Basic Properties of
Singular Fractional Order System with order (1,2)

Xiaogang Zhu, Jie Xu and Junguo Lu *

Abstract
This paper focuses on some properties, which include regularity, impulse, stability, admissibility and robust admissibility, of singular fractional order system (SFOS) with fractional order $1 < \alpha < 2$. The definitions of regularity, impulse-free, stability and admissibility are given in the paper. Regularity is analysed in time domain and the analysis of impulse-free is based on state response. A sufficient and necessary condition of stability is established. Three different sufficient and necessary conditions of admissibility are proved. Then, this paper shows how to get the numerical solution of SFOS in time domain. Finally, a numerical example is provided to illustrate the proposed conditions.

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* Jie Xu and Junguo Lu are with the School of Electronic Information and Electrical Engineering, Shanghai Jiao Tong University, Shanghai, 200240 China
1 Introduction

Fractional order systems can describe the real physical systems better than integer order systems because the real objects are generally fractional. A lot of systems have been studied via fractional order systems, such as wavelet transform [1], viscoelastic systems [2] and others ([3, 4, 5, 6, 7]).

Singular systems have been widely studied in many fields ([8, 9, 10]) because singular systems can describe real physical systems more directly than regular systems. However, very few researches have been studied on singular fractional order systems (SFOS), most of which are about stability. In [11], a sufficient and necessary condition for regularity is given; Based on regularity and free impulse, this paper also gives a sufficient condition for stability. Sufficient and necessary conditions for regularity and admissibility with fractional order $0 < \alpha < 1$ are given in [12], respectively. Some other papers study the stability of SFOS via linear matrix inequality (LMI) ([13, 14, 15]) and some study the stability of SFOS via transforming the SFOS into normal ones ([16, 17]).

However, none of them prove the regularity, free impulse and stability in time domain, which can prove these properties more directly. Moreover, to the best of our knowledge, there exists no research on free impulse and admissibility with fractional order $1 < \alpha < 2$. Therefore, in this paper we give the sufficient and necessary conditions of regularity, free impulse, stability and admissibility for SFOS with fractional order $1 < \alpha < 2$, respectively.

This paper is organized as follows.

In section II, the definition of Caputo’s fractional derivative and SFOS are recalled. And some useful lemma are provided. In section III, regularity and impulse are analysed in time domain. In section IV, sufficient and necessary conditions of stability and admissibility are proved, respectively. In section V, sufficient conditions of robust admissibility are presented. Finally, in section VI, numerical solution and example are illustrated. Conclusion will be given in section VII.

Notation 1. For a matrix $A$, its transpose and complex conjugate transpose are denoted by $A^T$ and $A^*$, respectively. $\mathbb{C}_- = \{ s \in \mathbb{C} \mid \text{Re}(s) < 0 \}$. $\text{Sym}(A)$ denotes $A + A^*$. Denote pair $(E_I, A_I)$ as the autonomous singular integer order system (SIOS) $E_I \dot{x}(t) = A_I x(t)$. Denote triplet $(E, A, \alpha)$ as the autonomous SFOS $E \dot{x}(t) = A x(t)$. The notation $\bullet$ stands for the symmetric component in matrix.

2 Preliminaries

In this paper, we use the Caputo’s fractional derivative, of which the Laplace transform allows utilization of initial values. The Caputo’s fractional derivative is defined as [18]
\[ aD^\alpha_t f(t) = \frac{1}{\Gamma(\alpha-n)} \int_a^t f^{(n)}(\tau) (t-\tau)^{\alpha+1-n} d\tau \]

where \( n \) is an integer satisfying \( 0 \leq n-1 < \alpha < n \); \( \Gamma(\cdot) \) is the Gamma function which is defined as

\[ \Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \]

In the following of the paper, \( aD^\alpha_t \) is denoted by \( D^\alpha \).

A two-parameter function of the Mittag-Leffler type is defined as [18]

\[ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \]

where \( \alpha > 0, \beta > 0 \).

And \( \delta^{(-\beta)}(t) \) means

\[ \delta^{(-\beta)}(t) = \begin{cases} \frac{t^{\beta-1}}{\Gamma(\beta)} & t > 0 \\ 0 & t < 0 \end{cases} \quad \beta \in \mathbb{R} \]

whose Laplace transform is

\[ L[\delta^{-\alpha}(t)] = s^{-\alpha}, \quad \text{Re}(s) > 0 \]

Consider the singular fractional order system (SFOS)

\[
\begin{cases}
ED^\alpha x(t) = Ax(t) + Bu(t) \\
y(t) = Cx(t) + Du(t)
\end{cases}
\]

(1)

where \( x(t) \in \mathbb{R}^n \) is the state of the system composed of state variables; \( u(t) \in \mathbb{R}^p \) is the control input; \( y(t) \in \mathbb{R}^q \) is the measure output; \( E, A \in \mathbb{R}^{n \times n}; B, C, D \) are constant matrices with appropriate dimensions; \( D^\alpha \) represents the Caputo fractional derivative; \( \alpha \) is the order of the SFOS and \( 1 < \alpha < 2 \).

The finite eigenvalues of SFOS is

\[ \lambda(E, A) = \{ s \mid s \in \mathbb{C}, |s| < \infty, \det(sE - A) = 0 \} \]

The finite pole set for the system is

\[ \sigma(E, A) = \{ s \mid s \in \mathbb{C}, |s| < \infty, \det(s^\alpha E - A) = 0, 0 < \alpha < 2 \} \]

and \( \sigma(I, A) \) will be specified as \( \sigma(A) \). Obviously, \( \lambda(E, A) = \sigma^\alpha(E, A) \).

The following lemmas and definitions will be useful.

**Lemma 2.** [19] For any two matrices \( E, A \in \mathbb{R}^{m \times n} \), there always exist two nonsingular matrices \( Q, P \) such that

\[
\begin{align*}
\tilde{E} & \triangleq QEP = \text{diag}(0, L_1, L_2, ..., L_p, L'_1, L'_2, ..., L'_q, I, N) \\
\tilde{A} & \triangleq QAP = \text{diag}(0, J_1, J_2, ..., J_p, J'_1, J'_2, ..., J'_q, A_1, I)
\end{align*}
\]

(2)
Consider the following initial-value problem:

\[ a \frac{D_t^{\sigma_n} y(t)}{D_t} + \sum_{j=1}^{n-1} p_j(t) a \frac{D_t^{\sigma_{n-j}} y(t)}{D_t} + p_n(t) y(t) = f(t) \quad (0 < t < T < \infty) \]

\[ [a \frac{D_t^{\sigma_k} y(t)}{D_t}]_{t=0} = b_k, \quad k = 1, 2, ..., n \quad (4) \]

where

\[ a \frac{D_t^{\sigma_k}}{D_t} \equiv a \frac{D_t^{\sigma_k}}{D_t} a \frac{D_t^{\sigma_{k-1}}}{D_t} ... a \frac{D_t^{\sigma_1}}{D_t} \]

\[ a \frac{D_t^{\sigma_k-1}}{D_t} \equiv a \frac{D_t^{\sigma_k-1}}{D_t} a \frac{D_t^{\sigma_{k-1}}}{D_t} ... a \frac{D_t^{\sigma_1}}{D_t} \]
\[ \sigma_k = \sum_{j=1}^{k} \alpha_j, \quad (k = 1, 2, ..., n) \]

and \( f(t) \in L_1(0, T) \), i.e.

\[ \int_0^T |f(t)| \, dt < \infty \]

**Lemma 3.** [18] If \( f(t) \in L_1(0, T) \), and \( p_j(t) \) \( (j = 1, 2, ..., n) \) are continuous functions in the closed interval \([0, T]\), then the initial-value problem [3]-[4] has a unique solution \( y(t) \in L_1(0, T) \).

**Definition 4.** [20] A subset \( D \) of the complex plane is called an LMI region if there exist a symmetric matrix \( \Phi \in \mathbb{R}^{d \times d} \) and a matrix \( \Psi \in \mathbb{R}^{d \times d} \) such that

\[ D = \{ z \in \mathbb{C} \mid f_D(z) < 0 \} \tag{5} \]

where \( f_D(z) = \Phi + z\Psi + \bar{z}\Psi^T \) and "<" stands for negative definite. When \( \Phi = 0 \), the LMI region is denoted by \( D_\Gamma \).

**Definition 5.** [20] If all the eigenvalues of \( A \in \mathbb{R}^{n \times n} \) take values in region \( D \), i.e. \( \lambda(A) \subset D \), then \( A \) is called \( D \)-stable.

**Lemma 6.** [20] Matrix \( A \) is \( D \)-stable if and only if there exists a symmetric real matrix \( X > 0 \) such that

\[ M_D(A, X) = \Phi \otimes X + \Psi \otimes (XA) + \Psi^T \otimes (AX)^T < 0 \]

**Lemma 7.** [21] System \( D^\alpha x(t) = Ax(t) + Bu(t) \) with fractional order \( 1 < \alpha < 2 \) is asymptotically stable if and only if there exists a matrix \( P > 0, P \in \mathbb{R}^{n \times n} \) such that

\[ \text{Sym}\{\Theta \otimes (AP)\} < 0 \tag{6} \]

where \( \Theta = \begin{bmatrix} \sin \frac{\pi}{2} \alpha & -\cos \frac{\pi}{2} \alpha \\ \cos \frac{\pi}{2} \alpha & \sin \frac{\pi}{2} \alpha \end{bmatrix} \]

**Lemma 8.** [22] Let \( X, Y, \Lambda \) be real matrices of suitable dimensions and \( \Lambda > 0 \), then

\[ X^T Y + Y^T X \leq X^T \Lambda X + Y^T \Lambda Y \tag{7} \]

**Definition 9.** For system (E, A, \( \alpha \)), the infinite eigenvectors \( v \), which are related to eigenvalue 0, are defined as follows

(1) The infinite eigenvector of order 1 satisfies \( Ev^1 = 0, \, v^0 = 0 \);  
(2) The infinite eigenvector of order \( k \) satisfies \( Ev^k = Av^{k-1}, \, k > 1 \).

**Remark 10.** The infinite eigenvector isn’t related to the index \( \alpha \), which implies it may have the same properties as the infinite eigenvectors of SIOS.
3 Solution of SFOS

3.1 Regularity of SFOS

The sufficient and necessary condition of regularity for SFOS have already been given in \[11, 12\]. But the systems in \[11, 12\] are a linear SFOS and the fractional order in \[12\] is \(0 < \alpha < 1\). In the following, a different definition of regularity is proposed. And based on this definition, we give a sufficient and necessary condition of regularity for nonlinear SFOS with fractional order \(1 < \alpha < 2\).

Let \(Bu(t) = g(t)\), then the system (1) can be rewritten as

\[
ED^\alpha x(t) = Ax(t) + g(t)
\]

where \(g(t)\) is nonlinear and assumed to be sufficiently differential; \(1 < \alpha < 2\). We will focus on the existence, uniqueness of (8).

**Definition 11.** If a SFOS has a unique solution, then the system is termed regular.

**Theorem 12.** System (8) is regular if and only if two nonsingular matrices \(Q\) and \(P\) may be chosen such that

\[
QEP = \text{diag}(I_{n_1}, N)
\]

\[
QAP = \text{diag}(A_1, I_{n_2})
\]

where \(A_1 \in \mathbb{R}^{n_1 \times n_1}; N \in \mathbb{R}^{n_2 \times n_2}\) is nilpotent; \(n_1 + n_2 = n\).

**Proof.** According to lemma 2, let \(x(t) = P\tilde{x}(t)\) and left multiply system (8) by a nonsingular \(Q\). Let \(\tilde{g}(t) = Qg(t)\), we get

\[
0D^\alpha x_{n_0}(t) = g_{m_0}(t)
\]

\[
L_iD^\alpha x_{m_i+1}(t) = J_ix_{m_i+1}(t) + g_{m_i}(t), \quad i = 1, 2, ..., p
\]

\[
L'_jD^\alpha x_{m_j}(t) = J'_jx_{m_j}(t) + g_{n_j+1}(t), \quad j = 1, 2, ..., q
\]

\[
N_{k_s}D^\alpha x_{k_s}(t) = x_{k_s}(t) + g_{k_s}(t), \quad s = 1, 2, ..., r
\]

\[
D^\alpha x_h(t) = A_1x_h(t) + g_h(t)
\]

where \(L_i, L'_j, J_i, J'_j\) are defined in \(2\) and \(x_k(t) \in \mathbb{R}^k; g_k(t) \in \mathbb{R}^k\),

\[
\tilde{x}^T(t) = [x_{n_0}^T, x_{m_1+1}^T, \cdots, x_{m_p+1}^T, x_{n_1}^T, \cdots, x_{n_0}^T, x_{k_1}^T, \cdots, x_{k_r}^T, x_{h}^T]
\]

\[
\tilde{g}^T(t) = [g_{m_0}^T, g_{m_1}^T, \cdots, g_{m_p}^T, g_{n_1+1}^T, \cdots, g_{n_0+1}^T, g_{k_1}^T, \cdots, g_{k_r}^T, g_{h}^T]
\]

System (9)-(13) is equivalent to system (8), thus we focus on the existence, uniqueness of system (9)-(13).

(1) If equation (9) can be solved, then \(g_{m_0}(t) = 0\) must be true. In this case, equation (9) is always true. Therefore, this equation has either no solution or an infinite number of solutions.
(2) Equation (10) is composed of a set of equations

\[
\begin{align*}
D^\alpha z_1 (t) &= z_2 (t) + g_1 (t) \\
D^\alpha z_2 (t) &= z_3 (t) + g_2 (t) \\
\vdots \\
D^\alpha z_{k-1} (t) &= z_k (t) + g_{k-1} (t)
\end{align*}
\]  \( \text{(14)} \)

According to lemma 3, for a certain \( z_k (t) \), \( z_1 (t) \), \( z_2 (t) \), ..., \( z_{k-1} (t) \) can be determined successively. Therefore, such equations have an infinite number of solutions.

(3) Rewrite equation (11) as

\[
\begin{align*}
D^\alpha z_1 (t) &= g_1 (t) \\
D^\alpha z_2 (t) &= z_1 (t) + g_2 (t) \\
\vdots \\
D^\alpha z_k (t) &= z_{k-1} (t) + g_k (t) \\
0 &= z_k (t) + g_{k+1} (t)
\end{align*}
\]

Except the last equation, \( z_1 (t) \), \( z_2 (t) \), ..., \( z_k (t) \) can be determined uniquely according to lemma 3. However, \( z_k (t) \) must satisfy the last equation, which means these equations have no solution unless \( g_{k+1} (t) \) satisfies the consistent condition \( z_k (t) + g_{k+1} (t) = 0 \).

(4) Expand equation (12) into the following form

\[
\begin{align*}
D^\alpha z_2 (t) &= z_1 (t) + g_1 (t) \\
D^\alpha z_3 (t) &= z_2 (t) + g_2 (t) \\
\vdots \\
D^\alpha z_k (t) &= z_{k-1} (t) + g_{k-1} (t) \\
0 &= z_k (t) + g_{k+1} (t)
\end{align*}
\]

Beginning with the last equation, \( z_1 (t) \), \( z_2 (t) \), ..., \( z_k (t) \) may be determined successively for sufficiently differentiable functions \( g_i (t) \) \( (i = 1, 2, ..., k) \). Therefore, equation (12) has a unique solution.

(5) Equation (13) is an ordinary fractional order differential equation, which has a unique solution since \( g(t) \) is sufficiently differential.

To sum up, the system \( QEP = diag(I_{n_1}, N) \)

\[ QAP = diag(A_1, I_{n_2}) \]

where \( N = diag(N_{k_1}, N_{k_2}, ..., N_{k_r}) \). The theorem is proved.

3.2 State response and impulse analysis

To the best of our knowledge, there exists no research which gives the entire state response for SFOS. The following gives an entire state response, based on which we give a sufficient and necessary condition of impulse-free for SFOS.
Consider the regular SFOS

\[ ED^\alpha x(t) = Ax(t) + Bu(t) \]  

where \( E \in \mathbb{R}^{n \times n}, \ 1 < \alpha < 2 \) and the initial condition \( x(0) = x_0, t \geq 0 \).

**Theorem 13.** When \( t \geq 0 \), the state response to SFOS (15) is

\[ x(t) = P \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \]  

where

\[
x_1(t) = E_{\alpha,1} \left( A_1 t^\alpha \right) x_{10} + \int_{t_0}^{t} (t - \tau)^{\alpha - 1} E_{\alpha,\alpha} \left( A_1 (t - \tau)^\alpha \right) B_1 u(\tau) d\tau
\]

\[
x_2(t) = -\sum_{k=1}^{h-1} A^k \left( \delta^{(k\alpha-1)} (t) x_{20} + \delta^{(k\alpha-2)} (t) x_{20}^{(1)} \right) - \sum_{k=0}^{h-1} N^k B_2 \left( D^{k\alpha} u(t) + \sum_{j=0}^{m-1} u_j^{(j)} \delta^{(k\alpha-j-1)} (t) \right)
\]

\( x_1(t) \in \mathbb{R}^{n_1}, \ x_2(t) \in \mathbb{R}^{n_2}, \ n_1 + n_2 = n, \) the initial condition \( x_1(0) = x_{10}, \ x_2(0) = x_{20}, \dot{x}_2(0) = x_{20}^{(1)}; \ N \in \mathbb{R}^{n_2 \times n_2} \) is nilpotent and the nilpotent index is denoted by \( h \); \( u(t) \) is \( h \) times piecewise continuously differentiable, the initial condition \( u^{(j)}(0) = u_0^{(j)}; \ m \) is an integer and \( m - 1 < k\alpha \leq m \). \( E_{\alpha,\beta} \) is the two-parameter function of the Mittag-Leffler type. \( P \) satisfies Theorem 12. When \( t > 0 \) and the initial condition

\[
x(0+) = P \begin{bmatrix} I \\ 0 \end{bmatrix} x_{10} - P \begin{bmatrix} 0 \\ I \end{bmatrix} \sum_{k=1}^{h-1} M^{-1} N^k \delta^{(k\alpha-2)} (0+) x_{20}^{(1)}
\]

\[ - P \begin{bmatrix} 0 \\ I \end{bmatrix} \sum_{k=0}^{h-1} M^{-1} N^k B_2 \left( D^{k\alpha} u(0+) + \sum_{j=0}^{m-1} u_j^{(j)}(0) \delta^{(k\alpha-j-1)} (0+) \right)
\]

is satisfied, then the solution (16) to system (15) is unique and \( M = I + \sum_{k=1}^{h-1} N^k \delta^{(k\alpha-1)} (0+) \).

**Proof.** Since the system is regular, two nonsingular matrices \( P, Q \) may be chosen and the system (15) is equivalent to

\[ D^{\alpha} x_1(t) = A_1 x_1(t) + B_1 u(t) \]  

\[ ND^{\alpha} x_2(t) = x_2(t) + B_2 u(t) \]  

where \( QB = \begin{bmatrix} B_1^T \\ B_2^T \end{bmatrix}^T \). Subsystems (17) and (18) are termed finite subsystem and infinite subsystem, respectively, which are similar to SIOS.
Finite subsystem (17) is an normal fractional order system. For the piecewise continuously differentiable input \( u(t) \), the state response to the subsystem (17) is

\[
x_1(t) = E_{\alpha,1} (A_1 t^\alpha) x_{10} + \int_{t_0}^{t} (t - \tau)^{\alpha - 1} E_{\alpha,\alpha} (A_1 (t - \tau)^\alpha) B_1 u(\tau) \, d\tau \quad (19)
\]

For the infinite subsystem (18), invoking Laplace transform and \((sN - I)^{-1} = -\sum_{k=0}^{h-1} s^k N^k\)

\[
X_2(s) = (s^\alpha N - I)^{-1} \left[ N \left( s^{\alpha-1} x_{20} + s^{\alpha-2} x_{20}^{(1)} \right) + B_2 U(s) \right]
= -\sum_{k=1}^{h-1} N^k \left( s^{k\alpha-1} x_{20} + s^{k\alpha-2} x_{20}^{(1)} \right) - \sum_{k=0}^{h-1} s^{k\alpha} N^k B_2 U(s) \quad (20)
\]

where \( X_2(s) = \mathcal{L}[x_2(t)] \), \( U(s) = \mathcal{L}[u(t)] \). For the piecewise continuously differentiable input \( u(t) \), by invoking inverse Laplace transform of (20), we get

\[
x_2(t) = -\sum_{k=1}^{h-1} N^k \left( \delta^{(k\alpha-1)}(t) x_{20} + \delta^{(k\alpha-2)}(t) x_{20}^{(1)} \right)
- \sum_{k=0}^{h-1} N^k B_2 \left( D^{k\alpha} u(t) + \sum_{j=0}^{m-1} u^{(j)}(0) \delta^{(k\alpha-j-1)}(t) \right) \quad (21)
\]

Now, we get state response (16) with equations (19) and (21).

For arbitrary initial conditions, some of them may not satisfy solution (21) at \( t = 0 \), which leads to discontinuous behavior at \( t = 0 \). Since discontinuous behavior is not desirable, the set of \( x(0) \) which does not result in discontinuous behavior at \( t = 0 \) is called the set of admissible initial conditions [9]. The following analyses the admissible initial conditions of system (15).

With \( t \to 0_+ \), equation (21) turns into

\[
x_2(0_+) = -\sum_{k=1}^{h-1} N^k \left( \delta^{(k\alpha-1)}(0_+) x_{20} + \delta^{(k\alpha-2)}(0_+) x_{20}^{(1)} \right)
- \sum_{k=0}^{h-1} N^k B_2 \left( D^{k\alpha} u(0_+) + \sum_{j=0}^{m-1} u^{(j)}(0) \delta^{(k\alpha-j-1)}(0_+) \right)
\]

i.e.

\[
\left[ I + \sum_{k=1}^{h-1} N^k \delta^{(k\alpha-1)}(0_+) \right] x_{20}
= -\sum_{k=1}^{h-1} N^k \delta^{(k\alpha-2)}(0_+) x_{20}^{(1)}
- \sum_{k=0}^{h-1} N^k B_2 \left( D^{k\alpha} u(0_+) + \sum_{j=0}^{m-1} u^{(j)}(0) \delta^{(k\alpha-j-1)}(0_+) \right)
\]
Let $M = \left[ I + \sum_{k=1}^{h-1} N^k \delta^{(k\alpha-1)}(0+) \right]$. $N^k$ is an upper triangular matrix and all the elements of main diagonal are zero since $N$ is nilpotent. On the other hand, if $(k\alpha - 1)$ is not an integer, then $\delta^{(k\alpha-1)}(0+)$ can not be zero and it’s a very large number but not an infinite number. Therefore, the matrix $M$ is invertible and the admissible initial conditions of system (15) is

$$x_{20} = -M^{-1} \left( \sum_{k=1}^{h-1} N^k \delta^{(k\alpha-2)}(0+) x^{(1)}_{20} + \sum_{k=0}^{h-1} N^k B_2 \left( D^{k\alpha} u(0+) + \sum_{j=0}^{m-1} u^{(j)}(0) \delta^{(k\alpha-j)}(0+) \right) \right)$$

The theorem is proved.

**Definition 14.** For arbitrary initial conditions, if the state response to SFOS does not include impulsive response, then the system is termed impulse-free.

Obviously, the state response to SFOS (15) is similar to the state response of SIOS. $x_1(t)$ is the state response to the finite subsystem, which is represented by Mittag-Leffler function. $x_2(t)$ is the state response to the infinite subsystem, which is composed of impulse function and input function. Based on state response (15), the following analyses the impulsive behavior of system (15). Because substate $x_1(t)$ is continuous, we focus on substate $x_2(t)$.

1. If $t = 0$
   
   Without loss of generality, let $u(t) = 0$. If $x_2(0) \neq 0$ and $x_2(0) \notin \ker(N)$, there holds
   
   $$x_2(t) = -\sum_{k=1}^{h-1} N^k \left( \delta^{(k\alpha-1)}(t) x_{20} + \delta^{(k\alpha-2)}(t) x^{(1)}_{20} \right)$$

   If $t \to 0$, then $\delta^{(\beta)}(t) \to \infty (\beta > 0)$. Thus, $x_{20}$ which doesn’t satisfy admissible initial condition may result in impulse.

2. If $t > 0$
   
   $x_2(t)$ can be represented as
   
   $$x_2(t) = -\sum_{k=1}^{h-1} N^k \left( \delta^{(k\alpha-1)}(t) x_{20} + \delta^{(k\alpha-2)}(t) x^{(1)}_{20} \right)$$

   $$-\sum_{k=0}^{h-1} N^k B_2 \left( D^{k\alpha} u(t) + \sum_{j=0}^{m-1} u^{(j)}(0) \delta^{(k\alpha-j)}(t) \right)$$

For $\delta^{(b)}$, if $b$ is a positive integer, the support set of $\delta^{(b)}(t)$ is $\{0\}$, which means $\delta^{(b)} = 0$; If $b$ is not a positive integer, the support set of $\delta^{(b)}(t)$ is not $\{0\}$, which means $\delta^{(b)}(t) \neq 0$ when $t > 0$. Therefore, if $N \neq 0$, then $x_{20}, x^{(1)}_{20}, u^{(j)}(0)$ participate in the dynamic process of $x_2(t)$ and the state response of SFOS includes impulsive response.
Remark 15. The $x_{20}$ and $u^{(s)}(0)$ of SFOS participate in the dynamic process of substate $x_2(t)$, which is very different from SIOS.

The final value theorem of fractional order system [23],

$$D^{α-1}x(∞) = \lim_{s\to 0} s^{α}X(s) \quad \text{Re}(s) > 0$$ (23)

Therefore, we get $δ^{(β)}(t) \to 0$ when $t \to ∞$. It implies that terms including $δ^{(β)}(t)$ on the right side of the equation (22) do not impact on the stability of the infinite subsystem [13], which is convenient when analysing the stability of SFOS.

From time 0 to time $t$, the input $u(t)$ always has an influence on the state $x_2(t)$ because of the properties of Caputo fractional derivative. Thus, change of $u(t)$ can not be reflected immediately by substate $x_2(t)$ at the time $t$ and jump behavior will not appear in the state response.

Remark 16. The input $u(t)$ will not give rise to the jump behavior of $x_2(t)$, which is also very different from SIOS.

To sum up, we get the following theorem.

Theorem 17. For arbitrary initial conditions, the regular SFOS [13] is impulse-free if and only if $N = 0$. $N$ comes from the decomposition

$$QEP = \text{diag}(I_{n_1}, N)$$
$$QAP = \text{diag}(A_1, I_{n_2})$$

Similar to paper [24], the following gives another condition of impulse-free.

Lemma 18. The following statements are equivalent:

1. the regular system $(E, A, α)$ is impulse-free;
2. there exist a vector $v \in \mathbb{R}^n$ and a vector $ω \in \mathbb{R}^n$ such that

$$Ev = 0$$
$$Av = Eω$$

then $v = 0$.

Proof. According to Theorem [12] regular system $(E, A, α)$ has the decomposition that

$$QEP = \text{diag}(I, N)$$
$$QAP = \text{diag}(A, I)$$
Thus,

\[
\begin{align*}
E\nu &= 0 \\
A\nu &= E\omega \\
\Leftrightarrow \begin{cases} 
QEP P^{-1} \nu &= 0 \\
QAP P^{-1} \nu &= QEP P^{-1} \omega 
\end{cases} \\
\Leftrightarrow \begin{bmatrix} I & 0 \\
0 & N \end{bmatrix} \begin{bmatrix} v_1 \\
v_2 \end{bmatrix} &= 0 \\
\begin{bmatrix} A & 0 \\
0 & I \end{bmatrix} \begin{bmatrix} v_1 \\
v_2 \end{bmatrix} &= \begin{bmatrix} I & 0 \\
0 & N \end{bmatrix} \begin{bmatrix} \omega_1 \\
\omega_2 \end{bmatrix} \\
v_1 &= 0 \\
Nv_2 &= 0 \\
Av_1 &= \omega_1 \\
v_2 &= N\omega_2
\end{align*}
\]

\(\omega_2\) is not specific, therefore, \(v_2 = 0\) if and only if \(N = 0\), which is the sufficient and necessary condition of impulse-free. Because \(P\) is nonsingular, we can conclude that \(v = 0\) if and only if \(N = 0\).

This ends the proof.

\[ \square \]

**Theorem 19.** The regular SFOS [15] is impulse-free if and only if there exists no infinite eigenvector of order 2, i.e. \(v^2\).

**Proof.** According to Lemma [18] the sufficient and necessary condition of impulse-free is

\[
Ev^1 = 0 \\
Ev^2 = Av^1 = 0
\]

which implies that the infinite eigenvector of order 2 does not exist.

This ends the proof.

\[ \square \]

**4 Stability and Admissibility Analysis**

**4.1 Asymptotic Stability**

Stability is very important in control theory. In [11], a sufficient condition of asymptotic stability is given, but the condition demands free impulse. Meanwhile, [12] also gives a sufficient condition of asymptotic stability, but its fractional order is \(0 < \alpha < 1\). The following gives a sufficient and necessary condition of asymptotic stability with fractional order \(1 < \alpha < 2\), which is simpler than the condition in [11]. Consider the autonomous regular SFOS

\[
ED^\alpha x(t) = Ax(t)
\] (24)
where $x(t) \in \mathbb{R}^n$, $1 < \alpha < 2$.

The following will analyse the asymptotic stability (stability for short) of SFOS. Firstly, the definition of the stability of SFOS is given as follows.

**Definition 20.** For arbitrary admissible initial condition $x(0)$, if regular SFOS (24) satisfies $\lim_{t \to +\infty} \|x(t)\| = 0$, then the SFOS (24) is called asymptotically stable.

The characteristic polynomial of system (24) is

$$\Delta (s) = \det (s^\alpha E - A) = a_{n^1} (s^\alpha)^{n^1} + \cdots + a_1 s^\alpha + a_0$$

(25)

It’s obvious that the polynomial $\Delta (s)$ is a multivalued function of $s$, of which the fractional degree is $n_1$ ($n_1 \leq n$). Let $s^\alpha = \omega$, then $\Delta (s)$ turns into a single-valued function $\Delta (\omega) = \det (\omega E - A)$. $\Delta (s)$ has a lot of roots but only the roots on the principal Riemann surface $\Omega = \{s \mid -\pi \leq \arg (s) < \pi\}$ decide the time-domain behavior and stability of fractional system (25, 26). Therefore, the physical domain of $\Delta (s)$ is defined on the principal Riemann surface. And the finite roots of $\Delta (s)$ on the principal Riemann surface $\Omega$ are defined as the finite roots of SFOS.

**Lemma 21.** Fractional order system [21]

$$D^\alpha x(t) = Ax(t), \quad 1 < \alpha < 2$$

(26)

is asymptotically stable if and only if $|\arg (\text{spec}(A))| > \alpha \pi / 2$, where $\text{spec}(A)$ is the spectrum (set of all eigenvalues) of $A$. Also, state vector $x(t)$ decays towards 0 and meets the following condition: $\|x(t)\| < Kt^{-\alpha}, t > 0, K > 0$.

**Theorem 22.** SFOS [24] is asymptotically stable if and only if

$$|\arg (\text{spec}(E, A))| > \frac{\pi}{2} \alpha$$

where $\text{spec}(E, A)$ is the spectrum (set of all eigenvalues) of $(E, A, \alpha)$.

**Proof.** Because the system [24] is regular, two nonsingular matrices $Q, P$ may be chosen such that system [24] is equivalent to

$$D^\alpha x_1(t) = A_1 x_1(t)$$

(27)

$$ND^\alpha x_2(t) = x_2(t)$$

(28)

where $[x_1^T(t) \hspace{1cm} x_2^T(t)]^T = P^{-1} x, QEP = \text{diag}(I_{n_1}, N), QAP = \text{diag}(A_1, I_{n-n_1})$.

According to theorem [13] the state response to system [24] is

$$x_1(t) = E_{\alpha,1} (A_1 t^\alpha) x_{10}$$

$$x_2(t) = - \sum_{k=1}^{h-1} N^k \left( \delta^{(k\alpha-1)} (t) x_{20} + \delta^{(k\alpha-2)} (t) x_{20}^{(1)} \right)$$

(29)
According to lemma 21, finite subsystem (27) is stable if and only if
\[ |\text{arg} (\text{spec}(A_1))| > \alpha \pi / 2 \]

For the infinite subsystem (28), according to the final value theorem (23), when \( t \to +\infty \), the state response \( x_2(t) \to 0 \). Thus, the infinite subsystem is essentially stable.

On the other hand, \( \det(sN - I_{n-n_1}) = (-1)^{n-n_1} \) because \( N \) is nilpotent. Thus, \( \text{spec}(N, I_{n-n_1}) = \emptyset \) and
\[
\text{spec}(E, A) = \text{spec}(\text{QEP}, QAP) = \text{spec}(\text{diag}(I_{n_1}, N), \text{diag}(A_1, I_{n-n_1})) = \text{spec}(I_{n_1}, A_1) \cup \text{spec}(N, I_{n-n_1}) = \text{spec}(A_1) \cup \emptyset = \text{spec}(A_1)
\]
i.e. \( \text{spec}(E, A) = \text{spec}(A_1) \). The theorem is proved.

4.2 Admissibility

In [12], a sufficient and necessary condition of admissibility with fractional order \( 0 < \alpha < 1 \) is given. The following gives a sufficient and necessary condition of admissibility with fractional order \( 1 < \alpha < 2 \).

Similarly to the admissibility of SIOS, the following gives the definition of admissibility for SFOS.

**Definition 23.** If a SFOS is regular, impulse-free and stable, then the SFOS is termed admissible.

From the above analysis, we can know that the sufficient and necessary conditions of regularity, free impulse, stability for SFOS are only related to matrices \( E, A \) and fractional order \( \alpha \). Thus, the admissibility of SFOS is only related to \( E, A \) and \( \alpha \).

According to theorem 22, SFOS (24) is stable if and only if all the finite eigenvalues of SFOS belong to the region \( \Lambda = \{ \lambda \in \mathbb{C} \mid |\text{arg}(\lambda)| > \pi \alpha / 2 \} \). When \( 1 < \alpha < 2 \), \( \Lambda \) is a LMI region. Thus, we can analyse the admissibility of system \( (E, A, \alpha) \) via \( \mathcal{D} \)-stable theorem.

**Definition 24.** If system \( (E, A, \alpha) \) is regular, impulse-free and all the finite eigenvalues belong to the region \( \mathcal{D} \), then system \( (E, A, \alpha) \) is termed \( \mathcal{D} \)-admissible.

**Theorem 25.** Let \( \text{rank}(E) = r \), \( E_0 \in \mathbb{R}^{n \times (n-r)} \) column full rank and \( E^TE_0 = 0 \). SFOS is \( \mathcal{D} \)-admissible if and only if there exist symmetric positive matrix \( P \in \mathbb{R}^{n \times n} \) and matrix \( Q \in \mathbb{R}^{(n-r) \times n} \) such that
\[
M_{\alpha}(E, A, P, Q) < 0
\]

\[ (30) \]
where

\[ M_s(E, A, P, Q) = \Phi \otimes (E^TPE) + \text{Sym} \left\{ \Psi \otimes (E^TPA) + I_d \otimes (Q^TE_0^TA) \right\} \]

\[ \Phi \in \mathbb{R}^{d \times d} \text{ is a symmetric matrix and } \Psi \in \mathbb{R}^{d \times d}, \]

\[ I_d \text{ has the dimension } d \times d. \]

**Proof.** Sufficiency.

We will prove it by contradiction. Assume that SFOS is impulsive. According to Theorem 19, there exists eigenvector \( v^2 \in \mathbb{R}^n \) such that \( Ev^2 = \lambda v^1 \) and \( Ev^1 = 0 \). By left multiplying \( (I_d \otimes v^1)^T \) and right multiplying \( (I_d \otimes v^1) \) on \( (30) \), we have

\[ (I_d \otimes v^1)^T M_s(E, A, P, Q) (I_d \otimes v^1) \leq 0 \]

i.e.

\[ I_d \otimes \left( (v^2)^T (E^TQ + Q^TE_0^T) v^2 \right) < 0 \] (31)

Because \( E^TQ = 0 \), the inequation (31) can’t be true. Thus, system \((E, A, \alpha)\) is impulse-free.

Let \( \lambda \) be the finite eigenvalue of system \((E, A, \alpha)\) and \( v \) be the eigenvector, then we get \( Av = \lambda Ev \) and \( v^* A^T = \bar{\lambda}v^* E^T \). From inequality (30), we get

\[ (I_d \otimes v^1)^T M_s(E, A, P, Q) (I_d \otimes v^1) \leq 0 \]

i.e.

\[ I_d \otimes \left( (v^2)^T (E^TQ + Q^TE_0^T) v^2 \right) < 0 \] (31)

Because \( P > 0 \), we can get \( f_D(\lambda) < 0 \). According to the definition (23), we can conclude that system \((E, A, \alpha)\) is D-admissible.

Necessity.

Because the system \((E, A, \alpha)\) is regular and impulse-free, there exist two nonsingular matrices \( M \) and \( N \) such that

\[ MEN = \begin{bmatrix} M_1 & M_2 \\ \end{bmatrix} E \begin{bmatrix} N_1 & N_2 \\ \end{bmatrix} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \\ \end{bmatrix} \]

\[ MAN = \begin{bmatrix} M_1 & M_2 \\ \end{bmatrix} A \begin{bmatrix} N_1 & N_2 \\ \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-r} \\ \end{bmatrix} \] (32)

where \( M_1 \in \mathbb{R}^{r \times n}, N_1 \in \mathbb{R}^{n \times r} \). Obivously, \( M_2AN = \begin{bmatrix} 0 & I_{n-r} \end{bmatrix} \), \( M_2E = 0 \).

The system \((E, A, \alpha)\) is D-admissible, thus we get \( \lambda(E, A) = \lambda(A_1) \), i.e. \( A_1 \) is D-stable. According to lemma 6, there exists a symmetric real matrix \( P_1 > 0 \) such that

\[ \Phi \otimes P_1 + \Psi \otimes (P_1A_1) + \Psi^T \otimes (A_1^TP_1) < 0 \]

Since it’s a strict inequality, there must exist a sufficiently small \( \varepsilon > 0 \) such that

\[ \Phi \otimes P_1 + \Psi \otimes (P_1A_1) + \Psi^T \otimes (A_1^TP_1) + I_d \otimes \left( \frac{\varepsilon}{2} N_1^T N_1 \right) < 0 \]
i.e.

\[
\Phi \otimes P_1 + \Psi \otimes (P_1 A_1) + \Psi^T \otimes (A_1^T P_1) + [I_d \otimes (\varepsilon N_1^T N_2)] ^{-1} [I_d \otimes (\varepsilon N_2^T N_1)] < 0
\] (33)

Invoking Schur complement, inequality (33) is equivalent to

\[
\begin{bmatrix}
\Phi \otimes P_1 + \Psi \otimes (P_1 A_1) + \Psi^T \otimes (A_1^T P_1) & -I_d \otimes (\varepsilon N_1^T N_2) \\
-I_d \otimes (2\varepsilon N_2^T N_1) & -I_d \otimes (2\varepsilon N_2^T N_2)
\end{bmatrix} < 0
\]

\[
\begin{bmatrix}
P_1 & 0 \\
0 & 0
\end{bmatrix} + \text{Sym}\left\{\Psi \otimes \begin{bmatrix}P_1 A_1 & 0 \\
0 & 0\end{bmatrix} + I_d \otimes \begin{bmatrix}0 \\
I_{n-r}\end{bmatrix}
\right\} < 0
\]

\[
\begin{bmatrix}
P_1 & 0 \\
0 & I_{n-r}
\end{bmatrix} + \text{Sym}\left\{\Psi \otimes \begin{bmatrix}0 \\
I_{n-r}\end{bmatrix} (-\varepsilon N_2^T N)
\right\} < 0
\]

Utilizing (32) and let \(\hat{P} = \begin{bmatrix}P_1 & 0 \\
0 & I_{n-r}\end{bmatrix} > 0\), we have

\[
\Phi \otimes (N^T E^T M^T \hat{P} M E N) + \text{Sym}\left\{\Psi \otimes (N^T E^T M^T \hat{P} M E N)\right\} + I_d \otimes (N^T A^T M_2^T (-\varepsilon N_2^T N)) < 0
\]

Let \(M^T \hat{P} M = P, M_2^T = E_0\) and \(-\varepsilon N_2^T = Q\), where \(E_0\) is column full rank and \(E^T E_0 = 0\). Since \(N\) is nonsingular, we get

\[
\Phi \otimes (E^T P E) + \Psi \otimes (E^T P A) + \Psi^T \otimes (A^T P E) + I_d \otimes (A^T E_0 Q + Q^T E_0^T A) < 0
\]

The theorem is proved. \(\square\)

The condition (30) is a strict linear matrix inequality. In order to analyse the Robust problems of SFOS conveniently, the nonstrict LMI condition is given as follows.

**Lemma 26.** \([27]\) SIOS \((E_1, A_1)\) is \(\mathcal{D}_1\) (when \(\Phi = 0\)) admissible if and only if there exists a matrix \(P \in \mathbb{R}^{n \times n}\) such that

\[
\text{Sym}\{\Psi \otimes (PA_1)\} < 0
\] (34)

\[
PE_1 = E_1^T P^T \geq 0
\] (35)

**Remark 27.** Note that conditions (30) and (37) do not have to be regular and impulse-free, thus they can be used generally. For \((E, A, \alpha)\), replace \(E_1, A_1\) by \(E, A\) respectively, the lemma [26] is also true because the eigenvalues of SFOS and SIOS are equivalent.
Thus, we get
\[ \hat{Q} = Q \]
where \( \Theta = \begin{bmatrix} \sin \frac{\pi}{2} \alpha & -\cos \frac{\pi}{2} \alpha \\ \cos \frac{\pi}{2} \alpha & \sin \frac{\pi}{2} \alpha \end{bmatrix} \), \( I_2 \) has the same dimension with \( \Theta \).

\textbf{Proof.} Sufficiency. Assume that \((E, A, \alpha)\) is impulsive, then there exists an infinite eigenvector of order 2, \( \upsilon \in \mathbb{R}^n \) such that \( E \upsilon^2 = A \upsilon^1, E \upsilon^1 = 0 \). By left multiplying \((I_2 \otimes \upsilon^1)^*\) and right multiplying \((I_2 \otimes \upsilon^1)\) on \((36)\), we have
\[
(I_2 \otimes \upsilon^1)^* \left[ \text{Sym} \left\{ \Theta \otimes (A^T PE) \right\} + I_2 \otimes (A^T QA) \right] (I_2 \otimes \upsilon^1) = I_2 \otimes ((\upsilon^1)^* A^T Q A \upsilon^1) = I_2 \otimes [(\upsilon^2)^* E^T Q E \upsilon^2] < 0
\]
Inequality \((38)\) can’t be true because \( E^T Q E \geq 0 \). Thus, system \((E, A, \alpha)\) is impulse-free and two matrices \( M \) and \( N \) may be chosen such that
\[
M E N = \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix}, \quad M A N = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-m} \end{bmatrix}
\]
Let \( Y = M^{-T} P M^{-1} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12}^* & Y_{22} \end{bmatrix} \). Obviously, there holds \( Y = Y^* > 0 \). Let \( \hat{Q} = M^{-T} Q M^{-1} = \begin{bmatrix} \hat{Q}_{11} & \hat{Q}_{12} \\ \hat{Q}_{12}^* & \hat{Q}_{22} \end{bmatrix} \). From \((37)\) we have
\[
N^T E^T Q E N = N^T E^T M^T \hat{Q} M E N
= \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{Q}_{11} & \hat{Q}_{12} \\ \hat{Q}_{12}^* & \hat{Q}_{22} \end{bmatrix} \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix}
= \begin{bmatrix} \hat{Q}_{11} & 0 \\ 0 & 0 \end{bmatrix} \geq 0
\]
Thus, we get \( \hat{Q}_{11} \geq 0 \).

Left multiply \((I_2 \otimes N)^T\) and right multiply \( I_2 \otimes N \) on \((36)\), we get
\[
(I_2 \otimes N)^T \left( \text{Sym} \left\{ \Theta \otimes (A^T PE) \right\} + I_2 \otimes (A^T QA) \right) (I_2 \otimes N) = \text{Sym} \left\{ \Theta \otimes (N^T A^T M^T PE) \right\} + I_2 \otimes (N^T A^T M^T QA)\]
\[
= \text{Sym} \left\{ \Theta \otimes \begin{bmatrix} A_1^T & 0 \\ 0 & I_{n-m} \end{bmatrix} \begin{bmatrix} Y_{11} & Y_{12} \\ \bullet & Y_{22} \end{bmatrix} \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix} \right\}
+ I_2 \otimes \begin{bmatrix} A_1^T & 0 \\ 0 & I_{n-m} \end{bmatrix} \begin{bmatrix} \hat{Q}_{11} & \hat{Q}_{12} \\ \bullet & \hat{Q}_{22} \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-m} \end{bmatrix}
= \text{Sym} \left\{ \Theta \otimes \begin{bmatrix} A_1^T Y_{11} & 0 \\ Y_{12} & 0 \end{bmatrix} \right\} + I_2 \otimes \begin{bmatrix} A_1^T \hat{Q}_{11} A_1 & A_1^T \hat{Q}_{12} \\ \bullet & \hat{Q}_{22} \end{bmatrix} < 0
\]
According to [28], inequality (41) is equivalent to

\[
Sym \left\{ \left[ \begin{array}{c} \Theta \otimes (A^T_1Y_{11}) \\ \Theta \otimes Y_{12}^T \\ \end{array} \right] 0 \right\} + \left[ \begin{array}{ccc} I_2 \otimes (A^T_1\hat{S}_{11}A_1) & I_2 \otimes (A^T_1\hat{Q}_{12}) \\ \end{array} \right] < 0 \quad (42)
\]

Thus, we get

\[
Sym \{ \Theta \otimes (A^T_1Y_{11}) \} + I_2 \otimes (A^T_1\hat{Q}_{11}A_1) < 0 \quad (43)
\]

Because \( \hat{Q}_{11} \geq 0 \), we have \( Sym \{ \Theta \otimes (A^T_1Y_{11}) \} < 0 \). According to lemma 7, \( A_1 \) is stable. Thus \( (E, A, \alpha) \) is stable. Finally, the admissibility of \( (E, A, \alpha) \) is achieved.

Necessary.

\( (E, A, \alpha) \) is admissible, thus there exist two nonsingular matrices \( M, N, Y_{11} \) such that

\[
MEN = \left[ \begin{array}{cc} I_m & 0 \\ 0 & 0 \\ \end{array} \right], \quad MAN = \left[ \begin{array}{cc} A_1 & 0 \\ 0 & I_{n-m} \end{array} \right],
\]

\[
Sym \{ \Theta \otimes (A^T_1Y_{11}) \} < 0.
\]

For a sufficiently small \( \varepsilon > 0 \), we have

\[
Sym \{ \Theta \otimes (A^T_1Y_{11}) \} + I_2 \otimes (A^T_1\varepsilon Y_{11}A_1) < 0 \quad (44)
\]

Note

\[
P = M^TYM = M^T \left[ \begin{array}{cc} Y_{11} & 0 \\ 0 & I_{n-m} \end{array} \right] M
\]

\[
Q = M^T\hat{Q}M = M^T \left[ \begin{array}{cc} \hat{Q}_{11} & 0 \\ 0 & \hat{Q}_{22} \end{array} \right] M
\]

where \( \hat{Q}_{11} = \varepsilon Y_{11} \) and \( \hat{Q}_{22} \) be any negative definite matrix. From (44) we get

\[
\left[ Sym \{ \Theta \otimes (A^T_1Y_{11}) \} 0 \\ 0 \right] + \left[ I_2 \otimes (A^T_1\varepsilon Y_{11}A_1) 0 \\ 0 \right] I_2 \otimes \hat{Q}_{22} < 0. \quad (45)
\]

which is equivalent to

\[
Sym \left\{ \Theta \otimes \left[ \begin{array}{cc} A^T_1Y_{11} & 0 \\ 0 & 0 \end{array} \right] \right\} + I_2 \otimes \left[ \begin{array}{cc} A^T_1\varepsilon Y_{11}A_1 & 0 \\ 0 & \hat{Q}_{22} \end{array} \right] < 0
\]

\[
\Leftrightarrow Sym \left\{ \Theta \otimes (N^T A^T M^T PMEN) \right\} + I_2 \otimes (N^T A^T M^T QMAN) < 0
\]

\[
\Leftrightarrow (I_2 \otimes N^T) [ Sym \{ \Theta \otimes (A^T PE) \} + I_2 \otimes (A^T QA) ] (I_2 \otimes N) < 0
\]

\[
\Leftrightarrow Sym \{ \Theta \otimes (A^T PE) \} + I_2 \otimes (A^T QA) < 0
\]
and

\[
\begin{bmatrix}
\hat{Q}_{11} & 0 \\
0 & 0 \\
\end{bmatrix} \geq 0
\]

\[\iff N^T E^T M^T \hat{Q} M E^T N \geq 0\]

\[\iff N^T E^T Q E^T N \geq 0\]

\[\iff E^T Q E \geq 0\]

This completes the proof. \(\square\)

**Theorem 29.** The following statements are equivalent

1. System \((E, A, \alpha)\) (1 < \(\alpha\) < 2) is admissible;
2. Assume that \(E_0 \in \mathbb{R}^{n \times (n-r)}\) is column full rank and \(E^T E_0 = 0\), there exist symmetric positive matrix \(P \in \mathbb{R}^{n \times n}\) and matrix \(Q \in \mathbb{R}^{(n-r) \times n}\) such that

\[
\text{Sym} \left\{ \Theta \otimes (E^T PA) + I \otimes (Q^T E_0^T A) \right\} < 0 \tag{46}
\]

3. There exists matrix \(P \in \mathbb{R}^{n \times n}\) such that

\[
\text{Sym} \{ \Theta \otimes (PA) \} < 0
\]

\[PE = E^T P^T \geq 0 \tag{47}\]

4. There exist symmetric positive matrix \(P \in \mathbb{R}^{n \times n}\) and symmetric matrix \(Q \in \mathbb{R}^{n \times n}\) such that

\[
\text{Sym} \left\{ \Theta \otimes (E^T PA) \right\} + I \otimes (A^T Q A) < 0 \tag{48}
\]

\[E^T Q E \geq 0\]

where \(\Theta = \begin{bmatrix}
\sin \frac{\pi \alpha}{2} & -\cos \frac{\pi \alpha}{2} \\
\cos \frac{\pi \alpha}{2} & \sin \frac{\pi \alpha}{2}
\end{bmatrix}\) and \(I\) has the same dimension with \(\Theta\).

**Proof.** When 1 < \(\alpha\) < 2, the stable region of SFOS is a LMI region, in which case \(\Phi = 0\) and \(\Psi = \begin{bmatrix}
\sin \frac{\pi \alpha}{2} & -\cos \frac{\pi \alpha}{2} \\
\cos \frac{\pi \alpha}{2} & \sin \frac{\pi \alpha}{2}
\end{bmatrix}\). For such \(\Phi\) and \(\Psi\) and according to Theorem [23] Lemma [24] and Lemma [28] the conclusion is achieved. \(\square\)

### 5 Robust admissibility analysis

To the best of our knowledge, there exists no research on robust admissibility of uncertain SFOS. In this section, sufficient conditions are given to check the robust admissibility of the uncertain SFOS.

Consider the following uncertain SFOS

\[
ED^\alpha x(t) = Ax(t) = (A_0 + D_A F_A E A)x(t) \tag{49}
\]
where \( 1 < \alpha < 2 \) and \( A_0 \in \mathbb{R}^{n \times n}, D_A \in \mathbb{R}^{n \times p}, E_A \in \mathbb{R}^{q \times n} \) are given certain matrices. The uncertain matrix \( F_A \in \mathbb{R}^{p \times q} \) satisfies

\[
F_AF_A^T < I_p
\]

**Theorem 30.** System (44) is robust admissible if there exist matrices \( X = X^T > 0, X \in \mathbb{R}^{n \times n}, S \in \mathbb{R}^{(n-m) \times n} \) such that

\[
\begin{bmatrix}
Z_{11} & Z_{12} & Z_{13} \\
\vdots & Z_{22} & Z_{23} \\
\vdots & \vdots & Z_{33}
\end{bmatrix} < 0
\]

where \( E^T E_0 = 0 \), \( I_2 \) is a \( 2 \times 2 \) matrix and \( \Theta = \begin{bmatrix} \sin \frac{\pi}{2} \alpha & -\cos \frac{\pi}{2} \alpha \\ \cos \frac{\pi}{2} \alpha & \sin \frac{\pi}{2} \alpha \end{bmatrix} \) and

\[
Z_{11} = \text{Sym} \left\{ \Theta \otimes (A_0^T X E) + I_2 \otimes (A_0^T E_0 S) \right\} + 2I_2 \otimes (E_A^T E_A)
\]

\[
Z_{12} = I_2 \otimes (E^T X D_A)
\]

\[
Z_{13} = I_2 \otimes (S^T E_0^T D_A)
\]

\[
Z_{22} = -I_2 \otimes I
\]

\[
Z_{23} = 0
\]

\[
Z_{33} = -I_2 \otimes I
\]

**Proof.** Invoking Schur complement, inequality (51) is equivalent to

\[
\text{Sym} \left\{ \Theta \otimes (A_0^T X E) + I_2 \otimes (A_0^T E_0 S) \right\} + 2I_2 \otimes (E_A^T E_A) + I_2 \otimes ((E^T X D_A D_A^T X E)) + I_2 \otimes (S^T E_0^T D_A D_A^T E_0 S) < 0
\]

From (50) and (51), we get

\[
\text{Sym} \left\{ \Theta \otimes (A_0^T X E) + I_2 \otimes (A_0^T E_0 S) \right\} + (\Theta \Theta^T) \otimes (E_A^T E_A) + 2I_2 \otimes (E_A^T E_A) + I_2 \otimes ((E^T X D_A D_A^T X E)) + I_2 \otimes (S^T E_0^T D_A D_A^T E_0 S) < 0
\]

According to lemma 8 and inequality (53), we get

\[
\text{Sym} \left\{ \Theta \otimes (A_0^T X E) + I_2 \otimes (A_0^T E_0 S) \right\}
\]

\[
+ \text{Sym} \left\{ \Theta \otimes ((D_A F_A E_A)^T X E) \right\} + \text{Sym} \left\{ I_2 \otimes ((D_A F_A E_A)^T (E_0 S)) \right\} < 0
\]

\[
\iff \text{Sym} \left\{ \Theta \otimes ((D_A F_A E_A)^T X E) \right\} + \text{Sym} \left\{ I_2 \otimes ((D_A F_A E_A)^T (E_0 S)) \right\} < 0
\]

\[
\iff \text{Sym} \left\{ \Theta \otimes (A^T X E) + I_2 \otimes (A^T E_0 S) \right\} < 0
\]

Thus, according to theorem 29 system (49) is robust admissible. The theorem is proved.

**Theorem 31.** System (44) is robust admissible if there exist matrices \( X = X^T > 0, X \in \mathbb{R}^{n \times n}, Y = Y^T > 0, Y \in \mathbb{R}^{n \times n} \) and \( S = S^T, S \in \mathbb{R}^{n \times n} \) such that

\[
\begin{bmatrix}
Z_{11} & Z_{12} & Z_{13} \\
\vdots & Z_{22} & Z_{23} \\
\vdots & \vdots & Z_{33}
\end{bmatrix} < 0
\]
where \( \Theta = \begin{bmatrix} \sin \frac{\pi}{2} \alpha & -\cos \frac{\pi}{2} \alpha \\ \cos \frac{\pi}{2} \alpha & \sin \frac{\pi}{2} \alpha \end{bmatrix} \) and

\[
\begin{align*}
Z_{11} &= \text{Sym} \left\{ (A_0 + D_A F_A E_A)^T X E \right\} + I_2 \otimes (A_0^T S A_0) \\
Z_{12} &= \Theta^T \otimes (E^T X D_A) + I_2 \otimes (A_0^T S D_A) \\
Z_{13} &= I_2 \otimes Y E_A^T \\
Z_{22} &= I_2 \otimes (S - Y) \\
Z_{23} &= 0 \\
Z_{33} &= -I_2 \otimes Y
\end{align*}
\]

**Proof.** According to theorem 29, uncertain system (49) is robust admissible if for any \( F_A \), there holds

\[
\text{Sym} \left\{ (A_0 + D_A F_A E_A)^T X E \right\} + I_2 \otimes (A_0^T S A_0) \leq 0
\]

i.e.

\[
(I_2 \otimes v)^T (\text{Sym} \left\{ (A_0 + D_A F_A E_A)^T X E \right\} + I_2 \otimes (A_0^T S A_0)) (I_2 \otimes v) < 0
\]

The inequality (56) can be rewritten as

\[
\begin{align*}
& \quad \left[I_2 \otimes v^T I_2 \otimes (F_A E_A v)^T \right] \left[I_2 \otimes [\text{Sym} \left\{ (A_0 + D_A F_A E_A)^T X E \right\} + I_2 \otimes (A_0^T S A_0)] - I_2 \otimes S \otimes (A_0^T S D_A)\right] \\
& \quad \times \left[I_2 \otimes v \right] \left[I_2 \otimes (F_A E_A v) \right] < 0
\end{align*}
\]

From inequality (56), we get

\[
(I_2 \otimes v)^T \left[I_2 \otimes (F_A E_A v)^T \right] \left[I_2 \otimes (E_A^T E_A) - I_2 \otimes S \right] (I_2 \otimes v) > 0
\]

By applying the \( S \)-procedure, inequalities (57) and (58) derive that there
exists some scalar $\tau > 0$ such that
\[
\begin{bmatrix}
\text{Sym} \{ \Theta \otimes (A_0^T X E) \} + I_2 \otimes (A_0^T S A_0) & \Theta^T \otimes (E^T X D_A) + I_2 \otimes (A_0^T S D_A) \\
0 & I_2 \otimes S
\end{bmatrix} + \tau \begin{bmatrix}
I_2 \otimes (E_A^T E_A) & 0 \\
0 & -I_2 \otimes I
\end{bmatrix} < 0
\]
\[
\Leftrightarrow \begin{bmatrix}
\text{Sym} \{ \Theta \otimes (A_0^T X E) \} + I_2 \otimes (A_0^T S A_0) & \Theta^T \otimes (E^T X D) + I_2 \otimes (A_0^T S D) \\
0 & I_2 \otimes S - I_2 \otimes (\tau I)
\end{bmatrix} + \tau \begin{bmatrix}
I_2 \otimes E_A^T & 0 \\
0 & I_2 \otimes E_A
\end{bmatrix} < 0
\]
(59)

Let $Y = \tau I$ and invoking Schur Complement, inequality (54) is obtained.

The theorem is proved. \qed

**Theorem 32.** System (59) is robust admissible if there exist matrix $X \in \mathbb{R}^{n \times n}$, such that
\[
\begin{bmatrix}
\text{Sym} \{ \Theta \otimes (A_0^T X) \} + I_2 \otimes (E_A^T E_A) & I_2 \otimes (X D_A) \\
0 & -I_2 \otimes I
\end{bmatrix} < 0
\]
(60)
\[
E^T X = X E \geq 0
\]
(61)

where $\Theta = \begin{bmatrix}
\sin \frac{\pi}{2} \alpha & -\cos \frac{\pi}{2} \alpha \\
\cos \frac{\pi}{2} \alpha & \sin \frac{\pi}{2} \alpha
\end{bmatrix}$.

**Proof.** Invoking Schur Complement, inequality (60) is equivalent to
\[
\text{Sym} \{ \Theta \otimes (A_0^T X) \} + I_2 \otimes (E_A^T E_A) + I_2 \otimes (X D_A D_A^T X) < 0
\]
(62)

From inequalities (50) and (62), we get
\[
\text{Sym} \{ \Theta \otimes (A_0^T X) \} + \Theta^T \otimes (E_A^T E_A) + I_2 \otimes (X D_A F_A F_A^T D_A^T X) < 0
\]
(63)

According to lemma 8 and inequality (63), we have
\[
\text{Sym} \{ \Theta \otimes (A_0^T X) \} + \text{Sym} \{ \Theta \otimes ((D_A F_A E_A)^T X) \} < 0
\]
\[
\Leftrightarrow \text{Sym} \{ \Theta \otimes ((A_0 + D_A F_A E_A)^T X) \} < 0
\]
\[
\Leftrightarrow \text{Sym} \{ \Theta \otimes (A^T X) \} < 0.
\]

Thus, according to theorem 29, the system (49) is robust admissible.

The theorem is proved. \qed

### 6 Numerical examples

#### 6.1 Numerical solution in time domain

Now we will get the numerical solution of SFOS.
System \((E, A, \alpha)\) can be decomposed into
\[
\begin{cases}
  D^\alpha x_1(t) = A_1 x_1(t) \\
  ND^\alpha x_2(t) = x_2(t)
\end{cases}
\]

Thus we have to get \(A_1\) and \(N\). N’Doye [11] has proved that system \((E, A, \alpha)\) is regular if and only if \(\text{det}(cE - A)\) is not identically zero. Thus, \((cE - A)^{-1}\) exists. Define
\[
\hat{E} = (cE - A)^{-1}E, \quad \hat{A} = (cE - A)^{-1}A
\]

Thus
\[
\hat{A} = (cE - A)^{-1}(cE + A - cE) = c(cE - A)^{-1}E - I = c\hat{E} - I
\]

According to standard Jordan matrix decomposition, there exists nonsingular matrix \(T\) such that
\[
T\hat{E}T^{-1} = \text{diag}(\hat{E}_1, \hat{E}_2)
\]
where \(T \in \mathbb{R}^{n \times n}; \hat{E}_1 \in \mathbb{R}^{n_1 \times n_1}\) is nonsingular; \(\hat{E}_2 \in \mathbb{R}^{n_2 \times n_2}\) is a nilpotent matrix. Thus, \(c\hat{E}_2 - I\) is nonsingular. Let
\[
Q = \text{diag}(\hat{E}_1^{-1}, (c\hat{E}_2 - I)^{-1})T(cE - A)^{-1}
\]
\[
P = T^{-1}
\]

Then, we get
\[
QEP = \text{diag}(\hat{E}_1^{-1}, (c\hat{E}_2 - I)^{-1})T(cE - A)^{-1}ET^{-1} = \text{diag}(I_{n_1}, (c\hat{E}_2 - I)^{-1}\hat{E}_2)
\]
and
\[
QAP = \text{diag}(\hat{E}_1^{-1}, (c\hat{E}_2 - I)^{-1})T(cE - A)^{-1}AT^{-1} = \text{diag}(cI_{n_1} - \hat{E}_1^{-1}, I_{n_2})
\]

Therefore, we finally get \(A_1\) and \(N\)
\[
A_1 = \hat{E}_1^{-1}(c\hat{E}_1 - I), \quad N = (c\hat{E}_2 - I)^{-1}\hat{E}_2
\]

Because \(A_1\) is a constant matrix, by using Riemann-Liouville fractional integral, we get (29 [18])
\[
x_1(t) - \sum_{k=0}^{m-1} x_1^{(k)}(0) \frac{t^k}{k!} = \frac{A_1}{\Gamma(\alpha)} \int_0^t x_1(\tau)(t - \tau)^{\alpha-1} d\tau
\]
where \(m - 1 < \alpha \leq m\).
And according to Diethelm [30]

\[
\int_0^{t_{n+1}} (t_{n+1} - \tau)^{\alpha-1} x_1(t) d\tau \approx \frac{z^\alpha}{\alpha(\alpha+1)} \sum_{j=0}^{n+1} a_{j,n+1} x_1(t_j)
\]

where \( z = t_{j+1} - t_j \) and

\[
a_{j,n+1} = \begin{cases} 
  n^{\alpha+1} - (n - \alpha)(n + 1)^\alpha, & \text{if } j = 0 \\
  (n - j + 2)^{\alpha+1} + (n - j)^{\alpha+1} - 2(n - j + 1)^{\alpha+1}, & \text{if } 1 \leq j \leq n \\
  1, & \text{if } j = n + 1
\end{cases}
\]

In order to calculate \( x_1(t_{n+1}) \), Diethelm [30] predicts the integral as

\[
\int_0^{t_{n+1}} (t_{n+1} - \tau)^{\alpha-1} x_1(\tau) d\tau \approx \sum_{j=0}^{n} b_{j,n+1} x_1(t_j)
\]

where \( b_{j,n+1} = \frac{z^\alpha}{\alpha} ((n + 1 - j)^\alpha - (n - j)^\alpha) \). Thus, \( x_1(t_{n+1}) \) can be calculated by

\[
x_1(t_{n+1}) = \sum_{k=0}^{m-1} x_1^{(k)}(0) \frac{t^k}{k!} + \frac{z^\alpha}{\Gamma(\alpha+2)} A_1 x_1^p(t_{n+1}) + \frac{z^\alpha}{\Gamma(\alpha+2)} A_1 \sum_{j=0}^{n} a_{j,n+1} x_1(t_j)
\]

where \( x_1^p(t_{n+1}) = \sum_{k=0}^{m-1} x_1^{(k)}(0) \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} A_1 \sum_{j=0}^{n} b_{j,n+1} x_1(t_j) \).

And according to solution (29), \( x_2(t_n) \) can directly be calculated by

\[
x_2(t_n) = -\sum_{k=1}^{h-1} N^k \left( \delta^{(k\alpha-1)}(t_n) x_{20} + \delta^{(k\alpha-2)}(t_n) x_{20}^{(1)} \right)
\]

Finally, we can get

\[
x(t_n) = P \begin{bmatrix} x_1(t_n) \\ x_2(t_n) \end{bmatrix}
\]

6.2 Numerical solution

In this section, we verify the inequality (46) of theorem 29 as an example.

Consider system \((E, A, \alpha)\) with parameters \( \alpha = 1.8 \),

\[
A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & -1 & -1 \end{bmatrix},
\]

\[
E = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}
\]

And \( E_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \) can be chosen to satisfy \( E^T E_0 = 0 \).

Then, by solving LMI (48), we get

\[
P = \begin{bmatrix} 1.7896 & -0.2755 & -0.5029 \\ -0.2755 & 0.8271 & -0.6667 \\ -0.5029 & -0.6667 & 1.5113 \end{bmatrix},
Q = \begin{bmatrix} -0.043 & 0.3709 & 0.3849 \end{bmatrix}\]

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Figure 1: Eigenvalues of the system

It means the system is admissible. Eigenvalues of the system is shown in figure 1. From figure 1 we can find that all the eigenvalues of \((E, A)\) lie in the stable area. State response of the system is shown in figure 2 which implies that the system is stable.

7 Conclusion

In this paper, singular fractional order system with fractional order \(1 < \alpha < 2\) has been studied. The regularity and impulse-free of SFOS are proved in time domain. Then, this paper analysed sufficient and necessary conditions of stability and admissibility, respectively. After that, sufficient conditions of robust admissibility were given. Finally, numerical example was illustrated to verify proposed theorem.

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Figure 2: State response of the system
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