Frobenius manifolds on orbit spaces of non-reflection representations

July 28, 2021

Zainab Al-Maamari \(^1\), Yassir Dinar \(^2\).

Abstract

We prove the orbit spaces of some non-reflection representations of finite groups posses Frobenius manifold structures.

Mathematics Subject Classification (2020) Primary 53D45; Secondary 16W22, 57R18, 14B05.

Keywords: Invariant rings, Frobenius manifold, Representations of finite groups, Flat pencil of metrics, Quotient singularities, Orbifolds.

Contents

1 Introduction 2

2 Flat pencil of metrics and Frobenius manifolds 3
  2.1 Frobenius manifolds ............................................................... 3
  2.2 Flat pencil of metrics .............................................................. 4
  2.3 A duality and inversion symmetry ........................................... 5

3 Coxeter groups 6
  3.1 Duality on orbits of reflection groups .................................... 6
  3.2 Sign times reflection representation ....................................... 9

4 Dihedral and dicyclic groups 13
  4.1 Dihedral groups .................................................................. 13
  4.2 Dicyclic groups .................................................................. 13
  4.3 Finite subgroups of \( SL_2(\mathbb{C}) \) ........................................ 16

5 Finite subgroups of \( SL_3(\mathbb{C}) \) 17

6 Conclusions 20

\(^1\) Sultan Qaboos University, Muscat, Oman, s100108@student.squ.edu.om
\(^2\) Sultan Qaboos University, Muscat, Oman, dinar@squ.edu.om
1 Introduction

Frobenius manifold is a geometric realization introduced by B. Dubrovin for a potential satisfying a system of partial differential equations known as Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations which describes the module space of two dimensional topological field theory. Remarkably, Frobenius manifolds are also recognized in many other fields in mathematics like invariant theory, quantum cohomology, integrable systems and singularity theory [5]. Briefly, a Frobenius algebra is a commutative associative algebra with unity $e$ and a nondegenerate bilinear form $\Pi$ invariant under product, i.e., $\Pi(a \cdot b, c) = \Pi(a, b \cdot c)$. A Frobenius manifold is a manifold with a smooth structure of a Frobenius algebra on the tangent space at any point with certain compatibility conditions. Globally, we require the metric $\Pi$ to be flat and the unity vector field $e$ is constant with respect to it. In this article, we will show that orbit spaces of some non-reflection representations of finite groups acquire Frobenius structures.

We use the following notations and facts for a finite group $G$ and a complex linear representation $\psi : G \to GL(V)$. We denote by $\mathbb{C}[V]$ the ring of polynomial functions on $V$, $\mathbb{C}[\psi]$ the subring of invariant polynomials in $\mathbb{C}[V]$, and $O(\psi)$ the orbit space of the action of $G$ on $V$. Then $\mathbb{C}[\psi]$ is finitely generated by homogeneous polynomials and $O(\psi)$ is a variety with coordinate ring $\mathbb{C}[\psi]$ ([17], [3]). By Chevalley–Shephard–Todd theorem, $\mathbb{C}[\psi]$ is a polynomial ring if and only if $\psi$ is generated by pseudo-reflections. Let $(x_1, \ldots, x_n)$ be linear coordinates on $V$ and $f \in \mathbb{C}[\psi]$. Then the Hessian $H(f) := \frac{\partial^2 f}{\partial x_i \partial x_j}$ defines a bilinear form on the tangent space of $O(\psi)$ and if $\det(H(f)) \neq 0$ then $f$ is a minimal degree invariant polynomial ([18], page 6). In this article, we will drop the word pseudo as all representations will be considered as complex representations.

Let $\mathcal{W}$ be a finite irreducible Coxeter group or Shephard group of rank $r$ and $\rho_{ref}$ is the standard reflection representation of $\mathcal{W}$. Boris Dubrovin proved that the orbit space $O(\rho_{ref})$ acquire a polynomial Frobenius structure ([4],[8]). This result led to the classification of irreducible semisimple polynomial Frobenius manifolds with positive degrees (see section 3.1 for more details). His method was used in [25] when $\mathcal{W}$ is a Coxeter group of type $B_r$ or $D_r$ to construct $r$ Frobenius manifolds on $O(\rho_{ref})$. However, all above-mentioned constructions seem like they depend on the fact that the invariant rings are polynomial rings. In this article, we report applying successfully Dubrovin’s method on some non-reflection representations, i.e., their invariant rings are not polynomial rings.

We mention that Dubrovin and his collaborators constructed Frobenius manifolds using invariant rings of infinite discrete groups being extensions of affine Weyl groups ([7], [10], [26]). However, also in these cases, the cardinality of a minimal set of generators equals the rank of the representation.

Let us fix a finite group $G$ and a linear representation $\psi : G \to GL(V)$ of rank $r$. Then we summarize Dubrovin’s method to prove existence of a Frobenius structure on $O(\psi)$ as follows:

1. Fix a homogenous invariant polynomial $f_1$ having the minimal positive degree.
2. Verify that
   
   \begin{equation}
   \text{the inverse of the Hessian } H(f_1) \text{ defines a contravariant flat metric } <\cdot, \cdot>_{2} \hspace{1cm} (1.1)
   \end{equation}
   
   on some open subset $U$ of $O(\psi)$. For example, this happens if $\psi$ is a real representation (in this case degree $f_1$ equals 2) [11] or $\psi$ is the standard reflection representation of a Shephard group [18].
3. Construct another contravariant metric $<\cdot, \cdot>_1$ which forms with $<\cdot, \cdot>_{2}$ a regular quasihomogenius flat pencil of metrics (abbr. regular QFPM) on $U$ ( see section 2.2 for details).
4. Using theorem 2.5 due to Dubrovin, we get a Frobenius structure on $U$ which depends on the representation $\psi$ of $G$ or $\mathbb{C}[\psi]$. 

2
By abuse of language, we say $\mathcal{O}(\psi)$ acquires a natural Frobenius structure.

We will prove the orbit spaces of the following representations possess Frobenius structures using Dubrovin’s method:

1. The standard reflection representation of a finite irreducible Coxeter group: We prove there is a natural rational Frobenius structure different from the ones constructed in [4] and [25]. We give details in section 3.1. Note that here the invariant ring is a polynomial ring.

2. The non-reflection irreducible representation of dimension $r$ of Coxeter group of type $A_r$: We show the orbit space carries the same Frobenius structures that can be obtained on orbit space of the standard reflection representation. We give details in section 3.2.

3. Irreducible representations of dihedral groups and dicyclic groups: These groups have only rank 1 and 2 irreducible representations. We will prove that any rank 2 representation has two Frobenius structures. Some of them acquire Frobenius structures with rational degrees. See section 4 for details.

4. All finite subgroups of the special linear group $SL_2(\mathbb{C})$: We get polynomial and rational Frobenius structures related to dihedral groups. See section 4.3.

5. All finite subgroups of the special linear group $SL_3(\mathbb{C})$ where the invariant rings are complete intersection: Dubrovin’s method fails on some of them. However we find a total of 8 types of Frobenius structures including a trivial one. Details are given in section 5.

We mention that except for the second item in the list above, there are enough information in the literature about the invariant rings in order to apply Dubrovin’s method. For the non-reflection irreducible representation of dimension $r$ of Coxeter group of type $A_r$, we have to construct $r$ algebraically independent invariant polynomials.

We emphasis that constructing a flat pencil of metric for a given contravariant metric is not a simple task (see for example [9] and [21] for details). In this article, we will find regular QFPMs using a simple criterion (see lemma 2.4 below). However, in some cases we will use a set of invariant polynomials different from the generators of the invariant ring.

To make the article as self-contained as possible, we recall in section 2.1 and 2.2 the definition of Frobenius manifold and its relation with flat pencils of metrics. In section 2.3 we review a duality theorem for certain class of flat pencil of metrics obtained in [1]. In section 3.1, we give some details about the work of Dubrovin and Zuo on standard reflection representations of finite irreducible Coxeter groups and the classification of polynomial Frobenius manifolds.

## 2 Flat pencil of metrics and Frobenius manifolds

### 2.1 Frobenius manifolds

Let $M$ be a Frobenius manifold with flat metric $\Pi$ and identity vector field $e$. In flat coordinates $(t_1, \ldots, t^r)$ for $\Pi$ where $e = \partial_{t^r}$ the compatibility conditions imply that there exists a function $F(t^1, \ldots, t^r)$ which encodes the Frobenius structure, i.e., the flat metric is given by

$$\Pi_{ij}(t) = \Pi(\partial_{t^i}, \partial_{t^j}) = \partial_{t^r} \partial_{t^i} \partial_{t^j} F(t)$$

(2.1)
and, setting $\Omega_1(t)$ to be the inverse of the matrix $\Pi(t)$, the structure constants of the Frobenius algebra are given by

$$C_{ij}^k(t) = \Omega_1^{kp}(t) \partial_{\tau^p} \partial_{\tau^q} \partial_{\tau^r} \mathbb{F}(t).$$

Here, and in what follows, summation with respect to repeated upper and lower indices is assumed. In this article, we assume the quasihomogeneity condition for $\mathbb{F}(t)$ takes the form

$$E \mathbb{F}(t) = d_i t^i \partial_{\tau^i} \mathbb{F}(t) = (3 - d) \mathbb{F}(t); \quad d_r = 1. \tag{2.2}$$

The vector field $E = d_i t^i \partial_{\tau^i}$ is known as Euler vector field and it defines the degrees $d_i$ and the charge $d$ of $M$. The associativity of the Frobenius algebra implies that the potential $\mathbb{F}(t)$ satisfies WDVV equations, i.e.,

$$\partial_{\tau^i} \partial_{\tau^j} \partial_{\tau^k} \mathbb{F}(t) = \partial_{\tau^i} \partial_{\tau^j} \partial_{\tau^k} \mathbb{F}(t) \Omega_1^{kp} \partial_{\tau^p} \partial_{\tau^q} \partial_{\tau^r} \mathbb{F}(t), \quad \forall i, j, q, n. \tag{2.3}$$

We say $M$ is polynomial (rational) if $\mathbb{F}(t)$ is polynomial (rational) function.

**Definition 2.1.** Let $M$ and $\tilde{M}$ be two Frobenius manifolds with flat metrics $\Pi$ and $\tilde{\Pi}$ and potentials $\mathbb{F}$ and $\tilde{\mathbb{F}}$, respectively. We say $M$ and $\tilde{M}$ are (locally) equivalent if there are open sets $U \subseteq M$ and $\tilde{U} \subseteq \tilde{M}$ with a local diffeomorphism $\phi : U \rightarrow \tilde{U}$ such that

$$\phi^* \tilde{\Pi} = c^2 \Pi, \tag{2.4}$$

for some nonzero constant $c$, and $\phi_* : T_1 U \rightarrow T_{\phi(t)} \tilde{U}$ is an isomorphism of Frobenius algebras.

Note that, if $M$ and $\tilde{M}$, are equivalent Frobenius structures then it is not necessary that $\phi^* \tilde{\mathbb{F}} = \mathbb{F}$.

### 2.2 Flat pencil of metrics

We review the relation between flat pencils of metrics and Frobenius manifolds outlined in [6].

Let $M$ be a smooth manifold of dimension $r$ and fix local coordinates $(u^1, \ldots, u^r)$ on $M$.

**Definition 2.2.** A symmetric bilinear form $(\ldots)$ on $T^*M$ is called a contravariant metric if it is invertible on an open dense subset $M_0 \subseteq M$. We define the contravariant Levi-Civita connection or Christoffel symbols $\Gamma^i_k$ for a contravariant metric $(\ldots)$ by

$$\Gamma^i_k := -g^{im} \Gamma^j_{mk} \tag{2.5}$$

where $\Gamma^j_{mk}$ are the Christoffel symbols of the metric $<\ldots>$ defined on $TM_0$ by the inverse of the matrix $\Omega^{ij}(u) = (du^i, du^j)$. We say the metric $(\ldots)$ is flat if $<\ldots>$ is flat.

Let $(\ldots)$ be a contravariant metric on $M$ and set $\Omega^{ij}(u) = (du^i, du^j)$. Then we will use $\Omega$ to refer to the metric and $\Omega(u)$ to its matrix in the coordinates. In particular, Lie derivative of $(\ldots)$ along a vector field $X$ will be written $\mathcal{L}_X \Omega$ while $X \Omega^{ij}$ means the vector field $X$ acting on the entry $\Omega^{ij}$.

**Definition 2.3.** [6] Let $\Omega_2$ and $\Omega_1$ be two flat contravariant metrics on $M$ with Christoffel symbols $\Gamma^{ij}_{2k}$ and $\Gamma^{ij}_{1k}$, respectively. Then they form a flat pencil of metrics (abbr. FPM) if $\Omega_\lambda := \Omega_2 + \lambda \Omega_1$ defines a flat metric on $T^*M$ for generic constant $\lambda$ and the Christoffel symbols of $\Omega_\lambda$ are given by $\Gamma^{ij}_{\lambda k} = \Gamma^{ij}_{2k} + \lambda \Gamma^{ij}_{1k}$.

Such flat pencil of metrics is called quasihomogenous of degree $d$ if there exists a function $\tau$ on $M$ such that the vector fields

$$E := \nabla_2 \tau, \quad E^i = \Omega^{ij}_{2k}(u) \partial_{u^k} \tau \tag{2.6}$$

$$e := \nabla_1 \tau, \quad e^i = \Omega^{ij}_{1k}(u) \partial_{u^k} \tau$$
satisfy the following properties

\[ [e, E] = e, \quad \mathcal{L}_E \Omega_2 = (d - 1) \Omega_2, \quad \mathcal{L}_e \Omega_2 = \Omega_1 \quad \text{and} \quad \mathcal{L}_e \Omega_1 = 0. \quad (2.7) \]

In addition, the quasihomogenous flat pencil of metrics (abbr. QFPM) is called **regular** if the (1,1)-tensor

\[ R^j_i = \frac{d - 1}{2} \delta^j_i + \nabla_i E^j \]  

is nondegenerate on \( M \).

We will use the following source for FPM.

Lemma 2.4. \([4]\) Let \( \Omega_2 \) be a contravariant flat metric on \( M \). Assume that in the coordinates \((u^1, ..., u^r)\), \( \Omega_2^{ij}(u) \) and \( \Gamma^{ij}_k(u) \) depend almost linearly on \( u^r \). Suppose that \( \Omega_1 := \mathcal{L}_{u^r} \Omega_2 = \partial_{u^r} \Omega_2(u) \) is nondegenerate. Then \( \Omega_2 \) and \( \Omega_1 \) form a FPM. The Christoffel symbols of \( \Omega_1 \) has the form \( \Gamma^{ij}_k(u) = \partial_{u^r} \Gamma^{ij}_k(u) \).

If \( M \) is a Frobenius manifold then \( M \) has a QFPM but it does not necessarily satisfy the regularity condition. In the notations of section 2.1, the QFPM consists of the intersection form \( \Omega_2(t) \) and the flat metric \( \Omega_1(t) \) with degree \( d \) where

\[ \Omega_2^{ij}(t) := (d - 1 + d_i + d_j) \Omega_1^m \Gamma_1^{im} \Omega_2^{jk} \partial_r \partial_{r+1} F. \]  

Furthermore, \( \tau = t^1 \) and \( E \) with \( e \) are defined by (2.6) and satisfy equations (2.7) \([6]\). The converse is given by the following theorem

Theorem 2.5. \([6]\) Let \( M \) be a manifold carrying a regular quasihomogenous flat pencil of metrics formed by \( \Omega_2 \) and \( \Omega_1 \). Denote by \( M_0 \subseteq M \) the subset of \( M \) where the metric \( \Omega_1 \) is invertible. Define the multiplication of 1-forms on \( M_0 \) putting

\[ u \cdot v := \Delta(u, R^{-1} v) \quad \text{where} \quad \Delta^{ijk} = \Omega_1^{im} \Gamma_1^{ik} - \Omega_2^{im} \Gamma_2^{jk}. \]  

Then there exists a unique Frobenius structure on \( M \) such that

1. The flat metric is given by the inverse of \( \Omega_1 \).
2. The product on the tangent space is dual to the product (2.10) under the metric \( \Omega_1 \).
3. The intersection form is \( \Omega_2 \).
4. The unity vector field \( e \) and the Euler vector field \( E \) are defined by (2.6).

2.3 A duality and inversion symmetry

Let \( M \) be a Frobenius manifold, we adapt the notations of section 2.1.

Theorem 2.6. \([5]\) Suppose \( \Pi_{rr}(t) = 0 \) and all the degrees \( d_i \)'s are distinct. Then there is a linear change of the coordinates \((t^1, ..., t^r)\) such that \( \Pi_{ij} = \delta_{i+j,r+1} \). In these flat coordinates, the potential \( F \) takes the form

\[ F(t) = \frac{1}{2} (t^r)^2 t^1 + \frac{1}{2} t^r \sum_{i=2}^{r-1} t^i t^{r-i+1} + G(t^1, ..., t^{r-1}) \]  

and the degrees satisfy the duality

\[ d_i + d_{r-i+1} = 2 - d. \]  

(2.12)
It is straightforward to prove under the hypothesis of theorem 2.6 that the associated flat pencil of metrics is regular if and only if $\frac{d_1}{2} + d_i$ for every $i$.

**Theorem 2.7.** Let $M$ be a Frobenius manifold with a flat coordinates $(t^1, \ldots, t^r)$ where the potential has the form (2.11). Assume the associated quasihomogenous flat pencil of metrics defined by the intersection form $\Omega_2$ and the flat metric $\Omega_1$ is regular of degree $d \neq 1$. Then there is another Frobenius structure on $M \setminus \{t^1 = 0\}$ having the flat coordinates

$$s^1 = -t^1, \quad s^i = t^j(t^1)^{\frac{d_2 - d_1}{2}}, \quad for \quad 1 < i < r, \quad s^r = \frac{1}{2} \sum_{i=1}^{r} t^i t^{r-i+1}(t^1)^{\frac{d_2}{2} - 1}$$

and its flat pencil of metrics is also regular and consists of the intersection form $\Omega_2$ and the flat metric $\bar{\Omega}_1$ where $\Omega_1$ is Lie derivative of $\Omega_2$ along the vector field $\partial_{s^r}$.

We prove in [1] that the potential $\bar{F}(s)$ of new Frobenius structure constructed in theorem 2.7 is nothing but the inversion symmetry of the solution $F(t)$ (see [4], appendix B for details about inversion symmetry). Furthermore, the potential $\bar{F}(s)$ has the form

$$\bar{F}(s) = t^1 \sum \left( F(t^1, \ldots, t^r) - \frac{1}{2} t^r \sum_{i=1}^{r} t^i t^{r-i+1} \right)$$

which will be rational in $s^1$. The degrees of the new Frobenius structure are

$$\bar{d}_1 = -d_1, \quad \bar{d}_r = 1, \quad \bar{d}_i = d_i - d_1 \quad for \quad 1 < i < r.$$ 

and the charge is $\bar{d} = 2 - d$. Form the point of view of this article, theorem 2.7 explains the appearance of some rational Frobenius structures in next sections.

Here is an example where theorem 2.7 can not apply. For convenience, we write in examples, indices of coordinates using subscripts instead of superscripts.

**Example 2.8.** A Frobenius manifold is called trivial if the structure constants does not depend on the point. In this case, all monomials in the potential are cubic, all degrees equal 1 and the charge is 0. Then the method given in this section will not give a dual Frobenius structure. An example in dimension 3 will encounter later is the trivial Frobenius manifold $T_3$ given by the potential

$$F = \frac{1}{2} t_2^2 t_1 + \frac{2}{3} t_1^3 - t_2 t_1^2 + 2t_2^2 t_1 - t_2 t_3^2,$$

where $e = \partial_{t_1}$. This has no dual since $F$ can not have the form (2.11).

### 3 Coxeter groups

#### 3.1 Duality on orbits of reflection groups

In this section, we recall the standard reflection representations of irreducible finite Coxeter groups and review the construction of Frobenius manifolds on their orbit spaces. Then we apply theorem 2.7 and give some examples.

We fix an irreducible finite Coxeter system $(W, S)$ of rank $r$, i.e.,

$$W = \langle S | (ss')^m(s,s') = 1; \forall s, s' \in S >, \quad r = |S|.$$ 

(3.1)
| Type of $\mathcal{W}$ | $\eta_1, \ldots, \eta_r$ |
|---------------------|------------------|
| $A_r$               | $2, 3, \ldots, r + 1$ |
| $B_r$               | $2, 4, 6, \ldots, 2r$ |
| $D_r$               | $2, 4, 6, \ldots, 2r - 2$ |
| $E_6$               | $2, 5, 6, 8, 9, 12$ |
| $E_7$               | $2, 6, 8, 10, 12, 14, 18$ |
| $E_8$               | $2, 8, 12, 14, 18, 20, 24, 30$ |
| $F_4$               | $2, 6, 8, 12$ |
| $H_3$               | $2, 6, 10$ |
| $H_4$               | $2, 12, 20, 30$ |
| $I_2(m)$            | $2, m$ |

Table 1: Degrees of invariant polynomials of $\rho_{\text{ref}}$

Let $V$ be the formal vector space over $\mathbb{C}$ with basis $\{\alpha_s \mid s \in S\}$. Then the standard reflection representation of $\mathcal{W}$ is defined by

$$
\rho_{\text{ref}} : \mathcal{W} \to GL(V), \ s \mapsto R_s, \ s \in S.
$$

$$
R_s(v) := v - 2B(\alpha_s, v)\alpha_s, \ v \in V, \ B(\alpha_s, \alpha_{s'}) := -\cos \frac{\pi}{m(s, s')}.
$$

Here $B$ is the standard positive-definite Hermitian form on $V$ which is invariant under $\rho_{\text{ref}}$.

By Chevalley–Shephard–Todd theorem, the invariant ring $\mathbb{C}[\rho_{\text{ref}}]$ is a polynomial ring generated by $r$ homogeneous polynomial. We fix generators $u^1, \ldots, u^r$ for $\mathbb{C}[\rho_{\text{ref}}]$. We assume $\deg u^i = \eta_i$ and

$$
2 = \eta_1 < \eta_2 \leq \eta_3 \leq \ldots \leq \eta_{r+1} < \eta_r.
$$

These degrees are an intrinsic property of the group $\mathcal{W}$ [14] and we list them in Table 3.1.

We set $u^1$ equals the quadratic from of $B$. Hence, the inverse of the Hessian of $u^1$ defines a flat contravariant metric $\Omega_2$ on $\mathcal{O}(\rho_{\text{ref}})$. It is easy to prove that $\Omega_2(u)$ is almost linear in $u^r$ by analysing the degrees of $\Omega_2^2(u)$. We fix the vector field $e = \partial_{u^r}$. Note that changing the generators of $\mathbb{C}[\rho_{\text{ref}}]$, $e$ is uniquely defined up to a constant factor. Dubrovin proved that $\Omega_2$ and $\Omega_1 := \mathcal{L}_e \Omega_2$ form a regular QFPM of charge $\frac{\eta_r - 2}{\eta_r}$ [6]. In this case, $\tau = \frac{1}{\eta_r} u^1_1$ and the vector field $E$ is given by $E = \frac{1}{\eta_r} \sum_i \eta_i u^i \partial_{u^i}$. This result initiated what we call Dubrovin’s method in the introduction. Below, we restate Dubrovin theorem. We observe that $E$ is uniquely defined and does not depend on the choice of invariants $u^i$. Also, we mention that the flat metric $\Omega_1$ was studied by K. Saito [20], [19] and his results was very important to the work [4].

**Theorem 3.1.** ([4], [6]) The FPM formed by $\Omega_2$ and $\Omega_1$ defines a unique (up to equivalence) polynomial Frobenius manifold on $\mathcal{O}(\rho_{\text{ref}})$ with degrees $\frac{\eta_i}{\eta_r}$ and charge $\frac{\eta_r - 2}{\eta_r}$.

The following theorem was conjectured by Dubrovin and proved by C. Hertling.

**Theorem 3.2.** [13] Any irreducible massive polynomial Frobenius manifold with positive degrees is isomorphic to a polynomial Frobenius manifold constructed by theorem 3.1 on the orbit space of the standard reflection representation of an irreducible finite Coxeter group.

Then theorem 2.7 gives another Frobenius structure on $\mathcal{O}(\rho_{\text{ref}})$. 

---

**Table 3.1:**

| Type of $\mathcal{W}$ | $\eta_1, \ldots, \eta_r$ |
|---------------------|------------------|
| $A_r$               | $2, 3, \ldots, r + 1$ |
| $B_r$               | $2, 4, 6, \ldots, 2r$ |
| $D_r$               | $2, 4, 6, \ldots, 2r - 2$ |
| $E_6$               | $2, 5, 6, 8, 9, 12$ |
| $E_7$               | $2, 6, 8, 10, 12, 14, 18$ |
| $E_8$               | $2, 8, 12, 14, 18, 20, 24, 30$ |
| $F_4$               | $2, 6, 8, 12$ |
| $H_3$               | $2, 6, 10$ |
| $H_4$               | $2, 12, 20, 30$ |
| $I_2(m)$            | $2, m$ |
Theorem 3.3. There is exist a unique (up to equivalence) rational Frobenius manifold structure on \( O(\rho_{ref}) \) with the intersection form \( \Omega_2 \) and flat metric \( \Sigma_\varepsilon \Omega_2 \) where \( \varepsilon = (u^1)^{\eta_r} \). The charge of this Frobenius manifold is \( \frac{\eta_r + 2}{\eta_r} \) and the degrees are

\[
\begin{align*}
  d_1 &= -\frac{2}{\eta_r}, & d_i &= \frac{\eta_r - 2}{\eta_r}; & i = 2, \ldots, r - 1, & \text{and} & d_r &= 1.
\end{align*}
\]

(3.3)

The potential in the flat coordinates \((t^1, \ldots, t^r)\) is polynomial in \( \frac{1}{\tau}, t^1, t^2, \ldots, t^r \).

Proof. From the work of K. Saito (see also [4]), there exist invariant polynomials \( t^1, \ldots, t^r \) which form a flat coordinates and the potential of the polynomial Frobenius structure on \( O(\rho_{ref}) \) has the form (2.11). Looking at the degrees, theorem 2.7 is applicable. The result is a rational Frobenius manifold in \( s^1 \) with the claimed data. Uniqueness follow from the uniqueness of polynomial Frobenius structures.

Let us assume \( W \) is of type \( B_r \). Then Dafeng Zuo obtained \( r \) Frobenius structures on \( O(\rho_{ref}) \) by fixing a certain generators \( z^1, \ldots, z^r \) of \( \mathbb{C}[\rho_{ref}] \) [25]. Under these generators, \( \Omega_2(z) \) and its Christoffel symbols \( \Gamma_{2k}^{ij}(z) \) are almost linear in each \( z^i, k = 1, 2, \ldots, r \). Then he proved lemma 2.4 can be applied and he constructed \( r \) rational Frobenius structure using the FPM formed by \( \Omega_2 \) and \( \Omega_k := \Sigma \partial_{\frac{\partial \varepsilon}{\partial z^k}} \Omega_2 \). He also proved that the same Frobenius manifolds can be constructed when we replace type \( B_r \) with \( D_r \).

We confirm that Zuo’s work falls into Dubrovin’s method as we can read from his work that the FPM is regular quasihomogenous of charge \( 1 - \frac{1}{r} \) with \( \tau = \frac{1}{\varepsilon} z^1 \). Thus, we can obtain these Frobenius manifolds equivalently using theorem 2.5. We state below Zuo’s theorem.

Theorem 3.4. [25] There exists a unique Frobenius structure for each \( 1 \leq k \leq r \) of charge \( d = 1 - \frac{1}{k} \) on the orbit space \( O(\rho_{ref}) \) when \( W \) is of type \( B_r \) and \( D_r \), polynomial in \( t^1, t^2, \ldots, t^r, \frac{1}{\tau} \) such that:

1. \( e = \frac{\partial}{\partial z^k} = \frac{\partial}{\partial \varepsilon} \).

2. \( E = \sum_{i=1}^{r} d_i t^i \partial_{\varepsilon} \), where

\[
\begin{align*}
  d_1 &= \frac{1}{k}, & d_i &= \frac{i}{k} \quad \text{for} \quad 2 \leq i \leq k, & d_i &= \frac{2k(r - i) + r}{2k(r - k)} \quad \text{for} \quad k + 1 \leq i \leq r.
\end{align*}
\]

3. The invariant flat metric and the intersection form of Frobenius structure coincide with the metric \( \Omega_k \) and \( \Omega_2 \) resp.

Theorem 3.5. There is exist a unique rational Frobenius manifold structure for each \( 2 \leq k \leq r \) of charge \( d = 1 + \frac{1}{k} \) on the orbit space when \( W \) is of type \( B_r \) and \( D_r \) with intersection form \( \Omega_2 \) and flat metric \( \Sigma_\varepsilon \Omega_2 \) where \( \varepsilon = (z^1)^{\eta_k} \partial_{\frac{\partial \varepsilon}{\partial z^k}} \). It has the degrees

\[
\begin{align*}
  d_1 &= -\frac{1}{k}, & d_i &= \frac{i - 1}{k} \quad \text{for} \quad 2 \leq i \leq k - 1, & d_k &= 1, & d_i &= \frac{2k(r - i + 1) - r}{2k(r - k)} \quad \text{for} \quad k + 1 \leq i \leq r.
\end{align*}
\]

(3.4)

The potential in the flat coordinates \((t^1, \ldots, t^r)\) is polynomial in \( \frac{1}{\tau}, \frac{1}{\tau}, t^1, t^2, \ldots, t^r \).

Proof. Similar to the proof of theorem 3.3, we use theorem 2.7 under the hypothesis of theorem 3.4.

We emphasis that Frobenius structures obtained by theorems 3.3 and 3.5 fall into Dubrovin’s method since they are obtained using the same intersection form \( \Omega_2 \).

Let \( K \) be the type of \( W \), then we say a Frobenius structure is of type \( K \) (rep. of type \( \bar{K} \)) if it is isomorphic to a Frobenius manifold constructed on \( O(\rho_{ref}) \) by theorem 3.1 (resp. theorem 3.3).
Example 3.6. We list in table 2 all Frobenius structures constructed on $O(\rho_{\text{ref}})$ when $W$ is of rank 3 using the above theorems. We borrow the potentials of Frobenius structures of type $A_3$, $B_3$ and $H_3$ from [6]. From these potentials, we find Frobenius structures of type $\overline{A}_3$, $\overline{B}_3$ and $\overline{H}_3$ using the formula (2.14). Then applying theorem 3.4 with Coxeter group of type $B_3$, we get a Frobenius manifold of type $\overline{B}_3$ (resp. $A_3$) when $k = 3$ (resp. $k = 2$). For $k = 1$, we get a rational Frobenius manifold $B_3^1$ which has no dual, i.e., the potential can not have the form (2.11).

| Notations | $\Psi(t_1, t_2, t_3)$ | $d_1, d_2, d_3, d$ |
|-----------|---------------------|------------------|
| $A_3$ | $\frac{1}{2}t_1^2t_1 + \frac{1}{2}t_2^2t_3 + \frac{1}{4}t_1^4t_2^2 + \frac{1}{60}t_1^6$ | $\frac{1}{2}, \frac{3}{4}, 1, \frac{1}{2}$ |
| $\overline{A}_3$ | $\frac{1}{2}t_2^2t_1 + \frac{1}{2}t_2^2t_3 + \frac{1}{4}t_1t_2^3 + t_1^2 - \frac{1}{607}$ | $\frac{1}{2}, \frac{1}{3}, 1, \frac{1}{2}$ |
| $B_3$ | $\frac{1}{2}t_2^3t_1 + \frac{1}{2}t_2^3t_3 + \frac{1}{6}t_2^5t_1 + \frac{1}{6}t_2^5t_1 + \frac{1}{210}t_1^6$ | $\frac{1}{3}, \frac{2}{3}, 1, \frac{1}{2}$ |
| $\overline{B}_3$ | $\frac{1}{2}t_3^3t_1 + \frac{1}{2}t_3^3t_3 + \frac{1}{8}t_3^5t_1 + \frac{1}{6}t_3^5t_1 + \frac{1}{240}t_1^6$ | $\frac{1}{3}, \frac{2}{3}, 1, \frac{1}{2}$ |
| $H_3$ | $\frac{1}{2}t_3^4t_1 + \frac{1}{2}t_3^4t_3 + \frac{1}{8}t_3^6t_1 + \frac{1}{6}t_3^6t_1 + \frac{1}{3960}t_1^8$ | $\frac{1}{5}, \frac{3}{5}, 1, \frac{1}{2}$ |
| $\overline{H}_3$ | $\frac{1}{2}t_3^4t_1 + \frac{1}{2}t_3^4t_3 + \frac{1}{8}t_3^6t_1 + \frac{1}{6}t_3^6t_1 + \frac{1}{3960}t_1^8$ | $\frac{1}{5}, \frac{3}{5}, 1, \frac{1}{2}$ |
| $B_3^1$ | $\frac{1}{2}t_3^2 + \frac{1}{2}t_1^3 + \frac{1}{8}t_1t_2t_3 + \frac{1}{16}t_1^2$ | $1, \frac{3}{4}, \frac{5}{4}, 0$ |

Table 2: Frobenius manifolds on orbits of reflection groups of rank 3

3.2 Sign times reflection representation

We keep the notations of the last section and we assume $W$ is of type $A_r$. We study an irreducible representation $\rho_{\text{new}}$ of $W$ which can be defined using the sign representation and the representation $\rho_{\text{ref}}$. The definition will enable us to construct $r$ invariant polynomials of $\rho_{\text{new}}$. We will prove the invariant ring $\mathbb{C}[\rho_{\text{new}}]$ is not a polynomial ring when $r > 2$.

We consider the sign representation of $W$, $\rho_{\text{sign}} : W \rightarrow \mathbb{C}^*$ defined by sending each element $s \in S$ to $-1$. Then we define the representation $\rho_{\text{new}}$ of $W$ by

$$\rho_{\text{new}} : W \rightarrow GL(\mathbb{C} \otimes V), \quad \rho_{\text{new}}(w) = \rho_{\text{sign}}(w) \otimes \rho_{\text{ref}}(w), \quad \forall w \in W.$$  

(3.5)

The following proposition proves $\rho_{\text{new}}$ is an irreducible representation. Note that $\rho_{\text{new}}$ is a real representation of rank $r$.

**Proposition 3.7.** The new representation $\rho_{\text{new}}$ is an irreducible representation of $W$. Moreover, $\rho_{\text{new}}$ and $\rho_{\text{ref}}$ are isomorphic when $r = 2$ and different otherwise.

**Proof.** Recall that if $\chi_\psi$ denotes the character of a representation $\psi$ of a finite group $G$, then $\psi$ is irreducible if and only if [22]

$$\frac{1}{|G|} \sum_{g \in G} \chi_\psi(g) \overline{\chi_\psi(g)} = 1.$$  

(3.6)

Note that $\rho_{\text{ref}}$ and $\rho_{\text{sign}}$ are irreducible representations and $\chi_{\rho_{\text{new}}}(w) = \chi_{\rho_{\text{sign}}}(w) \chi_{\rho_{\text{ref}}}(w)$. Then

$$\frac{1}{|W|} \sum_{w \in W} \chi_{\rho_{\text{new}}}(w) \overline{\chi_{\rho_{\text{new}}}(w)} = \frac{1}{|W|} \sum_{w \in W} (\chi_{\rho_{\text{sign}}}(w) \chi_{\rho_{\text{ref}}}(w)) (\chi_{\rho_{\text{sign}}}(w) \chi_{\rho_{\text{ref}}}(w))$$  

(3.7)

$$= \frac{1}{|W|} \sum_{w \in W} (\chi_{\rho_{\text{ref}}}(w) \chi_{\rho_{\text{ref}}}(w)) = 1.$$
For the second part, note that for any generator $s \in S$, $\chi_{\rho_{\text{new}}}(s) = -\chi_{\rho_{\text{ref}}}(s) = -(r - 2)$. Hence, the two representations are different when $r \neq 2$. When $r = 2$, $W$ is of type $I_2(m)$ for some integer $m$. Let $s$ and $s'$ be the two generators of the group. Setting $\sigma = ss'$, then elements of $W$ are in the form $\sigma^i$ and $s\sigma^i$ for some integer $i$. But then $\chi_{\rho_{\text{ref}}}(s\sigma^i) = \chi_{\rho_{\text{new}}}(s\sigma^i) = 0$, since $\rho_{\text{ref}}(s\sigma^i)$ is a reflection, and $\chi_{\rho_{\text{ref}}}(\sigma^i) = \chi_{\rho_{\text{new}}}(\sigma^i)$ since $\rho_{\text{sign}}(\sigma^i) = 1$. Thus, the two representation $\chi_{\rho_{\text{new}}}$ and $\chi_{\rho_{\text{ref}}}$ are isomorphic.

The Coxeter group of type $A_r$ is isomorphic to the symmetric group $S_{r+1}$. Thus, irreducible representations of $A_r$ are in one to one correspondence with the partition of $r + 1$. For a given partition $\lambda$ of $r + 1$, the corresponding irreducible representation can be constructed using Young tableaux associated to $\lambda$ [11]. Under this construction, the reflection representation $\rho_{\text{ref}}$ is associated with the partition $[r, 1]$, $\rho_{\text{sign}}$ is associated with the partition $[r + 1]$ while $\rho_{\text{new}}$ is associated with $[2, 1, 1, \ldots, 1]$. The character of each representation is given by Frobenius formula [11]. We use this formula to prove the following proposition.

For the remainder of this section we assume the rank $r > 2$.

**Proposition 3.8.** The irreducible representation $\rho_{\text{new}}$ is not a reflection representation. In particular the ring $\mathbb{C}[\rho_{\text{new}}]$ is not a polynomial ring.

**Proof.** Assume that $\rho_{\text{new}}$ is a reflection representation. Then, it is generated by a set involutions $w_1, \ldots, w_r$. Since $\rho_{\text{new}}$ is a real representation, we must have $\chi_{\rho_{\text{new}}}(w_i) = r - 2$. From $\rho_{\text{new}}(w_i) = \rho_{\text{sign}}(w_i)\rho_{\text{ref}}(w_i)$, $\rho_{\text{sign}}(w_i) = -1$, since if $\rho_{\text{sign}}(w_i) = 1$, then $\rho_{\text{ref}}(w_i)$ is a reflection and we get a contradiction. Thus, $\chi_{\rho_{\text{ref}}}(w_i) = 2 - r$. In the one-to-one correspondence between conjugacy classes of $S_{r+1}$ and partitions of $r + 1$, $w_i$ corresponds to a partition of the from $[\frac{r}{2}, \frac{r}{2}, \ldots, 1]$. Therefore, if $2p + q = r + 1$, $p > 0$. Using Frobenius formula, $\chi_{\rho_{\text{ref}}}(w_i)$ equals the coefficient of $x^{p+1}y$ in the expansion $(x-y)(x^2+y^2)^p(x+y)^{r+1-2p}$. Hence, $\chi_{\rho_{\text{ref}}}(w_i) = r - 2p$. Using the fact that $2p = r + 1$ and $\chi_{\rho_{\text{ref}}}(w_i) = 2 - r$ we get $r = 3$ is excluded by direct computations.

We study the ring $\mathbb{C}[\rho_{\text{new}}]$ in order to use Dubrovin’s method. We fix a basis $e_1, e_2, \ldots, e_r$ for $V$ and let $x^1, \ldots, x^r$ be the dual basis satisfying $x^i(e_j) = \delta^i_j$. Then $\bar{e}_i := 1 \otimes e_i, i = 1, \ldots, r$ form a basis of $\mathbb{C} \otimes V$ and we get a natural isomorphism

$$\theta : \mathbb{C} \otimes V \to V, \bar{e}_i \mapsto e_i. \quad (3.8)$$

Then the pullback $\bar{x}^i = \theta^*(x^i)$ defines the dual basis of $\bar{e}_i$. Let $w \in W$ and $a^{ij}$ be the matrix of $\rho_{\text{ref}}(w)$ under the basis $e_i$. Then $\rho_{\text{new}}(w)(\bar{e}_i) = \rho_{\text{sign}}(w)1 \otimes \rho_{\text{ref}}(w)e_i = \rho_{\text{sign}}(w)a^{ij} e_j$. Therefore, $\rho_{\text{new}}(w) = \rho_{\text{sign}}(w)\rho_{\text{ref}}(w)$.

**Lemma 3.9.** Let $w \in W$ with $\rho_{\text{sign}}(w)\rho_{\text{new}}(w) \neq \rho_{\text{ref}}(W)$ and $f \in \mathbb{C}[\rho_{\text{ref}}]$ be homogeneous polynomial. Then

$$w \cdot \theta^*(f) = (\rho_{\text{sign}}(w))^{\deg(f)} \theta^*(f). \quad (3.9)$$

In particular, if degree $f$ is even then $\theta^*(f) \in \mathbb{C}[\rho_{\text{new}}]$.

**Proof.** We obtain $\theta^*(f)$ simply by replacing the coordinate $x^i$ with $\bar{x}^i$. Therefore,

$$w \cdot \theta^*(f)(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n) = \theta^*(f)(\rho_{\text{new}}(w)\bar{x}^1, \rho_{\text{new}}(w)\bar{x}^2, \ldots, \rho_{\text{new}}(w)\bar{x}^n) = \theta^*(f)(\rho_{\text{sign}}(w)\rho_{\text{ref}}(w)\bar{x}^1, \rho_{\text{sign}}(w)\rho_{\text{ref}}(w)\bar{x}^2, \ldots, \rho_{\text{sign}}(w)\rho_{\text{ref}}(w)\bar{x}^n) = (\rho_{\text{sign}}(w))^{\deg(f)} \theta^*(f)(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n).$$
Let $z^1, ..., z^r$ be algebraically independent invariant polynomials of $\rho_{\text{new}}$ and $u^1, ..., u^r$ be the generators of $\mathbb{C}[\rho_{\text{ref}}]$ (in the notation of section 3.1). We assume $z^1 = \theta^*(u^1)$. Hence, the Hessian of $z^1$ defines a contravariant flat metric $\Omega_2$ on $O(\rho_{\text{new}})$. Examples show that the entries of $\Omega_2(z)$ are rational in general and it is hard to construct flat pencil of metrics. We overcome this problem by defining certain invariants for $\rho_{\text{new}}$ which also leads to the construction of Frobenius structures.

**Proposition 3.10.** There exist $r$ algebraically independent invariant polynomials $z^1, z^2, ..., z^r$ of $\rho_{\text{new}}$ with the degrees $\tilde{\eta_1}, ..., \tilde{\eta_r}$ listed in table 3.

**Proof.** We will use the invariants $u^1, ..., u^r$ of $\rho_{\text{ref}}$ to construct invariants of $\rho_{\text{new}}$. We set $I = \{ i : \eta_i \text{ is even} \}$ and $J = \{ j : \eta_j \text{ is odd} \}$. Using lemma 3.9, $\theta^*(u^i)$ is an invariant of $\rho_{\text{new}}$ for any $i \in I$. Let $\kappa$ be the minimal index in $J$. Then $\theta^*(u^j u^\kappa)$ is an invariant of $\rho_{\text{new}}$ for any $j \in J$. By this way we construct $r$ invariants polynomial, $z^1, ..., z^r$ for $\rho_{\text{new}}$ with the degrees given in table 3. Note that any polynomial in $z^1, ..., z^r$ can be written as a polynomial in $u^1, ..., u^r$. Hence, $z^1, ..., z^r$ are algebraically independent. □

**Remark 3.11.** We observe that the invariant polynomials constructed by proposition 3.10 do not necessarily form a set of primary invariant polynomials of $\rho_{\text{new}}$. According to the invariant theory [3], the product of the degrees of primary invariants is divisible by the order of the group. For example, when $W$ is type $A_4$, the degrees of $z^1$ are 2, 4, 6, 8. The product of these degrees is not divisible by the order 120 of the group.

We keep the notations $z^1, ..., z^r$ for the invariant polynomials of $\rho_{\text{new}}$ constructed in proposition 3.10.

**Theorem 3.12.** The orbits space $O(\rho_{\text{new}})$ has the structure of Frobenius manifolds isomorphic to Frobenius manifolds defined on $O(\rho_{\text{ref}})$ by theorem 3.1 and theorem 3.2.

**Proof.** We consider the map $(u^1, ..., u^r) \to (z^1, z^2, ..., z^r)$ given in proposition 3.10 as diffeomorphism on some open subset of $u^\kappa \neq 0$ where $\kappa$ is defined in proposition 3.10. Note that, under this diffeomorphism, the metric defined by the Hessian of $u^1$ is identified with the metric defined by the Hessian of $z^1$. Thus, we can transfer to $O(\rho_{\text{new}})$, any regular QFPM formed by $\Omega_2$ and $\Omega_1$ given by the theorems 3.1 and 3.1. In this way we obtain s Frobenius structures on $O(\rho_{\text{new}})$. □

**Example 3.13.** The irreducible reflection representation $\rho_{\text{ref}}$ of Coxeter group of type $A_4$ is generated by the matrices

$$\sigma = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad (3.10)$$

| Type of $\mathcal{W}$ | $\tilde{\eta_1}, ..., \tilde{\eta_r}$ |
|-----------------------|----------------------------------|
| $A_r$                 | $2, 4, 6, ..., 2\lfloor \frac{r+1}{2} \rfloor; 6, 8, ..., 2\lfloor \frac{r-1}{2} \rfloor$ |

Table 3: Degrees of invariant polynomials of $\rho_{\text{new}}$
The polynomial ring \( \mathbb{C}[^\rho] = \mathbb{C}[u_1, u_2, u_3, u_4] \) where

\[
\begin{align*}
    u_1 &= x_1^2 - \frac{1}{2}x_1x_2 - \frac{1}{2}x_1x_3 - \frac{1}{2}x_1x_4 + x_2^2 - \frac{1}{2}x_2x_3 + \frac{1}{2}x_2x_4 + x_3^2 - \frac{1}{2}x_3x_4 + x_4^2, \\
    u_2 &= x_1^3 - \frac{3}{4}x_1^2x_2 - \frac{3}{4}x_1^2x_3 + \frac{3}{4}x_1^2x_4 - \frac{3}{4}x_1x_2x_3 + \frac{3}{4}x_1x_2x_4 - \frac{3}{4}x_1x_3x_4 + \frac{3}{4}x_1x_4^2 + x_2^3 - \frac{3}{4}x_2x_3 + \frac{3}{4}x_2x_4 + x_3^3 - \frac{3}{4}x_3x_4 + x_4^3, \\
    u_3 &= x_1^4 - x_1^3x_2 - x_1^3x_3 + x_1^2x_2x_3 - x_1x_2x_3x_4 - x_1x_2x_4^2 + x_1x_3x_4^2 - x_1x_4 - x_2^4 - x_2^3x_3 + x_2^2x_3x_4 - x_2x_3x_4^2 - x_3^4 + x_3^3x_4 - x_3x_4^3 + x_4^4, \\
    u_4 &= x_1^5 - \frac{5}{4}x_1^4x_2 - \frac{5}{4}x_1^4x_3 - \frac{5}{4}x_1^4x_4 + \frac{3}{4}x_1^3x_2x_3 + \frac{5}{4}x_1^3x_2x_4 - \frac{3}{4}x_1^3x_3x_4 + \frac{3}{4}x_1^3x_4^2 + \frac{5}{4}x_1^2x_2x_3x_4 - \frac{5}{4}x_1^2x_2x_4^2 + \frac{5}{4}x_1^2x_3x_4^2 - \frac{5}{4}x_1^2x_4^3 + \frac{5}{4}x_1x_2^2x_3x_4 - \frac{5}{4}x_1x_2^2x_4^2 + \frac{5}{4}x_1x_3^2x_4^2 - \frac{5}{4}x_1x_4^3 + \frac{5}{4}x_2^3x_3x_4 - \frac{5}{4}x_2^3x_4^2 + \frac{5}{4}x_3^3x_4^2 - \frac{5}{4}x_2^3x_4^2 + \frac{5}{4}x_3^2x_4^3 - \frac{5}{4}x_2^2x_4^3 + \frac{5}{4}x_2x_3x_4^3 - \frac{5}{4}x_2x_4^4 + \frac{5}{4}x_3x_4^4 - \frac{5}{4}x_4^5.
\end{align*}
\]

The Frobenius manifold of type \( A_4 \) is a result of the regular QFPM consists of \( \Omega_2(u) \) and \( \Omega_4 = \Sigma \rho, \Omega_2(u) \) where \( \Omega_2(u) \) is defined by the Hessian of \( u_1 \). The representation \( \rho_{\text{new}} \) is generated by \( \tau \) and \( -\sigma \). Then the primary invariants of \( \rho_{\text{new}} \) have degrees 2, 4, 6, 10 while the secondary invariants have degrees 8, 13, 15. The Hessian of the degree 2 invariant \( z_1 \) leads to the flat contravariant metric \( \Omega_2 \) but it is hard to find a FPM. We fix the following 4 invariants polynomials for \( \mathcal{O}(\rho_{\text{new}}) \) of degrees 2, 4, 6 and 8: \( z_1 = u_1, \ z_2 = u_3, \ z_3 = u_2^2, \ z_4 = u_2u_4 \). Then the matrix of \( \Omega_2(z) \) consists of the columns

\[
    \Omega_2^1(z) = \begin{pmatrix}
    z_1 \\
    \frac{2}{z_2} \\
    3z_3 \\
    4z_4
\end{pmatrix}, \quad \Omega_2^2(z) = \begin{pmatrix}
    -64 & 6 & 68 \\
    62 & 2 & 62 \\
    64 & 2 & 62 \\
    62 & 2 & 62
\end{pmatrix}
\]

\[
    \Omega_2^3(z) = \begin{pmatrix}
    12 \\
    3z_2 \\
    26 \\
    45
\end{pmatrix}, \quad \Omega_2^4(z) = \begin{pmatrix}
    64 & 62 & 64 & 9 \\
    62 & 12 & 62 & 9 \\
    64 & 62 & 64 & 9 \\
    62 & 12 & 62 & 9
\end{pmatrix}
\]

and

\[
    \Omega_2^5(z) = \begin{pmatrix}
    214 & 214 & 214 & 214 \\
    52 & 52 & 52 & 52 \\
    625 & 625 & 625 & 625 \\
    2025 & 2025 & 2025 & 2025
\end{pmatrix}
\]

Therefore, on \( \mathcal{O}(\rho_{\text{new}}) \), we get the regular QFPM formed by \( \Omega_2(z) \) and \( \Omega_1(z) = \Sigma \Omega_2 \) where \( e = \sqrt{33}z_4 \). Of course, the result Frobenius manifold is of type \( A_4 \).

Remark 3.14. It is straightforward to generalized the results of this section to other types of Coxeter groups and we obtain Frobenius manifolds on \( \mathcal{O}(\rho_{\text{new}}) \). But we lack sorting when \( \rho_{\text{new}} \) is not a reflections group (i.e. see proposition 3.8). Robert Howett informed us that when \( W \) is of type \( E_6 \), the representation \( \rho_{\text{new}} \) is generated by reflections.
4 Dihedral and dicyclic groups

4.1 Dihedral groups

In this section we apply Dubrovin’s method to irreducible representations of the dihedral groups (Coxeter groups of type) $I_2(m)$, $m > 2$.

Irreducible representations of $I_2(m)$ are of rank 1 or 2. Let $\xi_m$ be a primitive $m$-th root of unity. The rank 2 representations are $\rho_k$, $k = 1, 2, \ldots, \frac{m-2}{2}$ for even $m$, and $k = 1, 2, \ldots, \frac{m-1}{2}$ for odd $m$. Here, $\rho_k$ is generated by the matrices

$$
\begin{pmatrix}
\xi_m^k & 0 \\
0 & \xi_m^{-k}
\end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
$$

(4.1)

When $k = 1$, we get the standard reflection representation of $I_2(m)$. It is straightforward to show that $\mathbb{C}[\rho_k]$ is the same as the invariant ring of the standard reflection representation of $I_2(h)$, $h = \frac{m}{\gcd(m,k)}$. Hence, applying Dubrovin’s method, we get the polynomial Frobenius manifold of type $I_2(h)$ and its dual $\tilde{I}_2(h)$. More precisely, fix the generators of $\mathbb{C}[\rho_k]$ to be

$$t_1 = \frac{1}{h} x_1 x_2, \quad t_2 = x_1^h + x_2^h;$$

we get Frobenius structures with the following data

| Notations | $\mathbb{C}(t_1,t_2)$ | $d_1, d_2$ | $d$ |
|-----------|----------------------|-------------|-----|
| $I_2(h)$  | $\frac{1}{h^2} t_1 t_2 + \frac{h^2}{n^2} t_1^h t_2^h$ | $\frac{2}{n^2}, 1$ | $\frac{h-2}{n}$ |
| $\tilde{I}_2(h)$ | $\frac{1}{h^2} t_1 t_2 + \frac{(-1)^{n-h}}{h^2} t_1^h t_2^h$ | $-\frac{2}{n^2}, 1$ | $\frac{h+2}{2}$ |

Table 4: Frobenius manifolds on orbit space of $\rho_k$.

4.2 Dicyclic groups

In this section we give results of applying Dubrovin’s method to all irreducible representation of dicyclic groups. We get a Frobenius structure with rational degrees.

We fix a natural number $m$. The dicyclic group $\text{Dic}_m$ is a group of order $4m$ defined by

$$\text{Dic}_m = \langle \sigma, \alpha | \sigma^{2m} = 1, \alpha^2 = \sigma^m, \alpha^{-1} \sigma \alpha = \sigma^{-1} \rangle.$$  \hspace{1cm} (4.2)

The irreducible representation of $\text{Dic}_m$ are of rank 1 or 2. The 2-dimensional irreducible representations are

$$\psi_k(\sigma) = \begin{pmatrix} \xi_m^k & 0 \\ 0 & \xi_m^{-k} \end{pmatrix}, \quad \psi_k(\alpha) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

(4.3)

and

$$\varphi_l(\sigma) = \begin{pmatrix} \xi_m^l & 0 \\ 0 & \xi_m^{-l} \end{pmatrix}, \quad \varphi_l(\alpha) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$  \hspace{1cm} (4.4)

Where $1 \leq k \leq \frac{m-2}{2}$ and $1 \leq l \leq m-1$ when $m$ is even while $1 \leq k \leq \frac{m-1}{2}$ and $1 \leq l \leq m-2$ when $m$ is odd. Then $\psi_1$ is the standard representation of $\text{Dic}_m$.

The invariant ring $\mathbb{C}[\varphi_l]$ is the same as $\mathbb{C}[\rho_k]$ where $\rho_k$ is the representation of $I_2(m)$ in section 4.1. Thus, the result of applying Dubrovin’s method to $\varphi_l$ is given in that section. We consider here the representations $\psi_k$. Let us fix the integer $k$ and set $h = \frac{m}{\gcd(m,k)}$. We define

$$u_1 = x_1^2 x_2^2, \quad u_2 = x_1^{2h} + x_2^{2h}, \quad u_3 = x_1 x_2 (x_1^{2h} - x_2^{2h}).$$

(4.5)
It is straightforward to verify that $u_1, u_2$ and $u_3$ are invariants under the action of $\psi_k$.

**Proposition 4.1.** The invariant ring $\mathbb{C}[\psi_k]$ is generated by $u_1, u_2$ and $u_3$.

**Proof.** A general homogeneous polynomial of degree $q$ have the form

$$f(x_1, x_2) = a_q x_1^q + a_{q-1} x_1^{q-1} x_2 + \cdots + a_1 x_1 x_2^{q-1} + a_0 x_2^q$$

(4.6)

where $a_0, \ldots, a_q \in \mathbb{C}$. Being invariant under $\psi_k(\alpha)$, we get

$$f = a_q x_1^q + a_{q-1} x_1^{q-1} x_2 + \cdots + a_1 x_1 x_2^{q-1} + a_0 x_2^q$$

implies $k \equiv m \pmod{2l}$ for all $i = 0, \ldots, q$. Similarly, the invariants of $f$ under $\psi_k(\alpha^2)$ implies $q$ is even. Hence, $f$ has the form

$$f = \sum_{i=0}^{k} a_{q-i}(x_1 x_2)^i \left[ x_1^{q-2i} + (-1)^{q-i} x_2^{q-2i} \right]$$

Moreover,

$$f = \psi_k(\sigma)f = \sum_{i=0}^{k} a_{q-i}(x_1 x_2)^i \left[ \xi^{-k(q-2i)} x_1^{q-2i} + (-1)^{q-i} \xi^{k(q-2i)} x_2^{q-2i} \right]$$

implies $k(q - 2i) = 0 \mod (2m)$. Then $q - 2i = 2hl$ for some integer $l$. Therefore, we can write

$$f = \sum_{q=2hl+2i} a_{q-i}(x_1 x_2)^i \left[ x_1^{2hl} + (-1)^{q-i} x_2^{2hl} \right].$$

(4.7)

Now we show that $f \in \mathbb{C}[u_1, u_2, u_3]$. Taking $i$ even or odd, it is sufficient to prove $\tilde{f}_i = x_1^{2hl} + x_2^{2hl}$ and $\tilde{f}_i = x_1 x_2(x_1^{2hl} - x_2^{2hl})$ are invariant for every natural number $l$. When $l = 1$, $\tilde{f}_i = u_2$ and $\tilde{f}_i = u_3$. For $l + 1$, we get

$$\tilde{f}_{l+1} = x_1^{2hl} x_2^{2hl} (x_1^{2hl} + x_2^{2hl}) = (x_1^{2hl} + x_2^{2hl})^{l+1} - \sum_{d=1}^{l+1} \binom{l+1}{d} x_1^{2hd} x_2^{2hl-d}$$

(4.8)

$$= (x_1^{2hl} + x_2^{2hl})^{l+1} - \sum_{d=1}^{l+1} \binom{l+1}{d} (x_1 x_2)^{2hd} x_1^{2hl-dd} + x_1^{2hl(l+1-2d)}$$

Since $d \geq 1$, we have $l + 1 - 2d \leq l - 1 < l$. Therefore, by the induction $\tilde{f}_{l+1} \in \mathbb{C}[u_1, u_2, u_3]$. Likewise $\tilde{f}_{l+1} \in \mathbb{C}[u_1, u_2, u_3]$ since

$$\tilde{f}_{l+1} = x_1 x_2 (x_1^{2hl} - x_2^{2hl}) (x_1^{2hl} + x_2^{2hl}) = x_1 x_2 (x_1^{2hl} - x_2^{2hl}) + (x_1 x_2)^{2hl} x_1^{2hl(l+1-2d)} + x_1^{2hl(l+1-2d)}$$

(4.9)

This proves the proposition.

We note that the invariant ring $\mathbb{C}[\psi_k]$ is the same as the invariant ring of the standard representation of $\text{Dic}_k$. A result of applying Dubrovin’s method is obtained in [2]. We summarize the construction here. The inverse of the Hessian of $u_1$ define a flat contravariant metric.
\[ \Omega_2(u) = \left( \begin{array}{c} \frac{4}{3} u_1 \text{ } 2h \frac{2}{3} u_2 - \frac{2h^2}{u_1} (u_2^2 - 6u_1^2) \end{array} \right) \] (4.10)

Then it turns out that for any vector field in the form \( e = f(u_1) \partial_{u_2} \) the Lie derivative \( \Omega_1^{ij}(u) := \mathcal{L}_e \Omega_1^{ij}(u) \) is flat. Here
\[ \Omega_1(u) = \left( \begin{array}{cc} 0 & \frac{1}{2} (hf - 2u_1 f') \\ \frac{2}{3} (hf - 2u_1 f') & \frac{1}{3} (h^2 u_2 f + h f_1 u_2 f') \end{array} \right) \] (4.11)

The choice of \( e \) was inspired by example 3.13. The two metrics \( \Omega_2 \) and \( \Omega_1 \) form a flat pencil of metrics. However, to be quasihomogenous, we must have \( \Sigma_\epsilon \Omega_1^{ij}(u) = 0 \) which leads to the following differential equation
\[ 2hu_1f' - 2u_2(f')^2 + h^2 f^2 = 0 \] (4.12)
which has two independent solutions
\[ f = \frac{4}{3}(1 + \sqrt{3})u_1. \] (4.13)

Let us take \( e = f_1 \partial_{u_2} = \frac{4}{3} (1 + \sqrt{3}) \partial_{u_2} \). In this case,
\[ \Omega_1(u) = \left( \begin{array}{cc} 0 & \frac{1}{3} (1 + \sqrt{3})h_1 \\ - \frac{2}{3} (1 + \sqrt{3})h_2 - \frac{1}{3} (1 + \sqrt{3}) & \frac{2}{3} (1 + \sqrt{3})h_1 - u_2 \end{array} \right) \] (4.14)

The two metrics \( \Omega_2 \) and \( \Omega_1 \) form a regular QFPM with charge \( d = \frac{2 + \sqrt{3h}}{\sqrt{3h}} \). Then \( \tau = \frac{-\sqrt{3}}{2h} u_1 \) and the Euler vector field
\[ E = - \frac{2}{\sqrt{3h}} u_1 \partial_{u_1} - \frac{1}{\sqrt{3}} u_2 \partial_{u_2}. \] (4.15)

The flat coordinates of \( \Omega_1 \) reads
\[ t_1 = - \frac{\sqrt{3}}{2h} u_1, \quad t_2 = \frac{u_2}{1 + \sqrt{3}}. \] (4.16)

The degrees of this Frobenius manifold are \(- \frac{2}{\sqrt{3h}} \) and 1. We denote the type of this manifold by \( \widetilde{Dic}_m \). It has the potential
\[ F = \frac{2^{-\sqrt{3h}} (1 + \sqrt{3h})}{3h^2 - 1} (-ht_1)^{1 - \sqrt{3h}} + \frac{1}{2} t_1^2 t_2^2. \] (4.17)

Using theorem 2.6, the dual of \( \widetilde{Dic}_m \) exists and we refer to it by \( Dic_m \). It can be obtained equivalently by fixing \( e = f_1 \partial_{u_2} = \frac{4}{3} (1 - \sqrt{3}) \partial_{u_2} \) in the computations above. It has the charge \( d = \frac{2 - \sqrt{3h}}{\sqrt{3h}} \), the degrees \( \frac{2}{\sqrt{3h}} \), 1 and the potential
\[ F = \frac{2^{\sqrt{3h}} (1 + \sqrt{3h})}{3h^2 - 1} (h t_1)^{\sqrt{3h} + 1} + \frac{1}{2} t_1^2 t_2^2. \] (4.18)

We observe that Dubrovin computed by ad hoc procedure all possible potentials of 2-dimensional Frobenius manifolds [5]. The potentials found in this article are equivalent to Frobenius manifolds of charge \( d \) having the potential
\[ F(z_1, z_2) = z_1^k + \frac{1}{2} z_2^2 z_1, \quad k = \frac{3 - d}{1 - d} \] (4.19)

However, finding them by using Dubrovin’s method on invariant rings is an interesting result.
4.3 Finite subgroups of $SL_2(\mathbb{C})$

In this section we use Dubrovin’s method on finite non trivial subgroups of $SL_2(\mathbb{C})$. They are classified up to conjugation and they are called binary polyhedral groups. They consist of the cyclic groups $C_m$ and binary dihedral groups $D_m$, binary tetrahedral group $T$, binary octahedral group $O$ and binary icosahedral group $I$. We treat them as representations of the corresponding groups. It is known that the invariant rings of these representations are not polynomial rings and the relations between the generators lead to the classification of simple hypersurface singularities. We use the sets of generators of the invariant rings listed in [15]. Applying Dubrovin’s method, we obtain polynomial Frobenius manifolds. We know from section 3.1 that all of them has a dual rational Frobenius manifold. Thus, we write below only the flat coordinates and the type of the resulting Frobenius structure. Note that the results are not apparent by just examining the invariant rings.

1. Cyclic groups $C_m$: Here $m \geq 2$ and the invarinat ring is generated by $xy, \ x^m, \ y^m$. We fix the following invariant polynomials

$$t_1 = \frac{1}{m} xy, \ t_2 = x^m + y^m$$

Then the ring generated by $t_1$ and $t_2$ is isomorphic to the invariant ring of the standard representation of the dihedral group $I_2(m)$. Thus, using Dubrovin’s method and $(t_1, t_2)$ as coordinates on the orbits space, we get Frobenius manifold of type $I_2(m)$ and its dual.

For the case of $t_1 = \frac{1}{m} xy$ and $t_2 = x^m$ then the Frobenius structure takes $\frac{1}{2} t_1 t_2^2$ of charge $m^2 - 2m$ and degrees $\frac{2}{m}, 1$. While by the change of coordinates $s_1 = -t_1, \ s_2 = t_2 t_1^{-w}$ the dual structure takes the form $\frac{1}{2} s_1^2 s_2^2$.

2. The binary dihedral group $D_m$: This is the standard representation of the dicyclic group $Dic_m$. A result of applying Dubrovin’s method is given in section 4.2.

3. The binary tetrahedral $T$: We fix the following set of generators of the invariant ring

$$t_1 = \frac{5}{12} xy (x^4 - y^4), \ t_2 = (x^4 + y^4)^3 - 36 x^4 y^4 (x^4 + y^4), \ t_3 = 16 x^4 y^4 + 2 (x^4 - y^4). \quad (4.20)$$

We choose $(t_1, t_2)$ as coordinates on the orbits space. Then the Hessian of $\frac{1}{2} t_1$ defines a flat metric $\Omega_2(t)$ linear in $t_2$. Here, lemma 2.4 is applicable and we get a regular QFPM of charge $d = \frac{1}{2}$ with $\tau = t_1$ consists of

$$\Omega_2^{ij}(t) = \begin{pmatrix} \frac{1}{4} t_1 & t_2 \left( \frac{t_2}{625} + \frac{478076}{625} t_1^3 \right) \end{pmatrix}, \Omega_1^{ij} = \delta_{ij} \Omega_2^{ij}(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (4.21)$$

The resulting Frobenius manifold is of type $I_2(4)$.

4. The binary octahedral $O$: Let us fix the generators

$$t_1 = \frac{7}{12} (16 x^4 y^4 + (x^4 - y^4)^2), \ t_2 = \left( xy (x^4 - y^4) \right)^2, \ t_3 = y x^{17} - 34 y^5 x^{13} + 34 y^{13} x^5 - y^{17} x \quad (4.22)$$

In the coordinates $(t_1, t_2)$, the metric $\Omega_2(t)$ defined by the Hessian of $\frac{12}{7} t_1$ is linear in $t_2$ and leads to a regular QFPM with charge $\frac{1}{3}$. The resulting Frobenius manifold is of type $I_2(3)$.

5. The binary icosahedral $I$: We fix the generators of the invariant ring

$$t_1 = \frac{11}{30} (x^{11} y + 11 x^6 y^6 - x y^{11}), \ t_2 = x^{20} - 228 x^{15} y^5 + 494 x^{10} y^{10} + 228 x^5 y^{15} + y^{20} \quad (4.23)$$

In the coordinates $(t_1, t_2)$, the Hessian of $\frac{12}{11} t_1$ leads to a metric $\Omega_2(t)$ linear in $t_2$. The regular QFPM formed by $\Omega_2$ and $\Omega_1 = \partial_{t_2} \Omega_2$ leads to a Frobenius manifold of type $I_2(5)$. 

16
5 Finite subgroups of \( SL_3(\mathbb{C}) \)

Finite subgroups of \( SL(3, \mathbb{C}) \) are classified into the families (\( A \)), (\( B \)), \ldots, (\( L \)) [24]. We treat them as representation of the corresponding groups. The fact that they are subset of \( SL_3(2, \mathbb{C}) \) means that they are not a reflection representation. Watanabe and Rotillon listed in [23] those subgroups where the invariant rings are complete intersections missing type (\( J \)) and (\( K \)). These missing groups were recognized by Yau and Yu [24]. In the end, there is a total of 29 types of finite subgroups of \( SL_3(\mathbb{C}) \) whose invariant rings are complete intersection and their sets of generators are known explicitly. We apply Dubrovin’s method to these groups and we summarize the result below. The set of generators is taken from [23] and we use the same numbering (1), (2), \ldots, (27) of the 27 subgroups listed there.

Recall from the introduction that in order to apply Dubrovin’s method, we must find a minimal degree invariant polynomial where the Hessian defines a flat contravariant metric. (5.1)

This condition excluded the following subgroups

1. (17) which are of type (\( A \)).
2. (3) – (8), (19) – (23) which are of type (\( B \)).
3. (10), (24) and (25) which are of type (\( C \)).
4. (13), (15), (16) and (27) which are of types (\( G \)), (\( L \)), (\( I \)) and (\( E \)), respectively.
5. The groups (\( J \)) and (\( K \)) which are not considered in [23].

Applying Dubrovin’s method to the remaining subgroups, we will find Frobenius structures of types \( A_3, B_3, H_3, B_3^1 \) (see example 3.6 ) and the trivial \( T_3 \) (see example 2.8). We know from section 3.1 that once an orbit space has Frobenius structure of type \( A_3, B_3 \) or \( H_3 \) then it also possesses the dual of that structure. Thus we will not mention explicitly the appearance of the dual structures \( \tilde{A}_3, \tilde{B}_3 \) and \( \tilde{H}_3 \). In all cases below, we construct the regular QFPM using lemma 2.4. However, sometimes we fix a set of invariant polynomials different from the standard set of generators of the invariant ring. In each case, we will mention the type of the resulting Frobenius structure and its flat coordinates.

(1) These groups are of type (\( A \)) depending on an integer \( m > 1 \). A minimal set of generators of the invariant rings consists of \( x^m, y^n, z^m, xyz \). The Hessian of \( xyz \) does not define a flat metric. Hence, condition (5.1) exclude the case \( m > 3 \).

For \( m = 2 \), we fix the invariant polynomials

\[ u_1 = x^2 + y^2 + z^2, \quad u_2 = x^2y^2 + z^2y^2 + x^2z^2, \quad u_3 = (xyz)^2. \quad (5.2) \]

Then \( \{u_1, u_2, u_3\} \) can be identified with the set of generators of the invariant ring of the standard representations of Coxeter groups of type \( B_3 \). Thus, applying Dubrovin’s method, we get 5 Frobenius structures of the types listed in example 3.6.

We also get the trivial Frobenius structure of type \( T_3 \) using the setting of the case (2) below. Thus we proved that the orbit spaces of this group has 6 different Frobenius structures.

For \( m = 3 \), we fix the invariant polynomials

\[ u_1 = x^3 + y^3 + z^3, \quad u_2 = x^3y^3 + y^3z^3 + z^3x^3, \quad u_3 = (xyz)^3. \quad (5.3) \]
The Hessian of $u_1$ defines a contravariant flat metric $\Omega_2$. This metric and its Christoffel symbols are almost linear in each variable $u_i$ and lemma 2.4 is applicable and we get three regular QFPM. Here, similar to the case $m = 2$, we get 5 Frobenius manifold structures equivalence to the ones given in example 3.6. For QFPM formed by $\Omega_2$ and $\mathcal{L}_{\partial u_3} \Omega_2$, we get the type $B_3$. It has flat coordinates

$$t_1 = \frac{2}{9}u_1, \ t_2 = -\frac{u_1^2}{6\sqrt{2}}, \ t_3 = \frac{7u_1^3}{216} - \frac{1}{6}u_2u_1 + u_3. \tag{5.4}$$

The FPM consists of $\Omega_2$ and $\mathcal{L}_{\partial u_2} \Omega_2$ leads to type $A_3$. It has flat coordinates

$$t_1 = \frac{1}{3}u_1, \ t_2 = u_2 - \frac{1}{8}u_1^2, \ t_3 = \sqrt{u_3}. \tag{5.5}$$

While for the FPM formed by $\Omega_2$ and $\mathcal{L}_{\partial u_1} \Omega_2$, we get the type $B_3^1$. We can fix the flat coordinates

$$t_1 = u_1, \ t_2 = u_2u_3^{-\frac{1}{2}}, t_3 = \frac{4}{3}u_3. \tag{5.6}$$

These are family of groups of type $(B)$ where $m \geq 1$. The polynomials $x^{2m+y^{2m}}, (xy)^2, xyz(x^{2m}-y^{2m})$ and $z^2$ form a minimal sets of generators for the invariant rings. However, because of condition (5.1), we need only to consider $m = 1$. In this case, we fix the invariant polynomials

$$u_1 = x^2 + y^2 + z^2, u_2 = z^2, u_3 = x^2 y^2$$

The metric $\Omega_2(u)$ defined by the Hessian of $u_1$ and its Christoffel symbols are linear in each variable $u_i$. However, lemma 2.4 is applicable only for $u_2$. The FPM consisting of $\Omega_2$ and $\Omega_1 := \mathcal{L}_{\partial u_2} \Omega_2$ is a regular quasihomogenous of charge 0 with $\tau = u_1$. We can fix the flat coordinates to be

$$t_1 = \frac{1}{2}u_1, \ t_2 = u_2, \ t_3 = u_3^{\frac{1}{2}} \tag{5.7}$$

which leads to a trivial Frobenius structure $T_3$.

These groups are of type $(C)$ with order $3m^2$ where $m \geq 1$. Minimal sets of generators of the invariant rings consists of $xyz, x^m+y^m+z^m, x^m y^m + x^m z^m + y^m z^m, (x^m-y^m) (y^m-z^m)$. The argument of applying Dubrovin’s method is the same as of type (1) above.

These groups are also of type $(C)$ where $m > 1$. Minimal sets of generators of the invariant rings consists of

$$u_1 = x^m + y^m + z^m, \ u_2 = x^m y^m + z^m, u_3 = x^m y^m + y^m z^m + z^m, u_4 = xyz(x^m-y^m)(z^m-x^m)(y^m-z^m). \tag{5.8}$$

Since the Hessian $u_2$ does not define a flat metric, we consider only $2 \leq m \leq 6$. The argument of applying Dubrovin’s method to the case $m = 2$ is similar to (1).

For $3 \leq m \leq 6$, the contravariant metric $\Omega_2^{ij}$ defined by the Hessian of $u_1$ and its Christoffel symbols are almost linear in $u_1$ and $u_3$ and we can apply lemma 2.4 to both variables.

The FPM formed by $\Omega_2$ and $\Omega_1 := \mathcal{L}_{\partial u_3} \Omega_2$ is regular quasihomogeneous of degree $\frac{1}{2}$ with $\tau = u_1$. We can fix the flat coordinates

$$t_1 = \frac{m-1}{2m}u_1, \ t_2 = u_2^{\frac{m}{2}}, t_3 = u_3 - \frac{1}{8}u_1^2. \tag{5.10}$$
The resulting Frobenius structure is polynomial of type $A_3$.

Also, the FPM consists of $\Omega_2$ and $\Omega_1 = \partial_{x_1} \Omega_2$ is regular quasihomogeneous of degree 0 with $\tau = \frac{m-1}{m} u_1$. We can fix the flat coordinates

$$t_1 = u_1, \ t_2 = u_2^{\frac{m}{5}}, \ t_3 = u_3 u_2^{\frac{-m}{5}}.$$  \hspace{1cm} (5.11)

The resulting Frobenius structure is of type $B_1^1$.

(12) This is a group of type ($\mathcal{F}$). A minimal set of generators of the invariant ring consists of

$$u_1 = (x^3 + y^3 + z^3)^2 - 12(x^3 y^3 + y^3 z^3 + z^3 x^3), \quad u_2 = (x^3 - y^3)(y^3 - z^3)(z^3 - x^3),$$  \hspace{1cm} (5.12)

$$u_3 = (xyz)^4 + 216(xyz)^3(x^3 + y^3 + z^3), \quad u_4 = ((x^3 + y^3 + z^3)^2 - 18(xyz)^2)^2.$$

The FPM formed by $\Omega_{ij}^{ij}$ and $\Omega_{ij}^{ij} := \partial_{x_3} \Omega_2$ is regular quasihomogeneous of degree $d = \frac{1}{2}$ with $\tau = \frac{5}{12} u_1$. Flat coordinates are given by

$$t_1 = \frac{5}{12} u_1, \ t_2 = 10 \sqrt{\frac{62}{41}} u_2, \ t_3 = u_3 - \frac{847}{1312} u_1^2.$$  \hspace{1cm} (5.13)

The resulting Frobenius structure is of type $A_3$.

(14) This is a group of type ($\mathcal{H}$) and it has a minimal set of generators of the invariant ring consists of

$$u_1 = x^2 + yz, \quad u_2 = 8 y z x^4 - 2 y^2 z^2 x^2 - (y^5 + z^5) x + y^3 z^3,$$

$$u_3 = y^{10} + 6 z^5 y^5 + 20 x^2 z^4 y^4 - 160 x^4 z^3 y^3 + 320 x^6 z^2 y^2 + z^{10} - 4 x (y^5 + z^5) \left( 32 x^4 - 20 y z x^2 + 5 y^2 z^2 \right).$$

The Hessian of $10 u_1$ leads to a regular QFPM with $\Omega_2$ and $\Omega_1 = \partial_{x_3} \Omega_2$ of charge $d = \frac{4}{5}$ with $\tau = \frac{1}{10} u_1$.

By fixing the flat coordinates

$$t_1 = \frac{1}{10} u_1, \ t_2 = \sqrt{2} u_2 - \sqrt{2} u_3^3, \ t_3 = 14 u_1^5 - 20 u_2 u_1^2 + u_3,$$

we arrive to a polynomial Frobenius structure of type $H_3$.

(18) A family of representation of type ($\mathcal{B}$) of order $8pq^2$ where $p \geq 1$ and $q \geq 2$. The minimal set of generators of the invariant rings consists of $(x^{2pq} + y^{2pq}), (xy)^{2q}, (xyz)^2$, $(x^{2pq} - y^{2pq})xyz, z^{2q}$. The Hessian of $(xyz)^2$ does not define a flat metric. Hence, because of condition (5.1), we consider only two cases when $q = 2$ or $q = 3$ with $p = 1$. In both cases we get 3 types of Frobenius structures as follows.

First Frobenius structure is obtained by the regular QFPM formed by $\Omega_2$ and $\Omega_1 = \partial_{x_3} \Omega_2$ of charge $d = 0$ with $\tau = t_1$. The flat coordinates are

$$t_1 = \frac{2q-1}{2q} (x^{2q} + y^{2q} + z^{2q}), \ t_2 = \frac{1}{2} z^{2q}, \ t_3 = (xy)^q.$$  \hspace{1cm} (5.14)

The resulting Frobenius manifold is a trivial of type $T_3$. Other Frobenius structures are obtained by fixing the invariants

$$u_1 = x^{2q} + y^{2q} + z^{2q}, \quad u_2 = (xyz)^2, \quad u_3 = -2 x^{2q} y^{2q} - 2 x^{2q} z^{2q} - 2 y^{2q} z^{2q}.$$  \hspace{1cm} (5.15)
We get regular QFPM with $\Omega_2$ and $\Omega_1 = \partial_1, \Omega_2$ of charge 0 with $\tau = \frac{2q-1}{2q} u_1$. We choose the flat coordinates 
\[ t_1 = \frac{2q-1}{2q} u_1, \ t_2 = u_2^2, \ t_3 = u_3 u_2^2 \] (5.16)

The resulting structure is of type $B_3^1$.

Finally, we have regular QFPM with $\Omega_2$ and $\Omega_1 = \partial_1, \Omega_2$ of charge $\frac{1}{2}$ with $\tau = \frac{2q-1}{4q} u_1$, we get type $A_3$ of the flat coordinates 
\[ t_1 = \frac{2q-1}{4q} u_1, \ t_2 = \frac{2\sqrt{q-1}}{\sqrt{q}} u_2^2, \ t_3 = u_3 + \frac{1}{4} u_1^2. \] (5.17)

(26) These are groups of type $(C)$ of order $9m^2$ for even $m \geq 2$. The set of minimal generators of invariant ring has 
\[ x^{3m} + y^{3m} + z^{3m}, (xyz)^2, x^{2m} y^m + x^m y^{2m} + y^m z^m + y^m z^{2m} + z^m x^m + z^m x^{2m}, \]
\[ xyz(x^m - y^m)(y^m - z^m)(z^m - x^m), (x^m - y^m)^2(y^m - z^m)^2(z^m - x^m)^2. \]
The only possible case for condition (5.1) is when $m = 2$. We choose the flat coordinates 
\[ t_1 = \frac{5}{12} (x^6 + y^6 + z^6), \ t_2 = \frac{20}{3} (xyz)^3, \ t_3 = x^{12} + y^{12} + z^{12} - \frac{3}{4} (x^6 + y^6 + z^6)^2. \]

Hence the Hessian of $\frac{42}{5} t_1$ leads to a regular QFPM with $\Omega_2$ and $\Omega_1 = \partial_1, \Omega_2$ of charge $d = \frac{1}{2}$ with $\tau = t_1$. Then the Frobenius structure is of type $A_3$.

On the other hand, the Hessian of $\frac{5}{6} t_1$ leads to a regular QFPM formed by $\Omega_2$ and $\Omega_1 = \partial_1, \Omega_2$ of degree 0 with $\tau = t_1$. We fix the flat coordinates 
\[ t_1 = \frac{5}{6}(x^6 + y^6 + z^6), \ t_2 = (xyz)^\frac{3}{2}, \ t_3 = \frac{5}{6}(xyz)^\frac{1}{2} \left( x^{12} + y^{12} + z^{12} - \frac{3}{4} (x^6 + y^6 + z^6)^2 \right). \] (5.18)

The resulting Frobenius structure is of type $B_3^1$.

6 Conclusions

In this article, we consider invariants rings (not only polynomial ones) of finite linear representations as a source for Frobenius manifolds. The construction at the moment depends on explicit knowledge of the sets of generators. Then, it will be important to classify or characterize those finite groups representations when the Hessian of a minimal degree invariant polynomial defines a flat contravariant metric. Recall that this condition is satisfied for real representations but not for complex reflection representations which are not Shephard groups [18].

Acknowledgement

The authors thank Robert Howett, Hans-Christian Herbig and Christopher Seaton for useful discussions. They very much appreciate the Magma program’s support team for their helpful cooperation. This work was funded by the internal grant of Sultan Qaboos University (IG/SCI/COMS/19/08).
References

[1] Al-Maamari, Z.; Dinar, Y., Inversion symmetry on Frobenius manifolds, arXiv:2106.08000 (2021).
[2] Al-Maamari Z.; Dinar, Y., Dicyclic groups and Frobenius manifolds, SQU Journal for Science, 25(2), 107-111, (2020).
[3] Derksen, H.; Kemper, G., Computational invariant theory, Springer, (2015).
[4] Dubrovin, B., Differential geometry of the space of orbits of a Coxeter group, Surveys in differential geometry IV: integrable systems, 181–211, (1998).
[5] Dubrovin, B., Geometry of 2D topological field theories, Integrable systems and quantum groups (Montecatini Terme, 1993), 120–348, Lecture Notes in Math., 1620, Springer, Berlin, (1996).
[6] Dubrovin B., Flat pencils of metrics and Frobenius manifolds, Integrable systems and algebraic geometry (Kobe/Kyoto, 1997), 47–72, World Sci. Publ., River Edge, NJ, (1998).
[7] Dubrovin B.; Zhang, Y., Extended affine Weyl groups and Frobenius manifolds, Compositio Math. 111, no. 2, 167–219 (1998).
[8] Dubrovin, B., On almost duality for Frobenius manifolds, Geometry, topology and mathematical physics, 75–132. Am. Math. Soc. Transl. Ser. 2, 212 (2004).
[9] Fordy, A. P.; Mokhov, O. I., ON a special class of compatible Poisson structures of hydrodynamic type, Physica D 152-153 475-490 (2001).
[10] Dubrovin, B.; Strachan, I. A. B.; Zhang, Y.; Zuo, D., Extended affine Weyl groups of BCD-type: their Frobenius manifolds and Landau-Ginzburg superpotentials, Adv. Math. 351, 897–946 (2019).
[11] Fulton, W.; Harris, J., Representation theory: A first course, V(129). Springer Science and Business Media, (2013).
[12] Howett, Robert; in personal communications.
[13] Hertling, C., Frobenius manifolds and moduli spaces for singularities, Cambridge Tracts in Mathematics, V(151), Cambridge University Press, (2002).
[14] Humphreys, J. E., Reflection groups and Coxeter groups, V(29), Cambridge University Press, (1990).
[15] Leuschke, G. J.; Wiegand, R, Cohen-Macaulay representations,Mathematical Surveys and Monographs, 181. American Mathematical Society, ISBN: 978-0-8218-7581-0 (2012).
[16] Mokhov, O. I., Compatible Dubrovin-Novikov Hamiltonian operators, the Lie derivative, and integrable systems of hydrodynamic type. (Russian) Teoret. Mat. Fiz. 133, no. 2, 279–288; translation in Theoret. and Math. Phys. 133 , no. 2, 1557–1564 (2002).
[17] Neusel, M., Invariant Theory, American Mathematical Society, Student Mathematical Library (Book 36), ISBN: 978-0821841327 (2006).
[18] Orlik, P.; Solomon, L., The Hessian map in the invariant theory of reflection groups, Nagoya Math. J. 109 , 1-21 (1988).
[19] Saito, K., On a linear structure of the quotient variety by a finite reflexion group. Publ. Res. Inst. Math. Sci. 29 , no. 4, 535–579 (1993).
[20] Saito, K.; Yano, T.; Sekeguchi, J., On a certain generator system of the ring of invariants of a finite reflection group, Comm. in Algebra 8(4) (1980).

[21] Sergyeyev, A. A simple way of making a Hamiltonian system into a bi-Hamiltonian one, Acta Applicandae Mathematica, 83(1), 183-197, 2004.

[22] Steinberg, B., Representation Theory of Finite Groups An Introductory Approach, Springer Science+Business Media, (2012).

[23] Watanabe, K. I.; Rotillon, D., Invariant subrings of \( \mathbb{C}[X,Y,Z] \) which are complete intersections, manuscripta mathematica, 39(2), 339-357, (1982).

[24] Yau, S. T.; Yau, S. S. T.; Yu, Y, Gorenstein quotient singularities in dimension three, (Vol. 505), American Mathematical Soc, (1993).

[25] Zuo, D., Frobenius Manifolds Associated to \( B_l \) and \( D_l \), Revisited, International Mathematics Research Notices, Vol. (2007).

[26] Zuo, D., Frobenius manifolds and a new class of extended affine Weyl groups \( \overline{W}^{(k,k+1)}(A_l) \), 2000 Mathematics Subject Classification, (2019).

Yassir Dinar
dinar@squ.edu.om

Zainab Al-Maamari
s100108@student.squ.edu.om

Depatment of Mathematics
College of Science
Sultan Qaboos University
Muscat, Oman