Clock Monte Carlo methods

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We propose the clock Monte Carlo technique for sampling each successive step in constant time, built on a recently proposed factorized transition filter. The core features include its O(1) computational complexity and its generality. We elaborate how it leads to the clock factorized Metropolis (clock FMet) method, and discuss its application in other update schemes. By grouping interaction terms into boxes of tunable sizes, we further formulate a variant of clock FMet algorithm, with the limiting case of a single box reducing to the standard Metropolis method. Theoretical analysis shows that an overall acceleration of O(Nκ) (0 ≤ κ ≤ 1) can be achieved compared to the Metropolis method, where N is the system size and the κ value depends on the nature of the energy extensivity. As a systematic test, we simulate long-range O(n) spin models in a wide parameter regime: for n = 1, 2, 3, with disordered algebraically decaying or oscillatory RKKY-type interactions and with/without external fields, and in spatial dimensions from d = 1, 2, 3 to mean-field. The O(1) computational complexity is demonstrated, and the expected acceleration is confirmed. Its flexibility and its independence from interaction range guarantee that the clock method would find decisive applications in systems with many interaction terms.

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Markov-chain Monte Carlo methods (MCMC) have proven to be powerful tools in many branches of science and engineering [1,8]. For instance, MCMC plays a crucial role in the recent success of AlphaGo [9], and appears as a keystone of the potential next deep learning revolution [10, 11]. To estimate high-dimensional integrals, MCMC generates a chain of random configurations, called samples. The stationary distribution is typically a Boltzmann distribution and the successive steps of the chain. As the factorized filter allows for an independent treatment of each interaction term, it suffices to sample only one rejecting term to decide the rejection of a proposed move (Fig. 1). Besides being simple and powerful, the clock technique is general for various update schemes. We formulate the clock factorized Metropolis (FMet) algorithm, and discuss its applications in the EC and cluster methods. Further, by grouping the interaction terms into boxes of tunable sizes, we obtain a variant of clock FMet algorithm for efficiency optimization; the limiting case of a single box recovers the Metropolis method. We classify the system into three types of strict, marginal and sub-extensivity, and show that an overall acceleration can be achieved up to O(N) for the strict extensivity and O(Nκ) (1 > κ ≥ 0) for the other two. This is supported by simulations of long-range O(n)-spin models in a wide parameter regime.

Clock FMet algorithm. Consider a system described by a collection of states S with Boltzmann weight \( \pi(S) \propto \exp(-\beta E(S)) \), with \( \beta = 1/k_B T \) the inverse temperature. The energy \( E(S) = \sum_i E_i(S) \) is the sum of all interaction terms that are pairwise or more generally in

\[ \begin{align*}
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& \quad \end{align*} \]
many-body groups. At each step, the Metropolis algorithm attempts to update a state $S$ into another $S'$ with acceptance probability
\[
P_{\text{Met}} = \min \left(1, \frac{\pi(S')}{\pi(S)} \right) = \exp(-\beta |\Delta E_{\text{tot}}|) \tag{1}\]
with $[x]^+ = \max(0,x)$. Evaluating the induced energy change $\Delta E_{\text{tot}} = \sum_i \Delta E_i$ requires a costly computation of all the involved interactions. Therefore, we focus now on the recent factorized Metropolis filter \[\text{(1)}\]
\[
P_{\text{fac}} = \prod_i p_i(S \rightarrow S') = \prod_i \exp(-\beta |\Delta E_i|^+) \tag{2}\]
which also satisfies the detailed-balance condition $\pi(S)p(S' \rightarrow S) = \pi(S')p(S' \rightarrow S)$. Hereinafter we omit the dependence on $S \rightarrow S'$ except in case it hinders the clarity. A crucial feature of Eq. \(2\) is the consensus rule: as the transition probability $P_{\text{fac}}$ is a product of independent factors $p_i$, an attempted move is accepted only if all the factors give the permission. This leads to a lower acceptance probability in Eq. \(2\) than in Eq. \(1\). However, we show here how it plays a key role in designing the clock technique that dramatically reduces the computational complexity from $O(N)$ to $O(1)$, and thus can greatly improve the overall performance.

Without loss of generality, we illustrate the clock FMet method in the example of a long-range $O(n)$ model of $N+1$ spins, with the Hamiltonian
\[
\mathcal{H} = -c(N) \sum_{i<j} J_{ij} S_i \cdot S_j \quad , \quad |S| = 1 \tag{3}
\]
with $S$ unit vectors in $\mathbb{R}^n$. For $n = 1, 2, 3$, one has the Ising, XY and Heisenberg models, respectively. The coupling strength $J_{ij}$ depends on distance $r_{ij}$, and can be ferromagnetic ($J_{ij} > 0$), anti-ferromagnetic ($J_{ij} < 0$), or even disordered. There are in total $N(N+1)/2$ interaction terms. The normalization constant $c(N)$, scaling typically in $1/N^n$ ($1 \geq \alpha \geq 0$), is to ensure the energy extensivity, which, e.g., is $1/N$ for a mean-field ferromagnet but $1/\sqrt{N}$ for the Sherrington-Kirkpatrick model \[\text{25, 26}\]. An attempted move is to flip/rotate a randomly-chosen spin $S_j$. This leads to an energy change $-c(N) \sum_{i \neq j} J_{ij} (S'_{j} - S_j) \cdot S_j$, which requests an $O(N)$ computation in the Metropolis algorithm.

A straightforward implementation of Eq. \(2\) is as follows. One orders the factor terms from $i = 1$, sequentially samples the rejection of each factor $i$ with probability $1 - p_i$, and stops at the first-rejecting factor $i_{\text{rej}}$: if no rejection is sampled till factor $N$, the move is accepted. This is analogous to an inhomogeneous Bernoulli process of rate $p_i$, as illustrated in Fig. \(1\) where the $i$th clock, with probability $P_{\text{rej}}(i) = (1 - p_i) \prod_{k=i+1}^{N-1} p_k$, represents the event for the $i$th factor to be first-rejecting. Instead of sequentially sampling each factor, one can also evaluate cumulative probability $P_k = \sum_{i=1}^{k} P_{\text{rej}}(k')$ and directly obtain the $i_{\text{rej}} = i$ value by solving $F_{i-1} < \nu \leq F_i$ with a single random number $\nu \in (0,1)$. Nevertheless, the individual probabilities $p_k$ depend prior on local configuration $(S_i, S_j)$, and each move still requires an average number $C \sim O(N)$ of $p_k$-evaluations.

To avoid these costly evaluations, we introduce a bound Bernoulli process with a configuration-independent probability $\tilde{p}_i$, so that $1 - \tilde{p}_i \geq 1 - p_i(S \rightarrow S')$. An actual rejection at a factor $i$ corresponds to a bound rejection once resampled with relative probability
\[
p_{i,\text{rel}} = (1 - p_i)/(1 - \tilde{p}_i) . \tag{4}\]
At each factor $i$, three events are possible: \(A_1\) bound acceptance with $p_{i,\text{acc}}^{A_1} = \tilde{p}_i$, \(A_2\) bound rejection and resampling with $p_{i,\text{acc}}^{A_2} = (1 - \tilde{p}_i)(1 - p_{i,\text{rel}})$, and \(R\) bound rejection and resampling acceptance with $p_{i,\text{acc}}^{R} = (1 - \tilde{p}_i)p_{i,\text{rel}}$. Sampling the $i$th clock, i.e. the first-rejection at factor $i$, is then replaced by sampling a random path of events \(A_1\) or \(A_2\) for $k \leq i-1$ and a first event \(R\) at $i$, as described by
\[
P_{\text{rej}}(i) = p_{i,\text{rel}}^{R} \prod_{k=1}^{i-1} (p_{k,\text{acc}}^{A_1} + p_{k,\text{acc}}^{A_2}) . \tag{5}\]
As the bound $\tilde{p}_i$s are configuration-independent, the bound cumulative probabilities $\tilde{F}_i$ can be analytically calculated or tabulated. Initializing $i_{\text{rej}} = 0$, the next bound rejection $i_{\text{rej}}$ is updated to $i$ by solving
\[
\tilde{F}_{i-1} < \nu(1 - \tilde{F}_{i_{\text{rej}}}) \leq \tilde{F}_i , \tag{6}\]
and the resampling is then applied. This is done within an $O(1)$ complexity. If no actual rejection occurs--i.e.,
event \(A_2\), the procedure is repeated until an event \(R\) (actual rejection) is sampled or until \(\nu(1 - \tilde{F}_{i_{\text{rev}}}) > \tilde{F}_N\) (actual acceptance). The overall complexity \(C\) identifies now with the average number of attempted bound rejections \(\sim O(\ln P_B / \ln P_{\text{Fac}}) \sim O(1)\) if the bound consensus probability \(P_B = \prod \tilde{p}_i\) scales with \(N\) as \(P_{\text{Fac}}\). For a homogeneous case \(\tilde{p}_i \equiv \tilde{p}\), Eq. (4) reduces to \(i = i_{\text{rev}} + 1\), which can be easily adapted to inhomogeneous bound probabilities by ordering the factors decreasingly with \(\tilde{p}_i\) and by replacing \(\tilde{p}\) by \(\tilde{p}_{i_{\text{rev}}} + 1\). Alternatively, one can directly generate the whole list of bound rejection events by the Walker method [25, 26] or its Fukui-Todo extension [21], and then sequentially apply the resampling.

Algorithm 1 summarizes a clock FMet method for a long-range spin system. For Hamiltonian \(H\), \(\tilde{p}_i\) can be taken as a decision of function \(r_i\) as \(\tilde{p}(r_i) = \exp(-2\beta c(N)|J_{ij}|)\).

Algorithm 1 Clock factorized Metropolis (Clock FMet)

\[
\text{Draw a random spin } j \text{ and a random move } S_j \rightarrow S'_j
\]

while True do

\[
i_{\text{rev}} \leftarrow 0 \quad \text{Sample bound rejection starting from } i_{\text{rev}}
\]

while \(i_{\text{rev}} < i\) do

\[
i_{\text{rev}} \leftarrow i \quad \text{Next bound rejection given by Eq. (4)}
\]

if \(i_{\text{rev}} > N\) then

\[
S_j \leftarrow S'_j \quad \text{Move accepted}
\]

Break

else \(\text{Decide whether it is an actual rejection}\)

\[
p_{i_{\text{rev}}, \text{rej}} \leftarrow \text{Eq. (4)}
\]

if \(\text{ran}(0,1) \leq p_{i_{\text{rev}}, \text{rej}}\) then

\[
\text{Break} \quad \text{Move rejected}
\]

Note that each factor in Eq. (2) can contain an arbitrary number of interactions. This observation immediately leads to a variant of clock FMet algorithm in which the interactions are grouped into “boxes” of tunable sizes, leading to new optimization possibilities. If all the interactions are in a single box, one recovers the standard Metropolis method.

The clock technique has two important ingredients: the consensus rule and the resampling. Both the ingredients are general: they do not depend either on any specific configurations, or on factor ordering, or on energy functions, or on update schemes. For systems in a continuous volume, as soft spheres, one can introduce a grid to which the clock technique is applied [24].

**Generalization to other update schemes.** We illustrate the generality of the clock method by discussing its application in the EC method for the \(O(n)\) spin model with \(n \geq 2\). [30, 31]. With an auxiliary lifting variable \(j\) that specifies the moving spin, the EC method proposes to rotate infinitesimally its angle as \(\phi_j \rightarrow \phi_j + d\phi\). Such a move is rejected by at most one spin \(i\), thanks to a continuous derivation of Eq. (2). This yields \(p_i \rightarrow 1 - \lambda_i d\phi\) and \(P_{\text{rej}}(i) \rightarrow \lambda_i d\phi\). For each factor \(i\), the rejection event, with distance \(d\phi\), is thus ruled by a Poisson process (PP) of rate \(\lambda_i\), continuous derivation of the standard Bernoulli process. The spin \(j\) is then rotated by the minimum distance \(d\phi = \min(\delta_i d\phi)\), and the associated factor becomes the moving spin—i.e., \(j \rightarrow i_{\text{min}}\). For long-range interactions, evaluating \(\delta_i d\phi\) for all the factors becomes costly.

To derive the clock method, we introduce a bound Poisson process of total rate \(\lambda = \sum \lambda_i (\lambda_i \geq \lambda_i)\), evaluate a random bound rotation \(d\phi = -\ln \nu/\lambda, \text{sample the rejecting bound factor } i_{\text{min}} = i \text{ with probability } \lambda_i/\lambda, \text{and resample it as an actual lift } j \rightarrow i \text{ according to } \lambda_i / \lambda_i \text{ (Eq. (4))}.

This comes down to the thinning method [22], already applied for soft-sphere systems [24] and for logistic regression in machine learning [33]. We also note that the cluster methods [15, 16] factorize each interaction term independently as in Eq. (2). The resampling procedures in the extended cluster algorithms for long-range interactions and for quantum spin systems [20, 23] can be understood as specific cases of the clock method.

**Performance analysis.** We expect and numerically confirm in Fig. 2 that the standard Metropolis and the clock FMet algorithms have the same physical dynamics. The overall acceleration \(A\) in the latter comes from the speeding-up in the complexity \(C\), corrected by the slowing-down \(\gamma\) due to a lower acceptance in the factorized filter [24], leading to \(A \sim O(N/C\gamma)\). Both effects can be characterized by the scaling of \(\sum_i \max |\Delta E_i|\) and \(\sum_i |\Delta E_i|\), as \(\gamma = P_{\text{Met}} / P_{\text{Fac}}\) can be written as

\[
\ln \gamma = \beta \left( \sum_i |\Delta E_i| - \frac{1}{N} \sum_i |\Delta E_i| \right),
\]

and as \(C \sim \ln P_B / \ln P_{\text{Fac}} \sim \sum_i \max |\Delta E_i| / \sum_i |\Delta E_i|\). Depending on the nature of the energy extensivity and phase of the system, the sum \(\sum_i |\Delta E_i|\) may diverge as size \(N \rightarrow \infty\), while the sum \(\sum_i |\Delta E_i|\) converges to a constant. This normally occurs in disordered systems with slowly decaying interactions, in which the “satisfied” and “unsatisfied” interaction terms compensate each other. The divergence of \(\gamma\) can be controlled by introducing enough compensation through a box size \(B\), increasing the complexity to \(O(B |\ln P_B|)\), but leading to an accel-
eration \sim O(N/(B \ln P_B))$. By definition, $B \propto N$ would ensure a maximal energy compensation but an $O(1)$ acceleration.

We classify the system into the three types of strict, marginal and sub-extensity, for which $\sum_i \max |\Delta E_i|$ respectively scales as $O(1)$, $O(\ln N)$ and $O(N^{\alpha})$ $(1 > \alpha > 0)$. We demonstrate that the clock FMet method of tunable box sizes $B$ might achieve an overall acceleration

- $\mathcal{A} \sim O(N)$ for strict extensivity, directly from $\gamma \sim O(1)$ and $C \sim O(1)$.
- $\mathcal{A} \sim O(N^\kappa)$ $(0 \leq \kappa < 1)$ for sub-extensivity. Depending on the phase, $\ln \gamma$ may diverge, up to $N^\kappa$. For the spin glass of algebraically decaying interaction as $1/r^\sigma$ $(\sigma < 1)$, we find that a box size $B \propto N/N^\kappa$, with a fine-tuning exponent $0 < \omega < 1$, gives a sufficient compensation and an $O(N^\kappa)$ $(\kappa \sim \omega - \alpha)$ acceleration.
- $O(N/(\ln N)^2) < A_{\text{margin}} \sim O(N/\ln N)$ for marginal extensivity. We observe that setting $B$ up to $\ln N$ can be necessary to control $\gamma$.

For frustrated systems, irrespective of which class they belong to, efficient cluster algorithms are normally unavailable due to the huge cluster sizes. Given the substantial acceleration for all the three classes of strict, marginal, and sub-extensity, the application of the clock FMet method is very promising.

**Simulations.** We simulate three typical systems in statistical physics, including long-range Ising spin glass, disordered $O(n)$ model with random external fields, and $O(n)$ model with RKKY-type interactions. We record the number of energy evaluations $C$ for each MC step, which for the Metropolis method is simply $C = N$. We measure the integrated correlation times $\tau$ for magnetic susceptibility $\chi$ in the units of energy evaluations, and compute the overall acceleration $\mathcal{A}$ as the inverse ratio $\mathcal{A} = \tau_{\text{other}}/\tau_{\text{FMet}}$, where $\tau_{\text{other}}$ is for the Metropolis or the Luijten-Blöte (LB) cluster method. For the Metropolis method, it comes down to $\mathcal{A} = N/(\gamma C)$.

**Long-range Ising spin glass.** We consider an one-dimensional (1D) spin glass defined by Eq. (3). The interactions decay algebraically as $J_{ij} = s_{ij}/r_{ij}^\sigma$ $(\sigma > 0)$, with $s_{ij} = \pm 1$ from a bimodal distribution. The normalization $c(N)$ is given by $c(N)^{-2} = \sum_{j > i} \langle J_{ij}^2 \rangle$. This system, with a tunable exponent $\sigma$, is particularly useful in revealing the crossover behavior from the low-dimensional to the mean-field spin glass \cite{32,33}. For simplicity, the simulation is made at the mean-field critical temperature $\beta = 1$ \cite{28}. Depending on the value of $\sigma$, we recover the three extensivity regimes i.e., strict $(\sigma > 1)$, marginal $(\sigma = 1)$ and sub-extensivity $(\sigma < 1)$. We group the interaction terms symmetrically, and set the box size as $B = 2$ $(\sigma > 1)$, $\ln N$ $(\sigma = 1)$, and $N^{2(1-\sigma)}$ $(\sigma < 1)$. The results are shown in Fig. 3. For $\sigma > 1$, the computational complexity $C$ converges to a constant, and a dramatic overall acceleration $\mathcal{A} \sim O(N)$ is achieved. For $\sigma = 1$, we have $C \sim \ln N^2$ and a significant acceleration $\mathcal{A}$ converging to $N/(\ln N)^2$. For $\sigma < 1$, both $C$ and $\mathcal{A}$ increase sub-linearly as $N$. The acceleration $\mathcal{A}$ drops as $\sigma$ becomes smaller. Nevertheless, given the simplicity of the clock FMet method, the gained improvement is still significant. These results are fully consistent with the performance analysis.

**RKKY-type interactions.** We then consider the 2D and 3D Heisenberg models with oscillatory Ruderman-Kittel-Kasuya-Yoshida (RKKY) interactions $J_{ij} = J_0(\cos(2\kappa r_{ij}))/r_{ij}^d \exp(-r_{ij}/\lambda)$, where $d$ is the spatial dimension, $k_F$ is the Fermi vector $(k_F \approx 4.91$ for spin-glass system CuMn), and $\lambda$ is the characteristic length in the damping term $3\pi^2$ \cite{44,45}. Due to their approximate description of real materials, rich behaviors, and important roles in bridging the experimental study of glassy materials and the spin-glass theory of short-range interactions \cite{35,44}, these systems are under extensive studies. For simplicity, we set $J_0 = 1$ and $k_F = \pi$, and take $\lambda = 3$ for 3D and $\lambda = \infty$ for 2D, so that the system is in the class of strict (3D) and marginal (2D) extensivity. The simulations are at $\beta(2D) = 1$ and $\beta(3D) = 0.693$, close to the critical temperature $\beta_c = 0.693 \pm 0.003(2)$ for the 3D pure Heisenberg model \cite{44}. Box sizes are set to 1 and the achieved acceleration is again $\mathcal{A} \sim O(N)$ for the strict extensivity, and $\mathcal{A} \sim O(N/\ln N)$ for the marginal extensivity.

**Disordered random-field model.** Finally, we study a disordered mean-field $O(n)$ model in a random external field. The interactions are partly disordered – i.e., $I_{ij} = 1$ for 90% of interactions while the remaining $I_{ij}$ are drawn from a normal distribution with $\langle I_{ij} \rangle = 0$ and $\langle I_{ij}^2 \rangle = 1$. A quenched random field is applied to each lattice site as $-h_i \cdot S_i$, where $h_i$ is drawn from a $n-$dimensional normal distribution. The normalization is $c(N) = 1/N$, and the system belongs to the class of strict extensivity. Random-field models have applications in a wide range of physics \cite{37,41}, including the pinning of vortices in superconductors, Coulomb glass, the metal-insulator transition, and hysteresis and avalanche physics. In general spatial dimensions, the thermodynamic properties and

![FIG. 3: Computational complexity $C$ and overall acceleration $A$ for the clock FMet algorithm for the 1D ($\beta = 1$) long-range Ising spin glass (left and middle) and for the 2D ($\beta = 1$) and 3D ($\beta = 0.693$) RKKY Heisenberg spin systems (right).](image-url)
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[1] D. M. Ceperley, Reviews of Modern Physics 67, 2:279–335 (1995).
[2] D. Frenkel and B. Smit, Understanding Molecular Simulation - From Algorithms to Applications., Academic Press, San Diego (1996).
[3] D. P. Landau and K. Binder, A Guide to Monte Carlo Simulations in Statistical Physics, Cambridge University Press, Cambridge (2000).
[4] T. Opplestrup, V. V. Bulatov, G. H. Gilmer, M. H. Kalos, and B. Sadigh, Phys. Rev. Lett. 97, 230602 (2006).
[5] D. W. O. Rogers, Fifty years of Monte Carlo Simulations for medical physics, Phys. Med. Biol. 51, R287 (2006).
[6] C. P. Robert and G. Casella, Monte Carlo Statistical Methods, Springer Texts in Statistics, Springer (1999).
[7] J. S. Liu, Monte Carlo Strategies in Scientific Computing, Springer Series in Statistics, Springer (2001).
[8] P. Glasserman, Monte Carlo Methods in Financial Engineering, Springer-Verlag, New York (2004).
[9] D. Silver, A. Huang, C. J. Maddison, A. Guez, L. Sifre, G. Van Den Driessche, J. Schrittwieser, I. Antonoglou, V. Panneershelvam, M. Lanctot and others, Nature 529, 7587, 484–489 (2016).
[10] R. M. Neal, Bayesian Learning for Neural Networks, Springer (1996).
[11] Z. Ghahramani, Nature 521, 7553, 452–459 (2015).
[12] N. Metropolis, A. W. Rosenbluth, M. N. Rosenbluth, A. H. Teller and E. Teller, J. Chem. Phys. 21, 1087 (1953).
[13] J. Dongarra and S. Sullivan, Comput. Sci. Eng. 2, 1, 22–23 (2000).
[14] R. Bardenet, A. Doucet and C. Holmes, preprint arXiv:1505.02827 (2015).
[15] R. H. Swendsen and J. S. Wang, Phys. Rev. Lett. 58, 86 (1987).
[16] U. Wolff, Phys. Rev. Lett. 62, 361 (1989).
[17] N. Prokof’ev and B. Svistunov, Phys. Rev. Lett. 87, 160601 (2001).
[18] E. P. Bernard, W. Krauth and D. B. Wilson, Phys. Rev. E 80, 056704 (2009).
[19] M. Michel, S. C. Kapfer and W. Krauth, J. Chem. Phys. 140, 054116 (2014).
[20] E. Luijten and H. W. J. Bloete, Int. J. Mod. Phys. C 06, 359 (1995).
[21] K. Fukui and S. Todo, Journal of Computational Physics 228, 7:2629 - 2642 (2008).
[22] E. Flores-Sola, M. Weigel, R. Kenna and B. Berche, Eur. Phys. J. Special Topics 226, 581-594 (2017).
[23] Y. Deng and H. W. J. Blöte, Phys. Rev. Lett. 88, 190602 (2002).
[24] S. C. Kapfer and W. Krauth, Physical Review E 94, 031302(R) (2016).
[25] A. J. Walker, ACM Trans. Math. Softw. 3, 253 (1977).
[26] G. Marsaglia, W. W. Tsang and J. Wang, Journal of

FIG. 4: Acceleration $A$ for the disordered mean-field $O(n)$ model in random fields at $\beta = n$ ($n = 1, 2, 3$). The insets are for complexity $C$. The clock FMet method exhibits important acceleration $A$, both compared to both the Metropolis (red circles) and the LB cluster algorithm (blue squares).

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[27] E. A. J. F. Peters and G. de With, *Phys. Rev. E* **85**, 026703 (2012).
[28] D. Sherrington and S. Kirkpatrick, *Phys. Rev. Lett.* **35**, 26:1792-1796 (1975).
[29] S. Kirkpatrick and D. Sherrington, *Phys. Rev. B* **17**, 4384 (1978).
[30] M. Michel, J. Mayer and W. Krauth, *EPL* **112**, 20003 (2015).
[31] Y. Nishikawa, M. Michel, W. Krauth and K. Hukushima, *Phys. Rev. E* **92**, 063306 (2015).
[32] P. A. W. Lewis and G. S. Shedler, *Naval Research Logistics* **26**, 403-413 (1979).
[33] A. Bouchard-Côté, S. J. Vollmer and A. Doucet, *ArXiv preprint*, arXiv:1510.02451 (2016).
[34] G. Kotliar, P. W. Anderson and D. L. Stein, *Phys. Rev. B* **27**, 602(R) (1983).
[35] A. J. Bray, M. A. Moore and A. P. Young, *Phys. Rev. Lett.* **56**, 2641 (1986).
[36] F. Beyer, M. Weigel and M. A. Moore, *Phys. Rev. B* **86**, 014431 (2012).
[37] A. Larkin, *Sov. Phys. JETP* **31**, 784 (1970).
[38] A.L. Efros and B.I Shklovskii, *J. Phys. C* **8**, L49 (1975).
[39] T. R. Kirkpatrick and D. Belitz, *Phys. Rev. Lett.* **73**, 862 (1994).
[40] J. P. Sethna, K. Dahmen, S. Kartha, J. A. Krumhansl, B. W. Roberts and J. D. Shore, *Phys. Rev. Lett.* **70**, 3347 (1993).
[41] M. L. Rosinberg, G. Tarjus and F. J. Perez-Reche, *J. Stat. Mech.*, P03003 (2009).
[42] F. Krzakala, F. Ricci-Tersenghi and L. Zdeborova, *Phys. Rev. Lett.* **104**, 207208 (2010).
[43] L. Leuzzi and G. Parisi, *Phys. Rev. B* **88**, 224204 (2013).
[44] F. Matsubara and M. Iguchi, *Phys. Rev. Lett.* **68**, 3781 (1992).
[45] D. J. Priour, Jr., E.H. Hwang, and S. Das Sarma, *Phys. Rev. Lett.* **95**, 037201 (2005).
[46] D. J. Priour, Jr. and S. Das Sarma, *Phys. Rev. Lett.* **97**, 127201 (2006).
[47] K. Szalowski and T. Balcerzak, *Phys. Rev. B* **77**, 115204 (2008).
[48] K. Kirkpatrick and T. Nawaz, *J. Stat. Phys.* **165**, 1114 (2016).
[49] Y. Deng, H. W. J. Blöte and M. P. Nightingale, *Phys. Rev. E* **72**, 016128 (2005).