BEZERIN SYMBOLS OF OPERATORS ON THE UNIT SPHERE OF $\mathbb{C}^n$

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CONTENTS

1. Introduction and summary 1
2. A class of holomorphic spaces 3
  2.1. Construction of measure $dm^h_{n,p}$. 3
  2.2. The space $\mathcal{E}_{n,p}$ 5
3. Unitary transform 7
4. Coherent states for $\mathcal{O}$ 9
5. Berezin symbolic calculus 11
  5.1. Asymptotic expansion of the Berezin’s symbol. 12
6. The star product 15
7. Appendix A. On the inner product on complex sphere 20
8. Appendix B. Asymptotic of the Integral kernel of $U_{n,-1}$ 21
References 22

ABSTRACT. We describe the symbolic calculus of operators on the unit sphere in the complex $n$-space $\mathbb{C}^n$ defined by the Berezin quantization. In particular, we derive an explicit formula for the composition of Berezin symbol and with that a noncommutative star product. In the way is necessary introduce a holomorphic spaces which admit a reproducing kernel in the form of generalized hypergeometric series.

1. INTRODUCTION AND SUMMARY

We start recalling some results of Berezin’s theory that will be used below. See Ref. [4] for details.

Let $H$ be a Hilbert space endowed with the inner product $(\cdot, \cdot)$ and $M$ some set with the measure $d\mu$. Let $\{e_\alpha \in H \mid \alpha \in M\}$ a family of functions in $H$ labelled by elements of $M$, such that satisfies the following properties:

a) The family $\{e_\alpha\}$ is complete, this is for any $f, g \in H$ Parseval’s identity is valid

$$(f, g) = \int_M (f, e_\alpha)(e_\alpha, g)d\mu(\alpha).$$

b) The map $f \rightarrow \hat{f}$, defining by $\hat{f}(\alpha) = (f, e_\alpha)$, is an embedding from $H$ into $L^2(M, d\mu)$.

In 1970’s, Berezin [4] introduced a general symbolic calculus for bounded linear operators on $H$. More specifically, for $A \in \mathcal{B}(H)$, the algebra of all
bounded linear operator on $H$, the Berezin symbol (or Berezin transform) of $A$ is the function on $M$ defined by
\[
\mathcal{B}(A)(\alpha) = \frac{\langle Ae_\alpha, e_\alpha \rangle}{\langle e_\alpha, e_\alpha \rangle}, \quad \alpha \in M.
\]
The prototypes of the spaces $H$ are the Bergman spaces of all holomorphic functions in $L^2(M, d\mu)$ on a bounded domain $M \subset \mathbb{C}^n$ with Lebesgue measure $d\mu$, or the Segal-Bargmann spaces of all entire functions in $L^2(\mathbb{C}^n, d\mu)$ for the Gaussian measure $d\mu(z) = (2\pi)^{-n}e^{-|z|^2/2}d\pi$, where $d\pi$ denotes Lebesgue measure on $\mathbb{C}^n$. In these cases the functions $e_\alpha$ are the reproducing kernels.

Moreover, for every $A, B \in \mathcal{B}(H)$ and $f \in H$, we have the following formulas
\[
\mathcal{B}(A + cB) = \mathcal{B}(A) + c\mathcal{B}(B), \quad \text{for all constant } c \in \mathbb{C},
\]
\[
\mathcal{B}(\text{Id}) = 1, \quad \text{with Id the identity operator}.
\]
If we suppose that the Berezin’s symbol may be extended in a neighbourhood of the diagonal $M \times M$ to the function
\[
\mathcal{B}(A)(\alpha, \beta) = \frac{\langle Ae_\alpha e_\beta, e_\alpha e_\beta \rangle}{\langle e_\alpha e_\beta, e_\alpha e_\beta \rangle},
\]
then we have the following formulas
\[
\mathcal{B}(A^*)(\alpha, \beta) = \overline{\mathcal{B}(A)(\beta, \alpha)}, \quad \alpha, \beta \in M.
\]
(1) \[\hat{A}\hat{f}(\alpha) = \int_M \hat{f}(\beta) \mathcal{B}(A)(\beta, \alpha) e_\alpha(\beta) e_\beta(\alpha) d\mu(\beta),\]
(2) \[\mathcal{B}(AB)(\alpha, \beta) = \int_M \mathcal{B}(A)(\gamma, \beta) \mathcal{B}(B)(\alpha, \gamma) \frac{\langle e_\alpha e_\beta, e_\gamma e_\gamma \rangle}{\langle e_\alpha e_\beta, e_\alpha e_\beta \rangle} d\mu(\gamma).
\]

The useful application of this symbolic calculus is that allows us to build a star product. In [4] Berezin applied this method to Kähler manifolds, in this case $H$ is the Hilbert space of functions in $L^2(M, d\mu)$ which are analytic so that the embedding from $H$ into $L^2(M, d\mu)$ is the inclusion, and the complete family $\{e_\alpha\}$ are the reproducing kernel with one variable fixed.

The main goal of the present paper is to introduce a Berezin symbolic calculus for the Hilbert space $O$ of all functions in $L^2(S^n)$ whose Poisson extension into the interior of $S^n$ is holomorphic, where $S^n = \{x \in \mathbb{C}^n \mid |x_1|^2 + \cdots + |x_n|^2 = 1\}$ and $L^2(S^n)$ denote the Hilbert space of the square integrable function with respect to the normalized surface measure $dS_n(x)$ on $S^n$ and endowed with the usual inner product
\[
\langle \phi, \psi \rangle_{S^n} = \int_{S^n} \overline{\phi(x)} \psi(x) dS_n(x), \quad \phi, \psi \in L^2(S^n).
\]

To achieve this, we build a family $\{e_\alpha\}$ of functions in $O$ satisfying a), since coherent states meet this property and based of our experience in $L^2(S^m)$, $m = 2, 3, 5$, where $S^m = \{y \in \mathbb{R}^{m+1} \mid y_1^2 + \cdots + y_{m+1}^2 = 1\}$ (see Ref. [4]), we propose in Sec. 2 our coherent states in $O$ as a suitable power series of the inner product $\sum_{\ell=1}^m x_\ell \bar{z}_\ell / h$ regarding $(x_1, \ldots, x_n) \in S^n$ and $(z_1, \ldots, z_n) \in \mathbb{C}^n$, where $h$ denoting the Planck’s constant. In addition, to ensure b) we define a new Hilbert space $L^2(\mathbb{C}^n, dm^h_{n,y})$, whose measure
\( dm^h_{n,p} \) is obtained such that the inner product of these coherent states is the reproducing kernel from range of mapping \( f \to \hat{f} \) (see Sec. 2.1 for details of how this measure was obtained and Eq. (12) for definition of \( dm^h_{n,p} \)).

In order to prove that the coherent states and the space \( L^2(\mathbb{C}^n, dm^h_{n,p}) \) constructed in Sec. 2 satisfy the conditions a) and b), in Sec. 3 we prove that the mapping \( f \to \hat{f} \), denoted by \( U_{n,p} \), is an isometry from \( \mathcal{O} \) to \( \mathcal{E}_{n,p} \subset L^2(\mathbb{C}^n, dm^h_{n,p}) \). In addition, in Sec. 4 we prove that the unitary transformation \( U_{n,p} \) applied to our coherent states gives the reproducing kernel of the space \( \mathcal{E}_{n,p} \), which allows us to prove that the family of coherent states form a complete system for \( \mathcal{O} \).

From this and according to Berezin’s theory, in Sec. 5 we describe the rules for symbolic calculus on \( \mathcal{B}(S^n) \), further more we obtain asymptotic expansions of Berezin’s symbol of Toeplitz operator.

Finally, in Sec. 6 we define a star product on the algebra out of Berezin’s symbol for bounded operators with domain in \( \mathcal{O} \) and will prove that this noncommutative star-product satisfies the usual requirement on the semi-classical limit.

It should be noted, since \( H = \mathcal{O} \) and \( L^2(M, d\mu) = L^2(\mathbb{C}^n, dm^h_{n,p}) \), in our construction there is no inclusion of \( H \) into \( L^2(M, d\mu) \), further more, the complete family \( \{e_{\alpha}\} \) is not obtained by the reproducing kernel. This situation is thus slightly different from Berezin’s one.

In the present paper we will be used throughout the text the following basic notation. For every \( z, w \in \mathbb{C}^n \), \( z = (z_1, \ldots, z_n) \), \( w = (w_1, \ldots, w_n) \), let

\[
z \cdot w = \sum_{\ell=1}^n z_\ell \overline{w_\ell}, \quad |z| = \sqrt{z \cdot z}.
\]

Furthermore, for every multi-index \( k = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n \) of length \( n \) where \( \mathbb{Z}_+ \) is the set of non negative integers, let

\[
|k| = \sum_{\ell=1}^n k_\ell, \quad k! = \prod_{\ell=1}^n k_\ell!, \quad z^k = \prod_{\ell=1}^n z_\ell^{k_\ell}.
\]

Given \( \omega \in \mathbb{C} \), let us denote its real and imaginary parts by \( \Re(\omega) \) and \( \Im(\omega) \) respectively.

\section{A class of holomorphic spaces}

In this section, we will give a description of how to define the special Hilbert space \( L^2(\mathbb{C}^n, dm^h_{n,p}) \).

\subsection{Construction of measure \( dm^h_{n,p} \)}

Based on the expression of coherent states obtained for spaces \( L^2(S^m) \), with \( m = 2, 3, 5 \) and \( S^m = \{ x \in \mathbb{R}^{m+1} \mid x_1^2 + \ldots + x_{m+1}^2 = 1 \} \) (See [2] for more details). We consider the coherent states on \( L^2(S^n) \) as the functions

\[
\Phi_w(x) = \sum_{\ell=0}^\infty \frac{c_\ell}{\ell!} \left( \frac{x \cdot w}{h} \right)^\ell, \quad x \in S^n, \ w \in \mathbb{C}^n,
\]

where the constants \( c_\ell \) are defined below, which will be obtained considering that \( \langle \Phi_z, \Phi_w \rangle _{S^n} \) is the reproducing kernel of some Hilbert space.
From Lemma 7.1, we have
\[
\langle \Phi_z, \Phi_w \rangle_{S^n} = \Gamma(n) \sum_{\ell=0}^{\infty} c_\ell^2 \frac{\langle z \cdot w \rangle^{\ell}}{\ell! \Gamma(\ell + n)},
\]
where \( \Gamma \) denote the Gamma function. If the constants \( c_\ell \) take the following values
\[
c_\ell^2 = \frac{C \Gamma(n + \ell)}{\Gamma(n + p + \ell) \Gamma(n)},
\]
with \( p \in \mathbb{R} \), and \( C \) a constant, then we have
\[
(5) \quad \langle \Phi_z, \Phi_w \rangle_{S^n} = C \langle z \cdot w \rangle^{\frac{1}{2}(1-p-n)} I_{n+p-1} \left( \frac{2\sqrt{z \cdot w}}{h} \right)
\]
with \( I_k \) denoting the modified Bessel function of the first kind of order \( k \) (see Secs. 8.4 and 8.5 of Ref. [8] for definition and expressions for this special function). We are taking the branch of the square root function such that
\[
\sqrt{z} = |z|^{1/2} \exp(\frac{i\theta}{2}),
\]
where \( \theta = \text{Arg}(z) \) and \(-\pi < \theta < \pi\).

The Aronszajn-Moore theorem [2] states that, every positive definite (Hermitian) function \( K \) on \( \mathbb{C}^n \times \mathbb{C}^n \) determines a unique Hilbert function space \( H \) for which \( K \) is the reproducing kernel. In order to construct the Hilbert space \( H \) for which the right side of Eq. (5) is its reproducing kernel, let \( \phi(z) = C(z/h)^{-\nu/2}I_\nu(2\sqrt{z/h}) = \sum_{\ell=0}^{\infty} b_\ell z^\ell \). Define the space \( H_\phi \) as the set of all holomorphic functions in \( \mathbb{C}^n \) equipped with inner product
\[
(6) \quad (f,g)_{H_\phi} = \sum_{\ell=0}^{\infty} \frac{1}{b_\ell} \sum_{|k|=\ell} \frac{k!}{\ell!} f_k \overline{g_k},
\]
where \( f_k, g_k \) are Taylor’s coefficients of \( f \) and \( g \), respectively.

Note that the function \( \phi(z \cdot w) \) belongs to \( H_\phi \) as a function of \( z \in \mathbb{C}^n \) for every fixed \( w \in \mathbb{C}^n \), and for any \( f \in H_\phi \)
\[
(f, \phi(\cdot \cdot w))_{H_\phi} = \sum_{\ell=0}^{\infty} \frac{1}{b_\ell} \sum_{|k|=\ell} \frac{k!}{\ell!} f_k \overline{b_k} = \sum_{\ell=0}^{\infty} \frac{1}{b_\ell} \sum_{|k|=\ell} \frac{k!}{\ell!} f_k w^k = f(w).
\]
Thus, \( \phi(z \cdot w) \) is the reproducing kernel of \( H_\phi \). From (6) we see that
\[
(7) \quad (z^k, z^k)_{H_\phi} = \frac{1}{b_{|k|}} \frac{k!}{|k|!}.
\]
We now assume that the inner product (6) can be expressed by
\[
(8) \quad (f,g)_{H_\phi} = \int_{\mathbb{C}^n} f(z) \overline{g(z)} \omega(|z|) dz d\overline{z}.
\]
Substituting the value of \( b_{|k|} \) in Eq. (7), using Eq. (8), polar coordinates and Lemma 7.1, we obtain
\[
\frac{1}{C^2 h^{2|k|} k!} \Gamma(\nu + |k| + 1) = (z^k, z^k)_{H_\phi} = \frac{2\pi^n k!}{(n - 1 + |k|!)} \int_0^\infty r^{2n - 1 + 2|k|} \omega(r) dr.
\]
So, we need find \( \omega(r) \) that satisfies the equation

\[
\int_0^\infty r^{2n-1+2|k|}|\omega(r)|^2dr = \frac{\hbar}{2C}\pi^n \Gamma(\nu + |k| + 1)\Gamma(n + |k|).
\]

Note that expression (9) becomes essentially the Mellin transform (see Refs. [8], [12]). From this simple observation and the formula 6.561-16 of Ref [8] we immediately obtain

\[
\omega(r) = \frac{2}{C(\pi\hbar^2)^n} \left( \frac{r}{\hbar} \right) \nu^{-n+1} K_{\nu-1} \left( \frac{2}{\hbar}r \right), \quad \nu > -1,
\]

with \( K_p \) denoting the MacDonal-Bessel functions of order \( p \) (see Secs. 8.4 and 8.5 of Ref. [8] for definitions and expressions for this special functions).

Taking the constant \( C \) such that (1.1) \( \Phi = 1 \) we obtain

\[
C = \Gamma(\nu + 1).
\]

2.2. The space \( \mathcal{E}_{n,p} \). In the previous section we constructed a Hilbert space whose inner product can be expressed in the form (8) and its reproducing kernel is the right side of Eq. (5); in this section we proves rigorously all those results. From Eqs. (10) and (11) let us consider the following measure

\[
dm_{n,p}(z) = \frac{1}{\Gamma(n + p)} \frac{2}{(\pi\hbar^2)^n} \left( \frac{|z|}{\hbar} \right)^p K_p \left( \frac{2|z|}{\hbar} \right) dzd\overline{z}, \quad p > -n
\]

with \( z = (z_1, \ldots, z_n) \in \mathbb{C}^n \) and \( dzd\overline{z} \) denoting Lebesgue measure on \( \mathbb{C}^n \).

Note that the measure \( dm_{n,p}^h \) is invariant under the rotation group SO\((n)\) (the group of \( n \times n \) orthogonal matrices with unit determinant and real entries). The action of SO\((n)\) on \( \mathbb{C}^n \) that we are considering is the natural extension of the usual action of SO\((n)\) on \( \mathbb{R}^n \), this is: for \( R \in \text{SO}(n) \) defining \( T_R : \mathbb{C}^n \rightarrow \mathbb{C}^n \) by \( T_R(z) = R \overline{R}(z) + iR \Im(z) \).

Let us consider the Hilbert space \( L^2(\mathbb{C}^n, dm_{n,p}^h) \) of square integrable functions on \( \mathbb{C}^n \) with respect to the measure \( dm_{n,p}^h \) and endowed with the inner product

\[
(f,g)_p = \int_{\mathbb{C}^n} f(z)\overline{g(z)} dm_{n,p}^h(z), \quad f, g \in L^2(\mathbb{C}^n, dm_{n,p}^h).
\]

Let us denote by \( \|f\|_p = \sqrt{(f,f)_p} \) the corresponding norm of \( f \in L^2(\mathbb{C}^n, dm_{n,p}^h) \).

Definition 2.1. For \( p > -n \), the space of entire functions \( f \) defined on \( \mathbb{C}^n \) such that \( \|f\|_p \) is finite is denoted by \( \mathcal{E}_{n,p} \).

Remark 1. For \( n = 1 \), the spaces \( \mathcal{E}_{1,p} \) were used by Karp Dmitrii to derive an analytic continuation formula for functions on \( \mathbb{R}^+ \) (see [10]).

Lemma 2.2. Let

\[
L := \int_{\mathbb{C}^n} z^k \overline{z}^m \overline{(z \cdot w)}^s (u \cdot z)^\ell dm_{n,p}^h(z),
\]
with $\ell, s \in \mathbb{Z}_+, k, m \in \mathbb{Z}_n^+$, and $w, u \in \mathbb{C}^n \setminus \{0\}$. Then $L = 0$ if $|k| + s \neq |m| + \ell$, and if $|k| + s = |m| + \ell$, then

$$L = \hbar^2 (|k| + s)(n)|k| + s(n + p)|k| + s \int_{S^n} x^k y^m (x \cdot w)^s (u \cdot x)^\ell dS_n(x).$$

Proof. From Eq. (12) and expressing $L$ in polar coordinates

$$L = \frac{|S_n|}{\pi^n \Gamma(n + p)} \int_0^\infty r^{2n - 1 + |m| + \ell + s + p} K_p(2r \hbar) dr \int_{S^n} x^k y^m (x \cdot w)^s (u \cdot x)^\ell dS_n(x),$$

where $|S_n| = \text{vol}(S^n)$. From the formula 6.561-16 of Ref [8] and Lemma 7.1 we conclude the proof of Lemma 2.2.

Proposition 2.3. Let $p > -n$

a) The space of analytic polynomials defined on $\mathbb{C}^n$ is dense on $E_{n,p}$.

b) For $\ell \in \mathbb{Z}_+$, let us denote by $W_\ell$ the space of homogeneous polynomials of degree $\ell$. Then

$$E_{n,p} = \bigoplus_{\ell=0}^\infty W_\ell.$$

Further, the set $\{ \Phi^\ell_{k,p} | k \in \mathbb{Z}_n^+ \}$ is orthonormal basis of $E_{n,p}$, where $\Phi^\ell_{k,p} : \mathbb{C}^n \rightarrow \mathbb{C}$ is defining by

$$\Phi^\ell_{k,p}(z) = \frac{1}{\hbar^{|k|} k!} \sqrt{\frac{1}{(n + p)!|k|!}} z^k,$$

where, for $b > 0$, the Pochhammer symbol is given by

$$(b)_\ell = \frac{\Gamma(b + \ell)}{\Gamma(b)}.$$

c) The space $E_{n,p}$ enjoys the property of having a reproducing kernel $T_{n,p} = T_{n,p}(z, w)$. Namely, for all $f \in E_{n,p}$ we have

$$f(z) = (f, T_{n,p}(\cdot, z)) = \int_{w \in \mathbb{C}^n} f(w) T_{n,p}(z, w) dm_{n,p}^\hbar(w), \quad \forall z \in \mathbb{C}^n$$

where the reproducing kernel is given by

$$T_{n,p}(z, w) = \Gamma(n + p) \left( \frac{z \cdot w}{\hbar^2} \right)^{\frac{1}{2}(-p-n+1)} I_{n+p-1} \left( \frac{2\sqrt{z \cdot w}}{\hbar} \right).$$

Even more

$$T_{n,p}(z, w) = \sum_{\ell=0}^\infty T_{n,p}^\ell(z, w),$$

where $T_{n,p}^\ell$ are the reproducing kernel of the subspace $W_\ell$

$$T_{n,p}^\ell(z, w) = \frac{1}{\ell!(n + p)_\ell} \left( \frac{z \cdot w}{\hbar^2} \right)^\ell.$$
Proof. a) Is a consequence of the Stone-Weiertrass theorem.

b) Note that the space $W_\ell$ is contained in $L^2(C^n, dm_{n,p})$. Then by a), we only need prove that the spaces $W_\ell$ and $W_{\ell'}$ are orthogonal for $\ell \neq \ell'$, which is a consequence of Lemma 2.2. The second part of this point to o is a direct consequence of Lemma 2.2, since

$$\left\{ \frac{1}{\hbar^\ell} \sqrt{\frac{1}{(n+p)\ell k!}} z^k \mid k \in \mathbb{Z}_+^n, \ |k| = \ell \right\}$$

is orthogonal basis of $W_\ell$.

c) By Eq. (15), it suffices to show that the function $T_{n,p}(z, w)$ is the reproducing kernel in $W_\ell$. That will appear as a simple consequence of Lemma 2.2.

As a consequence of the existence of the reproducing kernel and the Cauchy-Schwartz inequality, we obtain the following estimate for any $f \in \mathcal{E}_{n,p}$

$$|f(z)| \leq ||f||_p \| |T_{n,p}(. , z)||, \ \forall z \in \mathbb{C}^n.$$

3. UNITARY TRANSFORM

In this section we introduce a unitary transform $U_{n,p}$ from $\mathcal{O}$ onto Hilbert space $\mathcal{E}_{n,p}$, with $n \geq 2$ and $p > -n$. In order to define this transformation, we first recall some results about the space $\mathcal{O}$.

For $\ell \in \mathbb{Z}_+$, let us define the space $V_\ell$ of homogeneous polynomials of degree $\ell$ as the vector space of restrictions to the $n$-sphere of homogeneous polynomials of degree $\ell$ defined initially on the ambient space $\mathbb{C}^n$. We will use the fact that $\mathcal{O}$ is equal to the direct sum of the space $V_\ell$ (see [13])

$$\mathcal{O} = \bigoplus_{\ell=0}^{\infty} V_\ell.$$

Further, the set $\{ \phi_k \mid k \in \mathbb{Z}_+^n \}$ is orthogonal basis of $\mathcal{O}$, where $\phi_k : \mathbb{S}^n \to \mathbb{C}$ is defining by

$$\phi_k(x) = \sqrt{(n)!! \frac{|k|}{k!}} x^k.$$

For every $\ell \in \mathbb{Z}_+$, exist a natural transformation from $V_\ell$ to $W_\ell$, this is the linear extension of the assignment $\phi_k \to \Phi_{k,p}^\ell$, where $\phi_k$, $\Phi_{k,p}^\ell$ are defined in Eqs. (22) and (16) respectively, and $k \in \mathbb{Z}_+^n$ with $|k| = \ell$. Let us define this unitary transform by $U_{n,p}^\ell$.

We now consider the linear extension $U_{n,p} : \mathcal{O} \to \mathcal{E}_{n,p}$ of the operators $U_{n,p}^\ell$, defined as follows:

Given $\Psi \in \mathcal{O}$ written as $\Psi = \sum_{\ell=0}^\infty \Psi_\ell$ with $\Psi_\ell \in V_\ell$, we define the partial sums $S_\ell(\Psi) = \sum_{\ell=0}^{k=0} U_{n,p}^\ell \Psi_\ell$ and then

$$U_{n,p} \Psi := \lim_{k \to \infty} S_\ell(\Psi).$$

The operator $U_{n,p}$ is well defined and unitary due to the unitarity of the operators $U_{n,p}^\ell$ and the fact that every element $f$ in the space $\mathcal{E}_{n,p}$ can be written in a unique way as $f = \lim_{k \to \infty} \sum_{\ell=0}^{k} f_\ell$ with $f_\ell \in W_\ell$. (See Eq. (15)).
Theorem 3.1. The operator $U_{n,p}$ mapping $O$ isometrically onto the space $E_{n,p}$. Even more

$$(24) \quad U_{n,p} \Psi (z) = \int_{S^n} \Psi (x) \sum_{\ell=0}^{\infty} \frac{c_{\ell,p}}{\ell!} \left( \frac{x \cdot z}{\hbar} \right)^{\ell} \, dS_n(x), \quad \forall \Psi \in O, \, z \in \mathbb{C}^n,$$

where

$$(25) \quad (c_{\ell,p})^2 = \frac{(n)_\ell}{(n + p)_\ell}.$$

Proof. Note that for $z \in \mathbb{C}^n$ fixed and $x \in S^n$, the series

$$(26) \quad K_{n,p}^h(x, z) := \sum_{\ell=0}^{\infty} \frac{c_{\ell,p}}{\ell!} \left( \frac{x \cdot z}{\hbar} \right)^{\ell} \phi_k^h (x) \Phi (z) \, dS_n(x), \quad z \in \mathbb{C}^n, \quad \Phi \in V_{\ell}.$$

is bounded on $S^n$ and therefore it is a vector in $L^2(S^n)$ as well. Then, by the Cauchy-Schwartz inequality, the right side of Eq. (24) is a well defined function.

Let $\ell \in \mathbb{Z}_+$, is not difficult show that

$$(27) \quad U_{n,p} \Phi (z) = \int_{S^n} \sum_{|k|=\ell} \frac{1}{\hbar^{\ell}} \sqrt{\frac{(n)_\ell}{k!} \sqrt{\frac{1}{(n + p)_\ell}}} x^k z^k \, dS_n(x), \quad x \in S^n.$$

Even more, by Eqs (22) and (16)

$$(28) \quad \sum_{|k|=\ell} \frac{1}{\hbar^{\ell}} \sqrt{\frac{(n)_\ell}{k!} \sqrt{\frac{1}{(n + p)_\ell}}} x^k z^k = \frac{c_{\ell,p}}{\ell!} \left( \frac{x \cdot z}{\hbar} \right)^{\ell}.$$

Then, given $\Psi \in O$, written as $\Psi = \sum_{\ell=0}^{\infty} \Psi_{\ell} \Phi_{\ell}$ with $\Psi_{\ell} \in V_{\ell}$, we obtain the Eq. (24) from definition of $U_{n,p}$ (see Eq. (23)), the integral expression the operator $U_{n,p}$ (see Eqs. (27) and (28)) and the dominated convergence theorem. \qed

We end this section with an inversion formula for the operator $U_{n,p}$.

Theorem 3.2. Let $f \in E_{n,p}$, then

$$(U_{n,p})^{-1} f(x) = \int_{\mathbb{C}^n} \sum_{\ell=0}^{\infty} \frac{c_{\ell,p}}{\ell!} \left( \frac{x \cdot z}{\hbar} \right)^{\ell} f(z) \, dm_{n,p}^h(z), \quad x \in S^n.$$

Proof. Since the set $\{ \phi_k | k \in \mathbb{Z}_+^n \}$ is orthonormal basis of $O$, then the reproducing kernel, $H_n$, from $O$ is

$$H_n(x, y) = \sum_{\ell=0}^{\infty} \sum_{|k|=\ell} \phi_k(x) \phi_k(y) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \left( x \cdot y \right)^{\ell} = 2F_1(n, 1; 1; x \cdot y),$$

where $2F_1$ denoting the Gauss hypergeometric function (see Secs. 9.1 of Ref. [8] for definition and expression for this special function).

Even more, for $y \in S^n$ fixed, we obtain from Lemma 7.1

$$(29) \quad U_{n,p} H_n(\cdot, y) (z) = K_{n,p}^h (z, y) \quad \forall z \in \mathbb{C}^n.$$
Moreover, from the reproducing property of $\mathbf{H}_n$, the unitarity of the transform $U_{n,p}$ and the Eq. (29), we have

$$(U_{n,p})^{-1} f(x) = (\langle (U_{n,p})^{-1} f, \mathbf{H}_n(\cdot, x) \rangle_{S^n})_p = (f, U_{n,p} \mathbf{H}_n(\cdot, x))_p = (f, K_{n,p}^h(\cdot, x))_p$$

□

4. Coherent states for $\mathcal{O}$

We consider the set of functions $K = \{K_{n,p}^h(\cdot, z) | z \in \mathbb{C}^n \} \subset \mathcal{O}$ defined by Eq. (26). In this section shows that the functions in $K$ satisfy the conditions to define a Berezin symbolic calculus on $\mathcal{O}$.

First notice that the unitary operator $U_{n,p}$ is a coherent states transform because its action on a function $\Phi$, in $\mathcal{O}$, is a function in $E_{n,p}$ whose evaluation in $z \in \mathbb{C}^n$ is equal to the $L^2(S^n)$-inner product of $\Phi$ with the coherent states labeled by $z$. This is

$$U_{n,p} \Phi(z) = \langle \Phi, K_{n,p}^h(\cdot, z) \rangle_{S^n}.$$

Remark 2. From Eq. (30) and theorem 3.2 is not difficult to prove

$$\Psi(x) = \int_{\mathbb{C}^n} \langle \Psi, K_{n,p}^h(z) \rangle_{S^n} K_{n,p}^h(x, z) d^n m_{n,p}(z), \quad x \in S^n, \Psi \in \mathcal{O}.$$

Thus the system of coherent states provides a resolution of identity for $\mathcal{O}$, this is the so-called reproducing property of the coherent states.

Theorem 4.1. The unitary operator $U_{n,p}$ applied to $K_{n,p}^h(\cdot, z)$ gives the reproducing kernel of the Hilbert space $E_{n,p}$, $n \geq 2$, $p > -n$. Namely, for all $z, w$ in $\mathbb{C}^n$ we have

$$[U_{n,p} K_{n,p}^h(\cdot, z)](w) = T_{n,p}(w, z).$$

Proof. From theorem 3.1, the dominated converge theorem and Lemma 7.1 we obtain

$$[U_{n,p} K_{n,p}^h(\cdot, z)](w) = \int_{S^n} K_{n,p}^h(x, z) \mathbf{K}_{n,p}^h(x, w) dS_n(x) = \sum_{\ell=0}^\infty \frac{c_{\ell,p}}{\ell! h^\ell} \int_{S^n} (x \cdot z)^\ell (w \cdot x)^{\ell} dS_n(x) = \sum_{\ell=0}^\infty \frac{1}{(n+p)^\ell \ell!} \frac{1}{h^2} (w \cdot z)^{\ell}.$$

And this is the expression for the reproducing kernel $T_{n,p}$ given in Eq. (19) as an infinity series.

The proof of some theorems below uses an estimate of the norm of a coherent state $K_{n,p}^h(\cdot, z)$ which in turn is a consequence of the following estimate for the inner product of two coherent states:
Proposition 4.2. Let $z, w \in C^n$ fixed. Assume $z \cdot w \neq 0$ and $|\text{Arg}(z \cdot w)| < \pi$. Then for $h \to 0$

$$\langle K^h_{n,p}(\cdot, z), K^h_{n,p}(\cdot, w) \rangle^\Omega = \frac{\Gamma(n+p)}{2\sqrt{\pi}} \left(\frac{w \cdot z}{h^2}\right)^{\frac{1}{2}(p-n+\frac{1}{2})} e^{\frac{2}{\pi\sqrt{w \cdot z}}} \left[1 - \frac{(n+p-\frac{3}{2})(n+p-\frac{1}{2})}{4\sqrt{w \cdot z}} h + O(h^2) \right].$$

(31)

Proof. Since the transform $U_{n,p}$ is a unitary transformation, we obtain from theorem 4.1, the reproducing property of $T_{n,p}$ (see Eq. (17)) and the expression for the reproducing kernel $T_{n,p}$ given in Eq. (18) that

$$\langle K^h_{n,p}(\cdot, z), K^h_{n,p}(\cdot, w) \rangle^\Omega = T_{n,p}(w, z)$$

(32)

$$= \Gamma(n+p) \left(\frac{w \cdot z}{h^2}\right)^{\frac{1}{2}(p-n+1)} I_{n+p-1} \left(\frac{2\sqrt{w \cdot z}}{h}\right).$$

The modified Bessel function $I_\vartheta$, $\vartheta \in \mathbb{R}$, has the following asymptotic expression when $|\vartheta| \to \infty$ (see formula 8.451-5 of Ref. [8])

$$I_\vartheta(\varphi) = \frac{e^{\varphi}}{\sqrt{2\pi\varphi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2\varphi)^k} \frac{\Gamma(\vartheta + k + \frac{1}{2})}{k! \Gamma(\vartheta - k + \frac{1}{2})}, \quad |\text{Arg}(\varphi)| < \frac{\pi}{2}.$$  

(33)

From Eqs. (32) and (33) we conclude the proof of this proposition. \hfill \Box

Remark 3. We are mainly interested in using proposition 4.2 for the cases $z = w$ (and then $\text{Arg}(z \cdot w) = 0$) in this paper. The case when $|\text{Arg}(z \cdot w)| = \pi$ requires the use of an asymptotic expression valid in a different region than the one we are considering in proposition 4.3. Thus if we take the branch of the square root function given by $\sqrt{z} = |z|^{1/2} \exp(i \text{Arg}(z)/2)$ with $0 < \text{Arg}(z) < 2\pi$ then by using formula 8.451-5 in Ref. [8] we obtain the asymptotic expression,

$$\langle K^h_{n,p}(\cdot, z), K^h_{n,p}(\cdot, w) \rangle^\Omega = \frac{\Gamma(n+p)}{2\sqrt{\pi}} \left(\frac{w \cdot z}{h^2}\right)^{\frac{1}{2}(p-n+\frac{1}{2})}$$

$$\left[ \frac{e^{\frac{2}{\pi\sqrt{w \cdot z}}} + e^{-\frac{2}{\pi\sqrt{w \cdot z}}} e^{\pi(n+p-\frac{1}{2})}}{2} \right] [1 + O(h)].$$

(34)

Note that both asymptotic expressions in Eqs. (31) and (34) coincide up to an error of the order $O(h^\infty)$ in the common region where they are valid.

From the proposition 4.2 we obtain an estimate for the norm of a given coherent states

Proposition 4.3. For $z \in C^n - \{0\}$

$$||K^h_{n,p}(\cdot, z)||^2_{\mathcal{K}^h} = \frac{\Gamma(n+p)}{2\sqrt{\pi}} \left(\frac{|z|}{h}\right)^{-p-n+\frac{1}{2}} e^{\frac{2}{\pi}|z|}$$

$$\left[ 1 - \frac{(n+p-\frac{3}{2})(n+p-\frac{1}{2})}{4|z|} h + O(h^2) \right].$$

(35)

We end this section showing that the family of coherent states $\mathcal{K}$ is complete.
Proposition 4.4. The family \( \{ K_{n,p}^h(\cdot, z) \mid z \in \mathbb{C}^n \} \) form a complete system for \( \mathcal{O} \).

Proof. Let \( \Phi, \Psi \in \mathcal{O} \), since the transform \( U_{n,p} \) is a unitary transformation we have
\[
\langle \Phi, \Psi \rangle_{S_n} = \left( U_{n,p} \Phi, U_{n,p} \Psi \right)_p
= \int_{\mathbb{C}^n} U_{n,p} \Phi(z) \overline{U_{n,p} \Psi(z)} dm_{n,p}^h(z)
= \int_{\mathbb{C}^n} \langle \Phi, K_{n,p}^h(\cdot, z) \rangle_{S_n} \langle K_{n,p}^h(\cdot, z), \Psi \rangle_{S_n} dm_{n,p}^h(z),
\]
where we have used Eq. (30). \( \square \)

5. Berezin symbolic calculus

According to Berezin’s theory (see Ref. [4]), from Theorem 3.1, Eq. (30) and the proposition 4.4, we may consider the following

Definition 5.1. The Berezin’s symbol of a bounded linear operator \( A \) with domain in \( \mathcal{O} \) is defined, for every \( z \in \mathbb{C}^n \), by
\[
\mathfrak{B}_{h,p}(A)(z) = \frac{\langle AK_{n,p}^h(\cdot, z), K_{n,p}^h(\cdot, z) \rangle_{S_n}}{\langle K_{n,p}^h(\cdot, z), K_{n,p}^h(\cdot, z) \rangle_{S_n}}.
\]

From Eq. (32), we have that
\[
||K_{n,p}^h(\cdot, z)||_{S_n}^2 = \Gamma(n + p) \left( \frac{|z|}{\hbar} \right)^{-p-n+1} I_{n+p-1} \left( \frac{2|z|}{\hbar} \right) > 0,
\]
hence the coherent states are continuous, i.e. the map \( z \rightarrow |K_{n,p}^h(\cdot, z)| \) is continuous. Therefore, if \( A : \mathcal{O} \rightarrow \mathcal{O} \) is a bounded operator, its Berezin’s symbol, can be extended uniquely to a function defined on a neighbourhood of the diagonal in \( \mathbb{C}^n \times \mathbb{C}^n \) in such a way that it is holomorphic in the first factor and anti-holomorphic in the second. In fact, such an extension is given explicitly by
\[
\mathfrak{B}_{h,p}(A)(z, w) := \frac{\langle AK_{n,p}^h(\cdot, z), K_{n,p}^h(\cdot, w) \rangle_{S_n}}{\langle K_{n,p}^h(\cdot, z), K_{n,p}^h(\cdot, w) \rangle_{S_n}}.
\]

Remark 4. By Eqs. (72), the extended Berezin’s symbol has singularities in the zeros of the modified Bessel function \( I_{n+p-1}(z) \), which are well known (see Ref. [11 Sec. 5.13]) and where \( w \cdot z = 0 \). Then the extended Berezin’s symbol is defined on \( \mathbb{C}^n \times \mathbb{C}^n \setminus S \), with
\[
S = \{ (w, z) \in \mathbb{C}^n \times \mathbb{C}^n \mid w \cdot z = 0 \text{ or } 2\sqrt{w \cdot z/h} = i\lambda \}
\]
where \( \lambda \) is a negative real number that satisfies \( I_{n+p-1}(i\lambda) = 0 \).

We now give the rules for symbolic calculus.
Proposition 5.2. Let $A, B$ bounded linear operators with domain in $\mathcal{O}$. Then for $z, w \in \mathbb{C}^n$ and $\phi \in \mathcal{O}$ we have

$$\mathfrak{B}_{h,p}(\text{Id}) = 1, \quad \text{with Id the identity operator},$$

$$\mathfrak{B}_{h,p}(A^*) (z, w) = \overline{\mathfrak{B}_{h,p}(A)(w, z)},$$

$$\mathfrak{B}_{h,p}(AB)(z, w) = \frac{2}{(\pi h)^n} \int_{\mathbb{C}^n} \mathfrak{B}_{h,p}(B)(z, u) \mathfrak{B}_{h,p}(A)(u, w) \left[ \frac{u \cdot (z - w)}{w \cdot z} \right]^{\frac{1}{2}(-n-1)} \frac{I_{n+p-1} \left( \frac{2\sqrt{uw}}{h} \right) I_{n+p-1} \left( \frac{2\sqrt{uw}}{h} \right)}{I_{n+p-1} \left( \frac{2\sqrt{uw}}{h} \right) I_{n+p-1} \left( \frac{2\sqrt{uw}}{h} \right)} u^p K_p \left( \frac{2|u|}{h} \right) du d\bar{u} ,$$

(38)

$$U_{n,p}(A\phi)(z) = \frac{2}{(\pi h)^n} \int_{\mathbb{C}^n} U_{n,p}\phi(w) \mathfrak{B}_{h,p}(A)(w, z) (z \cdot w)^{\frac{1}{2}(-n-1)} \frac{I_{n+p-1} \left( \frac{2\sqrt{zw}}{h} \right) |w|^p K_p \left( \frac{2|w|}{h} \right)}{I_{n+p-1} \left( \frac{2\sqrt{zw}}{h} \right) I_{n+p-1} \left( \frac{2\sqrt{zw}}{h} \right) dwd\bar{w}} .$$

(39)

Proof. This is a direct consequence from the formulas in Ref. [4] (see Eqs. (31), (37), and the Eqs. (39) and (32)). □

Corollary 5.3. Let $A : L^2(S^n) \to L^2(S^n)$ be a bounded operator, and $A^\#$ the operator on $\mathcal{E}_{n,p}$ with Schwartz kernel the function $\mathcal{K}_A$ defined on $\mathbb{C}^n \times \mathbb{C}^n$, by

$$\mathcal{K}_A(z, w) := \langle AK_{n,p}^h(\cdot, w), K_{n,p}^h(\cdot, z) \rangle_{S^n} .$$

Then

(40)

$$A^\# f(z) := U_{n,p}A(U_{n,p})^{-1}f(z) \quad \forall f \in \mathcal{E}_{n,p}, \quad z \in \mathbb{C}^n.$$

Proof. Let $f \in \mathcal{E}_{n,p}$, and $\phi = (U_{n,p})^{-1} f$. From Eqs. (39), (37) and (32)

$$U_{n,p}A(U_{n,p})^{-1}f(z) = \int_{\mathbb{C}^n} f(w) \mathcal{K}_A(z, w) dm_{n,p}(w).$$

5.1. Asymptotic expansion of the Berezin’s symbol. In this section, we obtain asymptotic expansion of Berezin’s symbol of Toeplitz operator.

Let $P : L^2(S^n) \to \mathcal{O}$ the orthogonal projection and $A$ pseudo-differential operator on $S^n$ of order zero. The Toeplitz operator is defined as

$$T_A = PAP .$$

For $\psi \in C^\infty(S^n)$, with $C^\infty(S^n)$ be the algebra of complex-valued $C^\infty$ functions on $S^n$, let $M_\psi$ the operator of multiplication by $\psi$. For simplicity we use notation $T_\psi := T_{M_\psi}$.

Theorem 5.4. Let $k \in \mathbb{Z}_+^n$ be a multi-index. Then

$$\mathfrak{B}_{h,p}(T_k)(z, w) = \frac{\left( \frac{w \cdot z}{h^2} \right)^{\frac{1}{2}(n+p-1)} \left( \frac{w \cdot z}{h^2} \right)^{\sum_{\ell=0}^{\infty} \left( \frac{w \cdot z}{h^2} \right)^\ell}}{I_{n+p-1} \left( \frac{2\sqrt{nw}}{h} \right) I_{n+p-1} \left( \frac{2\sqrt{nw}}{h} \right)} \frac{1}{\ell! \Gamma(|k| + \ell + n + p)} \frac{c_{\ell,p}}{c_{|k|+\ell,p}} .$$
where $c_{l,p}$ was define in Eq. (34).

**Proof.** From Eq. (37) and properties the orthogonal projection

$$
\mathcal{B}_{h,p}(T_{x^n})(z, w) = \frac{(x^kK^h_{n,p}(\cdot, z), K^h_{n,p}(\cdot, z))_{S^n}}{(K^h_{n,p}(\cdot, z), K^h_{n,p}(\cdot, z))_{S^n}}.
$$

Using the dominated convergence theorem, the Lemma 7.1 and Eq. (32) we conclude the proof of this theorem. □

**Remark 5.** In the particular case when $p = 0$, we obtain from the last theorem

$$
\mathcal{B}_{h,0}(T_{x^n})(z, w) = \left(\frac{w}{\sqrt{w \cdot z}}\right)^{k} \frac{I_{n+|k|-1}((2\sqrt{|w |} |z |)/h)}{I_{n-1}((2\sqrt{|w |} |z |)/h)}.
$$

**Theorem 5.5.** Let $p = 0$, for any $z \neq 0$ and $\Phi$ a smooth function on $S^n$, the Berezin symbol $\mathcal{B}_{h,0}$ associated to the Toeplitz operator $T_{\Phi}$ has the asymptotic expansion

$$
(41) \quad \mathcal{B}_{h,0} T_{\Phi}(z) = \Phi(z/|z|) + O(h). \quad \text{(41)}
$$

**Proof.** First we observe from Eq. (26), since $p = 0$, that $K^h_{n,p}(x, z) = e^{x \cdot z/h}$.

Let $\Phi \in C^\infty(S^n)$, from Eq. (35) we have

$$
\begin{align*}
\mathcal{B}_{h,0} T_{\Phi}(z) &= \frac{2\sqrt{n}}{\Gamma(n)} \left(\frac{|z|}{h}\right)^{n-\frac{1}{2}} \int_{S^n} \exp\left(\frac{2}{h} |\Re(x \cdot z) - |z||\right) \Phi(x) \\
&= (1 + O(h)) dS_n(x).
\end{align*}
$$

We identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ in the usual way: for $(x_1, \ldots, x_n) \in S^n$, let $y = (y_1, \ldots, y_{2n})$ with $x_j = y_j + iy_{n+j}$. Note that $y \in S^{2n-1} = \{a \in \mathbb{R}^{2n} : |a| = 1\}$. Then, the argument of the exponential function in Eq. (42) is

$$
\frac{2}{h} \left(y \cdot (\Re(z), \Im(z)) - |z|\right).
$$

In order to estimate (41), we define

$$
A := \left(\frac{|z|}{\pi h}\right)^{n-\frac{1}{2}} \int_{S^{2n-1}} \Psi(y) \exp\left(\frac{2}{h} |y \cdot (\Re(z), \Im(z)) - |z||\right) d\Omega(y)
$$

where $d\Omega$ is the surface measure on $S^{2n-1}$ and $\Psi$ is a smooth function on $S^{2n-1}$.

Note that given $(\Re(z), \Im(z)) \in \mathbb{R}^{2n}$, there exist a rotation $R$ in $\text{SO}(2n)$ such that $(\Re(z), \Im(z)) = rR\hat{e}_1$ with $r = |z|$ and $\hat{e}_1$ is canonical unit vector in $\mathbb{R}^{2n}$. Thus we have

$$
A = \left(\frac{r}{\pi h}\right)^{n-\frac{1}{2}} \int_{\omega \in S^{2n-1}} \Psi(R\omega) \exp\left(\frac{r}{h} f_x(\omega)\right) d\Omega(\omega)
$$

where $f_x(\omega) = -2tr[\omega_1 - 1]$. 

Let us introduce spherical coordinates for the variables \((\omega_1, \ldots, \omega_{2n}) \in S^{2n-1}\):

\[
\begin{align*}
\omega_1 &= \sin(\theta_{2n-1}) \cdots \sin(\theta_2) \sin(\theta_1), \\
\omega_2 &= \sin(\theta_{2n-1}) \cdots \sin(\theta_2) \cos(\theta_1), \\
&\vdots \\
\omega_{2n-1} &= \sin(\theta_{2n-1}) \cos(\theta_{2n-2}), \\
\omega_{2n} &= \cos(\theta_{2n-1})
\end{align*}
\]

with \(\theta_1 \in (-\pi, \pi), \theta_2, \theta_3, \ldots, \theta_{2n-1} \in (0, \pi)\).

The function \(f_x\) appearing in the argument of the exponential function in Eq. (44) has a non-negative imaginary part and has only one critical point (as a function of the angles) \(\theta_0\) which contributes to the asymptotic given by \(\theta_j = \pi/2, j = 1, \ldots, 2n - 1\). In addition, since

\[
\frac{\partial^2 f_x}{\partial \theta_i \partial \theta_j} \bigg|_{\theta = \theta_0} = 2\pi \delta_{ij},
\]

with \(\delta_{ij}\) denoting the Kronecker symbol, then the determinate of the Hessian matrix of the function \(f_x\) evaluated at the critical point is equal to \((2\pi)^{2n-1}\).

From the stationary phase method (see Ref. [9]) we obtain

\[
(45) \quad A = [\Psi(R\omega(\theta))]_{\theta = \theta_0} + O(h) = \Psi((\Re(z), \Im(z))/|z|) + O(h).
\]

For \(x \in S^n\), we consider \(\Psi(\Re(x), \Im(x)) = \Phi(x)\). Then from Eqs. (42), (43) and (45) we conclude the proof of this theorem. \(\square\)

When \(p = -1\) we can give an asymptotic expression of the coherent states (see Appendix 3). This result will give us as a consequence an asymptotic expression for Berezin symbol \(\mathfrak{B}_{h^{-1}}\) to the Toeplitz operator.

**Theorem 5.6.** Let \(p = -1\), for any \(z \neq 0\) and \(\Phi\) a smooth function on \(S^n\), the Berezin transform \(\mathfrak{B}_{h^{-1}}\) associated to the Toeplitz operator \(T_{\Phi}\) has the asymptotic expansion

\[
(46) \quad \mathfrak{B}_{h^{-1}}T_{\Phi}(z) = \Phi(z/|z|) + O(h).
\]

**Proof.** Let us define the following regions on \(S^n\), with the constant \(C\) mentioned in the Lemma 5.1 taken greater than one

\[
(47) \quad W = \left\{ x \in S^n \mid C \frac{\Re(x \cdot z)}{|z|} \geq 1 \right\} \quad \text{and} \quad V = S^n - W.
\]

Note that \(x \in W\), implies \(|\Im(x \cdot z)| \leq |z| \leq C \Re(x \cdot z)\) and therefore we can use the asymptotic expression of coherent states (see proposition 5.2).

Let

\[
(48) \quad A = \int_{x \in W} \Phi(x) \frac{|K^{h}_{n-1}(x, z)|^2}{||K^{h}_{n-1}(\cdot, z)||^2_{S^n}} dS_n(x), \quad B = \int_{x \in V} \Phi(x) \frac{|K^{h}_{n-1}(x, z)|^2}{||K^{h}_{n-1}(\cdot, z)||^2_{S^n}} dS_n(x).
\]

We affirm that the integral on \(V\) is actually \(O(h^\infty)\). To try this, from Lemma 10.1 of Ref. [5] (specifically Eq. 10.4), the function \(K^{h}_{n-1}(\cdot, z)\) has a integral
Thus we get the estimate for $x$ for some constant $C$ in the algebra $A$ start product (see Refs. [1] for the standard definition of star-product) on we have

$$Hence B be denoted by $\ast$ in this section we verifies that this noncommutative star-product, which will

From the norm estimate for the coherent states in Eq. (35) we obtain

$$From the last equation we obtain

$$From the last equation we obtain

$$Note that, for $x \in V$

Thus we get the estimate for $x \in V$

$$From the norm estimate for the coherent states in Eq. (35) we obtain

$$Hence $B = O(h^\infty)$. On the other hand, from proposition 8.2 and Eq. 35 we have

$$Note that, for $x \in V, \frac{\Re(x)z}{|z|} - 1 < \frac{1}{C} - 1 < 0$. Thus, we can take the integral defining $A$ in Eq. 49 over the whole sphere with error $O(h^\infty)$. From Eq. 49, considering

$$and (45) we conclude the proof of theorem. □

6. THE STAR PRODUCT

In Ref. 4, Berezin show that the formula 38 will allow us to define a start product (see Refs. 11 for the standard definition of star-product) on the algebra $A_{h,p}$ out of Berezin’s symbol for bounded operators with domain in $\mathcal{O}$ (see Eq. 36), i.e.

$$In this section we verifies that this noncommutative star-product, which will be denoted by $\ast_p$, satisfies the usual requirement on the semiclassical limit, i.e. as $h \to 0$

$$f_1 \ast_p f_2(z) = f_1(z)f_2(z) + hB(f_1, f_2)(z) + O(h^2), \quad z \in \mathbb{C}^n, f_1, f_2 \in \mathcal{A}_{h,p}$,$$
where $B(\cdot, \cdot)$ is a certain bidifferential operator of first order.

**Theorem 6.1.** Let $n \geq 2$, $z \in \mathbb{C}^n$, $\beta$ a smooth function defined on $\mathbb{C}^n$, $p > -n$ and $\mu, \nu \in \mathbb{R}$ with $\mu, \nu > \frac{1}{2}$. Then for $h \to 0$

$$I(z) := \int_{\mathbb{C}^n} \beta(w) \frac{_{0}F_1(\nu, w \cdot z/h^2)_{0}F_1(\mu, w \cdot z/h^2)}{_{0}F_1(\mu, |z|^2/h^2)} \text{det} \frac{h}{i} \text{d}m_{n,p}(w)$$

$$= \left( \frac{|z|}{h} \right)^{n+p-\nu} \frac{\Gamma(\nu)}{\Gamma(n+p)} \left\{ \beta(z) + h \left( \frac{1}{4|z|}(p-\nu+1)(p+\nu-1) \beta(z) \right) + \frac{R}{2} \beta(z) + g^{i\bar{j}}(z) \left[ \frac{1}{2|z|^2} z_i(n+p-\nu) \partial_j + z_j(n+p-\mu) \partial_i \right] \beta(w) \right.$$

$$\left. + \partial_i \partial_j \beta(w) + \frac{\beta(z)}{|z|^{n+p-\nu}} \partial_i \left( \frac{\xi^\nu_{\mu}}{g} \right) \right|_{w=z} \right\} + O(h^2)$$

(51)

with $_{0}F_1$ denoting the generalized hypergeometric function (see Sec.9.14 of Ref. [11] for definition and expressions for this special function), $\partial_i = \partial/\partial w_i$ and $\partial_j = \partial/\partial \bar{w}_j$. Further, where $R$ is the scalar curvature defined by the Kähler metric $ds^2 = g_{ij} \text{d}z_i \wedge \text{d}z_i$, and $g_{ij}, g^{ij}, \xi^\nu_{\mu}$, $g$ are functions described subsequently.

**Proof.** Let us define the following two regions on $\mathbb{C}^n$

$$W = \left\{ w \in \mathbb{C}^n \mid \frac{\Re \sqrt{w \cdot z}}{|z|} \geq \frac{1}{4} \right\} \text{ and } V = \mathbb{C}^n - W.$$

The integral in Eq. (51) can be written as an integral on the region $W$ plus an integral on $V$ denoted by the letters $J_W$ and $J_V$ respectively. The integral on $W$ is the one giving us the main asymptotic and the integral on $V$ is actually $O(h^\infty)$. In order to estimate $J_V$, from the equality

$$\Gamma(\nu + 1) \left( \frac{1}{2} z \right)^{-\nu} I_\nu(z) = _0F_1(\nu + 1, z^2/4)$$

(see formula 9.6.47 of Ref. [3] for details), and the integral representation of the function $I_\lambda$ (see formula 8.431-1 of Ref. [8])

$$I_\lambda(z) = \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda+\frac{1}{2})} \int_{-1}^{1} (1-t^2)^{-\lambda-\frac{1}{2}} e^{zt} dt, \quad \Re(\lambda) > -\frac{1}{2}$$

we have

$$\left| _0F_1\left( \nu, \frac{w \cdot z}{h} \right) \right| = C \left| \int_{-1}^{1} (1-t^2)^{\nu-\frac{1}{2}} e^{\frac{2\Re\sqrt{w \cdot z}}{|z|} t} dt \right|$$

$$\leq C e^{\frac{|z|}{h}} \int_{-1}^{1} (1-t^2)^{\nu-\frac{1}{2}} \exp \left( \frac{|z|}{h} \left[ \frac{2\Re\sqrt{w \cdot z}}{|z|} - 1 \right] \right) dt.$$

for some constant $C$. Note that, for $w \in V$,

$$\frac{2\Re\sqrt{w \cdot z}}{|z|} - 1 < -\frac{1}{2}.$$
Thus we get the estimate for \( w \in V \)

\[
\left| {}_0F_1\left( \nu, \frac{w \cdot z}{\hbar} \right) \right| \leq C e^{\frac{1}{4} |z| e^{-\frac{1}{2}|z|}} \int_{-1}^1 (1 - t^2)^{\nu - \frac{3}{2}} dt \\
= C B \left( \frac{1}{2}, \nu - \frac{1}{2} \right) e^{\frac{1}{4} \|z\|} e^{-\frac{1}{2} |z|},
\]

where \( B(x, y) \) denoting the beta function (see 8.38 of Ref. [8] for definition and expression for this special function). From the equality \( {}_0F_1(\mu, |z|/\hbar^2) = \Gamma(\mu)|z|^{-\mu+1}\Gamma_{\mu-1}(2|z|/\hbar) \) (see Eq. (53)) and Eq. (54) we obtain for \( w \in V \)

\[
\left| {}_0F_1(\nu, w \cdot z/\hbar^2) \right| \leq C \left( \frac{|z|}{\hbar} \right)^{\mu - \frac{1}{2}} e^{-\frac{1}{4}|z|} (1 + O(\hbar)) .
\]

Hence \( J_V = O(\hbar^\infty) \). Let us now study the term \( J_W \), first we note that for \( w \in W, \text{Arg}(w \cdot z) < \pi \). So from Eqs. (53), (54), the expression for the measure \( dm_{n,p}^h \) (see Eq. (12)) and the asymptotic expression of the MacDonal-Bessel function of order \( p \), \( K_p \) (see formula 9.7.2 of Ref. [3]) we find that

\[
J_W(z) = \int_W \beta(\nu, w \cdot z/\hbar^2) {}_0F_1(\mu, z \cdot w/\hbar^2) d\mu_{n,p}(w) \\
= \frac{1}{2\pi^2} \frac{\hbar^\nu}{\hbar^{2n+p}} \frac{\Gamma(\nu)}{(n + p)} \int_W \beta(w) \zeta^{\nu,\mu}_z(w) \chi_z(w) \exp \left( \frac{1}{\hbar} f_z(w) \right) dwd\overline{w}
\]

with

\[
\zeta^{\nu,\mu}_z(w) = |z|^{\mu - \frac{1}{2}} |w|^{p - \frac{1}{2}} (z \cdot w)^{-\frac{\nu}{2} + \frac{1}{4}} (w \cdot z)^{-\frac{\nu}{2} + \frac{1}{4}},
\]

\[
\chi_z(w) = 1 + \frac{\hbar}{16} \left[ \frac{1 - 4(\mu - 1)^2}{\sqrt{z \cdot w}} + \frac{1 - 4(\nu - 1)^2}{\sqrt{w \cdot z}} + \frac{4p^2 - 1}{|w|} \right] + E(z, h)_+,
\]

\[
f_z(w) = 2 \left( \sqrt{z \cdot w} + \sqrt{w \cdot z} - |w| - |z| \right) = 2 \left( 2\Re(\sqrt{z \cdot w}) - |z| - |w| \right),
\]

where the error term \( E(z, h) \) in (56) is \( O(\hbar^2) \) uniformly with respect to \( w \in W \) because in such a region we have \( \frac{1}{|zw|} \leq \frac{1}{|z|} \) and \( \frac{1}{|w|} \leq \frac{4}{|z|} \).

Furthermore, from Eq. (54) \( f_z(w) \leq -|z| - |w| < 0 \) for \( w \in V \). Thus we can take the integral defining \( J_W \) over the whole space \( \mathbb{C}^n \) with an error \( O(\hbar^\infty) \).

Let us now write \( J_W(z) = B(z) + h C(z) + O(\hbar^2) \) with

\[
B(z) = \frac{1}{2\pi^2} \frac{\hbar^\nu}{\hbar^{2n+p}} \frac{\Gamma(\nu)}{(n + p)} \int_{\mathbb{C}^n} \beta(w) \zeta^{\nu,\mu}_z(w) \exp \left( \frac{1}{\hbar} f_z(w) \right) dwd\overline{w},
\]
and

\[
C(z) = \frac{1}{32\pi^2} \frac{\hbar^\nu}{\hbar^{2n+p}} \frac{\Gamma(\nu)}{\Gamma(n + p)} \int_{\mathbb{C}^n} \beta(w) \zeta_{\nu}^{\nu}(w) \left[ \frac{1 - 4(\mu - 1)^2}{\sqrt{w \cdot w}} + \frac{1 - 4(\nu - 1)^2}{\sqrt{w \cdot w}} \right] + 4p^2 - 1 + \frac{4(\mu - 1)^2 - 1}{|w|} \right] \exp \left( \frac{1}{\hbar} f_z(w) \right) \, dw dw.
\]

(58)

For obtain the asymptotic expansion \([51]\), we express the integrals \(B(z)\) and \(C(z)\), using the stationary phase method (see Ref. [9]), as \(B_1(z) + \hbar B_2(z) + O(\hbar^2), C_1(z) + O(\hbar)\) respectively.

For our purpose, note that, using Schwartz inequality, the phase function \(f_z(w)\) satisfies \(f_z(w) \leq -2(\sqrt{|z|} - \sqrt{|w|})^2 \leq 0\). Moreover \(f_z(w) = 0\) if and only if \(w = \bar{w} = z\). Thus \(f_z\) is smooth function on a neighbourhood of the critical point \(w = z\). We also have

\[
-\partial_j \partial_\bar{i} f_z(w) = \frac{1}{|w|} \left( \delta_{i,j} - \frac{w_j \bar{w}_i}{2|w|^2} \right).
\]

Using that fact that for any complex numbers \(a_i, b_i, i = 1, \ldots, n, \text{det} (\delta_{i,j} \pm a_i b_j) = 1 \pm \sum_\ell a_\ell b_\ell\), we obtain

\[
\text{det} (-\partial_j \partial_\bar{i} f_z(w)) = \frac{1}{2|w|^n}.
\]

From the stationary phase method, applied to the integral \(C(z)\), we deduce

\[
C_1(z) = \left( \frac{|z|}{\hbar} \right)^{n+p-\nu} \frac{\Gamma(\nu)}{\Gamma(n + p)} \frac{1}{4|z|} (p - \nu + 1)(p + \nu - 1) \beta(z).
\]

(60)

Furthermore, the computation of \(B_1(z)\) and \(B_2(z)\) in Eq. \([51]\) it is using theorem 3 in Ref. [6]. For this we consider the Kähler potential \(\Phi(w) = 2|w|\), and let \(\Phi(w, u) = 2\sqrt{w \cdot u}\) be a sesqui-analytic extension of \(\Phi(w) = \Phi(w, w)\) to a neighbourhood of the diagonal.

From it, we have

\[
f_z(w) = \Phi(z, w) + \Phi(w, z) - \Phi(w, w) - \Phi(z, z),
\]

and

\[
g(w) = \text{det} \left( g_{\bar{i}j}(w) \right),
\]

\[
g_{\bar{i}j} = \partial_j \partial_\bar{i} \Phi = -\partial_j \partial_\bar{i} f_z(w).
\]

From theorem 3 in Ref. [6] and Eq. \([59]\)

\[
B_1(z) = \frac{1}{2\pi^n} \frac{\hbar^\nu}{\hbar^{2n+p}} \frac{\Gamma(\nu)}{\Gamma(n + p)} \left[ (\pi \hbar)^n \right] g(z)^{n+p-\nu} \beta(z)
\]

(61)

and

\[
B_2(z) = \frac{1}{2\pi^n} \frac{\hbar^\nu}{\hbar^{2n+p}} \frac{\Gamma(\nu)}{\Gamma(n + p)} (\pi \hbar)^n \left[ L_1 \left( \frac{\beta(w)}{g(w)} \zeta_{\nu}^{\nu}(w) \right) \right]_{w=z} + R \frac{\beta(z)}{2 g(z)} |z|^{p-\nu}
\]

with \(R \leq 1\).
where \( L_1 := g^{ji} \partial_i \partial_j \) is the Laplace-Beltrami operator, with \( g^{ij} \) the coefficients of the inverse matrix of \([g_{ij}(w)]\), and \( R := L_1(\log g) \) is the scalar curvature. Then

\[
\left( \frac{|z|}{\hbar} \right)^{n+p-\nu} \frac{\Gamma(\nu)}{\Gamma(n+p)} \left\{ g^{ji}(z) \left[ \partial_i \partial_j \beta(w) + \frac{\beta(z)}{|z|^{n+p-\nu}} \partial_j \beta \left( \frac{\zeta^{\nu\mu}}{g}(w) \right) \right] \\
+ \frac{1}{2|z|^2} [\beta_j(n+p-\nu) \partial_j + z_j(n+p-\mu) \partial_i] \beta(w) \right\}_{w=z} + \frac{R}{2} \beta(z)
\]  

(62)

where we have used the definition \( L_1 \) and

\[
\frac{\partial \xi^{\nu\mu}}{\partial w_i} g \bigg|_{w=z} = |z|^{n+p-\nu-2(n+p-\nu)} \xi_i \\
\frac{\partial \xi^{\nu\mu}}{\partial w_j} g \bigg|_{w=z} = |z|^{n+p-\nu-2(n+p-\mu)} z_j
\]

We note that in Eq. (51) appears \( \partial_j \partial_i (\zeta^{\nu\mu}/g) \), which is not difficult to prove that

\[
\left. \partial_j \partial_i \left( \frac{\zeta^{\nu\mu}}{g} \right)(w) \right|_{w=z} = \frac{|z|^{n+p-\nu}}{|z|^2} \left[ \left( n+p - 1 \right) \left( -z_j \xi_i + \delta_{ij} |z|^2 \right) + z_j \xi_i (n+p-\nu)(n+p-\mu) \right]
\]

(63)

which results in later use.

We remember that for \( f_1, f_2 \in A^h_{n,p} \), the law of multiplication in \( A^h_{n,p} \) is

\[
(f_1 \ast_p f_2)(z) = \frac{2}{(\pi \hbar)^n} \int_{\mathbb{C}^n} f_1(u, z) f_2(z, u) \left( \frac{|u|}{|z|} \right)^{p-\nu-2} \frac{I_{n+p-1} \left( \frac{2 \cdot |z|}{|u|} \right) I_{n+p-1} \left( \frac{2 \cdot |u|}{|z|} \right) |u|^p K_p \left( \frac{2|u|}{\hbar} \right)}{I_{n+p-1} \left( \frac{2|z|}{|u|} \right)} \, dud\bar{u},
\]

(64)

where the functions \( f_j(z, w), j = 1, 2 \), which are the analytic continuation of \( f_j(z) \) to \( \mathbb{C}^n \times \mathbb{C}^n \) (see Eq. (61)).

**Theorem 6.2.** Let \( n \geq 2, p > -n \). The product \( \ast_p \) satisfies

a) \( f \ast_p 1 = 1 \ast_p f = f \), for all \( f \in A^h_{n,p} \).

b) is associative, and
c) for \( f_1, f_2 \in A^h_{n,p} \) we have the following asymptotic expression when \( h \to 0 \)

\[
f_1 \ast_p f_2(z) = f_1(z) f_2(z) + h \left[ g^{ji} \left[ \partial_i f_1(z, w) + \partial_j f_2(w, z) \right] \right]_{w=z} \\
+ f_1(z) f_2(z) \left( -\frac{1}{4|z|} (n-1)(n-1+2p) + \frac{R}{2} \right) \\
+ g^{ji}(z) \frac{p-n+\frac{1}{2}}{2|z|^2} (z_j - |z|^2 \delta_{ij})
\]

(65)
where $R$ and $g^{\bar{A}}$ was defined in the theorem 6.1.

Proof. a) Let $f \in A^h_{n,p}$, then $f(z) = \mathfrak{B}_{n,p}(A)(z)$ with $A \in \mathcal{B}(\mathcal{O})$. From Eq. (57) and proposition 4.4

$$f *_p 1(z) = \int_{\mathbb{C}^n} f(w, z) \left\langle \frac{(K^h_{n,p}(\cdot, z), K^h_{n,p}(\cdot, w))_{s^n}}{(K^h_{n,p}(\cdot, z), K^h_{n,p}(\cdot, z))_{s^n}} \right\rangle_{s^n} dm^h_{n,p}(w)$$

$$= \int_{\mathbb{C}^n} (AK^h_{n,p}(\cdot, w), K^h_{n,p}(\cdot, z))_{s^n} \frac{(K^h_{n,p}(\cdot, z), K^h_{n,p}(\cdot, w))_{s^n}}{(K^h_{n,p}(\cdot, z), K^h_{n,p}(\cdot, z))_{s^n}} dm^h_{n,p}(w)$$

$$= f(z).$$

Analogously $1 *_p f = f$.

b) The associativity follows from the fact that the composition in the algebra of all bounded linear operator on $\mathcal{O}$ is associative.

c) The asymptotic expression given in the Eq. (63) is a direct consequence from the integral expression of star product (see Eq. (64)), the Eq. (53) and theorem 6.1.

\[ \square \]

7. Appendix A. On the inner product on complex sphere

In this appendix we study the inner product of functions on $S^n$ from the form $\partial_z^{k,s}(x) = x^k(x \cdot z)^s$ with $z \in \mathbb{C}^n$, $x \in S^n$, $k \in \mathbb{Z}_+^n$ and $s \in \mathbb{Z}_+$. We introduce the following notation, for multi-indices $k, m \in \mathbb{Z}_+^n$ we will write $k \geq m$ if and only if $k_j \geq m_j$ for all $j = 1, \ldots, n$.

Lemma 7.1. Let

$$F := \langle \partial_z^{k,s}, \partial_z^{m,\ell} \rangle_{S^n} = \int_{S^n} x^k \bar{x}^m (x \cdot z)^s (w \cdot x)^\ell dS_n(x),$$

with $\ell, s \in \mathbb{Z}_+$, $k, m \in \mathbb{Z}_+^n$, and $z, w \in \mathbb{C}^n \setminus \{0\}$. Then $F = 0$ if $|k| + s \neq |m| + \ell$, furthermore

a) If $|k| + s = |m| + \ell$ and $m \geq k$, then

$$F = \frac{(n-1)!(|m| - |k| + \ell)!}{(n-1 + |m| + \ell)!} \frac{\bar{z}^m}{z^k} \sum_{|\beta| = \ell} \frac{\ell!}{\beta! (m - k + \beta)!} z^\beta w^\beta.$$ 

b) If $|k| + s = |m| + \ell$, and $k \geq m$, then

$$F = \frac{(n-1)!(|s + |k| - |m||)!}{(|k| + s + n - 1)!} \frac{w^k}{w^m} \sum_{|\alpha| = s} \frac{s!}{\alpha! (k - m + \alpha)!} z^\alpha w^\alpha.$$ 

Proof. We note that the function $\partial_z^{k,s}(x)$ is harmonic and homogeneous of order $|k| + s$, then $J = 0$ if $|k| + s \neq |m| + \ell$. We suppose now that
$|k| + s = |m| + \ell$, and $m \geq k$, then

$$F = \sum_{|\alpha| = s} \sum_{|\beta| = \ell} \frac{s! \ell!}{\alpha! \beta!} x^{\alpha + \beta} \int_{S^n} x^{k + \alpha} y^{m + \beta} dS_n(x)$$

$$= \sum_{|\beta| = \ell} \frac{(m - |k| + \ell)! \ell!}{(m - k + \beta)! \beta!} y^{m - k + \beta} \frac{(n - 1)!(m + \beta)!}{(n - 1 + |m| + \ell)!}$$

$$= \frac{z^{m - k} (n - 1)!(m - |k| + \ell)!}{(n - 1 + |m| + \ell)!} \sum_{|\beta| = \ell} \frac{(m + \beta)! \ell!}{(m - k + \beta)! \beta!} y^{\beta} w^\beta,$$

where we use $\int_{S^n} |x^\alpha|^2 dS_n(x) = \frac{(n - 1)! \alpha!}{(n - 1 + |\alpha|)!}$ (see Ref. [13] for details). Similarly we get the other formula. □

8. Appendix B. Asymptotic of the Integral kernel of $U_{n,-1}$

In the particular case when $p = -1$, we can obtain the asymptotic expression of the coherent states $K_{n,p}^h$. First, based on the definition of $K_{n,p}^h$ (see Eqs. (26) and (25)) let us define the function $g: \mathbb{C} \rightarrow \mathbb{C}$ by

$$g(z) = \sum_{\ell=0}^{\infty} \frac{\sqrt{a\ell^2 + 1}}{\ell!} z^\ell, \quad a = \frac{1}{n - 1}.$$

Note that the coherent states $U_{n,-1}$ are equal to the function $g$ evaluated at $(x \cdot z)/\hbar$. In this appendix we obtain the main asymptotic term for the function $g$.

**Lemma 8.1.** For $\Re(z) > 0$ and $|\Im(z)| \leq C \Re(z)$ with $C$ a positive constant and $\Re(z) \rightarrow +\infty$, $g$ has the following asymptotic expansion:

$$g(z) = \sqrt{a} z^{1/2} \exp(z) \left[ 1 + \frac{a_1}{z} + \frac{a_2}{z^2} + O(z^{-3}) \right],$$

with $a_1, a_2$ some constants.

**Proof.** This follows from Lemma 10.1, which appears in Ref. [5]. □

Using Lemma [8.1] with $z = x \cdot z/\hbar$ we obtain the following asymptotic expansion:

**Proposition 8.2.** Let $z \in \mathbb{C}^n - \{0\}$. Then for $\hbar \rightarrow 0$ and $|\Im(x \cdot z)| \leq C \Re(x \cdot z)$, with $C$ a positive constant, we have

$$K_{n,p}^h(x, z) = \left[ \frac{x \cdot z}{\hbar(n - 1)} \right]^{1/2} \exp \left( \frac{x \cdot z}{\hbar} \right) \left[ 1 + \frac{a_1}{x \cdot z} \hbar + O(\hbar^2) \right].$$
References

[1] Bayen, F., Flato, M., Fronsdal, C., Lichnerowicz, A., Sternheimer, D., Deformation theory and quantization, I. Annals of Physics, 111, 61-110, 1978.

[2] N. Aronszajn, Theory of reproducing kernels, Trans. Amer. Math. Soc., 68 337-404, 1950

[3] Abramowitz, M., Stegun, I. A. Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Dover, New York 1972.

[4] Berezin, F. A., Quantization, Math USSR-Izv., 8, 1116-1175 (1974)

[5] Diaz-Ortiz, E. I. and Villegas-Blas, C., Semiclassical properties of coherent states for $L^2(S^n)$, $n = 2, 3, 5$, Qual. Theory Dyn. Syst. 8(2), 279–317 (2009).

[6] Englis, M., The asymptotics of a Laplace integral on a Kähler manifold, Journal Fur Die Reine und angewandte Mathematik, 528, 1-39, 2000.

[7] Gel’fand, I. M. y Shilov, G. E., Generalized functions, Vol 1, Properties and Operators, Academic Press, New York and London, 1964.

[8] Gradshteyn, I.S. y Ryzhik, I.M. Table of integrals, series, and products, Fifth edition. Academic Press, United Kingdom (1994), editado por Alan Jeffrey.

[9] Hörmander, L. The analysis of Linear Partial Differential Operators, Vol I. Distribution Theory and Fourier Analysis, 2nd eds. Springer, Berlin, 1990.

[10] Karp, Dmitrii Hypergeometric reproducing kernels and analytic continuation from a half-line, Integral Transforms and Spec. Funct. 14 (2003), no. 6, 485–498.

[11] Lebedev, N. Special functions and their applications, Pretice-Hall, 1965.

[12] Prudnikov, A.P. Brychkov Yu. A. and Marichev, O. I. Integrals and series, Volume 3: More Special Functions, Gordon and Breach Science Publishers, 1990.

[13] Rudin, Walter Function theory in the unit ball on $C^n$, Classics Math. Springer-Verlag, Berlin, 2008, reprint of the 1980 edition.

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