On a Type of Para Kenmotsu Manifold

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Abstract: The object of this paper is to study a class of almost para contact metric manifold namely para Kenmotsu (briefly p-Kenmotsu) manifold in which \( R(X, Y) . C = 0 \) where C is the conformal curvature tensor of the manifold and R is the Riemannian curvature and \( R(X, Y) \) is considered as a derivation of the tensor algebra at each point of the manifold for tangent vectors X and Y.

Key words: Kenmotsu Manifold - Curvature Tensor - Ricci Tensor - Tangent Vector

INTRODUCTION

Sato [1] defined the notions of an almost para contact Riemannian manifold. After that, T. Adati and K. Matsumoto [2] defined and studied para-Sasakian and SP-Sasakian manifolds which are regarded as a special kind of an almost contact Riemannian manifolds. Before Sato, Kenmotsu [3] defined a class of almost contact Riemannian manifolds. In 1995, B. B. Sinha and K. L. Sai Prasad [4] have defined a class of almost para contact metric manifolds namely para Kenmotsu (briefly p-Kenmotsu) and SP-Kenmotsu manifolds. They also have studied the curvature properties of p-Kenmotsu manifold and the curvature properties of semi-symmetric metric connection of SP-Kenmotsu manifold.

Let \( M_n \) be an n-dimensional differentiable manifold equipped with structure tensors \((\phi, \xi, \eta)\) where \( \phi \) is a tensor of type (1,1), \( \xi \) is a vector field, \( \eta \) is a 1-form such that

\[
\phi^2 = 0, \ \eta(\phi X) = 0, \ \text{rank} \ \phi = n - 1
\] (1.4)

\[g(\phi X, \phi Y) = g(X, Y) - \eta(X) \eta(Y)\] (1.5)

Then the manifold \( M_n \) is said to admit a almost para contact Riemannian structure \((\phi, \xi, \eta, g)\).

A manifold of dimension ‘n’ with Riemannian metric ‘g’ admitting a tensor field ‘\( \phi \)’ of type (1,1), a vector field ‘\( \xi \)’ and a 1-form ‘\( \eta \)’ satisfying (1.1), (1.3) along with

\[(\forall X) \eta Y - (\forall Y) \eta X = 0\] (1.6)

\[(\forall X) \ (\forall Y) \eta Z = [\ - g(X, Z) + \eta(X) \eta(Z) \] \( \eta(Y) + [\ - g(X, Y) + \eta(X) \eta(Y)] \eta(Z)\] (1.7)

\[\forall X, \xi \phi^2 = \phi^2 X = X - \eta(X) \xi\] (1.8)

is called a para Kenmotsu manifold or briefly P-Kenmotsu manifold [4]. This paper deals with type of p-Kenmotsu manifold in which

\[R(X, Y) . C = 0\] (1.9)

where C is the conformal curvature tensor of the manifold and R is the Riemannian curvature.

Let ‘\( g \)’ be the Riemannian metric \( g \) satisfying such that, for all vector fields X and Y on M,

\[g(X, \xi) = \eta(X)\] (1.3)

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\( (\tau, \eta) Y = g(X, Y) - \eta(X) \eta(Y) \quad (1.10) \)

\[ g(X, \xi) = \eta(X) \text{ and } (\tau, \eta) Y = \varphi (\xi, Y) \text{ where } \varphi \text{ is an associate of } \Phi. \quad (1.11) \]

is called a special P-Kenmotsu manifold or briefly SP-Kenmotsu manifold [4]. In this paper it is proved that if in a P-Kenmotsu manifold \((M^*, g) \) \((n > 3)\) the relation (1.9) holds then the manifold is conformally flat and hence is an SP-Kenmotsu manifold. Also it is shown that a conformally symmetric P-Kenmotsu manifold \((M^*, g) \) is an SP-Kenmotsu manifold for \( n > 3 \). (since it is known that \( C = 0 \) when \( n = 3 \), it has taken that \( n > 3 \)).

It is known that [4] in a P-Kenmotsu manifold the following relations hold:

\[ S(X, \xi) = -(n - 1) \eta(X) \quad (1.12) \]

\[ g [R(X, Y)Z, \xi] = \eta [R(X, Y, Z)] = g(X, Z) \eta(Y) - \eta(X) \eta(Y) \quad (1.13) \]

\[ R(X, \xi) = -1 \quad (1.14) \]

\[ R(X, \xi, X) = \xi \quad (1.15) \]

\[ R(\xi, X, \xi) = X \quad (1.16) \]

\[ R(X, Y, \xi) = \eta(X)Y - \eta(Y)X; \text{ when } X \text{ is orthogonal to } \xi \quad (1.17) \]

where \( S \) is the Ricci tensor and \( R \) is the Riemannian curvature. Moreover, it is also known that a P-Kenmotsu manifold cannot be flat and a P-Kenmotsu manifold satisfying \( R(X, Y)W = 0 \) i.e., a projectively flat P-Kenmotsu manifold is said to be Einstein manifold with the constant curvature \(-n(n-1)\).

The above results will be used in the next section.

**P-kenmotsu Manifold Satisfying \( R(X, Y)C = 0 \):**

We have

\[ C(X,Y)Z = R(X, Y)Z - \frac{1}{n-2} [g(X, Z)QX - g(X, Z)QY + S(Y, Z)X - S(X, Z)Y] + \]

\[ \frac{r}{(n-1)(n-2)} [g(Y, Z)X - g(X, Z)Y] \quad (2.1) \]

where ‘\( r \)’ is the scalar curvature and ‘\( Q \)’ is the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor ‘\( S \)’ [5] i.e.,

\[ g(QX, Y) = S(X, Y). \quad (2.2) \]

Then

\[ \eta(C(X,Y)Z) = g(C(X,Y)Z, \xi) \]

\[ = \frac{1}{n-2} \left[ \frac{r}{(n-1)} + 1 \right] g(Y, Z)(\eta(X) - g(X, Z)\eta(Y)) - (S(Y, Z)\eta(X) - S(X, Z)\eta(Y)) \quad (2.3) \]

Putting \( Z = \xi \) in (2.3), we get

\[ \eta(C(X,Y)Z) = 0 \quad (2.4) \]

Again putting \( X = \xi \) in (2.3), we get
\begin{equation}
\eta(C(\xi, Y)Z) = \frac{1}{n-2} \left[ \left( \frac{r}{n-1} + 1 \right) g(X, Z) - S(Y, Z) - \left( \frac{r}{n-1} + n \right) \eta(Y)\eta(Z) \right]. \tag{2.5}
\end{equation}

Now

\[
(R(X, Y), C)(U, V) W = R(X, Y) C(U, V) W - C(R(X, Y) U, V) W - C(U, R(X, Y) V) W - C(U, V) R(X, Y) W.
\]

In virtue of (1.9) we get

\[
R(X, Y) C(U, V) W - C(R(X, Y) U, V) W - C(U, R(X, Y) V) W - C(U, V) R(X, Y) W = 0. \tag{2.6}
\]

Therefore

\[
g(R(\xi, Y) C(U, V) W, \xi) - g(C(R(\xi, Y) U, V) W, \xi) - g(C(U, R(\xi, Y) V) W, \xi) - g(C(U, V) R(\xi, Y) W, \xi) = 0. \tag{2.7}
\]

From this it follows that

\[
\tau(C(U, V) W, \xi) = 0. \tag{2.8}
\]

where

\[
\tau(C(U, V) W, \xi) = g(C(U, V) W, \xi).
\]

Putting \(Y = U\) in (2.8) we get

\[
\tau(C(U, V) W, \xi) = 0. \tag{2.9}
\]

Let \(\{ e_i \}, i = 1, 2, \ldots, n\) be an orthonormal basis of the tangent space at any point. Then the sum \(1 = i = n\) of the relation (2.9) for \(U = e_i\) gives

\[
\tau(C(\xi, V) W) = 0. \tag{2.10}
\]

By using (2.4), we have from (2.8)

\[
\tau(C(U, V) W) = 0. \tag{2.11}
\]

In virtue of (2.5) and (2.10) we have

\[
S(V, W) = \left( \frac{r}{n-1} + 1 \right) g(V, W) - \left( \frac{r}{n-1} + n \right) \eta(V)\eta(W). \tag{2.12}
\]

Using (2.3), (2.4) and (2.12) the relation (2.11) reduces to
\[ C(U, V, W, Y) = 0. \quad (2.13) \]

From (2.13) it follows that

\[ C(U, V) W = 0. \quad (2.14) \]

Thus we can state the following theorem:

**Theorem 1:** A P-Kenmotsu manifold \((M^n, g)\) \((n > 3)\) satisfying the relation \(R(X, Y)C = 0\) is conformally flat and hence is an SP-Kenmotsu manifold.

For a conformally symmetric Riemannian manifold, we have \(\nabla C = 0\) [6] and hence for such a manifold \(R(X, Y)C = 0\) holds. Thus we have the following corollary of the above theorem:

**Corollary 1:** A conformally symmetric P-Kenmotsu manifold \((M^n, g)\) \((n > 3)\) is an SP-Kenmotsu manifold.

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