ON THE EQUATION OF THE $p$-ADIC OPEN STRING FOR THE SCALAR TACHYON FIELD

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Abstract. We study the structure of solutions of the one-dimensional non-linear pseudodifferential equation describing the dynamics of the $p$-adic open string for the scalar tachyon field $p^{\frac{1}{2}}\partial_t^2 \Phi = \Phi^p$. We elicit the role of real zeros of the entire function $\Phi^p(z)$ and the behaviour of solutions $\Phi(t)$ in the neighbourhood of these zeros. We point out that discontinuous solutions can appear if $p$ is even. We use the method of expanding the solution $\Phi$ and the function $\Phi^p$ in the Hermite polynomials and modified Hermite polynomials and establish a connection between the coefficients of these expansions (integral conservation laws). For $p = 2$ we construct an infinite system of non-linear equations in the unknown Hermite coefficients and study its structure. We consider the 3-approximation. We indicate a connection between the problems stated and the non-linear boundary-value problem for the heat equation.

§ 1. Introduction

The dynamics of the open $p$-adic string for the scalar tachyon field is described by the non-linear pseudodifferential equation [1]–[9]

$$p^{\frac{1}{2}}\Box \Phi = \Phi^p, \tag{1.1}$$

where

$$\Box = \partial_t^2 - \partial_{x_1}^2 - \cdots - \partial_{x_{d-1}}^2, \quad t = x_0,$$

is the d’Alembert operator and $p$ is a prime number, $p = 2, 3, 5, \ldots$. In what follows $p$ is any positive integer. We consider only real solutions of equation (1.1), since only real solutions have physical meaning.

In the one-dimensional case ($d = 1$) we use the change

$$\varphi(t) = \Phi(t \sqrt{2 \ln p})$$

and write equation (1.1) in the following equivalent form:

$$e^{\frac{1}{2} \partial_t^2} \varphi = \varphi^p. \tag{1.2}$$

Equation (1.2) is a non-linear integral equation of the following form [9]:

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(t-\tau)^2} \varphi(\tau) \, d\tau = \varphi^p(t), \quad t \in \mathbb{R}. \tag{1.3}$$
Solutions of equation (1.3) are sought in the class of measurable functions $\varphi(t)$ such that

$$|\varphi(t)| \leq C \exp\{(1-\varepsilon)t^2\} \quad \text{for any } \varepsilon > 0, \quad t \in \mathbb{R}. \quad (1.4)$$

The following boundary-value problems for the solutions $\varphi$ of equation (1.3) have physical meaning:

$$\lim_{t \to -\infty} \varphi(t) = 0, \quad \lim_{t \to \infty} \varphi(t) = 1 \quad (1.5)$$

if $p$ is even, and

$$\lim_{t \to -\infty} \varphi(t) = -1, \quad \lim_{t \to \infty} \varphi(t) = 1 \quad (1.6)$$

if $p$ is odd.

In §2 we give the information on Hermite polynomials $H_n(t), \quad n = 0, 1, 2, \ldots,$ that will be used in the subsequent sections, introduce modified Hermite polynomials $V_n(t), \quad n = 0, 1, \ldots,$ and study their properties. In §3 we study the properties of the integral operator $K$ on the left side of equation (1.3), in the space $L^2_0,$ $0 < \alpha < 2.$ In §4 we expand the solution $\varphi(t)$ of equation (1.3) in series in polynomials $H_n$ and $V_n,$ expand $\varphi^p(t)$ in Taylor series and in series in $H_n,$ and establish relations between these expansions. In particular, we prove an important equality (4.5), which plays the role of integral conservation laws:

$$(\varphi^p, H_n)_1 = (\varphi, V_n)_{1/2}, \quad n = 0, 1, \ldots.$$  

In §5 we suggest a new method of study of equation (1.3), which enables us to reduce the problem of solving this equation to the problem of solving a non-linear boundary-value problem for the heat equation. We use this equivalence to prove Theorem 5.1. In §6 we consider the linear equation (1.3) $(p = 1),$ establish a connection between its solutions and the solutions of the heat equation (5.1) periodic with respect to $x$ and write these solutions in an explicit form. We prove a uniqueness theorem in the class $S'$ of tempered distributions (generalized functions of slow growth) (Theorem 6.1). We show that the spectrum of the integral operator $K$ is continuous and is concentrated in $[0, 1],$ and compute the (generalized) eigenfunctions. For the Hermite coefficients $a_n = (\varphi, H_n)_1, \quad n = 2, 3, \ldots,$ of the solution $\varphi$ we deduce an infinite triangular linear system of equations (6.8). In §7 we study the boundary-value problem (1.3), (1.5) with an even $p$ and investigate the behaviour of its solution in the neighbourhood of real zeros of the entire function $\varphi^p(z), \quad z = t + iy,$ and in the neighbourhood of discontinuities of the first kind of the solution $\varphi(t)$ (Theorem 7.1). It is still an open question whether this problem has solutions and whether a solution can have discontinuities of the first kind. We also study the ramification of zeros of the interpolating function $u(1 - \varepsilon, t - t_0)$ in the $\varepsilon$-neighbourhood, $\varepsilon > 0,$ of the zero $t_0$ of $\varphi^p(t)$ (Theorem 7.2). In §8 we consider the special case when $p = 2.$ We deduce an infinite non-linear system of equations (8.3) in the coefficients $a_n$ of solutions of equation (1.3). We consider the 3-approximation. In §9 we deduce a similar system of equations (9.4) for the solution of problem (1.3), (1.5) with $p = 2.$ We consider the 3-approximation. In §10 we study the structure of solutions of the boundary-value problem (1.3), (1.6) with an odd $p.$ In particular, we study their behaviour in the neighbourhood of real zeros of the entire function $\varphi^p(z)$ under the assumption that this problem has solutions (Theorem 10.1). We point out that problem (1.3), (1.6) [9] has an odd continuous solution.
If \( \varphi(t) \) is a solution of equation (1.3), then \( \varphi(t + t_0) \) also is a solution (for every \( t_0 \)). Therefore, this equation cannot have precisely one solution \( \varphi(t) \). The question whether the shifts of \( \varphi(t) \) by \( t \) (that is, the \( \varphi(t + t_0) \), \( t_0 \in \mathbb{R} \)) exhaust the set of solutions of equation (1.3), remains open.

It is most natural to investigate the integral equation (1.3), using the Hermite polynomials, since its kernel is the generating function for these polynomials (§4). This is the reason why we use these polynomials in our construction of solutions (exact and approximate) of equation (1.3). Let us note that Hermite polynomials were used in [13]–[15] in the theory of \( D \)-brane perturbations as well as in the study of more complicated problems concerning the tachyons for both open and closed strings.

We shall need the scale of weighted separable Hilbert spaces \( L_2^\alpha \), \( 0 < \alpha < \infty \), consisting of functions on \( \mathbb{R} \) measurable and square integrable with respect to the measure

\[
d\mu_\alpha(t) = \sqrt{\frac{\alpha}{\pi}} e^{-\alpha t^2} \, dt, \quad \int_{-\infty}^{\infty} d\mu_\alpha(t) = 1, \quad \alpha > 0,
\]

with the following scalar product and norm:

\[
(f, g)_\alpha = \int_{-\infty}^{\infty} f(t)\overline{g(t)} \, d\mu_\alpha(t), \quad \|f\|_\alpha = \sqrt{(f, f)_\alpha}, \quad f, g \in L_2^\alpha, \quad \alpha > 0.
\]

The embedding \( L_2^\alpha \subset L_2^\beta \), \( \alpha < \beta \), is dense and continuous, and

\[
\|f\|_\beta \leq \|f\|_\alpha, \quad f \in L_2^\alpha, \quad (1.7)
\]

The following assertion will be used in the study of problems (1.3), (1.5) and (1.3), (1.6).

**Assertion 1.1.** If \( \varphi \) is a solution of equation (1.3) such that

\[
\lim_{t \to \infty} \varphi(t) = a, \quad |a| < \infty,
\]

then \( a = 0 \) or \( a = 1 \) if \( p \) is even and \( a = 0 \) or \( a = \pm 1 \) if \( p \) is odd, \( \lim_{t \to \infty} (\varphi^p)'(t) = 0 \). If \( a \neq 0 \), then \( \lim_{t \to \infty} \varphi'(t) = 0 \).

**Proof.** We deduce from equation (1.3) the following chain of equalities:

\[
\lim_{t \to \infty} \varphi^p(t) = \left[ \lim_{t \to \infty} \varphi(t) \right]^p = a^p = \lim_{t \to \infty} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \varphi(\tau)e^{-(t-\tau)^2} \, d\tau
\]

\[
= \lim_{t \to \infty} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \varphi(t-u)e^{-u^2} \, du = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \lim_{t \to \infty} \varphi(t-u)e^{-u^2} \, du = a,
\]

whence \( a = 0 \) or \( a = 1 \) if \( p \) is even and \( a = 0, \pm 1 \) if \( p \) is odd. Further, we have

\[
\lim_{t \to \infty} (\varphi^p)'(t) = -2 \lim_{t \to \infty} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \varphi(\tau)(t-\tau)e^{-(t-\tau)^2} \, d\tau
\]

\[
= -2 \lim_{t \to \infty} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \varphi(t-u)ue^{-u^2} \, du = -\frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} \lim_{t \to \infty} \varphi(t-u)ue^{-u^2} \, du
\]

\[
= -\frac{2}{\sqrt{\pi}} a \int_{-\infty}^{\infty} ue^{-u^2} \, du = -\frac{2}{\sqrt{\pi}} a \cdot 0 = 0.
\]
If \( a \neq 0 \), then \( \lim_{t \to \infty} \phi'(t) = 0 \), since
\[
\lim_{t \to \infty} (\phi^p)'(t) = p \lim_{t \to \infty} \phi^{p-1}(t) \phi'(t) = p a^{p-1} \lim_{t \to \infty} \phi'(t) = 0.
\]
We passed to the limit under the integral sign, using Lebesgue’s theorem and estimate (1.4).

We shall write \( a \equiv b \) if the integers \( a \) and \( b \) are both even or both odd, and \( a \not\equiv b \) if one of them is even and the other is odd.

\section{2. Hermite polynomials}

Hermite polynomials are defined to be the polynomials [10]
\[
H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}, \quad n = 0, 1, \ldots, \quad (2.1)
\]
whence \( H_0(x) = 1, \ H_1(x) = 2x, \ H_2(x) = 4x^2 - 2, \ H_3(x) = 8x^3 - 12x, \ldots \). They form a complete orthogonal system in the Hilbert space \( L_2^1 \), and
\[
\|H_n\|_1^2 = \int_{-\infty}^{\infty} H_n^2(x) d\mu_1(x) = 2^n n!.
\]
(2.2)

Any \( f \in L_2^1 \) can be expanded in Hermite polynomials:
\[
f(x) = \sum_{n=0}^{\infty} (f, H_n)_1 \frac{H_n(x)}{2^n n!} \quad \text{in} \quad L_2^1, \quad (2.3)
\]
and the Parseval–Steklov equality holds:
\[
\|f\|_1^2 = \sum_{n=0}^{\infty} |(f, H_n)_1|^2 \frac{1}{2^n n!}.
\]
(2.4)

The following equalities hold:
\[
(x^m, H_n)_1 = \begin{cases} 
2^{n-m} m! \left( \frac{m-n}{2} \right)!^{-1}, & m \geq n \quad \text{and} \quad n \equiv m, \\
0, & m < n \quad \text{or} \quad n \not\equiv m.
\end{cases} \quad (2.5)
\]

Here we used formula 2.20.3.4 in [11] (p. 487 of the Russian version) with \( \alpha = m + 1, \ p = c = 1 \).

The values of the polynomials at \( x = 0 \) are given by the formulae
\[
H_{2n}(0) = (-1)^n \frac{(2n)!}{n!}, \quad H_{2n+1}(0) = 0, \quad n = 0, 1, \ldots. \quad (2.6)
\]

The expansion in powers of \( x \) (see [12], formula 8.950, p. 1047 of the Russian version) has the form
\[
H_n(x) = n! \sum_{m=0}^{n} c_{n,m} x^m, \quad n = 0, 1, \ldots, \quad (2.7)
\]
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where

$$c_{n,m} = \begin{cases} \frac{(-1)^{n-m}}{m!} \frac{2^m}{m} \left( \frac{n-m}{2} \right)!^{-1}, & m \equiv n \text{ and } m \leq n, \\ 0, & m \not\equiv n \text{ or } m > n. \end{cases}$$ \tag{2.8}

In particular, we have

$$c_{n,n} = \frac{2^n}{n!}, \quad c_{2n,0} = \frac{(-1)^n}{n!}, \quad c_{2n+1,1} = \frac{2(-1)^n}{n!},$$
$$c_{2n,2} = -2 \frac{(-1)^n}{(n-1)!}, \quad c_{2n+1,3} = -\frac{4(-1)^n}{3(n-1)!}. \tag{2.9}$$

The following asymptotic formulae hold:

$$H_n(x) = (2x)^n \left[ 1 + O(x^{-2}) \right], \quad x \to \pm \infty. \tag{2.10}$$

The modified Hermite polynomials are defined to be the polynomials

$$V_n(x) = 2^{-n/2} H_n\left( \frac{x}{\sqrt{2}} \right), \quad n = 0, 1, \ldots. \tag{2.11}$$

In particular, $V_0(x) = 1$, $V_1(x) = x$, $V_2(x) = x^2 - 1$, $V_3(x) = x^3 - 3x$, \ldots.

These polynomials form a complete orthogonal system in the Hilbert space $L^1_2$, and

$$\|V_n\|_{1/2}^2 = \int_{-\infty}^{\infty} V_n^2(x) d\mu_{1/2}(x) = n!. \tag{2.12}$$

Any $f \in L^1_2$ can be expanded in modified Hermite polynomials:

$$f(x) = \sum_{n=0}^{\infty} \langle f, V_n \rangle_{1/2} \frac{V_n(x)}{n!} \quad \text{in} \quad L^1_2, \tag{2.13}$$

and the Parseval–Steklov equality holds:

$$\|f\|_{1/2}^2 = \sum_{n=0}^{\infty} |\langle f, V_n \rangle_{1/2}|^2 \frac{1}{n!}. \tag{2.14}$$

We have the following equalities:

$$(H_m, V_n)_{1/2} = \begin{cases} 2^n m! \left( \frac{m-n}{2} \right)!^{-1}, & m \geq n \text{ and } n \equiv m, \\ 0, & m < n \text{ or } n \not\equiv m, \end{cases} \tag{2.15}$$

$$(H_n, V_m)_1 = \begin{cases} (-1)^{\frac{m-n}{2}} 2^{m-n} m! \left( \frac{m-n}{2} \right)!^{-1}, & m \geq n \text{ and } n \equiv m, \\ 0, & m < n \text{ or } n \not\equiv m. \end{cases} \tag{2.16}$$

We used here formula 2.20.16.4 in [11] (p. 502 of the Russian version) with $p = 1/2$, $b = 1$, $c = \frac{1}{\sqrt{2}}$ and $p = c = 1$, $b = \frac{1}{\sqrt{2}}$. 

We can express $V_n$ in terms of $H_n$ and vice versa, using formulae (2.3), (2.13), (2.15) and (2.16):

$$V_n(x) = 2^{-n} n! \sum_{m=0}^{n} (-1)^{\frac{n-m}{2}} \left( \frac{n-m}{2} \right)!^{-1} H_m(x) \frac{1}{m!}, \quad (2.17)$$

$$H_n(x) = n! \sum_{m=0}^{n} \frac{2^m}{m!} \left( \frac{n-m}{2} \right)!^{-1} V_m(x). \quad (2.18)$$

The integral representation for the modified Hermite polynomials has the form:

$$V_n(x) = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} H_n(\tau) e^{-2(t^2-\tau^2)} d\tau, \quad n = 0, 1, \ldots. \quad (2.19)$$

Equality (2.19) follows from formula 2.20.3.16 in [11] (p. 488 of the Russian version) with $p = 2, \ c = 1, \ y = \tau/2$ and formula (2.11) defining $V_n$.

Let $f \in L^1$ be a function. It follows from (1.2) that $f \in L^2$, and

$$\sum_{n=0}^{\infty} a_n \frac{H_n(x)}{2^n n!} = f(x) = \sum_{n=0}^{\infty} b_n \frac{V_n(x)}{n!} \quad \text{in} \quad L^1,$$

and $a_n = (f, H_n)_1$ can be expressed in terms of $b_n = (f, V_n)_{1/2}$ and vice versa by the formulae

$$a_n = \sum_{m \geq n} (-1)^{\frac{m-n}{2}} 2^{n-m} \left( \frac{m-n}{2} \right)!^{-1} b_m, \quad n = 0, 1, \ldots, \quad (2.21)$$

$$b_n = \sum_{m \geq n} 2^{n-m} \left( \frac{m-n}{2} \right)!^{-1} a_m, \quad n = 0, 1, \ldots. \quad (2.22)$$

These formulae follow from (2.19), (2.15) and (2.16).

§ 3. PROPERTIES OF THE OPERATOR $K$

We denote by $K$ the linear integral operator in equation (1.3):

$$\varphi \to (K\varphi)(t) \equiv \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(t^2-\tau^2)} \varphi(\tau) d\tau.$$

**Lemma 3.1.** The operator $K$ assigns to every function $f(t)$ satisfying condition (1.4) an entire function $(Kf)(z)$ with the estimate

$$|(Kf)(z)| \leq \frac{C}{\sqrt{\varepsilon}} \exp\left\{ y^2 + \left( \frac{1}{\varepsilon} - 1 \right) t^2 \right\}, \quad z = t + iy. \quad (3.1)$$

**Proof** follows immediately from (1.4):

$$|(Kf)(z)| \leq \frac{C}{\sqrt{\varepsilon}} \int_{-\infty}^{\infty} e^{(1-\varepsilon)t^2} |e^{-(t^2)}| d\tau$$

$$= \frac{C}{\sqrt{\varepsilon}} e^{y^2-t^2} \int_{-\infty}^{\infty} e^{-\varepsilon t^2 + 2it} d\tau = \frac{C}{\sqrt{\varepsilon}} e^{y^2+(1/\varepsilon-1)t^2}. $$
Lemma 3.2. The operator $K$ assigns to $f \in L^2_2$, $0 < \alpha < 2$, an entire function $(Kf)(z)$ with the estimate
\[
| (Kf)(z) | \leq \| f \|_\alpha (2 - \alpha)^{-1/4} \exp \left\{ y^2 + \frac{\alpha}{2 - \alpha} t^2 \right\}, \quad z = t + iy. \quad (3.2)
\]

Proof follows from the Cauchy–Bunyakovskii inequality (applied to $(Kf)(z)$) and the following estimates:

\[
| (Kf)(z) | \leq \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} | f(\tau) | e^{-\alpha \tau^2/2} \exp \left\{ -z^2 + 2z\tau - \left( 1 - \frac{\alpha}{2} \right) \tau^2 \right\} \, d\tau
\]
\[
\leq \| f \|_\alpha \left[ \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp \left\{ 2y^2 + 4t\tau - (2 - \alpha) \tau^2 \right\} \, d\tau \right]^{1/2}
\]
\[
\leq \| f \|_\alpha \left[ \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp \left\{ 2y^2 - 2t^2 + 4t\tau - (2 - \alpha) \tau^2 \right\} \, d\tau \right]^{1/2}
\]
\[
= \| f \|_\alpha \exp \left\{ y^2 + \frac{\alpha - 1}{2 - \alpha} t^2 \right\} \left[ \frac{1}{\pi} \int_{-\infty}^{\infty} \exp \left\{ (2 - \alpha) \left( \tau - \frac{2t}{2 - \alpha} \right)^2 \right\} \, d\tau \right]^{1/2}.
\]

Lemma 3.3. The operator $K : L^2_2 \to L^2_2$, $0 < \alpha < 2$, $\beta > \frac{2\alpha}{2-\alpha}$, is bounded, and
\[
\| Kf \|_\beta \leq \left( 2\alpha - \frac{2\alpha^2}{\beta} - \alpha^2 \right)^{-1/4} \| f \|_\alpha, \quad f \in L^2_2. \quad (3.3)
\]

Proof. We prove the lemma by writing the following chain of equalities and inequalities for $f \in L^2_2$:

\[
\| Kf \|_\beta^2 = \sqrt{\beta} \int_{-\infty}^{\infty} e^{-\beta t^2} |(Kf)(t)|^2 \, dt
\]
\[
= \sqrt{\beta} \int_{-\infty}^{\infty} e^{-(\beta+2) t^2} \left| \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(\tau) e^{-(\alpha/2) \tau^2 - (1-\alpha/2) \tau^2 + 2t\tau} \, d\tau \right|^2 \, dt
\]
\[
\leq \frac{1}{\pi} \sqrt{\beta} \int_{-\infty}^{\infty} e^{-(\beta+2) t^2} \int_{-\infty}^{\infty} |f(\tau)|^2 e^{-\alpha \tau^2} \, d\tau \int_{-\infty}^{\infty} e^{-(2-\alpha) \tau^2 + 4t\tau} \, d\tau \, dt
\]
\[
= \frac{1}{\pi} \sqrt{\beta} \| f \|^2_\alpha \int_{-\infty}^{\infty} e^{-(\beta+2-\frac{2\alpha}{\beta}) t^2} \, dt \int_{-\infty}^{\infty} e^{-(2-\alpha) \tau^2} \, d\tau
\]
\[
= (2\beta - 2\alpha - \alpha\beta)^{-1/2} \sqrt{\frac{\beta}{\alpha}} \| f \|^2_\alpha.
\]

Lemma 3.4. If $f \in L^1_2$, then its image $(Kf)(t)$ can be expanded in the Taylor series
\[
(Kf)(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}, \quad a_n = (f, H_n)_1, \quad (3.4)
\]
which converges uniformly on every compact set in $\mathbb{R}$. If $f \in L^1_2$, then

$$(Kf)(t) = \sum_{n=0}^{\infty} b_n \frac{H_n(t)}{2^n n!} \quad \text{in} \quad L^1_2, \quad b_n = (Kf, H_n)_1,$$

and

$$(Kf, H_n)_1 = (f, V_n)_{1/2}, \quad n = 0, 1, \ldots. \quad (3.6)$$

Proof. By Lemma 3.3, the function $(Kf)(t)$ is the trace of an entire function $(Kf)(z)$ for $y = 0$. Hence, it can be expanded in the Taylor series with the coefficients

$$\frac{d^n}{dt^n}(Kf)(t)|_{t=0} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(\tau) \frac{d^n}{dt^n} e^{-(t-\tau)^2} d\tau|_{t=0}$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(\tau)(-1)^n e^{-(t-\tau)^2} H_n(t-\tau) d\tau|_{t=0}$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(\tau)(-1)^n H_n(-\tau) e^{-t^2} d\tau = (f, H_n)_1 = a_n.$$ 

Here we used equality (2.1).

Further, if $f \in L^1_2$, then (3.5) holds by (2.20), since $Kf \in L^1_2$ by Lemma 3.3. Equalities (3.6) can be proved as follows:

$$(Kf, H_n)_1 = (f, K^*H_n)_{1/2} = (f, V_n)_{1/2}. \quad (4.1)$$

Here we used formula (2.19), which implies that $V_n = K^*H_n$, where $K^*$ is the operator adjoint to $K$.

§ 4. EXPANDING SOLUTIONS IN HERMITE POLYNOMIALS

Let $\varphi$ be a solution of equation (1.3) belonging to $L^1_2$, whence $\varphi^p = K\varphi$. Putting $a_n = (\varphi, H_n)_1$, we deduce from (2.3) and (2.4) that

$$\varphi(t) = \sum_{n=0}^{\infty} a_n \frac{H_n(t)}{2^n n!} \quad \text{in} \quad L^1_2, \quad \sum_{n=0}^{\infty} \frac{a_n^2}{2^n n!} = \|\varphi\|^2. \quad (4.1)$$

The function $\varphi^p(t)$ is the trace of the entire function $A(z) = (K\varphi)(z)$, for which (3.2) holds with $\alpha = 1$:

$$|A(z)| \leq \|\varphi\|_1 e^{|z|^2}, \quad z \in \mathbb{C}. \quad (4.2)$$

By Lemma 3.4, it can be expanded in the Taylor series (3.4):

$$\varphi^p(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}. \quad (4.3)$$

If $\varphi \in L^1_2$ is a solution of equation (1.3), then Lemma 3.3 implies that $\varphi^p \in L^1_2$ and

$$\|\varphi^p\|_1 \leq \sqrt{2}\|\varphi\|_{1/2}. \quad (4.4)$$
Lemma 3.4 implies that equalities in (3.6) (integral conservation laws) hold:

\[(\varphi^p, H_n)_1 = b_n = (\varphi, V_n)_{1/2}, \quad n = 0, 1, \ldots.\] (4.5)

Therefore, \(\varphi^p(t)\) can be expanded in Hermite polynomials

\[\varphi^p(t) = \sum_{n=0}^{\infty} b_n \frac{H_n(t)}{2^{n}n!} \quad \text{in} \quad L^1_2,\] (4.6)

\(\varphi(t)\) can be expanded in modified Hermite polynomials:

\[\varphi(t) = \sum_{n=0}^{\infty} b_n \frac{V_n(t)}{n!} \quad \text{in} \quad L^{1/2}_2,\] (4.7)

and the corresponding Parseval–Steklov equalities hold:

\[\sum_{n=0}^{\infty} \frac{b_n^2}{2^n n!} = \|\varphi^p\|_1^2, \quad \sum_{n=0}^{\infty} \frac{b_n^2}{n!} = \|\varphi^p\|_{1/2}^2.\] (4.8)

Let us note that the \(a_n\) and \(b_n\) are related by (2.21) and (2.22).

§ 5. CONNECTION BETWEEN \(\varphi(t)\) AND THE SOLUTIONS OF THE HEAT EQUATION

The integral equation (1.3) is equivalent to the following boundary-value problem for the heat equation:

\[u_x = \frac{1}{4} u_{tt}, \quad 0 < x \leq 1, \quad t \in \mathbb{R},\] (5.1)

\[u(0,t) = \varphi(t), \quad u(1,t) = \varphi^p(t), \quad t \in \mathbb{R}.\] (5.2)

A solution of problem (5.1), (5.2) is defined to be any measurable function \(u(x,t)\) for which (1.4) holds, where \(C\) does not depend on \(x\). We say that \(u(x,t)\) is an interpolating function between \(\varphi(t)\) and \(\varphi^p(t)\).

Let us note that if there is an interpolating function, it can be represented by Poisson’s formula for equation (5.1):

\[u(x,t) = \frac{1}{\sqrt{\pi x}} \int_{-\infty}^{\infty} \varphi(\tau) \exp\left\{ -\frac{(t-\tau)^2}{x} \right\} d\tau, \quad 0 < x \leq 1.\] (5.3)

If \(\varphi\) is such that

\[|\varphi(t)| \leq C \exp\{\varepsilon t^2\} \quad \text{for any} \quad \varepsilon > 0, \quad t \in \mathbb{R},\] (5.4)

then formula (5.3) gives its analytic continuation to the domain \(x > 1, \quad t \in \mathbb{R}\) and, further, its analytic continuation with respect to \((x,t)\) to the complex domain \(T^+ \times \mathbb{C}\), where \(T^+\) is the right half-plane \(\text{Re} \, \zeta = x > 0\).

Representation (5.3) implies that if \(\varphi(t)\) is a solution of problem (1.3), (1.5) or (1.3), (1.6), then \(|u(x,t)| < 1, \quad 0 \leq x, \quad t \in \mathbb{R}, \quad u(x,t)\) satisfies the boundary conditions (1.5) or (1.6), respectively, and \(u(x,t) > 0\) for \(x > 1\).
Example. For the solution
\[ \varphi(t) = p^{\frac{1}{2(p-1)}} \exp\left\{ \frac{p-1}{p} t^2 \right\} \]
of equation (1.3) we have the following interpolating function:
\[ u(x, t) = p^{\frac{1}{2(p-1)}} \left( 1 - x + \frac{x}{p} \right)^{-\frac{1}{2}} \exp\left\{ \frac{t^2(p-1)}{p-xp+x} \right\}. \]

**Theorem 5.1.** Let \( u(x, t) \) be an interpolating function between the solution \( \varphi \) and its power \( \varphi^p \) for problems (1.3), (1.5) and (1.3), (1.6). Then
\[ \int_{-\infty}^{\infty} \varphi^2(t) [1 - \varphi^{2p-2}(t)] dt = \frac{1}{2} \int_{0}^{1} \int_{-\infty}^{\infty} u_t^2(x, t) dt dx, \]  
\[ \int_{-\infty}^{\infty} [u(x, t) - \varphi(t)] dt = 0, \quad \int_{-\infty}^{\infty} [u(x + 1, t) - \varphi^p(t)] dt = 0, \quad x \geq 0 \]
(a conservation law).

**Proof.** Multiplying equation (5.1) by \( u(x, t) \), integrating both sides over the rectangle \( 0 \leq x \leq 1, \ a \leq t \leq b \), and taking into account (5.2), we obtain the following chain of equalities:
\[ \int_{0}^{1} \int_{a}^{b} uu_x dt dx = \frac{1}{2} \int_{a}^{b} [u^2(1, t) - u^2(0, t)] dt \]
\[ = \frac{1}{2} \int_{a}^{b} \varphi^2(t) [\varphi^{2p-2}(t) - 1] dt = \frac{1}{4} \int_{0}^{1} \int_{a}^{b} uu_t dt dx \]
\[ = -\frac{1}{4} \int_{0}^{1} \int_{a}^{b} u_t^2 dt dx + \frac{1}{4} \int_{0}^{1} [u(x, b)u_t(x, b) - u(x, a)u_t(x, a)] dx. \]  
Using (5.3), (1.5) and (1.6), we obtain that
\[ \lim_{b \to \infty} u(x, b) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \lim_{b \to \infty} \varphi(b - \sqrt{x} v)e^{-v^2} dv = 1, \]
\[ \lim_{b \to \infty} u_t(x, b) = -\frac{2}{\sqrt{x\pi}} \int_{-\infty}^{\infty} \lim_{b \to \infty} \varphi(b - \sqrt{x} v)ve^{-v^2} dv = 0, \]
\[ \|u_t(x, b)\| < \frac{2}{\sqrt{x\pi}}, \quad 0 < x \leq 1. \]
(The passage to the limit under the integral sign is possible by Lebesgue’s theorem.)
Similar relations hold for \( u(x, a) \) and \( u_t(x, a) \). Taking into account these relations and passing to the limit in (5.7), we obtain (5.5).
Equality (5.6) can be proved as follows. Integrating equation (5.1) over the rectangle \( 0 \leq x \leq X, \ a \leq t \leq b \), we obtain, as before, the chain of equalities

\[
\int_a^b [u(X,t) - \varphi(t)] \, dt = \frac{1}{4} \int_0^X [u_t(x,b) - u_t(x,a)] \, dx \\
= - \frac{1}{2\sqrt{\pi}} \int_0^X \frac{1}{x^{\frac{1}{2}}} \int_{-\infty}^{\infty} \left[ (b - \tau)e^{-\frac{(b-x)^2}{x}} - (a - \tau)e^{-\frac{(a-x)^2}{x}} \right] \varphi(\tau) \, d\tau \, dx \\
= - \frac{1}{2\sqrt{\pi}} \int_0^X \frac{1}{\sqrt{x}} \int_{-\infty}^{\infty} ve^{-v^2} [\varphi(b - \sqrt{v}) - \varphi(a - \sqrt{v})] \, dv \, dx.
\]

Passing here to the limit as \( a \to -\infty \) and \( b \to \infty \) and taking into account the boundary conditions (1.5) and (1.6), we obtain (5.6).

**Corollary 5.1.** (i) The integrals

\[
\int_{-\infty}^0 \varphi^2(t) \, dt, \quad \int_0^\infty (1 - \varphi^2(t)) \, dt \quad \text{for} \quad (1.3), (1.5),
\]

\[
\int_{-\infty}^\infty (1 - \varphi^2(t)) \, dt \quad \text{for} \quad (1.3), (1.6)
\]

converge if and only if

\[
\int_0^1 \int_{-\infty}^\infty u_t^2(x,t) \, dt \, dx < \infty.
\]

(ii) Inequality (5.6) with \( x = 1 \) implies that

\[
\int_{-\infty}^\infty [\varphi(t) - \varphi^p(t)] \, dt = 0.
\]

(iii) The following integrals converge for problem (1.3), (1.5):

\[
\int_0^\infty [1 - \varphi^{p-1}(t)] \, dt, \quad \int_0^\infty [1 - \varphi(t)] \, dt, \quad \int_0^\infty [1 - u(x,t)] \, dt, \quad x \geq 1.
\]

If \( \varphi(t) \) has constant sign for \( t < c \), then the following integrals also converge:

\[
\int_{-\infty}^0 \varphi(t) \, dt, \quad \int_{-\infty}^0 u(x,t) \, dt, \quad x \geq 1.
\]

(iv) The following integrals converge for problem (1.3), (1.6):

\[
\int_{-\infty}^\infty [1 - \varphi^{p-1}(t)] \, dt, \quad \int_0^\infty [1 - \varphi(t)] \, dt, \quad \int_{-\infty}^0 [1 + \varphi(t)] \, dt,
\]

\[
\int_0^\infty [1 - u(x,t)] \, dt, \quad \int_{-\infty}^0 [1 + u(x,t)] \, dt, \quad x \geq 0.
\]


§ 6. The linear case \((p = 1)\)

Consider the linear equation (1.3):

\[
\varphi(t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \varphi(\tau) e^{-(t-\tau)^2} d\tau.
\]  

(6.1)

It is natural to consider a more general problem for equation (6.1) – the spectral problem of finding generalized eigenfunctions for the operator \(K\) (see § 3):

\[
\lambda \varphi = K\varphi, \quad \varphi \in L^2, \quad 0 < \alpha < 2.
\]  

(6.2)

**Theorem 6.1.** The spectrum of \(K\) is continuous and concentrated in [0, 1], and to every eigenvalue \(\lambda_\xi = e^{-\xi^2/4}, \ -\infty < \xi < \infty\), there correspond two eigenfunctions:

\[
\varphi_\xi(t) = \begin{cases} 
\cos(\xi t), & \text{if } \xi \neq 0, \\
1, & \text{if } \xi = 0.
\end{cases}
\]  

(6.3)

These functions exhaust the solutions of equation (6.2) with \(\lambda = \lambda_\xi\) in the class \(S'\) of tempered distributions. The set of these functions is dense in \(L^2\), \(\alpha > 0\).

**Proof.** Let \(\varphi \in S'\) be a solution of equation (6.2) written as \(\lambda \varphi = \varphi * d\mu_1\). Passing to the Fourier transform \(\widehat{\varphi}(\xi)\) and using the theorem on the Fourier transform of a convolution [13], we obtain the equation

\[
\lambda \widehat{\varphi}(\xi) = e^{-\xi^2/4} \widehat{\varphi}(\xi),
\]  

(6.4)

whence we deduce that either \(\widehat{\varphi}(\xi) = 0\) or for \(\lambda = \lambda_{\xi_0}\) the support of \(\widehat{\varphi}\) consists of the points \(\pm \xi_0\) (if \(\xi_0 = 0\), then this support consists of the single point 0). In this case \(\widehat{\varphi}\) can be represented as

\[
\widehat{\varphi}(\xi) = \sum_{k=0}^{n} c_k \delta^{(k)}(\xi - \xi_0) + d_k \delta^{(k)}(\xi + \xi_0)
\]  

(6.5)

with some (complex) \(c_k\) and \(d_k\). Substituting (6.5) into (6.2) and using the fact that the distributions (generalized functions) \(\delta^{(k)}, \ k = 0, 1, \ldots, n\), are linearly independent, we obtain, after some natural transformations, that \(c_k = d_k = 0, \ k = 1, 2, \ldots, n\), if \(\xi_0 \neq 0\) and \(c_k = d_k = 0, \ k = 2, 3, \ldots, \ d_1 = 0\) if \(\xi_0 = 0\). Therefore,

\[
\widehat{\varphi}(\xi) = \begin{cases} 
c_0 \delta(\xi - \xi_0) + d_0 \delta'(\xi - \xi_0) & \text{if } \xi_0 \neq 0, \\
c_0 \delta(\xi) + c_1 \delta'(\xi) & \text{if } \xi_0 = 0,
\end{cases}
\]

which implies that (6.3) holds.

The set of solutions (6.3) is dense in \(L^2\), \(\alpha > 0\). Indeed, if there is an \(f \in L^2\) orthogonal to the functions defined by formula (6.3), that is,

\[
(f, \varphi_\xi) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(\tau) e^{-\alpha \tau^2} e^{i \xi \tau} d\tau = 0
\]

for all \(\xi \in \mathbb{R}\), then we have \(f(\tau) e^{-\alpha \tau^2} = 0\), since \(f(\tau) e^{-\alpha \tau^2} \in L_1(\mathbb{R})\). Hence, \(f(\tau) = 0\) almost everywhere in \(\mathbb{R}\).
Corollary 6.1. The operator $K$ is bounded, self-adjoint and positive in $L_2(\mathbb{R})$, and

$$\|Kf\| \leqslant \|f\|, \quad (Kf, f) \geqslant 0, \quad f \in L_2(\mathbb{R}).$$

By Theorem 6.1, equation (6.1) has two linearly independent solutions in $S'$: 1 and $t$. The solutions of equation (6.1) belonging to $L_1/L_2^2$, if there are any, can be expanded in series (4.1), (4.3), (4.6) and (4.8) with $a_n = b_n = (\varphi, H_n)_1, \ n = 0, 1, \ldots$. The infinite systems (2.21) and (2.22) take the form

$$\sum_{m=n+2}^{m=n+2} (-1)^{m-n} 2^{n-m} \binom{m-n}{2}^{-1} a_m = 0, \quad n = 0, 1, \ldots, \quad (6.6)$$

$$\sum_{m=n+2}^{m=n+2} 2^{n-m} \binom{m-n}{2}^{-1} a_m = 0, \quad n = 0, 1, \ldots. \quad (6.7)$$

The linear systems (6.6) and (6.7) are triangular, they do not contain $a_0$ and $a_1$ and are decomposed into four independent block-triangular systems corresponding to the values of the residue of $n$ modulo 4.

Indeed, adding systems (6.6) and (6.7) together and subtracting one of them from the other, we obtain four groups (for $\kappa = 0, 1, 2, 3$) of independent block-triangular systems in $a_m$, $m = 2 + 4k + 4l + \kappa, \ l, k = 0, 1, \ldots$:

$$\sum_{l=0}^{\infty} a_{2+4k+4l+\kappa} \frac{1}{2^{4l}(2l+1)!} = 0, \quad \sum_{l=0}^{\infty} a_{2+4k+4l+\kappa} \frac{1}{2^{4l}(2l+2)!} = 0. \quad (6.8)$$

It turns out that there are non-trivial solutions of system (6.8) (with the exception of the case when $a_0$ and $a_1$ are arbitrary and $a_2 = a_3 = \cdots = 0$) such that $\sum a_m^2/m! < \infty$, that is, equation (6.1) has the following solutions belonging to $L_2^{1/2}$:

$$\varphi_k^\pm(t) = e^{\pm 2\sqrt{k\pi} t} \cos(2\sqrt{k\pi} t), \quad k = 0, 1, \ldots. \quad (6.9)$$

This follows from the fact that the heat equation (5.1) has the following solutions periodic with respect to $x$ with period 1:

$$u_k^\pm(x, t) = e^{\pm 2\sqrt{k\pi} t} \cos(2\sqrt{k\pi} t \pm 2k\pi x), \quad k = 0, 1, \ldots. \quad (6.10)$$

We have $\varphi_k^\pm(t) = u_k^\pm(1, t)$.

§ 7. The case when $p$ is even ($p = 2q$)

Equation (1.3) takes the form

$$\varphi^{2q}(t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \varphi(\tau) e^{-(t-\tau)^2} d\tau, \quad q = 1, 2, \ldots. \quad (7.1)$$

If $\varphi(t)$ is a solution of equation (7.1), then $\varphi(-t)$ and $\varphi(t + t_0)$ also are solutions of this equation (for all $t_0$).
Assume that problem (7.1), (1.5) has a piecewise continuous solution \( \varphi(t) \). By Theorem 2 in [9], we have \( |\varphi(t)| < 1 \), and \( \varphi(t) \) satisfies the equation

\[
\varphi^{2q}(t) = A(t),
\]

(7.2)

where \( A(t) \geq 0 \) is the trace of an entire function \( A(z) \) for which estimate (4.2) holds. Equation (7.2) has two real solutions in the neighbourhood of every \( t \):

\[
\varphi(t) = \pm A^{\frac{1}{2q}}(t).
\]

(7.3)

Therefore, the global structure of the solution \( \varphi(t) \) depends on its points \( T_k \) of discontinuity of the first kind and on the real zeros \( t_k \) of the entire function \( A(z) \). The sets \{\( T_k \} \) and \{\( t_k \} \) are bounded above and at most countable.

By (7.2), the function \( \varphi^{2q}(t) \) has the following representation in the neighbourhood of \( t_k \):

\[
\varphi^{2q}(t) = \frac{a_{2\sigma_k}}{(2\sigma_k)!}(t - t_k)^{2\sigma_k}[1 + O(|t - t_k|)],
\]

(7.4)

where \( a_{2\sigma_k} > 0 \) and the multiplicity of the zero \( 2\sigma_k \) is an even number. Hence, \( \varphi(t) \) can be represented in the neighbourhood of \( t_k \) as follows:

\[
\varphi(t) = \pm \left[ \frac{a_{2\sigma_k}}{(2\sigma_k)!} \right]^\frac{1}{2q}[t - t_k]^{\frac{\sigma_k}{q}}[1 + O(|t - t_k|)],
\]

(7.5)

and

\[
\frac{2^{2\sigma_k}}{\sqrt{\pi}} \int_{-\infty}^{\infty} \varphi(\tau)(t_k - \tau)^n e^{-(t_k - \tau)^2} d\tau = \begin{cases} a_{2\sigma_k} > 0, & n = 2\sigma_k, \\ 0, & n = 0, 1, \ldots, 2\sigma_k - 1. \end{cases}
\]

(7.6)

By (7.3), the function \( \varphi(t) \) can be represented in the neighbourhood of the point \( T_k \) of discontinuity of the first kind as follows:

\[
\varphi(t) = \pm \text{sgn } t A^{\frac{1}{2q}}(t).
\]

(7.7)

Therefore, it has the saltus

\[
2A^{\frac{1}{2q}}(T_k + 0) \quad \text{or} \quad -2A^{\frac{1}{2q}}(T_k + 0)
\]

at \( T_k \). Consider the different cases.

(a) The function \( A(t) \) has no zeros, whence \( A(t) > 0 \) for all \( t \). By Theorem 3 in [9], \( \varphi(t) = A^{1/(2q)}(t) > 0 \) cannot be a solution of problem (7.1), (1.5). Therefore, its sign cannot be constant. Hence, there is a \( T_0 \) in whose neighbourhood \( \varphi(t) \) has representation (7.7).

If \( \varphi(t) \) has only one point of discontinuity of the first kind (at \( T_0 \)), that is, \( \varphi(t) > 0 \) for \( t > T_0 \) and \( \varphi(t) < 0 \) for \( t < T_0 \), then

\[
A'(T_0) = -\frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} \varphi(\tau)(T_0 - \tau)e^{-(T_0 - \tau)^2} d\tau > 0.
\]

(7.8)

(b) Assume that \( A(t) \) has only one zero \( t_0 \). Hence, \( A(t_0) = A'(t_0) = 0 \). Then problem (7.1), (1.5) has no continuous solution \( \varphi(t) \). Indeed, in this case \( \varphi(t) > 0 \)
for \( t > t_0 \) and \( \varphi(t) < 0 \) for \( t < t_0 \). Formula (7.8) with \( t_0 \) instead of \( T_0 \) implies that \( A'(t_0) > 0 \), which contradicts the relation \( A'(t_0) = 0 \). Hence, points of discontinuity of the first kind can exist in this case as well.

(c) Assume that \( A(t) \) has only two zeros: \( t_0 = 0 \) and \( t_1 < 0 \). Then \( A(0) = A'(0) = 0 \) and \( A(t_1) = A'(t_1) = 0 \). A priori the following three cases are possible for continuous solutions:

\[
\varphi(t) = \begin{cases} 
A^{\frac{1}{k}}(t), & t > 0, \\
\pm A^{\frac{1}{k}}(t), & t_1 < t < 0, \\
\pm A^{\frac{1}{k}}(t), & t < t_1.
\end{cases} \tag{7.9}
\]

We claim that in reality of all these four cases only the following case can occur:

\[
\varphi(t) = \begin{cases} 
A^{\frac{1}{k}}(t), & t > 0, \\
-A^{\frac{1}{k}}(t), & t_1 < t < 0, \\
A^{\frac{1}{k}}(t), & t < t_1.
\end{cases} \tag{7.10}
\]

Indeed, the case \((+, +)\) contradicts Theorem 3 in [9]. The case \((-,-)\) contradicts the relation \( A'(0) = 0 \) (see (7.8) with \( T_0 = 0 \)), and the case \((+,-)\) contradicts the relation \( A'(t_1) = 0 \) (see (7.8) with \( T_0 = t_1 \)).

(d) Assume that the solution \( \varphi(t) \) is continuous and the function \( \varphi^{2q}(t) \) has a zero \( t_0 \) of multiplicity \( 2n \). Then \( \varphi(t) \) has at least \( 2n \) sign changes.

Indeed, we can assume without loss of generality that \( t_0 = 0 \). By (7.6), we have

\[
\int_{-\infty}^{\infty} \varphi(\tau) \tau^k e^{-\tau^2} d\tau = 0, \quad k = 0, 1, \ldots, 2n - 1. \tag{7.11}
\]

Assume the contrary: let \( \varphi(t) \) have \( m < 2n \) sign changes. Let \((a_k, b_k), \ k = 1, 2, \ldots, m \leq 2n - 1\), be the intervals on which \( \varphi(t) \) is negative (on the complementary segments it is non-negative). There is a polynomial of degree \( m \) such that \( P(t) > 0 \) if \( t \in (a_k, b_k) \) and \( P(t) \geq 0 \) on the complementary segments. We have

\[
\int_{-\infty}^{\infty} \varphi(\tau) P(\tau) e^{-\tau^2} d\tau > 0,
\]

which contradicts (7.11).

The results obtained above can be stated as follows.

**Theorem 7.1.** If \( \varphi \) is a solution of problem (7.1), (1.5), then

(i) \( \varphi(t) \) can have discontinuity of the first kind at \( T_k \) with saltuses \( \pm 2\varphi(T_k + 0) \) if \( \varphi^{2q}(t) \) has at most one zero,

(ii) \( \varphi(t) \) has structure (7.10) if it is continuous and \( \varphi^{2q}(t) \) has only two zeros \( t_0 \) and \( t_1 \),

(iii) in the neighbourhood of \( t_k \) \( \varphi(t) \) has representation (7.5), and (7.6) holds,

(iv) if \( \varphi(t) \) is continuous, then the number of its sign changes does not exceed the number of zeros of \( \varphi^{2q}(t) \) and is greater than or equal to \( \sup_k 2\sigma_k \), where \( 2\sigma_k \) is the multiplicity of the zero \( t_k \),

(v) \( \varphi(t) \) is piecewise real-analytic everywhere, with the exception of its zeros and the points of discontinuity of the first kind.

The following theorem holds for the zeros of the interpolating function \( u(x, t) \).
Theorem 7.2 (on the branching of zeros). Let $u(x, t)$ be an interpolating function between the solution $\varphi$ of problem (5.1), (5.2) and $\varphi^{2q}$ such that $u(1, t)$ has a zero of multiplicity $2n$ at $t = 0$. Then the equation

$$u(1 - \varepsilon, t) = 0 \quad \text{as} \quad \varepsilon \to +0$$

(7.12)

has precisely $2n$ simple real roots

$$t_k^\pm(\varepsilon) = \frac{1}{2} \lambda_k^\pm \varepsilon + O(\varepsilon), \quad k = 1, 2, \ldots, n,$$

(7.13)

where $\lambda_k^\pm$ are the roots of the equation

$$\sum_{m=0}^{n} (-1)^m \frac{\lambda^{2n-2m}}{(2n-2m)! m!} = 0.$$  

(7.14)

Proof. The results of § 5 imply that $u(x, t)$ has a holomorphic continuation to $T^+ \times \mathbb{C}$. Since $u(1, t) = \varphi^{2q}(t) > 0$ is a positive function, the multiplicity of its zero $t = 0$ is even $(2n)$, whence

$$u(1, 0) = u'(1, 0) = \cdots = u^{(2n-1)}(1, 0) = 0, \quad u^{(2n)}(1, 0) = a > 0.$$  

(7.15)

We expand $u(1 - \varepsilon, t)$ in the Taylor series about $(1, 0)$:

$$u(1 - \varepsilon, t) = \sum_{m=0}^{2n} \frac{1}{m!} \sum_{s=0}^{m} C_s^m \frac{\partial^m u(1, 0)}{\partial x^s \partial t^{m-s}} (-\varepsilon)^s t^{m-s} + R_1(\varepsilon, t),$$

(7.16)

where the residual term $R_1 \in C^\infty$ can be computed by the formula

$$R_1 = \frac{1}{(2n+1)!} \sum_{s=0}^{2n+1} C_{2n+1}^s \frac{\partial^{2n+1} u(1 - \theta \varepsilon, \theta t)}{\partial x^s \partial t^{2n+1-s}} (-\varepsilon)^s t^{2n+1-s}$$

(7.17)

with some $\theta \in (0, 1)$. Taking into account that the derivatives of $u(x, t)$ are bounded in the neighbourhood of $(1, 0)$, we obtain the following estimate for $R_1$ in (7.17):

$$|R_1(\varepsilon, t)| \leq C_1 \sum_{s=0}^{2n+1} |\varepsilon|^s |t|^{2n+1-s}.$$  

Using (7.15) and the heat equation (5.1), we reduce (7.16) to the form

$$u(1 - \varepsilon, t) = a \sum_{m=0}^{n} (-1)^m \frac{t^{2n-2m}}{(2n-2m)! m!} \left(\frac{\varepsilon}{4}\right)^{n-m} + R_2(\varepsilon, t),$$

(7.18)

where $R_2$ is such that

$$|R_2(\varepsilon, t)| \leq C_2 \sum_{s=1}^{2n} \varepsilon^s t^{2n-s}.$$
Making the change of variable
\[ \lambda = \frac{2t}{\varepsilon}, \quad t = \frac{\lambda \varepsilon}{2}, \quad (7.19) \]
we transform (7.18) into the equation
\[ u(1 - \varepsilon, t) = a 4^{-n} \varepsilon^n \sum_{m=0}^{n} (-1)^m \frac{\lambda^{2n-2m}}{(2n-2m)! m!} + R_3(\varepsilon, \lambda), \quad (7.20) \]
\[ |R_3(\varepsilon, \lambda)| \leq C_3 \varepsilon^{n+1/2}. \quad (7.21) \]

Dividing (7.20) by \( a 4^{-n} \varepsilon^n \), we obtain the following equation for the zeros \( t(\varepsilon) = \lambda(\varepsilon) \sqrt{\varepsilon}/2 \) of \( u(1 - \varepsilon, t) \) (see (7.12)):
\[ \sum_{m=0}^{n} (-1)^m \frac{\lambda^{2n-2m}}{(2n-2m)! m!} = R_4(\varepsilon, \lambda), \quad |R_4(\varepsilon, \lambda)| \leq C_4 \sqrt{\varepsilon}. \quad (7.22) \]

The equation
\[ \sum_{m=0}^{n} (-1)^m \frac{\Lambda^{n-m}}{(2n-2m)! m!} = 0 \quad (7.23) \]
has \( n \) positive roots \( \Lambda_k, \ k = 1, 2, \ldots, n \). Hence, equation (7.14) has \( 2n \) real roots \( \lambda^\pm_\kappa = \pm \sqrt{\Lambda_k}, \ k = 1, 2, \ldots, n \). Equation (7.22) takes the following form in the neighbourhood of \( \lambda^+_n \):
\[ \lambda - \lambda^+_k = \frac{r(\varepsilon, \lambda)}{(\lambda - \lambda^-_k) \prod_{i \neq k} (\lambda^2 - \Lambda_i)} \equiv R_5(\varepsilon, \lambda). \quad (7.24) \]

By (7.23), the residual term \( R_5 \) has the following properties:
\[ |R_5(\varepsilon, \lambda)| \leq C_5 \sqrt{\varepsilon}, \quad |R_5(\varepsilon, \lambda) - R_5(\varepsilon, \lambda')| \leq C_5 \sqrt{\varepsilon} |\lambda - \lambda'|. \]
(In the neighbourhood of \( \lambda^-_k \) equation (7.22) takes a similar form). Using the contraction map principle, we obtain that equation (7.24) has precisely one continuous solution
\[ \lambda^+_k(\varepsilon) = \lambda^+_k + O(\sqrt{\varepsilon}) \]
in the neighbourhood of \( \lambda^+_k \). Using (7.19), we obtain that equation (7.12) has precisely \( 2n \) roots (7.13) in the \( \varepsilon \)-neighbourhood, \( \varepsilon > 0 \), of \( (1, 0) \).

**Example.** For \( n = 1 \) equation (7.14) takes the form \( \lambda^2 = 2 \). For \( n = 2 \) it takes the form
\[ \lambda^4 - 12 \lambda^2 + 12 = 0, \]
\[ \lambda^\pm_1 = \pm \sqrt{6 + 2\sqrt{6}}, \quad \lambda^\pm_2 = \pm \sqrt{6 - 2\sqrt{6}}. \]
For \( n = 3 \) it takes the form
\[ \lambda^6 - 30 \lambda^4 + 180 \lambda^2 - 120 = 0, \]
\[ \lambda^\pm_1 \approx \pm 0.87, \quad \lambda^\pm_2 \approx \pm 2.67, \quad \lambda^\pm_3 \approx \pm 4.70. \]
Assertion 7.1. If \( \varphi(t) \) is a solution of equation (7.1) such that (5.4) holds, then
\[
\frac{1}{\sqrt{\pi x}} \int_{-\infty}^{\infty} \varphi^2(\tau) \exp\left\{ -\frac{(t-\tau)^2}{x} \right\} \, d\tau \leq x^{\frac{q}{2q-1}} (1+x)^{-\frac{1}{4q-2}} \sqrt{\frac{2q-1}{2qx-x-1}}
\]  
(7.25)
for all \( x > 1/(2q-1) \).

**Proof.** Denoting the left-hand side of inequality (7.25) by \( J(x,t) \), using the boundary conditions (5.2), the properties of solutions of the heat equation and Hölder’s inequality, we obtain the following chain of relations for all \( x > 1/(2q-1) \):

\[
J(x,t) \equiv \frac{1}{\sqrt{\pi x}} \int_{-\infty}^{\infty} \varphi^2(\tau) \exp\left\{ -\frac{(t-\tau)^2}{x} \right\} \, d\tau \\
= \frac{1}{\sqrt{\pi(1+x)}} \int_{-\infty}^{\infty} \varphi(\tau) \exp\left\{ -\frac{(t-\tau)^2}{1+x} \right\} \, d\tau \\
\leq \frac{1}{\sqrt{\pi(1+x)}} \int_{-\infty}^{\infty} \varphi(\tau) \exp\left\{ -\frac{(t-\tau)^2}{2qx} \right\} \exp\left\{ -\frac{(t-\tau)^2(2qx-x-1)}{(1+x)2qx} \right\} \, d\tau \\
= \frac{1}{\sqrt{\pi(1+x)}} [\sqrt{\pi t J}]^{\frac{1}{2q}} \left( \int_{-\infty}^{\infty} \exp\left\{ -\frac{(t-\tau)^2(2qx-x-1)}{x(1+x)(2q-1)} \right\} \, d\tau \right)^{1-\frac{1}{2q}},
\]
whence
\[
J^{1-\frac{1}{2q}} \leq \frac{(\pi x)^{\frac{1}{2q}}}{\sqrt{\pi x(1+x)}} \left( \sqrt{\frac{\pi x(1+x)(2q-1)}{2qx-x-1}} \right)^{1-\frac{1}{2q}},
\]
which implies that (7.25) holds.

Corollary 7.1. For \( x = 1 \) estimate (7.25) with \( q = 2, 3, \ldots \) takes the form
\[
\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \varphi^2(\tau) \exp\left\{ -(t-\tau)^2 \right\} \, d\tau \leq 2^{-\frac{1}{4q-2}} \sqrt{\frac{2q-1}{2q-2}}.
\]  
(7.26)

§ 8. The case when \( p = 2 \). Solution of the equation

Equation (1.3) with \( p = 2 \) takes the form
\[
\varphi^2(t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \varphi(\tau)e^{-(t-\tau)^2} \, d\tau.
\]  
(8.1)

Let \( \varphi(t) \) be a solution of equation (8.1) belonging to \( L^1_\beta \). We are going to obtain a formally infinite system of non-linear equations in the coefficients \( a_n \) occurring in expansion (4.3) of \( \varphi^2 \) and expansion (4.1) of \( \varphi \). We obtain the Taylor series for \( \varphi(t) \).
from (4.1), using expansion (2.7) of Hermite polynomials in powers of $t$. We have

$$\varphi^2(t) = \sum_{m=0}^{\infty} \frac{a_m}{2^m m!} H_m(t) \sum_{s=0}^{\infty} \frac{a_s}{2^s s!} H_s(t)$$

$$= \sum_{m=0}^{\infty} \frac{a_m a_s}{2^{m+s}} \sum_{k=0}^{\infty} c_{m,k} t^k \sum_{i=0}^{\infty} c_{s,i} t^i$$

$$= \sum_{m=0}^{\infty} \frac{a_m a_s}{2^{m+s}} \sum_{n=0}^{\infty} t^n \sum_{k\equiv m}^{\infty} c_{m,k} c_{s,i}$$

$$= \sum_{n=0}^{\infty} t^n \sum_{m+s \geq n}^{\infty} \frac{a_m a_s}{2^{m+s}} \sum_{k\equiv m}^{\infty} c_{m,k} c_{s,i}$$

$$= \sum_{n=0}^{\infty} t^n \sum_{k+i=n}^{\infty} \left( \sum_{m=k}^{\infty} \frac{a_m}{2^m c_{m,k}} \right) \left( \sum_{s=i}^{\infty} \frac{a_s}{2^s c_{s,i}} \right). \quad (8.2)$$

Comparing (8.2) with (4.3), we obtain the desired system of non-linear equations in $a_n$:

$$a_n = n! \sum_{k+i=n}^{\infty} \left( \sum_{m=k}^{\infty} \frac{a_m}{2^m c_{m,k}} \right) \left( \sum_{s=i}^{\infty} \frac{a_s}{2^s c_{s,i}} \right), \quad n = 0, 1, \ldots. \quad (8.3)$$

Let us note that the series in (8.3) converge, which follows from the Parseval–Steklov equality (4.1) and the Cauchy–Bunyakowskii inequality.

Let us write the first four equations of system (8.3) ($n = 0, 1, 2, 3$), using equalities (2.9) for $c_{m,k}$. For $n = 0$ we have

$$a_0 = \left( \sum_{m=0}^{\infty} (-1)^m \frac{a_{2m}}{4^m m!} \right)^2, \quad (8.4)$$

which enables us to obtain the following linear equation in $a_{2m}$, $m \geq 1$:

$$\sum_{m=0}^{\infty} (-1)^m \frac{a_{2m}}{4^m m!} = \varepsilon \sqrt{a_0}, \quad \varepsilon = \pm 1. \quad (8.5)$$

First we consider the case when $a_0 > 0$. For $n = 1$ we have

$$a_1 = 2 \left( \sum_{m=0}^{\infty} (-1)^m \frac{a_{2m}}{4^m m!} \right) \left( \sum_{s=0}^{\infty} (-1)^s \frac{a_{2s+1}}{4^s s!} \right). \quad (8.6)$$

Using (8.5), we obtain the following linear equation in $a_{2m+1}$, $m \geq 0$:

$$\sum_{m=0}^{\infty} (-1)^m \frac{a_{2m+1}}{4^m m!} = \frac{a_1}{2 \varepsilon \sqrt{a_0}}. \quad (8.7)$$
For $n = 2$ we have
\[
  a_2 = 2 \left( \sum_{m=0}^{\infty} (-1)^m \frac{a_{2m+1}}{4^m m!} \right)^2 
  - 8 \left( \sum_{m=0}^{\infty} (-1)^m \frac{a_{2m}}{4^m m!} \right) \left( \sum_{s=1}^{\infty} (-1)^s \frac{a_{2s}}{4^s (s-1)!} \right). \tag{8.8}
\]

Combining this with (8.5) and (8.7), we deduce the following linear equation in $a_{2m}$, $m \geq 1$:
\[
  \sum_{m=1}^{\infty} (-1)^m \frac{a_{2m}}{4^m (m-1)!} = \frac{1}{8\varepsilon \sqrt{a_0}} \left[ \frac{a_1^2}{2a_0} - a_2 \right]. \tag{8.9}
\]

Finally, for $n = 3$ we have
\[
  a_3 = -8 \left( \sum_{m=0}^{\infty} (-1)^m \frac{a_{2m}}{4^m m!} \right) \left( \sum_{s=1}^{\infty} (-1)^s \frac{a_{2s+1}}{4^s (s-1)!} \right) 
  - 24 \left( \sum_{m=0}^{\infty} (-1)^m \frac{a_{2m+1}}{4^m m!} \right) \left( \sum_{s=1}^{\infty} (-1)^s \frac{a_{2s}}{4^s (s-1)!} \right). \tag{8.10}
\]

Combining this with (8.5), (8.7) and (8.9), we deduce the following linear equation in $a_{2m+1}$, $m \geq 1$:
\[
  \sum_{m=1}^{\infty} (-1)^m \frac{a_{2m+1}}{4^m (m-1)!} = -\frac{1}{8\varepsilon \sqrt{a_0}} \left( a_3 - \frac{3a_1}{2a_0} \left( \frac{a_1^2}{\varepsilon \sqrt{a_0}} - a_2 \right) \right). \tag{8.11}
\]

Hence, the non-linear system of equations (8.3) for $n = 0, 1, 2, 3$ is decomposed into four linear equations – equations (8.5) and (8.9) in $a_{2m}$, $m \geq 1$, and equations (8.7) and (8.11) in $a_{2m+1}$, $m \geq 1$. System (8.3) has a similar structure for all $n$.

Now assume that $a_0 = 0$. By (8.5), we have
\[
  \sum_{m=1}^{\infty} (-1)^m \frac{a_{2m}}{4^m m!} = 0. \tag{8.12}
\]

It follows from (8.6) that $a_1 = 0$, and (8.8) implies that
\[
  a_2 = 2 \left( \sum_{m=0}^{\infty} (-1)^m \frac{a_{2m+1}}{4^m m!} \right)^2 \geq 0. \tag{8.13}
\]

If $a_2 = 0$, which implies, by (8.13), that
\[
  \sum_{m=0}^{\infty} (-1)^m \frac{a_{2m+1}}{4^m m!} = 0, \tag{8.14}
\]

then (8.10) implies that $a_3 = 0$, and so on. Finally, we find an integer $\sigma \geq 1$ such that $a_{2\sigma} > 0$, whence $a_{2\sigma+1} = 0$ (if the solution is different from the identical zero).
Solving (8.3) with $a_0 = a_1 = \cdots = a_{2\sigma-1} = 0$, we construct (as in the case when $a_0 > 0$) a formal solution of equation (8.1), using formula (4.1).

In the case when $a_2 > 0$ formula (8.13) implies that

$$\sum_{m=0}^{\infty} (-1)^m \frac{a_{2m+1}}{4^m m!} = \pm \sqrt{\frac{a_2}{2}}.$$  \hfill (8.15)

Combining this with (8.10), we obtain that

$$a_3 = \pm 24 \sqrt{\frac{a_2}{2}} \sum_{m=1}^{\infty} (-1)^m \frac{a_{2m}}{4^m (m-1)!}.$$  \hfill (8.16)

To obtain an approximate solution of system (8.3), we have to put $a_n = a_{n+1} = \cdots = 0$ in the first $n$ equations (the $(n-1)$-approximation). In this way we obtain a system of $n$ equations in $n$ unknowns $a_0, a_1, \ldots, a_{n-1}$.

Consider the $n$-approximation, $n = 3$, for $a_0 > 0$. We obtain from (8.5), (8.7), (8.9) and (8.11) the following system of equations in $a_0, a_1, a_2$ and $a_3$:

\begin{align*}
a_0 &= \varepsilon \sqrt{a_0} + \frac{a_2}{4}, \quad a_0 \geq 0, \tag{8.170} \\
a_1 &= 2\varepsilon \sqrt{a_0} \left( a_1 - \frac{a_3}{4} \right), \tag{8.171} \\
a_2 &= \frac{a_2^2}{2a_0} + 2\varepsilon \sqrt{a_0} a_2, \tag{8.172} \\
a_3 &= 2\varepsilon \sqrt{a_0} a_3 + \frac{3a_1 a_2}{\varepsilon \sqrt{a_0}}. \tag{8.173}
\end{align*}

Consider the linear system of equations in $a_1$ and $a_3$ that consists of equations (8.171) and (8.173):

\begin{align*}
(1 - \varepsilon \sqrt{a_0}) a_1 + \frac{1}{2} \varepsilon \sqrt{a_0} a_3 &= 0, \\
-3 \frac{a_2 a_1}{\varepsilon \sqrt{a_0}} + (1 - 2\varepsilon \sqrt{a_0}) a_3 &= 0. \tag{8.18}
\end{align*}

The determinant of this system is equal to

$$D = 1 + 4a_0 - 4\varepsilon \sqrt{a_0} + \frac{3}{2} a_2.$$  \hfill (8.19)

Taking into account (8.170), we obtain that

$$D = 1 + 10a_0 - 10\varepsilon \sqrt{a_0}.$$  \hfill (8.19)

In a similar way, for $a_0 = 0$ and $a_2 > 0$ we obtain from (8.15) and (8.16) the following system of equations in $a_2$ and $a_3$:

$$a_3 = \pm 8a_2, \quad a_3 = \pm 3\sqrt{2} a_2^{3/2}. \tag{8.20}$$

The following assertion holds for the solutions of systems (8.17) and (8.20).
Assertion 8.1. If \( a_0 > 0 \) and \( D \neq 0 \), then
- (a) \( a_0 = 1 \) and \( a_1 = a_2 = a_3 = 0 \), which corresponds to the trivial solution \( \varphi(t) = 1 \);
- (b) \( a_0 = 1/4, \ a_1 = 0, \ a_2 = -1 \) and \( a_3 = 0 \) (in this case \( \varepsilon = 1 \)), which corresponds to the approximate solution
  \[ \varphi \approx \frac{1}{2} (1 - t^2). \] (8.21)

If \( a_0 > 0 \) and \( D = 0 \) (in this case \( \varepsilon = 1 \)), then
- (c) \( a_0 = 0, 4000 + \sqrt{0.15} \approx 0.7873, \ a_1 \approx \pm 0.6984, \ a_2 = -0.4000 \) and \( a_3 \approx \pm 1.219 \), which corresponds to the following two approximate solutions:
  \[ \varphi(t) \approx 0.7873 \pm 0.6984 t - 0.05000 H_2(t) \pm 0.02540 H_3(t) \]
  \[ = 0.8873 \pm 0.3936 t - 0.20000 t^2 \pm 0.2032 t^3. \] (8.22)

If \( a_0 = 0 \) (in this case \( a_1 = 0 \)), then
- (d) either \( a_2 = a_3 = 0 \) or \( a_2 = 2/3 \) and \( a_3 = \pm 4/\sqrt{3} \), which corresponds either to the trivial solution \( \varphi(t) = 0 \) or to the two approximate solutions
  \[ \varphi(t) \approx \frac{1}{12} H_2(t) \pm \frac{1}{12 \sqrt{3}} H_3(t). \] (8.23)

Proof. If \( a_0 > 0 \) and \( D \neq 0 \), then \( a_1 = a_3 = 0 \). In this case equations (8.170) and (8.172) give solutions (a) and (b) (and \( \varepsilon = 1 \)). If \( a_0 > 0 \) and \( D = 0 \), then \( \varepsilon = 1 \) and \( a_0 = 0, 4 \pm \sqrt{0.15} \). It follows from (8.170) that \( a_2 = -0.4 \). Equations (8.172) and (8.173) imply that \( a_0 = 0.4 + \sqrt{0.15}, \) and we have two approximate solutions (c) with \( a_1 \approx \pm 0.6984 \) and \( a_3 \approx \pm 1.219 \). If \( a_0 = a_1 = 0 \), then (8.20) implies that assertion (d) holds.

Remark 8.1. The question arises whether the approximate solutions (8.21)–(8.23) are “parasite” or they are the first terms of unknown solutions belonging to \( L^1_t \). We have verified up to the terms of order of \( t^4 \) the squares of solutions (8.22) coincide with the squares of these solutions
  \[ \varphi^2(t) \approx 0.7873 \pm 0.6984 t - 0.20000 t^2 \pm 0.2032 t^3 \] (8.24)
computed by formula (4.3).

§ 9. THE CASE WHEN \( p = 2 \). THE BOUNDARY-VALUE PROBLEM

The method of solving equation (8.1) described in § 8 can be applied to problem (8.1), (1.5).

Let \( \varphi \) be a solution of problem (8.1), (1.5). We seek this solution in the form
  \[ \varphi(t) = \varphi_0(t) + e^{-(\alpha^2-1)t^2} \sum_{m=0}^{\infty} c_m H_m(\alpha t), \] (9.1)
where \( \alpha \) is a parameter greater than 1 and
  \[ \varphi_0(t) = \frac{1}{2} + \frac{1}{2} \operatorname{erf}(t), \quad \operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2} dx. \] (9.2)
The right-hand side of (9.1) satisfies the boundary conditions (1.5).

We shall be able to use representation (9.1) if we verify that the system of functions

$$
\chi_n(t) = e^{-(\alpha^2 - 1)t^2} H_n(t), \quad n = 0, 1, \ldots,
$$

is a basis of the separable Hilbert space $L^1_2$. It is sufficient to prove that this system is complete, that is, to prove that $f = 0$ if $f \in L^1_2$ is such that

$$(f, \chi_n) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(t) e^{-(\alpha^2 - 1)t^2} H_n(t) \, dt = 0, \quad n = 0, 1, \ldots.$$ 

This follows immediately from the fact that the system of Hermite polynomials is complete in $L^1_2$.

Now we are going to compute the coefficients $a_n = (\varphi, H_n)$ of the expansion of $\varphi$ in Hermite polynomials. We have

$$
e_n = (\varphi_0, H_n) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \varphi_0(\tau) H_n(\tau) e^{-\tau^2} \, d\tau
= \frac{1}{2} \delta_{n0} + \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \text{erf}(\tau) H_n(\tau) e^{-\tau^2} \, d\tau
= \begin{cases}
\frac{1}{2\pi} (-1)^{n+1} 2^{n/2} \Gamma\left(\frac{n}{2}\right), & n \equiv 1, \\
0, & n \equiv 0, \quad n \neq 0, \\
\frac{1}{2}, & n = 0.
\end{cases}
$$

(9.3)

Here we used formula 2.20.10.1 in [11], p. 497 of the Russian version, with $b = c = 1$.

Using formula 2.20.16.4 in [11], p. 502 of the Russian version, with $p = \alpha^2$, $b = \alpha$, and $c = 1$, we compute the integral

$$
\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\alpha^2 \tau^2} H_m(\alpha \tau) H_n(\tau) \, d\tau
= \begin{cases}
\frac{1}{n!} 2^m \left(\frac{n - m}{2}!\right)^{-1} (1 - \alpha^2)^{\frac{n-m}{2}} \alpha^{-n-1}, & m \leq n \quad \text{and} \quad m \equiv n, \\
0, & m > n \quad \text{or} \quad m \neq n.
\end{cases}
$$

Combining this with (9.1) and (9.3), we obtain the following formula for the Hermite coefficients of $\varphi$:

$$
a_n = e_n + n! \alpha^{-n-1} \sum_{m=0}^{n} c_m 2^m \left(\frac{n - m}{2}!\right)^{-1} (1 - \alpha^2)^{\frac{n-m}{2}}. \quad (9.4)
$$

In particular,

$$
a_0 = \frac{1}{2} + \frac{c_0}{\alpha}, \quad a_1 = \frac{1}{\sqrt{2\pi}} + \frac{2c_1}{\alpha^2}, \quad a_2 = -2c_0 \frac{\alpha^2 - 1}{\alpha^3} + \frac{8c_2}{\alpha^4},
$$

$$
a_3 = -\frac{1}{\sqrt{2\pi}} - 12c_1 \frac{\alpha^2 - 1}{\alpha^4} + \frac{48c_3}{\alpha^4}. \quad (9.5)
$$
Substituting expressions (9.4) for \( a_n \) into (8.3), we obtain an infinite system of non-linear equations in the unknowns \( c_m \), which occur in formula (9.1) for \( \varphi \).

Considering the 3-approximation, we put in (9.5) for definiteness \( a_0 = 0.7873 \), \( a_1 = 0.6984 \), \( a_2 = -0.4000 \) and \( a_3 = 1.219 \) (see Assertion 7.1) and obtain the following equations in \( c_0 \), \( c_1 \), \( c_2 \) and \( c_3 \):

\[
\begin{align*}
0.7873 &= \frac{1}{2} + \frac{c_0}{\alpha}, \\
0.6984 &= \frac{1}{\sqrt{2\pi}} + \frac{2c_1}{\alpha^2}, \\
-0.4000 &= -2c_0 \frac{\alpha^2 - 1}{\alpha^3} + \frac{8c_2}{\alpha^3}, \\
1.219 &= -\frac{1}{\sqrt{2\pi}} - 12c_1 \frac{\alpha^2 - 1}{\alpha^4} + \frac{48c_3}{\alpha^4}.
\end{align*}
\]

Hence,

\[
\begin{align*}
c_0 &= 0.2873\alpha, \\
c_1 &= 0.1498\alpha^2, \\
c_2 &= 0.02182\alpha^3 - 0.07182\alpha, \\
c_3 &= 0.07120\alpha^4 - 0.03746\alpha^2.
\end{align*}
\]

(9.6)

Putting \( \alpha^2 = 1,1 \) in (9.6), we obtain that

\[
\begin{align*}
c_0 &= 0.3014, \\
c_1 &= 0.1648, \\
c_2 &= -0.05016, \\
c_3 &= 0.04494.
\end{align*}
\]

Substituting these numbers into (9.1), we obtain the following approximate solution of problem (8.1), (1.5):

\[
\varphi(t) \approx \frac{1}{2} + \frac{1}{2} \text{erf}(t) + e^{-0.1t^2} \left( 0.3014 + 0.1648H_1(\sqrt{1,1}t) + 0.05016H_2(\sqrt{1,1}t) + 0.04494H_3(\sqrt{1,1}t) \right) = \frac{1}{2} + \frac{1}{2} \text{erf}(t) + e^{-0.1t^2} \left( 0.4017 - 0.2200t - 0.2207t^2 + 0.4149t^3 \right). (9.7)
\]

The second solution can be obtained likewise if we consider \( a_1 \) and \( a_3 \) with minus sign.

Remark 9.1. Our choice \( \alpha = \sqrt{1,1} \approx 1,049 \) was an arbitrary one. The method does not depend on \( \alpha > 1 \) as long as it ranges between reasonable limits. In order to find the optimal value of \( \alpha \) one should perform supplementary calculations.

§ 10. THE CASE WHEN \( p \) IS ODD \((p = 2q + 1)\)

Equation (1.3) with \( p = 2q + 1 \) takes the form

\[
\varphi^{2q+1}(t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \varphi(\tau)e^{-(t-\tau)^2} d\tau, \quad q = 1, 2, \ldots.
\]

(10.1)

If \( \varphi(t) \) is a solution of equation (10.1), then \(-\varphi(t), \varphi(-t)\) and \( \varphi(t + t_0) \) with any \( t_0 \) also are solutions.

We can find solutions of equation (10.1), using the method of expanding in Hermite polynomials described in § 7–9, but the calculations involved would be bulky even in the case when \( p = 3 \). It is also possible to solve problem (10.1), (1.6), using the following substitution similar to (9.1):

\[
\varphi(t) = \text{erf}(t) + e^{-\left(\alpha^2-1\right)t^2} \sum_{m=0}^{\infty} c_m H_m(\alpha t).
\]
In [9] it was proved that problem (10.1), (1.6) has a continuous odd solution real-analytic for \( t \neq 0 \) that has precisely one real zero at \( t = 0 \), and

\[
\varphi(t) = (a_1 t)^{\frac{1}{2^{2q+1}+1}} [1 + O(|t|)], \quad t \to 0, \tag{10.2}
\]

where

\[
a_1 = \frac{4}{\sqrt{\pi}} \int_0^\infty \varphi(\tau) e^{-\tau^2} d\tau > 0. \tag{10.3}
\]

As in the case when \( p \) is even (see § 7), the following theorem holds.

**Theorem 10.1.** If \( \varphi(t) \) is a solution of problem (10.1), (1.6), then it is continuous, \( \varphi^{2q+1}(t) \) has finitely many zeros \( t_k \) of finite multiplicity \( \sigma_k \), \( k = 1, 2, \ldots, l \), \( \sum_{k=1}^{l} \sigma_k \) is an odd number, and

\[
\varphi(t) = \left[ \frac{a_{\sigma_k}}{(\sigma_k)!} \right]^{\frac{1}{2^{2q+1}+1}} (t - t_k)^{\frac{\sigma_k}{2^{2q+1}}} [1 + O(|t - t_k|)], \quad t \to t_k, \tag{10.4}
\]

where

\[
\sqrt{\frac{2}{\pi}} \int_{-\infty}^\infty \varphi(\tau)(\tau - t_k)^n e^{-(t_k - \tau)^2} d\tau = \begin{cases} a_{\sigma_k}, & n = \sigma_k, \\ 0, & n = 0, 1, \ldots, \sigma_k - 1. \end{cases} \tag{10.5}
\]

The number of sign changes of \( \varphi(t) \) is odd and coincides with that of \( \varphi^{2q+1}(t) \). This number is less than or equal to \( l \) and greater than or equal to \( \max_{1 \leq k \leq l} \sigma_k \).

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