Conductivity Tensor in a Holographic Quantum Hall Ferromagnet

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Abstract

The Hall and longitudinal conductivities of a recently studied holographic model of a quantum Hall ferromagnet are computed using the Karch-O’Bannon technique. In addition, the low temperature entropy of the model is determined. The holographic model has a phase transition as the Landau level filling fraction is increased from zero to one. We argue that this phase transition allows the longitudinal conductivity to have features qualitatively similar to those of two dimensional electron gases in the integer quantum Hall regime. The argument also applies to the low temperature limit of the entropy. The Hall conductivity is found to have an interesting structure. Even though it does not exhibit Hall plateaux, it has a flattened dependence on the filling fraction with a jump, analogous to the interpolation between Hall plateaux, at the phase transition.

Keywords: Holography, AdS/CFT correspondence, Hall effect, D branes

Quantum Hall ferromagnetism is an interesting example of dynamical symmetry breaking. It was predicted and observed in two dimensional electron gases formed by semiconductor heterojunctions \cite{1, 5} and it has more recently been observed in graphene in the integer quantum Hall regime \cite{6, 10}. When many-body interactions are weak, this ferromagnetism has a simple mechanism \cite{3, 5, 11}. For example, an electron with two spin states and negligible Zeeman interaction has two-fold degenerate Landau levels. When a 2-fold degenerate level is precisely half-filled, that is, at filling fraction $\nu = 1$, the electrons can minimize their Coulomb exchange energy by occupying those states which have only one of the two spin labels. The result is spontaneous breaking of spin symmetry and splitting of the degeneracy of the Landau level by the formation of a charge-gapped incompressible integer quantum Hall state at $\nu = 1$. A similar mechanism is thought to work for any integer filling fraction in a Landau level with higher degeneracy. Graphene has an emergent SU(4) symmetry which would result in four-fold degenerate Landau levels. This degeneracy is seen to be completely resolved in sufficiently strong magnetic fields. Evidence that the mechanism is dynamical symmetry breaking is seen in the magnitude of the energy gaps, which are too large to be accounted for by residual non-symmetric interactions and which are characteristic of the scale of the Coulomb interaction, which is very strong in graphene\cite{21}. This raises the question as to whether quantum Hall ferromagnetism can be understood at strong coupling.\textsuperscript{1}

Recently, a holographic model \cite{21, 22} where quantum Hall ferromagnetism persists in the strong coupling limit has been developed \cite{21, 22}. The model is a D3-probe-D5 brane system which is dual to a super-conformal defect field theory with $N_5$ complex fundamental representation hypermultiplets (where $N_5$ is the number of D5 branes) occupying a $2+1$ dimensional subspace of $3+1$ dimensional space-time. The system is Lorentz invariant, which can be regarded as analogous to the emergent Lorentz symmetry of graphene \cite{23}. The $3+1$-dimensional bulk contains $N = 4$ supersymmetric Yang-Mills theory with gauge group SU($N$). This theory is readily studied in the large $N$ planar limit and the probe limit where $N_5 \ll N$. The conformal field theory has a tuneable dimensionless coupling constant, the ’t Hooft coupling $\lambda = g^2_{YM} N$ of the $N = 4$ Yang-Mills theory.

One can introduce a non-zero temperature and a U(1) charge density and constant external magnetic field for the hypermultiplets. These deformations break supersymmetry. Moreover, in the limit of weak coupling, $\lambda \ll 1$, as discussed in \cite{22}, the low energy states of this system are fractional fillings of a $2N_5$-fold degenerate, charge neutral, fermionic Landau level.\textsuperscript{2} The weak coupling argument for quantum Hall ferromagnetism can be applied and one would expect that the $2N_5$-fold degeneracy is lifted and that incompressible, charge-gapped states appear at filling fractions $\nu = 0, \pm 1, \ldots, \pm N_5$. Analysis of the strong cou-

\textsuperscript{1}Some work in this direction considers systematic re-summations of perturbation theory which have been studied in the closely related framework known as magnetic catalysis of chiral symmetry breaking \cite{12, 13}. It suggests that magnetic catalysis, which is indistinguishable from quantum Hall ferromagnetism in this particular system, can still occur when many-body interactions are appreciable. Spontaneous symmetry breaking in a magnetic field in the charge neutral case is already well known for the holographic D3-D5 brane system \cite{19, 20}.

\textsuperscript{2}This counting of the degeneracy assumes that candidate ground states must be colour singlets, otherwise, there would be a further factor of $N$, the number of colour states of the fundamental representation fermion, in the degeneracy. With charge density $\rho$ and magnetic field $B$ (we always assume $B > 0$), we define the filling fraction as $\nu = \frac{\pi \rho B}{\pi \rho B}$ as if the charge comes in quanta of $N$ and one Landau level is completely filled when $\rho = N \frac{\pi \rho B}{\pi \rho B}$ and $\nu = 1$.\textsuperscript{1}
pling limit using the string theory dual, the D3-probe-D5 brane system, shows that, at least some of these states with smaller values of \(v\) are also there at strong coupling. In the strong coupling states, the D5 branes blow up to form a D7 brane. The D7 brane is capable of having incompressible integer Hall states at non-zero values of the U(1) charge density. For large values of \(N_5\), the phase diagram of this model was discussed in reference [22].

In this paper, we shall compute the conductivity and the low temperature limit of the entropy of the strongly coupled states that are found in the D3-D5 model at finite temperature \(T\), density \(\rho\) and in a magnetic field \(B\). We concentrate on an interval of filling fractions between the integer quantum Hall states, \(0 \leq v \leq 1\). Our main aim is to explore the consequences of the phase transition from the D5 to the D7 brane, which was found in references [21]-[22], for the electronic transport properties of the system. At the phase transition, which for the values of \(v\) that we consider here, occurs at a critical value of filling fraction \(v_c \sim 0.3 - 0.5\), the stack of \(N_5\) D5 branes, which are stable when \(v < v_c\), blows up to a single D7 brane which is the prefered state when \(v > v_c\). The two phases are distinguished by their symmetry breaking patterns, \(U(N_3) \times SO(3) \times SO(3) \to U(N_3) \times SO(2) \times SO(3)\) for the D5 branes and \(U(N_3) \times SO(3) \times SO(3) \to U(1) \times SO(3) \times SO(3)\) for the D7 brane. The D5 brane longitudinal conductivity, which we can find analytically in the limit where the parameter \(f = \frac{2\pi N_5}{\sqrt{\lambda}}\) is large, that is where \(N_5 > \frac{\sqrt{\lambda}}{2\pi}\), is

\[
\sigma_{xx}^{\text{D5}} = \frac{\frac{\sqrt{\lambda}}{2\pi} T^2}{1 + \left(\frac{\sqrt{\lambda}}{2\pi} T^2\right)^2} \cdot \frac{N\nu}{2\pi},
\]

This expression is a rather featureless linear function of the density, in particular, exhibiting no trace of the higher Landau level or the insulating behaviour which should occur at integer quantum Hall states. This is remedied by the phase transition. If we realize that, for large values of \(f\), the D5 branes are replaced by a D7 brane, the D7 brane conductivity should take over there. In the large \(f\) limit,

\[
\sigma_{xx}^{\text{D7}} = \frac{\frac{\sqrt{\lambda}}{2\pi} T^2}{1 + \left(\frac{\sqrt{\lambda}}{2\pi} T^2\right)^2} \cdot \frac{N(1 - \nu)}{2\pi},
\]

a decreasing function of \(v\) which reverts to an insulating state precisely when \(v = 1\). The result for \(\sigma_{xx}\) is depicted in the centre column of figures [1] and [2] (for two different temperatures). What we find for large values of \(f\) (bottom row) is qualitatively like the longitudinal conductivity that would be expected to appear between integer Hall plateaux. The first and second entries of the second columns in figures [1] and [2] show the behaviour for smaller values of \(f\), where the conductivity is discontinuous at the phase transition.

The low temperature entropy exhibits similar behaviour. If we first go to weak coupling and compute the zero temperature entropy of the many-electron state coming from the degeneracy, \(\{NBV/2\pi\}, \{NBV\nu/2\pi\}\) of a partially filled Landau level,

\[
s_{\nu \to 0} \approx B \frac{N\nu}{2\pi} \ln \frac{1}{\nu} + B \frac{N(1 - \nu)}{2\pi} \ln \frac{1}{1 - \nu}.
\]

Here, we have assumed that interactions have created the Hall ferromagnetic state, but are not strong enough to appreciably resolve the degeneracy of the partial fillings of the Landau level. To compare, we shall compute the low temperature entropy of the D5 brane (up to order \(T^3\)). The result is identical to the one reported in [24]

\[
s_{\nu}^{\text{D5}} = \sqrt{\lambda} B \frac{N\nu}{2\pi} = \sqrt{\lambda} B \frac{N(1 - \nu)}{2\pi}.
\]

Aside from the factor of \(\frac{\sqrt{\lambda}}{2\pi}\), which normally occurs in front of the entropy of a probe brane (see reference [24] for a discussion), this entropy increases linearly with the filling fraction. Now, again, we realize that at a critical \(v\), the D7 brane takes over. Our computation of the D7 brane entropy in the large \(f\) regime gives

\[
s_{\nu}^{\text{D7}} = \sqrt{\lambda} B \frac{N(1 - \nu)}{2\pi}.
\]

Interestingly, since \(v_c = 1/2\) in the large \(f\) limit, this restores the \(v \to 1 - v\) symmetry of the weak coupling limit. The plot of low temperature entropy versus \(v\) for a few values of \(f\) are displayed in figure [3]. As in the case of the longitudinal conductivity, for finite \(f\), they exhibit discontinuities at the phase transition.

The Hall conductivity does not exhibit integer Hall plateaux. Of course, in the translationally invariant system which we are considering here, the physics of impurity driven localization which is normally responsible for Hall plateaux is absent. Moreover, in a Lorentz covariant system, there is an argument that the zero temperature Hall conductivity is identical to its classical value, \(\sigma_{xy} = \frac{q}{2\pi}\). Our computation of the Hall conductivity at finite temperature nevertheless reveals an interesting dependence on \(v\). For example, in the large \(f\) limit, the Hall conductivities become

\[
\sigma_{xy}^{\text{D5}} = \frac{N\nu}{2\pi} + \frac{N\nu}{2\pi} \cdot \frac{\left(\frac{\sqrt{\lambda}}{2\pi} T^2\right)^2}{1 + \left(\frac{\sqrt{\lambda}}{2\pi} T^2\right)^2},
\]

\[
\sigma_{xy}^{\text{D7}} = \frac{N\nu}{2\pi} + \frac{N(1 - \nu)}{2\pi} \cdot \frac{\left(\frac{\sqrt{\lambda}}{2\pi} T^2\right)^2}{1 + \left(\frac{\sqrt{\lambda}}{2\pi} T^2\right)^2}.
\]

At the zero temperature limit, the second terms in (5) and (6) vanish and the Hall conductivity is identical to the classical Hall value, \(\lim_{T \to 0} \sigma_{xy} = \frac{q}{2\pi}\), as expected. At finite temperature, the thermal correction decreases the conductivity for the

\[\text{3}\]The charge density of a partially filled Landau level is \(\rho = \frac{N\nu}{2\pi}\). We can create a constant current \(i = \rho v\) by going to a reference frame with velocity \(v\). The accompanying boost of the magnetic field creates a transverse electric field \(E_i = -\epsilon_i \mu E_f\) and we have \(\mu = \frac{\sqrt{\lambda}}{2\pi} \epsilon_i E_f\) giving \(\sigma_{xy} = \frac{\sqrt{\lambda}}{2\pi}\).
Figure 1: The first, second and third columns are the deviation of the Hall conductivities $\sigma_{xy}$ from the classical Hall conductivity $\frac{eB}{2m}$, the longitudinal conductivity $\sigma_{xx}$ and the longitudinal resistivity $\rho_{xx}$, respectively, for three different values of $f$ and for $v \in [0, 1]$. The temperature is such that $T_b = 0.2$ where $T_b$ is defined in equation (13). The units of the $y$-axes are respectively $\frac{2\pi}{N_\nu}$ and for $v \in [0, 1]$.

Figure 2: The first, second and third columns are the deviation of the Hall conductivities $\sigma_{xy}$ from the classical Hall conductivity $\frac{eB}{2m}$, the longitudinal conductivity $\sigma_{xx}$ and the longitudinal resistivity $\rho_{xx}$, respectively, for three different values of $f$ and for $v \in [0, 1]$. The temperature is such that $T_b = 0.4$. The units of the $y$-axes are respectively $\frac{2\pi}{N_\nu}$, $\frac{2\pi}{N_\nu}$ and $\frac{2\pi}{N_\nu}$.
D5 and increases it for the D7 brane providing a jump at the phase transition and a flattening of the slope of the $\sigma_{xy}$ versus $\nu$ curve. If we could take the extreme high temperature limit, when $T^2 \gg 2B/\sqrt{\Lambda}$, in fact, $\sigma_{xy}^{D5} \to 0$ and $\sigma_{xy}^{D7} \to 1$ and we would have perfect Hall plateaux with the Hall step occurring at the phase transition. It is tantalizing to speculate that there is a strong coupling mechanism at play which, combined with temperature, gives a tendency toward plateau formation. However, in this system, we cannot take the large temperature limit. We are limited to very low temperatures, $\pi \sqrt{T^2/2B} < 0.16$, otherwise, the chiral symmetry is restored and there is no integer Hall state at all. As one can see in figure 3, the plateauing effect at this low temperature is miniscule. Whether there exists an elaboration of our model where the quantum Hall antiferromagnetic phase persists to higher temperature and the effect is more visible is an open question which we shall not pursue in this paper. The deviations of the Hall conductivities from the classical expression for three values of $f$ are displayed in the first column of figures 1 and 2.

We will now outline our computation of the conductivity. We follow a technique which was invented by Karch and O’Bannon 25, 26. Since there are several examples of how this technique is used in existing literature, we will be brief. We shall work with the D3-D5 and D3-D7 probe brane systems studied in references [21], [22] and follow the notation of those references. One difference will be the use of Penner Graham rather than Poincaré coordinates. The metric of the AdS black hole is

$$ds^2_{\text{AdS}} = \sqrt{\Lambda}^\alpha \left[ -\frac{(1 - \frac{\xi^2}{\zeta^2})^2}{\zeta^2(1 + \frac{\xi^2}{\zeta^2})} dt^2 + \frac{1 + \frac{\xi^2}{\zeta^2}}{\zeta^2} dx^2 + \frac{\zeta^2}{\zeta^2} dy^2 + \frac{\zeta^2}{\zeta^2} dz^2 + \frac{d\xi^2}{\zeta^2} \right].$$

where $\zeta = (x, y, w)$. The boundary of AdS $S$ is located at $z = 0$ and the horizon at $z_h$. The Hawking temperature of the horizon is $z_h = \frac{\sqrt{\Lambda}}{2\pi}$. We parametrize $S^3$ as

$$ds^2_{S^3} = \sqrt{\Lambda}^\alpha \left[ d\psi^2 + \sin^2 \psi d\Omega^2 + \cos^2 \psi d\Omega^2 \right],$$

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$, $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$. Our ansatz for the probe brane embedding is partially by symmetry. We will use volume coordinates $(t, x, y, z, \theta, \phi)$ for the the D7 brane with the ansatz that $w$ is constant and $\psi = \psi(z)$ depends only on $z$. For the D5 brane we use coordinates $(t, x, y, z, \theta, \phi)$ where $\theta, \phi$ are also constant. The world volume metrics are

$$ds^2_{S^5} = \sqrt{\Lambda}^\alpha \left[ -g_{tt} dt^2 + g_{xx} (dx^2 + dy^2) + g_{zz} dz^2 + \sin^2 \psi d\Omega^2 \right],$$

where the metric components are

$$g_{tt} = \frac{1}{\zeta^2} \left( 1 - \frac{\xi^2}{\zeta^2} \right)^2, \quad g_{xx} = \frac{1 + \frac{\xi^2}{\zeta^2}}{\zeta^2}, \quad g_{zz} = \frac{1}{\zeta^2} + \left( \frac{d\psi}{dz} \right)^2. \quad (8)$$

We make the ansatz for the world volume gauge fields,

$$2\pi \alpha' F = \frac{\sqrt{\Lambda}}{2\pi} \left[ \frac{d}{dz} a(z) dz \wedge dt + b dz \wedge dy - c dt \wedge dx \right. \left. + \frac{d}{dz} f(z) dz \wedge dx + \frac{d}{dz} f(z) dz \wedge dy - \frac{f}{2} \cos \theta dz \wedge d\phi \right],$$

with

$$B = \frac{\sqrt{\Lambda}}{2\pi} b, \quad E = \frac{\sqrt{\Lambda}}{2\pi} e, \quad \rho = \frac{1}{V_{2+1}} \frac{2\pi}{\sqrt{\Lambda}} \frac{\delta S}{\delta a(z)}, \quad (9)$$

where $E$ and $B$ are constant external electric and magnetic fields, $\rho$ is charge density and $V_{2+1} = \int dt dx dy$. The D5 and D7 branes have the same values of $E$, $B$ and $\rho$. We will consider $N_5$ D5 branes but always a single D7 brane. The parameter $f = \frac{2\pi}{\sqrt{\Lambda}} N_5$ is proportional to the number of D5 branes and it becomes a world-volume flux on the the D7 brane.

The probe geometries are fixed once we find the functions $\psi(z), a(z), f(z), f(z)$. These are determined by requiring that they extremize the Dirac-Born Infeld (DBI) action with the addition of a Wess-Zumino (WZ) term for the D5 or D7 brane.

We use the ansatz that $w = \frac{\Lambda}{2\pi} t$. With the above ansatz, these actions take the form

$$S_5 = -N_5 N_5 \int_0^{t_0} dz \sqrt{\bar{S}}, \quad (10)$$

$$S_7 = -N_5 \int_0^{t_0} dz \left[ \left( f^2 + 4 \cos^4 \psi \right)^{1/2} \sqrt{S} + 2 \left( a'(z) b - e f'(z) c(\psi) \right) \right],$$

where $\psi(z) = \frac{1}{2} \left( \sqrt{z^2 - r_0^2} \right)$ and $\bar{S} = \sqrt{z^2 - r_0^2}$. In particular, $z_h = \sqrt{\Lambda}/r_0$. 

Footnote 4: The radial coordinate $z$ is related to our previously used one, $r$, by the equation $z^2 = 2 \left( r^2 + \sqrt{r^2 - r_0^2} \right)$. In particular, $z_h = \sqrt{\Lambda}/r_0$. 

Figure 3: The low temperature limit of the entropy density (in units of $\sqrt{\Lambda} N B/(2\pi^2)$) is plotted on the vertical axis versus filling fraction on the horizontal axis for various values of $f$. 

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where \( N_5 = \frac{2 \sqrt{N_3}}{2\pi} V_{2+1} \), \( N_7 = \frac{4 \sqrt{N_3}}{2\pi} V_{2+1} \), and
\[
S = 4 \sin^4 \psi q_{[zz]} (g_{tt} (b^2 + g_{xx}^2) - g_{xx} c^2) - a'(z)^2 (b^2 + g_{xx}^2) + g_{xx} f_2(z) (f_1(z)^2 + f_1'(z)) + 2 a'(z) e b f_1'(z) - e^2 f_1'(z)^2 f_2(z)).
\]
The second term of eqn. (11) is from the WZ term and
\[
e(\psi) = \psi - \frac{1}{4} \sin 4 \psi - \frac{\pi}{2}.
\]
(12)
The functions \( a(z) \), \( f_1(z) \) and \( f_2(z) \) are cyclic variables and they can be eliminated in terms of their conserved momenta \((q, q_z, q_t)\) which are defined as
\[
q = \frac{\delta S_{\delta z}}{\delta a'(z)}, \quad q_z = -\frac{\delta S_{\delta z}}{\delta f_2(z)}, \quad q_t = -\frac{\delta S_{\delta z}}{\delta f_1'(z)}.
\]
The quantity \( q \) is proportional to the charge density \( \rho \) defined in (9) and \((q_z, q_t)\) to the current densities \((J_z, J_t)\). When the constants of integration \((q, q_z, q_t)\) are fixed, the appropriate functional energy is the Routhian which is obtained from a Legendre transformation of the action. The Routhians are
\[
R_5 = -N_5 N_5 \int dz R_5, \quad R_7 = -N_7 \int dz R_7,
\]
where
\[
R_5 = \sqrt{g_{zz}} \left[ 4 \sin^2 \psi (g_{tt} g_{xx}^2 + b^2 g_{tt} - e^2 g_{xx})
+ g_{tt} q_t^2 - g_{xx} q_x^2 - g_{xx} q_t^2
+ 2 b e q q_t
+ \frac{1}{g_{xx}} \left( b^2 q_x^2 + q_t^2 \right)^{1/2}
\right].
\]
The expression for \( R_5 \) follows from \( R_5 \) by replacing \( 4 \sin^4 \psi \rightarrow 4 \sin^4 \psi (f^2 + 4 \cos^2 \psi) \), \( q \rightarrow (q + 2 b e (\psi)) \) and \( q_t \rightarrow (q_t + 2 b e c (\psi)) \). The Karch-O’Bannon technique \([25, 26]\) now finds a relationship between the current densities \((J_z, J_t) \sim (q_z, q_t)\) and the electric field by requiring that the world-volume is nonsingular. To this end, we rewrite \( R_5 \) as
\[
R_{57} = \sqrt{g_{zz}} \left[ \frac{1}{2} \sqrt{B \cdot C - A^2},
\right]
\]
where in the case of \( R_5 \)
\[
A = q b g_{tt} - q_t q_e g_{xx},
B = g_{tt} g_{xx}^2 + b^2 g_{tt} - g_{xx} e^2,
C = 4 \sin^2 \psi g_{tt} g_{xx}^2 + g_{tt} b^2 - g_{xx} e^2 (q_x^2 + q_t^2).
\]
The expressions for \( R_7 \) follow by making the same replacements as explained just above. The expression \( B \) is negative at the horizon, \( z = z_h \), and positive at the asymptotic boundary of AdS, i.e. for \( z \rightarrow 0 \). It must therefore have at least one zero at some finite (positive) value of \( z \), which we denote by \( z^* \). Solving \( B = 0 \) one finds that there is only one positive real root,
\[
\frac{z_h^4}{\tilde{z}_h^4} = \tilde{z}_h^2 - \tilde{b}^2 + \sqrt{(\tilde{e}^2 - \tilde{b}^2)^2 + 2(\tilde{e}^2 + \tilde{b}^2) + 1} - 1,
\]
where \( \tilde{e} = \frac{e}{2}, \tilde{b} = \frac{b}{2} \). We note that \( z^* \rightarrow z_h \) as \( e \rightarrow 0 \). Like \( B \), \( C \) is negative at the horizon and positive at the boundary of AdS. It must therefore also have a zero for a finite value of \( z \) and, in order for the Routhian to stay real, this zero must coincide with the one of \( B \). Finally, \( A \) also has to vanish at the common zero of \( B \) and \( C \). In summary \( B(z = z^*) = 0 \) determines \( z^* \). Then \( C(z = z^*) = 0 \) and \( A(z = z^*) = 0 \) will determine \((q_z, q_t)\). This reasoning leads to
\[
q_{DS}^B = \frac{b q_{DS}^D}{b^2 + g_{xx}^2(z^*)} e,
q_{DS}^D = \frac{g_{xx}(z^*) e}{b^2 + g_{xx}^2(z^*)} \sqrt{4 \sin^4 \psi(z^*) (b^2 + g_{xx}^2(z^*))},
\]
and for the D7 brane
\[
q_{DS}^D = \frac{b q_{DS}^D - 2 e g_{xx}^2(z^*)}{b^2 + g_{xx}^2(z^*)} e,
q_{DS}^D = \frac{g_{xx}(z^*) e}{b^2 + g_{xx}^2(z^*)} \times \left[ \left( q_{DS}^D + 2 b e c (\psi(z^*)) \right)^2 + 4 \sin^4 \psi(z^*) (f^2 + 4 \cos^4 (\psi(z^*)) (b^2 + g_{xx}^2(z^*)) \right]^{1/2},
\]
where \( g_{xx}^2(z^*) \) can be expressed as
\[
g_{xx}^2(z^*) = \frac{2}{\tilde{z}_h^4} \left( 1 + \tilde{e}^2 - \tilde{b}^2 + \sqrt{(\tilde{e}^2 - \tilde{b}^2)^2 + 2(\tilde{e}^2 + \tilde{b}^2) + 1} \right).
\]
We recall the normalizations \( q_{DS}^D / b = \pi b / f \) and \( q_{DS}^D / b = \pi v \). The conductivities are defined as \( \sigma_{xx} = \frac{J_z}{E = 0}, \sigma_{xy} = \frac{J_t}{E = 0} \) and, with the normalization of the currents, \( D_{x,y}^{DS} = \frac{N_5 N_5 N_7}{2 \pi} q_{DS}^{x,y} \).
\[
J_{x,y}^{DS} = \frac{N_5}{\sqrt{N_7}} \sqrt{N_7} q_{DS}^{x,y} \text{ we obtain the Hall conductivities}
\]
\[
\sigma_{x,y}^{DS} = \left( 1 - \frac{\tilde{q}_h^4}{1 + \tilde{q}_h^4} \right) \frac{N_V}{2 \pi},
\]
\[
\sigma_{x,y}^{DS} = \frac{N_V}{2 \pi}
+ \frac{N}{2 \pi} \left( 1 - \nu \right) + \frac{1}{2 \pi} \sin 4 \psi(z_h) - \frac{2}{\pi} \psi(z_h)) \frac{\tilde{z}_h^4}{1 + \tilde{q}_h^4},
\]
where the horizon radius is related to the Hawking temperature in units of inverse magnetic length,
\[
\tilde{r}_h \equiv \frac{r_h}{\sqrt{b}} = \frac{\sqrt{2/\tilde{b}}}{z_h} = \sqrt{\frac{\nu \sqrt{4}}{2 B} T},
\]
and we are using natural units \((h = c = k_B = 1)\). For these formulae to be valid, the temperature must be low enough that
the quantum Hall ferromagnetic phase is stable, that is, \( r_h < 0.4 \). We have used \([22]\) as well as the fact that \( z^* \to 2z_h \) as \( e \to 0 \). We see that the conductivities are completely determined by the value of the angle \( \psi \) at the horizon when the electric field is set to zero. This angle depends on \( f \) and the other parameters and must be determined by numerical solution of the equation which determines \( \psi(z) \).

For a given filling fraction \( \nu \), the value of the Hall conductivity is larger for the D7 brane than for the D5 brane. (The two are equal only for the trivial solution \( \psi = \pi/2 \).) Hence, there is always an upwards jump in the Hall conductivity when the D5 brane ceases to be the favourable one and the D7 brane takes over, which for the values of \( f \) that we consider occurs in the vicinity of \( \nu \approx 0.3 - 0.5 \).

Finally, one finds numerically both for the D5 brane and the D7 brane that \( f^2 \sin^4 \psi(z_h) \to 0 \) as \( f \to \infty \). This allows us to find the limiting expression for the D7 brane Hall conductivity which we quoted in equation \([6]\). The D5 brane Hall conductivity quoted in equation \([9]\) is independent of \( f \). For \( f \to \infty \), the phase transition occurs at \( \nu = 1/2 \). At the phase transition, where the D7 brane takes over from the D5 brane, there is a jump of \( \nu/f \text{D}^7_h/(1 + \psi^2_0 h^2) \) in the Hall conductivity.

For the longitudinal conductivities we find

\[
\sigma_{xx}^{D5} = \frac{\hat{r}_h^2}{1 + \hat{r}_h^2} \frac{N f}{2 \pi^2} \sqrt{4 \sin^3 \psi(z_h)(1 + \hat{r}_h^2) + (\pi \nu f)^2},
\]

\[
\sigma_{xx}^{D7} = \frac{\hat{r}_h^2}{1 + \hat{r}_h^2} \frac{N}{2 \pi x} \times \left[ \left( \pi(1 - \nu) - 2 \psi(z_h) + \frac{1}{2} \sin 4 \psi(z_h) \right)^2 \right. \\
\left. + 4 \sin^4 \psi(z_h)f^2 + 4 \cos^4 \psi(z_h)(1 + \hat{r}_h^2) \right]^{1/2}.
\]

From here we find the analytic expressions for the limiting behaviours of the longitudinal conductivities quoted in equations \([1]\) and \([2]\). Obviously, using our results we can also calculate the longitudinal resistivity \( \rho_{xx} = \sigma_{xx}/(\sigma_{xx}^2 + \sigma_{xy}^2) \). In the limit of large \( f \) we find

\[
P_{xx}^{D5} = \hat{r}_h^2 \left( \frac{Nf}{2 \pi} \right)^{-1} \text{ as } f \to \infty,
\]

\[
P_{xx}^{D7} = \hat{r}_h^2 \left( \frac{Nf}{2 \pi} \right)^{-1} \frac{(1 - \nu)}{\sqrt{\nu^2 + \hat{r}_h^2}} \text{ as } f \to \infty.
\]

In order to find the conductivity when \( f \) is finite, we need to know the embedding angle at the horizon, i.e. \( \psi(z_h) \). In reference \([22]\) the equations of motion for the D5 and the D7 probe branes were solved numerically for \( E = 0 \) and \( \psi(z_h) \) can be extracted from that work. This allows us to compute \( (\sigma_{xy}, \sigma_{xx}) \).

In figure \([1]\) and \([2]\) we show the deviation of the Hall conductivities \( \sigma_{xy} \) from the classical Hall conductivity \( \frac{\pi}{2e} \), the longitudinal conductivity \( \sigma_{xx} \) and the longitudinal resistivity \( \rho_{xx} \), respectively, for three different values of \( f \), \( f = 1 \), \( f = 2 \) and \( f = 10 \), and for \( \nu \in [0, 1] \). Figure \([1]\) corresponds to \( \hat{r}_h = 0.2 \) and figure \([2]\) to \( \hat{r}_h = 0.4 \). For \( \hat{r}_h = 0.2 \) and \( f = 10 \) the curves are already indistinguishable from the corresponding curves for \( f \to \infty \). An interesting feature of the curves is that the deviation of the Hall conductivity from its classical value qualitatively shows a behaviour corresponding to the appearance of a Hall plateau. It is negative and decreasing for \( \nu \in [0, \nu_c] \) and changes discontinuously at \( \nu = \nu_c \) to becoming positive and decreasing for \( \nu \in [\nu_c, 1] \) where \( \nu_c \to 0.5 \) as \( f \to \infty \). In absolute value the plateau is, however, not very pronounced. The size of the observed quantum Hall effect is limited by the fact that the temperature for which the probe-brane solutions are stable turns out to be dynamically confined to \( \hat{r}_h \leq 0.4 \). In figure \([3]\) we show the actual Hall conductivity of our model together with the classical linear curve. All of the above pertains to the case \( \nu \in [0, 1] \). For \( 1 < \nu < 2 \) the favoured system is a composite system consisting of a single gapped D7 brane and either a set of un-gapped D5 branes or a single un-gapped D7 brane with the former system being relevant for the smaller values of \( \nu \) and the latter one for the larger values of \( \nu \). The total flux \( f_{\text{tot}} \) of the composite system must be distributed between the constituents in such a way that the energy is minimal. (A minimisation procedure to determine the distribution of the flux was implemented numerically in reference \([22]\).) The conductivity of a composite system is the sum of the conductivities of its various constituents. The Hall conductivity of the gapped D7 brane is independent of \( f \) and so is that of the D5 branes. The Hall conductivity of the single ungapped D7 brane does depend (indirectly via \( \psi(z_h) \)) on the value of \( f \) which due to the minimisation procedure must be smaller than \( f_{\text{tot}} \). All this means that the deviation from the classical Hall conductivity in the first part of the interval \( \nu \in [1, 2] \) looks as in the first part of the interval \( \nu \in [0, 1] \) and in the second part of the interval \( \nu \in [1, 2] \) the deviation looks like the deviation in the second part of the interval \( \nu \in [0, 1] \), but corresponding to a smaller value of the flux. This pattern continues as \( \nu \) increases but the region where D7+D5 is preferred over D7+D7 gets smaller and smaller.

The computation of the low temperature limit of the entropy is straightforward. The procedure follows the technique that is outlined in reference \([24]\). We consider the on-shell action, that is, \( R_5 \) or \( R_7 \) of the D5 or D7 brane, respectively. The Routhian is the relevant thermodynamic potential when the total charge is fixed and we identify it with the Helmholtz free energy. Then, the entropy is defined as the negative of the partial derivative of

![Figure 4: The Hall conductivity for \( f = 10 \) and \( \hat{r}_h = 0.4 \) for \( \nu \in [0, 1] \) in units of \( N/(2\pi) \). The red curve corresponds to the D5 brane and the blue one to the D7 brane. For comparison we have also plotted the linear curve (in green).](image-url)
the free energy by the temperature, and the entropy density is
\[ s^{D5,D7} = -\frac{1}{V_{2+1}} \frac{\partial}{\partial T} R_{5,7}. \]

The procedure is easiest if one reverts to the Poincaré coordinates for $AdS_5$ which were used in references [21], [22]. Then, the essential observation is that, because their equations of motion depend on temperature only by terms with $T^4$, the derivative of the embedding functions must be at least of order $T^3$. Similarly, the Routhian itself contains temperature only in terms of $T^4$ and its derivative is of order $T^3$. Then, finally, the low temperature limit picks up the integrand evaluated on the lower limit of the integral. The result for the low temperature limit of the entropy is what is quoted in equations (3) and (4) for large $f$ and displayed in figure 3 for other values of $f$.

We have computed the conductivity of the quantum Hall ferromagnetic states of the D3-D5 brane system and discussed some of the implications. We have assumed that the stable magnetic states of the D3-D5 brane system and discussed for some range of temperature and density, have instabilities to forming inhomogeneous condensates [28]-[30]. Whether the system we have examined can have such instabilities and how they would affect the electronic properties of the system is a fascinating subject which we leave for further work.

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