BLOW-UP BEHAVIOR FOR A QUASILINEAR PARABOLIC EQUATION WITH NONLINEAR BOUNDARY CONDITION

JONG-SHENQ GUO
Department of Mathematics, National Taiwan Normal University
88, S-4 Ting Chou Road, Taipei 116, Taiwan
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Abstract. In this paper, we study the solution of an initial boundary value problem for a quasilinear parabolic equation with a nonlinear boundary condition. We first show that any positive solution blows up in finite time. For a monotone solution, we have either the single blow-up point on the boundary, or blow-up on the whole domain, depending on the parameter range. Then, in the single blow-up point case, the existence of a unique self-similar profile is proven. Moreover, by constructing a Lyapunov function, we prove the convergence of the solution to the unique self-similar solution as \( t \) approaching the blow-up time.

1. Introduction. In this paper, we study the following initial boundary value problem (P):

\[
\begin{align*}
  u_t &= u^{1+\gamma}u_{xx}, \quad 0 < x < 1, \quad t > 0, \\
  u_x(0, t) &= -u^q(0, t), \quad u_x(1, t) = 0, \quad t > 0, \\
  u(x, 0) &= u_0(x), \quad 0 \leq x \leq 1,
\end{align*}
\]

where \( \gamma > 0 \) and \( q > 0 \) are given constants, and \( u_0 \) is a positive bounded smooth function defined on \([0, 1]\) such that \( u_0'(0) = -u_0^q(0) \) and \( u_0'(1) = 1 \).

The local existence and uniqueness of positive solution of (P) can be derived by the standard theory of parabolic equation. We say that a solution \( u \) blows up in finite time \( T \), if \( \limsup_{t \to T^-} \max_{x \in [0, 1]} u(x, t) = \infty \). The study of blow-up has attracted much attentions for past years. The typical questions are concerned about blow-up criteria, blow-up locations, blow-up rates, blow-up profiles, and so on. We refer the reader to the survey papers of Levine [18] and Deng-Levine [7], and the book by Samarskii-Galaktionov-Kurdyumov-Mikhailov [19]. Problem (P) with \( \gamma = 0 \) was studied by Ferreira-de Pablo-Rossi [8] for both the bounded interval and semi-infinite interval cases. For blow-up on the boundary, we refer the reader to the survey papers by Chlebík-Fila [4] and Fila-Filo [10].

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In [9], Ferreira-de Pablo-Quirós-Rossi studied the following initial boundary value problem \( \hat{P} \):

\[
\hat{u}_\tau = (\hat{u}^{m-1}\hat{u}_\xi)_{\xi}, \quad 0 < \xi < l, \quad \tau > 0,
\]

\[
(\hat{u}^{m-1}\hat{u}_\xi)(0, \tau) = \hat{u}^m(0, \tau), \quad (\hat{u}^{m-1}\hat{u}_\xi)(l, \tau) = 0, \quad \tau > 0,
\]

\[
\hat{u}(\xi, 0) = \hat{u}_0(\xi) > 0, \quad 0 \leq \xi \leq l,
\]

where \( m < 0 \) and \( l > 0 \) are constants. It is shown that the solution of (1.4)-(1.6) quenches, i.e., its minimum reaches zero in finite time. The first time when the minimum of the solution reaches zero is called the quenching time. Note that the phenomenon of quenching is different from the following dead-core problem (cf., e.g., [16] and the references cited therein):

\[
u_t = \nu_{xx} - \nu^p, \quad -1 < x < 1, \quad t > 0,
\]

\[
u(\pm 1, t) = k, \quad t > 0,
\]

\[
u(x, 0) = \nu_0(x) > 0, \quad -1 \leq x \leq 1,
\]

where \( 0 < p < 1 \) and \( k \) is a positive constant. For quenching, some derivative of the solution becomes singular at the quenching time. Indeed, for the problem (1.4)-(1.6), we have the time derivative blows up at the quenching time. On the other hand, in the dead-core problem, the solution stays regular whenever its minimum reaches zero in finite time and the solution can be continued for all time.

It is well-known that quenching problem is related to blow-up problem. Indeed, by setting

\[
u = \hat{u}^m, \quad \gamma = -1/m, \quad \xi = \gamma x, \quad \tau = \gamma^2 t,
\]

the problem (1.4)-(1.6) becomes (1.1)-(1.3) with \( q = 1 \) and spatial domain \([0, l/\gamma]\).

Therefore, in this paper we shall only consider the case when \( q \neq 1 \). Another related problem to (P) is about the blow-up behavior of the solution of the Cauchy problem for the equation

\[
u_t = \nu^\sigma (\Delta \nu + \nu^p), \quad x \in \mathbb{R}^n, \quad t > 0,
\]

where \( \sigma \geq 1 \) and \( p > 1 \). We refer the reader to [13, 14, 15] and the references cited therein. On the other hand, the Cauchy problem for the equation (1.1) in higher spatial dimension has been studied by Bertsch-Ughi [2] and Bertsch-Dal Passo-Ughi for nonnegative initial data. See also [20] for a more general equation.

In studying the blow-up behavior near the blow-up time, it is crucial to analyze the so-called (backward) self-similar solutions of (P). Let \( T < \infty \) be the blow-up time and assume that \( x = 0 \) is a blow-up point. For \( q > 1 \), we introduce the following self-similar change of variables:

\[
v(y, s) := (T - t)^\alpha u(x, t), \quad y := \frac{x}{(T - t)^{\beta}}, \quad s := -\ln(T - t),
\]

where the similarity exponents are given by

\[
\alpha := \frac{1}{\gamma + 2q - 1}, \quad \beta := (q - 1)\alpha.
\]

Note that \( \alpha > 0 \) and \( \beta > 0 \), if \( q > 1 \). It follows that \( u \) satisfies (1.1)-(1.3) if and only if \( v \) satisfies

\[
v_x = v^{1+\gamma}v_{yy} - \beta y v_y - \alpha v, \quad 0 < y < R(s) := e^{\beta s}, \quad s > s_0 := -\ln T,
\]

\[
v_y(0, s) = -v^\gamma(0, s), \quad v_y(R(s), s) = 0, \quad s > s_0,
\]

\[
v(y, s_0) = v_0(y) := T^{\alpha}\nu_0(yT^\beta), \quad 0 \leq y \leq 1/T^\beta.
\]
Note that $s \to \infty$ and $R(s) \to \infty$ as $t \uparrow T^-$. We expect that, as $s \to \infty$, the solution of (1.8)-(1.10) is stabilized. In this paper, we shall call a global solution of the following problem as a self-similar profile of (P):

$$
g'' - \beta y g^{-\gamma-1} g' - \alpha g^{-\gamma} = 0, \quad 0 < y < \infty, \quad (1.11)
g'(0) = -g^\delta(0). \quad (1.12)
$$

Since we are interested in the behavior of $u$ as $t \to T^-$, we shall be concerned with the positive global solution of (1.11)-(1.12). In particular, we are looking for a monotone decreasing positive global solution of (1.11)-(1.12).

This paper is organized as follows. In §2, we shall derive a blow-up criterion, prove the single point blow-up for monotone solutions when $q > 1$, and study the case when $q \in (0, 1)$. Motivated by a recent work [8], we shall prove in §3 that for $q > 1$ the self-similar profile exists and is unique, by using a phase plane analysis approach. Finally, by using a method of Zelenyak [21] (see also [15]), in §4, we shall prove the convergence of $v$, as $s \to \infty$, to the unique self-similar profile for $q > 1$.

2. Blow-up Criterion and Location. In this section, we first prove that the solution of the problem (P) always blows up in finite time.

**Theorem 2.1.** Suppose that $q > 0$. Then for every positive bounded smooth initial data $u_0$, there exists a finite time $T > 0$ such that

$$
\limsup_{t \to T^-} \max_{x \in [0,1]} u(x, t) = \infty. \quad (2.1)
$$

**Proof.** By assumption, there is a positive constant $\delta$ such that $u_0 \geq \delta$ in $[0,1]$. Then, by the maximum principle, $u(x, t) \geq \delta$ for the corresponding solution $u$ of (P).

We introduce the following quantity

$$
N(t) := \int_0^1 u^{-\gamma}(x, t) dx.
$$

By differentiating $N(t)$ and using (1.1)-(1.2), we get

$$
N'(t) = -\gamma u^\delta(0, t). \quad (2.2)
$$

Since $q > 0$, we get

$$
N'(t) \leq -\eta
$$

for some constant $\eta > 0$. Thus $N(t)$ should vanish at some finite time. Therefore, the solution $u$ cannot be bounded for all $t > 0$. This implies that there exists a finite $T > 0$ such that (2.1) holds and the theorem is proved.

In the following, we shall always assume that the solution $u$ of (P) blows up at time $T < \infty$. For simplicity, from now on we shall further assume that

$$
u_0 \leq 0, \quad u_0^\prime \geq 0 \quad \text{in} \ [0,1]. \quad (2.3)
$$

Using (2.3), it is easily seen by the strong maximum principle that $u_x < 0, u_{xx} > 0$ and $u_t > 0$.

We say that a point $x = a$ is a blow-up point, if there is a sequence $\{(x_n, t_n)\}$ such that $x_n \to a$, $t_n \to T^-$, and $u(x_n, t_n) \to \infty$ as $n \to \infty$. Note that $x = 0$ is always a blow-up point.
Theorem 2.2. Suppose that \( q > 1 \). Under the assumption \((2.3)\), \( x = 0 \) is the only blow-up point.

Proof. Suppose, for contradiction, that there exists another blow-up point \( a \in (0, 1] \). Then any point \( b \in [0, a] \) is also a blow-up point, since \( u_x < 0 \) and \( u_t > 0 \).

Now we fix any number \( b \in (0, a) \). Following [11], we consider the function
\[
J(x, t) := u_x(x, t) + \varepsilon h(x)u^q(x, t), \quad h(x) := (x - b)^2, \quad \varepsilon > 0.
\]

Then it is easy to compute that
\[
J_x - u^{1+q}J_{xx} = (1 + \gamma)u^\gamma u_x J_x
\]
\[= -\varepsilon(\gamma + q)qhu^{\gamma+q-1}u_x^2 - \varepsilon(1 + \gamma + 2q)h'u^{\gamma+q}u_x - \varepsilon h''u^{\gamma+q+1}
\]
\[\leq 0 \quad \text{in} \quad (0, b) \times (0, T),
\]
by using the properties of \( h \) and the fact that \( u_x < 0 \). Clearly, \( J(b, t) < 0 \) for all \( t \in (0, T) \). Moreover, \( J(0, t) = -u^q(0, t)(1 - \varepsilon b^2) \leq 0 \) for all \( t \in (0, T) \), if \( \varepsilon < 1/b^2 \).

By choosing \( \varepsilon \) small enough and using \( u_x(x, T/2) < 0 \) in \([0, b]\), we have \( J(x, T/2) \leq 0 \) for all \( x \in [0, b] \). Therefore, it follows from the maximum principle that \( J \leq 0 \) in \([0, b] \times [T/2, T] \), i.e.,
\[
-u^{-q}(x, t)u_x(x, t) \geq \varepsilon(x - b)^2, \quad x \in [0, b], \quad t \in [T/2, T).
\]

Integrating \((2.4)\) from 0 to \( b \), we obtain that
\[
[u^{1-q}(b, t) - u^{1-q}(0, t)]/(q - 1) \geq \varepsilon \int_0^b (x - b)^2 dx = \varepsilon b^3/3 \quad \forall t \in (T/2, T).
\]

Letting \( t \uparrow T^- \), we reach a contradiction. Thus the theorem follows.

For \( 0 < q < 1 \), since \( u_{xx} > 0 \), we have \( u_x(x, t) \geq u_x(0, t) = -u^q(0, t) \) and so
\[
u(x, t) \geq u(0, t) - xu^q(0, t) = u(0, t)[1 - xu^{q-1}(0, t)]
\]
\[(2.5)\]
for all \( x \in (0, 1] \). Since \( u(0, t) \to \infty \) as \( t \to T^- \) and \( 0 < q < 1 \), we conclude that \( u(x, t) \to \infty \) as \( t \to T^- \) for any \( x \in [0, 1] \). This means that we have the blow-up in the whole domain.

Moreover, we can estimate the blow-up rate for the case \( q \in (0, 1) \) as follows.

Theorem 2.3. Suppose that \( 0 < q < 1 \). Then, under the assumption \((2.3)\), there are positive constants \( c_1 \) and \( c_2 \) such that
\[
c_1(T - t)^{-1/(q+\gamma)} \leq u(0, t) \leq c_2(T - t)^{-1/(q+\gamma)}.
\]
\[(2.6)\]
for all \( t \in [0, T) \).

Proof. First, we choose \( t_0 \in (0, T) \) such that \( u^{q-1}(0, t_0) \leq 1/2 \). Then, using \( u_x < 0 \) and \((2.5)\), we have
\[
u(0, t)/2 \leq u(x, t) \leq u(0, t) \quad \forall x \in [0, 1], \quad t \in [t_0, T).
\]

Hence we obtain
\[
u^{-\gamma}(0, t) \leq N(t) \leq 2^\gamma u^{-\gamma}(0, t) \quad \forall t \in [t_0, T),
\]
i.e.,
\[
N^{-1/\gamma}(t) \leq u(0, t) \leq 2N^{-1/\gamma}(t) \quad \forall t \in [t_0, T).
\]
\[(2.7)\]
It follows from \((2.2)\) that
\[
-2^{\gamma}N^{-q/\gamma}(t) \leq N'(t) \leq -\gamma N^{-q/\gamma}(t) \quad \forall t \in [t_0, T).
\]
\[(2.8)\]
Then the estimate (2.6) follows by an integration of (2.8) from \( t \) to \( T \) and using (2.7).

3. Self-similar Profile for \( q > 1 \). In this section, we shall study the solution of the initial value problem (1.11)-(1.12):

\[
\begin{align*}
g'' - \beta y g^{-\gamma -1} g' - \alpha g^{-\gamma} &= 0, \quad y > 0, \quad (3.1) \\
g'(0) &= -g^2(0). \quad (3.2)
\end{align*}
\]

From the local existence and uniqueness theorem of ordinary differential equations, it follows that there is a unique positive local solution \( g \) of (3.1)-(3.2) for each given initial value \( g(0) > 0 \). For convenience, let \([0, R)\) be the maximum existence interval of \( g \). Note that \( g > 0 \) in \([0, R)\) and \( 0 < R \leq \infty \).

Since \( g'' = \alpha g^{-\gamma} > 0 \) when \( g = 0 \), we see that any critical point of \( g \) must be a local minimum point. Hence there is at most one critical point of \( g \). Moreover, if \( g \) has a critical point \( y_0 > 0 \), then \( g'(y) > 0 \) for any \( y \in (y_0, R) \) and \( g''(y) > 0 \) for any \( y \in [y_0, R) \).

For a given solution \( g \), define

\[
\rho(y) = \exp\left\{-\beta \int_0^y sg^{-\gamma -1}(s)ds\right\}.
\]

From (3.1) it follows that

\[
(\rho g')'(y) = \alpha \rho(y) g^{-\gamma}(y)
\]

and so

\[
g'(y) = \frac{g'(0) + \alpha \int_0^y g^{-\gamma}(s)\rho(s)ds}{\rho(y)}. \quad (3.3)
\]

Later on in §4, we shall need the following property.

**Lemma 3.1.** If there exists \( R < \infty \) such that \( g(R^-) = 0 \), then \( g'(y) \to -\infty \) as \( y \to R^- \).

**Proof.** Note that \( g \) must be monotone decreasing to zero, under the assumption of the lemma. Integrating (3.1) from 0 to \( y \), we get

\[
g'(y) = g'(0) - (\beta/\gamma) \int_0^y z(g^{-\gamma})'(z)dz + \alpha \int_0^y g^{-\gamma}(z)dz.
\]

Using integration by parts, we have

\[
g'(y) = g'(0) - (\beta/\gamma)yg^{-\gamma}(y) + (\alpha + \beta/\gamma) \int_0^y g^{-\gamma}(z)dz. \quad (3.4)
\]

Taking \( K = [1 + \alpha/(\alpha + \beta/\gamma)]R/2 \), from (3.4) it follows that

\[
g'(y) = g'(0) - (\beta/\gamma)yg^{-\gamma}(y) + (\alpha + \beta/\gamma) \left[ \int_0^K g^{-\gamma}(z)dz + \int_K^y g^{-\gamma}(z)dz \right]
\]

\[
\leq g'(0) - (\beta/\gamma)yg^{-\gamma}(y)
\]

\[
+ (\alpha + \beta/\gamma) \int_0^K g^{-\gamma}(z)dz + (\alpha + \beta/\gamma)g^{-\gamma}(y)(y - K)
\]

\[
= g'(0) + (\alpha + \beta/\gamma) \int_0^K g^{-\gamma}(z)dz + [\alpha y - (\alpha + \beta/\gamma)K]g^{-\gamma}(y)
\]

\[
\to -\infty \text{ as } y \to R^-.
\]
The lemma follows.

We shall need the asymptotic behavior as \( y \to \infty \) of any monotone decreasing positive global solution of (3.1)-(3.2) as follows.

**Lemma 3.2.** For any monotone decreasing positive global solution \( g \) of (3.1)-(3.2), we have \( g(y) \to 0 \), \( g'(y) \to 0 \) and \( [g'(y)/g(y)] \to -\alpha/\beta \) as \( y \to \infty \).

**Proof.** By assumption, we see that \( g(y) \to L \) as \( y \to \infty \) for some \( L \in [0, \infty) \). We claim that \( L = 0 \). If \( L \in (0, \infty) \), then there exists \( \{y_n\} \to \infty \) such that \( g'(y_n) \to 0 \) as \( n \to \infty \). Dividing (3.1) by \( y \) and integrating the resulting equation from 1 to \( y_n \), we obtain

\[
\int_1^{y_n} \frac{g''(s)}{s} ds + \frac{\beta}{\gamma} \int_1^{y_n} (g^{-\gamma}(s))' ds = \alpha \int_1^{y_n} \frac{g^{-\gamma}(s)}{s} ds. \tag{3.5}
\]

We compute that

\[
\int_1^{y_n} \frac{g''(s)}{s} ds = \frac{g'(y_n)}{y_n} - g'(1) + \int_1^{y_n} \frac{g'(s)}{s^2} ds,
\]

\[
0 > \int_1^{y_n} \frac{g'(s)}{s^2} ds \geq \int_1^{y_n} g'(s) ds = g(y_n) - g(1),
\]

\[
\int_1^{y_n} (g^{-\gamma}(s))' ds = g^{-\gamma}(y_n) - g^{-\gamma}(1).
\]

Hence the left-hand side of (3.5) is uniformly bounded for all \( n \). But, for \( K \) large enough, we have

\[
\int_1^{y_n} \frac{g^{-\gamma}(s)}{s} ds \geq \int_K^{y_n} \frac{2L^{-\gamma}}{s} ds \to \infty \text{ as } n \to \infty,
\]

a contradiction. Hence \( L = 0 \).

Next, we claim that \( g'(y) \to 0 \) as \( y \to \infty \). For this, we set

\[
I := \alpha \int_0^\infty g^{-\gamma}(s)\rho(s)ds.
\]

We claim that \( g'(0) + I = 0 \). Since \( g' < 0 \), by (3.3), \( g'(0) + I \leq 0 \). Since \( g(y) \to 0 \) as \( y \to \infty \), there exists a sequence \( \{y_n\} \) such that \( y_n \to \infty \) and \( g'(y_n) \to 0 \) as \( n \to \infty \). Since \( \rho(y) \to 0 \) as \( y \to \infty \), by (3.3), we must have \( g(0) + I = 0 \). Now, by L'Hôpital's Rule, we compute from (3.3) that

\[
\lim_{y \to \infty} g'(y) = \lim_{y \to \infty} \frac{g'(0) + \alpha \int_0^y g^{-\gamma}(s)\rho(s)ds}{\rho(y)} = \frac{-\alpha g(y)}{\beta y} = 0.
\]

Finally, by applying L'Hôpital's Rule to (3.3) again, we obtain that

\[
\lim_{y \to \infty} \frac{yg'(y)}{g(y)} = \lim_{y \to \infty} \frac{g'(0) + \alpha \int_0^y g^{-\gamma}(s)\rho(s)ds}{y^{-1}g(y)} = \frac{\alpha}{\beta} y^{-2}g^{-1}(y) + y^{-1}g^{-1}(y)g'(y) - \beta
\]

This completes the proof. \( \square \)
Following [1, 17, 8], we introduce the following variables:

\[ U := \frac{yg'(y)}{g(y)}, \quad V := y^2g^{-1}(y), \quad z := \ln y \]  

for any solution \( g \) of (3.1). Then it is easily to check that \((U, V)\) satisfies the first order autonomous system (Q):

\[
\begin{align*}
\frac{dU}{dz} &= U(1 - U) + V(\alpha + \beta U), \\
\frac{dV}{dz} &= V[2 - (1 + \gamma)U].
\end{align*}
\]

Note that there are two finite critical points \( A := (0, 0) \) and \( B := (1, 0) \) for the system (Q). Since the linearization of (Q) around \( A \) gives the matrix

\[
\begin{bmatrix}
1 & \alpha \\
0 & 2
\end{bmatrix},
\]

which has eigenvalues \( 1, 2 \) and corresponding eigenvectors \( \{ (1, 0), (\alpha, 1) \} \), we see that \( A \) is an unstable improper node. In particular, it follows from an easy phase plane analysis that every orbit near \( A \) in the second quadrant of \((U, V)\)-plane leaves \( A \) horizontally (see, e.g., [6]). Notice that orbits corresponding to monotone decreasing positive solutions of (3.1)-(3.2) lie in the second quadrant.

From Lemma 3.2 we see that a monotone decreasing positive global solution \( g \) of (3.1)-(3.2) corresponding to an orbit connecting from \( A \) to the point \( D := \left(-\frac{\alpha}{\beta}, 1\right) \) in \((U, V)\)-plane. To learn the behavior near \( D \), we choose the following new dependent variable \((U, W)\), \( W := \frac{1}{V(\alpha + \beta U)} \), and independent variable \( \tau := \int_0^z V(s)ds \). Then the system (Q) becomes the system (R):

\[
\begin{align*}
\frac{dU}{d\tau} &= WU(1 - U) + (\alpha + \beta U), \\
\frac{dW}{d\tau} &= -W^2[2 - (1 + \gamma)U].
\end{align*}
\]

Note that the critical point \( D \) of (Q) becomes the critical point \( E := \left(-\frac{\alpha}{\beta}, 0\right) \) of (R) in the \((U, W)\)-plane. It is easy to see that the linearization of (R) around \( E \) gives the matrix

\[
\begin{bmatrix}
\beta & -(1 + \alpha/\beta)(\alpha/\beta) \\
0 & 0
\end{bmatrix},
\]

which has eigenvalues \( \lambda_1 = \beta > 0, \lambda_2 = 0 \), and corresponding eigenvectors \( v_1 = (1, 0), v_2 = ((1 + \alpha/\beta)(\alpha/\beta), \beta) \). Hence the horizontal line is tangent to the unstable manifold of \( E \). Since the center manifold is tangent to the eigenspace spanned by \( v_2 \) and \( dW/d\tau < 0 \) for \((U, W) \in S \), where

\[ S := \{(U, W) \mid W > 0, \ -\alpha/\beta < U < 0\} \]

by a standard technique (see, e.g., [5]), there exists a unique orbit of the system (R) tending to \( E \) as \( \tau \to \infty \). This shows that there exists a unique orbit, call it as \( \Gamma^* \), of the system (Q) tending to the critical point \( D \) as \( z \to \infty \). Note that \( \Gamma^* \) lies in the strip \( S \) for all large \( z \). Since \( dU/dz < 0 \) on \( \{V > 0, U = -\alpha/\beta\} \), \( dU/dz > 0 \) on \( \{V > 0, U = 0\} \), and \( dV/dz > 0 \) in the second quadrant, the orbit \( \Gamma^* \) must tend to \( A \) as \( z \to -\infty \).

We thus have proved the following existence theorem.
Theorem 3.3. There exists a monotone decreasing positive global solution of (3.1)-(3.2).

We continue to prove the uniqueness of such solution. Note first that any orbit tending to $A$ has the behavior $V = bU^2 + O(U^3)$ as $U \to 0^-$ for some positive constant $b$ (which depending on each orbit). Let $b^*$ be the constant corresponding to the orbit $Γ^*$. Therefore, by the phase plane analysis, for each $b > b^*$ the corresponding orbit shall reach the positive $V$-axis in finite time and continue to stay in the first quadrant. These orbits are those solutions of (3.1)-(3.2) with exactly one critical point.

On the other hand, for each $b \in (0, b^*)$ the corresponding orbit shall reach the half-line $L := \{ V > 0, U = -α/β \}$. Suppose for contradiction that $g(y) > 0$ for all $y \in [0, y_1]$. Then $g'(y_1)$ is finite by (3.3). This implies that $U(z_1)$ is finite, a contradiction. Hence we have proved that $g(y) \to 0^+$ as $y \to y_1$.

Therefore, we are ready to prove the following uniqueness theorem.

Theorem 3.4. There exists a unique monotone decreasing positive global solution of (3.1)-(3.2).

Proof. Since we have a unique orbit in $(U, V)$-plane connecting the critical points $A$ and $D$, it remains to show the one-to-one correspondence of orbits with the positive solutions of (3.1)-(3.2). This is equivalent to show that different values of $g(0)$ give different orbits in $S$ leaving from $A$. Given a positive constant $b$ (which corresponding to an orbit in $S$ leaving from $A$). Since

$$b = \lim_{z \to -\infty} \frac{V(z)}{U^2(z)} = \lim_{y \to 0} \frac{g^{-γ+1}(y)}{(g')^2(y)} = g^{1-γ-q}(0),$$

by using (3.2), we obtain the one-to-one correspondence between $b$ and $g(0)$. Hence the theorem follows. □

In the following, we shall denote $g^*$ to be the unique monotone decreasing positive global solution of (3.1)-(3.2) and let $μ^* := g^*(0)$.

4. Asymptotic Behavior Near Blow-up Time for $q > 1$. In this section, we shall study the asymptotic behavior of the solution $u$ of (P) near the blow-up time $T$. This is equivalent to study the stabilization, as $s \to \infty$, of the solution $v$ of (1.8)-(1.10). More precisely, we shall prove the following main theorem of this section.

Theorem 4.1. Let $v$ be the solution of (1.8)-(1.10) and $g^*$ be the unique self-similar profile obtained in Theorem 3.4. Then, under the assumption (2.3), as $s \to \infty$, $v(y, s) \to g^*(y)$ uniformly for any compact subset of $[0, \infty)$.

To prove this theorem, we shall divide our discussions into a few subsections as follows.
4.1. Some a priori bounds. In this subsection, we shall derive some a priori bounds for $v$. First, we derive the following blow-up rate estimate.

Lemma 4.2. Under the assumption (2.3), there are positive constants $a, \kappa$ such that

$$a(T-t)^{-\alpha} \leq u(0, t) \leq \kappa(T-t)^{-\alpha} \tag{4.1}$$

for all $t \in (0, T)$.

Proof. The proof of this lemma is based on the so-called intersection comparison principle (cf. [19, 8]). We first recall from §3 that there exists a positive constant $\mu^*$ such that the solution $g$ of (3.1)-(3.2) with $g(0) \in (\mu^*, \infty)$ satisfying $g' < 0$ in $[0, R)$ and $g(y) \to 0$ as $y \to R^-$ for some finite $R$ depending on $g(0)$. Moreover, if $g(0) \in (0, \mu^*)$, then there exists a unique $y_0 > 0$ such that $g' < 0$ in $[0, y_0)$. We claim that $g(y) \to 0$ as $y \to \infty$.

For the upper bound, we compare $u$ with the function

$$U_1(x, t) := (T-t)^{-\alpha} g_1(x/(T-t)^{\beta}),$$

where $g_1$ is the solution of (3.1)-(3.2) with a very small $g_1(0)$ such that $u_0(0) > U_1(0, 0)$ and $u_0$ has at most one intersection with $U_1(x, 0)$ in $[0, 1]$. This is possible, since $u_0$ is positive and monotone decreasing. We claim that $u(0, t) > U_1(0, t)$ for all $t \in (0, T)$. Suppose, for contradiction, that $u(0, t) > U_1(0, t)$ for $t \in (0, t_0)$ and $u(0, t_0) = U_1(0, t_0)$ for some $t_0 \in (0, T)$. Since the number of intersections is non-increasing, we must have $u(x, t_0) < U_1(x, t_0)$ for all $x \in (0, 1)$. Then we have $u < U_1$ in the set

$$\{(x, t) \mid x(t) < x < 1, \ 0 < t \leq t_0\},$$

where $x(t)$ is the intersection point for each $t$. But, this leads to a contradiction with the Hopf Lemma. We thus have derived the lower bound that $u(0, t) > g_1(0)(T-t)^{-\alpha}$ for all $t \in (0, T)$.

For the upper bound, we compare $u$ with the function

$$U_2(x, t) := (T-t)^{-\alpha} g_2(x/(T-t)^{\beta}),$$

where $g_2$ is the solution of (3.1)-(3.2) with $g_2(0)$ very large so that $g_2$ is decreasing to zero at some finite $R$ and $u_0$ has at most one intersection with $U_2(x, 0)$. Note that this is possible, since, by Lemma 3.1, $g_2^2(y) \to -\infty$ as $y \to R^-$. Note also that $U_2$ is defined only in the set $\{(x, t) \mid 0 \leq x < R(T-t)^{\beta}, \ 0 \leq t < T\}$. Similar argument as above gives that $u(0, t) < g_2(0)(T-t)^{\alpha}$ for all $t \in (0, T)$. The lemma follows.

As a consequence of (4.1), we obtain the following estimate

$$0 < a \leq v(0, s) \leq \kappa < \infty \quad \text{for all} \quad s > s_0. \tag{4.2}$$

Since $u_{xx} > 0$, we have $v_{yy} > 0$. Hence, using (1.9) and (4.2), we obtain

$$v_y(y, s) \geq v_y(0, s) = -v^q(0, s) \geq -\kappa^q. \tag{4.3}$$

Also, $u_x < 0$ implies that $v_y < 0$ and so $v \leq \kappa$. Using (4.2) and (4.3), it is easy to see that there is a positive constant $\delta \in (0, 1)$ such that

$$v(y, s) \geq a/2 \quad \text{for} \quad 0 \leq y \leq \delta, \ s > s_0. \tag{4.4}$$

Moreover, we claim that

$$v(y, s) \geq \frac{a}{2} \left( \frac{y}{\delta} \right)^{-\alpha/\beta} \quad \text{for} \quad \delta < y \leq e^{\beta(s-s_0^+)}, \ s > s_0, \ s_0^+ := \max(s_0, 0). \tag{4.5}$$
Indeed, given \((y, s)\) with \(y \in (\delta, e^{\beta(s-s_0^+)})\), \(s > s_0\), we can find an \(l \in (s_0^+, s)\) such that \(y = \delta e^{\beta(s-l)}\). Since \(v_{yy} > 0\), it follows from (1.8) that \(v_s + \beta y v_y + \alpha v > 0\). Then
\[
\frac{d}{d\tau} v(z, \tau) = v_s(z, \tau) + \beta z v_y(z, \tau) \geq -\alpha v(z, \tau), \quad z := ye^{\beta(\tau-s)}.
\]
Hence (4.5) follows by an integration of the above inequality from \(\tau = l\) to \(\tau = s\).

From (4.4) and (4.5), we can derive that polynomial growth estimates in \(y\) for \(v_s\) and \(v_{yy}\), by applying the interior parabolic estimates to (1.8). More precisely, we have the following.

**Lemma 4.3.** Under the assumption (2.3), there is a positive constant \(C\) such that
\[
|v_s(y, s)| \leq C(1 + y^{2+\gamma/(q-1)}) \quad \forall \ y \in [0, e^{\beta s}/2], \ s \geq s_0.
\]

**Proof.** First, we derive the following estimate
\[
v(y, s) \leq C(1 + y)^{-\alpha \beta}, \quad y \in [0, e^{\beta s}/2], \ s > s_0,
\]
for some positive constant \(C\). Note that \(\alpha / \beta = 1/(q-1)\). We consider the function
\[
J(x, t) := u_x(x, t) + cu^q(x, t), \ c > 0.
\]
Then we have
\[
J_t - u^{1+\gamma}J_{xx} - (1 + \gamma)u^\gamma u_x J_x = -q(q + \gamma)cu^{\gamma+1-1}u_x^2.
\]
Also, \(J(0, t) = (-1 + c)u^q(0, t) < 0\) if \(c < 1\). Since \(x = 0\) is the only blow-up point and \(u_x < 0\) for \(x < 1\) and \(0 < t < T\), we have \(J(1/2, t) < 0\) for \(t \in [T/2, T)\) and \(J(x, T/2) < 0\) for \(x \in [0, 1/2]\), if \(c \ll 1\). It follows from the maximum principle that \(J < 0\) in \([0, 1/2] \times [T/2, T)\). Therefore, we obtain that
\[
v_y(y, s) \leq -cu^q(y, s), \quad y \in [0, e^{\beta s}/2], \ s \gg 1.
\]
By integrating (4.8), the estimate (4.7) follows.

To estimate \(v_s\) for a given \((\tilde{y}, \tilde{s})\) with \(\tilde{y} \gg 1\), as in [15], we make the following change of variables:
\[
V(y, s) := K v(\mu y + \tilde{y}, \mu^2 K^{1+\gamma} s + \tilde{s}), \ |y| \leq 1, \ -1 < s \leq 0,
\]
\[
K := k^{\alpha / \beta} = k^{1/(q-1)}, \quad \mu := k^{-1-(1+\gamma)/(q-1)},
\]
where \(k \geq 1\) is chosen so that \(2k \leq \tilde{y} \leq 4k\). Then \(V\) satisfies the equation
\[
V_s = V^{1+\gamma} V_{yy} - \mu K^{1+\gamma} (\mu y + \tilde{y}) \beta V_y - \mu^2 K^{1+\gamma} \alpha V.
\]
Note that, by the choices of \(K\) and \(\mu\), we have
\[
0 < \mu K^{1+\gamma}(\mu y + \tilde{y}) \leq 4\mu K^{1+\gamma} k \leq 4 \quad \text{for } |y| \leq 1,
\]
\[
0 < \mu^2 K^{1+\gamma} \leq 1.
\]
Also, by using (4.7) and (4.5), we have
\[
0 < c_0 \leq V \leq C_0 < \infty, \quad |y| \leq 1, \ -1 < s \leq 0,
\]
for some constants \(c_0\) and \(C_0\) which are independent of \((\tilde{y}, \tilde{s})\). By applying the interior Schauder estimate, we see that \(V_s(0,0)\) is bounded by a constant which is independent of \((\tilde{y}, \tilde{s})\). This gives the estimate (4.6) and the lemma is proved. 

From (1.8) and combining all the above estimates, the polynomial growth estimate in \(y\) for \(v_{yy}\) can also be derived.
4.2. Backward problem. To derive the convergence result, we need to construct a Lyapunov function. In constructing a suitable Lyapunov function, we first study the following backward initial value problem for a given \((y, v, \xi)\) with \(y > 0, v > 0, \xi \in \mathbb{R}\):

\[
\begin{align*}
g'' - \beta z g^{-\gamma - 1} g' - \alpha g^{-\gamma} &= 0, \quad z < y, \\
g(y) &= v, \quad g'(y) = \xi.
\end{align*}
\] (4.9) (4.10)

The local existence and uniqueness of the solution of (4.9)-(4.10) near \(y\) is trivial. We call this local backward solution as \(g(z; y, v, \xi)\) or simply \(g(z)\). As before, we define

\[
\rho(z) := \exp \left\{ \beta \int_{z}^{y} sg^{-\gamma - 1}(s; y, v, \xi) ds \right\}.
\]

Then \(\rho > 1, \rho' < 0,\) and

\[
g'(z) = \frac{1}{\rho(z)} \left\{ \xi - \alpha \int_{z}^{y} \rho(s) g^{-\gamma}(s) ds \right\}.
\] (4.11)

We first prove that this backward solution always stays bounded in \([0, y]\). Otherwise, if \(g(z) \to \infty\) as \(z \to z_0^+\) for some \(z_0 \in [0, y]\), then \(g'(z) \to -\infty\) as \(z \to z_0^+\). On the other hand, since \(g \geq \delta\) in \((z_0, y]\) for some constant \(\delta > 0\), \(\rho\) is uniformly bounded in \([z_0, y]\). It then follows from (4.11) that \(g'(z_0)\) is finite, a contradiction. Hence \(g\) remains bounded.

In particular, if \(\xi \leq 0\), then, by (4.11), \(g' < 0\) in \([0, y]\) and so \(g(z; y, v, \xi) \geq v\) for all \(z \in [0, y]\). We conclude that any local solution \(g(z; y, v, \xi)\) can be continued backward beyond \(z = 0\) as a positive solution of (4.9)-(4.10) defined in \([0, y]\) for any given \((y, v, \xi)\) with \(y > 0, v > 0, \xi \leq 0\).

We claim that

\[
g(z; y, v, \xi) \leq v + \alpha y^2 v^{-\gamma} - \xi y \quad \text{for} \quad z \in [0, y],
\] (4.12)

if \(\xi \leq 0\). Indeed, from (4.11) it follows that

\[
g'(z) = \frac{\xi}{\rho(z)} - \alpha \int_{z}^{y} [\rho(s)/\rho(z)] g^{-\gamma}(s) ds
\]

\[
\geq \xi - \alpha \int_{z}^{y} g^{-\gamma}(s) ds
\]

\[
\geq \xi - \alpha y v^{-\gamma}
\]

for \(z \in [0, y]\), by using \(\xi \leq 0, \rho > 1,\) and \(\rho' < 0\). Then for \(\xi \leq 0\) we have

\[
g(z) = v - \int_{z}^{y} g'(s) ds \leq v - y(\xi - \alpha y v^{-\gamma}) \quad \text{for} \quad z \in [0, y].
\]

The estimate (4.12) follows.

4.3. Lyapunov function. In this subsection, we shall construct a Lyapunov function by using a method of Zelenyak [21].

First, we define

\[
E[v](s) := \int_{0}^{s} \Phi(y, v(y, s), v_y(y, s)) dy - \frac{v^{q+1}(0, s)}{q+1},
\] (4.13)

where \(\Phi = \Phi(y, v, \xi)\) is to be determined later. Then, using (1.8) and an integration by parts, we compute that

\[
\frac{d}{ds} E[v](s) = J_0(s) + J_1(s) + J_2(s),
\] (4.14)
where

\[
J_0(s) = -\int_0^s \Phi_{\xi\xi}(y, v(y, s), v_y(y, s))v^{-\gamma-1}(y, s)v_y^2(y, s)dy,
\]

\[
J_1(s) = \Phi_{\xi}(s, v(s, s), v_y(s, s))v_y(s, s) - \Phi_{\xi}(0, v(0, s), v_y(0, s))v_y(0, s)
+ \Phi(s, v(s, s), v_y(s, s)) - v(y, 0)v_y(0, s),
\]

\[
J_2(s) = \int_0^s \left\{ \Phi_{\nu} - \Phi_{\xi\nu} - \Phi_{\xi v_y} \left[ \beta y v^{-\gamma-1} v_y + \alpha v^{-\gamma} \right] \right\} v_y(y, s)dy
:= \int_0^s K(y, v(y, s), v_y(y, s))v_y(y, s)dy.
\]

Next, we introduce

\[
\Phi(y, v, \xi) := \int_0^\xi (\xi - \sigma)P(y, v, \sigma)d\sigma + \int_\nu^\infty \alpha \mu^{-\gamma}P(y, \mu, 0)d\mu,
\]

\[
P(y, v, \sigma) := \exp\left\{ -\beta \int_0^y z g^{-\gamma-1}(z; y, v, \sigma)dz \right\}
\]

with the constant \(\kappa\) defined in (4.2) and \(g(z; y, v, \sigma)\) defined in §4.2. Then

\[
\Phi_{\xi}(y, v, \xi) = \int_0^\xi P(y, v, \sigma)d\sigma, \quad \Phi_{\xi\xi}(y, v, \xi) = P(y, v, \xi).
\]

Moreover, we compute that

\[
K(y, v, \xi) = \int_0^\xi \left\{ -\sigma P_v(y, v, \sigma) - P_y(y, v, \sigma)
+ \frac{\partial}{\partial \sigma} \left[ P(y, v, \sigma)(-\beta y v^{-\gamma-1} - \alpha v^{-\gamma}) \right]\right\} d\sigma
\]

\[
= \int_0^\xi \left\{ -\beta P(y, v, \sigma) \left[ \int_0^y (-\gamma - 1) z g^{-\gamma-2}(z; y, v, \sigma) \cdot 
- \sigma g_v(z; y, v, \sigma) - g_y(z; y, v, \sigma)
+ (-\beta y v^{-\gamma-1} - \alpha v^{-\gamma}) g_\sigma(z; y, v, \sigma) \right] dz \right\}d\sigma.
\]

Now, using (4.9)-(4.10), we can derive (cf., e.g., [15]) that

\[
g_y(z; y, v, \sigma) = -\sigma g_v(z; y, v, \sigma) + (-\beta y v^{-\gamma-1} - \alpha v^{-\gamma}) g_\sigma(z; y, v, \sigma).
\]

(4.15)

It follows from (4.15) that \(K(y, v, \xi) \equiv 0\) and hence \(J_2 = 0\).

Using (4.12), we find that

\[
P(y, v, \sigma) \leq \exp\left[ -\left(\beta/2\right)y^2(v + \alpha y^2 v^{-\gamma} - \sigma y)^{-\gamma-1} \right]
\]

(4.16)
for \( y \in [0, \infty), v > 0, \sigma \leq 0 \). Since \( P(0, v, \sigma) \equiv 1 \), we have \( \Phi (0, v, \xi) = \xi \). Also, it follows from (4.16) that for \( \xi \leq 0 \) and \( v \in (0, \kappa) \)

\[
\| \Phi (y, v, \xi) \| = \left| \int_0^\xi P(y, v, \sigma) d\sigma \right| \leq |\xi| \exp \left[ - (\beta/2) g^2 (v + \alpha y^2 v^{-\gamma} - \xi y)^{-\gamma-1} \right],
\]

\[
|\Phi (y, v, \xi)| \leq \frac{\xi^2}{2} \exp \left[ - (\beta/2) g^2 (v + \alpha y^2 v^{-\gamma} - \xi y)^{-\gamma-1} \right] + \alpha y^{-\gamma} \kappa \exp \left[ - (\beta/2) g^2 (\kappa + \alpha y^2 v^{-\gamma})^{-\gamma-1} \right].
\]

Note that from (1.9) it follows that

\[
J_1 (s) = \Phi (x(s), v(s), \sigma) v_x(s, s) + \Phi (x, v(s), v_y(s, s)).
\]

Since \(-\kappa^2 < v_y < 0, 0 < v < \kappa\), and \( v_y \) is bounded by polynomial in \( y \), we have

\[
|J_1 (s)| \leq C \exp (-\lambda s^2)
\]

for some small \( \lambda > 0 \). Combining all the above estimates, we obtain

\[
\int_0^\infty \int_0^s P(y, v(y, s), v_y(y, s)) v^{-\gamma} - 1 y^2 v_1^2 dy ds < \infty.
\]

Taking any sequence \( \{ s_n \} \) with \( s_n \to \infty \) as \( n \to \infty \), by the standard arguments (e.g., [12]), we conclude that a subsequence of the sequence \( \{ v_n (y, s) := v (y, s + s_n) \} \)

corverges to the unique monotone decreasing positive global solution \( g^* (y) \) of (3.1)-(3.2) as \( n \to \infty \). Since this limit is independent of the choice of \( \{ s_n \} \), we conclude that \( v (y, s) \to g^* (y) \) as \( s \to \infty \) uniformly for any compact subset of \([0, \infty)\). This completes the proof of Theorem 4.1.

**REFERENCES**

[1] G.I. Barenblatt, *On some unsteady motions of a liquid or a gas in a porous medium*, Prikl. Mat. Mekh. **16** (1952), 67–78. (In Russian).

[2] M. Bertsch, M. Ughi, *Positivity properties of viscosity solutions of a degenerate parabolic equation*, Nonlinear Analysis, TMA **14** (1990), 571–592.

[3] M. Bertsch, R. Dal Passo, M. Ughi, *Discontinuous "viscosity" solutions of a degenerate parabolic equation*, Trans. Amer. Math. Soc. **320** (1990), 779–798.

[4] M. Chlebík, M. Fila, *Some recent results on the blow-up on the boundary for the heat equation*, Banach Center Publ. **52** (2000), 61–71.

[5] S.-N. Chow, J.K. Hale, “Methods of Bifurcation Theory”, Springer-Verlag, New York, 1982.

[6] E.A. Coddington, N. Levinson, “Theory of Ordinary Differential Equations”, Krieger Publishing Co., Florida, 1987.

[7] K. Deng, H.A. Levine, *The role of critical exponents in blow-up theorems: the sequel*, J. Math. Anal. Appl. **243** (2000), 85–126.

[8] R. Ferreira, A. de Pablo, J.D. Rossi, * Blow-up for a degenerate diffusion problem not in divergence form*, Indiana U. Math. J. (to appear).

[9] R. Ferreira, A. de Pablo, F. Quirós, J.D. Rossi, *Superfast quenching*, J. Differential Equations **199** (2004), 189–209.

[10] M. Fila, J. Filo, *Blow-up on the boundary: A survey*, Singularities and Differential Equations, Banach Center Publ. **33** (1996), 67–78.

[11] A. Friedman, J.B. McLeod, *Blow-up of positive solutions of semilinear heat equations*, Indiana Univ. Math. J. **34** (1985), 425–447.

[12] Y. Giga, R.V. Kohn, *Asymptotically self-similar blow-up of semilinear heat equations*, Comm. Pure Appl. Math. **38** (1985), 297–319.

[13] J.-S. Guo, Y.-J. Guo, J.-C. Tsai, *Single-point blow-up patterns for a nonlinear parabolic equation*, Nonlinear Analysis, TMA **53** (2003), 1149–1165.
[14] J.-S. Guo, Y.-J. Guo, C.-J. Wang, Global and non-global solutions of a nonlinear parabolic equation, Taiwanese J. Math. 9 (2005), 187–200.
[15] J.-S. Guo, Bei Hu, Blowup behavior for a nonlinear parabolic equation of nondivergence form, Nonlinear Analysis, TMA 61 (2005), 577–590.
[16] J.-S. Guo, Ph. Souplet, Fast rate of formation of dead-core for the heat equation with strong absorption and applications to fast blow-up, Math. Ann. 331 (2005), 651–667.
[17] C.W. Jones, On reducible non-linear differential equations occurring in mechanics, Proc. Roy. Soc. London Ser. A 217 (1953), 327–343.
[18] H.A. Levine, The role of critical exponents in blow-up theorems, SIAM Rev. 32 (1990), 262–288.
[19] A.A. Samarskii, V.A. Galaktionov, S.P. Kurdyumov, A.P. Mikhailov, “Blow-up in Quasi-linear Parabolic Equations”, Translated from the Russian by Michael Grinfeld, Walter de Gruyter, Berlin, 1995.
[20] M. Winkler, Propagation versus constancy of support in the degenerate parabolic equation $u_t = f(u)\Delta u$, Rend. Istit. Mat. Univ. Trieste 36 (2004), 1–15.
[21] T. I. Zelenjak, Stabilization of solutions of boundary value problems for a second order parabolic equation with one space variable, Differential Equations 4 (1968), 17–22.

E-mail address: jsguo@math.ntnu.edu.tw