Quantum parameter estimation with imperfect reference frames

Dominik Šafářek, Mehdi Ahmadi* and Ivette Fuentes
School of Mathematical Sciences, University of Nottingham, University Park, Nottingham NG7 2RD, UK
* Author to whom any correspondence should be addressed.
E-mail: pmxdd@nottingham.ac.uk and mehdi.ahmadi@nottingham.ac.uk
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Abstract
Quantum metrology studies quantum strategies which enable us to outperform their classical counterparts. In this framework, the existence of perfect classical reference frames is usually assumed. However, such ideal reference frames might not always be available. The reference frames required in metrology strategies can either degrade or become misaligned during the estimation process. We investigate how the imperfectness of reference frames leads to noise which in general affects the ultimate precision limits in measurement of physical parameters. Moreover, since quantum parameter estimation can be phrased as a quantum communication protocol between two parties, our results provide deeper insight into quantum communication protocols with misaligned reference frames. Our framework allows for the study of noise on the efficiency of such schemes.

1. Introduction
In quantum metrology quantum properties such as squeezing and entanglement are employed to improve the precision with which physical quantities can be measured [1]. Quantum metrological techniques have been very fruitful in developing a new generation of quantum devices that can outperform their classical counterparts. In particular, the framework of quantum metrology is very useful for measuring physical quantities that do not have an associated operator in quantum theory such as time, phase, temperature, acceleration, etc. The process of quantum parameter estimation consists of three stages; the preparation of the probe state of the system, feeding the prepared state into the quantum channel which encodes the parameter of interest into the state of the system and finally decoding the parameter by performing a measurement on the system after it has gone through the channel. In order to reduce the statistical error in the estimation of a physical quantity, this whole process needs to be optimized. This is achieved by optimizing over both the preparation of the initial state of the system and the measurement of the final state of the system. For a given prepared state, the quantum Cramér–Rao bound provides us with the ultimate precision bound on the measurement of the physical quantities in quantum theory.

Holevo and Helstrom laid the foundations of quantum metrology by phrasing the problem in the context of a communication protocol between two parties [2, 3]. In this paradigm Alice chooses a quantum system to encode a message which she then sends to Bob. For example she might choose to use either a spin $-\frac{1}{2}$ system or a quantum harmonic oscillator as carrier of her message. The message can be encoded as a phase parameter in the state of the spin $-\frac{1}{2}$ particle or in the state of the quantum harmonic oscillator. Bob then performs a measurement on the system in order to decode the message, i.e. the encoded phase parameter. The quality of this communication protocol can be improved by optimization of both stages of the protocol, namely Alice’s encoding process and Bob’s decoding process. However, standard approaches to quantum communication, such as encoding qubits into polarization degree of freedom of photons, require that all parties have knowledge of a shared reference frame. This means that in the absence of such knowledge, the involved parties need to initially establish aligned reference frames. Despite the considerable amount of progress in the development of protocols for aligning reference frames such as clock synchronization and Cartesian frame alignment [4], maintaining aligned reference frames is still a large obstacle in achieving such tasks. For instance when the
parties are in relative motion with respect to each other, the relative orientation of their local reference frames can change in time [5]. However, quantum reference frames (QRF) enable us to circumvent this problem. Let us briefly explain what QRFs are and in what way they differ from classical reference frames (CRF).

Aharanov and Susskind in their seminal papers [6, 7], showed that the concept of reference frames can be suitably accommodated in quantum theory. In recent years, such a treatment of reference frames in quantum theory, i.e. as quantum objects has led to the formalism of ‘QRF’ [4, 8]. A QRF is different from its classical counterpart in two ways. First, due to its quantum nature, it has an inherent uncertainty and the measurement results are only an approximation of what would be obtained using a CRF. For instance, if the reference frame describes a continuum of orientations in space, then states with different orientations are not perfectly distinguishable. Second, each time the QRF is used, it suffers a back-action, which causes future measurements to be less accurate. Phase measurements of single-rail qubits relative to a QRF have been investigated in [9], while the degradation of a directional QRF has been analysed in [9–11]. In the past few years, the ‘resource theory of QRF’, also known as the ‘resource theory of asymmetry’, has been developed. This resource theory provides us with a very useful framework wherein the QRFs are the ‘resource states’ [12–15]. They enable us to achieve quantum information processing tasks without first establishing a shared reference frame. In such schemes, a QRF stands in for the possibility of performing tasks in the absence of a common CRF, in the same way that entangled states allow for the possibility of performing non-local quantum operations.

In this paper, we utilize the powerful machinery of quantum metrology to study the ultimate precision bounds in measurement of physical parameters with respect to QRFs. First we explain the connection between the quantum mechanical treatment of reference frames and quantum parameter estimation in the presence of noise. Then we investigate how the ultimate precision in measurement of a parameter decreases due to inaccessibility of a perfect CRF. In order to do so, we analyse the decrease in QFI as a result of not having access to a perfect reference frame for the physical quantity of interest. In particular, we provide necessary and sufficient conditions for two extreme cases that can occur in quantum parameter estimation with imperfect frames of reference. The first case is when the absence of a perfect reference frame does not affect the precision with which one can measure the parameter and the second case is when measurement of the parameter of interest is no longer possible due to not having access to a CRF. Motivated by this analysis we split the problem into two subproblems. The first case is when the encoding operator commutes with the operator representing the noise and the second is when it does not. Counter-intuitively, we show that the non-commuting case has some advantages over the commuting one. While the existence of ‘decoherence-free subspaces (DFS)’ is essential for encoding information in the commuting case [4, 16–18], for non-commuting operators the estimation is possible even in the absence of such subspaces. The trade-off is, however, that the precision will in general depend on the parameter to be estimated. In addition, we explain the connection between noisy quantum metrology and alignment-free quantum communication. In the end, we present three examples in order to further clarify different aspects of quantum metrology with imperfect frames of reference.

The structure of this paper is as follows: in section 2.1 we briefly review some of the mathematical tools from quantum metrology, in particular QFI and symmetric logarithmic derivative (SLD). In section 2.2 we represent the general scheme of the alignment-free communication protocols. Section 3 includes the main results of this paper, where we present the general framework for quantum parameter estimation in the absence of an ideal CRF and we discuss in detail how it relates to alignment-free communication protocols. In section 3.4 we present three examples in which we explain different aspects of quantum parameter estimation in the absence of aligned CRFs. Finally, in section 4 we discuss the results of this paper and we mention some of our research interests in quantum parameter estimation as possible future work.

2. Preliminaries

2.1. Quantum Cramér–Rao bound

The main goal in quantum metrology is to estimate an unknown parameter $\lambda$ of a completely positive trace-preserving quantum channel, $\mathcal{E}_\lambda$. In order to do so, first a probe state is prepared which is then fed into the channel of interest. Finally a measurement on the final state of the probe will enable us to estimate $\lambda$, where $\lambda$ is a physical quantity such as time, phase, temperature, acceleration, etc. Given a measurement strategy, the conditional probability of obtaining outcome $x$ when the initial state is $\rho_\lambda$ is given by $p(x|\lambda) = \text{Tr}\,\hat{O}_x\rho_\lambda$, where $\{\hat{O}_x\}$ are elements of a complete positive-operator valued measure (POVM) corresponding to the chosen measurement strategy (see figure 1). The lower bound on how precise we can estimate $\lambda$ is given by the ‘classical Cramér–Rao bound’, i.e. $\langle(\Delta\hat{\lambda})^2\rangle \geq \frac{1}{N\mathrm{F}(\hat{\lambda})}$, where the classical Fisher information $F(\rho_\lambda)$ is defined as
and $N$ is the number of repeated measurements. Due to its role in the Cramér–Rao bound, the classical Fisher informations can be viewed as an operational measure which tells us how much information we can gain about the unknown parameter $\lambda$ by choosing a certain measurement strategy.

Braunstein and Caves showed that optimization over all the possible quantum measurements provides an even more stringent lower bound [19], i.e.

$$\Delta \lambda \geq \frac{1}{N F(\rho_{\lambda})},$$

where $H(\rho_{\lambda})$ is the QFI. This quantity is closely related to the SLD $L(\rho_{\lambda})$ which is defined by $2d\rho_{\lambda}/d\lambda = L(\rho_{\lambda})\rho_{\lambda} + \rho_{\lambda}L(\rho_{\lambda})$. In particular in the basis $|\psi_{i}\rangle$ in which $\rho_{\lambda}$ is diagonalised, i.e. $\rho_{\lambda} = \sum_{i} p_{i} |\psi_{i}\rangle \langle \psi_{i}|$, the SLD and the QFI can be written as

$$L(\rho_{\lambda}) = 2 \sum_{ij} \frac{\langle \psi_{i} | \partial_{\lambda} \rho_{\lambda} | \psi_{j}\rangle}{p_{i} + p_{j}} |\psi_{i}\rangle \langle \psi_{j}|,$$

$$H(\rho_{\lambda}) = 2 \sum_{ij} \left( \frac{\langle \psi_{i} | \partial_{\lambda} \rho_{\lambda} | \psi_{j}\rangle}{p_{i} + p_{j}} \right)^{2},$$

with the relation

$$H(\rho_{\lambda}) = \text{Tr} \left( \partial_{\lambda} \rho_{\lambda} L(\rho_{\lambda}) \right).$$

The summations above do not include the terms where $p_{i} + p_{j} = 0$. The optimal POVM which achieves the quantum Cramér–Rao bound (2), can be constructed from the eigenstates of $L(\rho_{\lambda})$ [20].

Note that in this section the existence of ideal CRFs was assumed. This treatment of reference frames can lead to a great deal of confusion. For instance, in [21] the role of an external phase reference frame in interferometric setups has been analysed. In this paper we analyse the decrease in QFI as a result of not having access to a perfect reference frame for the physical quantity $\lambda$. In particular, we present necessary and sufficient conditions for two extreme cases; the case where the QFI does not decrease when a CRF is lacking and the case where the QFI vanishes due to imperfectness of the RF, i.e. one can no longer extract the parameter $\lambda$.

### 2.2. Alignment-free communication

As mentioned earlier, quantum parameter estimation can be phrased as a communication protocol between two parties. In this section we briefly review how QRFs have been employed in order to achieve alignment-free communication protocols [22].

Consider $g \in G$ to be the group element that describes the passive transformation from Alice’s to Bob’s reference frame. Furthermore, since Bob is completely unaware of the relation between his local RF and Alice’s local RF, we can assume that the group element $g$ is completely unknown. It follows that if Alice prepares a state $\rho_{\lambda}$ relative to her local reference frame, then relative to Bob’s RF this state is seen as

$$\rho_{\lambda}^g.$$  

\footnote{We will restrict our attention to Lie-groups that are compact, so that they possess a group-invariant (Haar) measure $dg$. We refer the readers for more details to [4].}
\[ \rho_B = G[\rho_A] = \int dg U(g) \rho_A U(g)^\dagger. \]

Therefore, lacking such a shared reference frame is equivalent to having a noisy completely positive trace-preserving map which is known as the ‘g-twirling map’, i.e. \( G(\rho_A) \).

However, despite the fact that Alice has no information about the group element \( g \) that relates her local RF to Bob’s local RF, she can still encode information in the so called ‘decoherence-free subspaces (DFS)’. These subsystems are resilient to the decoherence caused due to the lack of knowledge about the relative direction of the local reference frames. The efficiency of this protocol depends on the dimensionality of the largest DFS, i.e. the subsystem which possesses the largest number of degenerate eigenstates. Such communication scenarios in the absence of a shared Cartesian reference frame have been analysed before \[4\]. The idea is to encode logical qubits into rotationally invariant states of multiple physical qubits. In this case the number of logical qubits per number of physical qubits that can be transmitted scales as \( -\frac{1}{2} \log_2 M \), where \( M \) is the number of transmitted physical qubits. This remarkable result proves that in the limit of \( M \to \infty \) one logical qubit can be sent per one physical qubit. Therefore in this limit the efficiency of this scheme is the same as the scenario wherein the reference frames are aligned. This protocol has also been studied for the situation in which the parties do not share a common background phase reference frame \[22, 23\].

### 3. Parameter estimation with imperfect reference frames

As we explained in section 2.1, the standard scenarios considered in quantum metrology normally presume the existence of perfect CRF. In this section we investigate how the ultimate precision in measurement of a parameter decreases due to lack of access to a perfect CRF.

Let us first briefly explain the general picture of the estimation of a parameter in the absence of a perfect external RF. We consider the case where the parameter \( \lambda \) is encoded into the fiducial state via a unitary channel \( U_\lambda \). After this encoding process, we need to choose the optimal measurement in order to extract the maximum amount of information about \( \lambda \). We then need a suitable RF with respect to which we are able to perform the chosen measurement. For instance if we wish to measure time we need a clock or if we need to measure phase we will need a phase reference frame. The absence of such reference frames can be viewed as a noisy quantum channel, i.e.

\[ G[\rho_\lambda] = \lim_{T \to \infty} \frac{1}{T} \int_0^T dt \ U(t) \rho_\lambda U(t)^\dagger, \]

where \( \rho_\lambda = |\psi_\lambda\rangle \langle \psi_\lambda| \), \( U(\lambda) = e^{-i\hat{K}\lambda} \), where \( \hat{K} \) is the Hermitian encoding operator, and \( U(t) = e^{-i\hat{G} t} \). Note that \( \hat{G} \) and \( t \) are determined by the type of RF that is lacking. For instance if we need to measure phase of a quantum harmonic oscillator then \( t \) is a phase and \( \hat{G} = \hat{N} \), where \( \hat{N} \) is the total number operator \[4\].

Assuming the spectral decomposition \( \hat{G} = \sum_i \hat{P}_i \), where the \( \hat{P}_i \) s are the projectors into subspaces with eigenvalues \( G_i \) and \( \sum_i \hat{P}_i = 1 \), one can easily check that the state \( G[\rho_\lambda] \) in (7) can be written as

\[ G[\rho_\lambda] = \sum_i \hat{P}_i \rho_\lambda \hat{P}_i. \]

As depicted on figure 2 the problem of parameter estimation without a CRF can be phrased as the problem parameter estimation in the presence of noise. Note that in the special case of commuting \( \hat{K} \) and \( \hat{G} \) operators, i.e. when \( [\hat{K}, \hat{G}] = 0 \), this general noise reduces to the well-known ‘collective dephasing noise’ with uniform prior probability \[24\]. In fact in the quantum information protocols considered in \[4\] and the references therein it is assumed that \( [\hat{K}, \hat{G}] = 0 \).
Before we present the main theoretical results, let us define ‘QFI loss’ as

$$I(\rho, \hat{G}) = H(\rho) - H(\rho | \hat{G}).$$  \hfill (9)

This operational measure enables us to analyse how much information is lost if instead of ideal CRFs we only have access to imperfect frames of reference. We investigate the decrease in the accuracy of measurements in both cases of commuting and non-commuting $\hat{K}$ and $\hat{G}$.

3.1. General framework

We restrict our analysis to pure initial states of the system, i.e. $\rho = |\psi\rangle \langle \psi|$. Using equation (4) for QFI, the properties of the quantum channel $\mathcal{G}$ (8) and the Parseval identity (for more details see appendix A), we derive Bob’s QFI as

$$H(\mathcal{G} \rho) = 4 \sum_i \left( \text{Re} \left( \frac{\partial_j |\psi_i\rangle \langle \psi_j|}{\langle \psi_i | \hat{P} | \psi_j \rangle} \right)^2 \right),$$  \hfill (10)

where the summation is over the indices $i$ for which $p_i = \langle \psi_i | \hat{P} | \psi_i \rangle \neq 0$. Note that we will use this convention throughout the rest of the paper.

Differentiating $\langle \psi_i | \hat{P} | \psi_i \rangle = 1$ yields

$$\langle \psi_i | \hat{P} \partial_j | \psi_i \rangle + \langle \partial_j | \psi_i \rangle = 2 \text{Re} \left( \langle \psi_i | \partial_j | \psi_i \rangle \right) = 0,$$  \hfill (11)

and therefore $\langle \psi_i | \partial_j | \psi_i \rangle$ is purely imaginary. This fact will be used frequently in the rest of the paper.

Note that if we trivially choose $\sum_i \hat{P} = 1$, i.e. in the presence of a shared reference frame, we recover the result for the QFI of pure states

$$H(\rho) = 4 \sum_i \left( \text{Re} \left( \frac{\partial_j |\psi_i\rangle \langle \psi_j|}{\langle \psi_i | \hat{P} | \psi_j \rangle} \right)^2 \right);$$  \hfill (12)

as was proved in [20].

Using equations (10) and (12), we find the expression below for the QFI loss as defined by (9)

$$l(\rho, \hat{G}) = 4 \sum_i \left( \text{Re} \left( \frac{\partial_j |\psi_i\rangle \langle \psi_j|}{\langle \psi_i | \hat{P} | \psi_j \rangle} \right)^2 \right) - \left| \langle \psi_i | \partial_j | \psi_i \rangle \right|^2.$$  \hfill (13)

This quantity is always non-negative. This is expected because it simply means that the accuracy with which one can measure $\lambda$ in presence of a perfect CRF can not be less than the accuracy with which he/she can measure $\lambda$ in the absence of such CRFs. We formalise these bounds in the theorem below together with the necessary and sufficient conditions for two extreme cases. The first case is where the precision in measurement of $\lambda$ remains the same both in the absence or the presence of a perfect CRF and the second case is where the measurement of $\lambda$ is not possible anymore due to inaccessibility of such reference frames. For proof see appendix B. Note that from this point on we drop the subscript $\lambda$.

**Theorem 3.1.** Let $\rho = |\psi\rangle \langle \psi|$ be a pure initial state. Then the QFI loss is bounded, i.e. $0 \leq l(\rho, \hat{G}) \leq H(\rho)$. No precision is lost (no loss), i.e. $l(\rho, \hat{G}) = 0$, if and only if there exists a complex number $c$ such that

$$\partial_j |\psi_i\rangle = \sum_i \text{Im} \left( \frac{\langle \psi_i | \hat{P} \partial_j | \psi_i \rangle}{\langle \psi_i | \hat{P} | \psi_i \rangle} \right) = c \langle \psi_i | \hat{P} | \psi_i \rangle$$  \hfill (14)

or equivalently

$$\exists c \in \mathbb{C}, \forall i, \text{Im} \langle \psi_i | \hat{P} \partial_j | \psi_i \rangle = c \langle \psi_i | \hat{P} | \psi_i \rangle.$$  \hfill (15)

$\lambda$ cannot be estimated anymore (maximum loss), i.e. $l(\rho, \hat{G}) = H(\rho)$, if and only if

$$\langle \partial_j |\psi_i\rangle |\psi_i\rangle = \langle \partial_j |\psi_i\rangle |\psi_i\rangle,$$  \hfill (16)

or equivalently

$$\forall i, \text{Re} \langle \psi_i | \hat{P} \partial_j | \psi_i \rangle = 0 \quad \forall \langle \phi_j \rangle, \langle \phi_j \partial_j |\psi_i\rangle = 0,$$  \hfill (17)
where \( \{ \frac{\hat{P}_i \psi}{\sqrt{P_i}} , \phi_j \} \}_{i,j} \) is an orthonormal basis of the Hilbert space. Moreover, QFI loss can be written as

\[
I(\rho, \hat{G}) = 4 \langle \partial \rho | \partial \psi \rangle - 4 \langle \psi | \partial \rho \rangle^2.
\]  

(18)

Let us add three notes to this theorem. First, after summing over all the indices \( i \) in equation (15) and using equation (11), one can easily find that

\[
\text{Im} \langle \psi \rangle = \langle \partial \psi | \partial \rangle = -\langle \partial \psi | \partial \rangle.
\]

(18)

Second, without loss of generality in equation (15), we can restrict our analysis to the terms for which \( \psi = \langle \partial \psi | \partial \rangle \neq 0 \), since using the Cauchy–Schwarz inequality it can be checked that the condition (15) holds trivially if \( P_i = 0 \). Third, the set of states \( \{ \phi_j \} \) are orthonormal states which together with the set of normalized states \( \{ \psi \} \) make a complete set. We can always find the set of states \( \{ \phi_j \} \) via the Gram–Schmidt process for orthonormalization of a set of vectors.

If we assume that the eigenvectors of the operator \( \hat{G} \) span the whole Hilbert space, \( \{ \phi_j \} \) is exactly the set of the eigenvectors of \( \rho \) with the respective eigenvalue 0.

Using similar analysis we can find the SLD operator in (3) as

\[
L(\hat{G} (\rho)) = \sum_i |\psi_i \rangle \langle \psi_i | + |\phi_i \rangle \langle \phi_i |,
\]

(19)

where \( |\psi_i \rangle \) and \( |\phi_i \rangle \) are defined as

\[
|\psi_i \rangle = \frac{\hat{P}_i \psi}{\sqrt{P_i}},
\]

\[
|\phi_i \rangle = \frac{1}{\sqrt{P_i}} \left( 2\hat{P}_i \partial \psi - \langle \psi_i | \partial \psi \rangle \langle \phi_i | \right).
\]

(20)

Now we can use this SLD operator whenever we lack a perfect CRF in order to find the POVM that can optimally distinguish between the two neighbouring states \( \rho_{\lambda} \) and \( \rho_{\lambda+\delta \lambda} \), where \( \delta \lambda \) is an infinitesimal increment in the parameter \( \lambda \).

Since these projectors are constructed from the eigenvectors of the operator \( \hat{G} \), we can instead write every derived expression in terms of these eigenvectors (see appendix C).

3.2. Analysis of commutative and non-commutative noise due to lacking a perfect CRF

In this section we analyse QFI in terms of the hermitian operator \( \hat{K} \) which imprints the parameter \( \lambda \) into the fiducial state \( \psi_0 \) and \( \hat{G} \) which is the generator of the noisy channel. This way we split the problem into two different cases. The first case is where the encoding process in general does not commute with the noisy channel, i.e. \( [\hat{K}, \hat{G}] \neq 0 \). We call such noise non-commutative. The second is when when the noise is commutative\(^2\), i.e. \( [\hat{K}, \hat{G}] = 0 \). For commutative noise formulas usually simplify and are easier to interpret.

Using equation (10) we derive an alternative form for the QFI in the absence of a perfect RF as

\[
H(\hat{G} | \rho \rangle) = 4 \langle \psi_0 | \hat{K} \hat{K} \psi_0 \rangle - 4 \sum_i \langle \psi_i | \hat{P}_i \hat{K} \psi_i \rangle^2,
\]

(21)

where \( \{ \cdot , \cdot \} \) denotes anti-commutator. Noting that \( \hat{G} \) commutes with \( \hat{K} \) if and only if all the projectors \( \hat{P}_i \) commute with \( \hat{K} \), for the case of commuting \( \hat{G} \) and \( \hat{K} \) equation (21) reduces to

\[
H(\hat{G} | \rho \rangle) = 4 \langle \psi_0 | \hat{K} \psi_0 \rangle - 4 \sum_i \langle \psi_0 | \hat{P}_i \hat{K} \psi_0 \rangle^2.
\]

(22)

Now let us revisit the no-loss and maximum-loss conditions that we presented in theorem 3.1. These conditions can be written in terms of projectors \( \hat{P} \) and generator \( \hat{K} \) as

\[
I(\rho, \hat{G}) = 0 \iff \forall i, \{ \hat{P}_i, \hat{K} \}_{\rho} = 2 \langle \hat{K} \rangle_{\rho} \langle \hat{P}_i \rangle_{\rho},
\]

(23)

\(^2\)If the noise is commutative, it simply means that the noisy channel (7) commutes with the encoding process. In that case our results can be also applied on systems where the noise (8) precedes the encoding operation \( U(\lambda) \), or more specifically, systems with mixed fiducial state \( \rho_0 \).
where $\langle \cdot \rangle_\rho$ is the expectation value with respect to state $\rho$. If we assume that $[\hat{K}, \hat{G}] = 0$ and that the operator $\hat{G}$ has a non-degenerate spectrum, i.e., all the projectors $\hat{P}_i$ are rank-1 projections, then in the absence of a perfect CRF all the information about $\lambda$ will be lost (for proof see appendix C). We write this simple yet powerful result as a theorem.

**Theorem 3.2.** Let $\hat{G}$ have a non-degenerate spectrum. If $[\hat{K}, \hat{G}] = 0$, then $l(\rho, \hat{G}) = H(\rho)$ or, equivalently, $H(\rho) = 0$. That is, parameter $\lambda$ cannot be estimated anymore. This is no longer the case when the two operators do not commute. This means that, even though the DFS are crucial for successful encoding of parameter $\lambda$ in the commuting case, such subspaces are not necessary in the non-commuting case. As we will explain in section 3.3, this fact stands in for the possibility of alignment-free communication whenever the spectrum of $\hat{G}$ is non-degenerate. We will present examples of these two different cases, i.e., commutative and non-commutative noise in section 3.4.

We can write the no-loss condition (23) in a more intuitive way as

$$l(\rho, \hat{G}) = 0 \Leftrightarrow \forall i, \text{Cov}_\rho(\hat{P}_i, \hat{K}) = 0,$$

(25)

where the covariance of two observables $\hat{A}$ and $\hat{B}$ is defined as

$$\text{Cov}_\rho(\hat{A}, \hat{B}) = \frac{1}{2}(\langle \hat{A} - \langle \hat{A} \rangle_\rho \rangle_\rho - \langle \hat{B} - \langle \hat{B} \rangle_\rho \rangle_\rho),$$

and the variance can be written as

$$\text{Var}_\rho(\hat{A}) = \text{Cov}_\rho(\hat{A}, \hat{A}).$$

Covariance is a measure of correlations between two observables $\hat{A}$ and $\hat{B}$ with respect to the state $\rho$. Multiplying this equation by the eigenvalues $G_i$ and summing over all the indices $i$, we can write the necessary condition for not loosing any information about $\lambda$ as

$$l(\rho, \hat{G}) = 0 \Rightarrow \text{Cov}_\rho(\hat{G}, \hat{K}) = 0.$$  

(26)

This means that if operators $\hat{K}$ and $\hat{G}$ are correlated with respect to the pure initial state $\rho$, i.e. $\text{Cov}_\rho(\hat{G}, \hat{K}) \neq 0$, then some information is lost due to the imperfectness of RF, i.e. $l(\rho, \hat{G}) > 0$.

It is worth emphasizing that this condition is not sufficient. As an example consider the operators

$\hat{K} = |2\rangle\langle 2|, \hat{G} = 6 |0\rangle\langle 0| + 3 |1\rangle\langle 1| + 4 |2\rangle\langle 2|$, and the fiducial state $|\psi_\rho\rangle = \frac{1}{\sqrt{6}} |0\rangle + \frac{1}{\sqrt{3}} |1\rangle + \frac{1}{\sqrt{2}} |2\rangle$. In this example the covariance between $\hat{G}$ and $\hat{K}$ is zero, nevertheless, since $\hat{K}$ and $\hat{G}$ commute and the fact that no non-degenerate subspace exists, according to theorem 3.2 we will not be able to extract any information about $\lambda$.

Similar to the no-loss condition in (26), using equation (24) the necessary condition for the extreme case of loosing all the information can be written as

$$l(\rho, \hat{G}) = H(\rho) \Rightarrow \text{Cov}_\rho(\hat{G}, \hat{K}) = 0.$$  

(27)

This means that if for a given initial state $|\rho(\hat{G}, \hat{K})\rangle_\rho$ is non-zero, there is the possibility of extracting some information about parameter $\lambda$ even in the absence of a CRF.

The QFI of a unitary channel in the absence of any noise can be computed from equation (12) or alternatively by [20]

$$H_u(\rho) = 4 \left( \langle \hat{K}^2 \rangle_\rho - \langle \hat{K} \rangle_\rho^2 \right) = 4 \text{Var}_\rho(\hat{K}),$$

(28)

where $\hat{K}$ is again the generator of the unitary channel. We presented the QFI loss due to the quantum nature of the RF in (13), we can re-write this equation in terms of the projectors $\hat{P}_i$ and the generator $\hat{K}$ as

$$l(\rho, \hat{G}) = 4 \sum_i \left( \text{Cov}_\rho(\hat{P}_i, \hat{K}) \right)^2 / P_i.$$  

(29)

Using definition (9), this enables us to write the QFI in the absence of a perfect CRF in a form which is easier to compare to the QFI in the presence of classical frames of reference given in equation (28). We present this form of QFI in the following theorem.

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3 Here we use the symmetrized form of covariance. For other forms of covariance see [25].

4 We refer the readers to [26] for details of the relation between the correlations of two observables and the covariance of observables.
Theorem 3.3. For a fiducial pure state $|\psi_0\rangle$, a generator $\tilde{K}$ of a unitary operator $U(\lambda) = \exp(-i\tilde{K}\lambda)$ and projectors $\hat{P}_i$ of the $g$-twirling map in (8), the QFI of the state $\rho_{\lambda}$ is

$$H \left( \rho_{\lambda} \right) = 4 \text{Var}_{\rho_{\lambda}} \left( \hat{K} \right) - 4 \sum_i p_i \left[ \text{Cov}_{\rho_{\lambda}} \left( \hat{B}_i, \hat{K} \right) \right]^2,$$

(30)

where $p_i = \langle \psi_i \rangle \langle \psi_i \rangle$ and $p_i = \langle \hat{B}_i \rangle_{\rho_{\lambda}}$.

From equation (30) we deduce that the decrease in QFI is proportional to the mean of squared covariances between the normalized projectors $\hat{P}_i$ and the encoding operator $\hat{K}$. This means that the more projectors $\hat{P}_i$ are correlated with the encoding operator $\hat{K}$, the more precision is lost. Roughly speaking, in order to lose the minimum amount of precision one should choose an encoding operator $\hat{K}$ which is less correlated with the decoherence caused by the noisy channel $\mathcal{G}$. For an explicit example we refer the readers to the third example of section 3.4.

For the special case of a commutative noise, i.e. when $\hat{K}$ and $\hat{G}$ commute, the expression (30) can be further simplified. In this case we have $\text{Cov}(\hat{B}_i, \hat{K}) = p_i \langle \langle \hat{K} \rangle_{\rho_{\lambda}} - \langle \hat{K} \rangle_{\rho_{\lambda}} \rangle$, where $p_i = \langle \hat{B}_i \rangle_{\rho_{\lambda}}$. This causes the QFI in the absence of a perfect RF to reduce to $H \left( \rho_{\lambda} \right) = \sum_i p_i H_U \left( \rho_i \right)$, where $H_U$ is the QFI for the unitary channel as given in equation (28). For an explicit example, we refer the readers to the first example of section 3.4.

3.3. Relation to alignment-free communication protocols

As pointed out in [4], the problem of communication between two parties who do not share a common CRF can be mapped into the problem of communication between the parties via the noisy quantum channel in (8), while assuming that their local reference frames are aligned. As an example consider the case where Alice and Bob do not share a Cartesian reference frame as depicted in figure 3(a). Suppose Alice encodes a parameter $\lambda$ in a qubit plus a quantum sample of her local Cartesian RF, i.e. a quantum Cartesian RF. Also assume that the encoding process is done via a unitary channel $U(\lambda) = \exp(-i\tilde{K}\lambda)$, where $\tilde{K}$ is the generator of the unitary transformation. She then transmits the qubit together with the quantum token of her local Cartesian reference $^5\langle \hat{B}_i \rangle_{\rho_{\lambda}} = 1$. 

![Figure 3](image-url)

Figure 3. (a) Communication between two parties in the absence of aligned classical reference frames. (b) The effect of misalignment can be viewed as a noisy channel $\mathcal{G}$ in the presence of a shared CRF between Alice and Bob.
frame, namely the state $|\psi_{\lambda}\rangle = U(\lambda)|\psi_0\rangle \bigotimes |\psi_{\text{QRF}}\rangle$, where $|\psi_0\rangle$ and $|\psi_{\text{QRF}}\rangle$ are the initial state of qubit and the QRF respectively.

The state of the whole system will be decohered with respect to Bob’s local reference frame due to Bob’s lack of knowledge about the relative rotation that relates his local RF to Alice’s RF, i.e. $\rho_{K} = G[\rho]$. As explained in the previous section this decoherence effect can be taken into account by analysing the efficiency of communication in the presence of noise, i.e. we can assume that Alice and Bob have access to a shared CRF but they only have access to a noisy quantum channel as their means of communication, as depicted in figure 3(b). The type of noisy channel is dictated by the type of reference frame that the parties do not share. For instance, in the case of Cartesian reference frame, the generators $G$ are the generators of the group SO(3). Motivated by the analysis of the previous section, again we split the problem into two subproblems.

The first case is the case of commutative noise, i.e. when $\hat{K}$ and $\hat{G}$ commute. In this case the noisy quantum channel commutes with the unitary encoding process, i.e. $G[U_{\lambda}|\psi_{\lambda}\rangle \langle \psi_{\lambda}|U_{\lambda}^{\dagger}] = U_{\lambda}G[|\psi_{\lambda}\rangle \langle \psi_{\lambda}|]U_{\lambda}^{\dagger}$ for every $\lambda$. While this property prevents the parties to be able to communicate with encoding the message solely in the qubit, sending a sample of Alice’s local CRF together with the qubit makes the communication scheme plausible [27]. As mentioned earlier, this is due to the existence of DFSs. In fact in such cases, Alice has access to the states that remain invariant under the noisy channel $G$, i.e. states for which we have $G[|\psi_{\lambda}\rangle \langle \psi_{\lambda}|] = |\psi_{\lambda}\rangle \langle \psi_{\lambda}|$. As an example, consider the case where Alice and Bob do not share a phase reference frame. If Alice encodes $\lambda$ only using a single harmonic oscillator, then the states that she can prepare are of the general form $\sum_{n} c_{n} |n\rangle$ and the operator $G$ is the number operator $\hat{N}$. It is easy to check that in this case these states get completely decohered from Bob’s point of view. In contrast if Alice chooses two quantum harmonic oscillators as the carrier of her message and the operator $G = N \bigotimes 1 + 1 \bigotimes N$, then the states of the form $a \ket{0, n} + b \ket{1, n - 1}$ are invariant under the action of the $g$-twirling map. For an explicit example of this case we refer the readers to the first example given in section 3.4.

The second case is the case of non-commutative noise, i.e. when the operators $\hat{K}$ and $\hat{G}$ do not commute. This case is particularly interesting since estimation of the parameter is possible even in the absence of DFSs as explained in section 3.2. In the second and third example of section 3.4, we present two scenarios in which the absence of an ideal CRF results in non-commutative noise.

3.4. Examples

In the previous sections we analysed how QRFs modify our precision in measurement of physical parameters such as time, phase, direction in space, etc. We also explained how misalignment of local RFs is connected with commutative and non-commutative noise in quantum parameter estimation. We are now in place to present some explicit examples.

3.4.1. Example (i): two non-interacting quantum harmonic oscillators

The scenario that we consider in this example is as follows. Alice and Bob do not have access to synchronized clocks, i.e. they do not share a common classical RF for time. Alice prepares a state $|\psi_{\lambda}\rangle = U_{\lambda}|\psi_0\rangle$, where $U_{\lambda} = e^{i\hat{K}\lambda}$ and $\hat{K}$ is the operator which imprints the parameter $\lambda$ into the fiducial state $|\psi_0\rangle$. Since the local clocks of the parties are not synchronised, in Bob’s frame the state of the system is given by equation (6), where $U(t) = e^{-i\hat{H}t}$ and $\hat{G} \equiv \hat{H}$ is the Hamiltonian of the qubit and the QRF. The operators $\hat{P}$ are the projectors into subspaces with total energy $E_{\text{q}}$. We analyse the QFI of the state $\rho_{\text{B}} = G[\rho]$ which tells us how precise Bob will be able to measure $\lambda$.

Let us consider the example of two non-interacting quantum harmonic oscillators with the Hamiltonian $H = \hbar \omega (a^{\dagger}a + b^{\dagger}b)$, where $a$ and $a^{\dagger}$ are the creation and annihilation operators corresponding to the first quantum harmonic oscillator and $b$ and $b^{\dagger}$ are the creation and annihilation operators corresponding to the second quantum harmonic oscillator. The fiducial state is the product form $|\psi_{\lambda}\rangle = |\psi_q\rangle \bigotimes |\psi_{\text{QRF}}\rangle$, where $|\psi_q\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ and $|0\rangle$ and $|1\rangle$ are the eigenstates of number operator $\hat{N}_q = a a^{\dagger}$ with eigenvalues 0 and 1 respectively. We choose the generator of the unitary channel $\hat{K}$ to be $\hat{K} = a^{\dagger}a$. It is worth emphasizing at this point that in this example $[\hat{K}, \hat{H}] = 0$. Note that this example is similar to the quantum communication scheme between two parties when they do not have a common phase reference frame as was considered in [22].

Using equation (28), it is straightforward to find the QFI in Alice’s frame as $H(\rho) = 1$. Note that Alice’s QFI is independent of the state of the QRF. On the other hand, if we consider the state $|\psi_{\text{QRF}}\rangle = \sum_{n=0}^{N-1} c_{n} |n\rangle$, then using either equations (22) or (10), we find the QFI in Bob’s frame as

---

6 This is the definition of $G$-invariant states.
If Alice chooses a uniform superposition of Fock states, i.e. the state $|\psi_{\text{US}}\rangle = \sum_n |n\rangle$, then using (31) we can easily compute Bob’s QFI as

$$H_0(B) = 2 - 2 \left( \sum_{n=0}^{N-1} c_n^4 + c_{n+1}^4 \right).$$

(31)

The SLD provides us with the optimal observable to measure in order to minimize the statistical error in measurement of $\lambda$ and saturate the quantum Cramér–Rao bound. This can be easily verified by checking that $L(\rho_{B,\text{US}})$ satisfies the condition (5).

Let us next consider a squeezed, displaced vacuum state \cite{29} as the state of the QRF, i.e.

$$|\alpha, r\rangle = \frac{e^{-\frac{1}{2}(1 + \tanh r)}}{\sqrt{\cosh r}} \sum_{n=0}^{\infty} \frac{(\tanh r)^n}{\sqrt{2^n n!}} H_n\left(\frac{\gamma}{\sinh 2r}\right) |n\rangle,$$

(33)

where $H_n(x)$ is the Hermite polynomial, $r$ is the squeezing parameter, $\alpha$ is the displacement parameter and $\gamma = \alpha \exp (r)$. The mean energy of this state is equal to $\alpha^2 + \sinh^2 r$. We define parameter $x$ as the fraction of initial mean energy due to displacing the vacuum, i.e. $x = \frac{\alpha^2}{\langle \hat{N} \rangle}$. Note that with this definition, $x = 0$ and $x = 1$ represent a squeezed state and a coherent state respectively. In particular, noticing that in the Fock basis a squeezed state is of the form $|r\rangle = \sum_n a_n |2n\rangle$ together with equation (6), we find that $H(|\psi_r\rangle \langle r|, |\psi_r\rangle) = 0$, i.e. Bob won’t be able to decode $\hat{\lambda}$ if Alice prepares the QRF in a squeezed state.

In figure 4 we have plotted Bob’s QFI for the state $|\alpha, r\rangle$ in terms of $x$ and $\langle \hat{N} \rangle$. As can be seen in this figure, if we fix the mean energy of the QRF, then it is optimal to have zero squeezing in the initial state of the QRF, i.e. $x = 1$. This corresponds to preparing the QRF in a coherent state. Using equation (31) we find Bob’s QFI for a coherent state as

$$H(B) = 2 - \frac{\alpha^4}{1 + \alpha^4} M \left( 1, 2 + \alpha^4, -\alpha^4 \right),$$

(34)

where $M(a, b, z)$ is a confluent hypergeometric function. We derive the asymptotic expression for the limit of large mean energy, i.e. $|\alpha|^2 \to \infty$, as

$$H(B) \approx 1 - \frac{1}{4 \left( |\alpha|^2 + 1 \right)}.$$

(35)

In figure 5, we compare Bob’s QFI for different states chosen by Alice as a quantum sample of her local CRF. This figure shows that a coherent state outperforms the uniform superposition of Fock states. This is in complete agreement with the results of \cite{23} where it is shown that if Bob chooses the maximum-likelihood estimation...
process to decode $\lambda$, then choosing a coherent state as the initial state of the QRF instead of the state $\psi_{US}$ improves the efficiency of the communication protocol.

Also we maximize the QFI in (31) numerically, which provides us with the probability amplitudes of the optimal state for fixed $N$, i.e. the state that maximizes the QFI or minimizes Bob’s statistical error in measuring $\lambda$. The green square-shaped dots in figure 5 represent the QFI for the optimal state. As can be seen from the figure the coherent state is nearly optimal in this case.

3.4.2. Example (II): two interacting quantum harmonic oscillators

The authors of [28] considered a system of two non-interacting harmonic oscillators. They showed that if one of the harmonic oscillators is used as a quantum clock for the other one, the resultant dynamics will be an approximation to Schrödinger dynamics. In this section we investigate the quality of such quantum clocks when the system under study and the clock interact with each other. We analyse the accuracy with which the phase of a quantum system can be measured when we don’t have access to an ideal classical clock.

Let us consider the example of two interacting quantum harmonic oscillators with the total Hamiltonian

$$\hat{H} = \hbar \omega (a^\dagger a + b^\dagger b) + \hbar \kappa (a^\dagger b + b^\dagger a),$$

where $\kappa$ is the interaction strength. Similar to the example of two non-interacting quantum harmonic oscillators, we consider the generator of the unitary channel to be the number operator, i.e. $\hat{K} = a^\dagger a$. Note that the two operators $\hat{K}$ and $\hat{H}$ do not commute in this case, $[\hat{K}, \hat{H}] = \kappa (a^\dagger b - a b^\dagger)$. As mentioned earlier whenever these two operators do not commute, even in the absence of degenerate subspaces of total energy, we may still be able to estimate the parameter. For simplicity we assume that frequency $\omega$ is not a fraction of the interaction strength $\kappa$, i.e.

$$\forall P, R \in \mathbb{Z}, P\omega \neq R\kappa. \quad (37)$$

This assumption ensures that the Hamiltonian $\hat{H}$ does not possess any degenerate eigenvalues. In order to make the computations easier, we change the basis by defining a new set of annihilation operators as [30]

$$A = \frac{1}{\sqrt{2}} (a + b), \quad B = \frac{1}{\sqrt{2}} (a - b). \quad (38)$$

This change of basis allows us to write the Hamiltonian as $\hat{H} = \hbar (\omega + \kappa) A^\dagger A + \hbar (\omega - \kappa) B^\dagger B$ with the eigenvectors

$$|m, n\rangle = \frac{(A^\dagger)^m (B^\dagger)^n}{\sqrt{m!} \sqrt{n!}} |0, 0\rangle, \quad (39)$$

which using $|0, 0\rangle = |0, 0\rangle$ and applying the creation operators can be written in terms of the Fock basis as

$$|m, n\rangle = \sum_{k=0}^m \sum_{l=0}^n \binom{m}{k} \binom{n}{l} \frac{(k + l)! (m + n - k - l)!}{2^{m+n} m! n!} |k + l, m + n - k - l\rangle. \quad (40)$$

Now let us consider that the QRF is initially prepared in the uniform superposition of Fock states. Using equation (10), we derive the QFI of the averaged state as

![Figure 5. Bob’s QFI in terms of mean photon number in the initial state of the QRF for three different states. The solid-black, dashed-brown and dotted-green curves correspond to coherent state, uniform superposition state $\psi_{US}$, and the optimal state.](image-url)
where \( \lambda \) = \( \lambda \) + \( \lambda \) + \( \lambda \) - \( \lambda \), and \( \cdot \) is a floor function.

In this example since \( \hat{K} \) and \( \hat{G} \) do not commute, \( (\hat{G} [\rho]) \) is \( \lambda \)-dependent as opposed to the first example where Bob’s QFI was independent of the encoded parameter \( \lambda \). In figure 6 we have plotted the QFI \( H (\rho_B) \) in terms of \( \lambda \) for increasing values of the mean energy in the state of the QRF. The maximum and minimum of the QFI occurs at \( \lambda = \pm \frac{\pi}{2} \) and \( \lambda = 0, \pm \pi \) respectively. Note that, even for very large \( N \), the QFI does not approach the ideal case. In other words, even in the limit of very large mean energy in the initial state of the quantum clock, we can not estimate the phase parameter \( \lambda \) as precise as we could if we had access to a classical clock. This can be proved using the necessary conditions (26) and (27). One can easily check that

\[
\eta = \rho \omega G K \text{Cov} (\hat{G}, \hat{K}) \frac{4}{\pi},
\]

which means that independent of \( N \) and \( \lambda \), the QFI is always smaller than one, i.e.

\[
\rho < H (\rho_B) \frac{1}{\pi}.
\]

Similarly, since \( \eta \lambda \approx \frac{2i\hbar}{3} \sqrt{N} \sin \lambda \), we can deduce that independent of \( N \) for \( \lambda \neq -\pi, 0, \pi \), the QFI is always positive.

3.4.3. Example (III): direction indicator

In the first example we observed how using a QRF enables Alice and Bob to perform an alignment-free communication protocol in the presence of commutative noise, while in the second example we analysed how the absence of a perfect CRF can cause non-commutative noise which then reduces the precision with which a physical parameter can be estimated. Here, we present an example in which the noise caused due to Bob’s lack of knowledge about Alice’s local reference frame is non-commutative. We analyse how precise Bob can extract \( \lambda \) if Alice does not send him a quantum sample of her local CRF.

Let us start with the case where Alice wishes to both encode and decode a parameter herself. She chooses a spin \( -\frac{1}{2} \) particle as the physical system to encode a parameter \( \lambda \) and then she encodes this parameter using a unitary channel with the generator

\[
\hat{K} = \frac{1}{2} \vec{n} \cdot \vec{\sigma} = \frac{1}{2} (x\sigma_x + y\sigma_y + z\sigma_z).
\]

This is the generator of a general rotation in the Bloch sphere around the axis \( \vec{n} = (x, y, z) \), where \( x^2 + y^2 + z^2 = 1 \) and \( x, y, z \) are real parameters. For simplicity we choose the fiducial state to be the eigenstate of \( \sigma_z \) with eigenvalue 1, i.e. \( \ket{\psi_0} = \ket{0} \).

Using Euler’s formula for Pauli matrices\(^8\), we can write Alice’s prepared state as

\[
\ket{\psi_{\lambda}} = \left( \cos \left( \frac{\lambda}{2} \right) - i z \sin \left( \frac{\lambda}{2} \right) \right) \ket{0} + (y - i x) \sin \left( \frac{\lambda}{2} \right) \ket{1}.
\]

\(^8\) \( e^{-i\hat{K}t} = \cos \left( \frac{\hat{n}}{2} \right) \mathbb{I} - i \sin \left( \frac{\hat{n}}{2} \right) (\vec{n} \cdot \vec{\sigma}) \).
Then using equation (28), the QFI in Alice’s frame reads as

\[ \rho = -z_2 H(z) \] (44)

Note that for \( z = 1 \), the corresponding generator is \( \hat{K} = \frac{1}{2} \sigma_z \) which leaves the fiducial state invariant, i.e. \( \exp(-i\frac{\pi}{2})|0\rangle = |0\rangle \). Since the encoding process is not successful, the QFI, \( H(\rho) \), vanishes which simply means that a different generator needs to be used at the preparation stage. The QFI takes its maximum value when when the parameter \( \lambda \) is encoded via a rotation around any vector in the \( xy \)-plane, i.e. when \( z = 0 \).

Now suppose that Alice and Bob only share their \( z \)-axis, i.e. Bob is completely unaware of the relative angle \( t \) between his other two axes and Alice’s, as depicted in figure 7. In this case, \( \hat{G} \) is the generator of rotations around \( z \)-axis, i.e. \( \hat{G} = \frac{1}{2} \sigma_z \). Using equation (21), the QFI in Bob’s frame can be written as

\[ \rho_B = \frac{1 - z^2}{1 + z^2} \tan^2 \left( \frac{\lambda}{2} \right) \] (45)

Again note that for \( z = 1 \), the QFI is zero in Bob’s frame. This is expected, since Bob lacks some information with respect to Alice, therefore Alice’s inability in extracting information about \( \lambda \) means that Bob will not be able to decode the message either, i.e. \( \rho_B = 0 \). On the other hand, as can be seen from (45), when \( z = 0 \) the QFI is the same in Alice’s frame and Bob’s frame. Figure 8 depicts the two cases of \( \hat{n} = (1, 0, 0) \) and \( \hat{n} = (0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \). For the former case, the efficiency of communication is \( \lambda \)-independent, whereas for the latter case it is \( \lambda \)-dependent, as can be seen in figure 9. In this figure, we have plotted Bob’s QFI in terms of \( \lambda \) and \( z \) for general \( \hat{n} = (x, y, z) \). We observe that as \( \lambda \) approaches the value \( \pi \), the QFI approaches its minimum value, i.e.
$H(\rho_B) \rightarrow 0$. In other words, for the chosen encoding operator $\hat{K}$ and the fiducial state $|0\rangle$, Bob will not be able to distinguish $\rho_B$ form its neighbouring states $\rho_{\pi \epsilon}$, where $\epsilon$ is a very small change in $\lambda = \pi$.

Also after some algebra and with the aid of equation (19), we find the SLD operator that achieves the QFI in (45) as

$$L(\rho_B) = \frac{(z^2 - 1) \tan \left( \frac{\lambda}{2} \right)}{1 + z^2 \tan^2 \left( \frac{\lambda}{2} \right)} |0\rangle\langle 0| + \cot \left( \frac{\lambda}{2} \right) |1\rangle\langle 1|.$$  \hspace{1cm} (46)

Again the optimal POVM can be constructed from the eigenvectors of this operator, i.e. $|0\rangle\langle 0|$, $|1\rangle\langle 1|$. This simply means that the most informative measurement for Bob is the measurement in the computational basis.

In order to verify that this is in fact the case, we can either use the relation (5) or we can compute the classical Fisher information using equation (1) and show that it is equal to the QFI (45).

4. Discussions and outlook

In quantum metrological schemes the existence of a perfect CRF is often assumed. Here we have exploited the powerful mathematical tools from quantum metrology in order to analyse the modification of the ultimate precision limits due to the absence of such frames of reference. We considered the effects of commutative and non-commutative noise due to lack of a certain CRF. In doing so, we showed that the more the encoding process and the nature of the noise resemble each other, the more precision is lost (theorem 3.3). We also presented necessary and sufficient conditions for two extreme cases (theorem 3.1, (23), (24)). The first case is when the absence of an ideal RF does not reduce the accuracy of estimation and the second case is when the estimation of the parameter with respect to QRFs is no longer possible. Moreover, by explaining the connection between noisy parameter estimation protocols and alignment-free communication schemes [4], we shed light into different aspects of quantum communication in the absence of aligned reference frames. In particular, we showed that for an encoding operator which commutes with the non-degenerate noise channel we are not able estimate the parameter anymore (theorem 3.2), this is no longer true if we choose an encoding operator which does not commute.

Our future line of research includes incorporating other sources of noise in the alignment-free communication protocols. We are interested in the regimes where relativity starts to play a more significant role [31]. Relativistic effects such as the decoherence caused due to non-uniform motion [32] or the effects of the gravitational field of the Earth [33] are the possible sources of noise that yet need to be considered. Recently in [34, 35] techniques for the optimal estimation of parameters which appear in quantum field theory in curved spacetime have been presented. This enables the estimation of parameters such as proper acceleration, proper time, relative distance, amplitude of gravitational waves [36], as well as spacetime parameters of interest, such as the expansion rate of the Universe or the mass of a black hole. Finally, the reference frames of relativistic observers is inevitably misaligned with respect to non-relativistic observers, therefore it is crucial to consider the effect of relativistic noise on estimation of parameters of interest such as acceleration, time, phase, temperature, etc.
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Appendix A. Derivation of QFI and the SLD operator in the absence of perfect RFs

From equation (8), we immediately observe that eigenvalues of transformed density matrix $\rho_B$ are

$$\lambda_p \propto \langle \psi \psi \rangle$$

with respective normalized eigenvectors $\psi_p \hat{i}$.

Let $\phi_j$ be a set orthonormal eigenvectors of $\rho_B$ with respective eigenvalue 0. Using equation (4) we have

$$\sum \sum \sum \sum \sum \sum \rho \psi \rho \psi \psi \psi \rho \psi\hat{i} i i$$

and together with the Parseval identity, i.e.

$$\sum \sum \sum \sum \sum \rho \psi \rho \psi \psi \psi \rho \psi\hat{i} i i$$

we can remove the dependence on states $\phi_j$. Then $H(\rho_B)$ is

$$H(\rho_B) = \sum \sum \sum \sum \sum \sum \rho \psi \rho \psi \psi \psi \rho \psi\hat{i} i i$$

After substituting $\rho_p = |\psi_j \rangle \langle \psi_j |$

$$H(\rho_B) = \sum \sum \sum \sum \sum \sum \rho \psi \rho \psi \psi \psi \rho \psi\hat{i} i i$$

where $\partial_{\alpha \psi} = \sum_k (\partial_{\alpha} |k \rangle |k \rangle)$ for $k$-independent basis $|k \rangle$. The sum in (A.4) consists only of elements where $p_i \neq 0$, however, by differentiating $p_i = \langle \psi | \hat{P} | \psi \rangle = 0$ and using Cauchy–Schwarz inequality on $\langle \partial_{\alpha \psi} | \hat{P} | \psi \rangle$ we get $\langle \partial_{\alpha \psi} | \hat{P} | \partial_{\alpha \psi} | \hat{P} | \psi \rangle = 0$. Now summing over all $i$ and using the completeness relation $\sum \hat{P} = 1$ we get

$$H(\rho_B) = 4 \langle \partial_{\alpha \psi} | \partial_{\alpha \psi} \rangle - 4 \sum \sum \sum \sum \sum \sum \rho \psi \rho \psi \psi \psi \rho \psi\hat{i} i i$$

SLD (19) can be derived analogously, where instead of Parseval identity we use completeness relation $\sum \langle \phi_i | \langle \phi_j | = 1 - \sum \frac{\beta_i | \psi \rangle \langle \psi | \hat{P} \beta_i}{p_i}$.

Appendix B. Proof of theorem 3.1

Here we prove that $0 \leq l(\rho, G) \leq H(\rho)$ and the equality conditions. $l(\rho, G) \leq H(\rho)$ follows immediately from definition (9). Let us prove $l(\rho, G) \geq 0$. Looking at the expression for QFI loss, i.e. equation (9), we need to prove that

$$\sum \sum \sum \sum \sum \sum \rho \psi \rho \psi \psi \psi \rho \psi\hat{i} i i$$

where $\partial_{\alpha \psi} = \sum_k (\partial_{\alpha} |k \rangle |k \rangle)$ for $k$-independent basis $|k \rangle$. The sum in (A.4) consists only of elements where $p_i \neq 0$, however, by differentiating $p_i = \langle \psi | \hat{P} | \psi \rangle = 0$ and using Cauchy–Schwarz inequality on $\langle \partial_{\alpha \psi} | \hat{P} | \psi \rangle$ we get $\langle \partial_{\alpha \psi} | \hat{P} | \partial_{\alpha \psi} | \hat{P} | \psi \rangle = 0$. Now summing over all $i$ and using the completeness relation $\sum \hat{P} = 1$ we get

$$H(\rho_B) = 4 \langle \partial_{\alpha \psi} | \partial_{\alpha \psi} \rangle - 4 \sum \sum \sum \sum \sum \sum \rho \psi \rho \psi \psi \psi \rho \psi\hat{i} i i$$

where $\partial_{\alpha \psi} = \sum_k (\partial_{\alpha} |k \rangle |k \rangle)$ for $k$-independent basis $|k \rangle$. The sum in (A.4) consists only of elements where $p_i \neq 0$, however, by differentiating $p_i = \langle \psi | \hat{P} | \psi \rangle = 0$ and using Cauchy–Schwarz inequality on $\langle \partial_{\alpha \psi} | \hat{P} | \psi \rangle$ we get $\langle \partial_{\alpha \psi} | \hat{P} | \partial_{\alpha \psi} | \hat{P} | \psi \rangle = 0$. Now summing over all $i$ and using the completeness relation $\sum \hat{P} = 1$ we get

$$H(\rho_B) = 4 \langle \partial_{\alpha \psi} | \partial_{\alpha \psi} \rangle - 4 \sum \sum \sum \sum \sum \sum \rho \psi \rho \psi \psi \psi \rho \psi\hat{i} i i$$

SLD (19) can be derived analogously, where instead of Parseval identity we use completeness relation $\sum \langle \phi_i | \langle \phi_j | = 1 - \sum \frac{\beta_i | \psi \rangle \langle \psi | \hat{P} \beta_i}{p_i}$.
First, let us define $\bar{\partial}_\psi := \sum_{i, \tilde{P}_i \neq 0} 3m \frac{\langle \psi \mid \tilde{P}_i \rangle \hat{P}_i \psi}{\langle \psi \mid \tilde{P}_i \psi \rangle}$. Then using the fact that the state $|\psi\rangle$ is normalized, i.e. $\|\psi\| = 1$, the Cauchy–Schwarz inequality and that for any state $|\psi\rangle$, $\tilde{P}_i \equiv \langle \psi \mid \hat{P}_i \psi \rangle = 0$ if and only if $\hat{P}_i |\psi\rangle = 0$ and therefore $\langle \psi \mid \hat{P}_i \partial_\psi \rangle = 0$, together with the completeness relation $\sum_i \hat{P}_i = 1$, we have

$$\text{LHS} = ||\partial_\psi\rangle||^2 = \left( \sum_i \frac{3m \langle \psi \mid \hat{P}_i \partial_\psi \rangle}{\langle \psi \mid \hat{P}_i \psi \rangle} \right)^2 = \left( \frac{3m \langle \psi \mid \hat{P}_i \partial_\psi \rangle}{\langle \psi \mid \hat{P}_i \psi \rangle} \right)^2 = \|\langle \psi \mid \partial_\psi \rangle\|^2 = \left( \sum_i \frac{3m \langle \psi \mid \hat{P}_i \partial_\psi \rangle}{\langle \psi \mid \hat{P}_i \psi \rangle} \right)^2,$$

where for the last step we have used equation (11). Now, because Cauchy–Schwarz inequality is saturated if and only if there exists a complex number $c$ such that $\partial_\psi = c |\psi\rangle$, by re-writing the state $|\psi\rangle$ as $\sum_i \frac{\langle \psi \mid \hat{P}_i \rangle}{\sqrt{\langle \psi \mid \hat{P}_i \psi \rangle}} |\psi\rangle$ we find that equation (B.1) is saturated if and only if

$$\sum_i \frac{3m \langle \psi \mid \hat{P}_i \rangle}{\langle \psi \mid \hat{P}_i \psi \rangle} |\psi\rangle = 0,$$

which together with orthogonality condition for the projectors $\hat{P}_i$ leads to the no-loss condition (15), i.e.

$$l(\rho, G) = 0 \iff \exists c \in \mathbb{C}, \forall i, 3m \langle \psi \mid \hat{P}_i \partial_\psi \rangle = c \langle \psi \mid \hat{P}_i \psi \rangle.$$

Now let us derive the max-loss condition (17). From the definition of QFI loss and that $|\langle \psi \mid \partial_\psi \rangle| = |\langle \psi \mid \partial_\psi \rangle|$, we can write

$$l(\rho, G) = 4 \langle \psi \mid \partial_\psi \rangle \langle \partial_\psi \rangle - 4 \|\langle \psi \mid \partial_\psi \rangle\|^2 = 4 \|\langle \psi \mid \partial_\psi \rangle\|^2 - 4 \|\langle \psi \mid \partial_\psi \rangle\|^2.$$

Therefore by comparing (B.5) and (12) we have

$$l(\rho, \hat{G}) = H(\rho) \iff \langle \partial_\psi \rangle \langle \partial_\psi \rangle = \langle \partial_\psi \rangle \langle \partial_\psi \rangle.$$

Similar to the previous case we can write $|\partial_\psi \rangle$ in the complete orthonormal basis $\{ \frac{\hat{P}_i |\psi\rangle}{\sqrt{\langle \psi \mid \hat{P}_i \psi \rangle}}, |\phi_j\rangle \}_{i,j}$ as

$$|\partial_\psi \rangle = \sum_i \frac{\langle \psi \mid \hat{P}_i \rangle \hat{P}_i |\psi\rangle}{\sqrt{\langle \psi \mid \hat{P}_i \psi \rangle}} + \sum_j \langle \phi_j \partial_\psi \rangle |\phi_j\rangle.$$

Where $|\phi_j\rangle$ span the rest of the Hilbert space which is not spanned by vectors $\frac{\hat{P}_i |\psi\rangle}{\sqrt{\langle \psi \mid \hat{P}_i \psi \rangle}}$. After multiplying by $|\partial_\psi \rangle$ we get the Parseval identity, i.e.

$$\langle \partial_\psi \rangle |\partial_\psi \rangle = \sum_i \left( \frac{\langle \psi \mid \hat{P}_i \rangle |\partial_\psi \rangle}{\sqrt{\langle \psi \mid \hat{P}_i \psi \rangle}} \right)^2 + \sum_j \langle \phi_j \partial_\psi \rangle |\phi_j\rangle.$$

Comparing this with (B.6) we get condition for max-loss as $l(\rho, \hat{G}) = H(\rho) \iff$ \forall $i, \Re \langle \psi \mid \hat{P}_i \partial_\psi \rangle = 0$ and \forall $|\phi_j\rangle, \langle \phi_j \partial_\psi \rangle = 0$.

**Appendix C. Alternative description in terms of eigenvectors of $\hat{G}$**

Here we introduce alternative an description of QFI using eigenvectors of the operator $\hat{G}$. Let $\{ |\nu_j\rangle \}_{j}$ be a set of orthonormal eigenvectors of $\hat{G}$ with respective eigenvalue $G_j$. Then the projection operator into the eigenspace corresponding to eigenvalue $G_j$ can be written as
\[ \hat{P}_i = \sum_j \left| v_{ij} \right\rangle \left\langle v_{ij} \right| . \]  

(C.1)

Then we substitute this expression into (12) and find the QFI in terms of the eigenvectors \( \left| v_{ij} \right\rangle \) as

\[ H(\rho_B) = 4 \left\langle \partial_\psi \left| \partial_\psi \right| \psi \right\rangle - 4 \sum_i \left( \frac{3m}{\sum \left\langle \psi \left| v_{ij} \right| \psi \right\rangle} \right)^2 \left( \left| \psi \right\rangle \left\langle \psi \right| \right) \right] . \]  

(C.2)

In order to write this expression in a more compact way we define the un-normalized states \( \left| a_i \right\rangle \) and \( \left| b_i \right\rangle \) as

\[ \left| a_i \right\rangle = \left| v_{ij} \right\rangle, \quad \left| b_i \right\rangle = \left| \partial_\psi \right| \psi \right\rangle, \]  

(C.3)

where \( n_i \) denotes the dimension of the subspace corresponding to eigenvalue \( G_i \). This way we can write equation (C.2) as

\[ H(\rho_B) = 4 \left\langle \partial_\psi \left| \partial_\psi \right| \psi \right\rangle - \sum_i \left( \frac{3m}{\left| a_i \right|} \right)^2 \left| \psi \right\rangle \left\langle \psi \right| \right] , \]  

(C.4)

where \( \langle \cdot \left| \cdot \right\rangle \) is a standard inner product on \( \mathbb{C}^n_i \).

Let us here also prove that when \( \hat{K} \) and \( \hat{G} \) commute and the spectrum of \( \hat{G} \) is non-degenerate, then \( H(\rho_B) = 0 \), i.e. all the information about \( \lambda \) is lost due to the noise. This can be done either starting from (C.2) or by using max-loss condition (24). Here we choose the latter. Because \( \hat{G} \) has non-degenerate spectrum, the projectors \( \hat{P}_i \) are all rank-one projectors, i.e. \( \hat{P}_i = \left| v_i \right\rangle \left\langle v_i \right| \right] \). Also since \( \hat{K} \) and \( \hat{G} \) commute, then all projectors also commute with \( \hat{K} \), i.e. \( \left| \hat{K} \right| v_i \right\rangle \left\langle v_i \right| = 0 \right. \left. , \right| \right| \) is a complete basis. Consider a subset of this basis that does not have any overlap with the state \( \left| \psi \right\rangle \right. \left. , \right| \right| \), i.e. the subset \( \left| v_j \right\rangle \in \left( \left| v_i \right\rangle \right| \right. \left. \left| \psi \right\rangle = 0 \right. \left. , \right| \right| \). This set replaces \( \left| \phi_j \right\rangle \right. \left. \left| \psi \right\rangle = 0 \right. \left. , \right| \right| \) in the max-loss condition (24). Now we have

\[ \left| v_j \right\rangle \hat{K} \left| \psi \right\rangle = \left| v_j \right\rangle \left| v_j \right\rangle \left| \hat{K} \right| \left| \psi \right\rangle = \left| v_j \right\rangle \left| \hat{K} \right| \left| v_j \right\rangle \left| \psi \right\rangle = 0 , \]  

(C.5)

which makes the proof complete.

**Appendix D. Asymptotic scaling of \( H(\rho_B) \) in example (I) for the coherent state as the initial state of the QRF**

In the case that the two operators \( \hat{K} \) and \( \hat{G} \) commute and in the limit of large initial mean energy in the state of the QRF, the classical limit of a quantum reference frame should be recovered, i.e. \( H(\rho_B) \rightarrow 1 \), as was shown in the first example for the uniform superposition of Fock states. Here we analyse this asymptotic behaviour for a coherent state as the initial state of the QRF. We expect

\[ M \left( \left| \alpha \right| \right) = M \left( 1, 2 + \left| \alpha \right|, -\left| \alpha \right| \right) = \frac{1}{2} + f \left( \left| \alpha \right| \right) , \]  

(D.1)

such that \( \lim_{\left| \alpha \right| \rightarrow \infty} f \left( \left| \alpha \right| \right) = 0 \). As figure C1 suggests, this is true for large enough mean photon number in the initial state of the coherent state. Moreover, from this figure one can see that, for the mean photon number above a certain threshold, we can find constants \( C_1 \) and \( C_2 \) such that

\[ \frac{\left| \alpha \right|}{\left| \alpha \right| + C_1} M \left( \left| \alpha \right| \right) \leq \frac{1}{2} \leq \frac{\left| \alpha \right|}{\left| \alpha \right| + C_2} M \left( \left| \alpha \right| \right) . \]  

(D.2)

Re-arranging the expression above, we can find \( \alpha \)-dependent lower and upper bounds for the function \( f \) as

\[ \frac{C_1}{2 \left| \alpha \right|} \leq f \left( \left| \alpha \right| \right) \leq \frac{C_2}{2 \left| \alpha \right|}. \]  

(D.3)

Therefore the QFI in Bob’s reference frame is

\[ H(\rho_B) = 2 \frac{\left| \alpha \right|}{1 + \left| \alpha \right|} \left( \frac{1}{2} + f \left( \left| \alpha \right| \right) \right) \]  

\[ = 1 - g \left( \left| \alpha \right| \right), \]  

(D.4)

where \( \frac{1 - C_1}{1 + \left| \alpha \right|} \leq g \left( \left| \alpha \right| \right) \leq \frac{1 - C_1}{1 + \left| \alpha \right|} \). Choosing \( C_1 = 0.749999 \) and \( C_2 = 0.75 \), we tighten the lower and upper bounds.
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Figure C1. Function $\frac{|\alpha|^2}{|\alpha|^2 + M}$ in terms of $|\alpha|^2$. From top to bottom for $\epsilon = 0, 0.74, 0.75$ and 1.