Generalized Kontsevich Model
versus Toda hierarchy
and discrete matrix models

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ABSTRACT

We represent the partition function of the Generalized Kontsevich Model (GKM) in the form of a Toda lattice $\tau$-function and discuss various implications of non-vanishing "negative"- and "zero"-time variables: the appear to modify the original GKM action by negative-power and logarithmic contributions respectively. It is shown that so deformed $\tau$-function satisfies the same string equation as the original one. In the case of quadratic potential GKM turns out to describe forced Toda chain hierarchy and, thus, corresponds to a discrete matrix model, with the role of the matrix size played by the zero-time (at integer positive points). This relation allows one to discuss the double-scaling continuum limit entirely in terms of GKM, i.e. essentially in terms of finite-fold integrals.
1 Introduction

In [1] a new matrix model was introduced which interpolates between all the non-perturbative partition functions of Virasoro \((q, q')\)-minimal string models with \(c = 1 - \frac{6(q - q')^2}{qq'}\).

The partition function of GKM depends on two distinct sets of time-variables: one entering the “potential” \(V(X) = \sum_{p \geq 1} s_p X^p\), and the other one is connected by a kind of Miwa relation with auxiliary matrix \(M\):

\[
t_p = \frac{1}{p}[Tr M^{-p} + (p - 1)s_p]. \tag{1.1}
\]

For the particular choice of parameters (potential \(V(X) = \text{const} \cdot X^{q+1}\)) it coincides with the description of \((q, q')\)-series. Interpolation as given by GKM preserves both nice properties of the known non-perturbative partition functions: if considered as functions of \(t\)-variables these are usually \(\tau\)-functions of integrable KP hierarchy and usually satisfy “generalized string equation” \(L^{(V)}_1 \tau = 0\). These facts make GKM an appealing object to study in the context of non-perturbative and unified string theory.

This paper describes some new enlightening results about GKM. The question to be addressed is what is the meaning of GKM from the point of view of the Toda hierarchy. The point is that the KP hierarchy can be always embedded into the Toda lattice hierarchy, but the latter one has additional variables: “negative times” and “zero-time”, which are frozen in KP case. The natural thing to ask is whether GKM can be further generalized to include the dependence on these additional parameters so that non-perturbative partition function becomes \(\tau\)-function of the Toda lattice hierarchy.

Remarkably it appears possible not only to introduce the extra “zero-time” variable into GKM, but this is also a natural way to describe discrete matrix models, thus giving a chance to unify the entire theory of matrix models under a common roof of GKM.

The natural form in which the non-perturbative partition functions arise, when de-

\[1\]E.g. for \(q = 2\) – with Witten’s topological gravity [2], which has been represented in the form of a matrix model by Kontsevich [3].

\[2\]We use the term “non-perturbative partition function” for the generating functional of all the exact \(i.e.\) non-perturbative or appropriately summed over all orders of perturbation theory) correlation functions in string models. This object may be considered as a vacuum amplitude in the theory with the action, to which all the possible vertex operators (corresponding both to naive observables and to handle-
duced from the matrix models, is determinant of $N \times N$ matrix, probably with some extra simple (normalization) factors in front of it. Integrable $\tau$-functions are also representable through determinant formulae, but somewhat different expressions are adequately describing different hierarchies. This representation of KP $\tau$-function arises naturally in Miwa parameterization of time-variables $t_p$, i.e. essentially in terms of eigenvalues $\{m_i\}$ of the $N \times N$ matrix $M$, and looks like

$$\tau_{\{V\}}[t] = \frac{\det \phi_i(m_j)}{\Delta(m)},$$

(1.2)

where $\{\phi_i(m) = m^{i-1}(1 + O(\frac{1}{m}))\}$ is a Segal-Wilson basis, describing some point in the Sato’s Grassmannian (see all the details and references in [1]). On the other hand, the natural determinant formula for the Toda hierarchy is (see Appendix A, eq.(A.25))

$$\tau_n[t] = \det_{(ij)} H_{i+n,j+n}[t],$$

(1.3)

where $H_{ij}$ satisfy the following equations:

$$\frac{\partial H_{ij}}{\partial t_p} = H_{i,j-p} \text{ for “positive times” } t_p,$$

$$\frac{\partial H_{ij}}{\partial t_{-p}} = H_{i-p,j} \text{ for “negative times” } t_{-p},$$

(1.4)

and $n$ is integer-valued “zero-time”.

In what follows we need more statements from the Toda theory. The forced Toda-lattice hierarchy arises if

$$\tau_{-n}[t] = \delta_{n,0} \text{ for all } n \geq 0.$$  

(1.5)

The reduction of general Toda lattice hierarchy to the Toda chain one occurs, for example, when $H_{ij}$ depends only upon the difference $i-j$: $H_{ij} = H_{i-j}$ [3] (More details and references gluing operators) are added with arbitrary coefficients (which are nothing but the time-variables). Such quantity, though a priori constructed for one particular string model, a posteriori is naturally describing a family of models as large as the freedom, allowed in the deformations of the action. For a sufficiently rich family of deformations the quantity, describing the entire string theory (i.e. unification of all the string models), may be derived starting from any particular string model.

3The relations of this type, having a sense under the determinant, should be understood up to lower or upper triangle transformations admissible in the determinant.
concerning Toda hierarchies can be found in [4], see also (2.20) below).

The partition function of GKM has been already proved in [1] to be a KP $\tau$-function and can be explicitly represented in the form of (1.2). Our first goal below will be to bring it to the form of (1.3). Usually KP hierarchy can be embedded into Toda lattice hierarchy at fixed values of zero- and negative-times:

$$\tau_n[t_p; t_p] = \tau_{KP}^{\{nt_p\}}[t_p].$$  

(1.6)

In other words, the point of Grassmannian associated with the KP $\tau$-function at the r.h.s. depends on $t_p$. This is also clear from the free-field representation of $\tau$-functions (see Appendix A): a Toda-lattice $\tau$-function,

$$\tau_n^{\{g\}}[t_p; t_p] \sim <n|e^{-\sum t_p J_p g} e^{-\sum t_{-p} J_{-p}}|n>$$  

(1.7)

can be considered as a KP $\tau$-function,

$$\tau_{KP}^{\{g; nt_p\}}[t_p] \sim <n|e^{-\sum t_p J_p g_{KP}}|n>$$  

(1.8)

with $t_{-p}$-dependent $g_{KP} = g e^{-\sum t_{-p} J_{-p}}$. In GKM the point $g$ is specified by potential $V(X)$ (this relation is conventionally encoded in the form of the “string equation”). We shall see that $n$- and $t_{-p}$-dependencies, as defined by eqs.(1.6) and (1.8), can be imitated by additional logarithmic and negative-power terms in the action of GKM. This is a sort of deformation of GKM in the sense that the original string equation [1] still remains to be valid. A natural question is what are the restrictions on GKM, which imply the occurrence of the forced Toda hierarchy. Essentially these requirements are for all the functions $\phi_i(m)$ to be polynomial in $m$ (up to trivial factor, see, for example, eq.(3.24)), and this is the case if $V(X) = X^2/2$. The GKM as defined in [1] (i.e. with $n = t_{-p} = 0$)

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4We systematically use the notation [...] to denote the dependence of a family of arguments like eigenvalues of $M$ or all the time-variables (negative or positive). However, when the same objects are considered as parameters we put them into braces: {...}. For example, negative times $t_{-p}$ are arguments of Toda lattice $\tau$-function, but they are parameters of KP $\tau$-function (they define the point in Grassmannian, but are not the time variables which are conventionally considered as arguments of $\tau_{KP}$). The main purpose of the study matrix models is certainly to get rid of this delicate difference, i.e. to treat all the parameters (which label different string models) as arguments.
is trivial for such potential, but as we shall see it becomes more interesting at $n \neq 0$ and $t_{-p} \neq 0$.

The second big problem addressed in this paper is the representation of discrete matrix models in the form of GKM. One should suggest that such representation exists if we indeed want GKM to be a kind of a universal matrix model (encoding all the information about the KP hierarchy and Grassmannian, which is relevant for string theory). This problem is also related to the Toda theory, since partition functions of discrete matrix models are known to be $\tau$-functions of forced Toda-lattice hierarchy (or even its Toda-chain reduction in the one-matrix case) [6]. This problem is, in fact, a kind of inverse of the previous one: now we need to convert the models with characteristic Toda-type representation (1.3) into KP-related form (1.2), which is peculiar for GKM. As usually, such conversion is provided by Miwa transformation (1.1), moreover the resulting $\{\phi_i(m)\}$ in (1.2) are also orthogonal polynomials (up to trivial factor). As a particular result, we explicitly prove the equivalence of discrete Hermitean one-matrix model and GKM with $V(X) \sim X^2$ and the zero-time $n$ is identified with the size of the matrix in the discrete model.

The fact that discrete matrix models can be naturally embedded into GKM, allows one to formulate and study the “double”-scaling continuum limits as internal problem of GKM, where it becomes just a question about asymptotic formulas for families of integrals. We present some naive results about such interpretation of continuum limits, but its explicit relation to conventional Kazakov’s procedure (as described in full details in [7]) remains still a bit obscure.

2 Partition function of GKM as a Toda-lattice $\tau$-function

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5The particular model (with $t_{-p} = 0$ but $n \neq 0$) has been proposed and discussed to some extent in a very recent paper [5], which stimulated us to complete this research. Despite our disagreement with the main claim of that paper (that GKM with $V(X) \sim X^2$ and $n > 0$ describes $c = 1$ string model) we shall refer in appropriate place to the proper Ward identity (in fact, the system of discrete Virasoro constraints [6]), which is derived there.
2.1 Interrelation between KP and Toda-lattice $\tau$-functions

The purpose of this subsection is to describe without any special reference to GKM the explicit relation between KP-like (in Miwa variables),

$$\tau_{KP}[t_p] = \frac{\det_{ij} \phi_i(m_j)}{\Delta(m)}$$  \hfill (2.1)

and Toda-like,

$$\tau_n[t_{-p}, t_p] = \det_{ij} H_{i+j+n}[t_{-p}, t_p],$$ \hfill (2.2)

representations of $\tau$-functions, where

$$\Delta(m) = \prod_{i>j}(m_i - m_j),$$ \hfill (2.3)

$$\phi_i(m) = m^{i-1}(1 + \mathcal{O}(\frac{1}{m})),$$ \hfill (2.4)

$$t_p = \frac{1}{p} \sum_i m_i^{-p}, \ p > 0,$$ \hfill (2.5)

$$\partial H_{ij} / \partial t_p = H_{i,j-p}, \ j > p > 0,$$ \hfill (2.6)

and

$$\partial H_{ij} / \partial t_{-p} = H_{i-p,j}, \ i > p > 0.$$ \hfill (2.7)

(the origin of these formulas, concerning with Toda hierarchy, is explained in Appendix A).

Relation between (2.1) and (2.2) is formulated in terms of the Schur polynomials, which are defined by:

$$\mathcal{P}[z|t_p] \equiv \exp\{\sum_{p>0} t_p z^p\} = \sum z^k P_k[t],$$ \hfill (2.8)

e.g. $P_n = 0$ for any $n > 0$; $P_0[t] = 1$; $P_1[t] = t_1$; $P_2[t] = t_2 + \frac{1}{2}t_1^2$; $P_3[t] = t_3 + t_2t_1 + \frac{1}{6}t_1^3$

e.t.c. The crucial property of Schur polynomials is:
\[
\frac{\partial P_k}{\partial t_p} = P_{k-p}
\quad (2.9)
\]

(this is just because \(\frac{\partial \mathcal{P}}{\partial t_p} = z^p \mathcal{P}\)). This feature allows one to express all the dependence on time-variables of \(H_{ij}[t]\), which satisfies eqs. (2.6) and (2.7), through Schur polynomials (see (A.26) in Appendix).

\[
H_{ij}[t-p, t_p] = \sum_{k \leq i, l \geq -j} P_{i-k}[t-p] T_{kl} P_{l+j}[t_p],
\quad (2.10)
\]

where \(T_{kl} = H_{kl}[0,0]\) is already a \(t\)-independent matrix. Note that \(H_{ij}\) is defined by eq.(2.10) for all (positive or negative) integer values of \(i, j\). Somewhat different, the Grassmannian point, \(\{\phi_i(m)\}\), in KP-formula (2.1) is \textit{a priori} defined with \(i \geq 0\).

Therefore, we begin our consideration from the case, when all \(n = t_{-p} = 0\), then look what happens if \(n > 0\), discuss the continuation to negative values of \(n\) and introduction of \(t_{-p}\)-variables. At the end of the subsection we discuss the conditions for \textit{forced} and/or Toda-\textit{chain} reductions to occur.

Given the system of basic vectors \(\phi_i(m)\) for \(i > 0\), we put by definition

\[
H_{ij}[t_{-p} = 0, t_p] = \oint_{z \to 0} \phi_i(z) z^{-j} \mathcal{P}[z|t_p] dz, \quad i > 0.
\quad (2.11)
\]

The integration contour is around zero and it can be deformed to encircle infinity and the singularities of \(\mathcal{P}[z]\), if any. If we just substitute the definition (2.8) of \(\mathcal{P}[z]\) into (2.11), we get (2.10) with \(P_{k-i}[t_{-p} = 0] = \delta_{ki}\) and

\[
T_{kl} = \oint_{z \to 0} \phi_k(z) z^l dz.
\quad (2.12)
\]

In order to prove the identity between (2.1) and (2.2) under the condition (2.5), note that (2.5) implies that

\[
\mathcal{P}[z|t_p] = \frac{\det M}{\det(M - I z)} = \prod_i \frac{m_i}{(m_i - z)} = \left[ \prod_i m_i \right] \sum_k \frac{(-1)^k}{(z - m_k)} \Delta_k(m),
\]

where \(\Delta_k(m) \equiv \prod_{i>j, i,j \neq k} (m_i - m_j)\). Note that eigenvalues \(m_k = \infty\) do not contribute to \(\mathcal{P}[z|t_p]\). If there are exactly \(N\) finite eigenvalues \(m_k \neq \infty\), then the point \(z = \infty\) does not

\footnote{We manifestly write down the limits of this sum only for convenience as they are given automatically by properties of Schur polynomials.}
contribute to the integral (2.11), at least, for \( i \leq N \). It picks up contributions only from the poles of \( P[z|t_p] \) at the points \( m_k \):

\[
H_{ij}[t_p = 0, t_p] = \oint_{z \to 0} \phi_i(z) z^{-j} P[z|t_p] dz = \frac{\prod_i m_i}{\Delta(m)} \sum_k (-)^k \phi_i(m_k) \frac{\Delta_k(m)}{m_k^j}.
\]

The sum at the r.h.s. has a form of a matrix product and we conclude that

\[
\det H_{ij} = \det \phi_i(m_k) \prod_k \left[ \frac{\Delta_k(m)}{\Delta(m)} (-)^k \Delta_k(m) \right] \cdot \det \frac{1}{m_k^j}.
\]

The last determinant at the r.h.s. is equal to \( \Delta(1/m) = (-)^{N(N-1)/2} \Delta(m) \cdot \left[ \prod_k m_k^N \right]^{-1} \).

Note also that \( \prod_k \frac{\Delta_k(m)}{\Delta(m)} = \Delta(m)^{-2} \), and gathering all this together, we see that the r.h.s. of (2.12) is indeed equal to

\[
\det H_{ij} = \frac{\det \phi_i(m_j)}{\Delta(m)},
\]

as required.

Proceed now to introducing of zero- and negative-time variables. The zero-time \( n \) arises just as the simultaneous shifts of indices \( i \) and \( j \) of \( H_{ij} : H_{ij} \to H_{i+n,j+n} \), see (2.2). We can use eq.(2.11) to write:

\[
H_{i+n,j+n}[0, t_p] = \oint_{z \to 0} \phi_i^{(n)}(z) z^{-j} P[z] dz
\]

with

\[
\phi_i^{(n)}(z) = z^{-n} \phi_i(z)^n.
\]

This exhausts the problem of restoring the \( n \)-dependence for positive integer values of \( n \). As to \( n < 0 \), eq.(2.15) can be used only if original set \( \{ \phi_i(z) \} \) is enlarged to include \( \phi_i(z) = z^{i-1}(1 + O(1/z)) \) with negative \( i \). Such extension is not limited by any additional constraints and is not unique: this is exactly all the new information, introduced when

\[
\text{Let us note that the relations } H_{ij}^{(n)} = H_{i+n,j+n} \text{ and } \phi_i^{(n)} = z^{-n} \phi_i(z) \text{ are correct only for the special choice of } H \text{ and } \phi \text{ (remind that they are defined up to triangle transformation). Moreover, the choice of } H \text{ fixes } \phi \text{ unambiguously due to eq.(2.11): } H_{i+n,j+n} = (\phi_i^{(n)} z^{-j}) = (\phi_i^{(n-k)} z^{-k} z^{-j}) \text{ for any } j, \text{ i.e. all moments of functions } \phi_i^{(n)} \text{ and } \phi_i^{(n-k)} z^{-k} \text{ are coincide. This property, being evidently correct for GKM } \phi_i^{(n)}, \text{ singles out the latter.}
\]
proceeding from KP to Toda-lattice hierarchy\footnote{In the fermionic language of Appendix A this means that the form of KP \( \tau \)-function allows one to restore the element \( g \) of \( GL(\infty) \), which defines the point of Grassmannian, only up to a part, which cancels the vacuum \( |n\rangle \) \( (n = 0 \text{ for conventional KP}) \). Therefore, the choice of the negative \( n \) vacuum gives some additional information about \( g \) in compare with \( n = 0 \) (the same information can be, certainly, obtained from the form of the negative time dependence). Thus, generally speaking, there are plenty ways to continue \( g \) to “negative part”. But it is not the case if one respects particular reduction. For example, the condition of Toda chain reduction unambiguously fixes the continuation to negative \( n \) \cite{4}.}. If this extension is chosen, \( H_{ij}[t_p = 0, t_p] \) and the entire matrix \( T_{kl} \) with \textit{any} integer (positive or negative) \( i, j, k, l \) are defined. From the point of view of Grassmannian the integer-valued zero-time \( n \) labels connected components of Grassmannian, consisting of the vector sets \( \{ \phi_{i+n}(z), i \geq 0 \} \).

As for negative-times, as soon as \( T_{kl} \) is defined, they are introduced with the help of \eqref{eq:2.10} and

\begin{equation}
H_{i+n,j+n}[t_p, t_p] = \sum_{k \leq i} P_{i-k}[t_p] H_{k+n,j+n}[0, t_p] = \int_{z \to 0} \phi_i^{(t_p-n)}(z) z^{-j} P\left[\frac{1}{z} |t_p| \right] P[z | t_p] dz,
\end{equation}

with

\begin{equation}
\phi_i^{(t_p-n)}(z) = \left\{ P\left[\frac{1}{z} |t_p| \right] \right\}^{-1} \sum_{k \leq i} P_{i-k}[t_p] \phi_k^{[n]} = z^{-n} \exp \left\{ - \sum_{p \geq 0} t_p z^{-p} \right\} \sum_{k \leq i} P_k[t_p] \phi_{i+n-k}(z).
\end{equation}

The role of the exponential prefactor in \eqref{eq:2.17} is to guarantee the proper asymptotic behaviour

\begin{equation}
\phi_i^{(t_p-n)}(z) = z^{-i-1} \{ 1 + \mathcal{O}\left(\frac{1}{z}\right) \}.
\end{equation}

Note that the quantities defined by eqs.\eqref{eq:2.16} and \eqref{eq:2.17} depend crucially on \( \phi_i \) with \( i < 0 \) when “zero-time” \( n \) is negative.

Eqs.\eqref{eq:2.16} and \eqref{eq:2.17} provide a complete description of the interrelation between KP and Toda-lattice hierarchies. Given a KP \( \tau \)-function in the form of \eqref{eq:2.1} (\textit{i.e.} in Miwa coordinates \eqref{eq:2.5}), it can be interpreted as the Toda-lattice \( \tau \)-function at the vanishing values of \( n \) and \( \{ t_p \} \). When zero-time \( n \) is introduced, it corresponds to discrete shifts
along disconnected components of Grassmannian, while evolution along negative-times $t_{-p}$ is a smooth movement of a point in Grassmannian, explicitly described by (2.17). (In other words, at any given values of $n$ and $\{t_{-p}\}$ the Toda lattice $\tau$-function (2.2) as a function of positive-times $\{t_p\}$ can be considered as KP $\tau$-function, but the associated component and particular point of Grassmannian are different for different $n$ and $\{t_{-p}\}$.)

Further simplification of (2.17) can be achieved only for some particular choices of the basis $\{\phi_i(z), i \in \mathbb{Z}\}$. A drastic simplification arises for basises, relevant for matrix models (i.e. consistent with string equations). Then $\phi_i(z)$ are essentially of the form $\langle x^{i-1} \rangle_z$ (with certain linear averaging operation $< \ldots >_z$) (like contour integral representation)\footnote{This interpretation of string equation seems to be a very promising view on its meaning. Details are, however, beyond the scope of this paper.} and introducing of $n$ and $\{t_{-p}\}$ may be naturally described as a change of the “measure” from $< \ldots >_z$ to

$$\left\langle \ldots \left[ \frac{x}{z} \right]^n \exp \left\{ \sum_{p \geq 0} t_{-p} (x^{-p} - z^{-p}) \right\} \right\rangle_z.$$ (2.19)

Moreover, this formula allows one to consider $n$ as continuous rather than discrete parameter. This makes possible an “analytic continuation” in $n$ and, thus, implies a “natural” definition of $\phi_i$ with negative $i$. Before we proceed to a more detailed discussion of this situation in the next section, let us comment briefly on two important reductions of the Toda-lattice hierarchy.

The first reduction which is of importance as it is just the case in matrix models is already mentioned forced hierarchy. It was firstly introduced in [8,4] for the Toda chain but can be easily extended to Toda lattice case. The most manifest way to give this reduction is to constrain the element $g$ to give some subspace in Grassmannian. This is done in Appendix $B$, but here we would like to say some words in our previous framework.

That is, instead of half-infinite determinant in (2.2) let us consider finite determinant of the size $n \times n$ for $\tau_n$:

$$\tau_n = \det_{n \times n} H_{i,j}.$$ (2.20)
It just gives a \( \tau \)-function of forced hierarchy. Moreover, due to identities (2.6)–(2.7) it can be rewritten as

\[
\tau_n = \det_{n \times n} \partial^{i-1} \bar{\partial}^{j-1} H
\]  

(2.21)

with \( \partial \equiv \partial/\partial t_1, \bar{\partial} \equiv \partial/\partial t_{-1}, \) \( H \equiv H_{1,1} \) and \( H \) is constrained to satisfy \( \partial H/\partial t_p = \partial^p H, \) \( \partial H/\partial t_{-p} = \bar{\partial}^p H. \) This \( \tau \)-function can be produced in formalism of orthogonal polynomials implying polynomial Baker-Akhiezer function as we shall see in the sect.3.1. It explains why matrix models correspond to forced hierarchies.

For all these it is crucially to work in the sector with positive \( n. \) The problem of continuation of forced hierarchies to negative zero-time we have already discussed previously [4].

Another important reduction from Toda lattice is Toda chain (see, for example, [10]). It can be easily written both in terms of element \( g ( [g, J_k + J_{-k}] = 0 \text{ – see (A.29) and comments there} \) and in determinant form. Latter one merely implies the symmetry property:

\[
[ H , \Lambda + \Lambda^{-1} ] = 0,
\]  

(2.22)

where \( \Lambda \) is shift matrix \( \Lambda_{ij} \equiv \delta_{i,j-1}. \) This condition leads to \( \tau \)-function of Toda chain hierarchy (proper rescaled by exponential of bilinear form of times) which depends only on the sum of positive and negative times \( t_p + t_{-p}, \) but not on their difference (one can consider this as defining property of Toda chain hierarchy). Let us remark that one possible solution to constraint (2.22) is matrix \( H_{i,j} = H_{i-j}. \) We can combine both

\(^{10}\)Let us point out that this kind of solutions to Toda lattice hierarchy was firstly invented by Leznov and Saveliev [9]. Let us also note that solutions of Wronskian type [10] rather like forced ones turn out to be absolutely different in their properties.

\(^{11}\)Let us emphasize that usual notations correspond to the reduction to \textit{tau}-function independent of the sum of times. But due to non-standard sign of the first exponential in (1.7) which simplifies a lot of formulas in this paper, the reduction (2.25) corresponds to \( \tau \)-function which depends only on the sum of times.

\(^{12}\)It should not be certainly independent of the difference of times \( t_p - t_{-p}, \) but one can through out negative times as the final answer for complete objects like \( \tau \)-function should be really independent of this difference.
reductions to reproduce forced Toda chain hierarchy. In this case one can easily transform $H_{i-j}$ to matrix $\hat{H}_{i+j}$ by permutations of columns what does not effect to the determinant. This matrix just corresponds to one-matrix model case [4,6] (see also sect.3.1). Thus, we consider again the determinant of size $n \times n$, which now can be represent in the form:

$$\tau_n = \det_{n \times n} \partial_{i+j} H$$



(2.23)

where $\partial \equiv \partial/\partial t_1$, $\partial H/\partial t_p = \partial^p H$ and we canceled negative times (see footnote 12). Like the Toda lattice case, the forced Toda chain is unambiguously continuable to negative values of zero-time leading to (1.5)\textsuperscript{[13]}

\section{GKM in the context of Toda lattice hierarchy}

The purpose of this subsection is to introduce zero- and negative-time variables into GKM in such a way that its partition function becomes a $\tau$-function of the Toda lattice hierarchy.

Let us remind the definition of GKM at $n = t_{-p} = 0$. It was introduced in [1] by the following matrix integral:

$$Z_{\{V\}}[M] = \frac{\int e^{-Tr\{V(X+M) - V(M) - V'(M)X\}} dX}{\int e^{Tr\{-V_2(X,M)\}} dX}$$

(2.24)

with $N \times N$ Hermitean matrices $X, M$ and $V_2(X, M) \equiv \lim_{\epsilon \to 0} \epsilon^{-2}[V(M + \epsilon X) - V(M) - \epsilon V'(M)X]$. As explained in details in [1], the r.h.s. of (2.24) may be represented in the form of (2.1) and thus $Z_{\{V\}}[M]$ is the KP $\tau$-function, which has no explicit dependence on $N$ ($N$ is just the number of finite eigenvalues of the matrix $M$, when others can be considered as infinitely large). The relevant set of functions $\{\phi_i(m)\} – the point in Grassmannian – is given by the following integral formula:

$$\phi_i^{\{V\}}(m) = e^{V(m) - mV'(m)} \sqrt{V''(m)} \int dx \ x^{i-1} e^{-V(x) + xV'(m)} \equiv \phi_i \equiv s(m) \int dx \ x^{i-1} e^{-V(x) + xV'(m)} \equiv \left\langle x^{i-1} \right\rangle_m$$

(2.25)

\footnote{This continuation is unambiguous only with taking into account the negative times. With cancelled negative times, \textit{i.e.} in KP case, one can obtain plenty different continuations like, for example, CKP $\tau$-function which satisfies $\tau_n[t_k] = \tau_{-n}[(-)^k t_k]$ [4].}
(the integral in (2.25) is a contour integral, i.e. $x$ not a matrix). Dependence of $n$ and $t_{-p}$ is now introduced by the rule (2.19):

$$\phi_i^{\{V,n,t_{-p}\}}(m) \equiv \left\langle x^{i-1} \left[ \frac{x}{m} \right]^n \exp \left\{ \sum_{p \geq 0} t_{-p} (x^{-p} - m^{-p}) \right\} \right\rangle_m =$$

$$= \sqrt{V''(m)} e^{V(m) - mV''(m)} \int dx \ x^{n+i-1} e^{-V(x) + xV'(m)} \exp \left\{ \sum_{p \geq 0} t_{-p} (x^{-p} - m^{-p}) \right\} =$$

$$= e^\hat{V}(m - mV'(m)) \sqrt{V''(m)} \int dx \ x^{i-1} e^{-\hat{V}(x) + xV'(m)},$$

where

$$\hat{V}(X) \equiv V(X) - n \log X - \sum_{p \geq 0} t_{-p} X^{-p} = \sum_{p=-\infty}^{\infty} s_p \frac{X_p^p}{p}, \quad (2.27)$$

with

$$t_p = \frac{1}{p} [Tr M^{-p} + (p - 1)s_p],$$

$$t_0 = n = -s_0,$$

$$t_{-p} = -s_{-p}/p. \quad (2.28)$$

As to original potential $V(m)$ it can be identified with $\hat{V}_+$. We repeat that from the point of view of GKM $t_0 = -s_0$ does not need to be integer (though it is desirable to avoid a too complicated analytical structure of $\hat{Z}[M]$ – see (2.29) below). From (2.26) we immediately conclude (just repeating the arguments of [1] in the opposite direction) that the partition function of GKM, involving zero- and negative-times (and automatically being a Toda lattice $\tau$-function), is just:

$^{14}$Let us point out that the exponential of negative powers in normalization does not essentially effect to the KP $\tau$-function as it reduces to trivial exponential of bilinear form of times in front of $\tau$-function and corresponds to the freedom in its definition [1]. Indeed, $\tau \sim \det \{ \exp[\sum_k a_k z_i^{-k}] \phi_i(z_j) \} \sim \prod_i \exp[\sum a_k z_i^{-k}] \det \phi_i(z_j) \sim \exp[\sum k a_k t_k] \det \phi_i(m_j)$. This factor, certainly, effect to the string equation, however, results into trivial bilinear form of times (see subsection 2.3, eq.(2.37)).

$^{15}$All the integrals are certainly defined by analytic continuation and are not quite unambiguous because of Stokes phenomenon etc.
\[ \hat{Z}_{\{\hat{V}\}}[M] = e^{Tr\hat{V}(M)-TrMV'_+(M)} \frac{\int dX \ e^{-Tr\hat{V}(X) + Tr\hat{V}'(M)X}}{\int dX \ e^{-Tr\hat{V}+(X,M)}}. \]  

(2.29)

Occurrence of GKM in the somewhat unexpected form of eq.(2.29) (involving projection \( \hat{V} \rightarrow \hat{V}_+ \)) makes the whole subject even more intriguing than it was before. Our next purpose is to describe the associated modification of the second crucial ingredient of GKM: the string equation (the first ingredient is integrability).

### 2.3 Generalized string equation

Two ways to derive string equation and W- (in particular, Virasoro) constraints were suggested in [1]. One immediately leads to the string equation \( L_{-1}^{\{\hat{V}\}}Z = 0 \) and, as a consequence of it together with an additional information about \( Z \) (that it is a \( \tau \)-function of a specifically reduced KP hierarchy), one reproduce the set of all other W-constraints appropriate for the given reduction [11]. Another way [12] starts with exhaustive matrix Ward identity [13][14] which is transformed then into the set of appropriate W-constraints, without addressing to properties of \( Z \) at all. Below we concentrate on the first approach which is much simpler in general situation.

A \( \tau \)-function is parameterized by a point of Grassmannian. The role of string equation is to specify this dependence for the case of \( \tau \)-function given by the GKM partition function. Unlike the Toda case, if we look at \( Z_{\{\hat{V}\}} \) as at KP \( \tau \)-function, the point of Grassmannian depends on entire potential \( \hat{V}(X) \), i.e. not only on the polynomial piece \( V(X) = \hat{V}_+(X) \) but as well as on its negative and zero times (being included in definition of the point of Grassmannian in KP framework).

\[ \text{Note also that } \hat{Z}_{\{\hat{V}\}}[M] \neq Z_{\{\hat{V}\}}[M] \text{ because of occurrence of } \hat{V}_+ \text{ at some places. One more model can be defined by the rule } F_{\{\hat{V}\}}[\hat{V}'_+(M)] = F_{\{\hat{V}\}}[\hat{V}'(\hat{M})] \text{ (where } F \text{'s denote only integrals in the numerators of (2.24) and (2.29))}. \]  

Then \( \hat{M} \) is defined by the relation \( \hat{V}'_+(M) = V'(M) \), and \( Z_{\{\hat{V}\}}[\hat{M}] \) is a KP \( \tau \)-function of somewhat different time variables \( \hat{t}_p = \frac{1}{p}[Tr\hat{M}^{-p} + (p-1)\hat{s}_p] \), where \( \hat{s}_p \) are defined to be the coefficients of \( \hat{V} \) expansion in powers of \( \hat{M} \). In the simplest case of \( \hat{V}(X) = \frac{1}{2}X^2 - n \log X \) the definition of \( \hat{M} \) is \( M = \hat{M} - n\hat{M}^{-1} \). This is a rather familiar relation in Toda chain theory connecting its spectral parameter with that of KP hierarchy. We emphasize that while both \( \hat{Z}_{\{\hat{V}\}}[M] \) and \( Z_{\{\hat{V}\}}[M] \) are KP \( \tau \)-functions, only \( \hat{Z}_{\{\hat{V}\}} \) is also a Toda lattice \( \tau \)-function (i.e. possesses a proper dependence of \( t_0 \) and \( t_{-p} \)).

\[ \text{See also [14] and [15].} \]
In this subsection we shall shortly repeat the derivation of string equation\(^{18}\) addressing reader for more details to [1]. We shall demonstrate that this derivation crucially uses only the following information:

1) given asymptotics of \( \phi_i(\mu) \sim \mu^{i-1} \);

2) the manifest form of normalization factor depending on \( M \) in GKM integral;

3) the manifest form of linear term in \( X \) in exponential in integrand (2.25) (it is necessary in order to get correct expression (2.1) for \( Z \) implying \( Z \) to be \( \tau \)-function).

The main idea is to consider the following derivative of the GKM partition function

\[
Z = \frac{\det \phi_i(m_j)}{\Delta(m)} \equiv \frac{\det s(m_j)\phi_i(m_j)}{\Delta(m)} \quad \text{(see (2.24)–(2.25))}:
\]

\[
Tr \left\{ \frac{1}{V''(M)} \frac{\partial}{\partial M} \log Z \right\}
\]

and rewrite it as

\[
\sum_{p>0} Tr \frac{1}{V''(M)} \frac{\partial t_p}{\partial M} \frac{\partial \log Z}{\partial t_p} = -\sum_{p>0} Tr \frac{1}{V''(M)} \frac{1}{M^{p+1}} \frac{\partial \log Z}{\partial t_p} .
\] \hspace{1cm} (2.31)

On the other hand, the derivative (2.30) is equal to two pieces: the first one originates from the derivative of factors \( s(m) \) and \( \Delta(m)^{-1} \) (here we use the information of point 2) and is equal to:

\[
\frac{1}{2} \sum_{i,j} \frac{1}{V''(m_i)V''(m_j)} \frac{V''(m_i) - V''(m_j)}{m_i - m_j} .
\] \hspace{1cm} (2.32)

The remaining second piece can be transformed to derivative over \( t_1 \) essentially using the correct asymptotics of \( \phi_i(m) \) (point 1):

\[
Tr \left\{ \frac{1}{V''(M)} \frac{\partial}{\partial M} \log \det \phi_i(m_j) \right\} = \frac{\partial}{\partial t_1} \log Z .
\] \hspace{1cm} (2.33)

Now let us modify GKM partition function by introducing of negative- and zero-time variables, in accordance with (2.29). Then the only change (one should certainly to replace all \( V \) by \( \hat{V}_i \)) of string equation originates from the additional piece in (2.32):

\[^{18}\text{In this section we really deal with } L_{-1}\text{-constraint, the derivative of which over the first time variable is usually named string equation. Nevertheless, for the brevity, we call this Virasoro constraint as string equation too.}\]
\[ \frac{1}{V_+(M)} \frac{\partial \hat{V}_-(M)}{\partial M} \log Z = -t_0 \frac{1}{V_+(M)M} + \sum_{p>0} \rho_{t-p} \frac{1}{V_+(M)M^{p+1}}. \]  

(2.34)

Thus, we finally obtain the string equation in the form:

\[ L_{-1} \tau = 0, \]

\[ L_{-1}^{\{\hat{V}^+\}} = \sum_{p>0} T_p^{\{\hat{V}^+\}} \frac{\partial}{\partial t_p} + \frac{\partial}{\partial t_1} - T_{1}n^{\{\hat{V}^+\}} + \sum_{p>0} \rho T_p^{\{\hat{V}^+\}} t_{-p} \]  

\[ + \frac{1}{2} \sum_{i,j} \frac{1}{V_+(m_i)V_+(m_j)} \frac{\hat{V}_+''(m_i) - \hat{V}_+''(m_j)}{m_i - m_j}. \]  

(2.35)

with

\[ T_p^{\{\hat{V}^+\}} \equiv \frac{1}{V_+(M)M^p}. \]  

(2.36)

Let us note again (see footnote 14) that it is not necessary to include negative times into normalization factor \( \hat{s}(m) \). In this case, the string equation is not at all modified by negative times. Moreover, we can multiply GKM integrand by exponential of arbitrary series \( \sum_{-\infty}^k a_p X^p \) with the only restriction \( k < \) (the leading degree of \( \hat{V}(X) \)) to preserve the property 1 of correct asymptotics of \( \phi_i \). One can easily see from our previous discussion that this does not disturb three essential properties above and the form of the string equation is conserved. Thus, in KP framework we have plenty deformations of starting point of Grassmannian not changing the string equation. These modifications does certainly change the reduction condition, therefore, it emphasize the importance of fixing the reduction to define uniquely the \( \tau \)-function of (reduced) KP hierarchy by the string equation, in spirit of [11].

On the other hand, one can merely consider this as an extension of standard GKM viewpoint. Nevertheless, usual GKM representation is singled out by the possibility to do smooth transition to potentials of the other degrees corresponding to the double scaling continuum limit of the other multi-matrix models. In this sense, one can look at the string equation as giving some sort of universality class.

Now let us return to the string equation (2.35). For particular potential \( \hat{V}_+(X) = X^{K+1}/(K+1) \), \( T_{p+1} = (p+K)t_{p+K} \) and \( L_{-1} \)-constraint has a form:
\[ L^{(K)}_{-1} = \sum_{p>0} (p+K)t_{p+K} \frac{\partial}{\partial t_p} + K \frac{\partial}{\partial t_1} + \frac{1}{2} \sum_{a+b=K} at_a bt_b \] (2.37)

with \(a, b\) being any (positive and negative) integers and \(at_a\) should be substituted by \(n\) for \(a = 0\). For all \(t_{-p} = 0\) the term \(\partial/\partial t_1\) can be interpreted as resulting from the shift \(t_p \rightarrow t_p + \frac{K}{K+1} \delta_{p,K+1} = \hat{t}_p\). If also \(K = 1\),

\[ L^{(1)}_{-1} = \sum_{p>0} (p+1)\hat{t}_{p+1} \frac{\partial}{\partial \hat{t}_p} + nt_1 \] (2.38)

and this is similar to the string equation of discrete Hermitean one-matrix model, \(n\) being the size of the matrix. We shall return to this point a bit later. Here only note that, while in [5, 6, 16] it was proposed to interpret \(n\) as \(\partial/\partial t_0\), we see now it is more natural identify \(n\) with \(t_0\) itself, in the sense that it is \(n\) that plays the role of zero-time in Toda hierarchies (this was mentioned but not emphasized in [6]).

Somewhat alternative approach to the derivation of string equation and associated tower of \(W\)-like constraints could be to begin with the Ward identity for the integral

\[ F(\Lambda) = \int dX \ e^{-Tr\hat{V}(X)+Tr\Lambda X}, \] (2.39)

which results from the invariance under a shift \(X \rightarrow X + \epsilon\) (\(\epsilon\) being a matrix) of the integration variable:

\[ \langle Tr \ \epsilon \{\hat{V}'(X) - \Lambda\} \rangle = 0. \] (2.40)

Usually we can represent the positive powers of \(X\) coming from \(\hat{V}_+'(X)\) in the integrand (before averaging) in (2.40) by action of derivative operator \(\hat{V}_+'(\partial/\partial \Lambda_{tr})\) to already averaged expression. This is naively impossible for negative powers of \(X\) in \(\hat{V}_+'(X)\). However, be there only finite number of negative powers (i.e. if \(\hat{V}_+'(X)\) is a polynomial in \(X^{-1}\)), one could take additional \(\Lambda\)-derivatives of equation (2.40) in order to obtain more factors of \(X\) and eliminate its negative powers. This trick was applied in [5] in the particular case of all \(t_{-p} = 0\) with only non-zero \(n\). In this way they obtained the identity of the form:

\[ Tr\epsilon \left\{ \frac{\partial}{\partial \Lambda_{tr}} \hat{V}_+ ' \left[ \frac{\partial}{\partial \Lambda_{tr}} \right] + \frac{\partial}{\partial \Lambda_{tr}} \Lambda + n \right\} F(\Lambda) = 0 \] (2.41)
which, if rewritten in terms of $Z$ and in the limit of $N \to \infty$ (see details in [12] and [5]) turns into a set of Virasoro constraints

$$L_p Z = 0, \quad p \geq -1$$

with

$$L_p = \sum_{k=1}^{n} k \hat{t}_k \frac{\partial}{\partial k + n} + \sum_{k=1}^{n-1} \frac{\partial^2}{\partial t_k \partial t_{n-k}} + 2n \frac{\partial}{\partial t_p} \quad \text{for} \quad p \geq 1,$$

$$L_0 = \sum_{k=1}^{n} k \hat{t}_k \frac{\partial}{\partial k} + n^2 \quad \text{and} \quad L_{-1} \text{ as in (2.38)},$$

which are identified with Virasoro constraints for Hermitean one-matrix model. If some finite number of negative times are non-vanishing, one get instead a set of $\tilde{W}$-constraints [17].

3 Discrete models in the form of GKM

Since we devote this paper to discussion of Toda lattice hierarchies in the context of matrix models, we can not avoid touching the main conclusion of ref.[6] that all the discrete matrix models do correspond to particular cases of Toda hierarchies. In the simplest case of Hermitean one-matrix model one gets a Toda chain, other multi-matrix models correspond to other reductions of the Toda lattice hierarchy. Moreover, all discrete matrix models fall into the class of forced hierarchies [4].

3.1 Discrete models as forced Toda-lattice $\tau$-functions

In this subsection we show how partition functions of various matrix models can be rewritten in the KP-like form of eq.(2.1).

The first step in this direction is to reproduce them in the Toda like form of (2.2). This statement was already discussed in [6] and [4], but we present it more explicitly below. The second step is to notice that $H_{ij}$ which arises from discrete models are representable as averages, namely, finite dimensional integrals, or “matrices of moments”:

$$H_{ij}[t_{-p}, t_p] = \langle x^i y^j \mathcal{P}[x|t_p] \mathcal{P}[y|t_{-p}] \rangle.$$
The third step is to perform a Miwa transformation of times \( t_p \) with \( t_{-p} \) fixed, so that \( H_{ij} \)'s become averages of polynomial functions of \( X \). Then \( \det H_{ij} \) may be transformed with the help of orthogonal polynomials technique. These polynomials are, however, orthogonal with respect to the simple measure, defined by potential \( \tilde{V}(H) \equiv V(H) - HV'(H) \), which depends only on the \( s_p \)-variables and not on the times \( t_p \).

The main result of all these calculations is that partition functions arise in the form (2.1), with \( \phi_i(z) \) being proportional to orthogonal polynomials.

First, we shall briefly repeat what is known about discrete matrix models. In the case of the Hermitean one-matrix model the \( n \times n \) matrix integral

\[
Z_n^{(1)} = \{\text{Vol}_{U(n)n!}\}^{-1} \int DH \exp\{-\sum t_k spH^k\} =
\]

\[
= (n!)^{-1} \prod_i dh_i \Delta^2(h) \exp\{-\sum_{i,k} t_k h^k_i\}
\]

is taken by orthogonal polynomials for the arbitrary potential \( W(h) = \sum t_k h^k \)

\[
<P_i^{(1)}, P_j^{(1)}> = \int P_i^{(1)}(h) P_j^{(1)}(h) e^{-W(h)} dh = \delta_{ij} e^{\phi_i(t)}
\]

and equals

\[
Z_n^{(1)} = \prod_{i=0}^{n-1} e^{\phi_i(t)} = \tau_n^{(1)}(t)
\]

which is the \( \tau \)-function of the forced Toda chain hierarchy (being defined for negative \( n \) by (1.5)). In eq.(3.1) \( \text{Vol}_{U(n)} \) denotes the volume of unitary group \( U(n) \) (see (3.41) for explicit expression). On the other hand, it follows from (3.2) and the definition of orthogonal polynomials

\[
P_i^{(1)}(h) = \sum_{j \leq i} a^{(1)}_{ij} h^j
\]

\[
a^{(1)}_{ii} = 1
\]

that

\[
diag(e^{\phi_i(t)}) = A^{(1)} C^{(1)} A^{(1)T},
\]

where \( A^{(1)} = \|a^{(1)}_{ij}\|, A^T \) - transposed matrix, and \( C^{(1)} \) is so called matrix of moments
\[ C_{ij}^{(1)} = \int h^i \bar{h}^j e^{-W(h)} dh. \] (3.6)

Thus [4],

\[ \tau_n^{(1)}(t) = \det[\text{diag}(e^{\varphi_i(t)})] = \det A^{(1)}C^{(1)}A^{(1)T} = \det C^{(1)}. \] (3.7)

More or less the same is true for all discrete matrix models, only the form of \(A\) (which is unessential in (3.7)) and scalar product in the definition of \(C\) depend on the model. In the case of two Hermitean matrices and two potentials \(W(h) = \sum t_k h^k\) and \(\bar{W} = \sum \bar{t}_k \bar{h}^k\) we have

\[
Z^{(2)} = \{\text{Vol}_{U(n)}n!\}^{-1} \int DH \ D\bar{H} e^{-S_p[W(H)+\bar{W}(H)-HH]} = \\
= (n!)^{-1} \prod_i dh_i d\bar{h}_i \Delta(h)\Delta(\bar{h}) \exp\{-\sum_i [W(h_i) + \bar{W}(\bar{h}_i) - h_i \bar{h}_i]\} = \\
= \prod_{i=0}^{n-1} e^{\varphi_i(t,\bar{t})} = \tau_n^{(2)}(t,\bar{t}).
\] (3.8)

where

\[ e^{\varphi_i(t,\bar{t})}\delta_{ij} = \langle P^{(2)}_i, \bar{P}^{(2)}_j \rangle = \int P^{(2)}_i(h) \bar{P}^{(2)}_j(\bar{h}) e^{hh - W(h) - \bar{W}(\bar{h})} dh d\bar{h} \] (3.9)

and

\[ \tau_n^{(2)}(t,\bar{t}) = \det[\text{diag}(e^{\varphi_i(t,\bar{t})})] = \det A^{(2)}C^{(2)}A^{(2)T} = \det C^{(2)} \] (3.10)

for matrix of moments given by

\[ C_{ij}^{(2)} = \int h^i \bar{h}^j e^{hh-W(h)-W(h)} dh d\bar{h}. \] (3.11)

In generic \(K\)-matrix situation we can also use the formulas (3.10) and (3.11) if we take care of the integration measure. Namely,
$Z_{n}^{(K)} = \{Vol_{U(n)}n!\}^{-1} \int DH \, D\bar{H} \prod_{l=2}^{K-1} DH^{(l)} \exp\{-Sp[W(H) + \bar{W}(\bar{H}) + \sum_{l=2}^{K} V^{(l)}(H^{(l)}) - \sum_{l=2}^{K} c_l H^{(l)}(H^{(l+1)} - HH^{(2)} - H^{(K-1)} \bar{H})]\} = \quad (3.12)

= (n!)^{-1} \prod_{i} dh_i \bar{d}h_i \prod_{l=2}^{K-1} dh_i^{(l)} \Delta(h)\Delta(\bar{h}) \exp\{- \sum W(h_i) + \bar{W}(\bar{h}_i) + \sum W^{(l)}(h_i^{(l)}) - \sum c_l h_i^{(l)}(h_i^{(l+1)} - h_i h_i^{(2)} - h_i^{(K-1)} \bar{h}_i)\} = \quad (3.13)

= \prod_{i=0}^{n-1} e^{\varphi_i(t,\bar{t};W^{(l)},c_i)} = \tau_{n}^{(K;W^{(l)},c)}[t,\bar{t}] ,

where we distinguished $H^{(l)} \equiv H$ and $H^{(K)} \equiv \bar{H}$ among other integration variables, because the role of Toda time variables is played by $t^{(1)} \equiv t$ and $t^{(K)} \equiv \bar{t}$, while other times $t^{(l)}; l = 2, \ldots, K - 1$ (or potentials $W^{(l)}$) and the set of $\{c_i\}$ are considered as parameters. (The evolution along these frozen times can be described by the multi-component Toda lattice hierarchy.)

The orthogonal polynomials now look like

$$e^{\varphi_i(t,\bar{t};W^{(l)},c)} \delta_{ij} = <P_{i}^{(K)}, \bar{P}_{j}^{(K)} > = \int P_{i}^{(K)}(h) \bar{P}_{j}^{(K)}(\bar{h}) d\mu^{(K;W^{(l)},c)}[h,\bar{h}] ,$$

where

$$d\mu^{(K)}(h, \bar{h}) = dh \bar{d}h e^{-Sp(W(h) + \bar{W}(\bar{h}))} \prod_{2}^{K-1} dh^{(l)} \exp\{-Sp(\sum_{2}^{K-1} W^{(l)}(h^{(l)}) - \sum c_l h^{(l)}h^{(l+1)} - hh^{(2)} - h^{(K-1)} \bar{h})\} .$$

Again

$$\tau_{n}^{(K;W^{(l)},c)}[t,\bar{t}] = \det[\text{diag}(e^{\varphi_i(t,\bar{t};W^{(l)},c)})] = \quad (3.15)

= \det A^{(K)} C^{(K)} A^{(K)T} = \det C^{(K)}

and the matrix of moments is

$$C_{ij}^{(K)} = \int h^i \bar{h}^j d\mu^{(W^{(l)},c)}[h,\bar{h}] .$$

(3.16)
In the limit $K \to \infty$ this should correspond to the infinite-dimensional matrix model, whose first critical point has to describe (at least, formally) $c = 1$ conformal matter coupled to gravity [18]. Instead of the infinite sum we introduce the “quantum mechanical” integral and (3.12), (3.14) can be rewritten as

$$Z_n^{(\infty)} \sim \int_{H(0)=H}^{H(\Xi)=\bar{H}} \prod_\xi DH(\xi) \exp\{-Sp \int_0^\Xi d\xi \left\{ W[H(\xi)] + \left( \frac{\partial H}{\partial \xi} \right)^2 \right\} \} =$$

$$= \int \prod_i d\mu^{(\infty)}(h_i, \bar{h}_i) \Delta(h) \Delta(\bar{h}) =$$

$$= \prod_{i=0}^{n-1} e^{\rho_i[t, \bar{t}; W(\xi)]} = \tau_n^{(\infty; W(\xi))}[t, \bar{t}] \quad (3.17)$$

with

$$d\mu^{(\infty)}(h, \bar{h}) = dh d\bar{h} \int_{h(0)=h}^{h(\Xi)=\bar{h}} \prod_\xi Dh(\xi) \exp\{-Sp \int_0^\Xi d\xi \left\{ W[h(\xi)] + \left( \frac{\partial h}{\partial \xi} \right)^2 \right\} \} \ . \quad (3.18)$$

So far we considered the partition functions of various discrete matrix models as functions of Toda time-variables. Now we shall demonstrate that if one passes from times to Miwa variables

$$t_p = \frac{1}{p} Tr M^{-p} = \frac{1}{p} \sum_{i=1}^{N} m_i^{-p} + \frac{p-1}{p} s_p \quad (3.19)$$

($p > 0$) and/or

$$t_{-p} \equiv \bar{t}_p = \frac{1}{p} Tr \Lambda^{-p} = \frac{1}{p} \sum_{i=1}^{N} \lambda_i^{-p} + \frac{p-1}{p} \bar{s}_p \quad (3.20)$$

(note that $N$ – the size of matrices $M$ and $\Lambda$ has nothing to do with $n$ – the size of matrices $H$, being integrated over in (3.1), (3.8) and (3.12)) then the partition functions of discrete models $\{\tau_n^{(K)}\}$ acquire the form of KP $\tau$-function (2.1). Indeed,
\[ \tau_n^{(1)}(t) = (n!)^{-1} \int \prod_i dh_i \Delta^2(h) \exp\{-\sum_{i,k} t_k h_i^k\} = \]
\[ = (n!)^{-1} \int \prod_i dh_i \Delta^2(h) e^{-\tilde{V}(h_i)} \prod_i (1 - \frac{h_i}{m_a}) = \]
\[ = (n!)^{-1} \prod_a m_a^{-n} \int \prod_i dh_i e^{-\tilde{V}(h_i)} \Delta(h) \frac{\Delta(h, m)}{\Delta(m)} = \]
\[ = (n!)^{-1} \prod_a m_a^{-n} \Delta^{-1}(m) \int \prod_i dh_i e^{-\tilde{V}(h_i)} \times \]
\[ \times \det \tilde{P}^{(1)}_{i-1}(h_j) \bigg[ \begin{array}{cccc}
\tilde{P}^{(l)}_{i-1}(h_j) & \tilde{P}^{(l)}_{n+b-l}(h_j) \\
\vdots & \vdots \\
\tilde{P}^{(l)}_{i-1}(m_a) & \tilde{P}^{(l)}_{n+b-l}(m_a)
\end{array} \bigg] , \]

where \( i, j = 1, \ldots, n; a, b = 1, \ldots, N; \tilde{V}(h) = \sum \frac{k - 1}{k} s_k h^k \) and \( \{ \tilde{P}^{(1)}_i(h) \} \) are corresponding orthogonal polynomials with respect to deformed measure \( e^{-\tilde{V}} dh \).

\[ < \tilde{P}^{(1)}_i, \tilde{P}^{(1)}_j > = \int \tilde{P}^{(1)}_i(h) \tilde{P}^{(1)}_j(h) e^{-\tilde{V}(h)} dh = \delta_{ij} e^{\tilde{\psi}(s)} , \]

so that

\[ \tilde{P}^{(1)}_i(h) = h^i + O(h^{i-1}). \]

Computing determinants in (3.21) and using orthogonality condition (3.22) one obtains

\[ \tau_n^{(1)}[m|s] = \prod_a m_a^{-n} \Delta^{-1}(m) \det_{N \times N} \tilde{P}^{(1)}_{n+a-1}(m_b) \prod_i e^{\tilde{\psi}(s)} = \]
\[ = \prod_i e^{\tilde{\psi}(s)} \frac{\det_{ab} \phi^{(1,n)}_a(m_b)}{\Delta(m)} = \tau_n^{(1)}[\infty|s] \times \frac{\det_{ab} \phi^{(1,n)}_a(m_b)}{\Delta(m)} , \]

i.e. the \( \tau \)-function of the discrete Hermitean one-matrix model acquires the form of eq.(2.1) with

\[ \phi^{(1,n)}_a(m) = m^{-n} \tilde{P}^{(1)}_{n+a-1}(m) . \]

Below we shall see that (3.24) is natural representation for all discrete matrix models.

Let us remark that the expressions (3.21)-(3.24) does not depend on the quantity \( N \). It means that one can consider eq.(3.21) with \( N = 1 \) and reproduce well-known integral
representation for orthogonal polynomials [19] (the serious drawback of this representation is that its manifest form, and, moreover, the number of integrations depend on the degree of polynomial):

\[ P_{n}^{(1)}(m) = \int \frac{\prod_i dh_i \Delta^2(h) e^{-\tilde{V}(h)} \prod_i (m - h_i)}{\prod_i dh_i \Delta^2(h) e^{-\tilde{V}(h)}} . \]

In the case of two matrices instead of (3.21) one gets

\[ \tau_n^{(2)} (t, \bar{t}) = (n!)^{-1} \int \prod_i dh_i \Delta(h) \Delta(\bar{h}) \exp \left\{ - \sum_i [\tilde{W}(h_i) + \tilde{W}(\bar{h}_i) - h_i \bar{h}_i] \right\} \prod_i (1 - \frac{h_i}{m_a}) = \]

\[ = (n!)^{-1} \prod_a m_a^{-n} \int \prod_i dh_i \Delta(h) \exp \left\{ - \sum_i [\tilde{V}(h_i) + \tilde{V}(\bar{h}_i) - h_i \bar{h}_i] \right\} \Delta(h) \frac{\Delta(h, m)}{\Delta(m)} = \]

\[ = (n!)^{-1} \prod_a m_a^{-n} \Delta^{-1}(m) \int \prod_i dh_i \Delta(h) \exp \left\{ - \sum_i [\tilde{V}(h_i) + \tilde{V}(\bar{h}_i) - h_i \bar{h}_i] \right\} \times \]

\[ \times \det \tilde{P}_{i-1}^{(2)}(\bar{h}_j) \det \tilde{P}_{i-1}^{(2)}(h_j) \]

\[ = \left[ \prod_i e^{\tilde{P}_i(s, \bar{s})} \right] \frac{\det_{ab} \phi_a^{(2, n)}(m_b)}{\Delta(m)} , \]

where

\[ \phi_a^{(2, n)}(m) = m^{-n} \tilde{P}_{n+a-1}^{(2)}(m) , \]

\[ < \tilde{P}_i^{(2)}, \tilde{P}_j^{(2)} > = \int \tilde{P}_i^{(2)}(h) \tilde{P}_j^{(2)}(\bar{h}) e^{\tilde{V}(h) + \tilde{V}(\bar{h})} dh d\bar{h} = e^{\tilde{P}_i(s, \bar{s})} \delta_{ij} . \]

Note that, using symmetry of the two-matrix model, one can obtain the formula (3.25) using Miwa-transformed form of the other set of times (3.20), though its proper interpretation is not yet completely clear. In this case:

\[ \tau_n^{(2)} (t, \bar{t}) = \left[ \prod_i e^{\tilde{P}_i(s, \bar{s})} \right] \frac{\det_{ab} \phi_a^{(2, n)}(\lambda_b)}{\Delta(\lambda)} \]
with
\[
\tilde{\phi}_a^{(2,n)}(\lambda) = \lambda^{-n} \tilde{Q}_{n+a-1}^{(2)}(\lambda)
\]
where
\[
< \tilde{Q}_i^{(2)}, \tilde{Q}_j^{(2)} > = \int \tilde{Q}_i^{(2)}(\bar{h}) \tilde{Q}_j^{(2)}(\bar{h}) e^{h\bar{h} - \tilde{V}(h) - \tilde{V}(\bar{h})} dh d\bar{h} = e^{\tilde{\varphi}_i(t,\bar{t})} \delta_{ij}
\]
for \( \tilde{V}(\bar{h}) = \sum_{k=1}^M \frac{k-1}{k} \bar{s}_k \bar{h}^k \).

Let us point out that formulas (3.25), (3.28) look especially nice in the particular case of asymmetric two-matrix model [17,20] with one of the potential being a finite polynomial of fixed degree, say \( \tilde{W}(\bar{h}) = \sum_{k=1}^M \frac{k-1}{k} \bar{s}_k \bar{h}^k \). Then it is natural to take \( \tilde{V}(h) = W(h) \) and (3.25) turns to be
\[
\tau_{n,W}^{(2)}(m,s) = \left[ \prod_i e^{\tilde{\varphi}_i(s,\bar{t})} \right] \frac{\det_{ab} \phi_a^{(2,n,W)}(m_b)}{\Delta(m)} ,
\]
where
\[
\phi_a^{(2,n,W)}(m) = m^{-n} P^{(2,W)}_{n+a-1}(m) ,
\]
and
\[
< P_i^{(2,W)}, \tilde{P}_j^{(2,W)} > = \int P_i^{(2,W)}(h) \tilde{P}_j^{(2,W)}(\bar{h}) e^{h\bar{h} - W(h) - W(\bar{h})} dh d\bar{h} = e^{\varphi_i(s,\bar{t})} \delta_{ij}.
\]
As we have seen above orthogonal polynomials technique without serious changes can be applied to generic \( K \)-matrix model. For the analogues of (3.25), (3.28) it gives
\[
\tau^{(K)}_n(t,\bar{t}) = \left[ \prod_i e^{\tilde{\varphi}_i(s,\bar{t})} \right] \frac{\det_{ab} \phi_a^{(K,n)}(\lambda_b)}{\Delta(\lambda)} = \left[ \prod_i e^{\tilde{\varphi}_i(s,\bar{t})} \right] \frac{\det_{ab} \phi_a^{(K,n)}(\lambda_b)}{\Delta(\lambda)} ,
\]
where
\[
\phi_a^{(K,n)}(m) = m^{-n} \tilde{P}^{(K)}_{n+a-1}(m) ,
\]
\[
\langle \tilde{P}_i^{(K)}, \tilde{P}_j^{(K)} \rangle = \int \tilde{P}_i^{(K)}(h) \tilde{P}_j^{(K)}(\bar{h}) e^{h\bar{h} - \tilde{V}(h) - \tilde{V}(\bar{h})} d\mu^{(K)}(h, \bar{h}) = e^{\tilde{\varphi}_i(s, \bar{s})} \delta_{ij}, \quad (3.36)
\]

\[
\tilde{\phi}_a^{(K,n)}(\lambda) = \lambda^{-n} \tilde{Q}_{n+a-1}^{(K)}(\lambda). \quad (3.37)
\]

\[
\langle \tilde{Q}_i^{(K)}, \tilde{Q}_j^{(K)} \rangle = \int \tilde{Q}_i^{(K)}(h) \tilde{Q}_j^{(K)}(\bar{h}) e^{h\bar{h} - \tilde{V}(h) - \tilde{V}(\bar{h})} d\mu^{(K)}(h, \bar{h}) = e^{\tilde{\varphi}_i(t, \bar{t})} \delta_{ij}. \quad (3.38)
\]

### 3.2 Hermitean one-matrix model

In the previous section we demonstrated the manifest connection between \(\tau\)-functions of discrete matrix models and KP \(\tau\)-function in Miwa variables. However, in order to be representable as GKM they still need to arise in a somewhat specific form. Namely, components of the vector \(\{\phi_i(m)\}\) should possess a representation as “averages”,

\[
\phi_i(m) = \langle x_i^{-1} \rangle_m.
\]

Since in the study of discrete matrix models \(\phi_i(m)\) arise as orthogonal polynomials \(P_{n+i}(m)\), what is necessary is a kind of integral representation of these polynomials, with \(i\)-dependence coming only from the \(x_i^{-1}\)-factor in the integrand. It is an interesting problem to find out such kind of representation for various discrete models, but it is easily available only whenever orthogonal polynomials are associated with the Gaussian measure: the relevant Hermit polynomials are known to possess integral representation, which is exactly of the form which we need. We saw that the choice of the measure is rather arbitrary in the study of discrete matrix models, as soon as the complicated time-dependence is eliminated from the measure with the help of Miwa transformation. Therefore, it may be possible to make this measure Gaussian. The obstacle can arise either in multi-matrix case, if we do not want to apply Miwa transformation to “intermediate” times, or in the interesting version of two-matrix model with one potential fixed \([17,20]\), or in the models, involving non-Hermitean matrices. Each of these situations deserves special analysis. No more details are necessary in the simplest case of one-matrix Hermitean model, to be discussed below for illustrative purposes.
The statement to be discussed is that Hermitean one-matrix model with the matrix size $n$ is equivalent to GKM with $\hat{V}(X) = X^2/2 - n \log X$. We can come to this conclusion in various ways.

First of all, we can look at the complete set of Virasoro constraints for this model, following from the Ward-identity (2.41). It has been derived in [5] and shown to coincide with the set of Virasoro constraints for Hermitean one-matrix model, as discovered in [6,16,21].

Second, we can prove that such GKM corresponds to a forced Toda-chain hierarchy. This statement, if combined with the string equation (2.38), would also lead to the same conclusion.

Third, we can just verify the explicit identity between Gaussian matrix integrals,

$$\frac{\int dH_{n \times n} \det(I - H/M)e^{-SpH^2/2}}{\int dH_{n \times n}e^{-SpH^2/2}} = \frac{\int dX_{N \times N} \det(I - iX/M)^n e^{-TrX^2/2}}{\int dX_{N \times N}e^{-TrX^2/2}}. \quad (3.39)$$

Note that the size of matrix in the l.h.s. is $n \times n$ and in the r.h.s. is $N \times N$, and these parameters are absolutely independent. This identity is indeed true for any $n$ and $N$, as follows from the reasoning of the previous subsection 3.1, and integral representation of Hermit polynomials $\phi_j(\mu) = He_j(i\mu)$.

Remarkably, we could use this explicit proof (of eq.(3.39)) as an manifest verification of the commonly accepted belief that either (i) the string equation ($L_{-1}$-constraint) plus the fact that it is imposed on appropriately reduced Toda lattice $\tau$-function or (ii) the entire bunch of Virasoro (or $W$-, if necessary) constraints (without any a priori information about integrable structure) defines the partition function uniquely (and, in particular, predetermines the validness of all other $W$-constraints, suitable to the given reduction, in the case (i), or guarantees that partition function is, in fact, the appropriate Toda lattice $\tau$-function in the case (ii)). This belief is, of course, implicit in the first two of the above “derivation” of identity between Hermitean one-matrix model and GKM with $\hat{V}(X) = X^2/2 - n \log X$.

In the remaining part of this subsection we shall prove that partition function of such GKM is indeed a $\tau$-function of forced Toda chain reduction of Toda lattice hierarchy.

Let us begin with the word “forced”. Conceptually, the relevant hierarchy should be
forced just because we deal with discrete matrix models, orthogonal polynomials, and, thus, all $\phi_i(m)$ for $i > 0$ should be polynomials. It is, however, somewhat more complicated to prove the property (1.5). It even looks a bit intriguing, since partition function of GKM (the integral at the r.h.s. of (3.39)) does not seem to vanish for negative $n$. The resolution of the puzzle comes from the study of $n$-dependence of the coefficient of proportionality.

To do this, let us accurately restore all normalizations in $\tau$-function and use formula (3.39):

$$\tau_n = \frac{1}{n!Vol_{U(n)}} \int dH_{n \times n} e^{-Tr V(H)} = \frac{(2\pi)^{n^2/2}}{n!Vol_{U(n)}} \int dX_{N \times N} \det(I - X/M) e^{-Tr X^2/2} \int dX_{N \times N} e^{-Tr X^2/2},$$

where $Vol_{U(n)}$ denotes the volume of unitary group (it appears due to necessity of integration over angular variables),

$$Vol_{U(n)}^{-1} = (2\pi)^{-n(n-1)/2} \prod_{k=1}^{n} k! .$$

This formula can be obtained, for example, by immediate calculation of matrix Gaussian integral $I = \int dH e^{-Tr H^2} = (2\pi)^{n^2/2}$ and comparing it with one calculated in orthogonal polynomials technique: $I = Vol_{U(n)} \int \prod dm_i e^{-m^2} \Delta^2(m) = Vol_{U(n)} n! \prod_{k=1}^{n-1} (\text{norm of } He_k) = Vol_{U(n)} (2\pi)^{n/2} \prod_{k=1}^{n} k!$. Now we can continue it to negative $n$ using the formula (see for example [22]):

$$f(p) = \prod_{i=1}^{p} \phi(i), \text{ then } f(-p) = \prod_{i=0}^{p-1} \phi(-i)^{-1} .$$

It gives in our case

$$n!Vol_{U(n)} \to_{n < 0} (2\pi)^{n(n-1)/2} \prod_{i=0}^{[n]} \Gamma(-i) ,$$

what leads to the statement (1.5) due to singularities of $\Gamma$-function at negative integers.

Our next point is to prove that the GKM integral under consideration is indeed a Toda chain $\tau$-function, or, what is the same, $H_{ij} = H_{i-j}$. We neglect, for a moment, the negative-time variables. To calculate, we use the following Ward identity for the average with potential $V(X) = X^2/2$ (the logarithmic term is absorbed into the subscript of $\phi$):

$$\langle x^i \rangle_z = \langle x^{i-1} z \rangle_z + (i - 1) \langle x^{i-2} \rangle_z .$$
As $H_{ij} = \left\langle \frac{\phi_i(z)}{z^j} \right\rangle$ and $\phi_i(z) = \langle x^{i-1} \rangle_z$, one can write down:

$$H_{2j} = H_{1,j-1},$$
$$H_{3j} = H_{2,j-2} + H_{1,j},$$
$$H_{4j} = H_{3,j-1} + 2H_{2,j},$$
$$H_{5j} = H_{4,j-1} + 3H_{3,j},$$
$$\ldots,$$

Then, using the admissible triangle transformation

$$\tilde{H}_{1j} \equiv H_{1j},$$
$$\tilde{H}_{2j} \equiv H_{2j},$$
$$\tilde{H}_{3j} \equiv H_{3j} - H_{1j},$$
$$\tilde{H}_{4j} \equiv H_{4j} - 2H_{2j},$$
$$\tilde{H}_{5j} \equiv H_{5j} - 3H_{3j}, \ldots,$$

i.e.

$$\tilde{H}_{ij} = \sum_k A_{ik} H_{kj}$$

with lower-triangle matrix

$$A = \begin{pmatrix}
1 & 1 \\
1 & 1 & 1 \\
-1 & 1 & 1 & \ddots \\
-2 & 1 & \ddots & \ddots \\
-3 & 1 & \ddots & \ddots & \ddots \\
\end{pmatrix},$$

one obtains the property $\tilde{H}_{ij} = H_{i-j}$. It is evident that switching on the negative times as well as logarithmic term modifies Ward identity (3.44) by contributions of smaller powers in $x$ and can be also absorbed by proper lower-triangle transformation. It completes the proof.
4 Double-scaling continuum limit in terms of GKM

We demonstrated in the previous sections that the discrete Hermitean one-matrix model is equivalent to GKM with $\hat{V}(X) = X^2/2 + n \log X$, while from [1] (see also very instructive review [23]) and [12,14,15,24] we know that its double-scaling continuum limit is described by GKM with $V(X) = X^3/3$. Thus, we conclude that

$$\lim_{d.s. \ n \to \infty} Z_{\{\hat{V}\}} = Z_{\{V\}}^2.$$  \hspace{1cm} (4.1)

This relation should certainly be understandable just in terms of GKM itself. Moreover, since GKM is not sensitive to the size of the matrices $N$, it needs to become just a simple relation between finite dimensional integrals.

In this section we present some ideas about this relation, without going into details concerning its compatibility with conventional description of double-scaling limit [7].

To begin with, let us recall that double-scaling continuum limit for the model of interest implies that only even times $t_{2k} = \frac{1}{2k} Tr \frac{1}{M^{2k}}$ should remain non-zero, while all odd times $t_{2k+1} = 0$. This obviously implies that the matrix $M$ should be of block form:

$$M = \begin{pmatrix} \mathcal{M} & 0 \\ 0 & -\mathcal{M} \end{pmatrix}$$ \hspace{1cm} (4.2)

and, therefore, the matrix integration variable is also naturally decomposed into block form:

$$X = \begin{pmatrix} \mathcal{X} & \mathcal{Z} \\ \mathcal{Z} & \mathcal{Y} \end{pmatrix}.$$ \hspace{1cm} (4.3)

Then

$$Z_{\{\hat{V} = X^2/2 - n \log X\}} =$$
$$= \int d\mathcal{X} d\mathcal{Y} d^2 \mathcal{Z} \ det(\mathcal{X} \mathcal{Y} - \mathcal{Z} \mathcal{Y})^n e^{-Tr(|\mathcal{Z}|^2 + X^2/2 + Y^2/2 - M X + M Y)}.$$ \hspace{1cm} (4.4)

To take the limit $n \to \infty$, one should assume certain scaling behaviour of $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$. Moreover, the notion of double-scaling limit implies a specific fine tuning of this scaling behaviour. So we shall take
\[ X = \alpha (i \beta I + x), \]
\[ Y = \alpha (-i \beta I + y), \]
\[ Z = \alpha \zeta, \]
\[ M = \alpha^{-1} (i \gamma I + m) \]

with some large real \( \alpha, \beta \) and \( \gamma \). If expressed through these variables, the action becomes:

\[
\begin{align*}
\text{Tr} \{ |Z|^2 + X^2/2 + Y^2/2 - MX + MY - n \log(XY - \bar{Z}^1 Y) \} &= \\
= \gamma^2 Tr \{ (i \beta I + x)^2 + \gamma^2 (2i \beta I + x - y)^2 \} - Tr(i \alpha I + m)(2i \beta I + x - y) - \\
n Tr \log \beta^2 \gamma^2 \left( 1 - i \frac{x - y}{\beta} + \frac{xy}{\beta^2} - \frac{|\zeta|^2}{\beta^2} (1 + o(1/\beta)) \right) &= \\
= [2\alpha \beta - \beta^2 \gamma^2 - 2n \log \beta \gamma] Tr I - 2i \beta Tr m + \\
i(\beta \gamma^2 - \alpha + \frac{n}{\beta}) Tr (x - Tr y) + \frac{1}{2} (\gamma^2 - \frac{n}{\beta^2}) Tr x^2 + Tr y^2 + \\
+ (\gamma^2 - \frac{n}{\beta^2}) Tr |\zeta|^2 - \\
-Tr m x + Tr m y + \frac{in}{3 \beta^2} Tr (x^3 - y^3) + \\
+ O(n/\beta^4) + O(|\zeta|^2 \frac{n}{\beta^3}).
\end{align*}
\]

We want to adjust the scaling behaviour of \( \alpha, \beta \) and \( \gamma \) in such a way that only the terms in the line \( D \) survive. This goal is achieved in several steps.

The line \( A \) describes normalization of functional integral, it does not contain \( x \) and \( y \). Thus, it is not of interest for us at the moment.

Two terms in the line \( B \) are eliminated by adjustment of \( \alpha \) and \( \gamma \):

\[ \gamma^2 = \frac{n}{\beta^2}, \alpha = \frac{2n}{\beta}. \]  

As we shall see soon, \( \gamma^2 = n/\beta^2 \) is large in the limit of \( n \to \infty \). Thus, the term \( C \) implies that the fluctuations of \( \zeta \)-field are severely suppressed, and this is what makes the terms of the second type in the line \( E \) negligible. More general, this is the reason for the integral \( Z_{\{\bar{\psi}\}} \) to split into a product of two independent integrals leading to the
square of partition function in the limit $n \to \infty$ (this splitting is evident as, if $Z$ can be neglected, the only mixing term $\log \det \begin{pmatrix} \mathcal{X} & \mathcal{Z} \\ \mathcal{Z} & \mathcal{Y} \end{pmatrix}$ turns into $\log \mathcal{X} \mathcal{Y} = \log \mathcal{X} + \log \mathcal{Y}$).

Thus, we remain with a single free parameter $\beta$ which can be adjusted so that

$$\frac{\beta^3}{n} \to \text{const} \quad \text{as} \quad n \to \infty \quad (i.e. \quad \beta \sim \sqrt{n}),$$

making the terms in the last line $(E)$ vanishing and the third term in the line $(D)$ finite.

This proves the statement (4.1) in a rather straightforward manner, without addressing directly to the complicated matter like Kazakov’s change of time variables, reformulation of Virasoro constraints and so on [7]. We do not go into more details here, but point out one important detail. That is, the possibility to eliminate all the terms of original potential $\hat{V}_+(X)$ of degree $K = 2$ by the contributions coming from expansion of logarithm in such a way that the $(K + 1)$-th power of expansion survives is due to careful fine tuning of parameters of original $\hat{V}_+(X)$. This is just the idea involved in the notion of double-scaling limit and for higher-degree potentials it should be replaced by “$K$-scaling” limit which turns $Z_{\{V = X^K - n \log X\}}$ into $Z_{\{V = X^{K + 1}\}}^2$ as $n \to \infty$.

5 CONCLUSION

To conclude, let us first remind the main idea of the paper. The proposed Generalized Kontsevich Model seems now to be close to a unified theory of all (discrete and continuous) matrix models. By introducing of “negative-“ and “zero-“ time variables it results to be a $\tau$-function of the Toda lattice hierarchy and allows one to unify all matrix models in this framework. In particular, the original Hermitean one-matrix model is, in these terms, the GKM with “trivial” potential $X^2$ (non-trivial models start from original Kontsevich’s one with $X^3$ potential) but with non-vanishing zero-time. This must give an opportunity to study all subtle questions in this unified language (the continuum limit of discrete models among them – the sketch of such treatment was proposed in the sect.4). Besides, the introducing of zero-time variable leads to the notion of “natural basis” in Grassmannian with property $\phi_i^{(m)}(z) = z^{-n} \phi_i^{(0)}(z)$. This basis differs from the canonical one (which is singled out in the Segal–Wilson construction [27] and corresponds to standard represent-
tation of \( \phi_i \) through the fermionic averages in spirit of GKM [1]) and coincides with one inherited from GKM integral (2.25).

One of the most important items is the matter of the sect.2.3 concerning with (generalized) string equation. We see now that the string equation does not determine the point of infinite-dimensional Grassmannian uniquely – it allows one to deform it in accordance with negative-time Toda flows. In this sense, the introducing of the negative times would rather correspond to handle-gluing operators being added to action of string model, while the positive times to the motion in the space of string models. What is really spoilt by these negative times is the particular reduction condition, which is important in order to determine a particular string model with \( c = 1 - \frac{6(q - q')^2}{qq'} \). This condition is also spoilt by perturbation of the pure monomial potentials \( X^{K+1} \) by lower order irrelevant operators (in the sense of the universality class defined by the string equation), however the physical properties of these “deformed” models should be the same as those of the original one. In particular, it means that the choice of normalization of the basis in Grassmannian given by GKM \( \phi_i(z) \sim \int dx \ e^{-\tilde{V}(x)+V'(z)x} \) could be more flexible.

We have only touched in this paper the most serious problem of the continuum limit, which has been done correctly up to now only for the Hermitean one-matrix model [7]. As it has been shown in the sect.4, this continuum limit in principle can be done by purely “field-theory” technique in the framework of GKM, though some questions are still not completely clear. It allows us to hope that the proposed in [1] and above formalism will shed light on the serious problem of the continuum limit in multi-matrix models, among which the asymmetric two-matrix model proposed in [17,20] seems to be the most perspective one.

Finally, we have to repeat that we do not agree with the interpretation of [5] that the potential \( X^2 - n \log X \) describe consistently \( c = 1 \) string theory, or, equivalently, matrix quantum mechanics. However, we still hope that the limit \( c \to 1 \) can be found in the framework of GKM and, probably, by using of non-polynomial potentials.

All these problems deserve further investigation and we hope to return to them elsewhere.

We are grateful to L.Chekhov, A.Orlov, A.Sagnotti and A.Zabrodin for deep and very
Appendix A. Integrable hierarchies in the operator formalism

Here we would like to describe some basic ingredients of the integrable hierarchies in the terms of the massless fermions. This approach was initiated in the series of papers [25] and it appears to be very fruitful for description of the general structure of the large variety of nonlinear integrable equations like KP, Toda etc.

Let us define the fermionic operators on sphere

\[ \psi(z) = \sum_{k \in \mathbb{Z}} \psi^k z^k, \quad \psi^*(z) = \sum_{k \in \mathbb{Z}} \psi^{*-k}, \quad (A.1) \]

where fermionic modes satisfy the usual anti-commutation relations:

\[ \{\psi_k, \psi^*_m\} = \delta_{km}, \quad \{\psi_k, \psi_m\} = \{\psi^*_k, \psi^*_m\} = 0. \quad (A.2) \]

The Dirac vacuum \( |0\rangle \) is defined by the conditions:

\[ \psi_k |0\rangle = 0, \quad k < 0; \quad \psi^*_k |0\rangle = 0, \quad k \geq 0. \quad (A.3) \]

We also need to introduce the “shifted” vacua constructed from \( |0\rangle \) as follows

\[ |n\rangle = \left\{ \begin{array}{ll}
\psi_{n-1} \ldots \psi_0 |0\rangle, & n \geq 0 \\
\psi^*_n \ldots \psi^*_{-1} |0\rangle, & n < 0
\end{array} \right. \quad (A.4) \]

and satisfying the obvious conditions

\[ \psi_k |n\rangle = 0, \quad k < n; \quad \psi^*_k |n\rangle = 0, \quad k \geq n. \quad (A.5) \]

From fermionic modes one can built the \( U(1) \)–currents

\[ J_k = \sum_{i \in \mathbb{Z}} \psi_i \psi^*_{i+k}, \quad J_{-k} \equiv J_k, \quad k \in \mathbb{Z}_+ \quad (A.6) \]

and define “Hamiltonians”
\[ H(x) = \sum_{k=1}^{\infty} x_k J_k \ , \ \tilde{H}(x) = \sum_{k=1}^{\infty} y_k \tilde{J}_k \ , \]
(A.7)

where \( \{x_k\} \) and \( \{y_k\} \) are half-infinite sets of independent time variables ("positive" and "negative" times correspondingly; in main text \( \{x_k\} \equiv \{t_k\} \), \( \{y_k\} \equiv \{-t_{-k}\} \), \( k > 0 \)) which generate the evolution of nonlinear system.

Let \( g \) be an arbitrary element of the Clifford group which does not mixes the \( \psi \)- and \( \psi^\star \)- modes:

\[ g =: \exp[\sum A_{km} \psi_k \psi_m^\star] :, \]
(A.8)

where \( : \) \( : \) denotes the normal ordering with respect to the Dirac vacuum \( |0\rangle \). Then it is well known [10,25] that

\[ \tau_n(x, y) = \langle n | e^{H(x)} g e^{-\tilde{H}(y)} | n \rangle \]
(A.9)

solves the two-dimensional Toda lattice hierarchy, \( i.e. \) is the solution to the whole set of the Hirota bilinear equations. Any particular solution depends only on the choice of the element \( g \) (or, equivalently it can be uniquely described by the matrix \( A_{km} \)). From the eqs.(A.2) one can conclude that any element in the form (A.8) rotates the fermionic modes as follows

\[ g \psi_k g^{-1} = \psi_j R_{jk} \ , \ \ g \psi_k^\star g^{-1} = \psi_j^\star R_{kj}^{-1} , \]
(A.10)

where the matrix \( R_{jk} \) can be expressed through \( A_{jk} \) (see [26]). We will see below that the general solution (A.9) can be expressed in the determinant form with explicit dependence of \( R_{jk} \). In order to calculate \( \tau \)-function we need some more notations. Using commutation relations (A.2) one can obtain the evolution of \( \psi(z) \) and \( \psi^\star(z) \) in times \( \{x_k\} \), \( \{y_k\} \) in the form

\[ \psi(z, x) \equiv e^{H(x)} \psi(z) e^{-H(x)} = e^{\xi(x,z)} \psi(z) , \]
(A.11)

\[ \psi^\star(z, x) \equiv e^{H(x)} \psi^\star(z) e^{-H(x)} = e^{-\xi(x,z)} \psi^\star(z) ; \]
(A.12)
\( \psi(z, \bar{y}) \equiv e^{H(y)} \psi(z) e^{-\bar{H}(y)} = e^{\xi(y,z^{-1})} \psi(z) \), \quad (A.13)

\( \psi^*(z, \bar{y}) \equiv e^{\bar{H}(y)} \psi^*(z) e^{-\bar{H}(y)} = e^{-\xi(y,z^{-1})} \psi(z) \), \quad (A.14)

where

\( \xi(x, z) = \sum_{k=1}^{\infty} x_k z^k \). \quad (A.15)

Let us define the the Schur polynomials \( P_k(x) \):

\[ P[k \mid x] = e^{\xi(x,z)} = \sum_{k=0}^{\infty} P_k(x) z^k ; \quad (A.16) \]

then from eqs. (A.11)-(A.14) one can easily obtain the evolution of the fermionic modes:

\( \psi_k(x) \equiv e^{H(x)} \psi_k e^{-H(x)} = \sum_{m=0}^{\infty} \psi_{k-m} P_m(x) \), \quad (A.17)

\( \psi^*_k(x) \equiv e^{H(x)} \psi^*_k e^{-H(x)} = \sum_{m=0}^{\infty} \psi^*_{k+m} P_m(-x) \); \quad (A.18)

\( \psi_k(\bar{y}) \equiv e^{\bar{H}(y)} \psi_k e^{-\bar{H}(y)} = \sum_{m=0}^{\infty} \psi_{k+m} P_m(y) \); \quad (A.19)

\( \psi^*_k(\bar{y}) \equiv e^{\bar{H}(y)} \psi^*_k e^{-\bar{H}(y)} = \sum_{m=0}^{\infty} \psi^*_{k-m} P_m(-y) \). \quad (A.20)

It is useful to introduce the totally occupied state \( | - \infty \rangle \) (see definition (A.4) in the limit \( n \to -\infty \)) which satisfies the requirements

\( \psi^*_i | - \infty \rangle = 0 \), \( i \in \mathbb{Z} \). \quad (A.21)

Then any shifted vacuum can be generated from this state as follows:

\( |n\rangle = \psi_{n-1} \psi_{n-2} ... | - \infty \rangle \). \quad (A.22)

Note that the action of an any element \( g \) of the Clifford group (and, as consequence, the action of \( e^{-\bar{H}(y)} \) on \( | - \infty \rangle \) is very simple: \( g | - \infty \rangle \sim | - \infty \rangle \), so using (A.18) and (A.19) one can obtain from eq.(A.9):
\[ \tau_n(x, y) = \langle -\infty | \psi_{n-2}^*(-x) \psi_{n-1}^*(-x) g \psi_{n-1}(-\bar{y}) \psi_{n-2}(-\bar{y}) \ldots | -\infty \rangle \sim \]
\[ \sim \det [\langle -\infty | \psi_i^*(-x) g \psi_j(-\bar{y}) g^{-1} | -\infty \rangle]_{i,j \leq n-1} . \] (A.23)

Using (A.10) it is easy to see that
\[ g \psi_j(-\bar{y}) g^{-1} = \sum_{m,k} P_m(-y) \psi_k R_{k,j+m} \] (A.24)
and the “explicit” solution of the two-dimensional Toda lattice has the determinant representation:
\[ \tau_n(x, y) \sim \det \hat{H}_{i+n,j+n}(x, y) \big|_{i,j < 0} , \] (A.25)
where
\[ \hat{H}_{ij}(x, y) = \sum_{k,m} R_{km} P_{k-i}(x) P_{m-j}(-y) . \] (A.26)

The ordinary solutions to KP hierarchy [25] correspond to the case when the whole evolution depends only of positive times \( \{x_k\} \); negative times \( \{y_k\} \) serve as parameters which parameterize the family of points in Grassmannian and can be absorbed by re-definition of the matrix \( R_{km} \). Then \( \tau \)-function of (modified) KP hierarchy has the form
\[ \tau_n(x) = \langle n | e^{H(x)} g(y) | n \rangle \sim \det \left[ \sum_k R_{k,j+n}(y) P_{k-i-n}(x) \right]_{i,j < 0} , \] (A.27)
where \( g(y) \equiv g e^{-\hat{H}(y)} \) and
\[ R_{kj}(y) \equiv \sum_m R_{km} P_{m-j}(-y) . \] (A.28)

One can consider the reduction to the Toda chain hierarchy after imposing the condition on the element \( g \) [10]
\[ [J_k + \bar{J}_k, g] = 0 \] (A.29)
which is equivalent to constraint
\[ [\Lambda + \Lambda^{-1}, R] = 0 . \] (A.30)
In this case

$$ge^{-y_k J_k} = e^{-y_k J_k} e^{-y_k J_k} g e^{y_k J_k}$$

and $\tau$-function depends (up to the trivial factor) only on times $\{x_k - y_k\}$:

$$\tau_n(x, y) = e^{\sum k x_k y_k (n| e^{H(x-y)} g|n) . \quad (A.31)}$$

The reduction (A.30) has an important solution\footnote{Generally the solutions $R_{nk} = R_{n+k}$ and $R_{nk} = R_{n-k}$ are different, but for forced hierarchy, when $\tau$-function is the determinant of finite matrix, these two solutions are equivalent due to possibility to reflect matrix with respect to vertical axis without changing the determinant.}

$$R_{km} = R_{k+m} . \quad (A.32)$$

In this case the matrix $\hat{H}_{ij}$ defined by eq.(A.26) evidently satisfies the relations $\hat{H}_{ij} = \hat{H}_{i+j}$ and

$$(\partial_{x_k} + \partial_{y_k}) \hat{H}_{i+j} = 0 \text{ for any } k < n - i, k < n - j \quad (A.33)$$

due to the properties of Schur polynomials

$$\partial_{x_n} P_k(x) = P_{k-n}(x). \quad (A.34)$$

The property (A.33) certainly does not imply that the corresponding $\tau$-function depends only on difference of times because of restriction of values of $k$, but it restores correct dependence of times with taking into account of exponential in (A.31).

Now let us establish the correspondence between matrices $R_{ij}$ (eq.(A.10)), $\hat{H}_{ij}$ (eq.(A.26)) and $T_{ij}$ (eq.(2.12)), $H_{ij}$ (eq.(2.2)). These equations transform into each other under charge conjugation of fermions $\psi_k \rightarrow \psi^*_{-k-1}$ (i.e. vacua transform as follows: $|n\rangle \rightarrow | - n\rangle$) and correspondence: $T_{ij} = R_{-j-1,-i-1}^{-1}$ and $\hat{H}_{ij} = H_{ij}$.\footnote{Generally the solutions $R_{nk} = R_{n+k}$ and $R_{nk} = R_{n-k}$ are different, but for forced hierarchy, when $\tau$-function is the determinant of finite matrix, these two solutions are equivalent due to possibility to reflect matrix with respect to vertical axis without changing the determinant.}
Appendix B. Fermionic representation of the matrix models

Here we would like to discuss the fermionic language for (multi-) matrix models. Namely, we shall show that the \( \tau \)-function of the two-dimensional Toda lattice (A.9) describes the whole variety of the (multi-) matrix models for some specific choice of the element \( g \). Since one should reproduce the partition function of the matrix models from eq.(A.9), we shall deal with the forced hierarchies (see (1.5) and discussion in the sect.2.1), i.e.

\[ \tau_n = 0 \ , \ n < 0 \ . \]  

(B.1)

Therefore, it is reasonable to consider the point of the Grassmannian in the form

\[ g = g_0 P_+ , \]  

(B.2)

where \( P_+ \) is the projector onto positive states:

\[ P_+ |n\rangle = \theta(n) |n\rangle . \]  

(B.3)

There is exist a natural fermionic projector

\[ P_+ =: \exp[\sum_{i<0} \psi_i \psi^*_i] : \]  

(B.4)

with the properties

\[ P_+ \psi^*_{-k} = \psi_{-k} P_+ = 0 \ , \ k > 0 \ ; \]  

(B.5)

\[ [P_+, \psi_k] = [P_+, \psi^*_k] = 0 \ , \ k \geq 0 \ ; \]  

(B.6)

\[ P_+^2 = P_+ . \]  

(B.7)

The insertion of such projector into eq.(A.9) naturally leads us to conclusion that \( g_0 \) should depends only on \( \psi_k \) and \( \psi^*_k \) with \( k \geq 0 \). We use the choice
\[ g_0 =: \exp \left\{ \left( \int_\gamma A(z, w) \psi_+(z) \psi^*_+(w^{-1}) dz dw \right) - \sum_{i \geq 0} \psi_i \psi^*_i \right\} : ; \quad (B.8) \]

where \( \psi_+(z) = \sum_{k \geq 0} \psi_k z^k \), \( \psi^*_+(z) = \sum_{k \geq 0} \psi^*_k z^{-k} \) and \( \gamma \) is some contour of integration. In what follows we shall also use the projector

\[ P_- =: \exp \left[ - \sum_{i \geq 0} \psi_i \psi^*_i \right] : \quad (B.9) \]

with the properties

\[ P_- \psi_k = \psi^*_k P_- = 0 \quad (k \geq 0) ; \quad (B.10) \]

\[ [P_-, \psi_{-k}] = [P_-, \psi_{-k}^*] = 0 \quad (k > 0) ; \quad (B.11) \]

\[ P_-^2 = P_- . \quad (B.12) \]

Now one should calculate the state

\[ g_0 P_+ e^{-\tilde{H}(y)} |n\rangle \quad (B.13) \]

It is easy to see that this state vanishes when \( n < 0 \). Indeed, using eqs.(A.4) and (A.20) one can obtain that with \( n < 0 \) the state

\[ e^{-\tilde{H}(y)} |n\rangle = \psi^*_n (-\tilde{y}) \ldots \psi^*_1 (-\tilde{y}) e^{-\tilde{H}(y)} |0\rangle \]

contains only negative modes \( \psi^*_m (m > 0) \). Therefore the action of \( P_+ \) annihilates this state due to eq.(B.5). For \( n \geq 0 \) using eqs.(A.4), (A.19) and (B.6) we have

\[ P_+ e^{-\tilde{H}(y)} |n\rangle = \psi_{n-1} (-\tilde{y}) \ldots \psi_0 (-\tilde{y}) P_+ e^{-\tilde{H}(y)} |0\rangle . \quad (B.14) \]

Then we use the fact that

\[ P_+ e^{-\tilde{H}(y)} |0\rangle = |0\rangle . \quad (B.15) \]

Proof of eq.(B.15). Let us denote
\[ |y\rangle = P_+ e^{-R(y)} |0\rangle . \]

Then
\[ \frac{\partial}{\partial y_k} |y\rangle = P_+ e^{-R(y)} \sum_{i=0}^{k-1} \psi_i^* \psi_i |0\rangle = 0 \]
due to eqs. (A.3) and (B.5). Since \(|y\rangle |_{y_i = 0} = |0\rangle\), the eq. (B.15) is proved.

Therefore, we have
\[ g_0 P_+ e^{-R(y)} |n\rangle = g \psi(-\bar{y}) \ldots \psi(-\bar{y}) |0\rangle = \sum \frac{1}{m!} \int \prod_{i=1}^{m} A(z_i, w_i) dz_i dw_i \psi_+(z_1) \ldots \psi_+(z_m) \times \]
\[ \times P_- \psi_+^*(w_{m-1}^{-1}) \ldots \psi_+^*(w_{1}^{-1}) \psi_{n-1}(-\bar{y}) \ldots \psi_0(-\bar{y}) |0\rangle \]
with using of the eqs. (B.8) and (B.9). Now we shall see that only the term with \( m = n \) gives a non-zero contribution in the infinite sum (B.16). Indeed, for \( m > n \) the state \( \psi_+^*(w_{m-1}^{-1}) \ldots \psi_+^*(w_{1}^{-1}) \psi_{n-1}(-\bar{y}) \ldots \psi_0(-\bar{y}) |0\rangle \) vanishes, because in this case some positive modes in \( \psi_+^*(w_{i}^{-1}) \) will reach the vacuum \(|0\rangle\) and annihilate it. Vice versa, for \( m < n \) some positive modes in \( \psi_k(-\bar{y}) \) will reach the projector \( P_- \) and due to eq. (B.10) it is zero.

Therefore,
\[ g_0 P_+ e^{-R(y)} |n\rangle = \frac{1}{n!} \int \prod_{i=1}^{n} A(z_i, w_i) dz_i dw_i \psi_+(z_1) \ldots \psi_+(z_n) \times \]
\[ \times P_- \psi_+^*(w_{n-1}^{-1}) \ldots \psi_+^*(w_{1}^{-1}) \psi_{n-1}(-\bar{y}) \ldots \psi_0(-\bar{y}) |0\rangle . \]

Now we use the following proposition:
\[ \psi_+^*(w_{n-1}^{-1}) \ldots \psi_+^*(w_{1}^{-1}) \psi_{n-1}(-\bar{y}) \ldots \psi_0(-\bar{y}) |0\rangle = \]
\[ = \Delta(w) \exp[- \sum_{j=1}^{n} \xi(y, w_j)] |0\rangle . \]

\textbf{Proof.} Since the number of creation (w.r.t. to \(|0\rangle\)) operators \( \psi_i(-\bar{y}) \) equals to the number of annihilation operators \( \psi_+^*(w_{j}^{-1}) \), then it is obvious that after normal re-ordering
\[ \psi_+^*(w_{n-1}^{-1}) \ldots \psi_+^*(w_{1}^{-1}) \psi_{n-1}(-\bar{y}) \ldots \psi_0(-\bar{y}) |0\rangle = const \cdot |0\rangle \]
and, consequently,
\[ \text{const} = \langle 0 | \psi^*_+ (w^{-1}_n) \ldots \psi^*_+ (w^{-1}_1) \psi_{n-1} (-\bar{y}) \ldots \psi_0 (-\bar{y}) | 0 \rangle = \] \[ = \det [ \langle 0 | \psi^*_+ (w^{-1}_i) \psi_{j-1} (-\bar{y}) | 0 \rangle ]_{i,j=1,\ldots,n} \]

and using eqs. (A.20), (A.16) one can obtain

\[ \langle 0 | \psi^*_+ (w^{-1}_i) \psi_{j-1} (-\bar{y}) | 0 \rangle = w^{j-1}_i e^{\bar{H}(w)} ; \]

thus,

\[ \text{const} = \det [ w^{j-1}_i e^{-\xi(y,w_i)} ] = \Delta(w) \exp [ -\sum_{j=1}^{n} \xi(y, w_j) ] . \]

After substitution of eq. (B.18) into eq. (B.17) and using the obvious fact that \( P_- | 0 \rangle = | 0 \rangle \) and \( \psi_-(z_i)|0\rangle = 0 \) one can obtain

\[ \int_\gamma n! \prod_{i=1}^{n} A(z_i, w_i) e^{-\xi(y,w_i)} dz_i dw_i \Delta(w) \psi(z_1) \ldots \psi(z_n)|0\rangle. \]

Using one of the basic formulas [25] (which can be simply proved by bosonization technique)

\[ \psi(z_1) \ldots \psi(z_n)|0\rangle = \Delta(z) \exp [ \bar{H}(\sum_{i=1}^{n} \epsilon(z_i))]|0\rangle, \]

where \( \epsilon(z_i) \) is the vector with components \( \epsilon_k(z_i) = \frac{1}{k} z_i^k \), we have the desired result:

\[ \int_\gamma n! \prod_{i=1}^{n} A(z_i, w_i) e^{-\xi(y,w_i)} dz_i dw_i \Delta(w) \Delta(z) \exp [ \bar{H}(\sum_{i=1}^{n} \epsilon(z_i))]|0\rangle. \]

Thus, finally we obtain:

\[ \tau_n(x, y) = \langle n \mid e^{H(x)} g_0 P_+ e^{-H(y)} | n \rangle = \] \[ = \frac{1}{n!} \int_\gamma \Delta(w) \Delta(z) \prod_{i=1}^{n} A(z_i, w_i) e^{\xi(x,z_i) - \xi(y,w_i)} dz_i dw_i. \]

Let us consider some particular cases.
i) \( \gamma = (-\infty, +\infty) \), \( A(z, w) = \delta(z - w) \). Then one can recover the case of Hermitean one-matrix model.

ii) \( \gamma \) is the small circle around the origin in the complex \( z \)-plane, and \( A(z, w) = \frac{1}{2\pi i z} \delta(z - w^{-1}) \). Then \( g_0 = 1 \) and \( \tau_n(x - y) \) is the \( \tau \)-function for the symmetric unitary model.

iii) \( \gamma = (-\infty, +\infty) \), \( A(z, w) = e^{zw} \). This is Hermitean two-matrix model.

iv) \( \gamma = (-\infty, +\infty) \). Let us denote \( z_i \equiv z_i^{(1)} \), \( w_i \equiv z_i^{(p)} \), \( x \equiv -t^{(1)} \), \( y \equiv t^{(p)} \) and

\[
A(z_i^{(1)}, z_i^{(p)}) = -\int_0^{+\infty} \prod_{j=1}^{p-1} dz_j^{(j)} \exp\{-\sum_{j=2}^{p-1} \xi(t^{(j)}, z_i^{(j)}) + \sum_{j=1}^{p-1} z_i^{(j)} z_i^{(j+1)}\}. \tag{B.23}
\]

Then we obtain the partition function for Hermitean \( p \)-matrix model which originates from the matrix integrals

\[
\int DZ^{(1)} \ldots DZ^{(p)} \exp\{-Tr \sum_{j=1}^{p-1} \xi(t^{(j)}, Z^{(j)}) + Tr \sum_{j=1}^{p-1} Z^{(j)} Z^{(j+1)}\}. \tag{B.24}
\]

The only point is that in this case \( g \) is parameterized by the set of additional times \( t^{(j)} \), \( j = 2, \ldots, p - 1 \). One can obtain the multi-matrix model with time-independent \( g \) in the context of the multi-component Toda lattice hierarchy.

**Determinant representation.**

Using eqs.(A.4), (B.7) and the fact that \( [P_+, g_0] = 0 \), we have:

\[
\tau_n(x, y) = \langle 0 | \psi_0^* \ldots \psi_{n-1}^* e^{H(x)} g_0 P_+ e^{-H(y)} \psi_{n-1} \ldots \psi_0 | 0 \rangle =
\]

\[
= \langle 0 | e^{H(x)} \psi_0^*(-x) \ldots \psi_{n-1}^*(-x) P_+ g_0 P_+ \psi_{n-1}(-\bar{y}) \ldots \psi_0(-\bar{y}) e^{-H(y)} | 0 \rangle .
\]

Since \( \psi_i^*(-x) \) and \( \psi_i(-\bar{y}) \) contain only positive modes (see eqs.(A.18) and (A.19)), due to (B.6) and (B.15) one can obtain

\[
\tau_n(x, y) = \langle 0 | \psi_0^*(-x) \ldots \psi_{n-1}^*(-x) g_0 \psi_{n-1}(-\bar{y}) \ldots \psi_0(-\bar{y}) | 0 \rangle =
\]

\[
\det[\langle 0 | \psi_i^*(-x) g_0 \psi_j(-\bar{y}) | 0 \rangle] |_{i,j=0,\ldots,n-1} . \tag{B.25}
\]

The same arguments used for transition from eq.(B.16) to eq.(B.17) when applied to eq.(B.25) lead to conclusion that only linear term in \( A(z, w) \) contributes, so using eq.(B.10) we have
\begin{equation}
\langle 0 | \psi_1^*(-x) g \psi_j(-\bar{y}) | 0 \rangle = \int_\gamma A(z, w) dz dw \langle 0 | \psi_1^*(-x) \psi_+ (z) P_+ \psi_+^*(w^{-1}) \psi_j(-\bar{y}) | 0 \rangle = \\
= \int_\gamma z^{i}w^{j} A(z, w) e^{\xi(x,z)-\xi(y,w)} dz dw = \partial_x^i(-\partial_y)^j \int_\gamma A(z, w) e^{\xi(x,z)-\xi(y,w)} dz dw
\end{equation}

and, finally, one can obtain the expression for \( \tau \)-function in the determinant form:

\begin{equation}
\tau_n(x, y) = \det[\partial_{x_i}(-\partial_{y_i})^j \int_\gamma A(z, w) e^{\xi(x,z)-\xi(y,w)} dz dw] |_{i,j=0,\ldots,n-1}.
\end{equation}

Again, the consideration of particular choices of \( A(z, w) \) (see discussion below eq.(B.22)) leads to representation of \( \tau \)-function for Hermitean, unitary etc. models in the determinant form.

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