Bipartite Mixed States of Infinite-Dimensional Systems are Generically Nonseparable

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Given a bipartite quantum system represented by a Hilbert space \( \mathcal{H}_1 \otimes \mathcal{H}_2 \), we give an elementary argument to show that if either \( \dim \mathcal{H}_1 = \infty \) or \( \dim \mathcal{H}_2 = \infty \), then the set of nonseparable density operators on \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) is trace-norm dense in the set of all density operators (and the separable density operators nowhere dense). This result complements recent detailed investigations of separability, which show that when \( \dim \mathcal{H}_i < \infty \) for \( i = 1,2 \), there is a separable neighborhood (perhaps very small for large dimensions) of the maximally mixed state.

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I. INTRODUCTION

One key feature that distinguishes classical from quantum information theory is the availability, in quantum theory, of composite systems with entangled states. Often a composite system will itself form part of a larger system in a pure state wherein that composite system is entangled, and perhaps interacting in a noisy way, with some ‘environment’. In that case, the state of the composite system will necessarily be mixed. Unfortunately, the theoretical analysis of mixed entangled states — usually called nonseparable states — is somewhat more complicated than it is for the case of pure states. This is so even in the simplest case of a bipartite system represented by a tensor product of just two Hilbert spaces \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) with dimensions \( d_1, d_2 > 1 \).

For example, while there is a uniquely natural measure of entanglement for pure states, based on von Neumann entropy \([1]\), there are a number of apparently distinct yet useful measures for nonseparable states \([2]\). Moreover, there are nonseparable states that are ‘bound entangled’, i.e., no pure entanglement (in the form of singlet states) can be distilled from them \([3]\). At a more fundamental level, while every pure entangled state dictates nonlocal Bell correlations, i.e., violates some Bell inequality \([4]\), nonseparable states need not \([5]\). Bell-correlated states must always be nonseparable. However, there is no known characterization of the Bell-correlated states among those which are nonseparable, at least beyond the simplest \( d_1 = d_2 = 2 \) case \([6]\). There is not even an effective characterization of the nonseparable states themselves, amongst the set of all mixed states (but see \([7]\)). There is, however, a useful general necessary condition for separability, in terms of positivity of the partial transposition of the state’s density matrix \([8]\) (and in the cases \( d_1 = 2, d_2 = 2 \) or \( 3 \), this condition is sufficient for separability as well \([9]\)).

There is a further curious disanalogy between pure entangled and (mixed) nonseparable states. While the former are always norm-dense in the unit sphere of \( \mathcal{H}_1 \otimes \mathcal{H}_2 \), the nonseparable states are not dense in the set of all mixed states when \( d_1, d_2 < \infty \). This was brought to light recently in an investigation of whether the very noisy mixed states exploited by certain models of NMR quantum computing are truly nonseparable \([10]\). Moreover, determining just how generic nonseparable states are is motivated by the more fundamental question, “Is the world more classical or more quantum?” \([11]\).

More specifically, in the finite-dimensional case — either with \( d_1, d_2 < \infty \), or with an arbitrary finite number \( N \) of two-dimensional Hilbert spaces which can be grouped together to form a tensor product \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) with \( d_1, d_2 = \text{even} \) — it turns out that there is always an open neighborhood (or a set of nonvanishing measure) of separable states \([12,13,14]\). In particular, in such a case there is a natural maximally mixed state \( \frac{1}{N} \mathbb{I} \) (where \( n = d_1 d_2 \)), and there is a separable neighborhood of \( \frac{1}{N} \mathbb{I} \). It has been shown, however, that in the case of \( N \) qubits, the size of this neighborhood decreases with increasing \( N \) \([12,13]\). And in the case of two (arbitrary) finite-dimensional Hilbert spaces, numerical evidence has been obtained indicating a similar shrinking of the separable neighborhood \([12]\). These results have prompted the question, “Does the volume of the set of separable states
really go to zero as the dimension of the composite system grows, and how fast?” [11].

In particular, one might conjecture that at the limit, where \( d_1 = d_2 = \infty \), the separable states should be “negligible.” Our objective is to confirm this particular conjecture. In fact, we shall show that when \( d_1 = \infty \) or \( d_2 = \infty \), the set of separable states is nowhere dense (relative to the trace-norm topology).

II. DENSITY OF NONSEPARABLE STATES IN THE INFINITE CASE

Let \( \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2) \) denote the set of all (bounded) operators on \( \mathcal{H}_1 \otimes \mathcal{H}_2 \), and let \( \mathcal{I} \equiv \mathcal{I}(\mathcal{H}_1 \otimes \mathcal{H}_2) \) be the subset of (positive, trace-1 operators) density operators. Recall that \( \mathcal{I} \) is a convex set; that is, for any \( \{ D_i : i = 1, \ldots, n \} \subseteq \mathcal{I} \), and for any sequence \( \{ \lambda_i \} \) of positive real numbers summing to 1, \( \sum\lambda_i D_i \) is also in \( \mathcal{I} \). Throughout, we shall consider \( \mathcal{I} \) as endowed with the trace-norm topology, defined by \( \| A \|_T \equiv \text{Tr}((A^*A)^{1/2}) \) (reserving the notation \( \| A \| \) for the standard operator norm). For \( D \in \mathcal{I} \), \( D \) is said to be a product state just in case there is a \( D_1 \in \mathcal{I}(\mathcal{H}_1) \) and a \( D_2 \in \mathcal{I}(\mathcal{H}_2) \) such that \( D = D_1 \otimes D_2 \). The separable density operators are then defined to be all members of \( \mathcal{I} \) that may be approximated (in trace-norm) by convex combinations of product states \( \mathcal{I} \). In other words, the separable density operators are those in the closed convex hull of the set of all product states in \( \mathcal{I} \).

In what follows, we shall make use of a third, auxiliary Hilbert space \( \mathcal{H}_3 \) with dimension \( d_3 = \infty \). Let \( \mathcal{S} \) denote the closed unit sphere of \( \mathcal{H} \equiv \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \), endowed with the vector-norm topology. For \( v \in \mathcal{S} \), let \( \Phi(v) \) denote the unique reduced density operator \( D \in \mathcal{I}(\mathcal{H}_1 \otimes \mathcal{H}_2) \) such that \( (v, (A \otimes I)v) = \text{Tr}(DA) \) for all \( A \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2) \). It is not difficult to see that the reduction mapping \( \Phi : \mathcal{S} \to \mathcal{I} \) is both continuous and onto.

To see the continuity of \( \Phi \), set \( |C| = (C^*C)^{1/2} \). Using the polar decomposition of \( C \), we have \( |C| = V C \), where \( V \) is the partial isometry with initial space the closure of the range of \( C \) and final space the closure of the range of \( C^* \) (cf. [14], p. 4). If \( C \) is trace-class, then

\[
\|C\|_T = \text{Tr}(|C|) = |\text{Tr}(V C)|. \tag{1}
\]

Now, for \( u, v \in \mathcal{S} \), \( \Phi(u) - \Phi(v) \) is of trace-class. Thus, using [14] with \( C = \Phi(u) - \Phi(v) \),

\[
\|\Phi(u) - \Phi(v)\|_T = \left| \text{Tr}(V(\Phi(u) - \Phi(v))) \right| \tag{2}
\]
\[
= \left| \langle u, (V \otimes I)u \rangle - \langle v, (V \otimes I)v \rangle \right| \tag{3}
\]
\[
= \left| \langle u, (V \otimes I)u \rangle - \langle u, (V \otimes I)v \rangle + \langle u, (V \otimes I)v \rangle - \langle v, (V \otimes I)v \rangle \right| \tag{4}
\]
\[
\leq 2|u - v|, \tag{5}
\]

since \( ||V \otimes I|| = ||V|| = 1 \) and \( u, v \) are unit vectors. Thus, \( \Phi : \mathcal{S} \to \mathcal{I} \) is continuous. It follows from this that \( \Phi \) maps any dense set in \( \mathcal{S} \) onto a dense set in its image \( \Phi(\mathcal{S}) \).

To see that the mapping \( \Phi \) is onto, let \( D \in \mathcal{I} \). Then, \( D = \sum_{i=1}^{d_1 d_2} \lambda_i P_{x_i} \), where \( \{ x_i : i = 1, \ldots, d_1 d_2 \} \) is an orthonormal family in \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) (and \( P_{x_i} \) projects onto the ray \( x_i \) generates). Since \( d_3 = \infty \), there is always an orthonormal family \( \{ y_i : i = 1, \ldots, d_1 d_2 \} \) in \( \mathcal{H}_3 \). Thus

\[
v \equiv \sum_{i=1}^{d_1 d_2} \sqrt{\lambda_i} (x_i \otimes y_i) \tag{6}
\]
defines a vector in \( \mathcal{S} \) for which \( \Phi(v) = D \).

We shall denote the norm-closure of any subset \( \mathcal{R} \subseteq \mathcal{H} \) by \( [\mathcal{R}] \). For any vector \( v \in \mathcal{H} \), we say that \( v \) is 1-cyclic just in case the closed subspace defined by

\[
[(\mathcal{B}(\mathcal{H}_1) \otimes I \otimes I)v] = \{(A \otimes I \otimes I)v : A \in \mathcal{B}(\mathcal{H}_1)\} \tag{7}
\]
is the whole of \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \). With a 1-cyclic state vector, one can get arbitrarily close to any other vector in \( \mathcal{H} \) simply by acting on the vector with operators on the Hilbert space of system 1. Note that there is no restriction to acting only with unitary operators or projections. Nevertheless, states \( v \) that are 1-cyclic have an intuitive physical interpretation. Such states are all ‘maximally EPR-correlated’ in the following sense: For any state vector \( w \) of system 2 + 3 and any \( \epsilon > 0 \), there is a measurement one can perform on system 1 in state \( v \) such that, upon conditionalizing on an appropriate measurement outcome, the probability that system 2 + 3 is in state \( w \) will be greater than \( 1 - \epsilon \) [13]. This is reminiscent of Schrödinger’s [14] sardonic remark about the “sinister” possibility in quantum mechanics of steering a distant system (here, system 2 + 3) into any desired state by a suitable local measurement (on system 1). Our next task is to show that 1-cyclicity of a state \( v \in \mathcal{H} \) entails that it induces a nonseparable density operator on the first two systems.

We begin with a few elementary observations. Let \( \mathcal{H}' \) be any Hilbert space and let \( D \in \mathcal{I}(\mathcal{H}') \). Then, for any \( A \in \mathcal{B}(\mathcal{H}') \), \( ADA^* \) is a positive, trace-class operator. We may then define the operator \( D^A \) as follows:

\[
D^A \equiv \begin{cases} \frac{ADA^*}{\|ADA^*\|_T} & \text{when } \|ADA^*\|_T \neq 0, \\ 0 & \text{otherwise}. \end{cases} \tag{8}
\]

Thus, \( D^A \) is either a density operator or the zero operator. Suppose now that \( D \) is a convex combination,

\[
D = \sum_{i=1}^{n} \lambda_i D_i, \tag{9}
\]

where each \( D_i \in \mathcal{I}(\mathcal{H}') \). Then,
\[ ADA^* = \sum_{i=1}^{n} \lambda_i (AD_i A^*) \] (10)

and

\[ \sum_{i=1}^{n} \lambda_i ||AD_i A^*||_T = \sum_{i=1}^{n} \lambda_i \text{Tr}(AD_i A^*) \] (11)

\[ = \text{Tr}\left( \sum_{i=1}^{n} \lambda_i AD_i A^* \right) \] (12)

\[ = \text{Tr}(ADA^*) = ||ADA^*||_T. \] (13)

Thus, if \( ADA^* \neq 0 \), and we let

\[ \lambda^A_i \equiv \lambda_i \frac{||AD_i A^*||_T}{||ADA^*||_T}, \] (14)

then \( \sum_{i=1}^{n} \lambda^A_i = 1 \) and

\[ D^A \equiv \frac{ADA^*}{||ADA^*||_T} = \sum_{i=1}^{n} \lambda^A_i D^A_i. \] (15)

**Lemma 1** If \( v \in S \) is 1-cyclic, then \( \Phi(v) \) is nonseparable.

**Proof:** Let \( \mathcal{H}' = A_1 \otimes A_2 \) and let \( A \in \mathfrak{B}(\mathcal{H}_1) \) be such that \( ||(A \otimes I \otimes I)v|| = 1 \). A straightforward calculation—using the definition of a reduced density operator—shows that

\[ \Phi[(A \otimes I \otimes I)v] = (A \otimes I)\Phi(v)(A \otimes I)^* \in \mathfrak{I}. \] (16)

Suppose \( \Phi(v) \) is separable. (We show that \( v \) cannot be 1-cyclic.) Then, \( \Phi(v) = \lim_n W_n, \) where each \( W_n \in \mathfrak{I} \) is a convex combination of product states. But then,

\[ (A \otimes I)\Phi(v)(A \otimes I)^* = (A \otimes I) \lim_n W_n(A \otimes I)^* \] (17)

\[ = \lim_n (A \otimes I)W_n(A \otimes I)^* \] (18)

\[ = \lim_n W_n^A \otimes I. \] (19)

The penultimate equality follows since multiplication by a fixed element in \( \mathfrak{B}(\mathcal{H}') \) is trace-norm continuous [14, p. 39]. The final equality holds since

\[ \lim_n ||(A \otimes I)W_n(A \otimes I)^*||_T = ||(A \otimes I)\Phi(v)(A \otimes I)^*||_T = 1. \]

Now, for fixed \( n \),

\[ W_n = \sum_{i=1}^{k} \lambda_i (D_{1i} \otimes D_{2i}) \] (20)

and hence, from (15),

\[ W_n^A \otimes I = \sum_{i=1}^{k} \lambda_i^A (D_{1i}^A \otimes D_{2i}) \] (21)

\[ = \sum_{i=1}^{k} \lambda_i^A (D_{1i}^A \otimes D_{2i}). \] (22)

Thus, the density operator \((A \otimes I)\Phi(v)(A \otimes I)^*\) is again a limit of convex combinations of product states. Hence \( \Phi[(A \otimes I \otimes I)v] \) is separable for any \( A \in \mathfrak{B}(\mathcal{H}_1) \) where \( ||(A \otimes I \otimes I)v|| = 1 \).

Suppose, for reductio ad absurdum, that \( v \) is also 1-cyclic. If we let \( M \) denote the set of unit vectors in \( \mathcal{H} \) of the form \((A \otimes I \otimes I)v\), for some \( A \in \mathfrak{B}(\mathcal{H}_1) \), then \( M \) is dense in \( \mathcal{S} \). (If \( A_n v \to w \in \mathcal{S} \), then we may replace the sequence \( \{A_n\} \) with the sequence \( \{A_n/||A_n v||\} \).) However, from the argument of the previous paragraph, \( \Phi(M) \) consists entirely of separable density operators. Since the separable density operators are closed, the trace-norm closure \( \Phi(M)^{-} \) only contains separable states. Finally, since \( \Phi \) is both onto and continuous, and \( M \) is dense,

\[ \mathfrak{T} = \Phi(\mathcal{S}) = \Phi(M^-) \subseteq \Phi(M)^{-} \subseteq \mathfrak{T}. \] (23)

Therefore every density operator in \( \mathfrak{T} \) must be separable, which is absurd. It follows that, if \( \Phi(v) \) is separable, \( v \) cannot be 1-cyclic. \( \text{QED} \)

We shall need only one further lemma, for which we will require the following definition. A vector \( v \in \mathcal{H} \) is called separating for the subalgebra \( I \otimes \mathfrak{B}(\mathcal{H}_2 \otimes \mathcal{H}_3) \) if \( Av = 0 \), with \( A \in I \otimes \mathfrak{B}(\mathcal{H}_2 \otimes \mathcal{H}_3) \), entails that \( A = 0 \). The physical interpretation of a separating state vector \( v \) is simply that every possible outcome of every possible measurement on system 2 + 3 has a nonzero probability of being found.

**Lemma 2** If \( d_1 = \infty \), then the set of 1-cyclic vectors is dense in \( \mathcal{S} \).

**Proof:** First observe that if \( v \) is not 1-cyclic, then it cannot be separating for \( I \otimes \mathfrak{B}(\mathcal{H}_2 \otimes \mathcal{H}_3) \). Indeed, if we let \( P \) denote the orthogonal projection onto the closed subspace \( \{(\mathfrak{B}(\mathcal{H}_1) \otimes I \otimes I)v\} \), then it follows that \( Pv = v \) and \( P \) is in \( I \otimes \mathfrak{B}(\mathcal{H}_2 \otimes \mathcal{H}_3) \). (It is easy to see that \( P \) must commute with all elements in \( \mathfrak{B}(\mathcal{H}_1) \otimes I \otimes I \), because they leave the range of \( P \) invariant.) Moreover, if \( \{(\mathfrak{B}(\mathcal{H}_1) \otimes I \otimes I)v\} \neq \mathcal{H} \), then \( I - P \neq 0 \) yet \( (I - P)v = 0 \). With this in mind, it is sufficient to establish that the set of state vectors separating for \( I \otimes \mathfrak{B}(\mathcal{H}_2 \otimes \mathcal{H}_3) \) is dense in \( \mathcal{S} \).

Let \( v \in \mathcal{S} \). Since \( d_1 = d_2d_3 = \infty \), \( v \) has a Schmidt decomposition

\[ v = \sum_{i=1}^{\infty} a_i (x_i \otimes y_i), \] (24)

where \( \{x_i\} \subseteq \mathcal{H}_1 \), \( \{y_i\} \subseteq \mathcal{H}_2 \otimes \mathcal{H}_3 \), are orthonormal bases, and \( \{a_i\} \) is a sequence of coefficients (not necessarily all nonzero). If \( a_i \neq 0 \) for all \( i \), then it is clear that \( v \) is separating for \( I \otimes \mathfrak{B}(\mathcal{H}_2 \otimes \mathcal{H}_3) \) (since the terms in the expansion of \((I \otimes A)v\), for any \( A \in \mathfrak{B}(\mathcal{H}_2 \otimes \mathcal{H}_3) \), will again be orthogonal).

On the other hand, if \( a_i = 0 \) for at least one \( i \), then consider the new (normalized) state vector
where \( b_i = a_i \) if \( a_i \neq 0 \), \( b_i \neq 0 \) if \( a_i = 0 \), and the sequence \( \{ b_i : a_i = 0 \} \) is square-summable. Each such \( u \) is separating for \( I \otimes \mathcal{B}(\mathcal{H}_2 \otimes \mathcal{H}_3) \). Moreover, we can make \( u \) as close as we wish to \( v \) by choosing the coefficients \( \{ b_i : a_i = 0 \} \) arbitrarily small. Thus, the set of separating state vectors for \( I \otimes \mathcal{B}(\mathcal{H}_2 \otimes \mathcal{H}_3) \) is dense in \( S \). Since all separating vectors for \( I \otimes \mathcal{B}(\mathcal{H}_2 \otimes \mathcal{H}_3) \) are 1-cyclic, the set of 1-cyclic state vectors is dense in \( S \). QED

We turn, finally, to proving our main theorem:

**Theorem** If \( \dim \mathcal{H}_1 = \infty \) or \( \dim \mathcal{H}_2 = \infty \), then the set of nonseparable density operators on \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) is trace-norm dense, and the set of separable density operators nowhere dense.

**Proof:** If \( d_2 = \infty \) then we may interchange the roles of \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \). Thus, we may assume that \( d_1 = \infty \). Recall again that the continuous reduction map \( \Phi : S \to \mathfrak{T}(\mathcal{H}_1 \otimes \mathcal{H}_2) \) is onto. By Lemma 2, the set of 1-cyclic state vectors is dense in \( S \). Thus, \( \{ \Phi(v) : v \text{ is 1-cyclic} \} \) is dense in \( \mathfrak{T} \). However, Lemma 1 entails that each element of the latter set is nonseparable.

By definition, the set of separable states in \( \mathfrak{T} \) is closed. Since the set of nonseparable states is dense, the set of separable states has empty interior. Thus, the separable states are nowhere dense in \( \mathfrak{T} \). QED

### III. CONCLUSION

The fact that the states of an infinite-dimensional bipartite system are generically nonseparable may or may not find any direct practical application in quantum information theory. However, from the point of view of fundamental quantum physics, the implications appear profound. For example, it is well-known that one cannot have a finite-dimensional representation of the canonical (anti-)commutation relations for a single degree of freedom. It follows that the position-momentum state of any pair of spin-less particles, or, indeed, the position-momentum/spin state of a single particle, will be generically nonseparable (with similar conclusions applicable in the field-theoretic case).

Moreover, it would be interesting to know, again from the point of view of fundamental physics, whether Bell-correlated, and hence nonlocal, states are also generic in the infinite case. This conjecture is given some credence by the fact that the most widely studied case of states which are nonseparable but violate no Bell inequalities—so-called ‘Werner states’—all involve mixing the maximally mixed state with some nonseparable state, yet there is no (strictly) maximally mixed state in the infinite case. Assuming no obvious ‘no-Bell’ neighborhood is forthcoming, one might first attempt to prove the analogue of our main theorem for Bell states by establishing that density operators on \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) induced by 1-cyclic states violate a Bell inequality (noting that the Bell states, just like the nonseparable ones, form an open set in \( \mathfrak{T}(\mathcal{H}_1 \otimes \mathcal{H}_2) \)). And observe that while 1-cyclic of a state \( v \) implies that it induces a nonseparable reduced density operator on \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) (Lemma 1), 1-cyclicity is strictly stronger than nonseparability. (To see this, choose some state vector \( v \in S \) such that \( \Phi(v) \) is any entangled pure state in \( \mathfrak{T}(\mathcal{H}_1 \otimes \mathcal{H}_2) \). Then \( v \) reduces to a pure state in \( \mathfrak{T}(\mathcal{H}_3) \), is therefore not separating for \( I \otimes \mathcal{B}(\mathcal{H}_2 \otimes \mathcal{H}_3) \), and hence could not be 1-cyclic.) Finally, in the special case where both \( d_1 = d_2 = \infty \), it was shown in [13] that the set of all states that are simultaneously 1-, 2-, and 3-cyclic is also dense in the unit sphere of \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \). It would be surprising if even one such ‘tricyclic’ state were to induce a density operator on \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) without Bell correlations. Barring surprises, our conjecture is, then, that our main theorem holds (possibly only after replacing ‘or’ by ‘and’) for Bell-correlated states as well. If true, this would mean that there is essentially no difference in the infinite case between Bell-correlated and nonseparable states—a dramatic simplification over the finite case.

It should be emphasized, however, that while nonseparable states need not be Bell-correlated, this does not make them entirely devoid of nonlocal properties. As Popescu [1] has shown, a large class of Werner states contain “hidden locality,” in that they violate extended Bell inequalities that involve the performance of consecutive measurements on the two subsystems. Moreover, this violation becomes maximal approaching the limit \( d_1 = d_2 = \infty \). Werner states have also been said to contain “active nonlocality” at the level of each single particle pair, since they can realize teleportation with a fidelity better than any classical procedure [16]. Similar investigations have not been undertaken on infinite-dimensional Werner-like states (should there be any). However, it would now appear that, at least with regards to infinite-dimensional systems, the world is far more quantum than classical.

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