Energy transport and optimal design of noisy Platonic quantum networks

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Abstract
Optimal energy transport is one of the primary goals for designing efficient quantum networks. In this work, the maximum energy transport is investigated for three-dimensional quantum networks with Platonic geometries affected by dephasing and dissipative Markovian noise. The network and the environmental characteristics corresponding the optimal design are obtained and investigated for five Platonic networks with 4, 6, 8, 12, and 20 number of sites that one of the sites is connected to a sink site through a Markovian dissipative process. Such optimal designs could have various applications like switching and multiplexing in quantum circuits.

Introduction
Transport is an essential phenomena in atomic-scale devices and networks. The structure that hosts the energy or information carriers could be a continuous medium like metallic nanorods and waveguides [1, 2] or site-based structure like metallic nanoparticle arrays [3, 4], quantum dot arrays [5, 6], and many more. Discrete or site-based transport—considered in this work—has many applications such as quantum state transport through spin chains [7–10], quantum energy transport in chains of trapped ions, environment-assisted transport in networks of sites [11, 12], and switching with qubit arrays [13]. While much is known for ideal networks, the effect of noise on the desired properties are less understood. We study the case of noisy quantum networks here. A specific type of quantum network that has been proven to have exact theoretical solutions for site-based energy excitation transfer are called Fully Connected Networks (FCNs) [14]. FCNs are defined by the property that all sites are equally connected to each other and the last site is dissipatively connected to a sink site. In [15] we studied some three-dimensional Platonic configurations with distance dependent couplings, and proved that they have some similar properties as those of an N-site FCN, where in the corresponding Platonic network N − 1 would be the number of nearest neighbours of each site. For example, it was shown that the sink population—the energy excitation accumulated in the sink site—of the ‘noiseless’ Platonic quantum networks and FCNs are the same at the steady state (or infinite time). These identities are convenient and powerful tools for the study of quantum networks, and we find similar relations in the more relevant case of noisy quantum networks. In general, Platonic networks can implement FCNs made of distant-dependent (e.g. dipole-dipole) interacting qubits, with various number of sites that are 4, 6, 8, 12, and 20. The only possible implementation of FCN made of such interacting qubits is a 4-site network with Tetrahedron geometry (figure 1(a)), and using Platonic geometries (figure 1), we can have FCN network properties while having more number of distant-dependent interacting qubits (n ≥ 4).

In this work we prove that in different noisy Platonic quantum networks, the analytical solution of the steady sink population is the same as that of the equivalent FCN. In addition, we provide some relations among the couplings and noise rates corresponding the full transport parameter regimes. These relations will be used to optimally design noisy Platonic quantum networks with the ability of full transport to the sink site in the presence of Lindbladon environmental noises. At the steady state or the full transport conditions, the initially...
The corresponding transformations that leave them invariant, are as follows: For Tetrahedron called point symmetry groups. Their group dimensions or symmetry orders i.e. the number of symmetries or same number of faces meeting at each vertex. They are all centrally symmetric, and so in group theory, they are Platonic geometries are regular polyhedrons or 3D objects that possess identical faces, angles, edges, and the refractive index of the host material. Results

The schematic of the five Platonic geometries: (a) Tetrahedron- 4 vertices, (b) Cube- 8 vertices, (c) Octahedron- 6 vertices, (d) Dodecahedron- 20 vertices, and (e) Icosahedron- 12 vertices.

Injected energy excitation to the first site would be totally accumulated on the sink site that is connected to one of the main Platonic network sites. If we connect the first and the sink sites to different one-dimensional chains of qubits, the total incident energy from the first chain could be fully transferred to the second chain. This bents the energy flow towards different angles that are available in Platonic networks. In this case, any site of the Platonic network could be a potential output port. To change the direction of the flow of quantum energy, as in multiport switch boards possessing several inputs and outputs, one should be able to turn on and off the dissipative noise around an arbitrary last site in the vicinity of target sinks. To control the Lindbladian dissipation noise around the last qubit connected to the sink site and be able to transfer the released energy towards a specific direction, a separate study and accurate experimental designs are demanded that depend on the type of qubits of the Platonic networks. Here we can point some possible ways of creating the dissipative noise that might be used in some experimental designs. For the ion qubit implementation of Platonic networks positioned in three-dimensional optical lattices, the dissipative noise around the last site might be created by filling in the surrounding by some type of gas, to increase the inelastic collisions of the ion qubit and the gas molecules. For Platonic network implementation with ions doped in solids, it might be possible to turn on and off the dissipation noise around the target sink site by exposing the surrounding area to an electric field to change the refractive index of the host material.

Before presenting our results, we comment on two advantages of using Platonic quantum networks for such purposes. First, three-dimensional networks in general are more compact in comparison with two and one-dimensional networks having the same number of sites, and the nanoscale three-dimensional printing methods [16, 17] could help to overcome the three-dimensional construction difficulties. The three-dimensional structures might also be compatible with the physics or other constraints of some architectures. The other advantage of Platonic quantum networks is that they are proven to be a FCN network with reduced number of sites that has exact transport solutions, and thus ideal benchmarks for new techniques and demonstrations.

Results

Platonic geometries are regular polyhedrons or 3D objects that possess identical faces, angles, edges, and the same number of faces meeting at each vertex. They are all centrally symmetric, and so in group theory, they are called point symmetry groups. Their group dimensions or symmetry orders i.e. the number of symmetries or corresponding transformations that leave them invariant, are as follows: For Tetrahedron (figure 1(a)) it is 24, for the Cube and Octahedron (figures 1(b), (c) it is 48, and for Dodecaheron and Icosahedron (figures 1(d), (e)) it is 120. These symmetry groups consist of all transformations including the combinations of reflections and rotations. Some molecules like Icoborate \( B_{12}H_{12} \) and \( C_{60} \) fullerenes possess the symmetries of the Icosahedron symmetry group. Since the Hamiltonian of such molecules have known symmetries, their eigenstate and eigenvalues could be calculated using the symmetry-adapted bases obtained from the irreducible bases in the Icosahedron symmetry group [18, 19]. Calculating the eigenvalues and eigenstates by symmetry-adapted bases is an important application and is usually used for molecules [20–22].

A Platonic network in this work is defined as a group of interactive identical two-level systems (main sites), located on the vertices of one of the five Platonic geometries. Initially, some energy would be injected to one or several main sites, and one additional site, the target sink, is dissipatively connected with rate \( \Gamma' \), to a main site. A homogeneous environment is assumed surrounding the network so that all main qubits are affected by equal dissipation and dephasing Markovian local noises, with rates \( \Gamma'_{\text{diss}} \) and \( \gamma' \), respectively. Due to the presence of non-homogeneous \( \Gamma' \) noise, the central symmetry in Platonic networks is not fully preserved as in Platonic geometries and so the energy dynamics in the main sites of Platonic networks are not identical. In the noiseless case, the Platonic networks possess full Platonic group symmetries, and their eigenvalues and eigenstates will be easily found and presented. This is due to the fact that Platonic networks are not as complex as molecules which have some atoms on each of their vertices, and so the symmetry-adapted bases [18, 19] are not necessary to find their eigenstates and eigenvalues.
In the first, we aim to analytically find the variation of energy accumulation at the sink site of the defined Platonic networks. To reach this goal, we will numerically calculate the energy dynamics in all sites, and show that the dynamics is equal in some sites, due to the remained symmetries in the corresponding Platonic networks. We will then reduce the number of sites to that of with unique dynamics to simplify the analytical processes.

Now we start by defining the Hamiltonian of the Platonic networks (in units $h = 1$) as following:

$$\hat{H} = \sum_{i,j=1}^{N} J_{ij} \hat{\sigma}_i^x \hat{\sigma}_j^x,$$

(1)

where $N$ is the total number of main sites, $J_{ij}$ ($i \neq j$) is the distance dependent coupling rate between qubits $i$ and $j$, $J_t$ is the intrinsic energy of qubit $i$ that is identical for all qubits, and $\sigma^x_i$ is the annihilation operator. The dynamics of the total density matrix of this system dissipatively connected to a sink site and under the influence of Markovian dissipation and dephasing noises is found by the Lindbladian master equation [23] as follows:

$$\dot{\rho} = -i[\hat{H}, \rho] + \mathcal{L}_{\text{target}}(\rho) + \mathcal{L}_{\text{deph}}(\rho) + \mathcal{L}_{\text{diss}}(\rho);$$

$$\mathcal{L}_{\text{target}}(\rho) = \Gamma (2\hat{\sigma}_{\text{target}}^x \hat{\sigma}_{\text{target}}^x \rho_{\text{target}} - \{\hat{\sigma}_{\text{target}}^x \hat{\sigma}_{\text{target}}^x, \rho_{\text{target}}\}),$$

$$\mathcal{L}_{\text{deph}}(\rho) = \gamma \sum_{i=1}^{N} (2\hat{\sigma}_i^x \hat{\sigma}_i^x \rho_{\text{target}} - \{\hat{\sigma}_i^x \hat{\sigma}_i^x, \rho\}),$$

$$\mathcal{L}_{\text{diss}}(\rho) = \Gamma_{\text{diss}} (2\hat{\sigma}_{\text{target}}^x \hat{\sigma}_{\text{target}}^x \rho_{\text{target}} - \{\hat{\sigma}_{\text{target}}^x \hat{\sigma}_{\text{target}}^x, \rho_{\text{target}}\}),$$

(2)

where we assumed $\hat{\rho} = \rho_{\text{qubits}} \otimes \rho_{\text{target}} \otimes \rho_{\text{diss}}$, is the direct sum of the $N \times N$ density matrix of the Platonic network ($\rho_{\text{qubits}}$), the $1 \times 1$ population matrix of the sink site ($\rho_{\text{target}}$), and the $1 \times 1$ matrix representing the total population discharged to the environment by the dissipation noise ($\rho_{\text{diss}}$). $\gamma$, and $\Gamma_{\text{diss}}$ are the dephasing and dissipation noise rates from the network sites to their local environment. $\Gamma$ is the rate of irreversible energy transfer from the last site to the target site and $\hat{\sigma}_{\text{target}}^x$ is the creation (annihilation) operator of site $i$. The aim is to provide an analytical expression for the population of target sink i.e. $\rho_{\text{target}}(t) = \rho_{\text{target}}(t)$ in the equilibrium state at $t \rightarrow \infty$.

In order to simplify the analytical calculation, the dynamical characteristics of the Platonic networks are studied by numerically solving $\dot{\rho}(t)$ from equation (2). For the simulation, the ion qubits implementation of the Platonic networks is assumed in which the coupling rate, or the interaction energy between two dipoles of ion qubits $i$ and $j$ would be inversely proportional to the cube of their interconnecting distance ($L_{ij} = v / L_{ij}^3$, $v = 1$). In figures 2, 3, and 4 we plotted the sites’ populations ($\rho_{\text{site}}(t)$) of four Platonic networks in the noiseless case ($\gamma = \Gamma_{\text{diss}} = 0$). It can be seen that in some cases at the steady state, the populations of some sites vary inversely, while the other sites would have the same population dynamics. The characteristics shown in these figures (noiseless cases) would be the same as that of the noisy case unless the fact that in the presents of dephasing and/or dissipation noises, the oscillating patterns of populations would be evanescent and so the sink site would be fully populated at the equilibrium ($t \rightarrow \infty$). In [15] we found that the target population of the noiseless Platonic networks at the steady state are independent of their size and is only related to the number of neighboring sites ($N_e = 1$) and the number of surrounding sites ($N_e = 2$), i.e. $\rho_{\text{sink}} = 0.25, 0.33$, and $0.33$ for $N = 6, 8, 12, 20$, respectively. Later on, we conclude that the exact solution of the dynamics of Platonic networks at the steady state are the same as that of FCNs, in which all sites are equidistant from each other [14].

Figure 2 shows the dynamics of two Platonic networks with $N = 6$ and $8$. It can be seen that some pairs of sites are oscillating inversely at the steady state, while the populations of other pairs vary equally. Figures 3 and 4 show the dynamics of Platonic networks at the steady state, with $N = 12$ and 20 sites, respectively. Likewise the Platonic networks of figure 2, these graphs show that for the chosen initial energy excitation, the populations of some groups of sites vary similarly.

That some group of sites having similar or inversely oscillating populations at the steady state can be understood by the fact that the noiseless Platonic networks have eigenstate of the form of equation (9) and the anti-symmetric eigenstates of the form $|m\rangle - |\bar{m}\rangle$, where $m$ and $\bar{m}$ are the point-symmetry or mirroring sites (equations (2), (3) [15]). So at the steady state, according to the energy initialization pattern, the system would be trapped in the superposition of the eigenstates of the form $|m\rangle - |\bar{m}\rangle$, and would oscillate within the groups of two mirroring sites.

We simplify the analytical dynamics of the Platonic networks by assuming only a specific number of sites i.e. $N_e$ sites. If $\hat{\rho}(t)$ (or for simplicity of notation $\rho(t)$) would be the density matrix operator of a Platonic network of $N$ sites with rank $(N + 2)$ and elements $\rho_{ij}(t)$, we define the the symbolic or reduced density matrix $\hat{\rho}(t)$ of an equivalent network of $N_e$ super-sites, with rank $(N_e + 2)$ and symbolic elements of $\rho_{ij}(t)$ as follows. Note that in
Figure 2. This graph shows the simulated population dynamics of all sites ($\rho_{ii} \equiv \rho_i, i = 1...N = 6, 8$) of tetrahedron and cube noiseless networks with zero environmental noises ($\gamma = \Gamma_{diss} = 0$), where the sink site ($i = N + 1$) is dissipatively connected to the $N$th site with rate $\Gamma$. Parts (a) and (c) show the schematics of the tetrahedron and cube networks with 6 and 8 sites on the vertices. The sites coordinates for tetrahedron are $(0, 0, -\sqrt{2}/2), (0, \pm 1, 0), ($ and for the cube (graph (c)), site 1 is located at the origin of coordinates and its symmetrically positioned site (7) is located at $(1, 1, 1)$. In both networks, at $t = 0$, site No. 1 was charged by one excitation. Part (b) presents the oscillating populations of sites 1 and 2 in time that are positioned symmetrically with respect to the centre of the tetrahedron. Note that since the rate $\Gamma$ has dimension $1/T$, the quantity $\Gamma t$ is dimensionless. The population of the spherically symmetric sites (3, 4) at equilibrium i.e. $\Gamma t \gtrsim 20$ is constant (0.0625). The symmetrically positioned sites 5 and 6 are discharged at equilibrium while the sink site 7 is saturated to the population of 0.25. The graphs (d) and (e) show that the spherically symmetric sites (positioned symmetrically with respect to the centre of the cube), i.e. (1, 7), (2, 5) and (3, 6) are oscillating inversely at equilibrium i.e. $\Gamma t \gtrsim 20$. At equilibrium, the populations of the spherically symmetric sites 4 and 8 are zero since the population of the sink site (9) is saturated to 0.33.

Figure 3. This graph shows the simulated population dynamics of all sites ($\rho_{ii} \equiv \rho_i, i = 1...N = 12$) of a regular icosahedron noiseless network with zero environmental noises ($\gamma = \Gamma_{diss} = 0$), where the sink site ($i = N + 1 = 13$) is dissipatively connected to only one site (12) with rate $\Gamma$. The left graph shows the schematic of the icosahedron network where sites are located on the vertices with coordinates $(0, \pm \phi, \pm 1), (\pm 1, 0, \pm \phi), (\pm \phi, \pm 1, 0)$ where $\phi = \frac{1 + \sqrt{5}}{2}$. At $t = 0$ four sites $(1, 2, 3, 4)$ are equally charged by $1/4$ amount of excitation. The right graphs show that due to the various symmetries in Platonic geometries, some sites are oscillating similarly (sites ‘1, 2, ’3, 4, ’5, 7, ’6, 8, ’ and 10, 11’), at the equilibrium i.e. $\Gamma t \gtrsim 20$. In addition, the populations of the spherically symmetrically positioned sites of 12 and 9, are zero, since the population of the sink site (13) is saturated to 0.30 at equilibrium.
the legends of figures 2, 3, 4 and the two following formulas, we simplify the notation of indices as $\rho_\beta(t) \equiv \rho(t)$.

$$N = 6: \quad \tilde{p}_i(t) = \rho_1(t) + \rho_2(t), \quad \tilde{p}^\gamma_i(t) = \rho(t), \quad i = 1, 2$$

$$N = 8, 12: \quad \tilde{p}_i(t) = \rho(t), \quad \tilde{p}^\gamma_i(t) = \rho(t), \quad i = 1, 2$$

(3)

where $\gamma$ is the index of the spherical-symmetrically positioned site with respect to site $i$. So the number of super-sites of the Octahedron network ($N = 6$), is $N_c = 5$, and that of Cube ($N = 8$) and Icosahedron network ($N = 12$) are $N_c = 4$ and $N_c = 6$, respectively. For the Dodecahedron network ($N = 20$), according to figure 4 we choose:

$$N = 20: \quad \tilde{p}_1(t) = \rho_1(t) + \rho_5(t) + \rho_9(t) + \rho_{20}(t),$$

$$\tilde{p}_2(t) = \rho_2(t) + \rho_6(t) + \rho_{10}(t) + \rho_{19}(t),$$

$$\tilde{p}_3(t) = \rho_3(t) + \rho_7(t) + \rho_{12}(t) + \rho_{13}(t) + \rho_{17}(t),$$

$$\tilde{p}_4(t) = \rho_4(t) + \rho_8(t) + \rho_{14}(t) + \rho_{15}(t) + \rho_{16}(t),$$

(4)

So the number of super-sites chosen for the Dodecahedron network ($N = 20$) is $N_c = 4$. The above two formulas show that we assume $N_c = 5, 4, 6, 4$ super-sites with populations $\tilde{p}(t)$, for the five Platonic networks with $N = 6, 8, 12, 20$ sites, respectively. Note that the distance between the super-sites that are created according to the assumed initial conditions, are all equal, except for the Dodecahedron network. Now the coherences between the sites of the equivalent reduced networks would be defined according to the symbolic indices as

$$\tilde{p}_\beta(t) = \sum_{p=1}^{n} \sum_{q=1}^{m} \rho_{ij}(t),$$

(5)

where $n, m$ are the number of sites that define the symbolic sites $i, j$, respectively, that are $\tilde{p}_\beta(t) \equiv \tilde{p}_1(t) = \sum_{p=1}^{n} \rho_\beta(t)$ and $\tilde{p}_\gamma(t) \equiv \tilde{p}_1(t) = \sum_{q=1}^{m} \rho_\gamma(t)$. As an example, in a Dodecahedron network:

$$\tilde{p}_{12}(t) = \sum_{p=1,5,9,20, q=2,6,10,19} \rho_{ij}(t).$$

(6)

In future, for simplicity of notation, we substitute $\tilde{p}(t) \rightarrow \rho(t)$.

Using equations (3)–(6), we summarize the established assumption called reduced networks. In this assumption, we sum over the populations of groups of sites that oscillate inversely or, in some cases, equally.
Each bunch of sites represents a super-site with populations of $\hat{\rho}(t)$ as defined in equations (3)–(6). Then we ignore the rest of the network sites and solve the dynamics for $N_t$ remaining nearest neighboring super-sites. The reason of ignoring other sites is that at repeating time instances, e.g. $\Gamma \approx 155$ in figure 2 parts (d) and (e), all of the energy is stored within the nearest neighboring sites $(1, 3, 5, 8)$. Equation (10) also provides an explanation for such behaviour. The quantum energy hops between a site and its mirroring site, and we assign this hopping energy to one site. In Platonic configurations, we have a more equidistant number of sites. Since each site has the same set of equidistant neighboring sites, the dynamics of each site are calculated by assuming $N_t$ neighboring sites around it.

Note that in the Dodecahedron network, the group of super-sites is arranged according to the arbitrarily chosen pattern of initial energies. Another initial pattern of energies is given in figure 1 (a, d, 5) of [15], for which the equally oscillating sites are different from those in this work. With the current chosen initial energies, the supersites of equation (4) are not nearest neighbors, and the above illustration of the reduced networks would be slightly different for the Dodecahedron network. Later on, the nearest neighbor assumption will be stated by which the Dodecahedron network is better illustrated.

Using the established reduced network assumption, equation (2) can be simplified as follows.

$$
\dot{\rho}_i = -2\Gamma_{\text{dis}} \rho_i + i\gamma (R_i - \tilde{R}_i); \quad i \neq N,
$$

$$
\dot{\rho}_N = -2(\Gamma_{\text{dis}} + \gamma) \rho_N + i\gamma (R_N - \tilde{R}_N),
$$

$$
\dot{\rho}_{NN} = -2(\Gamma_{\text{dis}} + 2\gamma + \Gamma) \rho_{NN} + i\gamma (R_N - \tilde{R}_N),
$$

$$
\dot{\rho}_{00} = -2\Gamma_{\text{dis}} \rho_{NN},
$$

$$
\dot{\rho}_{\text{target}} = 2\Gamma \rho_{\text{target}},
$$

(7)

where $\rho_{00}$ corresponds to a virtual site that stores some fraction of initial energy discharged to the environment by dissipative noise rate $\Gamma_{\text{dis}} \rho_{NN}$ corresponds to the population of the last site that is being discharged dissipatively by rate $\Gamma$ towards the target site with population $\rho{(N+1)}/(N+1)$ and the definition of the collective variables are as follows:

$$
R_i(t) = \sum_{j = f_N(i)}^N \rho_j(t), \quad \Lambda_i = \sum_{j = f_N(i)}^N R_i,
$$

(8)

where $f_N(i)$ is the set of $N_t$ indices of the nearest neighbors of site $i$ plus itself for all Platonic networks; for the Dodecahedron, $f_N(i)$ is the series of indices of super-sites presented in equation (4).

Equation (7) could be obtained by another assumption that is considering the nearest neighbor interactions for solving the dynamics of Platonic networks. By this assumption, the density matrix elements in equation (7) do not correspond to a reduced number of sites but to all sites of the networks. Here are two supporting facts for this assumption. The first one is that the rank $N$ Hamiltonian of the noiseless Platonic networks has a degenerate eigenstate of the following form:

$$
|\phi\rangle = \sum_{l=1}^{N} |l\rangle = |\phi^{m}\rangle + |\phi^{\bar{m}}\rangle, \quad |\phi^{m}\rangle = \sum_{l = f_{m}(m)}^{N} |l\rangle, \quad m = 1 \ldots N.
$$

(9)

For the Platonic networks with $N = 4, 6, 8, 12$, $f_{N}(m) = f_{N}(m)$ is the set of nearest neighboring sites of the arbitrary site $m$, added by one, and $m$ is the point symmetry site of $m$. For the simplicity of notation, we consider $f_{N}(m) \rightarrow f(m)$. For a Dodecahedron network, $f_{N}(m)$ consist of the first and the second sets of nearest neighbors of arbitrary site $m$, that is 10, or the half of the total of sites. So in all Platonic networks, for any arbitrary site $m$, we have $f_{N}(m) + f_{m}(m) = N$. The eigenvalue of the eigenstate $|\phi\rangle$ can be found as follows:

$$
H\langle\phi\rangle = \sum_{i,j=1}^{N} J_{ij} |i\rangle \langle j| = \sum_{i=1}^{N} J_{i\pi} |i\rangle \langle \pi| = \sum_{i=1}^{N} \left( \sum_{l = f(m)}^{N} J_{il} |l\rangle \langle i| + \sum_{l = f(\bar{m})}^{N} J_{il} |l\rangle \langle i| \right)
$$

$$
= \sum_{i=1}^{N} \left( J_{i\pi} \sum_{l = f(m)}^{N} |i\rangle |l\rangle + J_{i\pi} \sum_{l = f(\bar{m})}^{N} |i\rangle |l\rangle \right) = \sum_{i=1}^{N} J_{i\pi} (|\phi^{m}\rangle + |\phi^{\bar{m}}\rangle) = \sum_{i=1}^{N} J_{i\pi} |\phi\rangle = NJ|\phi\rangle.
$$

(10)

The third equality in the above formula is due to the fact that in Platonic networks with $N = 4, 6, 8, 12$, if we choose an arbitrary site $m$, all network’s sites are the nearest neighbors of site $m$ or its point symmetry mirroring site $\bar{m}$, and their coupling rates are equal to $m$ or $\bar{m}$. Without the Platonic symmetries, the states of the form $|\phi^{m}\rangle + |\phi^{\bar{m}}\rangle$ could not be an eigenstate of the general Hamiltonian of equation (1). The state $|\phi\rangle$ is also an eigenstate of a FCN for which $J_{ij} = J[14]$. It can be expected from the eigenstate $|\phi\rangle = |\phi^{m}\rangle + |\phi^{\bar{m}}\rangle$ of Platonic networks that if a noisy Platonic network at the steady state would be partially formed according to this eigenstate, only the nearest neighboring sites of each arbitrary site $m$ would interact with each other, and the total
quantum energy is in a probabilistic manner in the state \( |\phi^m\rangle \) or \( |\phi^n\rangle \). The other fact by which we support the nearest neighbor assumption is that we have previously shown (figure 2 of [15]) that at the steady state and the noiseless case, the dynamics of energy accumulation at sink site is similar when all sites or nearest sites are interacting. In the current work, when adding environmental noises to the system, some energy would be extracted homogeneously from each site due to the dissipation noise, and the energy levels of all qubits would be randomly disturbed with an equal rate as a result of dephasing noise. So it is expected that the overall behaviour of the energy accumulation at the sink site would be the same that of noiseless networks, i.e. identical behaviour when all sites or nearest sites are interacting. Consequently, in some parameter regimes (e.g. \( t \to \infty \)), we can assume that the nearest neighbor interaction would be the intrinsic characteristic of the noisy and noiseless platonic networks. It is possible to numerically show the parameter regimes for which the sink dynamics of nearest and non-nearest neighbor networks are similar. If considering the nearest neighbor interaction, the rest of the proofs in this work should be amended accordingly and will have similar results.

Now we continue developing equation (7) using the reduced networks assumption. The equations of motion of the collective variables of equation (8) would be as following:

\[
\begin{align*}
\dot{R}_i &= -i\Lambda_i + iN_f R_i - 2(\Gamma_{diss} + \gamma)R_i - \Gamma \rho_N + 2\gamma \rho_{\text{dis}}, \\
\dot{R}_N &= -i\Lambda_N + iN_f R_N - (2\Gamma_{diss} + 2\gamma + \Gamma)R_N + (2\gamma - \Gamma)\rho_{NN}, \\
\dot{\Lambda}_i &= -(2\Gamma_{diss} + \gamma)\Lambda_i + \Gamma(R_N + \dot{R}_N) + 2\gamma \rho_{N,0}(\rho),
\end{align*}
\]

where

\[
\rho_{N,0}(\rho) = \sum_{i=f_{\text{seq}}(0)} \rho_i = \text{Tr}(\rho).
\]

This is due to the fact that the \( i \) indices corresponds to super-sites. Now from the rule of conservation of population we have:

\[
1 = \text{Tr}(\rho) + \rho_{00} + \rho_{\text{target}}.
\]

It indicates that the initial population would oscillate among all network sites, and partially accumulated in the surrounding environment and the target site, through the dissipation noise rates of \( \Gamma_{diss}, \gamma \) and \( \Gamma \), respectively. Considering \( R_N = x + iy \), equation (11) line 2 yields two first order differential equations for \( x \) and \( y \) which besides equations (7) lines 4–6, equation (11) line 3 and equation (13), form a close set of differential equations for variables \( x, y, \Lambda_N, \rho_{NN}, \rho_{00}, \rho_{\text{target}} \) as follows:

\[
\begin{align*}
\dot{\Lambda}_N &= -2(\Gamma_{diss} + \gamma)\Lambda_N - 2\gamma(1 - \rho_{00} - \rho_{\text{target}}), \\
\dot{x} &= -(2\Gamma_{diss} + 2\gamma + \Gamma)x + (2\gamma - \Gamma)\rho_{NN} - jN_f y, \\
\dot{y} &= -(2\Gamma_{diss} + 2\gamma + \Gamma)y + jN_f x - j\Lambda_N, \\
\dot{\rho}_{NN} &= -2(\Gamma_{diss} + \Gamma)\rho_{NN} - 2\gamma y, \\
\dot{\rho}_{00} &= 2\Gamma_{diss}(1 - \rho_{00} - \rho_{\text{target}}), \\
\dot{\rho}_{\text{target}} &= 2\Gamma \rho_{NN}.
\end{align*}
\]

According to equation (3) and figures 2–4, the initial conditions for different Platonic networks are assumed as:

\[
\begin{align*}
N &= 6, 8: \\
\rho_i(0) &\equiv \rho_{ij}(0) = 1, i, j = 1 \\
N &= 12: \\
\rho_i(0) &\equiv \rho_{ij}(0) = \rho_i(0) = \rho_q(0) = 1/4, i \neq j = 1, 2, 3, 4 \\
N &= 20: \\
\rho_i(0) &\equiv \rho_{ij}(0) = \rho_i(0) = 1/3, \rho_i(0) = 0, i \neq j = 1, 5, 9
\end{align*}
\]

The above initial conditions yield:

\[
\begin{align*}
R_N(0) &= \sum_{i \in f_{\text{seq}}(N)} \rho_{Ni} = 0, x(0) = y(0) = 0, \\
\Lambda_N(0) &= \sum_{i \in f_{\text{seq}}(N)} \rho_i(0) = R_N(0) + R_{i=[1\ldots4,N]}(0) + R_{i=[1,5,9,N]}(0) = 1.
\end{align*}
\]
Now by applying the Laplace transform to equations (14) i.e. \( t \to 1/s, \alpha(t) \to (\alpha, \dot{\alpha} = \alpha(0)) \), we obtain:

\[
\begin{align*}
(s + 2\Gamma_{\text{diss}} + 2\gamma)\bar{x}_N + 2\Gamma x + 2\gamma\rho_{\text{target}} + 2\gamma\rho_0 - 2\gamma/s - 1 &= 0, \\
(s + 2\Gamma_{\text{diss}} + 2\gamma + \Gamma)\bar{y} + (\Gamma - 2\gamma)\bar{\rho}_{\text{NN}} + JN\bar{x} &= 0, \\
(s + 2\Gamma_{\text{diss}} + 2\gamma + \Gamma)\bar{y} + (\Gamma - 2\gamma)\bar{\rho}_{\text{NN}} + JN\bar{x} &= 0, \\
(s + 2\Gamma_{\text{diss}})\bar{\rho}_0 + 2\Gamma_{\text{diss}}\rho_{\text{target}} - 2\Gamma_{\text{diss}}/s &= 0, \\
\dot{\rho}_{\text{target}} - 2\Gamma\bar{\rho}_{\text{NN}} &= 0.
\end{align*}
\]

Solving the complete set of equations (17), the target sink population will be found for Platonic networks in the presence of homogeneous local noises as following:

\[
\bar{\rho}_{\text{target}} = 4\Gamma J^2\frac{(\Gamma_b + s)(s + \Gamma)}{s\Delta(s)}
\]

\[
\Delta(s) = 8\Gamma J^2\gamma(\Gamma_b + s) + (s + 2\Gamma_{\text{diss}})
\]

\[
((\Gamma_A + s)(\Gamma_C + s)(\Gamma_b + s)^2
\]

\[
- 4\Gamma J^2(\Gamma - 2\gamma)
\]

\[
- 2\Gamma J^2N\gamma(\Gamma_A + s)(\Gamma - 2\gamma)
\]

\[
+ 2\Gamma J^2N\gamma(\Gamma_C + s)
\]

\[
+ J^2N\gamma(\Gamma_A + s)(\Gamma_C + s).
\]

where

\[
\begin{align*}
\Gamma_A &= 2\gamma + 2\Gamma_{\text{diss}}, \\
\Gamma_b &= 2\gamma + 2\Gamma_{\text{diss}} + \Gamma, \\
\Gamma_C &= 2\Gamma_{\text{diss}} + 2\Gamma.
\end{align*}
\]

This expression is equivalent to that of an FCN network, i.e. equations (A25), (A26) of [14], where the Platonic coordinate number \((N_c)\) is equivalent to the total number of sites. The final target population of the considered Platonic networks at the steady state is as following:

\[
\rho_{\text{target}}(t \to \infty) = \lim_{s \to 0} [s\bar{\rho}_{\text{target}}] = 4\Gamma J^2\frac{\Gamma_b\Gamma_A}{\Delta(0)}
\]

It can be seen that in the noiseless environment \((\gamma, \Gamma \to 0)\), the steady state target population is the same as the previous expression found in [15] i.e. \(1/(N_c - 1)\), which can be here achieved by first tending the local dephasing rate to zero. If first tending the local dissipation rate to zero, the target population tends to 1. This is due to the fact that the local dissipation noise would only discharge the excitation from each site to the environment and not to the other sites, however the dephasing noise provides new paths of energy transport within network sites, leading to discharge of all excitation to the target site through the \(N_c\) site.

From the numerical investigations we know that the maximum value of the target population in the presence of noises is one. To find the network-environment parameters corresponding the maximum excitation transport, we equate the target population of equation (20) to one, and find a relation among all network and environment parameters. The resulting equation can be used to find the optimal design variables. For example, the coupling rate \(J\) could be found in terms of other parameters. Figures 5(a), (b) show the relation between \(J\), the coupling rate between the nearest neighbors in Platonic networks, and the Markovian dephasing rate \(\gamma\), for different values of dissipation rates from each site to the environment \((\Gamma_{\text{diss}})\), and also the different values of dissipation rates from the \(N_c\) site to the sink site \((\Gamma)\). The network of consideration for both parts (a) and (b) is a cubic lattice with \(N = 8\) main sites, and the constant parameters are \(\Gamma_{\text{diss}} = 10\) and \(\Gamma = 10\), respectively. It can be seen in figure 5 that for the fixed chosen parameters, by increasing the dephasing rate, the coupling strength should be increased, so that the disturbing effect of dephasing noise would be compensated on energy transport towards the sink.

It can be also seen from figure 5(a) that for a fixed dephasing noise rate \(\gamma\) and the fixed chosen dissipation noise rate \(\Gamma_{\text{diss}} = 10\), to maintain the maximum transport, the nearest neighbors sites couplings should be increased, by increasing the dissipation rate to the target sink \((\Gamma)\). In other words, in the fixed environment with the same dephasing and dissipation noise rates, the higher sites’ couplings demands faster noise rates to the sink site to obtain the optimal design corresponding the maximum energy transfer. To understand this behaviour, note that the higher sites’ coupling rate results in stronger and more energy bouncing among the sites, that demands higher coupling rate towards the sink. It should be also taken into account that the relation among the parameters for the maximum transport is nonlinear.

In such Platonic noisy networks, since the nonzero dissipation rate of \(\Gamma_{\text{diss}} = 10\) will irreversibly transfer some energy to the environment, to reach the full energy transport, the coupling rate and the sink dissipation
rate should be high or fast enough to transfer all energy before any fraction of that would be dissipated towards the surrounding environment. Part of this process could be supported by dephasing-noise-assisted-transport [11, 12]. This fact can be seen in figure 5(b). It shows the relation of network-environment parameters for the fixed amount of $\Gamma = 10$. It can be seen that for a strong dissipation rate ($\Gamma_{\text{diss}} = 100$), the sites’ coupling rate should be increased to be able to transfer the energy fast enough before getting dissipated to the environment.

Figure 6 shows the optimal design graphs of Platonic networks with different number of sites. The constant parameters are chosen as $\Gamma_{\text{diss}} = \Gamma = 10$. It can be seen that for a fixed dephasing rate, by increasing $N_c$ of each pair of sites is less. This indicates that more sites with less coupling strength are equivalent to less number of sites with higher coupling rates.

Figure 5. The relation of network and environment variables in a noisy cubic Platonic network with optimal transport to the target site at steady state. Graph (a) shows the relation of coupling rate $J$ and dephasing noise $\gamma$ for different amounts of dissipation rate to the sink ($\Gamma$), while dissipation rate to the local environments are fixed at $\Gamma_{\text{diss}} = 10$. Graph (b) shows the same relations for different values of $\Gamma_{\text{diss}}$ and fixed amount of $\Gamma = 10$. It could be indicated from the graphs that the coupling strength should be increases by increasing other parameters to maintain the full transport. The values of parameters could be used for optimal design of Platonic networks.

Figure 6. This graph shows the relation of sites couplings and the environmental dephasing rate for different Platonic networks with optimal transport at steady state. It could be seen that for networks with more $N_c$ or number of nearest neighbours, the coupling rate $J$ of each pair of sites is less. This indicates that more sites with less coupling strength are equivalent to less number of sites with higher coupling rates.
Now we briefly investigate the effect of random geometrical errors of a cube Platonic network on the transport towards sink. We provide such geo-robustness test for only two sample parameter sets of ($\Gamma_{\text{target}} = 100; \gamma = 10; \Gamma_{\text{dis}} = 0.1$) and ($\Gamma_{\text{target}} = 100; \gamma = 10; \Gamma_{\text{dis}} = 0.01$) for which we have near full transport to the sink site, i.e. $p_{\text{target}} = 0.999$ and 0.9999. For each parameter set, we assign random errors of maximum 0.5 to all three coordinates of each site, where the distance of nearest neighbor qubits are unit. Then we numerically find the target populations of 200 different Platonic networks with randomly dislocated qubits. We finds that the mean values of these 200 target populations for both sets remains the same as non-dislocated networks. To design a Platonic network, one might choose one parameter set from figures 5 or 6, check the robustness of transport as above, and try to engineer experimental parameters accordingly.

In [15], we studied the effect of geometrical variations on the robustness of transport in one-dimensional four qubit networks with a different approach. In future, we can more rigorously study the geometrical robustness in three-dimensional Platonic networks in different parameter regimes.

**Conclusion**

Energy transport is an inevitable phenomenon in many atomic-scale networks. In this work, we numerically studied the characteristics of energy dynamics in Platonic quantum networks consists of 4, 6, 8, 12, and 20 qubits, located on vertices of centrally symmetric three-dimensional Platonic geometries. A target site was assumed to be dissipatively connected to one of the qubits. Due to the opposite or same oscillation patterns of qubits populations, we made an assumption of reducing the number of qubits of each network to an effective value which was equal to the number of one group of nearest neighbor sites within each network. We found the analytical expression for the target site population in the presence of environmental Markovian dephasing and dissipation noises. In addition, we investigated the optimal design characteristics of Platonic networks for maximum energy transport from the first site towards the target site. We plotted the relation between the coupling strength and the dephasing noise rate corresponding the maximum transport. The optimal designs of Platonic quantum networks could have several applications like switches or multiplexers in quantum devices. In the future, we hope the energy transport in three-dimensional Platonic devices be further analysed and their physical implementations be investigated.

**Data availability statement**

All data that support the findings of this study are included within the article.

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