Convexity and quasi-uniformizability of closed preordered spaces

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Abstract

In many applications it is important to establish if a given topological preordered space has a topology and a preorder which can be recovered from the set of continuous isotone functions. Under antisymmetry this property, also known as quasi-uniformizability, allows one to compactify the topological space and to extend its order dynamics. In this work we study locally compact \( \sigma \)-compact spaces endowed with a closed preorder. They are known to be normally preordered, and it is proved here that if they are locally convex, then they are convex, in the sense that the upper and lower topologies generate the topology. As a consequence, under local convexity they are quasi-uniformizable. The problem of establishing local convexity under antisymmetry is studied. It is proved that local convexity holds provided the convex hull of any compact set is compact. Furthermore, it is proved that local convexity holds whenever the preorder is compactly generated, a case which includes most examples of interest, including preorders determined by cone structures over differentiable manifolds. The work ends with some results on the problem of quasi-pseudo-metrizability. As an application, it is shown that every stably causal spacetime is quasi-uniformizable and every globally hyperbolic spacetime is strictly quasi-pseudo-metrizable.

1. Introduction

Topological preordered spaces are ubiquitous. They appear in the study of dynamical systems [1], general relativity [2], microeconomics [3, 4], thermodynamics [5] and computer science [6]. In these applications it is important to establish if a topological preordered space \((E, \mathcal{T}, \leq)\) is quasi-uniformizable, namely, if there is a quasi-uniformity \(\mathcal{U}\) such that \(\mathcal{T} = \mathcal{T}(\mathcal{U}^*)\) and \(G(\leq) = \bigcap \mathcal{U}\). Taking into account a characterization of quasi-uniformizability established by Nachbin [7], this problem is equivalent to that of establishing if the topology and the preorder of the space are determined by the family of continuous isotone functions.
Hausdorff quasi-uniformizable spaces are compactifiable [7, 8] and, in general, the possibility of restricting an analysis to the compact case brings several simplifications. In other circumstances, the boundary (remainder) involved in the compactification has special importance. For instance, a good definition of spacetime boundary in general relativity would allow us to identify the singular spacetime points [9].

Quasi-uniformizable spaces are $T_2$-preordered spaces, thus $\leq$ must be closed in order to have any chance to come from a quasi-uniformity. In the various fields that in one way or the other are connected with topological preordered spaces, it has been discovered that it is indeed very convenient to study some new closed preorder related to the original preorder. This is the strategy of ‘prolongations’ introduced by Auslander in dynamical systems [10], and rediscovered in a different setting in relativity theory, where Seifert [11, 12] introduced a closed relation related to the causal relation, and Sorkin and Woolgar [13] introduced the smallest closed relation containing the causal relation (see Sect. 1.1).

Given a $T_2$-preordered space $(E, \mathcal{T}, \leq)$ it is possible to infer preorder normality provided $(E, \mathcal{T})$ is a $k_\omega$-space [14]. We recall that a topological space is a $k_\omega$-space if there is a (admissible) sequence of compact sets $K_i, \bigcup_i K_i = E$, such that $O \subset E$ is open if and only if $O \cap K_i$ is open in $K_i$. It is not restrictive to assume $K_i \subset K_{i+1}$, and $K_1$ equal to any chosen compact set. In this work we shall use the fact that locally compact $\sigma$-compact spaces are $k_\omega$-spaces. Indeed, under local compactness the properties: hemicompact $k_\omega$-space, $k_\omega$-space, $\sigma$-compact, and Lindelöf are equivalent [13]. Since we do not assume that $E$ is Hausdorff, we remark that in our terminology a topological space is locally compact if each point has a compact neighborhood. It is strongly locally compact if at each point the neighborhood system of the point has a base made of compact neighborhoods (not necessarily closed).

Convex normally preordered spaces are quasi-uniformizable [8, Prop. 4.7] (i.e. they are completely regular preordered spaces [7]), and quasi-uniformizable spaces are convex closed preordered spaces. Unfortunately, although a $T_2$-preordered $k_\omega$-space is normally preordered, preorder normality does not imply quasi-uniformizability as convexity is missing. Indeed, we shall give an example of a $T_2$-preordered locally compact $\sigma$-compact space which is not convex (see example 1.5).

This work is devoted to the proof of the convexity and hence quasi-uniformizability of a large class of locally compact $\sigma$-compact closed preordered spaces.

The main result of this work is the proof that for these preordered spaces local convexity and convexity are equivalent (Cor. 2.14). We then proceed to study local convexity, showing that it follows from antisymmetry plus some other assumptions. We prove that local convexity holds for $k$-preserving spaces (Theor. 5.3), namely for those spaces for which the convex hull of any compact set is compact. The definition of $k$-preserving space is quite important for the connection with global hyperbolicity in relativity theory [10] (see Sect. 1.1).

Furthermore, we show that if the order is compactly generated then local convexity holds (Cor. 4.2). Joining this result with the previous one we infer
that if, roughly speaking, both the topology and order are generated ‘locally’ then convexity holds (Cor. 4.14). This case includes most examples of topological preordered spaces of interest, including those in which the preorder is induced by a distribution of tangent cones on a differentiable manifold [17]. We shall compare our findings with similar results obtained by Akin and Auslander in the study of dynamical systems [18].

Finally, under second countability we obtain a result on the quasi-pseudometrizability of the space which generalizes Urysohn’s theorem (Theor. 5.1), and under the $I$-space condition we are able to assure the strict quasi-pseudometrizability of the space (Theor. 5.3). As an application, we prove that globally hyperbolic spacetimes (see Sect. 1.1 for the definition) are strictly quasi-pseudometrizable.

1.1. Some reference results on mathematical relativity and causality theory

At places we shall illustrate our findings using the topological ordered space given by the spacetime manifold ordered with a causal relation. Therefore, it is worth recalling some definitions and result from this field. The reader can skip this section on first reading, returning to it whenever this application is mentioned.

Let $M$ be a Hausdorff, connected, paracompact $(C^{r+1}, r \geq 0)$ manifold and let $g : M \to T^*M \otimes T^*M$ be a $(C^r, r \geq 0)$ Lorentzian metric, namely a pseudo-Riemannian metric of signature $(-, +, \cdots, +)$. Non vanishing tangent vectors split into spacelike, lightlike or timelike depending on the sign of $g(v, v)$, $v \in TM$, respectively positive, null or negative. Lightlike or timelike vectors are called causal. Assume that a continuous timelike vector field can be defined over $M$, and call future the cone of causal vectors including it. If this is not possible there is always a double covering of $M$ with this property, thus this is not a severe restriction. Once such a choice of future cone has been made, the Lorentzian manifold is time oriented. A spacetime $(M, g)$ is a time oriented Lorentzian manifold. The simplest example of spacetime is the 2-dimensional Minkowski spacetime, namely $\mathbb{R}^2$ with coordinates $(t, x)$, metric $g = -dt^2 + dx^2$ and time orientation given by the global timelike vector $\partial_t$.

Let us observe that once a time orientation is given, any causal vector is either future directed or past directed depending on whether it belongs to the future cone. This terminology extends to $C^1$ curves depending on the character of their tangent vector, provided it is consistent throughout the curve.

The causal relation $J^+ \subset M \times M$ over $M$ is defined through: $(x, y) \in J^+$ if there is a future directed causal curve from $x$ to $y$ or $x = y$. The chronology relation $I^+ \subset M \times M$ over $M$ is defined through: $(x, y) \in I^+$ if there is a future directed timelike curve from $x$ to $y$. We have $J^+ \circ I^+ = I^+ \circ J^+ = I^+$, and $I^+$ is open in the product topology [19, 16]. Unfortunately, the causal relation is not necessarily closed, as can be easily realized considering the spacetime which is obtained removing a point from the 2-dimensional Minkowski spacetime.

The relation $K^+$ is by definition the smallest closed and transitive relation containing $J^+$ and it exists because $R := M \times M$ provides an example
of closed and transitive relation containing \( J^+ \). Unfortunately, it is difficult to work with \( K^+ \) since it is defined through its closure and transitivity properties rather than through the more intuitive notion of causal curve. Seifert [11] found another route to build a closed and transitive relation. Let us write \( g' > g \) if the timelike cones of \( g' \) contain the causal cones of \( g \), and let \( J^+_S \) be the causal relation for \((M, g')\). Seifert proved that \( J^+_S := \bigcap_{g' > g} J^+_g \) is indeed closed, transitive and contains \( J^+ \).

A spacetime \((M, g)\) is said to be causal if it does not contain any closed causal curve. It is stably causal if there is \( g' > g \) such that \((M, g')\) is causal, namely if it is possible to open the light cones everywhere over \( M \) without introducing closed causal curves. A relation \( R \) is antisymmetric if \((x, y) \in R \) and \((y, x) \in R \) implies \( x = y \). It can be proved that the spacetime is causal (resp. stably causal) iff \( J^+ \) (resp. \( J^+_S \)) is antisymmetric [12]. It also turns out [20] that stable causality holds iff \( K^+ \) is antisymmetric, and in this case \( K^+ = J^+_S \). Thus \( J^+_S \) is really the most natural closed and transitive relation that can be introduced in a stably causal spacetime.

Let us write \( J^+(x) := \{ y : (x, y) \in J^+ \} \) and \( J^-(y) := \{ x : (x, y) \in J^+ \} \), and if \( X \subset M \), let \( J^+(X) := \bigcup_{x \in X} J^+(x) \). A spacetime is causally continuous if the relation

\[
D^+ = \{ (x, y) : y \in J^+(x) \text{ and } x \in J^-(y) \},
\]

is antisymmetric (a property known as weak distinction) and coincides with \( J^+ \) (a property known as reflectivity). It is not hard to prove [21] that \( D^+ \) is transitive, thus under causal continuity \( J^+ \) is closed, transitive and contains \( J^+ \). As a consequence, it is the smallest relation with such properties, \( K^+ = J^+ \), and hence causal continuity implies stable causality.

A spacetime is causally simple if it is causal and \( J^+ \) is closed. Clearly, under causal simplicity \( D^+ = J^+ \), thus causal simplicity implies causal continuity (note that under causal simplicity we have also \( J^+ = K^+ = J^+_S \)).

Another important causality property is global hyperbolicity. A spacetime \((M, g)\) is globally hyperbolic if it is causal and for every compact set \( K \), its convex causal hull \( J^+(K) \cap J^-(K) \) is compact. It can be shown that every globally hyperbolic spacetime is causally simple [19]. These spacetimes are the most studied in mathematical relativity because a spacetime is globally hyperbolic iff it admits a Cauchy hypersurface, namely a topological hypersurface intersected by any inextendible (i.e. with no endpoint) causal curve in exactly one point [10]. Therefore, they are the spacetimes for which the Cauchy problem of general relativity and that of wave equations makes sense.

1.2. Preliminaries on topological preordered spaces

A topological preordered space is a triple \((E, \mathcal{F}, \leq)\) where \((E, \mathcal{F})\) is a topological space and \( \leq \) is a preorder on \( E \), namely a reflexive and transitive relation. A preorder is an order if it is antisymmetric (that is, \( x \leq y \) and \( y \leq x \) implies \( x = y \)). For a topological preordered space \((E, \mathcal{F}, \leq)\) our terminology follows Nachbin [7]. With \( i(x) = \{ y : x \leq y \} \) and \( d(x) = \{ y : y \leq x \} \) we denote the increasing and decreasing hulls, and we define \([x] = d(x) \cap i(x)\). The topological preordered
space is $T_1$-preordered (or semiclosed preordered) if $i(x)$ and $d(x)$ are closed for every $x \in E$, and it is $T_2$-preordered (or closed preordered) if the graph of the preorder $G(\leq) = \{(x, y) : x \leq y\}$ is closed.

Let $S \subset E$, we define $i(S) = \bigcup_{x \in S} i(x)$ and analogously for $d(S)$. A subset $S \subset E$, is called increasing if $i(S) = S$ and decreasing if $d(S) = S$. It is called monotone if it is increasing or decreasing. With $I(S)$ we denote the smallest closed increasing set containing $S$, and with $D(S)$ we denote the smallest closed decreasing set containing $S$. A subset $C$ is convex if it is the intersection of a decreasing and an increasing set in which case $C = d(C) \cap i(C)$. A subset $C$ is a $c$-set if it is the intersection of a closed decreasing and a closed increasing set in which case $C = D(C) \cap I(C)$. The neighborhood of a point which is a $c$-set is a $c$-neighborhood, and a $c$-set which is compact is a $c$-compact set. In the notation of this work the set inclusion $\subset$, is reflexive, i.e. $X \subset X$.

A topological preordered space is a normally preordered space if it is $T_1$-preordered and for every closed decreasing set $A$ and closed increasing set $B$ which are disjoint, $A \cap B = \emptyset$, it is possible to find an open decreasing set $U$ and an open increasing set $V$ which separate them, namely $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$.

Given a reflexive relation $R$ on $E$, a function $f : E \rightarrow \mathbb{R}$ such that $(x, y) \in R \Rightarrow f(x) \leq f(y)$ is an isotone function. An isotone function such that $(x, y) \in R$ and $(y, x) \notin R \Rightarrow f(x) < f(y)$ is a utility function.

In a normally preordered space, closed disjoint monotone sets as $A$ and $B$ above can be separated by a continuous isotone function $f : E \rightarrow [0, 1]$, that is $A \subset f^{-1}(0)$, $B \subset f^{-1}(1)$ (this is the preorder analog of Urysohn’s separation lemma, see [22, Theor. 1]). Normally preordered spaces are $T_2$-preordered spaces, and $T_2$-preordered spaces are $T_1$-preordered spaces.

A topological preordered space $E$ is convex at $x \in E$, if for every open neighborhood $O \ni x$, there are an open decreasing set $U$ and an open increasing set $V$ such that $x \in U \cap V \subset O$ (this definition is due to Nachbin [22] and is used in [24, 22, 25], though the terminology is not uniform in the literature). It is locally convex at $x \in E$ if the set of convex neighborhoods of $x$ is a base for the neighborhoods system of this point [22, 3]. It is weakly convex at $x \in E$ if the set of convex open neighborhoods of $x$ is a base for the neighborhoods system of this point [23, 26]. The topological preordered space $E$ is convex (locally convex, weakly convex) if it is convex (resp. locally convex, weakly convex) at every point. Clearly, convexity (at a point) implies weak convexity (at a point) which in turn implies local convexity (at a point). Notice that according to this terminology the statement “the topological preordered space $E$ is convex” differs from the statement “the subset $E$ is convex” (which is always true).

A quasi-uniformity $\mathcal{F}[X]$ is a pair $(X, \mathcal{U})$ such that $\mathcal{U}$ is a filter on $X \times X$, whose elements contain the diagonal $\Delta$, and such that if $V \in \mathcal{U}$ then there is $W \in \mathcal{U}$, such that $W \circ W \subset V$. A quasi-uniformity is a uniformity if $V \in \mathcal{U}$ implies $V^{-1} \in \mathcal{U}$, where $V^{-1} = \{(x, y) : (y, x) \in V\}$. To any quasi-uniformity $\mathcal{U}$ corresponds a dual quasi-uniformity $\mathcal{U}^{-1} = \{U : U^{-1} \in \mathcal{U}\}$.

From a quasi-uniformity $\mathcal{U}$ it is possible to construct a topology $\mathcal{T}(\mathcal{U})$ in such a way that a base for the filter of neighborhoods at $x$ is given by the sets
of the form $U(x)$ where $U(x) = \{y : (x, y) \in U\}$ with $U \in \mathcal{U}$. In other words, $O \in \mathcal{F}(\mathcal{U})$ if for every $x \in O$ there is $U \in \mathcal{U}$ such that $U(x) \subseteq O$.

Given a quasi-uniformity $\mathcal{U}$, the family $\mathcal{U}^*$ given by the sets of the form $V \cap W^{-1}$, $V, W \in \mathcal{U}$, is the coarsest uniformity containing $\mathcal{U}$. The symmetric topology of the quasi-uniformity is $\mathcal{T}(\mathcal{U}^*)$. Moreover, the intersection $\bigcap \mathcal{U}$ is the graph of a preorder on $X$ (see [7]), thus given a quasi-uniformity one naturally obtains a topological preordered space $(X, \mathcal{F}(\mathcal{U}^*), \bigcap \mathcal{U})$. The topology $\mathcal{T}(\mathcal{U}^*)$ is Hausdorff if and only if the preorder $\bigcap \mathcal{U}$ is an order [7].

Nachbin proves [7, Prop. 8] that a topological preordered space $(E, \mathcal{F}, \leq)$ comes from a quasi-uniformity $\mathcal{U}$, in the sense that $\mathcal{F} = \mathcal{F}(\mathcal{U}^*)$ and $G(\leq) = \bigcap \mathcal{U}$, if and only if $E$ is a completely regularly preordered space ($T_{3\frac{1}{2}}$-preordered space, Tychonoff-preordered space), namely if and only if the following two conditions hold:

(i) $\mathcal{F}$ coincides with the initial topology generated by the set of continuous isotone functions $g : E \to [0, 1]$;

(ii) $x \leq y$ if and only if for every continuous isotone function $f : E \to [0, 1]$, $f(x) \leq f(y)$.

Completely regularly preordered spaces are convex $T_2$-preordered spaces (convexity follows from (i) see [7, Prop. 6, Cap.II], and the closure of the preorder follows from (ii)). Contrary to what happens in the usual discrete-preorder case, normally preordered spaces need not be completely regularly preordered spaces (see example [1.5], nevertheless the preorder analog of Urysohn’s separation lemma implies that convex normally preordered spaces are completely regularly preordered spaces. Completely regularly ordered spaces admit the Nachbin’s $T_2$-ordered compactification $nE$ (see [8] and [27] for the preorder case).

### 1.3. Preliminary results on convexity

A theorem by Nachbin states that every compact $T_2$-ordered space is convex [6, p. 48]. Unfortunately, this theorem assumes the compactness of the space from the start, and hence it is not really useful in applications. There one would like to pass through convexity exactly to prove quasi-uniformizability, so as to introduce and work in the compactified space.

The most common strategy is then that of adding some additional conditions to the preorder such as the $C$-space and $I$-space conditions [28] (compare with the definitions of continuous and anti-continuous preorder in [26, 24]). A topological preordered space $E$ is a $C$-space ($I$-space) if for every closed (open) subset $S$, $d(S)$ and $i(S)$ are closed (resp. open).

The following theorem and proof are due to H.-P. Künzi [29, Lemma 2]. They are included for the reader convenience.

**Theorem 1.1.** Every normal $T_1$-ordered $C$-space $(E, \mathcal{F}, \leq)$ is convex.

**Proof.** Let $O$ be an open neighborhood of $x \in E$. The closed sets $d(x) \setminus O$ and $i(x) \setminus O$ are disjoint. By normality these sets can be separated by open sets, say
Consider $H_1$ and $H_2$, then $d(x) \subset H_1 \cup O$ and $i(x) \subset H_2 \cup O$. The set $E \setminus (H_1 \cup O)$ is closed and is disjoint from $d(x)$. By the $C$-space assumption $i(E \setminus (H_1 \cup O))$ is closed and is disjoint from $d(x)$ thus $U = E \setminus i(E \setminus (H_1 \cup O))$ is an open decreasing set such that $d(x) \subset U \subset H_1 \cup O$. Analogously there is an open increasing set $V$ such that $i(x) \subset V \subset H_2 \cup O$. Thus $x \in (U \cap V) \subset (H_1 \cup O) \cap (H_2 \cup O) \subset O$. Hence the space is convex.

Unfortunately the $C$-space condition is too strong as not even $\mathbb{R}^2$ with the product order is a $C$-space (consider the increasing hull of the closed set $S = \{(x,y): x < 0 , y > 0 , y = -1/x\}$.

Concerning the $I$-space property we have the following simplification.

**Theorem 1.2.** Every locally convex $I$-space $(E, \mathcal{T}, \leq)$ is convex.

**Proof.** Let $x \in E$ and let $O$ be an open neighborhood of $x$. By local convexity there are a convex set $C$ and an open set $O'$ such that $x \in O' \subset C \subset O$. Since $E$ is an $I$-space the sets $V = i(O')$ and $U = d(O')$ are respectively open increasing and open decreasing. Furthermore, $x \in U \cap V \subset d(C) \cap i(C) = C \subset O$, which proves that $E$ is convex.

In this connection, the next interesting result due to Burgess and Fitzpatrick [24, Cor. 4.4] is worth mentioning.

**Theorem 1.3.** Every locally compact convex $T_2$-ordered $I$-space is completely regularly ordered.

**Remark 1.4.** The $I$-space property is sometimes justified in applications. For instance, in general relativity (see Sect. 1.1) the closure $\overline{\mathcal{J}^\tau}$ of the causal relation in a causally continuous spacetime provides a preorder which turns spacetime into a topological closed preordered $I$-space [2].

In this work we shall try to avoid as much as possible the simplifying $C$-space and $I$-space assumptions, and we shall instead impose weak conditions on the preorder and the topology in order to attain convexity. We shall meet again the $I$-space assumption at the end of this work, where it is used in connection with strict quasi-pseudo-metrizability.

We end the section with examples which show that a normally preordered space need not be convex. An example can be found in [8, Example 4.9].

A locally compact $\sigma$-compact $T_2$-ordered space which is not locally convex can be found in [8, p. 59]. The next example is particularly interesting because the topology has nice properties.

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1 Proof: we have mentioned in Sect. 1.1 that under causal continuity $D^+ = \overline{\mathcal{J}^\tau} = K^+$ is antisymmetric, thus the spacetime is stably causal and $(M, \mathcal{T}, \overline{\mathcal{J}^\tau})$ is a closed ordered space. Let $O \subset M$ be an open set, and let $(x, y) \in D^+$ where $x \in O$, so that $x \in \overline{\mathcal{J}^\tau}(y)$. Pick $x' \in \mathcal{I}^+(x) \cap O$, then $x \in \mathcal{I}^+(x')$ and hence $y \in \mathcal{J}^+(x')$. Since $\mathcal{I}^+$ is open and $\mathcal{I}^+ \subset \overline{\mathcal{J}^\tau}$, we conclude that a neighborhood of $y$ is contained in $D^+(O)$ and hence that $D^+(O)$ is open.
**Example 1.5.** Let $E = (0, 1] \subset \mathbb{R}$ with the induced topology which we denote $\mathcal{T}$. The topology is particularly well behaved, it is connected, metrizable, locally compact, $\sigma$-compact, second countable. Define on $E$ the order $\preceq$ through the following increasing hulls

$$i(x) = \{ y \in (0, 1] : x \leq y \leq 1 - x \} \quad \text{if } 0 < x \leq 1/2,$$
$$i(x) = \{ x \} \quad \text{if } 1/2 < x < 1,$$
$$i(x) = (0, 1] \quad \text{if } x = 1.$$  

With this definition the decreasing hulls are

$$d(x) = \{ y \in (0, 1] : 0 < y \leq x \text{ or } y = 1 \} \quad \text{if } 0 < x \leq 1/2,$$
$$d(x) = \{ y \in (0, 1] : 0 < y \leq 1 - x \text{ or } y = x \text{ or } y = 1 \} \quad \text{if } 1/2 < x \leq 1.$$  

It is easy to check that $\preceq$ is reflexive, transitive and antisymmetric and hence an order. $E$ with this order is a $T_2$-ordered space, indeed let $x_n \preceq y_n$ with $(x_n, y_n) \to (x, y)$. If $x = 1$ then necessarily as $i(x) = E$, $y \in i(x)$. If $1/2 < x < 1$ then for sufficiently large $n$, $1/2 < x_n < 1$ thus $y_n = x_n$ and then $y = \lim y_n = \lim x_n = x$ that is $y \in i(x)$. If $0 < x \leq 1/2$ then we can assume, up to a subsequence, that either for all $n$, $1/2 < x_n < 1$ (and hence $x = 1/2$), or $0 < x_n \leq 1/2$. In the former case $y_n = x_n$ and then $y = x = 1/2$ thus $y \in i(x)$, while in the latter case passing to the limit the equation $x_n \preceq y_n \leq 1 - x_n$ we get $x \leq y \leq 1 - x$ that is $y \in i(x)$ which concludes the proof. Let us observe that $\mathcal{T}$ is second countable and locally compact which implies that $(E, \mathcal{T}, \preceq)$ is a normally ordered space\footnote{\cite{Nachbin}.}. Nevertheless, convexity does not hold at $x = 1$ and in fact even local convexity fails there because every convex neighborhood of 1 contains points ‘arbitrarily close to the lower edge at 0’.

### 2. From local convexity to convexity

The mentioned examples of $T_2$-preordered locally compact $\sigma$-compact spaces which are not convex are also non-locally convex. This fact suggests that, perhaps, we could obtain convexity by assuming local convexity plus some topological property. This is indeed the case and in this section we shall prove that a locally convex $T_2$-preordered locally compact $\sigma$-compact space is necessarily convex. This result is important because it is often much easier to prove local convexity than convexity. The next two sections will then show how to obtain local convexity for a large class of topological preordered spaces.

We need to state the next two propositions which generalize to preorders two corresponding propositions due to Nachbin\footnote{\cite{Nachbin}, Prop. 4.5, Chap. I}. Actually the proofs given by Nachbin for the order case work unaltered. For this reason they are omitted.

**Proposition 2.1.** Let $E$ be a $T_2$-preordered space. For every compact set $K \subset E$, we have $d(K) = D(K)$ and $i(K) = I(K)$, that is, the decreasing and increasing hulls are closed.
Proposition 2.2. Let $E$ be a $T_2$-preordered compact space. Let $F \subset V$ where $F$ is increasing and $V$ is open, then there is an open increasing set $W$ such that $F \subset W \subset V$. An analogous statement holds in the decreasing case.

We start with a convex analog to the previous proposition.

Lemma 2.3. Let $E$ be a normally preordered space, let $A$ be a closed decreasing set and let $B$ be a closed increasing set. Finally, let $S$ be a compact set and let $O$ be an open set such that $A \cap B \cap S \subset O$, then there are an open decreasing set $U \supset A$ and an open increasing set $V \supset B$, such that $D(U) \cap I(V) \cap S \subset O$.

Proof. The set $K = S \setminus O$ being a closed subset of a compact set is compact. Let $y \in K$, we know that $y \notin A$ or $y \notin B$. In the former case there is an open increasing set $M_y \ni y$ and an open decreasing set $U_y \supset A$ such that $U_y \cap M_y = \emptyset$. If $y \in A$ (and hence $y \notin B$) there is an open decreasing set $M_y \ni y$ and an open increasing set $V_y \supset B$ such that $V_y \cap M_y = \emptyset$. Since $K$ is compact there are some $y_i, i \in \Lambda, \Lambda = \{1, 2, \cdots, n\}$, such that the sets $M_{y_i}$ cover $K$. The index set $\Lambda$ splits into the disjoint union of the two subsets $\Lambda_d, \Lambda_i$, where $k \in \Lambda_d$ iff $y_k \notin A$. Let us define $U' = \bigcap_{j \in \Lambda_d} U_{y_j}$ and $V' = \bigcap_{j \in \Lambda_i} V_{y_j}$. The subsets $U', V'$ are such that $U' \supset A$ and $V' \supset B$. Let us prove that $U' \cap V' \cap S \subset O$. Indeed, suppose $z \in K = S \setminus O$, then $z$ is contained in some $M_{y_j}, j \in \Lambda$, that does not intersect $U'$ or $V'$ depending on whether $y_j \notin A$ or not, thus $z \notin U' \cap V'$ which implies $U' \cap V' \cap S \subset O$. By applying preorder normality we find $U$ open decreasing set such that $A \subset U \subset D(U) \subset U'$ and $V$ open increasing set such that $B \subset V \subset I(V) \subset V'$, thus $D(U) \cap I(V) \cap S \subset O$. \qed

Lemma 2.4. Let $E$ be a $T_2$-preordered compact space, let $A$ be a closed decreasing set and let $B$ be a closed increasing set. Finally, let $O$ be an open set such that $A \cap B \subset O$. Then there are an open decreasing set $U \supset A$ and an open increasing set $V \supset B$, such that $D(U) \cap I(V) \subset O$.

Proof. Since $E$ is a $T_2$-preordered compact space it is normally preordered [14, Theor. 2.4]. Setting $S = E$ the desired conclusion follows from lemma 2.3. \qed

It is well known that under Hausdorffness local compactness and strong local compactness are equivalent. Every $T_2$-ordered space is Hausdorff thus under antisymmetry these notions of local compactness coincide. We can actually prove that this equivalence holds at a single point.

Let $S$ be a subspace of $E$. In the next theorems with “on $S$” we shall mean “with respect to $S$ regarded as a subspace, namely with its induced topology and induced preorder”. On $S$ the increasing hull of a subset $H \subset S$ will be denoted $i_S(H)$ and analogously for the decreasing hull, $d_S(H)$, and for the corresponding closure versions, $I_S(H)$ and $D_S(H)$.

Proposition 2.5. Let $E$ be a $T_2$-preordered space. If $[x] \subset E$ admits a compact neighborhood then for every open set $O' \supset [x]$ there is a compact neighborhood of $[x]$ contained in $O'$. In particular, under antisymmetry at $x$, local compactness at $x$ implies strong local compactness at $x$. 9
Proof. Let $K$ be a compact neighborhood of $[x]$, $[x] \subset \text{Int}K$. Let $O'$ be an open set such that $[x] \subset O'$ and define $O = O' \cap \text{Int}K$. Let $A = d(x) \cap K$, $B = i(x) \cap K$. Since $[x] \subset O$, we have $d(x) \cap i(x) \subset O$ which implies $A \cap B \subset O$.

We work on the $T_2$-preordered compact space $K$ and apply lemma 2.4. There are an open decreasing set $U \supset A$ (on $K$) and an open increasing set $V \supset B$ (on $K$), such that $D_K(U) \cap I_K(V) \subset O$. Since $U \cap V \subset O \subset K$, the $K$-open set $U \cap V$ is actually open in $E$. The set $D_K(U) \cap I_K(V)$ being a closed subset of $K$ is compact, and containing $U \cap V$, it is actually a compact neighborhood of $x$ contained in $O$ and hence $O'$.

\[\square\]

**Lemma 2.6.** Let $E$ be a $T_2$-preordered space, $S$ a compact subset, $O$ an open set on $E$, $C$ a convex set on $E$, and $x$ a point in $S$, such that $[x]_S \subset O \subset C \subset S$ where $[x]_S = d_S(x) \cap i_S(x)$. Then $[x] = [x]_S$, and there are an open convex neighborhood (on $E$) of $[x]$ contained in $O$, and a $c$-compact neighborhood (on $E$) of $[x]$ contained in $O$.

**Proof.** If $z \in d(x) \cap i(x)$ then $z \in d(C) \cap i(C) = C \subset S$, thus $z \in d_S(x) \cap i_S(x) = [x]_S$, that is $[x] = [x]_S$.

Let $N$ be a $(S)$-convex neighborhood of $[x]$ contained in $O$, where convexity refers to the subspace $S$. We are going to prove that $N$ is convex in $E$. Indeed

\[d(N) \cap i(N) \subset d(C) \cap i(C) \subset C,\]

thus if $z \in d(N) \cap i(N)$ then $z \in S$ which implies

\[d(N) \cap i(N) = d_S(N) \cap i_S(N) = N,\]

by convexity of $N$ in $S$, thus $N$ is indeed convex in $E$.

Suppose that we prove the existence of an open convex neighborhood $N$ of $[x]$ contained in $O$ in the $T_2$-preordered subspace $S$. Since $N \subset O$ and $O \subset S$ is open in $E$, $N$ is open in $E$ and also convex by the above argument.

Analogously, suppose that we prove the existence of a $c$-compact neighborhood $N$ of $[x]$ contained in $O$ in the $T_2$-preordered subspace $S$ with the additional property that it contains an open convex (in the subspace $S$) neighborhood $N'$ of $[x]$ contained in $O$. Since $N$ is compact on $S$ it is compact on $E$. The equation $d(N) \cap i(N) = d_S(N) \cap i_S(N) = N$ proves that $N$ is closed in $E$, and that it is a $c$-compact set in $E$ (recall Prop. 2.1). Since it contains $N'$ which contains $[x]$ it is a $c$-compact neighborhood of $[x]$.

Thus for the remainder of the proof we can work in the $T_2$-preordered compact subspace $S$. Let $A = d_S(x)$ and $B = i_S(x)$, so that $A \cap B = [x]_S \subset O$. Lemma 2.4 proves that there are an open convex neighborhood $U \cap V$ of $[x]$ contained in $O$ (according to the subspace $S$), and a $c$-compact neighborhood $D_S(U) \cap I_S(V)$ of $[x]$ contained in $O$ (according to the subspace $S$), which finishes the proof. \[\square\]

**Lemma 2.7.** If local convexity holds at $x \in E$ then $[x]$ is compact and every open neighborhood of $x$ is also an open neighborhood of $[x]$. 

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Proof. Let \( O \) be an open neighborhood of \( x \) and let \( C \) be a convex set such that \( x \in C \subset O \), then \( [x] = d(x) \cap i(x) \subset d(C) \cap i(C) = C \subset O \), thus \( O \) is also an open neighborhood for \([x] \). Let us consider an open covering of \([x] \), then there is some open set of the covering which includes \( x \) and hence \([x] \), thus every open covering admits a subcovering of only one element. \( \square \)

**Proposition 2.8.** Let \( E \) be a \( T_2 \)-preordered space. If \( E \) is locally compact and locally convex at \( x \in E \), then the topology at \( x \) admits a base of \( c \)-compact neighborhoods, and a base of open convex neighborhoods (that is, weak convexity holds at \( x \)).

**Proof.** Let \( N \) be any open neighborhood of \( x \). By local compactness there are a compact set \( S \) and an open set \( O' \) such that \( x \in O' \subset S \cap N \). By local convexity (lemma 2.7) \([x] \subset O' \). By lemma 2.6 there are an open convex neighborhood of \( x \) contained in \( O \) (and hence \( N \)) and a \( c \)-compact neighborhood of \( x \) contained in \( O \) (and hence \( N \)). \( \square \)

**Corollary 2.9.** Every locally compact and locally convex closed preordered space is weakly convex.

**Lemma 2.10.** Let \( E \) be a normally preordered space, \( S \) a compact subset, \( A \) a closed decreasing set on \( S \) and \( B \) a closed increasing set on \( S \). Further, let \( O \) be an open and convex set on \( E \) (not necessarily contained in \( S \)) such that \( A \cap B \subset O \), then there are an open decreasing set \( U \) on \( S \) and an open increasing set \( V \) on \( S \), such that \( A \subset U \subset D_S(U) \subset \hat{U} \), \( B \subset V \subset I_S(V) \subset \hat{V} \) and

\[
d(D_S(U)) \cap i(I_S(V)) \subset O.
\]

**Proof.** Since \( S \) is a subspace and the \( T_2 \)-preorder property is hereditary, the subset \( S \), with the induced preorder and topology, is a \( T_2 \)-preordered space and, being compact, it is a normally preordered space [14].

By lemma 2.3 and by preorder normality of \( S \) there are \( \hat{U}, \hat{V} \subset S \), open decreasing sets on \( S \) and \( \hat{U}, \hat{V} \subset S \), open increasing sets on \( S \) such that \( \hat{U} \cap \hat{V} \subset O \cap S \) and

\[
A \subset \hat{U} \subset D_S(\hat{U}) \subset \hat{U}, \quad B \subset \hat{V} \subset I_S(\hat{V}) \subset \hat{V}.
\]

The set \( A \setminus \hat{V} \subset S \) is closed on \( S \) and hence compact on both \( S \) and \( E \), decreasing on \( S \) and disjoint from \( B \) thus, \( d(A \setminus \hat{V}) \cap i(B) = \emptyset \) where \( d(A \setminus \hat{V}) \) is closed decreasing on \( E \) and \( i(B) \) is closed increasing on \( E \). By preorder normality of \( E \) there are \( \hat{U}_A \), \( \hat{V}_A \) open decreasing on \( E \) and \( \hat{V}_A \) open increasing on \( E \), such that

\[
d(A \setminus \hat{V}) \subset \hat{U}_A, \quad i(B) \subset \hat{V}_A \quad \text{and} \quad \hat{U}_A \cap \hat{V}_A = \emptyset.
\]

Analogously, \( B \setminus \hat{U} \) is closed on \( S \), hence compact on both \( S \) and \( E \), increasing on \( S \) and disjoint from \( A \), \( i(B \setminus \hat{U}) \) is closed increasing in \( E \), \( d(A) \) is closed
decreasing on \( E \), we have \( i(B\setminus \tilde{U}) \cap d(A) = \emptyset \) and we find \( \tilde{U}_B \) open decreasing on \( E \), \( \tilde{V}_B \) open increasing on \( E \) such that
\[
d(A) \subset \tilde{U}_B, \quad i(B\setminus \tilde{U}) \subset \tilde{V}_B \quad \text{and} \quad \tilde{U}_B \cap \tilde{V}_B = \emptyset.
\]

Let us define the open subsets of \( S \)
\[
P_A = (O \cup \tilde{U}_A) \cap \tilde{U}_B \cap \tilde{U}, \quad P_B = (O \cup \tilde{V}_B) \cap \tilde{V}_A \cap \tilde{V}.
\]
We have \( A \subset P_A \) because \( A \subset \tilde{U}_B \cap \tilde{U} \) and if \( x \in A \setminus \tilde{V} \) then \( x \in \tilde{U}_A \) while if \( x \in \tilde{V} \cap A \subset \tilde{V} \cap \tilde{U} \subset O \) we have \( x \in O \). Analogously, \( B \subset P_B \).

Let us prove that \( d(P_A) \cap i(P_B) \subset O \). If \( z \in d(P_A) \cap i(P_B) \) then there are \( x \in P_A \) and \( y \in P_B \), such that \( y \leq z \leq x \). The possibility \( x \in \tilde{U}_A \) is excluded because \( y \in P_B \subset \tilde{V}_A \), and \( \tilde{U}_A \cap \tilde{V}_A = \emptyset \). Analogously, \( y \in \tilde{V}_B \) is excluded because \( x \in P_A \subset \tilde{U}_B \) and \( \tilde{V}_B \cap \tilde{U}_B = \emptyset \). Thus
\[
x \in P_A \setminus \tilde{U}_A \subset O \cap \tilde{U}_B \cap \tilde{U},
y \in P_B \setminus \tilde{V}_B \subset O \cap \tilde{V}_A \cap \tilde{V}.
\]
Since \( x, y \in O \) which is convex we obtain \( z \in O \), that is \( d(P_A) \cap i(P_B) \subset O \).

Using Prop. 2.2 since \( P_A \) is open in \( S \) and \( A \) is decreasing in \( S \) there is \( U' \) open decreasing on \( S \) such that \( A \subset U' \subset P_A \) and applying preorder normality of \( S \) there is \( U \), open decreasing on \( S \), such that \( A \subset U \subset D_S(U) \subset U' \subset P_A \).

Analogously, we find \( V, V' \), open increasing sets on \( S \), such that \( B \subset V \subset I_S(V) \subset V' \subset P_B \). We have
\[
d(D_S(U)) \cap i(I_S(V)) \subset d(P_A) \cap i(P_B) \subset O.
\]

\( \square \)

**Lemma 2.11.** Let \( E \) be a \( T_2 \)-preordered \( k_\omega \)-space, \( x \in E \), and let \( O \) be an open and convex neighborhood of \( x \). Then there are an open decreasing set \( U \) and open increasing set \( V \), such that \( x \in U \cap V \subset O \).

**Proof.** We already know that \( E \) is normally preordered \([14]\). Since \( O \) is convex \([x] \subset O \). Let \( K_i, K_i \subset K_{i+1} \), be an admissible sequence for the \( k_\omega \)-space \( E \).

Without loss of generality we can assume \( x \in K_1 \). Each \( K_i \) endowed with the induced topology and preorder is a compact \( T_2 \)-preordered space.

Let \( A_1 = d(x) \cap K_1, B_1 = i(x) \cap K_1 \). We have that \( A_1 \) is closed decreasing in \( K_1, B_1 \) is closed increasing in \( K_1 \) and \( A_1 \cap B_1 \subset O \). Since \( K_1 \) is compact, by lemma 2.10 (with \( S = K_1 \)) we can find \( U_1 \supset A_1 \), open decreasing set in \( K_1 \), and \( V_1 \supset B_1 \), open increasing set in \( K_1 \), such that \( d(D_1(U_1)) \cap i(I_1(V_1)) \subset O \), where \( D_1 \) and \( I_1 \) are the closed-hull maps of \( K_1 \). Observe that \( D_1(U_1) \) and \( I_1(V_1) \) being closed subsets of \( K_1 \) are compact in \( E \). We define \( A_2 = d_2(D_1(U_1)) \) and \( B_2 = i_2(D_1(V_1)) \), where \( A_2 \) is clearly closed decreasing in \( K_2 \), and \( B_2 \) is closed increasing in \( K_2 \). We have \( A_2 \cap B_2 \subset d(D_1(U_1)) \cap i(I_1(V_1)) \subset O \). We can proceed
applying again lemma 2.10 with \( S = K_2 \). Thus proceeding inductively, given \( A_i, B_i \subset K_i \), \( A_i \cap B_i \subset O \) we find \( U_i, V_i \) respectively open decreasing and open increasing subsets of \( K_i \) such that \( U_i \supset A_i, V_i \supset B_i, d(D_i(U_i)) \cap i(I_i(V_i)) \subset O \), and define \( A_{i+1} = d_{i+1}(D_i(U_i)) \) and \( B_{i+1} = i_{i+1}(D_i(V_i)) \).

Note that \( V_j \subset B_{j+1} \subset V_{j+1} \) and analogously, \( U_j \subset U_{j+1} \).

Let us define the sets \( U = \bigcup_{j=1}^{\infty} U_j \) and \( V = \bigcup_{j=1}^{\infty} V_j \). The set \( V \) is open because \( V \cap K_s = \bigcup_{j \geq 1} (V_j \cap K_s) = \bigcup_{j \geq s} (V_j \cap K_s) \), and the set \( V_j \subset K_j \) is open in \( K_j \) so that, since for \( j \geq s, K_s \subset K_j \), \( V_j \cap K_s \) is open in \( K_s \) and so is the union \( V \cap K_s \). The \( \omega \)-space property implies that \( V \) is open. Analogously, \( U \) is open.

Let us prove that \( V \) is increasing. Let \( w \in V \) then there is some \( j \geq 1 \) such that \( w \in V_j \subset K_j \). Let \( y \in i(w) \), then we can find some \( r \geq j \) such that \( y \in K_r \).

Since \( V_j \subset V_r \), \( w \in V_r \), and since \( V_r \) is increasing on \( K_r \), \( y \in V_r \) thus \( y \in V \). Analogously, \( U \) is decreasing. Finally, if \( z \in U \cap V \) then there are some \( j, k \geq 1 \) such that \( z \in U_j \cap V_k \) and setting \( r = \max(j, k) \), \( z \in U_r \cap V_r \) thus

\[
U \cap V \subset \bigcup_{r=1}^{\infty} (U_r \cap V_r) \subset O' \subset O.
\]

As an immediate consequence we obtain the desired result.

**Theorem 2.12.** Every weakly convex \( T_2 \)-preordered \( \omega \)-space is a convex normally preordered space (and hence quasi-uniformizable).

**Remark 2.13.** Actually we proved something more, namely that a \( T_2 \)-preordered \( \omega \)-space which is weakly convex at \( x \) is convex at \( x \). Thus, by Prop. 2.8 in a \( T_2 \)-preordered \( \omega \)-space \( E \), if local convexity and local compactness hold at \( x \), then convexity holds at \( x \).

**Corollary 2.14.** Every locally convex \( T_2 \)-preordered locally compact \( \sigma \)-compact space is a convex normally preordered space (and hence quasi-uniformizable).

**Proof.** Every locally compact \( \sigma \)-compact space is a \( \omega \)-space, and under local compactness local convexity and weak convexity are equivalent (Cor. 2.9). \( \Box \)

### 3. Convexity of \( k \)-preserving spaces

The next definition is inspired by the property of global hyperbolicity in Lorentzian geometry, see Sect. 1.1.

**Definition 3.1.** A \( T_2 \)-preordered space \( E \) is \( k \)-preserving if every compact set \( K \subset E \) has a compact convex hull \( d(K) \cap i(K) \).

**Proposition 3.2.** Let \( E \) be a \( T_2 \)-preordered space. If the topology does not distinguish the points of \( [x] \) (e.g. if \( E \) is locally convex at \( x \) or antisymmetry holds at \( x \)) and \( x \) admits a \( c \)-compact neighborhood, then \( x \) admits a base of \( c \)-compact neighborhoods and, moreover, \( E \) is weakly convex at \( x \).
\textit{Proof.} Let $K$ be a $c$-compact neighborhood of $x$ and hence $[x]$, and let $O \ni x$ be an open neighborhood of $x$ and hence $[x]$ which we can assume contained in $K$. We have to show that there is a compact $c$-set neighborhood $K'$ of $[x]$ such that $K' \subset O$, and analogously in the convex open neighborhood case. It suffices to apply lemma 2.6 with $S := K$, $C := K$. Observe that by local convexity (lemma 2.7) or antisymmetry at $x$, if $O$ is any open neighborhood of $x$ we have $[x] \subset O$.

Clearly a compact $T_2$-ordered space is $k$-preserving (Prop. 2.1). We know that the compact $T_2$-ordered spaces are convex [7]. We have the following interesting generalization

\textbf{Theorem 3.3.} Every $T_2$-preordered $k$-preserving $k_\omega$-space is convex at every point $x$ such that (i) the topology does not distinguish different points of $[x]$, (ii) local compactness holds at $x$ (e.g. wherever it is locally compact and antisymmetric).

In particular, every $k$-preserving $T_2$-ordered locally compact $\sigma$-compact space is convex (and hence quasi-uniformizable).

\textit{Proof.} Every $T_2$-preordered $k_\omega$-space is normally preordered [14]. By assumption there is a compact neighborhood $K$ of $[x]$. The set $d(K) \cap i(K)$ is a $c$-compact neighborhood of $x$. By Prop. 3.2 weak convexity holds at $x$, and by remark 2.13 convexity holds at $x$. \hfill $\square$

\textit{Remark 3.4.} Actually the $k$-preserving property could be dropped provided we replace (ii) with the requirement that the point $x$ has a $c$-compact neighborhood, or that local compactness holds at $x$ and the $k$-preserving property holds \textit{locally}.

4. Compactly generated $T_2$-preorders

In this section we study sufficient conditions for local convexity. The main idea is to consider preorders which, intuitively, are generated by relations which are limited, in the sense that do not connect arbitrarily ‘far away’ points (compactness is used to give a rigorous meaning to this concept). Thus we shall be basically concerned with topological preordered spaces for which both topology and preorder are generated from local information.

For this type of preorder and for a locally compact space, given two related ‘far away’ points $p, q$, there is some point $r$, $p \leq r \leq q$, at ‘reasonable distance’ but not too close to the original point $p$. From that it is possible to show that if local convexity is violated at $p$ then, by a limiting argument, some point $r' \neq p$ exists such that $p \leq r' \leq p$ and hence antisymmetry is violated at $p$. This strategy has been used in mathematical relativity theory to prove that the $K^+$ relation (the smallest closed preorder containing the causal relation $J^+$) is locally convex [13, Lemma 16] [12, Lemma 5.5].
Definition 4.1. A $T_2$-preordered space $(E, \mathcal{T}, \leq)$ is a $k$-$T_2$-preordered space (read 'compactly generated $T_2$-preordered space') if there is a relation $R \subseteq G(\leq)$ such that:

(i) for every compact set $K$ the set $R(K)$ is compact,

(ii) the preorder $\leq$ is the smallest closed preorder containing $R$.

We shall also say that $\leq$ is a compactly generated preorder.

Note that in (ii) the smallest closed preorder exists because the family of closed preorders containing $R$ is non-empty as $E \times E$ is a closed preorder which contains $R$. Note that if $R$ satisfies (i)-(ii) then also $\Delta \cup R$ satisfies them, thus $R$ can be chosen reflexive.

Remark 4.2. For applications in which $E$ is locally compact it is useful to observe that the condition

(i') every point $x \in E$ admits a closed and compact neighborhood $F(x)$ such that $R(F)$ is compact,

implies (i), and thus a space $E$ satisfying (i') and (ii) is compactly generated. Note that if $R$ satisfies (i')-(ii) then also $\Delta \cup R$ satisfies them, thus $R$ can be chosen reflexive.

Proposition 4.3. If $(E, \mathcal{T}, \leq)$ is a $T_2$-preordered compact space, then $\leq$ is compactly generated.

Proof. The conditions in the definition of compactly generated preorder are satisfied taking $R = G(\leq)$. 

The next result is worth mentioning although we shall not use it.

Theorem 4.4. Let $(E, \mathcal{T}, \leq)$ be a $k$-$T_2$-preordered space, and let $R$ be a reflexive relation as in definition [4]. The set of continuous isotone functions for $R$ coincides with the set of continuous isotone functions for $\leq$.

Proof. If $f$ is a continuous isotone function for $\leq$ and $(x, y) \in R$ we have, since $R \subseteq G(\leq)$, $(x, y) \in G(\leq) \Rightarrow f(x) \leq f(y)$ thus $f$ is a continuous isotone function for $R$. If $f$ is a continuous isotone function for $R$ the relation $R_f = \{(w, z) : f(w) \leq f(z)\}$ is a closed preorder containing $R$ thus $G(\leq) \subseteq R_f$ which implies $x \leq y \Rightarrow f(x) \leq f(y)$, that is, $f$ is a continuous isotone function for $\leq$.

This result is interesting because in those cases in which $E$ is also normally preordered (the $k_\omega$-space condition suffices [14]) this set of continuous isotone functions for $R$ allows us to recover $\leq$, that is, $x \leq y$ iff for all continuous isotone functions $f : E \to [0, 1]$, we have $f(x) \leq f(y)$.

\[\text{2}^\text{Compare with the definition of +proper relation $R$, and relation $\mathcal{GR}$ given in [18].}\]
Remark 4.5. It is worth to mention a recent work by Akin and Auslander on recurrence problems and compactifications in dynamical systems [18]. This section is very much related with their work, although we followed a different line of reasoning inspired by results in topological preordered spaces and relativity theory. In their paper they assume that $E$ is a separable locally compact metric space [18, p. 50], while in our work second countability and Hausdorffness are not assumed, and local compactness is used only where it is strictly needed. We do not use compactification arguments as in their article.

We usually work with a reflexive relation $R$ because this is the interesting case from the topological point of view, as the elements of a quasi-uniformity contain the diagonal. Furthermore, the application to cone structures seems to require a reflexive $R$. Observe that if $R$ is reflexive then the generalized recurrent set mentioned in [18, Theor. 11] is the whole space. Our theorem 4.14 will be similar but stronger than their [18, Theor. 14].

We find that our terminology concerning compactly generated $T_2$-preordered spaces is more appropriate, since relations do generalize functions but the term proper is used for maps such that the inverse images of compact subsets are compact, while we do not take any inverse here. Maps which send compact set to compact sets are sometimes called compact. Finally, observe that our terminology places the accent on $\leq$ rather than $R$. In applications there is often a natural choice for $R$ but, mathematically, it could be chosen with some freedom.

4.1. Some examples of compactly generated preorders

Most closed preorders appearing in applications are compactly generated. We give some examples proving conditions (i)-(ii) or (i')-(ii) of remark 4.2.

Example 4.6. Let us recall that in a spacetime $(M, g)$ (see Sect. 1.1) the relation $K^+$ is by definition the smallest closed and transitive relation containing $J^+$. Let $F_\alpha$ be a locally finite closed and compact covering of $M$ (it exist because of local compactness and [30, Theor. 20.7]) and let $R = \cup J^+ \cap (F_\alpha \times F_\alpha)$. Since each $F_\alpha$ is intersected only by a finite number of $F_\beta$, $\overline{R(F_\alpha)}$ is compact. Thus if $C$ is a compact set, $\overline{R(C)}$ is compact.

Clearly, $J^+$ is the smallest transitive relation containing $R$, thus $K^+$ is the smallest closed and transitive relation containing $R$. We conclude that $K^+$ is a $k$-$T_2$-preorder for which $R$ is a generating relation. As mentioned in Sect. 1.1 it coincides with the causal relation in causally simple spacetimes and with its closure in stably causal spacetimes.

Example 4.7. Let $E$ be a Hausdorff, connected, paracompact $(C^{r+1}, r \geq 0)$ manifold and let $v : E \to TE$ be a $(C^r, r \geq 0)$ vector field. We write $(x, y) \in J$ if there is some integral curve of $v$ which connects $x$ to $y$ in the forward direction. Let $F_\alpha$ be a locally finite closed and compact covering of $E$, and let $R = \cup J^+ \cap (F_\alpha \times F_\alpha)$. Clearly, $J$ is the smallest transitive relation containing $R$. One is interested in the smallest closed and transitive relation containing $J$, denoted $GJ$ by some authors [18], which is therefore the smallest closed and
transitive relation containing $R$. Arguing as in example 4.6 we obtain that $GJ$ is a $k$-$T_2$-preorder for which $R$ is a generating relation.

**Example 4.8.** Let $E$ be a Hausdorff, connected, compact manifold and let $f : E \to E$ be a continuous map. We write $(x, y) \in J$ if there is some integer $k \geq 0$ such that $y = f^k(x)$. Let $R = \{(x, f(x)), x \in E\}$. One is interested in the smallest closed and transitive relation containing $R$, which is clearly a $k$-$T_2$-preorder for which $R$ is a generating relation.

### 4.2. Antisymmetry and local convexity

The next two proofs generalize to the topological preordered case, ideas contained in [12, Lemma 5.3, 5.5] [13].

**Proposition 4.9.** Let $(E, \mathcal{T}, \leq)$ be a $k$-$T_2$-preordered space, let $R$ be a reflexive generating relation as in definition 4.7 and let $K$ be a compact set. If $x \leq z$ with $x \in \text{Int}(K)$ and $z \notin \overline{R(K)}$, then there is $y \in \overline{R(K)} \setminus \text{Int}(K)$ such that $x \leq y \leq z$.

**Proof.** Let us consider the relation, where $O = \text{Int}(K)$,

$$B = \{(x, z) \in G(\leq) : \text{if } "x \in O \text{ and } z \notin \overline{R(K)}" \text{ then there is } y \in \overline{R(K)} \setminus O \text{ such that } x \leq y \leq z"\}.$$

Suppose we prove that it is closed, reflexive, transitive and that $R \subset B \subset G(\leq)$. From the minimality property in the definition of $\leq$, we have $G(\leq) \subset B$, thus $B = G$ which is the thesis.

The inclusion $B \subset G$ is trivial, let us prove $R \subset B$. If $(x, z) \in R$ then, by the definition of $\leq$, $(x, z) \in G$. In the definition of $B$ the hypothesis "$x \in O$ and $z \notin \overline{R(K)}$" is necessarily false because if $x \in O$ then $z \in R(O) \subset R(K) \subset \overline{R(K)}$. As the hypothesis is false the implication in the definition of $B$ is true, thus $(x, z) \in B$ which proves $R \subset B$. Since $R$ is reflexive $B$ is reflexive.

Let us prove closure. Let $(x, z) \in \overline{B}$. If $x \notin O$ or $z \notin \overline{R(K)}$ then $(x, z) \in B$ because the hypothesis "if $x \in O$ and $z \notin \overline{R(K)}$" in the definition of $B$ is false and so the implication in the definition of $B$ is true. Thus we can consider the case $x \in O$ and $z \notin \overline{R(K)}$. Let $O_x \ni x, O_z \ni z$ be any open sets with $O_x \subset O, O_z \subset \overline{R(K)}$. By assumption we can find $x' \in O_x, z' \in O_z, (x', z') \in B$. Since $x' \in O$ and $z' \notin \overline{R(K)}$ there is $y' \in \overline{R(K)} \setminus O$ such that $x' \leq y' \leq z'$. This means that $i(O_x) \cap d(O_z) \cap \overline{R(K)} \setminus O$, with $O_x$ and $O_z$ running on the open neighborhoods of $x$ and $z$, gives a family of non-empty sets with the finite intersection property (in fact they are a base for a filter). As $\overline{R(K)} \setminus O$ is compact the associated filter admits a cluster point $y \in \overline{R(K)} \setminus O$, i.e. every neighborhood of $y$ intersects $i(O_x) \cap d(O_y)$ for every open sets $O_x \ni x, O_z \ni z$. But if it were $x \notin y$ then, by the closure of $G$ [4, Prop. 1], there would be a neighborhood of $O_x$ of $x$ and $O_y$ of $y$ such that $i(O_x) \cap d(O_y) = \emptyset$, since it does not hold we infer $x \leq y$ and analogously $y \leq z$. Thus $(x, z) \in B$.

Let us prove transitivity. Let $(x, w) \in B$ and $(w, z) \in B$. If "$x \in O$ and $z \notin \overline{R(K)}$" is false there is nothing to prove because, as the hypothesis in the implication defining $B$ is false, $(x, z) \in B$. If "$x \in O$ and $z \notin \overline{R(K)}$" is true...
and $w \in \overline{R(K) \setminus O}$ we have gain $(x, z) \in B$ (set $y = w$), thus let us assume $w \notin \overline{R(K) \setminus O}$ so that it either belongs to $O$ or to $E \setminus R(K)$. In the former case since $(w, z) \in B$ we infer that there is $y \in \overline{R(K) \setminus O}$ such that $x \leq w \leq y \leq z$. In the latter case since $(x, w) \in B$ we infer that there is $y \in \overline{R(K) \setminus O}$ such that $x \leq y \leq w \leq z$. We conclude $(x, z) \in E$. 

**Theorem 4.10.** Let $(E, \mathcal{F}, \leq)$ be a $k$-$T_2$-preordered space. Let $x \in IntK$ where $K$ is a compact set. There is an open convex neighborhood $C$ of $[x]$ such that $C \subset IntK$ or there is a point $p \notin IntK$ such that $x \leq p \leq x$.

**Proof.** The set $[x]$ is closed and we can assume that it is contained in $Int(K)$, otherwise we have finished.

Let $R$ be a reflexive relation as in the definition 4.1 and let us set $O = Int(K)$. Either (i) there is a neighborhood $N$ of $[x]$ such that $d(N) \cap i(N)$ is contained in the compact set $R(K)$ or (ii) for every neighborhood $M$ of $[x]$, $(d(M) \cap i(M)) \setminus \overline{R(K)} \neq \emptyset$.

In case (iii), if $z \in (d(M) \cap i(M)) \setminus \overline{R(K)}$ with $M \subset O$ neighborhood of $[x]$, then by Prop. 4.9 since $z \notin R(K)$ and $w \leq z$ for some $w \in M \subset O$, we have that there is $y \in \overline{R(K) \setminus O}$ such that $w \leq y \leq z \leq q$ for some $q \in M$. Thus for every neighborhood $M \subset O$ of $[x]$, the set $(d(M) \cap i(M)) \cap (R(K) \setminus O)$ is non-empty. Observe that varying $M$ we obtain a family of sets which satisfies the finite intersection property. As $\overline{R(K) \setminus O}$ is compact the associated filter admits a cluster point $p \in \overline{R(K) \setminus O}$, i.e. every neighborhood of $p$ intersects $d(M) \cap i(M)$ for every neighborhood $M$ of $[x]$. But if it were $x \notin p$ then by the closure of $G(\leq)$ and the compactness of $[x]$ there would be a neighborhood $O_x$ of $[x]$ and $O_p$ of $p$ such that $i(O_x) \cap O_p = \emptyset$, since it does not hold we infer $x \leq p$ and analogously $p \leq x$.

In case (i) there is a neighborhood $N$ of $[x]$ such that $d(N) \cap i(N)$ is contained in the compact set $\overline{R(K)}$. Assume that for every neighborhood $Y \subset N$ of $[x]$, $d(Y) \cap i(Y) \cap (\overline{R(K) \setminus O}) = \emptyset$. Varying $Y$ we get a family of non-empty subsets of the compact set $\overline{R(K) \setminus O}$ which satisfies the finite intersection property. There is a cluster point $p \in \overline{R(K) \setminus O}$ thus, arguing as above, $x \leq p \leq x$.

If for some neighborhood $Y' \subset N$ of $[x]$, $d(Y') \cap i(Y') \subset O$ we have that the set $C' = d(Y') \cap i(Y')$ is a convex neighborhood of $[x]$ contained in $O$. Let $O'$ be an open set such that $[x] \subset O' \subset C' \subset K$. Lemma 2.6 with $S = K$, implies the existence of an open convex neighborhood $C$ of $[x]$ contained in $O'$ and hence in $O$.

**Theorem 4.11.** Every $k$-$T_2$-preordered space is weakly convex at every point $x$ such that (i) the topology does not distinguish different points of $[x]$, (ii) local compactness holds at $x$ (e.g. wherever it is locally compact and antisymmetric).

**Proof.** Let $O$ be an open neighborhood of $x$ and hence $[x]$. By local compactness there is a compact neighborhood of $x$ and hence $[x]$. By proposition 2.6 there is a compact neighborhood $K'$ of $[x]$ contained in $O$. By theorem 4.10 there is an open convex neighborhood of $x$ contained in $K$ and hence $O$.

**Corollary 4.12.** Every $k$-$T_2$-ordered locally compact space is weakly convex.
Theorem 4.13. Every $k$-$T_2$-preordered $k_\omega$-space is convex at every point $x$ such that (i) the topology does not distinguish different points of $[x]$, (ii) local compactness holds at $x$ (e.g. wherever it is locally compact and antisymmetric).

Proof. By theorem 4.11 weak convexity holds at $x$. By remark 2.13 convexity holds at $x$.

Corollary 4.14. Every locally compact $\sigma$-compact $k$-$T_2$-ordered space is convex (and since they are normally ordered they are quasi-uniformizable).

Proof. Under local compactness the $k_\omega$-space property and $\sigma$-compactness are equivalent.

With reference to the example of compactly generated preorder given by Example 4.6 (see also Sect. 4.4) we have the following consequence.

Theorem 4.15. Let $(M,q)$ be a stably causal spacetime, let $\mathcal{T}$ be the manifold topology and let $J_+^2$ be the Seifert relation, then $(M, \mathcal{T}, J_+^2)$ is quasi-uniformizable and hence admits the Nachbin compactification.

With reference to Example 4.7 we obtain:

Theorem 4.16. Let $E$ be the dynamical system whose flow is generated by a vector field described in Example 4.7, let $\mathcal{T}$ be the manifold topology, and let $J$ be the reflexive relation there defined. If $GJ$ is antisymmetric then $(E, \mathcal{T}, GJ)$ is quasi-uniformizable and hence admits the Nachbin compactification.

5. Quasi-pseudo-metrizability

A quasi-pseudo-metric $[31, 32]$ on a set $X$ is a function $p : X \times X \to [0, +\infty)$ such that for $x, y, z \in X$

(i) $p(x, x) = 0$,

(ii) $p(x, z) \leq p(x, y) + p(y, z)$.

The quasi-pseudo-metric is called quasi-metric $[33]$ if (i) is replaced with (i'):

$p(x, y) = 0$ iff $x = y$. Other variations exist in the literature. The Albert’s quasi-metric $[34]$ is a special type of quasi-pseudo-metric which is obtained replacing (i) with (i'') $p(x, y) = p(y, x) = 0$ iff $x = y$.

The quasi-pseudo-metric is called pseudo-metric if $p(x, y) = p(y, x)$. If a quasi-metric is such that $p(x, y) = p(y, x)$, then it is a metric in the usual sense. If $p$ is a quasi-pseudo-metric then $p^{-1}$, defined by $p^{-1}(x, y) = p(y, x)$, is a quasi-pseudo-metric called conjugate of $p$. Each quasi-pseudo-metric $p$ generates a topology whose base is given by the $p$-balls, $B^p_\varepsilon(x) = \{y : p(x, y) < \varepsilon\}$.

A topological preordered space $(E, \mathcal{T}, \leq)$ is quasi-pseudo-metrizable if there is a pair of conjugate quasi-pseudo-metrics $p, q$, called admissible, such that $\mathcal{T}$ is the topology generated by the pseudo-metric $p + q$ (equivalently $p \lor p^{-1}$), and the graph of the preorder is given by $G(\leq) = \{(x, y) : p(x, y) = 0\}$. 

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In the literature on bitopological spaces \([31, 32]\), a bitopological space \((X, P, Q)\) is \textit{quasi-pseudo-metrizable} if there is a quasi-pseudo-metric \(p\) such that \(p\) generates \(P\) and \(p^{-1}\) generates \(Q\).

A topological preordered space \((E, T, \leq)\) is \textit{strictly quasi-pseudo-metrizable} if it is convex semiclosed preordered and there is a pair of conjugate quasi-pseudo-metrics \(p, q\) such that the topology associated to \(p\) is the upper topology \(T^\#\), and the topology associated to \(q\) is the lower topology \(T^\♭\). In other words, according to our terminology \((E, T, \leq)\) is strictly quasi-pseudo-metrizable iff it is convex semiclosed preordered and \((E, T^\#, T^\♭)\) is quasi-pseudo-metrizable.

Every strictly quasi-pseudo-metrizable preordered space is a quasi-pseudo-metrizable preordered space. Every quasi-pseudo-metrizable preordered space is a completely regularly preordered space \([35, \text{Prop. 2.3}]\).

Every \(T_2\)-ordered space is Hausdorff and "every second countable Hausdorff locally compact topological space is metrizable" by Urysohn's metrization theorem. The next result is a kind of order generalization, which reduces to the just given statement for the discrete order.

**Theorem 5.1.** Let \((E, T, \leq)\) be a \(T_2\)-ordered space such that \((E, T)\) is second countable and locally compact. If \((E, T, \leq)\) is \(k\)-preserving or compactly generated, then it is quasi-pseudo-metrizable. That is, there is a quasi-pseudo-metric \(p : E \times E \to [0, +\infty)\) (actually an Albert's quasi-metric) such that \(T\) is the topology induced by the metric \(p \lor p^{-1}\) and \(G(\leq) = \{(x, y) : p(x, y) = 0\}\).

**Proof.** Second countability implies the Lindelöf property which under local compactness is equivalent to \(\sigma\)-compactness. The topological ordered space is a completely regularly ordered space (quasi-uniformizable) by theorem 3.3 (in the \(k\)-preserving case) or theorem 4.14. Thus \(E\) is a separable quasi-pseudo-metric space by \([35, \text{Theor. 2.5}]\). The pseudo-metric \(p \lor p^{-1}\) is actually a metric by antisymmetry of \(\leq\), thus \(p\) is an Albert’s quasi metric \([34]\).

We remark that we are not claiming that the topology induced by \(p\) is the upper topology \(T^\#\) and that induced by \(p^{-1}\) is the lower topology \(T^\♭\) (which would be true if we could prove strict quasi-pseudo-metrizability \([32]\)).

### 5.1. Strict quasi-pseudo-metrization from the \(I\)-space condition

In order to prove the strict quasi-pseudo-metrizability of a topological preordered space we assume the \(I\)-space condition.

Let us recall that a topological preordered space is a \textit{regularly preordered space} if it is semiclosed preordered, (a) for every closed decreasing set \(A\) and closed increasing set \(B\) of the form \(B = i(x)\) which are disjoint, \(A \cap B = \emptyset\), it is possible to find an open decreasing set \(U\) and an open increasing set \(V\) which separate them, namely \(A \subset U\), \(B \subset V\), and \(U \cap V = \emptyset\), and (b) for every closed decreasing set \(A\) of the form \(A = d(x)\) and closed increasing set \(B\) which are disjoint, \(A \cap B = \emptyset\), it is possible to find an open decreasing set \(U\) and an open increasing set \(V\) which separate them, namely \(A \subset U\), \(B \subset V\), and \(U \cap V = \emptyset\). A completely regularly preordered space need not be regularly

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preordered [36, Example 1]. This is a crucial difference with respect to the usual
 discrete-preorder version.

The problem of quasi-pseudo-metrization of a bitopological space was con-
sidered in Kelly’s work [31] and has been extensively studied over the years
37, 38, 39, 40, 41, 42, 43, 44. For bitopological spaces Kelly [31, Theor. 2.8]
obtained a generalization of Urysohn’s metrization theorem which in our topo-
logical preordered space framework reads as follows

**Theorem 5.2.** (Kelly) Let \((E, T, \leq)\) be a convex regularly preordered space and
assume that both \(T^\#\) and \(T^\flat\) are second countable, then \((E, T, \leq)\) is strictly
quasi-pseudo-metrizable.

Under the I-space assumption it is possible to infer the second countability
of the coarser topologies \(T^\#\) and \(T^\flat\) given that of \(T\), and hence we are able to
prove the next result.

**Theorem 5.3.** Every second countable locally convex locally compact
\(T_2\)-preordered
I-space \((E, T, \leq)\) is strictly quasi-pseudo-metrizable (observe that local convexity
holds whenever the space is \(k\)-preserving or compactly generated, and the
preorder is antisymmetric, see Theor. 3.3 and Cor. 4.14).

Proof. By theorem 1.2 \(E\) is convex. Let us prove that \(E\) is a regularly preordered
space. Let \(B\) be a closed increasing set and let \(x \in E \setminus B\). By Prop. 2.5 strong
local compactness holds at \(x\), thus there are an open set \(O\) and a compact set
\(K\), such that \(x \in O \subset K \subset E \setminus B\). The open decreasing set \(d(O)\) is contained in
the closed decreasing set \(d(K)\) which is disjoint from \(B\). The proof in the dual
case is analogous, thus \(E\) is regularly preordered. Let \(\{O_i\}\) be a countable base
for \(T\), then \(\{i(O_i)\}\) is a countable base for \(T^\#\) and \(\{d(O_i)\}\) is a countable base
for \(T^\flat\). By theorem 5.2 \((E, T, \leq)\) is strictly quasi-pseudo-metrizable. \(\square\)

A relevant application of this theorem is (see Sect. 1.1 for definitions and
basic results in causality theory)

**Theorem 5.4.** Globally hyperbolic, causally simple, and causally continuous
spacetimes endowed with the manifold topology \(T\) and the order \(J^+\) are strictly
quasi-pseudo-metrizable topological ordered spaces.

Proof. Under causal continuity \(K^+ = J^+\) and \((M, T, K^+)\) is compactly gen-
erated (Sect. 4.1). Under causal continuity the relation \(J^+\) sends open sets to
open sets (Remark 1.4). Global hyperbolicity implies causal simplicity which
implies causal continuity (Sect. 1.1). \(\square\)

In other words, the strongest causality property met in causality theory (i.e.
global hyperbolicity) implies the strongest preorder-separability condition.

6. Conclusions

In many applications the underlying mathematical structure involves a topo-
logical space \((E, T)\) endowed with a preorder \(\leq\). If the preorder is not closed,
it is usually convenient to consider the smallest closed preorder containing it, and hence to work in the framework of closed preordered spaces.

Quasi-uniformizable topological preordered spaces are among the most well behaved topological preordered spaces. They admit completions and compactifications [38, 45, 7, 8], and under second countability they can be shown to be quasi-pseudo-metrizable [35].

In a previous work we established that every $T_2$-preordered locally compact $\sigma$-compact space is normally preordered, and hence that it is possible to obtain strong preorder-separability properties imposing some topological conditions on $E$. Unfortunately, normally preordered spaces are not necessarily quasi-uniformizable, a fact that distinguishes the theory of topological preordered spaces from the usual (discrete-preorder) topology. In order to obtain the quasi-uniformizability of the topological preordered space it is necessary to prove its convexity.

This property is trivially satisfied in the discrete preorder case and, as a consequence, results on the convexity of a topological preordered space are particularly interesting as they have no analog in the usual non-ordered topology.

We have proved that locally compact $\sigma$-compact locally convex $T_2$-preordered spaces are convex, that is, imposing good topological conditions on $E$ promotes local convexity to convexity. This result is non-trivial because convexity is a global property as it makes reference to the openness of some monotone sets over $E$.

Then we investigated conditions that guarantee local convexity under antisymmetry. We proved that if the ordered space is such that the convex hull of a compact set is compact ($k$-preserving) then convexity holds. We also considered compactly generated preorders proving that this condition together with the above topological assumption on $E$, implies convexity.

In most applications the preorder is compactly generated (Sect. 4.1), thus we have indeed succeeded in proving the quasi-uniformizability of the corresponding topological preordered space, and hence the possibility of compactifying it. For instance, a spacetime is stably causal if and only if the relation $K^+$ of example 4.1 is antisymmetric, in which case it coincides with the Seifert’s causal relation [21, 2]. From our results a stably causal spacetime endowed with this relation is quasi-uniformizable, Theor. 4.15 (and in fact quasi-pseudo-metrizable). The Nachbin compactification allows us to introduce a spacetime boundary and to extend the Seifert relation as a closed relation on the whole compactified space.

The paper ends with some results on (strict) quasi-pseudo-metrizability of second countable and locally compact closed preordered spaces.

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