $n \geq 0$, are the eigenvalues for the automorphism $S$ on the space $(I^Z, \theta_0[\nu])$;

5) the dynamical system $(I^Z, \theta_0[\nu], S)$ has a quotient isomorphic to the odometer $(I^N, m, \mathbb{O})$.

The following natural question arises: which measures on $I^Z$ can be projections of aperiodic measures in $Ex(\mathbb{M})$? By Lemma 3, these measures are ergodic with respect to the shift $S$. It turns out that the condition in part 5) of Corollary 6 is not only necessary but also sufficient.

**Lemma 6.** Let $\eta \in Ex(\mathcal{M}_0)$ and let the shift $S$ on the space $(I^Z, \eta)$ have a quotient isomorphic to the odometer. Then there is an aperiodic measure $\nu \in Ex(\mathbb{M})$ for which $\theta_0[\nu] = \eta$. Moreover, the set of all measures $\nu$ of this kind is naturally parametrized by the points $\alpha \in I^N$ (see Lemma 7).

**Proof.** For every $n \geq 0$ we can find a unique (up to constant) eigenfunction $f_n$ corresponding to the eigenvalue $\varrho_n$:

$$Sf_n = \varrho_n f_n.$$  

Multiplying these functions by suitable constants if necessary, we may assume that $f_0 = 1$ and $f_n = f_{n+1}^2$, $n \geq 0$. Then, almost everywhere with respect to the measure $\eta$, the values of the function $f_n$ coincide with the powers $\varrho_n^r$, $0 \leq r < 2^n - 1$, $n \geq 0$. Consider the level sets of the functions $f_n$:

$$B(r,n) = \{w \in I^Z: f_n(w) = \varrho_n^r\}, \quad 0 \leq n, \quad 0 \leq r < 2^n - 1.$$  

Obviously, for every fixed $n \geq 1$ the map $S$ permutes the sets $B(r,n)$, $r = 0, \ldots, 2^n - 1$, cyclically and therefore $\eta(B(r,n)) = 1/2^n$. 


Let \( \alpha \in I^\mathbb{N} \), \( \alpha = (\alpha_k)_{k \geq 1} \). We write \( r(n, \alpha) = \sum_{k=0}^{n-1} 2^k \alpha_{k+1} \), \( n \geq 0 \). For \( n \geq 0 \) we define a measure \( \theta_n^{(\alpha)} \) as the normalized restriction of the measure \( \eta \) to the set \( \mathcal{B}(n, \alpha) = B(r(n, \alpha), n) \):

\[
\theta_n^{(\alpha)} = 2^n \eta|_{\mathcal{B}(n, \alpha)}, \quad n \geq 0.
\]

It is clear that the measure \( \theta_n^{(\alpha)} \) is invariant under \( S^{2^n} \). It can readily be seen that the sequence of measures \( (\theta_n^{(\alpha)})_{n \geq 0} \) thus constructed satisfies relation (17). This follows immediately from the fact that the sets \( \mathcal{B}(n, \alpha) \) and \( \mathcal{B}(n + 1, \alpha) \) are connected by the equation

\[
\mathcal{B}(n, \alpha) = \{ f_{n+1}^2 = \theta_n^{(\alpha)} \} = \{ f_{n+1} = \theta_n^{(\alpha)} \} \cup \{ f_{n+1} = -\theta_n^{(\alpha)} \} = B(r(n, \alpha), n + 1) \cup B(2^n + r(n, \alpha), n + 1) = \mathcal{B}(n + 1, \alpha) \cup S^{2^n} \mathcal{B}(n + 1, \alpha).
\]

By Corollary 1, the sequence \( (\theta_n^{(\alpha)})_{n \geq 0} \) defines a measure \( \nu^{(\alpha)}[\eta] \in \mathbb{M} \) for which \( \theta_n^{(\alpha)}[\nu^{(\alpha)}[\eta]] = \theta_n^{(\alpha)}, \ n \geq 0 \). In particular,

\[
\theta_0^{(\alpha)}[\nu^{(\alpha)}[\eta]] = \theta_0^{(\alpha)} = \eta.
\]

It remains to prove that the measure \( \nu^{(\alpha)}[\eta] \) thus constructed is ergodic. By Lemma 3, its ergodicity is equivalent to the condition that the measure \( \theta_n^{(\alpha)} \) is ergodic under \( S^{2^n} \) for every \( n \geq 0 \). We now prove this fact.

Let a set \( B \subset I^\mathbb{Z} \) be invariant under \( S^{2^n} \). Then the function

\[
f = \sum_{j=0}^{2^n-1} \theta_n^j \chi_{S_j B}
\]

satisfies the equation \( Sf = \eta f \). All the eigenvalues of the operator \( S \) are multiplicity-free, and therefore \( f = C f_n \) almost everywhere with respect to the measure \( \eta \) for some constant \( C \in \mathbb{C} \). This implies that either \( \eta(B) = 0 \) or the set \( B \) coincides with one of the sets \( B(r, n), 0 \leq r \leq 2^n - 1 \). In particular, the set \( \mathcal{B}(n, \alpha) \) contains no non-trivial \( S^{2^n} \)-invariant subsets. Hence, the measure \( \theta_n^{(\alpha)} \) is ergodic for \( S^{2^n} \). This completes the proof of Lemma 6. \( \square \)

**Lemma 7.** The family of measures \( \nu^{(\alpha)}[\eta] \), \( \alpha \in I^\mathbb{N} \), constructed in the proof of Lemma 6 describes all the measures in \( \text{Ex}(\mathbb{M}) \) with the given projection \( \eta \). In other words, if \( \nu \in \mathbb{M} \) and \( \theta_0^{(\alpha)}[\nu] = \eta \), then \( \nu = \nu^{(\alpha)}[\eta] \) for some \( \alpha \in I^\mathbb{N} \).

Before passing to the proof of this lemma, we discuss a spectral property of measures of periodic type.

**Lemma 8.** Let \( k \geq 0 \), \( \eta \in \mathcal{M}_k \) and \( D_k[\eta] \in \text{Ex}(\mathbb{M}) \). Then the numbers \( \theta_n, n > k \), are not eigenvalues of the operator \( S \) on the space \( (I^\mathbb{Z}, \theta_0[D_k[\eta]]) \).

**Proof.** Let

\[
Sf = \theta_n f
\]

almost everywhere with respect to the measure \( \theta_0 = \theta_0[D_k[\eta]] \). We claim that \( f = 0 \) almost everywhere with respect to the measure \( \theta_0 \). It is clear that \( S^{2^n} f = f \)
almost everywhere with respect to the measures $S_j^\eta$, $0 \leq j \leq 2^k - 1$. By Lemmas 5 and 4, the measures $S_j^\eta$, $0 \leq j \leq 2^k - 1$, are ergodic for $\mathbb{S}^n$. Hence, the function $f$ is constant almost everywhere with respect to each of them. It follows that $f$ takes at most $2^k$ distinct values on some subset of full measure with respect to $\theta_0$. On the other hand, by relation (19), if $f$ is not almost everywhere zero, then it must take at least $2^n$ different values on sets of positive measure. Hence, $f = 0$ almost everywhere with respect to $\theta_0$. This completes the proof of Lemma 8. □

**Proof of Lemma 7.** We use the notation in the proof of Lemma 6. Let $\nu \in \mathbb{M}$ be ergodic and let $\theta_0[\nu] = \eta$. It follows from Lemma 8 that the measure $\nu$ cannot be of periodic type. By Corollary 6, the measures $\mathcal{P}_{r,n}[\nu]$, $0 \leq r \leq 2^n - 1$, are pairwise mutually singular for a fixed $n \geq 0$. Moreover, by the relations in Remark 10, the following equation holds:

$$\eta = \frac{1}{2^n} \sum_{r=0}^{2^n-1} \mathcal{P}_{r,n}[\nu].$$

Let a set $B_n \subset I^Z$ be such that $\mathcal{P}_{0,n}[\nu] = 2^n \eta\big|_{B_n}$. Then $\mathcal{P}_{r,n} = 2^n \eta\big|_{S^r B_n}$, $0 \leq r \leq 2^n - 1$, and the sets $S^r B_n$ are pairwise disjoint.

The function

$$g_n = \sum_{r=0}^{2^n-1} \varphi_n^r \chi_{S^r B_n}$$

satisfies the relation $S g_n = \varphi_n g_n$. Therefore, $g_n = c_n f_n$ for some constant $c_n \in \mathbb{C}$. This implies that $B_n = B(r_n, n)$ for some $r_n$, $0 \leq r_n \leq 2^n - 1$, and $c_n = \varphi_n^{-r_n}$.

It remains to find out how the numbers $r_n$ and $r_{n+1}$ are related. We note that the measure $\mathcal{P}_{0,n}$ is the semi-sum of the measures $\mathcal{P}_{0,n+1}$ and $\mathcal{P}_{2^n,n+1}$. Hence, $B_{n+1}$ is a subset of $B_n$. Thus, $B(r_{n+1}, n + 1) \subset B(r_n, n)$. By the definition of these sets, this means that $\varphi_{n+1}^{2r_{n+1}} = \varphi_n^{r_n}$, that is, $(r_{n+1} - r_n) : 2^n$. Hence, either $r_{n+1} = r_n$ or $r_{n+1} = r_n + 2^n$. We set $\alpha_{n+1} = 2^{-n}(r_{n+1} - r_n)$, $n \geq 0$. Then $r_n = r(n, \alpha)$ and $B_n = B(n, \alpha)$, which implies that $\theta_n[\nu] = \theta_n^\alpha$ and $\nu = \nu(\alpha)[\eta]$. This completes the proof of Lemma 7. □

5.3.3. Description of all ergodic measures. Finally, we have obtained the following description of the ergodic measures in $\mathbb{M}$.

**Theorem 4.** The ergodic measures in the set $\mathbb{M}$ are divided into two disjoint classes.

1) Ergodic measures of periodic type $k$, $k \geq 0$, of the form

$$D_k[\eta], \quad \text{where} \quad \eta \in \mathcal{M}_k \cap \text{Ex}(\mathcal{M}_{k+1}).$$

The projections of ergodic measures of periodic type to the space $I^Z$ admit the following description: a measure $\tilde{\eta}$ on $I^Z$ which is ergodic with respect to the shift $S$ is the projection of some ergodic measure of periodic type $k$ on $I^Z \times I^N$, $k \geq 0$, if and only if the operator $S$ on $(I^Z, \eta)$ has a quotient isomorphic to the shift on the set of $2^n$ points with uniform measure with $n = k$ rather than $n = k + 1$. 
2) Ergodic aperiodic measures of the form $\nu^{(\alpha)}[\eta]$, where the measure $\eta$ on the space $I^Z$ is invariant and ergodic with respect to the shift $S$ and, moreover, the operator $S$ on $(I^Z, \eta)$ has a quotient isomorphic to the odometer.$^3$

Remark 12. The ergodic measures in $M$ of periodic type 0 are direct products of the form $\eta \times m$, where $\eta$ stands for an $S$-invariant measure on $I^Z$ which is ergodic with respect to the transformation $S$.

The properties of ergodic aperiodic measures were described in detail in Corollary 6.

5.4. Additional information about the ergodic measures on $I^Z \times I^N$: the structure of conditional measures. Let an ergodic automorphism $T$ of the Lebesgue space $(X, \mu)$ have an invariant measurable partition $\varsigma$. In this case, the automorphism $T$ has a quotient automorphism $T_\varsigma$ acting on the quotient space $X_\varsigma$ with the quotient measure $\mu_\varsigma$. Then one says that $T$ is a skew product over the automorphism $T_\varsigma$ of the space $X_\varsigma$ with some system of automorphisms of the fibres (see, for example, [28]). In this case, it is usually assumed that, because of the ergodic property, the space $(X, \mu)$ is naturally isomorphic to the direct product of the space $(X_\varsigma, \mu_\varsigma)$ and the ‘generic fibre’ with the generic measure of the fibre, that is, a generic conditional measure on the elements of the partition $\varsigma$. However, especially if the space $X$ is equipped with the topology of a separable metric space, it is more convenient not to make a uniformization of the fibres but to assume that the fibres remain subsets of the original space $X$ with conditional measures as Borel measures on the fibres of the partition. Then the action of the automorphism is not changed, and only the problem of explicitly calculating the conditional measures remains. Usually, in Rokhlin theory, these measures are computed using a basis of the measurable partition $\varsigma$, and this is a general method.$^4$ In our case, the basis of the partition is determined by the structure of the quotient automorphism (the odometer), more precisely, by its spectrum. Thus, almost all the conditional measures (which are pairwise mutually singular) become limits of sequences of measures of the type of ergodic averagings (see below). We do not use any information about the conditional measures below. However, a similar method can be used for other skew products.

Definition 15. We say that a probability measure $\vartheta$ on $I^Z$ is averageable if for any $k \geq 1$ there is a measure $\vartheta_k$ on $I^Z$ which is the weak limit of the sequence of measures $\vartheta_{k,n}$, $n \to \infty$, where

$$\vartheta_{k,n} = \frac{1}{n} \sum_{j=0}^{n-1} S^{2^k j} \vartheta,$$

and the sequence of measures $\vartheta_k$ tends weakly to the measure $\vartheta$ as $k \to \infty$.

---

$^3$The authors are grateful to the referee for indicating another possible proof of this theorem using the general technique of (Furstenberg–Zimmer) automorphism extensions. While our proof is more specific, the alternative may possibly generalize in a simpler way to problems of describing the invariant measures for adic transformations.

$^4$We note that the family of conditional (canonical) measures corresponds to the kernel of the operator of conditional expectation (an orthogonal projection) onto the subalgebra corresponding to the partition $\varsigma$. 
The following lemma is a consequence of the ergodic theorem and the Lebesgue differentiation theorem.

**Lemma 9.** If $\nu \in \mathcal{M}$, then for almost all $\alpha \in I^\mathbb{N}$ the conditional measure on the fibre $I^\mathbb{Z} \times \{\alpha\}$ is averageable as a measure on $I^\mathbb{Z}$. Here the measure $\nu$ is uniquely determined by the generic conditional measure.

**Proof.** Let $\nu[\alpha]$ be the conditional measure on a fibre $I^\mathbb{Z} \times \{\alpha\}$. Then, since the measure $\nu$ is invariant under the transformation $S \times \mathcal{O}$, it follows that for almost all $\alpha \in I^\mathbb{N}$ we have

$$S^j \nu[\alpha] = \nu[\mathcal{O}^j \alpha], \quad j \in \mathbb{Z}.$$  

Let $\phi$ be the characteristic function of some cylindrical set in $I^\mathbb{Z}$. Consider the function $\psi$ on $I^\mathbb{N}$ defined almost everywhere by the formula

$$\psi(\alpha) = \int_{I^\mathbb{Z}} \phi(w) \, d\nu[\alpha](w),$$

that is, the integral of the function $\phi$ over the fibre. Let $k \geq 0$. Then for almost all $\alpha \in I^\mathbb{N}$ we have the equation

$$\frac{1}{n} \sum_{j=0}^{n-1} \psi(\mathcal{O}^j \alpha) = \int_{I^\mathbb{Z}} \phi(w) \, d\nu[\alpha]_{k,n}(w), \quad n \geq 1, \quad (21)$$

where the measure $\nu[\alpha]_{k,n}$ is the averaging (of the measure $\nu[\alpha]$) defined by formula (20). The transformation $\mathcal{O}^2$ is ergodic on each of the cylinders $A_{r,k}$, $0 \leq r \leq 2^k - 1$ (see Definition 8) with the Lebesgue measure, and therefore, by the ergodic theorem, for almost all $\alpha \in I^\mathbb{N}$ the left-hand side of equation (21) converges as $n \to \infty$ to the mean value of the function $\psi$ over the cylindrical set $A_{r,k}$ containing the point $\alpha$, that is,

$$\int_{I^\mathbb{Z}} \phi(w) \, d\nu[\alpha]_{k,n}(w) \xrightarrow{n \to \infty} \int_{(w,\beta) \in I^\mathbb{Z} \times A_{r,k}} \phi(w) \, d\nu(w,\beta) = \int_{I^\mathbb{Z}} \phi \, dP_{r,k}[\nu]$$

provided that $\alpha \in A_{r,k}$. Hence, for every $k \geq 0$ and almost every $\alpha \in I^\mathbb{N}$ the measures $\nu[\alpha]_{k,n}$ converge weakly to the measure $P_{r(k,\alpha),k}[\nu]$, where the number $r(k,\alpha)$ is chosen in such a way that $\alpha \in A_{r(k,\alpha),k}$.

Further, applying the Lebesgue differentiation theorem, we see that for almost every $\alpha \in I^\mathbb{N}$ the average value of the function $\psi$ over the cylindrical set $A_{r(k,\alpha),k}$ containing the point $\alpha$ converges to $\psi(\alpha)$ as $k \to \infty$. By choosing $\phi$ to be a countable family of cylinders, we get that for almost all $\alpha \in I^\mathbb{N}$ the measure $P_{r(k,\alpha),k}[\nu]$ converges to $\nu[\alpha]$. Hence, for almost all $\alpha \in I^\mathbb{N}$ the measure $\nu[\alpha]$ is averageable.

Moreover, for almost every $\alpha \in I^\mathbb{N}$ $\nu[\alpha]$ uniquely defines the sequence of measures $P_{r(k,\alpha),k}[\nu]$, $k \geq 0$ (as the limits of the averaging), and therefore it also determines the whole of the measure $\nu$. This completes the proof of Lemma 9. □

**Remark 13.** If $\nu \in \operatorname{Ex}(\mathcal{M})$, then for almost all $\alpha \in I^\mathbb{N}$ the conditional measures in the fibres $I^\mathbb{Z} \times \{\alpha\}$ are extreme points of the set of averageable measures on $I^\mathbb{Z}$.
§ 6. Evaluation of a scaling sequence

In this section we recall the definition of the scaling sequence of an action of a group (in our case, \( \mathbb{Z} \) or \( \mathbb{D} \)) and also the definition of the scaling sequence of a filtration. In [27], the scaling sequences of the action \( \kappa \) of \( \mathbb{D} \) and of the adic action of \( \mathbb{Z} \) on the path space \( T(\text{OP}) \) of the graph of ordered pairs \( \text{OP} \) with special measures \( \mu^\sigma \) were calculated. It turned out that the classes of scaling sequences of these actions coincide. In this section, we describe another way of calculating a scaling sequence for the action \( \kappa \) of \( \mathbb{D} \) on the space \((T(\text{OP}),\mu^\sigma)\). Moreover, we find the class of scaling sequences of the tail filtration \( \xi = (\xi_n)_{n\geq0} \) on the space \((T(\text{OP}),\mu^\sigma)\) and show that it coincides with the class of scaling sequences of the actions described previously.

6.1. Recalling the definitions and properties. We recall the part of the theory of admissible metrics we need (see, for example, [29], [30], [5], [6], [11], [31] and [27]).

6.1.1. Admissible semimetrics and \( \varepsilon \)-entropy.

**Definition 16.** Let \((X,\mu)\) be a Lebesgue space. A semimetric \( \rho \) on \( X \) (and the triple \((X,\mu,\rho)\)) is said to be admissible if \( \rho \) is measurable as a function of two variables with respect to the measure \( \mu^2 \) and there is a subset of full measure on which \( \rho \) is separable. If \( \rho \) is a metric on such a subset, then it is called an admissible metric. The cone of admissible summable (with respect to \( \mu^2 \)) semimetrics on \((X,\mu)\) is denoted by \( \text{Adm}(X,\mu) \).

To work with admissible semimetrics, the following norm on the space of functions of two variables is convenient. It has been referred to as an \( m \)-norm (see [11]).

**Definition 17.** For \( f \in L^1(X^2,\mu^2) \) we write

\[
\|f\|_m = \inf \left\{ \|\rho\|_{L^1(X^2,\mu^2)} : |f| \leq \rho \text{ almost everywhere with respect to } \mu^2, \right. \\
\left. \text{where } \rho \text{ is a measurable semimetric on } (X,\mu) \right\}.
\]

**Definition 18.** Let \( \rho \) be a measurable (as a function of two variables) semimetric on \((X,\mu)\) and let \( \varepsilon > 0 \). By the \( \varepsilon \)-entropy of the triple \((X,\mu,\rho)\) we mean the number \( H_\varepsilon(X,\mu,\rho) \) defined as the binary logarithm of the smallest positive integer \( k \) for which the space \( X \) can be partitioned into measurable sets \( X_0, \ldots, X_k \) in such a way that the set \( X_0 \) has small measure, \( \mu(X_0) < \varepsilon \), and the sets \( X_j, 1 \leq j \leq k \), have diameters less than \( \varepsilon \) in the semimetric \( \rho \). If there is no positive integer \( k \) of this kind, then \( H_\varepsilon(X,\mu,\rho) = +\infty \).

**Remark 14.** A measurable semimetric \( \rho \) on \((X,\mu)\) is admissible if and only if \( H_\varepsilon(X,\mu,\rho) < +\infty \) for every \( \varepsilon > 0 \).

**Lemma 10** (see [11] and [13]). \( \varepsilon \)-entropy has the following properties.

1) The \( \varepsilon \)-entropy \( H_\varepsilon(X,\mu,\rho) \) decreases with respect to \( \varepsilon \) and increases with respect to \( \rho \).

2) If \( \rho \in \text{Adm}(X,\mu) \) and \( \int_{X^2} \rho \, d\mu^2 < \varepsilon^2/2 \), then \( H_\varepsilon(X,\mu,\rho) = 0 \).
3) If \( \rho, \rho_1, \rho_2 \in \text{Adm}(X, \mu) \) and \( \rho \leq \rho_1 + \rho_2 \) almost everywhere with respect to \( \mu^2 \), then
\[
\mathbb{H}_{4\varepsilon}(X, \mu, \rho) \leq \mathbb{H}_{\varepsilon}(X, \mu, \rho_1) + \mathbb{H}_{\varepsilon}(X, \mu, \rho_2).
\]

4) If \( \rho_1, \rho_2 \in \text{Adm}(X, \mu) \) and \( \|\rho_1 - \rho_2\|_m < \varepsilon^2/32 \), then
\[
\mathbb{H}_{\varepsilon}(X, \mu, \rho_1) \leq \mathbb{H}_{\varepsilon/4}(X, \mu, \rho_2).
\]

6.1.2. Scaling sequences of a measure-preserving transformation.

**Definition 19.** Let \( (X, \mu) \) be a Lebesgue space and \( T \) a measure-preserving transformation on \( (X, \mu) \). Let \( \rho \) be a measurable semimetric on \( (X, \mu) \). By the **averaging** of the semimetric \( \rho \) under the action of \( T \) in \( n \) steps, \( n \geq 1 \), we mean the semimetric
\[
T_n^\text{av} \rho = \frac{1}{n} \sum_{j=0}^{n-1} T^j \rho.
\]

In what follows, we use the symbol \( \simeq \) for two sequences of positive numbers bounding each other with some constant:
\[
a_n \asymp b_n \iff 0 < \lim \inf_{n \to \infty} \frac{a_n}{b_n} \leq \lim \sup_{n \to \infty} \frac{a_n}{b_n} < \infty.
\]

**Definition 20.** Let \( T \) be a measure-preserving transformation of a Lebesgue space \( (X, \mu) \) and \( \rho \) an admissible semimetric on \( (X, \mu) \). A sequence \( h = (h_n)_{n \geq 1} \) of positive numbers is said to be **scaling for** \( \rho \) if the relation
\[
\mathbb{H}_{\varepsilon}(X, \mu, T_n^\text{av} \rho) \asymp h_n, \quad n \to \infty,
\]
holds for a sufficiently small \( \varepsilon > 0 \). The class of all scaling sequences for \( \rho \) is denoted by \( \mathcal{H}(X, \mu, T, \rho) \).

In [13] (see also the short communication [12]), the conjecture formulated by Vershik and claiming that a scaling sequence does not depend on the semimetric \( \rho \) was proved in a wide class of semimetrics.

**Definition 21.** A semimetric \( \rho \) on \( (X, \mu) \) is said to be (two-sided) **generating** for a transformation \( T \) if there is a subset \( X_0 \subset X \) of full measure such that for any two distinct points \( x, y \in X_0 \) there is an \( n \in \mathbb{Z} \) for which \( \rho(T^n x, T^n y) > 0 \).

Obviously, a measurable metric is a generating semimetric.

**Theorem 5** (see [13]). Let \( \rho_1, \rho_2 \in \text{Adm}(X, \mu) \) be generating semimetrics for a measure-preserving transformation \( T \). Then \( \mathcal{H}(X, \mu, T, \rho_1) = \mathcal{H}(X, \mu, T, \rho_2) \).

The proof of this theorem uses the following lemma.

**Lemma 11.** Let \( \rho_1, \rho_2 \in \text{Adm}(X, \mu) \) be two admissible metrics. Then for any \( \varepsilon > 0 \) there is a set \( X_0 \subset X \) such that \( \mu(X_0) > 1 - \varepsilon \) and the semimetrics \( \rho_1 \) and \( \rho_2 \) define the same topology on \( X_0 \).

Theorem 5 means that the class of scaling sequences is a characteristic of the transformation \( T \).
Definition 22. A sequence $h$ is said to be a scaling entropy sequence of a measure-preserving transformation $T$ of a Lebesgue space $(X, \mu)$ if $h \in \mathcal{H}(X, \mu, T, \rho)$ for some (and then for every) generating semimetric $\rho \in \mathcal{A}dm(X, \mu)$. We denote the class of scaling entropy sequences of $T$ by $\mathcal{H}(X, \mu, T)$.

The class of scaling entropy sequences is obviously a metric invariant.

It was proved in [20] that if the class of scaling sequences is non-empty, then it contains an increasing subadditive sequence. It was proved in [27] that there is a dynamical system with a prescribed scaling entropy sequence if this sequence is subadditive and increases. Thus, a characteristic of all possible scaling entropy sequences of measure-preserving transformations was given. Examples of dynamical systems with a given growth of scaling sequences were given by the adic transformation $A$ of the path space $T(OP)$ of the graph OP of ordered pairs with measures $\mu^\sigma$ for different sequences $\sigma$. Namely, the following theorem was proved.

Theorem 6 (see [27]). The sequence $h_n = 2^{\sum_{i=0}^{n-1} \sigma_i}, \; n \geq 1$, is a scaling sequence of the adic transformation $A$ of the space $(T(OP), \mu^\sigma)$.

6.1.3. Scaling sequences of the action of a group. By analogy with the definition for a single measure-preserving transformation, we can introduce the definition of a scaling sequence for the action of a group $G$. To do this, it is necessary to distinguish some equipment in $G$, namely, a family of subsets $G_n$, $n \in \mathbb{N}$, over which the averaging is to be carried out.

Definition 23. Let $G$ be a group of automorphisms of a Lebesgue space $(X, \mu)$ with equipment $G_n \subset G$, $n \geq 1$, and let $\rho$ be an admissible semimetric on $(X, \mu)$. A sequence $h = (h_n)_{n \geq 1}$ of positive numbers is called a scaling sequence for $\rho$ if the relation

$$\mathbb{H}_\varepsilon(X, \mu, T^{G_n}_\text{av}\rho) \asymp h_n, \quad n \to \infty,$$

holds for a sufficiently small $\varepsilon > 0$, where $T^{G_n}_\text{av}\rho = \frac{1}{|G_n|} \sum_{g \in G_n} \rho(g \cdot)$ is the averaging of $\rho$ with respect to the shifts in $G_n$.

We denote the class of all scaling sequences for a semimetric $\rho$ with respect to the action of $G$ by $\mathcal{H}(X, \mu, G, \rho)$.

As in the case of a single transformation, the scaling sequences of an action of an equipped group $G$ do not depend on the choice of the original metric.

Theorem 7 (see [27]). If $\rho_1, \rho_2 \in \mathcal{A}dm(X, \mu)$ are metrics, then

$$\mathcal{H}(X, \mu, G, \rho_1) = \mathcal{H}(X, \mu, G, \rho_2).$$

To replace the metrics in this theorem by generating semimetrics, it is necessary to impose some conditions on the sequence of sets $G_n$ (see [27]). As in the case of a single transformation, we introduce the notion of a scaling sequence of an action of an equipped group; this is a scaling sequence of an arbitrary admissible metric. The class of these sequences is a metric invariant of the action of the equipped group $G$ and is denoted by $\mathcal{H}(X, \mu, G)$.

The following assertion enables us to prove that some number sequence is scaling by testing this on a suitable sequence of semimetrics which need not be generating.
Lemma 12. Let a sequence of semimetrics \( \rho_k \in \text{Adm}(X, \mu) \), \( k \geq 1 \), together separate the points of a subset of full measure. Let a sequence \( h \) of positive numbers be such that \( h \in \mathcal{H}(X, \mu, G, \rho_k) \), \( k \geq 1 \). Then \( h \in \mathcal{H}(X, \mu, G) \).

Proof. One can choose a sequence of sufficiently small positive numbers \( c_k \), \( k \geq 1 \), in such a way that the function \( \rho = \sum c_k \rho_k \) is in \( \text{Adm}(X, \mu) \). Here \( \rho \) is an admissible metric. Using Lemma 10, we can readily see that \( h \in \mathcal{H}(X, \mu, G, \rho) \), and therefore \( h \in \mathcal{H}(X, \mu, G) \). \( \square \)

6.1.4. Scaling sequences of a filtration. The notion of the scaled entropy of a filtration can be found in [7] and [8], and also in the doctoral thesis of the first author (see also [5], [10], [21] and [29]). We recall one of the possible definitions of a scaling sequence of a filtration.

Let \( \varsigma = (\varsigma_n)_{n \geq 0} \) be a dyadic filtration on a Lebesgue space \( (X, \mu) \), where \( \varsigma_0 \) is the partition into points. Every element of the partition \( \varsigma_n \), \( n \geq 0 \), is naturally equipped with a dyadic hierarchy, namely, this element consists of two elements of the partition \( \varsigma_n-1 \), each of which also consists of two elements of the partition \( \varsigma_n-2 \), and so on up to the partitioning into points. The group \( T_n \) of automorphisms of the binary tree of height \( n \) (this tree has \( n+1 \) floors of vertices and \( n \) floors of edges) acts on the points of every element of the partition \( \varsigma_n \), and preserves the hierarchy (that is, the elements of previous partitions).

Let \( \rho \) be a semimetric on \( (X, \mu) \). For every \( n \geq 0 \) we construct a semimetric \( \mathcal{K}_n = \mathcal{K}_n[\rho] \) on the set of elements of the partition \( \varsigma_n \) as follows. Let \( c_1, c_2 \) be two elements of the partition \( \varsigma_n \), \( c_i = \{x_{i,j} : j = 1, \ldots, 2^n\} \), \( i = 1, 2 \). We write

\[
\mathcal{K}_n[\rho](c_1, c_2) = \inf_{S \in T_n} \frac{1}{2^n} \sum_{j=1}^{2^n} \rho(x_{1,j}, Sx_{2,j}).
\]

(22)

We note that the sequence \( \mathcal{K}_n[\rho] \), \( n \geq 0 \), of semimetrics can be constructed iteratively as follows. Let \( \mathcal{K}_0[\rho] = \rho \) be the semimetric on \( X = X/\varsigma_0 \). When \( n \geq 0 \) every point of the set \( X/\varsigma_{n+1} \) is an unordered pair of points in \( X/\varsigma_n \), and we can assign to this pair the semi-sum of the delta measures at the points. Thus, the set \( X/\varsigma_{n+1} \) is embedded in the space of measures on \( X/\varsigma_n \). There is a Kantorovich metric on the space of measures on the metric space \( (X/\varsigma_n, \mathcal{K}_n[\rho]) \), and it is this metric that defines the metric \( \mathcal{K}_{n+1}[\rho] \) on \( X/\varsigma_{n+1} \).

We note that the semimetric \( \mathcal{K}_n[\rho] \) defined on the quotient space \( (X/\varsigma_n, \mu/\varsigma_n) \) can also be treated as a semimetric on the original space \( (X, \mu) \) (one need only take its composite with the quotient map). In this case, the resulting semimetric triples are isomorphic, and the map preserving the measure and the semimetric is the quotient map.

Definition 24. A sequence of positive numbers \( h_n, n \geq 1 \), is called a scaling sequence for a semimetric \( \rho \) and a dyadic filtration \( \varsigma = (\varsigma_n)_{n \geq 0} \) of the space \( (X, \mu) \) if the following asymptotic relation holds for sufficiently small \( \varepsilon > 0 \):

\[
\mathbb{H}_\varepsilon(X/\varsigma_n, \mu/\varsigma_n, \mathcal{K}_n[\rho]) \asymp h_n, \quad n \to \infty.
\]

The class of scaling sequences for a semimetric \( \rho \) and a filtration \( \varsigma \) is denoted by \( \mathcal{H}(X, \mu, \varsigma, \rho) \).
As in the case of the entropy of an action, it turns out that the class of scaling sequences does not depend on the metric.

**Theorem 8.** Let $\varsigma = (\varsigma_n)_{n \geq 0}$ be a dyadic filtration on a Lebesgue space $(X, \mu)$ and let $\rho_1, \rho_2 \in \text{Adm}(X, \mu)$ be two metrics on $(X, \mu)$. Then

$$\mathcal{H}(X, \mu, \varsigma, \rho_1) = \mathcal{H}(X, \mu, \varsigma, \rho_2).$$

This theorem motivates the following definition.

**Definition 25.** A sequence of positive numbers $h_n, n \geq 1$, is said to be a scaling sequence of a dyadic filtration $\varsigma = (\varsigma_n)_{n \geq 0}$ of the space $(X, \mu)$ if it is scaling for some (and then also for every) summable admissible metric $\rho$ on $(X, \mu)$.

### 6.2. Independence of the metric of a scaling sequence of a filtration.

In this section we give a proof of Theorem 8 using the theory of admissible semimetrics. The following lemma is crucial in our proof.

**Lemma 13.** Let $\varsigma = (\varsigma_n)_{n \geq 0}$ be a dyadic filtration of a Lebesgue space $(X, \mu)$ and let $n \geq 0$ and $\rho_1, \rho_2 \in \text{Adm}(X, \mu)$. Then

$$\|K_n[\rho_1] - K_n[\rho_2]\|_m \leq 3\|\rho_1 - \rho_2\|_m.$$ 

In other words, the Kantorovich iteration $K_n$ is a Lipschitz function with respect to the $m$-norm with the Lipschitz constant equal to 3.

The proof of Lemma 13 uses the following observation which follows immediately from the triangle inequality.

**Assertion 1.** Let $N \geq 1$, let $\rho$ be a semimetric on a set $X$, and let $x_1, \ldots, x_N, y_1, \ldots, y_N \in X$. Then

$$\frac{1}{N} \sum_{j=1}^{N} \rho(x_j, y_j) \leq \frac{3}{N^2} \sum_{j=1}^{N} \sum_{k=1}^{N} \rho(x_j, y_k).$$

**Proof.** Let $k, m \in \{0, \ldots, N - 1\}$. Then for every $j, 1 \leq j \leq N$, it follows from the triangle inequality that

$$\rho(x_j, y_j) \leq \rho(x_j, y_{j+k}) + \rho(x_{j+k-m}, y_{j+k}) + \rho(x_{j+k-m}, y_j).$$

Summing this inequality over all $j, k, m$ modulo $N$ and dividing the result by $N^3$, we obtain the desired inequality. □

**Proof of Lemma 13.** Let $N = 2^n$ and let $c_1, c_2$ be two elements of the partition $\varsigma_n$. Moreover, let $c_1 = \{x_1, \ldots, x_N\}$ and $c_2 = \{y_1, \ldots, y_N\}$ and let $\rho$ be a measurable semimetric on $(X, \mu)$ such that $|\rho_2 - \rho_1| \leq \rho$. Then

$$K_n[\rho_2](c_1, c_2) \leq K_n[\rho + \rho](c_1, c_2) \leq K_n[\rho_1](c_1, c_2) + \max_{s \in S_N} \frac{1}{N} \sum_{j=1}^{N} \rho(x_j, y_{s(j)}).$$
where the maximum is taken over all permutations $s$ in the symmetric group $S_N$.

Estimating the last term using Assertion 1, we obtain the inequality

$$K_n[\rho_2](c_1, c_2) \leq K_n[\rho_1](c_1, c_2) + 3\bar{\rho}(c_1, c_2),$$

where $\bar{\rho}(c_1, c_2) = \frac{1}{N^2} \sum_{j=1}^{N} \sum_{k=1}^{N} \rho(x_j, y_k)$.

Obviously, the symmetric inequality also holds:

$$K_n[\rho_1](c_1, c_2) \leq K_n[\rho_2](c_1, c_2) + 3\bar{\rho}(c_1, c_2).$$

It only remains to note that the function $\tilde{\rho}$ defined in this way is a measurable semimetric on the space $X/\varsigma_n$ with the quotient measure $\mu/\varsigma_n$, and $\|\tilde{\rho}\|_{L^1(X^2, \mu^2)} = \|\rho\|_{L^1(X^2, \mu^2)}$. Hence,

$$\|K_n[\rho_1] - K_n[\rho_2]\|_m \leq 3\|\tilde{\rho}\|_{L^1(X^2, \mu^2)} = 3\|\rho\|_{L^1(X^2, \mu^2)}.$$

Passing to the infimum over all possible semimetrics $\rho$ dominating the modulus of the difference $|\rho_1 - \rho_2|$, we obtain the desired inequality. □

The following lemma is an analogue of Lemma 9 in [13].

**Lemma 14.** Let $\rho, \tilde{\rho} \in \text{Adm}(X, \mu)$, where $\rho$ is a metric. Then for every $\varepsilon > 0$ there are $\varepsilon_1 > 0$ and $C_1 > 0$ such that the following inequality holds for every $n \geq 0$:

$$H_\varepsilon(X, \mu, K_n[\tilde{\rho}]) \leq C_1 H_{\varepsilon_1}(X, \mu, K_n[\rho]). \quad (23)$$

**Proof.** Consider the set $M_\rho \subset \text{Adm}(X, \mu)$ consisting of the semimetrics $\tilde{\rho}$ for which the conclusion of the lemma holds. Our objective is to prove that $M_\rho = \text{Adm}(X, \mu)$.

The set $M_\rho$ obviously contains, together with every semimetric, all semimetrics dominated by that semimetric, that is, if $\rho_1 \in M_\rho$ and $\rho_2 \leq \rho_1$, then $\rho_2 \in M_\rho$.

We claim that $M_\rho$ is closed under the $m$-norm. Indeed, if a semimetric $\rho_1$ is contained in the closure of $M_\rho$, then for every $\varepsilon > 0$ there is a semimetric $\rho_2 \in M_\rho$ for which $\|\rho_1 - \rho_2\|_m < \varepsilon^2/96$. It follows from Lemma 13 that the inequality

$$\|K_n[\rho_1] - K_n[\rho_2]\|_m < \frac{\varepsilon^2}{32}$$

holds for every $n \geq 0$. By Lemma 10, the following inequality holds:

$$H_\varepsilon(X, \mu, K_n[\rho_1]) \leq H_{\varepsilon/4}(X, \mu, K_n[\rho_2]),$$

which implies that $\rho_1$ also belongs to $M_\rho$.

Let $f : (X, \mu) \rightarrow \mathbb{R}$ be a measurable function. We denote by $d[f]$ the semimetric constructed from $f$ as follows:

$$d[f](x, y) = |f(x) - f(y)|, \quad x, y \in X.$$

If $f$ is the characteristic function of a measurable set $A \subset X$, then $d[f]$ is called a cut semimetric (or simply a cut).
It can readily be seen that, if $f$ is a Lipschitz function with respect to $\rho$, then the semimetric $d[f]$ belongs to $M_\rho$. Indeed, if $|f(x) - f(y)| \leq C \rho(x, y)$ for some constant $C > 1$ and all $x, y \in X$, then $K_n[d[f]] \leq C K_n[\rho]$ for $n \geq 0$, and therefore
\[
H_\varepsilon(X, \mu, K_n[d[f]]) \leq H_{\varepsilon/C}(X, \mu, K_n[\rho]),
\]
which implies that $d[f] \in M_\rho$.

We can readily verify the inequality
\[
|d[f_1](x, y) - d[f_2](x, y)| \leq |f_1(x) - f_2(x)| + |f_1(y) - f_2(y)|, \quad x, y \in X,
\]
and the function on the right-hand side is a semimetric on $X$. This implies the inequality
\[
\|d[f_1] - d[f_2]\|_m \leq 2\|f_1 - f_2\|_{L^1(X, \mu)}.
\]
Let $f \in L^1(X, \mu)$. Approximating the function $f$ by functions Lipschitz with respect to $\rho$ in the $L^1$-norm and applying the closedness of $M_\rho$, we see that $d[f] \in M_\rho$.

It follows from what has been said that $M_\rho$ contains all semimetrics that are dominated by a finite sum of cut semimetrics. Semimetrics of this kind approximate, with respect to the $m$-norm, an arbitrary summable admissible semimetric (see the proof of Lemma 9 in [13]). Thus, it follows from the closedness of $M_\rho$ under the $m$-norm that $M_\rho = \text{Adm}(X, \mu)$. □

**Proof of Theorem 8.** By symmetry, it suffices to prove the inclusion $\mathcal{H}(X, \mu, \varsigma, \rho_1) \subset \mathcal{H}(X, \mu, \varsigma, \rho_2)$. If the first set is empty, then there is nothing to prove. Suppose that it is non-empty and that a sequence $h = (h_n)_{n \geq 0}$ of positive numbers is such that $h \in \mathcal{H}(X, \mu, \varsigma, \rho_1)$. We claim that $h \in \mathcal{H}(X, \mu, \varsigma, \rho_2)$.

It obviously follows from Lemma 14 applied to $\rho = \rho_1$ and $\tilde{\rho} = \rho_2$ that for every $\varepsilon > 0$ we have the relation
\[
\limsup_{n \to \infty} \frac{H_\varepsilon(X, \mu, K_n[\rho_2])}{h_n} < +\infty.
\]
It remains to be proved that the lower limit is bounded away from zero. To this end, we apply Lemma 14, transposing the roles of the metrics $\rho_1$ and $\rho_2$. Namely, when $\rho = \rho_2$ and $\tilde{\rho} = \rho_1$ and for every $\varepsilon > 0$ there are an $\varepsilon_1 > 0$ and a $C_1 > 0$ such that
\[
H_\varepsilon(X, \mu, K_n[\rho_1]) \leq C_1 H_{\varepsilon_1}(X, \mu, K_n[\rho_2]).
\]
If $\varepsilon$ is sufficiently small, then
\[
0 < \liminf_{n \to \infty} \frac{H_\varepsilon(X, \mu, K_n[\rho_1])}{h_n} \leq C_1 \liminf_{n \to \infty} \frac{H_{\varepsilon_1}(X, \mu, K_n[\rho_2])}{h_n}.
\]
Thus, if $\varepsilon < \varepsilon_1$, then
\[
0 < \liminf_{n \to \infty} \frac{H_\varepsilon(X, \mu, K_n[\rho_2])}{h_n}.
\]
This shows that $h \in \mathcal{H}(X, \mu, \varsigma, \rho_2)$ and completes the proof of the theorem. □
The following remark enables us to find a scaling sequence of a filtration by making the calculations for a suitable family of semimetrics.

**Assertion 2.** Let a sequence of summable admissible semimetrics \((\rho_k)_{k \geq 0}\) be non-decreasing and together separate the points of a space \((X, \mu)\) up to a set of measure zero. Let \(\varsigma = (\varsigma_n)_{n \geq 0}\) be a dyadic filtration on \((X, \mu)\). Suppose that a sequence \(h = (h_n)_{n \geq 0}\) of positive numbers is scaling for \(\varsigma\) and for every semimetric \(\rho_k, k \geq 0\). Then \(h\) is scaling for \(\varsigma\).

**Proof.** We first choose small positive coefficients \(C_k, k \geq 0\), in such a way that the function \(\rho = \sum_{k \geq 0} C_k \rho_k\) is a summable admissible metric. By Theorem 8 and Definition 25, it suffices to show that \(h \in H(X, \mu, \varsigma, \rho)\). The metric \(\rho\) obviously dominates every semimetric \(\rho_k, k \geq 0\), and therefore the bound for the lower limit is obvious. We now estimate the upper limit.

Let \(\varepsilon > 0\) and take an index \(l\) for which
\[
\left\| \rho - \sum_{k=0}^{l} C_k \rho_k \right\|_m = \left\| \sum_{k \geq l+1} C_k \rho_k \right\|_{L^1(X^2, \mu^2)} < \frac{\varepsilon^2}{96}.
\]

Then, by Lemma 13, for every \(n \geq 0\) we have the inequality
\[
\left\| \mathcal{K}_n[\rho] - \mathcal{K}_n \left[ \sum_{k=0}^{l} C_k \rho_k \right] \right\|_m < \frac{\varepsilon^2}{32},
\]
whence, by Lemma 10, the following inequality holds:
\[
\mathbb{H}_\varepsilon(X, \mu, \mathcal{K}_n[\rho]) \leq \mathbb{H}_{\varepsilon/4} \left( X, \mu, \mathcal{K}_n \left[ \sum_{k=0}^{l} C_k \rho_k \right] \right) \leq \mathbb{H}_{\varepsilon/4} \left( X, \mu, \sum_{k=0}^{l} C_k \cdot \mathcal{K}_n[\rho_l] \right) \leq C h_n
\]
for some constant \(C > 0\). This readily implies the desired bound for the upper limit. \(\square\)

For completeness we point out that the definition of the entropy of a homogeneous filtration (in particular, a dyadic one) and the original papers (such as [9]) used the metric entropies of some partitions of the orbit space of the automorphism group of a tree related to a filtration rather than the \(\varepsilon\)-entropy of the metrics. In comparable terms, the difference is that Kantorovich iterations of semimetrics determined by arbitrary functions with finitely many values were used. In essence, it was proved above that the supremum of the scaling sequences of entropies over all such semimetrics (that is, over all functions with finitely many values) can be replaced by an arbitrary metric. It seems that such a result holds for many inhomogeneous filtrations, for example, for semi-homogeneous ones (that is, for filtrations with respect to central measures on the path spaces of arbitrary graded graphs).

**6.3. Computation of scaling sequences of actions.** As already mentioned above (see Theorem 6), the class \(\mathcal{H}(T(\text{OP}), \mu^\sigma, A)\) was computed in [27]. In the same paper, the class \(\mathcal{H}(T(\text{OP}), \mu^\sigma, D)\) was computed for the action \(\kappa\) of the group \(D\) with the natural equipment \(D_n, n \geq 1\), and it was shown that these
classes coincide. In this subsection, we describe another way of computing the class \( \mathcal{H}(T(\text{OP}), \mu^\sigma, D) \).

We note the following general fact.

**Lemma 15.** Let an action of some equipped group \( G \) on a space \((X_1, \mu_1)\) have a scaling sequence \( h_n \), and its action on a space \((X_2, \mu_2)\) have a ‘discrete spectrum’ (that is, an invariant summable admissible metric). Then the direct product of the actions on the space \((X_1 \times X_2, \mu_1 \times \mu_2)\) also has a scaling sequence \( h_n \).

**Proof.** Let \( \rho_1 \) be a generating admissible semimetric on \((X_1, \mu_1)\) and \( \rho_2 \) an invariant admissible metric on \((X_2, \mu_2)\). Then the semimetric

\[
\rho\left((x_1, x_2), (y_1, y_2)\right) = \rho_1(x_1, y_1) + \rho_2(x_2, y_2)
\]

is admissible and generating for the direct product of the actions on the space \((X_1 \times X_2, \mu_1 \times \mu_2)\). Here the \( \varepsilon \)-entropies of finite averagings of this semimetric admit two-sided estimates, because of the inequality in Lemma 10, using the \( \varepsilon \)-entropies of an averaging of the semimetric \( \rho_1 \), since the metric \( \rho_2 \) is invariant and has finite \( \varepsilon \)-entropies. □

This implies an assertion, which was proved in [27].

**Corollary 7.** The sequence \( h_n = 2^{\sum_{i=0}^{n-1} \sigma_i} \) is scaling for the action diag of the group \( D \) on the space \((I^D \times I^N, \omega^\sigma)\), and therefore for the isomorphic canonical action \( \kappa \) of \( D \) on \((T(\text{OP}), \mu^\sigma)\).

**Proof.** The action of \( D \) on \((I^N, m)\) has an invariant metric, and therefore we arrive at the conditions of Lemma 15. Thus, the action of \( D \) on \((I^D \times I^N, \omega^\sigma)\) has the same scaling sequence as the action of \( D \) on \((I^D, m^\sigma)\). The measure \( m^\sigma \) is the pushforward of the Lebesgue measure on \( I^D/D^\sigma \) (see Definition 2), and therefore the scaling sequence of the action on \((I^D, m^\sigma)\) coincides with the scaling sequence of the action on \((I^D, m^\sigma)\) with the Lebesgue measure. The subgroup \( D^\sigma \) acts trivially on this space, and therefore the calculation reduces to that of a scaling sequence of the ‘effectively’ acting part of the group \( D \), that is, of the complementary subgroup

\[
\overline{D}^\sigma = \langle g_i : \sigma_i = 1, i \geq 0 \rangle.
\]

Let \( \rho \) be a cut along the first coordinate on \( I^{D^\sigma} = I^D/D^\sigma \), which is obviously an admissible semimetric that is generating for the action \( D \). Then the semimetric space \((I^{D^\sigma}, T_{av}^{D^\sigma} \rho)\) with the Lebesgue measure is isomorphic to the dyadic cube of dimension \( |D_n/(D^\sigma \cap D_n)| = 2^{\sum_{i=0}^{n-1} \sigma_i} \) with the uniform measure, which implies the result. □

Arguing in a similar way one can obtain the following result.

**Lemma 16.** Let \( H \subset D \) be a subgroup. Consider the measure \( m^H \) on \( I^D \) which is the pushforward of the Lebesgue measure on \( I^D/H \). Then the sequence \( h_n = |D_n/(H \cap D_n)| \) is scaling for the direct product of the actions of \( D \) (with the equipment \( D_n, n \geq 1 \)) on the space \((I^D \times I^N, m^H \times m)\).
6.4. Computing a scaling sequence of a filtration. In this subsection we compute a scaling sequence of the filtration $\zeta$ on $(I^D \times I^N, \omega^\sigma)$, and hence of the (isomorphic to $\zeta$) tail filtration $\xi$ on the path space $T(\text{OP})$ of the graph $\text{OP}$ of ordered pairs with the measure $\mu^\sigma$.

We first prove a lemma. Let $m \in \mathbb{Z}$, $m \geq 0$. We note that there is a dyadic hierarchy on the subgroup $D_m$, that of the cosets of the subgroups $D_j$, $0 < j < m$. The bijections of the set $D_m$ that preserve this hierarchy form a group isomorphic to the group $T_m$ of automorphisms of a binary tree of height $m$. For an arbitrary finite set $Q$, the action of $T_m$ on $Q^{D_m}$ by permuting the coordinates is trivially generated. This action obviously preserves the normalized Hamming metric $d_H$ on $Q^{D_m}$. Let

$$\text{dist}_m(w^{(1)}, w^{(2)}) = \min_{s \in T_m} d_H(w^{(1)}, Sw^{(2)}), \quad w^{(1)}, w^{(2)} \in Q^{D_m},$$

be the semimetric on $Q^{D_m}$ measuring the distance between orbits under the action of $T_m$.

**Lemma 17.** Let $Q$ be a finite set. Let $r, m \in \mathbb{Z}$, $0 \leq r \leq m$. Let $G_m$ be the subgroup of $D_m$ generated by some $r$ elements of $\{g_0, \ldots, g_{m-1}\}$. Let

$$W = \{w \in Q^{D_m} : w(g + \cdot) = w(\cdot) \ \forall g \in G_m\}$$

be the set of functions on $D_m$, with values in $Q$, that are invariant under the shift of the argument by elements of $G_m$. Let $\nu$ be the uniform measure on $W$. Then the $\varepsilon$-entropy of the triple $(Q^{D_m}, \text{dist}_m, \nu)$ admits a two-sided estimate for any sufficiently small positive $\varepsilon$ by $2^{m-r}$ with multiplicative constants depending on $\varepsilon$ and $|Q|$.

**Proof.** The measure $\nu$ is concentrated on $W$, which is the set of configurations that are invariant under shifts of the argument by elements of $G_m$. The set $W$ is not invariant under the action of $T_m$, and therefore the orbit of a function $w \in W$ under the action of $T_m$ can contain functions which are not in $W$. Nevertheless, the minimum of the distances in the Hamming metric between such orbits is achieved at elements of $W$. Thus, we can pass to the quotient by the action of $G_m$, under which the triple $(Q^{D_m}, \text{dist}_m, \nu)$ passes to an isomorphic triple. Hence, the $\varepsilon$-entropy of this triple is a function of $m - r$. We denote the $\varepsilon$-entropy under consideration by $h_{m-r}(\varepsilon) = h_{m-r}(\varepsilon, |Q|)$.

In what follows, we may assume that $r = 0$. Then $W = Q^{D_m}$. We seek two-sided bounds for the number $h_m(\varepsilon)$. The upper bound reduces to the standard entropy estimate. Indeed, this number does not exceed the $\varepsilon$-entropy of the uniform measure on the metric space $(Q^{D_m}, d_H)$. This space is a hypercube (with side $|Q|$) of dimension $2^m$ whose $\varepsilon$-entropy is well known and asymptotically proportional to the dimension of the cube, that is, to $2^m$.

The lower bound is found in a somewhat more subtle way and requires an analysis of the cases $|Q| = 2$ and $|Q| > 2$. In both cases, we estimate the size $M_m$ of a maximal orbit in the space $Q^{D_m}$ under the action of $T_m$. We shall find a constant $C_1 = C_1(\varepsilon, |Q|)$ such that the ball of radius $\varepsilon$ in the metric space $(Q^{D_m}, d_H)$
contains at most $C_1(\varepsilon, |Q|)^{2^m}$ points, and $C_1(\varepsilon, |Q|) \to 1$ as $\varepsilon \to 0$ for a fixed $|Q|$. The $\varepsilon$-neighborhood of every orbit contains at most $C_1(\varepsilon, |Q|)^{2^m}M_m$ points, and therefore the $\varepsilon$-entropy of the uniform measure admits the lower bound

$$h_m(\varepsilon) \geq \log\left((1 - \varepsilon)\frac{|Q|^{2^m}}{C_1(\varepsilon, |Q|)^{2^m}M_m}\right). \quad (24)$$

When $|Q| > 2$ it suffices to note that $M_m \leq |T_m| = 2^{2^m-1}$. Therefore, by inequality $(24)$ we have

$$h_m(\varepsilon) \geq 2^m \log\left(\frac{|Q|}{2C_1(\varepsilon, |Q|)}\right) + \log(1 - \varepsilon) \geq 2^mC_2(\varepsilon, |Q|)$$

for sufficiently small $\varepsilon$ and some positive constant $C_2(\varepsilon, |Q|)$.

When $|Q| = 2$, the number $M_m$ can be estimated by induction. Obviously, the inequality $M_{m+1} \leq 2M_m^2$ holds and we have $M_2 = 4$, and therefore $M_m \leq 2^{3 \cdot 2^{m-2} - 1}$. Applying this inequality together with $(24)$, we obtain the bound

$$h_m(\varepsilon) \geq 2^m \log\left(\frac{2^{1/4}}{C_1(\varepsilon, 2)}\right) + \log(1 - \varepsilon) \geq 2^mC_2(\varepsilon)$$

for sufficiently small $\varepsilon$ and some positive constant $C_2(\varepsilon)$.

Summarizing what has been said, we obtain that $h_m(\varepsilon) \asymp 2^m, m \to \infty$. □

**Theorem 9.** The sequence $h = (h_n)$, $h_n = 2\sum_{i=0}^{n-1} \sigma^i$, $n \geq 1$, is a scaling sequence of the filtration $\zeta = (\zeta_n)_{n \geq 0}$ on the space $I^D \times I^N$ with the measure $\omega^\sigma$ and also of the filtration $\xi = (\xi_n)_{n \geq 0}$ on the space $T(OP)$ with the measure $\mu^\sigma$, which is isomorphic to $\zeta$.

**Proof.** Let us compute the asymptotic behaviour of the entropies for the following sequence of semimetrics $\rho_k, k \geq 1$, on $I^D \times I^N$. Let $w^{(1)}, w^{(2)} \in I^D$ and $\alpha^{(1)}, \alpha^{(2)} \in I^N$. We write

$$\rho_k((w^{(1)}, \alpha^{(1)}), (w^{(2)}, \alpha^{(2)})) = \begin{cases} 0 & \text{if } w^{(1)}|D_k = w^{(2)}|D_k \text{ and } \alpha^{(1)}_i = \alpha^{(2)}_i \text{ for } i = 1, \ldots, k, \\ 1 & \text{otherwise}. \end{cases}$$

It is clear that this sequence of semimetrics is monotone increasing and separates the points of the space by $I^D \times I^N$. By Assertion 2, it suffices to show that the sequence $h$ is scaling for every semimetric $\rho_k, k \geq 1$.

Let $k$ be a fixed positive integer and let $n > k$. Our objective is to understand the structure of the semimetric $K_n[\rho_k]$ on the set of elements of the partition $\zeta_n$.

We define a map $\phi_n : I^D \times I^N \to I^D$ as follows. Every element of the partition $\zeta_n$ is the orbit of some point under the action of the group $D_n$, and this element contains a unique pair $(w, \alpha), w \in I^D, \alpha \in I^N$, for which $\alpha_i = 0$ for $i = 1, \ldots, n$. We say that this pair is the *representative* of this element of the partition and set $\phi_n = w|D_n$ on this orbit.
Let \((w^{(1)}, \alpha^{(1)})\) and \((w^{(2)}, \alpha^{(2)})\) be the representatives of elements \(c_1\) and \(c_2\), respectively, of the partition \(\zeta_n\). Then by (22) we have (recalling that the group \(T_n\) acts on \(D_n\) and preserves the hierarchy and that \(\tau\) embeds \(D\) in \(I^N\))

\[
\mathcal{K}_n[\rho_k](c_1, c_2) = \min_{S \in T_n} \left\{ \frac{1}{2^n} \sum_{g \in D_n} \rho_k \left( (w^{(1)}(g + \cdot), \tau(g) + \alpha^{(1)}), (w^{(2)}(Sg + \cdot), \tau(Sg) + \alpha^{(2)}) \right) \right\}.
\]  

(25)

Let \(D_{n,k}\) be the subgroup of \(D_n\) generated by the elements \(g_{k+1}, \ldots, g_n\), which is complementary to \(D_k\). The minimum in expression (25) is obviously achieved on the transformations \(S \in T_n\) for which \(Sg - g \in D_{n,k}\) for every \(g \in D_n\). Transformations of this kind can be represented in the form \(S(g + \tilde{g}) = g + \tilde{S}g\), where \(g \in D_k\), \(\tilde{g} \in D_{n,k}\), and the transformation \(\tilde{S}\) acts on \(D_{n,k}\), preserving the hierarchy. Therefore, we can rewrite formula (25) in the form

\[
\mathcal{K}_n[\rho_k](c_1, c_2) = \min_{S} \left\{ \frac{1}{2^n} \sum_{\tilde{g} \in D_{n,k}} \rho_k \left( (w^{(1)}(g + \tilde{g} + \cdot), \tau(g + \tilde{g}) + \alpha^{(1)}), (w^{(2)}(g + \tilde{g} + \cdot), \tau(g + \tilde{S}g) + \alpha^{(2)}) \right) \right\}
\]

\[
= \min_{S} \left\{ \frac{1}{2^{n-k}} \left\{ \tilde{g} \in D_{n,k} : w^{(1)}(\tilde{g} + \cdot) |_{D_k} \neq w^{(2)}(\tilde{S}g + \cdot) |_{D_k} \right\} \right\}.
\]

(26)

The restriction of a configuration \(w \in I^D\) to \(D_n\) can be viewed as an element of the space \(Q^{D_{n,k}}_k\) with \(Q_k = I^{D_k}\). We equip \(Q^{D_{n,k}}_k\) with the Hamming metric \(d_H\). The group \(T_{n-k}\) acts on this space, preserving the hierarchy. The last expression in formula (26) is precisely the distance between the orbits of this action that contain \(w^{(1)} |_{D_n} = \phi_n(w^{(1)}, \alpha^{(1)})\) and \(w^{(2)} |_{D_n} = \phi_n(w^{(2)}, \alpha^{(2)})\). In other words, the map \(\phi_n\) is an isometry taking the semimetric space \((I^D \times I^N)/\zeta_n, \mathcal{K}_n[\rho_k]\) to the metric space \((Q^{D_{n,k}}_k, d_H)\).

Let \(\nu_n\) be the pushforward of the measure \(\omega^\sigma\) under the map \(\phi_n\). This is a measure on \(I^{D_n} = Q^{D_{n,k}}_k\) concentrated and uniform on the set of configurations \(w \in I^{D_n}\) that satisfy the relation \(w(g_i + \cdot) = w(\cdot)\) for \(0 \leq i \leq n - 1, \sigma_i = 0\). Let \(Q = \{w \in I^{D_k} : w(\cdot + g_i) = w(\cdot) \text{ for } 0 \leq i \leq k - 1, \sigma_i = 0\}\).

Take \(m = n - k\). The group \(D_{n,k}\) is isomorphic to the group \(D_m\) (by a shift of the indexing of the generators). Let \(G_m\) be the subgroup of \(D_{n,k}\) generated by the elements \(g_i, k \leq i \leq n\), for which \(\sigma_i = 0\). We arrive at the conditions of Lemma 17, where \(\nu_n\) is the pushforward of \(\omega^\sigma\) under \(\phi_n\) and \(W\) is its support. Applying the lemma, we find the asymptotic behaviour of the \(\varepsilon\)-entropy of the quotient of \(\omega^\sigma\) with respect to the partition \(\zeta_n\) on the space \((I^D \times I^N)/\zeta_n\) with the semimetric \(\mathcal{K}_n[\rho_k]\) constructed from the semimetric \(\rho_k\):

\[
\mathbb{H}_\varepsilon((I^D \times I^N)/\zeta_n, \omega^\sigma/\zeta_n, \mathcal{K}_n[\rho_k]) = \mathbb{H}_\varepsilon(Q^{D_{n,k}}_k, \nu_n, d_H) \asymp \sum_{i=k}^{n} \sigma_i \asymp h_n, \quad n \to \infty.
\]
Thus, we have proved that the sequence \( h \) is scaling for the semimetric \( \rho_k, k \geq 1 \), and for the filtration \( \zeta \) on the space \((I^D \times I^N, \omega^\sigma)\). By Assertion 2, this means that \( h \) is a scaling sequence of the filtration \( \zeta \). This completes the proof of the theorem. \( \square \)

§ 7. Conclusion

1. The universal model of an adic action of the group \( \mathbb{Z} \) (and also of the infinite sum of groups of order 2) on the path space of the graph OP, which is suggested in the paper, can be compared with the generally accepted and well-known symbolic model of a group action, which is also universal. The advantage of the model suggested here is that it already has a canonical periodic approximation of the action, which is absent in the symbolic model.

2. The universal model assumes a description of all central measures on the path space of the graph. This description can be found in the paper, and it turns out to be highly visible modulo the description of the invariant measures in the symbolic model.

3. The scale of intermediate arbitrary sublinear asymptotic behaviours of the scaled entropy (the existence of these asymptotic behaviours has only been discovered recently) is associated with the scaled entropy of the filtrations on the path space of the graph, that is, finally, with the rate of periodic approximations of the adic action of groups. This coincidence reveals a deep connection between the theories of filtrations and periodic approximations. This connection will be investigated in future papers.

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