PROJECTIVE CURVES WITH SEMISTABLE NORMAL BUNDLE

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ABSTRACT. For every $m, g \geq 1$ and all large enough $e$ such that $e \equiv 1 - g \mod m$, there is a curve of genus $g$ and degree $e$ in $\mathbb{P}^{2m+1}$ with semistable normal bundle in $\mathbb{P}^{2m+1}$, and a curve of genus $g$ and degree $e(2m+2)$ on a general hypersurface $X$ of degree $2m+2$ in $\mathbb{P}^{2m+2}$ with semistable normal bundle in $X$. Analogous, less precise results are obtained for hypersurfaces of even degree $< n$ in $\mathbb{P}^n$ for any $n \geq 8$. Also a general curve of genus 1 and any large enough degree in any $\mathbb{P}^n$ has semistable normal bundle. Previous results were restricted to the case $g = 1$ and small $e$ or $m = 1$ and ambient space $\mathbb{P}^n$.

A vector bundle $E$ on a curve $C$ is said to be semistable if every subbundle $F \subset E$ has slope $\mu(F) := \deg(F)/\text{rk}(F) \leq \mu(E)$. It is a natural question whether for a sufficiently general curve $C$ on a given projective variety $X$, with given homology class or degree $e$ and genus $g \geq 1$, the normal bundle $N = N_{C/X}$ is semistable. This intuitively means $C$ moves evenly in all directions. In the case $X = \mathbb{P}^n$, this question has been studied by many mathematicians since the early 1980s. These include Ballico-Ellia [2], Coskun-Larson-Vogt [3], Ein-Lazarsfeld [4], Ellia [6], Ellingsrud-Hirschowitz [7], Ellingsrud-Laksov [8], Ghione-Sacchiero [9], Eisenbud-Van de Ven [5], Hartshorne [10], Hulek [11], Newstead [12], Sacchiero [15]. Notably, Ein and Lazarsfeld [4] have shown that an elliptic curve of (minimal) degree $n + 1$ in $\mathbb{P}^n$ has semistable normal bundle, and Coskun-Larson-Vogt [3] have shown recently that with few exceptions the same is true for Brill-Noether general curves in $\mathbb{P}^3$ (where $N$ has rank 2, and in fact $N$ is shown to be stable). Thus far (semi)stability results have been restricted to the case where $X = \mathbb{P}^n$ and either $g = 1$ or $n = 3$.

Along different lines, it was shown by Atanasov-Larson-Yang [1] that the normal bundle to a general nonspecial curve in $\mathbb{P}^n$ is interpolating in the sense that for a general effective divisor $D$ of any degree one has either $H^0(N(-D)) = 0$ or $H^1(N(-D)) = 0$.
Some interpolation results for hypersurfaces are in [14]. The interpolation property implies that \( N \) admits a (Harder-Narasimhan) filtration with semistable quotients whose slopes lie in an interval of length 1.

The purpose of this paper is to expand the collection of curves and ambient spaces with known semistable normal bundle. We will prove the following.

**Theorem.** (a) Given \( g, e, n \), suppose either

(i) \( g = 1 \) and \( e = 2n - 2 \) or \( e \geq 3n - 3 \);

or

(ii) \( g \geq 1, n = 2m + 1, e = \ell m - 1 + g, \ell \geq 2g + 2 \).

Then a general curve of genus \( g \) and degree \( e \) in \( \mathbb{P}^n \) has semistable normal bundle in \( \mathbb{P}^n \).

(b) Let \( X \) be a general hypersurface of degree \( n + 1 \) in \( \mathbb{P}^{n+1} \) and \( g, e \) be as in (ii).

Then there is a curve of genus \( g \) and degree \( (n + 1)e \) on \( X \) with semistable normal bundle in \( X \).

(c) Let \( X \) be a general hypersurface of even degree \( d \leq n \) in \( \mathbb{P}^n \), such that \((n + 1, d/2) = 1 = (n - 2, d - 3)\). Then given any \( g \geq 0 \) there is an arithmetic progression of \( e \) values such that \( X \) contains a smooth curve of genus \( g \) and degree \( e \) with semistable normal bundle.

To my knowledge these are the first cases beyond \( g = 1 \) or \( n = 3 \) (or the obvious \((d, \ldots, d)\) complete intersection) of genus-\( g \) curves in \( \mathbb{P}^n \) with known semistable normal bundle; and, again other than an obvious complete intersections, there are no such results for hypersurfaces.

Note that for \( n = 3 \) Coskun-Larson-Voigt have proven that the normal bundle is in fact stable rather than just semistable. This does not seem to be accessible by our methods.

As for the geometric meaning of a curve having semistable normal bundle, note that a smooth subvariety containing the curve yields a subbundle of the normal bundle, so from the standard formula for the degree of the normal bundle and the definition of semistability, we have:

**Lemma 1.** Let \( C \subset X \) be a smooth curve of genus \( g \) with semistable normal bundle on a smooth \( n \)-dimensional variety. Then for any subvariety \( Y \subset X \) of dimension \( m \) containing \( C \) and smooth along it, we have

\[
(n-1)C.(\mathcal{K}_Y) + (n-m)(2g-2) \leq (m-1)C.(-\mathcal{K}_X).
\]

Our strategy is to first prove, in Theorem 2, semistability for genus 1 using a ‘fish fang’ degeneration (compare [13], [14]) where the embedding \( C \subset \mathbb{P}^n \) degenerates to

\[
C_1 \cup_{p,q} C_2 \subset P_1 \cup_E P_2
\]

where \( P_1, P_2 \) are blowups of \( \mathbb{P}^n \) in a suitable \( \mathbb{P}^m \) resp \( \mathbb{P}^{n-1-m} \) with common divisor \( E = \mathbb{P}^m \times \mathbb{P}^{n-1-m} \) and where \( C_1, C_2 \) are rational with \( C_1 \cap E = C_2 \cap E = \{p, q\} \). We then extend the result by showing in Lemma 4, again using a suitable fish fang, that
in favorable cases the birational transform of \( C \) in the blowup on \( \mathbb{P}^n \) in a 1-secant \( \mathbb{P}^m \) also has semistable normal bundle. This result is then used to prove the main result for \( \mathbb{P}^n \), Theorem [5] by an induction on the genus, using another fang degeneration. The result for anticanonical hypersurfaces, Theorem [8](slightly more general than the above statement), is proven using the \( \mathbb{P}^n \) case plus a fan-quasi cone degeneration similar to the one used in [13]. Finally the result for lower-degree hypersurfaces, Theorem [10] is proven using a suitable fang as in [13], essentially putting the curve in a suitable projective bundle over a \( \mathbb{P}^m \), using semistability of the horizontal part of the normal bundle by Theorem [5] and proving semistability of the vertical part by another degeneration argument.

1. Semistability for genus 1

As mentioned above, Ein-Lazarsfeld [4] have proven semistability of the normal bundle of an elliptic normal curve, of degree \( n + 1 \) in \( \mathbb{P}^n \). Here we prove an analogous but logically independent result, showing semistability of the normal bundle of a general elliptic curve of degree \( 2n - 2 \) or \( \geq 3n - 3 \) in \( \mathbb{P}^n \). In the case \( n = 3 \), this result follows from that of Coskun-Larson-Voigt [3].

First, as a matter of terminology, the bidegree of a bundle \( E \) on a reducible curve \( C_1 \cup C_2 \) is by definition \((\deg(E|_{C_1}), \deg(E|_{C_2}))\).

**Theorem 2.** A general elliptic curve of degree \( e = 2n - 2 \) or \( e \geq 3n - 3 \) in \( \mathbb{P}^n, n \geq 3 \), has semistable normal bundle.

In view of Lemma [1] this implies:

**Corollary 3.** Notations as above, any smooth \( m \)-dimensional subvariety \( Y \) containing \( C \) must have

\[
C.(−K_Y) \leq \frac{(n + 1)(m - 1)e}{n - 1}.
\]

**Proof of Theorem.** We will do the case \( n = 2m \) even, \( n \geq 4 \) as the case \( n \) odd is similar and simpler (see comments at the end of the proof). Assume first that \( e = 2n - 2 \). Consider a fang degeneration

\[
P_0 = P_1 \cup_E P_2
\]

where

\[
P_1 = B_{\mathbb{P}^n} \mathbb{P}^n \supset E_1 = \mathbb{P}^m \times \mathbb{P}^{m-1} \\
P_2 = B_{\mathbb{P}^{m-1}} \mathbb{P}^n \supset E_2 = \mathbb{P}^m \times \mathbb{P}^{m-1}
\]

in which \( E_1 \subset P_1, E_2 \subset P_2 \) are exceptional divisors and \( P_0 \) is constructed via an isomorphism \( E_1 \cong E \cong E_2 \). There is a standard smoothing of \( P_0 \) to \( \mathbb{P}^n \). Consider curves

\[
C_1 \subset P_1, C_2 \subset P_2
\]
with each being a birational transform of a rational normal curve, such that
\[ C_1 \cap E = C_2 \cap E = \{p, q\}. \]

Then \( C_0 = C_1 \cup C_2 \) is a nodal, lci ‘fish’ curve in \( P_0 \) and smooths out to an elliptic curve \( C_* \) of degree \( 2n - 2 \) in \( \mathbb{P}^n \) whose normal bundle is a deformation of \( N_{C_0/P_0} = N_{C_1/P_1} \cup N_{C_2/P_2} \). Now using Lemma 31 of [14], we may assume each \( C_i \) is balanced in \( P_i, i = 1, 2 \), which means that
\[ N_{C_1/P_1} = \mathcal{O}(2m + 2) \oplus (2m - 2)\mathcal{O}(2m + 1), \]
\[ N_{C_2/P_2} = (2m - 2)\mathcal{O}(2m + 1) \oplus \mathcal{O}(2m). \]

Consequently,
\[ \wedge^i N_{C_1/P_1} = \binom{2m - 2}{i - 1} \mathcal{O}(i(2m + 1) + 1) \oplus \binom{2m - 2}{i} \mathcal{O}(i(2m + 1)), \]
\[ \wedge^i N_{C_2/P_2} = \binom{2m - 2}{i} \mathcal{O}(i(2m + 1)) \oplus \binom{2m - 2}{i - 1} \mathcal{O}(i(2m + 1) - 1). \]

Note that the slope of \( N_{C_0/P_0} \), hence that of \( N_{C_*/\mathbb{P}^n} \) equals \( 2n + 2 = 4m + 2 \). Now we have natural identifications
\[ N_{C_i/P_i}|_p = T_p E_i, i = 1, 2 \]
and likewise for \( q \). The blowdown map \( P_1 \to \mathbb{P}^n \) contracts the vertical factor \( \mathbb{P}^{m-1} \) of \( E_1 \) and because the upper subbundle \( \mathcal{O}(2m + 2) \subset N_{C_1/P_1} \) maps isomorphically to its image in \( N_{C_1/P_1} = (2m - 1)\mathcal{O}(2m + 2) \), it follows that the fibre of the upper subbundle at \( p \) is not contained in the vertical subspace \( T_p \mathbb{P}^{m-1} \subset T_p E_1 \), and likewise at \( q \).

Now I claim that with general choices, the upper subspaces of \( N_{C_1/P_1} \) and \( N_{C_2/P_2} \) at \( p \) are transverse, and likewise for the exterior powers. To this end we use automorphisms. The automorphisms of \( \mathbb{P}^m \) stabilizing \( \mathbb{P}^m \) lift to automorphisms of \( P_1 \) that send \( E_1 \) to itself and are compatible with the projection \( E_1 \to \mathbb{P}^m \) (i.e. mapping a fibre to a fibre). Now the automorphism group of \( P_1 \) fixing \( p, q \) maps surjectively to the automorphism group of \( (E_1/\mathbb{P}^m, p, q) \) and the latter acts transitively up to scalars on the ‘nonvertical pairs’, i.e. pairs \((v_p, v_q) \in T_p E_1 \oplus T_q E_2 \) such that \( v_p \notin T_p \mathbb{P}^m, v_q \notin T_q \mathbb{P}^m \). Such automorphisms of \( P_1 \) move \( C_1 \) through \( p, q \), compatibly with the isomorphism (1), hence also move the upper subspace \( \mathcal{O}(2m + 2) \subset N_{C_1/P_1} \) so its fibres at \( p \) and \( q \) are arbitrary nonvertical lines. Therefore, choosing \( C_1 \) sufficiently general fixing \( p \) and \( q \), we may assume the upper subspaces of \( N_{C_1/P_1}|_p \) and \( N_{C_2/P_2}|_p \) are complementary. Therefore \( \wedge^i N_{C_1/P_1}|_p \) and \( \wedge^i N_{C_2/P_2}|_p \) are in general position. This implies that \( \wedge^i N_{C_0/P_0} \) has no line subbundle of bidegree \((i(2m + 1) + 1, i(2m + 1)) \). It follows that the total degree of any line subbundle of \( \wedge^i N_{C_0/P_0} \) is at most \( i(4m + 2) \) which is equal to the slope of \( \wedge^i N_{C_0/P_0} \). Now a line
subbundle $M$ of $\wedge^i N_{C_*/P^n}$ specializes either to a line subbundle of $\wedge^i N_{C_0/P_0}$, or else to a non-invertible subsheaf $M_0$ of $\wedge^i N_{C_0/P_0}$ which has rank 1 on each component, and in the latter case we have

$$\deg(M) < \deg(M_0|_{C_1}/(\text{torsion})) + \deg(M_0|_{C_2}/(\text{torsion})).$$

Therefore in either case we have

$$\deg(M) \leq i(2m + 2).$$

Since this is true for all $i$, it follows that $N_{C_*/P^n}$ is (cohomologically) semistable (this under the assumption that $n$ is even and $e = 2n - 2$).

For the case $n$ even and $e \geq 2n - 3$ we take $C_1$ rational normal as above and $C_2$ rational of degree $e_2 \geq 2n - 1$, and again use the appropriate cases of Lemma 31 of [14]. Then the proof proceeds similarly.

Finally for $n = 2m + 1$ odd we just take $P_1 \simeq P_2 \simeq B_p = \mathbb{P}^n$ and proceed similarly. □

The following technical result extends semistability to elliptic curves in certain blowups of projective space, and plays a key role in the subsequent extension of semistability to higher genus.

**Lemma 4.** Let $n = 2m + 1 \geq 5$, $A \subset \mathbb{P}^n$ be either $\mathbb{P}^m$ or $\mathbb{P}^m \coprod \mathbb{P}^m$, $P^A = B_A \mathbb{P}^n$, and let $C^A \subset P^A$ be the birational transform of a general elliptic curve of degree $km$, $k \geq 4$ meeting each component of $A$ in 1 point. Then $N_{C^A/P^A}$ is semistable.

**Proof.** The proof is modeled after that of Theorem [2], with modifications. It is somewhat complicated because in general an elementary modification of a semistable bundle is not semistable. We first do the case $A = \mathbb{P}^m$, the only case needed in the sequel. Consider a fang degeneration with general fibre $\mathbb{P}^n$ and special fibre

$$P_0 = P_1 \cup_P P_1$$

where

$$P_i = B_{\mathbb{P}^n} \mathbb{P}^n, i = 1, 2.$$ 

with exceptional divisors $E_i = \mathbb{P}^m \times \check{\mathbb{P}}^m_i \subset P_i$, where $\check{\mathbb{P}}^m_i$ is the space of hyperplanes through $\mathbb{P}^m_i$. Thus $P_0$ is defined by an isomorphism $E_1 \simeq E_2$ interchanging the $\mathbb{P}^m_i$ (‘horizontal’) and $\check{\mathbb{P}}^m_i$ (‘vertical’) factors.

Let $P^A_2 \rightarrow P_2$ be the blowup of $A$ where $A \subset P_2$ is the inverse image of a $\mathbb{P}^m$ disjoint from $\mathbb{P}^m_2$, and let $E_A \subset P^A_2$ be the exceptional divisor of $P^A_2 \rightarrow P_2$, which is disjoint from $E_2$. Finally let $P^A_0 = P_1 \cup P^A_2$, which smooths out to $P^A$.

We assume first that $e = 2n - 2 = 4m$. Consider a curve

$$C_0 = C_1 \cup_{P_A} C_2 \subset P_0$$
where \( C_i \subset P_i \) is the birational transform of a rational normal curve \( C'_i \subset \mathbb{P}^n_i \) (degree \( n \)) meeting \( \mathbb{P}^m_i \) in 2 points and \( C'_2 \) meets \( A \) in 1 point, say \( a \). Let \( C^A = C^A_1 \cup C_2 \). Then \( C_0 \subset P_0 \) smooths to an elliptic curve \( C \subset \mathbb{P}^n \) of degree \( 2n - 2 \) and \( C^A_0 \subset P^A_0 \) smooths to an elliptic curve \( C^A \subset P^A \) mapping isomorphically to \( C \), and we claim that normal bundle \( N_{C^A/P^A} \) is semistable. Note that

\[
\mu(\Lambda^i N_{C/\mathbb{P}^n}) = (4m + 4)i, \quad \mu(\Lambda^i N_{C^A/P^A}) = (4m + 7/2)i = \mu(\Lambda^i N_{C/\mathbb{P}^n}) - i/2.
\]

By [14], Lemma 31, we have

\[
N_{C_i/P_i} \simeq 2mO(2m + 2), \quad i = 1, 2,
\]

\[
N_{C_2/P_2} \simeq mO(2m + 2) \oplus mO(2m + 1).
\]

We have isomorphisms

\[
H^0(N_{C_i/P_i}(-(2m - 2))) \overset{\psi_i}{\rightarrow} T_pE_i, \quad H^0(N_{C_i/P_i}(-(2m - 2))) \overset{\psi_i}{\rightarrow} T_qE_i.
\]

\[
H^0(N_{C_2/P_2}(-(2m - 2))) \cong N_{C_2/P_2}(-(2m - 2))|_a
\]

It is easy to check that \( \eta_{p,q}^i : e_p^i(e_p^{-1})^{-1} \) sends the vertical subspace \( T_p\mathbb{P}_i^m = T_p(E_i/\mathbb{P}^m_i) \subset T_pE_i \) to a ‘quasi-horizontal’ subspace of \( T_qE_i \), i.e. one having trivial intersection with the vertical subspace (though not necessarily equal to \( T_p\mathbb{P}_i^m \)).

As in the proof of Theorem 2 above, we can move the \( C_i \) through \( p, q \) by a suitable automorphism of \( P_i \), so it has a given vertical tangent direction \( k.w \in T_pE_i \) and any given quasi-horizontal tangent direction at \( q \), i.e. the induced map

\[
\omega_{p,q}^i = (\text{projection}) \circ \eta^i : T_p\mathbb{P}_i^m \rightarrow T_q\mathbb{P}_i^m
\]

is general up to scalar multiple. Likewise with \( p, q \) interchanged. Therefore

\[
\eta := \eta_{q,p}^1 \circ \eta_{p,q}^2 \in \text{End}(T_pE_2)
\]

may be assumed diagonalizable with distinct eigenvalues, and therefore has just finitely many invariant subspaces, namely subspaces generated by eigenvectors. Moreover \( A \) is general meeting \( C_2 \) , hence under the above identification of \( T_pE_2 \) with \( N_{C_2/P_2}|_a \), the image \( W \) of \( T_aA \) in the normal space \( N_{C_2/P_2}|_a \) is identified with a general subspace of \( T_pE_2 \) and in particular contains no \( \eta \)-invariant subspace.

Now let \( F \subset N := N_{C^A_0/P^A_0} \) be a subsheaf having rank \( i \) on each component. We want to show \( \mu(F) \leq \mu(N) \). Let \( Q = N/F \) and let \( \tau \subset Q \) be the torsion subsheaf, and let \( F^+ \subset N \) be the kernel of the natural map \( N \rightarrow Q/\tau \) (i.e. the ‘saturation’ of \( F \)). Then \( F^+ \) is locally free and \( F^+/F \simeq \tau \) so \( F^+ \) has larger slope than \( F \). The upshot if that we may assume \( F \) is locally free to begin with.
Now let $F_1 = F|_{C_1}, F_2 = F|_{C_2}$. We assume the upper subbundle of $F_j$ is $V^j \otimes \mathcal{O}(2m + 2), j = 1, 2$ with $i_j = \dim(V^j) > 0, j = 1, 2$, as the case where one of the upper subbundles has slope $< 2m + 2$ is similar and simpler. Note that $V^2$ may be considered a subspace of $W$. Let $V^j_p, V^j_q$ denote the respective upper subspaces. Then

$$V^j_p \cap \eta(V^j_p) \cap V^2_p \subset T_p E_1 = T_p E_2.$$  

If this is nonzero, would correspond to a $k\mathcal{O}(2m + 2)$ subbundle of $N_{C^2_2}/P_2$, which is an invariant subbundle of $N_{C_2}/P_2$, corresponding to an invariant subspace contained in $W$. However as noted above this does not exist by generality. Therefore the intersection is trivial and consequently

$$i_1 + i_2 \leq (3/2)\text{rk}(N) = (3/2)2m.$$  

Therefore $\mu(F) \leq \mu(N)$. This completes the proof in case $e = 2n - 2$.

In the general case $e = km$ we take for $C_2$ the same rational normal curve but for $C_2$ we take a rational curve of degree $e_1 = (k - 2)m + 1, k \geq 4$ whose normal bundle in both $\mathbb{P}^n$ and $P_1$ is perfect. Then the proof proceeds similarly. This completes the proof in case $A = \mathbb{P}^m$.

The case $A = A_1 \bigsqcup A_2 = \mathbb{P}^m \bigsqcup \mathbb{P}^m$ (which will not be needed in the sequel) proceeds similarly by putting each $A_i$ on $P_i$ as general 1-secant to $C_i$ and again using the fact that the normal bundle to the birational transform in the blowup of $A$ is perfect.

$\square$

2. HIGHER GENUS

The following result extends Theorem 2 and Lemma 4 to higher genus, using an induction on the genus. The second statement is included mainly for induction purposes.

**Theorem 5.** Let $n = 2m + 1, g \geq 1, \ell \geq 6g - 2$ and let $C$ be a general curve of degree $\ell m - g + 1$ and genus $g$ in $\mathbb{P}^{2m+1}$. Then

(i) $C$ has semistable normal bundle;

(ii) if $A$ is a general 1-secant $\mathbb{P}^m$ to $C$, then the birational transform of $C$ in the blowup $B_A \mathbb{P}^n$ has semistable normal bundle.

**Corollary 6.** Notations as above, if $Y \subset \mathbb{P}^n$ is a smooth $p$-dimensional variety containing $C$ then

$$C.(—K_Y) \leq ((p - 1)(n + 1)e - (n - p)(2g - 2))/(n - 1).$$

**Proof of Theorem.** We use induction on $g$, the case $g = 1$ being contained in Theorem 2 and Lemma 4. We focus on (ii) as (i) is similar and simpler. We use the same fang
degeneration of \( \mathbb{P}^n \) as in Lemma 4 but this time we put \( A \) on \( P_1 \). Again we consider a curve

\[ C_0 = C_1 \cup C_2 \subset P_0 = P_1 \cup E P_2. \]

For \( C_1 \) we use the birational transform in \( P_1 \) of a rational normal curve that is 1-secant to \( A \), with \( C_1 \cap E = \{ p, q \} \) and \( C_1^A \subset P_1^A \) is the birational transform of \( C_1 \). Thus we have perfect normal bundles

\[ N_{C_1/P_1} = 2m \mathcal{O}(2m + 2), N_{C_1^A/P_1^A} = 2m \mathcal{O}(2m + 1). \]

For \( C_2 \) we however use a disjoint union

\[ C_2 = C_{2,1} \bigsqcup C_{2,2} \]

of curves of respective genera 1, \( g - 1 \) and respective degrees

\[ e_{2,1} = 4m, e_{2,2} = (\ell - 6)m - g + 2, \]

(so that \( \ell - 6 \geq 6(g - 1) + 2 \)), such that

\[ C_{2,1} \cap E = C_{2,1} \cap C_1^A = p, C_{2,2} \cap E = C_{2,2} \cap C_1^A = q. \]

meeting the common divisor \( E \) and \( C_1 \) in \( p \) resp. \( q \). Thus \( C_0 \) has arithmetic genus \( g \) and is smoothable to a curve of genus \( g \) and degree \( e = 2m + 1 + e_{2,1} + e_{2,2} - 2 \) in \( \mathbb{P}^n \).

Similarly for \( C_0^A = C_1^A \cup C_2 \), By induction and Lemma 4, the normal bundles \( N_{C_{2,i}/P_2} \) are semistable for \( i = 1, 2 \). Therefore by the elementary Lemma below so is, in a suitable sense \( N_{C_0^A/P_0^A} \) and likewise the smoothing \( N_{C_0^A/P_0^A} \).

**Lemma 7.** Let \( E_0 \) be a vector bundle on a nodal curve \( C_0 \) that is the union of smooth components, such that the restriction of \( E_0 \) on each component is semistable. Then for any smoothing \( (C, E) \) of \( (C_0, E_0) \), \( E \) is semistable.

\[ \square \]

### 3. Hypersurfaces

The purpose of this section is to construct curves with semistable normal bundle on some general Fano hypersurfaces of dimension \( n \geq 3 \) in projective space. We begin with the case of anticanonical hypersurfaces (degree \( n + 1 \) in \( \mathbb{P}^{n+1} \)).

**Theorem 8.** Let \( X \) be a general hypersurface of degree \( n + 1 \) in \( \mathbb{P}^{n+1}, n \geq 3 \) and let \( g, e \) be given. Assume either

(i) \( g = 1 \) and either \( e = 2n - 2 \) or \( e \geq 3n - 3 \); or

(ii) \( n = 3, e \geq 7 \); or

(iii) \( n = 2m + 1, g \geq 1, e = \ell m - g + 1 \) where \( \ell \geq 2g - 2 \).

Then \( X \) contains a curve of genus \( g \) and degree \( (n + 1)e \) with semistable normal bundle.
Corollary 9. Notations as above, if the curve $C$ is contained in a smooth $p$-dimensional subvariety $Y \subseteq X$ then

$$C.(−K_Y) \leq ((p − 1)(n + 1)e − (n − p)(2g − 2))/(n − 1).$$

Proof of Theorem. We will use the same fan- quasi cone degeneration as in [13]. Thus we take

$$P_0 = P_1 \cup_E P_2$$

with $P_1$ the blowup of $\mathbb{P}^{n+1}$ at a point $p$, with exceptional divisor $E = \mathbb{P}^n$, and $P_2 = \mathbb{P}^{n+1}$ containing $E$ as a hyperplane. In $P_0$ we consider a Cartier divisor

$$X_0 = X_1 \cup X_2$$

with $X_1$ the blowup of a quasi-cone $\tilde{X}_1$, i.e. a hypersurface of degree $n + 1$ with multiplicity $n$ at $p$, and $X_2$ a general hypersurface of degree $n$ in $P_2 = \mathbb{P}^{n+1}$. Then $X_0 \subseteq P_0$ smooths out to a hypersurface $X \subseteq \mathbb{P}^{n+1}$ of degree $n + 1$. As noted in [13], $X_1$ may be realized as the blowup of $\mathbb{P}^n$ in an $(n, n + 1)$ complete intersection

$$Y = F_n \cap F_{n+1}$$

where $F_n$ corresponds to $X_1 \cap E = X_2 \cap E$ and $F_{n+1}$ (which is not unique) corresponds to a hyperplane section of $\tilde{X}_1$. Now as in [13] we consider a curve $C_0 \subseteq X_0$ of the form

$$C_0 = C_1 \cup C_2.$$

Here $C_1 \subseteq X_1$ is the birational (=isomorphic) transform of a curve $C'$ of genus $g$ and degree $e$ in $\mathbb{P}^n$, disjoint from $Y$, whose normal bundle $N_{C'/\mathbb{P}^n}$ is semistable. Such a curve exists by Theorem 2 resp. Theorem 5 in cases (i), (iii), or by the results of Coskun-Larson-Vogt [3] in case (ii). And as in [13], $C_2 \subseteq X_2$ is a disjoint union of lines with trivial (hence semistable) normal bundle, meeting $C_1$ in $C_1 \cap F_n$, As $C'$ is disjoint from $Y$, obviously

$$N_{C_1/X_1} = N_{C'/\mathbb{P}^n}$$

is semistable, and moreover

$$N_{C_0/P_0}|_{C_i} = N_{C_i/P_i}, i = 1, 2.$$

Therefore e.g. by Lemma 7 above or by an argument as in [13], a smoothing $C \subseteq X$ of $C_0 \subseteq X_0$ to a curve on a hypersurface of degree $n + 1$ in $\mathbb{P}^{n+1}$ has semistable normal bundle.

Next we take up the case of hypersurfaces of degree $d < n$ in $\mathbb{P}^n$.

Theorem 10. Let $X$ be a general hypersurface of even degree $d \leq n$ in $\mathbb{P}^n$. Assume

$$(n + 1, d/2) = 1 = (n − 2, d − 3).$$

\[\square\]
Then given any \( g \geq 0 \), there exists an arithmetic progression of \( e \) values with difference \((n - 2)d/2\) such that \( X \) contains a smooth curve of degree \( e \) and genus \( g \) with semistable normal bundle.

Example 11. For \( d = 10 \) the conditions are satisfied for all \( n \geq 10 \) with \( n \not\equiv 4 \mod 5 \) and \( n \not\equiv 2 \mod 7 \).

Proof. We will use the fang setup as in [13], §6. Thus we set \( m = d - 1 = 2m_0 + 1 \) and consider a limiting form of \( P^n \) which is a fang of the form

\[
Z_0 = Z_1 \cup Z_2
\]

where

\[
Z_1 = \mathbb{P}^m(1, 0^{n-m}), Z_2 = \mathbb{P}^{n-m-1}(1, 0^{m+1}).
\]

In \( Z_0 \) we consider a divisor which is a limiting form of a degree-\( d \) hypersurface in \( P^n \) and has the form

\[
X_0 = X_1 \cup X_2
\]

where \( X_1 = \mathbb{P}^n(G) \subset Z_1 \) and \( X_2 \subset Z_2 \) is fibred over \( \mathbb{P}^{n-m-1} \) with general fibre a general hypersurface of degree \( m \) in the \( \mathbb{P}^{m+1} \) fibre of \( Z_2 \). We recall that \( G \) is a rank-\( n - m \) bundle over \( \mathbb{P}^m \) which fits in an exact sequence

\[
0 \to \mathcal{O}(-d + 1) \to \mathcal{O}(1) \oplus (n - m)\mathcal{O} \to G \to 0.
\]

Then in \( X_0 \) we consider a connected lci curve of the form

\[
C_0 = C_1 \cup C_2
\]

where \( C_2 \subset X_2 \) is a disjoint union of lines with trivial (hence semistable) normal bundle while \( C_1 \subset X_1 \) is a suitable isomorphic lift of \( P^n \)-degree \( e \) of a smooth curve \( C_+ \subset \mathbb{P}^m \) of genus \( g \) with semistable normal bundle. Then \( C_0 \subset X_0 \) deforms to a smooth curve \( C \subset X \) of degree \( e \) and genus \( g \) on a general hypersurface of degree \( d \) and the normal bundle \( N_C/X \) will be semistable if \( N_{C_0}/X_0 \) is, which in turn will be true provided \( N_{C_1}/X_1 \) is semistable. Thus it would suffice to show that with suitable choices \( N_{C_1}/X_1 \) may be assume semistable. For convenience, let us call the \( P^n \) and \( P^m \) degrees of a curve \( C_1 \subset X_1 \) the upper and lower degrees, say \( e, e_0 \), and the pair \((e, e_0)\) the bidegree.

Assuming \( C_1 \) can be constructed we may proceed as in loc. cit. and consider the exact sequence

\[
0 \to T_{\nu}|_{C_1} \to N_{C_1/X_1} \to N_{C_+}/P^m \to 0.
\]

Now take \( m = 2m_0 + 1 \) odd and let \( C_+ \subset \mathbb{P}^m \) be as in Theorem[5] of genus \( g \) and degree \( e_0 = \ell m_0 - g + 1 \). Then \( N_{C_+}/P^m \) will be semistable and by an elementary calculation its slope is \( e_0 + \ell = \ell(m_0 + 1) - g + 1 \). Now note the following easy remark
Lemma 12. Let

\[ 0 \to E_1 \to E \to E_2 \to 0 \]

be an exact sequence of vector bundles on a curve such that \( E_1 \) and \( E_2 \) are semistable and have the same slope \( s \). Then \( E \) is semistable of slope \( s \).

We wish to apply Lemma 12 to the exact sequence (4). To this end note that our assumptions (2) imply

\[ ((n - 2)(m_0 + 1), n - 1 - 2m_0) = 1 \]

hence the congruence

\[ \ell((n - 2)(m_0 + 1)) \equiv (g - 1)n \mod n - 1 - 2m_0 \]

admits an arithmetic progression of solutions \( \ell \). Setting

\[ e = \frac{\ell((n - 2)(m_0 + 1)) - (g - 1)n}{n - 1 - 2m_0} \in \mathbb{Z} \]

one computes that for some constant \( a > 0 \), \( e/e_0 > 1 + a > 1 \) for all large \( \ell \). Consequently, for any constant \( b \) we have \( e > e_0 + b \) for large \( \ell \) (i.e. large \( e_0 \)).

Now the argument of [14], proof of Theorem 41 shows that if \( \ell \), hence \( e_0 \), is large enough then there exists a lift \( C_1 \subset X_1 \) of \( C_+ \subset \mathbb{P}^m \) whose \( \mathbb{P}^n \)-degree is \( e \) as in (5), and as in loc. cit. we have \( T_v = K^*(B) \) where \( K \) is the kernel of the surjection \( G|_{C_+} \to B \), and \( T_v|_{C_1} \) has slope

\[ \mu(T_v|_{C_1}) = e + (e - de_0)/(n - d) = \frac{(n - 1 - 2m_0)e - (2m_0 + 2)e_0}{n - 2 - 2m_0} \]

By straightforward arithmetic, our choice of \( e \) as in (5) ensures that \( T_v|_{C_1} \) and \( N_{C_+}/\mathbb{P}^m \) have the same slope on \( C_+ \simeq C_1 \). Therefore by the Lemma, in order to show \( N_{C_1}/X_1 \) semistable it will suffice to show that \( T_v|_{C_1} \) is semistable.

To that end we argue as in [14] by induction on the genus \( g \). For \( g = 0 \), it is proven in [13], Lemma 33, that for all \( e_0 \) large enough (in fact \( e_0 \geq m = 2m_0 + 2 \)) and any \( e \geq e_0 \), a twist \( K^*(B) \) is balanced. Since in our case the slope is an integer, it follows that \( T_v|_{C_1} \) is perfect, hence semistable. Note that in general the condition the \( T_v|_{C_1} \) have integer slope is

\[ e \equiv ne_0 \mod n - 2m_0 - 2. \]

Of course one must also have \( e \geq e_0 \). Now we call a bidegree \( (e, e_0) \) admissible (for given \( g, n \)) if (6) holds and also

\[ e > e_0 + gn, e_0 > (g + 1)n. \]

Thus we have that for \( g = 0 \) \( T_v|_{C_1} \) is semistable whenever its bidgree is admissible.
Next consider the case \( g = 1 \). Suppose given an admissible bidgree \((e, e_0)\). Thus 
\[ e = ne_0 + kn_0 \] 
where \( n_0 := n - 2m_0 - 2 \). We consider a for \( C_1 \) a nodal genus-1 curve of the form

\[ C_{10} = C_{11} \cup_{p,q} C_{12} \]

where \( C_{11}, C_{12} \) have genus 0 and bidegree \((e_i, e_{i0})\) chosen as follows:

(i) \( e, e_0 \) even: 
\[ e_i = e/2, e_{i0} = e_0/2, i = 1, 2. \]

(ii) \( e, e_0 \) odd: 
\[ e_1 = (e + n)/2, e_{10} = (e_0 + 1)/2, e_2 = (e - n)/2, e_{20} = (e_0 - 1)/2. \]

(iii) \( e \) even, \( e_0 \) odd (note that then \( k \) is odd):
\[ e_{10} = \frac{e_0 + 1}{2}, e_1 = ne_{10} + \frac{k - 1}{2}n_0, e_{20} = \frac{e_0 - 1}{2}, e_2 = ne_2 + \frac{k + 1}{2}n_0. \]

(iv) \( e \) odd, \( e_0 \) even (note that then again \( k \) is odd):
\[ e_{10} = e_0/2, e_1 = ne_{10} + \frac{k + 1}{2}n_0, e_{20} = e_1/2, e_2 = ne_2 + \frac{k - 1}{2}n_0. \]

By admissibility for \( g = 1, e \geq e_0 + n \) hence \((e_i, e_{i0})\) are admissible (for \( g = 0 \)). Then by the case \( g = 0 \) already considered, \( T_v|_{C_{11}}, T_v|_{C_{12}} \) are semistable, hence so is \( T_v|_{C_{10}} \). Then as above \( C_{10} \) smooths out to a genus-1 curve of bidegree \((e, e_0)\) on which \( T_v \) is semistable.

Finally for \( g \geq 2 \) we use a 1-node reducible curve

\[ C_{10} = C_{11} \cup p C_{12} \]

with components of genera \( \lceil g/2 \rceil, \lfloor g/2 \rfloor \) adding up to \( g \) and \( e \) degree distribution as above, to again conclude \( T_v|_{C_{10}} \) is semistable and proceed as above.

\[ \square \]
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