$\alpha$-cluster states and 4$\alpha$-particle Bose condensate in $^{16}$O

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Abstract. In order to explore the 4$\alpha$-particle condensate state in $^{16}$O, we solve a full four-body equation of motion based on the 4$\alpha$ OCM (Orthogonality Condition Model) in a large 4$\alpha$ model space spanned by Gaussian basis functions. A full spectrum up to the 0$^+_6$ state is reproduced consistently with the lowest six 0$^+$ states of experimental spectrum. The 0$^+_6$ state is obtained at 2 MeV above the 4$\alpha$ breakup threshold and has a dilute density structure, where the rms radius is more than 5 fm. The state has an appreciably large $\alpha$ condensate fraction 61%, and a highly amount of $\alpha + ^{12}$C(0$^+_2$) components, both of which are strong pieces of evidence of the state being the 4$\alpha$ condensate state.

Nuclear clustering plays a very important role in nuclear structures as well as mean-field-type of correlation. $\alpha$ particle, which is tightly bound and the first doubly magic nucleus, is the fundamental subunit as a cluster, and microscopic $\alpha$ cluster models have succeeded in describing many cluster states as well as shell-model-like states in light nuclei.

On the other hand, the importance of $\alpha$ cluster structures has also been discussed for infinite nuclear matter, where the $\alpha$-type condensation is expected in a low density region [1, 2], as an analog to the Bose-Einstein condensation of ultracold bosonic atoms in a magnetic trap. A particular interest has also been paid to the occurrence of $\alpha$ condensation in finite nuclei. It will be the most likely realized in excited states with a low density structure in the vicinity of $n\alpha$ threshold in self conjugate 4$n$ nuclei. As the familiar example of such an excited state having a dilute structure, the Hoyle state (the 0$^+_2$ state at 7.65 MeV in $^{12}$C) is known to have a loosely bound 3$\alpha$ cluster structure found by many calculations with the use of microscopic $\alpha$ cluster models [3, 4, 5, 6], while it is quite difficult to understand from the view point of shell model, for example, even from the recent no-core shell model calculation [7]. Recently the Hoyle state has thus been reinvestigated in terms of the $\alpha$ condensation, and now it is well established by many theoretical works as having the 3$\alpha$-particle condensate character, in which 3$\alpha$ particles occupy an identical 0$^S$ orbit forming a dilute gas-like configuration [8, 9, 10, 11, 12]. It should be mentioned that very interesting works on loosely bound $\alpha$-structures have also been performed [13, 14, 15, 16, 17, 18, 19, 20, 21, 22].
The establishment of the new aspect for the Hoyle state naturally leads us to the speculation on the 4α-particle condensation in $^{16}$O, on which this work focuses. The excited spectrum in $^{16}$O is very well reproduced up to $\sim 13$ MeV excitation energy including the ground state by taking a semi-microscopic approach using $\alpha+^{12}$C OCM (Orthogonality Condition Model) by Y. Suzuki [23]. In particular, he showed that the $0^+_1$ state at 6.05 MeV and $0^+_2$ state at 12.05 MeV have $\alpha+^{12}$C structures [24] where the remaining $\alpha$ particle orbits around the $^{12}$C core with the ground state in an $S$-wave and with the $2^+_1$ state in a $D$-wave, respectively. The consistent results were later obtained by the OCM calculation performed in more extended, i.e. 4$\alpha$ model space [25]. However, this calculation, together with the $\alpha+^{12}$C OCM, could not predict the 4$\alpha$ condensate state presumably because relative motions between clusters are expanded by harmonic oscillator basis, which hardly gives the gas-like structure of $\alpha$ particles with a large spatial extension. On the other hand, the 4$\alpha$-particle condensate state was investigated in Ref. [8] and its existence was predicted around the 4$\alpha$ threshold via an analysis with the use of a new type of microscopic wave function of $\alpha$ condensate character, which we hereafter call THSR wave function. While the THSR wave function can well describe the dilute $\alpha$ cluster states as well as shell-model-like states, the other structures such as $\alpha+^{12}$C clustering are smeared out and only incorporated in an average way. Since there is no calculation which reproduces both the dilute 4$\alpha$ and more compact $\alpha+^{12}$C cluster structures simultaneously, it is crucial to perform a full calculation, which will give a decisive remark on the existence of the 4$\alpha$-particle condensate state from a theoretical point of view.

The purpose of this work is to fully take into account the cluster structures composed of $\alpha$ particle. We therefore solve an equation of motion without any model assumption with respect to the $\alpha-\alpha$ relative motions in a huge model space being spanned by Gaussian basis functions. Dilute 4$\alpha$-particle configurations will be covered by this basis, unlike the case of choosing the harmonic oscillator basis. Such a full four-body equation of motion is given by taking the 4$\alpha$ OCM approach, in which the Hamiltonian is given as follows:

$$\mathcal{H} = \sum_{i}^{4} T_i - T_{cm} + \sum_{i<j}^{4} \left[ V_{2\alpha}^{(N)}(i,j) + V_{2\alpha}^{(C)}(i,j) + V_{2\alpha}^{(P)}(i,j) \right] + \sum_{i<j<k}^{4} V_{3\alpha}(i,j,k) + V_{4\alpha}(1,2,3,4),$$

where $T_i$, $V_{2\alpha}^{(N)}(i,j)$, $V_{2\alpha}^{(C)}(i,j)$, $V_{3\alpha}(i,j,k)$ and $V_{4\alpha}(1,2,3,4)$ stand for the operators of kinetic energy for the $i$-th $\alpha$ particle, two-body, Coulomb, three-body and four-body interaction forces between $\alpha$ particles. The center-of-mass kinetic energy $T_{cm}$ is subtracted from the Hamiltonian. $V_{2\alpha}^{(P)}(i,j)$ is the Pauli exclusion operator, by which the Pauli forbidden states between two $\alpha$ particles, $0S$, $0D$ and $1S$ states are eliminated, so that the ground state with the shell-model-like configuration can be described correctly [26, 27]. The $\alpha-\alpha$ effective interaction $V_{2\alpha}^{(N)}$ is constructed by folding procedure from the two kinds of effective two-nucleon force. One is the Modified Hasegawa-Nagata (MHN) force and the other is the Schmidt-Wildermuth (SW) force, both of which are adopted by Ref. [11] and [28], respectively. We should note that the folded $\alpha-\alpha$ potentials reproduce the $\alpha-\alpha$ scattering phase shift at least in a low energy region, the ground state energy of $^{8}$Be, and the ground state and Hoyle state energies in $^{12}$C. The same force parameter set as used in Refs. [11] or [28] is adopted in each case, except for the part of four-body force, which is newly introduced so as to fit the ground state energy of $^{16}$O.

By utilizing the Gaussian expansion method [29] for the choice of variational basis functions, the total wave function $\Psi$ is expanded in terms of Gaussian basis functions as follows:

$$\Psi = \sum_{\nu,\nu'} A_{\nu}(\nu) \Phi_{\nu}(\nu),$$

$$\Phi_{\nu}(\nu) = \hat{S} \left[ [\phi_{\nu_1}(r_1,\nu_1) \phi_{\nu_2}(r_2,\nu_2)]_{l_1,2} \phi_{\nu_3}(r_3,\nu_3) \right]_{J},$$

where $\phi_{\nu}(r,\nu)$ stands for the one-body Gaussian wave function.
R
we will give a detail in a coming full paper, it should be mentioned that the
for these states are also reproduced well by the present calculations.
are shown in units of fm and fm
$^{12}\text{C}$ threshold

Figure 1. Comparison of energy spectra between experiments and the present calculation. Two kinds of effective two-body nucleon-nucleon interaction force are adopted, Modified Hasegawa-Nagata (MHN) force and Schmidt-Wildermuth (SW). Experimental data are taken from Ref. [30], and Ref. [31] for the $0^+_1$ state.

Table 1. The rms radii $R$ and monopole transition matrix elements to the ground state $M(E0)$ are shown in units of fm and fm$^2$, respectively. The corresponding existing experimental data $R_{\text{exp.}}$ and $M(E0)_{\text{exp.}}$ are also shown.

|      | $R$  | $M(E0)$ | $R_{\text{exp.}}$ | $M(E0)_{\text{exp.}}$ |
|------|------|---------|-------------------|-----------------------|
|      | SW   | MHN     | SW               | MHN                   |
| $0^+_1$ | 2.7  | 2.7     | 2.69              |                       |
| $0^+_2$ | 3.0  | 3.0     | 4.1               | 3.9                   |
| $0^+_3$ | 2.9  | 3.1     | 2.6               | 2.4                   |
| $0^-_4$ | 4.0  | 4.0     | 3.0               | 2.4                   |
| $0^-_5$ | 3.1  | 3.1     | 3.0               | 2.6                   |
| $0^-_6$ | 5.0  | 5.6     | 0.5               | 1.0                   |

where $r_1$, $r_2$ and $r_3$ are the Jacobi coordinates describing internal motions of the $4\alpha$ system. $S$ stands for symmetrization operator acting on all $\alpha$ particles obeying Boson statistics. $\nu$ denotes the set of size parameters $\nu_1, \nu_2$ and $\nu_3$ of the Gaussian function, 

$$\varphi(r, \nu) = N_{\nu} r^J \exp(-\nu r^2) \Gamma_l (\tilde{r}),$$

and $c$ the set of relative orbital angular momentum channels $[[l_1, l_2{l_12}, l_3]J$ depending on either the coordinate type of $K$ or $H$ [29], where $l_1$, $l_2$ and $l_3$ are the orbital angular momenta with respect to the corresponding Jacobi coordinates. The coefficients $A_c(\nu)$ are determined by following the Rayleigh-Ritz variational principle.

The energy spectrum with $J^\pi = 0^+$ is shown in FIG. 1. In both cases of choosing the $\alpha-\alpha$ effective forces, we can reproduce the full spectrum of $0^+$ states, and tentatively make a one-to-one correspondence of those states with the six lowest $0^+$ states of the experimental spectrum. In view of the complexity of the situation, the agreement can be considered to be very satisfactory. We show in TABLE 1 the calculated rms radii and monopole matrix elements with the ground state, together with the corresponding experimental values. The $M(E0)$ values for the $0^+_1$ and $0^+_3$ states are consistent with the corresponding experimental values. Although we will give a detail in a coming full paper, it should be mentioned that the $\alpha+^{12}\text{C}$ structures for these states are also reproduced well by the present calculations. $R$ and $M(E0)$ values for
these states are indeed in a good agreement with those obtained in Refs. [23, 25]. We also mention that the ground state is described as having the shell-model configuration within the present OCM framework, and the calculated rms value agrees with the observed one 2.69 fm.

The realistic descriptions of these states allow us to safely discuss structures of the upper states.

The largest rms value is found for the $0^+_6$ state, which corresponds to a sufficiently low average density $\rho_0/10$. Compared with the relatively small rms radius of the $0^+_4$ and $0^+_5$ states, this larger size implies that the $0^+_6$ state is composed of a weakly interacting gas of $\alpha$ particles.

\[ \int dr' \rho(r, r') \phi^L(r') = \mu^L \phi^L(r), \]

where $\phi^L(r)$ is the single-$\alpha$ orbit with orbital angular momentum $L$. The one-body density matrix elements $\rho(r, r')$ are given as

\[ \rho(r, r') = \sum_{i=1}^{4} (\langle \Psi | \delta(r_i^{(cm)} - r') \rangle \langle \delta(r_i^{(cm)} - r) | \Psi \rangle, \]

where $r_i^{(cm)}$ is the coordinate operator on the $i$-th $\alpha$ particle which is measured from the total center-of-mass position. As a result of the calculation for $L = 0$ case, a large occupation probability $\mu^L = 0/4 = 61\%$ is found for the $0^+_6$ state, whereas the other five $0^+$ states all have smaller values, at most 25\%. The corresponding single-$\alpha S$ orbit is shown in FIG. 2. It has much spatially extended behaviour without any node. These indicate that $\alpha$ particles are condensed into the very much dilute $0S$ single-$\alpha$ orbit with the large condensate fraction. Namely the $0^+_6$

**Figure 2.** (Colors online) Single-$\alpha$ orbits with $L = 0$ belonging to the largest occupation number, for the ground and $0^+_6$ states. The radial part of the orbits are shown. The MHN force is adopted.

**Figure 3.** (Colors online) $r\mathcal{V}(r)$ for the $0^+_6$ state is shown, where $\mathcal{V}(r)$ is the $\alpha$-particle overlap amplitude defined by Eq. (6). $L$ denotes the orbital angular momentum of the remaining $\alpha$ particle coupling to $^{12}\text{C}$. The MHN force is adopted.
state clearly has the 4α condensate character. The α particles are loosely bound and moving in a mean-field-like potential with a large spatial extent. We should note that the orbit is very similar to the single-α S orbit for the Hoyle state, on which 70% of 3α particles concentrate [10, 11]. We also show in FIG. 2 the single-α orbit for the ground state. This shrinks very much and has 2S inner large oscillations, reflecting the shell-model configuration. To compare this orbit with the one for the 0^+_6 state is interesting. While for the ground state the effect of Pauli principle, which is strengthened by its small size, causes the inner oscillatory behaviour of the orbit, for the 0^+_6 state, due to its larger size, the effect is weakened, and therefore the inner oscillation disappears with the amplitude being spread out. It is surprising that this large structural difference is seen below as low as the ~ 16 MeV excitation energy, which corresponds to only ~ 1 MeV excess per one nucleon.

In order to further analyze the obtained wave functions, we calculate an overlap amplitude, which is defined as follows:

$$\mathcal{Y}(r) = \left< \left[ \frac{\delta(r' - r)}{r'^2} \Phi_L(12\text{C}) \right] \right|_0 \Psi. \quad (6)$$

Here, $\Phi_L(12\text{C})$ is the wave function of $^{12}\text{C}$ given by the 3α OCM calculation [11], and $r$ is a relative distance between the center-of-mass of $^{12}\text{C}$ and α particle. By this quantity we can see how much components in a certain α+$^{12}\text{C}$ channel are contained in an obtained wave function. The amplitudes for the 0^+_6 state are shown in FIG. 3. The state only has a large amplitude in the α+$^{12}\text{C}$(Hoyle state) channel, whereas the amplitudes in other channels are much suppressed in a whole region. There is no inner oscillation and a long tail stretched up to ~ 20 fm. This behaviour is very similar to that of single-α orbit for the 0^+_6 state previously discussed. These results mean that the α particle loosely couples to the $^{12}\text{C}$ core in the Hoyle state. As is mentioned, in the Hoyle state the 3α clusters weakly interact like a gas. Therefore it is again indicated that the 0^+_6 state has a dilute gas-like 4α-particle structure, and even the 4α condensate structure for this state is also supported since in general $n\alpha$ condensate state should have a large amount of $(n-1)\alpha$ condensate components.

Finally we mention the experimental situation for the 0^+_4, 0^+_5 and 0^+_6 states. The 0^+_4 state at 13.6 MeV was newly discovered recently via α inelastic scattering, and it has been considered the strong candidate for the 4α condensate state [31]. On the other hand, the 0^+_5 state at 14.01 MeV is strongly excited via monopole transition from the ground state [33]. Since the strong monopole transition implies a developed cluster structure [34], the state has also been regarded as one of the candidates [35]. However, our present calculations show that the 4α condensate state should not be assigned to the 0^+_4 or 0^+_5 states [36] but possibly to the 0^+_6 state, since as is previously discussed, the 0^+_6 state has a typical signature of the α-particle condensation, while the 0^+_5 and 0^+_4 states not. Here we just mention that the 0^+_7 and 0^+_8 states mainly have the α+$^{12}\text{C}(0^+_4)$ structure with higher nodal behaviour and the α+$^{12}\text{C}(1^-)$ structure, respectively. Further details for these states will be given in the forthcoming full paper.

In conclusion, we solved a full four-body equation of motion based on the 4α OCM, in which the 4α model space is extended largely enough to cover the 4α condensate structure of dilute density as well as the other structures such as the α+$^{12}\text{C}$ cluster and the shell-model-like compact structures. An experimental full spectrum up to the 0^+_6 state is reproduced in a very satisfactory way. This ensures the argument about the state near the 4α threshold without ambiguity. We showed that the 0^+_6 state newly found around the 4α threshold has the strong 4α condensate character. The state has a large condensate fraction 61%, which concentrates on a spatially extended 0S single-α orbit. It was also shown that the wave function has a large amplitude in the α+$^{12}\text{C}$(Hoyle state) channel only. These results give a strong support in concluding this newly found state can be regarded as the 4α-particle condensate state. Further experimental information is strongly desired.
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[1] G. Röpke, A. Schnell, P. Schuck and P. Nozières, Phys. Rev. Lett. 80, 3177 (1998).
[2] M. Beyer, S. A. Sofianos, C. Kuhrtts, G. Röpke, and P. Schuck, Phys. Lett. B 488, 247 (2000).
[3] H. Horiuchi, Prog. Theor. Phys. 51, 1266 (1974); 53, 447 (1975).
[4] Y. Fukushima and M. Kamimura, Proc. Int. Conf. on Nuclear Structure, Tokyo, 1977, ed. T. Marumori (Suppl. of J. Phys. Soc. Japan, 44, 225 (1978)); M. Kamimura, Nucl. Phys. A 351, 456 (1981).
[5] E. Uegaki, S. Okabe, Y. Abe, and H. Tanaka, Prog. Theor. Phys. 57, 1262 (1977); E. Uegaki, Y. Abe, S. Okabe, and H. Tanaka, Prog. Theor. Phys. 59, 1031 (1978); 62, 1621 (1979).
[6] P. Descouvemont and D. Baye, Phys. Rev. C 36, 54 (1987).
[7] B. R. Barrett, B. Mihaila, S. C. Pieper, and R. B. Wiringa, Nucl. Phys. News, 13, 17 (2003).
[8] A. Tohsaki, H. Horiuchi, P. Schuck and G. Röpke, Phys. Rev. Lett. 87 (2001), 192501.
[9] Y. Funaki, A. Tohsaki, H. Horiuchi, P. Schuck and G. Röpke, Phys. Rev. C 67, 051306(R) (2003).
[10] H. Matsumura and Y. Suzuki, Nucl. Phys. A 739, 238 (2004).
[11] T. Yamada and P. Schuck, Euro. Phys. J. A 26, 185 (2005).
[12] Y. Funaki, A. Tohsaki, H. Horiuchi, P. Schuck and G. Röpke, Eur. Phys. J. A 24, 321 (2005); 28, 259 (2006).
[13] T. Kokalova et al., Eur. Phys. J A 23, 19 (2005); T. Kokalova, N. Itagaki, W. von Oertzen, and C. Wheldon, Phys. Rev. Lett. 96, 192502 (2006).
[14] T. Kawabata et al., Phys. Lett. B 646, 6 (2007).
[15] M. Freer et al., Phys. Rev. C 71, 047305 (2005); 76, 034320 (2007).
[16] A. A. Ogloblin et al., Proceedings of the International Nuclear Physics Conference, Peterhof, Russia, June 28–July 2, 2005.
[17] M. W. Brenner et al, Proceedings of the International Conference “Clustering Phenomena in Nuclear Physics”, St. Petersburg, published in 'Physics of Atomic Nuclei (Yadernaya Fizika), 2000.
[18] T. Yamada, P. Schuck, Phys. Rev. C 69, 024309 (2004).
[19] S. Ohkubo and Y. Hirabayashi, Phys. Rev. C 70, 041602(R) (2004).
[20] M. Takashina and Y. Sakuragi, Phys. Rev. C 74, 054606 (2006).
[21] Y. Kanada-En'yo, Phys. Rev. C 75, 024302 (2007).
[22] M. Chernykh, H. Feldmeier, T. Neff, P. von Neumann-Cosel, and A. Richter, Phys. Rev. Lett. 98, 032501 (2007).
[23] Y. Suzuki, Prog. Theor. Phys. 55, 1751 (1976); 56, 111 (1976).
[24] H. Horiiuchi and K. Ikeda, Prog. Theor. Phys. 40 (1968), 277.
[25] K. Fukatsu and K. Kató, Prog. Theor. Phys. 87, 151 (1992).
[26] S. Saito, Prog. Theor. Phys. 40 (1968); 41, 705 (1969); Prog. Theor. Phys. Suppl. 62, 11 (1977).
[27] V. I. Kukulin, V. M. Krasnopol’sky, V. T. Voronchev, P. B. Sazanov, Nucl. Phys. A 417, 128 (1984).
[28] C. Kurakowa, K. Kato, Phys. Rev. C 71, 021301 (2005); Nucl. Phys. A 792, 87 (2007).
[29] M. Kamimura, Phys. Rev. A 38, 621 (1988); E. Hiyama, Y. Kino, and M. Kamimura, Prog. Part. Nucl. Phys. 51, 223 (2003).
[30] F. Ajzenberg-Selove, Nucl. Phys. A 46, 1 (1968).
[31] T. Wakasa et al., Phys Lett B 653, 173 (2007).
[32] Y. Suzuki and M. Takahashi, Phys. Rev. C 65, 064318, (2002).
[33] M. Stroetzel and A. Goldmann, Phys. Lett. 29B, 306 (1969).
[34] T. Yamada, H. Horiiuchi, K. Ikeda, Y. Funaki and A. Tohsaki, arXiv:nucl-th/0703045.
[35] Y. Funaki, H. Horiuchi, G. Röpke, P. Schuck, A. Tohsaki, T. Yamada, Nucl. Phys. News, 17 (04), 11 (2007).
[36] The reason why we previously tried to identify one of the two 0+ states at 13.6 MeV or at 14.01 MeV with the analog of the Hoyle state, was that the calculation with our condensate model wave function [8] gave a third 0+ state at 13.6 MeV as the highest 0+ state of our calculation. A new analysis, however, shows that in that publication a fourth 0+ state at 17.1 MeV with a very large radius of 6.0 fm was missed. We now identify this state with the four α-particle condensate state, corresponding to the one at 16.5 MeV obtained with the OCM calculation. We also will publish details of this calculation in [37].

[37] In preparation.