Nonvanishing quantum corrections to the mass and central charge of the $N = 2$ vortex and BPS saturation

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ABSTRACT

The one-loop quantum corrections to the mass and central charge of the $N = 2$ vortex in 2+1 dimensions are determined using supersymmetry-preserving dimensional regularization by dimensional reduction of the corresponding $N = 1$ model with Fayet-Iliopoulos term in 3+1 dimensions. Both the mass and the central charge turn out to have nonvanishing one-loop corrections which however are equal and thus saturate the Bogomolnyi bound. We explain BPS saturation by standard multiplet shortening arguments, correcting a previous claim in the literature postulating the presence of a second degenerate short multiplet at the quantum level.
1 Introduction and summary

In supersymmetric (susy) theories with solitons, topological quantum numbers appear as central extensions of the susy algebra. In the topological sectors there is then typically a lower bound to the mass spectrum determined by the central charge, and BPS (Bogomolnyi-Prasad-Sommerfield) states, which saturate this lower bound, form shortened multiplets when compared to the usual massive multiplets. They are of particular interest because this “multiplet shortening” ties the mass of these states to their topological quantum numbers such that the BPS saturation is protected from quantum corrections [1]. (This is sometimes overstated as implying that there are no quantum corrections to the classical mass spectrum at all, while it rather means that such quantum corrections have to affect mass and central charge by equal amounts.)

The simplest example of a susy theory with solitons, the 1+1 dimensional susy kink, seemed to be an exception in that the counterparts of short and long multiplets have an equal number of states (namely two) [1] according to standard representation theory of susy. Nevertheless, most of the explicit calculations found neither nontrivial corrections to the mass [2] nor to the central charge [3].

A couple of years ago, this issue was reopened when two of the present authors found [4] that the simple energy-momentum cutoff used explicitly or implicitly in most of the calculations that obtained a null result was inconsistent with the integrability of the bosonic sine-Gordon model [5]. Careful calculations using mode number cutoff such that boundary energy contributions are avoided subsequently established a nonzero result for the quantum corrections to the mass [6, 7, 8, 9] that agreed in fact with an older result by Schonfeld [10] and which has also been reproduced by different methods [11, 12, 13]. On the other hand, it was established by Shifman et al. [14], using a susy-preserving higher-derivative regularization method, that there is also an anomalous contribution to the central charge which still leads to BPS saturation at the quantum level\(^1\), and which was subsequently explained by the possibility of “super-short” single-state supermultiplets in 1+1 dimensions [16, 8] which have no definite fermion number.

In Ref. [17] we have recently shown that these results are most elegantly

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\(^1\)A novel derivation of the central charge anomaly using superspace methods to evaluate the Jacobian in the path integral with heat-kernel methods can be found in [15].
derived by employing dimensional regularization through susy-preserving dimensional reduction from a higher-dimensional model. The correct quantum corrections to the mass of the susy kink are obtained [18] without having to deal with energy located at boundaries introduced in other methods, and the anomalous contribution to the central charge can be obtained from corrections to the momentum operator in the extra dimensions, which in the case of a kink background leave a finite remainder in the limit of 2 dimensions [17].

In this paper we consider the Abrikosov-Nielsen-Olesen [19, 20, 21, 22] vortex solution of the abelian Higgs model in 2+1 dimensions which has a supersymmetric extension [23, 24] (see also [25, 26]) such that classically the Bogomolnyi bound [27] is saturated. We employ our variant of dimensional regularization to the $N = 2$ vortex by dimensionally reducing the $N = 1$ abelian Higgs model in 3 + 1 dimensions. We confirm the results of [23, 28, 29] that in a particular gauge (background-covariant Feynman-'t Hooft) the sums over zero-point energies of fluctuations in the vortex background cancel completely, but contrary to [23, 28] we find a nonvanishing quantum correction to the vortex mass coming from a finite renormalization of the expectation value of the Higgs field in this gauge [30, 29]. In contrast to [23], where a null result for the quantum corrections to the central charge was stated, we show that the central charge receives also a net nonvanishing quantum correction, namely from a nontrivial phase in the fluctuations of the Higgs field in the vortex background, which contributes to the central charge even though the latter is a surface term that can be evaluated far away from the vortex. The correction to the central charge exactly matches the correction to the mass of the vortex.

In Ref. [28], it was claimed that the usual multiplet shortening arguments in favor of BPS saturation would not be applicable to the $N = 2$ vortex since in the vortex background there would be two rather than one fermionic zero modes [31], leading to two short multiplets which have the same number of states as one long multiplet.\footnote{Incidentally, Refs. [28, 31] considered supersymmetric Maxwell-Chern-Simons theory, which contains the supersymmetric abelian Higgs model as a special case.} We show however that the extra zero mode postulated in [28] has to be discarded because its gaugino component is singular, and that only after doing so there is agreement with the results from index theorems [32, 31, 33]. For this reason, standard multiplet shortening arguments do apply, explaining the BPS saturation at the quantum level that
we observe in our explicit one-loop calculations.

2 The vortex in $3 \leq D \leq 4$ dimensions

The $N = 2$ susy vortex in 2+1 dimensions is the solitonic (finite-energy) solution of the abelian Higgs model which can be obtained by dimensional reduction from a 3+1-dimensional $N = 1$ model. We shall use the latter for the purpose of dimensional regularization of the 2+1-dimensional model by susy-preserving dimensional reduction from 3+1 dimensions (where the vortex has infinite mass but finite energy-density).

2.1 The model

The superspace action for the vortex in terms of 3+1-dimensionalsuperfields contains an $N = 1$ abelian vector multiplet and an $N = 1$ scalar multiplet, coupled as usual, together with a Fayet-Iliopoulos term but without superpotential,

$$\mathcal{L} = \int d^2 \theta W^\alpha W_\alpha + \int d^4 \theta e^V \Phi + \kappa \int d^4 \theta V.$$  \hfill (1)

In terms of 2-component spinors in 3+1 dimensions, the action reads

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 + \bar{\chi}^\alpha \sigma^\mu_{\alpha\beta} \partial_\mu \chi^\beta + \frac{1}{2} D^2 + (\kappa - e |\phi|^2) D - |D_\mu \phi |^2 + \bar{\psi}^\dot{\alpha} \sigma^\mu_\dot{\beta\dot{\alpha}} \partial_\mu \psi^\beta + |F|^2 + \sqrt{2} e \left[ \phi^\ast \chi^\alpha \psi^\alpha + \phi \bar{\chi}^\dot{\alpha} \bar{\psi}^\dot{\alpha} \right],$$  \hfill (2)

where $D_\mu = \partial_\mu - ie A_\mu$ when acting on $\phi$ and $\psi$, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Elimination of the auxiliary field $D$ yields the scalar potential $V = \frac{1}{2} D^2 = \frac{1}{2} e^2 (|\phi|^2 - \nu^2)^2$ with $\nu^2 \equiv \kappa / e$.

In 3+1 dimensions, this model has a chiral anomaly, and in order to cancel the chiral U(1) anomaly, additional scalar multiplets would be needed such that the sum over charges vanishes, $\sum_i e_i = 0$. In the present paper we shall consider only the dimensional reduction of the minimal model (2),

\footnote{Our conventions are $\eta^{\mu\nu} = (-1, +1, +1, +1)$, $\chi^\alpha = e^{\alpha\beta} \chi_\beta$ and $\bar{\chi}^\dot{\alpha} = e^{\dot{\alpha}\dot{\beta}} \bar{\chi}_{\dot{\beta}}$ with $e^{\alpha\beta} = e_{\alpha\beta} = -e^{\dot{\alpha}\dot{\beta}} = -e_{\dot{\alpha}\dot{\beta}}$ and $e^{12} = +1$. In particular we have $\bar{\psi}_\alpha = (\psi_\alpha)^\ast$ but $\bar{\psi}^\dot{\alpha} = -(\psi^{\dot{\alpha}})^\ast$. Furthermore, $\sigma^\mu = (-1, \sigma)$ with the usual representation for the Pauli matrices $\sigma$.}
postponing a discussion of anomaly-free 3+1 dimensional vortices \[34, 25\] to a forthcoming work \[35\].

In 2+1 dimensions, dimensional reduction gives an \( N = 2 \) model involving, in the notation of \[28\], a real scalar \( N = A_3 \) and two complex (Dirac) spinors \( \psi = (\psi^\alpha) , \chi = (\chi^\alpha) \).

Completing squares in the bosonic part of the classical Hamiltonian density one finds the Bogomolnyi equations and the central charge

\[
\mathcal{H} = \frac{1}{4} F_{kl}^2 + |D_k \phi|^2 + \frac{1}{2} e^2 (|\phi|^2 - v^2)^2 \\
= \frac{1}{2} |D_k \phi + i \epsilon_{kl} D_l \phi|^2 + \frac{1}{2} (F_{12} + e (|\phi|^2 - v^2))^2 \\
+ \frac{e}{2} v^2 \epsilon_{kl} F_{kl} - i \delta_k (\epsilon_{kl} \phi^* D_l \phi) 
\]

(3)

where \( k, l \) are the spatial indices in 2+1 dimensions. The classical central charge reads

\[
Z = \int d^2 x \epsilon_{kl} \partial_k (e v^2 A_l - i \phi^* D_l \phi) ,
\]

(4)

where asymptotically \( D_l \phi \) tends to zero exponentially fast. Classically, BPS saturation \( E = |Z| = 2\pi v^2 n \) holds when the BPS equations \( (D_1 \pm i D_2) \phi \equiv D_\pm \phi = 0 \) and \( F_{12} \pm e (|\phi|^2 - v^2) = 0 \) are satisfied, where the upper and lower sign corresponds to vortex and antivortex, respectively. The vortex solution with winding number \( n \) is given by

\[
\phi_V = e^{in} f(r) , \quad e A_+^V = -ie^{in} a(r) \frac{a - n}{r} , \quad A_+^V \equiv A_1^V \pm i A_2^V 
\]

(5)

where \( f'(r) = \frac{a}{r} f(r) \) and \( a'(r) = re^2 (f(r)^2 - v^2) \) with boundary conditions \[22\]

\[
a(r \to \infty) = 0 , \quad f(r \to \infty) = v , \\
a(r \to 0) = n + O(r^2) , \quad f(r \to 0) \propto r^n + O(r^{n+2}) .
\]

(6)

2.2 Fluctuation equations

For the calculation of quantum corrections to a vortex solution we decompose \( \phi \) into a classical background part \( \phi_V \) and a quantum part \( \eta \). Similarly, \( A_\mu \) is decomposed as \( A_\mu^V + a_\mu \), where only \( A_\mu^V \) with \( \mu = 1, 2 \) is nonvanishing.
We use a background $R_\xi$ [36] gauge fixing term which is quadratic in the quantum gauge fields,

$$L_{\text{g.fix}} = -\frac{1}{2\xi}(\partial_\mu a^\mu - ie\xi(\phi_V^*\eta - \phi_V^*\eta))^2. \quad (7)$$

The corresponding Faddeev-Popov Lagrangian reads

$$L_{\text{ghost}} = b(\partial^2 - e^2\xi\{2|\phi_V|^2 + \phi_V^*\eta + \phi_V^*\eta\}) c. \quad (8)$$

The fluctuation equations in 2+1 dimensions have been given in [23, 28] for the choice $\xi = 1$ (Feynman-'t Hooft gauge) which leads to important simplifications. We shall mostly use this gauge choice when considering fluctuations in the solitonic background, but will carry out renormalization in the trivial vacuum for general $\xi$ to exhibit some of the intermediate gauge dependences.

Because we are going to consider dimensional regularization by dimensional reduction from the 3+1 dimensional model, we shall need the form of the fluctuation equations with derivatives in the $x^3$ direction included. (This one trivial extra dimension will eventually be turned into $\epsilon \to 0$ dimensions.)

In the 't Hooft-Feynman gauge, the part of the bosonic action quadratic in the quantum fields reads

$$L^{(2)}_{\text{bos}} = -\frac{1}{2}(\partial_\mu a_\nu)^2 - e^2|\phi_V|^2a_\mu^2 - |D_\mu^V\eta|^2 - e^2(3|\phi_V|^2 - v^2)|\eta|^2$$
$$-2iea_\mu\left[\eta^*D_\mu^V\phi_V - \eta(D_\mu^V\phi_V)^*\right]. \quad (9)$$

In the trivial vacuum, which corresponds to $\phi_V \to v$ and $A^V_\mu \to 0$, the last term vanishes, but in the solitonic vacuum it couples the linearized field equations for the fluctuations $B \equiv (\eta, (a_+/\sqrt{2}))$ with $a_+ = a_1 + ia_2$ to each other according to $(k = 1, 2)$

$$\left(\partial_3^2 - \partial_k^2\right)B = \left(\begin{array}{cc}
(D_k^V)^2 + e^2(3|\phi_V|^2 - v^2) & i\sqrt{2}e(D_+\phi_V) \\
-i\sqrt{2}e(D_-\phi_V)^* & -\partial_k^2 + 2e^2|\phi_V|^2
\end{array}\right)B. \quad (10)$$

The quartet $(a_3, a_0, b, c)$ with $b, c$ the Faddeev-Popov ghost fields has diagonal field equations at the linearized level

$$\left(\partial^2 - 2e^2|\phi_V|^2\right)Q = 0, \quad Q \equiv (a_3, a_0, b, c). \quad (11)$$
For the fermionic fluctuations, which we group as $U = (\psi^1, \bar{\chi}^1), V = (\psi^2, \bar{\chi}^2)$, the linearized field equations read

$$LU = i(\partial_t + \partial_\theta)V, \quad L^\dagger V = i(\partial_t - \partial_\theta)U,$$

with

$$L = \begin{pmatrix} iD_+^V & \sqrt{2}e\phi_V \\ -\sqrt{2}e\phi_V & i\partial_+ \end{pmatrix}, \quad L^\dagger = \begin{pmatrix} iD_+^V & -\sqrt{2}e\phi_V \\ \sqrt{2}e\phi_V & i\partial_+ \end{pmatrix}. \quad (13)$$

Iteration shows that $U$ satisfies the same second order equations as the bosonic fluctuations $B$,

$$L^\dagger L U = (\partial_\theta^2 - \partial_t^2)U, \quad L^\dagger L B = (\partial_\theta^2 - \partial_t^2)B \quad (14)$$

$$L L^\dagger V = (\partial_\theta^2 - \partial_t^2)V, \quad (15)$$

with $L^\dagger L$ given by (10), whereas $V$ is governed by a diagonal equation with

$$L L^\dagger = \begin{pmatrix} -(D_k^V)^2 + e^2|\phi_V|^2 + e^2v^2 & 0 \\ 0 & -\partial_k^2 + 2e^2|\phi_V|^2 \end{pmatrix}. \quad (16)$$

(In deriving these fluctuation equations we used the BPS equations throughout.)

### 2.3 Renormalization

At the classical level, the energy and central charge of vortices are multiples of $2\pi v^2$ with $v^2 = \kappa/e$. Renormalization of tadpoles, even when only by finite amounts, will therefore contribute directly to the quantum mass and central charge of the $N = 2$ vortex, a fact that has been overlooked in the original literature [23, 28] on quantum corrections to the $N = 2$ vortex.\(^4\)

Adopting a “minimal” renormalization scheme where the scalar wave function renormalization constant $Z_\phi = 1$, the renormalization of $v^2$ is fixed by the requirement of vanishing tadpoles in the trivial sector of the 2+1 dimensional model. The calculation can be conveniently performed by using dimensional regularization of the 3+1 dimensional $N = 1$ model. For the calculation of the tadpoles we decompose $\phi = v + \eta \equiv v + (\sigma + i\rho)/\sqrt{2}$, where $\sigma$ is the Higgs field and $\rho$ the would-be Goldstone boson. The gauge fixing term

\(^4\)The nontrivial renormalization of $\kappa/e$ has however been included in [30, 29].
(7) avoids mixed $a_{\mu}\rho$ propagators, but there are mixed $\chi\psi$ propagators, which can be diagonalized by introducing new spinors $s = (\psi + i\chi)/\sqrt{2}$ and $d = (\psi - i\chi)/\sqrt{2}$ with mass terms $m(s_\alpha s^\alpha - d_\alpha d^\alpha) + h.c.$, where $m = \sqrt{2}v$. The part of the interaction Lagrangian which is relevant for $\sigma$ tadpoles to one-loop order is given by

$$L_{\sigma-tadpoles}^{int} = g(\chi_\alpha \psi^\alpha + \bar{\chi}_\dot{\alpha} \bar{\psi}^\dot{\alpha}) \sigma - \frac{e m}{2}(\sigma^2 + \rho^2) \sigma - e m(a_\mu^2 + \xi b c - \delta v^2) \sigma,$$

where $b$ and $c$ are the Faddeev-Popov fields.

The one-loop contributions to the $\sigma$ tadpole thus read

$$\{ -2\text{tr} I(m) + \frac{3}{2} I(m) + \frac{1}{2} I(\frac{1}{4m}) + [3 I(m) + \xi I(\frac{1}{4m})] - \xi I(\frac{1}{4m}) - \delta v^2 \},$$

where

$$I(m) = \int \frac{d^3k}{(2\pi)^3} \frac{-i}{k^2 + m^2} = -\frac{m^{1+\epsilon}}{4\pi^{1+\epsilon/2}} \frac{\Gamma(-\frac{1}{2} - \frac{\epsilon}{2})}{\Gamma(-\frac{1}{2})} = -\frac{m}{4\pi} + O(\epsilon).$$

Requiring that the sum of tadpole diagrams (18) vanishes fixes $\delta v^2$,

$$\delta v^2 = \frac{1}{2} \left( I(m) + I(\frac{1}{4m}) \right) \bigg|_{D=3} = -\frac{1 + \xi^2}{8\pi} m.$$
3 Quantum corrections to mass and central charge

3.1 Mass

At the one-loop level, the quantum mass of a solitonic state is given by

$$M = M_{cl} + \frac{1}{2} \sum \omega_{\text{bos}} - \frac{1}{2} \sum \omega_{\text{term}} + \delta M$$

(21)

where $M_{cl}$ is the classical mass expressed in terms of renormalized parameters, $\delta M$ represents the effects of the counter-terms to these renormalized parameters, and the sums are over zero-point energies in the soliton background (the zero-point energies in the trivial vacuum, which one should subtract in principle, cancel in a susy theory).

In the $\xi = 1$ gauge the sum over zero-point energies is formally

$$\frac{1}{2} \sum \omega_{\text{bos}} - \frac{1}{2} \sum \omega_{\text{term}} = \sum \omega_{\eta} + \sum \omega_{a_+} - \sum \omega_{U} - \sum \omega_{V}$$

$$= \sum \omega_{U} - \sum \omega_{V},$$

(22)

where the quartet $(a_3, a_0, b, c)$ cancels separately. (Note that in (22) all frequencies appear twice because all fields are complex.) Using dimensional regularization as developed in [18], these sums can be made well defined by replacing all eigens frequencies $\omega_k$ in 2+1 dimensions by $\omega_{k,\ell} = (\omega^2_k + \ell^2)^{1/2}$ where $\ell$ are the extra momenta, and integrating over $\ell$. The spectral densities are nontrivial only with respect to the 2-dimensional momenta $k$ and the former are not modified by dimensional regularization.

In [23, 28] it has been shown that $L^1 L$ and $LL^1$, which govern $U$ and $V$, respectively, are isospectral up to zero modes. In dimensional regularization, where the zero-mode contributions continue to give zero because scaleless integrals vanish, one can therefore conclude that in the $\xi = 1$ gauge there is a complete cancellation of the sums over zero-point energies. All that remains is the finite renormalization of $\delta v^2$ in that gauge:

$$E = 2\pi |n| (v^2 + \delta v^2 |_{\xi = 1}) = 2\pi |n| (v^2 - \frac{m}{4\pi}) \equiv |n| \left( \frac{\pi m^2}{e^2} - \frac{m}{2} \right)$$

(23)

(In gauges other than $\xi = 1$ the fluctuation equations for the $B$ fields, i.e. $\eta, a_+$, no longer match those of the $U$ fermions.)
This result agrees with [29], where however a careful analysis of boundary conditions in the heat-kernel approach was needed because the vortex had to be put in a box to discretize the spectrum. In dimensional regularization one does not need to put the system in a box, and as a consequence there is no need to study the contributions from these artificial boundaries.

3.2 Central charge

By starting from the susy algebra in 3+1 dimensions one can derive the central charge in 2+1 dimensions as the component \( T^0_3 \) of

\[
T^\mu_\nu = -\frac{i}{4} \text{Tr} \sigma^{\mu\alpha\dot{\alpha}} \{ \bar{Q}_\alpha, J^\nu_\dot{\alpha} \}
\]

where \( J^\nu_\dot{\alpha} \) is the susy Noether current.

The antisymmetric part of \( T^\mu_\nu \) gives the standard expression for the central charge density, while the symmetric part is a genuine momentum in the extra dimension:

\[
\langle Z \rangle = \int d^2 x \langle T^{03} \rangle = \left\langle \tilde{Z} + \tilde{P}_3 \right\rangle.
\]

(A similar decomposition is valid for the kink [17].)

\( \tilde{Z} \) corresponds to the classical expression for the central charge. Being a surface term, its quantum corrections can be evaluated at infinity:

\[
\langle \tilde{Z} \rangle = \int d^2 x \partial_k \epsilon_{kl} \langle \tilde{\zeta}_l \rangle = \int_0^{2\pi} d\theta \langle \tilde{\zeta}_\theta \rangle |_{r\to\infty}
\]

with \( \tilde{\zeta}_l = ev^2_0 A_l - i\phi^\dagger D_l \phi \) and \( v^2_0 = v^2 + \delta v^2 \).

Expanding in quantum fields \( \phi = \phi_V + \eta, A = A^V + a \) and using that the classical fields \( \phi_V \to ve^{i\theta}, A^V_\theta \to n/e, D^V_\theta \phi_V \to 0 \) as \( r \to \infty \), we obtain to one-loop order

\[
\langle \tilde{Z} \rangle = 2\pi n v^2_0 - i \int_0^{2\pi} d\theta \left( \langle \phi_V^* + \eta^\dagger \rangle (D^V_\theta - iea_\theta) \langle \phi_V + \eta \rangle \right) |_{r\to\infty}
\]

\[
= 2\pi n \{ v^2_0 - \langle \eta^\dagger \eta \rangle |_{r\to\infty} \} - i \int_0^{2\pi} d\theta \left\{ \langle \eta^\dagger \partial_\theta \eta \rangle - i e \phi_V^* \langle a_\theta \eta \rangle - i e \phi_V \langle a_\theta \eta^\dagger \rangle \right\} |_{r\to\infty}
\]

\[
\equiv Z_a + Z_b
\]

where we have used \( \langle \eta |_{r\to\infty} \rangle \to 0 \) (which determines \( \delta v^2 \)), \( \int_0^{2\pi} d\theta \langle a_\theta \rangle = 0 \), and \( \langle \eta^\dagger \eta a_\theta \rangle = O(\hbar^2) \).
The first contribution, $Z_a$, can be easily evaluated for arbitrary gauge parameter $\xi$, yielding
\[
Z_a = 2\pi n \left\{ v_0^2 - \frac{1}{2} \langle (\sigma \sigma) + (\rho \rho) \rangle \big|_{r \to \infty} \right\}
\]
\[
= 2\pi n \left\{ v_0^2 - \frac{1}{2} [I(m) + I(\xi \frac{1}{2} m)] \right\}
\]
\[
= 2\pi n (v_0^2 - \delta v^2) = 2\pi n v^2.
\]
(28)

If this was all, the quantum corrections to $Z$ would cancel, just as in the naive calculation of $Z$ in the susy kink [3, 4].

The second contribution in (27), however, does not vanish when taking the limit $r \to \infty$. This contribution is simplest in the $\xi = 1$ gauge, where the $\eta$ and $a_\theta$ fluctuations are governed by the fluctuation equations (10). In the limit $r \to \infty$ one has $|\phi_V| \to v$ and $D_\phi \phi_V \to 0$ exponentially. This eliminates the contributions from $\langle a_\theta \eta \rangle$. However, $D^2_k$, which governs the $\eta$ fluctuations, contains long-range contributions from the vector potential. Making a separation of variables in $r$ and $\theta$ one finds that asymptotically
\[
|D^V_k \eta|^2 \to |\partial_r \eta|^2 + \frac{1}{r^2} |(\partial_\theta - i n) \eta|^2
\]
so that the $\eta$ fluctuations have an extra phase factor $e^{i n \theta}$ compared to those in the trivial vacuum. We thus have, in the $\xi = 1$ gauge,
\[
Z_b = -i \int_0^{2\pi} d\theta \left\{ \langle \eta^\dagger \partial_\theta \eta \rangle - i e \phi_V^\dagger \langle a_\theta \eta \rangle - i e \phi_V \langle a_\theta \eta^\dagger \rangle \right\} \big|_{r \to \infty}
\]
\[
= -i \int_0^{2\pi} d\theta \langle \eta^\dagger \partial_\theta \eta \rangle \big|_{\xi = 1} = 2\pi n \langle \eta^\dagger \eta \rangle \big|_{\xi = 1, r \to \infty} = 2\pi n \delta v^2 \big|_{\xi = 1}.
\]
(30)

This is exactly the result for the one-loop correction to the mass of the vortex in eq. (23), implying saturation of the BPS bound provided that there are now no anomalous contributions to the central charge operator as there are in the case in the $N = 1$ susy kink [17].

In dimensional regularization by dimensional reduction from a higher-dimensional model such anomalous contributions to the central charge operator come from a finite remainder of the extra momentum operator [17]. The latter is given by [28]
\[
Z_c = \langle \tilde{P}_3 \rangle = \int d^2 x \langle F_0 F_{3i} + (D_0 \phi) \dagger D_3 \phi + (D_3 \phi) \dagger D_0 \phi - i \bar{\chi} \bar{\sigma}_0 \partial_3 \chi - i \bar{\psi} \bar{\sigma}_0 D_3 \psi \rangle.
\]
(31)
Inserting mode expansions for the quantum fields one immediately finds that the bosonic contributions vanish because of symmetry in the extra trivial dimension. However, this is not the case for the fermionic fields, which have a mode expansion of the form

$$
\left( \begin{array}{c} U \\ V \end{array} \right) = \int \frac{d^d \ell}{(2\pi)^{d/2}} \sum_k \frac{1}{\sqrt{2\omega}} \left\{ b_{k,\ell} e^{-i(\omega t - \ell z)} \begin{pmatrix} \sqrt{\omega - \ell^2} u_1 \\ \sqrt{\omega - \ell^2} u_2 \\ \sqrt{\omega + \ell^2} v_1 \\ \sqrt{\omega + \ell^2} v_2 \end{pmatrix} + d_{k,\ell}^\dagger \times (c.c.) \right\}, \quad (32)
$$

where we have not written out explicitly the zero-modes (for which $\omega^2 = \ell^2$). The fermionic contribution to $Z_c$ reads, schematically,

$$
Z_c = \langle \tilde{P}_3 \rangle = \int \frac{d^d \ell}{(2\pi)^{d/2}} \sum_k \frac{\ell^2}{2\omega} \int d^2 x \left[ |u_1|^2 + |u_2|^2 - |v_1|^2 - |v_2|^2 \right] \quad (33)
$$

where $\omega = \sqrt{\omega_k + \ell^2}$, so that the $\ell$ integral does give a nonvanishing result. However, the $x$-integration over the mode functions $u_{1,2}(k; x)$ and $v_{1,2}(k; x)$ produces their spectral densities, which cancel up to zero-mode contributions as we have seen above\(^5\), and zero-mode contributions only produce scaleless integrals which vanish in dimensional regularization. Hence, $Z_c = 0$ and $|Z| = |Z_a + Z_b| = E$, so that the BPS bound is saturated at the (one-loop) quantum level.

### 3.3 Fermionic zero modes and multiplet shortening

Massive representations of the Poincaré supersymmetry algebra for which the absolute value of the central charge equals the energy, i.e. when the BPS bound is saturated, contain as many states as massless representations, which is half of that of massive representations for which the BPS bound is not saturated. These results also apply in 2+1 dimensions for the $N = 2$ super-Poincaré algebra [28].

A particular multiplet of states is obtained by taking the vortex solution, and acting on it with the susy generators of the $N = 2$ susy algebra, which contains two complex charges $Q_+$, $Q_-$, and their hermitian conjugates $(Q_+)^\dagger$ and $(Q_-)^\dagger$. One of these charges, $Q_-$, annihilates the vortex solution, while

\(^5\)An explicit calculation which confirms these cancellations will be presented in [35].
the other one, $Q_+$, is to linear order in quantum operators proportional to
the annihilation operator of a fermionic zero mode.

However, if there indeed is a second fermionic zero mode in the model as
postulated in [28]6, in second quantization it would be present in the mode expansion of the
fermionic quartet $U$ and $V$,

$$
\begin{pmatrix}
U \\
V
\end{pmatrix}
= a_1 \begin{pmatrix}
\psi_1^1 \\
\bar{x}_1^1
\end{pmatrix} + a_H \begin{pmatrix}
\psi_H^1 \\
\bar{x}_H^1
\end{pmatrix} + \text{non-zero modes.}
$$

(34)

As a result, there would then be a quartet of BPS states

$$
|v\rangle, a_1^\dagger |v\rangle, a_H^\dagger |v\rangle, a_H^\dagger a_H^\dagger |v\rangle
$$

(35)

comprising two short multiplets of $N = 2$ susy, which are degenerate and
together have as many states as one long multiplet without BPS saturation.
As stressed in [28], the standard argument for stability of BPS saturation
under quantum corrections from multiplet shortening [1] thus would not be applicable.

However, we shall now show that there is in fact only a single fermionic
zero mode in a vortex background with winding number $n = 1$. To this end,
we first observe that the zero modes must lie in $U$, because $V$ is governed by
the operator $LL^\dagger$ of Eq. (16), whose only zero mode solution is $V_0 \equiv 0$. A
zero mode for $U$ must satisfy $LU = 0$, and to analyse this equation we follow
[28] and set $\psi^1(x, y) = -ie^{(j-\frac{1}{2}+n)}u(r)$ and $\bar{\chi}^1 = e^{i(j+\frac{1}{2})}d(r)$. The equation
$LU = 0$ reduces then to

$$
\begin{pmatrix}
\partial_r - \frac{a+j-\frac{1}{2}}{r} & \sqrt{2ef} \\
\sqrt{2ef} & \partial_r + \frac{j+\frac{1}{2}}{r}
\end{pmatrix}
\begin{pmatrix}
u \\
d
\end{pmatrix}
= 0,
$$

(36)

where $f = f(r)$ and $a = a(r)$ satisfy $f' = \frac{a}{r}f$ and $a' = re^2(f^2 - e^2)$. Iterating
this equation yields

$$
\left(\partial_r^2 + \frac{1}{r} \partial_r - \frac{(j-\frac{1}{2})^2}{r^2} - 2e^2 f^2\right) \frac{u}{f} = 0.
$$

(37)

---

6In the literature one can in fact find two different conventions for indicating the number of fermionic zero modes. Like Refs. [28, 33] we only count the number of zero modes in the fermionic quartet $(U, V)$ and not additionally those in the corresponding conjugated fields $(U^\dagger, V^\dagger)$. One zero mode in this way of counting then corresponds to a pair of creation/annihilation operators. Alternatively one may count the zero modes in both $(U, V)$ and $(U^\dagger, V^\dagger)$ and thus ascribe one zero mode to each creation or annihilation operator. The latter way of counting is employed for instance in Ref. [26].
Given a solution for $u$, the corresponding solution for $d$ follows from $LU = 0$.

For given $j$, this equation has two independent solutions, a linear combination of which yields solutions which decrease exponentially fast as $r \to \infty$. Hence, both solutions should be regular at $r = 0$. For $j \neq \frac{1}{2}$, one has, using $f(r \to 0) \sim r^n$,

$$
\psi^1 \sim u \sim r^n(C_1 r^{j - \frac{1}{2}} + C_2 r^{-(j - \frac{1}{2})}) \quad \text{for } r \to 0
$$

which selects for $n = 1$ only $j = -\frac{1}{2}$. This solution is the zero mode that is obtained by acting with $Q_+$ on the background solutions, which gives

$$
\psi^1 = -iD_- \phi_V/\sqrt{2} = -i\sqrt{2} f^\prime, \quad \bar{\chi}^1 = F_{12} = -e(f^2 - v^2).
$$

For $j = \frac{1}{2}$, one finds for $n = 1$ near $r = 0$

$$
\psi^1 \sim C_1 (x + iy) + C_2 (x + iy) \ln r.
$$

For large $r$, $\psi_1 \sim e^{-mr} e^{i\theta}$, as follows from (37). This solution corresponds to the second fermionic zero mode postulated in Ref. [28].

However, while (39) is regular at the origin, the associated gaugino component is not: (36) implies that

$$
\bar{\chi}^1 \sim C_2 e^{i\theta} / r,
$$

so this solution has to be discarded when $C_2 \neq 0$.

Similarly, one can show that for winding number $n > 1$ regularity of the gaugino component generically requires that $j \leq -\frac{1}{2}$ so that the correct quantization condition for normalizable fermionic zero modes is $-n + \frac{1}{2} \leq j \leq -\frac{1}{2}$. Hence, there are $n$ independent fermionic zero modes, not $2n$ as concluded in [28]. It is in fact only the former value that agrees with the results [31, 33] obtained from the index theorem [32].

As has been proved rigorously in [37], in the bosonic sector there are $2n$ zero modes, which are related to the above $n$ independent fermionic zero modes by supersymmetry. In the $R_{\xi = 1}$ background gauge $\partial_\mu a^\mu - ie(\phi_V \eta^* - \phi_V^* \eta) = 0$, the bosonic zero modes satisfy a set of equations completely equivalent to those for the fermionic zero modes [31]. But the linearly dependent solutions $(U^0_0)$ and $i(U^0_0)$ correspond to linearly independent solutions for the bosonic zero modes $a$ and $\eta$.$^7$ In particular, for $n = 1$, the $j = -\frac{1}{2}$ solution $(\psi^1 = -i u(r), \bar{\chi}^1 = d(r))$ with real $u(r)$ and $d(r)$ corresponds to the

$^7$For an analogous case see eq. (3.8) of Ref. [38].
bosonic zero mode $\eta(r) = -iu(r)$, $(a_1, a_2) = (\sqrt{2}d(r), 0)$, while multiplying the fermionic solution by $i$ corresponds to the bosonic zero mode $\eta(r) = u(r)$, $(a_1, a_2) = (0, \sqrt{2}d(r))$, which is evidently linearly independent of the former. For both solutions the $R_{\xi=1}$ gauge condition is satisfied due to the lower component of the field equation (36). Conversely, one can start from the classical vortex solution and find two independent bosonic zero modes by considering their derivatives with respect to the $x$ and $y$ coordinates. Performing a gauge transformation to satisfy the $R_{\xi=1}$ gauge condition leads one back to the above solutions. This additionally confirms that the above analysis has identified all fermionic zero modes in the quartet $(U, V)$.

We thus conclude that for the basic vortex (winding number $n = 1$) there is exactly one fermionic zero mode (corresponding to one pair of fermionic creation/annihilation operators) and this gives rise to a single short multiplet at the quantum level. Standard multiplet shortening arguments therefore do apply and explain the preservation of BPS saturation that we verified at one-loop order.

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