CLASSIFICATION OF THE STABLE SOLUTION TO BIHARMONIC PROBLEMS IN LARGE DIMENSIONS

JUNCHENG WEI, XINGWANG XU, AND WEN YANG

ABSTRACT. We give a new bound on the exponent for the nonexistence of stable solutions to the biharmonic problem

$$\Delta^2 u = u^p, \quad u > 0 \text{ in } \mathbb{R}^n$$

where $p > 1, n \geq 20$.

1. Introduction

Of concern is the following biharmonic equation

$$\Delta^2 u = u^p, \quad u > 0 \text{ in } \mathbb{R}^n \quad (1.1)$$

where $n \geq 5$ and $p > 1$. Let

$$\Lambda_u(\varphi) := \int_{\mathbb{R}^n} |\Delta \varphi|^2 dx - p \int_{\mathbb{R}^n} u^{p-1} \varphi^2 dx, \quad \forall \varphi \in H^2(\mathbb{R}^n). \quad (1.2)$$

The Morse index of a classical solution to (1.1), $\text{ind}(u)$ is defined as the maximal dimension of all subspaces of $E_{\mathbb{R}^n} := H^2(\mathbb{R}^n)$ such that $\Lambda_u(\varphi) < 0$ in $E_{\mathbb{R}^n} \{0\}$. We say $u$ is a stable solution to (1.1) if $\Lambda_u(\varphi) \geq 0$ for any test function $\varphi \in H^2(\mathbb{R}^n)$, i.e., the Morse index is zero.

In the first part, we obtain the following classification result of stable solution to (1.1).

**Theorem 1.1.** Let $n \geq 5$.

1. $n \leq 8$ and any $1 < p < \infty$, the equation (1.1) has no stable solution.
2. $9 \leq n \leq 19$, there exist $\varepsilon_n > 0$ such that for any $1 < p < \frac{n}{n-8} + \varepsilon_n$, the equation (1.1) has no stable solution.
3. $20 \leq n$ and $1 < p < 1 + \frac{6^{n-6}}{n-8}$, the equation (1.1) has no stable solution.

Here $p^*$ stands for the smallest real root which is greater than $\frac{n-4}{n-8}$ of the following algebraic equation

$$512(2-n)x^6 + 4(n^3 - 60n^2 + 670n - 1344)x^5 - 2(13n^3 - 424n^2 + 3064n - 5408)x^4 + 2(27n^3 - 572n^2 + 3264n - 5440)x^3 - (49n^3 - 772n^2 + 3776n - 5888)x^2 + 4(5n^3 - 66n^2 + 288n - 416)x - 3(n^3 - 12n^2 + 48n - 64) = 0.$$ 

Let us recall that for the second order problem

$$\Delta u + u^p = 0 \quad u > 0 \text{ in } \mathbb{R}^n, \quad p > 1, \quad (1.3)$$

Farina [3] gave a complete classification of all finite Morse index solutions. The main result of [3] is that no stable solution exists to (1.3) if either $n \leq 10, p > 1$ or $n \geq 11, p < p_{JL}$. Here $p_{JL}$ represents the well-known Joseph-Ludgren exponent, see [7]. On the other hand, stable radial solution exists for $p \geq p_{JL}$. 

1
In the fourth order case, the nonexistence of positive solutions to (1.1) are showed if \( p < \frac{n+4}{n-4} \), and all entire solutions are classified if \( p = \frac{n+4}{n-4} \), see [11], [14]. For \( p > \frac{n+4}{n-4} \), the radially symmetric solutions to (1.1) are completely classified in [4], [5] and [10]. The radial solutions are shown to be stable if and only if \( p \geq \rho_{JL}^{n} \) and \( n \geq 13 \) where \( \rho_{JL}^{n} \) stands for the corresponding Joseph-Lundgren exponent (see [4], [5]). In the general case, in [15] they showed the nonexistence of stable or finite Morse index solutions when either \( n \leq 8, p > 1 \) or \( n \geq 9, p \leq \frac{n-8}{n-8} \). In dimensions \( n \geq 9 \), a perturbation argument is used to show the nonexistence of stable solutions for \( p < \frac{n-8}{n-8} + \epsilon_{n} \) for some \( \epsilon_{n} > 0 \). No explicit value of \( \epsilon_{n} \) is given. The proof of [16] follows earlier idea of Cowan- Esposito- Ghoussoub [2] in which similar problem in a bounded domain was studied.

In the second order case, the proof of Farina uses basically the Moser iteration: namely multiply the equation (1.3) by the power of \( u \), like \( u^{4}, q > 1 \). Moser iteration works because of the following simple identity

\[
\int_{\mathbb{R}^{n}} u^{q}(-\Delta u) = \frac{4q}{(q+1)^{2}} \int_{\mathbb{R}^{n}} |\nabla u|^{2q}, \forall u \in C_{0}^{1}(\mathbb{R}^{n}).
\]

In the fourth order case, such equality does not hold, and in fact we have

\[
\int_{\mathbb{R}^{n}} u^{q}(\Delta^{2} u) = \frac{4q}{(q+1)^{2}} \int_{\mathbb{R}^{n}} |\Delta u|^{2q+1}, \forall u \in C_{0}^{1}(\mathbb{R}^{n}).
\]

The additional term \( \int_{\mathbb{R}^{n}} u^{q-3} |\nabla u|^{4} \) makes the Moser iteration argument difficult to use. In [15], they used instead the new test function \(-\Delta u\) and showed that \( \int_{\mathbb{R}^{n}} |\Delta u|^{2} \) is bounded. Thus the exponent \( \frac{n}{n-4} \) is obtained. In this paper, we use the Moser iteration for the fourth order problem and give a control on the term \( \int_{\mathbb{R}^{n}} u^{q-3} |\nabla u|^{4} \) (Lemma 2.3). As a result, we obtain a better exponent \( \frac{n}{n-8} + \epsilon_{n} \) where \( \epsilon_{n} \) is explicitly given. As far as we know, this seems to be the first result for Moser iteration for fourth order problem.

In the second part, we show that the same idea can be used to show the regularity of the extremal solutions to

\[
\begin{cases}
\Delta^{2} u = \lambda (u+1)^{p}, & \lambda > 0 \quad \text{in } \Omega \\
u > 0, & \text{in } \Omega \\
u = \Delta u = 0, & \text{on } \partial \Omega
\end{cases}
\]

(1.4)

where \( \Omega \) is a smooth and bounded convex domain in \( \mathbb{R}^{n} \).

For problem (1.4), it is known that there exists a critical value \( \lambda^{*} > 0 \) depending on \( p > 1 \) and \( \Omega \) such that

- If \( \lambda \in (0, \lambda^{*}) \), equation (1.4) has a minimal and classical solution which is stable;
- If \( \lambda = \lambda^{*} \), a unique weak solution, called the extremal solution \( u^{*} \) exists for equation (1.4);
- No weak solution of equation (1.4) exists whenever \( \lambda > \lambda^{*} \).

Our second result is the following.

**Theorem 1.2.** The extremal solutions \( u^{*} \), the unique solution of (1.4) when \( \lambda = \lambda^{*} \) is bounded provided that

1. \( n \leq 8 \) and any \( 1 < p < \infty \),
2. \( 9 \leq n \leq 19, \) there exist \( \epsilon_{n} > 0 \) such that for any \( 1 < p < \frac{n}{n-8} + \epsilon_{n} \),
3. \( 20 \leq n \) and \( 1 < p < 1 + \frac{8p^{*}}{n-4} \). (where \( p^{*} \) is defined as above)
This paper is organized as follows. We prove Theorem 1.1 and Theorem 1.2 respectively in section 2 and section 3. Some technical inequalities are given in the appendix.

2. Proof of Theorem 1.1

In this section, we prove Theorem 1.1 through a series of Lemmas. First of all, we have following.

Lemma 2.1. For any \( \varphi \in C^4_c(\mathbb{R}^n) \), \( \gamma > 1 \) and \( \varepsilon > 0 \) arbitrary small number, we have

\[
\int_{\mathbb{R}^n} (\Delta (u^\gamma \varphi^\gamma))^2 \leq \int_{\mathbb{R}^n} \left( (\Delta u^\gamma \varphi^\gamma)^2 + \varepsilon |\nabla u^\gamma \varphi^\gamma|^4 - C u^\gamma |\nabla^4 (\varphi^\gamma)| \right),
\]

where \( C \) is a positive number only depends on \( \gamma, \varepsilon \) and \( \|\nabla^4 (\varphi^\gamma)\| \) is defined by

\[
\|\nabla^4 (\varphi^\gamma)\|^2 = \varphi^{-2\gamma} |\nabla \varphi|^4 + |\varphi^\gamma (\Delta^2 \varphi^\gamma)| + |\nabla^2 \varphi^\gamma|^2.
\]

Proof. Since \( \varphi \) has compact support, we can freely use the integration by parts without mentioning the boundary term. First, by direct calculations, we get

\[
(\Delta (u^\gamma \varphi^\gamma))^2 = (\Delta u^\gamma \varphi^\gamma)^2 + 4 \nabla u^\gamma \nabla \varphi^\gamma \Delta u^\gamma \varphi^\gamma + 4 \nabla u^\gamma \nabla \varphi^\gamma \Delta u^\gamma \varphi^\gamma + 4 (\nabla^2 u^\gamma \nabla \varphi^\gamma)^2 + 2 \Delta u^\gamma u^\gamma \Delta \varphi^\gamma \varphi^\gamma + u^\gamma (\Delta \varphi^\gamma)^2.
\]

We now may deal with the third and fifth term on the right hand side of the above equality up to the integration both sides.

For the third term,

\[
\int_{\mathbb{R}^n} \Delta u^\gamma \nabla u^\gamma \nabla \varphi^\gamma \varphi^\gamma = - \int_{\mathbb{R}^n} (u^\gamma)^i_i (u^\gamma)^i_j (\varphi^\gamma)^j \varphi^\gamma
\]

\[- \int_{\mathbb{R}^n} (u^\gamma)^i_j (u^\gamma)^j_i (\varphi^\gamma) \varphi^\gamma - \int_{\mathbb{R}^n} (u^\gamma)^i_i (u^\gamma)^j_j (\varphi^\gamma)^i (\varphi^\gamma)^j,
\]

where \( f_i = \frac{\partial f}{\partial x_i} \) and \( f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j} \). (Here and in the sequel, we use the Einstein summation convention: an index occurring twice in a product is to be summed from 1 up to the space dimension, e.g., \( u_{i} v_{i} = \sum_{i=1}^{n} u_{i} v_{i} \), \( \partial_{i}(u_{i} u_{j} \varphi_{j}) = \sum_{1 \leq i, j \leq n} \partial_{i}(u_{i} u_{j} \varphi_{j}) \).)

The first term on the right hand side of the previous equation can be estimated as

\[
2 \int_{\mathbb{R}^n} (u^\gamma)^i_i (u^\gamma)^j_j (\varphi^\gamma)^i (\varphi^\gamma)^j = \int_{\mathbb{R}^n} \partial_{i}(u^\gamma)^i_i (u^\gamma)^j_j (\varphi^\gamma)^i (\varphi^\gamma)^j - \int_{\mathbb{R}^n} ((u^\gamma)^i_i)^2 (\varphi^\gamma)^i (\varphi^\gamma)^j,
\]

\[- \int_{\mathbb{R}^n} ((u^\gamma)^i_i)^2 (\varphi^\gamma)^i (\varphi^\gamma)^j.
\]
Combining these two equalities, we get
\[
2 \int_{\mathbb{R}^n} \Delta u^\gamma \nabla u^\gamma \nabla \varphi^\gamma = - \int_{\mathbb{R}^n} \partial_j ((u^\gamma)_i (u^\gamma)_j (\varphi^\gamma)_j) - \int_{\mathbb{R}^n} 2(u^\gamma)_i (u^\gamma)_j (\varphi^\gamma)_j \partial_j - \int_{\mathbb{R}^n} 2(u^\gamma)_i (u^\gamma)_j (\varphi^\gamma)_j (\varphi^\gamma)_i + \int_{\mathbb{R}^n} ((u^\gamma)_i)^2 (\varphi^\gamma)_j (\varphi^\gamma)_j + \int_{\mathbb{R}^n} ((u^\gamma)_i)^2 (\varphi^\gamma)_j (\varphi^\gamma)_j.
\]

Up to the short form of notation, thus we obtain
\[
4 \int_{\mathbb{R}^n} \Delta u^\gamma \nabla u^\gamma \nabla \varphi^\gamma = 2 \int_{\mathbb{R}^n} |\nabla u^\gamma|^2 \Delta \varphi^\gamma + 2 \int_{\mathbb{R}^n} |\nabla u^\gamma|^2 |\nabla \varphi^\gamma|^2 - 4 \int_{\mathbb{R}^n} (u^\gamma)_i (u^\gamma)_j (\varphi^\gamma)_j \partial_j - 4 \int_{\mathbb{R}^n} (<\nabla u^\gamma, \nabla \varphi^\gamma>)^2.
\]  
(2.5)

For the fifth term on the right hand side of Equation (2.4) we have
\[
\int_{\mathbb{R}^n} \Delta u^\gamma u^\gamma \Delta \varphi^\gamma = - \int_{\mathbb{R}^n} u^\gamma <\nabla u^\gamma, \nabla (\Delta \varphi^\gamma)> \varphi^\gamma - \int_{\mathbb{R}^n} \nabla u^\gamma, \nabla \varphi^\gamma > u^\gamma \Delta \varphi^\gamma - \int_{\mathbb{R}^n} |\nabla u^\gamma|^2 \Delta \varphi^\gamma \varphi^\gamma.
\]  
(2.6)

Combining Equations (2.4), (2.5) and (2.6), one obtains
\[
\int_{\mathbb{R}^n} (\Delta (u^\gamma \varphi^\gamma))^2 - \int_{\mathbb{R}^n} (\Delta u^\gamma)^2 \varphi^{2\gamma}
= 2 \int_{\mathbb{R}^n} |\nabla u^\gamma|^2 |\nabla \varphi^\gamma|^2 - 4 \int_{\mathbb{R}^n} \varphi^\gamma (\nabla^2 \varphi^\gamma (\nabla u^\gamma, \nabla u^\gamma)) + \int_{\mathbb{R}^n} u^{2\gamma} \varphi^\gamma \Delta^2 (\varphi^\gamma) - 4 \int_{\mathbb{R}^n} u^{2\gamma} (\Delta \varphi^\gamma)^2.
\]  
(2.7)

Now by the Young equality, for any \( \epsilon > 0 \), there exists a constant \( C(\epsilon) \) such that
\[
|\nabla u^\gamma|^2 |\nabla \varphi^\gamma|^2 \leq \frac{\epsilon}{4} |\nabla u^\gamma|^4 u^{-2\gamma} \varphi^{2\gamma} + C(\epsilon) |\nabla \varphi^\gamma|^4 u^{2\gamma} \varphi^{-2\gamma}
\]
and
\[
|\varphi^\gamma (\nabla^2 \varphi^\gamma (\nabla u^\gamma, \nabla u^\gamma))| \leq \frac{\epsilon}{8} |\nabla u^\gamma|^4 u^{-2\gamma} \varphi^{2\gamma} + C(\epsilon) u^{2\gamma} |\nabla^2 \varphi^\gamma|^2.
\]

Thus by the equation (2.4), together with these two estimates, one gets:
\[
|\int_{\mathbb{R}^n} (\Delta (u^\gamma \varphi^\gamma))^2 - \int_{\mathbb{R}^n} (\Delta u^\gamma)^2 \varphi^{2\gamma}| \leq \epsilon \int_{\mathbb{R}^n} |\nabla u^\gamma|^4 u^{-2\gamma} \varphi^{2\gamma} + 6C(\epsilon) \int_{\mathbb{R}^n} u^{2\gamma} |\nabla^2 \varphi^\gamma|^2.
\]

Thus the estimates (2.1) and (2.2) follow from this easily by observing that we always have \( |\Delta \varphi^\gamma|^2 \leq |\nabla^2 \varphi^\gamma|^2 \).

Now observe that \( |\nabla^2 u^\gamma|^{2\gamma} \varphi^{2\gamma} = \frac{1}{2} |\Delta |\nabla u^\gamma|^2 - <\nabla u^\gamma, \nabla \Delta u^\gamma> |\varphi^{2\gamma}. \)
Thus up to the integration by parts, with the help of the equation (2.5) and the estimates we just proved, the estimate (2.3) also follows except one should notice that \( \int_{\mathbb{R}^n} (\Delta (u^\gamma \varphi^\gamma))^2 = \int_{\mathbb{R}^n} |\nabla^2 (u^\gamma \varphi^\gamma)|^2. \) Thus Lemma 2.1 follows.

Let us return to the equation
\[
\Delta^2 u = u^p, \quad u > 0 \text{ in } \mathbb{R}^n.
\]  
(2.8)
Fix $q = 2\gamma - 1 > 0$ and $\gamma > 1$. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$. Multiplying (2.8) by $u^q \varphi^{2\gamma}$ and integration by parts, we obtain

$$\int_{\mathbb{R}^n} \Delta u \Delta (u^q \varphi^{2\gamma}) = \int_{\mathbb{R}^n} u^{p+q} \varphi^{2\gamma}.$$  \hfill (2.9)

For the left hand side of (2.9), we have the following lemma.

**Lemma 2.2.** For any $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\varphi \geq 0$, for any $\varepsilon > 0$ and $\gamma$ with $q$ defined above, there exists a positive constant $C$ such that

$$\int_{\mathbb{R}^n} \frac{\gamma^2}{q} \Delta u \Delta (u^q \varphi^{2\gamma}) \geq \int_{\mathbb{R}^n} (\Delta u^\gamma \varphi)^2 - \int_{\mathbb{R}^n} Cu^{2\gamma} ||\nabla^4 (\varphi^{2\gamma})|| - \int_{\mathbb{R}^n} (\gamma^2 (\gamma - 1)^2 + \varepsilon) u^{2\gamma-4} |\nabla u|^4 \varphi^{2\gamma}. \hfill (2.10)$$

**Proof.** First, by direct computations, we obtain

$$\Delta u \Delta (u^{2\gamma-1} \varphi^{2\gamma}) = \Delta u ((2\gamma - 1)u^{2\gamma-2} \Delta u \varphi^{2\gamma} + 2(2\gamma - 1)u^{2\gamma-2} \nabla u \nabla (\varphi^{2\gamma}) + (2\gamma - 1)(2\gamma - 2)u^{2\gamma-3} |\nabla u|^2 \varphi^{2\gamma} + u^{2\gamma-1} \Delta \varphi^{2\gamma}),$$

$$(\Delta u^\gamma \varphi)^2 = \gamma^2 u^{2\gamma-2} (\Delta u)^2 \varphi^{2\gamma} + \gamma^2 (\gamma - 1)^2 u^{2\gamma-4} |\nabla u|^4 \varphi^{2\gamma} + 2(\gamma - 1)^2 u^{2\gamma-3} |\nabla u|^2 \Delta u \varphi^{2\gamma}.$$ Combining the above two identities, we get

$$\frac{\gamma^2}{q} \Delta u \Delta (u^q \varphi^{2\gamma}) = (\Delta u^\gamma \varphi)^2 + 2\gamma^2 u^{2\gamma-2} \Delta u \nabla u \nabla \varphi^{2\gamma} + \frac{\gamma^2}{q} u^{2\gamma-1} \Delta u \Delta \varphi^{2\gamma} - \gamma^2 (\gamma - 1)^2 u^{2\gamma-4} |\nabla u|^4 \varphi^{2\gamma}. \hfill (2.11)$$

For the term $u^{2\gamma-2} \Delta u \nabla u \nabla \varphi^{2\gamma}$, we have

$$u^{2\gamma-2} \Delta u \nabla u \nabla \varphi^{2\gamma} = \partial_j (u^{2\gamma-2} u_i u_j (\varphi^{2\gamma})) - (2\gamma - 2) u^{2\gamma-3} (u_i)^2 u_j (\varphi^{2\gamma}))_j - u^{2\gamma-2} u_i (\varphi^{2\gamma})_j - u^{2\gamma-2} u_i u_j (\varphi^{2\gamma})_i.$$ We can regroup the term $u^{2\gamma-2} u_i u_j (\varphi^{2\gamma})_j$ as

$$2u^{2\gamma-2} u_i u_j (\varphi^{2\gamma})_j = \partial_j (u^{2\gamma-2} (u_i)^2 (\varphi^{2\gamma}))_j - (2\gamma - 2) u^{2\gamma-3} u_i u_j (\varphi^{2\gamma})_j - u^{2\gamma-2} (u_i)^2 (\varphi^{2\gamma})_j.$$ Therefore we get

$$2u^{2\gamma-2} \Delta u \nabla u \nabla \varphi^{2\gamma} = 2\partial_j (u^{2\gamma-2} u_i u_j (\varphi^{2\gamma}))_j - \partial_j (u^{2\gamma-2} (u_i)^2 (\varphi^{2\gamma}))_j - (2\gamma - 2) u^{2\gamma-3} (u_i)^2 u_j (\varphi^{2\gamma})_j + u^{2\gamma-2} (u_i)^2 (\varphi^{2\gamma})_j,$$

$$- 2u^{2\gamma-2} u_i u_j (\varphi^{2\gamma})_j.$$ For the last three terms on the right hand side of (2.12), applying Young's inequality, we get

$$|u^{2\gamma-3} (u_i)^2 u_j (\varphi^{2\gamma})_j| \leq \frac{\varepsilon}{6\gamma^2 (\gamma - 1)} u^{2\gamma-4} |\nabla u|^4 \varphi^{2\gamma} + Cu^{2\gamma} ||\nabla^4 (\varphi^{2\gamma})||,$$

$$|u^{2\gamma-2} (u_i)^2 (\varphi^{2\gamma})_j| \leq \frac{\varepsilon}{6\gamma^2} u^{2\gamma-4} |\nabla u|^4 \varphi^{2\gamma} + Cu^{2\gamma} ||\nabla^4 (\varphi^{2\gamma})||,$$

$$|u^{2\gamma-2} u_i u_j (\varphi^{2\gamma})_j| \leq \frac{\varepsilon}{6\gamma^2} u^{2\gamma-4} |\nabla u|^4 \varphi^{2\gamma} + Cu^{2\gamma} ||\nabla^4 (\varphi^{2\gamma})||.$$

CLASSIFICATION OF THE STABLE SOLUTION TO BIHARMONIC PROBLEMS IN LARGE DIMENSIONS
By the above three inequalities and (2.12), we have
\[
\int_{\mathbb{R}^n} 2\gamma^2 u^{2\gamma-4} \Delta u \nabla u \nabla \varphi^{2\gamma} \geq -\frac{\varepsilon}{2} \int_{\mathbb{R}^n} u^{2\gamma-4} |\nabla u|^{4} \varphi^{2\gamma} - C \int_{\mathbb{R}^n} u^{2\gamma} \|\nabla^4 (\varphi^{2\gamma})\|. \tag{2.13}
\]
Similarly we get
\[
\int_{\mathbb{R}^n} \frac{\gamma^2}{q} u^{2\gamma-1} \Delta u \Delta \varphi^{2\gamma} \geq -\frac{\varepsilon}{2} \int_{\mathbb{R}^n} u^{2\gamma-4} |\nabla u|^{4} \varphi^{2\gamma} - C \int_{\mathbb{R}^n} u^{2\gamma} \|\nabla^4 (\varphi^{2\gamma})\|. \tag{2.14}
\]
The inequality (2.10) follows from (2.11), (2.13) and (2.14).

Lemma 2.3. This is the key step in proving Theorem 1.1.

As a result of (2.1) and (2.10), we have
\[
\int_{\mathbb{R}^n} \frac{\gamma^2}{q} \Delta u \Delta (u^p \varphi^{2\gamma}) \geq \int_{\mathbb{R}^n} (\Delta (u^p \varphi^{2\gamma}))^2 - \int_{\mathbb{R}^n} C u^{2\gamma} \|\nabla^4 (\varphi^{2\gamma})\| \tag{2.15}
\]
Next we estimate the most difficult term \( \int_{\mathbb{R}^n} u^{2\gamma-4} |\nabla u|^{4} \varphi^{2\gamma} \) appeared in (2.15). This is the key step in proving Theorem 1.1.

**Lemma 2.3.** If \( u \) is the classical solution to the bi-harmonic equation \((2.8)\), and \( \varphi \) is defined as above, then for any sufficiently small \( \varepsilon > 0 \), we have the following inequality
\[
\left(\frac{1}{2} - \varepsilon\right) \int_{\mathbb{R}^n} u^{2\gamma-4} |\nabla u|^{4} \varphi^{2\gamma} \leq \frac{2}{\gamma^2} \int_{\mathbb{R}^n} (\Delta (u^\gamma \varphi^{2\gamma}))^2 + \int_{\mathbb{R}^n} C u^{2\gamma} \|\nabla^4 (\varphi^{2\gamma})\| - \int_{\mathbb{R}^n} \frac{4}{(4\gamma - 3 + p)(p + 1)} u^{2\gamma+p-1} \varphi^{2\gamma}. \tag{2.16}
\]

**Proof.** It is easy to see that
\[
\int_{\mathbb{R}^n} u^{2\gamma-4} |\nabla u|^{4} \varphi^{2\gamma} = \frac{1}{\gamma^4} \int_{\mathbb{R}^n} u^{-2\gamma} |\nabla u^\gamma|^{4} \varphi^{2\gamma}, \tag{2.17}
\]
and
\[
\int_{\mathbb{R}^n} u^{-2\gamma} |\nabla u^\gamma|^{4} \varphi^{2\gamma} = \int_{\mathbb{R}^n} u^{-2\gamma} |\nabla u^\gamma|^2 \nabla u^\gamma \nabla u^\gamma \varphi^{2\gamma} = \int_{\mathbb{R}^n} -\nabla u^{-\gamma} |\nabla u^\gamma|^2 \nabla u^\gamma \varphi^{2\gamma} = \int_{\mathbb{R}^n} |\nabla u^\gamma|^2 \Delta u^\gamma \varphi^{2\gamma} + \int_{\mathbb{R}^n} \nabla(|\nabla u^\gamma|^2) \nabla u^\gamma \varphi^{2\gamma} + \int_{\mathbb{R}^n} \frac{\nabla(|\nabla u^\gamma|^2) \nabla u^\gamma \varphi^{2\gamma}}{u^\gamma}, \tag{2.18}
\]
where the last step is integration by parts. For the first term in the last part of the above equality, we have
\[
\int_{\mathbb{R}^n} \frac{|\nabla u^\gamma|^2 \Delta u^\gamma \varphi^{2\gamma}}{u^\gamma} = \gamma^3 \int_{\mathbb{R}^n} ((\gamma - 1)u^{2\gamma-4} |\nabla u|^{4} \varphi^{2\gamma} + u^{2\gamma-3} |\nabla u|^2 \Delta u \varphi^{2\gamma}). \tag{2.19}
\]
The first term on the right hand side of (2.20) can be estimated as

\[
\int_{\mathbb{R}^n} u^{2\gamma-4} |\nabla u|^4 \varphi^{2\gamma} = \int_{\mathbb{R}^n} \frac{1}{\gamma^3} \nabla (|\nabla u^\gamma|^2) \nabla u^\gamma \varphi^{2\gamma} + \int_{\mathbb{R}^n} u^{2\gamma-3} (|\nabla u|^2) \Delta u \varphi^{2\gamma} + \int_{\mathbb{R}^n} \frac{1}{\gamma^3} (|\nabla u^\gamma|^2) \nabla u^\gamma \varphi^{2\gamma}.
\]

(2.20)

The first term on the right hand side of (2.20) can be estimated as

\[
u^{-\gamma} \nabla (|\nabla u^\gamma|^2) \nabla u^\gamma = 2u^{-\gamma}((u^\gamma)_i(u^\gamma)_j)
\]

\[
\leq 2\gamma(u^\gamma)_i(u^\gamma)_j + \frac{u^{-2\gamma}}{2\gamma} (u^\gamma)_i(u^\gamma)_j (u^\gamma)_j
\]

\[
= 2\gamma |\nabla^2 u^\gamma|^2 + \frac{u^{-2\gamma}}{2\gamma} |\nabla u^\gamma|^4.
\]

As a consequence, we have

\[
\int \frac{1}{\gamma^3} \nabla (|\nabla u^\gamma|^2) \nabla u^\gamma \varphi^{2\gamma} \leq \int \frac{2}{\gamma^2} |\nabla^2 u^\gamma|^2 \varphi^{2\gamma} + \int \frac{1}{2\gamma^4} |\nabla u^\gamma|^4 \varphi^{2\gamma}
\]

\[
\leq \int \frac{2}{\gamma^2} |\nabla^2 (u^\gamma \varphi^{\gamma/2})|^2 + \int \frac{C u^{2\gamma}}{2\gamma^2} \| \nabla^4 (\varphi^{2\gamma}) \|
\]

\[
+ \int \frac{1 + 4\gamma^2 \varepsilon}{2\gamma^2} |\nabla u^\gamma|^4 \varphi^{2\gamma}
\]

\[
= \int \frac{2}{\gamma^2} (\Delta (u^\gamma \varphi^{\gamma/2}))^2 + \int \frac{C u^{2\gamma}}{2\gamma^2} \| \nabla^4 (\varphi^{2\gamma}) \|
\]

\[
+ \int \frac{1 + 4\gamma^2 \varepsilon}{2\gamma^2} |\nabla u^\gamma|^4 \varphi^{2\gamma},
\]

(2.21)

where we used (2.23) in the last step.

For the second term on the right hand side of (2.20), applying the estimate (2.3) from [16], i.e., \((\Delta u)^2 \geq \frac{2}{\gamma^2} u^{\gamma+1}\), and the fact that \(\Delta u < 0\) from [13] or [16], we have

\[
\int \frac{1}{\gamma^3} (|\nabla u|^2) \Delta u \varphi^{2\gamma} \leq -\int \frac{2}{\gamma^2} u^{2\gamma-3 + \frac{\varepsilon+1}{2}} (|\nabla u|^2) \varphi^{2\gamma}
\]

\[
+ \int \frac{2}{\gamma^2} u^{2\gamma-2 + \frac{\varepsilon+1}{2}} \Delta u \varphi^{2\gamma}
\]

\[
+ \int \frac{2}{\gamma^2} u^{2\gamma-2 + \frac{\varepsilon+1}{2}} \nabla u \nabla \varphi^{2\gamma}.
\]

(2.22)

Using the inequality \(-\Delta u \geq \sqrt{\frac{2}{\gamma^2}} u^{\frac{\varepsilon+1}{2}}\), we get

\[
\int \frac{2}{\gamma^2} u^{2\gamma-2 + \frac{\varepsilon+1}{2}} \Delta u \varphi^{2\gamma} \leq -\int \frac{2}{\gamma^2} u^{2\gamma-p+1} \varphi^{2\gamma}.
\]

(2.23)
On the other hand, for the second term on the right hand side of (2.22), we have
\[
\int_{\mathbb{R}^n} u^{2\gamma-2+\frac{p+1}{2}} \nabla u \nabla \varphi^2 = -\int_{\mathbb{R}^n} \frac{1}{L} u^{2\gamma-1+\frac{p+1}{2}} \Delta \varphi^2
\]
\[
= -\int_{\{x|\Delta \varphi^2 > 0\}} \frac{1}{L} u^{2\gamma-1+\frac{p+1}{2}} \Delta \varphi^2 + \int_{\{x|\Delta \varphi^2 \leq 0\}} \frac{1}{L} u^{2\gamma-1+\frac{p+1}{2}} \Delta \varphi^2, \tag{2.24}
\]
where the first equality follows from integration by parts and \( L = 2\gamma - 1 + \frac{p+1}{2} \). As for the first term on the last part of (2.24), using the inequality \( \Delta u \leq -\sqrt{\frac{2}{p+1}} u^{\frac{p+1}{2}} < 0 \), we have
\[
\int_{\{x|\Delta \varphi^2 > 0\}} \frac{1}{L} u^{2\gamma-1} \Delta u \varphi^2 \leq -\int_{\{x|\Delta \varphi^2 > 0\}} \frac{1}{L} u^{2\gamma-1+\frac{p+1}{2}} \Delta \varphi^2. \tag{2.25}
\]
Similar to the proof of Lemma 2.1, it is easy to get
\[
| \int_{\{x|\Delta \varphi^2 > 0\}} \frac{1}{L} u^{2\gamma-1} \Delta u \varphi^2 | \leq \varepsilon \int_{\mathbb{R}^n} u^{2\gamma-4} |\nabla u|^4 \varphi^2 + \int_{\mathbb{R}^n} Cu^{2\gamma} \|\nabla^4 (\varphi^2)\|. \tag{2.26}
\]
By (2.25) and (2.26), we have
\[
| \int_{\{x|\Delta \varphi^2 > 0\}} \frac{1}{L} u^{2\gamma-1+\frac{p+1}{2}} \Delta \varphi^2 | \leq \varepsilon \int_{\mathbb{R}^n} u^{2\gamma-4} |\nabla u|^4 \varphi^2 + \int_{\mathbb{R}^n} Cu^{2\gamma} \|\nabla^4 (\varphi^2)\|. \tag{2.27}
\]
Similarly, we also obtain
\[
| \int_{\{x|\Delta \varphi^2 \leq 0\}} \frac{1}{L} u^{2\gamma-1+\frac{p+1}{2}} \Delta \varphi^2 | \leq \varepsilon \int_{\mathbb{R}^n} u^{2\gamma-4} |\nabla u|^4 \varphi^2 + \int_{\mathbb{R}^n} Cu^{2\gamma} \|\nabla^4 (\varphi^2)\|. \tag{2.28}
\]
By (2.24), (2.27) and (2.28), we have
\[
| \int_{\mathbb{R}^n} u^{2\gamma-2+\frac{p+1}{2}} \nabla u \nabla \varphi^2 | \leq \varepsilon \int_{\mathbb{R}^n} u^{2\gamma-4} |\nabla u|^4 \varphi^2 + \int_{\mathbb{R}^n} Cu^{2\gamma} \|\nabla^4 (\varphi^2)\|. \tag{2.29}
\]
Combining (2.22), (2.23) and (2.29), we get the following inequality
\[
\int_{\mathbb{R}^n} u^{2\gamma-3} |\nabla u|^2 \Delta u \varphi^2 \leq \varepsilon \int_{\mathbb{R}^n} u^{2\gamma-4} |\nabla u|^4 \varphi^2 + \int_{\mathbb{R}^n} Cu^{2\gamma} \|\nabla^4 (\varphi^2)\|
\]
\[
- \int_{\mathbb{R}^n} \frac{4}{(4\gamma-3+p)(p+1)} u^{2\gamma+p-1} \varphi^2. \tag{2.30}
\]
Finally, we apply Young’s inequality to the third term on the right hand side of (2.20), and get
\[
\int_{\mathbb{R}^n} \frac{(\nabla u^2)^2}{u^{2\gamma-3}} \nabla u^2 \nabla \varphi^2 \leq \varepsilon \int_{\mathbb{R}^n} u^{2\gamma-4} |\nabla u|^4 \varphi^2 + \int_{\mathbb{R}^n} Cu^{2\gamma} \|\nabla^4 (\varphi^2)\|. \tag{2.31}
\]
By (2.20), (2.21), (2.30) and (2.31), we finally obtain
\[
\left(\frac{1}{2} - \varepsilon\right) \int_{\mathbb{R}^n} u^{2n-4} \left| \nabla u \right|^4 (\varphi^{2\gamma})^2 \leq \frac{2}{\gamma^2} \int_{\mathbb{R}^n} (\Delta (u^\gamma \varphi)^2) + \int_{\mathbb{R}^n} Cu^{2\gamma} \|\nabla^4 (\varphi^{2\gamma})\| \\
- \int_{\mathbb{R}^n} \frac{4}{(4\gamma - 3 + p)(p + 1)} u^{2\gamma + p - 1} \varphi^{2\gamma}. 
\]

By (2.19), (2.16) and (2.15), since the number \( \varepsilon \) is arbitrary small in those three places, we have for \( \delta > 0 \) sufficiently small, the following inequality holds
\[
\int_{\mathbb{R}^n} (1 - 4(\gamma - 1)^2 - \delta)(\Delta (u^\gamma \varphi)^2)^2 - \int_{\mathbb{R}^n} \left(\frac{\gamma^2}{2\gamma - 1} - \frac{8\gamma^2(\gamma - 1)^2}{(4\gamma - 3 + p)(p + 1)}\right) u^{2\gamma - 2\gamma - 1} \varphi^{2\gamma} \\
\leq \int_{\mathbb{R}^n} C_\delta \|\varphi^{4\gamma}\|, \tag{2.32}
\]

where \( C_\delta \) is a positive constant depends on \( \delta \) only. Here, we need \( 1 - 4(\gamma - 1)^2 > 0 \), since we have assumed that \( \gamma > 1 \) in Lemma (2.1). So \( \gamma \) is required be in \((1, \frac{1}{2})\). While we can choose \( \delta \) small enough to make \( 1 - 4(\gamma - 1)^2 - \delta \) positive, by the stability property of function \( u \), we obtain
\[
\int_{\mathbb{R}^n} (E - p\delta) u^{pq} \varphi^{2\gamma} \leq \int_{\mathbb{R}^n} C_\delta u^{2\gamma} \|\nabla^4 (\varphi^{2\gamma})\|, \tag{2.33}
\]
where \( E \) is defined to be
\[
E = p(1 - 4(\gamma - 1)^2) - \frac{\gamma^2}{q} + \frac{8\gamma^2(\gamma - 1)^2}{(4\gamma - 3 + p)(p + 1)}. \tag{2.34}
\]

Now we take \( \varphi = \eta^m \) with \( m \) sufficiently large, and choose \( \eta \) a cut-off function satisfying \( 0 \leq \eta \leq 1 \), \( \eta = 1 \) for \( |x| < R \) and \( \eta = 0 \) for \( |x| > 2R \). By Young’s inequality again, we have
\[
\int_{\mathbb{R}^n} u^{2\gamma} \|\nabla^4 (\varphi^{2\gamma})\| \leq C_\delta R^{-4} \int_{\mathbb{R}^n} u^{2\gamma} \eta^{2\gamma m - 4} \\
\leq C_{\delta, \epsilon} R^{-\frac{4\gamma m}{1 - \theta}} \int_{\mathbb{R}^n} u^{2\gamma} \eta^{2\gamma m - 4} + \epsilon C_\delta \int_{\mathbb{R}^n} u^{2\gamma + p - 1} \eta^{2\gamma m}, \tag{2.35}
\]

where \( C_{\delta, \epsilon} \) is a positive constant depends on \( \delta, \epsilon, \theta \) is a number such that \( 2(1 - \theta) + (2\gamma + p - 1)\theta = 2\gamma \) so that \( 0 < \theta < 1 \) for \( 2 < 2\gamma < 2\gamma + p - 1 \). By (2.33) and (2.35), we get
\[
(E - p\delta - \epsilon C_\delta) \int_{\mathbb{R}^n} u^{p\gamma + 2\gamma - 1} \eta^{2\gamma m} \leq C_{\delta, \epsilon} R^{-\frac{4\gamma m}{1 - \theta}} \int_{\mathbb{R}^n} u^{2\gamma} \eta^{2\gamma m - \frac{4\gamma m}{1 - \theta}}. \tag{2.36}
\]

Since \( \theta \) is strictly less than \( 1 \) and will be fixed for given \( \gamma, p \), we can choose \( m \) sufficiently large to make \( 2\gamma m - \frac{4\gamma m}{1 - \theta} > 0 \). On the other hand, if \( E > 0 \), we can find small \( \delta \) and then small \( \epsilon \), such that \( E - p\delta - \epsilon C_\delta > 0 \). Therefore, by the definition of function \( \eta \) and (2.36), we obtain
\[
(E - p\delta - \epsilon C_\delta) \int_{B_{2R}} u^{p\gamma + 2\gamma - 1} \leq C_{\delta, \epsilon} R^{-\frac{4\gamma m}{1 - \theta}} \int_{B_{2R}} u^{2\gamma}. \tag{2.37}
\]

By (15), we have \( \int_{B_{2R}} u^2 \leq CR^{-\frac{8}{p - 1}} \), as a result, the left hand side of (2.37) is less equal than \( C_{\delta, \epsilon} R^{-\frac{4\gamma m}{1 - \theta} - \frac{8}{p - 1}} \), which tends to 0 as \( R \) tends to \( \infty \), provided
the power $n - \frac{s}{p-1} - \frac{4}{1-q}$ is negative. By the definition of $\theta$, this is equivalent to $(p + 2\gamma - 1) > (p - 1)\frac{n}{4}$. So, if $(p + 2\gamma - 1) > (p - 1)\frac{n}{4}$ and $E - p\delta - C_\delta \epsilon > 0$, we have that $u \equiv 0$.

Thus, we have proved the nonexistence of the stable solution to (2.8) if $p$ satisfies the condition $(p + 2\gamma - 1) > (p - 1)\frac{n}{4}$ and $E > 0$ (for $\delta, \epsilon$ are arbitrarily small).

By Lemma 4.1, the power $p$ can be in the interval $(\frac{n}{n-8}, 1 + \frac{8p^*}{n-4})$. Combining with Theorem 1.1 of [15], we prove the third part of Theorem 1.1, i.e., for any $1 < p < 1 + \frac{8p^*}{n-4}$, $n \geq 20$, equation (2.8) has no stable solution. The first and second part of Theorem 1.1 is contained in Theorem 1.1 of [15].

3. Proof of Theorem 1.2

In this section, we give the proof of Theorem 1.2. We note that it is enough to consider stable solutions $u_\lambda$ to (1.4) since $u^* = \lim_{\lambda \to \lambda^*} u_\lambda$. Now we give a uniform bound for the stable solutions to (1.4) when $0 < d < \lambda < \lambda^*$, where $d$ is a fixed positive constant from $(0, \lambda^*)$.

First, we need to analyze the solution near the boundary.

3.1. Regularity of the solution on the boundary. In this subsection, we establish the regularity of the stable solution with its derivative near the boundary of the following equation:

$$\begin{cases}
\Delta^2 u = \lambda(u + 1)^p, & \lambda > 0 \text{ in } \Omega \\
u > 0, & \text{in } \Omega \\
u = \Delta u = 0, & \text{on } \partial \Omega
\end{cases}$$

Theorem 3.1. Let $\Omega$ be a bounded, smooth, and convex domain. Then there exists a constant $C$ (independent of $\lambda, u$) and small positive number $\delta$, such that for stable solution $u$ to (3.1) we have

$$u(x) < C, \quad \forall x \in \Omega \epsilon := \{z \in \Omega : d(z, \partial \Omega) < \epsilon\}. \quad (3.2)$$

Proof. This result is well-known. For the sake of completeness, we include a proof here. By Lemma 3.5 of [2], we see that, there exists a constant $C$ independent of $\lambda, u$, such that

$$\int_{\Omega} (1 + u)^p dx \leq C. \quad (3.3)$$

We write Equation (3.1) as

$$\begin{cases}
\Delta u + v = 0, & \text{in } \Omega \\
\Delta v + \lambda(1 + u)^p = 0, & \text{in } \Omega \\
u = v = 0, & \text{on } \partial \Omega
\end{cases}$$

If we denote $f_1(u, v) = v$, $f_2(u, v) = \lambda(u + 1)^p$, we see that $\frac{\partial f_1}{\partial u} = 1 > 0$ and $\frac{\partial f_2}{\partial u} = \lambda p(u + 1)^{p-1} > 0$. Therefore, the convexity of $\Omega$, Lemma 5.1 of [12], and the moving plane method near $\partial \Omega$ (as in the appendix of [6]) imply that there exist $t_0 > 0$ and $\alpha$ which depends only on the domain $\Omega$, such that $u(x - tv)$ and $v(x - tv)$ are nondecreasing for $t \in [0, t_0]$, $v \in R^N$ satisfying $|v| = 1$ and
(\nu, \nu(x)) \geq \alpha \text{ and } x \in \partial \Omega. \text{ Therefore, we can find } \rho, \epsilon > 0 \text{ such that for any } x \in \Omega_\epsilon := \{z \in \Omega : d(z, \partial \Omega) < \epsilon\} \text{ there exists a fixed-sized cone } \Gamma_x \text{ (with } x \text{ as its vertex) with}

- \text{meas}(\Gamma_x) \geq \rho,
- \Gamma_x \subset \{z \in \Omega : d(z, \partial \Omega) < 2\epsilon\}, \text{ and}
- u(y) \geq u(x) \text{ for any } y \in \Gamma_x.

Then, for any } x \in \Omega_\epsilon, \text{ we have

\[ (1 + u(x))^p \leq \frac{1}{\text{meas}(\Gamma_x)} \int_{\Gamma_x} (1 + u)^p \leq \frac{1}{\rho} \int_{\Omega} (1 + u)^p \leq C. \]

This implies that \((1 + u(x))^p \leq C\), therefore \(u(x) \leq C\). \qed

**Remark:** By classical elliptic regularity theory, \(u(x)\) and its derivative up to fourth order are bounded on the boundary by a constant independent of \(u\). See [13] for more details.

### 3.2. Proof of Theorem 1.2

In the following, we will use the idea in Section 2 to prove Theorem 1.2. First of all, multiplying \((1.4)\) by \((u + 1)^q\) and integration by parts, we have

\[ \int_{\Omega} \lambda(u + 1)^{p+q} = \int_{\Omega} \Delta^2 u(u + 1)^q = \int_{\partial \Omega} \frac{\partial (\Delta u)}{\partial n} + \int_{\Omega} \Delta(u + 1)\Delta(u + 1)^q. \quad (3.4) \]

Setting \(v = u + 1\), by direct calculations, we get

\[ \int_{\Omega} (\Delta v)^2 = \int_{\Omega} \gamma^2 v^{2\gamma - 2}(\Delta v)^2 + \int_{\Omega} \gamma^2 (\gamma - 1)^2 v^{2\gamma - 4}|\nabla v|^4
\]

\[ + 2 \int_{\Omega} \gamma^2 (\gamma - 1)^2 v^{2\gamma - 3}\Delta v|\nabla v|^2, \quad (3.5) \]

\[ \int_{\Omega} \Delta v\Delta v^q = \int_{\Omega} q(\Delta v)^2 v^{q-1} + \int_{\Omega} q(q - 1)|\nabla v|^2\Delta v v^{q-2}. \quad (3.6) \]

From \((3.4), (3.5)\) and \((3.6)\), we obtain

\[ \int_{\Omega} \frac{q}{\gamma^2}(\Delta v)^2 - q(\gamma - 1)^2|\nabla v|^4 v^{2\gamma - 4} + \frac{\partial (\Delta v)}{\partial n} = \int_{\Omega} \lambda v^{p+q}. \quad (3.7) \]

For the second term in \((3.7)\), we have

\[ \int_{\Omega} |\nabla v|^4 v^{2\gamma - 4} = \frac{1}{\gamma^4} \int_{\Omega} v^{-2\gamma |\nabla v|^4} = \frac{1}{\gamma^4} \int_{\Omega} |\nabla v|^{2\gamma} v^{\gamma(-\nabla v^{\gamma})}
\]

\[ = \frac{1}{\gamma^4} \int_{\Omega} (-\nabla(|\nabla v|^2 v^{\gamma})) + \frac{\nabla(|\nabla v|^2 v^{\gamma})}{v^{\gamma}} + \frac{|\nabla v|^2\Delta v^{\gamma}}{v^{\gamma}})
\]

\[ = \frac{1}{\gamma^4} \int_{\Omega} \nabla(|\nabla v|^2 v^{\gamma}) + |\nabla v|^2\Delta v^{\gamma} v^{\gamma} - \frac{1}{\gamma} \int_{\partial \Omega} v^{2\gamma - 3}|\nabla v|^2\frac{\partial v}{\partial n}. \quad (3.8) \]

Since the simple calculation implies that

\[ \frac{1}{\gamma^4} \int_{\Omega} \frac{|\nabla v|^{2\gamma} \Delta v^{\gamma}}{v^{\gamma}} \leq \frac{\gamma - 1}{\gamma} \int_{\Omega} v^{2\gamma - 4}|\nabla v|^4 + \frac{1}{\gamma} \int_{\Omega} v^{2\gamma - 3}|\nabla v|^2 \Delta v, \quad (3.9) \]
by substituting (3.9) into (3.8), we get

\[ \int_{\Omega} |\nabla v|^4 v^{2\gamma - 4} = \int_{\Omega} v^{2\gamma - 3} |\nabla v|^2 \Delta v + \frac{1}{\gamma^3} \int_{\Omega} \frac{\nabla(|\nabla v|^2) \nabla v^\gamma}{v^\gamma} - \int_{\partial\Omega} |\nabla v|^2 \frac{\partial v}{\partial n} \]  

(3.10)

We now estimate the second term appeared on the right hand side of (3.10). From the proof of Lemma 2.3 together with the identity \( \frac{1}{2} \Delta |\nabla v|^2 = |\nabla^2 v|^2 + < \nabla\Delta v^1, \nabla v^1 > \), the following inequality holds

\[ \frac{1}{\gamma^3} \int_{\Omega} \frac{\nabla(|\nabla v|^2) \nabla v^\gamma}{v^\gamma} \leq \frac{1}{2} \int_{\Omega} |\nabla v|^4 v^{2\gamma - 4} + \frac{2}{\gamma^2} \int_{\Omega} (\Delta v^\gamma)^2 + \frac{1}{2} \int_{\partial\Omega} \frac{\partial |\nabla v|^2}{\partial n} - \frac{2}{\gamma^2} \int_{\partial\Omega} (\Delta v^\gamma) \frac{\partial v^\gamma}{\partial n}. \]  

(3.11)

By (3.10) and (3.11), thanks to the convexity of the domain \( \Omega \), we get

\[ \frac{1}{2} \int_{\Omega} |\nabla v|^4 v^{2\gamma - 4} \leq \int_{\Omega} v^{2\gamma - 3} |\nabla v|^2 \Delta v + \frac{2}{\gamma^2} \int_{\Omega} (\Delta v^\gamma)^2 - (2\gamma - 1) \int_{\partial\Omega} |\nabla v|^2 \frac{\partial v}{\partial n}. \]  

(3.12)

For the first term on the right hand side of (3.12), since \( v = u + 1 \), we have \( \Delta v = \Delta u < 0 \) by maximal principle, and the inequality \( \Delta v < -\sqrt{\frac{2\lambda}{p+1}} v^{\frac{p+1}{p+2}} < 0 \) by Lemma 3.2 of [2]. Thus

\[ \int_{\Omega} v^{2\gamma - 3} |\nabla v|^2 \Delta v \leq \int_{\Omega} -\sqrt{\frac{2\lambda}{p+1}} v^{2\gamma - 3 + \frac{p+1}{p+2}} |\nabla v|^2. \]

Moreover, we have

\[ \int_{\Omega} -\sqrt{\frac{2\lambda}{p+1}} v^{2\gamma - 3 + \frac{p+1}{p+2}} |\nabla v|^2 = - \int_{\Omega} \frac{\sqrt{\frac{2\lambda}{p+1}}}{2\gamma - 2 + \frac{p+1}{2}} \nabla (v^{2\gamma - 2 + \frac{p+1}{2}} \nabla v) \]

\[ + \int_{\Omega} \frac{\sqrt{\frac{2\lambda}{p+1}}}{2\gamma - 2 + \frac{p+1}{2}} v^{2\gamma - 2 + \frac{p+1}{2}} \Delta v. \]

For the second term on the right hand side of the above equality, using the inequality \( \Delta v < -\sqrt{\frac{2\lambda}{p+1}} v^{\frac{p+1}{p+2}} < 0 \) again, we have

\[ \int_{\Omega} \frac{\sqrt{\frac{2\lambda}{p+1}}}{2\gamma - 2 + \frac{p+1}{2}} v^{2\gamma - 2 + \frac{p+1}{2}} \Delta v \leq - \int_{\Omega} \frac{2\lambda}{p+1} v^{2\gamma - p - 1}. \]

Hence, we obtain

\[ \int_{\Omega} v^{2\gamma - 3} |\nabla v|^2 \Delta v \leq - \int_{\partial\Omega} \frac{\sqrt{\frac{2\lambda}{p+1}}}{2\gamma - 2 + \frac{p+1}{2}} \frac{\partial v}{\partial n} - \int_{\Omega} \frac{2\lambda}{p+1} v^{2\gamma - p - 1}, \]  

(3.13)

where we used \( v|_{\partial\Omega} = u + 1|_{\partial\Omega} = 1 \), for the boundary term appeared in (3.3), (3.12) and (3.13). By the remark after Theorem 3.1 we find that there exists a constant \( C \) (the constant \( C \) appeared now and later in this section is independent of \( u \)), such that

\[ \int_{\partial\Omega} (|\nabla v|^2 |\frac{\partial u}{\partial n}| + |\frac{\partial (\Delta u)}{\partial n}| + |\frac{\partial u}{\partial n}|) \leq C. \]  

(3.14)
Combining (3.12), (3.13) and (3.14), we get
\[
(1 - 4(\gamma - 1)^2) \int_\Omega (\Delta(u + 1)')^2 + \left( \frac{8\lambda^2(\gamma - 1)^2}{(4\gamma + p - 3)(p + 1)} - \frac{\lambda^2}{q} \right) \int_\Omega (u + 1)^{p+q} \leq C.
\]
(3.15)

If \(1 - 4(\gamma - 1)^2 > 0\), \(p(1 - 4(\gamma - 1)^2) + \frac{8\lambda^2(\gamma - 1)^2}{(4\gamma + p - 3)(p + 1)} - \frac{\lambda^2}{q} > 0\) and \(u\) is a stable solution to the equation (1.4), we have
\[
(p(1 - 4(\gamma - 1)^2) + \frac{8\lambda^2(\gamma - 1)^2}{(4\gamma + p - 3)(p + 1)} - \frac{\lambda^2}{q}) \int_\Omega (u + 1)^{p+q} \leq C.
\]

This leads to \(u + 1 \in L^{p+q}\).

If \(p + q > \frac{(p-1)n}{4}\), then classical regularity theory implies that \(u \in L^\infty(\Omega)\).

Therefore we have established the bound of extremal solutions of (1.4) if
\[
p(1 - 4(\gamma - 1)^2) + \frac{8\lambda^2(\gamma - 1)^2}{(4\gamma + p - 3)(p + 1)} - \frac{\lambda^2}{q} > 0
\]
and
\[
p < \frac{8\gamma + n - 4}{n - 4}.
\]

By Lemma 4.1 and Theorem 3.8 of [15], we prove the extremal solution \(u^*\), the unique solution of equation (1.4) (where \(\lambda = \lambda^*\)) is bounded provided that

1. \(n \leq 8\), \(p > 1\),
2. \(9 \leq n \leq 19\), there exists \(\varepsilon_n > 0\) such that for any \(1 < p < \frac{n}{n-8} + \varepsilon_n\),
3. \(n \geq 20\), \(1 < p < 1 + \frac{8p^*}{n-4}\). (\(p^*\) is defined as before)

4. APPENDIX

In this appendix, we study the following inequalities
\[
p(1 - 4(\gamma - 1)^2) - \frac{\gamma^2}{2\gamma - 1} + \frac{8\gamma^2(\gamma - 1)^2}{(4\gamma - 3 + p)(p + 1)} > 0,
\]
(4.1)

\[
p < \frac{8\gamma + n - 4}{n - 4}.
\]
(4.2)

In order to get a better range of the power \(p\) from (4.1) and (4.2), it is necessary for us to study the following equation (Letting \(p = \frac{8n+n-4}{n-4}\) in (4.1)):
\[
\frac{8\gamma + n - 4}{n - 4}(1 - 4(\gamma - 1)^2) - \frac{\gamma^2}{2\gamma - 1} + \frac{8\gamma^2(\gamma - 1)^2}{(4\gamma - 3 + \frac{8n+n-4}{n-4})(\frac{8n+n-4}{n-4})} = 0.
\]
(4.3)

We can only consider the behavior of (4.3) for \(\gamma \in (1, \frac{3}{2})\). Through tedious computations, we see the following equation which appeared in the introduction is the simplified form of (4.3).

\[
512(2 - n)\gamma^6 + 4(n^3 - 60n^2 + 670n - 1344)\gamma^5 - 2(13n^3 - 424n^2 + 3064n - 5408)\gamma^4 + 2(27n^3 - 572n^2 + 3264n - 5440)\gamma^3 - (49n^3 - 772n^2 + 3776n - 5888)\gamma^2 + 4(5n^3 - 66n^2 + 288n - 416)\gamma - 3(n^3 - 12n^2 + 48n - 64) = 0.
\]
(4.4)
We denote the left hand side of the equation (4.3) by \( h(\gamma) \). Notice that if \( \gamma = \frac{n-4}{n-8} \), then \( p = \frac{n}{n-8} \) and \( \gamma - 1 = \frac{4}{n-8} \). Hence
\[
h(n - 4) - \frac{8}{n-8} [h^4 - 18n^3 - 56n^2 + 384n - 512].
\]
In fact, if \( n = 20 \), then \( h(\frac{4}{8}) = 512 > 0 \). On the other hand, it is also easy to see that \( h(\frac{4}{8}) < 0 \), while it is obvious that \( (4\gamma - 3 + \frac{8\gamma n - 4}{n-8})(\frac{2\gamma n - 1}{n-4} + 1) > 0 \) and \( (2\gamma - 1) > 0 \) when \( \gamma \in (\frac{n-4}{n-8}, \frac{3}{2}) \). Therefore, by continuity, equation (4.3) possesses a root in \( (\frac{n-4}{n-8}, \frac{3}{2}) \). We denote the smallest root of (4.3) which is greater than \( \frac{n-4}{n-8} \) by \( p^* \). Once we pick out a \( \gamma \) from the interval \( (\frac{n-4}{n-8}, p^*) \), \( h(\gamma) \) is of course positive. By continuity, we can find a small positive number \( \delta \) such that, the inequality \( p(1 - 4(\gamma - 1)^2) - \frac{\gamma^2}{2\gamma - 1} + \frac{8\gamma(\gamma - 1)^2}{(4\gamma - 3 + p)(p + 1)} > 0 \) holds when \( p \in (\frac{8\gamma n - 4}{n-8} - \delta, \frac{8\gamma n - 4}{n-8}) \).

So, we conclude that when \( \gamma \) runs in the whole interval \( (\frac{n-4}{n-8}, p^*) \), the power \( p \) can be in the whole interval \( (\frac{n}{n-8}, 1 + \frac{8\gamma}{n-4}) \). We summarize the result as follows:

Lemma 4.1. When \( n \geq 20 \), we have \( p \) which satisfies (4.1) and (4.2) can range in \( (\frac{n}{n-8}, 1 + \frac{8\gamma}{n-4}) \) and this interval is not empty.

Acknowledgments: The first author was supported from an Earmarked grant (“On Elliptic Equations with Negative Exponents”) from RGC of Hong Kong.

References

[1] S. Agmon; A. Douglis and L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations, satisfying general boundary conditions, I. Comm. Pure Appl. Math. 12 (1959), 623-727.

[2] C. Cowan, P. Espesito and N. Ghoussoub, Regularity of extremal solutions in fourth order nonlinear eigenvalue problems on general domains. DCDS-A. 28 (2010), 1033-1050.

[3] A. Farina, On the classification of solutions of the Lane-Emden equation on unbounded domain of \( \mathbb{R}^N \). J. Math. Pures Appl. 87 (2007), 537-561.

[4] A. Ferro, H. Ch. Grunau and P. Karageorgis, Supercritical biharmonic equations with power-like nonlinearity. Ann. Mat. Pura Appl. 188 (2009), 171-185.

[5] F. Gazzola and H. Ch. Grunau, Radial entire solutions for supercritical biharmonic equations. Math. Ann. 334 (2006), 905-936.

[6] Z. M. Guo and J. R. L. Webb, Large and small solutions of a class of quasilinear elliptic eigenvalue problems. J. Differential Equations. 180 (2002), 1-50.

[7] C. Gui, W. M. Ni and X. F. Wang, On the stability and instability of positive steady states of a semilinear heat equation in \( \mathbb{R}^N \). Comm. Pure Appl. Math. Vol.XLV (1992), 1153-1181.

[8] Gilbarg and Trudinger, Elliptic Partial differential Equations of Second Order, 3rd Edition, Springer-Verlag.

[9] Z. M. Guo and J. Wei, On a fourth order nonlinear elliptic equations with negative exponent. SIAM. J. Math. Anal. 40 (2009), 2034-2054.

[10] Z. M. Guo and J. Wei, Qualitative properties of entire radial solutions for biharmonic equations with supercritical nonlinearity. Proc. American Math. Soc. 138 (2010), 3957-3964.

[11] C. S. Lin, A classification of solutions to a conformally invariant equations in \( \mathbb{R}^N \). Comm. Math. Hele. 73 (1998), 206-231.

[12] W. C. Troy, Symmetry properties in systems of semilinear elliptic equations. J. Differential Equations. 42 (1981), 400-413.
[13] J. Wei, Asymptotic behavior of a nonlinear fourth order eigenvalue problem. *Comm. Partial Differential Equations* **9** (1996), 1451-1467.

[14] J. Wei and X. Xu, Classification of solutions of high order conformally invariant equations. *Math. Ann.* **313**(2) (1999), 207-228.

[15] J. Wei and D. Ye, Liouville theorems for finite morse index solutions of biharmonic problem. Preprint.

[16] X. Xu, Uniqueness theorem for the entire positive solutions of biharmonic equations in $\mathbb{R}^n$. *Proceedings of the Royal Society of Edinburgh*, **130A** (2000), 651-670.

Department of Mathematics, Chinese University of Hong Kong, Shatin, Hong Kong
*E-mail address: weimin.math.cuhk.edu.hk*

Department of Mathematics, National University of Singapore, Singapore 119076, Republic of Singapore
*E-mail address: matxuxw@nus.edu.sg*

Department of Mathematics, Chinese University of Hong Kong, Shatin, Hong Kong
*E-mail address: wyang@math.cuhk.edu.hk*