Dual Iterative Hard Thresholding: From Non-convex Sparse Minimization to Non-smooth Concave Maximization

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Abstract

Iterative Hard Thresholding (IHT) is a class of projected gradient descent methods for optimizing sparsity-constrained minimization models, with the best known efficiency and scalability in practice. As far as we know, the existing IHT-style methods are designed for sparse minimization in primal form. It remains open to explore duality theory and algorithms in such a non-convex and NP-hard setting. In this article, we bridge the gap by establishing a duality theory for sparsity-constrained minimization with $\ell_2$-regularized objective and proposing an IHT-style algorithm for dual maximization. Our sparse duality theory provides a set of sufficient and necessary conditions under which the original NP-hard/non-convex problem can be equivalently solved in a dual space. The proposed dual IHT algorithm is a super-gradient method for maximizing the non-smooth dual objective. An interesting finding is that the sparse recovery performance of dual IHT is invariant to the Restricted Isometry Property (RIP), which is required by all the existing primal IHT without sparsity relaxation. Moreover, a stochastic variant of dual IHT is proposed for large-scale stochastic optimization. Numerical results demonstrate that dual IHT algorithms can achieve more accurate model estimation given small number of training data and have higher computational efficiency than the state-of-the-art primal IHT-style algorithms.

Key words. Sparse Learning, Dual Methods

1 Introduction

Sparse learning has emerged as an effective approach to alleviate model overfitting when feature dimension outnumbers training sample. Given a set of training samples $\{(x_i, y_i)\}_{i=1}^N$ in which $x_i \in \mathbb{R}^d$ is the feature representation and $y_i \in \mathbb{R}$ the corresponding label, the following sparsity-constrained $\ell_2$-norm regularized loss minimization problem is often considered in high-dimensional analysis:

$$\min_{\|w\|_0 \leq k} P(w) := \frac{1}{N} \sum_{i=1}^N l(w^\top x_i, y_i) + \frac{\lambda}{2} \|w\|^2.$$  (1.1)

Here $l(\cdot, \cdot)$ is a convex loss function, $w \in \mathbb{R}^d$ is the model parameter vector and $\lambda$ is the regularization strength. For example, the squared loss $l(a, b) = (b - a)^2$ is used in linear regression and the hinge loss $l(a, b) = \max\{0, 1 - ab\}$ in support vector machines. Due to the presence of cardinality
constraint \(|w|_0 \leq k\), the problem (1.1) is simultaneously non-convex and NP-hard in general, and thus is challenging for optimization. A popular way to address this challenge is to use proper convex relaxation, e.g., \(\ell_1\) norm (Tibshirani, 1996) and \(k\)-support norm (Argyriou et al., 2012), as an alternative of the cardinality constraint. However, the convex relaxation based techniques tend to introduce bias for parameter learning.

In this paper, we are interested in algorithms that directly minimize the non-convex formulation in (1.1). Early efforts mainly lie in compressed sensing for signal recovery, which is a special case of (1.1) with squared loss. Among others, a family of the so called Iterative Hard Thresholding (IHT) methods (Blumensath & Davies, 2009; Foucart, 2011) have gained significant interests and they have been witnessed to offer the fastest and most scalable solutions in many cases. More recently, IHT-style methods have been generalized to handle generic convex loss functions (Beck & Eldar, 2013; Yuan et al., 2014; Jain et al., 2014) as well as structured sparsity constraints (Jain et al., 2016). The common theme of these methods is to iterate between gradient descent and hard thresholding to enforce sparse solutions.

Although IHT-style methods have been extensively studied, the state-of-the-art is only designed for the primal formulation (1.1). It remains an open problem to investigate the feasibility of solving the original NP-hard/non-convex formulation in a dual space that might potentially further improve computational efficiency. Inspired by the recent success of dual methods in regularized learning problems, we systematically build a sparse duality theory and propose an IHT-style algorithm along with its stochastic variant for dual optimization.

Overview of our contribution. The core contribution of this work is two-fold in theory and algorithm. As the theoretical contribution, we have established a novel sparse Lagrangian duality theory for the NP-hard/non-convex problem (1.1) which to the best of our knowledge has not been reported elsewhere in literature. We provide in this part a set of sufficient and necessary conditions under which one can safely solve the original non-convex problem through maximizing its concave dual objective function. As the algorithmic contribution, we propose the dual IHT (DIHT) algorithm as a super-gradient method to maximize the non-smooth dual objective. Generally, DIHT iterates between dual gradient ascent and primal hard thresholding pursuit until convergence. A stochastic variant of DIHT is proposed to handle large-scale learning problems. For both algorithms, we provide non-asymptotic convergence results on parameter estimation error, support recovery, and primal-dual gap as well. In contrast to the existing analysis for primal IHT-style algorithms, our analysis is not relying on Restricted Isometry Property (RIP) conditions and thus is less restrictive in real-life high-dimensional statistical settings. Numerical results on synthetic datasets and machine learning benchmark datasets demonstrate that dual IHT significantly outperforms the state-of-the-art primal IHT algorithms in accuracy and efficiency. The theoretical and algorithmic contributions of this article are highlighted in below:

- Sparse Lagrangian duality theory: we established a sparse saddle point theorem (Theorem 1), a sparse mini-max theorem (Theorem 2) and a sparse strong duality theorem (Theorem 3).

- Dual optimization: we proposed an IHT-style algorithm along with its stochastic extension for non-smooth dual maximization. These algorithms have been shown to converge at the rate of \(\frac{1}{\epsilon} \ln \frac{1}{\epsilon}\) in parameter estimation error and primal-dual gap (see Theorem 4 and Theorem 5). These guarantees are invariant to RIP conditions which are typically required by primal IHT.

Notation. Before continuing, we define some notation to be used. Let \(x \in \mathbb{R}^d\) be a vector and \(F\) be an index set. We denote \(H_F(x)\) is a truncation operator that restricts \(x\) to the set \(F\). \(H_k(x)\) is a
truncation operator which preserves the top \( k \) (in magnitude) entries of \( x \) and forces the remaining to be zero. The notation \( \text{supp}(x) \) represents the index set of nonzero entries of \( x \). We conventionally define \( \|x\|_\infty = \max_i |[x]_i| \) and define \( x_{\min} = \min_{i \in \text{supp}(x)} |[x]_i| \). For a matrix \( A \), \( \sigma_{\text{max}}(A) \) (\( \sigma_{\text{min}}(A) \)) denotes its largest (smallest) singular value.

**Organization.** The rest of this paper is organized as follows: In §2 we briefly review the relevant work. In §3 we develop a Lagrangian duality theory for sparsity-constrained minimization problems. The dual IHT-style algorithms along with convergence analysis are presented in §4. The numerical evaluation results are reported in §5. Finally, the concluding remarks are made in §6. All the technical proofs are deferred to appendix sections.

2 Related Work

For generic convex objective beyond quadratic loss, the rate of convergence and parameter estimation error of IHT-style methods were established under proper RIP (or restricted strong condition number) bound conditions (Blumensath, 2013; Yuan et al., 2014, 2016). In (Jain et al., 2014), several relaxed variants of IHT-style algorithms were presented for which the estimation consistency can be established without requiring the RIP conditions. In (Bahmani et al., 2013), a gradient support pursuit algorithm is proposed and analyzed. In large-scale settings where a full gradient evaluation on all data becomes a bottleneck, stochastic and variance reduction techniques have been adopted to improve the computational efficiency of IHT (Nguyen et al., 2014; Li et al., 2016; Chen & Gu, 2016).

Dual learning methods have gained significant attention in recent years (Shalev-Shwartz & Zhang, 2013a,b). Dual ascent type algorithms have been widely used in various learning tasks including SVM (Hsieh et al., 2008) and multi-task learning (Lapin et al., 2014). To further improve computational efficiency, some primal-dual methods are developed to alternately minimize the primal objective and maximize the dual objective. The successful examples of primal-dual methods include learning total variation regularized model (Chambolle & Pock, 2011) and generalized Dantzig selector (Lee et al., 2016). More recently, a number of stochastic variants (Zhang & Xiao, 2015; Yu et al., 2015) and parallel variants (Zhu & Storkey, 2016) were developed to make the primal-dual algorithm more scalable and efficient.

3 A Sparse Lagrangian Duality Theory

In this section, we establish weak and strong duality theory that guarantees the original non-convex and NP-hard problem in (1.1) can be equivalently solved in a dual space. The results in this part build the theoretical foundation of our dual IHT methods.

From now on we abbreviate \( l_i(w^\top x_i) = l(w^\top x_i, y_i) \). The convexity of \( l(w^\top x_i, y_i) \) implies that \( l_i(u) \) is also convex. Let \( l_i^*(\alpha_i) = \max_u \{ \alpha_i u - l_i(u) \} \) be the convex conjugate of \( l_i(u) \) and \( \mathcal{F} \subseteq \mathbb{R} \) be the feasible set of \( \alpha_i \). According to the fact of \( l_i(u) = \max_{\alpha_i \in \mathcal{F}} \{ \alpha_i u - l_i^*(\alpha_i) \} \), the problem (1.1) can be reformulated into the following mini-max formulation:

\[
\min_{\|w\|_0 \leq k} \left\{ \frac{1}{N} \sum_{i=1}^N \max_{\alpha_i \in \mathcal{F}} \{ \alpha_i w^\top x_i - l_i^*(\alpha_i) \} + \frac{\lambda}{2} \|w\|^2 \right\}
\]  

(3.1)
The following Lagrangian form will be useful in analysis:

\[ L(w, \alpha) = \frac{1}{N} \sum_{i=1}^{N} \left( \alpha_i w^\top x_i - l_i'(\alpha_i) \right) + \frac{\lambda}{2} \|w\|^2, \]

where \( \alpha = [\alpha_1, ..., \alpha_N] \in \mathcal{F}^N \) is the vector of dual variables. We now introduce the following concept of sparse saddle point which is a restriction of the conventional saddle point to the setting of sparse optimization.

**Definition 1 (Sparse Saddle Point).** A pair \((\bar{w}, \bar{\alpha}) \in \mathbb{R}^d \times \mathcal{F}^N\) is said to be a \(k\)-sparse saddle point for \(L\) if \(\|\bar{w}\|_0 \leq k\) and the following holds for all \(\|w\|_0 \leq k, \alpha \in \mathcal{F}^N:\)

\[ L(\bar{w}, \alpha) \leq L(\bar{w}, \bar{\alpha}) \leq L(w, \bar{\alpha}). \tag{3.2} \]

In contrast to the conventional definition of saddle point, the \(k\)-sparse saddle point only requires the inequality \(3.2\) holds for any arbitrary \(k\)-sparse vector \(w\). The following result is a basic sparse saddle point theorem for \(L\). Throughout the paper, we will use \(f'(\cdot)\) to denote a sub-gradient (or super-gradient) of a convex (or concave) function \(f(\cdot)\), and use \(\partial f(\cdot)\) to denote its sub-differential (or super-differential).

**Theorem 1 (Sparse Saddle Point Theorem).** Let \(\bar{w} \in \mathbb{R}^d\) be a \(k\)-sparse primal vector and \(\bar{\alpha} \in \mathcal{F}^N\) be a dual vector. Then the pair \((\bar{w}, \bar{\alpha})\) is a sparse saddle point for \(L\) if and only if the following conditions hold:

(a) \(\bar{w}\) solves the problem in \((1.1)\);

(b) \(\bar{\alpha} \in [\partial l_1(\bar{w}^\top x_1), ..., \partial l_N(\bar{w}^\top x_N)]\),

(c) \(\bar{w} = H_k \left( -\frac{1}{N} \sum_{i=1}^{N} \bar{\alpha}_i x_i \right)\);

A proof of this theorem can be found in Appendix A.1.

**Remark 1.** Theorem 1 shows that the conditions (a)–(c) are sufficient and necessary to guarantee the existence of a sparse saddle point for the Lagrangian form \(L\). This result is different from the traditional saddle point theorem which requires the use of the Slater Constraint Qualification to guarantee the existence of saddle point.

**Remark 2.** Let us consider \(P'(\bar{w}) = \frac{1}{N} \sum_{i=1}^{N} \bar{\alpha}_i x_i + \lambda \bar{w} \in \partial P(\bar{w})\). Denote \(\bar{F} = \text{supp}(\bar{w})\). It is easy to verify that the condition (c) in Theorem 1 is equivalent to

\[ H_{\bar{F}}(P'(\bar{w})) = 0, \quad \bar{w}_{\min} \geq \frac{1}{\lambda} \|P'(\bar{w})\|_\infty. \]

The following sparse mini-max theorem guarantees that the min and max in \((3.1)\) can be safely switched if and only if there exists a sparse saddle point for \(L(w, \alpha)\).

**Theorem 2 (Sparse Mini-Max Theorem).** The mini-max relationship

\[ \max_{\alpha \in \mathcal{F}^N} \min_{\|w\|_0 \leq k} L(w, \alpha) = \min_{\|w\|_0 \leq k} \max_{\alpha \in \mathcal{F}^N} L(w, \alpha) \tag{3.3} \]

holds if and only if there exists a sparse saddle point \((\bar{w}, \bar{\alpha})\) for \(L\).
A proof of this theorem can be found in Appendix A.2.

The sparse mini-max result in Theorem 2 provides sufficient and necessary conditions under which one can safely exchange a min-max for a max-min, in the presence of sparsity constraint. The following corollary is a direct consequence of applying Theorem 1 to Theorem 2.

**Corollary 1.** The mini-max relationship

$$\max_{\alpha \in \mathcal{F}} \min_{\|w\|_0 \leq k} L(w, \alpha) = \min_{\|w\|_0 \leq k} \max_{\alpha \in \mathcal{F}} L(w, \alpha)$$

holds if and only if there exist a k-sparse primal vector $\bar{w} \in \mathbb{R}^d$ and a dual vector $\bar{\alpha} \in \mathcal{F}^N$ such that the conditions (a)~(c) in Theorem 1 are satisfied.

The mini-max result established in Theorem 2 can be used as a basis for sparse duality theory. Indeed, we have already shown

$$\min_{\|w\|_0 \leq k} \max_{\alpha \in \mathcal{F}} L(w, \alpha) = \min_{\|w\|_0 \leq k} P(w).$$

This is called the *primal* minimization problem and it is the min-max side of the sparse mini-max theorem. The other side, the max-min problem, will be called as the *dual* maximization problem with dual objective function $D(\alpha) := \min_{\|w\|_0 \leq k} L(w, \alpha)$, i.e.,

$$\max_{\alpha \in \mathcal{F}^N} D(\alpha) = \max_{\alpha \in \mathcal{F}^N} \min_{\|w\|_0 \leq k} L(w, \alpha). \tag{3.4}$$

The following Lemma 1 shows that the dual objective function $D(\alpha)$ is concave and explicitly gives the expression of its super-differential.

**Lemma 1.** The dual objective function $D(\alpha)$ is given by

$$D(\alpha) = \frac{1}{N} \sum_{i=1}^{N} -l_i^*(\alpha_i) - \frac{\lambda}{2} \|w(\alpha)\|^2,$$

where $w(\alpha) = H_k \left( -\frac{1}{N\lambda} \sum_{i=1}^{N} \alpha_i x_i \right)$. Moreover, $D(\alpha)$ is concave and its super-differential is given by

$$\partial D(\alpha) = \frac{1}{N} [w(\alpha)^\top x_1 - \partial l_1^*(\alpha_1), ..., w(\alpha)^\top x_N - \partial l_N^*(\alpha_N)].$$

Particularly, if $w(\alpha)$ is unique at $\alpha$ and $\{l_i^*\}_{i=1,...,N}$ are differentiable, then $\partial D(\alpha)$ is unique and it is the super-gradient of $D(\alpha)$.

A proof of this result is given in Appendix A.3.

Based on Theorem 1 and Theorem 2, we are able to further establish a sparse strong duality theorem which gives the sufficient and necessary conditions under which the optimal values in the primal and dual problems coincide.

**Theorem 3** (Sparse Strong Duality Theorem). Let $\bar{w} \in \mathbb{R}^d$ is a k-sparse primal vector and $\bar{\alpha} \in \mathcal{F}^N$ be a dual vector. Then $\bar{\alpha}$ solves the dual problem in (3.4), i.e., $D(\bar{\alpha}) \geq D(\alpha), \forall \alpha \in \mathcal{F}^N$, and $P(\bar{w}) = D(\bar{\alpha})$ if and only if the pair $(\bar{w}, \bar{\alpha})$ satisfies the conditions (a)~(c) in Theorem 1.
A proof of this result is given in Appendix A.4.

We define the sparse primal-dual gap $\epsilon_{PD}(w, \alpha) := P(w) - D(\alpha)$. The main message conveys by Theorem 3 is that the sparse primal-dual gap reaches zero at the primal-dual pair $(\bar{w}, \bar{\alpha})$ if and only if the conditions (a)~(c) in Theorem 1 hold.

The sparse duality theory developed in this section suggests a natural way for finding the global minimum of the sparsity-constrained minimization problem in (1.1) via maximizing its dual problem in (3.4). Once the dual maximizer $\bar{\alpha}$ is estimated, the primal sparse minimizer $\bar{w}$ can then be recovered from it according to the primal-dual connection $\bar{w} = H_k \left( -\frac{1}{\lambda N} \sum_{i=1}^{N} \bar{\alpha}_i x_i \right)$ as given in condition (c). Since the dual objective function $D(\alpha)$ is shown to be concave, its global maximum can be estimated using any convex/concave optimization method. In the next section, we present a simple projected super-gradient method to solve the dual maximization problem.

4 Dual Iterative Hard Thresholding

Generally, $D(\alpha)$ is a non-smooth function since: 1) the conjugate function $l_i^*$ of an arbitrary convex loss $l_i$ is generally non-smooth and 2) the term $\|w(\alpha)\|^2$ is non-smooth due to the truncation operation involved in computing $w(\alpha)$. Therefore, smooth optimization methods are not applicable here and we resort to sub-gradient-type methods to solve the non-smooth dual problem in (3.4).

4.1 Algorithm

The Dual Iterative Hard Thresholding (DIHT) algorithm, as outlined in Algorithm 1, is essentially a projected super-gradient method for maximizing $D(\alpha)$. The procedure generates a sequence of primal-dual pairs $(w^{(0)}, \alpha^{(0)}), (w^{(1)}, \alpha^{(1)}), \ldots$ from an initial pair of $w^{(0)} = 0$ and $\alpha^{(0)} = 0$. At the $t$-th iteration, the dual update step S1 conducts the projected super-gradient ascent as in (4.1) to update $\alpha^{(t)}$ from $\alpha^{(t-1)}$ and $w^{(t-1)}$. Then in the primal update step S2, the corresponding primal variable $w^{(t)}$ is constructed from $\alpha^{(t)}$ using a $k$-sparse hard thresholding operation (4.2).

When a batch estimation of super-gradient $D'(\alpha)$ becomes expensive in large scale applications, it is natural to consider the stochastic implementation of DIHT, namely SDIHT, as outlined in Algorithm 2. Different from the batch computation in Algorithm 1, the dual update step S1 in Algorithm 2 randomly selects a block of samples (from a given block partition of samples) and update their corresponding dual variables according to (4.3). Then in the primal update step S2.1, we incrementally update an intermediate accumulation vector $\tilde{w}^{(t)}$ which records $-\frac{1}{\lambda N} \sum_{i=1}^{N} \bar{\alpha}_i^{(t)} x_i$ as a weighted sum of samples. In S2.2, the primal vector $w^{(t)}$ is updated by applying $k$-sparse truncation on $\tilde{w}^{(t)}$. The SDIHT is essentially a block-coordinate super-gradient method for the dual problem. Particularly, in the extreme case $m = 1$, SDIHT reduces to the batch DIHT. At the opposite extreme end with $m = N$, i.e., each block contains one sample, SDIHT becomes a stochastic coordinate-wise super-gradient method for the dual problem.

The dual update (4.3) in SDIHT is much more efficient than DIHT as the former only needs to access a small subset of samples at a time. If the hard thresholding operation in primal update becomes a bottleneck, e.g., in high-dimensional settings, we suggest to use SDIHT with relatively smaller number of blocks so that the hard thresholding operation in S2.2 can be much less frequently called.
Algorithm 1: Dual Iterative Hard Thresholding (DIHT)

**Input**: Training set \( \{ x_i, y_i \}_{i=1}^N \). Regularization strength parameter \( \lambda \). Cardinality constraint \( k \). Step-size \( \eta \).

**Initialization** \( w^{(0)} = 0, \alpha_i^{(0)} = ... = \alpha_N^{(0)} = 0 \).

**for** \( t = 1, 2, ... \) **do**

(S1) Dual projected gradient ascent: \( \forall i \in [1, 2, ..., N], \)

\[
\alpha_i^{(t)} = \text{P}_F \left( \alpha_i^{(t-1)} + \eta^{(t-1)} g_i^{(t-1)} \right),
\]

where \( g_i^{(t-1)} = \frac{1}{N} (x_i^\top w^{(t-1)} - l_i'(\alpha_i^{(t-1)})) \) is the supergradient and \( \text{P}_F(\cdot) \) is the Euclidian projection operator with respect to feasible set \( F \).

(S2) Primal hard thresholding:

\[
w^{(t)} = H_k \left( -\frac{1}{\lambda N} \sum_{i=1}^N \alpha_i^{(t)} x_i \right).
\]

**end**

**Output**: \( w^{(t)} \).

4.2 Convergence analysis

We now analyze the non-asymptotic convergence behavior of DIHT and SDIHT. In the following analysis, we will denote \( \bar{\alpha} = \arg \max_{\alpha \in \mathcal{F}} D(\alpha) \) and use the abbreviation \( \epsilon_{PD}^{(t)} := \epsilon_{PD}(w^{(t)}, \alpha^{(t)}) \).

Let \( r = \max_{a \in \mathcal{F}} |a| \) be the bound of the dual feasible set \( \mathcal{F} \) and \( \rho = \max_{i,a \in \mathcal{F}} |l_i'(a)| \). For example, such quantities exist when \( l_i \) and \( l_i^* \) are Lipschitz continuous (Shalev-Shwartz & Zhang, 2013b).

We also assume without loss of generality that \( \|x_i\| \leq 1 \). Given an index set \( \mathcal{F} \), we denote \( X_F \) as the restriction of \( X \) with columns restricted to \( \mathcal{F} \). The following quantities will be used in our analysis:

\[
\sigma_{\text{max}}^2(X, s) = \sup_{u,F} \left\{ u^\top X_F^\top X_F u \mid |F| \leq s, \|u\| = 1 \right\},
\]

\[
\sigma_{\text{min}}^2(X, s) = \inf_{u,F} \left\{ u^\top X_F^\top X_F u \mid |F| \leq s, \|u\| = 1 \right\}.
\]

Particularly, \( \sigma_{\text{max}}(X, d) = \sigma_{\text{max}}(X) \) and \( \sigma_{\text{min}}(X, d) = \sigma_{\text{min}}(X) \). We say a univariate differentiable function \( f(x) \) is \( \gamma \)-smooth if \( \forall x, y, \)

\[
f(y) \leq f(x) + \langle f'(x), y - x \rangle + \frac{\gamma}{2} |x - y|^2.
\]

The following is our main theorem on the dual parameter estimation error, support recovery and primal-dual gap of DIHT.

**Theorem 4.** Assume that \( l_i \) is 1/\( \mu \)-smooth. Set \( \eta^{(t)} = \frac{\lambda N^2}{(\lambda N \mu + \sigma_{\text{min}}(X,k))^{(t+1)}} \). Define constants \( c_1 = \frac{N^3 (r + \lambda \rho)^2}{(\lambda N \mu + \sigma_{\text{min}}(X,k))^2} \) and \( c_2 = (r + \lambda \rho)^2 \left( 1 + \frac{\sigma_{\text{max}}(X,k)}{\mu N} \right)^2 \).

(a) **Parameter estimation error:** The sequence \( \{\alpha^{(t)}\}_{t \geq 1} \) generated by Algorithm 1 satisfies
Algorithm 2: Stochastic Dual Iterative Hard Thresholding (SDIHT)

Input: Training set \( \{x_i, y_i\}_{i=1}^N \). Regularization strength parameter \( \lambda \). Cardinality constraint \( k \). Step-size \( \eta \). A block disjoint partition \( \{B_1, ..., B_m\} \) of the sample index set \([1, ..., N]\).

Initialization \( w^{(0)} = \tilde{w}^{(0)} = 0 \), \( \alpha_1^{(0)} = ... = \alpha_N^{(0)} = 0 \).

for \( t = 1, 2, ... \) do
  (S1) Dual projected gradient ascent: Uniform randomly select an index block \( B_i^{(t)} \in \{B_1, ..., B_m\} \). For all \( j \in B_i^{(t)} \) update \( \alpha_j^{(t)} \) as
  \[
  \alpha_j^{(t)} = P_F \left( \alpha_j^{(t-1)} + \eta^{(t-1)} g_j^{(t-1)} \right).
  \] (4.3)

  Set \( \alpha_j^{(t)} = \alpha_j^{(t-1)} \), \( \forall j \notin B_i^{(t)} \).
  (S2) Primal hard thresholding:
  - (S2.1) Intermediate update \( \tilde{w}^{(t)} = \tilde{w}^{(t-1)} - \frac{1}{\lambda N} \sum_{j \in B_i^{(t)}} (\alpha_j^{(t)} - \alpha_j^{(t-1)}) x_j \). (4.4)

  - (S2.2) Hard thresholding: \( w^{(t)} = H_k(\tilde{w}^{(t)}) \).

end

Output: \( w^{(t)} \).

The following estimation error inequality:
\[
\|\alpha^{(t)} - \bar{\alpha}\|^2 \leq c_1 \left( \frac{1}{t} + \frac{\ln t}{t} \right),
\]

(b) Support recovery and primal-dual gap: Assume additionally that \( \hat{\epsilon} := \hat{\epsilon}_{\min} - \frac{1}{\lambda} \|P'(\hat{\omega})\|_\infty > 0 \). Then, \( \text{supp}(w^{(t)}) = \text{supp}(\tilde{w}) \) when
\[
t \geq t_0 = \left[ \frac{12c_1\sigma_{\max}^2(X)}{\lambda^2 N^2 \bar{\epsilon}^2} \ln \frac{12c_1\sigma_{\max}^2(X)}{\lambda^2 N^2 \bar{\epsilon}^2} \right].
\]

Moreover, for any \( \epsilon > 0 \), the primal-dual gap satisfies \( \epsilon_{PD}^{(t)} \leq \epsilon \) when \( t \geq \max\{t_0, t_1\} \) where \( t_1 = \left[ \frac{3c_2\bar{\epsilon}}{\lambda^2 N^2 \bar{\epsilon}^2} \ln \frac{3c_2\bar{\epsilon}}{\lambda^2 N^2 \bar{\epsilon}^2} \right] \).

A proof of this theorem can be found in Appendix A.5

Remark 3. The theorem allows \( \mu = 0 \) when \( \sigma_{\min}(X, k) > 0 \). If \( \mu > 0 \), then \( \sigma_{\min}(X, k) > 0 \) is allowed to be zero and thus the step-size can be set as \( \eta^{(t)} = \frac{N}{\mu(t+1)} \). The examples of \( 1/\mu \)-smooth convex loss functions include square loss \( l_i(a) = \frac{1}{2\mu}(y_i - a)^2 \) and logistic loss \( l_i(a) = \frac{1}{\mu} \log(1 + \exp(-y_ia)) \).

Remark 4. The part(b) of Theorem 4 naturally suggests a termination criterion for DIHT: the primal-dual gap \( \epsilon_{PD}^{(t)} \) is sufficiently small and \( \text{supp}(w^{(t)}) \) becomes stable.
Consider primal sub-optimality $\epsilon_P^{(t)} := P(w^{(t)}) - P(\bar{w})$. Since $\epsilon_P^{(t)} \leq \epsilon_{PD}^{(t)}$ always holds, the convergence rates in Theorem 4 are applicable to the primal sub-optimality as well. An interesting observation is that these convergence results on $\epsilon_P^{(t)}$ are not relying on the Restricted Isometry Property (RIP) (or restricted strong condition number) which is required in most existing analysis of IHT-style algorithms (Blumensath & Davies, 2009; Yuan et al., 2014). In (Jain et al., 2014), several relaxed variants of IHT-style algorithms are presented for which the estimation consistency can be established without requiring the RIP conditions. Our results in Theorem 4 are stronger than those in (Jain et al., 2014) in the sense that we do not require the sparsity level $k$ to be relaxed.

For SDIHT, we can establish similar non-asymptotic convergence results as summarized in the following theorem.

**Theorem 5.** Assume that $l_i$ is $1/\mu$-smooth. Set $\eta^{(t)} = \lambda m N^2 (\lambda N^2 \sigma_{\min}(X,k)) (t+1)$.

(a) **Parameter estimation error**: The sequence $\{\alpha^{(t)}\}_{t \geq 1}$ generated by Algorithm 2 satisfies the following inequality:

$$E[\|\alpha^{(t)} - \bar{\alpha}\|^2] \leq mc_1 \left( \frac{1}{t} + \ln \frac{t}{\delta} \right),$$

(b) **Support recovery and primal-dual gap**: Assume additionally that $\bar{\epsilon} := \bar{w}_{\min} - \frac{1}{\lambda} \|P'(\bar{w})\|_\infty > 0$. Then, for any $\delta \in (0,1)$, with probability at least $1 - \delta$, it holds that $\text{supp}(w^{(t)}) = \text{supp}(\bar{w})$ when

$$t \geq t_2 = \left\lceil \frac{12mc_1 \sigma_{\max}^2(X)}{\lambda^2 \delta^2 N^2 \epsilon^2 \ln \frac{12mc_1 \sigma_{\max}^2(X)}{\lambda^2 \delta^2 N^2 \epsilon^2}} \right\rceil.$$

Moreover, with probability at least $1 - \delta$, the primal-dual gap satisfies $\epsilon_{PD}^{(t)} \leq \epsilon$ when $t \geq \max\{4t_2, t_3\}$ where $t_3 = \left\lceil \frac{12mc_1 \sigma_{\max}^2(X)}{\lambda^2 \delta^2 N^2 \epsilon^2 \ln \frac{12mc_1 \sigma_{\max}^2(X)}{\lambda^2 \delta^2 N^2 \epsilon^2}} \right\rceil$.

A proof of this theorem can be found in Appendix A.6.

**Remark 5.** Theorem 5 shows that, up to scaling factors, the expected or high probability iteration complexity of SDIHT is almost identical to that of DIHT. The scaling factor $m$ reflects a trade-off between the decreased per-iteration cost and the increased iteration complexity.

## 5 Experiments

This section dedicates in demonstrating the accuracy and efficiency of DIHT and SDIHT algorithm. We first conduct experiment on sparse ridge regression model learning problem using synthetic datasets. This experiment aims to show the superior model estimation accuracy of DIHT algorithm. After that we evaluate algorithm efficiency on sparse $\ell_2$-regularized Huber loss and Hinge loss model training tasks using real-world datasets.

### 5.1 Model parameter estimation accuracy evaluation

A synthetic model is generated with sparse model parameter $\bar{w} = [1,1,\ldots,1,0,0,\ldots,0]$. Each of the $N$ training data examples $\{x_i\}_{i=1}^N$, $x_i \in \mathbb{R}^d$ is designed to have two components. The first
component is the top $\bar{k}$ feature dimensions drawn from multivariate Gaussian distribution $N(\mu_1, \Sigma)$. Each entry in $\mu_1 \in \mathbb{R}^k$ independently follows unit Gaussian distribution. The entries of covariance $\Sigma_{ij} = \begin{cases} 1 & i = j \\ 0.2 & i \neq j \end{cases}$. The second component consists the left $d - \bar{k}$ feature dimensions. It follows $N(\mu_2, I)$ where each entry in $\mu_2 \in \mathbb{R}^{d-\bar{k}}$ is drawn from unit Gaussian distribution. We simulate two data parameter settings: (1) $d = 500, \bar{k} = 100$; (2) $d = 300, \bar{k} = 100$. For each data parameter setting 150 random data copies are produced independently. The task is to solve the following $\ell_2$-regularized sparse linear regression model:

$$\min_{\|w\| \leq k} \frac{1}{N} \sum_{i=1}^{N} l_{sq}(y_i, w^\top x_i) + \frac{\lambda}{2} \|w\|^2,$$

where $l_{sq}(y_i, w^\top x_i) = (y_i - w^\top x_i)^2$. The responses $\{y_i\}_{i=1}^{N}, y_i \in \mathbb{R}$ are produced by $y_i = w^\top x_i + \varepsilon_i$, where $\varepsilon_i \sim N(0,1)$. The convex conjugate of $l_{sq}(y_i, w^\top x_i)$ is known as $l_{sq}^*(\alpha_i) = \frac{\alpha_i^2}{4} + y_i \alpha_i$. We consider solving the problem under the sparsity level $k = \bar{k}$. Two measurements are calculated for model parameter estimation accuracy evaluation. The first is parameter estimation error $\frac{\|w - \bar{w}\|}{\|w\|}$. Apart from it we calculate the percentage of successful support recovery (PSSR) as the second performance metric. A successful support recovery is obtained if $\text{supp}(\bar{w}) = \text{supp}(w)$. The evaluation is conducted on the generated batch data copies to calculate the percentage of successful support recovery. 50 data copies are used as validation set to select the parameter $\lambda$ from $\{10^{-6}, ..., 10^{2}\}$ and the percentage of successful support recovery is evaluated on the other 100 data copies.

Hard thresholding pursuit (HTP) [Foucart (2011)] is used as the baseline primal algorithm. The parameter estimation error and percentage of successful support recovery comparison under varying training size are illustrated in Figure 5.1. We can observe that compared to HTP, the proposed dual space algorithm DIHT achieves more accurate model parameter solution in terms of lower model estimation error and higher successful support recovery rate. It is noticeable that when training size is comparable or even smaller than the model parameter sparsity level $\bar{k}$, the proposed DIHT algorithm has significantly better capability to exactly recover $\text{supp}(\bar{w})$ and produce more accurate solution than HTP algorithm. This confirms the prediction of Theorem 4 that DIHT is free of RIP condition.

Figure 5.1: Model parameter estimation performance comparison between HTP and DIHT on the two synthetic dataset settings. The varying number of training sample is denoted by $N$. 

![Figure 5.1](image-url)
5.2 Model training efficiency evaluation

5.2.1 Huber loss model learning

We evaluate the proposed algorithms on $\ell_2$-regularized sparse Huber loss minimization problem:

$$\min_{\|w\|_0 \leq k} \frac{1}{N} \sum_{i=1}^{N} l_{Huber}(y_i x_i^\top w) + \frac{\lambda}{2} \|w\|^2,$$

(5.1)

where

$$l_{Huber}(y_i x_i^\top w) = \begin{cases} 0 & \text{if } y_i x_i^\top w \geq 1 \\ 1 - y_i x_i^\top w - \frac{\gamma}{2} & \text{if } y_i x_i^\top w < 1 - \gamma \\ \frac{1}{2\gamma} (1 - y_i x_i^\top w)^2 & \text{otherwise} \end{cases}$$

It is known that

$$l'_{Huber}(\alpha_i) = \begin{cases} y_i \alpha_i + \frac{\gamma}{2} \alpha_i^2 & \text{if } y_i \alpha_i \in [-1, 0] \\ +\infty & \text{otherwise} \end{cases}.$$

Two binary benchmark datasets from LibSVM data repository\footnote{https://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/binary.html} RCV1 ($d = 47,236$) and News20 ($d = 1,355,191$) are used for algorithm efficiency evaluation and comparison. We select 0.5 million samples from RCV1 dataset for model training ($N \gg d$). For news20 dataset, all of the 19,996 samples are used as training data ($d \gg N$).
We evaluate the algorithm efficiency by comparing the running time with two primal domain baseline algorithms, HTP and gradient hard thresholding with stochastic variance reduction (SVR-GHT) (Li et al., 2016). We first run HTP by setting its convergence criterion to be \(|P(w^{(t)}) - P(w^{(t-1)})| \leq 10^{-4}\) or maximum number of iteration is reached. After that we test the time cost spent by other algorithms to make the primal loss reach \(P(w^{(t)})\). The parameter update step-size of all considered algorithms is tuned by grid search from \(\{10^{-6}, ..., 10^3\}\) to achieve the optimum efficiency. The parameter \(\gamma\) is set to be 1. For the two stochastic algorithms SDIHT and SVR-GHT we randomly partition the training data into \(|B| = 10\) mini-batches.

The running time comparison on both datasets given varying sparsity degree constraint \(k\) and \(\lambda = 10^{-3}, 10^{-4}\) is shown in Figure 5.2. It is obvious that under all tested \((k, \lambda)\) configurations on both datasets, the proposed dual space algorithms, DIHT especially SDIHT need far less time than the primal space baseline algorithms, HTP and SVR-GHT to reach the same primal sub-optimality. Figure 5.3 shows the relationship between primal-dual gap \(\epsilon_{PD}\) convergence and the number of training data pass. For DIHT, one data pass corresponds to updating the whole dual variables so it is actually iteration number. For SDIHT, every \(|B|\) minibatch dual variable updates is counted as one data pass. The result of Figure 5.3 supports our analysis in Theorem 4 and 5 that \(\epsilon_{PD}\) converges non-asymptotically and larger value of \(\lambda\) leads to faster convergence.
5.2.2 Hinge loss model learning

Last but not the least we test our algorithms on non-smooth model learning problem through $\ell_2$-regularized sparse Hinge loss model training task:

$$\min_{\|w\|_0 \leq k} \frac{1}{N} \sum_{i=1}^{N} l_{\text{Hinge}}(y_i x_i^\top w) + \frac{\lambda}{2}\|w\|^2,$$

where $l_{\text{Hinge}}(y_i x_i^\top w) = \max(0, 1 - y_i w^\top x_i)$. It is standard to know

$$l^*_{\text{Hinge}}(\alpha_i) = \begin{cases} y_i \alpha_i & \text{if } y_i \alpha_i \in [-1, 0] \\ +\infty & \text{otherwise} \end{cases}.$$

We adopt the same experiment setting as in §5.2.1 to compare with baseline algorithms on the benchmark datasets. The time cost comparison is illustrated in Figure 5.4 and the prima-dual gap sub-optimality is illustrated in Figure 5.5. The results indicate that the developed algorithms has remarkable efficiency advantage over baseline algorithms and the established theoretical results apply to such non-smooth model learning problem.

6 Conclusion

In this paper, we systematically investigate the duality theory and algorithm for solving the sparsity-constrained minimization problem which is NP-hard and non-convex in its primal form. As a
theoretical contribution, we develop a sparse Lagrangian duality theory which guarantees strong duality in sparse settings, under mild sufficient and necessary conditions. This theory opens the gate to solve the original NP-hard/non-convex problem equivalently in a dual space. We then propose DIHT as a first-order method to maximize the non-smooth dual concave formulation. The algorithm is characterized by dual gradient ascent and primal hard thresholding. To further improve iteration efficiency in large-scale settings, we propose SDIHT as a block stochastic variant of DIHT. For both algorithms we have proved sub-linear primal-dual gap convergence rate when the primal loss is smooth, without assuming RIP-style conditions. Based on our theoretical findings and numerical results, we conclude that DIHT/SDIHT is a theoretically sound and computationally attractive alternative to the conventional primal IHT algorithms, especially when the sample size is smaller than feature dimensionality.

Figure 5.5: Hinge loss: The primal-dual gap evolving curves of DIHT and SDIHT.
A Technical Proofs

A.1 Proof of Theorem 1

Proof. “⇐”: If the pair \((\bar{w}, \bar{\alpha})\) is a sparse saddle point for \(L\), then from the definition of conjugate convexity and inequality (3.2) we have

\[
P(\bar{w}) = \max_{\alpha \in \mathcal{F}^N} L(\bar{w}, \alpha) \leq L(\bar{w}, \bar{\alpha}) \leq \min_{\|w\|_0 \leq k} L(\bar{w}, \bar{\alpha}).
\]

On the other hand, we know that for any \(\|w\|_0 \leq k\) and \(\alpha \in \mathcal{F}^N\)

\[
L(w, \alpha) \leq \max_{\alpha' \in \mathcal{F}^N} L(w, \alpha') = P(w).
\]

By combining the preceding two inequalities we obtain

\[
P(\bar{w}) \leq \min_{\|w\|_0 \leq k} L(w, \bar{\alpha}) \leq \min_{\|w\|_0 \leq k} P(w) \leq P(\bar{w}).
\]

Therefore \(P(\bar{w}) = \min_{\|w\|_0 \leq k} P(w)\), i.e., \(\bar{w}\) solves the problem in (1.1), which proves the necessary condition (a). Moreover, the above arguments lead to

\[
P(\bar{w}) = \max_{\alpha \in \mathcal{F}^N} L(\bar{w}, \alpha) = L(\bar{w}, \bar{\alpha}).
\]

Then from the maximizing argument property of convex conjugate we know that \(\bar{\alpha}_i \in \partial l_i(\bar{w}^\top x_i)\). Thus the necessary condition (b) holds. Note that

\[
L(\bar{w}, \bar{\alpha}) = \lambda \left\| w + \frac{1}{N\lambda} \sum_{i=1}^{N} \bar{\alpha}_i x_i \right\|^2 - \frac{1}{N} \sum_{i=1}^{N} l_i^*(\bar{\alpha}_i) + C,
\]

where \(C\) is a quantity not dependent on \(w\). Let \(\bar{F} = \text{supp}(\bar{w})\). Since the above analysis implies \(L(\bar{w}, \bar{\alpha}) = \min_{\|w\|_0 \leq k} L(w, \bar{\alpha})\), it must hold that

\[
\bar{w} = H_F \left(- \frac{1}{N\lambda} \sum_{i=1}^{N} \bar{\alpha}_i x_i \right) = H_k \left(- \frac{1}{N\lambda} \sum_{i=1}^{N} \bar{\alpha}_i x_i \right).
\]

This validates the necessary condition (c).

“⇒”: Conversely, let us assume that \(\hat{w}\) is a \(k\)-sparse solution to the problem (1.1) (i.e., condition (a)) and let \(\bar{\alpha}_i \in \partial l_i(\hat{w}^\top x_i)\) (i.e., condition (b)). Again from the maximizing argument property of convex conjugate we know that \(l_i(\hat{w}^\top x_i) = \hat{\alpha}_i \hat{w}^\top x_i - l_i^*(\hat{\alpha}_i)\). This leads to

\[
L(\hat{w}, \alpha) \leq P(\hat{w}) = \max_{\alpha \in \mathcal{F}^N} L(\hat{w}, \alpha) = L(\hat{w}, \bar{\alpha}).
\]

The sufficient condition (c) guarantees that \(\hat{F}\) contains the top \(k\) (in absolute value) entries of \(- \frac{1}{N\lambda} \sum_{i=1}^{N} \bar{\alpha}_i x_i\). Then based on the expression in (A.1) we can see that the following holds for any \(k\)-sparse vector \(w\)

\[
L(\hat{w}, \alpha) \leq L(w, \alpha). \tag{A.3}
\]

By combining the inequalities (A.2) and (A.3) we get that for any \(\|w\|_0 \leq k\) and \(\alpha \in \mathcal{F}^N\),

\[
L(\hat{w}, \alpha) \leq L(\hat{w}, \bar{\alpha}) \leq L(\hat{w}, \bar{\alpha}) \leq L(w, \bar{\alpha}).
\]

This shows that \((\hat{w}, \bar{\alpha})\) is a sparse saddle point of the Lagrangian \(L\). □
A.2 Proof of Theorem 2

Proof. “⇒”: Let \((\tilde{w}, \tilde{\alpha})\) be a saddle point for \(L\). On one hand, note that the following holds for any \(k\)-sparse \(w'\) and \(\alpha' \in \mathcal{F}^N\):

\[
\min_{\|w\|_0 \leq k} L(w, \alpha') \leq L(w', \alpha') \leq \max_{\alpha' \in \mathcal{F}^N} L(w', \alpha),
\]

which implies

\[
\max_{\alpha \in \mathcal{F}^N} \min_{\|w\|_0 \leq k} L(w, \alpha) \leq \min_{\|w\|_0 \leq k} \max_{\alpha \in \mathcal{F}^N} L(w, \alpha). \tag{A.4}
\]

On the other hand, since \((\tilde{w}, \tilde{\alpha})\) is a saddle point for \(L\), the following is true:

\[
\min_{\|w\|_0 \leq k} L(w, \alpha) \leq \max_{\alpha \in \mathcal{F}^N} L(\tilde{w}, \tilde{\alpha}) \leq \min_{\|w\|_0 \leq k} \max_{\alpha \in \mathcal{F}^N} L(w, \alpha) \tag{A.5}
\]

By combining (A.4) and (A.5) we prove the equality in (3.3).

“⇐”: Assume that the equality in (3.3) holds. Let us define \(\bar{w}\) and \(\bar{\alpha}\) such that

\[
\max_{\alpha \in \mathcal{F}^N} L(\bar{w}, \alpha) = \min_{\|w\|_0 \leq k} L(w, \bar{\alpha}) \leq \max_{\alpha \in \mathcal{F}^N} L(\bar{w}, \bar{\alpha}) \leq \min_{\|w\|_0 \leq k} \max_{\alpha \in \mathcal{F}^N} L(w, \alpha).
\]

Then we can see that for any \(\alpha \in \mathcal{F}^N\),

\[
L(\bar{w}, \bar{\alpha}) \geq \min_{\|w\|_0 \leq k} L(w, \bar{\alpha}) = \max_{\alpha' \in \mathcal{F}^N} L(\bar{w}, \alpha') \geq L(\bar{w}, \alpha),
\]

where the “⇐” is due to (3.3). In the meantime, for any \(\|w\| \leq k\),

\[
L(\bar{w}, \bar{\alpha}) \leq \max_{\alpha \in \mathcal{F}^N} L(\bar{w}, \alpha) = \min_{\|w\| \leq k} \max_{\alpha \in \mathcal{F}^N} L(w, \alpha) \leq L(\bar{w}, \bar{\alpha}).
\]

This shows that \((\bar{w}, \bar{\alpha})\) is a sparse saddle point for \(L\). \hfill \Box

A.3 Proof of Lemma 1

Proof. For any fixed \(\alpha \in \mathcal{F}^N\), then it is easy to verify that the \(k\)-sparse minimum of \(L(w, \alpha)\) with respect to \(w\) is attained at the following point:

\[
w(\alpha) = \arg \min_{\|w\|_0 \leq k} L(w, \alpha) = H_k \left( -\frac{1}{N \lambda} \sum_{i=1}^{N} \alpha_i x_i \right).
\]

Thus we have

\[
D(\alpha) = \min_{\|w\|_0 \leq k} L(w, \alpha) = L(w(\alpha), \alpha)
\]

\[
= \frac{1}{N} \sum_{i=1}^{N} \left( \alpha_i w(\alpha)^T x_i - l_i^*(\alpha_i) \right) + \frac{\lambda}{2} \|w(\alpha)\|^2
\]

\[
= \frac{1}{N} \sum_{i=1}^{N} -l_i^*(\alpha_i) - \frac{\lambda}{2} \|w(\alpha)\|^2,
\]

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where “$\zeta_1$” follows from the above definition of $w(\alpha)$.

Now let us consider two arbitrary dual variables $\alpha', \alpha'' \in \mathcal{F}^N$ and any $g(\alpha'') \in \frac{1}{N}[w(\alpha'')^\top x_1 - \partial l_1^1(\alpha''), ..., w(\alpha'')^\top x_N - \partial l_N^\alpha(\alpha'')]$. From the definition of $D(\alpha)$ and the fact that $L(w, \alpha)$ is concave with respect to $\alpha$ at any fixed $w$ we can derive that
\[
D(\alpha') = L(w(\alpha'), \alpha') \\
\leq L(w(\alpha''), \alpha') \\
\leq L(w(\alpha''), \alpha'') + \langle g(\alpha''), \alpha' - \alpha'' \rangle.
\]
This shows that $D(\alpha)$ is a concave function and its super-differential is as given in the theorem.

If we further assume that $w(\alpha)$ is unique and $\{l_i^\alpha\}_{i=1,...,N}$ are differentiable at any $\alpha$, then
\[
\partial D(\alpha) = \frac{1}{N}[w(\alpha)^\top x_1 - \partial l_1^1(\alpha_1), ..., w(\alpha)^\top x_N - \partial l_N^\alpha(\alpha_N)]
\]
becomes unique, which implies that $\partial D(\alpha)$ is the unique super-gradient of $D(\alpha)$.

\section{Proof of Theorem 3}

\begin{proof}

\textit{$\Rightarrow$}: Given the conditions in the theorem, it can be known from Theorem 1 that the pair $(\bar{w}, \bar{\alpha})$ forms a sparse saddle point of $L$. Thus based on the definitions of sparse saddle point and dual function $D(\alpha)$ we can show that
\[
D(\bar{\alpha}) = \min_{\|w\|_0 \leq k} L(w, \bar{\alpha}) \geq L(\bar{w}, \bar{\alpha}) \geq L(\bar{w}, \alpha) \geq D(\alpha).
\]
This implies that $\bar{\alpha}$ solves the dual problem in (3.4). Furthermore, Theorem 2 guarantees the following
\[
D(\bar{\alpha}) = \max_{\alpha \in \mathcal{F}^N} \min_{\|w\|_0 \leq k} L(w, \alpha) = \min_{\|w\|_0 \leq k} \max_{\alpha \in \mathcal{F}^N} L(w, \alpha) = P(\bar{w}).
\]
This indicates that the primal and dual optimal values are equal to each other.

\textit{$\Leftarrow$}: Assume that $\bar{\alpha}$ solves the dual problem in (3.4) and $D(\bar{\alpha}) = P(\bar{w})$. Since $D(\bar{\alpha}) \leq P(w)$ holds for any $\|w\|_0 \leq k$, $\bar{w}$ must be the sparse minimizer of $P(w)$. It follows that
\[
\max_{\alpha \in \mathcal{F}^N} \min_{\|w\|_0 \leq k} L(w, \alpha) = D(\bar{\alpha}) = P(\bar{w}) = \min_{\|w\|_0 \leq k} \max_{\alpha \in \mathcal{F}^N} L(w, \alpha).
\]

This “$\Leftarrow$” argument in the proof of Theorem 2 and Corollary 1 we get that the conditions (a)~(c) in Theorem 1 should be satisfied for $(\bar{w}, \bar{\alpha})$.
\end{proof}

\section{Proof of Theorem 4}

We need a series of technical lemmas to prove this theorem. The following lemmas shows that under proper conditions, $w(\alpha)$ is locally smooth around $\bar{w} = w(\bar{\alpha})$.

\begin{lemma}
Let $X = [x_1, ..., x_N] \in \mathbb{R}^{d \times N}$ be the data matrix. Assume that $\{l_i\}_{i=1,...,N}$ are differentiable and
\[
\bar{\varepsilon} := \bar{w}_{\min} - \frac{1}{\lambda} \|P'(\bar{w})\|_\infty > 0.
\]
If $\|\alpha - \bar{\alpha}\| \leq \frac{\lambda N}{5\sigma_{\max}(X)}$, then $\text{supp}(w(\alpha)) = \text{supp}(\bar{w})$ and
\[
\|w(\alpha) - \bar{w}\| \leq \frac{\sigma_{\max}(X, k)}{N\lambda} \|\alpha - \bar{\alpha}\|.
\]
\end{lemma}
Proof. For any $\alpha \in F^N$, let us define
\[
\tilde{w}(\alpha) = -\frac{1}{N\lambda} \sum_{i=1}^{N} \alpha_i x_i.
\]
Consider $\bar{F} = \text{supp}(\tilde{w})$. Given $\bar{\epsilon} > 0$, it is known from Theorem 3 that $\bar{w} = H_{\bar{F}}(\tilde{w}(\bar{\alpha}))$ and $\frac{d^2}{d\lambda^2}(\bar{\lambda}(\bar{\alpha})) = H_{\bar{F}}(\tilde{w}(\bar{\alpha}))$. Then $\bar{\epsilon} > 0$ implies $\bar{F}$ is unique, i.e., the top $k$ entries of $\tilde{w}(\bar{\alpha})$ is unique. Given that $\|\alpha - \bar{\alpha}\| \leq \lambda N \bar{\epsilon}$, it can be shown that
\[
\|\tilde{w}(\alpha) - \tilde{w}(\bar{\alpha})\| = \frac{1}{N\lambda} \|X(\alpha - \bar{\alpha})\| \leq \frac{\sigma_{\max}(X)}{N\lambda} \|\alpha - \bar{\alpha}\| \leq \frac{\bar{\epsilon}}{2}.
\]
This indicates that $\bar{F}$ still contains the (unique) top $k$ entries of $\tilde{w}(\alpha)$. Therefore,
\[
\text{supp}(w(\alpha)) = \bar{F} = \text{supp}(\tilde{w}).
\]
Then it must hold that
\[
\|w(\alpha) - w(\bar{\alpha})\| = \|H_{\bar{F}}(\tilde{w}(\alpha)) - H_{\bar{F}}(\tilde{w}(\bar{\alpha}))\|
\leq \frac{1}{N\lambda} \|X_{\bar{F}}(\alpha - \bar{\alpha})\|
\leq \frac{\sigma_{\max}(X,k)}{N\lambda} \|\alpha - \bar{\alpha}\|.
\]
This proves the desired bound.

The following lemma bounds the estimation error $\|\alpha - \bar{\alpha}\| = O(\sqrt{\langle D'(\alpha), \bar{\alpha} - \alpha \rangle})$ when the primal loss $\{l_i\}_{i=1}^{N}$ are smooth.

Lemma 3. Assume that the primal loss functions $\{l_i(\cdot)\}_{i=1}^{N}$ are $1/\mu$-smooth. Then the following inequality holds for any $\alpha, \alpha'' \in F$ and $g(\alpha'') \in \partial D(\alpha'')$:
\[
D(\alpha') \leq D(\alpha'') + \langle g(\alpha''), \alpha' - \alpha'' \rangle - \frac{\lambda N\mu + \sigma_{\min}^2(X,k)}{2\lambda N^2} \|\alpha' - \alpha''\|^2.
\]
Moreover, $\forall \alpha \in F$ and $g(\alpha) \in \partial D(\alpha)$,
\[
\|\alpha - \bar{\alpha}\| \leq \sqrt{\frac{2\lambda N^2 \langle g(\alpha), \bar{\alpha} - \alpha \rangle}{\lambda N\mu + \sigma_{\min}^2(X,k)}}.
\]

Proof. Recall that
\[
D(\alpha) = \frac{1}{N} \sum_{i=1}^{N} -l_i^*(\alpha_i) - \frac{\lambda}{2} \|w(\alpha)\|^2,
\]
Now let us consider two arbitrary dual variables $\alpha', \alpha'' \in F$. The assumption of $l_i$ being $1/\mu$-smooth...
implies that its convex conjugate function $l_i^*$ is $\mu$-strongly-convex. Let $F'' = \text{supp}(w(\alpha''))$. Then

$$D(\alpha') = \frac{1}{N} \sum_{i=1}^{N} -l_i^*(\alpha_i') - \frac{\lambda}{2} \|w(\alpha')\|^2$$

$$= \frac{1}{N} \sum_{i=1}^{N} -l_i^*(\alpha_i') - \frac{\lambda}{2} \left\| H_k \left( -\frac{1}{N\lambda} \sum_{i=1}^{N} \alpha_i' x_i \right) \right\|^2$$

$$\leq \frac{1}{N} \sum_{i=1}^{N} \left( -l_i^*(\alpha_i'') - l_i''(\alpha_i')(\alpha_i' - \alpha_i'') - \frac{\mu}{2} (\alpha_i' - \alpha_i'')^2 \right) - \frac{\lambda}{2} \left\| H_{F''} \left( -\frac{1}{N\lambda} \sum_{i=1}^{N} \alpha_i' x_i \right) \right\|^2$$

$$\leq \frac{1}{N} \sum_{i=1}^{N} \left( -l_i^*(\alpha_i'') - l_i''(\alpha_i')(\alpha_i' - \alpha_i'') - \frac{\mu}{2} (\alpha_i' - \alpha_i'')^2 \right) - \frac{\lambda}{2} \|w(\alpha'')\|^2 + \frac{1}{N} \sum_{i=1}^{N} x_i^T w(\alpha'') (\alpha_i' - \alpha_i'')$$

$$\leq \frac{1}{2 \lambda N^2} (\alpha' - \alpha'')^\top X_{F''} X_{F''} (\alpha' - \alpha'')$$

$$\leq D(\alpha'') + \langle g(\alpha''), \alpha' - \alpha'' \rangle - \frac{\lambda N \mu + \sigma_{\min}(X, k)}{2 \lambda N^2} \|\alpha' - \alpha''\|^2.$$

This proves the first desirable inequality in the lemma. By invoking the above inequality and using the fact $D(\alpha) \leq D(\bar{\alpha})$ we get that

$$D(\bar{\alpha}) \leq D(\alpha) + \langle g(\alpha), \bar{\alpha} - \alpha \rangle - \frac{\lambda N \mu + \sigma_{\min}(X, k)}{2 \lambda N^2} \|\alpha - \bar{\alpha}\|^2$$

$$\leq D(\bar{\alpha}) + \langle g(\alpha), \bar{\alpha} - \alpha \rangle - \frac{\lambda N \mu + \sigma_{\min}(X, k)}{2 \lambda N^2} \|\alpha - \bar{\alpha}\|^2,$$

which leads to the second desired bound.

The following lemma gives a simple expression of the gap for properly related primal-dual pairs.

**Lemma 4.** Given a dual variable $\alpha \in F^N$ and the related primal variable

$$w = H_k \left( -\frac{1}{N\lambda} \sum_{i=1}^{N} \alpha_i x_i \right).$$

The primal-dual gap $\epsilon_{PD}(w, \alpha)$ can be expressed as:

$$\epsilon_{PD}(w, \alpha) = \frac{1}{N} \sum_{i=1}^{N} \left( l_i(w^\top x_i) + l_i^*(\alpha_i) - \alpha_i w^\top x_i \right).$$

**Proof.** It is directly to know from the definitions of $P(w)$ and $D(\alpha)$ that

$$P(w) - D(\alpha)$$

$$= \frac{1}{N} \sum_{i=1}^{N} l_i(w^\top x_i) + \frac{\lambda}{2} \|w\|^2 - \frac{1}{N} \sum_{i=1}^{N} \left( \alpha_i w^\top x_i - l_i^*(\alpha_i) \right) + \frac{\lambda}{2} \|w\|^2$$

$$= \frac{1}{N} \sum_{i=1}^{N} \left( l_i(w^\top x_i) + l_i^*(\alpha_i) - \alpha_i w^\top x_i \right).$$

This shows the desired expression.
Based on Lemma 4, we can derive the following lemma which establishes a bound on the primal-dual gap.

**Lemma 5.** Consider a primal-dual pair \((w, \alpha)\) satisfying

\[
w = H_k \left( -\frac{1}{N\lambda} \sum_{i=1}^{N} \alpha_i x_i \right).
\]

Then the following inequality holds for any \(g(\alpha) \in \partial D(\alpha)\) and \(\beta \in [\partial l_1(w^\top x_1), ..., \partial l_N(w^\top x_N)]\):

\[
P(w) - D(\alpha) \leq \langle g(\alpha), \beta - \alpha \rangle.
\]

**Proof.** For any \(i \in [1, ..., N]\), from the maximizing argument property of convex conjugate we have

\[
l_i(w^\top x_i) = w^\top x_i l_i^*(w^\top x_i) - l_i(l_i^*(w^\top x_i)),
\]

and

\[
l_i^*(\alpha_i) = \alpha_i l_i'(\alpha_i) - l_i(l_i^*(\alpha_i)).
\]

By summing both sides of above two equalities we get

\[
l_i(w^\top x_i) + l_i^*(\alpha_i) \\
\leq w^\top x_i l_i'(w^\top x_i) + \alpha_i l_i'(\alpha_i) - l_i(l_i^*(\alpha_i))
\]

where \(\zeta_1\) follows from Fenchel-Young inequality. Therefore

\[
\langle g(\alpha), \beta - \alpha \rangle \\
= \frac{1}{N} \sum_{i=1}^{N} (w^\top x_i - l_i^{**}(\alpha_i))(l_i'(w^\top x_i) - \alpha_i)
\]

\[
= \frac{1}{N} \sum_{i=1}^{N} \left( w^\top x_i l_i'(w^\top x_i) - l_i^{**}(\alpha_i)l_i'(w^\top x_i) - \alpha_i w^\top x_i + \alpha_i l_i'(\alpha_i) \right)
\]

\[
\geq \frac{1}{N} \sum_{i=1}^{N} (l_i(w^\top x_i) + \alpha_i l_i^*(\alpha_i) - w^\top x_i)
\]

where \(\zeta_2\) follows from \((A.6)\) and \(\zeta_3\) follows from Lemma 4. This proves the desired bound. \(\square\)

The following simple result is also needed in our iteration complexity analysis.

**Lemma 6.** For any \(\epsilon > 0\),

\[
\frac{1}{t} + \frac{\ln t}{t} \leq \epsilon
\]

holds when \(t \geq \max \left\{ \frac{3}{\epsilon} \ln \frac{3}{\epsilon}, 1 \right\}\).
Proof. Obviously, the inequality $\frac{1}{r} + \frac{\ln t}{t} \leq \epsilon$ holds for $\epsilon \geq 1$. When $\epsilon < 1$, it holds that $\ln(\frac{3}{\epsilon}) \geq 1$. Then the condition on $t$ implies that $\frac{1}{t} \leq \frac{\epsilon}{3}$. Also, we have

$$\frac{\ln t}{t} \leq \frac{\ln(3\ln \frac{3}{\epsilon})}{\frac{3}{\epsilon} \ln \frac{3}{\epsilon}} \leq \frac{(\frac{2}{\epsilon})^2}{\frac{3}{\epsilon} \ln \frac{3}{\epsilon}} = \frac{2\epsilon}{3},$$

where the first “≤” follows the fact that $\ln t/t$ is decreasing when $t \geq 1$ while the second “≤” follows $\ln x < x$ for all $x > 0$. Therefore we have $\frac{1}{t} + \frac{\ln t}{t} \leq \epsilon$. 

We are now in the position to prove the main theorem.

**Part(a):** Let us consider $g(t) \in \partial D(\alpha(t))$ with $g(t) = \frac{1}{N}(x_1^{\top} w(t) - l_i'(\alpha_i(t)))$. From the expression of $w(t)$ we can verify that $\|w(t)\| \leq r/\lambda$. Therefore we have

$$\|g(t)\| \leq c_0 = \frac{r + \lambda \rho}{\sqrt{2N}}.$$

Let $h(t) = \|\alpha(t) - \bar{\alpha}\|$ and $v(t) = \langle g(t), \bar{\alpha} - \alpha(t) \rangle$. The concavity of $D$ implies $v(t) \geq 0$. From Lemma 3 we know that $h(t) \leq \sqrt{2N}v(t)/\lambda^2 \|\alpha(t) - \bar{\alpha}\|$. Then

$$(h(t))^2 = \|P_{\mathcal{F}N}(\alpha(t-1) + \eta(t-1) g(t-1)) - \bar{\alpha}\|^2$$

$$\leq \|\alpha(t-1) + \eta(t-1) g(t-1) - \bar{\alpha}\|^2$$

$$= (h(t-1))^2 - 2\eta(t-1) v(t-1) + (\eta(t-1))^2 \|g(t-1)\|^2$$

$$\leq (h(t-1))^2 - \frac{\eta(t-1)(\lambda N + \sigma_{\min}(X, k))}{\lambda^2} (h(t-1))^2 + (\eta(t-1))^2 c_0^2.$$

Let $\eta(t) = \frac{\lambda N^2}{(\lambda N + \sigma_{\min}(X, k))(t+1)}$. Then we obtain

$$(h(t))^2 \leq \left(1 - \frac{1}{t}\right) (h(t-1))^2 + \frac{\lambda^2 N^4 c_0^2}{(\lambda N + \sigma_{\min}(X, k))^2(t+1)^2}.$$ 

By recursively applying the above inequality we get

$$(h(t))^2 \leq \frac{\lambda^2 N^4 c_0^2}{(\lambda N + \sigma_{\min}(X, k))^2} \left(1 + \frac{\ln t}{t}\right) = c_1 \left(1 + \frac{\ln t}{t}\right).$$

This proves the desired bound in part(a).

**Part(b):** Let us consider $\epsilon = \frac{\lambda N^t}{2\sigma_{\max}(X)}$. From part(a) and Lemma 6 we obtain

$$\|\alpha(t) - \bar{\alpha}\| \leq \epsilon$$

after $t \geq t_0 = \frac{3c_1}{\epsilon^2} \ln \frac{3c_1}{\epsilon^2}$. It follows from Lemma 2 that $\text{supp}(w(t)) = \text{supp}(\bar{w})$.

Let $\beta(t) := \{l_1'(((w(t))^{\top} x_1), ..., l_N'(((w(t))^{\top} x_N))\}$. According to Lemma 5 we have

$$\epsilon_{PD}^t = P(w(t)) - D(\alpha(t))$$

$$\leq \langle g(t), \beta(t) - \alpha(t) \rangle$$

$$\leq \|g(t)\| (\|\beta(t) - \bar{\alpha}\| + \|\bar{\alpha} - \alpha(t)\|).$$
Since \( \epsilon = \bar{w}_{\text{min}} - \frac{1}{X} \|P'(\bar{w})\|_\infty > 0 \), it follows from Theorem 2 that \( \bar{\alpha} = [l_1'(\bar{w}^\top x_1), ..., l_N'(\bar{w}^\top x_N)] \). Given that \( t \geq t_0 \), from the smoothness of \( l_i \) and Lemma 2 we get

\[
\| \beta^{(t)} - \bar{\alpha} \| \leq \frac{1}{\mu} \| w^{(t)} - \bar{w} \| \leq \frac{\sigma_{\max}(X, k)}{\mu \lambda N} \| \alpha^{(t)} - \bar{\alpha} \|.
\]

where in the first “\( \leq \)” we have used \( \| x_i \| \leq 1 \). Therefore, the following is valid when \( t \geq t_0 \):

\[
\epsilon^{(t)}_{PD} \leq \| g^{(t)} \| (\| \beta^{(t)} - \bar{\alpha} \| + \| \bar{\alpha} - \alpha^{(t)} \|)
\]

\[
\leq c_0 \left( 1 + \frac{\sigma_{\max}(X, k)}{\mu \lambda N} \right) \| \alpha^{(t)} - \bar{\alpha} \|.
\]

Since \( t \geq t_1 \), from part(a) and Lemma 6 we get \( \| \alpha^{(t)} - \bar{\alpha} \| \leq \frac{\epsilon}{c_0 \left( 1 + \frac{\sigma_{\max}(X, k)}{\mu \lambda N} \right)} \), which according to the above inequality implies \( \epsilon^{(t)}_{PD} \leq \epsilon \). This proves the desired bound.

\section*{A.6 Proof of Theorem 5}

\textbf{Proof. Part(a):} Let us consider \( g^{(t)} \) with \( g_j^{(t)} = \frac{1}{N} (x_j^\top w^{(t)} - l_j^*(\alpha^{(t)})) \). Let \( h^{(t)} = \| \alpha^{(t)} - \bar{\alpha} \| \) and \( v^{(t)} = \langle g^{(t)}, \bar{\alpha} - \alpha^{(t)} \rangle \). The concavity of \( D \) implies \( v^{(t)} \geq 0 \). From Lemma 3 we know that \( h^{(t)} \leq \sqrt{2\lambda N^2 v^{(t)}/(\lambda N \mu + \sigma_{\min}(X, k))} \). Let \( g^{(t)}_{B_i} = H_{B_i}(g^{(t)}) \) and \( v^{(t)}_{B_i} = \langle g^{(t)}_{B_i}, \bar{\alpha} - \alpha^{(t)} \rangle \) Then

\[
(h^{(t)})^2 = \| P_{\mathcal{F}^N} (\alpha^{(t-1)} + \eta^{(t-1)} g^{(t-1)}_{B_i}) - \bar{\alpha} \|^2
\]

\[
\leq \| \alpha^{(t-1)} + \eta^{(t-1)} g^{(t-1)}_{B_i} - \bar{\alpha} \|^2
\]

\[
= (h^{(t-1)})^2 - 2\eta^{(t-1)} v^{(t-1)} + \eta^{(t-1)}^2 \| g^{(t-1)}_{B_i} \|^2.
\]

By taking conditional expectation (with respect to uniform random block selection, conditioned on \( \alpha^{(t-1)} \)) on both sides of the above inequality we get

\[
\mathbb{E}[ (h^{(t)})^2 | \alpha^{(t-1)} ]
\]

\[
\leq (h^{(t-1)})^2 - \frac{1}{m} \sum_{i=1}^{m} 2\eta^{(t-1)} v^{(t-1)}_{B_i} + \frac{1}{m} \sum_{i=1}^{m} \eta^{(t-1)}^2 \| g^{(t-1)}_{B_i} \|^2
\]

\[
= (h^{(t-1)})^2 - 2\eta^{(t-1)} v^{(t-1)} + \frac{\eta^{(t-1)}^2 \| g^{(t-1)} \|^2}{m}
\]

\[
\leq (h^{(t-1)})^2 - \frac{\eta^{(t-1)} (\lambda N \mu + \sigma_{\min}(X, k))}{\lambda m N^2} (h^{(t-1)})^2 + \frac{\eta^{(t-1)}^2}{m} c_0^2.
\]

Let \( \eta^{(t)} = \frac{\lambda m N^2}{(\lambda N \mu + \sigma_{\min}(X, k)) (t+1)} \). Then we obtain

\[
\mathbb{E}[ (h^{(t)})^2 | \alpha^{(t-1)} ] \leq \left( 1 - \frac{1}{t} \right) (h^{(t-1)})^2 + \frac{\lambda^2 m N^2 c_0^2}{(\lambda N \mu + \sigma_{\min}(X, k))^2 t^2}.
\]

By taking expectation on both sides of the above over \( \alpha^{(t-1)} \), we further get

\[
\mathbb{E}[ (h^{(t)})^2 ] \leq \left( 1 - \frac{1}{t} \right) \mathbb{E}[ (h^{(t-1)})^2 ] + \frac{\lambda^2 m N^2 c_0^2}{(\lambda N \mu + \sigma_{\min}(X, k))^2 t^2}.
\]
This recursive inequality leads to
\[
\mathbb{E}[(h(t))^2] \leq \frac{\lambda^2 m N^4 c_0^2}{(\lambda N \mu + \sigma_{\text{min}}(X, k))^2} \left( \frac{1 + \ln t}{t} \right) = c_2 \left( \frac{1 + \ln t}{t} \right).
\]
This proves the desired bound in part (a).

**Part (b):** Let us consider \( \epsilon = \frac{\lambda N \bar{\epsilon}}{2\sigma_{\text{max}}(X)} \). From part (a) and Lemma 6 we obtain
\[
\mathbb{E}[\|\alpha(t) - \bar{\alpha}\|] \leq \delta \epsilon
\]
after \( t \geq t_2 = \frac{3c_2}{\epsilon^2} \ln \frac{3c_2}{\epsilon^2} \). Then from Markov inequality we know that \( \|\alpha(t) - \bar{\alpha}\| \leq \mathbb{E}[\|\alpha(t) - \bar{\alpha}\|]/\delta \leq \epsilon \) holds with probability at least \( 1 - \delta \). Lemma 2 shows that \( \|\alpha(t) - \bar{\alpha}\| \leq \epsilon \) implies \( \text{supp}(w(t)) = \text{supp}(\bar{w}) \). Therefore when \( t \geq t_2 \), the event \( \text{supp}(w(t)) = \text{supp}(\bar{w}) \) occurs with probability at least \( 1 - \delta \).

Similar to the proof arguments of Theorem 4(b) we can further show that when \( t \geq 4t_2 \), with probability at least \( 1 - \delta/2 \)
\[
\|\alpha(t) - \bar{\alpha}\| \leq \frac{\lambda N \epsilon}{2\sigma_{\text{max}}(X)},
\]
which then leads to
\[
\epsilon_{PD}(t) \leq c_0 \left( 1 + \frac{\sigma_{\text{max}}(X, k)}{\mu \lambda N} \right) \|\alpha(t) - \bar{\alpha}\|.
\]
Since \( t \geq t_3 \), from the arguments in part (a) and Lemma 6 we get that \( \|\alpha(t) - \bar{\alpha}\| \leq \frac{\epsilon}{c_0 \left( 1 + \frac{\sigma_{\text{max}}(X, k)}{\mu \lambda N} \right)} \) holds with probability at least \( 1 - \delta/2 \). Let us consider the following events:

- \( \mathcal{A} \): the event of \( \epsilon_{PD}(t) \leq \epsilon \);
- \( \mathcal{B} \): the event of \( \|\alpha(t) - \bar{\alpha}\| \leq \frac{\lambda N \epsilon}{2\sigma_{\text{max}}(X)} \);
- \( \mathcal{C} \): the event of \( \|\alpha(t) - \bar{\alpha}\| \leq \frac{\epsilon}{c_0 \left( 1 + \frac{\sigma_{\text{max}}(X, k)}{\mu \lambda N} \right)} \).

When \( t \geq \max\{4t_2, t_3\} \), we have the following holds:
\[
\mathbb{P}(\mathcal{A}) \geq \mathbb{P}(\mathcal{A} \mid \mathcal{B}) \mathbb{P}(\mathcal{B}) \geq \mathbb{P}(\mathcal{C} \mid \mathcal{B}) \mathbb{P}(\mathcal{B}) \geq (1 - \delta/2)^2 \geq 1 - \delta.
\]
This proves the desired bound.

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