An Introduction to Hyperbolic Analysis

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## Contents

1. Introduction 3
2. The hyperbolic algebra as a bidimensional Clifford algebra 4
3. Limits and series in the hyperbolic plane 8
4. The hyperbolic Euler formula 10
5. Analytic functions in the hyperbolic plane 11
6. Multivalued functions on the hyperbolic plane and hyperbolic riemann surfaces 24
7. Physical application to the vibrating string 30
8. Hyperbolic Analysis as the (1,0)-case of Clifford Analysis 31
9. Acknowledgements 42
10. References 42
I. INTRODUCTION

Let us consider the ring $G$ of the numbers of the form $z = a + ib$, $a, b \in \mathbb{R}$, with $i$ satisfying the following equation:

$$i^2 = 1 \quad (1.1)$$

The elements of such a ring has been called in the mathematical literature with different names (cfr. [1] and references therein): hyperbolic numbers, double numbers, split complex numbers, perplex numbers, and duplex numbers $^1$. We will call them hyperbolic numbers and we will refer to $i$ as to the hyperbolic imaginary unit.

Hyperbolic numbers emerged in the research of one of the authors [3], [4], [5] as the underlying number system of a mathematical theory, "Hyperbolic Quantum Mechanics", axiomatized in the mentioned papers.

Since, as we will show in section II the complex field $\mathbb{C}$ and the hyperbolic ring $G$ are the two bidimensional Clifford algebras:

$$\mathbb{C} = Cl_{0,1} \quad (1.2)$$
$$G = Cl_{1,0} \quad (1.3)$$

the investigations about Hyperbolic Calculus can be developed along two different lines:

1. one can analyze which of the mirabilities of Complex Calculus survives passing to the hyperbolic case
2. one can directly consider Clifford Calculus [6], [7] in its generality and to apply it to the particular $(1,0)$ case

In this paper we will try to pursue both these strategies.

Remark I.1

ON OUR USE OF THE LOCUTION "HYPERBOLIC PLANE"

We advise the reader that our adoption of the locution "Hyperbolic Analysis" follows the terminology of [8]. Consequentially the locution "hyperbolic plane" is used here to denote $G$ and has no relation with the more common use of such a locution to denote the Riemannian manifold $\{(x, y) \in \mathbb{R}^2 : y > 0, \frac{dx \otimes dx + dy \otimes dy}{y^2}\}$.

$^1$ We invite, by the way, the reader to pay attention to the fact that, contrary to what is claimed in [1], the locution unipodal numbers is used by Garrett Sobczyk [2] to denote the more extended number system $U$ of the numbers of the form $z = a + ib$, $a, b \in \mathbb{C}, i^2 = 1$. 

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II. THE HYPERBOLIC ALGEBRA AS A BIDIMENSIONAL CLIFFORD ALGEBRA

Let us consider the ring $G$ of the hyperbolic numbers, i.e. of the numbers of the form $z = a + ib$, $a, b \in \mathbb{R}$, with the hyperbolic imaginary unit $i$ satisfying the following equation:

$$i^2 = +1$$  \hspace{1cm} (2.1)

$G$ may be seen as a bidimensional Clifford algebra as we will show in this paragraph.

Given a linear space $V$ over the real field:

**Definition II.1**

**TENSORS OF TYPE $(r,s)$ OVER $V$:**

$$T^r_s(V) := \{ T^r_s : \times_{i=1}^r V \times_{j=1}^s V^* \mapsto \mathbb{R} \text{ multilinear} \}$$  \hspace{1cm} (2.2)

Let us introduce in particular the following:

**Definition II.2**

**TENSOR ALGEBRA OVER $V$:**

$$T(V) := \bigoplus_{n \in \mathbb{N}_+} T^n_0(V)$$  \hspace{1cm} (2.3)

Given an $(r,s)$-tensor $T^r_s \in T^r_s(V)$ over $V$:

**Definition II.3**

$T^r_s$ IS SKEW-SYMMETRIC IN THE INDICES $(i,j)$:

$$T^r_s \mapsto -T^r_s \text{ under permutation of } i \text{ and } j$$  \hspace{1cm} (2.4)

**Definition II.4**

$T^r_s$ IS SKEW-SYMMETRIC

$$T^r_s \text{ skew-symmetric in } (i,j) \forall(i,j)$$  \hspace{1cm} (2.5)

**Definition II.5**

**EXTERIOR ALGEBRA OVER $V$:**

$$\wedge^* V := (\bigcup_{p \in \mathbb{N}} \wedge^p V, \wedge)$$  \hspace{1cm} (2.6)

where:

$$\wedge^p V := \{ T^p_0 \in T^p_0(V) : T^p_0 \text{ skew-symmetric} \}$$  \hspace{1cm} (2.7)

with:

$$A \wedge B := \Lambda(A \otimes B) \in \wedge^{r+s} V \ A \in \wedge^r V, B \in \wedge^s V$$  \hspace{1cm} (2.8)

$$\Lambda(T)(v^1, \cdots, v^r) := \frac{1}{r!} \sum_{p \in S_r} \text{sign}(p)T(v^{p(1)}, \cdots, v^{p(r)}) \ T \in T^p_0(V), v_1, \cdots, v_r \in V$$  \hspace{1cm} (2.9)

Given a scalar product $q : V \times V \mapsto \mathbb{R}$ over $V$:

**Definition II.6**
\[ \mathcal{I}_q := \{ x \otimes v \otimes v + q(v, v) \otimes y \mid x, y \in T(V), v \in V \} \]  

(2.10)

One has that:

**Theorem II.1**

\[ T(V) = \bigwedge^* V \oplus \mathcal{I}_q \]  

(2.11)

Denoted by \( \pi_q : T(V) \to \bigwedge^* V \) the projection induced by the direct sum decomposition of theorem II.1 let us introduce the following:

**Definition II.7**

**CLIFFORD ALGEBRA ON** \( V \) **W.R.T.** \( q \): 
the algebra \( \text{Cl}_q(V) := (\bigwedge^* V, \cdot) \):

\[ \alpha \cdot \beta := \pi_q(s \otimes t) \mid s \in \pi_q^{-1}(\alpha), t \in \pi_q^{-1}(\beta) \]  

(2.12)

Let us recall that, given a scalar product \( q \) on \( V \), one can introduce the following:

**Definition II.8**

**QUADRATIC FORM W.R.T.** \( q \): 
the map \( Q : V \to \mathbb{R} \):

\[ Q(v) := q(v, v) \]  

(2.13)

Let us recall, furthermore, the following:

**Definition II.9**

\( q \) **IS NONDEGENERATE**: 
\[ q(v, w) = 0 \quad \forall v \in V \Rightarrow w = 0 \]  

(2.14)

We can at last present the following:

**Theorem II.2**

**SYLVESTER’S THEOREM:**  
**HP:**

\[ \dim_{\mathbb{R}} V = n \]  

(2.15)

\[ B := \{ e_1, \cdots, e_n \} \text{ basis of } V \]  

(2.16)

\[ q : V \times V \to \mathbb{R} \text{ scalar product } : Q(x) = \sum_{i=1}^{p} x_i^2 - \sum_{i=1}^{r} x_i^2 x = \sum_{i=1}^{n} x_i e_i p + r = n \]  

(2.17)

**TH:**

\[ \text{sign}(V, q) := (p, r) \text{ is } B\text{-independent} \]  

(2.18)

**Definition II.10**
Cl_{p,r} := Cl_q(\mathbb{R}^{p+r}) : \text{sign}(\mathbb{R}^{p+r}, q) = (p, r) \quad (2.19)

**Definition II.11**

**PAULI MATRICES:**

\[
\sigma_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.20)
\]

\[
\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (2.21)
\]

\[
\sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad (2.22)
\]

\[
\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (2.23)
\]

**Theorem II.3**

**CLASSIFICATION OF BIDIMENSIONAL CLIFFORD ALGEBRAS**

1. \[ Cl_{1,0} = \mathbb{G} = \mathbb{R} \oplus \mathbb{R} \quad (2.24) \]

2. \[ Cl_{0,1} = \mathbb{C} \quad (2.25) \]

**PROOF:**

1. Introduced the algebra:

\[ S_{1,0} := \{ x := x_1\sigma_0 + x_2\sigma_1 \mid x_1, x_2 \in \mathbb{R} \} \quad (2.26) \]

(with sum and product given, respectively, by matricial sum and matricial multiplication) and the linear map \( I_{S_{1,0}} : \mathbb{R}^2 \mapsto S_{1,0} \):

\[
I_{S_{1,0}}(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}) := x_1\sigma_0 + x_2\sigma_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_1 \end{pmatrix} \quad (2.27)
\]

one has that:

\[
I_{S_{1,0}}(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}) \cdot I_{S_{1,0}}(\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}) = \begin{pmatrix} x_1 \\ x_2 \\ x_1 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_1y_1 + x_2y_2 + x_1y_2 + x_2y_1 \\ x_1y_2 + x_2y_1 + x_1y_1 + x_2y_2 \end{pmatrix} \quad (2.28)
\]

is an algebra isomorphism among \((S_{1,0}, +, \cdot)\) and \((\mathbb{R} \oplus \mathbb{R}, +, \cdot, _{1,0})\) where in the latter algebra the sum is the usual sum of vectors in \(\mathbb{R}^2\) while the product is given by:

\[
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \cdot_{1,0} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1y_1 + x_2y_2 \\ x_1y_2 + x_2y_1 \end{pmatrix} \quad (2.29)
\]

From the other side the isomorphism among \((\mathbb{R} \oplus \mathbb{R}, +, \cdot, _{1,0})\) and \((\mathbb{G}, +, \cdot)\) appears evident as soon as one makes the identification:

\[
i \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (2.30)
\]

In particular:

\[
i^2 \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot_{1,0} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = +1 \equiv \sigma_1 \cdot \sigma_1 \quad (2.31)
\]
2. Introduced the algebra:

\[ S_{0,1} := \{ x := x_1 \sigma_0 + x_2 (i \sigma_2) \mid x_1, x_2 \in \mathbb{R} \} \quad (2.32) \]

where \( i \) is the usual complex imaginary unit such that \( i^2 = -1 \) (with sum and product given, respectively, by matricial sum and matricial multiplication) and the linear map \( I_{S_{0,1}} : \mathbb{R}^2 \rightarrow S_{0,1} \):

\[ I_{S_{0,1}} \left( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) := x_1 \sigma_0 + x_2 (i \sigma_2) = \begin{pmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{pmatrix} \quad (2.33) \]

one has that:

\[ I_{S_{0,1}} \left( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) \cdot I_{S_{0,1}} \left( \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right) = \begin{pmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{pmatrix} \cdot \begin{pmatrix} y_1 & y_2 \\ -y_2 & y_1 \end{pmatrix} = \begin{pmatrix} x_1 y_1 - x_2 y_2 & x_1 y_2 + x_2 y_1 \\ -x_1 y_2 - x_2 y_1 & x_1 y_1 - x_2 y_2 \end{pmatrix} \quad (2.34) \]

is an algebra isomorphism among \((S_{0,1}, +, \cdot)\) and \((\mathbb{R}^2, +, \cdot_{0,1})\) where in the latter algebra the sum is the usual sum of vectors in \( \mathbb{R}^2 \) while the product is given by:

\[ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \cdot_{0,1} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 y_1 - x_2 y_2 \\ x_1 y_2 + x_2 y_1 \end{pmatrix} \quad (2.35) \]

From the other side the isomorphism among \((\mathbb{R}^2, +, \cdot_{0,1})\) and \((\mathbb{C}, +, \cdot)\) appears evident as soon as one makes the identification:

\[ i \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (2.36) \]

In particular:

\[ i^2 \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot_{0,1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -1 \equiv (i \sigma_2) \cdot (i \sigma_2) \quad (2.37) \]

According to theorem II.3 both Complex Analysis and Hyperbolic Analysis may be seen as Analysis over a bidimensional Clifford algebra, the passage from the former to the latter corresponding to the ansatz:

\[ \text{sign}(\mathbb{R}^2, q) = (0, 1) \rightarrow \text{sign}(\mathbb{R}^2, q) = (1, 0) \quad (2.38) \]
III. LIMITS AND SERIES IN THE HYPERBOLIC PLANE

Let us consider the ring $G$ of the hyperbolic numbers, i.e. of the numbers of the form $z = a + ib \ a, b \in \mathbb{R}$, with the hyperbolic imaginary unit $i$ satisfying the following equation:

$$i^2 = +1$$ (3.1)

**Definition III.1**

**HYPERBOLIC CONJUGATE OF** $z = a + bi \ a, b \in \mathbb{R}$

$$z^* := a - ib$$ (3.2)

Given $z = a + bi \in G$:

**Definition III.2**

$$G_+ := \{z \in G : |z| := \sqrt{zz^*} \geq 0\}$$ (3.3)

**Definition III.3**

$$G^*_+ := \{z \in G : |z| := \sqrt{zz^*} > 0\}$$ (3.4)

Let us now introduce on $G$ the following norm:

**Definition III.4**

$$\|x + iy\| := \sqrt{x^2 + y^2} \ x, y \in \mathbb{R}$$ (3.5)

and the associated metric:

**Definition III.5**

$$d_{\|\cdot\|}(z_1, z_2) := \|z_1 - z_2\|$$ (3.6)

**Remark III.1**

THE NONISOMORPHISM OF THE ALGEBRAIC STRUCTURES $(G, +, \cdot, \| \cdot \|)$ AND $(\mathbb{C}, +, \cdot, \| \cdot \|)$

One could suspect that the adoption on $G$ of the norm of definition III.4 results in the collapse to the complex case. That this is not the case, anyway, may be immediately realized as soon as one introduces the complex algebraic structure $(\mathbb{C}, +, \cdot, \| \cdot \|)$ and compares it with the hyperbolic one $(G, +, \cdot, \| \cdot \|)$ by introducing the following:

**Definition III.6**

$1^{TTH}$ KIND COMPLEX TRANSFORM

the map $T : G \mapsto \mathbb{C}$:

$$T(x_1 + ix_2) := x_1 + jx_2 \ \forall x_1, x_2 \in \mathbb{R}$$ (3.7)

where $j \in \mathbb{C}$ is the complex imaginary unit such that:

$$j^2 := j \cdot j = -1$$ (3.8)

One has of course that:

**Theorem III.1**
1. \[ |T(z)| = \|z\| \quad \forall z \in G \quad (3.9) \]

2. \[ |T(z_1 + z_2)| = \|z_1 + z_2\| \quad \forall z_1, z_2 \in G \quad (3.10) \]

Let us observe anyway that:

\[ |T(z_1 \cdot z_2)| \neq \|z_1 \cdot z_2\| \quad (3.11) \]

The \( d\|\cdot\| \) allows to define limits on \( G \) in the usual way \([10]\).

Given a map \( f : G \mapsto G \) and two points \( z_1, z_2 \in G \):

**Definition III.7**

\( f \) TENDS TO \( z_2 \) IN THE LIMIT \( z \to z_1 \) \((\lim_{z \to z_1} f(z) = z_2)\):

\[ \forall \epsilon > 0, \exists \delta > 0 : \, d\|\cdot\|(z, z_1) < \delta \Rightarrow \, d\|\cdot\|(f(z), z_2) < \epsilon \quad (3.12) \]

Given a sequence \( \{a_n\}_{n \in \mathbb{N}} \) of hyperbolic numbers and a point \( z \in G \) in the hyperbolic plane:

**Definition III.8**

\( a_n \) TENDS TO \( z \) IN THE LIMIT \( n \to \infty \) \((\lim_{n \to \infty} a_n = z)\):

\[ \forall \epsilon > 0, \exists N \in \mathbb{N} : n > N \Rightarrow \, d\|\cdot\|(a_n, z) < \epsilon \quad (3.13) \]

**Definition III.9**

\[ \sum_{n=0}^{\infty} a_n z^n := \lim_{n \to \infty} \sum_{i=0}^{n} a_i z^i \quad (3.14) \]

(of course provided the limit exists).
IV. THE HYPERBOLIC EULER FORMULA

Let us introduce the following maps defined in the points of the hyperbolic plane where the series converges:

Definition IV.1

\[ \exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!} \]  
(4.1)

Definition IV.2

\[ \cosh(z) := \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \]  
(4.2)

Definition IV.3

\[ \sinh(z) := \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \]  
(4.3)

One has that:

Theorem IV.1

HYPERBOLIC EULER FORMULA:

\[ \exp(i\theta) = \cosh(\theta) + i \sinh(\theta) \quad \forall \theta \in \mathbb{R} \]  
(4.4)

PROOF:

\[ \exp(i\theta) = \sum_{n=0}^{\infty} \frac{(i\theta)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(i\theta)^{2n+1}}{(2n+1)!} \]  
(4.5)

Since:

\[ (i)^{2n} = 1 \quad \forall n \in \mathbb{N} \]  
(4.6)

\[ (i)^{2n+1} = i \quad \forall n \in \mathbb{N} \]  
(4.7)

the thesis immediately follows ■
V. ANALYTIC FUNCTIONS IN THE HYPERBOLIC PLANE

Given a map \( f : \mathbb{G} \to \mathbb{G} \) and a point \( z_0 \in \mathbb{G} \) of the hyperbolic plane:

**Definition V.1**

**DERIVATIVE OF \( f \) IN \( z_0 \):**

\[
f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{5.1}
\]

**Theorem V.1**

**CAUCHY-RIEMANN CONDITIONS IN THE HYPERBOLIC PLANE:**

**HP:**

\[ z_0 = x_0 + iy_0 \in \mathbb{G} \]

\[ f : \mathbb{G} \to \mathbb{G} : \exists f'(z_0) \]

\[ f(x + iy) = u(x,y) + iv(x,y) \tag{5.2} \]

**TH:**

\[
\frac{\partial u}{\partial x}(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0) \\
\frac{\partial u}{\partial y}(x_0,y_0) = \frac{\partial v}{\partial x}(x_0,y_0)
\]

**PROOF:**

By definition V.1 we have that:

\[
f'(z_0) = \lim_{\Delta x \to 0, \Delta y \to 0} \left( \frac{u(x_0 + i\Delta x_0, y_0 + i\Delta y_0) - u(x_0,y_0)}{\Delta x + i\Delta y} + i \frac{v(x_0 + i\Delta x_0, y_0 + i\Delta y_0) - v(x_0,y_0)}{\Delta x + i\Delta y} \right) \tag{5.3}
\]

Since this limit has to be always the same for all the paths going to \( z_0 \) it has in particular to exist for the two particular paths in which, respectively, \( \Delta x = 0 \) and \( \Delta y = 0 \). For the first path we get:

\[
f'(z_0) = \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x_0, y_0) - u(x_0,y_0)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x_0 + \Delta x_0, y_0) - v(x_0,y_0)}{\Delta x} = \frac{\partial u}{\partial x}(x_0,y_0) + i \frac{\partial v}{\partial x}(x_0,y_0) \tag{5.4}
\]

while for the second path we obtain:

\[
f'(z_0) = \lim_{\Delta y \to 0} \frac{u(x_0,y_0 + \Delta y_0) - u(x_0,y_0)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x_0,y_0 + \Delta y_0) - v(x_0,y_0)}{i\Delta y} = \frac{\partial u}{\partial y}(x_0,y_0) + \frac{\partial v}{\partial y}(x_0,y_0) \tag{5.5}
\]

where in the last passage we have used the fact that in the hyperbolic plane:

\[
\frac{1}{i} = i \tag{5.6}
\]

Equating the real and imaginary part of eq.5.4 and eq.5.5 the thesis follows \( \blacksquare \)
Corollary V.1

HP:

\[ z_0 = x_0 + iy_0 \in \mathcal{G} \]

\[ f : \mathcal{G} \rightarrow \mathcal{G} : \exists f'(z_0) \]

\[ f(x + iy) = u(x, y) + iv(x, y) \tag{5.7} \]

TH:

\[ f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0) = \frac{\partial u}{\partial y}(x_0, y_0) + i \frac{\partial v}{\partial y}(x_0, y_0) \tag{5.8} \]

Corollary V.2

HP:

\[ z_0 = x_0 + iy_0 \in \mathcal{G} \]

\[ f : \mathcal{G} \rightarrow \mathcal{G} : \exists f'(z_0) \]

\[ f(x + iy) = u(x, y) + iv(x, y) \tag{5.9} \]

TH:

1. \[ \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right) u(x_0, y_0) = 0 \tag{5.10} \]

2. \[ \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right) v(x_0, y_0) = 0 \tag{5.11} \]

PROOF:

1. Differentiating the first hyperbolic Cauchy-Riemann equation w.r.t. x one obtains:

\[ \frac{\partial^2}{\partial x^2} u(x_0, y_0) = \frac{\partial}{\partial x} \frac{\partial}{\partial y} v(x_0, y_0) \tag{5.12} \]

Differentiating the second hyperbolic Cauchy-Riemann equation w.r.t. y one obtains:

\[ \frac{\partial^2}{\partial y^2} u(x_0, y_0) = \frac{\partial}{\partial x} \frac{\partial}{\partial y} v(x_0, y_0) \tag{5.13} \]

Subtracting the eq.5.13 from the eq.5.12 one obtains the thesis
2. Differentiating the first hyperbolic Cauchy-Riemann equation w.r.t. y one obtains:

$$\partial_y \partial_x u(x_0, y_0) = \partial_y^2 v(x_0, y_0) \quad (5.14)$$

Differentiating the second hyperbolic Cauchy-Riemann equation w.r.t. x one obtains:

$$\partial_x \partial_y u(x_0, y_0) = \partial_x^2 v(x_0, y_0) \quad (5.15)$$

Subtracting the eq.5.14 from the eq.5.15 one obtains the thesis

Definition V.2

f IS ANALYTIC IN $z_0 \in \mathbb{G}$

$$\exists O \in \mathcal{T}_G : (z_0 \in O) \land (\exists f'(z) \forall z \in O) \quad (5.16)$$

Remark V.1

Let us observe that according to definition V.2 it is not required that $u, v \in C^\infty(\mathbb{R}^2)$ as, in our opinion erroneously, is made in [11]

Example V.1

$$f(z) := z^2 = (x + iy)^2 = x^2 + y^2 + 2ixy = u(x, y) + iv(x, y) \quad (5.17)$$

$$u(x, y) = x^2 + y^2 \quad (5.18)$$

$$v(x, y) = 2xy \quad (5.19)$$

$$\partial_x u = 2x = \partial_y v \quad (5.20)$$

$$\partial_y u = 2y = \partial_x v \quad (5.21)$$

It follows that $f(z)$ is analytic in the whole hyperbolic plane.

Example V.2

$$f(z) := \exp(z) = \exp(x)(\cosh(y) + i\sinh(y)) = u(x, y) + iv(x, y) \quad (5.22)$$

$$u(x, y) = \exp(x) \cosh(y) \quad (5.23)$$

$$v(x, y) = \exp(x) \sinh(y) \quad (5.24)$$

$$\partial_x u = \exp(x) \cosh(y) = \partial_y v \quad (5.25)$$

$$\partial_y u = \exp(x) \sinh(y) = \partial_x v \quad (5.26)$$

It follows that $f(z)$ is analytic in the whole hyperbolic plane.
Example V.3

Let us consider the map $f : G \rightarrow Diagonals$:

$$Diagonals := \{x + iy \in G : y = \pm x\} \quad (5.27)$$

$$f(z) := \frac{1}{z} = \frac{x - iy}{x^2 - y^2} = u(x, y) + iv(x, y) \quad (5.28)$$

$$u(x, y) = \frac{x}{x^2 - y^2} \quad (5.29)$$

$$v(x, y) = \frac{-y}{x^2 - y^2} \quad (5.30)$$

$$\partial_x u = -\frac{x^2 + y^2}{(x^2 - y^2)^2} = \partial_y v \quad (5.31)$$

$$\partial_y u = \frac{2xy}{(x^2 - y^2)^2} = \partial_x v \quad (5.32)$$

It follows that $f$ is analytic in its whole domain of definition $G - Diagonals$.

Since:

Lemma V.1

$$z \cdot z^* = Q_{1,1}(\begin{pmatrix} x \\ y \end{pmatrix}) = x^2 - y^2 \forall z = x + iy \in \mathbb{G} \quad (5.33)$$

it would appear more natural to look at $\mathbb{G}$ as the couple $(\mathbb{R}^2, q_{1,1})$. So, for prudence, we will consider all the possible signatures of the bilinear symmetric form one is implicitly assuming in the definition of the notion of angles among two vectors.

So, given two curves $\gamma_1, \gamma_2$ in $G$ intersecting in a point $z_0$ and given $n, m \in \mathbb{N} : n + m = 2$:

Definition V.3

(n,m)-ANGLE AMONG $\gamma_1$ and $\gamma_2$ IN $z_0$

$$(n, m) - angle_{z_0}(\gamma_1, \gamma_2) := q_{n,m}(\hat{e}_1, \hat{e}_2) \quad (5.34)$$

where $\hat{e}_i$ is the tangent vector to $\gamma_i$ in $z_0$ $i = 1, 2$.

Given a function $f : \mathbb{G} \rightarrow \mathbb{G}$:

Definition V.4

f IS (n,m)-CONFORMAL in $z_0 \in \mathbb{G}$

$$\gamma_1, \gamma_2 \text{ curves in } \mathbb{G} \text{ intersecting in } z_0 \Rightarrow (n, m) - angle_{f(z_0)}(\gamma'_1, \gamma'_2) = (n, m) - angle_{z_0}(\gamma_1, \gamma_2) \quad (5.35)$$

where:

$$\gamma'_i = f(\gamma_i) \quad i = 1, 2 \quad (5.36)$$

Contrary to the analogous situation in Complex Analysis one has that:

Theorem V.2
1. 

\[(f \text{ analytic in } z_0 \land f'(z_0) \neq 0) \Rightarrow f(2,0)\text{-conformal in } z_0\]  

(5.37)

2. 

\[(f \text{ analytic in } z_0 \land f'(z_0) \neq 0) \Rightarrow f(1,1)\text{-conformal in } z_0\]  

(5.38)

PROOF:

1. By definition:

\[(2,0) - \text{angle}_{z_0}(\gamma_1, \gamma_2) = q_{2,0}(\hat{e}_1, \hat{e}_2)\]  

(5.39)

where \(\hat{e}_i\) is the unit tangent vector to \(\gamma_i\) in \(z_0\) \(i = 1, 2\).

Similarly:

\[(2,0) - \text{angle}_{f(z_0)}(\gamma'_1, \gamma'_2) = q_{2,0}(\hat{e}'_1, \hat{e}'_2)\]  

(5.40)

where \(\hat{e}'_i\) is the unit tangent vector to \(\gamma'_i\) in \(f(z_0)\) \(i = 1, 2\). Since an infinitesimal displacement along \(\gamma_i\) may be expressed formally as \(\delta x_i \hat{e}_x + \delta y_i \hat{e}_y\) one has that:

\[
\hat{e}_i = \frac{\delta x_i \hat{e}_x + \delta y_i \hat{e}_y}{\sqrt{(\delta x_i)^2 + (\delta y_i)^2}} \quad i = 1, 2
\]  

(5.41)

\[
\hat{e}'_i = \frac{\delta x'_i \hat{e}_x + \delta y'_i \hat{e}_y}{\sqrt{(\delta x'_i)^2 + (\delta y'_i)^2}} \quad i = 1, 2
\]  

(5.42)

Therefore:

\[
q_{2,0}(\hat{e}_1, \hat{e}_2) = \frac{\delta x_1 \delta x_2 + \delta y_1 \delta y_2}{\prod_{i=1}^2 \sqrt{(\delta x_i)^2 + (\delta y_i)^2}}
\]  

(5.43)

\[
q_{2,0}(\hat{e}'_1, \hat{e}'_2) = \frac{\delta x'_1 \delta x'_2 + \delta y'_1 \delta y'_2}{\prod_{i=1}^2 \sqrt{(\delta x'_i)^2 + (\delta y'_i)^2}}
\]  

(5.44)

Substituting the relations:

\[
\delta x'_i = \partial_x u \delta x_i + \partial_y u \delta y_i \quad i = 1, 2
\]  

(5.45)

\[
\delta y'_i = \partial_x v \delta x_i + \partial_y v \delta y_i \quad i = 1, 2
\]  

(5.46)

into the eq \ref{5.44} one obtains:

\[
\hat{e}'_1 \cdot \hat{e}'_2 = \frac{\prod_{i=1}^2 \sqrt{[(\partial_x u)^2 + (\partial_y v)^2] \delta x_1 \delta x_2 + [(\partial_y u)^2 + (\partial_x v)^2] \delta y_1 \delta y_2 + (\partial_x u \partial_y v + \partial_y v \partial_x u)(\delta x_1 \delta y_2 + \delta x_2 \delta y_1)}}{\prod_{i=1}^2 \sqrt{[(\partial_x u)^2 + (\partial_y v)^2] \delta x_1 \delta x_2 + [(\partial_y u)^2 + (\partial_x v)^2] \delta y_1 \delta y_2 + 2(\partial_x u \partial_y v + \partial_y v \partial_x u) \delta x_1 \delta y_1}}
\]  

(5.47)

that using the hyperbolic Cauchy-Riemann equations reduces to:

\[
\hat{e}'_1 \cdot \hat{e}'_2 = \frac{(\nabla u)^2 (\delta x_1 \delta x_2 + \delta y_1 \delta y_2) + (\nabla v \cdot \nabla v)(\delta x_1 \delta y_2 + \delta x_2 \delta y_1)}{\prod_{i=1}^2 \sqrt{[(\nabla u)^2 + (\nabla y)^2] \delta x_1 \delta x_2 + [(\nabla y)^2 + (\nabla x)^2] \delta y_1 \delta y_2 + 2\nabla u \cdot \nabla v \delta x_1 \delta y_1}} \neq \frac{\delta x_1 \delta x_2 + \delta y_1 \delta y_2}{\prod_{i=1}^2 \sqrt{[(\nabla u)^2 + (\nabla y)^2] \delta x_1 \delta x_2 + [(\nabla y)^2 + (\nabla x)^2] \delta y_1 \delta y_2}}
\]  

(5.48)

2. Since an infinitesimal displacement along \(\gamma_i\) may be again expressed formally as \(\delta x_i \hat{e}_x + \delta y_i \hat{e}_y\) one has that again:

\[
\hat{e}_i = \frac{\delta x_i \hat{e}_x + \delta y_i \hat{e}_y}{\sqrt{(\delta x_i)^2 + (\delta y_i)^2}} \quad i = 1, 2
\]  

(5.49)
\[ \hat{e}_i' = \frac{\delta x'_i \hat{e}_x + \delta y'_i \hat{e}_y}{\sqrt{(\delta x'_i)^2 + (\delta y'_i)^2}} \quad i = 1, 2 \] (5.50)

Let us now observe that:

\[ q_{1,1}(\hat{e}_1', \hat{e}_2') = \frac{\delta x_1 \delta x_2 - \delta y_1 \delta y_2}{\prod_{i=1}^{2} \sqrt{(\delta x'_i)^2 - (\delta y'_i)^2}} \] (5.51)

\[ q_{1,1}(\hat{e}_1', \hat{e}_2') = \frac{\delta x'_1 \delta x'_2 - \delta y'_1 \delta y'_2}{\prod_{i=1}^{2} \sqrt{(\delta x'_i)^2 - (\delta y'_i)^2}} \] (5.52)

Substituting the relations:

\[ \delta x'_i = \partial_x u \delta x_i + \partial_y u \delta y_i \quad i = 1, 2 \] (5.53)

\[ \delta y'_i = \partial_x v \delta x_i + \partial_y v \delta y_i \quad i = 1, 2 \] (5.54)

into the eq. (5.52) one obtains:

\[ \hat{e}_1' \cdot \hat{e}_2' = \frac{(\partial_x u \delta x_1 + \partial_y u \delta y_1)(\partial_x v \delta x_2 + \partial_y v \delta y_2) - (\partial_x v \delta x_1 + \partial_y v \delta y_1)(\partial_x u \delta x_2 + \partial_y u \delta y_2)}{\sqrt{(\partial_x u \delta x_1 + \partial_y u \delta y_1)^2 - (\partial_x v \delta x_1 + \partial_y v \delta y_1)^2}} \sqrt{(\partial_x u \delta x_2 + \partial_y u \delta y_2)^2 - (\partial_x v \delta x_2 + \partial_y v \delta y_2)^2}} \] (5.55)

that after a tedious computation using the hyperbolic Cauchy-Riemann conditions may be expressed as:

\[ \hat{e}_1' \cdot \hat{e}_2' = \frac{[(\partial_x u)^2 - (\partial_y u)^2] \delta x_1 \delta x_2 + [(\partial_y v)^2 - (\partial_x v)^2] \delta y_1 \delta y_2 + (\partial_x u \partial_y u - \partial_x v \partial_y v)(\delta x_1 \delta y_2 + \delta x_2 \delta y_1)}{\prod_{i=1}^{2}[(\partial_x u)^2 - (\partial_y u)^2] \delta y_i^2 - \delta y_i^2 + 2(\partial_x u \partial_y u - \partial_x v \partial_y v) \delta x_i \delta y_i} \neq \hat{e}_1 \cdot \hat{e}_2 \] (5.56)

Given a function \( f : \mathbb{G} \rightarrow \mathbb{G} \) such that:

\[ f(z) = f(x + iy) = u(x, y) + iv(x, y) \] (5.57)

and a continuous curve \( C \) in \( \mathbb{G} \):

**Definition V.5**

**INTEGRAL OF \( f \) ALONG \( C \):**

\[ \int_C f(z) dz = \int_C (u(x, y) + iv(x, y)) \, dx + i \, dy \] (5.58)

One has the following:

**Theorem V.3**

**HYPERBOLIC CAUCHY-GOURSAT THEOREM:**

\[ G \subset \mathbb{G} : \pi_1(G) = \{1\} : \text{f is analytic in } G \] (5.59)

\[ C \subset G \text{ closed smooth curve not self-linking} \] (5.60)

**TH:**
\[ \oint_C f(z)dz = 0 \] (5.61)

PROOF:

Let us consider \( G \) as the xy-plane of \( \mathbb{R}^3 \).

Introduced the vector fields:

\[ \vec{A}_1 = (u, v, 0) \] (5.62)
\[ \vec{A}_2 = (v, u, 0) \] (5.63)

one has that:

\[ \oint_C f(z)dz = \oint_C \vec{A}_1 \cdot d\vec{r} + i \oint_C \vec{A}_2 \cdot d\vec{r} \] (5.64)

By Stokes Theorem one has that:

\[ \oint_C \vec{A}_i \cdot d\vec{r} = 0 \iff \nabla \times \vec{A}_i (x, y) = 0 \forall \, z = x + iy \in \partial^{-1} C \, \, \, i = 1, 2 \] (5.65)

Since:

\[ \nabla \times \vec{A}_1 = \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \hat{e}_z \] (5.66)
\[ \nabla \times \vec{A}_2 = \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) \hat{e}_z \] (5.67)

(where \( \{ \hat{e}_x := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \hat{e}_y := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \hat{e}_z := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \} \) is the canonical basis of \( \mathbb{R}^3 \)) the thesis immediately follows from theorem V.1 \( \blacksquare \)

Example V.4

\[ c_{a, b} : [0, 2\pi) \to \mathbb{G} \, \, \, a, b \in \mathbb{R}_+ : c_{a, b}(t) := a \cos(t) + ib \sin(t) \] (5.68)

\[ f(z) := z^2 = u(x, y) + iv(x, y) \] (5.69)

\[ u(x, y) = x^2 + y^2 \] (5.70)

\[ v(x, y) = 2xy \] (5.71)

\[ \oint_{c_{a, b}} f(z)dz = \oint_{c_{a, b}} (x^2 + y^2)dx + 2xydy + i \oint_{c_{a, b}} 2xydx + (x^2 + y^2)dy \] (5.72)

\[ \oint_{c_{a, b}} f(z)dz = \int_0^{2\pi} [(a^2 \cos^2(t) + b^2 \sin^2(t)) \cdot (-a \sin(t)) + 2a \cos(t)b \sin(t) \cos(t)]dt + \\
\quad i \int_0^{2\pi} (-2a^2 b \cos(t) \sin^2(t) + a^2 b \cos^3(t) + b^3 \sin^2(t) \cos(t)) = 0 \] (5.73)
Example V.5
\[ c_{a,b} : [0, 2\pi) \mapsto \mathbb{C} \quad a, b \in \mathbb{R}_+ : c_{a,b}(t) := a \cos(t) + ib \sin(t) \quad (5.74) \]
\[ f(z) := \exp(z) = u(x, y) + iv(x, y) \quad (5.75) \]
\[ u(x, y) = \exp(x) \cosh(y) \quad (5.76) \]
\[ v(x, y) = \exp(x) \sinh(y) \quad (5.77) \]
\[ \oint_{c_{a,b}} f(z) dz = \oint_{c_{a,b}} \exp(x) \cosh(y) dx + \exp(x) \sinh(y) dy + i \oint_{c_{a,b}} \exp(x) \sinh(y) dx + \exp(x) \cosh(y) dy \quad (5.78) \]
\[ \oint_{c_{a,b}} f(z) dz = \int_0^{2\pi} \left[ \exp(a \cos(t)) \cosh(b \sin(t))(-a \sin(t)) + \exp(a \cos(t)) \sinh(b \sin(t))(b \cos(t)) \right] dt + \]
\[ i \int_0^{2\pi} \left[ \exp(a \cos(t)) \sinh(b \sin(t))(-a \sin(t)) + \exp(a \cos(t)) \cosh(b \sin(t))(b \cos(t)) \right] = 0 \quad (5.79) \]

Example V.6
\[ c_{a,b} : [0, 2\pi) \mapsto \mathbb{C} \quad a := 1, b := \frac{1}{2} : c_{a,b}(t) := 4 + a \cos(t) + i(2 + b \sin(t)) \quad (5.80) \]
\[ f(z) := \frac{1}{z} = u(x, y) + iv(x, y) \quad (5.81) \]
\[ u(x, y) = \frac{x}{x^2 - y^2} \quad (5.82) \]
\[ v(x, y) = \frac{-y}{x^2 - y^2} \quad (5.83) \]
\[ \oint_{c_{a,b}} f(z) dz = \oint_{c_{a,b}} \frac{x}{x^2 - y^2} dx + \frac{-y}{x^2 - y^2} dy + i \oint_{c_{a,b}} \frac{-y}{x^2 - y^2} dx + \frac{x}{x^2 - y^2} dy \quad (5.84) \]
\[ \oint_{c_{a,b}} f(z) dz = \int_0^{2\pi} \left[ \frac{(4 + a \cos(t))}{(4 + a \cos(t))^2 - (2 + b \sin(t))^2}(-a \sin(t)) - \frac{(2 + b \sin(t))}{(4 + a \cos(t))^2 - (2 + b \sin(t))^2}(b \cos(t)) \right] dt + \]
\[ i \int_0^{2\pi} \left[ \frac{-2 + b \sin(t)}{(4 + a \cos(t))^2 - (2 + b \sin(t))^2}(-a \sin(t)) + \frac{(4 + a \cos(t))}{(4 + a \cos(t))^2 - (2 + b \sin(t))^2}(b \cos(t)) \right] dt = 0 \quad (5.85) \]

Let us observe that as in the complex case one has that:

Lemma V.2
\[\gamma \text{ curve in } \mathbb{G} \text{ not self-linking} \quad (5.86)\]

\[f : \mathbb{G} \to \mathbb{G} \text{ continuous : } (\exists M \in \mathbb{R}_+ : \|f(z)\| \leq M \forall z \in \gamma) \quad (5.87)\]

\[\|
\int_{\gamma} f(z)dz \| \leq ML_{\gamma} \quad (5.88)\]

where \(L_{\gamma}\) denotes the length of the curve \(\gamma\)

**PROOF:**

\[
\|
\int_{\gamma} f(z)dz \| = \lim_{N \to \infty, \Delta z_i \to 0} \sum_{i=1}^{N} \| f(z_i) \Delta z_i \| = \lim_{N \to \infty, \Delta z_i \to 0} \sum_{i=1}^{N} \| f(z_i) \Delta z_i \| \leq \lim_{N \to \infty, \Delta z_i \to 0} \sum_{i=1}^{N} \| f(z_i) \| \| \Delta z_i \| \leq M \sum_{i=1}^{N} \| \Delta z_i \| = ML_{\gamma} \quad (5.89)
\]

We are now ready to analyze a great difference among Complex Calculus and Hyperbolic Calculus: in the latter case the Cauchy Integral formula doesn’t hold.

Let us recall that:

**Theorem V.4**

**COMPLEX CAUCHY INTEGRAL FORMULA:**

**HP:**

\[G \subset \mathbb{C} : \pi_1(G) = \{I\} : f \text{ is analytic in } G \quad (5.90)\]

\[C \subset G \text{ closed smooth curve not self-linking} \quad (5.91)\]

\[z_0 \in \text{Interior}(\partial^{-1}C) \quad (5.92)\]

**TH:**

\[f(z_0) = \frac{1}{2\pi j} \oint_C \frac{f(z)}{z - z_0} \quad (5.93)\]

That theorem doesn’t hold if one replace \(\mathbb{C}\) with \(\mathbb{G}\) may be verified through the following:

**Example V.7**
Let us consider the following function analytic in the whole hyperbolic plane $f : \mathbb{G} \to \mathbb{G}$:

$$f(z) := z^2 + a \quad a \in \mathbb{G} : a \neq 0$$

(5.94)

and the circle:

$$c : [0, 2\pi) \to \mathbb{G} : c(t) := \cos(t) + i\sin(t)$$

(5.95)

One has that:

$$f(0) = a \neq \frac{1}{2\pi i} \oint_{c} f(z) z - 0 = \frac{1}{2\pi i} \left( \oint_{c} z + \oint_{c} \frac{dz}{z} \right) = \frac{a}{2\pi i} \left( \oint_{c} -\frac{x}{x^2 - y^2} dx - \frac{y}{x^2 - y^2} dy + i \oint_{c} -\frac{y}{x^2 - y^2} dx + \frac{x}{x^2 - y^2} dy \right)$$

(5.96)

$$f(0) = a \neq \frac{a}{2\pi i} \left\{ -2 \int_{0}^{2\pi} \frac{\sin(t)\cos(t)}{\cos^2(t) - \sin^2(t)} dt + i \int_{0}^{2\pi} \frac{dt}{\cos^2(t) - \sin^2(t)} \right\} = -2I_1 + iI_2$$

(5.97)

where:

$$I_1 := \int_{0}^{2\pi} \frac{\sin(t)\cos(t)}{\cos^2(t) - \sin^2(t)} dt = -\frac{1}{4} \lim_{\epsilon \to 0} \left[ \log |\cos(2t)| \right]_{0}^{\frac{\pi}{4} - \epsilon} + \left[ \log |\cos(2t)| \right]_{\frac{\pi}{4} + \epsilon}^{\frac{3\pi}{4} - \epsilon} +$$

$$\left[ \log |\cos(2t)| \right]_{\frac{3\pi}{4} + \epsilon}^{\frac{7\pi}{4} - \epsilon} + \left[ \log |\cos(2t)| \right]_{\frac{7\pi}{4} + \epsilon}^{\frac{9\pi}{4} - \epsilon} + \left[ \log |\cos(2t)| \right]_{\frac{9\pi}{4} + \epsilon}^{\frac{11\pi}{4} - \epsilon} = 0$$

(5.98)

and:

$$I_2 := \int_{0}^{2\pi} \frac{dt}{\cos^2(t) - \sin^2(t)} = \lim_{\epsilon \to 0} \left[ \arctanh(\tan(t)) \right]_{\frac{\pi}{4} - \epsilon}^{\frac{\pi}{4} + \epsilon} + \left[ \arctanh(\tan(t)) \right]_{\frac{3\pi}{4} - \epsilon}^{\frac{\pi}{4} + \epsilon} +$$

$$\left[ \arctanh(\tan(t)) \right]_{\frac{5\pi}{4} - \epsilon}^{\frac{\pi}{4} + \epsilon} + \left[ \arctanh(\tan(t)) \right]_{\frac{7\pi}{4} - \epsilon}^{\frac{3\pi}{4} + \epsilon} = 0$$

(5.99)

so that:

$$f(0) = a \neq 0$$

(5.100)

To understand the difference existing among the complex and the hyperbolic cases let us recall the proof of theorem 5.3 in that one consider the curve $C'$ coinciding with $C$ apart from a little deformation in which $C$ go nearest to $z_0$ along an horizontal segment $L_1$, makes a circle $\gamma_\delta$ of radius $\delta$ with orientation opposite to that of $C$ (clockwise) and then returns to $C$ along an horizontal segment $L_2$ equal to $L_1$ but with opposite direction.

In the complex case one has that the function $\phi(z) := \frac{f(z) - f(z_0)}{z - z_0}$ is analytic in the region $\partial^{-1}C'$ so that, by the Cauchy-Goursat theorem, one has that:

$$\oint_{C'} \phi(z) = \oint_{C} \phi(z) + \oint_{L_1} \phi(z) + \oint_{L_2} \phi(z) - \oint_{\gamma_\delta} \phi(z) = 0$$

(5.101)

and since:

$$\oint_{L_2} \phi(z) = -\oint_{L_1} \phi(z)$$

(5.102)

one can infer that:

$$\oint_{C} \phi(z) = \oint_{\gamma_\delta} \phi(z)$$

(5.103)

that applying the complex analogous of theorem 5.2 to the function $\frac{f(z) - f(z_0)}{z - z_0}$ immediately leads to the thesis.

Contrary, in the hyperbolic case, the function $\phi(z) := \frac{f(z) - f(z_0)}{z - z_0}$ is not analytic in the region $\partial^{-1}C'$ since:

$$\partial^{-1}C' \cap \{ x + iy \in \mathbb{G} : y - y_0 = \pm(x - x_0) \} \neq 0$$

(5.104)
Remark V.2

One would be tempted to state the following:

**Conjecture V.1**

\[ \int_{\mathbb{C}^*} f(z) \, dz = 0 \quad \forall f \text{ analytical} \]

Conjecture V.1 is, anyway, false as it is shown by the following:

**Example V.8**

Let us consider the following function \( f : \mathbb{G} \rightarrow \mathbb{G} \):

\[ f(z) := \exp(z) = u(x, y) + i v(x, y) \quad (5.105) \]

\[ u(x, y) = \exp(x) \cosh(y) \quad (5.106) \]

\[ v(x, y) = \exp(x) \sinh(y) \quad (5.107) \]

and the two curves \( c_i : (-\infty, +\infty) \rightarrow \mathbb{G} \quad i = 1, 2; \)

\[ c_1(t) = (\cosh(t), \sinh(t)) \quad (5.108) \]

\[ c_2(t) = (-\cosh(t), -\sinh(t)) \quad (5.109) \]

One has that:

\[ \int_{c_1} f(z) \, dz = \int_{c_1} \exp(x) \cosh(y) \, dx + \exp(x) \sinh(y) \, dy + i \int_{c_1} \exp(x) \sinh(y) \, dx + \exp(x) \cosh(y) \, dy \quad (5.110) \]

and hence:

\[ \int_{c_1} f(z) \, dz = \int_{-\infty}^{+\infty} \exp(\cosh(t)) \cosh(\sinh(t)) + \exp(\cosh(t)) \sinh(\sinh(t)) \cosh(t) \]

\[ + i \int_{-\infty}^{+\infty} \exp(\cosh(t)) \sinh(\sinh(t)) \sinh(t) + \exp(\cosh(t)) \cosh(\sinh(t)) \cosh(t) \, dt \]

\[ = \left[ \exp(\cosh(t)) \cosh(\sinh(t)) \right]_{-\infty}^{+\infty} + i \left[ \exp(\cosh(t)) \sinh(\sinh(t)) \right]_{-\infty}^{+\infty} \quad (5.111) \]

and:

\[ \int_{c_2} f(z) \, dz = \int_{c_2} \exp(x) \cosh(y) \, dx + \exp(x) \sinh(y) \, dy + i \int_{c_2} \exp(x) \sinh(y) \, dx + \exp(x) \cosh(y) \, dy \quad (5.112) \]

and hence:

\[ \int_{c_2} f(z) \, dz = \int_{-\infty}^{+\infty} \exp(-\cosh(t)) \cosh(-\sinh(t)) + \exp(-\cosh(t)) \sinh(-\sinh(t)) (-\cosh(t)) \]

\[ + i \int_{-\infty}^{+\infty} \exp(-\cosh(t)) \sinh(-\sinh(t)) (-\sinh(t)) + \exp(-\cosh(t)) \cosh(-\sinh(t)) (-\cosh(t)) \, dt \]

\[ = \left[ \exp(-\cosh(t)) \cosh(\sinh(t)) \right]_{-\infty}^{+\infty} + i \left[ -\exp(-\cosh(t)) \sinh(\sinh(t)) \right]_{-\infty}^{+\infty} \quad (5.113) \]

from which it follows that:

\[ \int_{c_1} f(z) \, dz \pm \int_{c_2} f(z) \, dz \neq 0 \quad (5.114) \]
**Example V.9**

Let us consider the following function $f : \mathbb{G} \mapsto \mathbb{C}$:

$$f(z) := z^2 = u(x, y) + i v(x, y)$$

(5.115)

$$u(x, y) = x^2 + y^2$$

(5.116)

$$v(x, y) = 2xy$$

(5.117)

and the two curves $c_i : (-\infty, +\infty) \mapsto \mathbb{G}$ $i = 1, 2$:

$$c_1(t) = (\cosh(t), \sinh(t))$$

(5.118)

$$c_2(t) = (-\cosh(t), -\sinh(t))$$

(5.119)

One has that:

$$\int_{c_1} z^2 dz = \int_{c_1} (x^2 + y^2) dx + 2xy dy + i \int_{c_1} 2xy dx + (x^2 + y^2) dy$$

(5.120)

and hence:

$$\int_{c_1} z^2 dz = \int_{-\infty}^{+\infty} (\cosh^2(t) + \sinh^2(t)) \sinh(t) + 2 \cosh^2(t) \sinh(t) dt$$

$$+ i \int_{-\infty}^{+\infty} 2 \cosh(t) \sinh^2(t) + (\cosh^2(t) + \sinh^2(t)) \cosh(t) dt$$

$$= \left[ \frac{1}{3} \cosh(t) \right]_{-\infty}^{+\infty} + i \left[ \frac{1}{3} \sinh(t) \right]_{-\infty}^{+\infty}$$

(5.121)

and:

$$\int_{c_2} z^2 dz = \int_{c_2} (x^2 + y^2) dx + 2xy dy + i \int_{c_2} 2xy dx + (x^2 + y^2) dy$$

(5.122)

and hence:

$$\int_{c_2} z^2 dz = \int_{-\infty}^{+\infty} (\cosh^2(t) + \sinh^2(t)) (-\sinh(t)) + 2 \cosh^2(t) (-\sinh(t)) dt$$

$$+ i \int_{-\infty}^{+\infty} 2 \cosh(t) (-\sinh^2(t)) + (\cosh^2(t) + \sinh^2(t)) (-\cosh(t)) dt$$

$$= \left[ -\frac{1}{3} \cosh(3t) \right]_{-\infty}^{+\infty} + i \left[ -\frac{1}{3} \sinh(t) \right]_{-\infty}^{+\infty}$$

(5.123)

from which it follows that:

$$\int_{c_1} f(z) dz = \int_{c_2} f(z) dz \neq 0$$

(5.124)

With analogy to Conjecture V.1 one would be tempted to state the following:

**Conjecture V.2**

$$f(z_0) = \frac{1}{2\pi i} \int_{(z-z_0)(z^* - z_0) = 1} \frac{f(z)}{z - z_0} \forall f \text{ analytic}$$

Conjecture V.2 is, anyway, false as it is proved by the following:
Example V.10

Given the function:

\[ f(z) := z^2 + c \text{ with } c \neq 0 \]  

(5.125)

and the two curves \( c_i : (-\infty, +\infty) \mapsto G \ i = 1, 2; \)

\[ c_1(t) = (\cosh(t), \sinh(t)) \]  

(5.126)

\[ c_2(t) = (-\cosh(t), -\sinh(t)) \]  

(5.127)

we will show that:

\[ \frac{1}{2\pi i} \left( \int_{c_1} f(z) \pm \int_{c_2} f(z) \right) \neq 0 \]  

(5.128)

At this purpose let us observe that:

\[ \frac{1}{2\pi i} \left( \int_{c_1} z^2 + c \mp \int_{c_2} z^2 + c \right) = \]

\[ \frac{1}{2\pi i} \left( \int_{c_1} xdx + ydy + i \int_{c_1} ydx + xdy + c \left( \int_{c_1} \frac{x}{x^2 - y^2} dx - \frac{y}{x^2 - y^2} dy + i \int_{c_1} \frac{-y}{x^2 - y^2} dx + \frac{x}{x^2 - y^2} dy \right) \right) \]

\[ \pm \frac{1}{2\pi i} \left( \int_{c_2} xdx + ydy + i \int_{c_2} ydx + xdy + c \left( \int_{c_2} \frac{x}{x^2 - y^2} dx - \frac{y}{x^2 - y^2} dy + i \int_{c_2} \frac{-y}{x^2 - y^2} dx + \frac{x}{x^2 - y^2} dy \right) \right) \]  

(5.129)

and hence:

\[ \frac{1}{2\pi i} \left( \int_{c_1} z^2 + c \mp \int_{c_2} z^2 + c \right) = \]

\[ \frac{1}{2\pi i} \left( \int_{-\infty}^{+\infty} 2 \cosh(t) \sinh(t) \pm \int_{-\infty}^{+\infty} 2 \cosh(t) \sinh(t) + i \left( \int_{-\infty}^{+\infty} dt (\sinh^2(t) + \cosh^2(t)) \right) \right) \]

\[ \pm \int_{-\infty}^{+\infty} dt (\sinh^2(t) + \cosh^2(t))) + \frac{c}{2\pi} \left( \int_{-\infty}^{+\infty} dt \pm \int_{-\infty}^{+\infty} dt \right) \]  

(5.130)

so that:

\[ \frac{1}{2\pi i} \left( \int_{c_1} z^2 + c \mp \int_{c_2} z^2 + c \right) = \mp \infty \neq c \]  

(5.131)

\[ \frac{1}{2\pi i} \left( \int_{c_1} z^2 + c \mp \int_{c_2} z^2 + c \right) = 0 \neq c \]  

(5.132)
VI. MULTIVALEUED FUNCTIONS ON THE HYPERBOLIC PLANE AND HYPERBOLIC RIEMANN SURFACES

The introduction of multi-valued functions in Complex Calculus realizes an unexpected bridge between Analysis and Differential Geometry through the double nature of the notion of riemann surface as a collection of riemann sheets \[12\] and as a one-dimensional compact orientable complex manifold \[13\].

Following \[14\] let us start introducing the following map:

**Definition VI.1**

\[
|| \cdot || : \mathbb{G} \to [0, \infty):
\]

\[
||x + iy|| := \sqrt{|x^2 - y^2|}
\]  \hspace{1cm} (6.1)

Drawn in figure\[1\]

![FIG. 1: the map $|| \cdot ||$](image)

Clearly:

**Proposition VI.1**

\[
||z|| = |z| \ \forall z \in \mathbb{G}_+
\]

Given \( r \in [0, \infty) \):

**Definition VI.2**

\[
Hyp_r := \{ z \in \mathbb{G} : ||z|| = r \}
\]

Clearly:

**Theorem VI.1**

\[
Hyp_0 = Diagonals
\]

The set \( Hyp_1 \) is drawn in figure\[2\]

One has clearly that:

**Theorem VI.2**

\[
\mathbb{G} = \bigcup_{r \in [0, \infty)} Hyp_r
\]

Furthermore introduced the following:

**Definition VI.3**
HYPERBOLIC QUADRANTS

\[ H_1 := \{ x + iy \in \mathbb{G} : |y| < x, x > 0 \} \]
\[ H_2 := \{ x + iy \in \mathbb{G} : |y| > x, y > 0 \} \]
\[ H_3 := \{ x + iy \in \mathbb{G} : |y| < x, x < 0 \} \]
\[ H_4 := \{ x + iy \in \mathbb{G} : |y| > x, y < 0 \} \]

one has clearly that:

**Theorem VI.3**

\[ \mathbb{G} = \bigcup_{i=1}^{4} H_i \cup \text{Diagonals} \]

\[ \mathbb{G}^*_+ = H_1 \cup H_3 \]

Introduced the following:

**Definition VI.4**

HYPERBOLIC ARGUMENT OF \( z = x + iy \in \mathbb{G} - \text{Diagonals} \):

\[ \theta(z) := \begin{cases} 
\arctanh\left( \frac{x}{y} \right), & \text{if } z \in H_1 \cup H_3; \\
\arctanh\left( \frac{x}{y} \right), & \text{if } z \in H_2 \cup H_4 .
\end{cases} \]

we can finally state the following:

**Theorem VI.4**

EXPONENTIAL REPRESENTATION OF \( z = x + iy \in \mathbb{G} - \text{Diagonals} \):

\[ z = \begin{cases} 
re^{i\theta}, & \text{if } z \in H_1; \\
ir^{i\theta}, & \text{if } z \in H_2; \\
-r^{i\theta}, & \text{if } z \in H_3; \\
-ir^{i\theta}, & \text{if } z \in H_4 .
\end{cases} \]

Let us recall, now, that in Complex Analysis, given a map \( f : \mathbb{C} \mapsto \mathbb{C} \) and a point \( z_0 \in \mathbb{C} \):
Definition VI.5

\( z_0 \) IS A BRANCH POINT OF \( f \):

\[ f(r_0, \theta_0) \neq f(r_0, \theta_0 + 2\pi) \] for every closed curve \( C \) encircling \( z_0 \)

Remark VI.1

Branch points emerge in Complex Calculus owing to the arbitrariness, up to a multiple of \([0, 2\pi)\) of the argument. Since such an arbitrariness doesn’t occur as to the hyperbolic argument, the definition VI.5 has no analogue in Hyperbolic Analysis where the source of multi-valued functions lies elsewhere.

Example VI.1

MULTIVOCITY OF THE SQUARE-ROOT AND ITS CONSEQUENCES

Given \( z_1, z_2 \in \mathbb{G} \):

Definition VI.6

\( z_1 \) IS A SQUARE-ROOT OF \( z_2 \) (\( z_1 = \sqrt{z_2} \))

\[ z_1^2 = z_2 \quad (6.2) \]

One has clearly that:

Lemma VI.1

ON THE BIVOCITY OF THE SQUARE ROOT:

\[ z_1 = \sqrt{z_2} \iff i z_1 = \sqrt{z_2} \quad (6.3) \]

Lemma VI.2

ON THE NON-EXISTENCE OF THE SQUARE ROOT OF A NEGATIVE NUMBER

\[ \nexists \sqrt{-x} \; \forall x \in \mathbb{R}_+ \quad (6.4) \]

Lemma VI.1 and Lemma VI.2 imply the following

Theorem VI.5

NO-GO FUNDAMENTAL THEOREM OF HYPERBOLIC ALGEBRA

\( \text{HP:} \)

\[ a_0, \ldots, a_n \in \mathbb{G} \quad (6.5) \]

\( \text{TH:} \)

\[ \neg (\exists ! (z_1 \cdots z_n) \in \mathbb{G}^n : \sum_{i=0}^{n} a_i z_i = 0) \quad (6.6) \]

Example VI.2
SECOND DEGREE EQUATIONS ON THE HYPERBOLIC PLANE

Given the equation:

\[ az^2 + bz + c = 0 \quad a, b, c \in \mathbb{G}, a \neq 0 \]  \hspace{1cm} (6.7)

one has that:

1. if the discriminant \( \Delta := b^2 - 4ac > 0 \) then eq.6.7 has 4 solutions:

\[
\begin{align*}
    z_1 &= \frac{-b + (\sqrt{\Delta})_1}{2a} \quad (6.8) \\
    z_2 &= \frac{-b + (\sqrt{\Delta})_1}{2a} \quad (6.9) \\
    z_3 &= \frac{-b + (\sqrt{\Delta})_2}{2a} \quad (6.10) \\
    z_4 &= \frac{-b + (\sqrt{\Delta})_2}{2a} \quad (6.11)
\end{align*}
\]

where \((\sqrt{\cdot})_i\) denote the \(i^{th}\) branch of the square root \(i = 1, 2\).

2. if the discriminant \( \Delta := b^2 - 4ac = 0 \) then eq.6.7 has 1 solutions:

\[
    z_1 = \frac{-b}{2a} \quad (6.13)
\]

3. if the discriminant \( \Delta := b^2 - 4ac < 0 \) then eq.6.7 has no solution

As in the complex case it appears natural to introduce (hyperbolic) Riemannian surfaces for multi-valued functions over \( \mathbb{G} \).

**Example VI.3**

**THE HYPERBOLIC RIEMANN SURFACE OF THE SQUARE ROOT:**

the hyperbolic riemannian surface associated to the square roots: it is made of four sheets \( \{\mathbb{G}^{(i)}\}_{i=1}^4 \) such that:

1. any \( \mathbb{G}^{(i)} \) is cutted along the semi-axis \( \mathbb{R}_+ \)
2. the lower border of the cut in \( \mathbb{G}^{(1)} \) is pasted with the upper border of the cut in \( \mathbb{G}^{(2)} \)
3. the upper border of the cut in \( \mathbb{G}^{(2)} \) is pasted with the lower border of the cut in \( \mathbb{G}^{(3)} \)
4. the upper border of the cut in \( \mathbb{G}^{(3)} \) is pasted with the lower border of the cut in \( \mathbb{G}^{(4)} \)
5. the upper border of the cut in \( \mathbb{G}^{(4)} \) is pasted with the lower border of the cut in \( \mathbb{G}^{(1)} \)

\[
    \sqrt{x} \in \mathbb{R}^{(1)}_+ \quad \forall x \in \mathbb{R}^{(1)}_+ \quad (6.14)
\]

6.

\[
    \sqrt{x} \in \mathbb{R}^{(2)}_- \quad \forall x \in \mathbb{R}^{(2)}_+ \quad (6.15)
\]

7.

\[
    \sqrt{x} \in i\mathbb{R}^{(3)}_+ \quad \forall x \in \mathbb{R}^{(3)}_+ \quad (6.16)
\]

8.

\[
    \sqrt{x} \in i\mathbb{R}^{(4)}_- \quad \forall x \in \mathbb{R}^{(4)}_+ \quad (6.17)
\]
9. If \( z \) starts from \( z_0 \in \mathbb{G}^{(i)} \) and describes a closed contour containing the origin, then \( \sqrt{z} \) passes from the \( i^{th} \) sheet to the \((i+1)^{th}\) sheet and thus the point on the hyperbolic Riemann surface passes from \( \mathbb{G}^{(i)} \) to \( \mathbb{G}^{(i+1)} \) for \( i=1,2,3 \).

10. If \( z \) starts from \( z_0 \in \mathbb{G}^{(4)} \) and describes a closed contour containing the origin, then \( \sqrt{z} \) passes from the 4th sheet to the 1st sheet and thus the point on the hyperbolic Riemann surface passes from \( \mathbb{G}^{(4)} \) to \( \mathbb{G}^{(1)} \).

Let us now look at hyperbolic Riemann surfaces from a differential-geometric viewpoint.

Given a topological space \( M \):

**Definition VI.7**

**HYPERBOLIC ANALYTIC ATLAS ON** \( M \)

A family \( \{(U_\alpha, \phi_\alpha)\}_{\alpha \in I} \) (where \( I \) is some arbitrary index set) such that:

1. \( \{U_\alpha\}_{\alpha \in I} \) is an open covering of \( M \), i.e. \( \cup_{\alpha \in I} U_\alpha = M \)
2. \( \phi_\alpha \) is an homeomorphism from \( U_\alpha \) to an open subset \( U'_\alpha \) of \( \mathbb{G}^m \),
3. Given \( U_\alpha, U_\beta \) such that: \( U_\alpha \cap U_\beta \neq \emptyset \) the map \( \psi_{\beta\alpha} : \phi_\alpha(U_\alpha \cap U_\beta) \mapsto \phi_\beta(U_\alpha \cap U_\beta) : \)

\[
\psi_{\beta\alpha} := \phi_\beta \phi_\alpha^{-1}
\]

is analytic.

We will denote the set of all the hyperbolic analytic atlas of \( M \) as \( \text{HYP-ATLAS}(M) \).

Given \( x_1, x_2 \in \text{HYP-ATLAS}(M) \):

**Definition VI.8**

\( x_1 \) AND \( x_2 \) ARE COMPATIBLE (\( x_1 \sim_C x_2 \)):

\[
x_1 \cup x_2 \in \text{HYP-ATLAS}(M)
\]

It may be easily proved that \( \sim_C \) is an equivalence relation.

**Definition VI.9**

**HYPERBOLIC STRUCTURES OVER** \( M \):

\[
HS(M) := \frac{\text{HYP-ATLAS}(M)}{\sim_C}
\]

(6.18)

M endowed with an hyperbolic structure will be called an hyperbolic manifold.

**Definition VI.10**

**HYPERBOLIC RIEMANN SURFACE**

A one-dimensional compact orientable hyperbolic manifold

**Example VI.4**

**HYPERBOLIC STRUCTURES ON THE TORUS:**

Let us consider two hyperbolic numbers \( \omega_1, \omega_2 \in \mathbb{G} \) such that:

\[
\frac{\omega_2}{\omega_1} \notin \mathbb{R} \quad \text{and} \quad \text{Im}(\frac{\omega_2}{\omega_1}) > 0
\]

(6.19)

Introduced the lattice:

\[
L(\omega_1, \omega_2) = \{n_1\omega_1 + n_2\omega_2 \mid n_1, n_2 \in \mathbb{Z}\}
\]

(6.20)

Let us observe that the hyperbolic structure of \( \mathbb{G} \) induces an hyperbolic structure over \( \frac{\mathbb{G}}{L} = T^{(2)} \).

Let us observe that there are many pairs \( (\omega_1, \omega_2) \) which give rise to the same hyperbolic structure on \( T^{(2)} \).

The problem of the classification of the hyperbolic structures of the torus is under investigation.
Example VI.5

Let us consider the stereographic coordinates of a point $P(x, y, z) \in S^2 - \{\text{North Pole}\}$ projected from the North Pole:

$$ (X, Y) := \left( \frac{x}{1 - z}, \frac{y}{1 - z} \right) \quad (6.21) $$

and the stereographic coordinates of a point $P(x, y, z) \in S^2 - \{\text{South Pole}\}$ projected from the South Pole:

$$ (U, V) := \left( \frac{x}{1 + z}, \frac{-y}{1 + z} \right) \quad (6.22) $$

Introduced the hyperbolic coordinates:

$$ Z := X + iY \quad (6.23) $$
$$ W := U - iV \quad (6.24) $$

one has that:

$$ W = \frac{x - iy}{1 + z} = \frac{1 - z}{1 + z} (X - iY) = \frac{X - iY}{X^2 + Y^2} \neq \frac{1}{Z} = \frac{X - iY}{X^2 - Y^2} \quad (6.25) $$

Let us now observe that the function $F(Z) = \alpha(X, Y) + i\beta(X, Y)$:

$$ \alpha(X, Y) := \frac{X}{X^2 + Y^2} \quad (6.26) $$
$$ \beta(X, Y) := \frac{-Y}{X^2 + Y^2} \quad (6.27) $$

doesn’t obey the hyperbolic Cauchy-Riemann condition and isn’t hence an hyperbolic analytic function.

So stereographic coordinates don’t allow to define an hyperbolic structure on $S^2$.

Let us consider instead the hyperboloid:

$$ H^{(2)} := \{ (x, y, z) \in \mathbb{R}^3 : x^2 - y^2 + z^2 = 1 \} \quad (6.28) $$

Introduced again the coordinates:

$$ (X, Y) := \left( \frac{x}{1 - z}, \frac{y}{1 - z} \right) \quad (6.29) $$
$$ (U, V) := \left( \frac{x}{1 + z}, \frac{-y}{1 + z} \right) \quad (6.30) $$

$$ Z := X + iY \quad (6.31) $$
$$ W := U - iV \quad (6.32) $$

one has that:

$$ W = \frac{x - iy}{1 + z} = \frac{1 - z}{1 + z} (X - iY) = \frac{X - iY}{X^2 - Y^2} = \frac{1}{Z} \quad (6.33) $$

So we have that $W$ is an analytical function everywhere in $\mathbb{R}^3$ outside $\Delta := \{ (x, y, z) \in \mathbb{R}^3 : y = \pm x \}$.

Let us introduce the set:

$$ H^{(2)}_{\text{cut}} := H^{(2)} - \Delta \quad (6.34) $$

In [11] A.E. Motter and M.A.F. Rosa endow $H^{(2)}$ with an hyperbolic structure and propose the resulting hyperbolic manifold as the natural candidate to the role of Hyperbolic Riemann Sphere defying the readers to make a better proposal.

According to our modest opinion what is lacking in Motter’s and Rosa’s proposal is a clear illustration on how the infinity point emerges from their stuff.

This is not the case as to the hyperbolic manifold $H^{(2)}_{\text{cut}}$ endowed with the hyperbolic structure induced by the previously defined coordinates $Z$ and $W$.

Since such a hyperbolic manifold may be identified with $\mathbb{G} \cup \{\infty\} - \Delta$ we propose it as alternative candidate to the role of Hyperbolic Riemann Sphere.
VII. PHYSICAL APPLICATION TO THE VIBRATING STRING

Let \( f : S \mapsto \mathbb{G} \) be a function analytic in its domain of definition \( S \subset \mathbb{G} \) that we will suppose to be connected and simply-connected.

By the Corollary V.2 it follows that the real and imaginary parts of \( f(z) = f_1(x,y) + i f_2(x,y) \) obey on \( S \) the one-dimensional wave equation:

\[
(\partial_t^2 - \partial_x^2) f_i(x,y) = 0 \quad i = 1, 2
\]

(7.1)

This implies that, on \( S \) \( f_i(x,y) \) \( i = 1, 2 \) is of the form:

\[
f_i(x,y) = c_{i,1} g_i(x+y) + c_{i,2} g_i(x-y) \quad c_{i,1}, c_{i,2} \in \mathbb{R}, i = 1, 2
\]

(7.2)

where \( g_i \in C^1(\mathbb{R}) \) \( i = 1, 2 \).

Furthermore the solution of the Cauchy problem for the 1-dimensional wave equation \cite{15} implies that if we know the value and the rate of change of \( f_i \) on one of the coordinates axes we can infer the values of \( f_i \) on the whole hyperbolic plane in the following way:

\[
[(f_i(0, y) = g_i(y)) \wedge (\partial_y f_i(0, y) = h_i(y)) \Rightarrow (f_i(x,y) = \frac{1}{2}(g_i(y-x) + g_i(y+x)) + \int_{y-x}^{y+x} h_i(s) ds) \forall g_i, h_i \in C^1(\mathbb{R}), i = 1, 2
\]

(7.3)

\[
[(f_i(x, 0) = g_i(x)) \wedge (\partial_x f_i(x, 0) = h_i(x)) \Rightarrow (f_i(x,y) = \frac{1}{2}(g_i(x-y) + g_i(x+y)) + \int_{x-y}^{x+y} h_i(s) ds) \forall g_i, h_i \in C^1(\mathbb{R}), i = 1, 2
\]

(7.4)

As in the complex case the fact that the real and imaginary part of an analytic function obey the Laplace equation may be used to apply Complex Analysis to Electrostatics, in the hyperbolic case the fact that the real and imaginary part of an analytic function obey the 1-dimensional wave equation should allow to apply Hyperbolic Analysis to the physics of a vibrating string.
VIII. HYPERBOLIC ANALYSIS AS THE (1,0)-CASE OF CLIFFORD ANALYSIS

It is interesting to investigate whether the formalization of Hyperbolic Analysis performed in the previous sections is compatible with the more general Clifford calculus.

Following [6] let us introduce first of all the following notation:

Definition VIII.1

\[ \mathbb{R}^{(p,q)} := (\mathbb{R}^{p+q}, \tilde{q}) : \text{sign}(\tilde{q}) = (p, q) \]  

(8.1)

Denoted with \( \{ \tilde{e}_1, \cdots, \tilde{e}_{p+q} \} \) the canonical basis of \( \mathbb{R}^{(p,q)} \) one introduces the following operators:

Definition VIII.2

DIRAC OPERATOR ON \( \mathbb{R}^{(p,q)} \):

\[ D : C^1(\mathbb{R}^{(p,q)}, Cl_{p,q}) \mapsto C^1(\mathbb{R}^{(p,q)}, Cl_{p,q}) : D := \sum_{i=1}^{p+q} \tilde{e}_i \partial_i \]  

(8.2)

Definition VIII.3

CAUCHY-FUETER OPERATOR ON \( \mathbb{R} \otimes \mathbb{R}^{(p,q)} \):

\[ \partial : C^1(\mathbb{R} \otimes \mathbb{R}^{(p,q)}, Cl_{p,q}) \mapsto C^1(\mathbb{R} \otimes \mathbb{R}^{(p,q)}, Cl_{p,q}) : \partial := \partial_0 + D \]  

(8.3)

Definition VIII.4

ADJOINT DIRAC OPERATOR ON \( \mathbb{R}^{(p,q)} \):

\[ \bar{D} : C^1(\mathbb{R}^{(p,q)}, Cl_{p,q}) \mapsto C^1(\mathbb{R}^{(p,q)}, Cl_{p,q}) : \bar{D} := \sum_{i=1}^{p+q} \bar{\tilde{e}}_i \partial_i \]  

(8.4)

where:

\[ \bar{x} := (-1)^{\text{degree}(x) \left( \text{degree}(x) + 1 \right)} \]  

(8.5)

is the conjugate of \( x \in Cl_{p,q} \).

Definition VIII.5

ADJOINT CAUCHY-FUETER OPERATOR ON \( \mathbb{R} \otimes \mathbb{R}^{(p,q)} \):

\[ \bar{\partial} : C^1(\mathbb{R} \otimes \mathbb{R}^{(p,q)}, Cl_{p,q}) \mapsto C^1(\mathbb{R} \otimes \mathbb{R}^{(p,q)}, Cl_{p,q}) : \bar{\partial} := \partial_0 - D \]  

(8.6)

Given \( f \in C^1(\mathbb{R}^{(p,q)}, Cl_{p,q}) \):

Definition VIII.6

\( f \) IS \( Cl_{p,q} \)-REGULAR:

\[ Df = 0 \]  

(8.7)

Given \( f \in C^1(\mathbb{R} \otimes \mathbb{R}^{(p,q)}, Cl_{p,q}) \):

Definition VIII.7

\( f \) IS \( Cl_{p,q} \)-HOLOMORPHIC

\[ \partial f = 0 \]  

(8.8)

One has that \( \boxed{\text{6}} \):
Theorem VIII.1

GURLEBECK-SPROSSIG’S (0,n)-CAUCHY GOURSAT THEOREM:

HP:

\[ G \subset \mathbb{R}^n \]

\[ u \in C^1(G, Cl_{0,n}) \cap C(\bar{G}, Cl_{0,n}) \]

\[ u \in \text{Ker}D \]

\[ S \subset G \text{ surface} \]

\[ n_S(y) \text{ outward pointing unit normal to } S \text{ at } y \]

TH:

\[ \int_S u(y)n_S(y)dS_y = 0 \]

Let us now introduce the following maps:

**Definition VIII.8**

SURFACE AREA OF THE UNIT SPHERE IN $\mathbb{R}^n$:

\[ \sigma_n := \int_{S^{n-1}} dS = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \]

where $\Gamma(x) := \int_0^\infty \exp(t)t^{x-1}dt$ is Euler’s Gamma function.

Assumed that $n > 2$:

**Definition VIII.9**

$E : \mathbb{R}^n \to \mathbb{R}$:

\[ E(x) := \frac{1}{\sigma_n} \frac{1}{2-n} |x|^{-(n-2)} \]

**Definition VIII.10**

$e : \mathbb{R}^n \to \mathbb{R}$:

\[ e(x) := \bar{D}E(x) = \frac{-x}{\sigma_n|x|^n} \quad (8.9) \]

Given $G \subset \mathbb{R}^n$ domain with Liapunov boundary $S := \partial G$ and $u \in C^1(G, Cl_{0,n}) \cap C(\bar{G}, Cl_{0,n})$:

**Definition VIII.11**

CAUCHY-BITSADZE OPERATOR:

\[(F_Su)(x) := \int_S e(x - y)u(y)n_S(y)dS_y\quad (8.10)\]

one has the following
Theorem VIII.2

GURLEBECK-SPROSSIG’S $(0,n)$-CAUCHY INTEGRAL FORMULA:

HP:

$G \subset \mathbb{R}^n$ domain with Liapunov boundary $S := \partial G$

$u \in \text{Ker}D$

TH:

$$(F_S u)(x) = \begin{cases} u(x), & \text{if } x \in G; \\ 0, & \text{if } x \in \mathbb{R}^n - \bar{G}. \end{cases}$$

Remark VIII.1

Let us observe that theorem VIII.2, requiring that $n > 2$, doesn’t allow to recover the complex Cauchy integral formula as the $(0,1)$-case. So it is not so clear, at least to us, in which sense theorem VIII.2 is a generalization of Cauchy’s integral formula.

Let us observe, furthermore, that not contemplating the $(n,0)$-cases, such a theorem doesn’t allow to recover an hyperbolic Cauchy’s integral formula as the $(1,0)$-case.

A more advanced generalization of Cauchy’s integral formula has been presented in 7, 10.

Since the axiomatization of Clifford algebras therein introduced differs from that presented in section II we will briefly discuss their interrelations.

Definition VIII.12

GEOMETRIC ALGEBRA:
a set $\mathcal{G}$, whose elements are called multivectors

- endowed with two binary internal operations, the sum and the multiplication (called geometric product), such that:

1. the addition is commutative:

   $$A + B = B + A \quad \forall A, B \in \mathcal{G}$$

2. the addition and the multiplication are associative:

   $$(A + B) + C = A + (B + C) \quad \forall A, B, C \in \mathcal{G}$$

   $$(AB)C = A(BC) \quad \forall A, B, C \in \mathcal{G}$$

3. the multiplication is distributive w.r.t. addition:

   $$A(B + C) = AB + AC \quad \forall A, B, C \in \mathcal{G}$$

   $$(B + C)A = (BA + CA) \quad \forall A, B, C \in \mathcal{G}$$

4. existence and uniqueness of additive and multiplicative identities:

   $$\exists! 0 \in \mathcal{G} : A + 0 = A \quad \forall A \in \mathcal{G}$$

   $$\exists! 1 \in \mathcal{G} : 1A = A \quad \forall A \in \mathcal{G}$$
5. existence and uniqueness of the additive inverse:
\[ \forall A \in \mathcal{G}, \exists! -A \in \mathcal{G} : A + (-A) = 0 \]

6. grade-decomposition of a multivector \( A \in \mathcal{G} \)
\[ A = \sum_{r \in \mathbb{N}} <A>_r \]
where \( <A>_r \) is called the \textit{r-vector part} of \( A \) and where the \textit{r-grade operator} \( < \cdot >_r : \mathcal{G} \to \mathcal{G}^r \) (\( \mathcal{G}^r \) being the subalgebra of \( \mathcal{G} \) formed by the \( r \)-vectors, i.e. the set of the multivectors such that \( A = <A>_r \), with the assumption that \( \mathcal{G}^0 = \mathbb{R} \)), is such that:
\[ <A + B>_r = <A>_r + <B>_r \quad \forall A, B \in \mathcal{G}, \forall r \in \mathbb{N} \]
\[ <\lambda A>_r = \lambda <A>_r = <A>_r \lambda \quad \forall A \in \mathcal{G}, \forall \lambda \in \mathcal{G}^0, \forall r \in \mathbb{N} \]
\[ <\langle A \rangle>_r = <A>_r \quad \forall A \in \mathcal{G}, \forall r \in \mathbb{N} \]

7. the pseudo-euclidean condition:
\[ a^2 := aa = <a^2>_0 \quad \forall a \in \mathcal{G}^1 \]

8. simple \( r \) vectors
\[ \forall A \in \mathcal{G}^r_S : A \neq 0 \exists a \in \mathcal{G}^1 : a \neq 0 \text{ and } Aa \in \mathcal{G}^{r+1} \]
where the set \( \mathcal{G}^r_S \) of the \textit{simple} \( r \)-vectors is defined as the set of the \( r \)-vectors that can be expressed as the product of \( r \) mutually anticommuting 1-vectors

- endowed with a \textit{reversion operator} \( \cdot \dagger : \mathcal{G} \to \mathcal{G} \) such that:
\[ (AB)\dagger = B\dagger A\dagger \quad \forall A, B \in \mathcal{G} \]
\[ (A + B)\dagger = A\dagger + B\dagger \quad \forall A, B \in \mathcal{G} \]
\[ <A\dagger>_0 = <A>_0 \quad \forall A \in \mathcal{G} \]
\[ a\dagger = a \quad \forall a \in \mathcal{G}^1 \]

The name \textit{reversion operator} is justified by the following:

\textbf{Theorem VIII.3}

\[ (a_1 \cdots a_n)\dagger = a_n \cdots a_1 \quad \forall a_1, \cdots, a_n \in \mathcal{G}^1 \]

Given \( A \in \mathcal{G}^r \) and \( B \in \mathcal{G}^s \):

\textbf{Definition VIII.13}

OUTER PRODUCT OF A AND B:
\[ A \wedge B := <AB>_{r+s} \]

The outer product of two arbitrary multivectors \( A, B \in \mathcal{G} \) may be then defined as:

\textbf{Definition VIII.14}
OUTER PRODUCT OF A AND B:

\[ A \wedge B := \sum_{r \in \mathbb{N}} \sum_{s \in \mathbb{N}} < A >_r \wedge < B >_s \]

Given \( A, B \in \mathcal{G} \):

**Definition VIII.15**

SCALAR PRODUCT OF A AND B:

\[ A \star B := < AB >_0 \]

One has that:

**Theorem VIII.4**

HP:

\[ A \in \mathcal{G}^n_S \]

TH:

\[ \exists a_1, \ldots, a_n \in \mathcal{G}^1 : A = a_1 \cdots a_n \text{ and } A^\dagger \star A = a^2_1 \cdots a^2_n \]

Theorem VIII.4 allows to introduce the following:

**Definition VIII.16**

\( A \in \mathcal{G}^S_S \) HAS SIGNATURE (p,r):

\[ A = a_1 \cdots a_n \text{ and } A^\dagger \star A = a^2_1 \cdots a^2_n \text{ and } \text{card}(\{a_i : a^2_i > 0\}) = p \text{ and } \text{card}(\{a_i : a^2_i < 0\}) = r \quad (8.11) \]

We will denote the set of all the simple vectors with signature (p,r) by \( \mathcal{G}^{(p,r)}_S \).

Given \( A \in \mathcal{G}^{(p,r)}_S \):

**Definition VIII.17**

\[ A_{p,r} := \{ a \in \mathcal{G}^1 : a \wedge A = 0 \} \]

**Definition VIII.18**

\[ \mathcal{G}(A_{p,r}) := (\mathcal{G}, +, \text{ geometric product }, \cdot^\dagger)|_{A_{p,r}} \]

The link with definition II.10 is then given by the following:

**Theorem VIII.5**

\[ \mathcal{G}(A_{p,r}) = Cl_{(p,r)} \quad \forall p, r \in \mathbb{N} \]

**Corollary VIII.1**
Remark VIII.2

ON THE METRIC STRUCTURE OF THE GEOMETRIC ALGEBRA

Let us suppose to replace the pseudo-euclidean condition in definition VIII.12 with the more restrict euclidean condition:

$$a^2 := aa = <a^2 >_0 \in (0, \infty) \ \forall a \in \mathcal{G}^1 : a \neq 0$$

Is then possible to introduce the following:

**Definition VIII.19**

MAGNITUDE OF $a \in \mathcal{G}^1$

$$|a| := \sqrt{a^2}$$

Theorem VIII.3 can then be used to infer that:

**Theorem VIII.6**

$$A^\dagger \star A \geq 0 \ \forall A \in \mathcal{G}$$

**PROOF:**

Given:

$$A = \prod_{i=1}^{n} a_i : a_ia_j = -a_ja_i \ \forall i \neq j \quad (8.12)$$

one has that:

$$(a_1 \cdots a_n)^\dagger \star (a_1 \cdots a_n) = (a_n \cdots a_1) \star (a_1 \cdots a_n) = \prod_{i=1}^{n} |a_i|^2 \geq 0 \quad (8.13)$$

So, introduced the notation:

$$\mathcal{G}_S := \cup_{n \in \mathbb{N}} \mathcal{G}_S^n \quad (8.14)$$

we have proved that:

$$A^\dagger \star A \geq 0 \ \forall A \in \mathcal{G}_S$$

The case of non-simple multivectors can then be reduced to that of simple multivectors to get the thesis ■

It is then possible to introduce the following:

**Definition VIII.20**

MAGNITUDE OF $A \in \mathcal{G}$

$$|A| := \sqrt{A^\dagger \star A}$$

and the induced distance:

**Definition VIII.21**

$$d : \mathcal{G} \times \mathcal{G} \mapsto [0, +\infty)$$

$$d(A, B) := |A - B|$$

endowed with which the geometric algebra is a metric space on which limits may be defined in the usual way [10].

Contrary, if, as we did in definition VIII.12, one doesn’t assume the euclidean condition but only the pseudo-euclidean condition, $a^2 = <a^2 >_0 a \in \mathcal{G}^1$ and hence $A^\dagger \star A \in \mathcal{G}$ may become negative.

In this case definition VIII.20 has to be replaced with the following:
Definition VIII.22
MAGNITUDE OF $A \in G$

$$|A| := \sqrt{|A^* A|}$$

Theorem VIII.7
HP:

$\mathcal{A}_n \subset G$ n-dimensional linear subspace

TH:

$$\exists (+I, -I)I \in G_n^0 : \forall a_1, \cdots, a_n \in G^1 \cap \mathcal{A}_n, \exists \lambda \in G^0 : |I| = 1 \text{ and } a_1 \wedge \cdots \wedge a_n = \lambda I$$

Given $\mathcal{A}_n \subset G$ n-dimensional linear subspace:

Definition VIII.23
PSEUDOSCALARS OF $\mathcal{A}_n$:

$$PS(\mathcal{A}_n) := \{ \lambda I, \lambda \in G^0 : \lambda \neq 0 \}$$

Theorem VIII.8
HP:

$\mathcal{A}_n \subset G$ n-dimensional linear subspace

$$a_1, \cdots, a_n \in G^1 \cap \mathcal{A}_n$$

TH:

$$a_1 \wedge \cdots \wedge a_n \in PS(\mathcal{A}_n) \iff a_1, \cdots, a_n \text{ linearly independent}$$

Given a set $\mathcal{M} \subset G^1$, let us demand to $[7]$ and $[17]$ as to the determination of the conditions under which $\mathcal{M}$ is said to be a vector manifold.

Given a vector manifold $\mathcal{M}$, a point $x \in \mathcal{M}$ and a 1-vector $a(x) \in G^1$:

Definition VIII.24

a(x) IS TANGENT TO $\mathcal{M}$ in x:

$$\exists C : [0, 1] \rightarrow \mathcal{M} \text{ curve} : C(0) = x \text{ and } \frac{dC(\tau)}{d\tau} |_{\tau=0} = a(x)$$

We will denote the set of all the tangent vectors to $\mathcal{M}$ in x by $\mathcal{A}(x)$.

Let us assume that $\mathcal{A}(x)$ is nonsingular, i.e. that it possesses a unit pseudoscalar I(x) which we will call the unit pseudo-scalar of $\mathcal{M}$ in x.
Definition VIII.25
\( \mathcal{M} \) IS CONTINUOUS:

\[ I(x) \text{ continuous in } x \ \forall x \in \mathcal{M} \]

Definition VIII.26
ORIENTATION ON \( \mathcal{M} \)
the assignment on \( \mathcal{M} \) of a continuous pseudoscalar.

Definition VIII.27
\( \mathcal{M} \) IS ORIENTABLE
its unit pseudoscalar \( I(x) \) is single valued

Definition VIII.28
\( \mathcal{M} \) IS SMOOTH
its unit pseudoscalar \( I(x) \) has derivative of all orders \( \forall x \in \mathcal{M} \)

Given an m-dimensional smooth oriented vector manifold \( \mathcal{M} \) an a map \( f : \mathcal{M} \rightarrow \mathcal{G} \):

Definition VIII.29
DIRECTED INTEGRAL OF \( f \) OVER \( \mathcal{M} \):

\[ \int_{\mathcal{M}} dX f := \lim_{n \rightarrow \infty} \sum_{i=1}^{n} \Delta X(x_i) f(x_i) \]

where the limit on the right side is to be understood in the usual sense of Riemann integration theory.

Definition VIII.30
HESTENES-SOBCZYK DERIVATIVE OF \( f \):

\[ \partial f(x) := \lim_{|\mathcal{R}| \rightarrow 0} \frac{I^{-1}(x)}{|\mathcal{R}|(x)} \oint_{\partial \mathcal{R}} dS f \]

where:

1. \( \mathcal{R} \) is an open smooth m-dimensional submanifold of \( \mathcal{M} \) with \( x \) as an interior point
2. the directed integral of \( f \) is taken over the boundary of \( \mathcal{R} \). The \( (\dim \mathcal{M} - 1) \)-vector \( dS \) representing a directed volume element of \( \partial \mathcal{R} \) is oriented so that:

\[ dS(x') n(x') = I(x')|dS(x')| \]

where \( n(x') \) is the outward unit normal vector at a point \( x' \in \partial \mathcal{R} \)
3. the limit is taken by shrinking \( \mathcal{R} \) and hence its volume \( |\mathcal{R}| \) to zero at the point \( x \); we allow the limit to be proportional to a delta-function or its derivatives, so that it is well defined in the sense of distribution theory [10].

Definition VIII.31
\( f \) IS HESTENES-SOBCZYK ANALYTIC ON \( \mathcal{M} \):

\[ \partial f(x) = 0 \ \forall x \in \mathcal{M} \]

Let us now consider the following:

Definition VIII.32
GREEN FUNCTION OF THE HESTENES-SOBCZYK DERIVATIVE

\[ g : (\mathcal{M} \cup \partial \mathcal{M}) \times (\mathcal{M} \cup \partial \mathcal{M}) \mapsto \mathcal{G} \]

\[ \partial g = \delta \quad (8.16) \]

where the Dirac \( \delta \) distribution is defined by:

\[ \int_{\mathcal{R}} |dX(x')| \delta(x - x') F(x') = F(x) \quad (8.17) \]

with \( \mathcal{R} \) being a subregion of \( \mathcal{M} \) containing \( x \) and with \( F : \mathcal{R} \mapsto \mathcal{G} \) continuous.

We have at last all the necessary notions to present the following:

**Theorem VIII.9**

HESTENES-SOBCZYK'S GENERALIZED CAUCHY INTEGRAL FORMULA:

**HP:**

\[ f : \mathcal{M} \mapsto \mathcal{G} \text{ analytic} \]

**TH:**

\[ f(x) = \frac{(-1)^{\text{dim}\mathcal{M}}}{I(x)} \oint_{\partial \mathcal{M}} g(x, x') f(x') dS(x') \quad \forall x \in \mathcal{M} \]

Let us now analyze what Theorem VIII.9 tells us as to the particular case \( \mathcal{M} := \mathcal{G}(A_1, 0) = \mathbb{G} \) and \( f : \mathcal{M} \mapsto \mathcal{M} \).

At this purpose it is sufficient to follow step by step the analysis concerning the geometric algebra of the plane \( \mathbb{G}_2 \) performed in [17], adapting it to the case of the bidimensional Minkowski spacetime algebra, i.e. the four-dimensional Clifford algebra \( \mathbb{G}_{2^{hyp}} \) spanned by the basis set consisting in:

1. 1 one scalar
2. \( \{e_0, e_1\} \) two 1-vector
3. \( I := e_0 \wedge e_1 \) one 2-vector

where \( e_0 \) and \( e_1 \) are 1-vectors such that:

\[ e_0^2 = -1 \quad (8.18) \]

\[ e_1^2 = 1 \quad (8.19) \]

\[ e_0 \cdot e_1 = 0 \quad (8.20) \]

One has that:

\[ e_0 e_1 = e_0 \cdot e_1 + e_0 \wedge e_1 = e_0 \wedge e_1 = -e_1 \wedge e_0 \quad (8.21) \]

i.e. \( e_0 \) and \( e_1 \) anticommute.

Let us observe furthermore that:

\[ I^2 = e_0 e_1 e_0 e_1 = -e_0^2 e_1^2 = +1 \quad (8.22) \]

so that:

\[ \mathbb{G} = \{ x + I y, x, y \in \mathbb{R} \} \subset \mathbb{G}_{2^{hyp}} \quad (8.23) \]
One has that:

\[ Ie_0 = e_0e_1e_0 = -e_0^2e_1 = e_1 \]  
\[ Ie_1 = e_0e_1e_1 = e_0e_1^2 = e_0 \]  
\[ e_0I = e_0e_0e_1 = e_0^2e_1 = -e_1 \]  
\[ e_1I = e_1e_0e_1 = -e_1^2e_0 = -e_0 \]  

from which it follows that \( I \) anticommutes with \( e_0 \) and \( e_1 \).

Observing preliminarily that:

\[ \mathbb{R}^2 = \{xe_0 + ye_1, x, y \in \mathbb{R}\} \subset \mathcal{G}_2^{hyp} \]  

let us introduce the following map \( F : G \rightarrow \mathbb{R}^2 \):

\[ F(z) := ze_0 \]  

One has that:

**Theorem VIII.10**

\[ F(x + Iy) = xe_0 + ye_1 \quad \forall x, y \in \mathbb{R} \]  

**PROOF:**

\[ (x + Iy)e_0 = xe_0 + yIe_0 \]  

The thesis follows by eq. 8.24 □

**Corollary VIII.2**

\[ F^{-1}(xe_0 + ye_1) = x + Iy = -(xe_0 + ye_1)e_0 \quad \forall x, y \in \mathbb{R}e_0 \]  

Let us now introduce the following functional \( \mathcal{F} : MAP(G, G) \rightarrow MAP(\mathbb{R}^2, \mathbb{R}^2) \):

\[ \mathcal{F}[\psi] := F \circ \psi \circ F^{-1} \]  

One has that:

**Theorem VIII.11**

**HP:**

\[ f = u + Iv \in MAP(G, G) \]  

**TH:**

\[ f \text{ is analytic} \Leftrightarrow \mathcal{F}[f] \text{ is Hestenes-Sobczyk analytic} \]  

**PROOF:**
Let us observe, first of all, that the Hestenes-Sobczyk derivative of definition VIII.30 reduces in our case to:

\[ \nabla := e_0 \frac{\partial}{\partial x} + e_1 \frac{\partial}{\partial y} \quad (8.34) \]

One has that:

\[ \nabla F[f] = (e_0 \partial_x + e_1 \partial_y)(ue_0 + ve_1) = -\partial_x u + \partial_y v + I(\partial_x v - \partial_y u) \quad (8.35) \]

and hence:

\[ \nabla f = 0 \iff \partial_x u = \partial_y v \text{ and } \partial_x v = \partial_y u \quad (8.36) \]

By theorem V.1 the thesis immediately follows.

Let us now observe that the key point of the Cauchy Integral Formula of theorem V.4, expressed in terms of the geometric algebra \( G_2 \), is that the Cauchy kernel \( \frac{1}{z - z_0} \) is the Green function of the Hestenes-Sobczyk derivative:

\[ \mathcal{F}' \left[ \frac{1}{z' - z_0} \right] = \frac{r' - r_0'}{(r' - r_0')^2} \quad (8.37) \]

\[ \nabla \frac{r' - r_0'}{(r' - r_0')^2} = 2\pi \delta(r' - r_0') \quad (8.38) \]

where the basis \( \{1, e_0', e_1', I' := e_0' \wedge e_1'\} \) spanning \( G_2 \) is defined by the conditions:

\[ e_0'^2 = e_1'^2 = 1 \text{ and } e_0' \cdot e_1' = 0 \quad (8.39) \]

where:

\[ \mathbb{C} = \{x + I'y, x, y \in \mathbb{R}\} \subset G_2 \quad (8.40) \]

where:

\[ r' := xe_0' + ye_1' \quad (8.41) \]

\[ r_0' = x_0e_0' + y_0e_1' \quad (8.42) \]

and where \( \mathcal{F}' : MAP(\mathbb{C}, \mathbb{C}) \mapsto MAP(\mathbb{R}^2, \mathbb{R}^2) \) is such that:

\[ \mathcal{F}'[\psi] := F' \circ \psi \circ F'^{-1} \quad (8.43) \]

with \( F' : \mathbb{C} \mapsto \mathbb{R}^2 \) such that:

\[ F'(x + I'y) := xe_0' + ye_1' \quad (8.44) \]

In the case of \( G_2^{hyp} \), contrary, one has that:

\[ \mathcal{F}[\frac{1}{z - z_0}] = \frac{(x - x_0)e_0 - (y - y_0)e_1}{(x - x_0)^2 - (y - y_0)^2} \quad (8.45) \]

and hence:

\[ \nabla \mathcal{F}[\frac{1}{z - z_0}] \neq 2\pi \delta(r - r_0) \quad (8.46) \]

In analogy with what we saw for the complex case, it is natural, anyway, to suppose that an Hyperbolic Cauchy Integral formula may be obtained by determining the Green’s function of the \( G_2^{hyp} \)’s Hestenes-Sobczyk derivative. Such a subject is under investigation.
IX. ACKNOWLEDGEMENTS

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