BIRATIONAL COBORDISM INVARIANCE OF UNIRULED SYMPLECTIC MANIFOLDS

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1. Introduction

Birational geometry has always been an important topic in algebraic geometry. In the 80’s, an industry called Mori’s birational geometry program was created for the birational classification of algebraic manifolds of dimension three. In the early 90’s, the last author observed that some aspects of this extremely rich program of Mori can be extended to symplectic geometry via the newly created Gromov-Witten theory [R]. Further, he speculated that in fact there should be a symplectic birational geometric program. Such

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a program is important in two ways. The flexibility of symplectic geometry should give a better understanding of birational algebraic geometry in the same way that the Gromov-Witten theory gave a much better understanding of the role of rational curves in Mori theory. Secondly, such a symplectic birational geometry should be the first step towards a classification of symplectic manifolds [R2]. During the last ten years, there was virtually no progress in this direction. There are many reasons for lack of this progress, one of which was the difficulty to carry out computations in Gromov-Witten theory. Fortunately, a great deal of progress has been made to remedy this aspect of the problems. Many techniques have been developed to calculate the Gromov-Witten invariants. It seems to be a good time now to restart a real push for the symplectic birational geometry. This is the first of a series of papers by the last two authors to treat this new subject of symplectic birational geometry. Our treatment is by no means complete. On the contrary, there are many more problems being discovered than answered.

For a long time, it was not really clear what is an appropriate notion of birational equivalence in symplectic geometry. Simple birational operations such as blow-up/blow-down were known in symplectic geometry for a long time [GS, MS]. But there is no straightforward generalization of the notion of a general birational map in the flexible symplectic category. The situation changed a great deal when the weak factorization theorem was established recently (see the lecture notes [M] and the reference therein) that any birational map between projective manifolds can be decomposed as a sequence of blow-ups and blow-downs. This fundamental result resonates perfectly with the picture of the wall crossing of symplectic reductions analyzed by Guillemin-Sternberg in the 80’s. Therefore, we propose to use their notion of cobordism in [GS] as the symplectic analogue of the birational equivalence (see Definition 2.1). To avoid confusion with other notions of cobordism in the symplectic category, we would call it symplectic birational cobordism.

A fundamental concept of birational geometry is uniruledness. Algebraically, it means that the manifold is covered by rational curves. Notice that, by [Ltj], it is not meaningful to define this notion by simply mimicking the definition in algebraic geometry and requiring that there is a symplectic sphere in a fixed class through each point. Otherwise, every simply connected manifold would be uniruled. On the other hand, by a theorem of Kollar-Ruan [K], [R1], a uniruled projective manifold has a nonzero genus zero GW-invariant with a point insertion. Therefore, we call a symplectic manifold \((M, \omega)\) (symplectically) uniruled if there is a nonzero genus zero GW invariant involving a point constraint. Then, it is a fundamental problem in symplectic birational geometry to prove that symplectic uniruledness is a birational invariant. It is obviously deformation invariant. The main purpose of this paper is to prove

**Theorem 1.1.** Symplectic uniruledness is invariant under symplectic blow-up and blow-down.
This theorem follows from a general Relative/Absolute correspondence for a symplectic manifold together with a symplectic submanifold. Such a correspondence was first established in [MP] when the submanifold is of codimension 2, i.e., a symplectic divisor.

Clearly our theorem can also be viewed as a kind of blow-up/down formula of GW invariants. It is rare to be able to obtain a general blow-up formula. For the last ten years, only a few limited cases were known [H1, H2, H3, HZ, Ga]. Although our technique is more powerful, for instance, it still does not work for rational connectivity. Here a symplectic manifold is said to be rationally connected if there is a nonzero genus zero GW invariant involving two point insertions. We do speculate that symplectic rational connectivity is invariant under blow-up and blow-down.

A natural question is whether Hamiltonian $S^1$ manifolds are uniruled. Based on our blowup technique as well as Seidel representation of $\pi_1$ of the Hamiltonian group in the small quantum homology, McDuff [Mc3] gave an affirmative answer to this question.

For the reader's convenience, we review in sections two and three some background materials scattered in the literature. In section two, we will review the definition of birational cobordism in the symplectic geometry and describe blow-up, blow-down and $\mathbb{Z}$-linear deformation as birational cobordisms. In section three, we will briefly review the relative GW-invariant and the degeneration formula which is an essential tool for us. In section four, we will give the definition of symplectic uniruledness and some elementary properties of uniruled symplectic manifolds. The core of the paper consists of sections five and six. In section five, we will describe and partially verify the Relative versus Absolute Correspondence for GW-invariants, which generalizes early works of Maulik-Okounkov-Pandharipande [OP], [MP]. Then, in section six, we apply our Relative versus Absolute correspondence to prove the main theorem. In section seven, we will complete the proof of the Relative versus Absolute correspondence by computing certain relative GW invariants of $(\mathbb{P}^n, \mathbb{P}^{n-1})$.

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2. Birational cobordism

The basic reference for this section is [GS]. We start with the definition which is essentially contained in [GS].

**Definition 2.1.** Two symplectic manifolds $(X, \omega)$ and $(X', \omega')$ are birational cobordant if there are a finite number of symplectic manifolds $(X_i, \omega_i), 0 \leq i \leq k$, with $(X_0, \omega_0) = (X, \omega)$ and $(X_k, \omega_k) = (X', \omega')$, and for each $i$,
\[(X_i, \omega_i)\] and \[(X_{i+1}, \omega_{i+1})\] are symplectic reductions of a semi-free Hamiltonian \(S^1\) symplectic manifold \(W_i\) (of 2 more dimension).

Here an \(S^1\) action is called semi-free if it is free away from the fixed point set.

There is a related notion in dimension 4 in [OO]. However we remark that the cobordism relation studied in this paper is quite different from some other notions of symplectic cobordisms, see [EGH], [EH], [Gi], [GGK].

According to [GS], we have the following basic factorization result.

**Theorem 2.2.** A birational cobordism can be decomposed as a sequence of elementary ones, which are modeled on blow-up, blow-down and \(\mathbb{Z}\)-linear deformation of symplectic structure.

Comparing with the weak factorization theorem, we then have

**Theorem 2.3.** Two birational projective manifolds with any polarizations are birational as symplectic manifolds.

We will now review each of these elementary birational cobordisms in Theorem 2.2.

### 2.1. Coupling form and linear deformations.

#### 2.1.1. Universal construction.

Let us first review the Sternberg-Weinstein universal construction. Let \(\pi : P \to X\) be a principal bundle with structure group \(G\) over a symplectic manifold \(X\) with symplectic form \(\omega\). If \(g\) denote the Lie algebra of \(G\), then a connection on \(P\) gives rise to a \(g\) valued 1-form on \(P\) corresponding to the projection onto the vertical. Let \(\text{Ver}\) be the vertical bundle of the fibration. A \(G\)-invariant complementary subbundle \(F\) is nothing but a connection of \(P\). It also embeds \(P \times g^*\) into \(T^*P\).

The desired 1-form at \((p, \tau)\) is given by \(\tau \cdot A\), where we use \(\cdot\) to denote the pairing between \(g\) and \(g^*\). Denote this 1-form by \(\omega_A\) as well. Notice that it is the restriction of the canonical 1-form on \(T^*P\). Therefore \(d(\tau \cdot A)\) is non-degenerate on the fibers of \(P \times g^*\).

Then \(\omega_A = \pi^*\omega + d(\tau \cdot A)\) is sometimes called the coupling form of \(A\). The \(G\)-action on \(P \times g^*\) given by

\[g(p, \zeta) = (g^{-1}p, \text{Ad}(g)^*\zeta),\]

preserves \(\tau \cdot A\) and hence \(\omega_A\).

Notice that at any point in \(P \times \{0\}\), \(\tau = 0\) and \(\omega_A\) is equal to \(\pi^*\omega + d\tau \cdot A\), hence it is symplectic there.

**Lemma 2.4.** \(\omega_A\) is a symplectic form on \(P \times W_A\) for some \(G\)-invariant neighborhood \(W_A\) of \(0 \in g^*\). The projection onto \(W_A\) is a moment map on \(P \times W_A\).

This lemma is well-known. Notice that the vertical bundle \(\text{Ver}\) is the bundle of the null vectors of \(\pi^*\omega\). So as explained in [GS], the construction is just a special case of the coisotropic embedding theorem. It follows
from the uniqueness part of the coisotropic embedding theorem that the symplectic structure $\omega_A$ on $P \times W_A$ near $P \times 0$ is independent of $A$ up to symplectomorphisms.

More generally, if $(F, \omega_F)$ is a symplectic manifold with a Hamiltonian $G$ action, we can form the associated bundle $P_F = P \times G F$. Let $\mu_F : F \to g^*$ be a moment map. Furthermore, assume that

(1) $\mu_F(F) \subset W_A$.

Then there is a symplectic structure $\omega_{F,A}$ on $P_F$ which restricts to $\omega_F$ on each fiber.

To construct $\omega_{F,A}$ consider the 2-form $\omega_A + \omega_F$ on $P \times g^* \times F$. It is invariant under the diagonal $G$–action and is symplectic on $P \times W_A \times F$.

The $G$–action is in fact Hamiltonian with $\Gamma_{W_A} = \pi g^* + \mu_F : P \times W_A \times F \to g^*$ as a moment map.

Furthermore, by (1), for any $f \in F$, we have $\mu_F(f) \in W_A$; thus

$\Gamma^{-1}_{W_A}(0) = \{(p, -\mu_F(f), f)\}$.

In particular, $\Gamma^{-1}_{W_A}(0)$ is $G$–equivariantly diffeomorphic to $P \times F$, and the symplectic reduction at 0 yields the desired symplectic form $\omega_{F,A}$ on $P_F$.

In fact when $(F, \omega_F) = (T G^*, \omega_{can})$, then $P \times g^* = P \times G TG^*$.

2.1.2. $\mathbb{Z}$–linear deformation.

Definition 2.5. A $\mathbb{Z}$–linear deformation is a path of symplectic form $\omega + t\kappa$, $t \in I$, where $\kappa$ is a closed 2–form representing an integral class and $I$ is an interval. Two symplectic forms are $\mathbb{Z}$–linear deformation equivalent if they are joined by a finite number of $\mathbb{Z}$–linear deformations.

Let $P$ be the principal $S^1$–bundle whose Chern class is $[\kappa]$. Let $A$ be a connection 1–form such that $dA = \pi^*\kappa$. Write $\omega_A = \pi^*\omega + d(tA)$, where $t$ is the linear coordinate on $g^* = \mathbb{R}^*$. It follows from Lemma $\omega_A$ is symplectic at $(x, \theta, t) \in P \times \mathbb{R}$ if and only if $\omega + tk$ is symplectic at $x \in X$. Thus the Duistermaat–Heckman Theorem can be interpreted as saying that a $\mathbb{Z}$–linear deformation is a birational cobordism.

It seems to be more natural to consider a general deformation of symplectic structures. However the following lemma shows that the two notions are essentially the same.

Lemma 2.6. Let $\omega_t$, $t \in [0, 1]$, be a path of symplectic forms with $[\omega_0] - [\omega_1]$ being rational. Then $\omega_0$ and $\omega_1$ are $\mathbb{Z}$–linear deformation equivalent.

Proof. Observe that for any convex open neighborhood of the space of symplectic forms the convex linear combination of any two symplectic forms with relative rational period is a $\mathbb{Z}$–linear deformation. Observe also that any $\omega$ has a convex neighborhood.
For each $x \in [0, 1]$ let $I_x$ be a neighborhood such that the family $\omega_t, t \in I_x$, is in a convex neighborhood $V_{\omega_x}$. Let $I_{x_1}, \ldots, I_{x_n}$ with $0 = x_0 < \cdots < x_n = 1$ be a finite subcover of $[0, 1]$. Since $V_{\omega_{x_0}}$ and $V_{\omega_{x_1}}$ are both convex and rational symplectic forms are dense in the nonempty set $V_{\omega_{x_0}} \cap V_{\omega_{x_1}}$, $\omega_0$ is $\mathbb{Z}$–linear deformation equivalent to any symplectic form $\omega \in V_{\omega_{x_1}}$ with relative rational period. By repeating this argument we find that $\omega_0$ is $\mathbb{Z}$–linear deformation equivalent to $\omega_1$.

\[ \square \]

**Remark 2.7.** For any symplectic form $\omega$ and $J$ tamed by $\omega$, there is a nearby $\omega'$ with rational period and still tamed by $J$. Further, $\omega'$ also tames any $J'$ near $J$. Thus given any path of symplectic forms $\omega_t, t \in [0, 1]$ and $J_t$ be a path of almost complex structures such that $J_t$ is tamed by $\omega_t$ for each $t$, we can find finitely many symplectic forms with rational period such that each $J_t$ is tamed by one of them.

### 2.2. Blow-up and blow-down.

Suppose that $X$ is a closed symplectic manifold of dimension $2n$ and $Y \subset X$ is a submanifold of $X$ of codimension $2k$.

As we are going to apply the degeneration formula, it is convenient to phrase it in terms of symplectic cut [Le], which we now review.

#### 2.2.1. Symplectic cut and normal connected sum.

Suppose that $X_0 \subset X$ is an open codimension zero submanifold with a Hamiltonian $S^1$–action. Let $H : X_0 \to \mathbb{R}$ be a Hamiltonian function with 0 as a regular value. If $H^{-1}(0)$ is a separating hypersurface of $X_0$, then we obtain two connected manifolds $X^{\pm}_0$ with boundary $\partial X^{\pm}_0 = H^{-1}(0)$, where the + side corresponds to $H < 0$. Suppose further that $S^1$ acts freely on $H^{-1}(0)$. Then the symplectic reduction $Z = H^{-1}(0)/S^1$ is canonically a symplectic manifold of dimension 2 less. Collapsing the $S^1$–action on $\partial X^{\pm} = H^{-1}(0)$, we obtain closed smooth manifolds $\overline{X}^{\pm}$ containing respectively real codimension 2 submanifolds $Z^{\pm} = Z$ with opposite normal bundles. Furthermore $\overline{X}^{\pm}$ admits a symplectic structure $\omega^{\pm}$ which agrees with the restriction of $\omega$ away from $Z$, and whose restriction to $Z^{\pm}$ agrees with the canonical symplectic structure $\omega_Z$ on $Z$ from symplectic reduction. The pair of symplectic manifolds $(\overline{X}^{\pm}, \omega^{\pm})$ is called the symplectic cut of $X$ along $Y$.

This is neatly shown by considering $X_0 \times \mathbb{C}$ equipped with appropriate product symplectic structures and the product $S^1$–action on $X_0 \times \mathbb{C}$ where $S^1$ acts on $\mathbb{C}$ by complex multiplication. The extended action is Hamiltonian if we use the standard symplectic structure $\sqrt{-1}dw \wedge d\bar{w}$ or its negative on the $\mathbb{C}$ factor.

The normal connected sum operation ([G], [MW]), or the fiber sum operation is the inverse operation of the symplectic cut. Given two symplectic manifolds containing symplectomorphic codimension 2 symplectic submanifolds with opposite normal bundles, the normal connected sum operation...
produces a new symplectic manifold by identifying the tubular neighborhoods.

Notice that we can apply the normal connected sum operation to the pairs
\((X^+, \omega^+|Z^+), (X^-, \omega^-|Z^-)\)
to recover \((X, \omega)\).

2.2.2. Blow-up and blow-down. Now we apply the symplectic cut to construct the blow-up along \(Y\).

The normal bundle \(N_Y\) is a symplectic vector bundle, i.e. a bundle with fiber \((\mathbb{R}^{2k}, \omega_{\text{std}})\). Picking a compatible almost complex structure on \(N_Y\), we then have an Hermitian bundle. Let \(P\) be the principal \(U(k)\) bundle.

Now pick a unitary connection \(A\) for \(P\), and let \(W_A\subset u(k)^*\) be as in Lemma 2.4. Let \(D_{\epsilon_0} \subset C^k = \mathbb{R}^{2k}\) be the closed \(\epsilon_0\)-ball such that its image under the moment map lies inside \(W_A\).

Apply the universal construction to \(P\) and \(D_{\epsilon_0}\), we get a symplectic form \(\omega_{\epsilon_0, A}\) on the disc bundle \(N_Y(\epsilon_0)\) which restricts to \(\omega_{\text{std}}\) on each fiber, and restricts to \(\omega|Y\) on the zero section.

By the symplectic neighborhood theorem, and by possibly taking a smaller \(\epsilon_0\), a tubular neighborhood \(N_{\epsilon_0}(Y)\) of \(Y\) in \(X\) is symplectomorphic to the disc bundle \(N_Y(\epsilon_0)\) with the symplectic form \(\omega_{\epsilon_0, A}\). Let \(\phi : N_{\epsilon_0}(Y) \rightarrow N_Y(\epsilon_0)\) be such a symplectomorphism.

Consider the Hamiltonian \(S^1\) action on \(X_0 = N_{\epsilon_0}(Y)\) by complex multiplication. Fix \(\epsilon\) with \(0 < \epsilon < \epsilon_0\) and consider the moment map
\[H(u) = |\phi(u)|^2 - \epsilon, \quad u \in N_Y(\epsilon_0),\]
where \(|\phi(u)|\) is the norm of \(\phi(u)\) considered as a vector in a fiber of the Hermitian bundle \(N_Y\). Here \(X_0 \times C\) is just \(N_{\epsilon_0}(Y) \oplus C\).

Write the hypersurface \(P = H^{-1}(0)\) in \(X\) corresponding to the sphere bundle of \(N_Y\) with radius \(\epsilon\).

We cut \(X\) along \(P\) to obtain two closed symplectic manifolds \(\overline{X}^+\) and \(\overline{X}^-\). Notice that \(Y\) is contained in the + side.

\(\overline{X}^-\) is called the blow-up of \(X\) along \(Y\). Notice that the construction depends on \(\epsilon\), the connection \(A\) and the symplectomorphism \(\phi : N_{\epsilon_0}(Y) \rightarrow N_Y(\epsilon_0)\). However, as remarked in p. 250 in [MS], given two different choices \(A, \phi\) and \(A', \phi'\), for sufficiently small \(\epsilon\), the resulting symplectic forms are isotopic. We will often denote the blow-up by \(\tilde{X}\) ignoring various choices.

Denote the codimension 2 symplectic submanifold \(Z\) by \(E\). We will call \(E\) the exceptional divisor.

Blowing down is the inverse operation of blowing up. More precisely, if \(\tilde{X}\) is the blow up of \(X\) along \(Y\) with the exceptional divisor \(E\). Then, as remarked, the normal connected sum of \(\tilde{X}\) and \(\overline{X}^+\) along \(E\) gives back \(X\). This process from \(\tilde{X}\) to \(X\) is called the blow down along \(E\).

Now we set up to describe the topology of the blow-up.
As smooth manifolds, $E$ is diffeomorphic to the projectivization of $N_Y$, and $\overline{\mathcal{X}}^+$ is the projectivization of $N_Y \oplus \mathbb{C}$.

Observe that $\tilde{X} = (X - N_\epsilon(Y)) \cup_{\phi} N_Y(\epsilon_0)$. We can define a map
\[(2) \quad p : \tilde{X} \to X \]
which is identity away from $N_\epsilon(Y)$. Such a map can be constructed by identifying $N_\epsilon(Y) - N_\epsilon(0)$ with the deleted neighborhood $N_\epsilon(Y) - Y$ using a diffeomorphism from $(\epsilon, \epsilon_0)$ to $(0, \epsilon_0)$. Such a $p$ is not unique, but the induced maps $p_*$ and $p^*$ on homology and cohomology are the same for different choices.

In particular, if $Y$ is of codimension 2, then $E = Y$. And $\tilde{X}$ is diffeomorphic to $X$, although the symplectic structures are not quite the same.

It is important to observe that the pair $(\tilde{X}, E)$ is the common piece of the symplectic cuts of $X$ and $\tilde{X}$. More explicitly, $X$ degenerates into $(\tilde{X}, E)$ and $(\mathbb{P}(N_Y \oplus \mathbb{C}), E)$, while $\tilde{X}$ degenerates into $(\tilde{X}, E)$ and $(\mathbb{P}(N_{E|\tilde{X}} \oplus \mathbb{C}), E)$.

Finally we mention that it is also shown in [GS] that blowing up/down can be explicitly realized as a birational cobordism.

3. Relative GW invariants and the degeneration formula

Li and Ruan [LR] first introduced the moduli space of relative stable maps and constructed its virtual fundamental class. Integrating against the virtual fundamental class, they first defined the relative Gromov-Witten invariants (see [IP] for a different version and [Li] for the algebraic treatment). They are the main tool of the paper. We want to review briefly the construction.

3.1. GW-invariants. Suppose that $(X, \omega)$ is a compact symplectic manifold and $J$ is a tamed almost complex structure.

**Definition 3.1.** A stable $J$–holomorphic map is an equivalence class of pairs $(\Sigma, f)$. Here $\Sigma$ is a connected nodal marked Riemann surface with arithmetic genus $g$, $k$ smooth marked points $x_1, \ldots, x_k$, and $f : \Sigma \to X$ is a continuous map whose restriction to each component of $\Sigma$ (called a component of $f$ in short) is $J$–holomorphic. Furthermore, it satisfies the stability condition: if $f|_{S^2}$ is constant (called a ghost bubble) for some $S^2$–component, then the $S^2$–component has at least three special points (marked points or nodal points). $(\Sigma, f)$, $(\Sigma', f')$ are equivalent, or $(\Sigma, f) \sim (\Sigma', f')$, if there is a biholomorphic map $h : \Sigma' \to \Sigma$ such that $f' = f \circ h$.

An essential feature of Definition 3.1 is that, for a stable $J$–holomorphic map $(\Sigma, f)$, the automorphism group
\[\text{Aut}(\Sigma, f) = \{h | h \circ (\Sigma, f) = (\Sigma, f)\}\]
is finite. We define the moduli space $\overline{\mathcal{M}}^X_A(g, k, J)$ to be the set of equivalence classes of stable $J$–holomorphic maps such that $[f] = f_*[\Sigma] = A \in \mathbb{Z}$. 
The virtual dimension of $\overline{\mathcal{M}}_A^X(g, k, J)$ is computed by index theory,

$$\text{virdim}_{\mathbb{R}}\overline{\mathcal{M}}_A^X(g, k, J) = 2c_1(A) + 2(n - 3)(1 - g) + 2k,$$

where $n$ is the complex dimension of $X$.

Unfortunately, $\overline{\mathcal{M}}_A^X(g, k, J)$ is highly singular and may have larger dimension than the virtual dimension. There are several ways to extract invariants ([FO], [LT], [ST], [R1]), the one we use is the virtual neighborhood method in [R1].

First, we drop the $J$-holomorphic condition from the previous definition and require only each component of $f$ be smooth. We call the resulting object a stable map or a $C^\infty$-stable map. Denote the corresponding space of equivalence classes by $\overline{B}_X^A(g, k, J)$.

$\overline{B}_X^A(g, k, J)$ is clearly an infinite dimensional space. It has a natural stratification given by the topological type of $\Sigma$ together with the fundamental classes of the components of $f$. The stability condition ensures that $\overline{B}_X^A(g, k, J)$ has only finitely many strata such that each stratum is a Frechet orbifold. Further one can use the pregluing construction to define a topology on $\overline{B}_X^A(g, k, J)$ which is Hausdorff and makes $\overline{\mathcal{M}}_A^X(g, k, J)$ a compact subspace (see [R1]).

We can define another infinite dimensional space $\Omega^{0,1}$ together with a map

$$\pi : \Omega^{0,1} \to \overline{B}_X^A(g, k, J)$$

such that the fiber is $\pi^{-1}(\Sigma, f) = \Omega^{0,1}(f^*TX)$. The Cauchy-Riemann operator is now interpreted as a section of $\pi : \Omega^{0,1} \to \overline{B}_X^A(g, k, J)$,

$$\overline{\partial}_J : \overline{B}_X^A(g, k, J) \to \Omega^{0,1}$$

with $\overline{\partial}_J^{-1}(0)$ nothing but $\overline{\mathcal{M}}_A^X(g, k, J)$.

At each $(\Sigma, f) \in \overline{\mathcal{M}}_A^X(g, k, J)$, there is a canonical decomposition of the tangent space of $\Omega^{0,1}$ into the horizontal piece and the vertical piece. Furthermore, by choosing a compatible Riemannian metric on $X$ we can linearize $\overline{\partial}_J$ with respect to deformations of stable maps and project the to the vertical piece to obtain an elliptic complex over $\overline{B}_A^X(g, k, J)$,

$$L_{\Sigma, f} : \Omega^0(f^*TX) \to \Omega^{0,1}(f^*TX).$$

(3)

Since $\overline{\mathcal{M}}_A^X(g, k, J)$ is compact, we can construct a global orbifold bundle $\mathcal{E}$ over $\overline{B}_A^X(g, k, J)$ together with a bundle map $\eta : \mathcal{E} \to \Omega^{0,1}$ supported in a neighborhood $\mathcal{U}$ of $\overline{\mathcal{M}}_A^X(g, k, J)$.

Consider the finite dimensional vector bundle over $\mathcal{U}$, $p : \mathcal{E}|_{\mathcal{U}} \to \mathcal{U}$. The stabilizing equation $\overline{\partial}_J + \eta$ can be interpreted as a section of the bundle $p^*\Omega^{0,1} \to \mathcal{E}|_{\mathcal{U}}$. By construction this section

$$\overline{\partial}_J + \eta : \mathcal{E}|_{\mathcal{U}} \to p^*\Omega^{0,1}$$

is transverse to the zero section of $p^*\Omega^{0,1} \to \mathcal{E}|_{\mathcal{U}}$. 
The set $U^X_{S_e} = (\partial J + \eta)^{-1}(0)$ is called a virtual neighborhood in [R1]. The heart of [R1] is to show that $U^X_{S_e}$ has the structure of a $C^1$–manifold.

Notice that $U^X_{S_e} \subset E |_U$. Over $U^X_{S_e}$ there is the tautological bundle $E^X = p^*(E |_U)$.

It comes with the tautological inclusion map $S^X: U^X_{S_e} \hookrightarrow E^X$, $((\Sigma', f'), e) \mapsto e$, which can be viewed as a section of $E^X$. It is easy to check that $S^{-1}_X(0) = \overline{M}_A^X(g, k, J)$.

Furthermore, one can show that $S^X$ is a proper section.

There are evaluation maps $ev_i: \overline{B}_A(g, k, J) \rightarrow X$, $(\Sigma, f) \mapsto f(x_i)$, for $1 \leq i \leq k$. $ev_i$ induces a natural map from $U_{S_e} \rightarrow X^k$, which can be shown to be smooth.

Let $\Theta$ be the Thom form of the finite dimensional bundle $E^X \rightarrow U^X_{S_e}$.

**Definition 3.2.** The (primitive and primary) GW invariant is defined as

\[ \langle \alpha_1, \cdots, \alpha_k \rangle^X_{g, A} = \int_{U^X_{S_e}} S^*_X \Theta \wedge \prod_i ev_i^* \alpha_i, \]

where the $\alpha_i$ are classes in $H^*(X; \mathbb{R})$ and are called primary insertions. For the genus zero case, we also sometimes write $\langle \alpha_1, \cdots, \alpha_k \rangle^X_A$ for $\langle \alpha_1, \cdots, \alpha_k \rangle^X_{0,A}$.

**Definition 3.3.** For each marked point $x_i$, we define an orbifold complex line bundle $L_i$ over $\overline{B}_A(g, k, J)$ whose fiber is $T^*_x \Sigma$ at $(\Sigma, f)$. Such a line bundle can be pulled back to $U^X_{S_e}$ (still denoted by $L_i$). Denote $c_1(L_i)$, the first Chern class of $L_i$, by $\psi_i$.

**Definition 3.4.** The descendent GW invariant is defined as

\[ \langle \tau_{d_1} \alpha_1, \cdots, \tau_{d_k} \alpha_k \rangle^X_{g, A} = \int_{U^X_{S_e}} S^*_X \Theta \wedge \prod_i \psi_i^{d_i} \wedge ev_i^* \alpha_i, \]

where $\alpha_i \in H^*(X; \mathbb{R})$.

**Remark 3.5.** In the stable range $2g + k \geq 3$, one can also define non-primitive GW invariants (See e.g. [R1]). Recall that there is a map $\pi: \overline{B}_A(g, k, J) \rightarrow \overline{M}_{g,k}$ contracting the unstable components of the source Riemann surface. We can bring in a class $\kappa$ from the Deligne-Mumford space via $\pi$ to define the ancestor GW invariants

\[ \langle \kappa \parallel \prod_i \alpha_i \rangle^X_{g, A} = \int_{U^X_{S_e}} S^*_X \Theta \wedge \pi^* \kappa \wedge \prod_i ev_i^* \alpha_i. \]

The particular class we will use is the point class in $\overline{M}_{0,k}$ (see Theorem 4.2).
All GW invariants are invariants of \((X,\omega)\). In fact they are invariant under deformations \(\omega_t\) of \(\omega\) by Remark 2.7.

**Remark 3.6.** For computational purpose we would use the following variation of the virtual neighborhood construction in Section 6. Suppose \(1 : D \subset X\) is a submanifold. For \(\alpha \in H^*(D;\mathbb{R})\) we define \(1!(\alpha) \in H^*(X;\mathbb{R})\) via the transfer map \(1! = PD_X \circ 1_s \circ PD_D\). One can construct GW-invariants with an insertion of the form \(1!(\alpha)\) as follows. Apply the virtual neighborhood construction to the compact subspace

\[ \overline{M}_A(g,k,J) \cap ev_1^{-1}(D) \]

in \(\overline{B}^X_A(g,k,J,D) = ev_1^{-1}(D)\) to obtain a virtual neighborhood \(U_\mathcal{S}_e(D)\) together with the natural map \(ev_D : U_\mathcal{S}_e(D) \to D\). It is easy to show that

\[ \langle \tau_{d_1}1!(\alpha), \tau_{d_2}\beta_2, \cdots, \tau_{d_k}\beta_k \rangle^X_{g,A} = \int_{U_\mathcal{S}_e(D)} S^\Theta \wedge ev_D^*\alpha \wedge \prod_{i=2}^k \psi_i^{d_i}ev_1^*\beta_i. \]

**Remark 3.7.** For each \(\langle \tau_{d_1}\alpha_1, \cdots, \tau_{d_k}\alpha_k \rangle^X_{g,A}\), we can conveniently associated a simple graph \(\Gamma\) of one vertex decorated by \((g,A)\) and a tail for each marked point. We then further decorate each tail by \((d_i,\alpha_i)\) and call the resulting graph \(\Gamma^*\{((d_i,\alpha_i))\}\) a weighted graph. Using the weighted graph notation, we denote the above invariant by \(\langle \Gamma^*\{((d_i,\alpha_i))\}\rangle^X\). We can also consider the disjoint union \(\Gamma^*\{\Gamma\}\) of several such graphs and use \(A_{\Gamma^*}\) to denote the total homology class and total arithmetic genus. Here the total arithmetic genus is \(1 + \sum(g_i - 1)\). Then, we define \(\langle \Gamma^*\{((d_i,\alpha_i))\}\rangle^X\) as the product of GW invariants of the connected components.

### 3.2. Relative GW-invariants

In this section, we will review the relative GW-invariants. The readers can find more details in the reference [LR].

Let \(Z \subset X\) be a real codimension 2 symplectic submanifold. Suppose that \(J\) is an \(\omega\)-tamed almost complex structure on \(X\) preserving \(TZ\), i.e., making \(Z\) an almost complex submanifold. The relative GW invariants are defined by counting the number of stable \(J\)-holomorphic maps intersecting \(Z\) at finitely many points with prescribed tangency. More precisely, fix a \(k\)-tuple \(T_k = (t_1, \cdots, t_k)\) of positive integers, consider a marked pre-stable curve

\[ (C, x_1, \cdots, x_m, y_1, \cdots, y_k) \]

and stable \(J\)-holomorphic maps \(f : C \to X\) such that the divisor \(f^*Z\) is

\[ f^*Z = \sum_i t_iy_i. \]

One would like to consider the moduli space of such curves and apply the virtual neighborhood technique to construct the relative invariants. But this scheme needs modification as the moduli space is not compact. It is true that for a sequence of \(J\)-holomorphic maps \((\Sigma_n, f_n)\) as above, by possibly passing to a subsequence, \(f_n\) will still converge to a stable \(J\)-holomorphic
map \((\Sigma, f)\). However the limit \((\Sigma, f)\) may have some \(Z\)-components, i.e. components whose images under \(f\) lie entirely in \(Z\).

To deal with this problem the authors in [LR] adopt the open cylinder model. Choose a Hamiltonian \(S^1\) function \(H\) in a closed \(\epsilon\)-symplectic tubular neighborhood \(X_0\) of \(Z\) as in 2.2.2 with \(H(X_0) = [\epsilon, 0] \) and \(Z = H^{-1}(-\epsilon)\). Next we need to choose an almost complex structure with nice properties near \(Z\). An almost complex structure \(J\) on \(X\) is said to be tamed relative to \(Z\) if \(J\) is \(\omega\)-tamed, \(S^1\)-invariant for some \((X_0, H)\), and such that \(Z\) is an almost complex submanifold. The set of such \(J\) is nonempty and forms a contractible space. With such a choice of almost complex structure, \(X_0\) can be viewed as a neighborhood of the zero section of the complex line bundle \(N_{Z|X}\) with the \(S^1\) action given by the complex multiplication \(e^{2\pi i\theta}\). Now we remove \(Z\). The end of \(X - Z\) is simply \(X_0 - Z\). Recall that the punctured disc \(D - \{0\}\) is biholomorphic to the half cylinder \(S^1 \times [0, \infty)\). Therefore, as an almost complex manifold, \(X_0 - Z\) can be viewed as the translation invariant almost complex half cylinder \(P \times [0, \infty)\) where \(P = H^{-1}(0)\). In this sense, \(X - Z\) is viewed as a manifold with almost complex cylinder end.

Now we consider a holomorphic map in the cylinder model where the marked points intersecting \(Z\) are removed from the domain surface. Again we can view a punctured neighborhood of each of these marked points as a half cylinder \(S^1 \times [0, \infty)\). With such a \(J\), a \(J\)-holomorphic map of \(X\) intersecting \(Z\) at finitely many points then exactly corresponds to a \(J\)-holomorphic map to the open manifold \(X - Z\) from a punctured Riemann surface which converges to (a multiple of) an \(S^1\)-orbit at a punctured point.

Now we reconsider the convergence of \((\Sigma_n, f_n)\) in the cylinder model. The creation of a \(Z\)-component \(f_\nu\) corresponds to disappearance of a part of \(im(f_n)\) into the infinity. We can use translation to rescale back the missing part of \(im(f_n)\). In the limit, we may obtain a stable map \(\tilde{f}_\nu\) into \(P \times \mathbb{R}\). When we obtain \(X\) from the cylinder model, we need to collapse the \(S^1\)-action at the infinity. Therefore, in the limit, we need to take into account maps into the closure of \(P \times \mathbb{R}\). Let \(Q\) be the projective completion of the normal bundle \(N_{Z|X}\), i.e. \(Q = \mathbb{P}(N_{Z|X} \oplus \mathbb{C})\). Then \(Q\) has a zero section \(Z_0 = \mathbb{P}(0 \oplus \mathbb{C})\) and an infinity section \(Z_\infty = \mathbb{P}(N_{Z|X} \oplus 0)\). One can further show that \(\tilde{f}_\nu\) indeed is a stable map into \(Q\) with the stability specified below.

To form a compact moduli space of such maps we thus must allow the target \(X\) to degenerate as well (compare with [Li]). For any non-negative integer \(m\), construct \(Q_m\) by gluing together \(m\) copies of \(Q\), where the infinity section of the \(i^{th}\) component is glued to the zero section of the \((i + 1)^{th}\) component for \(1 \leq i \leq m\). Denote the zero section of the \(i^{th}\) component by \(Z_i,0\), and the infinity section by \(Z_i,\infty\), so \(SingQ_m = \cup_{i=1}^{m} Z_i,\infty\). We will also denote \(Z_{m,\infty}\) by \(Z_\infty\) if there is no possible confusion. Define \(X_m\) by gluing \(X\) to \(Q_m\) along \(Z \subset X\) and \(Z_{0,0} \subset Q_m\). In particular, \(X_0 = X\) will
be referred to as the root component and the other irreducible components will be called the bubble components.

$Z \subset X$ can be thought of as the infinity section $Z_{0,\infty}$ of the 0-th component (which is $X$), thus $\text{Sing}X_m = \bigcup_{i=0}^{m-1} Z_{i,\infty}$. Let $\text{Aut}_Z Q_m$ be the group of automorphisms of $Q_m$ preserving $Z_{i,0}$, $Z_{i,\infty}$, and the morphism to $Z$. And let $\text{Aut}_Z X_m$ be the group of automorphisms of $X_m$ preserving $X$ (and $Z$) and with restriction to $Q_m$ being contained in $\text{Aut}_Z Q_m$ (so $\text{Aut}_Z X_m = \text{Aut}_Z Q_m \cong (\mathbb{C}^*)^m$, where each factor of $(\mathbb{C}^*)^m$ dilates the fibers of the $i$-th $\mathbb{P}^1$-bundle). Denote by $\pi_m : X_m \rightarrow X$ the map which is the identity on the root component $X_0$ and contracts all the bubble components to $Z = Z_{1,0}$ via the fiber bundle projections.

Now consider a nodal curve $C$ mapped into $X_m$ by $f : C \rightarrow X_m$ with specified tangency to $Z_{m,\infty}$. There are two types of marked points:

(i) absolute marked points whose image under $f$ lie outside $Z_{m,\infty}$ labeled by $x_i$,

(ii) relative marked points which are mapped into $Z_{m,\infty}$ by $f$ labeled by $y_j$.

A relative $J$-holomorphic map $f : C \rightarrow X_m$ is said to be pre-deformable if $f^{-1}(Z_{i-1,\infty} = Z_{i,0})$ consists of a union of nodes so that for each node $p \in f^{-1}(Z_{i-1,\infty} = Z_{i,0}), i = 1, 2, \ldots, m$, the two branches at the node are mapped to different irreducible components of $X_m$ and the orders of contact to $Z_{i-1,\infty} = Z_{i,0}$ are equal.

An isomorphism of two such $J$-holomorphic maps $f$ and $f'$ to $X_m$ consists of a diagram

\[
\begin{array}{ccc}
(C, x_1, \cdots, x_l, y_1, \cdots, y_k) & \xrightarrow{f} & X_m \\
\downarrow h & & \downarrow t \\
(C', x'_1, \cdots, x'_l, y'_1, \cdots, y'_k) & \xrightarrow{f'} & X_m
\end{array}
\]

where $h$ is an isomorphism of marked curves and $t \in \text{Aut}_Z(X_m)$. With the preceding understood, a relative $J$-holomorphic map to $X_m$ is said to be stable if it has only finitely many automorphisms.

We introduced the notion of a weighted graph in Remark. We need to refine it for relative stable maps to $(X, Z)$. A (connected) relative graph $\Gamma$ consists of the following data:

1. a vertex decorated by $A \in H_2(X; \mathbb{Z})$ and genus $g$,
2. a tail for each absolute marked point,
3. a relative tail for each relative marked point.

**Definition 3.8.** Let $\Gamma$ be a relative graph with $k$ (ordered) relative tails and $T_k = (t_1, \cdots, t_k)$, a $k$-tuple of positive integers forming a partition of $A \cdot [Z]$. A relative $J$-holomorphic map of $(X, Z)$ with type $(\Gamma, T_k)$ consists of a marked curve $(C, x_1, \cdots, x_l, y_1, \cdots, y_k)$ and a map $f : C \rightarrow X_m$ for some non-negative integer $m$ such that
(i) \( C \) is a connected curve (possibly reducible) of arithmetic genus \( g \), (ii) the map 
\[
\pi_m \circ f : C \rightarrow X_m \rightarrow X
\]
satisfies \((\pi_m \circ f)_\ast [C] = A\),
(iii) the \( x_i, 1 \leq i \leq l \), are the absolute marked points,
(iv) the \( y_i, 1 \leq i \leq k \), are the relative marked points,
(v) \( f^{-1}(Z_{m, \infty}) \) consists of precisely the points \( \{y_1, \ldots, y_k\} \) and \( f \) has order \( t_i \) at each \( y_i \).

Let \( \overline{M}_{\Gamma, T_k}(X, Z, J) \) be the moduli space of equivalence classes of pre-deformable relative stable \( J \)-holomorphic maps with type \((\Gamma, T_k)\). Notice that for an element \( f : C \rightarrow X_m \) in \( \overline{M}_{\Gamma, T_k}(X, Z, J) \) the intersection pattern with \( \text{Sing} \ X_m \) is only constrained by the genus condition and the pre-deformability condition.

Consider the configuration space \( B_{\Gamma, T_k}(X, Z, J) \) of equivalence classes of smooth pre-deformable relative stable maps to \( X_m \) for all \( m \geq 0 \). Here, for each \( m \), we still take the equivalence class under \( \text{Aut} \ Z X_m \). In particular, the subgroup of \( \text{Aut} \ Z X_m \) fixing such a map is required to be finite. The maps are required to intersect the \( Z_i, \infty \) only at finitely many points in the domain curve. Further, at these points, the map is required to have a holomorphic leading term in the normal Taylor expansion for any local chart of \( X \) taking \( D \) to a coordinate hyperplane and being holomorphic in the normal direction along \( D \). Thus the notion of contact order still makes sense, and we can still impose the pre-deformability condition and contact order condition at the \( y_i \) being governed by \( T_k \).

With the preceding understood, by choosing a unitary connection on the normal complex line bundles of the \( Z_i, \infty \), we can define the analog of [3],
\[
I_{X, f}^{X, Z} : \Omega_r^0 \rightarrow \Omega_r^{0, 1},
\]
taking into account the pre-deformability condition and the contact condition \( T_k \) along \( Z_{m, \infty} \). Now we can apply the virtual neighborhood technique to construct \( U_{X, Z}^S, E_{X, Z}, S_{X, Z} \) as in section 3.1.

In addition to the evaluation maps on \( \overline{B}_{\Gamma, T_k}(X, Z, J) \),
\[
ev_i^X : \overline{B}_{\Gamma, T_k}(X, Z, J) \rightarrow X, \quad 1 \leq i \leq l,
\]
\[
\pi_m \circ f(x_i),
\]
there are also the evaluations maps
\[
ev_j^Z : \overline{B}_{\Gamma, T_k}(X, Z, J) \rightarrow Z, \quad 1 \leq j \leq k,
\]
\[
f(y_j),
\]
where \( Z = Z_{m, \infty} \) if the target of \( f \) is \( X_m \).

**Definition 3.9.** Let \( \alpha_i \in H^*(\Xi; \mathbb{R}), 1 \leq i \leq l \), \( \beta_j \in H^*(Z; \mathbb{R}), 1 \leq j \leq k \).
Define the relative GW invariant
\[
\langle \Pi_i \tau_{d_i} \alpha_i | \Pi_j \beta_j \rangle_{\Gamma, T_k} = \frac{1}{|\text{Aut}(T_k)|} \int_{U_{X, Z}^S} S_{X, Z}^* \Theta \wedge \Pi_i \psi_i^{d_i} \wedge (\ev_i^X)^* \alpha_i \wedge \Pi_j (\ev_j^Z)^* \beta_j.
\]
where $\Theta$ is the Thom class of the bundle $\mathcal{E}_{X,Z}$ and $\text{Aut}(T_k)$ is the symmetry group of the partition $T_k$. Denote by $T_k = \{(t_j, \beta_j) \mid j = 1, \cdots, k\}$ the weighted partition of $A \cdot [Z]$. If the vertex of $\Gamma$ is decorated by $(g, A)$, we will sometimes write

$$\langle \Pi_{i=1}^{g} \tau_{d_i \alpha_i} \mid T_k \rangle_{X,Z}^{X,Z}$$

for $\langle \Pi_{j=1}^{g} \tau_{d_j \beta_j} \mid T_k \rangle_{X,Z}^{\Gamma,T_k}$.

**Remark 3.10.** In [LR] only invariants without descendent classes were considered. But it is straightforward to extend the definition of [LR] to include absolute descendent classes.

We can decorate the tail of a relative graph $\Gamma$ by $(d_i, \alpha_i)$ as in the absolute case. We can further decorate the relative tails by the weighted partition $T_k$. Denote the resulting weighted relative graph by $\Gamma \{(d_i, \alpha_i)\}|T_k$. In [LR] the source curve is required to be connected. We will also need to use a disconnected version. For a disjoint union $\Gamma^\bullet$ of weighted relative graphs and a corresponding disjoint union of partitions, still denoted by $T_k$, we use $\langle \Gamma^\bullet \{(d_i, \alpha_i)\}|T_k \rangle^{X,Z}_{X,Z}$ to denote the corresponding relative invariants with a disconnected domain, which is simply the product of the connected relative invariants. Notice that although we use $\bullet$ in our notation following [MP], our disconnected invariants are different. The disconnected invariants there depend only on the genus, while ours depend on the finer graph data.

### 3.3. Degeneration formula

Now we describe the degeneration formula of GW-invariants under symplectic cutting.

As an operation on topological spaces, the symplectic cut is essentially collapsing the circle orbits in the hypersurface $H^{-1}(0)$ to points in $Z$. Thus we have a continuous map

$$\pi : X \rightarrow \overline{X}^+ \cup_Z \overline{X}^-.$$

As for the symplectic forms, we have $\omega^+|_Z = \omega^-|_Z$. Hence, the pair $(\omega^+, \omega^-)$ defines a cohomology class of $\overline{X}^+ \cup_Z \overline{X}^-$, denoted by $\overline{[\omega^+ \cup_Z \omega^-]}$.

It is easy to observe that

$$\pi^*([\omega^+ \cup_Z \omega^-]) = [\omega].$$

Let $B \in H_2(X;Z)$ be in the kernel of

$$\pi_* : H_2(X;Z) \rightarrow H_2(\overline{X}^+ \cup_Z \overline{X}^-;Z).$$

By (1) we have $\omega(B) = 0$. Such a class is called a vanishing cycle. The geometric description of the vanishing cycle is rim tori. For each simple closed curve $\gamma$ in $Z$, $\pi^{-1}(\gamma)$ is a torus in $H^{-1}(0)$, i.e., rim tori. It is easy to see that each vanishing cycle can be represented by a rim tori. In particular, in the case of blowup along a complex codimension two submanifold $S$, $H^{-1}(0)$ is the sphere bundle of the normal bundle to $S$ in $X$. Since the fiber
is simply connected, there are no rim tori. Therefore, there are no vanishing cycle. For \( A \in H_2(X; \mathbb{Z}) \) define \([A] = A + \text{Ker}(\pi_*)\) and

\[
\langle \Pi_i \tau_d, \alpha_i \rangle_{g,A}^X = \sum_{B \in [A]} \langle \Pi_i \tau_d, \alpha_i \rangle_{g,B}^X.
\]

Notice that \( \omega \) has constant pairing with any element in \([A]\). It follows from the Gromov compactness theorem that there are only finitely many such elements in \([A]\) represented by \( J \)-holomorphic stable maps. Therefore, the summation in (5) is finite.

The degeneration formula expresses \( \langle \Pi_i \tau_d, \alpha_i \rangle_{g,A}^X \) in terms of relative invariants of \((X^+, Z)\) and \((X^-, Z)\) possibly with disconnected domains.

To begin with, we need to assume that the cohomology class \( \alpha_i \) is of the form

\[
\alpha_i = \pi^*(\alpha_i^+ \cup_Z \alpha_i^-).
\]

Here \( \alpha_i^+ \in H^*(X^+; \mathbb{R}) \) are classes with \( \alpha_i^+ |_Z = \alpha_i^- |_Z \) so that they give rise to a class \( \alpha_i^+ \cup_Z \alpha_i^- \in H^*(X^+ \cup_Z X^-; \mathbb{R}) \).

Next, we proceed to write down the degeneration formula. We first specify the relevant topological type of a marked Riemann surface mapped into \( X^+ \cup_Z X^- \) with the following properties:

(i) Each connected component is mapped either into \( X^+ \) or \( X^- \) and carries a respective degree 2 homology class;

(ii) The marked points are not mapped to \( Z \);

(iii) Each point in the domain mapped to \( Z \) carries a positive integer (representing the order of tangency).

By abusing language we call the above data a \((X^+, X^-)\)-graph. Such a graph gives rise to two relative graphs of \((X^+, Z)\) and \((X^-, Z)\), each possibly being disconnected. We denote them by \( \Gamma_+^\bullet \) and \( \Gamma_-^\bullet \) respectively. From (iii) we also get two partitions \( T_+ \) and \( T_- \). We call a \((X^+ \cup X^-)\)-graph a degenerate \((g, A, l)\)-graph if the resulting pairs \((\Gamma_+^\bullet, T_+)\) and \((\Gamma_-^\bullet, T_-)\) satisfy the following constraints: the total number of marked points is \( l \), the relative tails are the same, i.e. \( T_+ = T_- \), and the identification of relative tails produces a connected graph of \( X \) with total homology class \( \pi_* [A] \) and arithmetic genus \( g \).

Let \( \{\beta_a\} \) be a self-dual basis of \( H^*(Z; \mathbb{R}) \) and \( \eta^{ab} = \int_Z \beta_a \cup \beta_b \). Given \( g, A \) and \( l \), consider a degenerate \((g, A, l)\)-graph. Let \( T_k = T_+ = T_- \) and \( T_k \) be a weighted partition \( \{t_j, \beta_{a_j}\} \). Let \( \tilde{T}_k = \{t_j, \beta_{a_j}\} \) be the dual weighted partition.

The degeneration formula for \( \langle \Pi_i \tau_d, \alpha_i \rangle_{g,A}^X \) then reads as follows,

\[
\langle \Pi_i \tau_d, \alpha_i \rangle_{g,A}^X = \sum (\Gamma^\bullet \{ (d_i, \alpha_i^+) \}) |T_k|^{X^+, Z} \Delta(T_k) \langle \Gamma^\bullet \{ (d_i, \alpha_i^-) \} |\tilde{T}_k|^{X^-, Z},
\]

\( g, A, l \).
where the summation is taken over all degenerate \((g, A, l)\)-graphs, and
\[
\Delta(T_k) = \prod_j t_j \text{Aut}(T_k).
\]

4. Uniruled manifolds

In this section all GW invariants are of genus 0 and we will omit the subscript for the genus. This convention will be used in section 6 as well.

4.1. Uniruledness in algebraic geometry. Let us first recall the notion of uniruledness in algebraic geometry.

**Definition 4.1.** A smooth projective variety \(X\) (over \(\mathbb{C}\)) is called (projectively) uniruled if for every \(x \in X\) there is a morphism \(f : \mathbb{P}^1 \to X\) satisfying \(x \in f(\mathbb{P}^1)\), i.e. \(X\) is covered by rational curves.

Rational curves on uniruled projective varieties have the following nice property \([KMM, K1]\): for a very general point \(x\), if \(g : \mathbb{P}^1 \to X\) is a morphism such that \(g_*[\mathbb{P}^1] = A\) and \(g(y_0) = x\), then
\[
H^1(\mathbb{P}^1, g^*TX) = 0. \tag{7}
\]

The characterization of uniruled varieties is very important in classification theory. Here we only review the following beautiful result which is due to Kollár and Ruan. We sketch a proof as it may not be so well known.

**Theorem 4.2.** \([K], [R1]\): A projective manifold \((X, J, \omega)\) is uniruled if and only if there exist a homology class \(A \in H_2(X; \mathbb{Z})\) and cohomology classes \(\alpha_2, \ldots, \alpha_{p+1} \in H^*(X; \mathbb{R})\) such that
\[
\langle [pt] \parallel [pt], \alpha_2, \ldots, \alpha_{p+1} \rangle_X^X \neq 0,
\]
where the first \([pt]\) represents the Poincaré dual of the point class of \(\overline{\mathcal{M}}_{0,k}\).

**Proof.** Suppose there is a nonzero invariant \(\langle [pt] \parallel [pt], \alpha_2, \ldots, \alpha_{p+1} \rangle_X^X\). Then for the given (integrable) complex structure \(J\), through any point \(x\), the corresponding moduli space of stable \(A\)-rational curves cannot be empty. Otherwise the invariant is zero. Since each domain component of a stable rational curve is \(\mathbb{P}^1\), there is a morphism \(g : \mathbb{P}^1 \to X\) such that \(x \in g(\mathbb{P}^1)\). So \(X\) is uniruled.

Suppose \(X\) is uniruled. Fix a sufficiently general point \(x\) and a very ample divisor \(H\). Let \(g : \mathbb{P}^1 \to X\) be a morphism such that \(x \in g(\mathbb{P}^1)\). Such a \(g\) exists as \(X\) is assumed to be uniruled. Furthermore assume that the class \(A = g_*[\mathbb{P}^1]\) has the property that the pairing \(H(A)\) with the ample divisor \(H\) is minimal among all such classes. Then every rational curve through \(x\) is irreducible, i.e. any stable \(A\)-rational curve through \(x\) is of the form \(f : \mathbb{P}^1 \to X\). Therefore the moduli space of \(A\)-rational curves through \(x\) is the same as the compactified moduli space of stable \(A\)-rational curves through \(x\). We denote this compact moduli space simply by \(\mathcal{M}_x\). Moreover
if we invoke the property (7) we conclude that $\mathcal{M}_x$ is smooth of expected dimension.

Let $\mathcal{E} = f^*TX$. Then $\mathcal{E}$ is a convex holomorphic bundle over $\mathbb{P}^1$, and, as a complex bundle, it is independent of $f$ since $f_*[\mathbb{P}^1] = A$. For any $f \in \mathcal{M}_x$, the tangent space $T_f\mathcal{M}_x$ is identified with $\{v \in H^0(\mathcal{E})|v(x) = 0\}$. Observe that there are finitely many points $y_2, \cdots, y_{p+1} \in \mathbb{P}^1$ such that for any holomorphic section $v \in H^0(\mathcal{E})$, $v = 0$ if and only if $v(y_i) = 0$ for each $2 \leq i \leq p + 1$.

Consider the holomorphic evaluation map $\Xi_p: \mathcal{M}_x \to X^p, f \to (f(y_2), \cdots, f(y_{p+1}))$. Its differential is simply given by $\delta(\Xi_p)_f(v) = (v(y_2), \cdots, v(y_{p+1}))$.

Therefore, by our choice of the $y_i$, $\Xi_p$ is a holomorphic immersion at $(g, y_2, \cdots, y_{p+1})$. Now $\Xi_p(\mathcal{M}_x) \subset X^p$ is a compact complex subvariety of the same dimension as that of $\mathcal{M}_x$. In particular, it represents a nonzero homology class $[\Xi_p(\mathcal{M}_x)] \in H_*(X^p; \mathbb{Z})$. Furthermore, $$(\Xi_p)_*[\mathcal{M}_x] = \lambda[\Xi_p(\mathcal{M}_x)]$$ for some $\lambda > 0$. There are cohomology classes $\alpha_2, \cdots, \alpha_{p+1}$ coming from very ample divisors such that $$(\prod_{i=2}^{p+1} \alpha_i)([\Xi_p(\mathcal{M}_x)]) \neq 0.$$ Notice that the points $y_i, 2 \leq i \leq p + 1$ are fixed, so the invariant $$(\langle [pt] \parallel [pt], \alpha_2, \cdots, \alpha_{p+1}\rangle^X_A = \int_{\mathcal{M}_x} \pi^*[\overline{\mathcal{M}}_{0,p}]\Xi_p^*(\prod_{i=2}^{p+1} \alpha_i) = \prod_{i=2}^{p+1} \alpha_i([\Xi_p(\mathcal{M}_x)]) \neq 0.$$ Observe that we can choose some special $\alpha_i$ in the following fashion. The simplest is $\Omega^l$, where $\Omega$ is a Kähler form of $X^p$ and $l$ is the dimension of $\Xi_p(\mathcal{M}_x)$. We can choose $\Omega$ as the product of a Kähler form $\omega$ on $X$. It follows that each $\alpha_i$ can be chosen to be some power of $\omega$. Finally, the nonzero invariant of Theorem 4.2 can be decomposed as the sum of products of invariants with 3 insertions by associativity. It implies that (see also Proposition 7.3 in [Lu2])

**Corollary 4.3.** A projective manifold is projectively uniruled if and only if $$(\langle [pt], \omega^p, \alpha\rangle^X_A \neq 0$$ for some $A \neq 0, p, \alpha$, where $\omega$ is a Kähler form.
4.2. **Symplectic uniruledness.** Let $X$ be a closed symplectic manifold. Let $E$ be a smooth symplectic divisor, possibly empty.

**Definition 4.4.** Let $A \in H_2(X; \mathbb{Z})$ be a nonzero class. $A$ is said to be a uniruled class if there is a nonzero GW invariant

$$\langle [pt], \alpha_2, \cdots, \alpha_k \rangle^X_A,$$

where $\alpha_i \in H^*(X; \mathbb{R})$. $A$ is said to be a uniruled class relative to a divisor $E$ if there is a nonzero relative invariant

$$\langle [pt], \alpha_2, \cdots, \alpha_k | \beta_1, \cdots, \beta_l \rangle^X_E$$

where $\alpha_i \in H^*(X; \mathbb{R})$ and $\beta_j \in H^*(E; \mathbb{R})$.

**Definition 4.5.** $X$ is said to be (symplectically) uniruled if there is a uniruled class. $(X, E)$ is said to be uniruled if there is a uniruled class relative to $E$.

**Remark 4.6.** It is easy to see that we could well use the more general disconnected GW invariants to define this concept. This flexibility is important for the proof of the birational cobordism invariance.

This notion has been studied in the symplectic context by G. Lu (see [Lu1-3]). Notice that, by [Ltj], it is not meaningful to define this notion by requiring that there is a symplectic sphere in a fixed class through every point, otherwise every simply connected manifold would be uniruled.

**Remark 4.7.** According to Corollary 4.3 a projectively uniruled manifold is symplectically uniruled, in fact strongly symplectically uniruled. Here $X$ is said to be strongly uniruled if there is a nonzero invariant of the form (10) with $k = 3$.

In dimension 4 it follows from [Mc1], [LL1], [LL2], [LM] that the converse is essentially true. While in higher dimensions it follows from [G] (see also [Lu3]) that there are uniruled symplectic manifolds which are not projective, and it follows from [R3] that there could be infinitely many distinct uniruled symplectic structures on a given smooth manifold.

4.3. **Minimal descendent invariants.** In this paper an absolute descendent invariant is called strict if one of the insertions is of the form $\tau_k(\gamma)$ with $k \geq 1$. In this section we want to replace a GW-invariant in the definition of uniruledness with only primary insertions and a connected domain by a descendent GW-invariant with a possibly disconnected domain. But the point insertion will be kept to be $[pt]$, rather than a descendent one $\tau_d([pt])$ with $d > 0$.

This additional flexibility is very important in our proof of the birational cobordism invariance of uniruledness. The key ingredient is the fact that, as in algebraic geometry (see e.g. [Ge]), a (absolute) descendent class $\psi$ in the moduli space of genus zero stable maps can always be expressed as a sum of boundary classes. This is true because, on one hand, such a relation is well known on the Deligne-Mumford moduli space of genus zero stable curves, and on the other hand, the difference of the descendent class...
from the pull-back of the corresponding descendent class on the genus zero Deligne-Mumford moduli space (the so called ancestor class) is a boundary class as well.

The following well-known fact in the case $A = 0$ will be often used.

**Lemma 4.8.** Let $A$ be the zero class. Then

a) A nonzero GW invariant with only primary insertions must have exactly 3 insertions. In particular, if one insertion is the point class, then the invariant is essentially of the form $\langle [pt], [X], [X] \rangle^X = 1$ (see e.g. p.230 of [MS2]).

b) Any nonzero strictly descendent GW invariant must also have 3 primary insertions with the total degree equal to the dimension of $X$. In particular, there are at least 4 marked points.

In the next lemma we assume all the invariants are connected.

**Lemma 4.9.** Let $\gamma, \alpha$ be cohomology classes of $X$ and $0 \neq A \in H_2(X; \mathbb{Z})$.

- If $\langle [pt], \tau_i(\gamma), \tau_j(\alpha) \rangle^X = 0$ for some $i \geq 1, j \geq 1$, then there is a homology class $A' \neq 0$ and a cohomology class $\beta$ such that $\langle [pt], \tau_i(\gamma), \beta \rangle^X = 0$.

- If $\langle [pt], \tau_i(\gamma), \alpha \rangle^X = 0$ for some $i \geq 1$, then either $X$ is uniruled or $\langle [pt], \tau_{i-1}(\gamma) \rangle^X = 0$.

- If $\langle [pt], \tau_i(\gamma) \rangle^X = 0$ for some $i \geq 1$ then $X$ is uniruled.

**Proof.** First of all, recall that when there are $k \geq 3$ marked points $x_1, \ldots, x_k$, $\psi_2$ can be expressed as a sum of boundary divisors (see the comments in the beginning of the subsection):

$$\psi_2 = \sum_{A = A_1 + A_2} D_{(1,3), A_1(2), A_2}.$$  

Here $D_{(1,3), A_1(2), A_2}$ denotes the divisor consisting of all stable maps with at least two components, one in class $A_1$ containing $x_1$ and $x_3$, and the other in class $A_2$ containing $x_2$. (9) is understood as an identity in the cohomology of a virtual neighborhood of $\overline{M}_A^X(0, k, J)$.

When $k = 3$ it follows from the relation (9) and the splitting axiom that

$$\langle [pt], \tau_i(\gamma), \tau_j(\alpha) \rangle^X = \sum_{A = A_1 + A_2} \sum_{\mu, \nu} \langle [pt], \tau_j(\alpha), e_\mu \rangle_{A_1}^X g^{\mu\nu} \langle e_\nu, \tau_{i-1}(\gamma) \rangle_{A_2}^X.$$  

Fix $i \geq 1$. Let $A' \neq 0$ be a class with $\langle [pt], \tau_i(\gamma), \tau_j(\alpha) \rangle^X = 0$ and the smallest pairing with $\omega$ among all strictly descendent invariants with a point insertion and 2 other insertions. Then we have either $A_1 = A', A_2 = 0$, or
\(A_1 = 0, A_2 = A'.\) Therefore
\[
\langle [pt], \tau_i(\gamma), \tau_j(\alpha) \rangle_{A'}^X
\]
\[
= \sum_{\mu, \nu} \langle [pt], \tau_j(\alpha), e_\mu \rangle_{A'}^X g^{\mu \nu} \langle e_{\nu}, \tau_{i-1}(\gamma) \rangle_{0}^X
\]
\[
+ \sum_{\mu, \nu} \langle [pt], \tau_j(\alpha), e_\mu \rangle_{0}^X g^{\mu \nu} \langle e_{\nu}, \tau_{i-1}(\gamma) \rangle_{A'}^X.
\]

Next, we want to prove that \(j = 0.\) Suppose first that \(j \geq 1.\) Then by Lemma \(\textbf{4.8}\) \(\langle [pt], \tau_j(\alpha), e_\mu \rangle_{0}^X = 0.\) Moreover \(\langle e_{\nu}, \tau_{i-1}(\gamma) \rangle_{0}^X\) is always zero. Thus \(\langle [pt], \tau_i(\gamma), \tau_j(\alpha) \rangle_{A'}^X = 0.\) This contradicts the assumption \(\langle [pt], \tau_i(\gamma), \tau_j(\alpha) \rangle_{A'}^X \neq 0.\) So we have \(j = 0, i.e.
\]
\[
\langle [pt], \tau_i(\gamma), \alpha \rangle_{A'}^X \neq 0.
\]
This proves the first part of the lemma.

Suppose now that \(j = 0.\) Then either
\[
\langle [pt], \alpha, e_\mu \rangle_{A_1}^X \neq 0
\]
for some \(A_1 \neq 0,\) or
\[
\langle [pt], \alpha, e_\mu \rangle_{0}^X g^{\mu \nu} \langle e_{\nu}, \tau_{i-1}(\gamma) \rangle_{A}^X \neq 0.
\]

In the former case, since the insertions \(\alpha\) and \(e^\mu\) are primary, \(A_1\) is a uniruled class.

In the latter case we must have \(\alpha = e^\mu = [X]\) and \(e^\nu = pt.\) Hence we have \(\langle [pt], \tau_{i-1}(\gamma) \rangle_{A}^X \neq 0.\) Notice that \(A\) is actually a uniruled class if \(i - 1 = 0.\)

Finally we deal with the last case, a descendent invariant with 2 insertions.
As \(A \neq 0\) there is a class \(\alpha\) in \(H^2(X; \mathbb{R})\) with \(\alpha(A) \neq 0.\) By the divisor axiom for a descendent invariant, if \(i \geq 1,\)
\[
\alpha(A) \langle [pt], \tau_i(\gamma) \rangle_{A}^X = \langle [pt], \tau_i(\gamma), \alpha \rangle_{A}^X - \langle [pt], \tau_{i-1}(\gamma \cup \alpha) \rangle_{A}^X.
\]
We have either
\[
\langle [pt], \tau_i(\gamma), \alpha \rangle_{A}^X \neq 0,
\]
or
\[
\langle [pt], \tau_{i-1}(\gamma \cup \alpha) \rangle_{A}^X \neq 0.
\]
In the former case, notice that \(\alpha\) is non-descendent, so we conclude by the 2nd part that, either \(X\) is uniruled or \(\langle [pt], \tau_{i-1}(\gamma) \rangle_{X}^X \neq 0.\) Therefore we find that either \(X\) is uniruled or there is a nonzero invariant with 2 insertions and less descendent power. We can repeat this process to show that either \(X\) is uniruled, or finally \(\langle [pt], \tau_{0}(\gamma \cup \alpha^I) \rangle_{X}^X \neq 0\) for some \(0 \leq l \leq i.\) But \(X\) is obviously uniruled in the last case as well.

\[\Box\]

The main result in this subsection is
Theorem 4.10. A symplectic manifold $X$ is uniruled if and only if there is a nonzero, possibly disconnected genus zero descendent GW invariant

$$
\langle [pt], \tau_{j_2}(\alpha_2), \cdots, \tau_{j_k}(\alpha_k) \rangle^X_A
$$

such that the component with the $[pt]$ insertion has nonzero curve class.

Proof. First of all notice that an invariant with disconnected domain is the product of the invariants of the connected components. Thus the only if part is clear. And to prove the if part we can assume that the nonzero descendent invariant is actually connected and $A \neq 0$.

The case of $k \leq 3$ has been dealt with in Lemma 4.9. We now examine descendent invariants with $k \geq 4$ insertions. Let $A$ be a class with a nonzero invariant $\langle [pt], \tau_{j_2}(\alpha_2), \cdots, \tau_{j_k}(\alpha_k) \rangle^X_A$ with $j_i > 0$ for some $i$ and $k \geq 4$. Assume that $A$ is such a class with the smallest $\omega(A)$. Further assume that $k$ is the smallest among such an $A$.

Since $k \geq 4$, we can apply the boundary relation (10) to obtain

$$
\sum \sum_{A_1+A_2} \sum_{\mu, \nu} \langle [pt], \tau_{j_2}(\alpha_2), \tau_{j_3}(\alpha_3), \tau_{j_4}(\alpha_4), \cdots, \tau_{j_{k-2}}(\alpha_{k-2}) \rangle^{X}_{A_1} \cdot \langle [pt], \tau_{j_{k-1}}(\alpha_{k-1}), \tau_{j_{k-2}}(\alpha_{k-2}) \rangle^{X}_{A_2} = 0.
$$

where the leftmost sum is over all partitions of $\{4, \cdots, k\} = \{i_1, \cdots, i_{k-1}\} \cup \{i_{k-1}, \cdots, i_k\}$. By the minimality of $\omega(A)$ we have either $A_1 = A$ or $A_2 = 0$.

Suppose the sum with $A_1 = 0$ in (11) is nonzero. Consider a nonzero product in this sum. For the first factor we must have $k_1 \geq 1$, $\alpha_3 = e^\mu = [X]$ by Lemma 4.8. Therefore $e_\nu = [pt]$ and $k - 2 - k_1 \leq k - 3$. Hence the second factor

$$
\langle [pt], \tau_{j_2}(\alpha_2), \tau_{j_{k-1}}(\alpha_{k-1}), \cdots, \tau_{j_{k-2}}(\alpha_{k-2}) \rangle^{X}_{A_2 = A}
$$

has at most $k - 1$ many insertions. By our assumption of the minimality of $k$ among all strictly descendent invariants for the class $A$, this is a nonzero non-descendent invariant, which shows that $X$ is uniruled.

The case where the sum with $A_2 = 0$ in (11) is nonzero is similar. Consider again a nonzero product in this sum. We must have $k_1 + 1 \leq k$, or equivalently, $k_1 \leq k - 1$, by Lemma 4.8. By the minimality of $k$ we claim as above that there is a nonzero invariant of the form $\langle [pt], \alpha_3, \alpha_{i_1}, \cdots, \alpha_{i_{k-1}}, e^\mu \rangle_{A_1 = A}$. Therefore in any case we have that $X$ is uniruled.

\[\square\]

Remark 4.11. We can actually replace (10) in Theorem 4.10 by a more general descendent invariant of the form

$$
\langle \tau_{j_1}(\alpha_2), \tau_{j_2}(\alpha_2), \cdots, \tau_{j_k}(\alpha_k) \rangle^X_A.
$$

Such a generalization will be useful in our next paper [LR].
5. Blow-up Correspondence for GW-invariants

We assume in this section that $X$ is a compact symplectic manifold. Let $S \subset X$ be a symplectic submanifold of $X$ of codimension $2k$, $\tilde{X}$ the blow up of $X$ along $S$, and $E$ the exceptional divisor, which is a $\mathbb{P}^{k-1}$-bundle over $S$. Let $p: \tilde{X} \to X$ be the map defined in 2.2.2. In this section, we will obtain a correspondence between the relative GW-invariant of $(\tilde{X}, E)$ and certain absolute GW-invariants of $X$. Notice that we allow $E = S$. Thus the correspondence of this section can be viewed as a generalization of the correspondence of Maulik-Pandharipande [MP] in the case $k = 1$. The main method used in the proof of our main theorems is the degeneration formula reviewed in section 3.3.

5.1. A refined partial ordering. In this subsection, we want to order the graphs of certain relative GW invariants of $(\tilde{X}, E)$ following [MP]. The new feature here is that we refine the order in the case $k > 1$.

The partial order is defined in terms of several preliminary partial orders. We first deal with those involving classes of the $\mathbb{P}^{k-1}$-bundle $E$.

Let $\theta_1, \cdots, \theta_{m_S}$ be a self-dual basis of $H^*(S; \mathbb{R})$ with respect to the intersection pairing, where $1$ is the distinguished degree 0 class. $E$ is a $\mathbb{P}^{k-1}$-bundle over $S$, so it has a basis of the form

\[
\{\pi_S^*\theta_i \cup [E]^j\}, \quad 1 \leq i \leq m_S, 0 \leq j \leq k-1.
\]

Here $[E]$ is understood to be the first Chern class of the tautological line bundle over the projective bundle $\mathbb{P}(N_{E|\tilde{X}})$. Denote this basis of $E$ by $\Theta = \{\delta_t\}, 1 \leq t \leq km_S$ with $\delta_1 = 1$. Notice that the basis $\Theta$ is still self-dual.

**Definition 5.1.** A standard (relative) weighted partition $\mu$ with length $l(\mu)$ is a partition weighted by classes of $E$ from $\Theta$, i.e.

\[
\mu = \{(\mu_1, \delta_{K_1}), \cdots, (\mu_{l(\mu)}, \delta_{K_{l(\mu)}})\},
\]

where $\mu_i$ and $k_i$ are positive integers with $K_i \leq km_S$.

**Definition 5.2.** For $\delta = \pi^*\theta \cup [E]^j \in H^*(E; \mathbb{R})$, we define

\[
\deg_S(\delta) = \deg(\theta), \quad \deg_f(\delta) = 2j.
\]

For a standard weighted partition $\mu$, we define

\[
\deg_S(\mu) = \sum_{i=1}^{l(\mu)} \deg_S(\delta_{K_i}).
\]

**Definition 5.3.** The set of pairs $(m, \delta)$ where $m \in \mathbb{Z}_{>0}$ and $\delta \in H^*(E; \mathbb{R})$ is partially ordered by the following size relation:

\[
(m, \delta) > (m', \delta')
\]

- if $m > m'$, or
- if $m = m'$ and $\deg_S(\delta) > \deg_S(\delta')$, or
- if $m = m', \deg_S(\delta) = \deg_S(\delta')$ and $\deg_f(\delta) > \deg_f(\delta')$. 

We may place the pairs of $\mu$ in decreasing order by size, i.e. by (14).

**Definition 5.4.** A lexicographic ordering on weighted partitions is then defined as follows:

$$\mu \succ \mu'$$

if, after placing the pairs in $\mu$ and $\mu'$ in decreasing order by size, the first pair for which $\mu$ and $\mu'$ differ in size is larger for $\mu$.

Next we introduce a relevant partial orders on the curve classes of $\tilde{X}$.

**Definition 5.5.** A class $A \in H_2(\tilde{X};\mathbb{Z})$ is called $\omega$-effective if $A$ is represented by a pseudo-holomorphic stable map for all $\omega$-tamed almost complex structures. For effective classes $A$ and $A'$ in $H_2(\tilde{X};\mathbb{Z})$, we say that $A' < A$ if $p_* A - p_* A' \in H_2(X;\mathbb{Z})$ has positive pairing with the symplectic form on $X$.

Let $p : \tilde{X} \to X$ be the continuous blow down map defined as in (2), and

$$\sigma_1, \ldots, \sigma_{m_X}$$

be a basis of $H^*(X;\mathbb{R})$. Then, as observed in [Mc2], $p^* : H^*(X;\mathbb{R}) \to H^*(\tilde{X};\mathbb{R})$ is an injection, and

$$\gamma_j = p^* \sigma_j, \quad 1 \leq j \leq m_X,$$

$$\gamma_{j+m_X} = \delta_j \cup [E], \quad 1 \leq j \leq km_S,$$

generate $H^*(\tilde{X};\mathbb{R})$. Here we use $[E]$ to denote both the homology class of $E$ and the dual degree 2 cohomology class in $H^2(\tilde{X};\mathbb{R})$. This $[E]$ actually restricts to the class in $H^2(E;\mathbb{R})$ that we called $[E]$ in (13). In particular, for $1 \leq i \leq km_S$, the Poincaré Dual of $\gamma_{i+m_X}$ in $\tilde{X}$ is represented by a cycle lying inside $E$ which is Poincaré dual to $\delta_i$ in $E$.

We would order the following type of connected relative invariants.

**Definition 5.6.** A connected standard relative GW invariant of $(\tilde{X},E)$ is of the form

$$\langle \varpi | \mu \rangle_{\tilde{X},E} = \langle \gamma_{L_1}, \ldots, \gamma_{L_n} | \mu \rangle_{\tilde{X},E},$$

where $A$ is $\omega$-effective, $\mu$ is a standard weighted partition with $\sum_j \mu_j \leq A \cdot [E]$, and $L_i \leq m_X$, i.e. $\gamma_{L_i} = p^* \sigma_{L_i}$.

In this section we use $\Gamma(\varpi)|\mu$, instead of $\Gamma(\{(0, \gamma_{L_i})\})|\mu$, to denote the (connected) relative weighted graph of the relative invariant in Definition 5.6. We partially order such weighted graphs in the following way.

**Definition 5.7.**

$$\Gamma(\varpi')|\mu' \prec \Gamma(\varpi)|\mu$$

if one of the conditions below holds

(1) $A' < A$,

(2) equality in (1) and the arithmetic genus satisfies $g' < g$,

(3) equality in (1-2) and $||\varpi'|| < ||\varpi||$. 

(4) equality in (1-3) and $\text{deg}_{S}(\mu') > \text{deg}_{S}(\mu)$,
(5) equality in (1-4) and $\mu' \overset{l}{\prec} \mu$,
where $\|\varpi\|$ denotes the number of insertions of $\varpi$.

All these inequalities are designed so that the dimension of the moduli space satisfying the larger constraint/condition is larger. This explains the seemingly strange conditions (4) and (5) where the inequalities are reversed.

The relative invariants obtained by taking disjoint unions of the connected invariants in Definition 5.6 are called the standard relative invariants of $(\tilde{X}, E)$ (with a disconnected domain). We extend the partial ordering to a disjoint union $\Gamma^* (\varpi) | \mu$ in the obvious way.

**Remark 5.8.** It is easy but important to observe that this extended partial order $<\overset{\text{ext}}{\prec}$ is preserved under disjoint union.

**Remark 5.9.** If we are only interested in genus zero invariants, then we can replace $g' < g$ in (2) by the inequality of the number of connected components, $n' > n$.

**Lemma 5.10.** Given a standard relative invariant, there are only finitely many standard relative invariants lower in the partial ordering. In particular, there is a minimal standard invariant with nonzero value.

**Proof.** There is a lower bound on $\omega(A)$ among all effective classes. \hfill $\square$

We call a partially order set with the above property lower bounded.

### 5.2. Trading insertions

In this subsection, we associate an absolute descendent invariant of $X$ to each standard relative descendent invariant of $(\tilde{X}, E)$. The critical step is to trade a relative insertion with an absolute descendent insertion of the blown-down manifold. This can be thought as the relative-absolute correspondence for insertions.

For a relative insertion $(m, \delta)$ with $\delta = \pi^* \theta_i \cup [E] \tilde{j} \in H^*(E; \mathbb{R})$, we associate the absolute descendent insertion on $X$ supported on $S$, where

$$\tilde{\delta} = \theta_i \cup [S],$$
$$d(m, \delta) = km - k + \frac{1}{2} \text{deg}_{T}(\delta) = km - k + j.$$

Here $[S]$ denote the class of a Thom form of the normal bundle to $S$ in $X$.

Notice that when $k = 1$, $S = E$ and $j$ is always zero, so we simply have

$$\tilde{\delta} = \delta_i \cup [E], \quad d(m, \delta) = m - 1.$$

It is convenient to view $[E]$ as the class of a Thom form of the normal bundle to $E$ in $\tilde{X}$ supported near the symplectic divisor $E$. In terms of homology constraints, $\delta$ and $\delta$ correspond to the same cycle lying inside $E$ as previously remarked.
Definition 5.11. Given a standard (relative) weighted partition $\mu$, let
\[ d_i(\mu) = d(\mu_i, \delta_{K_i}) = k\mu_i - k + \frac{1}{2}\deg_f(\delta_{K_i}), \]
and
\[ (18) \tilde{\mu} = \{\tau_{d_1(\mu)}(\tilde{\delta}_{K_1}), \ldots, \tau_{d_l(\mu)}(\tilde{\delta}_{K_l(\mu)})\}. \]

Definition 5.12. The absolute descendent invariant associated to a standard relative invariant
\[ \langle \Gamma^\bullet(\varpi)\mid \mu \rangle_{\tilde{X}, E} \]
is then
\[ \langle \tilde{\Gamma}^\bullet(\varpi, \tilde{\mu}) \rangle^X. \]
Here $\tilde{\Gamma}^\bullet$ is an absolute graph, obtained from the relative graph $\Gamma^\bullet$ by changing the homology class for each vertex from $A$ to $p_*(A)$. And the insertions $\gamma_{L_j} = p^*\sigma_{L_j}$ in $\varpi$ are replaced by $\sigma_{L_j}$.

We therefore consider the following absolute descendent invariants of $X$.

Definition 5.13. An absolute descendent invariant of $X$ is said to be an absolute descendent invariant relative to $S$ if its descendent insertions are supported on $S$, i.e. have the form $\tau_d(\tilde{\delta})$ where $\tilde{\delta} = \theta_i \cup [S]$. Such an invariant is called a colored standard absolute descendent invariant of $X$ relative to $S$ if the insertions are divided into two collections, $\varpi$ and $\tilde{\mu}$, with each insertion in $\varpi$ being of the form $\sigma_L$, and each insertion in $\tilde{\mu}$ being of the form $\tau_d\tilde{\delta}_K$.

When $k = 1$, a colored absolute descendent invariant of $X$ relative to $S$ is called admissible if $\sum_j \mu_j = A \cdot [E]$. Notice that in this case, we simply have $d_i(\mu) = \mu_i - 1$, and $S = E$, so
\[ \tilde{\mu} = \{\tau_{\mu_i-1}(\theta_{K_1} \cup [E]), \ldots, \tau_{\mu_l(\mu)-1}(\theta_{K_l(\mu)} \cup [E])\}. \]
Therefore $\langle \tilde{\Gamma}^\bullet(\varpi, \tilde{\mu}) \rangle^X$ agrees with the one in [MP]. Thus every relative invariant gives rise to an admissible absolute invariant relative to $S$. Notice that, after possibly adding a number of $E$ insertions, every non-descendent absolute invariant is admissible for at least one separation.

The following observation is crucial.

Lemma 5.14. If $\mu \neq \mu'$, then $\tilde{\mu} \neq \tilde{\mu}'$. Therefore there is a natural bijection between the set of colored weighted absolute graphs $\tilde{\Gamma}^\bullet(\varpi, \tilde{\mu})$ relative to $S$ and the set of weighted relative graphs in $\tilde{X}$ relative to the exceptional divisor $E$ in the blow up $\tilde{X}$ of $X$ along $S$ in the case $k > 1$, and it is still true in the case $k = 1$ if we restrict to the admissible ones.

Proof. It suffices to show that $d(m, \delta)$ uniquely determines $m$ and $\deg_f(\delta)$. The point is that $0 \leq \deg_f(\delta) \leq k - 1$. Thus $m$ and $\deg_f(\delta)$ are uniquely determined by the integral division of $d(m, \delta)$ by $k$. \qed
Hence we can and will order the set of colored weighted absolute graphs $\tilde{\Gamma}(\varpi, \tilde{\mu})$, and in the case $k = 1$, the admissible ones, in the same way as the weighted relative graphs. \footnote{Notice that it is possible that several relative invariants correspond to the same absolute invariant, because some $\sigma_i$ might be expressed as $\theta_i \cup [S]$.}

Let $I$ be the partially order set of standard weighted relative graphs $\Gamma^*(\varpi)|\mu$. Consider the infinite dimensional vector space $R^I_{\tilde{X}, E}$, whose coordinates are ordered in the way compatible with the partial order. Given a standard weighted graph $\Gamma^*(\varpi)|\mu$, we have the relative invariant $\langle \Gamma^*(\varpi)|\mu \rangle^{\tilde{X}, E}$.

From the numerical values we can form a vector $v_{\tilde{X}, E} \in R^I_{\tilde{X}, E}$

given by the numerical values. By Lemma 5.14, $I$ is also the partially ordered set of colored standard weighted absolute graphs $\tilde{\Gamma}(\varpi, \tilde{\mu})$ in the case $k > 1$ and the admissible ones in the case $k = 1$. Hence we also have the corresponding vector space $R^I_{X,S}$ and a vector $v_{X,S} \in R^I_{X,S}$

given by the numerical values of the absolute invariants of $X$ relative to $S$, $\langle \tilde{\Gamma}(\varpi, \tilde{\mu}) \rangle^{X}$. 

5.3. Full correspondence. In this subsection we prove the Relative/Absolute correspondence in the following form.

**Theorem 5.15.** There is an invertible lower triangular linear transformation

$$A_S : R^I_{\tilde{X}, E} \rightarrow R^I_{X,S}$$

such that (i) the coefficients of $A_S$ are local in the sense of being dependent only on $S$ and its normal bundle; (ii) $A_S(v_{\tilde{X}, E}) = v_{X,S}$.

In particular, $v_{\tilde{X}, E}$ and $v_{X,S}$ determine each other.

Moreover, if $I_{pt} \subset I$ denotes the subset indexed by the standard relative invariants of $(\tilde{X}, E)$ with the first insertion being the point insertion, then $A_S$ restricts to an invertible lower triangular transformation from $R^{I_{pt}}_{\tilde{X}, E}$ to $R^{I_{pt}}_{X,S}$.

Finally, if $I_{0,pt} \subset I_{pt}$ denotes the subset of genus zero invariants, then $A_S$ further restricts to an invertible lower triangular transformation from $R^{I_{0,pt}}_{\tilde{X}, E}$ to $R^{I_{0,pt}}_{X,S}$.

**Proof.** The idea is as follows. Since the disjoint union preserves the order of graphs, it is enough to prove such a correspondence for connected invariants.

We hence consider a connected relative invariant $\langle \Gamma(\varpi)|\mu \rangle^{\tilde{X}, E}$ of $(\tilde{X}, E)$ and the associated absolute invariant $\langle \tilde{\Gamma}(\varpi, \tilde{\mu}) \rangle^{X}$ of $X$ relative to $S$. As
mentioned in \[2.2.2\] \(X^- = X^-,\) the \(-\) piece of the symplectic cut of \(X\) along a normal sphere bundle over \(S\). Thus we have the degeneration of \(X\) into \((\tilde{X} = X^-, E)\) and \((\mathbb{P}(N_{S|X} \oplus \mathbb{C}), E) = (\mathbb{X}^+, E)\). We apply the degeneration formula to this connected absolute invariant \((\tilde{\Gamma}(\varpi, \tilde{\mu}))^X\) of \(X\) distributing all the \(\tilde{\mu}\) insertions to the \(\mathbb{P}^k\) bundle side. Then, the degeneration formula can be immediately interpreted as expressing the absolute invariant \((\tilde{\Gamma}(\varpi, \tilde{\mu}))^X\) as a linear combination of relative invariants of \((\tilde{X}, E)\) with the coefficients being essentially certain relative invariants of the projective bundle. With the preferred distribution of insertions, the original graph \(\Gamma(\varpi)|\mu\) turns out to be the largest weighted relative graph appearing in the linear combination.

We only prove the assertion for the case with the first insertion being the point insertion, i.e. the case of \(I_{pt}\). The proof of the general case is the same (and easier).

The proof will consists of several steps. In the first step we set up how the degeneration formula is applied.

**Step I–Set up.** We begin with a connected standard weighted relative graph

\[\Gamma([pt], \varpi)|\mu\]

with the vertex decorated by \((g, A)\). The associated connected colored standard absolute descendent invariant of \(X\) relative to \(S\) can be written as

\[\langle [pt], \varpi, \tilde{\mu} \rangle^X_{g, p_* (A)}\]

To apply the degeneration formula let us first explicitly make the preferred distribution of insertions mentioned above.

The classes \(\tilde{\delta}_{K_i}\) are supported on \(S\) and so can be represented by forms with support near \(S\). Hence we just distribute all of them to the \((\mathbb{P}(N_{S|X} \oplus \mathbb{C}), E)\) side, i.e. we set \(\tilde{\delta}^-_{K_i} = 0\) in \((\mathbb{X}^+, E)\). Recall \(\varpi\) consists of insertions of the form \(\sigma_{L_j}\) for \(L_j \leq m_X\) \((\text{cf } (15))\). For each such an insertion we set on the \((\tilde{X}, E)\),

\[\sigma^-_{L_j} = \gamma_{L_j},\]

with an appropriate extension \(\sigma^+_{L_j}\) to the positive side. In particular, the relative invariants of \((\tilde{X}, E)\) appearing in the degeneration formula are all standard invariants as in Definition \[5.13\] and so can be ordered.

It is easy to see that there are no vanishing cycles in this case, hence \([\pi_*(A)] = \pi_*(A)\). When \(k \geq 2\) this is observed in \[LR\]. The point is that any degree 2 homology class \(B\) of \(X\) can be represented by a surface away from \(S\). Thus \(\pi_*(B)\) is represented by the same surface viewed as a surface in \(\tilde{X}\). We can extend \(p : \tilde{X} \to X\) from \(\tilde{X}\) to \(\tilde{X} \cup E \mathbb{P}^k\) bundle by collapsing the \(\mathbb{P}^k\) fibers of the \(\mathbb{P}^k\) bundle and still denote it by \(p\). Composing with this \(p\), we conclude that \(p_* \pi_*(B) = B\). Hence \(\pi_*(B) = 0\) if and only if \(B = 0\). In the case \(k = 1\), \(\tilde{X} = X\) and \(\pi_* = \phi_*\) where \(\phi\) is the inclusion of \(X\) into its union with the \(\mathbb{P}^1\) bundle. Thus the composition \(p_* \pi_*\) is actually the identity map on \(H_*(X; \mathbb{R})\).
Therefore, by the degeneration formula, we have

\[(19) \quad \langle [pt], \varpi, \mu \rangle^X = \sum \langle \Gamma \circ ([pt], \varpi_1) \rangle^\Delta(\eta) \langle \Gamma_+ (\varpi_2, \tilde{\mu}) \rangle^{p(N_S|X + \mathbb{C})_E}.\]

The sum on the right is over all \((g, p_\ast, ||\varpi|| + ||\tilde{\mu}|| + 1)\)-graphs, including all distributions of the insertions \(\varpi\) and all standard intermediate weighted partitions \(\eta\). Here \(\Delta(\eta) = \prod r \eta \cdot |\text{Aut}(\eta)|\), and \(\tilde{\eta}\) is the dual partition of \(\eta\).

Since the basis \(\{\delta_i\}\) is self-dual, \(\tilde{\eta}\) is still a standard weighted partition. The relative GW invariants on the right are possibly disconnected.

As mentioned our main claim is that \(\Gamma([pt], \varpi)\) is the largest weighted relative graph among (connected or not) weighted relative graphs appearing in the linear combination \(19\). Of course we are only interested in terms with nonzero coefficients. Our strategy, following \([MP]\), involves finding conditions for which the relevant relative invariants of \((\mathbb{P}(\mathcal{N}_S|X + \mathbb{C}), E)\) are nonzero. We use a fibred almost complex structure \(J\) on \(\mathbb{P}(\mathcal{N}_S|X + \mathbb{C})\) to evaluate such invariants.

Let \(f_- : \mathbb{C}^- \to \tilde{X}\) and \(f_+ : \mathbb{C}^+ \to \mathbb{P}(\mathcal{N}_S|X + \mathbb{C})\) be elements of the relative moduli spaces. Both \(\mathbb{C}^-\) and \(\mathbb{C}^+\) might be disconnected.

**Step II–The coefficient of** \(\Gamma([pt], \varpi)\). In this step we show that the coefficient \(C_0\) of \(\Gamma([pt], \varpi)\) is nonzero.

In this case the splitting of \(p_\ast A\) is

\[(20) \quad A_- = A, \quad A_+ = (A \cdot [E]) F\]

where \(F\) is the fiber class of \(\mathbb{P}(\mathcal{N}_S|X + \mathbb{C})\).

Thus for each connected component on \(\mathbb{P}(\mathcal{N}_S|X + \mathbb{C})\) side, we are reduced to the following kind of connected relative invariants of the fiber \((\mathbb{P}^k, \mathbb{P}^{k-1})\),

\[(21) \quad \langle \tau_{nd-1-j}[pt] \mid D^j\rangle_{\mathbb{P}^k, \mathbb{P}^{k-1}}^{\mathbb{P}^k, \mathbb{P}^{k-1}}\]

with \(j = \deg f(\delta_{K_i})\), \(d\) a positive integer, and \(D\) the hyperplane class of \(H^*(\mathbb{P}^{k-1}; \mathbb{Z})\).

For the above relative invariants \(21\) of \((\mathbb{P}^k, \mathbb{P}^{k-1})\) the answer is known when \(k = 1\) (see \([OP]\)). It follows that in this case the coefficient \(C_0\) is given by (see \([MP]\))

\[\prod_j \frac{1}{(\mu_j - 1)!} (A \cdot [E])^{1(\mu)} \neq 0,\]

where \(1(\mu)\) is the number of \((1, \delta_1) = (1, 1)\) in \(\mu\).

The computation of \(21\) in the general case is completed in Theorem 7.1. Consequently, when \(k > 1\), \(C_0\) is the product of rational numbers of the form

\[\frac{1}{d^{k-j}(d-1)!^k}.\]

In particular, \(C_0\) is nonzero as well.

**Step III–Curve configuration.** In this step we conclude that any configuration that might occur with nonzero coefficient is no bigger than the
extremal configuration where \( C_\pm \) is a connected genus \( g \) curve, and \( C_\pm \) consists of \( l(\eta) \) rational connected components each having exactly one relative marking (i.e. totally ramified over the zero section \( E_0 \)).

We first argue the class \( A_+ \) of a largest weighted graph must be a multiple of the fiber class. One splitting of \( p_*A \) is \( A_- = A, A_+ = (A \cdot [E])F \) where \( F \) is the fiber class of \( \mathbb{P}(N_{S|X} \oplus \mathbb{C}) \). Any other splitting differs by a class \( \beta \) in \( H_2(E; \mathbb{Z}) \). An equivalent description of the splitting is that

\[
p_*A_- + t_*t_*A_+ = p_*A,
\]

where \( t : S \to X \) is the inclusion and \( t : \mathbb{P}(N_{S|X} \oplus \mathbb{C}) \to S \) is the projection.

Assume \( A_+ = (A \cdot [E])F + \beta \) is an effective curve class of \( \mathbb{P}(N_{S|X} \oplus \mathbb{C}) \). Since the projection \( t : \mathbb{P}(N_{S|X} \oplus \mathbb{C}) \to S \) is pseudo-holomorphic, \( t_*((A \cdot [E])F + \beta) = t_*t_*(\beta) \) is either the zero class or an effective curve class of \( S \) with respect to \( J|_S \). In the latter case, since \( J|_S \) is compatible with \( \omega|_S \), \( t_*\beta \) has positive symplectic area in \( S \) with respect to \( \omega|_S \) and hence has positive symplectic area in \( X \) with respect to \( \omega \); i.e. \( A_- = A - \beta \) is smaller than \( A \).

Such a term on the right of (19) involves only standard relative weighted graphs of \((X, E)\) lower in the partial order than \( \Gamma([pt], \varpi)|\mu \).

Let us focus on the terms with \( t_*\beta = 0 \). In this case \( \beta \) is a multiple of the fiber class \( F \). Therefore \( A_+ \) is also such a class.

Fix a splitting of \( p_*A \) with \( A_+ \) a multiple of the fiber class. Since the map \( t \circ f_+ : C_+ \to S \) is holomorphic, it maps every component of \( C_+ \) to either a point or a holomorphic curve in \( S \). Since they together represent the zero class in \( S \), each image must be a point. Hence the restriction of \( f_+ \) to each connected component of \( C_+ \) also represents a multiple of the fiber class.

Next we show that \( C_- \) must be a connected curve of genus \( g \). Since \( C_- \) is a disjoint union of connected components which forms a part of a degenerate genus \( g \) Riemann surface, the sum of the arithmetic genera will be less than or equal to \( g \). If \( C_- \) has more than one connected component, its arithmetic genus will hence be strictly smaller than \( g \) and therefore the graph is of lower order.

**Step IV—\( \varpi \) insertions.** In this short step we deal with the distribution of \( \varpi \) insertions. If any of the insertions of \( \varpi_2 \) is not empty, then the weighted relative graph of \((X, E)\) on the right of (19) is smaller than \( \Gamma([pt], \varpi)|\mu \).

**Step V—\( \mu \) insertions.** In this step we show that \( \eta \) in the largest weighted relative graph with possibly nonzero coefficient is equal to \( \mu \). This step is a bit long and so we break into 4 sub-steps.

**V.1 Set up.** It remains to analyze the case in which the only non-relative insertions on the \( \mathbb{P}^k \)-bundle side are given by \( \tilde{\mu} \). From now on we focus on the projective bundle side.

The distribution of the \( l(\mu) \) insertions of \( \tilde{\mu} \) among the \( l(\eta) \) rational components of \( C_+ \) decomposes the relative insertion \( \mu \) into \( l(\eta) \) cohomology weighted partitions

\[
\pi^{(1)}, \ldots, \pi^{(l(\eta))},
\]
By summing over all ramified with order \( \eta \) have total degree. Since the to say that
\[
\deg_S(\pi^{(r)}) + \deg_S(\rho_r) = \dim S
\]
for each \( r \).

Notice that the curves in the fiber class live in the \( \mathbb{P}^k \) fibers. Then, for each \( r \), in order for the multiple fiber class relative invariant of the \( \mathbb{P}^k \)-bundle with the insertions \( < \tilde{\pi}^{(r)}| (\eta_r, \tilde{\rho}_r) > \) to be nonzero, the projections of cycles representing the Poincaré dual of \( \delta \)'s and \( \tilde{\rho}_r \) have to intersect in \( S \). This is to say that
\[
\deg_S(\tilde{\pi}^{(r)}) = \sum \deg_S(\delta_{i_j}^{(r)}) \leq \dim(S) - \deg_S(\tilde{\rho}_r) = \deg_S(\rho_r).
\]
By summing over all \( r \), we conclude that \( \deg_S(\mu) \leq \deg_S(\eta) \), and equality holds if and only if \( \deg_S(\pi^{(r)}) = \deg_S(\rho_r) \) for all \( r \).

Thus if \( \deg_S(\rho_r) > \deg_S(\pi^{(r)}) \) for some \( r \), then the relative invariant of \( (\tilde{X}, E) \) on the right of \( \text{(19)} \) is smaller than \( \langle [pt], \varpi | \mu \rangle_{\tilde{X}, E} \). It remains to analyze the case in which
\[
\text{(22)} \quad \deg_S(\rho_r) = \deg_S(\pi^{(r)}) = \sum \deg_S(\delta_{i_j}^{(r)}),
\]
i.e. \( \deg_S(\pi^{(r)}) + \deg_S(\rho_r) = \dim(S) \) for all \( r \).

**V.3 Dimension formula.** Since the \( r \)-th component of \( C_+ \) is totally ramified with order \( \eta_r \), it represents the class \( \eta_r F \). Hence the dimension of the moduli space for the \( \eta_r \)-totally ramified relative invariant of the \( \mathbb{P}^k \)-bundle is
\[
2 < c_1(\mathbb{P}^k, \text{bdl}), \eta_r F > + \dim_{\mathbb{R}}(X) - 6 + 2 - 2\eta_r + 2l(\pi^{(r)})
\]
\[
= (2k + 2)\eta_r + \dim_{\mathbb{R}}(X) - 4 - 2\eta_r + 2l(\pi^{(r)})
\]
\[
= 2k\eta_r + \dim_{\mathbb{R}}(X) - 4 + 2l(\pi^{(r)}).
\]
On the other hand, notice that
\[
\deg(\tilde{\delta}_i) = \deg_S(\delta_i) + 2k.
\]
Thus the insertions
\[
< \tilde{\pi}^{(r)}| (\eta_r, \tilde{\rho}_r) >
\]
have total degree
\[
\deg(\tilde{\pi}^{(r)}) + \deg(\tilde{\rho}_r) + \sum_{j=1}^{l(\pi^{(r)})} 2d_j(\mu)
\]
\[
= 2kl(\pi^{(r)}) + \deg_S(\pi^{(r)}) + \deg_S(\tilde{\rho}_r) + \deg_f(\tilde{\rho}_r)
\]
\[
+ \sum_{j=1}^{l(\pi^{(r)})} 2(k\mu_{n_j}^{(r)} - k + \frac{1}{2} \deg_f(\delta_{K_j}))
\]
\[
= 2\sum_{j=1}^{l(\pi^{(r)})} k\mu_{n_j}^{(r)} + \dim(S) + \deg_f(\pi^{(r)}) + \deg_f(\tilde{\rho}_r).
\]
Comparing the two formulas and using $\dim \mathbb{R} X = \dim \mathbb{R} S + 2k$, we conclude that

\begin{equation}
2k\eta_r + 2k - 4 + 2l(\pi^{(r)}) = \sum_{j=1}^{l(\pi^{(r)})} 2k\mu^{(r)}_{n_j} + \deg f(\pi^{(r)}) + \deg f(\tilde{\rho}_r).
\end{equation}

In fact, since we have assumed that (22), we can arrive at the same formula by simply computing the relevant relative invariant of $(\mathbb{P}^k, \mathbb{P}^{k-1})$.

**V.4 The case of $k = 1$.** For a $\mathbb{P}^1$-bundle, we have $\deg f = 0, \deg S = \deg$ and $k = 1$. The equation (23) simplifies to

\begin{equation}
\eta_r - 1 = \sum_{j=1}^{l(\pi^{(r)})} (\pi^{(r)}_j - 1) = \sum_{j=1}^{l(\pi^{(r)})} (\mu^{(r)}_{n_j} - 1)
\end{equation}

for each $r$. Notice that when $\pi^{(r)}$ is empty, $l(\pi^{(r)}) = 0$ and the right hand of formula (24) is understood to be 0.

Now consider the weighted partition $\pi^{(r)}$ containing a maximal element $(\mu_1, \delta_{i_1})$ of $\mu$ in the size ordering. Notice that $\mu^{(r)}_{n_j} \geq 1$ by the definition of relative invariants. Hence, by formula (21), either $\eta_r > \mu_1$, or

$\eta_r = \mu_1$ and all the other pairs of $\pi^{(r)}$ are of the form $(1, \delta)$.

In the second case, according to (22), either $\deg(\rho_r) > \deg(\delta_{i_1})$, or

$\deg(\rho_r) = \deg(\delta_{i_1}), \tilde{\rho}_r$ is dual to $\delta_{i_1}$,

and all other pairs of $\pi^{(r)}$ are of the form $(1, 1)$. In fact, since the basis $\{\delta_i\}$ is self-dual, we must have $\rho_r = \delta_{i_1}$.

Therefore either $\eta$ is larger than $\mu$ in the lexicographic ordering and corresponds to a weighted relative graph of $(\tilde{X}, E)$ strictly lower than $\Gamma([pt], \varpi)|\mu$ in the $<$ ordering, or the maximal pairs of $\eta$ and $\mu$ agree.

We now repeat the above analysis for the next largest element of $\mu$ and continue until all the elements of $\mu$ not equal to the smallest pair $(1, \delta_1)$ are exhausted.

Now let us understand how the smallest terms $(1, \delta_1)$ in the lexicographic orderings in $\mu$ are distributed. We observe that formula (24) sums to

\begin{equation}
\sum_r \eta_r - l(\eta) = \sum_j \mu_j - l(\mu).
\end{equation}

Since $\sum_j \mu_j = A \cdot [E]$, we have $\sum_r \eta_r = \sum_j \mu_j$. Any pair of the form $(1, \delta_1)$ also corresponds to some $(\eta_r, \rho_r)$. Thus $C_+$ has exactly $l(\eta) = l(\mu)$ connected components and $\eta = \mu$, i.e. we recover $\Gamma([pt], \varpi)|\mu$ on the right of (19).

**V.5 The case of $k \geq 2$.** In this case $\pi^{(r)}$ cannot be empty by (23). This is because $\eta_r \geq 1$ and hence $2k\eta_r + 2k - 4 \geq 4k - 4$, while $\deg f \leq 2k - 2$. 

If \( \pi^{(r)} \) contains only one pair \((\mu_p, \delta_{ip})\), then we have \( \deg_S(\rho_r) = \deg_S(\delta_{ip}) \) by (22), and in fact \( \hat{\rho}_r \) is \( S \)-dual to \( (\delta_{ip}) \). If we write \( \rho_r = \pi^* \theta \cup [E]^{\deg_f(\rho_r)} \) and \( \delta_{ip} = \pi^* \theta' \cup [E]^{\deg_f(\delta_{ip})} \), what we have shown is that
\[
\theta = \theta'
\]
as \( \theta \) and \( \theta' \) are elements of the self-dual basis of \( S \).

In addition, by (23),
\[
\deg_f(\rho_r) = \deg_f(\delta_{ip}) + \deg_f(\hat{\rho}_r).
\]

Since \( 0 \leq \deg_f(\delta_{ip}) + \deg_f(\hat{\rho}_r) \leq 2k - 2 + 2k - 2 \),
we have
\[
-(2k - 2) \leq 2k(\eta_r - \mu_p) \leq (2k - 2).
\]
Therefore
\[
\eta_r = \mu_p.
\]
Furthermore, by (27) it implies
\[
\deg_f(\delta_{ip}) + \deg_f(\hat{\rho}_r) = 2k - 2.
\]
It follows that \( \deg_f(\rho_r) = \deg_f(\delta_{ip}) \). Therefore by (26) we must have
\[
\rho_r = \delta_{ip}.
\]

Since \( \pi^{(r)} \) cannot be empty, if each \( \pi^{(r)} \) contains at most one pair, we recover the weighted relative graph \( \Gamma([pt], \varpi) | \mu \) on the right of (19).

We argue now that if some \( \pi^{(r)} \) contains more than one pair, then the corresponding weighted relative graph of \((\tilde{X}, E)\) has either larger maximal tangency or has the same maximal tangency but larger \( \deg_f \).

Now consider the weighted partition \( \pi^{(r)} \) containing a maximal element \((\mu_1, \delta_{i_1})\) of \( \mu \) in the size ordering. We rewrite (23) as
\[
2k(\eta_r - \mu_1) = \sum_{l(\pi^{(r)}) \geq 3} 2k\mu_{n_1}^{(r)} - 1 + (2k\mu_{n_2} - 2k) + \deg_f(\pi^{(r)}) + \deg_f(\hat{\rho}_r).
\]
Each term is non-negative since \( \mu_i \geq 1 \) and \( k \geq 1 \). As \( k \) is assumed to be at least 2, all the terms are zero only if
\[
l(\pi^{(r)}) = 2, \quad \mu_{n_2} = 1, \quad \deg_f(\pi^{(r)}) = \deg_f(\hat{\rho}_r) = 0.
\]
In particular, \( \deg_f(\delta_1) = 0 \). Therefore the relative invariant with insertions \( <pt, \varpi|(\eta_r, \rho_r)> \) on the right of (19) is smaller since
\[
\deg_f(\rho_r) = k - 1 - \deg_f(\hat{\rho}_r) = k - 1 > 0 = \deg_f(\delta_1).
\]

**Step VI–Conclusion.** In summary we have shown so far that
\[
\langle \tilde{\Gamma}(\{pt\}, \varpi, \tilde{\mu}) \rangle_X = C_0 \langle \Gamma([pt], \varpi)|\mu \rangle_X^X, \hat{X},E + \text{lower order terms,}
\]
where \( C_0 \neq 0 \).
Viewing the invariants as the coordinates of the respective $R_{Ip}$, then (28) defines a transformation
$$A_S : R_{Ip}^{X,E} \to R_{Ip}^{\tilde{X},S}.$$ 
As $Ip$ is indexed by the partial order, this matrix is lower triangular with the $C_0$ as the diagonal entries. As the partial ordered is lower bounded, this lower triangular matrix is invertible. □

Remark 5.16. When $k = 1$ and $\sum_j \mu_j < A \cdot [E]$, the relative insertions of the largest invariant on the right of (28) are $\mu$ followed by $A \cdot [E] - \sum_j \mu_j$ pairs of $(1, 1)$. Notice that when $\sum_j \mu_j < A \cdot [E]$, the relative invariant $\langle [pt], \varpi | \mu \rangle_{g,A}^{\tilde{X},E}$ is zero by definition. What Theorem 5.15 says in this case is that $\langle [pt], \varpi, \tilde{\mu} \rangle_{g,p,A}^{X} \neq 0$.

6. Birational invariance

In this section let $\tilde{X}$ be the blow up of $X$ along a symplectic submanifold $Y$. Since GW invariants are unchanged under deformation, our main theorem is equivalent to the following theorem.

Theorem 6.1. $\tilde{X}$ is uniruled if and only if $X$ is uniruled.

Proof. Suppose $X$ is uniruled. Then there is a nonzero homology class $A \in H_2(X; \mathbb{Z})$, together with cohomology classes $\alpha_2, \cdots, \alpha_p \in H^*(X; \mathbb{R})$ such that the connected invariant
$$\langle [pt], \alpha_2, \cdots, \alpha_p \rangle_{0,A}^{X} \neq 0.$$ 
By the multi-linearity of GW-invariants of $X$, we can assume that $\alpha_i$ is one of the basis elements $\sigma_j$ as in (15). Consider the degeneration of $X$ into $(\tilde{X}, E)$ and $\mathbb{P}(NS_{X|X} \oplus \mathbb{C}), \mathbb{P}(NS_{X|X}))$ and apply the degeneration formula of 3.3 to this invariant. If we put the point insertion on the $(\tilde{X}, E)$ side and set
$$\alpha_i^- = \sigma_j^- = \gamma_j,$$
we find that $\tilde{X}$ is uniruled relative to $E$. In fact, there is a nonzero, possibly disconnected genus 0 relative invariant of the form
$$\langle \Gamma^*([pt], \varpi) | \mu \rangle_{\tilde{X},E},$$
where any insertion in $\varpi$ is of the form $\gamma_j$ with $1 \leq j \leq m_X$, $\mu$ is a standard weighted partition, and the connected component containing the $[pt]$ insertion has nonzero curve class. The relative invariant gives rise to a nonzero component of the vector $\nu_{\tilde{X},E}$ in $R_{Ip}^{\tilde{X},E}$. Apply the $I_{0,Ip}$ version of Theorem 5.15 with $S = E$, together with the fact that the blowup of $\tilde{X}$ along $E$ is still...
\(\tilde{X}\) with a deformation equivalent symplectic structure, we find that there is a nonzero, possibly disconnected genus 0 absolute descendant invariant of \(\tilde{X}\) with a \([pt]\) insertion in a connected component with nonzero curve class. Hence \(\tilde{X}\) is uniruled by Theorem 4.10.

Conversely, suppose that \(\tilde{X}\) is uniruled. Then there exists a homology class \(B \in H_2(\tilde{X};\mathbb{Z})\) and cohomology classes \(\beta_2, \ldots, \beta_p \in H^*(\tilde{X};\mathbb{R})\) such that the connected invariant

\[
\langle [pt], \beta_2, \ldots, \beta_p \rangle_{\tilde{X},B} \neq 0.
\]

By the multi-linearity of GW invariants of \(\tilde{X}\), we can assume that \(\beta_i\) is of the form \(\gamma_j\) as in (16). Furthermore, assume that \(1 \leq j_i \leq m_X\) if \(2 \leq i \leq l\), and \(j_i \geq m_X + 1\) if \(l + 1 \leq i \leq p\).

Now apply the degeneration formula to the degeneration of \(\tilde{X}\) along \(E\) into \((\tilde{X},E)\) and \((\mathbb{P}^1\text{-bundle},E)\), distributing the first \([pt]\) insertion to the \((\tilde{X},E)\) side and the insertion \(\beta_i\) with \(i \geq l + 1\) to the \(\mathbb{P}^1\text{-bundle}\) side, i.e. setting \([pt]^+ = 0\) and \(\beta_i^- = 0\) for \(i \geq l + 1\). For \(\beta_i\) with \(2 \leq i \leq l\) we set \(\beta_i^- = \beta_i\). Notice that, for such a \(\beta_i\), \(\beta_i^+\) is not necessarily zero. Then in the degeneration formula there must be a nonzero relative invariant of \((\tilde{X},E)\) of the form (29) where \(\varpi\) has at most \(l\) insertions of the form \(\gamma_j\) with \(1 \leq j \leq m_X\), \(\mu\) is a standard weighted partition, and the connected component containing the \([pt]\) insertion has nonzero curve class.

Hence the vector \(v_{\tilde{X},E} \in \mathbb{R}^{I_0,\text{pt}}_{\tilde{X},E}\) is nonzero. Now again apply the \(I_0,\text{pt}\) version of Theorem 5.15 but this time with \(S = Y\), we similarly conclude that \(X\) is uniruled.

\[\square\]

Remark 6.2. We could actually prove the following more precise result. If a connected genus 0 relative invariant \(\langle \Gamma([pt],\varpi)|\mu\rangle_{\tilde{X},E} \neq 0\) and has smallest positive symplectic area among all nonzero invariants, then \(\langle \tilde{\Gamma}([pt],\varpi,\tilde{\mu})\rangle_{\tilde{X}}\) is also nonzero. Moreover, \(A_{\Gamma}\) is the smallest uniruled class of \(\tilde{X}\).

Remark 6.3. It would be interesting to see whether the notion of strongly uniruled in Remark 4.7 is a birational cobordism property. There is also the notion of genus \(g\) uniruled in \([Lu2]\). We could not prove this is a birational cobordism property when \(g\) is positive. What we can show is that the notion of at most genus \(g\) uniruled is a birational cobordism property. \[2\]

Remark 6.4. There is another important birational property of projective manifolds, the rational connectedness, which could be defined via the existence of a (connected, possibly reducible) rational curve through any two given points. We could also define the symplectic analogue using connected genus zero GW invariant with two point insertions. But it is not yet known

\[2\]There is also the notion of strongly genus \(g\) uniruled in \([Lu2]\) and it is shown there that it is actually equivalent to strongly uniruled.
whether a projective rational connected manifold is symplectic rational connected. Moreover, we could not show that this notion is invariant under symplectic birational cobordisms. Our technique only proves the disconnected version. But for rational connectedness, unlike uniruledness, the disconnected version is strictly weaker.

### 7. Computing certain relative GW invariants of \((\mathbb{P}^n, \mathbb{P}^{n-1})\)

Suppose that \(D\) is the hyperplane class of \(\mathbb{P}^{n-1}\). In this section, we compute the connected genus zero relative GW invariants of \((\mathbb{P}^n, \mathbb{P}^{n-1})\),

\[
\langle \tau_{nd-1-j}[pt] \mid D^j \rangle_{dL}^{\mathbb{P}^n, \mathbb{P}^{n-1}},
\]

which are factors of the diagonal entries of \(A_S\).

**Theorem 7.1.** If \(D\) is the hyperplane class of \(\mathbb{P}^{n-1}\) and \(L\) is the line class of \(\mathbb{P}^n\) respectively, then for \(d \geq 1\) and \(0 \leq j \leq n-1\),

\[
\langle \tau_{nd-1-j}[pt] \mid D^j \rangle_{dL}^{\mathbb{P}^n, \mathbb{P}^{n-1}} = \frac{1}{d^{n-j}((d-1)!)^n}.
\]

We now outline the proof of Theorem 7.1. Choose homogeneous coordinates \([z_0 : z_1 : \cdots : z_n]\) on \(\mathbb{P}^n\). Let \((d)\) be the trivial partition of \(d\) of length one. Denote by \(H = \{z_0 = 0\}\) the infinity hyperplane \(\mathbb{P}^{n-1}_\infty\). Denote by \(\overline{M}_{0,1}^{\mathbb{P}^n, H}(d,(d))\) the moduli space of genus zero relative stable maps to \((\mathbb{P}^n, H)\) with one absolute marked point and one relative marked point with tangential order \((d)\). Here we simply choose the standard integrable complex structure \(J_{st}\) on \(\mathbb{P}^n\). It is well-known that the moduli spaces of genus zero (absolute) stable maps to \(\mathbb{P}^n\) with this choice of almost complex structure are actually smooth orbifolds of the correct dimension, i.e. they represent the virtual moduli cycle (see e.g. [FP]). Similar arguments show that the genus zero moduli spaces of relative stable maps to \((\mathbb{P}^n, H)\) with \(J_{st}\) are smooth orbifolds of corrected dimension as well. \(^{3}\) In particular, \(\overline{M}_{0,1}^{\mathbb{P}^n, H}(d,(d))\) is itself the virtual moduli cycle and hence

\[
\langle \tau_{nd-1-j}[pt] \mid D^j \rangle_{dL}^{\mathbb{P}^n, \mathbb{P}^{n-1}} = \int_{\overline{M}_{0,1}^{\mathbb{P}^n, H}(d,(d))} \psi^{nd-1-j} \wedge ev^{\mathbb{P}^n}[pt] \wedge ev^H D^j,
\]

where \(ev^{\mathbb{P}^n}\) is the evaluation map at the only absolute marked point \(x_1\).

Inspired by a calculation in the case \(n = 1\) in [OP], we consider

\[
V_1 = \overline{M}_{0,1}^{\mathbb{P}^n, H}(d,(d)) \cap (ev^{\mathbb{P}^n})^{-1}(p_0),
\]

where \(p_0 = [1 : 0 : \cdots : 0]\). We then identify in Lemma 7.2 a smooth orbifold \(V_d\) in \(V_1\) and transform the computation to an integral over \(V_d\) in Lemma 7.3. Finally we apply the (ordinary) localization technique to evaluate the integral in Lemma 7.4.

\[^{3}\text{We are indebted to R. Vakil for confirmation on this.}\]
Recall that $\mathcal{L}_1$ is the orbifold cotangent line bundle whose fiber is the cotangent line at the unique absolute marked point. We will simply denote it by $L$.

**Lemma 7.2.** Let $V_i \subset V_1, 1 \leq i \leq d$, be the subspace of relative stable $J$-holomorphic maps with ramification order at least $i-1$ at the absolute marked point $x_1$. Then $V_{i+1} \subset V_i$ is the zero locus of a transverse section $s^{(i)}$ of $\oplus_n \mathcal{L}^i$ over $V_i$, where $\oplus_n \mathcal{L}^\infty$ is the direct sum of $n$ copies of the $i$-th power of $\mathcal{L}$. In particular, the smooth orbifold $V_d$ represents the class

$$\frac{1}{[(d-1)!]^n} c_1(\mathcal{L}_1)^{n(d-1)}$$

in $V_1$.

**Proof.** $\overline{\mathcal{M}}_{0,1}^{n,H}(d, (d))$ consists of two types of relative stable maps: the ones with the rigid target $\mathbb{P}^n$, and the ones with a non-rigid target $\mathbb{P}^n[m]$ for some $m \geq 1$. Recall that $\mathbb{P}^n[m]$ is obtained by gluing $\mathbb{P}^n$ with $m$ copies of the projective bundle $\mathbb{P}(\mathcal{O}_H(1) \oplus \mathcal{O})$ along the zero sections and the infinity sections. Denote by $H_\infty$ the last infinity divisor of $\mathbb{P}^n[m]$, which is just $H$ in the case $m = 0$.

If $f \in V_d$ is mapped to a non-rigid target $\mathbb{P}^n[m]$ for some $m \geq 1$, due to the maximal ramification requirement at $x_1$ and $y_1$, the rubber part must be a degree $d$ covering onto a chain of $\mathbb{P}^1$—fibers of $\mathbb{P}^n[m]$. But such maps are invariant under $\text{Aut}_{\mathbb{P}^n[m]}$, which are in fact $\mathbb{Z}_d$—orbidol parametrized by lines connecting $p_0$ to points in $H = \mathbb{P}^n_{\infty}$.

For a relative stable map $(C, x_1, y_1; f) \in V_1$ with $x_1$ and $y_1$ as the absolute and relative marked points respectively, $x_1$ is mapped to $p_0$ and the contact order of $f$ to $H_\infty$ at $y_1$ is $d$.

Let $C_1$ be the irreducible component of $C$ containing $x_1$. Then the fiber $T_{x_1} C$ of the orbifold complex line bundle $L$ at $(C, x_1, y_1; f)$ is naturally identified with $T_{x_1} C_1$. If $C_1$ is contracted by $f$ to $p_0$, then,

(i) $C_1$ must meet $\overline{C \setminus C_1}$ at no less than 2 points by stability, and

(ii) $\overline{C \setminus C_1}$ must be connected by the imposed monodromy $(d)$ at $H_\infty$.

However conditions (i) and (ii) violate the genus constraint $g(C) = 0$. Thus the restriction of $f$ to $C_1$ is not a constant map for any $f \in V_1$.

For $f \in V_1$ we have the pull-back map on the cotangent spaces:

$$f^* : T_{p_0}^* \mathbb{P}^n \to T_{x_1}^* C_1 = T_{x_1}^* C = L | f$$

Consider homogeneous coordinates $[w_0 : w_1]$ for $C_1 \cong \mathbb{P}^1$ with $x_1 = [1 : 0]$. Take the complex coordinates $[1 : Z_1 : \cdots : Z_n]$ where $Z_i = \frac{w_i}{w_0}$ over the neighborhood $U_0 = \{z_0 \neq 0\} \subset \mathbb{P}^n$ of $p_0$ and the complex coordinate $w = \frac{w_1}{w_0}$ over the neighborhood $C_1 - [0 : 1]$ of $x_1$. Then we have the canonical isomorphisms,

$$T_{p_0}^* \mathbb{P}^n \cong \mathbb{C}^n = \text{span}_\mathbb{C} \{dz_1, \cdots, dz_n\}, \quad T_{x_1}^* C_1 \cong \text{span}_\mathbb{C} \{dw\}.$$
Write the restriction of \( f \) to \( C_1 - [0 : 1] \) as \( f(w) = (Z_1(w), \cdots, Z_n(w)) \). Observe that the map \( f^* \) in (32) is the dual of the differential of \( f(w) \) at \( w = 0 \). Thus the pullback map \( f^* \in \text{Hom}(C^n, \mathcal{L} | f) \) in (32) yields the following section \( s^{(1)} \) of the direct sum \( \oplus_n \mathcal{L} \) of \( n \) copies of \( \mathcal{L} \),

\[
s^{(1)} : \quad V_1 \longrightarrow \oplus_n \mathcal{L}
\]

\[
(C, x_1, y_1; f) \longrightarrow (Z'_1(0)dw, \cdots, Z'_n(0)dw).
\]

Denote the zero locus \( Z(s^{(1)}) \) by \( V_2 \). Clearly \( V_2 \subset V_1 \) is the subspace of maps which have ramification order at least 1 at \( x_1 \).

Denote by \( Ds^{(1)} \) the linearization of \( s^{(1)} \), which is independent of the choice of local trivializations over \( V_2 = Z(s^{(1)}) \).

For \( f \in V_1 \) consider \( n \) holomorphic tangent vector fields \( \xi_i \in T_f V_1 \), \( 1 \leq i \leq n \) vanishing at \( x_1 \), and when restricted to \( C_1 - [0 : 1] \), \( \xi'_i(w) = (0, \cdots, 1, \cdots, 0) \) with the nonzero component only in the \( i \)-th entry.

Then we have \( (Ds^{(1)})_f(\xi_i) = (0, \cdots, dw, \cdots, 0) \) with the nonzero component only in the \( i \)-th component. Thus \( Ds^{(1)} \) at a point of \( Z(s^{(1)}) \) is surjective. Therefore the cycle \( Z(s^{(1)}) \) represents \( c_1(\mathcal{L})^{\otimes n} \cap [V_1] \). When restricted to \( Z(s^{(1)}) \), the pull-back map on the second differential of \( f \) at \( x_1 \) will give rise to a canonical section \( s^{(2)} \in H^0(Z(s), \oplus_n \mathcal{L}^{\otimes 2}) \), i.e.

\[
s^{(2)} : \quad V_2 \longrightarrow \oplus_n \mathcal{L}^{\otimes 2}
\]

\[
(C, x_1, y_1; f) \longrightarrow (Z''_1(0)dw \otimes dw, \cdots, Z''_n(0)dw \otimes dw),
\]

where \( \oplus_n \mathcal{L}^{\otimes 2} \) stands for the direct sum of \( n \) copies of \( \mathcal{L}^{\otimes 2} \). It is easy to see that \( Z(s^{(2)}) \subset Z(s^{(1)}) \) is the subspace where \( x_1 \) has ramification order at least 2 over \( p_0 \). Hence the cycle \( Z(s^{(2)}) \) represents the class \( [2c_1(\mathcal{L})]^n \).

After iterating the above construction, we conclude that \( [(d-1)!]^n \psi^{n(d-1)} \) is represented by \( V_d \).

\( \square \)

Theorem 7.1 is an immediate consequence of Lemmas 7.3 and 7.4.

**Lemma 7.3.** We have

\[
\langle \tau_{nd-1-j}[pt] \mid D^j \rangle_{dL} = \frac{1}{[(d - 1)!]^n} \int_{V_d} \psi^{n-1-j} \wedge \text{ev}^H \wedge D^j,
\]

where \( \psi = c_1(\mathcal{L}) \) and \( \text{ev}^H \) is the evaluation map at the only relative marked point \( y_1 \).
Proof. By Lemma 7.2 we have

\[
\langle \tau_{nd-1-j}[pt] \mid D^j_{dL} \rangle_{\mathbb{P}^n, \mathbb{P}^{n-1}} = \int_{\overline{\mathcal{M}}_{0,1}^{\mathbb{P}^n,H}(d,(d))} \psi^{nd-1-j} \wedge ev_{\mathbb{P}^n}^*[pt] \wedge ev^{H*}D^j = \int_{V_d} \psi^{n-1-j} \wedge ev^{H*}D^j = \frac{1}{[(d-1)!]^n} \int_{V_d} \psi^{n-1-j} \wedge ev^{H*}D^j.
\]

□

To evaluate the last integral over the smooth orbifold $V_d$ we use localization.

Lemma 7.4. We have

\[
\int_{V_d} \psi^{n-1-j} \wedge ev^{H*}D^j = \frac{1}{d^{n-j}}.
\]

Proof. Consider the action of the group $T \cong (\mathbb{C}^*)^{n+1}$ of diagonal matrices on $\mathbb{P}^n$ with pairwise distinct weights $\lambda_0, \cdots, \lambda_n$. The $T$–action on $\mathbb{P}^n$ has $n + 1$ fixed points $p_i = [0 : \cdots : 0 : 1 : 0 \cdots : 0], \ i = 0, 1, \cdots, n$. Denote by $\ell_{ij} = \ell_{ji}, \ i \neq j$, the line in $\mathbb{P}^n$ passing through $p_i$ and $p_j$. Since the infinity hyperplane $H$ is invariant under the $T$–action, there is a natural $T$–action on $\overline{\mathcal{M}}_{0,1}^{\mathbb{P}^n,H}(d,(d))$ by translating the image of a relative stable map. The smooth orbifold $V_d \subset \overline{\mathcal{M}}_{0,1}^{\mathbb{P}^n,H}(d,(d))$ is invariant under this $T$–action, thus we may use the localization technique to compute the integral. If we represent the Poincaré dual of $D^j$ by the $T$–invariant projective subspace of $\mathbb{P}^{n-1}$ generated by $p_1, \cdots, p_{n-j}$ (in some sense localizing $D^j$ first), then we may replace $V_d$ by the projective space $\mathbb{P}^{n-1-j} \subset V_d$ parameterizing lines in $\mathbb{P}^n$ connecting $p_0$ and a point in $\mathbb{P}^{n-1-j}$ generated by $p_1, p_2, \cdots, p_{n-j}$.

With the preceding understood, each fixed point in $V_d$ contributing to the integral corresponds to the following graph:

\[
\begin{array}{c}
0 \\
\overline{d} \\
\hline
i \end{array} \quad \{1\}
\]

Figure 1. Graph $\Gamma_i$ with $i \in \{1, 2, \cdots, n-j\}$

where the label $i$ indicates that the relative marked point is mapped to the fixed point $p_i$. Therefore the normal bundle to the fixed point corresponding to the graph $\Gamma_i$ is the tangent space of $\mathbb{P}^{n-1-j}$ at $p_i$. This tangent space is generated by the tangent vectors of the line $\ell_{ia}, \ a \in \{1, 2, \cdots, i, \cdots, n-j\}$. Then the contribution of the virtual normal bundle to the fixed point
associated with the graph $\Gamma^i$ is

$$\prod_{\alpha=1, \alpha \neq i}^{n-j} \frac{1}{\lambda_i - \lambda_\alpha}.$$ 

So by the Atiyah-Bott localization formula, we have

$$\int_{V_d} \psi^{n-1-j} \wedge ev^* D^j = \frac{1}{d} \sum_{i=1}^{n-j} \frac{(\lambda_i - \lambda_0)^{n-1-j}}{\prod_{\alpha=1, \alpha \neq i}^{n-j} (\lambda_i - \lambda_\alpha)}$$

$$= \frac{1}{d^{n-j}} \sum_{i=1}^{n-j} \frac{(\lambda_i - \lambda_0)^{n-1-j}}{\prod_{\alpha=1, \alpha \neq i}^{n-j} (\lambda_i - \lambda_\alpha)} = \frac{1}{d^{n-j}}.$$  

The last equality follows from the expansion of the Vandermonde determinant. □

In a sequel paper we would need to compute some other relative invariants, and for that purpose we need to apply the symplectic analogue of the virtual localization in [GV1, GP].

We end this section with the sketch of another argument of Lemma 7.4. Let $V_{d,j}$ be the subspace of $V_d$ cut down by $D^j$. Then $V_{d,j}$ can be identified with $P^{n-1-j}$, and Lemma 7.4 follows from the claim that $L^\otimes d$ is $O(1)$ over $V_{d,j}$. To prove this claim notice that $L^\otimes d$ is the cotangent line at $p_0$ of the $\mathbb{P}^1$ connecting $p_0$ and a point $x$, which corresponds to the plane $\mathbb{C}(p_0) \oplus \mathbb{C}(x)$. And the tangent space at $p_0$ of this $\mathbb{P}^1$ is the quotient of $\mathbb{C}(p_0) \oplus \mathbb{C}(x)$ by $\mathbb{C}(p_0)$, which is isomorphic to $\mathbb{C}(x)$. Thus the dual of $L^\otimes d$ is the tautological line bundle $O(-1)$ on $\mathbb{P}^{n-1-j}$.

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