On type II\textsubscript{0} $E_0$-semigroups induced by boundary weight doubles

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Abstract

Powers has shown that each spatial $E_0$-semigroup can be obtained from the boundary weight map of a $CP$-flow acting on $B(K \otimes L^2(0, \infty))$ for some separable Hilbert space $K$. In this paper, we define boundary weight maps through boundary weight doubles $(\phi, \nu)$, where $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a $q$-positive map and $\nu$ is a boundary weight over $L^2(0, \infty)$. These doubles induce $CP$-flows over $K$ for $1 < \dim(K) < \infty$ which then minimally dilate to $E_0$-semigroups by a theorem of Bhat. Through this construction, we obtain uncountably many mutually non-cocycle conjugate $E_0$-semigroups for each $n > 1, n \in \mathbb{N}$.

Keywords: $E_0$-semigroup, $CP$-flow, completely positive map

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1. Introduction

Let $H$ be a separable Hilbert space, denoting its inner product by the symbol $(\cdot, \cdot)$ which is conjugate-linear in its first coordinate and linear in its second. A result of Wigner in [16] shows that every weakly continuous one-parameter group of $\ast$-automorphisms $\{\alpha_t\}_{t \in \mathbb{R}}$ of $B(H)$ is implemented by a strongly continuous unitary group $\{U_t\}_{t \in \mathbb{R}}$ in that $\alpha_t(A) = U_tAU_t^*$ for all $A \in B(H)$ and $t \in \mathbb{R}$. This leads us to pursue the more general task of classifying all suitable semigroups of $\ast$-endomorphisms of $B(H)$:

Definition 1.1. We say a family $\{\alpha_t\}_{t \geq 0}$ of $\ast$-endomorphisms of $B(H)$ is an $E_0$-semigroup if:

1. $\alpha_{s+t} = \alpha_s \circ \alpha_t$ for all $s, t \geq 0$, and $\alpha_0(A) = A$ for all $A \in B(H)$.
2. For each $f, g \in H$ and $A \in B(H)$, the inner product $(f, \alpha_t(A)g)$ is continuous in $t$.
3. $\alpha_t(I) = I$ for all $t \geq 0$ (in other words, $\alpha$ is unital).
We have two different notions of what it means for two $E_0$-semigroups to be the same, namely conjugacy and cocycle conjugacy, the latter of which arises from Alain Connes’ definition of outer conjugacy.

**Definition 1.2.** Let $\alpha$ and $\beta$ be $E_0$-semigroups on $B(H_1)$ and $B(H_2)$, respectively. We say that $\alpha$ and $\beta$ are conjugate if there is a $*$-isomorphism $\theta$ from $B(H_1)$ onto $B(H_2)$ such that $\theta \circ \alpha_t = \beta_t \circ \theta$ for all $t \geq 0$. We say that $\alpha$ and $\beta$ are cocycle conjugate if $\alpha$ is conjugate to $\beta'$, where $\beta'$ is an $E_0$-semigroup on $B(H_2)$ satisfying the following condition: For some strongly continuous family of unitaries $U = \{U_t : t \geq 0\}$ acting on $H_2$ and satisfying $U_{t+s} = U_t U_s$ for all $s, t \geq 0$, we have $\beta_t'(A) = U_t \beta_t(A) U_t^*$ for all $A \in B(H_2)$ and $t \geq 0$. Such a family of unitaries is called a unitary cocycle for $\beta$.

$E_0$-semigroups are divided into three types based upon the existence, and structure of, their units. More specifically, let $\alpha$ be an $E_0$-semigroup on $B(H)$. A unit for $\alpha$ is a strongly continuous semigroup of bounded operators $U = \{U(t) : t \geq 0\}$ such that $\alpha_t(A)U(t) = U(t)A$ for all $A \in B(H)$. Let $\mathcal{U}_\alpha$ be the set of all units for $\alpha$. We say $\alpha$ is spatial if $\mathcal{U}_\alpha \neq \emptyset$, while we say that $\alpha$ is completely spatial if, for each $t \geq 0$, the closed linear span of the set $\{U_1(t_1) \cdots U_n(t_n)f : f \in H, t_i \geq 0 \text{ and } U_i \in \mathcal{U}_\alpha \ \forall i, \sum t_i = t\}$ is $H$. If an $E_0$-semigroup $\alpha$ is completely spatial, we say it is of type I. If $\alpha$ is spatial but is not completely spatial, we say $\alpha$ is of type II. If $\alpha$ has no units, we say it is of type III.

If $\alpha$ is of type I or II, we may further assign an integer $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ to $\alpha$, in which case we say $\alpha$ is of type $I_n$ or $II_n$. We call $n$ the index of $\alpha$. It was initially defined in different ways in [12] and [2], and the connection between these definitions was explored in [13]. The index of $\alpha$ is the dimension of a particular Hilbert space associated to its units, and it is perhaps the most fundamental cocycle conjugacy invariant for spatial $E_0$-semigroups. Arveson showed in [2] that the type I $E_0$-semigroups are entirely classified (up to cocycle conjugacy) by their index: the type $I_n E_0$-semigroups are semigroups of $*$-automorphisms, while for $n \in \mathbb{N} \cup \{\infty\}$, every type $I_n E_0$-semigroup is cocycle conjugate to the CAR flow of rank $n$.

However, at the present time, we do not have such a classification for those of type II or III. The first type II and type III examples were constructed by Powers in [11] and [13]. Through Arveson’s theory of product systems, Tsirelson became the first to exhibit uncountably many mutually non-cocycle conjugate $E_0$-semigroups of types II and III (see [13]). A dilation theorem of Bhat in [3] shows that every unital $CP$-flow $\alpha$ can be dilated to an $E_0$-semigroup, and that there is a minimal dilation $\alpha^2$ of $\alpha$ which is unique up to conjugacy. Using Bhat’s result, Powers proved in [8] that every spatial $E_0$-semigroup can be obtained from the boundary weight map of a $CP$-flow over a separable Hilbert space $K$. In [9], he constructed spatial $E_0$-semigroups using boundary weights over $K$ when $\dim(K) = 1$ and then began to investigate the case when $\dim(K) = 2$.

Our goal is to use boundary weight maps to induce unital $CP$-flows over $K$ for $1 < \dim(K) < \infty$ and to classify their minimal dilations to $E_0$-semigroups up to cocycle conjugacy. To do so, we define a natural boundary weight map $\rho \to \omega(\rho)$ using a unital completely positive map $\phi$ and a normalized boundary weight $\nu$ over $L^2(0, \infty)$. The necessary and sufficient condition that this map induce a unital $CP$-flow $\alpha$ is that $\phi$ satisfies a definition of $q$-positive analogous to that from [8] (see Definition 3.1 and Proposition 3.2), in which case we say that $\alpha$ is the $CP$-flow induced by the boundary weight double $(\phi, \nu)$. We develop a comparison theory for boundary weight doubles $(\phi, \nu)$.
and \((\psi, \nu) (\phi \text{ and } \psi \text{ unital})\) in the case that \(\nu\) is a normalized unbounded boundary weight over \(L^2(0, \infty)\) of the form \(\nu(\sqrt{T - \Lambda(1)}B\sqrt{T - \Lambda(1)}) = (f, Bf)\), finding that the doubles induce cocycle conjugate \(E_0\)-semigroups if and only if there is a hyper maximal \(q\)-corner from \(\phi\) to \(\psi\) (see Definition 4.4 and Proposition 4.6).

The problem of determining hyper maximal \(q\)-corners from \(\phi\) to \(\psi\) becomes much easier if we focus on a particular class of \(q\)-positive maps, called the \(q\)-pure maps, which have the least possible \(q\)-subordinates (Definition 4.2). Given a \(q\)-positive map \(\phi\) acting on \(M_n(\mathbb{C})\) and a unitary \(U \in M_n(\mathbb{C})\), we can form a new map \(\phi_U\) by \(\phi_U(A) = U^*\phi(UAU^*)U\).

We describe the order isomorphism between the \(q\)-subordinates of \(\phi\) and those of \(\phi_U\), which in turn leads to the existence of a hyper maximal \(q\)-corner from \(\phi\) to \(\phi_U\) if \(\phi\) is unital and \(q\)-pure (Proposition 4.5). With this result in mind, we begin the task of classifying the unital \(q\)-pure maps. We find that the rank one unital \(q\)-pure maps \(\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})\) are precisely the maps \(\phi(A) = \rho(A)I\) for faithful states \(\rho\) on \(M_n(\mathbb{C})\) (Proposition 4.5). That these maps give us an enormous class of mutually non-cocycle conjugate \(E_0\)-semigroups in one of our main results (Theorem 5.4). Furthermore, for \(n > 1\), none of the \(E_0\)-semigroups constructed from boundary weight doubles satisfying the conditions of Theorem 5.4 are cocycle conjugate to any of the \(E_0\)-semigroups obtained from one-dimensional boundary weights by Powers in [3] (Corollary 5.5).

We turn our attention to the unital \(q\)-pure maps that are invertible. These maps are best understood through their (conditionally negative) inverses. In Theorem 6.11 we find a necessary and sufficient condition for an invertible unital \(\phi\) on \(M_n(\mathbb{C})\) to be \(q\)-pure. In this case, however, if \(\nu\) is a normalized unbounded boundary weight of the form \(\nu(\sqrt{T - \Lambda(1)}B\sqrt{T - \Lambda(1)}) = (f, Bf)\), then the \(E_0\)-semigroup induced by the boundary weight double \((\phi, \nu)\) is entirely determined by \(\nu\). This \(E_0\)-semigroup is the one induced by \(\nu\) in the sense of [3].

2. Background

2.1. Completely positive maps

Let \(\phi : \mathfrak{M} \to \mathfrak{B}\) be a linear map between \(C^*\)-algebras. For each \(n \in \mathbb{N}\), define \(\phi_n : M_n(\mathfrak{M}) \to M_n(\mathfrak{B})\) by

\[
\phi_n \left( \begin{array}{ccc}
A_{11} & \cdots & A_{1n} \\
\vdots & \ddots & \vdots \\
A_{n1} & \cdots & A_{nn}
\end{array} \right) = \left( \begin{array}{ccc}
\phi(A_{11}) & \cdots & \phi(A_{1n}) \\
\vdots & \ddots & \vdots \\
\phi(A_{n1}) & \cdots & \phi(A_{nn})
\end{array} \right).
\]

We say that \(\phi\) is completely positive if \(\phi_n\) is positive for all \(n \in \mathbb{N}\). A linear map \(\phi : B(H_1) \to B(H_2)\) is completely positive if and only if for all \(A_1, \ldots, A_n \in B(H_1), f_1, \ldots, f_n \in H_2,\) and \(n \in \mathbb{N}\), we have

\[
\sum_{i,j=1}^{n} (f_i, \phi(A_i^*A_j)f_j) \geq 0.
\]

Stinespring’s Theorem asserts that if \(\mathfrak{M}\) is a unital \(C^*\)-algebra and \(\phi : \mathfrak{M} \to B(H)\) is a unital completely positive map, then \(\phi\) dilates to a \(*\)-homomorphism in that there is a
Hilbert space $K$, a $*$-homomorphism $\pi : \mathcal{U} \to B(K)$, and an isometry $V : H \to K$ such that

$$\phi(A) = V^* \pi(A)V$$

for all $A \in \mathcal{U}$.

From the work of Choi (4) and Arveson (11), we know that a normal linear map $\phi : B(H_1) \to B(H_2)$ is completely positive if and only if it can be written in the form

$$\phi(A) = \sum_{i=1}^{n} S_i A S_i^*$$

for some $n \in \mathbb{N} \cup \{\infty\}$ and maps $S_i : H_1 \to H_2$ which are linearly independent over $\ell_2(\mathbb{N})$ in the sense that if $\sum_{i=1}^{r} z_i S_i = 0$ for a sequence $\{z_i\}_{i=1}^{r} \in \ell_2(\mathbb{N})$, then $z_i = 0$ for all $i$. With these hypotheses satisfied, the number $n$ is unique. We will use the above conditions for complete positivity interchangeably.

2.2. Conditionally negative maps

We say a self-adjoint linear map $\psi : B(K) \to B(K)$ is conditionally negative if, whenever $\sum_{i=1}^{m} A_i f_i = 0$ for $A_1, \ldots, A_m \in B(K)$, $f_1, \ldots, f_m \in K$, and $m \in \mathbb{N}$, we have $\sum_{i=1}^{m} (f_i, A_i^* A_j f_j) \leq 0$. If $K = \mathbb{C}^n$, then from the literature (see, for example, Theorem 3.1 of [10]) we know that $\psi$ has the form

$$\psi(A) = sA + YA + AY^* - \sum_{i=1}^{p} \lambda_i S_i A S_i^*,$$

where $s \in \mathbb{R}$, $tr(Y) = 0$, and for all $i$ and $j$ we have $\lambda_i > 0$, $tr(S_i) = 0$ and $tr(S_i^* S_j) = n\delta_{ij}$, where $p \leq n^2$ is independent of the maps $S_i$.

This form for $\psi$ is unique in the sense that if $\psi$ is written in the form

$$\psi(A) = tA + ZA + AZ^* - \sum_{i=1}^{p} \mu_i T_i A T_i^*,$$

where $t \in \mathbb{R}$, $tr(Z) = 0$, and for all $i$ and $j$ we have $\mu_i > 0$, $tr(T_i) = 0$, and $tr(T_i^* T_j) = n\delta_{ij}$, then $s = t$, $Z = Y$, and $\sum_{i=1}^{p} \lambda_i S_i A S_i^* = \sum_{i=1}^{p} \mu_i T_i A T_i^*$ for all $A \in M_n(\mathbb{C})$. Indeed, let $\{e_k\}_{k=1}^{n}$ be any orthonormal basis for $\mathbb{C}^n$, let $h_k = e_k/\sqrt{n}$ for each $k$, let $f \in \mathbb{C}^n$ be arbitrary, and for $k = 1, \ldots, n$, define $A_k \in M_n(\mathbb{C})$ by $A_k = f h_k^*$. Using the trace conditions, we find

$$\sum_{k=1}^{n} \psi(A_k) h_k = \sum_{k=1}^{n} (h_k, h_k) s f + \sum_{k=1}^{n} (h_k, h_k) Y f + \sum_{k=1}^{n} (h_k, Y^* h_k) f - \sum_{k=1}^{n} \left( \sum_{i=1}^{p} \lambda_i (h_k, S_i^* h_k) S_i f \right) = sf + Yf + 0 - \sum_{i=1}^{p} \left( \sum_{k=1}^{n} \lambda_i (h_k, S_i^* h_k) S_i f \right) = sf + Yf - \sum_{i=1}^{p} \lambda_i (0) S_i f = sf + Yf.$$
An analogous computation shows that \( \sum_{k=1}^{\alpha} \psi(A_k)h_k = tf + Zf \). Since \( f \in C^n \) was arbitrary, we conclude \((t-s)I = Y - Z \). Therefore, \( tr((t-s)I) = tr(Y - Z) = 0 \), so \( t = s \) and \( Y = Z \). Consequently, \( \sum_{i=1}^{\alpha} \lambda_i S_i A_{S_i}^* = \sum_{i=1}^{\alpha} \mu_i T_i A_{T_i}^* \) for all \( A \in M_n(\mathbb{C}) \).

### 2.2. CP-flows and Bhat’s theorem

Let \( K \) be a separable Hilbert space and let \( H = K \otimes L^2(0, \infty) \). We identify \( H \) with \( L^2([0, \infty); K) \), the space of \( K \)-valued measurable functions on \((0, \infty)\) which are square integrable. Under this identification, the inner product on \( H \) is

\[
(f, g) = \int_{0}^{\infty} (f(x), g(x))dx.
\]

Let \( U = \{U_t\}_{t \geq 0} \) be the right shift semigroup on \( H \), so for all \( t \geq 0 \) and \( f \in H \) we have \((U_t f)(x) = f(x-t) \) for \( x > t \) and \((U_t f)(x) = 0 \) otherwise. Let \( \Lambda : B(K) \to B(H) \) be the map defined by \( (\Lambda(A)f)(x) = e^{-x}Af(x) \) for all \( A \in B(K), f \in H \).

**Definition 2.1.** Assume the above notation. A strongly continuous semigroup \( \alpha = \{\alpha_t : t \geq 0\} \) of completely positive contractions of \( B(H) \) into itself is a CP-flow if \( \alpha_t(A)U_t = U_tA \) for all \( A \in B(K) \).

A theorem of Bhat in \( [3] \) allows us to generate \( E_0 \)-semigroups from unital CP-flows, and, more generally, from strongly continuous completely positive semigroups of unital maps on \( B(H) \), called CP-semigroups. We give a reformulation of Bhat’s theorem (see Theorem 2.1 of \( [3] \)):

**Theorem 2.2.** Suppose \( \alpha \) is a unital CP-semigroup of \( B(H_1) \). Then there is an \( E_0 \)-semigroup \( \alpha^d \) of \( B(H_2) \) and an isometry \( W : H_1 \to H_2 \) such that

\[
\alpha_t(A) = W^* \alpha_t^d(WAW^*)W
\]

and \( \alpha_t(WW^*) \geq WW^* \) for all \( t > 0 \). If the projection \( E = WW^* \) is minimal in that the closed linear span of the vectors

\[
\alpha_t^d(EA_1E) \cdots \alpha_t^d(EA_nE)Ef
\]

for \( f \in K, A_i \in B(H_1) \) and \( t_i \geq 0 \) for all \( i = 1, 2, \ldots, n \) and \( n = 1, 2, \ldots \) is \( H_2 \), then \( \alpha^d \) is unique up to conjugacy.

In \( [8] \), Powers showed that every spatial \( E_0 \)-semigroup acting on \( B(\mathcal{H}) \) (for \( \mathcal{H} \) a separable Hilbert space) is cocycle conjugate to an \( E_0 \)-semigroup which is a CP-flow, and that every CP-flow over \( K \) arises from a boundary weight map over \( H = K \otimes L^2(0, \infty) \). The boundary weight map \( \rho \to \omega(\rho) \) of a CP-flow \( \alpha \) associates to every \( \rho \in B(K) \), a boundary weight, that is, a linear functional \( \omega(\rho) \) acting on the null boundary algebra

\[
\mathfrak{A}(H) = \sqrt{I_H - \Lambda(I_K)}B(H)\sqrt{I_H - \Lambda(I_K)}
\]

which is normal in the following sense: If we define a linear functional \( \ell(\rho) \) on \( B(H) \) by

\[
\ell(\rho)(A) = \omega(\rho)\left(\sqrt{I_H - \Lambda(I_K)}A\sqrt{I_H - \Lambda(I_K)}\right),
\]
then $\ell(\rho) \in B(H)_*$. If $\omega(\rho)(I_H - \Lambda(I_K)) = \rho(I_K)$ for all $\rho \in B(K)_*$, then $\alpha$ is unital. For the sake of neatness, we will omit the subscripts $H$ and $K$ from the previous sentence when they are clear. Let $\delta$ be the generator of $\alpha$, and define $\Gamma : B(H) \to B(H)$ by $\Gamma(A) = \int_0^\infty e^{-t}U_tAU_t^*$. The resolvent $R_\alpha := (I - \delta)^{-1}$ of $\alpha$ satisfies $R_\alpha(A) = \int_0^\infty e^{-t}\alpha_t(A)dt$ for all $A \in B(H)$. Its associated predual map $\hat{R}_\alpha$ is given by

$$\hat{R}_\alpha(\eta) = \hat{\Gamma}(\omega(\hat{\Lambda}\eta) + \eta)$$

for all $\eta \in B(H)_*$.

A CP-flow $\alpha$ over $K$ is entirely determined by a set of normal completely positive contractions $\pi^\# = \{\pi^\#_t : t > 0\}$ from $B(H)$ into $B(K)$, called the generalized boundary representation of $\alpha$. Its relationship to the boundary weight map is as follows. For each $t > 0$, denote by $\hat{\pi}_t : B(K)_* \to B(H)_*$ the predual map induced by $\pi^\#_t$. For the truncated boundary weight maps $\rho \to \omega_t(\rho) \in B(H)_*$ defined by

$$\omega_t(\rho)(A) = \omega(\rho)(U_t^*AU_t),$$

we have $\hat{\pi}_t = \omega_t(I + \hat{\Lambda}\omega_t)^{-1}$ and $\omega_t = \hat{\pi}_t(I - \hat{\Lambda}\hat{\pi}_t)^{-1}$ for all $t > 0$. The maps $\{\pi^\#_t\}_{t > 0}$ have a $\sigma$-strong limit $\pi^\#_0$ as $b \to 0$ for each $A \in \bigcup_{t > 0} U_t B(H)U_t^*$, called the normal spine of $\alpha$. If $\alpha$ is unital, then the index of $\alpha^d$ as an $E_0$-semigroup is equal to the rank of $\pi^\#_0$ as a completely positive map (Theorem 4.49 of [9]).

Having seen that every CP-flow has an associated boundary weight map, we would like to approach the situation from the opposite direction. More specifically, under what conditions is a map $\rho \to \omega(\rho)$ from $B(K)_*$ to weights acting on $\mathfrak{A}(H)$ the boundary weight map of a CP-flow over $K$? Powers has found the answer (see Theorem 3.3 of [9]):

**Theorem 2.3.** If $\rho \to \omega(\rho)$ is a completely positive mapping from $B(K)_*$ into weights on $B(H)$ satisfying $\omega(\rho)(I - \Lambda(I_K)) \leq \rho(I_K)$ for all positive $\rho \in B(K)_*$, and if the maps $\hat{\pi}_t := \omega_t(I + \hat{\Lambda}\omega_t)^{-1}$ are completely positive contractions from $B(K)_*$ into $B(H)_*$ for all $t > 0$, then $\rho \to \omega(\rho)$ is the boundary weight map of a CP-flow over $K$. The CP-flow $\rho \to \omega(\rho)$ is unital if and only if $\omega(\rho)(I - \Lambda(I_K)) = \rho(I_K)$ for all $\rho \in B(K)_*$.

If $\dim(K) = 1$, the boundary weight map is just $c \in \mathbb{C} \to \omega(c) = \omega(1)$, so we may view our boundary weight map as a single positive boundary weight $\omega := \omega(1)$ acting on $\mathfrak{A}(L^2(0, \infty))$. Since the functional $\ell$ defined on $B(H)$ by

$$\ell(A) = \omega\left(\sqrt{I - \Lambda(1)}A\sqrt{I - \Lambda(1)}\right)$$

is positive and normal, it has the form $\ell(A) = \sum_{k=1}^n (f_k, A f_k)$ for some mutually orthogonal vectors $\{f_k\}_{k=1}^n \subseteq \mathbb{R}^\infty$, so

$$\omega\left(\sqrt{I - \Lambda(1)}A\sqrt{I - \Lambda(1)}\right) = \sum_{k=1}^n (f_k, A f_k)$$

for all $A \in B(H)$. If $\omega$ is normalized (that is, $\omega(I - \Lambda(1)) = 1$), then $\sum_{k=1}^n ||f_k||^2 = 1$. In [9], Powers induced $E_0$-semigroups using normalized boundary weights over $L^2(0, \infty)$.
The type of $E_0$-semigroup $\alpha^d$ induced by a normalized boundary weight
$\omega(\sqrt{I - \Lambda(t)}A\sqrt{I - \Lambda(t)}) = \sum_{k=1}^n (f_{\cdot k}, A_{f_{\cdot k}})$ depends on whether $\omega$ is bounded in the
sense that for some $r > 0$ we have $|\omega(B)| \leq r ||B||$ for all $B \in \mathcal{A}(H)$. Results from
[8] imply that $\alpha^d$ is of type $I_0$ if $\omega$ is bounded and of type $\Pi_0$ if $\omega$ is unbounded. If $\omega$
is unbounded, then both $\omega_t(I)$ and $\omega_t(\Lambda(t))$ approach infinity as $t$ approaches zero.
We will focus on normalized unbounded boundary weights over $L^2(0, \infty)$ of the form
$\omega(\sqrt{I - \Lambda(t)}A\sqrt{I - \Lambda(t)}) = (f, Af)$. We note that, as discussed in detail in [7], such
boundary weights are not normal weights.

If $\alpha$ and $\beta$ are $CP$-flows, we say that $\alpha \geq \beta$ if $\alpha_t - \beta_t$ is completely positive for all $t \geq 0$. The subordinates of a $CP$-flow are entirely determined by the subordinates of its
generalized boundary representation (see Theorem 3.4 of [9]):

**Theorem 2.4.** Let $\alpha$ and $\beta$ be $CP$-flows over $K$ with generalized boundary representa-
tions $\pi^# = \{\pi^#_t\}$ and $\xi^# = \{\xi^#_t\}$, respectively. Then $\beta$ is subordinate to $\alpha$ if and only if
$\pi^#_t - \xi^#_t$ is completely positive for all $t > 0$.

Given two unital $CP$-flows $\alpha$ and $\beta$, it is natural to ask when their minimally dilated
$E_0$-semigroups are cocycle conjugate. The following definition from [8] provides us with a key:

**Definition 2.5.** Let $\alpha$ and $\beta$ be $CP$-flows over $K_1$ and $K_2$, respectively, where $H_1 = K_1 \otimes L^2(0, \infty)$ and $H_2 = K_2 \otimes L^2(0, \infty)$. We say that a family of linear maps $\gamma = \{\gamma_t : t \geq 0\}$ from $B(H_2, H_1)$ into itself is a flow corner from $\alpha$ to $\beta$ if the family of maps
$\Theta = \{\Theta_t : t \geq 0\}$ defined by

$$
\Theta_t \left( \begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array} \right) = \left( \begin{array}{cc} \alpha_t(A_{11}) \quad \gamma_t(A_{12}) \\ \gamma_t^*(A_{21}) \quad \beta_t(A_{22}) \end{array} \right)
$$

is a $CP$-flow over $K_1 \otimes K_2$.

If $\gamma$ is a flow corner from $\alpha$ to $\beta$, we consider subordinates $\Theta' = \{\Theta'_t : t \geq 0\}$ of $\Theta$
that are $CP$-flows of the form

$$
\Theta'_t \left( \begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array} \right) := \left( \begin{array}{cc} \alpha'_t(A_{11}) \quad \gamma_t(A_{12}) \\ \gamma_t^*(A_{21}) \quad \beta'_t(A_{22}) \end{array} \right).
$$

We say that $\gamma$ is a hyper maximal flow corner from $\alpha$ to $\beta$ if, for every such subordinate
$\Theta'$ of $\Theta$, we have $\alpha = \alpha'$ and $\beta = \beta'$.

Our results will involve type $\Pi_0 E_0$-semigroups. These are spatial $E_0$-semigroups
which are not semigroups of $\ast$-automorphisms and have only one unit $V = \{V_t\}_{t \geq 0}$ up
to scaling by $e^{t \lambda}$ for $\lambda \in \mathbb{C}$. In the case that unital $CP$-flows $\alpha$ and $\beta$ minimally dilate
to type $\Pi_0 E_0$-semigroups, we have a necessary and sufficient condition for $\alpha^d$ and $\beta^d$ to
be cocycle conjugate (Theorem 4.56 of [8]):

**Theorem 2.6.** Suppose $\alpha$ and $\beta$ are unital $CP$-flows over $K_1$ and $K_2$ and $\alpha^d$ and $\beta^d$
are their minimal dilations to $E_0$-semigroups. Suppose $\gamma$ is a hyper maximal flow corner from $\alpha$
to $\beta$. Then $\alpha^d$ and $\beta^d$ are cocycle conjugate. Conversely, if $\alpha^d$ is a type $\Pi_0$ and $\alpha^d$ and $\beta^d$
are cocycle conjugate, then there is a hyper maximal flow corner from $\alpha$ to $\beta$.

We will later use this theorem to determine a necessary and sufficient condition for
some of the $E_0$-semigroups we construct to be cocycle conjugate (see Definition 1.3 and
Proposition 1.6).
3. Our boundary weight map

Recall that a completely positive linear map \( \phi \) can have negative eigenvalues. Moreover, even if \( I + t \phi \) is invertible for a given \( t \), it does not necessarily follow that \( \phi(I + t \phi)^{-1} \) is completely positive. In our boundary weight construction, we will require a special kind of completely positive map:

**Definition 3.1.** A linear map \( \phi : M_n(\mathbb{C}) \to M_n(\mathbb{C}) \) is \( q \)-positive if \( \phi \) has no negative eigenvalues and \( \phi(I + t \phi)^{-1} \) is completely positive for all \( t \geq 0 \).

Henceforth, we naturally identify a finite-dimensional Hilbert space \( K \) with \( \mathbb{C}^n \) and \( B(K \otimes L^2(0, \infty)) \) with \( M_n(B(L^2(0, \infty))) \). Under these identifications, the right shift \( t \) units on \( K \otimes L^2(0, \infty) \) is the matrix whose \( ij \)th entry is \( \delta_{ij} V_t \) for \( V_t \) the right shift on \( L^2(0, \infty) \). The map \( \Lambda_{n \times n} : B(K) \to B(K \otimes L^2(0, \infty)) \) sends an \( n \times n \) matrix \( B = (b_{ij}) \in M_n(\mathbb{C}) \) to the matrix \( \Lambda_{n \times n}(B) \) whose \( ij \)th entry is \( b_{ij} \Lambda(1) \in B(L^2(0, \infty)) \). The null boundary algebra \( \mathfrak{A}(H) \) is simply \( M_n(\mathfrak{A}(L^2(0, \infty))) \).

Given a boundary weight \( \nu \) over \( L^2(0, \infty) \), we write \( \Omega_{\nu,n \times k} \) for the map that sends an \( n \times k \) matrix \( A = (A_{ij}) \in M_{n \times k}(\mathfrak{A}(L^2(0, \infty))) \) to the matrix \( \Omega_{\nu,n \times k}(A) \in M_{n \times k}(\mathbb{C}) \) whose \( ij \)th entry is \( \nu(A_{ij}) \). We will suppress the integers \( n \) and \( k \) when they are clear, writing the above maps as \( \Omega_{\nu} \) and \( \Lambda \). In the proposition and corollary that follow, we show how to construct a \( CP \)-flow using a \( q \)-positive map \( \phi : M_n(\mathbb{C}) \to M_n(\mathbb{C}) \), a normalized boundary weight \( \nu \) over \( L^2(0, \infty) \), and the map \( \Omega_{\nu} := \Omega_{\nu,n \times n} : \mathfrak{A}(H) \to M_n(\mathbb{C}) \). The map \( \Omega_{\nu} \) is completely positive since \( \nu \) is positive.

**Proposition 3.2.** Let \( H = \mathbb{C}^n \otimes L^2(0, \infty) \). Let \( \phi : M_n(\mathbb{C}) \to M_n(\mathbb{C}) \) be a unital completely positive map with no negative eigenvalues, and let \( \nu \) be a normalized unbounded boundary weight over \( L^2(0, \infty) \). Then the map \( \rho \to \omega(\rho) \) from \( M_n(\mathbb{C})^* \) into boundary weights on \( \mathfrak{A}(H) \) defined by

\[
\omega(\rho)(A) = \rho(\phi(\Omega_{\nu}(A))).
\]

is completely positive. Furthermore, the maps \( \tilde{\pi}_t := \omega_t(I + \hat{\Lambda}_{\omega_t})^{-1} \) define normal completely positive contractions \( \pi_t^\nu \) of \( B(H) \) into \( M_n(\mathbb{C}) \) for all \( t > 0 \) if and only if \( \phi \) is \( q \)-positive.

**Proof.** The map \( \rho \to \omega(\rho) \) is completely positive since it is the composition of two completely positive maps. Before proving either direction, we let \( s_t = \nu_t(\Lambda(1)) \) for all \( t > 0 \) and prove the equality

\[
\tilde{\pi}_t(\rho) = \rho(\phi(I + s_t \phi)^{-1}\Omega_{\nu_t})
\]

for all \( \rho \in M_n(\mathbb{C})^* \). Denoting by \( U_t \) the right shift on \( H \) for every \( t > 0 \), we claim that \( (I + \hat{\Lambda}_{\omega_t})^{-1} = (I + s_t \phi)^{-1} \). Indeed, for arbitrary \( t > 0, B \in M_n(\mathbb{C}), \) and \( \rho \in M_n(\mathbb{C})^* \), we have

\[
\hat{\Lambda}_{\omega_t}(\rho)(B) = \rho\bigg(\phi\bigg(\Omega_{\nu_t}(U_tU_t^*\Lambda(B)U_tU_t^*)\bigg)\bigg) = \rho\bigg(\phi\bigg(\Omega_{\nu_t}\big(\Lambda(B)\big)\bigg)\bigg) = s_t \rho(\phi(B)),
\]

hence \( \hat{\Lambda}_{\omega_t} = s_t \phi \) and \( (I + \hat{\Lambda}_{\omega_t})^{-1} = (I + s_t \phi)^{-1} \).
For any $t > 0$ and $A \in B(H)$, we have

$$
\hat{\pi}_t(\rho)(A) = \omega_t(I + \hat{\Lambda}t)^{-1}(\rho)(A) = \left( (I + \hat{\Lambda}t)^{-1} \right) \left( \phi(\Omega_{\nu_t}(A)) \right)
$$

$$
= \left( (I + s_t \phi)^{-1}(\rho) \right) \left( \phi(\Omega_{\nu_t}(A)) \right) = \rho \left( (I + s_t \phi)^{-1}(\Omega_{\nu_t}(A)) \right)
$$

$$
= \rho \left( \phi(I + s_t \phi)^{-1}(\Omega_{\nu_t}(A)) \right),
$$

establishing (3).

Assume the hypotheses of the backward direction and let $t > 0$. By construction, $\hat{\pi}_t$ maps $M_n(\mathbb{C})^*$ into $B(H)_*$. It is also a contraction, since for all $\rho \in M_n(\mathbb{C})^*$ we have

$$
||\hat{\pi}_t(\rho)|| = \left| \left| \rho \left( \phi(I + s_\lambda \phi)^{-1}\Omega_{\nu_t} \right) \right| \right| \leq ||\rho|| \left| \left| \phi(I + s_t \phi)^{-1}\Omega_{\nu_t} \right| \right|
$$

$$
= ||\rho|| \left| \left| \phi(I + s_\lambda \phi)^{-1}\Omega_{\nu_t} \right| \right| = ||\rho|| \left| \left| \phi(I + s_t \phi)^{-1}\left( \nu_t(I)I_{\mathbb{C}} \right) \right| \right|
$$

$$
= ||\rho|| \left| \left| \frac{\nu_t(I)}{1 + s_t} I_{\mathbb{C}} \right| \right| = ||\rho|| \left| \left| \frac{\nu_t(I)}{1 + \nu_t(\Lambda(1))} \right| \right| \leq ||\rho||,
$$

where the last inequality follows from the fact that

$$
\nu_t(I - \Lambda(1)) \leq \nu(I - \Lambda(1)) = 1.
$$

Therefore, for every $t > 0$, $\hat{\pi}_t$ defines a normal contraction $\pi_t^#$ from $B(H)$ into $M_n(\mathbb{C})$ satisfying $\hat{\pi}_t(\rho) = \rho \circ \pi_t^#$ for all $\rho \in M_n(\mathbb{C})^*$. From Eq. (3) we see $\pi_t^# = \phi(I + s_t \phi)^{-1}\Omega_{\nu_t}$, so $\pi_t^#$ is the composition of completely positive maps and is thus completely positive for all $t > 0$.

Now assume the hypotheses of the forward direction. By unboundedness of $\nu$, the (monotonically decreasing) values $\{\nu_t\}_{t>0}$ form a set equal to either $(0, \infty)$ or $[0, \infty)$. Choose any $t > 0$ such that $s_t > 0$. Let $T \in B(H)$ be the matrix with $ij$th entry $(1/\nu_t(I))I$, and let $\kappa_t : M_n(\mathbb{C}) \rightarrow B(H)$ be the map that sends $B = (b_{ij}) \in M_n(\mathbb{C})$ to the matrix $\kappa_t(B) \in B(H)$ whose $ij$th entry is $(b_{ij}/\nu_t(I))I$. We note that $\kappa_t$ is the Schur product $B \rightarrow B \cdot T$, which is completely positive since $T$ is positive. For all $B \in M_n(\mathbb{C})$, we have

$$
\phi(I + s_t \phi)^{-1}(B) = \pi_t^#(\kappa_t(B)),
$$

so $\phi(I + s_t \phi)^{-1}$ is the composition of completely positive maps and is thus completely positive. As noted above, the values $\{s_t\}_{t>0}$ span $(0, \infty)$, so $\phi$ is $q$-positive. \qed

**Corollary 3.3.** The map $\rho \rightarrow \omega(\rho)$ in Proposition 3.2 is the boundary weight map of a unital CP-flow $\alpha$ over $\mathbb{C}^n$, and the Bhat minimal dilation $\alpha^d$ of $\alpha$ is a type $II_0$ E0-semigroup.

**Proof.** The first claim of the corollary follows immediately from Theorem 2.3 and Proposition 3.2 since

$$
\omega(\rho)(I - \Lambda(I_{\mathbb{C}}^n)) = \rho(\phi(I_{\mathbb{C}}^n)) = \rho(I_{\mathbb{C}}^n)
$$

(4)
for all \( \rho \in M_n(\mathbb{C})^* \). For the second assertion, we note that by Theorem 4.49 of [3], the index of \( \alpha^d \) is equal to the rank of the normal spine \( \pi_0^# \) of \( \alpha \), where \( \pi_0^# \) is the \( \sigma \)-strong limit of the maps \( \{ \pi_b^# \}_{b > 0} \) for each \( A \in \bigcup_{t > 0} U_t B(H) U_t^* \). Fix \( t > 0 \), and let \( A \in U_t B(H) U_t^* \). From formula (3),

\[
\pi_b^#(A) = \phi(I + \nu_t(\Lambda(1))\phi)^{-1}(\Omega_{\nu_b}(A)).
\]

For all \( b < t \) we have \( ||\Omega_{\nu_b}(A)|| = ||\Omega_{\nu_b}(A)|| < \infty \). Since \( \nu_b(\Lambda(1)) \rightarrow \infty \) as \( b \rightarrow 0 \), we conclude \( \lim_{b \rightarrow 0} ||\pi_b^#(A)|| = 0 \), hence \( \pi_0^# = 0 \) and the index of \( \alpha \) is zero. However, \( \alpha^d \) is not completely spatial since \( \alpha \) is not derived from the zero boundary weight map (see Lemma 4.37 and Theorem 4.52 of [3]), so \( \alpha^d \) is of type II_0.

Given a \( q \)-positive \( \phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C}) \) and a normalized unbounded boundary weight \( \nu \) over \( L^2(0, \infty) \), we call \((\phi, \nu)\) a boundary weight double. As we have seen, if \( \phi \) is unital then the boundary weight double naturally defines a boundary weight map through the construction of Proposition 5.2 inducing a type II_0 \( \mathcal{E}_0 \)-semigroup \( \alpha^d \) which is unique up to conjugacy by Theorem 2.2. We should note that it is not necessary for \( \phi \) to be unital in order for the boundary weight double to induce a \( CP \)-flow: If \( \phi \) is any \( q \)-positive contraction such that \( ||\nu_b(I)(\phi + \nu_b(\Lambda(1))\phi)^{-1}|| \leq 1 \) for all \( t > 0 \), then the arguments given in the proofs of Proposition 5.2 and Corollary 5.3 show that the boundary weight double \((\phi, \nu)\) induces a \( CP \)-flow \( \alpha \). However, if \( \phi \) is not unital, then by Eq (3) and Theorem 2.2 neither is \( \alpha \).

Motivated by [3], we make the following definition:

**Definition 3.4.** Suppose \( \alpha : B(H_1) \rightarrow B(K_1) \) and \( \beta : B(H_2) \rightarrow B(K_2) \) are normal and completely positive. Write each \( A \in B(H_1 \otimes H_2) \) as \( A = (A_{ij}) \), where \( A_{ij} \in B(H_j, H_i) \) for each \( i, j = 1, 2 \). We say a linear map \( \gamma : B(H_2, H_1) \rightarrow B(K_2, K_1) \) is a corner from \( \alpha \) to \( \beta \) if \( \psi : B(H_1 \otimes H_2) \rightarrow B(K_1 \otimes K_2) \) defined by

\[
\psi \left( \begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array} \right) = \left( \begin{array}{cc} \alpha(A_{11}) & \gamma(A_{12}) \\ \alpha^\star(A_{21}) & \beta(A_{22}) \end{array} \right)
\]

is a normal completely positive map.

We will repeatedly use the following lemma, which gives us the form of any corner between normal completely positive contractions of finite index. We believe that this result is already present in the literature, but we present a proof here for the sake of completeness:

**Lemma 3.5.** Let \( H_1, H_2, K_1 \), and \( K_2 \) be separable Hilbert spaces. Let \( \alpha : B(H_1) \rightarrow B(K_1) \) and \( \beta : B(H_2) \rightarrow B(K_2) \) be normal completely positive contractions of the form

\[
\alpha(A_{11}) = \sum_{i=1}^{n} S_i A_{11} S_i^\star, \quad \beta(A_{22}) = \sum_{j=1}^{p} T_j A_{22} T_j^\star,
\]

where \( n, p \in \mathbb{N} \) and the sets of maps \( \{S_i\}_{i=1}^{n} \) and \( \{T_j\}_{j=1}^{p} \) are both linearly independent. A linear map \( \gamma : B(H_2, H_1) \rightarrow B(K_2, K_1) \) is a corner from \( \alpha \) to \( \beta \) if and only if for all \( A_{12} \in B(H_2, H_1) \) we have

\[
\gamma(A_{12}) = \sum_{i,j} c_{ij} S_i A_{12} T_j^\star,
\]

where \( C = (c_{ij}) \in M_{n \times p}(\mathbb{C}) \) is any matrix such that \( ||C|| \leq 1 \).
PROOF. For the backward direction, let $C = (c_{ij}) \in M_{n \times p}(\mathbb{C})$ be any contraction, and define a linear map $\gamma : B(H_2, H_1) \to B(K_2, K_1)$ by $\gamma(A) = \sum_{i,j} c_{ij} S_i A T_j^*$. We need to show that the map

$$L \left( \begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array} \right) = \left( \begin{array}{c} \alpha(A_{11}) \gamma(A_{12}) \\ \gamma^*(A_{21}) \beta(A_{22}) \end{array} \right)$$

is normal and completely positive. To prove this, we first assume that $n \geq p$ and note that by Polar Decomposition we may write $C_{n \times p} = V_{n \times p} T_{p \times p}$, where $V_{n \times p}$ is a partial isometry of rank $p$ and $T$ is positive. Unitarily diagonalizing $T$ we see $C_{n \times p} = V_{n \times p} W_{p \times p}^* D_{p \times p} W_{p \times p}$. We may easily add columns to $V_{n \times p} W_{p \times p}^*$ to form a unitary matrix in $M_n(\mathbb{C})$, which we call $U^*$. Defining $\hat{D} = (d_{ij}) \in M_{n \times p}(\mathbb{C})$ to be the matrix obtained from $D$ by adding $n-p$ rows of zeroes, we see $U^* \hat{D} = V_{n \times p} W_{p \times p}^* D_{p \times p}$, so $C_{n \times p} = U^* \hat{D} W_{p \times p}$ and

$$UC_{n \times p} W_{p \times p}^* = \hat{D}.$$

In other words,

$$\sum_{i,j} c_{ij} u_{ki} w_{lj} = \left\{ \begin{array}{cl} \delta_{ki} d_{ij} & \text{if } k \leq p \\ 0 & \text{if } k > p \end{array} \right\}.$$

Next, define $\{S'_i\}_{i=1}^n : H_1 \to K_1$ and $\{T_j\}_{j=1}^p : H_2 \to K_2$ by

$$S'_i = \sum_{k=1}^n u_{ki} S_k, \quad T'_j = \sum_{l=1}^p w_{lj} T_l,$$

so $S_i = \sum_{k=1}^n u_{ki} S'_k$ and $T_j = \sum_{l=1}^p w_{lj} T'_l$ for all $i$ and $j$.

Since $U$ and $W$ are unitary, it follows that $||D|| = ||C|| \leq 1$ and that the maps $\{S'_i\}_{i=1}^n$ are linearly independent, as are the maps $\{T_j\}_{j=1}^p$. We observe that for any $A_{11} \in B(H_1)$ and $A_{22} \in B(H_2)$,

$$\sum_{i=1}^n S_i A_{11} S'_i = \sum_{i=1}^n S'_i A_{11} (S'_i)^* \quad \text{and} \quad \sum_{j=1}^p T_j A_{22} T'_j = \sum_{j=1}^p T'_j A_{22} (T'_j)^*.$$

Finally, for any $A_{12} \in B(H_2, H_1)$, we use our above computations to find that

$$\sum_{i,j} c_{ij} S_i A_{12} T_j^* = \sum_{i,j,k,l} c_{ij} u_{ki} w_{lj} S'_k A_{12} (T'_l)^* = \sum_{k,l} \left( \sum_{i,j} c_{ij} u_{ki} w_{lj} S'_k A_{12} (T'_l)^* \right)$$

$$= \sum_{(k \leq p), l} \left( \sum_{i,j} c_{ij} u_{ki} w_{lj} S'_k A_{12} (T'_l)^* \right)$$

$$+ \sum_{(k > p), l} \left( \sum_{i,j} c_{ij} u_{ki} w_{lj} S'_k A (T'_l)^* \right)$$

$$= \sum_{k \leq p} d_{kk} S'_k A_{12} (T'_k)^* + 0 = \sum_{k=1}^p d_{kk} S'_k A (T'_k)^*.$$

We have shown that
\[ L(A) = \left( \sum_{i=1}^{n} S_i A_{11}(S_i^*) \right) \left( \sum_{i=1}^{p} d_{ii} S_i A_{12}(T_i^*) \right) \left( \sum_{i=1}^{p} \sum_{i=1}^{p} d_{ii} S_i A_{22}(T_i^*) \right) \]

for all
\[ A = \left( \begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array} \right) \in B(H_1 \oplus H_2). \]

For each \( i = 1, \ldots, p \), define \( Z_i : H_1 \oplus H_2 \to K_1 \oplus K_2 \) by
\[ Z_i = \left( \begin{array}{cc} d_{ii} S_i & 0 \\ 0 & T_i^* \end{array} \right), \]

so
\[ L(A) = \sum_{i=1}^{p} Z_i A Z_i^* + \sum_{i=1}^{p} \left( (1 - |d_{ii}|^2) S_i A_{11} S_i^* \right) + \sum_{i=p+1}^{n} \left( S_i A_{11} S_i^* \right). \]

Since \(||D|| \leq 1\), the line above shows that \( L \) is the sum of three normal completely positive maps and is thus normal and completely positive. Therefore, \( \gamma \) is a corner from \( \alpha \) to \( \beta \). If, on the other hand, \( n < p \), then the same argument we just used shows that \( \gamma^* \) is a corner from \( \beta \) to \( \alpha \), which is equivalent to showing that \( \gamma \) is a corner from \( \alpha \) to \( \beta \).

For the forward direction, suppose that \( \gamma \) is a corner from \( \alpha \) to \( \beta \), so the map \( \Upsilon : B(H_1 \oplus H_2) \to B(K_1 \oplus K_2) \) defined by
\[ \Upsilon \left( \begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array} \right) = \left( \begin{array}{cc} \sum_{i=1}^{n} S_i A_{11} S_i^* & \gamma(A_{12}) \\ \gamma^*(A_{21}) & \sum_{j=1}^{p} T_j A_{22} T_j^* \end{array} \right) \]

is normal and completely positive. Therefore, for some \( q \in \mathbb{N} \cup \{ \infty \} \) and maps \( Y_i : H_1 \oplus H_2 \to K_1 \oplus K_2 \) for \( i = 1, 2, \ldots \), linearly independent over \( \ell_2(\mathbb{N}) \), we have
\[ \Upsilon(A) = \sum_{i=1}^{q} Y_i \tilde{A} Y_i^* \]

for all \( \tilde{A} \in B(H_1 \oplus H_2) \). For \( i = 1, 2 \), let \( E_i \in B(H_1 \oplus H_2) \) be projection onto \( H_i \), and let \( F_i \in B(K_1 \oplus K_2) \) be projection onto \( K_i \). Since \( \alpha \) and \( \beta \) are contractions we have \( \Upsilon(E_1) \leq F_1 \) and \( \Upsilon(E_2) \leq F_2 \), so \( Y_i E_i Y_i^* \leq F_j \) for each \( i \) and \( j \). It follows that each \( Y_i, i = 1, \ldots, q \), can be written in the form
\[ Y_i = \left( \begin{array}{cc} \tilde{S}_i & 0 \\ 0 & \tilde{T}_i \end{array} \right) \]

for some \( \tilde{S}_i \in B(H_1, K_1) \) and \( \tilde{T}_i \in B(H_2, K_2) \).

Note that \( \alpha(A_{11}) = \sum_{i=1}^{n} S_i A_{11} S_i^* = \sum_{i=1}^{q} \tilde{S}_i A_{11} \tilde{S}_i^* \) for all \( A_{11} \in B(H_1) \). For each \( \tilde{S}_i \), define a completely positive map \( L_i \) by \( L_i(A) = \tilde{S}_i A \tilde{S}_i^* \) for \( A \in B(H_1) \). Since \( \alpha - L_i \) is completely positive, it follows from the work of Arveson in [1] that \( \tilde{S}_i \) can be written as
\[ \tilde{S}_i = \sum_{j=1}^{n} r_{ij} S_j \]
for some complex coefficients \( \{ r_{ij} \}_{j=1}^n \). The same argument shows that for each \( \tilde{T}_i \) we have

\[
\tilde{T}_i = \sum_{j=1}^p b_{ij} T_j
\]

for some coefficients \( \{b_{ij}\}_{j=1}^p \). It now follows from linear independence of the maps \( \{Y_i\}_{i=1}^n \) that \( q \leq n + p \). Let \( R = (r_{ij}) \in M_{q \times n}(\mathbb{C}) \) and \( B = (b_{ij}) \in M_{q \times p}(\mathbb{C}) \), and let \( A \in B(H_1) \). We calculate

\[
\sum_{i=1}^n S_i A S_i^* = \sum_{i=1}^q S_i A S_i^* = \sum_{i=1}^q \left( \sum_{j,k=1}^n r_{ij} r_{jk} S_j S_k^* \right) = \sum_{j,k=1}^n \left( \sum_{i=1}^q r_{ij} r_{jk} S_j S_k^* \right). \tag{5}
\]

Let \( M = R^T (R^T)^* \in M_n(\mathbb{C}) \), so its \( jk \)th entry is \( m_{jk} = \sum_{i=1}^q r_{ij} r_{ik} \). Unitarily diagonalizing \( M \) as \( U M U^* = D \) for some diagonal \( D \) and defining maps \( \{ S_i' \}_{i=1}^n \) by \( S_i' = \sum_{k=1}^n \delta_{ik} S_k \), we see that Eq. (5) and the same linear algebra technique from the proof of the backward direction yield

\[
\sum_{i=1}^n S_i'^* A S_i'^* = \sum_{i=1}^n S_i A S_i^* = \sum_{j,k=1}^n m_{jk} S_j S_k^* = \sum_{i=1}^n d_{ii} S_i'^* S_i'^*.
\]

Therefore \( D = I \) and consequently \( M = I \), hence \( ||R|| = 1 \). An identical argument shows that \( ||B|| = 1 \).

Let

\[
\tilde{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in B(H_1 \oplus H_2)
\]

be arbitrary. Let \( C = (c_{jk}) \in M_{n \times p}(\mathbb{C}) \) be the matrix \( C = (B^* R)^T \), noting that \( ||C|| \leq 1 \).

A straightforward computation of \( Y(\tilde{A}) = \sum_{i=1}^n Y_i \tilde{A} Y_i^* \) yields

\[
\gamma(A_{12}) = \sum_{i=1}^q S_i' A_{12} T_i^* = \sum_{j=1}^q \left( \sum_{i=1}^n a_{ij} S_j \right) A_{12} \left( \sum_{k=1}^p b_{ik} T_k^* \right)
\]

\[
= \sum_{j,k} \left( \sum_{i=1}^n a_{ij} b_{jk} \right) S_j A_{12} T_k^* = \sum_{j,k} c_{jk} S_j A_{12} T_k^*,
\]

hence \( \gamma \) is of the form claimed.

\[\square\]

4. Comparison theory for \( q \)-positive maps

Just as in the general study of various classes of linear operators, it is natural to impose, and examine, an order structure for \( q \)-positive maps. If \( \phi \) and \( \psi \) are \( q \)-positive maps acting on \( M_n(\mathbb{C}) \), we say that \( \phi \) \( q \)-dominates \( \psi \) (and write \( \phi \geq_q \psi \)) if \( \phi(I + t\phi)^{-1} - \psi(I + t\psi)^{-1} \) is completely positive for all \( t \geq 0 \). We would like to find the \( q \)-positive
maps with the least complicated structure of \( q \)-subordinates. That last statement is not as simple as it seems. We might think to define a \( q \)-positive map \( \phi \) to be “\( q \)-pure” if \( \phi \geq q \psi \geq q 0 \) implies \( \psi = \lambda \phi \) for some \( \lambda \in [0,1] \), but there exist \( q \)-positive maps \( \phi \) such that for every \( \lambda \in (0,1) \) we have \( \phi \not\geq q \lambda \phi \). One such example is the Schur map \( \phi \) on \( M_2(\mathbb{C}) \) given by

\[
\phi \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & \frac{1}{2}(a_{12} + a_{12}^*) \\ (\frac{1}{2}a_{21}^*) & a_{22} \end{pmatrix}.
\]

As it turns out, every \( q \)-positive map is guaranteed to have a one-parameter family of \( q \)-subordinates of a particular form:

**Proposition 4.1.** Let \( \phi \geq q 0 \). For each \( s \geq 0 \), let \( \phi^{(s)} = \phi(I + s\phi)^{-1} \). Then \( \phi^{(s)} \geq q 0 \) for all \( s \geq 0 \). Furthermore, the set \( \{\phi^{(s)}\}_{s \geq 0} \) is a monotonically decreasing family of \( q \)-subordinates of \( \phi \), in the sense that \( \phi^{(s_1)} \geq q \phi^{(s_2)} \) if \( s_1 \leq s_2 \).

**Proof.** For all \( s \geq 0 \) and \( t \geq 0 \), we have

\[
\phi^{(s)}(I + t\phi^{(s)})^{-1} = \phi(I + s\phi)^{-1}(I + t\phi(I + s\phi)^{-1})^{-1} = \phi(I + t\phi(I + s\phi))(I + s\phi)^{-1} = \phi(I + (s+t)\phi)^{-1},
\]

which is completely positive by \( q \)-positivity of \( \phi \). Therefore, \( \phi^{(s)} \geq q 0 \) for all \( s \geq 0 \).

To prove that \( \phi^{(s_1)} \geq q \phi^{(s_2)} \) if \( s_1 \leq s_2 \), we let \( t \geq 0 \) be arbitrary and examine the map

\[
\Phi := \phi^{(s_1)}(I + t\phi^{(s_1)})^{-1} - \phi^{(s_2)}(I + t\phi^{(s_2)})^{-1}.
\]

Letting \( t_1 = s_1 + t \) and \( t_2 = s_2 + t \), we make the following observations:

\[
\phi^{(s_j)}(I + t\phi^{(s_j)})^{-1} = \phi^{(t_j)} \text{ for } j = 1, 2,
\]

\[
\phi^{(t_1)} - \phi^{(t_2)} = (I + t_2\phi)^{-1}((I + t_2\phi)\phi - \phi(I + t_1\phi))(I + t_1\phi)^{-1}. \tag{6}
\]

Equations (6) and (7) give us

\[
\Phi = (I + t_2\phi)^{-1}((I + t_2\phi)\phi - \phi(I + t_1\phi))(I + t_1\phi)^{-1} = (I + t_2\phi)^{-1}((t_2 - t_1)\phi^2)(I + t_1\phi)^{-1} = (t_2 - t_1)((I + t_2\phi)^{-1}(\phi(I + t_1\phi)^{-1}).
\]

The last line is a non-negative multiple of a composition of completely positive maps and is thus completely positive. We conclude that \( \phi^{(s_1)} \geq q \phi^{(s_2)} \). \( \square \)

We now have the correct notion of what it means to be \( q \)-pure:

**Definition 4.2.** Let \( \phi : M_n(\mathbb{C}) \to M_n(\mathbb{C}) \) be unital and \( q \)-positive. We say that \( \phi \) is \( q \)-pure if its set of \( q \)-subordinates is precisely \( \{0\} \cup \{\phi^{(s)}\}_{s \geq 0} \).
Lemma 4.3. Let \( \nu \) be a normalized unbounded boundary weight over \( L^2(0, \infty) \) of the form
\[
\nu(\sqrt{I - \Lambda(1)}B\sqrt{I - \Lambda(1)}) = (f, Bf).
\]
Let \( \phi : M_n(\mathbb{C}) \to M_n(\mathbb{C}) \) be a \( q \)-positive contraction such that \( ||\nu_t(I)\phi(I+\nu_t(\Lambda(1))\phi^{-1})|| \leq 1 \) for all \( t > 0 \), and let \( \alpha \) be the CP-flow derived from the boundary weight double \((\phi, \nu)\), with boundary generalized representation \( \pi = \{\pi_t^\#\}_{t>0} \).

Let \( \beta \) be any CP-flow over \( \mathbb{C}^n \), with generalized boundary representation \( \xi^\# = \{\xi_t^\#\}_{t>0} \) and boundary weight map \( \rho \to \eta(\rho) \). Then \( \alpha \geq \beta \) if and only if \( \beta \) is induced by the boundary weight double \((\psi, \nu)\), where \( \psi : M_n(\mathbb{C}) \to M_n(\mathbb{C}) \) is a \( q \)-positive map satisfying \( \phi \geq \psi \).

Proof. As before, for each \( t > 0 \) we let \( s_t = \nu_t(\Lambda(1)) \). Assume the hypotheses of the backward direction. Then \( \xi_t^\# = \psi(I + s_t\psi)^{-1}\Omega_{\nu_t} \), and the direction now follows from Theorem 2.4 since the line below is completely positive for all \( t > 0 \):
\[
\pi_t^\# - \xi_t^\# = (\phi(I + s_t\phi)^{-1} - \psi(I + s_t\psi)^{-1})\Omega_{\nu_t}.
\]

Now assume the hypotheses of the forward direction. Recall that by construction of \( \nu \), the set \( \{s_t\}_{t>0} \) is decreasing. If \( s_t > 0 \) for all \( t > 0 \) we define \( P = \infty \). Otherwise, we define \( P \) to be the smallest positive number such that \( s_P = 0 \). Fix any \( t_0 \in (0, P) \). Notationally, write each \( g \in H := \mathbb{C}^n \otimes L^2(0, \infty) \) in its components as \( g(x) = (g_1(x), \ldots, g_n(x)) \), and write \( f_{t_0} \) for the function \( V_{t_0}V_{t_0}^*f \in L^2(0, \infty) \), where \( V_{t_0} \) is the right shift \( t_0 \) units on \( L^2(0, \infty) \). Let \( U_{t_0} \) be the right shift \( t_0 \) units on \( H \). Under our identifications, \( U_{t_0}U_{t_0}^* \) is the diagonal matrix in \( M_n(B(L^2(0, \infty))) \) with \( ii^\# \) entries \( V_{t_0}V_{t_0}^* \). Define \( S : H \to \mathbb{C}^n \) by
\[
Sg = ((f_{t_0}, g_1), \ldots, (f_{t_0}, g_n)),
\]
noting that \( \Omega_{\nu_{t_0}}(A) = SAS^* \) for all \( A \in B(H) \). Since \( \phi(I + s_{t_0}\phi)^{-1} \) is completely positive, we know it has the form \( \phi(I + s_{t_0}\phi)^{-1}(M) = \sum_{i=1}^m R_iMR_i^* \) for some \( R_1, \ldots, R_m \in M_n(\mathbb{C}) \). Therefore,
\[
\pi_{t_0}^\#(A) = \left(\phi(I + s_{t_0}\phi)^{-1}\right)(\Omega_{\nu_{t_0}}(A)) = \sum_{i=1}^m R_iSAS^*R_i^*.
\]
The map \( \xi_{t_0}^\# \) is a subordinate of \( \pi_{t_0}^\# \), so from Arveson’s work in metric operator spaces in \( \mathbb{C}^n \), we know that \( \xi_{t_0}^\# \) has the form
\[
\xi_{t_0}^\#(A) = \sum_{i,j=1}^m c_{ij} R_iSAS^*R_j^*,
\]
for some complex numbers \( \{c_{ij}\} \). Let \( L_{t_0} \) be the map \( L_{t_0}(M) = \sum_{i,j} c_{ij} R_iMR_j^* \), noting that \( \xi_{t_0}^\#(A) = L_{t_0}(SAS^*) = L_{t_0}(\Omega_{\nu_{t_0}}(A)) \) for all \( A \in B(H) \).

Defining \( \psi_{t_0} : M_n(\mathbb{C}) \to M_n(\mathbb{C}) \) by \( \psi_{t_0} = (I - \xi_{t_0}^\#)^{-1}L_{t_0} \), we find that for arbitrary \( A \in B(H) \) and \( \hat{A} \in M_n(\mathbb{C}) \),
\[ \eta_t(\rho)(A) = \left( \xi_t(I - \hat{\Lambda}t)(\rho)(A) = \rho \left( (I - \xi_t^\# \Lambda)^{-1}(\xi_t^\#(A)) \right) \right) \]
\[ = \rho \left( (I - \xi_t^\# \Lambda)^{-1}L_t(\Omega_{\nu_{t0}}(A)) \right) = \rho(\psi_{t0}(\Omega_{\nu_{t0}} A)) \] (8)

and
\[ \hat{\Lambda}\eta_t(\rho)(\hat{A}) = \eta_t(\rho)(\Lambda(\hat{A})) = \rho(\psi_{t0}(\Omega_{\nu_{t0}}(\Lambda(\hat{A})))) = s_{t0}\rho(\psi_{t0}(\hat{A})), \] (9)

so \( \hat{\Lambda}\eta_t = s_{t0}\psi_{t0} \).

Using formulas (8) and (9) and the fact that \( \xi_t = \eta_t(\hat{\Lambda})^{-1} \), we find
\[ \rho(\xi_t^\#) = \xi_t(\rho) = \eta_t(I + \hat{\Lambda})^{-1}(\rho) = (I + \hat{\Lambda})^{-1}(\rho)(\psi_{t0}\Omega_{\nu_{t0}}) \]
\[ = \left( (I + s_{t0}\psi_{t0})^{-1}(\rho) \right)(\psi_{t0}\Omega_{\nu_{t0}}) = \rho \left( (I + s_{t0}\psi_{t0})^{-1}\psi_{t0}\Omega_{\nu_{t0}} \right) \]
\[ = \rho(\psi_{t0}(I + s_{t0}\psi_{t0})^{-1}\Omega_{\nu_{t0}}) \]
for all \( \rho \in M_n(\mathbb{C})^* \), hence \( \xi_t^\# = \psi_{t0}(I + s_{t0}\psi_{t0})^{-1}\Omega_{\nu_{t0}} \).

We now show that the maps \( \{\psi_t\}_{t \geq 0} \) are constant on the interval \((0, P)\). Let \( t \in [t_0, P) \) be arbitrary. For each \( \hat{A} = (a_{ij}) \in M_n(\mathbb{C}) \), let \( A \in B(H) \) be the matrix with \( ij \)th entry \( (a_{ij}/\nu(I))V_iV_j^* \). Let \( \rho \in M_n(\mathbb{C})^* \). Straightforward computations using formula (2) yield \( \Omega_t(A) = \Omega_t(\hat{A}) = \hat{A} \) and \( \eta_t(\rho)(\hat{A}) = \eta_t(\rho)(A) \). Combining these equalities gives us
\[ \rho(\psi_{t0}(\hat{A})) = \rho(\psi_{t0}(\psi_{t0}\psi_{t0})(A)) = \eta_t(\rho)(A) \]
\[ = \eta_t(\rho)(A) = \rho(\psi_{t0}(\Omega_{\nu_{t0}}(\hat{A}))) = \rho(\psi_{t0}(\hat{A})). \]

Since the above formula holds for every \( \hat{A} \in M_n(\mathbb{C}) \) and \( \rho \in M_n(\mathbb{C})^* \), we have \( \psi_{t0} = \psi_t \). But both \( t_0 \in (0, P) \) and \( t \in [t_0, P) \) were chosen arbitrarily, so the previous sentence shows that \( \psi_t = \psi_{t0} \) for all \( t \in (0, P) \).

Letting \( \psi = \psi_{t0} \), we have
\[ \xi_t^\# = \psi(I + s_t\psi)^{-1}\Omega_{\nu_t} \] (10)
for all \( t \in (0, P) \). Defining \( \kappa_t \) as in the proof of Proposition 3.2, we observe that \( \psi(I + s_t\psi)^{-1} = \xi_t^\# \kappa_t \), for all \( t \in (0, P) \), where the right hand side is completely positive by hypothesis. Since every \( t \in (0, \infty) \) can be written as \( t = s_{t'} \) for some \( t' \in (0, P) \), it follows that \( \psi(I + t\psi)^{-1} \) is completely positive for all \( t > 0 \). Furthermore, \( \psi(I + s_t\psi)^{-1} \rightarrow \psi \) in norm as \( t \rightarrow \infty \), hence \( \psi \geq_q 0 \). Similarly, since \( \pi_t^\# - \xi_t^\# \) is completely positive for all \( t > 0 \) by assumption, it follows from our formula
\[ \phi(I + s_t\phi)^{-1} - \psi(I + s_t\psi)^{-1} = (\pi_t^\# - \xi_t^\#)\kappa_t \]
that \( \phi(I + s_t\phi)^{-1} - \psi(I + s_t\psi)^{-1} \) is completely positive for all \( t > 0 \), and so its norm limit (as \( t \rightarrow \infty \)) \( \phi - \psi \) is completely positive. Therefore, \( \phi \geq_q \psi \). Finally, since the CP-flow \( \beta \) is entirely determined by its generalized boundary representation \( \xi^\# \), which itself is determined by any sequence \( \{\psi_{t_n}\} \) with \( t_n \) tending to 0 (see the remarks preceding Theorem 4.29 of [8]), it follows from (10) that \( \beta \) is induced by the boundary weight double \( (\psi, \nu) \). \( \square \)
In a manner analogous to that used by Powers in [9] and [8], we define the terms $q$-corner and hyper maximal $q$-corner:

**Definition 4.4.** Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ and $\psi : M_k(\mathbb{C}) \to M_k(\mathbb{C})$ be $q$-positive maps. A corner $\gamma : M_{n\times k}(\mathbb{C}) \to M_{n\times k}(\mathbb{C})$ from $\phi$ to $\psi$ is said to be a $q$-corner from $\phi$ to $\psi$ if the map

$$\Upsilon \left( \begin{array}{cc} A_{n\times n} & B_{n\times k} \\ C_{k\times n} & D_{k\times k} \end{array} \right) = \left( \begin{array}{cc} \phi(A_{n\times n}) & \gamma(B_{n\times k}) \\ \gamma^*(C_{k\times n}) & \psi(D_{k\times k}) \end{array} \right)$$

is $q$-positive. A $q$-corner $\gamma$ is called hyper maximal if, whenever

$$\Upsilon' = \left( \begin{array}{cc} \phi' & \gamma' \\ \gamma^* & \psi' \end{array} \right) \geq_q 0,$$

we have $\Upsilon = \Upsilon'$.

**Proposition 4.5.** For any $q$-positive $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ and unitary $U \in M_n(\mathbb{C})$, define a map $\phi_U$ by

$$\phi_U(A) = U^* \phi(UAU^*) U.$$

1. The map $\phi_U$ is $q$-positive, and there is an order isomorphism between $q$-positive maps $\beta$ such that $\phi \geq_q \beta$ and $q$-positive maps $\beta_U$ such that $\phi_U \geq \beta_U$. In particular, $\phi$ is $q$-pure if and only if $\phi_U$ is $q$-pure.

2. If $\phi$ is unital and $q$-pure, then there is a hyper maximal $q$-corner from $\phi$ to $\phi_U$.

**Proof.** To prove the first assertion, we define a completely positive map $\zeta$ on $M_n(\mathbb{C})$ by $\zeta(A) = U^* A U$, noting that $\zeta^{-1}$ is also completely positive. For every $t \geq 0$ and $A \in M_n(\mathbb{C})$, we find that $(I + t\phi_U)^{-1}(A) = U^* (I + t\phi)^{-1}(UAU^*) U$ and

$$\phi_U(I + t\phi_U)^{-1}(A) = U^* \phi \left( U(U^*(I + t\phi)^{-1}(UAU^*) U) U^* \right) U$$

$$= U^* \phi(U(I + t\phi)^{-1}(UAU^*) U) U$$

$$= \zeta \circ \phi(I + t\phi)^{-1} \circ \zeta^{-1}(A),$$

so $\phi_U \geq_q 0$. Given any $q$-positive map $\beta$ such that $\phi \geq_q \beta$, define $\beta_U(A) = U^* \beta(UAU^*) U$. Then $\beta_U$ is $q$-positive by (11), and for each $t \geq 0$ we have

$$\phi_U(I + t\phi)^{-1} - \beta(U(I + t\beta)^{-1}) = \zeta \circ \phi(I + t\phi)^{-1} - \beta(I + t\beta)^{-1} \circ \zeta^{-1},$$

hence $\phi_U \geq_q \beta_U$. Of course, since $\phi = (\phi_U)_{U^*}$, the argument just used gives an identical correspondence between $q$-subordinates $\alpha$ of $\phi_U$ and $q$-subordinates $\alpha_U^*$ of $\phi$. Our first assertion now follows.

To prove the second statement, we define $\gamma : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by $\gamma(A) = \phi(AU^*) U$. By Lemma 5.3, $\gamma$ is a corner from $\phi$ to $\phi_U$, so the map

$$\Theta \left( \begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array} \right) = \left( \begin{array}{cc} \phi(A_{11}) & \gamma(A_{12}) \\ \gamma^*(A_{21}) & \phi_U(A_{22}) \end{array} \right)$$

is completely positive. We calculate $\gamma(I + t\gamma)^{-1}(A) = \phi(I + t\phi)^{-1}(AU^*) U$, so for each $t \geq 0$ and $\tilde{A} = (A_{ij}) \in M_{2n}(\mathbb{C})$, we have

$$

\begin{align*}
\phi(U(I + t\phi)^{-1}(UAU^*) U) U & = U^* \phi(U(I + t\phi)^{-1}(UAU^*) U) U \\
& = U^* \phi(U(I + t\phi)^{-1}(UAU^*) U) U \\
& = (U(I + t\phi)^{-1} \circ \zeta^{-1}(A)) U.
\end{align*}

\]
\[ \Theta(I + t\Theta)^{-1}(\tilde{A}) = \begin{pmatrix} \phi(I + t\phi)^{-1}(A_{11}) & \phi(I + t\phi)^{-1}(A_{12}) \\ U^*\phi(I + t\phi)^{-1}(A_{21}) & \phi_U(I + t\phi_U)^{-1}(A_{22}) \end{pmatrix}. \]

This shows that \( \gamma(I + t\gamma)^{-1} \) is a corner from \( \phi(I + t\phi)^{-1} \) to \( \phi_U(I + t\phi_U)^{-1} \) for all \( t \geq 0 \), so \( \gamma \) is a \( q \)-corner. Finally, if

\[ \Theta'( \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} ) = \begin{pmatrix} \alpha(A_{11}) & \gamma(A_{12}) \\ \gamma^*(A_{21}) & \beta(A_{22}) \end{pmatrix} \]

is \( q \)-positive and \( \Theta \geq \Theta' \), then since \( \phi \) and \( \phi_U \) are \( q \)-pure we have \( \alpha = \phi(I + t\phi)^{-1} \) for some \( t \geq 0 \) and \( \beta = \phi_U(I + s\phi_U)^{-1} \) for some \( s \geq 0 \). Complete positivity of \( \Theta' \) implies that

\[ \Theta'( \begin{pmatrix} I & U \\ U^* & I \end{pmatrix} ) = \begin{pmatrix} \frac{1}{1+t}I & U \\ \frac{1}{1+s}U^* & I \end{pmatrix} \geq 0, \]

so \( s = t = 0 \) and \( \Theta = \Theta' \), hence \( \gamma \) is hyper maximal.

We have arrived at the key result of the section, which tells us that, under certain conditions, the problem of determining whether two \( E_0 \)-semigroups induced by boundary weight doubles are cocycle conjugate can be reduced to the much simpler problem of finding hyper maximal \( q \)-corners between \( q \)-positive maps:

**Proposition 4.6.** Let \( \nu \) be a normalized unbounded boundary weight over \( L^2(0, \infty) \) which has the form \( \nu(\sqrt{1 - \Lambda(1)}B\sqrt{1 - \Lambda(1)}) = (f, Bf) \). Let \( \phi \) and \( \psi \) be unital \( q \)-positive maps on \( M_n(\mathbb{C}) \) and \( M_k(\mathbb{C}) \), respectively, and induce \( CP \)-flows \( \alpha \) and \( \beta \) through the boundary weight doubles \( \phi, \nu \) and \( \psi, \nu \).

Then \( \alpha^d \) and \( \beta^d \) are cocycle conjugate if and only if there is a hyper maximal \( q \)-corner from \( \phi \) to \( \psi \).

**Proof.** Let \( N = n + k \). For the forward direction, suppose \( \alpha^d \) and \( \beta^d \) are cocycle conjugate. Since \( \alpha^d \) and \( \beta^d \) are of type \( I_0 \), we know from Theorem 2.6 that there is a hyper maximal flow corner \( \sigma \) from \( \alpha \) to \( \beta \), with associated \( CP \)-flow

\[ \Theta = \begin{pmatrix} \alpha & \sigma \\ \sigma^* & \beta \end{pmatrix}. \]

Let \( \Pi^\# = \{\Pi_i^\#\}, \pi^\# = \{\pi_i^\#\} \), and \( \xi^\# = \{\xi_i^\#\} \) be the generalized boundary representations for \( \Theta, \alpha, \) and \( \beta \), respectively. Define \( s_t = \nu_t(\Lambda(1)) \) for all positive \( t \), so for each \( t > 0 \) there is some \( 3_t \) such that

\[ \Pi_t^\# = \begin{pmatrix} \pi_t^\# & 3_t \\ 3_t^* & \xi_t^\# \end{pmatrix} = \begin{pmatrix} \phi(I + s_t\phi)^{-1} \circ \Omega_{\nu_t,n \times n} \\ \psi(I + s_t\psi)^{-1} \circ \Omega_{\nu_t,k \times k} \end{pmatrix} \]

Since each \( 3_t \) is a corner from \( \phi(I + s_t\phi)^{-1} \circ \Omega_{\nu_t,n \times n} \) to \( \psi(I + s_t\psi)^{-1} \circ \Omega_{\nu_t,k \times k} \), we have \( 3_t = L_t \circ \Omega_{\nu_t,n \times k} \) for some \( L_t \). Define \( B_t \) for each \( t > 0 \) by

\[ B_t = \begin{pmatrix} \phi(I + s_t\phi)^{-1} & L_t \\ L_t^* & \psi(I + s_t\psi)^{-1} \end{pmatrix}. \]
We observe that $\Pi^t_\# = B_t \circ \Omega_{\nu_t,N \times N}$ for all $t > 0$, whereby the same argument given in the proof of Lemma 4.3 shows that each $B_t$ has the form $B_t = W_t(I + s_tW_t)^{-1}$ for some $W_t : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ and that the maps $W_t$ are independent of $t$. Therefore, for some $\gamma : M_{n \times k}(\mathbb{C}) \to M_{n \times k}(\mathbb{C})$, we have

$$Z_t = \gamma(I + s_t\gamma)^{-1} \circ \Omega_{\nu_t,n \times k}$$

for all $t > 0$. Define $\kappa_{t,N \times N} : M_N(\mathbb{C}) \to B(H)$ as in Proposition 3.2. Letting

$$\vartheta = \left( \begin{array}{c} \phi \\ \gamma \\ \psi \end{array} \right),$$

we observe for each $t$ that $\vartheta(I + s_t\vartheta)^{-1} = \Pi^t_\# \circ \kappa_{t,N \times N}$ is the composition of completely positive maps and is thus completely positive, hence $\vartheta \geq q$ 0. Suppose that for some map $\vartheta'$ we have

$$\vartheta \geq q \vartheta' = \left( \begin{array}{c} \phi' \\ \gamma' \\ \psi' \end{array} \right) \geq q 0.$$

As in Proposition 3.2, the boundary weight map $\rho \in M_N(\mathbb{C})^* \to L(\rho)$ defined by $L(\rho)(C) = \rho(\vartheta'(\Omega_{\nu,N \times N}(C)))$ induces a CP-flow $\Theta'$ over $\mathbb{C}^N$, where for some CP-flows $\alpha'$ over $\mathbb{C}^n$ and $\beta'$ over $\mathbb{C}^k$, we have

$$\Theta' = \left( \begin{array}{c} \alpha' \\ \sigma' \\ \beta' \end{array} \right).$$

By Lemma 4.3 we have $\Theta \geq \Theta'$ since $\vartheta \geq q \vartheta'$. But $\Theta$ is a hyper maximal flow corner, so $\Theta = \Theta'$. Our formulas for the generalized boundary representations imply that $\varphi(I + t\varphi)^{-1} = \varphi'(I + t\varphi')^{-1}$ and $\psi(I + t\psi)^{-1} = \psi'(I + t\psi')^{-1}$ for all $t > 0$, hence $\varphi = \varphi'$ and $\psi = \psi'$. We conclude that $\gamma$ is a hyper maximal $q$-corner.

For the backward direction, suppose there is a hyper maximal $q$-corner $\gamma$ from $\phi$ to $\psi$, so the map $\Upsilon : M_N(\mathbb{C}) \to M_N(\mathbb{C})$ defined by

$$\Upsilon \left( \begin{array}{cc} A_{n \times n} & B_{n \times k} \\ C_{k \times n} & D_{k \times k} \end{array} \right) = \left( \begin{array}{cc} \varphi(A_{n \times n}) & \gamma(B_{n \times k}) \\ \gamma'(C_{k \times n}) & \psi'(D_{k \times k}) \end{array} \right)$$

is $q$-positive. By Proposition 3.2, the boundary weight map $\rho \in M_N(\mathbb{C})^* \to \Xi(\rho)$ defined by

$$\Xi(\rho)(A) = \rho(\Upsilon(\Omega_{\nu,N \times N}(A)))$$

is the boundary weight map of a CP-flow $\theta$ over $\mathbb{C}^N$, where for some $\Sigma$ we have

$$\theta = \left( \begin{array}{c} \alpha \\ \Sigma \end{array} \right).$$

Let

$$\theta' = \left( \begin{array}{c} \alpha' \\ \Sigma' \end{array} \right)$$

be any CP-flow such that $\theta \geq \theta'$. Letting $Z_t = \gamma(I + s_t\gamma)^{-1} \circ \Omega_{\nu_t,n \times k}$ for all $t > 0$, we see the generalized boundary representations $\Pi^t = \{\Pi^t_t\}$ and $\Pi' = \{\Pi'_t\}$ for $\theta$ and $\theta'$ satisfy
for all $t > 0$. Lemma 4.3 implies that for some $\phi'$ and $\psi'$ with $\phi \geq_0 \phi' \geq_0 0$ and $\psi \geq_0 \psi' \geq_0 0$ we have $\pi'_t = \phi'(I + st\phi')^{-1} \circ \Omega_{\nu',n \times n}$ and $\xi'_t = \psi'(I + st\psi')^{-1} \circ \Omega_{\nu,k \times k}$ for all $t > 0$. Defining $\Upsilon' : M_N(\mathbb{C}) \to M_N(\mathbb{C})$ by

$$
\Upsilon'(A_{n \times n} B_{n \times k} C_{k \times n} D_{k \times k}) = \left( \phi'(A_{n \times n}) \quad \gamma(B_{n \times k}) \quad \gamma^*(C_{k \times n}) \quad \psi'(D_{k \times k}) \right),
$$

we observe that $\Pi'_t \circ \kappa_{n,N \times N} = \Upsilon'(I + s_t \Upsilon')^{-1}$ for all $s_t > 0$, hence $\gamma$ is a $q$-corner from $\phi'$ to $\psi'$. Hyper maximality of $\gamma$ implies $\phi = \phi'$ and $\psi = \psi'$, thus $\theta = \theta'$. Therefore, $\sigma$ is a hyper maximal flow corner from $\alpha$ to $\beta$, so $\alpha^d$ and $\beta^d$ are cocycle conjugate by Theorem 2.8.

\[ \blacksquare \]

5. $E_0$-semigroups obtained from rank one unital $q$-pure maps

Any unital linear map $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ of rank one is of the form $\phi(A) = \tau(A)I$ for some linear functional $\tau$. If $\phi$ is positive, then $\tau$ is positive and $\tau(I) = 1$, so $\tau$ is a state. On the other hand, given any state $\rho$, the map $\phi$ defined by $\phi(A) = \rho(A)I$ is unital and completely positive. Furthermore, $\phi$ is $q$-positive since $\phi(I + t\phi)^{-1} = (1/(1 + t))\phi$ for all $t > 0$. The rank one unital $q$-positive maps are therefore precisely the maps $A \to \rho(A)I$ for states $\rho$.

The goal of this section is to determine when such maps are $q$-pure, and then to determine when the $E_0$-semigroups induced by $(\phi, \nu)$ and $(\psi, \nu)$ are cocycle conjugate, where $\phi$ and $\psi$ are rank one unital $q$-pure maps and $\nu$ is a normalized unbounded boundary weight of the form $\nu(\sqrt{I - A(1)}B\sqrt{I - \Lambda(1)}) = (f, Bf)$ (Theorem 5.4). We also obtain a partial result for comparing $E_0$-semigroups induced by $(\phi, \nu)$ and $(\psi, \mu)$ for rank one unital $q$-pure maps $\phi$ and $\psi$ and any normalized unbounded boundary weights $\nu$ and $\mu$ over $L^2(0, \infty)$ (Corollary 5.5).

We begin with a lemma:

**Lemma 5.1.** Let $\rho$ be a faithful state on $M_n(\mathbb{C})$, and define a unital $q$-positive map $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by $\phi(A) = \rho(A)I$. For any non-zero positive linear functional $\tau$ on $M_n(\mathbb{C})$ and non-zero positive operator $C \in M_n(\mathbb{C})$, define $\psi_{\tau,C} : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by $\psi_{\tau,C}(A) = \tau(A)C$.

Then $\psi_{\tau,C}$ is $q$-positive, and $\phi \geq_0 \psi_{\tau,C}$ if and only if $\psi_{\tau,C} = \lambda \phi$ for some $\lambda \in (0, 1]$.

**Proof.** Note that for all $A \in M_n(\mathbb{C})$ and $t \geq 0$, we have $(I + t\psi_{\tau,C})^{-1}(A) = A - \tau(A)/(1 + t\tau(C))C$, so

$$
\psi_{\tau,C}(I + t\psi_{\tau,C})^{-1}(A) = \frac{\tau(A)}{1 + \tau(C)}C,
$$

hence $\psi_{\tau,C}$ is $q$-positive. It follows from (12) that $\phi(I + t\phi)^{-1}(A) = (\rho(A)/(1 + t))I$ for all $A \in M_n(\mathbb{C})$.
Assume the hypotheses of the forward direction. Since $\phi \geq_q \psi_\tau,C$, we have

$$\frac{\rho(A)I}{1+t} \geq \frac{\tau(A)C}{1+t\tau(C)}$$

for all $t \geq 0$ and $A \geq 0$. This is impossible if $\tau(C) = 0$, so we may assume $\tau(C) \neq 0$. Letting $t \to \infty$ in (13) yields

$$\rho(A)I \geq \frac{\tau(A)C}{\tau(C)}$$

for all $A \geq 0$. Setting $A = C$ in (14), we see $\rho(C)I - C \geq 0$, yet

$$\rho\left(\rho(C)I - C\right) = \rho(C) - \rho(C) = 0,$$

hence $C = \rho(C)I$ by faithfulness of $\rho$. Rewriting (14) as

$$\rho(A)I \geq \frac{\tau(A)}{\tau(\rho(C)I)\rho(C)}\rho(C)I = \frac{\tau(A)}{||\tau||}I$$

for all $A \geq 0$, we see that $\rho - \tau/||\tau||$ is a positive linear functional. Therefore,

$$\left\|\rho - \frac{\tau}{||\tau||}\right\| = \rho(I) - \frac{\tau(I)}{||\tau||} = 1 - 1 = 0,$$

hence $\tau = ||\tau||\rho$. Setting $t = 0$ and $A = I$ in (13) gives us $||\tau|| = \tau(I) = \lambda/\rho(C)$ for some $\lambda \in (0,1]$. Therefore,

$$\psi_\tau,C(A) = \tau(A)C = ||\tau||\rho(A)\rho(C)I = \lambda\rho(A)I = \lambda\phi(A)$$

for all $A \in M_n(\mathbb{C})$, proving the forward direction.

The backward direction follows from Proposition 4.1 since $\lambda\phi = \phi^{(-1+1/\lambda)}$ for every $\lambda \in (0,1]$.

**Remark**: Let $\psi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a non-zero q-positive contraction such that the maps $L_{\psi_t} := t\psi(I + t\psi)^{-1}$ satisfy $||L_{\psi_t}|| < 1$ for all $t > 0$. By compactness of the unit ball of $B(M_n(\mathbb{C}))$, the maps $L_{\psi_t}$ have some norm limit as $t \to \infty$. This limit is unique: Pick any orthonormal basis with respect to the trace inner product $(A,B) = tr(A^*B)$ of $M_n(\mathbb{C})$, and let $M_t$ be the $n^2 \times n^2$ matrix of $L_{\psi_t}$ with respect to this basis. From the cofactor formula for $(I + t\psi)^{-1}$, we know that the $ij$th entry of $M_t$ is a rational function $r_{ij}(t)$. Uniqueness of $\lim_{t \to \infty} L_{\psi_t}$ now follows from the fact that each $r_{ij}(t)$ has a unique limit as $t \to \infty$. We call this limit $L_{\psi_t}$. Noting that

$$t\psi = L_{\psi_t}(I - L_{\psi_t})^{-1} = L_{\psi_t} + L_{\psi_t}^2 + \ldots$$

for each $t > 0$, we claim that $L_{\psi_t}$ fixes a positive element $T$ of norm one. To prove this, we first observe for each $k \in \mathbb{N}$ and $t > 0$ that

$$t||\psi|| = t||\phi(I)|| \leq ||L_{\psi_t}(I)|| + \ldots + ||(L_{\psi_t})^{k-1}(I)|| + k \sum_{n=1}^{\infty} ||(L_{\psi_t})^{kn}(I)||$$

$$< (k-1) + k \sum_{n=1}^{\infty} ||(L_{\psi_t})^k(I)||^n,$$
Let $s$ suffices in showing that a large class of maps is $q$-pure:  

For the forward direction, we prove the contrapositive. If $\rho$ is not faithful, then for some $k < n$ and mutually orthogonal vectors $f_1, \ldots, f_k$ with $\sum_{i=1}^{k} ||f_i||^2 = 1$, we have $\rho(A) = \sum_{i=1}^{k} (f_i, A f_i)$ for all $A \in M_n(C)$. Let $P$ be the projection onto the $k$-dimensional subspace of $C^n$ spanned by the vectors $f_1, \ldots, f_k$, and define a $q$-positive map $\psi : M_n(C) \to M_n(C)$ by $\psi(A) = \rho(A)P$. For each $t \geq 0$ and $A \in M_n(C)$, we find 

$$
(\phi^{(t)} - \psi^{(t)})(A) = \frac{1}{1 + t}(\phi(A) - \psi(A)) = \frac{1}{1 + t}\rho(A)(I - P),
$$

so $\phi \geq_q \psi$. Obviously, $\psi \neq \phi^{(s)}$ for any $s \geq 0$, so $\phi$ is not $q$-pure.

To prove the backward direction, suppose $\phi \geq_q \psi \geq_q 0$ for some $\psi \neq 0$, and form $L_\phi$ and $L_\psi$. Since $L_\phi = (t/(1 + t))\phi$ for each $t > 0$, we have $L_\phi = \phi$. The map $L_\phi - L_\psi$ is completely positive for all $t$, so by taking its limit as $t \to \infty$ we see $\phi - L_\phi$ is completely positive. By the remarks preceding this proposition, we know that $L_\phi$ fixes a positive $T$ with $||T|| = 1$. But $(\phi - L_\psi)(T) = \rho(T)I - T \geq 0$, so $\rho(T) = 1$, hence $T = I$ by faithfulness of $\rho$.

By complete positivity of $\phi - L_\psi$, we have $||\phi - L_\psi|| = ||\phi(I) - L_\psi(I)|| = 0$, so $\phi = L_\psi$. Therefore, 

$$
0 = \lim_{t \to \infty} (\phi - L_\psi)(\frac{I}{t} + \psi) = \lim_{t \to \infty} (\phi(\frac{I}{t} + \psi) - L_\psi(\frac{I}{t} + \psi)) \\
= \lim_{t \to \infty} (\frac{\phi}{t} + \phi \psi - t\psi(I + t\psi)^{-1}(\frac{I}{t} + \psi)) \\
= \phi \psi - \psi.
$$

(15)

Letting $\tau$ be the positive linear functional $\tau = \rho \circ \psi$, we conclude from (15) that $\psi(A) = \rho(\psi(A))I = \tau(A)I$ for all $A \in M_n(C)$. Lemma 5.1 implies that $\psi = \lambda \phi = \phi^{(-1 + 1/\lambda)}$ for some $\lambda \in (0, 1]$. \hfill $\square$

To prove the main result of the section, we need the following:  

**Lemma 5.3.** Let $\phi : M_n(C) \to M_n(C)$ and $\psi : M_k(C) \to M_k(C)$ be rank one unital $q$-pure maps, and let $\nu$ and $\mu$ be normalized unbounded boundary weights over $L^2(0, \infty)$. If the boundary weight doubles $(\phi, \nu)$ and $(\psi, \mu)$ induce cocycle conjugate $E_\alpha$-semi-groups $\alpha^\phi$ and $\beta^\psi$, then there is a corner $\gamma$ from $\phi$ to $\psi$ such that $||\gamma|| = 1$.  


PROOF. By construction, \( \alpha^d \) and \( \beta^d \) are type \( \Pi_0 \) \( E_0 \)-semigroups. If they are cocycle conjugate, then by Theorem 2.6 there is a hyper maximal flow \( \sigma \) from \( \alpha \) to \( \beta \) with associated \( CP \)-flow \( \Theta \) over \( K_1 \oplus K_2 \), where

\[
\Theta = \left( \begin{array}{cc}
\alpha & \sigma^* \\
\sigma & \beta
\end{array} \right).
\]

Let \( H_1 = \mathbb{C}^n \otimes L^2(0, \infty) \) and \( H_2 = \mathbb{C}^k \otimes L^2(0, \infty) \). Write the boundary representation \( \Pi = \{ \Pi^\#_t \} \) for \( \Theta \) as

\[
\Pi^\#_t = \left( \begin{array}{c}
\frac{1}{1 + \tau_1(1)} \phi \circ \Omega_{\mu_{n \times n}} \\
\frac{1}{1 + \tau_2(1)} \phi \circ \Omega_{\mu_{k \times k}}
\end{array} \right)
\]

for some maps \( \{ \tau_1 \}_{t_1 > 0} \) from \( B(H_2, H_1) \) into \( B(K_2, K_1) \). Let \( \rho_{11} \to \omega(\rho_{11}) \) and \( \rho_{22} \to \eta(\rho_{22}) \) denote the boundary weight maps for \( \alpha \) and \( \beta \), respectively. Let \( \rho \to \Xi(\rho) \) be the boundary weight map for \( \Theta \), so for some map \( \rho_{12} \to \ell(\rho_{12}) \) from \( M_{n \times k}(\mathbb{C})^* \) to weights on \( B(H_2, H_1) \) we have

\[
\Xi \left( \begin{array}{cc}
\rho_{11} & \rho_{12} \\
\rho_{21} & \rho_{22}
\end{array} \right) = \left( \begin{array}{cc}
\omega(\rho_{11}) & \ell(\rho_{12}) \\
\ell^*(\rho_{21}) & \eta(\rho_{22})
\end{array} \right).
\]

Denote by \( U_t \) the right shift \( t \) units on \( H \), and let \( \pi^\# \) and \( \xi^\# \) be the generalized boundary representations for \( \alpha \) and \( \beta \), respectively. For every \( A = (A_{ij}) \in \bigcup_{t > 0} U_t B(H)U_t^* \) and bounded family of functionals \( \{ \rho(t) = (\rho_{ij}(t)) \}_{t > 0} \) in \( M_{n \times k}(\mathbb{C})^* \), we observe that the argument used in Corollary 3.3 to show that \( \pi_0^\# = \xi_0^\# = 0 \) implies

\[
\lim_{t \to 0} \omega_t(I + \hat{\Lambda}_t)^{-1}(\rho_{11}(t))(A_{11}) = \lim_{t \to 0} \eta_t(I + \hat{\Lambda}_t)^{-1}(\rho_{22}(t))(A_{22}) = 0,
\]

so by complete positivity of the generalized boundary representation, we have

\[
\lim_{t \to 0} \ell_t(I + \hat{\Lambda}_t)^{-1}(\rho_{12}(t))(A_{12}) = 0. \tag{16}
\]

We claim that \( \rho_{12} \to \ell(\rho_{12}) \) is unbounded. If \( \ell \) is bounded, then for each \( \rho_{12} \in M_{n \times k}(\mathbb{C})^* \), the family \( \rho_{12}(t) := (I + \hat{\Lambda}_t)(\rho_{12}) \) is bounded, and it follows from \( \text{[16]} \) that

\[
\lim_{t \to 0} \ell_t(\rho_{12})(A_{12}) = 0. \tag{17}
\]

for each \( A_{12} \in \bigcup_{t > 0} W_t B(H_2, H_1)X_t^* \), where \( W_t \) and \( X_t \) are the right shift \( t \) units on \( H_1 \) and \( H_2 \), respectively. Let \( A_{12} \in \bigcup_{t > 0} W_t B(H_2, H_1)X_t^* \), so \( A_{12} = W_sBX_t^* \) for some \( s > 0 \) and \( B \in B(H_2, H_1) \). For all \( b < s \), we have

\[
\ell_b(\rho_{12})(A_{12}) = \ell_b(\rho_{12})(W_sBX_t^*) = \ell(\rho_{12})(W_bW_sBX_t^*) = \ell(\rho_{12})(W_bW_sBX_t^*) = \ell(\rho_{12})(A_{12}).
\]

Therefore, by equation \( \text{[17]} \) we have \( \ell(\rho_{12})(A_{12}) = 0 \). Let \( A \in B(H_2, H_1) \), \( \rho_{12} \in M_{n \times k}(\mathbb{C})^* \), and \( t > 0 \) be arbitrary. From above we have

\[
\ell_t(\rho_{12})(A) = \ell(\rho_{12})(W_tAX_t^*) = 0,
\]

and this completes the proof.
hence $\ell_t \equiv 0$ for all $t > 0$. We conclude from uniqueness of the generalized boundary representation that $\rho_{12} \rightarrow \ell(\rho_{12})$ is the zero map. The boundary weight map $\rho \rightarrow \Xi(\rho)$ defined by

$$\Xi\left(\begin{array}{cc}
\rho_{11} & \rho_{12} \\
\rho_{21} & \rho_{22}
\end{array}\right) = \left(\begin{array}{cc}
\omega(\rho_{11}) & 0 \\
0 & 0
\end{array}\right)$$

gives rise to the $CP$-flow

$$\Theta' = \left(\begin{array}{cc}
\alpha & \sigma \\
\sigma^* & \beta'
\end{array}\right)$$

where $\beta'$ is the non-unital $CP$-flow $\beta'_t(A_{22}) = X_t A_{22} X_t^*$. Trivially, $\Theta \neq \Theta'$ and $\Theta \geq \Theta'$, contradicting hyper maximality of $\sigma$. Therefore, the map $\rho_{12} \rightarrow \ell(\rho_{12})$ is unbounded.

Since $\Pi^\sharp_t$ is a contraction for every $t > 0$, so is $3_t$, hence the map $3_t \circ \Lambda : M_{n \times k}(\mathbb{C}) \rightarrow M_{n \times k}(\mathbb{C})$ is a contraction for each $t > 0$. A compactness argument shows that $3_t \circ \Lambda$ has a norm limit $\gamma$ for some sequence $\{t_n\}$ tending to zero, where $||\gamma|| \leq 1$. From unboundedness of $\ell$ and the formula $\ell_t = 3_t(I - \Lambda 3_t)^{-1}$ for all $t > 0$, it follows that $I - \gamma$ is not invertible, so $||\gamma|| \geq 1$, hence $||\gamma|| = 1$. We claim that $\gamma$ is a corner from $\phi$ to $\psi$. Indeed, for the family of completely positive maps $\{R_t\}_{t > 0}$ defined by $R_t = \Pi^\sharp_t \circ \Lambda$, we have

$$\lim_{n \rightarrow \infty} R_{t_n} = \lim_{n \rightarrow \infty} \left(\frac{\nu_n(\Lambda(1))}{1 + \nu_n(\Lambda(1))} \left(\begin{array}{cc}
3_{t_n} \circ \Lambda & \gamma \\
\gamma^* & \psi
\end{array}\right)
\right) = \left(\begin{array}{cc}
\phi & \gamma \\
\gamma^* & \psi
\end{array}\right).$$

If $\nu$ is a normalized unbounded boundary weight over $L^2(0, \infty)$ of the form $\nu(\sqrt{1 - \Lambda(1)}B\sqrt{1 - \Lambda(1)}) = (f, Bf)$ and if $\phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ is unital and $q$-pure, we know from Propositions 4.5 and 4.6 and 4.8 that the condition $\psi = \phi^*$ is sufficient for the boundary weight doubles $(\phi, \nu)$ and $(\psi, \nu)$ to induce cocycle conjugate $E_0$-semigroups.

In the case that $\phi$ is a rank one unital $q$-pure map, this condition is also necessary:

**Theorem 5.4.** Let $\phi_1 : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ and $\phi_2 : M_k(\mathbb{C}) \rightarrow M_k(\mathbb{C})$ be rank one unital $q$-pure maps. Let $\nu$ be a normalized unbounded boundary weight over $L^2(0, \infty)$ of the form $\nu(\sqrt{1 - \Lambda(1)}B\sqrt{1 - \Lambda(1)}) = (f, Bf)$.

Then the boundary weight doubles $(\phi_1, \nu)$ and $(\phi_2, \nu)$ induce cocycle conjugate $E_0$-semigroups if and only if $n = k$ and $\phi_2 = (\phi_1)_U$ for some unitary $U \in M_n(\mathbb{C})$.

**Proof.** The backward direction follows immediately from Propositions 4.5 and 4.6. Assume the hypotheses of the forward direction. Since $\phi_1$ and $\phi_2$ are rank one, unital, and $q$-pure, there exist faithful states $\rho_1$ on $M_n(\mathbb{C})$ and $\rho_2$ on $M_k(\mathbb{C})$ such that $\phi_1(M) = \rho_1(M)I_{n \times n}$ and $\phi_2(B) = \rho_2(B)I_{k \times k}$ for all $M \in M_n(\mathbb{C})$, $B \in M_k(\mathbb{C})$.

By Lemma 5.2 there is a corner $\gamma$ from $\phi_1$ to $\phi_2$ such that $||\gamma|| = 1$. Therefore, for some $A_0 \in M_{n \times k}(\mathbb{C})$ of norm one and unit vectors $f_0 \in \mathbb{C}^n$ and $g_0 \in \mathbb{C}^k$, we have $||f_0, \gamma(A_0)g_0|| = 1$. Define $\omega \in M_{n \times k}(\mathbb{C})^*$ by $\omega(A) = (f_0, \gamma(A)g_0)$, noting that $||\omega|| = ||\omega(A_0)|| = 1$. We claim that the map $\psi : M_{n+k}(\mathbb{C}) \rightarrow M_2(\mathbb{C})$ defined by

$$\psi\left(\begin{array}{cc}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{array}\right) = \left(\begin{array}{cc}
\rho_1(A_{11}) & \omega(A_{12}) \\
\omega^*(A_{21}) & \rho_2(A_{22})
\end{array}\right)$$




is completely positive. To see this, let \{\tilde{F}_i\}_{i=1}^\ell$ be arbitrary vectors in $\mathbb{C}^2$, writing each $F_i$ as
\[
\tilde{F}_i = \begin{pmatrix} \lambda_{i1} \\ \lambda_{i2} \end{pmatrix}
\]
for some complex numbers \{\lambda_{i1}\}_{i=1}^\ell$ and \{\lambda_{i2}\}_{i=1}^\ell$.

Since the map $\psi : M_{n+k}(\mathbb{C}) \to M_{n+k}(\mathbb{C})$ defined by
\[
\psi \left( \begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array} \right) = \begin{pmatrix} \rho_1(A_{11})I & \gamma(A_{12}) \\ \gamma^*(A_{21}) & \rho_2(A_{22})I \end{pmatrix}
\]
is completely positive by assumption, we know that for any $A_1, \ldots, A_\ell \in M_{n+k}(\mathbb{C})$ and the vectors
\[
F_i = \begin{pmatrix} \lambda_{i1}f_0 \\ \lambda_{i2}g_0 \end{pmatrix} \in \mathbb{C}^{n+k}, \quad i = 1, \ldots, k,
\]
we have
\[
\sum_{i,j=1}^\ell \left( F_i, \psi(A_i^*A_j)F_j \right) \geq 0.
\]
However, for each $i$ and $j$ we find that
\[
\left( F_i, \psi(A_i^*A_j)F_j \right)_{C^{n+k}} = \overline{\lambda_{i1}\lambda_{j1}}\rho_1((A_i^*A_j)_{11}) + \overline{\lambda_{i1}\lambda_{j2}}\omega((A_i^*A_j)_{12})
+ \overline{\lambda_{i2}\lambda_{j1}}\omega([((A_i^*A_j)_{21})^*]) + \overline{\lambda_{i2}\lambda_{j2}}\rho_2((A_i^*A_j)_{22})
= \left( \tilde{F}_i, \tilde{\psi}(A_i^*A_j)\tilde{F}_j \right)_{\mathbb{C}^2}.
\]
Therefore, for all $\ell \in \mathbb{N}$, $A_1, \ldots, A_\ell \in M_{n+k}(\mathbb{C})$, and $\tilde{F}_1, \ldots, \tilde{F}_\ell \in \mathbb{C}^2$, we have
\[
\sum_{i,j=1}^\ell \left( \tilde{F}_i, \tilde{\psi}(A_i^*A_j)\tilde{F}_j \right) \geq 0,
\]
so $\tilde{\psi} : M_{2n}(\mathbb{C}) \to M_{2}(\mathbb{C})$ is completely positive. Since $\rho_1$ and $\rho_2$ are positive linear functionals (hence completely positive maps), $\omega$ is a corner from $\rho_1$ to $\rho_2$.

By faithfulness of $\rho_1$ and $\rho_2$, there exist monotonically increasing sequences of strictly positive numbers \{\lambda_i\}_{i=1}^n$ and \{\mu_j\}_{j=1}^k with $\sum_{i=1}^n \lambda_i^2 = \sum_{j=1}^k \mu_j^2 = 1$, along with orthonormal sets of vectors \{\{f_i\}_{i=1}^n\} and \{\{g_j\}_{j=1}^k\}, such that $\rho_1(M) = \sum_{i=1}^n \lambda_i f_i^*Mf_i$ and $\rho_2(M) = \sum_{j=1}^k \mu_j g_j^*Mg_j$ for all $M \in M_n(\mathbb{C})$, $B \in M_k(\mathbb{C})$. Given $A \in M_{n \times k}(\mathbb{C})$, let $\tilde{A}$ be the matrix whose $ij$th entry is $(f_i, Ag_j)$, observing that $||\tilde{A}|| = ||A||$. Let $D_{\lambda}$ and $D_{\mu}$ be the diagonal matrices whose $i$th entries are $\lambda_i$ and $\mu_i$, respectively, for all $i$, and let $D_{\lambda^2}$ and $D_{\mu^2}$ be the diagonal matrices whose $i$th entries are $\lambda_i^2$ and $\mu_i^2$, respectively, observing that $D_{\lambda^2} = (D_{\lambda})^2$ and $D_{\mu^2} = (D_{\mu})^2$.

By Proposition 3.3, $\omega$ has the form
\[
\omega(A) = \sum_{i,j} c_{ij} \lambda_i \mu_j \langle f_i, Ag_j \rangle = \text{tr}(CD_{\mu} \tilde{A}D_{\lambda}) = \text{tr}\left( CD_{\mu}(D_{\lambda} \tilde{A}^* \right)\right)
\]
for some $C = (c_{ij}) \in M_{n \times k}(\mathbb{C})$ such that $||C|| \leq 1$. 


By the Cauchy-Schwartz inequality for the inner product \( (B, A) = \text{tr}(AB^*) \) on \( M_{n \times k}(\mathbb{C}) \), we have
\[
1 = |\omega(A_0)|^2 = |\text{tr}(CD_\mu(D_\lambda \tilde{A}_0^*)^2)|^2 = |(CD_\mu, D_\lambda \tilde{A}_0^*)|^2 \\
\leq (CD_\mu, CD_\mu)(D_\lambda \tilde{A}_0^*, D_\lambda \tilde{A}_0^*) = \text{tr}(D_\mu C^* CD_\mu)\text{tr}(D_\lambda \tilde{A}_0^* \tilde{A}_0 D_\lambda) \\
\leq \text{tr}(D_\mu I_k)\text{tr}(D_\lambda I_n) \leq 1 * 1 = 1. \tag{18}
\]
Since equality holds in all the inequalities above, we have \( mCD_\mu = D_\lambda \tilde{A}_0^* \) for some \( m \in \mathbb{C} \). It follows from (18) that \( |m| = 1 \) since \( \|CD_\mu\|_\text{tr} = \|D_\lambda \tilde{A}_0^*\|_\text{tr} = 1 \). Furthermore, since equality holds in (18) and the trace map is faithful, we have \( C^*C = I_k \) and \( \tilde{A}_0^* \tilde{A}_0 = I_n \). But \( C \in M_{n \times k}(\mathbb{C}) \) and \( \tilde{A}_0^* \in M_{n \times k}(\mathbb{C}), \) so \( n = k \), hence \( C \) and \( \tilde{A}_0 \) are unitary.

Writing \( D_\lambda = mCD_\mu \tilde{A}_0 = (mCA_0)(\tilde{A}_0^* D_\mu A_0) \), we observe that \( mCA_0 \) is unitary and \( \tilde{A}_0^* D_\mu A_0 \) is positive. Uniqueness of the right Polar Decomposition for the invertible matrix \( D_\lambda \) implies
\[
D_\lambda = \tilde{A}_0^* D_\mu A_0.
\]
Since the diagonal entries in \( D_\lambda \) and \( D_\mu \) are listed in increasing order, it follows that \( D_\lambda = D_\mu \), hence \( \rho_2 \) is of the form \( \rho_2(M) = \sum_{i=1}^n \lambda_i^2(g_i, M g_i) \). Defining a unitary \( U \in M_n(\mathbb{C}) \) by letting \( U g_i = f_i \) for all \( i \) and extending linearly, we observe that
\[
\rho_2(M) = \sum_{i=1}^n \lambda_i^2(U^* f_i, MU^* f_i) = \sum_{i=1}^n \lambda_i^2(f_i, UMU^* f_i) = \rho_1(UMU^*) \]
for all \( M \in M_n(\mathbb{C}) \). In other words, \( \phi_2 = (\phi_1)_U \). \( \square \)

In [3], Powers constructed \( E_0 \)-semigroups using boundary weights over \( L^2(0, \infty) \). It is routine to check that in our notation, these are the \( E_0 \)-semigroups arising from the boundary weight doubles \( (\iota_C, \eta) \), where \( \iota_C \) is the identity map on \( \mathbb{C} \) and \( \eta \) is any boundary weight over \( L^2(0, \infty) \).

**Corollary 5.5.** Let \( \phi : M_n(\mathbb{C}) \to M_n(\mathbb{C}) \) and \( \psi : M_k(\mathbb{C}) \to M_k(\mathbb{C}) \) be unital rank one \( q \)-pure maps, and let \( \nu \) and \( \eta \) be normalized unbounded boundary weights over \( L^2(0, \infty) \). Denote by \( \alpha^d \) and \( \beta^d \) the Bhat minimal dilations of the \( CP \)-flows induced by the boundary weight doubles \( (\phi, \nu) \) and \( (\psi, \mu) \), respectively.

If \( n \neq k \), then \( \alpha^d \) and \( \beta^d \) are not cocycle conjugate. In particular, if \( n \neq 1 \), then \( \alpha^d \) is not cocycle conjugate to the \( E_0 \)-semigroup induced by \( (\iota_C, \mu) \).

**Proof.** From the proof of Theorem 5.3, we know that every corner \( \gamma \) from \( \phi \) to \( \psi \) satisfies \( ||\gamma|| < 1 \) since \( n \neq k \). The result now follows from Lemma 5.3 \( \square \)

### 6. Invertible unital \( q \)-pure maps

Now that we have classified the unital \( q \)-pure maps on \( M_n(\mathbb{C}) \) of rank one, we explore the unital \( q \)-pure maps \( \phi \) which are invertible. In a stark contrast to the rank one case, we find that for a given normalized unbounded boundary weight of the form \( \nu(\sqrt{T - \Lambda(1)}B \sqrt{T - \Lambda(1)}) = (f, Bf) \) on \( L^2(0, \infty) \), the doubles \( (\phi, \nu) \) and \( (\psi, \nu) \) always induce cocycle conjugate \( E_0 \)-semigroups if \( \phi \) and \( \psi \) are unital invertible \( q \)-pure maps on
The inside of the integral above is the composition of completely positive maps, so we have

\[ e = \int_0^\infty e^{-s\psi}ds \]

Proof. Assume the hypotheses of the proposition, and let \( f \) be an invertible unital \( q \)-positive map.

\[ t(\psi + I)^{-1} = t\psi^{-1}(I + t\psi)^{-1} = (I + \frac{\psi}{t})^{-1} = I - \frac{\psi}{t} + \frac{\psi^2}{t^2} - \ldots \]

To prove the second statement, let \( \Psi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C}) \) be any unital conditionally negative map. Since \( \Psi \) is conditionally negative, it follows from a result of Evans and Lewis in \([5]\) that \( e^{-s\Psi} \) is completely positive for all \( s \geq 0 \). Therefore, \( ||e^{-s\Psi}|| = ||e^{-s\Psi}(I)|| = ||e^{-s}I|| = e^{-s} \) for all \( s \geq 0 \), and the integral \( \int_0^\infty e^{-s\Psi}ds \) converges. Observing that \( (d/ds)(-e^{-s\Psi}) = \Psi e^{-s\Psi} \), we find that

\[ \Psi\left(\int_0^\infty e^{-s\Psi}ds\right) = \int_0^\infty \Psi e^{-s\Psi}ds = \lim_{s \to \infty} (-e^{-s\Psi})_{0}^{s} = I, \]

so \( \Psi \) is invertible and \( \Psi^{-1} = \int_0^\infty e^{-s\Psi}ds \). Since \( \Psi^{-1} \) is the integral of completely positive maps, it is completely positive. Furthermore, we find that \( tI + \Psi \) is invertible for every \( t > 0 \) and that \( \Psi^{-1} \geq 0 \), since the following holds for all \( t > 0 \):

\[ \int_0^\infty e^{-st}e^{-s\Psi}ds = \int_0^\infty e^{-st(t+\Psi)}ds = (tI + \Psi)^{-1} = \Psi^{-1}(I + t\Psi^{-1})^{-1}. \]

Examine the inverse of a unital invertible \( q \)-positive map \( \phi \) is the key to finding the invertible \( q \)-subordinates of \( \phi \), as we find in the following proposition and corollary:

Proposition 6.1. If \( \phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C}) \) is an invertible unital \( q \)-positive map, then \( \phi^{-1} \) is conditionally negative. On the other hand, if \( \Psi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C}) \) is a unital conditionally negative map, then \( \Psi \) is invertible and \( \Psi^{-1} \) is \( q \)-positive.

Proof. Let \( \psi = \phi^{-1} \). Since \( \phi \) is self-adjoint, so is \( \psi \), and the first statement of the proposition now follows from the fact that for large positive \( t \) we have

\[ t\phi(I + t\phi)^{-1} = t\psi^{-1}(I + t\psi)^{-1} = t(\psi + tI)^{-1} = \left(I + \frac{\psi}{t}\right)^{-1} = I - \frac{\psi}{t} + \frac{\psi^2}{t^2} - \ldots \]

Proposition 6.2. Let \( \phi_1 : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C}) \) and \( \phi_2 : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C}) \) be an invertible unital \( q \)-positive map, and let \( \psi_1 = \phi_1^{-1} \). Suppose \( \psi_2 : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C}) \) is conditionally negative and \( \phi_2 - \psi_1 \) is completely positive. Then \( \psi_2 \) is invertible, and \( \phi_2 := (\psi_2)^{-1} \) satisfies \( \phi_1 \geq q \phi_2 \geq 0 \).

Proof. Assume the hypotheses of the proposition, and let \( s > 0 \) be arbitrary. Define a function \( f \) on \( \mathbb{R} \) by \( f(t) = e^{-s\psi_1}e^{(t-1)s\psi_2} \). The equality below is \( f(1) - f(0) = \int_0^1 f'(t)dt \):

\[ e^{-s\psi_1} - e^{-s\psi_2} = \int_0^1 se^{-t\psi_1}(\psi_2 - \psi_1)e^{(t-1)s\psi_2}dt. \]

The inside of the integral above is the composition of completely positive maps, so \( e^{-s\psi_1} - e^{-s\psi_2} \) is completely positive. This implies \( e^{-s\psi_1}(I) - e^{-s\psi_2}(I) \geq 0 \), so

\[ ||e^{-s\psi_2}|| = ||e^{-s\psi_2}(I)|| \leq ||e^{-s\psi_1}(I)|| = ||e^{-s}(I)|| = e^{-s}. \]
Now the argument given in the previous proposition shows that \( \int_0^\infty e^{-s\psi_2} ds \) converges and is equal to \( \psi_2^{-1} \). Letting \( \phi_2 = \psi_2^{-1} \), we observe that \( \phi_1 \geq_q \phi_2 \) since the quantity below is completely positive for every \( t \geq 0 \):

\[
\phi_1(I + t\phi_1)^{-1} - \phi_2(I + t\phi_2)^{-1} = \int_0^\infty e^{-st}(e^{-s\psi_1} - e^{-s\psi_2}) ds.
\]

\[ \square \]

**Corollary 6.3.** Let \( \phi_1 : M_n(\mathbb{C}) \to M_n(\mathbb{C}) \) be an invertible unital \( q \)-positive map, and let \( \phi_2 : M_n(\mathbb{C}) \to M_n(\mathbb{C}) \) be linear and invertible.

Then \( \phi_1 \geq_q \phi_2 \geq 0 \) if and only if \( \phi_2^{-1} - \phi_1^{-1} \) is conditionally negative and \( \phi_2^{-1} - \phi_1^{-1} \) is completely positive.

**Proof.** The backward direction follows from Proposition 6.2. Assume the hypotheses of the forward direction and let \( \psi_1 = \phi_1^{-1} \) and \( \psi_2 = \phi_2^{-1} \). Since \( \phi_2 \) is self-adjoint, so is \( \psi_2 \). For sufficiently large positive \( t \) we have

\[
t\phi_2(I + t\phi_2)^{-1} = \left(I + \frac{\psi_2}{t}\right)^{-1} = I - \frac{\psi_2}{t} + \frac{\psi_2^2}{t^2} - \ldots
\]

and

\[
t^2(\phi_1(I + t\phi_1)^{-1} - \phi_2(I + t\phi_2)^{-1}) = \psi_2 - \psi_1 + \left(\frac{\psi_2 - \psi_1}{t} + \frac{\psi_2^2 - \psi_1^2}{t^2} + \ldots\right).
\]

The first equation shows that \( \phi_2^{-1} - \phi_1^{-1} \) is conditionally negative, while the second shows that \( \phi_2^{-1} - \phi_1^{-1} \) is completely positive. \( \square \)

Now that we know how to find all invertible \( q \)-subordinates of an invertible unital \( q \)-positive map \( \phi \), we ask if there can be any other \( q \)-subordinates of \( \phi \). We will find that the answer is no (see Proposition 6.9). Proving this will require the use of some machinery (notably Lemma 6.8), which we now build.

**Definition 6.4.** For every \( \phi : M_n(\mathbb{C}) \to M_n(\mathbb{C}) \) and \( \epsilon \in [0, 1] \), we define a map \( \phi_\epsilon \) by \( \phi_\epsilon = \epsilon I + (1 - \epsilon)\phi \).

If \( \phi \) is \( q \)-positive, then \( \phi_\epsilon \) is invertible for all \( \epsilon \in (0, 1] \). In the lemmas that follow, we make frequent use of the fact that for all \( t \geq 0 \) we have

\[
t\phi(I + t\phi)^{-1} = I - (I + t\phi)^{-1}. \tag{19}
\]

We present a quick consequence of (19) for all \( a \geq 0 \) and \( b \geq 0 \):

\[
a(I + bt\phi)^{-1} = aI - abt\phi(I + bt\phi)^{-1}. \tag{20}
\]

**Lemma 6.5.** Let \( \phi : M_n(\mathbb{C}) \to M_n(\mathbb{C}) \) be completely positive. If \( \phi_{\epsilon_k} \geq_q 0 \) for some monotonically decreasing sequence \( \{\epsilon_k\} \) of positive real numbers tending to 0, then \( \phi \geq_q 0 \).
Proof. Assume the hypotheses of the lemma. Let \( k \) be arbitrary. Since \( \phi_{\epsilon_k} \geq q 0 \), we know \( I - (I + \phi_{\epsilon_k})^{-1} \) is completely positive for all \( t \geq 0 \). Noting that
\[
I - (I + t\phi_{\epsilon_k})^{-1} = I - \left( (1 + t\epsilon I) + (1 - \epsilon) t\phi \right)^{-1} = I - \frac{1}{1 + t\epsilon} \left( I + \frac{t(1 - \epsilon)}{1 + t\epsilon} \phi \right)^{-1}
\]
and substituting \( t' = t(1 - \epsilon_k)/(1 + \epsilon_k) \), we see
\[
I - (I + t\phi) = I - \frac{1}{1 + \left( \frac{t}{1 - \epsilon_k + 1/\epsilon_k} \right) t'} \left( I + t'\phi \right)^{-1}.
\]
Varying \( t \) throughout \([0, \infty)\), we find that the above equation is completely positive for all \( t' \in [0, -1 + 1/\epsilon_k] \). Of course, for any \( t' \in [0, -1 + 1/\epsilon_k] \), we have \( t' \in [0, -1 + 1/\epsilon_k] \) for all \( \ell \geq k \) by monotonicity of the sequence \( \{\epsilon_n\} \). Therefore, we may repeat the same argument to conclude that for any \( t' \in [0, -1 + 1/\epsilon_k] \), the map
\[
I - \frac{1}{1 + \epsilon_k t'} \left( I + t'\phi \right)^{-1}
\]
is completely positive for all \( \ell \geq k \).

Now fix any \( t' > 0 \), so \( t' \in (0, -1 + 1/\epsilon_k) \) for some \( k \in \mathbb{N} \). A straightforward computation shows that the sequence \( \{\epsilon_n\} \) defined by \( \epsilon_n = \epsilon_n/(1 - \epsilon_n + t'\epsilon_n) \) monotonically decreases to 0. From the previous paragraph, we know that the map
\[
I - \frac{1}{1 + \epsilon_k t'} \left( I + t'\phi \right)^{-1}
\]
is completely positive for all \( \ell \geq k \). Since \( \epsilon_n \downarrow 0 \) it follows that
\[
I - (I + t'\phi)^{-1}
\]
is completely positive. In other words, \( t'\phi(I + t'\phi)^{-1} \) is completely positive. Since \( t' > 0 \) was chosen arbitrarily and \( \phi \) is completely positive, the lemma follows. \( \square \)

**Lemma 6.6.** If \( \phi : M_n(\mathbb{C}) \to M_n(\mathbb{C}) \) and \( \phi \geq q 0 \), then \( \phi_{\epsilon} \geq q 0 \) for all \( \epsilon \in [0, 1) \).

Proof. Suppose that \( \phi \geq q 0 \), and let \( \epsilon \in [0, 1) \) be arbitrary. For each \( t > 0 \), we apply formula (20) to \( a = 1/(1 + t\epsilon) \) and \( b = t(1 - \epsilon)/(1 + t\epsilon) \) to find
\[
I - (I + t\phi_{\epsilon})^{-1} = I - \frac{1}{1 + t\epsilon} \left( I + \frac{t(1 - \epsilon)}{1 + t\epsilon} \phi \right)^{-1} = \left( 1 - \frac{1}{1 + t\epsilon} \right) I + \frac{t(1 - \epsilon)}{(1 + t\epsilon)^2} \phi \left( I + \frac{t(1 - \epsilon)}{1 + t\epsilon} \phi \right)^{-1},
\]
where both terms on the last line are completely positive by assumption. Furthermore, \( \phi_{\epsilon} \) is completely positive, hence \( \phi_{\epsilon} \geq q 0 \). \( \square \)

**Corollary 6.7.** Let \( \phi : M_n(\mathbb{C}) \to M_n(\mathbb{C}) \) be a completely positive map. Then \( \phi \geq q 0 \) if and only if \( \phi_{\epsilon} \geq q 0 \) for all \( \epsilon \in (0, 1) \).
Lemma 6.8. Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ and $\psi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be $q$-positive maps. Then $\phi \geq_q \psi$ if and only if $\phi_\epsilon \geq_q \psi_\epsilon$ for all $\epsilon \in (0, 1)$.

**Proof.** For any $\epsilon \in (0, 1)$ we have $\phi_\epsilon - \psi_\epsilon = \epsilon(\phi - \psi)$, so $\phi - \psi$ is completely positive if and only if $\phi_\epsilon - \psi_\epsilon$ is completely positive for all $\epsilon \in (0, 1)$. For all $t' > 0$ we have

$$t'(\phi(I + t'\phi)^{-1} - \psi(I + t'\psi)^{-1}) = (I + t'\psi)^{-1} - (I + t'\phi)^{-1},$$

(21)

and for all $t > 0$ we have

$$t(\phi, I + t\phi)^{-1} - \psi(I + t\psi)^{-1} = \left( I - (I + t\phi)^{-1} \right) - \left( I - (I + t\psi)^{-1} \right)$$

$$= \frac{1}{1 + t\epsilon} \left( I + t(1 - \epsilon) \psi \right)^{-1} - (I + t(1 - \epsilon) \phi)^{-1).$$

(22)

Assume the hypotheses of the forward direction. Showing that $\phi_\epsilon \geq_q \psi_\epsilon$ for all $\epsilon \in (0, 1)$ is equivalent to proving that (22) is completely positive for every $t \in (0, \infty)$ and $\epsilon \in (0, 1)$. But this follows from complete positivity of (21) since $t(1 - \epsilon)/(1 + t\epsilon) \in (0, \infty)$ for every $\epsilon \in (0, 1)$ and $t \in (0, \infty)$. Now assume the hypotheses of the backward direction. Any $t' \in (0, \infty)$ can be written as $t(1 - \epsilon)/(1 + t\epsilon)$ for some $\epsilon \in (0, 1)$ and $t \in (0, \infty)$, so complete positivity of (22) for all such $\epsilon$ and $t$ implies that (21) is completely positive for all $t' > 0$, hence $\phi \geq_q \psi$. □

We are now in a position to prove what is perhaps the most striking result of the section:

**Proposition 6.9.** Let $\xi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be an invertible unital $q$-positive map. If $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is $q$-positive and $\xi \geq_q \phi$, then $\phi$ is either invertible or identically zero.

**Proof.** For every $\epsilon \in (0, 1)$, form $\xi_\epsilon$ and $\phi_\epsilon$ as in Definition 6.1 and let $\psi_\epsilon := (\phi_\epsilon)^{-1}$. By Lemma 6.8 we have $\xi_\epsilon \geq_q \phi_\epsilon$ for each $\epsilon$, so $\psi_\epsilon$ is conditionally negative and $\psi_\epsilon - (\xi_\epsilon)^{-1}$ is completely positive by Corollary 6.3. We first examine the case when the norms $||\psi_\epsilon||$ remain bounded as $\epsilon \to 0$. More precisely, suppose that for all $\epsilon$ sufficiently small we have $||\psi_\epsilon|| < r$ for some $r > 0$. By compactness of the closed unit ball of radius $r$ in $B(M_n(\mathbb{C}))$, there is a decreasing sequence $\{\epsilon_k\}_{k \in \mathbb{N}}$ converging to 0 such that $\{\psi_{\epsilon_k}\}_{k \in \mathbb{N}}$ has a (bounded) norm limit $\psi$ as $k \to \infty$. Noting that

$$I - \phi \psi = \phi_{\epsilon_k} \psi_{\epsilon_k} - \phi \psi = (\phi_{\epsilon_k} - \phi)(\psi_{\epsilon_k} - \psi) + \phi(\psi_{\epsilon_k} - \psi) + (\phi_{\epsilon_k} - \phi)\psi$$

and then applying the triangle inequality, we find that

$$||I - \phi \psi|| = ||\phi_{\epsilon_k} \psi_{\epsilon_k} - \phi \psi|| \leq ||\phi_{\epsilon_k} - \phi|| ||\psi_{\epsilon_k} - \psi|| + ||\phi|| ||\psi_{\epsilon_k} - \psi|| + ||\phi_{\epsilon_k} - \phi|| ||\psi||$$

for all $k \in \mathbb{N}$. But $\phi$ and $\psi$ are bounded maps while $\psi_{\epsilon_k} \to \psi$ in norm and $\phi_{\epsilon_k} \to \phi$ in norm, so the above equation tends to 0 as $k \to \infty$. We conclude that $\phi \psi = I$. Similarly $\psi \phi = I$, hence $\phi$ is invertible and $\psi = \phi^{-1}$. 30
If the first case does not hold, then for some decreasing sequence \( \{ \epsilon_k \} \) tending to zero, the norms \( \| \psi_{\epsilon_k} \| \) form an unbounded sequence. For each \( k \in \mathbb{N} \), we write

\[
(\xi_{\epsilon_k})^{-1}(A) = s_k A + Y_k A + A Y_k^* - \sum_{i=1}^{m_k} S_{k_i} A S_{k_i}^*,
\]

and

\[
\psi_{\epsilon_k}(A) = t_k A + Z_k A + A Z_k^* - \sum_{i=1}^{\ell_k} T_{k_i} A T_{k_i}^*,
\]

where \( m_k, \ell_k \leq n^2 \), \( s_k, t_k \in \mathbb{R} \), \( t_r(Y_k) = tr(Z_k) = 0 \), \( tr(S_{k_i}) = 0 \) and \( tr(S_{k_i}^* S_{k_i}) \) is non-zero if and only if \( i = j \) \((i, j \leq m_k)\), and \( tr(T_{k_i}) = 0 \) and \( tr(T_{k_i}^* T_{k_i}) \) is non-zero if and only if \( i = j \) \((i, j \leq \ell_k)\).

Since \( \psi_{\epsilon_k} - (\xi_{\epsilon_k})^{-1} \) is completely positive for all \( k \in \mathbb{N} \), we know that for each \( k \), there exist \( p_k \leq n^2 \), complex numbers \( \{ x_{k_i} \}_{i=1}^{p_k} \), and maps \( \{ X_{k_i} \}_{i=1}^{p_k} \) with \( tr(X_{k_i}) = 0 \), such that for all \( A \in M_n(\mathbb{C}) \),

\[
(\psi_{\epsilon_k} - (\xi_{\epsilon_k})^{-1})(A) = \sum_{i=1}^{p_k} (X_{k_i} + x_{k_i} I) A (X_{k_i} + x_{k_i} I)^* \]

\[
= \left( \sum_{i=1}^{p_k} |x_{k_i}|^2 \right) A + \left( \sum_{i=1}^{p_k} \overline{x_{k_i}} X_{k_i} \right) A + A \left( \sum_{i=1}^{p_k} \overline{x_{k_i}} X_{k_i} \right)^* \]

\[
+ \sum_{i=1}^{p_k} X_{k_i} A X_{k_i}^*. \quad (23)
\]

Simultaneously, for all \( A \in M_n(\mathbb{C}) \) we have

\[
(\psi_{\epsilon_k} - (\xi_{\epsilon_k})^{-1})(A) = (t_k - s_k) A + (Z_k - Y_k) A + A (Z_k - Y_k)^* \]

\[
+ \left( \sum_{i=1}^{m_k} S_{k_i} A S_{k_i}^* - \sum_{i=1}^{\ell_k} T_{k_i} A T_{k_i}^* \right). \quad (24)
\]

We claim that

\[
\left\| \sum_{i=1}^{p_k} X_{k_i} A X_{k_i}^* \right\| \leq \left\| \sum_{i=1}^{m_k} S_{k_i} A S_{k_i}^* \right\|. \quad (25)
\]

for all \( k \in \mathbb{N} \). To prove this, we let \( \{ v_j \}_{j=1}^{n} \) be any orthonormal basis for \( \mathbb{C}^n \), let \( h_j = v_j / \sqrt{n} \) for each \( i \), let \( f \in \mathbb{C}^n \) be arbitrary, and define maps \( A_j \) for \( j = 1, \ldots, n \) by \( A_j = f h_j^* \). Using the trace conditions on the maps \( Y_k, Z_k, \{ T_{k_i} \}, \{ S_{k_i} \}, \) and \( \{ X_{k_i} \} \), we find that

\[
\sum_{j=1}^{n} (\psi_{\epsilon_k} - (\xi_{\epsilon_k})^{-1})(A_j) h_j = (t_k - s_k) f + (Z_k - Y_k) f \]

\[
= \left( \sum_{i=1}^{p_k} |x_{k_i}|^2 \right) f + \left( \sum_{i=1}^{p_k} \overline{x_{k_i}} X_{k_i} \right) f. \]
Since $f$ was arbitrary, it follows that
\[
(t_k - s_k - \sum_{i=1}^{p_k} |x_{k_i}|^2) I = \left( \sum_{i=1}^{p_k} x_{k_i} X_{k_i} \right) - (Z_k - Y_k).
\]
Taking the trace of both sides yields
\[
0 = tr\left( \left( \sum_{i=1}^{p_k} x_{k_i} X_{k_i} \right) - (Z_k - Y_k) \right) = tr\left( (t_k - s_k - \sum_{i=1}^{p_k} |x_{k_i}|^2) I \right),
\]
so $t_k - s_k = \sum_{i=1}^{p_k} |x_{k_i}|^2$ and $Z_k - Y_k = \sum_{i=1}^{p_k} x_{k_i} X_{k_i}$. Formulas (23) and (24) now imply that
\[
\sum_{i=1}^{p_k} X_{k_i} A X_{k_i}^* = \left( \sum_{i=1}^{m_k} S_{k_i} A S_{k_i}^* - \sum_{i=1}^{\ell_k} T_{k_i} A T_{k_i}^* \right).
\]
Therefore, the map $A \rightarrow \sum_{i=1}^{m_k} S_{k_i} A S_{k_i}^* - \sum_{i=1}^{\ell_k} T_{k_i} A T_{k_i}^*$ is completely positive, and
\[
\left\| \sum_{i=1}^{p_k} X_{k_i} X_{k_i}^* \right\| = \left\| \sum_{i=1}^{m_k} S_{k_i} S_{k_i}^* - \sum_{i=1}^{\ell_k} T_{k_i} T_{k_i}^* \right\| \leq \left\| \sum_{i=1}^{m_k} S_{k_i} S_{k_i}^* \right\|,
\]
establishing (25).

We now show that there exists some $M \in \mathbb{N}$ such that
\[
||X_{k_i}|| \leq M \tag{26}
\]
for all $k \in \mathbb{N}$ and $i \in \{1, \ldots, p_k\}$. To do this, we first note that since the sequence of invertible maps $\{\xi_k\}_{k \in \mathbb{N}}$ converges in norm to the invertible map $\xi$, the sequence $\{\xi^{-1}_k\}_{k \in \mathbb{N}}$ converges in norm to $\xi^{-1}$. Write $\xi^{-1}$ in the form
\[
\xi^{-1}(A) = s A + Y A + A Y^* - \sum_{i=1}^{m} S_i A S_i^*,
\]
where $m \leq n^2$, $s \in \mathbb{R}$, $tr(Y) = 0$, and for all $i$ and $j$, $tr(S_i) = 0$ and $tr(S_i S_j^*)$ is non-zero if and only if $i = j$. Let $f \in \mathbb{C}^n$ be arbitrary, and define vectors $\{h_j\}_{j=1}^n$ and maps $\{A_j\}_{j=1}^p$ exactly as we did earlier in the proof. Then $\sum_{j=1}^{n} \xi^{-1}_k(A_j) h_j = s_k f + Y_k f$ for all $k \in \mathbb{N}$ and $\sum_{j=1}^{n} \xi^{-1}_k(A_j) h_j = s f + Y f$. Since $(\xi^{-1}_k)^{-1}$ converges to $\xi^{-1}$ as $k \to \infty$, we see that $(s_k - s) f + (Y_k - Y) f$ converges to $0$ as $k \to \infty$. But $f$ was arbitrary, so
\[
\lim_{k \to \infty} \left( (s_k - s) I + Y_k - Y \right) = 0.
\]
The limit of the trace of the above equation must also be zero, so $s_k$ converges to $s$ and consequently $Y_k$ converges to $Y$. This implies that not only are the sequences of complex numbers $\{s_k\}_{k=1}^\infty$ and maps $\{Y_k\}_{k=1}^\infty$ both bounded, but that the sequence of linear maps $\{W_k\}_{k=1}^\infty$ defined by $W_k(A) = \sum_{i=1}^{m_k} S_{k_i} A S_{k_i}^*$ is bounded and converges to the map $W(A) = \sum_{i=1}^{m} S_i A S_i^*$. Choose $M \in \mathbb{N}$ so that $M^2 \geq n^2 \sup_{k \in \mathbb{N}} \{|W_k||\}$. For
We note that for all \( k \), we find that for every \( k \) and only if \( k \) every Eq.(29) is positive, we have

\[
\text{Proposition 6.10.} \quad \text{An invertible unital linear map} \quad \phi \in M_n(\mathbb{C}) \quad \text{is q-pure if and only if} \quad \phi^{-1} \quad \text{is of the form}
\]

\[
\phi^{-1}(A) = A + YA + AY^* \quad \text{for some} \quad Y = -Y^* \in M_n(\mathbb{C}) \quad \text{such that} \quad \text{tr}(Y) = 0.
\]

for some \( Y = -Y^* \in M_n(\mathbb{C}) \) such that \( \text{tr}(Y) = 0. \)
Proof. Let $\psi = \phi^{-1}$. Assume the hypotheses of the forward direction. Write

$$\psi(A) = sA + YA + AY^* = \sum_{i=1}^{k} \lambda_i X_i AX_i^*,$$

where $s \in \mathbb{R}$, $\text{tr}(Y) = 0$, and for each $i$ and $j$ we have $\lambda_i \geq 0$, $\text{tr}(X_i) = 0$, and $\text{tr}(X_i^* X_j) = n\delta_{ij}$.

Defining $\psi' : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by

$$\psi'(A) = sA + YA + AY^*,$$

we note that $\psi'$ is conditionally negative, and $\psi' - \psi$ is completely positive since $(\psi' - \psi)(A) = \sum_{j=1}^{k} \lambda_j X_j AX_j^*$ for all $A$. By Lemma 6.2, it follows that $\psi'$ is invertible and that $\phi' := (\psi')^{-1}$ satisfies $\phi \geq q$, $\phi' \geq q$, 0.

Since $\phi$ is $q$-pure, there is some $t_0 \geq 0$ such that $\phi' = \phi(t_0)$, hence

$$\psi' = (\phi')^{-1} = \left(\phi(I + t_0 \phi)^{-1}\right)^{-1} = \left(\psi^{-1}(I + t_0 \psi)^{-1}\right)^{-1} = \left((t_0 I + \psi)^{-1}\right)^{-1} = t_0 I + \psi.$$

Therefore, for all $A \in M_n(\mathbb{C})$ we have

$$\psi'(A) = \psi(A) + \sum_{j=1}^{k} \lambda_j X_j AX_j^* = \psi(A) + t_0 A,$$

so the map $L : A \to \lambda_j X_j AX_j^*$ satisfies $L = t_0 I$. We repeat a familiar argument: Let $f \in \mathbb{C}^n$ be arbitrary, choose an orthonormal basis $\{v_k\}_{k=1}^{n} \subset \mathbb{C}^n$, define $h_k = v_k / \sqrt{\langle f^* h_k \rangle}$ for each $k$, and form $\{A_k\}_{k=1}^{n}$ by $A_k = f h_k^*$. The trace conditions for the maps $\{X_j\}$ imply that $\sum_{k=1}^{n} L(A_k) h_k = 0$. However, since $L = t_0 I$, we must also have $\sum_{k=1}^{n} L(A_k) h_k = t_0 f$. From arbitrariness of $f$, we conclude $t_0 = 0$. Therefore, $\psi'$ has the form $\psi(A) = sA + YA + AY^*$. Since $\psi(I) = I = sI + Y + Y^*$ and $\text{tr}(Y) = 0$, we have $s = 1$ and consequently $Y = -Y^*$.

Now assume the hypotheses of the backward direction. Note that $\psi$ is conditionally negative and unital, hence $\phi$ is q-positive by Proposition 5.1. Let $\Phi$ be any non-zero $q$-positive map such that $\phi \geq q$, $\Phi$, so by Corollary 6.5 and Proposition 5.3 $\Phi$ is invertible and $\Psi := (\Phi)^{-1}$ is a conditionally negative map such that $\Psi - \psi$ is completely positive. Write $\Psi$ in the form

$$\Psi(A) = s'A + ZA + AZ^* - \sum_{i=1}^{m} \mu_i T_i AT_i^*,$$

where $s' \in \mathbb{R}$ and for all $i$ and $j$, $\mu_i > 0$, $\text{tr}(T_i) = 0$, and $\text{tr}(T_i^* T_j) = n\delta_{ij}$. Writing $C = Z - Y$, we have

$$(\Psi - \psi)(A) = (s' - 1)A + CA + AC^* - \sum_{i=1}^{m} \mu_i T_i AT_i^*.$$ 

By a familiar argument, complete positivity of $\Psi - \psi$ and the trace conditions for the above maps imply that $s' \geq 1$, $C = 0$, and $T_i = 0$ for all $i$. Therefore $\Psi = \psi + (s' - 1)I$, so $\Phi = \Psi^{-1} = \phi(s' - 1)$. We conclude that $\phi$ is $q$-pure. \qed
Let the matrices \( \{ e_{jk} \}_{j,k=1}^n \) denote the standard basis for \( M_n(\mathbb{C}) \), writing each \( A = (a_{jk}) \in M_n(\mathbb{C}) \) as \( A = \sum_{j,k} a_{jk} e_{jk} \). The following theorem classifies all unital invertible \( q \)-pure maps on \( M_n(\mathbb{C}) \):

**Theorem 6.11.** An invertible unital linear map \( \phi : M_n(\mathbb{C}) \to M_n(\mathbb{C}) \) is \( q \)-pure if and only if for some unitary \( U \in M_n(\mathbb{C}) \), the map \( \phi_U \) is the Schur map

\[
\phi_U(a_{jk} e_{jk}) = \begin{cases} 
\frac{a_{jk}}{1 + i(\lambda_j - \lambda_k)} e_{jk} & \text{if } j < k \\
\frac{a_{jk}}{1 - i(\lambda_j - \lambda_k)} e_{jk} & \text{if } j = k \\
\frac{a_{jk}}{1 - i(\lambda_j - \lambda_k)} e_{jk} & \text{if } j > k
\end{cases}
\]

for all \( A = (a_{jk}) \in M_n(\mathbb{C}) \) and \( j, k = 1, \ldots, n \), where \( \lambda_1, \ldots, \lambda_n \in \mathbb{R} \) and \( \lambda_1 + \ldots + \lambda_n = 0 \).

**Proof.** Assume the hypotheses of the forward direction. By the previous proposition, \( \psi := \phi^{-1} \) has the form \( \psi(A) = A + \bar{Y} A + A \bar{Y}^* \) for some \( \bar{Y} \in M_n(\mathbb{C}) \) with \( \bar{Y} = -\bar{Y}^* \) and \( \text{tr}(\bar{Y}) = 0 \). Let \( B = -i\bar{Y} \), so \( B = B^* \). Defining \( Y := (1/2) I + \bar{Y} = (1/2) I + iB \), we find \( \psi(A) = Y A + A Y^* \) for all \( A \in M_n(\mathbb{C}) \). Since \( B \) is self-adjoint, there is some unitary \( U \in M_n(\mathbb{C}) \) such that \( U^* BU \) is a diagonal matrix \( D \). For each \( k \in \{ 1, \ldots, n \} \) let \( \lambda_k \in \mathbb{R} \) be the \( kk \) entry of \( D \). Note that since \( \text{tr}(B) = 0 \) we have \( \sum_{k=1}^n \lambda_k = 0 \), and that \( U^* Y U \) is the diagonal matrix \( M \) whose \( kk \) entry is \( 1/2 + i\lambda_k \). Defining a map \( \psi_U \) by \( \psi_U(A) = U^* \psi(U A U^*) U \) for all \( A \in M_n(\mathbb{C}) \), we find that

\[
\psi_U(A) = U^* (Y A U^* + U A U^*) U = (U^* Y U) A + A (U^* Y U)^* = MA + A M^*.
\]

A quick calculation shows that this is just the Schur map

\[
\psi_U(a_{jk} e_{jk}) = \begin{cases} 
(1 + i(\lambda_j - \lambda_k))a_{jk} e_{jk} & \text{if } j < k \\
a_{jk} e_{jk} & \text{if } j = k \\
(1 - i(\lambda_j - \lambda_k))a_{jk} e_{jk} & \text{if } j > k
\end{cases}
\]

and so \( (\psi_U)^{-1} \) has the form

\[
(\psi_U)^{-1}(a_{jk} e_{jk}) = \begin{cases} 
\frac{a_{jk}}{1 + i(\lambda_j - \lambda_k)} e_{jk} & \text{if } j < k \\
\frac{a_{jk}}{1 - i(\lambda_j - \lambda_k)} e_{jk} & \text{if } j = k \\
\frac{a_{jk}}{1 - i(\lambda_j - \lambda_k)} e_{jk} & \text{if } j > k
\end{cases}
\]

It is straightforward to verify that \( (\psi_U)^{-1} \) is the map \( \phi_U(A) = U^* \phi(U A U^*) U \).

Assume the hypotheses of the backward direction. Let \( T \) be the diagonal matrix whose \( kk \)th entry is \( \lambda_k \) for every \( k = 1, \ldots, n \). We observe that \( \text{tr}(T) = 0 \) and \( T = T^* \). Now let \( C = iT \), and let \( \bar{T} = (1/2) I + C \). We routinely verify that \( C = -C^* \) and \( \text{tr}(C) = 0 \), and that \( (\phi_U)^{-1} \) satisfies \( (\phi_U)^{-1}(A) = \bar{T} A + A \bar{T}^* = A + CA + AC^* \) for all \( A \in M_n(\mathbb{C}) \). Proposition 6.10 implies that \( \phi_U \) is \( q \)-pure, whereby \( \phi \) is \( q \)-pure by Proposition 4.5. \( \square \)

As it turns out, boundary weight doubles \( (\phi, \nu) \) for invertible unital \( q \)-pure maps \( \phi : M_n(\mathbb{C}) \to M_n(\mathbb{C}) \) and normalized unbounded boundary weights \( \nu \) over \( L^2(0, \infty) \) of the form \( \nu(\sqrt{T - \lambda(1)} B \sqrt{T - \lambda(1)}) = (f, B f) \) give us nothing new in terms of \( E_0 \)-semigroups:

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Theorem 6.12. Let \( \phi : M_n(\mathbb{C}) \to M_n(\mathbb{C}) \) be unital, invertible, and \( q \)-pure, and let \( \nu \) be a normalized unbounded boundary weight over \( L^2(0, \infty) \) of the form \( \nu(\sqrt{T - \Lambda(1)B\sqrt{T - \Lambda(1)}) = (f,B,f) \). Then \( (\phi, \nu) \) and \( (\iota_\mathbb{C}, \nu) \) induce cocycle conjugate \( E_0 \)-semigroups.

Proof. By Theorem 6.11 and Propositions 4.5 and 4.6, we may assume that \( \phi \) is the Schur map

\[
\phi(a_{jk}e_{jk}) = \begin{cases} \frac{a_{jk}}{1+1(\lambda_j-\lambda_k)}e_{jk} & \text{if } j < k \\ a_{jk}e_{jk} & \text{if } j = k \\ \frac{a_{jk}}{1+1(\lambda_j-\lambda_k)}e_{jk} & \text{if } j > k \end{cases}
\]

for some \( \lambda_1, \ldots, \lambda_n \in \mathbb{R} \) with \( \sum_{k=1}^n \lambda_k = 0 \). By Proposition 4.6, it suffices to find a hyper maximal \( q \)-corner from \( \phi \) to \( \iota_\mathbb{C} \). For this, define \( \gamma : M_{n+1}(\mathbb{C}) \to M_{n+1}(\mathbb{C}) \) by

\[
\gamma \left( \begin{array}{c} b_1 \\ b_2 \\ \vdots \\ b_n \end{array} \right) = \left( \begin{array}{c} \frac{b_1}{1+1(\lambda_1-\lambda_2)}b_2 \\ \frac{b_1}{1+1(\lambda_2-\lambda_3)}b_3 \\ \vdots \\ \frac{b_1}{1+1(\lambda_n-\lambda_1)}b_n \end{array} \right).
\]

Now define \( \Upsilon : M_{n+1}(\mathbb{C}) \to M_{n+1}(\mathbb{C}) \) by

\[
\Upsilon \left( \begin{array}{c} A_{n\times n} \\ B_{n\times 1} \\ C_{1\times n} \end{array} \right) = \left( \begin{array}{c} \phi(A_{n\times n}) \\ \gamma(C_{1\times n}) \end{array} \right).
\]

Letting \( \lambda_{n+1} = 0 \), we observe that \( \Upsilon \) is the Schur map satisfying

\[
\Upsilon(a_{jk}e_{jk}) = \begin{cases} \frac{a_{jk}}{1+1(\lambda_j-\lambda_k)}e_{jk} & \text{if } j < k \\ a_{jk}e_{jk} & \text{if } j = k \\ \frac{a_{jk}}{1+1(\lambda_j-\lambda_k)}e_{jk} & \text{if } j > k \end{cases}
\]

for all \( j, k = 1, \ldots, n+1 \) and \( A = (a_{jk}) \in M_n(\mathbb{C}) \). Since \( \sum_{k=1}^{n+1} \lambda_k = \sum_{k=1}^n \lambda_k = 0 \), it follows from Theorem 6.11 that \( \Upsilon \) is \( q \)-positive (in fact, \( q \)-pure), hence \( \gamma \) is a \( q \)-corner from \( \phi \) to \( \iota_\mathbb{C} \). Now suppose that \( \Upsilon' \geq_q \Upsilon \geq_0 \) for some \( \Upsilon' \) of the form

\[
\Upsilon' \left( \begin{array}{c} A_{n\times n} \\ B_{n\times 1} \\ C_{1\times n} \end{array} \right) = \left( \begin{array}{c} \phi'(A_{n\times n}) \\ \gamma'(C_{1\times n}) \end{array} \right).
\]

Since \( \Upsilon \) is \( q \)-pure and \( \Upsilon' \) is not the zero map, we know that \( \Upsilon' = \Upsilon^{(t)} \) for some \( t \geq 0 \), and a quick calculation gives us

\[
\Upsilon' \left( \begin{array}{c} A_{n\times n} \\ B_{n\times 1} \\ C_{1\times n} \end{array} \right) = \left( \begin{array}{c} \phi^{(t)}(A_{n\times n}) \\ (\gamma^t)(C_{1\times n}) \end{array} \right).
\]

By inspecting the two formulas for \( \Upsilon' \) we see \( \gamma = \gamma^{(t)} \). But \( \gamma^{(t)} \) has the form

\[
\gamma^{(t)} \left( \begin{array}{c} b_1 \\ b_2 \\ \vdots \\ b_n \end{array} \right) = \left( \begin{array}{c} \frac{b_1}{1+1+1}(\lambda_1-\lambda_2)b_2 \\ \frac{b_1}{1+1+1}(\lambda_2-\lambda_3)b_3 \\ \vdots \\ \frac{b_1}{1+1+1}(\lambda_n-\lambda_1)b_n \end{array} \right),
\]

hence \( t = 0 \). Therefore, \( \Upsilon' = \Upsilon \), and we conclude the \( q \)-corner \( \gamma \) is hyper maximal. \( \Box \)
In conclusion, we approach the broader question of simply finding all unital $q$-pure maps $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$, as they provide us with the simplest way to construct and compare $E_0$-semigroups through boundary weight doubles. We believe that all $q$-pure maps are invertible or have rank one. For $n = 2$, we find in [6] that this conjecture holds: There is no unital $q$-pure map $\phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ of rank 2, and there is no unital $q$-positive map $\phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ of rank 3. It seems that for $n = 3$, the key to classifying unital $q$-pure maps is through investigation of the limits $L_\phi = \lim_{t \to \infty} t\phi(I + t\phi)^{-1}$, though the situation becomes very complicated if $n > 3$.

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[1] W.B. Arveson, The Index of a Quantum Dynamical Semigroup, J. Funct. An. 146 (1997), 557-588.
[2] W.B. Arveson, Continuous Analogues of Fock space, Memoirs Amer. Math. Soc. 80, no. 409 (1989).
[3] B.V.R. Bhat, An index theory for quantum dynamical semigroups, Trans. A.M.S. 348 (1996), no. 2, 561-583.
[4] M. Choi, Completely positive linear maps on complex matrices, Lin. Alg. Appl. 10 (1975), 285-290.
[5] D.E. Evans and J.T. Lewis, Dilations of irreversible evolutions in algebraic quantum theory, Comm. Dubl. Inst. Adv. Studies Ser A 24 (1977).
[6] C. Jankowski, Unital $q$-pure maps on $M_2(\mathbb{C})$, in preparation.
[7] D. Markiewicz and R.T. Powers, Local unitary cocycles of $E_0$-semigroups, J. Funct. An. 256 (2009), no. 5, 1511-1543.
[8] R.T. Powers, Continuous spatial semigroups of completely positive maps of $B(H)$, New York J. Math. 9 (2003), 165-269.
[9] R.T. Powers, Construction of $E_0$-semigroups of $B(\mathcal{H})$ from CP-flows, Advances in Quantum Dynamics, Contemp. Math. 335, Amer. Math. Soc., Providence, RI (2003), 57-97.
[10] R.T. Powers, Induction of semigroups of endomorphisms of $B(\mathcal{H})$ from completely positive semigroups of $(n \times n)$ matrix algebras, Internat. J. Math. 10 (1999), no. 7, 773-790.
[11] R.T. Powers, New examples of continuous spatial semigroups of $*$-endomorphisms of $B(\mathcal{H})$, Internat. J. Math. 10 (1999), no. 2, 215-288.
[12] R.T. Powers, An index theory for semigroups of $*$-endomorphisms of $B(\mathcal{H})$ and type $II_1$ factors, Can. Jour. Math. 40 (1988), 86-114.
[13] R.T. Powers, A nonspatial continuous semigroup of $*$-endomorphisms of $B(\mathcal{H})$, Publ. Res. Inst. Math. Sci. 23 (1987), no. 6, 1053-1069.
[14] R.T. Powers and G. Price, Continuous spatial semigroups of $*$-endomorphisms of $B(\mathcal{H})$, Trans. A.M.S. 321 (1990), 347-361.
[15] B. Tsirelson, Non-isomorphic product systems, Advances in Quantum Dynamics, Contemp. Math. 335, Amer. Math. Soc., Providence, RI (2003), 273-328.
[16] E.P. Wigner, On unitary representations of the inhomogeneous Lorentz group, Ann. of Math. 40 (1939), 149-204.
7. Notes on Version 2

The text of this version should be identical to that of the version appearing in the Journal of Functional Analysis.

There is an update for reference [6]:
C. Jankowski, Unital \( q \)-positive maps on \( M_2(\mathbb{C}) \) and a related \( E_0 \)-semigroup result, arXiv:1005.4404v1.

The author apologizes that some typos remain:

In Theorem 2.2, the equation of minimality should read “\( \alpha d_1(WA_1W^*) \cdots \alpha d_n(WA_nW^*)Wf, \)” and we should have “\( f \in H_1 \)” rather than “\( f \in K \).”

In the last equation of the proof of Lemma 3.5, the \( S'_i \) and \( T'_i \) terms should be \( \tilde{S}_i \) and \( \tilde{T}_i \) terms. In this same equation, the numbers \( a_{ij} \) should be \( r_{ij} \).

The last line of page 23 should read “\( \ell((\rho_{12})'(A) = \ell((\rho_{12})\left(W_i[W_i^*AX_i]X_i^*\right) = 0. \)”