We extend our previous investigations of fermionic condensation in broad Feshbach resonances by using the two-channel model developed for narrow Feshbach resonances. We investigate two crossovers: the BCS-BEC crossover by changing the s-wave scattering length and the crossover from a narrow to a broad resonance by changing the atom-molecule coupling. At zero temperature we analyze, as a function of both atom-molecule coupling and s-wave scattering length, the chemical potential, the energy gap, and the condensate fraction of atoms. In particular, we predict the contribution of Cooper pairs and preformed molecules to the total condensate density along the two crossovers.

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I. INTRODUCTION

Nowadays there is a remarkable experimental control over ultracold alkali atoms interacting through a Feshbach resonance. Manipulation of the binding energy through external magnetic fields enables experimentalists to evolve clouds of two-component fermionic atoms from the weakly coupled BCS-like behavior of Cooper pairs to the strongly coupled Bose-Einstein condensation (BEC) of molecules [1]. The transition is characterized by a crossover in which, most simply, the s-wave scattering length $a_s$ diverges as it changes sign $2,3$. Recently, a considerable theoretical effort $4, 12$ has been expended on studying the condensate fraction of Cooper pairs of such a tunable superfluid. The behavior of tunable gases can differ according as the Feshbach resonance is broad or narrow, the former essentially describing a one-channel system, the latter a two-channel system $18–21$. In two experiments $13, 14$ with a broad Feshbach resonance the condensate fraction of Cooper pairs $15$ has been studied in two hyperfine component Fermi vapours of $^6$Li atoms. The experimental data of the condensate fraction, which is directly related to the off-diagonal-long-range order of the two-body density matrix of fermions $16, 17$, are in quite good agreement with broad-resonance mean-field theoretical predictions $4, 5$ at zero temperature, while at finite temperature beyond-mean-field corrections are needed $16$.

In this paper we extend our previous investigations of fermionic condensation in broad Feshbach resonances $4, 7, 9, 12$ by analyzing the condensate fraction of Cooper pairs and the condensate fraction of preformed molecules in the case of a narrow Feshbach resonance. We find that by increasing the resonant atom-molecule coupling the condensate fraction of preformed molecules is strongly reduced in the BEC regime while the condensate fraction of Cooper-paired atoms grows reaching a maximum value close to the unitarity limit (infinite scattering length). Finally, for very large values of the resonant coupling we recover the broad-resonance regime where there are no more preformed molecules and the BCS-BEC crossover is entirely due to Cooper-paired atoms.

II. EFFECTIVE ACTION AND SADDLE-POINT APPROXIMATION

We adopt the path integral formalism $22$ and consider a mixture of fermionic atoms and molecular bosons, in which the fermions, described by the complex Grassmann fields $\psi_\sigma (r, \tau)$, with spin $\sigma = (\uparrow, \downarrow)$, can be bound into a molecular boson described by the complex scalar field $\phi (r, \tau)$ through a Feshbach resonance $13, 20$. The partition function $Z$ of the system at temperature $T$ can be written as

$$Z = \int D\psi_\sigma D\psi_\sigma^* D\phi D\phi^* \exp (-S/\hbar),$$

where the Euclidean action functional $S$ is given by

$$S = \int_0^{\beta \hbar} d\tau \int_V (\mathcal{L}_F + \mathcal{L}_B + \mathcal{L}_{FB})$$

with

$$\mathcal{L}_F = \sum_{\sigma = \uparrow, \downarrow} \psi_\sigma^* \left[ \hbar \frac{\partial}{\partial \tau} - \frac{\hbar^2}{2m} \nabla^2 - \mu \right] \psi_\sigma$$

the Lagrangian density of free fermionic atoms,

$$\mathcal{L}_B = \phi^* \left[ \hbar \frac{\partial}{\partial \tau} - \frac{\hbar^2}{4m} \nabla^2 - 2\mu + \epsilon_0 \right] \phi$$

the Lagrangian density of free bosonic molecules, and

$$\mathcal{L}_{FB} = g \left( \phi^* \psi_\uparrow + \phi \psi_\downarrow^* \right)$$

the Lagrangian density of fermion-molecule coupling. Notice that $\beta = 1/(k_B T)$ with $k_B$ the Boltzmann constant and $V$ is the volume of the system. Moreover, the bound molecular bosons of Feshbach resonance have
twice the mass of the fermions and a tunable binding energy $\epsilon_0$.

The action functional $S$ is quadratic in the fermionic fields $\psi_n(r, \tau)$, which can be then integrated out exactly obtaining

$$Z = \int D\phi D\phi^* \exp(-S_c/h),$$

where $S_c$ is the effective action, given by

$$S_c = -h Tr[\ln (\mathcal{G}^{-1})] + \int_0^{h\beta} d\tau \int d^3r \mathcal{L}_B$$

with $\mathcal{G}^{-1}(r, r', \tau, \tau')$ the inverse Green-Nambu function, defined as

$$\mathcal{G}^{-1}(r, r', \tau, \tau') = \frac{1}{h} \left( \frac{h}{g} \frac{\partial}{\partial \tau} - \frac{\hbar^2 \nabla^2}{2m} - \mu - \int d^3k \left( g \phi(r, \tau) \right) \frac{\hbar}{2} \frac{\partial}{\partial \tau} + \frac{\hbar^2 \nabla^2}{2m} \right) \delta(r-r') \delta(\tau-\tau').$$

The total number density $n$ of atoms in the system is instead obtained from the thermodynamic formula $n = -\partial \Omega_{sp}/\partial \mu$, which gives

$$n = \int \frac{d^3k}{(2\pi)^3} \left[ 1 - (\bar{u}_k^2 - \bar{v}_k^2) \tanh (\beta E_k/2) \right] + 2\phi_0^2.$$  (15)

The broad-resonance regime is easily obtained in the limit $g \to +\infty$ with the constraint of a finite energy gap $\Delta$ which implies $\phi_0 \to 0$. Notice that in the literature it is common to study models that in addition to the two-channel Feshbach resonant interaction considered above, a featureless nonresonant four-Fermi atomic interaction is also included. Guraire and Radzihovsky have shown that in three dimensions doing so does not add any new physics to the pure two-channel model considered here.

### III. CONDENSATION

The total condensate density $n_0$ of the system is the sum of two contributions, namely $n_0 = n_{F,0} + n_{B,0}$, where $n_{F,0} = 2(\langle |\psi_1(r, \tau)|^2 \rangle |\psi_2(r, \tau)|^2)$ is the condensate density of Cooper-paired atoms and $n_{B,0} = 2(\langle |\phi(r, \tau)|^2 \rangle)^2$ is the condensate density of atoms in the preformed bosonic molecules. Within the saddle-point approximation we have $n_{B,0} = 2\phi_0^2$ and taking into account previous results on the condensate fraction of Cooper-paired atoms we finally obtain

$$n_0 = \int \frac{d^3k}{(2\pi)^3} \left[ u_k^2 v_k^2 \tanh^2 (\beta E_k/2) \right] + 2\phi_0^2.$$  (17)
As discussed in \[19\], at zero temperature the saddle-point approximation gives reliable results while at finite temperature it is necessary to include fluctuations about the saddle point \(\phi_0\); fluctuations are particularly important close to the critical temperature \(T_c\) of the super-to-normal phase transition. For this reason we analyze Cooper pairing and Bose-Einstein condensation in narrow Feshbach resonances by using Eqs. (14), (15) and (16) only at zero temperature, where they reduce to the following ones:

\[
y = -\frac{2}{\pi} I_1\left(\frac{\mu}{\Delta}\right),
\]

\[
1 = 3 \frac{\gamma}{2} \frac{\Delta}{2} I_2\left(\frac{\mu}{\Delta}\right) + 2 \frac{\Delta^2}{\gamma^2},
\]

\[
n_0/n = -\frac{3\pi}{2\sqrt{\Delta}^{3/2}} I_2\left(\frac{\mu}{\Delta}\right) + 2 \frac{\Delta^2}{\gamma^2},
\]

where \(\mu = \mu/\epsilon_F\) is the adimensional chemical potential with \(\epsilon_F = \hbar^2 k_F^2/(2m)\) the Fermi energy and \(k_F = (3\pi^2 n)^{1/3}\) is the Fermi wavenumber, \(\Delta\) is the adimensional energy gap, \(y = 1/(k_F a_s)\) is the adimensional inverse scattering length, \(\gamma = gn^{1/2}/\epsilon_F\) is the adimensional resonant atom-molecule coupling, \(I_1(x)\) and \(I_2(x)\) are the two monotonic functions

\[
I_1(x) = \int_0^{+\infty} z^2 \left(\frac{1}{\sqrt{(z^2 - x)^2 + 1}} - \frac{1}{z^2}\right) dz,
\]

\[
I_2(x) = \int_0^{+\infty} z^2 \left(1 - \frac{z^2 - x}{\sqrt{(z^2 - x)^2 + 1}}\right) dz,
\]

which can be expressed in terms of elliptic integrals.

We have solved Eqs. (13) and (15) numerically obtaining the scaled (adimensional) chemical potential \(\mu/\epsilon_F\) and the scaled energy gap \(\Delta/\epsilon_F\) as a function of \(y = 1/(k_F a_s)\) for different values of \(\gamma = gn^{1/2}/\epsilon_F\). The results are shown in Fig. 1. Remarkably the chemical potential \(\mu\) does not depend very much on the resonant atom-molecule coupling \(\gamma\) apart close to the unitarity region \((-1 < y < 1)\). Instead the behavior the energy gap \(\Delta\) depends strongly on \(\gamma\) in the BEC region \((y > 1)\). In particular, in the limit \(y \rightarrow +\infty\) the energy gap \(\Delta\) goes to a value which is obtained as the solution of Eq. (19) with \(I_2(-\infty) = 0\), namely \(\Delta/\epsilon_F = \gamma/\sqrt{2}\). Taking into account Eq. (19), this result means that in the limit \(y \rightarrow +\infty\) one always has \(n_{B,0} = n\), for any finite value of \(\gamma\).

In the upper panel of Fig. 2 we plot the total condensate fraction \(n_0/n\) as a function of the scaled inverse scattering length \(1/(k_F a_s)\). Middle panel: Condensate fraction \(n_{F,0}/n\) of Cooper paired atoms as a function of the scaled inverse scattering length \(1/(k_F a_s)\). Lower panel: Condensate fraction \(n_{F,0}/n\) of atoms in preformed molecules as a function of the scaled inverse scattering length \(1/(k_F a_s)\). The curves correspond to different values of the scaled atom-molecule coupling \(\gamma = gn^{1/2}/\epsilon_F\).
sate fraction \( n_0/n \), obtained from Eq. (21), as a function of the scaled inverse scattering length \( 1/(k_Fa_s) \) for four values of \( \gamma \). For a very small atom-molecule coupling, i.e. \( \gamma = 0.1 \) (dot-dot-dashed curves), the total condensate fraction becomes equal to 1 at unitarity \( y = 0 \). Indeed in this regime of weak-coupling the fraction \( n_0/n \) is mainly due to preformed molecules, as clearly shown in the middle panel, where we plot the condensate fraction \( n_{F,0}/n \) of Cooper paired atoms, and in the lower panel, where we plot the condensate fraction \( n_{B,0}/n \) of atoms in preformed molecules. By increasing the resonant coupling \( \gamma \), see for instance dot-dashed \( (\gamma = 1) \) and dashed \( (\gamma = 10) \) curves, the value of the total condensate fraction at unitarity \( (y = 0) \) decreases. Moreover, by increasing \( \gamma \) in the BEC region \( (y > 1) \) it appears a finite condensate fraction of Cooper-paired (and bound) atoms while the fraction of condensed preformed molecules is strongly reduced. Finally, for very large values of \( \gamma \), e.g. \( \gamma = 50 \) (solid curves) of Fig. 2 in practice (i.e. in a large range of \( y \) ) one recovers the broad-resonance limit where there are not preformed molecules and the BCS-BEC crossover is entirely due to Cooper-paired atoms.

Within the zero-temperature saddle-point approximation we have used, the condensate density \( n_{F,0} \) of fermions is smaller than the total density \( n_F \) of fermions, while the condensate density \( n_{B,0} \) of preformed molecules coincides with the total density \( n_B \) of preformed molecules. In general, as shown by Werner, Tarrell, and Castin [24], for the two-channel model the density \( n_B \) (but not \( n_{B,0} \) nor \( n_{F,0} \) ) can be directly related to the derivative of the energy gap with respect to the inverse scattering length. The zero-temperature condensate depletion \( (n_B - n_{B,0})/n_B \) of bosonic molecules can be instead obtained going beyond the saddle-point approximation by including quadratic fluctuations of \( \phi(r,\tau) \) around \( \phi_0 \) in the action functional.

**IV. CONCLUSIONS**

In this paper we have used the two-channel model of atomic fermions coupled to preformed bosonic molecules to study two crossovers: the BCS-BEC crossover driven by the s-wave scattering length and the crossover from a narrow to a broad resonance controlled by the atom-molecule coupling. Within the saddle-point approximation of the path integral formalism we have derived an extended gap equation which depends on both the s-wave scattering length and the atom-molecule coupling. Only for a very large atom-molecule coupling the extended gap equation reduces to the one of broad Feshbach resonances. At zero temperature, we have calculated the chemical potential and the energy gap of the system. Remarkably, in the deep BEC region of the BCS-BEC crossover the energy gap \( \Delta \) goes to the asymptotic value \( g n^{1/2}/\sqrt{2} \), where \( g \) the atom-molecule coupling and \( n \) is the total atomic density. We have then analyzed the condensate density and the condensate fraction of the system. We have found that by increasing the resonant atom-molecule coupling the condensate fraction of preformed molecules is strongly reduced in the BEC regime with a corresponding growth in the condensate fraction of Cooper-paired atoms. As expected, for very large values of the resonant coupling one recovers the broad-resonance regime characterized by the absence of preformed molecules.

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