Massless particles plus matter in the rest-frame instant form of dynamics

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Abstract
After introducing the parametrized Minkowski theory describing a positive-energy scalar massless particle, we study the rest-frame instant form of dynamics of such a particle in the presence of another massive one (to avoid the front form of dynamics). Then we describe massless particles with Grassmann-valued helicity and their quantization.

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1. Introduction
The problem of the description of relativistic particles with a complete control of Poincaré group has recently been fully solved by means of parametrized Minkowski theories [1–4] and of the associated inertial rest-frame instant form of dynamics [5, 6], subsequently extended to non-inertial frames [7]. The basic idea is to use admissible 3+1 splittings of Minkowski spacetime to define global non-inertial frames [8, 9].

In the 3+1 point of view [9] we assign:
(a) the world-line of an arbitrary time-like observer;
(b) an admissible 3+1 splitting of Minkowski spacetime, namely a nice foliation with space-time instantaneous 3-spaces (i.e. a clock synchronization convention) tending to the same space-like hyperplane at spatial infinity (so that there are always asymptotic inertial observers to be identified with the fixed stars). See [7] for the meaning of the Møller conditions [8, 9] defining the admissible foliations.

This allows one to define a global non-inertial frame centered on the observer and to use observer-dependent Lorentz-scalar radar 4-coordinates $\sigma^A = (\tau; \sigma')$, where $\tau$ is a monotonically increasing function of the proper time of the observer and $\sigma'$ are the curvilinear 3-coordinates on the 3-space $\Sigma_{\tau}$, having the observer as origin. If $x^a \mapsto \sigma^A(x)$ is the
coordinate transformation from the inertial cartesian 4-coordinates $x^\mu$ to radar coordinates, its inverse $\sigma^A \mapsto x^\mu = z^\mu(\tau, \sigma')$ defines the embedding functions $z^\mu(\tau, \sigma')$ describing the instantaneous 3-spaces $\Sigma_r$ as embedded 3-manifold into Minkowski spacetime. The induced 4-metric on $\Sigma_r$ is the following functional of the embedding $g_{AB}(\tau, \sigma') = \left[ z^\mu_\eta_{\mu\nu} z^\nu_\eta(\tau, \sigma') \right](\tau, \sigma')$, where $z^\mu_\eta = \partial z^\mu / \partial \sigma^\eta$ and $\eta_{\mu\nu} = \epsilon(+- --)$ is the flat metric ($\epsilon = \pm 1$ according to either the particle physics $\epsilon = 1$ or the general relativity $\epsilon = -1$ convention). While the 4-vectors $z^\mu(\tau, \sigma')$ are tangent to $\Sigma_r$, so that the unit normal $l^\mu(\tau, \sigma')$ is proportional to $\epsilon^\mu_\alpha \eta_{\alpha\nu} \left[ z^\alpha_\eta z^\nu_\eta \right](\tau, \sigma')$, we have $z^\mu_\eta(\tau, \sigma') = \left[ N^{\mu} + N^{\nu} \right](\tau, \sigma')(N(\tau, \sigma') = \epsilon(z^\mu_\mu(\tau, \sigma'))$ and $N(\tau, \sigma') = -\epsilon g_{\tau\tau}(\tau, \sigma')$ are the lapse and shift functions.

The 4-metric $g_{AB}(\tau, \sigma')$ on $\Sigma_r$ has the components $\epsilon g_{\tau\tau} = N^2 - h^{rs} N_r N_s$, $-\epsilon g_{\tau\tau} = N_r = h_{rs} N^s$, $-\epsilon g_{\tau\tau} = h_{rs}$. The inverse of $h_{rs}$, whose signature is $(++ -)$, is $h^{rs} (h^{rs} h_{rs} = \delta^r_s)$ and we have $z^\mu_\eta = N^{\mu} + h^{rs} N_r z^\nu_\eta$ and $\eta^{\mu\nu} = \epsilon(l^{\mu} l^\nu - z^\mu z^\nu)$. The components $g_{AB}$ play the role of the inertial potentials generating the relativistic apparent forces in the non-inertial frame [7].

Let us now consider any isolated system (particles, strings, fields, fluids) admitting a Lagrangian description. Then the coupling to an external gravitational field allows the determination of the matter energy–momentum tensor and of the ten conserved Poincaré generators $P^\mu$ and $J^{\mu\nu}$ (assumed finite) of every configuration of the system. Let us replace the external gravitational 4-metric in the coupled Lagrangian with the 4-metric $g_{AB}(\tau, \sigma')$ of an admissible 3+1 splitting of Minkowski spacetime and let us replace the matter fields with new ones knowing the instantaneous 3-spaces $\Sigma_r$.

For instance a Klein–Gordon field $\phi(x)$ will be replaced with $\phi(\tau, \sigma') = \phi(z(\tau, \sigma'))$, the same for every other field.

Instead for a relativistic particle with world-line $x^\mu(\tau)$ we must make a choice of its energy sign and it will be described by 3-coordinates $\eta(\tau)$ defined by the intersection of the world-line with $\Sigma_r$: $x^\mu(\tau) = z^\mu(\tau, \eta(\tau))$.

In this way we get a Lagrangian depending on the given matter and on the embedding $z^\mu(\tau, \sigma')$ and this formulation is called parametrized Minkowski theories [1, 8, 9]. These theories are invariant under frame-preserving diffeomorphisms, so that there are four first-class constraints (an analog of the super-Hamiltonian and super-momentum constraints of canonical theories) implying that the embeddings $z^\mu(\tau, \sigma')$ are gauge variables. As a consequence, all the admissible non-inertial frames are gauge equivalent, namely physics does not depend on the clock synchronization convention and/or the choice of the 3-coordinates in $\Sigma_r$: only the appearances of phenomena change by changing the notion of instantaneous 3-space.

A particular case of this description is the inertial rest-frame instant form of dynamics for isolated systems [1, 8, 9] which is done in the intrinsic inertial rest frame of their configurations: instantaneous 3-spaces, named Wigner 3-space due to the fact that the 3-vectors inside them are Wigner spin-1 3-vectors, are orthogonal to the conserved 4-momentum of the configuration (in [7] there is the extension to non-inertial rest frames).

In these rest frames there are only three notions of collective variables, which can be built by using only the Poincaré generators (they are non-local quantities knowing the whole $\Sigma_r$) [1]: the canonical non-covariant Newton–Wigner center of mass (or center of spin), the non-canonical covariant Fokker–Pryce center of inertia and the non-canonical non-covariant Møller center of energy. All of them tend to the Newtonian center of mass in the non-relativistic limit. See [9] for the Møller non-covariance world-tube around the Fokker–Pryce 4-vector identified by these collective variables. As shown in [4–6] these three variables can be expressed as known functions of the rest time $\tau$, of canonically conjugate Jacobi data (frozen Cauchy data) $\bar{z} = M c \Sigma_{xw}(0) (\bar{\Sigma}_{xw}(\tau)$ is the standard Newton–Wigner 3-position) and $\dot{h} = P/M_c$, of the invariant mass $M c = \sqrt{\epsilon P^2}$ of the system and of its rest spin $\hat{S}$. 

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As a consequence, every isolated system (i.e. a closed universe) can be visualized as a decoupled non-covariant collective (non-local and therefore un-observable) pseudo-particle described by the frozen Jacobi data \( \vec{z}, \vec{h} \), carrying a pole–dipole structure, namely the invariant mass and the rest spin of the system, and with an associated external realization of the Poincaré group.

The universal breaking of Lorentz covariance is connected to this decoupled non-local collective variable and is irrelevant because all the dynamics of the isolated system lives inside the Wigner 3-spaces and is Wigner covariant. Inside these Wigner 3-spaces the system is described by an internal 3-center of mass with a conjugate 3-momentum and by relative variables and there is an unfaithful internal realization of the Poincaré group [5]: the internal 3-momentum, conjugate to the internal 3-center of mass, vanishes due the rest-frame condition. To avoid a double counting of the center of mass, i.e. an external one and an internal one, also the internal (interaction-dependent) Lorentz boosts vanish. As shown in [5] the only non-zero internal generators are the invariant mass and the rest spin and the dynamics is re-expressed only in terms of internal Wigner-covariant relative variables. Moreover this construction implies that the time-like observer, origin of the 3-coordinates on the Wigner 3-spaces, must be identified with the Fokker–Pryce inertial observer [5], so that the embedding describing the Wigner 3-spaces is

\[
\begin{align*}
\epsilon^\mu (\vec{h}) &= (-h^2; \vec{h}) , \\
\epsilon^\mu (\vec{h}) &= \frac{\Delta h_z}{1 + \sqrt{1 + \vec{h}^2}} , \\
Y^\mu (0) &= \left( \sqrt{1 + \vec{h}^2} \vec{h} \cdot \vec{z}_m^0 + \vec{h} \cdot \vec{z}_m + \vec{\bar{S}} \times \vec{h} \right) \Delta t + \vec{z}_m^0 + \vec{h} \cdot \vec{z}_m \Delta t + \vec{\bar{S}} \times \vec{h} \Delta t ,
\end{align*}
\]

In the case of relativistic particles the reconstruction of their world-lines requires a complex interaction-dependent procedure delineated in [4]. See [5] for the comparison with the other formulations of relativistic mechanics developed for the study of the problem of relativistic bound states.

This allows one to get a relativistic formulation of atomic physics as an effective theory below the threshold of pair production. As a consequence it is possible to include the ‘light-cone’ in atomic physics. While this is irrelevant for experiments on Earth, it is fundamental for space physics in the Solar system. The ACES mission of ESA will make the first precision measurement of the gravitational redshift near Earth (deviation of the null geodesics for light rays from the Minkowski ones at the \( 1/c^2 \) order) by putting the Pharao atomic clock on the space station [10]. Protocols for teleportation from Earth to the space station [11] will require ‘photon’ (eikonal approximation to light rays) propagating along null geodesics and a relativistic formulation of entanglement [12]. Finally this framework will also be needed for the search of gravitational waves by means of atom interferometers [13].

What is still missing is the description of massless particles in the rest-frame instant form of dynamics. This will be described in this paper, including quantization. This is needed for going from quantum optics with non-relativistic two-level atoms [14], used in the experiments on non-relativistic entanglement where strictly speaking photons do not exist (only their polarization and not their world-line is described), to a relativistic theory in which both atoms and photons can coexist. It will allow one to arrive at a relativistic formulation of entanglement experiments with laser beams with a fixed number of photons.

In section 1 we describe a scalar positive-energy massless particle by means of a parametrized Minkowski theory. In section 2, after the addition of a decoupled positive-energy massive particle, to avoid the front form of dynamics, we define the rest-frame instant form of the dynamics of the isolated system of a massless particle plus a massive one. In section 3 we give the pseudo-classical description of a massless particle carrying a Grassmann-valued helicity. Then in section 4 we quantize the system to get a quantum particle with helicity \( \pm 1 \). In the conclusions we make some final comments. In the appendix there is a review of the light-like polarization vectors.
2. A scalar positive-energy massless particle

Let the massless particle have the world-line $x^\mu(\tau) = z^\mu(\tau, \vec{\eta}(\tau))$, so that it is identified by the 3-coordinates $\eta^r(\tau)$ inside the instantaneous 3-spaces $\Sigma_1\tau$. Usually it is described with the singular Lagrangian

$$S = \int d\tau \dot{x}^2(\tau)(\tau, \eta^a(\tau)),$$

which is a functional depending on the Lagrangian variables $z^\mu(\tau, \sigma^u)$, $\eta^a(\tau)$ and $\mu(\tau)$. The Euler–Lagrange equation

$$\delta S = 0$$

implies $\dot{x}^2(\tau) = 0$. On the Wigner 3-spaces, where $x^\mu(\tau) = Y^\mu(0) + h^\mu + \epsilon^\mu(\vec{h})\eta^a(\tau)$, this implies $\dot{\vec{\eta}}^2(\tau) = 1$.

The canonical momenta are

$$\kappa_r(\tau) = \frac{\partial L(\tau)}{\partial \dot{\eta}^r(\tau)} = \frac{2}{\mu(\tau)N(\tau, \eta(\tau))}(g_{\tau\tau} + 2g_{\tau r}\dot{\eta}^r(\tau))(\tau, \eta^a(\tau)),$$

$$\rho(\tau, \sigma^u) = \frac{\partial L(\tau)}{\partial \dot{z}^\mu(\tau, \sigma^u)} = \epsilon l_\mu \frac{z^{\mu + \eta^a}}{N}(\tau, \sigma^v),$$

$$\pi(\tau) = \epsilon l_\mu \frac{z^{\mu + \eta^a}}{N}(\tau, \sigma^v).$$

The canonical Hamiltonian vanishes, $H_c = 0$.

Therefore, we have the following primary constraints:

$$\mathcal{H}_\mu(\tau, \sigma^u) = \rho(\tau, \sigma^u) - \frac{\partial^3}{\partial \sigma^u \partial \eta^a(\tau)}
\times \left[ l_\mu \left( \frac{1}{\mu(\tau)} + \frac{\mu(\tau)}{4} h^\tau(\tau) \kappa_r(\tau, \eta^a) \right) \approx 0, \right.$$}

$$\pi(\tau) \approx 0.\right)$$

Therefore the Dirac Hamiltonian is

$$H_D = \lambda(\tau)\pi(\tau) + \int d^3\sigma \lambda^\mu(\tau, \sigma^u)\mathcal{H}_\mu(\tau, \sigma^u),$$
The preservation in $\tau$ of the primary constraints ($\{H_\mu(\tau, \sigma^u), H_D\} = 0$, $\{\pi(\tau), H_D\} = 0$) implies the following secondary constraint:

$$\chi(\tau) = \frac{1}{\mu^2(\tau)} - \frac{1}{4} h^{\alpha\beta}(\tau, \eta^u(\tau))\kappa_\alpha(\tau)\kappa_\beta(\tau) \approx 0,$$

(2.5)

so that we have $\mu(\tau) \approx \sqrt{\frac{1}{\chi(\tau)}}$.

The $\tau$-preservation of the secondary constraint $\chi(\tau) \approx 0$ determines the Dirac multiplier $\lambda(\tau)$

$$\lambda(\tau) \approx \left( \frac{1}{2} (h^{\alpha\beta}(\tau, \kappa_\alpha(\tau)) \kappa_\beta(\tau))^{1/2} \int \mu \sqrt{h^{\alpha\beta} \kappa_\alpha(\tau) \kappa_\beta(\tau)} - \epsilon \epsilon^\mu \kappa_\beta(\tau) \right) \times (\tau, \eta^u(\tau)) \frac{\partial \lambda(\tau, \eta^u(\tau))}{\partial \eta^u}.$$

(2.6)

In conclusion the constraints $H_\mu(\tau, \sigma^u) \approx 0$ are of first class. Instead the two constraints $\pi(\tau) \approx 0$ and $\chi(\tau) \approx 0$ are second class, $\{\chi(\tau), \pi(\tau)\} = \frac{2}{\mu(\tau)} \neq 0$.

If we eliminate the variables $\mu(\tau)$ and $\pi(\tau)$ by going to Dirac brackets, the first class constraints take the following form:

$$H_\mu(\tau, \sigma^u) = \rho_\mu(\tau, \sigma^u) - \delta^3(\sigma^u - \eta^u(\tau)) [\mu \sqrt{h^{\alpha\beta} \kappa_\alpha(\tau) \kappa_\beta(\tau)} - \epsilon \epsilon^\mu \kappa_\beta(\tau)](\tau, \sigma^u) \approx 0.$$

(2.7)

Since, as shown in [5], the Poincaré generators generated by the action (2.1) are $P^\mu = \int d^3\sigma \rho^\mu(\tau, \sigma^u)$ and $J^{\mu\nu} = \int d^3\sigma (\omega^\mu \rho^\nu - \omega^\nu \rho^\mu)(\tau, \sigma^u)$, we have that equation (2.7) implies $P^\mu \approx \sqrt{h^{\alpha\beta}(\tau, \eta^u(\tau))\kappa_\alpha(\tau)\kappa_\beta(\tau)} - \epsilon \epsilon^\mu h^{\alpha\beta}(\tau, \eta^u(\tau))\kappa_\alpha(\tau)$, and $P^\mu \approx 0$. This implies that for an isolated massless particle we cannot have the description in the rest-frame instant form of dynamics, requiring $e \rho^2 > 0$, but only a front (null) form [15] (see also [16] and its bibliography).

However if we add massive matter it is possible to have the massless particle described in the rest-frame instant form of the overall system. To this end let us add a decoupled positive-energy scalar massive particle $\chi_1^a(\tau) = \chi^a(\tau, \eta^u(\tau))$ with mass $m_1$. The action (2.1) is replaced by the following one:

$$S = \int d\tau d^3\sigma \left( \frac{-m_1 c}{\sqrt{g}} \right) \delta^3(\sigma^u - \eta^u(\tau)) \sqrt{\epsilon g_{\tau\tau} + 2g_{\tau\eta^u} \dot{\eta}^u(\tau) + g_{\eta^u\eta^u} \dot{\eta}^u(\tau)}(\tau, \sigma^u)$$

$$+ \frac{\delta^3(\sigma^u - \eta^u(\tau))}{\mu(\tau)} \frac{g_{\tau\tau} + 2g_{\tau\eta^u} \dot{\eta}^u(\tau) + g_{\eta^u\eta^u} \dot{\eta}^u(\tau)}{e \lambda(\tau)(\tau, \sigma^u) \frac{\partial \lambda(\tau)(\tau, \sigma^u)}{\partial \eta^u}}. \right.$$

(2.8)

Besides equations (2.2) there is the new momentum

$$\kappa_\tau(\tau) = -e m_1 c \sqrt{g_{\tau\tau}(\tau, \eta^u(\tau)) + g_{\eta^u\eta^u}(\tau, \eta^u(\tau)) \dot{\eta}^u(\tau)}$$

(2.9)

and the first class constraints (2.3) become

$$H_\mu(\tau, \sigma^u) = \rho_\mu(\tau, \sigma^u)$$

$$- l_{\mu}(\tau, \sigma^u) \left[ \delta^3(\sigma^u - \eta^u(\tau)) \left( \frac{1}{\mu(\tau)} + \frac{\mu(\tau)}{4} \right) h^{\alpha\beta}(\tau, \sigma^u)\kappa_\alpha(\tau)\kappa_\beta(\tau) \right]$$

$$+ \delta^3(\sigma^u - \eta^u(\tau)) \left[ \frac{m_1 c^2}{\sqrt{g}} + h^{\alpha\beta}(\tau, \sigma^u)\kappa_\alpha(\tau)\kappa_\beta(\tau) \right]$$

$$+ e(\dot{\eta}^u h^{\alpha\beta})(\tau, \sigma^u) \left[ \delta^3(\sigma^u - \eta^u(\tau))\kappa_\alpha(\tau) + \delta^3(\sigma^u - \eta^u(\tau))\kappa_\beta(\tau) \right] \approx 0.$$
The following form of the internal Poincaré generators \( P_{\mu} \) is used:

\[
\rho_{\mu}(\tau, \sigma^u) = \delta^3(\sigma^u - \eta^u(\tau)) \sqrt{h^{\tau\tau}(\tau, \sigma^u)} \kappa_\tau(\tau) \kappa_\tau(\tau) + \delta^3(\sigma^u - \eta^u(\tau)) \kappa_{1\tau}(\tau) \approx 0.
\]

(2.11)

Therefore we get

\[
P_{\mu} = \int \delta^3 \rho_{\mu}(\tau, \sigma^u) = 1 \mu(\tau, \sigma^u) \approx \rho_{p}(\tau, \sigma^u) + \rho_{1}(\tau, \sigma^u).
\]

(2.12)

Since we have \( \rho_{p} = 0 \) and \( \rho_{1} = 0 \) so that the Dirac Hamiltonian \( H_{D} \approx 0 \) of equation (2.4) is replaced by the internal invariant mass \( H_{D} \approx 0 \) as the effective Hamiltonian for the evolution of the internal variables inside the Wigner 3-space.

From [5], as said in the introduction, we get the following form of the external Poincaré generators

\[
P_{\mu} = M \hbar c \rho_{\mu}(\tau, \sigma^u) = M c(\sqrt{1 + \frac{\hbar^2}{E_{\mu}^2}}, \hbar c),
\]

(2.14)

and the following form of the internal Poincaré generators

\[
\rho_{\mu}(\tau, \sigma^u) = \sqrt{\bar{k}_{1}^2 + m_{i}^2 c^2},
\]

(2.15)

As shown in [5, 7–9], the explicit \( \tau \)-dependence of the gauge fixing \( \zeta^u(\tau, \sigma^u) \approx 0 \), defining the inertial rest-frame instant form, implies that the Dirac Hamiltonian \( H_{D} \approx 0 \) of equation (2.4) is replaced by the internal invariant mass \( M c = 1 \bar{\mathcal{E}}_{(\text{int})} = \sqrt{p^2} \) as the effective Hamiltonian for the evolution of the internal variables inside the Wigner 3-space.
3. The pseudo-classical photon: a massless particle with helicity

To describe the helicity of a pseudo-classical photon we follow the method of [16]. Let us associate two complex Grassmann 4-vector \( \theta^\mu(\tau), \theta^\nu(\tau) \) \((\theta^\mu \theta^\nu + \theta^\nu \theta^\mu = 0\) for each value of \( \mu \)) to the massless particle.

If \( \zeta(\tau), \xi(\tau) \) are Grassmann Lagrange multipliers and \( \nu(\tau) \) a Lagrange multiplier, action (2.8) is replaced by the following one:

\[
S = \int \mathrm{d}^4 \sigma \left( -m_1 e \frac{\delta^3(\sigma^\mu - \eta^\mu(\tau))}{\delta \eta^\nu(\tau)} \sqrt{g_{\tau\tau} + 2g_{\tau r} \eta^r(\tau) + g_{\tau \xi}(\tau) \eta^\xi(\tau)} \right)(\tau, \sigma^\nu) + \frac{\mu(\tau)}{\delta \mu(\tau)} \left( \frac{\delta S}{\delta \mu(\tau)} - \frac{\delta S}{\delta \eta^\nu(\tau)} \right)(\tau, \sigma^\nu) \]

The Grassmann momenta are \( \Pi^\nu_\mu = -\frac{\partial L}{\partial \eta^\nu} = -\frac{1}{2} \theta^\nu \), \( \Pi^\mu_\nu = -\frac{\partial L}{\partial \eta^\mu} = -\frac{1}{2} \theta^\mu \) with Poisson brackets \( \{\Pi^\mu_\nu, \theta^\rho\} = \{\Pi^\rho_\nu, \theta^\mu\} = \eta^\rho_\mu\). Since they imply second class constraints, the Grassmann momenta can be eliminated by using the Dirac brackets \( \{\theta^\mu, \theta^\nu\} = i\eta^\mu_\nu \) (we use the notation \( \ldots \) also for \( \ldots \) for simplicity).

Besides equation (2.9) the other canonical momenta are

\[
\kappa_\tau(\tau) = \frac{\partial L(\tau)}{\partial \eta^\tau} = -\frac{1}{2} \frac{N_\tau + h_{\tau r} \eta^r(\tau) - e z_{r\mu}(\tau) \theta^\mu(\tau) - \zeta \theta^\mu(\tau)}{l_\mu \left( \zeta^\tau + \zeta^\tau \theta^\mu - \zeta \theta^\mu \right)}(\tau, \eta^\tau(\tau))
\]

\[
\rho_{\mu}(\tau, \sigma^\nu) = \frac{\delta S}{\delta \xi^\mu(\tau, \sigma^\nu)} = \frac{\delta^3(\sigma^\mu - \eta^\mu(\tau))}{\delta \mu(\tau)} \left( \frac{\delta^3(\sigma^\mu - \eta^\mu(\tau))}{\delta \eta^\nu(\tau)} \right)(\tau, \sigma^\nu)
\]

\[
\times \frac{\left( \left( \zeta^\rho + \zeta^\rho \theta^\tau - \zeta \theta^\tau \right) \left( \left( \xi^\rho + \xi^\rho \theta^\tau - \xi \theta^\tau \right) \right) \right)}{l_\mu \left( \zeta^\tau + \zeta^\tau \theta^\mu - \zeta \theta^\mu \right)}(\tau, \sigma^\nu)
\]

\[
\pi(\tau) = \frac{\partial L(\tau)}{\partial \mu(\tau)}, \quad \pi_{\mu}(\tau) = \frac{\partial L(\tau)}{\partial \xi^\mu(\tau, \sigma^\nu)} = 0,
\]

\[
\pi_{\tau}(\tau) = \frac{\partial L(\tau)}{\partial \nu(\tau)} = 0, \quad \pi_{\nu}(\tau) = \frac{\partial L(\tau)}{\partial \nu(\tau)} = 0.
\]

The primary constraints are \( \pi(\tau) \approx 0, \quad \pi_{\tau}(\tau) \approx 0, \quad \pi_{\nu}(\tau) \approx 0, \quad \pi_{\nu}(\tau) \approx 0 \), and the contraints \( H_{\mu}(\tau, \sigma^\mu) \approx 0 \), given in equation (2.10).

The Dirac Hamiltonian is

\[
H_D(\mathbf{r}, \mathbf{p}) = \int \mathrm{d}^3 \sigma \lambda(\tau, \sigma^\nu) H_{\mu}(\tau, \sigma^\nu) + \lambda(\tau) \pi(\tau) + \lambda_{\nu}(\tau) \pi_{\nu}(\tau) + \xi(\tau) \pi_{\nu}(\tau) + \lambda_{\nu}(\tau) \pi_{\nu}(\tau) + \lambda_{\nu}(\tau) \pi_{\nu}(\tau) + \xi(\tau) \pi_{\nu}(\tau) + \psi(\tau) \pi_{\nu}(\tau) + \nu(\tau) \theta^\nu(\tau) \theta^\mu(\tau).
\]
where
\[ \theta^\mu_\mu'(\tau) \theta^\mu(\tau) \approx 0, \]
\[ \psi(\tau) = \left[ l_\mu \left( \frac{1}{\mu(\tau)} + \frac{\mu(\tau)}{4} h^{r^s} k_s(\tau) k_r(\tau) \right) - \varepsilon z_{r\mu} h^{r^s} k_s(\tau) \right] (\tau, \eta^\mu(\tau)) \theta^\mu(\tau) \approx 0, \]
\[ \psi^*(\tau) = \left[ l_\mu \left( \frac{1}{\mu(\tau)} + \frac{\mu(\tau)}{4} h^{r^s} k_s(\tau) k_r(\tau) \right) - \varepsilon z_{r\mu} h^{r^s} k_s(\tau) \right] (\tau, \eta^\mu(\tau)) \theta^\mu(\tau) \approx 0, \]
are the secondary constraints implied by the \( \tau \)-preservation of the primary constraints
\[ \pi_\pi(\tau) \approx 0, \]
\[ \pi_\pi^*(\tau) \approx 0, \]
\[ \pi(\nu)(\tau) \approx 0. \]
In obtaining this result we used the following consequence of the Grassmann nature of the variables:
\[ l_{\mu}(\theta^\mu - \eta^\mu) \approx 0. \]

The \( \tau \)-preservation of the primary constraints implies the secondary constraint of equation (2.5), so that we can eliminate the pair of variables \( \mu(\tau) \) and \( \pi(\tau) \).

The final form of the first class constraints is
\[ H_\mu(\tau, \sigma^\mu) = \rho_\mu(\tau, \sigma^\mu) H^\mu(\tau, \sigma^\mu) + \lambda(\nu)(\tau) \pi(\nu)(\tau) + \lambda(\nu)(\tau) \pi(\nu)(\tau) + \lambda(\nu)(\tau) \pi(\nu)(\tau) + \lambda(\nu)(\tau) \pi(\nu)(\tau) + \lambda(\nu)(\tau) \pi(\nu)(\tau) + \lambda(\nu)(\tau) \pi(\nu)(\tau) + \lambda(\nu)(\tau) \pi(\nu)(\tau) = 0, \]
\[ \pi_\pi(\tau) \approx 0, \]
\[ \pi_\pi^*(\tau) \approx 0, \]
\[ \pi(\nu)(\tau) \approx 0. \]

The Dirac Hamiltonian (3.3) becomes
\[ H_D = \int d^3 \sigma \lambda^\mu(\tau, \sigma^\mu) H^\mu(\tau, \sigma^\mu) \]
\[ + \gamma^\mu(\tau) \psi^*(\tau) + \gamma(\tau) \psi(\tau) + \beta(\tau) \theta^\mu(\tau) \theta^\mu(\tau), \]
which takes into account all the constraints of equations (3.4).

In the rest-frame instant form on the Wigner instantaneous 3-spaces with \( z^\mu(\tau, \sigma^\nu) = z^\mu_{\text{W}}(\tau, \sigma^\nu) \), where \( l^\mu(\tau, \sigma^\mu) = h^\mu \), the constraints on the \( \theta \) variables can be rewritten in the form
\[ \psi(\tau) = P_{\mu \pi} \theta^\mu(\tau) \approx 0, \]
\[ \psi(\tau) = P_{\mu \pi} \theta^\mu(\tau) \approx 0, \]
\[ \theta^\mu(\tau) \theta^\mu(\tau) \approx 0, \]
where \( P_{\mu \pi} \) was defined in equation (2.13).

Since the Grassmann variables are Lorentz 4-vectors, the ten Poincaré generators generated by action (3.1) are now \( P^\mu = \int d^3 \sigma \rho^\mu(\tau, \sigma^\mu) \) and \( J^{\mu \nu} = \int d^3 \sigma (z^\mu \rho^\nu - z^\nu \rho^\mu)(\tau, \sigma^\mu) + S^{\mu \nu} \) with
\[ S^{\mu \nu} = -i (\theta^\mu \theta^\nu - \theta^\nu \theta^\mu), \]
\[ P_{\mu \pi} S^{\mu \nu} \approx 0. \]

Let us remark that often in the literature one uses an extended Hamiltonian
\[ H_E = \int d^3 \sigma \lambda^\mu(\tau, \sigma^\mu) H^\mu(\tau, \sigma^\mu) + \gamma^\mu(\tau) \psi^*(\tau) + \gamma(\tau) \psi(\tau) + \beta(\tau) \theta^\mu(\tau) \theta^\mu(\tau), \]
which takes into account all the constraints of equations (3.4).
As shown in [16], the gauge fixings to the transversality constraints $\psi^*(\tau) \approx 0$ and $\psi(\tau) \approx 0$ are
\[
\phi^*(\tau) = K_0(P_\tau) \cdot \theta^* \approx 0, \quad \phi(\tau) = K_0(P_\tau) \cdot \theta \approx 0,
\] (3.10)
where the null 4-vector $K_0(P_\tau)$ is defined in the appendix.

In this way we get two pairs of second class constraints with the following Poisson brackets:
\[
\{\psi, \phi^*\} = \{\psi^*, \phi\} = \frac{i}{P_\alpha} \cdot K_0(P_\tau) = i, \quad \{\psi, \phi^*\} = \{\psi^*, \phi\} = 0,
\]
(3.11)
so that the elimination of these constraints implies the following Dirac brackets [16]:
\[
\{A, B\} = \{A, B\} + \frac{i}{P_\alpha} \cdot K_0(P_\tau)\{[A, \psi]\} \cdot \{A, B\}
\]
(3.12)
As shown in [16], the variables $\theta^*\mu(\tau)$ and $\theta^*\nu(\tau)$ can be now replaced by the following ones:
\[
\bar{\theta}_\lambda(\tau) = \theta^*\mu(\tau)e^\mu_{\lambda}(P_\tau), \quad \bar{\theta}_\lambda^*(\tau) = \theta^*\mu(\tau)e^\mu_{\lambda}(P_\tau), \quad \lambda = 1, 2,
\]
\[
\{\bar{\theta}_\lambda, \bar{\theta}_\lambda^*\} = -i \delta_{\lambda\lambda}'', \quad \theta^*\nu(\tau) \cdot \theta(\tau) = -\sum_{\lambda=1}^{2} \tilde{\theta}_\lambda^*(\tau)\tilde{\theta}_\lambda(\tau) \approx 0,
\] (3.13)
where the polarization vectors $\epsilon^\mu_{\lambda}(P_\tau)$ are defined in equations (A.2).

However, since $P^\mu_\tau$ depends on the momenta $\vec{k}(\tau)$ and $\vec{h}$, also the variables $\vec{\eta}(\tau)$ of the massless particle and the Jacobi data $\vec{z}$ have to be modified
\[
\vec{z} = \vec{z} + \frac{1}{2} \left[ P_{\sigma\alpha} \frac{\partial P_{\alpha\rho}}{\partial h} + K_{\rho\alpha} \frac{\partial K_{\rho\sigma}}{\partial h} - \sum_{\lambda} \epsilon_{\lambda\sigma} \frac{\partial \epsilon_{\lambda\rho}}{\partial h} \right] S_{\sigma\rho}^{\alpha\rho}, \quad \{z^i, h^j\}^* = 6^{ij},
\]
(3.14)
\[
\vec{\eta} = \vec{\eta} + \frac{1}{2} \left[ P_{\sigma\alpha} \frac{\partial P_{\alpha\rho}}{\partial k} + K_{\rho\alpha} \frac{\partial K_{\rho\sigma}}{\partial k} - \sum_{\lambda} \epsilon_{\lambda\sigma} \frac{\partial \epsilon_{\lambda\rho}}{\partial k} \right] S_{\sigma\rho}^{\alpha\rho}, \quad \{\eta^r, k^s\}^* = 6^{rs}.
\]

Equations (3.13) imply $(\epsilon_{\lambda\lambda}'') = 1$:
\[
S_{\sigma\rho}^{\mu\nu} = \sum_{\lambda\lambda'} \epsilon^\mu_{\lambda}(P_\tau) \epsilon^\nu_{\lambda'}(P_\tau) S_{\lambda\lambda'},
\]
\[
S_{\lambda\lambda'} = -i (\tilde{\theta}_\lambda^* \tilde{\theta}_{\lambda'} - \tilde{\theta}_{\lambda'}^* \tilde{\theta}_\lambda) = \epsilon_{\lambda\lambda}, \Sigma,
\]
(3.15)
\[
\Sigma = S_{\theta\mu\nu} \epsilon^\mu_{\lambda}(P_\tau) \epsilon^\nu_{\lambda}(P_\tau) = -i (\tilde{\theta}_\lambda^* \tilde{\theta}_{\lambda'} - \tilde{\theta}_{\lambda'}^* \tilde{\theta}_\lambda).
\]

Now we have $S_{\sigma\rho}^{\mu\nu} = 0$ and $(S_{\theta} = \frac{1}{2} \epsilon_{ik} S_{ik}^{\theta})$
\[
\vec{S}_{\theta} = \frac{\vec{P}_\alpha}{|P_\alpha|} \Sigma, \quad \Sigma = \frac{\vec{P}_\alpha \cdot \vec{S}_{\theta}}{|P_\alpha|}.
\] (3.16)

Therefore $\Sigma$ describes the helicity of the massless photon, which has the spin collinear with the 3-momentum.

The $\tau$-preservation of the gauge fixings (3.10) implies the vanishing of the corresponding Dirac multipliers $\gamma^*(\tau) = \gamma(\tau) = 0$ in the Hamiltonian $H_E$ of equation (3.7), which in the
inertial rest frame is replaced by the effective Hamiltonian
\[ H = Mc - \beta(\tau) \sum_k \theta^*_k(\tau) \theta_k(\tau), \]
\[ Mc = \frac{1}{c} \mathcal{E}_{\text{(int)}} = \sqrt{\kappa^2 + m^2c^2 + \kappa_1^2}. \]
\[ (3.17) \]

The external Poincaré generators have the form of equation (2.14) with \( Mc \) of equation (3.17) and internal spin
\[ \vec{S} = \vec{J}_{\text{(int)}} = \vec{n} \times \vec{k} + \vec{n}_1 \times \vec{k}_1 + \vec{S}_\theta. \]
\[ (3.18) \]

The other internal Poincaré generators, explicitly satisfying the Poincaré algebra, are
\[ \vec{P}_{\text{(int)}} = \vec{k} + \vec{k}_1 \approx 0, \]
\[ \vec{K}_{\text{(int)}} = -\vec{n} \sqrt{\kappa^2} - \vec{n}_1 \sqrt{m^2c^2 + \kappa^2_1 + \kappa_1} \approx 0. \]
\[ (3.19) \]

Their vanishing eliminates the internal 3-center of mass inside the Wigner 3-space and its conjugate momentum. The form of the internal Lorentz boosts, with the correct Poisson brackets with the other generators, has been guessed, since action (3.1) contains Lagrange multipliers which make difficult to find an energy–momentum tensor independent from them.

4. Quantization

Let us add some remarks about the quantization of the rest-frame instant form.

The quantization of the bosonic variables \( \vec{z}, \vec{h}, \vec{n}, \vec{k}, \vec{n}_1, \vec{k}_1 \), and \( \vec{n}, \vec{k} \) is done with the new rest-frame quantization scheme for relativistic quantum mechanics introduced in [12] in an un-physical Hilbert space \( \mathcal{H} = \mathcal{H}_{\text{com}} \otimes \mathcal{H}_{\vec{h}_1} \otimes \mathcal{H}_{\vec{n}} \), where \( \mathcal{H}_{\text{com}} \) is the Hilbert space of the frozen Jacobi data of the external decoupled center of mass. The physical states and the associated physical Hilbert space have to be identified by solving the restrictions
\[ \langle \Phi_{\text{phys}} | \vec{P}_{\text{(int)}} | \Phi_{\text{phys}} \rangle = \langle \Phi_{\text{phys}} | \vec{K}_{\text{(int)}} | \Phi_{\text{phys}} \rangle = 0 \] (quantization of the rest-frame conditions (3.19) eliminating the internal center of mass inside the Wigner 3-space). The resulting physical Hilbert space should have the structure \( \mathcal{H}_{\text{phys}} = \mathcal{H}_{\text{com}} \otimes \mathcal{H}_{\text{rel}}, \) where \( \mathcal{H}_{\text{rel}} \) is the internal Hilbert space associated with the Wigner-covariant relative variables \( \vec{r}, \vec{p}, \vec{r}_1, \vec{p}_1, \) describing the isolated system of the massive plus massless particles in the rest frame.

As shown in [12] the quantization of the frozen Jacobi data \( \vec{z}, \vec{h}, \vec{n}, \vec{k} \), is done in the preferred \( \vec{h} \)-basis (definition of the rest frame) in the momentum representation: \( h' \mapsto h', \vec{z}' \mapsto i\hbar \frac{\vec{z}}{2} - i\hbar \frac{\vec{h}}{2} \). The center-of-mass wavefunction with 3-velocity \( \vec{k} \) is defined by \( \psi_{\vec{k}}(\vec{h}) = \delta^3(\vec{h} - \vec{k}) \) and the scalar product is
\[ \langle \psi_1 | \psi_2 \rangle = \int \frac{d^3h}{2\pi^3} \psi^*_1(\vec{h}) \psi_2(\vec{h}). \]

The unphysical Hilbert space \( \mathcal{H}_{\vec{n}} \), with scalar product \( \langle \phi_1, \phi_2 \rangle = \int d^3\eta \phi_1^*(\tau, \vec{\eta}) \phi_2(\tau, \vec{\eta}) \) in the coordinate representation, is the standard space of a massive particle whose positive energy operator \( \sqrt{m^2c^2 + \vec{k}^2} \) is a pseudo-differential operator defined in [19].

Instead the unphysical Hilbert space \( \mathcal{H}_{\vec{h}} \) of the massless particle is a suitable limit for zero mass of the precedent Hilbert space. The delicate point is to see whether the methods of [20] (with a smoothing around \( \vec{k} = 0 \)) allow one to define a massless pseudo-differential operator \( \sqrt{\vec{k}^2} \), such that the velocity operator \( \frac{d}{d\tau} \vec{h}'(\tau) = [\vec{h}', (\tau), \sqrt{\vec{k}^2}] \) is well defined and satisfies
\[ \sum_{\tau} \left( \frac{d}{d\tau} \vec{h}'(\tau) \right)^2 = 1. \]

See [17] for the quantization of spinning massive particles.
If this problem has a well-defined solution and if the physical Hilbert space \( \mathcal{H}_{\text{rel}} \) with its scalar product can be identified, then the physical Hamiltonian and the rest spin will be operators depending only on the relative variables \( \vec{p}_{12}, \vec{\pi}_{12} \), and corresponding to equations (3.17) and (3.18) restricted by equations (3.19). This problem will be studied elsewhere.

Instead we add the quantization of the Grassmann-valued helicity of a massless particle by enlarging its Hilbert space \( \mathcal{H}_{\eta} \) in the following way.

As shown in [16], the quantization rule \( \theta^\mu \mapsto \hat{b}^\mu, \theta^{\ast \mu} \mapsto \hat{b}^{\ast \mu} \) leads to a four-dimensional Fermi oscillator (\( [\hat{a}, \hat{b}]_+ = ab + ba \) denotes the anti-commutator):

\[
[b^\mu, b^{\ast \nu}]_+ = -\eta^\mu\nu, \quad [b^\mu, b^\nu]_+ = [b^{\ast \mu}, b^{\ast \nu}]_+ = 0. \tag{4.1}
\]

Therefore we have a 16-dimensional Hilbert space \( \mathcal{H}_{\text{pol}} \), describing the polarization of the massless particle, spanned by the basis \( |0\rangle, |\mu\rangle = \hat{b}^{\ast \mu} |0\rangle, |\mu\nu\rangle = \hat{b}^{\ast \mu} \hat{b}^{\ast \nu} |0\rangle \).

However we have the quantum analog of constraints (3.6) to take into account: they will restrict \( \mathcal{H}_{\text{pol}} \) to a physical two-dimensional Hilbert space \( \mathcal{H}_{\text{helicity}} \) describing the \((1, 0) + (0, 1)\) helicity representation of the Poincaré group. The first stage of the reduction is done by applying the quantum transversality constraints (replacing the classical ones (3.8)) with the Gupta–Bleuler method

\[
\hat{P}_o \cdot b |\psi\rangle_{\text{phys}} = 0, \quad \text{phys} \langle \psi | \hat{P}_o \cdot b |0\rangle = 0. \tag{4.2}
\]

In this way only four states of the basis survive \( |0\rangle, A_\mu(P_o) b^{\ast \mu} |0\rangle \) (with \( P^\mu_\alpha A_\mu(P_o) = 0 \)) and \( F_{\mu\nu}(P_o) b^{\ast \mu} b^{\ast \nu} |0\rangle \) (with \( P^\mu_\alpha F_{\mu\nu}(P_o) = 0, F_{\mu\nu}(P_o) = -F_{\nu\mu}(P_o) \)).

As shown in [16] the last constraint in equations (3.13) is \( (b^{\ast \mu} \cdot b + \delta) |\psi\rangle_{\text{phys}} = 0 \), where \( \delta \) is a c-number present due to ordering problems. To select the photon-like state \( A_\mu(P_o) b^{\ast \mu} |0\rangle \) (with \( P^\mu_\alpha A_\mu(P_o) = 0 \)) with two helicity levels one must choose an ordering corresponding to \( \delta = 1 \) [16].

5. Conclusions

In this paper we have included in the rest-frame instant form of the dynamics of isolated systems in Minkowski spacetime the description of positive-energy massless particles both without and with helicity. To avoid the front form of dynamics we must include at least an additional positive-energy scalar massive particle (or any other type of matter). This allows us to find a classical background for the particle description of a ray of light in Minkowski spacetime.

We have also added some comments on how to apply to this isolated system, the procedure of quantization based on the recently developed rest-frame form of relativistic quantum mechanics [12]. If a suitable definition for a pseudo-differential operator corresponding to \( \sqrt{\vec{\kappa}^2} \) exists, we have a description of an isolated photon, to be compared with the one-particle approximations used for a photon in [18].

Appendix: The light-like polarization vectors

By using the future-pointing null vector \( P^\mu_o = \sqrt{\vec{\kappa}^2} h^\mu - \epsilon^\mu_l \vec{h} k_l \) of equations (2.12), we can rewrite appendix A of [16] in our formalism.

For null vectors we have \( P^\mu_o = L(P_o, \hat{P}_o)^\mu, \hat{P}_o^{\mu, \nu} \) with \( P^{\mu, \nu} \) being a 4-vector with components \( \vec{P}_o = \omega(1; 0, 0, 1) \), where \( L(P_o, \hat{P}_o) \) is the standard boost for null vectors.
whose expression is

\[
L(P_\alpha, \vec{P}_\alpha) = \left( \frac{1}{2} \left( \frac{\vec{P}_\alpha}{m} + \frac{\omega}{|\vec{P}_\alpha|} \right) \right) \epsilon^\mu_\alpha(\vec{P}_\alpha),
\]

where \( \epsilon^\mu_\alpha(\vec{P}_\alpha) \) is the standard basis \((1; 0, 0, 0), (0; 1, 0, 0), (0; 0, 1, 0), (0; 0, 0, 1)\).

Instead of this basis it is convenient to use the following one:

\[
P_\alpha^\mu = \delta^\mu_\alpha + \frac{P^\mu_\alpha P_\lambda}{|P_\alpha|^2} \frac{P_\lambda}{|P_\alpha|}, \quad \alpha, \lambda = 1, 2, \quad P_\alpha^2 = K_\alpha^\mu(\vec{P}_\alpha) = 0, \quad \epsilon_\alpha \cdot K_\alpha(\vec{P}_\alpha) = 1,
\]

\[
K_\alpha^\mu(\vec{P}_\alpha) = \frac{1}{2 \omega} \left( \epsilon^\mu_\alpha(\vec{P}_\alpha) - \epsilon^\mu_\alpha(\vec{P}_\alpha) \right) = \left( \frac{1}{|P_\alpha|^2} \right) \epsilon_\alpha(\vec{P}_\alpha), \quad \alpha, \lambda = 1, 2,
\]

\[
\eta^{\mu\nu} = \epsilon^\mu_\alpha(\vec{P}_\alpha) \eta^{\alpha\beta} \epsilon^\nu_\beta(\vec{P}_\alpha),
\]

where \( \epsilon^\mu_\alpha(\vec{P}_\alpha) \) is the standard basis \((1; 0, 0, 0), (0; 1, 0, 0), (0; 0, 1, 0), (0; 0, 0, 1)\).

The light-like polarization vectors are just the columns of this matrix

\[
\eta^{\mu\nu} = \epsilon^\mu_\alpha(\vec{P}_\alpha) \eta^{\alpha\beta} \epsilon^\nu_\beta(\vec{P}_\alpha),
\]

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