POSITIVE DEHN TWIST EXPRESSIONS FOR SOME ELEMENTS OF FINITE ORDER IN THE MAPPING CLASS GROUP

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Abstract. Positive Dehn twist products for some elements of finite order in the mapping class group of a 2-dimensional closed, compact, oriented surface \( \Sigma_g \), which are rotations of \( \Sigma_g \) through \( 2\pi/p \), are presented. The homeomorphism invariants of the resulting simply connected symplectic 4-manifolds are computed.

Introduction

Positive Dehn twist expressions for a new set of involutions in the mapping class group of 2-dimensional closed, compact, oriented surfaces were presented in [1]. In a later article the idea was extended to bounded surfaces and from that, the positive Dehn twist expressions for involutions on closed surfaces that are obtained by joining several copies of bounded surfaces together were also obtained, [2]. It is the purpose of this article to present positive Dehn twist expressions for some elements of finite order in \( M_g \), which are rotations through \( 2\pi/p \) on the surface \( \Sigma_g \). This is shown in section 2 using the expressions for the involutions described in [1]. In section 3 the homeomorphism invariants of the resulting simply connected symplectic 4-manifolds that are realized as Lefschetz fibrations are computed.

1. Review

Let \( i \) represent the hyperelliptic (horizontal) involution and \( s \) represent the vertical involution as shown in Figure 1.

![Figure 1. The vertical and horizontal involutions](image)

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If \( i \) is the horizontal involution on a surface \( \Sigma_h \) and \( s \) is the vertical involution on a surface \( \Sigma_k \), \( k \)–even, then let \( \theta \) be the horizontal involution on the surface \( \Sigma_g \), where \( g = h + k \), obtained as in Figure 2.

Figure 2. The involution \( \theta \) on the surface \( \Sigma_{h+k} \)

Figure 3 shows the cycles that are used in expressing \( \theta \) as a product of positive Dehn twists which is stated in the next theorem.

**Theorem 1.0.1.** The positive Dehn twist expression for the involution \( \theta \) on the surface \( \Sigma_{h+k} \) shown in Figure 3 is given by

\[
\theta = c_{2i+2} \cdots c_{2h} c_{2h+1} c_{2i} \cdots c_2 c_1 b_0 c_{2h+1} c_{2h} \cdots c_{2i+2} c_1 c_2 \cdots c_2 b_1 b_2 \cdots b_{k-1} b_k c_{2i+1}.
\]
See [1] for the proof.

The order of the twists is from right to left, i.e., $c_{2i+1}$ is applied first.

2. Main Results

It is very easy to define many of the elements of finite order in the mapping class group qualitatively using geometry. In this section we will consider those which are realized as rotations on the surface $\Sigma_{1+p}$ through an angle of $2\pi/p$, Figure 4.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4.png}
\caption{2\pi/p rotation $\phi_p$ on $\Sigma_{1+p}$}
\end{figure}

It is a very difficult task in general, however, to find a positive Dehn twist expression for a given finite order element in the mapping class group. In this section we will find explicit positive Dehn twist expressions for rotations of the kind described above.

To achieve that, we will make use of the expression for the involution $\theta$ that is given in Theorem 1.0.1. $p$ represents a positive odd integer throughout this section.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5.png}
\caption{Definition of the involutions $\theta_1^p$ and $\theta_2^p$ on $\Sigma_{1+3}$}
\end{figure}

Let $\phi_p$ denote the $2\pi/p$ rotation on the surface $\Sigma_{1+p}$ as shown in Figure 4.

Let $\theta_1^p$ be the involution defined as $180\degree$ rotation about the axis through the center of the surface $\Sigma_{1+p}$ and the hole number $(p + 1)/2$, Figure 5(a). Let $\theta_2^p$
be the involution defined as $180^\circ$ rotation about the axis through the center of the surface $\Sigma_{1+p}$ and the hole number 1, Figure 5 (b).

It’s not difficult to see that application of $\theta_1^p$ followed by $\theta_2^p$ results in $2\pi/p$ rotation about the center of the surface. Application of $\theta_1^p$ on the surface gives the result shown in Figure 7 (b), and $\theta_2^p$ takes this figure to the result shown in Figure 8 (b).

Note that Figure 8 (b) and Figure 6 (b) are the same, namely the result of successive applications of the involutions $\theta_1^p$ and $\theta_2^p$ is the same as the action of $\phi_p$ on the surface $\Sigma_{1+p}$. From this we conclude that

$$\phi_p = \theta_2^p \theta_1^p$$
on the surface $\Sigma_{1+p}$.

![Figure 6. $2\pi/3$ rotation $\phi_3$ on $\Sigma_{1+3}$](image)

Now the question is how to find a positive Dehn twist product for $\phi_p$. The answer is simply juxtaposing the positive Dehn twist expressions for $\theta_1^p$ and $\theta_2^p$. Therefore, we first need to get those expressions for $\theta_1^p$ and $\theta_2^p$ using Theorem 1.0.1.

Figure 9 shows the cycles that realize $\theta_1^3$ on the surface $\Sigma_{1+3}$ using Theorem 1.0.1. Similarly, Figure 10 shows the cycles that realize $\theta_2^3$ on the surface $\Sigma_{1+3}$ using the same theorem.

According to Theorem 1.0.1 we have

$$\theta_1^3 = c_1^4 c_3 c_2 c_1 b_0^1 c_1 c_2 c_3 c_1^1 b_1^1 b_2^1 c_3^1$$

and

$$\theta_2^3 = c_1^2 c_3^2 c_2^2 b_0^1 c_1^2 c_3^2 c_2^2 c_1^1 b_2^1 c_5^2.$$ 

Therefore

$$\phi_3 = \theta_2^3 \theta_1^3 = c_1^2 c_3 c_2 c_1 b_0^1 c_1 c_2 c_3 c_1^1 b_0^1 b_2^2 c_5^1 c_3 c_2 c_1 b_0^1 c_1 c_2 c_3 c_1^1 b_0^1 b_2^2 c_5^1.$$
Figure 7. Involution $\theta_1^3$ applied to $\Sigma_{1+3}$

Figure 8. Involution $\theta_2^3$ applied to the result from Figure 7

Since $\phi_3^3 = 1$, we have

$$(c_1^2 c_2^2 c_3^2 b_5^2 c_1^1 c_2 c_3^1 b_2^1 c_5^1 c_1^1 c_2^1 c_3^1 b_0^1 c_2^1 c_3^1 c_4^1 b_1^1 b_2^1 c_5^1)^3 = 1$$

in $M_4$, the mapping class group of the surface $\Sigma_4$.

Next, we will demonstrate the cycles used in the expression for $\phi_3$ on a regular genus 4 surface $\Sigma_4$ and generalize it. For this, we need an identification between the two surfaces $\Sigma_4$ and $\Sigma_{1+3}$. Figure 11 shows that identification.

Figure 12 shows the same identification for general case. The hole with no number on the right is the hole in the center on the left.
Figure 9. Involution $\theta^3_1$ on $\Sigma_{1+3}$ and the cycles that realize it

Figure 10. Involution $\theta^3_2$ on $\Sigma_{1+3}$ and the cycles that realize it
Figure 11. Identification between the two surfaces $\Sigma_4$ and $\Sigma_{1+3}$.

Figure 12. Identification between the two surfaces for general case.

Figure 13 shows the cycles in Figure 9 (b) via the identification established in Figure 11. Similarly, Figure 14 shows the cycles in Figure 10 (b) via the identification established in Figure 11.

Figure 13. The cycles realizing $\theta_1^3$ on $\Sigma_4$. 
In general, the cycles realizing $\phi_p$ on the round surface $\Sigma_{1+p}$ are carried onto the regular surface $\Sigma_{p+1}$ using the identification in Figure 12.

3. Applications

Let $\Sigma_g$ be the 2-dimensional, closed, compact, oriented surface of genus $g > 0$ and $M_g$ be its mapping class group, the group of isotopy classes of all orientation-preserving diffeomorphisms $\Sigma_g \to \Sigma_g$. It is a well-known fact that any word in the mapping class group $M_g$ that contains positive exponents only and is equal to the identity element defines a symplectic Lefschetz fibration $X^4 \to S^2$. In this section we compute the homeomorphism invariants of the symplectic Lefschetz fibrations that are defined by the words $\phi_p^p = 1$, $p$– odd, in $M_{p+1}$.

First, we will show that the 4– manifold $X$ carrying the symplectic Lefschetz fibration structure $X^4 \to S^2$ is simply connected. A well known fact in theory of Lefschetz fibrations states that $\pi_1(X)$ is isomorphic to the quotient of $\pi_1(\Sigma_{p+1})$. 
by the normal subgroup generated by the vanishing cycles, the cycles about which positive Dehn twists in the expression $\phi_p^p = 1$ are performed. The vanishing cycles are seen as elements of $\pi_1(\Sigma_{p+1})$ in this section, not as elements of $M_{g+1}$.

Let $\{\alpha_1, \beta_1, \ldots, \beta_{p+1}, \alpha_{p+1}\}$ be the set of standard $2p+2$ generators of $\pi_1(\Sigma_{p+1})$ as shown in Figure 15. We will show that the subgroup of $\pi_1(\Sigma_{p+1})$ generated by the vanishing cycles includes all the generators of $\pi_1(\Sigma_{p+1})$ by showing that each generator is equal to 1 in the quotient group. We will use the elements in $\pi_1(\Sigma_{p+1})$ that are shown in Figure 16 along with the standard generators to express the vanishing cycles as elements of $\pi_1(\Sigma_{p+1})$. The expressions should be read from right to left.

\[ \phi_1^p = \gamma_1^2 \gamma_2^2 \gamma_3^2 \gamma_4^2 \gamma_5^2 \gamma_6^2 \gamma_7^2 \gamma_8^2 \gamma_9^2 \]

and we have the following expressions in $\pi_1(\Sigma_4)$ for the vanishing cycles shown in Figures 13 and 14:

\[ c_1 = \alpha_1, c_2 = \beta_1, c_3 = \gamma_1, c_4 = \beta_2, c_5 = \beta_3 \alpha_3^{-1} \beta_3^{-1} \gamma_2^{-1} \]
\[ b_0 = \beta_4^{-1} \beta_3^{-1} \beta_2^{-1} \beta_1^{-1} \]
\[ b_1 = \alpha_3 \beta_4^{-1} \gamma_3^{-1} \beta_3^{-1} \gamma_2^{-1} \]
\[ b_2 = \beta_3 \alpha_3 \gamma_3^{-1} \beta_3^{-1} \gamma_2^{-1} \]
\[ c_1 = \alpha_3, c_2 = \beta_3, c_3 = \gamma_2, c_4 = \beta_2, c_5 = \gamma_1^{-1} \beta_1^{-1} \alpha_1 \]
\[ b_0 = \beta_4^{-1} \beta_3^{-1} \beta_2^{-1} \beta_1^{-1} \]
\[ b_1 = \gamma_1 \beta_2 \beta_3 \beta_4^{-1} \gamma_3^{-1} \beta_3^{-1} \gamma_2^{-1} \beta_2^{-1} \gamma_1^{-1} \beta_1^{-1} \alpha_1 \]
\[ b_2 = \gamma_1 \beta_2 \beta_3 \gamma_3^{-1} \beta_3^{-1} \gamma_2^{-1} \beta_2^{-1} \gamma_1^{-1} \beta_1^{-1} \alpha_1 \beta_1 \]

Let $q = \frac{p+1}{2}$, i.e., $p = 2q - 1$. Then for $p \geq 5$, and hence $q \geq 3$,

\[ \phi_2^p = \gamma_1^2 \gamma_2^2 \gamma_3^2 \gamma_4^2 \gamma_5^2 \gamma_6^2 \gamma_7^2 \gamma_8^2 \gamma_9^2 \]

and we have the following expressions in $\pi_1(\Sigma_{p+1})$ for the vanishing cycles:

\[ \phi_1^p = \gamma_1^2 \gamma_2^2 \gamma_3^2 \gamma_4^2 \gamma_5^2 \gamma_6^2 \gamma_7^2 \gamma_8^2 \gamma_9^2 \]
\[ b_1^0 = \beta_{p+1}^{-1} \beta_p^{-1} \cdots \beta_2^{-1} \beta_1^{-1} \]

\[ b_1^1 = \alpha_3 \beta_{p+1}^{-1} \gamma_1^{-1} \beta_1^{-1} \cdots \gamma_3^{-1} \beta_3^{-1} \gamma_2^{-1} \]

\[ b_{2m} = \beta_3 \cdots \beta_{m+1} \beta_{m+2} \alpha_{m+2} \gamma_{p-m+1} \beta_{p-m+1}^{-1} \gamma_{p-m}^{-1} \cdots \beta_4^{-1} \gamma_3^{-1} \beta_3^{-1} \gamma_2^{-1}, \quad q-1 \geq m \geq 1 \]

\[ b_{2m-1} = \beta_3 \cdots \beta_{m} \beta_{m+1} \alpha_{m+2} \gamma_{p-m+1} \beta_{p-m+1}^{-1} \gamma_{p-m}^{-1} \cdots \beta_4^{-1} \gamma_3^{-1} \beta_3^{-1} \gamma_2^{-1}, \quad q-1 \geq m \geq 2 \]

\[ \theta_2^p \text{ cycles:} \]

\[ c_1^2 = \alpha_{q+1}, c_2^2 = \beta_{q+1}, c_3^2 = \beta_2, c_4^2 = \gamma_1^{-1} \beta_1 \]

\[ c_3^2 = \beta_3 \cdots \beta_{q-1} \beta_q \gamma_q^{-1} \beta_q^{-1} \cdots \gamma_3^{-1} \beta_3^{-1} \gamma_2^{-1} \]

\[ b_0^2 = \beta_{p+1}^{-1} \beta_p^{-1} \cdots \beta_2^{-1} \beta_1^{-1} \]

\[ b_{p-1}^2 = \gamma_1 \beta_2 \beta_3 \cdots \beta_{p-1} \beta_p \gamma_p^{-1} \beta_p^{-1} \gamma_{p-1}^{-1} \cdots \gamma_2^{-1} \beta_1 \]

\[ b_{p-2}^2 = \gamma_1 \beta_2 \beta_3 \cdots \beta_{p-2} \beta_p \beta_{p+1} \gamma_{p+1}^{-1} \beta_p^{-1} \gamma_{p-1}^{-1} \cdots \gamma_2^{-1} \beta_1 \]

\[ b_{2m}^2 = \beta_{1+q-m}^{-1} \gamma_{q-m}^{-1} \cdots \beta_4^{-1} \gamma_3^{-1} \beta_3^{-1} \gamma_2^{-1} \beta_2 \beta_3 \cdots \beta_{q+m} \alpha_{q+m+1} \cdots \gamma_{p-1} \gamma_p^{-1} \beta_p^{-1} \gamma_p^{-1} \cdots \gamma_1^{-1} \beta_1^{-1} \alpha_1, \quad q-2 \geq m \geq 1 \]

\[ b_{2m}^2 = \gamma_{q-m}^{-1} \beta_{q-m}^{-1} \cdots \gamma_3^{-1} \beta_3^{-1} \gamma_2^{-1} \beta_2 \beta_3 \cdots \beta_{q+m} \gamma_{q+m+1} \cdots \gamma_{p-1} \gamma_p^{-1} \beta_p^{-1} \gamma_p^{-1} \cdots \gamma_1^{-1} \beta_1^{-1} \alpha_1, \quad q-2 \geq m \geq 1. \]

All of the vanishing cycles above are equal to 1 in the quotient $\pi_1 (\Sigma_{p+1}) / N$, where $N$ is the normal subgroup generated by the vanishing cycles. Therefore we can cancel some of the standard generators right away: $\alpha_1, \beta_1, \beta_2, \alpha_{q+1}, \beta_{q-1}$ can be cancelled because they are equal to the cycles $c_1^1, c_2^1, c_1^2, c_2^2$, respectively. Also, $\alpha_2$ can be cancelled because $c_3^1 = \gamma_1 = \alpha_1 \alpha_2^{-1}$. There remains $2p - 4$ elements, which are

\[
 (3.0.2) \quad \alpha_3, \alpha_4, \cdots, \alpha_q, \alpha_{q+2}, \cdots, \alpha_p, \alpha_{p+1} \beta_3, \beta_4, \cdots, \beta_q, \beta_{q+2} \cdots, \beta_p, \beta_{p+1}.
\]

Now,
Similarly, and give

\[ b_1 = \alpha_3 \beta_{p+1}^{-1} \gamma_p^{-1} \beta_p^{-1} \cdots \beta_4^{-1} \gamma_3^{-1} \beta_3^{-1} \gamma_2^{-1} = 1 \]

and

\[ b_2 = \beta_3 \alpha_3 \gamma_p^{-1} \beta_p^{-1} \cdots \beta_4^{-1} \gamma_3^{-1} \beta_3^{-1} \gamma_2^{-1} = 1 \]

give

\[ \alpha_3 \beta_{p+1}^{-1} = \beta_3 \alpha_3, \text{ i.e., } \beta_3 = \alpha_3 \beta_{p+1}^{-1} \alpha_3^{-1}. \]

Similarly,

\[ b_3 = \beta_3 \alpha_4 \gamma_p^{-1} \beta_p^{-1} \cdots \beta_4^{-1} \gamma_3^{-1} \beta_3^{-1} \gamma_2^{-1} = 1 \]

and

\[ b_4 = \beta_3 \beta_4 \alpha_4 \gamma_p^{-1} \beta_p^{-1} \cdots \beta_4^{-1} \gamma_3^{-1} \beta_3^{-1} \gamma_2^{-1} = 1 \]

give

\[ \beta_3 \beta_4 \alpha_4 = \beta_3 \alpha_4 \beta_p^{-1}, \text{ i.e., } \beta_4 = \alpha_4 \beta_p^{-1} \alpha_4^{-1}. \]

In general

\[ b_{2m-1}^1 = \beta_3 \cdots \beta_{m+1} \alpha_m \beta_{m+1}^{-1} \cdots \beta_{p-m+2} \gamma_{p-m+1}^{-1} \cdots \beta_4^{-1} \gamma_3^{-1} \beta_3^{-1} \gamma_2^{-1} = 1 \]

and

\[ b_{2m}^1 = \beta_3 \cdots \beta_{m+2} \alpha_{m+2} \gamma_{p-m+1}^{-1} \beta_{p-m+1}^{-1} \cdots \beta_4^{-1} \gamma_3^{-1} \beta_3^{-1} \gamma_2^{-1} = 1 \]

give

\[ \beta_{m+2} = \alpha_{m+2} \beta_{p-m+2}^{-1} \alpha_{m+2}^{-1}, \text{ q - 1 \geq m} \geq 1. \]

A similar conjugation relation can be obtained by looking at the cycles of \( \theta_2^p \) in pairs as well. The first pair is \( b_1^2 \) and \( b_2^2 \):

\[ b_1^2 = \beta_q^{-1} \gamma_{q-1}^{-1} \cdots \gamma_3^{-1} \beta_3^{-1} \gamma_2^{-1} \beta_2^{-1} \cdots \beta_q^{-1} \gamma_{q+2}^{-1} \cdots \gamma_p^{-1} \beta_{p+1}^{-1} \gamma_{p-1}^{-1} \beta_{p-1}^{-1} \cdots \gamma_1^{-1} \beta_1^{-1} \alpha_1 = 1 \]

and

\[ b_2^2 = \gamma_{q-1}^{-1} \beta_q^{-1} \cdots \gamma_3^{-1} \beta_3^{-1} \gamma_2^{-1} \beta_2 \cdots \beta_q^{-1} \gamma_{q+2}^{-1} \cdots \gamma_p^{-1} \beta_{p+1}^{-1} \gamma_{p-1}^{-1} \beta_{p-1}^{-1} \cdots \gamma_1^{-1} \beta_1^{-1} \alpha_1 = 1 \]
result in

\[ \beta_q^{-1}\gamma_{q-1}^{-1} \cdots \gamma_3^{-1}\beta_3^{-1}\gamma_2^{-1}\beta_2 \cdots \beta_{q+1} = \gamma_{q-1}^{-1}\beta_{q-1}^{-1} \cdots \gamma_3^{-1}\beta_3^{-1}\gamma_2^{-1}\beta_2 \cdots \beta_{q+1}\beta_{q+2}, \]

which can be written as

\[ (\gamma_{q-1}^{-1}\beta_{q-1}^{-1} \cdots \gamma_3^{-1}\beta_3^{-1}\gamma_2^{-1}\beta_2 \cdots \beta_{q+1})^{-1} \beta_q^{-1} (\gamma_{q-1}^{-1} \cdots \gamma_3^{-1}\beta_3^{-1}\gamma_2^{-1}\beta_2 \cdots \beta_{q+1}) = \beta_{q+2}. \]

Similarly, from

\[ b_3^2 = \beta_{q-1}^{-1}\gamma_{q-2}^{-1} \cdots \gamma_3^{-1}\beta_3^{-1}\gamma_2^{-1}\beta_2 \cdots \beta_{q+2}\gamma_{q+3} \cdots \gamma_p\beta_{p+1}^{-1}\beta_p^{-1}\gamma_{p-1}^{-1} \cdots \gamma_1^{-1}\beta_1^{-1}\alpha_1 = 1 \]

and

\[ b_4^2 = \gamma_{q-2}^{-1}\beta_{q-2}^{-1} \cdots \gamma_3^{-1}\beta_3^{-1}\gamma_2^{-1}\beta_2 \cdots \beta_{q+3}\gamma_{q+3} \cdots \gamma_p\beta_{p+1}^{-1}\beta_p^{-1}\gamma_{p-1}^{-1} \cdots \gamma_1^{-1}\beta_1^{-1}\alpha_1 = 1 \]

we obtain

\[ \beta_{q-1}^{-1}\gamma_{q-2}^{-1} \cdots \gamma_3^{-1}\beta_3^{-1}\gamma_2^{-1}\beta_2 \cdots \beta_{q+2} = \gamma_{q-2}^{-1}\beta_{q-2}^{-1} \cdots \gamma_3^{-1}\beta_3^{-1}\gamma_2^{-1}\beta_2 \cdots \beta_{q+2}\beta_{q+3}, \]

which can be written as

\[ (\gamma_{q-2}^{-1}\beta_{q-2}^{-1} \cdots \gamma_3^{-1}\beta_3^{-1}\gamma_2^{-1}\beta_2 \cdots \beta_{q+2})^{-1} \beta_{q-1}^{-1} (\gamma_{q-2}^{-1} \cdots \gamma_3^{-1}\beta_3^{-1}\gamma_2^{-1}\beta_2 \cdots \beta_{q+2}) = \beta_{q+3}. \]

In general, from

\[ b_{2m-1}^2 = \beta_{1+q-m}^{-1}\gamma_{q-m}^{-1} \cdots \gamma_3^{-1}\beta_3^{-1}\gamma_2^{-1}\beta_2 \cdots \beta_{q+m} \gamma_{q+m+1} \cdots \gamma_p\beta_{p+1}^{-1}\beta_p^{-1}\gamma_{p-1}^{-1} \cdots \gamma_1^{-1}\beta_1^{-1}\alpha_1 = 1 \]

and

\[ b_{2m}^2 = \gamma_{q-m}\beta_{q-m}^{-1} \cdots \gamma_3^{-1}\beta_3^{-1}\gamma_2^{-1}\beta_2 \cdots \beta_{q+m+1}\gamma_{q+m+1} \cdots \gamma_p\beta_{p+1}^{-1}\beta_p^{-1}\gamma_{p-1}^{-1} \cdots \gamma_1^{-1}\beta_1^{-1}\alpha_1 = 1 \]

we obtain

\[ \beta_{1+q-m}^{-1}\gamma_{q-m}^{-1} \cdots \gamma_3^{-1}\beta_3^{-1}\gamma_2^{-1}\beta_2 \cdots \beta_{q+m} = \gamma_{q-m}\beta_{q-m}^{-1} \cdots \gamma_3^{-1}\beta_3^{-1}\gamma_2^{-1}\beta_2 \cdots \beta_{q+m}\beta_{q+m+1} \]

which can be written as

\[ (3.0.4) \]

\[ \beta_{1+q-m} = (\gamma_{q-m}^{-1} \cdots \gamma_3^{-1}\beta_3^{-1}\gamma_2^{-1}\beta_2 \cdots \beta_{q+m}) \beta_{q+m+1}^{-1} (\gamma_{q-m}\beta_{q-m}^{-1} \cdots \gamma_3^{-1}\beta_3^{-1}\gamma_2^{-1}\beta_2 \cdots \beta_{q+m})^{-1}, \]
\[ q - 2 \geq m \geq 1 \]

Setting \( m = 1 \) in \ref{3.0.3} we see that \( \beta_4 \) and \( \beta_{p+1}^{-1} \) are conjugate. On the other hand substituting \( m = q - 2 \) in \ref{3.0.3} we see that \( \beta_3 \) and \( \beta_p^{-1} \) are conjugate. Likewise setting \( m = 2 \) in \ref{3.0.3} and \( m = q - 3 \) in \ref{3.0.3} we see that \( \beta_4, \beta_p^{-1} \) and \( \beta_{p+1}^{-1} \) are conjugate to each other. Continuing this way we get the following list of triples.

\[
\begin{align*}
\beta_3 & \quad \beta_{p+1}^{-1} & \beta_p^{-1} \\
\beta_4 & \quad \beta_p^{-1} & \beta_{p-1}^{-1} \\
\beta_5 & \quad \beta_{p-1}^{-1} & \beta_{p-2}^{-1} \\
\ldots & \quad \ldots & \quad \ldots \\
\beta_q & \quad \beta_{q+2}^{-1} & \beta_{q+2}^{-1}
\end{align*}
\]

Each row in this list contains elements that are conjugate to each other. It is easy now to see that \( \beta_3, \beta_4, \ldots, \beta_q, \beta_{q+2}^{-1}, \beta_{q+3}^{-1}, \ldots, \beta_{p+1}^{-1}, \beta_{p+1}^{-1} \) are all conjugate to each other. Finally, setting \( m = q - 1 \) in \ref{3.0.3} we see that \( \beta_{q+1} \) and \( \beta_{q+1}^{-1} \) are conjugate. Since \( c_2 = \beta_{q+1} = 1 \) we conclude that the second half of the generators in \ref{3.0.2} are all identity in the quotient. Instead of \( \beta_{q+1} = 1 \) here we could have used \( \beta_{p+1} = 1 \), which is obtained from the pair of cycles \( b_{p-1}^2, b_{p-2}^2 \):

\[ b_{p-1}^2 = \gamma_1 \beta_2 \cdots \beta_{p-1} \beta_p \gamma_p \gamma_p^{-1} \beta_p^{-1} \gamma_p^{-1} \gamma_p^{-1} \gamma_1^{-1} \beta_1^{-1} \alpha_1 \beta_1 = 1 \]

and

\[ b_{p-2}^2 = \gamma_1 \beta_2 \cdots \beta_{p-1} \beta_p \beta_{p+1} \gamma_p \gamma_p^{-1} \beta_p^{-1} \gamma_p^{-1} \gamma_p^{-1} \gamma_1^{-1} \beta_1^{-1} \alpha_1 = 1 \]

give

\[ \gamma_1 \beta_2 \cdots \beta_{p-1} \beta_p = \gamma_1 \beta_2 \cdots \beta_{p-1} \beta_p \beta_{p+1}^{-1}, \text{ i.e., } \beta_{p+1} = 1 \]

using the fact that \( c_2^2 = \beta_1 = 1 \).

Therefore, the list of generators reduces to

\[ \alpha_3, \alpha_4, \ldots, \alpha_q, \alpha_{q+2}, \ldots, \alpha_p, \alpha_{p+1}. \]

Before we proceed further let’s recall that \( \gamma_i = \alpha_i \alpha_{i+1}^{-1} \). Therefore

\[
\begin{align*}
\gamma_i \cdots \gamma_j & = \alpha_i \alpha_{i+1}^{-1} \alpha_{i+1} \alpha_{i+2}^{-1} \cdots \alpha_{j-1} \alpha_j^{-1} \alpha_j \alpha_{j+1}^{-1} = \alpha_i \alpha_{j+1}^{-1} \\
\gamma_j^{-1} \cdots \gamma_i^{-1} & = \alpha_{j+1} \alpha_j^{-1} \alpha_j \alpha_{j-1}^{-1} \cdots \alpha_{i+2} \alpha_{i+1}^{-1} \alpha_{i+1} \alpha_i^{-1} = \alpha_{j+1} \alpha_i^{-1}
\end{align*}
\]

for \( j > i \). Now,

\[ b_1 = \alpha_3 \beta_{p+1} \gamma_p^{-1} \cdots \beta_4^{-1} \gamma_3^{-1} \beta_3^{-1} \gamma_2^{-1} = \alpha_3 \gamma_1^{-1} \cdots \gamma_3^{-1} \gamma_2^{-1} = \alpha_3 \alpha_{p+1} \alpha_2^{-1} = \alpha_3 \alpha_{p+1} = 1. \]
In general we have

\[ b_{2m-1} = \beta_3 \beta_{m+1} \alpha_{m+2} \beta_{p-m+2}^{-1} \gamma_{m+1}^{-1} \beta_3^{-1} \gamma_2^{-1} = \alpha_{m+2} \gamma_{m+1}^{-1} \beta_3^{-1} \gamma_2^{-1} \]

(3.0.5) \[ = \alpha_{m+2} \alpha_{p-m+2}^{-1} = \alpha_{m+2} \alpha_{p-m+2} = 1, \quad q - 1 \geq m \geq 2. \]

A similar list of products can be obtained using the cycles of \( \theta_p^2 \) as follows:

\[ b_1^2 = \beta_{q-1}^{-1} \gamma_{q-1}^{-1} \gamma_3^{-1} \beta_2 \cdot \beta_{q+1} \gamma_{q+2} \cdot \beta_p \beta_{p+1}^{-1} \gamma_p^{-1} \beta_{p-1}^{-1} \gamma_{p-1}^{-1} \gamma_1^{-1} \beta_1^{-1} \alpha_1 \]

\[ = \gamma_{q-1}^{-1} \cdot \gamma_2^{-1} \gamma_{q+1}^{-1} \cdot \gamma_p^{-1} \gamma_{p-1}^{-1} \gamma_1^{-1} \]

\[ = \alpha_q \alpha_2^{-1} \alpha_q \alpha_4^{-1} \]

\[ = \alpha_q \alpha_{q+2} = 1 \]

In general

\[ b_{2m-1} = \beta_{1+q-m}^{-1} \gamma_{q-m} \cdot \gamma_3^{-1} \beta_3^{-1} \gamma_2^{-1} \beta_2 \cdot \beta_{q+m+1} \gamma_{q+m+1} \cdot \beta_p \beta_{p+1}^{-1} \gamma_p^{-1} \beta_{p-1}^{-1} \gamma_{p-1}^{-1} \gamma_1^{-1} \beta_1^{-1} \alpha_1 \]

\[ = \gamma_{q-m}^{-1} \cdot \gamma_2^{-1} \gamma_{q+m+1}^{-1} \cdot \gamma_p^{-1} \gamma_{p-1}^{-1} \gamma_1^{-1} \]

\[ = \alpha_{q-m+1} \alpha_2^{-1} \alpha_{q+m+1} \alpha_1^{-1} \]

(3.0.6) \[ = \alpha_{q-m+1} \alpha_{q+m+1} = 1 \]

\[ q - 2 \geq m \geq 1 \]

The product relations in 3.0.5 and 3.0.6 can be summarized as

\[ \alpha_3 \alpha_{p+1} = 1 \]

\[ \alpha_3 \alpha_p = 1 \]

\[ \vdots \]

\[ \alpha_q \alpha_{q+3} = 1 \]

\[ \alpha_{q+1} \alpha_{q+2} = 1 \]

and

\[ \alpha_q \alpha_{q+2} = 1 \]

\[ \alpha_{q-1} \alpha_{q+3} = 1 \]

\[ \vdots \]

\[ \alpha_4 \alpha_{p-1} = 1 \]

\[ \alpha_3 \alpha_p = 1 \]
respectively. Using $c_1^2 = \alpha_{q+1} = 1$ we see that $\alpha_{q+2} = 1$. If $\alpha_{q+2} = 1$, then $\alpha_q = 1$ and if $\alpha_q = 1$, then $\alpha_{q+3} = 1$, etc. Continuing this way we can conclude that $\alpha_3 = \alpha_4 = \cdots = \alpha_{p+1} = 1$ in the quotient $\pi_1 (\Sigma_{p+1}) / N$, which is isomorphic to $\pi_1 (X)$, and this concludes the proof.

We do not know a topological argument to determine what those simply connected symplectic 4-manifolds are but we can compute their homeomorphism invariants using the symplectic Lefschetz fibration structure that they support. This will be accomplished by computing the Euler characteristics and the signatures of those Lefschetz fibrations.

Another fact that we will borrow from the theory of Lefschetz fibrations is the fact that the Euler characteristic of such a fibration $\mathbb{X}^4 \to S^2$ is given by the formula

$$\chi(X) = 4 - 4g + s,$$

where $s$ is the number of singular fibers, i.e., the number of vanishing cycles.

From $\mathbb{X}^4$ we see that each of $\theta_1^p$ and $\theta_2^p$ consists of $p + 9$ cycles. Therefore $\phi_p^p = (\theta_2^p \theta_1^p)^p$ consists of

$$2p (p + 9)$$

cycles and the Euler characteristic is found to be

$$\chi(X) = 4 - 4(p + 1) + 2p (p + 9)$$

$$= 2p^2 + 14p$$

$$= 2p (p + 7).$$

In order to compute the signature $\sigma(X)$ we wrote a Matlab program using the algorithm described in [3]. The computations that we have done using this program point to the closed formula

$$\sigma(X) = -12p$$

for the signature. Taking this formula for granted we can conclude that

$$c_1^2 (X) = 3\sigma (X) + 2\chi (X)$$

$$= 3(-12p) + 2 \cdot 2p (p + 7)$$

$$= 4p (p - 2)$$

and

$$\chi_h (X) = \frac{\sigma(X) + \chi(X)}{4}$$

$$= \frac{-12p + 2p (p + 7)}{4}$$

$$= \frac{1}{2}p (p + 1).$$
\( \chi_h(X) \) in the above computation makes sense because \( X \) has almost complex structure.

### 4. Appendix

We include the actual outputs of signature computations for some small values of \( p \).

\[
\begin{align*}
p = 3: & \\
0, 0, 0, 0, 0, 0, -1, -1, -1, -1, -1, -1, 0, 0, -1, -1, -1, -1, 0, 0, 0, 0, \\
0, 0, 0, 0, -1, -1, -1, -1, 0, 0, -1, -1, 0, 0, -1, -1, -1, 0, 0, 0, \\
0, 0, 0, 0, -1, -1, -1, -1, 0, 0, -1, -1, 0, 0, -1, -1, -1, 0, 0, 0, 0.
\end{align*}
\]

\[
\begin{align*}
p = 5: & \\
0, 0, 0, 0, 0, 0, 0, -1, -1, -1, -1, -1, -1, 0, 0, 0, 0, -1 - 1, -1, -1, 0, 0, 0, 0, \\
0, 0, 0, 0, 0, -1, -1, -1, -1, 0, 0, -1, -1, 0, 0, 0, 0, -1 - 1, -1, -1, 0, 0, 0, \\
0, 0, 0, 0, 0, -1, -1, -1, -1, 0, 0, -1, -1, 0, 0, 0, 0, -1 - 1, -1, -1, 0, 0, 0, \\
0, 0, 0, 0, 0, -1, -1, -1, -1, 0, 0, -1, -1, 0, 0, 0, 0, -1 - 1, -1, -1, 0, 0, 0, \\
0, 0, 0, 0, 0, -1, -1, -1, -1, 0, 0, -1, -1, 0, 0, 0, 0, -1 - 1, -1, -1, 0, 0, 0, 0.
\end{align*}
\]

\[
\begin{align*}
p = 7: & \\
0, 0, 0, 0, 0, 0, 0, 0, 0, -1, -1, -1, -1, -1, -1, 0, 0, 0, 0, 0, -1, -1, -1, -1, 0, 0, 0, 0, \\
0, 0, 0, 0, 0, 0, -1, -1, -1, -1, 0, 0, -1, -1, 0, 0, 0, 0, -1, -1, -1, 0, 0, 0, 0, \\
0, 0, 0, 0, 0, 0, -1, -1, -1, -1, 0, 0, -1, -1, 0, 0, 0, 0, -1, -1, -1, 0, 0, 0, 0, \\
0, 0, 0, 0, 0, 0, -1, -1, -1, -1, 0, 0, -1, -1, 0, 0, 0, 0, -1, -1, -1, 0, 0, 0, 0, \\
0, 0, 0, 0, 0, 0, -1, -1, -1, -1, 0, 0, -1, -1, 0, 0, 0, 0, -1, -1, -1, 0, 0, 0, 0, 0.
\end{align*}
\]

\[
\begin{align*}
p = 9: & \\
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, -1, -1, -1, -1, -1, -1, 0, 0, 0, 0, 0, 0, 0, -1, -1, -1, -1, 0, 0, 0, 0, \\
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, -1, -1, -1, 0, 0, -1, -1, 0, 0, 0, 0, 0, 0, -1, -1, -1, -1, 0, 0, 0, 0, \\
0, 0, 0, 0, 0, 0, 0, 0, 0, -1, -1, -1, -1, 0, 0, -1, -1, 0, 0, 0, 0, 0, -1, -1, -1, -1, 0, 0, 0, 0, \\
0, 0, 0, 0, 0, 0, 0, 0, -1, -1, -1, -1, 0, 0, -1, -1, 0, 0, 0, 0, -1, -1, -1, -1, 0, 0, 0, 0, \\
0, 0, 0, 0, 0, 0, 0, 0, -1, -1, -1, -1, 0, 0, -1, -1, 0, 0, 0, 0, -1, -1, -1, -1, 0, 0, 0, 0, \\
0, 0, 0, 0, 0, 0, 0, 0, -1, -1, -1, -1, 0, 0, -1, -1, 0, 0, 0, 0, -1, -1, -1, -1, 0, 0, 0, 0, \\
0, 0, 0, 0, 0, 0, 0, 0, -1, -1, -1, -1, 0, 0, -1, -1, 0, 0, 0, 0, -1, -1, -1, -1, 0, 0, 0, 0, \\
0, 0, 0, 0, 0, 0, 0, 0, -1, -1, -1, -1, 0, 0, -1, -1, 0, 0, 0, 0, -1, -1, -1, -1, 0, 0, 0, 0, \\
0, 0, 0, 0, 0, 0, 0, 0, -1, -1, -1, -1, 0, 0, -1, -1, 0, 0, 0, 0, -1, -1, -1, -1, 0, 0, 0, 0, \\
0, 0, 0, 0, 0, 0, 0, 0, -1, -1, -1, -1, 0, 0, -1, -1, 0, 0, 0, 0, -1, -1, -1, -1, 0, 0, 0, 0, \\
0, 0, 0, 0, 0, 0, 0, 0, -1, -1, -1, -1, 0, 0, -1, -1, 0, 0, 0, 0, -1, -1, -1, -1, 0, 0, 0, 0, \\
0, 0, 0, 0, 0, 0, 0, 0, -1, -1, -1, -1, 0, 0, -1, -1, 0, 0, 0, 0, -1, -1, -1, -1, 0, 0, 0, 0, \\
0, 0, 0, 0, 0, 0, 0, 0, -1, -1, -1, -1, 0, 0, -1, -1, 0, 0, 0, 0, -1, -1, -1, -1, 0, 0, 0, 0, \\
0, 0, 0, 0, 0, 0, 0, 0, -1, -1, -1, -1, 0, 0, -1, -1, 0, 0, 0, 0, -1, -1, -1, -1, 0, 0, 0, 0, \\
0, 0, 0, 0, 0, 0, 0, 0, -1, -1, -1, -1, 0, 0, -1, -1, 0, 0, 0, 0, -1, -1, -1, -1, 0, 0, 0, 0, \\
0, 0, 0, 0, 0, 0, 0, 0, -1, -1, -1, -1, 0, 0, -1, -1, 0, 0, 0, 0, -1, -1, -1, -1, 0, 0, 0, 0, \\
0, 0, 0, 0, 0, 0, 0, 0, -1, -1, -1, -1, 0, 0, -1, -1, 0, 0, 0, 0, -1, -1, -1, -1, 0, 0, 0, 0, \\
0, 0, 0, 0, 0, 0, 0, 0, -1, -1, -1, -1, 0, 0, -1, -1, 0, 0, 0, 0, -1, -1, -1, -1, 0, 0, 0, 0, \\
0, 0, 0, 0, 0, 0, 0, 0, -1, -1, -1, -1, 0, 0, -1, -1, 0, 0, 0, 0, -1, -1, -1, -1, 0, 0, 0, 0.
\end{align*}
\]
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