On the Erraticity in Random-Cascading $\alpha$ Model

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Abstract

The erraticity in the random-cascading $\alpha$ model is revisited. It is found that in contrary to the previous expectation, even in the pure single-$\alpha$ random-cascading model without putting in any particle there exists erraticity behavior and the corresponding entropy indices do not vanish. This means that the dynamical fluctuations in a pure single-$\alpha$ model already fluctuate event-by-event. Models with multiple-$\alpha$ strengthen this fluctuation. Taking double-$\alpha$ model as example, the variation of the event-space fluctuation strength with the mixing ratio of the two $\alpha$’s is studied in some detail. The influence of particle number on the results when particles are putted into the system is also investigated.

PACS: 13.85Hd

Keywords: Dynamical fluctuation  random cascading $\alpha$ model  erraticity
            entropy index

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1Supported in part by the NSFC under Grant No 19975021
2The authors are students of the State-Level Personnel Training Base for Research and Teaching in Fundamental Sciences (Physics).
Since the observation of large local fluctuations in a high multiplicity nucleus-nucleus collision event induced by cosmic ray in 1983\cite{1}, similar phenomenon has also been observed in the accelerator experiments\cite{2}. Since then, extensive attentions are paid to the non-linear phenomena in high energy collisions\cite{3}.

Due to the finiteness of particle number, the particle distribution fluctuates with respect to the probability distribution obtained from theoretical calculation. This is the so called statistical fluctuations. For a long time the popular idea was that the probability distribution obtained from theoretical calculation is smooth and the actual particle distribution fluctuates statistically on top of the smooth probability distribution. However, is it really the case? Is it possible that there are non-linear dynamics which causes the dynamical probability distribution itself to be non-smooth? This question is still to be answered. Although a systematic quantum non-linear theory has not been constructed yet and a definite reply to this question is still impossible by now, non-linear phenomena have already been observed in high energy experiments\cite{4–6} as stated above.

In studying the non-linear dynamical fluctuations, besides the factorial moments averaged over event sample, the importance of the fluctuations of single-event moments in the event samples has also been noticed\cite{7}. It is shown that these fluctuations are associated with the system’s chaotic behavior. A parameter $\mu$, called entropy index, is introduced to describe the chaotic behavior. An entropy index greater than zero is the signal of chaos. This method is called “erraticity analysis”.

The random-cascading $\alpha$ model is a simple model which can simulate the non-linear dynamical fluctuations. It has been used in Ref.\cite{8–11} to study the dynamical fluctuations in event space, i.e. erraticity. Since the value of parameter $\alpha$ determines the strength of dynamical fluctuations, it was argued in Ref.\cite{9} that there is no erraticity in the model with a single parameter $\alpha$, and therefore a model with Gaussian-distributed $\alpha$ was proposed instead.

In this letter the erraticity of $\alpha$ model is revisited. It is found that there is erraticity already in a pure single-$\alpha$ model. The corresponding entropy index is small but does not vanish. It increases with the increasing of $\alpha$. The mixing of different $\alpha$’s makes the entropy index increases further. The dependence of entropy index on the mixing ratio is studied. It is shown through putting in particles that in all cases the entropy index calculated by particle number is bigger than that in pure $\alpha$ model and tends to the latter only when the particle number tends to infinity. This reflects the effect of statistical fluctuations.
Let us first briefly recall the definition of erraticity and entropy index\textsuperscript{[7]}. The single-event factorial moments \( F_q^{(e)} \) and probabilistic moments \( C_q^{(e)} \) are defined as

\[
F_q^{(e)} = \frac{1}{M} \sum_{m=1}^{M} n_m(n_m - 1) \cdots (n_m - q + 1) \left( \frac{1}{M} \sum_{m=1}^{M} n_m \right)^q, \tag{1}
\]

\[
C_q^{(e)} = M^{q-1} \sum_{m=1}^{M} \left( p_m^{(e)} \right)^q, \tag{2}
\]

respectively, where a phase space region is divided into \( M \) equivalent sub-regions, \( p_m \) and \( n_m \) are correspondingly the probability and particle numbers in the \( m \)th sub-region.

The fluctuation of moments \( F_q^{(e)} \) and \( C_q^{(e)} \) in the event space can be characterized by their event-space moments

\[
C_{p,q}^{(F)} = \langle (F_q^{(e)})^p \rangle / \langle (F_q^{(e)})^p \rangle, \quad C_{p,q}^{(C)} = \langle (C_q^{(e)})^p \rangle / \langle (C_q^{(e)})^p \rangle. \tag{3}
\]

If \( C_{p,q}^{(F)} \) or \( C_{p,q}^{(C)} \) has power law behavior

\[
C_{p,q}^{(F)} \propto M^{\psi_q^{(F)}(p)}; \quad C_{p,q}^{(C)} \propto M^{\psi_q^{(C)}(p)}, \tag{4}
\]

when \( M \) is big then we say that there is erraticity in the system. It can be characterized quantitatively by the entropy index

\[
\mu_q = \left. \frac{d}{dp} \psi_q(p) \right|_{p=1}. \tag{5}
\]

Another method for calculating entropy index is to express \( C_{p,q} \) as

\[
C_{p,q} = \langle \Phi_q^{(e)^p} \rangle, \tag{6}
\]

where

\[
\Phi_q^{(e)} = F_q^{(e)}/\langle F_q^{(e)} \rangle, \quad \Phi_q^{(e)} = C_q^{(e)}/\langle C_q^{(e)} \rangle. \tag{7}
\]

Define

\[
\Sigma_q = \langle \Phi_q^{(e)} \ln \Phi_q^{(e)} \rangle, \tag{8}
\]

then in the scaling region, i.e. in the region where \( \Sigma_q \) depends on \( \ln M \) linearly, we can write

\[
\mu_q = \left. \frac{\partial \Sigma_q}{\partial \ln M} \right|_{\ln M}. \tag{9}
\]

In the following we will study the erraticity in the random-cascading \( \alpha \) model\textsuperscript{[12]}. In this model the \( M \) partition of the phase space region \( \Delta \) is realized in \( \nu \) steps. At the first
step, it is divided into two equal parts; at the second step, each part is further divided into two parts, \( \ldots \), and so on. The steps are repeated until \( M = \Delta Y/\delta y = 2^\nu \). In this step-by-step partition how particles are distributed between the two parts of a given phase space cell is determined by the value of independent random variable \( \omega_{\nu j_\nu} \), where \( j_\nu \) is the position of the sub-cell \( (1 \leq j_\nu \leq 2^\nu) \), \( \nu \) is the number of steps. The value of the random variable \( \omega_{\nu j_\nu} \) is given by:

\[
\omega_{\nu,2j-1} = \frac{1 + \alpha r}{2}, \quad \omega_{\nu,2j} = \frac{1 - \alpha r}{2}, \quad j = 1, \ldots, 2^{\nu-1}
\]

where \( r \) is a random number distributed uniformly in the interval \([-1,1]\), \( \alpha \) is a positive number less than or equal to unity, which determines the value-region of the random variable \( \omega \), and thus describes the strength of dynamical fluctuations in the model. After \( \nu \) steps, the probability in the \( m \)th sub-cell \( (m = 1, \ldots, M) \) is

\[
p_m = \omega_{1,j_1} \omega_{2,j_2} \cdots \omega_{\nu j_\nu}.
\]

Using these probabilities the probability moment \( C_q^{(e)} \) in each event is calculated according to (2), and the entropy index \( \mu_q \) is obtained through Eq’s (2)-(9).

Besides the single-\( \alpha \) model described above, the double-\( \alpha \)-mixing model will also be used later. It is constructed by letting \( r_1N \) events among the total \( N \) events have \( \alpha = \alpha_1 \), and the remaining \( (1 - r_1)N \) events have \( \alpha = \alpha_2 \).

In order to study the influence of limited particle number, we put a certain number \( N \) of particles into the \( M \) subcells according to the Bernauli distribution

\[
B(n_1, \ldots, n_M | p_1, \ldots, p_M) = \frac{N!}{n_1! \cdots n_M!} p_1^{n_1} \cdots p_M^{n_M}.
\]

The particle numbers \( n_m \ (m = 1, \ldots, M) \) in every window are thus obtained and the event factorial moments \( F_q^{(e)} \) can be calculated. The entropy index \( \mu_q \) is then obtained through Eq’s. (2)–(9).

Eq.’s (5) and (9) are two equivalent methods for getting entropy index. We now discuss which one is more suitable for our purpose.

What we are interested in is the case when the partition number \( M \) of phase space is very big. When the number of particles is small and that of sub-cells is big, it is probable that the particle number in every window is 0 or 1. It can be seen from Eq.(1) that this will result in \( F_q^{(e)} = 0 \) for all \( q > 1 \), and then lead to \( \Phi_q^{(e)} \) and \( \Sigma_q^{(e)} \) both vanish. Therefore, the second method fails in this case. So we choose the first method to get \( \mu_q \).
When using the first method, it can be seen from Eq.(5) that $\mu_q$ is obtained through partially differentiating $\psi_q^{(e)}$ with respect to $p$ near $p = 1$. Then how to choose the range of $p$ value near $p = 1$ to fit the partial derivative is also a question which we must answer. A too small range will lead to a wrong result due to the limitation of computer precision. To make the choice we show in Fig.1 the $\psi$-$p$ plot. It can be seen from the figure that within the range $0.5 \leq p \leq 2$ $\psi$ versus $p$ is approximately linear. Therefore, we take it as the right range for fitting the partial derivative $\partial \psi / \partial p$.

Fig.1 $\psi \sim p$ relation

In real calculation when $\alpha$ is very small (e.g. $\alpha=0.1$) the value of single event probability moment $C_q^{(e)}$ is close to zero. This results in big errors in numerical calculation. In order to solve this problem, we multiply the value of single event probability moments by a large number, e.g. 1000. This extra factor will appear in both the denominator and the numerator in the formula for calculating $C_{p,q}$, Eq. (3), and will not effect the final results.

The calculating results of single $\alpha$ model show that there exists erraticity. As example, in Fig.2 is shown the variation of $C_{p,q}$ with $M$ for $\alpha=0.3$. It shows a typical characteristic of erraticity. The entropy indices $\mu_2$ in this model for various values of $\alpha$ are plotted at the right side of Fig.3. It can be seen that $\mu_2 > 0$, indicating the existence of erraticity. The value of entropy index $\mu_2$ increases with the increasing of $\alpha$.

Fig.2 The variation of $C_{p,q} \sim M$ for $\alpha = 0.3$

Fig.3 Entropy index for single $\alpha$ model
On the left of Fig. 3 is shown the variation of entropy index with the number of particles when particles are putted into the model. It can be seen that when the number of particles is small the corresponding entropy index is much bigger than that of the pure α model without particle. The entropy index decreases with the increasing of the number of particles, tends to the value of the pure α model when the number of particles tends to infinity.

The reason why erraticity exists also in the single-α model is that, in the expression Eq. (10) for the elementary probability in the model, there is a random number \( r \), causing the window probability to fluctuate inspite of the fact that the values of \( \alpha \) are the same in different events.

It is clear that if the \( \alpha \)'s are different in different events, the fluctuation will be strengthened. We take the double-α model as example to study this situation.

The two \( \alpha \)'s are taken to be \( \alpha_1 = 0.2, \alpha_2 = 0.8 \). There are in total \( N \) events among which \( r_1N \) events have \( \alpha_1 = 0.2 \) and the remaining \( (1-r_1)N \) events have \( \alpha_1 = 0.8 \).

In Fig.4 are shown the \( \log_{10}\mu \) versus the number of particles \( n \) for \( r_1 = 0, 0.1, 0.3, 0.5, 0.7, 0.9, 0.95, 0.98, 1 \), respectively. It can be seen that the entropy index \( \mu \) firstly increases with the increasing of \( r_1 \) starting from \( r_1 = 0 \), but when \( r_1 \) is big (near to unity) it turns to decreasing. The reason is that \( r_1 = 0 \) and \( r_1 = 1 \) correspond to single-α model with \( \alpha = 0.2 \) and 0.8 respectively. The entropy indices for these two cases are in general smaller than that of the cases with the two \( \alpha \) values mixed. The only exception is when only a very little fraction (\(< 0.1\)) of big \( \alpha \) (0.8) is mixed in. In this case, the double-α-mixing model will have a smaller entropy index than the single-α model with \( \alpha = 0.8 \).

![Fig.4 Entropy index versus multiplicity for different double-α model](image)

The variation of entropy index in double-α-mixing model with the mixing ratio \( r_1 \) is plotted in Fig.5 for \( \alpha_1, \alpha_2 = 0.1, 0.9; 0.2, 0.8; 0.3, 0.7; 0.4, 0.6 \), respectively. The increasing of \( \mu \) with \( r_1 \) when the latter is not very big and the decreasing of \( \mu \) when \( r_1 \) near to unity can be seen in all the cases.

In order to show the fine structure near \( r_1 = 1 \) the curves for \( r_1, r_2 = 0.1, 0.9 \) (case-A)
and 0.4, 0.6 (case-B) are magnified and shown in the small figure inside Fig.5. The crossing of these two curves is understandable. This is because $\alpha_1^{(A)} < \alpha_1^{(B)}$ while $\alpha_2^{(A)} > \alpha_2^{(B)}$.

Therefore, when $r_1$ is small (1-$r_1$ is big) $\mu^{(A)} > \mu^{(B)}$, but when $r_1$ is big (1-$r_1$ is small) $\mu^{(B)} > \mu^{(A)}$.

In summary, it is found in this letter that even in the pure single-$\alpha$ random-cascading model without putting in any particle there exists erraticity behavior and the corresponding entropy indices do not vanish. This observation is in contrary to the previous expectation. It means that the dynamical fluctuations in a pure single-$\alpha$ model already fluctuate event-by-event.

This fluctuation is strengthened when different values of $\alpha$ are mixed in the model. Taking double-$\alpha$-mixing model as example, the variation of the entropy index with the mixing ratio of the two $\alpha$’s is studied. It is found that the entropy index $\mu$ firstly increases with the increasing of $r_1$ and then decreases. The maximum is located near $r_1 = 1$ when $\alpha_1 < \alpha_2$.

The influence of particle number on the results when particles are putted into the system is also investigated. In all cases the entropy index calculated by particle number is bigger than that in pure $\alpha$ model and tends to the latter only when the particle number tends to infinity.

Acknowledgement

The authors show sincere thanks to the patient direction given by Liu Lianshou and the warm help given by Liu Fuming.
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