Torus fibrations, gerbes, and duality

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1 Introduction

1.1 Duality for elliptic fibrations

In this paper we are concerned with categories of sheaves on varieties fibered by genus one curves. For an elliptic fibration on $X$, by which we always mean a genus one fibration $\pi : X \to B$ admitting a holomorphic section $\sigma : B \to X$, there is by now a well understood theory of the Fourier-Mukai transform $\text{[Muk81, BBRP98, Bri98, BM02]}$. The basic result is:

**Theorem [BM02]** Let $\xymatrix{ X \ar[r]^-\pi \ar[d]^-\sigma & B }$ be an elliptic fibration with smooth total space. Then the integral transform (Fourier-Mukai transform)

$$ FM : D^b(X) \longrightarrow D^b(X) $$

$$ F \longrightarrow Rp_2^*(L_{p_1}^*F \otimes \mathcal{P}), $$

induced by the Poincare sheaf $\mathcal{P} \to X \times_B X$, is an auto-equivalence of the bounded derived category $D^b(X)$ of coherent sheaves on $X$.

An important feature of $FM$ is that it transforms geometric objects in an interesting way:

$$ \begin{cases} 
\text{Bundle data:} & \text{vector bundles on } X, \text{ semistable of degree zero on the} \\
\text{generic fiber of } \pi. 
\end{cases} $$

$$ \xymatrix{ \{ \text{Bundle data: vector bundles on } X, \text{ semistable of degree zero on the} \\
\text{generic fiber of } \pi. \} \ar[d]^-{FM} }

$$ \begin{cases} 
\text{Spectral data:} & \text{sheaves on } X \text{ with the numerics of a line bundle on a} \\
\text{‘spectral’ divisor } C \subset X, \text{ finite over } B. 
\end{cases} $$

This spectral construction was used to study general compactifications of heterotic string theory and their moduli, and especially the duality with F-theory $\text{[FMW97, BJPS97, Don97, AD98, FMW98, Don99, FMW99]}$. It was used also to construct special bundles on elliptic Calabi-Yau manifolds which lead to more-or-less realistic compactified theories $\text{[DOPW01a, DOPW01b]}$.

1.2 Partial duality for genus one fibrations

In many applications (see e.g. $\text{[DOPW01b, DOPW01a]}$) one is also interested in constructing bundles on genus one fibrations $\pi : Y \to B$ which do not necessarily admit a section. From the viewpoint of the spectral construction one expects that vector bundles on $Y$ should correspond to spectral data supported on a divisor $C \subset X := \underleftarrow{\text{Pic}}^0(Y/B)$, where $\underleftarrow{\text{Pic}}^0(Y/B)$ is the compactified relative Jacobian of $\pi : Y \to B$. However it is unrealistic to expect that
this spectral data should again be a sheaf on \( C \). One problem is that \( X \) is not a fine moduli space of objects on \( Y \) and so we do not have a Poincare sheaf on \( Y \times_B X \). Another problem is that in the passage from \( Y \) to \( X \) we seem to be losing information. Indeed, there can be many different \( Y \)'s with the same Jacobian \( X \), and there is no obvious way to recover \( Y \) from spectral data consisting of a divisor on \( X \) and an ordinary sheaf supported on this divisor.

The resolution of this problem is suggested by string theory. Namely, in the transition from \( Y \) to \( X \) one should add an extra piece of data corresponding to the physicists' \( B \)-field. Mathematically the holomorphic version of this data is encoded in an \( \mathcal{O}^\times \)-gerbe on \( X \). A detailed discussion of \( \mathcal{O}^\times \)-gerbes and their geometric properties can be found in [Gir71, Bry93, Bre94, Hit01] and in our section 2.1. A baby version of our result is that:

- \( Y \) determines a non-trivial \( \mathcal{O}_X^\times \)-gerbe \( Y \cdot X \) on \( X \).
- There is a gerby Fourier-Mukai transform exchanging

\[
\begin{align*}
\text{Bundle data: vector bundles on } Y, \text{ semistable of degree zero on the} & \\
\text{generic fiber of } Y \to B. & \\
\end{align*}
\]

\[\text{FM} \]

\[
\begin{align*}
\text{Spectral data: sheaves on the gerbe } Y \cdot X \text{ with the numerics of a line} & \\
\text{bundle on a ‘spectral’ divisor } (Y \cdot X) \times_X C, \text{ with } C \subset X \text{ finite over } B. & \\
\end{align*}
\]

This statement is asymmetric - we only consider vector bundles on the variety \( Y \), and the spectral data appears only for the gerbe \( Y \cdot X \). The symmetry can be restored by extending this gerby spectral construction to a full Fourier-Mukai equivalence of derived categories.

One peculiarity of \( \mathcal{O}^\times \)-gerbes is that the categories of coherent sheaves on them, and hence also the derived categories, admit an orthogonal decomposition by subcategories labeled by the characters of \( \mathcal{O}^\times \), i.e. by the integers (see section 2.1). For any \( k \in \mathbb{Z} \), we will write \( D^b_k(Y \cdot X) \subset D^b(Y \cdot X) \) for the \( k \)-th summand and we will call \( D^b_k(Y \cdot X) \) the derived category of weight \( k \) sheaves on \( Y \cdot X \). It may be helpful to note here that when \( Y \cdot X \) comes from an Azumaya algebra, the weight \( k \) corresponds to the central character of the action of this algebra.

Related partial dualities were considered previously in [C˘ al02, C˘ al01] in the context of Fourier-Mukai transforms and in [DG02] in the context of the spectral construction. A detailed analysis of the corresponding moduli spaces and the duality transformation between them was recently carried out, for the particular case of Hopf-like surfaces, in [BM03b, BM03a].
1.3 Main results: duality for genus one fibrations

For any elliptic fibration \( X \xrightarrow{\pi} B \), the twisted versions of \( X \) are parameterized by the analytic Tate-Shafarevich group \( \text{III}_{an}(X) \) (see section 2.2). For a given \( \beta \in \text{III}_{an}(X) \), let \( \pi_\beta : X_\beta \to B \) denote the corresponding genus one fibration. On the other hand for any analytic space \( X \), the analytic \( \mathcal{O}_X^{\times} \)-gerbes on \( X \) are parameterized by the analytic Brauer group \( Br'_an(X) = H^2_{an}(\mathcal{O}_X^{\times}) \). For any \( \alpha \in Br'_an(X) \) we denote the corresponding gerbe by \( \alpha X \) and the bounded derived category of coherent sheaves on \( \alpha X \) by \( D^b(\alpha X) \). The latter decomposes naturally as the orthogonal direct sum of “pure weight” subcategories \( D^b_1(X) \) indexed by characters of \( \mathcal{O}_X^{\times} \), i.e. by integers \( k \in \mathbb{Z} \). For all \( \alpha \in Br'_an(X) \) and all \( k \in \mathbb{Z} \) there is a canonical equivalence \( D^b_k(\alpha X) \cong D^b_1(kaX) \). This follows immediately by comparing representations of \( \alpha X \) of pure weight \( k \) with representations of \( kaX \) of pure weight 1 or by the appropriate identifications with the category of twisted sheaves on \( X \) (see section 2.1.1).

Consider first the case when \( X \) is a surface. We will see in section 2.3 that for a higher dimensional \( X \) sense for arbitrary choices of \( \alpha, \beta \) defined whenever \( \alpha \) exists a natural pairing \( \langle \cdot, \cdot \rangle : \text{III}_{an}(X) \otimes_\mathbb{Z} \text{III}_{an}(X) \to H^3_{an}(\mathcal{O}_B^{\times}) \) and that \( \alpha X_\beta \) can be defined whenever \( \alpha, \beta \) are complementary, i.e. \( \langle \alpha, \beta \rangle = 0 \). The natural generalization of Theorem A is the following (see Conjecture 2.19):

**Theorem A** Let

\[
X \xrightarrow{\pi} B = \mathbb{P}^1
\]

be a non-isotrivial elliptic fibration on a smooth complex surface \( X \). Assume that \( \pi \) has \( I_1 \) fibers at worst. Let \( \alpha, \beta \in \text{III}_{an}(X) \) be two elements such that \( \beta \) is torsion. Then there is an equivalence

\[
FM : D^b_1(\alpha X_\beta) \to D^b_{-1}(\beta X_\alpha)
\]

of the derived category of weight 1 coherent sheaves on the gerbe \( \alpha X_\beta \) and the derived category of weight \((-1)\) coherent sheaves on the gerbe \( \beta X_\alpha \). Equivalently \( FM \) can be thought of as an equivalence of the derived categories \( D^b_1(\alpha X_\beta) \) and \( D^b_{-1}(\beta X_\alpha) \).

We will see in section 2.3 that for a higher dimensional \( X \), the gerbes \( \alpha X_\beta \) may not make sense for arbitrary choices of \( \alpha, \beta \in \text{III}_{an}(X) \). However in section 2.3 we show that there exists a natural pairing \( \langle \cdot, \cdot \rangle : \text{III}_{an}(X) \otimes_\mathbb{Z} \text{III}_{an}(X) \to H^3_{an}(\mathcal{O}_B^{\times}) \) and that \( \alpha X_\beta \) can be defined whenever \( \alpha, \beta \) are complementary, i.e. \( \langle \alpha, \beta \rangle = 0 \). The natural generalization of Theorem A is the following (see Conjecture 2.19):

**Main Conjecture** For any complementary pair \( \alpha, \beta \in \text{III}_{an}(X) \), there exists an equivalence

\[
D^b_1(\alpha X_\beta) \cong D^b_{-1}(\beta X_\alpha)
\]

of the bounded derived categories of sheaves of weights \( \pm 1 \) on \( \alpha X_\beta \) and \( \beta X_\alpha \) respectively.
We cannot prove this in full generality, mostly due to our inability to handle the general singular fibers. We are able to settle the conjecture in the non-singular case, under the somewhat more restrictive condition that $\alpha$ is $m$-divisible and $\beta$ is $m$-torsion for some integer $m$. This of course implies that $\alpha, \beta$ are complementary, so $\alpha X_\beta$ is well defined. For a smooth $\pi$ our main result is:

**Theorem B** Let

\[
\begin{array}{ccc}
\alpha & \pi & \beta \\
\downarrow & \downarrow & \downarrow \\
\alpha X & \to & \beta X
\end{array}
\]

be a smooth elliptic fibration on an algebraic variety $X$ over a smooth algebraic base $B$. Assume that $Br'_\text{an}(B) = 0$. Fix a positive integer $m$ and let $\alpha, \beta \in \mathbb{H}_{\text{an}}(X)$ be two elements such that $\alpha$ is $m$-divisible and $\beta$ is $m$-torsion. Then there is an equivalence

$$FM : D^b_1(\alpha X_\beta) \to D^b_{-1}(\beta X_\alpha)$$

of the derived category of weight 1 coherent sheaves on the gerbe $\alpha X_\beta$ with the derived category of weight $(-1)$ coherent sheaves on the gerbe $\beta X_\alpha$. Equivalently $FM$ can be thought of as an equivalence of $D^b_1(\alpha X_\beta)$ and $D^b_1(-\beta X_\alpha)$.

In fact, the proof of Theorem B is quite a bit easier than that of Theorem A, so we give it first, in section 3. The proof is based on the construction of two explicit presentations: the lifting presentation of $\alpha X_\beta$, in section 3.1.1, and the extension presentation of $\beta X_\alpha$, in section 3.1.2, together with the construction of an explicit Fourier-Mukai duality between them, in section 3.4.

Our two main theorems and their proofs have fairly straightforward analogues asserting the equivalence of the derived categories of quasi-coherent sheaves and, in the algebraic case, asserting the equivalence of appropriate categories of algebraic coherent sheaves. Indeed, if the class $\alpha$ happens to be torsion as well, then the spaces $X_\alpha$ and $X_\beta$ are algebraic and the gerbes $\alpha X_\beta$ and $\beta X_\alpha$ are algebraic stacks in the sense of Artin. We will see in the proofs of the two theorems that the gerby Fourier-Mukai transform in this case will correspond to a kernel object which is algebraic and so will give rise to an equivalence of the derived categories of weight one algebraic coherent sheaves.

### 1.4 Duality for commutative group stacks

As was pointed out by Arinkin, our Theorem B (but not Theorem A) fits very naturally in the context of commutative group stacks (cgs). The $\mathcal{O}^\times_X$-gerbe $\alpha X$ is a family of cgs over $B$ which is an extension:

$$0 \to B \mathbb{G}_m \to \alpha X_0 \to X \to 0$$

of $X$ by the classifying stack of $\mathbb{G}_m$. The torsor $X_\beta$, on the other hand, is not a cgs over $B$; but it does determine one, namely the extension:

$$0 \to X \to \tilde{X}_\beta \to \mathbb{Z} \to 0$$
of \( \mathbb{Z} \) by \( X \), where \( X_\beta \) is recovered as the inverse image of \( 1 \in \mathbb{Z} \). Similarly, the gerbe \( \alpha X_\beta \) which we construct, using either the lifting presentation (when \( \beta \) is \( m \)-torsion) or the extension presentation (when \( \alpha \) is \( m \)-torsion), determines a cgs \( \mathcal{X} = \alpha \tilde{X}_\beta \) which has a two-step filtration, with sub \( B \mathbb{G}_m \), middle subquotient \( X \), and quotient \( \mathbb{Z} \). This can be considered either as an \( \mathcal{O}^\times \)-gerbe over \( X_\beta \), or dually as a torsor over \( \alpha X \). In particular, the derived category \( D^b(\alpha X_\beta) \) is graded by \( \mathbb{Z} \times \mathbb{Z} \).

Now quite generally, such a cgs \( \mathcal{X} \) has a dual cgs \( \mathcal{X}^\vee \) which has a similar two-step filtration with the roles of the sub and the quotient interchanged. There is a Poincare sheaf \( \mathcal{P} \) which is a biextension of \( \mathcal{X}^\vee \times \mathcal{X} \) by \( \mathbb{G}_m \), and it induces a Fourier-Mukai transform which is an equivalence of categories \( D^b(\mathcal{X}^\vee) \cong D^b(\mathcal{X}) \) interchanging the two \( \mathbb{Z} \) gradings.

Our Theorem [D] can therefore be interpreted as saying that the cgs dual to \( \alpha \tilde{X}_\beta \) is \( \beta \tilde{X}_\alpha \); the previous version is recovered by restricting the equivalence to the piece of bidegree \((1, -1)\). This is explained in some more detail in the Appendix [A] which D. Arinkin kindly wrote for us.

This duality picture has a straightforward extension to more general cgs \( \mathcal{X} \) over \( B \): these are again endowed with a two step filtration \( W_{-2} \mathcal{X} \subset W_{-1} \mathcal{X} \subset W_0 \mathcal{X} = \mathcal{X} \), where \( \text{gr} \subset_2 = W_{-2} \mathcal{X} = BT \) for some affine torus bundle \( T \to B \), the middle subquotient \( \text{gr} \subset_1 \) is some abelian scheme \( A \to B \) and the last quotient \( \text{gr}_0 \) is some bundle \( \Lambda \to B \) of finite rank free abelian groups over \( B \). The dual cgs \( \mathcal{X}^\vee \) is again the stack of homomorphisms \( \text{Hom}_{\text{cgs}}(\mathcal{X}, B \mathbb{G}_m) \) and one expects that in good cases the duality gives rise to an equivalence of the appropriate categories of representations.

Arinkin calls the cgs described in the previous paragraph ‘Beilinson’s one motives’ since they were considered by Beilinson (unpublished) in the context of the theory of mixed motives. The cgs \( \mathcal{X} \) are formally very similar to the classical one motives studied by Deligne in [D74]. The one motives of [D74] can be viewed either as certain mixed Hodge structures of level \( \leq 0 \) or as cgs \( \mathcal{M} \) defined over \( \mathbb{C} \). As a commutative group stack, every Deligne’s one motive \( \mathcal{M} \) is equipped with a two step filtration \( W_{-2} \mathcal{M} \subset W_{-1} \mathcal{M} \subset W_0 \mathcal{M} = \mathcal{M} \), for which \( \text{gr} \subset_2 = T \) for some affine torus \( T \), \( \text{gr} \subset_1 = A \) is a polarized abelian variety, and \( \text{gr}_0 = B \Lambda \) for some free abelian group \( \Lambda \) of finite rank. If we now look at families of Deligne’s one motives defined over some base \( B \) we arrive at cgs over \( B \) which are of essentially the same shape as the Beilinson’s one motives, but with the stackiness appearing at a different subquotient of the filtration. Furthermore, as explained in [D74], the dual of a Deligne’s one motive \( \mathcal{M} \) is the cgs \( \mathcal{M}^\vee := \text{Hom}_{\text{cgs}}(\mathcal{M}, B \mathbb{G}_m) \), which is again a one motive of the same type with \( \text{gr} \subset_2 \mathcal{M}^\vee = \text{Hom}(\Lambda, \mathbb{G}_m) \), \( \text{gr} \subset_1 \mathcal{M}^\vee = \hat{A} \) (the dual abelian variety to \( A \)), and \( \text{gr}_0 \mathcal{M}^\vee = B \text{Hom}(T, \mathbb{G}_m) \).

In fact, we can view \( \text{Hom}_{\text{cgs}}(\bullet, B \mathbb{G}_m) \) as a transformation acting on commutative group stacks, which preserves the two natural families of Deligne’s and Beilinson’s one motives and induces a duality on each of these families. Moreover, since in both cases the duality is realized in terms of suitable biextensions of \( \mathcal{X} \times \mathcal{X}^\vee \) and \( \mathcal{M} \times \mathcal{M}^\vee \), one expects that the duality of cgs will give rise to an equivalence of the corresponding categories of representations.
of cgs. For the specific stacks $\tilde{X}_\beta$ this is precisely the content of our Theorem B.

### 1.5 The non-commutative aspect

Results having the general shape of Theorem A were anticipated in the physics literature. In fact, Ganor-Mihailov-Saulina have conjectured in [GMS00] that when $Y$ is a genus one fibered $K3$ surface, there should exist a non-commutative deformation $\gamma X$ of $X = \text{Pic}^0(Y/B)$ and a categorical equivalence between instantons on $\gamma X$ and spectral data on $Y$. This is a special case of Theorem A.

This statement admits an intriguing interpretation in terms of non-commutative geometry, a topic currently of high interest to physicists [NS98, KKO01]. According to the general yoga of deformation quantization (see e.g. [Kon01]), any symplectic (or Poisson) structure on $X$ is the first term in a non-commutative deformation of its structure sheaf. In a suitable algebro-geometric context, e.g. on a $K3$, a symplectic structure $\theta$ has three incarnations: as a real 2-form $\theta_R$, a holomorphic 2-form $\theta^{2,0}$, or an antiholomorphic 2-form $\theta^{0,2}$. Then $\theta_R$ determines a "non-commutative four-manifold", and $\theta^{2,0}$ determines a "non-commutative $K3". The third incarnation, $\theta^{0,2}$, gives both the element $X_\theta$ in the Tate-Shafarevich group $\text{III}(X)$ and the $\mathcal{O}_X$-gerbe $\theta^X$. In this sense, Theorem A can be viewed as an affirmative answer and a generalization of the [CMS00] conjecture.

### 1.6 Modified T-duality and the SYZ conjecture

The celebrated work of Strominger, Yau and Zaslow [SYZ96] interprets mirror symmetry of Calabi-Yau spaces in terms of special Lagrangian (SLAG) torus fibrations. If a CY manifold $X$ (with "large complex struture") has mirror $X'$, [SYZ96] conjecture the existence of fibrations $\pi : X \to B$ and $\pi' : X' \to B$ whose generic fibers are SLAG tori dual to each other: each parameterizes $U(1)$ flat connections on the other. In particular, each of these fibrations admits a SLAG zero-section, corresponding to the trivial connection on the dual fibers. The analogy with the theorem of [BM02] is clear: the SLAG torus fibration on the Calabi-Yau threefold replaces the elliptic fibration on the surface, and mirror symmetry (interchanging D-branes of type B with D-branes of type A) replaces the Fourier-Mukai transform (which interchanges vector bundles with spectral data).

Our work suggests that the SYZ conjecture should be extended to a SLAG analogue of Theorem A or of the Main Conjecture, in which the physical B-fields $\alpha \in H^2(X, \mathbb{R}/\mathbb{Z})$ play the role of our gerbes. This extension leads to an integrable system structure on the moduli space underlying mirror symmetry. We give an informal discussion of these matters in section 5 and we hope to return to them in future work.

### 1.7 Modularity

As often happens in physics, the Fourier-Mukai functor $FM : D^b(\alpha X_\beta) \to D^b(-\beta X_\alpha)$ is just one particular element of a whole collection of dualities. For simplicity consider only the case of a projective elliptic surface $\pi : X \to B$. In this case, our Fourier-Mukai duality works
for any pair of elements \((\alpha, \beta) \in \Sha(X) \times \Sha(X)\) in the algebraic Tate-Shafarevich group. Thus the Fourier-Mukai functor corresponds to the action of the matrix

\[
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\in \text{SL}(2, \mathbb{Z})
\]

on the Cartesian square \(\Sha(X)^{\times 2}\) of the abelian group \(\Sha(X)\). Moreover, one can show (see e.g. section 2.3) that for surfaces the natural map \(T_\beta : \Sha(X) \to Br'(X_\beta)\), used to define our gerbes, has kernel generated by the element \(\beta \in \Sha(X)\). In particular, \(T_\beta(\alpha + \beta) = T_\beta(\alpha)\) and so the gerbes \(\alpha + \beta X_\beta\) and \(\alpha X_\beta\) are isomorphic. A choice of such an isomorphism gives rise to an equivalence \(D^b_1(\alpha + \beta X_\beta) \cong D^b_1(\alpha X_\beta)\) which corresponds to the action of the matrix

\[
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
\in \text{SL}(2, \mathbb{Z})
\]

on \(\Sha(X)^{\times 2}\). Since these two matrices generate \(\text{SL}(2, \mathbb{Z})\), it will be very interesting to investigate which braid group extension of \(\text{SL}(2, \mathbb{Z})\) acts on \(\bigcup_{\alpha, \beta \in \Sha(X)} D^b_1(\alpha X_\beta)\), lifting the action on \(\Sha(X)^{\times 2}\). In the case when the Mordell-Weil group of \(X\) is trivial we expect this extension to be central and to be related to the extensions appearing in [Pol96, Pol02, Orl02] and [ST01]. We do not discuss this question here but hope to return to it in a future work.

1.8 Twisted sheaves

Another context in which Theorem A turns out to be relevant is the theory of twisted sheaves on a complex space which admits a genus one fibration. Recall [C˘ ald˘ araru 2000] that for any \(\mathcal{O}^{\times}\)-valued Čech 2-cocycle \(\alpha\) on a complex space \(X\), one can consider the abelian category of \(\alpha\)-twisted sheaves on \(X\) and its derived category \(D^b(X, \alpha)\). By definition, an \(\alpha\)-twisted sheaf on \(X\) is a collection of coherent sheaves defined over open sets in \(X\), together with gluing data on overlaps which satisfy the \(\alpha\)-twisted cocycle condition on triple overlaps (see [C˘ ald˘ araru 2000] or our section 2.1.1 for details). Refining the open covering or changing the cocycle by a coboundary results in an equivalent category of twisted sheaves. Twisted sheaves on Calabi-Yau manifolds, and in particular on genus one fibered Calabi-Yau manifolds, were recently studied by A.C˘ ald˘ araru [C˘ ald˘ araru 2000, C˘ ald˘ araru 2002]. In particular, he observed [C˘ ald˘ araru 2001] that in the case of a K3 surface, the derived category of \(\alpha\)-twisted sheaves possesses certain natural Fourier-Mukai partners. The starting point of his analysis is the observation that if \(X\) is a smooth projective K3 surface, then every element \(\alpha \in H^2_{\et}(X, \mathcal{O}^{\times})\) can be interpreted as a homomorphism \(\alpha : T_X \to \mathbb{Q}/\mathbb{Z}\), where \(T_X\) denotes the transcendental lattice of \(X\) (see [C˘ ald˘ araru 2001] or section 2.1.1). This interpretation suggests the following:

C˘ ald˘ araru’s Conjecture Let \(X\) and \(Y\) be two projective K3 surfaces and let \(\alpha \in H^2_{\et}(X, \mathcal{O}^{\times})\) and \(\beta \in H^2_{\et}(Y, \mathcal{O}^{\times})\). Then the derived categories \(D^b(X, \alpha)\) and \(D^b(Y, \beta)\) are equivalent as triangulated categories iff the lattices \(\ker(\alpha) \subset T_X\) and \(\ker(\beta) \subset T_Y\) are Hodge isometric.
When both $\alpha$ and $\beta$ are zero, the conjecture is true in view of a theorem of D. Orlov [Orl97] asserting that two smooth projective K3 surfaces have equivalent derived categories iff their transcendental lattices are Hodge isometric. This has been extended by Căldăraru, who used Mukai’s quasi-universal sheaves for non-fine moduli spaces of sheaves on K3 surfaces to deduce that the conjecture holds whenever one of the classes, say $\beta$, is trivial. The algebraic case of our Theorem A proves Căldăraru’s Conjecture in a series of new cases, with both $\alpha$ and $\beta$ non-zero.

Indeed, if $\pi : X \to B$ is an elliptic K3 surface and if $\alpha, \beta \in \Sha(X)$ are two elements in the algebraic Tate-Shafarevich group, then the natural identification $\Sha(X) = H^2_{\text{et}}(X, O_X)$ coming from the Leray spectral sequence allows us to view both $\alpha$ and $\beta$ as homomorphisms $T_X \to \mathbb{Q}/\mathbb{Z}$. Using this interpretation one checks immediately that the transcendental lattices of the K3 surfaces $X, X_\alpha$ and $X_\beta$ satisfy $T_{X_\alpha} = \ker(\alpha) \subset T_X$ and $T_{X_\beta} = \ker(\beta) \subset T_X$, where it is understood that all the equalities are Hodge isometries. Let $T_\beta(\alpha) \in Br^*_an(X_\beta) = H^2_{an}(X_\beta, O_X^*)$ denote the class of the gerbe $\alpha X_\beta$. Assuming that $\alpha$ and $\beta$ are in general position in $H^2_{et}(X, O_X^*)$, i.e. that the cyclic subgroups generated by $\alpha$ and $\beta$ intersect only at zero, we have natural identifications of Hodge lattices:

\[
\ker \left[ T_{X_\alpha} \xrightarrow{T_\alpha(\beta)} \mathbb{Q}/\mathbb{Z} \right] = \ker(\alpha) \cap \ker(\beta) \subset T_X
\]

\[
\ker \left[ T_{X_\beta} \xrightarrow{T_\beta(\alpha)} \mathbb{Q}/\mathbb{Z} \right] = \ker(\alpha) \cap \ker(\beta) \subset T_X.
\]

In other words - the hypothesis of Theorem A implies the hypothesis of Căldăraru’s conjecture. Combined with the remark that $D^b(X_\alpha, T_\alpha(\beta)) \cong D^b_1(\alpha X_\beta)$ and $D^b(X_\beta, -T_\beta(\alpha)) \cong D^b_2(\beta X_\alpha)$, this shows that Theorem A implies an interesting new case of Căldăraru’s conjecture (see Corollary 4.6 for a slightly more general statement). Note that the condition (required in the statement of Theorem A) that $\beta$ should be torsion, is vacuous in this case, since for a smooth projective surface $X$, both the cohomological Brauer group $H^2_{et}(X, O_X^*)$ and the Tate-Shafarevich group $\Sha(X)$ are torsion groups.

The paper is organized as follows. In Section 2 we recall some standard facts about the geometry of $O_X^*$ gerbes and genus one fibrations. We also derive the compatibility condition between two Tate-Shafarevich classes and state a general conjecture on the equivalence of derived categories for gerbes over genus one fibrations. In Section 3 we introduce the main characters appearing in the proofs of the two theorems stated above. Working in the setup of Theorem B, we define two geometric presentations - the lifting and extension presentations - for the gerbes $\alpha X_\beta$ and $\beta X_\alpha$. Furthermore, we construct an integral transform between the corresponding atlases. We show that this integral transform sends descent data to descent data and gives rise to an equivalence of the derived categories of the gerbes, thus proving
Theorem \[\textbf{B}\] Section \[\textbf{4}\] deals with the case of surfaces. We show how, in the case of a surface, one can extend the lifting and extension presentations across the singular fibers and produce a Fourier-Mukai transform between the corresponding gerbes. We again check that this transform is an equivalence, which proves Theorem \[\textbf{A}\]. Finally, in Section \[\textbf{5}\] we discuss the analogy between algebraic gerbes over genus one fibrations and flat gerbes over SLAG 3-torus fibrations on Calabi-Yau threefolds. We describe a conjectural picture which amends the Strominger-Yau-Zaslow version of mirror symmetry to incorporate non-trivial B-fields on both sides of the mirror correspondence.

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2 The Brauer group and the Tate-Shafarevich group

We need some basic facts relating elements of the Brauer group to elements of the Tate-Shafarevich group of an elliptic fibration. We discuss $O^\times$-gerbes and the Brauer groups which classify them in section \[\text{2.1}\] then genus-1 fibrations and the Tate-Shafarevich group which classifies them, in section \[\text{2.2}\]. For an elliptic fibration there is a simple, direct relation between these two groups. The extension to genus-1 fibrations though is more delicate, and is defined only when a certain alternating pairing vanishes. This is discussed in section \[\text{2.3}\].

2.1 Brauer groups and $O^\times$-gerbes

In this section we review the notions of $O^\times$-gerbe and presentation, and discuss the relationship between $O^\times$-gerbes and elements in the Brauer group.

2.1.1 $\mathcal{H}$-gerbes

Let $\mathcal{H}$ be a sheaf of abelian groups on a topological space (or a site) $X$. The case of main interest for us is when $(X, O_X)$ is a ringed space and $\mathcal{H} = O_X^\times$ is the sheaf of invertible elements in the structure sheaf. In fact, most of the time we will have $\mathcal{H} = O_X^\times$ in either the etale or the analytic topologies on a complex scheme (or an algebraic or analytic space) $X$. In section \[\text{2.3}\] we will be interested also in the case when $\mathcal{H}$ is the sheaf of germs of smooth maps from a $C^\infty$ manifold $X$ to the circle $S^1$.

An $\mathcal{H}$-gerbe on $X$ is a global structure on $X$ which “locally looks like the quotient of $X$ by the trivial action of $\mathcal{H}$”. More precisely “the quotient of $X$ by the trivial action of $\mathcal{H}$” is the classifying object $B\mathcal{H}$. For example, in case $\mathcal{H}$ is the sheaf of holomorphic maps from $X$ to a fixed group $H$, $B\mathcal{H}$ is the sheaf of sections of $X \times BH$ over $X$, where $BH$ is the classifying space of $H$. In the general case, $B\mathcal{H}$ can be interpreted either as a topological
space over $X$ (defined up to homotopy), or as a stack in groupoids over $X$ (see [LMB00 §3] for the definition). We adopt the second approach and treat $B\mathcal{H}$ as a stack (‘sheaf of categories’): over any open set $V$, the objects of $B\mathcal{H}(V)$ are the $\mathcal{H}$-torsors on $V$ and the morphisms are the isomorphisms of torsors. In particular, the automorphisms of the trivial torsor $1_V$ are given by elements in $\mathcal{H}(V)$. Note that $B\mathcal{H}$ is in fact a commutative group stack over $X$ with a group structure given by convolution of $\mathcal{H}$-torsors. Explicitly, for any two $\mathcal{H}$-torsors $A'$ and $A''$ over $V$ the convolution $A' \otimes A''$ is defined as the $\mathcal{H}$-torsor $(A' \times A'')/\ker(m_{\mathcal{H}})$, where $m_{\mathcal{H}} : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ is the multiplication map.

**Definition 2.1** An $\mathcal{H}$-gerbe on $X$ is a $B\mathcal{H}$ torsor, i.e. a stack of groupoids $\mathcal{A}X$ over $X$, which is equipped with a principal homogeneous action of $B\mathcal{H}$.

**Remark 2.2** • Explicitly, a stack of groupoids $\mathcal{A}X$ over $X$ is an $\mathcal{H}$-gerbe if for any open $V \subset X$ and any object $s$ of $\mathcal{A}X(V)$ we have chosen isomorphisms $\mathcal{H}(V) \cong \text{Aut}_{\mathcal{A}X(V)}(s)$, compatible with pullbacks.

• In the literature [Gir71], [Bre90] one encounters a more general notion of an $\mathcal{H}$-gerbe, namely - a stack $\mathcal{T}$ of groupoids on $X$, which is locally isomorphic to $B\mathcal{H}$. These more general gerbes are classified by the first cohomology of $X$ with coefficients in the 1-truncated simplicial abelian group $\mathcal{H} \to \text{Aut}(\mathcal{H})$ [Bre90]. They are intimately related to the forms of $\mathcal{H}$, i.e. to sheaves of groups on $X$ which are only locally isomorphic to $\mathcal{H}$. This relationship is based on the identification $\text{Out}(\mathcal{H}) = \text{Aut}_X(B\mathcal{H})$: to any $\mathcal{T} \to X$, which is an $\mathcal{H}$-gerbe in this more general sense, one naturally associates an $\text{Out}(\mathcal{H})$-torsor $\text{band}(\mathcal{T}) := \text{Isom}_X(\mathcal{T}, B\mathcal{H})$ - the band of the gerbe $\mathcal{T}$ [Bre90]. A gerbe $\mathcal{T}$ is said to be banded by $\mathcal{H}$ if it is equipped with a trivialization of the torsor $\text{band}(\mathcal{T})$. When $\mathcal{H}$ is abelian, this condition is equivalent to requiring that for any open $V$ and any $s \in \mathcal{T}$ we have chosen isomorphisms $\mathcal{H}(V) \cong \text{Aut}_{\mathcal{T}}(s)$ in a way compatible with pullbacks. In other words, the more restrictive notion of an $\mathcal{H}$-gerbe that we have adopted in this paper is the same as the standard notion of an $\mathcal{H}$-banded gerbe (at least for an abelian $\mathcal{H}$). We will casually ignore this distinction and will call all our gerbes simply $\mathcal{H}$-gerbes.

In case $\mathcal{H} = \mathcal{O}_X^\times$ (in the relevant topology), the classifying stack $BO_X^\times$ is the sheaf of Picard categories $\mathcal{Pic}(X)$: for an open $U$, the objects of $\mathcal{Pic}(X)(U)$ are by definition the line bundles on $U$, and for two objects $L, M \in \text{ob}(\mathcal{Pic}(X)(U))$ the set $\text{Hom}_{\mathcal{Pic}(X)}(L, M)$ is defined to be $\text{Isom}(L, M)$. An $\mathcal{O}_X^\times$-gerbe $\mathcal{A}X$ assigns to each open $U$ a $\mathcal{Pic}(X)(U)$-torsor, denoted $\mathcal{Pic}_a(U)$, with a compatibility of the assignments to different $U$’s. We can thus think of a section of an $\mathcal{O}_X^\times$-gerbe as a twisting of the notion of a line bundle on $X$. More generally the sections in an $\mathcal{H}$-gerbe are twistings of the notion of an $\mathcal{H}$-torsor on $X$: simply replace in the previous discussion each appearance of $\mathcal{Pic}$ with $\text{Tors}^{\mathcal{H}}$ - the group of $\mathcal{H}$-torsors. This interpretation suggests that the group classifying $\mathcal{H}$-gerbes should be $H^2(X, \mathcal{H})$. When $\mathcal{H}$ is abelian this statement can be made precise via the standard
cohomological machinery [Mil80 IV.2] or [Gir71], [Bre90] (but keep in mind that our \( \mathcal{H} \)-gerbes are the \( \mathcal{H} \)-banded gerbes of loc. cit.).

In more down to earth terms the interpretation of the elements in \( H^2(X, \mathcal{H}) \) as equivalence classes of gerbes can be seen as follows. Assume that we are in the good situation when the cohomology of \( \mathcal{H} \) can be computed in Čech terms. Let \( \{ \alpha_{ijk} \} \) be an \( \mathcal{H} \)-valued Čech 2-cocycle w.r.t. an open cover \( \{ U_i \} \) of \( X \). An object \( L \) of \( \mathcal{T}_{ors, \alpha} \) is defined to be an assignment of an \( \mathcal{H}(U_i) \)-torsor \( L(U_i) \) to each \( U_i \), together with transition functions

\[
g_{ij} : L(U_i) \otimes_{\mathcal{H}(U_i)} \mathcal{H}(U_{ij}) \to L(U_j) \otimes_{\mathcal{H}(U_j)} \mathcal{H}(U_{ij})
\]

satisfying the twisted cocycle condition:

\[
g_{ij} \circ g_{jk} \circ g_{ki} = \alpha_{ijk}
\]

on triple intersections. A morphism between two \( \alpha \)-twisted \( \mathcal{H} \)-torsors \( L' \) and \( L'' \) is given by a compatible collection of isomorphisms \( L'(U_i) \to L''(U_i) \).

Similarly we define the category \( \mathcal{T}_{ors, \alpha}(U) \) for any open \( U \). The resulting sheaf of categories (=stack) \( \mathcal{T}_{ors, \alpha} \) on \( X \) is by definition a torsor over \( B_{\mathcal{H}} = \mathcal{T}_{ors, 1} \), i.e. an \( \mathcal{H} \)-gerbe, which we denote by \( _\alpha X \). Clearly two cocycles which represent the same cohomology class in \( \check{H}^2(X, \mathcal{H}) \) define isomorphic gerbes. Conversely, if sheaf cohomology on \( X \) can be computed in Čech terms, any \( \mathcal{H} \)-gerbe arises this way from some \( \alpha \) w.r.t. a sufficiently refined cover [Gir71], [Bre90, Bre94].

**Notation:**
- Given an \( \mathcal{H} \)-gerbe \( \mathcal{T} \) over \( X \) we write \([\mathcal{T}] \in H^2(X, \mathcal{H})\) for the element that classifies it.
- The base space \( X \) for an \( \mathcal{H} \)-gerbe \( \mathcal{T} \to X \) is called the *coarse moduli space* of \( \mathcal{T} \). This terminology reflects the fact that \( X \) represents the sheaf of sets \( \pi_0(\mathcal{T}) \), i.e. the sheaf of isomorphism classes of sections in \( \mathcal{T} \).

**Basic construction:** Starting with an algebraic (or analytic) space \( X \) and a short exact sequence of sheaves of groups

\[
1 \to \mathcal{H} \to \mathcal{G} \to \mathcal{K} \to 1,
\]

with \( \mathcal{H} \)-abelian, we get a coboundary map \( \delta : H^1(X, \mathcal{K}) \to H^2(X, \mathcal{H}) \). This admits the following lift on the level of torsors and gerbes: a \( \mathcal{K} \)-torsor \( \mathcal{C} \) with class \([\mathcal{C}] \in H^1(X, \mathcal{K})\) determines an \( \mathcal{H} \)-gerbe \( \delta(\mathcal{C}) \) with class \( \delta([\mathcal{C}]) \in H^2(X, \mathcal{H}) \). Explicitly, for an open \( U \), \( \delta(\mathcal{C})(U) \) is the category of pairs \((\mathcal{D}, i)\) where \( \mathcal{D} \) is a \( \mathcal{G} \)-torsor on \( U \) and \( i : \mathcal{D} \times_{\mathcal{G}} \mathcal{K} \to \mathcal{C} \) is an isomorphism of \( \mathcal{K} \)-torsors on \( U \).

A familiar special case involves the sequence

\[
1 \to \mathcal{O}_X^\times \to GL_n(\mathcal{O}_X) \to \mathbb{P}GL_n(\mathcal{O}_X) \to 1.
\]
It says that every projective bundle on $X$ gives rise to an $\mathcal{O}_X^\times$-gerbe which is trivial if and only if the projective bundle is a projectivization of a vector bundle.

If $(X, \mathcal{O}_X)$ is a nice ringed space for which cohomology can be computed in Čech terms, then the choice of $\alpha \in H^2(X, \mathcal{O}_X^\times)$ gives rise to the notion of $\alpha$-twisted sheaves on $X$. More precisely, let $\mathcal{U} = \{U_i\}$ be an open cover of $X$ (in the topology under consideration) and let

$$\underline{\alpha} = \{\alpha_{ijk}\} \in \check{H}^2(\mathcal{U}, \mathcal{O}_X^\times)$$

be a 2-cocycle representing $\alpha \in H^2(X, \mathcal{O}_X)$. One defines an $\underline{\alpha}$-twisted sheaf on $X$ as a collection $\{F_i\}$ of sheaves $F_i \to U_i$ of $\mathcal{O}_X$-modules, together with a collection of gluing isomorphisms

$$\varphi_{ij} : F_j|_{U_{ij}} \to F_i|_{U_{ij}}$$

satisfying $\varphi_{ii} = \text{id}$, $\varphi_{ij} = \varphi_{ji}^{-1}$, and $\varphi_{ij} \circ \varphi_{jk} \circ \varphi_{ik} : F_j|_{U_{ijk}} \to F_i|_{U_{ijk}}$ is given by multiplication by $\alpha_{ijk}$. Given two $\underline{\alpha}$-twisted sheaves $F = \{F_i, \varphi_{ij}\}$ and $G = \{G_i, \gamma_{ij}\}$ we define a homomorphism $f : F \to G$ to be a collection $f = \{f_i\}$ of sheaf morphisms $f_i : F_i \to G_i$ satisfying $f_i \circ \varphi_{ij} = \gamma_{ij} \circ f_j$. Composition is defined in an obvious way and so we obtain a category of $\underline{\alpha}$-twisted sheaves, which depends both on the cover $\mathcal{U}$ and on the cocycle $\underline{\alpha}$. It can be checked that the operations of passing to a refinement $\mathcal{U}'$ of $\mathcal{U}$ and of replacing $\alpha$ by a cohomologous cocycle $\alpha'$, give rise to an equivalent category of $\underline{\alpha}'$-twisted sheaves. Thus for any $\alpha \in H^2(X, \mathcal{O}_X^\times)$ we get category $(\mathcal{O}_X, \alpha)$-$\text{mod}$ of $\alpha$-twisted sheaves on $X$ (defined only up to a non-canonical equivalence). An $\alpha$-sheaf $F$ on $X$ is called quasi-coherent (respectively coherent) if each $F_i$ is quasi-coherent (respectively coherent). We will write $\text{QCoh}(X, \alpha)$ and $\text{Coh}(X, \alpha)$ for the categories of quasi-coherent and coherent $\alpha$-twisted sheaves. Note that $(\mathcal{O}_X, \alpha)$-$\text{mod}$, $\text{QCoh}(X, \alpha)$ and $\text{Coh}(X, \alpha)$ are all abelian categories.

More intrinsically the $\alpha$-twisted sheaves on $X$ can be interpreted as weight one sheaves on $\alpha X$, where a sheaf on $\alpha X$ is understood as a representation of the sheaf of groupoids $\alpha X \to X$. To spell what this means, let us denote by $\mathcal{Z} \text{Coh}_X$ the stack of quasi-coherent sheaves on the space $X$. Let $\mathcal{Z} \to X$ be any fibered category over $X$. Recall (see e.g. [Del90 Section 3.3] or [LMB00 Definition 13.3.3]) that a representation of $\mathcal{Z}$ is a morphism $F : \mathcal{Z} \to \mathcal{Z} \text{Coh}_X$ of fibered categories defined over $X$. Explicitly this means that for any algebraic (or analytic) space $T \to X$ we are given a functor $F_T : \mathcal{Z}(T) \to \text{QCoh}(T)$ so that $F_T$ is compatible with base changes.

In particular, if $\alpha X \to X$ is an $\mathcal{O}_X^\times$-gerbe, then a representation of $\alpha X$ is a $X$-functor $F : \alpha X \to \mathcal{Z} \text{Coh}_X$. Given an integer $n$ we say that $F$ is a pure $\alpha X$-representation of weight $n$ if for any open $U \subset X$ and any section $L \in \alpha X(U)$ the natural sheaf homomorphism $\text{Aut}_U(L) \to \text{Aut}_U(F(L))$ induced by $F$ factors as

$$
\begin{array}{ccc}
\text{Aut}_U(L) & \longrightarrow & \text{Aut}_U(F(L)) \\
\downarrow & & \downarrow \\
\mathcal{O}_U^\times & \longrightarrow & \mathcal{O}_U^\times \\
\bigcirc^n & \downarrow & \\
\mathcal{O}_U^\times & \longrightarrow & \mathcal{O}_U^\times
\end{array}
$$
where in the bottom row the map is the raising into power \( n \). It is instructive to point out that when we are dealing with the trivial gerbe \( _0X \) on \( X \), a representation of \( _0X \) is nothing but a quasicoherent sheaf \( F \) on \( X \) equipped with a direct sum decomposition \( F = \oplus_{n \in \mathbb{Z}} F_n \) into quasicoherent sheaves \( F_n \) so that a locally defined function \( f \in \mathcal{O}_X \) acts on \( F \) as multiplication by \( f^n \) on \( F_n \). The reader can check as an exercise that the category of representations of \( _\alpha X \) of pure weight one is equivalent to the category \( \text{QCoh}(X, \alpha) \) and that the category of representations of \( _\alpha X \) of pure weight \( n \) is equivalent to the category \( \text{QCoh}(X, n\alpha) \).

2.1.2 Geometric gerbes and their presentations

In this section we recall a more geometric approach to \( \mathcal{H} \)-gerbes which involves gluing of certain good local models. This exploits the standard idea that various geometric objects can be conveniently presented in terms of an atlas modulo certain gluing relations on it. For example, for a a manifold \( X \), an atlas \( U \) can be taken to be the disjoint union \( U = \coprod_i U_i \) of coordinate charts, and the gluing can be specified by the closed subset of relations

\[
R := U \times_X U \subset U \times U,
\]

which comes together with two maps \( s, t : R \to U \) (corresponding to the two projections of \( U \times U \) onto \( U \)) each of which is a local diffeomorphism. Analogously, presentations can be used to define schemes, algebraic spaces and analytic spaces.

Formally, a presentation by objects in a (fibered) category \( \mathcal{C} \) (or a groupoid in \( \mathcal{C} \)) consists of the following data:

- (atlas) an object \( U \) of \( \mathcal{C} \)
- (relations) an object \( R \) of \( \mathcal{C} \)
- (source-target maps) \( \xrightarrow{s} R \)
- (composition map) \( R \times_U R \xrightarrow{m} R \)
- (inversion map) \( R \xrightarrow{i} R \)
- (identity map) \( U \xrightarrow{e} R \).

These data are subject to the obvious analogues of the group axioms, applied to the maps \( m, i \) and \( e \).

Note that any morphism \( \gamma : U \to X \) in \( \mathcal{C} \) determines a presentation \( (\mathcal{R}, U, m, i, e) \) in \( \mathcal{C} \), where: \( \mathcal{R} := U \times_X U \); the maps \( s, t \) are the two projections; the composition map \( m \) sends \( (a, b) \times (b, c) \) to \( (a, c) \), \( i \) sends \( (a, b) \) to \( (b, a) \); and \( e \) is the diagonal map. In this situation we identify \( X \) with the quotient \( U/\mathcal{R} \) and we will say that \( \mathcal{R} \xrightarrow{s} U \) is generated by \( \gamma \).

Let now \( \gamma : U \to X \) be a morphism of complex schemes and let

\[
\xrightarrow{s} \gamma \xrightarrow{t} X
\]
be the presentation of $X$ generated by $\gamma$. Let $p_1, p_2, m : R \times_U R \to R$ denote the two projections and the multiplication map respectively. Let $H$ be an abelian group scheme over $X$, $\mathcal{H} \to X$ its sheaf of sections, $\mathcal{H}$ its pullback to $R$ via $\gamma \circ s = \gamma \circ t$, and let $\pi : R \to R$ be an $\mathcal{H}$-torsor over $R$. In order for $R \xrightarrow{\epsilon} U$, with $s := s \circ \pi, t := t \circ \pi$, to be a groupoid we need a biextension isomorphism

$$p_1^*R \otimes p_2^*R \to m^*R$$

of torsors over $R \times_U R$, as well as a lifting

$$e : U \to R$$

of $\epsilon$, i.e. a trivialization of $\epsilon^*R$ on $U$. Given these data we obtain a new presentation

$$(R \xrightarrow{\epsilon} U, m, i, e).$$

We claim that this presentation determines an $\mathcal{H}$-gerbe $[U/R]$ on $X$, which we can interpret as the (stacky) quotient of $U$ by $R$.

Indeed, for any open $V$ in $X$, define $[U/R]'(V)$ to be the category of pairs $(T, j)$, where $T$ is an $H$-torsor over $U|_V$ and

$$j : t^*T \xrightarrow{\cong} s^*T \otimes R$$

is an isomorphism of $\mathcal{H}$-torsors on $R$. As $V$ varies, this gives a prestack $[U/R]'$ of groupoids over $X$. We define $[U/R]$ to be the stackification (see Remark 2.3 below) of $[U/R]'$. By construction $[U/R]'(V)$ is a torsor over the tensor category of pairs $(T_0, j_0)$ where $T_0$ is an $H$-torsor over $U|_V$ and

$$j_0 : t^*T_0 \xrightarrow{\cong} t^*T_0$$

is an isomorphism of $\mathcal{H}$-torsors on $R|_V$. The stackification of the latter is identified, via descent, with $B\mathcal{H}$ and so $[U/R]$ is indeed an $\mathcal{H}$-gerbe on $X$.

**Remark 2.3** The necessity of taking stackification in the above construction is dictated by the subtlety of the conditions required to have a ‘sheaf of categories’. Let $\mathcal{X} \to X$ be a category fibered in groupoids. Recall that there are two types of sheaf-like conditions on can impose on $\mathcal{X}$:

1. For any open $V \subset X$ and any two objects $\xi, \eta \in \mathcal{X}(V)$, the presheaf of sets $(\text{open } W \subset V) \mapsto \text{Hom}_\mathcal{X}(W)(\xi|_W, \eta|_W)$ is required to be a sheaf.
2. If $V = \bigcup_i W_i$ is an open covering of $V$ and we have
\[ \xi_i \in \text{ob}(\mathcal{X}(W_i)); \]

\[ \varphi_{ij} : \xi_{j|W_{ij}} \xrightarrow{\sim} \xi_{i|W_{ij}} \text{ isomorphisms satisfying the cocycle condition;} \]

then we require the existence of an object \( \xi \in \text{ob}(\mathcal{X}(V)) \) together with isomorphisms \( \psi_i : \xi_{|W_i} \xrightarrow{\sim} \xi_i \) so that \( \varphi_{ij} = \psi_i \circ \psi_j^{-1} \).

Now if \( \mathcal{X} \) satisfies (1) we say that \( \mathcal{X} \) is a prestack over \( X \) and if it satisfies (1) + (2), then we say that \( \mathcal{X}(X) \) is a stack.

Given any prestack \( \mathcal{X} \) one shows (see [LMB00, §3] for details) that there is a unique (up to equivalence) stack \( \mathcal{X}^a \to X \) together with a map \( \mathcal{X} \to \mathcal{X}^a \) which is fully faithful and locally on \( X \) is essentially bijective. The stack \( \mathcal{X}^a \) is called the stackification of \( \mathcal{X} \) and is completely analogous to the sheaf one associates with a presheaf of sets.

**Remark 2.4** (i) We say that a groupoid \( (R \xrightarrow{s,t} U, m, i, e) \) of algebraic (or analytic) spaces is smooth (respectively etale) if the structure maps \( s \) and \( t \) are smooth (respectively etale). The stacks \( \mathcal{X} \to X \) over \( X \) which admit a smooth or etale groupoid presentation (lifting a presentation for \( X \)) are the stacks which are closest to schemes and on which one can do geometry in essentially the same way as on spaces. In fact if \( \mathcal{X} \) is a stack which admits a smooth (respectively etale) presentation, then \( \mathcal{X} \) is called an Artin algebraic stack (respectively Deligne-Mumford stack) and is the main object of study in the algebraic geometry of stacks [LMB00].

We consider only stacks for which the diagonal map \( \mathcal{X} \to \mathcal{X} \times_X \mathcal{X} \) is affine. Note that for an \( \mathcal{H} \)-gerbe \( \alpha X \to X \), the condition of having an affine diagonal is equivalent to \( H \to X \) being an affine group scheme. In particular \( \alpha X \) is an algebraic stack (in the sense of Artin) if and only if \( \alpha X \) has a groupoid presentation.

(ii) The case of main interest for us is when \( H = \mathbb{G}_m \), so \( \mathcal{G} = \mathcal{O}_X^\times \). In this case \( R \) is the total space of a (punctured) line bundle on \( \mathfrak{X} \) and \( j, j_0 \) are isomorphisms of line bundles.

(iii) The above discussion has an obvious analogue where schemes are replaced by (algebraic or analytic) spaces or manifolds.

(iv) Not every presentation of \( X \) will lift to a presentation of a given gerbe \( \alpha X \). For example, only the trivial gerbe \( \alpha X = B\mathcal{H} \to X \) can be presented by a lift of the trivial presentation \( X \xrightarrow{\alpha} X \to X \) generated by \( \text{id}_X : X \to X \). However if \( \alpha = \{\alpha_{ijk}\} \) is an \( \mathcal{H} \)-valued Čech cocycle w.r.t. an open covering \( \{U_i\} \) of \( X \), then the presentation of \( X \) generated by \( \gamma : U = \coprod U_i \to X \) can be lifted to a presentation

\[ \xymatrix{ R \ar[r]^s \ar@{>->}[r]_t & U \ar[r]^-\gamma & X } \]

for an \( \mathcal{H} \)-gerbe \( \alpha X \), whose classifying element is the class \( \alpha = [\underline{\alpha}] \in H^2(X, \mathcal{H}) \). To define this presentation we take \( R := \coprod_{i,j} U_{ij} \times \mathcal{H} \) with its natural projections \( s \) and \( t \) onto...
$U = \bigsqcup_i U_i$. The multiplication map $m : R \times_U R \to R$ sends a point

$$(x; a, b) \in U_{ijk} \times_X H \times_X H = (U_{ij} \times_X H) \times_{U_i} (U_{jk} \times_X H) \subset R \times_U R$$

to the point $(x; \alpha_{ijk} \cdot a \cdot b) \in U_{ik} \times_X X_{H \times X}$. The inversion $i : R \to R$ sends $(x, a) \in U_{ij} \times_X H$ to the point $(x, a^{-1})$ and the identity $e : U \to R$ sends $x \in U_i$ to the point $(x, x; 1) \in U_{ii} \times_X H$.

Slightly more generally: the same reasoning shows that for any map of complex spaces $\gamma : U \to X$ and any $\alpha \in \check{H}^2(X, \mathcal{H})$ it follows that the presentation of $X$ generated by $\gamma$ can be lifted to a presentation for the $\mathcal{H}$-gerbe $a_X$ if and only if $\gamma^* \alpha = 0 \in \check{H}^2(U, \gamma^* \mathcal{H})$.

**Basic construction, continued:** Assume we are given a short exact sequence of sheaves of groups

$$1 \to \mathcal{H} \to \mathcal{G} \to \mathcal{K} \to 1$$

on an algebraic (analytic) space $X$. Suppose that $\mathcal{H}$ is commutative and that the sheaves $\mathcal{H}$, $\mathcal{G}$ and $\mathcal{K}$ are represented by group schemes $H$, $G$ and $K$ respectively. In section 2.1.1 we associated to every $\mathcal{K}$ torsor $T \to X$ an $\mathcal{H}$-gerbe $\delta(T)$ with class $\delta([T]) \in H^2(X, \mathcal{H})$. In this situation the gerbe $\delta(T)$ comes with a natural presentation:

$$R \xrightarrow{s} U \xrightarrow{t} \delta(T).$$

Here $\gamma : U \to X$ is the scheme representing $T$ and $R$ is defined as follows. The presentation of $X$ generated by $\gamma$ has $\mathcal{R} = U \times_X U = U \times_X K$ since $U$ is an $H$-torsor. Furthermore, under the identification $\mathcal{R} = U \times_X K$ the structure maps $s$ and $t$ become the projection on $U$ and the action of $K$ respectively. In other words $\mathcal{R} \xrightarrow{s} U$ is the transformation groupoid for the action of $K$ on $U$ and $X = U/\mathcal{R} = U/K$. To get the presentation $R \xrightarrow{s} U$ of $\delta(T)$ we can just take the transformation groupoid for the action of $G$ on $U$ (where $H$ acts trivially), i.e. take $R := U \times_X G$ with $s$ and $t$ being again the projection and the action maps. Equivalently we may take $R \to \mathcal{R}$ to be the trivial $\mathcal{G}$-torsor and check that it satisfies the biextension and trivialization conditions. In particular we get that $\delta(T)$ is a quotient gerbe - it is identified as the quotient

$$\delta(T) = [U/R] = [U/G]$$

of the space $U$ by the group scheme $G$, where $G$ acts with a stabilizer $H$ at each point.

**Example 2.5** As a special case of the above we obtain the Azumaya presentation of an $\mathcal{O}_X$-gerbe. Let $P$ be a $\mathbb{P}^{n-1}$ bundle on a scheme $X$ and let $\mathfrak{P}$ denote the corresponding sheaf of sections. The bundle $P$ is associated to a unique $\mathbb{P}GL_n$-bundle $U \to X$ (the frame bundle
of $P$) whose sheaf of sections $\mathcal{U}$ is naturally a torsor over $\mathbb{P}GL_n(\mathcal{O}_X)$. The image of $\mathcal{U}$ under the coboundary map for the sequence

$$1 \to \mathcal{O}_X^\times \to GL_n(\mathcal{O}_X) \to \mathbb{P}GL_n(\mathcal{O}_X) \to 1$$

is an $\mathcal{O}_X^\times$-gerbe $\mathfrak{p}X$ on $X$ which comes together with a right Azumaya presentation of $\mathfrak{p}X$:

$$R^r \xrightarrow{s} U \xrightarrow{t} \mathfrak{p}X$$

where $R^r := U \times GL_n(\mathbb{C})$, $s : R^r \to U$ is the projection and $t : R^r \to U$ is the (right) action of $GL_n(\mathbb{C})$ on $U$. Alternatively one may consider the sheaf $\mathcal{A}_\mathfrak{p}$ of Azumaya algebras corresponding to $\mathfrak{p}$. The subsheaf $\mathcal{A}_\mathfrak{p}^\times \subset \mathcal{A}_\mathfrak{p}$ of invertible elements in $\mathcal{A}_\mathfrak{p}$ is representable by an affine group scheme $\mathcal{A}_\mathfrak{p}^\times \to X$ which acts simply transitively on the left on the frame bundle $U \to X$. Using this group scheme we get a left Azumaya presentation of $\mathfrak{p}X$:

$$R^l \xrightarrow{s} U \xrightarrow{t} \mathfrak{p}X$$

where $R^l := A_{\mathfrak{p}}^\times \times_X U$, $s : R^l \to U$ is the projection and $t : R^l \to U$ is the (left) action of $A_{\mathfrak{p}}^\times$ on $U$.

The same gerbe $\mathfrak{p}X$ has yet another presentation, called the Brauer-Severi presentation. Here the atlas is $P$ itself and the relations are the total space of the punctured line bundle $\mathcal{O}(1,-1)^\times$ on $P \times_X P$.

It is often useful to describe the sheaves on a gerbe as cartesian sheaves on the simplicial space generated by a presentation or equivalently as descent datum for a presentation. Concretely, given a flat presentation

$$R \xrightarrow{s} U \xrightarrow{t} \alpha X$$

of an $\mathcal{H}$-gerbe $\alpha X$ on $X$ we have a simple interpretation (see e.g. [Del90 Section 3.3] or [LMB00 Propositions 12.8.2 and 13.2.4] for details and proofs) for the category of sheaves on $\alpha X$: a sheaf of $\mathcal{O}_{\alpha X}$-modules (respectively: a line bundle, a vector bundle) on $\alpha X$ can be identified with a pair $(F, j)$, where $F$ is a sheaf of $\mathcal{O}_U$-modules (respectively: a line bundle, a vector bundle, a complex of sheaves, etc.) on $U$, and $j : s^*F \xrightarrow{\sim} t^*F$ is an isomorphism of sheaves on $R$ satisfying the cocycle condition

$$\begin{align*}
(p_2^*j) \circ (p_1^*j) &= m^*j
\end{align*}$$

To write this condition one uses the natural identifications

$$p_1^* s^* F \xrightarrow{p_1^*j} p_1^* t^* F = p_2^* s^* F \xrightarrow{p_2^*j} p_2^* t^* F,$$

and

$$p_1^* s^* F = m^* s^* F \xrightarrow{m^*j} m^* t^* F = p_2^* t^* F,$$

provided by the groupoid axioms.
on $R \times_{t,U} R$ and the normalization $e^*(j) = \text{id}_F$ on $U$. Assume now that $\mathcal{H} = H(\mathcal{O}_X)$ for some complex reductive abelian group $H$. In this situation we can recast the above description of sheaves on $\alpha X$ in terms of the presentation $\mathfrak{M} \xrightarrow{s} t \rightarrow U$, and the normalization $e^*(\gamma) = \text{id}$. Assume now that $H = H(O_X)$ for some complex reductive abelian group $H$. In this situation we can recast the above description of sheaves on $\alpha X$ in terms of the presentation $R \rightarrow t \rightarrow U$ generated by $\gamma: U \rightarrow X$ and the $H$-torsor $\pi: R \rightarrow \mathfrak{M}$. Given a sheaf of modules $(F, j)$ on $\alpha X$ we can use the map $\pi: R \rightarrow R$ to push the isomorphism $j$ down to $R$. Decomposing according to the characters $\hat{H}$ of $H$ we see that $\pi^*(j)$ corresponds to a family $\{j_\chi\}$ of isomorphisms, where $j_\chi: s^*F \otimes (\pi^*\mathcal{O}_R)_\chi \xrightarrow{\sim} t^*F$.

The category of sheaves of modules on $\alpha X$ is therefore “graded” by the character group $\hat{H}$. A sheaf of weight $0 \in \hat{H}$ is just a sheaf of modules on $X$. In case $H = \mathbb{G}_m$ we have $\hat{H} = \mathbb{Z}$ and the sheaves of weight $n$ are precisely the sheaves of weight $n$ in the sense of section 2.1.1. In particular, the sheaves of weight $1$ are the $\alpha$-twisted sheaves on $X$. This observation leads to a very concrete description of the weight one sheaves on a $\mathbb{G}_m$-gerbe. Starting with a presentation (2.1) of a $\mathbb{G}_m$-gerbe $\alpha X$ on $X$ write $\mathcal{L} \rightarrow \mathfrak{M} = U \times_X U$ for the line bundle associated to the $\mathbb{G}_m$-torsor $R \rightarrow \mathfrak{M}$ via the tautological character $\text{id}: \mathbb{G}_m \rightarrow \mathbb{G}_m$. The groupoid condition on the presentation (2.1) gives us a biextension isomorphism $p_{12}^*\mathcal{L} \otimes p_{23}^*\mathcal{L} = p_{13}^*\mathcal{L}$ on $U \times_X U \times_X U$ and so a sheaf on $\alpha X$ is the same thing as a sheaf $F$ on $U$ equipped with an $\mathcal{L}$-twisted descent datum on $U \times_X U$, i.e. with an isomorphism

$$p_1^*F \xrightarrow{j} p_2^*F \otimes \mathcal{L},$$

of sheaves on $U \times_X U$, satisfying the cocycle condition

$$(2.3) \quad p_{13}^*j = (p_{23}^*j \otimes \text{id}_{p_{23}^*\mathcal{L}}) \circ p_{12}^*j$$
on $U \times_X U \times_X U$. Note that in writing (2.3) we had to use the biextension isomorphism for $\mathcal{L}$.

**Example 2.6** Specializing the previous discussion to the case of the Azumaya gerbe $\mathfrak{P}X$ of example 2.5 we get a natural identification of the category $D_n^b(\mathfrak{P}X)$ with the derived category of complexes of quasi-coherent sheaves on $X$ equipped with an action of the Azumaya algebra $\mathfrak{P}A$ and such that the center $\mathcal{O}_X^\times$ of $\mathfrak{P}A^\times$ acts on the cohomology sheaves with character $n$.

### 2.1.3 Brauer groups

Since the $\mathcal{O}^\times$ gerbes are naturally classified by elements in cohomological Brauer groups, it will be helpful to have an overview of the different variants of the Brauer group of a complex space before discussing properties of individual gerbes.

Below we are going to discuss three versions of the Brauer group of a ringed space $Z$: Azumaya ($Br(Z)$), geometric ($Br_{\text{geom}}(Z)$), and cohomological ($Br'(Z)$). Each of these makes sense in either the etale or the analytic topology on $Z$. In particular, for a complex algebraic
space $Z$ we have a diagram:

$$
\begin{array}{ccc}
Br(Z) & \rightarrow & Br(Z)_{\text{geom}} \\
& \downarrow & \downarrow \\
Br_{\text{an}}(Z) & \rightarrow & Br_{\text{an}}(Z)_{\text{geom}}
\end{array}
\rightarrow
\begin{array}{ccc}
& \rightarrow & \rightarrow \\
& \downarrow & \downarrow \\
& \rightarrow & \rightarrow \\
Br_{\text{an}}(Z) & \rightarrow & Br_{\text{an}}(Z)_{\text{geom}} \\
& \downarrow & \downarrow \\
& \rightarrow & \rightarrow \\
Br_{\text{an}}(Z) & \rightarrow & Br_{\text{an}}(Z).
\end{array}
$$

When $Z$ is smooth, the following facts are known:

(i) all the maps in this diagram are injective;

(ii) $Br'(Z)$ is torsion by the purity theorem from [Gro68c];

(iii) $\text{im}[Br'(Z) \rightarrow Br'_{\text{an}}(Z)]$ coincides (see [Mil80]) with the torsion subgroup of $Br'_{\text{an}}(Z)$;

Grothendieck has conjectured that the inclusion $Br(Z) \hookrightarrow Br'(Z)$ is an isomorphism for all smooth quasi-projective schemes. This may hold also for separated normal $Z$. The validity of the conjecture was established in the algebraic setting in [Gab81, Hoo82, Sch01] for arbitrary curves, for normal separated algebraic surfaces, for abelian varieties, for smooth toric varieties and for separated unions of two affine varieties. The analogous conjecture in the analytic case is virtually unexplored. The only general result to date [HS03] concerns analytic K3 surfaces and asserts that every torsion class in $Br'_{\text{an}}(X)$ of an analytic K3 surface $X$ comes from an Azumaya algebra on $X$.

**Remark 2.7** As a corollary of fact (i) and Grothendieck’s conjecture, we get that $Br(Z) = Br(Z)_{\text{geom}}$ for a smooth $Z$. This corollary is known to hold [EHKV01] in many cases in which the Grothendieck conjecture is still unknown. In fact, for a normal Noetherian scheme, the result of [EHKV01] Theorem 3.6] characterizes the image of $Br(Z)$ in $Br(Z)_{\text{geom}}$ as the algebraic-geometric gerbes for which one can find a flat presentation

$$\left( \frac{R}{s-t}U, m, i, e, \gamma \right)$$

with a projective structure map $\gamma : U \rightarrow Z$, or equivalently as the classes of $\mathbb{G}_m$ gerbes of quotient type (i.e. a quotient of an algebraic space by an affine algebraic group). In general this characterization seems to be optimal since there are examples of quotient gerbes on non-separated surfaces whose isomorphism class is not represented by an Azumaya algebra, and examples of infinite order elements in $Br'(Z)$ for a normal separated $Z$ which are represented by algebraic-geometric gerbes but are not quotient gerbes [EHKV01 Examples 2.21 and 3.12].

If $Z$ is a complex scheme, then the *Azumaya Brauer group* $Br(Z)$, is defined [Gro68a] as the group of Morita equivalence classes of sheaves of Azumaya algebras on $Z$. Recall
that an Azumaya algebra on \( Z \) is a coherent sheaf of algebras which locally in the etale topology on \( Z \) is isomorphic to the endomorphisms algebra of an algebraic vector bundle on \( Z \). Two Azumaya algebras \( A \) and \( B \) are called Morita equivalent if etale locally on \( Z \) we can find vector bundles \( E \) and \( F \) so that the sheaves of algebras \( A \otimes \mathcal{E}nd(E) \) and \( B \otimes \mathcal{E}nd(F) \) are isomorphic. Morita equivalence classes of Azumaya algebras form a commutative group under the operation of tensoring over \( \mathcal{O}_Z \); the inverse is given by the opposite algebra.

The Skolem-Noether theorem \([Mil80, Proposition 2.3]\) implies that the Azumaya algebras of rank \( n^2 \) are classified by elements in \( H^1_{et}(Z, \mathbb{P}GL(n)) \). The short exact sequence of groups schemes over \( Z \):

\[
1 \to \mathbb{G}_m \to GL(n) \to \mathbb{P}GL(n) \to 1,
\]

gives rise to a coboundary map

\[
(2.4) \quad H^1_{et}(Z, \mathbb{P}GL(n)) \to H^2_{et}(Z, \mathbb{G}_m).
\]

The image of \( a \in H^1_{et}(Z, \mathbb{P}GL(n)) \) under this coboundary map is an \( n \)-torsion class in \( H^2_{et}(Z, \mathbb{G}_m) \) which is the obstruction to representing \( a \) by the endomorphism algebra of a rank \( n \) vector bundle. In particular the map \((2.4)\) induces a homomorphism

\[
(2.5) \quad Br(Z) \to H^2_{et}(Z, \mathbb{G}_m) \subset H^2_{et}(Z, \mathbb{G}_m).
\]

When \( Z \) is smooth, the homomorphism \((2.5)\) is known to be injective \([Mil80, Theorem IV.2.5]\). This suggests that the Brauer classes are intimately related to elements in \( H^2_{et}(Z, \mathbb{G}_m) \) and so one defines the algebraic cohomological Brauer group:

\[
Br'(Z) := H^2_{et}(Z, \mathbb{G}_m).
\]

Recall that by fact (ii) the group \( H^2_{et}(Z, \mathbb{G}_m) \) is purely torsion. As explained in section \( \ref{2.1.2} \) Azumaya algebras give rise to groupoid presentations of \( \mathbb{G}_m \)-gerbes on \( Z \). In other words, for a smooth \( Z \) the inclusion \((2.5)\) can be refined to a sequence of inclusions:

\[
Br(Z) \hookrightarrow Br(Z)_{geom} \hookrightarrow Br'(Z) = H^2_{et}(Z, \mathbb{G}_m) = H^2_{et}(Z, \mathbb{G}_m)_{tor},
\]

where \( Br(Z)_{geom} \) denotes the group of equivalence classes of algebraic-geometric \( \mathbb{G}_m \)-gerbes on \( Z \). Recall that a \( \mathbb{G}_m \)-gerbe is algebraic geometric if it is an algebraic stack in the sense of Artin, i.e. if it admits a flat (equivalently, a smooth) groupoid presentation \([Art74]\).

By analogy we define the analytic Azumaya Brauer group \( Br_\text{an}(Z) \), and the analytic geometric Brauer group \( Br'_\text{an}(Z)_{geom} \) of an analytic space \( Z \), as the groups of Morita equivalent classes of analytic Azumaya algebras on \( Z \) and of isomorphism classes of analytic geometric \( \mathcal{O}_Z^{\times} \)-gerbes respectively. The isomorphism type of an \( \mathcal{O}_Z^{\times} \)-gerbe is determined by a class in the analytic cohomological Brauer group:

\[
Br'_\text{an}(Z) := H^2_\text{an}(Z, \mathcal{O}_Z^{\times}).
\]
In other words, the classifying map

\[ Br_{an}(Z)_{\text{geom}} \hookrightarrow Br'_{an}(Z) \]

is injective.

The analytic cohomological Brauer group can be studied via the exponential sequence:

\[ 0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^\times \to 1. \]

The corresponding cohomology sequence gives

\[ \begin{array}{ccccccccc}
\text{ } & 0 & \to & H^2_{\text{an}}(Z, \mathcal{O}_Z) / H^2(Z, \mathbb{Z}) & \to & Br'_{an}(Z) & \to & \ker [H^3(Z, \mathbb{Z}) \to H^3_{\text{an}}(Z, \mathcal{O}_Z)] & \to & 0.
\end{array} \]

This of course is analogous to the usual description of the Picard group:

\[ \begin{array}{ccccccccc}
0 & \longrightarrow & H^1_{\text{an}}(Z, \mathcal{O}_Z) / H^1(Z, \mathbb{Z}) & \longrightarrow & \text{Pic}(Z) & \longrightarrow & \ker [H^2(Z, \mathbb{Z}) \to H^2_{\text{an}}(Z, \mathcal{O}_Z)] & \longrightarrow & 0.
\end{array} \]

In addition, if \( Z \) is compact and Kähler, the Hodge theorem implies that

\[ \text{im}[H^1(Z, \mathbb{Z}) \to H^1_{\text{an}}(Z, \mathcal{O}_Z)] \subset H^1_{\text{an}}(Z, \mathcal{O}_Z), \]

is a discrete subgroup of maximal rank. Hence, we can identify the connected component of Pic(Z) with the quotient of its tangent space \( H^1_{\text{an}}(Z, \mathcal{O}_Z) \) by \( H^1(Z, \mathbb{Z}) \). In the case of \( Br'_{an}(Z) \), there is still a ‘tangent space’: \( H^2_{\text{an}}(Z, \mathcal{O}_Z) \), but it is divided by the typically non-discrete subgroup

\[ \text{im}[H^2(Z, \mathbb{Z}) \to H^2_{\text{an}}(Z, \mathcal{O}_Z)] \subset H^2_{\text{an}}(Z, \mathcal{O}_Z), \]

and so there is no good (=separated) topology on \( Br'_{an}(Z) \).

In the special case when \( Z \) is a K3 surface, we get that \( Br_{an}(Z)_{\text{geometric}} = Br'_{an}(Z) \) is the quotient of the one dimensional vector space \( H^1_{\text{an}}(Z, \mathcal{O}_Z) \) by the lattice dual to the transcendental lattice of \( Z \), i.e. by \( H^2(Z, \mathbb{Z})/H^1_{Z, \text{trans}} \). Notice that for a very general analytic K3 this lattice has rank 22 and for a very general algebraic K3 it has rank 21.

More precisely, one defines the transcendental lattice \( T_Z \) of a K3 surface \( Z \) by the short exact sequence:

\[ 0 \to T_Z \to H^2(Z, \mathbb{Z}) \to H^{1,1}_{Z, \text{trans}} \to 0. \]

In other words, \( T_Z \) is the sublattice of \( H^2(Z, \mathbb{Z}) \) consisting of classes perpendicular to all classes of curves in \( Z \). The dual sequence reads:

\[ 0 \to H^{1,1}_{Z, \text{trans}} \to H^2(Z, \mathbb{Z}) \to T_Z^\vee \to 0, \]
and we have a natural map

\[ \text{Hom}_\mathbb{Z}(T_Z, \mathbb{R}) \cong H^2(\mathbb{Z}, \mathbb{R})/(H^1_\mathbb{Z}(\mathbb{Z}) \otimes \mathbb{R}) \to H^2(\mathbb{Z}, \mathbb{R})/H^1_\mathbb{R}(\mathbb{Z}) \cong H^2_{\text{an}}(\mathbb{Z}, \mathcal{O}_\mathbb{Z}). \]

This leads to the following commutative diagram with exact rows and columns:

\[
\begin{array}{ccccccc}
0 & \to & H^1_\mathbb{R}(\mathbb{Z})/(H^1_\mathbb{Z}(\mathbb{Z}) \otimes \mathbb{R}) & \to & \text{Hom}_\mathbb{Z}(T_Z, \mathbb{Z}) & \to & H^2_{\text{an}}(\mathbb{Z}, \mathcal{O}_\mathbb{Z}) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & H^1_\mathbb{R}(\mathbb{Z})/(H^1_\mathbb{Z}(\mathbb{Z}) \otimes \mathbb{R}) & \to & \text{Hom}_\mathbb{Z}(T_Z, \mathbb{R}) & \to & 0 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \text{Hom}_\mathbb{Z}(T_Z, \mathbb{R}/\mathbb{Z}) & \to & 0 & \to & 0 & \to & 0 \\
\end{array}
\]

The bottom row explicates \( Br_{\text{an}}(\mathbb{Z}) = Br'_{\text{an}}(\mathbb{Z}) \) as the quotient of the real torus \( \text{Hom}_\mathbb{Z}(T_Z, \mathbb{R}/\mathbb{Z}) \) by the vector space \( H^1_\mathbb{R}(\mathbb{Z})/(H^1_\mathbb{Z}(\mathbb{Z}) \otimes \mathbb{R}) \), embedded in it as a (usually dense) subgroup. Note that this vector space does not contain any torsion points of the torus. Equivalently the restricted map

\[ \text{Hom}_\mathbb{Z}(T_Z, \mathbb{Q}/\mathbb{Z}) \to Br_{\text{an}}(\mathbb{Z})_{\text{tor}} \]

is an isomorphism. When \( Z \) happens to be an algebraic K3 surface we have a natural identification \( Br(Z) = Br_{\text{an}}(Z)_{\text{torsion}} \) and so we recover the standard interpretation of elements of the algebraic Brauer group of \( Z \) as a homomorphism from the transcendental lattice of \( Z \) to \( \mathbb{Q}/\mathbb{Z} \) (see e.g. [C˘ al00, Lemma 5.4.1] or [C˘ al01]).

### 2.2 Tate-Shafarevich groups and genus one fibrations

In this section we review some basic facts about twisted forms of a given elliptic fibration over an analytic space \( B \). For more details the reader is referred to the excellent references [DG94] and [Nak01]. First we recall some terminology and set up the notation.

For us a \textit{genus one fibration} will always mean a holomorphic map \( \pi : X \to B \) between normal analytic varieties whose generic fiber is a smooth curve of genus one. We define an \textit{elliptic fibration} to be a genus one fibration equipped with a holomorphic section \( \sigma : B \to X \) of \( \pi \). Note that this is slightly more restrictive than the conventional notion of an elliptic fibration used in say [DG94], [Nak01], where only the existence of a meromorphic section of \( \pi \) is required. A genus one fibration will be called (relatively) minimal if \( X \) has at most terminal singularities and if the canonical class \( K_X \) is \( \pi \)-nef.
Let now $X$ and $B$ be normal analytic varieties and let

\[
X \xrightarrow{\pi} B
\]

be an elliptic fibration on $X$. Let $D \subset B$ denote the discriminant divisor of $\pi$ and let $B^\circ := B - D$, $B^{\circ\circ} := B - \text{Sing}(D)$. The corresponding inclusions are denoted by $\iota : D \hookrightarrow B$, $j^0 : B^\circ \hookrightarrow B$ and $j^{\circ\circ} : B^{\circ\circ} \hookrightarrow B$. We also put $X^\circ := X \times_B B^\circ$, $X^{\circ\circ} := X \times_B B^{\circ\circ}$ and $\pi^0 := \pi|_{X^\circ}$, $\pi^{\circ\circ} := \pi|_{X^{\circ\circ}}$.

Sometimes we may need to require the additional genericity assumption that $X$ is smooth and that $\pi : X \to B$ is Weierstrass.

**Remark 2.8** (i) When $X$ is a surface, the genericity assumption implies in particular that all the singular fibers of $\pi$ are of Kodaira types $I_1$ or $II$, i.e. they are nodes and cusps.

(ii) In this paper we will always deal with a situation in which $X$ is smooth and either $\pi$ is smooth or $X$ is a surface and $\pi$ has at worst $I_1$ fibers. We have included in the present discussion the more general case of an arbitrary Weierstrass $\pi$ with a smooth total space, because of the potential applications of our duality construction to genus one fibered Calabi-Yau manifolds of arbitrary dimension. This however goes beyond the scope of the present work and will be the subject of a future paper.

Let $X^\sharp \subset X$ denote the regular locus of $\pi$, viewed as an abelian group scheme over $B$. Denote by $\mathcal{X}_{an}$ the corresponding sheaf of abelian groups in the analytic topology on $B$. When $B$ and $X$ happen to underly complex algebraic varieties we will write $\mathcal{X}$ for the etale sheaf of sections of $X^\sharp \to B$.

The **analytic Weil-Châtelet group** $WC_{an}(X)$ of $X$ is the group of bimeromorphism classes of analytic genus one fibrations $Y \to B$ such that:

- $Y \times_B B^\circ \to B^\circ$ is bimeromorphic to a smooth genus one fibration;
- The relative Jacobian fibration $\text{Pic}^0(Y/B)$ is bimeromorphic to $X^\sharp$ (and hence to $X$).

Note that this definition makes sense since for a suitably chosen dense open subset $U \subset B$ the (sheafification of the) presheaf $\mathcal{P}ic^0(Y/U)$ of relative Picard groups along the fibers of $Y \times_B U \to U$ is representable by an analytic space.

The **analytic Tate-Shafarevich group** $\text{III}_{an}(X)$ of $X$ is the subgroup of $WC_{an}(X)$ consisting of elements $\alpha \in WC_{an}(X)$ such that for any representative $Y \to B$ of $\alpha$ and any point $b \in B$ one can find an analytic neighborhood $b \in U \subset B$ so that $Y \times_B U \to U$ has a meromorphic section. This implies that $Y \to B$ has no multiple fibers in codimension one.
The group $\mathbb{III}_{an}(X)$ can be described cohomologically as follows. Assume that $X^{\infty}$ is a smooth space. Then by Proposition 5.5.1, the natural classifying map

$$\mathbb{III}_{an}(X) \to H^1_{an}(B, j_*^{\infty}j^{oo^*} \mathcal{X}_{an})$$

is injective. Furthermore if $B^{\infty} = B$, or if $\pi$ is Weierstrass with a smooth total space, then the map (2.6) is an isomorphism. In addition one knows (see e.g. Theorem 5.4.9) that under the same assumptions, the sheaf $j_*^{oo^*}j^{oo^*} \mathcal{X}_{an}$ fits in a short exact sequence

$$0 \to \mathcal{X}_{an} \to j_*^{oo^*}j^{oo^*} \mathcal{X}_{an} \to (R^2\pi_* \mathcal{O}_X/\mathcal{I}_*\mathcal{I}^!R^1\pi_* \mathcal{O}_X^*)_{torsion} \to 0.$$ 

Since by definition the sheaf $(R^2\pi_* \mathcal{O}_X/\mathcal{I}_*\mathcal{I}^!R^1\pi_* \mathcal{O}_X^*)_{torsion}$ is supported on the multiple fiber sublocus of $D$, it follows that in the absence of multiple fibers, i.e. under our definition of an elliptic fibration we have an isomorphism:

$$\mathbb{III}_{an}(X) \cong H^1_{an}(B, \mathcal{X}_{an}).$$

In the remainder of this paper we will always assume tacitly that the isomorphism holds, in fact we will assume that either $\pi$ is smooth or that $X$ is a surface.

Because of this cohomological interpretation we can view the elements in $\mathbb{III}_{an}(X)$ simply as $\mathcal{X}_{an}$-torsors. This definition of $\mathbb{III}_{an}(X)$ is consistent with the usual definition of the algebraic Tate-Shafarevich group and $\mathbb{III}_{an}(X)$ can be viewed as algebraic spaces $Y \to B$ which are $\mathcal{X}$-torsors.

The algebraic Weil-Châtelet and Tate-Shafarevich groups $WC(X)$ and $\mathbb{III}(X)$ are defined in a similar manner with the etale topology replacing the analytic one. Furthermore, the analysis carried out in Section 1] implies, that under the assumption that $X$ and $B$ are both smooth and that $\pi$ has a regular section, the algebraic Tate-Shafarevich group can be interpreted cohomologically as

$$\mathbb{III}(X) = H^1_{et}(B, \mathcal{X}),$$

i.e. the elements in $\mathbb{III}(X)$ can be viewed as algebraic spaces $Y \to B$ which are $\mathcal{X}$-torsors.

Given an element $\alpha \in \mathbb{III}_{an}(X)$ (or $\alpha \in \mathbb{III}(X)$) we denote by $X^\sharp_\alpha$ the analytic (or algebraic) space representing the torsor $\alpha$ and by $\pi^\sharp_\alpha : X^\sharp_\alpha \to B$ the corresponding projection. Following we say that a morphism of analytic (algebraic) spaces $Y \to B$ is a good model for $\alpha$ if $Y \to B$ is bimeromorphic to $X^\sharp_\alpha \to B$, $Y$ is smooth and the map $Y \to B$ is proper and flat.
Remark 2.9 Note that when $\pi$ is smooth $X^\sharp$ is itself a good model for $\alpha$ and when $X$ is an arbitrary smooth surface we always have a preferred good model for $\alpha$, namely the relatively minimal model of a compactification of $X^\sharp$. When $X$ is of dimension three the good models of elements in $\Pi(X)$ have been analyzed in detail, see e.g. [Mir83, Gra91, DG94]. In this case the good model exists (possibly after blowing up $B$ at finitely many points) but is not unique. However all good models of a given $\alpha$ are related by flops and in particular have equivalent derived categories of coherent sheaves (see e.g. [BO95, Bri02, Kaw02]).

In the cases when $\pi : X \to B$ is smooth or $X$ is a surface we put $\pi_\alpha : X_\alpha \to B$ for the canonical good model of $\alpha$. In particular, if $\pi : X \to B \cong \mathbb{P}^1$ is an elliptic $K3$ surface we have that $X_\alpha$ is a well defined analytic (respectively algebraic) $K3$ surface for any element $\alpha \in \Pi_{an}(X)$ (respectively $\alpha \in \Pi(X)$).

The meromorphic action of the analytic group space $X^\sharp \to B$ on $X_\alpha$ induces a natural meromorphic action map $a_\alpha : X \times_B X_\alpha \dashrightarrow X$.

Furthermore, given a positive integer $n$ we can consider the sheaf of groups $\mathcal{X}_{an}[n] \to B$ consisting of the $n$-torsion points in $\mathcal{X}_{an}$. The sheaf $\mathcal{X}_{an}[n]$ is represented by a group space $X^\sharp[n]$ which is quasi-finite over $B$. We will write $X[n]$ for the closure of $X^\sharp[n]$ in $X$ and by an abuse of notation we will denote the meromorphic map $X[n] \times_B X_\alpha \dashrightarrow X_\alpha$ again by $a_\alpha$.

Since $X^\sharp[n]$ is finite over a dense open set in $B$ we can form the quotient $X_\alpha/X^\sharp[n]$ which as an analytic space is well defined up to a bimeromorphism which respects the genus one fibration. Moreover $X_\alpha/X^\sharp[n]$ is naturally a $\mathcal{X}_{an}$-torsor at the general point and so represents an element in $\Pi_{an}(X)$. It is not hard to calculate this element in terms of $\alpha$ and $n$ only. In fact it is clear that $X_\alpha/X^\sharp[n]$ is tautologically the same as the quotient $X_\alpha^\times B^n/K$,

which by definition represents the element $n\alpha \in \Pi_{an}(X)$.

Here $K = \ker[(X^\sharp)^\times B^n \times X^\sharp]$

is the kernel of the natural product map corresponding the group law on $X^\sharp$, and the action of $K$ is induced from the component-wise action of $(X^\sharp)^\times B^n$ on $(X_\alpha)^\times B^n$.

In particular we have a bimeromorphism $X_\alpha/X^\sharp[n] \dashrightarrow X_{n\alpha}$ which is unique up to an auto-bimeromorphism of $X_{n\alpha}$, compatible with the genus one fibration. However, as one can see from the proof of [Nak01, Lemma 5.3.3], if we assume that $\pi$ is relatively minimal with a smooth total space, then all such auto-bimeromorphisms are holomorphic and are
translations by sections in $\mathcal{X}_{an}$. So, under this the genericity assumption, we will have a bimeromorphic identification $X_{na} = X_α/X^2[n]$ and hence a well defined meromorphic map

$$q_α^n : X_α \rightarrow X_{na}.$$  

If in addition we assume that the fibration $π : X \rightarrow B$ has a trivial Mordel-Weil group, then the meromorphic map $q_α^n$ is canonical and does not depend on any choices.

The *index* of an element $α \in \mathbb{III}_{an}(X)$ is defined to be the minimal degree of a global multisection of $π_α$. We will denote the index by $\text{ind}(α)$.

Assume now that $B$ and $X$ are quasi-projective. Since the element $0 \in \mathbb{III}_{an}(X)$ is represented by the algebraic elliptic fibration $π : X \rightarrow B$, it follows that for each $α$ of finite index the space $X_α$ admits a dominant meromorphic map

$$q_α^{\text{ind}(α)} : X_α \rightarrow X$$

to the algebraic variety $X$. In fact Nakayama \[Nak01\] Proposition 5.5.4] proves that such a $X_α$ is bimeromorphic to an algebraic variety and so must be an algebraic space. Furthermore in the case of surfaces Kodaira shows \[Kod63\] that $X_α$ is quasi-projective if and only if $α$ is torsion in $\mathbb{III}_{an}(X)$.

### 2.3 Complementary fibrations

Let $X$ be smooth and let

$$X \xrightarrow{\pi} B$$

be a relatively minimal elliptic fibration. Consider an element $α \in \mathbb{III}_{an}(X)$ and a good representative $π_α : X_α \rightarrow B$ for $α$. Our goal in this section is to describe the cohomological Brauer group $Br_{an}(X_α)$ in terms of the Tate-Shafarevich group $\mathbb{III}_{an}(X)$. For this we need to analyze the relationship between the sheaf $\mathcal{X}_{an}$ and the relative Picard sheaf of $π_α$.

If all the fibers of $π$ are integral, then $Pic(X_α/B)$ is representable and we have a short exact sequence of abelian sheaves in the analytic topology:

\[
0 \rightarrow \mathcal{X} \xrightarrow{\deg_α} Pic(X_α/B) \xrightarrow{\deg_α} \mathbb{Z} \rightarrow 0,
\]

where $\deg_α$ is the map assigning to each $L \in Pic(π_α^{-1}(U))/π_α^*Pic(U)$ its degree along a smooth fiber.

**Remark 2.10** If we want to allow non-integral fibers for $π$, then $Pic(X_α/B)$ becomes non-representable, but it has a maximal representable quotient $\mathcal{Z}_α$ as shown in e.g. \[Ray70\] and \[DG94\] in the algebraic case and \[Nak01\] in the analytic case. The sheaf of groups $\mathcal{Z}_α$ is defined as:

$$\mathcal{Z}_α := Pic(X_α/B)/\mathcal{E}_α,$$
where $E_\alpha \subset \mathcal{Pic}(X_\alpha/B)$ is a subsheaf generated by local components of the preimage $\pi^{-1}_\alpha(D)$ of the discriminant $D \subset B$ (see [DG94, Proposition 1.13] for the precise statement). Note that when all fibers of $\pi$ are integral we have $E_\alpha = \emptyset$.

In this generality, the short exact sequence (2.8) is replaced by a commutative diagram with exact rows and columns:

$$
\begin{array}{cccc}
0 & 0 \\
\mathbb{Z} & \mathbb{Z} \\
\downarrow \deg_\alpha & \downarrow \\
0 & \mathcal{K}_an & \mathcal{Z}_\alpha & R^2\pi_\alpha^*\mathbb{Z}/E_\alpha \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & \ker(\deg_\alpha) & (R^2\pi_\alpha^*\mathbb{Z}/E_\alpha)_{torsion} & 0 \\
\end{array}
$$

Note also that $(R^2\pi_\alpha^*\mathbb{Z}/E_\alpha)_{torsion}$ is supported on $D$ and that its fibers at smooth points of a component of $D$ parameterizing Kodaira fibers of type $I_n$ are isomorphic to $\mathbb{Z}/n$.

Fix now an element $\alpha \in \mathcal{III}_{an}(X)$. Under some mild assumptions on $\alpha$ we will construct a natural map $T_\alpha : \mathcal{III}_{an}(X) \to Br'_an(X_\alpha)$, which will allow us to compare the Tate-Shafarevich and Brauer groups. The existence of $T_\alpha$ is established in the following lemma.

**Lemma 2.11** Assume that $X$ is smooth, $\pi$ has integral fibers and $Br'_an(B) = 0$. Assume further that

$$
(2.9) \quad \ker \left( H^3_{an}(B, \mathcal{O}_B^\times) \overset{\pi_\alpha^*}{\longrightarrow} H^3_{an}(X_\alpha, \mathcal{O}_{X_\alpha}^\times) \right) = 0
$$

Then there is a canonical homomorphism $T_\alpha$ which fits in an exact sequence of abelian groups:

$$
0 \to \mathbb{Z}/\text{ind}(\alpha) \to \mathcal{III}_{an}(X) \overset{T_\alpha}{\longrightarrow} Br'_an(X_\alpha) \to H^1(B, \mathbb{Z}).
$$

**Proof.** The long exact sequence of (2.8) gives

$$
\begin{array}{cccc}
H^0(B, \mathbb{Z})/H^0_{an}(B, \mathcal{Pic}(X_\alpha/B)) & \longrightarrow & H^1_{an}(B, \mathcal{K}) & \longrightarrow & H^1_{an}(B, \mathcal{Pic}(X_\alpha/B)) & \longrightarrow & H^1(B, \mathbb{Z}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{Z}/\text{ind}(\alpha) & & \mathcal{III}_{an}(X) & & & &
\end{array}
$$
So it suffices to find an identification

\[(2.10) \quad H^1_{\text{an}}(B, \mathcal{P}ic(X_\alpha/B)) \cong Br_{\text{an}}'(X_\alpha).\]

Consider now the Leray spectral sequence for \( \pi_\alpha : X_\alpha \to B \) and the sheaf \( \mathcal{O}_{X_\alpha}^x \), which has only two non-zero rows, so it also becomes a long exact sequence:

\[
Br_{\text{an}}'(B) \to Br_{\text{an}}'(X_\alpha) \to H^1_{\text{an}}(B, \mathcal{P}ic(X_\alpha/B)) \to \ker \left( H^3_{\text{an}}(B, \mathcal{O}_B^x) \xrightarrow{\pi^*_\alpha} H^3_{\text{an}}(X_\alpha, \mathcal{O}_{X_\alpha}^x) \right).
\]

The assumption \( \ker \left( H^3_{\text{an}}(B, \mathcal{O}_B^x) \xrightarrow{\pi^*_\alpha} H^3_{\text{an}}(X_\alpha, \mathcal{O}_{X_\alpha}^x) \right) = 0 \) thus immediately implies the identification \((2.10)\) and so the lemma is proven. \( \square \)

The lemma has the following immediate corollary:

**Corollary 2.12** Assume that \( X \) is a smooth projective surface and \( \pi \) has integral fibers. Then the map \( T_\alpha : \mathbb{III}_{\text{an}}(X) \to Br_{\text{an}}'(X_\alpha) \) exists for all \( \alpha \in \mathbb{III}_{\text{an}}(X) \).

**Proof.** The existence of \( T_\alpha \) is an immediate consequence of Lemma 2.11 since in this case \( B \) is a smooth curve and so \( H^2_{\text{an}}(B, \mathcal{O}_B^x) = H^3_{\text{an}}(B, \mathcal{O}_B^x) = 0 \) for dimension reasons. \( \square \)

If the vanishing assumption \((2.9)\) does not hold, we can still construct a variant of the map \( T_\alpha \) which is defined only on a part of the group \( \mathbb{III}_{\text{an}}(X) \):

**Lemma 2.13** Assume that \( X \) is smooth, \( \pi \) has integral fibers and \( Br_{\text{an}}'(B) = 0 \). Then:

(i) if \( \alpha \) is \( m \)-torsion in \( \mathbb{III}_{\text{an}}(X) \), then there is a group homomorphism (compatible with \( T_\alpha \) when the latter exists)
\[
m\mathbb{III}_{\text{an}}(X) \to Br_{\text{an}}'(X_\alpha),
\]
from the subgroup \( m\mathbb{III}_{\text{an}}(X) \subset \mathbb{III}_{\text{an}}(X) \) of \( m \)-divisible elements in \( \mathbb{III}_{\text{an}}(X) \) to the cohomological Brauer group of \( X_\alpha \);

(ii) if \( \alpha \) is \( m \)-divisible in \( \mathbb{III}_{\text{an}}(X) \), then there is a group homomorphism (compatible with \( T_\alpha \) when the latter exists)
\[
\mathbb{III}_{\text{an}}(X)[m] \to Br_{\text{an}}'(X_\alpha)
\]
from the subgroup \( \mathbb{III}_{\text{an}}(X)[m] \subset \mathbb{III}_{\text{an}}(X) \) of \( m \)-torsion elements in \( \mathbb{III}_{\text{an}}(X) \) to the cohomological Brauer group of \( X_\alpha \);
Proof. For any given $\alpha \in \mathcal{X}(X)$ we have a composition map

$$
\begin{array}{ccc}
\mathcal{X}(X) & \xrightarrow{d_\alpha} & H^3_{an}(B, \mathcal{O}_B^\times) \\
& & \downarrow \\
& & H^1_{an}(B, \mathcal{P}ic(X_\alpha/B))
\end{array}
$$

The assignment $\alpha \mapsto d_\alpha$ gives rise to a group homomorphism

$$
d : \mathcal{X}(X) \to \text{Hom}_{\mathbb{Z}}(\mathcal{X}(X), H^3_{an}(B, \mathcal{O}_B^\times)).
$$

In particular, if $\alpha$ is $m$-torsion, then $d_\alpha(m \cdot \xi) = d_{m \cdot \alpha}(\xi) = 0 \in H^3_{an}(B, \mathcal{O}_B^\times)$ and so the image of $m\mathcal{X}(X)$ in $H^1_{an}(B, \mathcal{P}ic(X_\alpha/B))$ must be contained in $Br'_{an}(X_\alpha)$. This proves (i).

Similarly, if $\alpha = m \cdot \varphi$ is $m$-divisible, then for any $\xi$ we have $d_\alpha(\xi) = d_{m \cdot \varphi}(\xi) = d_\varphi(m \cdot \xi)$ and so $d_\alpha$ vanishes identically on $\mathcal{X}(X)[m]$. Thus the image of $\mathcal{X}(X)[m]$ in $H^1_{an}(B, \mathcal{P}ic(X_\alpha/B))$ must be contained in $Br'_{an}(X_\alpha)$ which completes the proof of (ii) and the lemma.

We will denote the maps in items (i) and (ii) of Lemma 2.13 again by $T_\alpha$. Since by construction these maps are compatible with the map $T_\alpha$ from Lemma 2.11 whenever the latter exists, this abuse of notation can not lead to any confusion.

Let us examine in more detail the map $d : H^1_{an}(B, \mathcal{X}) \to \text{Hom}_{\mathbb{Z}}(H^1_{an}(B, \mathcal{X}), H^3_{an}(B, \mathcal{O}_B^\times))$ given in (2.11). This map can be rewritten as a bilinear pairing

$$
\langle \cdot, \cdot \rangle : H^1_{an}(B, \mathcal{X}) \otimes_{\mathbb{Z}} H^1_{an}(B, \mathcal{X}) \to H^3_{an}(B, \mathcal{O}_B^\times).
$$

The proof of Lemma 2.13 shows that for every $\alpha \in \mathcal{X}(X)$ we have a well defined homomorphism

$$
T_\alpha : \alpha^\perp \to Br'_{an}(X_\alpha),
$$

where $\alpha^\perp \subset \mathcal{X}(X)$ is the orthogonal complement of $\alpha$ with respect to $\langle \cdot, \cdot \rangle$.

**Definition 2.14** Two genus one fibrations $\alpha, \beta \in \mathcal{X}(X)$ will be called complementary if $\langle \alpha, \beta \rangle = 0$. We will call $\alpha$ and $\beta$ $m$-compatible if one of them is $m$-divisible and the other one is $m$-torsion.

Note that using the pairing $\langle \cdot, \cdot \rangle$, Lemma 2.13 follows from the obvious observation that every $m$-compatible pair $\alpha, \beta$ is complementary.

For future reference we spell out the special case when $\alpha = 0$:
Corollary 2.15 Assume that $X$ is smooth, the fibers of $\pi$ are integral, and $Br'_{an}(B) = 0$. Then we have an isomorphism $H^1_{an}(B, \mathcal{P}ic(X/B)) \cong Br'_{an}(X)$ and we have an exact sequence of abelian groups

$$0 \to \Pi_{an}(X) \xrightarrow{T_0} Br'_{an}(X) \to H^1(B, \mathbb{Z}).$$

Proof. Since $\sigma : B \to X$ is a section of $\pi$ it follows that the composition

$$H^i_{an}(B, \mathcal{O}_B^\times) \xrightarrow{\pi^*} H^i_{an}(X, \mathcal{O}_X^\times) \xrightarrow{\sigma^*} H^i_{an}(B, \mathcal{O}_B^\times)$$

is the identity. Thus $\ker \left( H^i_{an}(B, \mathcal{O}_B^\times) \xrightarrow{\pi^*} H^i_{an}(X, \mathcal{O}_X^\times) \right) = 0$ and so $H^1_{an}(B, \mathcal{P}ic(X/B)) \cong Br'_{an}(X)$. Combined with the fact that $\text{ind}(0) = 1$ this gives the short exact sequence of groups above. The corollary is proven. \hfill \square

Our pairing $\langle \cdot, \cdot \rangle$ can be explicitly described as follows. Every element $\alpha \in \Pi_{an}(X) = H^1_{an}(B, \mathcal{X})$ has two different incarnations:

- $\alpha$ can be interpreted as a group extension of $\mathbb{Z}_B$ by $\mathcal{X}$. Concretely this is just the sheaf of groups $\mathcal{P}ic(X_{\alpha}/B)$ as it fits in the extension (2.8) viewed as an element $e(\alpha)$ in $\text{Ext}^1_{\mathbb{Z}_B}(\mathbb{Z}_B, \mathcal{X})$.

- $\alpha$ can be interpreted as an extension of $\mathcal{X}$ by $\mathcal{O}_B^\times[1]$. Concretely this is the amplitude one object $\alpha_{\mathcal{X}}$ in the derived category of abelian sheaves on $B$ which is the pullback of the extension class of

$$1 \to \mathcal{O}_B^\times[1] \to R\pi_{alpha} \mathcal{O}_{X_{alpha}}^\times[1] \to R^1\pi_{alpha} \mathcal{O}_{X_{alpha}}^\times \to 1$$

via the natural inclusion $\mathcal{X} \to \mathcal{P}ic(X_{\alpha}/B) = R^1\pi_{alpha} \mathcal{O}_{X_{alpha}}^\times$. Alternatively, $\alpha_{\mathcal{X}}$ can be thought of as a sheaf of commutative group stacks on $B$ which is just the sheaf of all maps from $B$ to the $\mathcal{O}^\times$-gerbe on $X$ whose characteristic class is $T_0(\alpha)$. Note that this gerbe is well defined in view of Corollary 2.15. We will write $g(\alpha) \in \text{Ext}^2_{\mathbb{Z}_B}(\mathcal{X}, \mathcal{O}_B^\times[1]) = \text{Ext}^2_{\mathbb{Z}_B}(\mathcal{X}, \mathcal{O}_B^\times)$ for the extension class of $\alpha_{\mathcal{X}}$. For more on the relevance of commutative group stacks see Section A.1.

With this notation it is now clear that $\langle \alpha, \beta \rangle$ is just the Yoneda product $g(\beta) \circ e(\alpha)$.

Lemma 2.16 The bilinear pairing

$$\langle \cdot, \cdot \rangle : H^1_{an}(B, \mathcal{X}) \otimes_{\mathbb{Z}} H^1_{an}(B, \mathcal{X}) \to H^3_{an}(B, \mathcal{O}_B^\times)$$

is skew-symmetric.

Proof. The Poincare sheaf $\mathcal{P} \to X \times_B X$ satisfies the biextension property and so can be interpreted functorially (see [SGA7-I, Exposé VII,Corollary 3.6.5]) as an object $\mathcal{E}(\mathcal{P}) \in$...
ob $D^b(\mathcal{Z}_B \text{-mod})$ in the derived category of abelian sheaves on $B$, which is an extension of $\mathcal{X} \otimes \mathcal{X}$ by $\mathcal{O}_B^\times$. In other words, $\mathcal{L}(\mathcal{P})$ fits in a distinguished triangle

$$\mathcal{O}_B^\times \to \mathcal{L}(\mathcal{P}) \to \mathcal{X} \otimes \mathcal{X} \to \mathcal{O}_B^\times[1]$$

of complexes of abelian sheaves. Let $p \in \text{Ext}_B^1(\mathcal{X} \otimes \mathcal{X}, \mathcal{O}_B^\times) = \text{Hom}(\mathcal{X} \otimes \mathcal{X}, \mathcal{O}_B^\times[1])$ be the corresponding extension class. From the definition of the homomorphisms

$$e(\alpha) \in \text{Hom}_{D^b(\mathcal{Z}_B \text{-mod})}(\mathcal{Z}_B, \mathcal{X}[1]),$$

and

$$g(\alpha) \in \text{Hom}_{D^b(\mathcal{Z}_B \text{-mod})}(\mathcal{X}, \mathcal{O}_B^\times[2])$$

one can easily check that $g(\alpha)$ can be identified with the composition

$$\mathcal{X} = \mathcal{Z}_B \otimes \mathcal{X} \xrightarrow{e(\alpha) \otimes \text{id}_\mathcal{X}} \mathcal{X} \otimes \mathcal{X}[1] \xrightarrow{p} \mathcal{O}_B^\times[1].$$

Indeed, observe that both $g(\alpha)$ and $p \circ (e(\alpha) \otimes \text{id}_\mathcal{X})$ can naturally be interpreted as amplitude one objects in the derived category of abelian sheaves on $B$. Since any amplitude one object in $D^b(\mathcal{Z}_B \text{-mod})$ can be viewed as a stack over $B$ it suffices to show the equivalence of the categories fibered in groupoids corresponding to $g(\alpha)$ and $p \circ (e(\alpha) \otimes \text{id}_\mathcal{X})$ respectively. Let as before $\alpha_\mathcal{X} \to B$ denote the fibered category corresponding to $g(\alpha)$. Since by construction $\alpha_\mathcal{X}$ comes from the push-forward $R\pi_{\alpha*}\mathcal{O}_{X_\alpha}^\times$ we can identify explicitly the groupoid of sections of $\alpha_\mathcal{X}$ over an open set $U$ in $B$ as the groupoid of all line bundles $L$ on $(X_\alpha \times_B X)_U$ having the property that for any point $b \in U$ and any $x \in X_b$ we have that $L_{|(X_\alpha)_b \times \{x\}} \cong \mathcal{O}_{X_b}(x - \sigma(b))$. Finally, using the description of the complex $\mathcal{L}(\mathcal{P})$ in terms of fibered categories given in [SGA7-I, Expos`e VII] we see immediately that this groupoid is precisely the groupoid of sections over $U$ of the fibered category corresponding to $p \circ (e(\alpha) \otimes \text{id}_\mathcal{X})$.

Now taking into account that $g(\alpha) = p \circ (e(\alpha) \otimes \text{id}_\mathcal{X})$, we see that for any two elements $\alpha, \beta \in H^1(\mathcal{Z}_B, \mathcal{X})$ the product $\langle \alpha, \beta \rangle \in H^3(\mathcal{Z}_B, \mathcal{O}_B^\times)$ can be rewritten as the Yoneda product

$$\langle \alpha, \beta \rangle = p \circ (\alpha \uplus \beta),$$

where $\alpha \uplus \beta \in H^2(\mathcal{Z}_B, \mathcal{X} \otimes \mathcal{X})$ is the external cup product of $\alpha$ and $\beta$.

To understand the symmetry properties of $\langle \bullet, \bullet \rangle$ it only remains to notice that

$$\langle \beta, \alpha \rangle = p \circ (\beta \uplus \alpha) = p \circ \text{sw}(\alpha \uplus \beta),$$

where $\text{sw} : \mathcal{X} \otimes \mathcal{X} \to \mathcal{X} \otimes \mathcal{X}$ is the involution switching the two factors. However recall that $\mathcal{P}$ is a normalized Poincare bundle and so can be explicitly described as the rank one divisorial sheaf

$$\mathcal{P} = \mathcal{O}_{X \times_B X}(\Delta - \sigma \times_B X - X \times_B \sigma - \varpi^*N_{\sigma/X})$$

where $\varpi : X \times_B X \to B$ is the natural projection and $N_{\sigma/X}$ is the normal bundle to the section $\sigma \subset X$. In particular $\text{sw}^*(\mathcal{P}) = \mathcal{P}$ and so $\mathcal{P} \to X \times_B X$ is a symmetric biextension. This
shows that $\mathfrak{p} \circ \mathfrak{sw} = \mathfrak{p}$. Combined with the fact that $\mathfrak{sw}(\beta \cup \alpha) = (-1)^{|\alpha||\beta|} \alpha \cup \beta = -\alpha \cup \beta$ we conclude that $\langle \beta, \alpha \rangle = -\langle \alpha, \beta \rangle$. The lemma is proven. □

An immediate corollary of the skew-symmetry of $\langle \cdot, \cdot \rangle$ is that $T_{\alpha}(\beta) \in Br_{an}(\mathbb{X})$ is well defined iff $T_{\beta}(\alpha) \in Br_{an}(\mathbb{X})$ is well defined.

In the case of surfaces we get the following:

**Corollary 2.17** Suppose that $\mathbb{X}$ is a smooth surface and that $\pi$ is non-isotrivial with all fibers integral. Then $\mathbb{X}$ is infinitely divisible and so any $\alpha \in \mathbb{X}_{an}(\mathbb{X})$ is $m$-compatible with all elements in $\mathbb{X}_{an}(\mathbb{X})[m]$.

**Proof.** To show that $\mathbb{X}$ is infinitely divisible note that since $\mathcal{B}$ is a curve we can apply Corollary 2.15 to conclude that the map $T_0$ fits in a short exact sequence

$$0 \to \mathbb{X}_{an}(\mathbb{X}) \to Br_{an}(\mathbb{X}) \to H^1(\mathbb{B}, \mathbb{Z})$$

where the last map is the composition of the identification $Br_{an}(\mathbb{X}) \cong H^1_{an}(\mathbb{B}, \mathcal{Pic}(\mathbb{X}/\mathbb{B}))$ coming from the Leray spectral sequence and the map $H^1_{an}(\mathbb{B}, \mathcal{Pic}(\mathbb{X}/\mathbb{B})) \to H^1(\mathbb{B}, \mathbb{Z})$ corresponding to the degree morphism $deg : \mathcal{Pic}(\mathbb{X}/\mathbb{B}) \to \mathbb{Z}$. Since by assumption $\pi$ has only integral fibers, we have natural identifications

$$\mathcal{Pic}(\mathbb{X}/\mathbb{B}) = R^1\pi_*\mathcal{O}_\mathbb{X}^\times, \quad \text{and} \quad \mathbb{Z} = R^2\pi_*\mathbb{Z}_X$$

under which the degree map $deg : \mathcal{Pic}(\mathbb{X}/\mathbb{B}) \to \mathbb{Z}$ becomes the coboundary homomorphism $\delta : R^1\pi_*\mathcal{O}_\mathbb{X}^\times \to R^2\pi_*\mathbb{Z}_X$ in the long exact sequence of higher direct images associated to the exponential sequence

$$0 \to \mathbb{Z}_X \to \mathcal{O}_\mathbb{X} \to \mathcal{O}_\mathbb{X}^\times \to 1$$

and the map $\pi : \mathbb{X} \to \mathbb{B}$.

In particular, this implies that the map $Br_{an}(\mathbb{X}) \to H^1(\mathbb{B}, \mathbb{Z})$ fits in the commutative diagram:

$$\begin{array}{ccc}
Br_{an}(\mathbb{X}) & \longrightarrow & H^1(\mathbb{B}, \mathbb{Z}) \\
\cong \downarrow & & \cong \downarrow \\
H^1_{an}(\mathbb{B}, R^1\pi_*\mathcal{O}_\mathbb{X}^\times) & \overset{\delta}{\longrightarrow} & H^1(\mathbb{B}, R^2\pi_*\mathbb{Z}_X) \\
\theta \uparrow & & \eta \uparrow \\
H^2_{an}(\mathbb{X}, \mathcal{O}_\mathbb{X}^\times) & \overset{\delta}{\longrightarrow} & H^3(\mathbb{X}, \mathbb{Z}).
\end{array}$$

Here the maps $\theta$ and $\eta$ between the third and second rows come from the Leray spectral sequences for the map $\pi : \mathbb{X} \to \mathbb{B}$ and the sheaves $\mathcal{O}_\mathbb{X}^\times$ and $\mathbb{Z}_X$, which give:

$$\begin{array}{c}
H^2_{an}(\mathbb{B}, \mathcal{O}_\mathbb{B}^\times) \longrightarrow H^2_{an}(\mathbb{X}, \mathcal{O}_\mathbb{X}^\times) \overset{\theta}{\longrightarrow} H^1_{an}(\mathbb{B}, R^1\pi_*\mathcal{O}_\mathbb{X}^\times) \longrightarrow 0
\end{array}$$

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and

\[ H^2(B, R^1\pi_*\mathcal{O}_X) \rightarrow H^3(X, \mathbb{Z}) \rightarrow H^1(B, R^2\pi_*\mathcal{O}_X) \rightarrow 0. \]

Now \( H^2_{an}(B, \mathcal{O}^*_B) = 0 \) since \( B \) is one dimensional, and \( H^2(B, R^1\pi_*\mathcal{O}_X) = 0 \) by the irreducibility of monodromy. This implies that \( \mathbb{II}_{an}(X) = \ker[Br'_{an}(X) \to H^1(B, \mathbb{Z})] = \text{im}[H^2_{an}(X, \mathcal{O}_X) \to H^2_{an}(X, \mathcal{O}^*_X)] \) and so \( \mathbb{II}(X) \) is divisible. The corollary is proven. \( \square \)

**Definition 2.18** For any complementary pair \( \alpha, \beta \in \mathbb{III}_{an}(X) \) we denote the \( \mathcal{O}^* \)-gerbe on \( X_\beta \) classified by \( T_\beta(\alpha) \) by \( \alpha X_\beta \).

**Conjecture 2.19** For any complementary pair \( \alpha, \beta \in \mathbb{III}_{an}(X) \), there exists an equivalence

\[ D^b_{-1}(\alpha X_\beta) \cong D^b_{1}(\beta X_\alpha) \]

of the bounded derived categories of sheaves of pure weights \( \pm 1 \) on \( \alpha X_\beta \) and \( \beta X_\alpha \) respectively.

In section 3.4 we will prove this conjecture in any dimension under the additional assumptions that \( \pi \) is smooth and that \( \alpha \) and \( \beta \) are \( m \)-compatible. In section 4 we will prove it unconditionally when \( X \) is a surface.

### 3 Smooth genus one fibrations

In this section we will consider smooth genus one fibrations over smooth bases of arbitrary dimension and \( \mathcal{O}^* \)-gerbes over them.

#### 3.1 \( \mathcal{O}^* \)-gerbes

In this section we work with a fixed smooth elliptic fibration

\[ X \xrightarrow{\pi} B, \]

and two genus one fibrations \( X_\alpha, X_\beta \) corresponding to two \( m \)-compatible elements \( \alpha, \beta \in \mathbb{III}_{an}(X) \). Recall that \( m \)-compatibility means that one of the elements, say \( \beta \), is actually algebraic, i.e. \( \beta \) is a torsion element of some order \( m \) in \( \mathbb{III}_{an}(X) \), while \( \alpha \) is an \( m \)-divisible element. Choose an element \( \varphi \in \mathbb{III}_{an}(X) \) such that \( m\varphi = \alpha \). We will use this data to construct presentations for gerbes \( \beta E_\alpha \) over \( X_\alpha \) and \( \alpha L_\beta \) over \( X_\beta \). Different choices of the root \( \varphi \) give rise to different but Morita equivalent presentations of the same gerbes.
3.1.1 The lifting presentation

Recall from Section 2 that a gerbe presentation over a variety $X$ is a diagram

\[
\begin{array}{ccc}
R & \downarrow s & \rightarrow U \\
& t & \rightarrow \downarrow \quad U \\
& & p_1 \rightarrow p_2 \rightarrow U \\
& & \quad u \rightarrow X
\end{array}
\]

where $R \rightarrow U \times_X U$ is a $\mathbb{C}^\times$-bundle satisfying the biextension condition. We define a gerbe $\alpha \mathcal{L}_\beta$ on $X_\beta$ via the lifting presentation:

\[
\begin{array}{ccc}
\alpha LR_\beta & \downarrow s & \rightarrow \quad \alpha LU_\beta \\
& t & \rightarrow \quad \alpha LU_\beta \times_X \alpha LU_\beta \\
& & \quad p_1 \rightarrow p_2 \rightarrow \alpha LU_\beta \\
& & \quad \downarrow \quad \rightarrow \alpha LU_\beta \times_X \alpha LU_\beta \\
& & \quad \downarrow \quad \rightarrow \alpha LU_\beta \times_X \alpha LU_\beta \\
& & \quad \downarrow \quad \rightarrow X_\beta
\end{array}
\]

where

\[
\alpha LU_\beta := X_\varphi \times_B X_\beta,
\]

\[
p_2 : \alpha LU_\beta = X_\varphi \times_B X_\beta \rightarrow X_\beta
\]

is the second projection, and

\[
\alpha LR_\beta := \text{tot}(\mathcal{P}^x_{1-2,m-3}) \rightarrow (\alpha LU_\beta) \times_X (\alpha LU_\beta).
\]

Here $\mathcal{P}^x_{1-2,m-3} \rightarrow (\alpha LU_\beta) \times_X (\alpha LU_\beta)$ is the pullback via the map

\[
p_{1-2,m-3} : \quad X_\varphi \times_B X_\varphi \times_B X_\beta \rightarrow X \times_B X
\]

\[
(a,b,x) \rightarrow (a - b, m \cdot x),
\]

of the Poincare bundle

\[
\mathcal{P} := \mathcal{O}_{X \times_B X}(\Delta - \sigma \times_B X - X \times_B \sigma - \varpi^* c_1(B))
\]

on $X \times_B X$. As usual we denote the natural projection $X \times_B X \rightarrow B$ by $\varpi$. The required biextension property for $\mathcal{P}^x_{1-2,m-3}$ follows immediately from the see-saw principle.

For future reference we note that under the obvious identification

\[
(\alpha LU_\beta) \times_X (\alpha LU_\beta) = X_\varphi \times_B X_\varphi \times_B X_\beta
\]

the lifting presentation (3.1) can be rewritten as

\[
\begin{array}{ccc}
\text{tot}(\mathcal{P}^x_{1-2,m-3}) & \downarrow s & \rightarrow \quad X_\varphi \times_B X_\varphi \times_B X_\beta \\
& t & \rightarrow \quad \downarrow \quad X_\varphi \times_B X_\varphi \times_B X_\beta \\
& & \quad p_{13} \rightarrow p_{23} \rightarrow \quad \rightarrow \downarrow \quad \rightarrow X_\varphi \times_B X_\beta \\
& & \quad \downarrow \quad \rightarrow \quad \rightarrow \downarrow \quad \rightarrow X_\varphi \times_B X_\beta \\
& & \quad \downarrow \quad \rightarrow \quad \rightarrow \downarrow \quad \rightarrow X_\beta.
\end{array}
\]
3.1.2 The extension presentation

Similarly, we define a gerbe $\beta E_{\alpha}$ on $X_\alpha$ via the extension presentation:

\[(3.3)\]

\[
\begin{array}{ccc}
\beta E R_{\alpha} & \xrightarrow{s} & \beta E U_{\alpha} \\
\downarrow & & \downarrow \\
\beta E U_{\alpha} \times_{X_\alpha} \beta E U_{\alpha} & \xrightarrow{p_1} & \beta E U_{\alpha} \\
\downarrow p_2 & & \downarrow \\
\beta E U_{\alpha} & \xrightarrow{q_\varphi} & X_\alpha
\end{array}
\]

where

$$
\beta E U_{\alpha} := X_\varphi,
q_\varphi : X_\varphi \rightarrow X_\alpha \text{ is the multiplication by } m \text{ map, and}
\beta E R_{\alpha} := \text{tot}(\Phi_\beta) \rightarrow X_\varphi \times_{X_\alpha} X_\varphi.
$$

Here $\Phi_\beta$ could be taken as the line bundle $d^*(M)$ on $X_\varphi \times_{X_\alpha} X_\varphi$, where $d : X_\varphi \times_{X_\alpha} X_\varphi \rightarrow X[m]$ is the difference map and $M$ is any line bundle on $X[m]$ whose punctured total space gives a group extension:

$$
1 \rightarrow \mathcal{O}_B^\times \rightarrow \text{tot}(M^\times) \rightarrow X[m] \rightarrow 0.
$$

We will explain below how to construct a relative line bundle $\Sigma_\beta \in \Gamma(B, \mathcal{P}ic^m(X_\beta/B))$ and a global line bundle $M_\beta \rightarrow X[m]$, determined by the condition that its punctured total space is the theta group $G_\beta$:

$$
1 \rightarrow \mathcal{O}_B^\times \rightarrow G_\beta \rightarrow X[m] \rightarrow 0
$$

of $\Sigma_\beta$. The simplest choice would be to take $M := M_\beta$. However, we will see later that in order to achieve duality with the lifting gerbe, the correct choice is to take $M := M_\beta \otimes M_0^{-1}$, where $M_0$ is determined by the condition that its punctured total space is the theta group $G_0$:

$$
1 \rightarrow \mathcal{O}_B^\times \rightarrow G_0 \rightarrow X[m] \rightarrow 0
$$

corresponding to the similarly defined relative line bundle $\Sigma_0 \in \Gamma(B, \mathcal{P}ic^m(X/B))$.

To define $\Sigma_\beta$, consider first two genus one curves $E'$ and $E''$ with the same Jacobian $E$. Let $q : E' \rightarrow E''$ be a map which induces the multiplication by $m$ map $E \rightarrow E$. For any two points $a, b \in E'$ such that $q(a) = q(b)$, we have that $\mathcal{O}_{E'}(m \cdot a) \cong \mathcal{O}_{E'}(m \cdot b)$. This determines a map $E'' \rightarrow \text{Pic}^m(E')$. Applying this to our map $q_\beta : X_\beta \rightarrow X$ we get a well defined morphism $X \rightarrow \text{Pic}^m(X_\beta/B)$. The image of $\sigma \subset X$ is our relative line bundle $\Sigma_\beta$. Similarly we get $\Sigma_0$ from the multiplication by $m$ map $q_0^m : X \rightarrow X$. Note that by construction the relative line bundle $\Sigma_0 \in \text{Pic}^m(X/B)$ is induced by a global line bundle $\mathcal{O}_X(m\sigma)$. Similarly, if $Br'_an(B) = 0$, then the relative line bundle $\Sigma_\beta \in \text{Pic}^m(X/B)$ is induced by some global line bundle $\mathcal{O}_{X_\beta}(m\sigma_\beta)$, where $\sigma_\beta \subset X_\beta$ is an $m$-section of $\pi_\beta$. 

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Given any relative line bundle $\Sigma \in \Gamma(B, \text{Pic}^m(Y/B))$, where $\pi_Y : Y \to B$ is in $\text{III}(X)$, we get a global line bundle $M$ on $X[m]$ equipped with an isomorphism

$$\mu^* M \cong \text{pr}_1^* M \otimes \text{pr}_2^* M$$

satisfying the usual biextension property. Here $\mu : X[m] \times_B X[m] \to X[m]$ is multiplication, and $\text{pr}_1$, $\text{pr}_2$ are the projections.

Locally on $B$, choose an actual line bundle $\Sigma' \in \text{Pic}(Y)$ lifting $\Sigma$. We define $M' := p_1^* (a^* \Sigma' \otimes p_2^* \Sigma'^{-1})$. Here $a : X[m] \times_B Y \to Y$ is the action and $p_1$, $p_2$ are the projections. Note that $a^* \Sigma' \otimes p_2^* \Sigma'^{-1}$ is trivial on each fiber of $p_1$, so $M'$ is a line bundle. Let $M''$ be the line bundle determined by another lift $\Sigma''$ of $\Sigma$. Any such $\Sigma''$ is necessarily of the form $\Sigma' \otimes \pi_Y^* L$ for some line bundle $L$ on $B$. The equality $\pi_Y \circ a = \pi_Y \circ p_2$ therefore induces a canonical isomorphism

$$a^* \Sigma' \otimes p_2^* \Sigma'^{-1} \to a^* \Sigma'' \otimes p_2^* \Sigma''^{-1},$$

so the locally defined line bundles $M'$, $M''$ glue to a global line bundle $M$ on $X[m]$.

The biextension property of $M$ is equivalent to saying that its punctured total space $G := \text{tot}(M^\times)$ has the structure of a group scheme over $B$ called the theta group of $\Sigma$. It is a central extension of $X[m]$ by $\mathbb{G}_m$. Explicitly a local section of $G$ is a pair $(x, \lambda)$, where $x$ is a local section of $X[m] \to B$ and $\lambda : t_x^* \Sigma \to \Sigma$ is an isomorphism.

Applying these constructions to our relative line bundles $\Sigma_\beta$ and $\Sigma_0$ produces the desired line bundles $M_\beta$ and $M_0$ and theta groups $G_\beta$ and $G_0$.

For future reference we note that under the obvious isomorphism

$$d \times p_2 : X_\varphi \times_{X_\alpha} X_\varphi \to X[m] \times_B X_\varphi$$

the extension presentation (3.3) can be rewritten in the equivalent form:

$$\text{tot}(\Phi_\beta^\varphi)$$

where $a_\varphi : X[m] \times_B X_\varphi \to X_\varphi$ denotes the action and $p_2$ denotes the second projection.

### 3.1.3 Coboundary realizations

Although we constructed $\alpha L_\beta$ and $\beta E_\alpha$ directly, it is worth noting that the lifting and extension presentations are both special cases of the coboundary construction described in Sections 2.1.1 and 2.1.2. Recall that the input for the coboundary construction for a gerbe presentation on a variety $Y$ consists of a short exact sequence of group schemes over $Y$

$$1 \to \mathbb{G}_m \to G \to K \to 1$$
together with a $K$ torsor $U$.

The lifting presentation is obtained from the short exact sequence

$$1 \to \mathbb{G}_m \to \text{tot}(p_{1,m,2}^*\mathcal{P}^\times) \to \pi_\beta^*(X) \to 1$$

and the $\pi_\beta^*(X) = X \times_B X_\beta$-torsor $X_\varphi \times_B X_\beta$. Note that the group structure on $\text{tot}(p_{1,m,2}^*\mathcal{P}^\times)$ in the above sequence comes from the biextension property of the Poincare bundle (for the group law on $\pi_\beta^*(X)$).

The extension presentation is obtained from the short exact sequence

$$1 \to \mathbb{G}_m \to \text{tot}(\Phi^x) \to \pi_\alpha^*(X[m]) \to 1$$

and the $\pi_\alpha^*(X[m]) = X[m] \times_B X_\alpha$-torsor $X_\varphi$.

In other words, we can write $\alpha L \beta$ and $\beta E \alpha$ as quotient gerbes:

$$\alpha L \beta = [X_\varphi \times_B X_\beta/ \text{tot}(p_{1,m,2}^*\mathcal{P}^\times)]$$

$$\beta E \alpha = [X_\varphi/ \text{tot}(\Phi^x)].$$

### 3.2 The class of the lifting gerbe

In this section we continue to assume that $\pi : X \to B$ is smooth, $\beta \in \text{III}_{an}(X)$ is of finite order $m$ and $\alpha \in \text{III}_{an}(X)$ is $m$-divisible.

**Theorem 3.1** The class $[\alpha L \beta]$ of the lifting gerbe equals $T_\beta(\alpha)$. In other words $\alpha L \beta$ is a model for $\alpha X_\beta$.

**Proof.** The proof is in two steps. In step (1) we show that a cocycle representing the class $T_\beta(\alpha)$ which defines $\alpha X_\beta$ becomes a coboundary $\delta(c)$ when pulled back to $L := \alpha L U_\beta$. In step (2) we check that the line bundle defined by $c$ on $L \times X_\beta$ $L$ coincides with the Poincare bundle $\alpha LR_\beta$.

We need to show the isomorphism of two gerbes on the smooth space $X_\beta$.

We will be working with the Cartesian product

$$L \xrightarrow{\lambda_\varphi} X_\varphi \xrightarrow{\pi_\varphi} B \xleftarrow{\pi_\beta} X_\beta \xleftarrow{\lambda_\beta} L.$$

Recall from section 2.1 that the class of $\alpha X_\beta$ is $T_\beta(\alpha) \in H^2(X_\beta, \mathcal{O}^\times)$. By the Leray spectral sequence for $\pi_\beta : X_\beta \to B$, this group equals $H^1(B, \mathcal{P}\text{ic}(X_\beta/B))$. Explicitly, $T_\beta(\alpha)$ is the class of the $\mathcal{P}\text{ic}(X_\beta/B)$-torsor induced from the $\mathcal{P}\text{ic}^0(X_\beta/B) = X$-torsor $X_\beta$.

Similarly, the Leray spectral sequence for $\lambda_B : L \to B$ gives an injection

$$H^1(B, \mathcal{P}\text{ic}(L/B)) \hookrightarrow H^2(L, \mathcal{O}^\times).$$
The $\lambda_\beta$-pullback of $T_\beta(\alpha)$ is in the image of $\lambda_\beta^\ast$ and is the $\mathcal{P}ic(L/B)$-torsor induced from $X_\beta$ via $\lambda_\beta^\ast : \mathcal{P}ic^0(X_\beta/B) \to \mathcal{P}ic(L/B)$.

For step (1), consider the short exact sequence of sheaves of groups on $B$:

$$0 \to \mathcal{P}ic(X_\beta/B) \xrightarrow{\lambda_\beta^\ast} \mathcal{P}ic(L/B) \xrightarrow{\ev} \mathcal{H}om_B(X_\beta, \mathcal{P}ic(X_\beta/B)) \to 0.$$  

Here ev sends a line bundle on $L$ to the family of its restrictions on pt $\times_B X_\varphi$, and it is surjective because $\pi_\beta : X_\beta \to B$ is smooth.

We are claiming that $\lambda_\beta^\ast(T_\beta(\alpha)) = 0$, so we need to show that $T_\beta(\alpha)$ is in the image of the coboundary

$$\partial : H^0(B, \mathcal{H}om_B(X_\beta, \mathcal{P}ic(X_\beta/B))) \to H^1(B, \mathcal{P}ic(X_\beta/B)).$$

In $H^0(B, \mathcal{H}om_B(X_\beta, \mathcal{P}ic(X_\beta/B)))$ we have a natural element

$$q : X_\beta \to X_{m\beta} = X = \mathcal{P}ic^0(X_\beta/B) \subset \mathcal{P}ic(X_\beta/B),$$

depending on the choice of a trivialization $\Sigma_\beta$ of $X_{m\beta}$.

We will see that $\partial(q) = T_\beta(\alpha)$.

Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an analytic open cover of $B$ for which we have trivializations $s_i : U_i \to X_\varphi$ of the $X$-torsor $X_\varphi$. In order to calculate $\partial(q)$ we first lift $q$ to an element $c \in C^0(\mathcal{U}, \mathcal{P}ic(L/B))$. This lift is given in terms of the map

$$t_{-s_i} \times q : L|_{U_i} = X_\varphi \times_{U_i} X_\beta \to X \times_{U_i} X$$

by $c_i := (t_{-s_i} \times q)^* \mathcal{P}$, where $\mathcal{P}$ is the standard Poincare bundle on $X \times_B X$.

The Čech differential $\delta(c) \in Z^1(\mathcal{U}, \mathcal{P}ic(L/B))$ is given by $\{c_i \otimes c_i^{-1}\}_{i,j \in I}$. It comes from $Z^1(\mathcal{U}, \mathcal{P}ic(X_\beta/B))$, and is represented there by $\{O_{\varphi^{-1}(U_i)}(m(s_j - s_i))\}_{i,j \in I}$. On the other hand, $ms_i$ can be interpreted as a section of $X_\alpha = X_{m\varphi}$ over $U_i$, so this cocycle represents our $T_\beta(\alpha)$.

For step (2), consider the cochain

$$\{p_i^* c_i \otimes p_i^* c_i^{-1}\}_{i \in I} \in C^0(\mathcal{U}, \mathcal{P}ic(L \times_{X_\beta} L/B)).$$

From the discussion in section 3.1 we know that this cochain is in fact a global section $L$ of $\mathcal{P}ic(L \times_{X_\beta} L/B)$. We need to show that $L = \mathcal{P}ic(1-2, m-3)$. As usual we identify $L \times_{X_\beta} L$ with $X_\varphi \times_B X_\varphi \times_B X_\beta$, so $p_1$ and $p_2$ become $p_{13}$ and $p_{23}$.

It suffices to show the equality $L|_{\varphi^{-1}(U_i)} = p_i^* c_i \otimes p_i^* c_i^{-1}$ for each open set $U_i$. This follows by the theorem of the cube from the identifications:

$$p_i^* c_i = p_{13}^* \circ (t_{-s_i} \times q)^* \mathcal{P}$$
$$p_i^* c_i = p_{23}^* \circ (t_{-s_i} \times q)^* \mathcal{P}$$
$$L = p_{1-2, m-3}^* \mathcal{P}.$$ 

This finishes the proof of the theorem. □
3.3 The class of the extension gerbe

In this section we again assume that \( \pi : X \to B \) is a smooth elliptic fibration, that \( Br_{an}(B) = 0 \) and that \( \alpha, \beta \in \text{III}_{an}(X) \) are \( m \)-compatible with \( \beta \) of order \( m \).

**Theorem 3.2** The class \( [\beta \mathcal{E}_\alpha] \) of the extension gerbe equals \( T_\alpha(\beta) \). In other words \( \beta \mathcal{E}_\alpha \) is a model for \( \beta X_\alpha \).

**Proof.** Recall that the assumption \( Br_{an}(B) = 0 \) together with the Leray spectral sequence for \( \pi : X_\alpha \to B \) give us an injection

\[
H^2_{an}(X_\alpha, \mathcal{O}^\times) \to H^1_{an}(B, R^1\pi_*\mathcal{O}^\times) = H^1(B, \mathcal{P}ic(X_\alpha/B)).
\]

In terms of this inclusion, \( T_\alpha(\beta) \) can be identified with the isomorphism class of the \( \mathcal{P}ic(X_\alpha/B) \)-torsor associated to the \( \mathcal{P}ic^0(X_\alpha/B) = \mathcal{X} \) torsor \( X_\beta \). In order to show that \( [\beta \mathcal{E}_\alpha] = T_\alpha(\beta) \) we must first check that \( T_\alpha(\beta) \) pulls back to the trivial element in \( H^2_{an}(\beta EU_\alpha, \mathcal{O}^\times) \).

Recall from Section 3.1.2 that the atlas \( \beta EU_\alpha \) for the extension presentation is defined by fixing an element \( \varphi \in \text{III}_{an}(X) \) such that \( m \cdot \varphi = \alpha \) and then taking \( \beta EU_\alpha := X_\varphi \). With this definition the structure morphism \( \beta EU_\alpha \to X_\alpha \) is identified with the map \( q := q^m_\varphi : X_\varphi \to X_\alpha \) of multiplication by \( m \) along the fibers.

The pullback via \( q \) of relative line bundles defined on the fibers of \( \pi_\alpha : X_\alpha \to B \) gives rise to a morphism of sheaves of groups

\[
Q : \mathcal{P}ic(X_\alpha/B) \to \mathcal{P}ic(X_\varphi/B).
\]

Since \( q \) corresponds to multiplication by \( m \), it follows that \( Q \) fits in a commutative diagram

\[
\begin{array}{ccc}
\mathcal{P}ic(X_\alpha/B) & \xrightarrow{Q} & \mathcal{P}ic(X_\varphi/B) \\
\downarrow & & \downarrow \\
\mathcal{P}ic^0(X_\alpha/B) & \xrightarrow{\mathcal{X}} & \mathcal{P}ic^0(X_\varphi/B) \\
\downarrow \mult_m & & \downarrow \mult_m \\
\mathcal{X} & \xrightarrow{\mathcal{X}} & \mathcal{X}
\end{array}
\]

Also, since \( \pi_\alpha \circ q = \pi_\varphi \), it follows that the pullback map

\[
q^* : H^2_{an}(X_\alpha, \mathcal{O}^\times) \to H^2_{an}(X_\varphi, \mathcal{O}^\times)
\]
is compatible with the Leray spectral sequences for $\pi_\alpha$ and $\pi_\varphi$, and so fits in a commutative diagram

\[
\begin{array}{ccc}
H^2_{an}(X_\alpha, \mathcal{O}^\times) & \xrightarrow{q^*} & H^2_{an}(X_\varphi, \mathcal{O}^\times) \\
\downarrow & & \downarrow \\
H^1_{an}(B, \mathcal{P}ic(X_\alpha/B)) & \xrightarrow{h^1(Q)} & H^1_{an}(B, \mathcal{P}ic(X_\varphi/B)) \\
\downarrow & & \downarrow \\
H^1(B, \mathcal{X}) & \xrightarrow{h^1(mult_m)} & H^1(B, \mathcal{X})
\end{array}
\]

Thus we can identify $q^*(T_\alpha(\beta))$ with the class of the $\mathcal{P}ic(X_\varphi/B)$-torsor which is induced from the $\mathcal{P}ic^0(X_\varphi/B) = \mathcal{X}$-torsor $h^1(mult_m)(X_\beta) = X_{m,\beta}$. However by assumption $m \cdot \beta = 0$ and so $X_{m,\beta}$ is trivial as a $\mathcal{X}$-torsor. Therefore $q^*(T_\alpha(\beta)) = 0$ as promised.

To complete the proof of the theorem we need to realize the cocycle $q^*(T_\alpha(\beta))$ as a coboundary:

$$q^*(T_\alpha(\beta)) = \partial(\psi)$$

for some $\psi \in C^1_{an}(X_\varphi, \mathcal{O}^\times)$, and then check that the line bundle defined by $\psi$ on $X_\varphi \times X_\alpha$ is isomorphic to $\Phi_\beta$.

In terms of the inclusion $H^2_{an}(X_\varphi, \mathcal{O}^\times) \subset H^1_{an}(B, \mathcal{P}ic(X_\alpha/B))$ this amounts to writing the class $q^*(T_\alpha(\beta)) \in H^1_{an}(B, \mathcal{P}ic(X_\varphi/B))$ as the coboundary of some Čech cochain $\psi \in C^0_{an}(B, \mathcal{P}ic(X_\varphi/B))$ and then showing that the global section of $\mathcal{P}ic(X_\varphi \times X_\alpha B)$ determined by $\psi$ coincides with the global section given by $\Phi_\beta$. To carry this out we will need to first choose a cocycle representing of $T_\beta(\alpha) \in H^1_{an}(B, \mathcal{P}ic(X_\alpha/B))$ or equivalently a cocycle representing for the $\mathcal{X}$-torsor $X_\beta$.

Let $\mathcal{U} = \{U_i\}$ be an analytic open covering of $B$ which trivializes $X_\beta$ as an $X$-torsor. Choose trivializing sections $s_i \in \Gamma(U_i, X_\beta)$ over each $U_i$. Then $T_\alpha(\beta) \in H^1_{an}(B, \mathcal{P}ic(X_\alpha/B))$ is represented by the Čech cocycle

$$\{\mathcal{O}_{X_\beta}(s_j - s_i)\} \in Z^1(\mathcal{U}, \mathcal{X}) = Z^1(\mathcal{U}, \mathcal{P}ic^0(X_\alpha/B)) \to Z^1(\mathcal{U}, \mathcal{P}ic(X_\alpha/B)).$$

Here $\mathcal{O}_{X_\beta}(s_j - s_i)$ is viewed as a line bundle of degree zero along the fibers of $\pi_\alpha : X_\alpha|_{U_{ij}} \to U_{ij}$ via the canonical identification $\mathcal{P}ic^0(X_\beta/B) = \mathcal{X} = \mathcal{P}ic^0(X_\alpha/B)$. In particular $q^*(T_\alpha(\beta)) \in H^1_{an}(B, \mathcal{P}ic(X_\varphi/B))$ is represented by the cocycle

$$\{\mathcal{O}_{X_\beta}(s_j - s_i)\} \in Z^1(\mathcal{U}, \mathcal{X}) = Z^1(\mathcal{U}, \mathcal{P}ic^0(X_\varphi/B)) \to Z^1(\mathcal{U}, \mathcal{P}ic(X_\varphi/B)).$$

In order to write this cocycle as a coboundary we will have to trivilize the $\mathcal{X}$-torsor $X_{m,\beta}$.

Recall that in the construction of the line bundle $\Phi_\beta$ we used a particular trivialization of $X_{m,\beta}$, namely the relative line bundle

$$\Sigma_\beta \in \Gamma_{an}(B, \mathcal{P}ic^m(X_\beta/B)).$$
Using $\Sigma_\beta$ we can construct a cochain

$$\psi = \{\psi_i\} \in \mathcal{C}^0(\Omega, \mathcal{P}ic^0(X_\varphi/B))$$

with $\psi := \mathcal{O}_{X_\beta}(-m \cdot s_i) \otimes \Sigma_\beta$. By construction, $\psi$ determines a global section

$$p_1^*\psi \otimes p_2^*\psi^{-1} : B \to \text{Pic}^0(X_\varphi \times_{X_\alpha} X_\varphi/B),$$

namely, the section locally given by $p_1^*\psi_i \otimes p_2^*\psi_i^{-1} \in \Gamma_{an}(U_i, \mathcal{P}ic^0(X_\varphi \times_{X_\alpha} X_\varphi/B))$. On the other hand the section corresponding to $\Phi_\beta \to X_\varphi \times_{X_\alpha} X_\varphi$ can be described as follows.

Recall from section 3.1.2 that $\Phi_\beta = d^*(M_\beta \otimes M_0^{-1})$ where $d : X_\varphi \times_{X_\alpha} X_\varphi \to X[m]$ is the natural difference map and $M_\beta, M_0$ are line bundles on $X[m]$ satisfying

$$d_\beta^*M_\beta \cong p_1^*\Sigma_\beta \otimes p_2^*\Sigma_\beta^{-1}$$
$$d_0^*M_0 \cong p_1^*\Sigma_0 \otimes p_2^*\Sigma_0^{-1}.$$ Here again

$$X_\beta \times_{q_\beta \times X_\alpha} X_\beta \xrightarrow{d_\beta} X[m]$$
$$X \times_{q_0 \times X_\alpha} X \xrightarrow{d_0} X[m]$$

stand for the difference maps.

By construction $p_1^*\psi_i \otimes p_2^*\psi_i^{-1}$ lives naturally in $\Gamma(U_i, \mathcal{P}ic^0(X_\varphi \times_{X_\alpha} X_\varphi/B))$ (which we have identified with $\Gamma(U_i, \mathcal{P}ic^0(X_\varphi \times_{X_\alpha} X_\varphi/B))$). In view of this, it will be convenient if we rewrite all objects as line bundles on $X_\beta \times_X X_\beta$. To that end, choose a local section $s : U_i \to X_\beta$ and let $t_{-s}$ be the induced isomorphism

$$X_{\beta|U_i} \xrightarrow{t_{-s}} X_{|U_i} \xleftarrow{U_i}$$

of translation by $s$ along the fibers. With this notation we have a commutative diagram

$$X[m]_{|U_i} \xrightarrow{d_\beta} (X_\beta \times_X X_\beta)_{|U_i} \xrightarrow{p_1} X_{\beta|U_i}$$
$$\downarrow \quad \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \quad \downarrow$$
$$X[m]_{|U_i} \xrightarrow{d_0} (X \times_X X)_{|U_i} \xrightarrow{p_1} X_{|U_i}$$

and thus

$$d_\beta^*M_0 = (t_s \times t_{-s})^* \circ d_0^*M_0 = p_1^*\mathcal{O}_{X_\beta}(ms) \otimes p_2^*\mathcal{O}_{X_\beta}(-ms).$$
Therefore, in order to compare \( p_1^* \psi_i \otimes p_2^* \psi_i^{-1} \) and \( d^*_\beta(M_\beta \otimes M_0^{-1}) \), we need to show that on \( (X_\beta \times X_\beta)|_{U_i} \) we have

\[
p_1^*(\Sigma_\beta(-ms_i)) \otimes p_2^*(\Sigma_\beta(ms_i)) \cong p_1^*(\Sigma_\beta(-ms)) \otimes p_2^*(\Sigma_\beta(ms))
\]
for every section \( s : U_i \to X_\beta \).

Equivalently, it suffices to show that

\[
(3.6) \quad p_1^* O_{X_\beta}(m(s-s_i)) \otimes p_2^* O_{X_\beta}(m(s_i-s)) \cong O.
\]

On the other hand we have a commutative diagram

\[
\begin{array}{cccc}
X_\beta \times X_\beta & \xrightarrow{p_1} & X_\beta & \xleftarrow{p_2} \\
\text{ } & \downarrow{\mu} & \text{ } & \text{ } \\
X_\beta & \xrightarrow{q_\beta^m} & X & \xleftarrow{q_\beta^m} \\
\end{array}
\]

and since \( O_{X_\beta}(m(s-s_i)) \) is of degree zero along the fibers of \( \pi_\beta \) we have

\[
(3.7) \quad O_{X_\beta}(m(s-s_i)) = (q_\beta^m)^* O_X(s-s_i).
\]

Here \( O_X(s-s_i) \in \Gamma(U_i, \mathcal{P}ic^0(X/B)) \) denotes the relative line bundle on \( X \) corresponding to \( O_{X_\beta}(s-s_i) \in \Gamma(U_i, \mathcal{P}ic^0(X_\beta/B)) \) under the canonical identification \( \mathcal{P}ic^0(X/B) = \mathcal{X}^* = \mathcal{P}ic^0(X_\beta/B) \).

The formula (3.7) implies that

\[
\begin{align*}
p_1^* O_{X_\beta}(m(s-s_i)) & \cong \mu^* O_X(s-s_i) \\
p_2^* O_{X_\beta}(m(s_i-s)) & \cong \mu^* O_X(s_i-s)
\end{align*}
\]

and so (3.6) holds. The theorem is proven. \( \Box \)

3.4 Duality between the lifting and extension presentations

We are now ready to prove Theorem \( \ref{thm:lifting-extension} \) for a smooth elliptic fibration

\[
\begin{array}{ccc}
X & \xrightarrow{\pi} & B \\
\sigma & \xleftarrow{\phantom{\pi}} & \phantom{\pi} \\
\end{array}
\]

over a smooth space \( B \) satisfying \( Br'_\alpha(B) = 0 \).

Let \( \alpha, \beta, \varphi \in \Pi_{an}(X) \) satisfy \( m\beta = 0, m\varphi = \alpha \) (in particular \( \alpha \) and \( \beta \) are \( m \)-compatible). We want to compare the derived categories of coherent sheaves on \( \alpha X_\beta \) and \( \beta X_\alpha \).
3.4.1 The gerby Fourier-Mukai transform

Let $D^b_1(\alpha X_\beta)$ and $D^b_{-1}(\beta X_\alpha)$ denote the derived categories of coherent sheaves of weight one and minus one on the gerbes $\alpha X_\beta$ and $\beta X_\alpha$ respectively. Alternatively, as explained at the end of Section 2.1.2, we can view $D^b_1(\alpha X_\beta)$ and $D^b_{-1}(\beta X_\alpha)$ as derived categories of $T_\beta(\alpha)$-twisted sheaves on $X_\beta$ and $T_\alpha(-\beta)$ twisted sheaves on $X_\alpha$ respectively.

We want to construct a Fourier-Mukai functor

$$FM : D^b_1(\alpha X_\beta) \to D^b_{-1}(\beta X_\alpha)$$

which is an equivalence. To achieve this we will work with the models $\alpha L_\beta$ and $\beta E_\alpha$ for $\alpha X_\beta$ and $\beta X_\alpha$ respectively. The idea is to use the explicit presentations for these models of the gerbes and construct the functor $FM$ in terms of data on the atlases.

Since $\alpha L_\beta = [\alpha LU_\beta/\alpha LR_\beta]$ and $\beta E_\alpha = [\beta EU_\alpha/\beta ER_\alpha]$ we have natural structure morphisms

$$\gamma_L : \alpha LU_\beta \longrightarrow \alpha L_\beta$$

$$\gamma_E : \beta EU_\alpha \longrightarrow \beta E_\alpha$$

for the lifting and extension presentations. The (derived) pullback by $\gamma_L$ gives a natural functor

$$\gamma_L^* : D^b_1(\alpha L_\beta) \to D^b(\alpha LU_\beta),$$

which sends complexes of sheaves on $\alpha L_\beta$ to objects in $D^b(\alpha LU_\beta)$ preserved by the relations.

Explicitly, for a $\mathcal{L} \in D^b_1(\alpha L_\beta)$, the pullback $\gamma_L^*\mathcal{L}$ is given by a pair $(L, f)$ where:

- $L$ is a bounded complex of sheaves on the atlas $\alpha LU_\beta = X_\varphi \times_B X_\beta$.
- $f : p_{13}\mathcal{L} \overset{q}{\longrightarrow} p_{23}\mathcal{L} \otimes \mathcal{P}_{1-2,m,3}$ is a quasi-isomorphism of complexes on $X_\varphi \times_B X_\varphi \times_B X_\beta$, satisfying the cocycle condition $(2.3)$.

Here $p_{ij}$ is the projection of $X_\varphi \times_B X_\varphi \times_B X_\beta$ onto the product of the $i$-th and $j$-th components.

Under the gerby Fourier-Mukai transform, $\mathcal{L}$ should go to an object $\mathcal{Q} \in D^b_{-1}(\beta E_\alpha)$. To produce this object we will perform an integral transform from the derived category of the atlas $\alpha LU_\beta$ to the derived category of the atlas $\beta EU_\alpha$. Again, we would like to use the fact that the pullback by $\gamma_E$ gives a functor

$$\gamma_E^* : D^b_{-1}(\beta E_\alpha) \to D^b(\beta EU_\alpha),$$

which sends complexes on $\beta E_\alpha$ to objects in $D^b(\beta EU_\alpha)$ preserved by the relations. In principle this is all we can say about the images of $\gamma_E^*$ since even for schemes the derived categories of coherent sheaves do not necessarily glue, see e.g. [Har66]. However in the case of $\beta E_\alpha$ we can be much more precise. It is known [Pol96, Theorem A] that given a scheme $S$ and a finite flat morphism $p : U \to S$, the derived category of coherent sheaves on $S$ is equivalent to the category of pairs $(F, \phi)$, where $F \in D^b(U)$ and $\phi : p_1^*F \overset{\sim}{\longrightarrow} p_2^*F$ is an isomorphism in
$D^b(U \times_S U)$ satisfying the cocycle condition in $D^b(U \times_S U \times_S U)$. If in addition we are given a $\mathbb{C}^\times$ bundle $R \to U \times_S U$ equipped with a biextension isomorphism, so that $[U/R]$ is a $\mathcal{O}^\times$-gerbe on $S$, we can repeat the reasoning of [Pol96, Theorem A] verbatim to conclude that the category $D^b([U/R])$ is equivalent to the category of pairs $(G, \psi)$, where $G \in D^b(U)$ and $\psi : p_1^*G \to p_2^*G \otimes R$ is an isomorphism in $D^b(U \times_S U)$ satisfying the cocycle condition (2.3) in $D^b(U \times_S U \times_S U)$. In particular, since by construction the morphism $\beta EU_\alpha = X_\varphi \to X_\alpha$ is finite and flat, we conclude that $\gamma^*_E$ identifies $D^b_{-1}(\beta \mathcal{E}_\alpha)$ with the category of pairs $(Q, \mathbf{g})$ where:

- $Q$ is a bounded complex of sheaves on the atlas $\beta EU_\alpha = X_\varphi$.

- $\mathbf{g} : \alpha^*_\varphi Q \xrightarrow{\eta} p_2^*Q \otimes p_1^*(M^{-1}_\beta \otimes M_0)$ is a quasi-isomorphism of complexes on $X[m] \times_B X_\varphi$, satisfying the cocycle condition (2.3).

Here $p_i$ is the projection of $X[m] \times_B X_\varphi$ onto the $i$-th component.

**Remark 3.3** The reader may wish to focus on the case when $\mathcal{L}$ is a line bundle on $\alpha \mathcal{L}_\beta$ of fiber degree zero, i.e. $L \to X_\varphi \times_B X_\beta$ is a line bundle with $\deg(L|_{\{x\} \times_B X_\beta}) = 0$ and the existence of $f$ is equivalent to having isomorphisms $L|_{X_\varphi \times_B \{y\}} \cong \mathcal{O}(q\underset{\beta}{\text{y}}) \otimes \Sigma^{-1}_\beta$ for all $y \in X_\beta$.

In this case the object $Q$ should be a spectral datum on the gerbe $\beta \mathcal{E}_\alpha$ whose support is of degree one over $B$, i.e. $Q$ is a torsion sheaf on $X_\varphi$ supported on $(q\underset{\varphi}{\text{y}})^{-1}(s)$ for some section $s \subset X_\alpha$.

We will construct the functor $FM$ by first constructing a functor between the derived categories on the atlases and then checking that this functor preserves the relations.

On the level of atlases, consider the functor

$$p_1^* : D^b(X_\varphi \times_B X_\beta) \to D^b(X_\varphi).$$

We now have the following:

**Proposition 3.4** The functor $p_1^* : D^b(X_\varphi \times_B X_\beta) \to D^b(X_\varphi)$ preserves the relations defining $\alpha \mathcal{L}_\beta$ and $\beta \mathcal{E}_\alpha$ and so descends to a well defined exact functor

$$FM := (\gamma^*_E)^{-1} \circ p_1^* \circ \gamma^*_L : D^b_{1}(\beta X_\beta) \to D^b_{-1}(\beta X_\alpha).$$

**Proof.** Let $\gamma^*_L \mathcal{L} = (L, f)$ be as above. Let $Q := p_1^*L$. We need to construct a quasi-isomorphism of complexes on $X[m] \times_B X_\varphi$

$$g : \alpha^*_\varphi Q \to p_2^*Q \otimes p_1^*(M^{-1}_\beta \otimes M_0) \tag{3.8}$$

which depends functorially on $f$. 

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We start by noting that there is a natural commutative diagram

\[
\begin{array}{ccc}
X \times_B X \times_B X & \xrightarrow{p_{13}} & X \times_B X \\
\uparrow & & \downarrow \\
X \times_X X \times_B X & \xrightarrow{pr_{13}} & X \times_B X \\
\downarrow^{pr_{12}} & & \downarrow^{p_1} \\
X \times_X X & \xrightarrow{pr_{13}} & X \\
\end{array}
\]

Since the two bottom squares are fiber products we have the base change formulas:

\[
\begin{align*}
pr_1^* p_1_* L &= p_{12}^* \, pr_{13}^* L \\
pr_2^* p_1_* L &= p_{12}^* \, pr_{23}^* L.
\end{align*}
\]

Also, using the isomorphism \( a_\phi \times \text{id} : X[m] \times_B X \phi \to X \times_X X \phi \) we reduce the problem of finding the map (3.8) to the equivalent problem of constructing a map

\[
(3.10) \quad pr_1^* p_1_* L \to pr_2^* p_1_* L \otimes \Phi_\beta.
\]

of complexes on \( X \times_X X \phi \). Now using (3.9) and adjunction, this becomes

\[
(3.11) \quad pr_{13}^* L \to pr_{23}^* L \otimes pr_{12}^* \Phi_\beta.
\]

Therefore, in order to reconstruct from \( f \) a map (3.11) (equivalently \( g \)), it suffices to exhibit a canonical isomorphism

\[
(3.12) \quad \mathcal{P}_{1-2,m-3|X_\phi \times_X X_\phi \times_B X_\beta} \cong pr_{12}^* \Phi_\beta,
\]

on \( X \times_X X \phi \times_B X_\beta \). As a first step in establishing (3.12) we note that both sides are pullbacks of sheaves on \( X[m] \). On the right hand side, \( \Phi_\beta \) was defined as \( d^*(M_{\beta}^{-1} \otimes M_0) \) for the difference map \( d : X \times_X X \phi \to X[m] \). On the left hand side, it suffices (in view of the see-saw principle) to argue that, for a point \( \xi \in X[m] \), the restriction

\[
\mathcal{P}_{1-2,m-3|\xi \times_X X_\phi \times_B X_\beta}
\]

is trivial. But by the definition of \( \mathcal{P}_{1-2,m-3} \) (see Section 3.1.1), this restriction can be identified with \( \xi^{\otimes m} \). Since \( \xi \) has order \( m \), we are done.
To conclude the construction of the map (3.12) and the proof of the proposition, we need to show that the direct image $R^0 p_{1,2*} (\mathcal{P}_{1,2,m|X_\varphi \times X_\alpha, X_\varphi \times B X_\beta})$ is isomorphic to the line bundle $M_\beta^{-1} \otimes M_0$ on $X[m]$. For this it is useful to identify $X_\varphi \times X_\alpha, X_\varphi \times B X_\beta$ with $X[m] \times_B X_\varphi \times_B X_\beta$. Under that identification $\mathcal{P}_{1,2,m|X_\varphi \times X_\alpha, X_\varphi \times B X_\beta}$ becomes the pullback of $\mathcal{P}|X[m] \times_B X$ under the natural map

$$X[m] \times_B X_\varphi \times_B X_\beta \xrightarrow{} X[m] \times_B X$$

$$((\xi, x, y)) \xrightarrow{} (\xi, q_\beta(y)).$$

Thus, the existence of the isomorphism (3.12) is equivalent to the existence of an isomorphism

(3.13)  
$$p^*_{1,m,2}(\mathcal{P}|X[m] \times_B X_\beta) \cong p^*_{1}(M_\beta^{-1} \otimes M_0),$$

where $p_1 : X[m] \times_B X_\beta \rightarrow X[m]$ is the projection and $p_{1,m,2} : X[m] \times_B X_\beta \rightarrow X[m] \times_B X$ is the map given by $(\xi, x) \mapsto (\xi, q_\beta(x))$.

Recall from Section 3.1.2 that by definition we have

$$p^*_{1} M_\beta = a^*_\beta \Sigma_\beta \otimes p^*_2 \Sigma_\beta^{-1}.$$

Here $a_\beta : X[m] \times_B X_\beta$ is the action, $p_2 : X[m] \times_B X_\beta \rightarrow X_\beta$ is the projection on the second factor and $\Sigma_\beta$ is a line bundle on $X_\beta$ of fiber degree $m$, which corresponds to the ‘multiplication by $m$’ map $q_\beta : X_\beta \rightarrow X$.

Look at the embedding $X[m] \times_B X_\beta \subset X \times_B X_\beta$. The projections $p_1, p_2$ and the maps $p_{1,m,2}$ and $a_\beta$ extend to the natural projections $X \times_B X_\beta \rightarrow X$ and $X \times_B X_\beta \rightarrow X_\beta$ and maps $X \times_B X_\beta \rightarrow X \times_B X$ and $X \times_B X_\beta \rightarrow X_\beta$, which we will denote by the same letters.

With this notation we have

**Lemma 3.5** Let $\mathcal{P} \rightarrow X \times_B X$ denote the standard Poincare bundle. Then we have a natural isomorphism

$$p^*_{1,m,2} \mathcal{P} \otimes a^*_\beta \Sigma_\beta \otimes p^*_2 \Sigma_\beta^{-1} \cong p^*_1 \mathcal{O}_X(m\sigma).$$

**Proof of the lemma.** We will use the see-saw principle. Let $\xi \in X$ and let $b = \pi(\xi) \in B$. Then by viewing $\xi$ as a line bundle of degree zero on $(X_\beta)_b$ and using the fact that $\Sigma_\beta|_{X_b}$ is of degree $m$ we compute

$$p^*_{1,m,2} \mathcal{P}_{|(\xi) \times (X_\beta)_b} = q^*_\beta(\xi) = \xi^{\otimes m},$$

$$(_{a^*_\beta \Sigma_\beta \otimes p^*_2 \Sigma_\beta^{-1})}_{(\xi) \times (X_\beta)_b} = t^*_\xi L_\beta \otimes L_\beta^{-1} = \xi^{\otimes -m}.$$

Thus for every $\xi$ we have

$$(p^*_{1,m,2} \mathcal{P} \otimes a^*_\beta \Sigma_\beta \otimes p^*_2 \Sigma_\beta^{-1})_{(\xi) \times (X_\beta)_b} \cong \mathcal{O}_{(X_\beta)_b} = \mathcal{O}_{(X_\beta)_b}$$

and so by the see-saw principle $D_\beta := R^0 p_{1*}(p^*_{1,m,2} \mathcal{P} \otimes a^*_\beta \Sigma_\beta \otimes p^*_2 \Sigma_\beta^{-1})$ is a line bundle on $X$ satisfying

$$p^*_{1,m,2} \mathcal{P} \otimes a^*_\beta \Sigma_\beta \otimes p^*_2 \Sigma_\beta^{-1} \cong p^*_1 D_\beta.$$
To compute the bundle \( D_\beta \) we consider a point \( x \in X_\beta \). Let \( b = \pi_\beta(x) \). Restricting to 
\( X_b \times \{x\} \) we get

\[
\begin{align*}
 p^*_{1,m_2} \mathcal{P}(X_b \times \{x\}) &= q_\beta(x) \quad \text{(considered as a line bundle of degree zero on } X_b) \\
 a^*_\beta \Sigma_\beta|_{X_b \times \{x\}} &= t^*_\Sigma_\beta \\
 p^*_2 \Sigma_\beta|_{X_b \times \{x\}} &= \mathcal{O}_{X_b}.
\end{align*}
\]

Next, by the defining relationship between \( q_\beta \) and \( \Sigma_\beta \) we have that \( q_\beta(x) \) is the line bundle of degree zero on \( X \) corresponding to \( \mathcal{O}_{(X_\beta)_b}(mx) \otimes \Sigma^{-1}_\beta \) under the identification \( \text{Pic}^0((X_\beta)_b) = \text{Pic}^0(X_b) \). Also, by the definition of a translation we have that \( t^*_x \Sigma_\beta \) is the tensor product of \( \mathcal{O}_{X_\beta}(m\sigma(b)) \) with the line bundle of degree zero on \( X_b \) corresponding to \( \mathcal{O}_{(X_\beta)_b}(-mx) \otimes \Sigma_\beta \) under the identification \( \text{Pic}^0((X_\beta)_b) = \text{Pic}^0(X_b) \).

In other words, we have

\[
(p^*_{1,m_2} \mathcal{P} \otimes a^*_\beta \Sigma_\beta \otimes p^*_2 \Sigma^{-1}_\beta)|_{X_b \times \{x\}} \cong \mathcal{O}_{X_\beta}(m\sigma(b)),
\]

for all \( x \in X_\beta \). This implies that up to a twist by a pullback of a line bundle on \( B \) we have \( D_\beta \cong \mathcal{O}_X(m\sigma) \). Finally, to fix the choice of this line bundle on \( B \) we look at the restriction of \( p^*_{1,m_2} \mathcal{P} \otimes a^*_\beta \Sigma_\beta \otimes p^*_2 \Sigma^{-1}_\beta \) on \( \sigma \times_B X_\beta \) which is clearly isomorphic to \( \mathcal{O}_{X_\beta} \). Hence the line bundle on \( B \) is trivial and the lemma is proven.

In view of Lemma 3.5, the only thing left to check in order to establish the isomorphism \( (3.13) \) is that the line bundle \( M_0 \) on \( X[m] \) is isomorphic to the restriction \( \mathcal{O}_X(m\sigma)|_{X[m]} \). However applying the same reasoning we used in the proof of Lemma 3.5 to the projections \( p_1, p_2 : X \times_B X \to X \) and the obvious maps \( p_{1,m_2} : X \times_B X \to X \times_B X \) and \( a_0 : X \times X \to X \), we see that \( p^*_{1,m_2} \mathcal{P} \otimes a^*_0 \mathcal{O}_X(m\sigma) \otimes p^*_2 \mathcal{O}_X(-m\sigma) \cong p^*_0 \mathcal{O}_X(m\sigma) \). On the other hand, from the definition of the Poincare bundle we have that \( p^*_{1,m_2} \mathcal{P}|_{X[m] \times_B X} \cong \mathcal{O} \) and so \( M_0 \cong \mathcal{O}_X(m\sigma)|_{X[m]} \). This finishes the proof of the existence of \( (3.12) \). To complete the proof of the proposition, it only remains to note that since \( (Q,g) \) was constructed from \( (L,f) \) by means of the pushforward via \( X_\varphi \times_B X_\beta \to X_\beta \) and the fixed isomorphism \( (3.12) \), it follows that \( g \) will satisfy the cocycle condition whenever \( f \) does. The proposition is proven.

### 3.4.2 Categorical yoga for equivalences

We have constructed a functor \( \mathbf{FM} : D^b_1(\alpha X_\beta) \to D^b_{-1}(\beta X_\alpha) \). We are going to prove that it is an equivalence. In order to do this, it is convenient to recall some general criteria, due to Bondal-Orlov and Bridgeland, for equivalences of triangulated categories.

Throughout this subsection we let \( \mathbf{F} : \mathcal{A} \to \mathcal{B} \) be an exact functor between triangulated categories. A class \( \Omega \) of objects in \( \mathcal{A} \) is called a spanning class if for every \( a \in \text{ob} \mathcal{A} \), the left orthogonality condition

\[
\text{Hom}'_{\mathcal{A}}(a, \omega) = 0, \quad \text{for all } i \in \mathbb{Z}, \omega \in \Omega
\]
implies that \(a = 0\), and similarly on the right. Recall the following

**Theorem [BO95]** Assume that \(\Omega\) is a spanning class for \(\mathcal{A}\) and that the functor \(F: \mathcal{A} \to \mathcal{B}\) has left and right adjoints. Then \(F\) is fully faithful if and only if it is orthogonal:

\[
F: \text{Hom}^i_{\mathcal{A}}(\omega_1, \omega_2) \cong \text{Hom}^i_{\mathcal{B}}(F\omega_1, F\omega_2), \quad \text{for all } i \in \mathbb{Z}, \omega_1, \omega_2 \in \Omega.
\]

Assume now that our triangulated category \(\mathcal{A}\) is linear. A functor \(S_{\mathcal{A}}: \mathcal{A} \to \mathcal{A}\) is called [BK90] a Serre functor for \(\mathcal{A}\) if it is an exact equivalence and induces bifunctorial isomorphisms

\[
\text{Hom}_{\mathcal{A}}(a, b) \to \text{Hom}_{\mathcal{A}}(b, S_{\mathcal{A}}a)^\vee, \quad \text{for all } a, b \in \text{ob } \mathcal{A},
\]
satisfying compatibility with compositions. The basic example of a Serre functor is \(S: D^b(X) \to D^b(X), S\omega := a \otimes \omega_X[n]\), where \(X\) is a smooth \(n\)-dimensional projective variety and \(\omega_X\) is the canonical bundle of \(X\). If a Serre functor exists, it is unique up to an isomorphism of functors. We are now ready to state the main equivalence criterion we will be using:

**Theorem [Bri99, BKR01]** Assume that \(\mathcal{A}\) and \(\mathcal{B}\) have Serre functors \(S_{\mathcal{A}}, S_{\mathcal{B}}\), that \(\mathcal{A} \neq 0\), \(\mathcal{B}\) is indecomposable, and that \(F: \mathcal{A} \to \mathcal{B}\) has a left adjoint. Then \(F\) is an equivalence if it is fully faithful and it intertwines the Serre functors: \(F \circ S_{\mathcal{A}}(\omega) = S_{\mathcal{B}} \circ F(\omega)\) on all elements \(\omega \in \Omega\) in the spanning class.

We want to show that our gerby Fourier-Mukai functor \(FM: D^b_1(\alpha X_\beta) \to D^b_1(\beta X_\alpha)\) is an equivalence. The results above suggest that we should first exhibit Serre functors for \(D^b_1(\alpha X_\beta)\) and \(D^b_1(\beta X_\alpha)\), and find a suitable spanning class for \(D^b_1(\alpha X_\beta)\). These results, which do not involve \(FM\), are carried out in section 3.4.3 below. In section 3.4.4 we then complete the argument by showing that our \(FM\) preserves orthogonality and intertwines the Serre functors.

### 3.4.3 Serre functors and spanning classes for \(\mathcal{O}_X^X\)-gerbes

Let \(X\) be an \(n\)-dimensional smooth projective variety. Let \(c: \alpha X \to X\) be an \(\mathcal{O}_X^X\)-gerbe corresponding to an element \(\alpha \in H^2(X, \mathcal{O}_X^X)\).

**Claim 3.6** The functor

\[
S: D^b_1(\alpha X) \longrightarrow D^b_1(\alpha X)
\]

\[
a \longrightarrow a \otimes c^*\omega_X[n]
\]

is a Serre functor.
\textbf{Proof.} For any \(a, b \in D^b_1(\alpha X)\), we need a natural isomorphism
\[
\text{Hom}_{D^b_1(\alpha X)}(a, b) \to \text{Hom}_{D^b_1(\alpha X)}(b, Sa)^\vee.
\]
Since \(a, b\) have weight 1, \(R\text{Hom}_{\alpha X}(a, b)\) has weight 0, so there exists a unique \(H(\alpha X)(a, b) \in D^b(X)\) such that
\[
R\text{Hom}_{\alpha X}(a, b) = c^*H(\alpha X)(a, b).
\]
It follows that
\[
\text{Hom}_{D^b_1(\alpha X)}(a, b) = R\Gamma_X(H(\alpha X)(a, b)).
\]
Similarly,
\[
R\text{Hom}_{\alpha X}(b, Sa) = c^*(H(b, a) \otimes \omega_X[n]),
\]
so
\[
\text{Hom}_{D^b_1(\alpha X)}(b, Sa)^\vee = R\Gamma_X(H(b, a) \otimes \omega_X[n])^\vee = R\Gamma_X(H(b, a)^\vee),
\]
where the last step uses the usual Serre duality. So all we need is the identification
\[
H(\alpha X)(a, b) \cong H(b, a)^\vee,
\]
or a non-degenerate pairing on \(H(\alpha X)(a, b) \times H(b, a)^\vee\). But since
\[
c^*: D^b(X) \cong D^b_0(\alpha X)
\]
is an equivalence of categories, this follows immediately from the non-degenerate pairing on \(R\text{Hom}_{\alpha X}(a, b) \times R\text{Hom}_{\alpha X}(b, a)\) given by the trace. \(\Box\)

Since our functor \(F = FM\) was constructed as a push-forward on the atlases, it has an obvious left adjoint \(G\) corresponding to the pullback functor on the atlases. Therefore \(FM\) also has a right adjoint, namely \(S_{D^b_1(\alpha X)} \circ G \circ S_{D^b_1(\beta X)}^{-1}\).

Fix a point \(p \in X_\beta\). Since the restriction of \(\alpha X_\beta\) to \(p\) is the trivial gerbe on \(p\) for any \(\alpha\), the torsion sheaf \(O_p\) can be considered as a weight one sheaf on \(\alpha X_\beta\) for any \(\alpha\). For our spanning class \(\Omega\) we take the structure sheaves of points on the space \(X_\beta\), viewed as sheaves of weight one on the stack \(\alpha X_\beta\).

\textbf{Claim 3.7} Let \(c: \alpha X \to X\) be an \(O_X^\times\)-gerbe on a smooth projective \(X\). Then the class \(\Omega\) consisting of structure sheaves \(O_p\) of points on \(X\), viewed as sheaves of weight one on \(\alpha X\), is a spanning class for \(D^b_1(\alpha X)\).

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Proof. In order to show that the class $\Omega$ is a spanning class, we need to show that for every $a \in \text{ob } D_{1}^{b}(\alpha X)$, the left orthogonality condition

$$\text{Hom}_{D_{1}^{b}(\alpha X)}^{i}(a, \mathcal{O}_{p}) = 0, \quad \text{for all } i \in \mathbb{Z}, p \in X$$

implies that $a = 0$. We also need the analogous result on the right, but this follows using the Serre functor. We can also reduce to the case that $a$ is represented by a single sheaf on $\alpha X$, i.e. $a$ is an $\alpha$-twisted sheaf on $X$. Now such an $a$ is specified in terms of its sections on an appropriate etale atlas $U$ plus some $\alpha$-twisted gluing. In order to conclude that $a = 0$, it suffices to show that every $p \in X$ has a neighborhood $U'$ on which $a = 0$. But by restricting to a small enough neighborhood $U'$ of $p$ in $U$, we can get the class $a$ to vanish. The restriction of $a$ to $U'$ and the $\mathcal{O}_{p}$ for $p \in U'$ become ordinary sheaves. The group $\text{Hom}_{D_{1}^{b}(\alpha X)}^{i}(a, \mathcal{O}_{p})$ can be computed on either $U$ or $U'$. Therefore, the orthogonality condition forces $a$ to vanish on $U'$, which is what we need. \qed

3.4.4 Orthogonality and intertwining

Now that we have a Serre functor and a spanning class, we are ready to apply the general results of subsection 3.4.3 to our gerby Fourier-Mukai functor $FM$.

Claim 3.8 The gerby Fourier-Mukai functor $FM : D_{1}^{b}(\alpha X) \rightarrow D_{-1}^{b}(\beta X)$ is orthogonal on $\Omega$:

$$FM : \text{Hom}_{D_{1}^{b}(\alpha X)}^{i}(\mathcal{O}_{x_1}, \mathcal{O}_{x_2}) \rightarrow \text{Hom}_{D_{-1}^{b}(\beta X)}^{i}(F\mathcal{O}_{x_1}, F\mathcal{O}_{x_2}), \quad \text{for all } i \in \mathbb{Z}, x_1, x_2 \in X_{\beta}.$$

Proof. Recall (3.4.1) that our functor $FM$ descends from $p_{1*} : D^{b}(X_{\varphi} \times_{B} X_{\beta}) \rightarrow D^{b}(X_{\varphi})$. Let $b_1, b_2 \in B$ be the images of $x_1, x_2 \in X_{\beta}$. If $b_1 \neq b_2$ then the supports are disjoint, so the $\text{Hom}$ on both sides clearly vanish. Assume then that $b_1 = b_2 = b$. In this case, the structure sheaf $\mathcal{O}_{\alpha, \mathcal{L}_{\beta}|x_i}$ is supported on the fiber $C_{x_i} = X_{\varphi} \times_{B} (x_i)$, and both supports map isomorphically to $C_{b} = X_{\varphi} \times_{B} (b) \subset X_{\varphi}$. Now $FM(\mathcal{O}_{\alpha, \mathcal{L}_{\beta}|x_i})$ is the line bundle $\mathcal{L}_{\beta}(-mx_i)$ on $C_{b}$, so

$$\text{Hom}_{D_{1}^{b}(\alpha X)}^{i}(F\mathcal{O}_{x_1}, F\mathcal{O}_{x_2}) = \text{Hom}_{C_{b}}(\mathcal{L}_{\beta}(-mx_1), \mathcal{L}_{\beta}(-mx_2))$$

$$= \text{Hom}_{D_{1}^{b}(\alpha X)}^{i}(\mathcal{O}_{x_1}, \mathcal{O}_{x_2}),$$

completing the proof. \qed

We note that both sides of the claim vanish unless $x_1, x_2$ differ by a point of $X[m]$, in which case they define isomorphic sheaves. The spanning class $\Omega$ may therefore be taken to be parametrized by $X = X_{\beta}/X[m]$ rather than by $X_{\beta}$.

Claim 3.9 The gerby Fourier-Mukai functor $FM : D_{1}^{b}(\alpha X) \rightarrow D_{-1}^{b}(\beta X)$ intertwines the Serre functors, i.e.: $FM \circ S_{\alpha, X}^{b}(\mathcal{O}_{p}) = S_{\beta, X}^{b} \circ FM(\mathcal{O}_{p})$ for all points $p \in X_{\beta}$.

Proof. Follows immediately from the fact that $FMO_{x}$ is supported on $C_{b}$ and that the canonical sheaf of $X_{\beta}$ restricts to the trivial line bundle on $C_{b}$. \qed
4 Surfaces

In case $X$ is a surface, we can refine the previous results to include the singular fibers. On a surface, any pair of classes $\alpha, \beta \in \pi_{an}(X)$ are complementary in the sense of subsection 2.3 by Corollary 2.17 so the gerbes $\alpha X_{\beta, \beta} X_\alpha$ are always well-defined. When $m\beta = 0$ we will construct the lifting presentation of $\alpha X_\beta$ and the extension presentation of $\beta X_\alpha$. Then we will exhibit a Fourier-Mukai transform $\mathbf{FM}$ between these presentations. Finally, we will show that $\mathbf{FM}$ is an equivalence of categories by verifying the criterion of Bondal-Orlov and Bridgeland.

We assume throughout that $X$ is a smooth surface, $B$ is a smooth curve, and the elliptic fibration $\pi : X \to B$ has at most singular fibers of type $I_1$. Since every such elliptic surface is uniquely determined by its monodromy representation it is clear that we can always extend $X$ to a smooth compact relatively minimal elliptic surface whose base curve is a suitable compactification of $B$. Furthermore, by Kodaira’s classification of compact complex surfaces it follows that every smooth compact elliptic surface is Kähler (in fact algebraic). Therefore $X$ must be Kähler as well.

4.1 The lifting presentation

Our first goal is to construct the lifting presentation of $\alpha X_\beta$, in a way that restricts to the previously constructed presentation on the non-singular fibers. We start with the second projection $p_2 : X_\varphi \times_B X_\beta \to X_\beta$. Unfortunately, this is not an atlas for the gerbe $\alpha X_\beta$. The problem can be traced back to the fact that the threefold $X_\varphi \times_B X_\beta$ is singular. So let

$$Y := \frac{X_\varphi \times B}{X_\beta}$$

be a small resolution of $X_\varphi \times_B X_\beta$. Now $Y$ is smooth and equipped with flat morphisms $\nu_1 : Y \to X_\varphi$ and $\nu_2 : Y \to X_\beta$ which lift $p_1$ and $p_2$ respectively. There is an induced map

$$\nu_2^* : Br_{an}(X_\beta) \to Br_{an}(Y).$$

We claim that $Y$ is an atlas for $\alpha X_\beta$, i.e. that $\alpha X_\beta$ has a presentation:

\begin{equation}
\begin{array}{c}
\begin{tikzpicture}
\node (a) at (0,0) {$\alpha\text{LU}_\beta \times X_\beta$};
\node (b) at (0,-1) {$\alpha\text{LR}_\beta$};
\node (c) at (1,-1) {$\alpha\text{LU}_\beta$};
\node (d) at (2,-1) {$X_\beta$};
\node (e) at (1,0) {$\alpha\text{LU}_\beta$};
\shade[bottom color=white, top color=red] (a) circle (1.5);\end{tikzpicture}
\end{array}
\end{equation}

where

$$\alpha\text{LU}_\beta := Y,$$

$$\nu_2 : Y \to X_\beta$$

is the second projection, and

$$\alpha\text{LR}_\beta := Y \times_{\alpha X_\beta} Y.$$

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We call (4.1) the \textit{Lifting Presentation} of $\alpha X_\beta$. By Remark 2.4 (iv), the fact that (4.1) is indeed a presentation follows from:

\textbf{Lemma 4.1} $\nu_2^*(T_\beta(\alpha)) = 0$.

\textbf{Proof.} Let $B^o \subset B$ be the complement of the discriminant, and let $Y^o \subset Y, X^o \subset X$ be the inverse images of $B^o$. The maps

$$Y^o \xrightarrow{i} Y \xrightarrow{\nu_2} X_\beta$$

lead via the exponential sequence to the diagram:

$$
\begin{array}{ccc}
H^2_{an}(X_\beta, \mathcal{O}^\times) & \xrightarrow{\partial} & H^3(X_\beta, \mathbb{Z}) = 0 \\
\downarrow \nu_2 & & \downarrow \nu_2 \\
0 & \xrightarrow{\exp_{\nu_2}} & H^2_{an}(Y, \mathcal{O}^\times) \\
\downarrow i_Y^* & & \downarrow i_Y^* \\
0 & \xrightarrow{\exp_{Y^o}} & H^2_{an}(Y^o, \mathcal{O}^\times) & \rightarrow & H^3(Y^o, \mathbb{Z}).
\end{array}
$$

We have

$$\partial \nu_2^*(T_\beta(\alpha)) = \nu_2^* \partial(T_\beta(\alpha)) = \nu_2^* 0 = 0,$$

so

$$\nu_2^*(T_\beta(\alpha)) = \exp_Y(a)$$

for some $a \in H^2_{an}(Y, \mathcal{O}_Y)/H^2(Y, \mathbb{Z})$. We know from 3.1 that $Y^o$ is an atlas for the restriction of $T_\beta(\alpha)$ to $X^o$, so

$$0 = i_{\mathcal{O}^\times}^* \nu_2^*(T_\beta(\alpha)) = i_{\mathcal{O}^\times}^* \exp_Y(a) = \exp_{Y^o} i_{\mathcal{O}^\times}^*(a).$$

But $H^2_{an}(Y, \mathcal{O}_Y)/H^2(Y, \mathbb{Z})$ is a birational invariant, so $i_{\mathcal{O}^\times}$ is an isomorphism. Since $\exp_{Y^o}$ is injective, we see that $a = 0$, so we are done. \qed

\textbf{Counter Example 4.2} Let $A^o \subset A$ be an open subset in a smooth variety. By the birational invariance of cohomological Brauer groups \cite{Mil80}, the restriction map:

$$H^2_{el}(A, \mathcal{O}_A^\times) \rightarrow H^2_{el}(A^o, \mathcal{O}_{A^o}^\times)$$

is injective. Nevertheless, the analogous map:

$$H^2_{an}(A, \mathcal{O}_A^\times) \rightarrow H^2_{an}(A^o, \mathcal{O}_{A^o}^\times)$$

may fail to be injective. In our situation, all we were able to prove, and fortunately all that was needed, was that $H^2_{an}(Y, \mathcal{O}_Y^\times) \rightarrow H^2_{an}(Y^o, \mathcal{O}_{Y^o}^\times)$ is injective on the image of $H^2_{an}(X_\beta, \mathcal{O}_{X_\beta}^\times)$. \hfill 53
As an example, let $C \subset \mathbb{P}^3$ be a smooth curve of genus \( \geq 2 \), let $A$ be the blowup of $\mathbb{P}^3$ along $C$, and let $A^o := \mathbb{P}^3 \setminus C$. Then by the exponential sequence,

\[
H^2_{an}(A^o, \mathcal{O}^x_{A^o}) = H^3(A^o; \mathbb{Z}), \\
H^2_{an}(A, \mathcal{O}^x_A) = H^3(A; \mathbb{Z}),
\]

but by excision, $H^3(A^o; \mathbb{Z}) \cong H^3(\mathbb{P}^3, C; \mathbb{Z}) \cong \mathbb{Z}$, while $H^3(A; \mathbb{Z}) \cong H^1(C, \mathbb{Z})$.

### 4.2 The extension presentation

Next, we want to construct the extension presentation of $\beta X_\alpha$, in a way that restricts to the previously constructed extension presentation on the complement of the singular fibers. Fix $\varphi \in \mathfrak{I}_{an}(X)$ satisfying $m \cdot \varphi = \alpha$. Let $X^\alpha_\alpha, X^o_\alpha$ be the inverse images in $X_\alpha, X_\varphi$ of $B^o$, the complement of the discriminant in $B$. In the non-singular case, our atlas was given by the multiplication-by-$m$ map $q_\varphi : X^o_\varphi \to X^\alpha_\alpha$. Unfortunately, this does not extend to a morphism $q_\varphi : X_\varphi \to X_\alpha$. Instead, we will construct another (singular!) surface $\tilde{X}_\varphi$ with a birational morphism $\tilde{X}_\varphi \to X_\varphi$ and a flat morphism $\tilde{q}_\varphi : \tilde{X}_\varphi \to X_\alpha$ which restricts to the previous $q_\varphi^o$.

This data gives a commutative diagram:

\[
\begin{array}{ccc}
H^2_{an}(X^\alpha_\alpha, \mathcal{O}^x) & \xrightarrow{\tilde{q}_\varphi^*} & H^2_{an}(\tilde{X}_\varphi, \mathcal{O}^x) \\
\downarrow\cong & & \downarrow\cong \\
H^2_{an}(X^o_\alpha, \mathcal{O}^x) & \xrightarrow{(q_\varphi^o)^*} & H^2_{an}(X^o_\varphi, \mathcal{O}^x)
\end{array}
\]

The exponential sequence shows that the two vertical maps are isomorphisms, as in the proof of Lemma \ref{lemma:extension} this uses the fact that $H^2_{an}(\tilde{X}_\varphi, \mathcal{O}_{\tilde{X}_\varphi})/H^2(\tilde{X}_\varphi, \mathbb{Z})$ is a birational invariant, and that $\ker[H^3(\tilde{X}_\varphi, \mathbb{Z}) \to H^3(\tilde{X}_\varphi, \mathbb{R})] = 0$, $\ker[H^3(X_\alpha, \mathbb{Z}) \to H^3(X_\alpha, \mathbb{R})] = 0$, which in turn follows from the observation that the third cohomology of a smooth 4-manifold has no torsion and that $\tilde{X}_\varphi$ and $X_\alpha$ are Kähler surfaces. Since $(q_\varphi^o)^*$ kills all classes of order $m$, it follows that so does $\tilde{q}_\varphi^*$, so $\tilde{X}_\varphi$ is indeed an atlas.

In order to construct $\tilde{X}_\varphi$ we have to resolve the rational map $q_\varphi : X_\varphi \dashrightarrow X_\alpha$. For that we can work locally in the complex topology on $B$ near a point $p \in B \setminus B^o$, i.e. we can replace $B$ by a small disc centered at $p$. Over this disc the group scheme $X^2[m]$ has a subgroup scheme $I \subset X^2[m]$ of cycles invariant under the local monodromy around $p$. Since by assumption the singular fibers of $X$ are of type $I_1$, the group scheme $I$ is isomorphic to $B \times (\mathbb{Z}/m)$ and consists of all the sections in $X[m]$ over the disc that pass through smooth points of the fiber $X_p$. Translations by such sections give rise to a well defined action of $I$ on $X_\varphi$ which fixes the singular point $x_p$ of the fiber $(X_\varphi)_p$. Therefore over our disk the rational
map $q_\phi$ decomposes as

$$
\begin{array}{ccc}
X_\phi & \xrightarrow{q_\phi} & X_\alpha \\
\downarrow s & & \downarrow \\
X_\phi / I & & X_\alpha
\end{array}
$$

where $s$ denotes the quotient map. The surface $X_\phi / I$ has a unique singularity at the image of the point $x_p$ and the map $X_\phi / I \to B$ is a flat genus one fibration. A straightforward local computation at the singular point of the $I_1$ fiber of $X_\phi$ shows that in suitably chosen local coordinates $(z, w)$ near $x_p$ the generator of $I$ acts as $(z, w) \mapsto (\zeta z, \zeta w)$, where $\zeta$ is a primitive $m$-th root of unity. This implies that the singularity of $X_\phi / I$ is of type $A_{m-1}$. The minimal resolution $\tilde{X}_\phi / I \to X_\phi / I$ of $X_\phi / I$ is a flat genus one fibration over $B$ with a single $I_m$ fiber over $p$.

On the other hand, over our small disk, $X_\alpha$ retracts to the singular fiber $(X_\alpha)_p$ whose fundamental group is $\mathbb{Z}$. Therefore there is a unique $m$-sheeted etale cover $\tilde{X}_\alpha \to X_\alpha$. By construction the covering map commutes with the projections $\tilde{\pi}_\alpha : \tilde{X}_\alpha \to B$ and $\pi_\alpha : X_\alpha \to B$ and so the fiber $(\tilde{X}_\alpha)_p$ of $\tilde{\pi}_\alpha$ over the point $p \in B$ is a Kodaira fiber of type $I_m$. The minimal resolution $\tilde{X}_\phi / I \to X_\phi / I$ of $X_\phi / I$ can be identified with the Hilbert scheme of $I$-clusters in $X_\phi$. In the spirit of [BKR01] consider the universal closed subscheme $Z \subset X_\phi \times \tilde{X}_\phi / I$ with its natural projection to $X_\phi$ and $\tilde{X}_\phi / I \cong \tilde{X}_\alpha$. There is a commutative diagram of spaces

$$
\begin{array}{ccc}
Z & \xrightarrow{} & X_\phi \\
\downarrow & & \downarrow \\
\tilde{X}_\alpha & \xrightarrow{} & \tilde{X}_\phi / I \\
\downarrow & & \downarrow \\
X_\phi / I & \xrightarrow{} & X_\alpha
\end{array}
$$

where the morphism $Z \to X_\phi$ is birational and surjective and the morphism $Z \to \tilde{X}_\alpha$ is finite and flat. In particular the composite map

$$(4.2)\quad Z \to \tilde{X}_\alpha \to X_\alpha$$

is a finite and flat morphism which extends the multiplication-by-$m$ map $q_\phi^o : X_\phi^o \to X_\phi^o$. It is also helpful to observe that the two intermediate maps appearing in the construction of (4.2) are both Galois covers with Galois group isomorphic to $\mathbb{Z} / m$. The first one is just the quotient of $X_\phi \times_{X_\phi / I} \tilde{X}_\alpha$ by $I$ and the second is the etale Galois cover $\tilde{X}_\alpha \to X_\alpha$.
Our extension atlas $\hat{X}_\varphi$ is obtained by gluing the local surfaces $Z$ defined over small discs centered at discriminant points $p \in B \setminus B^o$ to $X_\varphi^o$. We will write $\varepsilon : \hat{X}_\varphi \to X_\varphi$ for the contraction map and $\hat{q}_\varphi : \hat{X}_\varphi \to X_\alpha$ for the finite flat map gluing each to $q_\varphi^o$. Note that by construction the surface $\hat{X}_\varphi$ is singular. It has isolated toroidal singularities sitting over the singular points of the singular fibers of $\hat{X}_\alpha$. In particular $\hat{X}_\varphi$ is a normal analytic surface.

4.3 Duality for gerby genus one fibered surfaces

With the description of the global lifting and extension presentations for gerby genus one fibered surfaces in place, we are now ready to construct the Fourier-Mukai functor between the derived categories of pure weight one and to show that it is an equivalence. The key property of our gerby surfaces, which makes the construction possible is the fact that the gerbes appearing in the picture become trivial when we restrict our attention to a piece of the surface sitting over a sufficiently small open disk in $B$.

We recall our convention that all our direct and inverse images, as well as the tensor product, are taken in the derived category. Thus for any space $Z$, we will simply write $\otimes$ for the derived tensor product $\otimes^L$ on $D^b(Z)$ and for any map of spaces $p : Z \to T$ we will write $p_\ast, p^\ast, p_!$, for the corresponding derived functors (whenever these functors make sense on the bounded derived categories).

Following the pattern of the proof in section 3.4.1 we will construct a Fourier-Mukai functor

$$FM : D^b_1(\alpha X_\beta) \to D^b_{-1}(\beta X_\alpha)$$

by exhibiting an integral transform between the derived categories on the atlases of the presentations $\alpha L_\beta$ and $\beta E_\alpha$ and then checking that this functor preserves the relations.

To avoid cumbersome notation we will write $Y := \hat{X}_\varphi \times_B X_\beta$ and $S := \hat{X}_\varphi$ for the atlases of the presentations $\alpha L_\beta$ and $\beta E_\alpha$ respectively.

With this notation the relations of $\alpha L_\beta$ are given by the total space

$$\text{tot}(\mathcal{P}_{1-2,m3}^\times) \longrightarrow Y \times_{X_\beta} Y \longrightarrow Y.$$ 

Here $\mathcal{P}_{1-2,m3}$ denotes an appropriate line bundle on $Y \times_{X_\beta} Y$ which extends the line bundle $p_{1-2,m3}^\ast \mathcal{P} \to X_\varphi^o \times_{B^o} X_\beta^o$ discussed in section 3.1.1. Note that we know that such a line bundle exists due to Lemma 4.1. Explicitly, the total space $\text{tot}(\mathcal{P}_{1-2,m3}^\times)$ is isomorphic to the stacky fiber product $Y \times_{X_\beta} Y$ (this product is a space again by Lemma 4.1). We will write $Y^o = X_\varphi^o \times_{B^o} X_\beta^o$ for the part of $Y$ sitting over $B^o$, and we put $\mathcal{P}_{1-2,m3}^o := \mathcal{P}_{1-2,m3} \otimes^L Y^o = p_{1-2,m3}^\ast \mathcal{P}$.

Similarly the relations of the presentation $\beta E_\alpha$ are given by the total space

$$\text{tot}(\Phi_\beta^\times) \longrightarrow S \times_{X_\alpha} S \longrightarrow S.$$
Here $\Phi_\beta$ denotes an appropriate line bundle which extends the line bundle $d^*(M_\beta \otimes M_0^{-1}) \to X_\varphi \times X_\varphi$ discussed in section 3.1.2. As before, the existence of the line bundle $\Phi_\beta$ is guaranteed by the observation that the class $T_\alpha(\beta)$ of the extension gerbe vanishes (see section 1.2) when we pull it back to $S$. We will write $S^0 = X_\varphi$ for the part of $S$ sitting over $B^o$ and we put $\Phi_\beta := \Phi_\beta|S^0 = d^*(M_\beta \otimes M_0^{-1})$.

Using the above setup we can now identify the category $D^b(\alpha X_\beta)$ with the category of objects $L \in D^b(Y)$ equipped with descent datum $f : p_1^*L \xrightarrow{q,i} p_2^*L \otimes \mathcal{P}_{1-2,m}$ on $Y \times_{X_\beta} Y$, satisfying the cocycle condition described at the end of section 2.1.2. Similarly we identify $D^b_{1-1}(\beta X_\alpha)$ with the category of objects $Q \in D^b(S)$ equipped with descent datum $g : p_1^*Q \xrightarrow{q,i} p_2^*Q \otimes \Phi^{-1}_{\beta}$ on $S \times_{X_\alpha} S$ satisfying the same cocycle condition as in section 2.1.2. These identifications reduce the problem of constructing the Fourier-Mukai functor $FM : D^b(\alpha X_\beta) \to D^b_{1-1}(\beta X_\alpha)$ to the problem of constructing a functor $F : D^b(Y) \to D^b(S)$ which maps descent data to descent data.

We will define the functor $F$ as an integral transform with respect to a suitable kernel object $\Pi \in D^b(Y \times S)$. We proceed to construct $\Pi$ by gluing together certain locally defined coherent sheaves on $Y \times S$. We carry out this gluing in the analytic topology to obtain the general functor $F$ we need. Note that in the algebraic case, the kernel $\Pi$ still produces the correct functor $\Pi$ in view of the GAGA principle.

First we look at the smooth part of the genus one fibrations $X_\alpha$, $X_\beta$, $X_\varphi$, etc. As usual, we write $B^o \subset B$ for the complement of the discriminant of the map $\pi : X \to B$. Similarly, for any space (or stack) $Z \to B$ mapping to $B$, we put $Z^o := Z \times_B B^o$. The atlases $Y^o$ and $S^0$ for the gerbes $\alpha \mathcal{L}^o$ and $\beta \mathcal{E}^o$ can be described simply as

$Y^o = X^o_\varphi \times_{B^o} X^o_\beta$

$S^0 = X^o_\varphi$

and so over $B^o$ we recover the setup analyzed in section 3.7.4. In this setup the integral transform we need was defined as the pushforward $p_{1*} : D^b(X^o_\varphi \times_{B^o} X^o_\beta) \to D^b(X^o_\varphi)$ with respect to the projection on the first factor. Equivalently, we can view this functor as the integral transform whose kernel is the sheaf $\Delta_{\varphi*} \mathcal{O}_{X^o_\varphi \times_{B^o} X^o_\beta}$ on $X^o_\varphi \times_{B^o} X^o_\varphi \times_{B^o} X^o_\beta$, where $\Delta_{\varphi}(x,y) := (x,x,y)$ for all $(x,y) \in X^o_\varphi \times_{B^o} X^o_\beta$. In view of this we set:

$\Pi^o := \mathcal{O}_{X^o_\varphi \times_{B^o} X^o_\beta} \subset \text{Coh}(X^o_\varphi \times_{B^o} X^o_\beta) = \text{Coh}(Y^o \times X^o_\varphi S^0)$.

Let now $p \in B \setminus B^o$. Choose small analytic discs $p \in U^p \subset B$ around each such $p$ so that $U^p \cap U^q = \emptyset$ for $p \neq q$ and the genus one fibrations $X_\varphi$ and $X_\beta$ both admit analytic sections over each $U^p$. For any space (or stack) $Z \to B$ mapping to $B$ we will write $Z^p$ for the restriction $Z \times_B U^p$. Note that

$Y \times_{X_\varphi} S = (Y^o \times_{X^o_\varphi} S^o) \bigcup \left( \prod_{p \in B \setminus B^o} (Y^p \times_{X^o_\varphi} S^p) \right)$. 

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and so, in order to extend $\Pi^p$ to a globally defined sheaf on $Y \times X_\varphi S$, it suffices to construct coherent analytic sheaves $\Pi^p$ on each $Y^p \times X^p_\varphi S^p$ and isomorphisms

$$\Pi^p_{(Y^p \times X^p_\varphi S^p) \cap (Y^o \times X^o_\varphi S^o)} \cong \Pi^p_{(Y^p \times X^p_\varphi S^p) \cap (Y^o \times X^o_\varphi S^o)}.$$

Fix a $p \in B \setminus B^o$ and let $s_\varphi : U^p \to X^p_\varphi$ and $s_\beta : U^p \to X^p_\beta$ be local holomorphic sections. Since the rational map $q_\varphi : X_\varphi \dashrightarrow X_\alpha$ is a well defined morphism when restricted to the complement $X^p_\varphi$ of the singular points of the singular fibers of $\pi_\varphi : X_\varphi \to B$, and since $s_\varphi(U^p) \subset X^p_\varphi$, it follows that $s_\varphi$ induces also a well defined section $s_\alpha := q_\varphi \circ s_\varphi : U^p \to X^p_\alpha$ of $X^p_\alpha$. Consider now the natural map

$$\nu_1 \times \hat{q}_\varphi : Y^p \times X^p_\varphi S^p \to X^p_\beta \times_U^p Y^p \times_U^p X^p_\alpha$$

and the isomorphism

$$t_{-s_\beta} \times t_{-s_\alpha} : X^p_\beta \times_U^p X^p_\alpha \to X^p \times_U^p X^p.$$

Let $\mathcal{P}^p$ denote the pullback

$$\mathcal{P}^p := (\nu_1 \times \hat{q}_\varphi)^* \circ (t_{-s_\beta} \times t_{-s_\alpha})^* (\mathcal{P}|_{X^p \times_U^p X^p}).$$

Note that the composite map $(t_{-s_\beta} \times t_{-s_\alpha}) \circ (\nu_1 \times \hat{q}_\varphi) : Y^p \times S^p \to X^p \times_U^p X^p$ factors through a small resolution $X^p \times \hat{X}^p \to X^p \times_U^p X^p$. This implies that the derived pullback of the torsion free sheaf $\mathcal{P}|_{X^p \times_U^p X^p}$ via $(t_{-s_\beta} \times t_{-s_\alpha}) \circ (\nu_1 \times \hat{q}_\varphi)$ is equal to the pullback as a coherent sheaf. Indeed, the pullback of $\mathcal{P}|_{X^p \times_U^p X^p}$ to the small resolution is locally free and hence it will pullback to a coherent sheaf $\mathcal{P}^p$ (in fact a line bundle) $Y^p \times X^p_\varphi S^p$.

The next step is to observe that without a loss of generality we may assume that the gerbes $\alpha \mathcal{L}^p_\beta \to X^p_\beta$ and $\beta \mathcal{E}^p_\alpha \to X^p_\alpha$ are trivial. Explicitly this means that we can find line bundles $\mathcal{L}^p \in \text{Pic}(Y^p)$ and $\mathcal{E}^p \in \text{Pic}(S^p)$ on $Y^p$ and $S^p$ respectively, together with isomorphisms

$$\mathcal{P}_{1-2,m-3|Y^p \times X^p_\varphi Y^p} \cong p_1^*(\mathcal{L}^p) \otimes p_2^*(\mathcal{L}^p)^{-1}$$

$$\Phi_{\beta|m|S^p \times X^p_\varphi S^p} = p_1^*(\mathcal{E}^p) \otimes p_2^*(\mathcal{E}^p)^{-1},$$

satisfying the obvious cocycle conditions.

We will work with particular trivializations of $\alpha \mathcal{L}^p_\beta \to X^p_\beta$ and $\beta \mathcal{E}^p_\alpha \to X^p_\alpha$ which we build in terms of the sections $s_\varphi$ and $s_\beta$ respectively. To construct $\mathcal{P}^p$ we use the rational map

$$Y^p \overset{\nu}{\longrightarrow} X^p_\varphi \times_U^p X^p_\alpha \to X^p \times_U^p X^p.$$

Note that $(t_{-s_\varphi} \times q^o_\beta) \circ \nu$ is a morphism from the complement of the exceptional curve $n^p$ for the small resolution $\nu$ to the complement of the unique singular point in $X^p \times_U^p X^p$. Since the standard Poincare sheaf $\mathcal{P}$ is a rank one torsion free sheaf on $X \times_B X$ which fails to be locally free only at the singular points of $X \times_B X$, it follows that the pullback of $\mathcal{P}$ via
the map \((t_{-s_β} \times q^m) \circ ν\) makes sense as a line bundle defined on \(Y^p \setminus n^p\). Combined with the observation that \(Y^p\) is smooth and that \(n^p \subset Y^p\) is of codimension two, it follows that this pullback extends to a unique line bundle \(ℙ^p\) on all of \(Y^p\). We can now use the seesaw principle in the same way we did in the proof of Theorem 3.1 to show that the isomorphism (4.3) exists and satisfies the cocycle condition.

Similarly, to construct \(ψ^p\) we note that the section \(s_β\) gives rise to a relative line bundle \(Σ_β ⊗ O_{X^p}(-m \cdot s_β) ∈ ℙic^0(\mathcal{X}^p_{β}/U^p)\). Under the canonical identification \(ℙic^0(\mathcal{X}^p_{β}/U^p) = ℙic^0(\mathcal{X}^p_{β}/U^p)\), this relative bundle corresponds to a relative line bundle \(ρ^p\) of degree zero along the fibers of \(\mathcal{X}^p_{β} \to U^p\) and hence to a globally defined line bundle \(\tilde{ψ}^p\) on \(\mathcal{X}^p_{β}\) which restricts to \(\rho^p \in ℙic^0(\mathcal{X}^p_{β}/U^p)\). We normalize \(\tilde{ψ}^p\) by choosing a trivialization \(s^*_β \tilde{ψ}^p \cong O_{U^p}\). Now the argument used in Theorem 3.2 implies that on \((S^p × X^p_{β} S^p)\) we can find an isomorphism \(Φ_β \cong p_1^* \tilde{ψ}^p \otimes p_2^*(\tilde{ψ}^p)^{-1}\) which (after possibly rescaling the trivialization \(s^*_β \tilde{ψ}^p \cong O_{U^p}\)) will also satisfy the cocycle condition. Now since \(S\) is an atlas for the gerbe \(β δ_α\), we conclude that \(\tilde{ψ}^p\) on \((S^p × X^p_{β} S^p)\) extends to a unique line bundle \(ψ^p\) on \(S^p × X^p_{β} S^p\) equipped with an isomorphism (4.4) satisfying the cocycle condition.

We now define the coherent sheaf \(Π^p \in Coh(Y^p × X^p_{β} S^p)\) by setting

\[Π^p := ℙ^p ⊗ p_Y^*(ℙ^p)^{-1} ⊗ p_S^*(ψ^p)^{-1},\]

where \(p_Y : Y × S \to Y\) and \(p_S : Y × S \to S\) denote the natural projections.

With this notation we now have

**Lemma 4.3** For any \(p ∈ B \setminus B^0\) there is an isomorphism

\[Π^p_{((Y^o × X^p_{β} S^p) \cap (Y^p × X^p_{β} S^p))} \cong Π^p_{((Y^o × X^p_{β} S^p) \cap (Y^p × X^p_{β} S^p))} .\]

In particular the sheaves \(\{Π^p\}_{p ∈ B \setminus B^0}\) glue to the sheaf \(Π^p\) to yield a globally defined analytic coherent sheaf \(Π\) on \(Y × S\).

**Proof.** We have to show that \(Π^p\) is naturally isomorphic to the trivial line bundle on \((Y^o × X^p_{β} S^p) \cap (Y^p × X^p_{β} S^p)\).

Write \(U^{po}\) for the punctured disc \(U^p \setminus \{p\}\) and for any space or stack \(Z \to B\) write \(Z^{po}\) for the fiber product \(Z ×_B U^{po}\). Using the isomorphisms \(t_{-s_β}\) and \(t_{-s_β}\) we can now identify \(π_β : X^p_{β} \to U^{po}\) and \(π_β : X^p_{β} \to U^{po}\) with the smooth elliptic fibration \(π : X^{po} \to U^{po}\). Under these identifications the intersection \((Y^o × X^p_{β} S^o) \cap (Y^p × X^p_{β} S^p)\) gets identified with the fiber product \(X^{po} ×_{U^{po}} X^{po}\). Using the same trivializations to recast \(ℙ^p, ℙ^p\) and \(ψ^p\) as sheaves on \(X^{po} ×_{U^{po}} X^{po}\) we get that \(ℙ^p\) becomes \(p_1^* ℙ^p\), \(ℙ^p\) becomes \(p^*_m ℙ\) and \(ψ^p\) becomes \(O\). In particular we get that \(Π^p\) corresponds to \(p_{1,m} ℙ \otimes p^*_m ℙ^{-1} \cong O_{X^{po} ×_{U^{po}} X^{po}}\).
This however follows immediately from the universal property of $\mathcal{P}$ on $X \times_B X$ and the seesaw theorem. 

The sheaf $\Pi$ gives rise to a well defined integral transform

\[
\begin{array}{c}
D^b(Y) \\ \downarrow L \\
\rightarrow \\
D^b(S) \\ \downarrow p_S^*(p_Y^* L \otimes \Pi)
\end{array}
\]

between the derived categories of analytic coherent sheaves on $Y$ and $S$ respectively. If in addition $\alpha \in \mathcal{III}(X) \subset \mathcal{III}_{an}(X)$ is also algebraic, then the atlases $Y$ and $S$ of the lifting and extension gerbes are proper separated algebraic spaces and so we can invoke the GAGA theorem \[Art70, Corollary 7.15\] to conclude that $\Pi$ is an algebraic coherent sheaf on $Y \times_S S$. This implies that in the algebraic context $F$ makes sense as an integral transform between algebraic coherent sheaves.

We now have the following

**Proposition 4.4** The functor $F : D^b(Y) \rightarrow D^b(S)$ defined by (4.5) maps the descent data for the lifting presentation to the descent data for the extension presentation and thus defines a functor $FM : D^b_1(\alpha \beta) \rightarrow D^b_1(\beta \alpha)$.

**Proof.** Let $L$ be an object in $D^b_1(\alpha \beta)$ represented by descent datum $(L, f)$ for the presentation $\alpha \mathcal{L} \beta$. In other words, $L$ is an object in $D^b(Y)$ and $f : p_1^* L \rightarrow p_2^* L \otimes \mathcal{P}_{1-2,m-3}$ is a quasi-isomorphism on $Y \times_{X_\alpha} Y$ satisfying the cocycle condition on $Y \times_{X_\alpha} Y \times_{X_\beta} Y$.

Consider the object $FL \in D^b(S)$. To prove the proposition we need to construct a quasi-isomorphism $g : p_1^* FL \rightarrow p_2^* FL \otimes \Phi^{-1}$ on $S \times_{X_\alpha} S$ which depends functorially on $f$ and satisfies the cocycle condition on $S \times_{X_\alpha} S \times_{X_\alpha} S$.

Let $\Gamma := Y \times_{X_\alpha} S$ and let $p_Y : \Gamma \rightarrow Y$ and $p_S : \Gamma \rightarrow S$ denote the natural projections. Then $FL = p_S^*(p_Y^* L \otimes \Pi)$, and so our problem boils down to finding a quasi-isomorphism

$$g : p_1^* p_S^*(p_Y^* L \otimes \Pi) \xrightarrow{\cong} p_2^* p_S^*(p_Y^* L \otimes \Pi) \otimes \Phi^{-1}$$

in $D^b(S \times_{X_\alpha} S)$, which depends functorially on $f$. In other words we want to compare the objects $p_1^* p_S^*(p_Y^* L \otimes \Pi)$ and $p_2^* p_S^*(p_Y^* L \otimes \Pi)$ on $S \times_{X_\alpha} S$. 

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Since we have an obvious commutative diagram

\[
\begin{array}{c}
Y \times_{X, \beta} Y \xrightarrow{p_1} Y \\
p_Y \times p_Y \\
\Gamma \times_{X, \beta} \Gamma \xrightarrow{p_1} \Gamma \\
p_{S \times S} \\
S \times_B S \xrightarrow{pr_1} S \\
S \times_{X, \alpha} S \xrightarrow{p_1} S \\
\end{array}
\]

we see that equivalently we can compare the objects \( \iota^* \text{pr}_1^* p_{S*}(p_{Y, L} \otimes \Pi) \) and \( \iota^* \text{pr}_2^* p_{S*}(p_{Y, L} \otimes \Pi) \). To compute these objects, we would like to perform a base change in the commutative squares

\[
\begin{array}{c}
\Gamma_{X, \beta} \xrightarrow{p_1} \Gamma \\
p_{S \times S} \\
S \times_B S \xrightarrow{pr_1} S \\
\end{array}
\quad
\begin{array}{c}
\Gamma_{X, \beta} \xrightarrow{p_2} \Gamma \\
p_{S \times S} \\
S \times_B S \xrightarrow{pr_2} S \\
\end{array}
\]

Unfortunately these squares are not cartesian and so we do not have the base change property on the nose. However we have the following useful

**Remark 4.5** Let \( X, Y, Z \) and \( T \) be analytic (or algebraic) spaces and let

\[
\begin{array}{c}
Z \xrightarrow{p} X \\
q \\
Y \xrightarrow{g} T
\end{array}
\]

be a commutative square of proper maps. Suppose further that the natural map \( u : Z \to Y \times_T X \) satisfies \( u_* \mathcal{O}_Z = \mathcal{O}_{Y \times_T X} \). Then for every \( F \in D^b(X) \) we have a base change identification \( q_* p^* F = g^* f_* F \) in \( D^b(Y) \). Indeed, we can complete the above square to a commutative diagram
Since $q_\ast \bar{p}^* F = g^* f_* F$ we get

$$q_\ast p^* F = q_\ast u_\ast u^* \bar{p}^* F = q_\ast (\bar{p}^* F \otimes u_\ast O_Z) = q_\ast \bar{p}^* F = g^* f_* F.$$  

(transitivity)  
(projection formula)  
(assumption)  
(base change)

In view of the previous remark we will be able to treat the squares (4.7) as base change squares if we can show that for the maps

$$u_1 : \Gamma \times_{X_\beta} \Gamma \to (S \times_B S) \times_{p_1, p_2} \Gamma \quad \text{and} \quad u_2 : \Gamma \times_{X_\beta} \Gamma \to (S \times_B S) \times_{p_2, p_2} \Gamma$$

we have $u_1_\ast O = O$ and $u_2_\ast O = O$.

To check this, note that

$$\Gamma \times_{X_\beta} \Gamma = (S \times_B S) \times_{X_\beta} Y \times_{X_\beta} Y \quad \text{and} \quad (S \times_B S) \times_{p_1, p_2} \Gamma = (S \times_B S) \times_{\varepsilon_{\op_1}, \varepsilon_{\op_2}, \nu_1} Y$$

and so $u_i, i = 1, 2$ are given explicitly by

$$u_i : (S \times_B S) \times_{X_\beta} (Y \times_{X_\beta} Y) \to (S \times_B S) \times_{\varepsilon_{\op_1}, \varepsilon_{\op_2}, \nu_1} Y$$

$$((a_1, a_2), (b_1, b_2)) \quad \text{for} \quad i = 1, 2.$$

This implies in particular that the maps $u_i$ fit in the cartesian squares

$$\begin{array}{ccc}
(S \times_B S) \times_{X_\beta} (Y \times_{X_\beta} Y) & \xrightarrow{p_{2\op_Y \times X_\beta}} & Y \\
\downarrow u_1 & & \downarrow \nu \\
(S \times_B S) \times_{\varepsilon_{\op_1}, \varepsilon_{\op_2}, \nu_1} Y & \xrightarrow{(\varepsilon_{\op_1} \times_{\op_2} S) \times (\nu_2 \times Y)} & X_\beta \times_B X_\beta
\end{array}$$

and

$$\begin{array}{ccc}
(S \times_B S) \times_{X_\beta} (Y \times_{X_\beta} Y) & \xrightarrow{p_{1\op_Y \times X_\beta}} & Y \\
\downarrow u_2 & & \downarrow \nu \\
(S \times_B S) \times_{\varepsilon_{\op_2}, \nu_1} Y & \xrightarrow{(\varepsilon_{\op_1} \times_{\op_2} S) \times (\nu_2 \times Y)} & X_\beta \times_B X_\beta
\end{array}$$

where $\nu : Y \to X_\beta \times_B X_\beta$ is the small resolution map defining $Y$.

Since the pullback of $O$ by any morphism is again $O$, it suffices to check that there is a canonical isomorphism $u_\ast O_Y = O_{X_\beta \times_B X_\beta}$. This is obvious by the cohomology and base change theorem. Thus $u_\ast O = O$ and so the squares (4.7) have the base change property.
In particular, in $D^b(S \times_B S)$ we get identifications:
\[
pr_i^* p_S^*(p_Y^* L \otimes \Pi) = (p_S \times p_S)_* p_i^*(p_Y^* L \otimes \Pi),
\]
for all $L \in D^b(Y)$ and for $i = 1, 2$. Furthermore since
\[
(S \times_{X, \alpha} S) \times_{S \times_B S} (\Gamma \times_{X, \beta} \Gamma) = \Gamma \times_{X, \alpha \times_B X, \beta} \Gamma,
\]
we can identify $i^*(p_S \times p_S)_* p_i^*(p_Y^* L \otimes \Pi)$ with $p_{S \times_{X, \alpha} S}^* p_i^*(p_Y^* L \otimes \Pi)$ where $p_{S \times_{X, \alpha} S} : \Gamma \times_{X, \alpha \times_B X, \beta} \Gamma \to S \times_{X, \alpha} S$ is the natural projection.

Now, using the commutativity of the top double square in (4.6) we get
\[
(p_S \times_{X, \alpha} S)_* p_1^*(p_Y^* L \otimes \Pi) = (p_S \times_{X, \alpha} S)_* (((p_Y \times p_Y)^* p_1^* L) \otimes (p_1^* \Pi)),
\]
and
\[
(p_S \times_{X, \alpha} S)_* p_2^*(p_Y^* L \otimes \Pi) = (p_S \times_{X, \alpha} S)_* (((p_Y \times p_Y)^* p_2^* L) \otimes (p_2^* \Pi)).
\]
If in addition $L \in D^b(Y)$ is part of a descend datum $(L, f)$ defining an object in $D^b_{f}(\alpha X, \beta)$, we can use $f$ to obtain an identification
\[
(p_S \times_{X, \alpha} S)_* p_1^*(p_Y^* L \otimes \Pi) = (p_S \times_{X, \alpha} S)_* (((p_Y \times p_Y)^* p_1^* L \otimes ((p_Y \times p_Y)^* \mathcal{P}_{1-2, m, 3}) \otimes (p_1^* \Pi)).
\]
Also
\[
((p_S \times_{X, \alpha} S)_* p_2^*(p_Y^* L \otimes \Pi)) \otimes \Phi^{-1} = (p_S \times_{X, \alpha} S)_* (((p_Y \times p_Y)^* p_2^* L) \otimes ((p_S \times_{X, \alpha} S \Phi^{-1}) \otimes (p_2^* \Pi)),
\]
and so, in order to get the desired isomorphism $g : p_1^* FL \to p_2^* FL \otimes \Phi^{-1}$ in $D^b(S \times_{X, \alpha} S)$ we only have to construct a canonical identification
\[
(4.8) \quad p_1^* \Pi \otimes (p_Y \times p_Y)^* \mathcal{P}_{1-2, m, 3} = p_2^* \Pi \otimes p_S \times_{X, \alpha} S \Phi^{-1}
\]
of coherent sheaves on $\Gamma \times_{X, \alpha \times_B X, \beta} \Gamma$.

We will construct the desired map (4.8) by gluing some locally defined but canonical identifications. Note that at this point we have completely eliminated the derived category from the picture. In particular, we are left with a question about sheaves, not complexes, and so gluing is a relatively simple matter.

(i) Over the part of $\Gamma \times_{X, \alpha \times_B X, \beta} \Gamma$ sitting over $B^o \subset B$, the identification (4.8) is just the one established in section 3.4. Indeed, by definition $\Pi_{|B^o} = \mathcal{O}_{\Gamma}$ and so constructing (4.8) over $B^o$ becomes equivalent to constructing a canonical identification (3.4). Such a construction was carried out in the proof of Proposition 3.4.

(ii) Over the part of $\Gamma \times_{X, \alpha \times_B X, \beta} \Gamma$ sitting over $U^p \subset B$, $p \in B \setminus B^o$, we can use again the line bundles $\mathcal{L}^p \to Y^p$ and $\psi^p \to S^p$ appearing in the construction of $P^p = P_{|U^p}$ to trivialize our gerbes. Recall that $\mathcal{L}^p$ and $\psi^p$ come equipped with the natural isomorphisms

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and \((4.4)\). Furthermore, \(p\ln_1 P_1 \otimes p_2^* Y \otimes (Q_\beta^{-1} \otimes S_2) \psi_2^{-1} \otimes S_1 \psi_1^{-1} \otimes Y Q_\beta^{-1} \otimes 2\).

or equivalently, after the obvious cancellations, an identification

\[ p_1^* P = p_2^* P. \]

However, \(\mathcal{P} \in \text{Coh}(\Gamma^p)\) was defined as a pullback of a sheaf on \(X_\alpha^p \times_{U^p} X_\beta^p\) and so we get a canonical identification \(p_1^* \mathcal{P} = p_2^* \mathcal{P}\) on \(\Gamma^p \times_{X_\alpha^p \times_{U^p} X_\beta^p} \Gamma^p\). This yields the desired canonical identification \((4.8)\) over the part of \(\Gamma \times_{X_\alpha \times_B X_\beta} \Gamma\) sitting over \(U^p\).

Finally it only remains to observe that the isomorphisms chosen in \((4.3)\) and \((4.4)\) were the ones used in the proof of Lemma \((4.3)\) to glue \(\Pi^p\) and \(\Pi^o\) on the overlap \(\Gamma^p \cap \Gamma^o\). Therefore the identifications in items (i) and (ii) above glue on the overlaps \((\Gamma \times_{X_\alpha \times B X_\beta} \Gamma) \times_B U^{p^o}\) and so we have found our global identification \((4.8)\). This finishes the proof of the proposition.

We are now ready to complete the

**Proof of Theorem** \(\mathbb{A}\). The only thing left to show is that the gerby Fourier-Mukai transform \(FM : D^b(\alpha X_\beta) \to D^b(\beta X_\alpha)\) constructed in Proposition \((4.3)\) is an equivalence of categories. We will again use the criterion of Bondal-Orlov and Bridgeland applied to the spanning class \(\Omega\) of gerby points in \(\alpha X_\beta\) described in Claim \((3.7)\). As before we need to show that \(FM\) intertwines the Serre functors on the sheaves \(O_x \in \Omega\) and that \(FM\) satisfies the orthogonality property

\[ FM : \text{Hom}_{D^b(\alpha X_\beta)}^i(O_{x_1}, O_{x_2}) \to \text{Hom}_{D^b(\beta X_\alpha)}^i(F O_{x_1}, F O_{x_2}). \]

for all \(i \in \mathbb{Z}, x_1, x_2 \in X_\beta\).

Since by definition the Fourier-Mukai image \(FM O_x\) is supported on the fiber \((\beta X_\alpha)_b\) of \(\beta X_\alpha\) over the point \(b = \pi_\beta(x) \in B\), it suffices to check the intertwining and orthogonality properties of \(FM\) locally in the base \(B\).

Over \(B^o\) these properties were established in Claims \((3.8)\) and \((3.9)\). To check the properties for the parts of our gerbes sitting over \(U^p \subset B\) we note that the proof of Lemma \((4.3)\) shows
that over $U^p$ the functor $FM$ fits in a commutative diagram of functors

$$
\begin{array}{ccc}
D^b_1(\alpha X^p) & \xrightarrow{FM} & D^b_{-1}(\beta X^p) \\
\nu_1^a(\bullet) \otimes \mathcal{Z}^p & \downarrow & \nu_2^a(\bullet) \otimes (\rho^p)^{-1} \\
D^b(X^p) & \downarrow & D^b(X^p) \\
t^*_{s_{\beta}} & \downarrow & t^*_{s\alpha} \\
D^b(X^p) & \xrightarrow{P_2(\nu_1^a(\bullet) \otimes \mathcal{Z}^p)} & D^b(X^p)
\end{array}
$$

where the vertical arrows are equivalences. However the bottom arrow is the usual integral transform with respect to the Poincare sheaf on an elliptic surface having at most $I_1$ fibers. Such a transform is an equivalence, e.g. by [BM02]. Finally, the functor $(\nu_1^a(\bullet) \otimes \mathcal{Z}^p) \circ t^*_{s\beta}$ transforms a structure sheaf of a point $x \in X^p$ into a sheaf in the spanning class $\Omega^p$ for $\alpha X^p$ and clearly every sheaf in $\Omega^p$ is obtained this way. This implies that $FM$ has the orthogonality and intertwining properties for sheaves in $\Omega^p$. The theorem is proven. \hfill \Box

From the statement of Theorem A one can derive a whole sequence of new cases of Căldăraru’s conjecture. Indeed, suppose $X$ is an elliptic K3 surface whose singular fibers are of type $I_1$ only. Note that for any element $\alpha \in \text{III}(X) = \text{Hom}(T_X, \mathbb{Q}/\mathbb{Z})$ we have a natural Hodge isometry $T_{X_\alpha} \cong \ker(\alpha)$ induced by the isogeny of $X_\alpha$ and $X$. In terms of the identifications $\text{III}(X) = Br'(X) = \text{Hom}(T_X, \mathbb{Q}/\mathbb{Z})$ and $\text{Br}'(X_\alpha) = \text{Hom}(\ker(\alpha), \mathbb{Q}/\mathbb{Z})$, the surjective map $T_{\alpha \text{Br}} : \text{III}(X) \to \text{Br}'(X_\alpha)$ sends a homomorphism $a : T_X \to \mathbb{Q}/\mathbb{Z}$ to its restriction $d|_{\ker(\alpha)} : \ker(\alpha) \to \mathbb{Q}/\mathbb{Z}$. Now we have:

**Corollary 4.6** Let $X$ be an elliptic K3 surface whose singular fibers are of type $I_1$ only. Let $\alpha, a \in \text{III}(X) = \text{Hom}(T_X, \mathbb{Q}/\mathbb{Z})$ and let $b, \beta \in \text{III}(X)^{\times 2}$ be in the $\text{SL}(2, \mathbb{Z})$ orbit of $(\alpha, a)$. Then

(a) $\ker\left[\ker(\alpha) \xrightarrow{a} \mathbb{Q}/\mathbb{Z}\right]$ and $\ker\left[\ker(\beta) \xrightarrow{b} \mathbb{Q}/\mathbb{Z}\right]$ are Hodge isometric;

(b) $D^b_1(\alpha X_\alpha)$ and $D^b_1(b X_\beta)$ are equivalent.

**Proof.** Part (b) follows from the fact that for every $(a, \alpha)$ we have equivalences $D^b_1(\alpha X_\alpha) \cong D^b_1(-\alpha X_\alpha)$ (by Theorem A) and $D^b_1(a X_\alpha) \cong D^b_1(a + \alpha X_\alpha)$ (since $T_\alpha(\alpha) = 0$).

For part (a) observe that by our identifications we have isometries of Hodge lattices $\ker\left[\ker(\alpha) \xrightarrow{a} \mathbb{Q}/\mathbb{Z}\right] = \ker(a) \cap \ker(\alpha)$ and $\ker\left[\ker(\beta) \xrightarrow{b} \mathbb{Q}/\mathbb{Z}\right] = \ker(b) \cap \ker(\beta)$. Since $(a, \alpha) : T_X \to (\mathbb{Q}/\mathbb{Z})^2$ and $(b, \beta) : T_X \to (\mathbb{Q}/\mathbb{Z})^2$ differ by an element of $\text{SL}(2, \mathbb{Z})$ acting on $(\mathbb{Q}/\mathbb{Z})^2$, it follows that $\ker(a) \cap \ker(\alpha) = \ker(b) \cap \ker(\beta)$ as sublattices in $T_X$. \hfill \Box
5 Modified $T$-duality and the SYZ conjecture

The celebrated work of Strominger, Yau and Zaslow \cite{SYZ96} interprets mirror symmetry of Calabi-Yaus in terms of special Lagrangian (SLAG) torus fibrations. If a CY manifold $X$ (with \textquoteleft\textquoteleft large complex structure\textquoteright\textquoteright) has mirror $X'$, \cite{SYZ96} conjecture the existence of fibrations $\pi : X \rightarrow B$ and $\pi' : X' \rightarrow B$ whose generic fibers are SLAG tori dual to each other: each parameterizes $U(1)$ flat connections on the other. In particular, each of these fibrations admits a SLAG zero-section, corresponding to the trivial connection on the dual fibers. The analogy with the situation considered in the main part of our work is clear: the SLAG torus fibrations replace the elliptic fibrations, and mirror symmetry (interchanging D-branes of type B with D-branes of type A) replaces the Fourier-Mukai transform (which interchanges vector bundles with spectral data).

In this context, the analogue of our gerbes and the Brauer group is given by the \"B-fields\" $\alpha \in H^2(X, \mathbb{R}/\mathbb{Z})$. On the other hand, the SLAG analogue of the Tate-Shafarevich group of $X'$ is given by $H^1(B, X')$, which over the locus where $\pi, \pi'$ are smooth can be identified with $H^1(B, R^1\pi_*(\mathbb{R}/\mathbb{Z}))$. As in the proof of lemma 2.11, $H^2(X, \mathbb{R}/\mathbb{Z})$ is related via a Leray spectral sequence to the three groups $H^i(B, R^{2-i}\pi_*(\mathbb{R}/\mathbb{Z}))$, for $i = 1, 2, 3$. Now for $i = 0$, the local system $H^0(B, R^2\pi_*(\mathbb{R}/\mathbb{Z}))$ can be identified, over the locus where $\pi$ is smooth, with the group of homotopy classes of sections of $X \rightarrow B$. Therefore, if the fibration $X \rightarrow B$ is good in the sense of Gro98, Gro99, $H^0(B, R^2\pi_*(\mathbb{R}/\mathbb{Z}))$ should be thought of as the analogue of the Mordell-Weil group of $X \rightarrow B$. Assume that the SLAG fibration $X \rightarrow B$ is generic, in the sense that the local system $R^2\pi_*(\mathbb{R}/\mathbb{Z})$ has no global sections.

The Leray spectral sequence therefore gives a Brauer-to-Tate-Shafarevich map:

$$H^2(X, \mathbb{R}/\mathbb{Z}) \rightarrow H^1(B, R^1\pi_*(\mathbb{R}/\mathbb{Z})).$$

We therefore may as well start with a pair of B-fields $\alpha \in H^2(X, \mathbb{R}/\mathbb{Z}), \beta \in H^2(X', \mathbb{R}/\mathbb{Z})$ on $X$ and $X'$ respectively. Since mirror symmetry involves the B-field in an essential way, this suggests that the SYZ conjecture, which is a SLAG analogue of the Fourier-Mukai transform for elliptic fibrations, must be supplemented by an analogue of our Theorem A.

Let $\mathcal{M}$ be the CY moduli space on which mirror symmetry acts. The emerging picture is that $\mathcal{M}$ looks, at least in some approximate sense, like an integrable system. The base is a real submanifold $\mathcal{M}_{\mathbb{R}} \subset \mathcal{M}$. The normal directions are parametrized by the B-fields $\alpha, \beta$, so they form a torus isomorphic to $H^2(X, \mathbb{R}/\mathbb{Z}) \times H^2(X', \mathbb{R}/\mathbb{Z})$. The original SYZ conjecture holds along $\mathcal{M}_{\mathbb{R}}$. As we move in normal directions, $X$ becomes gerby along $B$-directions on $X$, while along $B$-directions on $X'$, the SLAG fibration on $X$ loses its SLAG zero-section. Mirror symmetry interchanges these two behaviors.

Indeed, the moduli space $\mathcal{M}$ parameterizes pairs $(X, \alpha)$ where $X$ is a complex manifold together with a Calabi-Yau metric, and $\alpha \in H^2(X, \mathbb{R}/\mathbb{Z})$ is a B-field on it. Mirror symmetry is an involution $MS: \mathcal{M} \rightarrow \mathcal{M}$, presumably defined in a neighborhood of the \textquoteleft\textquoteleft large complex structure\textquoteright\textquoteright point, taking $(X, \alpha)$ to $(X', \beta)$. We let $\mathcal{M}_{\mathbb{R}} \subset \mathcal{M}$ denote the locus where $\alpha = \beta = 0$. It is a component of the fixed locus of the antilinear involution which reverses the signs of $\alpha$ and $\beta$. We denote a point of $\mathcal{M}_{\mathbb{R}}$ by the mirror pair $X, X'$. 

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Now a B-field $\beta \in H^2(X', \mathbb{R}/\mathbb{Z})$ on $X'$ determines a point $(X', \beta)$ of $\mathcal{M}$, hence a mirror point $X_\beta := \text{MS}(X', \beta)$. For small $\beta$, this $X_\beta$ is a deformation of $X$, so the additional B-field $\alpha \in H^2(X, \mathbb{R}/\mathbb{Z})$ on $X$ determines a corresponding B-field $T_\beta(\alpha) \in H^2(X_\beta, \mathbb{R}/\mathbb{Z})$ on $X_\beta$.

**Conjecture 5.1**

- The SYZ picture holds (near the large complex structure/large volume limit) on $\mathcal{M}_{\mathbb{R}}$.
- For a B-field $B'$ on $X'$, the deformed Calabi-Yau $X_{B'}$ admits a SLAG fibration (generally without a section) whose Jacobian (i.e. double dual) is the original SLAG fibration (with section) on $X$.
- Mirror symmetry preserves the integrable system structure: for any pair $\alpha, \beta$, the mirror of $(X_\beta, T_\beta(\alpha))$ is $(X_\alpha, T_\alpha(\beta))$.

We note that this modification of \cite{SYZ96} is consistent with recent interpretations in the literature (see \cite{Hit01} and references therein) of D-branes on Calabi-Yaus in the presence of a B-field: A D-brane of type B on $X$ is a coherent sheaf on the gerbe given by the B-field $\beta$, while a D-brane of type A on $X'$ is, roughly, a flat $U(1)$ connection on the restriction of the gerbe $\alpha$ to a SLAG submanifold in $X'$. The third part of this conjecture is the exact SLAG translation of Theorem A.

**Appendix A (by D.Arinkin) Duality for representations of 1-motives**

In this appendix, we sketch a different approach to the Fourier-Mukai transform for $\mathcal{O}^\times$-gerbes over smooth genus one fibrations (Theorem B). In this approach, Theorem B claims that the dual commutative group stacks (of a certain type) have equivalent derived categories of coherent sheaves. Let us review the duality for commutative group stacks (sometimes called the duality for generalized 1-motives).

Recall that the dual $X^\vee$ of an abelian variety $X$ is the moduli space of line bundles with zero first Chern class on $X$. Equivalently, $X^\vee$ parametrizes the extensions of the algebraic group $X$ by $\mathbb{G}_m$. In this form, the definition immediately generalizes to stacks: for a commutative group stack $\mathcal{X}$, its dual $\mathcal{X}^\vee$ is the moduli stack of extensions of commutative group stacks

$$1 \rightarrow \mathbb{G}_m \rightarrow G \rightarrow \mathcal{X} \rightarrow 0.$$  

The sum of extensions defines a group operation on $\mathcal{X}^\vee$; actually, $\mathcal{X}^\vee$ is naturally a commutative group stack.
Remark A.1 For technical reasons, we use a slightly different definition of the dual stack (Definition A.2). This allows to avoid the discussion of short exact sequences of group stacks; also, the group structure on $\mathcal{X}$ seems somewhat more natural.

Let $\mathcal{P} \to \mathcal{X}^\vee \times \mathcal{X}$ be the universal $\mathcal{X}^\vee$-family of extensions of $\mathcal{X}$ by $\mathbb{G}_m$; in particular, $\mathcal{P}$ is a $\mathbb{G}_m$-torsor on $\mathcal{X}^\vee \times \mathcal{X}$ (in fact, $\mathcal{P}$ is a biextension of $\mathcal{X}^\vee \times \mathcal{X}$ by $\mathbb{G}_m$). Notice that we can also view $\mathcal{P}$ as a $\mathcal{X}$-family of extensions of $\mathcal{X}$ by $\mathbb{G}_m$; this defines a morphism $\mathcal{X} \to (\mathcal{X}^\vee)^\vee$. The main idea of the Fourier-Mukai transform for commutative group stacks can be informally stated as follows:

\begin{equation} \tag{A.1} \end{equation}

For a “good” commutative group stack $\mathcal{X}$, the morphism $\mathcal{X} \to (\mathcal{X}^\vee)^\vee$ is an isomorphism, and the Fourier-Mukai transform defined by $\mathcal{P}_\mathcal{C}$ is an equivalence $\text{FM} : D^b(\mathcal{X}) \to D^b(\mathcal{X}^\vee)$. Here $\mathcal{P}_\mathcal{C}$ is the line bundle on $\mathcal{X}^\vee \times \mathcal{X}$ associated to the $\mathbb{G}_m$-torsor $\mathcal{P}$.

Now let us explain how Theorem B fits into the framework of the duality for commutative group stacks. First, we notice that the $\mathcal{O}^\times$-gerbe $\alpha X_0$ over $X$ is a group stack. Then we see that $\alpha X_\beta$ is a torsor over the group stack $\alpha X_0$; more precisely, the gerbes constructed using the lifting presentation and the extension presentation (from Section 3.1) have a natural structure of $\alpha X_0$-torsors.

Torsors over a group stack can be thought of as extensions of $\mathbb{Z}$ by this group stack; in this way, $\alpha X_\beta$ defines a commutative group stack $\alpha \tilde{X}_\beta$ that fits into an exact sequence

$$0 \to \alpha X_0 \to \alpha \tilde{X}_\beta \to \mathbb{Z} \to 0.$$  

The argument in section 3.4 shows that the constructions of the lifting presentation and the extension presentation are dual, so $\alpha \tilde{X}_\beta$ and $\pm \beta \tilde{X}_\alpha$ are dual commutative group stacks (provided that we use the lifting presentation for one of the two stacks and the extension presentation for the other). Moreover, these stacks are “good” in the sense of (A.1), and so the Fourier-Mukai transform gives an equivalence between $D^b(\alpha \tilde{X}_\beta)$ and $D^b(\pm \beta \tilde{X}_\alpha)$. The Fourier-Mukai transform of Theorem B is the restriction of this equivalence to direct summands in the derived categories (see Section A.2).

In the rest of the appendix, we discuss the notion of the dual of a group stack (Section A.1) and the special case when the group stack is an extension of $\mathbb{Z}$ (Section A.2). No proofs are given, but most statements are almost obvious.

I learned about the duality for commutative group stacks from A. Beilinson, and I am deeply grateful to him for the explanation.

A.1 Duality for commutative group stacks

From now on, the word ‘stack’ means an algebraic stack locally of finite type over a fixed base scheme $B$. All results also have an analytic version.
Definition A.2 For a commutative group stack \( \mathcal{X} \), the dual stack \( \mathcal{X}^\vee \) parametrizes 1-morphisms of commutative group stacks from \( \mathcal{X} \) to \( B\mathbb{G}_m \) (the classifying stack of \( \mathbb{G}_m \)). Thus, for a \( B \)-scheme \( S \), the category \( \mathcal{X}^\vee(S) \) is the category of 1-morphisms of commutative group stacks \( \mathcal{X} \times_B S \to B\mathbb{G}_m \times S \). Notice that \( \mathcal{X}^\vee \) does not have to be algebraic.

Remark A.3 For the definition to make sense, we need certain smallness assumptions. Indeed, if \( \mathcal{X} \) and \( \mathcal{Y} \) are stacks on a site \( B \), the 1-morphisms from \( \mathcal{X} \) to \( \mathcal{Y} \) form a stack only if \( \mathcal{X} \), \( \mathcal{Y} \), and \( B \) are small. However, this problem can be avoided if we assume that \( \mathcal{X} \) is an algebraic stack which is locally of finite type and replace the category of finitely presented \( B \)-schemes by an equivalent small category.

Example A.4 If \( \mathcal{X} \) is an abelian scheme over \( B \), then \( \mathcal{X}^\vee \) is the dual abelian scheme.

Example A.5 Let \( \mathcal{X} = G \) be an affine (or ind-affine) abelian group (over \( \mathbb{C} \)). Then \( \mathcal{X}^\vee \) is the classifying stack of the Cartier dual of \( G \). In particular, if \( \mathcal{X} = \mathbb{Z} \), we have \( \mathcal{X}^\vee = B\mathbb{G}_m \).

Another example is provided by the stacks \( _\alpha \widetilde{X} \) (constructed using either the lifting presentation or the extension presentation). It is clear from the construction that locally on \( B \), the torsor is trivial, so \( \widetilde{X} \) is isomorphic to \( \mathcal{X} \times \mathbb{Z} \). Since both \( \mathcal{X} \) and \( \mathbb{Z} \) are “good”, so is \( \widetilde{X} \). The dual stack \( \widetilde{X}^\vee \) is isomorphic to \( \mathcal{X}^\vee \times B\mathbb{G}_m \) locally on \( B \) (globally, it contains a substack isomorphic to \( B\mathbb{G}_m \), and the quotient equals \( \mathcal{X}^\vee \)). In particular, if \( \mathcal{X}^\vee \) is actually a space (rather than a stack), then \( \mathcal{X}^\vee \) is an \( \mathcal{O}^\times \)-gerbe over the space. The following statement is clear:

Proposition A.6 \( _\alpha \widetilde{X} \) is “good” in the sense of (A.1).

Proof. The property of being “good” is local on \( B \), so it is enough to notice that the stacks \( \mathcal{X}, B\mathbb{G}_m, \) and \( \mathbb{Z} \) are “good”.

In particular, we see that the Fourier-Mukai transform gives an equivalence between \( D^b(\alpha \widetilde{X}) \) and \( D^b(-\beta \widetilde{X}) \).

A.2 Duality for torsors

Now suppose \( \mathcal{X} \) is a commutative group stack which is “good”, and let \( \mathcal{X}' \) be a torsor over \( \mathcal{X} \). Denote by \( \widetilde{\mathcal{X}} \) the corresponding extension of \( \mathbb{Z} \) by \( \mathcal{X} \): it fits into the exact sequence

\[
0 \to \mathcal{X} \to \widetilde{\mathcal{X}} \to \mathbb{Z} \to 0
\]

and \( \mathcal{X}' \) is identified with the preimage of 1 \( \in \mathbb{Z} \). Notice that locally on \( B \), the torsor is trivial, so \( \widetilde{\mathcal{X}} \) is isomorphic to \( \mathcal{X} \times \mathbb{Z} \). Since both \( \mathcal{X} \) and \( \mathbb{Z} \) are “good”, so is \( \widetilde{\mathcal{X}} \). The dual stack \( \widetilde{\mathcal{X}}^\vee \) is isomorphic to \( \mathcal{X}^\vee \times B\mathbb{G}_m \) locally on \( B \) (globally, it contains a substack isomorphic to \( B\mathbb{G}_m \), and the quotient equals \( \mathcal{X}^\vee \)). In particular, if \( \mathcal{X}^\vee \) is actually a space (rather than a stack), then \( \mathcal{X}^\vee \) is an \( \mathcal{O}^\times \)-gerbe over the space. The following statement is clear:
Proposition A.7 The Fourier-Mukai functor $\mathbf{FM}: \mathcal{D}^b(\tilde{\mathcal{X}}) \to \mathcal{D}^b(\tilde{\mathcal{X}}^\vee)$ induces an equivalence $\mathcal{D}^b(\mathcal{X}^\vee) \to \mathcal{D}^b(\tilde{\mathcal{X}}^\vee)$, where $\mathcal{D}^b_1(\tilde{\mathcal{X}}^\vee) \subset \mathcal{D}^b(\tilde{\mathcal{X}}^\vee)$ is the complete subcategory of objects $F$ such that the action of $\mathbb{G}_m$ on $H^1(F)$ is tautological. Here the action is induced by the morphism $BG_m \to \tilde{\mathcal{X}}^\vee$.

In the case of duality between $\alpha \tilde{X}_\beta$ and $-\beta \tilde{X}_\alpha$, both stacks are torsors (over $\alpha X_\beta$ and $-\beta X_\alpha$, respectively), and so we get equivalences

$$\mathcal{D}^b(\alpha X_\beta) \to \mathcal{D}^b_1(-\beta \tilde{X}_\alpha)$$

and

$$\mathcal{D}^b_1(\alpha \tilde{X}_\beta) \to \mathcal{D}^b(-\beta X_\alpha).$$

They induce an equivalence

$$\mathcal{D}^b(\alpha X_\beta) \cap \mathcal{D}^b_1(\alpha \tilde{X}_\beta) = \mathcal{D}^b_1(\alpha X_\beta) \to \mathcal{D}^b_1(-\beta X_\alpha).$$

This is exactly what Theorem B claims.

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