Matrix Graph Grammars and Monotone Complex Logics

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Abstract. Graph transformation is concerned with the manipulation of graphs by means of rules. Graph grammars have been traditionally studied using techniques from category theory. In previous works, we introduced Matrix Graph Grammars (MGGs) as a purely algebraic approach for the study of graph grammars and graph dynamics, based on the representation of graphs by means of their adjacency matrices. MGGs have been successfully applied to problems such as applicability of rule sequences, sequentialization and reachability, providing new analysis techniques and generalizing and improving previous results. Our next objective is to generalize MGGs in order to approach computational complexity theory and static properties of graphs out of the dynamics of certain grammars. In the present work, we start building bridges between MGGs and complexity by introducing what we call Monotone Complex Logic, which allows establishing a (bijective) link between MGGs and complex analysis. We use this logic to recast the formulation and basic building blocks of MGGs as more proper geometric and analytic concepts (scalar products, norms, distances). MGG rules can also be interpreted – via operators – as complex numbers. Interestingly, the subset they define can be characterized as the Sierpinski gasket.

1 Introduction

Graph transformation [16] is concerned with the manipulation of graphs by means of rules. Similar to Chomsky grammars for strings, a graph grammar is made of a set of rules, each having a left and a right hand side (LHS and RHS) graphs and an initial host graph, to which rules are applied. The application of a rule to a host graph is called a derivation step and involves the deletion and addition of nodes and edges according to the rule specification. Roughly, when an occurrence of the rule’s LHS is found in the graph, then it can be replaced by the RHS. Graph transformation has been successfully applied in many areas of computer science, for example, to express the valid structure of graphical languages, for the specification of system behaviour, visual programming, visual simulation, picture processing and model transformation (see [3]). In particular, graph grammars have been used to specify computations on graphs, as well as to define graph languages (i.e. sets of graphs with certain properties), thus being possible to “translate” static properties of graphs such as coloring into equivalent properties of dynamical systems (grammars).

In previous work [12,13,14,15] we developed a new approach to the transformation of simple digraphs. Simple graphs and rules can be represented with Boolean matrices
and vectors and the rewriting can be expressed using Boolean operators only. One important point of MGGs is that, as a difference from other approaches [16], it explicitly represents the rule dynamics (addition and deletion of elements), instead of only the static parts (pre- and post-conditions). Apart from the practical implications, this fact facilitates new theoretical analysis techniques such as for example checking independence of a sequence of arbitrary length and a permutation of it, or obtaining the smallest graph able to fire a sequence. See [15] for a detailed account.

In [14] we improved our framework with the introduction of the nihilation matrix, which makes explicit some implicit information in rules: elements that, if present in the host graph, disable a transformation step. These are all edges not included in the left hand side, adjacent to nodes deleted by the rule (which would become dangling) and edges that are added by the production, as in simple digraphs parallel edges are forbidden. In this paper, we further develop this idea, as it is natural to consider that a production transforms pairs of graphs, a “positive” one with elements that must exist (identified by the LHS), and a “negative” one, with forbidden elements (identified by the nihilation matrix).

Complexity theory [6,11] is concerned with the study of the intrinsic complexity of computational tasks. Traditionally, it has been studied through abstract devices able to represent the notion of algorithm, such as Turing Machines or Boolean Circuits [17]. Our proposal is to use MGGs instead, as its algebraic nature allows using results from different branches of mathematics such as logics, group theory and Boolean algebra.

In this paper we give a first step in the direction of approaching complexity theory with MGGs, by introducing Monotone Complex Logic (MCL). Similar to complex numbers, a complex formula in MCL contains a certainty and a nihil part, both Boolean propositional formulas. We use MCL terms to encode the “positive” part of a simple digraph (the LHS) and the elements that cannot be found (e.g. the nihilation matrix). Using a rational encoding of adjacency matrices, we can express complex terms referring to simple digraphs into the unit interval of complex numbers $\mathbb{C}$. Interestingly, the set of complex numbers defined by valid MCL terms on simple digraphs is the well-known Sierpinski gasket fractal [8]. The rational encoding allows using geometric and analytic concepts, for example, we have defined a xor-based norm for MCL terms which can be interpreted as the number of elementary operations needed to transform one digraph into another.

Thus, we can use MCL terms to redefine and extend all concepts of MGGs. In this paper we introduce the encoding of productions in both its static and dynamic formulations. In the dynamic formulation of a production, the rule dynamics (element addition and deletion) are also represented as an MCL term, and thus belong to the Sierpinski gasket too. We also show the generalization of the main MGG concepts, like coherence, compatibility, initial digraphs, image of sequences and G-congruence using MCL.

**Paper organization.** Section 2 gives a brief overview of the basic concepts of MGGs. Section 3 introduces MCL, used to establish a link between MGGs and complex analysis. Section 4 encodes graphs as complex numbers, and a scalar product, a norm and a notion of distance are introduced. Section 5 encodes productions as complex numbers and completes the link between MGGs and complex numbers. Sections 6 and 7 gen-
eralize the main sequential results of MGGs such as coherence, compatibility, initial digraphs, image of sequences and G-congruence. Finally, Sec. 8 ends with the conclusions and further research.

2 Matrix Graph Grammars: Basic Concepts

In this section we give a very brief overview of some of the basics of MGGs, for a detailed account and accessible presentation, the reader is referred to [15].

Graphs and Rules. We work with simple digraphs, which we represent as \((M, V)\) where \(M\) is a Boolean matrix for edges (the graph adjacency matrix) and \(V\) a Boolean vector for vertices or nodes. We explicitly represent the nodes of the graph with a vector because rules may add and delete nodes, and thus we mark the existing nodes with a 1 in the corresponding position of the vector. Although nodes and edges can be assigned a type (as in [14]), here we omit it for simplicity.

A production, or rule, \(p : L \rightarrow R\) is a partial injective function of simple digraphs. Using a static formulation, a rule is represented by two simple digraphs that encode the left and right hand sides.

**Definition 1 (Static Formulation of Production).** A production \(p : L \rightarrow R\) is statically represented as \(p = (L = (L^E, L^V); R = (R^E, R^V))\), where \(E\) stands for edges and \(V\) for vertices.

A production adds and deletes nodes and edges; therefore, using a dynamic formulation, we can encode the rule’s pre-condition (its LHS) together with matrices and vectors to represent the addition and deletion of edges and nodes.

**Definition 2 (Dynamic Formulation of Production).** A production \(p : L \rightarrow R\) is dynamically represented as \(p = (L = (L^E, L^V); e^E, e^V; r^E, r^V)\), where \(e^E\) and \(e^V\) are the deletion Boolean matrix and vector, \(r^E\) and \(r^V\) are the addition Boolean matrix and vector (with a 1 in the position where the element is deleted or added respectively).

The output of rule \(p\) is calculated by the Boolean formula \(R = p(L) = r \lor \overline{\sigma} L\), which applies to nodes and edges (the \(\land\) (and) symbol is usually omitted in formulae).

**Example.** Fig. 1 shows an example rule and its associated matrix representation, in its static (right upper part) and dynamic (right lower part) formulations.

In MGGs, we may have to operate graphs of different sizes (i.e. matrices of different dimensions). An operation called completion [12] rearranges rows and columns (so that the elements that we want to identify match) and inserts zero rows and columns as needed. For example, if we need to operate with graphs \(L_1\) and \(R_1\) in Fig. 1 completion adds a third row and column to \(R^E\) (filled with zeros) as well as a third element (a zero) to vector \(R^V\).

**Compatibility.** A graph \((M, V)\) is compatible if \(M\) and \(V\) define a simple digraph, i.e. if there are no dangling edges (edges incident to nodes that are not present in the graph). A rule is said to be compatible if its application to a simple digraph yields a simple digraph (see [15] for the conditions). A sequence of productions \(s_n = p_n; \ldots; p_1\) (where the rule application order is from right to left) is compatible if the image of \(s_m = p_m; \ldots; p_1\) is compatible, \(\forall m \leq n\).
Nihilation Matrix. In order to consider the elements in the host graph that disable a rule application, rules are extended with a new graph $K$. Its associated matrix specifies the two kinds of forbidden edges: those incident to nodes deleted by the rule and any edge added by the rule (which cannot be added twice, since we are dealing with simple digraphs).

According to the theory developed in [15], no extra effort is needed from the grammar designer to derive the nihilation matrix, as $K = p(D)$ with $D = e^\tau \otimes e^\nu$, where $\otimes$ is the tensor product, which sums up the covariant and contravariant parts and multiplies every element of the first vector by the whole second vector [14]. Transposition will be represented by $t$. Please note that given an arbitrary LHS $L$, a valid nihilation matrix $K$ should satisfy $L^E K = 0$, that is, the LHS and the nihilation matrix should not have common edges.

Example. The left of Fig. 2 shows, in the form of a graph, the nihilation matrix of the rule depicted in Fig. 1. It includes all edges incident to node 3 that were not explicitly deleted and all edges added by $p_1$. To its right we show the full formulation of $p_1$ which includes the nihilation matrix.

$P_1 = \left( \begin{array}{c} L^E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \quad L^\tau = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad R^E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad R^\tau = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$

Fig. 1. Simple Production Example (left). Matrix Representation, Static and Dynamic (right).

Fig. 2. Nihilation Graph (left). Full Formulation of Production (center). Evolution of $K$ (right).

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1 In [15], $K$ is written $N_L$ and $Q$ is written $N_R$. We shall use subindices when dealing with sequences in Sec. 7 hence the change of notation. In the definition of production, $L$ stands for left and $R$ for right. The letters that preceed them in the alphabet ($K$ and $Q$) have been chosen.

2 Nodes are not considered because their addition does not generate conflicts of any kind.
As proved in [15] (Prop. 7.4.5), the evolution of the nihilation matrix is fixed by the production. If \( R = p(L) = r \lor \tau L \) then

\[
Q = p^{-1}(K) = e \lor \tau K,
\]

being \( Q \) the nihilation matrix of the right hand side of the production \( p \). Hence, we have that \((R, Q) = (p(L), p^{-1}(K))\). Notice that \( Q \neq \emptyset \) in general though it is true that \( \emptyset \subset Q \).

**Example.** The right of Fig. 2 shows the change in the nihilation matrix of \( p_1 \) when the rule is applied. As node 3 is deleted, no edge is allowed to stem from it. Self-loops from nodes 1 and 2 are deleted by \( p \) so they cannot appear in the resulting graph.\footnote{In [12] we introduced a functional notation for rules inspired by the Dirac or bra-ket notation [1]. Thus, we can depict a rule \( p : L \rightarrow R \) as \( R = p(L) = \langle L, p \rangle \), splitting the static part (initial state, \( L \)) from the dynamics (element addition and deletion, \( p \)). Using such formulation, the ket operators (i.e. those to the right side of the bra-ket) can be moved to the bra (left side) by using their adjoints. In this work we recast this notation more properly through MCL.}

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**Direct Derivation.** A direct derivation consists on applying a rule \( p : L \rightarrow R \) to a graph \( G \), through a match \( m : L \rightarrow G \) yielding a graph \( H \). In MGGs we use injective matchings, so given \( p : L \rightarrow R \) and a simple digraph \( G \) any \( m : L \rightarrow G \) total injective morphism is a match for \( p \) in \( G \). The match is one of the ways of completing \( L \) in \( G \).

In MGGs we do not only consider the elements that should be present in the host graph \( G \) (those in \( L \)) but also those that should not be (those in the nihilation matrix, \( K \)). Hence two morphisms are sought: \( m_L : L \rightarrow G \) and \( m_K : K \rightarrow \neg G \), where \( \neg G \) is the complement of \( G \), which in the simplest case is just its negation (see \([14,15]\)).

**Definition 3 (Direct Derivation).** Given rule \( p : L \rightarrow R \) and graph \( G = (G^E, G^V) \) as in Fig. 3(a), \( d = (p, m) \) with \( m = (m_L, m_K) \) is called a direct derivation with result \( H = p^*(G) \) if the following conditions are fulfilled:

1. There exist \( m_L : L \rightarrow G \) and \( m_K : K \rightarrow \neg G \) total injective morphisms.
2. \( m_L(n) = m_K(n), \forall n \in L^V \).
3. The match \( m_L \) induces a completion of \( L \) in \( G \). Matrices \( e \) and \( r \) are then completed in the same way to yield \( e^* \) and \( r^* \). The output graph is calculated as \( H = p^*(G) = r^* \lor e^* G \).

**Remarks.** The square in Fig. 3(a) is a pushout. Item 2 is needed to ensure that \( L \) and \( K \) are matched to the same nodes in \( G \).

**Example** The right of Fig. 3 depicts a direct derivation example using rule \( p_1 \) shown in Fig. 1 which is applied to a graph \( G \) yielding graph \( H \). A morphism from the nihilation matrix to the complement of \( G \), \( m_K : K \rightarrow \neg G \), must also exist for the rule to be applied.\footnote{We have followed the mathematical style instead of the one commonly used in physics, which should have been \( \langle p | L \rangle \).}

**Analysis Techniques.** In [12,13,14,15] we developed some analysis techniques for MGGs. One of our goals was to analyze rule sequences independently of a host graph.
For its analysis, we complete the sequence by identifying the nodes across rules which are assumed to be mapped to the same node in the host graph (and thus rearrange the matrices of the rules in the sequences accordingly). Once the sequence is completed, our notion of sequence coherence [12] allows to know if, for the given identification, the sequence is potentially applicable (i.e. if no rule disturbs the application of those following it).

Given a completed sequence, the minimal initial digraph (MID) is the smallest graph that allows applying it. Conversely, the negative initial digraph (NID) contains all elements that should not be present in the host graph for the sequence to be applicable. Therefore, the NID is a graph that should be found in $G$ for the sequence to be applicable (i.e. none of its edges can be found in $G$). If the sequence is not completed (i.e. no overlapping of rules is decided), we can give the set of all graphs able to fire such sequence or spoil its application.

Other concepts aim at checking sequential independence (i.e. same result) between a sequence of rules and a permutation of it. $G$-congruence detects if two sequences (one permutation of the other) have the same MID and NID. It returns two matrices and two vectors, representing two graphs, which are the differences between the MIDs and NIDs of each sequence respectively. Thus if zero, the sequences have the same MID and NID. Two coherent and compatible completed sequences that are $G$-congruent are sequential independent.

All these concepts have been characterized using operators $\triangle$ and $\nabla$. They extend the structure of sequence, as explained in [15]. Their definition is included here for future reference:

$$
\triangle_{t_1}^{t_0} (F(x, y)) = \bigvee_{y=t_0}^{t_1} \left( \bigwedge_{x=y}^{t_2} (F(x, y)) \right)
$$

$$
\nabla_{t_1}^{t_0} (G(x, y)) = \bigwedge_{y=t_0}^{t_1} \left( \bigvee_{x=y}^{t_2} (G(x, y)) \right).
$$

Productions are the building blocks of sequences and sequences are the basic construction to study graph dynamics. All these concepts are further studied and generalized in the present contribution.

Some other important notions such as application conditions, graph constraints or reachability are just sketched or not even mentioned and left for further research.
3 MCL: Monotone Complex Logic

In this section we introduce Monotone Complex Logic (MCL), Preliminary Monotone Complex Algebra (PMCA) and Preliminary Monotone Matrix Algebra (PMMA). The term “logic” in the title should be understood as in fist-order logic or propositional logic (arguably, “calculus” might be more appropriate). It has been called complex to resemble the similarities with complex numbers and how they are defined out of the real numbers. Monotone because we are not defining the negation of complex terms (see below).

Monotone complex logic is in our opinion of interest by itself, but it is introduced here due to its usefulness for Matrix Graph Grammars (MGGs). First, it permits a compact reformulation of grammar rules. Second, the numerical representation that will be introduced in Def. 8 establishes a link between graphs in MGGs and $\mathbb{Q}[t]$, although the operations we are interested in are not addition and multiplication. Also, any production $p$ induces the evolution of a pair of graphs $(L, K) \xrightarrow{p} (R, Q) = (p(L), p^{-1}(K))$. Productions will be reinterpreted by encoding them as complex formulas and representing their actions as a “Hermite product”. MCL will allow us to measure the size of graphs via a natural norm. Finally, sequential notions of MGGs such as independence, initial digraphs, coherence, etcetera, will be thus recasted and extended.

Definition 4 (Complex Formula). A complex formula $z = (a, b)$ consists of a certainty part ‘$a$’ plus a nihil part ‘$b$’, where $a$ and $b$ are propositional logic formulas using adjacency matrices as propositional variables. Two complex formulas $z_1 = (a_1, b_1)$ and $z_2 = (a_2, b_2)$ are equal, $z_1 = z_2$, if and only if $a_1 = a_2$ and $b_1 = b_2$.

Monotone Complex Logic is the formal system whose propositional variables are complex formulas with logical connectives $\lor$, $\land$. We will not go further because we are more interested in an algebraic development of the theory.

Throughout the present contribution, complex formula, complex term and Boolean complex will be used as synonyms. Next, some basic operations on Boolean complexes are introduced.

Definition 5 (Basic Complex Operations). Let $z = (a, b)$, $z_1 = (a_1, b_1)$ and $z_2 = (a_2, b_2)$ be complex terms. The following operations are defined componentwise:

- Addition: $z_1 \lor z_2 = (a_1 \lor a_2, b_1 \lor b_2)$.
- Multiplication: $z_1 \land z_2 = z_1 \cdot z_2 = (a_1 a_2 \lor b_1 b_2, a_1 b_2 \lor a_2 b_1)$.
- Conjugation: $z^\# = (b, a)$.
- Dot Product: $\langle z_1, z_2 \rangle = z_1 \cdot z_2^\#$.

The notation $\langle \cdot, \cdot \rangle$ is used for two reasons. First, we would like to highlight the similarities with scalar products. There is however no underlying linear space so this is just a convenient notation. Second, we will see that it coincides with the functional notation introduced in 1215.

The dot product of two Boolean complexes is zero (orthogonal) if and only if any element of the first complex term is included in both the certainty and nihil parts of the second complex term. Otherwise stated, if $z_1 = (a_1, b_1)$ and $z_2 = (a_2, b_2)$, then
\[ \langle z_1, z_2 \rangle = 0 \iff a_1 \overline{a}_2 = a_1 \overline{b}_2 = b_1 \overline{a}_2 = b_1 \overline{b}_2 = 0. \]

Let’s say that \( a < b \) if \( ab = a \), i.e. whenever \( a \) has a \( 1 \) \( b \) also has a \( 1 \) (graph \( a \) is contained in graph \( b \)). Previous identities can be rephrased as \( a_1 < a_2, a_1 < b_2, a_1 < a_2 \) and \( b_1 < b_2 \). This is equivalent to \((a_1 \lor b_1) < (a_2 \lor b_2)\). Orthogonality is directly related to the common elements of the certainty and nihil parts.

A particular relevant case is when we consider the dot product of one element \( z = (a, b) \) with itself. In this case we get \((a \lor b) < (ab)\), which is possible if and only if \( a = b \). We shall come back to this issue.

**Definition 6 (Preliminary Monotone Complex Algebra, PMCA).** The set \( \mathcal{F}' = \{ z \mid z \) is a complex formula\} together with the basic operations introduced in Def. 5 will be known as preliminary monotone complex algebra.

We will get rid of the term “preliminary” in Def. 11 when not only the adjacency matrix is considered but also the vector of nodes that make up a simple digraph.

We introduce a subalgebra of the preliminary monotone complex algebra to be known as **preliminary monotone matrix algebra** (PMMA). It is useful due to its relationship with MGGs.

**Definition 7 (Preliminary Monotone Matrix Algebra, PMMA).** Let \( z_1 = (a_1, b_1) \in \mathcal{F}' \). Define the equivalence relation \( z_1 \sim z_2 \iff \exists c = (c, c) \mid a_2 = a_1 \lor c \) and \( b_2 = b_1 \lor c \). Then,

\[ s_1' = \mathcal{F}' / \sim \]

is the preliminary monotone matrix algebra.

A graph and any of its possible nihilation matrices in MGG do not share any edge \((L^E K = 0)\), refer to [15]). So when representing the left hand side of a production in MCL, its complex term is made of a digraph in the certainty part and some valid nihilation matrix in the nihil part. Intuitively, PMMA is made of the valid complex terms in this sense (i.e. those that do not share any edge).

It is not difficult to check reflexivity, symmetry and transitivity for \( \sim \). The equivalence relation permits the simplification of those elements that appear in both the certainty and nihil parts (eliminating non-valid complex terms).

A more complex-analytical representation can be handy in some situations and in fact will be preferred for the rest of the present contribution: \( z = (a, b) \mapsto z = a \lor i b \).

Define one element \( i \), that we will name nil term or nihil term, with the property \( i \land i = 1 \), being \( i \) itself not equal to 1. Then, the basic operations of Def. 5 following the same notation, can be rewritten: \( z_1 \lor z_2 = (a_1 \lor a_2) \lor i (b_1 \lor b_2) \), \( z_1 \land z_2 = (a_1 \lor ib_1) \land (a_2 \lor ib_2) \), \( z^* = \overline{a} \lor i \overline{a} \) and the same for the dot product.

Notice that the conjugate of a complex term \( z \in \mathcal{F}' \) that consists of certainty part only is \( z^* = (a \lor i0)^2 = 1 \lor i \). Similarly for one that consists of nihil part alone: \( z^* = (0 \lor ib)^2 = \overline{b} \lor i \). If \( z \in \mathcal{F}' \) then they further reduce to \( a \lor i0 \) and \( 0 \lor ib \), respectively, i.e. they are invariant. Also, the multiplication reduces to the standard and operation if there are no nihil parts: \( (a_1 \lor i0)(a_2 \lor i0) = a_1 a_2 \).

\[ \text{Notice that } 1 \lor i \overline{a} = (a \lor i \overline{a}) \lor i \overline{a} = a \lor i0 \text{ and } \overline{b} \lor i(1 - \overline{b} \lor i (b \lor \overline{b}) - 0 \lor ib). \]
Proposition 1. Let \( x, y, z \in \mathfrak{G}' \) and \( z_1, z_2 \in \mathfrak{S}' \). Then, \( \langle x \lor y, z \rangle = \langle x, z \rangle \lor \langle y, z \rangle \). \( \langle z_1, z_2 \rangle = \langle z_2, z_1 \rangle^* \) and \( (z_1 z_2)^* = z_1^* z_2^* \).

Proof

The first identity is fulfilled by any complex term and follows directly from the definition. The other two need the equivalence relation (simplification), i.e. they hold in \( \mathfrak{S}' \) but not necessarily in \( \mathfrak{G}' \). For the second equation just write down the definition of each side of the identity:

\[
\langle z_1, z_2 \rangle = (a_1 \overline{r}_2 \lor \overline{r}_2 b_1) \lor i (a_1 \overline{r}_2 \lor b_1 \overline{r}_2)
\]

\[
\langle z_2, z_1 \rangle^* = [a_1 \overline{r}_2 \lor \overline{r}_2 b_1 \lor (a_1 b_1 \lor \overline{r}_2 \overline{r}_2)] \lor i [a_1 \overline{r}_2 \lor b_1 \overline{r}_2 \lor (a_1 b_1 \lor \overline{r}_2 \overline{r}_2)].
\]

Terms \( a_1 b_1 \lor \overline{r}_2 \overline{r}_2 \) vanish as they appear in both the certainty and nihil parts. The third identity is proved similarly.

Notice however that \( (z_1 \lor z_2)^* \neq z_1^* \lor z_2^* \). It can be checked easily as \( (z_1 \lor z_2)^* = [(a_1 \lor a_2) \lor i (b_1 \lor b_2)]^* = \overline{b}_1 \overline{b}_2 \lor i \overline{a}_1 \overline{a}_2 \) but \( z_1^* \lor z_2^* = (\overline{b}_1 \lor \overline{b}_2) \lor i (\overline{a}_1 \lor \overline{a}_2) \). This implies that, although \( \langle z_1 \lor z_2, z \rangle = \langle z_1, z \rangle \lor \langle z_2, z \rangle \), we no longer have sesquilinearity, i.e. it is not linear in its second component taking into account conjugacy:

\[
z (\overline{b}_1 \lor \overline{b}_2) \lor i (\overline{a}_1 \lor \overline{a}_2)] = \langle z, z_1 \lor z_2 \rangle \neq \langle z, z_1 \rangle \lor \langle z, z_2 \rangle = z [\overline{b}_1 \overline{b}_2 \lor i \overline{a}_1 \overline{a}_2].
\]

4 Numerical Representation, Norm and Distance

This section introduces an application \( \ell \) that assigns a complex number in the unit interval \( \mathbb{C}([0, 1]) \) to any Boolean complex. It is not a homomorphism: neither \( \ell(x \lor y) = \ell(x) + \ell(y) \) nor \( \ell(xy) = \ell(x)\ell(y) \) hold. Application \( \ell \) provides some geometric intuition. A norm and a conditional norm are defined out of the dot product of Def. 5.

Finally we will define the distance between two complex terms.

Definition 8 (Rational Encoding). Let \( g = (g^i_j)_{i,j \in \{1, \ldots, n\}} \) be a simple digraph. Its rational encoding is given by

\[
\ell(g) = \sum_{k=1}^{n^2} (2^{-k} i_k^j),
\]

(5)

where \( i_k = \lfloor \frac{k}{n} \rfloor \) and \( j_k = k - n \lfloor \frac{k}{n} \rfloor \).

It has become customary to represent the lowest integer above \( m \) as \( [m] \) and the biggest integer below \( m \) as \( \lfloor m \rfloor \). These functions are known as ceiling and floor, respectively. The indices \( i_k \) and \( j_k \) are just the integer quotient and the remainder. They are a convenient way to visit all the elements of the adjacency matrix ordered by columns.

As the elements of \( \mathfrak{G}' \) are adjacency matrices we can define \( \mathfrak{G}' = \ell (\mathfrak{G}') \). If \( z = (a, b) \in \mathfrak{G}' \), then \( \ell(z) = \ell(a) + i\ell(b) \). Analogously, \( \mathfrak{S}' = \ell (\mathfrak{S}') \).
Example. \( \ell(L_1) = 0'101011_2 = 0'671875_{10} \), where \( L_1 \) is the LHS of \( p_1 \) in Fig. 1. If
\[
L = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \lor i \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}
\]
then \( \ell(L) = 0'0112 + i0'12 = 0'375_{10} + i0'5_{10} \). Subindices indicate the numbering system: base 2 or base 10. 

The rational encoding is similar to the standard one given in the literature (which we will call natural encoding). The main difference is that we use negative powers. This, in the limit, would make us consider the interval \([0, 1] \) as the underlying space instead of the natural numbers \( \mathbb{N} \).

In the present contribution we will deal only with finite graphs. Hence, the codomain of \( \ell \) is \([0, 1] \cap \mathbb{Q} \), in fact only terminal rationals. Nonetheless, a small digression seems appropriate as at some point in the future we will be interested in the asymptotic behaviour of algorithms and, hence, it will be more convenient to consider the interval \([0, 1] \). Note that 0 is the symmetric element with respect to the \( \lor \) operation. This is the graph with no edges. It should be also natural to include 1 because it should be the neutral or identity element with respect to the \( \land \) operation (the graph that has any possible edge) if the assumption \( 1 = 0'1_2 \) was made. However, this assumption is not adequate for MGGs. For example, if we want to consider the graph \( g \) that only has the self-edge \((1, 1)\) and no other one, and the number of nodes of the graph was countable, we would be tempted to write \( g = 0'1_2 \lor 0'01_2 \). But \( 0'01_2 = 0'1_2 \) with the standard operations in \( \mathbb{C} \). We would thus be asking for the self-edge \((1, 1)\) to be present and not to be present, according to the interpretation of the certainty and nihil parts in MGG. Seemingly, the Archimedian property fails as for graphs there are nontrivial infinitesimals. This implies for example that any diagonal in Fig. 4 does not belong to the MGG characteristic set.

Throughout the present contribution we may make the following abuse of notation. Let \( A \) be the adjacency matrix of some simple digraph. When we make the operation \( A \lor \overline{A} \) we do not obtain 1 but \( 1_A \), which is the matrix with all its elements 1 according to vector \( A^V \), i.e. \( A \lor \overline{A} = A^V \otimes A^V \). This has a clear relationship with the definition of the nihilation matrix in [15]. See the definition of conditional norm below. Also, we can interpret \( 1_A \) as the characteristic or indicator function of all edges potentially incident to nodes of digraph \( A \) (the smallest complete digraph that contains \( A \)). The notation \( \chi_A(x) \) is also standard. It is defined \( \chi_A(x) = 1 \) if \( x \in A^V \otimes A^V \) and zero otherwise.

The rational encoding in Def. 8 establishes an injection from the preliminary monotone complex algebra \( \mathcal{O}^1 \) into the complex numbers \( \mathbb{C} \) where the certainty part becomes the real part, the nihil part becomes the imaginary part and the nihil term becomes \( i = \sqrt{-1} \). Figure 4 represents the preliminary monotone matrix algebra \( g^\ell \) as a proper subset of \( \mathbb{C} \) which is a well known fractal.

\(^5\) By hypothesis, binary representation of rationals will always be terminating because any dyadic rational number \( 1/2^n \) has a terminating binary numeral (although other rational numbers recur). To fix the notation: a terminal number is \( 0'01101 \) while a recurring number is \( 0'0\overline{11} = 0'011011011 \ldots \)

\(^6\) Though, as commented in this section, just as sets. The morphism is an injection if the number of nodes is fixed. For example, \( 0'1010_2 \) may be the graph with two nodes and edges \((1, 1)\) and \((1, 2)\) or \( 0'101000000_2 \), the graph with three nodes and edges \((1, 1)\) and \((3, 1)\).
**Fig. 4.** Rational Encoding of the Preliminary Matrix Algebra as a Subset of \( \mathbb{C} \)

**Proposition 2.** The characteristic function of the rational encoding of the preliminary matrix algebra approaches the Sierpinski gasket as the number of nodes increases.

**Proof**

The set corresponds to the zeros of the and function (coloring zeros in black). It is the Sierpinski gasket due to the Lucas correspondence theorem [5] (see also [18]) which can be used to compute the binomial coefficient \( \binom{L}{K} \mod 2 \) with bitwise operations: \( L \land K \). This tells us that the parity of the function \( \binom{L}{K} \) (this is what the function \( \mod 2 \) does) is the same as that of \( L \land K \). In our case \( L \) is the abscissa. Its negation just reverts the order (it is a symmetry) and does not change the shape of the figure. As commented above in this section, we do not want the diagonals to belong to the set in the limit. ■

The dot product introduced in Sec. 3 induces a “norm” \( \| \cdot \| \) in \( \mathcal{G} \). It is not a norm in the sense of linear algebra (no underlying vector space) but shares some of its properties and will play a similar role: it will be used as a means to measure the size of a graph.

**Definition 9 (Norm – Conditional Norm).** Let \( y, z \in \mathcal{G} \). Its norm is the application \( \| \cdot \| : \mathcal{G} \to [0, 1] \) defined by

\[
\|z\| = \ell \left( \langle z, z \rangle \right) .
\] (6)

The conditional norm of \( z \) with respect to \( y \) is given by

\[
\|z|y\| = \|z\| y = \frac{\|z \land y\|}{\|y\|} .
\] (7)

The following identities show that the norm (before applying \( \ell \)) returns what one would expect. Equation (10) is particularly relevant as it states that in MCL the certainty and nihil parts are in some sense mutually exclusive, which together with eq. (11) suggest the definition of \( \mathcal{S} \mathcal{G} \) as introduced in Sec. 3. Notice that this fits perfectly
well with the interpretation of $L$ and $K$ given in [15].

$$\langle (a, 0), (a, 0) \rangle = (a, 0) (1, \overline{a}) = (a \lor 0 \overline{a}, a \lor 0 1) = a$$

(8)

$$\langle (0, b), (0, b) \rangle = (0, b) (\overline{b}, 1) = (0 \lor b \overline{b}, b \lor 0 \overline{b}) = b$$

(9)

$$\langle (c, c), (c, c) \rangle = (c, c) (\overline{c}, \overline{c}) = (c \lor c \overline{c}, c \lor c \overline{c}) = 0.$$  

(10)

The dot product of one element with itself gives rise to the following useful identity:

$$\langle z, z \rangle = z z^* = (a \overline{b} \lor ib) \lor i (b \overline{a} \lor a\overline{b}) = a \oplus b.$$  

(11)

Equation (11) admits two readings. In first place, it tells how to factorize one of the basic Boolean operations, $\text{xor}$. Secondly, and more relevant to us, it justifies the use of $\text{xor}$ as a norm for complex terms. Besides, it follows directly from the definition that $\|i\| = 1$ and $\|z^*\| = \|z\|$. 

The conditional norm $\| \cdot | y \|$ reduces the total space to graph $y$. If $z = a_1 \lor ib_1 \in \mathcal{G}$ and $y = a_2 \lor ib_2 \in \mathcal{G}$, after some simple algebraic operations

$$\|z|y\| = \ell \left( \frac{(a_1 \lor b_1) (a_2 \lor b_2)}{\ell (a_2 \lor b_2)} \right)$$

(12)

is obtained. If we further have that $y, z \in \mathcal{G}$, then

$$\|z|y\| = \ell \left( \frac{(a_1 \lor b_1) (a_2 \lor b_2)}{\ell (a_2 \lor b_2)} \right).$$

(13)

In particular this implies that $\|z|y\| = 1 \iff y$ is a subgraph of $z$. The following proposition highlights the similarities between norms in linear spaces and the one introduced in Def. [9]

**Proposition 3.** Let $y, z \in \mathcal{G}$ and $z_1, z_2 \in \mathcal{G}$. Then, for norms and conditional norms we have that

$$\|z\| = 0 \iff z = 0,$$

(14)

$$\|yz\| = \|y\| \land \|z\|,$$

(15)

$$\|z_1 \lor z_2\| \leq \|z_1\| \lor \|z_2\|.$$  

(16)

**Proof**

Identity (14) is derived from (11). Some simple manipulations prove (15) (where and is performed bitwise). Inequality (16) is not difficult either where, again, or on the right hand side is applied bitwise:

$$\|z_1\| \lor \|z_2\| = \ell(a_1 \overline{b}_1) \lor \ell(a_2 \overline{b}_2) \lor \ell(\overline{a}_1 b_1) \lor \ell(\overline{a}_2 b_2)$$

$$\|z_1 \lor z_2\| = \ell(a_1 \overline{b}_1 \overline{b}_2) \lor \ell(a_2 \overline{b}_1 \overline{b}_2) \lor \ell(b_1 \overline{a}_1 \overline{a}_2) \lor \ell(b_2 \overline{a}_1 \overline{a}_2).$$

Comparing term by term, the inequality follows. For example, there must be at least the same numbers of 1’s in $a_1 \overline{b}_1$ than in $a_1 \overline{b}_1 \overline{b}_2$, so $\ell(a_1 \overline{b}_1) \geq \ell(a_1 \overline{b}_1 \overline{b}_2).$

$^7$ Recall that in complex analysis we have $z z^* - |z|^2$. 
The xor operation $z = z_1 \oplus z_2$ is the number of distinct elements in $z_1$ and $z_2$. The number of ones that appear in $z$ tells the number of atomic operations that must be performed in order to transform $z_1$ in $z_2$ (or viceversa) in the sense of MGG productions. Therefore, it seems natural to define the distance between two complex terms as follows:

**Definition 10.** Let $z_j = a_j \vee ib_j$ with $z_j \in \mathcal{C}'$, $j = 1, 2$, and define $w_1 = a_1 \vee ia_2$ and $w_2 = b_1 \vee ib_2$. Then, the distance between $z_1$ and $z_2$ is given by

$$d(z_1, z_2) = \|w_1\| \oplus \|w_2\|.$$  

(17)

It is an easy exercise to check that $d$ fulfills the axioms of a metric: $d(x, y) \geq 0$, $d(x, y) = 0 \iff x = y$, $d(x, y) = d(y, x)$ and the triangle inequality $d(x, z) \leq d(x, y) + d(y, z)$. The triangle inequality follows from $d(x, z) = d(x, y) \oplus d(y, z)$ and the fact that $\forall a > 0, \forall b > 0$ we have that $a \oplus b \leq a + b$. See [9] for an application of the xor metric.

Notice that in $\mathcal{H}$ only one of the terms contributes to the norm if $z_2 = p(z_1)$ for some production because the actions on the certainty part completely determine the nihil part, as proved in [15] (Prop. 7.4.5).

### 5 Production Encoding

In this section we introduce the Monotone Complex Algebra and the Monotone Matrix Algebra, that not only consider edges but also nodes. Compatibility issues may appear so we study compatibility for a simple digraph and also for a single production (compatibility for sequences will be addressed in Sec. [7]). Next we turn to one of the main topics in this paper: how to characterize MGG productions using MCL and the dot product of Def. 5. The section ends introducing swaps and providing some geometric interpretations.

To get rid of the “preliminary” term in the definitions of $\mathcal{C}'$ and $\mathcal{H}'$ (Defs. [6] and [7] resp.) we shall consider an element as being composed of a matrix term and a vector of nodes. Hence, we have that $L = (L^E \vee iK^E, L^V \vee iK^V)$ where $E$ stands for edge and $V$ for vertex. Notice that $L^E \vee iK^E$ are matrices and $L^V \vee iK^V$ are vectors.

**Definition 11 (Monotone Complex and Matrix Algebras).** $\mathcal{C}$ and $\mathcal{H}$ (the Monotone Complex and Matrix Algebras, resp.) are defined as their preliminary counterparts – see Defs. [6] and [7] – but considering elements of the form:

$$L = (L^E \vee iK^E, L^V \vee iK^V).$$  

(18)

We also introduce $\mathcal{G} = \ell (\mathcal{C})$ and $\mathcal{H} = \ell (\mathcal{H})$.

Concerning $\mathcal{G}$, a production $p : \mathcal{G} \to \mathcal{G}$ consists of two independent productions $p = (p_1, p_2)$ – being $p_i$ MGG productions as those introduced in [15] – one acting on the certainty part and the other on the nihil part:

$$R = p(L) = p_C(L) \vee ip_N(K) = R \vee iQ.$$  

(19)

\[^8\] If an equation is applied to both edges and nodes then the superindices will be omitted. They will also be omitted if it is clear from context which one we refer to.
As there are no restrictions on $p_C$ and $p_N$ if we stick to $\emptyset$, it is true that $\forall g_1, g_2 \in \emptyset, \exists p$ such that $p(g_1) = g_2$. Recall from Sec.2 that productions in MGG have $\mathcal{F}$ as domain and codomain. Moreover, they must fulfill $p_N = p^{-1}_C$. Unless otherwise stated, we will concentrate on MGG productions for the rest of the paper.

We want $p_N$ to be a production so we must split it into two parts: the one that acts on edges and the one that acts on vertices. Otherwise there would probably be dangling edges in the nihil part as soon as the production acts on nodes. The point is that the image of the nihil part with the operations specified by productions are not graphs in general, unless we restrict to edges and keep nodes apart. This behaviour is unimportant and should not be misleading.

![Fig. 5. Potential Dangling Edges in the Nihilation Part](image)

**Example.** To the left of Fig. 5 we have drawn the certainty part of a production $p$ that deletes node 1 (along with two incident edges) and adds node 3 (and two incident edges). Its nihil counterpart for edges is depicted to the right of the same figure. Notice that node 1 should not be included in $K$ because it appears in $L$ and we would be simultaneously demanding its presence and its absence. Therefore, edges $(1, 3), (2, 1)$ and $(3, 1)$ – those with a red dotted line – would be dangling in $K$ (red dotted edges do belong to the graphs they appear on). The same reasoning shows that something similar happens in $Q$ but this time with edges $(1, 3), (3, 1), (3, 2)$ and $(3, 3)$ and node 3.

This is the reason to consider nodes and edges independently in the nihil parts of graphs and productions. In $K$, as nodes 1 and 3 belong to $L$, it should not make much sense to include them in $K$ too, for if $K$ dealt with nodes we would be demanding their presence and their absence. In $Q$ the production adds node 3 and something similar happens.

Now that nodes are considered compatibility issues in the certainty part may show up. The determination of compatibility for a simple digraph under MCL is almost straightforward. Let $g = (g^E_C \vee i g^E_N, g^V_C \vee i g^V_N) \in \mathcal{F}$. Potential dangling edges are given by $\overline{D}_g = g^E_C \otimes g^V_N$, so the graph $g$ will be compatible if $g^E_C \overline{D}_g = 0$. If $g \in \mathcal{F}$ there are no common elements between the certainty and nihil parts and $\overline{D}_g < g^E_N$.

A production $p(L) = p(L \vee i K) = R \vee i Q = R$ is compatible if it preserves compatibility, i.e. if it transforms a compatible digraph into a compatible digraph. This amounts to saying that $RQ = 0$.

Recall from Sec.2 that grammar rules actions are specified through *erasing* and *addition* matrices, $e$ and $r$ respectively. Because $e$ acts on elements that must be present
and $r$ on those that should not exist, it seems natural to encode a production as

$$ p = e \lor ir. $$

(20)

Our next objective is to use the dot product – see Def. 5 – to represent the application of a production. This way, a unified approach would be obtained. To this end define the operator $P : \mathcal{G} \rightarrow \mathcal{G}$ by

$$ p = e \lor ir \mapsto P(p) = \tau \tau \lor i(e \lor r). $$

(21)

**Proposition 4 (Production).** Let $L$ and $R$ be the left and right hand sides, resp., as in Def. 11, eq. (18), and $P$ as defined in eq. (21). Then,

$$ R = \langle \mathcal{L}, P(p) \rangle. $$

(22)

**Proof**
The proof is a short exercise that makes use of some identities which are detailed right afterwards:

$$ \langle \mathcal{L}, P(p) \rangle = \langle (L, K), (\tau \tau, e \lor r) \rangle = \\
= (\tau \tau L \lor (e \lor r)K, \tau \tau K \lor (e \lor r)L) = \\
= (r \lor \tau L, e \lor rK) = (p(L), p^{-1}(K)) = R. $$

(23)

Apart from equation (4.13) of Prop. 4.1.4 in [15] which states that $\tau L = L$, we have used the following identities:

$$(e \lor r)K = eK \lor rK = rK = r(r \lor \tau \tau D) = r.$$

$$ \tau \tau K = \tau (\tau e \lor r \tau D) = \tau K.$$

$$(e \lor r)L = eL \lor rL = eL = e.$$

We have also used that $r\tau = r$ (again Prop. 4.1.4 in [15]), $rD = r$ due to compatibility and $rL = 0$ almost by definition. Besides, Prop. 7.4.5 in [15] has also been used, which proves that $Q = p^{-1}(K)$.]

The production is defined through operator $P$ instead of directly as $p = \tau \tau \lor i(e \lor r)$ for several reasons. First, eq. (20) and its interpretation seem more natural. Second, $P(p)$ is self-adjoint, i.e. $P(p)^\ast = P(p)$, which in particular implies that $\|P(p)\|_2 = 1$, $\forall p$, being $z$ the “total” graph, i.e. the graph with respect to which completions are performed (for completion refer to [15], Sec. 4.2). Therefore, the norm would not measure the size of productions (interpreted as graphs according to eq. (20)) and we would be forced to introduce a new norm. This is because

$$ \langle P(p), P(p) \rangle = (\tau \tau \lor i(e \lor r))(\tau \tau \lor i(e \lor r))^\ast = \tau \tau \lor e \lor r = 1_\mathcal{G}. $$

By way of contrast, $\|p\| = e \otimes r = e \lor r$. With operator $P$ the size of a production is the number of changes it specifies, which is appropriate for MGGs. Moreover, due to eq.

9 One of our objectives is to look for an appropriate measure of the number of actions that would eventually transform one graph into another.
geometrically, grammar rules are also the Sierpinski gasket. In fact, the codomain of operator $P$ is the diagonal $e + r = 1$ (refer to Figs. 4 and 7).

Complex logic encoding puts into a single expression the application of a grammar rule, both $L$ and $K$. Also, it links the functional notation introduced in [15] and the dot product of Sec. 3.

**Theorem 1 (Surjective Morphism).** There exists a surjective morphism from the set of MGG productions on to the set of self-adjoint graphs in $\mathcal{H}$.

**Proof.** It is not difficult to check that $z$ is self-adjoint if and only if $\|z\| = 1_z$: on the one hand, if $z = a \lor \bar{a}$ then $\langle z, z \rangle = z z^* = (a \lor \bar{a})(a \lor \bar{a}) = a \lor \bar{a} = 1_z$. On the other hand, if we have $z = a \lor ib$ and $\|z\| = a \oplus b = 1_z$ then $a = \bar{b}$. Note that $\|z\| = 1_z$ is equivalent to asking for the conditional norm to be equal to 1 with respect to $Z = z \otimes z^t$:

$$\|z|Z\| = \|zZ\| = \|z\| = 1.$$

The surjective morphism is given by operator $P$. Clearly, $P$ is well-defined for any production. To see that it is surjective, fix some graph $g = g_1 \lor i g_2$ such that $\|g\| = 1_g$. Then, $g = g_1 \lor i g_1$. Any partition of $g_1$ as or of two disjoint digraphs would do. Recall that productions (as graphs) have the property that their certainty and nihil parts must be disjoint.

The operator $P$ is surjective but not necessarily injective. It defines an equivalence relation and the corresponding quotient space. We will be led to a reinterpretation of the notion of production in Matrix Graph Grammars.

**Definition 12 (Swap).** The swap space is defined as $\mathcal{W} = \mathcal{H}/P(\mathcal{H})$. An equivalence class in the swap space will be called a swap. The swap $w$ associated to production $p : \mathcal{H} \rightarrow \mathcal{H}$ is $w = w_p = P(p)$, i.e. $p \in \mathcal{H} \mapsto w_p \in \mathcal{W}$.

![Fig. 6. Example of Productions](image)

**Example.** Let $p_2$ and $p_3$ be two productions as those depicted in Fig. 6. Their images in $\mathcal{H}$ are:

$$P(p_2) = P(p_3) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \lor t \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = w.$$  \hspace{1cm} (24)

The tensor (Kronecker) product in this contribution will always be used on nodes. The $V$ superindex will be omitted in this case: $z \otimes z^t \equiv z^V \otimes (z^t)^V$, where $t$ stands for transposition.

According to eq. (20), any element in $\mathcal{H}$ can be interpreted as a production and vice versa.
They appear to be very different if we look at their defining matrices $L_2, L_3$ and $R_2, R_3$ or at their graph representation. Also, they seem to differ if we look at their erasing and addition matrices:

$$
e_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad r_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad r_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

However, they are the same swap as eq. (24) shows, i.e. they belong to the same equivalence class. Notice that both productions act on edges $(1, 1), (2, 2)$ and $(1, 2)$ and none of them touches edge $(2, 1)$. This is precisely what eq. (24) says as we will promptly see.

Swaps will be of help in studying and classifying the productions of a grammar. For example, there are 16 different simple digraphs with 2 nodes. Hence, there are 256 different productions that can be defined. However, there are only 16 different swaps. From the point of view of the number of edges that can be modified, there is 1 swap that does not act on any element (which includes 16 productions), 4 swaps that act on 1 element, 6 swaps that act on 2 elements, 4 swaps that act on 3 elements and 1 swap that acts on all elements.

There is a simple geometrical interpretation of $P$ in terms of the rational encoding of the productions $p$ and the way they are transformed. Let the principal diagonal be the closest line to $x + y = 1$. Operator $P$ assigns the same element $P(p)$ in the principal diagonal to any element $p$ of a parallel line (to the left of Fig. 7 three parallel diagonals are represented). $P(p)$ has the same certainty part as the biggest certainty part of any $p$ in the parallel line. See Fig. 7 for the transformation of three sets of productions (circles) into their associated swaps (squares). Geometrically, $P$ can be thought of as the composition of two projections: one along the corresponding diagonal and another parallel to the abscissa axis.

Fig. 7. Rational Encoding of the Transformation of Productions via Operator $P$

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12 “Closest” because the line $x + y = 1$ is approached as the number of nodes in the graphs tend to $\infty$. 
The name swap has been chosen because of the way they act on elements in $\mathcal{S}$. Following previous example, let’s consider the swap $w = \text{0}'0100 \lor \text{0}'1011$. Let’s also consider a generic element $L = \text{0}'L_1^1L_2^2L_3^3L_4^4 \lor \text{0}'K_1^1K_2^2K_3^3K_4^4$. The image $w(L) = \text{0}'K_1^1L_2^2K_3^3K_4^4 \lor \text{0}'L_1^1K_2^2L_3^3L_4^4$ swaps the elements that appear in the nihil part of $w$ and keeps unaltered those that appear in its certainty part. Swaps summarize the dynamics of a production, independently of its left hand side. Notice that, because swaps are self-adjoint, it is enough to keep track of the certainty or nihil parts. So one production is fully specified by, for example, its left hand side and the nihil part of its associated swap.

We can reinterpret actions specified by productions in Matrix Graph Grammars under MCL: instead of adding and deleting elements, they interchange elements between the certainty and nihil parts.

The geometrical interpretation of the actions of swap $w_1 = P(p) = \text{0}'0\overline{1}_2 \lor \text{0}'1_2$ (first element is swapped and the rest remain unaltered) is a reflection with respect to the diagonal $y = x$ in the Sierpinski gasket $\mathcal{S}$. See the right of Fig. 7. We are considering those Boolean complexes that are complementary in the places where the swap interchanges elements. Swap $w_2 = \text{0}'01\overline{1}_2 \lor \text{0}'01_2$ divides $\mathcal{S}$ into two regions, $S_1$ and $S_2$, along the line $y = x$. It acts again as a reflection, but independently in $S_1$ and $S_2$. The reflection is defined with respect to the line $y = x - \frac{1}{2}$ in $S_1$ and with respect to $y = x + \frac{1}{2}$ in $S_2$. The swap $w_{12} = \text{0}'00\overline{1}_2 \lor \text{0}'01_2$ is the composition of $w_1$ and $w_2$. The order does not matter. Geometrically, $w_{12}$ corresponds to a reflection with respect to $y = x$ and another reflection with respect to $y = x \pm \frac{1}{2}$ (only one of them applies).

---

13 Given a swap and a complex term $L$, it is straightforward to calculate the production having $L$ as left hand side and whose actions agree with those of the swap.
14 The notation $\overline{1}$ stands for “as many ones as nodes”. In this contribution we deal with finite graphs.
15 For example, if $w_1$ acts on $z_1 = \text{0}'0110 \lor \text{0}'0001$ then $w_1(z_1) = z_1$. If the first element in $z_1$ is the same in both graphs (in fact zero) then $z_1$ is a fixed element of $w_1$. 

---

Fig. 8. Image of Simple Digraphs via Swaps
Let’s consider those digraphs \( L \) whose certainty and nihil parts are complementary. Figure 8 has \( L \) on the \( x \)-axis and the certainty part of the swaps \( w \) in the \( y \)-axis. The \( z \)-axis represents its image \( \hat{w}(L) = w \oplus L \). Both figures are the same surface, but the second has been rotated \( \pi/4 \) radians clockwise around the \( z \)-axis. Notice that the more elements of \( L \) that we fix as zero (instead of as the complement of its certainty part) the closer \( \hat{w}(L) \) gets to \( \hat{w}(L) \).

6 Coherence and Initial Digraph

So far we have extended MGGs by defining the transformations in \( \mathfrak{G} \) and \( \mathfrak{H} \). The theory will be more interesting if we are able to develop the necessary concepts to deal with sequences of applications rather than productions alone. Among the two most basic notions are coherence and the initial digraph. We shall reformulate and extend the concepts introduced in [15].

Recall that coherence of the sequence \( s = p_n; \ldots; p_1 \) guarantees that the actions of one production \( p_i \) do not prevent the actions of those sequentially behind it: \( p_{i+1}, \ldots, p_n \) (the first production to be applied in \( s \) is \( p_1 \) and the last one is \( p_n \); the order is as in composition, from right to left).

**Theorem 2 (Coherence).** The sequence of productions \( s = p_n; \ldots; p_1 \) is coherent if the Boolean complex \( C \equiv C^+ \lor iC^- = 0 \), where

\[
C^+ = \bigvee_{j=1}^n \left( R_j \land_{j+1}^n (\tau_x r_y) \lor L_j \triangle_{j+1}^1 (e_y \tau_x) \right)
\]  

and

\[
C^- = \bigvee_{j=1}^n \left( Q_j \land_{j+1}^n (e_y \tau_x) \lor K_j \triangle_{j+1}^1 (r_y \tau_x) \right).
\]

**Proof**

\( nC^+ \lor iC^- = 0 \iff C^+ = C^- = 0 \). The certainty part \( C^+ \) is addressed in [15] and \( C^- = 0 \) can be proved similarly. The reader is invited to consult the proof of Th. 4.3.5 in [15] plus Lemma 4.3.3 and the explanations that follow Def. 4.3.2 in the same reference. Next, a sequence of two productions \( s = p_2; p_1 \) is considered to show the way to proceed for \( C^- \).

In order to decide whether the application of \( p_1 \) does not exclude \( p_2 \) (regarding elements that appear in the nihil parts) the following conditions must be demanded:

1. No common element is deleted by both productions:

\[
e_1 e_2 = 0.
\]

2. Production \( p_2 \) does not delete any element that the production \( p_1 \) demands not to be present and that besides is not added by \( p_1 \):

\[
e_2 K_1 \tau_1 = 0.
\]
3. The first production does not add any element that is demanded not to exist by the second production:

\[ r_1K_2 = 0. \]  

Altogether we can write \( e_1e_2 \lor r_1e_2K_1 \lor r_1K_2 = e_2(e_1 \lor r_1K_1) \lor r_1K_2 = e_2Q_1 \lor r_1K_2 = 0 \), which is equivalent to

\[ e_2r_2Q_1 \lor r_1K_2 = 0 \]  

(29)

due to basic properties of MGG productions (see Prop. 4.1.4 in [15]). For a sequence that consists of three productions, \( s = p_3; p_2; p_1 \), the procedure is to apply the same reasoning to subsequences \( p_2; p_1 \) (restrictions on \( p_2 \) actions due to \( p_1 \)) and \( p_3; p_2 \) (restrictions on \( p_3 \) actions due to \( p_1 \)) and or them. Finally, we have to deduce what has to be imposed on \( p_3 \) actions due to \( p_1 \), but this time taking into account that \( p_2 \) is applied in between. Altogether:

\[ Q_1(e_2 \lor r_2e_3) \lor Q_2e_3 \lor K_2r_1 \lor K_3(r_1r_2 \lor r_2). \]  

(30)

We may proceed similarly for four productions. Equation (26) can be deduced applying induction on the number of productions.

To see that eq. (30) implies coherence we only need to enumerate all possible actions on the nihil parts. It might be easier if we think in terms of the negation of a potential host graph to which both productions would be applied (\( \overline{G} \)) and check that any problematic situation is ruled out. See table 1 where \( D \) is deletion of one element from \( G \) (i.e., the element is added to \( G \)), \( A \) is addition to \( G \) and \( P \) is preservation. For example, action \( A_2; A_1 \) tells that in first place \( p_1 \) adds one element \( \varepsilon \) to \( G \). To do so this element has to be in \( e_1 \) (or be incident to a node that is going to be deleted). After that, \( p_2 \) adds the same element, deriving a conflict between the rules.

| \( D_3; D_1 \) | \( D_2; P_1 \) | \( D_3; A_1 \) | \( P_2; A_1 \) |
|---------------|---------------|---------------|---------------|
| \( P_2; D_1 \) | \( P_2; P_1 \) | \( P_2; A_1 \) | \( A_2; A_1 \) |
| \( A_2; D_1 \) | \( A_2; P_1 \) | \( A_2; A_1 \) | \( A_2; A_1 \) |

Table 1. Possible Actions for Two Productions

This proves \( C^- = 0 \) for the case \( n = 2 \). When the sequence has three productions, \( s = p_3; p_2; p_1 \), there are 27 possible combinations of actions. However, some of them are considered in the subsequences \( p_2; p_1 \) and \( p_3; p_2 \). Table 2 summarizes them.

There are four forbidden actions, \( A_3 \lor A_1 \lor A_2 \). Let’s consider the first one, which corresponds to \( r_1r_3 \) (the first production adds the element – it is erased from \( \overline{G} \) – and the same for \( p_3 \)). In Table 3 we see that related conditions appear

16 Preservation means that the element is demanded to be in \( \overline{G} \) because it is demanded not to exist by the production (it appears in \( K_1 \)) and it remains as non-existent after the application of the production (it appears also in \( Q_1 \)).

17 Those actions appearing in table 1 updated for \( p_3 \).
The condition $r_3 r_1$ taking into account the presence of $p_2$ in the middle in eq. (31) is contained in $K_3 r_1 e_2$, which includes $r_1 e_2 r_3$. This must be zero, i.e. it is not possible for $p_1$ and $p_3$ to remove from $G$ one element if it is not added to $G$ by $p_2$. The other three forbidden actions can be checked similarly.

The proof can be finished by induction on the number of productions. The induction hypothesis leaves again four cases: $D_n; D_1$, $A_n; P_1$, $P_n; D_1$ and $A_n; A_1$. The corresponding table changes but it is not difficult to fill in the details.  

There are some duplicated conditions, so it could be possible to “optimize” $C$. The form considered in Th. 2 is preferred because we may use $\triangle$ and $\triangledown$ to synthesize the expressions. Notice that eq. (27) is already in $C$ through eq. (25), which demands $e_1 L_2 = 0$ (as $e_2 < L_2$ we have that $e_1 L_2 = 0 \Rightarrow e_1 e_2 = 0$). Condition (28) is $e_2 K_1 e_1 = e_2 e_1 r_1 \triangledown e_2 e_1 r_1 D_1 = e_2 e_1 D_1$, where we have used that $K_1 = p (D_1)$.

Note that those $e_1 D_1 \neq 0$ are the dangling edges not deleted by $p_1$. Finally, equality (29) is $r_1 K_2 = r_1 p_2 (D_2) = r_1 (r_2 \triangledown e_2 D_2) = r_1 r_2 \triangledown r_1 e_2 D_2$. The first term $(r_1 r_2)$ is already included in $C$ and the second term is again related to dangling edges. Potential dangling edges appear in coherence and this may seem to indicate a possible link between coherence and compatibility\[18\].

![Table 2. Possible Actions for Three Productions](image)

**Example.** Let’s consider the sequence $s = p_5; p_4$. Recall that the order of application is from right to left so $p_4$ is applied first and $p_5$ right afterwards. Let $p_4$ and $p_5$ be

---

\[18\] Compatibility for sequences is characterized in Sec. 7. Coherence takes into account dangling edges, but only those that appear in the “actions” of the productions (in matrices $e$ and $r$).
those productions depicted in Fig. 9. Once simplified, its coherence term is

\[ C(s) = C^+(s) \lor iC^-(s) = (R_4r_5 \lor L_5e_4) \lor i(Q_4e_5 \lor K_5r_4) = \]

\[
= \left( \begin{bmatrix}
0 & 1 & 1 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
\end{bmatrix} \right) \lor \left( \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1 \\
\end{bmatrix} \right) \lor i \left( \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0 \\
\end{bmatrix} \right)
\]

\[
\lor \left( \begin{bmatrix}
1 & 0 & 1 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
\end{bmatrix} \right) \left( \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
\end{bmatrix} \right) \lor i \left( \begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
\end{bmatrix} \right).
\]

Coherence problems appear in this example for several reasons. Edge \( (2, 3) \) is added twice while self-loop \( (3, 3) \) is first deleted in \( p_4 \) and then used in \( p_5 \). Edge \( (1, 3) \) becomes dangling because production \( p_5 \) deletes node 1. Edge \( (2, 3) \) appears in \( C^- \) for the same reason that makes it appear in \( C^+ \).

The minimal initial digraph \( M(s) \) for a completed sequence \( s = p_n; \ldots; p_1 \) was introduced in [15] as a simple digraph that permits all operations of \( s \) and that does not contain a proper subgraph with the same property. The negative initial digraph has a similar definition but for the nihil part. Theorem 5 encodes as a complex term the minimal and negative initial digraphs, renaming it to initial digraph.

Now we are interested in what elements will be forbidden and which ones will be available once every production is applied. Matrix \( D \) specifies what edges can not be present because at least one of their incident nodes have been deleted. Let’s introduce the dual concept:

\[ T = (\tau \otimes \tau') \land (\tau \otimes \tau'). \tag{32} \]

\( T \) are the newly available edges after the application of the production because of the addition of nodes. The first term, \( \tau \otimes \tau' \), has a one in all edges incident to a vertex that is added by the production. We have to remove those edges that are incident to some node deleted by the production, which is what \( \tau \otimes \tau' \) does.

\[ L \quad q \quad R \]

\[
\begin{bmatrix}
1 \\
2 \\
\end{bmatrix} \quad \begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 1 \\
0 & 1 & 1 \\
\end{bmatrix} \quad \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 1 \\
1 & 1 & 1 \\
\end{bmatrix}
\]

\[ \otimes_\delta = \begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 1 \\
0 & 1 & 1 \\
\end{bmatrix} \quad \otimes_{\delta^2} = \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 1 \\
1 & 1 & 1 \\
\end{bmatrix} \quad \tau = \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 1 \\
0 & 1 & 1 \\
\end{bmatrix}
\]

Fig. 10. Available and Unavailable Edges After the Application of a Production

---

19 Recall that whenever the tensor (Kronecker) product is used, we refer to the vector of nodes so the \( V \) superscript is omitted. For example \( R \otimes R' \equiv R^V \otimes (R^V)^t \). The \( t \) stands for transposition.

20 This is why \( T \) does not appear in the calculation of the coherence of a sequence: coherence takes care of real actions \((e, r)\) and not of potential elements that may or may not be available \((D, T)\).
Example. Figure 10 depicts to the left a production \( q \) that deletes node 1 and adds node 3. Its nihil term and its image are

\[
K = q(\overline{D}) = r \lor \tau D = \begin{bmatrix}
1 & 1 & 1 \\
1 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix}
\]

\[
Q = q^{-1}(K) = e \lor \tau K = \begin{bmatrix}
1 & 1 & 1 \\
1 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix}
\]

To the right of Fig. 10 matrix \( T \) is included. It specifies those elements that are not forbidden once production \( q \) has been applied.

Matrices \( D \) and \( T \) do not tell actions of the production to be performed in the complement of the host graph, \( \overline{G} \). Actions of productions are specified exclusively by matrices \( e \) and \( r \).

**Theorem 3 (Initial Digraph).** The initial digraph \( M(s) \) for the completed coherent sequence of productions \( s = p_n; \ldots; p_1 \) is given by

\[
M(s) = \nabla^n \left( \tau_x L_y \lor i \tau_x T_x K_y \right). \tag{33}
\]

**Proof (sketch)**

The proof proceeds along the lines of that for Th. 4.4.2 in [15], which in essence starts with a big enough graph and removes as many elements as possible. However, for edges, besides the actions of the productions on edges we need to keep track of the actions of the productions on nodes because some potential dangling edges may become available (if their incident nodes are added by some grammar rule).

Notice that Th. 4.4.2 in [15] proves that the certainty part of the initial digraph is the one that appears in eq. (33). For the nihil term \( \nabla^n \tau_x T_x K_y \) it is easier to think in what must be or must not be found in \( \overline{G} \).

We proceed by induction on the number of productions. For the time being, for simplicity, we omit the effect of adding nodes which may turn potential dangling edges into available ones. In a sequence with a single production it should be obvious that \( K_1 \) (and only \( K_1 \)) needs to be demanded.

For a sequence of two productions \( s_2 = p_2; p_1 \), \( K_1 \) is again necessary. It is clear that \( K_1 \lor K_2 \) with \( K_1^V K_2^V = 0 \) – i.e. all nodes and hence edges unrelated – would be enough, but it may include more elements than strictly needed. Among them, those already deleted by \( p_1 \) (once they are deleted they belong to \( \overline{G} \)) and those that already appear in \( K_1 \) and that are not added by \( p_1 - \overline{\tau_1} K_1 - \). If these elements of \( K_2 \) are not going to be considered, we need to and their negation: \( \overline{\tau_1 (\overline{\tau_1} K_1) K_2} \). Altogether we get \( K_1 \lor \overline{\tau_1 (\overline{\tau_1} K_1) K_2} \).

Some simple manipulations prove that:

\[
K_1 \lor K_2 \tau_1 (\overline{\tau_1} K_1) = K_1 \lor K_2 \tau_1 (K_1) = K_1 \lor K_2 \tau_1 (K_1) = K_2 Q_1. \tag{34}
\]

Minimality is inferred by construction. If any other element was removed then either \( p_1^{-1} \) or \( p_2^{-1} \) could not be applied (and still consider dangling edges). It is not difficult to check that the sequence \( p_2^{-1}; p_1^{-1} \) can be applied to \( K_1 \lor K_2 Q_1 \). The expressions for
sequences of three, four, $n$ productions are:

$$N_3 = N_2 \lor K_3 \overline{B_2 Q_1 Q_2}$$

$$N_4 = N_3 \lor K_3 \overline{B_3 Q_1 Q_2 Q_3 Q_4}$$

$$\ldots$$

$$N_n = K_1 \lor r_1 \bigoplus_{j=1}^{n-1} \left( \overline{Q_x K_y} \right) \lor \bigvee_{j=2}^{n} \left[ K_j \Delta_i^{j-1} \left( \overline{Q_x r_y} \right) \right]$$

There are two tricky steps. The first one is how to derive $N_n$ in eq. (37) and the second is how to obtain its equivalent expression (38). The reader is referred again to the aforementioned proof in [15] where detailed explanations are given for a similar case.

Once we get here it is easy to obtain $\bigoplus_{1}^{n} \left( \overline{Q_x K_y} \right)$. First, note that the sequence is coherent so the third term in eq. (38) is zero. Second, as $K_1 = K_1 \lor r_1$, the $r_1$ can be simplified because $a \lor \overline{a} = a$ in propositional logic.

Finally, the same reasoning applies for those nodes that are added. So we do not only need to remove elements erased by previous productions but also edges that are not incident to any non-existent edge, $\bigoplus_{1}^{n} \left( \overline{Q_x K_y} \right) \leftrightarrow \bigoplus_{1}^{n} \left( \overline{Q_x T_x K_y} \right)$.■

Fig. 11. sequence of Two Productions
Example. Figure 11 includes two productions with their nihilation matrices $K_1$ and $K_2$. The initial digraph of the sequence $s = q_2; q_1$ is

$$M(s_2) = \nabla_2^{2} (r_x L_y \lor i r_x \overline{T}_x K_y) = (r_1 L_1 \lor r_1 r_2 L_2) \lor i (r_1 T_1 K_1 \lor r_1 r_2 T_1 T_2 K_2) =$$

$$= (L_1 \lor r_1 L_2) \lor i (T_1 K_1 \lor r_1 T_1 T_2 K_2) = \left[ \begin{array}{ccc}
1 & 0 & 0 \\
0 & 0 & 1 \\
1 & 1 & 0
\end{array} \right] \lor \left[ \begin{array}{ccc}
1 & 1 & 1 \\
1 & 1 & 1 \\
0 & 0 & 0
\end{array} \right] = \left[ \begin{array}{ccc}
1 & 1 & 1 \\
1 & 1 & 1 \\
0 & 0 & 0
\end{array} \right]$$

We have represented $M_C(s_2) \lor M_N(s_2)$ to the left of Fig. 12 together with its evolution as well as the final state, $s_2 (M(s_2))$. To the right of the same figure there is the same evolution but limited to edges and from the point of view of swaps. With black solid line we have represented the edges that are present and with red dotted line those that are absent. Recall that swaps interchange them.

![Fig. 12. Initial Digraph of a Sequence of Two Productions Together with its Evolution](image)

A final remark is that $T$ makes the number of edges in $\overline{G}$ as small as possible. For example, in $r_1 r_2 T_1 T_2 K_2$ we are in particular demanding $r_1 T_1 T_2 r_2$ (because $K_2 = r_2 \lor r_2 \overline{T}_2$). If we start with a compatible host graph, it is not necessary to ask for the absence of edges incident to nodes that are added by a production (we called them potentially available above). Notice that these edges could not be in the host graph as they would be dangling edges or we would be adding an already existent node).

### 7 Compatibility and Congruence

This section revises some more sequential results, adapting and extending them via MCL. The notions we cope with are the image of a sequence, compatibility\footnote{Compatibility for a single production has been tackled in Prop. 4, Sec. 4} and G-
congruence. By the end of the section we will very briefly touch on sequential independence, application conditions and graph constraints.

The image of a sequence of productions $s = p_0; \ldots; p_1$ acting on its initial digraph $M(s) = M_C(s) \vee iM_N(s)$ is given by:

$$s(M(s)) = \left( \bigwedge^n_{j=1} x_j \vee \bigwedge^n_{j=1} y_j M_C(s) \right) \vee i \left( \bigwedge^n_{j=1} x_j e_y \vee \bigwedge^n_{j=1} y_j M_N(s) \right). \quad (39)$$

Equation (39) is deduced by simply applying each production to the initial digraph. We would like to interpret the shape of the image of a sequence as a production by setting $s = e(s) \vee ir(s)$, with $e(s) = \bigvee^n_{i=1} e_i$ and $r(s) = \bigvee^n_{i=1} x_i r_y$. We could then put it as $\langle M(s), s \rangle$ and for example calculate its associated swap:

$$w(s) = \bigwedge^n_{j=1} (x_j r_j) \vee i \bigvee^n_{j=1} (e_j \vee r_j).$$

Unfortunately this is not possible because although in the certainty part of eq. (39) $r(s) = \bigvee^n_{i=1} x_i r_y$ and $e(s) = \bigvee^n_{i=1} e_i$, in the nihil part we find $\bigwedge^n_{j=1} x_j \neq r_j$ and $\bigwedge^n_{j=1} x_j e_y \neq e(s)$.

Compatibility asks for “closedness” of the space (graphs) with respect to the specified operations. In essence, it demands the lack of dangling edges. Definitions of compatibility for increasingly general concepts can be found in [13]: single simple digraph, production and sequence. According to Prop. [4] productions act on edges and on vertices. They are obviously related but this relation has not been demonstrated. It is of importance in order to study the evolution of the nihil part of complex terms. What one production forbids, another production may need or even can make accessible again.

**Proposition 5 (Compatibility).** Let $s = p_0; \ldots; p_1$ be a sequence made up of compatible productions. If

$$\bigvee^n_{i=1} (e_i x_i M_C(s_x) M_N(s_x)) = 0 \quad (40)$$

then $s$ is compatible, where $M_C(s_m)$ and $M_N(s_m)$ are the certainty and nihil parts of the initial digraphs of $s_m = p_m; \ldots; p_1$, $m \in \{1, \ldots, n\}$.

**Proof (Sketch)**

Equation (40) is a restatement of the definition of compatibility for a sequence of productions. The condition appears when the certainty and nihil parts are demanded to have no common elements. Compatibility of each production is used to simplify terms of the form $L_i K_i$. Compatibility and coherence are related notions but only to some extent. Coherence deals with actions of productions while compatibility with potential presence or absence of elements. This is better understood if we think in derivations: when the left hand side $L \vee iK$ of rule $p$ is matched in a host graph $G \vee iG$, all elements of $L$ must be found in $G$ and all edges of $K$ must be found in $G$. When $p$ is applied a new graph

22 The idea behind would be the composition of a sequence of productions to derive a single production: $s_n = p_n; \ldots; p_1 \mapsto c_n = p_n \circ \ldots \circ p_1$. 


$H \triangleright \mathcal{I}$ is derived. Again, all elements of $R$ have to be found in $H$ and all edges in $Q$ will be in $\mathcal{H}$, no matter if some of them are now potentially usable (say $p$ adds some nodes and some potentially dangling edges are not dangling edges anymore).

Now we turn to G-congruence, which studies equality of initial digraphs for a sequence $s = p_n; \ldots; p_1$ and a permutation of it, $\sigma(s)$. All the job for advancement and delaying of productions – permutations $\phi$ and $\delta$, where advancement is $\phi = (1 \ 2 \ \ldots \ n \ n-1 \ \ldots \ 2 \ 1)$ and delaying is $\delta = (n \ n-1 \ \ldots \ 2 \ 1)$, i.e $\phi(s) = p_{n-1}; p_{n-2} \ldots ; p_1; p_n$ and $\delta(s) = p_1; p_2; \ldots ; p_2$ – is done in [15], Sec. 6.1, so we state the result without proof.

**Theorem 4 (G-congruence).** With notation as above, sequences $s$ and $\phi(s)$ are G-congruents if $F \equiv F^+ \lor iF^- = 0$, where

$$F^+ = L_n \nabla e_y K_y (r_y \lor e_n)$$

and

$$F^- = K_n \nabla e_y L_y (e_y \lor r_n).$$

Also, $s$ and $\delta(s)$ are G-congruents if $D \equiv D^+ \lor iD^- = 0$, with

$$D^+ = L_1 \nabla e_y K_y (r_y \lor e_1)$$

and

$$D^- = K_1 \nabla e_y L_y (e_y \lor r_1).$$

**Proof**

An easy remark is that the complex term $C^+ \lor iC^-$ in Th. 2 provides more information than just settling coherence as it measures non-coherence: *Problematic* elements (i.e. those that prevent coherence) would appear as ones and the rest as zeros. The same holds for $F^\lor iF^-$ and $D^\lor iD^-$ in Th. 4 for congruence and eq. (40) in Prop. 5 for compatibility.

There are some relevant topics that we have not mentioned such as sequential independence, application conditions and graph constraints. We briefly discuss how they could be handled with MCL.

With respect to the image of a sequence and sequential independence, recall that MCL naturally uses swaps rather than productions. This abstraction has its effects on the interpretation of operations. On the positive side, among many other things, swaps are a nice redefinition and generalization of productions that take into account the certainty and nil parts; on the negative side, our intuition needs to be adjusted. For example, consider a production $p$ that only deletes edge $(1, 2)$ and does nothing else. Suppose that it is applied twice to the graph $G$ that consists of nodes $1, 2$ and edge $(1, 2)$. In this case $p; p(G) = G$ which is algebraically correct. However, it does not encode “delete edge $(1, 2)$ twice”. Of course, the point here is that of completion: we would rather have considered its application to $G'$, made up of nodes $1, 1'$ and $2$ and edges $(1, 2)$ and $(1', 2)$. A similar reasoning shows that sequential independence is “granted” if we rely only on algebraic operations and do not pay attention to completion:

$$p_2; p_1(L) = \langle \langle L, P(p_1) \rangle, P(p_2) \rangle = \mathcal{L} P(p_1) P(p_2) = \mathcal{L} P(p_2) P(p_1) = p_1; p_2(L).$$

23 Numbers in the permutation refers to the position that the production occupies inside the sequence, not to its subindex.
Previous comments highlight some of the reasons why coherence, compatibility, initial digraph and G-congruence are so valuable, justifying their inclusion and also linking the present and previous sections to Sec. 5.

Regarding application conditions and graph constraints, they are not difficulty related to what has been presented so far. Recall from Sec. 5 that swaps transform elements in the same diagonal of the Sierpinski gasket. If they are allowed to be applied to $g \in \mathcal{G}$ instead of the restricted case that we have studied ($\mathcal{H}$), we may impose limits on what elements can not be added nor deleted by sequences of productions (swaps). This is because if one edge is in the certainty part and in the nihil part, it can not be deleted by any swap. On the contrary, if one edge does not appear neither in the certainty nor in the nihil parts, it is not possible for a swap to add it.

If we call any of these situations a swap restriction, it can be guaranteed that a sequence will not add nor delete (or both) some element, despite the actual definition of the productions that make up the sequence or the grammar. Again, geometrically, we are choosing the diagonal inside the Sierpinski gasket in which all operations will take place.

8 Conclusions and Future Work

In this paper we have introduced Monotone Complex Logic (MCL) which, in our opinion, is an interesting topic in itself. With respect to Matrix Graph Grammars (MGGs), MCL allows the encoding of simple digraphs and grammar rules using complex terms. We believe it is a natural representation in the MGGs context, as productions act on pairs of graphs $p : (L, K) \rightarrow (p(L), p^{-1}(K))$. Relevant algebraic structures for their study have been introduced (PMCA, PMMA, $\mathcal{G}$, $\mathcal{H}$). Swaps allow studying and classifying productions according to their dynamic behaviour, defining a surjective morphism into the self-adjoint graphs in $\mathcal{H}$.

The rational encoding of Boolean complexes gives an embedding of MGGs into a subset of the complex numbers: the Sierpinski gasket. Using such representation we have been able to introduce standard geometric and analytic concepts such as a scalar product, a norm and a notion of distance in MGGs. This generalizes the theory developed in [15] and opens the door to the study of dynamics of infinite graphs with a countable number of nodes (this topic is left for future research). Finally, some of the most relevant concepts of MGGs have been expressed and reinterpreted using MCL: coherence, initial digraph, image of a sequence, compatibility and G-congruence.

Our main interest is complexity theory so we have to introduce a measure of complexity. The natural proposal seems to be the geodesic distance, which measures the cost of reaching one element from another one through elements of the MGG (rules of the grammar). However, the natural distance here is not the Euclidean one, but the xor metric restricted by available operations.

One foreseen advantage of the results in this paper is that there is a lot of interest and current research activity on the Sierpinski gasket [2,7,18]. In the mid-long term we plan to continue our work towards computational complexity theory through MGGs. It is our opinion that one of the main “problems” of current approaches to complexity theory is that there are very few links to other branches of mathematics (there are some
exceptions though, such as [10]). As MGGs are a compact and path connected fractal, it seems promising to introduce harmonic and functional analysis and noncommutative geometry techniques, apart from those already available in MGGs. Two main research directions will be explored in the future: measurable Riemannian geometry as in [7] and noncommutative geometry as in [2].

Notice that it is not difficult to interpret MGGs as a model of computation (we are preparing a paper on this topic). Also, it might be of interest to encode properties of graphs (such as coloring) using graph grammars, translating static properties into equivalent dynamic properties of associated sequences.

Another point of interest might be the introduction of stochastic analysis. This is closely related to MGGs as a model of computation and the way grammar rules are selected (a source of non-determinism). Other source of non-determinism appears in case there are several places in a host graph to which a production can be applied.

There are many more topics for further research, e.g. graph constraints, derivations, applicability, reachability, dynamic encoding of static properties and infinite graphs some of which we have already commented on.

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References

1. Braket notation intro: http://en.wikipedia.org/wiki/Bra-ket_notation
2. Christensen, E., Ivan, C., Lapidus, M. 2007. \textit{Dirac operators and spectral triples for some fractal sets built on curves}. arXiv:math/0610222v2 [math.MG]
3. Ehrig, H., Engels, G., Kreowski, H.-J., Rozenberg, G. 1999. \textit{Handbook of Graph Grammars and Computing by Graph Transformation}. Vol. 2 (Applications, Languages and Tools). World Scientific.
4. Ehrig, H., Ehrig, K., Prange, U., Taentzer, G. 2006. \textit{Fundamentals of Algebraic Graph Transformation}. Springer.
5. Fine, N. J. 1947. \textit{Binomial Coefficients Modulo a Prime}. Amer. Math. Monthly 54, pp. 589-592.
6. Goldreich, O. 2008. \textit{Computational Complexity: A Conceptual Approach}. Cambridge University Press.
7. Kigami, J. 2007. \textit{Measurable Riemannian geometry on the Sierpinski gasket: the Kusuoka measure and the Gaussian heat kernel estimate}. Mathematische Annalen. Vol. 340 (4), pp. 781-804. Springer.
8. Mandelbrot, B.B. 1982. \textit{The Fractal Geometry of Nature}. W.H. Freeman and Company.
9. Maymounkov, P., Mazières, D. 2002. \textit{Kademlia: A Peer-to-Peer Information System Based on the XOR Metric}. 1st Int. Workshop on Peer-to-peer Systems.
10. Mulmuley, K., Sohoni, M. A. 2001. \textit{Geometric Complexity Theory I: An Approach to the P vs. NP and Related Problems}. SIAM J. Comput. 31(2): 496-526.
11. Papadimitriou, C. 1994. \textit{Computational Complexity}. Addison-Wesley.
12. Pérez Velasco, P. P., de Lara, J. 2006. \textit{Matrix Approach to Graph Transformation: Matching and Sequences}. LNCS 4178, pp.:122-137. Springer.
13. Pérez Velasco, P. P., de Lara, J. 2006. Petri Nets and Matrix Graph Grammars: Reachability. EC-EAAS(2).
14. Pérez Velasco, P. P., de Lara, J. 2007. Using Matrix Graph Grammars for the Analysis of Behavioural Specifications: Sequential and Parallel Independence. ENTCS 206, pp.:133-152. Elsevier.
15. Pérez Velasco, P. P. 2008. Matrix Graph Grammars. E-book available at: http://www.mat2gra.info/, CoRR abs/0801.1245.
16. Rozenberg, G. (ed.) 1997. Handbook of Graph Grammars and Computing by Graph Transformation. Vol.1 (Foundations), World Scientific.
17. Vollmer, H. 1999. Introduction to Circuit Complexity. A Uniform Approach. Springer.
18. Weisstein, E. Sierpiński Sieve. “From MathWorld–A Wolfram Web Resource”. http://mathworld.wolfram.com/SierpinskiSieve.html