

A 3-Variable Bracket

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Abstract

Kauffman’s bracket is an invariant of regular isotopy of knots and links which since its discovery in 1985 it has been used in many different directions: (a) it implies an easy proof of the invariance of (in fact, it is equivalent to) the Jones polynomial; (b) it is the basic ingredient in a completely combinatorial construction for quantum 3-manifold invariants; (c) by its fundamental character it plays an important role in some theories in Physics; it has been used in the context of virtual links; it has connections with many objects other objects in Mathematics and Physics. I show in this note that, surprisingly enough, the same idea that produces the bracket can be slightly modified to produce algebraically stronger regular isotopy and ambient isotopy invariants living in the quotient ring $R/I$, where the ring $R$ and the ideal $I$ are:

$$R = \mathbb{Z}[\alpha, \beta, \delta],\ I = <p_1, p_2>, $$

with $p_1 = \alpha^2\delta + 2\alpha\beta\delta^2 - \delta^2 + \beta^2\delta,\ p_2 = \alpha\beta\delta^3 + \alpha^2\delta^2 + \beta^2\delta^2 + \alpha\beta\delta - \delta$.

It is easy to prove that any pair of links distinguished by the usual bracket is also distinguishable by the new invariant. The contrary is not necessarily true. However, an explicit example of a pair of knots not distinguished by the bracket and distinguished by this new invariant is an open problem.

1 The brackets $\langle D \rangle$ and $[D]$ of a link diagram $D$

(1.1) Definition. Kauffman’s bracket maps a link diagram $D$ to $\langle D \rangle \in \mathbb{Z}[\alpha^{-1}, \alpha]$ and is characterized by the following properties:

$$(i)\ \langle \bigcirc \rangle = 1,\quad (ii)\ \langle D \cup \bigcirc \rangle = (-\alpha^{-2} - \alpha^2)\langle D \rangle,\quad (iii)\ \langle \bigotimes \rangle = \alpha\langle \bigotimes \rangle + \beta\langle \bigotimes \rangle.$$

In this definition, $\bigcirc$ is a diagram of the unknot with no crossing. $D \cup \bigcirc$ is a diagram consisting of the diagram $D$ together with an extra closed curve $\bigcirc$ that contains no crossing at all, either with itself or with $D$. In property (iii) the three link diagrams are the same except near the crossing where they are smoothed in the way shown. Observe that the crossing at the left of (iii) is not invariant under a 90° rotation and thus $A$ and $A^{-1}$ at its right can not be interchanged. The bracket polynomial of a link diagram with $n$ crossings can be calculated by expressing it as the sum of $2^n$ diagrams with no crossing by using (iii). By (i) and (ii) it follows that a link diagram with $k$ components without crossings has $(-\alpha^{-2} - \alpha^2)^{k-1}$ as its bracket polynomial.

As a matter of fact, the bracket is designed to be blind under Reidemeister move of type II, namely: $\langle \bigotimes \rangle = \langle \bigotimes \rangle$. Let the free variables $\alpha, \beta, \delta$ satisfy $\langle D \cup \bigcirc \rangle = \delta\langle D \rangle$ and $\langle \bigotimes \rangle = \alpha\langle \bigotimes \rangle + \beta\langle \bigotimes \rangle$. We seek restrictions on these variables to make $\langle \bigotimes \rangle = \langle \bigotimes \rangle$ true. Consider the expansion:

$$\langle \bigotimes \rangle = \alpha \langle \bigotimes \rangle + \beta \langle \bigotimes \rangle = \alpha^2 \langle \bigotimes \rangle + \alpha\beta \langle \bigotimes \rangle + \beta\alpha \langle \bigotimes \rangle + \beta^2 \langle \bigotimes \rangle = \alpha\beta \langle \bigotimes \rangle + (\alpha^2 + \beta^2 + \alpha\beta \delta) \langle \bigotimes \rangle.$$
By making (a) $\beta = \alpha^{-1}$ and (b) $\delta = -A^{-2} - A^2$ and normalizing it to satisfy $\langle \bigcirc \rangle = 1$, we get the bracket, which is insensitive under Reidemeister move II. A wonderful feature of this scheme is that invariance of the bracket under Reidemeister move III is for free and it behaves multiplicatively under Reidemeister moves I: $\langle \bigvee \rangle = -\alpha^3\langle \bigwedge \rangle$ and $\langle \bigtriangleup \rangle = -\alpha^3\langle \bigcap \rangle$. See Chapter 3 of Lickorish’s book [10] or the Kauffman’s original paper [3]. Because of the simple behavior of $\langle \bigcirc \rangle$ under move I, to obtain an ambient isotopy invariant of links just define $[D] = (-\alpha^3)^{-w(D)}\langle D \rangle$.

2 The brackets $\langle D \rangle_3$ and $[D]_3$ of a link diagram $D$

My inspiration for this work comes from King [3]. The basic observation is that the above assignments are too restrictive. Another line of thought provides invariance of moves II and III under less restrictive assumptions than (a) and (b). The idea is to consider the exterior of a crossing and fully expand it. Then we get a more intimate relationship among the three variables. In fact, after fully expanding the complement of a pair $m$ of crossings forming the left side of a move II in a link diagram $D$, there are well defined polynomials $D_1^m = (D \setminus m)_1(\alpha, \beta, \delta),\ D_2^m = (D \setminus m)_2(\alpha, \beta, \delta) \in \mathbb{Z}[\alpha, \beta, \gamma]$ such that the exterior is expressible as the sum of two 2-tangles:

$$Ext(m, D) = \langle \square \rangle_3 = D_1^m(\langle \bigvee \rangle_3 + D_2^m) \langle \bigcap \rangle_3.$$ 

Consider also the two 2-tangles coming from the expansion of $m$ minus the horizontal 2-tangle (the right side of move II):

$$\langle m \rangle - \langle \bigvee \rangle_3 - \langle \bigcap \rangle_3 = (\alpha \beta - 1)(\bigvee) + (\alpha^2 + \beta^2 + \alpha \beta \delta) \langle \bigcap \rangle_3.$$

By matching the corresponding output lines in the product of the above 2-tangles we get the polynomials which must be zero to make move II unseen by the new bracket:

$$0 = (\alpha \beta - 1)D_1^m(\langle \bigvee \rangle_3 + (\alpha \beta - 1)D_2^m(\langle \bigvee \rangle_3 + (\alpha^2 + \beta^2 + \alpha \beta \delta)D_1^m(\langle \bigvee \rangle_3 + (\alpha^2 + \beta^2 + \alpha \beta \delta)D_2^m = (\alpha \beta - 1)\delta^2 D_1^m + (\alpha \beta - 1)\delta D_2^m + (\alpha^2 + \beta^2 + \alpha \beta \delta)\delta D_1^m + (\alpha^2 + \beta^2 + \alpha \beta \delta)\delta D_2^m = [(\alpha \beta - 1)\delta^2 + (\alpha^2 + \beta^2 + \alpha \beta \delta)\delta]D_1^m + [(\alpha \beta - 1)\delta + (\alpha^2 + \beta^2 + \alpha \beta \delta)\delta^2]D_2^m.$$

Define

$$p_1 = (\alpha \beta - 1)\delta^2 + (\alpha^2 + \beta^2 + \alpha \beta \delta)\delta = \alpha^2 \delta + 2\alpha \beta \delta^2 - \delta^2 + \beta \delta,$$

and

$$p_2 = (\alpha \beta - 1)\delta + (\alpha^2 + \beta^2 + \alpha \beta \delta)\delta^2 = \alpha \beta \delta^3 + \alpha^2 \delta^2 + \beta \delta^2 + \alpha \beta \delta - \delta.$$ 

If $p_1$ and $p_2$ are zero, then move II is unseen by $\langle \bigcap \rangle_3$. The algebraic variety defined by the solution set of the polynomial system of equations consisting of these two polynomials has 33 branches (easily obtained with Mathematica [11]):

$$sol_1 = \{ \beta \to \frac{1}{\alpha}, \delta \to -\frac{\alpha^3 - 1}{\alpha} \},$$

$$sol_2 = \{ \delta \to -1, \beta \to \alpha - i \},$$

$$sol_3 = \{ \delta \to -1, \beta \to \alpha + i \},$$

$$sol_4 = \{ \delta \to 1, \beta \to -\alpha - 1 \},$$

$$sol_5 = \{ \delta \to 1, \beta \to 1 - \alpha \},$$

$$sol_6 = \{ \beta \to -\sqrt[3]{-1}, \delta \to -1, \alpha \to (1)^{5/6} \},$$

$$sol_7 = \{ \beta \to \sqrt[3]{-1}, \delta \to -1, \alpha \to (1)^{5/6} \},$$

$$sol_8 = \{ \beta \to -\sqrt[3]{-1}, \delta \to 1, \alpha \to (1)^{2/3} \},$$

$$sol_9 = \{ \beta \to \sqrt[3]{-1}, \delta \to -1, \alpha \to (1)^{2/3} \},$$

$$sol_{10} = \{ \beta \to -i - \sqrt[3]{-1}, \delta \to -1, \alpha \to -\sqrt[3]{-1} \},$$

$$sol_{11} = \{ \beta \to i + \sqrt[3]{-1}, \delta \to -1, \alpha \to -\sqrt[3]{-1} \},$$

$$sol_{12} = \{ \beta \to 2i - \sqrt[3]{-1}, \delta \to -1, \alpha \to (1)^{5/6} \},$$

$$sol_{12} = \{ \beta \to -i + \sqrt[3]{-1}, \delta \to -1, \alpha \to \sqrt[3]{-1} \},$$

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Proof. By introducing polynomials are:

\[ \text{sol}_{13} = \{ \beta \rightarrow i + \sqrt[3]{-1}, \delta \rightarrow -1, \alpha \rightarrow \sqrt{-1} \}, \]
\[ \text{sol}_{14} = \{ \beta \rightarrow -2i + \sqrt[3]{-1}, \delta \rightarrow -1, \alpha \rightarrow (-1)^{5/6} \}, \]
\[ \text{sol}_{15} = \{ \beta \rightarrow -1 - \sqrt[3]{-1}, \delta \rightarrow 1, \alpha \rightarrow \sqrt{-1} \}, \]
\[ \text{sol}_{16} = \{ \beta \rightarrow 1 - \sqrt[3]{-1}, \delta \rightarrow 1, \alpha \rightarrow \sqrt{-1} \}, \]
\[ \text{sol}_{17} = \{ \beta \rightarrow 2 - \sqrt[3]{-1}, \delta \rightarrow 1, \alpha \rightarrow (-1)^{2/3} \}, \]
\[ \text{sol}_{18} = \{ \beta \rightarrow -2 + \sqrt[3]{-1}, \delta \rightarrow 1, \alpha \rightarrow (-1)^{2/3} \}, \]
\[ \text{sol}_{19} = \{ \beta \rightarrow -1 + \sqrt[3]{-1}, \delta \rightarrow 1, \alpha \rightarrow -\sqrt{-1} \}, \]
\[ \text{sol}_{20} = \{ \beta \rightarrow 1 + \sqrt[3]{-1}, \delta \rightarrow 1, \alpha \rightarrow -\sqrt{-1} \}, \]
\[ \text{sol}_{21} = \{ \delta \rightarrow -1, \beta \rightarrow -2i, \alpha \rightarrow -i \}, \]
\[ \text{sol}_{22} = \{ \delta \rightarrow -1, \beta \rightarrow 2i, \alpha \rightarrow i \}, \]
\[ \text{sol}_{23} = \{ \delta \rightarrow -1, \beta \rightarrow i - \sqrt[3]{-1}, \alpha \rightarrow -\sqrt{-1} \}, \]
\[ \text{sol}_{24} = \{ \delta \rightarrow -1, \beta \rightarrow -i + \sqrt[3]{-1}, \alpha \rightarrow \sqrt{-1} \}, \]
\[ \text{sol}_{25} = \{ \delta \rightarrow -1, \beta \rightarrow i - (-1)^{5/6}, \alpha \rightarrow -(-1)^{5/6} \}, \]
\[ \text{sol}_{26} = \{ \delta \rightarrow -1, \beta \rightarrow -i + (-1)^{5/6}, \alpha \rightarrow -(-1)^{5/6} \}, \]
\[ \text{sol}_{27} = \{ \delta \rightarrow 1, \beta \rightarrow -2, \alpha \rightarrow 1 \}, \]
\[ \text{sol}_{28} = \{ \delta \rightarrow 1, \beta \rightarrow 2, \alpha \rightarrow -1 \}, \]
\[ \text{sol}_{29} = \{ \delta \rightarrow 1, \beta \rightarrow 1 - \sqrt[3]{-1}, \alpha \rightarrow \sqrt{-1} \}, \]
\[ \text{sol}_{30} = \{ \delta \rightarrow 1, \beta \rightarrow -1 + \sqrt[3]{-1}, \alpha \rightarrow -\sqrt{-1} \}, \]
\[ \text{sol}_{31} = \{ \delta \rightarrow 1, \beta \rightarrow -1 - (-1)^{2/3}, \alpha \rightarrow -(-1)^{2/3} \}, \]
\[ \text{sol}_{32} = \{ \delta \rightarrow 1, \beta \rightarrow 1 + (-1)^{2/3}, \alpha \rightarrow -(-1)^{2/3} \}, \]
\[ \text{sol}_{33} = \{ \delta \rightarrow 0 \}. \]

The first of these branches corresponds to the usual bracket. Let \( I \) be the ideal of \( R = \mathbb{Z}[\alpha, \beta, \delta] \) generated by \( p_1, p_2 \), that is, \( I = \langle p_1, p_2 \rangle \).

(2.1) Theorem. The class \( I + \langle D \rangle_3 \in R/I \) is a regular isotopy invariant of the link diagram \( D \).

Proof. The invariance under Reidemeister move II has been established in the above discussion, because the hypothesis makes the expression to become zero for arbitrary polynomials \( D^w(\alpha, \beta, \delta) \), \( i = 1, 2 \). The invariance under move III holds exactly as in the case of the simple bracket, see [10]. □

There exists a very simple Gröbner basis ([1]) for the ideal \( I \) in the lexicographic ordering of the monomials with the 3 variables taken in the order \( \alpha > \beta > \delta \) is given by \( \mathcal{G} = \langle q_1, q_2, q_3 \rangle \). The \( q \) polynomials are:

\[ q_1 = \delta^3\beta^4 - \delta^2\beta^4 + \delta^4\beta^2 - \delta^2\beta^2 + \delta^3 - \delta, \]
\[ q_2 = \beta\delta^4 + \beta^3\delta^3 + \alpha\delta^3 - \beta^2\delta - \beta^3\delta - \alpha\delta, \]
\[ q_3 = \delta\alpha^2 + 2\beta\delta^2\alpha - \delta^2 + \beta^2\delta. \]

Note that \( \langle q_1, q_2, q_3 \rangle = I = \langle p_1, p_2 \rangle \). An important property of Gröbner basis for an ideal \( J \) that each class \( J + p \) has a distinguished representative which depends on the Gröbner basis, on the monomial ordering and on an ordering of the variables ([1]). Such a normal form or reduced polynomial is very quick to compute. Henceforth we define \( \langle D \rangle_3 \) to mean this distinguished element of \( I + \langle D \rangle_3 \), modulo \( \mathcal{G} \) in the lexicographic monomial ordering relative to the choice \( \alpha > \beta > \delta \). The behavior of \( \langle D \rangle_3 \) under Reidemeister moves \( I \) are also fairly simple:

\[ \langle C \rangle_3 = \langle (\beta\delta^2 + \alpha\delta) \rangle_3 \]
and
\[ \langle \Theta \rangle_3 = \langle (\alpha\delta^2 + \beta\delta) \rangle_3 \]

(2.2) Theorem. Assume the link diagram \( D \) has non-negative writhe \( w \). Then, \( [D]_3 = \langle (\alpha\delta^2 + \beta\delta)^w D \rangle_3 \) is an invariant of the ambient isotopy class of \( D \). If the writhe is negative, then \( [D]_3 = \langle (\beta\delta^2 + \alpha\delta)^{-w} D \rangle_3 \) is an invariant of the ambient isotopy class of \( D \).

Proof. By introducing \( |w| \) curls of the opposite sign of \( \text{sgn}(w) \) we normalize the links to have writhe zero, and the Theorem follows. □

(2.3) Conjecture. There exist pairs of links distinguished by \( [D]_3 \) and not by \( |D| \).

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