Simplicial-like Identities for The Paths and The Regular Paths on Discrete Sets

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Abstract

Simplicial identities play an important and fundamental role in simplicial homotopy theory. On the other hand, the study of the paths and the regular paths on discrete sets is the foundation for the path-homology theory of digraphs. In this paper, by investigating some weighted face maps and weighted co-face maps on the space of the paths as well as the space of the regular paths, we prove some simplicial-like identities for the paths and the regular paths on discrete sets.

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1 Introduction

Simplicial identities are important tools in simplicial homotopy theory. So far, topologists applied simplicial methods in algebraic topology systematically and developed the simplicial homotopy theory. For instances, Edward B. Curtis [2] in 1971, Goerss, Paul G. and Jardine, John [6] in 2009, Jie Wu [15] in 2010, etc. Moreover, essential applications of simplicial homotopy theory have been found in various areas. For example, F. R. Cohen and J. Wu [11] in braids, Fengchun Lei, Fengling Li, Jie Wu [14] in framed links, Fedor Pavutnitskiy and Jie Wu [15] in Adams spectral sequence, etc.

In 1990s, A. Dimakis and F. Müller-Hoissen [3, 4, 5] initiated the study of discrete differential calculus and discrete Riemannian geometry with a motivation in theoretical physics. During 2010s, based on the study of A. Dimakis and F. Müller-Hoissen [3, 4, 5], Alexander Grigor’yán, Yong Lin and Shing-Tung Yau [7], Alexander Grigor’yán, Yong Lin, Yuri Muranov and Shing-Tung Yau [8, 9, 10] and Alexander Grigor’yán, Yuri Muranov and Shing-Tung Yau [11, 12] developed a path-homology theory of digraphs. The theoretical foundation for the path-homology in [7] - [12] relies on the theory of the paths and the regular paths on discrete sets.

In this paper, we study some weighted face maps and some weighted co-face maps for the paths and the regular paths on discrete sets. We prove some simplicial-like identities for the paths and the regular paths on discrete sets. The main results of this paper are Theorem 2.5 and Theorem 3.1.

As by-products, we derive some weighted boundary operators from the weighted face maps and derive some weighted co-boundary operators from the weighted co-face maps. We prove the anti-commutative rule and the Newton-Leibniz rule for the weighted boundary operators and the weighted co-boundary operators of the paths on discrete sets, in Subsection 2.3. We calculate the anti-commutators for the weighted boundary operators and the weighted co-boundary operators of the regular paths on discrete sets, in Subsection 3.2.
2 Simplicial-Like Identities for Paths on Discrete Sets

Let $V$ be a finite set whose elements are called vertices. Let $n \geq 0$ be a non-negative integer. An elementary $n$-path on $V$ is an ordered sequence $v_0 v_1 \ldots v_n$ where $v_0, v_1, \ldots, v_n$ are vertices in $V$ (cf. [8, Definition 2.1]). Here for any $0 \leq i < j \leq n$, the vertices $v_i$ and $v_j$ are not assumed to be distinct. Let $\mathbb{R}$ be the real numbers. A linear combination

$$\sum_{v_0, v_1, \ldots, v_n \in V} r_{v_0} v_0 v_1 \ldots v_n v_0 v_1 \ldots v_n \in \mathbb{R},$$

of elementary $n$-paths on $V$ is called an $n$-path on $V$ (cf. [8, Definition 2.2]). Let $\Lambda_n(V)$ be the vector space consisting of all the $n$-paths on $V$ (cf. [8, Definition 2.2]). Note that for any $n \geq 0$, the space $\Lambda_n(V)$ is of dimension $(\#V)^{n+1}$. In particular, $\Lambda_0(V)$ is the vector space spanned by all the vertices in $V$ thus is of dimension $\#V$. We consider the graded vector space

$$\Lambda_*(V) = \bigoplus_{n=0}^{\infty} \Lambda_n(V)$$

with the canonical addition $+$ and the canonical (real) scalar multiplication. Let $n$ and $m$ be two non-negative integers. We take an $n$-path

$$\xi = \sum_{v_0, v_1, \ldots, v_n \in V} r_{v_0} v_0 v_1 \ldots v_n$$

in $\Lambda_n(V)$ and take an $m$-path

$$\eta = \sum_{u_0, u_1, \ldots, u_m \in V} t_{u_0} u_0 u_1 \ldots u_m$$

in $\Lambda_m(V)$. We define their join $\xi \ast \eta$ to be an $(n + m + 1)$-path in $\Lambda_{n+m+1}(V)$ by letting (cf. [8, Subsection 2.2, Join of paths])

$$\xi \ast \eta = \sum_{v_0, \ldots, v_n, u_0, \ldots, u_m \in V} r_{v_0} v_0 v_1 \ldots v_n v_0 u_0 u_1 \ldots u_m.$$  \hfill (2.3)

We extend $\ast$ bilinearly over $\Lambda_*(V)$. It is direct to verify the following laws:

(i). bilinear law: $(\lambda \xi) \ast (\mu \eta) = (\lambda \mu)(\xi \ast \eta)$ for any $\lambda, \mu \in \mathbb{R}$ and any $\xi, \eta \in \Lambda_*(V)$;

(ii). associative law: $(\xi \ast \eta) \ast \theta = \xi \ast (\eta \ast \theta)$ for any $\xi, \eta, \theta \in \Lambda_*(V)$;

(iii). distributive law: $\xi \ast (\eta + \theta) = \xi \ast \eta + \xi \ast \theta$ and $(\eta + \theta) \ast \xi = \eta \ast \xi + \theta \ast \xi$ for any $\xi, \eta, \theta \in \Lambda_*(V)$.

We give the next definition.

**Definition 1.** We call the graded vector space $\Lambda_*(V)$ equipped with the join $\ast$ the path algebra on $V$ with coefficients in the real numbers.

2.1 The Partial Derivatives and the Partial Differentiations for Paths

Let $v \in V$. In this subsection, we define the partial derivative as well as the partial differentiation on $\Lambda_*(V)$, with respect to $v$. 

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Definition 2. For each \( n \geq 0 \), we define the partial derivative with respect to \( v \) to be a linear map

\[
\frac{\partial}{\partial v} : \Lambda_n(V) \longrightarrow \prod_{n+1} \Lambda_{n-1}(V)
\]

by

\[
\frac{\partial}{\partial v} = \left( \frac{\partial_0}{\partial v}, \frac{\partial_1}{\partial v}, \ldots, \frac{\partial_n}{\partial v} \right). \tag{2.4}
\]

Here in (2.4), for each \( 0 \leq i \leq n \), the \( i \)-th coordinate of (2.4) is a linear map

\[
\frac{\partial_i}{\partial v} : \Lambda_n(V) \longrightarrow \Lambda_{n-1}(V)
\]

given by

\[
\frac{\partial_i}{\partial v}(v_0v_1 \ldots v_n) = (-1)^i \delta(v, v_i)v_0 \hat{v}_i \ldots v_n \tag{2.5}
\]

for any elementary \( n \)-path \( v_0v_1 \ldots v_n \) on \( V \).

Remark 1: We give a remark on the notation in (2.4). For any \( v, u \in V \), the notation \( \delta(v, u) \) is defined to be 1 if \( v = u \) and is defined to be 0 if \( v \neq u \).

Definition 3. For each \( n \geq 0 \), we define the partial differentiation with respect to \( v \) to be a linear map

\[
\tilde{d}v : \Lambda_n(V) \longrightarrow \prod_{n+2} \Lambda_{n+1}(V)
\]

by

\[
\tilde{d}v = (d_0v, d_1v, \ldots, d_{n+1}v). \tag{2.6}
\]

Here in (2.6), for each \( 0 \leq i \leq n+1 \), the \( i \)-th coordinate of (2.6) is a linear map

\[
d_i v : \Lambda_n(V) \longrightarrow \Lambda_{n+1}(V)
\]

given by

\[
d_i v(v_0v_1 \ldots v_n) = (-1)^i v_0 \ldots v_{i-1}v_i \hat{v}_i \ldots v_n \tag{2.7}
\]

for each elementary \( n \)-path \( v_0v_1 \ldots v_n \) on \( V \).

Remark 2: We give a remark for Definition 3. Letting \( i = 0 \) in (2.7), we have

\[
d_0 v(v_0v_1 \ldots v_n) = (-1)^0 v_0 \ldots v_n; \tag{2.8}
\]

letting \( i = n+1 \) in (2.7), we have

\[
d_{n+1} v(v_0v_1 \ldots v_n) = (-1)^{n+1} v_0 \ldots v_n v. \tag{2.9}
\]

Hence for each elementary \( n \)-path \( v_0v_1 \ldots v_n \) on \( V \), by (2.7), (2.8) and (2.9), its image under \( d\tilde{v} \) is a vector

\[
\tilde{d}v(v_0v_1 \ldots v_n) = \left( (-1)^0 v_0v_1 \ldots v_n, (-1)^1 v_0v_1 \ldots v_n, \ldots, (-1)^i v_0 \ldots v_{i-1}v_i \hat{v}_i \ldots v_n, \ldots, (-1)^{n+1} v_0 \ldots v_n v \right).
\]
By Definition 2 and Definition 2 we have the next three lemmas.

**Lemma 2.1.** Let $v, u \in V$. Let $n \geq 0$. Then for any possible $i$ and $j$ with $i < j$, we have

$$\frac{\partial_i}{\partial u} \circ \frac{\partial_j}{\partial v} (v_0 v_1 \ldots v_n) = \frac{\partial_{j-1}}{\partial v} \circ \frac{\partial_i}{\partial u} (v_0 v_1 \ldots v_n).$$

**Proof.** Let $v, u \in V$ and $n \geq 0$. For any $0 \leq j \leq n$ and any $0 \leq i \leq n - 1$, it follows from a direct calculation that

$$\frac{\partial_i}{\partial u} \circ \frac{\partial_j}{\partial v} (v_0 v_1 \ldots v_n) = \frac{\partial_i}{\partial u} \left((-1)^j \delta(v, v_j) v_0 \ldots \hat{v}_j \ldots v_n\right) = (-1)^j \delta(v, v_j) \frac{\partial_i}{\partial u} (v_0 \ldots \hat{v}_j \ldots v_n) = \begin{cases} (-1)^{i+j} \delta(v, v_j) \delta(u, v_i) v_0 \ldots \hat{v}_i \ldots \hat{v}_j \ldots v_n, & 0 \leq i \leq j - 1; \\ (-1)^{i+j} \delta(v, v_j) \delta(u, v_{i+1}) v_0 \ldots \hat{v}_i \ldots \hat{v}_{i+1} \ldots v_n, & j \leq i \leq n - 1. \end{cases}$$

Exchanging $u$ and $v$, we have

$$\frac{\partial_i}{\partial v} \circ \frac{\partial_j}{\partial u} (v_0 v_1 \ldots v_n) = \begin{cases} (-1)^{i+j} \delta(u, v_i) v_0 \ldots \hat{v}_i \ldots \hat{v}_j \ldots v_n, & 0 \leq i \leq j - 1; \\ (-1)^{i+j} \delta(u, v_{i+1}) v_0 \ldots \hat{v}_i \ldots \hat{v}_{i+1} \ldots v_n, & j \leq i \leq n - 1. \end{cases}$$

Now we suppose $i < j$. It follows from the above two equations that for any elementary $n$-path $v_0 v_1 \ldots v_n$ on $V$, we have

$$\frac{\partial_i}{\partial u} \circ \frac{\partial_j}{\partial v} (v_0 v_1 \ldots v_n) = \frac{\partial_{j-1}}{\partial v} \circ \frac{\partial_i}{\partial u} (v_0 v_1 \ldots v_n).$$

By the linear property of $\frac{\partial_i}{\partial u} \circ \frac{\partial_j}{\partial v}$ and $\frac{\partial_{j-1}}{\partial v} \circ \frac{\partial_i}{\partial u}$, the lemma follows.

**Lemma 2.2.** Let $v, u \in V$. Let $n \geq 0$. Then for any possible $i$ and $j$ we have

$$\frac{\partial_i}{\partial u} \circ d_j v = \begin{cases} -d_{j-1} v \circ \frac{\partial_i}{\partial u}, & i < j; \\ \delta(u, v) \text{id}, & i = j; \\ -d_j v \circ \frac{\partial_{j-1}}{\partial u}, & i > j. \end{cases}$$

**Proof.** Let $v, u \in V$ and $n \geq 0$. Let $0 \leq i, j \leq n + 1$. Take an elementary $n$-path $v_0 v_1 \ldots v_n$ on $V$.

We consider three cases:

**Case 1.** $i < j$. Then

$$\frac{\partial_i}{\partial u} \circ d_j v (v_0 v_1 \ldots v_n) = \begin{cases} (-1)^{i+j} \delta(u, v_i) v_0 \ldots \hat{v}_i \ldots v_{j-1} v v_j \ldots v_n, & 0 \leq i \leq j - 1; \\ -(-1)^{i+j-1} \delta(u, v_i) v_0 \ldots \hat{v}_i \ldots v_{j-1} v v_j \ldots v_n, & j \leq i \leq n - 1. \end{cases}$$

Thus by the linear property of $\frac{\partial_i}{\partial u} \circ d_j v$ and $d_{j-1} v \circ \frac{\partial_i}{\partial u}$, we have

$$\frac{\partial_i}{\partial u} \circ d_j v = -d_{j-1} v \circ \frac{\partial_i}{\partial u}.$$
Thus by the linear property of $\frac{\partial}{\partial u} \circ d_j v$, we have
\[
\frac{\partial}{\partial u} \circ d_j v = \delta(u, v) \text{id}.
\]

**Case 3.** $i > j$. Then
\[
\frac{\partial}{\partial u} \circ d_j v(v_0v_1 \ldots v_n) = (-1)^{i+j} \delta(u, v_i)v_0 \ldots v_{j-1}uv_{j} \ldots v_{i-1} \ldots v_n
\]
\[
= -(-1)^{(i-1)+j} \delta(u, v_i)v_0 \ldots v_{j-1}uv_{j} \ldots v_{i-1} \ldots v_n
\]
\[
= -d_j v \circ \frac{\partial_{i-1}}{\partial u}(v_0v_1 \ldots v_n).
\]

Thus by the linear property of $\frac{\partial}{\partial u} \circ d_j v$ and $d_j v \circ \frac{\partial}{\partial u}$, we have
\[
\frac{\partial}{\partial u} \circ d_j v = -d_j v \circ \frac{\partial_{i-1}}{\partial u}.
\]

Summarizing all the three cases, we obtain the lemma.

**Lemma 2.3.** Let $v, u \in V$. Let $n \geq 0$. Then for any possible $i$ and $j$ with $i \leq j$, we have
\[
d_i u \circ d_j v = -d_{j+1} v \circ d_i u.
\]

**Proof.** Let $v, u \in V$ and $n \geq 0$. Take an elementary $n$-path $v_0v_1 \ldots v_n$ on $V$. Then for any possible $i$ and $j$ we have
\[
d_i u \circ d_j v(v_0v_1 \ldots v_n) = (-1)^i d_i u(v_0 \ldots v_{j-1}uv_j \ldots v_n)
\]
\[
= \begin{cases} (-1)^{i+j}v_0 \ldots v_{j-1}uv_j \ldots v_iuv_i \ldots v_n, & i < j; \\
v_0 \ldots v_{j-1}uv_i \ldots v_n, & i = j; \\
-v_0 \ldots v_{j-1}uv_i \ldots v_n, & i = j + 1; \\
(-1)^{i+j}v_0 \ldots v_{j-1}uv_j \ldots v_iuv_i \ldots v_n, & i > j + 1. 
\end{cases}
\]

Exchanging $u$ and $v$ we have
\[
d_j v \circ d_i u(v_0v_1 \ldots v_n) = \begin{cases} (-1)^{i+j}v_0 \ldots v_{j-1}uv_j \ldots v_iuv_i \ldots v_n, & i < j; \\
v_0 \ldots v_{j-1}uv_i \ldots v_n, & i = j; \\
-v_0 \ldots v_{j-1}uv_i \ldots v_n, & i = j + 1; \\
(-1)^{i+j}v_0 \ldots v_{j-1}uv_j \ldots v_iuv_i \ldots v_n, & i > j + 1. 
\end{cases}
\]

Now we suppose $i \leq j$. Then it follows from the above two equations that
\[
d_i u \circ d_j v(v_0v_1 \ldots v_n) = -d_{j+1} v \circ d_i u(v_0v_1 \ldots v_n).
\]

By the linear property of $d_i u \circ d_j v$ and $d_{j+1} v \circ d_i u$, the lemma follows.

By the end of this subsection, we summarize Lemma 2.1, Lemma 2.2 and Lemma 2.3 in the following list:

- $\frac{\partial}{\partial u} \circ \frac{\partial}{\partial v} = -\frac{\partial_{i-1}}{\partial v} \circ \frac{\partial}{\partial u}$ for $i < j$;
- $\frac{\partial}{\partial u} \circ d_j v = \begin{cases} -d_{j-1} v \circ \frac{\partial}{\partial u}, & i < j; \\
\delta(u, v) \text{id}, & i = j; \\
d_j v \circ \frac{\partial_{i-1}}{\partial u}, & i > j; \end{cases}$
- $d_i u \circ d_j v = -d_{j+1} v \circ d_i u$ for $i \leq j$.  

5
2.2 The Weighted Face Maps and the Weighted Co-Face Maps for Paths

Let \( f : V \rightarrow \mathbb{R} \) be a real valued function on \( V \) which assigns a real number \( f(v) \) to each vertex \( v \in V \). Let \( n \geq 0 \). In this subsection, we define the \( f \)-weighted face maps as well as the \( f \)-weighted co-face maps on \( \Lambda_n(V) \). Then we prove some simplicial-like identities.

**Definition 4.** For each \( 0 \leq i \leq n \), we define the \( i \)-th \( f \)-weighted face map to be a linear map

\[
\sum_{v \in V} f(v) \frac{\partial}{\partial v} : \Lambda_n(V) \rightarrow \Lambda_{n-1}(V).
\]

For simplicity, we use the notation

\[ \partial^f_i := \sum_{v \in V} f(v) \frac{\partial}{\partial v} \] (2.10)

for the \( i \)-th \( f \)-weighted face map.

**Definition 5.** For each \( 0 \leq i \leq n + 1 \), we define the \( i \)-th \( f \)-weighted co-face map to be a linear map

\[
\sum_{v \in V} f(v) d_i v : \Lambda_n(V) \rightarrow \Lambda_{n+1}(V).
\]

For simplicity, we use the notation

\[ d^f_i := \sum_{v \in V} f(v) d_i v \] (2.11)

for the \( i \)-th \( f \)-weighted co-face map.

**Definition 6.** For any real functions \( f \) and \( g \) on \( V \), their inner product with respect to \( V \) is defined by

\[ \langle f, g \rangle = \langle f, g \rangle^V := \sum_{v \in V} f(v) g(v). \]

**Definition 7.** Taking \( f = g \) in Definition 6 for any real function \( f \) on \( V \), the \( L^2 \)-norm of \( f \) is defined by

\[ \|f\|_2 = \|f\|_2^V := \sqrt{\langle f, f \rangle} = \sqrt{\sum_{v \in V} |f(v)|^2}. \]

The next proposition gives the explicit formulas for \( \partial^f_i \) and \( d^f_i \) on the elementary \( n \)-paths.

**Proposition 2.4.** Let \( f \) be a real function on \( V \). Let \( n \geq 0 \) and let \( v_0 v_1 \ldots v_n \) be an elementary \( n \)-path on \( V \). Then for any \( 0 \leq i \leq n \) we have

\[ \partial^f_i (v_0 v_1 \ldots v_n) = (-1)^i f(v_i) v_0 \ldots \hat{v}_i \ldots v \]

and for any \( 0 \leq j \leq n + 1 \) we have

\[ d^f_j (v_0 v_1 \ldots v_n) = \sum_{v \in V} (-1)^j f(v) v_0 \ldots v_{j-1} v v_{j+1} \ldots v_n. \]
Proof. Let $0 \leq i \leq n$ and $0 \leq j \leq n + 1$. By a straight-forward calculation,

$$
\partial_i^f(v_0v_1 \ldots v_n) = \sum_{v \in V} f(v) \frac{\partial_i}{\partial v}(v_0v_1 \ldots v_n)
$$

$$
= \sum_{v \in V} (-1)^i \delta(v, v_i) f(v)v_0 \ldots \hat{v}_i \ldots v
$$

and

$$
d_j^f(v_0v_1 \ldots v_n) = \sum_{v \in V} f(v) d_j v(v_0v_1 \ldots v_n)
$$

$$
= \sum_{v \in V} (-1)^j f(v_0 \ldots v_{j-1}v_j \ldots v_n).
$$

The proposition follows.

The next theorem follows with the help of Subsection 2.1.

**Theorem 2.5** (Main Result I: The simplicial-like identities for paths on discrete sets). Let $f$ and $g$ be two real functions on $V$. Then for any $n \geq 0$, we have

(i). $\partial_i^f \circ \partial_j^g = -\partial_j^{g-1} \circ \partial_i^f$ for any $i < j$;

(ii). $\partial_i^f \circ d_j^g = \begin{cases} -d_j^{g-1} \circ \partial_i^f, & i < j; \\
(f, g) \text{ id}, & i = j; \\
-d_j^g \circ \partial_i^{f-1}, & i > j; \end{cases}$

(iii). $d_i^f \circ d_j^g = -d_j^{g+1} \circ d_i^f$ for $i \leq j$.

Proof. (i). Suppose $i < j$. By a straight-forward calculation and Lemma 2.1, we have

$$
\left( \sum_{v \in V} f(v) \frac{\partial_i}{\partial v} \right) \circ \left( \sum_{v \in V} g(v) \frac{\partial_j}{\partial v} \right) = \left( \sum_{u \in V} f(u) \frac{\partial_i}{\partial u} \right) \circ \left( \sum_{v \in V} g(v) \frac{\partial_j}{\partial v} \right)
$$

$$
= \sum_{u, v \in V} f(u)g(v) \frac{\partial_i}{\partial u} \circ \frac{\partial_j}{\partial v}
$$

$$
= -\sum_{u, v \in V} f(u)g(v) \frac{\partial_{j-1}}{\partial v} \circ \frac{\partial_i}{\partial u}
$$

$$
= -\left( \sum_{v \in V} g(v) \frac{\partial_{j-1}}{\partial v} \right) \circ \left( \sum_{u \in V} f(u) \frac{\partial_i}{\partial u} \right)
$$

$$
= -\left( \sum_{v \in V} g(v) \frac{\partial_{j-1}}{\partial v} \right) \circ \left( \sum_{v \in V} f(v) \frac{\partial_i}{\partial v} \right).
$$

Thus we obtain

$$
\left( \sum_{v \in V} f(v) \frac{\partial_i}{\partial v} \right) \circ \left( \sum_{v \in V} g(v) \frac{\partial_j}{\partial v} \right) = -\left( \sum_{v \in V} g(v) \frac{\partial_{j-1}}{\partial v} \right) \circ \left( \sum_{v \in V} f(v) \frac{\partial_i}{\partial v} \right).
$$

By the notations (2.10) and (2.11), we obtain (i).
(ii). By a straight-forward calculation Lemma 2.2 we have
\[
\left( \sum_{v \in V} f(v) \frac{\partial}{\partial v} \right) \circ \left( \sum_{v \in V} g(v) d_j v \right) = \left( \sum_{u \in V} f(u) \frac{\partial}{\partial u} \right) \circ \left( \sum_{v \in V} g(v) d_j v \right) = \sum_{u, v \in V} f(u) g(v) \frac{\partial}{\partial u} \circ d_j v
\]
\[
= \begin{cases} 
- \sum_{u, v \in V} f(u) g(v) d_{j-1} v \circ \frac{\partial}{\partial u}, & i < j, \\
\sum_{u, v \in V} f(u) g(v) \delta(u, v) \text{id}, & i = j, \\
- \sum_{u, v \in V} f(u) g(v) d_j v \circ \frac{\partial}{\partial u}, & i > j.
\end{cases}
\]

By a similar calculation in (i), we have
\[
\left( \sum_{v \in V} f(v) d_{j-1} v \right) \circ \left( \sum_{v \in V} g(v) \frac{\partial}{\partial v} \right) = \sum_{u, v \in V} f(u) g(v) d_{j-1} v \circ \frac{\partial}{\partial v},
\]
\[
(f, g) \text{id} = \left( \sum_{v \in V} f(v) g(v) \right) \text{id} = \sum_{u \in V, v \in V} f(u) g(v) \delta(u, v) \text{id},
\]
\[
\left( \sum_{v \in V} f(v) d_j v \right) \circ \left( \sum_{v \in V} g(v) \frac{\partial}{\partial v} \right) = \sum_{u \in V, v \in V} f(u) g(v) d_j v \circ \frac{\partial}{\partial v}.
\]

Therefore, it follows from the above four equations that
\[
\left( \sum_{v \in V} f(v) \frac{\partial}{\partial v} \right) \circ \left( \sum_{v \in V} g(v) d_j v \right) = \begin{cases} 
- \left( \sum_{v \in V} f(v) d_{j-1} v \right) \circ \left( \sum_{v \in V} g(v) \frac{\partial}{\partial v} \right), & i < j, \\
\left( \sum_{v \in V} f(v) g(v) \right) \text{id}, & i = j, \\
- \left( \sum_{v \in V} f(v) d_j v \right) \circ \left( \sum_{v \in V} g(v) \frac{\partial}{\partial v} \right), & i > j.
\end{cases}
\]

With the help of the notations \((2.10)\) and \((2.11)\), we obtain (ii).

(iii). Suppose \(i \leq j\). By a straight-forward calculation and Lemma 2.2 we have
\[
\left( \sum_{v \in V} f(v) d_i v \right) \circ \left( \sum_{v \in V} g(v) d_j v \right) = \sum_{u, v \in V} f(u) g(v) d_i v \circ d_j v
\]
\[
= - \sum_{u, v \in V} f(u) g(v) d_{j+1} v \circ d_i v
\]
\[
= - \left( \sum_{v \in V} g(v) d_{j+1} v \right) \circ \left( \sum_{v \in V} f(v) d_i v \right).
\]

Thus we obtain
\[
\left( \sum_{v \in V} f(v) d_i v \right) \circ \left( \sum_{v \in V} g(v) d_j v \right) = - \left( \sum_{v \in V} g(v) d_{j+1} v \right) \circ \left( \sum_{v \in V} f(v) d_i v \right).
\]

By the notations \((2.10)\) and \((2.11)\), we obtain (iii).

Taking \(f = g\) and \(i = j\), the next corollary follows from Theorem 2.5 (ii).

**Corollary 2.6.** For any real function \(f\) on \(V\) and any \(0 \leq i \leq n\), we have
\[
\partial_i^f \circ d_i^f = ||f||_2^2 \text{id}.
\]

\[\square\]
2.3 The Weighted Boundary Operators and the Weighted Co-Boundary Operators for Paths

Let \( f : V \to \mathbb{R} \) be a real function on \( V \). Let \( n \geq 0 \). In this subsection, we define the \( f \)-weighted boundary operator as the sum of the \( f \)-weighted face maps and define the \( f \)-weighted co-boundary operator as the sum of the \( f \)-weighted co-face maps. Then we prove the anti-commutative properties for the weighted boundary operators and the weighted co-boundary operators.

**Definition 8.** We define the \( f \)-weighted boundary vector as a linear map

\[
\sum_{v \in V} f(v) \frac{\partial}{\partial v} : \Lambda_n(V) \to \prod_{n+1} \Lambda_{n-1}(V).
\]

For simplicity, we use the notation

\[
\bar{\partial}^f := \sum_{v \in V} f(v) \frac{\partial}{\partial v}.
\]

With the help of \( (2.10) \) we can write the \( f \)-weighted boundary vector as

\[
\bar{\partial}^f = (\partial^f_0, \partial^f_1, \ldots, \partial^f_n).
\] (2.12)

**Definition 9.** Taking the sums of the coordinates in the \( f \)-weighted boundary vector \( (2.12) \), we define the \( f \)-weighted boundary operator as

\[
\partial^f = \sum_{i=0}^n \partial^f_i : \Lambda_n(V) \to \Lambda_{n-1}(V).
\]

**Definition 10.** We define the \( f \)-weighted co-boundary vector as a linear map

\[
\sum_{v \in V} f(v) \bar{d}v : \Lambda_n(V) \to \prod_{n+1} \Lambda_{n+2}(V).
\]

For simplicity, we use the notation

\[
\bar{d}^f := \sum_{v \in V} f(v) \bar{d}v.
\]

With the help of \( (2.11) \) we can write the \( f \)-weighted co-boundary vector as

\[
\bar{d}^f = (d^f_0, d^f_1, \ldots, d^f_{n+1}).
\] (2.13)

**Definition 11.** Taking the sums of the coordinates in the \( f \)-weighted co-boundary vector \( (2.13) \), we define the \( f \)-weighted co-boundary operator as

\[
d^f = \sum_{i=0}^{n+1} d^f_i : \Lambda_n(V) \to \Lambda_{n+1}(V).
\]

**Definition 12.** As a particular case of Definition 9 and Definition 11, we take any \( v \in V \) and take \( f \) to be the characteristic function \( \chi_v \) of \( v \) given by

\[
\chi_v(u) = \delta(v, u), \quad u \in V.
\]

For any \( v \in V \), we let

\[
\frac{\partial}{\partial v} = \partial^{\chi_v}, \quad d\bar{v} = d^{\chi_v}.
\] (2.14)
Remark 3: We give a remark on Definition \[12\]. It is direct to see that for any elementary \( n \)-path \( v_0 v_1 \ldots v_n \) on \( V \), we have

\[
\frac{\partial}{\partial v}(v_0 v_1 \ldots v_n) = \sum_{i=0}^{n} (-1)^i \delta(v, v_i)v_0 \ldots \hat{v}_i \ldots v_n,
\]

\[
dv(v_0 v_1 \ldots v_n) = \sum_{i=0}^{n+1} (-1)^i v_0 v_1 \ldots v_{i-1} v_i v_{i+1} \ldots v_n.
\]

By Definition \[9\], Definition \[11\] and Definition \[12\] we have the next lemma.

**Lemma 2.7.** For any \( u, v \in V \) we have

\[
\frac{\partial}{\partial u} \circ \frac{\partial}{\partial v}(v_0 v_1 \ldots v_n) = -\frac{\partial}{\partial v} \circ \frac{\partial}{\partial u}, \quad dv \circ du = -du \circ dv. \tag{2.15}
\]

**Proof.** For any \( n \geq 0 \) and any elementary \( n \)-path \( v_0 v_1 \ldots v_n \in \Lambda_n(V) \), we have

\[
\frac{\partial}{\partial u} \circ \frac{\partial}{\partial v}(v_0 v_1 \ldots v_n) = \frac{\partial}{\partial u}\left( \sum_{j=0}^{n} (-1)^j \delta(v, v_j)v_0 \ldots \hat{v}_j \ldots v_n \right)
\]

\[
= \sum_{j=0}^{n} (-1)^j \delta(v, v_j) \frac{\partial}{\partial u}(v_0 \ldots \hat{v}_j \ldots v_n)
\]

\[
= \sum_{j=0}^{n} (-1)^j \delta(v, v_j) \sum_{i=0}^{j-1} (-1)^i \delta(u, v_i)(v_0 \ldots \hat{v}_i \ldots \hat{v}_j \ldots v_n)
\]

\[
+ \sum_{j=0}^{n} (-1)^j \delta(v, v_j) \sum_{i=j+1}^{n} (-1)^{i-1} \delta(u, v_i)(v_0 \ldots \hat{v}_i \ldots \hat{v}_j \ldots v_n)
\]

\[
= \sum_{0 \leq i < j \leq n} (-1)^{i+j} \delta(u, v_i) \delta(v, v_j)(v_0 \ldots \hat{v}_i \ldots \hat{v}_j \ldots v_n)
\]

\[
+ \sum_{0 \leq i < j \leq n} (-1)^{i+j-1} \delta(u, v_i) \delta(v, v_j)(v_0 \ldots \hat{v}_j \ldots \hat{v}_i \ldots v_n)
\]

and

\[
\frac{\partial}{\partial v} \circ \frac{\partial}{\partial u}(v_0 v_1 \ldots v_n) = \sum_{0 \leq j < i \leq n} (-1)^{i+j} \delta(u, v_i) \delta(v, v_j)(v_0 \ldots \hat{v}_i \ldots \hat{v}_j \ldots v_n)
\]

\[
+ \sum_{0 \leq j < i \leq n} (-1)^{i+j-1} \delta(u, v_i) \delta(v, v_j)(v_0 \ldots \hat{v}_j \ldots \hat{v}_i \ldots v_n).
\]

Therefore, for any elementary \( n \)-path \( v_0 v_1 \ldots v_n \) on \( V \), we have

\[
\frac{\partial}{\partial u} \circ \frac{\partial}{\partial v}(v_0 v_1 \ldots v_n) = -\frac{\partial}{\partial v} \circ \frac{\partial}{\partial u}(v_0 v_1 \ldots v_n).
\]

Consequently, by the linear property of \( \frac{\partial}{\partial u} \circ \frac{\partial}{\partial v} \) and \( \frac{\partial}{\partial v} \circ \frac{\partial}{\partial u} \), we obtain the first identity in (2.15).
Similarly, for any \( n \geq 0 \) and any elementary \( n \)-path \( v_0v_1\ldots v_n \in \Lambda_n(V) \), we have

\[
d u \circ dv(v_0v_1\ldots v_n) = d u \left( \sum_{i=0}^{n+1} (-1)^i v_0 \ldots v_{i-1} v_{i+1} \ldots v_n \right)
\]

\[
= \sum_{i=0}^{n+1} (-1)^i d u(v_0 \ldots v_{i-1} v_{i+1} \ldots v_n)
\]

\[
= \sum_{i=0}^{n+1} (-1)^i \left( \sum_{j=0}^{i-1} (-1)^j v_0 \ldots v_{j-1} v_{j+1} \ldots v_{i-1} v_{i+1} \ldots v_n \right)
+ (-1)^i v_0 \ldots v_{i-1} v_{i+1} v_{i+2} \ldots v_n + (-1)^{i+1} v_0 \ldots v_{i-1} v_{i+2} \ldots v_n
+ \sum_{j=i+1}^{n+1} (-1)^j v_0 \ldots v_{i-1} v_{i+2} \ldots v_{j-1} v_{j+1} \ldots v_n
\]

and

\[
d v \circ du(v_0v_1\ldots v_n) = \sum_{i=0}^{n+1} (-1)^i \left( \sum_{j=0}^{i-1} (-1)^j v_0 \ldots v_{j-1} v_{j+1} \ldots v_{i-1} v_{i+1} \ldots v_n \right)
+ (-1)^i v_0 \ldots v_{i-1} v_{i+1} v_{i+2} \ldots v_n + (-1)^{i+1} v_0 \ldots v_{i-1} v_{i+2} \ldots v_n
+ \sum_{j=i+1}^{n+1} (-1)^j v_0 \ldots v_{i-1} v_{i+2} \ldots v_{j-1} v_{j+1} \ldots v_n
\]

Thus

\[
d u \circ dv(v_0v_1\ldots v_n) = -dv \circ du(v_0v_1\ldots v_n)
\]

for any \( n \geq 0 \) and any elementary \( n \)-path \( v_0v_1\ldots v_n \in \Lambda_n(V) \). Consequently, by the linear property of \( d u \circ dv(v_0v_1\ldots v_n) \) and \( dv \circ du(v_0v_1\ldots v_n) \), we obtain the second identity in (24.15).

The next corollary is a re-statement of Lemma 2.7.

**Corollary 2.8.** For any \( u, v \in V \) we have

\[
\sum_{i,j=0}^{n} \frac{\partial}{\partial v} \circ \frac{\partial}{\partial u} = -\sum_{i,j=0}^{n} \frac{\partial}{\partial u} \circ \frac{\partial}{\partial v}
\]

\[
\sum_{i,j=0}^{n} d_i v \circ d_j u = -\sum_{i,j=0}^{n} d_j u \circ d_i v.
\]

**Proof.** The first identity follows from Lemma 2.7 by

\[
\sum_{i,j=0}^{n} \frac{\partial}{\partial v} \circ \frac{\partial}{\partial u} = \left( \sum_{i=0}^{n} \frac{\partial}{\partial v} \right) \circ \left( \sum_{j=0}^{n} \frac{\partial}{\partial u} \right) = \frac{\partial}{\partial v} \circ \frac{\partial}{\partial u}
\]

\[
= -\frac{\partial}{\partial u} \circ \frac{\partial}{\partial v} = -\left( \sum_{j=0}^{n} \frac{\partial}{\partial u} \right) \circ \left( \sum_{i=0}^{n} \frac{\partial}{\partial v} \right) = -\sum_{i,j=0}^{n} \frac{\partial}{\partial u} \circ \frac{\partial}{\partial v}
\]

The second identity follows from Lemma 2.7 by

\[
\sum_{i,j=0}^{n} d_i v \circ d_j u = \left( \sum_{i=0}^{n} d_i v \right) \circ \left( \sum_{j=0}^{n} d_j u \right) = dv \circ du
\]

\[
= -du \circ dv = -\left( \sum_{j=0}^{n} d_j u \right) \circ \left( \sum_{i=0}^{n} d_i v \right) = -\sum_{i,j=0}^{n} d_j u \circ d_i v.
\]

The corollary is proved.
With the help of Corollary 2.8 we generalize Lemma 2.7 in the next proposition.

**Proposition 2.9 (The anti-commutative properties).** Let \( f \) and \( g \) be two real functions on \( V \). Then

\[
\partial f \circ \partial g = -\partial g \circ \partial f, \quad d f \circ d g = -d g \circ d f.
\]

In particular, taking \( f = \chi_u \) and \( g = \chi_v \), we obtain (2.15).

**Proof.** Let \( n \geq 0 \). Since

\[
\partial f = \sum_{i=0}^{n} \sum_{v \in V} f(v) \frac{\partial}{\partial v}^i,
\]

by a straight-forward calculation we have

\[
\partial f \circ \partial g = \left( \sum_{i=0}^{n} \sum_{v \in V} f(v) \frac{\partial}{\partial v}^i \right) \circ \left( \sum_{j=0}^{n} \sum_{u \in V} g(u) \frac{\partial}{\partial u}^j \right)
\]

\[
= \sum_{v,u \in V} f(v)g(u) \sum_{i,j=0}^{n} \frac{\partial}{\partial v}^i \circ \frac{\partial}{\partial u}^j,
\]

and

\[
\partial g \circ \partial f = \left( \sum_{j=0}^{n} \sum_{u \in V} g(u) \frac{\partial}{\partial u}^j \right) \circ \left( \sum_{i=0}^{n} \sum_{v \in V} f(v) \frac{\partial}{\partial v}^i \right)
\]

\[
= \sum_{v,u \in V} f(v)g(u) \sum_{i,j=0}^{n} \frac{\partial}{\partial v}^i \circ \frac{\partial}{\partial u}^j.
\]

Hence with the help of the first identity in Corollary 2.8 we have \( \partial f \circ \partial g = -\partial g \circ \partial f \). Similarly, since

\[
d f = \sum_{i=0}^{n} d f_i = \sum_{i=0}^{n} \sum_{v \in V} f(v) d_i v,
\]

by a straight-forward calculation we have

\[
d f \circ d g = \left( \sum_{i=0}^{n} \sum_{v \in V} f(v) d_i v \right) \circ \left( \sum_{j=0}^{n} \sum_{u \in V} g(u) d_j u \right)
\]

\[
= \sum_{v,u \in V} f(v)g(u) \sum_{i,j=0}^{n} d_i v \circ d_j u
\]

and

\[
d g \circ d f = \left( \sum_{j=0}^{n} \sum_{u \in V} g(u) d_j u \right) \circ \left( \sum_{i=0}^{n} \sum_{v \in V} f(v) d_i v \right)
\]

\[
= \sum_{v,u \in V} f(v)g(u) \sum_{i,j=0}^{n} d_j u \circ d_i v.
\]

Hence with the help of the second identity in Corollary 2.8 we have \( d f \circ d g = -d g \circ d f \). The proposition follows. □
With respect to the canonical inner product \(( , )\) on \(\Lambda_*(V)\) given by

\[
\langle v_0 v_1 \ldots v_n, u_0 u_1 \ldots u_n \rangle = \prod_{i=0}^{n} \delta(v_i, u_i), \quad n = 0, 1, 2, \ldots.
\] (2.16)

the linear operator \(dv\) is adjoint to \(\frac{\partial}{\partial v}\) for any \(v \in V\). Precisely, we have the next lemma.

**Lemma 2.10.** For any \(n \geq 1\), any \(\xi \in \Lambda_{n-1}(V)\), and any \(\eta \in \Lambda_n(V)\), we have

\[
\langle \frac{\partial}{\partial v}(\eta), \xi \rangle = \langle \eta, dv(\xi) \rangle.
\] (2.17)

**Proof.** We take \(\eta\) to be an elementary \(n\)-path \(v_0 v_1 \ldots v_n \in \Lambda_n(V)\) and take \(\xi\) to be an elementary \((n-1)\)-path \(u_0 u_1 \ldots u_{n-1} \in \Lambda_{n-1}(V)\). Then

\[
\langle \frac{\partial}{\partial v}(v_0 v_1 \ldots v_n), u_0 u_1 \ldots u_{n-1} \rangle = \langle \sum_{i=0}^{n} (-1)^i \delta(v_i, v_0) \cdots \hat{v}_i \cdots v_n, u_0 u_1 \ldots u_{n-1} \rangle
\]

\[
= \sum_{i=0}^{n} (-1)^i \delta(v_i, v_0) \prod_{j=0}^{i-1} \delta(v_j, u_j) \prod_{j=i}^{n-1} \delta(v_{j+1}, u_j).
\]

Consequently, if we use \((\frac{\partial}{\partial v})^*\) to denote the adjoint linear operator of \(\frac{\partial}{\partial v}\), then we have

\[
(\frac{\partial}{\partial v})^*(u_0 u_1 \ldots u_{n-1}) = \sum_{v_0, v_1, \ldots, v_n \in V} \langle v_0 v_1 \ldots v_n, (\frac{\partial}{\partial v})^*(u_0 u_1 \ldots u_{n-1}) \rangle v_0 v_1 \ldots v_n
\]

\[
= \sum_{v_0, v_1, \ldots, v_n \in V} \langle \frac{\partial}{\partial v}(v_0 v_1 \ldots v_n), u_0 u_1 \ldots u_{n-1} \rangle v_0 v_1 \ldots v_n
\]

\[
= \sum_{v_0, v_1, \ldots, v_n \in V} \left( \sum_{i=0}^{n} (-1)^i \delta(v_i, v_0) \prod_{j=0}^{i-1} \delta(v_j, u_j) \prod_{j=i}^{n-1} \delta(v_{j+1}, u_j) \right) v_0 v_1 \ldots v_n
\]

\[
= \sum_{i=0}^{n} (-1)^i \left( \sum_{v_0, v_1, \ldots, v_n \in V} \delta(v_i, v_0) \prod_{j=0}^{i-1} \delta(v_j, u_j) \prod_{j=i}^{n-1} \delta(v_{j+1}, u_j) \right) v_0 v_1 \ldots v_n
\]

\[
= \sum_{i=0}^{n} (-1)^i u_0 u_1 \ldots u_{i-1} v_0 u_i u_{i+1} \ldots u_{n-1}
\]

\[
= dv(u_0 u_1 \ldots u_{n-1}).
\]

Therefore, by the linear property of \((\frac{\partial}{\partial v})^*\) and \(dv\), we have

\[
(\frac{\partial}{\partial v})^* = dv.
\]

The lemma is proved.

In general, the linear operator \(df\) is adjoint to \(\partial f\) for any real function \(f\) on \(V\). The next proposition follows from Lemma 2.10.

**Proposition 2.11** (The adjoint property). Let \(f\) be any real function on \(V\). Then for any \(n \geq 1\), any \(\xi \in \Lambda_{n-1}(V)\), and any \(\eta \in \Lambda_n(V)\), we have

\[
\langle \partial f(\eta), \xi \rangle = \langle \eta, df(\xi) \rangle.
\] (2.18)
Proof. Let $n \geq 1$. Let $f$ be any real function on $V$. Let $\xi \in \Lambda_{n-1}(V)$ and $\eta \in \Lambda_n(V)$. Then
\[
\langle \partial f(\eta), \xi \rangle = \sum_{i=0}^{n} \langle \partial f^i(\eta), \xi \rangle = \sum_{i=0}^{n} \left( \sum_{v \in V} f(v) \frac{\partial}{\partial v} \right)(\eta, \xi) = \sum_{v \in V} f(v) \sum_{i=0}^{n} \left( \frac{\partial}{\partial v} \right)(\eta, \xi) = \sum_{v \in V} f(v) \langle \eta, d\nu(\xi) \rangle = \langle \eta, d^f(\xi) \rangle.
\]
We obtain (2.18). \hfill \Box

We prove some Newton-Leibniz-type rules in the next proposition.

**Proposition 2.12** (The Newton-Leibniz-type rules). Let $f$ be a real function on $V$. Let $n, m \geq 0$. Let $\xi \in \Lambda_n(V)$ be given in (2.1) and let $\eta \in \Lambda_m(V)$ be given in (2.2). Then
\[
\partial^f(\xi \ast \eta) = \partial^f(\xi) \ast \eta + (-1)^{n+1} \xi \ast \partial^f(\eta), \quad (2.19)
\]
\[
d^f(\xi \ast \eta) = d^f(\xi) \ast \eta + (-1)^{n+1} \xi \ast d^f(\eta), \quad (2.20)
\]

Proof. By a straight-forward calculation and with the help of (2.3), we have
\[
\partial^f(v_0v_1 \ldots v_nu_0u_1 \ldots u_m) = \sum_{i=0}^{n+m+1} \partial^f_i(v_0v_1 \ldots v_nu_0u_1 \ldots u_m) = \sum_{i=0}^{n+m+1} \sum_{v \in V} f(w) \frac{\partial}{\partial w} (v_0v_1 \ldots v_nu_0u_1 \ldots u_m) = \sum_{v \in V} f(v) \delta(v, v_i)(v_0 \ldots \hat{v}_i \ldots v_n) \ast (u_0u_1 \ldots u_m) + \sum_{i=0}^{m} \sum_{u \in V} \left( -1 \right)^{n+i+1} f(u) \delta(u, u_i)(v_0v_1 \ldots v_n) \ast (u_0 \ldots \hat{u}_i \ldots u_m) = \sum_{i=0}^{n} \frac{\partial^f_i(v_0v_1 \ldots v_n) \ast (u_0u_1 \ldots u_m)}{(-1)^{n+1} \sum_{i=0}^{m} \frac{\partial^f_i(v_0v_1 \ldots v_n) \ast \partial^f_i(u_0u_1 \ldots u_m)}} = \partial^f(v_0v_1 \ldots v_n) \ast (u_0u_1 \ldots u_m) + (-1)^{n+1} \sum_{i=0}^{m} \frac{\partial^f_i(v_0v_1 \ldots v_n) \ast \partial^f_i(u_0u_1 \ldots u_m)}{(-1)^{n+1} (v_0v_1 \ldots v_n) \ast \partial^f(u_0u_1 \ldots u_m)}.
\]
Theorem 2.5. This subsection is supplementary to Subsection 2.1 - Subsection 2.3.

In this subsection, we re-state the usual simplicial identities of simplicial sets, in contrast with

2.4 A Contrast with The Usual Simplicial Identities

\[ \partial^f(\xi \ast \eta) = \sum_{n_0, v_1, \ldots, v_n \in V; \ n_0u_1, \ldots, u_m \in V} \partial^f(v_0v_1 \ldots v_n) \ast (u_0u_1 \ldots u_m) \]

\[ = \sum_{n_0, v_1, \ldots, v_n \in V; \ n_0u_1, \ldots, u_m \in V} \partial^f(v_0v_1 \ldots v_n) \ast (u_0u_1 \ldots u_m) \]

\[ + (-1)^{n+1}(v_0v_1 \ldots v_n) \ast \partial^f(u_0u_1 \ldots u_m) \]

\[ = \partial^f(\xi) \ast \eta + (-1)^{n+1} \xi \ast \partial^f(\eta). \]

We obtain (2.19). On the other hand, it follows from a straight-forward calculation that

\[ d^f(v_0v_1 \ldots v_nu_0u_1 \ldots u_n) = \sum_{i=0}^{n+1} d^f(v_0v_1 \ldots v_n) \ast (u_0u_1 \ldots u_n) \]

\[ + (-1)^{n+1} \sum_{i=0}^{m+1} (v_0v_1 \ldots v_n) \ast d^f(u_0u_1 \ldots u_n). \]

Thus by a formal calculation analogous with the above proof of (2.19), we obtain (2.20). The proposition follows.

By the end of this subsection, we summarize Proposition 2.9 and Proposition 2.12. We give the exterior algebra \( T_*(V) \) generated by all the weighted boundary operators and the exterior algebra \( T^*(V) \) generated by all the weighted co-boundary operators on the discrete set \( V \), in the following list.

- Let \( T_*(V) = \bigoplus_{k \geq 0} T_k(V) \) be the exterior algebra spanned by \( \partial^f \) for all real functions \( f \) on \( V \). Then we have all of the followings:
  - (i). for each \( k \geq 0 \) and for each \( \alpha \in T_k(V) \), we have a graded linear map \( \alpha : \Lambda_n(V) \rightarrow \Lambda_{n-k}(V) \), \( n = 0, 1, 2 \ldots \);
  - (ii). the exterior product in \( T_*(V) \) is the composition of the graded linear maps in (i);
  - (iii). the operations of \( T_*(V) \) on \( \Lambda_*(V) \) satisfy the Newton-Leibniz-type law (2.19).
- Let \( T^*(V) = \bigoplus_{k \geq 0} T^k(V) \) be the exterior algebra spanned by \( d^f \) for all real functions \( f \) on \( V \). Then we have all of the followings:
  - (i)’. for each \( k \geq 0 \) and for each \( \omega \in T^k(V) \), we have a graded linear map \( \omega : \Lambda_n(V) \rightarrow \Lambda_{n+k}(V) \), \( n = 0, 1, 2 \ldots \);
  - (ii)’. the exterior product in \( T^*(V) \) is the composition of the graded linear maps in (i)’;
  - (iii)’. the operations of \( T^*(V) \) on \( \Lambda_*(V) \) satisfy the Newton-Leibniz-type law (2.20).

2.4 A Contrast with The Usual Simplicial Identities

In this subsection, we re-state the usual simplicial identities of simplicial sets, in contrast with

Theorem 2.5. This subsection is supplementary to Subsection 2.1 - Subsection 2.3.

Consider the special function \( f \equiv 1 \), that is, \( f(v) \) takes the constant value 1 for all \( v \in V \). We denote this function \( f \) as 1. We denote the corresponding \( f \)-weighted face maps as \( \partial^1_i \) for \( 0 \leq i \leq n \).
and denote the corresponding $f$-weighted co-face maps as $d_i^f$ for $0 \leq i \leq n + 1$. Let $v_0v_1 \ldots v_n$ be an elementary $n$-path on $V$. It follows that
\[
\partial_1^j(v_0v_1 \ldots v_n) = (-1)^jv_0 \ldots \widehat{v}_i \ldots v_n.
\]
Thus the linear map $(-1)^j\partial_1^j : \Lambda_n(V) \rightarrow \Lambda_{n-1}(V)$ given by
\[
(-1)^j\partial_1^j(v_0v_1 \ldots v_n) = v_0 \ldots \widehat{v}_i \ldots v_n
\]
is the usual face map of simplicial sets (cf. [2, pp. 110-111]). Moreover, for $0 \leq i \leq n$, we use the linear map $s_i : \Lambda_n(V) \rightarrow \Lambda_{n+1}(V)$ to denote the $i$-th degeneracy (cf. [2, pp. 110-111]) given by
\[
s_i(v_0v_1 \ldots v_n) = v_0 \ldots v_{i-1}v_i^i v_{i+1}v_n \ldots v_n.
\]
Then for any elementary $n$-path $v_0v_1 \ldots v_n$ on $V$, we have
\[
s_i(v_0v_1 \ldots v_n) = (-1)^i d_i^j(v_0v_1 \ldots v_n) = (-1)^i d_i^{k+1}(v_0v_1 \ldots v_n)
\]
and
\[
s_i(v_0v_1 \ldots v_n) = (-1)^{i+1} d_{i+1}^j(v_0v_1 \ldots v_n) = (-1)^{i+1} d_{i+1}^{k+1}(v_0v_1 \ldots v_n).
\]
With the help of (2.21) - (2.24), the simplicial identities (cf. [2, p. 110]) can be re-stated in the next proposition.

**Proposition 2.13** (The usual simplicial identities, in contrast with Theorem 2.5). Let $n \geq 0$ and $0 \leq i, j \leq n$. Then we have the first simplicial identity
\[
\partial_1^i \partial_1^j = -\partial_1^{j-1} \partial_1^i,
\]
the second simplicial identity
\[
\partial_1^i s_j = \begin{cases} 
s_{j-1} \partial_1^i, & i < j, \\
\text{id}, & i = j, j + 1, \\
-s_j \partial_1^{i-1}, & i > j + 1,
\end{cases}
\]
and the third simplicial identity
\[
s_i s_j = s_{j+1} s_i, \quad i \leq j.
\]

3 Simplicial-Like Identities for Regular Paths on Discrete Sets

Let $V$ be a discrete set. Let $n \geq 0$. An elementary $n$-path $v_0 \ldots v_n$ on $V$ is called regular if $v_{i-1} \neq v_i$ for all $1 \leq i \leq n$ and is called irregular otherwise (cf. [5, Definition 2.7]). For each $n \geq 0$, let $I_n(V)$ be the subspace of $\Lambda_n(V)$ spanned by all the irregular elementary $n$-paths on $V$. We have a graded subspace
\[
I_n(V) = \bigoplus_{n=0}^\infty I_n(V)
\]
of $\Lambda_n(V)$. Consider the quotient space $\mathcal{R}_n(V) = \Lambda_n(V)/I_n(V)$. Then $\mathcal{R}_n(V)$ is the vector space spanned by all the regular elementary $n$-paths on $V$ (cf. [8, Definition 2.8]). An element in $\mathcal{R}_n(V)$ is called a regular $n$-path on $V$. We take the direct sum

$$\mathcal{R}_n(V) = \bigoplus_{n=0}^{\infty} \mathcal{R}_n(V).$$

### 3.1 The Weighted Face Maps and the Weighted Co-Face Maps for Regular Paths

Let $f$ be a real function on $V$. In this subsection, we define the $f$-weighted face maps and the $f$-weighted co-face maps for regular paths on $V$ and prove some simplicial-like identities.

**Definition 13.** For each $0 \leq i \leq n$, it follows from the argument in [8, Subsection 2.3, Regular paths] that if we modulo the terms in $I_{n-1}(V)$ of the image of $\partial^f_i : \Lambda_n(V) \rightarrow \Lambda_{n-1}(V)$ then the $f$-weighted face map $\partial^f_i$ from $\Lambda_n(V)$ to $\Lambda_{n-1}(V)$ induces a linear map

$$\tilde{\partial}^f_i := \partial^f_i / I_i(V) : \mathcal{R}_n(V) \rightarrow \mathcal{R}_{n-1}(V).$$

We call $\tilde{\partial}^f_i$ the $i$-th $f$-weighted regular face map.

We give the explicit expression for the $i$-th $f$-weighted regular face map $\tilde{\partial}^f_i$ defined in Definition 13. Let $v_0v_1\ldots v_n$ be an arbitrary regular elementary $n$-path on $V$. For any $u, v \in V$, we let

$$\epsilon(u,v) = 1 - \delta(u,v).$$

With the help of the first formula in Proposition 2.4 we have

$$\tilde{\partial}^f_i(v_0v_1\ldots v_n) = \begin{cases} f(v_0)v_1\ldots v_n, & i = 0, \\ (-1)^i \epsilon(v_{i-1}, v_{i+1}) f(v_i)v_0\ldots \hat{v}_i\ldots v_n, & 1 \leq i \leq n-1, \\ (-1)^n f(v_n)v_0\ldots v_{n-1}, & i = n. \end{cases}$$

(3.1)

For convenience, for all $0 \leq i \leq n$ we write (3.1) as

$$\tilde{\partial}^f_i(v_0v_1\ldots v_n) = (-1)^i \epsilon(v_{i-1}, v_{i+1}) f(v_i)v_0\ldots \hat{v}_i\ldots v_n$$

for short, by an abuse of the following notations

$$\epsilon(v_{-1}, v_1) = \epsilon(v_{n-1}, v_{n+1}) = 1.$$

**Definition 14.** For any $0 \leq i \leq n+1$, if we modulo the terms in $I_i(V)$ of the image, then the $f$-weighted co-face map $d^f_i$ from $\Lambda_n(V)$ to $\Lambda_{n+1}(V)$ induces a linear map

$$\tilde{d}^f_i := d^f_i / I_i(V) : \mathcal{R}_n(V) \rightarrow \mathcal{R}_{n+1}(V).$$

We call $\tilde{d}^f_i$ the $i$-th $f$-weighted regular co-face map.

1by saying "modulo the terms in $I_{n-1}(V)$ of the image of $\partial^f_n$, it means that we take the canonical projection from the image of $\partial^f_n$ in $\Lambda_{n-1}(V)$ to the orthogonal complement of $I_{n-1}(V)$ in $\Lambda_{n-1}(V)$ with respect to the inner product $\epsilon$.
We prove (i), (ii), and (iii) separately.

For convenience, for all (Main Result II: The simplicial-like identities for regular paths on discrete sets)

(iii).

Let \( v_0v_1\ldots v_n \) be an arbitrary regular elementary \( n \)-path on \( V \). With the help of the second formula in Proposition 2.4 we have

\[
\tilde{d}^f_i (v_0v_1\ldots v_n) = \begin{cases} 
\sum_{v\in V\setminus\{v_0\}} f(v)v_0\ldots v_n, & i = 0, \\
\sum_{v\in V\setminus\{v_{i-1}, v_i\}} (-1)^i f(v)v_0\ldots v_{i-1}v_i\ldots v_n, & 0 \leq i \leq n, \\
\sum_{v\in V\setminus\{v_n\}} (-1)^{n+1} f(v)v_0\ldots v_nv, & i = n + 1.
\end{cases}
\]  

(3.2)

For convenience, for all \( 0 \leq i \leq n + 1 \) we write \( \tilde{d}^f_i \) as

\[
\tilde{d}^f_i (v_0v_1\ldots v_n) = \sum_{v \neq v_{i-1}, v_i} (-1)^i f(v)v_0\ldots v_{i-1}v_i\ldots v_n
\]

for short, by an abuse of the following notations

\[
v \neq v_{-1}, v_0 \iff v \neq v_0, \quad v \neq v_n, v_{n+1} \iff v \neq v_n.
\]

It follows from Theorem 2.5 Definition 13, Definition 14 and (3.2) that the regular face maps and regular co-face maps also satisfy some simplicial-like identities partially:

**Theorem 3.1 (Main Result II: The simplicial-like identities for regular paths on discrete sets).** Let \( f \) and \( g \) be two real functions on \( V \). Then for any \( n \geq 0 \), we have

(i). \( \tilde{d}^f_i \circ \tilde{d}^g_j = -\tilde{d}^g_j \circ \tilde{d}^f_i \) for any \( i \leq j - 2 \);

(ii). \( \tilde{d}^f_i \circ \tilde{d}^g_j = \begin{cases} 
-\tilde{d}^g_{j-1} \circ \tilde{d}^f_i, & i \leq j - 2, \\
(f, g)^{V\setminus\{v_{j-1}, v_j\}} \id, & i = j, \\
-\tilde{d}^g_j \circ \tilde{d}^f_{i-1}, & i \geq j + 2;
\end{cases} \)

(iii). \( \tilde{d}^f_i \circ \tilde{d}^g_j = -\tilde{d}^g_{j+1} \circ \tilde{d}^f_i \) for \( i \leq j - 1 \).

**Proof.** Let \( v_0v_1\ldots v_n \) be a regular elementary \( n \)-path on \( V \). By the definition of the regular paths on \( V \), for any \( 0 \leq i \leq n \) we have

\[
e(\varepsilon_i, \varepsilon_{i+1}) = e(\varepsilon_i, \varepsilon_{i-1}) = 1.
\]

(3.3)

We prove (i), (ii), and (iii) separately.

(i). Suppose \( i < j \). By a straight-forward calculation, we have

\[
\tilde{d}^f_i \circ \tilde{d}^g_j (v_0v_1\ldots v_n) = \begin{cases} 
(-1)^{i+j} f(v_j)g(v_j)e(\varepsilon_{j-1}, \varepsilon_{j+1})e(\varepsilon_{i-1}, \varepsilon_{i+1})v_0\ldots \varepsilon_i\ldots \varepsilon_j\ldots v_n, & i \leq j - 2, \\
-f(\varepsilon_j)e(\varepsilon_{j-1}, \varepsilon_{j+1})e(\varepsilon_{i-2}, \varepsilon_{i-1})v_0\ldots \varepsilon_{i-1}\varepsilon_i\ldots v_n, & i = j - 1.
\end{cases}
\]

On the other hand,

\[
\tilde{d}^g_{j-1} \circ \tilde{d}^f_i (v_0v_1\ldots v_n) = \begin{cases} 
(-1)^{i+j-1} f(v_i)g(v_i)e(\varepsilon_{i-1}, \varepsilon_{i+1})e(\varepsilon_{i-1}, \varepsilon_{i+1})v_0\ldots \varepsilon_i\ldots \varepsilon_j\ldots v_n, & i \leq j - 2, \\
f(\varepsilon_{i-1})e(\varepsilon_{i-2}, \varepsilon_{i-1})e(\varepsilon_{i-1}, \varepsilon_{i-2})v_0\ldots \varepsilon_{i-1}\varepsilon_i\ldots v_n, & i = j - 1.
\end{cases}
\]

(3.4)
Therefore, we obtain (i) for \( i \leq j - 2 \).

(ii) By a straight-forward calculation, for any possible \( i \) and \( j \) we have

\[
\tilde{d}_i^j \circ \tilde{d}_j^i (v_0 v_1 \ldots v_n)
= \tilde{d}_i^j \left( \sum_{v \neq v_{j-1}, v_j} (-1)^j g(v) v_0 \ldots v_{j-1} vv_j \ldots v_n \right)
\]

\[
\begin{cases}
\sum_{v \neq v_{j-1}, v_j} (-1)^{j+1} g(v) f(v) (v_{j-1} + 1) v_0 \ldots v_{j-1} vv_j \ldots v_n, & i \leq j - 2, \\
- \sum_{v \neq v_{j-1}, v_j} g(v) f(v) (v_{j-1}) v_0 \ldots v_{j-1} vv_j \ldots v_n, & i = j - 1, \\
\sum_{v \neq v_{j-1}, v_j} g(v) f(v) v_0 \ldots v_n, & i = j, \\
- \sum_{v \neq v_{j-1}, v_j} g(v) f(v) (v_{j+1}) v_0 \ldots v_{j-1} vv_j \ldots v_n, & i = j + 1, \\
\sum_{v \neq v_{j-1}, v_j} (-1)^{j+1} g(v) f(v) (v_{j-1}) (v_{j-2}, v) v_0 \ldots v_{j-1} vv_j \ldots v_n, & i \geq j + 2.
\end{cases}
\]

Therefore, we obtain (ii) for the case \( i = j \). Moreover, for \( i < j \) we have

\[
\tilde{d}_i^j \circ \tilde{d}_j^i (v_0 v_1 \ldots v_n)
= \tilde{d}_i^j \left( (-1)^i \epsilon(v_{i-1}, v_{i+1}) f(v_i) v_0 \ldots v_n \right)
\]

\[
\begin{cases}
\sum_{v \neq v_{j-1}, v_j} (-1)^{j-1} \epsilon(v_{j-1}, v_{j+1}) f(v) (v_{j-1}) v_0 \ldots v_{j-1} vv_j \ldots v_n, & j \geq i + 2, \\
\sum_{v \neq v_{i-1}, v_i} \epsilon(v_{i-1}, v_{i+1}) f(v) v_0 \ldots v_n, & j = i + 1.
\end{cases}
\]

Therefore, we obtain (ii) for the case \( i \leq j - 2 \). Furthermore, for \( i > j \) we have

\[
\tilde{d}_i^j \circ \tilde{d}_j^i (v_0 v_1 \ldots v_n)
= (-1)^{i-1} \epsilon(v_{i-2}, v_i) f(v_{i-1}) f(v_i) v_0 \ldots v_n
\]

\[
\begin{cases}
\sum_{v \neq v_{j-1}, v_j} (-1)^{i-1} \epsilon(v_{i-2}, v_i) f(v_{i-1}) g(v) (v_{i-1}) v_0 \ldots v_{i-1} vv_i \ldots v_n, & j \leq i - 2, \\
\sum_{v \neq v_{i-1}, v_i} \epsilon(v_{i-1}, v_i) f(v_{i-1}) g(v) v_0 \ldots v_{i-2} vv_i \ldots v_n, & j = i - 1.
\end{cases}
\]

Therefore, we obtain (ii) for the case \( i \geq j + 2 \). Summarizing all the above, we obtain (ii).

(iii) Suppose \( i \leq j \). By a straight-forward calculation, we have

\[
\tilde{d}_i^j \circ \tilde{d}_j^i (v_0 v_1 \ldots v_n)
= \sum_{v \neq v_{j-1}, v_j} (-1)^{i+j} f(u) g(v) (v_{j-1}) v_0 \ldots v_{j-1} uu_j \ldots v_n, & i \leq j - 1, \\
\sum_{v \neq v_{j-1}, v_j} \epsilon(u) g(v) v_0 \ldots v_{j-1} uu_j \ldots v_n, & i = j.
\]

On the other hand,

\[
\tilde{d}_j^{i+1} \circ \tilde{d}_j^i (v_0 v_1 \ldots v_n)
= \sum_{u \neq v_{i-1}, v_i} \sum_{v \neq v_{j-1}, v_j} (-1)^{i+j+1} f(u) g(v) (v_{j-1}) v_0 \ldots v_{j-1} uu_j \ldots v_n, & i \leq j - 1, \\
- \sum_{u \neq v_{i-1}, v_i} \sum_{v \neq v_{j-1}, v_j} f(u) g(v) v_0 \ldots v_{i-1} uu_i \ldots v_n, & i = j.
\]

Therefore, we obtain (iii) for \( i \leq j - 1 \).

\[\square\]

**Remark 5:** The simplicial-like identities for the regular paths given in Theorem 3.1 are slightly different from the simplicial-like identities for the paths given in Theorem 2.5. The difference is that generally in Theorem 3.1 the first simplicial identity does not hold in the case \( i = j - 1 \), the second simplicial identity does not hold in the case \( i = j - 1 \), and the third simplicial identity does not hold in the case \( i = j \).
3.2 The Weighted Boundary Operators and The Weighted Co-Boundary Operators for Regular Paths

Let $f$ be a real function on $V$. Let $n \geq 0$. In this subsection, we investigate the $f$-weighted boundary maps and the $f$-weighted co-boundary maps for regular paths on $V$.

**Definition 15.** For any $v \in V$, we use the notation

$$\tilde{\partial}^f := \sum_{i=0}^{n} \tilde{\partial}^f_i.$$

Then we have a graded linear map

$$\tilde{\partial}^f : \mathcal{R}_n(V) \rightarrow \mathcal{R}_{n-1}(V), \quad n \geq 0.$$

We call $\tilde{\partial}^f$ the $f$-weighted boundary map for regular paths.

**Definition 16.** For any $v \in V$, we use the notation

$$\tilde{d}^f := \sum_{i=0}^{n+1} \tilde{d}^f_i.$$

Then we have a graded linear map

$$\tilde{d}^f : \mathcal{R}_n(V) \rightarrow \mathcal{R}_{n+1}(V), \quad n \geq 0.$$

We call $\tilde{d}^f$ the $f$-weighted co-boundary map for regular paths.

**Definition 17.** For any $v \in V$, as a particular case of Definition 15 and Definition 16, we take $f$ to be the characteristic function $\chi_v$. We define the reduced partial derivative with respect to $v$ as

$$\tilde{\partial} : \mathcal{R}_n(V) \rightarrow \mathcal{R}_{n-1}(V), \quad n \geq 0$$

and define the reduced partial differentiation with respect to $v$ as

$$\tilde{d} : \mathcal{R}_n(V) \rightarrow \mathcal{R}_{n+1}(V), \quad n \geq 0.$$

We give the explicit expressions of Definition 17. By Definition 17, we have graded linear maps

$$\tilde{\partial} : \mathcal{R}_n(V) \rightarrow \mathcal{R}_{n-1}(V), \quad n \geq 0$$

and

$$\tilde{d} : \mathcal{R}_n(V) \rightarrow \mathcal{R}_{n+1}(V), \quad n \geq 0.$$

For any regular elementary $n$-path $v_0v_1 \ldots v_n$ on $V$, we have

$$\frac{\partial}{\partial v}(v_0v_1 \ldots v_n) = \sum_{i=0}^{n} (-1)^i \delta(v, v_i) \epsilon(v_{i-1}, v_{i+1})v_0 \ldots \hat{v}_i \ldots v_n$$

and

$$\frac{d}{d v}(v_0v_1 \ldots v_n) = \sum_{i=0}^{n+1} (-1)^i \epsilon(v, v_{i-1}) \epsilon(v, v_i)v_0 \ldots \hat{v}_{i-1} \hat{v}_i \ldots v_n.$$

Here we abuse the notation by writing

$$\epsilon(v, v_{-1}) = 1, \quad \epsilon(v, v_{n+1}) = 1.$$
Definition 18. We define

(i). the anti-commutator \( \left( \frac{\partial}{\partial v}, \frac{\partial}{\partial u} \right) \) of \( \frac{\partial}{\partial v} \) and \( \frac{\partial}{\partial u} \) by

\[
\left( \frac{\partial}{\partial v}, \frac{\partial}{\partial u} \right) := \frac{\partial}{\partial v} \circ \frac{\partial}{\partial u} + \frac{\partial}{\partial u} \circ \frac{\partial}{\partial v}.
\]

(ii). the anti-commutator \( (\delta v, \delta u) \) of \( \delta v \) and \( \delta u \) by

\[
(\delta v, \delta u) := \delta v \circ \delta u + \delta u \circ \delta v.
\]

We have the next lemma.

Lemma 3.2. Let \( u, v \in V \). Let \( n \geq 0 \). Let \( v_0v_1 \ldots v_n \) be a regular elementary \( n \)-path on \( V \). Then

\[
\left( \frac{\partial}{\partial v} \frac{\partial}{\partial u} \right)(v_0v_1 \ldots v_n) = \sum_{i=0}^{n} \left( \delta(v, v_i)\delta(u, v_{i-1}) - \delta(u, v_i)\delta(v, v_{i-1}) \right)
\]

\[
\epsilon(v_{i-1}, v_{i+1})\epsilon(v_{i-2}, v_{i+2})v_0 \ldots \tilde{v}_{i-1} \tilde{v}_i \ldots v_n + \left( \delta(u, v_i)\delta(v, v_{i+1}) - \delta(v, v_i)\delta(u, v_{i+1}) \right)
\]

\[
\epsilon(v_{i-1}, v_{i+1})\epsilon(v_{i-2}, v_{i+2})v_0 \ldots \tilde{v}_{i-1} \tilde{v}_{i+1} \ldots v_n \tag{3.4}
\]

and

\[
(\delta v, \delta u)(v_0v_1 \ldots v_n) = \sum_{i=0}^{n+1} \left( \epsilon(u, v_{i-1}) - \epsilon(v, v_i) \right)\epsilon(u, v_i)\epsilon(v, v_{i-1})\epsilon(v, u)
\]

\[
v_0 \ldots v_{i-2}u \hat{v}_{i-1} \hat{v}_i \ldots v_n + \left( \epsilon(v, v_{i-1}) - \epsilon(u, v_i) \right)\epsilon(u, v_i)\epsilon(v, v_{i-1})\epsilon(v, u)
\]

\[
v_0 \ldots v_{i-2}u \hat{v}_{i-1} \hat{v}_{i+1} \ldots v_n \tag{3.5}
\]

Proof. Let \( u, v \in V, n \geq 0 \), and \( v_0v_1 \ldots v_n \) be a regular elementary \( n \)-path on \( V \). It follows from a straight-forward calculation that

\[
\frac{\partial}{\partial v} \circ \frac{\partial}{\partial u}(v_0v_1 \ldots v_n) = \sum_{j=0}^{n} (-1)^j \delta(u, v_i)\epsilon(v_{i-1}, v_{i+1}) \left( \sum_{j=0}^{i-2} (-1)^j \delta(v, v_j)\epsilon(v_{j-1}, v_{j+1}) \right)
\]

\[
v_0 \ldots \hat{v}_j \ldots \hat{v}_i \ldots v_n + \sum_{j=i+1}^{n-1} (-1)^j \delta(v, v_j+1)\epsilon(v_{j+1}, v_{j+2})v_0 \ldots \hat{v}_j \hat{v}_{j+1} \ldots v_n + (-1)^{-1} \delta(v, v_{i-1})\epsilon(v_{i-2}, v_{i+1})v_0 \ldots \hat{v}_{i-1} \hat{v}_i \ldots v_n
\]

\[
+ (-1)^j \delta(v, v_{i+1})\epsilon(v_{i-1}, v_{i+2})v_0 \ldots \hat{v}_i \hat{v}_{i+1} \ldots v_n \tag{3.6}
\]

and

\[
\frac{\partial}{\partial u} \circ \frac{\partial}{\partial v}(v_0v_1 \ldots v_n) = \sum_{j=0}^{n} (-1)^j \delta(v, v_j)\epsilon(v_{j-1}, v_{j+1}) \left( \sum_{i=0}^{j-2} (-1)^i \delta(u, v_i)\epsilon(v_{i-1}, v_{i+1}) \right)
\]

\[
v_0 \ldots \hat{v}_i \ldots \hat{v}_j \ldots v_n + \sum_{i=j+1}^{n-1} (-1)^i \delta(u, v_i+1)\epsilon(v_{i+1}, v_{i+2})v_0 \ldots \hat{v}_i \hat{v}_{i+1} \ldots v_n + (-1)^{-1} \delta(u, v_{j-1})\epsilon(v_{j-2}, v_{j+1})v_0 \ldots \hat{v}_{j-1} \hat{v}_j \ldots v_n
\]

\[
+ (-1)^j \delta(u, v_{j+1})\epsilon(v_{j-1}, v_{j+2})v_0 \ldots \hat{v}_j \hat{v}_{j+1} \ldots v_n \tag{3.7}
\]
Here we abuse the notations by writing

\[
\delta(v, v_{-1}) = \delta(v, v_{n+1}) = \delta(u, v_{-1}) = \delta(u, v_{n+1}) = 0,
\]

\[
\epsilon(v_{-1}, v_2) = \epsilon(v_{n-2}, v_{n+1}) = 1.
\]

Summing up (3.6) and (3.7), it follows that

\[
\left( \frac{\partial}{\partial v}, \frac{\partial}{\partial u} \right) (v_0 v_1 \ldots v_n)
= \sum_{i=0}^{n} \left( -\delta(u, v_i) \delta(v, v_{i-1}) \epsilon(v_{i-1}, v_{i+1}) \epsilon(v_{i-2}, v_{i+1}) v_0 \ldots \hat{v}_i v_{i+1} \ldots v_n 
+ \delta(u, v_i) \delta(v, v_{i+1}) \epsilon(v_{i-1}, v_{i+1}) \epsilon(v_{i-2}, v_{i+1}) v_0 \ldots \hat{v}_i v_{i+1} \ldots v_n \right)
- \sum_{j=0}^{n} \left( -\delta(v, v_j) \delta(u, v_{j-1}) \epsilon(v_{j-1}, v_{j+1}) \epsilon(v_{j-2}, v_{j+1}) v_0 \ldots \hat{v}_{j-1} v_{j+1} \ldots v_n 
+ \delta(v, v_j) \delta(u, v_{j+1}) \epsilon(v_{j-1}, v_{j+1}) \epsilon(v_{j-2}, v_{j+1}) v_0 \ldots \hat{v}_{j-1} v_{j+1} \ldots v_n \right)
= \sum_{i=0}^{n} \left( \delta(v, v_i) \delta(u, v_{i-1}) - \delta(u, v_i) \delta(v, v_{i-1}) \right) \epsilon(v_{i-1}, v_{i+1}) \epsilon(v_{i-2}, v_{i+1}) v_0 \ldots \hat{v}_{i-1} v_i \ldots v_n
+ \left( \delta(u, v_i) \delta(v, v_{i+1}) - \delta(v, v_i) \delta(u, v_{i+1}) \right) \epsilon(v_{i-1}, v_{i+1}) \epsilon(v_{i-2}, v_{i+2}) v_0 \ldots \hat{v}_i v_{i+1} \ldots v_n.
\]

We obtain (3.8).

On the other hand, it also follows from a straight-forward calculation that

\[
\tilde{d}v \circ \tilde{d}u(v_0 v_1 \ldots v_n)
= \sum_{j=0}^{n+1} (-1)^j \epsilon(u, v_{j-1}) \epsilon(u, v_j) \left( \sum_{i=0}^{j-1} (-1)^i \epsilon(v, v_{j-1}) \epsilon(v, v_j) v_0 \ldots v_{j-1} v v_{j-1} \ldots v_n 
+ \sum_{i=j+2}^{n+2} (-1)^i \epsilon(v, v_{j-2}) \epsilon(v, v_{j-1}) v_0 \ldots v_{j-1} v v_{j-2} \ldots v_{j-1} v v_{j-1} \ldots v_n 
+ (-1)^j \epsilon(v, v_{j-1}) \epsilon(v, v_j) v_0 \ldots v_{j-1} v v v_{j-1} \ldots v_n 
+ (-1)^{j+1} \epsilon(v, u) \epsilon(v, v_j) v_0 \ldots v_{j-1} u v v v_{j-1} \ldots v_n \right)
\tag{3.8}
\]

and

\[
\tilde{d}u \circ \tilde{d}v(v_0 v_1 \ldots v_n)
= \sum_{j=0}^{n+1} (-1)^j \epsilon(v, v_{j-1}) \epsilon(v, v_j) \left( \sum_{i=0}^{j-1} (-1)^i \epsilon(u, v_{j-1}) \epsilon(u, v_j) v_0 \ldots v_{j-1} u v_{j-1} \ldots v_n 
+ \sum_{i=j+2}^{n+2} (-1)^i \epsilon(u, v_{j-2}) \epsilon(u, v_{j-1}) v_0 \ldots v_{j-1} u v_{j-2} \ldots v_{j-1} u v_{j-1} \ldots v_n 
+ (-1)^j \epsilon(u, v_{j-1}) \epsilon(u, v_j) v_0 \ldots v_{j-1} u v v_{j-1} \ldots v_n 
+ (-1)^{j+1} \epsilon(v, u) \epsilon(u, v_j) v_0 \ldots v_{j-1} v v v_{j-1} \ldots v_n \right),
\tag{3.9}
\]
Summing up (3.8) and (3.9), it follows that

\[
(\tilde{dv}, \tilde{du})(v_0v_1\ldots v_n) = \sum_{i=0}^{n+1} \left( \epsilon(u, v_{i-1})\epsilon(u, v_i)\epsilon(v, v_{i-1})\epsilon(v, u)v_0\ldots v_{i-1}uvv_i\ldots v_n \right. \\
- \epsilon(u, v_{i-1})\epsilon(u, v_i)\epsilon(v, u)v_0\ldots v_{i-1}uvv_i\ldots v_n \\
\left. + \sum_{j=0}^{n+1} \epsilon(v, v_{j-1})\epsilon(v, v_j)\epsilon(u, v_{j-1})\epsilon(v, u)v_0\ldots v_{j-1}uvv_j\ldots v_n \right) \\
- \epsilon(v, v_{j-1})\epsilon(v, v_j)\epsilon(u, v_j)v_0\ldots v_{j-1}uvv_j\ldots v_n \\
+ \left( \epsilon(v, v_{i-1}) - \epsilon(u, v_i) \right) \epsilon(v, v_i)\epsilon(v, u)v_0\ldots v_{i-1}uvv_i\ldots v_n.
\]

We obtain (3.10).

Let \( f \) and \( g \) be two real functions on \( V \). We note that

\[
\tilde{D} = \sum_{v \in V} f(v) \tilde{\partial}_v, \quad \tilde{d} = \sum_{v \in V} f(v)\tilde{dv}
\]

and the same identities hold for \( g \) as well. Moreover, if we write the anti-commutators as

\[
(\tilde{D}, \tilde{\partial}) = \tilde{D} \circ \tilde{\partial} + \tilde{\partial} \circ \tilde{D}, \quad (\tilde{d}, \tilde{d}) = \tilde{d} \circ \tilde{d} + \tilde{\partial} \circ \tilde{\partial},
\]

then we have

\[
(\tilde{D}, \tilde{\partial}) = \sum_{v, u \in V} f(v)g(u)\left( \frac{\tilde{\partial}}{\partial v} \frac{\tilde{\partial}}{\partial u} \right) = \sum_{v, u \in V} f(v)g(u)(\tilde{dv}, \tilde{du}). \tag{3.10}
\]

and

\[
(\tilde{d}, \tilde{d}) = \sum_{v, u \in V} f(v)g(u)(\tilde{dv}, \tilde{du}). \tag{3.11}
\]

The next proposition follows from Lemma 3.2.

**Proposition 3.3.** Let \( f \) and \( g \) be real functions on \( V \). Let \( n \geq 0 \) and let \( v_0v_1\ldots v_n \) be a regular elementary \( n \)-path on \( V \). Then

\[
(\tilde{D}, \tilde{\partial})(v_0v_1\ldots v_n) = \sum_{v, u \in V} f(v)g(u) \left[ \sum_{i=0}^{n} \left( \delta(v, v_i)\delta(u, v_{i-1}) - \delta(u, v_i)\delta(v, v_{i-1}) \right) \\
\epsilon(v_{i-1}, v_{i+1})\epsilon(v_{i-2}, v_{i+1})v_0\ldots \tilde{v}_{i-1}v_{i+1}\ldots v_n \\
+ \left( \delta(u, v_i)\delta(v, v_{i+1}) - \delta(v, v_i)\delta(u, v_{i+1}) \right) \\
\epsilon(v_{i-1}, v_{i+1})\epsilon(v_{i-2}, v_{i+2})v_0\ldots \tilde{v}_{i-1}v_{i+1}\ldots v_n \right] \tag{3.12}
\]

and

\[
(\tilde{d}, \tilde{d})(v_0v_1\ldots v_n) = \sum_{v, u \in V} f(v)g(u) \left[ \sum_{i=0}^{n+1} \left( \epsilon(v, v_{i-1}) - \epsilon(v, v_i) \right)\epsilon(u, v_i)\epsilon(v, v_{i-1})\epsilon(v, u) \\
v_0\ldots v_{i-1}uvv_i\ldots v_n \\
+ \left( \epsilon(v, v_{i-1}) - \epsilon(u, v_i) \right)\epsilon(u, v_{i-1})\epsilon(v, v_{i-1})\epsilon(v, u) \\
v_0\ldots v_{i-1}uvv_i\ldots v_n \right]. \tag{3.13}
\]
Proof. The expression (3.12) follows from (3.4) and (3.10). And the expression (3.13) follows from (3.5) and (3.11).

Remark 6: By Proposition 3.3 the operators $\tilde{\partial}^I$ and $\tilde{\partial}^p$ as well as the operators $\tilde{\partial}^J$ and $\tilde{\partial}^q$ on the regular paths are not anti-commutative, in general. The anti-commutative property holds in Proposition 2.9 and does not hold in Proposition 3.3.

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