Support theorem for a singular semilinear stochastic partial differential equation

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Abstract

We consider the generalized parabolic Anderson equation (gPAM) in 2 dimensions with periodic boundary. This is an example of a singular semilinear stochastic partial differential equations, solutions of which require renormalization and have only be understood recently via Hairer’s regularity structures and, in some cases equivalently, paracontrolled distributions due to Gubinelli, Imkeller and Perkowski. In the present paper we describe the law of gPAM, by establishing a Stroock–Varadhan type support theorem in suitable Hölder–Besov spaces.

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1 Introduction

In a major recent advance, carried out independently (and with different techniques) by Hairer [9] and Gubinelli, Imkeller and Perkowski [8] it was understood how to make rigorous sense of a number of important singular semi-linear stochastic partial differential equations (SPDEs) arising in mathematical physics. While Hairer’s theory of regularity structures can handle more general classes of such highly irregular SPDEs, the two theories yield essentially equivalent results in a number of interesting cases, including the (generalized) parabolic Anderson model.
in a spatial continuum of dimension 2, on which this article will focus. More specifically, we consider a solution \( u : \mathbb{R}^+ \times \mathbb{T}^2 \to \mathbb{R} \) to the following SPDE (cf. Theorem 2.9 below)

\[
\begin{aligned}
\mathcal{L} u &= f(u) \xi \\
u(0, x) &= u_0(x) \in C^a(\mathbb{T}^2)
\end{aligned}
\]  

(1)

where \( \mathcal{L} = \partial_t - \Delta \), \( \Delta \) is the Laplacian on the two dimensional torus \( \mathbb{T}^2 \), \( C^a(\mathbb{T}^2) \) is the Besov space \( B^{\alpha}_{\infty, \infty} \) (see (5) for the exact definition), \( f \in C^3_b(\mathbb{R}) \) (or \( f(u) = u \)) is a three times differentiable function, bounded with bounded derivatives, and at last \( \xi \) is a zero mean spatial white noise, i.e. a centred Gaussian field with

\[
\mathbb{E}[\xi(x)\xi(y)] = \delta(x-y), \quad \int_{\mathbb{T}^2} \xi(x)dx = 0.
\]

The aim of this paper is to give a characterization for the topological support of the law of the solution \( u \) in a suitable Hölder-Besov space.

Even in the well-understood and classical context of stochastic differential equations (SDEs), such a “support theorem” is a deep result and was first obtained in a seminal paper by Stroock–Varadhan [20], many extensions and alternative proofs followed. Let us recall ([20], and essentially all later works on support theorems) that Wong–Zakai approximations give the “easy” inclusion in the support theorem. Most relevant for us, Lyons’ rough path theory [15–17] has provided a “robust” view on SDE theory which subsequently led to decisive proofs of the Stroock–Varadhan support theorem: it “suffices” to establish the support characterization for the enhanced noise, in sufficiently strong topologies, upon which the solution depends in a continuous fashion. This strategy of proof was carried out first by Ledoux, Qian and Zhang [14], later strengthened by Friz, Lyons and Stroock [6]. Much more can be found in the monograph [7, Ch.19], and the references therein.

The theories of regularity structures and paracontrolled distributions, both inspired by rough path theory, provide an equally “robust” view on the classes of SPDEs, which they helped to define in the first place. A similar route towards support characterizations should then be possible. To this end, a number of technical problems need to be overcome, the perhaps most immediate being the divergence of “Wong–Zakai” approximations due to infinite-dimensionality of the noise.\(^1\)

Among all support theorems for SPDEs available in the literature, let us point in particular to the work of Bally, Millet and Sanz-Solé [1] which appears to us closest in spirit, although of course both techniques and classes of SPDEs considered are completely different.

We believe that our approach will work for large classes of SPDEs where the theory of regularity structures or paracontrolled distributions can be employed, notably the three dimensional stochastic quantization equation [4,9], the KPZ equation [12] and [5, Ch.15] and its generalizations presently studied by Hairer–Zambotti. That said, in the present paper all this will be implemented in the case of the generalized Parabolic Anderson model [8,9].

The problem with this model, given by equation (1) is that \( \xi \) is too rough to have a well-defined product \( f(u)\xi \). Indeed, it is well-known that \( \xi \in C^{-1-\delta}(\mathbb{T}^2) \) a.s. for all \( \delta > 0 \), and no better, which implies at least formally that at best \( u(t,.) \in C^{1-\delta} \) a.s., in view of regularization

\(^1\) Very recently, such approximations were studied, from a regularity structure point of view, for one-dimensional parabolic nonlinear stochastic PDEs driven by space-time white noise. See Hairer-Pardoux [10].
properties of the heat flow. It is well-known from harmonic analysis that Schwartz distributions in such Besov–Hölder spaces can be multiplied only if the exponents add up to a positive number, which is plainly not the case here and leaves one with the ill-defined product \( f(u)\xi \). If one proceeds by brute force approximations arguments, one quickly finds that the limiting equations is not (1) but of the (non-sensical) form

\[
\begin{align*}
\mathcal{L} u &= f(u)\xi - \infty(\ldots) \\
u(0, x) &= u_0(x) \in C^\alpha(T^2).
\end{align*}
\]

This problem has been treated in the simple case of \( f(u) = u \) in [19] by interpreting the product as the Wick product generated by the Gaussian structure of the white noise and then a chaos expansion of the solution is obtained. For the case of general \( f \), no such trick will work.

As already mentioned, recently two different approaches have been developed to deal with this singular SPDE. One is based on the theory of regularity structure due to M.Hairer [9], the other one the paracountrolled distribution approach due to Gubinelli, Imkeller and Perkowski [8]. In the latter, the authors use the Bony paraproduct (see [2,3]) to obtain a space of distributions which admit some sort of Taylor expansion where in a sense the pointwise product is replaced by the Bony paraproduct, and which is ultimately seen to contain the solution \( u \) of the equation, with good properties vis à vis to the afore-mentioned multiplication \( f(u)\xi \).

We note that a global existence result, i.e. non-explosion, for (1) has been in established in [8], complementing the uniqueness and local existence results previously obtained ([9] and an earlier arXiv version of [8]). Although not essential, this removes the need for attaching a cemetery state to the space in which the solution lives. See Hairer–Weber [11] for a situation, in the context of large deviations for singular semilinear SPDEs, where this is necessary. We have chosen to work in the paracontrolled setting here, but there is no doubt that everything can be implemented in an essentially equivalent fashion in the framework of regularity structures. Let us now state the main result of this paper.

**Theorem 1.1.** Let \( T > 0,\alpha \in (2/3,1), u_0 \in C^\alpha(T^d) \) and \( f \in C^3_k(\mathbb{R}) \). If we denote by \( u \) the solution of the Cauchy problem (1) given by the Theorem 2.9 and by \( u_\mathbb{P} \) the law of \( u \) in \( C([0,T],C^\alpha(T^2)) \) then we have, with the closure below taken in \( C([0,T],C^\alpha(T^2)) \),

\[
\text{supp}(u_\mathbb{P}) = \{\mathcal{F}(u_0,h,c), \ h \in \mathcal{H}, c > 0\},
\]

where \( \mathcal{F}(u_0,h,c) = u \) is the classical solution to (cf. Proposition 5.1 in the Appendix)

\[
\begin{align*}
\mathcal{L} u &= f(u)h - cf'(u)f(u) \\
u(0, x) &= u_0(x) \in C^\alpha(T^2),
\end{align*}
\]

and \( \mathcal{H} \) is the Cameron-Martin space associated to \( \xi \) i.e. the set of \( f \in L^2(T^2) \) with zero-mean, \( \int_{\mathbb{T}^d} f(x)dx = 0 \).

We remark that the infinite term in (2) is replaced by a finite expression, of the form \( cf'(u)f(u) \) and this is a key aspect in the analysis. It should be noted that, in general, the constant \( c \) which appears in (3) ranging over all positive reals$^2$ cannot be omitted; cf. Lemma 5.3 (and also Lemma 3.13). This is in contrast to the results of Bally, Millet and Sanz-Solé [1] where an infinite constant was effectively set to zero. The underlying reason is that they deal with space-time white noise whereas in PAM case we have purely spatial noise.

$^2$What actually matters is that \( c \) ranges over a set which has \( +\infty \) as accumulation point.
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2 Well-posedness result for the parabolic Anderson equation

2.1 Besov spaces and Bony paraproduct

Before stating the main result of [8] about the parabolic Anderson equation let us collect some definition and basics facts about the Besov space. Let $\chi$ and $\rho$ be a nonnegative smooth radial functions such that

1. The support of $\chi$ is contained in a ball and the support of $\rho$ is contained in an annulus;
2. $\chi(\xi) + \sum_{j \geq 0} \rho(2^{-j}\xi) = 1$ for all $\xi \in \mathbb{R}^d$;
3. $\text{supp}(\chi) \cap \text{supp}(\rho(2^{-j} \cdot)) = \emptyset$ for $i \geq 1$ and $\text{supp}(\rho(2^{-i} \cdot)) \cap \text{supp}(\rho(2^{-j} \cdot)) = \emptyset$ when $|i-j| > 1$.

(for the existence of such a function see [2], Proposition 2.10.). Then the Littlewood-Paley blocks are defined by:

$$\Delta_{-1} u = \mathcal{F}^{-1}(\chi \mathcal{F} u)$$
and for $j \geq 0$, $\Delta_j u = \mathcal{F}^{-1}(\rho(2^{-j} \cdot) \mathcal{F} u)$.

Where $\mathcal{F} f$ is the Fourier transform of a distribution $f \in \mathcal{S}'(\mathbb{R}^d)$.

We define the Besov space of distributions by:

$$\mathcal{B}^{s}_{p,q} = \left\{ u \in \mathcal{S}'(\mathbb{R}^d); \quad ||u||_{\mathcal{B}^{s}_{p,q}} = \sum_{j \geq -1} 2^{jq \alpha} ||\Delta_j u||_{L^p} < +\infty \right\}.$$  \hspace{1cm} (5)

In the sequel we will deal extensively with the special case of $C^\alpha := \mathcal{B}^{\alpha}_{\infty,\infty}$ and the Sobolev space $H^\alpha := \mathcal{B}^{\alpha}_{2,2}$ and we write $||u||_{C^\alpha} = ||u||_{\mathcal{B}^{\alpha}_{\infty,\infty}}$. Let us also introduce the space $H^\alpha(\mathbb{R}^d)$ (respectively $C^\alpha(\mathbb{R}^d)$) of distributions $f \in H^\alpha(\mathbb{R}^d)$ (respectively $f \in C^\alpha(\mathbb{R}^d)$) such that $\hat{f}(0) = 0$ equipped with the norm of $H^\alpha(\mathbb{R}^d)$ (respectively $C^\alpha(\mathbb{R}^d)$) and we remark that $H^0 = \mathcal{H}$. At some point we will deal with stochastic objects and the trick is to work with Besov spaces with finite indexes and then go back to the space $C^\alpha$. For that we have the following useful Besov embedding.

**Proposition 2.1** (Besov embedding). Let $1 \leq p_1 \leq p_2 \leq +\infty$ and $1 \leq q_1 \leq q_2 \leq +\infty$. For all $s \in \mathbb{R}$ the space $\mathcal{B}^{s}_{p_1,q_1}$ is continuously embedded in $\mathcal{B}^{s-d\left(1/p_1-1/p_2\right)}_{p_2,q_2}$. In particular we have $||u||_{C^\alpha} \lesssim ||u||_{\mathcal{B}^{\alpha}_{p,q}}$.

Taking $f \in C^\alpha$ and $g \in C^\beta$ we can formally decompose the product as

$$fg = f \prec g + f \circ g + f \succ g$$

with

$$f \prec g = g \succ f = \sum_{j \geq -1} \sum_{i<j} \Delta_i f \Delta_j g$$ (Paraproduct term)
and
\[ f \circ g = \sum_{j \geq -1} \sum_{|i-j| \leq 1} \Delta_i f \Delta_j g \quad \text{(Resonating term)} \]

With these notations the following results hold.

**Proposition 2.2** (Bony estimates [3]). Let \( \alpha, \beta \in \mathbb{R} \)

1. \[ \|f \prec g\|_\beta \lesssim \|f\|_\infty \|g\|_\beta \] for \( f \in L^\infty \) and \( g \in C^\beta \)

2. \[ \|f \succ g\|_{\alpha+\beta} \lesssim \|f\|_\alpha \|g\|_\beta \] for \( \beta < 0 \), \( f \in C^\alpha \) and \( g \in C^\beta \)

3. \[ \|f \circ g\|_{\alpha+\beta} \lesssim \|f\|_\alpha \|g\|_\beta \] for \( \alpha + \beta > 0 \) and \( f \in C^\alpha \) and \( g \in C^\beta \). Moreover if we have that \( f \in C^\alpha \) and \( g \in H^\beta \) with \( \alpha + \beta > 0 \) then
\[ \|f \circ g\|_{\alpha+\beta-4/2} \lesssim \|f\|_\alpha \|g\|_{H^\beta} \]

We finish this section by describing the action of the Fourier multiplier operator on the Besov spaces.

**Proposition 2.3** (Schauder estimate ). Let \( m \in \mathbb{R} \) and \( \psi \) a infinitely differentiable function on \( \mathbb{R}^d - \{0\} \) such that \( |D^k \psi(x)| \lesssim |x|^{-m-k} \) for all \( k \). Then the following bound
\[ \|\psi(D)f\|_{\alpha+m} \lesssim \|f\|_\alpha \]
for \( f \in C^\alpha \) with \( \psi(D)f = \mathcal{F}^{-1}(\psi \hat{f}) \).

**Remark 2.4.** We note that all the above facts about Besov space can be stated on the Torus \( \mathbb{T}^d \) for detail see [18].

### 2.2 Convergence of the mollified equation

Let us now discuss the known (global!) existence and uniqueness results for the gPAM. Similar to the resolution of SDEs via rough path theory, the problem is divided in two parts.

- A first part which is purely analytic in which the PDE driven by smooth \( \xi \) is extended to “rougner” driving noise, with values in a “bigger space” \( \mathcal{X}^\alpha \).

- A second purely stochastic step in which it is shown that the white noise \( \xi \) can be enhanced in an element \( \Xi_{pam} \in \mathcal{X}^\alpha \)

Let us write \( \mathcal{L} = \partial_t - \Delta \) for the heat-operator. \( \mathcal{C}^\infty \) denotes the space of smooth functions, with zero mean, on the torus. We have
Proposition X Moreover by a direct computation we get:

\[ S(\xi) = \text{extension of } \xi = (\xi_1, \xi_2) \]

\[ \text{where } \xi \text{ enhancement (or lift) of } \Xi = (\Xi_1, \Xi_2) \]

Finally we denote by

\[ \mathcal{M}(\theta, c) = (\theta; \theta \circ K\theta - c) \]

where \( K\theta := (-\Delta)^{-1}\theta \) is the (unique) smooth, zero-mean solution to \((-\Delta)u = \theta \in \mathcal{C}^\infty \), cf. Proposition 2.5.

Let us now be more precise about the “enhanced noise space” \( \mathcal{X}^\alpha \):

**Definition 2.6.** Let \( \mathcal{H}^\alpha := C^{\alpha-2}(\mathbb{T}^2) \times C^{2\alpha-2}(\mathbb{T}^2) \) and \( ||F||_{\mathcal{H}^\alpha} \) denote the norm in this Banach space. Now we define the set \( \mathcal{X}^\alpha \) by the following identity:

\[ \mathcal{X}^\alpha := \{ (\theta, \theta \circ K\theta - c); \theta \in \mathcal{C}^\infty(\mathbb{T}^2), c \in \mathbb{R} \} \]

Finally we denote by \( \Xi = (\Xi^1, \Xi^2) \) a generic element in \( \mathcal{X}^\alpha \). Whenever \( \Xi^1 = \xi \), we call \( \Xi \) an enhancement (or lift) of \( \xi \).

We have the following alternative description of \( \mathcal{X}^\alpha \). Recall that \( \mathcal{H} = \mathcal{L}^2 \), the space of zero-mean square-integrable functions on the torus, is precisely the Cameron–Martin space for our spatial zero-mean white-noise \( \xi \).

**Lemma 2.7.** For \( \alpha < 1 \), the following set identity holds,

\[ \mathcal{X}^\alpha = \{ (\theta, \theta \circ K\theta - c); \theta \in \mathcal{H}, c \in \mathbb{R} \} \]

\[ ||\theta \circ K\theta||_{\mathcal{H}^\alpha}^2 = \sum_{k \in \mathbb{Z}^2} |k|^{2\gamma} \sum_{k_1 + k_2 = k, k_1 \neq k, |k_1| \leq 1} \frac{1}{|k_2|^2} \mathcal{F}(\Delta, \theta)(k_1, \mathcal{F}(\Delta, \theta))(k_2) \]

\[ \lesssim \sum_{k \in \mathbb{Z}^2} |k|^{2\gamma} \sum_{k_1 + k_2 = k, k_1 \neq k_2} \frac{1}{|k_2|^2} \hat{\theta}(k_1)\hat{\theta}(k_2) \]

\[ \lesssim ||\theta||_{L^2(\mathbb{T}^2)}^4 \sum_{k \in \mathbb{Z}^2} |k|^{2\gamma-4} < \infty \]

(8)
if \( \gamma < 1 \). Now using the Besov embedding once again we get that \( \| \theta \circ K \theta \|_{L^2(\mathbb{T}^2)} \lesssim \| \theta \|_{L^2(\mathbb{T}^2)}^2 \) for all \( \gamma < 1 \) and in particular if we take \( 2\alpha - 2 \leq \gamma < 1 \) we get that \( \| \theta \circ K \theta \|_{L^2(\mathbb{T}^2)} \lesssim \| \theta \|_{L^2(\mathbb{T}^2)}^2 \). Then if we take the limit in this equation we obtain immediately that

\[
\| \theta^\varepsilon - \theta \|_{L^2(\mathbb{T}^2)} \to 0 \quad \text{as} \quad \varepsilon \to 0
\]

then we obtain the convergence of \( (\theta^\varepsilon, \theta^\varepsilon \circ K \theta^\varepsilon) \) to \( (\theta, \theta \circ K \theta) \) in \( \mathcal{H}^\alpha \) and this for every \( \theta \in \mathcal{H} \) which said us that every element in \( \mathcal{H} \) can be lifted in an rough distribution in \( \mathcal{D}^\alpha \) or in other word the following identity:

\[
\mathcal{D}^\alpha = \{ (\theta, \theta \circ K \theta - c); \quad \theta \in \mathcal{H}, c \in \mathbb{R} \}
\]

hold.

**Remark 2.8.** The extension property (7) was given for all smooth zero-mean function on the torus. But it extends to all elements of \( \mathcal{H} \) and this can be seen as follows.

Let \( \theta \in \mathcal{H} \) and \( \theta^\varepsilon \) a regularization of \( \theta \) such that \( \| \theta^\varepsilon - \theta \|_{L^2(\mathbb{T}^2)} \to 0 \) then due to the previous lemma we know that \( (\theta^\varepsilon, \theta^\varepsilon \circ K \theta^\varepsilon - c) \) converge to \( (\theta, \theta \circ K \theta - c) \) in \( \mathcal{H}^\alpha \) and this gives the convergence of \( \mathcal{S}_r(u_0, (\theta^\varepsilon, \theta^\varepsilon \circ K \theta^\varepsilon - c)) \) to \( \mathcal{S}_r(u_0, (\theta, \theta \circ K \theta - c)) \) in \( C(\mathbb{R}^+, \mathcal{C}^\alpha(\mathbb{T}^2)) \). Now taking the classical solution \( \mathcal{I}_c(u_0, \theta^\varepsilon, c) \) to

\[
\mathcal{L} u^\theta = f(u)\theta^\varepsilon - cf(u^\theta) f(u^\theta), \quad u(0, x) = u_0(x),
\]

we know that by definition it satisfies the relation

\[
\mathcal{S}_r(u_0, (\theta^\varepsilon, \theta^\varepsilon \circ K \theta^\varepsilon - c)) = \mathcal{I}_c(u_0, \theta^\varepsilon, c).
\]

And then taking the limit in this equation we obtain immediately that

\[
\lim_{\varepsilon \to 0} \mathcal{I}_c(u_0, \theta^\varepsilon, c) = \mathcal{S}_r(u_0, (\theta, \theta \circ K \theta - c)).
\]

Moreover we know by the Proposition 5.1 that the map \( \theta \mapsto \mathcal{S}_c(u_0, \theta, c) \) is continuous from \( \mathcal{H} \) to \( C(\mathbb{R}^+, L^2(\mathbb{T}^2)) \), then finally we get the relation

\[
\mathcal{S}_c(u_0, \theta, c) = \mathcal{I}_c(u_0, (\theta, \theta \circ K \theta - c))
\]

for all \( \theta \in \mathcal{H} \) and \( c \in \mathbb{R} \).

Recall from the introduction that \( \xi \) denotes zero mean spatial white noise on the two-dimensional torus. We consider a mollification of this noise. Let \( \psi \) be a radial bounded function with compact support which is continuous at the origin, with \( \psi(0) = 1 \), and set

\[
\xi^\varepsilon := \xi^\varepsilon(\psi) := \sum_{k \neq 0} \psi(\varepsilon k) \xi(k)e_k
\]

where \( (e_k) \) is the Fourier basis of \( L^2(\mathbb{T}^2) \) then at this point we have the following convergence result.
Theorem 3.2. \[ \text{Let } \alpha < 1. \text{ Then, with (diverging!) constants } c_\varepsilon = c_\varepsilon(\psi) \in \mathbb{R} \text{ given by} \]

\[
c_\varepsilon = \sum_{k \neq 0} \frac{|\psi(\varepsilon k)|^2}{|k|^2}
\]

we have that \( \Xi^\varepsilon = M(\xi^\varepsilon, c_\varepsilon(\psi)) \) converges in \( L^p(\Omega, \mathcal{F}) \) for all \( p > 1 \) and almost surely to some element \( \Xi^{\text{pam}} \in \mathcal{F}^\alpha \) such that \( (\Xi^{\text{pam}})^1 = \xi \). Moreover, this limit \( \Xi^{\text{pam}} \) is independent on the function \( \psi \) used to mollify the noise.

Now, following [8], the point is that with \( u^\varepsilon := \mathcal{A}(u_0, \Xi^\varepsilon) \), due to the constants \( c_\varepsilon(\psi) \), the function \( u^\varepsilon \) does not satisfy the classical equation but a modified equation given by

\[
\mathcal{L} u^\varepsilon = f(u^\varepsilon)\xi^\varepsilon - c_\varepsilon f(u^\varepsilon)f'(u^\varepsilon).
\]

One then deduces that \( u^\varepsilon \) converges to \( u = \mathcal{A}(u_0, \Xi) \) in \( C([0, T], C^\alpha(\mathbb{T}^2)) \) where the convergence is in \( L^p(\Omega, \mathcal{F}) \) for all \( p > 1 \) and almost surely.

3 Support theorem

3.1 Description of the strategy and support theorem for the white noise

Now to obtain the support theorem for our equation we begin by obtaining the result for the rough distribution \( \Xi^{\text{pam}} \) associated to the white noise and then we transfer our result to \( u \) by using the continuity of the map \( \mathcal{A} \). For that let us begin by characterizing the support of the white noise \( \xi \) in the Besov-Hölder space. Let us denote by \( \mathcal{C}^{0,\beta}(\mathbb{T}^2) \) the closure of smooth functions on the torus in the \( \beta \)-Besov-Hölder norm. Similarly, write \( \mathcal{C}^{0,\beta}(\mathbb{T}^2) \) for the closure of zero-mean, smooth functions. One easily sees that, for \( \beta < -1 \),

\[
\mathcal{C}^{0,\beta}(\mathbb{T}^2) \equiv \mathcal{C}^{\infty}(\mathbb{T}^2) = \mathcal{H}
\]

where all closures are with respect to \( \beta \)-Besov-Hölder norm and we recall that \( \mathcal{H} \) is the Cameron–Martin space associated to zero-mean, spatial white noise. This space is, trivially, a closed subspace of \( \mathcal{C}^\alpha(\mathbb{T}^2) \). Then we have this first result

Proposition 3.1. Let \((\Omega, \mathcal{F}, \mathbb{P})\) the abstract probability space associated to the white noise \( \xi \) and \( \xi^*\mathbb{P} \) the law of \( \xi \) viewed as probability measure of \( \mathcal{C}^{\alpha-2}(\mathbb{T}^2) \). Then the following equality

\[
\text{supp}(\xi^*\mathbb{P}) = \mathcal{C}^{0,\alpha-2}(\mathbb{T}^2)
\]

hold for all \( \alpha < 1 \).

Let \( \xi^\varepsilon \) defined as in the equation (9) then we know that \( \xi^\varepsilon \) converge almost surely to \( \xi \) in \( \mathcal{C}^{\alpha-2} \) for all \( \alpha < 1 \), then \( \xi \in \mathcal{C}^{0,\alpha-2} \) almost surely and this gives immediately the first inclusion \( \text{supp}(\xi^*\mathbb{P}) \subseteq \mathcal{C}^{0,\alpha-2}(\mathbb{T}^2) \). Now to prove the other inclusion let us introduce the translation operator \( T_h : \mathcal{F}(\mathbb{T}^2) \to \mathcal{F}(\mathbb{T}^2) \) for \( h \in \mathcal{H} \) which is defined by \( T_h \psi = h + \psi \), then at this point is not difficult to see that \( T_h \) is a continuous invertible operator from \( \mathcal{C}^{0,\alpha-2} \) to \( \mathcal{C}^{0,\alpha-2} \) with inverse \( T_{-h} \). Now to show our second inclusion we will need the following well-known Cameron Martin theorem.

Theorem 3.2. Let \( h \in \mathcal{H} \) then the law of \( \xi \) and the law of \( T_h \xi \) are equivalent.
A simple consequence of this theorem is that the support of the law of \( \xi \) is invariant by \( T_h \). Indeed let \( x \in \text{supp}(\xi \ast \mathbb{P}) \) then by definition we know that for any open set \( U \) of \( \mathcal{C}^{\alpha - 2}(\mathbb{T}^2) \) such that \( x \in U \) we have
\[
\mathbb{P}(\xi \in U) > 0
\]
Now let \( V \) an open set such that \( T_h x \in V \), then by continuity of \( T_h \) we know that there exist an open set \( U \) such that \( x \in U \) and \( T_h U \) is contained in \( V \) which gives :
\[
\mathbb{P}(\xi \in V) \geq \mathbb{P}(T_h \xi \in U) \sim \mathbb{P}(\xi \in U) > 0
\]
where we have used the Cameron Martin theorem. And then we can conclude that \( T_h x \in \text{supp}(\xi \ast \mathbb{P}) \). Let us now summarize our result

**Lemma 3.3.** Let \( h \in \mathcal{H} \) then \( T_h \text{supp}(\xi \ast \mathbb{P}) \subseteq \text{supp}(\xi \ast \mathbb{P}) \).

Now let \( x \in \text{supp}(\xi \ast \mathbb{P}) \) then we know that there exist \((x_n)\) a sequence of function in \( \mathcal{H} \) such that \( \lim_{n \to +\infty} x_n = x \) in the space \( \mathcal{C}^{\alpha - 2}(\mathbb{T}^2) \) or similarly \( T_{-x^n} x \to 0 \), then the invariance of the support of \( \xi \ast \mathbb{P} \) by the translation operator gives \( T_{-x^n} x \in \text{supp}(\xi \ast \mathbb{P}) \) and using the fact the support is a closed set in \( \mathcal{C}^{\alpha - 2}(\mathbb{T}^2) \) we obtain immediately that \( 0 \in \text{supp}(\xi \ast \mathbb{P}) \) and then \( \mathcal{H} \subseteq \text{supp}(\xi \ast \mathbb{P}) \) which gives the second inclusion.

### 3.2 Support theorem for the enhanced white noise

The goal of this section is to characterize the support of the law of \( \Xi_{\text{pam}} \), namely we have the following result :

**Theorem 3.4.** Let \( \alpha \in (2/3, 1) \) and let \( (\Xi_{\text{pam}}) \ast \mathbb{P} \) the law of \( \Xi_{\text{pam}} \) viewed as a probability measure in the space \( \mathcal{H}^{\alpha} \) (introduced in the Definition 2.6) then we have the following characterization :
\[
\text{supp}((\Xi_{\text{pam}}) \ast \mathbb{P}) = \left\{ (\theta, \theta \circ K \theta - c) : \theta \in \mathcal{H}, c \in \mathbb{R} \right\}^{\mathbb{R}^{\alpha}}
\]

Before giving the proof of this result let us recall that
\[
\Xi_{\text{pam}} = \lim_{\varepsilon \to 0} \mathcal{H}(\xi, \varepsilon) \]
where \( \varepsilon = \mathbb{E}[\xi \circ K \xi] = \sum_{k \neq 0} |\psi(\varepsilon k)|^2 |k|^{-2} \) where the convergence as we said before is a.s in the space \( \mathcal{H}^{\alpha} \). And from this we get immediately
\[
\text{supp}((\Xi_{\text{pam}}) \ast \mathbb{P}) \subseteq \mathcal{X}^{\alpha}
\]
To obtain the other inclusion we will need the following lemma which can be seen in the context of rough paths, as analogue of highly oscillatory approximations to the so called pure area rough path (will also be used in Lemma 3.12 below.)

**Lemma 3.5.** Let \( c \geq 0 \) then there exist \((X^{n,c})_{n \in \mathbb{N}}\) a sequence of smooth function such that :
1. \( ||X^{n,c}||_{\alpha - 2} \to N^{-\alpha + \infty} 0 \)
2. \( ||X^{n,c} \circ K X^{n,c} - c||_{2\alpha - 2} \to N^{-\alpha + \infty} 0 \)
for \( \alpha < 1 \). Moreover we can choose \( X^{n,c} \) such that
\[
X^{n,c}(x) = c^{1/2} 2^{n+1} \cos(2^n(z,x))
\]
with \( z = (1, 1) \).
Proof. Let \( Y^n(x) = 2^n e^{i(2^nz,x)} \). Then we see that by a simple computation that
\[
\Delta_1 Y^n(x) = \rho(2^{-\alpha} n) e^{i(2^nz,x)}, \quad \Delta_{-1} Y^n(x) = \chi(2^nz) 2^n e^{i(2^nz,x)}
\]
for all \( q \geq 0 \) and then we get easily that
\[
||Y^n||_{\alpha-2} \lesssim \max(2^{-\alpha}, 2^n \chi(2^nz)) \to n \to \infty 0
\]
And by a direct computation we see also that
\[
Y^n \circ KY^n = \frac{1}{2^{2n\alpha} |z|^2} Y^n \circ Y^n = |z|^{-2} e^{i2^{n+1}(z,x)}
\]
and then we get that
\[
||Y^n \circ KY^n||_{2\alpha-2} \lesssim \max(2^{-\alpha}, \chi(2^nz)) \to n \to \infty 0
\]
Now when \( c > 0 \) we can take \( X^{n,c}(x) = e^{1/2}(Y^n(x) + \overline{Y^n(x)}) = e^{1/2} 2^{n+1} \cos(2^n(z,x)) \) where \( \overline{Y^n} \) is the complex conjugate of \( Y^n \) and then obviously we have that :
\[
||X^{n,c} \circ KX^{n,c} - c||_{2\alpha-2} \lesssim ||Y^n \circ KY^n||_{2\alpha-2}
\]
Moreover we have the following equality
\[
X^{n,c} \circ KX^{n,c} = 2\mathcal{R}e(Y^n \circ KY^n) + 2Y^n \circ \overline{KY^n} = 2\mathcal{R}e(Y^n \circ KY^n) + c
\]
with \( \mathcal{R}e(Y^n \circ KY^n) \) is the real part of \( Y^n \circ KY^n \). And then we obtain immediately that
\[
||X^{n,c} \circ KX^{n,c} - c||_{2\alpha-2} \lesssim ||X^n \circ KY^n||_{2\alpha-2}
\]
And then we obtain immediately the following result, to be compared with Lemma 2.7.

Proposition 3.6. Let \( \alpha \in (2/3,1) \) then we have that
\[
\mathcal{H}_{\alpha}^\alpha = \{(\theta, \theta \circ K\theta - c) : \theta \in \mathcal{H}, c > 0 \}
\]
Proof. Let \( \theta \in \mathcal{H}, a \in \mathbb{R}, c > \max(0,a) \) and take \( X^{n,c-a} \) as in the Lemma 3.5 and define \( \theta^n = \theta + X^{n,c-a} \) then of course we have that
\[
||\theta^n - \theta||_{2\alpha-2} = ||X^{n,c-a}||_{2\alpha-2} \to n \to \infty 0
\]
and a quick computation gives
\[
\theta^n \circ K\theta^n = \theta \circ K\theta + X^{n,c-a} \circ K\theta + \theta \circ KX^{n,c-a} + X^{n,c-a} \circ KX^{n,c-a}
\]
And using the Bony estimate for the resonating term we get that
\[
||\theta \circ KX^{n,c-a}||_{2\alpha-2} \lesssim ||\theta \circ KX^{n,c-a}||_{\alpha-1} \lesssim ||\theta||_{L^2(T^2)} ||KX^{n,c-a}||_{\alpha} \to n \to \infty 0
\]
and by the same way we show that \( ||K\theta \circ X^{n,c-a}||_{2\alpha-2} \) vanish when \( n \) go to the infinity. Then we have shown that \( (\theta^n, \theta^n \circ K\theta^n - c) \) converge to \( (\theta, \theta \circ K\theta - a) \) which gives :
\[
(\theta, \theta \circ K\theta - a) \in \{(h, h \circ Kh - c) : h \in \mathcal{H}, c > 0 \}
\]
and finally we get
\[
\mathcal{H}_{\alpha}^\alpha \subseteq \{(\theta, \theta \circ K\theta - c) : \theta \in \mathcal{H}, c > 0 \}
\]
of course the other inclusion is an obvious fact.
\[\square\]
We get from this result that:
\[
\text{supp}(\Xi_{\text{pam}}) \subseteq \{(\theta, \theta \circ K \theta - c) : \theta \in \mathcal{H}, c > 0\} = \mathcal{X}^\alpha
\]

Now let us focus on the other inclusion. As in the case of the white noise we will need to introduce an appropriate translation operator on the space $\mathcal{X}^\alpha$. Let $h \in \mathcal{H}$ and we define for $\Xi = (\Xi^1, \Xi^2) \in \mathcal{H}^\alpha$ the following translation operator:
\[
T_h \Xi = (\Xi^1 + h, \Xi^2 + h \circ Kh + h \circ K \Xi^1 + \Xi^1 \circ Kh).
\]
Is not difficult to see that $T_h$ is a continuous invertible map on $\mathcal{H}^\alpha$, the inverse given by $T_{-h}$.

More precisely, we have

**Proposition 3.7.** Let $h \in \mathcal{H}$. Then we have
\[
||T_h \Xi_1 - T_h \Xi_2||_{\mathcal{X}^\alpha} \leq 2(||h||_\mathcal{H} + 1)||\Xi_1^1 - \Xi_2^1||_{\alpha-2} + ||\Xi_1^2 - \Xi_2^2||_{2\alpha-2}.
\]

**Proof.** By definition we have that:
\[
||T_h \Xi_1 - T_h \Xi_2||_{\mathcal{X}^\alpha} \leq ||\Xi_1^1 - \Xi_2^1||_{\alpha-2} + ||\Xi_1^2 - \Xi_2^2||_{2\alpha-2} + ||h \circ (K \Xi_1^1 - K \Xi_2^1)||_{\alpha-2} + ||(\Xi_1^1 - \Xi_2^1) \circ Kh||_{2\alpha-2}.
\]
Using the Bony estimates for the resonating term (see (2.2)) and the Schauder estimate for $K$ (Proposition 5.2) we obtain that:
\[
||h \circ (K \Xi_1^1 - K \Xi_2^1)||_{\alpha-2} \lesssim ||h \circ (K \Xi_1^1 - K \Xi_2^1)||_{\alpha-1} \lesssim ||\Xi_1^1 - \Xi_2^1||_{\alpha-2}||h||_\mathcal{H}.
\]
Now by the same argument we get:
\[
||K \circ (\Xi_1^1 - \Xi_2^1)||_{\alpha-2} \lesssim ||\Xi_1^1 - \Xi_2^1||_{\alpha-2}||h||_\mathcal{H}
\]
which completes the proof. $\square$

Now we have the following proposition which is the equivalent of the cameron martin theorem for $\Xi_{\text{pam}}$

**Proposition 3.8.** We have that
\[
\mathbb{P}(\{\omega \in \Omega; \quad T_h \Xi_{\text{pam}}(\omega) = \Xi_{\text{pam}}(\omega + h) \quad \text{for all } h \in \mathcal{H}\}) = 1
\]
As a consequence (of the standard Cameron–Martin theorem for Gaussian measures) the laws of $\Xi_{\text{pam}}$ and $T_h \Xi_{\text{pam}}$ are equivalent.

**Proof.** Without loss of generality we can assume that $\Omega = \mathcal{S}'(\mathbb{T}^2)$ and $\mathbb{P}$ is the law of the white noise with zero mean and that $\xi$ is given by the projection process (ie: for $\omega \in \Omega$, $\xi(\phi) = \omega(\phi)$ for all $\phi \in \mathcal{S}(\mathbb{T}^2)$). Let us now define $\Xi_{\text{pam},\epsilon}$ by
\[
\Xi_{\text{pam},\epsilon}(\omega) := (\omega^\epsilon, \omega^\epsilon \circ K \omega^\epsilon - c_\epsilon)
\]
with $\omega^\epsilon := \sum_k \psi(\epsilon k) \hat{\omega}(k) e_k$ then we know that there exist a measurable set $\mathcal{A}$ with $\mathbb{P}(\mathcal{A}) = 1$ and such that for all $\omega \in \mathcal{A}$ the convergence $\Xi_{\text{pam},\epsilon}(\omega) \to \Xi_{\text{pam}}(\omega)$ holds in $\mathcal{X}^\alpha$. Now taking $\omega \in \mathcal{A}$ using the fact that
\[
\Xi_{\text{pam},\epsilon}(\omega + h) = T_h \Xi_{\text{pam}}(\omega)
\]
and the continuity of the translation operator \((h, \Xi) \mapsto T_h\Xi\) we see that \(\Xi^{\text{pam}}(\omega + \varepsilon)\) is also convergent to \(T_h\Xi^{\text{pam}}(\omega)\) on the other hand we have that
\[
\Xi^{\text{pam}}(\omega + h) = \lim_{\varepsilon \to 0} \Xi^{\text{pam}, \varepsilon}(\omega + h)
\]
thanks to the fact that the limit in the r.h.s exist and the fact that very realization of \(\Xi^{\text{pam}}\) is the limit of \(\Xi^{\text{pam}, \varepsilon}\). This of course allow us to identify
\[
T_h\Xi^{\text{pam}}(\omega) = \Xi^{\text{pam}}(\omega + h)
\]
\[\square\]

Now as in the case of the white noise this last result allows to get the invariance of the support by \(T_h\). Indeed we have the following corollary

**Corollary 3.9.** Let \(h \in \mathcal{H}\) and take \(\Xi \in \text{supp}(\langle \Xi^{\text{pam}} \rangle, \mathbb{P})\) then \(T_h\Xi^{\text{pam}} \in \text{supp}(\langle \Xi^{\text{pam}} \rangle, \mathbb{P})\)

Now to proceed as in the white noise case we will show that the support of \(\Xi^{\text{pam}}\) contain the 0 element. This exactly the propose of the next proposition :

**Proposition 3.10.** Given \(a \in \mathbb{R}\) then there exist \(\Xi \in \text{supp}(\langle \Xi^{\text{pam}} \rangle, \mathbb{P})\) and \(h^k_n \in \mathcal{H}\) such that
\[
T_{-h^k_n}\Xi \to \varepsilon \to 0 \ (0, -a) \ \text{ in } \mathcal{H}^\alpha
\]

**Proof.** The proof of this proposition is based on the following lemma

**Lemma 3.11.** Let \(b_\varepsilon\) defined by
\[
b_\varepsilon := \sum_{k \in \mathbb{Z}^2, k \neq 0} \frac{|\psi(\varepsilon k)|}{k^2}
\]
then the following convergence
\[
\xi \circ K_\varepsilon - b_\varepsilon \to \varepsilon \to 0 \int_{\Omega} \Xi^{2, \text{pam}}, \ \ \xi^\varepsilon \circ K_\varepsilon - b_\varepsilon \to \varepsilon \to 0 \int_{\Omega} \Xi^{2, \text{pam}}
\]
hold in the space \(L^p(\Omega, C^{2\alpha - 2}(\mathbb{T}^2))\) for every \(p > 1\).

**Proof.** We have by a direct computation that :
\[
b_\varepsilon = \sum_{|i - j| \leq 1, k \in \mathbb{Z}^2: k_{12} = k} \rho(2^{-i}k_1)\rho(2^{-j}k_2)\mathbb{E}[\xi(k_1)\xi(k_2)]|\psi(\varepsilon k_1)|k_2|^2 |e_k - \sum_{k_1} \psi(\varepsilon k_1)|k_1|^{-2}
\]
where we wrote \(k_{12}\) instead of \(k_1 + k_2\) for shorter notation. In view of the convergence giving in the Theorem 2.9 it suffice to prove that :
\[
(\xi^\varepsilon \circ K_\varepsilon - b_\varepsilon) - (\xi^\varepsilon \circ K_\varepsilon - c_\varepsilon) \to \varepsilon \to 0
\]
in \(L^p(\Omega, C^{\delta}(\mathbb{T}^2))\). A quick computation gives
\[
|\Delta q((\xi^\varepsilon \circ K_\varepsilon - b_\varepsilon) - (\xi^\varepsilon \circ K_\varepsilon^\varepsilon - c_\varepsilon))(x)|^2
= \sum_{|i_1 - i_2| \leq 1, k_{12} = k} \rho(2^{-i}k_1)\rho(2^{-j}k_2)\Pi_{i=1}^2 \rho(2^{-i}k_i)\rho(2^{-i}k_i')
\times \psi(\varepsilon k_1)\psi(\varepsilon k_2)(1 - \psi(\varepsilon k_2))(1 - \psi(\varepsilon k_2'))
\times (\xi(k_1)\xi(k_2) - \varepsilon \mathbb{E}[\xi(k_1)\xi(k_2)])(\xi(k_1')\xi(k_2') - \varepsilon \mathbb{E}[\xi(k_1')\xi(k_2')])e_{k-k'}(x).
\]
Then using the Wick theorem we obtain that:

\[ E[|\Delta_q((\xi^\epsilon \circ K \xi - b_\epsilon) - (\xi^\epsilon \circ K \xi^\epsilon - c_\epsilon))(x)|^2] = J_1^\epsilon + J_2^\epsilon \]

with

\[ J_1^\epsilon = \sum_{q \leq j_1 \sim j_2 \sim j_1 \sim j_2} |\rho(2^{-q}k)|^2 \Pi_{i=1}^2 (\rho(2^{-j_1}k_i) \rho(2^{-j_2}k_i)) |\psi(\varepsilon k_1)|^2 |k_2|^{-4} |\psi(\varepsilon k_2) - 1|^2 \]

and remarking that this sum is restricted to the frequency \(|k| \lesssim |k_2| \sim |j_1|\) we get easily:

\[ \sum_{k_2 \neq 0} |k_2|^{-2+\delta} |\psi(\varepsilon k_2) - 1|^2 \to^{\varepsilon} 0 \]

by dominate convergence. Then putting this last bound in the definition of \(J_1^\epsilon\) allows to obtain the following inequality:

\[ J_1^\epsilon \lesssim r(\varepsilon)^{2q_2 \delta}. \]

To finish our argument let us observe that:

\[ J_2^\epsilon = \sum_{q \leq j_1 \sim j_2 \sim j_1 \sim j_2} |\rho(2^{-q}k)|^2 \Pi_{i=1}^2 (\rho(2^{-j_1}k_i) \rho(2^{-j_2}k_i)) |\psi(\varepsilon k_1)| |\psi(\varepsilon k_2)| |\psi(\varepsilon k_2) - 1| |\psi(\varepsilon k_1) - 1| |k_1|^{-2} |k_2|^{-2} \]

and then due to the fact that the sum is over the frequency \(|k_1| \sim |k_2|\) we see that \(J_1^\epsilon \sim J_2^\epsilon\) from which we can conclude the following bound:

\[ E[|\Delta_q((\xi^\epsilon \circ K \xi - b_\epsilon) - (\xi^\epsilon \circ K \xi^\epsilon - c_\epsilon))(x)|^2] \lesssim r(\varepsilon)^{2q_2 \delta} \]

for all \(\delta > 0\) and \(\rho < \delta\) and then using the Gaussian hypercontractivity (see [13]) and the Besov embedding we get:

\[ \|((\xi^\epsilon \circ K \xi - b_\epsilon) - (\xi^\epsilon \circ K \xi^\epsilon - c_\epsilon))\|_{L^p(\Omega, C^{-2q_2 \delta})} \lesssim \|((\xi^\epsilon \circ K \xi - b_\epsilon) - (\xi^\epsilon \circ K \xi^\epsilon - c_\epsilon))\|_{L^p(\Omega, B^{-2q_2 \delta}_{p,p})} \]

\[ \lesssim \sum_{q \geq 1} 2^{-2q_2 \delta} \int_{T^d} E[|\Delta_q((\xi^\epsilon \circ K \xi - b_\epsilon) - (\xi^\epsilon \circ K \xi^\epsilon - c_\epsilon))(x)|^2] \frac{dx}{2} \lesssim r(\varepsilon)^{p/2} \]

which finishes the proof of the lemma.

\[ \square \]

**Lemma 3.12.** Let us define \(\xi^n\) and \(c_n\)

\[ \xi^n := \sum_{|k| \leq \nu 2^n} \hat{\xi}(k) e_k, \quad c_n = \sum_{|k| \leq \nu 2^n} \frac{1}{|k|^2} \]
for \( \nu > 0 \). Then for \( \nu \) large enough (depending only on the annulus and the ball given in the Littlewood-Paley decomposition) the following convergence

\[
\lim_{n \to +\infty} T_{\xi^n + X^n c_n - a} \Xi_{pam} = (0, -a)
\]

holds in \( \mathcal{H}^\alpha \) in probability. Where \( X^n c_n - a \) is given by the Lemma 3.5. Then to obtain the statement of the Proposition 3.10 is suffice to take \( \Xi = \Xi_{pam}(\omega) \) and \( h^k = \xi^n(\omega) - X^n c_n - a \) with \( \omega \) is fixed in the set of probability one on which the last convergence hold almost surely along the subsequence \( T_{\xi^n + X^n c_n - a} \Xi_{pam} \).

**Proof.** We have by definition that

\[
(T_{\xi^n + X^n c_n - a} \Xi_{pam})^1 = \xi^n - \xi^n + X^n c_n - a.
\]

To prove that the right hand side of this equality converge to 0 is suffice to remark that \( ||\xi^n - \xi^n||_{\alpha - 2} \to_n +\infty 0 \) and then it suffice to prove that

\[
||X^n c_n - a||_{\alpha - 2} \to_n +\infty 0
\]

but following the proof of the Lemma 3.5 we see that

\[
||X^n c_n - a||_{\alpha - 2} \lesssim (c_n)^{1/2} \max(2^{-\alpha(1 - \alpha)}, 2^n k).
\]

Then recall in that

\[
c_n = \sum_{|k| \leq 2^n k \in \mathbb{Z}^2} |k|^{-2} \lesssim \nu n
\]

we deduce easily that

\[
||X^n c_n - a||_{\alpha - 2} \lesssim n^{1/2} \max(2^{-\alpha(1 - \alpha)}, 2^n \chi(2^n k)) \to 0.
\]

Which is gives the needed convergence for the first component of \( T_{\xi^n + X^n c_n} \Xi_{pam} \). Now by definition we have that

\[
(T_{\xi^n + X^n c_n - a} \Xi_{pam})^2 = \Xi_{pam,2} + \xi^n \circ K \xi^n - \xi^n \circ K \xi^n - \xi \circ K \xi^n + (\xi - \xi^n) \circ K X^n c_n - a
\]

\[+ X^n c_n - a \circ K(\xi - \xi^n) + X^n c_n - a \circ K X^n c_n - a.
\]

And let us remark that

\[
\text{supp}(\mathcal{F} (\xi^n - \xi)) \subseteq \{|k| > \nu 2^n\}
\]

and that

\[
\text{supp}(\mathcal{F} (X^n c_n - a)) \subseteq \{|k| = 2^n |z|\}
\]

and then we can choose \( \nu \) large enough (depending only on the size of the annulus and the ball which given in the definition of \( \chi \) and \( \rho \) ) such that

\[
\Delta_i(\xi^n - \xi) \Delta_j(X^n c_n - a) = 0
\]

for \( |i - j| \leq 1 \). And then we get immediately that

\[
X^n c_n - a \circ K(\xi - \xi^n) = (\xi - \xi^n) \circ X^n c_n - a = 0
\]
for all \( n \). Then we see that
\[
(T_{-\xi^n+X^n,c_n-a}^{\text{pam}})^2 = \Xi^{\text{pam},2} + (\xi^n \circ K\xi^n - c_n) + (c_n - \xi^n \circ K\xi) + (c_n - \xi \circ K\xi^n) + (X^n,c_n-a \circ KX^n,c_n-a - (c_n-a)) - a.
\]

Now using the Lemma 3.11 we can see that
\[
||(|\xi^n \circ K\xi^n - c_n) + (c_n - \xi^n \circ K\xi) + (c_n - \xi \circ K\xi^n)||_{2a} \to n \to +\infty 0
\]
in probability. To obtain the needed convergence is suffice to show that
\[
||X^n,c_n-a \circ KX^n,c_n-a - (c_n-a)||_{2a} \to n \to +\infty 0.
\]

Once again following the argument given in the Lemma 3.5 we see easily that
\[
||X^n,c_n \circ KX^n,c_n - (c_n-a)||_{2a} \lesssim n \max(2^{-2n(1-a)}, \chi(2^n|z|)) \to n \to +\infty 0
\]
we have used the fact that \( c_n \lesssim n \). This completes the proof.

Take \( \Xi \in \text{supp}(\Xi^{\text{pam}}) \) and \( h^k \in \mathcal{H} \) such that :
\[
T_{-h^k} \Xi \to^{k \to +\infty} (0,-a) \quad \text{in} \quad \mathcal{H}^\alpha
\]
(this is possible thanks the Proposition 3.10). Moreover we know that the support of \( \Xi^{\text{pam}} \) is invariant by translation, then \( T_{h^k}\Xi^{\text{pam}} \in \text{supp}(\Xi^{\text{pam}})_\text{p} \) for all \( k \) which give us that \( (0,-a) \in \text{supp}(\Xi^{\text{pam}})_\text{p} \). Once again by the invariance by translation we get that
\[
\mathcal{X}^\alpha \subseteq \text{supp}(\Xi^{\text{pam}})_\text{p}
\]
which finish the proof of the Th. 3.4.

Before going into the proof of the Theorem 1.1 let us observe the fact that the constant \( c \) can’t dropped from the space \( \mathcal{X}^\alpha \), indeed we claim that

**Lemma 3.13.** Given \( \alpha \in (2/3,1) \), then the closure of the set
\[
\{(h, h \circ Kh), \quad h \in \mathcal{H}\}
\]
in the space \( \mathcal{H}^\alpha \) is strictly embedded in \( \mathcal{X}^\alpha \).

**Proof.** Let assume that there exist \( h_n \) in \( \mathcal{H} \) such that \( (h_n, h_n \circ Kh_n) \) converge in \( \mathcal{H}^\alpha \) to \( (0,-1) \).

Then the point is that now
\[
\Delta(Kh_n)^2 = 2|\nabla Kh_n|^2 - 2h_n Kh_n
\]
with \( \nabla \) is the gradient operator. Then using the fact that \( h_n \to^{n \to +\infty} 0 \) in \( \mathcal{C}^{\alpha-2} \) and thus \( \Delta(Kh_n)^2 \to^{n \to +\infty} 0 \) in \( \mathcal{C}^{\alpha-2}(\mathbb{T}^2) \). On the other side using the Bony estimates (2.2) and the fact that \( h_n \circ Kh_n \to^{n \to +\infty} -1 \) we obtain easily that :
\[
h_n Kh_n = h_n < Kh_n + h_n > Kh_n + h_n \circ Kh_n \to^{n \to +\infty} -1
\]
in the space \( \mathcal{C}^{\alpha-2}(\mathbb{T}^2) \). Which allow us to conclude that \( 2|\nabla Kh_n|^2 \to -1 \) in the space \( \mathcal{C}^{\alpha-2}(\mathbb{T}^2) \) which is of course impossible and thus a such sequence can’t exist which end the proof due to the fact \( (0,-1) \in \mathcal{X}^\alpha \).
4 Proof of the Theorem 1.1

We know by construction that \( u = \mathcal{S}(u_0, \Xi_{\text{pam}}) \) and that \( \Xi_{\text{pam}} \in \mathcal{S}^\alpha \) a.s. then we can conclude that there exist \( \theta^n \in \mathcal{H} \) and such that \( \Xi_{\text{pam}} = \lim_n (\theta^n, \theta^n \circ K \theta^n - c_n) \) which by the continuity of the map \( \mathcal{S} \) give

\[
u = \lim_n \mathcal{S}(u_0, (\theta^n, \theta^n \circ K \theta^n - c_n)) = \lim_n \mathcal{S}(u_0, \theta^n, c_n)
\]
a.s in \( C([0, T], \mathcal{C}^\alpha (T^2)) \). We then have that

\[
\text{supp}(u*P) \subseteq \{\mathcal{S}(u_0, h, c), \ h \in \mathcal{H}, c > 0\} ^{C([0, T], \mathcal{C}^\alpha (T^2))}
\]

The other inclusion is more interesting. Now,

\[
\text{supp}(\Xi_{\text{pam}}) = \{\theta, \theta \circ K \theta - c \in \mathcal{H}, c > 0\} ^{\mathcal{S}^\alpha}
\]

ensures that for any \( \eta > 0, c > 0 \) and \( \theta \in L^2 (T^2) \) we have:

\[
P (||\Xi_{\text{pam}} - \mathcal{M}(\theta, c)||_{\mathcal{S}^\alpha} < \eta) > 0.
\]

Let \( \delta > 0 \) then by the continuity of \( \mathcal{S} \) there exist \( \eta := \eta (\delta, \theta, c) > 0 \) such that \( ||\Xi_{\text{pam}} - \mathcal{M}(\theta, c)||_{\mathcal{S}^\alpha} \leq \eta \Rightarrow ||u - \mathcal{S}(u_0, \theta, c)||_{C([0, T], \mathcal{C}^\alpha (T^2))} \leq \delta \) and then

\[
P (||u - \mathcal{S}(u_0, h, c)||_{C([0, T], \mathcal{C}^\alpha (T^2))} \leq \delta) \geq P (||\Xi_{\text{pam}} - \mathcal{M}(\theta, c)||_{\mathcal{S}^\alpha} < \eta) > 0.
\]

The proof is then finished.

5 Appendix

**Proposition 5.1.** Let \( T > 0 \). Given \( f \in C^2_R \), \( u_0 \in L^2 (T^2) \) and \( h \in \mathcal{H} \) then there exists a unique global solution \( v \in C([0, T]; L^2 (T^2)) \) to the equation:

\[
\mathcal{L} v = f(v)h - cf'(v)f(v), \ u(0, x) = u_0 (x)
\]

Moreover the map \( h \mapsto v \) is continuous from \( \mathcal{H} \) to \( C(R^+, L^2 (T^2)) \)

**Proof.** Let \( a, b \in L^2 (T^2) \) then by a direct computation we get:

\[
|\mathcal{F}((f(a) - f(b))h)(k)| = \sum_{k_1 + k_2 = k} |\mathcal{F}(f(a) - f(b))(k_1)\mathcal{F}(h)(k_2)| \lesssim ||f(a) - f(b)||_{L^2 (T^2)} ||h||_{\mathcal{H}}
\]

moreover we have that \( ||f(a) - f(b)||_{L^2} \lesssim ||f'||_{L^\infty (R)} ||a - b||_{L^2 (T)} \). And then

\[
||f(a) - f(b)||_{H^\gamma} \lesssim ||f'||_{L^\infty (R)} ||h||_{\mathcal{H}} ||a - b||_{L^2 (T^2)}
\]

for all \( \gamma < -1 \). Now is suffice to remark that if \( h \in C([0, T], H^\gamma (T^2)) \) and denoting by \( \mathcal{P} = e^{t\Delta} \) the heat flow then the following bound:

\[
\left\| \int_0^t \mathcal{P}_{t-s} h_s ds \right\|_{L^2 (T^2)} \lesssim \int_0^t ||\mathcal{P}_{t-s} h_s||_{L^2} ds \lesssim \int_0^t (t-s)^{\gamma/2} ds ||h||_{C([0, T], H^\gamma (T^2))} \lesssim T^{1+\gamma/2} ||h||_{C([0, T], H^\gamma (T^2))}
\]
hold for $\gamma > -2$. Introducing the map $\Gamma : C([0, T], L^2(\mathbb{T}^2)) \to C([0, T], L^2(\mathbb{T}^2))$ defined by

$$\Gamma_T(v) = P_t u_0 + \int_0^t \mathcal{P}_{t-s} f(v_s) h ds - c \int_0^t \mathcal{P}_{t-s} f'(v_s) f(v_s) ds$$

Due to the last computation this map is well defined moreover it satisfy the following bound:

$$\|\Gamma_T(u) - \Gamma_T(v)\|_{C([0, T], L^2(\mathbb{T}^2))} \lesssim T^{1+\gamma/2}(1 + \|f\|_{L^\infty(\mathbb{R})} + \|f'\|_{L^\infty(\mathbb{R})} + \|f''\|_{L^\infty(\mathbb{R})})^2 \|h\|_H \|u - v\|_{C([0, T], L^2(\mathbb{T}^2))}$$

for $T < 1$ and some $\gamma \in (-2, -1)$. Then choosing $T^*$ small enough we can see that $\Gamma$ become a contraction on $C([0, T], L^2(\mathbb{T}^2))$ into itself and then it admit a unique fix point $v^h$. Due to the fact that $T^*$ does not depend on $\|u_0\|_{L^2}$ we can iterate our result to obtain a global solution.

Now we will us focus on the continuity of the map $h \mapsto v^h$ and let $h_1, h_2 \in \mathcal{H}$ and $R > 0$ such that $\|h_1\| + \|h_2\| \leq R$ then we have by definition that:

$$v^{h_1} - v^{h_2} = \int_0^t ds \mathcal{P}_{t-s} f(h_1 - h_2) + \int_0^t ds \mathcal{P}_{t-s} (f(v^{h_2}) - f(v^{h_1})) h_2 + c \int_0^t ds \mathcal{P}_{t-s} (g(v^{h_1}) - g(v^{h_2}))$$

with $g = f'f$. Due to the estimate used to stand the fixed point argument we easily get that:

$$\|v^{h_1} - v^{h_2}\|_{C([0, T], L^2(\mathbb{T}^2))} \lesssim T \|f\|_{L^\infty(\mathbb{T}^2)} \|h_2 - h_1\|_H + T^{1/2+1}(R \|f'\|_{L^\infty(\mathbb{R})} + c \|g'\|_{L^\infty(\mathbb{R})}) \|v^{h_1} - v^{h_2}\|_{C([0, T], L^2(\mathbb{T}^2))}$$

then choosing $T_1 > 0$ small enough such that $T_1^{1/2+1}(R \|f'\|_{L^\infty(\mathbb{R})} + c \|g'\|_{L^\infty(\mathbb{R})}) < 1/2$ allow to obtain that:

$$\|v^{h_1} - v^{h_2}\|_{C([0, T_1], L^2(\mathbb{T}^2))} \lesssim T_1 \|f\|_{L^\infty(\mathbb{T}^2)} \|h_2 - h_1\|_H$$

Now iterating this procedure allow to get finally that:

$$\|v^{h_1} - v^{h_2}\|_{C([0, T], L^2(\mathbb{T}^2))} \lesssim R \|h_2 - h_1\|_H$$

for every $T > 0$, which end the proof.

Recall that $\hat{\cdot}$ indicates zero-mean of elements in the appropriate function spaces.

**Proposition 5.2.** Let $T > 0$, the map:

$$\theta \in \dot{H}^\alpha \mapsto -\Delta \theta \in \dot{H}^{\alpha-2}$$

is invertible. In particular, its inverse

$$K : \dot{H}^\alpha \to \dot{H}^{\alpha+2}$$

is well-defined and is a continuous linear operator. The same statement holds if we replace the space $\dot{H}^\alpha(\mathbb{T}^d)$ by $C^\alpha(\mathbb{T}^d)$.

**Proof.** For $f \in S'(\mathbb{T}^2)$ with $\hat{f}(0) = 0$ the equation

$$-\Delta \theta = f, \quad \hat{\theta}(0) = 0$$

admit a unique solution $\theta \in \mathcal{S}'(\mathbb{T}^2)$ defined by $\hat{\theta}(k) = |k|^{-2} \hat{f}(k)$ for $k \neq 0$ and $\hat{\theta}(0) = 0$. Moreover by a direct computation we see that if $f \in H^\alpha$ then $\|\theta\|_{H^{\alpha+2}} \approx \|f\|_{H^\alpha}$ which gives the statement for the Sobolev space. Now if $f \in C^\alpha(\mathbb{T}^d)$ we have by a direct application of the Proposition 2.3 that $\|\theta\|_{\alpha+2} \lesssim \|f\|_\alpha$ and this finishes the proof.
Now the following lemma ensure that the constant $c$ can’t be set it to zero without cost.

**Lemma 5.3.** Let $\alpha < 1$, $f$ the identity function and $u^0 \equiv 1$ then in this case the closure of the set
\[
\{ \mathcal{F}(u^0, h, 0), \quad h \in \mathcal{H} \}
\]
in the space $C([0, T], C^\alpha(\mathbb{T}^2))$ is strictly contained in the support of the law of $u$ characterized in the Theorem 1.1.

**Proof.** Let $v$ the unique solution of the equation
\[
\mathcal{L}v = cv, \quad v(0, x) = 1
\]
for some fixed $c < 0$. Of course $v$ have the explicit formula $v(t, x) = e^{ct}$. Now let $v_n$ a sequence of function which converge to $v$ in $C([0, T], C^\alpha(\mathbb{T}^2))$ and such that
\[
\mathcal{L}v_n = v_nh_n, \quad v_n(0, x) = 1
\]
for some $h_n \in \mathcal{H}$. By the Feynman-Kac formula we have immediately that
\[
v_n(t, x) = \mathbb{E} \left[ e^{\int_0^t h_n(B_s + x) ds} \right] > 0
\]
with $B$ is a Brownian motion. Then if we set $\tilde{v}_n = \log v_n$ we can see that $\tilde{v}$ satisfy the following equation
\[
\mathcal{L}\tilde{v}_n = |\nabla \tilde{v}_n|^2 + h_n
\]
By passing to the integral on $[0, T] \times \mathbb{T}^2$ in this last equation and observing that $\int_{\mathbb{T}^2} h_n = 0$ we get
\[
\tilde{v}_n(T, x) = \int_0^T \int_{\mathbb{T}^2} |\nabla \tilde{v}_n(t, x)|^2 dx dt \geq 0
\]
Now the point is that $\tilde{v}_n(T, x)$ converge to $cT < 0$ and then we conclude that $v$ can’t be approximated by sequence which satisfy the equation (5) which end the proof. 

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