Abstract

From a supersymmetry covariant source extension of $N = 2$ SYM we study non-trivial thermodynamical limits thereof. Using an argument by one of us about the solution of the strong CP problem and the uniqueness of the QCD ground state we find that the dependence of the effective potential on the defining field operators is severely restricted. In contrast to the solution by Seiberg and Witten an acceptable infrared behavior only exists for broken supersymmetry while the gauge symmetry remains unbroken.

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1 Introduction

In the last few years numerous new results considering supersymmetry in a field theoretical or string background have been derived. Although supersymmetry and superstrings are the theoretical favorites for new physics and for a consistent quantization of gravity, phenomenologically interesting models have a serious problem: at the time present no supersymmetry breaking mechanism in a field theoretical context is known.

As perturbative breaking mechanisms are excluded and since non-perturbative regions are not available for exact calculations the situation is mostly unclear. Several arguments have been given that rigid supersymmetry does not break non-perturbatively. Witten [1] argued that supersymmetry does not break in certain classes of interesting models. Veneziano and Yankielowicz [2] concluded for pure $N = 1$ Yang-Mills theory that supersymmetry remains unbroken after the breakdown of chiral symmetry and Shifman and Vainshtein [3] calculated the gluino condensate of $SU(N)$ theories exactly. Considering extended supersymmetry Seiberg and Witten used the duality argument to derive an exact Wilsonian low-energy effective action of $N = 2$ SYM with and without matter [4, 5]. They concluded that supersymmetry remains unbroken while the gauge group is broken leading to non-trivial monopole configurations.

In [6, 7] it has been argued that the conclusion of Veneziano and Yankielowicz might be wrong and that chiral symmetry breaking induces supersymmetry breaking. The key leading to this different result is an old observation by one of us [8, 9] that the uniqueness of the non-perturbative ground state can solve the strong CP problem setting the vacuum angle to zero. As a complete argument thereof never has been published we discuss this topic in the Appendix of this paper. After this modification the Witten index calculation breaks down and the Veneziano-Yankielowicz effective action gets modified in such a way that supersymmetry breaks together with chiral symmetry.

In this paper we want to extend the work of [6, 7] to gauge theories with two supersymmetries. Without going into the details of $N = 1$ we also want to clarify some points that have been omitted in [6, 7]. The aim is to calculate a thermodynamical limit of our theory leading to relations among vacuum expectation values of different composite operators. Three steps lead to this result: After a short review of classical and perturbative aspects of supersymmetry we define an external field expansion of our system (section 3). This expansion necessarily breaks the symmetries of the theory (especially supersymmetry) but is done in a supersymmetry covariant way. In section 4 we define an effective action in terms of the operators associated to the external sources. Finally we relax the external fields in the thermodynamical limit and we obtain different consistency conditions among possible spontaneous parameters (section 5). The main result is similar to $N = 1$: Unbroken supersymmetry does not allow for any condensates that can be attached to its Lagrangian in a supersymmetry covariant way. This relates in $N = 1$ the gluino condensate, in $N = 2$ all scalar condensates to supersymmetry breaking. In contrast to [4] we conclude that supersymmetry is broken while the gauge symmetry remains unbroken.
2 Basics about Pure $N = 2$ Yang-Mills Theory

We briefly want to review some basic facts about extended supersymmetric Yang-Mills theories. The anti-commutators among the charges of extended supersymmetry are given by

$$\{Q^i_\alpha, \bar{Q}^j_{\dot{\alpha}}\} = \delta^i_j P_{\alpha\dot{\alpha}} \quad \{Q^i_\alpha, Q^j_\beta\} = \varepsilon_{\alpha\beta} Z^{ij} \quad \{Q_{i\dot{\alpha}}, \bar{Q}^j_{j\dot{\alpha}}\} = \varepsilon_{\alpha\beta} \bar{Z}^{ij}$$  \hspace{1cm} (1)

To get theories with unbroken gauge-symmetry at tree-level the defining algebra must have vanishing central charges. Then the algebra has an internal $U(1) \otimes SU(2)$ symmetry that we represent according to

$$[Q^i_\alpha, R] = Q^i_\alpha \quad [Q^i_\alpha, I_r] = \frac{(\tau_r^i)^j}{2} Q^j_\alpha \quad [I_r, I_s] = i\varepsilon_{rst} I_t \hspace{1cm} (2)$$

In superspace (central basis) charges and covariant derivatives can be represented as

$$Q^i_\alpha = i\partial^i_\alpha - \frac{1}{2} \sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{i\dot{\alpha}} \partial_\mu \quad \bar{Q}^i_{i\dot{\alpha}} = -i\bar{\partial}_{i\dot{\alpha}} + \frac{1}{2} \theta^{i\dot{\alpha}} \sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu \hspace{1cm} (3a)$$

$$D^i_\alpha = \partial^i_\alpha - \frac{i}{2} \sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{i\dot{\alpha}} \partial_\mu \quad \bar{D}_{i\dot{\alpha}} = -\bar{\partial}_{i\dot{\alpha}} + \frac{i}{2} \theta^{i\dot{\alpha}} \sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu \hspace{1cm} (3b)$$

The canonical variables are then $z^M = (x^\mu, \theta^i_\alpha, \bar{\theta}^i_{\dot{\alpha}})$ with the following conjugation and multiplication rules

$$(\theta_{i\alpha})^* = \bar{\theta}_{i\dot{\alpha}} \quad (\theta^i_\alpha)^* = -\bar{\theta}^i_{i\dot{\alpha}} \hspace{1cm} (4a)$$

$$\theta^i_\alpha \theta^j_\beta = -\varepsilon_{ij} \theta^{\alpha\beta} - \varepsilon^{\alpha\beta} \theta^i_\alpha \theta^j_\beta \quad \theta^4 = \frac{1}{12} \theta^i_\alpha \theta^j_\beta \theta^j_\beta \quad \bar{\theta}^i_{i\dot{\alpha}} = \bar{D}^i_{\dot{\alpha}} \theta^4 \hspace{1cm} (4b)$$

where we use the spinorial metric $\varepsilon_{\alpha\beta} = \varepsilon^{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Supersymmetric field strengths and the classical invariant action can be written in terms of a gauge chiral, scalar superfield $W$ that is subject to the constraint $[\nabla^i j, W] = [\bar{\nabla}^i j, \bar{W}]$ \hspace{1cm} \hspace{1cm} (4), where $\nabla^i j$ is the gauge covariant and symmetric version of the quadratic covariant derivative. The gauge transformations of $W$ and its conjugate are

$$W^\dagger = e^{-iX} \bar{W} e^{iX} \quad W \rightarrow e^{i\Lambda} W e^{-i\Lambda} \quad W^\dagger \rightarrow e^{i\bar{\Lambda}} W^\dagger e^{-i\bar{\Lambda}} \quad e^{iX} \rightarrow e^{i\Lambda} e^{iX} e^{-i\bar{\Lambda}} \hspace{1cm} (5)$$

As in $N = 1$ the component expansion is best written in a special gauge similar to WZ gauge. In this gauge $W$ is not only gauge-chiral but also chiral. Moreover the purely $\theta$ or $\bar{\theta}$ dependent terms of $X$ can be gauged away \hspace{1cm} \hspace{1cm} (4). The expansion then reads

$$W(x, \theta^i_\alpha) = \sqrt{2} C(x) + \sqrt{2} \theta^i_\alpha \lambda^i_\alpha(x) + \theta^{\alpha\beta} v_{\alpha\beta}(x) + \theta_{ij} H^{ij}(x) + \theta^i_\alpha \chi^i_\alpha(x) + \theta^4 D(x) \hspace{1cm} (6)$$
with the two highest components

\[ \chi_i^\alpha = \sqrt{2i} \bar{\sigma}_\mu^{\alpha}[D^\mu, \bar{\lambda}_i] + i[\bar{C}, \chi_i^\alpha] \quad \text{and} \quad D = \sqrt{2}[D^\mu, [D_\mu, \bar{C}]] - \frac{1}{\sqrt{2}}[\bar{C}, [\bar{C}, C]] - i\{\bar{\lambda}_i^\alpha, \lambda_i^\beta\} \quad (7) \]

To define a proper action we introduce the complex coupling constant \( \tau = \frac{1}{g^2} + \frac{ig}{8\pi^2} \) and set \( S = -\frac{1}{8G_C} \int d^4x \left( \tau \text{Tr}(W^2) + \text{h.c.} \right) = -\frac{1}{8G_C} \int d^4x \left( \tau \text{Tr}(W^2) \right) \) with the Lie-algebra invariant \( \text{Tr} t^a t^b = C(G)\delta^{ab} \). The complete on-shell Lagrangian then reads \[ (10) \]

\[ \mathcal{L} = \frac{1}{C(G)} \text{Tr} \left( \frac{1}{g^2} [D_\mu, \bar{C}][D^\mu, C] + \frac{i}{g^2} \bar{\lambda}^\alpha \sigma^\mu_{\alpha\dot{\alpha}} [D^\mu, \bar{\lambda}^\dot{\alpha}] \right) - \frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} - \frac{\partial}{32\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} \]

\[ + \frac{1}{4g^2} H_{(ij)} H^{(ij)} + \frac{1}{g^2} C[C, \bar{C}]C + \frac{i}{\sqrt{2}g^2} \{\bar{\lambda}^i_{\dot{\alpha}}, \bar{\lambda}^j_{\dot{\beta}}\} - \frac{i}{\sqrt{2}g^2} \{\bar{\lambda}^i_{\dot{\alpha}}, \lambda^j_\alpha\} \quad (8) \]

To quantize the theory we have to express it in terms of unconstrained superfields. This has been done by Howe et al. \[ (11) \] using central basis. A simpler but still manifestly \( N = 2 \) invariant formulation has been given by Galperin et al. \[ (12, 13, 14) \] with the concept of harmonic superspace. According to the non-renormalization theorem of extended supersymmetry \[ (13) \] Howe et al. derived from the ghost-structure that \( N = 2 \) SYM is perturbatively finite above one loop. Using background fields in harmonic superspace \[ (16) \] a more rigorous proof of this statement has been given in \[ (17) \]. Due to the existence of \( N = 2 \) SYM to any order in perturbation theory the defining superfield \( W \) has a definite meaning in quantum theory when replacing the classical fields by properly renormalized ones. This will allow us to write the effective potential in terms of the latter. Though we do not indicate this explicitly we will assume in the following all fields to be renormalized.

The symmetries of a supersymmetric theory can be expressed in superspace by means of the supercurrent. Extending the formulation of \[ (18) \] to \( N = 2 \) the current conservation including a chiral anomaly field can be written as

\[ w^i_j \Gamma = -\frac{i}{4} (\bar{\sigma}^{\mu})^{\alpha\dot{\alpha}} \partial_\mu [D^i_\alpha, \bar{D}_{j\dot{\alpha}}] V - \delta^i_j (D^4 S - \bar{D}^4 S) \quad (9) \]

where \( W^i_j = \int d^4x \: w^i_j \) is the covariant operator superfield of \( N = 2 \) supersymmetry

\[ W^i_j = -\frac{1}{2} \delta^i_j R + (\tau^i_j)_{\alpha\dot{\alpha}} I_{\alpha\dot{\alpha}} - i\theta^{\alpha\dot{\alpha}} Q_{\alpha\dot{\alpha}} + i\bar{\theta}_{\dot{\alpha}\alpha} Q^{\dot{\alpha}\alpha} + \theta^{\alpha\dot{\alpha}} P_{\alpha\dot{\alpha}} \bar{\theta}^j \quad (10) \]

and \( V \) and \( S \) are supercurrent and anomaly introduced by Sohnius \[ (19) \]. Using the component structure thereof \[ (20) \]

\[ V = \theta^i_\alpha \left( -\frac{1}{2} \delta^i_j \sigma^\mu_{\alpha\dot{\alpha}} R^\mu + (\tau^i_j)_{\alpha\dot{\alpha}} \sigma_{\alpha\dot{\alpha}}^\mu (I_{\gamma\alpha\dot{\alpha}}) \bar{\theta}^{i\dot{\alpha}} + (\theta_i \sigma^\mu \bar{\theta}^j) ((\theta_j \chi^j_\mu) + (\bar{\theta}^j \bar{\lambda}_j)) \right) \]

\[ + \frac{1}{2} (\theta_i \sigma^\mu \bar{\theta}^j)(\theta_j \sigma^{\mu\nu} \bar{\theta}^j) \eta_{\mu\nu} + \ldots \]

\[ D^4 S - \bar{D}^4 S = F - \bar{F} - i(\bar{\theta}_i \tilde{\sigma}_\mu \psi^i) - i(\theta^i \sigma^\mu \partial_\mu \bar{\psi}_i) + \frac{i}{2} (\theta_i \sigma^\mu \bar{\theta}^j) \partial_\mu (F + \bar{F}) + \ldots \]

3
current conservations and anomalies are given by
\[
\partial^\mu R_\mu = i w^\mu \Gamma + 4 \text{Im } F \\
\partial^\mu (I_r)_\mu = i w^\mu \Gamma
\] (12a)
\[
v_{\mu\nu} = - T_{\mu\nu} + \frac{1}{2} g_{\mu\nu} T^\lambda \lambda \\
\partial^\nu T_{\mu\nu} = i w^\mu \Gamma \\
T^\lambda \lambda = - 2 \text{Re } F
\] (12b)
\[
Q_{\mu i} = - i \chi_{\mu i} + i \frac{\sigma_\mu \sigma_\nu \chi_{\nu i}}{2} \\
\partial^\mu Q_{\mu i} = i w^\mu \chi_{\nu i} \\
\sigma^\mu Q_{\mu i} = - 2 i \bar{\psi}_i
\] (12c)

In pure supersymmetric Yang-Mills theory the supercurrent is
\[
V = - \frac{1}{C(G)} \text{Tr } WW
\]
while
\[
\text{the anomaly is proportional to the action superfield. The singlet axial current anomaly of}
\]
\[
\text{this system reads}
\]
\[
\frac{1}{C(G)} \partial_\mu \text{Tr} (\lambda^i \sigma^\mu \bar{\lambda}_i) = - \frac{N_c}{8\pi^2 C(G)} \text{Tr} (F \tilde{F})
\] (13)

The trace anomaly on the other hand is given by
\[
T^\mu \mu = \frac{\beta}{2 g^3 C(G)} \text{Tr} (F^2)
\]
\[
\beta\text{-function of } N = 2 \text{ SYM is given to all orders in perturbation theory by [21, 22, 23]} \beta = - \frac{N_c}{8\pi^2 g^3}. \text{ Noting that } R_\mu = - J_5^\mu + \ldots \text{ we then get } F = \frac{N_c}{32\pi^2 C(G)} \text{Tr} (F_{\mu\nu} F^{\mu\nu} + i F_{\mu\nu} \tilde{F}^{\mu\nu}) \text{ and thus}
\]
\[
S = \frac{N_c}{32\pi^2 C(G)} \text{Tr } W^2 = - \frac{\beta}{4 g^3 C(G)} \text{Tr } W^2
\] (14)

Chiral, trace and supertrace anomaly become:
\[
\partial^\mu R_\mu = \frac{N_c}{8\pi^2 C(G)} \text{Tr} \left( (F_{\mu\nu} \tilde{F}^{\mu\nu} - 2i \partial^\mu (C[D_\mu, \bar{C}] - [D_\mu, C] \bar{C}) - 2D^\mu (\lambda^i \sigma^\mu \bar{\lambda}_i) \right)
\]
\[
T^\mu \mu = \frac{\beta}{g} \mathcal{L}
\]
\[
\sigma^\mu Q_{\mu i} = \frac{\sqrt{2} \beta}{g^3 C(G)} \text{Tr} \left( \frac{1}{2} \lambda_i \sigma^{\mu\nu} F_{\mu\nu} - i \chi_i C - i \lambda^j H_{ij} \right)
\] (15)

For a detailed discussion of the component structure of the $N = 2$ SYM current see e.g. [24].

### 3 The Minimal Source Extension of the $N = 2$-System

In this section we want to discuss how to introduce supersymmetry covariant sources to the $N = 2$ SYM-Lagrangian to be able to study thermodynamical limits of composite operators. As unbroken supersymmetry exists in a finite volume only with trivial boundary conditions, these sources break the SUSY invariance of the Lagrangian as the highest component of a superfield. We can nevertheless introduce SUSY covariant sources by replacing the complex
coupling constant $\tau$ by a complete superfield. In $N = 1$ SYM this has been discussed in
detail in [6, 7]. Considering the thermodynamical limit we choose the sources non-vanishing
in a volume $V_{sub} \subset V$ and take the limit $V_{sub} \subset V \to \infty$. We call a source-term global if it is
non-vanishing and constant inside $V_{sub}$ during this limiting process. For a detailed discussion
of this and other possible limits see [6, 7].

We start with the invariant Lagrangian in $N = 2$ superspace (8):

$$\mathcal{L}_0 = \int d^4 \theta \tau \Phi + h.c. \quad \tau = \frac{1}{g^2} + \frac{i \partial}{8\pi^2} \quad \Phi = -\frac{1}{8C(G)} \text{Tr} W^2 \quad \text{Tr} t^a t^b = C(G) \delta^{ab}$$

Of course we have to add a gauge-fixing Lagrangian when considering the theory as a quan-
tum theory. As we do not add external sources to any of the operators appearing therein we
suppress this part of the action. The invariant Lagrangian can also be written as an inte-
gral over full superspace which is important when considering the renormalization property
thereof. The above form however allows us to introduce a chiral source multiplet instead
of a full one. We will see that this minimal version is enough to get sources of all relevant
composite operators.

We thus replace the coupling constant $\tau$ by a chiral $N = 2$ multiplet $J$ and the covariant
source Lagrangian is then given by $\mathcal{L}_J = \int d^4 \theta \ J \Phi + h.c.$ The full superfield $\Phi$ reads

$$\Phi = -\frac{1}{C(G)} \text{Tr} \left[ \frac{1}{4} C^2 + \frac{1}{2} \theta_i^\alpha \lambda^i_\alpha C + \frac{1}{4} \theta_{ij} (\sqrt{2} H^{ij} C - \lambda^i_\alpha \lambda^j_\alpha) + \frac{1}{4} \theta^{\alpha \beta} \left( \frac{i}{\sqrt{2}} \sigma_{\alpha \beta} F_{\mu \nu} C - \lambda_\alpha \lambda_\beta \right) \right]
\frac{1}{2} \theta^4 L$$

and we write for the chiral source multiplet:

$$J(x) = \tau(x) + \theta_i^\alpha \zeta^i_\alpha - 4 \theta_{ij} m^{ij}(x) + \theta^{\alpha \beta} w_{\alpha \beta} + \theta_i^\alpha n^i_\alpha + 4 \theta^4 M^2(x)$$

The non-scalar sources are needed to keep SUSY covariance. As these sources break Poincaré
invariance, their thermodynamical limit must be trivial and we thus suppress them in the
following. The highest component of our minimal source-Lagrangian then becomes

$$\mathcal{L}_J = \frac{1}{C(G)} \text{Tr} \left[ \text{Re}(\tau L) - M^2 C^2 - M^2 C^2 - m_{ij} (\sqrt{2} H^{ij} C - \lambda^i_\alpha \lambda^j_\alpha) - \bar{m}_{ij} (\sqrt{2} H^{ij} C - \bar{\lambda}^i_{\bar{\alpha}} \bar{\lambda}^j_{\bar{\alpha}}) \right]$$

The first term is the source of the quantum mechanical Lagrangian (9). The complete
source-extended Lagrangian is now given as $\mathcal{L}_{tot} = \mathcal{L}_0 + \mathcal{L}_J + \mathcal{L}_{GF}$.

The source multiplet is subject to the constraint $J(x) \to 0 \ (x \to \infty)$. SUSY covariance
enforces to take the limit of all components in $J$ simultaneously while the relative normalizations
thereof can be changed by an appropriate supersymmetry transformation. Considering
only the highest components of all relevant superfields we can however modify this picture and include $\mathcal{L}_o$ in $\mathcal{L}_J$ by changing the boundary conditions to

$$\lim_{x \to \infty} \tau(x) = \tau \quad \lim_{x \to \infty} j(x) = 0 \quad \text{for all other components of } J(x) \quad (20)$$

In presence of a non-trivial source the auxiliary field $H_{ij}$ does not vanish, but the full auxiliary-field Lagrangian reads

$$\mathcal{L}_{aux} = -\frac{1}{8g^2} H_A H_A^a + \frac{1}{\sqrt{2}} m_A H_A^a C^a + \frac{1}{\sqrt{2}} \bar{m}_A H_A^a \bar{C}^a \quad H^i_j = \frac{(\tau_A)_j^i}{2} H_A^a t^a \quad (21)$$

In this basis $H_A$ is anti-hermitian $(H_A)^\dagger = -H_A$ and $m_A = -\bar{m}_A$. Eliminating the auxiliary fields we get

$$\mathcal{L}_{aux} = -\frac{2g^2}{C(G)} \text{Tr}[(m^{ij} C + \bar{m}^{ij} \bar{C})(m_{ij} C + \bar{m}_{ij} \bar{C})] \quad (22)$$

Starting with a non-trivial source-configuration we can in principle obtain any other configuration by applying a suitable SUSY transformation. This is not problematic when considering local sources only. Arbitrary global sources however can lead to unstable configurations. To avoid this problem we have to introduce the following constraints on the lowest and highest component of $J$ ($\mu^2 = -m_A m_A$, $\rho^2 = -m_A \bar{m}_A$):

$$\text{Re}(\tau) \geq 0 \quad g^2 \rho^2 \geq |M^2 + g^2 \mu^2| \quad (23)$$

We get the second constraint by noting that the eigenvalues of the scalar mass matrix are given by $m_{1,2} = 2(g^2 \rho^2 \pm |M^2 + g^2 \mu^2|)$ and that the “$\Phi^4$” term $\text{Tr}(C[C, \bar{C}][C])$ cannot stabilize negative mass terms.

Due to these constraints we can no longer apply any finite supertranslation to our Lagrangian. But as long as the constraints are inequalities we can still apply an arbitrary finite translation with small enough parameters.

4 The Static Effective Action and its Symmetries

In order to be able to perform the thermodynamical limit of the the source-extended system we must formulate the effective action in terms of the composite operators needed. The SUSY covariance of this effective action can then be used to derive relations between the thermodynamical limits of different operators. Formally this is done by a Legendre transformation of the energy-functional:

$$Z[J, \bar{J}] = \int \mathcal{D}X \exp(iS_0 + iS_J) = \exp(iW[J, \bar{J}])$$

$$\Gamma[\bar{J}, \bar{J}] = \int d^4x \left(J(x) \frac{\delta W[J]}{\delta J(x)} + \text{h. c.}\right) - W[J] \quad (24)$$
The variations and thermodynamical limiting conditions are
\[ \frac{\delta}{\delta J(x)} \Gamma[\tilde{J}, \tilde{\bar{J}}] = J(x) \to 0 \quad \frac{\delta}{\delta \tilde{J}(x)} W[J, \tilde{J}] = \tilde{J}(x) \to \tilde{J}^*(x) \] (25)
where a non-zero component of \( \tilde{J}^*(x) \) indicates the appearance of a spontaneous parameter (vacuum expectation value).

When transferring the coupling constant \( \tau \) to the boundary conditions we may define instead of (24)
\[ \Gamma[\tilde{J}, \tilde{\bar{J}}] = \int d^4x \left( (J(x) + \tau) \frac{\delta W[J]}{\delta J(x)} + \text{h. c.} \right) - W[J] \] (26)

The internal symmetries of the supersymmetry algebra define two Ward-Identities, one of them being anomalous:
\[ W^I_5(x) \Gamma = 0 \quad W^R_5(x) \Gamma \sim \tilde{F} \tilde{F} \cdot \Gamma \quad W^R(x) \Gamma \sim \left( F \tilde{F} + (\text{tot. der.}) \right) \cdot \Gamma \] (27)
As explained in the Appendix the anomalous chiral symmetry gets restored when evaluated with respect to the ground-state, which sets the (local and global) variations of \( W[J, \tilde{J}] \) with respect to the vacuum-angle to zero:
\[ W^I_5(x) \Gamma_{\text{th.dyn.}} = W^R(x) \Gamma_{\text{th.dyn.}} = 0 \quad \frac{\delta W[J, \tilde{J}]}{\delta \vartheta(x)} = \frac{\partial W[J, \tilde{J}]}{\partial \vartheta} = 0 \] (28)
This restored symmetry then implies that \( \text{Im} \tau^*(x) \sim \text{Im} \langle \Omega | L | \Omega \rangle = \text{Im} L_{\text{cl}} = 0 \).

4.1 The Effective Potential as Static Part of a Nonlinear \( \sigma \)-Model

Besides the formal definition given above we can get \( \Gamma_{\text{th.dyn.}} \) or the effective potential by extracting the static part from the most general Lagrangian obeying the symmetries of our theory, i.e. the most general Lagrangian of the chiral \( N = 2 \) multiplet of equation (17). We extend this problem and derive the Lagrangian of \( k \) chiral multiplets.

The most general Lagrangian of \( k \) chiral \( N = 2 \) superfields

The nonlinear \( \sigma \)-models of \( k \) chiral \( N = 1 \) multiplets with or without an additional second supersymmetry are well known to lead to hyper-Kähler and Kähler manifolds, respectively \[25, 26\]. The situation we are dealing with here is slightly different. This can easily be seen when reducing a scalar chiral \( N = 2 \) superfield to the \( N = 1 \) superfield formulation:
\[ \Phi = \Sigma_1(x, \theta_1) + \theta_2^a \phi_a(x, \theta_1) + \theta_2^2 \Sigma_2(x, \theta_1) \] (29)
The most general Lagrangian is of the form

\[ \mathcal{L} = \int \mathrm{d}^8 \theta \mathcal{K}(\Phi^A, \bar{\Phi}^A) + \left( \int \mathrm{d}^4 \theta \mathcal{W}(\Phi^A) + \text{h. c.} \right) \quad A = 1, \ldots, k \]  

(30)

Using the component expansion

\[ \Phi = \varphi + \theta^\alpha \Lambda_\alpha^i + \theta^{\alpha \beta} v_{\alpha \beta} + \theta^\alpha \eta^i_\alpha + \theta^4 D \]  

(31)

we can write the nonlinear \( \sigma \)-model (after integrating out superspace) in the following short form

\[
\mathcal{K}(\Phi, \bar{\Phi})|_{\theta^8} = \left[ D^A \partial_A + \Phi_0^{AB} \partial_{AB} + \Phi_0^{ABC} \partial_{ABC} + \Phi_0^{ABCD} \partial_{ABCD} \right] \left[ \text{h. c.} \right] \mathcal{K}(\varphi, \bar{\varphi}) + i \left[ (\eta^{\alpha A}) \partial_A + (\Phi_1^{AB})^\alpha \partial_{AB} + (\Phi_1^{ABC})\partial_{ABC} \right] \sigma^{\mu \alpha} \partial_\mu \\
+ \left[ (\bar{\eta}_i^A) \bar{\partial}_A + (\bar{\Phi}_1^{AB})^\alpha \partial_{AB} + (\bar{\Phi}_1^{ABC}) \partial_{ABC} \right] \mathcal{K}(\varphi(x), \bar{\varphi}(x)) \\
- \left[ (C^{ij})^A \partial_A + (\Phi_1^{ij A}) \partial_{AB} \right] \square \left[ (\bar{C}_{ij})^A \partial_A + (\bar{\Phi}_1^{ij A}) \partial_{AB} \right] \mathcal{K}(\varphi(x), \bar{\varphi}(x)) \\
- \left[ (v^{\alpha \beta})^A \partial_A + (\Phi_1^{AB})^{\alpha \beta} \partial_{AB} \right] \sigma^{\mu \alpha \mu} \sigma^{\nu \beta \nu} \partial_\mu \partial_\nu \\
+ \left[ (\bar{v}^{\alpha \beta})^A \partial_A + (\bar{\Phi}_1^{AB})^{\alpha \beta} \partial_{AB} \right] \mathcal{K}(\varphi(x), \bar{\varphi}(x)) \\
- i(\Lambda^{A \alpha})^A \partial_A \sigma^{\alpha \alpha} \partial_\mu \Box (\bar{\lambda}_i^A) A \partial_A \mathcal{K}(\varphi(x), \bar{\varphi}(x)) \\
+ \varphi^A \partial_A (\Box \Box \bar{\varphi}^A) \partial_A \mathcal{K}(\varphi(x), \bar{\varphi}(x))
\]  

(32)

with the components

\[
\Phi_0^{AB} = \frac{1}{2} (v^{\alpha \beta})^A (v_{\alpha \beta})^B - \frac{1}{2} (C_{ij})^A (C^{ij})^B - (\eta_i^A)^A (\Lambda_i^A)^B
\]

\[
\Phi_0^{ABC} = \frac{1}{2} \left( (C_{ij})^A (\Lambda_i^A)^B (\Lambda_j^A)^C - (v^{\alpha \beta})^A (\Lambda_i^A)^B (\Lambda_j^C)^B \right)
\]

\[
(\Phi_1^{AB})^\alpha = - \left( (v^{\alpha \beta})^A (\Lambda_j^A)^B + (\Lambda_i^A)^A (C^{ij})^B \right)
\]

\[
(\Phi_1^{ij A})^\alpha = - \frac{1}{2} (\Lambda_i^A)^A (\Lambda_j^B)^B
\]

\[
(\Phi_1^{ABCD})^\alpha = \frac{1}{12} (\Lambda_i^A \Lambda_i^A \Lambda_j^A \Lambda_j^A)^{ABCD}
\]

\[
(\Phi_1^{ABC})^\alpha = \frac{1}{3} (\Lambda_i^A \Lambda_j^A \Lambda_j^A)^{ABC}
\]

\[
(\Phi_1^{AB})^\alpha = - \frac{1}{2} (\Lambda_i^A)^A (\Lambda_j^B)^B
\]
\[ \partial_{A_1 \ldots A_n} \text{stands for } \frac{\partial^n}{\partial \varphi_{A_1} \ldots \partial \varphi_{A_n}} \text{ and the vertical line in } \mathcal{K}(\varphi(x), \bar{\varphi}(x)) \text{ indicates that the space-time derivatives only act on the } \varphi - \text{fields inside } \mathcal{K}(\varphi, \bar{\varphi}). \]

In a completely analogous way we get the expression for the highest component of the superpotential:

\[ \mathcal{W}(\Phi)_{\theta^4} = \left[ D^A \partial_A + \Phi_0^{AB} \partial_{AB} + \Phi_0^{ABC} \partial_{ABC} + \Phi_0^{ABCD} \partial_{ABCD} \right] \mathcal{W}(\varphi) \quad (33) \]

Although the complete Lagrangian of this model is rather complicated when expressed on component level we can immediately read off some important properties:

- The terms quadratic in the fields \( D, \eta_i^\alpha, C^{ij} \) and \( v_{\alpha \beta} \) are given by

\[ L_2 = \frac{\partial^2 \mathcal{K}}{\partial \varphi \partial \bar{\varphi}} (D^A D^A + i(\eta^i A)^A \sigma^\mu_{\alpha \alpha} \partial_\mu (\bar{\eta}_i^A)^A - (C^{ij})^A \square (C_{ij})^A
\]

\[ - (v^{\alpha \beta})^A \sigma^\mu_{\alpha \alpha} \sigma^\nu_{\beta \beta} \partial_\mu \partial_\nu (\bar{v}^{\alpha \beta})^A) + \ldots \quad (34) \]

To get stable \( p^2 \)-fluctuations of the dynamical fields in the above equation, \( \mathcal{K} \) must be Kählerian and \( g_{AA} = \frac{\partial^2 \mathcal{K}}{\partial \varphi \partial \bar{\varphi}} \) defines the hermitian metric of the manifold. In contrast to the \( N = 2 \) matter fields the manifold need not be hyper-Kählerian. This is in fact easy to understand: Both \( N = 2 \) matter fields, the Howe-Stelle-Townsend as well as the Fayet-Sohnius hypermultiplet, are in a non-trivial representation of the internal \( SU(2) \) symmetry. This induces the quaternionic structure of a hyper-Kähler manifold when constructing a nonlinear \( \sigma \)-model. Our superfield transforms with respect to the trivial representation and consequently such a structure is not needed.

- \( L_2 \) does not generate correct \( p^2 \)-terms for the remaining fields \( \varphi \) and \( \Lambda^{\alpha}_{\beta} \). This can already be seen from dimensional considerations and leads to important constraints on the dynamics of the system, as such a term must be produced from higher order components. We can indeed read off from the \( \partial_{AB\bar{A}B} \mathcal{K} \) component the expression:

\[ L_4 = \frac{\partial^4 \mathcal{K}}{\partial \varphi \partial \bar{\varphi} \partial \bar{\varphi} \partial \bar{\varphi}} (i(C_{ji})^B (\bar{C}^{ik})^B (\Lambda^{\lambda})^A \sigma^\mu_\mu (\bar{\Lambda}_k^{\lambda})^\lambda
\]

\[ - (C_{ji})^B (\bar{C}^{ij})^B (\varphi^A \square \bar{\varphi}^A) + \ldots \quad (35) \]

This expression defines a second metric of our system which does not transform trivially under the internal \( SU(2) \). It is given by:

\[ (\tilde{g}_{AA})^A_k = (C_{ji})^B g_{AA, BB} (\bar{C}^{ik})^B \quad (36) \]

It immediately follows that the \( p^2 \)-fluctuations of this system are only stable if at least one operator \( (C_{ji})^B (\bar{C}^{ik})^B \) has a non-trivial vacuum-expectation value. The hermiticity
of this metric follows trivially. Notice that from equation (32) a similar expression
\[ \sim v_\alpha^\beta \bar{v}^\dot{\alpha}\dot{\beta} \]
containing the correct number of derivatives can be obtained. But due to its Lorentz structure it cannot contribute to this metric.

The most significant consequence of this second metric and of the associated vacuum-expectation value is not its pure existence, but the fact that it breaks the internal SU(2) symmetry of the theory. We will discuss this point again when specializing to our SYM-model.

- We want to eliminate the auxiliary fields. From (32) and (33) we can read off the following expression for the variation with respect to \( \bar{D}^\dot{A} \):

\[
g_{A\dot{A}}D^A = -g_{A\dot{A},B}\Phi_0^{AB} - g_{A\dot{A},BC}\Phi_0^{ABC} - g_{A\dot{A},BCD}\Phi_0^{ABCD} - \frac{\partial \bar{W}}{\partial \phi^A} \quad (37)
\]

Now we specialize to a single chiral superfield and extract the static part of the Lagrangian given above. Suppressing again all contributions from non Lorentz-scalars we get

\[
\mathcal{L}_{th.dyn.} = Dg_{\phi \bar{\phi}} \bar{D} - \frac{1}{2} DC_{kl} \bar{C}^{kl} g_{\phi \bar{\phi},\bar{\phi}} - \frac{1}{2} C_{ij} C^{ij} Dg_{\phi \bar{\phi},\phi} + \frac{1}{4} C_{ij} C^{ij} \bar{C}_{kl} \bar{C}^{kl} g_{\phi \bar{\phi},\phi} \\
+ DW_{\phi} + \bar{D}W_{\bar{\phi}} - \frac{1}{2} C_{ij} C^{ij} W_{\phi \bar{\phi}} - \frac{1}{2} \bar{C}_{kl} \bar{C}^{kl} W_{\phi \bar{\phi}} \quad (38)
\]

The effective potential we are looking for is (up to a sign) the above expression when identifying the components with the classical fields \( \langle \Omega \| \Phi \| \Omega \rangle \), where \( \Phi \) has been defined in equation (17).

Finally we want to define the geometrical objects of our Kähler manifold. As the dual metric of a one-dimensional manifold is simply given by \( g^{\phi \bar{\phi}} = \frac{1}{g_{\phi \bar{\phi}}} \), connection and curvature are:

\[
\Gamma^\phi_{\phi \bar{\phi}} = \Gamma = \frac{g_{\phi \bar{\phi},\bar{\phi}}}{g_{\phi \bar{\phi}}} \quad \Gamma_{\bar{\phi} \phi \bar{\phi}} = \bar{\Gamma} = \frac{g_{\phi \bar{\phi},\phi}}{g_{\phi \bar{\phi}}} \quad R_{\phi \bar{\phi} \phi \bar{\phi}} = R = g_{\phi \bar{\phi},\phi \bar{\phi}} - g_{\phi \bar{\phi},\phi} g_{\phi \bar{\phi},\bar{\phi}} \quad (39)
\]

5 SUSY (Non-)Breaking Conditions from the Static Effective Action

Finally we combine the results of the previous sections. Using the component structure of the effective potential and the symmetries of the latter from its formal definition we get constraints on possible spontaneous parameters. It is well known that the order parameter of supersymmetry breaking is the vacuum energy density due to the relation

\[
\frac{1}{2} \langle \Omega \| \left( (\sigma_{\mu})_{\alpha \dot{\alpha}} \{ Q^{\alpha \dot{\alpha}}, \bar{S}_{j \nu} \} + \epsilon^{\alpha \dot{\alpha}} \{ \bar{Q}_{j \dot{\alpha}}, S^{i}_{\nu \alpha} \} \right) \| \Omega \rangle = \delta_j^i \langle \Omega \| T_{\mu \nu} \| \Omega \rangle = E_0 \delta_j^i g_{\mu \nu} \quad (40)
\]
which is directly connected to \( \langle \Omega | L | \Omega \rangle \) via equation (15). As \( L_{cl} \) is one of our defining variables, we can easily control possible SUSY-breaking effects. According to equation (14) it seems to be impossible to break rigid supersymmetry partially. But in a spontaneously broken theory we can consider the algebra of currents only. The latter can be modified such that partial supersymmetry breaking is indeed possible in some models [27, 28, 29, 30, 31]. However we will not discuss this here.

5.1 The Chiral-Weight Puzzle

As the chiral (or R-) symmetry is unbroken in classical SYM, every classical superfield of this theory has a well-defined chiral weight. In quantum theory however chiral symmetry is broken. In the static limit it gets restored at least for the Lagrangian multiplet and it should thus re-inherit its classical chiral weight\(^2\).

When assigning chiral weight +1 to the left-handed super-generator \( \chi[Q_i^\alpha] = +1 \), we get the classical weights: \( \chi[W] = -2 \), \( \chi[\Phi] = -4 \), \( \chi[J] = 0 \). The first two weights follow from the fact that the classical Lagrangian has vanishing chiral weight, the last from \( \chi[\tau] = 0 \). Restoration of the chiral symmetry now tells us that \( \chi[\Phi_{cl}] = -4 \), especially \( \chi[L_{cl}] = 0 \) which follows from equation (28) that can alternatively be written as \( \text{Im} L_{cl} \equiv 0 \). More complicated is the behavior of the source-multiplet under quantization. Considering the coupling constant \( \tau \) we can define:

\[
\tau_{QM} = \frac{1}{g^2} + \frac{i \theta}{8 \pi^2} \quad \tau_V = -\frac{i \theta_V}{8 \pi^2} \quad \tau(x) = \frac{1}{g^2(x)} + \frac{i \theta(x)}{8 \pi^2}
\]

The first \( \tau \) is the usual renormalized coupling constant, the second one the vacuum angle and the last one the source term. Due to the dynamical equations (83) and (86) in the Appendix the effective quantum-mechanical coupling constant \( \tau_{eff} = \tau_{QM} + \tau_V \) transforms under chiral rotations as:

\[
Q \rightarrow e^{i\alpha} Q \quad \tau_{eff} \rightarrow \tau_{eff} + \frac{2i \alpha}{8 \pi^2}
\]

Thus \( \tau_{eff} \) has a logarithmic weight: \( \chi[\exp(\tau_{eff})] = \frac{1}{4\pi^2} \). After transforming the effective coupling constant to the boundary conditions the lowest component of the source-multiplet therefore has a special weight.

It is now easy to read off the chiral weights of the static Kähler- and superpotential. Noting that \( \chi[\varphi] = -4 \) and thus \( \frac{\partial}{\partial \varphi} = +4 \) we get

\[
\chi[\Gamma_{th.dyn.}] = 0 \quad \chi[K] = 0 \quad \chi[W] = -4
\]

\(^2\)Some points in the subsequent discussion have been omitted in [6, 7]. Especially the chiral weight of the superpotential has been left open and the possibility of a non-trivial dependence on the imaginary part of the coupling constant has not been discussed. All arguments given here also hold –mutatis mutandis– in \( N = 1 \).
The fact that $\Gamma_{\text{th.dyn.}}$ has a defined chiral weight restricts the dependence of the latter on the classical fields. Before going into details we however want to eliminate the auxiliary field of the nonlinear $\sigma$-model.

5.2 Elimination of the Second-Generation Auxiliary Fields

Our model includes two types of auxiliary fields: $D \sim L_{\text{cl}}$ and the auxiliary field of the underlying theory $H^{ij}$. We will refer to them as 2nd- and 1st-generation auxiliary fields respectively. The 2nd-generation auxiliary fields are eliminated using equation (37). Taking the static part we obtain

$$D = \langle \Omega | \frac{g^2}{2} L | \Omega \rangle = \frac{1}{2} \Gamma C^{ij} C_{ij} - \frac{W_{\varphi \bar{\varphi}}}{g_{\varphi \bar{\varphi}}}$$

$$\Gamma_{\text{th.dyn.}} = \frac{|W_{\varphi |}{2} g_{\varphi \bar{\varphi}}} + \frac{1}{2} \left( C^{ij} \left( W_{\varphi \bar{\varphi}} - \Gamma \right) + \text{h.c.} \right) - \frac{1}{4} C_{ij} C^{ij} \bar{C}_{kl} \bar{C}^{kl} R$$

After elimination the auxiliary field $\Gamma_{\text{th.dyn.}}$ can depend on the $SU(2)$ singlets $C^{ij}, C_{ij}, \varphi$ and its hermitian conjugate only. Besides the combinations of these fields with chiral weight zero we could also construct terms of the form $\exp(16\pi^2 \tau_{\text{eff}}) \varphi$. Such a term is however excluded by the invariance of the theory under a global change of the $\varphi$-angle, as from (28)

$$\frac{\partial W[J, \bar{J}]}{\partial \vartheta} = 0 \Rightarrow \frac{\partial \Gamma[\Phi_{\text{cl}}]}{\partial \vartheta} = 0$$

which may be checked explicitly by using the inverse of equation (24) and noting that

$$\int d^4x \left( \int d^4\theta \Phi_{\text{cl}} \frac{\partial}{\partial \vartheta} J(x) + \text{h.c.} \right) = 0$$

Also note that it is irrelevant which $\vartheta$ (quantum-mechanical, vacuum-angle or global source) we choose in the above equation, as the latter only appear in the specific combination of the equations (28) and (29) respectively. We thus conclude that after eliminating $D$ and after turning off all sources the static effective action can only depend on

$$\Gamma_{\text{th.dyn.}}[\Phi_{\text{cl}}] = \Gamma_{\text{th.dyn.}} \left[ (\varphi)^2, C_{ij} \bar{C}^{ij}, C_{ij} C^{ij} \bar{C}_{kl} \bar{C}^{kl}, \{ \varphi \bar{C}_{kl} \bar{C}^{kl}, \frac{C_{ij} C^{ij}}{\varphi}, \text{and h.c.} \} \right]$$

We have already seen that the fields $C_{ij}$ cannot acquire a vacuum expectation value due to the internal $SU(2)$ symmetry. This observation contradicts the conclusion made in section 4.1. We want to argue here that our model nevertheless exists in the thermodynamical limit. The assumptions we are making are the existence of the underlying theory ($N = 2$ SYM) as non-perturbative field theory and its stability with respect to small perturbations of all composite operators considered here. Then the effective action in terms of these composite
operators (the components of \( J(x) \)) must indeed exist and is given by the limiting process \( J(x) \to 0 \). If \( J(x) \) is non-zero the non-equilibrium effective action has the form of the discussed nonlinear \( \sigma \)-model due its SUSY covariance. Together with the above assumption the limiting process must be defined and leads to the correct effective action. What does this mean for the dependence of \( \Gamma_{\text{th.dyn.}} \) on \( C_{ij} \)? If \( \varphi \neq 0 \) in the limit, \( \Gamma_{\text{th.dyn.}} \) can only depend on positive powers of \( C_{ij} \). If \( \varphi = 0 \) a definite limit of \( \frac{C_{ij}}{\varphi} \) could in principle exist. Besides the fact that such a term immediately sets all scalar condensates to zero to ensure the stability of the effective action under variations (see next section), SUSY-transformations on the non-equilibrium system can e.g. set \( m_{ij}(x) \) to zero while \( M^2(x) \neq 0 \), which excludes the dependence on such a fraction. Thus \( \Gamma_{\text{th.dyn.}} \) can only depend on positive powers of \( C_{ij} \).

### 5.3 Elimination of the First-Generation Auxiliary Fields

The elimination of the 1\textsuperscript{st}-generation auxiliary fields is somewhat different from the usual procedure, as they appear inside the expression of a classical composite operator. This situation is similar to a (non-supersymmetric) theory of two scalar fields, one of them being auxiliary. The Lagrangian shall be given by:

\[
\mathcal{L} = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi + \frac{1}{2} F^2 - V(\phi) \quad \langle \Omega | \phi^2 | \Omega \rangle_{\text{pert. theory}} = 0 \quad \langle \Omega | \phi^2 | \Omega \rangle_{\text{non-pert.}} \neq 0 \quad (48)
\]

By attaching a source \((F\phi)(x)\) we get under the basic assumption that the non-perturbative vacuum does not change under the elimination of the auxiliary fields

\[
\langle \Omega | (F\phi)(x) | \Omega \rangle_{m(x)} = m(x) \langle \Omega | (\phi^2)(x) | \Omega \rangle \quad \to 0 \quad \text{as} \quad m(x) \to 0 \quad (49)
\]

The variation of \( \langle \Omega | (F\phi)(x) | \Omega \rangle_{m(x)} \) however is non-vanishing by the assumption made above. Thus we get the following variation of the energy-functional

\[
\frac{\delta W[m]}{\delta m(x)} \Big|_{m(x) \to 0} = (F\phi)_{cl} = 0
\]

\[
\frac{\delta^2 W[m]}{\delta m(x) \delta m(y)} \Big|_{m(x) \to 0} = \frac{\delta}{\delta m(y)} (F\phi)_{cl} \Big|_{m(x) \to 0} = \delta(x - y)(\phi^2)_{cl} \quad (50a)
\]

\[
\frac{\delta^n W[m]}{\delta m(x_1) \ldots \delta m(x_n)} = \frac{\delta^n \Gamma[(F\phi)_{cl}]}{\delta m(x_1) \ldots \delta m(x_n)} \quad (51b)
\]

On the other hand we may calculate the variation of \( \Gamma[(F\phi)_{cl}] \) with respect to the "wrong" variable \( m(x) \): \( \frac{\delta}{\delta m(x)} \Gamma[(F\phi)_{cl}] = (F\phi)_{cl} \). Thus all variations of \( W \) and \( \Gamma \) with respect to \( m(x) \) are equivalent and we get the non-vanishing condensate as second variation of the effective action:
Of course we could also extract the physical condensate when varying the effective action with respect to its defining variable \( (F\phi)_{cl} \). This relation reads:

\[
\frac{\delta^2 \Gamma[(F\phi)_{cl}]}{\delta (F\phi)_{cl}(x)\delta (F\phi)_{cl}(y)} = -\frac{\delta(x-y)}{(\phi^2)_{cl}(x)}
\]  

(52)

Finally we want to note that all formal relations given above remain true when introducing any new fields and attaching any new sources.

To translate this to our SUSY-model we first rewrite the latter according to equation (21). Then source-extension and Legendre transform of the \( H^A \) dependent \( C^A \)-fields read:

\[
L_{m^A} = -2(m^A C^A + \bar{m}^A \bar{C}^A) \quad \Gamma[C^A_{cl}] = -2 \int d^4x (m^A C^A_{cl} + \bar{m}^A \bar{C}^A_{cl}) - W[m^A]
\]  

(53)

The variations of the field and the effective action thus become:

\[
\frac{\delta C^B(z)}{\delta m^A(x)} = -g^2 \delta^B_A \delta(x-z)(C^2)_{cl} \quad \frac{\delta C^B(z)}{\delta \bar{m}^A(x)} = -g^2 \delta^B_A \delta(x-z)(\bar{C}C)_{cl} \]  

(54a)

\[
\frac{\delta \bar{C}^B(z)}{\delta m^A(x)} = -g^2 \delta^B_A \delta(x-z)(C \bar{C})_{cl} \quad \frac{\delta \bar{C}^B(z)}{\delta \bar{m}^A(x)} = -g^2 \delta^B_A \delta(x-z)(\bar{C}^2)_{cl} \]  

(54b)

\[
\frac{\delta \Gamma[C^A_{cl}]}{\delta m^A(x)} = -2C^A_{cl}(x) \quad \frac{\delta \Gamma[C^A_{cl}]}{\delta \bar{m}^A(x)} = -2\bar{C}^A_{cl}
\]  

(54c)

\[
\frac{\delta^2 \Gamma[C^A_{cl}]}{\delta m^A(x)\delta m^B(z)} = 2g^2 \delta^{AB} \delta(x-z)(C^2)_{cl} \quad \frac{\delta^2 \Gamma[C^A_{cl}]}{\delta \bar{m}^A(x)\delta \bar{m}^B(z)} = 2g^2 \delta^{AB} \delta(x-z)(\bar{C}^2)_{cl}
\]  

(54d)

\[
\frac{\delta^2 \Gamma[C^A_{cl}]}{\delta m^A(x)\delta \bar{m}^B(z)} = 2g^2 \delta^{AB} \delta(x-z)(\bar{C}C)_{cl}
\]  

(54e)

Besides these formal relations which follow directly from the Legendre transformation we want to calculate the same variations using our explicit effective potential. To do this we have to distinguish more carefully two different variations with respect to the source \( m^A(x) \).

On one hand the effective action may depend on the source as a function of the classical fields \( m^A = m^A[\Phi_{cl}(x)] \) even without eliminating the \( 1^\text{st} \)-generation auxiliary fields, on the other hand the \( C^A \) depend explicitly on \( m^A(x) \) after the elimination. Therefore we expand the effective action to second order in the fields and sources \( \psi^A_i = \{C^A, \bar{C}^A, m^A, \bar{m}^A\} \). This variation of the effective action (and completely analogous of the Kähler- and the superpotential) is given by

\[
\Gamma[C^A, m^A, \ldots] = \Gamma^0 + \frac{1}{2} \int d^4x d^4y \frac{\delta^2 \Gamma}{\delta \psi^A_i(x)\delta \psi^B_j(y)} \psi^A_i(x) \psi^B_j(y)
\]  

(55)

As \( \Gamma \) (and all other functions involved) is a \( SU(2) \)-singlet it only depends on the quadratic combinations \( \zeta_i(x, y) = \{C^A(x)C^A(y), m^A(x)C^A(y), m^A(x)m^A(y), \ldots\} \). Using the vanishing
thermodynamical limits of the source and of $C^A$ we can then rewrite the second variation as

$$
\int d^4x d^4y \frac{\delta^2 \Gamma}{\delta \psi_i^A \delta \psi_j^B} \psi_i^A \psi_j^B = \int d^4x d^4y \frac{\delta \Gamma}{\delta \zeta_k(u, v)} \frac{\delta^2 \zeta_k(u, v)}{\delta \psi_i^A(x) \delta \psi_j^B(y)} \psi_i^A(x) \psi_j^B(y) 
$$

Using equations (54a) and (54b) the variations of our effective potential (44) can now easily be calculated.

5.4 Non-Vanishing Condensates and Supersymmetry Breaking

We are now ready to discuss the restrictions on unbroken supersymmetry from the effective potential and its variations. First of all equation (44) reads after dropping all trivial terms

$$
\Gamma_{th.dyn.} (|\varphi|^2) = \frac{|W_{\varphi\varphi}|^2}{g_{\varphi\varphi}} D = \langle \Omega | \frac{g^2}{2} \mathcal{L} | \Omega \rangle = - \frac{\bar{W}_{\bar{\varphi}\bar{\varphi}}}{g_{\varphi\varphi}}
$$

Unbroken Supersymmetry

Completely analogous to the discussion of the $N = 1$ case [3-7], the effective potential attains its minimum along a circle in the complex plane of the lowest component of the classical Lagrangian superfield. Again analyticity of the superpotential $W_{\varphi\varphi}$ implies that unbroken supersymmetry $(\mathcal{L}_{cl} \sim \bar{W}_{\bar{\varphi}\bar{\varphi}} = 0)$ can only exist non-trivially at $\varphi = 0$. All other solutions are trivial the sense that $W_{\varphi\varphi} \equiv 0 \forall \varphi$ which is unacceptable when perturbing the system with a source $M^2(x)$ while keeping $C_{ij} = 0$.

Considering the variations of the effective action (54d) and (54e) the curvature-term does only contribute to fourth and higher order variations and consequently we only have to look at

$$
\Gamma_{th.dyn.} = \frac{|W_{\varphi\varphi}|^2}{g_{\varphi\varphi}} - \frac{1}{4} \left( C^A C^A (W_{\varphi\varphi} - \bar{\Gamma}) + \bar{C}^A \bar{C}^A (\bar{W}_{\bar{\varphi}\bar{\varphi}} - \bar{\Gamma}) \right)
$$

From the first term we have to take its quadratic expansion as discussed above, the second is non-trivial when varying both $C^A$-fields only. The former will vanish as all terms of the expansion are still $\sim W_{\varphi\varphi}$ or $\sim \bar{W}_{\bar{\varphi}\bar{\varphi}}$ which vanishes by assumption of unbroken SUSY. The second term is $\sim (W_{\varphi\varphi} - \Gamma)$, which is the derivative of a function with chiral weight zero with respect to $\varphi$. As such a function cannot have a term linear in $\varphi$ and as all non-constant terms vanish in the thermodynamical limit, this derivative must vanish at $\varphi = 0$. Thus it immediately follows that $(C^2)_{cl} = (\bar{C}\bar{C})_{cl} = 0$.

This leads to the main conclusion of this paper: Unbroken supersymmetry does not allow for any non-trivial condensates that can be attached to its Lagrangian in a SUSY covariant way.
Broken Supersymmetry

If supersymmetry is broken ($W, \phi \neq 0$) the chiral weight restricts the superpotential to be of the form $W = a + b\phi$ and thus $W, \phi = \text{constant}, W, \phi\phi \equiv 0$ independently of the value of $W, \phi$ at the minimum. Thus the minimum of the effective potential is completely defined by the maximum of the Kähler metric. Our formalism only tells us that this maximum must be on a circle in the complex $\phi$-plane, but we cannot decide whether $\phi = 0$ is the correct solution or not. In contrast to $N = 1$ where the lowest component of the Lagrangian-multiplet is the essentially non-zero gaugino-condensate, such a restriction does not exist in $N = 2$. Especially our formalism does not exclude the possibility of $N = 2$ SYM being described in the thermodynamical limit by a simple manifold with a single maximum at the origin.

If $\phi = 0$ the variations (54d) and (54e) lead to constraint-equations on the expansion-coefficients of $W$ and $K$, as $\frac{|W, \phi|^2}{g, \phi\phi}$ leads to non-trivial contributions only. Lack of a detailed knowledge of the the dependence of these two functions on the source $m^A$ in the vicinity of the thermodynamical limit a non-trivial condensate of $CC\bar{C}$ is possible but not required. Of course these consistency-conditions become much more complicated when considering $\phi \neq 0$ and they do not allow for any conclusions at this level of calculations.

Besides the condensates discussed here other non-trivial vacuum expectation values are of course possible. On one hand we have all non-renormalizable operators which we do not want to discuss. On the other hand there are renormalizable operators that do not appear in the Lagrangian superfield. Of main interest is the question of a possible Higgs phenomenon due to a non-vanishing vacuum expectation value of the scalars, leading to magnetic monopoles. Under the assumption of unbroken supersymmetry this has been discussed by Seiberg and Witten [4]. In contrast to their calculation the field-strength tensor gets a non-trivial vacuum expectation value when supersymmetry is broken. Striebel [32] showed that magnetic monopole configurations cannot exist in a constant background field. We thus conclude that supersymmetry breaks without touching any other symmetries. Consequently the only Goldstone modes are the two Goldstone fermions from SUSY breaking.

6 Summary and Conclusions

Using the covariant source-extension non-trivial thermodynamical limits of supersymmetric theories can be studied in a supersymmetry covariant way. For pure ($N = 1$ and $N = 2$) SYM theories the effective potential can be derived. Together with the uniqueness of the ground state of non-Abelian gauge theories with respect to the variation of the vacuum angle this links in $N = 2$ the condensate of the Lagrangian to those of the scalars. An acceptable infrared behavior is consistent with broken supersymmetry only. Supersymmetry breaking then lifts the classical and perturbative vacuum degeneracy at a fixed modulus and monopole configurations disappear. Consequently the gauge symmetry remains unbroken.

Interesting questions are left open: The existence of massless Goldstone fermions is re-
stricted by phenomenological results. The coupling of the theory to supergravity thus has to be studied. Moreover only two special models have been considered yet. A general statement about supersymmetry breaking in a wide class of interesting models is not yet possible. Of particularly interest is the fate of perturbatively finite theories like $N = 4$ SYM under non-perturbative quantum corrections.

APPENDIX

QCD, Strong CP and Thermodynamical Limits

In this Appendix we want to explain in detail our arguments that allow us to set in a QCD-like theory without explicit CP-violation except for the topological term any CP-violating phase to zero. The calculations are basically old ideas by one of us [8, 9] (see also [33, 34]). As none of these citations contains a complete discussion of all arguments, we give a rather detailed Appendix considering this problem here.

A.1 Non-Trivial Topology and the Singlet Anomaly

As all calculations in this Appendix are –up to some uninteresting constants– independent of the representation of the fermions we consider QCD with $N_f$ quark flavors only. The free (Minkowskian) QCD Lagrangian is given by

$$\mathcal{L}_0 = -\frac{1}{4C(G)g^2} \text{Tr}(F_{\mu\nu}F^{\mu\nu}) + \frac{i}{2} \bar{\psi}_i \gamma^\mu \partial_\mu \psi_i \quad i = 1, \ldots, N_f$$

The most general (complex) mass-Lagrangian in QCD is given by

$$\mathcal{L}_m = \bar{\psi}_i M_{ik} \frac{1 + \gamma_5}{2} \psi_k + \bar{\psi}_i \tilde{M}_{ik} \frac{1 - \gamma_5}{2} \psi_k \quad (M_{ik})^\dagger = \tilde{M}_{ki} \quad \frac{1 + \gamma_5}{2} \psi_L \psi = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix}$$

Gauge-fixing is done using the standard BRST procedure. We do not derive this in detail here.

Non-Abelian gauge theories have a non-trivial topology. A topological invariant of the Euclidean $SU(N)$ gauge theory is given by (Pontrjagin index, second Chern-number, Atiyah-Singer index theorem)

$$Q = -C_2 = \text{index}D_+ = \frac{1}{16\pi^2 C(G)} \int_{S^4} \text{Tr} F^2 = -\frac{1}{48\pi^2 C(G)} \int_{S^4} \text{Tr}(h_+dh_+)^3 = n$$

In the “physical” language this represents the instanton number and reads (Euclidean space)

$$Q = -\frac{1}{32\pi^2 C(G)} \int d^4x \text{ Tr}(F_{\mu\nu}\tilde{F}^{\mu\nu}) \quad Q \in \mathbb{Z}$$
The physical meaning of the instantons is usually studied in the temporal gauge $A^0 = 0$. In the vacuum states $F_{\mu\nu} = 0$ the connections are then spatially pure gauge $A_i = e^{-\alpha(\vec{x})} \partial_i e^{\alpha(\vec{x})}$, where $\alpha(\vec{x})$ are traceless, anti-hermitian $N \times N$ matrices. In this formulation the Pontrjagin index is equivalent to

$$Q = N_+ - N_- \quad N_{\pm} = \frac{-1}{48\pi^2 C(G)} \int d^3 x \, \varepsilon^{ijk} \text{Tr} \left( (e^\alpha \partial_i e^{-\alpha})(e^\alpha \partial_j e^{-\alpha})(e^\alpha \partial_k e^{-\alpha}) \right) \big|_{t = \pm \infty} \quad (63)$$

Gauge transformations changing the topological sector of a given field configuration are called large gauge transformations. As the operators of the large gauge transformations $H$ must be of the form $[35, 36]$

$$|\theta\rangle = \sum_{N} e^{iN\theta} |N\rangle \quad U(g^k)|N\rangle = e^{ikN}|N\rangle \quad (64)$$

Different values of $\theta$ represent different sectors of the theory in the sense that $\langle \theta'|B|\theta\rangle = 0$ ($\theta \neq \theta'$) for any gauge-invariant operator $B$. Without proof we note that the above structure of the Yang-Mills vacuum is present in quantum theory. Of course the exact realization of both states $|\theta\rangle$ and $|N\rangle$ is then unknown but also unimportant here.

Finally we get for the Euclidean generating functional

$$\lim_{t \to \infty} \langle \theta'|e^{-Ht}|\theta\rangle = 2\pi \delta(\theta - \theta')Z_{\theta} \quad Z_{\theta} = \sum_{Q} e^{-iQ\theta} \int \mathcal{D}X_Q \exp(-S_{Euc}) \quad (65)$$

Treating $\theta$ as a free parameter its effect is to add the term

$$\mathcal{L}_{Euc} \to \mathcal{L}_{Euc} + \frac{i\theta}{32\pi^2 C(G)} \text{Tr} F_{\mu\nu} \tilde{F}^{\mu\nu} \quad \mathcal{L}_{Mink} \to \mathcal{L}_{Mink} - \frac{\theta}{32\pi^2 C(G)} \text{Tr} F_{\mu\nu} \tilde{F}^{\mu\nu} \quad (66)$$

to the Lagrangian and again using the path integral measure over all instanton configurations.

In the chiral limit QCD is classically invariant under $SU(N_f)_R \times SU(N_f)_L \times U(1) \times U(1)_A$. However $U(1)_A$ is anomalous with the (Minkowskian) anomaly $[37, 38, 35, 36, 39]$

$$\partial_\mu J^\mu_5 = \partial_\mu (\bar{\psi} \gamma^\mu \gamma_5 \psi_i) = 2N_f q + i\bar{\psi} \tilde{M} \psi \quad \tilde{M} = \gamma_5 (M + \bar{M}) + (M - \bar{M}) \quad (67)$$

where $Q = \int d^4 x \, q(x)$ is the Pontrjagin index. The anomaly allows to rotate complex phases of the mass matrix away by means of $\theta \to \tilde{\theta} = \theta + \text{arg} \det M$.

The anomalous term is a total divergence of the Chern-Simons form: $\text{Tr} F^2 = dQ_3$ with $Q_3 = \text{Tr}(AdA + \frac{2}{3} A^3)$. Thus we can construct a new conserved current

$$F^\mu_5 = J^\mu_5 + 2N_f K^\mu \quad K^\mu = \frac{1}{16\pi^2 C(G)} \varepsilon^{\mu\nu\rho\sigma} \text{Tr}(A_\nu \partial_\rho A_\sigma + \frac{2}{3} A_\nu A_\rho A_\sigma) \quad (68)$$

with $\partial_\mu F^\mu_5 = 0$ in the chiral limit. However the new conserved charge $Q_5 = \int d^3 x \, F^0_5$ is not gauge invariant, but $U_5 = \exp(-i\frac{Q_5}{N_f})$ generates the discrete symmetry corresponding to a shift $\theta \to \theta + 2\pi$.

\footnote{Our terminology allows a theory to have several vacua (e.g. $|N\rangle$) but one unique ground state $|\Omega\rangle$ only.}
A.2 Thermodynamical Constraints on the θ-Parameter

We now want to study the generating functional (65) more in detail. First we fix the remaining gauge freedom in the standard way by imposing the constraints \( A_i|_{t=-\infty} = 0, e^{i\alpha(x)} \to 1, (\vec{x} \to \infty) \). Then the generating functional \( Z_{\theta_V} \) is the sum over the vacuum-to-vacuum transition amplitudes from \( N_-=0 \) to an arbitrary \( N_+=N \):

\[
Z_{\theta_V} = \sum_N \exp(iN\theta_V) \langle N| \int \mathcal{D}X \exp(-S_{\text{Euc}})|0\rangle
\]

\[
= \sum_N \exp(iN(\theta_V - \bar{\theta})) \langle N| \int \mathcal{D}X \exp(-S_0)|0\rangle = \sum_N \langle N| \int \mathcal{D}X \exp(-S_{\theta_V})|0\rangle
\]

In the above equation \( \langle N| \) and \( |0\rangle \) indicate the index of the states at \( t = \pm \infty \) and \( \bar{\theta} \) is the effective coupling constant \( \bar{\theta} = \theta + \text{arg det } M \). Without loss of generality we will assume in the following that all phases of the mass matrix have been rotated into the \( \theta \) parameter and thus \( \bar{\theta} = \theta \). Then the action \( S_0 \) is the usual QCD action without \( \theta \)-term and \( S_{\theta_V} \) is given by

\[
S_{\theta_V} = S_0 + \frac{i(\theta - \theta_V)}{2\pi^2 C(G)} \int d^4x \text{ Tr } F \tilde{F}.
\]

The crucial point in our discussion is the interpretation of the two parameters \( \theta_V \) and \( \theta \). The way we defined them, \( \theta \) is the coupling constant of a renormalizable and gauge invariant operator as the quark masses or the Yang-Mills coupling constant. Of course this coupling constant may be chosen arbitrarily but fixed. The parameter \( \theta_V \) on the other hand is a free phase of an off-diagonal S-matrix element. A priori it is also arbitrary but the dynamics of the system may determine its value uniquely for a given set of coupling constants. In our opinion \( \theta_V \) actually has to be dynamical: For a given set of external parameters (i.e. coupling constants) a theory must have an unique ground-state. In order to satisfy this uniqueness \( \theta_V \) must either be irrelevant or dynamical as it does not belong to the set of external parameters. Thus the overall coupling constant \( \theta - \theta_V \) of the CP-violating operator \( F \tilde{F} \) may indeed be subject to dynamical constraints. Calculating the thermodynamical limit of the associated operator we want to show that this is the case and that the dynamical value of the \( \theta_V \)-parameter in a theory without explicit CP-breaking (except for the \( F \tilde{F} \) term) is \( \theta - \theta_V = 0 \).

The interpretation of the different \( \theta_V \)-vacua as being caused by tunneling between topological vacua \( |N\rangle \) is a gauge-dependent interpretation restricted to the usage of temporal gauge \[40\]. However the existence of a free phase \( \theta_V \) is not, as shown in \[41\]. The existence of this free parameter is the important difference between the “normal” coupling constants \( (M_{ik}, g) \) and the “topological” coupling constant \( \theta \).

Including the possible sources the generating functional may be written as

\[
Z[J] = \int \mathcal{D}X \ e^{-S_{\text{Euc}}} \quad S_{\text{Euc}} = S_{\text{QCD}} + S_{\text{GF}} + S_J
\]
where we include a term $\sim \text{Tr} F \tilde{F}$ with an arbitrary “coupling constant” $\theta - \theta_V$ in $\mathcal{L}_{QCD}$. Considering the sources we are mainly interested in those for local composite operators:

$$\mathcal{L}_J = -\frac{1}{4C(G)} \text{Tr}(F_{\mu\nu}F_{\mu\nu}) - \frac{1}{4C(G)} \vartheta \text{Tr}(F_{\mu\nu}\tilde{F}_{\mu\nu}) + \bar{\psi}_i \sigma^{ij} \psi_j + \ldots$$  \hspace{1cm} (71)

All sources are subject to the boundary conditions $\lim_{x \to \infty} J(x) = 0$. For the gauge boson and the quark field configurations we introduce the boundary conditions

$$2 \frac{\delta}{\delta \tau(x)} Z[J] = \langle \frac{1}{2C(G)} \text{Tr}(F_{\mu\nu}F_{\mu\nu}) e^{-S} \rangle = N_c B^2 Z[J]$$

$$2 \frac{\delta}{\delta \vartheta(x)} Z[J] = \langle \frac{1}{2C(G)} \text{Tr}(F_{\mu\nu}\tilde{F}_{\mu\nu}) e^{-S} \rangle = N_c B \tilde{B} Z[J]$$

$$\frac{\delta}{\delta \sigma^{ij}(x)} Z[J] = -\langle \bar{\psi}_i \psi_j e^{-S} \rangle = M_{ij}^S Z[J]$$  \hspace{1cm} (72)

The parameters $B$ and $\tilde{B}$ are related by the inequality $|\tilde{B}| \leq B$. In terms of the energy functional and of the effective action we get the following variations with respect to the sources and associated operators respectively:

$$\frac{\delta}{\delta J(x)} W[J] = -\tilde{J}(x) \hspace{1cm} \frac{\delta}{\delta \tilde{J}(x)} \Gamma[\tilde{J}] = -J(x)$$  \hspace{1cm} (73)

In our case the associated operators are given by

$$\tilde{\tau}(x) = \langle \Omega | \frac{1}{4C(G)} \text{Tr}(F_{\mu\nu}F_{\mu\nu}) | \Omega \rangle \hspace{1cm} \tilde{\vartheta}(x) = \langle \Omega | \frac{1}{4C(G)} \text{Tr}(F_{\mu\nu}\tilde{F}_{\mu\nu}) | \Omega \rangle \hspace{1cm} \ldots$$  \hspace{1cm} (74)

As discussed in detail in [33] a non-vanishing vacuum expectation value of a given local operator for vanishing associated source leads to a spontaneous parameter. We indicate such a parameter by a star. The spontaneous parameter $B^*$ associated with the gluon condensate would then be

$$\tilde{\tau}(x) = \langle \Omega | \frac{1}{4C(G)} \text{Tr}(F_{\mu\nu}F_{\mu\nu}) | \Omega \rangle \hspace{1cm} \frac{\delta}{\delta \tilde{\tau}} \Gamma = \left. \frac{\partial}{\partial B^*} \Gamma \right|_{B^{*}=0} = 0$$  \hspace{1cm} (75)

Operators are in general subject to renormalization. Considering quantum effects we thus assume the above operators to be renormalized. The spontaneous parameters are then functions of the (renormalization group invariant) generalized coupling constants: $B^* = B^*(\Lambda_{QCD}, m_s, \theta)$, $\tilde{B}^* = \tilde{B}^*(\Lambda_{QCD}, m_s, \theta)$. Since the above operator gives (by hypothesis) the ground-state of the theory, $B^*$ and $\tilde{B}^*$ must be of the form

$$B^* = C(\theta)(\Lambda_{QCD})^2 \hspace{1cm} \tilde{B}^* = \tilde{C}(\theta)(\Lambda_{QCD})^2$$  \hspace{1cm} (76)
From the integrated variation with respect to $\vartheta$

$$\int V d^4x \tilde{\vartheta}(x) = -\frac{\partial}{\partial \vartheta} W[J]$$  \hspace{1cm} (77)

we get the thermodynamical limit determining the value of $\tilde{B}^*$:

$$\frac{1}{2} N_c B^* \tilde{B}^*(\vartheta) = -\lim_{V \to \infty} \frac{1}{V} \frac{\partial}{\partial \vartheta} W[\vartheta, \ldots] \bigg|_{J \to 0}$$ \hspace{1cm} (78)

The explicit calculation of this limit is now similar to a limiting process of a spin chain in statistical mechanics: We consider a spin-chain in a magnetic field $\vec{B}$ of an arbitrary but fixed direction. The question is whether the angle between the spins and the magnetic field may be chosen non-trivially in the limit of an infinitely large chain and zero temperature leading to the possibility of CP-violating configurations. Calculating the non-trivial limit, the stabilization of this situation during the limiting process would require an infinite amount of energy with respect to a dynamical variable. Consequently non-trivial phases relax leading to a non-equilibrium state violating the (approximate) translation invariance during the limiting process. This flipping of spins in the limit $T \to 0$ also takes place when considering the ground-state of a finite chain. There is however an important difference between the situations with $l = \text{(finite)}$ and $l \to \infty$ respectively. While the transverse susceptibility is well defined at a finite length $\frac{\partial^2}{\partial \vartheta^2} Z$ leads out of the Hilbert space for $l \to \infty$. Thus the translation-invariance is getting restored at the trivial angle and the CP-violating configurations do not exist in the limit, although we may choose an arbitrary angle $\theta$ between the spin and the $\vec{B}$-field direction at the beginning.

In the case of QCD we must be careful to choose a well-defined limit which forces to have non-vanishing values for both $B$ and $M^S$. To ensure this we choose sources $\tilde{\tau} \neq 0$ and $\tilde{\sigma} \neq 0$ inside a sub-volume $V_{sub}$, but vanishing sources on the complement $V \setminus V_{sub}$ and then take the infinite volume limit $V_{sub} \subset V \to \infty$, as discussed in detail in [6]. Of course we have to take the limit $J \to 0$ for all sources in the end, as the equilibrium conditions demand both equations in (73) to be valid simultaneously. Then the spontaneous parameter $(M^S)^*$ and $B^*$ may vanish again. We do not want to discuss this purely dynamical problem here, as it does not change our conclusions.

Considering additional infrared problems due to infinite correlation lengths we just note that $L$ and $L'$ defined as

$$\langle \Omega | F(x) \mathcal{O}_S F(y) | \Omega \rangle \sim \exp(-\frac{z^2}{L^2}) \hspace{1cm} \langle \Omega | F \tilde{F}(x) F \tilde{F}(y) | \Omega \rangle \sim \exp(-m_{qf} L')$$ \hspace{1cm} (79)

with $z = x - y$, are both finite.

The leading terms of the energy-density and of $\tilde{\tau}$ are given by [38, 35] (dropping a possible constant independent of $\theta$)

$$\frac{E(\theta)}{V} = -2 Ke^{-S} \cos \theta \hspace{2cm} \tilde{\tau} = -64 \pi^2 i Ke^{-S} \sin \theta$$ \hspace{1cm} (80)
where $K$ is a constant independent of $\theta$ and non-zero due to our choice of sources. Replacing $\theta$ again by $\theta - \theta_V$ the absolute minima are at

$$\theta_V = \theta \mod 2\pi$$

and in the thermodynamical limit the free parameter takes one of these values removing all spontaneous CP violation and thus setting $\tilde{B}^*$ to zero. Thus we get the following system of equilibrium conditions (still under assumption of a real mass term)

$$\frac{\delta\Gamma}{\delta J(x)} = 0 \quad \theta - \theta_V = 0 \quad \frac{\delta W}{\delta \vartheta(x)} = 0 \quad \frac{\delta W}{\delta \tau(x)} = -\frac{1}{2}N_c(B^*)^2$$

(82)

If we allow again for a general mass matrix the effective action can only depend on the combination $(\theta - \theta_V) + \arg\det M$. The dynamical equations for $\theta_V$ are thus:

$$\frac{\delta W}{\delta \vartheta(x)} = 0 \quad (\theta - \theta_V) + \arg\det M = 0$$

(83)

The dynamical constraints now restore the chiral invariance when evaluated with respect to the ground-state and we get

$$\langle \Omega|\partial_\mu J_\mu^5|\Omega \rangle = 0 = i(M + \bar{M})\langle \Omega|\bar{\psi}\gamma_5\psi|\Omega \rangle + i(M - \bar{M})\langle \Omega|\bar{\psi}\psi|\Omega \rangle$$

(84)

which is a nontrivial dynamical equation for the quark-condensates. By defining the operators $\tilde{\sigma}_L$ and $\tilde{\sigma}_R$ as $\tilde{\sigma}_{L/R} = \bar{\psi}\frac{1+\gamma_5}{2}\psi$ the above constraint becomes

$$M\frac{\delta W}{\delta \sigma_L} = \bar{M}\frac{\delta W}{\delta \sigma_R}$$

(85)

It is again obvious that the dynamics minimize CP violation, i.e. it minimizes the relative angle between the mass-matrix and the quark-condensates. Moreover equations (83) and (84) or (85) directly connect the phase of the quark-condensates to the effective $\theta$-parameter appearing in the Lagrangian. To be explicit equation (85) tells us that $\arg\det M = \arg\det \tilde{\sigma}_R = -\arg\det \tilde{\sigma}_L$ and thus we may rewrite equation (83) as

$$\frac{\delta W}{\delta \vartheta(x)} = 0 \quad (\theta - \theta_V) + \arg\det \tilde{\sigma}_R = 0$$

(86)

If we are considering a theory with massless quarks, we can still attach non-trivial sources $\sigma_{L/R}$ and thus the thermodynamical limit still connects the $\theta$-angle to the phase of the quark-condensate as given in the equation above. As all calculations in this section are independent of the representation of the quarks, the results are also applicable to supersymmetric gauge theories.

Of course the question arises whether the partition function derived here $Z(\theta \equiv 0) = Z_0$ is related to the partition function of the standard interpretation restricted to the value $\theta = 0$
$Z(\theta)|_{\theta=0}$. The discussion of this point is now completely analogous to the above example of a spin chain. In the standard interpretation of global topological objects $Z(\theta)$ is defined for all values of $\theta$ with a unique spectrum (in the case of massless fermions it is even independent of $\theta$). Thus we may safely travel along the circle of the $\theta$-phase and consequently all variations $\frac{\delta}{\delta \theta} Z(\theta)$ are globally and locally well defined (of course we assume here physically relevant changes of the $\theta$ angle and not just variable transformations in the anomaly term).

The situation is however completely different in our calculation: At non-trivial values of $\theta$ the path integral does not converge and thus the partition function need not even be defined. Therefore global changes of the $\theta$-parameter lead out of the Hilbert-space. This just means that a global change of the coupling-constant $\theta$ leads to a dynamical reaction of $\theta_V$ such that the effective parameter remains zero. Formally this can be written as a constraint on the global variation with respect to $\theta$ (or $\theta_V$), namely $\frac{\delta}{\delta \theta} W \equiv 0$ – together with (78) this is just another way to see that $\tilde{B}^*$ must be zero.

This completely different mathematical behavior makes it reasonable that the two limiting processes $\lim V \to \infty$ and $\lim \theta \to 0$ need not be interchangeable and thus in general $Z_0 \neq Z(\theta)|_{\theta=0}$.

### A.3 Concluding Remarks

We have discussed in this Appendix how non-perturbative dynamics lead to a natural solution of the strong CP problem without re-introducing the $U(1)$ problem. Considering this point we want to make two remarks. Witten [42] and Veneziano [43] argued that there exists a relation between the mass of the $\eta'$ and the topological susceptibility $\frac{\delta^2}{\delta \theta^2} W|_{\theta=0}$. In principle such a relation does not stand in contradiction to our analysis, however the direct meaning of $\frac{\delta^2}{\delta \theta^2} W$ is far from clear, as we are leaving the Hilbert space when going over to non-vanishing $\theta$’s. Notwithstanding the mass-square of the $\eta'$ can be obtained through local variations $\frac{\delta}{\delta \theta}$.

It has also been proven by Shifman, Vainshtein and Zakharov [44] that, if it were possible to start from a QCD Lagrangian with a non-trivial $\theta$-term, dynamics can not resolve both the $U(1)$ and the strong CP problem. However our procedure removes all $\theta$-angles ab initio through a complete analysis of the thermodynamical limit showing that this assumption is not valid.

Our last remark considers a different suggestion to solve the strong CP problem. It has been shown by Banerjee, Mitra and Chatterjee [45] that complex phases in the mass term may be decoupled from the $\theta$-term by using a representation of Euclidean fermions different from the usual Osterwalder-Schrader scenario. Although this apparently resolves the fine-tuning problem in the Standard Model we are left with the unsatisfactory situation that there would exist two fundamentally different version of QCD. Our analysis shows that this need not be the case because in this alternative version of QCD non-trivial $\theta$-angles relax thermodynamically, too. Though the two versions may be different technically, they are equivalent after studying non-perturbative dynamics.
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