Standard conjectures in model theory, and categoricity of comparison isomorphisms: A model theory perspective

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ABSTRACT
We formulate two conjectures about étale cohomology and fundamental groups motivated by categoricity conjectures in model theory. One conjecture says that there is a unique \( \mathbb{Z} \)-form of the étale cohomology of complex algebraic varieties, up to \( \text{Aut}(\mathbb{C}) \)-action on the source category; put differently, each comparison isomorphism between Betti and étale cohomology comes from a choice of a topology on \( \mathbb{C} \). Another conjecture says that each functor to groupoids from the category of complex algebraic varieties which is similar to the topological fundamental groupoid functor \( \pi_{1}^{\text{top}} \), in fact factors through \( \pi_{1}^{\text{top}} \), up to a field automorphism of the complex numbers acting on the category of complex algebraic varieties. We also try to present some evidence towards these conjectures, and show that some special cases seem related to Grothendieck standard conjectures and conjectures about motivic Galois group.

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1. Introduction

We consider the following question as it would be understood by a model theorist

**Question.** Is there a purely algebraic definition of the notion of singular (Betti) cohomology or the topological fundamental groupoid of a complex algebraic variety?

and formulate precise conjectures proposing that the comparison isomorphism of étale cohomology/fundamental groupoid admits such a purely algebraic definition (characterization).

Note that an algebraic geometer might interpret the question differently from a logician and in that interpretation, the answer is well-known to be negative.

For a logician this is a question about the desirable property of **categoricity:** a theory (i.e. a collection of properties expressed in a language) of a model is called **categorical** iff any two models of the theory (i.e. any two structures satisfying the properties) are necessarily isomorphic. In the approach of logically perfect structures [19, 22] constructs a number of categorical theories associated with complex analytic functions and particularly exponentiation [7], or rather their conjectural properties related to the field structure, such as Schanuel-type conjectures on transcendence of values, Kummer theory and Galois \( \ell \)-adic representations (seen as being about the field automorphism action on the sequences of form \( e^\pi, e^{\pi/2}, e^{\pi/3}, \ldots \)). In algebraic geometry, this kind of conjectures occurs in the conjectural theory of the motivic Galois group and the conjectural theory of pure motives enabled by Grothendieck Standard Conjectures, e.g. Mumford-Tate and Grothendieck periods conjecture. Hence “standard conjectures in model theory” in the title refer...
both to the Grothendieck Standard Conjectures and to the Zilber’s standard conjectures in the approach of logically perfect structures claiming categoricity of certain theories associated with complex analytic functions.

We now explain our motivation in two essentially independent ways. Section 1.1 explains how a model theorist would interpret the question above; Section 1.2 views these conjectures as continuation of work in model theory on the complex field with pseudoexponentiation [7, 8, 18, 22] and its main goal is to make the reader aware of the possibilities offered by methods of model theory.

1.1. How to interpret the question

Let us now explain the difference between how an algebraic geometer and a model theorist might interpret the question.

Let $H_{\text{top}}$ be a functor defined on the category $\mathcal{V}ar$ of algebraic varieties (say, separated schemes of finite type) over the field $\mathbb{C}$ of complex numbers; we identify this category with a subcategory of the category of topological spaces. We shall be interested in the case when $H_{\text{top}}$ is either the functor $H_{\text{sing}} : \mathcal{V}ar \rightarrow \text{Ab}$ of singular cohomology or the fundamental groupoid functor $\pi_{1}^{\text{top}} : \mathcal{V}ar \rightarrow \text{Groupoids}$.

An algebraic geometer might reason as follows. A purely algebraic definition applies both to $H_{\text{top}}$ and $H_{\text{top}} \circ \sigma$ where $\sigma : \mathbb{C} \rightarrow \mathbb{C}$ is a field automorphism. Hence, to answer the question in the negative, it is enough to find a field automorphism $\sigma : \mathbb{C} \rightarrow \mathbb{C}$ such that $H_{\text{top}}$ and $H_{\text{top}} \circ \sigma$ differ. And indeed, [20] constructs an example of a projective algebraic variety $X$ and a field automorphism $\sigma$ such that $X(\mathbb{C})$ and $X^{\sigma}(\mathbb{C})$ have non-isomorphic fundamental groups.

A model theorist might reason as follows. A purely algebraic definition applies both to $H_{\text{top}}$ and $H_{\text{top}} \circ \sigma$ where $\sigma : \mathbb{C} \rightarrow \mathbb{C}$ is a field automorphism. Hence, we should try to find purely algebraic description (possibly involving extra structure) of $H_{\text{top}}$ which fits precisely functors of form $H_{\text{top}} \circ \sigma, \sigma \in \text{Aut}(\mathbb{C})$ with the extra structure. We say that such a purely algebraic description determines $H_{\text{top}}$ (with the extra structure) uniquely up to an automorphism of $\mathbb{C}$.

For $H_{\text{top}} = H_{\text{sing}}$ the singular (Betti) cohomology theory, a model theorist might continue thinking as follows. The singular (Betti) cohomology theory admits a comparison isomorphism to a cohomology theory defined purely algebraically, say $\ell$-adic étale cohomology theory. This is an algebraic description in itself. However, note that it considers the $\ell$-adic étale cohomology theory and the comparison isomorphism as part of structure. Thus, an appropriate conjecture (see Section 2) gives a purely algebraic description of the family of comparison isomorphisms coming from a choice of topology on $\mathbb{C}$

$$H_{\text{sing}}(X(\mathbb{C})_{\text{top}}, \mathbb{Z}) \otimes \mathbb{Z}_{\ell} \xrightarrow{\sigma} H_{\text{et}}(X(\mathbb{C}), \mathbb{Z}_{\ell}), \quad \sigma \in \text{Aut}(\mathbb{C})$$

For $H_{\text{top}} = \pi_{1}^{\text{top}}$ a model theorist might continue thinking as follows. The profinite completion of the topological fundamental groupoid functor is the étale fundamental groupoid defined algebraically. This is an algebraic property of the topological fundamental groupoid on which we can base our purely algebraic description if we include the étale fundamental groupoid as part of structure. Essentially, this describes subgroupoids of the étale fundamental groupoids. Category theory suggests to consider a related universality property (see Section 3): up to $\text{Aut}(\mathbb{C})$ action on the category of complex algebraic varieties, there is a universal functor among those whose profinite completion embeds into the étale fundamental groupoid, and it is the topological fundamental groupoid. Some technicalities may be necessary to ignore non-residually finite fundamental groups.
1.1.1. Structure of the paper

In the introduction, we explain our motivation in two essentially independent ways. Section 1.1 explains how a model theorist would interpret the question above; Section 1.2 views these conjectures as a continuation of work in model theory on the complex field with pseudoexponentiation [1–9, 13, 22], see also [18, Ch. X.6.11-6.19] and its main goal is to make the reader aware of the possibilities offered by methods of model theory. In Section 2.1, we begin by formulating two conjectures on étale cohomology. Conjecture 2.1 says there is a unique there is a unique \( \mathbb{Z} \)-form of the étale cohomology of complex algebraic varieties, up to \( \text{Aut}(\mathbb{C}) \)-action on the source category; put differently, each comparison isomorphism between Betti and étale cohomology comes from a choice of topology on \( C \). Conjecture 2.2 states a similar property for the Weil cohomology theory restricted to a category generated by a single pure motive. In Section 2.2, Proposition 2.3 and Statement 2.4 give examples of similar properties which hold for a subcategory generated by a single Abelian variety. Section 2.3 contains a concise exposition of several conjectures on the motivic Galois group which appear related to our conjectures; it follows [21, 1.3]. Section 2.4 contains some speculations about the model theoretic point of view on these conjectures. Section 3 states analogous conjectures for the Etale fundamental group. In Section 3.1, we define the notion of a \( \pi_1 \)-like functor and state two conjectures saying there is a universal \( \pi_1 \)-like functor up to a field automorphism. In section 3.2, we state two conjectures which we hope to be within reach of current methods. In Section 3.3, we list several partial positive results which are implicit in model theoretic literature. We end by Section 3.4 which mentions mathematical facts which go into the proofs of these results.

1.2. Pseudo-exponentiation, Schanuel conjecture, and categoricity theorems in model theory

Complex topology allows to construct a number of objects with good algebraic properties e.g. a group homomorphism \( \exp : \mathbb{C}^+ \rightarrow \mathbb{C}^* \), singular (Betti) cohomology theory and the topological fundamental groupoid of varieties of complex algebraic varieties.

A number of theorems and conjectures says that such an object constructed topologically or analytically is “free” or “generic,” for lack of better term, in the sense that it satisfies algebraic relations only, or mostly, for “obvious” reasons of algebraic nature.

Sometimes such a conjecture is made precise by saying that a certain automorphism group is as large as possible subject to some “obvious obstructions or relations” imposed by functoriality and/or homotopy theory. Such an automorphism group may involve values of functions or spaces defined analytically or topologically.

A natural question to ask is whether these conjectures are “consistent” in the sense that there do exist such “free” objects with the conjectured properties, not necessarily of analytic or topological origin.

Methods of model theory allow to build such objects by an elaborate transfinite induction. In what follows we shall sketch results of Refs. [7, 22] which does this for the complex exponential function and Schanuel conjecture.

Let us now explain what we mean by showing how to view Kummer theory, Hodge conjecture, conjectural theory of the motivic Galois group, and Schanuel conjecture in this way, i.e. as saying certain objects satisfies algebraic relations only, or mostly, for “obvious” reasons of algebraic nature.

1.2.1. Grothendieck standard conjectures

In words of Grothendieck [14], “[these conjectures] would form a basis of the so-called theory of “pure motives” which is a systematic theory of “arithmetic properties” of algebraic varieties, as
embodied in their group of classes of cycles for numerical equivalence.' For the purposes of this paper, it is important that these conjectures imply that there are many automorphisms of the Betti cohomology theory of the complex algebraic varieties, and are needed to formulate the conjectural theory of the motivic Galois group Aut \((H_{\text{sing}})\), cf. Section 1.2.4.

A logician would wish to interpret the words of Grothendieck as follows. View a cohomology theory as a language which extends the usual theory ACF of algebraically closed fields by being able to talk about classes of algebraic cycles up to numerical equivalence; Grothendieck says this language is also able to talk about arithmetical properties of algebraic varieties. In model theory it is natural to assume (or wish) that such a structure is homogeneous and perhaps also categorical, which is a much stronger property. In this way, it is tempting to think of the motivic Galois group as the automorphism group of such a structure, and therefore quite large. However, it is unclear how to make this point of view precise.

In this paper, we do not need the precise statements of the Standard Conjectures; see Refs. [14, 17] for details.

### 1.2.2. Kummer theory

An “obvious way” to make \(e^{x_1/N}, \ldots, e^{x_n/N}, N > 0\) satisfy a polynomial relation is to pick \(x_1, \ldots, x_n\) such that they satisfy a \(\mathbb{Q}\)-linear relation over \(2\pi i\), which is preserved by \(\exp\), or such that that \(e^{x_1/M}, \ldots, e^{x_n/M}\) satisfy a polynomial relation for some other \(M\).

Kummer theory tells you these are the only reasons for polynomial relations between these numbers. This is stated precisely in terms of automorphisms groups as follows:

For any \(\mathbb{Q}\)-linearly independent numbers \(x_1, \ldots, x_n \in \mathbb{C}\) there is \(N > 0\) such that for any \(m > 0\) it holds

\[
\text{Gal}(\mathbb{Q}(e^{\frac{2\pi i}{N}}, \ldots, e^{\frac{2\pi i}{N}}, e^{2\pi i \mathbb{Q}}) / \mathbb{Q}(e^{\frac{2\pi i}{N}}, \ldots, e^{\frac{2\pi i}{N}}, e^{2\pi i \mathbb{Q}})) \approx (\mathbb{Z}/m\mathbb{Z})^n
\]

### 1.2.3. Hodge conjecture

Consider the Hodge theory of a nonsingular complex projective manifold \(X(\mathbb{C})\). By Chow theory we know that \(X\) is in fact a complex algebraic variety and an easy argument using harmonic forms shows that an algebraic subvariety \(Z(\mathbb{C})\) defines an element of \(H(X, \mathbb{Q}) \cap H^{p, p}(X, \mathbb{C})\) where \(H^{p, p}(X, \mathbb{C})\) is a certain linear subspace of \(H^{2p}(X, \mathbb{C})\) defined analytically.

A topological cycle in \(X(\mathbb{C})\) defines an element of \(H(X, \mathbb{Q})\) which may lie in \(H^{p, p}(X, \mathbb{C})\). An “obvious reason” for this is that it comes from an algebraic subvariety, or a \(\mathbb{Q}\)-linear combination of such. Hodge conjecture tells you that this is the only reason why it could happen.

### 1.2.4. \(\ell\)-adic Galois representations and motivic Galois group Aut \((H_{\text{sing}})\)

Remarks below are quite vague but we hope some readers might find them helpful. In Section 2.3 we sketch several definitions and conjectures in the conjectural theory of motivic Galois group following [21].

We would like to think that these conjectures say that the singular (Betti) cohomology theory of complex algebraic varieties is “free” in the sense that it satisfies algebraic relations only, or mostly, for “obvious” reasons of algebraic nature. The theory of the motivic Galois group assumes that there are many automorphisms of the singular cohomology theory of complex algebraic varieties, and they form a pro-algebraic, in fact pro-reductive ([21, Conjecture 2.1?], group. Conjectures on \(\ell\)-adic Galois representations, e.g. [21, Conjecture 3.2?], describe the image of Galois action as being dense or open in a certain algebraic group defined by the cohomology classes which the Galois action has to preserve (or is conjectured to preserve).
Let us very briefly sketch some details.

The conjectural theory of the motivic Galois group [21], also cf. Section 2.3, assumes that the following is a well-defined algebraic group:

\[ G_E = \text{Aut}^\otimes (H_{\text{sing}} : \langle E \rangle \to \mathbb{Q}\text{-Vect}) \]

Here \( \sigma : k \to \mathbb{C} \) is an embedding of a number field \( k \) into the field of complex numbers, \( E \) is a pure motive in the conjectural category \( \text{Mot}_k \) of pure motives defined over \( k \), and \( \langle E \rangle \) is the least Tannakian subcategory of \( \text{Mot}_k \) containing \( E \), and \( H_{\text{sing}} \) is the fiber functor on \( \langle E \rangle \) corresponding to the singular cohomology of complex algebraic varieties and embedding \( \sigma : k \to \mathbb{C} \). This is well-defined if we assume certain conjectures, e.g. Standard Conjectures and Hodge conjecture [14, 17, 21].

[21, Conjecture 3.1?] says that \( G_E \) is the subgroup of \( GL(H_{\text{sing}}(E)) \) preserving the tensors corresponding to morphisms \( 1 \to E^{\otimes r} \otimes E' ^{\otimes s}, r, s \geq 0 \). Think of these tensors as “obvious relations” which have to be preserved.

[21, Conjecture 3.2? and Conjecture 9.1?] describe the image of \( \ell \)-adic Galois representations in \( G_E(\mathbb{Q}_\ell) \).

Both say it is dense or open in the group of \( \ell \)-adic points of a certain algebraic subgroup of \( GL_{N_\mathbb{Q}} \); we think of this subgroup as capturing “obvious obstructions or relations” imposed by functoriality of \( H_{\text{sing}} \).

### 1.2.5. Schanuel conjecture: Questions

Schanuel conjecture says that for \( \mathbb{Q} \)-linearly independent \( x_1, \ldots, x_n \in \mathbb{C} \), the transcendence degree of \( x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n} \) is at least \( n \):

\[ \text{tr.deg.}_\mathbb{Q}(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n}) \geq \text{lin.deg.}_\mathbb{Q}(x_1, \ldots, x_n) \quad (\text{SC}) \]

The bound becomes sharp if we use surjectivity to pick \( x_1 = e^{x_1}, \ldots, x_{i-1} = e^{x_{i-1}}, x_i = e^{x_{i-1}} \) and \( e^{x_i} \in \mathbb{Q} \):

\[ \text{tr.deg.}_\mathbb{Q}(x_1, e^{x_1}, e^{x_2}, \ldots, e^{x_{n-1}}, e^{x_n}) \leq n \]

Here, “an algebraic relation” is a polynomial relation between \( x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n} \); an obvious way to make these numbers satisfy such a relation is to pick \( x_i \) such that either \( x_i = a_1 x_1 + \cdots + a_{i-1} x_{i-1} \) or \( e^{x_i} = a_1 x_1 + \cdots + a_{i-1} x_{i-1} \) or \( x_i = e^{a_1 x_1 + \cdots + a_{i-1} x_{i-1}} \) where \( a_1, \ldots, a_i \in \mathbb{Q} \) are rational.

Is Schanuel conjecture “consistent” in the sense that there is a pseudo-exponentiation, i.e. a group homomorphism \( \exp : \mathbb{C}^+ \to \mathbb{C}^+ \) satisfying conjectural properties of complex exponentiation, in particular Schanuel conjecture? Does there exist such a “free” pseudo-exponentiation \( \exp : \mathbb{C}^+ \to \mathbb{C}^+ \), e.g. such that a system of exponential-polynomial equations has a zero only if it does not contradict Schanuel conjecture? Can we build such an algebraic “free” object without recourse to topology?

Does every such “free” object come from a choice of topology on \( \mathbb{C} \), i.e. is the complex exponential \( \exp : \mathbb{C}^+ \to \mathbb{C}^+ \) up an automorphism of \( \mathbb{C} \)?

Note that the last question is the only one which mentions topology. It turns out this difference is crucial: model theory says nothing about this question while giving fairly satisfactory positive answers to the previous ones.

### 1.2.6. Schanuel conjecture and pseudoexponentiation: Answers

The following theorem of Ref. [22] provides a positive answer for \( \exp : \mathbb{C}^+ \to \mathbb{C}^+ \). For a discussion of the theorem and surrounding model theory see [18, 6.16]; for a proof, detailed statements
and generalizations to other analytic functions see [7, Theorem 1.2, Theorem 1.6; Theorem 9.1; also Theorem 8.2; Theorem 9.3] and references therein.

**Theorem 1.1** (Zilber [22]). Let $K$ be an uncountable algebraically closed field of characteristic 0.

Up to $\text{Aut}(K)$, there is a unique surjective group homomorphism $\text{ex} : K^+ \rightarrow K^*$

(SK) (Standard Kernel) $\text{Ker ex}$ is the infinite cyclic group generated by a transcendental element

(SC) (Schanuel Property) $\text{ex} : K^+ \rightarrow K^*$ satisfies Schanuel conjecture

(SEAC) (Strong exponential-algebraic closedness) any system of $n$ independent exponential-polynomial equations in $n$ variables that does not directly contradict Schanuel conjecture has a regular zero, but not more than countably many

Call this unique group homomorphism $\text{ex} : K^+ \rightarrow K^*$ pseudoexponentiation defined on field $K$.

In somewhat more detail, this can also be expressed as follows.

Let $K$ and $K'$ be two uncountable algebraically closed fields of characteristic 0, and let $\text{ex} : K^+ \rightarrow K^*$ and $\text{ex}' : K'^+ \rightarrow K'^*$ be group homomorphisms satisfying the properties above.

Then if there is a bijection $\sigma_0 : K \approx K'$, then there is a bijection $\sigma : K \approx K'$ preserving $+,-,\cdot$, and $\text{ex}$, i.e. such that for each $x,y \in K$ it holds

$$\sigma(x+y) = \sigma(x) + \sigma(y), \sigma(xy) = \sigma(x)\sigma(y), \sigma(\text{ex}(x)) = \text{ex}'(\sigma(x))$$

**Conjecture 1.2.** (Zilber [22]) If $\text{card } K = \text{card } \mathbb{C}$, then $(K, +, \cdot, \text{ex})$ is isomorphic to $(\mathbb{C}, +, \cdot, \exp)$.

Our conjectures are direct analogs of the Theorem and Conjecture above stated in the language of functors. Instead of the complex exponentiation we consider the comparison isomorphisms between topological and étale cohomology, resp. fundamental groupoid functor. We hope that model theoretic methods used by [22] may be of use in proving these conjectures.

**1.2.7. Remarks about the proof**

We adapt [18, 6.11–6.16]; see also [7] for a detailed exposition in a more general case using different terminology. Pseudoexponentiation is constructed by an elaborate transfinite induction. We start with an algebraically closed field $K_{\text{base}} \subset K$ and a partial group homomorphism $\text{ex} : K_{\text{base}}^+ \rightarrow K_{\text{base}}$ and try to extend the field and the group homomorphism such that it is related to the field in as free a way as possible.

Informally the freeness condition is described as follows:

(Hr) the number of independent explicit basic dependencies *added* to a subset $X \cup \text{ex}(X)$ of $K$ by the new structure is at most the dimension of $X \cup \text{ex}(X)$ in the old structure.

This is made precise in the following way.

*The new structure* is the group homomorphism $\text{ex} : K^+ \rightarrow K^*$; *explicit basic dependencies in $X \cup \text{ex}(X)$ added by the new structures* are defined as equations $\text{ex}(x) = y$ where $x \in X$. For example, for $X = \{x\}$ where $\text{ex}(\text{ex}(x)) = x$, we do not regard $\text{ex}(\text{ex}(x)) = x$ as a explicit basic dependency in $X \cup \text{ex}(X) = \{x, \text{ex}(x)\}$.

*The number of independent basic explicit dependencies* is the $\mathbb{Q}$-linear dimension $\dim_{\mathbb{Q}}(x_1, \ldots, x_n)$; the *dimension of $X$ in the old structure* is its transcendence degree which is equal to $\text{tr.deg.}(x_1, \ldots, x_n, \text{ex}(x_1), \ldots, \text{ex}(x_n))$.

With this interpretation, (Hr) becomes Schanuel conjecture (SC).
Define Hrushovski predimension \( \delta(X) := \text{tr.deg.}(X \cup \text{ex}(X)) - \text{lin.dim.}_\mathbb{Q}(X) \). Say a partial group homomorphism \( \text{ex} : K^+ \to K^+ \) satisfies Hrushovski inequality with respect to Hrushovski predimension \( \delta \) iff for any finite \( X \subset K \) it holds \( \delta(X) \geq 0 \). An extension \((K, \text{ex}_K) \subset (L, \text{ex}_L)\) of fields equipped with partial group homomorphisms is strong, write \( K \leq_{\delta} L \), iff all dependencies between elements of \( K \) occurring in \( L \) can be detected already in \( K \), i.e. for every finite \( X \subset K \),

\[
\min \{ \delta(Y) : Y \text{ finite}, \ X \subset Y \subset K \} = \min \{ \delta(Y) : Y \text{ finite}, \ X \subset Y \subset L \}
\]

We then build a countable algebraically closed field \((K_{\text{base}}, \text{ex}_{K_{\text{base}}})\) by taking larger and larger strong extensions \( K_{\text{base}} \leq_{\delta} K_1 \leq_{\delta} K_2 \leq_{\delta} \ldots \) of finite degree. If we do this with enough care, we obtain a countable algebraically closed field \( K_{\text{base}} = \bigcup K_n \) and a group homomorphism \( \text{ex}_{K_{\text{base}}} : K_{\text{base}}^+ \to K_{\text{base}}^+ \) defined everywhere which satisfies (SC) and other conditions of Theorem 1.1. For details see [7, section 5] where it is described in terms of taking Fraisse limit along a category of strong extensions.

Building an uncountable model requires deep model theory; see [7, section 6] and [5] Let us say a couple of words about this. In the inductive construction above, being countable is essential: if we start with an uncountable field, we can no longer hope to obtain an algebraically closed field after taking union of countably many extensions of finite degree. Very roughly, it turns out that we can construct composites of countable linearly disjoint algebraically closed fields this way, and this helps to build an uncountable field with pseudoexponentiation and prove it is unique in its cardinality up to isomorphism.

1.2.8. Generalizations and speculations
Bays-Kirby [7] generalizes the considerations above in a number of ways. In particular, they construct pseudo-exponential maps of simple Abelian varieties, including pseudo-\( \varphi \)-functions for elliptic curve. [Proposition 10.1, Section 10, ibid.] relates the Schanuel property of these to the André-Grothendieck conjecture on the periods of 1-motives. They suspect that for Abelian varieties the predimension inequality \( \delta(X) \geq 0 \) also follows from the André-Grothendieck periods conjecture, but there are more complications because the Mumford-Tate group plays a role and so have not been able to verify it. [Section 9.2, ibid.] says it is possible to construct a pseudoexponentiation incorporating a counterexample to Schanuel conjecture, by suitably modifying the Hrushovski predimension and thus the inductive assumption (Hr). [Section 9.7, also Theorem 1.7, ibid.] considers differential equations.

We intentionally leave the following speculation vague.

Speculation 1.3 Can one build a pseudo-singular, or pseudo-de Rham cohomology theory, or a pseudo-topological fundamental group functor of complex algebraic varieties, or an algebra of pseudo-periods which satisfies a number of conjectures such as the Standard Conjectures, the conjectural theory of the motivic Galois group, the conjectures on the image of \( \ell \)-adic Galois representations, André-Grothendieck periods conjecture, Mumford-Tate conjecture, etc.? 

2. Uniqueness property of comparison isomorphism of singular and \( \text{étale} \) cohomology of a complex algebraic variety

2.1. Statement of the conjectures
A \( \mathbb{Z} \)-form of a functor \( H_t : \mathcal{V} \to \mathbb{Z}_l\text{-Mod} \) is a pair \((H, \tau)\) consisting of a functor \( H : \mathcal{V} \to \mathbb{Z}\text{-Mod} \) and an isomorphism 

\[
H \otimes_{\mathbb{Z}} \mathbb{Z}_l \xrightarrow{\tau} H_t
\]

of functors.
An example of a $\mathbb{Z}$-form we are interested in is given by the comparison isomorphism between étale cohomology and Betti cohomology, see [SGA 4, XVI, 4.1], also [16, p. 23] for the definitions and exact statements.

Let $H_{\text{et}} : \text{Schemes} \to \mathbb{Z}\text{-Mod}$ be the functor of $\ell$-adic étale cohomology, and let $H_{\text{sing}} : \text{Top} \to \mathbb{Z}\text{-Mod}$ be the functor of singular cohomology. For $X$ a separated $\mathbb{C}$-scheme of finite type there is a canonical comparison isomorphism

$$H_{\text{sing}}(X(\mathbb{C}), \mathbb{Z}) \otimes \mathbb{Z}_l \approx H_{\text{et}}(X, \mathbb{Z}_l).$$

This defines a $\mathbb{Z}$-form of the functor of $\ell$-adic étale cohomology $H_{\text{et}}(-, \mathbb{Z}_l)$ restricted to the category of separated $\mathbb{C}$-schemes of finite type.

Let $K$ be an algebraically closed field, let $\mathcal{V}_K$ be a category of varieties over $K$. A field automorphism $\sigma : K \to K$ acts $X \mapsto X^\sigma$ on the category $\mathcal{V}_K$ by automorphisms. Moreover, for each variety $X$ defined over $K$, a field automorphism $\sigma$ defines an isomorphism $\sigma_X : X \to X^\sigma$ of schemes (over $\mathbb{Z}$ or $\mathbb{Z}/p\mathbb{Z}$), and hence

$$H_{\text{et}}(X, \mathbb{Z}_l) \xrightarrow{\sigma_X} H_{\text{et}}(X^\sigma, \mathbb{Z}_l).$$

This defines an action of $\text{Aut}(K)$ on the $\mathbb{Z}$-forms of $H_l$:

$$(H, \tau) \mapsto (H^\sigma \tau, \tau^\sigma \tau^{-1})$$

$$H(X^\sigma) \xrightarrow{\tau^\sigma \tau^{-1}} H_{\text{et}}(X, \mathbb{Z}_l).$$

We conjecture that, up to action of $\text{Aut}(\mathbb{C})$ defined above, the comparison isomorphism between singular and $\ell$-adic cohomology of is the only $\mathbb{Z}$-form of the $\ell$-adic cohomology theory $H_{\text{et}}(-, \mathbb{Z}_l)$:

**Conjecture 2.1.** $(Z(H_{\text{sing}}, H_l))$ Up to $\text{Aut}(\mathbb{C})$ action, there is a unique $\mathbb{Z}$-form of the $\ell$-adic cohomology theory functor $H_{\text{et}}(-, \mathbb{Z}_l)$ restricted to the category of separated $\mathbb{C}$-schemes of finite type which respects the cycle map and Kunneth decomposition.

In other words, every comparison isomorphism of a $\mathbb{Z}$- and the $\ell$-adic cohomology theory of separated $\mathbb{C}$-schemes of finite type is, up to a field automorphism of $\mathbb{C}$, the standard comparison isomorphism

$$H_{\text{sing}}(X(\mathbb{C}), \mathbb{Z}) \otimes \mathbb{Z}_l = H_{\text{et}}(X, \mathbb{Z}_l)$$

The conjecture is intended to be too optimistic; it is probably more reasonable to conjecture uniqueness of $\mathbb{Z}$-form of the torsion-free part of the $\ell$-adic cohomology.

Assume Grothendieck Standard Conjectures and that, in particular the $\ell$-adic cohomology theory factors via the category $\mathcal{M}ot_k$ of pure motives over a field $k$. Then, a Weil cohomology theory (cf. [17]) corresponds to a tensor fiber functor from the category of pure motives, and we may ask how many $\mathbb{Z}$-forms does have the fiber functor corresponding to the $\ell$-adic cohomology theory. Moreover, we may formulate a “local” version of the conjecture restricting the functor to a subcategory generated by a single motive.

**Conjecture 2.2.** $(Z(H_{\text{et}}, \langle E \rangle_k))$ Let $k$ be a number field. Assume Grothendieck Standard Conjectures and that, in particular, $\ell$-adic cohomology factors via the category of pure numerical motives $\mathcal{M}ot_k$ over $k$.

Let $E$ be a motive and let $\langle E \rangle$ be the subcategory of $\mathcal{M}ot_k$ generated by $E$, i.e. the least Tannakian subcategory of $\mathcal{M}ot_k$ containing $E$. Up to $\text{Aut}(\bar{k}/k)$-action, the functor $H_{\text{et}}(- \otimes \bar{\mathbb{Q}}, \mathbb{Z}_l) : \langle E \rangle_k \to \mathbb{Z}_l\text{-Mod}$ has at most finitely many $\mathbb{Z}$-forms.

Moreover, if $E$ has finitely many $\mathbb{Z}$-forms [21, 10.2?], then the functor $H_{\text{et}}(- \otimes \bar{\mathbb{Q}}, \hat{\mathbb{Z}}) : \langle E \rangle_k \to \hat{\mathbb{Z}}\text{-Mod}$ has at most finitely many $\mathbb{Z}$-forms.
2.2. An example: An Abelian variety

Let us give an example of a particular case of the conjecture which is easy to prove.

**Proposition 1.** Let $A$ be an Abelian variety defined over a number field $k$. Assume that the Mumford-Tate group of $A$ is the maximal possible, i.e. the symplectic group $\text{MT}(A) = \text{GSp}_{2g}$ where $\dim A = g$, and that the image of Galois action on the torsion has finite index in the group $\text{GSp}_{2g}(\mathbb{Z}_l)$ of $\mathbb{Z}_l$-points of the symplectic group.

Then there are at most finitely many $\mathbb{Z}$-forms of the $\ell$-adic cohomology theory $\text{H}^0_{\text{et}}(\mathcal{A}/\mathcal{C}_0, \mathbb{Z}_l)$ restricted to the category $\langle A \rangle_k$, up to $\text{Aut}(\mathcal{A}/\mathcal{C}_2)$.

**Proof (sketch).** The Weil pairing corresponds to the divisor corresponding to an ample line bundle over $A$, and by compatibility with the cycle class map of a $\mathbb{Z}$-form and $\text{H}^0_{\text{et}}(\mathcal{A}/\mathcal{C}_0, \mathbb{Z}_l)$ the non-degenerate Weil pairing $\omega : (\text{H}^1^1(A))^2 \to \text{H}^0_{\text{et}}(A) = \mathbb{Z}_l$ restricts to a pairing $\omega : (\text{H}^1(A))^2 \to \text{H}^0(A) = \mathbb{Z}$, which is easily seen to be non-degenerate.

Now let $H_i$ be a $\mathbb{Q}$-form for $i = 1, 2$.

Let $(x_1, \ldots, x_i, \ldots, y_i)$ be a symplectic basis for $\text{H}^1_1(A)$. Then each is also a symplectic basis for $\text{H}^1_1(A)$ and so some $\sigma \in \text{GSp}(\text{H}^1_1(A), \omega)$ maps $\text{H}^1_1(A)$ to $\text{H}^1_2(A)$. The assumption on the Mumford-Tate group precisely means that such a $\sigma$ extends to $\sigma \in \text{Aut}(\text{H}^1_{\text{et}}_{\text{et}}(A))$, and it follows from the fact that the cohomology of an Abelian variety is generated by $\text{H}^1$ that $\sigma(\text{H}^1) = \text{H}^2$.

Finally, use the assumption on the Galois representation to see that there are at most finitely many $\mathbb{Z}$-forms.

It might be true that a reader familiar with the theory of the motivic Galois group may find that a straightforward generalization of the argument above leads to the following.

**Statement 2.4** (a generalization of the example). Let $A$ be a motive of a smooth projective variety defined over a number field $k$. Assume the Mumford-Tate group $G = \text{Aut}^\circ(\text{H}^1_{\text{et}}(\mathcal{A}/\mathcal{C}_0, \mathbb{Z}_l))$ has the following property:

if $V_1$ and $V_2$ are Abelian subgroups of $\text{H}^1_{\text{et}}(A \otimes \mathbb{Z}_l)$ which are both dense and of the same rank and such that $GL(V_i) \cap G(\mathbb{Z}_l)$ is dense in $G(\mathbb{Z}_l)$ for $i = 1, 2$,

then there is a $g \in G(\mathbb{Z}_l)$ such that $gV_1 = V_2$ (setwise).

Then the conjectures [2.1?, 3.1?, 3.2?, 9.1?] of [21] imply that there are at most finitely many $\mathbb{Z}$-forms of $\text{H}^1_{\text{et}}(A \otimes \mathbb{Z}_l)$.

Conjectures [2.1?, 3.1?, 3.2?, 9.1?] have analogs the cohomology theories with coefficients in the ring of finite adeles $\mathcal{A}$, cf. [21, 11.4?], [11.5?], cf. also [21, 10.2?, 10.6?].

2.3. Standard conjectures and motivic Galois group

Now we try to give a self-contained exposition of several conjectures on motivic Galois group which appear related to our conjectures. Our exposition follows [21, Section 1, Section 3]
Let $k$ be a field of characteristic 0 which embeds into the field $\mathbb{C}$ of complex numbers; pick an embedding $\sigma : k \rightarrow \mathbb{C}$.

Assume Standard Conjectures and Hodge conjecture [14, 17]. Let $\text{Mot}_k$ denote the category of pure motives over $k$ defined with the help of numerical equivalence of algebraic cycles (or the homological equivalence, which should be the same by Standard Conjectures). $\text{Mot}$ is a semi-simple category.

Let $E \in \text{ObMot}$ be a motive; let $\langle E \rangle$ denote the least Tannakian subcategory of $\text{Mot}_k$ containing $E$.

A choice of embedding $\sigma : k \rightarrow \mathbb{C}$ defines an exact fiber functor $\text{Mot}_k \rightarrow \mathbb{Q}\text{-Vect}$ corresponding to the Betti realization

$$H_{\text{sing}\sigma} : \text{Mot}_k \rightarrow \mathbb{Q}\text{-Vect}, E \mapsto H_{\text{sing}}(E_{\sigma}(\mathbb{C}), \mathbb{Q}).$$

The scheme of automorphisms $\text{MGal}_{k,\sigma} = \text{Aut}^\otimes(H_{\text{sing}\sigma} : \text{Mot} \rightarrow \mathbb{Q}\text{-Vect})$ of the functor preserving the tensor product is called motivic Galois group of $k$. It is a linear proalgebraic group defined over $\mathbb{Q}$. Its category of $\mathbb{Q}$-linear representations is equivalent to $\text{Mot}$. The group depends on the choice of $\sigma$.

The motivic Galois group of a motive $E$ is $\text{Aut}^\otimes(H_{\text{sing}\sigma} : \langle E \rangle \rightarrow \mathbb{Q}\text{-Vect})$.

We now list several conjectures from Ref. [21].

Conjecture (2.1?). The group $\text{Aut}^\otimes(H_{\text{sing}\sigma} : \text{Mot}_k \rightarrow \mathbb{Q}\text{-Vect})$ is proreductive, i.e. a limit of liner reductive $\mathbb{Q}$-groups.

Let $1$ denote the trivial morphism of rank 1, i.e. the cohomology of the point Spec $k$.

Conjecture (3.1?). The group $\text{Aut}^\otimes(H_{\text{sing}\sigma} : \langle E \rangle \rightarrow \mathbb{Q}\text{-Vect})$ is the subgroup of $\text{GL}(H_{\text{sing}\sigma}(E))$ preserving the tensors corresponding to morphisms $1 \rightarrow E^\otimes r \otimes E^\vee s \otimes r, s \geq 0$.

It is also conjectured that this group is reductive. Via the comparison isomorphism of étale and singular cohomology,

$$H_{\text{sing}}(E(\mathbb{C}), \mathbb{Q}) \otimes \mathbb{Q}_l = H_{\text{et}}(E \otimes \mathbb{Q}, \mathbb{Q}_l),$$

the $\mathbb{Q}_l$-points of $\text{Aut}^\otimes(H_{\text{sing}\sigma} : \langle E \rangle \rightarrow \mathbb{Q}\text{-Vect})(\mathbb{Q}_l)$ act on the étale cohomology $H_{\text{et}}(E \otimes \mathbb{Q}, \mathbb{Q}_l)$. On the other hand, the Galois group $\text{Gal}(\overline{\mathbb{Q}}/k)$ acts on $\mathbb{Q}$ and therefore on $E \otimes \mathbb{Q}$. By functoriality, the Galois group acts by automorphisms of the functor of étale cohomology. Hence, this gives rise to $\ell$-adic representation associated to $E$

$$\rho_{k,1} : \text{Gal}(\overline{\mathbb{Q}}/k) \rightarrow \text{Aut}^\otimes(H_{\text{sing}\sigma} : \langle E \rangle \rightarrow \mathbb{Q}\text{-Vect})(\mathbb{Q}_l).$$

Conjecture (3.2?). Let $k$ be a number field. The image of the $\ell$-adic representation associated with $E$ is dense in the group $\text{Aut}^\otimes(H_{\text{sing}\sigma} : \langle E \rangle \rightarrow \mathbb{Q}\text{-Vect})(\mathbb{Q}_l)$ in the Zariski topology.

Conjecture (9.1?). Let $k$ be a number field. The image

$$\text{Im}(\rho_{k,1} : \text{Gal}(\overline{\mathbb{Q}}/k) \rightarrow \text{Aut}^\otimes(H_{\text{sing}\sigma} : \langle E \rangle \rightarrow \mathbb{Q}\text{-Vect})(\mathbb{Q}_l))$$

is open in $\text{Aut}^\otimes(H_{\text{sing}\sigma} : \langle E \rangle \rightarrow \mathbb{Q}\text{-Vect})(\mathbb{Q}_l)$.

Conjecture (9.3?). Let $k$ be a number field. $H_{\text{et}}(E \otimes \mathbb{Q}, \mathbb{Q}_l)$ is semi-simple as a $\text{Gal}(\overline{\mathbb{Q}}/k)$-module.

We suggest that the conjectures [2.1?,3.1?,3.2?,9.1?,10.2?,10.3?,10.4?,10.7?,10.8?] may be interpreted as saying there are only finitely many $\mathbb{Z}$-forms of the étale cohomology $H_{\text{et}}(-, \mathbb{Z}_l) : \langle E \rangle \otimes_k \overline{\mathbb{Q}} \rightarrow \mathbb{Q}\text{-Vect}$, up to Galois action. There are similar conjectures for finite adeles instead of $\mathbb{Q}_l$, cf. [21, 11.4?(ii), 11.5?], also [21, 10.2?, 10.6?].
2.4. Speculations and remarks

Standard conjectures claim there are algebraic cycles corresponding to various cohomological constructions. Model-theoretically it should mean that something is definable in ACF and it is natural to expect that such properties be useful in a proof of categoricity, i.e. in the characterization of the $\mathbb{Q}$-forms of étale cohomology theory.

We wish to specifically point out the conjectures and properties involving smooth hyperplane sections, namely weak and strong Lefschetz theorems and Lefschetz Standard Conjecture, cf. [17, p. 11, p. 14]. Weak Lefschetz theorem describes part of the cohomology ring of a smooth hyperplane section of a variety. Perhaps such a description can be useful in showing that a $\mathbb{Q}$-form extends uniquely to $\text{Mot}/K$ from the subcategory $\text{Mot}/\bar{Q}$. An analog of the weak Lefschetz theorem for the fundamental group was used in a similar way in [12, Lemma V.III.3.2.1], see 3.4.3 for some details. Namely, as is well-known, the fundamental group of a smooth hyperplane section of a smooth projective variety is essentially determined by the fundamental group of the variety. Ref. [12, III.2.2] extends this to a somewhat technical weaker statement about arbitrary generic hyperplane sections. An arbitrary variety can be represented as a generic hyperplane section of a variety defined over $\bar{Q}$ and this implies that, in some sense, the fundamental groupoid functor on the subcategory of varieties defined over $\bar{Q}$ “defines” its extension to varieties defined over larger fields. The word “defines” is used in a meaning similar to model theoretic meaning of one first-order language definable in another.

**Question 1.** Find a characterization of the following families of functors:

\[
H_{\text{sing}}(X(K_\tau), \mathbb{Q}) : \text{Var}/K \to \mathbb{Q}\text{-Vect}, \\
H_{\text{sing}}(X(K_\tau), \mathbb{C}) : \text{Var}/K \to \mathbb{Q}\text{-Hodge}
\]

where $\tau$ varies through isomorphisms of $K$ to $\mathbb{C}$, or, almost equivalently, though locally compact locally connected topologies on $K$.

Note that Zilber [22] unconditionally constructs a pseudo-exponential map $\text{ex} : \mathbb{C}^+ \to \mathbb{C}^*$ which satisfies the Schanuel conjecture. Of course, this map is not continuous (not even measurable). Hence we ask:

**Question 2.** Construct a pseudo-singular cohomology theory which satisfies an analog of the Schanuel conjecture and some other conjectures.

2.5. Model theoretic conjectures

Define model theoretic structures corresponding to the cohomology theories.

**Conjecture 2.5.** The field is purely embedded into the structures corresponding to functors

(i) $H_{\text{sing}} : \text{Var}/\bar{Q} \to \mathbb{Q}\text{-Vect}$
(ii) $H_{\text{sing}}(-, \mathbb{Q}) : \text{Var}/\mathbb{C} \to \mathbb{Q}\text{-Vect}$
(iii) $H_{\text{sing}}(-, \mathbb{C}) : \text{Var}/\mathbb{C} \to \mathbb{Q}\text{-Hodge}$

Moreover, the structure (ii) is an elementary extension of (i) and the cohomology ring $H_{\text{sing}}(V, \mathbb{Q})$ is definable for every variety over $\mathbb{C}$.

Several of the Standard Conjectures [17, section 4, p. 11/9] claim that certain cohomological cycles (construction) correspond to algebraic cycles. This feels related to many of the conjectures above, in particular to the purity conjectures.
3. Uniqueness properties of the topological fundamental groupoid functor of a complex algebraic variety

3.1. Statement of the conjectures

Let $\mathcal{V}$ be a category of varieties over a field $K$, let $\pi$ be a functor to groupoids such that $\text{Ob } \pi(X) = X(K)$ is the functor of $K$-points. For $\sigma \in \text{Aut}(K)$, define $\sigma(\pi)$ by

$$\text{Ob } \sigma(\pi) = \text{Ob } \pi(X) = X(K), \sigma(\text{Mor } (x,y)) = \text{Mor } (\sigma(x), \sigma(y)),$$

$$\text{source}(\gamma) = \sigma(\text{source}(\gamma)), \text{target}(\gamma) = \sigma(\text{target}(\gamma)).$$

For $K = \mathbb{C}$, an example of such a functor is the topological fundamental groupoid functor $\pi_1^{\text{top}}(X(\mathbb{C}))$ of the topological space of complex points of an algebraic variety, and $\{\sigma(\pi_1^{\text{top}}) : \sigma \in \text{Aut}(\mathbb{C})\}$ is the family of all the topological fundamental groupoid functors associated with different choices of a locally compact locally connected topology on $\mathbb{C}$. (Such a topology determines a field automorphism, uniquely up to conjugation).

$\text{Aut } (K)$ acts by automorphisms of the source category, hence all these (possibly nonequivalent!) functors have the same properties in the language of functors, in particular

1. Ob $\pi(X) = X(K)$ is the functor of $K$-points of an algebraic variety $X$
2. preserve finite limits, i.e. $\pi(X \times Y) = \pi(X) \times \pi(Y)$
3. $\pi(X)$ is connected if $X$ is geometrically connected (i.e. the set of points $X(K)$ equipped with Zariski topology is a connected topological space)
4. for $f : \tilde{X} \to X$ étale, the map $\pi(\tilde{X}) \xrightarrow{\pi(f)} \pi(X)$ of groupoids has the path lifting property of topological covering maps, namely
   for $x = f(\tilde{x}), \tilde{x} \in \tilde{X}(K)$, for every path $\gamma \in \pi(X)$ starting at $x$, there is a unique path $\tilde{\gamma} \in \pi(\tilde{X})$ such that $\text{source}(\gamma) = \tilde{x}$ and $\pi(f)(\tilde{\gamma}) = \gamma$.
5. whenever $Y$ is connected, $f : X \to Y$ is a proper and separable morphism with geometrically connected fibers satisfying assumptions of Corollary 1.4 [SGA1, X], there is a short exact sequence
   $$p_1(X, x, x) \to p_1(X, x, x) \to \pi_1(Y, f(x), f(x))$$
   where $X_x$ is the geometric fibre over a geometric point $x$ of $X$.

Definition 3.1. A $\pi_1$-like functor over a field $K$ is a functor from a category of varieties to the category of groupoids satisfying (0-4) above. A $\pi_1$-like functor is a $\pi_1$-like functor over some field.
Note that by (0) a \( \pi_1 \)-functor comes equipped with a forgetful natural transformation to the functor of \( K \)-points.

**Conjecture 3.2.** \( (Z(\pi_1^{\text{top}}, \text{Var}_C)) \). Each \( \pi_1 \)-like functor on the category of smooth quasi-projective complex varieties factors through the topological fundamental groupoid functor, up to a field automorphism.

In detail: Let \( \text{Var}_C \) be the category of smooth quasi-projective varieties over the field of complex numbers \( C \). For each \( \pi_1 \)-like functor \( \pi : \text{Var}_C \to \text{Groupoids} \) there is a field automorphism \( \sigma : C \to C \) and a natural transformation \( \varepsilon : \pi_1^{\text{top}} \Rightarrow \pi^\sigma \) such that the induced natural transformation \( \text{Ob } \pi_1^{\text{top}} \Rightarrow \text{Ob } \pi^\sigma \) on the functor of \( C \)-points is identity.

**Conjecture 3.3.** \( (Z(\pi_1, K, \text{Var}_K)) \). Let \( K \) be an algebraically closed field. Let \( \text{Var}_K \) be the category of smooth quasi-projective varieties over \( K \).

There is a functor \( \pi_1 : \text{Var}_K \to \text{Groupoids} \) such that for each \( \pi_1 \)-like functor \( \pi : \text{Var}_K \to \text{Groupoids} \) there is a field automorphism \( \sigma : K \to K \) and a natural transformation \( \varepsilon : \pi_1 \Rightarrow \pi^\sigma \) such that the induced natural transformation \( \text{Ob } \pi_1 \Rightarrow \text{Ob } \pi^\sigma \) on the functor of \( K \)-points is identity.

These conjectures are direct analogs of categoricity conjectures in model theory, hence we shall refer to these conjectures as categoricity conjectures for the fundamental groupoid functor.

**Remark 3.4.** As stated, these conjectures are likely too optimistic. To get more plausible and manageable conjectures, replace \( \text{Var}_K \) by a smaller category and add additional conditions on the \( \pi_1 \)-like functors. The conclusion can also be weakened to claim there is a finite family of functors, rather than a single functor, through which \( \pi_1 \)-like functors factor up to field automorphism.

It may also be necessary to put extra structure on the fundamental groupoids.

**Remark 3.5.** In model theory, it is more convenient to work with universal covering spaces rather than fundamental groupoids. Accordingly, model theoretic results are stated in the language of universal covering spaces, sometimes with extra structure.

The conjectures above are motivated by questions and theorems about categoricity of certain structures.

**Remark 3.6.** It is tempting to think that the right generalization of the conjectures above should make use of the short exact sequence of étale fundamental groups (see [SGA1, XIII.4.3; XII.4.4])

\[
1 \longrightarrow \pi_1^{\text{alg}}(X \times_{\text{Spec } k} \text{Spec } k^{\text{sep}}, x) \longrightarrow \pi_1^{\text{alg}}(X, \bar{x}) \longrightarrow \pi_1(\text{Spec } k, \text{Spec } k^{\text{sep}}) = \text{Gal}(k^{\text{sep}}/k) \longrightarrow 1
\]

where \( X \) is a scheme over a field \( k \), \( k^{\text{sep}} \) is a separable closure of \( k \), and \( x : \text{Spec } k^{\text{sep}} \longrightarrow X \times_{\text{Spec } k} \text{Spec } k^{\text{sep}} \) is a geometric point of \( X \times_{\text{Spec } k} \text{Spec } k^{\text{sep}} \), and \( \bar{x} : \text{Spec } k^{\text{sep}} \longrightarrow X \times_{\text{Spec } k} \text{Spec } k^{\text{sep}} \longrightarrow X \) is the corresponding geometric point of \( X \).

In fact such a sequence could be associated with a morphism \( X \longrightarrow S \) admitting a section and satisfying certain assumptions [SGA 1, XIII.4].

These short exact sequences come from pullback squares

\[
\begin{array}{ccc}
\pi_1^{\text{alg}}(X \times_{\text{Spec } k} \text{Spec } k^{\text{sep}}, x) & \longrightarrow & \pi_1^{\text{alg}}(X, \bar{x}) \\
\downarrow & & \downarrow \\
\pi_1(\text{Spec } k, \text{Spec } k^{\text{sep}}) = \text{Gal}(k^{\text{sep}}/k) & \longrightarrow & 1 \\
\downarrow & & \downarrow \\
X \times_{\text{Spec } k} \text{Spec } k^{\text{sep}} & \longrightarrow & \text{Spec } k^{\text{sep}} \ \\
\downarrow & & \downarrow \\
X & \longrightarrow & \text{Spec } k \\
\downarrow & & \downarrow \\
X_s & \longrightarrow & s \\
\downarrow & & \downarrow \\
X & \longrightarrow & S
\end{array}
\]
3.2. Conjectures within reach

In this section, we give two conjectures which we hope to be within reach. Their proof requires a combination of methods of model theory and algebraic geometry. In the next subsection we list several partial positive results towards conjectures above which are implicit in model theoretic literature.

We find the following conjecture plausible and hope its statement clarifies the arithmetic nature of our conjectures. It is perhaps the simplest conjecture not amendable to model theoretic analysis because it uses bundles, a notion from geometry rather than model theory.

For a variety \( X \), let \( (X)_K \) denote the category whose objects are the finite Cartesian powers of \( X \), and morphisms are morphisms of algebraic varieties defined over \( K \); we let \( X^0 \) denote a variety consisting of a single \( K \)-rational point.

Conjecture 3.7. \((Z(\pi_1, L_A^\lambda))\). Let \( K \) be an algebraically closed field of zero characteristic, \( A \) an Abelian variety defined over a number field \( k \). Let \( L_A \) be an ample line bundle over \( A \) and \( L_A^\lambda \) be the corresponding \( \mathbb{G}_m \)-bundle. Further assume that the Mumford-Tate group of \( A \) is the maximal possible, i.e. the general symplectic group,

\[
\text{MT}(A) = \text{GSp}_Z
\]

and that the image of Galois action on the torsion has finite index in the group of \( \hat{\mathbb{Z}} \)-points of the symplectic group.

Then there is a finite family of \( \pi_1 \)-like functors \( \Pi_1 \) such that each \( \pi_1 \)-like functor on the full subcategory \( (L_A^\lambda)_K \) consisting of the Cartesian powers of the \( \mathbb{G}_m \)-bundle \( L_A^\lambda \), factors via an element of \( \Pi_1 \).

These functors in \( \Pi_1 \) correspond to different embeddings of the field of definition of \( A \) into the field of complex numbers.

The following conjecture is probably within reach, at least if we replace the fundamental groupoid functor by its residually finite part.

Model theoretic methods of [5] are likely enough to show that it is enough to prove this conjecture for a countable algebraically closed subfield. Methods of Ref. [12, III.5.4.7], cf. Section 3.4.3 and particularly Statement 3.13, are likely enough to reduce the remaining part of the conjecture to certain properties of complex analytic topology and normalization of varieties.

Conjecture 3.8. \((Z(\pi_1, \mathbb{Q} \subset \mathbb{C}, \text{Var}_\mathbb{Q} \subset \text{Var}_\mathbb{C}))\). Let \( \text{Var}_\mathbb{C} \) be the category of smooth quasi-projective varieties over \( \mathbb{C} \), and let \( \text{Var}_\mathbb{Q} \) be its subcategory consisting of varieties and morphisms defined over \( \mathbb{Q} \).

Assume that \( \pi : \text{Var}_\mathbb{C} \longrightarrow \text{Groupoids} \) is a \( \pi_1 \)-like functor which coincides with the topological fundamental groupoid \( \pi_1^{\text{top}} \) for varieties and morphisms defined over \( \mathbb{Q} \), i.e. for each variety \( V \) in \( \text{Var}_\mathbb{Q} \) and each morphism \( V \xrightarrow{f} W \) in \( \text{Var}_\mathbb{Q} \) it holds \( \pi(V) = \pi_1^{\text{top}}(V) \) and \( \pi(f) = \pi_1^{\text{top}}(f) \).

Then there exist a field automorphism \( \sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q}) \) such that \( \pi^\sigma \sigma \) and \( \pi_1^{\text{top}} \) are equivalent.

There are a number of theorems and conjectures which can be seen as saying that, up to finite index, the Galois action is described by geometric, algebraic or topological structures; our conjectures can also be seen in this way.

3.3. Partial positive results

These conjectures are closely related to categoricity theorems in model theory, and this has led to several partial positive results uniqueness of \( \pi_1 \)-like functor restricted to certain full subcategories of the category of varieties over an algebraically closed field: the full subcategories \( \langle K^* \rangle \) of algebraic tori in arbitrary characteristic, \( \langle E \rangle \) powers of an elliptic curve over a number field, a weaker
result about \( \langle A \rangle \) powers of an Abelian variety over a number field, a still weaker result about \( \langle V \rangle \) powers of a smooth projective variety whose fundamental group satisfies a group theoretic property of being subgroup separable, a strengthening of residually finite.

Note that the first three categories are linear in the sense that the groups \( \text{Aut}(K^*) \), \( \text{Aut}_{\text{End}}(E(K)) \), and \( \text{Aut}_{\text{End}}(A(K)) \) act on the set of \( \pi_1 \)-like functors on the respective categories \( \langle K^* \rangle \), \( \langle E \rangle_K \), and \( \langle A \rangle_K \). This is so because these groups act on these categories.

Let us now list several statements translated from categoricity theorems available in model theory literature. We give references but do not explain the translation.

A reader familiar with algebraic geometry may find that there is a better way to formulate these results. In the case of a countable field \( K \), their proofs do not require elaborate techniques of model theory, and such a reader may find it easier to reprove these results in a familiar language while only drawing inspiration from the proofs in the literature.

For a variety \( V \) over a field \( K \), let \( \langle V \rangle_K \) denote the full subcategory of the category of varieties defined over \( K \) whose objects are all finite Cartesian powers of \( V \) and a \( K \)-rational point.

**Statement 3.9** ([8, Th. 2.1; 13]). Let \( K \) be an algebraically closed field of char \( K = 0 \). Then there is a \( \pi_1 \)-like functor \( \pi \) over \( K \) defined on the category \( \langle K^* \rangle_K \) of algebraic tori such that for any \( \pi_1 \)-like functor \( \pi' \) over \( K \) defined on the same category there is a field automorphism \( \sigma : K \rightarrow K \) and a natural transformation \( \pi \Rightarrow \sigma(\pi') \) such that the induced natural transformation \( \text{Ob}\pi \Rightarrow \text{Ob}\sigma(\pi') \) is identity.

Similarly, **Statement 3.10** is a particular case of Conjecture 3.3((\( Z(\pi_1, K, \text{Var}_K) \)) when we take \( \text{Var}_K = \langle K^* \rangle_K \) to be the category of algebraic tori. Note this implies Conjecture 3.2((\( Z(\pi_1, K, \langle E \rangle_C) \)) weakened in a similar way.

**Statement 3.10.** ([9, Th. 4.4.1; 13, Prop.2]). Let \( K \) be an algebraically closed field of char \( K = 0 \), and let \( E \) be an elliptic curve defined over a number field \( k \) with a \( k \)-rational point \( 0 \in E(k) \).

Then there are finitely many \( \pi_1 \)-like functors \( \pi^1, ..., \pi^n \) on the category \( \langle E \rangle_K \) such that for any functor \( \pi \) such that

\[
-\pi(E, 0, 0) \approx \mathbb{Z}^2
\]

there is a field automorphism \( \sigma : K \rightarrow K \) and a natural transformation \( \pi \Rightarrow \sigma(\pi^i) \) for some \( i \) such that the induced natural transformation \( \text{Ob}\pi \Rightarrow \text{Ob}\sigma(\pi^i) \) is identity.

**Statement 3.11** is a particular case of Conjecture 3.3((\( Z(\pi_1, K, \langle E \rangle_K) \)) when we take \( K \) to be an algebraically closed field of prime characteristic, \( \text{Var}_K = \langle K^* \rangle_K \) to be the category of algebraic tori, and weaken the conclusion by placing additional restrictions on \( \pi_1 \)-functors.

These restrictions are necessary because there are too few automorphisms of the algebraic closure \( \overline{F}_p \) of the field with \( p \) elements. However, we may view this as a hint towards Remark 3.6 suggesting we should replace below \( \text{Aut}(K/\overline{F}_p) \) by the Etale fundamental group of an appropriate scheme over \( \overline{F}_p \).

**Statement 3.11.** ([8, Th. 2.2]). Let \( K \) be an algebraically closed field of char \( K = p \). Then there is a \( \pi_1 \)-like functor \( \pi_1 : \langle K^* \rangle \rightarrow \text{Groupoids} \) such that each \( \pi_1 \)-like functor \( \pi : \langle K^* \rangle \rightarrow \text{Groupoids} \) factors via \( \pi_1 : \langle K^* \rangle \rightarrow \text{Groupoids up to } \text{Aut}(K/\overline{F}_p) \) provided
• $\pi(K^*, 1, 1) \simeq \mathbb{Z}[1/p]$

• the restrictions $\pi_1|_{\overline{\mathbb{F}}_p}$ and $\pi_i|_{\overline{\mathbb{F}}_p}$ to $\overline{\mathbb{F}}_p$-rational points coincide:

$$\pi_i|_{\overline{\mathbb{F}}_p} = \pi'_i|_{\overline{\mathbb{F}}_p}$$

Statement 3.12 is a particular case of Conjecture 3.3 ($Z(\pi_1, K, \langle A \rangle_K)$), for $A$ is an Abelian variety over a number field, weakened by requiring the fundamental group (rather than groupoid) functors of the $\pi_1$-like functors to coincide.

**Statement 3.12.** (BaysDPhil, Th.4.4.1). Let $K$ be an algebraically closed field of char $K = 0$. Let $A$ be an Abelian variety defined over a number field $k$ with a $k$-rational point $0 \in A(k)$.

Then there is a $\pi_1$-like functor $\pi_1 : \langle A \rangle \to \text{Groupoids}$ such that each $\pi_1$-like functor $\pi : \langle A \rangle \to \text{Groupoids}$ factors via $\pi_1 : \langle K^* \rangle \to \text{Groupoids}$ up to $\text{Aut}(K/\mathbb{Q}(A_{\text{tors}}))$ whenever $0$.

• $\pi(A, 0, 0) = \mathbb{Z}^{2\dim A}$

• for any two functors $\pi, \pi'$ in $\mathcal{F}$, the corresponding fundamental group functors coincide

$$\pi(A, 0, 0) = \pi'(A, 0, 0)$$

and further, for $\varphi : \tilde{A} \to A$ is étale, $\gamma \in \pi(A, 0, 0) = \pi'(A, 0, 0)$, $\pi(\gamma_{\pi}) = \pi'(\gamma_{\pi'}) = \gamma$, it holds that

source$(\gamma_{\pi}) = \text{source}(\gamma_{\pi'})$ implies target$(\gamma_{\pi}) = \text{target}(\gamma_{\pi'})$.

Statement 3.13 is a particular case of Conjecture 3.8 ($Z(\pi_1, \mathbb{Q} \subset \mathbb{C}, \mathcal{V}_{\text{ar}} \subset \mathcal{V}_{\text{arC}})$) when $\mathcal{V}_{\text{ar}} = \langle V \rangle_\mathbb{Q}$ and $\mathcal{V}_{\text{arC}} = \langle V \rangle_\mathbb{C}$ are the subcategories generated by a variety with certain properties.

Recall a group $G$ is subgroup separable iff for each finitely generated subgroup $H < G$ and $h \notin H$ there is a morphism $f : G \to G_H$ into a finite group $G_H$ such that $h \notin f(H)$.

**Statement 3.13.** ([12, III.5.4.7]). Let $K$ be an algebraically closed field of char $K = 0$ and of the least uncountable cardinality $\aleph_1$. Let $V$ be an smooth projective variety defined over a number field $k$ with a $k$-rational point $0 \in V(k)$ such that the universal covering space of $V(\mathbb{C})$ is holomorphically complex, for some embedding $K \to \mathbb{C}$, and its fundamental groups $\pi_1(V(\mathbb{C}), 0)$ are subgroup separable for each $n > 0$.

Then there is a $\pi_1$-like functor $\pi_1 : \langle V \rangle \to \text{Groupoids}$ such that a $\pi_1$-like functor $\pi : \langle V \rangle \to \text{Groupoids}$ factors via $\pi_1 : \langle V \rangle \to \text{Groupoids}$ up to $\text{Aut}(K/\mathbb{Q})$ provided

• $\pi(V, 0, 0) \simeq \pi_{1\text{top}}(V(\mathbb{C}), 0, 0)$

• the restrictions $\pi_1|_{\mathbb{Q}}$ and $\pi'|_{\mathbb{Q}}$ to $\mathbb{Q}$-rational points coincide:

$$\pi_1|_{\mathbb{Q}} = \pi'|_{\mathbb{Q}}$$

We wish to mention the work of Refs. [10, 15] and particularly [11] on Shimura curves, which does not quite fit in our framework. To interpret their results, one needs to consider $\pi_{1\text{top}}$ as a functor to groupoids with extra structure.

**Remark 3.14.** The conjectures on independence of Galois representations of non-isogenous curves likely imply our conjectures for the full subcategory $\langle E_1 \times \cdots \times E_n \rangle_K$ generated by a finite product of elliptic curves $E_1, \ldots, E_n$ over a number field.

Consider the family of $\pi_1$-like functors with Abelian fundamental groups.

Is it true that each such “abelianised” $\pi_1$-like functor factors via $\pi_{1\text{top}}$ up to a field automorphisms? Is this equivalent to Conjecture 3.2 ($Z(\pi_{1\text{top}}, \text{AbVar})$) for the subcategory $\text{AbVar}$ of all Abelian varieties?
3.4. Mathematical meaning of the conjectures: Elements of proof of the conjectures

Here, we try to explain the arithmetic and geometric meaning of the conjectures. In a sense, the conjectures say that \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) and \( \text{Aut}(K/\mathbb{Q}) \) are large enough. We try to show below in what sense, by showing possible obstructions/difficulties in proof.

3.4.1. Galois action on roots of Unity and Kummer theory

Consider the infinite sequence \( \exp(2\pi i/n) \) of roots of unity. This sequence can be obtained topologically: take the loop \( \gamma \) generating \( \pi(\mathbb{C}^*, 1, 1) \approx \mathbb{Z} \), the étale morphism \( z^n : \mathbb{C}^* \to \mathbb{C}^* \) and lift \( \gamma \) uniquely to a path \( \tilde{\gamma}_n \) starting at \( 1 \in \mathbb{C}^* \). Then \( \exp(2\pi i/n) \) is the end-point of \( \gamma_n \). This construction shows that a \( \pi_1 \)-like functor on the category \( \langle K^* \rangle \) determines a distinguished sequence \( \xi_n, n \geq 0 \) of roots of unity. Hence, our conjectures require that the Galois group acts transitively on the set of sequences of roots of unity associated with \( \pi_1 \)-like functors.

Consider a \( \pi_1 \)-like functor on the category \( \langle K^* \rangle \). As noted above, group automorphisms \( \text{Aut}(K^*) \) of \( K^* \) act on the set of these functors. Hence, item (9) requires that multiplicative group automorphisms \( \text{Aut}(K^*) \) and field automorphisms \( \text{Aut}(K/\mathbb{Q}) \) have the same orbits on the sequences \( \xi_n, (\xi_{mn})^m = \xi_n, m, n > 0 \) of roots of unity.

Kummer theory arises in a similar way if we consider endpoints of liftings of paths joining 1 and arbitrary elements \( a_1, \ldots, a_n \).

3.4.2. Elliptic curves and Abelian varieties: Kummer theory and Serre’s open image theorem for elliptic curves

Kummer theory for elliptic curves and Abelian varieties arises in the same way if we consider \( \pi_1 \)-like functors on the category \( \langle A \rangle \) generated by an Abelian variety.

Similarly, our conjectures about \( \pi_1 \)-like functors on \( \langle A \rangle_K \) require that the action of \( \text{Aut}_{\text{End}A-\text{mod}}(A(K)) \) and \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) on the torsion points do not differ much. This is true for elliptic curves but fails for Abelian varieties of dim \( A > 1 \), hence the extra assumption in (4) on the family of \( \pi_1 \)-like functors.

3.4.3. Arbitrary variety: Étale topology and an analog of Lefshetz theorem for the fundamental group

To prove Statement 3.13, we need several facts about étale topology. Most of these facts are well-known for smooth varieties; what we use is that they hold “up to finite index” for arbitrary (not necessarily smooth or normal) subvarieties of a smooth projective variety.

Consider the inverse limit \( \lim \tilde{V}(\mathbb{C}) \) of finite étale covers \( \tilde{V}(\mathbb{C}) \to V(\mathbb{C}) \) of a complex algebraic variety \( V \). The universal analytic covering map \( U \to V(\mathbb{C}) \) gives rise to covering maps \( U \to \tilde{V}(\mathbb{C}) \) and hence a map \( U \to \lim \tilde{V}(\mathbb{C}) \). Zariski topology on the étale covers makes \( \lim \tilde{V}(\mathbb{C}) \) into a topological space. Hence there are two topologies on \( U – \) the complex analytic topology and the “more algebraic” topology on \( U \) induced from the map \( U \to \lim \tilde{V}(\mathbb{C}) \). Call the latter étale topology on \( U \).

**Definition 3.15.** The étale topology on the universal covering space of the topological space of complex points of an algebraic variety is defined as the topology induced from the map to the inverse limit of the spaces of complex points of finite étale covers of the variety equipped with the Zariski topology. The étale topology on a covering space of the topological space of complex points of an algebraic variety is defined similarly.

To prove Statement 3.13 we use that these two topologies are similar and nicely related. In particular,
Lemma 3.16. Assume $V$ satisfies the assumptions of Statement 3.13.

- Closed irreducible sets in étale topology are closed irreducible in complex analytic topology (by definition).

- For a set closed in étale topology, its irreducible components in complex analytic topology are also closed in étale topology [12, III.1.4.1(4,5)].

- The image of an étale closed irreducible subset of $U \times \cdots \times U$ under a coordinate projection is étale closed [12, III.2.2.1].

Note that this is easy to see that connected components of a set closed in étale topology are also closed in étale topology, and hence that the properties above holds for smooth or normal closed sets.

Let $f : W \to V$ be a morphism of varieties, and let $f_* : U_W \to U_V$ be the map of the universal covering spaces of $W(\mathbb{C})$ and $V(\mathbb{C})$. We may assume that $V$ is smooth projective but it is essential that $W$ is arbitrary. In applications, $W$ is an arbitrary closed subvariety of a Cartesian power of a fixed variety $V$.

Lemma 3.17. Under the assumptions above, if $f : W \to V$ is proper, then the image $f(U_W)$ is closed in $U_V$ in étale topology.

This is related to the following well-known geometric fact:

Lemma 3.18.

- If $f : W(\mathbb{C}) \to V(\mathbb{C})$ is a morphism of smooth normal algebraic varieties, $g$ a generic point of $V(\mathbb{C})$ and $W_g = f^{-1}(g)$ then

$$\pi_1(W_g, w, w) \to \pi_1(W, w, w) \to \pi_1(V, g, g)$$

is exact up to finite index.

- Moreover, if $f(W(\mathbb{C}))$ is dense in $V(\mathbb{C})$, then $\pi_1(W, w, w) \to \pi_1(V, g, g)$ is surjective.

In fact we need to use a generalization of this, namely that it holds up to finite index for arbitrary varieties if one considers the image of the fundamental group in the ambient smooth projective variety, see [12, V.3.3.6, V.3.4.1] for details.

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