FOURIER TRANSFORM AND RIGIDITY OF CERTAIN DISTRIBUTIONS

BINYONG SUN AND CHEN-BO ZHU

Abstract. Let \( E \) be a finite dimensional vector space over a local field, and \( F \) be its dual. For a closed subset \( X \) of \( E \), and \( Y \) of \( F \), consider the space \( D^{-\xi}(E;X,Y) \) of tempered distributions on \( E \) whose support are contained in \( X \) and support of whose Fourier transform are contained in \( Y \). We show that \( D^{-\xi}(E;X,Y) \) possesses a certain rigidity property, for \( X, Y \) which are some finite unions of affine subspaces.

1. Introduction and main result

One of the most well-known results in Euclidean harmonic analysis is the uncertainty principle. As a meta-theorem, it states that a nonzero function and its Fourier transform cannot both be sharply localized. For an insightful survey of various manifestations of this principle, see the article of G. Folland and A. Sitarams [FS].

We fix a finite dimensional vector space \( E \) and its dual \( F \). For a closed subset \( X \) of \( E \), and \( Y \) of \( F \), consider the space \( D^{-\xi}(E;X,Y) \) of tempered distributions on \( E \) whose support are contained in \( X \) and support of whose Fourier transform are contained in \( Y \). The general thrust of the current note is to examine to what extent one is able to separate the support and support of the Fourier transform for distributions in \( D^{-\xi}(E;\cup X, \cup Y) \), where \( X \) and \( Y \) are finite sets of affine subspaces. This is the meaning of rigidity in the title, which the authors consider as another manifestation of the uncertainty principle.

One key observation, which likely has been noted by others before us, is the general importance of relative position of the pair \((X,Y)\), now assumed to be affine subspaces. As it turns out, there will be three different circumstances, which we respectively call thin, perfect, thick. As an indication of the relevance of these concepts, we have: (a) \( D^{-\xi}(E;X,Y) = 0 \) if \((X,Y)\) is a thin pair; (b) \( D^{-\xi}(E;X,Y) \) can be explicitly described in terms of a linear basis if \((X,Y)\) is a perfect pair. This is

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similar to the classical result of L. Schwartz on the structure of distributions supported on a single point; (c) If \((X,Y)\) is a thick pair, then \(\mathcal{D}^{-\xi}(E;X,Y)\) contains (in a non-canonical fashion) the space of tempered distributions on a nonzero subspace of \(E\) and is thus not rigid in any reasonable sense.

Our main result (Theorem A in this section) is that \(\mathcal{D}^{-\xi}(E;\cup\mathcal{X},\cup\mathcal{Y})\) possesses the afore-mentioned rigidity property as long as there is no pair \((X,Y)\in \mathcal{X} \times \mathcal{Y}\) which is thick. Very roughly the idea goes as follows: By applying a good multiplier operator (a suitable function which vanishes on a part of the support of the Fourier transform), one may cut off that part of the support in the Fourier transform side. At the distribution side, the process will generally yield a distribution with additional support. If no thick pairs are involved, then this process can be carried out in such a way that the additional support is very much controlled.

The result of this note was motivated by certain representation-theoretic issues arising from the proof of archimedean multiplicity-one theorems [SZ]. More specifically the rigidity statement of Theorem A allows us to establish the semi-simplicity and non-negativity of an Euler vector field, crucial in certain reduction step involving the so-called distinguished nilpotent orbits. For applications to representation theory of algebraic groups over non-archimedean local fields, we will state (and prove) our results over an arbitrary local field, rather than over \(\mathbb{R}\) or \(\mathbb{C}\). We apologize to readers interested in harmonic analysis who may view these generalities as unnecessarily cumbersome.

We now introduce necessary notation for this note.

Let \(k\) be an arbitrary local field, and let \(\psi : k \to \mathbb{C}^\times\) be a fixed nontrivial unitary character. Let \(E\) be a finite dimensional \(k\)-vector space. Denote by

\[
C^\times(E) \subset C^{-\xi}(E), \quad D^\varsigma(E) \subset D^{-\xi}(E)
\]

the (complex) spaces of Schwartz functions, tempered generalized functions, Schwartz densities, and tempered distributions on \(E\), respectively. Thus \(C^{-\xi}(E)\) (resp., \(D^{-\xi}(E)\)) is the dual of \(D^\varsigma(E)\) (resp., \(C^\varsigma(E)\)).

Let \(F\) be another finite-dimensional \(k\)-vector space which is dual to \(E\), i.e., a non-degenerate bilinear map

\[
\langle , \rangle : E \times F \to k
\]

is given. The Fourier transform

\[
D^\varsigma(F) \to C^\varsigma(E)
\]

\[
\omega \mapsto \hat{\omega}
\]

is the linear isomorphism given by

\[
\hat{\omega}(x) := \int_F \psi(\langle x, y \rangle) \omega(y), \quad x \in E.
\]
Dually for every $D \in D^{-\xi}(E)$, its Fourier transform $\hat{D} \in C^{-\xi}(F)$ is given by

$$\hat{D}(\omega) := D(\hat{\omega}), \quad \omega \in D^{\xi}(F).$$

For every closed subset $X$ of $E$, and $Y$ of $F$, denote

$$D^{-\xi}(E; X) := \{ D \in D^{-\xi}(E) \mid \text{supp}(D) \subset X \},$$

and

$$D^{-\xi}(E; X, Y) := \{ D \in D^{-\xi}(E; X) \mid \text{supp}(\hat{D}) \subset Y \}.$$

Let $X$ be an affine subspace of $E$, and $Y$ an affine subspace of $F$. Denote

$$L(X) := \{ u - v \mid u, v \in X \}$$

the subspace associated to $X$, and likewise $L(Y)$ for $Y$. We say that the pair $(X, Y)$ is thick, perfect or thin according as

$$L(X)^\perp \subsetneq L(Y), \quad L(X)^\perp = L(Y), \quad \text{or} \quad L(X)^\perp \not\subset L(Y),$$

or what is the same,

$$L(Y)^\perp \subsetneq L(X), \quad L(Y)^\perp = L(X), \quad \text{or} \quad L(Y)^\perp \not\subset L(X),$$

respectively.

**Example**: Take $F$ to be a non-degenerate quadratic space. Suppose that $F_0$ is a non-degenerate nonzero subspace of $F$ and

$$(F_0)^\perp = F^+ \oplus F^-$$

is a decomposition into totally isotropic subspaces $F^+$ and $F^-$. Then the pairs $(F^+, F^+)$, $(F^+ \oplus F_0, F^+)$, and $(F^+ \oplus F_0, F^+ \oplus F_0)$ are thin, perfect and thick, respectively.

**Remark**: If $(X, Y)$ is a perfect pair, then

$$D^{-\xi}(E; X, Y) = \begin{cases} \mathbb{C} \mu_{X, y_0}, & \text{if } k \text{ is nonarchimedean,} \\ (\mathbb{C}[X] \mu_{X, y_0}) \otimes D^{-\xi}(L'; \{0\}), & \text{if } k \text{ is archimedean.} \end{cases}$$

Here $y_0 \in Y$ is an arbitrary element, $\mu_{X, y_0} \in D^{-\xi}(X)$ is the product of the function $\psi(\langle \cdot , -y_0 \rangle)$ with a fixed $L(X)$-invariant positive Borel measure on $X$. In the archimedean case, $L'$ is an arbitrary subspace of $E$ such that $L(X) \oplus L' = E$. Through addition we have a decomposition $E = X \times L'$. The space $\mathbb{C}[X]$ is then the algebra of (complex valued) polynomial functions on $X$, viewed as a real affine space.

We now state the main result of this note.
**Theorem A.** Let $\mathcal{X}$ be a finite set of affine subspaces of $E$, and $\mathcal{Y}$ a finite set of affine subspaces of $F$. Assume that there is no pair $(X,Y) \in \mathcal{X} \times \mathcal{Y}$ which is thick. Then

$$D^{-\xi}(E; \cup \mathcal{X}, \cup \mathcal{Y}) = \bigoplus_{(X,Y) \in \mathcal{X} \times \mathcal{Y} \text{ which is perfect}} D^{-\xi}(E; X, Y).$$

**Remarks:** (i) Theorem A asserts in particular that $D^{-\xi}(E; \cup \mathcal{X}, \cup \mathcal{Y}) = 0$, if every pair $(X,Y) \in \mathcal{X} \times \mathcal{Y}$ is thin. (ii) In the archimedean case, $D^{-\xi}(E; X, Y)$ is a module for the Weyl algebra of $E$ (consisting of (complex) polynomial coefficient differential operators on $E$). We note that for a perfect pair $(X,Y)$, the Weyl algebra module $D^{-\xi}(E; X, Y)$ is irreducible.

The strategy to prove Theorem A is to control support and it goes as follows. For every vector $u \in E$, define the following function on $F$:

$$\phi_u := \psi(\langle u, \cdot \rangle).$$

Take a distribution $D \in D^{-\xi}(E; \cup \mathcal{X}, \cup \mathcal{Y})$. For any $Y \in \mathcal{Y}$, pick a nonzero $u_Y \in \text{L}(Y)^\perp$. The function $\phi_{u_Y}$ takes a constant value on $Y$, which we denote by $c_Y$. Thus $\phi_{u_Y} - c_Y$ will vanish on $Y$, and multiplying a high power of $\phi_{u_Y} - c_Y$ will cut $Y$ out of the support of the Fourier transform of $D$. The result is the Fourier transform of a new distribution which is a linear combination of $D$ and translates of $D$ by multiples of $u_Y$. Doing this consecutively for different $Y$’s in $\mathcal{Y}$ will thus yield a distribution which is a linear combination of $D$ and translates of $D$ by elements of the lattice in $\text{L}(Y)^\perp$ generated by $u_Y$’s, and which has a significantly reduced support for its Fourier transform. If $X \in \mathcal{X}$ is thin with respect to some $Y$’s, then one could arrange the $u_Y$’s so that the lattice generated by $u_Y$’s is in a favorable position relative to $X$, resulting in an excellent control on the support of the new distribution.

Here are some words on the organization of this note. In Section 2, we show that if $X \in \mathcal{X}$ has the property that $(X,Y)$ is thin for every $Y \in \mathcal{Y}$, then $X$ in fact does not appear in the support of any $D \in D^{-\xi}(E; \cup \mathcal{X}, \cup \mathcal{Y})$. This is a form of the uncertainty principle. In Section 3, we show that the rigidity property as claimed in Theorem A holds in the case of pure affine pairs, namely when $\mathcal{X}$ (resp. $\mathcal{Y}$) is a finite set of translations of a subspace $X_0$ of $E$ (resp. a subspace $Y_0$ of $F$). Section 4 is devoted to the induction step towards general affine pairs. Theorem A will then follow immediately from the case of pure affine pairs in Section 3 and the induction result of Section 4 just alluded to.
2. Thin pairs and elimination of support

Let \( \mathcal{X} \) be a finite set of affine subspaces of \( E \), and let \( \mathcal{Y} \) be a finite set of affine subspaces of \( F \), as in the Introduction. Assume that both \( \mathcal{X} \) and \( \mathcal{Y} \) are nonempty.

Note that an integer may also be considered as an element of \( k \). For any family \( a = \{a_Y\}_{Y \in \mathcal{Y}} \), denote \([a]\) the corresponding element of \( k^\mathcal{Y} \).

**Lemma 2.1.** Assume that \( X_1 \in \mathcal{X} \) and that \((X_1, Y)\) is thin for every \( Y \in \mathcal{Y} \). Let \( x_1 \in X_1 \setminus \cup(\mathcal{X} \setminus \{X_1\}) \). Then there is a family \( \{u_Y\}_{Y \in \mathcal{Y}} \) of vectors in \( E \) with the following property: for every \( a = \{a_Y\}_{Y \in \mathcal{Y}} \in Z^\mathcal{Y} \) with \([a]\) \neq 0, we have

\[
\sum_{Y \in \mathcal{Y}} a_Y u_Y \notin \cup \mathcal{X} - x_1.
\]

**Proof.** For every \( X \in \mathcal{X} \) and every \( a = \{a_Y\}_{Y \in \mathcal{Y}} \in Z^\mathcal{Y} \) with \([a]\) \neq 0, put

\[
S_{X, a} := \left\{ \{u_Y\} \in \prod_{Y \in \mathcal{Y}} L(Y)^\perp \mid \sum_{Y \in \mathcal{Y}} a_Y u_Y \in X - x_1 \right\}.
\]

If \( X \neq X_1 \), then \( 0 \notin S_{X, a} \), and \( S_{X, a} \) is a proper affine subspace of \( \prod_{Y \in \mathcal{Y}} L(Y)^\perp \). If \( X = X_1 \), then \( S_{X, a} \) is a subspace of \( \prod_{Y \in \mathcal{Y}} L(Y)^\perp \), and is proper due to the hypothesis that \((X_1, Y)\) is thin for every \( Y \in \mathcal{Y} \). In any case each \( S_{X, a} \) is a measure zero set of \( \prod_{Y \in \mathcal{Y}} L(Y)^\perp \), and so is the (countable) union \( \cup_{X \in \mathcal{X}, [a] \neq 0} S_{X, a} \). We finish the proof by taking a vector in

\[
(\prod_{Y \in \mathcal{Y}} L(Y)^\perp) \setminus (\cup_{X \in \mathcal{X}, [a] \neq 0} S_{X, a}).
\]

\( \square \)

For every vector \( u \in E \), denote by \( T_u : D^{-\xi}(E) \rightarrow D^{-\xi}(E) \) the push forward of the translation by \( u \), and write

\[
(T_u f)(\omega) := f(T_u \omega), \quad f \in C^{-\xi}(E), \omega \in D^\varsigma(E).
\]

Similar notation applies for \( v \in F \).

The following is a form of the uncertainty principle.

**Proposition 2.2.** Assume that \( X_1 \in \mathcal{X} \) and that \((X_1, Y)\) is thin for every \( Y \in \mathcal{Y} \). Then we have

\[
D^{-\xi}(E; \cup \mathcal{X}, \cup \mathcal{Y}) = D^{-\xi}(E; \cup (\mathcal{X} \setminus \{X_1\}), \cup \mathcal{Y}).
\]

Consequently we have

\[
D^{-\xi}(E; \cup \mathcal{X}, \cup \mathcal{Y}) = D^{-\xi}(E; \cup \mathcal{X}', \cup \mathcal{Y}'),
\]

where

\[
\mathcal{X}' := \{X \in \mathcal{X} \mid (X, Y) \text{ is not thin for some } Y \in \mathcal{Y}\},
\]

\[
\mathcal{Y}' := \{Y \in \mathcal{Y} \mid (X, Y) \text{ is not thin for some } X \in \mathcal{X}\}.
\]
Proof. Let $x_1 \in X_1 \setminus \bigcup(X \setminus \{X_1\})$ and $\{u_Y \in L(Y)^\perp\}_{Y \in \mathcal{Y}}$ be as in Lemma 2.1. Let $\phi_{u_Y}$ be as in (1), and $c_Y$ be its common value on $Y$, as in the Introduction.

Let $D \in D^\xi(E; \cup X, \cup \mathcal{Y})$. Then

$$\prod_{Y \in \mathcal{Y}} (\phi_{u_Y} - c_Y)^m \hat{D} = 0,$$

where $m = 1$ if $k$ is nonarchimedean, and $m$ is a sufficiently large positive integer if $k$ is archimedean. The above equality is equivalent to

$$\prod_{Y \in \mathcal{Y}} (T_{u_Y} - c_Y)^m D = 0,$$

or what is the same

$$\sum_{a \in \{a_Y\} \in \{0, 1, \ldots, m\}^\mathcal{Y}} c_a T_{u_a} D = 0,$$

where

$$c_a := \prod_{Y \in \mathcal{Y}} \left( \frac{m}{a_Y} \right) (-c_Y)^{m-a_Y},$$

and

$$u_a := \sum_{Y \in \mathcal{Y}} a_Y u_Y.$$

The choice of $\{u_Y\}$ ensures that $-u_a \notin \cup X - x_1$ whenever $[a]$ is nonzero. Let $U$ be an open neighborhood of 0 in $E$, small enough so that

$$(-u_a + U) \cap (\cup X - x_1) = \emptyset, \quad \forall [a] \neq 0.$$

Since $D$ is supported in $\cup X$, this implies that

$$(T_{u_a} D)|_{x_1 + U} = 0 \text{ when } [a] \text{ is nonzero}. $$

Together with (2), this implies that $D|_{x_1 + U} = 0$. Since $x_1$ is arbitrary, we conclude that $D$ is supported in $\cup(X \setminus \{X_1\})$. \qed

3. The case of pure affine pairs

**Proposition 3.1.** If $\mathcal{X}$ is a finite set of translations of a subspace $X_0$ of $E$, and $\mathcal{Y}$ is a finite set of translations of a subspace $Y_0$ of $F$, then

$$D^\xi(E; \cup \mathcal{X}, \cup \mathcal{Y}) = \bigoplus_{(X,Y) \in \mathcal{X} \times \mathcal{Y}} D^\xi(E; X, Y).$$
Proof. Assume that $X_0 \subseteq Y_0$, i.e., $(X_0, Y_0)$ is not a thin pair. Otherwise both sides of (3) are 0 and there is nothing to prove.

Let $D \in \mathcal{D}^{-\xi}(E; \cup X, \cup Y)$. For every $X \in \mathcal{X}$, denote by $D_X \in \mathcal{D}^{-\xi}(E; X)$ the distribution which coincides with $D$ on a neighborhood of $X$. Then

$$D = \sum_{X \in \mathcal{X}} D_X.$$ 

Note that $\cup X$ is a disjoint union, by our assumption on $\mathcal{X}$.

Assume that $k$ is archimedean. Let $P$ be a real polynomial function whose zero locus is $\cup Y$. Then there is a positive integer $k$ such that $P^k \hat{D} = 0$, or equivalently, $(\hat{P})^k D = 0$, where $\hat{P}$ is a certain constant coefficient differential operator, acting on the space $\mathcal{D}^{-\xi}(E)$. Therefore for all $X \in \mathcal{X}$, $(\hat{P})^k D_X = 0$, which implies that $D_X \in \mathcal{D}^{-\xi}(E; X, \cup Y)$. This proves that

$$D^{-\xi}(E; \cup X, \cup Y) = \bigoplus_{X \in \mathcal{X}} \mathcal{D}^{-\xi}(E; X, \cup Y).$$

Now assume that $k$ is nonarchimedean. Fix $X_1 \in \mathcal{X}$. For any $X \in \mathcal{X} \setminus \{X_1\}$, choose a vector $v_X \in X_0 \perp$ such that

$$\psi(\langle X, v_X \rangle) \neq \psi(\langle X_1, v_X \rangle).$$

Here $\psi(\langle X, v_X \rangle)$ stands for $\psi(\langle u, v_X \rangle)$, which is independent of $u \in X$, and $\psi(\langle X_1, v_X \rangle)$ is defined similarly. Then we have that

$$\left( \prod_{X \in \mathcal{X} \setminus \{X_1\}} (\phi_{v_X} - \psi(\langle X, v_X \rangle)) \right) D = \left( \prod_{X \in \mathcal{X} \setminus \{X_1\}} (\psi(\langle X_1, v_X \rangle) - \psi(\langle X, v_X \rangle)) \right) D_{X_1}.$$ 

(This is not true when $k$ is archimedean.)

The Fourier transform of the left hand side of the above equality is

$$\left( \prod_{X \in \mathcal{X} \setminus \{X_1\}} (T_{v_X} - \psi(\langle X, v_X \rangle)) \right) \hat{D}.$$ 

Since $v_X \in X_0 \perp \subseteq Y_0$, and since $\cup Y$ is invariant under translation by elements of $Y_0$, the above generalized function is again supported in $\cup Y$. Therefore the Fourier transform of $D_{X_1}$ is also supported in $\cup Y$. This proves (4) in the nonarchimedean case.

Applying (4) to the pair $Y$ and $\{X\}$, we have that

$$D^{-\xi}(E; X, \cup Y) = \bigoplus_{Y \in \mathcal{Y}} \mathcal{D}^{-\xi}(E; X, Y).$$

We finish the proof by combining (4) and (5).
4. Towards general affine pairs

**Proposition 4.1.** Let $X_0$ be a subspace of $E$ and $Y_0$ a subspace of $F$. Write

$$X_0 := \{ X \in X : L(X) = X_0 \} \text{ and } \mathcal{Y}_0 := \{ Y \in \mathcal{Y} : L(Y) = Y_0 \}.$$ 

Assume that $(X', Y_0)$ and $(X_0, Y')$ are thin for all \( X' \in X' := X \setminus X_0 \) and all \( Y' \in Y' := Y \setminus Y_0 \).

Then

$$D^\xi(E; \cup X, \cup Y) = D^\xi(E; \cup X_0, \cup Y_0) \oplus D^\xi(E; \cup X', \cup Y').$$

**Proof.** Proposition 2.2 implies that the right hand side of (6) is a direct sum.

Let \( \{ u_{Y'} \in L(Y') \} \) be a family of vectors in $E$ such that for every family \( a = \{ a_{Y'} \} \in \mathbb{Z}^{Y'} \) with \([ a ] \neq 0\), we have

$$\sum_{Y' \in Y'} a_{Y'} u_{Y'} \notin \bigcup_{X_1, X_2 \in X_0} (X_1 \setminus X_2).$$

The existence of such a family is proved along the same line as that of Lemma 2.1. Let \( \phi_{u_{Y'}} \) be as in (1), and \( c_{Y'} \) be its common value on \( Y' \), as in the Introduction.

Let $D \in D^\xi(E; \cup X, \cup Y)$. We take \( m \) to be a sufficiently larger positive integer if \( k \) is archimedean and \( m = 1 \) if \( k \) is non-archimedean. Then the generalized function

$$\left( \prod_{Y' \in Y'} (\phi_{u_{Y'}} - c_{Y'})^m \right) \hat{D}$$

is supported in $\cup \mathcal{Y}_0$. It is the Fourier transform of the distribution

$$D' := \left( \prod_{Y' \in Y'} (T_{u_{Y'}} - c_{Y'})^m \right) D.$$ 

By expansion, we have

$$D' = \sum_{a = \{ a_{Y'} \} \in \{0,1,\ldots,m\}^{Y'}} c_a T_{u_a} D,$$

where

$$c_a := \prod_{Y' \in Y'} \left( \begin{array}{c} m \\ a_{Y'} \end{array} \right) (-c_{Y'})^{m-a_{Y'}},$$

and

$$u_a := \sum_{Y' \in Y'} a_{Y'} u_{Y'}.$$
For every set $Z$ of affine subspaces of $E$, we put
\[ \tilde{Z} := \left\{ u_a + Z \mid a \in \{0, 1, \cdots, m\}^Y, Z \in Z \right\}. \]
Then $D'$ is clearly supported in $\cup \tilde{X}$. Its Fourier transform is supported in $\cup \mathcal{Y}_0$, and since $(X', Y_0)$ is thin for all $X' \in \mathcal{X}'$, Proposition 2.2 implies that it is supported in $\cup \tilde{X}_0$.
Now the choice of $\{u_{Y'}\}$ ensures that
\[ \tilde{X}_0 = X_0 \sqcup X_1 \quad \text{(a disjoint union)}, \]
where
\[ X_1 := \left\{ u_a + X \mid a \in \{0, 1, \cdots, m\}^Y, [a] \neq 0, X \in X_0 \right\}. \]
By Proposition 3.1, we may write
\begin{equation}
(8) \quad D' = D_0 + D_1,
\end{equation}
where $D_0 \in D^{-\xi}(E; \cup X_0, \cup \mathcal{Y}_0)$ and $D_1 \in D^{-\xi}(E; \cup X_1, \cup \mathcal{Y}_0)$.
Let
\[ x_0 \in \cup X_0 \setminus \cup \tilde{X}'. \]
The disjointness of $X_0$ and $X_1$ allows us to choose an open neighborhood $U$ of $0$ in $E$, small enough so that
\[ (x_0 + U) \cap (\cup \tilde{X}') = \emptyset, \]
and
\begin{equation}
(9) \quad (x_0 + U) \cap (\cup X_1) = \emptyset.
\end{equation}
For all nonzero $[a]$, we thus have
\[ (x_0 + U - u_a) \cap (\cup X) = \emptyset \]
and therefore
\[ (T_{u_a} D)|_{x_0+U} = 0. \]
Then (7) implies that
\[ D'|_{x_0+U} = c_0 D|_{x_0+U}, \quad \text{with} \quad c_0 = \prod_{Y' \in Y'} (-c_{Y'})^m, \]
and (8) and (9) implies that
\[ (D' - D_0)|_{x_0+U} = D_1|_{x_0+U} = 0. \]
We thus conclude from the last two equalities that $D - D_0/c_0$ vanishes on $x_0 + U$.
Since $x_0$ is arbitrary, we see that $D - D_0/c_0$ is supported in
\[ (\cup X') \setminus (\cup X_0 \setminus \cup \tilde{X}') \subset (\cup X') \cup (\cup (X_0 \setminus \tilde{X}')). \]
where
\[ \mathcal{X}_0 \land \tilde{\mathcal{X}}' = \{ X_0 \cap X' \mid X_0 \in \mathcal{X}_0, X' \in \tilde{\mathcal{X}}' \}. \]

Since every pair in \((\mathcal{X}_0 \land \tilde{\mathcal{X}}') \times \mathcal{Y}\) is thin, and every pair in \(\mathcal{X}' \times \mathcal{Y}_0\) is thin, we see that \(D - D_0/c_0\) actually belongs to
\[ D^{-\xi}(E; (\cup \mathcal{X}') \cup (\cup(\mathcal{X}_0 \land \tilde{\mathcal{X}}')) \cup \mathcal{Y}) = D^{-\xi}(E; \cup \mathcal{X}', \cup \mathcal{Y}) = D^{-\xi}(E; \cup \mathcal{X}', \cup \mathcal{Y}'), \]
by two applications of Proposition 2.2. This finishes the proof of the current proposition. \(\square\)

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Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, 100190, P.R. China

E-mail address: sun@math.ac.cn

Department of Mathematics, National University of Singapore, Block S17, 10 Lower Kent Ridge Road Singapore 119076

E-mail address: matzhucb@nus.edu.sg