On the use of tent spaces for solving PDEs: A proof of the Koch-Tataru theorem

by

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Abstract

In these notes we will present (a part of) the parabolic tent spaces theory and then apply it in solving some PDE’s originated from the fluid mechanics. In more details, to our most interest are the incompressible homogeneous Navier–Stokes equations. These equations have been investigated mathematically for almost one century. Yet, the question of proving well-posedness (i.e. existence, uniqueness and regularity of solutions) lacks satisfactory answer.

A large part of the known positive results in connection with Navier–Stokes equations are those in which the initial data $u_0$ is supposed to have a small norm in some critical or scaling invariant functional space. All those spaces are embedded in the homogeneous Besov space $\dot{B}_{\infty,\infty}^{-1}$. A breakthrough was made in the paper [16] by Koch and Tataru, where the authors showed the existence and the uniqueness of solutions to the Navier–Stokes system in case when the norm $\|u_0\|_{\text{BMO}^{-1}}$ is small enough. The principal goal of these notes is to present in detail a new proof of the theorem by Koch and Tataru on the Navier–Stokes system using the tent spaces theory. We do not mean new in the sense simpler but we hope that after having read these notes, the reader will be convinced that the theory of tent spaces is highly likely to be useful in the study of other equations in fluid mechanics.

These notes are mainly based on the content of the article [1] by P. Auscher and D. Frey. However, in [1] the authors deal with a slightly more general system of parabolic equations of Navier–Stokes type. Here we have chosen to write down a self-contained text treating only the relatively easier case of the classical incompressible homogeneous Navier–Stokes equations.

**Keywords:** Incompressible homogeneous Navier–Stokes equations, Tent spaces, Hardy spaces.
Chapter 1

Introduction

1.1 Some history

Let us more rigorously formulate the main result of these notes. To this end, we first introduce a number of definitions.

Definition 1.1.1. Let \( u \) be a vector-field with components in the Schwartz class \( S(\mathbb{R}^n) \). Leray’s projector \( \mathbb{P}(u) \) is defined by \( \mathbb{P}(u) := u - \nabla \Delta^{-1}(\nabla \cdot u) \).

In other words, this projector is the operator with the Fourier multiplier matrix symbol
\[
M_{\mathbb{P}}(\xi) := \left( \delta_{i,j} - \frac{\xi_i \xi_j}{|\xi|^2} \right)_{1 \leq i,j \leq n}.
\]

Remark 1.1.1. There is an equivalent definition of the operator \( \mathbb{P} \) which is:
\[
\mathbb{P}(u) := u + (\mathcal{R} \otimes \mathcal{R})u \quad \text{with} \quad \mathcal{R} := (\mathcal{R}_1, \ldots, \mathcal{R}_n),
\]
where \( \mathcal{R}_i \) for \( i = 1, \ldots, n \) stands for the Riesz transform in \( \mathbb{R}^n \).

Remark 1.1.2. Note that if \( u \) is a vector-field, then \( \text{div}(\mathbb{P}(u)) = 0 \).

We are ready to introduce the incompressible homogeneous Navier–Stokes equations:

\[
\begin{cases}
\partial_t u + (u \cdot \nabla)u - \Delta u + \nabla p = 0, \\
\text{div} u = 0, \\
u(0, \cdot) = u_0(\cdot).
\end{cases}
\]

(1.1)

Here is the physical meaning of the terms above: \( p : \mathbb{R}^n \to \mathbb{R} \) is the pressure of the “ideal” fluid, \( u : \mathbb{R}^n \to \mathbb{R}^n \) is its velocity vector-field and \( u_0 \) is the initial value of the velocity. The first line in (1.1) is called momentum equation. The equation \( \text{div} u = 0 \) means that the fluid is incompressible.

We shall consider in what follows the following Duhamel’s formulation (also called mild formulation) of the Navier–Stokes equations:

\[
u(t, \cdot) = e^{t\Delta}u_0(\cdot) - \int_0^t e^{(t-s)\Delta}\mathbb{P}\text{div}(u(s, \cdot) \otimes u(s, \cdot))\, ds.
\]

(1.2)
We remind the reader that for vector-fields \( u \) and \( v \) the notation \( u \otimes v \) stands for the matrix valued function, obtained by multiplying each coordinate function of \( u \) by each coordinate function of \( v \), namely \((u \otimes v)_{i,j}(x) := u_i(x)v_j(x)\). For a matrix valued function \( A(x) = (a_1(x), \ldots, a_n(x))^T \) (where \( a_j(x) \) is the \( j \)-th row vector of the matrix \( A \)), its divergence is defined by \( \text{div}(A) := (\text{div}a_1, \ldots, \text{div}a_n) \).

Let us stress that the equation (1.2) is equivalent to the system (1.1) under a very mild assumptions on \( u_0 \). This is proved for instance in [17]. See also [10]. The solutions to the system (1.2) are called mild solutions.

In connection with the system (1.2), we define the bilinear form \( B \) as follows:

\[
B(u, v)(t, \cdot) = \int_0^t e^{(t-s)\Delta} \text{div}(u(s, \cdot) \otimes v(s, \cdot)) \, ds. \tag{1.3}
\]

We are now in position to formulate the Koch and Tataru theorem.

**Theorem 1.1.1.** If the norm \( \|u_0\|_{\text{BMO}^{-1}} \) is small enough, then Equations (1.1) admit a unique global solution in a ball of the functional space \( X \) (see Definition 1.2.2).

We shall give a rigorous definition of the space \( \text{BMO}^{-1} \) later on. For the time being, the reader should think about this space as the set of distributions \( u_0 \) such that \( u_0 = \text{div}(v_0) \) for some vector-field \( v_0 \) with components in the space \( \text{BMO}(\mathbb{R}^n) \). By \( \text{BMO}(\mathbb{R}^n) \), we mean the space of functions whose mean oscillation is bounded, namely \( v \in \text{BMO}(\mathbb{R}^n) \) if and only if

\[
\sup_B \frac{1}{|B|} \int_B |v(x) - v_B| \, dx < \infty \quad \text{with} \quad v_B := \frac{1}{|B|} \int_B v,
\]

where \( B \) is a (Euclidean) open ball, \( |B| \) is its Lebesgue measure and the supremum is taken over all balls \( B \subset \mathbb{R}^n \). The space \( \text{BMO}(\mathbb{R}^n) \) is very important in the modern harmonic and Fourier analysis, see for instance the books [18] and [14] and the article [7].

Several remarks are in order. First, note that the system (1.1) is scale invariant under the following parabolic scaling:

\[
(u, p)(t, x) \sim (\lambda u, \lambda^3 p)(\lambda^2 t, \lambda x) \quad \text{and} \quad u_0(x) \sim \lambda u_0(\lambda x), \quad \lambda > 0.
\]

Consequently, one can expect optimal functional spaces for solving this system by means of the Picard fixed point theorem to have the above invariance, for all \( \lambda > 0 \). In particular, ‘critical’ spaces for initial data have homogeneity \(-1\). Second, the space \( \text{BMO}^{-1} \) is contained in the homogeneous Besov space \( \dot{B}^{-1}_{\infty,\infty}_\infty \), both spaces are critical for the Navier–Stokes system, and the latter one is the largest critical space. However, it was shown by Bourgain and Pavlovic, see [5] that the equations (1.1) are ill posed in \( \dot{B}^{-1}_{\infty,\infty}_\infty \), and this is our third remark here. Finally, we would like to cite some previous results in the direction of the well-posedness of the system (1.1). Cannone (see [6]) considered \( u_0 \in \dot{B}^{-1+\alpha}_{p, \infty}(\mathbb{R}^n) \). Earlier, Fujita and Kato in [11] proved well-posedness (in the three-dimensional case) for \( u_0 \) in the Sobolev space \( \dot{H}^{1/2}(\mathbb{R}^3) \), and Kato in the paper [15] examined the case of small initial data in \( L^n(\mathbb{R}^n) \) (see also [13]).

\(^1\)By \( \text{BMO}^{-1} \) here and everywhere after in this text we mean the homogeneous version of this space.
Taking into account the mild formulation (1.2), it does not come up as a surprise that the proof of well-posedness is based on a fixed point argument. Namely, this will be the Picard contraction principle applied in a very special functional context. In the following section, we introduce the functional spaces in which we will solve the Navier-Stokes equations and some related definitions and results. These spaces will allow us to formulate and prove a theorem which is the core of our proof of the Koch and Tataru result.

1.2 Main definitions and auxiliary results

Here we collect the most important definitions and results that we shall need afterwards. We begin with the tent spaces.

**Definition 1.2.1.** Let $\mathbb{R}_{+}^{n+1} := \{(t, x), t > 0 \text{ and } x \in \mathbb{R}^n\}$. We shall say that a measurable function $\alpha : \mathbb{R}_{+}^{n+1} \to \mathbb{C}^n \otimes \mathbb{C}^n$ belongs to the $(\infty, p)$-parabolic tent space $T^{\infty,p}(\mathbb{R}_{+}^{n+1}, \mathbb{C}^n \otimes \mathbb{C}^n)$ where $1 \leq p < \infty$ if

$$
\left\| \alpha \right\|_{T^{\infty,p}(\mathbb{R}_{+}^{n+1}, \mathbb{C}^n \otimes \mathbb{C}^n)} := \sup_{x_0 \in \mathbb{R}^n, R > 0} \frac{1}{B(x_0, R)} \int_{B(x_0, R) \times [0, R^2]} |\alpha(t, x)|^p \, dx \, dt < \infty.
$$

By $B(x, R)$ we designate the open Euclidean ball with center $x$ and radius $R$.

**Remark 1.2.1.** In analogy with this definition one can define tent spaces with different target space. Everywhere in this text, we shall simply denote by $T^{\infty,p}$ the space $T^{\infty,p}(\mathbb{R}_{+}^{n+1}, \mathbb{C}^n \otimes \mathbb{C}^n)$, unless otherwise specified.

Tent spaces were first introduced by Coifman, Meyer and Stein in the paper [8], in the elliptic setting.

Define further the so-called “path” spaces that will play an important role in what follows.

**Definition 1.2.2.** We shall say that a measurable function $u : \mathbb{R}_{+}^{n+1} \to \mathbb{C}^n \otimes \mathbb{C}^n$ belongs to the space $X$ if

$$
\|u\|_X := \|(t, x) \mapsto t^{1/2}u(t, x)\|_{L^\infty(\mathbb{R}_{+}^{n+1}, \mathbb{C}^n \otimes \mathbb{C}^n)} + \|u\|_{T^{\infty,2}(\mathbb{R}_{+}^{n+1}, \mathbb{C}^n \otimes \mathbb{C}^n)} < \infty.
$$

Note that $X$ is a Banach space.

**Definition 1.2.3.** For $t > 0$, define the heat kernel $\Phi_t : \mathbb{R}^n \to \mathbb{R}$, by the following formula $\Phi_t(x) := (4\pi t)^{-n/2} e^{-|x|^2/4t}$. If we denote by $\Delta$ the Laplace operator in $\mathbb{R}^n$, the function given by $U(t, \cdot) = e^{t\Delta}u_0 := \Phi_t * u_0$ is a solution of the heat equation

$$
\partial_t U = \Delta U, \quad U(0, \cdot) = u_0.
$$

By definition, we say that $u_0 \in \text{BMO}^{-1}$ if $U \in T^{\infty,2}$.

**Remark 1.2.2.** Note that the just given definition of the space BMO$^{-1}$ coincides with the one discussed after the formulation of Theorem 1.1.1. The proof of this fact is contained in the article [16] and is based on the “caloric extension” characterization of the space BMO$(\mathbb{R}^n)$.
1.2 Main definitions and auxiliary results

**Definition 1.2.4.** We shall say that a measurable function \( \alpha : \mathbb{R}^{n+1}_+ \to \mathbb{C}^n \otimes \mathbb{C}^n \) belongs to the space \( Y \) if

\[
\| \alpha \|_Y := \|(t, x) \mapsto t \alpha(t, x)\|_{L^\infty_t(\mathbb{R}^{n+1}_+, \mathbb{C}^n \otimes \mathbb{C}^n)} + \| \alpha \|_{T^\infty,1(\mathbb{R}^{n+1}_+, \mathbb{C}^n \otimes \mathbb{C}^n)} < \infty.
\]

Note that \( Y \) is a Banach space.

One of the possible approaches to use while trying to solve the Navier–Stokes system is “separation of time and space”. This is called the maximal regularity setting.

**Definition 1.2.5.** The operator \( M^+ \) that is defined below is called the maximal regularity operator:

\[
M^+ f(t, \cdot) := \int_0^t e^{(t-\tau)A} \Delta f(\tau, \cdot) \, d\tau.
\]

This is well-defined for \( f \in D = L^2(\mathbb{R}^{n+1}_+) \cap L^1(\mathbb{R}_+, H^2(\mathbb{R}^n)) \).

Next, we state the de Simon theorem, which says that the maximal regularity operator is bounded on \( L^2 \).

**Theorem 1.2.1.** There exists a constant \( C \) depending only on \( n \) such that for all \( f \in D \) holds

\[
\| M^+ f \|_{L^2(\mathbb{R}^{n+1}_+)} \leq C \| f \|_{L^2(\mathbb{R}^{n+1}_+)}.
\]

For the reader’s convenience, we state a slightly more general result, that is taken from the paper [9].

**Theorem 1.2.2.** Let \( H \) be a Hilbert space and let \( A \) be an operator such that the operator \((-A)\) generates an analytic semigroup bounded by the constant \( m_0 \). Consider the maximal regularity operator associated with \( A \), namely

\[
M_A f(t, \cdot) := \int_0^t e^{-(t-\tau)A} Af(\tau, \cdot) \, d\tau, \quad t \in \mathbb{R}, \quad f \in L^2(\mathbb{R}_+, H) \cap L^1(\mathbb{R}_+, D(A)).
\]

For all \( f \in L^2(\mathbb{R}_+, H) \cap L^1(\mathbb{R}_+, D(A)) \), there holds

\[
\| M_A f \|_{L^2(\mathbb{R}_+, H)} \leq (m_0 + 1) \| f \|_{L^2(\mathbb{R}_+, H)}.
\]

**Proof.** Let us extend \( f \) by 0 on \( \mathbb{R}_- \) and introduce

\[
u(t) := \int_0^t e^{-(t-\tau)A} f(\tau, \cdot) \, d\tau.
\]

For \( z \in \mathbb{C} \) such that \( \text{Im}(z) < 0 \), the Laplace transform of \( \nu \) reads:

\[
\tilde{\nu}(z) := \int_{\mathbb{R}} e^{-izt} \nu(t) \, dt.
\]

Note that if \( z = \xi + i\eta \) and \( \eta < 0 \) then

\[
\tilde{\nu}(z) := \int_{\mathbb{R}_-} \int_0^t e^{-izt} e^{-(t-\tau)A} f(\tau) \, d\tau \, dt = \int_{\mathbb{R}_-} \int_{\tau}^{+\infty} e^{(-iz-\eta)A} f(\tau) \, dt \, d\tau,
\]
where in the second equality we performed a change of variables and used the fact that $f(\tau) = 0$ when $\tau < 0$. Since $\eta < 0$, the inner integral of the exponential function with respect to the variable $t$ in the right-hand side of the last line above converges. In fact, this integral can be calculated explicitly, giving:

$$\hat{u}(z) = \int_{\mathbb{R}} [-iz - A]^{-1}(-e^{-iz\tau})f(\tau)\,d\tau = [iz + A]^{-1}\hat{f}(z).$$

From the relation $z = \xi + i\eta$, we infer the following formula:

$$iz\hat{u}(z) = iz[iz + A]^{-1}F(e^{\eta(\cdot)}f(\cdot))(\xi),$$

(1.4)

where $F$ stands for the usual Fourier transform on $\mathbb{R}$.

Since $-A$ generates an analytic semigroup bounded by $m_0$, we have that

$$\|iz[iz + A]^{-1}\| \leq m_0,$$

(1.5)

once again for all $z = \xi + i\eta$ satisfying $\eta < 0$.

Observe that if $z = \xi + i\eta$, then (1.4) yields

$$iz\hat{u}(z) = \int_{\mathbb{R}} e^{-izt}u'(t)\,dt = \int_{\mathbb{R}} e^{-i\xi t}e^{\eta t}u'(t)\,dt = F(e^{\eta(\cdot)}u'(\cdot))(\xi).$$

(1.6)

The last formula follows from one of the basic properties of the Laplace transform (Laplace transform of derivative).

Fix $\eta < 0$. Since $f \in L^2$, the Fourier-Plancherel theorem and lines (1.4), (1.5) and (1.6) yield

$$2\pi\|e^{\eta(\cdot)}u'(\cdot)\|_{L^2} \leq \|F(e^{\eta(\cdot)}u'(\cdot))\|_{L^2} = \|iz\hat{u}\|_{L^2} \leq m_0\|F(e^{\eta(\cdot)}f(\cdot))\|_{L^2} = 2\pi m_0\|e^{\eta(\cdot)}f(\cdot)\|_{L^2}.$$

To conclude, it suffices to let $\eta$ tend to 0 and to observe that $M_Af = -u' + f$.

Further properties of maximal regularity operators acting on tent spaces can be found for instance in the paper [4].

In the end of this chapter we recall some estimates that concern the Oseen kernel (i.e. the kernel of the operator $e^{t\Delta\mathcal{P}}$) and its derivatives. The reason why we will need these estimates is that they imply the so-called off-diagonal estimates and the latter are going to be very important for our goals.

**Theorem 1.2.3.** Let $\sigma_t$ denote the kernel of the operator $e^{t\Delta\mathcal{P}}$. For all $\beta \in \mathbb{N}^n, x \in \mathbb{R}^n$ and $t > 0$, there holds

$$|t^{\beta/2}\partial^\beta_x \sigma_t(x)| \leq Ct^{-n/2} \left(1 + \frac{|x|}{t^{1/2}}\right)^{-n-|\beta|},$$

(1.7)

where $C$ is a constant depending on $n$ and $\beta$ only.

The proof of this theorem can be found for example in [17, Prop. 11.1].
Finally, let us present the mentioned above off-diagonal estimates.

**Definition 1.2.6.** A family of bounded linear operators \((T_t)_{t>0}\) on \(L^2(\mathbb{R}^n)\) is said to satisfy off-diagonal estimates of order \(M\), with homogeneity \(m\), if there exists a constant \(C\) such that for all Borel sets \(E, F \subset \mathbb{R}^n\), all \(t > 0\), and all \(f \in L^2(\mathbb{R}^n)\), there holds:

\[
\|1_E T_t 1_F f\|_{L^2} \leq C \left(1 + \frac{\text{dist}(E, F)^m}{t}\right)^{-M} \|1_F f\|_{L^2}.
\]  

(1.8)

It is well known that for many differential operators \(L\) of order \(m\) (such as, for \(m = 2\), divergence form elliptic operators with bounded measurable complex coefficients), the family \((t L e^{-tL})_{t \geq 0}\) satisfies off-diagonal estimates of any order, with homogeneity \(m\). This is proved for instance in [2].

Throughout the rest of the text \(C\) denotes a “harmless” constant. The sign \(\lesssim\) indicates that the left-hand part of an inequality is less than the right-hand part multiplied by a constant \(C\) as above.
Chapter 2

Proof of the Koch and Tataru theorem via tent spaces

2.1 Some preliminary observations

First of all, let us state the principal result, from which Koch and Tataru’s theorem will follow easily via Picard’s contraction principle.

**Theorem 2.1.1.** Let X be the Banach space from Definition 1.2.2. The bilinear operator $B : X \times X \to X$ is bounded.

**Proof.** The first observation is that instead of working with the bilinear form $B$, we can consider the linear operator $A : Y \to X$ (where $Y$ is the Banach space introduced in Definition 1.2.4) defined by

$$A(\alpha)(t, \cdot) := \int_0^t e^{(t-s)\Delta} \text{div} \alpha(s, \cdot) ds.$$ 

Indeed, we have $A(\alpha) := B(u, v)$ for $\alpha = u \otimes v$ and it is clear that $(u, v) \mapsto u \otimes v$ maps $X \times X$ to $Y$ since, by Cauchy-Schwarz inequality,

$$\|u \otimes v\|_Y = \|t(u \otimes v)\|_{L^\infty_{t,x}} + \|u \otimes v\|_{T^\infty_{1,1}} \lesssim \|t^{1/2} u\|_{L^\infty_{t,x}} \|t^{1/2} v\|_{L^\infty_{t,x}} + \|u\|_{T^\infty_{2,2}} \|v\|_{T^\infty_{2,2}} \lesssim \|u\|_X \|v\|_X.$$

Therefore, Theorem 2.1.1 just stems from the following pointwise inequality:

$$|A(\alpha)(t, x)| \lesssim t^{-1/2} \|\alpha\|_Y \quad \text{for all } (t, x) \in \mathbb{R}^{n+1}_+, \quad (2.1)$$

and the tent space bound:

$$\|A(\alpha)\|_{T^\infty_{2,2}} \leq \|\alpha\|_{T^\infty_{2,1}} + \|t^{1/2} \alpha\|_{T^\infty_{2,2}}, \quad (2.2)$$

since, obviously,

$$\|t^{1/2} \alpha\|_{T^\infty_{2,2}} \leq \|t \alpha\|^{1/2}_{L^\infty_{t,x}} \|\alpha\|^{1/2}_{T^\infty_{2,1}} \leq \|\alpha\|_Y.$$

We shall establish the estimate (2.1) (which turns out to be easier) in this section and the estimate (2.2) will be proved in the sections 2.2, 2.3 and 2.4. Remark that the quantity $\|t^{1/2} \alpha\|_{T^\infty_{2,2}}$ is not used in [16].
In order to prove estimate (2.1), we denote by \( k_\tau \) the kernel of the operator \( e^{\tau \Delta \mathbb{P} \text{div}} \) and split the integrals in the definition of the operator \( A \) as follows:

\[
|A(\alpha)(t,x)| \leq \left| \int_0^t \int_{\mathbb{R}^n \setminus B(x,\sqrt{t})} k_{t-s}(x,y)\alpha(s,y) \, dy \, ds \right| \\
+ \left| \int_{t/2}^t \int_{B(x,\sqrt{t})} k_{t-s}(x,y)\alpha(s,y) \, dy \, ds \right| \\
+ \left| \int_0^{t/2} \int_{B(x,\sqrt{t})} k_{t-s}(x,y)\alpha(s,y) \, dy \, ds \right|. \tag{2.3}
\]

Let us denote by \( I_1(t,x) \), \( I_2(t,x) \) and \( I_3(t,x) \) the three summands above. As a consequence of (1.7), we have

\[
|k_{t-s}(x,y)| \lesssim (\sqrt{t-s} + |x-y|)^{-n-1}
\]

and we shall use this estimate differently in each case.

In order to estimate \( I_1(t,x) \), consider for all \( i \in \mathbb{Z}^n \), the points \( x_i := x + i \sqrt{t} \) and the balls \( B(x_i, \sqrt{nt}) \). Note that these balls cover \( \mathbb{R}^n \). Using \( |k_{t-s}(x,y)| \lesssim |x-y|^{-n-1} \leq \min(|x-y|^{-n-1}, (\sqrt{t})^{-n-1}) \) for \( y \in \mathbb{R}^n \setminus B(x, \sqrt{t}) \) and forgetting about the range of integration, it follows that

\[
I_1(t,x) \leq C \int_0^t \int_{\mathbb{R}^n} \min(|x-y|^{-1}, (\sqrt{t})^{-1}) |\alpha(s,y)| \, dy \, ds \\
\lesssim \sum_{i \in \mathbb{Z}^n} \frac{1}{|B(x_i, \sqrt{nt})|} \int_0^t \int_{B(x_i, \sqrt{nt})} |\alpha(s,y)| \, dy \, ds \, (nt)^{n/2} (\max(|i| - \sqrt{n}, 1)t^{1/2})^{-n-1} \\
\lesssim t^{-1/2} ||\alpha||_{T^{\infty,1}}. \tag{2.4}
\]

In order to bound \( I_3(t,x) \), one can further observe that \( |k_{t-s}(x,y)| \lesssim |t|^{-(n+1)/2} \) once \( s \in (0, t/2) \) and \( |x-y| \leq \sqrt{t} \). Hence,

\[
I_3(t,x) \lesssim \frac{t^{-1/2}}{t^{n/2}} \int_0^t \int_{B(x,\sqrt{t})} |\alpha(s,y)| \, dy \, ds \\
\lesssim t^{-1/2} ||\alpha||_{T^{\infty,1}}. \tag{2.5}
\]

Finally, since we also have \( |k_{t-s}(x,y)| \lesssim |x-y|^{-n+1/2}(t-s)^{-3/4} \), one can bound \( I_2(t,x) \) as follows:

\[
I_2(t,x) \lesssim \left( \int_{t/2}^t (t-s)^{-3/4} ds \right) \left( \int_{B(x,\sqrt{t})} |x-y|^{-n+1/2}dy \right) ||\alpha||_{\infty} \\
\lesssim t^{1/4} \left( \int_0^{\sqrt{t}} \frac{1}{r^{n-1/2}} r^{n-1} dr \right) ||\alpha||_{\infty} \\
\lesssim \sqrt{t} ||\alpha||_{\infty}. \tag{2.6}
\]

Putting Inequalities (2.4), (2.5) and (2.6) together gives (2.1).
2.2 Estimate of the term $A_1$

In order to prove estimate (2.2), we shall use the decomposition

$$A(\alpha) = A_1(\alpha) + A_2(\alpha) + A_3(\alpha)$$

with

$$A_1(\alpha)(t, \cdot) := \int_0^t e^{(t-s)\Delta} \Delta(s\Delta)^{-1}(I - e^{2s\Delta})s^{1/2} \text{div} s^{1/2} \alpha(s, \cdot) \, ds,$$  \hspace{1cm} (2.7)

$$A_2(\alpha)(t, \cdot) := \int_0^\infty e^{(t+s)\Delta} \text{div} \alpha(s, \cdot) \, ds,$$  \hspace{1cm} (2.8)

$$A_3(\alpha)(t, \cdot) := \int_t^\infty e^{(t+s)\Delta} \text{div} \alpha(s, \cdot) \, ds.$$  \hspace{1cm} (2.9)

There are two reasons for such a decomposition. First, $A_1$ can be handled via the maximal regularity techniques. Second, the form of the integral $A_2$ will allow us to use a duality argument together with the fact that the Leray projector commutes with the Laplacian. This is where tools from harmonic analysis come into play. Finally, the term $A_3$ should be thought of as a remainder.

2.2 Estimate of the term $A_1$

In this section we shall bound from above the term $A_1$. Namely, our main goal is to prove the estimate

$$\|A_1(\alpha)\|_{T^{\infty,2}} \lesssim \|s^{1/2} \alpha\|_{T^{\infty,2}}.$$  \hspace{1cm} (2.10)

First of all, note that $A_1(\alpha)(t, x) = M^+ Z(s^{1/2} \alpha)(t, x)$, where $M^+$ is the maximal regularity operator defined in Definition 1.2.5 and the operator $Z$ in turn is defined by $Z F(s, \cdot) := T_s F(s, \cdot)$ with

$$T_s f := \mathcal{P} s^{1/2} \text{div}(s\Delta)^{-1}(I - e^{2s\Delta}) f.$$

It is obvious that we are done once we prove that operators $M^+ : T^{\infty,2} \to T^{\infty,2}$ and $Z : T^{\infty,2} \to T^{\infty,2}$ are bounded.

Claim 1. $M^+ : T^{\infty,2} \to T^{\infty,2}$ is a bounded operator.

Proof. Let $F \in T^{\infty,2}$ and fix $x_0 \in \mathbb{R}^n$ and $R > 0$. Write $\mathbf{1}_{[0, R^2] \times \mathbb{R}^n} F$ as

$$\mathbf{1}_{[0, R^2] \times \mathbb{R}^n} F = \sum_{j \geq 0} F_j,$$

where

$$F_j := F \mathbf{1}_{[0, R^2] \times (B(x_0, 2^{j+1} R) \setminus B(x_0, 2^j R))} \quad \text{for} \quad j \geq 1, \quad \text{and} \quad F_0 := F \mathbf{1}_{[0, R^2] \times B(x_0, 2R)}.$$

We first rule out $M^+ F_0$ using de Simon’s theorem that ensures that

$$I_0^2 := \int_{[0, R^2] \times B(x_0, R)} |M^+ F_0|^2 \, ds \, dx \leq C \int_{\mathbb{R}^n_+} |F_0|^2 \, ds \, dx.$$
Next, we study the case when \( j \geq 1 \). Denote

\[
I_j^2 := \int_{[0,R^2] \times B(x_0,R)} |M^+ F_j|^2 \, ds \, dx.
\]

Note that \( M^+ F_j(t,x) = \sum_{k \geq 1} F_{j,k}(t,x) \), where, for \( k \geq 1 \),

\[
F_{j,k}(t,x) = \int_{t/2^k}^{t/2^{k-1}} e^{(t-s)\Delta} \Delta F_j(s,x) \, ds.
\]

From the triangle inequality, we infer the estimate

\[
I_j \leq \sum_{k \geq 1} \|F_{j,k}\|_{L^2([0,R^2] \times B(x_0,R))},
\]

and hence it suffices to bound from above the \( L^2 \) norms of the functions \( F_{j,k} \). Note that for \( k \geq 2 \) one has

\[
\|F_{j,k}\|_{L^2([0,R^2] \times B(x_0,R))}^2 = \int_0^R \int_{B(x_0,R)} \left| \int_{t/2^k}^{t/2^{k-1}} (t-s)e^{(t-s)\Delta} \Delta F_j(s,x) \frac{ds}{t-s} \right|^2 \, dx \, dt
\]

\[
\leq \int_0^R \int_{B(x_0,R)} \frac{2^{-k}k}{t^2} \int_{t/2^k}^{t/2^{k-1}} \left| (t-s)e^{(t-s)\Delta} \Delta F_j(s,x) \right|^2 \, ds \, dx \, dt
\]

\[
\leq \int_0^R \int_{t/2^k}^{t/2^{k-1}} 2^{-k}k \left( 1 + \frac{(2^j R)^2}{t-s} \right)^{-2M} \|F_j(s, \cdot)\|_{L^2(\mathbb{R}^n)}^2 \, ds \, dx \, dt,
\]

where in the first inequality we used the Cauchy–Schwarz inequality for the integral with respect to \( s \) and the Fubini theorem to exchange the integrals in \( s \) and in \( x \). In the second one we used the off-diagonal estimates for the family \((t-s)e^{(t-s)\Delta}\), see Definition 1.2.6. Note that here we can take \( M \) as big as we want.

We continue the estimate, now exchanging the integrals in \( s \) and in \( t \) and also using the fact that \( t \lesssim t-s \) for \( s \) and \( t \) as in the integrals above, getting

\[
\|F_{j,k}\|_{L^2([0,R^2] \times B(x_0,R))}^2 \leq \int_0^R \frac{2^{-k}k}{t^2} \int_{t/2^k}^{t/2^{k-1}} 2^{-k}k \left( 1 + \frac{(2^j R)^2}{t-s} \right)^{-2M} \|F_j(s, \cdot)\|_{L^2(\mathbb{R}^n)}^2 \, ds \, dx \, dt
\]

\[
\leq 2^{-4Mj} 2^{-k} \|F_j\|_{L^2([0,R^2] \times \mathbb{R}^n)}^2.
\]

In the case \( k = 1 \), one can get the same estimate on the norm \( \|F_{j,1}\|_{L^2([0,R^2] \times B(x_0,R))}^2 \) in an almost identical way as above. Indeed we first obtain the following estimate

\[
\|F_{j,1}\|_{L^2([0,R^2] \times B(x_0,R))}^2 = \int_0^R \int_{B(x_0,R)} \left| \int_{t/2}^{t} (t-s)e^{(t-s)\Delta} \Delta F_j(s,x) \frac{ds}{t-s} \right|^2 dx \, dt
\]

\[
\leq \int_0^R \int_{t/2}^{t} \frac{t}{(t-s)^2} \int_{B(x_0,R)} \left| (t-s)e^{(t-s)\Delta} \Delta F_j(s,x) \right|^2 dx \, ds \, dt
\]

\[
\leq \int_0^R \int_{t/2}^{t} \frac{t}{(t-s)^2} \left( 1 + \frac{(2^j R)^2}{t-s} \right)^{-2M} \|F_j(s, \cdot)\|_{L^2(\mathbb{R}^n)}^2 \, ds \, dt,
\]
thanks to the fact that the $L^2$ norm of an integral is bounded from above by the integral of its $L^2$ norm, to the Fubini theorem and to the off-diagonal estimates. Hence, referring to the fact that in this case $t \lesssim s$ we find that

$$\|F_j\|_{L^2([0,R^2] \times B(x_0,R))}^2 \leq \int_0^{R^2} \|F_j(s, \cdot)\|_{L^2(\mathbb{R}^n)}^2 \int_s^{2s} \frac{R^2}{(t-s)^2} \left(1 + \frac{(2jR)^2}{t-s}\right)^{-2M} dt \,ds$$

$$\lesssim 2^{-4Mj}\|F_j\|_{L^2([0,R^2] \times \mathbb{R}^n)}^2.$$

As a consequence we get for all integer $M$ that $I_j \lesssim 2^{-2Mj}\|F_j\|_{L^2([0,R^2] \times \mathbb{R}^n)}$. Taking $M > n/4$, one can thus conclude that :

$$\left(\int_{[0,R^2] \times B(x_0,R)} |M^+F|^2 \,ds \,dx\right)^{1/2} \leq \sum_j I_j$$

$$\lesssim R^{n/2}\|F\|_{T^{\infty,2}} \sum_j 2^{(n/2-2M)j}$$

$$\lesssim R^{n/2}\|F\|_{T^{\infty,2}},$$

which completes the proof of our claim.

\[ \square \]

**Claim 2.** $Z : T^{\infty,2} \rightarrow T^{\infty,2}$ is a bounded operator.

**Proof.** Note that $T_sF$ is an integral operator and denote by $\kappa_s$ its kernel. In order to achieve our goal, we need the following estimate on the function $\kappa_s$ (the proof of which is similar to that of a similar lemma).

**Lemma 2.2.1.** There exists $C > 0$ such that for all $(s,x) \in \mathbb{R}^{n+1}_+$ with $|x| \geq s^{1/2}$, we have

$$|\kappa_s(x)| \leq Cs^{-n/2}\left(\frac{|x|}{s^{1/2}}\right)^{-n-1}.$$

Now we can easily prove the following important property of the family $T_s$, which will be used in a moment.

**Lemma 2.2.2.** Let $F_j$ be as above and let $s > 0$. Then,

$$\|T_sF_j(s, \cdot)\|_{L^\infty(B(x_0,R))} \lesssim s^{-n/4}\left(\frac{s^{1/2}}{2jR}\right)^{n/2+1}\|F_j(s, \cdot)\|_{L^2(\mathbb{R}^n)}.$$

**Proof.** If $x \in B(x_0,R)$, then we deduce using the previous lemma that

$$\left|\int_{\mathbb{R}^n} \kappa_s(x-y)F_j(s,y) \,dy\right| \lesssim s^{-n/2}\left(\frac{2jR}{s^{1/2}}\right)^{-n-1}\int_{B(x_0,2j+1R) \setminus B(x_0,2jR)} |F_j(s,y)| \,dy$$

$$\lesssim s^{-n/2}\left(\frac{2jR}{s^{1/2}}\right)^{-n-1}\left(\int_{B(x_0,2j+1R) \setminus B(x_0,2jR)} |F(s,y)|^2 \,dy\right)^{1/2},$$

where in the second estimate we have used the H"{o}lder inequality. Hence Lemma 2.2 follows.

\[ \square \]
2.3 Estimate of the term $A_3$

As in the previous claim, we first concentrate on the function $F_0$. We use the fact that the operators $T_s$ are uniformly bounded in $L^2$ with respect to $s$ so as to write:

$$
\int_{[0,R^2] \times B(x_0,R)} |ZF_0|^2 \, ds \, dx \leq \int_{[0,R^2] \times \mathbb{R}^n} |T_s F_0|^2 \, ds \, dx
$$

$$
\leq C \int_{[0,R^2] \times \mathbb{R}^n} |F_0|^2 \, ds \, dx = C \int_{[0,R^2] \times B(x_0,R)} |F|^2 \, ds \, dx
$$

$$
\leq C \|F\|_{T^{\infty,2}}^2 |B(x_0,R)|. \tag{2.11}
$$

Next we turn to the off-diagonal terms, i.e. we consider indices $j \in \mathbb{N}$ such that $j \geq 1$. Fix $s > 0$. The Hölder inequality and Lemma 2.2.2 imply that

$$
\|T_s F_j(s, \cdot)\|_{L^2(B(x_0,R))} \leq R^{n/2} \|T_s F_j(s, \cdot)\|_{L^\infty(B(x_0,R))}
$$

$$
\leq CR^{n/2} s^{-n/4} \left( \frac{s^{1/2}}{2^j R} \right)^{n/2+1} \|F_j(s, \cdot)\|_{L^2(\mathbb{R}^n)}. \tag{2.12}
$$

Hence, from the fact that $s^{1/2} \leq R$, we infer the estimates

$$
\frac{1}{R^n} \int_{(0,R^2) \times B(x_0,R)} |T_s F_j|^2 \, ds \, dx \leq \frac{2^{-2j}}{(2^j R)^n} \int_{(0,R^2) \times (B(x_0,2^j R),B(x_0,2^j R))} |T_s F_j|^2 \, ds \, dx
$$

$$
\leq 2^{-2j} \frac{1}{|B(x_0,2^j+1 R)|} \int_{(0,2^j+1 R^2) \times B(x_0,2^j+1 R)} |F|^2 \, ds \, dx
$$

$$
\leq 2^{-2j} \|F\|_{T^{\infty,2}}^2. \tag{2.13}
$$

Putting he lines (2.11) and (2.13) together completes the proof the second claim. \hfill \square

As pointed out above, estimate (2.10) follows obviously from the first and the second claims.

### 2.3 Estimate of the term $A_3$

Our next goal is to prove

$$
\|A_3(\alpha)\|_{T^{\infty,2}} \lesssim \|s^{1/2} \alpha\|_{T^{\infty,2}}. \tag{2.14}
$$

Recall that the operator $A_3$ is defined by

$$
A_3 \alpha(t, \cdot) = \int_t^\infty e^{(t+s)\Delta} \text{div} \alpha(s, \cdot) \, ds.
$$

Observe that $A_3(\alpha) = \mathcal{R}(s^{1/2} \alpha)$, where

$$
(\mathcal{R} F)(t, \cdot) := \int_t^\infty K_{t,s} F(s, \cdot) \, ds,
$$

and the operator $K_{t,s}$ is defined by $K_{t,s} := e^{(t+s)\Delta} \mathbb{P} s^{-1/2} \text{div}$ for $s, t > 0$.

Note that $K_{t,s}$ is a kernel operator for all $s, t > 0$, with kernel $k_{t,s}$ satisfying, according to (1.7),

$$
|k_{t,s}(x)| \leq Cs^{-1/2}(t+s)^{-1/2-n/2}(1 + (t+s)^{-1/2}|x|)^{-n-1}, \tag{2.15}
$$

for all $x \in \mathbb{R}^n$ and $s, t > 0$. 
2.3 Estimate of the term $A_3$

We shall first prove that $\mathcal{R}$ is a bounded operator on the space $L^2(\mathbb{R}^{n+1}_+)$. This follows from

$$\|K_{t,s}\|_{L^2(\mathbb{R}^n)} \leq Cs^{-1/2}(t+s)^{-1/2},$$

(2.16)

which, in turn, is an easy consequence of the estimate (2.15).

In order to prove the boundedness of $\mathcal{R}$, pick some $\beta \in (-1/2,0)$, set $p(t) = t^\beta$ and observe that the function $k(t,s) := \|K_{t,s}\|_{L^2(\mathbb{R}^n)}$ satisfies

$$\int_0^\infty k(t,s)p(t)\,dt \lesssim \int_0^s s^{-1/2}t^{-1/2}t^{\beta} \,dt \lesssim s^\beta = p(s), \quad \text{for all } s > 0,$$

$$\int_0^\infty k(t,s)p(s)\,ds \lesssim \int_t^\infty s^{-1/2}t^{-1/2}s^{\beta} \,ds \lesssim t^{\beta} = p(t), \quad \text{for all } t > 0.$$

The desired $L^2(\mathbb{R}^{n+1}_+)$ boundedness now follows since, applying the Minkowski inequality and the Schur test, we have:

$$\|RF\|_{L^2(\mathbb{R}^{n+1}_+)}^2 = \int_{\mathbb{R}_+} \int_{\mathbb{R}^n} \int_t^\infty K_{t,s}F(s,x)\,ds \,dx \,dt$$

$$\leq \int_{\mathbb{R}_+} \left( \int_{\mathbb{R}^n} \int_t^\infty |K_{t,s}F(s,x)|^2 \,dx \,ds \right)^{1/2} \,dt$$

$$\lesssim \int_{\mathbb{R}_+} \left( \int_{\mathbb{R}^n} k(t,s)\|F(s,\cdot)\|_{L^2(\mathbb{R}^n)}^2 \,ds \right)^{1/2} \,dt$$

$$\lesssim \|F\|_{L^2(\mathbb{R}^{n+1})}^2.$$

Next, we concentrate on the boundedness of $\mathcal{R}$ on $T^{\infty,2}$. As a first, observe that Inequality (2.15) readily implies the following $L^2 - L^\infty$ off-diagonal estimate for all disjoint Borel sets $E, \hat{E} \subseteq \mathbb{R}^n$ and $s, t > 0$:

$$\|\mathbf{1}_E K_{t,s} \mathbf{1}_{\hat{E}}\|_{L^2(\mathbb{R}^n)} \leq s^{-1/2}(t+s)^{-1/2-n/4} \left(1 + (t+s)^{-1/2}\operatorname{dist}(E, \hat{E})\right)^{-n/2-1}.$$  

(2.17)

Let $F \in T^{\infty,2}$ and fix $(R, x_0) \in \mathbb{R}^{n+1}$. Define $B_j := (0, 2^j R^2) \times B(x_0, 2^j R)$ for $j \geq 0$ and $C_j := B_j \setminus B_{j-1}$ for $j \geq 1$. Then, set $F_0 := \mathbf{1}_{B_0}F$ and $F_j := \mathbf{1}_{C_j}F$ for $j \geq 1$. Using the Minkowski inequality we deduce that

$$\left( R^{-n} \int_0^{R^2} \|(RF)(t,\cdot)\|^2_{L^2(B(x_0,R))} \,dt \right)^{1/2}$$

$$\lesssim \sum_{j \geq 0} \left( R^{-n} \int_0^{R^2} \|(RF_j)(t,\cdot)\|^2_{L^2(B(x_0,R))} \,dt \right)^{1/2} =: \sum_{j \geq 0} I_j.$$  

For a natural number $j$ such that $j \leq 2$, the boundedness of $\mathcal{R}$ on $L^2$ yields the estimate $I_j \lesssim \|F\|_{T^{\infty,2}}$. For $j \geq 3$, split $C_j$ as follows:

$$C_j = (0, 2^j R^2) \times (B(x_0, 2^j R) \setminus B(x_0, 2^{j-1} R))$$

$$\cup (2^{j-1} R^2, 2^j R^2) \times B(x_0, 2^j R) =: C_j^{(0)} \cup C_j^{(1)}.$$
Denote $F_j^{(0)} := 1_{C_j^0} F$ and $F_j^{(1)} := 1_{C_j^1} F$ and, correspondingly, $I_j^{(0)}$ and $I_j^{(1)}$.

For $I_j^{(0)}$, split the integral in $s$ and use the Hölder inequality to obtain

$$I_j^{(0)} \lesssim \sum_{k \geq 0} \left( R^{-n} \int_0^{2^{k+1} t} 2^k t \|K_{t,s} F_j^{(0)}(s, \cdot)\|_{L^2(B(x_0,R))}^2 ds \right)^{1/2}.$$

Now observe that for $j \geq 3, k \geq 0, t \in (0, R^2)$ and $s \in (2^k t, 2^{k+1} t)$, Hölder’s inequality and the $L^2 - L^\infty$ off-diagonal estimate (2.17) above yield for any $\delta \in (0, 1]$

$$\|K_{t,s} F_j^{(0)}(s, \cdot)\|_{L^2(B(x_0,R))} \lesssim R^{n/2} \|K_{t,s} F_j^{(0)}(s, \cdot)\|_{L^\infty(B(x_0,R))} \lesssim R^{n/2} s^{-1/2} (t+s)^{-1/2-n/4} \left(1 + \frac{2^{j-1} R - R}{(t+s)^{1/2}}\right)^{-n/2+\delta} \|F_j(s, \cdot)\|_{L^2} \lesssim (2^j)^{-n/4-\delta/2} R^{-\delta} (2^j)^{-1/2} \|F_j(s, \cdot)\|_{L^2}.$$  

Combining this estimate with the previous one, interchanging the order of integration and choosing $\delta < 1$ gives

$$\sum_{j \geq 1} I_j^{(0)} \lesssim \sum_{j \geq 1} \sum_{k \geq 0} 2^{-j\delta/2 - k(1/2 - \delta/2)} \left( (2^j R^2)^{-n/2} \int_0^{2^j R^2} \|F_j(s, \cdot)\|_{L^2}^2 ds \right)^{1/2} \lesssim \|F\|_{T^\infty,2}.$$  

For $I_j^{(1)}$ it is enough to use the $L^2 - L^\infty$ bound for the operator $K_{t,s}$ instead of the off-diagonal estimates. For $s \in (2^{j-1} R^2, 2^{j} R^2)$ and $0 < t < R^2$ one thus obtains

$$\|K_{t,s} F_j^{(1)}(s, \cdot)\|_{L^2(B(x_0,R))} \lesssim R^{n/2} \|K_{t,s} F_j^{(1)}(s, \cdot)\|_{L^\infty(B(x_0,R))} \lesssim R^{n/2} s^{-1/2} (t+s)^{-1/2-n/4} \|F_j(s, \cdot)\|_{L^2} \lesssim (2^j)^{-n/4} (2^j R^2)^{-1} \|F_j(s, \cdot)\|_{L^2}.$$  

Plugging this into $I_j^{(1)}$ yields

$$I_j^{(1)} \lesssim \left( R^{-n} \int_0^{2^{j+1} t} 2^j R^2 \|K_{t,s} F_j^{(1)}(s, \cdot)\|_{L^2(B(x_0,R))}^2 ds \right)^{1/2} \lesssim \left( (2^j R^2)^{-n/2} \int_0^{2^j R^2} (2^j R^2)^{-1/2} R \|F_j(s, \cdot)\|_{L^2}^2 ds \right)^{1/2} \lesssim 2^{-j/2} \|F\|_{T^\infty,2}.$$  

Summing up over $j$ gives $\|R F\|_{T^\infty,2} \lesssim \|F\|_{T^\infty,2}$, whence (2.14) is proved.

### 2.4 Estimate of the term $A_2$

We now wish to prove

$$\|A_2(\alpha)\|_{T^\infty,2} \lesssim \|\alpha\|_{T^\infty,1}. \quad (2.18)$$

This is where we need tools coming from harmonic analysis. Let us first introduce some terminology that is very important in this section.
Definition 2.4.1. We shall say that a continuous function \( u : \mathbb{R}^{n+1} \to \mathbb{C} \) belongs to the tent space \( T^{1,\infty} \) if \( \| u \|_{T^{1,\infty}} := \| N(u) \|_{L^1(\mathbb{R}^n)} < \infty \), where
\[
N(u)(x) := \sup_{\{(t,y) : y \in B(x,\sqrt{t})\}} |u(t,y)|,
\]
is the parabolic non-tangential maximal function and
\[
\lim_{t \to 0, y \in B(x, \sqrt{t})} u(t,y) \text{ exists for almost every } x \in \mathbb{R}^n.
\]

Definition 2.4.2. We shall say that a measurable function \( u \) belongs to the tent space \( T^{1,2} \) if \( \| u \|_{T^{1,2}} := \| S(u) \|_{L^1(\mathbb{R}^n)} < \infty \), where
\[
S(u)(x) := \left( \int \int_{\{(t,y) : y \in B(x,\sqrt{t})\}} |u(t,y)|^2 \frac{dy \, dt}{t^{n/2}} \right)^{1/2}
\]
is the parabolic square function.

Definition 2.4.3. Let \( x \in \mathbb{R}^n \). The parabolic cone \( \Gamma(x) \) with vertex \( x \) is defined by
\[
\Gamma(x) := \{(t,y) : y \in B(x,\sqrt{t})\}.
\]

Definition 2.4.4. Let \( O \subset \mathbb{R}^n \) be an open set. The tent \( \hat{O} \) (also called the parabolic tent above the set \( O \)) is defined by \( \hat{O} := \{(t,y) : \text{dist}(y, O^c) \geq \sqrt{t}\} \) (the tent above a ball is pictured in red in the figure just below):

![Diagram of a parabolic tent above a ball](image)

Definition 2.4.5. A Borel measure \( \mu \) on \( \mathbb{R}^{n+1} \) is called a Carleson measure if
\[
\sup_{B \subset \mathbb{R}^n - \text{open ball}} \frac{\mu(B)}{|B|} < \infty.
\]

Let \( B(x_0, r) \subset \mathbb{R}^n \) be an open ball. Note that
\[
\left[ 0, \left(\frac{r}{2}\right)^2 \right] \times B\left(x_0, \frac{r}{2}\right) \subset \overline{B(x_0, r)} \subset [0, r^2] \times B(x_0, r).
\]
Thus (recall Definition 1.2.1) \( F \in T^{\infty,1} \) if and only if \( |F| \, dy \, dt \) is a Carleson measure.
2.4 Estimate of the term $A_2$

Introduce for $x \in \mathbb{R}^n$ the following important function:

$$ C(\mu)(x) := \sup_{x \in B,B-'an open ball} \frac{\mu(B)}{|B|} $$

and the corresponding norm $\|\mu\|_C := \|C(\mu)\|_\infty$. We shall further write $C(F)$ instead of $C(|F| \, dy \, dt)$ in the case where $F$ is a function.

The strategy to prove the boundedness $A_2 : T^{\infty,1} \to T^{\infty,2}$ comprises three main steps. The first one is the embedding property $T^{\infty,1} \subset (T^{1,\infty})^*$, the second one is the fact that $T^{\infty,2} = (T^{1,2})^*$ and the third one is that $A_2^* : T^{1,2} \to T^{1,\infty}$, where $A_2^*(G)(s,\cdot) := e^{s\Delta} \int_0^\infty \nabla^2 e^{t\Delta} G(t,\cdot) \, dt$. Indeed, suppose that these steps are proved. Then, for $F \in T^{\infty,1}$ and $G \in T^{1,2}$, using

$$ \int_{\mathbb{R}^{n+1}_+} A_2(F)G \, dy \, dt = \int_{\mathbb{R}^{n+1}_+} F A_2^*(G) \, dy \, dt $$

we deduce that

$$ \|A_2(F)\|_{T^{\infty,2}} \leq \sup_{\|G\|_{T^{1,2}=1}} \|F\|_{T^{\infty,1}} \|A_2^*(G)\|_{T^{1,\infty}} \lesssim \|F\|_{T^{\infty,1}}. $$

**Step 1.** The main idea here is to use the Carleson embedding:

$$ \int_{\mathbb{R}^{n+1}_+} |H| \, d\mu \leq C_n \int_{\mathbb{R}^n} N(H)(x)C(\mu)(x) \, dx, \quad (2.19) $$

whenever $H \in T^{1,\infty}$ and $\mu$ is a Carleson measure for some constant $C_n$ depending only on the dimension $n$ of the ambient space.

Let us explain how the first step follows from (2.19). Indeed, if $F \in T^{\infty,1}$ and $d\mu := |F| \, dy \, dt$, then the mapping $H \mapsto \int HF \, dy \, dt$ is a bounded functional on $T^{1,\infty}$.

So, let us focus on the proof of inequality (2.19). Since $N(H)$ is lower semi-continuous with $N(H) \in L^1(\mathbb{R}^n)$, for $\lambda > 0$ the set $O_\lambda = \{x \in \mathbb{R}^n : N(H)(x) > \lambda\}$ is open with $|O_\lambda| < \infty$ (and hence $O_\lambda \subseteq \mathbb{R}^n$). Consider the Whitney decomposition of the set $O_\lambda$: there exists a constant $c_n \in (0,1)$ and a sequence $(B_j)_j$ of open balls contained in $O_\lambda$ such that $O_\lambda \subseteq \bigcup B_j$, $4B_j \not\subseteq O_\lambda$ and the balls $c_n B_j$ are mutually disjoint. We want an estimate on the value $\mu(\{(t,y) : |H(t,y)| > \lambda\})$. Note that $|H(t,y)| > \lambda \Rightarrow B(y,\sqrt{t}) \subseteq O_\lambda \Rightarrow (t,y) \subset \widehat{\Omega}_\lambda$. Hence,

$$ \mu(\{(t,y) : |H(t,y)| > \lambda\}) \leq \mu(\widehat{\Omega}_\lambda). $$

Since $(t,y) \in \widehat{\Omega}_\lambda$, there exists $i$ such that $y \in B_i(=B(x,r_i))$ and there exists $z \notin O_\lambda$ such that $z \in 4B_i$. It follows that

$$ \sqrt{t} \leq \text{dist}(y,O_\lambda) \leq |y-z| \leq |y-x_i| + |x_i-z| \leq r_i + 4r_i = 5r_i \leq \text{dist}(y,\bigcup_i 6B_i). $$
Hence, \((t, y) \in 6B_i\). Now, we see that there holds
\[
\mu(\hat{O}_\lambda) \leq \sum_i \mu(6B_i) \leq \sum_i |6B_i| \inf_{x \in 6B_i} C(\mu)(x)
\leq \left(\frac{6}{c_n}\right)^n \sum_i |c_n B_i| \inf_{x \in c_n B_i} C(\mu)(x)
\leq \left(\frac{6}{c_n}\right)^n \int_{\bigcup_i c_n B_i} C(\mu)(x) \, dx
\leq \left(\frac{6}{c_n}\right)^n \int_{\Omega_{\lambda}} C(\mu)(x) \, dx.
\]
This allows us to complete the proof of the estimate (2.19) by integrating in \(\lambda > 0\) and thus that of the first step.

Step 2. The key to the proof of the embedding \(T^{\infty, 2} \subset (T^{1, 2})^*\) is the following estimate :
\[
\int_{\mathbb{R}^n+1} |F||G| \, dy \, dt \leq C \int_{\mathbb{R}^n} C_2(F)(x) S(G)(x) \, dx,
\]
where \(C_2(F) := (C(|F|^2))^{1/2}\), for all \(F \in T^{\infty, 2}\) and \(G \in T^{1, 2}\). Indeed, the inequality (2.20) obviously yields
\[
\left| \int_{\mathbb{R}^n+1} FG \, dy \, dt \right| \lesssim \|C_2(F)\|_{L^\infty(\mathbb{R}^n)} \|S(G)\|_{L^1(\mathbb{R}^n)} = \|F\|_{T^{\infty, 2}} \|G\|_{T^{1, 2}},
\]
which realizes \(F\) as a bounded linear functional on the space \(T^{1, 2}\).

Remark 2.4.1. Note that an easier inequality
\[
\int_{\mathbb{R}^n+1} |F||G| \, dy \, dt \leq \int_{\mathbb{R}^n} S(F)(x) S(G)(x) \, dx
\]
follows from the Cauchy–Schwarz inequality. Alas, \(S(F)\) is not comparable to \(C_2(F)\) when \(F \in T^{\infty, 2}\).

In order to prove (2.20) we shall use a stopping time argument. To this end, we recall a couple of definitions.

Definition 2.4.6. Let \(h > 0\) and let \(x \in \mathbb{R}^n\). The truncated cone \(\Gamma_h(x)\) is defined by
\[
\Gamma_h(x) := \{(t, y) \in \mathbb{R}^n+1 : y \in B(x, \sqrt{t}) \text{ and } \sqrt{t} \leq h\}.
\]
The corresponding average is given by
\[
S_h(G)(x) := \left( \int_{\Gamma_h(x)} |G|^2 \, dy \, dt \right)^{1/2}.
\]
We also need to define the function \(\mathfrak{h}\) on \(\mathbb{R}^n\) by the formula
\[
\mathfrak{h}(x) := \sup\{h > 0 : S_h(F)(x) \leq \nu C_2(F)(x)\},
\]
where \(\nu\) is a large universal constant to be disclosed in a moment (\(\nu = 3^n \cdot 100\) will work).
Let $\phi : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ be a nonnegative function. The Fubini theorem yields the formula

$$I := \int_{\mathbb{R}^n} \left( \int_{\Gamma_{\mathbf{b}}(x)} \phi(t, y) dy dt \right) dx = \int_{\mathbb{R}^{n+1}_+} \phi(t, y) \left( \int_{x \in B(y, \sqrt{t})} \frac{dx}{\sqrt{t}} \right) dy dt. \quad (2.21)$$

The function $\phi$ will be chosen in a moment. Denote until the end of this step $B := B(y, \sqrt{t})$. Consider the set $E := \{ x \in B(y, \sqrt{t}) : \mathbf{b}(x) \geq \sqrt{t} \}$. Suppose that we know that $|E| > c|B|$ for some constant $c \in (0, 1)$ independent of $y$ and $t$. How to conclude then? Indeed, if it is the case, then choose $\phi := |F||G|/\sqrt{t^n}$ and note that (2.21) gives $I \geq c \int_{\mathbb{R}^{n+1}_+} \phi(t, y) \sqrt{t^n} dy dt$, whence

$$\int_{\mathbb{R}^{n+1}_+} |F||G| dy dt \leq \frac{1}{c} \int_{\mathbb{R}^n} \left( \int_{\Gamma_{\mathbf{b}}(x)} |F||G| \frac{dy dt}{t^{n/2}} \right) dx
\leq \frac{1}{c} \int_{\mathbb{R}^n} S_{\mathbf{b}}(F)(x) S_{\mathbf{b}}(G)(x) dx
\leq \frac{\nu}{c} \int_{\mathbb{R}^n} C_2(F)(x) S(G)(x) dx.$$

It is left to show that $|E| > c|B|$ for some $c \in (0, 1)$. It is sufficient to prove that $|B \setminus E| \leq (1 - c)|B|$. Take a point $x \in B \setminus E$. As $x \in E^c, \tau := \sqrt{t} > \mathbf{b}(x)$ and $S_{\tau}(F)(x) > \nu C_2(F)(x)$. On the other hand,

$$\frac{1}{|B|} \int_B S^2_{\tau}(F)(x) dx \leq \frac{1}{|B|} \int_B \int_{\Gamma_{\tau}(x)} |F|^2 \frac{dy dt}{t^{n/2}} dx
\leq \frac{1}{|B|} \int_{3B} |F|^2 dy dt
\leq \frac{3|B|}{|B|} \inf_{x \in 3B} C(|F|^2)(x)
\leq 3^n \inf_{x \in E \setminus E} C(|F|^2)(x)
\leq \frac{3^n}{\nu \cdot |B \setminus E|} \int_{B \setminus E} S^2_{\tau}(F)(x) dx
\leq \frac{3^n}{\nu \cdot |B \setminus E|} \int_B S^2_{\tau}(F)(x) dx.$$

Hence, $\nu \cdot |B \setminus E| \leq 3^n|B|$ and with the choice $\nu = 3^n \cdot 100$ revealed above, one can take $c = 0.99$.

Now, why $T^{\infty, 2} \supseteq (T^{1, 2})^*$? In order to prove this, we shall need one important type of $T^{1, 2}$ functions, called atoms, that are introduced in the following definition.

**Definition 2.4.7.** We say that a function $\alpha \in T^{1, 2}$ is an atom if there exists a ball $B \subset \mathbb{R}^n$ such that $\text{supp}(\alpha) \subset \hat{B}$ and $(\int_{\hat{B}} |\alpha|^2 dy dt)^{1/2} \leq |B|^{-1/2}$.

**Remark 2.4.2.** One of the main advantages of $T^{1, 2}$ atoms is that they are compactly supported. More than that, the $L^2$ norm of an atom is controlled by the size of its support. We shall soon see that any function in the space $T^{1, 2}$ admits a representation in terms of an infinite converging linear combination of $T^{1, 2}$ atoms.
Remark 2.4.3. Observe that if $a$ is a $T^{1,2}$ atom, then $\|a\|_{T^{1,2}} \lesssim 1$.

So, first of all, note that if a function $f$ is supported in some compact set $K \subset \mathbb{R}^{n+1}$ and if $f \in L^2(K)$, then also $f \in T^{1,2}$ with $\|f\|_{T^{1,2}} \leq C(K)\|f\|_{L^2(K)}$. Suppose further that $\ell$ is a bounded linear functional on $T^{1,2}$. Hence we see that $\ell$ is also a linear functional on $L^2(K)$. As a consequence of the Riesz representation theorem, we deduce that there exists a function $g_K \in L^2(K)$ such that $\ell(f)$ is representable by $\int_K f g_K \, dt \, dy$. Taking an exhaustive sequence of compacts, we obtain a function $g \in L^2_{\text{loc}}(\mathbb{R}^{n+1})$ such that $\ell(f) = \int_{\mathbb{R}^{n+1}} f g \, dt \, dy$ once $f \in T^{1,2}$ and $f$ has a compact support. Next, if we test the functional $\ell$ against all atoms supported in a tent $\hat{B}$ then we obtain $\|\ell\|^2 \geq 1/|B| \int_B |g|^2 \, dt \, dy$, which proves the assertion. This representation is extendable to all functions in $T^{1,2}$ since compactly supported functions are dense in $T^{1,2}$. This proves $T^{1,2} \supset (T^{1,2})^*$.

**Step 3.** To complete the proof, it is left to show that the adjoint operator

$$A_2^*(G)(s, \cdot) = e^{s\Delta} \int_0^\infty \nabla P e^{t\Delta} G(t, \cdot) \, dt$$

acts from the space $T^{1,2}$ to the space $T^{1,\infty}$. We shall use here the atomic decomposition of the space $T^{1,2}$ that follows.

**Lemma 2.4.1.** For any function $G \in T^{1,2}$ there exist atoms $a_j$ and numbers $\lambda_j \in \mathbb{C}$ such that $\sum_j |\lambda_j| < \infty$, satisfying $G = \sum_j \lambda_j a_j$. On top of that, it holds that

$$\sum_j |\lambda_j| \lesssim \|G\|_{T^{1,2}} \lesssim \sum_j |\lambda_j|.$$  

**Proof.** Let us sketch the proof of this decomposition. Let $k$ be a natural number and denote $O_k := \{x \in \mathbb{R}^n : S(G)(x) > 2^k\}$ the corresponding level set of the function $S(G)$. Let $\mathcal{M}$ denote the Hardy–Littlewood maximal function: $\mathcal{M} f(x)$ is the supremum of the averages of $\frac{1}{|B|} \int_B |f|$ taken over all open balls $B$ that contain $x$. Consider further the sets $O_k^* := \{x \in \mathbb{R}^n : \mathcal{M} 1_{O_k}(x) > 1 - \gamma\}$ with $\gamma$ sufficiently close to one (to be revealed in a moment). Then, as $O_k$ is open and using the weak type $(1,1)$ of $\mathcal{M}$, we have

$$O_k \subset O_k^*, \quad |O_k^*| \leq c(\gamma)|O_k|, \quad \hat{O}_k \subset \hat{O}_k^*$$

and the set $\bigcup_k \hat{O}_k^*$ contains the support of the function $G$. Consider a Whitney decomposition of $O_k^*$: $O_k^* = \bigcup_j Q_j^k$ with the cubes $Q_j^k$ having the property that their diameters are comparable with the distance from $Q_j^k$ to the complement of the set $O_k^*$. Next, consider a ball $B_j^k$ centered at the center of the cube $Q_j^k$ and having the radius equal to some large constant times the edge length of $Q_j^k$. If this constant is large enough, then we can write the following disjoint union:

$$\Delta^k = \bigcup_j \Delta_j^k,$$

where $\Delta^k := \hat{O}_k^* - \hat{O}_{k+1}^*$ and

$$\Delta_j^k := \hat{B}_j^k \cap ((0, \infty) \times Q_j^k) \cap (\hat{O}_k^* - \hat{O}_{k+1}^*).$$
We are now in position to define the desired atomic decomposition. Denote
\[ a_j^k := G \Delta_j^k |B_j^k|^{-1/2} (\mu_j^k)^{-1/2} \quad \text{with} \quad \mu_j^k := \int_{\Delta_j^k} |G(t,y)|^2 \, dt \, dy. \]
Set \( \lambda_j^k = |B_j^k|^{1/2} (\mu_j^k)^{1/2} \). We then have
\[ G = \sum_{k,j} \lambda_j^k a_j^k. \]
Note that the functions \( a_j^k \) are atoms associated to the balls \( B_j^k \).

Recall that \( \|a_j^k\|_{T^{1,2}} \lesssim 1 \). Hence, \( \|S(G)\|_{L^1(\mathbb{R}^n)} \lesssim \sum \lambda_j^k \). Therefore, it is left to show that
\[ \sum_{j,k} \lambda_j^k \lesssim \|S(G)\|_{L^1(\mathbb{R}^n)}. \quad (2.22) \]

First of all, it is easy to see that
\[ \mu_j^k = \int_{\Delta_j^k} |G(t,y)|^2 \, dy \, dt \leq \int_{B_j^k \cap (O_{k+1})^c} |G(t,y)|^2 \, dy \, dt. \]

Second of all, we shall prove that
\[ \int_{B_j^k \cap (O_{k+1})^c} |G(t,y)|^2 \, dy \, dt \leq \int_{B_j^k \cap (O_{k+1})^c} |S(G)(x)|^2 \, dx. \quad (2.23) \]

For a set \( E \subset \mathbb{R}^n \), introduce the notation \( \Gamma(E) := \bigcup_{x \in E} \Gamma(x) \). Note that \( \Gamma(E) \) is a subset of \( \mathbb{R}^{n+1}_+ \) that consists of all parabolic cones centered at points of \( E \). Apply the Fubini theorem:
\[ \int_{B_j^k \cap (O_{k+1})^c} |S(G)(x)|^2 \, dx = \int_{B_j^k \cap (O_{k+1})^c} \int_{\mathbb{R}^{n+1}_+} 1_{B(x, \sqrt{t})} |f(t,y)|^2 \frac{dt \, dy}{t^{n/2}} \, dx \]
\[ = \int_{\Gamma(B_j^k \cap (O_{k+1})^c)} |B(y, \sqrt{t}) \cap B_j^k \cap (O_{k+1})^c|^{1/2} |f(t,y)|^2 \, dy \, dt \quad \text{in} \quad \mathbb{R}^{n/2}. \]

Inequality \( (2.23) \) will follow, if we show that for all \( (t,y) \in \Gamma(B_j^k \cap (O_{k+1})^c) \), there holds
\[ t^{n/2} \lesssim |B(y, \sqrt{t}) \cap B_j^k \cap (O_{k+1})^c|. \quad (2.24) \]

Denote \( F := B_j^k \cap (O_{k+1})^c \). Note that \( (t,y) \in \Gamma(B_j^k \cap (O_{k+1})^c) \) implies that there exists \( x \in B_j^k \cap (O_{k+1})^c \) such that \( |x-y| \leq \sqrt{t} \). It can be easily seen from geometric observations that there exists a universal constant \( \varepsilon < 1 \) depending only on \( n \) such that:
\[ |B(x, \sqrt{t}) \cap B(y, \sqrt{t})^c| \leq \varepsilon |B(x, \sqrt{t})|. \]

Observe that
\[ (F \cap B(y, \sqrt{t})) \cup (B(x, \sqrt{t}) \cap B(y, \sqrt{t})^c) \supset F \cap B(x, \sqrt{t}), \]
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and hence
\[
|F \cap B(y, \sqrt{t})| \geq |F \cap B(x, \sqrt{t})^c| - |B(y, \sqrt{t}) \cap B(y, \sqrt{t})^c|
\]
\[
\geq |F \cap B(x, \sqrt{t})^c| - \varepsilon |B(x, \sqrt{t})|
\]
\[
\geq (1 - \gamma) |B(x, \sqrt{t})| - \varepsilon |B(x, \sqrt{t})|,
\]
where the last inequality follows from the fact that $x \in B_k \subset O_k$ (and hence $\mathfrak{M} \mathbb{1}_{O_k}(x) > 1 - \gamma$). So, the lines (2.23) and (2.24) follow if only we take $\gamma$ sufficiently close to 1.

Using the lower bound (2.24) and the definition of the set $O_{k+1}$, we see that
\[
\mu_{j,k} \lesssim |Q_{j,k}|^{1/2}
\]
With help of this inequality, we now complete the proof of the estimate (2.22):
\[
\sum_{j,k} \lambda_{j,k} = \sum_{j,k} |B_j^k|^{1/2} (\mu_{j,k})^{1/2}
\]
\[
\lesssim \sum_{j,k} |B_j^k|^{1/2} |Q_j^k|^{1/2} 2^k
\]
\[
\lesssim \sum_{j,k} |Q_j^k| 2^k
\]
\[
\lesssim \sum_k |Q_j^k| 2^k \leq \sum_k |O_k^k| 2^k \lesssim \|S(G)\|_{L^1(\mathbb{R}^n)},
\]
and the lemma is proven. \hfill \Box

We shall next show that the operator
\[
\mathcal{M}G(\cdot) := \int_0^\infty \nabla \mathbb{P} e^\Delta G(t, \cdot) \, dt,
\]
which is already defined and bounded from $L^2(\mathbb{R}^{n+1}, \mathbb{C}^n \otimes \mathbb{C}^n)$ to $L^2(\mathbb{R}^n, \mathbb{C}^n \otimes \mathbb{C}^n)$ (see the argument below when calculating $\mathcal{M}a$) can be shown to extend from the tent space $T^{1,2}$ to the Hardy space $H^1(\mathbb{R}^n, \mathbb{C}^n \otimes \mathbb{C}^n)$. For the reader’s convenience, we add a definition and some important properties of Hardy spaces.

**Definition 2.4.8.** Let $f$ be a bounded tempered distribution on $\mathbb{R}^n$ and let $0 < p < \infty$. Let $\Psi$ be a positive function in the Schwartz class such that $\int_{\mathbb{R}^n} \Psi \, dx = 1$ and denote $\Psi_t(x) = t^{-n} \Psi(x/t)$. We say that $f$ lies in the Hardy space $H^p(\mathbb{R}^n)$ if the non-tangential maximal function
\[
M^* f(x) := \sup_{t > 0} \sup_{y \in B(x,t)} |\Psi_t \ast f(y)|
\]
belongs to the space $L^p(\mathbb{R}^n)$. The corresponding norm is given by $\|f\|_{H^p(\mathbb{R}^n)} := \|M^* f\|_{L^p(\mathbb{R}^n)}$

**Remark 2.4.4.** The space defined above does not depend on the particular choice of $\Psi$. The proof of this statement can be found, for instance, in the book [14], Theorem 6.4.4.

**Remark 2.4.5.** The spaces $H^p(\mathbb{R}^n)$ coincide with $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. However for $0 < p \leq 1$ this is no longer true, see [14], Theorem 6.4.3.

**Remark 2.4.6.** The dual of the space $H^1(\mathbb{R}^n)$ is the space $\text{BMO}(\mathbb{R}^n)$, see [14], Theorem 6.4.3.
Remark 2.4.7. Based on the definition of the space $H^1(\mathbb{R}^n)$, one can define the space $H^1(\mathbb{R}^n, \mathbb{C}^n \otimes \mathbb{C}^n)$ in the component-wise way.

Hardy spaces also admit atomic decompositions. However, we shall not use them in this text. On the other hand, the connected notion of a Hardy molecule will be very useful in the end of the proof of the third step. These Hardy molecules should be thought of as “sums of Hardy atoms”. In other words, a Hardy molecule can have (in comparison to a Hardy atom) an unbounded support. However, it must satisfy a “decay at infinity” property as we shall observe in the following definition.

Definition 2.4.9. Let $0 < p \leq 1 < q \leq \infty$ and $b > 1/p - 1/q$. Then, a $(p, q, b)$ – molecule centered at $x_0 \in \mathbb{R}^n$ is a real-valued function $m$ defined on $\mathbb{R}^n$ and satisfying :

$$|||m||| := \|m\|_{q,\theta}^{1-\theta} \| \cdot - x_0 \|_1^{nb} \|m\|_{q,\theta} < \infty,$$

where $\theta = (1/p - 1/q)/b$, (so that $0 < \theta < 1$) and

$$\int_{\mathbb{R}^n} x^\beta m(x) \, dx = 0$$

for every multi-index $\beta$ such that $|\beta| \leq [n(1/p - 1)]$.

Remark 2.4.8. The definition above is taken from the book [12] (see page 328, Definition 7.13). In the very same book, it is proven that each function $m$ as in Definition 2.4.9 belongs to the Hardy space $H^p(\mathbb{R}^n)$ with the norm control $\|m\|_{H^p(\mathbb{R}^n)} \lesssim |||m|||$ (see Theorem 7.16 at page 330).

We go back to the proof of the third step. Let $G \in T^{1,2}$. Thanks to the atomic decomposition of the space $T^{1,2}$ in Lemma 2.4.1, we can write

$$G = \sum_i \lambda_i a_i$$

where $a_i$ are $T^{1,2}$ atoms and $\sum |\lambda_i| \lesssim \|G\|_{T^{1,2}}$. Hence

$$\sum_j |\lambda_j| \|Ma_j\|_{H^1(\mathbb{R}^n, \mathbb{C}^n \otimes \mathbb{C}^n)} \lesssim \sum_j |\lambda_j| \lesssim \|G\|_{T^{1,2}},$$

provided we know that $\|Ma\|_{H^1(\mathbb{R}^n, \mathbb{C}^n \otimes \mathbb{C}^n)} \lesssim 1$ for any atom $a \in T^{1,2}$. Remark at this stage that such atoms are $L^2$ functions so that $Ma$ was already defined. This shows that the series $\sum_j \lambda_j Ma_j$ converges in the Banach space $H^1(\mathbb{R}^n, \mathbb{C}^n \otimes \mathbb{C}^n)$ and we can conclude provided we identify its limit as $MG$. It will suffice to do it for $G$ in a dense class.

We begin with the uniform estimate. Let $B$ be a ball in $\mathbb{R}^n$ and let $a \in T^{1,2}$ be an atom supported in the tent $\tilde{B}$. First, we need to prove the estimate

$$\int_{2B} |Ma(x)|^2 \, dx \leq \int_{\mathbb{R}^n} |Ma(x)|^2 \, dx \lesssim \frac{1}{|B|}.$$  \hspace{1cm} (2.25)

Note that the Plancherel theorem yields :

$$\int_{\mathbb{R}^n} |Ma(x)|^2 \, dx = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\mathcal{F}Ma(\xi)|^2 \, d\xi.$$
Hence, using the definition of $M$ and denoting by $M_\mathcal{P}$ the matrix symbol of $\mathcal{P}$, we get

$$
\int_{\mathbb{R}^n} |Ma(x)|^2 \, dx = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left| \int_0^\infty \xi \otimes e^{-t|\xi|^2} \mathcal{F}a(t, \xi) \, dt \right|^2 \, dx
$$

whence, by using Cauchy-Schwarz inequality and the fact that $M_\mathcal{P}$ is bounded by 1,

$$
\int_{\mathbb{R}^n} |Ma(x)|^2 \, dx \leq \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left( \int_0^\infty |\xi|^2 e^{-2t|\xi|^2} \, dt \right) \left( \int_0^\infty |\mathcal{F}a(t, \xi)|^2 \, dt \right) \, d\xi
\leq \frac{1}{2} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_0^\infty |\mathcal{F}a(t, \xi)|^2 \, dt \, dx
= \frac{1}{2} \int_{\mathbb{R}^n} \int_0^\infty |a(t, x)|^2 \, dt \, dx \leq \frac{1}{2} |B|^{-1},
$$
since $a$ is an atom supported in $\hat{B}$. Hence, the estimate (2.25) follows.

Second, fix $q \in (1, n/(n-1))$ and denote $b := 2(q-1)/q$. Note that $q \leq 2$ once $n \geq 2$.

We shall show the inequality

$$
\int_{\mathbb{R}^n} |Ma(x)|^q |x - x_0|^{qn_b} \, dx \lesssim R^{n(qb + 1 - q)}, \tag{2.26}
$$

where $x_0$ is the center of the ball $B$ and $R$ is its radius. As a consequence of the estimate (2.25), thanks to the Hölder inequality we get the bound

$$
\int_{2B} |Ma(x)|^q |x - x_0|^{qn_b} \, dx \lesssim |B|^{qb} \int_{2B} |Ma(x)|^q \, dx \lesssim R^{n(qb + 1 - q)}. \tag{2.27}
$$

Denote for $j \geq 1$ the dyadic annuli $\Omega_j := 2^{j-1}B \setminus 2^j B$. The decay at infinity of the kernel $q_t$ and the Cauchy–Schwarz inequality yield

$$
\int_{\Omega_j} |Ma(x)|^q |x - x_0|^{qn_b} \, dx \leq |B|^{qb + jn_b} \int_{\Omega_j} |Ma(x)|^q \, dx
\leq |B|^{qb + jn_b} \int_{\Omega_j} \left( \int_{\overline{B}} |q|^{2} \right)^{q/2} \cdot \left( \int_{\overline{B}} \left| q_t(x - y) \right|^2 \, dy \, dt \right)^{q/2} \, dx
\leq |B|^{qb - q + 2j n_b} \int_{\Omega_j} \left( \int_{\overline{B}} \left| q_t(x - y) \right|^2 \, dy \, dt \right)^{q/2} \, dx
\lesssim R^{nq_b - \frac{nq}{2} + jn_n b} |B|^{2j (2j)} \Omega_j (2j R)^{-q(n+1)}
= R^{nq_b - \frac{nq}{2} + jn_n b} \Omega_j (2j R)^{-q(n+1)} 2^j (q_n b - q_n + q + n), \tag{2.28}
$$

where the last inequality above is justified by the fact that $|x - y| \lesssim (2j R)^{-(n+1)}$ once $x \in \Omega_j$ and $(t, y) \in B$. Note that

$$
qn_b - q_n/2 + (n + 2)q/2 + n - q(n+1) = n(qb + 1 - q).
$$

This means that the estimates (2.27) and (2.28) are of the same order in $R$. Since $q < n/(n-1)$, recalling the definition of $b$ one gets $qn_b - q_n - q + n = q(n-1) - n < 0$. So,
the corresponding sum over integer \( j \geq 0 \) is convergent. Hence the estimate (2.26) follows. Arguing in the same way as in the estimate of the term (2.28) one gets

\[
\int_{(2B)^c} |Ma(x)|^q \, dx \lesssim R^{n(1-q)}.
\]

This and the second part of the inequality (2.27) give

\[
\int_{\mathbb{R}^n} |Ma(x)|^q \, dx \lesssim R^{n(1-q)}.
\] (2.29)

Third, observe that

\[
\int_{\mathbb{R}^n} Ma(x) \, dx = 0.
\]

This follows from the observation that

\[
\int_{\mathbb{R}^n} Ma(x) \, dx = \int_0^\infty \int_{\mathbb{R}^n} \nabla \mathbb{P} e^{t\Delta} a(t, x) \, dx \, dt
\]

and the divergence theorem gives

\[
\int_{\mathbb{R}^n} \nabla \mathbb{P} e^{t\Delta} a(t, x) \, dx = 0.
\]

This observation together with the inequalities (2.26) and (2.29) signify (by definition) that \( Ma \) is a \((1, q, b)\) matrix valued Hardy molecule (see Definition 2.4.9) and in turn that \( \|Ma\|_{H^1(\mathbb{R}^n, \mathbb{C}^n)} \lesssim 1 \).

It remains to do the identification of the series as \( MG \) for \( G \) in a dense class of \( T^{1,2} \). We adapt an argument of [3] for the convenience of the reader. The subspace of those \( T^{1,2} \) functions having compact support in \( \mathbb{R}^{n+1}_+ \) is dense in \( T^{1,2} \). Remark that this space is also contained in \( L^2(\mathbb{R}^{n+1}_+, \mathbb{C}^n \otimes \mathbb{C}^n) \). Let \( G \) be in such a space and pick as before an atomic decomposition \( G = \sum_i \lambda_i a_i \). The convergence is in \( T^{1,2} \). If the series was to converge also in \( L^2(\mathbb{R}^{n+1}_+, \mathbb{C}^n \otimes \mathbb{C}^n) \) we would be done as \( M \) is continuous from \( L^2(\mathbb{R}^{n+1}_+, \mathbb{C}^n \otimes \mathbb{C}^n) \) to \( L^2(\mathbb{R}^n, \mathbb{C}^n \otimes \mathbb{C}^n) \) and the equality \( MG = \sum_i \lambda_i Ma_i \) would follow (with \( L^2 \) convergence). To see this equality with the appropriate interpretation, we proceed with a further truncation. For \( k \in \mathbb{N} \), let \( \chi_k(t, x) = \mathbb{1}_{B(0, 2^k)}(x) \mathbb{1}_{[2^{-k-1}, 2^{-k})}(t) \). Then for \( k \) large enough, \( G = G\chi_k = \sum_i \lambda_i a_i \chi_k \) with convergence in \( T^{1,2} \). Since \( a_i \) is supported in a tent \( \tilde{B}_i \), if the radius of \( B_i \) is too small, then \( a_i \chi_k = 0 \). Thus the radii are bounded below in this series and this, together with \( T^{1,2} \) convergence, implies \( L^2 \) convergence. It follows that \( MG = \sum_i \lambda_i Ma_i \) for \( k \) large enough. The calculations above implies that \( M(a_i \chi_k) \) are uniformly bounded in \( H^1 \) and also that for \( i \) fixed, \( M(a_i \chi_k) \to Ma_i \) in \( H^1 \) as \( k \to \infty \). It easily follows that \( \sum_i \lambda_i Ma_i \) converges \( H^1 \) as \( k \to \infty \) and we are done.

This finishes the proof of the fact that the operator \( M \) extends boundedly from the tent space \( T^{1,2} \) to the Hardy space \( H^1(\mathbb{R}^n, \mathbb{C}^n \otimes \mathbb{C}^n) \).

We shall further prove that for a function \( h \) in the Hardy space \( H^1(\mathbb{R}^n) \), there holds that \((s, x) \to e^{s\Delta} h(x) \in T^{1,\infty}(\mathbb{R}^n, \mathbb{C}) \) with the norm estimate

\[
\|N(e^{s\Delta} h)\|_{L^1(\mathbb{R}^n)} \lesssim \|h\|_{H^1(\mathbb{R}^n)}.
\]
2.4 Estimate of the term $A_2$

Let $x \in \mathbb{R}^n$. Note that by definition

$$N(e^{s\Delta}h)(x) = \sup_{\{(s,y):y \in B(x,\sqrt{s})\}} \left| \int_{\mathbb{R}^n} \Phi_s(y - z)h(z)\,dz \right|$$

$$= \sup_{\{(\xi,y):y \in B(x,\xi)\}} \left| \int_{\mathbb{R}^n} \Phi_{\xi^2}(z)h(y - z)\,dz \right|,$$

where $\Phi_s$ for $s > 0$ stands for the heat kernel, see Definition 1.2.3. Since the function

$$\Phi_{\xi^2}(z) = (4\pi\xi^2)^{-n/2}e^{-|z|^2/(4\xi^2)}$$

satisfies conditions of the definition of the Hardy space from Remark 2.4.4, the assertion $N(e^{s\Delta}h) \in L^1(\mathbb{R}^n)$ follows. Furthermore $(s,x) \mapsto e^{s\Delta}h(x)$ is continuous on $\mathbb{R}^{n+1}_+$ and has the desired non-tangential almost everywhere limit by standard arguments. The same of course holds component-wise for $\mathbb{C}^n \otimes \mathbb{C}^n$ valued Hardy functions.

So, we can conclude that the operator $A_2^*$ is bounded from $T^{1,2}$ to $T^{1,\infty}$ which means that the third step is proven. \qed
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