Relativistic stellar aberration for the
Space Interferometry Mission: parallax analysis

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ABSTRACT

This paper is a continuation of our previous analysis (i.e. Turyshev 2002a, 2002b) of the relativistic stellar aberration requirements for the Space Interferometry Mission (SIM). Here we have considered a problem of how the expected astrometric accuracy of parallax determination will constrain the accuracy of the spacecraft navigation. We show that effect of the spacecraft’s navigational errors on the accuracy of parallax determination with SIM will be negligible. We discuss the implication of the results obtained for the future mission analysis.

Subject headings: astrometry; techniques: interferometric, SIM; methods: analytical; solar system; relativity

1. Introduction

It is expected that SIM will achieve the $\sigma_\pi = 4 \mu$as mission accuracy in determination of parallaxes to a nearby stars. (3) have demonstrated that future astrometric model for SIM will have to account not only for the effects introduced by the motion of the solar system bodies, but also for those that are generated by the motion of the spacecraft and corresponding errors in the spacecraft’s navigation. Here we would like to analyze whether or not the future SIM astrometric accuracy will be compromised by the spacecraft navigational errors.

Analysis of tolerable errors for position determination in a general case is a complicated problem and should be addressed by using a formal numerical treatment. However, we may simplify the task by analyzing a special case which is expected to provide the most stringent requirements on the positional accuracy. The three-dimensional vector of parallax, $\vec{\pi}$, is given by the expression:

$$\vec{\pi} = \frac{\vec{s} \times (\vec{r} \times \vec{s})}{D},$$  (1)
where \( \mathbf{s} \) and \( D \) is the direction and distance to the observed source correspondingly; and \( \mathbf{r} \) is the barycentric position of the observer. The overall three-dimensional geometry of the problem presented in Figure 1. The involved vectors may be given in the spherical coordinate system by their magnitudes and two corresponding sky-angles:

\[
\mathbf{s} = \left( \cos \alpha \cos \delta, \ \sin \alpha \cos \delta, \ \sin \delta \right), \\
\mathbf{r} = r \left( \cos \lambda \cos \chi, \ \sin \lambda \cos \chi, \ \sin \chi \right). \tag{2}
\]

![Fig. 1.— Geometry of the problem.](image)

This parameterization allows one to present the expression Eq.(1) in the following form:

\[
\overline{\pi} = \frac{r}{D} \overline{P}, \tag{3}
\]

with vector \( \overline{P} \) given by

\[
\overline{P} = \mathbf{e}_x \left( \sin \alpha \cos^2 \delta \cos \chi \sin(\alpha - \lambda) - \sin \delta \left[ \cos \alpha \cos \delta \sin \chi - \sin \delta \cos \lambda \cos \chi \right] \right) - \\
- \mathbf{e}_y \left( \cos \alpha \cos^2 \delta \cos \chi \sin(\alpha - \lambda) + \sin \delta \left[ \sin \alpha \cos \delta \sin \chi - \sin \delta \sin \lambda \cos \chi \right] \right) + \\
+ \mathbf{e}_z \left( \cos^2 \delta \sin \chi - \sin \delta \cos \delta \cos \chi \cos(\alpha - \lambda) \right). \tag{4}
\]
2. Covariance matrix

The expressions Eqs.(3)-(4) may now be used to answer the question - how the positional errors \( \sigma_r, \sigma_\lambda, \sigma_\chi \) will affect the accuracy of parallax determination, \( \sigma_\pi \). To do this one needs to take the first order variations of the right-hand side of the expression (3) with respect to positional errors \( \delta r, \delta \lambda \) and \( \delta \chi \). The goal here is to present the obtained result in the form:

\[
(\delta \vec{\pi})^2 = \left(\frac{\delta r}{D}\right)^2 \vec{P}^2 + \frac{r^2}{D^2} (\delta \vec{P})^2 + \frac{2\delta r}{D} \frac{r}{D} (\vec{P} \cdot \delta \vec{P}).
\]

This expression is quite difficult for analytical description in a general case, however, for the purposes of present study, it may be significantly simplified. To do this, we assume that motion of the spacecraft is almost in the plane of ecliptic, thus \( \chi \sim 0 \), but \( \delta \chi \neq 0 \). We will not go into the lengthy derivations, but will present here only the final results for this algebraic exercise:

\[
\begin{align*}
\vec{P}^2 &= 1 - \cos^2 \delta \cos^2(\alpha - \lambda), \\
(\delta \vec{P})^2 &= \delta \lambda^2 \left[ 1 - \cos^2 \delta \sin^2(\alpha - \lambda) \right] + \delta \chi^2 \cos^2 \delta - \delta \chi \delta \lambda \sin 2\delta \sin(\alpha - \lambda) \\
(\vec{P} \cdot \delta \vec{P}) &= \delta \lambda \cos^2 \delta \sin(\alpha - \lambda) \cos(\alpha - \lambda) - \delta \chi \sin^3 \delta \cos \delta \cos(\alpha - \lambda).
\end{align*}
\]

Assuming that the errors \( \delta r, \delta \lambda \) and \( \delta \chi \) are normally distributed and uncorrelated, one may average the obtained expressions over the period of the spacecraft’s barycentric orbit. To further simplify the analysis we will assume that the orbit is circular. The latitude argument for circular motion coincides with the mean anomaly, e.q. \( \lambda = nt \), where \( n = \frac{2\pi}{T} \) and \( T \) is the period of orbital motion of the spacecraft. As a result, one obtains the expression for contribution of the navigational errors to the accuracy of parallax determination:

\[
\sigma^2_\pi = \frac{\sigma^2_r}{D^2} \frac{1}{2} (1 + \sin^2 \delta) + \frac{r^2}{D^2} \left[ \frac{\sigma^2_\lambda}{2} (1 + \sin^2 \delta) + \sigma^2_\chi \cos^2 \delta \right].
\]

3. Obtained results and conclusions

The obtained expression represents the error in parallax determination depending on the accuracy of SIM navigation. We have considered the motion of the spacecraft almost in the plane of ecliptic. The three terms on the right-hand side of the expression (7) represent the contributions of the radial, \( \sigma_r \) and tangential, \( \sigma_\lambda, \sigma_\chi \), positional errors. The obtained expression valid for any source, but depends only on \( \delta \) - the declination angle of the source’s position. One may see that positional errors are most influencing determination of parallax
for the case when the source is near the poles, or \( \delta \sim \pm \frac{\pi}{2} \). Thus, leading us to the final expression:

\[
\sigma^2_\pi = \frac{\sigma_r^2}{D^2} + \frac{r^2 \sigma_\lambda^2}{D^2}. \tag{8}
\]

The expression Eq.(8) is somewhat familiar and it may be obtained by a much simpler way. However, the second term on the right-hand side of it is very important. This term allows one to quickly obtain the necessary result. Thus, one may show that the tangential error, \( \sigma_\lambda \), is related to the radial one as follows \( \sigma_\lambda = \sigma_r/r \). One may also expect that this longitudinal error in the spacecraft’s position will be the same as the error in the spacecraft’s velocity sky-angle \( \sigma_\psi=\sigma_\lambda \). (3) have shown that this error was related to the velocity error by \( \sigma_\psi = \sigma_v/v \). This is why we may use the following relation \( \sigma_r/r = \sigma_v/v \) to obtain the maximum error in the SIM’s radial distance from the Sun for the entire mission. As a result, this relation suggests that the barycentric distance of the spacecraft during the whole mission should be known with an accuracy equal or better then \( \sigma_r = 20 \) km.

Finally, one may obtain the expression for the expected parallax errors induced by the inaccuracies in determination of the SIM’s solar system’s barycentric position. For example, for a source located at the distance \( D = k \) pc from the Sun, the corresponding expression reads:

\[
\sigma_\pi = \frac{r \sigma_v}{D v} = \frac{1.5 \times 10^{13} \text{ cm}}{3 \times 10^{18} \text{ cm}} \frac{4 \text{ mm/s}}{3 \times 10^7 \text{ mm/s}} = \frac{0.1}{k} \mu \text{as}. \tag{9}
\]

Thus, provided that the accuracy of the spacecraft’s velocity determination (driven by the need to correct for the relativistic stellar aberration) will be at the level of \( \sigma_v = 4 \) mm/s, the influence of the spacecraft’s barycentric position errors will be negligible for the parallax determination.

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