INTRODUCTION TO A THEORY OF $b$-FUNCTIONS

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We give an introduction to a theory of $b$-functions, i.e. Bernstein-Sato polynomials. After reviewing some facts from $D$-modules, we introduce $b$-functions including the one for arbitrary ideals of the structure sheaf. We explain the relation with singularities, multiplier ideals, etc., and calculate the $b$-functions of monomial ideals and also of hyperplane arrangements in certain cases.

1. D-modules.

1.1. Let $X$ be a complex manifold or a smooth algebraic variety over $\mathbb{C}$. Let $\mathcal{D}_X$ be the ring of partial differential operators. A local section of $\mathcal{D}_X$ is written as

$$\sum_{\nu \in \mathbb{N}^n} a_{\nu} \partial_1^{\nu_1} \cdots \partial_n^{\nu_n} \in \mathcal{D}_X \quad \text{with} \quad a_{\nu} \in \mathcal{O}_X,$$

where $\partial_i = \partial / \partial x_i$ with $(x_1, \ldots, x_n)$ a local coordinate system.

Let $F$ be the filtration by the order of operators i.e.

$$F_p \mathcal{D}_X = \{ \sum_{|\nu| \leq p} a_{\nu} \partial_1^{\nu_1} \cdots \partial_n^{\nu_n} \},$$

where $|\nu| = \sum_i \nu_i$. Let $\xi_i = \text{Gr}^F_1 \partial_i \in \text{Gr}^F_1 \mathcal{D}_X$. Then

$$\text{Gr}^F \mathcal{D}_X := \bigoplus_p \text{Gr}^F_p \mathcal{D}_X = \bigoplus_p \text{Sym}^p \Theta_X \quad (= \mathcal{O}_X[\xi_1, \ldots, \xi_n] \text{ locally}),$$

$$\text{Spec}_X \text{Gr}^F \mathcal{D}_X = T^* X.$$  \hfill (1.1.1)

1.2 Definition. We say that a left $\mathcal{D}_X$-module $M$ is coherent if it has locally a finite presentation

$$\bigoplus \mathcal{D}_X \to \bigoplus \mathcal{D}_X \to M \to 0.$$  \hfill (1.2)

1.3. Remark. A left $\mathcal{D}_X$-module $M$ is coherent if and only if it is quasi-coherent over $\mathcal{O}_X$ and locally finitely generated over $\mathcal{D}_X$. (It is known that $\text{Gr}^F \mathcal{D}_X$ is a noetherian ring, i.e. an increasing sequence of locally finitely generated $\text{Gr}^F \mathcal{D}_X$-submodules of a coherent $\text{Gr}^F \mathcal{D}_X$-module is locally stationary.)

1.4. Definition. A filtration $F$ on a left $\mathcal{D}_X$-module $M$ is good if $(M, F)$ is a coherent filtered $\mathcal{D}_X$-module, i.e. if $F_p \mathcal{D}_X F_q M \subset M_{p+q}$ and $\text{Gr}^F M := \bigoplus_p \text{Gr}^F_p M$ is coherent over $\text{Gr}^F \mathcal{D}_X$.

1.5. Remark. A left $\mathcal{D}_X$-module $M$ is coherent if and only if it has a good filtration locally.
1.6. Characteristic varieties. For a coherent left $D_X$-module $M$, we define the characteristic variety $CV(M)$ by
\[
CV(M) = \text{Supp} \ \text{Gr}^F M \subset T^* M,
\]
 taking locally a good filtration $F$ of $M$.

1.7. Remark. The above definition is independent of the choice of $F$. If $M = D_X/I$ for a coherent left ideal $I$ of $D_X$, take $P_i \in F_k I$ such that the $\rho_i := \text{Gr}_k^F P_i$ generate $\text{Gr}^F I$ over $\text{Gr}^F D_X$. Then $CV(M)$ is defined by the $\rho_i \in O_X[\xi_1, \ldots, \xi_n]$.

1.8. Theorem (Sato, Kawai, Kashiwara [39], Bernstein [2]). We have the inequality $\dim CV(M) \geq \dim X$. (More precisely, $CV(M)$ is involutive, see [39].)

1.9. Definition. We say that a left $D_X$-module $M$ is holonomic if it is coherent and $\dim CV(M) = \dim X$.

2. De Rham functor.

2.1. Definition. For a left $D_X$-module $M$, we define the de Rham functor $DR(M)$ by
\[
M \to \Omega^1_X \otimes_{O_X} M \to \cdots \to \Omega^{\dim X} \otimes_{O_X} M,
\]
 where the last term is put at the degree 0. In the algebraic case, we use analytic sheaves or replace $M$ with the associated analytic sheaf $M^{an} := M \otimes_{O_X} O_X^{an}$ in case $M$ is algebraic (i.e. $M$ is an $O_X$-module with $O_X$ algebraic).

2.2. Perverse sheaves. Let $D^b_c(X, C)$ be the derived category of bounded complexes of $C_X$-modules $K$ with $\mathcal{H}^j K$ constructible. (In the algebraic case we use analytic topology for the sheaves although we use Zariski topology for constructibility.) Then the category of perverse sheaves $\text{Perv}(X, C)$ is a full subcategory of $D^b_c(X, C)$ consisting of $K$ such that
\[
\dim \text{Supp} \mathcal{H}^{-j} K \leq j, \quad \dim \text{Supp} \mathcal{H}^{-j} DK \leq j,
\]
where $DK := R\mathcal{H}om(K, C[2 \dim X])$ is the dual of $K$, and $\mathcal{H}^j K$ is the $j$-th cohomology sheaf of $K$.

2.3. Theorem (Beilinson, Bernstein, Deligne [1]). $\text{Perv}(X, C)$ is an abelian category.

2.4. Theorem (Kashiwara). If $M$ is holonomic, then $DR(M)$ is a perverse sheaf.

Outline of proof. By Kashiwara [19], we have $DR(M) \in D^b_c(X, C)$, and the first condition of (2.2.1) is verified. Then the assertion follows from the commutativity of the dual $D$ and the de Rham functor $DR$.

2.5. Example. $DR(O_X) = C_X[\dim X]$.

2.6. Direct images. For a closed immersion $i : X \to Y$ such that $X$ is defined by $x_i = 0$ in $Y$ for $1 \leq i \leq r$, define the direct image of left $D_X$-modules $M$ by
\[
i_+ M := M[\partial_1, \ldots, \partial_r].\]
(Globally there is a twist by a line bundle.) For a projection $p : X \times Y \to Y$, define

$$p_+ M = R p_* DR_X(M).$$

In general, $f_+ = p_+ i_+$ using $f = pi$ with $i$ graph embedding. See [4] for details.

2.7. Regular holonomic $D$-modules. Let $M$ be a holonomic $D_X$-module with support $Z$, and $U$ be a Zariski-open of $Z$ such that $DR(M)|_U$ is a local system up to a shift. Then $M$ is regular if and only if there exists locally a divisor $D$ on $X$ containing $Z \setminus U$ and such that $M(*D)$ is the direct image of a regular holonomic $D$-module ‘of Deligne-type’ (see [11]) on a desingularization of $(Z, Z \cap D)$, and $\text{Ker}(M \to M(*D))$ is regular holonomic (by induction on $\dim \text{Supp} M$).

Note that the category $M_{rh}(D_X)$ of regular holonomic $D_X$-modules is stable by subquotients and extensions in the category $M_h(D_X)$ of holonomic $D_X$-modules.

2.8. Theorem (Kashiwara-Kawai [24], [22], Mebkhout [28]).
(i) The structure sheaf $O_X$ is regular holonomic.
(ii) The functor $DR$ induces an equivalence of categories

$$(2.8.1) \quad DR : M_{rh}(D_X) \xrightarrow{\sim} \text{Perv}(X, \mathbb{C}).$$

(See [4] for the algebraic case.)

3. $b$-Functions.

3.1. Definition. Let $f$ be a holomorphic function on $X$, or $f \in \Gamma(X, O_X)$ in the algebraic case. Then we have

$$D_X[s]f^s \subset O_X[\frac{1}{s}][s]f^s \quad \text{where } \partial_i f^s = s(\partial_i f)f^{s-1},$$

and $b_f(s)$ is the monic polynomial of the least degree satisfying

$$b_f(s)f^s = P(x, \partial, s)f^{s+1} \quad \text{in } O_X[\frac{1}{s}][s]f^s,$$

with $P(x, \partial, s) \in D_X[s]$. Locally, it is the minimal polynomial of the action of $s$ on

$$D_X[s]f^s/D_X[s]f^{s+1}.$$

We define $b_{f,x}(s)$ replacing $D_X$ with $D_{X,x}$.

3.2. Theorem (Sato [38], Bernstein [2], Bjork [3]). The $b$-function exists at least locally, and exists globally in the case $X$ affine variety with $f$ algebraic.

3.3. Observation. Let $i_f : X \to \tilde{X} := X \times \mathbb{C}$ be the graph embedding. Then there are canonical isomorphisms

$$M := i_f_* O_X = O_X[\partial_i] \delta(f - t) = O_X \times \mathbb{C} \left[ \frac{1}{\partial_i} \right] / O_X \times \mathbb{C},$$

where the action of $\partial_i$ on $\delta(f - t)$ ($= \frac{1}{\partial_i}$) is given by

$$\partial_i \delta(f - t) = - (\partial_i f) \partial_t \delta(f - t).$$

Moreover, $f^s$ is canonically identified with $\delta(f - t)$ setting $s = -\partial_t$, and we have a canonical isomorphism as $D_X[s]$-modules

$$D_X[s]f^s = D_X[s] \delta(f - t).$$
3.4. **V-filtration.** We say that $V$ is a filtration of Kashiwara-Malgrange if $V$ is exhaustive, separated, and satisfies for any $\alpha \in \mathbb{Q}$:

(i) $V^\alpha \widehat{M}$ is a coherent $\mathcal{D}_X[s]$-submodule of $\widehat{M}$.

(ii) $tV^\alpha \widehat{M} \subset V^{\alpha+1} \widehat{M}$ and $= \text{ holds for } \alpha \gg 0$.

(iii) $\partial_t V^\alpha \widehat{M} \subset V^{\alpha-1} \widehat{M}$.

(iv) $\partial_t - \alpha$ is nilpotent on $\text{Gr}_V^\alpha \widehat{M}$.

If it exists, it is unique.

3.5. **Relation with the $b$-function.** If $X$ is affine or Stein and relatively compact, then the multiplicity of a root $\alpha$ of $b_f(s)$ is given by the minimal polynomial of $s - \alpha$ on

$$\text{Gr}_V^\alpha(\mathcal{D}_X[s]f^s/\mathcal{D}_X[s]f^{s+1}),$$

using $\mathcal{D}_X[s]f^s = \mathcal{D}_X[s]\delta(f - t)$ with $s = -\partial_t$.

Note that $V^\alpha \widehat{M}$ and $\mathcal{D}_X[s]f^{s+i}$ are ‘lattices’ of $\widehat{M}$, i.e.

$$V^\alpha \widehat{M} \subset \mathcal{D}_X[s]f^{s+i} \subset V^{\beta} \widehat{M} \text{ for } \alpha \gg i \gg \beta,$$

and $V^\alpha \widehat{M}$ is an analogue of the Deligne extension with eigenvalues in $[\alpha, \alpha + 1]$. The existence of $V$ is equivalent to the existence of $b_f(s)$ locally.

3.6. **Theorem** (Kashiwara [21], [23], Malgrange [27]). *The filtration $V$ exists on $\widehat{M} := i_{f+}M$ for any holonomic $\mathcal{D}_X$-module $M$.*

3.7. **Remarks.** (i) There are many ways to prove this theorem, since it is essentially equivalent to the existence of the $b$-function (in a generalized sense). One way is to use a resolution of singularities and reduce to the case where $CV(M)$ has normal crossings, if $M$ is regular.

(ii) The filtration $V$ is indexed by $\mathbb{Q}$ if $M$ is quasi-unipotent.

3.8. **Relation with vanishing cycle functors.** Let $\rho : X_t \to X_0$ be a ‘good’ retraction (using a resolution of singularities of $(X, X_0)$), where $X_t = f^{-1}(t)$ with $t \neq 0$ sufficiently near 0. Then we have canonical isomorphisms

$$\psi_fC_X = R\rho_*C_{X_t}, \quad \varphi_fC_X = \psi_fC_X/C_{X_0},$$

where $\psi_fC_X, \varphi_fC_X$ are nearby and vanishing cycle sheaves, see [13].

Let $F_x$ denote the Milnor fiber around $x \in X_0$. Then

$$\mathcal{H}^j\psi_fC_X)_x = H^j(F_x, \mathcal{C}), \quad (\mathcal{H}^j\varphi_fC_X)_x = \tilde{H}^j(F_x, \mathcal{C}).$$

For a $\mathcal{D}_X$-module $M$ admitting the $V$-filtration on $\widehat{M} = i_{f+}M$, we define $\mathcal{D}_X$-modules

$$\psi_fM = \bigoplus_{0 < \alpha < 1} \text{Gr}_V^\alpha \widehat{M}, \quad \varphi_fM = \bigoplus_{0 \leq \alpha < 1} \text{Gr}_V^\alpha \widehat{M}.$$

3.9. **Theorem** (Kashiwara [23], Malgrange [27]). *For a regular holonomic $\mathcal{D}_X$-module $M$, we have canonical isomorphisms

$$\text{DR}_X\psi_f(M) = \psi_f\text{DR}_X(M)[-1],$$

$$\text{DR}_X\varphi_f(M) = \varphi_f\text{DR}_X(M)[-1],$$
and $\exp(-2\pi i \partial_t t)$ on the left-hand side corresponds to the monodromy $T$ on the right-hand side.

3.10. Definition. Let
\[ R_f = \{ \text{roots of } b_f(-s) \}, \]
\[ \alpha_f = \min R_f, \]
\[ m_{\alpha} : \text{the multiplicity of } \alpha \in R_f. \]
(Similarly for $R_{f,x}$, etc. for $b_{f,x}(s)$.)

3.11. Theorem (Kashiwara [20]). $R_f \subset \mathbb{Q}_{>0}$.
(This is proved by using a resolution of singularities.)

3.12. Theorem (Kashiwara [23], Malgrange [27]).
(i) $e^{-2\pi i R_f} = \{ \text{the eigenvalues of } T \text{ on } H^j(F_x, \mathbb{C}) \text{ for } x \in X_0, j \in \mathbb{Z} \}$;
(ii) $m_{\alpha} \leq \min \{ i \mid N^i \psi_{f,\lambda} C_X = 0 \}$ with $\lambda = e^{-2\pi i \alpha}$, where $\psi_{f,\lambda} = \ker(T_s - \lambda) \subset \psi_f$, $N = \log T_u$ with $T = T_s T_u$.
(This is a corollary of the above Theorem (3.9) of Kashiwara and Malgrange.)

4. Relation with other invariants.

4.1. Microlocal $b$-function. We define $\tilde{R}_f, \tilde{m}_{\alpha}, \tilde{\alpha}_f$ with $b_f(s)$ replaced by the microlocal (or reduced) $b$-function
\[ \tilde{b}_f(s) := b_f(s)/(s + 1). \]
This $\tilde{b}_f(s)$ coincides with the monic polynomial of the least degree satisfying
\[ \tilde{b}_f(s) \delta(f - t) = \tilde{P} \partial_t^{-1} \delta(f - t) \quad \text{with } \tilde{P} \in \mathcal{D}_X[s, \partial_t^{-1}]. \]
Put $n = \dim X$. Then

4.2. Theorem. $\tilde{R}_f \subset [\tilde{\alpha}_f, n - \tilde{\alpha}_f], \quad \tilde{m}_{\alpha} \leq n - \tilde{\alpha}_f - \alpha + 1$.
(The proof uses the filtered duality for $\varphi_f$, see [35].)

4.3. Spectrum. We define the spectrum by $\text{Sp}(f, x) = \sum \alpha n_{\alpha} t^\alpha$ with
\[ n_{\alpha} := \sum_j (-1)^j \dim \text{Gr}_F^p \tilde{H}^j(F_x, \mathbb{C})_\lambda, \]
where $p = [n - \alpha]$, $\lambda = e^{-2\pi i \alpha}$, and $F$ is the Hodge filtration (see [12]) of the mixed Hodge structure on the Milnor cohomology, see [44]. We define
\[ E_f = \{ \alpha \mid n_{\alpha} \neq 0 \} \quad \text{(called the exponents)}. \]

4.4. Remarks. (i) If $f$ has an isolated singularity at the origin, then $\tilde{\alpha}_{f,x}$ coincides with the minimal exponent as a corollary of results of Malgrange [26], Varchenko [45], Scherk-Steenbrink [41].
(ii) If $f$ is weighted-homogeneous with an isolated singularity at the origin, then by Kashiwara (unpublished)
\[ \tilde{R}_f = E_f, \quad \max \tilde{R}_f = n - \tilde{\alpha}_f, \quad \tilde{m}_{\alpha} = 1 \quad (\alpha \in \tilde{R}_f). \]
If \( f = \sum_i x_i^2 \), then \( \tilde{\alpha}_f = n/2 \) and this follows from the above Theorem (4.2).

By Steenbrink [42], we have moreover
\[
(4.4.2) \quad \text{Sp}(f, x) = \prod_i (t - t^{w_i})/(t^{w_i} - 1),
\]
where \((w_1, \ldots, w_n)\) is the weights of \( f \), i.e. \( f \) is a linear combination of monomials \( x_1^{m_1} \cdots x_n^{m_n} \) with \( \sum_i w_i m_i = 1 \).

4.5. Malgrange’s formula (isolated singularities case). We have the Brieskorn lattice [5] and its saturation defined by
\[
(4.5.1) \quad H''_f = \Omega^n_{X,x}/df \wedge d\Omega^{n-2}_{X,x}, \quad \tilde{H}_f'' = \sum_{i \geq 0} (t \partial_t)^i H''_f \subset H''_f[t^{-1}].
\]
These are finite \( \mathbb{C}\{t\} \)-modules with a regular singular connection.

4.6. Theorem (Malgrange [26]). The reduced \( b \)-function \( \tilde{b}_f(s) \) coincides with the minimal polynomial of \(-\partial_t\) on \( H''_f/t\tilde{H}_f'' \).

(The above formula of Kashiwara on \( b \)-function (4.4.1) can be proved by using this together with Brieskorn’s calculation.)

4.7. Asymptotic Hodge structure (Varchenko [45], Scherk-Steenbrink [41]). In the isolated singularity case we have
\[
(4.7.1) \quad \text{F}^p H^{n-1}(F_x, \mathbb{C})_\lambda = \text{Gr}^{\alpha}_V H''_f,
\]
using the canonical isomorphism
\[
(4.7.2) \quad H^{n-1}(F_x, \mathbb{C})_\lambda = \text{Gr}^{\alpha}_V H''_f[t^{-1}],
\]
where \( p = [n - \alpha], \lambda = e^{-2\pi i \alpha}, \) and \( V \) on \( H''_f[t^{-1}] \) is the filtration of Kashiwara and Malgrange.

(This can be generalized to the non-isolated singularity case using mixed Hodge modules.)

4.8. Reformulation of Malgrange’s formula. We define
\[
(4.8.1) \quad \tilde{\text{F}}^p H^{n-1}(F_x, \mathbb{C})_\lambda = \text{Gr}^{\alpha}_V \tilde{H}_f''
\]
using the canonical isomorphism (4.7.2), where \( p = [n - \alpha], \lambda = e^{-2\pi i \alpha}. \) Then
\[
(4.8.2) \quad \tilde{m}_\alpha = \text{the minimal polynomial of } N \text{ on } \text{Gr}^{\alpha}_F H^{n-1}(F_x, \mathbb{C})_\lambda.
\]

4.9. Remark. If \( f \) is weighted homogeneous with an isolated singularity, then
\[
(4.9.1) \quad \tilde{\text{F}} = F, \quad \tilde{R}_f = E_f \text{ (by Kashiwara)}.
\]
If \( f \) is not weighted homogeneous (but with isolated singularities), then
\[
(4.9.2) \quad \tilde{R}_f \subset \bigcup_{k \in \mathbb{N}} (E_f - k), \quad \tilde{\alpha}_f = \text{min } \tilde{R}_f = \text{min } E_f.
\]

4.10. Example. If \( f = x^5 + y^4 + x^3 y^2 \), then
\[
E_f = \left\{ \frac{i}{5} + \frac{j}{4} : 1 \leq i \leq 4, 1 \leq j \leq 3 \right\}, \quad \tilde{R}_f = E_f \cup \left\{ \frac{11}{20} \right\} \setminus \left\{ \frac{31}{20} \right\}.
\]
More generally, if \( f = g + h \) with \( g \) weighted homogeneous and \( h \) is a linear combination of monomials of higher degrees, then \( E_f = E_g \) but \( \tilde{R}_f \neq \tilde{R}_g \) if \( f \) is a non trivial deformation.

### 4.11. Relation with rational singularities [34]

Assume \( D := f^{-1}(0) \) is reduced. Then \( D \) has rational singularities if and only if \( \tilde{\alpha}_f > 1 \). Moreover, 

\[
\omega_D/\rho_* \omega_{\tilde{D}} \cong F_{1-n\varphi f} \mathcal{O}_X,
\]

where \( \rho : \tilde{D} \to D \) is a resolution of singularities.

In the isolated singularities case, this was proved in 1981 (see [31]) using the coincidence of \( \tilde{\alpha}_f \) and the minimal exponent.

### 4.12. Relation with the pole order filtration [34]

Let \( P \) be the pole order filtration on \( \mathcal{O}_X(\ast D) \), i.e. \( P_i = \mathcal{O}_X((i+1)D) \) if \( i \geq 0 \), and \( P_i = 0 \) if \( i < 0 \). Let \( F \) be the Hodge filtration on \( \mathcal{O}_X(\ast D) \). Then \( F_i \subset P_i \) in general, and \( F_i = P_i \) on a neighborhood of \( x \) for \( i \leq \tilde{\alpha}_{f,x} - 1 \).

(For the proof we need the theory of microlocal \( b \)-functions [35].)

### 4.13. Remark

In case \( X = \mathbb{P}^n \), replacing \( \tilde{\alpha}_{f,x} \) with \( [(n-r)/d] \) where \( r = \dim \text{Sing} \ D \) and \( d = \deg \ D \), the assertion was obtained by Deligne (unpublished).

### 5. Relation with multiplier ideals.

#### 5.1. Multiplier ideals

Let \( D = f^{-1}(0) \), and \( \mathcal{J}(X, \alpha D) \) be the multiplier ideals for \( \alpha \in \mathbb{Q} \), i.e.

\[
\mathcal{J}(X, \alpha D) = \rho_* \omega_{\tilde{X}/X}(\sum [am_i] \tilde{D}_i),
\]

where \( \rho : (\tilde{X}, \tilde{D}) \to (X, D) \) is an embedded resolution and \( \tilde{D} = \sum_i m_i \tilde{D}_i := \rho^* D \).

There exist jumping numbers \( 0 < \alpha_0 < \alpha_1 < \cdots \) such that

\[
\mathcal{J}(X, \alpha_j D) = \mathcal{J}(X, \alpha D) \neq \mathcal{J}(X, \alpha_{j+1} D) \quad \text{for} \quad \alpha_j \leq \alpha < \alpha_{j+1}.
\]

Let \( V \) denote also the induced filtration on

\[
\mathcal{O}_X \subset \mathcal{O}_X[\partial_t] \delta(f-t).
\]

#### 5.2. Theorem (Budur, S. [10])

If \( \alpha \) is not a jumping number,

\[
\mathcal{J}(X, \alpha D) = V^\alpha \mathcal{O}_X.
\]

For \( \alpha \) general we have for \( 0 < \varepsilon \ll 1 \)

\[
\mathcal{J}(X, \alpha D) = V^{\alpha + \varepsilon} \mathcal{O}_X, \quad V^\alpha \mathcal{O}_X = \mathcal{J}(X, (\alpha - \varepsilon)D).
\]

Note that \( V \) is left-continuous and \( \mathcal{J}(X, \alpha D) \) is right-continuous, i.e.

\[
V^\alpha \mathcal{O}_X = V^{\alpha - \varepsilon} \mathcal{O}_X, \quad \mathcal{J}(X, \alpha D) = \mathcal{J}(X, (\alpha + \varepsilon)D).
\]

The proof of (5.2) uses the theory of bifiltered direct images [32], [33] to reduce the assertion to the normal crossing case.

As a corollary we get another proof of the results of Ein, Lazarsfeld, Smith and Varolin [16], and of Lichtin, Yano and Kollár [25]:
5.3. Corollary.
(i) \{Jumping numbers \leq 1\} \subset R_f$, see [16].
(ii) $\alpha_f = \text{minimal jumping number}$, see [25].

Define $\alpha'_{f,x} = \min_{y \neq x} \{\alpha_{f,y}\}$. Then

5.4. Theorem. If $\xi f = f$ for a vector field $\xi$, then

$$R_f \cap (0, \alpha'_{f,x}) = \{\text{Jumping numbers}\} \cap (0, \alpha'_{f,x}).$$

(This does not hold without the assumption on $\xi$ nor for $[\alpha'_{f,x}, 1]$.)

For the constantness of the jumping numbers under a topologically trivial deformation of divisors, see [14].

6. $b$-Functions for any subvarieties.

6.1. Let $Z$ be a closed subvariety of a smooth $X$, and $f = (f_1, \ldots, f_r)$ be generators of the ideal of $Z$ (which is not necessarily reduced nor irreducible). Define the action of $t_j$ on

$$\mathcal{O}_X \left[ \frac{1}{f_1 \cdots f_r} \right] [s_1, \ldots, s_r] \prod_i f_i^{s_i},$$

by $t_j(s_i) = s_i + 1$ if $i = j$, and $t_j(s_i) = s_i$ otherwise. Put $s_{i,j} := s_i t_i^{-1} t_j$, $s = \sum_i s_i$. Then $b_f(s)$ is the monic polynomial of the least degree satisfying

$$b_f(s) \prod_i f_i^{s_i} = \sum k \prod k \prod_i f_i^{s_i},$$

where $P_k$ belong to the ring generated by $\mathcal{D}_X$ and $s_{i,j}$.

Here we can replace $\prod_i f_i^{s_i}$ with $\prod_i \delta(t_i - f_i)$, using the direct image by the graph of $f : X \to \mathbb{C}^r$. Then the existence of $b_f(s)$ follows from the theory of the $V$-filtration of Kashiwara and Malgrange. This $b$-function has appeared in work of Sabbah [30] and Gyoja [18] for the study of $b$-functions of several variables.

6.2. Theorem (Budur, Mustață, S. [8]). Let $c = \text{codim}_X Z$. Then $b_Z(s) := b_f(s - c)$ depends only on $Z$ and is independent of the choice of $f = (f_1, \ldots, f_r)$ and also of $r$.

6.3. Equivalent definition. The $b$-function $b_f(s)$ coincides with the monic polynomial of the least degree satisfying

$$b_f(s) \prod_i f_i^{s_i} \in \sum_{|c| = 1} \mathcal{D}_X[s] \prod_{c_i < 0} \left( -\frac{s_i}{c_i} \right) \prod_i f_i^{s_i + c_i},$$

where $c = (c_1, \ldots, c_r) \in \mathbb{Z}^r$ with $|c| := \sum c_i = 1$. Here $\mathcal{D}_X[s] = \mathcal{D}_X[s_1, \ldots, s_r]$.

This is due to Mustață, and is used in the monomial ideal case. Note that the well-definedness does not hold without the term $\prod_{c_i < 0} \left( -\frac{s_i}{c_i} \right)$.

We have the induced filtration $V$ by

$$\mathcal{O}_X \subset i_{f,\ast} \mathcal{O}_X = \mathcal{O}_X[\partial_1, \ldots, \partial_r] \prod_i \delta(t_i - f_i).$$

6.4. Theorem (Budur, Mustață, S. [8]). If $\alpha$ is not a jumping number;

$$\mathcal{J}(X, \alpha Z) = V^\alpha \mathcal{O}_X.$$
For $\alpha$ general we have for $0 < \varepsilon \ll 1$
\begin{equation}
(6.4.2) \quad \mathcal{J}(X, \alpha Z) = V^{\alpha+\varepsilon} \mathcal{O}_X, \quad V^{\alpha} \mathcal{O}_X = \mathcal{J}(X, (\alpha - \varepsilon)Z).
\end{equation}

6.5. Corollary (Budur, Mustată, S. [9]). We have the inclusion
\begin{equation}
(6.5.1) \quad \{\text{Jumping numbers}\} \cap [\alpha_f, \alpha_f + 1) \subset R_f.
\end{equation}

6.6. Theorem (Budur, Mustată, S. [8]). If $Z$ is reduced and is a local complete intersection, then $Z$ has only rational singularities if and only if $\alpha_f = r$ with multiplicity $1$.

7. Monomial ideal case.

7.1. Definition. Let $a \subset \mathbb{C}[x] := \mathbb{C}[x_1, \ldots, x_n]$ a monomial ideal. We have the associated semigroup defined by
\[
\Gamma_a = \{u \in \mathbb{N}^n \mid x^u \in a\}.
\]
Let $P_a$ be the convex hull of $\Gamma_a$ in $\mathbb{R}_{\geq 0}^n$. For a face $Q$ of $P_a$, define
\[
M_Q : \text{the subsemigroup of } \mathbb{Z}^n \text{ generated by } u - v \text{ with } u \in \Gamma_a, v \in \Gamma_a \cap Q.
\]
\[
M'_Q = v_0 + M_Q \text{ for } v_0 \in \Gamma_a \cap Q (\text{this is independent of } v_0).
\]
For a face $Q$ of $P_a$ not contained in any coordinate hyperplane, take a linear function with rational coefficients $L_Q : \mathbb{R}^n \to \mathbb{R}$ whose restriction to $Q$ is 1. Let
\[
V_Q : \text{the linear subspace generated by } Q.
\]
\[
e = (1, \ldots, 1).
\]
\[
R_Q = \{L_Q(u) \mid u \in (e + (M_Q \setminus M'_Q)) \cap V_Q\},
\]
\[
R_a = \{\text{roots of } b_a(-s)\}.
\]

7.2. Theorem (Budur, Mustată, S. [9]). We have $R_a = \bigcup_Q R_Q$ with $Q$ faces of $P_a$ not contained in any coordinate hyperplanes.

Outline of the proof. Let $f_j = \prod_i x_i^{a_{i,j}}, \ell_i(s) = \sum_j a_{i,j} s_j$. Define
\[
g_c(s) = \prod_{c_i < 0} (-c_i) \prod_{\ell_i(c) > 0} (\ell_i(s) + \ell_i(c)).
\]
Let $I_a \subset \mathbb{C}[s]$ be the ideal generated by $g_c(s)$ with $c \in \mathbb{Z}^r$, $\sum_i c_i = 1$. Then

7.3. Proposition (Mustată). The $b$-function $b_a[s]$ of the monomial ideal $a$ is the monic generator of $\mathbb{C}[s] \cap I_a$, where $s = \sum_i s_i$.

Using this, Theorem (7.2) follows from elementary computations.

7.4. Case $n = 2$. Here it is enough to consider only 1-dimensional $Q$ by (7.2).

Let $Q$ be a compact face of $P_a$ with $\{v^{(1)}, v^{(2)}\} = \partial Q$, where $v^{(i)} = (v^{(i)}_1, v^{(i)}_2)$ with $v^{(1)}_1 < v^{(2)}_1, v^{(1)}_2 > v^{(2)}_2$. Let
\[
G : \text{the subgroup generated by } u - v \text{ with } u, v \in Q \cap \Gamma_a.
\]
\[
v^{(3)} \in Q \cap \mathbb{N}^2 \text{ such that } v^{(3)} - v^{(1)} \text{ generates } G.
\]
\[
S_Q = \{(i, j) \in \mathbb{N}^2 \mid i < v^{(3)}_1, j < v^{(3)}_2\}.
\]
Then
\[ R_Q = \{ L_Q(u + e) - k \mid u \in S_Q^{[k]} (k = 0, 1) \}. \]
In the case \( Q \subset \{ x = m \} \), we have \( R_Q = \{ i/m \mid i = 1, \ldots, m \} \).

7.5. Examples. (i) If \( a = (x^a y, x^b) \), with \( a, b \geq 2 \), then
\[ R_a = \left\{ \frac{(b-1)i + (a-1)j}{ab-1} \mid 1 \leq i \leq a, 1 \leq j \leq b \right\}. \]
(ii) If \( a = (x^5 y, x^3 y^2, x^5 y) \), then \( S_Q^{[1]} = \emptyset \) and
\[ R_a = \left\{ \frac{5}{13}, \frac{i}{13} (7 \leq i \leq 17), \frac{19}{13} \right\}. \]
(iii) If \( a = (x^5 y, x^3 y^2, x^4 y) \), then \( S_Q^{[1]} = \{ (2, 4) \} \) for \( \partial Q = \{ (1, 5), (3, 2) \} \) with \( L_Q(v_1, v_2) = (3v_1 + 2v_2)/13 \), and
\[ R_a = \left\{ \frac{i}{13} (5 \leq i \leq 17), \frac{j}{5} (2 \leq j \leq 6) \right\}. \]
Here 19/13 is shifted to 6/13.

7.6. Comparison with exponents. If \( n = 2 \) and \( f \) has a nondegenerate Newton polygon with compact faces \( Q \), then by Steenbrink [43]
\[ E_f \cap (0, 1] = \bigcup Q E_{Q}^{\leq 1} \quad \text{with} \quad E_{Q}^{\leq 1} = \{ L_Q(u) \mid u \in \overline{0 \bigcup Q \cap \mathbb{Z}_{\geq 0}} \}, \]
where \( \overline{0 \bigcup Q} \) is the convex hull of \( \{0\} \bigcup Q \). Here we have the symmetry of \( E_f \) with center 1.

7.7. Another comparison. If \( a = (x_1^{a_1}, \ldots, x_n^{a_n}) \), then
\[ R_a = \left\{ \sum p_i/a_i \mid 1 \leq p_i \leq a_i \right\}. \]
On the other hand, if \( f = \sum x_i^{p_i} \), then
\[ \tilde{R}_f = E_f = \left\{ \sum p_i/a_i \mid 1 \leq p_i \leq a_i - 1 \right\}. \]

8. Hyperplane arrangements.

8.1. Let \( D \) be a central hyperplane arrangement in \( X = \mathbb{C}^n \). Here, central means an affine cone of \( Z \subset \mathbb{P}^{n-1} \). Let \( f \) be the reduced equation of \( D \) and \( d := \deg f > n \). Assume \( D \) is not the pull-back of \( D' \subset \mathbb{C}^{n'} (n' < n) \).

8.2. Theorem. (i) \( \max R_f < 2 - \frac{1}{d} \). (ii) \( m_1 = n \).

Proof of (i) uses a partial generalization of a solution of Aomoto’s conjecture due to Esnault, Schechtman, Viehweg, Terao, Varchenko ([17], [40]) together with a generalization of Malgrange’s formula (4.8) as below:
8.3. Theorem (Generalization of Malgrange’s formula) [36]. There exists a pole order filtration $P$ on $H^{n-1}(F_0, C)_\lambda$ such that if $(\alpha + N) \cap R'_f = \emptyset$, then

(8.3.1) $\alpha \in R_f \Leftrightarrow \text{Gr}_P^p H^{n-1}(F_0, C)_\lambda \neq 0,$

with $p = \lfloor n - \alpha \rfloor$, $\lambda = e^{-2\pi i \alpha}$, where $R'_f = \cup_{x \neq 0} R_{f,x}$.

This reduces the proof of (8.2)(i) to

(8.3.2) $P^i H^{n-1}(F_0, C)_\lambda = H^{n-1}(F_0, C)_\lambda,$

for $i = n - 1$ if $\lambda = 1$ or $e^{2\pi i/d}$, and $i = n - 2$ otherwise.

8.4. Construction of the pole order filtration $P$. Let $U = \mathbf{P}^{n-1} \setminus Z$, and $F_0 = f^{-1}(0) \subset C^n$. Then $F_0 = \tilde{U}$ with $\pi : \tilde{U} \to U$ a $d$-fold covering ramified over $Z$. Let $L^{(k)}$ be the local systems of rank 1 on $U$ such that $\pi_* C = \bigoplus_{0 \leq i < d} L^{(k)}$ and $T$ acts on $L^{(k)}$ by $e^{-2\pi ik/d}$. Then

(8.4.1) $H^1(U, L^{(k)}) = H^3(F_0, C)_{e(-k/d)},$

and $P$ is induced by the pole order filtration on the meromorphic extension $\mathcal{L}^{(k)}$ of $L^{(k)} \otimes_C \mathcal{O}_U$ over $\mathbf{P}^{n-1}$, see [15], [36], [37]. This is closely related to:

8.5. Solution of Aomoto’s conjecture ([17], [40]). Let $Z_i$ be the irreducible components of $Z$ ($1 \leq i \leq d$), $g_i$ be the defining equation of $Z_i$ on $\mathbf{P}^{n-1} \setminus Z_d$ ($i < d$), and $\omega := \sum_{i < d} \alpha_i \omega_i$ with $\omega_i = dg_i/g_i$, $\alpha_i \in C$. Let $\nabla$ be the connection on $\mathcal{O}_U$ such that $\nabla u = du + \omega \wedge u$. Set $\alpha_d = -\sum_{i < d} \alpha_i$. Then $H^\bullet_{DR}(U, (\mathcal{O}_U, \nabla))$ is calculated by

$$(A^\bullet, \omega \wedge) \quad \text{with} \quad A^\bullet = \sum C \omega_1 \wedge \cdots \wedge \omega_p,$$

if $\sum_{Z \ni L} \alpha_i \notin N \setminus \{0\}$ for any dense edge $L \subset Z$ (see (8.7) below). Here an edge is an intersection of $Z_i$.

For the proof of (8.2)(ii) we have

8.6. Proposition. $N^{n-1} \psi_{f,\lambda} C \neq 0$ if $\text{Gr}_W^{2n-2} H^{n-1}(F_x, C)_\lambda \neq 0$.

(Indeed, $N^{n-1} : \text{Gr}_W^{2n-2} \psi_{f,\lambda} C \xrightarrow{\sim} \text{Gr}_0^{W} \psi_{f,\lambda} C$ by the definition of $W$, and the assumption of (8.6) implies $\text{Gr}_W^{W} \psi_{f,\lambda} C \neq 0$.)

Then we get (8.2)(ii), since $\omega_{i_1} \wedge \cdots \wedge \omega_{i_{n-1}} \neq 0$ in $\text{Gr}_W^{2n-2} H^{n-1}(\mathbf{P}^{n-1} \setminus Z, C) = \text{Gr}_W^{W} H^{n-1}(F_x, C)_1$.

8.7. Dense edges. Let $D = \cup_i D_i$ be the irreducible decomposition. Then $L = \cap_{i \in I} D_i$ is called an edge of $D$ ($I \neq \emptyset$).

We say that an edge $L$ is dense if $\{D_i/L \mid D_i \supset L\}$ is indecomposable. Here $C^n \supset D$ is called decomposable if $C^n = C^{n'} \times C^{n''}$ such that $D$ is the union of the pull-backs from $C^{n'}, C^{n''}$ with $n', n'' \neq 0$.

Set $m_L = \# \{D_i \mid D_i \supset L\}$. For $\lambda \in C$, define

$$\mathcal{D}E(D) = \{\text{dense edges of } D\}, \quad \mathcal{D}E(D, \lambda) = \{L \in \mathcal{D}E(D) \mid \lambda^{m_L} = 1\}.$$
We say that $L$, $L'$ are strongly adjacent if $L \subset L'$ or $L \supset L'$ or $L \cap L'$ is non-dense. Let

$$m(\lambda) = \max\{|S| \mid S \subset DE(D, \lambda) \text{ such that any } L, L' \in S \text{ are strongly adjacent}\}.$$ 

8.8. Theorem [37]. $m_\alpha \leq m(\lambda)$ with $\lambda = e^{-2\pi i \alpha}$.

8.9. Corollary. $R_f \subset \bigcup_{L \in DE(D)} \mathbb{Z}m_L^{-1}$.

8.10. Corollary. If $\text{GCD}(m_L, m_{L'}) = 1$ for any strongly adjacent $L, L' \in DE(D)$, then $m_\alpha = 1$ for any $\alpha \in R_f \setminus \mathbb{Z}$.

Theorem 2 follows from the canonical resolution of singularities $\pi : (\tilde{X}, \tilde{D}) \to (\mathbb{P}^{n-1}, D)$ due to [40], which is obtained by blowing up along the proper transforms of the dense edges. Indeed, $\text{mult} \tilde{D}(\lambda)_{\text{red}} \leq m(\lambda)$, where $\tilde{D}(\lambda)$ is the union of $\tilde{D}_i$ such that $\lambda \tilde{m}_i = 1$ and $\tilde{m}_i = \text{mult} \tilde{D}_i \tilde{D}$.

8.11. Theorem (Mustaţă [29]). For a central arrangement, 

$$(8.11.1) \quad J(X, \alpha D) = I_0^k \text{ with } k = [da] - n + 1 \text{ if } \alpha < \alpha_f,$$

where $I_0$ is the ideal of 0 and $\alpha_f = \min_{x \neq 0}\{\alpha_{f,x}\}$.

(This holds for the affine cone of any divisor on $\mathbb{P}^{n-1}$, see [36].)

8.12. Corollary. We have $\dim F^{n-1}H^{n-1}(F_0, \mathbb{C})_{\mathbb{C}(-k/d)} = \binom{k-1}{n-1}$ for $0 < \frac{k}{d} < \alpha_f$, and the same holds with $F$ replaced by $P$.

8.13. Corollary. $\alpha_f = \min(\alpha_{f}, \frac{n}{d}) < 1$.

(Note that $\alpha_f$ coincides with the minimal jumping number.)

8.14. Generic case. If $D$ is a generic central hyperplane arrangement, then

$$(8.14.1) \quad b_f(s) = (s + 1)^{n-1} \prod_{j=1}^{2d-2} (s + \frac{j}{d})$$

by U. Walther [46] (except for the multiplicity of $-1$). He uses a completely different method.

Note that Theorems (8.2) and (8.8) imply that the left-hand side divides the right-hand side of (8.14.1), and the equality follows using also (8.12).

8.15. Explicit calculation. Let $\alpha = k/d$, $\lambda = e^{-2\pi i \alpha}$ for $k \in \{1, \ldots, d\}$. If $\alpha \geq \alpha_f$, we assume there is $I \subset \{1, \ldots, d-1\}$ such that $|I| = k - 1$, and the condition of [40]

$$(8.15.1) \quad \sum_{Z_i \supset L} \alpha_i \notin \mathbb{N} \setminus \{0\} \text{ for any dense edge } L \subset Z,$$

is satisfied for

$$(8.15.2) \quad \alpha_i = 1 - \alpha \text{ if } i \in I \cup \{d\}, \text{ and } -\alpha \text{ otherwise}.$$

Let $V(I)$ be the subspace of $H^{n-1}A^*_{n}$ generated by

$$\omega_{i_1} \wedge \cdots \wedge \omega_{i_n-1} \text{ for } \{i_1, \ldots, i_{n-1}\} \subset I.$$

8.16. Theorem. Let $\alpha = k/d$, $\lambda = e^{-2\pi i \alpha}$ for $k \in \{1, \ldots, d\}$. Then
(a) If \( k = d - 1 \) or \( d \), then \( \alpha \in R_f, \alpha + 1 \not\in R_f \).

(b) If \( \alpha < \alpha_f \), then \( \alpha \in R_f \iff k \geq d \).

(c) If \( \binom{k-1}{n-1} < \dim H^{n-1}(F_0, C)_\lambda \), then \( \alpha + 1 \in R_f \).

(d) If \( \alpha < \alpha_f \), \( \alpha \not\in R_f + Z \) and \( \binom{k-1}{n-1} = \chi(U) \), then \( \alpha + 1 \not\in R_f \).

(e) If \( \alpha \geq \alpha_f \) and \( V(I) \neq 0 \), then \( \alpha \in R_f \).

(f) If \( \alpha \geq \alpha_f \) and \( V(I) = H^{n-1}A^*_n \), then \( \alpha + 1 \not\in R_f \).

8.17. Theorem [37]. Assume \( n = 3 \), mult.\( Z \leq 3 \) for any \( z \in Z \subset P^2 \), and \( d \leq 7 \).

Let \( \nu_3 \) be the number of triple points of \( Z \), and assume \( \nu_3 \neq 0 \). Then

\[
(8.17.1) \quad b_f(s) = (s + 1) \prod_{i=2}^{d} (s + \frac{i}{3}) \prod_{j=3}^{d} (s + \frac{j}{4}),
\]

with \( r = 2d - 2 \) or \( 2d - 3 \). We have \( r = 2d - 2 \) if \( \nu_3 < d - 3 \), and the converse holds for \( d < 7 \). In case \( d = 7 \), we have \( r = 2d - 3 \) for \( \nu_3 > 4 \), however, for \( \nu_3 = 4 \), \( r \) can be both \( 2d - 2 \) and \( 2d - 3 \).

8.18. Remarks. (i) We have \( \nu_3 < d - 3 \) if and only if

\[
(8.18.1) \quad \chi(U) = \frac{(d-2)(d-3)}{2} - \nu_3 > \frac{(d-3)(d-4)}{2} = \binom{d-3}{2}.
\]

(ii) By (8.4.1) we have \( \chi(U) = h^2(F_0, C)_\lambda - h^1(F_0, C)_\lambda \) if \( \lambda^d = 1 \) and \( \lambda \neq 0 \).

(iii) Let \( \nu'_i \) be the number of \( i \)-ple points of \( Z' := Z \cap C^2 \). Then by [6]

\[
(8.18.2) \quad b_0(U) = 1, \quad b_1(U) = d - 1, \quad b_2(U) = \nu'_2 + 2\nu'_3,
\]

8.19. Examples. (i) For \( (x^2-1)(y^2-1) = 0 \) in \( C^2 \) with \( d = 5 \), (8.17.1) holds with \( r = 7 \), and \( 8/5 \not\in R_f \). In this case we do not need to take \( I \), because \( (d - 2)/d = 3/5 < \alpha_f = 2/3 \). We have \( b_1(U) = b_2(U) = 4 \) and \( h^2(F_0, C)_\lambda = \chi(U) = 1 \) if \( \lambda^5 = 1 \) and \( \lambda \neq 1 \). So \( j/5 \in R_f \) for \( 3 \leq j \leq 7 \) by (a), (b), (c), and \( 8/5 \not\in R_f \) by (d).

(ii) For \( (x^2-1)(y^2-1)(x + y) = 0 \) in \( C^2 \) with \( d = 6 \), (8.17.1) holds with \( r = 9 \), and \( 10/6 \not\in R_f \). In this case we have \( b_1(U) = 5, b_2(U) = 6, \chi(U) = 2, h^1(F_0, C)_\lambda = 1, h^2(F_0, C)_\lambda = 3 \) for \( \lambda = e^{\pm 2\pi i/3} \). Then \( 4/6 \in R_f \) by (e) and \( 10/6 \not\in R_f \) by (f), where \( I^c \) corresponds to \( (x + 1)(y + 1) = 0 \). For other \( j/6 \), the argument is the same as in (i).

(iii) For \( (x^2 - y^2)(x^2 - 1)(y + 2) = 0 \) in \( C^2 \) with \( d = 6 \), (8.17.1) holds with \( r = 10 \), and \( 10/6 \in R_f \). In this case we have \( b_1(U) = 5, b_2(U) = 9, \chi(U) = 5, h^1(F_0, C)_\lambda = 0, h^2(F_0, C)_\lambda = 5 \) for \( \lambda = e^{\pm 2\pi i/3} \). Then \( 4/6 \in R_f \) by (e) and \( 10/6 \in R_f \) by (c), where \( I^c \) corresponds to \( (x + 1)(y + 2) = 0 \).

(iv) For \( (x^2 - y^2)(x^2 - 1)(y^2 - 1) = 0 \) in \( C^2 \) with \( d = 7 \), (8.17.1) holds with \( r = 11 \), and \( 12/7 \not\in R_f \). In this case we have \( b_1(U) = 6, b_2(U) = 9, \chi(U) = 4, h^2(F_0, C)_\lambda = 4 \) if \( \lambda^7 = 1 \) and \( \lambda \neq 1 \). Then \( 5/7 \in R_f \) by (e) and \( 12/7 \not\in R_f \) by (f), where \( I^c \) corresponds to \( (x + 1)(y + 1) = 0 \). Note that \( 5/7 \) is not a jumping number.
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