Quasiperiods of biinfinite Sturmian words

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Abstract

We study the notion of quasiperiodicity, in the sense of “coverability”, for biinfinite words. All previous work about quasiperiodicity focused on right infinite words, but the passage to the biinfinite case could help to prove stronger results about quasiperiods of Sturmian words. We demonstrate this by showing that all biinfinite Sturmian words have infinitely many quasiperiods, which is not quite (but almost) true in the right infinite case, and giving a characterization of those quasiperiods.

The main difference between right infinite and the biinfinite words is that, in the latter case, we might have several quasiperiods of the same length. This is not possible with right infinite words because a quasiperiod has to be a prefix of the word. We study in depth the relations between quasiperiods of the same length in a given biinfinite quasiperiodic word. This study gives enough information to allow to determine the set of quasiperiods of an arbitrary word.

1 Introduction

A finite word $q$ is a quasiperiod of a word $w$ if and only if each position of $w$ is covered by an occurrence of $q$. A word $w$ with a quasiperiod $q \neq w$ is called quasiperiodic. For instance, $abaababaabaaba$ is quasiperiodic and has two quasiperiods: $aba$ and $abaaba$. Likewise, an infinite word may have several, or even infinitely many quasiperiods; in the latter case, we call it multi-scale quasiperiodic. The study of quasiperiodicity began on finite words in the context of text algorithms [1,6], and was subsequently generalized to right infinite words [5,7,10], to symbolic dynamical systems [11], and to two-dimensional words [3] where it is a special case of the tiling problem. Finally, a previous article [4] provided a method to determine the set of quasiperiods of an arbitrary right infinite word. It also characterized periodic words and standard Sturmian words in terms of quasiperiods. This is interesting, because periodic words are the simplest possible infinite words, and Sturmian words are a widely studied class [7,8,12] which could be defined as the least complex non-periodic words. These results suggest that quasiperiodicity has some expressive power, and that the set of quasiperiods is an interesting object to study in order to get information about infinite words.

The current paper extends to the biinfinite case ($\mathbb{Z}$-words) some results from [4]. The motivations for this are threefold.

In the two-dimensional case, quasiperiodic $\mathbb{N}^2$-words and $\mathbb{Z}^2$-words behave quite differently [3]. This difference is not specific to the dimension 2, so it seems natural to start by understanding the differences in quasiperiodicity between N-words and Z-words.

Quasiperiodicity have been considered not only on infinite words, but also on subshifts [11]. However the shift map does not preserve quasiperiodicity in the right infinite case and this leads to annoying technicalities. The biinfinite case is sometimes considered more natural for subshifts because it turns the shift map into a bijection. Moreover, it also turns the shift map into a quasiperiodicity-preserving map, which makes the study of quasiperiodic subshifts much more convenient.

Finally, a previous article [4] gave a characterization of standard Sturmian words in terms of quasiperiods. Intuitively, the condition “standard” was only needed because of problems at the origin. By moving

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to the biinfinite case, we remove the origin so we can hope for a characterization of all Sturmian words. (We did not achieve this yet, but it is a possible continuation of our work.)

The current article makes a first step toward the resolution of these questions: it generalizes the method to study the set of quasiperiods of an arbitrary word from $\mathbb{N}$ to the biinfinite case. This is not a trivial task because, by contrast with the right infinite case, we might have several quasiperiods with the same length. (In the right infinite case, all quasiperiods are prefixes, thus there may be only one quasiperiod of a given length.) Therefore we need to determine not only the lengths of the quasiperiods, but also for each length which factors are quasiperiods and which are not.

Many natural results about quasiperiodicity on $\mathbb{N}$-words turned out to be surprisingly difficult to generalize to $\mathbb{Z}$-words because of this problem. In addition to show how to determine the set of quasiperiods of an arbitrary $\mathbb{Z}$-word, we investigate the relations existing between two quasiperiods of the same length inside a given biinfinite word. More precisely, we show that the following conditions are decidable, given two words $q, r$ of the same length:

(a) there exists a biinfinite word both $q$ and $r$-quasiperiodic;

(b) each $q$-quasiperiodic biinfinite word contains infinitely many occurrences of $r$;

(c) each $q$-quasiperiodic biinfinite word is also $r$-quasiperiodic;

(d) in any word with quasiperiods $q$ and $r$, the derivated sequences of $q$ and $r$ are equal.

Derived sequences are a tool previously used to build examples and counter-examples of quasiperiodic words and to show independence results [11]. A derived sequence can be thought as a normal form for quasiperiodic words. Intuitively, when two derivated sequences are equal, the considered quasiperiods contain the same information about $w$.

Finally, we give a complete description of the set of quasiperiods of each biinfinite Sturmian word. In particular, we show that each biinfinite Sturmian word has infinitely many quasiperiods. This contrasts with the right infinite case, where two Sturmian words of each slope have no quasiperiods.

The paper is structured as follows.

In Section 2, we provide a method to study the quasiperiods of an arbitrary biinfinite word, i.e., a description of the set of quasiperiods of an arbitrary word.

In Section 3, we define three relations over couples of words: compatible, definite, and positive. Those relations are decidable by an algorithm. We show that the couple $(q, r)$ is compatible if and only if there exists a biinfinite word $w$ having both $q$ and $r$ as quasiperiods (Item (a) above). Moreover, the couple $(q, r)$ is definite and positive if and only if all $q$-quasiperiodic words are also $r$-quasiperiodic (Item (c)).

In Section 4, we show that the couple $(q, r)$ is positive if and only if in any word $w$ which is both $q$ and $r$-quasiperiodic, the derived sequences along $q$ and $r$ are the equal (Item (d)). We also prove that $(q, r)$ is definite if and only if each $q$-quasiperiodic word contains infinitely many copies of $r$ (Item (b)).

In Section 5, we determine the set of quasiperiods of each biinfinite Sturmian word. In the process we show that all biinfinite Sturmian words have infinitely many quasiperiods.

Finally in Section 6, we conclude with a few related open questions and state our acknowledgements.

Figure 1 below shows the implications proven in Sections 3 and 4.

Figure 1: Implications proved in Sections 3 and 4
2 Determining the quasiperiods of biinfinite words

We quickly review classical definitions and notation. Let $u, v$ denote two finite words and $w$ a finite or infinite word. As usual, $|u|$ denotes the length of $u$ and $uv$ the concatenation $u$ and $v$. We note $w(i)$ the $i^{th}$ letter of $w$; letters are often considered as words of length 1. We write $\varepsilon$ for the empty word. If $u$ is of length $n$ and satisfies $u = w(i)w(i + 1)\ldots w(i + n - 1)$, then we say that $u$ is a factor of $w$ which occurs at position $i$ and which covers positions $i$ to $i + n - 1$ (included). The word $u$ is a quasiperiod of $w$ if each position of $w$ is covered by an occurrence of $u$. In particular, if $w$ is finite or right infinite, then $u$ is a prefix of $w$. If $u$ is a word and $\alpha, \beta$ two different letters such that $u\alpha$ and $u\beta$ are both factors of $w$, we say that $u$ is right special in $w$. Symmetrically, if $\alpha u$ and $\beta u$ are factors of $w$, then $u$ is left special in $w$. If $\alpha u\beta$ is a factor of $w$, then we say that $u\beta$ is a successor of $\alpha u$, and conversely that $\alpha u$ is a predecessor of $u\beta$ in $w$. A word has a unique successor (resp. predecessor) if and only if it is not right (resp. left) special. Finally, $|u|_\alpha$ denotes the number of occurrences of $\alpha$ in $u$. Unless stated otherwise, all infinite words are biinfinite, i.e. indexed by $\mathbb{Z}$.

We now have enough vocabulary to state the main theorem of [4], adapted to the biinfinite case.

**Theorem 2.1.** Let $w$ denote an infinite word, $q$ a factor of $w$ and $\alpha$ a letter.

1. Suppose $q$ is a quasiperiod and $q\alpha$ is a factor of $w$. The word $q\alpha$ is a quasiperiod if and only if $q$ is not right special.

2. Suppose $q$ is a quasiperiod and $q\alpha$ is a factor of $w$. The word $\alpha q$ is a quasiperiod if and only if $q$ is not left special.

3. Suppose $q\alpha$ is a quasiperiod of $w$. The word $q\alpha$ is a quasiperiod if and only if either $u = q\alpha q\alpha$ is not a factor of $w$, or if $q$ occurs at least 3 times in $u$.

4. Suppose $q\alpha$ is a quasiperiod of $w$. The word $q\alpha$ is a quasiperiod if and only if either $u = q\alpha q\alpha$ is not a factor of $w$, or if $q$ occurs at least 3 times in $u$.

A proof of Theorem 2.1 can be found in [4] in the right infinite case; the adaptation to the biinfinite case is immediate. That theorem basically states that it is enough to study the set of right special factors and square factors which are also prefixes to get the set of quasiperiods of a given right infinite word. As special and square factors are well-understood in combinatorics on words, it generally little additional word to get the set of quasiperiods of a given right infinite word. We will comment on the biinfinite version of the theorem, which we just stated, in a few paragraphs.

We can extend this theorem a bit further, but to do so we need the notion of overlap.

**Definition 2.2.** Let $q$ denote a finite word. An overlap of $q$ is a word $w$ having $q$ as a prefix and as a suffix, such that $|q| < |w| \leq 2|q|$. More generally, a $k$-overlap of $q$ is a word of the form $uwv$, where $u$ is a $(k - 1)$-overlap and $v$ is such that $qv$ is an overlap of $q$.

The quantity $2|q| - |w|$ is called the span of the overlap. If $q$ is fixed, then an overlap is uniquely determined by its span, thus we note $V_q(m)$ the overlap of $q$ having span $m$ (if it exists). We write $V_q(n_1, n_2, \ldots, n_k)$ the $k$-overlap built from overlaps $V_q(n_1), V_q(n_2), \text{etc.}$ and we call $n_i$ the $i^{th}$ span of this overlap.

An overlap (without any explicit $k$) is thus a 2-overlap. An infinite word $w$ is $q$-quasiperiodic if and only if two consecutive occurrences of $q$ in $w$ always form an overlap.

In general, we might have more than two occurrences of $q$ in an overlap of $q$. For instance, $V_{aaa}(1) = aaaaa$ contains 3 occurrences of $aaa$. We say that $w$ is a proper $k$-overlap of $q$ if $w$ is a $q$-quasiperiodic word containing exactly $k$ occurrences of $q$. We write $V_q(n_1, \ldots, n_k)$ when we mean that $V_q(n_1, \ldots, n_k)$ is a proper $k$-overlap of $q$. A proper overlap is implicitly a proper 2-overlap.

**Lemma 2.3.** Let $u$ denote a word and $\alpha, \beta$ letters. If $u\beta$ is a factor of an overlap of $u\alpha$, then $\alpha = \beta$.

**Proof.** Let $w$ denote an overlap of $u\alpha$; by definition of an overlap, there exist words $p, s$ (possibly empty) such that $u = ps$ and $w = u\alpha\alpha = ps\alpha\alpha$. If $u\beta$ is a factor of $w$, then $s\beta$ is a factor of $s\alpha\alpha$. Let $x, y$ denote the words such that $s\alpha\alpha = xs\beta y$. Observe that $|xy| = |s\alpha|$, that $x$ is a prefix and $y$ a suffix of $s\alpha$ to conclude that $xy = s\alpha$. Thus we can simplify $|s\alpha\alpha|_\alpha = |xs\beta y|_\alpha$ into $|s\alpha|_\alpha = |s\alpha|_\alpha$, which implies $|\alpha|_\alpha = |\beta|_\alpha$ and $\alpha = \beta$. \qed
Proposition 2.4. Let \( w \) denote an infinite word, and \( q \) a quasiperiod of length \( n \) of \( w \). A successor of \( q \) is a quasiperiod of \( w \) if and only if \( q \) is not right special. A predecessor of \( q \) is a quasiperiod of \( w \) if and only if \( q \) is not left special.

Proof. Let \( \alpha, \beta \) denote letters and \( u \) denote a word such that \( \alpha u \) is a quasiperiod and \( \alpha u \beta \) a factor of \( w \). If \( u \beta \) is a quasiperiod of \( w \) and \( v \gamma \) is also factor of \( w \) for a letter \( \gamma \neq \beta \), then \( u \gamma \) is a factor of an overlap of \( u \beta \). Lemma 2.3 shows that \( \beta = \gamma \): a contradiction. Conversely if \( \alpha u \) is not right special, then every occurrence of \( \alpha u \) continues into an occurrence of \( u \beta \); since \( \alpha u \) covers \( w \), so does \( u \beta \). The left special case is symmetric. \( \square \)

Theorem 2.1 and Proposition 2.4 together imply that, in order to understand the set of quasiperiods of a biinfinite word, it is enough to know its set of special factors and its set of square factors. These two types of factors are already well-studied and well-understood in combinatorics on words, therefore we can reuse this knowledge when we need to get the set of quasiperiods of an infinite word.

Proposition 2.4 has another interesting consequence: if an infinite, aperiodic word \( w \) has a quasiperiod of some length \( n \), then it also has a left-special quasiperiod \( \ell \) and a right special quasiperiod \( r \) of length \( n \). More precisely, the set of quasiperiods of some length \( n \) is given by a union of chains of the form \( \{ u_1, \ldots, u_k \} \), where \( u_1 \) is left special, \( u_k \) is right special, no other \( u_i \) is special, and \( u_{i+1} \) is the (unique) successor of \( u_i \) for each \( 1 \leq i < k \). If \( q \) belongs to such a chain, we call \( u_1 \) its left-special predecessor and \( u_k \) its right special successor.

After working out several examples, one may conjecture that there is at most one right special (and thus one left special) quasiperiod of a given length in any biinfinite word. In this case, there would be at most one chain of quasiperiods of a given length, so it would be easy to determine the set of quasiperiods of an arbitrary biinfinite word. Unfortunately the following example disproves this conjecture. Let \( q = aba\,aba\,ba\), \( r = aba\,ba\,a\) and \( w \) be defined by:

\[
w = w'(a^{-1}q) \cdot (q)^w = \cdots \, baaba\,baaba\,baaba \cdot baaba\,baaba\,abaaba \cdots
\]

where the end of each occurrence of \( q \) in \( w \) is showed by a space. The definition of \( w \) makes it clear that \( q \) is a quasiperiod of \( w \). As the excerpt of \( w \) suggests, \( r \) is also a quasiperiod of \( w \): since the word is ultimately periodic, the same behaviour repeats to the left and to the right. It can be directly observed in the excerpt that both \( q \) and \( r \) are right special. This example is the simplest “pathological case” which we mentioned in the introduction.

3 Checking implications between two quasiperiods

In this section we show that it is decidable to check, given two finite words \( q \) and \( r \) of the same length, which of the following is true:

1. Any \( q \)-quasiperiodic biinfinite word is also \( r \)-quasiperiodic;
2. there exists an infinite word which is \( q \)- and \( r \)-quasiperiodic, and another one which is just \( q \)-quasiperiodic;
3. no infinite word may have both quasiperiods \( q \) and \( r \) at the same time.

First we develop a bit of vocabulary to state the conditions in a convenient way.

Lemma 3.1. Let \( q, r \) denote two different words of the same length and \( w \) a proper overlap of \( q \). The word \( w \) has at most one occurrence of \( r \).

Proof. By a classical lemma [8 Prop. 1.3.4], there exist finite words \( x, y \) and an integer \( k \) satisfying \( q = (xy)^k x \) and \( w = (xy)^{k+1} x \). Moreover, \( xy \) is a primitive word. If it were not, call \( z \) its primitive root and observe that an occurrence of \( q \) would start at position \( |z| \) in \( w \), yielding three occurrences of \( q \) in \( w \), a contradiction. Additionally, we have either \( k \geq 1 \) or \( y = \varepsilon \). Indeed, if \( k = 0 \) and \( |y| \geq 1 \), we would have \( q = x \) and \( w = xyx \), implying \(|w| > 2|q| \), a contradiction with the definition of an overlap. We treat the cases \( k \geq 1 \) and \( y = \varepsilon \) separately.
First, suppose \( k \geq 1 \). As \( |q| = |r| \), all occurrences of \( r \) in \( w \) must start at positions between 1 and \( |xy| \) (included). Call \( u \) the prefix of length \( |xy| \) of \( r \). The word \( u \) is a factor of \( xyxy \). Because \( xy \) is primitive, each factor of length \( |xy| \) occurs only once in \( xyxy \), except \( xy \) itself \footnote{Prop. 1.3.2}. This means that there can only be one occurrence of \( u \), and therefore of \( r \), starting in the first \( |xy| \) letters of \( w \).

Now suppose \( k = 0 \). By the previous remarks, this implies \( y = x \), thus \( q = x \) and \( q \) is primitive. As a consequence, each factor of length \(|q|\) in \( qq = w \) occurs only once, excepted \( q \) itself (otherwise, \( q = q_1q_2 = q_2q_1 \) for some finite words \( q_1, q_2 \), and \footnote{Proposition 1.3.2} contradicts primitivity). In particular, \( r \), if it occurs at all, occurs only once.

\[ \text{Definition 3.2.} \] Let \( q, r \) denote finite nonempty words of the same length and \( m, n \) natural integers. If the proper overlap \( V_q^r(m) \) exists and contains \( r \) as a factor, then we write \( \text{occ}(q, r, m) \) for the position of \( r \) in \( V_q^r(m) \); otherwise \( \text{occ}(q, r, m) \) is not defined. (Lemma \footnote{3.1} ensures that if \( \text{occ}(q, r, m) \) exists, then it is unique.) If both \( \text{occ}(q, r, m) \) and \( \text{occ}(q, r, n) \) exist, then we define the quantity

\[
\text{f}_{q, r}(m, n) = m + \text{occ}(q, r, m) - \text{occ}(q, r, n)
\]

otherwise, \( \text{f}_{q, r}(m, n) \) is undefined.

We insist on the fact that \( \text{occ}(q, r, m) \) is defined only where \( V_q^r(m) \) is defined and contains an occurrence of \( r \). If \( V_q^r(m) \) is not a proper overlap (i.e. it contains more than two occurrences of \( q \), like \( V_{aaa}(1) \)), then \( \text{occ}(q, r, m) \) is not defined. The quantity \( \text{f}_{q, r}(m, n) \) is defined if and only if both \( \text{occ}(q, r, m) \) and \( \text{occ}(q, r, n) \) are. Moreover, \( q \) and \( r \) are not symmetric: \( \text{f}_{q, r} \neq \text{f}_{r, q} \).

Here is the intuitive interpretation of \( \text{f}_{q, r} \). Let \( m, n \) denote natural integers such that \( w = V_q^r(m, n) \) is a proper 3-overlap of \( q \). By Lemma \footnote{3.1} the word \( w \) has at most 2 occurrences of \( r \). Suppose it has exactly two. If these two occurrences form an overlap of \( r \), then \( \text{f}_{q, r}(m, n) \) is the span of this overlap. If these two occurrences do not overlap, then there exists a nonempty word \( s \) such that \( rsr \) is a factor of \( w \); in this case, \( \text{f}_{q, r}(m, n) = |s| \). If \( w \) has less than two occurrences of \( q \), then \( \text{f}_{q, r}(m, n) \) is not defined.

\[ \text{Example.} \] In Equation \footnote{1} we had \( q = aba \) and \( r = aba \); in this case the function \( \text{f}_{q, r} \) is given by:

\[
\begin{array}{c|ccc}
  f & 0 & 1 & 3 \\
  \hline
  0 & 0 & -2 & 0 \\
  1 & 3 & 1 & 3 \\
  3 & 3 & 1 & 3 \\
\end{array}
\]

Computing \( \text{f}_{q, r} \) given two finite words \( q \) and \( r \) of the same length can be done in \( O(|q|^3) \) time. For each \( m \) and for each \( n \) between 0 and \( |q| \) (included), compute \( V_q^r(m) \) and \( V_q^r(n) \); in each of them, test whether \( r \) appears as a factor; if so, use Equation \footnote{2} to compute the value of \( f(m, n) \). Otherwise, \( f(m, n) \) is not defined. The computation of \( V_q^r(m) \) and \( V_q^r(n) \), and the search for \( r \), can be done in \( O(|q|) \) time using an optimal string-searching algorithm.

\[ \text{Lemma 3.3.} \] Let \( q, r \) be finite words and \( \{a_1, \ldots, a_k\} \) the set of integers such that the proper overlap \( V_q^r(a_i) \) exists and contains one occurrence of \( r \). Then, for all \( s_1, \ldots, s_n \) in \( \{a_1, \ldots, a_k\} \), the following equation holds:

\[
s_1 + \cdots + s_k = f(s_1, s_2) + \cdots + f(s_k, s_1).
\]

In particular, for all integers \( k, l, m, n \) in \( \{a_1, \ldots, a_k\} \) we have: \( f(m, m) = m \); the relation \( f(m, n) = m \) implies that \( f(n, m) = n \); and the relation \( f(m, k) = f(m, l) \) implies that \( f(n, k) = f(n, l) \).

\[ \text{Proof.} \] Since \( V_q^r(a_i) \) contains exactly one occurrence of \( r \) for all \( i \), Lemma \footnote{3.1} implies that \( V_q^r(a_i, a_j) \) contains exactly two occurrences of \( r \) for all \( i, j \). As a consequence, \( \text{f}_{q, r}(a_i, a_j) \) is always defined. Conversely, \( \text{f}_{q, r}(m, n) \) is not defined if \( \{m, n\} \not\subseteq \{a_1, \ldots, a_k\} \), since \( V_q^r(m, n) \) does not contain two occurrences of \( r \).

For Equation \footnote{3}, first we compute:

\[
f(a_1, a_2) + f(a_2, a_1) = a_1 + \text{occ}(a_1) - \text{occ}(a_2) + a_2 + \text{occ}(a_2) - \text{occ}(a_1) = a_1 + a_2
\]

and then the result is easily proved by induction. The three other facts of this lemma are immediate consequences of Equation \footnote{3} and of the definition of \( f \). \hfill \square
Now we have enough machinery to state conditions on \((q, r)\) which characterize situations where \(q\)-quasiperiodicity implies \(r\)-quasiperiodicity, or implies non-\(r\)-quasiperiodicity.

**Definition 3.4.** Let \(q, r\) denote finite nonempty words of the same length. The couple \((q, r)\) is:

- **compatible** if there exist integers \(m, n\) such that \(f_{q,r}(m, n)\) is defined;
- **definite** if \(f_{q,r}(m, n)\) is defined wherever \(V^*_q(m, n)\) is;
- **positive** if \(f_{q,r}(m, n)\) is defined at least on one couple and is nonnegative wherever it is defined.

Since \(f_{q,r}\) is computable in time \(O(|q|^3)\), those relations are testable with the same time complexity.

**Theorem 3.5.** Let \(q, r\) denote two finite, nonempty words of the same length.

1. The couple \((q, r)\) is non-compatible if and only if \(q\)-quasiperiodicity implies non-\(r\)-quasiperiodicity.
2. The couple \((q, r)\) is definite and positive if and only if \(q\)-quasiperiodicity implies \(r\)-quasiperiodicity.
3. The couple \((q, r)\) is compatible, but not definite positive if and only if there exists a biinfinite word with quasiperiods \(q\) and \(r\), and another biinfinite word with only quasiperiod \(q\).

**Proof.** We prove the three statements separately.

**Statement 1.** Let \(w\) denote \(q\)-quasiperiodic word. It contains a factor of the form \(u = V^*_q(m, n)\). By hypothesis \(f_{q,r}(m, n)\) is not defined, which means that \(u\) contains either 0 or 1 occurrences of \(r\). By Lemma 3.3, at least one position in \(w\) is not covered by \(r\), so, \(w\) is not \(r\)-quasiperiodic.

Conversely, suppose that each \(q\)-quasiperiodic word is non-\(r\)-quasiperiodic. Let \(m, n\) denote a pair of integers such that the proper 3-overlap \(V^*_q(m, n)\) exists and consider the infinite periodic word given by \(w = V_q(\ldots m, n, m, n, n, n, \ldots)\). Since \(w\) is not \(r\)-quasiperiodic, either \(V^*_q(m, n)\) or \(V^*_q(n, m)\) (or both) contains less than two occurrences of \(r\). In other terms, either \(V^*_q(m, n)\) or \(V^*_q(n, m)\) is not defined, and by Lemma 3.3, the other one is not defined either. Since this reasoning holds for any \(m, n\) where \(V^*_q(m, n)\) exists, the function \(f_{q,r}\) is nowhere defined.

**Statement 2.** Suppose \((q, r)\) is definite and positive and consider \(w\) a \(q\)-quasiperiodic biinfinite word. Any position in \(w\) is covered by an occurrence of \(q\); let \(m, n\) denote the integers such that this occurrence is the middle one in the proper 3-overlap \(V^*_q(m, n)\). By hypothesis, \(f_{q,r}(m, n)\) is defined and positive, so \(V^*_q(m, n)\) contains a proper overlap of \(r\). Lemma 3.3 implies that this proper overlap of \(r\) covers the middle occurrence of \(q\). Consequently, any position in \(w\) is covered by an occurrence of \(r\).

Conversely, suppose that \(q\)-quasiperiodicity implies \(r\)-quasiperiodicity. Let \(m, n\) denote an arbitrary pair of integers \(m, n\) such that the proper 3-overlap \(V^*_q(m, n)\) exists and \(w\) denote the periodic biinfinite word given by \(w = V^*_q(\ldots m, n, m, n, m, n, \ldots)\). By hypothesis this word is \(r\)-quasiperiodic, so by Lemma 3.3, the word \(V^*_q(m, n)\) contains a proper overlap of \(r\). Consequently, \(f_{q,r}(m, n)\) is defined and positive.

**Statement 3.** The proof is immediate as this statement exhausts all possibilities not covered by Statements 1 and 2.

\[\square\]

4 On compatible and positive couples of quasiperiods

In this section, we investigate what the property “compatible and positive” implies for a couple of words (not necessarily definite). We get a characterization in terms of derived sequences, and another one in terms of chains of quasiperiods.

The concept derived sequence originates from Mouchard’s work on quasiperiodic finite words \(6\), and was later used by Marcus and Monteil to establish independence results between quasiperiodicity and other properties on right infinite words \(11\). We start by recalling the definition.

**Definition 4.1.** Let \(w\) denote a biinfinite word and \(q\) one of its quasiperiods. The sequence of positions of \(q\) in \(w\) is the sequence \((q_n)_{n \in \mathbb{Z}}\) of positions of occurrences of \(q\) in \(w\), in increasing order, such that \(q_0\) is the position of the leftmost occurrence covering the position 0. If \((q_n)_{n \in \mathbb{Z}}\) is the sequence of positions of \(q\) in \(w\), then \((q_{n+1} - q_n)_{n \in \mathbb{Z}}\) is called the derived sequence of \(w\) along \(q\).
For example, in Equation (1), the derivated sequence of \( w \) along \( q \) is \( \omega(7)(8)^{\omega} \) and the derivated sequence along \( r \) is \( \omega(7)(5)(8)^{\omega} \). Observe that a word is \( q \)-quasiperiodic if and only if its derivated sequence along \( q \) is bounded by \( |q| \). In this case, the derivated sequence contains enough information to reconstruct the initial word.

Chains of quasiperiods were already mentioned in Section 2. Recall the following consequence of Theorem 2.1 and Proposition 2.4 in an infinite word \( w \), the set of quasiperiods of some length \( n \) is given by a union of chains of the form \( \{u_1, \ldots, u_k\} \), where \( u_1 \) is left special, \( u_k \) is right special, no other \( u_i \) is special, and \( u_{i+1} \) is the (unique) successor of \( u_i \) for each \( 1 \leq i < k \). Equation (1) shows an example of a word having two such chains for length 8.

**Theorem 4.2.** Let \( w \) denote a biinfinite word and \( q, r \) denote two quasiperiods of \( w \) of the same length. The following statements are equivalent:

1. the couple \((q, r)\) is compatible and positive;
2. for all word \( w \) having quasiperiods \( q \) and \( r \), the derivated sequences along \( q \) and \( r \) are equal;
3. for all word \( w \) having quasiperiods \( q \) and \( r \), those quasiperiods belong to the same chain.

We actually prove something slightly more precise: the next proposition implies Theorem 4.2.

**Proposition 4.3.** Let \( w \) denote a biinfinite word and \( q, r \) denote two quasiperiods of \( w \) of the same length. The following statements are equivalent:

1. for each integers \( m, n \) such that the proper 3-overlap \( V_q^r(m, n) \) exists and is a factor of \( w \), we have \( f_{q, r}(m, n) \geq 0 \).
2. the derivated sequences of \( w \) along \( q \) and along \( r \) are equal up to a shift of one position;
3. the quasiperiods \( q \) and \( r \) belong to the same chain in \( w \).

The next lemma gives \( 2 \implies 1 \) in Proposition 4.3 because \( x \) and \( y \) are nonnegative integers. However it is actually more general and we will also reuse it later in the proof.

**Lemma 4.4.** Let \( w \) denote an infinite word and \( q, r \) two quasiperiods of \( w \) of the same length. The derivated sequences of \( q \) and \( r \) in \( w \) are the same if and only if: for each pair of natural integers \((x, y)\) such that \( f(x, y) \) is defined, we have \( f(x, y) = x \); or for each such pair \((x, y)\), we have \( f(x, y) = y \).

**Proof.** Call \((q_n)_{n \in \mathbb{Z}}\) the sequence of positions of \( q \) in \( w \), and similarly \((r_n)_{n \in \mathbb{Z}}\) the sequence of positions of \( r \) in \( w \); observe that \( r_{n+1} - r_n = f(q_n - q_{n-1}, q_{n+1} - q_n) \). The fact that \( f(x, y) = x \) and \( f(x, y) = y \) respectively translate to

\[
 f(q_{n+1} - q_n, q_n - q_{n-1}) = q_{n+1} - q_n, \quad \text{and} \quad f(q_{n+1} - q_n, q_n - q_{n-1}) = q_n - q_{n-1}.
\]

By replacing in the previous equation, we get the two possibilities

\[
 r_{n+1} - r_n = q_{n+1} - q_n, \quad \text{and} \quad r_{n+1} - r_n = q_n - q_{n-1},
\]

which both imply that the derivated sequences are equal up to a shift. The converse argument work symmetrically.

The next lemma proves \( 1 \implies 2 \) in Proposition 4.3 which is the most technical part.

**Lemma 4.5.** Let \( w \) denote an infinite word and \( q, r \) two quasiperiods of \( w \) of the same length. Suppose that for each pair of integers \((m, n)\) such that the proper 3-overlap \( V_q^r(m, n) \) exists and is a factor of \( w \), we have \( f_{q, r}(m, n) \geq 0 \). Then the derivated sequences of \( w \) along \( q \) and along \( r \) are identical, up to a shift of one position.
Proof. Let \(\tau_1, \tau_2, \ldots, \tau_m\) denote all the integers such that the proper overlap \(V^*_q(\tau_i)\) exists and is a factor of \(w\); sort the \(\tau_i\) by increasing length. If \(x, y\) are integers such that the proper 3-overlap \(V_q(x, y)\) exists and is a factor of \(w\), then we call \((x, y)\) an occurring couple.

We only need to prove that either for each \(x\) such that \((\tau_1, x)\) is an occurring couple, we have \(f(\tau_1, x) = \tau_1\); or that for each such \(x\), we have \(f(x, \tau_1) = \tau_1\). Indeed, if \(f(\tau_1, x) = \tau_1\) for all \(x\) (or the opposite one), then for each occurring couple \((x, y)\) we have \(f(\tau_1, x) = f(\tau_1, y)\); by Lemma 3.3 we deduce that \(f(x, y) = f(x, x) = x\); subsequently Lemma 4.4 shows that this is sufficient to finish our proof. Therefore now we argue that \(f(\tau_1, x) = \tau_1\) for each \(x\) such that \((\tau_1, x)\) is an occurring couple.

Let \(\sigma_1 = \tau_1\) and \(\sigma_2, \ldots, \sigma_n\) all the integers such that \((\sigma_1, \sigma_i)\) is an occurring couple; sort the \(\sigma_i\) by increasing length (in particular \((\sigma_1)_{1\leq i \leq n}\) is a subsequence of \((\tau_1)_{1\leq i \leq m}\)). The couple \((\sigma_1, \sigma_1)\) is not necessarily an occurring couple, but Lemma 3.3 guarantees that \((\sigma_1, \sigma_1)\) is well-defined and that \(f(\sigma_1, \sigma_1) = \sigma_1\). We can assume that \(f(\sigma_1, \sigma_2) = \sigma_1\); if it is not the case, then \(f(\sigma_2, \sigma_1) = \sigma_1\) by Lemma 3.3 and without loss of generality we consider the function \(f(x, y) = f(x, x)\) instead of \(f\). Now reason by contradiction and consider the smallest integer \(j\) such that \((\sigma_1, \sigma_j) \neq \sigma_1\). By hypothesis we can rule out \(f(\sigma_1, \sigma_j) < 0\), so it remains three cases to analyse.

Case 1. If \(f(\sigma_1, \sigma_j) > \sigma_j\), then use Lemma 3.3 to write \(\sigma_1 + \sigma_j = f(\sigma_1, \sigma_j) + f(\sigma_j, \sigma_j)\), which is equivalent to \(f(\sigma_j, \sigma_1) = \sigma_1 + \sigma_j - f(\sigma_1, \sigma_j)\). The quantity \(\sigma_j - f(\sigma_1, \sigma_j)\) is negative so we have \(f(\sigma_1, \sigma_j) < \sigma_1\), which is a contradiction since \(\sigma_j\) is the smallest possible span.

Case 2. If \(\sigma_1 < f(\sigma_1, \sigma_j) = x < \sigma_j\), then consider the \(q\)-quasiperiodic word whose derived sequence is \(\omega(x \sigma_1 \sigma_j)\); it would have \(f(\sigma_1, x) = \sigma_1\) and \(f(x, \sigma_1) = x\). Thus \(f_q(x, x) = \sigma_1\), but Lemma 3.3 implies \(f_q(x, x) = x\): we have a contradiction.

Case 3. Finally, suppose we have \(f(\sigma_1, \sigma_j) = \sigma_j\). Recall that \(j\) is minimal and that \(j > 2\). By Lemma 3.3 we have \(f(\sigma_1, \sigma_j) = \sigma_1\), and by Lemma 3.3 the relations \(f(\sigma_1, \sigma_1) = f(\sigma_1, \sigma_2)\) and \(f(\sigma_2, \sigma_1) = \sigma_1\) imply that \(f(\sigma_j, \sigma_2) = \sigma_1\) as well. Therefore we have four 3-overlaps of \(q\) with spans \((\sigma_1, \sigma_1); (\sigma_1, \sigma_2); (\sigma_j, \sigma_1); (\sigma_j, \sigma_2); \) all with the same induced \(r\)-overlap, which has span \(\sigma_1\).

![Figure 2: Illustration of Case 3 in the proof of Lemma 4.5](image)

From \(f(\sigma_1, \sigma_1) = f(\sigma_1, \sigma_2) = \sigma_1\) we deduce that \(V_q(\sigma_1, \sigma_1)\) and \(V_q(\sigma_1, \sigma_2)\) have a common factor \(V_q(\sigma_1)\). In either case, the middle occurrence of \(q\) is contained in this factor and last occurrence of \(q\) starts in this factor. This situation is displayed on Figure 2. There exist words \(s_1, s_2\) such that \(Q(\sigma_1)\) is a suffix of \(V_q(\sigma_1)\), and \(V_q(\sigma_2)\) is a suffix of \(V_q(\sigma_1)\). Call \(p_1, p_2\) the words satisfying \(p_1 s_1 = p_2 s_2 = q\), and without loss of generality suppose that \(|p_2| > |p_1|\). Observe that both \(p_1\) and \(p_2\) are suffixes of the same word \(V_q(\sigma_1)\), and \(|p_1| = |p_2| + \sigma_2 - \sigma_1\). As \(p_1, p_2\) are also both prefixes of \(q\), we deduce that \(p_2\) has a period \(\sigma_2 - \sigma_1\). From \(f(\sigma_1, \sigma_2) = f(\sigma_2, \sigma_2) = \sigma_1\) the same argument proves that there exist words \(p'_1, p'_2\) such that \(V_q(\sigma_1)\) is a prefix of \(p'_1 V_q(\sigma_1)\), and \(V_q(\sigma_2)\) is a prefix of \(p'_2 V_q(\sigma_1)\). Call \(s'_1, s'_2\) the words satisfying \(p'_1 s'_1 = p'_2 s'_2 = q\), and without loss of generality suppose that \(|s'_2| > |s'_1|\). Then remark that \(s'_2\) has a period \(\sigma_1 - \sigma_1\). In order to simplify notation in the rest of the proof, let \(\alpha = p_2\) and \(\beta = s'_1\).

We have \(|V_q(\sigma_1)| = 2|q| - \sigma_1 = |q| + |\alpha| + |\beta| - \sigma_2 - \sigma_1\). Therefore \(|q| + \sigma_2 + \sigma_1 = |\alpha| + |\beta|\). Since \(\alpha\) is a prefix and \(\beta\) a suffix of \(q\), by a length argument \(\alpha\) has a non-empty suffix which is a prefix of \(\beta\); call it \(\theta\). We have \(|\theta| = |\alpha| + |\beta| - |q| = \sigma_2 - \sigma_1 \geq (\sigma_j - \sigma_1) + (\sigma_2 - \sigma_1)\). By the Fine-Wilf Theorem
Let \( w \) denote a biinfinite word and \( q, r \) denote two quasiperiods of \( w \) of the same length. The derivated sequences of \( w \) along \( q \) and along \( r \) are equal if and only if, up to swapping \( q \) and \( r \), there exists a chain of quasiperiods \( u_1, \ldots, u_k \) with \( u_1 = q \) and \( u_k = r \), such that \( u_{i+1} \) is the successor of \( u_i \).

Proof. Let \( (q_n)_{n \in \mathbb{Z}} \) and \( (r_n)_{n \in \mathbb{Z}} \) denote the sequences of positions of \( q \) and \( r \) in \( w \). If the derivated sequences are equal up to a shift of one position, then there exists an integer \( k \) in \( \{1, \ldots, |q| - 1\} \) such that for all \( n \) we have \( r_n = q_n + k \). In particular \( r \) always starts at the same position inside \( q \). Differently put, this means that there exists a word \( s \) of length \( k \) such that \( r \) is a suffix of \( qs \) and each occurrence of \( q \) in \( w \) is the prefix of an occurrence of \( qs \). Set \( u_i = (qs)(i, \ldots, i + |r| - 1) \) and the implication is proved.

Conversely suppose that there is a family of quasiperiods \( u_0, u_1, \ldots, u_{k-1} \) such that \( u_0 = q \) and \( u_{k-1} = r \) and \( u_{i+1} \) is the unique successor of \( u_i \) for each \( 0 \leq i < k - 1 \). By Proposition 4.4 none of the \( u_i \) is right special except maybe \( u_{k-1} \). Therefore, there is an occurrence of \( r \) exactly \( k - 1 \) positions after each occurrence of \( q \) in \( w \). Lemma 4.1 ensures that no other occurrences of \( r \) appear in \( w \), therefore we can conclude that \( r_n = q_n + k - 1 \) for each integer \( n \).

Finally, the next lemma gives \( 2 \iff 3 \) in Proposition 4.3.

**Lemma 4.6.** Let \( w \) denote a biinfinite word and \( q, r \) denote two quasiperiods of \( w \) of the same length. The derivated sequences of \( w \) along \( q \) and along \( r \) are equal if and only if, up to swapping \( q \) and \( r \), there exists a chain of quasiperiods \( u_1, \ldots, u_k \) with \( u_1 = q \) and \( u_k = r \), such that \( u_{i+1} \) is the successor of \( u_i \).

**Proposition 4.7.** Let \( q, r \) denote two words of the same length. The couple \((q, r)\) is definite if and only if each \( q \)-quasiperiodic biinfinite word contains infinitely many occurrences of \( r \).

Proof. If \((q, r)\) is definite, then \( f_{q,r}(m, n) \) is defined whenever the proper 3-overlap \( w = V^*_q(m, n) \) exists; therefore each proper overlap of \( q \) contains one occurrence of \( r \). If a biinfinite word is \( q \)-quasiperiodic, then it contains infinitely many occurrences of proper overlaps of \( q \), and therefore infinitely many occurrences of \( r \).

Conversely, suppose that each \( q \)-quasiperiodic infinite word contains infinitely many occurrences of \( r \). Suppose that \( m \) is an integer such that the proper overlap \( V^*_q(m) \) exists, but does not contain an occurrence of \( r \). Then the periodic biinfinite word given by \( w = V^*_{\bar{q}}(\ldots, m, m, \ldots) \) does not contain any occurrence of \( r \); but it should also contain infinitely many occurrences of \( r \) by hypothesis. Therefore we have a contradiction and each proper overlap \( V^*_q(m) \) contains an occurrence of \( r \), which shows that \((q, r)\) is definite.

**5 Quasiperiods of biinfinite Sturmian words**

If \( w \) is an infinite word (either indexed by \( \mathbb{N} \) or by \( \mathbb{Z} \)), then \( P_w(n) \) denotes the number of distinct factors of length \( n \) in \( w \) and \( P_w(n) \) is the complexity function of \( w \). An infinite word is Sturmian if and only if it is not ultimately periodic and satisfies \( P_w(n) = n + 1 \) for each integer \( n \). Equivalently, a word is Sturmian if and only if it is not ultimately periodic and has exactly one right special factor and one left special factor of each length. Sturmian words are an important and well-studied class of infinite words (see [9] Chapter 2 and [2] Chapter 6)). Now we determine the set of quasiperiods of any biinfinite Sturmian word. To this end, if \( w \) is a biinfinite word, let \( Q_w(n) \) denote the number of quasiperiods of length \( n \) in \( w \).

**Theorem 5.1.** Let \( w \) denote a biinfinite Sturmian word and \( n \) a nonnegative integer.
1. We have $Q_w(n) = 0$ if and only if $w$ has a nonempty bispecial factor of length $n - 1$.

2. If $Q_w(n) > 0$ and $s$ denotes the shortest bispecial factor of $w$ with $|s| \geq n$, then quasiperiods of length $n$ in $w$ are exactly the factors of length $n$ in $s$.

Proof. We prove the two statements separately.

Statement 1. If $w$ has a nonempty bispecial factor of length $n - 1$, say $u$, then there exists a letter $\alpha$ such that the (unique) right special factor of length $n$ in $w$ is $\alpha u$, because any suffix of a right special factor is also right special. By Theorem 2.1 the word $\alpha u$ has no quasiperiod of length $n$ had a quasiperiod of length $n$ such that the (unique) right special factor of length $n$ have a bispecial factor of length $n$. Consequently $w$ has no quasiperiod of length $n$.

Conversely suppose that $w$ has no bispecial factor of length $n - 1$. Since $w$ is Sturmian, it has exactly one right-special and one left-special factor of length $n$, so its set of factors of length $n$ may be written $\{c_0, \ldots, c_{k-1}\} \cup \{d_0, \ldots, d_{\ell-1}\} \cup \{e_0, \ldots, e_{m-1}\}$, where $c_{k-1}$ is right special and has successors $d_0$ and $e_0$; both $d_{\ell-1}$ and $e_{m-1}$ have successor $c_0$, which is left special; each other $c_i$, $d_i$ and $e_i$ has respectively $c_{i+1}$, $d_{i+1}$ and $e_{i+1}$ as an (unique) successor. We have $k \geq 1$, but we might have $m = 0$ or $\ell = 0$. Figure 3 shows a graph of the “successor” relation. Observe that $k + \ell + m = n + 1$ and $k > 2$ (otherwise, we would have a bispecial factor of length $n - 1$). As a consequence, the maximal distance between two consecutive occurrences of $c_0$ is $\max(k + \ell, k + m)$, which is bounded by $n$. In other terms, $c_0$ is a quasiperiod of $w$.

![Figure 3: Successor graph (usually called Rauzy graph) of factors of length $n$ of a Sturmian word](image)

Statement 2. Let $w$ denote a biinfinite Sturmian word and suppose that $Q_w(n) > 0$ for some integer $n$. The left special and the right special factors of length $n$ of $w$, call them $\ell$ and $r$, are both quasiperiods by Proposition 2.4. Call $s$ the shortest factor of $w$ having $\ell$ as a prefix and $r$ as a suffix. The set of factors of length $n$ of $s$ is given by a sequence $u_1, u_2, \ldots, u_k$, where $u_1 = \ell$ and $u_k = r$, such that $u_{i+1}$ is the successor of $u_i$ for each $1 \leq i < k$. By Proposition 2.4 again, the set $\{u_1, \ldots, u_k\}$ is the set of quasiperiods of length $n$ of $w$. Observe that $s$ is, by definition, exactly the shortest bispecial factor of $w$ not shorter than $n$.

Since any Sturmian word has infinitely many bispecial factors, whose difference between consecutive lengths are unbounded, we have:

**Corollary 5.2.** Each biinfinite Sturmian word $w$ has infinitely many quasiperiods. Moreover, $Q_w$ is unbounded.

## 6 Conclusion

As explained in the introduction, the biinfinite case may give nicer results about quasiperiodicity of subshifts and of Sturmian words. This paper provided a toolbox to study the quasiperiods of biinfinite words, but many questions are still to be answered.

1. An $N$-word $w$ is periodic if and only if $Q_w(n) > 0$ for each large enough $n$. Is it possible to characterize ultimately periodic $Z$-words in terms of quasiperiods?
2. An $N$-word $w$ is standard Sturmian if and only if it satisfies $Q_w(n) = 0$ exactly when there is a bispecial factor of length $n - 1$ in $w$. Is there a characterization of biinfinite Sturmian words in terms of quasiperiods?

3. What about other families of low-complexity sequences, such as episturmian or Arnoux-Rauzy sequences?

4. If an $N$-word is multi-scale quasiperiodic, then it is uniformly recurrent \[11\]. It is easy to construct a $\mathbb{Z}$-word which is multi-scale quasiperiodic but not uniformly recurrent: the word $\omega^*(ba) \cdot (ab)^n$ has quasiperiod $a(ba)^n$ for each positive $n$, but the factor $aa$ occurs only once. Are all such words ultimately periodic? If not, how can they be characterized?

5. If a biinfinite word has infinitely many quasiperiods, does it necessarily have quasiperiodic derivated sequences? If not, can we build a counter-example which is uniformly recurrent?

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