ANOTHER NOTE ON FOCUS-FOCUS SINGULARITIES

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Abstract. We show a natural relation between the monodromy formula for focus-focus singularities of integrable Hamiltonian systems and a formula of Duistermaat-Heckman, and extend the main results of our previous note [16] (S\textsuperscript{1}-action, monodromy, and topological classification) to the case of degenerate focus-focus singularities. We also consider the non-Hamiltonian case, local normal forms, etc.

1. Introduction

The monodromy of an integrable Hamiltonian system has been introduced by Duistermaat in [7] as an obstruction to the existence of global action-angle variables, with a non-trivial example - the spherical pendulum - provided by Cushman. Since then, this monodromy phenomenon has gained the attention of many people, from different points of view (see e.g. [1, 5, 6, 14, 15] and references therein). For quite some time, and even in some very recent papers like [13], the monodromy had been calculated mainly by brute force, involving many pages of complicated computations of action functions. However, most of the integrable systems for which these computations have been done have a simple common feature: they possess so-called focus-focus singularities. And many authors have in fact calculated, case by case, the monodromy of the Liouville torus fibration around these singularities. Recall that, a point \( x \) on a 4-dimensional symplectic manifold \((M^4, \omega)\) with an integrable system given by a moment map \( F = (F_1, F_2) : (M^4, \omega) \to \mathbb{R}^2 \) is called a nondegenerate focus-focus singular point if \( dF_1(x) = dF_2(x) = 0 \), and the quadratic parts of \( F_1 \) and \( F_2 \) at \( x \) can be written as

\[
F_1^{(2)} = a(x_1 y_1 + x_2 y_2) + b(x_1 y_2 - x_2 y_1), \quad F_2^{(2)} = c(x_1 y_1 + x_2 y_2) + d(x_1 y_2 - x_2 y_1)
\]

in some symplectic system of coordinates \((x_1, y_1, x_2, y_2)\), with \( ad - bc \neq 0 \). The level sets of the moment map form a singular Lagrangian fibration. If a singular fiber contains a focus-focus singular point, then one says that it is a focus-focus singular fiber. Similarly, one can define corank-2 focus-focus singular points in integrable Hamiltonian systems with more than two degrees of freedom (see e.g. [17]). The following simple formula has been obtained in [10, 17]: the monodromy around a nondegenerate focus-focus singular fiber that contains \( k \geq 1 \) focus-focus singular points (and no hyperbolic singular point) in an integrable Hamiltonian system with two degrees of freedom is given by the following matrix:

\[
\begin{pmatrix}
1 & k \\
0 & 1
\end{pmatrix}
\]

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In this note, we will make Formula (1.1) and other results of our first note on focus-focus singularities \cite{16} easier to use by extending them to the (possibly) degenerate case. We will also show a direct relation between Formula (1.1) and a Duistermaat-Heckman formula.

**Definition 1.1.** Let $N$ be a compact singular fiber (i.e. singular level set of the moment map $F = (F_1, F_2) : (M^4, \omega) \to \mathbb{R}^2$) of an integrable Hamiltonian system with two degrees of freedom. Then $N$ is called a possibly degenerate focus-focus singular fiber, and its singular points are called possibly degenerate focus-focus points, if $N$ satisfies the following conditions:

a) There is a relatively compact saturated connected neighborhood $U(N)$ of $N$ in which $N$ is the only singular fiber.

b) $N$ contains only a finite number of singular points of the system, and $N$ minus these singular points is homeomorphic to a non-empty union of cylinders $S^1 \times \mathbb{R}^1$.

**Remark.** Of course, if $N$ is a compact singular fiber whose singular points are all nondegenerate focus-focus, then $N$ will satisfy the conditions of the above definition, see e.g. \cite{16}. Conversely, if $N$ is a possibly degenerate focus-focus fiber, and $x \in N$ is a nondegenerate singular point, then it is easy to see that $x$ must be of nondegenerate focus-focus type.

**Example.** Consider a Lagrange top under a potential energy field which is not given by the usual linear function of height, but say by a quadratic function. By varying the coefficients of this function, one will get degenerate focus-focus singularities at some parameters.

**Theorem 1.2.** If $N$ is a possibly degenerate focus-focus fiber in an integrable Hamiltonian system with two degrees of freedom, then there is a neighborhood $V(N)$ of $N$, such that in $V(N)$ there is a Hamiltonian $S^1$-action with the following properties:

a) This action preserves the system (i.e. the moment map).

b) This action is free outside the singular points of the singular fiber $N$, and fixes these singular points.

c) The action has weights $(1, -1)$ at singular points. In other words, near each singular point of $N$ there is a local symplectic system of coordinates $(x_1, y_1, x_2, y_2)$ in which the action is generated by the Hamiltonian vector field \( (x_1 \partial / \partial y_1 - y_1 \partial / \partial x_1) - (x_2 \partial / \partial y_2 - y_2 \partial / \partial x_2) \) of Hamiltonian function \( \frac{x_1^2 + y_1^2}{2} - \frac{x_2^2 + y_2^2}{2} \).

**Theorem 1.3.** If $N$ and $U(N)$ satisfy the conditions of Definition 1.1, then $U(N) \setminus N$ is fibred by regular tori, and the monodromy of this fibration around $N$ is given by the matrix \( \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \), where $k$ is the number of singular points in $N$.

**Theorem 1.4.** The only topological invariant of a possibly degenerate focus-focus singularity is its number of singular points. In other words, if $N$ and $N'$ are two possibly degenerate focus-focus fibers of two integrable Hamiltonian systems with two degrees of freedom, and the number of singular points in $N$ is equal to the number of singular points in $N'$, then there is a homeomorphism from a neighborhood of $N$ to a neighborhood of $N'$ which sends $N$ to $N'$ and which preserves the singular foliation by Liouville tori.

The above theorems generalize the main results of \cite{16} to the degenerate case. In particular, Theorem 1.4 means that degenerate focus-focus singularities are homeomorphic to nondegenerate ones. And since monodromy is a topological invariant,
Theorem 1.3 may be seen as a consequence of Theorem 1.4 and the monodromy formula for nondegenerate focus-focus singularities. However, we will give in this note a proof of Theorem 1.3 which doesn’t make use of Theorem 1.4, but which uses Theorem 1.2, a Duistermaat-Heckman formula with respect to a symplectic $S^1$-action instead. In our opinion, this natural relation between the monodromy and a Duistermaat-Heckman formula is as interesting as the monodromy formula itself.

The rest of this note is organized as follows: in Section 2 we prove Theorem 1.2 and by the way give a new proof of the local normal form, due to Vey 13, and Eliasson 9, of nondegenerate focus-focus points. In Section 3 we prove Theorem 1.3, and in Section 4 we prove Theorem 1.4. Section 5 is devoted to the non-Hamiltonian focus-focus case, first studied by Cushman and Duistermaat 5.

Various final observations and remarks are given in Section 6, the last section of this note.

2. The $S^1$-action

Proof of Theorem 1.2. Let us recall how to find the $S^1$-action using standard arguments 10, 14. Let $N$ be a degenerate focus-focus singular fiber, with a saturated connected neighborhood $U(N)$ as in Definition 1.1. Denote by $F_1$ and $F_2$ the two components of the moment map, and by $X_1 = X_{F_1}$ and $X_2 = X_{F_2}$ their corresponding Hamiltonian vector field. Let $m \in N$ be a non-singular point in $N$. By definition $m$ lies in a cylinder in $N$, implying that there are two real numbers $k(m), h(m)$ such that the vector field $(k(m))X_1 + (h(m))X_2$ is periodic of exact period $2\pi$ at $m$. The numbers $k(m), h(m)$ do not depend on the position of $m$ inside its cylinder in $N$, though a-priori they may depend on the cylinder itself (in fact they don’t, as will be shown). Let $D$ be a sufficiently small smooth 2-dimensional disk that intersects $N$ transversally at $m$, and by $V$ the union of regular Liouville tori which intersect with $D$. Since $U(N)$ is saturated by assumptions, we have that $V \subset U(N)$. Using the implicit function theorem, for each point $z$ in $D$ we find a unique pair of numbers $(k(z), h(z))$ close to $(k(m), h(m))$ such that the orbit starting at $z$ of the vector field $X = k(z)X_1 + h(z)X_2$ is periodic of exact period $2\pi$ (i.e. the period is exactly $2\pi$ but not a fraction of it). Consider $k$ and $h$ as functions of $z$. In fact, since the moment map $(F_1, F_2)$ is regular at $m$, we may take $(F_1, F_2)$ as a coordinate system on $D$, and so we may consider $k$ and $h$ as functions of two variables $k(F_1, F_2)$ and $h(F_1, F_2)$ on $D$, and then extend them to a neighborhood of $N$ (which contains $V$) via composition with the moment map. Due to the fact that $X_1$ commutes with $X_2$ (and together generate a standard flat affine structure on each Liouville torus), the vector field $X = k(F_1, F_2)X_1 + h(F_1, F_2)X_2$ is periodic of exact period $2\pi$ not only at the points in $D$, but in the whole $V$. In order to show that $X = k(F_1, F_2)X_1 + h(F_1, F_2)X_2$ is periodic in a neighborhood of $N$, it suffices to show that the closure of $V$ contains an open neighborhood of $N$.

Let $y$ be a point which is sufficiently close to $N$ but which does not belong to $N$. Since the codimension of $N$ in $U(N)$ is 2, we have that $U(N) \setminus N$ is connected, and therefore there is a path $\gamma$ going from $y$ to a point $z$ inside $D$ ($z$ close to $m$), such that $\gamma(0) = y$, $\gamma(1) = z$, and for any $t \in [0, 1]$ we have that $\gamma(t)$ is close to $N$ but does not belong to $N$. Then one checks that the whole path $\gamma$ lies in $V$, because the intersection of $\gamma$ with $V$ is an non-empty open closed subset of $\gamma$. Open because if $\gamma(t) \in V$ for some $t$, then the regular Liouville torus which contains
\( \gamma(t) \) will intersect \( D \) at a point inside \( D \) (because the value of the moment map at \( \gamma(t) \) is very close to its value at \( N \), i.e. at \( m \)), and hence nearby Liouville tori (which form an open neighborhood of this particular Liouville torus) also belong to \( V \). Closed because if \( \gamma(t) \) lies in the closure of \( V \), then there are Liouville tori in \( V \) which lie arbitrarily close to the regular Liouville torus which contains \( \gamma(t) \) (by regularity, if a point lies near \( \gamma(t) \) then the Liouville torus which contains this point is everywhere close to the Liouville torus which contains \( \gamma(t) \)). It implies that there are points in \( D \) which are arbitrarily close to the Liouville torus which contains \( \gamma(t) \). In particular, there is a point in the closure of \( D \) which lies in the same Liouville torus as \( \gamma(t) \). But considering the fact that the value of the moment map at this point, which is the same as its value at \( \gamma(t) \), is very near its value at \( m \), this point must in fact lie inside \( D \), which means that \( \gamma(t) \) belongs to \( V \). Thus the whole path \( \gamma \) belongs to \( V \), and in particular \( y = \gamma(0) \in V \), for any \( y \) close enough to \( N \) but not belonging to \( N \). We have proved that the closure of \( V \) contains a neighborhood of \( N \).

The vector field \( X = k(F_1, F_2)X_1 + h(F_1, F_2)X_2 \) has been shown to be periodic in a neighborhood of \( N \), and is of period exactly \( 2\pi \) outside \( N \). It is clear that this vector field fixes the singular points of \( N \), where \( X_1 = X_2 = 0 \). Let us show that at regular points in \( N \), the period of \( X = k(F_1, F_2)X_1 + h(F_1, F_2)X_2 \) is also exactly \( 2\pi \). Let \( m' \) be a regular point of \( N \). Then by construction, the value of functions \( k \) and \( h \) at \( m' \) is the same as their value at \( m \). If the period of \( X \) at \( m' \) is smaller than \( 2\pi \), then it means that there is a positive integer \( p \geq 2 \) such that \( k(m)X_1 + h(m)X_2 \) is periodic of period \( 2\pi/p \) at \( m' \). Repeat the above process of finding \( X \), but starting at point \( m' \) instead of at point \( m \), and with period \( 2\pi/p \) instead of \( 2\pi \). In the end we will get that \( k(m)X_1 + h(m)X_2 \) is periodic of period \( 2\pi/p \) (or smaller) at \( m \), which is contradictory to the assumptions about \( k(m) \) and \( h(m) \). Thus the period of \( X \) at \( m' \) must be exactly \( 2\pi \) also.

By integrating the above periodic vector field \( X \), we obtain an \( S^1 \)-action in a neighborhood of \( N \). The fact that this vector field is Hamiltonian follows from (the proof of) Arnold-Liouville theorem on the existence of action-angle variables, see [1]. It is clear that this \( S^1 \)-action preserves the system, is smooth (resp., analytic, finitely differentiable) if the system is smooth (resp., analytic, finitely differentiable), and it is free outside the singular points of \( N \).

Near each singular point \( x \in N \), the above \( S^1 \)-action can be linearized, i.e. there is a symplectic system of coordinates \((x_1, y_1, x_2, y_2)\) in which the action is generated by the Hamiltonian function \( f = a(x_1^2 + y_1^2) + b(x_2^2 + y_2^2) \), i.e.

\[
X = a(x_1 \partial / \partial y_1 - y_1 \partial / \partial x_1) + b(x_2 \partial / \partial y_2 - y_2 \partial / \partial x_2),
\]

where \( a, b \in \mathbb{Z} \). If say \( |a| \geq 2 \) then the \( S^1 \)-action is not free outside \( x \) (it has points of period \( 2\pi/|a| \)), and if \( a = 0 \) then \( x \) is not an isolated fixed point of the action. The remaining cases are \( a, b = \pm 1 \). If \((a, b) = (1, 1)\) or \((a, b) = (-1, -1)\) (which is the same up to an orientation) then the Hamiltonian \( f \) generating the \( S^1 \)-action has compact level sets near \( x \) (the 3-spheres), and it’s easy to check that on each of these 3-spheres the must exist singular points of the integrable system, due to the fact a 3-sphere cannot be foliated by regular tori. Thus the only possibility is \((a, b) = (1, -1), \) up to a permutation. \( \Diamond \)

Remark. In fact, most examples provided in [10] of systems with focus-focus singularities admit a global \( S^1 \)-action, except perhaps the Manakov system on \( so(4) \)
As a corollary of the existence of an $S^1$-action, we propose here to give a new proof of the following local normal form theorem due to Eliasson.

**Theorem 2.1** ([9]). Let $m \in \left( M^4, \omega \right)$ be a focus-focus singular point in a smooth integrable Hamiltonian system with two degree of freedom given by a moment map $F = (F_1, F_2) : (M^4, \omega) \rightarrow \mathbb{R}^2$. Then there exists a local smooth symplectic system of coordinates $(x_1, y_1, x_2, y_2)$ near $m$ (the symplectic form is $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$) in which the local singular Lagrangian fibration of the system is given by the two quadratic functions $f_1 = x_1 y_1 + x_2 y_2$ and $f_2 = x_1 y_2 - x_2 y_1$ (in other words, we have $df_1 \wedge df_1 \wedge df_2 = df_2 \wedge df_1 \wedge df_2 = 0$).

**Proof.** It is a corollary of the following three facts: a) existence of a local Hamiltonian $S^1$-action ([10], [11]); b) existence of a formal Birkhoff normalization (the classical result of Birkhoff); and c) the Hamiltonian equivariant Sternberg theorem (for an $S^1$-invariant vector field) ([2], [3]).

Using the existence of a formal Birkhoff normalization, and Borel’s theorem on the approximation of formal mappings by smooth mappings, we can normalize the system up to a flat term. In other words, we may assume that

$$F_1 = G_1(f_1, f_2) + \text{flat}, \quad F_2 = G_2(f_1, f_2) + \text{flat}$$

(flat means a flat term), where $(G_1, G_2) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a local smooth map, and

$$f_1 = x_1 y_1 + x_2 y_2, \quad f_2 = x_1 y_2 - x_2 y_1$$

is the standard quadratic focus-focus moment map in a smooth symplectic system of coordinates $(x_1, y_1, x_2, y_2)$. The symplectic form is $\omega_0 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$.

By the non-degeneracy assumptions, this map is a local diffeomorphism. Since we are interested in the singular Lagrangian fibration but not the moment map itself, we can compose $(F_1, F_2)$ with the inverse of $(G_1, G_2)$ to get a new moment map which satisfies

$$F_1 = f_1 + \text{flat}, \quad F_2 = f_2 + \text{flat}.$$

Notice that the Hamiltonian vector field of $f_2 = x_1 y_2 - x_2 y_1$ is periodic. In this normal form, the Hamiltonian $S^1$-action around the focus-focus singular point will be also generated by a function of the type $f_2 + \text{flat}$, so we may assume (by changing the moment map without changing the fibration) that $F_2$ is in fact the generator of our $S^1$-action. Since $F_2 = f_2 + \text{flat}$, this action is formally equal to the linear action generated by $f_2$. Thus, using averaging, we can linearize this action by a symplectomorphism which is formally equal to the identity map. Using this linearization, we can assume that:

$$F_1 = f_1 + \text{flat}, \quad F_2 = f_2.$$

The Hamiltonian vector field $X_1$ of $F_1$ is now formally equal to the linear Hamiltonian vector field $X_{f_1}$; it is invariant under the linear $S^1$-action degenerated by $X_2 = (x_1 \partial / \partial x_2 - x_2 \partial / \partial x_1) + (y_1 \partial / \partial y_2 - y_2 \partial / \partial y_1)$, and it does not have purely imaginary eigenvalues. Hence one can apply the equivariant Hamiltonian Sternberg theorem with respect to an $S^1$-action ([2], [3]) to find a local smooth symplectomorphism that maps $X_1$ to $X_{f_1}$ (and $X_2$ to itself, of course). Thus under this

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1See Eliasson’s Ph.D. thesis dated 1984, rather than his paper dated 1990, for his proof. In the analytic case this result is due to Vey ([2]).
symplectomorphism, the system becomes linear:

\[ X_1 = (x_1 \partial/\partial x_1 + x_2 \partial/\partial x_2) - (y_1 \partial/\partial y_1 + y_2 \partial/\partial y_2), \]
\[ X_2 = (x_1 \partial/\partial x_2 - x_2 \partial/\partial x_1) + (y_1 \partial/\partial y_2 - y_2 \partial/\partial y_1), \]

and the singular Lagrangian foliation is given by two quadratic functions \( f_1 = x_1y_1 + x_2y_2, \quad f_2 = x_1y_2 - x_2y_1. \) 

3. Monodromy and a Duistermaat-Heckman formula

Proof of Theorem 1.3. We will use the notations introduced in the previous sections: \( N, U(N), \ldots \) The proof will use the Duistermaat-Heckman measure \( \mathfrak{m} \) to show the role played by the \( S^1 \)-action given by Theorem 1.2. Without loss of generality, we can assume that \( F_2 \) (the second component of the moment map) is the generator of this \( S^1 \)-action. For simplicity, we will localize the degenerate focus-focus singularity. Using integrable surgery \( [3] \), we can add elliptic singularities to the boundary of \( U(N) \) (provided \( U(N) \) is chosen appropriately) to get an integrable system on a compact symplectic 4-manifold with the following property: on each compact level set \( Q_c = \{ F_2 = c \} \) with \( c \neq 0 \) and near 0, the \( S^1 \)-action generated by the periodic vector field \( X_c = X_{F_2} \) is free, and the set of singular points of the system on \( Q_c \) consists of two circles \( S_{1,c}, S_{2,c} \) of nondegenerate corank-1 elliptic singularities. On the singular level set \( Q_0 \subset N \), the \( S^1 \)-action is still free outside the set of singular points in \( N \), and the set of singular points of the system on \( Q_0 \) consists of two circles \( S_{1,0}, S_{2,0} \) of nondegenerate corank-1 elliptic singularities plus the singular set of \( N \). Additionally, we can assume that, for some small number \( \delta > 0 \), \( \delta \) is the maximum value of \( F_2 \) and \( -\delta \) is the minimum value of \( F_2 \), and the sets \( Q_\delta, Q_{-\delta} \) are two-dimensional sphere consisting of elliptic singular points of the system. For the \( S^1 \)-action, there are \( k \) fixed points of weights \((1, -1)\) (lying in \( N \)), two spheres of fixed points \( Q_\delta \) and \( Q_{-\delta} \), and the action is free outside these sets. The orbit space of the localized integrable system is topologically a square with a particular (degenerate focus-focus) singular point inside. The edges of the square correspond to corank-1 nondegenerate elliptic singularities, and the corners of the square correspond to nondegenerate corank-2 elliptic singularities.

Denote by \( V(c) \) the symplectic volume (area) of the quotient of \( Q_c \) by the \( S^1 \)-action, with the reduced symplectic form. Then a special case of a formula due to Duistermaat and Heckman \( [3] \) measures how the behavior of the function \( V(c) \) changes when one passes through the fixed points of the \( S^1 \)-action (here the point 0 in the variable \( c \)):

\[ V(c) + V(-c) = 2V(0) - kc. \]

(The minus sign on the right hand side is due to the fact that the weights at the fixed points are \((1, -1)\)). On the other hand, the orbit space of our localized integrable system has a stratified integral affine structure \( [3] \), and \( V(c) \) can be viewed as the affine length of the interval \( \{ F_2 = c \} \) on the orbit space with respect to this integral affine structure. Formula 3.1 then explains the change in the behavior of the integral affine structure in the orbit space when one passes through the focus-focus point: around this point, the integral affine structure can be obtained from the standard flat structure in \( \mathbb{R}^2 = \{(x, y)\} \) near the origin \( O \) by cutting out the angle \( \angle((0, 1), (-k, 1)) \) and gluing the edges of the rest together by the integral
linear transformation \((x, y) \mapsto (x + ky, y)\). And this change corresponds to the monodromy formula in action variables, which by duality is the same thing as the monodromy in the torus fibration (see e.g.\[3\]). \(\diamondsuit\)

4. Topological classification

Proof of Theorem 1.4. Denote by \(P\) and \(V(P)\) the quotient of \(N\) and \(U(N)\) by the Hamiltonian \(S^1\)-action given by Theorem 1.2. Due to the fact that the action has weights \((1, -1)\) at each fixed point, \(V(P)\) is a topological 3-manifold, and \(P\) is a simple closed curve in it. Since the moment map is preserved by the \(S^1\)-action, it can be projected to a map \((\tilde{F}_1, \tilde{F}_2) : V(P) \to \mathbb{R}^2\), which turns \(V(P)\) into a topologically trivial circle bundle. (The singular points in \(P\) are singular in the sooth sense, but non-singular in the homeomorphic sense.) Thus if \((N, U(N))\) and \((N', U'(N'))\) denote two possibly degenerate focus focus singularities with the same number of singular points (which we will denote by \(k\)), then at least their reductions by the \(S^1\)-action are topologically equivalent: there is a homeomorphism \(\psi\) from \(V(P)\) to \(V'(P')\) which preserves the circle fibrations, and which sends the singular points in \(P\) (i.e. the images of the singular points in \(N\) under projection) to the singular points in \(P'\).

Now one can lift the map \(\psi : V(P) \to V'(P')\) to an \(S^1\)-equivariant map from \(U(N)\) to \(U'(N')\): first do it for \(N\), then for small neighborhoods of singular points of \(N\), then extend to the rest of \(U(N)\). It can be done by using “local sections”, and is an exercise in elementary topology. There are no obstructions. \(\diamondsuit\)

Remark. In \[16\], I wrongly suggested that the above homeomorphism in the nondegenerate case can always be made smooth, i.e. the number \(k\) of focus-focus singular points in the focus-focus singular fiber is the only smooth invariant - after a more careful analysis, Bolsinov found out that there are some other smooth invariants in the case \(k > 1\) (see \[3\]). A topological classification of more general nondegenerate singularities of integrable Hamiltonian systems is given in \[17\].

5. The non-Hamiltonian case

In \[3\], Cushman and Duistermaat generalized the results of \[16\] to the case of integrable non-Hamiltonian systems with focus-focus singularities. They assign to each focus-focus point a sign, either plus or minus, depending on some orientation (in the Hamiltonian case, all focus-focus points have plus sign). They again obtained the existence of an \(S^1\)-action (which is not surprising), and the same monodromy matrix as given by Formula 1.1, but with \(k\) now being the number of positive focus-focus points minus the number of negative focus-focus points. Naturally, the results in the previous sections of this note can also be extended to the non-Hamiltonian case, so we also get a generalization of Cushman-Duistermaat’s results.

Definition 5.1. If we have two vector fields \(X_1, X_2\) and two functions \(F_1, F_2\) on a 4-manifold \(M^4\) such that \([X_1, X_2] = 0\), \(X_1(F_1) = X_1(F_2) = X_2(F_1) = X_2(F_2) = 0\), then we say that we have an integrable non-Hamiltonian system of bi-index \((2, 2)\) (i.e. 2 commuting fields and 2 common first integrals). A point \(x \in M^4\) is called singular for such an integrable system, if \(X_1 \wedge X_2(x) = 0\) or \(dF_1 \wedge dF_2(x) = 0\). A connected common level set \(N\) of the first integrals \(F_1, F_2\) is called a possibly degenerate focus-focus singular fiber, and its singular points called possibly degenerate focus-focus singular points, if \(N\) satisfies the following conditions:
a) There is a relatively compact neighborhood $U(N)$ of $N$, which is saturated by connected common level sets of $F_1$ and $F_2$.

b) $U(N)$ contains only a finite number of singular points of the system and they all lie in $N$, and $N$ minus these singular points is homeomorphic to a non-empty union of cylinders $\mathbb{S}^1 \times \mathbb{R}^1$.

**Theorem 5.2.** If $N$ is a possibly degenerate focus-focus fiber in an integrable non-Hamiltonian system of bi-index $(2, 2)$, then in a neighborhood $U(N)$ of $N$ there is an $\mathbb{S}^1$-action with the following properties:

a) This action preserves the system (i.e. the two commuting vector fields and the two first integrals)

b) This action is free outside the singular points of the singular fiber $N$, and fixes these singular points.

c) The action has weights $(1, \pm 1)$ at singular points. In other words, near each singular point of $N$ there is a local symplectic system of coordinates $(x_1, x_2, x_3, x_4)$ in which the action is generated by the vector field $(x_1 \partial/\partial x_2 - x_2 \partial/\partial x_1) \pm (x_3 \partial/\partial x_4 - x_4 \partial/\partial x_3)$.

The proof of Theorem 5.2 is absolutely similar to that of Theorem 1.2. The reason why we write weights $(1, \pm 1)$ instead of $(1, -1)$ in the above theorem is that there is no distinction between $(1, 1)$ and $(1, -1)$, before an orientation is introduced. Note that $U(N)$ is orientable. For example, one can choose an orientation as follows: Choose a Riemannian metric in $U(N)$. If $m \in U(N)$ is regular point of the system, then the vectors $X_1(m), X_2(m), \nabla F_1(m), \nabla F_2(m)$ (where $\nabla$ denotes the gradient of a function) are linearly independent at $m$, and one can use these vector fields to orientate $U(N)$.

**Definition 5.3.** Suppose that $N$ is a possibly degenerate focus-focus singular fiber as in Definition 5.1, and that the ambient 4-manifold $M^4$ is oriented. Then a singular point $x \in N$ is called positive if there is a positively oriented local system of coordinates $(x_1, x_2, x_3, x_4)$ in which the $\mathbb{S}^1$-action given by Theorem 5.2 is generated by $(x_1 \partial/\partial x_2 - x_2 \partial/\partial x_1) - (x_3 \partial/\partial x_4 - x_4 \partial/\partial x_3)$, and negative otherwise.

Remark. Of course, a singular point in $N$ is either positive or negative but never both. The above definition is an extension of a definition of Cushman and Duistermaat to the case of possibly degenerate focus-focus points. It is also clear that in the Hamiltonian case, all possibly degenerate focus-focus singular points are of plus sign. If we change the orientation of the ambient manifold $M^4$, then positive points become negative and vice versa.

**Theorem 5.4.** If $N$ and $U(N)$ satisfy the conditions Definition 5.1, then $U(N) \setminus N$ is fibred by regular tori, and the monodromy of this fibration around $N$ is given by the matrix

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix},$$

where $k$ is the number of positive singular points in $N$ minus the number negative singular points in $N$.

Theorem 5.4 may be viewed as a generalization of Theorem 1.3 to the non-Hamiltonian case, or also as a generalization of the results of [2] to the degenerate case. It can be proved, for example, by combining the results of [2] and the non-Hamiltonian version of Theorem 1.4: In the non-Hamiltonian case, the full topological invariant is made up of the cyclic order of the signs of the singular points in $N$. 
6. Some final remarks

6.1. **Topological torus fibrations.** From the topological point of view, the above
topological classification and monodromy formulas (both in the Hamiltonian case
and the non-Hamiltonian case) may be seen, after some preparatory results, as
special cases of the theory of singular torus fibrations in 4-manifolds developed by
Matsumoto and other people, see [10] and references therein. This theory is in
turn inspired by Kodaira’s theory of elliptic fibrations in complex surfaces. Un-
fortunately or fortunately, I didn’t know of Matsumoto’s paper when writing [16] -
otherwise that note would have been written differently, or would not appear at
all. Focus-focus singular points correspond to what are denoted in [10] as $I_+$ (the
case of positive sign) and $I_-$ (the case of negative sign, which can only happen
in non-Hamiltonian systems). The $S^1$-action does not seem to be present in the
above-mentioned work of Matsumoto nor in the earlier papers by Kodaira; they
didn’t need it. However, I would like to stress here again that this action (and local
torus actions in general) is very useful in the study of integrable systems. In [16], we
showed how to use this action to perturb an integrable system with two degrees of
freedom into an integrable system whose focus-focus singular fibers contain exactly
one singular focus-focus point. Of course, the same result holds in the degenerate
focus-focus case.

6.2. **Monodromy without focus-focus singularities.** Locally, among nonde-
generate singularities, only focus-focus components create non-trivial monodromy
(elliptic and hyperbolic components don’t). But globally, there may be domains in
the orbit space of a nondegenerate integrable Hamiltonian system, which does not
border any focus-focus singular point, and which still has non-trivial monodromy.
A simple example is the following: consider a spherical pendulum of radius 1, cen-
tered at the origin of $\mathbb{R}^3$, not with the standard potential energy function $z$
(where $z$ is the coordinate of the vertical axis), but a potential energy function of the type
$z - z^2/2R$, where $R$ is a positive constant slightly smaller than 1. The maximum of
the potential is achieved at the level $z = R$, slightly below the highest possible level
($z = 1$) of the pendulum. Such a spherical pendulum doesn’t have a focus-focus
point, but still has monodromy similar to the usual spherical pendulum.

6.3. **Systems with many degrees of freedom.** It follows easily from the results
of [7] that, if $N$ is a corank-2 focus-focus singular fiber of an integrable Hamiltonian
system with $n$ degrees of freedom, $n > 2$, with some additional nondegeneracy
condition (called “topological stability” in [17]) then a tubular neighborhood of
$N$ together with the singular Lagrangian fibration is homeomorphic to the direct
product of a tubular neighborhood of a regular torus in an integrable system with
$n - 2$ degrees of freedom with a tubular neighborhood of a focus-focus singular
fiber of an integrable system with two degrees of freedom. Thus, corank-2 focus-
focus singularities in higher dimensions are the same as focus-focus singularities in
dimension 4. for example, in the case with 3 degrees of freedom, the monodromy
matrix is

$$
\begin{pmatrix}
1 & k & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
$$

(6.1)

Still, one must be careful about the bases in which these matrices are written. For
example, Bates and Zou in a paper published in 1993 made a mistake of using
2 × 2 matrices when studying monodromy around two focus-focus singularities in a 3-degrees-of-freedom system, and arrived at a wrong multiplication formula.

6.4. Integral affine structure of the base space. By duality, the monodromy of the torus fibration of an integrable Hamiltonian system is the same as the monodromy of the integral affine structure on the base space. In fact, when we consider the whole base space with its singularities, then it has a stratified integral affine structure, and an affine monodromy sheaf associated with it, see [15]. This sheaf is a free Abelian sheaf which is locally constant on each stratum, and the structure of this sheaf is a topological invariant of the system. One can argue that, by Bohr-Sommerfeld rules, the quantum joint spectrum of a semi-classical integrable system is nothing but a discretization of the integral affine structure of the base space of the corresponding classical integrable system, and that’s the main reason why quantum integrable systems have the same monodromy as its classical counterpart, as observed in many places (see e.g. [6, 14, 15] and references therein). In fact, the quantum joint spectrum mimics the integral affine structure not only near focus-focus singularities, but near elliptic and hyperbolic singularities as well.

6.5. Focus-focus singularities in geometry. The Strominger-Yau-Zaslow conjecture in mirror symmetry [11] says that on each “dualizable” Calabi-Yau manifold there is a special Lagrangian fibration and the dual Calabi-Yau manifold corresponds to the dual special Lagrangian fibration. From the point of view of integrable Hamiltonian systems, these fibrations are integrable systems with corank-2 focus-focus singularities, and higher-corank singularities which must be similar in some sense to focus-focus ones. Hence, a good understanding of focus-focus singularities and its higher-dimensional degenerate sisters may be helpful in the SYZ construction of mirror symmetry. Though many authors have written many papers about the SYZ construction, at present the situation is still not clear, especially in what concerns singularities, to my very limited knowledge.

Another application of focus-focus singularities is in the geometry of 4-manifolds. For example, using a symplectic (integrable) surgery involving focus-focus singularities, Symington constructed so-called generalized symplectic rational blow-downs, which are useful for creating interesting 4-manifolds and calculating their invariants, see [12] and references therein.

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