On the spectral properties of $L_{\pm}$ in three dimensions

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Abstract

This paper is part of the radial asymptotic stability analysis of the ground state soliton for either the cubic nonlinear Schrödinger or Klein–Gordon equations in three dimensions. We demonstrate by a rigorous method that the linearized scalar operators which arise in this setting, traditionally denoted by $L_{\pm}$, satisfy the gap property, at least over the radial functions. This means that the interval $(0,1]$ does not contain any eigenvalues of $L_{\pm}$ and that the threshold 1 is neither an eigenvalue nor a resonance. The gap property is required in order to prove scattering to the ground states for solutions starting on the centre-stable manifold associated with these states. This paper therefore provides the final instalment in the proof of this scattering property for the cubic Klein–Gordon equation in the radial case, see the recent theory of Nakanishi and the third author. The method developed here is quite general, and applicable to other spectral problems which arise in the theory of nonlinear equations.

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1. Introduction

1.1. The nonlinear context

Consider the nonlinear Schrödinger equation

$$i\partial_t \psi - \Delta \psi = \pm |\psi|^{p-1} \psi \quad (t, x) \in \mathbb{R}^{1+d}$$

(1)

with powers $1 < p < 2^* - 1$ where $2^* = \frac{2d}{d-2}$ in dimensions $d \geq 3$ and $2^* = \infty$ in dimensions $d = 1, 2$. Assuming that $\psi(t, x)$ is a smooth solution of sufficient spatial decay, one verifies...
by differentiating under the integral sign that mass and energy are conserved:

\[
M[\psi(t)] := \frac{1}{2} \| \psi(t) \|_2^2 = M[\psi(0)],
\]

\[
E[\psi(t)] := \int_{\mathbb{R}^d} \left( \frac{1}{2} |\nabla \psi(t)|^2 \mp \frac{1}{p+1} |\psi(t)|^{p+1} \right) \, dx = E[\psi(0)].
\]

The range \( p < 2^* - 1 \) is referred to as energy subcritical regime due to the fact that in the conserved energy the nonlinear term \( \| \psi \|_{p+1}^{p+1} \) is controlled by the \( H^1 \)-norm of \( \psi(t) \) via Sobolev embedding.

The choice of sign in front of the nonlinearity of (1) is crucial: the \(-\) sign (known as the defocusing nonlinearity) leads to a positive definite conserved energy and one has global existence and scattering to a free wave for any data in \( H^1(\mathbb{R}^d) \), see [8, 29, 30] for an account of these classical results. Scattering here refers to the property that there exists \( \psi_0 \in H^1 \) so that with the associated free solution \( \psi_0(t) := e^{-it\Delta} \psi_0 \)

\[
\| \psi(t) - \psi_0(t) \|_{H^1} \to 0 \quad t \to \infty.
\]

On the other hand, the focusing sign \( + |\psi|^{p-1}\psi \) on the nonlinearity renders the energy indefinite and finite-time blowup may occur, for example for all data of negative energy and finite variance, see Glassey [15]. Blowup here refers to the property that \( \| \psi(t) \|_{H^1} \to \infty \) as \( t \to T^- < \infty \). In addition, the focusing nonlinearity admits special stationary wave solutions of the form \( e^{-it\alpha^2} \phi(x) \) with \( \alpha \neq 0 \), where

\[
- \alpha^2 \phi + \Delta \phi = |\phi|^{p-1} \phi.
\]}

Existence of nontrivial decaying solutions to this equation has been known for a long time, see for example Strauss [28] and Berestycki and Lions [5]. On the one-dimensional line, there are exactly two nonzero decaying solutions which are given by

\[
Q(x) = \pm \alpha \cosh^{-\tau} (\beta x), \quad \alpha = \left( \frac{p+1}{2} \right)^{\frac{1}{p-1}}, \quad \beta = \frac{p-1}{2}.
\]

The existence and uniqueness can be read off from the phase-portrait in the \((\phi, \phi')\)-plane. In higher dimensions no explicit formulae exist and one obtains existence via variational arguments. Moreover, uniqueness in the strong sense as in one dimension fails, as it is known that there are infinitely many solutions [5]. However, there exists exactly one positive, radial solution called the ground state. In fact, any positive decaying solution of (2) is necessarily radial about some point (by Gidas et al [14]) as well as exponentially decaying. This unique solution is called ground state and it is the one we consider in this paper.

The orbital stability analysis of this ground state standing wave was settled many years ago and depends on the power of the nonlinearity, see [5, 9, 16, 31, 32]: for \( p < p_2 := \frac{d}{2} + 1 \) (the latter being called \( L^2 \)-critical power) the ground state is stable, whereas for \( p_2 \leq p < 2^* - 1 \) the ground state is unstable in the orbital sense. In fact, the instability is very strong: arbitrarily small perturbations of initial data \( Q \) with respect to the \( H^1 \)-topology can lead to finite-time blowup, see [4, 8, 29] (and for Klein–Gordon [21]). The transition at the power \( p_2 \) can be seen at the linearized level. More precisely, linearizing about the standing wave with \( \alpha = 1 \) (which we may assume by scaling) and splitting into real and imaginary parts yields the matrix operator

\[
\mathcal{H} := \begin{bmatrix} 0 & L_- \\ -L_+ & 0 \end{bmatrix}
\]

where

\[
L_- = -\Delta + 1 - Q^{p-1}, \quad L_+ = -\Delta + 1 - pQ^{p-1}.
\]
One then finds that for $p \leq p_2$ the spectrum of $\mathcal{H}$ lies entirely on the real axis, whereas for $p > p_2$ there exists a pair of imaginary simple eigenvalues, signifying exponential linear instability.

The more difficult asymptotic stability problem was considered in Buslaev and Perelman [6, 7], Soffer and Weinstein [25], Cuccagna [11], but it would take us too long to review the literature on this topic. More relevant for our purposes is the conditional asymptotic stability problem which refers to the following question: in the orbitally unstable regime, does the ground state remain asymptotically stable in forward time under a finite co-dimension condition on the perturbation? In fact, due to the structure of the spectrum in that case one might expect that a co-dimension 1 condition should suffice in order to stabilize the ground state. This is indeed the case, as shown in the orbital stability sense by Bates and Jones [1] for the nonlinear Klein–Gordon equation and for the NLS equation in [13] (following Bates and Jones). [1] implemented the Hadamard (or graph transform) method in the infinite-dimensional setting given by nonlinear dispersive Hamiltonian PDEs such as NLS and Klein–Gordon. The graph transform together with the Lyapunov–Perron fixed point approach constitute the only two known methods available for the construction of invariant manifolds, and they were both intensely developed in finite dimensions (in other words, for ODEs). See the introduction of [20] and the references cited there.

The asymptotic stability question for the cubic NLS in three dimensions was studied in [3, 23], where the existence of a centre-stable manifold near the ground state was established on which the solutions remain asymptotically stable and scatters to the ground state. See [17] for the one-dimensional case, and [20, 26] for the Klein–Gordon equation. Finally, in the monograph by Nakanishi and the third author [20] (see the references there for the original papers) it was shown that these centre-stable manifolds divide a small ball around the ground state into two halves which respectively give rise to blowup in finite positive time on the one hand, and global existence in forward time and scattering to zero, on the other hand.

The most delicate part of the conditional asymptotic stability analysis turns out to be the scattering property of solutions starting on the centre-stable manifold. This refers to the fact that solutions starting on the manifold decompose into a ground state standing wave (with slightly different parameters—this is the phenomenon of modulation) plus a free wave plus a term which is $o(1)$ in the energy space as $t \to +\infty$. By some dispersive PDE machinery this is equivalent to the property that the perturbation of the modulated standing wave satisfies global dispersive estimates, such as pointwise decay (as in [23]) or Strichartz estimates as in [3].

The reason that such global dispersive estimates can be considered delicate lies with the fact that they appear to require detailed knowledge of the entire spectrum of the linearized operator, including the behaviour of the resolvent at the thresholds of the essential spectrum. In [3, 23] one therefore needed to assume the gap property of $L_\pm$ for the cubic power nonlinearity (i.e. $p = 3$) in $\mathbb{R}^3$. As already mentioned before, this refers to the fact that $L_\pm$ have no eigenvalues in $(0, 1]$ and that 1 is not a threshold resonance. And finally, the gap property which we verify in this paper implies via the Lyapunov–Perron method that solutions on the centre-stable manifold scatter to the ground state, see [20].

Demanet and the third author [12] implemented the Birman–Schwinger method numerically and showed that this assumption is indeed correct (in the general non-radial setting). Moreover, they found that the gap property is even more delicate than expected: it fails if the power $p$ on the nonlinearity is lowered slightly below $p = 3$. This is somewhat surprising, as Krieger and the third author [17], based on Perelman [22], had shown by analytical arguments that the gap property holds in the entire $L^2$-supercritical regime in one dimension. This was facilitated by the explicit form of the ground state in dimension 1 and one finds, moreover,
that $L_{\pm}$ retain the gap property for all powers down to the completely integrable cubic NLS, where a threshold resonance appears.

While [12] appears to be accurate on all empirical accounts, the numerical method implemented there is not a proof since it seems very difficult—if not impossible—to give rigorous error bounds for all numerical approximations and calculations required by the Birman–Schwinger method. For example, an approximate soliton is computed numerically, but without any rigorous bounds on the error introduced by this approximation.

The purpose of this paper is to offer a completely rigorous, albeit rather computational, proof which confirms the gap property of $L_{\pm}$ in the cubic radial case in $\mathbb{R}^3$.

1.2. The main equations

Over the radial functions, the equation for the ground state reduces to

$$-y''(r) - \frac{2}{r} y'(r) + y(r) - y^3(r) = 0.$$  \quad (3)

By Coffman’s theorem [10] there is a unique, positive decaying solution of (3) which is smooth on $[0, \infty)$. It is denoted by $Q$ and called the ground state. The eigenvalue problem over the radial subspace now becomes

$$L_+ u = \lambda u; \quad \text{where } L_+ = -\frac{d^2}{dr^2} - \frac{2}{r} \frac{d}{dr} + 1 + 3Q^2$$  \quad (4)

and

$$L_- y = \lambda y; \quad \text{where } L_- = -\frac{d^2}{dr^2} - \frac{2}{r} \frac{d}{dr} + 1 - Q^2.$$  \quad (5)

We show that in the gap $[0, 1]$ the operator $L_+$ has no eigenvalues or resonance, and the same is true for $L_-$ on $(0, 1]$.

1.3. Technical approach

Equation (3) is likely nonintegrable, and no useful closed form representation for $Q$ is known. However, in order to resolve the aforementioned gap problem, an exact expression of $Q$ or even the exact values of $Q(r)$ are clearly not required: a sufficiently accurate approximation, which we denote by $\tilde{Q}$, suffices. Unfortunately, we have found that the required accuracy for $\|Q/\tilde{Q} - 1\|_{\infty}$ in our problem is on the order of $10^{-4}$. This is a reflection of the phenomenon seen in [12], namely that the gap property is only barely correct. More mathematically, it must mean that $L_{\pm}$ have complex resonances very close to the real axis that become eigenvalues in the gap once the power is lowered slightly below $p = 3$. We note that this phenomenon may also account for the failure of ‘softer’ approaches based on bounds on the number of eigenvalues, etc.

We proceed as follows. We find a suitable approximation $\tilde{Q}$ in the form of a piecewise explicit function; for $r \geq 5/2$ it is given by

$$\tilde{Q}(r) = y_3(r; \beta) = r^{-1} \beta_1 e^{-r} + \beta_2 g(r); \quad g(r) := r^{-1} (2e^{r} \text{Ei}(-4r) - e^{-r} \text{Ei}(-2r))$$  \quad (6)

for specific $\beta_1$, $\beta_2$, see section 2.1. On $[0, 5/2)$, the reciprocal $\frac{1}{\tilde{Q}}$ is a piecewise polynomial. The coefficients of the polynomials arising in this construction are listed in the first two lines of the table on page 160, and we refer the reader to section 2 for more details of the construction, see especially definition 2.3.

The representation of $\tilde{Q}$ described above is found in the following manner. For $r \geq 5/2$ we iterate the Volterra equation once to obtain $y_3$. On $[0, 5/2)$, we construct by Taylor series a
solution which is well behaved at zero and then we extend it by matched Taylor series up to $5/2$. We take the value $\tilde{Q}(0)$ as a parameter and determine it so that $\tilde{Q}(5/2^-) = y_3(5/2^+)$. We then optimize the polynomial representation by sampling points from the collection of reciprocals of the aforementioned Taylor series and using least-squares fitting with three polynomials\(^1\). We then rationalize the coefficients of the polynomials by suitably accurate truncated continued fractions. This is the procedure by which the rational numbers listed in the tables at the end of the paper are obtained.

The next step is to show that $\tilde{Q}$ is close to $Q$. At this stage the problem is already, in some sense, perturbative: $\delta = Q - \tilde{Q}$ is very small. We thus proceed in a natural way, by solving a contractive equation for $\delta$. A slight hurdle arises at this point since Green’s function $G(r, r')$ in the integral equation for $\delta$ is not explicit either: $G$ solves a linear second order ODE with nontrivial coefficients (combinations of exponential integrals and polynomials). We overcome this by finding a nearby equation with explicit solutions, and contract out the difference between the two equations.

Estimating the remainder as a result of replacing $Q$ by $\tilde{Q}$ in (3) reduces to bounding rational functions with rational coefficients. This is done rigorously, as the degrees of the polynomials in the denominator and numerator are manageable. There are many ways to estimate polynomials. Perhaps the most straightforward one is to place absolute values on all coefficients, or at least on all coefficients of powers higher than three, say: a polynomial with positive coefficients is easy to bound by monotonicity; it is largest at the largest argument. However, inspection of the tables at the end reveals that this method cannot be applied here, as the coefficients of higher powers are not within the small interval we need (which is $<10^{-4}$). In order to overcome this problem, we re-expand the polynomials at a number of intermediate points selected so that the coefficients of the monomials of degree exceeding 3 are small enough to be discarded modulo small errors. Polynomials of degree 3 of course have explicit extrema. The re-expansion refers to nothing other than passing from $P(r) \in \mathbb{Q}[r]$ to the new polynomial $P(r_0 + az)$, where we always keep $|z| \leq 1$. The values of $r_0$ and $a$ are always stated explicitly in the text (we use the notation of ‘partitions’ for this purpose). See note 2.5 for more on this issue.

After having obtained $Q$ up to explicitly controlled errors, we then analyse the spectra of $L_\bar{a}$. This is done essentially in the same way, by finding an accurate Jost quasi-solution for $r \geq 5/2$, and a well-behaved one on $[0, 5/2]$ (which simply refers to the requirement that the solution remains bounded with a horizontal tangent at the origin). The quasi-solutions are explicit combinations of exponential integrals and polynomials. The way the quasi-solutions are obtained also mirrors the soliton approach: iterating the Volterra equation for large $r$ and using orthogonally projected Taylor series in the complement. This time around, we need the solution $u_1$ only on $[0, 5/2]$ and $u_2$ on $[5/2, \infty]$. However, since the quasi-solutions depend on the spectral parameter $\lambda$, as well, the calculations are more involved. We then check that $\inf_{\lambda \in [0,1]} |W(\lambda)| > 0$ (for $L^+$) and $\inf_{\lambda \in [0,1]} |W(\lambda)/\lambda| > 0$ for $L^-$, respectively) where $W$ is the Wronskian of $u_1$, $u_2$ at $5/2$, and this concludes the proof\(^2\).

We emphasize that all coefficients involved are in $\mathbb{Q}$, the calculations are exact, and the proof, tedious at places—for instance in having to repeatedly re-expand polynomials at numerous intermediate points—is fully rigorous.

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\(^1\) (i) This is essentially the discrete version of $L^2([a, b])$ orthogonal projection using Legendre polynomials. Projecting on Chebyshev polynomials would provide an even more economical representation, but we prefer the simplicity of least-squares fitting; (ii) we look at the reciprocals since they are smoother in that the singularities in $\mathbb{C}$ are farther away from the real axis and result in more efficient representations.

\(^2\) A solution bounded near 0 must be a multiple of $u_1$ since solutions linearly independent of $u_1$ are unbounded at zero; similarly any solution bounded as $r \to \infty$ is a multiple of $u_2$. 

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In addition, the integral operators upon which the contractive mappings are based have small norm (see, e.g., (14)), allowing for the calculation of the solutions rapidly and, in principle, with arbitrary accuracy. Therefore, this approach is useful numerically as well, to obtain rapidly convergent approximants.

While our approach can in principle be carried out by hand, it is of course unrealistic to attempt this in praxis as the calculations in their current form are too numerous as well as too long. While we organized section 2 in such a way that the calculations can in practice be done by hand, in later sections we preferred to use the computer algebra packages Maple and Mathematica to perform basic operations (such as multiplications of polynomials with rational coefficients and solving quadratic equations). The later sections involve longer calculations, but of the simple type mentioned above. The exponential integral, Ei, is the only transcendental function needed; we estimate it using the asymptotic inequalities it obeys and/or by integrating inequalities satisfied by its derivative, an elementary function. Once more, there are no numerical calculations involved (such as numerical integration or numerical location of zeroes), and with substantially more optimization effort, it is likely that every step could have been done by hand; we felt there is little to gain from this, as human error has a considerably higher chance to occur in such a setting.

We wish to emphasize again that all calculations that were carried out by either Maple or Mathematica are completely error free as they only involve finitely many algebraic operations in the polynomial ring \( \mathbb{Q}[r] \).

Let us also emphasize that the concrete implementation of the method as it appears below is by no means the only possible one. As mentioned in footnote 1, one may substantially reduce the amount of computations required (as well as the length of the tables in the appendix) by relying on Chebyshev polynomials instead of least-squares fitting in order to carry out the aforementioned projections of the matched Taylor series. We intend to present this simpler implementation, together with the nonradial gap property in a future publication.

Finally, we would like to mention that the approach developed in this paper is by no means restricted to the specific problem that we study here. Among the problems that will be analysed using this method is the Dubrovin conjecture in integrable systems concerning the regularity of the tritronquées, a class of solutions of the Painlevé equation P1 important in applications.

Furthermore, the methods developed here should also allow one to verify the absence of real spectrum other than the zero eigenvalue for the radial linearized cubic nonlinear Schrödinger Hamiltonian; recall that this was recently demonstrated by Marzuola and Simpson [18] with the aid of some numerics.

2. The approximate soliton

In this section we find an approximation of \( Q \) by means of simple functions. Let

\[
\begin{align*}
 r_1 & := 3/10, & r_2 & := 17/25, & r_3 & := 9/10, & r_4 & := 1, & r_5 & := 3/2, & r_6 & := 12/5, & r_7 & := 5/2,
\end{align*}
\]

These increasing numbers define the partition points relative to which we will define the piecewise approximations. We denote the characteristic function \( \chi((r_i, r_j)) \) by \( \chi_{ij}, j \neq 7 \) and \( \chi_{i7} = \chi([r_i, r_7]) \); similarly \( \chi([0, r_j]) =: \chi_{0j} \) and \( \chi([r_j, \infty)) =: \chi_{jr} \). We also denote \( \chi_{77} = \chi([r_7, r_j]) \).

2.1. Solving (3) from \( \infty \)

The following lemma describes all possible solutions of (3) which decay at \( \infty \) (at least within a certain range of parameters chosen to suit our needs, and up to some error). In what follows,
we shall repeatedly encounter the exponential integral
\[ \text{Ei}(x) := PP \int_{-\infty}^{x} \frac{e^u}{u} \, du; \quad x \in \mathbb{R} \]
where \( PP \) denotes the Cauchy principal value of the integral. Define the nonlinear operator
\[ (N(f))(r) = \int_{r}^{\infty} \sinh(r - s)s^{-2}f^3(s) \, ds. \quad (7) \]
With \( g \) as in (6) one checks that
\[ rg(r) = N(e^{-r}). \quad (8) \]

The exponential integral admits the following asymptotic expansions.

**Lemma 2.1.**

(i) For each positive integer \( N \) one has
\[ e^{-x} \sum_{k=0}^{2N-1} \frac{k!}{x^{k+1}} (-1)^k < -\text{Ei}(-x) < e^{-x} \sum_{k=0}^{2N} \frac{k!}{x^{k+1}} (-1)^k \quad \forall x > 0 \quad (9) \]
and
\[ 0 > g(r) = \frac{e^{-3r}}{r^3} \left( -\frac{1}{8} + \frac{3}{16r} - \frac{21}{64r^2} + \frac{45}{64r^3} \right) + \frac{15e^r}{64r} \int_{r}^{\infty} \frac{(e^{-4s} - 16e^{-2(r+s)})}{s^6} \, ds \]
\[ > \frac{e^{-3r}}{r^3} \left( -\frac{1}{8} + \frac{3}{16r} - \frac{21}{64r^2} + \frac{45}{64r^3} - \frac{465}{256r^4} \right). \quad (10) \]

In particular \( 0 > g(r) > -e^{-3r}/8r^3 \) for \( r \geq r_7 \).

(ii) The function \( h(r) = -re^rg(r) \) is positive and decreasing.

(iii) Define the norm \( \|\psi\| = \sup_{r \geq r_7} |\psi(r)e^r| \) on continuous functions on \([r_7, \infty)\). Then we have
\[ \|N(e^{-r})\| \leq 1/8900. \quad (11) \]

**Proof.**

(i) Both (9) and (10) follow by means of repeated integrations by parts.

(ii) \( h \) is manifestly positive, while
\[ h'(r) = -e^{2r} \int_{r}^{\infty} e^{-4s}s^{-2} \, ds < 0. \]

(iii) By (ii), \( \sup_{r \geq r_7} |e^rN(e^{-r})| \) is reached at \( r = r_7 \) and (11) is now immediate from (10).

\( \square \)

Henceforth we assume \( 0 < \beta \leq 3 \), which is sufficient for our purposes. Also, recall the definition of \( y_3 \), see (6):
\[ y_3(r; \beta) = r^{-1}\beta e^{-r} + \beta^3 g(r). \]
By the lemma, this defines a positive function for all \( 0 < \beta \leq 3 \) and \( r \geq r_7 \).

**Lemma 2.2.** There exists a unique positive solution \( y(r; \beta) \) to (3) with the property that \( y(r; \beta) \sim \beta r^{-1}e^{-r} \) as \( r \to \infty \). It satisfies
\[ \left| \frac{y(r; \beta)}{y_3(r; \beta)} - 1 \right| < 3.2 \cdot 10^{-6} \quad \forall r \geq r_7 \quad (12) \]
uniformly in \( 0 < \beta \leq 3 \).
Proof. Setting \( z(r) = ry(r) \) (and suppressing the \( \beta \)-dependence for notational convenience) yields the ODE

\[
- z''(r) + z(r) = r^{-2} z^3(r).
\]

Let \( z_0(r) = \beta e^{-r} \). For solutions \( z \) with \( z(r) e^r \) bounded for large \( r \), (13) can be written as

\[
z(r) = z_0(r) + \mathcal{N}(z)(r) =: \mathcal{M}(z)(r) \quad r > 0.
\]

Taking the ball \( B = \{ h : \| h \| \leq \alpha \beta \} \) and choose \( \alpha \) so that \( \mathcal{M} B \subset B \); the latter condition gives

\[
1 + \beta^2 \alpha^3 \| \mathcal{N} \| - \alpha \leq 0,
\]

which is satisfied for \( \beta \in [0, 3] \) if \( \alpha = 1 + 1/985 \), for example. Using (11) again, expanding out \( (z + \delta)^3 - z^3 = \delta (3z^2 + 3\delta + \delta^2) \) and estimating each term in the last parentheses by its largest norm in \( B \) (such as \( \| \delta \| \leq 2\alpha \beta \)) we obtain

\[
\| \mathcal{M}(z + \delta) - \mathcal{M}(z) \| \leq \frac{13 \alpha^2 \beta^2 \| \delta \|}{8900} \leq \frac{\| \delta \|}{76}.
\]

Thus \( \mathcal{M} \) is contractive in \( B \) and (14) has a unique solution \( z_\delta \) there. First, since \( \mathcal{N}(e^{-r}) = O(e^{-3r}) \) for large \( r \), we have \( y(r; \beta) \sim \beta r^{-1} e^{-r} \), as claimed. It remains to show (12). Note that \( ry_3(r; \beta) = \mathcal{M}(z_0)(r) \). Since \( \delta := z_\delta - z_0 = \mathcal{N}(z_\delta) \) we have

\[
\| \delta \| \leq \| \mathcal{N}(z_\delta) \| \leq \beta^3 \alpha^3 \| \mathcal{N}(e^{-r}) \| \leq \frac{\alpha \beta}{500}.
\]

Using this estimate to improve on (15) we conclude that

\[
\left| \frac{y(r; \beta)}{y_3(r; \beta)} - 1 \right| = \left| \frac{z_\delta - \mathcal{M}(z_\delta)}{\mathcal{M}(z_0)} \right| \leq \frac{1}{327} \frac{\beta^3 \alpha^3 \| \mathcal{N}(e^{-r}) \|}{\beta - \beta^3 \| \mathcal{N}(e^{-r}) \|} < 3.2 \cdot 10^{-6}
\]

as claimed. \( \square \)

2.2. Solving (3) on \([0, \infty)\) up to a small error

In this section, we study an approximate solution of (3). For the heuristics behind this construction, we refer the reader to the introduction. In particular, we specialize the value of \( \beta \) in \( y_3(r; \beta) \) in lemma 2.2 so as to most closely approximate the ground state \( Q(r) \).

Definition 2.3. The approximate soliton \( \tilde{Q} \) is defined as follows. First, set

\[
p_1(r) := q_1(r),
p_2(r) := q_2(r) + p_1(r_3) - q_2(r_3) + (p_1'(r_3) - q_2'(r_3))(r - r_3),
p_3(r) := Ae^{-r} + Br g(r),
\]

where \( q_1, q_2 \) are the explicit polynomials \( q_i(r) = \sum_{j=0}^{11} a_{ij} r^j \) with coefficients as in table 1, and \( A, B \) are chosen so that \( p_2, p_3 \) match up in a \( C^1 \) fashion at \( r = r_7 \); \( p_3 \) is very close to the solution in lemma 2.2, see (20). Finally, set

\[
\tilde{Q} := x_{03} / p_1 + x_{37} / p_2 + x_{7\infty} / p_3.
\]
The specific form of $p_3$ of course originates with the exact solution from (12). From (19) one verifies that
\begin{equation}
\frac{217}{80} < A < \frac{350}{129}, \quad |B - A^3| < 33 \cdot 10^{-5}, \quad B < 20. \tag{20}
\end{equation}
We now need to show that $\tilde{Q}$ from (18) is close to the actual unique ground state $Q$. We begin by checking that $\tilde{Q}$ satisfies (3) up to a small error. Below we denote by $C^2_p$ the space of piecewise $C^2$ functions.

**Lemma 2.4.** As defined above, $\tilde{Q}$ satisfies the following properties:

(i) It is decreasing for $r \in [0, r_3]$, and $0 < \tilde{Q}(r) < 22/5$.

(ii) It belongs to $C^1((0, \infty)) \cap C^2_p((0, \infty))$, and $\tilde{Q}'(0) = 0$.

(iii) It satisfies the bounds
\begin{equation}
\tilde{Q}(r) < 5(1 + r)e^{-2r}\chi_{02}(r) + \frac{11}{2}e^{-8r/5}\chi_{27}(r) \quad \forall r > 0 \tag{21}
\end{equation}
and
\begin{equation}
\frac{187}{69} e^{-r} < \tilde{Q}(r) < \frac{350}{129} e^{-r}, \quad \text{for } r \geq r_7. \tag{22}
\end{equation}

(iv) In the complement of the three-point set $\{r_3, r_4, r_7\}$ the error
\[R(r) := -\tilde{Q}'(r) - \frac{2}{r} \tilde{Q}'(r) + \tilde{Q}(r) - \tilde{Q}(r)^3\]
satisfies the bound
\begin{equation}
|R(r)| < \rho_1(11/10 - r)\chi_{04}(r) + \rho_2(13/5 - r)\chi_{47}(r) + \frac{e^{-3r}}{25r^2}\chi_{7\infty}(r) \tag{23}
\end{equation}
where $\rho_1 := 15 \cdot 10^{-6}$ and $\rho_2 := 25 \cdot 10^{-8}$.

**Note 2.5.**

(i) To estimate a higher order polynomial $P$ on an interval, we partition the interval and Taylor-re-expand $P$ in each subinterval. We define the partitions so that $P$ equals the first four terms plus a small error. There are of course other ways to estimate polynomials, but this method leads to straightforward calculations. We represent the partition by a vector $\pi$, whose components $\pi_i$ are precisely the partition points. Unless otherwise specified, in each interval $[\pi_i, \pi_{i+1})$ we shall write
\[\ell_i(z) = \frac{1}{2}(1 - z)\pi_i + \frac{1}{2}(1 + z)\pi_{i+1}, \quad -1 \leq z \leq 1\]
and re-expand $P(\ell_i(z))$ around $z = 0$.

(ii) We bound a polynomial $P(z)$ from below on $|z| \leq 1$ by the minimum of the cubic polynomial $P \mod O(z^4)$ minus the sum of the absolute value of the remaining coefficients. Likewise, to obtain an upper bound, we take the maximum of the cubic polynomial $P \mod O(z^4)$ plus the sum of the absolute value of the remaining coefficients.

**Proof of lemma 2.4.** For property (i) we first note by inspection that $p_1'(r)$ has the following property: the coefficient of its first term ($r$ term) is bigger than 1 while the sum of absolute values of the remaining coefficients is less than 1, implying that $p_1'(r) > 0$ for $r \in [0, r_3]$. Since obviously $p_1(0) > 0$, we see that $1/p_1(r)$ is decreasing and positive for $r \in [0, r_3]$. All coefficients of $p_2'(r_3 + z)$ are positive. Thus $1/p_2(r)$ is decreasing for $r \geq r_3$. In particular $\tilde{Q}(r) < \tilde{Q}(0) < 22/5.$
Property (ii) is immediate by construction. Indeed, note that (17) defines a $\tilde{Q} \in C^2_p$ in such a way that the values of the function and the value of its first derivative match up at $r_3, r_7$.

The vanishing $\tilde{Q}(0) = 0$ is a consequence of the fact that $p_1(r)$ has no linear component ($a_1 = 0$, see Table 1).

For (iii) we start with the interval $[0, r_2]$, where we will show

$$(1 + r)e^{-2r} p_1(r) > 1/5.$$ 

Instead of showing this directly, we first note that

$$e^{-2r} \left( p_1(0) + \frac{17r}{100} + r^2 \right) > 1/5$$

by explicitly finding the maximum via differentiation. Thus it is sufficient to show

$$(1 + r)p_1(r) > p_1(0) + \frac{17r}{100} + r^2$$

or equivalently

$$p_{11}(r) := ((1 + r)p_1(r) - p_1(0))/r - \frac{17}{100} = r > 0.$$ 

For this purpose we use the partition $\pi = (0, r_2)$ and note 2.5 to re-expand $p_{11}$ in $z$. The coefficients of $z^j$, $j > 3$, are all very small and a direct calculation shows that $p_{11} > 1/50 > 0$ on $[0, r_2]$. Therefore $\tilde{Q}(r) < 5(1 + r)e^{-2r}$ for $r \in [0, r_2]$.

Similarly, one can show $e^{-3r/5} p_1(r)$ is increasing on $[r_2, r_3]$ using the partition $\pi = (r_2, r_3)$ and estimating $e^{8r/5} (e^{-8r/5} p_1(r))'$ using note 2.5.

In the interval $[r_3, r_5]$ we consider the polynomial $\tilde{P}_2(r) := e^{8r/5} (e^{-8r/5} p_2(r))'$. Using the partition $\pi = (r_3, r_7)$ we obtain $\tilde{P}_2(r) < -1/5 < 0$ and thus $e^{-3r/5} p_2(r)$ is concave. This, combined with the fact that $e^{-3r/5} p_2(r) > 2/11$ for $k = 3, 7$, shows that $\tilde{Q}(r) < \frac{11}{2} e^{-3r/5}$ for $r \in [r_3, r_7]$.

For $r \geq r_7$ we have using (9)

$$r e^{r}/p_3(r) = Br^{-1}e^{-2r} - 4Be^{2r}Ei(4r) > 0.$$ 

Therefore $r e^{r}/p_3(r)$ is increasing. Since obviously $\lim_{r \to \infty} r e^{r}/p_3(r) = A < \frac{350}{129}$ and $r_7 e^{r}/p_3(r) > \frac{187}{29}$, the estimate follows.

For (iv), we first introduce the notation $I_1 := [0, r_3]$ and $I_2 := [r_3, r_7]$. For $j = 1, 2$ we let $y_j(r) = 1/p_j(r)$ and define

$$R_j(r) = -y_j''(r) - 2r^{-1}y_j'(r) + y_j(r) - y_j'(r), \quad r \in I_j$$

and consider the polynomial $M_j(r)$ in $Q[r]$ given by

$$M_j(r) := r^{j-1} p_j(r) R_j(r).$$

We introduce the partitions $\pi_1 = (0, \frac{1}{10}, \frac{1}{5}, \frac{3}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{9}{10})$ and $\pi_2 = (\frac{9}{10}, \frac{11}{10}, \frac{13}{10}, \frac{31}{10}, \frac{31}{10}, \frac{3}{2})$ and define

$$M_{jk}(z) := M_j(\ell_{jk}(z)); \quad p_{jk}(z) := p_j(\ell_{jk}(z)), \quad j = 1, 2, \quad |z| \leq 1$$

(24)

(where $\ell_{jk}$ is the $k$th component of $\ell_j$). We proceed to estimate $M_{jk}(z)$ on the unit disc as described in note 2.5.

This yields, for each $1 \leq k \leq 7$ and for any $-1 \leq z \leq 1$,

$$\left| \frac{M_{jk}(z)}{p_{jk}(z)^3} \right| = \sup \frac{|M_{jk}(z)|}{p_{jk}(-1)^3} < p_1 \left( \frac{11}{10} - \ell_{1k}(1) \right) \leq p_1 \left( \frac{11}{10} - \ell_{1k}(z) \right).$$

(25)
On the interval $[r_3, r_4]$ we have
\[ \left| \frac{M_{21}(z)}{\ell_{14}(z) p_{21}(z)^3} \right| \leq \frac{10 \sup |M_{21}(z)|}{9 p_{21}(-1)^3} < \frac{1}{10} \frac{\rho_1}{\ell_{21}(z)} \leq \frac{11}{10} \left( 1 - \ell_{14}(z) \right). \] (26)

On the interval $[r_4, r_7]$ we have for each $k = 2, ..., 6$
\[ \left| \frac{M_{2k}(z)}{\ell_{2k}(z) p_{2k}(z)^3} \right| \leq \sup |M_{2k}(z)| \leq \rho_2 \left( \frac{13}{5} - \ell_{2k}(1) \right) \leq \frac{13}{5} \ell_{2k}(z). \] (27)

These give the desired estimates of $|R(r)|$ for $r \in [0, r_7]$.

Finally, for $r \in I_1 := [r_7, \infty)$ we write, with $A$ and $g(r)$ as in (19),
\[ y_3(r) := \frac{1}{p_3(r)} = y_3(r; A) + (B - A^3)g(r) = A \frac{e^{-r}}{r} + A^3 g(r) + (B - A^3)g(r). \]

Then the error $R(r)$ has the form, with $\varepsilon := B - A^3$,
\[ R(r) = R_0(r) + \varepsilon R_1(r) + \varepsilon^2 R_2(r; \varepsilon), \]
where
\[ R_0(r) := -\frac{y''_3(r; A)}{r} - \frac{2}{r} y'_3(r; A) + y_3(r; A) - y_3(r; A)^3, \] (28)
\[ R_1(r) := -\frac{r}{r} \frac{g(r)}{3} y_3(r; A) + g(r) - 3 y_3(r; A)^2 g(r) = \frac{e^{-3r}}{r^3} - 3 g(r) \left( A \frac{e^{-r}}{r} + A^3 g(r) \right)^2, \] (29)
\[ R_2(r; \varepsilon) := -3 A g^2(r) \left( \frac{e^{-r}}{r} + A^2 g(r) + \frac{\varepsilon g(r)}{3A} \right). \] (30)

First, using lemma 2.1 we obtain $h(r_7) \leq 9 \cdot 10^{-5}$ (with $h$ defined in that lemma) and thus
\[ |R_0(r)| \leq 3 A^5 e^{-3r} h(r) \left( 1 - A^2 h(r) + \frac{1}{3} A^4 h^2(r) \right) \leq 3 A^5 \left( \frac{5}{2} \right) \frac{e^{-3r}}{r^3}. \] (31)

By (10), $0 > g > -e^{-3r}/(8r^3)$ and thus, noting that $-r^{-1} (rg)' + g = e^{-3r}/r^3$, we obtain
\[ |R_1(r)| < \frac{e^{-3r}}{r^3} \left( 1 + \frac{3 A^2 e^{-3r}}{8r^2} \right) < \frac{11}{10} \frac{e^{-3r}}{r^3} \quad \forall r \geq r_7. \] (32)

Finally,
\[ |R_2(r)| < \frac{e^{-3r}}{r^3} \frac{3 A e^{-3r}}{64r^3} \left( \frac{e^{-r}}{r} + \frac{A^2 e^{-3r}}{8r^3} + \frac{\varepsilon e^{-3r}}{8Ar^3} \right) < \frac{3}{2} \cdot 10^{-7} \frac{e^{-3r}}{r^3} \quad \forall r \geq r_7. \] (33)

Combining (28), (32) and (33) with (19) yields
\[ |R(r)| < \left( 3 A^5 h(5/2) + 11 \varepsilon/10 + 3 \cdot 10^{-7} \varepsilon^2/2 \right) \frac{e^{-3r}}{r^3} < \frac{1}{25} \frac{e^{-3r}}{r^3} \]
for all $r \geq r_7$ as stated. \hfill \Box

3. The exact soliton

3.1. Finding the exact ground state $Q$ near the approximate one $\tilde{Q}$

We now need to show that $|1 - \tilde{Q}/Q|$ is small. Equation (3) implies
\[ -\delta''(r) - 2 \frac{\delta'(r)}{r} + (1 - 3 \tilde{Q}^2(r)) \delta(r) = -R(r) + 3 \tilde{Q}(r) \delta^2(r) + \delta^3(r) \quad (\delta := Q - \tilde{Q}). \] (34)
The boundary conditions are \( \delta(0) = 0, \delta(\infty) = 0 \). We shall describe, again via polynomials, how to find an approximate fundamental system for (34). The challenge here is of course that we cannot hope to find an exact fundamental system for this equation, but require something close to it in order to set up a contraction for \( \delta \). We find it technically convenient to find an exact fundamental system for a homogeneous equation which is slightly different from the one in (34).

3.1.1. An approximate Green’s function

**Definition 3.1.** We define \( \varphi_1(r), \varphi_2(r) \) as follows. Set \( J_1 := [0, r_1), J_2 := [r_1, r_4), J_3 := [r_4, r_7), J_4 := [r_7, \infty) \) and let

\[
\varphi_j(r) := q_{j}\chi_0 + q_{j}\chi_1 + q_{j}(r - 1/2)\chi_{14} + q_{j}\chi_2 + q_{j}\chi_{37} + a_j e^{\sigma_j r} x_7
\]

for \( j = 1, 2 \), where \( a_j = 1, a_j = 1/2, \sigma_j = -1, \sigma_j = 1, q_{j}, q_{j} \) are of the form

\[
q_{j}(r) = \sum_{k=1}^{13} b_k e^{r t}
\]

where the \( b_k \) are given in table 1. The factor \( \frac{1}{2} \) in front of \( \sigma_j \) in the definition of \( \varphi_j \) is chosen so as to normalize a Wronskian to 1. Finally, set \( g_j = \varphi_j \) on \( J_4 \) and

\[
g_j(r) := \varphi_j(r) + g_j(r_j) - \varphi_j(r_j) \quad \forall r \in J_{\ell-1}
\]

for all \( \ell = 2, 3, 4 \), where \( J_{\ell-1} = [r_{\ell-1}, r_j) \) (we use this notation only for (36)).

We note that the jumps appearing in (36) are very small; more precisely,

\[
\max(|\varphi_j(r_j) - \varphi_j(r_{\ell-1})|, |\varphi_j(r_{\ell-1}) - \varphi_j(r_{\ell-1})|) < 3 \cdot 10^{-4} \quad \forall \ell = 2, 3, 4, \ j = 1, 2.
\]

The functions \( g_1, g_2 \) from the previous definition satisfy an ODE which is a perturbation of our main Sturm–Liouville equation.

**Lemma 3.2.** The functions \( g_1, g_2 \) are in \( C^1([0, \infty)) \cap C^2_p \), and solve the ODE

\[
-y''(r) + U(r)y'(r) + V(r)y(r) = 0,
\]

where \( U \) and \( V \) are piecewise rational functions which obey the estimates

\[
\|U\|_\infty < U^\ast := \frac{1}{165}, \quad \|V - 1 + 3\hat{Q}^2\|_\infty < V^\ast := \frac{1}{36}
\]

as well as \( U = 0, V = 1 \) on \( J_4 \).

**Proof.** That \( q_j \) are in \( C^1([0, \infty)) \cap C^2_p \) is clear by construction. One has for \( r \in J_4, 1 \leq \ell \leq 4 \),

\[
U(r) = g_2(r)g_1(r) - g_2(r)g_1(r),
\]

\[
V(r) = g_2(r)g_1(r) - g_2(r)g_1(r),
\]

which are manifestly rational functions. Moreover, \( U = 0, V = 1 \) on \( J_4 \). To obtain the bounds of (38) we rely on note 2.5. Let \( \pi = (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13) \).

To estimate \( U \), one writes within each subinterval \( U(r) = M(z)/P(z) \) with polynomials \( M, P \in \mathbb{Q}[e^r, e^{-r}][z] \) and obtain

\[
|U(r)| \leq \frac{M(1)}{|P(0)| - P(1)},
\]

where \( M(z) \) is obtained by placing absolute values on all coefficients of \( M \), whereas \( P \) is the result of the same procedure applied to \( P(z) - P(0) \). For \( V \), one proceeds similarly, by first writing \( p_j(r)V - 1 + 3\hat{Q}^2 \) as a rational function on \( (0, r_j) \) with \( j = 1, 2 \) depending on whether \( r < r_3 \), or \( r > r_3 \), respectively. Substituting the same affine change of variable as for \( U \) into the numerator and denominator of this rational function now establishes the desired bound. \( \square \)
We shall need to modify the system \( g_j \) to accommodate the \( \frac{1}{r^2} \) term in the three-dimensional radial Laplacian. In the following lemma, note that \( g_0 \) is regular at \( r = 0 \), but grows exponentially, whereas \( g_\infty \) decays exponentially, but is singular at \( r = 0 \).

**Lemma 3.3.** The functions

\[
  g_0(r) := r^{-1} \left( g_2(r) - \frac{g_2(0)}{g_1(0)} g_1(r) \right), \quad g_\infty(r) := r^{-1} g_1(r)
\]

form a fundamental system for the equation

\[
  - y''(r) + \left( \frac{U(r) - 2}{r^2} \right) y'(r) + \left( V(r) + \frac{U(r)}{r} \right) y(r) = 0,
\]

and their Wronskian satisfies the estimate

\[
  W(r) = r^{-2} \quad \text{if} \quad r \geq r_7,
\]

\[
  \frac{9}{10} < r^2 |W(r)| \leq 1 \quad \forall 0 < r < r_7.
\]

One has the following pointwise bounds:

\[
  |g_\infty(r)| \leq \left( \frac{1}{4r} + \frac{3}{20} \right) \chi_{04}(r) + \frac{e^{-r}}{r} \chi_{4\infty}(r) \quad \text{and} \quad |g'_\infty(r)| \leq \frac{1}{2r^2} \chi_{04}(r) + \frac{1 + r}{r^2} e^{-r} \chi_{4\infty}(r)
\]

as well as

\[
  |g_0(r)| < \frac{13}{2} \chi_{[0,1]}(r) + \frac{7 e^r}{5} \frac{1}{1 + r} \chi_{[1,\infty)}(r) \quad \text{and} \quad |g'_0(r)| < 18 \chi_{04}(r) + \frac{e^r}{2r} \chi_{4\infty}(r).
\]

**Proof.** The first claim follows immediately from (37). Let \( \tilde{g}_j(r) := r^{-1} g_j(r) \) for \( j = 1, 2 \). Then the Wronskian \( W(r) = g_0(r)g'_\infty(r) - g'_0(r)g_\infty(r) \) of \( g_\infty \) and \( g_0 \) satisfies

\[
  W(r) := \tilde{g}_1(r) \tilde{g}_2'(r) - \tilde{g}_1'(r) \tilde{g}_2(r) = r^{-2} \chi_{[0,1]}(r).
\]

This follows from the fact that \( W \) is continuous by lemma 3.2, and

\[
  W'(r) = (U(r) - 2/r)W(r), \quad W(r) = r^{-2} \quad \text{if} \quad r \geq r_7
\]

by (39). From (38) one now obtains (40).

For the estimates of \( g_0 \) and \( g_\infty \) we use re-expansions and the following simple observation.

**Note 3.4.** Estimates of functions \( f \) for which \( f' \) is a quadratic polynomial multiplied by a monomial of any degree or by an exponential are elementary. We can use this in our estimates as follows: if \( f = \sum_{j=1}^{m} f_j \) and \( J \subset \mathbb{R} \) then, clearly,

\[
  \inf_{f} f \geq \sum_{j=1}^{m} \inf_{f_j} f_j; \quad \sup_{f} f \leq \sum_{j=1}^{m} \sup_{f_j} f_j.
\]  

If \( f \) is a polynomial we write \( f \) as a sum of subpolynomials \( f_j \), the first one containing the monomials of degree \( \leq 3 \) and the others consist of the monomials of degrees \( \in [3l + 1, 3l + 3] \) for all \( l \) with \( 3l + 1 \leq \deg f \). If \( f(r) = e^{\alpha r} P_1(r) \), the same applies to a decomposition \( e^{\alpha r} P_j \) in which \( P_1 \) has degree two and the \( P_j, j \geq 2 \) consist of the monomials of degrees \( 2j + 1 \) and \( 2j + 2 \).

To establish \( |g_1(r)| = |rg_\infty(r)| < \frac{1}{4} + \frac{2e}{50} \) on \( (0, 1) \) let \( m_1(r) = \frac{1}{4} + \frac{2e}{50} \). The result follows using note 3.4 for the polynomials \( g_1 \pm m_1 \) taking \( r = \frac{3}{20} + \frac{3e}{20} \) and re-expanding (as a polynomial in \( s \)) \( s \in [-1, 1] \) and then \( r = \frac{1}{2} + s, s \in [-\frac{3}{20}, \frac{3}{20}] \).
To establish $|g_\infty'(r)| < \frac{1}{2} e^r$ on $(0,r_4)$, one proves the equivalent

$$|rg_\infty'(r) - g_1(r)| < \frac{1}{2} \quad 0 < r < r_4.$$  

(43)

The result follows once more from note 3.4, re-expanding $rg_\infty'(r) - g_1(r)$ via $r = \frac{3}{20} + \frac{3}{20} s$ for $|s| < 1$, $r = \frac{1}{2} + s$, $s \in [-\frac{1}{2}, \frac{1}{2}]$, and $r = \frac{3}{20} + s$, $s \in [0, \frac{3}{20}]$.

Next, we turn to the estimate $|g_\infty'(r)| < \frac{e^r}{r}$ on $r \geq r_4$. On $r \geq r_7$ we have an exact equality to $\frac{e^r}{r}$ so it suffices to deal with $r_4 \leq r < r_7$. We note that the inequality is sharp, and we need a different method: we let $\psi = e^r g_1$ and look at $\psi''$. Here we use note 3.4 and expansions at $r_0 + s$, $|s| \leq \frac{1}{10}$, and $r = 2 + s, s \in [-\frac{1}{2}, r_1]$ and $r = r_1 + s, s \in [-\frac{1}{2}, r_1]$, to see that $\psi'' < 0$. Since $\psi'(r_4) > 0$ and $\psi'(r_7) = 0$, we see that $\psi' > 0$. Since $\varphi(r_7) = 1$, the property follows.

Finally, one has the bound $|g_\infty'(r)| \leq (1 + r)e^{-r}/r^2$ for $r > r_4$. In view of the exact expression one has for $r \geq r_7$ it suffices to deal with $r_4 \leq r < r_7$. Thus, we need to verify that

$$e^r|rg_\infty'(r) - g_1(r)| \leq 1 + r, \quad r_4 \leq r < r_7.$$

We let $\varphi_1(r) := e^r(rg_\infty'(r) - g_1(r))$. Using the partition $\pi = (r_4, 3/2, 9/20, 21/20, r_7)$, explicitly minimizing the leading cubic polynomial and taking the $\ell_1$ norm of the rest, we see that $\varphi_1'(r) > 0$. Thus, by monotonicity, $\varphi_1'(r_7) < \varphi_1'(r_4) < -1$. This implies that $\varphi_1(r) < 0$ on $[r_4, r_7]$. Moreover, $\varphi_1(r) + 1 + r \geq 0$ on $[r_4, r_7]$ since $\varphi_1(r) + r$ is decreasing and $\varphi(r_7) + r_7 + 1 \geq 0$.

For $g_0$ one proceeds in a similar fashion. We begin with the bound $|g_0(r)| < 1/3(2/7)$ on $(0, 1/2)$. On $[0, r_1]$ we apply note 3.4 for $f$ with $r = \frac{1}{3} + \frac{1}{70} s$ while on $[r_1, r_4]$ we look at the polynomials $r g_0(r) \pm \frac{1}{20}$ with $r = \frac{1}{20} + s$ and $s \in [-\frac{1}{20}, \frac{1}{20}]$.

Next, one verifies that $|g_0(r)| < \frac{1}{5} e^r(r) < \frac{1}{5} e^r$ on $r \geq 1/2$. On the interval $r \geq r_7$ one checks the explicit expression $g_0(r) = \frac{e^r}{2} + \frac{k}{re^r}$ where $k = -\frac{\ln(20)}{\ln(10)} \in (0, 4)$ and thus

$$g_0(r) = \frac{e^r}{2} + \frac{k}{re^r} = \frac{e^r}{2} + \frac{1 + r}{r} \left[ \frac{1 + r}{2} + \frac{k}{e^{2r}} \right] \leq \frac{7}{5} \left( \frac{1 + r}{2} + \frac{k}{100} \right) \frac{e^r}{1 + r} \leq \frac{7}{5} e^r.$$

On the interval $[1/2, r_7]$ one can check via the partition $\pi = (1/2, 1, 9/20, 5/2)$ that $g_0 > 0$. Furthermore, we apply the same partition to the expression

$$5(g_2(r) + kg_1(r))(1 + r) - 7re^r,$$

which is of the form admitted by note 3.4. One then sees that it is negative.

For the bound $|g_0'(r)| < 18$ on $(0, r_1)$, we multiply through by $r^2$. We then use the substitution $r = \frac{1}{2} + s$. On $[r_1, r_4]$ one uses $r = \frac{1}{2} + s$.

Finally, we verify $|g_0'(r)| < \frac{e^r}{2r}$ for $r \geq r_4$. On $r \geq r_7$ one checks that

$$re^{-r} g_0'(r) = \frac{1}{2} - \alpha e^{-2r} = \frac{1}{2e^r} - \frac{\beta}{r} e^{-2r}
$$

with $\alpha, \beta > 0$. The right-hand side is clearly increasing in $r$ and $< \frac{1}{2}$. Since $g_0'(r_7) > 0$, the claim holds for $r \geq r_7$. On $r_4 \leq r < r_7$ we re-expand $r e^{-r} g_0'(r)$ via $r = \frac{1}{2} + s$ with $|s| \leq \frac{1}{2}$.$\square$

Now we come to the main result of this section, which is the estimate of the relative error between $\hat{Q}$ and $Q$.

**Proposition 3.5.** Let $Q$ be the exact ground state of (3) and $\hat{Q}$ be the approximate one given in definition 2.3. Then one has the error bound

$$|\hat{Q}(r) - Q(r)| \leq \varepsilon_0 \frac{e^{-r}}{1 + r} \quad \forall r \geq 0; \quad \varepsilon_0 := 7 \cdot 10^{-5}. \quad (44)$$
Proof. Rewrite (34) in the form
\[- \delta''(r) + \left( U(r) - \frac{2}{r} \right) \delta'(r) + \left( V(r) + \frac{U(r)}{r} \right) \delta(r) = h_1(\delta, r), \] (45)
where
\[h_1(\delta, r) := -R(r) + U(r) \delta'(r) + \left[ V(r) - 1 + 3 \tilde{Q}^2(r) + \frac{U(r)}{r} \right] \delta(r) + 3 \tilde{Q}(r) \delta'(r) + \delta^3(r).\]
We seek a solution to (45) which obeys the boundary conditions
\[\delta(0+) \in \mathbb{R}, \quad \delta(\infty) = 0.\]
In fact, this solution is unique and is of the form \( \delta = H(\delta) \) where
\[H(\delta)(r) = g_\infty(r) \int_0^r \frac{g_0(s) h_1(\delta, s)}{W(s)} \, ds + g_0(r) \int_r^\infty \frac{g_\infty(s) h_1(\delta, s)}{W(s)} \, ds\] (46)
in terms of the fundamental system from lemma 3.3. We also have
\[[H(\delta)(r)]' = : H' = g_\infty'(r) \int_0^r \frac{g_0(s) h_1(\delta, s)}{W(s)} \, ds + g_0'(r) \int_r^\infty \frac{g_\infty(s) h_1(\delta, s)}{W(s)} \, ds\]
\[= : H'_1(\delta)(r) + H'_2(\delta)(r).\] (47)
Lemma 3.6. \( H \) is a contraction in the ball
\[X := \{ f \in C^1((0, \infty)) \mid \| f \|_X \leq \varepsilon_0 \}\] (48)
with norm
\[\| f \|_X := \sup_{r \geq 0} (r + 1) e^r \left( | f(r) | + \frac{1}{3} | f'(r) | \right).\]
Thus there is a unique fixed point \( \delta_0 \in X \).

Corollary 3.7. Therefore, \( y := \tilde{Q} + \delta_0 > 0 \) solves (3) on \((0, \infty),\) remains bounded as \( r \to 0^+ \),
and decays as \( r \to \infty \). By Coffman’s theorem [10], this uniquely characterizes \( Q \) whence \( Q - \tilde{Q} = \delta_0 \).

3.2. Proof of lemma 3.6
For any \( r \in [0, r_\gamma] \), denoting \( \omega(r) := e^{-r}/(1 + r) \), any \( \delta \in X \) satisfies
\[|h_1(\delta, r)| \leq |R(r)| + \| U \|_\infty |\delta'(r)| + (\| V - 1 + 3 \tilde{Q}^2 \|_\infty + r^{-1} \| U \|_\infty) |\delta(r)| + (3\| \tilde{Q} \|_\infty + \| \delta \|_\infty) |\delta'(r)| \leq |R(r)|\]
\[+ \frac{\varepsilon_0}{33} + \left( \frac{1}{16} + r^{-1} \frac{1}{165} \right) \varepsilon_0 + 14 \varepsilon_0^2 + \varepsilon_0^3 \] \( \omega(r) \leq |R(r)| + \left[ \frac{3\varepsilon_0}{50} + \frac{\varepsilon_0}{165r} \right] \omega(r).\] (49)
For \( r > r_\gamma \) we state a different bound, see the case \( r > r_\gamma \) in lemmas 3.2 and 2.4: with \( \rho_3 := \frac{a}{25r^3} \),
\[|h_1(\delta, r)| \leq \frac{e^{-3r}}{25r^3} + 3 \tilde{Q}^2(r) |\delta'(r)| + 3 \tilde{Q}(r) \delta^2(r) + |\delta^3(r)| \leq (\rho_3 + 24 \varepsilon_0) \frac{e^{-3r}}{r^3} \quad \forall r > r_\gamma \] (50)
We now show that \( H \) takes \( X \) to itself, which is based on the estimates of the following lemma. This is the most technical part of the contraction argument for \( \delta \), and it is proved by inserting (49) and (50) into (46) and using the estimates from lemma 3.3. We suppress the argument \( r \) for the most part in the formulation of the following lemma.

3 Since \( y \in C^1 \) by construction, it is a weak solution of (3) and by standard Sturm–Liouville theory therefore also a smooth one.
Lemma 3.8. Let $\delta \in X$ be fixed and consider (46). For all $r > 0$ one has the estimates
\[
|H_1(\delta)| < \left( \frac{11}{10} \cdot 10^{-5} r^2 + \frac{\epsilon_0}{50} \right) \chi_{(0,\frac{1}{2})} + \left( \frac{9}{10} - \frac{1}{2} r \right) \left( 8 \cdot 10^{-6} + \frac{\epsilon_0}{25} \right) \chi_{(\frac{1}{2},1)}
\]
\[+ \frac{1}{r} \left( \frac{3}{2} \cdot 10^{-6} (3 - r) + \frac{13\epsilon_0}{1000} \right) \chi_{47}
\]
\[+ \frac{e^{-r}}{r} \left( 10^{-4} \left( \frac{33}{100} - 36e^{-2r} \right) + \frac{7\epsilon_0}{50} - \frac{48\epsilon_0}{25} e^{-2r} \right) \chi_{7\infty}, \tag{51}
\]
\[
|H'_1(\delta)| < \left[ 10^{-5} \left( 2 - \frac{6r}{5} \right) + \frac{\epsilon_0}{20} \right] \chi_{(0,\frac{1}{2})} + \left( \frac{7}{2} - 3r \right) \left( 8 \cdot 10^{-6} + \frac{\epsilon_0}{25} \right) \chi_{(\frac{1}{2},1)}
\]
\[+ \frac{2}{r} \left( \frac{3}{2} \cdot 10^{-6} (3 - r) + \frac{13\epsilon_0}{1000} \right) \chi_{47}
\]
\[+ \frac{(r+1)e^{-r}}{r^2} \left( 10^{-4} \left( \frac{33}{100} - 36e^{-2r} \right) + \frac{7\epsilon_0}{50} - \frac{48\epsilon_0}{25} e^{-2r} \right) \chi_{7\infty}, \tag{52}
\]
\[
|H_2(\delta)| < \left[ 10^{-5} \left( \frac{23}{25} - \frac{6r}{5} \right) + \frac{\epsilon_0}{15} \right] \chi_{(0,\frac{1}{2})} + \frac{7}{10} \left[ 10^{-6} \left( \frac{11}{50} + 2 (1 - r) \right) + \frac{1}{100} \frac{\epsilon_0}{\rho} \right] \chi_{(\frac{1}{2},1)}
\]
\[+ \frac{7}{5(r+1)} \left( \frac{11}{10} \cdot 10^{-6} + 10^{-2} \left( 6 - \frac{11}{5} r \right) \frac{\rho}{\rho_0} \right) \chi_{47}
\]
\[+ \frac{21e^{-3r} (3 + 1600\epsilon_0)}{4000r^2(r+1)} \chi_{7\infty}, \tag{53}
\]
\[
|H'_2(\delta)| < \frac{36}{13} \left[ 10^{-5} \left( \frac{23}{25} - \frac{6r}{5} \right) + \frac{\epsilon_0}{15} \right] \chi_{(0,\frac{1}{2})} + 18 \left[ 10^{-6} \left( \frac{11}{50} + 2 (1 - r) \right) + \frac{1}{100} \frac{\epsilon_0}{\rho} \right] \chi_{(\frac{1}{2},1)}
\]
\[+ \frac{1}{r+1} \left( \frac{11}{10} \cdot 10^{-6} + 10^{-2} \left( 6 - \frac{11}{5} r \right) \frac{\rho}{\rho_0} \right) \chi_{47}
\]
\[+ \frac{3e^{-3r} (3 + 1600\epsilon_0)}{1600r^3} \chi_{7\infty}. \tag{54}
\]

Proof. To avoid working with $\text{Ei}(x)$, we write $\omega(s) \leq e^s/(a + 1)$ inside every definite integral from $a$ to $b$, and $e^{-3s}/s^3 \leq 4e^{-3s}/(25s)$ for $s \geq r$. Also, estimating sums or products is of course elementary: for instance, $e^s + ax^2 + bx + c < d$ is equivalent to $e^{-s}(d - (ax^2 + bx + c)) - 1 > 0$—checked by examining the derivative. Using this, the result follows by straightforward calculations of the integrals using lemma 3.3, (49), and (50). See appendix A for details.

Now we can show that $H(\delta)$ takes the $\epsilon_0$-ball of $X$ to itself.

Corollary 3.9. Let $(H_0(\delta))(r) = |H(\delta)(r)| + \frac{1}{2}|H'(\delta)(r)|$. We have
\[
(r + 1)e^r (H_0(\delta))(r) \leq \epsilon_0 \quad \forall r \in \mathbb{R}^+. \tag{55}
\]

Proof. This is a straightforward consequence of lemma 3.8; the details are given in appendix A.3.
Lemma 3.10. For bounds on $h(56)$. Hence, with the replacements above, all contractivity calculations shadow those for the $-$. For the last one we note that $\delta$ Let $3.3.\ Contractivity\ of\ the\ map$

On the spectral properties of $L$.

It then follows from (50) that $|h_1(\delta_1, r) - h_1(\delta_2, r)| \leq \left( \|V - 1 + 3 \tilde{Q}^2 \|_{\infty} + r^{-1} \|U\|_{\infty} \right) \delta_0(r) |

+ (6) | \tilde{Q} \| \delta_0\|_{\infty} + 3 \| \delta_0\|_{2}^{2} | \delta_0(r) | + \|U\|_{\infty} \| \delta_0'(r) | \leq \left( \frac{3}{50} + \frac{1}{165r} \right) \omega(r) \| \delta_0 \|.

\begin{equation}
(56)
\end{equation}

If we replace $\epsilon_0$ by $\| \delta_0 \|$ and set $R = 0$, the last term in (49) is identical to the last term in (56). Hence, with the replacements above, all contractivity calculations shadow those for the bounds on $h_1$. Thus, with virtually the same proof as that of corollary 3.9, see appendix A.3, one derives the estimate

$(r + 1)e^{\int \left( \frac{3}{50} + \frac{1}{165r} \right) \omega(r) \| \delta_0 \|}

\begin{equation}
(57)
\end{equation}

where the final bounds follow by differentiating in $r$ for the first three and in $1/r$ for the last. For the last one we note that $-\frac{23}{10} + \frac{3}{5r^2} + \frac{431}{50r^2} - \frac{67}{25r}$ is negative and decreasing. \qed

3.4.\ Further\ estimates

In the study of $L_\pm$ we need a sharper estimate of $Q - \tilde{Q}$.

Lemma 3.10. For $r \geq \frac{3}{2} = r_7$, we have

$$\delta(r) = Q(r) - \tilde{Q}(r) = b_1e^{-r}/r + b_2(r),$$

where $|b_1| < 5 \cdot 10^{-5}$ and $|b_2(r)| < \frac{3}{50} e^{-3r}$.

**Proof.** It follows directly from (46) that

$$\delta(r) = g_\infty(r) \int_0^{\infty} \frac{g_0(s)h_1(\delta, s)}{W(s)} ds + g_\infty(r) \int_0^{r} \frac{g_0(s)h_1(\delta, s)}{W(s)} ds

+ g_0(r) \int_r^{\infty} \frac{g_\infty(s)h_1(\delta, s)}{W(s)} ds.$$

(58)

It then follows from (50) that

$$\left| \int_0^{\infty} \frac{g_0(s)h_1(\delta, s)}{W(s)} ds \right| \leq 5 \cdot 10^{-5}; \quad \left| g_\infty(r) \int_0^{r} \frac{g_0(s)h_1(\delta, s)}{W(s)} ds \right| \leq \frac{1}{25} e^{-3r}

\begin{equation}
(59)
\end{equation}

and

$$\left| g_0(r) \int_r^{\infty} \frac{g_\infty(s)h_1(\delta, s)}{W(s)} ds \right| \leq \frac{1}{50} e^{-3r},$$

(60)

which concludes the proof. \qed
4. The operator \( L_+ \)

In this section we prove the first of our main results, namely the gap property of \( L_+ \).

**Theorem 4.1.** The operator \( L_+ \) has no \((L^2(\mathbb{R}^+))\) eigenvalue or resonance in \([0, 1]\).

Standard ODE analysis shows that there are two solutions \( u_1(r; \lambda) \) and \( u_2(r; \lambda) \) of (4) with the properties \( u_1(0; \lambda) = 1 \) and \( u_2(0; \lambda) = r^{-1}e^{-\sqrt{\lambda^2 + \epsilon}}(1 + o(1)), \ r \to \infty \). These are, up to constants, the only ones acceptable at zero and infinity respectively, see appendix A.1.

Let \( W = u_1u_2' - u_2u_1' \) be the Wronskian of the two special solutions \( u_{1, 2} \). As mentioned in section 1.3, the existence of an eigenvalue or resonance of \( L_+ \) is equivalent to \( W = 0 \) for some \( \lambda \). Note that \( W \) is not constant due to the first order derivative in \( L_+ \). However, \( W(r) = -\frac{2}{r}W(r) \) whence \( W(r_0) = 0 \) at one point implies that \( W(r) = 0 \) everywhere. Theorem 4.1 is a corollary of the following result.

**Proposition 4.2.** We have the following estimate

\[
\inf_{r \in [0, 1]} |W(r; \lambda)| \geq 43 \cdot 10^{-4}. \tag{61}
\]

4.1. Proofs

As mentioned in section 1.3, we construct the quasi-solution \( \tilde{w}_1 \), a piecewise polynomial of two variables \( r \) and \( \lambda \), on the interval \([0, r_7]\), and \( \tilde{w}_2 \) whose expression involves exponential integrals on the interval \([r_7, \infty)\). To show that \( \tilde{w}_1 \) is close to \( u_1 \) and \( \tilde{w}_2 \) close to \( u_2 \), we use the same contraction mapping strategy we used for \( Q - \tilde{Q} \) outlined in section 1.3: the only difference is that the equations and solutions depend on the parameter \( \lambda \). The method for obtaining the estimates is explained in note 4.3. Finally, to estimate the Wronskian, we first approximate \( \tilde{w}_2(r) \) and \( \partial_r \tilde{w}_2(r) \) by polynomials of degree 7 in \( \lambda \), and the calculation then reduces to estimating a polynomial using note 2.5.

4.1.1. A polynomial quasi-solution for \( r \leq r_7 \). To build the quasi-solution, we first define

\[
\tilde{w}_1(r) = \sum_{(j,k,l) \in S} \chi_j c_{kl} j^l z^k, \tag{62}
\]

where \( S = \{j, k, l \mid 1 \leq j \leq 3, 0 \leq k \leq M_j, 0 \leq l \leq M_j\} \), \( c_{klj} \) are given in the appendix, \( \chi_j, \ j = 1, 2, 3 \), are the characteristic functions of \([0, r_2), [r_2, r_5)\), and \([r_5, r_7]\) respectively, \( M_j \leq 15 \) and \( z \) depends on \( r \) as specified in the top rows of the tables in the appendix. For the most part, we suppress the \( \lambda \)-dependence in our notations. To ensure that \( \tilde{w}_1 \in C^1 \) we next let

\[
\begin{align*}
\tilde{w}_1(r) &= \tilde{w}_1(r)\chi_{S7} + \left[ \tilde{w}_1(r) + \tilde{w}_1'(r) - \tilde{w}_1(r) - \tilde{w}_1'(r) \right](r - r_3) \chi_{25} \\
&+ \left[ \tilde{w}_1(r) + \tilde{w}_1'(r) - \tilde{w}_1(r) - \tilde{w}_1'(r) \right](r^2 - r^2_2) \chi_{02}. \tag{63}
\end{align*}
\]

**Note 4.3.** In this section, we divide \([a, b] \times [0, 1]\) ‘vertically’, using a partition \( \pi \) of \([a, b]\) in \( r \) and the trivial partition \((0, 1)\) in \( \lambda \). Clearly, such partitions are determined by \( \pi \).

(i) Let \( P(y, z) \) be a polynomial with \((y, z) \in [-1, 1] \) and \( P_1 \) defined to be the sub-polynomial consisting of all terms of \( P \) of the form \( c_{k,0} z^k, 0 \leq k \leq 3 \) and \( c_{0,j} y^j, 1 \leq j \leq 3 \).

We bound \( P \) above (below) by the maximum (minimum, respectively) of \( P_1 \) plus (minus, respectively) the sum of the absolute value of the coefficients of \( P - P_1 \). Since \( P_1 \) is the sum of a cubic polynomial in \( z \) and a cubic polynomial in \( y \), the extrema calculations reduce to solving one variable quadratic equations. We proceed in this way for simplicity, given that the mixed
Lemma 4.4. We have \( |w_1^*(r)| \leq M(r) := \frac{11(1 + r)}{10} X_{02} + \frac{9}{20} X_{27}. \)

Proof. We introduce the partition of \([0, r_1] \times [0, 1]\) induced by \(\pi = (0, 3/10, 17/25, 3/2, 5/2)\). The lemma now follows by applying the method in note 4.3 to \(w_1^*(\ell_k(z)), \frac{1}{2}(1 + y) \leq M(\ell_k(z))\) for \(1 \leq k \leq 4\).

Now we invoke the estimates for \(|Q - \tilde{Q}|\) (proposition 3.5) and \(\tilde{Q}\) (lemma 2.4) to obtain
\[
|3 \tilde{Q}^2(r) - 3 \tilde{Q}^3(r)| \leq 3(2 \tilde{Q} + (|Q - \tilde{Q}|)/Q - \tilde{Q})(r)
\]
\[
\leq \frac{1}{10000} \left[ 21e^{-3r} X_{02} + \frac{116}{5}(1 + r)^{-1} e^{-13r/5} X_{27} \right] + \frac{3}{2} 10^{-8} e^{-2r}
\]
\[
\leq \frac{1}{5000} \left[ 11e^{-3r} X_{02}(r) + 12(1 + r)^{-1} e^{-13r/5} X_{27}(r) \right].
\]
\[
(64)
\]
We shall use this estimate repeatedly, for example in the following bound on the remainder
\[
R_1 := -w_1^{**} - 2w_1^*/r + (1 - \lambda - 3 \tilde{Q}^2)w_1^*.
\]
\[
(65)
\]
Lemma 4.5. We have
\[
|R_1(r)| \leq 10^{-4} \left\{ \frac{33}{10} (11 - 35r + 34r^2) X_{02} + \frac{11 - 6r}{5} X_{25} + \frac{39 - 9r}{100} X_{57} \right\}.
\]
\[
(66)
\]
Proof. We first consider \(R_0 = -w_1^{**} - 2w_1^*/r + (1 - \lambda - 3 \tilde{Q}^2)w_1^*\) and the piecewise polynomial
\[
\tilde{R}_0(r) = [R_0(r)/\tilde{Q}^2(r)] X_{02} + [r R_0(r)/\tilde{Q}^3(r)] X_{27}.
\]
On \([0, r_2]\) we use the partition \(\pi = (0, \frac{2}{25}, \frac{21}{100}, \frac{7}{50}, \frac{12}{25}, \frac{49}{100}, \frac{33}{50}, r_2)\). Since \(\tilde{Q}\) is positive and decreasing, in each subinterval we only need to show that \(|\tilde{R}_0(\ell_k(z))|\tilde{Q}^2(\ell_k(-1))\) is bounded by the right-hand side of (66). It is easy to see that this can be reduced to applying the method in note 4.3 for \(\tilde{R}_0(\ell_k(z))\tilde{Q}^2(\ell_k(-1)) \leq 10^{-4}(11 - 43r + 54r^2).\) In this fashion, we obtain the estimate
\[
|R_0(\ell_k(z))| \leq |\tilde{R}_0(\ell_k(z))|\tilde{Q}^2(\ell_k(-1)) < 10^{-4}(11 - 43r + 54r^2)
\]
for all \(1 \leq k \leq 7\).

Next we introduce the partition of \([r_2, r_3]\) provided by \(\pi = (r_2, \frac{37}{50}, r_3, \frac{57}{50}, \frac{69}{50}, r_3, r_5)\) to estimate \(|R_0(\ell_k(z))|\tilde{Q}^2(\ell_k(-1)) = 2 \cdot 10^{-2} \ell_k(z).\) This yields the estimate
\[
\ell_k(z)|R_0(\ell_k(z))| \leq |\tilde{R}_0(\ell_k(z))|\tilde{Q}^2(\ell_k(-1)) < 2 \cdot 10^{-5} \ell_k(z) \quad \forall 1 \leq k \leq 5.
\]
For the interval \([r_5, r_7]\) we use the partition induced by \(\pi = (r_5, \frac{17}{10}, \frac{19}{10}, \frac{52}{25}, \frac{56}{25}, \frac{59}{25}, \frac{61}{25}, r_7)\) and conclude that
\[
\ell_k(z)|R_0(\ell_k(z))| \leq |\tilde{R}_0(\ell_k(z))|\tilde{Q}^2(\ell_k(-1)) < 15 \cdot 10^{-6} \ell_k(z).
\]
We have \(|R_1 - R_0| = 3|\tilde{Q}^2 - \tilde{Q}^2|\left|w_1^*\right|\). Combining these results with (64) and lemma 4.4 to estimate \(|3 \tilde{Q}^2 - 3 \tilde{Q}^2|\left|w_1^*\right|\), we obtain
\[
|R_1(r)| \leq 10^{-4} \left\{ \frac{11 - 43r + 54r^2}{5} e^{-3r} X_{02} + \left( 1 + \frac{54e^{-13r/5}}{1 + r} \right) \frac{X_{25}}{5} \right\}.
\]
\[
(68)
\]
The right-hand side of (66) minus the right-hand side of (68) is positive; this follows using note 4.3 (ii), after the substitution of \(5e^{-3r} X_{02}\) by the bound \((5 - 11r + 7r^2)X_{02}. \)
4.1.2. A quasi-fundamental system of solutions on \([0, r_7]\). To show there is an actual solution \(u_1\) of (4) on \([0, r_7]\) with \(\delta := u_1 - w_1^*\) small, we construct two functions \(\tilde{g}_1, g_1^*\) approximating two linearly independent solutions of (4); define first

\[
\tilde{g}_1^*(r) := \sum_{(j,k,l) \in S} d^*_i \chi_j \lambda^j \zeta^l, \quad g_2^*(r) := \sum_{(j,k,l) \in S} e^*_i \chi_j \lambda^j \zeta^l. \tag{69}
\]

where \(S = \{(j, k, l) \mid 1 \leq j \leq 3, 0 \leq k \leq M_j, 0 \leq l \leq 15\}\) and \(\chi_j, j = 1, 2, 3\), are the characteristic functions of \([0, r_1]\), \([r_1, r_3]\), and \([r_3, r_7]\), respectively. The coefficients, the intervals corresponding to \(j = 1, 2, 3\), and the expressions \(z = z(r)\) are given in the appendix.

To ensure \(C^1\) behaviour we use the same method as in definition 3.1 to set \(g^*_j = \hat{g}^*_j\) on \(J_1\) and

\[
\hat{g}^*_j(\ell) := \tilde{g}^*_j(\ell) + \frac{1}{(\ell - r)}(\tilde{g}^*_j(\ell) - \tilde{g}^*_j(r))(r - \ell) \quad \forall \ell \in J_\ell \tag{70}
\]

for all \(\ell = 2, 3, 4\) and \(\hat{g}^*_j(\ell) = \hat{g}^*_j(\ell)/r\). In contrast to definition 3.1 where the \(C^1\)-matching is done from right to left, we find it necessary to carry out the matching by going from left to right.

We construct a second order equation satisfied by \(g^*_1, g^*_2\) in the form

\[
-g'' + (A(r) - 2/r)g' + (B(r) + A(r)/r)g = 0, \tag{71}
\]

where

\[
A(r) = \frac{\hat{g}''_1^*(r) \hat{g}''^*_2(r) - \hat{g}'_1^*(r) \hat{g}'_2^*(r)}{\hat{g}''_2^*(r) \hat{g}'_1^*(r) - \hat{g}'_2^*(r) \hat{g}'_1^*(r)}, \quad B(r) = \frac{\hat{g}'_1^*(r) \hat{g}'_2^*(r) - \hat{g}'_1^*(r) \hat{g}'_2^*(r)}{\hat{g}''_2^*(r) \hat{g}'_1^*(r) - \hat{g}'_2^*(r) \hat{g}'_1^*(r)}. \]

Lemma 4.6. We have \(|A(r)| \leq \frac{1}{1000}(\frac{1}{2} \chi_{06} + 4 \chi_{07})\) and \(|B(r) - 1 + \lambda + 3 \hat{Q}^2(r)| \leq \frac{1}{2000} \chi_{06} \chi_{07} \chi_{17} \chi_{18} \chi_{19}

Proof. We use a partition \(\pi = (0, \frac{7}{50}, \frac{7}{25}, r_1, \frac{9}{25}, \frac{1}{2}, \frac{18}{25}, \frac{46}{50}, r_4, \frac{7}{5}, \frac{19}{10}, \frac{11}{7}, r_6, r_7)\) and estimate above/below, by re-expansion, the absolute value of the numerator (denominator, respectively).

We rewrite the equation for \(\delta\) (see beginning of section 4.1.2) in the form

\[
-\delta'' + (A(r) - 2/r)\delta' + (B(r) + A(r)/r)\delta = R_1(\delta) + A(\delta)\delta'(r)
\]

\[
+ [B(r) - 1 + \lambda + 3 \hat{Q}(r)]^2 + A(r)/r + 3(\hat{Q}(r)^2 - \hat{Q}(r)^2)](\delta) =: h_2.
\]

Note. In the following we write \(\|f\|\) for \(\sup_{[0, r_7]} |f|\).

Lemma 4.7. We have

\[
|h_2(\delta, r)| \leq 10^{-4} \left[33(11 - 35r + 34r^2)/10 + 5 \|\delta\| + (82 + 5/r)\|\delta\| \chi_{02} + 10^{-4} \left[(11 - 6r)/5 + 70\|\delta\| + 5\|\delta\|^2 \chi_{25} + 10^{-4}[(39 - 9r)/100 + 64\|\delta\|^2 + 5\|\delta\|^2 \chi_{50} + 10^{-4}[(39 - 9r)/100 + 78\|\delta\|^2 + 40\|\delta\|^2 \chi_{67}] \right) \right]. \tag{72}
\]

Proof. This is obtained by combining (66), lemma 4.6 and (64) and using the monotonicity of the coefficients containing exponentials.

Lemma 4.8. We have \(\|g^*_1\| \leq \frac{11}{100} \chi_{01} + \frac{1}{2} \chi_{17}\) and \(\|g^*_2\| \leq \frac{11}{100} + \frac{17}{100}\).

4 We note that \(g^*_1\) is approximately \(w^*_1\) redefined on the same interval as \(g^*_2\) for convenience.
Proof. We use the partition induced by \( \pi = (0, r_1, \frac{3}{10}, r_4, 2, r_7) \) and apply the method in note 4.3 to \( \chi_{01}(\frac{11}{10} \pm r g_1^1), \chi_{17}(\frac{5}{10} \pm r g_1^1) \) and \( \frac{11}{10} \pm \frac{17}{10} r g_1^2 \).

In addition, one has the following.

Lemma 4.9. There are the bounds \( |g_1^1(r)| \leq 3 \chi_{02} + \chi_{25} + \frac{3}{10} \chi_{57}; \quad |g_1^2(r)| \leq \frac{27}{100\pi} \chi_{02} + \frac{4}{7\pi} \chi_{25} + \frac{17}{10} \chi_{57} \).

Proof. Similar, using \( \pi = (0, r_1, r_2, r_4, r_5, \frac{17}{10}, r_7, \frac{21}{10}, r_7) \). \( \square \)

4.1.3. The actual smooth solution on \([0, r_7]\). Let

\[
H_0(\delta) = g_2^1(r) \int_0^r \frac{g_1^1(s) h_2(\delta, s)}{W(s)} \, ds - g_1^1(r) \int_0^r \frac{g_2^1(s) h_2(\delta, s)}{W(s)} \, ds.
\]

(73)

Clearly, we have \( \delta = H_0(\delta) \) and \( \delta' = H_0' \) where

\[
H_0'(\delta) = g_2^1(r) \int_0^r \frac{g_1^1(s) h_2(\delta, s)}{W(s)} \, ds - g_1^1(r) \int_0^r \frac{g_2^1(s) h_2(\delta, s)}{W(s)} \, ds.
\]

(74)

Lemma 4.10. There is the bound

\[
|H_0(\delta)(r)| + |H_0'(\delta)(r)|/5 \leq 1/1200 + \|\delta\|/10 + \|\delta'\|/80.
\]

Thus \( H \) is a contraction in the ball

\[
X := \{ f \in C^1((0, r_7)) \mid \| f \|_X \leq 1/1080 \},
\]

(76)

where \( \| f \|_X := \sup_{r \in [0, r_7]} (|f(r)| + \frac{1}{2} |f'(r)|) \) (cf also footnote 3 on p 139).

Proof. We crudely estimate the quantities \( H_0 \) and \( H_0' \) by placing absolute values on all terms, and using the bounds already calculated for \( g_1^1, g_2^1, \) etc; \( |\delta| \) and \( |\delta'| \) are estimated by their supremum norms, written as \( \| \cdot \| \). Since \( W(r) := g_1^1(r) g_2^1(r) - g_1^1(r) g_2^1(r) = r^{-2} \exp \left( \int_0^r A(s) \, ds \right) \geq r^{-2} \exp \left( - \int_0^r |A(s)| \, ds \right) \) (we note that \( W(r)r^2 \to 1 \) as \( r \to 0 \)) we have

\[
1/|W(r)| < \frac{626}{625} \frac{r^2}{2} < \frac{51r^2}{50}.
\]

On the first interval, \([0, r_1]\) a direct calculation shows that \( \tilde{H}_0(\delta)(r) := |H_0(\delta)| + |H_0'(\delta)'(r)|/5 \) is majorized by

\[
\max \left( |r P_2(r)| + |r P_2'(r)| \|\delta\| + |P_3(r)| \|\delta\| \right) \leq 2 \cdot 10^{-6} (200 + 1300 \|\delta\| + 57 \|\delta'\|),
\]

(77)

where \( P_j \) are polynomials of degree \( j \), easily maximized since they are increasing on this interval (all have positive coefficients except \( r P_4; (r P_2)' \) has positive coefficients after we replace \( r^2 \) by \( r r_1 \) and \( r^3 \) by \( r r_1^2 \); thus \( r P_4 \) is increasing.

Calculating \( \tilde{H}_0(\delta)(r) \) for \( r \in [r_1, r_2] \), we obtain a rational function; we first replace \( 1/r, 1/r^2 \) by \( 1/r_1, 1/r_1^2 \) respectively (their coefficients are positive) and we obtain an expression of the form \( P_3(r) + P_3(r) \|\delta\| + P_3(r) \|\delta'\| \), with the same convention as above for the polynomials (different from the \( P_s \) on the previous interval). Once more, all polynomials except \( P_3 \) have positive coefficients. In \( P_3 \) we first replace \( r^2 \) (whose coefficient is positive) by \( r^2 r_2 \); the derivative of the new polynomial has an explicit positive minimum. Thus the maximum of \( P_3 \) is reached at \( r = r_2 \). Thus, by taking \( r = r_2 \), we obtain, for \( r \in [r_1, r_2] \),

\[
\tilde{H}_0(\delta)(r) \leq \frac{19}{25000} + \frac{\|\delta\|}{125} + \frac{\|\delta'\|}{2500},
\]

(78)
On the interval \([r_2, r_5]\) we proceed in the same way, replacing \(1/r, 1/r^2\) by \(1/r_2, 1/r_5^2\) respectively. This results in an expression of the form \(P_4 + P_3\|\delta\| + P_3\|\delta'\|\) with the same properties and conventions as above. Now the derivative of \(P_4\) can be minimized explicitly: it is positive and thus \(P_4\) is maximal at the right-hand endpoint. We obtain the following majorization of \(\tilde{H}_0(\delta)(r)\)

\[
\tilde{H}_0(\delta)(r) \leq \frac{1}{1200} + \frac{\|\delta\|}{34} + \frac{\|\delta'\|}{515} \quad \forall r \in [r_2, r_5].
\]  

(79)

On the interval \([r_5, r_6]\) we replace \(1/r\) by \(1/r_5\) and obtain an expression very similar to the one on \([r_2, r_5]\). It is dealt with in the same way, whence

\[
\tilde{H}_0(\delta)(r) \leq 1/1250 + \|\delta\|/11 + \|\delta'\|/164 \quad \forall r \in [r_5, r_6].
\]  

(80)

Finally, on \([r_6, r_7]\), after replacing \(1/r\) by \(1/r_6\) in the positive terms and by \(1/r_7\) in the negative ones, we reduce \(\tilde{H}_0(\delta)(r)\) to the form \(P_4 + P_3\|\delta\| + P_3\|\delta'\|\). \(P_4\) and \(P_3\) are manifestly increasing and \(\min_{(r_6, r_7)} P_4 > 0\). Thus the maximum is reached at \(r_7\) and we obtain

\[
\tilde{H}_0(\delta)(r) \leq 1/1200 + \|\delta\|/10 + \|\delta'\|/80 \quad \forall r \in [r_6, r_7].
\]  

(81)

Of all estimates, the worst bounds are in (81); contractivity as well as preservation of the ball thus follow from (81).

\[\square\]

Corollary 4.11. The function \(w^*_1\) differs from an actual solution \(u_0\) of \(L_\lambda u = \lambda u\) by at most \(1/1080\) in \(\|\cdot\|_X\). Furthermore,

\[|\delta'(r_7)| \leq 1/1800.\]

\[\text{Proof.}\] The first part is just (76). The second part comes from direct substitution in \(|H_0'|\) followed by lemma 4.10:

\[|H_0(\delta')\langle r_7\rangle| \leq 1/2080 + \|\delta\|_X/17 \leq 1/2080 + 1/17 \cdot 1/1080\]

as claimed.

\[\square\]

Lemma 4.12. We have \(|w^*_1(\bar{r}_7)| < 19/47, |w^{\prime\prime}_1(\bar{r}_7)| < 4/19.\]

\[\text{Proof.}\] Viewed as quintic polynomials in \(\lambda\), \(w^*_1(\bar{r}_7)\) and \(w^{\prime\prime}_1(\bar{r}_7)\) are simply estimated as in note 4.3 (reduced here to one variable \(y\) where \(\lambda = \frac{1}{2}(1 + y)\)).

\[\square\]

4.1.4. The quasi-solution bounded on \([\bar{r}_7, \infty)\). Let \(\sigma = \sqrt{1 - \lambda}\) and \(\sigma_1 = 1 + \sigma\). We let \(a_0 = 1413/64\). In this region, we look for \(u_\infty\) in the form \(w^*_2 + \bar{b}_0\) where

\[
w^*_2(r) = \frac{e^{-\sigma_r}}{r} \left(1 + a_0 f_1(r)\right); \quad \sigma f_1(r) := -E_i(-2r) + e^{2\sigma r} \sigma_1 E_i(-2\sigma r)\]

(82)

is close to an exponentially decaying solution of (4).\(^5\) Then, \(w^*_2\) satisfies

\[-w^*_2'' - 2w^*_2/r + \left(1 - \lambda - a_0 \frac{e^{-2r}}{r^2}\right) w^*_2 = -a_0^2 \frac{e^{-2\sigma r}}{r^3} f_1(r) =: R_2.\]

\[\text{Lemma 4.13. We have (i) \(|w^*_2(r)| < e^{-\sigma r}/r\) and (ii) \(|R_2(r)| < \frac{2e^{-\sigma r}}{2\sigma r}.\)}\]

\(^5\) To obtain this approximation, we rewrite (4) as \((-\frac{d}{dr} - \frac{2}{r} + 1 - \lambda)u = 3Q^2u\), replace \(Q\) by the leading term of \(\bar{Q}\), and iterate the associated integral equation.
Proof. By straightforward algebra, using the bounds in (9) one obtains
\[ f_i(r) = 2\sigma_1 e^{2\sigma r} \text{Ei}(-2\sigma_1 r) + \frac{e^{-2r}}{r} > 0. \]  
(84)
Since \( \lim_{r \to \infty} f_i(r) = 0 \), we clearly have \( f_i(r_7) \leq f_i(r) < 0 \). Now
\[ f_1(r; \sigma) = \int_{-\infty}^{r} \frac{e^u(u + 5)}{u(u - 5\sigma)} \, du \geq f_1(r; 0) = 6\text{Ei}(-5) + e^{-5} > -\frac{1}{6500} \]
and the estimates follow. \( \square \)

4.1.5. The equation for \( \delta_0 \). The difference \( u_\infty - w^*_\infty = \delta_0 \) satisfies the equation
\[ -\delta''_0 - 2\delta'_0/r = R_2 + \left( \lambda - 1 + a_0 \frac{e^{-2r}}{r^2} \right) \delta_0 + \left( 3Q^2 - a_0 \frac{e^{-2r}}{r^2} \right) (w^*_0 + \delta_0) =: h_3(r). \]  
(85)

Note 4.14. One has
(i) \( |Q^2(r) - \tilde{Q}^2(r)| < (2\tilde{Q}(r) + |Q(r) - \tilde{Q}(r)||Q(r) - \tilde{Q}(r)| \leq 4 \cdot 10^{-4} e^{-2r} \).
(ii) \( |3Q^2(r) - a_0 \frac{e^{-2r}}{r^2}| \leq |3Q^2(r) - 3\tilde{Q}^2(r)| + |a_0 \frac{e^{-2r}}{r^2} - 3\tilde{Q}^2(r)| < \frac{1}{20} \frac{e^{-2r}}{r^2} \).

Indeed, (i) follows from (22) and (44). (ii) uses (22) and (44) and (i).

Now using (85), Lemma 4.13 and note 4.14 we have
\[ |h_3(r)| \leq |R_2| + \left| 3\tilde{Q}^2 - a_0 \frac{e^{-2r}}{r^2} \right| |w^*_0(r)| + |\lambda - 1 + 3\tilde{Q}^2| |\delta_0(r)| < \frac{13e^{-2r}}{100r^2} + |\delta_0(r)|. \]

Looking for exponentially decreasing solutions, we write (85) in the integral form
\[ \delta_0 = H_1(\delta_0) := -\int_r^{\infty} \frac{dt}{t^2} \int_t^{\infty} s^3 h_3(s) \, ds. \]  
(86)

4.1.6. The actual solution on \([r_7, \infty)\)

Lemma 4.15. \( H_1 \) is contractive in the ball \( \{ f \mid \| f \| \leq 13/300 \} \) in the Banach space \( \{ f \mid \| f \| = \sup_{r_7 < r} r^\lambda e^{2r} |f(r)| < \infty \} \).

Proof. Noting that \( (s/t)^3 > 1 \) and \( t > r \) we write
\[ \|H_1(\delta_0)\| \leq \frac{1}{r^3} \int_r^{\infty} \int_t^{\infty} s^3 h_3(s) \, ds \leq \frac{e^{-2r}}{r^3} \left( \frac{13}{400} + \frac{\|\delta_0\|}{4} \right) \]  
(87)
whence the claim. \( \square \)

Hence \( \|H_1(\delta_0)\| \leq 13/400 + \|\delta_0\|/4 \) and the claim follows.

Corollary 4.16. We have
\[ |u_2(r_7) - w^*_2(r_7)| = |\delta_2(r_7)| \leq \frac{13}{300} \frac{2^{3/5}}{5^3} e^{-5} < 2 \cdot 10^{-5}, \]
\[ |u'_2(r_7) - w'^*_2(r_7)| = |\delta'_2(r_7)| < 4 \cdot 10^{-5}. \]  
(88)

Proof. Only \( \delta' \) needs to be estimated; this is immediate:
\[ |\delta'_0(r)| = |(H_1\delta_0)'(r)| \leq \frac{1}{r^3} \int_r^{\infty} s^3 h_3(s) \, ds \leq \frac{e^{-2r}}{r^3} \left( \frac{13}{200} + \frac{\|\delta_0\|}{2} \right) < 4 \cdot 10^{-5} \]
and we are done. \( \square \)
4.1.7. The Wronskian of the well-behaved quasi-solutions. Equation (82) implies
\[
\begin{align*}
\tilde{w}_2^* (r_7) &= \frac{2e^{-r_7}}{5\sigma} (\sigma - a_0 \text{Ei}( -5) + a_0 e^{5\sigma} \text{Ei}( -5\sigma)); \\
\tilde{w}_3^* (r_7) &= \frac{2e^{-r_7}}{25\sigma} (\sigma(2a_0 e^{-5} - 2 - 5\sigma) + \text{Ei}( -5) a_0 (2 + 5\sigma) \\
&\quad + a_0 e^{5\sigma}(5\sigma^2 + 3\sigma - 2)\text{Ei}( -5\sigma)).
\end{align*}
\]  
(89)

Lemma 4.17.

(i) Let \( z = 2\sigma - 1 \). The functions \( w_1^* \) and \( w_1'^* \) satisfy the estimates
\[
\left| \frac{79}{691} - \frac{737}{2580} z^2 + \frac{147}{412} z^4 \quad \frac{9}{165} z^4 - \frac{103}{25} z^4 + \frac{13}{340} z^6 - \frac{131}{25} z^6 + \frac{6}{445} z^7 - w_2^* (r_7) \right| \\
< 3 \cdot 10^{-5} - \frac{52}{509} + \frac{26}{185} z^4 - \frac{10}{313} z^4 - \frac{80}{857} z^7 + \frac{307}{325} z^7 - \frac{3}{392} z^8 - w_2'(r_7) \right| < 6 \cdot 10^{-5}.
\]  
(90)

(ii) We have \( |w_1^* (r_7)| \leq 21/50 \) and \( |w_2^* (r_7)| < 19/100 \), which will be used in estimating the Wronskian of the two possible eigenfunctions.

Proof. The polynomials in (90) are simply truncates of the Taylor series of the functions involved. These, and the estimates, are obtained as follows. Denoting \( w_1^* (\sigma) = \sigma w_1^* (r_7; \sigma) \) we have \( w_1^* (r_7; \sigma) = \int_0^\sigma w_1'^* (s) \, ds \). We then approximate \( w_1'^* (\sigma) \) using a Taylor polynomial around \( \sigma = \frac{1}{2} \) with rigorous bounds for the remainder, using Cauchy’s formula. This can be obtained by expanding the exponential functions and expanding the integrands and then integrating the series term by term. For example, \( e^{z^4} \) differs from the sum of the first 12 terms of its Taylor series expansion in \( u \) at \( u = \frac{1}{2} \) by no more than \( 2 \cdot 10^{-7} \) (this follows from by Cauchy’s integral formula). Hence \( \text{Ei}( -5 - 5\sigma) = \text{Ei}( -10) + \int_{5\sigma}^{5\sigma} e^{-u} \, du \) differs from the integral of the Taylor polynomial by no more than \( 10^{-6} \). After obtaining a polynomial approximation of \( w_1^* (r_7) \) in this way, we re-expand it in \( z = 2\sigma - 1 \), and it so turns out that the coefficients of \( z^k \) for \( k > 7 \) are manifestly small. Discarding them and using rational approximations of the remaining coefficients we obtain the polynomial in lemma 4.17. The result for \( w_1'^* (r_7) \) follows in a similar way. The proof of (ii) follows from (i) using note 2.5 with the partition \( \pi = (0, 1/2, 1) \).

Corollary 4.18. Let \( W[w_1^*, w_2^*](r_7) = w_1^* (r_7)w_2'^* (r_7) - w_2^* (r_7)w_1'^* (r_7) \). We have
\[
\sup_{u \in [0, 1]} \left| W[w_1^*, w_2^*] \right| \geq 48 \cdot 10^{-4}.
\]  
(91)

Proof. We substitute \( \lambda = 1 + \sigma^2 \) in (63), and use the polynomials in (90) to calculate the Wronskian within an accuracy of \( \pm 4 \cdot 10^{-5} \) (obtained by crudely bounding away the effects of the errors on the right-hand side of (90)). For estimating the resulting polynomials, we write \( \sigma = \frac{1}{2} + \frac{1}{2}z \), re-expand and use note 2.5.

4.1.8. End of the proof of proposition 4.2: the Wronskian of the actual solutions.

Lemma 4.19. The Wronskian \( W[u_1, u_2](r_7) \) of the actual solutions satisfies
\[
|W[u_1, u_2](r_7) - W[w_1^*, w_2^*](r_7)| \leq 5 \cdot 10^{-5}.
\]  
(92)
Proof. This follows by estimating $W[u_1,u_2]$ via (91). This is straightforward and uses corollary 4.11, lemma 4.12, corollary 4.16 and lemma 4.17(ii) to bound $|u_{1,2} - w_{1,2}^+|$, $|u_{1,2}' - \partial_r w_{1,2}^+|$ as well as $|u_{1,2}|$, $|u_{1,2}'|$, $|w_{1,2}^+|$ and $|\partial_r w_{1,2}^+|$. □

5. The operator $L_-$

The second main result of this paper is the following one, which establishes the gap property for $L_-$.

Theorem 5.1. The operator $L_-$ has no eigenvalue or resonance for $\lambda$ in the interval $(0, 1]$ in $L_2(\mathbb{R}^+)$. 

As in the case of $L_+$, there are two solutions $y_1(r; \lambda)$ and $y_2(r; \lambda)$ of the equation (5) with the properties $y_1(0; \lambda) = 1$ and $y_2(r; \lambda) = e^{-r\sqrt{1-\lambda}}(1 + o(1)), \ r \to \infty$. Let $W[y_1, y_2](r; \lambda) = y_1 y_2' - y_2 y_1'$ be the Wronskian of these two special solutions. Theorem 5.1 is a corollary to the following result.

Proposition 5.2. One has the lower bound

$$\sup_{\lambda \in (0,1)} \lambda^{-1} |W[y_1, y_2](r; \lambda)| \geq 17/1000.$$  \hspace{1cm} (93)

Therefore $y_1$ and $y_2$ are linearly independent for all $\lambda \in (0, 1]$.

5.1. Proofs

Let

$$\tilde{u}_1(r) = \lambda^{-1} \left[ y_1(r) - \frac{Q(r)}{Q(0)} \right]; \quad \tilde{u}_2(r) = \frac{1}{\lambda} \left[ y_2(r) - \frac{1}{A_1} Q(r) \right]$$

where $A_1 = A + b_1$ \hspace{1cm} (94)

(see lemma 3.10). Then $\tilde{u}_1$ satisfies

$$-u'' - 2u'/r + (1 - \lambda - Q^2)u = \frac{Q}{Q(0)}.$$  \hspace{1cm} (95)

We construct a pair of functions that agree with $y_1$ and $y_2$ within relatively small errors. To this effect, we first define $g_{1,2}$ in the following way. Consider the piecewise polynomials

$$
\tilde{g}_j^-(r) = \sum_{(j,k,l) \in S} d_{kl,j} \chi_j x_k^l; \quad \tilde{g}_j^+(r) = \sum_{(j,k,l) \in S} e_{kl,j} \chi_j x_k^l,
$$  \hspace{1cm} (96)

where $S = \{(j, k, l) \mid 1 \leq j \leq 3, 0 \leq k \leq M_j, 0 \leq l \leq 15\}, d_{kl,j}$, $e_{kl,j}$ are given in the appendix, $\chi_j, j = 1, 2, 3$, are the characteristic functions of $[0, r_3], [r_5, r_1]$ and $[r_1, r_7]$ respectively, $M_j \leq 15$ and $\chi$ depends on $r$ as specified in the top rows of the tables in the appendix. To ensure that $g_{1,2}$ are $C^1$ we next let

$$
\hat{g}_j^- = \tilde{g}_j^-(r) \chi_{14} + \left[ \tilde{g}_j^-(r) + \tilde{g}_j^-(1) - \tilde{g}_j^-(1-)(r-1-)(r-1) + (\tilde{w}_1'(1) - \tilde{w}_1'(1-))(r-1) \right] \chi_{14}
$$

$$+ \left[ \tilde{g}_j^+(r) + \tilde{g}_j^+(r_1) - \tilde{g}_j^+(r_1-)(r_2-)(r_1-r_1) + 5(\tilde{g}_j^+(r_1) - \tilde{g}_j^+(r_1-))(r_2-r_1^2)/3 \right] \chi_{01},
$$  \hspace{1cm} (97)

where $j = 1, 2$. We let $\hat{g}_j^- = \hat{g}_j^-(r)/r^{-1}$ and $w_j(r) = (\hat{g}_j^- - \hat{g}_j^+(r)|_{r=0})/\lambda$.  \hspace{1cm} (98)

Note that for $L_+$, we constructed $w_j^+(r)$ using a different piecewise representation due to the high accuracy required. Here, however, it is sufficient to use $g_j^+(r)$ to define $w_j^-(r)$.  \hspace{1cm} (99)
Lemma 5.3. The following bounds hold:
\[ |w_1^+(r)| \leq 1/100 + 3r/25 \quad \forall 0 \leq r \leq r_5, \]  
\[ r|g_1^+(r)| \leq \frac{11r}{10} \chi_0 + \frac{r + 3}{10} \chi_7; \quad r|g_2^+(r)| \leq \frac{11r}{10} \chi_0 + \frac{23r}{10} \chi_8 + \frac{22r}{5} \chi_7. \]  
\[ |g_1^-(r)| \leq \frac{13}{10} \chi_0 + \frac{1}{2} \chi_7; \quad r^2|g_2^-(r)| \leq \frac{8}{5} \chi_0 + 3r^2 \chi_7. \]

Proof. We check this using note 4.3 and the partition \( \pi = (0, r_1, \frac{3}{5}, r_4, r_5, 2, r_7) \). For \( g_1^- \) in \( [0, r_1] \) we first divide (99) by \( r \).

Define
\[ R_1 = -w_1^--2w_1^-/r + (1 - \lambda - \hat{Q}^2)w_1^- - \hat{Q}/Q(0) \]
and
\[ \tilde{R}_0 = -w_1^- - 2w_1^-/r + (1 - \lambda - \hat{Q}^2)w_1^- - \tilde{Q}/Q(0). \]

Lemma 5.4. We have \( |\tilde{R}_0| \leq 10^{-3}(3\chi_0 + 2\chi_7) \) and \( |R_1| \leq 10^{-3}(6\chi_0 + 3\chi_7) \).

Proof. Note that
\[ |R_1| \leq |\tilde{R}_0| + |(\hat{Q}^2 - \hat{Q})w_1^-| + |Q - \hat{Q}|/Q(0). \]

We start with \( \tilde{R}_0 \), for which we use the same idea as in lemma 4.5 (we kept the same notations although the functions are different; this should cause no confusion). Due to the monotonicity of \( \hat{Q} \) and the fact that \( \tilde{R}_0(r)/\hat{Q}(r)^2 \) is a polynomial we have
\[ |\tilde{R}_0(\ell_k(z))| \leq |\tilde{R}_0(\ell_k(z))\hat{Q}^2(\ell_k(-1))/\hat{Q}(\ell_k(z))^2| \leq 3 \cdot 10^{-5} \]
using the method in note 4.3 and the partition given by \( \pi = (0, \frac{3}{5}, \frac{6}{7}, \frac{3}{10}) \).

Similarly on \( [r_1, r_7] \) we use the partition \( \pi = (r_1, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{9}{15}, \frac{1}{2}, \frac{3}{5}, \frac{10}{15}, \frac{11}{5}, r_6, r_7) \) and obtain
\[ |\tilde{R}_0(\ell_k(z))| \leq |\ell_k(z)/\ell_k(-1)\hat{Q}^2(\ell_k(-1))/\hat{Q}(\ell_k(z))^2\tilde{R}_0(\ell_k(z))| \leq 2 \cdot 10^{-5}. \]

The rest of the proof is relatively straightforward, and the details are given in appendix A.4.

The functions \( g_1^-, g_2^- \) solve a second order equation
\[ g'' + (A(r) - 2/r)g' + (B(r) + A(r)/r)g = 0, \]
where
\[ A(r) = \frac{\hat{g}_2^-(r)\hat{g}_2^-(r) - \hat{g}_1^-(r)\hat{g}_1^-(r)}{\hat{g}_2^-(r)\hat{g}_1^-(r) - \hat{g}_1^-(r)\hat{g}_2^-(r)} \]
and
\[ B(r) = \frac{\hat{g}_1^-(r)\hat{g}_2^-(r) - \hat{g}_2^-(r)\hat{g}_1^-(r)}{\hat{g}_2^-(r)\hat{g}_1^-(r) - \hat{g}_1^-(r)\hat{g}_2^-(r)}. \]

Lemma 5.5. We have the following bounds on \([0, r_7]\)
\[ |A(r)| \leq 2 \cdot 10^{-4}(\chi_{[0, \frac{3}{5}]}(r) + 3\chi_{[\frac{3}{5}, r_5]}(r)); \quad |B(r) - 1 + \lambda + \hat{Q}^2| < 3/2500. \]

Proof. We use the partition \( \pi = (0, \frac{1}{10}, \frac{1}{5}, r_1, \frac{8}{25}, \frac{3}{5}, \frac{1}{2}, \frac{3}{5}, \frac{4}{5}, \frac{9}{15}, \frac{1}{2}, \frac{3}{5}, \frac{4}{5}, \frac{9}{15}, \frac{1}{2}, \frac{3}{5}, \frac{11}{15}, r_6, r_7) \)
and as usual maximize the numerators and minimize the denominators.

Note 5.6. We now look for a nearby actual solution \( \bar{u}_1 = w_1^- - \delta, \delta(0) = 0 \). In the following we write \( \|f\| \) for \( \sup_{[0, r_1]} |f| \).
By (95) and (101) $\delta$ satisfies $-\delta'' - \frac{2}{r} \delta' + (1 - Q^2 - \lambda) \delta + R_1 = 0$, or
\[
-\delta'' + (A(r) - 2/r) \delta' + (B(r) + A(r)/r) \delta = h_2(r),
\]
where $h_2(r) = R_1(r) + A(r) \delta'(r) + \left[ B(r) - 1 + \lambda + A(r)/r + Q(r)^2 \right] \delta(r).

**Lemma 5.7.** We have
\[
|h_2(\delta, r)| \leq 10^{-4} \left\{ \left( \frac{3}{5} + 2\|\delta'\| + \left( \frac{20 + 2}{r} \right) \|\delta\| \right) X_{01} + \left( \frac{3}{5} + 2\|\delta'\| + 22\|\delta\| \right) \chi_{14} \right. \\
\left. \left( \frac{3}{10} + 2\|\delta'\| + 10\|\delta\| \right) X_{[r, \bar{r}]} + \left( \frac{3}{10} + 6\|\delta'\| + 15\|\delta\| \right) \chi_{[\bar{r}, r]} \right\}.
\]

**Proof.** We write $h_2(r) \leq |R_1(r)| + |A(r)|\|\delta'(r)\| + \left[ |B(r)| - 1 + \lambda + \tilde{Q}(r)^2 \right] |\delta(r)|.

The estimates now follow from the corresponding bounds satisfied by $Q^2 - \tilde{Q}^2$, $R_1$, $A$ and $B(r) - 1 + \lambda + \tilde{Q}^2(r)$, which are established in (64), lemma 5.4 and lemma 5.5. On every subinterval we use the monotonicity of the exponential and replace it by the value at the left endpoint. On the intervals $[r_1, r_2]$, $[r_4, 11/5]$, $[11/5, r_7]$ we simply replace $1/r$ by its left endpoint value. The resulting rational functions are of low degree and are maximized explicitly.

As in the $L_+$ case $\delta$ satisfies
\[
\delta = H_0(\delta), \quad \delta' = H'_0(\delta), \tag{107}
\]
where $H_0, H'_0$ are of the form given in (73) and (74) with $g_+^j$ replaced by $g_-^j$.

**Lemma 5.8.**

(i) We have
\[
N_0 := |H_0(\delta)(r)| + |H_0(\delta)'(r)| / 5 \leq 5 \cdot 10^{-4} + \frac{23\|\delta\|_{\infty}}{1000} + 5 \cdot 10^{-3}\|\delta'\|_{\infty}. \tag{108}
\]

Consequently, $(H_0, H'_0)$ is a contraction in the ball
\[
X := \left\{ f \in C^4((0, r_7)) \mid \|f\|_X \leq 6 \cdot 10^{-4} \right\},
\]
where $\|f\|_X := \sup_{r \in (0, r_7)} \left( |f(r)| + \frac{1}{r} |f'(r)| \right)$. Thus there is an actual solution $\hat{u}_1$ within $6 \cdot 10^{-4}$ in $\|\cdot\|_X$ of $w_1$. (iii) $|w_1''(r_7) - \hat{u}_1''(r_7)| = \|\hat{\delta}(r_7)\| < 6 \cdot 10^{-4}$ (cf again footnote 3 on p 139).

**Proof.** Since $1/W(r) := (g_1^r(r)g_2''(r) - g_1''(r)g_2^r(r))^{-1} = r^2 \exp \left( \int_{r_1}^{r} A(s) \, ds \right)$, we have
\[
1/|W(r)| < 51r^2/50.
\]

The proof follows from a piecewise analysis of $N_0$. This is done by straightforward integration of the polynomials involved in $N_0$ and then maximization of the rational functions (whose numerators are of degree at most 5) multiplying $\|\delta\|, \|\delta'\|$ and of the free term in the result of the integration. Taking the derivative of the rational functions and re-expanding the numerators at the left endpoint of each interval, we see that all these re-expanded polynomials have positive coefficients. Thus the maximum is obtained by evaluating $N_0$ at the right endpoint of each
Proposition 5.12. Thus $N$ interval. We provide the intermediate bounds; the largest one, from which the result follows, $(ii)$ This follows in the same way as (108), by estimating $H'_0$ on $[11/5, r_7]$. □

Note 5.9. We have $\tilde{u}_1(0) = w_7^{-}(0) = P_5(\lambda)$, a quintic polynomial. Substituting $\lambda = 1/(1+z)$ and using note 2.5 to estimate this quintic, we obtain

| $|\tilde{u}_1(0)| < 2 \cdot 10^{-6}$.

Thus $Q/Q(0) + \lambda \tilde{u}_1$ is close to but not exactly equal to the function $y_1$ defined immediately after theorem 5.1. We will modify it multiplicatively to make up for the discrepancy.

Lemma 5.10. The function

$$\tilde{u}_1(r) = \hat{u}_1(r) - \tilde{u}_1(0) \frac{Q(r)/Q(0) + \lambda \hat{u}_1(r)}{1 + \lambda \hat{u}_1(0)}$$

is the solution of (95) with $\tilde{u}_1(0) = 0$ (and thus $Q/Q(0) + \lambda \tilde{u}_1 = y_1$).

Proof. Clearly, $\tilde{u}_1(0) = 0$. Using (3), it is straightforward to check that $\tilde{u}_1$ solves (95). □

Lemma 5.11. One has the estimates $|w_7^- (r_7)| < \frac{3}{2}$ and $|w_7^- (r_7)| < \frac{1}{2}$.

Proof. The bound for $w_7^- (r_7)$ follows from lemma 5.3. We estimate the quintic polynomial $w_7^- (r_7)$ in $\lambda$ by taking $\lambda = 1/(1+z)$ using again note 2.5. □

Proposition 5.12. The solution $\tilde{u}_1$ satisfies

$$|w_7^- (r_7) - \tilde{u}_1(r_7)| < 7 \cdot 10^{-4}; \quad |w_7^- (r_7) - \tilde{u}_1'(r_7)| < 7 \cdot 10^{-4}.$$ 

Proof. Note that lemma 3.6 and the definition of $\hat{Q}$ imply $|Q(r_7)/Q(0)| < \frac{3}{100}$ and $|Q(r_7)/Q(0)| < \frac{3}{100}$, and (109) and (110) imply the (crude) bounds $|\tilde{u}_1(r_7) - \hat{u}_1(r_7)| < 3 \cdot 10^{-6}(\frac{1}{100} + |\tilde{u}_1(r_7)|)$ and $|\tilde{u}_1'(r_7) - \hat{u}_1'(r_7)| < 3 \cdot 10^{-6}(\frac{1}{100} + |\tilde{u}_1'(r_7)|)$. The rest is straightforward from lemmas 5.8 and 5.11. □

5.1.1. The region $r > r_7$: the quasi-solution bounded for large $r$. Let $\sigma = \sqrt{1-\lambda}$, and $a_1 = (6839/2521)^2$, $\sigma_1 = 1 + \sigma$, $a_2 = 1 - \sigma$. We consider

$$g_3(r) = (\sigma r)^{1 - e^{-r\sigma}}(\sigma - a_1)Ei(-2r); \quad w_2'(r) = \lambda^{-1}(g_3'(r) - g_3(r)),$$

where

$$g_3(r) = g_3(r)|_{\lambda=0} = e^{-r}/r + a_1 g(r).$$

Since $\lambda = 1 - \sigma^2$ we have

$$R_2(r) := -w_2' - 2w_2'/r + (1 - \lambda - a_1 r^2 e^{-2r})w_2 - g_3 = \frac{a_1^2 e^{-r}}{\sigma_1 r^3} \tilde{R}_2,$$

where

$$\tilde{R}_2(r) = (\sigma \sigma_2)^{-1} e^{-\sigma r} (2\sigma e^{3\sigma r}) Ei(-4r) + (e^{2r} - \sigma e^{\sigma r}) Ei(-2r) - e^{2\sigma r} \sigma_1 Ei(-2\sigma_1 r).$$

(114)
Lemma 5.13. We have $|R_2(r)| < \frac{3}{500} e^{-2r}/r^3$ and $|w_2^-(r)| < \frac{31}{50} e^{-\sigma r}$.

**Proof.** Rewriting $\text{Ei}(-r)$ as $e^{-r} \int_{-\infty}^{0} e^t/(s - r) \, dt$ we obtain

$$
\tilde{R}_2(r) = e^{-r} \int_{-\infty}^{0} e^t \left(1 - e^{\sigma r} \right) s^2 + 2rs \left(2(e^{\sigma r} - 1) + \sigma_2 \right) \sigma_2(s - 4r)(s - 2r)(s - 2\sigma_1 r) \, ds.
$$

We use the following inequalities to bound $\tilde{R}_2(r)$: For $\sigma \in (0, 1), r > 0$, we have $(e^{\sigma r} - 1) \leq \sigma e^r$, furthermore, $s < 0$, and thus $|s - 2\sigma_1 r| \geq |s - 2r|$ and $|r/(s - 4r)| < 1/4$; we are in the range $r \geq r_7$, thus $e^{-r} \leq e^{-\sigma r}$, and with $m > 0$, $|s - m r| \geq |s - mr_7|$. Therefore,

$$
|\tilde{R}_2(r)| \leq \int_{-\infty}^{0} \frac{s^2 e^r}{(s - 10)(s - 5)2} + \frac{2(2 + e^{-r})se^r}{4(s - 5)^2} \, dx = -4e^{10} \text{Ei}(-10) + 2e^5 \text{Ei}(-5)
$$

Using (111) and (114) we obtain $re^r \sigma_1 w_2^- (r) = (e^{\sigma r} - 1)/\sigma_2 - a_1 e^{-r} \tilde{R}_2$. Therefore, since $\forall \sigma_2 > 0$ we have $e^{\sigma r} - 1 \leq \sigma_2 e^{\sigma r}$ we obtain, using the estimate for $\tilde{R}_2$ in (115),

$$
|w_2^-(r)| \leq \frac{e^{-\sigma r} + e^{-2r} a_1 |\tilde{R}_2|}{r \sigma_1} < |w_2^-(r)| \leq \frac{e^{-\sigma r} + e^{-\sigma r - 5/2} a_1 |\tilde{R}_2|}{r \sigma_1} \leq \frac{51 e^{-\sigma r}}{50}
$$

and we are done.

Since $Q$ satisfies (3) and $y_2$ is a solution of (5), the definition of $\tilde{u}_2$ (see (94) and (19)) implies that $\tilde{u}_2$ satisfies the equation

$$
-\tilde{u}_2'' - 2\tilde{u}_2'/r + (1 - \lambda - Q^2)\tilde{u}_2 = Q/A_1.
$$

Writing $\tilde{u}_2 = w_2^- + \delta_0$, we obtain from (113) and (117) after regrouping the terms,

$$
-\delta''_0 - 2\delta'/r = -R_2 + \left(\lambda - 1 + Q^2\right)\delta_0 + \left(Q^2 - a_1 e^{2r}/r^2\right) w_2^- + \frac{Q}{A_1} - g_3^- =: h_3.
$$

Looking for exponentially decreasing solutions, we write (118) in the integral form

$$
\delta_0 = H_1(\delta_0) = T_1(h_3) \quad \text{where} \quad [T_1(f)](r) = -\int_{r}^{\infty} \frac{dt}{t^2} \int_{r}^{\infty} s^2 f(s) \, ds.
$$

Using (22) and note 4.14 we obtain

$$
\left|Q^2(r) - a_1 e^{-2r}/r^2\right| \leq \left|Q(r)^2 - \tilde{Q}(r)^2\right| + \left|a_1 e^{-2r}/r^2 - \tilde{Q}(r)^2\right| < \frac{3}{200} e^{-2r}.
$$

Using lemma 3.10 to estimate $Q$ in terms of $\tilde{Q}$ and to bound $b_{1,2}$, definition 2.3 and (18) to estimate $\tilde{Q}$, (10) to estimate $g$, (20) for $A$ and $B$, (94) for $A_1$ and (112) we obtain, for $r \geq 5/2$,

$$
\left|Q(r)/A_1 - g_3(r)\right| = \left|B g(r) + b_2(r)/A_1\right| - a_1 g(r) \leq \left|b_3(r)/A_1\right| + \left|B A_1 - a_1\right| g(r) \leq \frac{1}{20} e^{-3r}.
$$

In the same way it is shown that for $r \geq r_7$ we have $Q^2 < \frac{1}{12}$ which implies $|1 - \lambda - Q^2| \leq 1$.

Therefore denoting $\|f\| = \sup_{r \geq r_7} r^2 e^{2r} |f(r)|$ we use lemma 5.13, (118), (120) and (121) to obtain

$$
|b_3(r)| < \left(\frac{11}{500} + \|\delta_0\|\right) e^{-2r}.
$$
Lemma 5.14. \( H_1 \) is contractive in the ball \( \{ f : \| f \| \leq 1/125 \} \). Furthermore, \( |\tilde{u}_2(r_7) - w_2(r_7)| < 10^{-5} \) and \( |\tilde{u}_2'(r_7) - w_2'(r_7)| < 2 \cdot 10^{-5} \).

**Proof.** We use the crude estimate \( |T_1(e^{-2r}/r^2)| < r^{-2} \int_0^\infty dr \int_0^\infty e^{-2s} ds = \frac{1}{4} \) and obtain from (122)

\[
|H_1(\delta_0)(r)| \leq \frac{e^{-2r}}{r^2} \left( \frac{11}{2000} + \frac{1}{4} \| \delta_0 \| \right).
\] (123)

We simply bound \( |H_1'(\delta)| = \int_0^\infty (s/r)^2 h_3(s) ds \) using \( |H_1'(\exp(-2s)/s^2)| = \frac{1}{2} r^{-2} e^{-2r} \) and obtain

\[
|H_1'(\delta_0)(r)| \leq \left( \frac{11}{1000} + \frac{1}{2} \| \delta_0 \| \right) e^{-2r}/r^2.
\] (124)

It follows from (123) and (124) that \( |\delta_0(r_7)| = |H_1(\delta_0)(r_7)| < 10^{-5}, |\delta_0'(r_7)| = |H_1'(\delta_0)(r_7)| < 2 \cdot 10^{-5} \).

5.1.2. The Wronskian

The formulae for \( w^-_2 \) and \( w^-_2' = \partial_r w^-_2 \) are

\[
w^-_2(r_7) = \frac{2e^{-r_7}}{5\lambda} (e^{\sigma_2 - 1} - \frac{4a_1}{5\lambda} - e^{\sigma_7} - \frac{2a_1 e^{-r_7}}{5\lambda \sigma} (e^{\sigma_7} - \sigma) Ei(-5) + \frac{2a_1 \sigma}{5\lambda \sigma} e^{\sigma_7} Ei(-5\sigma_1),
\] (125)

\[
w^-_2'(r_7) = -\frac{2}{25\lambda} (2e^{-r_7\sigma} - 7e^{-\sigma_7} + 2a_1 e^{-3\sigma_7} + 5\sigma e^{-r_7\sigma} - 2a_1 e^{-5r_7\sigma}) - \frac{12a_1}{25\lambda} e^{-r_7\sigma}
+ \frac{2a_1 \sigma}{25\lambda \sigma} (-7\sigma e^{-r_7\sigma} + 2e^{-r_7\sigma} + 5\sigma e^{-r_7\sigma}) + \frac{2a_1 \sigma}{25\lambda \sigma} e^{\sigma_7} (5\sigma - 2) Ei(-5\sigma_1).
\] (126)

It is useful to estimate these in terms of polynomials.

**Lemma 5.15.**

(i) With \( z = 2\sigma - 1 \) we have

\[
\begin{vmatrix}
61 & 139 z^2 & 97 z^3 & 124 z^4 & 199 z^5 & 73 z^6 & 71 z^7 \\
560 & 588 z & 316 z^2 & 409 z^3 & 786 z^4 & 383 z^5 & 526 z^6 & 173 z^7 \\
-23 & 52 z & -3 z^2 & -13 z^3 & 19 z^4 & 22 z^5 & 21 z^6 & 13 z^7 \\
303 & 587 z & -103 z^2 & -311 z^3 & 235 z^4 & 257 z^5 & 292 z^6 & 242 z^7 \\
59 & 1563 z^8 & -16z^9 & 3 \cdot 10^{-4}
\end{vmatrix} < 3 \cdot 10^{-4}
\] (127)

(ii) We also have \( |w^-_2(r)| < 1/2 \) and \( |w^-_2'(r)| < 1/5 \).
Proof. We consider
\[
\begin{align*}
w_{\lambda}^-(\sigma) &= \frac{2e^{-\tau}}{5}(e^{\sigma \tau} - 1) - \frac{4a_1}{5}Ei(-10)e^{\tau} - \frac{2a_1}{5}(e^{\sigma \tau} - \sigma)Ei(-5) \\
&\quad + \frac{4a_1}{5}e^{\sigma \tau}Ei(-5\sigma_1)
\end{align*}
\]

and
\[
w_{\lambda}^- (\sigma) = -\frac{2a_1}{5\sigma_1}(e^{\sigma \tau} - \sigma)Ei(-5) + \frac{2a_1}{5\sigma_1}e^{\sigma \tau}Ei(-5\sigma_1).
\]

A direct calculation shows that \(w_{\lambda}^-(r_7; \sigma) = \lambda^{-1}w_{\lambda}^-(\sigma)+\sigma^{-1}w_{\lambda}^-(\sigma) = \sigma^{-1} \int_0^1 w_{\lambda}^-(s)\,ds + \int_0^1 w_{\lambda}^- (s)\,ds\). The rest of the proof is the same as that of lemma 4.17. The calculations for \(w_{\lambda}^-\) are similar. □

Note 5.16. To estimate the Wronskian \(W[y_1, y_2](r; \lambda)\), we express it in terms of \(\tilde{u}_{1,2}\) using (94),
\[
\lambda^{-1}W[y_1, y_2](r; \lambda) = \frac{\tilde{u}_1(r)Q'(r)}{\lambda} + \frac{Q(r)\tilde{u}_2(r) - Q'(r)\tilde{u}_1(r)}{\lambda}.
\]

At the first stage, we look for a nearby quantity solely containing polynomials with rational coefficients; we find such an approximation by replacing in (128) \(\tilde{u}_{1,2}\) by \(w_{1,2}\), \(Q\) by \(\tilde{Q}\), and \(A_1\) by \(A\) (cf. (94)). Let thus
\[
\lambda^{-1}\tilde{W}(\lambda) := \frac{w_{\lambda}^-(r)\tilde{Q}(r) - w_{\lambda}^-(r)\tilde{Q}(r)}{A} + \frac{\tilde{Q}(r)(w_{\lambda}^-(r) - \tilde{u}_{\lambda}^-(r))}{\lambda}.
\]

Corollary 5.17. The following lower bound holds:
\[
\sup_{\lambda \in [0,1]} |\lambda^{-1}\tilde{W}(\lambda)| \geq \frac{1}{55}.
\]

Proof. By (20), we have \(1/A = \frac{296}{803} + A_2\) where \(|A_2| < 10^{-4}\). Using the polynomials in lemma 5.15, we can write \(\tilde{W}\) as a polynomial with explicit rational coefficients plus a remainder whose absolute value is smaller than \(2 \cdot 10^{-4}\). Then we re-expand the resulting polynomial approximation of \(\tilde{W}\) at \(\sigma = \frac{1}{2}\) using note 2.5. □

5.1.3. The Wronskian of the actual solutions

Lemma 5.18. The Wronskian of the actual solutions \(y_1, y_2, W[y_1, y_2](r; \lambda)\), satisfies
\[
|W[y_1, y_2](r; \lambda) - \tilde{W}(\lambda)| < \frac{\lambda}{1000} \forall \lambda \in [0, 1].
\]

Proof. The difference between (128) and (129) (we omit its long and clearly straightforward expression) is bounded simply using the triangle inequality. The terms that need to be estimated in the difference are \(\tilde{u}_{1,2} - w_{1,2}\) and \(\tilde{u}_2 - \tilde{d}_1 w_{1,2}\) for which we use lemma 5.8 (ii), proposition 5.12 and lemma 5.14, \(Q/A_1 - \tilde{Q}/A_1\) and \(Q/Q(0) - \tilde{Q}/\tilde{Q}(0)\) which we estimate using (20), lemma 3.6 and lemma 3.10, \(|w_{1,2}|\) and \(|d_1 w_{1,2}|\) which are bounded in lemma 5.11 and proposition 5.12. □
5.1.4. End of the proof of proposition 5.2. Proposition 5.2 follows from corollary 5.17 and (131). Therefore $W[y_1, y_2](r_\ell; \lambda) \neq 0$ for $\lambda \in (0, 1]$, implying proposition 5.2.

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Appendix A. Further details of proofs

A.1. The general solutions of (4) and (5)

We have shown existence of two solutions $v_0$ and $v_\infty$ of (4) (and separately of (5)) which belong to $L^2$ near $r = 0$ and $r = \infty$, respectively. Here we address the question of uniqueness.

Lemma 6.1. Let $u_0$ and $u_\infty$ be the solutions of (4) described in sections 4.1.3 and 4.1.4. Let $v$ be any solution of (4). Then either the Wronskian of $v$ with respect to $u_0$ ($u_\infty$) is zero or else $v$ is not in $L^2$ near zero (infinity, respectively).

Proof. The functions $U_{0,\infty} := ru_{0,\infty}(r)$ satisfy the linear equation

$$-U'' + (1 - \lambda - 3Q^2)U = 0.$$  \hspace{1cm} (132)

The Wronskian of any two solutions of (132) is constant (since the coefficient of $U'$ is zero). If the Wronskian of $v$ with respect to $u_0$ ($u_\infty$) is not zero, then obviously the Wronskian $W_{0,\infty}$ of $V_1 = rv$ with respect to $U_0$ ($U_\infty$) is a nonzero constant. Now it follows from the expression for $w_{1,2}$, corollary 4.11, (86) and lemma 4.15 that $U_0(r) \to 0$, $U_\infty'(r) \to u_1(0) \neq 0$ as $r \to 0$, and $U_\infty'(r) \to \text{ const}$ for $U_\infty''(r) \to 0$ ($U_\infty''$ decays exponentially) as $r \to \infty$. Thus if $W_0 \neq 0$ then either $V_1(r) \to \text{ const} \neq 0$ or $V_1'(r) \sim \text{ const}/r$ as $r \to 0$, and if $W_\infty \neq 0$ then $V_1(r)$ must increase to $\infty$ either exponentially or like const.$r$ as $r \to \infty$. The conclusion then follows.

\[\square\]

Lemma 6.2. Let $y_1$ and $y_2$ be the solutions of (5) described in the beginning of section 5. Let $v$ be any solution of (5). Then either the Wronskian of $v$ with respect to $u_0$ ($u_\infty$) is zero or else $v$ is not in $L^2$ near zero (infinity, respectively).

Proof. Essentially the same as the proof of lemma 6.1, using (94), lemma 5.8, (119) and lemma 5.14. \[\square\]

A.2. Detailed proof of lemma 3.8

In the following, we use the bounds in lemma 2.4 (2.4), (46), (49), (50) and lemma 3.3.

Note 6.3. We use various bounds to find estimates for the integrals in terms of exponentials and/or polynomials of low degree, avoiding special functions. (i) We replace $\omega(s)$, $s^{-2}$ and $1/(s+1)$ by their left endpoint values and $\frac{1}{2\pi}$ by their values at the right end, $\frac{1}{2} + \frac{3}{20}$ by $\frac{9}{16} - \frac{1}{2}r$ on $[1/2, 1]$. (ii) We use the partition $\pi = (\frac{1}{2}, \frac{5}{6}, 1)$ (chosen so that the coefficient of the 4th power is small enough not to alter the desired estimate) for some quartic polynomials. (iii) A function $f$ of the form quadratic plus an exponential is estimated by explicitly finding the
zeros of the second derivative and inferring the monotonicity properties of \( f \). (iv) We also note that \( H_{1,2} \) are different from \( H'_{1,2} \) only by the prefactor function. The estimates of \( H'_{1,2} \) are thus obtained easily from those for \( H_{1,2} \). (v) In some integrals where \( \omega(s) \) would introduce exponentials in the final result, we simply bound \( \omega(s) \) by the value at the left endpoint. (vi) Polynomials of the form \( x^n q(x) \) with \( q \) quadratic can be clearly maximized explicitly.

A.2.1. \( r \in [0, 1/2) \). We start with \( H_2 \) and \( H'_2 \). We break the integral at \( \frac{1}{2}, 1 \) and \( \frac{5}{2} \) (\( \frac{1}{2} \) is introduced for better bounds; it is not otherwise a special point; in the second integral we bound \( \omega(s) \) by \( 2e^{-s} / 3 \)). Using note 6.3 (on \( \frac{1}{2}, 1 \)) we use (v) to bound the integrands we obtain

\[
|H_2(\delta)(r)| \leq \frac{65}{9} \int_r^{\frac{1}{2}} (1/4 + 3s/20) [\rho_1 (11/10 - s) s + \varepsilon_0 (3s/50 + 1/165)] ds
+ \frac{65}{9} \int_{\frac{1}{2}}^1 (1/4 + 3s/20) \left[ \rho_1 (11/10 - s) s + \frac{2}{3} \varepsilon_0 e^{-s} (3s/50 + 1/165) \right] ds
+ \frac{65}{9} \int_{\frac{1}{2}}^1 \left[ \rho_2 (13/5 - s) s + \varepsilon_0 \left( \frac{3s}{100} + \frac{1}{330} \right) e^{-s} \right] ds
+ \frac{13}{2} \int_{\frac{1}{2}}^\infty e^{-s} (\rho_3 + 24\varepsilon_0) \frac{4e^{-3s/2}}{25} ds.
\]  

(133)

The right-hand side of (133) is a quartic polynomial bounded above and below by cubic polynomials, obtained by noting that, in the present interval, \( r^4 \in [0, \frac{5}{2} r^3) \). We obtain

\[
|H_2(\delta)(r)| < 10^{-5} (23/25 - 6r^2/5) + \varepsilon_0/15
\]  

(134)

by explicit extremization of the cubic polynomials. Using note 6.3 (iv) we obtain

\[
|H'_2(\delta)(r)| \leq \frac{36}{13} \left[ 10^{-5} \left( \frac{23}{25} - \frac{6r^2}{5} \right) + \varepsilon_0/15 \right].
\]  

(135)

To estimate \( H_1 \) we replace \( \omega(s) = e^{-s}(s + 1)^{-1} \) by the upper bound 1 so that the integral evaluates to a polynomial; we obtain

\[
|H_1(\delta)(r)| < \left( \frac{1}{4r} + \frac{3}{20} \right) \frac{5}{9} \int_0^r 13s \left( \rho_1 \left( \frac{11}{10} - s \right) s + \varepsilon_0 \left( \frac{3s}{50} + \frac{1}{165} \right) \right) ds
\leq \frac{11}{10} \cdot 10^{-5} r^2 + \frac{1}{50} \varepsilon_0.
\]  

(136)

After evaluating the integral, the part without \( \varepsilon_0 \) is maximized using note 6.3 (vi). For the derivative we use note 6.3 (iv):

\[
|H'_1(\delta)(r)| < 10^{-5} r \left( 2 - \frac{6r}{5} \right) + \frac{\varepsilon_0}{20}.
\]  

A.2.2. \( r \in [\frac{1}{2}, 1) \). Using note 6.3 we obtain

\[
|H_2(\delta)(r)| \leq \frac{7e}{10} \int_0^{1/2} \left( \frac{1}{4} + \frac{3}{20} s \right) \left( \rho_1 \left( \frac{11}{10} - s \right) s + \varepsilon_0 \left( \frac{3s}{50} + \frac{1}{165} \right) \frac{2e^{-1/2}}{3} \right) ds
+ \frac{10}{9} \int_{1/2}^1 e^{-s} \left( \rho_2 (13/5 - s) s + \varepsilon_0 \left( \frac{3s}{50} + \frac{1}{165} \right) \frac{e^{-s}}{2} \right) ds + \int_{1/2}^\infty e^{-s} (\rho_3 + 24\varepsilon_0) \frac{4e^{-3s/2}}{25} ds
\]

\[
< \frac{7e}{10} \left( 10^{-6} \left( \frac{11}{50} + 2 (1 - r) \right) + \frac{1}{100} \varepsilon_0 \right).
\]  

(137)
and
\[ |H'_2(\delta)(r)| \leq 18 \left( 10^{-6} \left( \frac{11}{50} + 2(1-r) \right) + \frac{1}{100} \epsilon_0 \right). \] (138)

For \( H_1 \) we distribute \( e^r \) inside the integral and then use note 6.3. We obtain
\[ |H_1(\delta)(r)| \leq \left( \frac{1}{4r} + \frac{3}{20} \right) \left( \frac{65}{9} \int_0^r \left( \rho_1 \left( \frac{11}{10} - s \right) s + \epsilon_0 \left( \frac{3s}{50} + \frac{1}{165} \right) e^{-s} \right) s \, ds \right) \]
\[ + \frac{14}{9} \int_{r'}^r \frac{2s}{3} \left( e^r \rho_1 \left( \frac{11}{10} - s \right) s + \frac{2\epsilon_0}{3} \left( \frac{3s}{50} + \frac{1}{165} \right) \right) \, ds \]
\[ \leq \left( \frac{9}{10} - \frac{r}{2} \right) \left( 8 \cdot 10^{-6} + \frac{\epsilon_0}{25} \right). \] (139)

For \( H'_1 \) we obtain
\[ |H'_1(\delta)(r)| \leq \left( \frac{1}{2r^2} \right) \left( 8 \cdot 10^{-6} + \frac{1}{25} \epsilon_0 \right) \leq \left( \frac{7}{2} - 3r \right) \left( 8 \cdot 10^{-6} + \frac{1}{25} \epsilon_0 \right). \] (140)

**A.2.3. \( r \in [r_4, r_5] \)**. Here we use note 6.3 except for the re-expansions; other estimates are explained below. Let \( B_1 = \frac{11}{10} \cdot 10^{-6} + 10^{-2}(6 - \frac{13}{4})\epsilon_0 \). We obtain
\[ \frac{5}{7}(r+1)|H_2(\delta)(r)| \leq f_1(r) := \frac{10e^r}{9} \int_r^{r'} e^{-s} \left( \rho_2(3/5 - s) s + \epsilon_0 \left( \frac{3s}{50} + \frac{1}{165} \right) e^{-s/2} \right) \, ds \]
\[ + e^r \int_{r'}^{\infty} e^{-s} \left( \rho_3 + 24\epsilon_0 \right) \frac{4e^{-3s/2}}{25} \, ds < B_1. \] (141)

The coefficients of \( \delta_0, n = 0, 1 \) of \( f_1 - B_1 \) have the form in note 6.3 (iii) and are estimated as explained there. A nearly identical calculation yields
\[ |H'_2(\delta)(r)| < \frac{B_1}{2r} \leq \frac{B_1}{r+1}. \] (142)

In evaluating \( H_1 \), we break the interval of integration as follows: \([0, \frac{1}{2}], [\frac{1}{2}, 1] \) and \([1, r] \). To further simplify the result, in the integral on \([\frac{1}{2}, 1] \) we first replace \( s/(s+1) \) by its (alternating) Taylor series at \( s = \frac{3}{2} \mod O(s - \frac{1}{2})^4 \):
\[ \frac{s}{s+1} \leq \frac{3}{7} + \frac{16}{49} (s - 3/4) - \frac{64}{343} (s - 3/4)^2 + \frac{256}{2401} (s - 3/4)^3. \]

On the same interval, we also bound \( \omega(s) \leq \frac{1}{2} e^{-s} \). In the integral on \([1, r] \), we first bound \( s^2/(s+1) \) by \( 5s/7 \) (which holds for \( s \leq 5/2 \)) and then apply the bounds in note 6.3; for example, we use that \( \omega(s) \leq \frac{1}{2} e^{-s} \). After evaluation of the integrals the result is of the form described in note 6.3, (iii) and we obtain
\[ r |H_1(\delta)(r)| \leq \frac{3}{2} \cdot 10^{-6} (3-r) + \frac{13}{1000} \epsilon_0. \] (143)

Using note 6.3 (iv) we obtain
\[ |H'_1(\delta)(r)| < \frac{2}{r} \left( \frac{3}{2} \cdot 10^{-6} (3-r) + \frac{13}{1000} \epsilon_0 \right). \] (144)
A.2.4. $r \gg r_7$. The bounds for $H_2$ and $H'_2$ are straightforward: we obtain

$$|H_2(\delta)(r)| \leq \frac{7e^{-r}}{5^2(r + 1)} \int_r^\infty e^{-s} (\rho_3 + 24\epsilon_0) e^{-3s} \, ds \leq \frac{21e^{-3r} (3 + 1600\epsilon_0)}{4000r^3(r + 1)} \quad (145)$$

$$|H'_2(\delta)(r)| \leq \frac{3e^{-3r} (3 + 1600\epsilon_0)}{1600r^3}. \quad (146)$$

In the estimate for $H_1$ we proceed as in appendix A.2.3 (iii) above, except for the interval $[1, r_7]$ where instead of the Taylor series of $s/(s + 1)$ we use the simple inequality $\frac{1}{s+1} \leq \frac{1}{16} - \frac{2}{50}$.

After integration, the estimates are elementary and we obtain

$$|H_1(\delta)(r)| \leq \frac{e^{-r}}{r} \left( 10^{-4} \left( \frac{33}{100} - 36e^{-2r} \right) + \frac{7}{50} - \frac{48}{25} e^{-2r} \right) \epsilon_0, \quad (147)$$

$$|H'_1(\delta)(r)| \leq \frac{(r + 1)e^{-r}}{r^2} \left( 10^{-4} \left( \frac{33}{100} - 36e^{-2r} \right) + \frac{7}{50} - \frac{48}{25} e^{-2r} \right) \epsilon_0. \quad (148)$$

A.3. Proof of corollary 3.9

We first note that, by definition, $H_0 \geq 0$. Throughout this proof, we use the bounds (134)–(137); sometimes we replace them by nearby polynomials with simpler coefficients, or majorize them by close enough lower order polynomials, which are easier to maximize.

A.3.1. $r \in J_1 := [0, \frac{1}{2})$. Here we write

$$H_0(\delta)(r) < B_1(r) := 10^{-5} \left( \frac{3}{2} + 3r - r^2 \right) + 4\frac{1}{2}\epsilon_0 \quad \forall 0 \leq r < \frac{1}{2}. \quad (149)$$

Since $(r + 1)B_1 < 35 \cdot 10^{-6}$ in $J_1$, we have

$$| (r + 1)e^r H_0(\delta)(r) | \leq 35e^{1/2} \cdot 10^{-6} < 7 \cdot 10^{-5}. \quad (150)$$

A.3.2. $r \in J_2 := [\frac{1}{2}, 1)$. Here

$$H_0(\delta)(r) < B_2(r) := 10^{-5} \left( \frac{27}{10} - \frac{11}{10}r \right) \quad \forall \frac{1}{2} \leq r < 1 \quad (151)$$

and thus $(r + 1)e^r H_0 \leq \max_{J_2}(r + 1)e^r B_2(r) < \epsilon_0$. Note that the extrema of $(r + 1)e^r B_2(r)$ can be found explicitly.

A.3.3. $r \in J_3 := [r_4, r_7)$. Here we obtain

$$\frac{10^{-3}e^r}{r} \left( 10^{-3} \left( \frac{63}{10} + 6r - 21r^2 \right) + \frac{91}{5} + \frac{571}{5}r - \frac{176r^2}{5} + \frac{3}{2}(r - 1)^2 \right) \epsilon_0 \leq \epsilon_0. \quad (152)$$

We artificially added the term $\frac{1}{2}(r - 1)^2$ to $f(r)$ so that $f'(r)$ is of the form $e^r r^2 P_3(r)$ with $P_3$ a positive cubic polynomial on $J_3$, as it can be checked by studying its derivative. Thus the maximum in (152), which is $< \epsilon_0$, is attained at $r = r_7$. 

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A.3.4. \( r \in J = [r_7, \infty) \). In this region, \((r + 1)e^{-r} H_0(\delta)(r)\) is bounded by

\[
10^{-5} \left( \frac{99}{25} + \frac{231}{50r} + \frac{33}{50r^2} \right) + 10^{-3} e^{-2r} \left( -\frac{108}{25} - \frac{126}{25r} + \frac{404}{25r^2} + \frac{113}{100r^3} \right) + \varepsilon_0 \left( \frac{21}{125} + \frac{49}{250r} + \frac{7}{250r^2} + e^{-2r} \left( -\frac{23}{10} - \frac{67}{25r} + \frac{431}{50r^2} + \frac{3}{5r^3} \right) \right) < 10^{-5} \left( \frac{257}{50} + \frac{6}{r} + \frac{107}{125r_7r} \right) + 10^{-3} e^{-2r} \left( -\frac{112}{25} - \frac{26}{5r} + \frac{84}{5r_7r} + \frac{59}{50r_7r^2} \right) =: t_1 + t_2 < \varepsilon_0, \tag{153} \]

where the first inequality above follows after replacing \( \varepsilon_0 \) by \( 7 \cdot 10^{-5} \), simple term-by-term comparison, and then using the fact that \( r \geq r_7 \) in this region. For the very last bound we first check that \( t_2 < 0 \) and then show that \((t_1 - \varepsilon_0)/t_2 > -1 \) by finding the maximum \((> -3/10)\) of this ratio by an elementary computation.

A.4. Details of the proof of lemma 5.4

Combining the result for \( R_0 \) with lemma 2.4, lemma 5.3 and proposition 3.5, we obtain

\[
|R_1(r)| \leq 10^{-5} \left\{ \begin{array}{ll}
3 + 73(1/100 + 3r/25)e^{-3r}/(1 + r) & , \quad r \in [0, r_2), \\
2 + 80(1/100 + 3r/25)e^{-13r/5}/(1 + r) + (7/4)e^{-r}/(1 + r) & , \quad r \in [r_2, r_7].
\end{array} \right.
\tag{154} \]

On the first interval we use the estimate \( e^{-3r} \leq 1/(1 + r)^j \), \( j = 1, 3 \) to obtain a rational function whose numerator is a linear function and whose denominator is \((1 + r)^3\). This linear function is estimated in an elementary way, using its derivative. On the second interval, since the derivative of \((1/100 + 3r/25)e^{-13r/5}/(1 + r) + (7/4)e^{-r}/(1 + r)\) is negative, the function \(80(1/100 + 3r/25)e^{-13r/5}/(1 + r) + (7/4)e^{-r}/(1 + r)\) is decreasing. Its upper bound on the interval \([r_2, r_3]\) is 3 and on interval \([r_3, r_7]\) it is 1.

Appendix B. Polynomial quasi-solutions for the soliton

Table 1. \( q_1(r) \) and \( q_2(r) \).

| \( \ell \) | \( a_0^1 \) | \( a_1^1 \) | \( a_2^1 \) | \( a_3^1 \) | \( a_4^1 \) | \( a_5^1 \) | \( a_6^1 \) | \( a_7^1 \) | \( a_8^1 \) | \( a_9^1 \) | \( a_{10}^1 \) | \( a_{11}^1 \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | \( \frac{1}{10} \) | \( \frac{1}{20} \) | \( \frac{1}{30} \) | \( \frac{1}{40} \) | \( \frac{1}{50} \) | \( \frac{1}{60} \) | \( \frac{1}{70} \) | \( \frac{1}{80} \) | \( \frac{1}{90} \) | \( \frac{1}{100} \) | \( \frac{1}{110} \) | \( \frac{1}{120} \) |
| 2 | \( \frac{1}{20} \) | \( \frac{1}{30} \) | \( \frac{1}{40} \) | \( \frac{1}{50} \) | \( \frac{1}{60} \) | \( \frac{1}{70} \) | \( \frac{1}{80} \) | \( \frac{1}{90} \) | \( \frac{1}{100} \) | \( \frac{1}{110} \) | \( \frac{1}{120} \) | \( \frac{1}{130} \) |

\( q_3(r) \)

| \( r \in b_0^3 \) | \( b_1^3 \) | \( b_2^3 \) | \( b_3^3 \) | \( b_4^3 \) | \( b_5^3 \) | \( b_6^3 \) | \( b_7^3 \) | \( b_8^3 \) | \( b_9^3 \) | \( b_{10}^3 \) | \( b_{11}^3 \) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| \( J_1 \) | \( \frac{1}{10} \) | \( \frac{1}{20} \) | \( \frac{1}{30} \) | \( \frac{1}{40} \) | \( \frac{1}{50} \) | \( \frac{1}{60} \) | \( \frac{1}{70} \) | \( \frac{1}{80} \) | \( \frac{1}{90} \) | \( \frac{1}{100} \) | \( \frac{1}{110} \) |
| \( J_2 \) | \( \frac{1}{20} \) | \( \frac{1}{30} \) | \( \frac{1}{40} \) | \( \frac{1}{50} \) | \( \frac{1}{60} \) | \( \frac{1}{70} \) | \( \frac{1}{80} \) | \( \frac{1}{90} \) | \( \frac{1}{100} \) | \( \frac{1}{110} \) | \( \frac{1}{120} \) |
| \( J_3 \) | \( \frac{1}{30} \) | \( \frac{1}{40} \) | \( \frac{1}{50} \) | \( \frac{1}{60} \) | \( \frac{1}{70} \) | \( \frac{1}{80} \) | \( \frac{1}{90} \) | \( \frac{1}{100} \) | \( \frac{1}{110} \) | \( \frac{1}{120} \) | \( \frac{1}{130} \) |

\( q_4(r) \)

| \( r \in b_0^4 \) | \( b_1^4 \) | \( b_2^4 \) | \( b_3^4 \) | \( b_4^4 \) | \( b_5^4 \) | \( b_6^4 \) | \( b_7^4 \) | \( b_8^4 \) | \( b_9^4 \) | \( b_{10}^4 \) | \( b_{11}^4 \) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| \( J_1 \) | \( \frac{1}{10} \) | \( \frac{1}{20} \) | \( \frac{1}{30} \) | \( \frac{1}{40} \) | \( \frac{1}{50} \) | \( \frac{1}{60} \) | \( \frac{1}{70} \) | \( \frac{1}{80} \) | \( \frac{1}{90} \) | \( \frac{1}{100} \) | \( \frac{1}{110} \) |
| \( J_2 \) | \( \frac{1}{20} \) | \( \frac{1}{30} \) | \( \frac{1}{40} \) | \( \frac{1}{50} \) | \( \frac{1}{60} \) | \( \frac{1}{70} \) | \( \frac{1}{80} \) | \( \frac{1}{90} \) | \( \frac{1}{100} \) | \( \frac{1}{110} \) | \( \frac{1}{120} \) |
| \( J_3 \) | \( \frac{1}{30} \) | \( \frac{1}{40} \) | \( \frac{1}{50} \) | \( \frac{1}{60} \) | \( \frac{1}{70} \) | \( \frac{1}{80} \) | \( \frac{1}{90} \) | \( \frac{1}{100} \) | \( \frac{1}{110} \) | \( \frac{1}{120} \) | \( \frac{1}{130} \) |
### Appendix C. Polynomial quasi-solutions for $L_*$ and $L_-$

#### $c_{kl, j}$ for $r \in \{0, r_1\}, z = \frac{3r}{2} - 1$

| $l : k$ | 0 | 1 | 2 | 3 | 4 | 5 |
|-------|---|---|---|---|---|---|
| 0     | 0 | 0 | 0 | 0 | 0 | 0 |
| 1     | 0 | 0 | 0 | 0 | 0 | 0 |
| 2     | 0 | 0 | 0 | 0 | 0 | 0 |
| 3     | 0 | 0 | 0 | 0 | 0 | 0 |
| 4     | 0 | 0 | 0 | 0 | 0 | 0 |
| 5     | 0 | 0 | 0 | 0 | 0 | 0 |

#### $c_{kl, j}$ for $r \in \{r_2, r_3\}, z = r - 1$

| $l : k$ | 0 | 1 | 2 | 3 | 4 | 5 |
|-------|---|---|---|---|---|---|
| 0     | 0 | 0 | 0 | 0 | 0 | 0 |
| 1     | 0 | 0 | 0 | 0 | 0 | 0 |
| 2     | 0 | 0 | 0 | 0 | 0 | 0 |
| 3     | 0 | 0 | 0 | 0 | 0 | 0 |
| 4     | 0 | 0 | 0 | 0 | 0 | 0 |
| 5     | 0 | 0 | 0 | 0 | 0 | 0 |

#### $c_{kl, j}$ for $r \in \{r_3, r_7\}, z = r - 2$

| $l : k$ | 0 | 1 | 2 | 3 | 4 | 5 |
|-------|---|---|---|---|---|---|
| 0     | 0 | 0 | 0 | 0 | 0 | 0 |
| 1     | 0 | 0 | 0 | 0 | 0 | 0 |
| 2     | 0 | 0 | 0 | 0 | 0 | 0 |
| 3     | 0 | 0 | 0 | 0 | 0 | 0 |
| 4     | 0 | 0 | 0 | 0 | 0 | 0 |
| 5     | 0 | 0 | 0 | 0 | 0 | 0 |

#### $d_{kl, j}$ for $r \in \{0, r_1\}, z = 3r$

| $l : k$ | 0 | 1 | 2 | 3 | 4 | 5 |
|-------|---|---|---|---|---|---|
| 0     | 0 | 0 | 0 | 0 | 0 | 0 |
| 1     | 0 | 0 | 0 | 0 | 0 | 0 |
| 2     | 0 | 0 | 0 | 0 | 0 | 0 |
| 3     | 0 | 0 | 0 | 0 | 0 | 0 |
| 4     | 0 | 0 | 0 | 0 | 0 | 0 |
| 5     | 0 | 0 | 0 | 0 | 0 | 0 |

#### $d_{kl, j}$ for $r \in \{r_1, r_4\}, z = r - \frac{1}{2}$

| $l : k$ | 0 | 1 | 2 | 3 | 4 | 5 |
|-------|---|---|---|---|---|---|
| 0     | 0 | 0 | 0 | 0 | 0 | 0 |
| 1     | 0 | 0 | 0 | 0 | 0 | 0 |
| 2     | 0 | 0 | 0 | 0 | 0 | 0 |
| 3     | 0 | 0 | 0 | 0 | 0 | 0 |
| 4     | 0 | 0 | 0 | 0 | 0 | 0 |
| 5     | 0 | 0 | 0 | 0 | 0 | 0 |

#### $d_{kl, j}$ for $r \in \{r_4, r_7\}, z = r - 2$

| $l : k$ | 0 | 1 | 2 | 3 | 4 | 5 |
|-------|---|---|---|---|---|---|
| 0     | 0 | 0 | 0 | 0 | 0 | 0 |
| 1     | 0 | 0 | 0 | 0 | 0 | 0 |
| 2     | 0 | 0 | 0 | 0 | 0 | 0 |
| 3     | 0 | 0 | 0 | 0 | 0 | 0 |
| 4     | 0 | 0 | 0 | 0 | 0 | 0 |
| 5     | 0 | 0 | 0 | 0 | 0 | 0 |
\[ e_{ij}^r, \text{ for } r \in (0, r_1), z = 3r \]
\[ e_{ij}^r, \text{ for } r \in (r_1, r_2), z = r - \frac{1}{2} \]
\[ e_{ij}^r, \text{ for } r \in (r_2, r), z = r - 2 \]

\[ l \times k \]

\[ 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \]
\[ 64 \quad 88 \quad 77 \quad 95 \quad 224 \quad 0 \]
\[ 6 \quad 10 \quad 9 \quad 11 \quad 28 \quad 0 \]

\[ 1 \quad 2 \quad 3 \quad 4 \]

\[ l \times k \]

\[ 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \]
\[ 662 \quad 584 \quad 715 \quad 571 \quad 224 \quad 0 \]
\[ 9 \quad 11 \quad 8 \quad 10 \quad 28 \quad 0 \]

\[ 1 \quad 2 \quad 3 \quad 4 \]

\[ d_{ij}^r, \text{ for } r \in [0, r_1), z = 3r \]
\[ d_{ij}^r, \text{ for } r \in [r_1, r_2), z = r - \frac{1}{2} \]
\[ d_{ij}^r, \text{ for } r \in [r_2, r), z = r - 2 \]

\[ l \times k \]

\[ 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \]
\[ 88 \quad 48 \quad 64 \quad 40 \quad 18 \quad 0 \]
\[ 10 \quad 5 \quad 7 \quad 5 \quad 2 \quad 0 \]

\[ 1 \quad 2 \quad 3 \quad 4 \]

\[ l \times k \]

\[ 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \]
\[ 584 \quad 715 \quad 571 \quad 224 \quad 0 \]
\[ 10 \quad 8 \quad 10 \quad 28 \quad 0 \]

\[ 1 \quad 2 \quad 3 \quad 4 \]
c_{u,l} for r \in [0, r_1], z = 3r, c_{u,l} for r \in [r_1, r_3], z = r - \frac{1}{2}

| l \cdot k | 0 | 1 | 2 | 3 | 4 | 5 |
|------------|---|---|---|---|---|---|
| 0          | 0 | 1 | 0 | 0 | 0 | 0 |
| 1          | 0 | 0 | 0 | 1 | 1 | 1 |
| 2          | 0 | -2 | 0 | 8 | 2 | 2 |
| 3          | -2 | 0 | 8 | 2 | 2 | 2 |
| 4          | -2 | 0 | 8 | 2 | 2 | 2 |
| 5          | -2 | 0 | 8 | 2 | 2 | 2 |
| 6          | -2 | 0 | 8 | 2 | 2 | 2 |
| 7          | -2 | 0 | 8 | 2 | 2 | 2 |
| 8          | -2 | 0 | 8 | 2 | 2 | 2 |
| 9          | -2 | 0 | 8 | 2 | 2 | 2 |
| 10         | -2 | 0 | 8 | 2 | 2 | 2 |
| 11         | -2 | 0 | 8 | 2 | 2 | 2 |
| 12         | -2 | 0 | 8 | 2 | 2 | 2 |

References

[1] Bates P W and Jones C K R T 1989 Invariant Manifolds for Semilinear Partial Differential Equations (Dynamics Reported: Series in Dynamical Systems and their Applications vol 2) (Chichester: Wiley) pp 1–38

[2] Bates P, Lu K and Zeng C 1998 Existence and persistence of invariant manifolds for semiflows in Banach space Mem. Am. Math. Soc. 135

[3] Bates P, Lu K and Zeng C 1999 Persistence of overflowing manifolds for semiflow Commun. Pure Appl. Math. 52 983–1046

[4] Bates P, Lu K and Zeng C 2008 Approximately invariant manifolds and global dynamics of spike states Invent. Math. 174 355–433

[5] Bécout M 2009 A critical centre-stable manifold for the Schrödinger equation in three dimensions Commun. Pure Appl. Math. at press, arXiv:0902.1643

[6] Berestycki H and Cazenave T 1981 Instabilité des états stationnaires dans les équations de Schrödinger et de Klein–Gordon non linéaires C. R. Acad. Sci. Paris Sér. I Math. 293 489–92

[7] Berestycki H and Lions P-L 1983 Nonlinear scalar field equations: I. Existence of a ground state Arch. Rational Mech. Anal. 82 313–45

[8] Berestycki H and Lions P-L 1983 Nonlinear scalar field equations: II. Existence of infinitely many solutions Arch. Rational Mech. Anal. 82 347–75

[9] Buslaev V S and Perelman G S 1992 Scattering for the nonlinear Schrödinger equation: states that are close to a soliton Algebra Anal. 4 63–102 (in Russian)

[10] Buslaev V S and Perelman G S 1993 St. Petersburg Math. J. 4 1111–42 (Engl. transl.)

[11] Buslaev V S and Perelman G S 1995 On the Stability of Solitary Waves for Nonlinear Schrödinger Equations. Nonlinear Evolution Equations (American Mathematical Society Translation Series 2 vol 164) (Providence, RI: American Mathematical Society) pp 75–98

[12] Cazenave T 2003 Semilinear Schrödinger Equations (Courant Lecture Notes in Mathematics vol 10) (New York University, Courant Institute of Mathematical Sciences) (Providence, RI: American Mathematical Society)

[13] Cazenave T and Lions P-L 1982 Orbital stability of standing waves for some nonlinear Schrödinger equations Commun. Math. Phys. 85 549–61

[14] Cazenave T and Lions P-L 1982 Orbital stability of standing waves for some nonlinear Schrödinger equations Arch. Rat. Mech. Anal. 80 1–128

[15] Cuccagna S 2001 Stabilization of solutions to nonlinear Schrödinger equations Commun. Pure Appl. Math. 54 101–45
[12] Demanet L and Schlag W 2006 Numerical verification of a gap condition for a linearized nonlinear Schrödinger equation *Nonlinearity* **19** 829–52

[13] Gesztesy F, Jones C K R T, Latushkin Y and Stanislavova M 2000 A spectral mapping theorem and invariant manifolds for nonlinear Schrödinger equations *Indiana Univ. Math. J.* **49** 221–43

[14] Gidas B, Ni W M and Nirenberg L 1979 Symmetry and related properties via the maximum principle *Commun. Math. Phys.* **68** 209–43

[15] Glassey R T 1977 On the blowing up of solutions to the Cauchy problem for nonlinear Schrödinger equation *J. Math. Phys.* **18** 1794–7

[16] Grillakis M, Shatah J and Strauss W 1990 Stability theory of solitary waves in the presence of symmetry: I *J. Funct. Anal.* **74** 165–97

Grillakis M, Shatah J and Strauss W 1990 Stability theory of solitary waves in the presence of symmetry: II *J. Funct. Anal.* **94** 308–48

[17] Krieger J and Schlag W 2006 Stable manifolds for all monic supercritical focusing nonlinear Schrödinger equations in one dimension *J. Am. Math. Soc.* **19** 815–920

[18] Marzuola J and Simpson G 2011 Spectral analysis for matrix Hamiltonian operators *Nonlinearity* **24** 389–429

[19] McLeod K 1993 Uniqueness of positive radial solutions of $Δ u + f(u) = 0$ in $\mathbb{R}^n$: II *Trans. Am. Math. Soc.* **339** 495–505

[20] Nakamshi K and Schlag W 2011 *Invariant Manifolds and Dispersive Hamiltonian Evolution Equations* (Zürich Lectures in Advanced Mathematics) (Zürich: European Mathematical Society)

[21] Payne L E and Sattinger D H 1975 Saddle points and instability of nonlinear hyperbolic equations *Israel J. Math.* **22** 273–303

[22] Perelman G 2001 On the formation of singularities in solutions of the critical nonlinear Schrödinger equation *Ann. Henri Poincaré* **2** 605–73

[23] Schlag W 2009 Stable manifolds for an orbitally unstable nonlinear Schrödinger equation *Ann. Math.* **169** 139–227

[24] Shatah J 1985 Unstable ground state of nonlinear Klein–Gordon equations *Trans. Am. Math. Soc.* **290** 701–10

[25] Soffer A and Weinstein M 1990 Multichannel nonlinear scattering for nonintegrable equations *Commun. Math. Phys.* **133** 119–46

Soffer A and Weinstein M 1992 Multichannel nonlinear scattering, II. The case of anisotropic potentials and data *J. Diff. Eqns* **98** 376–90

[26] Stanislavova M and Stefanov A 2009 On precise center stable manifold theorems for certain reaction–diffusion and Klein–Gordon equations *Physica D* **238** 2298–307

[27] Strauss W A 1977 Existence of solitary waves in higher dimensions *Commun. Math. Phys.* **55** 149–62

[28] Strauss W A 1989 *Nonlinear Wave Equations* (CBMS Regional Conference Series in Mathematics vol 73) (Published for the Conference Board of the Mathematical Sciences, Washington, DC) (Providence, RI: American Mathematical Society)

[29] Sulem C and Sulem P-L 1999 *The Nonlinear Schrödinger Equation. Self-focusing and Wave Collapse* (Applied Mathematical Sciences vol 139) (New York: Springer)

[30] Tao T 2006 Nonlinear dispersive equations *Local and Global Analysis* (CRMS Regional Conference Series in Mathematics vol 106) (Providence, RI: American Mathematical Society)

[31] Weinstein M I 1985 Modulational stability of ground states of nonlinear Schrödinger equations *SIAM J. Math. Anal.* **16** 472–91

[32] Weinstein M 1986 Lyapunov stability of ground states of nonlinear dispersive evolution equations *Commun. Pure Appl. Math.* **39** 51–67