A NOTE ON KÄHLER-RICCI FLOW

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ABSTRACT. Let \( g(t) \) with \( t \in [0, T) \) be a complete solution to the Kaehler-Ricci flow: \( \frac{d}{dt}g_{i\bar{j}} = -R_{i\bar{j}} \) where \( T \) may be \( \infty \). In this article, we show that the curvatures of \( g(t) \) is uniformly bounded if the solution \( g(t) \) is uniformly equivalent. This result is stronger than the main result in Šešum [6] within the category of Kähler-Ricci flow.

1. Introduction

Let \( M^n \) be a Kähler manifold of complex dimension \( n \). Let \( g(t) \) with \( t \in [0, T) \) be a complete solution to the Kähler-Ricci flow:

\[
\frac{\partial g_{i\bar{j}}}{\partial t} = -R_{i\bar{j}} + cg_{i\bar{j}},
\]

where \( T \) may be \( \infty \) and \( c \) is a real number. We assume that \( g(t) \) satisfies Shi’s estimate (Ref. Shi [7]):

\[
\|\nabla^k Rm\|^2(x, t) \leq \frac{C(k, s)}{t^k}
\]

for any nonnegative integer \( k \), any \( s \in (0, T) \) and any \( (x, t) \in M \times (0, s] \), where \( C(k, s) \) is a positive constant depending on \( k \) and \( s \). When \( M^n \) is a compact Kähler manifold, this assumption is superfluous because a solution to the Kähler-Ricci flow (1.1) will automatically satisfy Shi’s estimate (1.2). By the uniqueness result of Chen-Zhu [3], such a solution to Kähler-Ricci flow (1.1) is uniquely determined by its initial metric \( g(0) \).

In this article, we obtain the following main result.

**Theorem 1.1.** Let \( g(t) \) with \( t \in [0, T) \) be a complete solution to the Kähler-Ricci flow (1.1) satisfying Shi’s estimate (1.2). Suppose that there is a positive constant \( C \) such that

\[
C^{-1}g(0) \leq g(t) \leq Cg(0).
\]
Then, for any nonnegative integer $k$, there are two positive constants $A_k$ and $B_k$ such that
\[ \| \nabla^k Rm \|^2(x,t) \leq A_k + \frac{B_k}{t^k} \]
for any $(x,t) \in M \times [0,T)$.

In particular, when $k = 0$, this result is stronger than the main result in Šešum [6] within the category of Kähler-Ricci flow.

This result is useful for obtaining long time existence for Kähler-Ricci flow. As an application, we will give a simple long time existence of Kähler-Ricci flow which implies Cao’s result (Ref. [1]) on long time existence of Kähler-Ricci flow on compact Kähler manifolds.

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2. A maximum principle

Proposition 2.1. Let $g(t)$ with $t \in [0,T]$ be a smooth family of complete Kähler metrics on $M$. Suppose that the sectional curvatures of $g(0)$ are bounded and there is positive constant $C_0$ such that
\[ C_0^{-1} g(0) \leq g(t) \leq C_0 g(0) \]
for any $t \in [0,T]$. Let $h \in C^\infty(M \times [0,T])$ be such that
\[ \sup_M h(0) < \sup_{M \times [0,T]} h < \infty. \]

Then, there is a sequence $(x_k, t_k) \in M \times (0,T]$ such that
\[ \lim_{k \to \infty} h(x_k, t_k) = \sup_{M \times [0,T]} h, \quad \lim_{k \to \infty} \| \nabla h \|(x_k, t_k) = 0, \quad \frac{\partial h}{\partial t}(x_k, t_k) \geq 0, \]
and
\[ \lim_{k \to \infty} \sup_M \Delta h(x_k, t_k) \leq 0. \]
Proof. By adding a constant to $h$, we can suppose that $\sup_{M \times [0,T]} h = 1$. Let $p$ be a fixed point. Let $\rho$ be a smooth function on $M$, such that
$$
\begin{cases}
C_1^{-1}(1 + r_0(p, x)) \leq \rho(x) \leq C_1(1 + r_0(p, x)) \\
\|\nabla_0 \rho\| \leq C_1 \\
|\Delta_0 \rho| \leq C_1
\end{cases}
$$
all over $M$, where $C_1$ is some positive constant. (c.f. Theorem 3.6 in Shi [7].)

Then
$$
|\Delta_t \rho| = |g^{i\bar{j}}(t) \rho_{i\bar{j}}| \leq C_0 |g^{i\bar{j}}(0) \rho_{i\bar{j}}| = C_0 |\Delta_0 \rho| \leq C_2.
$$
all over $M \times [0, T]$ for some positive constant $C_2$.

Let $\eta$ be a smooth function on $[0, \infty)$ such that $\eta = 1$ on $[0, 1]$, $\eta = 0$ on $[2, \infty)$, and $-2 \leq \eta' \leq 0$.

For any $\epsilon \in (0, 1/2)$, let $(x_0, t_0) \in M \times (0, T]$ be such that
$$
\max\{1 - \epsilon, \sup_M h(0)\} < h(x_0, t_0) \leq 1.
$$
Let $R > \rho(x_0)$ be a constant to be determined. Let $\phi = \eta(\rho/R)$ and let $(\bar{x}, \bar{t})$ be a maximum point of $\phi h$. It is clear that $\bar{t} > 0$ since
$$
h(\bar{x}, \bar{t}) \geq (\phi h)(\bar{x}, \bar{t}) \geq (\phi h)(x_0, t_0) = h(x_0, t_0) > \sup_M h(x, 0).
$$
Moreover, we have
$$
1 - \epsilon \leq (\phi h)(\bar{x}, \bar{t}) \leq 1, \nabla (\phi h)(\bar{x}, \bar{t}) = 0, \frac{\partial (\phi h)}{\partial t}(\bar{x}, \bar{t}) \geq 0
$$
and
$$
\Delta (\phi h)(\bar{x}, \bar{t}) \leq 0.
$$
By the first inequality and that $\sup_{M \times [0, s]} h = 1$, we know that
$$
\frac{1}{2} \leq 1 - \epsilon \leq h(\bar{x}, \bar{t}) \leq 1, \quad \text{and} \quad \frac{1}{2} \leq (1 - \epsilon) \leq \phi(\bar{x}, \bar{t}) \leq 1.
$$
Then,
$$
|\nabla h(\bar{x}, \bar{t})| = \frac{h|\nabla \phi|}{\phi}(\bar{x}, \bar{t}) \leq \frac{C_3}{R},
$$
$$
\frac{\partial h}{\partial t}(\bar{x}, \bar{t}) \geq 0,
$$
and
\[ \Delta h(\bar{x}, \bar{t}) \leq -\frac{h\Delta \phi + 2\langle \nabla h, \nabla \phi \rangle}{\phi}(\bar{x}, \bar{t}) \leq \frac{C_4}{R}. \]
By choosing \( R > \max\{\rho(x_0), \frac{C_2}{\epsilon}, \frac{C_4}{\epsilon}\} \), we get
\[ \|\nabla h(\bar{x}, \bar{t})\| \leq \epsilon, \text{ and } \Delta h(\bar{x}, \bar{t}) \leq \epsilon. \]
Therefore, by choosing a sequence \( \epsilon_k \to 0^+ \), we get a sequence \((x_k, t_k)\) satisfying our requirements. \( \square \)

3. CURVATURES ESTIMATES

**Lemma 3.1.** Let \( g(t) \) be a solution to the Kähler-Ricci flow (1.1) on \([0, T]\) with \( T < \infty \) satisfying the following assumption:

\[ \|\nabla^k Rm\| \leq C_k \]
all over \( M \times [0, T] \). Then, for any nonnegative integer \( k \), there is positive constant \( A_k \), such that
\[ \|\nabla^0 g\|_{g_0} \leq A_k \]
all over \( M \times [0, T] \).

**Proof.** Because the curvatures are uniformly bounded, by the Kähler-Ricci flow equation (1.1), all the metrics are uniformly equivalent. So, the lemma is true for \( k = 0 \).

In the follows, ”;” means taking covariant derivatives with respect to \( g_0 \). By equation (1.1), we have
\[ \frac{\partial g_{i;:j;k}}{\partial t} = -R_{i;:j;k} + cg_{i;:j;k} \]
(3.2)
\[ = -\left( \nabla_k R_{\bar{j};i} - (g_0)^{\bar{j}\bar{a}}(g_0)_{i\bar{a}} \nabla_k (g_0)_{\bar{a}\bar{j}} \right) + cg_{i;:j;k} \]
\[ = -\left[ \nabla_k R_{\bar{i};j} + (g_0)^{\bar{j}\bar{a}}(g_0)_{\bar{a}\bar{b}}(g_0)_{\bar{b}\bar{g};i;k} \right] + cg_{i;:j;k}. \]
Moreover, by the assumption on curvatures and the case that \( k = 0 \),
\[ \frac{\partial \|\nabla g\|^2_{g_0}}{\partial t} \leq C_1 + C_2 \|\nabla g\|^2_{g_0} \]
where \( C_1 \) and \( C_2 \) are some positive constants. Therefore, the lemma is true for \( k = 1 \) and \( \| \nabla_0 Rm \| \) is also uniformly bounded since \( \nabla_0 Rm \) can be expressed as a combination of \( \nabla Rm \) and \( \nabla_0 g \).

Computing further on step by step by taking more covariant derivatives with respect to \( g_0 \) on both sides of equation (3.2), we know that the lemma is true for all nonnegative integer \( k \).

\[ \square \]

**Theorem 3.1.** Let \( g(t) \) be a solution to the Kähler-Ricci flow (1.1) on \([0, T)\) satisfying the following assumption:

\[
\| \nabla Rm \| \leq C(k, s)
\]

all over \( M \times [0, s] \), for any nonnegative integer \( k \) and any \( s \in (0, T) \), where \( T \) may be \( \infty \) and \( C(k, s) \) is a positive constant depending on \( k \) and \( s \). Moreover, suppose that there is a positive constant \( C_0 \) such that

\[
C_0^{-1} g_0 \leq g(t) \leq C_0 g_0
\]

for any \( t \in [0, T) \). Then, there is a positive constant \( C \) such that

\[
\| \nabla_0 g \| \leq C
\]

all over \( M \times [0, T) \).

**Proof.** In the follows, covariant derivatives are taken with respect to the initial metric \( g_0 \) and normal coordinates are chosen with respect to \( g_0 \).

Let \( S = (g_0)^{\tilde{i}}\tilde{j} g_{i\tilde{j}} \) and

\[
Q = g^{q\bar{p}} g^{p\bar{k}} g_{\bar{i}\bar{j}k} g_{p\bar{q}} \bar{x}.
\]

We want to get an estimate of \( Q \). By the assumptions, we have

\[
n C_0^{-1} \leq S \leq n C_0.
\]

By direct computation, we have

\[
g_{i\bar{j},k\bar{l}} = [(g_0)_{i\bar{p}} ((g_0)^{\bar{p}q} g_{\bar{q}j})]_{k\bar{l}} = g_{i\bar{j},k\bar{l}} + (g_0)_{i\bar{p}} ((g_0)^{\bar{p}q} g_{\bar{q}j})_{k\bar{l}} = g_{i\bar{j},k\bar{l}} + (R_0)_{i\bar{p}k\bar{l}} g_{\bar{q}j},
\]

\[
(3.4)
\]
and

\[ R_{ijkl} = -g_{ij,kl} + g^{\rho\mu}g_{i\rho,j}g_{\mu;j,\bar{l}} = -g_{ij,kl} + g^{\rho\mu}g_{i\rho,j}g_{\mu;j,\bar{l}} + (R_0)_{i\bar{\mu}kJ\bar{\mu}j}, \]

where a comma means a partial derivative. Hence

\[ g^{\bar{k}}g_{ij,kl} = -R_{ij} + g^{\bar{k}}g^{\rho\mu}g_{i\rho,j}g_{\mu;j,\bar{l}} + (R_0)_{i\bar{\mu}kJ\bar{\mu}j}g^{\bar{k}}. \]

We are now ready to compute the evolution equation of \( S \):

\[
\left( \frac{\partial}{\partial t} - \Delta \right) S
= (g_0)_{\bar{j}i}(-R_{ij} + cg_{ij}) - g^{\bar{k}l}S_{\bar{k}i} \\
= - (g_0)_{\bar{j}i}(-R_{ij} + cg_{ij}) - g^{\bar{k}l}(g_0)_{\bar{j}i}g_{ij,kl} \\
= - (g_0)_{\bar{j}i}(-R_{ij} + cg_{ij}) - (g_0)_{\bar{j}i}(-R_{ij} + g^{\bar{k}l}g^{\rho\mu}g_{i\rho,j}g_{\mu;j,\bar{l}} + (R_0)_{i\bar{\mu}kJ\bar{\mu}j}g^{\bar{k}}) \\
= - (g_0)_{\bar{j}i}g^{\bar{k}l}g^{\rho\mu}g_{i\rho,j}g_{\mu;j,\bar{l}} + cS - (R_0)_{i\bar{\mu}kJ\bar{\mu}j}g^{\bar{k}} \\
\leq - c_1Q + C_2,
\]

since the curvatures of \( g_0 \) is bounded and that the metrics are uniformly equivalent.

A similar computations as in Appendix A of Yau \([10] \) (See Remark 3.1) give us the following evolution equation of \( Q \).

\[
\left( \frac{\partial}{\partial t} - \Delta \right) Q
= - \sum_{i,j,k,l} \frac{1}{\lambda_i\lambda_j\lambda_k\lambda_l} \left| g_{ik;jl} - \sum_{\gamma} \frac{1}{\lambda_\gamma} g_{i\gamma;j}g_{\gamma;k,l} \right|^2 \\
- \sum_{i,q,k,\mu} \frac{1}{\lambda_i\lambda_k\lambda_q\lambda_\mu} \left| g_{iq;k\mu} - \sum_{\alpha} \frac{1}{\lambda_\alpha} (g_{i\alpha};i g_{\alpha;k,\mu} + g_{\alpha;q\mu} g_{i\alpha;k}) \right|^2 + \Re
\]

with

\[ |\Re| \leq C_3Q + C_4, \]

where we have chosen a normal coordinate of \( g_0 \) such that \( g_{ij} = \lambda_i\delta_{ij} \). Then, by choosing \( C_5 \) be such that \( c_1 C_5 - C_3 = 1 \),

\[
\left( \frac{\partial}{\partial t} - \Delta \right) (Q + C_5S) \leq -(Q + C_5S) + C_6.
\]
Fixed $s \in (0, T)$, by Lemma 3.1 $Q + C_5S$ is a bounded function on $M \times [0, s]$.

By Proposition 2.1 on any closed interval $[0, s]$, if

$$
\sup_{M \times [0, s]} (Q + C_5S) > \sup_{M} (Q + C_5S)(\cdot, 0) = nC_5,
$$

there exists a sequence $(x_k, t_k) \in M \times (0, s]$, such that

$$
\lim_{k \to \infty} (Q + C_5S)(x_k, t_k) = \sup_{M \times [0, s]} (Q + C_5S), \quad \frac{\partial (Q + C_5S)}{\partial t}(x_k, t_k) \geq 0,
$$

and

$$
\limsup_{k \to \infty} \Delta (Q + C_5S)(x_k, t_k) \leq 0.
$$

Then, by (3.8),

$$
0 \leq \liminf_{k \to \infty} \left( \frac{\partial}{\partial t} - \Delta \right) (Q + C_5S)(x_k, t_k)
\leq - \lim_{k \to \infty} (Q + C_5S)(x_k, t_k) + C_6
= - \sup_{M \times [0, s]} (Q + C_5S) + C_6.
$$

That is,

$$
\sup_{M \times [0, s]} (Q + C_5S) \leq C_5.
$$

Since $s \in (0, T)$ is arbitrary,

$$
Q \leq Q + C_5S \leq \max\{C_6, nC_5\}
$$

all over $M \times [0, T]$.

Remark 3.1. We explain the computation of the evolution equation (3.7) of $Q$ in more details. Note that, locally, we have

$$
g_{ij} = u_{ij} + (g_0)_{ij}.
$$

Then, in a local coordinate, the Kähler-Ricci flow (1.1) becomes

$$
(u_t)_{ij} = \left( \log \frac{\det(u_{kl} + (g_0)_{kl})}{\det(g_0)_{ij}} \right)_{ij} + (R_0)_{ij} + c(g_0)_{ij}
$$

which makes our settings the same as in §8 of Chau [2]. An detailed computation can be found in Shi [7].
Theorem 3.2. Let assumptions be the same as in the last theorem. Then, for any nonnegative integer $k$, there is a positive constant $A_k$, such that

$$
\|\nabla^k_0 g\| \leq A_k
$$

all over $M \times [0, T)$.

Proof. We prove it by induction on $k$. When $k = 0$, it is by assumptions. When $k = 1$, it is just the last theorem. Suppose that the inequality is true for $k = 0, 1, \ldots, m - 1$, we want to get the inequality for $k = m$. Because the metrics are uniformly equivalent, it suffices to give an estimate to the quantity

$$
Q_m = \|\nabla^m_0 g\|_{g_0}^2 = (g_0)^{\beta_1} \cdots (g_0)^{\beta_m} (g_0)^{\alpha_1} \cdots (g_0)^{\alpha_m} (g_0)^{\beta_1} \cdots (g_0)^{\beta_m} g_{k_1} \beta_1 \cdots \beta_m,
$$

where $\alpha_i$'s and $\beta_j$'s belong to $\{1, 2, \ldots, n, 1, 2, \ldots, n\}$.

$$
\left( \frac{\partial}{\partial t} - \Delta \right) Q_m
$$

$$
= 2(g_0)^{\beta_1} \cdots (g_0)^{\beta_m} (g_0)^{\alpha_1} \cdots (g_0)^{\alpha_m} (g_0)^{\beta_1} \cdots (g_0)^{\beta_m} (-R_{i\bar{j}; \alpha_1 \alpha_2 \cdots \alpha_m} + c g_{i\bar{j}; \alpha_1 \alpha_2 \cdots \alpha_m}) g_{k_1} \beta_1 \beta_2 \cdots \beta_m
$$

$$
- 2g^{\beta \lambda} (g_0)^{\beta_1} \cdots (g_0)^{\beta_m} (g_0)^{\alpha_1} \cdots (g_0)^{\alpha_m} (g_0)^{\beta_1} \cdots (g_0)^{\beta_m} g_{\bar{\lambda} i j; \alpha_1 \alpha_2 \cdots \alpha_m} g_{k_1} \beta_1 \beta_2 \cdots \beta_m
$$

$$
- 2g^{\mu \lambda} (g_0)^{\beta_1} \cdots (g_0)^{\beta_m} (g_0)^{\alpha_1} \cdots (g_0)^{\alpha_m} (g_0)^{\beta_1} \cdots (g_0)^{\beta_m} g_{k_1} \beta_1 \beta_2 \cdots \beta_m
$$

$$
\leq - 2(g_0)^{\beta_1} \cdots (g_0)^{\beta_m} (g_0)^{\alpha_1} \cdots (g_0)^{\alpha_m} (g_0)^{\beta_1} \cdots (g_0)^{\beta_m} R_{i\bar{j}; \alpha_1 \alpha_2 \cdots \alpha_m} g_{k_1} \beta_1 \beta_2 \cdots \beta_m
$$

$$
- 2g^{\beta \lambda} (g_0)^{\beta_1} \cdots (g_0)^{\beta_m} (g_0)^{\alpha_1} \cdots (g_0)^{\alpha_m} (g_0)^{\beta_1} \cdots (g_0)^{\beta_m} g_{i\bar{j} \lambda; \alpha_1 \alpha_2 \cdots \alpha_m} g_{k_1} \beta_1 \beta_2 \cdots \beta_m
$$

$$
+ C_1 Q_m + C_2 Q_m^\frac{1}{2} - \frac{1}{C_0} Q_m + 1
$$

$$
\leq - 2(g_0)^{\beta_1} \cdots (g_0)^{\beta_m} (g_0)^{\alpha_1} \cdots (g_0)^{\alpha_m} (g_0)^{\beta_1} \cdots (g_0)^{\beta_m} R_{i\bar{j}; \alpha_1 \alpha_2 \cdots \alpha_m} g_{k_1} \beta_1 \beta_2 \cdots \beta_m
$$

$$
- 2(g_0)^{\beta_1} \cdots (g_0)^{\beta_m} (g_0)^{\alpha_1} \cdots (g_0)^{\alpha_m} (g_0)^{\beta_1} \cdots (g_0)^{\beta_m} g_{i\bar{j} \lambda; \alpha_1 \alpha_2 \cdots \alpha_m} g_{k_1} \beta_1 \beta_2 \cdots \beta_m
$$

$$
+ C_6 Q_m^\frac{1}{2} Q_m^\frac{1}{2} + C_5 Q_m^\frac{1}{2} + C_4 Q_m + C_3 Q_m^\frac{1}{2} - \frac{1}{C_0} Q_m + 1
$$

where we have used the Ricci identity and the induction hypothesis.
Substituting identity (3.6) into the last inequality, we get

\[
\left(\frac{\partial}{\partial t} - \Delta\right) Q_m 
\leq - 2(g_0)^{\beta_1 \alpha_1} \cdots (g_0)^{\beta_m \alpha_m} (g_0)^{\bar{\iota}_1} (g_0)^{\bar{\jmath}_k} R_{i\bar{j};\alpha_1 \alpha_2 \cdots \alpha_m} g_{k\bar{l};\beta_1 \beta_2 \cdots \beta_m} 
- 2(g_0)^{\beta_1 \alpha_1} \cdots (g_0)^{\beta_m \alpha_m} (g_0)^{\bar{\iota}_k} (g_0)^{\bar{\gamma}_l} (-R_{i\bar{j}} + g^{\bar{\beta}_\gamma} g^{\mu\bar{\nu}_\gamma} g_{\mu\bar{\nu}_\gamma} g_{\mu\bar{\nu}_\gamma}) 
+ (R_0)_{i\bar{i} \gamma \delta \bar{j}} g_{\mu \bar{l}}^{\jmath} \cdot \alpha_1 \alpha_2 \cdots \alpha_m \cdot g_{k\bar{l};\beta_1 \beta_2 \cdots \beta_m} 
+ C_6 Q_m^{\frac{1}{m}} Q_m^{\frac{1}{m+1}} + C_5 Q_m^{\frac{3}{m}} + C_4 Q_m + C_3 Q_m^{\frac{1}{m}} - \frac{1}{C_0} Q_{m+1} 
\leq C_{10} Q_m^{\frac{1}{m}} Q_m^{\frac{1}{m+1}} + C_9 Q_m^{\frac{3}{m}} + C_8 Q_m + C_7 Q_m^{\frac{1}{m}} - \frac{1}{C_0} Q_{m+1} 
\leq C_{11} Q_m^{\frac{3}{m}} + C_{12} - c_{13} Q_{m+1}.
\]

The same computation using the induction hypothesis provides us

\[
\left(\frac{\partial}{\partial t} - \Delta\right) Q_{m-1} \leq C_{14} - c_{15} Q_m.
\]

Moreover, note that

\[
\left| \langle \nabla Q_{m-1}, \nabla Q_{m-1} \rangle \right| 
= |g^{\mu \lambda}(Q_{m-1})_{\lambda}(Q_m)_{\bar{\mu}}| 
= |g^{\mu \lambda}(g_0)^{\beta_1 \alpha_1} \cdots (g_0)^{\beta_{m-1} \alpha_{m-1}} (g_0)^{\bar{\iota}_1} (g_0)^{\bar{\jmath}_k} (g_{i\bar{j};\alpha_1 \alpha_2 \cdots \alpha_{m-1}} g_{k\bar{l};\beta_1 \beta_2 \cdots \beta_{m-1})_{\lambda} \times 
(g_0)^{\beta'_1 \alpha'_1} \cdots (g_0)^{\beta'_m \alpha'_m} (g_0)^{\bar{\iota'}_1} (g_0)^{\bar{\jmath'}_k} (g_{i\bar{j};\alpha'_1 \alpha'_2 \cdots \alpha'_m} g_{k\bar{l};\beta'_1 \beta'_2 \cdots \beta'_m}_{\lambda} \bar{\mu}| 
\leq C_{16} Q_m Q_m^{\frac{1}{m+1}}
\]

where we have used that the metrics are uniformly equivalent and the induction hypothesis.
Let $Q = (Q_{m-1} + A)Q_m$ with $A$ a positive constant to be determined. We have,

\[
\left(\frac{\partial}{\partial t} - \Delta\right)Q = Q_m\left(\frac{\partial}{\partial t} - \Delta\right)Q_{m-1} + (Q_{m-1} + A)\left(\frac{\partial}{\partial t} - \Delta\right)Q_m - 2\text{Re}\{\langle \nabla Q_{m-1}, \nabla Q_m \rangle\}
\]

\[
\leq Q_m(-c_{15}Q_m + C_{14}) + (Q_{m-1} + A)(-c_{13}Q_{m+1} + C_{11}Q_m^3 + C_{12}) + 2C_{16}Q_mQ_{m+1}^{\frac{1}{2}}
\]

\[
\leq -\frac{c_{15}}{2}Q_m^2 + C_{14}Q_m + (Q_{m-1} + A)(-c_{13}Q_{m+1} + C_{11}Q_m^3 + C_{12}) + \frac{2C_{16}^2}{C_{15}}Q_{m+1}
\]

\[
\leq -\frac{c_{15}}{2}Q_m^2 + C_{14}Q_m + (Q_{m-1} + A)(C_{11}Q_m^3 + C_{12})
\]

\[
\leq -c_{16}Q_m^2 + C_{17}
\]

if we chose $A$ be such that $-Ac_{13} + \frac{2c_{16}^2}{c_{15}} = 0$, where we have used the induction hypothesis.

Similar as in the proof of Theorem 3.1 using Proposition 2.1, we have

\[
Q \leq \sqrt{\frac{C_{17}}{C_{16}}}
\]

Therefore, $Q_m \leq A^{-1}\sqrt{\frac{C_{17}}{C_{16}}}$. This completes the proof. \qed

Remark 3.2. The trick used in the proof is basically due to Chau [2] (§9). We only simplify it by using norms with respect to the initial metric $g_0$ instead of using $g$ in Chau [2].

Corollary 3.1. Let assumptions be the same as in Theorem 3.1. Then, for any nonnegative integer $k$, there is a positive constant $A_k$ such that

\[
\|\nabla^k Rm\| \leq A_k
\]

all over $M \times [0, T)$.

Proof. Note the by equation (3.5), we have

\[
R_{ijkl} = -g_{ij;kl} + g^\mu_{i\nu}g_{\nu;k}g_{\mu;j;l} + (R_0)_{ijkl}g_{ij}
\]

where covariant derivatives "$;$" and normal coordinates are taken with respect to $g_0$. Hence, by the last theorem, for any nonnegative integer $k$, there is a positive
constant $C_k$ such that

$$\|\nabla^k Rm\| \leq C_k$$

all over $M \times [0, T)$. In particular, the corollary is true for $k = 0$.

When $k = 1$, note that

$$\nabla \lambda R_{ijk\bar{l}} = (g^{\beta\alpha} g^{\bar{\gamma}\bar{\delta}} R_{\alpha \bar{\gamma} i \beta \bar{l} \lambda}) g_i g_{\beta} g_{\delta}$$

$$= R_{ij\bar{k}l;\lambda} - g^{\beta\alpha} R_{\alpha \bar{k}l i \beta \lambda} - g^{\bar{\gamma}\bar{\delta}} R_{i\bar{j}\bar{\gamma} \bar{l} g_{\delta} \lambda}.$$

Hence the corollary is true for $k = 1$.

Computing step by step, we can express $\nabla^k Rm$ as a combination of covariant derivatives of $g$ and $Rm_0$. Therefore, the corollary is true for all nonnegative integer $k$.

We come to prove the main result of this article.

**Theorem 3.3.** Let $g(t)$ with $t \in [0, T)$ be a complete solution to the Kähler-Ricci flow (1.1) satisfying Shi's estimate (1.2). Suppose that there is a positive constant $C$ such that

$$C^{-1} g(0) \leq g(t) \leq C g(0).$$

Then, for any nonnegative integer $k$, there are two positive constants $A_k$ and $B_k$ such that

$$\|\nabla^k Rm\|^2(x, t) \leq A_k + \frac{B_k}{t^k}$$

for any $(x, t) \in M \times [0, T)$.

**Proof.** Fixed $\epsilon \in (0, T)$. Then, the family $g(t)$ with $t \in [\epsilon, T)$ is a solution the Kähler-Ricci flow (1.1) satisfies the assumption of Corollary 3.1. Hence,

$$\|\nabla^k Rm\|^2(x, t) \leq A_k$$

for any $(x, t) \in M \times [\epsilon, T)$. Combining this with Shi's estimate, we complete the proof.

□
Remark 3.3. Theorem 3.2 is sometimes more useful than Theorem 3.3. For example, when $T = \infty$, we can extract convergent subsequence of metrics by Theorem 3.2.

Remark 3.4. The method will not work for the real case. The first obstacle is that we do not have an identity which is similar with (3.6).

4. An application.

**Theorem 4.1.** Let $\tilde{g}$ be a complete Kähler metric on $M^n$ with bounded sectional curvature. Suppose that

$$-\tilde{R}_{ij} + c\tilde{g}_{ij} = f_{ij}$$

with $f$ a smooth bounded function on $M^n$. Then, the Kähler-Ricci flow \((1.1)\) with initial metric $\tilde{g}$ has a long time solution satisfying Shi’s estimate \((1.2)\).

**Proof.** Let $g(t)$ with $[0, T)$ be maximal complete solution to the Kähler-Ricci flow \((1.1)\) with initial data $g(0) = \tilde{g}$ satisfying Shi’s estimate \((1.1)\). It suffices to show that $T = \infty$. Suppose that $T < \infty$, we want to get a contradiction.

As in Chau [2], the Kähler-Ricci flow \((1.1)\) is related to to the parabolic Monge-Ampère equation:

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \log \frac{\det(\tilde{g}_{ij} + u_{ij})}{\det \tilde{g}_{ij}} + cu + f \\
(u(x, 0) &= 0.
\end{aligned}
\]

\((4.1)\)

$g_{ij} = \tilde{g}_{ij} + u_{ij}$ is a solution of the Kähler-Ricci flow \((1.1)\) if $u$ is a solution to equation \((4.1)\).

Let $w = ut$. Then,

\[
\begin{aligned}
\frac{\partial w}{\partial t} &= \Delta w + cw \\
w(x, 0) &= f(x).
\end{aligned}
\]

By maximum principle, we know that

$$|w|(x, t) \leq e^{\epsilon t} \sup |f| \leq e^{\epsilon T} \sup |f| := C_1.$$

for any $(x, t) \in M \times [0, T)$. Moreover

$$|u|(x, t) \leq C_1 T := C_2.$$
for any \((x, t) \in M \times [0, T)\). By the same argument as in second order estimate of Monge-Ampère equation as in Yau [10] (see also Chau [2]), we know that

\[
\tilde{g}^{ii} g_{ij} = n + \Delta u \leq C_3.
\]

By equation (4.1),

\[
\frac{\det g_{ij}}{\det \tilde{g}_{ij}} \geq c_4.
\]

Let \(\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n\) be the \(n\) eigenvalues of \(g_{ij}\) with respect to \(\tilde{g}_{ij}\). Then, we have

\[
\lambda_1 + \lambda_2 + \cdots + \lambda_n \leq C_3 \text{ and } \lambda_1 \cdot \lambda_2 \cdots \lambda_n \geq c_4.
\]

Therefore \(\lambda_n \leq C_3\) and \(\lambda_1 \geq \frac{c_4}{\lambda_2 \lambda_n - \lambda_3} \geq c_4 C_3^{-n} \geq \lambda_3\), which implies that \(g(t)\) is uniformly equivalent to \(\tilde{g}\). By Theorem 3.3, curvatures of \(g(t)\) is uniformly bounded. This violates that \(g(t)\) is a maximal solution. \(\Box\)

Remark 4.1. When \(M^n\) is a compact Kähler manifold, if the potential \(f\) exists for the initial metric, then \(f\) is automatically bounded. Hence, the Kähler-Ricci flow (1.1) has a long time solution in this case. This is just the long time existence result in Cao [1].

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