A remark on the Laplacian flow and the modified Laplacian co-flow in \(G_2\)-geometry

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Abstract
We give a shorter proof of the well-posedness of the Laplacian flow in \(G_2\)-geometry. This is based on the observation that the DeTurck–Laplacian flow of \(G_2\)-structures introduced by Bryant and Xu as a gauge fixing of the Laplacian flow can be regarded as a flow of (not necessarily closed) \(G_2\)-structures, which fits in the general framework introduced by Hamilton in J Differ Geom 17(2):255–306, 1982. A similar application is given for the modified Laplacian co-flow.

Keywords Laplacian flow · \(G_2\)-geometry · Short-time existence

1 Introduction

In\(^1\) Bryant introduced a geometric flow in \(G_2\)-geometry which evolves an initial closed \(G_2\)-structure \(\varphi_0\) in the direction of its Laplacian.

Given a compact seven-dimensional manifold with a closed \(G_2\)-structure \((M, \varphi_0)\), a Laplacian flow is a solution to the evolution equation

\[
\frac{\partial}{\partial t} \varphi_t = \Delta \varphi_t, \quad d\varphi_t = 0, \quad \varphi_{t=0} = \varphi_0.
\]

The well-posedness of Eq. (1) is proved in\(^2\) by applying the Nash–Moser theorem to the gauge fixing

\[
\frac{\partial}{\partial t} \varphi_t = \Delta \varphi_t + \mathcal{L}_{V(\varphi_t)} \varphi_t, \quad d\varphi_t = 0, \quad \varphi_{t=0} = \varphi_0,
\]

where \(\mathcal{L}\) is the Lie derivative and \(V : \infty(M, \Lambda^3_+) \to \infty(M, TM)\) is a first-order differential operator which depends on the choice of a connection on \(M\). Here, \(\Lambda^3_+\) denotes the open
subbundle of $\Lambda^3$ of $G_2$-structures on $M$. A solution to (2) is usually called a DeTurck–Laplacian flow.

A DeTurck–Laplacian flow $\varphi_t$ is also a solution to

$$\frac{\partial}{\partial t} \varphi_t = dd^* \varphi_t + d\iota_{V(\varphi_t)} \varphi_t, \quad \varphi_{|t=0} = \varphi_0.$$  (3)

In the present note, we observe that Eq. (3) fits in the general framework introduced by Hamilton in[4]. As a direct consequence, we have the following theorem which in particular implies the well-posedness of (2)

**Theorem 1.1** Let $(M, \varphi_0)$ be a compact seven-dimensional manifold with a $G_2$-structure. Then, Eq. (3) has a unique short-time solution.

In[5] Karigiannis, McKay and Tsui introduced the Laplacian co-flow as the solution to the evolution equation

$$\frac{\partial}{\partial t}(*_{\varphi_t} \varphi_t) = -\Delta_{\varphi_t} *_{\varphi_t} \varphi_t, \quad d *_{\varphi_t} \varphi_t = 0, \quad \varphi_{|t=0} = \varphi_0,$$  (4)

where in this case $\varphi_0$ is supposed to be co-closed with respect to the metric induced by itself. The well-posedness of this last equation is still an open problem and Grigorian introduced in[3] the following modification

$$\frac{\partial}{\partial t}(*_{\varphi_t} \varphi_t) = \Delta_{\varphi_t} *_{\varphi_t} \varphi_t + 2d((A - \text{Tr}(T(\varphi_t)))\varphi_t), \quad d *_{\varphi_t} \varphi_t = 0, \quad \varphi_{|t=0} = \varphi_0,$$  (5)

where $A$ is a constant and $T(\varphi_t)$ is the torsion of $\varphi_t$. In[3], the well-posedness of (5) is proved following the same approach of Bryant in[1] by applying the Nash–Moser theorem to the gauge fixing

$$\frac{\partial}{\partial t}(*_{\varphi_t} \varphi_t) = \Delta_{\varphi_t} *_{\varphi_t} \varphi_t + 2d((A - \text{Tr}(T(\varphi_t)))\varphi_t) + \mathcal{L}_{V(\varphi_t)} \varphi_t, \quad d *_{\varphi_t} \varphi_t = 0, \quad \varphi_{|t=0} = \varphi_0.$$  (6)

Any solution to Eq. (6) satisfies

$$\frac{\partial}{\partial t}(*_{\varphi_t} \varphi_t) = dd^* *_{\varphi_t} \varphi_t + 2d((A - \text{Tr}(T(\varphi_t)))\varphi_t) + d\iota_{V(\varphi_t)} \varphi_t, \quad \varphi_{|t=0} = \varphi_0,$$  (7)

Analogously to Theorem 1.1, we have

**Theorem 1.2** Let $(M, \varphi_0)$ be a compact seven-dimensional manifold with a $G_2$-structure. Then, Eq. (6) has a unique short-time solution.

## 2 Proof of the results

Both Theorems 1.1 and 1.2 can be proved by using the following setup introduced by Hamilton in [4].

Let $M$ be an oriented compact manifold, $F$ a vector bundle over $M$, $U$ an open subbundle of $F$ and

$$E : C^\infty(M, U) \to C^\infty(M, F)$$
a second-order differential operator. For $f \in C^\infty(M, U)$, we denote by $DE(f) : C^\infty(M, F) \to C^\infty(M, F)$ the linearization of $E$ at $f$ and by $\sigma DE(f)$ the principal symbol of $DE(f)$.

**Definition 2.1** An integrability condition for $E$ is a first-order linear differential operator $L : C^\infty(M, F) \to C^\infty(M, G)$, where $G$ is another vector bundle over $M$, such that $L(E(f)) = 0$ for all $f \in C^\infty(M, U)$, and for every $(x, \xi)$ in $T^*M$ all the eigenvalues of $\sigma DE(f)(x, \xi)$ restricted to $\ker \sigma L(x, \xi)$ have strictly positive real part.

**Theorem 2.1** (Hamilton [4, Theorem 5.1]) Assume that $E$ admits an integrability condition. Then, for every $f_0 \in C^\infty(M, U)$ the geometric flow

$$\frac{df}{dt} = E(f), \quad f(0) = f_0,$$

has a unique short-time solution.

Now we can focus on the setup of Theorem 1.1. Here, we consider

$$F = \Lambda^3, \quad U = \Lambda^3_+, \quad G = \Lambda^4, \quad E(\varphi) = dd^\ast \varphi + dV(\varphi) \varphi, \quad L = d : C^\infty(M, \Lambda^3) \to C^\infty(M, \Lambda^4).$$

From [2], it follows that for every $\varphi \in C^\infty(M, U)$ and every closed $\psi \in C^\infty(M, \Lambda^3)$, we have

$$DE(\varphi)(\psi) = -\Delta_\varphi \psi + \text{l.o.t.}$$

Hence, all the assumptions of Hamilton’s Theorem 2.1 are satisfied and Theorem 1.1 follows.

Notice that if the starting form $\varphi_0$ is closed, then the solution to (3) is closed for every $t$ since

$$d\frac{d\varphi}{dt} = 0.$$

Therefore, if $\varphi_0$ is closed, the unique solution $\varphi_t$ to (3) solves also the DeTurck–Laplacian flow (2) and the short-time existence of the DeTurck–Laplacian flow (2) can be deduced from Theorem 1.1.

About the proof of Theorem 1.2, we set

$$F = \Lambda^4, \quad U = \Lambda^4_+, \quad G = \Lambda^5, \quad E(\ast_\varphi \varphi) = dd^\ast \varphi + dV(\varphi) + 2d((A - \text{Tr}(T(\varphi)))\varphi),$$

$$L = d : C^\infty(M, \Lambda^4) \to C^\infty(M, \Lambda^5).$$

From [3], it follows

$$DE(\ast_\varphi \varphi)(\psi) = -\Delta_\varphi \psi + \text{l.o.t.}$$

for every closed $\psi \in C^\infty(M, \Lambda^4)$ and the proof of Theorem 1.2 follows.

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