1 Introduction

Despite recent trends towards ever larger computers, I believe that accurate and reliable lattice simulations of QCD are possible on quite modest computers, possibly even on the workstation or PC sitting on your desk. If true, this will have a profound effect on the way we deal with strong interactions, both experimentally and theoretically. In these lectures I outline the reasons behind my belief.

The central issue is the reliability of lattice simulations of QCD. Lattice simulations, like experiments, have two sources of error: statistical errors due to our use of Monte Carlo methods for doing the simulations; and systematic errors due to the various approximations we make to simplify the theory to the point that it is numerically tractable. Statistical errors are pretty much understood, and under control; I do not discuss them further in these lectures. Instead, I concentrate on systematic errors. It is the control of systematic errors that determines the scale and reliability of contemporary QCD simulations.

I begin in Section 2 with a discussion of the main sources of systematic error in lattice simulations. Of these errors those due to the finite lattice spacing are the most important. I describe a general strategy for systematically improving lattice lagrangians to remove lattice-spacing errors. This is an alternative to simply reducing the lattice spacing that promises to be much more efficient. This strategy relies upon weak-coupling perturbation theory, and so in Section 3 I discuss the extent to which perturbation theory can be trusted in lattice QCD. Contrary to much conventional lore lattice perturbation theory seems to work well even at distances as large as 1/2 fm. In Section 4 I discuss another obstacle to the perturbative improvement of lattice QCD. The problem is with

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“tadpole” terms that arise in perturbation theory when computing corrections to the QCD lagrangian. The tadpoles are usually too large to be dealt with perturbatively. All such contributions have a common origin, and I describe a simple nonperturbative procedure for dealing with them. In Section 5 I illustrate all of these techniques with recent simulation results for the \( \Upsilon \) family of mesons — an area where lattice QCD has already largely succeeded. Finally, in Section 6, I summarize the central result of these lectures.

2 Systematic Errors in Lattice QCD

There are three important sources of systematic error in lattice QCD:

a) finite volume errors that arise because simulations cover only a finite volume \( V = L^4 \) in space-time;

b) errors due to extrapolations from large quark masses, where the simulations are easy, down to realistic quark masses for \( u \) and \( d \) quarks, or, equivalently, from large \( m_\pi^2 \propto m_{u,d} \) to \( m_\pi^2 = (140 \text{ MeV})^2 \);

c) errors due to the finite lattice spacing \( a \) in the grid used to approximate continuous space-time.

The central question in numerical QCD then is: How large must \( L \) be, and how small \( m_\pi \) and \( a \), for simulations to be realistic (and useful)? The answer to this question determines the cost and feasibility of lattice simulations.

The cost of a lattice simulation is at least proportional to the number of sites in the grid,

\[
\text{number of sites} = \left( \frac{L}{a} \right)^4.
\]

Usually, however, it is more costly because of a phenomenon called “critical slowing down” that tends to slow the convergence of the iterative algorithms used in lattice simulations. A more realistic estimate of the cost is

\[
\text{cost} \propto \left( \frac{L}{a} \right)^4 \left( \frac{1}{a^2} \right)^\alpha \left( \frac{1}{m_\pi^2 a} \right)^\beta
\]

where typically \( \alpha \) and \( \beta \) are somewhere in the range 0–1 (usually closer to 1 than 0). Obviously this cost is highly sensitive to the values of \( a \), \( m_\pi \), and \( L \); for example, in the worst case, the cost is proportional to the seventh power of \( 1/a \). Thus it is critically important to make \( a \) and \( m_\pi \) as large as possible, and \( L \) as small as possible. We cannot afford to be sloppy or to “play it safe.”

Here I discuss each of these systematic errors in turn. However my emphasis is on the finite-\( a \) errors as these are by far the most crucial determinants of the cost (see Eq. (2)), and they are also the most misunderstood of the systematic errors.
2.1 Finite $L$

Clearly when we simulate a hadron, like the pion or the proton, that is roughly 2 fm in diameter, we must have $L$ at least as large as 2 fm. The nature of the systematic error when $L \geq 2$ fm is easily understood. Lattice simulations generally use periodic boundary conditions to define the differential operators in the action. This means that one is never studying just a single hadron on a grid, but rather an infinite crystal of hadrons consisting of the hadron and images of it, induced by the boundary conditions, at intervals of $L$ in each direction. The properties of our hadron are affected by interactions with all of its images. Its energy, for example, is shifted by the interaction energy between the hadron and its images. The interaction energy between two hadrons a distance $r$ apart is

$$K \frac{e^{-m_\pi r}}{r}.$$  \hspace{1cm}(3)

Thus the finite volume error falls off exponentially quickly with increasing $L$ provided $L$ is large enough. This will not be the case when $L$ is so small that the hadron overlaps with its image; then one finds powerlaw effects related to the geometrical volume of the overlap. But for $L \geq 2$ fm we expect

$$\text{finite } L \text{ errors } \propto e^{-m_\pi L} \approx e^{-L/1 \text{ fm}}$$  \hspace{1cm}(4)

This formula suggests that finite volume errors for hadrons ought to be 10% or less for $L$’s of order 2 fm or larger, and this seems to be the case in current simulations. Even if 2 fm is too small, the exponentially-falling errors vanish so quickly with increasing $L$ that it is almost certain that 2.5 or 3 fm will suffice. The combination of exponentially small errors and powerlaw cost means that finite-volume effects are both tractable and fairly well understood.

2.2 Large $m_\pi$

The numerical intractability of small quark masses means that simulations rarely have pion masses smaller than about 500 MeV. (Remember that $m_\pi^2 \propto m_{u,d}$ is a measure of the quark mass.) As a consequence, it is standard practice in lattice simulations to present results as functions of $m_\pi$, extrapolating down to the physical mass (140 MeV). The reliability of this procedure depends upon how sensitive the simulation results are to the pion mass.

We have, in chiral perturbation theory, a reasonably reliable tool for exploring the $m_\pi$ dependence of different quantities. With chiral perturbation theory we can compute the $m_\pi$ dependence of various quantities. Some, like the nucleon mass, are quite sensitive to $m_\pi$ when $m_\pi \approx 500$ MeV. Others, like the pion decay constant $f_\pi$, are relatively insensitive. This suggests that extrapolations in $m_\pi$ ought to work well for $f_\pi$, while extrapolations of the nucleon mass could lead to problems.
Chiral perturbation theory can help us decide which quantities are least sensitive to $m_\pi$, and we might wish to concentrate our attention on these. More importantly, we should push on to smaller $m_\pi$’s. Large $m_\pi$ errors are typically be of order
\[
\text{large } m_\pi \text{ errors } \approx \left( \frac{m_\pi}{700 \text{ MeV}} \right)^2.
\]
(5)
This suggests that $m_\pi$’s of order 250 MeV or less ought to result in systematic errors that are less than 10%. Given that the cost grows only as $1/m_\pi^2$, it should be possible to work at these lower masses, especially if we are able to use significantly larger lattice spacings.

2.3 Finite $a$ — Classical Field Theory

In thinking about errors due to the finite lattice spacing, I find it useful to look first at a classical field theory on a lattice. Consider, for example, Poisson’s equation:
\[
\partial^2 \phi(r) = \rho(r).
\]
(6)
On a grid, we replace derivatives by differences, for example
\[
\partial^2_x \phi(r) \to \frac{\phi(r + a\hat{x}) - 2\phi(r) + \phi(r - a\hat{x})}{a^2},
\]
(7)
and specify $\phi$ only at the nodes of a lattice with spacing $a$.

Obviously $a$ must be small compared with the size of any important structure in $\rho$ or $\phi$. If this is the case, the finite-$a$ errors fall off like a power of $a$. This is easily demonstrated. The difference operator that replaces the second derivative is defined by
\[
a^2 \Delta^{(2)}_x \phi(r) = \phi(r + a\hat{x}) - 2\phi(r) + \phi(r - a\hat{x})
= \left( e^{a\partial_x} - 2 + e^{-a\partial_x} \right) \phi(r).
\]
(8)
Expanding the exponentials, we find that
\[
\Delta^{(2)}_x \phi(r) \approx \left( \partial^2_x + \frac{a^2 \partial^4_x}{12} + \cdots \right) \phi(r).
\]
(9)
Thus by replacing $\partial^2$ with $\Delta^{(2)}$ we make errors of order
\[
\text{finite } a \text{ errors } \approx \frac{a^2 \overline{p}^2}{12}
\]
(10)
where $\overline{p}$ is the typical wavenumber (ie, momentum) in the Fourier transform of $\phi(x)$.

There are two lessons to learn from this example. The first is that only a small number of lattice points is needed to obtain precision of order 10%. For
example, if \( \phi \) consists of a single smooth bump of width \( w \), then a lattice spacing \( a \approx w/3 \) gives roughly 10% accuracy: \( \bar{\mu} \approx 2\pi/\lambda \), where \( \lambda \approx 2w \approx 6a \), implies that \( a^2\bar{\mu}^2/12 \approx 0.08 \). A general rule of thumb is that it takes roughly \( 3^d \) grid points per \( d \)-dimensional bump to begin to model that bump accurately on a grid.

The second lesson is that one should not reduce the lattice spacing much below \( w/3 \) when trying to improve the accuracy of the grid approximation. This is because reducing the lattice spacing greatly increases the cost of the simulation while only modestly improving the errors. For example, to reduce 10% errors to 1% errors using our discretization of the three-dimensional Poisson’s equation one would have to decrease \( a \) by a factor of 3, thereby increasing the number of grid points by a factor of \( 3^3 = 27 \). Taking account of critical slowing down, the cost of the simulation would increase by a factor of 80–200.

A far more efficient way to reduce errors is to improve the discretization, keeping the lattice spacing large. For example, from our discussion above,

\[
\partial_x^2 = \Delta^{(2)}_x - \frac{a^2\partial_x^4}{12} - \frac{a^4\partial_x^6}{360} - \cdots
\]

\[
= \Delta^{(2)}_x - \frac{a^2(\Delta^{(2)}_x)^2}{12} + \frac{a^4\partial_x^6}{90} + \cdots
\]

and this implies that the lattice equation

\[
\sum_i \left( \Delta^{(2)}_i - \frac{a^2(\Delta^{(2)}_i)^2}{12} \right) \phi(x) = \rho(x)
\]

is accurate up to corrections of order \( (a\bar{\mu})^4/90 \). This improved lattice equation gives roughly 1% accuracy with \( a = w/3 \). The added complexity of the equation increases the cost of simulating, but only by a factor of 1.5 or 2.

Thus for classical fields we conclude that coarse lattices are optimal — perhaps \( 3^d \) grid points per bump in the answer. High precision is obtained by improving the lattice operators from which the field equations are constructed, not by reducing the lattice spacing.

**Exercise:** Compare exact derivatives with lattice derivatives for \( \phi(x) = \exp(-x^2/2) \) in one dimension. Try lattice spacings of \( a = 0.5 \) and 1, and study the second derivative at \( x = 0 \). Compare the errors obtained using the leading-order lattice derivative and the improved derivative. Check how these errors scale with \( a \).

The lattice derivatives can be corrected to all orders in \( a \) by continuing the process outlined above. Compute the error in the infinite-order lattice approximation of \( \partial_x^2 \phi(0) \). In general, for what sorts of function \( \phi(x) \) will the error be exponentially small, power-law suppressed, or identically zero? (Hint: Use Fourier transforms. Fourier transforming the lattice operators is simple. In particular, the infinite-order lattice approximation to \( \partial_x^2 \) has transform \( -k^2 \), where \( k \) is the wavenumber.)
This last exercise shows that our lattice derivatives can never be perfect; typically there are residual non-powerlaw differences between them and continuum derivatives. Examine the possibility of removing even these errors when $r \gg a$ in our lattice Poisson’s equation with a source $\rho(r)$ that is localized around $r = 0$. In particular, consider adding local counterterms to the source: $\rho(r) \rightarrow \rho + c_0 \delta(r) + c_2 \partial^2 \delta(r) + \cdots$. (Note the analogy with multipole expansions.)

**Exercise:** Compute the ground-state energy of a nonrelativistic particle trapped in a box of length $L$ using the Shrödinger equation with discretized derivatives of the sort discussed above. (This is easily done analytically using Fourier series for the eigenfunctions.) Verify the “3 points per bump” rule and explore the utility of improving the discretization as we did above.

As a variation, consider adding a harmonic potential to the hamiltonian and solving the eigenvalue problem numerically. Defined on a grid, the hamiltonian operator becomes a matrix, and the energy eigenvalue problem a simple matter of determining the eigenvalues of that matrix. Again explore the relative merits of reducing the lattice spacing versus improving the discretization. Try the problem in 2 or more dimensions as well.

### 2.4 Finite $a$ — Quantum Field Theory

The analysis of finite-$a$ errors for quantum theories is complicated by the fact that quantum fields are rough at all length scales. This roughness is due to quantum fluctuations. In perturbation theory, it is responsible for the ultraviolet sensitivity and infinities of loop diagrams. More generally, it seems to call into question the validity of a discrete approximation. Indeed it is hard to imagine defining continuum derivatives for an infinitely rough field, let alone discrete approximations to them.

Somewhat surprisingly, finite-$a$ errors in quantum theories are rather similar to those in classical theories. This is because only the long-wavelength structure is physical on a lattice. In general, any long-wavelength, low-momentum probe is sensitive only to fields averaged over a region of order the size of the probe. This averaging smooths out quantum fluctuations. Consequently the long-wavelength or infrared behavior of a quantum theory is insensitive to the details of its short-wavelength or ultraviolet behavior; and thus there are infinitely many theories, each with different ultraviolet dynamics, that give identical or nearly identical infrared physics. We take advantage of this fact when we construct our lattice theory: we replace the correct ultraviolet dynamics of the continuum theory with the incorrect, but numerically tractable, dynamics of the lattice. By changing its bare coupling constants, we are able to adjust the dynamics of the lattice theory so that its infrared behavior is the same as in continuum up to corrections that vanish with the lattice spacing $a$ — just as in classical lattice theories.
To see how this works consider a $\phi^4$ field theory defined on a lattice:

$$L_{\text{lat}} = -\frac{1}{2} \sum_\mu \phi \Delta^{(2)} \phi + \frac{1}{2} m^2 \phi^2 + \frac{1}{4!} \lambda \phi^4$$

(13)

This lagrangian, used in a path integral, defines a euclidean lattice approximation to the continuum quantum theory. Classically, large distortions due to the grid are restricted to the ultraviolet modes; but, in the quantum theory, the infrared sector is strongly affected as well since it couples to the ultraviolet modes through quantum fluctuations (i.e., loops in perturbation theory). For example, distortion of the ultraviolet modes changes the mass renormalization, thereby shifting the renormalized mass of the $\phi$ particle—a significant modification of the infrared behavior of the theory.

Introducing a grid causes $O(1)$ changes in the behavior of the quantum theory at all length scales, but the continuum behavior at large distances can always be restored by shifting or “renormalizing” the bare coupling constants. The renormalized mass of the $\phi$ particle, for example, is easily restored to its proper value by adjusting the bare mass $m$ in the lattice action. The grid is nothing more than an ultraviolet cutoff, restricting the path integral to momenta smaller than $\pi/a$. Renormalization theory tells us that the errors caused in, say, a scattering amplitude by introducing a finite cutoff $\Lambda$ are suppressed by powers of $p_{\text{ext}}/\Lambda$, where $p_{\text{ext}}$ is the largest external momentum in the amplitude. This is true provided the coupling constants of the theory are shifted appropriately from their continuum values. Thus, for our lattice $\phi^4$ theory, there are particular values of $m$ and $\lambda$ such that the lattice result for any physical quantity is the same as in the continuum theory up to corrections that vanish as powers of $a p$, where $p$ is a momentum associated with the classical (as opposed to quantum) size and scales relevant to that quantity. Generally the values needed for $\lambda$ and $m$ depend upon the lattice spacing $a$ (the couplings are said to “run” with varying $a$).

The problem of dealing with the effects of complex short-distance structure on long-distance behavior arises in many contexts, both quantum and classical. A classical example is the formulation of electrostatics inside a dielectric medium (a piece of glass, for example). The microscopic electric field inside a dielectric is very complicated and fluctuates rapidly even over distances as short as 1 Å. Usually, however, it is not the microscopic field that is physically relevant. Any macroscopic probe of the electric field sees only the average field, averaged over a large region that includes many atoms. This averaging smoothes out the rapid fluctuations, producing a relatively smooth macroscopic field that is described by simple equations. The macroscopic equations are universal in form; the type and nature of the atoms making up the dielectric are largely irrelevant. The only information needed about the microscopic structure of the material is contained in the numerical value of its dielectric constant. The dielectric constant is analogous to the bare couplings in our lattice theory; and
the standard formalism for dielectrics comes from a straightforward application of renormalization ideas to the classical problem.

Despite quantum fluctuations, the finite-\(a\) errors in our quantum theory are quite similar to those in the classical theory. Also as in the classical case, we can correct the theory to systematically reduce these errors. Renormalization theory tells us that all errors of order \((a^n)^n\) can be removed by adding local interactions of dimension \(4 + n\) and lower to the lattice lagrangian with appropriate couplings (and readjusting any couplings already present in the lagrangian). The correction terms are local because they are correcting for the physics at distances shorter than the lattice spacing. Only terms that preserve the symmetries of the theory need be added, and so there aren’t very many for small \(n\).

Our lattice \(\phi^4\) theory, for example, has no possible correction term of dimension five. This means that the leading finite-\(a\) errors are quadratic in \(a\). There are only two dimension six correction terms, and so the \(O(a^2)\) errors in \(L_{\text{lat}}\) are removed by adding

\[
\delta L_{\text{lat}} \equiv \frac{a^2 c(a)}{24} \sum_\mu \phi (\Delta^{(2)}_\mu) \phi + \frac{a^2 d(a)}{6!} \phi^6
\]  

(14)

and by further shifting \(m(a)\) and \(\lambda(a)\).

**Exercise:** There are actually two other dimension-6 operators that are consistent with the symmetries of the lattice \(\phi^4\) theory: \( (\Delta^{(2)} \phi)^2\) and \(\phi^3 \Delta^{(2)} \phi\). We do not include these in our corrected lagrangian because they are redundant. These operators can always be removed from the lagrangian by making the replacement

\[
\phi(x) \rightarrow \hat{\phi}(x) \equiv \hat{\phi}(x) + a^2 f_1 \phi^3(x) + a^2 f_2 \Delta^{(2)} \phi(x)
\]  

(15)

in the lagrangian, where \(f_1\) and \(f_2\) are suitably chosen functions of the bare mass and couplings. The path integral for the lattice theory is an ordinary multidimensional integral, and this field redefinition is merely a change of integration variable; it cannot alter the physical content of the theory.

Show that \(f_1\) and \(f_2\) can always be adjusted so that both redundant operators cancel when \(\phi\) is transformed in the lagrangian. This is easy to see at tree-level (ie, in the classical theory). What complications arise in one-loop order? For example, what happens to the jacobian associated with the field transformation?

While a field transformation of this sort has no effect on physical quantities, it usually changes off-shell Green’s functions. Thus the Green’s functions of our improved theory may still deviate from the continuum Green’s functions in order \(a^2\). However particle masses, on-shell scattering amplitudes, and the like will all be accurate through order \(a^4\). This is all that we need. Examine the differences between on-shell and off-shell quantities using tree-level perturbation theory for equivalent theories with and without the redundant operators.

An important step in discretizing a quantum field theory for nonperturbative analyses is to verify that the lattice version of the theory is stable. This is an issue when improving the lagrangian. For example, if it happened that the
\(\phi^6\) coupling, \(d(a)\), was negative, we would have to worry about stabilizing the improved theory against \(\phi \to -\infty\). Luckily we have tremendous latitude in the design of our corrections. For example, we can freely add \(\phi^8, \phi^{10}\ldots\) terms to the lagrangian without affecting simulation results through order \((a\bar{p})^4\) (provided all couplings are readjusted appropriately). Such terms could be used to stabilize the discrete theory. The presence of an instability in a lattice theory usually becomes obvious when one tries to simulate the theory numerically.

We conclude that there exists a set of values for the couplings \(m(a), \lambda(a), c(a),\) and \(d(a)\) such that the lattice theory gives the same physical results as the continuum theory up to corrections of order \((a\bar{p})^4\) when \(\bar{p} \ll \pi/a\). In the classical limit of our lattice theory, only the \(c(a)\) coupling is necessary: taking \(c(a) = 1\) cancels the \(\mathcal{O}(a^2)\) errors from the derivative operator in the original lagrangian. In the quantum theory, \(c(a)\) must be shifted, and a nonzero \(d(a)\) added to cancel \(\mathcal{O}(a^2)\) errors induced by quantum fluctuations. This raises the problem of how we should determine the correct values for these couplings. In principle we might run simulations for many different values of the couplings, and numerically search for the set that gives correct physics. But a defect of our lattice theory is that it has many more coupling constants than the continuum theory — usually far too many to tune numerically. This is a serious obstacle to the use of improved lagrangians. In many theories, including QCD, the situation is salvaged by using perturbation theory to express the extra couplings in terms of the original two couplings, \(m(a)\) and \(\lambda(a)\). In perturbation theory, we expect

\[
\begin{align*}
 c(a) & = 1 + c_2(am) \lambda(a)^2 + \cdots, \\
 d(a) & = d_3(am) \lambda(a)^3 + \cdots.
\end{align*}
\]

(16)

The radiative corrections for such couplings involve only momenta of order \(\pi/a\) and larger since these are the momenta where the lattice theory needs correcting. In asymptotically free theories, like QCD, the fundamental coupling constant is small and perturbation theory is valid when the lattice spacing is small. In such situations perturbation theory can be used to correct the lagrangian, even if the long-distance behavior of the theory is highly nonperturbative.

To summarize, finite-\(a\) errors in quantum lattice theories are very similar to those for classical lattice theories. In both cases the errors can be removed, order by order in \(a\), by correcting the lattice operators. The main difference in the quantum case is that the coefficients of the correction terms in expressions like Eq. (11) are renormalized by \(a\)-dependent quantum loop effects that are very specific to the particular theory and context in which the operator is used.

**Exercise:** When the lattice couplings are perturbative, they may be computed by matching perturbative results from the lattice theory against the corresponding results generated with the continuum theory. If one has \(n\) couplings to compute, one chooses \(n\) low-momentum physical quantities to match. Each quantity is computed, order by order in perturbation theory, both in the lattice theory and in the continuum theory. The lattice couplings are then adjusted so that
the lattice results agree with the continuum results. There are no ultraviolet
infinities in the lattice calculations, since the theory has a finite cutoff, and
therefore the bare couplings are all finite numbers.

The coupling $d(a)$ is easily computed by examining the one-loop scattering am-
plitudes for $\phi\phi \rightarrow \phi\phi\phi\phi$; and the coupling $c(a)$ can be obtained from the disper-
sion relation, relating the energy and momentum of a $\phi$ particle, that is obtained
by locating the poles in the $\phi \rightarrow \phi$ propagator. Sketch these calculations to one-
loop order.

## 2.5 QCD on a Lattice

Our discussion in the previous section indicates that there are really two re-
strictions on the maximum lattice spacing $a$ that is useful for numerical work
in QCD:

1) $a$ must be smaller than any important scale in the hadronic system under
study. Since light hadrons have diameters between 1.5 and 2 fm, the
lattice spacing should probably be no larger than about 0.5 fm. Such
a lattice spacing should suffice for calculations of static properties, like
masses, magnetic moments, charge radii, etc. Smaller lattice spacings are
needed if the hadrons have nonzero momenta.

2) $a$ must be sufficiently small that perturbation theory works well at dis-
tances smaller than $a$ (or momenta larger than $\pi/a$). If this is the case
we can use perturbation theory to compute the correction terms needed
in the lagrangian to obtain high precision from coarse lattices; in effect,
we can fill in the cracks in the lattice.

The second of these restrictions is the more controversial. There is much con-
ventional wisdom suggesting that lattice spacings as small as 0.1–0.05 fm might
be required to satisfy this condition. I will argue, in subsequent sections, that
$a$’s as large as 0.5 fm are probably alright. This means that lattices can be much
coarser than is usual today. Before pursuing this crucial point, we must first
discuss the formulation of QCD on a lattice.

The continuum lagrangian for QCD is

$$ \mathcal{L} = -\frac{1}{2} \text{Tr} F_{\mu\nu}^2 $$

(17)

where

$$ F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + i g [A_\mu, A_\nu] $$

(18)

is the field tensor, a traceless $3 \times 3$ hermitian matrix. The defining characteristic
of the theory is its invariance with respect to gauge transformations where

$$ F_{\mu\nu} \rightarrow \Omega(x) F_{\mu\nu} \Omega(x)^{-1} $$

(19)

and $\Omega(x)$ is an arbitrary $x$-dependent SU$_3$ matrix.
The standard discretization of this theory seems perverse at first sight. Rather than specifying the gauge field by the values of $A_\mu(x)$ at the nodes of the lattice, the field is specified by variables on the “links” joining the nodes. In the classical theory, the “link variable” on the link joining a node at $x$ to one at $x + a\hat{\mu}$ is determined by the line integral of $A_\mu$ along the link:

$$U_\mu(x) \equiv \text{P exp} \left( -ig \int_{x}^{x + a\hat{\mu}} A \cdot dy \right)$$  \hspace{1cm} (20)

where the P-operator path-orders the $A_\mu$’s along the integration path. We use $U_\mu$’s in place of $A_\mu$’s on the lattice, because it is impossible to formulate a lattice version of QCD directly in terms of $A_\mu$’s that has exact gauge invariance. The $U_\mu$’s, on the other hand, transform very simply under a gauge transformation:

$$U_\mu(x) \rightarrow \Omega(x + a\hat{\mu}) U_\mu(x) \Omega(x)^{-1}.$$  \hspace{1cm} (21)

This makes it easy to build a discrete theory with exact gauge invariance.

**Exercise:** Show that a gauge invariant object can be associated with any closed path on the lattice by forming the product of $U_\mu$’s and $U^\dagger_\mu$’s associated with the links of the path, and taking a trace. In general $U_\mu(x)$ is associated with a link leaving site $x$ in (positive) direction $\mu$, while $U^\dagger_\mu(x)$ is associated with a link entering site $x$ from direction $\mu$. The trace of a product of link variables along a closed path is known as a “Wilson loop.”

You might wonder why we go to so much trouble to preserve gauge invariance when we quite willing give up Lorentz invariance, rotation invariance, etc. The reason is quite practical. With gauge invariance, the quark-gluon, three-gluon, and four-gluon couplings in QCD are all equal, and the bare gluon mass is zero. Without gauge invariance, each of these couplings must be tuned independently and a gluon mass introduced if one is to recover QCD. Tuning this many parameters in a numerical simulation is very expensive. This is not much of a problem in the classical theory, where approximate gauge invariance keeps the couplings approximately equal; but it is serious in the quantum theory because quantum fluctuations (loop-effects) renormalize the various couplings differently in the absence of exact gauge invariance. So while it is quite possible to formulate lattice QCD directly in terms of $A_\mu$’s, the resulting theory would have only approximate gauge invariance, and thus would be prohibitively expensive to simulate. Symmetries like Lorentz invariance can be given up with little cost because the symmetries of the lattice, though far less restrictive, are still sufficient to prevent the introduction of new interactions with new couplings (at least to lowest order in $a$).

We must now build a lattice lagrangian from the link operators. We require that the lagrangian be gauge invariant, local, and symmetric with respect to axis interchanges (which is all that is left of Lorentz invariance). The most local nontrivial gauge invariant object one can build from the link operators is the
"plaquette operator," which involves the product of link variables around the smallest square at site \( x \) in the \( \mu \nu \) plane:

\[
\text{Tr} \ U_{\mu \nu}^{\text{plaq}}(x) \equiv \text{Tr} \ (U_{\mu}(x)U_{\nu}(x + a\hat{\mu})U_{\mu}^\dagger(x + a\hat{\mu} + a\hat{\nu})U_{\nu}^\dagger(x)).
\] (22)

To see what this object is, consider evaluating the plaquette centered about a point \( x_0 \) for a very smooth weak classical \( A_\mu \) field. In this limit,

\[
\text{Tr} \ U_{\mu \nu}^{\text{plaq}} \approx 3 \quad (23)
\]
since

\[
U_\mu \approx e^{-igaA_\mu} \approx 1. \quad (24)
\]

Given that \( A_\mu \) is slowly varying, its value anywhere on the plaquette should be accurately specified by its value and derivatives at \( x_0 \). Thus the corrections to Eq. (23) should be a polynomial in \( a \) with coefficients formed from gauge-invariant combinations of \( A_\mu(x_0) \) and its derivatives. Thus we expect

\[
\text{Tr} \ U_{\mu \nu}^{\text{plaq}} = 3 - c_1 a^4 \text{Tr} (gF_{\mu \nu}(x_0))^2 - c_2 a^6 \text{Tr} (gF_{\mu \nu}(x_0)(D_\mu^2 + D_\nu^2)gF_{\mu \nu}(x_0)) + \mathcal{O}(a^8) \quad (25)
\]

where \( c_1 \) and \( c_2 \) are constants, and \( D_\mu \) is a gauge-covariant derivative. The leading correction is order \( a^4 \) because \( F_{\mu \nu}^2 \) is the lowest-dimension gauge-invariant combination of derivatives of \( A_\mu \), and it has dimension 4. It is a simple exercise to show that \( c_1 = 1/2 \) and \( c_2 = 1/24 \) (in the classical limit).

The expansion in Eq. (25) is the classical analogue of an operator product expansion. It provides the classical relation between the plaquette operator and the local gauge-invariant operators of the continuum theory. This relationship is preserved in the quantum theory except that: a) the expansion parameters \( c_i \) are renormalized by quantum fluctuations; and b) additional terms enter at order \( a^6 \) and higher. So the lattice equivalent of the continuum lagrangian in the quantum theory is

\[
\frac{3}{g^2 a^4 u_0^2} \text{Re} \left( \frac{4}{3} \text{Tr} U_{\mu \nu}^{\text{plaq}} - 1 \right) \equiv -\frac{1}{2} \text{Tr} F_{\mu \nu}^2.
\] (26)

up to corrections of order \( a^2 \). Here \( u_0 \) accounts for the quantum renormalization of the classical relation; it is usually absorbed into the definition of the coupling:

\[
g_{\text{lat}} \equiv g u_0^2. \quad (27)
\]

Thus the lattice action for QCD is

\[
S = \beta \sum_{x,\mu \nu} \text{Re} \left( \frac{4}{3} \text{Tr} U_{\mu \nu}^{\text{plaq}} \right) + \text{constant} \quad (28)
\]
where

\[ \beta \equiv \frac{6}{g_{\text{lat}}}. \]  

(29)

Notice that the lattice spacing has disappeared from the action. It is customary in lattice simulations to use “lattice units” for which \( a = 1 \), thereby completely removing \( a \) from the simulation. The lattice spacing enters only implicitly, through the numerical value of the parameter \( \beta \); different \( \beta \)’s correspond to different \( a \)’s since the bare coupling constant is a function of the lattice spacing.

**Exercise:** Find correction terms for the classical lattice lagrangian that remove the order \( a^2 \) errors. Consider, for example, adding a term that involves \( a \times 2a \) rectangular Wilson loops.

**Exercise:** In the classical lattice theory there is only one order \( a^2 \) correction. When quantum effects are included, other operators appear in this order. What operators are possible? Remember that these operators must be local, gauge invariant, and respect lattice symmetries.

### 3 Perturbation Theory: Where are the Limits in Lattice QCD?

As we discussed in the previous section, the cost of numerical simulations of QCD is dramatically reduced as the lattice spacing \( a \) is increased. The resulting loss in accuracy can be minimized by improving the lattice lagrangian, using perturbation theory to compute the corrections. Such perturbative improvement is only possible if perturbation theory is applicable at momenta of order \( \pi/a \). Thus the critical question as we increase \( a \) is whether or not perturbation theory works at momenta of \( \pi/a \) and larger.

Empirical evidence suggests that perturbative QCD applied to continuum quantities works down to momenta as low as 1 GeV; for example, \( \tau \) decay provides one of the most accurate determinations of the strong coupling constant. Thus lattice spacings as large as 0.5 fm ought to work. This result is very encouraging, but it is important to confirm it with detailed numerical tests using lattice simulations. Indeed, given the tremendous control we have over simulations, it is far easier to test perturbative QCD against nonperturbative simulations than against experimental data.

One common procedure for testing perturbation theory using simulations is to compute the \( \beta \) function — a perturbative quantity — by comparing values for some physical quantity, like the glueball mass or the string tension, computed at various values of the bare coupling. This gives a nonperturbative determination of the \( \beta \) function to compare with the perturbative prediction. In my opinion, this is a bad way to test perturbation theory since failure in such a test could reflect one of two possible problems: either perturbation theory is not working, or there are large \( \mathcal{O}(a, a^2) \) errors in the calculation of the physical quantity used...
to determine the $\beta$ function. It is fairly difficult to rule out the second option numerically, and so a negative result is ambiguous.

A better procedure is to follow phenomenologists, who identify short-distance quantities that can be both measured nonperturbatively and calculated in perturbation theory. As lattice theorists, we have a much simpler job than the phenomenologist since we can easily design dozens of quantities that are very ultraviolet and that are easy to measure in simulations; we don’t have to worry about experimental cuts, Monte Carlo corrections for hadronization, and all the other problems associated with experimental data. The idea is to compute various short-distance quantities—for example, vacuum matrix elements of local operators like $\text{Tr} U_{\mu\nu}^{\text{plaq}}$—using numerical simulations and then to compare the results with predictions from perturbation theory.

Careful testing of perturbation theory requires a careful definition of the expansion parameter, the strong coupling constant. This is as true in lattice calculations as it is in continuum calculations. There are two aspects to the problem. First we need to define what we mean by the running coupling constant $\alpha_s(q)$ (“the scheme”), and then we need a procedure for choosing a $q$ (“the scale”) appropriate to the quantity of interest. We deal with these issues in the next section. Then, in the following section, we compare perturbative results with results from nonperturbative simulations. Note that our choice of scheme and scale for $\alpha_s$ is completely automatic; there is no fine tuning of the perturbative results we use in our comparison with simulations.

### 3.1 Defining $\alpha_s$

Given one definition $\alpha_1(q)$ it is easy to create other definitions: for example, $\alpha_2(q) \equiv \alpha_1(2q)$ or $\alpha_3(q) \equiv \alpha_1(q) + 3 \alpha_1(q)^2 + \cdots$. In principle, any of these can be used as the expansion parameter for perturbation theory; in practice some definitions will be more reliable than others. For example, if $\alpha_1(q)$ is a reliable choice then $\alpha_4(q) \equiv \alpha_1(q) + 100 \alpha_1(q)^2$ will probably be useless. Expansions in terms of either coupling will be the same in lowest order, but higher orders will have large negative corrections when $\alpha_4$ is used. To minimize such problems, I like to use a physical quantity to define the coupling. One such definition that is particularly convenient uses the potential between a static quark and antiquark:

$$V(q) = - \frac{C_F}{q^2} \frac{4\pi \alpha_V(q)}{q^2}$$

where $C_F = 4/3$ and $q$ is the momentum transferred between the quark and antiquark. I like this definition because there is no ambiguity about the relationship between the argument of $\alpha_V$ and the momentum scale in the process:

$$\alpha_V(q) = \text{the coupling strength of a gluon with momentum } q.$$
This is a useful definition both for continuum and for lattice results. It is easy to convert expansions in terms of $\alpha_{\overline{\text{MS}}}$ to $\alpha_V$ expansions by using

$$\alpha_{\overline{\text{MS}}}(q) = \alpha_V(e^{5/6} q) \{ 1 + 2\alpha_V/\pi + \cdots \}.$$  \hspace{1cm} (32)

Having settled on $\alpha_V(q)$, we now need some way of specifying the scale $q$. The procedure is best understood through an example. Consider the vacuum expectation value

$$\langle 1 - \frac{1}{3} \Tr U_\mu \rangle_{\text{LG}} = \alpha_V(q^*) 2\pi a^2 \int_{-\pi/a}^{\pi/a} \frac{d^4q}{(2\pi)^4} \frac{1}{q^2} + \mathcal{O}(\alpha_V^2)$$  \hspace{1cm} (33)

where $q_\mu \equiv (2/a) \sin(\alpha q_\mu/2)$, and $1/q^2$ comes from a gluon propagator. The question is what values for $q^*$ makes sense. Given that $\alpha_V$ is the coupling for a gluon of momentum $q$, we really want

$$\alpha_V(q^*) \int_{-\pi/a}^{\pi/a} \frac{d^4q}{q^2} = \int_{-\pi/a}^{\pi/a} \frac{d^4q}{q^2} \alpha_V(q)$$  \hspace{1cm} (34)

except that the right-hand side diverges at small $q$. To see why it diverges, we can rewrite that side of the equation in terms of $\alpha_V(\mu)$ for some fixed value of $\mu$:

$$\int_{-\pi/a}^{\pi/a} \frac{d^4q}{q^2} \left\{ \alpha_V(\mu) - \beta_0 \alpha_V^2(\mu) \ln(q^2/\mu^2) - \beta_2 \alpha_V^3(\mu) \ln^2(q^2/\mu^2) - \cdots \right\}.$$  \hspace{1cm} (35)

Each term in this expansion separately is finite, but the sum diverges. This is not surprising since perturbation theory yields only an asymptotic expansion. Furthermore it is inconsistent to keep an infinite number of radiative corrections in this expansion while only keeping only one or two orders elsewhere in a calculation. So, for present purposes, we should replace the $\alpha_V$’s in Eq. (34) by the first two terms in their expansion in terms of $\alpha_V(\mu)$. It is then easy to solve for $q^*$ to obtain:

$$\ln(q^*) = \frac{\int_{-\pi/a}^{\pi/a} \frac{d^4q}{q^2} \ln(q)}{\int_{-\pi/a}^{\pi/a} \frac{d^4q}{q^2}}.$$  \hspace{1cm} (36)

Evaluating the integrals one obtains $q^* = 2.8/a$, and a final prediction of

$$\langle 1 - \frac{1}{3} \Tr U_\mu \rangle_{\text{LG}} = 0.97 \alpha_V(2.8/a)$$  \hspace{1cm} (37)

\footnote{In practice there is little difference between using $\alpha_{\overline{\text{MS}}}$ or using $\alpha_V$, but the latter suggests a simple procedure for setting the scale.}
The analogous quantity in the continuum has a quadratic ultraviolet divergence, and so it is not surprising that the average momentum scale \( q^* \) in this process very nearly equals the maximum momentum allowed on the lattice.

This example illustrates a general procedure. A quantity whose first order contribution has the form

\[
\alpha_V(q^*) \int_{-\pi/a}^{\pi/a} d^4 q f(q),
\]

where \( q \) is the gluon momentum, has scale

\[
\ln(q^*) = \frac{\int_{-\pi/a}^{\pi/a} d^4 q f(q) \ln(q)}{\int_{-\pi/a}^{\pi/a} d^4 q f(q)}.
\]

Each perturbative quantity has its own characteristic scale; the more ultraviolet a quantity is, the larger is its \( q^* \). And, in an asymptotically free theory, the larger \( q^* \) is, the more accurate is perturbation theory.

### 3.2 Testing Perturbation Theory

To probe the validity of perturbation theory, we need simulation results for several short-distance quantities and we need perturbative predictions for those quantities. Here we focus on logarithms of the expectation values of planar \( m \times n \) Wilson loops,

\[
\ln W_{mn} \equiv \ln \left( \frac{1}{Z} \text{Tr} U_{m \times n} \right),
\]

because these are easy to simulate and have been analyzed in perturbation theory through second order. (We use the logarithms of the loops rather than the loops themselves because the logarithms have more convergent expansions — ultraviolet self-energies proportional to the length of the loops exponentiate and so are better handled by taking a logarithm.) Small Wilson loops are among the most ultraviolet quantities there are in lattice gauge theory. To probe lower momenta, we examine “Creutz ratios” of the loop values,

\[
\chi_{mn} \equiv -\ln \left( \frac{W_{mn} W_{m-1 n-1}}{W_{m n-1} W_{m-1 n}} \right).
\]

The most ultraviolet parts of the loops cancel in Creutz ratios and so these tend to be more infrared than the \( W \)’s (ie, have lower \( q^* \)’s). In general quantities with larger loops and/or smaller \( q^* \)’s are more likely to have significant nonperturbative contributions.

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2This definition needs modification if there are nearly canceling contributions coming from very different scales. Luckily this situation does not arise often, and not at all in the applications we are considering here.
\[ -\ln W_{11} = \frac{4\pi}{3} \alpha_V(3.4/a) \left\{ 1 - (1.19 + 0.02n_f) \alpha_V + O(\alpha_V^2) \right\}, \] (42)

where scale \( q^* = 3.4/a \) is determined using the methods of the previous section, and \( n_f \) is the number of light quark flavors in the simulation (\( n_f = 0 \) for the results we show here). This relation can be taken as a definition of \( \alpha_V \). The coupling is determined nonperturbatively by inverting the series to determine \( \alpha_V(3.4/a) \) from simulated values for \( W_{11} \). Having a value for \( \alpha_V(3.4/a) \), we obtain \( \alpha_V(q) \) at any other scale using two-loop perturbative evolution:

\[ \alpha_V^{-1}(q) = \beta_0 \ln(q^2/\Lambda_V^2) + \beta_1/\beta_0 \ln \ln(q^2/\Lambda_V^2) \] (43)

where \( \beta_0 = (11 - \frac{2}{3}n_f)/4\pi \), \( \beta_1 = (102 - \frac{16}{3}n_f)/16\pi^2 \), and \( \Lambda_V \) is a constant (the "scale parameter").

Simulation results for \( \alpha_V \) at several different \( \beta \)'s are given in Table 1. Values for \( a\Lambda_V \) are also listed. Since \( \Lambda_V \) is independent of \( \beta \), these last numbers can be used to compute ratios of the lattice spacings at different \( \beta \)'s. The inverse lattice spacing at \( \beta = 5.7 \) is about 1 GeV so that \( a \approx 0.2 \) fm.

As a first test of perturbation theory, consider the Creutz ratio \( \chi_{22} \). It has the perturbative expansion

\[ \chi_{22} = 1.2 \alpha_V(1.1/a) - 0.4 \alpha_V^2 + O(\alpha_V^3) \] (44)

Table 1: Monte Carlo data for logarithm of the plaquette, together with the coupling constant values and the \( \alpha_V \) scale parameter in lattice units. The inverse lattice spacing at \( \beta = 5.7 \) is \( a^{-1} \approx 1 \) GeV. All results are for the standard lattice action with \( n_f = 0 \).
Table 2: Values of $\alpha_V(\pi/a)$ as determined by comparing three-loop perturbation theory with nonperturbative simulation results for the expectation values of small planar Wilson loops.

| $\alpha_V(\pi/a)$ | $\beta = 5.4$ | $\beta = 5.7$ | $\beta = 6.0$ | $\beta = 6.2$ |
|-------------------|---------------|---------------|---------------|---------------|
| $-\ln W_{11}$     | 0.271         | 0.188         | 0.156         | 0.143         |
| $-\ln W_{12}$     | 0.269         | 0.190         | 0.156         | 0.143         |
| $-\ln W_{22}$     | 0.271         | 0.194         | 0.156         | 0.143         |
| $-\ln W_{33}$     | 0.293         | 0.205         | 0.159         | 0.145         |

where, as expected, its scale $q^* = 1.1/a$ is significantly smaller than that for the plaquette. Monte Carlo simulation results are compared with first-order and second-order perturbative predictions for several $\beta$'s in Figure 1. Perturbation theory gives excellent results throughout the range shown.

For comparison results using $\alpha_{\text{MS}}(q^*)$ in place of $\alpha_V(q^*)$ are also shown; they are roughly comparable. Results are also shown for perturbation theory in terms of the bare coupling, $\alpha_{\text{lat}} \equiv g_{\text{lat}}^2/4\pi$. These are terrible. I show these last results only because it is a common practice in lattice perturbation theory to expand in terms of the bare coupling. This is a serious mistake—one that has in large measure crippled earlier attempts at using lattice perturbation theory. Using $\alpha_{\text{lat}}$ is wrong for two reasons. First, it ignores the fact that the Creutz ratio is significantly less ultraviolet than most Wilson loops and so should be expressed in terms of the running coupling evaluated at a smaller scale. Second, it ignores the renormalization in the relation between the lattice coupling and the continuum coupling (Eq. (27)), and, as we discuss in the next section, this renormalization is large. Expansions in $\alpha_{\text{lat}}$ almost always underestimate perturbative effects.

Our choice of the plaquette for determining $\alpha_V$ is for convenience. In principle one ought to be able to use any short-distance quantity to measure $\alpha_V$ in a simulation. If perturbation theory is working, the results obtained from different quantities at a given $\beta$ should agree. In Table 2 we list numerical values of $\alpha_V(\pi/a)$ as determined from a variety of small Wilson loops at several values of $\beta$. In each case $\alpha_V(q^*)$ is adjusted so that the three-loop perturbative prediction agrees with simulation results, and then $\alpha_V$ is evolved to scale $q = \pi/a$ using the two-loop beta function. The agreement between different determinations of $\alpha_V$ is good (even excellent for the smallest loops) for all $\beta$'s listed, including $\beta = 5.4$ where the lattice spacing is almost 1/2 fm. (Larger loops give poorer results because they are more strongly affected by nonperturbative effects.)

The $q$-dependence of $\alpha_V(q)$ can be determined directly from simulations. The data in Figure 2 are obtained by fitting second-order expansions in $\alpha_V(q^*)$
Figure 1: Results for Creutz ratio $\chi_{22}$ at different couplings $\beta$ from Monte Carlo simulations (circles), and from perturbation theory (using $\alpha_V(q^*)$ (diamonds), $\alpha_{MS}(q^*)$ (boxes), and $\alpha_{lat}$ (crosses)). The first plot shows perturbation theory through one-loop order, and the second through two-loop order. Statistical errors in the Monte Carlo results are negligible.
to simulation results for the six smallest Creutz ratios and the six smallest Wilson loops. The value of $\alpha_V(q^*)$ obtained from each fit is plotted versus the $q^*$ for that quantity. The $q^*$’s for the twelve quantities used here range from $0.4/a$ (for $\chi_{44}$) to $3.4/a$ (for $-\ln W_{11}$). These simulation results may be compared with the two-loop perturbative prediction for $\alpha_V(q)$ (solid line), arbitrarily normalized so that the curve passes through the point for $-\ln W_{22}$. The Monte Carlo data are quite consistent with perturbation theory, even at $\beta = 5.7$. The smallest $q^*$’s shown are on the order of several hundred MeV.

These data, and much more (see later sections), show that lattice perturbation theory for QCD is very effective even at momenta as low as 1 GeV. Indeed it is about as effective as continuum perturbation theory. Perturbation theory seems to work so long as $\alpha_V(q) < 1$. Possibly the most important consequence of this result is that it becomes likely that perturbative improvement of the QCD lagrangian will work even with lattice spacings as large as $a = 1/2$ fm. However, there remains one problem: perturbative expansions for the couplings in the corrected lagrangian tend to have very large coefficients. The origin of this problem and its cure are the subject of the next section.

4 Tadpole Improvement

The very large renormalization factor relating the bare lattice coupling $\alpha_{\text{lat}}$ to the continuum coupling $\alpha_V$ is one of many examples where a large renormalization is required to relate a lattice quantity to its continuum analogue. These large renormalizations all have a common source: the compact nature of the link variable used in lattice theories.

We design lattice operators by mapping them onto analogous operators in the continuum theory. For gauge fields, the mapping is based upon the expansion

$$U_\mu(x) \equiv \exp(-igaA_\mu) \rightarrow 1 - i gaA_\mu(x).$$

(45)

This mapping seems plausible when $a$ is small, but it is misleading in the quantum theory since further corrections do not vanish as powers of $a$. Instead one gets terms like $\frac{1}{2}(igaA_\mu)^2$ whose vacuum expectation value is suppressed only by $g^2$ — the explicit $a^2$ cancels a $1/a^2$ from the ultraviolet divergent $\langle A_\mu^2 \rangle$. These corrections turn out to be uncomfortably large. For example,

$$\langle \frac{1}{2} \text{Tr} U_\mu \rangle_{\text{LG}} \approx 0.82$$

(46)

at $\beta = 5.7$, while $\langle \frac{1}{2} \text{Tr} (1 - i gaA_\mu) \rangle = 1$. These very divergent loop contributions are referred to as “tadpole” contributions.

The tadpoles spoil our intuition about the connection between lattice operators and the continuum, leading to unexpected renormalizations. To regain this intuition we need to sharpen the relation between $U_\mu$ and the continuum gauge
Figure 2: Values of $\alpha_V(q)$ for a range of $q$'s as determined from lattice QCD measurements at various $\beta$'s. The data points (circles) are measured values (with negligible statistical errors) obtained by fitting second-order perturbation theory to Monte Carlo simulation data for various short-distance quantities. The solid line shows the variation in $\alpha_V(q)$ expected from two-loop perturbation theory.
field $A_\mu$. Results like that in Eq. (46) suggest that the appropriate connection with the continuum is

$$U_\mu(x) \to u_0 (1 - i g a A_\mu(x))$$

(47)

where $u_0$ can be thought of as the mean value of the link operator (gauge invariance requires that it be an overall factor). This suggests that lattice operators will be more continuum-like if we replace every link operator $U_\mu$ by a “tadpole improved” operator $\tilde{U}_\mu$:

$$U_\mu \to \tilde{U}_\mu \equiv \frac{U_\mu}{u_0}.$$  

(48)

The precise definition of $u_0$ is somewhat arbitrary. One could use $\langle \frac{1}{2} \text{Tr} U_\mu \rangle_{L,G}$, but it is more convenient (and not very different) to define $u_0$ in terms of the expectation value of the plaquette:

$$u_0 \equiv \langle \frac{1}{3} \text{Tr} U_\text{plaq} \rangle^{1/4}.$$  

(49)

Plaquette values for several $\beta$’s are given in Table 1. At $\beta = 5.7$, $u_0$ is 0.86, making $\tilde{U}_\mu$ about 15\% smaller than $U_\mu$. This difference is considerable, particularly in operators that involve products of several link operators.

The tadpole-improved operators $\tilde{U}_\mu$ are much closer in their behavior to their continuum analogues than the unimproved operators; large tadpole renormalizations are canceled by the $u_0$. To see what effect tadpole improvement has on the design of lattice operators we now examine two applications.

4.1 The gluon action

Our new prescription for building continuum-like operators on the lattice suggests that

$$\tilde{S}_\text{gluon} = \sum \frac{6}{g_\text{lat}^2 u_0} \text{Re} \left( \frac{1}{3} \text{Tr} U_\mu^{\text{plaq}} \right),$$

(50)

with four powers of $u_0$ to correct for the four link operators, is a better action for lattice QCD. In particular, perturbation theory in $\bar{\alpha}_\text{lat} \equiv g_\text{lat}^2/4\pi$ should be more like continuum perturbation theory (ie, no tadpoles). Of course, this action becomes the same as the standard action if we identify

$$\alpha_\text{lat} = u_0^4 \tilde{\alpha}_\text{lat}.$$  

(51)

This relation is important because it explains why perturbative expansions in the bare coupling $\alpha_\text{lat}$ fail so badly. Our tadpole analysis indicates that $\tilde{\alpha}_\text{lat}$, not $\alpha_\text{lat}$, is the correct expansion parameter for the theory. The difference is significant: for example, $\tilde{\alpha}_\text{lat} \approx 1.8 \alpha_\text{lat}$ at $\beta = 5.7$. Using $\alpha_\text{lat}$ almost guarantees that perturbation theory will give results that are much too small.

3\ An exception is the expectation value of the plaquette itself (without a logarithm). Because this operator is identical to those in the lagrangian, tadpole corrections cancel in its
If this analysis is correct, the tadpole-improved bare coupling $\tilde{\alpha}_{\text{lat}}$ should be roughly equal to the continuum coupling $\alpha_V(\pi/a)$. We can check this in perturbation theory which gives a relation between $\alpha_V$ and the usual bare coupling:

$$\alpha_V(\pi/a) = \alpha_{\text{lat}} \{ 1 + 4.7\alpha_V + \mathcal{O}(\alpha_V^2) \}.$$  \hspace{1cm} (52)

We tadpole improve this expansion by dividing out the perturbative expansion for $1/a^0 = 1/\langle 1/3 \, \text{Tr} \, U_{\mu \nu}^{\text{plaq}} \rangle$ (Eq. (42)) on the left-hand side:

$$\alpha_V(\pi/a) = \frac{\alpha_{\text{lat}}}{\langle 1/3 \, \text{Tr} \, U_{\mu \nu}^{\text{plaq}} \rangle} \{ 1 + 0.513\alpha_V + \mathcal{O}(\alpha_V^2) \} \quad (53)$$

$$\approx \tilde{\alpha}_{\text{lat}} \{ 1 + 0.513\alpha_V + \mathcal{O}(\alpha_V^2) \}.$$  \hspace{1cm} (54)

This equation implies that $\alpha_V(\pi/a)$ and $\tilde{\alpha}_{\text{lat}}$ agree to within 10% at $\beta = 5.7$, while $\alpha_{\text{lat}}$ is almost a factor of two smaller. Thus perturbation theory confirms that almost all of the large renormalization factor relating $\alpha_V$ to $\alpha_{\text{lat}}$ is due to tadpoles.

Tadpole improvement is very important when improving lattice actions, since the correction terms tend to have lots of link operators. If you completed the Exercises in the earlier sections, you found that the order $a^2$ errors in the standard lattice action for classical QCD are removed by replacing

$$U_{\mu \nu}^{\text{plaq}} \rightarrow U_{\mu \nu}^{\text{plaq}} - \frac{1}{20} (U_{\mu \nu}^{1x2} + U_{\nu \mu}^{1x2}) \quad (\text{classical}),$$

where $U_{\mu \nu}^{1x2}$ is the $1 \times 2$ planar Wilson loop. Since $U_{\mu \nu}^{1x2}$ has two more link operators than $U_{\mu \nu}^{\text{plaq}}$, the tadpole improved correction in the quantum theory is given by

$$U_{\mu \nu}^{\text{plaq}} \rightarrow U_{\mu \nu}^{\text{plaq}} - \frac{1}{20 u_0^2} (U_{\mu \nu}^{1x2} + U_{\nu \mu}^{1x2}) \quad (\text{tadpole-improved}).$$

The extra factor of $1/u_0^2$ is important, particularly at the large lattice spacings we want to use; without it the correction is too small and order $a^2$ errors are only partially removed.

### 4.2 The quark action

As a second example, consider the tadpole-improved version of Wilson’s action for heavy quarks:

$$\tilde{S}_q = \sum_x \bar{\psi}_x \gamma_5 \psi_x - \bar{\psi}_x \left\{ \gamma_5 \left( 1 + \gamma_\mu \right) \frac{U_\mu}{u_0} \right\} \psi_x + \text{h.c.}.$$  \hspace{1cm} (57)
Again, this action is identical to the usual one if we relate the modified parameters, here the “hopping parameter” \( \tilde{\kappa} \), to the usual ones by rescaling with \( u_0 \):

\[
\tilde{\kappa} = \kappa u_0.
\] (58)

**Exercise:** Setting \( U_\mu / u_0 = 1 \), show that the quark mass in the free-quark theory is given by \( m a = 1 / 2 \kappa - 4 \).

The modified hopping parameter should be more continuum-like than the usual one; for example, the free-quark value that gives massless quarks, \( \kappa_c = 1 / 8 \), should be roughly correct for interacting quarks as well. Thus a nonperturbative formula for the critical value of the conventional hopping parameter is

\[
\kappa_c \approx 1 / 8 u_0,
\] (59)

which implies that the critical value of the bare quark mass is

\[
m_c a \equiv 1 / 2 \kappa_c - 4 \approx 4(u_0 - 1).
\] (60)

Again we turn to perturbation theory to check the validity of this result. In perturbation theory, \( m_c \) is given by

\[
m_c a = -5.457 \alpha_V(2.6 / a) + \mathcal{O}(\alpha_V^2)
\] (61)

where the scale of \( \alpha_V \) is determined as in Section 3.1. Pulling out a \( 4(u_0 - 1) \) for the tadpole contribution, this becomes

\[
m_c a = 4 \left( \frac{1}{4} \text{Tr} U_{\mu\nu}^{\text{plaquet}} \right)^{1/4} - 1.268 \alpha_V(1.0 / a) + \mathcal{O}(\alpha_V^2).
\] (62)

Thus perturbation theory confirms that the bulk of the renormalization of \( m_c \) is due to tadpoles. Notice also that the scale in \( \alpha_V \) is smaller with tadpole improvement; this is because the very ultraviolet tadpole contributions have been removed.

By removing the tadpole contributions, we make the perturbation theory more convergent, and therefore more accurate. The value of \( u_0 \) is obtained directly from the simulation (from the measured plaquette expectation value), and so we need never deal with tadpole effects in perturbation theory. The accuracy of this procedure is illustrated in Table 3 where we list values for \( m_c a \) at different \( \beta \)’s. We give results from the ordinary perturbative expansion (Eq. (61)), from the tadpole-improved expansion (Eq. (62)), and from nonperturbative Monte Carlo simulations. The tadpole-improved results agree very well with the simulation results, and are about as accurate.

Because there are link operators in the kinetic part of the quark action, tadpole improvement of the links affects the relation between the lattice quark field and the continuum quark field. In the continuum limit, the tadpole-improved lagrangian for massless quarks becomes

\[
2 \tilde{\kappa} \bar{\psi} \gamma_\mu \partial^\mu \psi + \mathcal{O}(a).
\] (63)
This indicates that the tadpole-improved lattice quark operator is

\[ \tilde{\psi} = \sqrt{2\tilde{\kappa}_c} \psi = \psi / 2 \] (massless quarks),

where we use the fact that \( \tilde{\kappa}_c \approx 1/8 \). This lattice operator has roughly the same normalization as the continuum field; in particular, there are no large tadpole contributions to the renormalization constant relating them. This is important in designing new lattice operators involving quark fields. For example, if one wants to calculate matrix elements of the continuum current \( \bar{\psi} \gamma^\mu \gamma^5 \psi \), then one should simulate with the lattice operator

\[ \bar{\tilde{\psi}} \gamma^\mu \gamma^5 \tilde{\psi} = \frac{1}{4} \bar{\psi} \gamma^\mu \gamma^5 \psi. \] (65)

Exercise: Compute the tree-level correction needed to remove order \( a \) errors from the Wilson action. Tadpole-improve your result.

5 Nonrelativistic QCD

As an illustration of the ideas developed in these lectures we end with a review of recent simulation results concerning the upsilon family of \( b \bar{b} \) mesons. This is an attractive system to study because so much is known about it. The quark potential model, for example, provides an accurate phenomenological model of the internal structure of the mesons. Also there is much experimental data for these mesons. The low-lying states are largely insensitive to light-quark vacuum polarization — for example, the \( \Upsilon, \Upsilon', \chi_b \ldots \) are all far below the threshold for decays into \( B \) mesons — and therefore can be accurately simulated with \( n_f = 0 \). Furthermore, these mesons are very small, the \( \Upsilon \) being about five times smaller than a light hadron. This makes them an excellent testing ground for our ideas concerning large lattice spacings.

The \( \Upsilon \) spectrum indicates that the \( b \)-quarks in the meson are nonrelativistic. This means that the most important momentum scales governing the meson’s dynamics are smaller than the quark’s mass \( M_b \). We can take advantage of
this fact by choosing an inverse lattice spacing of order the quark mass, thereby excluding relativistic states from the theory. Then it is efficient to analyze the heavy-quark dynamics using a nonrelativistic lagrangian (NRQCD). The lagrangian used to generate the results shown below was

\[
\mathcal{L}_{\text{NRQCD}} = \psi \left( 1 - \frac{aH_0}{2n} \right)^n \left( 1 - \frac{a\delta H}{2n} \right) \psi - \psi^\dagger \psi, \tag{66}
\]

where \( n = 2, H_0 \) is the nonrelativistic kinetic-energy operator,

\[
H_0 = -\frac{\Delta^{(2)}}{2M_0^\alpha}, \tag{67}
\]

\( M_0^\alpha \) is the bare quark mass, and \( \delta H \) is the leading relativistic and finite-lattice-spacing correction,

\[
\delta H = -\frac{(\Delta^{(2)})^2}{8(M_0^\alpha)^3} \left( 1 + \frac{aM_0^\alpha}{2n} \right) + \frac{a^2\Delta^{(4)}}{24M_0^\alpha} \\
- \frac{g}{2M_0^\alpha} \sigma \cdot B + \frac{ig}{8(M_0^\alpha)^2} (\Delta \cdot E - E \cdot \Delta) \\
- \frac{g}{8(M_0^\alpha)^2} \sigma \cdot (\Delta \times E - E \times \Delta). \tag{68}
\]

Here \( \Delta \) and \( \Delta^{(2)} \) are the simple gauge-covariant lattice derivative and laplacian, while \( \Delta^{(4)} \) is a lattice version of the continuum operator \( \sum D_i^4 \). The chromoelectric and chromomagnetic fields, \( E \) and \( B \), are defined in terms of different components of

\[
U_{\mu}^{\text{plaq}} - U_{\mu}^{\text{plaq}}. \tag{69}
\]

The entire action was tadpole improved by dividing every link operator \( U_\mu \) by \( u_0 \). Potential models indicate that corrections beyond \( \delta H \) contribute only of order 5–10 MeV to \( \Upsilon \) energies. The simulation results shown below used gluon fields generated at \( \beta = 6 \), which corresponds to a lattice spacing of about 1/12 fm or about half the radius of an \( \Upsilon \).

The details of this lagrangian are unimportant to us here. What matters is that \( \delta H \) consists of correction terms just like the ones we have been analyzing for other theories, the only difference here being that we are correcting both for finite-\( a \) and for the absence of relativity (ie, order \( \nu^2/c^2 \) errors). The coefficients of the correction terms were determined with tree-level perturbation theory and tadpole improvement, using precisely the techniques outlined in the earlier sections. So the extent to which \( \delta H \) improves the simulation results is a measure of the efficacy of all the techniques discussed in these lectures.

The \( \Upsilon \) spectrum is very well described by the simulation. Simulation results for the low-lying \( ^3S_1 \) and \( ^1P_1 \) energies are shown in Figure 3. These compare well with experimental results (the horizontal lines), as they should since systematic
errors are estimated to be less than 20–40 MeV. It is important to realize that these are calculations from first principles. The only inputs are the lagrangians describing gluon and quark dynamics, and the only parameters are the bare coupling constant and the bare quark mass. In particular, these results are not based on a phenomenological quark-potential model. These are among the most accurate lattice results to date.

The corrections in $\delta H$ have only a small effect on the overall spectrum, but the spin structure is strongly affected. The lagrangian without $\delta H$ is spin independent, and gives no spin splittings at all. Simulation results for the spin structure of the lowest lying $P$ state are shown in Figure 4. Again these compare very well with the data, giving strong evidence that corrected lagrangians work. Systematic errors here are estimated to be of order 5 MeV. Note that the spin terms in $\delta H$ all involve either chromoelectric or chromomagnetic field operators. These operators are built from products of four link operators and so tadpole improvement increases their magnitude by almost a factor of two at $\beta = 6$. Without tadpole improvement of the spin splittings from this simulation would have been much too small.

The correction terms also affect the simulation mass of the $\Upsilon$. Our non-
relativistic action gives only part of the total energy or mass of the meson. The full mass is obtained by adding the masses of the quarks to the meson’s nonrelativistic energy $E_{NR}$:

$$M_\Upsilon = 2 \left( Z_m M^0_b - E_0 \right) + E_{NR} \quad (70)$$

where $Z_m$ and $E_0$ are ultraviolet renormalizations that are computed using perturbation theory, $M^0_b$ is a tunable parameter of the theory, and $E_{NR}$ is computed in the simulation. The bare quark mass for the simulation discussed here was tuned so that this formula gave $M_\Upsilon = 9.5(1)$ GeV (the correct value is 9.46 GeV). The $\Upsilon$ mass was also determined a second way in the simulation by computing the nonrelativistic energy of an $\Upsilon$ as a function of its momentum $p$,

$$E_\Upsilon(p) = E_{NR} + \frac{p^2}{2M_{\text{kin}}} + \cdots.$$ \quad (71)

This determined the kinetic mass $M_{\text{kin}}$ of the meson. In a purely nonrelativistic system the kinetic mass equals the sum of the quark masses. Only when relativistic corrections are included is this mass shifted to include the binding energy, giving $M_\Upsilon$. The simulation without $\delta H$ gave $M_{\text{kin}} = 8.2(1)$ GeV, which is quite different from the upsilon mass determined using Eq. (70). With $\delta H$, the simulation gave $M_{\text{kin}} = 9.5(1)$ GeV, which is in excellent agreement. All of the spin-independent pieces of $\delta H$ contribute to the shift in $M_{\text{kin}}$; once again we have striking evidence that corrected actions work.

This simulation has only two parameters: the bare coupling constant and the bare quark mass. These were tuned to fit experimental data. From the

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**Figure 4**: Simulation results for the spin structure of the lowest lying $P$-state in the $\Upsilon$ family. The dashed lines are the experimental values for the triplet states, and the experimental spin average of all states for the singlet ($h_b$).
bare parameters we can compute the renormalized coupling and mass. This simulation implies that the renormalized or “pole mass” of the $b$-quark is

\[ M_b = 4.94(15) \text{ GeV}. \]  

(72)

The renormalized coupling that is obtained corresponds to

\[ \alpha_{\text{MS}}^{(5)}(M_Z) = 0.112(5), \]  

(73)

which agrees with results from high-energy phenomenology and is about as accurate. This last result is striking: it shows that the QCD of hadronic structure and the QCD of high-energy quark and gluon jets are really the same theory.

6 Conclusion

The $\Upsilon$ simulations described in the last section show that lattice QCD can produce accurate results even when the lattice spacing is as large as half the radius of a meson. For light hadrons this would correspond to $a \approx 0.4$ fm or $\beta \approx 5.4$ (for simulations with $n_f = 0$). Our analysis shows that perturbation theory is still reliable at such distances, and so perturbative improvement of the lagrangians used in simulations probably still works. Given that finite-volume errors become manageable for lattice sizes of order 2–3 fm, it seems likely that reliable simulations of full QCD are possible on lattices as small as $6^4$. Simulations on such small lattices are literally a thousand times faster than simulations on the $20^4$ lattices commonly used today. If coarse lattices really do work, the shift to small lattices and improved lagrangians will have a revolutionary effect on numerical QCD.

7 Acknowledgements

Many of the ideas and opinions presented here grow out of my long-standing collaboration with Paul Mackenzie. The numerical results in Sections 3 and 4 are from our work on lattice perturbation theory. The $\Upsilon$ simulations in Section 5 were by myself and my collaborators in the NRQCD collaboration. This work is supported by a grant from the National Science Foundation.

8 Bibliography

The central point in these lectures is that errors due to finite lattice spacing should be removed by improving the lagrangian, and not by decreasing the lattice spacing. Improvement schemes have a long history; see, for example,

K. Symanzik, *Nucl. Phys.* B226, 187(1983);
M. Lüscher and P. Weisz, *Comm. Math. Phys.* 97, 59(1985).

The implementation of these ideas has been held back by persistent worries about the validity and utility of perturbation theory; the improvement program is much more complicated without perturbation theory. Renormalized perturbation theory (Section 3) and tadpole improvement (Section 4) appear to resolve these problems. These techniques are developed at greater length in

G. P. Lepage and P. B. Mackenzie, *Phys. Rev.* D48, 2250 (1993).

My own interest in improved lagrangians and renormalization group strategies came from studying very different problems. Bill Caswell and I discovered that renormalization techniques provide a powerful tool for high precision analyses of QED boundstates like positronium. This led to the development of non-relativistic versions of QED and eventually also of QCD. These developments are discussed in

W. E. Caswell and G. P. Lepage, *Phys. Lett.* 167B, 437 (1986).

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G. P. Lepage, L. Magnea, C. Nakhleh, U. Magnea, and K. Hornbostel, *Phys. Rev.* D46, 4052(1992).

G. P. Lepage, "What is Renormalization?" in *From Actions to Answers*, edited by T. DeGrand and D. Toussaint (World Scientific, Singapore, 1989).

The last of these papers has a pedagogical discussion of the ideas from renormalization theory that are used in the present lectures. The NRQCD results presented in Section 6 are described in a series of papers to be published in the near future (by C. Davies et al).

For an elementary introduction to lattice QCD see

T.-P. Cheng and L.-F. Li, *Gauge theory of elementary particle physics* (Clarendon Press, Oxford, 1991).

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Useful references concerning the Feynman rules for weak-coupling perturbation theory in lattice QCD and some standard results are

H. Kawai, R. Nakayama and K. Seo, *Nucl. Phys.* B189, 40 (1981).

U. Heller and F. Karsch, *Nucl. Phys.* B251, 254 (1985).

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A. Ukawa, *Nucl. Phys. B (Proc. Suppl.)* **30**, 3(1993).

P.B. Mackenzie, in the *Proceedings of the 1993 Lepton-Photon Symposium*, edited by P. Drell and D. Rubin (AIP Press, New York, 1994).

The first of these is from one of the annual meetings on lattice field theory. The proceedings from these meetings are primary references for recent developments in the field.