Gravitational Thermodynamics

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To appear in: Complexity

Abstract

The gravitational N-body problem, for \( N > 2 \), is the oldest unsolved problem in mathematical physics. Some of the most ideal examples that can be found in nature are globular star clusters, with \( N \sim 10^6 \). In this overview, I discuss six types of fundamental sources of unpredictability, each of which poses a different challenge to attempts to determine the long-term behavior of these systems, governed by a peculiar type of thermodynamics.
1. Introduction

The theory of the gravitational dynamics of celestial bodies was first put on a firm footing by Newton in 1687, and was subsequently applied, both by him and by many others, to phenomena in the solar system. These particular applications, that form the discipline of celestial mechanics, were very specialized. Planets and asteroids move in orbits that mostly lie near a common plane, and their mutual perturbations are small compared to the dominant gravitational attraction of the much more massive Sun. And while comets describe orbits with a higher eccentricity and inclination, their masses are so small that they are effectively test particles, reacting to the force field around them but exerting no appreciable forces on their environment.

It took two and a half centuries before gravitational dynamics grew up, out of this cradle of celestial mechanics, into a full-fledged theory of stellar dynamics, able to describe self-gravitating configurations of particles without any a-priori restrictions. Indeed, a typical star cluster consists of a group of stars without any central dominant body such as the sun, and without any preferred plane of motion, such as that which characterizes the motion of the planets. A stellar dynamical description of a star cluster thus needs to be statistical in nature, and as such, it had to await the advent of thermodynamics, two centuries after Newton.

As we will see, the greater simplicity of stellar interactions over molecular interactions paradoxically created extra difficulties in attempts to apply laboratory thermodynamics to self-gravitating star systems. As a result, it was only toward the middle of this century that significant progress was made on a theoretical level. Actual experimentation had to wait even longer, until computers were powerful enough. Some of the most provocative theoretical ideas are just now becoming amenable to thorough testing in large-scale simulations of the full gravitational many-body problem, with particle numbers in the range 10,000 to 100,000.¹

¹ These numbers are the state of the art for simulations where each particle interacts with each other particle, for many relaxation times. Much more specialized calculations, such as those modeling structure formation in the early universe, can make use of various approximations that significantly reduce computer time, allowing total particle numbers up to $10^7$. 
2. Fundamental Sources of Uncertainty

The gravitational many-body problem addresses the behavior of an isolated system of point masses held together by Newtonian gravitational forces. There are several fundamental sources of uncertainty that stand in the way of a detailed description of the evolution of such a system. Three of those sources are directly related to predictability, while three others exist that limit an effective thermodynamic treatment of a self-gravitating system of point particles.

On a microscopic level, neighboring orbits show an exponential divergence with a Lyapunov time scale less than the time it takes for a typical particle to cross the system. In addition, bound pairs of particles, when encountering other particles or particle pairs, typically engage in complex interactions that exhibit an extreme sensitivity to initial conditions. On a macroscopic level, chaotic core oscillations in the dense center of the system introduce extra degrees of uncertainty into our attempts to model the detailed evolution of the system.

One might have hoped that these three sources of uncertainty would still admit an average overall understanding of the system in the form of a thermodynamic treatment. However, there are three other obstacles that stand in the way of a full-fledged theory of gravitational thermodynamics. For starters, we cannot define a thermodynamic limit, since gravitational binding energy grows faster than the mass of the system, when we increase the scale (in particle physics terminology, an infrared divergence). Furthermore, the system can exhibit an effectively negative heat capacity, which is related to the fact that a close pair of particles can generate an arbitrary amount of energy (an ultraviolet divergence). Finally, a self-gravitating system has only one coupling constant, which does not allow decoupling and fine-tuning of microscopic and macroscopic behavior separately. As a result, we are left with no degrees of freedom to play with. There are no dials that one can turn in order to study the response of the system.

In this overview, in §§3-8 each of these six characteristics will be discussed briefly in order to give some flavor of the difficulties involved. Notwithstanding these difficulties, gravitational many-body systems are realized in nature, and can indeed be modeled to high accuracy in computer simulations. These will be briefly reviewed in §§9,10.
3. Exponential Instability

In order to explore the behavior of a self-gravitating system of point masses, let us start with a thought experiment. Let us call our point particles ‘stars’, in order to anticipate astrophysical applications, even though at this point we neglect any physical properties the particles may have other than their mass.

Take a number of stars, a million say (a modest number, given that a typical galaxy contains $10^{11}$ stars). If we throw them into empty space, far away from disturbing tidal forces due to neighboring galaxies, the initial clump of stars may well look quite irregular at first. Since this arbitrary configuration is not likely to be near an equilibrium state, the whole system will undergo a few oscillations, and perhaps expand or contract considerably, depending on the initial kinetic energy given to the particles. However, after a few crossing times (the time $t_{cr}$ it takes for a typical star to cross the bulk of the star system) these oscillations will be damped out, largely through phase mixing [1]

From this point onwards, the resulting star cluster will look quite smooth. It may still be flattened or elongated considerably, and it may exhibit some bulk rotation. Subsequent two-body relaxation, exchange of energy and angular momentum between pairs of stars passing each other, tends to erase any such memory of the initial conditions. These effects take place on a two-body relaxation time scale $t_{rel}$, which can be shown to be related to the crossing time $t_{cr}$ as

$$t_{rel} \sim 0.1 \frac{N}{\ln N} t_{cr}.$$  

For a system of a million particles, the two time scales are thus well-separated, by some four orders of magnitude.

Analytical estimates, as well as more detailed computer simulations, can predict the overall qualitative and semi-quantitative effects of the process of two-body relaxation. However, a really accurate modeling is not feasible, even on the fastest computers. The problem is that the slightest change in the initial position or velocity of a single particle leads to an exponentially growing divergence of the trajectories of all particles, compared to the evolution of the original system. The e-folding time scale $t_e$ for this divergence was recently determined [2] to be

$$t_e \sim 0.2 \frac{1}{\ln \ln N} t_{cr},$$
nearly, but not quite, independent of the total number of stars.

The evolution of a star cluster takes place on a time scale far longer than $t_e$. Heat is transported through the cluster, as a consequence of many two-body encounters, on the time scale $t_{rel}$. On longer time scales, any self-gravitating star system is unstable. While some stars in the Maxwellian tail of the velocity distribution are lost through escape, other stars tend to congregate in the central regions which grow denser at an ever-increasing rate, since higher density implies more frequent encounters and hence a faster two-body relaxation.

This run-away redistribution of energy and mass leads to a phenomenon called gravothermal collapse, often called core collapse for short, which takes place on a time scale $t_{cc} \sim 10 t_{rel}$. Predicted analytically in the sixties, seen in approximate numerical simulations in the seventies, and verified in direct $N$-body simulations in the eighties, core collapse is a fundamental feature of long-term stellar-dynamical evolution. Its occurrence is related to the fact that series of self-gravitating equilibrium models exhibit a maximum entropy for a finite central concentration. There is no room here to go very deeply into the fascinating physics behind this phenomenon. For a general description, and references to the literature, Spitzer’s concise monograph on globular clusters would be a good starting point [3].

Suffice to say here that simple estimates, based on two-body encounters alone, would predict an infinitely high central density to develop after a time $t_{cc}$. In physics, any prediction giving infinite numbers suggests that some additional physics is needed. In this case simultaneous three-body encounters can save the situation, as we will see in the next section.

How serious is the uncertainty introduced by the exponential divergence of neighboring trajectories? If we want to simulate the evolution of a star cluster on a computer, all the way to core collapse and beyond, we have to overcome a number of e-folding times of order

$$t_{cc} \sim 5 N \frac{\ln \ln N}{\ln N} \sim N,$$

where the last approximate equality holds to within a factor two for particle numbers throughout the astronomically interesting range $10^2 \sim 10^{12}$. Since we lose roughly one digit of accuracy for every two e-folding times, a simulation of core-collapse of a self-gravitating $N$-body system thus requires us to calculate each orbit to a fantastic accuracy. Rather than working in
the usual double-precision mode, where each number is given to about 15 digits, we need a word-length of \( \sim N/2 \). Even if we use a 128-bit word length, we can only accurately model systems with up to 60 particles. On the fastest computers available, a simulation to core collapse thus turns out to be totally infeasible for any realistic value of \( N > 10^3 \), let alone \( N = 10^6 \) as is appropriate for globular star clusters, the primary examples of isolated star systems.

4. Three-Body Effects

Even in an idealized system of self-gravitating point particles, core collapse will be halted before an infinite central density is reached. When the central density is high enough, occasional close encounters between three unrelated particles will form bound pairs (double stars in the case of star clusters), with the third particle carrying off the excess kinetic energy required to leave the other two particles bound. Subsequent encounters between such pairs and other single particles tend to increase the binding energy of these pairs, which leads to a heating of the surrounding system of single particles.

When enough pairs have been formed this way, the resulting energy production will reverse the process of core collapse. After reaching a minimum radius and maximum density, the core region will expand again. An accurate modeling of this process of deep collapse and subsequent re-expansion is even more taxing than a collapse simulation. Besides the exponentially unstable trajectories of single particles, we now have to model the encounters between singles and pairs, as well as those between two pairs and occasionally even more complicated interactions.

Most frequent are the three-body interactions resulting from a pair-single encounter. In many of these cases, the third body is caught into a bound orbit, and the whole group of three begins an erratic gravitational dance stretching out over an extended period, before one of the three particles is ejected. During this dance, close encounters between two of the three particles often number in the tens or hundreds, if not thousands, and any small deviation in the initial conditions will result in a vastly different outcome. Typical amplification factors are \( 10^{10} \sim 10^{20} \), and factors exceeding \( 10^{50} \) are not uncommon [4]. A faithful simulation of these temperamental situations would require far larger word lengths than the standard 64-bit
(15-digit) ‘double precision’ word length implemented in most computers.

5. Gravothermal Oscillations

Core collapse, when threatened to occur by the collective effects of two-body relaxation, can thus be narrowly averted by a handful of crucial three-body or four-body reactions in the very dense core of a near-collapsed cluster. What will happen next depends on the total number $N$ of particles in the system. If this number is sufficiently small, $N < 10,000$ or thereabouts, the whole system will slowly and steadily expand. In this case an equilibrium can be found between the steady energy production in three-body encounters in the center, and the continuous loss of energy through the outskirts of the system.

If the total number of particles somewhat exceeds $10^4$, however, a different behavior emerges. The more particles there are in the system, the higher the central density has to become in deep core collapse, to be able to hold the initial contraction. As a result, the post-collapse phase features a very short central relaxation time, far shorter than the relaxation time in the outer regions, where most of the particles can be found. From the point of view of the inner dynamics, the bulk of the mass further out seems almost frozen. It is this discrepancy in time scales that can cause the inner core to become ‘impatient’, and to revert to a local collapse, triggered by the slightest fluctuation in the direction of the energy flow produced by stochastic three- and four-body interactions.

What happens then is that the inner one percent or so of the total number of particles will go once more into a coherent collapse, locally reminiscent of the original deep collapse.\footnote{Note that here and elsewhere the established jargon is somewhat misleading: the so-called ‘collapse’ occurs on a local relaxation time scale which even in the center is far larger than the local crossing time, except around core bounce – little happens during the time in which a typical particle traverses the system, since it is the thermal equilibrium that is perturbed, not the dynamical equilibrium.} As before, bound pairs of particles spring into action, generate energy, and manage to reverse the collapse in the nick of time, preventing an infinite central density from building up. This process repeats itself, leading to irregular oscillations. The chaotic
nature of these bulk oscillations forms a third source of fundamental uncertainty, preventing detailed predictions to be made of the behavior of a self-gravitating system of point masses, even on a macroscopic level.

Although we are dealing here with the oldest unsolved problem in mathematical physics, the behavior of a system of $N$ gravitationally interacting particles for $N > 2$, the existence of these oscillations was completely unknown until 1983, when they were first found in approximate simulations [5]. Dubbed ‘gravothermal oscillations’, they were subsequently analyzed in detail with semi-analytic methods [6]. Their occurrence was confirmed in a variety of approximate numerical simulations [7], and shown to correspond to low-dimensional chaos for large $N$ values [8]. Finally, their existence was proven beyond the shadow of a doubt when they were seen in the very first direct $N$-body simulation that was able to incorporate $N$ values well beyond $N = 10,000$ [9]. Interestingly, it took the construction of a special-purpose computer, with a peak speed of 1 Teraflops, to pull off this feat [10].

6. No Thermodynamic Limit

Quite apart from the three problems raised above, there are more fundamental problems that prevent a full thermodynamic treatment of a self-gravitating system of point particles. First of all, gravity exhibits what is known in particle physics as an infrared divergence. This means that the effect of long-distance interactions cannot be neglected, even though gravitational forces fall off with the inverse square of the distance.

Take a large box containing a homogeneous swarm of stars. Now enlarge the box, keeping the density and temperature of the star distribution constant. The total mass $M$ of the stars will then scale with the size $R$ of the box as $M \propto R^3$, and the total kinetic energy $E_{\text{kin}}$ will simply scale with the mass: $E_{\text{kin}} \propto M$. The total potential energy $E_{\text{pot}}$, however, will grow faster: $E_{\text{pot}} \propto M^2/R \propto M^{5/3}$. Unlike intensive thermodynamic variables that stay constant when we enlarge the system, and unlike extensive variables that grow linearly with the mass of the system, $E_{\text{pot}}$ is a superextensive variable, growing faster than linear.

As a consequence, the specific gravitational potential energy of the system, the total potential energy of the system divided by the particle
number $N$, grows without bounds when we increase $N$. This causes various problems. For example, the kinetic energy of a stable self-gravitating system is directly related to the gravitational potential energy through the so-called virial theorem. Therefore, we have to make a choice when enlarging a self-gravitating system. Either we increase the temperature steadily while increasing $N$, in order to increase the specific kinetic energy enough to satisfy the virial theorem and guarantee stability. Or we keep the temperature constant, and quickly lose stability when enlarging our system. In the latter case, the system will ‘curdle’: it will fall apart in more and more subclumps, and the original homogeneity will be lost quickly.

In conclusion, there is no way that we can reach an asymptotic thermodynamic limit, which the system size becoming arbitrarily large while holding the intensive variables fixed. Therefore, the traditional road to equilibrium thermodynamics is blocked. There are no arbitrarily large homogeneous distributions of stars. As soon as the Universe became old and cold enough to let matter condense out of the original fire ball into ‘islands’ in the form of galaxy clusters and galaxies, the original homogeneity was lost. And each individual clump of self-gravitating material, be it a galaxy or a star cluster, is ultimately unstable against evaporation, and will fall apart into a bunch of escaping particles. Most of these escapers will be single, some will escape as stable pairs, and a few will even manage to form stable triples or higher-number multiples of particles.

7. Negative Heat Capacity

Another problem interfering with a standard thermodynamic treatment of the gravitational many-body problem is what is known in particle physics as an ultraviolet divergence. Since the individual particles are mass points with no spatial extension, they can come arbitrarily close, and therefore their negative gravitational binding energy can become arbitrarily large. Just one pair of particles can therefore provide an unlimited amount of positive energy to the rest of the system.

This is dramatically illustrated in the following thought experiment. Let us confine $N$ point masses to a box with reflecting walls, while we neglect the gravitational effects due to the finite mass of the walls. If we wait long enough, a simultaneous close three-body encounter will produce a tightly bound pair. It can be shown that from that moment on, the prob-
ability is overwhelmingly large that this pair will grow tighter and tighter, on average, giving off more and more energy. This energy is converted to kinetic energy of all the other particles, including the kinetic energy of the center-of-mass motion of the bound pair.

In this thought experiment, after closing the lid of the box, we would notice that the walls of the box would get hotter, without bound. Even if we would slowly extract heat from the box, its temperature would keep rising. The cause of this paradoxical behavior is due to the fact that a self-gravitating system can effectively have a negative heat capacity. The tightly bound particle pair in our experiment is one example of such a system. Its orbital motion will be much faster than the movement of the single particles that it encounters. In an attempt to reach equipartition, the bound particles will try to convey some of their rapid motion to the single particles, speeding the latter up in the process. The bound particles themselves, however, while attempting to slow down, will find themselves falling to an even tighter orbit. A shorter distance in the gravitational two-body problem implies a higher orbital velocity, so the net effect is that both the single particles and the bound particles will speed up as a result of their interactions.

8. A Single Coupling Constant

The motion of the stars in a star cluster can be described in a way that is analogous to the treatment of the motion of molecules in a gas studied in a laboratory. One important difference is that a swarm of stars forms an open system, while a body of gas in a lab has to be contained. Typical textbook experiments in thermodynamics show the gas to reside inside a cylinder, with a movable piston that allows the experimenter to change the volume of the gas. In a star cluster, there are no cylinder and piston. Instead, the stars are confined by their collective gravitational field.

The structural simplicity of a star cluster thus allows far less experimentation than is the case for a body of gas in a lab situation. Whether in thought experiments, computer simulations, or in actual tabletop experiments, the macroscopic parameters of a laboratory gas can be changed freely, independent of the microscopic parameters governing the attraction and repulsion between individual molecules. Temperature, density, and size of the system all can be varied at will. In contrast, once the number of
particles in a self-gravitating system has been chosen, we are left with no
degree of freedom at all, apart from trivial scalings in the choice of unit of
length, time, and mass.

The fact that there are no dials that can be turned in a self-gravitating
experiment, apart from the choice of the total number of stars, is directly re-
lated to the ultraviolet and infrared divergences of classical gravity. Having
a simple shape for the gravitational potential energy well, with an energy in-
versely proportional to distance, leaves no room for preferred length scales.
In contrast, molecular interactions show far more complicated forces, typi-
cally strongly repulsive at shorter distances and weakly attractive at larger
separations between the molecules. This change in behavior automatically
specifies particular length scales, for example the distance at which repul-
sion changes into attraction. In contrast, gravity is attractive everywhere,
at least in the classical Newtonian approximation.

9. Astrophysical Applications

Gravitation is a purely attractive force. Even though electromagnetism
is far stronger than gravity, on microscopic length scales as well as on
normal human scales, electromagnetic effects tend to cancel out on much
larger distances. As a result, gravitation plays a dominant role in almost
all astrophysical objects, from planets to stars to star clusters to galaxies
and clusters of galaxies.

Not all of these objects can be described accurately with the simple
models that we have discussed so far. Galaxies, for example, contain mas-
sive gas clouds whose hydrodynamic interactions are as important as their
gravitational properties. Furthermore, the individual galaxies in a cluster
of galaxies are significantly extended, in comparison with the intergalactic
distances, and therefore cannot be treated very well as point particles.

The ideal type of astrophysical objects, from the point of view of study-
ing gravitational thermodynamics, are the globular clusters (fig. 1). Each
one contains of order a million stars. Other than that, they contain very
little gas, are far removed from the bulk of the stars in our galaxy, and
are among the oldest objects in the Universe. All these properties make
them ideal laboratories for stellar dynamics [3]. There are about a hundred
of these clusters circling our own galaxy. While many other galaxies are
accompanied by similar numbers of globulars, some galaxies are adorned by thousands or more of them.

Fig. 1. M15, a typical globular cluster with a collapsed core, as seen by the Hubble Space Telescope (in the V band)[11].

Recently, the Hubble Space Telescope has allowed us a few long-awaited peeks into the very densest inner regions of some of the core-collapsed globular clusters (fig. 2). Fortunately, we are ready to interpret this wealth of new data. As discussed above, we have made great progress in our theoretical understanding of the long-term evolution of self-gravitating systems during the eighties. In addition, computers have now become powerful enough to enable realistic simulations to be carried out.
Fig. 2. A high-resolution view of the dense central regions of M15; the previous figure, but with three colors (U, B, and V), covering only the central 9" by 9".[11]

10. Computer Simulations

In the late eighties, detailed analysis made it clear that realistic simulations of globular clusters on a star by star basis, necessary to investigate gravothermal oscillations, would remain beyond the power of even the most advanced computers for at least a decade [12]. Fortunately, a small group
of astrophysicists at Tokyo University decided to build their own specialized computer for stellar dynamical calculations. Recently, their completed product (fig. 3), the GRAPE-4 [10], reached a speed of 1 Teraflops, which made it the fastest computer in the world.

Fig. 3. The GRAPE-4, a special-purpose computer for stellar dynamics simulations, operating at a speed of 1 Teraflops, record holder of the fastest computations in 1995 and 1996, for which two Gordon Bell prizes were awarded to Junichiro Makino, seen here in the photograph, together with Makoto Taiji (in 1995) and Toshiyuki Fukushige (in 1996).

This computer allowed an unambiguous demonstration of the reality of gravothermal oscillations (figs. 4,5). The next generation special-purpose computer, the GRAPE-6, is planned to have a speed which is far higher than the GAPE-4, in the range of 100-1000 Teraflops. This will enable realistic simulations of up to a million point particles. If funding can be found, the GRAPE-6 could be built within a few years.
Fig. 4. The evolution of the central density in a 32,000-body problem, the result of a Teraflops-week calculation on the GRAPE-4. For comparison, calculations for smaller number of particles are shown as well, in which case the peak densities reached are lower.
Fig. 5. The evolution of central density $\rho_c$ and temperature (proportional to $v_c^2$, where $v_c$ is the velocity dispersion), for the 32,000-body calculation of Fig. 4, showing the first direct evidence for the existence of gravothermal oscillations.

While point-particle simulations are of great interest from the point of view of mathematical physics, astrophysics requires a more realistic treatment. Real stars have finite radii, and therefore can run into each other and merge. Also, they age and die, ending their life either in a violent supernova explosion, which may produce a neutron star or black hole or lead to a total annihilation of the star, or by settling down into a more quiescent final state as a white dwarf. These more realistic simulations are currently underway [13,14].
11. Conclusions

The fact that a self-gravitating system of point masses is governed by a law with only one coupling constant has important consequences. In contrast to most macroscopic systems, there is no decoupling of scales. We do not have at our disposal separate dials that can be set in order to study the behavior of local and global aspects separately. The only real freedom we have, when modeling a self-gravitating system of point masses, is our choice of the value of the dimensionless number $N$, the number of particles in the system.

The value of $N$ determines a large number of seemingly independent characteristics of the system: its granularity and thereby its speed of internal heat transport and evolution; the size of the central region of highest density after the system settles down in an asymptotic state; the nature of the oscillations that may occur in this central region; and to a surprisingly weak extent the rate of exponential divergence of nearby trajectories in the system. Independent of particle number, however, any self-gravitating system in or near equilibrium exhibits a most unusual thermodynamic characteristic, in the form of a negative heat capacity.

Acknowledgments. I thank Douglas Heggie for comments on the manuscript. This work was supported in part by a grant from the Alfred P. Sloan foundation, for research on limits to scientific knowledge. I thank the Santa Fe Institute for their hospitality during my visits in March and June 1996.

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