In this paper we construct an invariant probability measure concentrated on 
$H^2(K) \times H^1(K)$ for a general cubic Klein-Gordon equation (including the case of
the wave equation). Here $K$ represents both the 3-dimensional torus or a bounded
domain with smooth boundary in $\mathbb{R}^3$. That allows to deduce some corollaries on
the long time behaviour of the flow of the equation in a probabilistic sense. We also
establish qualitative properties of the constructed measure. This work extends the
Fluctuation-Dissipation-Limit (FDL) approach to PDEs having only one (coercive)
conservation law.

Keywords: Klein-Gordon equation, wave equation, invariant measure, fluctuation-
dissipation, inviscid limit.

Classification: 28D05, 60H30, 35B40, 35L05, 35L71.

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1 Introduction

The Klein-Gordon (KG) equation

\[ \frac{\partial^2}{\partial t^2} u - \Delta u + m_0^2 u + u^3 = 0, \quad (t, x) \in \mathbb{R}_+ \times K, \tag{1.1} \]

is a model of evolution of a relativistic massive particle. Here \( u \) is a real-valued function, \( m_0^2 \in \mathbb{R} \) is the square of the mass of the particle and \( K \subset \mathbb{R}^3 \) is the physical space.

The KG equation is a Hamiltonian PDE, with the Hamiltonian

\[ E(u, \partial_t u) = \frac{1}{2} \int (|\partial_t u|^2 + |\nabla u|^2 + m_0^2 |u|^2) \, dx + \frac{1}{4} \int u^4 \, dx. \tag{1.2} \]

The natural phase space is then the Sobolev product space \( H^1(K) \times L^2(K) \) containing the vectors \( y = [u, \partial_t u] \).

Our purpose is to construct an invariant measure and to study some of its qualitative properties. The motivations of such a problem are discussed below as well as the difficulties of the question in the context of (1.1). Moreover, a panorama of applications coming from general ergodic theorems is presented in Section 2. Here, we consider both the periodic and the bounded domain setting.

Both on \( T^3 \) or on a domain \( D \) (with boundary conditions \( u|_{\partial D} = 0 \)), we denote by \( (\lambda_j, e_j)_{j \in \mathbb{N}} \) the couples (eigenvalue, eigenfunction) of the Laplacian operator \( -\Delta \). Remark that \( \lambda_0 = 0 \) only when the problem is posed on a torus and that, in both cases, \( (\lambda_j) \) is a sequence of non-negative real numbers increasing to infinity like \( j^2 \) (Weyl asymptotics). We define the Sobolev space of order \( m \in \mathbb{R} \) by

\[ H^m = \left\{ u = \sum_{j=0}^{\infty} u_j e_j : \| u \|_m^2 := \sum_{j=0}^{\infty} (m_0^2 + \lambda_j)^m u_j^2 < \infty \right\}, \]

where \( m_0^2 > -\lambda_0 \). The space \( H^0 \) is also denoted by \( L^2 \), and \( \| . \|_0 \) by \( \| . \| \). The inner product on \( H^m \) corresponding to the norm \( \| . \|_m \) is denoted by \( ( . , . )_m \) and \( ( . , . )_0 \) is simply written \( ( . , . ) \). We have the following embedding inequality:

\[ \| u \|_m^2 \geq (m_0^2 + \lambda_0)^{m-s} \| u \|_s^2 \quad \text{for any } m \geq s \text{ in } \mathbb{R}. \tag{1.3} \]

The product Sobolev space \( H^m \times H^n \) is denoted by \( \mathcal{H}^{m,n} \) and endowed with the norm defined, for any vector \( [u, v] \in \mathcal{H}^{m,n} \), by

\[ \| [u, v] \|_{m,n}^2 := \| u \|_m^2 + \| v \|_n^2, \]

and the corresponding inner product is denoted by

\[ ([u_1, v_1], [u_2, v_2])_{m,n} := (u_1, u_2)_m + (v_1, v_2)_n. \]
Set $\Delta_0 := \Delta - m_0^2$, then the equation and its Hamiltonian are rewritten as

$$\partial_{tt}^2 u - \Delta_0 u + u^3 = 0,$$

$$E(u, \partial_t u) = \frac{1}{2} \|u, \partial_t u\|_{H^1}^2 + \frac{1}{4} \int u^4.$$

Notice that

$$\|u\|_m = \|(-\Delta_0)^{m/2} u\|, \quad (u, v)_m = \langle (-\Delta_0)^{m/2} u, (-\Delta_0)^{m/2} v \rangle.$$

### 1.1 Invariant measures for PDEs: Motivations and approaches

Solving the Cauchy problem for a PDE is equivalent to specifying a phase space $E$ and a (semi-) group of continuous maps $\phi_t : E \to E$ which governs the evolution in time of the phase-vectors. The couple $(E, \phi_t)$ defines a dynamical system. One of the important questions in qualitative theory of PDEs is to describe the long time behavior of $\phi_t$. A Borel measure $\mu$ on $E$ is called invariant for $\phi_t$ if for any Borel set $\Gamma \subset E$, for any $t$, we have

$$\phi_t \ast \mu (\Gamma) = \mu (\phi_t^{-1} (\Gamma)) = \mu (\Gamma).$$

Existence of such a measure allows to draw some conclusions on long time properties for the system $(E, \phi_t)$ (see e.g. Birkhoff, Poincaré and von Neumann theorems in Section 2). The concept of invariant measure plays also an important role in probabilistic global wellposedness for PDEs by providing a way to control globally the induced flow.

There are, at least, two approaches to construct invariant measures; for finite-dimensional equations representing the evolution of a divergence free vector-field, the so-called Liouville theorem states that the Lebesgue measure defined on the associated phase space is preserved along the time. This result covers indeed the finite-dimensional Hamiltonian flows and their theory of Gibbs measures. The question of infinite-dimensional Gibbs measures (for Hamiltonian PDEs) is not directly implied by this general theorem, but is studied in many works with its help. The other result is given by the Krylov-Bogoliubov theorem for dynamical systems under some compactness assumptions. A method has been developed with use of this argument to approach more general PDEs.

Let us briefly present the general philosophy of two approaches of the PDEs invariant measures problem and compare them on some of their characteristic points.

**Gibbs measures theory for PDE.** For a PDE having a "nicely structured" conservation law $E(u)$, we can expect that, under proper definition, the expression $\langle e^{-E(u)} du \rangle$ could be an invariant measure. An approach consists in projecting the PDE on finite dimensional subspaces of increasing dimension. Then a sequence of ordinary differential equations are considered and the idea is to use the Liouville theorem. We get, then, a sequence (w.r.t. the dimension) of invariant measures (having a density w.r.t. Gaussian measures). An accumulation point is the measure we look for.

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1Let us mention the paper by Burq and Tzvetkov [BT14] introducing new approach to study probabilistic wellposedness which does not use an invariance property.
FDL measures theory. The Fluctuation-Dissipation-Limit approach consists in approximating the Hamiltonian dynamics by some kind of "compact" ones. The Krylov-Bogoliubov theorem provides then a sequence of invariant measures whose accumulation point could be invariant for the limiting equation. Namely, a damping term (given by a negative operator) and a stochastic forcing are added to the equation. The former should give the compactness in question while the latter is intended to maintain the evolution that the damping tends to attenuate:

"PDE = αDamping + s(α)(Forcing).

The function s will be chosen so that there will be a balance between the contributions of the added two terms and to ensure then the tightness of the sequence of constructed invariant measures in order to get the existence of the desired measure. Here again, a leading role is played by conservation laws.

Gibbs measures vs FDL measures. The first remarkable difference between the two approaches is that Gibbs measures reduce the regularity of the underlying conservation law, that is, their supports are less regular than the conservation law used in the construction (with reduction of 1/2+), this fact imposes systematically a threshold of regularity to the support. Whereas the FDL measures increase (by 1, if damped by Δ) the initial regularity; the construction does not impose directly a threshold on the "living space" of these measures. This makes the Gibbs measures particularly adapted to approach some spaces of low regularity and to give a probabilistic alternative to the Cauchy theory for PDEs. However, FDL measures can approach some high regularity spaces, seemingly inaccessible by the formers, to establish long time behavior properties of PDEs (see [Sy16]). The intermediate situation is common to both.

For Klein-Gordon related equations, Gibbs measures are constructed both in finite or in infinite volumes, see for instance [BT07, BB14, dS14, Xu14]. These measures concern radial solutions (in the 3D case) and are then concentrated on $H^{1/2-\lambda_0}$, where the nonlinearity would become problematic. In contrast with the loss of regularity inherent to the Gibbs measure approach, the FDL method proceeds by regularization. In that approach, the nonlinearity is still tractable even in a non radial context. However, as we will see it later on, the uniqueness of a coercive conservation law gives rise to some difficulties in the method.

1.2 Statement of the main result and comments

To present the main result of the paper, recall that $\lambda_0$ denotes the first eigenvalue of $-\Delta$ in both settings considered in this work.

Theorem 1.1. Let $m_0^2 > -\lambda_0$, then, in both settings, there is an invariant measure $\mu$ for (1.1) defined on $\mathcal{H}^{1,0}$ and satisfying:

- $\mu(\mathcal{H}^{2,1}) = 1$;
• $0 < \int_{\mathcal{H}^{1,0}} \|y\|_{2,1}^2 \mu(dy) < \infty$;

• there is $\sigma > 0$ such that

$$\int_{\mathcal{H}^{1,0}} e^{\sigma E(y)} \mu(dy) < \infty,$$

consequently $\mu$ enjoys a Gaussian control property w.r.t. the norm $\mathcal{H}^{1,0}$;

• the distribution under $\mu$ of the Hamiltonian $E(y)$ has a density w.r.t. the Lebesgue measure on $\mathbb{R}$.

Remark 1.2. In fact we have a family of invariant measures for \((1.1)\) on $\mathcal{H}^{2,1}$, one can see that after parametrising the diffusion constants associated to the approximation problem \((1.4)\).

The Poincaré recurrence theorem (see Section 2) implies

**Corollary 1.3.** For $\mu$-almost any $y = [u, v]$ in $\mathcal{H}^{2,1}$, there is a sequence $t_k$ going to infinity as $k \to \infty$ such that

$$\lim_{k \to \infty} \|\phi(t_k)y - y\|_{2,1} = 0.$$

Here $\phi_t$ denotes the flow of \((1.1)\) on $\mathcal{H}^{2,1}$.

Let us make some comments on the results. First, remark that by Sobolev embedding, the solutions concerned by our results are, in particular, continuous in the $x$ variable. Second, in the case where the equation is posed on a bounded domain, $\lambda_0$ is positive; then the massless case, i.e. the wave equation, is covered by our result. Moreover, $m_0$ is also allowed to be an imaginary number, in that situation \((1.1)\) is associated to a particle with imaginary mass. Such hypothetical particles, named tachyons, are used in some areas of theoretical physics.

To obtain these results, an additional difficulty compared to the earlier works is the fact that we know only one coercive conservation law for KG (in the case of the torus we have also the momentum which is not coercive), that implies a "lack of estimates". Notice also that the FDL approach was developed for Hamiltonian PDEs having at least two "good" conservation laws [Kuk04, KS04, KS12, Sy16]. The present paper is also intended to extend this approach to Hamiltonian PDEs having one conservation law.

In order to confront the "lack of conservation" present in our context, we introduce what we call almost conservation laws associated to \((1.1)\). These quantities play essentially the same role in the construction of an invariant measure as the two conservation laws, however, they cannot be used in studying its qualitative properties. That problem is solved by an approximation argument combined with the approach of [Shi11, KS12]. Notice that the concept of "almost conservation laws" is the main ingredient in the so-called $I$-method technique, overcoming the lack of conservation in the study of well-posedness and asymptotic behavior of dispersive PDEs (see e.g. [CKS+02, Tao06]). However, while in the $I-$method theory the modification consists in damping the high frequencies, in our situation we opt for an additive regular perturbation which accommodates better with our damping scheme. In Section 3 we define precisely our understanding of that concept, then we introduce two of such quantities and derive their respective dissipation rates whose statistical control along the time and the viscosity...
parameter takes the central place in our analysis. Notice also that an argument of modification of energy was developed in \cite{Tzv15, OT15} in the context of quasi-invariant measures theory for Hamiltonian PDEs.

We now describe the Fluctuation/Dissipation scheme that we apply to the Klein-Gordon equation in our work. Consider the stochastic PDE

$$\frac{\partial^2}{\partial t^2} u - \Delta_0 u + u^3 = \alpha \Delta_0 \frac{\partial}{\partial t} u + \sqrt{\alpha} \eta,$$

(1.4)

where

$$\eta(t, x) = \frac{d}{dt} \xi(t, x) = \frac{d}{dt} \sum_{m=0}^{\infty} a_m e_m(x) \beta_m(t).$$

Here $\beta_m$ are independent standard Brownian motions and $a = (a_m)$ is a sequence of real numbers. For $n \geq 0$, define the number

$$A_n = \sum_{m=0}^{\infty} a_m^2 \lambda_m^n.$$

The vector $y_t = [u, \partial_t u]$ is a random variable on a complete probability space $(\Omega, \mathcal{F}, P)$ with range in Sobolev spaces. We assume that the filtration $\mathcal{F}$ associated to $\xi$ is right continuous and augmented w.r.t. $(\mathcal{F}, P)$.

For given positive quantities $A, B$ satisfying $A \leq c_1 B$, we write $A \lesssim B$ or $A \lesssim c_1 B$. The vectors in $H^m \times H^n =: \mathcal{H}^{m,n}$ are denoted with the symbol $[,]$ while the symbol $(,)$ represents the inner product in $L^2$.

## 2 Ergodic theorems and some consequences

In this section we discuss some details about the PDE’s motivations of invariant measures theory via some general results from ergodic theory. We can also see the introduction of \cite{Tho16}.

### 2.1 Ergodic theorems

Consider the measurable dynamical system $(X, \phi, \mu)$ constructed from an evolution equation, here the probability measure $\mu$ is invariant under the flow $\phi$. In the case of a reversible dynamics (e.g. Hamiltonian equations), the transformations $(\phi_t)_{t \in \mathbb{R}}$ form a group and $\phi_t^{-1} = \phi_{-t}$, we adopt this hypothesis in the present section although all the results we are discussing here can be adapted to the semi-group case by classical ways.

In \cite{Koo31}, Koopman observes that the (a priori) nonlinear transformations $(\phi_t)$ induce linear ones on the space $L^2(X, \mu)$. These induced transformations $U_t$ are defined for any function $f : L^2(X, \mu) \to \mathbb{R}$ by

$$U_t f(w) = f(\phi_t w) \quad \forall w \in X.$$

The linearity and group property of $(U_t)$ are clear, and for any $t \in \mathbb{R}$, $U_t$ defines an isometry on $L^2(X, \mu)$. In fact

$$\|U_t f\|^2_{L^2} = \int_X |U_t f(w)|^2 \mu(dw) = \int_X |f(\phi_t w)|^2 \mu(dw).$$

A standard approximation (by simple functions) argument combined with the invariance of $\mu$ establishes the desired property. We also remark that $U_t^{-1} = U_{-t}$. A message
contained in Koopman’s observation is that, provided that an invariant measure is given, the ”nonlinear description” of the evolution of the states can be replaced by a ”linear
description” on the observables. We then pass from a nonlinear ”microscopic” study
to a linear ”macroscopic” one. In the latter setting, general theorems such as Von Neu-
mann and Birkhoff ergodic theorems, can be used to obtain some statistical properties
of the dynamics. Let us present a version of these theorems (for their proofs and more
results concerning them see [Kre85, Cou13]). Let $T > 0$, set the Birkhoff average
$$S_T f(w) = \frac{1}{T} \int_0^T U_t f(w) dt,$$
and the following invariants of the evolution
$$I_1 = \{ h \in L^2(X, \mu) : U_t h = h, \forall t \},$$
$$I_2 = \{ A \in \text{Bor}(X) : \phi_t^{-1} A = A, \forall t \},$$
where $\text{Bor}(X)$ is the Borel $\sigma$–algebra of $X$. $I_1$ and $I_2$ are related by the fact that
$$A \in I_2 \iff 1_A \in I_1.$$

**Theorem 2.1** (Von Neumann). For all $f \in L^2(X, \mu)$, we have, as $T \to \infty$,
$$S_T f \to P_{I_1} f \text{ in } L^2(X, \mu),$$
where $P_{I_1}$ denotes the orthogonal projection onto $I_1$.

**Theorem 2.2** (Birkhoff). For all $f \in L^1(X, \mu)$, we have as $T \to \infty$
$$S_T f \to E_{I_2} f \text{ in } L^1(X, \mu),$$
where $E_{I_2}$ denotes the conditional expectation w.r.t. $I_2$. This convergence holds also $\mu$–almost surely on $X$.

### 2.2 Consequences

This subsection is an ”adaptation” to continuous dynamical systems of the idea con-
tained in [Cou13] (Chapter 1, Exercise 9), one can also see the proof given in [Tao08]
and the discussion of [Tho16].

Let $A$ and $B$ be two Borel sets in $X$, $1_A$ and $1_B$ are the indicator functions of $A$ and $B$
respectively, we denote the orthogonal projection onto $I_1$ just by $P$. We have

**Proposition 2.3.**

$$\frac{1}{t} \int_0^t \mu(A \cap \phi_s^{-1} B) ds \to \langle P1_A, P1_B \rangle \text{ as } t \to \infty. \quad (2.1)$$

In particular,

$$\frac{1}{t} \int_0^t \mu(A \cap \phi_s^{-1} A) ds \to \| P1_A \|^2 \text{ as } t \to \infty, \quad (2.2)$$

and

$$\mu(A)^2 \leq \lim_{t \to \infty} \frac{1}{t} \int_0^t \mu(A \cap \phi_s^{-1} A) ds \leq \| P1_A \|_{L^2(X)} \sqrt{\mu(A)}. \quad (2.3)$$
Proof. We prove (2.2) by taking $B = A$ in (2.1). Now (2.3) is derived from (2.2) as follow
\[ \mu(A) = E_\mu 1_A = E_\mu E_{\phi_t}(1_A) = E_\mu (P_1A) = \|P_1A\|_{L^1} \leq \|P_1A\|_{L^2}. \]

On the other hand, we use the property $P^*P = P^2 = P$ and the Cauchy-Schwarz inequality to find
\[ \|P_1A\|_{L^2} \leq \|P_1A\|_{L^2}(X) \sqrt{\mu(A)}. \]

It remains to prove (2.1). To this end, we use the von Neumann ergodic theorem, which establishes convergence in the $L^2$-norm. It follows that we also have weak convergence, and therefore, as $t \to \infty$,
\[ \langle 1_A, \frac{1}{t} \int_0^t 1_B(\phi_s)ds \rangle \to \langle 1_A, P_1B \rangle = \langle P_1A, P_1B \rangle, \]
but
\[ \langle 1_A, \frac{1}{t} \int_0^t 1_B(\phi_s)ds \rangle = \frac{1}{t} \int_0^t \langle 1_A, 1_B(\phi_s) \rangle ds \]
\[ = \frac{1}{t} \int_0^t \langle 1_A, 1_{\phi^{-1}_s(B)} \rangle ds. \]

Now it is clear that
\[ \langle 1_A, 1_{\phi^{-1}_s(B)} \rangle = \mu(A \cap \phi^{-1}_s B). \]
That finishes the proof. \[ \square \]

We have the quantitative version of the Poincaré recurrence theorem

**Proposition 2.4** (Poincaré recurrence theorem).
\[ \lim_{t \to +\infty} \mu(A \cap \phi_{-1}^t A) \geq \mu(A)^2. \] (2.4)

Accordingly, if $\mu(A) > 0$, then $A \cap \phi_{-1}^t A$ is non empty for a sequence $(t_k)$ converging to infinity with $k$.

**Proof.** For $t > 1$, we write
\[ \frac{1}{t^2} \int_0^t \mu(A \cap \phi_{-1}^t A) ds \leq \frac{1}{t^2} \int_0^t \mu(A \cap \phi_{-1}^s A) ds + \frac{1}{t^2} \int_{s \geq t} \sup_{s \geq t} \mu(A \cap \phi_{-1}^s A) ds \]
\[ \leq \frac{1}{t} + \frac{1}{t} \sup_{s \geq t} \mu(A \cap \phi_{-1}^s A). \]

Passing to the limit $t \to \infty$ and using the left-hand inequality in (2.3), we obtain (2.4). \[ \square \]
3 Almost conservation laws for KG

In the context of the FDL approach, there is some kind of "algebraic structure" that a functional in hand has to respect to be fruitful in the construction of an invariant measure. Indeed, one needs uniform in $\alpha$ controls in the passage to the limit from the stochastic model towards the Hamiltonian PDE. In the estimation procedure, terms interacting with the damping (of order $\alpha$) are added with terms interacting with the forcing (of order $\alpha$ after taking the quadratic variation) and "order 1" terms. To get the needed uniformity, the order 1 terms must vanish under expectation w.r.t. an invariant measure. In the case of a conservation law, this requirement is satisfied because of the special "algebraic relations" that the latter shares with the equation. We call almost conservation law any (non preserved) functional that satisfies this requirement. Such a functional must depend on the damping model. Now we state a precise definition of our understanding of almost conservation law:

Consider a PDE
\[ \partial_t u = f(u), \] (3.1)
and a functional $V(u)$, then, formally, we have for any solution $u$ that
\[ \partial_t V(u) = (\nabla_u V(u), f(u)). \]

We call the quantity $(\nabla_u V(u), f(u))$ by the evolution rate of $V$ under the equation (3.1). It is clear that this term is zero iff $V$ is a conservation law for this equation. Now consider a linear perturbation of (3.1)
\[ \partial_t u = f(u) + \alpha L u, \] (3.2)
then
\[ \partial_t V(u) = (\nabla_u V(u), f(u)) + \alpha (\nabla_u V(u), Lu). \]

In the case where $V$ is a conservation law for the equation, then $\partial_t V(u)$ is of "order" $\alpha$, i.e. $\partial_t V(u)$ is of the form $\alpha h(u)$ where $h$ does not depend on $\alpha$.

A functional $V$ is called almost conservation law for (3.1) relatively to (3.2) if
- $V$ is not a conservation law,
- for a solution $u$ to (3.2), $\partial_t V$ remains of order $\alpha$.

**Remark 3.1.** The evolution rate $\partial_t V$ for an almost conservation law $V$ must vanish when $\alpha = 0$, therefore $V$ has to be a perturbation of a conservation law.

In the present work, the damping scheme for (1.1) we consider is
\[ \partial^2_t u - \Delta_0 u + u^3 = \alpha \Delta_0 \partial_t u, \quad \alpha \in (0, 1). \] (3.3)

Let us introduce the following quantities:
\[ G_1(y) = E(y) + \alpha \frac{(m_0^2 + \lambda_0)}{2} \int u \partial_t u + \frac{\alpha^2 (m_0^2 + \lambda_0)}{4} ||u||_1^2, \]
\[ G_2(y) = E(y) + \frac{\alpha}{2} \int \partial_t u \Delta_0 u + \frac{\alpha^2}{4} ||u||_2^2. \]
With use of (1.3) (for $m = 1, s = 0$) and $y$ is the vector $[u, \partial_t u]$. We remark that

$$G_1(y) \geq E(y) - \frac{\alpha^2 (m_0^2 + \lambda_0)^2}{4} \|u\|^2 - \frac{1}{4} \|\partial_t u\|^2 + \frac{\alpha^2 (m_0^2 + \lambda_0)}{4} \|u\|^2 \geq E(y) - \frac{1}{4} \|\partial_t u\|^2 \geq \frac{1}{4} E(y).$$

(3.4)

$$G_2(y) \geq E(y) - \frac{\alpha^2}{4} \|\Delta u\|^2 - \frac{1}{4} \|\partial_t u\|^2 + \frac{\alpha^2}{4} \|u\|_2^2 = E(y) - \frac{1}{4} \|\partial_t u\|^2 \geq \frac{1}{4} E(y).$$

(3.5)

Hence, in particular, the positivity of $G_1$ and $G_2$.

We have that the functionals $G_1$ and $G_2$ are almost conservation laws for (1.1) relatively to our dissipation scheme. The following controls (obtained in Proposition 3.3) express this fact:

$$G_1(y_1) + \alpha \int_0^t L_1(y_s) ds \leq G_1(y_0),$$

(3.6)

$$G_2(y_1) + \alpha \int_0^t L_2(y_s) ds \leq G_2(y_0) + \alpha C \int_0^t \|u\|_{L^6}^6 ds,$$

(3.7)

where $C$ is a constant independent of $\alpha$, and we set

$$L_1(y) = \frac{1}{2} \left\{ (m_0^2 + \lambda_0) \|u\|_1^2 + 2 \|\partial_t u\|^2 - (m_0^2 + \lambda_0) \|\partial_t u\|^2 + (m_0^2 + \lambda_0) \|u\|_4^4 \right\},$$

$$L_2(y) = \frac{1}{2} \left( 1^- \|u\|_2^2 + \|\partial_t u\|_1^2 \right).$$

and $1^- = 1 - \varepsilon$, with $\varepsilon > 0$ arbitrarily close to 0.

We give some useful estimates:

**Proposition 3.2.** For all $[u, v] \in H^{1, 1}$, we have

$$G_1(u, v) \leq \frac{2 + m_0^2 + \lambda_0}{2\kappa^2} L_1(u, v),$$

(3.8)

$$G_2(u, v) \leq \frac{5E(u, v) + L_2(u, v)}{4},$$

(3.9)

where $\kappa = \min(m_0^2 + \lambda_0, 1)$.

The proof of the above proposition is straightforward.

Denoting by $y_0$ the positive number $2\kappa^2 / (2 + m_0^2 + \lambda_0)$, we infer from (3.6) and (3.8) that, for any solution $y_t \in H^{1, 1}$ to (3.3), we have

$$G_1(y_t) \leq e^{-\gamma_0 t\alpha} G_1(y_0).$$

Taking this inequality to the power $p > 0$, we get

$$G_1^p(y_t) \leq e^{-\gamma_0 t\alpha} G_1^p(y_0).$$

Combining this with the embedding $H^1 \subset L^6$, we obtain

$$G_2(y_t) + \alpha \int_0^t L_2(y_s) ds \leq G_2(y_0) + \alpha G_1(y_0)^3.$$

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Proposition 3.3. We have the inequalities

\[ L_1(u,v) \geq \frac{\kappa}{2} (\| [u,v] \|_{1,1}^2 + \| u \|_{L^2}^4), \quad (3.10) \]
\[ L_2(u,v) \geq \frac{\delta}{2} \| [u,v] \|_{2,1}^2 \quad \text{for any } \delta < 1. \quad (3.11) \]

Proof. The bound (3.11) is straightforward and (3.10) is obtained with use of the lemma 3.4 below where we take \( m = 1, s = 0. \)

Lemma 3.4. For \( w \in H^m \) with \( m \in \mathbb{R}, \) we have for any \( s \leq m \)

\[ \| v \|_{m}^2 - \frac{(m_0^2 + \lambda_0)^{m-s}}{2} \| v \|_s^2 \geq \frac{1}{2} \| v \|_m^2 + \frac{(m_0^2 + \lambda_0)^{m-s}}{2} \| v \|_s^2. \quad (3.12) \]

Proof. Using (1.3), we have

\[ \| v \|_{m}^2 = \frac{1}{2} \| v \|_m^2 + \frac{1}{2} \| v \|_m^2 \geq \frac{1}{2} \| v \|_1^2 + \frac{(m_0^2 + \lambda_0)^{m-s}}{2} \| v \|_s^2. \]

That finishes the proof.

4 Global wellposedness for the damped KG

In this section we consider the following equation

\[ \partial_t^2 v - \Delta_0 v + (v + f)^3 = \alpha \Delta_0 \partial_t v, \quad (4.1) \]

where \( f \) satisfies

\[ \sup_{t \in [0,T]} \sup_{x \in K} |f(t,x)| < \infty \quad \forall T > 0. \quad K = D \text{ or } \mathbb{T}^3. \]

4.1 A-priori analysis

Choice of the spaces. Let us, first, consider the damped linear equation

\[ \partial_t^2 v - \Delta_0 v = \alpha \Delta_0 \partial_t v. \quad (4.2) \]

Both on the periodic or the bounded domain setting, the non-damped equation \( (\alpha = 0) \) preserves the following quantities:

\[ M_m(r,s) = \frac{1}{2} \| [r,s] \|_{m,m-1}^2 \quad m = 1,2. \]

Now, let us introduce the following "perturbed" versions:

\[ N_1(r,s) = \frac{1}{2} \| [r,s] \|_{1,0}^2 + \frac{\alpha (m_0^2 + \lambda_0)}{2} \int rs + \frac{\alpha^2 (m_0^2 + \lambda_0)}{4} \| r \|_1^2. \]
\[ N_2(r,s) = \frac{1}{2} \| [r,s] \|_{2,1}^2 - \frac{\alpha (m_0^2 + \lambda_0)}{2} \int s \Delta_0 r + \frac{\alpha^2 (m_0^2 + \lambda_0)}{4} \| r \|_2^2. \]
By a standard procedure, we see that a solution \([v, \partial_t v]\) of the damped linear equation \ref{eq:1.4} satisfies the following dissipation estimates:

\[
\begin{align*}
N_1(v, \partial_t v) + \frac{\alpha}{2} \int_0^t \left\{ (m_0^2 + \lambda_0)\|v\|_1^2 + 2\|\partial_t v\|_1^2 - (m_0^2 + \lambda_0)\|\partial_v v\|_2^2 \right\} ds &= N_1(v(0), \partial_t v(0)), \\
N_2(v, \partial_t v) + \frac{\alpha}{2} \int_0^t \left\{ (m_0^2 + \lambda_0)\|v\|_3^2 + 2\|\partial_t v\|_3^2 - (m_0^2 + \lambda_0)\|\partial_v v\|_1^2 \right\} ds &= N_2(v(0), \partial_t v(0)).
\end{align*}
\]

Then we use the inequality \ref{ineq:3.12} to infer the following controls:

\[
N_m(v, \partial_t v) + \frac{\alpha \kappa}{2} \int_0^t \|\partial_t v\|_{m,m}^2 ds \leq N_m(v(0), \partial_t v(0)) \quad m = 1, 2.
\]

In view of these estimates, the natural spaces for studying wellposedness of \ref{eq:1.4} are

\[
Z_m^T = C([0, T], H^{m,m-1}) \cap L^2_{loc}([0, T); H^{m,m}) \quad \text{for } T \in (0, +\infty],
\]

endowed with the norm defined by

\[
\|u, v\|_{Z_m^T} = \sup_{t \in [0, T]} \left( \|u\|_{m,m-1}^2 + \alpha \kappa \int_0^t \|u\|_{m,m}^2 ds \right)^{\frac{1}{2}}.
\]

Definition.

**Definition 4.1.** The equation \ref{eq:1.4} is said to be stochastically (globally) well-posed in \(H^{m,m-1}\) if for all \(T > 0\)

1. for any random variable \(u_0\) in \(H^{m,m-1}\) which is independent of \(\mathcal{F}_t\), we have, for almost all \(\omega \in \Omega\),
   
   (a) (Existence) there exists \(u := u^\omega \in \Lambda_T := C([0, T]; H^{m,m-1}) \cap L^2(0, T; H^{m,m})\) satisfying the following relation in \(H^{m,m-2}\):
   
   \[
   [u, \partial_t u] = [u_0, \partial_t u_0] + \int_0^t [\partial_t u, \Delta_0 u] + u^3 + \alpha \Delta_0 \partial_t u] ds + [0, 1] \zeta(t) \quad \text{for all } t \in [0, T],
   \]

   we denote this solution by \(y(t, u_0) := y^\omega(t, y_0)\), where \(y_0\) is the initial vector data.

   (b) (Uniqueness) if \(y_1, y_2 \in \Lambda_T\) are two solutions in the sense of \ref{eq:4.3}, then \(y_1 \equiv y_2\) on \([0, T]\).

2. (Continuity w.r.t. initial data) for almost all \(\omega\), we have
   
   \[
   \lim_{y_0 \to y'_0} y(\cdot, y_0) = y(\cdot, y'_0) \quad \text{in} \ \Lambda_T,
   \]

   here \(y_0\) and \(y'_0\) are deterministic data;

3. the process \((\omega, t) \mapsto y^\omega(t)\) is adapted to the filtration \(\sigma(y_0, \mathcal{F}_t)\).

Let \(V \subset H \subset V^*\) be three separable Hilbert spaces, with densely embeddings and where \(V^*\) is the dual of \(V\) w.r.t. \(H\). Then \((V, H, V^*)\) is called a Gelfand triple.

**Definition 4.2.** We say that the equation \ref{eq:1.4} has the Ito property on the Gelfand triple \((H^{m,m-2}, H^{m,m-1}, H^{m,m})\) if
1. it is stochastically wellposed on $H$;
2. the process $h := [\partial_t u, \Delta_0 u - u^3 + \alpha \Delta_0 \partial_t u]$ is $\mathcal{F}_t$-adapted and

$$\mathbb{P}\left( \int_0^t (\|y_t\|_{m,m}^2 + \|h_t\|_{m,m-2}^2) ds < \infty, \forall t > 0 \right) = 1, \|\zeta(t)\|_{m,m-1} < \infty. \quad (4.5)$$

To such an Ito process we can apply an Ito formula proved in Section A.7 (Theorem A.7.5 and Corollary A.7.6) of [KS12].

A-priori estimates for the nonlinear equation (4.1).

**Proposition 4.3.** Set $\gamma_1 = 1 + m_0^2 + \lambda_0$. For any solution $q_t = [v_t, \partial_t v_t]$ to (4.1) starting at $q_0 = [v_0, \partial_t v_0]$ with $G_1(q_0) < \infty$ and $G_2(q_0) < \infty$, we have

$$G_1(q_t) + \alpha \int_0^t L_1(q_s) ds \leq e^{\gamma_1 R(f) ds} \left( G_1(q_0) + \gamma_1 \int_0^t \|f\|^4_{L^4} ds \right), \quad (4.6)$$

$$G_2(q_t) + \alpha \int_0^t L_2(q_s) ds \leq e^{\gamma_1 R(f) ds} \left( G_2(q_0) + \frac{1}{4} \int_0^t (2\|f\|^4_{L^4} + \alpha C \|v + f\|^6_{L^6}) ds \right), \quad (4.7)$$

where $R(f) = 2(24(\|f\|_{L^8}^2 + \|f\|^2_{L^6}) + \|f\|^2_{L^8})$ and $C$ is universal.

**Proof.** Rewrite (4.1) into

$$\partial_t^2 v - \Delta_0 v + v^3 = \alpha \Delta_0 \partial_t v - 3v^2 f - 3vf^2 - f^3,$$

and $G_1(q)$ as $E(q) + I_4(q)$. Since $E(q)$ is preserved by KG, we have

$$\partial_t E(q) = -(\partial_t^2 v, 3v^2 f + 3vf^2 + f^3) + \alpha(\partial_t v, \Delta_0 \partial_t v)$$

$$= -3(f \partial_t v, vf + v^2) - (\partial_t v, f^3) - \alpha \|\partial_t v\|^2$$

$$\leq 3(||f||_{L^4} + ||f||^2_{L^6}) \|\partial_t v\| (||v|| + ||v||^2_{L^2}) + \|\partial_t v\| \|f\|^3_{L^6} - \alpha \|\partial_t v\|^2$$

$$\leq 3(||f||_{L^4} + ||f||^2_{L^6}) \left( \frac{1}{2} \|\partial_t v\|^2 + 2\|v\|^2 + 2\|v\|^4_{L^4} \right) + \frac{1}{2} \|\partial_t v\|^2 \|f\|^2_{L^6} + \frac{1}{2} \|f\|^4_{L^6}$$

$$- \alpha \|\partial_t v\|^2$$

$$\leq (24(\|f\|_{L^8}^2 + \|f\|^2_{L^6}) + \|f\|^2_{L^8}) E(q) + \frac{1}{2} \|f\|^4_{L^6} - \alpha \|\partial_t v\|^2.$$

Now

$$\partial_t I_4(q) = \frac{\alpha(m_0^2 + \lambda_0)}{2} (\partial_t v, \partial_t v) + \frac{\alpha^2 (m_0^2 + \lambda_0)}{4} ||v||^2$$

$$+ \frac{\alpha(m_0^2 + \lambda_0)}{2} (vf, \Delta_0 v - (v + f)^3 + \alpha \Delta_0 \partial_t v)$$

$$= \frac{\alpha(m_0^2 + \lambda_0)}{2} \|\partial_t v\|^2 + \frac{\alpha^2 (m_0^2 + \lambda_0)}{4} ||v||^2 - \frac{\alpha(m_0^2 + \lambda_0)}{2} (\|v||^2 + ||v||^4_{L^4})$$

$$- \frac{\alpha^2 (m_0^2 + \lambda_0)}{4} \|\partial_t v||^2$$

$$+ \frac{\alpha(m_0^2 + \lambda_0)}{2} (vf, 3v^2 + 3vf + f^2).$$
Let us notice that
\[ A = \frac{\alpha(m_0^2 + \lambda_0)}{2} \left\{ (24\|f\|_{L^\infty} + \|f\|_{L^6}^2) + \|f\|_{L^6}^2 E(q) + \frac{1}{2}\|f\|_{L^6}^4 \right\}. \]

Finally, we obtain
\[ \partial G_1(q) = \partial E(q) + \partial I_s(q) \leq -\alpha L_1(q) + \left( 1 + \frac{\alpha m_0^2 + \lambda_0}{2} \right) \left\{ (24\|f\|_{L^\infty} + \|f\|_{L^6}^2)G_1(q) + \|f\|_{L^6}^4 \right\}. \]

Applying Gronwall lemma we obtain \( (4.6) \).

To prove \( (4.7) \), let us compute
\[
\partial G_2(v) = (\partial v, \alpha \Delta_0 \partial v - (3v^2f + 3vf^2 + f^3)) - \frac{\alpha}{2} (\Delta_0 v, \Delta_0 v - (v + f)^3)) + \frac{\alpha}{2} \|\partial v\|_2^2 \]
\[ + \frac{\alpha}{2} \partial \|v\|_2^2 - \frac{\alpha}{2} (\Delta_0 u, \alpha \Delta_0 \partial u) \]
\[ = (\partial v, - (3v^2f + 3vf^2 + f^3)) - \frac{\alpha}{2} (\Delta_0 v, \Delta_0 v - (v + f)^3)) - \frac{\alpha}{2} \|\partial v\|_2^2 \]
\[ = I + II + III. \]

By the first part of the proof, we have
\[ I \leq (24\|f\|_{L^\infty} + \|f\|_{L^6}^2)E(q) + \frac{1}{2}\|f\|_{L^6}^4, \]

combining that with \( (4.9) \), we get
\[ I \leq 2(24\|f\|_{L^\infty} + \|f\|_{L^6}^2)G_2(q) + \frac{1}{2}\|f\|_{L^6}^4 =: R(f)G_2(q) + \frac{1}{2}\|f\|_{L^6}^4 \]

Now for any \( \varepsilon > 0 \), we have
\[ II \leq -\frac{\alpha}{2} \|v\|_2^2 - \frac{\alpha}{2} \|v\|_2 \|v + f\|_{L^6}^3 \]
\[ \leq -\frac{\alpha}{2} \|v\|_2^2 + \frac{\alpha}{2} \varepsilon \|v\|_2^2 + \frac{\alpha}{8\varepsilon} \|v + f\|_{L^6}^6. \]

Combining all this we obtain, for any \( \varepsilon > 0 \),
\[ \partial G_2(q_t) + \frac{\alpha}{2} \left( \|\partial v\|_2^2 + \|v\|_2 \|v + f\|_{L^6}^3 \right) \leq G_2(q_t)R(f) + \frac{\alpha}{8\varepsilon} \|v + f\|_{L^6}^6 + \frac{1}{2}\|f\|_{L^6}^4, \]

it remains to apply Gronwall lemma to arrive at the claim. \( \square \)

The following result will be used in the proof of Proposition \( 7.2 \).

**Proposition 4.4.** For any \( T > 0 \), any \( \varepsilon > 0 \), we have the a-priori estimate
\[
E(q_T) + \alpha \int_0^T \|\partial v\|_2^2 ds \leq e^{-\frac{\alpha T}{2}} + \frac{T}{\varepsilon} \|f\|_{L^2}^2 \left( E(q_0) + \frac{1}{2} \int_0^T \|f\|_{L^2}^2 + \frac{T}{\varepsilon} \|f\|_{L^4}^4 \right) ds. \tag{4.8}
\]
4.2 Existence and uniqueness for the nonlinear equation [4.1]

Setting \( w = \tilde{\partial} v \), we rewrite (4.1) into

\[
\partial_t \begin{pmatrix} v \\ \tilde{\partial} v \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \Delta_0 & \alpha \Delta_0 & 0 \\ 0 & 0 & B(v+f) \end{pmatrix} \begin{pmatrix} v \\ \tilde{\partial} v \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ (v+f)^3 \end{pmatrix}.
\]

(4.9)

We denote by \( S(t) \) the semi-group generated by \( A \) thanks to the Hille-Yosida theorem.

**Proposition 4.5.** For any \( q_0 = [v_0, \partial v_0] \in H^{1,0}, \) for any \( T > 0, \) there is a unique \( q(t, q_0) \in Z_T^1 \) satisfying (4.1) with the condition \( q(0, q_0) = q_0. \) Moreover, the map \( q_0 \rightarrow q(., q_0) \) is continuous with respect to the underlying norms.

**Proof.** For a given \( q_0 \in H^{1,0}, \) for \( T > 0, \) we set the map \( \psi : Z_T^1 \rightarrow Z_T^1: \)

\[
\psi q(t) = S(t)q_0 - \int_0^t S(t-s)B(v+f)ds.
\]

(4.10)

Let \( R > 0, \) consider the ball \( B_R \) in \( Z_T^1 \) centred at 0 and of radius \( R, \) we show by standard arguments the following estimates:

\[
\|\psi q\|_{Z_T^1} \leq \tau_{T,R} \|q_0\|_{Z_T^1},
\]

\[
\|\psi q_1 - \psi q_2\|_{Z_T^1} \leq \tau_{T,R} \|q_1 - q_2\|_{Z_T^1},
\]

where \( \tau_{T,R} \) decreases to 0 with \( T. \) Thus for arbitrary \( R, \) the time \( T = T(R) \) can be choosen so that \( \psi \) be a contraction as map from \( B_R \) to \( B_R, \) we have then a local in time
existence that we can globalize by iteration using the estimate \(4.6\).
For given two solutions \(v_1, v_2\) to \(4.1\), set \(w = v_1 - v_2\). Then \(w\) satisfies the equation
\[
\partial^2_{tt} w - \Delta_0 w = \alpha \Delta_0 w - w \{ (v_1 + f)^2 + (v_2 + f)^2 + (v_1 + f)(v_2 + f) \},
\]
it is not difficult to derive the following
\[
N_t (w, \partial_t w) \lesssim v_{1,2} f N_t (w(0), \partial_t w(0)). \tag{4.11}
\]
In fact \(4.11\) establishes, at the same time, the continuity (in space) for the solution.

5 The stochastic linear KG equation, exponential control

In this section, we present a treatment of the following equation:
\[
\partial_t [z, \partial_z z] = [\partial_t z, \Delta_0 z + \alpha \Delta_0 \partial_z z] + \sqrt{\alpha} [0, \eta]
= A [z, \partial_z z] + \sqrt{\alpha} \partial_t \xi.
\]
with null initial condition, supplemented with the Dirichlet condition in the case where the equation is considered on a bounded domain. Then the solution is given by the following stochastic convolution (see Section 4.1 for a definition and some properties):
\[
[z, \partial_z z](t) = \sqrt{\alpha} \int_0^t S(t - s) d\xi(s).
\]
In view of the discussion in Section 4.1, \([z, \partial_z z]\) belongs to \(H^{1,0}\) (resp. \(H^{2,1}\)) if \(A_0\) (resp. \(A_1\)) is finite. In what follows we suppose that \(A_1\) is finite. An exponential control for \([z, \partial_z z]\) is given for any \(t\) by the Fernique theorem, here we prove an exponential control on the time-averaged norm. Such a control will be used to prove Proposition 7.2.

Proposition 5.1. Let \(0 < \varepsilon \leq \kappa / (2A_1 \varepsilon)\), we have
\[
\mathbb{E} e^{\frac{\varepsilon}{2} t} \| [z, \partial_z z] \|_{L^2}^2 dt \leq 3, \tag{5.1}
\]
where \(\kappa = \min(1, m_0^2 + \lambda_0)\).

Proof. Step 1: The finite-dimensional approximating equation and estimation of the moments. Let \(P_N\) be the projection on the finite-dimensional space \(E_N := \text{span}\{e_0, \ldots, e_N\}\). Set \(z^N = P_N z, A^N = P_N A, \xi^N = P_N \xi, \Delta^N = P_N \Delta\) and \(A_{m,N} = \sum_{j=0}^N (m_j^2 + \lambda_j) m_j a_j^2\). Then we have
\[
\partial_t [z^N, \partial_z z^N] = A^N [z^N, \partial_z z^N] + \sqrt{\alpha} \partial_t \xi^N.
\]
It is a matter of direct verification that the norm \(f(z^N, \partial_z z^N) := \| [z^N, \partial_z z^N] \|_{L^2}^2\) is still preserved by the approximating equation in which we take \(\alpha = 0\). The function \(f\) belongs to \(C^2(E^N \times E^N, \mathbb{R})\), then we can apply the finite-dimensional Itô formula:
\[
df(z^N, \partial_z z^N) = \alpha \left( \frac{A_1 \varepsilon}{2} - \| [z^N, \partial_z z^N] \|_{L^2}^2 \right) dt + \sqrt{\alpha} \sum_{m=0}^N a_m (z^N \partial_z z^N - \Delta^N)^{1/2} \partial_z z^N \partial_m \beta_m.
\]
Now let $p > 1$, we have that $f^p$ still belongs to $C^2(E^N \times E^N, \mathbb{R})$, the Itô formula gives that
\[
d f^p(z^N, \partial_z z^N) = p f^{p-1} f + \frac{\alpha p (p-1)}{2} \sum_{j=0}^{N} \partial^2_{z^N} f (\Delta^N (\partial z^N)_{j} (\partial z^N - (\partial z^N)^{1/2} e_m)) dt
\]
\[=: (1) + (2)\]

\[
(1) = \alpha p \| [z^N, \partial_z z^N] \|^2_{2,1} \left( \frac{A_{1,N}}{2} - \| [z^N, \partial_z z^N] \|^2_{2,2} \right) dt + \theta(t),
\]
where $\theta(t)$ is the stochastic integrand and verifies $\mathbb{E} \int_0^t \theta(s) = 0$. We see that
\[
(1) \leq \alpha p \| [z^N, \partial_z z^N] \|^2_{2,1} \frac{A_{1,N}}{2} - \alpha p \| [z^N, \partial_z z^N] \|^2_{2,1} + \theta(t).
\]

On the other hand,
\[
(2) \leq \frac{\alpha p (p-1)}{2} \| [z^N, \partial_z z^N] \|^2_{2,1} A_{1,N}.
\]

Then, with use of the Young inequality,
\[
d f^p(z^N, \partial_z z^N) - \theta(t) \leq -\alpha p \| [z^N, \partial_z z^N] \|^2_{2,1} dt + \frac{\alpha p^2}{2} \| [z^N, \partial_z z^N] \|^2_{2,1} A_{1,N} dt
\]
\[\leq -\alpha p \| [z^N, \partial_z z^N] \|^2_{2,1} dt + \frac{\alpha p^2}{2} \| [z^N, \partial_z z^N] \|^2_{2,1} dt + \frac{\alpha A_{1} p^{p+1}}{2} dt.
\]

After integrating in $t$, taking the expectation, and using the Gronwall lemma, we get
\[
\mathbb{E} \| [z^N, \partial_z z^N] \|^2_{2,1} \leq \frac{A_1 p^p}{k^p}.
\]

**Step 2: Passage to the limit $N \to \infty$.** Using Fatou’s lemma, we get the estimations for $[z, \partial_z z]$.
\[
\mathbb{E} \| [z, \partial_z z] \|^2_{2,1} \leq \frac{A_1 p^p}{k^p}.
\] (5.2)

**Step 3: Exponential control.** Integrating in $t$, we find
\[
\mathbb{E} \left( \frac{1}{t} \int_0^t \| [z, \partial_z z] \|^2_{2,1} ds \right) \leq \frac{A_1 p^p}{k^p}.
\]

Thanks to Jensen’s inequality, we infer
\[
\mathbb{E} \left( \frac{1}{t} \int_0^t \| [z, \partial_z z] \|^2_{2,1} ds \right)^p \leq \frac{A_1 p^p}{k^p}.
\]

Now, let $0 < \varepsilon \leq \kappa/(2A_1 t)$, then we have
\[
\mathbb{E} \left( \frac{t}{p} \int_0^t \| [z, \partial_z z] \|^2_{2,1} ds \right)^p \leq \frac{p^p}{2^p e^p p!}.
\]

We recall that for any integer $p > 0$, we have that $p! \geq (\frac{p}{e})^p$, then we arrive at the claimed result.

**Remark 5.2.** One could use directly the infinite-dimensional Itô formula, the requirements of the latter are satisfied using the Ferniique theorem.
6 Global dynamics of the damped-perturbed KG and well structuredness

In what follow we use the following notations:
\[
\begin{align*}
    y_i &= [u_i, \partial_t u_i], \\
    g &= [\partial_t u, \Delta_0 u - u^3 + \alpha \Delta_0 \partial_t u] =: [g_1, g_2], \\
    \xi(t) &= \sum_{m=0}^{\infty} a_m e_m \beta_m(t), \\
    e_m &= [0, e_m].
\end{align*}
\]

In the rest of the paper we suppose that the quantities \( A_i = \sum_{j=0}^n a_m^2 (m_0^2 + \lambda_m)^i \) are finite for \( i = 0, 1 \).

**Proposition 6.1.** Let \( m \in \{1, 2\} \). The equation \( (\mathcal{H}^{m-2}, \mathcal{H}^{m-1}, \mathcal{H}^{m+2}) \) is well structured on \( \mathcal{H}^{1,0} \) in the sense of Definition 4.2.

**Proof.** First, we prove the stochastic global wellposedness on \( \mathcal{H}^{1,0} \) in the following steps:

**Step 1:** Splitting the problem. In order to solve the initial value problem of the equation \( (6.4) \), we split the latter into the following two equations:
\[
\begin{align*}
    \partial_t^2 z_{\alpha} - \Delta_0 z_{\alpha} &= \alpha \Delta_0 \delta z_{\alpha} + \sqrt{\alpha} \eta, \\
    \partial_t^2 v - \Delta_0 v + (v + z_{\alpha})^3 &= \alpha \Delta_0 \partial_t v.
\end{align*}
\]

Under the initial conditions \( [z_{\alpha}, \partial_t z_{\alpha}]_{|t=0} = [0, 0] \) and \( [v, \partial_t v]_{|t=0} = [u, \partial_t u]_{|t=0} \), we have that \( u = v + z_{\alpha} \) solves \( (1.4) \). Therefore it suffices to solve each of these two equations.

**Step 2:** The linear stochastic problem. The linear equation \( (6.1) \), supplemented by the initial data \( z_{\alpha}|_{t=0} = \partial_t z_{\alpha}|_{t=0} = 0 \), is solved by the following stochastic convolution
\[
[z_{\alpha}, \partial_t z_{\alpha}]_{(t)} = \sqrt{\alpha} \int_0^t S(t-s) d\xi(s).
\]

The solution \( [z_{\alpha}, \partial_t z_{\alpha}] \) is almost surely in \( Z_1^{\alpha} \) when \( A_0 \) is finite (see Subsection 4.1 for properties of the stochastic convolution).

**Step 3:** The nonlinear deterministic problem. The initial value problem of the nonlinear equation \( (6.2) \) is solved by a deterministic way. Suppose \( A_0 \) finite. Fix \( \omega \) for which \( z_{\alpha} \in Z_1^{\alpha} \), we can then take \( f = z_{\alpha} \) in \( (4.1) \) and the problem is solved in view of Proposition 4.5.

**Step 4:** Progressive measurability and continuity. By the definition of \( z_{\alpha} \) and \( v \), we have that the solution \( u = v + z_{\alpha} \) is \( \sigma(u_0, \partial_t u_0, F) \)-adapted and is continuous in time (with values in \( H^1 \)). Then using the Proposition 1.13 of [KS91], we get the progressive measurability for \( u \).

The continuity (w.r.t. initial data) property follows that established for the "nonlinear solution" \( v \), since the "linear solution" \( z \) does not depend on the initial data.

Now, we prove that the solution \( y_i \) satisfies the assumptions \( (4.5) \) on the Gelfand triples \( (\mathcal{H}^{m-2}, \mathcal{H}^{m-1}, \mathcal{H}^{m+2}) \) for \( m = 1, 2 \).

1. \( y_i \) is a Itô process in \( \mathcal{H}^{m-2} \) since the process \( g(y) := [\partial_t u, \Delta_0 u - u^3 + \alpha \Delta_0 \partial_t u] \)
   is \( F_t \)-adapted, and we infer from Proposition 4.3 the following
   \[
   \mathbb{P} \left( \int_0^t \|g(y_s)\|^2_{\mathcal{H}^{m-2}} ds < \infty \quad \text{for all } \ t \geq 0 \right) = 1.
   \]

Notice that this kind of decomposition appears in literature of stochastic PDEs (see e.g. [KS12]) and of dispersive PDEs (see e.g. [BTT14] for the context of cubic wave equation).
2. The quantities \( A_0, A_1 \) are finite by assumption;
3. Again, we use the estimates of Proposition 4.3 to see that
   \[
P\left( \int_0^t \| y_s \|_{\mathcal{H}^{m,m}}^2 ds < \infty \text{ for all } t \geq 0 \right) = 1.
   \]

The proof is complete.

In view of the wellposedness established above, we are able to define the flow map of (1.4)
\[
\phi^\alpha_t w = y^\alpha(r, w),
\]
the transition function
\[
P^\alpha_t(w, E) = P(\phi^\alpha_t w \in E),
\]
and the Markov semigroups, with use of the Feller property induced by the continuity of the flow,
\[
\mathcal{Q}^\alpha_t f(w) = \int f(v)P^\alpha_t(w, dv), \quad C_b(\mathcal{H}^{1,0}) \to C_b(\mathcal{H}^{1,0});
\]
\[
\mathcal{Q}^{\alpha*}_t v(E) = \int P^\alpha_t(v, E)v(dv), \quad p(\mathcal{H}^{1,0}) \to p(\mathcal{H}^{1,0}),
\]
where \( p(H) \) is the set of probability measures on \( H \). The functions \( \mathcal{Q}^\alpha_t \) and \( \mathcal{Q}^{\alpha*}_t \) verify the duality relation
\[
(\mathcal{Q}^\alpha_t f, v) = (f, \mathcal{Q}^{\alpha*}_t v).
\]

7 Stationary measures for the damped-perturbed KG

We suppose that \( A_0 \) is finite and recall the notation \( \gamma_0 = \frac{2\kappa^2}{2\gamma + \lambda_0} \).

**Theorem 7.1.** For any \( \alpha \in (0, 1) \), the problem (1.4) admits an invariant measure \( \mu_\alpha \) defined concentrated on \( \mathcal{H}^{2,1} \). The invariant measures \( \mu_\alpha \) of (1.4) satisfy the following properties

1. For any \( \alpha \in (0, 1) \)
   \[
   \int_{\mathcal{H}^{1,0}} L_1(y)\mu_\alpha(dy) = \frac{A_0}{2}. \tag{7.1}
   \]

2. For any \( p \geq 1 \), we have
   \[
   \int_{\mathcal{H}^{1,0}} G_1^p(y)\mu_\alpha(dy) \leq \left( \frac{2pA_0}{\gamma_0} \right)^p. \tag{7.2}
   \]

3. There is \( C \) independent of \( \alpha \) such that
   \[
   \int_{\mathcal{H}^{1,0}} \| y \|_{\mathcal{H}^{2,1}}^2\mu_\alpha(dy) \leq C. \tag{7.3}
   \]
7.1 Step 1: Statistical controls

We would like to apply Itô’s formula ((Theorem A.7.5 and Corollary A.7.6 of [KS12]))
to the functionals $G_1$ w.r.t. the triple $(\mathcal{H}^{1,-1}, \mathcal{H}^{1,0}, \mathcal{H}^{1,1})$ and $G_2$ w.r.t. $(\mathcal{H}^{2,0}, \mathcal{H}^{2,1}, \mathcal{H}^{2,2})$.

The polynomial structure of these functionals allows to fill the conditions of Theorem A.7.5 of [KS12]. Here, we wish to apply the Itô formula with a deterministic time, then we have to verify the condition of the Corollary A.7.6 of the same book, namely the finiteness of the quadratic variation of each of these functionals.

**Proposition 7.2.** Suppose that $\mathbb{E}E_1^t(y_0) < \infty$ with $t > 1$. Then the quantities $G_1(y_t)$ and $G_2(y_t)$ have finite quadratic variations on any finite interval.

**Proof.** We have for $i = 1, 2,$

$$\sum_{m \geq 0} a_m^2 \mathbb{E} \int_0^t |\nabla G_i(y; \hat{e}_m)|^2 ds \lesssim \sum_{m \geq 0} a_m^2 \mathbb{E} \int_0^t |(\partial_t u + \frac{a}{2} (\Delta_0)^{i-1} u; e_m)|^2 ds$$

$$\lesssim \sum_{m \geq 0} a_m^2 \mathbb{E} \int_0^t \left\{ |(\partial_t v + \frac{a}{2} (\Delta_0)^{i-1} v; e_m)|^2 + |(\partial_t z + \frac{a}{2} (\Delta_0)^{i-1} z; e_m)|^2 \right\} ds$$

$$\lesssim \sum_{m \geq 0} a_m^2 \mathbb{E} \int_0^t E(q_i) ds + \mathbb{E} \int_0^t \left\{ \|z, \partial_t z\|_{L_{i-1,0}}^2 \right\} ds.$$

One see, with use of estimates (5.2), that

$$\mathbb{E} \int_0^t \left\{ \|z, \partial_t z\|_{L_{i-1,0}}^2 \right\} ds < \infty \text{ for any } t \geq 0.$$

Now we use estimate (4.8), that, for any $\varepsilon > 0,$

$$\mathbb{E} \int_0^t E(q_i) ds \leq \int_0^t e^{\frac{2^{\frac{1}{1+\varepsilon}} t}{2^{\frac{1}{1+\varepsilon}} t}} \mathbb{E} \left[ e^{|\tilde{\beta}||f||_{L^6}^2} \left( E(q_0) + \frac{1}{2} \int_0^t \left\| \frac{1}{2} \|f||_{L^6}^6 + \frac{T}{\varepsilon} \|f||_{L^4}^4 \right\| dr \right) \right] ds.$$

By the Young inequality, we have for any $\varepsilon > 0,$

$$\mathbb{E} \int_0^t E(q_i) ds \leq \int_0^t e^{\frac{2^{\frac{1}{1+\varepsilon}} t}{2^{\frac{1}{1+\varepsilon}} t}} \mathbb{E} \left[ e^{\frac{1}{1+\varepsilon} \frac{1}{t^{\frac{1}{1+\varepsilon}}} \|f||_{L^6}^6} + \left( G_1(q_0) + \frac{1}{2} \int_0^t \left\| \frac{1}{2} \|f||_{L^6}^6 + \frac{T}{\varepsilon} \|f||_{L^4}^4 \right\| dr \right) \right] ds.$$

One uses the estimate (5.2) and the Jensen inequality to bound $\mathbb{E}R(s)$ by in $C(1 + s^{\frac{1}{1+\varepsilon}}).$

And, for small enough $\varepsilon > 0$ (here $\varepsilon$ depends indeed on the infinitesimal parameter entering the definition of $1^{\frac{1}{1+\varepsilon}}$), the estimate (5.1) and the embedding $H^2 \subset L^6$ allow to get the bound

$$\mathbb{E} e^{\frac{1}{1+\varepsilon} \frac{1}{t^{\frac{1}{1+\varepsilon}}} \|f||_{L^6}^6} \leq 3.$$

Then we get

$$\mathbb{E} \int_0^t E(q_i) ds \leq \int_0^t e^{\frac{2^{\frac{1}{1+\varepsilon}} t}{2^{\frac{1}{1+\varepsilon}} t}} (1 + s^{\frac{1}{1+\varepsilon}}) ds < \infty \text{ for all } t \geq 0.$$

The proof is finished. \(\square\)

Taking the inequality (4.8) to the power $p > 1$ and repeating the above argument, we show that $G_1^p$ satisfies (??) as soon as $\mathbb{E}E_{R}^{p+}(y_0)$ is finite.
Proposition 7.3. Let $\alpha \in (0,1)$. Suppose $\mathbb{E}G_1^p(y_0) < +\infty$ for any $p > 1$. Then the solution $y_t$ of (1.4) starting at $y_0$ satisfies the following

$$\mathbb{E}G_1^p(y_t) \leq e^{-\alpha n^2 t} \mathbb{E}G_1^p(y_0) + 2 \left( \frac{2pA_0}{\gamma_0} \right)^p.$$  \hspace{1cm} (7.5)

Proof. For a functional $F(y)$, we denote by $\nabla_u F$ and $\nabla_v F$ the derivatives w.r.t. the first and the second variable respectively. Let us compute

$$\nabla_y G_1(y,g) = \nabla_y G_1(y,g_1) + \nabla_y G_1(y,g_2) = \nabla_y E(y,g_1) + \nabla_y E(y,g_2) + I + II$$

$$I = \frac{\alpha}{2} \|
abla u\|^2,$$

$$II = \frac{\alpha}{2} \|u\|_1^2 + \frac{\alpha^2}{4} \|
abla u\|^2 = -\frac{\alpha}{2} (\|u\|^2 + \|u\|_1^4),$$

thus

$$\nabla_y G_1(y,g) = \frac{\alpha}{2} (m_0^2 + \lambda_0) \|u\|_1^2 + 2 \|\nabla u\|^2 - (m_0^2 + \lambda_0) \|\nabla u\|^2 + (m_0^2 + \lambda_0) \|u\|_1^4) = -\alpha L_1(y).$$

On the other hand

$$\nabla_y^2 G_1(y,g) = \nabla_y^2 G_1(y,0) + \nabla_y^2 G_1(y,e_m) = (e_m,e_m) = 1.$$ 

Then, by Itô formula (see [KS12], Theorem A.7.5 and Corollary A.7.6), we have

$$dG_1(y_t) = -\alpha L_1(y)dt + \frac{\alpha}{2} A_0 dt + \Theta_1(t),$$

where

$$\Theta_1(t) = \sum_{m=0}^\infty a_m \nabla_y G_1(y,e_m) d\beta_m(t) = \sum_{m=0}^\infty a_m (\nabla u + \frac{\alpha}{2} u, e_m) d\beta_m(t).$$

Remark that, by an Itô integral property, $\mathbb{E} \int_0^t \Theta_1(s) \, ds = 0$, then we arrive at (7.4). Now, let $p > 1$, we have, by Itô formula,

$$dG_1^p(y) = pG_1^{p-1}(y) \, dt + \frac{\alpha p(p-1)}{2} \sum_{m=0}^\infty a_m^2 G_1^{p-2}(y) (\nabla y G_1(y,e_m))^2 dt$$

$$= pG_1^{p-1}(y) \, dt + \frac{\alpha p(p-1)}{2} \sum_{m=0}^\infty a_m^2 G_1^{p-2}(y) (\nabla u, e_m)^2 dt.$$ 

whence it follows that

$$\mathbb{E}G_1^p(y_t) + \int_0^t \mathbb{E}f_\alpha(s) \, ds \leq \mathbb{E}G_1^p(y_0),$$

where

$$f_\alpha(t) = p \frac{\alpha}{2} G_1^{p-1} (2L_1(y) - A_0) - \frac{\alpha p(p-1)}{2} G_1^{p-2}(y) \sum_{m=0}^\infty a_m^2 (\nabla u, e_m)^2.$$


We have, with the use of the inequalities (3.8), (3.4) and the inequality $2p^2 - 2p \leq 2p^2 - p/2$ for $p \geq 0$,

$$f_a(t) \geq p \frac{\alpha}{2} G_1^{p-1}(y) (2L_1(y) - A_0) - 2p(p - 1)\alpha A_0 G_1^{p-1}(y)$$

$$\geq p\alpha G_1^{p-1}(y)L_1(y) - 2\alpha p^2 A_0 G_1^{p-1}(y)$$

$$\geq \alpha \gamma_0 p G_1^p(y) - 2\alpha p^2 A_0 G_1^{p-1}(y)$$

$$\geq \alpha \gamma_0 p G_1^p(y) - \frac{2p p^{p+1}}{\gamma_0} A_0^p.$$ 

Finally

$$E G_1^p(y) + \frac{\alpha \gamma_0 p}{2} \int_0^t E G_1^p(y) ds \leq E G_1^p(y_0) + \alpha \frac{2p p^{p+1}}{\gamma_0} A_0^p t.$$ 

Thus we get (7.5) after applying the Gronwall lemma. \hfill \Box

**Proposition 7.4.** Let $\alpha \in (0, 1)$. Suppose $E G_2^1(y_0) < +\infty$. Then the solution $y_i$ of (1.4) starting at $y_0$ satisfies the following

$$E G_2(y_i) + \alpha \int_0^t E L_2(y_i) ds \leq E G_2(y_0) + c_1 \alpha \left( t + \int_0^t E \|u_i\|_{L^2} ds \right), \quad (7.6)$$

$$E G_2(y_i) \leq e^{-c_2 t} E G_2(y_0) + c_3, \quad (7.7)$$

where $c_i, \ i = 1, 2, 3$ are universal positive constants.

We remark in (7.6) we need to control $E \|u_i\|_{L^6}$. But one can see that this control holds true if one combines the embedding $H^1 \subset L^6$ and the estimate (7.5).

**Proof.** Set $J_a(y) = -\frac{D}{2} \int \partial_t u \Delta u + \frac{\alpha}{2} \|u\|_2^2$ so that $G_2 = E + J_a$. In order to apply the Itô formula used in the previous estimations, let us compute

$$\nabla_y G_2(y, dy) = \nabla_y E + \nabla_y J_a(y, g_1) + \nabla_y J_a(y, g_2) + \Theta_2 =: \nabla_y E + \Theta_1 + \Theta_2,$$

where

$$\Theta_2(t) = \sqrt{\alpha} \sum_{m \geq 0} a_m (\partial_t u - \frac{\alpha}{2} \Delta u, e_m) dB_m(t).$$

We see without any difficulty that

$$\nabla_y E(y, g) = -\alpha \|\partial_t u\|_1^2.$$ 

On the other hand, we have

$I = \frac{\alpha}{2} \|\partial_t u\|_1^2 + \frac{\alpha^2}{4} \|\partial_u\|_2^2$.

$II = -\frac{\alpha}{2} (\Delta u, \Delta u - u^3 + \alpha \Delta u \partial_t u)$

$$= -\frac{\alpha}{2} \|u\|_3^2 + \frac{\alpha}{2} (\Delta u, u^3) - \frac{\alpha^2}{3} \|\partial_u\|_2^2$$

$$\leq -\frac{\alpha}{2} (1 - \varepsilon) \|u\|_3^2 + \frac{\alpha}{8\varepsilon} \|u\|_{L^6}^6 - \frac{\alpha^2}{4} \|\partial_u\|_2^2.$$

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Summarizing all this, we have
\[ \nabla_y G_2(y, dy) \leq -\alpha_2(\|\partial_t u\|_1^2 + (1-\varepsilon)\|u\|_2^2) + \alpha_2 \frac{\|u\|_6^6}{8\varepsilon} + \Theta_2, \]
where \( \Theta_2 \) satisfies \( \mathbb{E} \int_0^T \Theta_2(s) = 0 \) for any \( t > 0 \). Now
\[ \nabla^2 y G_2(y, \hat{e}_m) = 1, \]
then for any \( \varepsilon \in (0, 1) \), we have
\[ \mathbb{E} G_2(y_t) + \alpha_2 \int_0^t \mathbb{E}(\|\partial_t u\|_1^2 + (1-\varepsilon)\|u\|_2^2) ds \leq \mathbb{E} G_2(y_0) + \alpha_2 \int_0^t \mathbb{E}(\|u\|_6^6) ds + \frac{\alpha_2}{2} A_0 t, \]
that is \((7.6)\). To prove \((7.7)\), we remark that
\[ L_2 \gtrsim G_2 - E. \]
Injecting that into \((7.6)\), we find
\[ \mathbb{E} G_2(y) + c_2\alpha \int_0^t \mathbb{E}(\|\partial_t u\|_1^2 + (1-\varepsilon)\|u\|_2^2) ds \leq \mathbb{E} G_2(y_0) + \alpha \int_0^t \mathbb{E}(E(u) + \|u\|_6^6) ds. \]
Then we use \((7.5)\) and Gronwall lemma to conclude.

From the estimate \((7.6)\), we infer, for \( y_0 = 0 \) a.e. that
\[ \frac{1}{t} \int_0^t \mathbb{E}(\|y_s\|_{\mathcal{H}^1,0}^2) ds \leq C, \quad (7.8) \]
where \( C \) is independent of \( t \).

### 7.2 Step 2: Stationary measures and their estimations

**Existence of a stationary measure on \( \mathcal{H}^{1,0} \) for any fixed \( \alpha \).** Let \( \delta_0 \) be the Dirac measure on \( \mathcal{H}^{1,0} \) concentrated at \( 0 \). We define the time-averaged measures
\[ \bar{\lambda}_T = \frac{1}{T} \int_0^T \mathbb{P}_{T}^{\alpha} \delta_0 dt. \quad (7.9) \]

Let us fix \( \alpha \). Now, we are going to establish (uniform) tightness of the sequence \( \{\bar{\lambda}_T, T > 0\} \). Let \( R > 0 \) and \( B_R \) be the ball of \( \mathcal{H}^{2,1} \) of center \( 0 \) and radius \( R \). \( B_R \) is compact in \( \mathcal{H}^{1,0} \) and, thanks to \((7.5)\), Chebychev inequality implies
\[ \bar{\lambda}_T(B_R) \geq 1 - \frac{C}{R^2}, \]
where \( C \) is independent of \( T \). Then \( \{\bar{\lambda}_T, T \} \) is tight on \( \mathcal{H}^{1,0} \). By Prokhorov theorem, there is an accumulation point on \( \mathcal{H}^{1,0} \), then the Bogoliubov-Krylov argument establishes that the latter is invariant for \((1.1)\). We denote this stationary measure by \( \mu_\alpha \).
Uniform (in \( \alpha \)) estimates for the measures \( \mu_\alpha \). Now we prove the estimates \((7.1)\), \((7.2)\), \((7.3)\). Let \( R \) be a positive number. Consider a \( C^\infty \) function \( \chi_R \) on \( \mathbb{R} \) defined by
\[
\chi_R(x) = \begin{cases} 
1, & \text{if } x \leq R, \\
0, & \text{if } x \geq R + 1.
\end{cases}
\]

Let us prove \((7.2)\), the proof of \((7.3)\) is similar and \((7.1)\) follows the finiteness of \( E G_1(y) \) and \((7.4)\). For any \( p \geq 1 \), we have
\[
\int_{\mathbb{H}^{1,0}} G_1^p(y) \chi_R(\|y\|_{1,0}) \mu_\alpha(dy) = \int_{\mathbb{H}^{1,0}} \mathbb{E}[G_1^p(y(t,v)) \chi_R(\|y(t,v)\|_{1,0})] \mu_\alpha(dv). 
\]
(7.10)
Passing to the limit \( t \to \infty \) in \((7.10)\) with use of \((7.5)\), we arrive at
\[
\int_{\mathbb{H}^{1,0}} G_1^p(y) \chi_R(\|y\|_{1,0}) \mu_\alpha(dy) \leq 3 p^2 A_0^p,
\]
and it remains to apply the Fatou lemma to finish. Now the control on the \( \mathcal{H}^{2,1} \) norm implies
\[
\mu_\alpha(\mathcal{H}^{2,1}) = 1.
\]

8 Invariant measure for KG and estimates

**Theorem 8.1.** There is an accumulation point \( \mu \) of \( \{\mu_\alpha\} \) as \( \alpha \to 0 \), in the weak topology of \( \mathcal{H}^{1,0} \), satisfying the following properties:
1. \( \mu \) is invariant under the flow of the KG equation \((1.1)\) defined on \( \mathcal{H}^{1,0} \);
2. \( \mu(\mathcal{H}^{2,1}) = 1 \); (8.1)
3. \( \int_{\mathbb{H}^{1,0}} \|y\|_{\mathcal{H}^{2,1}}^2 \mu(dy) < +\infty \); (8.2)
4. for any \( p \geq 1 \),
\[
\int_{\mathbb{H}^{1,0}} E_p(y) \mu(dy) \leq 2 \left( \frac{2 p A_0}{\gamma_0} \right)^p,
\]
(8.3)
where \( A_0 = \sum_{m=0}^\infty a_m^2 \) and \( \gamma_0 = \frac{2 \min(1, m_0^2 + \lambda_0)}{2 + m_0^2 + \lambda_0} \).

In what follows \( B_R \) denotes the ball in \( \mathcal{H}^{2,1} \) centred at zero and of radius \( R \), unless otherwise specified. \( \phi_\alpha^w \) and \( \phi^w \) denote respectively the flows of \((1.2)\) and \((1.1)\) on \( \mathcal{H}^{1,0} \). The Markov semi-groups associated to \((1.1)\) are denoted by \( \Psi_t \) and \( \Psi_t^w \).

For \( w \in \mathcal{H}^{1,0} \), \( v(t,w) \) and \( u(t,w) \) are the corresponding solutions to the nonlinear equation \((6.2)\) and the KG equation \((1.1)\). Set \( f = v - u \), then \( f|_{t=0} = 0 \). We have

**Lemma 8.2.** Let \( T, R \) and \( r \) be positive numbers, we have
\[
\sup_{t \in [0,T]} \sup_{w \in B_R} \mathbb{E} \left[ E(f, \partial_t f) 1_{\|z\|_{\mathcal{H}^{2,1}} \leq \sqrt{\alpha r}} \right] = O_{T,R,r}(\alpha). \] (8.4)
Proof. Consider the following equations
\[
\begin{align*}
\partial_t^2 v - \Delta_0 v + v^3 &= \alpha \Delta_0 \partial_t v - 3v^2 z_\alpha - 3v z_\alpha^2 - z_\alpha^3, \\
\partial_t^2 u - \Delta_0 u + u^3 &= 0.
\end{align*}
\]
Taking the difference between the above two equations, we get
\[
\begin{align*}
\partial_t^2 f - \Delta_0 f + f^3 &= \alpha \Delta_0 \partial_t v + 3u^2 v - 3v^2 z_\alpha - 3v z_\alpha^2 - z_\alpha^3
\end{align*}
\]
Let \( t \in [0,T] \) and \( w \in B_R \). Thanks to the preservation of \( E \) by the solution of (1.1) and the results of Proposition 4.3, we get
\[
\begin{align*}
\partial_t E(f, \partial_t f) &= -3(f \partial_t f, u v) + \alpha(\partial_t f, \Delta_0 \partial_t v) - (\partial_t f, 3v^2 z_\alpha + 3v z_\alpha^2 + z_\alpha^3) \\
&\leq \|\partial_t f\|_2 \|f\|_2 \|uv\|_2 + \alpha \|\partial_t f\|_1 \|\partial_t v\|_1 + \|\partial_t f\|_3 \|\partial_t f\|_3 + O_{R,r}(\sqrt{\alpha}) \\
&\leq (\|\partial_t f\|_2^2 + \|f\|_2^2) \|uv\|_2 + \|\partial_t v\|_2 + 1 + O_{R,r}(\alpha) \\
&\leq E(f, \partial_t f) \|\partial_t f\|_2^2 + \|\partial_t f\|_2 + 1 + O_{R,r}(\alpha) \\
&\leq T_{R,r}(1 + L_2(q)) E(f, \partial_t f) + \alpha.
\end{align*}
\]
Now Proposition 4.3 ensures the boundedness of \( \int_{0}^{T} L_2(q) ds \) where \( q = [v, \partial_t v] \). We apply Gronwall lemma to get the claimed result.

Since for a.a. \( \omega z_\alpha \) converges to 0 as \( \alpha \to 0 \), we have

**Corollary 8.3.** For any \( T, R, r > 0 \) and almost all \( \omega \in \Omega \),
\[
\sup_{t \in [0,T]} \sup_{w \in B_R} \mathbb{E} \left[ \| \phi_t^w w - \phi_w w \|_{L_{1,0}}^2 1_{\{\|z_\alpha\|_{L^2} \leq \sqrt{\alpha} r\}} \right] = O_{T,R,r}(\alpha).
\]

**Proof of Theorem 8.1.** The family \( \{\mu_\alpha\} \) is tight on \( \mathcal{H}_{1,0} \) w.r.t. \( \alpha \) by (7.3), then passing to a subsequence, we have a limiting measure \( \mu \). In what follow the subscript \( k \) is related to \( \alpha_k \), the \( k \)th term of the above subsequence.

1. **Estimates for the inviscid limit.** Let \( \chi_R \) be a bump function on \( [0,R] \) for \( R > 0 \).
   By (7.1), we have
   \[
   \int_{\mathcal{H}_{1,0}} L_1(y) \chi_R(\|y\|_{1,0}) \mu_k(dy) \leq \frac{A_0}{2}.
   \]
   We pass to the limits \( k \to \infty, R \to \infty \) (in this order, with the use of Fatou’s lemma in the second limit), we get
   \[
   \int_{\mathcal{H}_{1,0}} L_1(y) \mu(dy) \leq \frac{A_0}{2}.
   \]
   A similar procedure applied to (7.3) and (7.2) gives (8.2) and (8.3). And (8.2) implies (8.1).

2. **Inviscid limit and its invariance under KG.** The following diagram is the general scheme of the proof.

\[
\begin{array}{ccc}
\mathcal{P}_k \mu & \overset{(I)}{\twoheadrightarrow} & \mu_k \\
\downarrow \overset{(III)}{=} & & \\
\mathcal{P}_k \mu & \overset{(IV)}{\twoheadrightarrow} & \mu
\end{array}
\]
The point \( (I) \) is just the invariance of \( \mu_k \) under \( \phi^t_k \). The point \( (II) \) is in weak sense. The point \( (IV) \) follows immediately \( (III) \). Let us, then, prove \( (III) \).

Let \( f \) be a bounded Lipschitz function on \( \mathcal{H}^{1,0} \), suppose, without loss of generality, that \( f \) is bounded by 1 and denote by \( C_f \) its Lipschitz constant. We have

\[
\langle \Psi^k f, \mu_k \rangle = \langle \mu_k, \Psi^k f \rangle - \langle \mu, \Psi f \rangle = (\mu_k, \Psi^k f - \Psi f) = A + B.
\]

By weak convergence of \( \mu_k \) towards \( \mu \) as \( k \to \infty \), we have that \( B \to 0 \) as \( k \to \infty \).

Now since the measures \( \mu_k \) and \( \mu \) are concentrated on \( \mathcal{H}^{2,1} \), we can restrict the integrals on this space.

\[
|A| \leq \int_{B_R} \mathbb{E} |f(\phi^k w) - f(\phi w)| \mu_k(dw) + \int_{\mathcal{H}^{2,1} \setminus B_R} \mathbb{E} |f(\phi^k w) - f(\phi w)| \mu_k(dw)
= I_1 + I_2.
\]

Recalling that \( f \) is bounded by 1, we use Chebyshev inequality to find

\[
I_2 \leq \frac{C}{R^2}.
\]

Now

\[
I_1 = \int_{B_R} \mathbb{E} \left[ |f(\phi^k w) - f(\phi w)| \mathbb{1}_{\|z\alpha\|_{L^\infty} \leq r \sqrt{\alpha}} \right] \mu_k(dw)
+ \int_{B_R} \mathbb{E} \left[ |f(\phi^k w) - f(\phi w)| \mathbb{1}_{\|z\alpha\|_{L^\infty} > r \sqrt{\alpha}} \right] \mu_k(dw)
= I^1_1 + I^2_1.
\]

By Chebyshev inequality we get

\[
I^2_1 \leq \frac{C}{r^2}.
\]

Recall that \( f \) is Lipschitz, then using Corollary 8.3 we find

\[
I^1_1 \leq C_f \sup_{w \in B_R} \mathbb{E} \left[ ||\phi^k w - \phi w|| \mathbb{1}_{\|z\alpha\|_{L^\infty} \leq r \sqrt{\alpha}} \right] \leq C_f, r, R \to \infty.
\]

Now take in the good order the limits \( k \to \infty, r, R \to \infty \) to finish the argument.

The proof is complete.

9 Qualitative properties for the distribution of the Hamiltonian

The proof of Theorem 9.1 below is inspired by the method developed in [Kuk08, Shi11, KS12], however the general argument is modified because of the lack of conservation laws present in our situation. In the case of the Schrödinger and Euler equations [Shi11, KS12].
the combination of two conservation laws allowed to control the measure uniformly around zero, such a control was a useful step in the proof of some absolute continuity properties whose strategy relies in part on a splitting argument. Here, without such uniform control around zero, we show that the final conclusion is still true by using furthermore an approximation argument.

**Theorem 9.1.** Suppose \( a_m \neq 0 \) for any \( m \geq 0 \), then the distribution of the Hamiltonian \( E(y) \) under \( \mu \) has a density w.r.t. the Lebesgue measure on \( \mathbb{R} \).

Before proving the above result let us prove some balance type relations. For a continuous function \( h: \mathbb{R} \to \mathbb{R} \), set

\[
H(x) = \int_0^x h(r)dr.
\]

**Proposition 9.2.** Let \( \alpha \in (0, 1) \) and \( \mu_\alpha \) the invariant measure constructed for the problem (1.4). Let \( h \in C^\infty_0(\mathbb{R}) \), we have

\[
E_{\mu_\alpha} \left[ H(E)(A_0 - \| \partial_t u \|^2_2) \right] + \frac{1}{2} E_{\mu_\alpha} \left[ h(E) \sum_m a_m^2 (\partial_t u, e_m)^2 \right] = 0,
\]

where \( E_\mu \) denotes integral with respect to \( \mu \) and \( E \) is the Hamiltonian of the Klein-Gordon equation (1.1).

**Proof.** Consider the second order linear ODE

\[
-\Phi_\lambda'' + \lambda \Phi_\lambda = h \quad \lambda \in (0, 1), \tag{9.1}
\]

with initial data \( \Phi(0) = \Phi'(0) = 0 \). Then it is a matter of direct verification that its solution is

\[
\Phi_\lambda(x) = \frac{1}{2\sqrt{\lambda}} \int_0^x \left( e^{-(x-y)\sqrt{\lambda}} - e^{(x-y)\sqrt{\lambda}} \right) h(y)dy.
\]

The good behaviour of \( \Phi_\lambda(x) \) at \( x \to \infty \) allows to apply the Itô formula (Theorem A.7.5 and Corollary A.7.6 of [KS12]) to \( \Phi_\lambda \circ E(y) \):

\[
\Phi_\lambda(E(u)) = \Phi_\lambda(E(u_0)) + \int_0^t \left( \Phi_\lambda(E(u)) \left\{ (\nabla_y E, g) + \frac{\alpha}{2} \sum_m a_m^2 (\nabla_{y}^2 E, e_m) \right\} + \Phi'_\lambda(E(u)) \sum_m a_m^2 (\nabla_{y} E, e_m)^2 \right) ds
\]

\[
+ \sum_m a_m \int_0^t \Phi'_\lambda(E(u))(\nabla_y E, e_m)d\beta_m(s),
\]

where

\[
g = [g_1, g_2] = [\partial_t u, \Delta_0 u - u^3 + \alpha \Delta_0 \partial_t u].
\]

Taking the expectation with respect to \( \mu_\alpha \) and using the stationarity of the latter, we are led to

\[
E_{\mu_\alpha} \left[ \Phi'_\lambda(E(s)) (A_0 - \| \partial_t u \|^2_2) \right] + \frac{1}{2} E_{\mu_\alpha} \left[ \Phi''_\lambda(E(s)) \sum_m a_m^2 (\partial_t u, e_m)^2 \right] = 0. \tag{9.2}
\]

Now, we have that
Then we see, using the equation (9.1) and the Lebesgue dominated convergence theorem, that, as $\lambda \to 0$,

$$
\Phi'_\lambda(x) \to -\int_0^x h(y)dy = -H(x),
$$

$$
\Phi''_\lambda(x) \to -h(x).
$$

It remains to apply again the Lebesgue dominated convergence theorem in (9.2) to arrive at the claim.

By a standard approximation argument one can pass from $C^\infty_0$-functions to indicators on intervals functions, then using the monotone class theorem we arrive at:

**Corollary 9.3.** For any Borel set $\Gamma \subset \mathbb{R}$, we have

$$
\mathbb{E}_{\mu_{\alpha}} \left[ 1_{\Gamma} (E \sum_m a_m^2 (\partial_t u, e_m)^2) \right] \leq C(\Gamma) \quad (9.3)
$$

**Proof of Theorem 9.1.** Thanks to the Portmanteau theorem the proof is restricted to the invariant measures $\mu_{\alpha}$ associated to the stochastic problem as long as the resulting estimates are uniform in $\alpha$. It then consists of the following two steps:

**Absolute continuity on the interval $[0, +\infty]$.** By the regularity property, it suffices to consider the intervals $[\varepsilon, +\infty]$, where $\varepsilon > 0$ is arbitrarily small. Let’s define the sets

$$
I_{\varepsilon} = \{ [u, \partial_t u] \in \mathcal{H}^{2,1}, \|\partial_t u\| \leq \varepsilon, \|\partial_t u\|_1 \leq R \} \subset I_{\varepsilon, R}.
$$

Now write

$$
\sum_{m \geq 0} a_m^2 (\partial_t u, e_m)^2 \geq \sum_{m \leq N} a_m^2 (\partial_t u, e_m)^2
$$

$$
\geq a_N^2 \sum_{m \leq N} (\partial_t u, e_m)^2
$$

$$
= a_N^2 \left( \sum_{m \geq 0} (\partial_t u, e_m)^2 - \sum_{m > N} (\partial_t u, e_m)^2 \right)
$$

$$
\geq a_N^2 (\|\partial_t u\|^2 - (m_0^2 + \lambda_N)^{-1} \|\partial_t u\|_1^2),
$$

where $a_N := \min\{a_m, 0 \leq m \leq N\}$. Consider the set

$$
I_{\varepsilon, R} = \{ \|\partial_t u\| \geq \varepsilon, \|\partial_t u\|_1 \leq R \} \subset I_{\varepsilon}.
$$

We have on $I_{\varepsilon, R}$

$$
\sum_{m \geq 0} a_m^2 (\partial_t u, e_m)^2 \geq a_N^2 (\varepsilon^2 - (m_0^2 + \lambda_N)^{-1} R^2) \geq \frac{1}{C_{N, R, \varepsilon}}. \quad (9.4)
$$

Remark that, since $\lambda_N \to \infty$, for any $\varepsilon > 0$, any $R > 0$, we can choose $N$ so that $C_{N, R, \varepsilon}$ be positive. Then combining (9.3) and (9.4), we find

$$
\mu_{\alpha} (E^{-1}(\Gamma) \cap I_{\varepsilon, R}) \lesssim C_{N, R, \varepsilon} l(\Gamma),
$$
on the other hand, we have by Chebyshev inequality
\[ \mu_\varepsilon(E^{-1}(\Gamma) \cap (I_k \setminus I_{\varepsilon,R})) \lesssim R^{-2}, \]
then, for any \( \varepsilon > 0 \), we have, with use of (9.3),
\[ \mathbb{P}(E(y) \in \Gamma \cap [\varepsilon, \infty)) \lesssim R^{-2} + C_{N,R,\alpha} \ell(\Gamma), \quad \forall R > 0. \]

**Now we prove that** \( \mathbb{P}(u \equiv 0) = 0 \). It suffices to show that for some \( m \), for any \( \alpha > 0 \), \( \mathbb{P}(u_m = 0) = 0 \), where \( u_m \) is the projection of \( u \) on the direction \( e_m \). So consider the projected equation:
\[ y_m(t) = y_m(0) + \int_0^t g_m(s)ds + \xi_m(t), \]
where
\[ y_m = [u_m, \partial_t u_m], \quad g_m = [\partial_t u_m, (\Delta_0 u - u^3 + \alpha \Delta_0 \partial_t u, e_m)], \quad \xi_m(t) = \alpha \beta_m(t). \]

An estimate of the form (9.3) can be derived in a same manner, we use it in the mind of the above procedure. It is clear that the quadratic variation of \( u_m \) is bounded from below, it remains to control the drift term. Namely, it suffices to have that \( \mathbb{E}[\parallel u, \partial_t u ||^2_1 + \parallel u^3, e_m \parallel^2] \) for all \( \alpha > 0 \) to finish the proof, but this is ensured by (7.2) and (7.3). The proof is finished. \( \square \)

**Remark 9.4.** One could derive an inequality of type (9.3) by using the local time approach. We, first, apply the Itô formula to \( E(y) \):
\[ E(y(t)) = E(y(0)) + \alpha \int_0^t \left( \frac{A_0}{2} - ||\partial_t u||^2_1 \right) ds + \sqrt{\alpha} \sum_{m \geq 0} u_m \int_0^t (\partial_t u, e_m) d\beta_m(s). \]

Using the stationarity of \( y \), the local time \( \Lambda_t(a, \omega) \) of \( E(y) \) satisfies
\[ \mathbb{E}[\Lambda_t(a)] = -\alpha \mathbb{E} \left[ \left( \frac{A_0}{2} - ||\partial_t u(0)||^2_1 \right) \mathbb{1}_{(a, +\infty)}(E(y)) \right]. \quad (9.5) \]

Now let \( \Gamma \) be a Borel set of \( \mathbb{R} \), the local time identity (??) evaluated to the process \( E(y) \) at the function \( 1_\Gamma \) yields
\[ \int_\Gamma \Lambda_t(a) da = \alpha \sum_{m \geq 0} a_m^2 \int_0^t 1_\Gamma(E(y))(\partial_t u, e_m)^2 ds. \]

Again, the stationarity of \( u \) implies
\[ \int_\Gamma [\mathbb{E}\Lambda_t(a)] da = \alpha t \sum_{m \geq 0} a_m^2 \mathbb{E}[1_\Gamma(E(y))(\partial_t u, e_m)^2]. \quad (9.6) \]

Combining (9.5) and (9.6), we get
\[ \mathbb{E} \left[ 1_\Gamma(E(y)) \sum_{m \geq 0} a_m^2 (\partial_t u, e_m)^2 \right] = \int_\Gamma \mathbb{E} \left[ (2 ||\partial_t u(0)||^2_1 - A_0) \mathbb{1}_{(a, +\infty)}(E(y)) \right] da, \]

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then, with use of \((7.1)\), we find

\[
\mathbb{E} \left[ \mathbb{1}_t(E(y)) \sum_{m \geq 0} a_m^2 (\partial_t a, e_m)^2 \right] \leq C \mathbb{I}(\Gamma),
\]

where \(C\) is a universal constant. We recognize the needed inequality.

**Proposition 9.5.** Let \(a > 1\), set \(\sigma = \gamma_0 (2ae\sigma_0)^{-1}\), then

\[
\mathbb{E} e^{\sigma E(y)} = \int_{\mathbb{R}^1} e^{\sigma E(y)} \mu(dy) < +\infty. \tag{9.7}
\]

Consequently, for any \(R > 0\) we have

\[
\mathbb{P}(E(y) \geq R) \lesssim e^{-\sigma R}.
\]

**Proof.** From \((8.3)\), we write

\[
\mathbb{E} \frac{E^p}{p!} \leq 2 \left( \frac{2p\sigma_0)^p}{\gamma_0^p} \right),
\]

then, with use of the Stirling formula, we get

\[
\mathbb{E} \frac{(\sigma E)^p}{p!} \leq \frac{2p^p}{a^p e^p p!} \sim_{p \to \infty} \frac{\sqrt{2}}{a^p \sqrt{p \pi}}
\]

Since \(a > 1\), the serie of general term \(\mathbb{E} \frac{(\sigma E)^p}{p!}\) is convergent and we get \((9.7)\). Now, we use the Chebyshev inequality to derive the other claim.

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**References**

[BB14] J. Bourgain and A. Bulut. Invariant Gibbs measure evolution for the radial nonlinear wave equation on the 3d ball. *J. Funct. Anal.*, 266(4):2319–2340, 2014.

[BT07] N. Burq and N. Tzvetkov. Invariant measure for a three dimensional nonlinear wave equation. *Int. Math. Res. Not. IMRN*, (22):Art. ID mm108, 26, 2007.

[BT14] N. Burq and N. Tzvetkov. Probabilistic well-posedness for the cubic wave equation. *J. Eur. Math. Soc. (JEMS)*, 16(1):1–30, 2014.

[CKS+02] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. Almost conservation laws and global rough solutions to a nonlinear Schrödinger equation. *Math. Res. Lett.*, 9(5-6):659–682, 2002.
[Cou13] Y. Coudène. *Théorie ergodique et systèmes dynamiques*. EDP sciences, 2013.

[dS14] A-S. de Suzzoni. Invariant measure for the Klein-Gordon equation in a non periodic setting. *arXiv preprint arXiv:1403.2274*, 2014.

[Koo31] B. Koopman. Hamiltonian systems and transformation in hilbert space. *Proceedings of the National Academy of Sciences*, 17(5):315–318, 1931.

[Kre85] U. Krengel. *Ergodic theorems*, volume 59. Cambridge Univ Press, 1985.

[KS91] I. Karatzas and S. E. Shreve. *Brownian motion and stochastic calculus*. Graduate texts in Mathematics, Springer-Verlag, New York, 1991.

[KS04] S. Kuksin and A. Shirikyan. Randomly forced CGL equation: stationary measures and the inviscid limit. *Journal of Physics. A. Mathematical and General*, 37:3805–3822, 2004.

[KS12] S. Kuksin and A. Shirikyan. *Mathematics of Two-Dimensional Turbulence*. Cambridge University Press, Cambridge, 2012.

[Kuk04] S. Kuksin. The Eulerian limit for 2D statistical hydrodynamics. *J. Statist. Phys.*, 115(1-2):469–492, 2004.

[Kuk08] S. Kuksin. On distribution of energy and vorticity for solutions of 2d Navier-Stokes equation with small viscosity. *Communications in Mathematical Physics*, 284(2):407–424, 2008.

[OT15] T. Oh and N. Tzvetkov. Quasi-invariant Gaussian measures for the cubic fourth order nonlinear schrödinger equation. *Probability Theory and Related Fields*, pages 1–48, 2015.

[Shi11] A. Shirikyan. Local times for solutions of the complex Ginzburg-Landau equation and the inviscid limit. *J. Math. Anal. Appl.*, 384(1):130–137, 2011.

[Sy16] M. Sy. Invariant measure and long time behavior of regular solutions of the Benjamin-Ono equation. *arXiv preprint arXiv:1601.05055*, 2016.

[Tao06] T. Tao. *Nonlinear dispersive equations: local and global analysis*, volume 106. American Mathematical Soc., 2006.

[Tao08] T. Tao. Lectures on ergodic theory. *https://terrytao.wordpress.com/category/teaching/254a-ergodic-theory/*, 2008.

[Tho16] L. Thomann. Invariant Gibbs measures for dispersive PDEs. 2016.

[Tzv15] N. Tzvetkov. Quasienvariant Gaussian measures for one-dimensional Hamiltonian partial differential equations. *Forum Math. Sigma*, 3:e28, 35, 2015.

[Xu14] S. Xu. Invariant Gibbs measure for 3D NLW in infinite volume. *arXiv preprint arXiv:1405.3856*, 2014.