Auxiliary fields rescaling in higher–derivative supergravity

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Abstract

We study higher–derivative supergravity with curvature squared terms in different bases. Performing a Weyl rescaling only on the metric or on all the superfield components does not allow to obtain a normalized kinetic Einstein term from a $\mathcal{R} + \mathcal{R}^2$ theory. It is necessary to combine a Legendre transformation and a Weyl rescaling on a $\mathcal{R} + \mathcal{R}^2$ theory to arrive at a theory of supergravity coupled to matter. This mechanism is applied to supergravity coupled to a general function $k(R, R^\dagger, \Phi, \bar{\Phi})$, where $R$ is one of the supergravity chiral superfields and $\Phi$ a chiral matter superfield.

Keywords: Supergravity, higher–derivative auxiliary fields, supersymmetry breaking.

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1 Introduction

Recently, it has been shown [1] that a new mechanism for breaking supersymmetry can occur from higher–derivative supergravity. Precisely, this mechanism is based on the equivalence between $\mathcal{R} + \mathcal{R}^2$ theories and gravity coupled to a scalar $\Phi$. The breaking of supersymmetry clearly appears when we analyse the scalar potential which can be done after a Weyl rescaling. The delicate point is this Weyl rescaling that we want to discuss here. Indeed, if the Weyl rescaling is obvious in bosonic gravity, it becomes more complicated in higher–derivative supergravity because of the auxiliary fields $M$. These fields, needed in the theory to close the algebra of local supersymmetry transformations, are non–trivially related to the metric $g_{mn}$ and the gravitino field $\psi_m^\alpha$. Moreover, higher–derivative supergravity provides kinetic terms for auxiliary fields $M, \bar{M}$. These terms describe new propagating degrees of freedom. In this context, it is important to know whether auxiliary fields have to be rescaled or not.

Our paper is organized as follows. In the second chapter we review ”old minimal” supergravity. The introduction of matter can be realized after a Weyl rescaling. Although auxiliary fields can be eliminated by their equations of motion, it is interesting to learn what are their transformations under a Weyl rescaling. This will be useful because it will be not possible to eliminate auxiliary fields in higher–derivative supergravity since they appear with kinetic terms and since the equations of motion are not linear.

The third chapter is devoted to supergravity coupled to a general function $k(R, R^\dagger, \Phi, \bar{\Phi})$ where $R$ and $R^\dagger$ are two covariant superfields which describe torsion and curvature $\mathcal{R}$, $\Phi$ and $\bar{\Phi}$ are respectively chiral and antichiral superfields. We display bosonic components of the supersymmetric lagrangian. It is shown that the scalar curvature is not only coupled to matter but also to auxiliary fields. In this general case, we perform a Weyl rescaling on the metric and on all fields of the supergravity multiplet. These two rescaling do not allow for building a linear theory of gravity without scalar curvature squared terms.

In the chapter 4 we take place in the framework of $U(1)_K$ superspace $\mathcal{I}$ and consider a general Kähler potential $K(\phi, \bar{\phi})$ where $\phi$ has a $U(1)_K$ weight $\omega(\phi)$. Working in the $U(1)_K$ superspace means that one considers a superfield rescaling. It is a convenient formulation for describing matter coupled to supergravity, but it is not well–defined when $\phi$ is replaced by $R$ with $\omega(R) = 2$.

Finally, in the last chapter, we explain how a $\mathcal{R} + \mathcal{R}^2$ theory of supergravity coupled to matter can be reduced to a $\mathcal{R} + \Pi + \Lambda$ theory of supergravity coupled to matter. First, a Legendre transformation is performed in order that the new degrees of freedom appear explicitly. Then one can perform a Weyl rescaling on the metric, this will give a normalized Einstein term. As an example, the case of

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1We are interested in ”old minimal” supergravity in four dimensions. Auxiliary fields are two complex scalar fields $M, \bar{M}$ and one real vector field $b_a$.

2The second derivative of $M$ is related to the scalar curvature $\mathcal{R}$ among other terms.
a toy lagrangian is treated in component formulation.

2 Rescaling in supergravity with matter

Old minimal supergravity is described by the set of fields \((e_m^a, \psi_m^a, M, \bar{M}, b_a)\) where \(e_m^a\) is the vielbein field, \(\psi_m^a\) is the Rarita–Schwinger field also called gravitino, and \(M, \bar{M}, b_a\) are auxiliary fields. An invariant lagrangian is built by taking the superspace volume \([1]\)

\[ L_{sugra} = -3 \int d^4 \theta E , \]  

and can be written as

\[ L_{sugra} = -3 \int d^2 \Theta \ 2 \mathcal{E} R - 3 \int d^2 \bar{\Theta} \ 2 \bar{\mathcal{E}} \bar{R} , \]  

where \(\Theta\) is a covariant variable and \(\mathcal{E}\) is the chiral density, with the definitions

\[ 2 \mathcal{E} = e \left( 1 + i \Theta \sigma^m \bar{\psi}_m - \Theta \bar{\Theta} (M + \bar{M} \sigma^{mn} \bar{\psi}_n) \right) , \]  

\[ -6 R = M + \Theta \left( \sigma^m \bar{\sigma}^n \psi_{mn} - i \sigma^m \bar{\psi}_m M + i b^m \psi_m \right) \]
\[ + \Theta \bar{\Theta} \left( \frac{1}{2} \bar{\psi}^2 M - \frac{1}{2} \bar{\psi}_m \sigma^m \bar{\psi}_n b^n \right) \]
\[ + \frac{1}{8} \epsilon^{mnpq} \left( \bar{\psi}_m \bar{\sigma}_n \psi_{pq} + \psi_m \sigma_n \bar{\psi}_{pq} \right) . \]  

\( \mathcal{R} \) is the scalar curvature, \(\psi_{mn}^a\) is defined as follows

\[ \psi_{mn}^a \equiv D_m \psi_n^a - D_n \psi_m^a , \]  

with

\[ D_m \psi_n^a \equiv \partial_m \psi_n^a + \psi_n^\beta w_{m\beta}^a , \]  

where \(w_{m\beta}^a\) is the Lorentz connection. Developing \([1]\) in component field formulation yields

\[ L_{sugra} = - \frac{1}{2} e \mathcal{R} + \frac{1}{2} \epsilon \epsilon_{mnpq} \left( \bar{\psi}_m \bar{\sigma}_n D_p \psi_q - \psi_m \sigma_n D_p \bar{\psi}_q \right) \]
\[ - \frac{1}{3} e M \bar{M} + \frac{1}{3} e b^a b_a . \]  

This lagrangian describes supersymmetric Einstein gravity. It is clear that both graviton and gravitino are propagating while \(M, \bar{M}\) and \(b_a\) are not dynamical fields. The lagrangian \([1]\) can be generalized by \(3\)

\[ \mathcal{L} = \frac{1}{2} \int \frac{E}{R} r + \frac{1}{2} \int \frac{E}{R^4} r , \]  

\(^3\)In the following, the volume element \(d^4 \theta\) is omitted.
where $r$ is a chiral superfield with the components
\[ r| = r , \quad D_\alpha r| = \sqrt{2} r_\alpha , \quad D^2 r| = -4s . \]  

The component field formulation of the lagrangian (8) is
\[ \mathcal{L} = \frac{1}{2} \int \frac{E}{R} r + \frac{1}{2} \int \frac{E}{R^0} \tilde{r} \]
\[ = \int d^2 \Theta 2 \mathcal{E} r + \int d^2 \tilde{\Theta} 2 \tilde{\mathcal{E}} \tilde{r} \]
\[ = -\frac{1}{4} D^2(2\mathcal{E} r) - \frac{1}{4} \tilde{D}^2(2\tilde{\mathcal{E}} \tilde{r}) \]
\[ = -\frac{1}{4} D^2(2\mathcal{E}) r - \frac{1}{4} (2\mathcal{E}) \tilde{D}^2 r - \frac{1}{2} D^\alpha(2\mathcal{E}) D_\alpha r + h.c. \]
\[ = -e r(\bar{M} + \tilde{\psi}_m \tilde{\sigma}^m \tilde{\psi}_n) + es + \frac{i}{\sqrt{2}} e(\bar{\phi}_m \tilde{\sigma}^m \phi_n) r_\alpha + h.c. \]

The coupling of chiral matter to supergravity can be realized by considering a real function $\Omega$ of chiral and anti–chiral superfields $\Phi$ and $\bar{\Phi}$
\[ \mathcal{L}_{\text{cin+mat}} = -3 \int E \Omega(\Phi, \bar{\Phi}) , \]

which gives in components\footnote{We use (10) with $r = \frac{1}{4} (D^2 - 8R) f \Omega(\Phi, \bar{\Phi})$ .}
\[ e^{-1} \mathcal{L}_{\text{cin+mat}} = \frac{1}{2} \Omega \left( -\frac{1}{2} R + \frac{1}{3} b^m b_m - \frac{1}{3} M \bar{M} \right) \]
\[ + \Omega_i (M F_i + ib^m \partial_m A_i) \]
\[ + \Omega_i (\bar{M} F_i - ib^m \partial_m \bar{A}_i) \]
\[ + \Omega_{ij} \left( -3 F_i F_j + 3 \partial^m A_i \partial_m \bar{A}_j \right) , \]

using the definitions
\[ A_i \equiv \Phi_i| , \quad \chi_{i\alpha} \equiv \frac{1}{\sqrt{2}} D_\alpha \Phi_i| , \quad F_i \equiv -\frac{1}{4} D^2 \Phi_i| , \]

and
\[ \Omega_i \equiv \frac{\partial \Omega}{\partial A_i} , \quad \Omega_{i} \equiv \frac{\partial \Omega}{\partial \bar{A}_i} , \quad \Omega_{ij} \equiv \frac{\partial^2 \Omega}{\partial A_j \partial A_i} . \]

One can show that the lagrangian (12) does not have a kinetic normalized Einstein term. A right description of Einstein gravity needs the presence of $-\frac{1}{2} e R$ which can be obtained by performing a rescaling on the vielbein\footnote{The prime denotes rescaled fields.}
\[ e_m^a = e^\lambda e_m^a \]
\[ e^\lambda = \frac{1}{3} \left( D^2 - 8R \right) \Omega(\Phi, \bar{\Phi}) . \]
where $\lambda$ is expressed in terms of matter fields, i.e

$$\lambda = -\frac{1}{2} \ln \Omega . \quad (16)$$

This rescaling modifies the scalar curvature such that

$$\mathcal{R} = \Omega (\mathcal{R}' + 6(\partial^a \lambda C'_a + \partial'^a \lambda \partial'_a \lambda + \partial'^a \partial'_a \lambda)) , \quad (17)$$

where

$$C'_a \equiv e'_a m e'_b \left( \partial'_m e'_n b - \partial'_n e'_m b \right) . \quad (18)$$

Matter fields are invariant under this Weyl rescaling. This implies that the lowest components of $\Phi_i$ are invariant

$$A_i = A'_i , \quad (19)$$

but not highest components

$$\chi^\alpha_i = e^{-\frac{1}{2} \lambda} \chi'^\alpha_i , \quad F_i = e^{-\lambda} F'_i . \quad (20)$$

At this stage, there are two ways to treat the auxiliary fields $M, \overline{M}$ and $b_a$. Either one can eliminate them by taking their equations of motion \[4\] or one can consider them as supergravity multiplet components $(e_m^a, \psi_m^\alpha, M, \overline{M}, b_a)$. In this case a superfield rescaling acts on all component fields and one obtains non-trivial relations between rescaled and old fields \[5\]

$$M = \Omega^{\frac{1}{2}} \left( M' + 3 \Omega^{-1} \Omega_j F'_j \right) ,
\overline{M} = \Omega^{\frac{1}{2}} \left( \overline{M}' + 3 \Omega^{-1} \Omega_i F'_i \right) ,
b_a = \Omega^{\frac{1}{2}} \left( b'_a - \frac{i}{2} \left( -3 \Omega^{-1} \Omega_i \partial'_a A_i + 3 \Omega^{-1} \Omega_j \partial'_a A_j \right) \right) . \quad (21)$$

Then the rescaled lagrangian can be expressed in terms of new fields

$$\mathcal{L}_{vin+mat} = -\frac{1}{2} e' \mathcal{R}' + \frac{1}{3} e' b'_m b'_m - \frac{1}{3} e' M' \overline{M}' + e' \left( -3 \Omega^{-2} \Omega_i \Omega_j + 3 \Omega^{-2} \Omega_j \Omega_i \right) \left( \partial'^m A_i \partial'_m \overline{A}_j - F'_i \overline{F}'_j \right) . \quad (22)$$

This lagrangian reproduces Einstein gravity with supersymmetric counterpart as well as a kinetic matter term. Since $M'$ and $b'_a$ verify equations of motion

$$M' = 0 ,
b'_a = 0 , \quad (23)$$

\[6\] Note that $\partial_m = \partial'_m$ and $\partial_a = e^{-\lambda} \partial'_a$.

\[7\] We shall only keep bosonic fields.
one finds the well known lagrangian [4]

\[ \mathcal{L}_{\text{cin+mat}} = -\frac{1}{2} e' \mathcal{R}' - e' K_{ij} \partial' m A_i \partial' m A_j + e' K_{ij} F_i F_j', \tag{24} \]

with

\[ K = -3 \ln \Omega , \]
\[ K_{ij} = \frac{\partial^2 K}{\partial A_i \partial A_j} . \tag{25} \]

Then one concludes that the two ways are equivalent since they give the same result. Hence, if \( M, \varpi \) and \( b_a \) cannot be replaced by their equations of motion, relations (21) will be useful to rescale a non-normalized Einstein lagrangian. In the following chapter we shall point out the difficulty of rescaling a lagrangian with higher-derivative terms.

3 Coupling \( k(R, R^\dagger, \Phi, \bar{\Phi}) \) to supergravity

It is well known that old minimal supergravity is completely described by a set of superfields [4]:

\[ R, \ R^\dagger, \ G_a, \ W_{\alpha\beta\gamma}, \ \bar{W}^{\dot{\alpha}\dot{\beta}\dot{\gamma}}. \tag{26} \]

In a recent paper [6], we analysed curvature squared terms arising from these superfields. It appears that the highest superfield components of the combination \( RR^\dagger \) yields a \( \mathcal{R}^2 \) term, where \( \mathcal{R} \) is the scalar curvature term. In this chapter, we generalize our study of Weyl rescaling by considering the general lagrangian

\[ \mathcal{L} = -3 \int E \ k \left( \Phi, \bar{\Phi}, R, R^\dagger \right) , \tag{27} \]

where \( k \) is a real function, \( \Phi \) a chiral superfield and \( R \) the superfield defined above. This lagrangian includes the following particular cases

\[ \mathcal{L} = \int E \ \Omega(\Phi, \bar{\Phi}) f(R, R^\dagger) , \tag{28} \]
\[ \mathcal{L} = \int E \ f(R, R^\dagger) , \tag{29} \]
\[ \mathcal{L} = \int E \ e^{-K(\Phi, \bar{\Phi})}. \tag{30} \]

Computing (27) gives

\[ e^{-1} \mathcal{L} = k_{ij} \left\{ -\frac{3}{16} \mathcal{D}^2 \phi^i \mathcal{D}^2 \bar{\phi}^j + 3 \mathcal{D}^m \phi^i \mathcal{D}^m \bar{\phi}^j \right\} \]
\[ + k_i \left\{ \frac{3}{2} R \bar{D}^2 \phi^i - 3iG^m \bar{D}_m \phi^i \right\} \\
+ k_j \left\{ \frac{3}{2} \bar{R}^i \bar{D}^2 \phi^j + 3iG^m \bar{D}_m \phi^j \right\} \\
+ k \left\{ \frac{3}{4} \bar{D}^2 R + \frac{3}{4} \bar{D}^2 \bar{R}^i - 36 \bar{R} \bar{R}^i \right\}, \quad (31) \]

with the definitions
\[ \phi_i = (\Phi, R), \quad k_i = \frac{\partial k}{\partial \phi^i}, \quad k_{ij} = \frac{\partial^2 k}{\partial \phi^i \partial \phi^j}, \quad (32) \]
and
\[ \bar{D}_m \Phi = \partial_m \Phi, \quad (33) \]
\[ \bar{D}_m R = \partial_m R + ib_m R. \quad (34) \]

The superfield \( R| = -\frac{M}{6} \) will produce the scalar curvature term, and the chiral superfields \( \Phi \) will generate, among other terms, kinetic terms and interacting terms.

One derives the component field expression for (31)
\[ e^{-1} \mathcal{L} = -\frac{1}{2} \bar{R} \left( k + M k_M + \bar{M} k_{\bar{M}} - 2k_{\bar{M}} M \bar{M} - 4k_{\bar{M}} M \bar{M} \bar{M} - 3 \left( \frac{k_{\bar{M}} \Phi}{M} + k_{\bar{M}} \bar{F}^\Phi \right) + \right) \]
\[ - \frac{3}{4} k_{\bar{M}} M \bar{M} R^2 + 3k_{\bar{M}} M \bar{M} \partial^m M \partial_m \bar{M} - 3k_{\bar{M}} M \left( e_a^m \bar{D}_m b^a \right)^2 \\
+ 3k_{\bar{M}} \Phi \partial^m M \partial_m A^\Phi + 3k_{\bar{M}} \Phi \partial^m M \partial_m \bar{A}^\Phi \\
+ ib^m \left( k_M \partial_m M - k_{\bar{M}} \partial_m \bar{M} + k_{\bar{M}} \partial_m A^\Phi - k_{\bar{M}} \partial_m \bar{A}^\Phi \right) \\
- i \left( e_a^m \bar{D}_m b^a \right) \left( k_M M - k_{\bar{M}} \bar{M} - 3k_{\bar{M}} M \Phi \bar{F}^\Phi + 3k_{\bar{M}} \Phi \bar{F}^\Phi \right) + \right) \]
\[ - \frac{1}{3} M \bar{M} \left( k - 2k_M M - 2k_{\bar{M}} \bar{M} + 4k_{\bar{M}} M \bar{M} \bar{M} + 6k_{\bar{M}} M \bar{F}^\Phi + 6k_{\bar{M}} \Phi \bar{F}^\Phi \right) + \right) \]
\[ + \frac{1}{3} b^a b_a \left( k + k_M M + k_{\bar{M}} \bar{M} - 4k_{\bar{M}} M \bar{M} M M - k_{\bar{M}} \bar{M} b^c b_c - 3k_{\bar{M}} M \Phi \bar{F}^\Phi - 3k_{\bar{M}} \Phi \bar{F}^\Phi \right) - \right) \]
\[ - 3k_{\bar{M}} \Phi \Phi \bar{F}^\Phi \bar{F}^\Phi + 3k_{\bar{M}} \Phi \partial_m A^\Phi \partial_m \bar{A}^\Phi + k_{\bar{M}} \Phi \bar{F}^\Phi + k_{\bar{M}} \Phi \bar{F}^\Phi M + k_{\bar{M}} \Phi \bar{F}^\Phi \bar{M}, \quad (35) \]

with
\[ \bar{D}_m b^a = \partial_m b^a + b^e w_{mc}^a. \quad (36) \]

It is clear that taking \( k = \Omega(\Phi, \bar{\Phi}) \) in (33) reproduces (11). Starting with (35), a Weyl rescaling can be performed. We will consider two rescalings; either we make the conformal transformation only on the metric
\[ e_m^a = e^\lambda e_m^a, \quad (37) \]
keeping auxiliary fields inert, or we make a superconformal transformation on all fields, i.e

\begin{align*}
  e_m^a &= e^\lambda e'_m^a, \\
  M &= e^{-\lambda} \left( M' + 3k^{-1}k_j F^j \right), \\
  \overline{M} &= e^{-\lambda} \left( \overline{M} + 3k^{-1}k_i F^i \right), \\
  b_a &= e^{-\lambda} \left( b'_a - \frac{i}{2} \left(-3k^{-1}k_i \partial'_a \overline{A} + 3k^{-1}k_i \partial'_a A^i \right) \right),
\end{align*}

(38)

with the choice

\[ e^{2\lambda} = k^{-1}. \]

(39)

In the case of supergravity coupled to matter, these two conformal transformations give the same result. But in general it is not true, since auxiliary fields cannot be eliminated by equations of motion in higher-derivative supergravity. As an example, consider the term

\[ L_R = \frac{1}{2} e M \overline{M} R + e b^a b_a R. \]

(40)

Under transformation (37), this yields

\[ L_R = \frac{1}{2} e' e^{2\lambda} \left( M' + 2b^a b_a \left( R' + 6\Delta' \right) \right), \]

(41)

where

\[ \Delta' = \frac{3}{4} k^{-2} (\partial'^a k \partial'_a k) - \frac{1}{2} k^{-1} (\partial'^a \partial'_a k) - \frac{1}{2} k^{-1} (\partial'^a k) e'^m_a e'^b_n \left( \partial'_m e'^a_n \partial'_n e'^b_m \right). \]

(42)

The same lagrangian (40), under the transformation (38), leads to a mixing of auxiliary fields with scalar curvature

\[ L_R = \frac{1}{2} e' \left( M' + 3k^{-1}k_j F^j \right) \left( \overline{M}' + 3k^{-1}k_i F^i \right) (R' + 6\Delta') \]

\[ + e' \left( b'_a - \frac{i}{2} \left(-3k^{-1}k_i \partial'_a \overline{A} + 3k^{-1}k_i \partial'_a A^i \right) \right)^2 (R' + 6\Delta'). \]

(43)

And whatever transformation we take, (37) or (38), the curvature squared term \( R^2 \) generates squared kinetic terms. The lagrangian

\[ L_{R^2} = -\frac{3}{4} k'_M \overline{R} R^2, \]

(44)

becomes under (37)

\[ L_{R^2} = -\frac{3}{4} k'_M \left( R^2 + 12\Delta' R' + 36\Delta'^2 \right). \]

(45)
This lagrangian contains fourth order derivative terms in $\Phi$, which is not satisfying. Furthermore, the kinetic term for the graviton field is not normalized, and a Weyl rescaling increases the number of couplings between the curvature and the other fields. Moreover, this component field formulation shows us that a rescaling cannot absorb $R^2$ terms in the metric. Thus one has to find another way in order to rearrange higher-derivative supergravity. Another possibility consists in extending the symmetry group to a $U(1)$, namely $U(1)_K$ supergravity [5].

\section{Superfield rescaling in $U(1)_K$ superspace}

In addition to the vielbein $E^A$ and the Lorentz connection $\phi_B^A$, the $U(1)_K$ superspace contains a $U(1)$ connection $A$ which is a one-form in superspace: 

$$ A = d z^M A_M .$$

$T^A$, the usual torsion of old minimal supergravity, is modified as follows

$$ T^A = d E^A + E^B \phi_B^A + \omega(E^A) E^A A ,$$

and we define another two-form, the $U(1)$ fieldstrength $F$

$$ F = d A .$$

In $U(1)_K$ supergravity, our lagrangian (1) becomes

$$ \mathcal{L} = -3 \int E 

= - \frac{1}{2} e \mathcal{R} - \frac{1}{3} e M \mathcal{M} + \frac{1}{3} e b^a b_a 

+ \frac{1}{2} e \epsilon^{mnpq} (\bar{\psi}_m \tilde{\sigma}_n \bar{D}_p \psi_q - \psi_m \sigma_n \bar{D}_p \bar{\psi}_q ) 

+ \frac{i}{2} \epsilon (\bar{\psi}_m \tilde{\sigma}^m X + \psi_m \sigma^m \bar{X} ) - \frac{1}{2} \epsilon (\mathcal{D}^\alpha X_\alpha ) ,$$

where we define the $U(1)$ covariant derivative $\tilde{D}_m X$

$$ \tilde{D}_m X = \partial_m X + \omega(X) \tilde{A}_m X ,$$

with

$$ \tilde{A}_m = A_m + \frac{i}{2} b_m .$$

Matter is included in the components of $X, \mathcal{D}X$ and $\tilde{A}_m$. Indeed, one has the relation

$$ X_\alpha = - \frac{1}{8} (\mathcal{D}^2 - 8 \mathcal{R} ) \mathcal{D}_\alpha K(\phi, \bar{\phi} ) ,$$

$$ \bar{X}^\dot{\alpha} = - \frac{1}{8} (\mathcal{D}^2 - 8 \mathcal{R}^\dot{\alpha} ) \tilde{\mathcal{D}}^{\dot{\alpha}} K(\phi, \bar{\phi} ) .$$
Generally, matter superfields have a $U(1)_K$ weight which is zero. One can ask what happens if we consider chiral superfields of weight $\omega(\phi)$. Relation (52) can be written as

$$X_\alpha = \left(1 - \frac{\omega(\phi)}{2} K_\phi \phi \right)^{-1} \left( -\frac{1}{8} K_{\dot{\phi}\phi}(\bar{D}\phi)^2 D_\alpha \phi - \frac{1}{8} K_{\dot{\phi}\phi} \bar{D}^2 \phi D_\alpha \phi - \frac{i}{2} K_{\dot{\phi}\phi} \bar{D}^\alpha \phi \bar{D}_{\alpha\alpha} \phi \right).$$  

(55)

A similar relation is obtained for its complex conjugate

$$\bar{X}_{\dot{\alpha}} = \left(1 + \frac{\omega(\phi)}{2} K_{\dot{\phi}} \phi \right)^{-1} \left( -\frac{1}{8} K_{\phi\dot{\phi}}(D\phi)^2 \bar{D}_{\dot{\alpha}} \phi - \frac{1}{8} K_{\phi\dot{\phi}} D^2 \phi \bar{D}_{\dot{\alpha}} \phi + \frac{i}{2} K_{\phi\dot{\phi}} \phi \bar{D}^\alpha \phi \bar{D}_{\alpha\dot{\alpha}} \phi \right).$$  

(56)

The lagrangian (49) contains a $D^\alpha X_\alpha$ term which can be expressed as

$$D^\alpha X_\alpha \left(1 - \frac{\omega(\phi)}{2} K_\phi \phi \right) = -\frac{1}{8} K_{\dot{\phi}\phi\phi}(D\phi)^2 (\bar{D}\phi)^2$$

$$+ K_{\dot{\phi}\phi\phi} \left( -\frac{i}{2} \bar{D}_{\alpha\dot{\alpha}} \bar{D}^\alpha \phi \bar{D}^\alpha \phi \bar{D} \phi - \frac{1}{8} (\bar{D}\phi)^2 D^2 \phi \right)$$

$$+ K_{\dot{\phi}\phi\phi} \left( -\frac{i}{2} \bar{D}_{\dot{\alpha} \alpha} \bar{D}^\alpha \phi \bar{D}^\alpha \phi \bar{D} \phi - \frac{1}{8} (\bar{D}\phi)^2 D^2 \phi \right)$$

$$+ K_{\phi\dot{\phi}} \left( -\frac{i}{2} \bar{D}_{\dot{\alpha} \dot{\alpha}} \bar{D}^\alpha \phi \bar{D}^\alpha \phi \bar{D} \phi \right)$$

$$+ \frac{1}{2} (\omega(\phi) K_\phi + \omega(\bar{\phi}) K_{\dot{\phi}} \phi - \omega(\phi) K_{\dot{\phi}} \bar{\phi} ) (D^\alpha \phi X_\alpha)$$

$$+ \left(\omega(\phi) K_{\phi\phi} \phi \right) \left( \bar{D}_{\alpha} \phi \bar{X}_{\alpha} \right).$$  

(57)

Substituting (53) and (54) in (57) yields $D^\alpha X_\alpha$ in terms of derivatives of the chiral superfield $\phi$. Finally, $\dot{A}_m$, the last component of the gauge multiplet, must

*Our goal is to couple $R$ and $R^\dagger$ to $U(1)_K$ supergravity. In this perspective, one has to remember that $U(1)_K$ weight of $R$ is 2.
be defined. Using the definition of the fieldstrength\footnote{We follow the definition of \[\mathbf{[4, 5, 6]}\].}

\[ \tilde{F}_{BA} = D_B \tilde{A}_A - (-)^{ab} D_A \tilde{A}_B - T_{BA}^C \tilde{A}_C , \]

with the constraint

\[ \tilde{F}_{B\bar{a}} = 0 , \]

one deduces the expression of the vector \( \tilde{A}_m \),

\[ \tilde{A}_m = \frac{1}{4} \left( 1 - \frac{\omega(\phi)}{2} K_{\phi} \phi \right)^{-1} \left( K_{\dot{\phi}} \partial_m \phi - K_{\bar{\phi}} \partial_m \bar{\phi} + \frac{i}{2} K_{\phi\bar{\phi}} (D\phi \sigma_m D\bar{\phi}) \right) . \]

For this last relation, we used the fact that \( K_{\phi} \phi = K_{\bar{\phi}} \bar{\phi} \). Relations [53] to [57] and [61] allow us to express the lagrangian [10] in terms of supergravity fields and derivatives of chiral field \( \phi \). One finds

\[ e^{-1} \mathcal{L} = -\frac{1}{2} R - \frac{1}{3} \mathcal{M} \mathcal{M} \frac{1}{3} b^a b_a 
+ \frac{1}{2} e^{mnq} (\tilde{\psi}_m \tilde{\sigma}_n \tilde{D}_p \psi_q - \psi_m \sigma_n \tilde{D}_p \tilde{\psi}_q)
+ \left( \frac{i}{2} \lambda_{\phi} \tilde{\psi}_m \tilde{\sigma}^{m\dot{\alpha}} \phi \nabla_{\dot{\alpha}} + \Lambda^{\alpha} \right) \left( -\frac{1}{8} K_{\phi\bar{\phi}} (D\phi)^2 D\alpha \phi - \frac{1}{8} K_{\phi\phi} D^2 \phi D\alpha \phi - \frac{i}{2} K_{\phi\bar{\phi}} D^2 \phi \bar{D} \alpha \phi \right)
+ \left( \frac{i}{2} \bar{\lambda}_{\bar{\phi}} \tilde{\psi}_m \sigma_{m\dot{\alpha}} \phi \nabla_{\dot{\alpha}} + \bar{\Lambda}_{\dot{\alpha}} \right) \left( -\frac{1}{8} K_{\bar{\phi}\phi} (D\bar{\phi})^2 \bar{D} \dot{\alpha} \bar{\phi} - \frac{1}{8} K_{\bar{\phi}\bar{\phi}} D^2 \bar{\phi} \bar{D} \dot{\alpha} \bar{\phi} + \frac{i}{2} K_{\bar{\phi}\phi} D \phi \bar{D} \dot{\alpha} \bar{\phi} \right)
+ \frac{1}{2} \lambda_{\phi} \left\{ -\frac{1}{8} K_{\phi\phi\bar{\phi}} (D\phi)^2 (\bar{D} \bar{\phi})^2 
+ K_{\phi\phi\bar{\phi}} \left( -\frac{i}{2} \bar{D}_{\alpha\dot{\alpha}} \bar{\phi} D^\alpha \phi \bar{D} \dot{\alpha} \bar{\phi} - \frac{1}{8} (D\bar{\phi})^2 D^2 \bar{\phi} \right) 
+ K_{\phi\phi\bar{\phi}} \left( -\frac{i}{2} \bar{D}_{\alpha\dot{\alpha}} \phi D^\alpha \phi \bar{D} \dot{\alpha} \bar{\phi} - \frac{1}{8} (D\phi)^2 \bar{D}^2 \bar{\phi} \right) 
+ K_{\phi\bar{\phi}} \left( -\frac{i}{2} \bar{D}_{\alpha\dot{\alpha}} \bar{\phi} D^\alpha \phi \bar{D} \dot{\alpha} \bar{\phi} + \frac{i}{2} \bar{D} \phi \bar{D}_{\alpha\dot{\alpha}} D^\alpha \phi 
- \frac{1}{8} (D^2 \phi) (D^2 \bar{\phi}) - D_{\alpha\dot{\alpha}} \bar{\phi} \bar{D} \alpha \phi 
+ \frac{3}{2} G_{\alpha\dot{\alpha}} \bar{D} \alpha \phi \bar{D} \dot{\alpha} \phi \right) \right\} . \]
Thus one obtains, in the $U(1)_K$ superspace, a lagrangian with a normalized Einstein kinetic term. Since this result is general, it can be applied to any chiral superfield $\phi$ of $U(1)_K$ weight $\omega(\phi)$. What does (61) give if we decide to choose the particular case $\phi = R$ with $\omega(\phi) = 2$? In this case, lagrangian (61) will contain $R|, D_\alpha R|$ and $D^2 R|$ terms with the definitions

$$
R| = -\frac{M}{6}, \quad (66)
$$

$$
D_\alpha R| = -\frac{1}{6} (\sigma^n \bar{\sigma}^m)_\alpha ^\gamma (\bar{\cal D}_n \psi_{m\gamma} - \bar{\cal D}_m \psi_{n\gamma})
- \frac{i}{6} b^m \tilde{\psi}_{mn} + \frac{i}{6} (\sigma^m \bar{\psi}_m)_\alpha M - \frac{1}{3} X_\alpha, \quad (67)
$$

$$
D^2 R| = -\frac{1}{3} R + \frac{2}{9} b^a b_a + \frac{4}{9} M \tilde{M} - \frac{2i}{3} e_a^m \bar{D}_m b^a
+ \frac{1}{3} \tilde{\psi}_{mn} \bar{\psi}_n \sigma_m \psi_q + \tilde{\psi}_m \bar{\psi}_n \sigma_q \psi_m
- \frac{1}{3} \partial^a X_\alpha
+ \frac{i}{12} \epsilon^{mnpq} (\tilde{\psi}_{mn} \sigma_q \bar{\psi}_p + \psi_m \sigma_n \bar{\psi}_p q) - \frac{1}{3} \partial^a X_\alpha
+ \frac{i}{3} \left(X \sigma^m \bar{\psi}_m - \bar{X} \sigma^m \psi_m\right). \quad (68)
$$

It is clear that the two last terms will generate $\partial^a X_\alpha, \bar{X}_\alpha$ and $X_\alpha$ terms in the lagrangian which we do not want.

The problem comes from the expression of $X_\alpha$ (57), $\bar{X}_\alpha$ (58) and $\partial^a X_\alpha$ (59) which are not defined for $\phi = R$. Indeed, one has

$$
X_\alpha = (1 - K_R R)^{-1} \times
- \frac{1}{8} K_{RR} R! (\partial R!)^2 \partial_\alpha R - \frac{1}{8} K_{RR} R! \partial^2 R! \partial_\alpha R
- \frac{i}{2} K_{RR} R! \partial^a R! \partial_{a\alpha} R
\right). \quad (69)
$$

In this expression, the right hand side involves terms such as $\bar{X}^2, X_\alpha$ and $\partial^a X_\alpha$. Thus, one can not explicitly express quantities $X_\alpha, \bar{X}_\alpha$ and $\partial^a X_\alpha$ as well as $\bar{A}_m$ in terms of independent fields. Conformal transformation is not possible in $U(1)_K$ superspace with a function $K(R, R^!)$ because $R$ and $R^!$ superfields can not be treated at the same time as matter fields and as fields belonging to the supergravity multiplet.

5 Legendre transformation and superfield re-definition

Our purpose is to obtain a normalized Einstein term from higher-derivative supergravity coupled to matter. Since a Weyl rescaling and a superfield rescaling
are not allowed in higher–derivative supergravity, another approach has to be found. A correct description of such theories can be realized by a combination of two transformations: a Legendre transformation and a super field redefinition. This was already done in [7]. We generalize to the case of supergravity coupled to a general function \( k(\Phi, \bar{\Phi}, R, R^\dagger) \). We start from a \( \mathcal{R} + \mathcal{R}^2 \) theory of supergravity \(^{10}\)

\[
\mathcal{L}_1 = -3 \int E \, k(\Phi, \bar{\Phi}, R, R^\dagger)
\]

which is equivalent to a theory of supergravity coupled to two more matter superfields \( \Pi \) and \( \Lambda \),

\[
\mathcal{L}_2 = -3 \int E \, k(\Phi, \bar{\Phi}, \Pi, \bar{\Pi}) + \left[ -3 \int \frac{E}{R} \Lambda(R - \Pi) + h.c. \right]
\]

in the sense that superfield equations of motion give

\[
\Pi = R
\]

This second theory has a non–normalized Einstein term and can be reduced to a third theory \(^{11}\)

\[
\mathcal{L}_3 = -\frac{1}{2} \epsilon' \mathcal{R}' - \epsilon' K_{ij} \partial^m A_i \partial^j \bar{\Pi} + \epsilon' K_{ij} F_i F_j
\]

with a field redefinition

\[
\Phi' = \Phi, \quad \Pi' = \Pi, \quad \Lambda' = \Lambda
\]

and

\[
\epsilon' m^a = e^{-\lambda} e_m^a
\]

with

\[
\lambda = -\frac{1}{2} \ln k
\]

At this stage we would like to make an important remark. In the previous section we tried to rescale higher–derivative supergravity with scalar curvature squared term. We have seen two problems. The first one was the presence of a \( \mathcal{R}^2 \) term which could not be absorbed. A Legendre transformation solves this problem by introducing new fields. The other problem was the rescaling of a \( K(R, R^\dagger) \) function. In this formulation there is no such question, since the superfield \( R \) is hidden in \( \Pi \), and \( \Pi \) is treated as a matter superfield. Thus one can say that the transformation \( \mathcal{L}_2 \rightarrow \mathcal{L}_3 \) is not a superWeyl rescaling but a field redefinition where some of supergravity fields are rescaled \( (e_m^a, \psi_m^a, M, \bar{M}, b_a) \), but a certain combination of them is invariant because \( \Pi = R = -\frac{M}{6} \) is considered as matter

\(^{10}\)This lagrangian is displayed in (35).

\(^{11}\)We take \( \phi_i = (\Phi, \Pi, \Lambda) \) and \( A_i = \phi_i \), \( F_i = -\frac{1}{4} D^2 \phi_i \).
field and is an invariant superfield \((\Pi \longrightarrow \Pi\) under this redefinition).

As an example, we propose to construct a particular function \(f(R, R^\dagger)\) coupled to supergravity. We consider the case \([1]\)

\[
f(R, R^\dagger) = f(R) + f(R^\dagger) ,
\]

(77)

An invariant lagrangian is

\[
\mathcal{L}_1 = -3 \int E f(R, R^\dagger) .
\]

(78)

Taking \(r = -6RF(R)\) in the generic construction \([10]\), one derives the component field expression

\[
e^{-1}\mathcal{L}_1 = -\frac{1}{2} \mathcal{R} \left( f + M f_M + \overline{M} f_{\overline{M}} \right) + \frac{1}{3} b^a b_a \left( f + M f_M + \overline{M} f_{\overline{M}} \right) \\
- \frac{1}{3} \overline{M} M \left( f - 2M f_M - 2\overline{M} f_{\overline{M}} \right) + ib^m \left( f_M \partial_m M - f_M \partial_m \overline{M} \right) \\
+ ie_a^m \mathcal{D}_m b^a \left( f_M \overline{M} - f_M M \right) .
\]

(79)

In this lagrangian the scalar curvature is coupled to auxiliary fields \(M\) and \(\overline{M}\). Moreover, derivative terms of \(M, \overline{M}\) and \(b_a\) are present, but we can not consider them as propagating fields for instance. One can treat the \(b_a\) field as a purely auxiliary field, it means that \(b_a\) can be replaced by its equations of motion. We start by integrate \(\mathcal{D}_m b_a\) term. One has the relation

\[
e e_a^m \mathcal{D}_m v^a = \partial_m (ev^a e_a^m) + \frac{ie}{2} v^a (e_b^n e_a^m - e_b^m e_a^n) (\psi_n \sigma^b \overline{\psi}_m) ,
\]

(80)

where we choose

\[
v^a = b^a (f_M \overline{M} - f_M M) .
\]

(81)

Since we are only interested in bosonic terms, we simply have

\[
e e_a^m \mathcal{D}_m \left( b^a \left( f_M \overline{M} - f_M M \right) \right) = e e_a^m \mathcal{D}_m b^a \left( f_M \overline{M} - f_M M \right) \\
+ e e_a^m b^a \left( f_M \partial_m M + f_M \overline{M} \partial_m \overline{M} - f_M \partial_m M - f_M M \partial_m M \right) \\
= \text{total derivative} ,
\]

(82)

i.e:

\[
e e_a^m \mathcal{D}_m b^a \left( f_M \overline{M} - f_M M \right) = -e e_a^m b^a \left( f_M \partial_m M + f_M \overline{M} \partial_m \overline{M} - f_M \partial_m M - f_M M \partial_m M \right) .
\]

(83)

Substituting (83) in (79) gives

\[
e^{-1}\mathcal{L}_1 = -\frac{1}{2} \mathcal{R} \left( f + M f_M + \overline{M} f_{\overline{M}} \right) + \frac{1}{3} b^a b_a \left( f + M f_M + \overline{M} f_{\overline{M}} \right) \\
- \frac{1}{3} \overline{M} M \left( f - 2M f_M - 2\overline{M} f_{\overline{M}} \right) + 2ib^m \left( f_M \partial_m M - f_M \partial_m \overline{M} \right) \\
+ ib^m \left( f_M M \partial_m M - f_M \overline{M} \partial_m \overline{M} \right) .
\]

(84)
As we consider $b_a$ as an auxiliary field, one can diagonalize contributions of this field. One defines
\[ \hat{b}_m \equiv b_m + \frac{3i}{2} \left( f + M f_M + \overline{M} f_{\overline{M}} \right)^{-1} \left( (f_M M + 2f_M) \partial_m M - (f_{\overline{M} M} \overline{M} + 2f_{\overline{M}}) \partial_m \overline{M} \right), \]
and the Lagrangian becomes
\[
e^{-1} \mathcal{L}_1 = -\frac{1}{2} h \mathcal{R} - \frac{3}{4} h^{-1} \left( (f_M M + 2f_M) \partial_m M - (f_{\overline{M} M} \overline{M} + 2f_{\overline{M}}) \partial_m \overline{M} \right)^2 - \frac{1}{3} M \overline{M} \left( f - 2Mf_M - 2\overline{M}f_{\overline{M}} \right) + \frac{1}{3} h \hat{b}^a \hat{b}_a \\
+ \frac{3}{2} h^{-1} \left( (f_M M + 2f_M) \partial_m M - (f_{\overline{M} M} \overline{M} + 2f_{\overline{M}}) \partial_m \overline{M} \right)^2, \tag{86}
\]
where
\[ h \equiv f + M f_M + \overline{M} f_{\overline{M}}. \tag{87} \]

In order to study the breaking of supersymmetry, one has to compute the scalar potential. This can be done after a Weyl rescaling, because this Lagrangian clearly exhibits a non-normalized Einstein term. The conformal transformation
\[ e_m^a \equiv e^\lambda e'_m^a, \tag{88} \]
with
\[ e^{2\lambda} \equiv h^{-1} \left( f + B f_B + \bar{B} f_{\bar{B}} \right)^{-1}, \tag{89} \]
where we take $B = M$, gives a correct Einstein term $-\frac{1}{2} e' \mathcal{R}'$. Following the method of [1], i.e. performing a Weyl rescaling only on the metric and not on auxiliary fields, one obtains
\[
e'^{-1} \mathcal{L}_3 = -\frac{1}{2} \mathcal{R}' - 3h^{-2} \left( (f_B B + 2f_B) \partial'_m B (f_{BB} B + 2f_B) \partial'^m B \right) \\
+ \frac{1}{3} h^{-1} \hat{b}^a \hat{b}_a - \frac{1}{3} B \bar{B} h^{-2} \left( f - 2Bf_B - 2\bar{B} f_{\bar{B}} \right). \tag{90}
\]

As a remark we can say that this transformation is a field redefinition with $B = M$ and $e'_m^a \equiv e^{-\lambda} e_m^a$. In this sense it is not a Weyl rescaling but a field redefinition.

In other words, this Lagrangian can be written as
\[ \mathcal{L}_3 = -\frac{1}{2} e' \mathcal{R}' - e' K_{BB} \partial'^m B \partial'_m \bar{B} - e' \mathcal{V}(B, \bar{B}) + e' \frac{1}{3} h^{-1} \hat{b}^a \hat{b}_a, \tag{91} \]
with the following definitions
\[ K(B, \bar{B}) = -3 \ln h, \tag{92} \]
\[ K_B = -3h^{-1} h_B, \tag{93} \]
\[ K_{BB} = 3h^{-2} h_B h_B - 3h^{-1} h_{BB}. \tag{94} \]
\( \mathcal{V}(B, \bar{B}) \) is the scalar potential and is given by

\[
\mathcal{V}(B, \bar{B}) = \frac{1}{3} h^{-2} B \bar{B} \left( f - 2 B f_B - 2 \bar{B} f_{\bar{B}} \right).
\] (95)

Thus one deduces that lagrangian (78) is equivalent to supergravity coupled to a complex scalar field, namely \( B \).

At the superfield level and in this particular case, where \( f(R, R^\dagger) = f(R) + f(R^\dagger) \), one can say that this theory is equivalent to supergravity coupled to only one chiral superfield \( \Pi \). Indeed, the superfield formulation of this example is more simple than (71). On the one hand, the chiral matter superfields \( \Phi \) are absent, and on the other hand the \( \Lambda \) superfield can be expressed in terms of \( \Pi \) and \( f(\Pi) \) in this special case \([1]\). This reduction to one superfield instead of two chiral superfields \( \Pi \) and \( \Lambda \), can be understood in component formulation: one of the auxiliary fields, \( b_a \), can be eliminated by its equations of motion (83). This is the case when the lagrangian does not contain any fourth order derivative such as \((D_b)^2\) or \((\partial M)^2\).

In general both auxiliary fields, \( b_a \) and \( M \), can not be replaced by their equations of motion in higher–derivative supergravity. This is why two chiral superfields are needed to construct a theory of supergravity with a normalized kinetic Einstein term.

6 Conclusion

We emphasized the fact that different bases can be used in higher–derivative supergravity with scalar curvature squared term. An appropriate formulation consists in performing not only a Weyl rescaling, but also a Legendre transformation. We built a general theory of higher–derivative supergravity coupled to matter and displayed the bosonic component lagrangian. This theory is equivalent to Einstein supergravity coupled to matter with two additional chiral superfields. It is better to work in this formulation since the scalar potential can be easily calculated in order to see whether supersymmetry is spontaneously broken or not.

We have shown that the auxiliary fields \((M, \bar{M}, b_a)\) are not invariant under a Weyl rescaling. When we work at the superfield level the whole set of fields \((e_m^a, \psi_m^\alpha, M, \bar{M}, b_a)\) has to be modified if a Weyl rescaling is performed. Furthermore, we studied \(U(1)_K\) supergravity where we have constructed a general lagrangian with a chiral superfield \( \phi \) of weight \( \omega(\phi) \). As it well known, the formulation of supergravity in \(U(1)_K\) superspace presents a natural framework for describing matter coupled to supergravity \([5]\). In this formulation \( R \) has a weight two, and, unfortunately, when we replace this field in our general calculation, one can see that the fields \( X, D_X \) and \( \tilde{A}_m \) can not be expressed in terms of the supergravity multiplet \((e_m^a, \psi_m^\alpha, M, \bar{M}, b_a)\).
We expect that higher-derivative supergravity will help us to understand the role played by supergravity auxiliary fields in supersymmetry breaking.

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