Parameter estimation for Vasicek model driven by a general Gaussian noise

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Abstract This paper developed an inference problem for Vasicek model driven by a general Gaussian process. We construct a least squares estimator and a moment estimator for the drift parameters of the Vasicek model, and we prove the consistency and the asymptotic normality. Our approach extended the result of Xiao and Yu (2018) for the case when noise is a fractional Brownian motion with Hurst parameter $H \in [1/2, 1)$.

Keywords Vasicek model · product formula · Gaussian process · fourth moment theorem

1 Introduction

We are interested in the statistical interference for the Vasicek model defined by the following stochastic differential equation (SDE)

$$dX_t = k(\mu - X_t)dt + \sigma dG_t, t \in [0, T], T \geq 0, x_0 = 0,$$

(1.1)

where $(G_t)_{t \geq 0}$ is a general one-dimensional centered Gaussian process. We noticed that the volatility parameter $\sigma > 0$ can be estimated by the power variation method. Without loss of generality, we assume that $\sigma = 1$. Assuming that there is only one trajectory $(X_t, t \geq 0)$, we construct a least squares estimator and a moment estimator, and study its asymptotic behavior.

The Vasicek model of Vasicek [1977] has a wide range of applications in many fields, such as economics, finance, biology, medical and environmental sciences. In the economic field, it has been used to describe the fluctuation of interest rates, please refer to Huang and Huang [2012]. In the financial field, it can also be used as a random investment model in Wu et al. [2020].

If the parameter in the drift function of model is unknown, an important problem is to estimate the drift coefficient based on the observation. When the noise is Brownian motion, the statistical inference for Vasicek process are well studied in the literature, e.g. a maximum likelihood method was proposed in Fergusson and Platen [2015], whereas Yang [2013] studied a least squares approach.

The research methods are different when the drift parameter $k > 0$ or $k < 0$. When the Brownian motion in the Vasicek model was replaced by the fractional Brownian motion Xiao and Yu [2017], with the Hurst parameter greater than or equal to one-half, the asymptotic theory for $k$ was proved, the stationary case for $k > 0$, the explosive case for $k < 0$, and the null recurrent case for $k = 0$, respectively. In these cases, the least squares method is considered, and when $k > 0$, the moment estimation method of Hu and Nualart [2010] is also considered.
Based on this, Xiao et al. [2018] extended their work to Vasicek-type models driven by subfBm. For the case of non-ergodic and null recurrent, the least squares method was studied. In addition, it can be extended to a more general self-similar process, such as Hermite process, (see Nourdin and Tran [2019]).

Moreover, when the Brownian motion is replaced by a Gaussian process with self-similarity Yu [2020], based on some conditions of $G$, the least squares method was studied and its asymptotic behavior was completed with non-ergodic case $k < 0$.

In this paper, we consider the Vasicek model driven by a general Gaussian process that fails to be self-similar or have stationary increments, when the persistence parameter $k$ is positive.

This paper refers to Chen and Zhou [2020] and makes the following assumptions about the second-order partial derivative form of the covariance function of a general Gaussian process.

**Assumption 1.1** For $\beta \in \left(\frac{1}{2}, 1\right)$, the covariance function $R(t, s) = E[G_tG_s]$ for any $t \neq s \in [0, \infty)$

\[
\frac{\partial^2}{\partial t \partial s} R(t, s) = C_\beta |t-s|^{2\beta-2} + \Psi(t, s),
\]

where the constants $C_\beta > 0, C'_\beta > 0$ do not depend on $T$. Moreover, for any $t \geq 0$, $R(0, t) = 0$.

We can see that fractional Brownian motion and some other Gaussian processes satisfy Assumption 1.1. From this assumption, we obtain the result as follow. When $k > 0$, the estimator of $\mu$ is continuous-time sample mean, (see Hu and Nualart [2010]).

\[
\hat{\mu} = \frac{1}{T} \int_0^T X_t \, dt.
\]

Moreover, following Xiao and Yu [2017], when $k > 0$, the second moment estimator is given by

\[
\hat{k} = \left[ \frac{T \int_0^T X_t^2 \, dt - (\frac{T}{T} \int_0^T X_t \, dt)^2}{C_\beta T^{(2\beta - 1)}} \right]^{\frac{k}{2\beta}}.
\]

The LSE is motivated by the argument of minimize a quadratic function of $k$ and $\mu$, respectively

\[
L(k, \mu) = \int_0^T (X_t - k(\mu - X_t))^2 \, dt.
\]

Solving the equations, we can obtain the LSE of $k$ and $\mu$, denoted by $\hat{k}_{LS}$ and $\hat{\mu}_{LS}$, respectively.

\[
\hat{k}_{LS} = \frac{X_T \int_0^T X_t \, dt - T \int_0^T X_t \, dt \int_0^T X_t \, dt}{T \int_0^T X_t^2 \, dt - (\int_0^T X_t \, dt)^2},
\]

\[
\hat{\mu}_{LS} = \frac{X_T \int_0^T X_t^2 \, dt - (\int_0^T X_t \, dt)^2}{X_T \int_0^T X_t \, dt - T \int_0^T X_t \, dt \int_0^T X_t \, dt}.
\]

where the integral $\int_0^T X_t \, dt \, dX_t$ can be interpret as an Itô-Skorohod integral (Xiao and Yu [2017]).

In this paper, we will prove the strong consistency and the central limit theorems for the four estimators, these results are stated in the following theorems.

**Theorem 1.2** When assumption 1.1 is satisfied, both the least squares estimator and the moment estimator of $\mu$ and $k$ are strongly consistent, i.e.

\[
\lim_{T \to \infty} \hat{\mu} = \mu, \lim_{T \to \infty} \hat{\mu}_{LS} = \mu, \ a.s..\]

\[
\lim_{T \to \infty} \hat{k} = k, \lim_{T \to \infty} \hat{k}_{LS} = k, \ a.s..\]
Theorem 1.3 Assume assumption 1.1 is satisfied. When $\beta \in (1/2, 1)$, both $T^{1-\beta} (\hat{\mu} - \mu)$ and $T^{1-\beta} (\hat{\mu}_{LS} - \mu)$ are asymptotically normal as $T \to \infty$, namely,

$$
T^{1-\beta} (\hat{\mu} - \mu) \xrightarrow{law} N(0, \frac{1}{k^2}), T^{1-\beta} (\hat{\mu}_{LS} - \mu) \xrightarrow{law} N(0, \frac{1}{k^2}).
$$

(1.11)

when $\beta \in (\frac{1}{2}, \frac{3}{4})$, both $\sqrt{T}(\hat{k} - k)$ and $\sqrt{T}(\hat{k}_{LS} - k)$ are asymptotically normal as $T \to \infty$, namely,

$$
\sqrt{T}(\hat{k}_{LS} - k) \xrightarrow{law} N(0, 4ka^2 \sigma^2_\beta), \sqrt{T}(\hat{k} - k) \xrightarrow{law} N(0, \sigma^2_\beta k/4\beta^2)
$$

(1.12)

where $a = C_\beta \Gamma(2\beta - 1)k^{-2\beta}$, $\sigma^2_\beta = (4\beta - 1)[1 + \frac{\Gamma(3-4\beta)\Gamma(4\beta-1)}{\Gamma(2\beta)\Gamma(2-2\beta)}].$

The outline of the paper is the following. First, we provide some basic elements of stochastic calculus with respect to the Gaussian process which are helpful for some of the arguments we use and some of the technical results used in various proofs. In Sect. 3 and 4 we derive our estimator, prove consistency and asymptotic normality respectively.

2 Preliminary

In this section, we describe some basic facts on stochastic calculus with respect to the Gaussian process and recall the main results in Nualart et al. [2005] concerning the central limit theorem for multiple integrals, for more complete presentation on the subject can be find in Chen and Zhou [2020].

Defined on a complete probability space $(\Omega, \mathcal{F}, P)$, the $\mathcal{F}$ is generated by the Gaussian family $G$. Denote $G = G_t, t \in [0, T]$ as a continuous centered Gaussian process, and suppose in addition that the covariance function $R$ is continuous.

$$
E(G_sG_t) = R(s,t), s, t \in [0, T],
$$

(2.1)

let $\mathcal{E}$ denote the space of all real valued step functions on $[0, T]$. The Hilbert space $\mathcal{H}$ is defined as the closure of $\mathcal{E}$ endowed with the inner product.

$$
\langle 1_{[a,b]}, 1_{[c,d]} \rangle_{\mathcal{H}} = E((G_b - G_a)(G_d - G_c)).
$$

(2.2)

If $G = G_h, h \in \mathcal{H}$ as the isonormal Gaussian process on the probability space, indexed by the elements in the Hilbert space $\mathcal{H}$, $G$ is a Gaussian family of random variables as follows

$$
E(G_gG_h) = \langle g, h \rangle_{\mathcal{H}}, \forall g, h \in \mathcal{H}.
$$

(2.3)

The following proposition is an extension of Theorem 2.3 of Jolis [2007], which gives the inner products representation of the Hilbert space $\mathcal{H}$ and the References therein.

Proposition 2.1 Denote $V_{[0, T]}$ as the set of bounded variation functions on $[0, T]$, then $V_{[0, T]}$ in dense in $V$ and we have

$$
\langle f, g \rangle_{\mathcal{H}} = \int_{[0, T]^2} R(t, s) v_f(dt)v_g(ds), \forall f, g \in V_{[0, T]},
$$

(2.4)

where $v_g$ is the Lebesgue-Stieltjes signed measure associated with $g^0$ defined as

$$
g^0 = \begin{cases} 
g(x), & \text{if } x \in [0, T]; \\ 0, & \text{otherwise}. \end{cases}
$$

(2.5)

Furthermore, if covariance function $R(t, s)$ satisfies Assumption 1.1, then

$$
\langle f, g \rangle_{\mathcal{H}} = \int_{[0, T]^2} R(t, s) \frac{\partial^2 R(t, s)}{\partial t \partial s} dt ds, \forall f, g \in V_{[0, T]}.
$$

(2.6)
Corollary 2.2 If Assumption 1.1 is satisfied, there exists a constant $C > 0$ independent of $T$, such that for all $s, t \geq 0$,

$$\mathbb{E}(G_t - G_s)^2 \leq C|t - s|^{2\beta},$$

(2.7)

and when $s = 0$, we have $\mathbb{E}(G_t^2) \leq C\beta t^{2\beta}$.

Proof

$$\mathbb{E}(G_t - G_s)^2 = \int_{[s,t]^2} \frac{\partial^2 R(u,v)}{\partial u \partial v} \, du \, dv$$

$$\leq \int_{[s,t]^2} |u - v|^{2\beta - 2} \, du \, dv + \int_{[s,t]^2} |uv|^{\beta - 1} \, du \, dv \leq C|t - s|^{2\beta}.$$  

(2.8)

Hence, we deduce the desired result.

Remark 2.3 Denote $\mathcal{S}^{\otimes p}$ and $\mathcal{S}^{\mathcal{O}p}$ as the pth tensor product and the pth symmetric tensor product of the Hilbert space $\mathcal{S}$. Let $H_p$ be the Wiener-Itô stochastic integral. Then the map $I_p$ provides a linear isometry between $\mathcal{S}^{\otimes p}$ and $H_p$. Here $H_0 = R$ and $I_0(x) = x$ by the convention.

We choose $e_k$ to be a complete orthonormal system in the Hilbert space $\mathcal{S}$. The q-th contraction between $f \in \mathcal{S}^{\otimes n}$ and $g \in \mathcal{S}^{\mathcal{O}n}$ is an element in $\mathcal{S}^{|n+n-2q|}$ that is defined by

$$f \otimes_g q = \sum_{i_1, \ldots, i_q=1}^{\infty} \langle f, e_{i_1} \otimes \cdots \otimes e_{i_q} \rangle_{\mathcal{S}^{\otimes q}} \otimes \langle g, e_{i_1} \otimes \cdots \otimes e_{i_q} \rangle_{\mathcal{S}^{\mathcal{O}q}}, \text{ for } q = 1, \ldots, m \wedge n.$$  

(2.10)

Then we have the following product formula for the multiple integrals,

$$I_p(g)I_q(h) = \sum_{r=0}^{p+q} r! \binom{p}{r} \binom{q}{r} I_{p+q-2r}(\hat{g} \otimes \hat{h}).$$  

(2.11)

The following theorem 2.3, known as the fourth moment theorem, provides necessary and sufficient conditions for the asymptotic theory of the persistent parameter $k$, (see Nualart et al. [2005]).

Theorem 2.4 Let $n \geq 2$ be a fixed integer, consider a collection of elements $f_T, T > 0$ such that $f_T \in \mathcal{S}^{\otimes n}$ for every $T > 0$. Assume further that

$$\lim_{T \to \infty} \mathbb{E}[|I_n(f_T)|^2] = \lim_{T \to \infty} n! ||f_T||_{\mathcal{S}^{\otimes n}}^2 = \sigma^2.$$  

(2.12)

then the following conditions are equivalent:

1) $\lim_{T \to \infty} \mathbb{E}[|I_n(f_T)|^4] = 3\sigma^4$.

2) For every $q=1, \ldots, n-1$, $\lim_{T \to \infty} \mathbb{E}[|f_T \otimes \cdots \otimes f_T|_{\mathcal{S}^{\otimes 2(n-q)}}] = 0$.

3) As $T$ tends to infinity, the $n$-th multiple integrals $\{I_n(f_T), T \geq 0\}$ converge in distribution to a Gaussian random variable $N(0, \sigma^2)$. 


3 Strong consistency

3.1 The moment estimator

If $k > 0$, we can consider estimators of $k$ and $\mu$. The estimators are motivated by Hu and Nualart [2010] where the stationary and ergodic properties of a process were used to construct a new estimator for $k$ in the FOU model. Then we first consider strong consistency of $\hat{\mu}$, the solution of the model in (1.1) is given by

$$X_t = \mu(1 - e^{-kt}) + \int_0^T e^{-k(t-s)}dG_s,$$

so the estimator of $\mu$ is the continuous-time sample mean

$$\hat{\mu} = \frac{1}{T} \int_0^T X_t dt.$$

Combining (3.1) and (3.2), we can rewrite $\hat{\mu}$ as,

$$\hat{\mu} = \frac{1}{T} \int_0^T (1 - e^{-kt})dt + \frac{1}{T} \int_0^T (\int_0^t e^{-k(t-s)}dG_s)dt.$$

The next two propositions are about the asymptotic behavior of the second moment of $\hat{\mu}$.

Remark 3.1 (see Chen and Zhou [2020]) For a function $\phi(r) \in \mathcal{V}_{[0,T]}$, we define two norms as

$$\|\phi\|_{\mathcal{A}_1}^2 = C_\beta \int_{[0,T]^2} \phi(r_1,\phi(r_2))|r_1 - r_2|^{2\beta-2}d_1d_2,$n
$$\|\phi\|_{\mathcal{B}_1}^2 = C_\beta' \int_{[0,T]^2} |\phi(r_1,\phi(r_2))|(r_1 - r_2)^{\beta-1}d_1d_2.$$

(3.5)

For a function $\varphi(s) \in [0,T]^2$, define an operator from $\mathcal{V}_{[0,T]}\otimes^2$ to $\mathcal{V}_{[0,T]}$ as follows,

$$(K\varphi)(r) = \int_0^T |\varphi(r,u)|u^{\beta-1}du.$$

(3.6)

Proposition 3.2 (see Chen and Zhou [2020]) Suppose that Assumption 1.1 holds, then for any $\phi(r) \in \mathcal{V}_{[0,T]}$,

$$|\|\phi\|_{\mathcal{A}_1}^2 - \|\phi\|_{\mathcal{B}_1}^2| \leq \|\phi\|_{\mathcal{B}_1}^2,$$

(3.7)

and for any $\varphi, \psi \in ((V)_{[0,T]} \otimes^2$,

$$|\|\phi\|_{\mathcal{B}_2}^2 - \|\phi\|_{\mathcal{B}_1}^2| \leq \|\phi\|_{\mathcal{B}_1}^2 + 2C_\beta\|K\varphi\|_{\mathcal{B}_1}^2,$$

$$|\langle \varphi, \psi \rangle_{\mathcal{B}_2} - \langle \varphi, \psi \rangle_{\mathcal{B}_1}| \leq \|\varphi\|_{\mathcal{B}_1}^2 + 2C_\beta\|K\varphi, K\psi\|_{\mathcal{B}_1}|.$$

(3.8)

The next two propositions are about the asymptotic behaviors of the second moment of $F_T$ and the increment $F_t - F_s$ with $0 \leq t, s \leq T$, respectively. First, we need a technical lemma as follows.
Lemma 3.3 Assume $\beta \in (0, 1)$, there exists a constant $C > 0$ such that for any $T \in [0, \infty)$,
\begin{equation}
    e^{-kT} \int_0^T e^{kr^{\beta-1}} dr \leq C(1 \land T^{\beta-1}),
\end{equation}
(see lemma 3.3 of Chen and Zhou [2020]).

Proposition 3.4 When $\beta \in (\frac{1}{2}, 1)$, we can find that
\begin{equation}
    E(F_T^2) \leq C_2 T^{2\beta}.
\end{equation}

Proof By Itô isometry, we have
\begin{equation}
    \mathbb{E}[|F_T|^2] = \|F_T\|_0^2,
\end{equation}
the inequality (3.7) implies that
\begin{equation}
    \|F_T\|_0^2 - \|F_T\|_{\beta_1}^2 \leq \|F_T\|_{\beta_2}^2,
\end{equation}
clearly, we have
\begin{equation}
    0 \leq \int_0^T (1 - e^{-k(T-u)}) u^{\beta-1} du \leq \int_0^T u^{\beta-1} du \leq CT^{\beta},
\end{equation}
so,
\begin{equation}
    \|F_T\|_{\beta_2}^2 = \left( \int \left( 1 - e^{-k(T-u)} \right) u^{\beta-1} du \right)^2 \leq CT^{2\beta}.
\end{equation}

Meanwhile, we have
\begin{align}
    \|F_T\|_{\beta_1}^2 &= \frac{1}{k^2} \int_{[0,T]^2} (1 - e^{-k(T-u)})(1 - e^{-k(T-v)}) |u - v|^{2\beta-2} dudv \\
    &\leq \frac{1}{k^2} \int_{[0,T]^2} |u - v|^{2\beta-2} dudv + \int_{[0,T]^2} e^{-k(T-u)} e^{-k(T-v)} |u - v|^{2\beta-2} dudv \\
    &\leq \frac{1}{k^2} \left[ \frac{T^{2\beta}}{(2\beta-1)^{\beta}} + \frac{\Gamma(2\beta-1)}{k^{2\beta}} \right].
\end{align}

Hence, $\|F_T\|_0^2 \leq C_2 T^{2\beta}$, we obtain the desired result in (3.10).

Proposition 3.5 Assume that assumption 1.1 holds, there exists a constant $C > 0$ independent of $T$ such that for all $s, t \geq 0$,
\begin{equation}
    E(|F_t - F_s|^2) \leq C_{0, \beta} |t - s|^{2\beta}.
\end{equation}

Proof Firstly, the equality (3.4) implies that
\begin{equation}
    E(|F_t - F_s|^2) \leq 2E(|G_t - G_s|^2) + 2E(|Z_t - Z_s|^2).
\end{equation}
From lemma 1.2, we have $E(G_t - G_s)^2 \leq C_2 |t - s|^{2\beta}$. Furthermore, we have
\begin{align}
    E(|Z_t - Z_s|^2) &= E[\int_s^t e^{-k(t-u)} dG_u - \int_s^t e^{-k(s-v)} dG_v]^2 \\
    &= E[e^{-kt} \int_s^t e^{ku} dG_u + (e^{-kt} - e^{-ks}) \int_s^t e^{ku} dG_v]^2 \\
    &\leq E[e^{-kt} \int_s^t e^{ku} dG_u]^2 + E[(e^{-kt} - 1)e^{-ks} \int_s^t e^{ku} dG_v]^2.
\end{align}
For the second term in (3.18), we have
\begin{equation}
    E[(e^{-kt} - 1)e^{-ks} \int_s^t e^{ku} dG_v]^2 \leq C |t - s|^{2\beta}.
\end{equation}
Meanwhile, we have
\[
E[e^{-kt} \int_s^t e^{ku} dG_u]^2 \leq \int_s^t e^{-k(t-u) - k(t-v)} |u-v|^{2\beta - 2} du dv + \left( \int_s^t e^{-k(t-u)u^{\beta-1}} du \right)^2
\]
\[
\leq C' |t-s|^{2\beta}.
\] (3.20)

Hence, we obtain the desired result.

**Proposition 3.6** Under the hypothesis 1.1, and \( \gamma > \beta \) we can obtain that \( \lim_{T \to \infty} \frac{\tilde{F}_n}{n^\gamma} = 0 \) almost surely.

**Proof** The proof is similar as Chen et al. [2017]. When \( \beta \in (\frac{1}{2}, 1) \), Chebyshev's inequality, the hypercontractivity of multiple Wiener-Itô integrals imply that for any \( \varepsilon > 0 \) and \( p(\gamma - \beta) > -1 \),
\[
p\left( \frac{F_n}{n^\gamma} > \varepsilon \right) \leq \frac{EF_n^p}{n^{\gamma p} p} \leq \frac{C(\gamma - \beta)^{p/2}}{n^{\gamma p}} \leq \frac{C}{n^{\gamma p(\gamma - \beta)}}. \] (3.21)

The Borel-Cantelli lemma implies for \( \beta \in (\frac{1}{2}, 1) \),
\[
\lim_{n \to \infty} \frac{F_n}{n^\gamma} = 0, a.s. \] (3.22)

Second, there exist two constants \( \alpha \in (0, 1) \), \( C_{\alpha, \beta} > 0 \) independent of T such that any \( |t-s| \leq 1 \),
\[
E[|F_T - F_s|^2] \leq C_{\alpha, \beta} |t-s|^{2\beta}. \] (3.23)

Then the Garsia-Rumsey inequality implies that for any real number \( p > \frac{1}{\alpha}, q > 1 \), and integer \( n \geq 1 \),
\[
|F_T - F_s| \leq R_{p,q} |n|^{q/p}, \forall t, s \in [n, n+1], \] (3.24)

where \( R_{p,q} \) is a random constant independent of \( n \). Finally, since \( \frac{|F_T - F_n|}{n} \leq \frac{1}{n} |F_T - F_n| + \frac{q}{p} \frac{|F_n|}{n} \), where \( n = \lfloor T \rfloor \) is the biggest integer less than or equal to a real number \( T \), we have \( \frac{F_T}{\tilde{F}_n} \) converges to 0 almost surely as \( T \to \infty \).

**Proposition 3.7** Let \( X_T \) be given by (3.1), then
\[
\frac{1}{T} \int_0^T X_t^2 dt \to C_\beta \Gamma(2\beta - 1) k^{-2\beta} + \mu^2, \] (3.25)

almost surely, as \( T \) tends to infinity.

**Proof** From the expression of \( X_T \) in (3.1), we obtain
\[
\frac{1}{T} \int_0^T X_t^2 dt = \frac{1}{T} \int_0^T [\mu(1 - e^{-kt}) + \int_0^t e^{-k(t-s)} dG_s]^2 dt
\]
\[
= \frac{1}{T} \int_0^T [\mu(1 - e^{-kt})]^2 dt + \frac{1}{T} \int_0^T [\int_0^t e^{-k(t-s)} dG_s]^2 dt + \frac{2}{T} \int_0^T [\mu(1 - e^{-kt}) \int_0^t e^{-k(t-s)} dG_s] dt
\]
\[
=: I_1 + I_2 + I_3. \] (3.26)

For the term \( I_1 \), it is easy to see that
\[
I_1 = \frac{1}{T} \int_0^T [\mu(1 - e^{-kt})]^2 dt \overset{a.s.}{\to} \mu^2. \] (3.27)

Using an argument similar to that in (3.4) of Chen and Zhou [2020], we have
\[
I_2 = \frac{1}{T} \int_0^T [\int_0^t e^{-k(t-s)} dG_s]^2 dt \overset{a.s.}{\to} C_\beta \Gamma(2\beta - 1) k^{-2\beta}. \] (3.28)
We can deduce that
\[
I_3 = \frac{2\mu}{T} \int_0^T \left( \int_0^t e^{-k(t-s)} dG_s \right) dt - \frac{2\mu}{T} \int_0^T e^{-kt} \left( \int_0^t e^{-k(t-s)} dG_s \right) dt,
\]
a standard calculation yields
\[
\frac{2\mu}{T} \int_0^T e^{-kt} \left( \int_0^t e^{-k(t-s)} dG_s \right) dt = \frac{2\mu}{T} \int_0^T dG_s \int_s^T e^{-k(2t-s)} dt
\]
\[
= \frac{2\mu}{T} \int_0^T \frac{1}{2k} (e^{-ks} - e^{-k(2T-s)}) dG_s
\]
\[
= \frac{\mu}{T} \int_0^T \frac{1}{k} e^{-ks} dG_s - \frac{\mu}{T} \int_0^T \frac{1}{k} e^{-k(2T-s)} dG_s.
\]
For the first term in (3.30), set \( \frac{1}{T} \int_0^T e^{-ks} dG_s = M_T \), we have
\[
E[M_T^2] = \int_{[0,T]^2} e^{-ks} e^{-kr} \beta^2 R(s,r) ds dr
\]
\[
\leq \int_{[0,T]^2} e^{-ks} e^{-kr} |s-r|^{2\beta - 2} ds dr + \left( \int_{[0,T]} e^{-ks} s^{-1} ds \right)^2 \leq C_\beta T^{2\beta}.
\]
When \( \beta \in \left( \frac{2}{3}, \frac{2}{5} \right) \) and \( p(\beta - 1) < -1 \),
\[
P\left( \frac{|M_n|}{n} > \epsilon \right) \leq \frac{E[|M_n|^p]}{n^{p(\beta - 1)}} \leq n^{p(\beta - 1)},
\]
we can obtain \( \lim_{n \to \infty} \frac{M_n}{n} = 0 \), a.s.. Second, there exist a constant \( C_\beta > 0 \) independent of \( T \) such that any \( |t - s| \leq 1 \),
\[
E[|M_t - M_s|^2] = \int_{[s,t]^2} e^{-ks} e^{-kr} \frac{\partial^2 R(m,m)}{\partial m \partial n} dmdn
\]
\[
\leq \int_{[s,t]^2} e^{-ks} e^{-kr} m - n|^{2\beta - 2} dmdn + \left( \int_{[s,t]} e^{-ks} e^{-kr} m|^{\beta - 1} dmdn \right)^2 \leq E[|G_t - G_s|^2] \leq C_\beta (t-s)^{2\beta}.
\]
Then the Garsia-Rumsey inequality implies that for any real number \( p > \frac{2}{5}, q > 1, \) and integer \( n \geq 1 \),
\[
|M_t - M_s| \leq R_{p,q} n^{q/p}, \forall t, s \in [n, n+1],
\]
where \( R_{p,q} \) is a random constant independent of \( n \). Finally, since \( \frac{M_T}{n} \leq \frac{1}{T} |M_T - M_n| + \frac{n}{T} \frac{|M_n|}{n} \), where \( n = [T] \) is the biggest integer less than or equal to a real number \( T \), we have \( \frac{M_T}{n} \) converges to 0 almost surely as \( T \to \infty \), we imply that
\[
\frac{\mu}{T} \int_0^T \frac{1}{k} e^{-ks} dG_s \xrightarrow{a.s.} 0.
\]
For the last term in (3.30), we obtain
\[
\frac{\mu}{T} \int_0^T \frac{1}{k} e^{-k(2T-s)} dG_s = e^{-kT} \frac{\mu}{T} \int_0^T \frac{1}{k} e^{-k(T-s)} dG_s \xrightarrow{a.s.} 0,
\]
where the last step is from Chen and Zhou [2020]. Combining the above result, we obtain
\[
\frac{2\mu}{T} \int_0^T e^{-kt} \left( \int_0^t e^{-k(t-s)} dG_s \right) dt \xrightarrow{a.s.} 0.
\]
This implies that
\[
I_3 = \frac{2}{T} \int_0^T \left[ \mu (1 - e^{-kt}) \int_0^t e^{-k(t-s)} dG_s \right] dt \xrightarrow{a.s.} 0. \tag{3.38}
\]

By (3.26)-(3.38), as \( T \) tends to infinity, we imply that
\[
\frac{1}{T} \int_0^T X_t^2 dt \xrightarrow{a.s.} C_\beta \Gamma(2\beta - 1) k^{-2\beta} + \mu^2. \tag{3.39}
\]
Hence, the second moment estimator \( \hat{k} \) is strongly consistent, namely, \( \hat{k} \xrightarrow{a.s.} k \).

3.2 The least squares estimator

**Proposition 3.8** Let \((X_t, t \in [0, T])\) be given by (3.1), then
\[
\frac{1}{T} \int_0^T X_t dX_t \xrightarrow{a.s.} C_\gamma, \tag{3.40}
\]
as \( T \) tends to infinity, where \( C_\gamma \) denotes a suitable positive constant.

**Proof** By (1.1), we represent the stochastic integral \( \int X_t dX_t \) as
\[
\frac{1}{T} \int_0^T X_t dX_t = \frac{k \mu}{T} \int_0^T X_t dt - \frac{k}{T} \int_0^T X_t^2 dt + \frac{1}{T} \int_0^T X_t dG_t. \tag{3.41}
\]

By (3.1), we have that
\[
\frac{1}{T} \int_0^T X_t dG_t = \frac{1}{T} \int_0^T \mu (1 - e^{-kt}) dG_t + \frac{1}{T} \int_0^T e^{-k(t-s)} dG_s dG_t. \tag{3.42}
\]

For the first term in (3.42), by (3.35), we have
\[
\frac{1}{T} \int_0^T \mu (1 - e^{-kt}) dG_t = \frac{\mu}{T} \int_0^T dG_t - \frac{1}{T} \int_0^T e^{-k(t-s)} dG_s dG_t \xrightarrow{a.s.} 0. \tag{3.43}
\]

From proposition 3.7 in Chen and Zhou [2020], we know that \( \frac{1}{T} \int_0^T e^{-k(t-s)} dG_s dG_t \) converges almost surely, as \( T \) tends to infinity to 0. Then, combining (3.42) and (3.43), it’s suffices to show that
\[
\frac{1}{T} \int_0^T X_t dG_t \xrightarrow{a.s.} 0. \tag{3.44}
\]

Meanwhile, by proposition 3.6, we imply that
\[
\frac{k \mu}{T} \int_0^T X_t dt - \frac{k}{T} \int_0^T X_t^2 dt \xrightarrow{a.s.} C_\gamma. \tag{3.45}
\]

Therefore, we can obtain the desired result in (3.40).

**Proposition 3.9** Let \( \beta \in \left[ \frac{1}{2}, 1 \right) \), then we have \( \hat{\mu}_{LS} \xrightarrow{a.s.} \mu \).
Proof Recall that,
\[
\hat{\mu}_{LS} - \mu = \frac{\sum_{t=1}^{T} X_{t}^2 dt - \frac{1}{T} \int_{0}^{T} X_{t} dX_{t} \cdot \frac{1}{T} \int_{0}^{T} X_{t} dt - \mu \left( \frac{\sum_{t=1}^{T} X_{t} dX_{t} - \frac{1}{T} \int_{0}^{T} X_{t} dX_{t}}{\frac{1}{T} \int_{0}^{T} X_{t} dt} \right)}{\sum_{t=1}^{T} X_{t}^2 dt - \frac{1}{T} \int_{0}^{T} X_{t} dX_{t}}. 
\] (3.46)
Denote part I as follow, and according to proposition 3.8, we have
\[
\text{part I} = \frac{1}{T} \int_{0}^{T} X_{t} dX_{t} \left[ \frac{1}{T} \int_{0}^{T} X_{t} dt - \mu \right] \xrightarrow{a.s} 0. 
\] (3.47)
Denote part II as follow, by proposition 3.6, it’s easy to see that
\[
\text{part II} = \frac{X_{T} \left( \frac{1}{T} \int_{0}^{T} X_{t}^2 dt - \frac{1}{T} \int_{0}^{T} X_{t} dt \right)}{\frac{1}{T} \int_{0}^{T} X_{t} dt} \xrightarrow{a.s} 0. 
\] (3.48)
Finally, we obtain the desired result.

Proposition 3.10 Let \( \beta \in \left[ \frac{1}{2}, 1 \right) \), then we have \( \hat{k}_{LS} \xrightarrow{a.s} k \).

Proof Since the expression for \( \hat{k}_{LS} \) in (1.7), we write
\[
\hat{k}_{LS} - k = \frac{\frac{1}{T} \int_{0}^{T} X_{t}^2 dt - \frac{k}{T} \int_{0}^{T} X_{t} dt}{\frac{1}{T} \int_{0}^{T} X_{t} dt} \xrightarrow{a.s} 0.
\] (3.49)
First, denote part A as follow,
\[
\text{part A} = k \left( \frac{1}{T} \int_{0}^{T} X_{t} dt \right)^2 - k \mu \frac{1}{T} \int_{0}^{T} X_{t} dt \xrightarrow{a.s} 0. 
\] (3.50)
Then, denote part B as follow,
\[
\text{part B} = \frac{1}{T^2} X_{T} \int_{0}^{T} X_{t} dt = \frac{X_{T}}{T} \frac{1}{T} \int_{0}^{T} X_{t} dt. 
\] (3.51)
From proposition 3.7 in Chen and Zhou [2020] and (3.2), we obtain
\[
\frac{1}{T^2} X_{T} \int_{0}^{T} X_{t} dt \xrightarrow{a.s} 0. 
\] (3.52)
Combining (3.44), (3.50) and (3.52), we proves the claim of the Proposition.

4 Asymptotic behaviors for \( k > 0 \)

4.1 The moment estimator

We need several lemmas, providing sufficient conditions to prove the asymptotic normality of \( \hat{\mu} \).

Lemma 4.1 When \( \beta \in \left( \frac{1}{2}, 1 \right) \),
\[
T^{\beta - \frac{1}{2}} \frac{1}{T} \int_{0}^{T} X_{t} dt - \mu \rightarrow 0, 
\] (4.1)
almost surely as \( T \rightarrow \infty \).
Proof

\[ T^{1-\beta} \left( \frac{1}{T} \int_0^T X_t dt - \mu \right) = T^{1-\beta} \left( \frac{1}{T} \int_0^T \mu(1 - e^{-kt})dt + \frac{1}{T} \int_0^T e^{-k(t-s)} dG_s dt - \mu \right) = T^{1-\beta} \left( \frac{1}{T} \int_0^T \int_0^t e^{-k(t-s)} dG_s dt \right) \]

\[ = \frac{1}{T^{1-\beta}} \int_0^T \int_0^t e^{-k(t-s)} dG_s dt \]

\[ = \frac{k}{T^{1-\beta}} \int_0^T (1 - e^{-k(T-t)}) dG_s \]

\[ = \frac{k}{T^{1-\beta}} \mathbb{E} \left[ G_t - k \| f_T \| \frac{I_t(f_T)}{\| f_T \|} \right]. \tag{4.2} \]

where \( \frac{3}{2} - \beta > \beta \frac{G_T}{T^{1-\beta}} \) and \( \frac{I_t(f_T)}{\| f_T \|} \) are also the standard normal distribution of random variables, which together with Proposition 3.5 proves the claim of the lemma.

**Proposition 4.2** For \( \beta \in \left[ \frac{1}{2}, 1 \right] \), we obtain for \( \hat{\mu} \) defined by (1.4).

\[ T^{1-\beta} (\hat{\mu} - \mu) \xrightarrow{a.s.} N(0, \frac{1}{k^2}). \tag{4.3} \]

Proof First, we have

\[ T^{1-\beta} (\hat{\mu} - \mu) = T^{1-\beta} \left( \frac{1}{T} \int_0^T \mu(1 - e^{-kt})dt + \frac{1}{T} \int_0^T \int_0^t e^{-k(t-s)} dG_s dt - \mu \right) \]

\[ = \frac{1}{T^{\beta}} \int_0^T \int_0^t e^{-k(t-s)} dG_s dt \]

\[ = \frac{G_T}{kT^{\beta}} - \frac{k}{T^{1-\beta}} \int_0^T e^{-k(T-t)} dG_s. \tag{4.4} \]

From Proposition 3.8 in Chen and Zhou [2020], we have \( \frac{1}{T} \int_0^T e^{-k(T-t)} dG_s \to 0 \), and \( \frac{G_T}{T^{\beta}} \) is a standard normal distribution. Finally, by the Slutsky’s theorem we get the asymptotic normality (4.3) holds.

**Proposition 4.3** Denote a constant that depends on \( k \) and \( \beta \) as \( \alpha := C_3 \Gamma(2\beta-1)k^{-2\beta} \), then for \( \beta \in \left[ \frac{1}{2}, \frac{3}{4} \right] \) and \( T \to \infty \), we have

\[ \sqrt{T} (k - k) \xrightarrow{a.s.} N(0, a^2 \sigma_\beta^2/4\beta^2), \tag{4.5} \]

where \( \sigma_\beta^2 = (4\beta - 1) [1 + \frac{\Gamma(3-2\beta)\Gamma(4\beta-1)}{\Gamma(2\beta+1)\Gamma(1-2\beta)}] \).

Proof First, we obtain

\[ \sqrt{T} \left( \frac{1}{T} \int_0^T X_t dt - \left( \frac{1}{T} \int_0^T X_t dt \right)^2 - a \right) = \sqrt{T} \left( \frac{1}{T} \int_0^T \left[ \mu(1 - e^{-kt}) \right]^2 dt + \frac{1}{T} \int_0^T \left( \int_0^t e^{-k(t-s)} dG_s \right)^2 dt \right) \]

\[ + \frac{2}{T} \int_0^T \left[ \mu(1 - e^{-kt}) \right] \int_0^t e^{-k(t-s)} dG_s dt - \left( \frac{1}{T} \int_0^T X_t dt \right)^2 - a \right). \tag{4.6} \]

In fact, we have

\[ \frac{1}{T} \int_0^T \left[ \mu(1 - e^{-kt}) \right]^2 dt - \left( \frac{1}{T} \int_0^T X_t dt \right)^2 \xrightarrow{a.s.} 0. \tag{4.7} \]

Meanwhile, by (3.35), we can write

\[ \frac{\mu}{\sqrt{T}} \int_0^T \frac{1}{k} e^{-ks} dG_s \xrightarrow{a.s.} 0. \tag{4.8} \]
From Proposition 3.8 in Chen and Zhou [2020], we have
\[
\frac{\mu}{\sqrt{T}} \int_0^T e^{-k(2T-t)}dG_s \xrightarrow{a.s.} 0.
\] (4.9)

Those two facts now imply that,
\[
\sqrt{T} \left( \frac{1}{T} \int_0^T X_t^2 dt - \left( \frac{1}{T} \int_0^T X_t dt \right)^2 - a \right) \xrightarrow{law} N(0, \sigma^2_k/2).
\] (4.10)

Finally, since the delta method implies that the asymptotic normality holds.

### 4.2 The least squares estimator

Let us now discuss the asymptotic normality of LSE \( \hat{\mu}_{LS} \) and \( \hat{k}_{LS} \).

**Proposition 4.4** For \( \beta \in \left[\frac{1}{2}, 1\right) \), and \( T \to \infty \),
\[
T^{1-\beta} (\hat{\mu}_{LS} - \mu) \xrightarrow{law} N(0, \frac{1}{k^2}).
\] (4.11)

**Proof** By the representation of (3.46),
\[
T^{1-\beta} (\hat{\mu}_{LS} - \mu) = \frac{T^{1-\beta} (\text{partI + partII})}{\sqrt{\frac{\mu}{T} \int_0^T X_t dt - \frac{1}{T} \int_0^T X_t dt X_t}}.
\] (4.12)

First, combining (3.40) and Proposition 4.1, \( T^{1-\beta} \cdot \text{partI} \) can write as,
\[
T^{1-\beta} \frac{1}{T} \int_0^T X_t dt \int_0^T X_t dt - \mu \xrightarrow{law} N(0, \frac{1}{k^2}).
\] (4.13)

Using arguments similar to strong convergence of \( \hat{\mu} \), we can easily obtain
\[
T^{1-\beta} \cdot \text{partII} \xrightarrow{a.s.} 0.
\] (4.14)

Then, applying Slutsky’s theorem, we obtain the desired result.

**Proposition 4.5** when \( k > 0 \) and \( \beta \in \left(\frac{1}{4}, \frac{3}{4}\right) \), then the following convergence results hold true
\[
\sqrt{T} (\hat{k}_{LS} - k) \xrightarrow{law} N(0, 4ka^2 \sigma^2_k).
\] (4.15)

**Proof** From (3.49), we have
\[
\sqrt{T} (\hat{k}_{LS} - k) = \left( \sqrt{T} \frac{1}{T} \int_0^T X_t dt - k\mu \frac{1}{T} \int_0^T X_t dt - \frac{1}{T} \int_0^T X_t dt X_t dt + \frac{1}{T} \int_0^T X_t dt \right) \left( \frac{1}{T} \int_0^T X_t dt \right).
\] (4.16)

where we first consider only two terms of it, namely, the \( \sqrt{T} \cdot \text{partA} \) can write as
\[
\sqrt{T} \left( \frac{1}{T} \int_0^T X_t dt \right)^2 - \frac{k\mu}{T} \int_0^T X_t dt = \sqrt{T} \left( \frac{k}{T} \int_0^T X_t dt - k\mu \frac{1}{T} \int_0^T X_t dt \right)
\]
\[
= \sqrt{T} \left[ \frac{k}{T} \int_0^T \mu(1-e^{-kt})dt + \frac{k}{T} \int_0^T e^{-k(t-s)}dG_s dt - k\mu \frac{1}{T} \int_0^T X_t dt \right]
\]
\[
\left[ \frac{k}{\sqrt{T}} \int_0^T e^{-kt} dt + \frac{k}{\sqrt{T}} \int_0^T e^{-k(t-s)}dG_s dt \right] \frac{1}{T} \int_0^T X_t dt.
\] (4.17)
where $\frac{k}{\sqrt{T}} \int_{0}^{T} e^{-kt}dt \Rightarrow 0$, we can imply that

$$\frac{k}{\sqrt{T}} \int_{0}^{T} e^{-k(t-s)}dG_sdt = \frac{k}{\sqrt{T}} \int_{0}^{T} dG_s \int_{s}^{T} e^{-k(t-s)} dt = \frac{1}{\sqrt{T}} \int_{0}^{T} dG_s - \frac{1}{\sqrt{T}} \int_{0}^{T} e^{-k(T-s)}dG_s$$

$$= \frac{G_{T}}{\sqrt{T}} - \frac{1}{\sqrt{T}} \int_{0}^{T} e^{-kt} 0$$

Using the Proposition3.1, we can obtain that $\frac{1}{\sqrt{T}} \int_{0}^{T} e^{-kt} 0$, we deduce that

$$\sqrt{T}(\hat{k}_{LS} - k) = -\frac{1}{\sqrt{T}} \int_{0}^{T} X_t dG_t + (\mathcal{X}_{T} + \mathcal{G}^D) \frac{1}{\sqrt{T}} \int_{0}^{T} X_t dt$$

$$= -\frac{1}{\sqrt{T}} \int_{0}^{T} \mu(1 - e^{-kt})dG_t + \frac{1}{\sqrt{T}} \int_{0}^{T} e^{-k(t-s)}dG_t dG_s$$

$$= -\frac{\mu}{\sqrt{T}} G_{T} + \frac{\mu}{\sqrt{T}} \int_{0}^{T} e^{-kt} dG_t - \frac{1}{\sqrt{T}} \int_{0}^{T} \int_{0}^{T} e^{-k(t-s)}dG_s dt$$

$$= \frac{G_{T}}{\sqrt{T}} \frac{1}{T} \int_{0}^{T} X_t dt - \mu$$

$$= \frac{G_{T}}{\sqrt{T}} \frac{1}{T} \int_{0}^{T} e^{-k(t-s)}dG_s dt$$

$$= \frac{G_{T}}{\sqrt{T}} \frac{1}{T} \int_{0}^{T} e^{-k(t-s)}dG_s dt + \frac{\mathcal{X}_{T}}{\sqrt{T}} + \mathcal{G}^D \frac{1}{\sqrt{T}} \int_{0}^{T} X_t dt$$

$$= \frac{G_{T}}{\sqrt{T}} \frac{1}{T} \int_{0}^{T} e^{-k(t-s)}dG_s dt$$

(4.19)

It’s also clear that $\frac{1}{\sqrt{T}} \int_{0}^{T} e^{-kt} 0$, Using the lemma4.1, we can imply that $\frac{G_{T}}{\sqrt{T}} \frac{1}{T} \int_{0}^{T} X_t dt - \mu \Rightarrow 0$. Hence we have

$$\sqrt{T}(\hat{k}_{LS} - k) = -\frac{1}{\sqrt{T}} \int_{0}^{T} e^{-k(t-s)}dG_s dt = -\frac{1}{\sqrt{T}} \int_{0}^{T} e^{-k(t-s)}dG_s dt$$

(4.20)

where $\sigma^2 = (4\beta - 1)[1 + T(3 - 4\beta)I(4\beta - 1)]$.

From (4.8) in Chen and Zhou [2020], we know that $\frac{1}{\sqrt{T}} \int_{0}^{T} e^{-k(t-s)} \Rightarrow N(0, 4k\alpha^2\sigma^2)$. Thus, combining with (3.25), the Slutsky’s theorem implies that the asymptotic normality holds.

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