PRINCIPAL SERIES COMPONENT OF GELFAND-GRAEV REPRESENTATION

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Abstract. Let $G$ be a connected reductive group defined over a non-archimedean local field $F$. Let $B$ be a minimal $F$-parabolic subgroup with Levi factor $T$ and unipotent radical $U$. Let $\psi$ be a non-degenerate character of $U(F)$ and $\lambda$ a character of $T(F)$. Let $(K, \rho)$ be a Bushnell-Kutzko type associated to the Bernstein block of $G(F)$ determined by the pair $(T, \lambda)$. We study the $\rho$-isotypical component $(c\text{-}\text{ind}_{U(F)}^{G(F)}(c\text{-}\text{ind}_{U(F)}^{G(F)}(\psi)))$ of the induced space $c\text{-}\text{ind}_{U(F)}^{G(F)}(\psi)$ of functions compactly supported mod $U(F)$. We show that $(c\text{-}\text{ind}_{U(F)}^{G(F)}(\psi))$ is cyclic module for the Hecke algebra $\mathcal{H}(G, \rho)$ associated to the pair $(K, \rho)$. When $T$ is split, we describe it more explicitly in terms of $\mathcal{H}(G, \rho)$. We make assumptions on the residue characteristic of $F$ and later also on the characteristic of $F$ and the center of $G$ depending on the pair $(T, \lambda)$. Our results generalize the main result of Chan and Savin in [6] who treated the case of $\lambda = 1$ for $T$ split.

1. Introduction

Let $G$ be a connected reductive group defined over a non-archimedean local field $F$. Fix a minimal $F$-parabolic subgroup $B = TU$ of $G$ with unipotent radical $U$ and whose Levi factor $T$ contains a maximal $F$-split torus of $G$. Let $\psi$ be a non-degenerate character of $U(F)$ and consider the induced representation $c\text{-}\text{ind}_{U(F)}^{G(F)}(\psi)$ realized by functions whose support is compact mod $U(F)$. It is a result of Bushnell and Henniart [3] that the Bernstein components of $c\text{-}\text{ind}_{U(F)}^{G(F)}(\psi)$ are finitely generated.

Now let $\lambda$ be a character of $T(F)$. The pair $(T, \lambda)$ determines a Bernstein block $R_{T, \lambda}^{G}(G(F))$ in the category of smooth representations $R(G(F))$ of $G(F)$. Bushnell-Kutzko types are known to exist for Bernstein blocks under suitable residue characteristic hypothesis [5,11]. Let $(K, \rho)$ be a $[T, \lambda]_{G}$-type and let $\mathcal{H}(G, \rho)$ be the associated Hecke algebra. We show in Theorem 1 that the $\rho$-isotypical component $(c\text{-}\text{ind}_{U(F)}^{G(F)}(\psi))$ is a cyclic $\mathcal{H}(G, \rho)$-module.

Now assume that $T$ is split and $\psi$ is a non-degenerate character of $U(F)$ of generic depth zero (see §4.2). If $\lambda \neq 1$, then assume further that the center of $G$ is connected. In that case, $\mathcal{H}(G, \rho)$ is an Iwahori-Hecke algebra. It contains a finite subalgebra $\mathcal{H}_{W, \lambda}$. The algebra $\mathcal{H}_{W, \lambda}$ has a one dimensional representation $\text{sgn}$. We show in Theorem 2 that the $\mathcal{H}(G, \rho)$-module $(c\text{-}\text{ind}_{U(F)}^{G(F)}(\psi))$ is isomorphic to $\mathcal{H}(G, \rho) \otimes \text{sgn}$. If $\lambda$ is positive, Theorem 2 assumes that $F$ has characteristic $0$ and its residue characteristic is not too small.

Theorems 1 and 2 generalize the main result of Chan and Savin in [6] who treat the case $\lambda = 1$ for $T$ split, i.e., unramified principal series blocks of split groups.

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Our proofs benefit from the ideas in [6]. However they are quite different. The existence of a generator in \((c\text{-ind}_{U(F)}^{G(F)})^o\) is concluded by specializing quite general results in [3, 4]. For Theorem 3 instead of computing the effect of intertwiners on the generator as in [6], we make a reduction to depth-zero case and then to a finite group analogue of the question. There it holds by a result of Reeder [12, §7.2].

2. Notations

Throughout, \(F\) denotes a non-archimedean local field. Let \(O\) denote the ring of integers of \(F\). We denote by \(q\), the cardinality of the residue field \(F_q\) of \(F\) and by \(p\) the characteristic of \(F_q\).

3. Preliminaries

We use this section to recall some basic theory and also fix notation.

3.1. Bernstein decomposition. Let \(R(G(F))\) denote the category of smooth complex representations of \(G(F)\). The Bernstein decomposition gives a direct product decomposition of \(R(G(F))\) into indecomposable subcategories:

\[
R(G(F)) = \prod_{s \in \mathcal{B}(G)} R^s(G(F)).
\]

Here \(\mathcal{B}(G)\) is the set of inertial equivalence classes, i.e., equivalence classes \([L, \sigma]_G\) of cuspidal pairs \((L, \sigma)\), where \(L\) is an \(F\)-Levi, \(\sigma\) is a supercuspidal of \(L(F)\) and where the equivalence is given by conjugation by \(G(F)\) and twisting by unramified characters of \(L(F)\). The block \(R^{[L, \sigma]_G}(G(F))\) consists of those representations \(\pi\) for which each irreducible constituent of \(\pi\) appears in the parabolic induction of some supercuspidal representation in the equivalence class \([L, \sigma]_G\).

3.2. Hecke algebra \([4]\). Let \((\tau, V)\) be an irreducible representation of a compact open subgroup \(J\) of \(G(F)\). The Hecke algebra \(H(G, \tau)\) is the space of compactly supported functions \(f : G(F) \to \text{End}_C(V^\vee)\) satisfying,

\[
f(j_1 gj_2) = \tau^\vee(j_1)f(g)\tau^\vee(j_2) \quad \text{for all } j_1, j_2 \in J \text{ and } g \in G(F).
\]

Here \((\tau^\vee, V^\vee)\) denotes the dual of \(\tau\). The standard convolution operation gives \(H(G, \tau)\) the structure of an associative \(C\)-algebra with identity.

Let \(R_\tau(G(F))\) denote the subcategory of \(R(G(F))\) whose objects are the representations \((\pi, V)\) of \(G(F)\) generated by the \(\tau\)-isotypic subspace \(V^\tau\) of \(V\). There is a functor

\[
M_\tau : R_\tau(G(F)) \to H(G, \tau)-\text{Mod},
\]

given by

\[
\pi \mapsto \text{Hom}_J(\tau, \pi).
\]

Here \(H(G, \tau)-\text{Mod}\) denotes the category of unital left modules over \(H(G, \tau)\).

For \(s \in \mathcal{B}(G)\), the pair \((J, \tau)\) is an \(s\)-type if \(R_\tau(G(F)) = R^s(G(F))\). In that case, the functor \(M_\tau\) gives an equivalence of categories.
3.3. **G-cover** \( [1] \). Let \((K_M, \rho_M)\) be a \([M, \sigma]_M\)-type. Let \((K, \rho)\) be a pair consisting of a compact open subgroup \(K\) or \(G(F)\) and an irreducible representation \(\rho\) of \(K\). Suppose that for any opposite pair of \(F\)-parabolic subgroups \(P = MN\) and \(\bar{P} = M\bar{N}\) with Levi factor \(M\) and unipotent radicals \(N\) and \(\bar{N}\) respectively, the pair \((K, \rho)\) satisfies the following properties:

1. \(K\) decomposes with respect to \((N, M, \bar{N})\), i.e.,
   \[ K = (K \cap N), (K \cap M), (K \cap \bar{N}) \]
2. \(\rho|K_M = \rho_M\) and \(K \cap N, K \cap \bar{N} \subset \ker(\rho)\).
3. For any smooth representation \((\pi, V)\) of \(G(F)\), the natural projection \(V\) to the Jacquet module \(V_N\) induces an injection on \(V^\rho\).

The pair \((K, \rho)\) is then called the **G-cover** of \((K_M, \rho_M)\). See \([2\) Theorem 1\] for this reformulation of the original definition of G-cover due to Bushnell and Kutzko \([4\] §8.1\) (see also \([11\] §4.2\]). If \((K, \rho)\) is a G-cover of \((K_M, \rho_M)\), then it is an \([M, \sigma]_G\)-type.

Suppose \((K, \rho)\) is a G-cover of \((K_M, \rho_M)\), then for any \(F\)-parabolic subgroup \(P' = MN'\) with Levi factor \(M\) and unipotent radical \(N'\), there is a \(\mathbb{C}\)-algebra embedding \([4\] §8.3\]
\[
(3.1) \quad t_{P'} : \mathcal{H}(M, \rho_M) \rightarrow \mathcal{H}(G, \rho),
\]
with the property that for any smooth representation \(\Upsilon\) of \(G(F)\),
\[
(3.2) \quad M_{\rho_M}(\Upsilon_{N'}) \cong t_{P'}(M_{\rho}(\Upsilon)).
\]

Here \(t_{P'} : \mathcal{H}(G, \rho)\)-mod \(\rightarrow \mathcal{H}(M, \rho_M)\)-mod induced by \(t_{P'}\).

Kim and Yu \([11\], using Kim’s work \([10\], showed that Yu’s construction of super-cuspidals \([14\] can be used to produce G-covers of \([M, \sigma]_M\)-types for all \([M, \sigma]_G \in \mathfrak{B}(G)\), assuming \(F\) has characteristic 0 and the residue characteristic \(p\) of \(F\) is suitably large. Recently Fintzen \([8\], using an approach different from Kim, has shown the construction of types for all Bernstein blocks without any restriction on the characteristic of \(F\) and assuming only that \(p\) does not divide the order of the Weyl group of \(G\).

4. **Gelfand-Graev spaces**

Let \(G\) be a connected reductive group defined over \(F\). Fix a maximal \(F\)-split torus \(S\) in \(G\) and let \(T\) denote its centralizer. Then \(T\) is the Levi factor of a minimal \(F\)-parabolic subgroup \(B\) of \(G\). We denote the unipotent radical of \(B\) by \(U\). A smooth character
\[
\psi : U(F) \rightarrow \mathbb{C}^\times
\]
is called non-degenerate if its stabilizer in \(S(F)\) lies in the center \(Z\) of \(G\).

The Gelfand-Graev representation \(\text{c-ind}^{G(F)}_{U(F)}\psi\) of \(G(F)\) is provided by the space of right \(G(F)\)-smooth compactly supported \(U(F)\) functions \(f : G(F) \rightarrow \mathbb{C}\) satisfying:
\[
f(ug) = \psi(u)f(g), \quad \forall u \in U(F), g \in G(F).
\]

Let \(M\) be a \((B, T)\)-standard \(F\)-Levi subgroup of an \(F\)-parabolic \(P = MN\) of \(G\), i.e., \(M\) contains \(T\) and \(P\) contains \(B\). Then \(B \cap M\) is a minimal parabolic subgroup of \(M\) with unipotent radical \(U_M := U \cap M\). Also, \(\psi_M := \psi|_M\) is a non-degenerate character of \(U_M(F)\). As before, denote by \(\bar{P} = M\bar{N}\), the opposite
parabolic subgroup. We also have an isomorphism of $M(F)$ representations [3] §2.2, Theorem]

\[c\text{-}
\text{ind}^{M(F)}_{U_M(F)}(\psi_M) \cong (c\text{-}
\text{ind}^{G(F)}_{U(F)}(\psi))^\rho_N.\]

Now let $\sigma$ be a character of $T(F)$. Let $(K_T, \rho_T)$ be a $[T, \sigma]_T$-type and let $(K, \rho)$ denote its $G$-cover. We assume that the residue characteristic $p$ does not divide the order of the Weyl group of $G$, so that $(K, \rho)$ exists by [8]. Let $\bar{B} = TU$ denote the opposite Borel. View $\mathcal{H}(T, \rho_T)$ as a subalgebra of $\mathcal{H}(G, \rho)$ via the embedding:

\[t_B : \mathcal{H}(T, \rho_T) \rightarrow \mathcal{H}(G, \rho), \]

of Equation (3.1).

**Theorem 1.** There is an isomorphism

\[(c\text{-}
\text{ind}^{G(F)}_{U(F)}(\psi))^\rho \cong \mathcal{H}(T, \rho_T)\]

of $\mathcal{H}(T, \rho_T)$-modules. Consequently, $(c\text{-}
\text{ind}^{G(F)}_{U(F)}(\psi))^\rho$ is a cyclic $\mathcal{H}(G, \rho)$-module.

**Proof.** Putting $M = T$ in Equation (4.1) and observing that in this case, $U_M = 1$, we get an isomorphism of $T(F)$ representations

\[(c\text{-}
\text{ind}^{G(F)}_{U(F)}(\psi))^\rho_U \cong c\text{-}
\text{ind}^{T(F)}_1 \mathbb{C} \cong C^\infty_c(T(F)).\]

Consequently,

\[(c\text{-}
\text{ind}^{G(F)}_{U(F)}(\psi))^\rho_T \cong C^\infty_c(T(F))^\rho_T \cong \mathcal{H}(T, \rho_T)\]

as $\mathcal{H}(T, \rho_T)$-modules. Now by Equation (3.2),

\[(c\text{-}
\text{ind}^{G(F)}_{U(F)}(\psi))^\rho_T \cong (c\text{-}
\text{ind}^{G(F)}_{U(F)}(\psi))^\rho_U\]

as $\mathcal{H}(T, \rho_T)$-modules. The result follows.

\[\square\]

5. Principal series component

5.1. Some results of Roche. In this subsection, we summarize some results of Roche in [13].

Let the notations be as in Section 4. We assume further that $S = T$, so that $B = TU$ is now an $F$-Borel subgroup of $G$ containing the maximal $F$-split torus $T$. The pair $(B, T)$ determines a based root datum $\Psi = (X, \Phi, \Pi, X^\vee, \Phi^\vee, \Pi^\vee)$. Here $X$ (resp. $X^\vee$) is the character (resp. co-character) lattice of $T$ and $\Pi$ (resp. $\Pi^\vee$) is a basis (resp. dual basis) for the set of roots $\Phi = \Phi(G, T)$ (resp. $\Phi^\vee$) of $T$ in $G$.

For the results of this subsection, we assume that $F$ has characteristic 0 and the residue characteristic $p$ of $F$ satisfies the following hypothesis.

**Hypothesis 2.** If $\Phi$ is irreducible, $p$ is restricted as follows:

- for type $A_n$, $p > n + 1$
- for type $B_n, C_n, D_n$, $p \neq 2$
- for type $F_4$, $p \neq 2, 3$
- for types $G_2, E_6$, $p \neq 2, 3, 5$
- for types $E_7, E_8$, $p \neq 2, 3, 5, 7$
If $\Phi$ is not irreducible, then $p$ excludes primes attached to each of its irreducible factors.

We let $T = T(O)$ denote the maximal compact subgroup of $T(F)$, $N_G(T)$ to be the normalizer of $T$ in $G$ and $W = W(G,T) = N_G(T)/T = N_G(T)(F)/T(F)$ denote the Weyl group of $G$.

Let $\chi^\#$ be a character of $T(F)$ and put $\chi = \chi^\#|T(F)_0$, where $T(F)_0$ denotes the maximal compact subgroup of $T(F)$. Then $(T(F)_0, \chi)$ is a $[T, \chi^\#]_T$-type.

Let $N_G(T)(F)$ (resp. $N_G(T)(O)$) denote the subgroup of $N_G(T)(F)$ (resp. $N_G(T)(O)$) which fixes $\chi$. The group $N_G(T)(F)$ contains $T(F)$ and we have $W_\chi = N_G(T)(F)/T(F)$. Denote by $\mathcal{W} = \mathcal{W}(G,T) = N_G(T)(F)/T$, the Iwahori-Weyl group of $G$. There is an identification $N_G(T)(F) = X^\vee \rtimes N_G(T)(O)$ given by the choice of a uniformizer of $F$. Since $N_G(T)(O)/T = N_G(T)/T$, this identification also gives an identification $\mathcal{W} = X^\vee \rtimes W$. Let $\mathcal{W}_\chi = X^\vee \rtimes W_\chi$ be the subgroup of $\mathcal{W}$ which fixes $\chi$.

Let

$$\Phi' = \{ \alpha \in \Phi | \chi \circ \alpha^\vee|_{O^\times} = 1 \}.$$ 

Then $\Phi'$ is a sub-root system of $\Phi$. Let $s_\alpha$ denote the reflection on the space $A = X^\vee \otimes \mathbb{R}$ associated to a root $\alpha \in \Phi$ and write $W' = \langle s_\alpha | \alpha \in \Phi' \rangle$ to be the associated Weyl group. Let $\Phi^+$ (resp. $\Phi^-$) be the system of positive (resp. negative) roots determined by the choice of the Borel $B$ and let $\Phi'^+ = \Phi^+ \cap \Phi'$. Then $\Phi'^+$ is a positive system in $\Phi'$. Put

$$C_\chi = \{ w \in W_\chi | w(\Phi'^+) = \Phi'^+ \}.$$ 

Then we have,

$$W_\chi = W' \rtimes C_\chi.$$ 

The character $\chi$ extends to a $W_\chi$-invariant character $\hat{\chi}$ of $N_G(T)(O)_\chi$. Denote by $\tilde{\chi}$, the character of $N_G(T)(F)_\chi$ extending $\chi$ trivially on $X^\vee$.

Roche’s construction produces a $[T, \chi^\#]_G$-type $(K, \rho)$. The pair $(K, \rho)$ depends on the choice of $B, T, \chi$ but not on the extension $\chi^\#$ of $\chi$. Denote by $I_{G(F)}(\rho)$, the set of elements in $G(F)$ which intertwine $\rho$. Equivalently, $g \in I_{G(F)}(\rho)$ iff the double coset $KgK$ supports a non-zero function in $\mathcal{H}(G, \rho)$. We have an equality

$$I_{G(F)}(\rho) = KW_\chi K.$$ 

For an element $w \in W_\chi$, choose any representative $n_w$ of $w$ in $N_G(T)(F)_\chi$ and let $f_{\tilde{\chi}, w}$ be the unique element of the Hecke algebra $\mathcal{H}(G, \rho)$ supported on $Kn_wK$ and having value $q^{-\ell(w)/2}\tilde{\chi}(n_w)^{-1}$. Here $\ell$ is the length function on the affine Weyl group $W$. The functions $f_{\tilde{\chi}, w}$ for $w \in W_\chi$ form a basis for the $C$-vector space $\mathcal{H}(G, \rho)$. Denote by $\mathcal{H}_{W, \chi}$ the subalgebra of $\mathcal{H}(G, \rho)$ generated by $\{ f_{\tilde{\chi}, w} | w \in W' \}$. Also, identify $\mathcal{H}(T, \chi)$ as a subalgebra of $\mathcal{H}(G, \rho)$ using the embedding $t_g$. When $G$ has connected center, $C_\chi = 1$ assuming Hypothesis 2. In that case, $\mathcal{H}_{W, \chi}$ and $\mathcal{H}(T, \chi)$ together generate the full Hecke algebra $\mathcal{H}(G, \rho)$.

5.2. Statement of Theorem. We continue to assume that $G$ is split. Extend the triple $(G, B, T)$ to a Chevalley-Steinberg pinning of $G$. This determines a hyperspecial point $x$ in the Bruhat-Tits building which gives $G$ the structure of a Chevalley group. With this identification, $(G, B, T)$ are defined over $O$ and the hyperspecial subgroup $G(F)_{x,0}$ at $x$ is $G(O)$. Let $G(F)_{x,0+}$ denote the pro-unipotent radical of $G(F)_{x,0}$. Then $G(F)_{x,0}/G(F)_{x,0+} \cong G(F_q)$. We say that
a non-degenerate character \( \psi \) of \( U(F) \) is of generic depth zero at \( x \) if \( \psi|U(F) \cap G(F)_{x,0} \) factors through a generic character of \( \overline{\psi} \) of \( U(F) \cap G(F)_{x,0}/U(F) \cap G(F)_{x,0} \) (see [2] §1 for a more general definition). Note that if \( G \) has connected center, then all non-degenerate characters of \( U(F) \) form a single orbit under \( T(F) \).

Let \( sln \) denote the one dimensional representation of \( \mathcal{H}_{W,\chi} \) in which \( f_{x,w} \) acts by the scalar \((-1)^\ell(w)\). Here \( \ell \) denotes the length function on \( W' \).

**Theorem 3.** Let \( \psi \) be a non-degenerate character of \( U(F) \) of generic depth zero at \( x \). If \( \chi \neq 1 \), then assume that the center of \( G \) is connected. If \( \chi \) has positive depth, then assume further that \( F \) has characteristic 0 and the residue characteristic satisfies Hypothesis [3]. Then \( \mathcal{H}(G,\rho) \)-module \( (c\text{-}\text{ind}^{G(F)}_{U(F)}\psi)^\rho \) is isomorphic to \( \mathcal{H}(G,\rho) \otimes_{\mathcal{H}_W,\chi} \text{sgn} \).

**6. Proof of Theorem 3**

We retain the notations introduced in Sections 4 and 5.

6.1. **Reduction to depth-zero.** It follows from the the proof of [13] Theorem 4.15] (see also loc. cit., page 385, 2nd last paragraph), that there exists a standard \( F \)-Levi subgroup \( M \) of \( G \) which is the Levi factor of a standard parabolic \( P = MN \) of \( G \) and which is minimal with the property that

\[
I_{G(F)}(\rho) \subset KM(F)K.
\]

Put \( (K_M,\rho_M) = (K \cap M, \rho|_{M(F)}) \). From [4] Theorem 7.2(ii)], it follows that \((K,\rho)\) satisfies the requirements [4] §8.1] of being \( G \)-cover of \((K_M,\rho_M)\). It also follows from [4] Theorem 7.2(ii)] that there is a support preserving Hecke algebra isomorphism

\[
(6.1) \quad \Psi^M : \mathcal{H}(M,\rho_M) \cong \mathcal{H}(G,\rho).
\]

By Equation (6.1), we have an isomorphism of \( \mathcal{H}(M,\rho_M) \)-modules

\[
(6.3) \quad (c\text{-}\text{ind}^{M(F)}_{U_M(F)}\psi_M)^{\rho_M} \cong (\text{c}\text{-}\text{ind}^{G(F)}_{U(F)}\overline{\psi})_{\overline{N}(F)}^{\rho_M}.
\]

And by Equation (3.2), we have a \( \Psi^M \)-equivariant isomorphism

\[
(6.4) \quad (c\text{-}\text{ind}^{G(F)}_{U(F)}\psi)^{\rho_M} \cong (c\text{-}\text{ind}^{G(F)}_{U(F)}\psi)^{\rho}.
\]

Combining Equations (6.3) and (6.4), we get a \( \Psi^M \)-equivariant isomorphism

\[
(6.5) \quad (c\text{-}\text{ind}^{M(F)}_{U_M(F)}\psi_M)^{\rho_M} \cong (c\text{-}\text{ind}^{G(F)}_{U(F)}\psi)^{\rho}.
\]

Also it is shown in the proof of [13] Theorem 4.15] that for such an \( M \), there is a character \( \chi_1 \) of \( M(F) \) such that \( \chi \chi_1 \) viewed as a character of \( T(F)_0 \) is depth-zero. We then have an isomorphism

\[
(6.5) \quad \psi_{\chi_1} : f \in \mathcal{H}(M,\rho_M) \cong f_{\chi_1} \in \mathcal{H}(M,\rho_M \chi_1).
\]

This gives a \( \Psi_{\chi_1} \)-equivariant isomorphism

\[
(6.6) \quad (c\text{-}\text{ind}^{M(F)}_{U_M(F)}\psi_M)^{\rho_M} \cong (c\text{-}\text{ind}^{M(F)}_{U_M(F)}\psi_M)^{\rho_M \chi_1}.
\]

We thus have a \( \Psi^M \circ \psi^{-1}_{\chi_1} \)-equivariant isomorphism

\[
(6.6) \quad (c\text{-}\text{ind}^{G(F)}_{U(F)}\psi)^{\rho} \cong (c\text{-}\text{ind}^{G(F)}_{U(F)}\psi)^{\rho}.
\]
By Equations (5.1) and (6.1), it follows that $\Psi^M$ restricts to an algebra isomorphism

$$\mathcal{H}_{W(G,T),\chi} \cong \mathcal{H}_{W(M,T),\chi}.$$  

From the proof of [13, Theorem 4.15], $W(M,T)_\chi = W(M,T)^{\chi,1}$, and therefore $\Psi_{\chi,1}$ restricts to an isomorphism

$$\mathcal{H}_{W(M,T),\chi} \cong \mathcal{H}_{W(M,T)^{\chi,1}}.$$  

Thus $\Psi^M \circ \Psi_{\chi,1}^{-1}$ restricts to an isomorphism

$$\mathcal{H}_{W(M,T)^{\chi,1}} \cong \mathcal{H}_{W(G,T),\chi}.$$  

Note that if $G$ has connected center, then so does $M$ (see proof of [5, Proposition 8.1.4] for instance for this fact). Thus, from Equations (6.6) and (6.8), it follows that to prove Theorem 3, we can and do assume without loss of generality that $\chi$ has depth-zero.

**Remark 4.** For a much more general statement of the isomorphism $\Psi^M \circ \Psi_{\chi,1}^{-1}$, see [11 §8].

### 6.2. Proof in depth-zero case.

For results of this section, no restriction on characteristic or residue characteristic is imposed.

Let $I$ be the Iwahori subgroup of $G$ which is in good position with respect to $(B,T)$ (note here that we are taking opposite Borel) and let $I_{0+}$ denote its pro-unipotent radical. Put $T(F)_{0+} = I_{0+} \cap T(F)_0$. Then $I/I_{0+} \cong T(F)_0/T(F)_{0+}$. Since $\chi$ is depth-zero, it factors through $T(F)_0/T(F)_{0+}$ and consequently lifts to a character of $I$ which we denote by $\rho$. The pair $(I,\rho)$ is then a $G$-cover of $(T(F)_0,\chi)$ [9].

Define $\phi : G(F) \to \mathbb{C}$ to be the function supported on $U(F),(I \cap \bar{B})$ such that $\phi(u) = \psi(u)\chi(i)$ for $u \in U(F)$ and $i \in I \cap \bar{B}$. There is an isomorphism of $G(F)_\chi$-spaces

$$(c\text{-}\text{ind}^{G(F)}_{U(F)} \psi)^{G(F)_{\chi,0}} \cong \text{ind}^{G(F)}_{U(F)} \psi.$$  

Under this isomorphism, $\phi$ maps to a function $\phi_\chi : G(F) \to \mathbb{C}^\times$ which is supported on $U(F),B(F)$ and such that $\phi_\chi(ub) = \omega(u)\chi(b)$ for $u \in U(F)$ and $b \in B(F)$. Now $(\text{ind}^{G(F)}_{U(F)} \psi)^{\chi}$ is an irreducible $\mathcal{H}_{W,\chi}$-module isomorphic to the $\chi$-isotypical component of the irreducible $\psi$-generic constituent of $\text{ind}^{G(F)}_{B(F)} \psi$. Observe that $\phi_\chi \in (\text{ind}^{G(F)}_{U(F)} \psi)^{\chi}$. If $\chi$ is trivial then $(\text{ind}^{G(F)}_{U(F)} \psi)^{\chi}$ corresponds to the Steinberg constituent of $\text{ind}^{G(F)}_{B(F)} \psi$. If $G$ has connected center, then it is shown in [12 §7.2, 2nd last paragraph] that $(\text{ind}^{G(F)}_{U(F)} \psi)^{\chi} \cong \text{sgn}$. Thus in either case, the 1-dimensional space spanned by $\phi_\chi$ affords the $\text{sgn}$ representation of $\mathcal{H}_{W,\chi}$. Consequently, the 1-dimensional space spanned by $\phi$ affords the $\text{sgn}$ representation $\mathcal{H}_{W,\chi}$.

It is readily checked that $\phi$ maps to 1 under the isomorphism of Theorem 1. It follows that $\phi$ is a generator of the cyclic $\mathcal{H}(G,\rho)$-module $(c\text{-}\text{ind}^{G(F)}_{U(F)} \psi)^{\rho}$. We have by Frobenius reciprocity:

$$\text{Hom}_{\mathcal{H}_{W,\chi}}(\text{sgn}, (c\text{-}\text{ind}^{G(F)}_{U(F)} \psi)^{\rho}) \cong \text{Hom}_{\mathcal{H}(G,\rho)}(\mathcal{H}(G,\rho) \otimes \text{sgn}, (c\text{-}\text{ind}^{G(F)}_{U(F)} \psi)^{\rho}).$$  

This isomorphism sends $1 \mapsto \phi$ to the element $1 \otimes 1 \mapsto \phi$. Theorem 3 now follows from the fact that $\mathcal{H}(G,\rho) \otimes \text{sgn}$ and $(c\text{-}\text{ind}^{G(F)}_{U(F)} \psi)^{\rho}$ are free $\mathcal{H}(T,\chi)$-modules generated by $1 \otimes 1$ and $\phi$ respectively.
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