A note on Schanuel’s conjectures for exponential logarithmic power series fields

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Abstract. In Ax (Ann. Math. 93(2):252–268, 1971), J. Ax proved a transcedency theorem for certain differential fields of characteristic zero: the differential counterpart of the still open Schanuel conjecture about the exponential function over \( C \) (Lang, Introduction to transcendental numbers, 1966). In this article, we derive from Ax’s theorem transcendency results in the context of differential valued exponential fields. In particular, we obtain results for exponential Hardy fields, Logarithmic-Exponential power series fields, and Exponential-Logarithmic power series fields.

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In [1], J. Ax proves the following conjecture (SD) due to S. Schanuel, which is the differential counterpart of the still open Schanuel conjecture about the exponential function over \( C \) [11, page 30]. Let \( F \) be a field of characteristic 0 and \( D \) a derivation of \( F \), i.e. \( D(x + y) = D(x) + D(y) \) and \( D(xy) = xD(y) + yD(x) \). We assume that the field of constants \( C \) contains \( \mathbb{Q} \). Below \( \text{td} \) denotes the transcendence degree.

\[ \text{(SD)} \text{ Let } y_1, \ldots, y_n, z_1, ..., z_n \in F^\times \text{ be such that } Dy_k = \frac{Dz_k}{z_k} \text{ for } k = 1, \ldots, n. \]

If \( \{Dy_k ; k = 1, \ldots, n\} \) is \( \mathbb{Q} \)-linearly independent, then

\[ \text{td}_C(y_1, \ldots, y_n, z_1, ..., z_n) \geq n + 1. \]

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Consider the field of Laurent series $\mathbb{C}((t))$, endowed with term by term derivation. The field of constants is $\mathbb{C}$. Let $\mathbb{C}[[t]]$ denote the ring of formal power series with complex coefficients in the variable $t$, i.e. $\mathbb{C}[[t]]$ is the ring of Laurent series with nonnegative exponents. It is a valuation ring (Definition 5). For any series $y \in \mathbb{C}((t))$, let $y(0)$ denote its constant term. The exponential map $\exp$ on $\mathbb{C}[[t]]$ is given by the Taylor series expansion $\sum_{k \geq 0} \frac{y^k}{k!}$ and satisfies $Dy = \frac{D\exp(y)}{\exp(y)}$.

A corollary to (SD) which appears in Ax’s paper is:

**Corollary B.** Let $y_i \in \mathbb{C}[[t]]$ be such that $y_i - y_i(0)$ are $\mathbb{Q}$-linearly independent, $i = 1, \cdots, n$. Then $\text{td}_C\mathbb{C}(y_1, \ldots, y_n, \exp(y_1), \cdots, \exp(y_n)) \geq n + 1$.

We rephrase (SD) as follows:

**Theorem A.** Let $y_1, \ldots, y_n, z_1, \ldots, z_n \in F^\times$ be such that $Dy_k = \frac{Dz_k}{z_k}$ for $k = 1, \ldots, n$. If $\text{td}_C\mathbb{C}(y_1, \ldots, y_n, z_1, \ldots, z_n) \leq n$, then $\sum_{i=1}^{n} m_iy_i \in \mathbb{C}$ for some $m_1, \ldots, m_n \in \mathbb{Q}$ not all zero.

**Definition 1.** A differential valued exponential field $K$ is a field of characteristic $0$, equipped with a derivation $D : K \to K$, a valuation $v : K^\times \to G$ with value group $G$, and an exponential map $\exp : K \to K^\times$ which satisfy the following:

- $\forall x \forall y (\exp(x + y) = \exp(x) \exp(y))$,
- $\forall x (Dx = \frac{D\exp(x)}{\exp(x)})$,
- The field of constants $\mathbb{C}$ is isomorphic to the residue field of $v$.

We denote the valuation ring by $\mathcal{O}_v$, the maximal ideal by $\mathcal{M}_v$, and the residue field by $\overline{K} = \mathcal{O}_v/\mathcal{M}_v$. We thus require that the field of constants $\mathbb{C} \subseteq K$ is (isomorphic to) $\overline{K}$, i.e. every element $y$ of $\mathcal{O}_v$ has a unique representation $y = c + \epsilon$, where $c \in \mathbb{C}$ and $\epsilon \in \mathcal{M}_v$ (so $\mathcal{O}_v = \mathbb{C} \oplus \mathcal{M}_v$). For $y \in \mathcal{O}_v$ we write $\overline{y}$ for the residue of $y$ which is the above $c \in \mathbb{C}$. In particular $\overline{c} = c$ for $c \in \mathbb{C}$.

In this note, we generalize Corollary B to arbitrary elements of a differential valued exponential field $K$, that is, to elements $y_1, \ldots, y_n$ not necessarily in the valuation ring $\mathcal{O}_v$ (see Corollary 3). In particular we apply Corollary 3 to exponential-logarithmic series and other examples (examples 4 to 8). Since $K$ is not in general a field of series, we need to find an abstract substitute for the “constant term” $y(0)$ of a series. Since the additive group of $K$ is a $\mathbb{Q}$-vector space and $\mathcal{O}_v$ is a subspace, we choose and fix a vector space complement $A$ such that $K = A \oplus \mathbb{C} \oplus \mathcal{M}_v$. For $y \in K^\times$ we define $\text{co}_A(y) := (y - a)$ for the uniquely determined $a \in A$ satisfying that $(y - a) \in \mathcal{O}_v$. Note that $\text{co}_A(y) = \overline{y}$ if $y \in \mathcal{O}_v$.

In this setting we observe:

**Lemma 2.** Let $y_1, \ldots, y_n \in K$ be such that $\sum_{i=1}^{n} m_iy_i \in \mathbb{C}$ for some $m_i \in \mathbb{Q}$, then $\sum_{i=1}^{n} m_i(y_i - \text{co}_A(y_i)) = 0$. 

