The influence of atomic collisions on collective spontaneous emission from an $f$-deformed Bose–Einstein condensate

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Abstract

In this paper, we investigate the spontaneous emission of an $f$-deformed Bose–Einstein condensate of a gas with $N$ identical two-level atoms immersed in a single-mode ideal cavity with $s$ atoms initially excited. We apply an $f$-deformed quantum model in which Gardiner’s phonon operators are deformed by an operator-valued function $f(\hat{n})$ of the particle number operator $\hat{n}$. We consider the collisions between the atoms as a special kind of $f$-deformation where the collision rate $\kappa$ is regarded as a corresponding deformation parameter. The time evolution of the expectation value of the atomic inversion is presented, the phenomenon of collective collapses and revivals is shown and the effects of deformation on the cooperative behaviour of the system are discussed.

1. Introduction

The process of spontaneous emission of time-dependent quantum systems is one of the fundamental concepts of quantum mechanics. It is a manifestation of the dynamical interaction between matter and the electromagnetic vacuum and depends in part on its environment through the density of states and local strength of the electromagnetic modes. In 1950s, Purcell [1] discovered the enhancement of spontaneous emission rates of atoms when they were matched in a resonant cavity. It has been predicted [2] that photonic crystals, artificially created, three-dimensional arrangements of dielectric material that are periodic on a length scale comparable to the wavelength of light, can be used to modify spontaneous emission. The enhancement of spontaneous emission in finite-thickness Si photonic layer-by-layer crystals has been studied in [3]. Many experiments have been devoted to studies of spontaneous emission of molecules embedded in different photonic crystals [4, 5]. The effect of suppression of the spontaneous emission inside the stop-band (i.e. a forbidden gap actual for one direction) region has been experimentally demonstrated for the molecules embedded in the volume of photonic crystals [6]. In [7], controlling the dynamics of the spontaneous emission from semiconductor quantum dots embedded in inverse opal photonic crystals has been reported. In any case, controlling the spontaneous emission would provide an interesting relation between atomic physics and macroscopic electrodynamics.

The collective decay of an excited population of atoms via spontaneous emission of photons was first predicted by Dicke [8]. Several authors have studied the process of collective spontaneous emission in the Dicke model [9–18]. The exact solution for the spontaneous emission from $N$ identical two-level atoms prepared initially in a symmetrical Dicke state with only one atom excited has been discussed, in cases of both a perfectly tuned damped cavity [9] and a detuned damped cavity [10]. It has been shown that the super-radiant collective effects can be suppressed appreciably by enlarging the cavity detuning and number of atoms $N$, and the cavity detuning can be used for the regulation of the radiation rate in a damped cavity. Also, Cummings and Dorri [11] have presented an exact solution for spontaneous emission in the presence of $N$ atoms in a cavity with one atom initially inverted. They have considered nonsymmetrical initial excitation of the atomic system and have shown that the effect of radiation trapping holds for atoms in an equivalent as well as an inequivalent
The presence of initially excited two-level atoms radiate spontaneously in the cooperative spontaneous emission in the Dicke model when they are placed in a high-$Q$ single-mode cavity. They have found that for sufficiently small values of $c$, it is possible to neglect the losses. In this situation, the model becomes exactly solvable for $s \leq 8$ [17]. For $s = 3$, the complete solution for the time evolution of the atomic system has been presented in [18]. Moreover, for $s \geq 3$, the phenomenon of collective collapses and revivals has been discussed [17, 18]. The above-mentioned studies reveal the importance of further investigation of spontaneous emission in the Dicke model.

The theory of the interaction of many two-level atoms with a quantized resonator mode was originated from the work of Dicke [8]. Because the Bose–Einstein condensate (BEC) is a collection of $N$ atoms, the interaction of the BEC with a single mode of the electromagnetic field may be described conveniently by the Dicke model. However, there are also some differences between a BEC and the Dicke model. For example, a BEC is a macroscopically large number of identical bosons occupying a single quantum state, while this is not a necessary condition to describe an arbitrary many-atom system by the Dicke model. The central tool in the description of the BEC is the Bogoliubov approximation [19] in which the condensate operators are replaced by a $c$-number. This approximation destroys the conservation of the total atomic number. In order to preserve the conservation of the total atomic number, Gardiner [20] suggested a modified Bogoliubov approximation by introducing the exciton operators which satisfy an $f$-deformed commutation relation such that as $N \to \infty$, the standard bosonic commutation relation is regained. The case of a finite number of atoms has been recently investigated [21]. It provides a physical and natural realization of the $f$-deformed bosons by applying Gardiner’s phonon operators for the description of a BEC.

Since the fundamental works by Sklyanin [22], Kulish and Reshetikhin [23] and Drinfed [24] on algebraic models, a considerable amount of increasing interest has been devoted to the study of $q$-deformation and quantum groups [25–27]. Quantum groups appear in several research areas of physics and mathematics such as exactly solvable statistical models [28], non-commutative geometry [29], nuclear quantum many-body problems [30] and rational conform field theories [31]. The important property of quantum deformation is its relation to a specific type of nonlinearity [32, 33]. For example, $q$-deformed oscillators are interpreted as nonlinear oscillators with a specific type of nonlinearity, in which the frequency of vibration depends on the energy of vibration [32]. In the framework of a $q$-deformed boson, there have been solvable models which permit the applications of quantum algebras [28, 34]. In [34], the $q$-deformation of the quantum harmonic oscillator has been used to describe the generalized Jaynes–Cummings (JC) model and the time evolution of the expectation value of the atomic inversion has been calculated. It has been found that with the increasing value of the deformation parameter $q$, the destruction of periodic revivals of atomic inversion becomes more pronounced. There are other types of nonlinearity for which the frequency of oscillations varies with the amplitude by means of a generic function, $f(\hat n)$. Such oscillators may be called $f$-deformed oscillators [32]. The relation between the $f$-deformed radiation field and a nonlinear quantum optical process has also been studied [35]. Mancini and coworkers [36] have considered a physical important model of the interaction of a deformed radiation field with $N$ two-level atoms placed in a lossless single-mode cavity and have shown how the deformation affects the collective phenomenon. Recently, we have studied [37] the influence of $f$-deformation on light propagation in an $f$-deformed BEC. By considering the atomic collisions within the condensate as a special kind of $f$-deformation of atomic operators, we have shown the possibility of tunable control of the group velocity of a weak probe field propagating through a deformed BEC from subluminal to superluminal. In fact by applying the $f$-deformation to the atomic operators of the BEC medium, it is possible to obtain a large nonlinearity that leads to enhanced subluminal and superluminal propagation [37]. Now, it is useful to ask how the atomic collision phenomenon, as a special kind of $f$-deformation, affects the properties of spontaneous emission of a BEC in the framework of the $f$-deformed Dicke model.

In this paper, we are interested in spontaneous emission by an $f$-deformed BEC of $N$ two-level atoms placed in a lossless single-mode cavity when a part $s$ of $N$ atoms radiates spontaneously in the presence of $N - s$ unexcited atoms. The system under consideration is an $f$-deformed BEC of a gas with two-level atoms in which Gardiner’s phonon operators for a BEC are deformed by an operator-valued function $f(\hat n)$. By considering the effect of collisions between the atoms within the condensate, we analyse the properties of collective spontaneous emission and show that the nonlinearities due to $f$-deformation affect the collective phenomenon.

The scheme of the paper is as follows. In section 2, we present our model and by using the $f$-deformed algebra, we study the BEC with a large but finite number of atoms. We show that a physical and natural realization of the $f$-deformed boson is provided by using Gardiner’s phonon operators for a description of the BEC. The model is an intrinsically deformed one in which the deformation parameter is determined by the total number of atoms $N$. We show that the atomic collisions within the condensate can be regarded as an extra deformation on the intrinsically deformed Gardiner’s phonon operators for the BEC. In section 3, we study the time evolution of the expectation value of the atomic inversion for the lower three values of $s$ ($s = 1, 2, 3$). Finally, we summarize our results in section 4.
2. The physical model and \( f \)-deformed bosonic algebra for Gardiner’s phonon operators

We consider a system consisting of a weakly interacting BEC of two-level atoms coupled to a single-mode cavity, when a part \( s = N \) atoms initially prepared in excited states radiate spontaneously in the presence of \( N - s \) unexcited atoms. The Hamiltonian in the rotating-wave approximation is given by

\[
\hat{H} = \hat{H}_0 + \hat{V},
\]

\[
\hat{H}_0 = \omega_0 \hat{\alpha}^{\dagger} \hat{\alpha} + \alpha \hat{S}_3,
\]

\[
\hat{V} = g (\hat{\alpha}^{\dagger} \hat{\delta}_{-} + \tilde{\alpha} \hat{\delta}_{+}),
\]

where the photon creation and annihilation operators \( \hat{\alpha}^{\dagger} \) and \( \tilde{\alpha} \) satisfy the standard bosonic commutation relation (\( [\hat{\alpha}, \hat{\alpha}^{\dagger}] = 1 \)). The operators \( \hat{S}_3 = \sum_{j=1}^{N} \hat{S}_{3j} \) and \( \hat{S}_1 = \sum_{j=1}^{N} \hat{S}_{1j} \) describe the total dipole moment for the atoms in the BEC (\( \hat{S}_{3j}, \hat{S}_{1j} \) are pseudospin lowering, raising and inversion operators of the \( j \)th atom, respectively), \( \omega_0 \) denotes the frequency of the field mode and \( \omega \) is the atomic transition frequency. In the small sample approximation, the coupling constant \( g \) is the same for all the atoms. Furthermore, we assume exact resonance and choose the scale in a way that \( \omega_0 = \omega = 1 \).

The basis states of the system under consideration are as follows [16–18]:

\[
|s, m\rangle = |s - m\rangle_a \otimes |m\rangle_t,
\]

where \( |m\rangle_t \) denotes the Fock state of the field and \( |s - m\rangle_a \) is the normalized symmetric Dicke state [9, 10, 38] of the atomic subsystem with \( s - m \) atoms excited. The initial condition corresponds to \( |s, 0\rangle \), i.e., \( s \) atoms are initially excited and no photon is present.

Now, we consider the second quantization in the atomic degrees of freedom of the above model. Let \( \hat{b}_{(e/g)}^{(a)} \) and \( \hat{b}_{(e/g)}^{(c)} \) denote the annihilation and creation operators of the atoms in the excited (ground) state, respectively. Therefore, the Hamiltonian (1) can be written as

\[
\hat{H} = (\hat{\alpha}^{\dagger} \hat{\alpha} + \hat{b}_{e}^{\dagger} \hat{b}_e - \hat{b}_{c}^{\dagger} \hat{b}_c) + g (\hat{\alpha}^{\dagger} \hat{b}_e^{\dagger} \hat{b}_c + \text{h.c.}).
\]

The above Hamiltonian, the total number of atoms \( \hat{N} = \hat{b}_e^{\dagger} \hat{b}_e + \hat{b}_c^{\dagger} \hat{b}_c \) is conserved (\( \{\hat{N}, \hat{H}\} = 0 \)). In order to deal with the dynamics of the BEC, the Bogoliubov approximation [19] is an efficient approach, in which the ground-state operators \( \hat{b}_{e}, \hat{b}_{c} \) are replaced by a \( c \)-number \( \sqrt{\hat{N}_c} \), where \( \hat{N}_c \) is the average number of the initial condensate atoms. In this case, the Hamiltonian (3) describes a system of two coupled harmonic oscillators:

\[
\hat{H}_b = (\hat{\alpha}^{\dagger} \hat{\alpha} + \hat{b}_e^{\dagger} \hat{b}_e + \hat{b}_c^{\dagger} \hat{b}_c + \sqrt{\hat{N}_c} (\hat{\alpha}^{\dagger} \hat{b}_c + \text{h.c.}).
\]

However, this approximation destroys the conservation of the total number of atoms (\( \{\hat{N}, \hat{H}_b\} \neq 0 \)). In order to preserve the property of the initial model, we consider the following Gardner’s phonon operators [20]:

\[
\hat{b}_q = \frac{1}{\sqrt{\hat{N}_c}} \hat{b}_c, \quad \hat{b}_q^{\dagger} = \frac{1}{\sqrt{\hat{N}_c}} \hat{b}_c^{\dagger}.
\]

These operators obey the deformed commutation relation

\[
[\hat{b}_q, \hat{b}_q^{\dagger}] = 1 - \frac{2}{\hat{N}_c} \hat{b}_c^{\dagger} \hat{b}_c = 1 - 2\eta \hat{b}_e^{\dagger} \hat{b}_e,
\]

with \( \eta = \frac{1}{\sqrt{\hat{N}_c}} \). When \( \eta \to 0 \) (\( \hat{N} \to \infty \)), the standard (nondeformed) bosonic commutation relation is regained. In general, the deformed operator \( \hat{b}_q \) is related to a nondeformed operator \( \hat{b} \) through an operator-valued function \( f_i \) as

\[
\hat{b}_q = \hat{b} f_1 (\hat{b}^{\dagger} \hat{b}_e; \eta).
\]

In our particular case, we have

\[
f_1 (\hat{b}^{\dagger} \hat{b}_e; \eta) = \sqrt{1 - \eta (\hat{b}^{\dagger} \hat{b}_e - 1)}.
\]

Now, we intend to illustrate the effect of atomic collisions within the condensate as a special kind of \( f \)-deformation. For this purpose, we first make a brief review on an \( f \)-deformed oscillator algebra in the following subsection.

2.1. \( f \)-Deformed oscillator algebra

The \( q \)-deformed boson algebra is realized by defining the \( q \)-deformed creation and annihilation operators \( \hat{A}^{\dagger}, \hat{A} \) and the number operator \( \hat{N} \) which satisfy the commutation relations

\[
[\hat{N}, \hat{A}] = -\hat{A}, \quad [\hat{N}, \hat{A}^{\dagger}] = \hat{A}^{\dagger}
\]

and the nonlinear relations

\[
\hat{A}^{\dagger} \hat{A} = [\hat{N}], \quad \hat{A}^{\dagger} \hat{A} = [\hat{N} + 1],
\]

where the notation \( [x] \) is defined as

\[
[x] = \frac{q^x - q^{-x}}{q - q^{-1}},
\]

and \( q \) is the deformation parameter.

A four-parameter generalized \( q \)-deformed Heisenberg–Weyl algebra has been introduced in [39]:

\[
\hat{A}^{\dagger} \hat{A} - q^{\alpha \beta} \hat{A}^{\dagger} \hat{A} = q^{\alpha \beta} \hat{A}^{\dagger} \hat{A} - q^{\alpha \beta},
\]

where \( \alpha, \beta \) and \( \gamma \) are real parameters. A realization for the creation and annihilation operators \( \hat{A}^{\dagger}, \hat{A} \), in terms of the standard creation and annihilation operators \( \hat{a}^{\dagger}, \hat{a} \) by means of a nonlinear function \( f (\hat{N}) \) specific to each \( q \)-deformed oscillator algebra, has been introduced in [40, 41]:

\[
\hat{A} = \hat{a} f (\hat{N}), \quad \hat{A}^{\dagger} = f^{\dagger}(\hat{N}) \hat{a}^{\dagger}, \quad \hat{N} = \hat{a}^{\dagger} \hat{a}.
\]

We can show that for all \( m \in \mathbb{N} \),

\[
\hat{A}(\hat{A}^{\dagger})^m = q^{m \gamma} (\hat{A}^{\dagger})^m \hat{A} = F^{(\gamma)}_{\dagger}(m; q) (\hat{A}^{\dagger})^{m-1} q^{\alpha \hat{N}},
\]

where
with
\[ F_{\alpha,\beta}(m; q) = \begin{cases} \frac{q^\alpha q^m - q^\gamma}{q^\mu - q^\gamma}, & \alpha \neq \gamma, \\ m^\beta(\rho^{(m-1)}), & \alpha = \gamma. \end{cases} \] (16)

By using the same method as in [42], we obtain the following expression for \( f(\tilde{N}) \):
\[ f(\tilde{N}) = \begin{cases} \frac{\sqrt{\tilde{N}} q^\alpha \tilde{N} - q^\gamma \tilde{N}}{q^\mu - q^\gamma}, & \alpha \neq \gamma, \\ q^{[\beta + (\beta-1)]/2}, & \alpha = \gamma. \end{cases} \] (17)

As an example, we consider the Hamiltonian for the free \( f \)-oscillator (\( \hbar = 1 \)):
\[ \hat{H}(\tilde{N}) = \frac{\alpha_0}{2}\left(\hat{A}^\dagger \hat{A} + \hat{A} \hat{A}^\dagger\right) 
= \frac{\alpha_0}{2}\left[|f(\tilde{N} + 1)|^2(\tilde{N} + 1) + |f(\tilde{N})|^2 \tilde{N}\right]. \] (18)

By introducing new deformation parameters \( q = e^\gamma, \alpha = \nu + \mu, \gamma = \nu - \mu \), and by using equation (17), the Hamiltonian of the free \( f \)-oscillator can be written as
\[ \hat{H} = \frac{\alpha_0}{2} e^{(\nu + \mu) \tilde{N}} \left\{ \frac{\sinh(\tau \mu \tilde{N} + 1)}{\sinh(\tau \mu)} + e^{-(\nu \mu)} \frac{\sinh(\tau \mu \tilde{N})}{\sinh(\tau \mu)} \right\}. \] (19)

We shall apply these results in the following subsection to illustrate the atomic collisions within the BEC as a specific kind of \( f \)-deformation.

2.2. An example: the atomic collisions effect

As a particular physical example, we consider the effect of collisions between the atoms within the condensate. The effective interaction Hamiltonian contains a nonlinear term proportional to \( (\hat{b}^\dagger \hat{b})^2 \) [43]:
\[ \hat{H}_I = \frac{\kappa}{2} (\hat{b}^\dagger \hat{b})^2, \] (20)
where the collision rate is denoted by \( \kappa \). In sensible experimental situation, for a condensate composed of rubidium-87 atoms at a temperature of 180 nK, with a density \( \rho = 10^{13} \text{ cm}^{-3} \) [44], we can estimate the collision rate to be \( \kappa \approx \rho \pi a^2 v_{\text{rms}} \), where \( a \) is the scattering length and \( v_{\text{rms}} \) is the root-mean square speed of the rubidium atoms. We obtain a collision rate of about one collision per second for these parameters, while we can adjust the value of the collision rate \( \kappa \) by changing the temperature. By expanding the Hamiltonian (18) and considering small values of \( \nu \) and \( \mu \), we obtain
\[ \hat{H}(\tilde{N}) \approx \frac{\alpha_0}{2} \left[ (2\tilde{N} + 1) + \frac{1}{6} \mu^2 \tilde{N} + \left(\frac{1}{2} \mu^2 + 2\nu \right) \tilde{N}^2 \right. 
+ O(\nu^2, \nu^2 \mu^2, \mu^4) \right]. \] (21)

where the interaction Hamiltonian reads as
\[ \hat{H}_I(\tilde{N}) \approx \frac{\alpha_0}{2} \left[ \left(\frac{1}{6} \mu^2 \tilde{N} + \left(\frac{1}{2} \mu^2 + 2\nu \right) \tilde{N}^2 \right) \right]. \] (22)

The Hamiltonian (22) reproduces the Hamiltonian (20) by setting \( \mu^2 = 0, \nu = \frac{e^\gamma - 1}{e^\gamma + 1} \) and \( \tilde{N} = \hat{b}^\dagger \hat{b} \). Thus, we see that the collision effect transforms the standard (nonlinear) harmonic oscillator model into an \( f \)-deformed one. Alternatively, we could set, \( ab \text{ initio} \), in equation (18), \( f(\tilde{N}) = \sqrt{\kappa \tilde{N} + (1 - \kappa)} \) with \( \kappa' = \frac{\kappa}{\kappa_0} \) (by choosing the scale in such a way \( \omega_0 = 1 \), we get \( \kappa' = \frac{\kappa}{\kappa_0} \)) to obtain the Hamiltonian (20). Therefore, the parameter of the generalized deformed algebra is related to the rate of atomic collisions \( \kappa \).

Subsequently, by considering the effects of collisions between the atoms within the condensate, we can apply the extra deformation to the intrinsically deformed Gardiner’s phonon operators for the BEC by an operator-valued function \( f_{\lambda}(\tilde{N}) = \sqrt{\kappa \tilde{N} + (1 - \kappa)} \) of the particle number operator \( \hat{n} \). Here, the nonlinearity is related to the collisions between the atoms within the condensate. The operator-valued function \( f_{\lambda}(\tilde{N}) \) reduces to 1 as soon as \( \kappa \to 0 \). It means that the deformation increases with the collision rate \( \kappa \). Therefore, the deformed version of the interaction Hamiltonian \( \hat{V} \) in equation (1) can be written as
\[ \hat{V} = g \sqrt{\tilde{N}_c} (\hat{a}^\dagger \hat{B} + \text{h.c.}), \] (23)
where
\[ \hat{B} = \hat{b}_{\lambda} f_{\lambda}(\tilde{N}) = \hat{b}_{\lambda} \sqrt{1 + \kappa'(\tilde{N}_{\lambda} - 1)}, \]
\[ \hat{B}^+ = f_{\lambda}^*(\tilde{N}) \hat{b}_{\lambda}^+ = \sqrt{1 + \kappa'(\tilde{N}_{\lambda} - 1)} \hat{b}_{\lambda}^*, \] (24)
\[ \hat{n}_{\lambda} = \hat{b}_{\lambda}^\dagger \hat{b}_{\lambda}. \]

By assuming that all the atoms are initially in the condensate \( (N = \tilde{N}_c) \) and by using equation (24), the deformed Hamiltonian (23) can be expressed in terms of the nondeformed operators \( \hat{b}^\dagger \) and \( \hat{b} \) as follows:
\[ \hat{V} = k \left[ \hat{a}^\dagger \hat{b} \sqrt{1 + \kappa'} \left( \hat{b}^\dagger \hat{b} - \frac{1}{\tilde{N}} \hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b} - 1 \right) 
- \frac{\kappa}{2N} \hat{b}^\dagger \hat{b}^\dagger \hat{b} \sqrt{1 + \kappa'} \left( \hat{b}^\dagger \hat{b} - \frac{1}{\tilde{N}} \hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b} - 1 \right) + \text{h.c.} \right], \] (25)
with \( k = g \sqrt{\tilde{N}} \).

It is evident that equation (25) describes the photon–atom interactions. We see that the Hamiltonian (25) reproduces the Hamiltonian (1c) by setting both \( \kappa \) and \( \eta \) (=1/\tilde{N}) equal to zero.

3. Time evolution of the atomic inversion

In this section, for simplicity, we consider the case of the lower three values of \( s \). For \( s = 1 \), the wavefunction of the system under consideration can be written in the form
\[ |\psi(t)\rangle = C_0(t)|1, 1\rangle + C_1(t)|1, 0\rangle, \] (26)
where \( C_0 \) and \( C_1 \) are the probability amplitudes for finding the system in states \( |1, 1\rangle \) and \( |1, 0\rangle \) respectively. By using the Schrödinger equation in the interaction picture \( i\frac{d}{dt}\langle \psi| = \hat{H}|\psi\rangle \), the equations of motion for the amplitudes \( C_0 \) and \( C_1 \)
with the initial condition $C_1(t = 0) = 1$ may be written as

$$i \frac{\partial}{\partial t} C_0 = k C_1,$$

(27a)

$$i \frac{\partial}{\partial t} C_1 = k C_0.$$

(27b)

We see that the $f$-deformation does not affect the oscillatory solutions. In the case of $s = 2$, the total wavefunction of the system reads as

$$|\psi(t)\rangle = C_0(t)|2, 2\rangle + C_1(t)|2, 1\rangle + C_2(t)|2, 0\rangle.$$  (28)

The equations of motion for the probability amplitudes $C_i(t), i = 0, 1, 2$, with the initial condition $C_2(t = 0) = 1$ read as

$$i \frac{\partial}{\partial t} C_0 = \sqrt{2} k C_1,$$

(29a)

$$i \frac{\partial}{\partial t} C_1 = \sqrt{2} k C_0 + k \left(1 - \frac{1}{2N}\right) \sqrt{2 + 2 \kappa'} (1 - \frac{2}{N}) C_2,$$

(29b)

$$i \frac{\partial}{\partial t} C_2 = k \left(1 - \frac{1}{2N}\right) \sqrt{2 + 2 \kappa'} (1 - \frac{2}{N}) C_1.$$  (29c)

The solutions for the probability amplitudes are given by

$$C_0(t) = \sqrt{2} k^2 \left(1 - \frac{1}{2N}\right) \sqrt{2 + 2 \kappa'} \left(1 - \frac{2}{N}\right) \frac{\cos(\Omega t) - 1}{\Omega^2},$$

(30a)

$$C_1(t) = -i k \left(1 - \frac{1}{2N}\right) \sqrt{2 + 2 \kappa'} \left(1 - \frac{2}{N}\right) \frac{\sin(\Omega t)}{\Omega},$$

(30b)

$$C_2(t) = k^2 \left(1 - \frac{1}{2N}\right)^2 \left[2 + 2 \kappa' (\frac{2}{N})\right] \cos(\Omega t) + 2 k^2.$$

(30c)

where the generalized Rabi frequency $\Omega$ is defined as

$$\Omega = k \sqrt{2 + \left(1 - \frac{1}{2N}\right)^2 \left[2 + 2 \kappa' (\frac{2}{N})\right]}.$$  (31)

The expectation value of the collective operator $\hat{S}_3$ is obtained by using the standard formula [18]

$$\langle s, m | \hat{S}_3 | s, m \rangle = -\frac{N}{2} + s - m.$$  (32)

This represents the inversion of the atomic energy of the whole system of $N$ atoms. By using the operator $\hat{S}_3 = \hat{S}_3 + \frac{s}{2} - \frac{1}{2}$, we obtain the inversion of the atomic energy for the group of $s$ atoms [18]:

$$\langle s, m | \hat{S}_3 | s, m \rangle = \frac{s}{2} - m.$$  (33)

The time evolution of the expectation value of the atomic inversion can be written as

$$E_{at}^s(t) = \langle s, 0 | e^{i \hat{H} t} \hat{S}_3 e^{-i \hat{H} t} | s, 0 \rangle.$$  (34)

For the case of $s = 2$, we get

$$E_{at}^{s=2}(t) = C_2^2(t) - C_0^2(t).$$  (35)
the energy between the atoms and the radiation field in the cavity. This result becomes clear from equation (31). To examine the influence of the atomic collisions on the collective spontaneous emission, we plot the atomic inversion \( E_{\text{at}}(t) \) as a function of \( g t \) for three different values of the deformation parameter \( \kappa' \) in figure 2. We set \( N = 100 \) and investigate the effects of the atomic collisions within the condensate. As is seen, the \( f \)-deformation reduces the period of oscillations and the minimum of the amplitude of these oscillations tends towards the upper (non-zero) values. This means that, due to the deformation, there are no times at which all initially excited atoms release photons. In the case of extremely high deformation (low number of atoms \( N \) and strong atomic collision effect) due to the strongly nonlinearity, the period of oscillations tends to zero; hence, the inversion of the group of \( s \) atoms tends to remain constant \( \frac{1}{2} \) and beating of the energy between the atoms and radiation field disappears. These results are in agreement with those given in [36].

For the case of \( s = 3 \), the wavefunction becomes

\[
|\psi(t)\rangle = C_0(t)|3, 3\rangle + C_1(t)|3, 2\rangle + C_2(t)|3, 1\rangle + C_3(t)|3, 0\rangle.
\] (36)

By using the corresponding Schrödinger equation, we get

\[
i \frac{\partial}{\partial t} C_0 = \sqrt{3}k C_1,
\] (37a)

\[
i \frac{\partial}{\partial t} C_1 = \sqrt{3}k C_0 + 2k \left(1 - \frac{1}{2N}\right) \sqrt{1 + \kappa' \left(1 - \frac{2}{N}\right)} C_2,
\] (37b)

\[
i \frac{\partial}{\partial t} C_2 = 2k \left(1 - \frac{1}{2N}\right) \sqrt{1 + \kappa' \left(1 - \frac{2}{N}\right)} C_1
\]

\[
+ \sqrt{3}k \left(1 - \frac{1}{N}\right) \sqrt{1 + \kappa' \left(2 - \frac{6}{N}\right)} C_3,
\] (37c)
The energy inversion referred to the group of three atoms is

\[ E_{s=3}^{\text{at}}(t) = \frac{3}{2} - 3|C_0(t)|^2 - 2|C_1(t)|^2 - |C_2(t)|^2. \]  

(38)

In the following, we discuss the time evolution of the system with \( s = 3 \) for different values of the deformation parameter \( \kappa' \) and the total number of atoms \( N \). The case of \( s = 3 \) is characterized by inequidistant eigenvalue spectra [45]. Since the system oscillates with different Rabi frequencies, the phenomenon of quantum beats leads to modulated oscillations and the phenomenon of collective collapses and revivals of the oscillations of the system manifests itself. We demonstrate the effects of the total number of atoms \( N \) and the atomic collision phenomenon on the spontaneous emission in the case of \( s = 3 \).

Figure 5. (a) The oscillations of the atomic inversion before and after the first collapse region for \( s = 3 \) and the total number of atoms \( N = 50 \) in the case of no collisions (\( \kappa' = 0 \)). (b) The envelope of the atomic inversion for \( s = 3 \) and the total number of atoms \( N = 50 \) in the case of no collisions (\( \kappa' = 0 \)).

\[ \frac{\partial}{\partial t} C_3 = \sqrt{3k} \left( 1 - \frac{1}{N} \right) \left( 1 + \kappa' \left( 2 - \frac{6}{N} \right) \right) C_2. \]  

(37d)

The energy inversion referred to the group of three atoms during the first collapse are shown. The envelope of the atomic inversion for \( s = 3 \), \( N = 10 \) and \( \kappa' = 0 \) is presented in figure 3(b). Figure 4(a) gives the form of the oscillations in the first collapse region and in its vicinity for \( s = 3 \), \( N = 20 \) and \( \kappa' = 0 \). The corresponding envelope of the oscillations is presented in figure 4(b). In figures 5(a) and (b), we show the oscillations during the first collapse and envelope of the oscillations for \( s = 3 \), \( N = 50 \) and \( \kappa' = 0 \), respectively.

In the case of \( s = 3 \), characterized by nonequidistant spectra of the eigenvalues of the Hamiltonian, the phenomenon of collective collapses and revivals appears and it tends to disappear with an increasing deformation parameter \( \eta = \frac{1}{N} \). A comparison of figures 3–5 shows that an increasing deformation parameter \( \eta = \frac{s}{N} \) gives shorter revival and collapse periods. This result agrees with [18]. In figures 6–8, we study the effect of collisions within the condensate in the case of \( s = 3 \) and \( N = 100 \). Figures 6(a) and (b), respectively, give the form of the oscillations and envelope of the oscillations during the first collapse.
for $s = 3$, $N = 100$ and $\kappa' = 0$. The analogues of figure 6 are shown in figure 7 for $s = 3$, $N = 100$, $\kappa' = 0.5$ and in figure 8 for $s = 3$, $N = 100$, $\kappa' = 1$. From a comparison of figures 6–8, it can be seen that the nonlinearity described by $f$-deformation influences the properties of spontaneous emission of the BEC. In fact, by introducing the $f$-deformation (atomic collisions effect) the phenomenon of collapses and revivals occurs. However, with increasing the deformation parameter $\kappa'$ the periods of collapses and revivals reduce and the collapses and revivals tend to disappear. From numerical calculation of the atomic population inversion in the case of $s = 3$ for different values of $\eta$ and $\kappa'$ (figures 3–8), it can be concluded that the phenomenon of collective collapses and revivals occurs with different periods if the atomic operators of the BEC are deformed. The periodic collapses and revivals of the model are destroyed more for increasing deformation parameters $\eta$ and $\kappa'$, and by increasing the strength of the deformation a larger departure from perfect revivals is seen.

4. Summary and conclusions

In summary, we have studied the properties of spontaneous emission of an $f$-deformed BEC of $N$ identical two-level atoms in an ideal cavity in the framework of the $f$-deformed boson model. We have seen that the particle-number conservation in the BEC requires a deformation on the bosonic field, where the corresponding deformation parameter is defined by the total number of atoms $N$. We have considered the collision effect within the condensate, as a special kind of $f$-deformation for which the collision rate $\kappa$ is regarded as the corresponding deformation parameter. These deformations introduce new parameters $\eta = \frac{1}{N}$ and $\kappa' = \frac{\kappa}{2}$ into the Dicke Hamiltonian. By using analytic and numerical results, the effects of deformation parameters $\eta$ and $\kappa'$ on the collective spontaneous emission have been discussed. We have found that the presence of the deformation parameters $\eta$ and $\kappa'$ introduces nonlinearities, which may lead to inhibition of
the collective spontaneous emission. In fact, the collective spontaneous emission can be suppressed by increasing the deformation parameters $\eta$ and $\kappa'$. We have investigated the time evolution of the expectation value of the atomic inversion. For the case $s = 3$, the phenomenon of collapses and revivals occurs, if we apply the deformation on the atomic operators of the BEC medium. The collective collapses and revivals are very sensitive to the total number of atoms $N$ and the atomic collision phenomenon. The periods of the collapses and revivals become shorter with growing deformation parameters $\eta$ and $\kappa'$ and they tend to disappear by increasing the strength of the deformation. We have considered the case of the lower three values of $s$ ($s = 1, 2, 3$) for simplicity. We expect that by increasing the value of $s$ ($s > 3$), the many-body correlation effects would be enhanced. We guess that in the case of ($s > 3$), the system oscillates with different frequencies leading to their collapses and revivals and increasing the strength of the deformation gives shorter collapse and revival periods.

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