Chasing a Drunk Robber in Many Classes of Graphs

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Abstract
A cops and robbers game (CR) on graphs was originated in 1983 by Quilliot and by Nowakowski and Winkler independently. This game has been applied in many problems in the area of theoretical computer science such as information seeking, robot motion planning or network security as evidenced by many conferences and publications. The CR game has also been extensively studied and varied to many versions. In this paper, we focus on CR game under the condition that the robber performs a random walk. Namely, he drunkenly, or randomly, chooses his next move to escape the cop, while the cop plays optimally in order to minimize the expected drunk capture times $dct(G)$ of a graph $G$. We apply the concepts of expected hitting time in Markov Chain and combinatorial technique to find $dct(G)$ for graphs $G$ that have been used in many applications which are cycles, complete multipartite graphs, grid graphs and prism graphs.

Keywords Pursuit–evasion · Cops and robbers · Random walks

AMS Subject Classification 05C57 · 05C8 · 91A43

1 Introduction and Motivation
All graphs in this paper are finite connected and simple. For a graph $G$, we let $V(G)$ be the vertex set and $E(G)$ be the edge set. The order of $G$ is $|V(G)|$. For a vertex $v \in V(G)$, the neighbor set $N(v)$ of $v$ in $G$ is the set of vertices which are adjacent to $v$. A subset $S$ of $V(G)$ is independent if every pair of vertices in $S$ are not adjacent. A $k$-complete multipartite graph...
A grid graph if it is either \((A_i)\) subsets let grid graph \(P_n\) disjoint cycles \(\leq\) and \(yb\) \(yb\) by joining \(c_i\) to \(c_i'\) for all \(0 \leq i \leq n - 1\). For paths \(P_n = x_1, x_2, \ldots, x_n\) and \(P_m = y_1, y_2, \ldots, y_m\), we let grid graph \(P_n \square P_m\) be the graph having the vertex set \{(a, b) : a \in \{1, 2, \ldots, n\}\) and \(b \in \{1, 2, \ldots, m\}\) and any two vertices \((a, b)\) and \((a', b')\) adjacent if and only if \(a = a'\) and \(y_b y_b' \in E(P_m)\) or \(x_a x_a' \in E(P_n)\) and \(b = b'\). A vertex \((a, b)\) is said to be a corner of a grid graph if it is either \((1, 1)\), \((1, m)\), \((n, 1)\) or \((n, m)\), further, \((a, b)\) is on the boundary if \(a = 1, b = 1, a = n\) or \(b = m\).

Cop and Robber game on graphs(CR), which is a subclass of a famous game so-called pursuit and evasion, was introduced independently by Nowakowski and Winkler [17] and Quilliot [20] in 1983. The game is played on finite, connected, undirected graphs. In their classic version of CR, when the game starts, the cop lands on a vertex as her initial position and then, so does the robber. Then, the cop moves to adjacent vertices or stays in her place, after that, the robber does the same. This is counted one turn. Both players alternate move with full information from the others. The cop aims to capture the robber by moving on the same vertex as the robber is. The robber tries to escape the cop. The cop wins if she has a strategy to catch the robber in a finite number of moves, while the robber wins if he has a strategy to escape from the cop forever. A graph is called cop-win iff a single cop has a strategy which guarantees capture of a single robber, no matter how the robber plays. Otherwise, the graph is called robber-win and we need more than one cop to capture the robber. In the robber-win graphs, we usually find the minimum number of cops that are needed to catch the robber. This is called the cop number of graphs, \(c(G)\). Another direction to study CR is to find \(k\)-capture time which is the number of turns that \(k\) cops need to spend until catching the robber.

Many CR variations have been introduced by changing the environment of the dragnet, see [4, 5, 15] for example, or modifying some rules of moves, see [6, 14] for example. A classic result proved by Aigner and Fromme [1] is that on planar graphs three cops suffice to catch the robber. That is \(c(G) = 3\) if a graph \(G\) is planar. (This theorem is applied in many studies of CR in different topological graphs such as [2, 7].) Further, Pisantechakool and Tan [19] proved that three cops on planar graphs of order \(n\) need at most \(2n\) moves to catch the robber. Another example of finding a capture time is provided by Mehrabian [16] who proved that two cops need at most \(\frac{n+m}{2} - 1\) turns to catch the robber when we play on grid graphs \(P_n \square P_m\). That is:

**Theorem 1** [16] Let \(G\) be a grid graph \(P_n \square P_m\). Then the 2-capture time of \(G\) is at most \(\frac{n+m}{2} - 1\).

One well-known study CR with the different rules of moves is to consider when robber performs a random walk to escape from cops, while cops play optimally to minimize an expected value of capture time. This game is called Cops and Drunk Robbers, CDR and was introduced by Kehagias and Prałat [10]. In this paper, we aim to find a drunk capture time of one cop to catch one drunk robber, or just drunk from now on, in prism graphs, grid graphs, complete multipartite graphs and cycles. We may establish the structure of our game type in the following.

**Structure of the Game** Now, we provide more details of our CDR rules. Mainly, we use notations from Kehagias and Prałat [10] and Komarov and Winkler [13]. In the work of Kehagias and Prałat [10], for integers \(i, j, k\) such that \(j \geq 0\) and \(1 \leq i \leq k\), the authors applied stochastic process to find expected value of drunk capture time. They let \(x_j^i\) be the
position of \(i^{th}\) cop at the time \(j\) and let \(X_j = (x^{1}_{j}, x^{2}_{j}, \ldots, x^{k}_{j})\) be the vector of the positions of all \(k\) cops at time \(j\). Similarly, they let \(y_j\) be the position of the \textit{drunk} at time \(j\). When the game starts the cops choose their initial positions \(X_0 \in V(G)^k\), then the drunk chooses his initial position \(y_0 \in V(G)\) with probability \(1/|V(G)|\). For the \(j^{th}\) turn when \(j \geq 1\), cops move from \(X_{j-1}\) to \(X_j\), then the drunk moves from \(y_{j-1}\) to \(y_j\) with probability \(1/|N(v)|\) where \(v = y_{j-1}\). Since each player moves along an edge, we have that \(\{x^{i}_{j-1}, x^{i}_{j}\} \in E(G)\) and \(\{y_{j-1}, y_j\} \in E(G)\). Further, cops and the drunk need to change their positions in every turn. The cops will catch the drunk if some of the cops and the drunk are on the same vertex. Because cops move first in each turn, the catching possibilities are (1) one cop moves to the drunk’s position, that is \(x^{i}_{j} = y_{j-1}\) or (2) the drunk moves to some of the cop’s positions, that is \(x^{i}_{j} = y_j\) for some \(1 \leq i \leq k\). By this assumption, the capture time \(T\) is a random variable. Further, for any \(x \in V(G)^k\) and \(y \in V(G)\), they let

\[ dct_{x,y}(G, k) = \mathbb{E}(T | X_0 = x, y_0 = y \text{ and } k \text{ cops play optimally}) \]

Cops play optimally means they move with a strategy that minimizes \(dct_{x,y}(G, k)\). We may let the capture time when \(k\) cops land on the vertices of vector \(x\) be:

\[ dct_x(G, k) = \sum_{y \in V(G)} \frac{dct_{x,y}(G, k)}{|V(G)|} \]

Hence, we define the \textit{expected \(k\)-drunk capture time} to be:

\[ dct(G, k) = \min_{x \in V(G)^k} \sum_{y \in V(G)} \mathbb{P}(y_0 = y) \cdot \mathbb{E}(T | X_0 = x, y_0 = y \text{ and } k \text{ cops play optimally}) \]

\[ = \min_{x \in V(G)^k} \sum_{y \in V(G)} \frac{dct_{x,y}(G, k)}{|V(G)|} \]

\[ = \min_{x \in V(G)^k} dct_x(G, k) \]

When \(k = 1\), we write \(dct(G)\) instead of \(dct(G, 1)\) and we call \(dct(G)\) the \textit{drunk capture time}, further, \(x\) will denote a vertex rather than a vector of vertices. Interestingly, Kehagias and Pralat [10] proved that only one cop suffices to catch the drunk even the graphs are robber-win. That is they proved that:

\textbf{Theorem 2} [10] \textit{If \(G\) is a finite connected graph, then one cop suffices to catch the drunk.}

So, in this study, we assume to have only one cop in CDR. Further, it is worth noting that, when the robber is drunk, the optimal strategy for the cop to minimize the capture time needs not be the same as when she chases the normal robber, the robber who is not drunk. For example, the optimal strategy for the cop in the Graph \(G_1\) in Fig. 1, informed in [18], is to locate herself at the vertex \(x_2\), we have \(dct_{x_2}(G) = 13/12\). But if she lands at the vertex \(x_3 \in V(G_1)\) (which is not an optimal strategy for original CR), we have \(dct_{x_3}(G) = 163/168 < 13/12 = dct_{x_2}(G)\). Therefore, we are more likely to obtain \(dct(G)\) by landing the cop at \(x_3\) than at \(x_2\).

Recently, Komarov and Winkler [13] introduced the powerful method of chasing the drunk, so-called \textit{Gross and Fine}, to show that the single cop usually spends at most \(n + o(n)\) moves to catch the drunk where \(n\) is the order of graphs. By this strategy, Komarov and Winkler [13] proved that:

\textbf{Theorem 3} [13] \textit{On a connected simple graph of order \(n\). The single cop will capture the drunk in expected time at most \(n + o(n)\). Namely, \(dct(G) \leq n + o(n)\).}
This paper is organized as follows. We discuss the background and motivation of cops and drunk robbers game in Sect. 1. In Sect. 2, we state all the main theorems concerning expected capture times when the single cop chases the drunk robber in many classes of well-known graphs, while the proofs are given in Sect. 3. In each of the main theorems, we also provide numerical results which are run by two algorithms, one of which has been developed by using Monte Carlo simulation, while the other has been modified by the algorithm so-called CADR, (Cop Against Drunk Robber) from Kehagias and Prałat [9]; note that this is an exact algorithm, i.e., it produces the correct optimal \( dct \) value. All the codes have been uploaded in https://github.com/nuttanon19701/CDRMCG.git.

### 2 Main Results

In this paper, we aim to find the drunk capture time of the single cop when we play on prism graphs \( C_n \square P_2 \), grid graphs \( P_n \square P_m \), \( k \)-complete multipartite graphs \( K_{m_1,m_2,...,m_k} \), and Cycles \( C_n \). By quickly applying Theorem 3, we know that

\[
dct(C_n \square P_2) \leq \frac{n}{4} + \frac{5}{6} + o(1)
\]

Firstly, we employ the technique of recurrence relations to reduce \( \text{dct}(C_n \square P_2) \) to

\[
\begin{cases}
\frac{n}{4} + \frac{5}{6} - \frac{1}{9m} + \frac{1}{9m} \left( \frac{1}{2} \right) \frac{n}{2} & \text{if } \frac{n}{2} \text{ is even}, \\
\frac{n}{4} + \frac{5}{6} - \frac{1}{9m} - \frac{1}{9m} \left( \frac{1}{2} \right) \frac{n}{2} & \text{if } \frac{n}{2} \text{ is odd}, \\
\frac{n}{4} + \frac{5}{6} - \frac{13}{36n} - \frac{7\sqrt{2}}{9n} \left( \frac{1}{4} \right) \frac{n}{2} & \text{if } \frac{n-1}{2} \text{ is odd}, \\
\frac{n}{4} + \frac{5}{6} - \frac{13}{36n} - \frac{2\sqrt{2}}{9n} \left( \frac{1}{4} \right) \frac{n}{2} & \text{if } \frac{n-1}{2} \text{ is even}.
\end{cases}
\]

**Remark 1** For the graph \( C_3 \square P_2 \), the cop will choose her initial position to be the vertex 0. The probability that the robber lands his initial position at the vertex 0 is \( \frac{1}{6} \) and the capture
Table 1  Comparison of the results in prism graphs

| n  | 4      | 5      | 6      | 7      | 10     | 25     | 100    | 175    | 250    |
|----|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| Theorem 4 | 1.8125 | 2.0000 | 2.3125 | 2.5179 | 3.3219 | 7.0689 | 25.8322 | 63.3329 |        |
| Simulation | 1.8145 | 1.9969 | 2.3026 | 2.5182 | 3.3233 | 7.0727 | 25.8609 | 63.2810 |        |
| CADR    | 1.8125 | 2.0000 | 2.3125 | 2.5179 | 3.3219 | 7.0689 | 25.8322 | 63.3329 |        |

If the robber lands at vertices 1, 2 and 1’, then the capture time will be 1 since the cop can catch the robber in the next move. This will occur with probability $\frac{1}{2}$. For the other 2 vertices, the cop will move to the vertex 1’ and the capture time will be

$$dct_{0,2'}(C_3 □ P_2) = 1 + \frac{1}{3}dct_{1',1'}(C_3 □ P_2) + \frac{1}{3}dct_{1',1'}(C_3 □ P_2) + \frac{1}{3}dct_{1',2'}(C_3 □ P_2)$$

Because $dct_{1',1'}(C_3 □ P_2) = 0$, $dct_{1',1'}(C_3 □ P_2) = dct_{0,2'}(C_3 □ P_2)$ and $dct_{1',2'}(C_3 □ P_2) = dct_{0,2'}(C_3 □ P_2)$, it follows that

$$\frac{2}{3}dct_{0,2'}(C_3 □ P_2) = 1 + \frac{1}{3}$$

$$dct_{0,2'}(C_3 □ P_2) = 2.$$

Since the graph is symmetric, $dct_{0,2'}(C_3 □ P_2) = dct_{0,3'}(C_3 □ P_2)$. Hence, the drunk capture time of the graph $C_3 □ P_2$ is

$$dct(C_3 □ P_2) \leq \frac{7}{6}$$

which less than

$$\frac{3}{4} + \frac{5}{6} - \frac{13}{36(3)} - \frac{7\sqrt{2}}{9(3)} \left(\frac{1}{4}\right)^{\frac{3}{2}}$$

from Theorem 4 when $\frac{n-1}{2}$ is odd as the cop will chase the drunk robber with a different strategy which will be detailed in the proof in Sect. 3. However, Table 1 illustrates that the results from Theorem 4 agree with all the two numerical computations. Hence, we (informally) claim that our strategy in Sect. 3 appears to be optimal.

In grid graphs $P_n □ P_m$, by applying the method from Komarov and Winkler [13] together with selecting an appropriate vertex to land the initial position of the cop, we prove that:

**Theorem 5**  Let $G$ be a grid graph $P_n □ P_m$. Then

$$dct(G) \leq t + \frac{9\sqrt{t}}{2} + 1 + 2\ln(4\sqrt{2}q^3) + 11 = t + o(t)$$

where $q = \lceil \frac{n}{2} \rceil + \lceil \frac{m}{2} \rceil$ and

$$t = \begin{cases} 
\frac{s+p}{2} & \text{if } n = 2s \text{ and } m = 2p, \\
\frac{s+p}{2} + \frac{s}{4s+2} + \frac{p}{4p+2} & \text{if } n = 2s + 1 \text{ and } m = 2p + 1, \\
\frac{s+p}{2} + \frac{s}{4s+2} & \text{if } n = 2s + 1 \text{ and } m = 2p.
\end{cases}$$
Theorem 6
Let \( K_{m} \) where \( M \) is an integer, \( m \) is a positive integer.

Interestingly, \( \text{dct}(P_n \square P_m) \) from Theorem 5 is roughly \((n + m)/4\) which is only half of the number of moves from the one that 2 cops need in Theorem 1 when they chased one non-drunk robber on grid graphs \( P_n \square P_m \) with the classic rule. Table 2 illustrates the comparison between Theorem 5 and the numerical results.

For \( k \)-complete multipartite \( K_{m_1,m_2,\ldots,m_k} \) when \( k \geq 3 \) without loss of generality \( m_1 \leq m_2 \leq \cdots \leq m_k \). Recall that \( M = \sum_{i=1}^{k} m_i \). We prove the following theorems.

**Theorem 6** Let \( K_{m_1,m_2,\ldots,m_k} \) be a \( k \)-complete multipartite graph, where \( m_i \leq m_j \) for \( 1 \leq i < j \leq k \). The drunk capture time is

\[
\text{dct}(K_{m_1,m_2,\ldots,m_k}) = \frac{M_1}{M} + \frac{m_1 - 1}{MM_1(1 - x)} \left[ \frac{2(M - 1)}{M_2(1 - x)} \left( \frac{M_1,2 + m_2 - 1}{M_1} \right) + 1 \right]
\]

where \( M_i = M - m_i \), \( M_{i,j} = M - m_i - m_j \) and \( x = \frac{(m_{i-1} - 1)(m_2 - 1)}{M_1 M_2} \).

**Theorem 7** Let \( K_{m_1,m_2,\ldots,m_k} \) be a \( k \)-complete multipartite graph where \( m_1 = m_2 = \cdots = m_1 = n \) for \( 2 \leq l \leq k \) and \( m_j > n \) for all \( j > l \). The drunk capture time is

\[
\text{dct}(K_{m_1,m_2,\ldots,m_k}) = \frac{M^2 - M - Mn + 2n - n^2}{M(M - 2n + 1)}.
\]

By Theorem 7, when \( k = l \), we have the following corollary.

**Corollary 1** For a \( k \)-complete multipartite graph \( K_{n,n,\ldots,n} \), the drunk capture time is

\[
\text{dct}(K_{n,n,\ldots,n}) = \frac{nk - n - 1}{nk - 2n + 1} - \frac{n - 2}{k(nk - 2n + 1)}.
\]

Table 3 illustrates the comparison between Theorem 6 and the numerical results.

We also find \( \text{dct}(G) \) when \( G \) is a complete bipartite graphs.

**Theorem 8** For a complete bipartite graph \( K_{m,n} \) where \( m \leq n \), the drunk capture time is

\[
\text{dct}(K_{m,n}) = \frac{2m^2n - m^2 + n^2 - mn - n + m}{(m + n - 1)(m + n)}.
\]
Table 4 Comparison of the results in complete bipartite graphs

| (m, n) | (2, 2) | (3, 15) | (5, 5) | (10, 15) | (20, 20) | (20, 50) | (50, 50) |
|--------|--------|---------|--------|----------|---------|---------|---------|
| Theorem 8 | 1.0000 | 1.4020 | 2.5000 | 4.9500 | 10.0000 | 8.5031 | 25.0000 |
| Simulation | 1.0020 | 1.3899 | 2.5110 | 4.9046 | 9.9754 | 8.5208 | 24.8860 |
| CADR | 1.0000 | 1.4020 | 2.5000 | 4.9499 | 9.9999 | 8.5030 | 24.9999 |

Table 5 Comparison of the results in cycles

| n | 3 | 4 | 5 | 6 | 10 | 25 | 100 | 175 | 250 |
|---|---|---|---|---|----|----|-----|-----|-----|
| Theorem 9 | 0.6667 | 1.0000 | 1.2000 | 1.5000 | 2.5000 | 6.2400 | 25.0000 | 43.7486 | 62.5000 |
| Simulation | 0.6673 | 0.9966 | 1.2019 | 1.5056 | 2.4979 | 6.2396 | 24.9412 | 43.7371 | 62.4055 |
| CADR | 0.6667 | 1.0000 | 1.2000 | 1.5000 | 2.5000 | 6.2400 | 25.0000 | 43.7486 | 62.5000 |

Corollary 2 For a complete bipartite graph $K_{n,n}$, the drunk capture time is

$$dct(K_{n,n}) = \frac{n}{2}.$$

Proof From Theorem 8, when $m = n$, we have that

$$dct(K_{n,n}) = \frac{2n^3 - n^2}{(2n - 1)(2n)} = \frac{n}{2}.$$  

It is worth noting that Corollary 2 can be proved by using Corollary 1 with $l = k = 2$ too. Table 4 illustrates the comparison between Theorem 8 and the numerical results. From Tables 3 and 4, the results from Theorems 6–8 agree with the numerical computations. We may claim that the strategies of these theorems, which will be introduced in Sect. 3, appear to be optimal.

Finally, we prove that the cop usually moves roughly $1/4$ of the order when she chases the drunk in $C_n$.

Theorem 9 For a cycle $C_n$, the drunk capture time is

$$dct(C_n) = \begin{cases} \frac{n}{4} & \text{if } n \text{ is even,} \\ \frac{n}{4} - \frac{1}{4n} & \text{if } n \text{ is odd.} \end{cases}$$

The following table illustrates the comparison between Theorem 9 and the numerical results (Table 5).

3 Proofs

We provide proofs of Theorems 4–9 in this sections. Remark that the cop plays by our strategies. Hence, there is no probability from the cop. All the proofs obtain probabilities of moving from the drunk only.
Proof of Theorem 4. First of all, for the sake of convenience, we may write a vertex in $C_n \square P_2$ by only its index. That is, we let $\{0, 1, \ldots, n-1\}$ be the vertex set of $C_n$ and $\{0', 1', \ldots, (n-1)’\}$ be the vertex set of $C_n'$. Since $C_n \square P_2$ is symmetric, we assume that the initial position of the cop is at the vertex 0. Further, for vertices of $C_n'$, it is more convenience to calculate an upper bound of $dct(G)$ in this proof if we let index of $i'$ be the same as the distance between itself and the vertex 0 for $1 \leq i \leq n/2$. Hence, for each vertex $i'$ of $C_n'$, we rename it to be $(i + 1)'$. Therefore, $V(C_n') = \{1', 2', \ldots, n'\}$ and the vertex $i$ of $C_n$ is adjacent to the vertex $(i + 1)'$ of $C_n'$ for all $0 \leq i \leq n - 1$. Recall that $y_0$ is the initial position of the drunk. The cop’s strategy is to minimize the capture time by moving on the cycle $C_n$ in the direction that is closer to the drunk until (1) she is at the vertex $i$ and the drunk moves to vertices $(i + 1)'$ or $i + 1$, both of which the drunk will be arrested in the next turn of the cop or (2) she is at the vertex $i$ and the drunk moves to her. Although the results in this theorem agree with the numerical computations as shown in Table 1, we do not prove that our strategy is an optimal and all the results are upper bounds of $dct(C_n \square P_2)$.

In the following, we apply recurrence relations to count $\mathbb{E}(T|y_0 = i)$ and $\mathbb{E}(T|y_0 = i')$. First, we may assume that $y_0 = i$ as illustrated in Fig. 2 where the white square vertex is the initial position of the cop and black square vertex is the initial position of the drunk. □

Clearly, after the cop finished moving her first turn, the drunk can move to vertices $i - 1$, $(i + 1)'$ and $i + 1$, all of which with probability $1/3$. If he moves to $i - 1$, the distance between two players will be $i - 2$ and the capture time will be $\mathbb{E}(T|y_0 = i - 2)$. If he moves to $(i + 1)'$, the capture time will be the same as when the initial positions of the cop and the drunk are 0 and $i'$, respectively. Hence, the capture time is $\mathbb{E}(T|y_0 = i')$. Finally, if he moves to $i + 1$, the distance between them is the same and both players are on the same cycle which implies that the capture time is $\mathbb{E}(T|y_0 = i)$. Since all the cases occur after the first move of the cop, we have that

$$\mathbb{E}(T|y_0 = i) = 1 + \frac{1}{3}\mathbb{E}(T|y_0 = i - 2) + \frac{1}{3}\mathbb{E}(T|y_0 = i') + \frac{1}{3}\mathbb{E}(T|y_0 = i)$$

which implies that

$$\mathbb{E}(T|y_0 = i) = \frac{3 + \mathbb{E}(T|y_0 = i - 2) + \mathbb{E}(T|y_0 = i')}{2}. \quad (1)$$

Similarly, we have that

$$\mathbb{E}(T|y_0 = i') = \frac{3 + \mathbb{E}(T|y_0 = (i - 2)') + \mathbb{E}(T|y_0 = i - 2)}{2} \quad (2)$$

with initial conditions $\mathbb{E}(T|y_0 = 0) = 0$, $\mathbb{E}(T|y_0 = 1) = \mathbb{E}(T|y_0 = 1') = 1$ and $\mathbb{E}(T|y_0 = 2)$, $\mathbb{E}(T|y_0 = 2')$ can be calculated by

$$\mathbb{E}(T|y_0 = 2) = 1 + \frac{1}{3}\mathbb{E}(T|y_0 = 0) + \frac{1}{3}\mathbb{E}(T|y_0 = 2') + \frac{1}{3}\mathbb{E}(T|y_0 = 2).$$
\[ \mathbb{E}(T|y_0 = 2') = 1 + \frac{1}{3}\mathbb{E}(T|y_0 = 0) + \frac{1}{3}\mathbb{E}(T|y_0 = 2') + \frac{1}{3}\mathbb{E}(T|y_0 = 2') \]

We get \( \mathbb{E}(T|y_0 = 2) = \mathbb{E}(T|y_0 = 2') = 3 \). Since \( C_n \sqcap P_2 \) is symmetric, we have that

\[ \mathbb{E}(T|y_0 = i) = \mathbb{E}(T|y_0 = n - i) \quad \text{and} \quad \mathbb{E}(T|y_0 = i') = \mathbb{E}(T|y_0 = (n + 2 - i)'). \]

Hence, it suffices to consider only when \( y_0 = 0, 1, \ldots, \frac{n}{2} \) and \( y_0 = 1', 2', \ldots, (\frac{n}{2} + 1)' \).

By solving the system of linear non-homogeneous recurrence relations (1) and (2), we obtain 4 expected values of \( T \) depending on the parity of the position of \( y_0 \).

\[
\mathbb{E}(T|y_0 = i) = \begin{cases} 
  i + 1 - \frac{2}{3} - \frac{4}{3} \left( \frac{1}{4} \right)^{\frac{i}{2} + \frac{1}{2}} & \text{if } i \text{ is odd,} \\
  i + \frac{4}{3} - \frac{4}{3} \left( \frac{1}{4} \right)^{\frac{i}{2}} & \text{if } i \text{ is even.}
\end{cases}
\]

\[
\mathbb{E}(T|y_0 = j') = \begin{cases} 
  j + 1 - \frac{5}{3} + \frac{8}{3} \left( \frac{1}{4} \right)^{\frac{j}{2} + \frac{1}{2}} & \text{if } j \text{ is odd,} \\
  j + \frac{1}{3} + \frac{8}{3} \left( \frac{1}{4} \right)^{\frac{j}{2}} & \text{if } j \text{ is even.}
\end{cases}
\]

The drunk capture time is calculated by the expected value by

\[
\text{dct}(C_n \sqcap P_2) \leq \mathbb{E}(T) = \sum_{v \in V(C_n \sqcap P_2)} \mathbb{P}(y_0 = v) \mathbb{E}(T|y_0 = v)
= \sum_{i=0}^{n-1} \mathbb{P}(y_0 = i) \mathbb{E}(T|y_0 = i) + \sum_{j=1}^{n} \mathbb{P}(y_0 = j') \mathbb{E}(T|y_0 = j')
\]

**Case 1** For \( n \) even (Fig. 3)

\[
\mathbb{E}(T) = \sum_{i=0}^{n-1} \mathbb{P}(y_0 = i) \mathbb{E}(T|y_0 = i) + \sum_{j=1}^{n} \mathbb{P}(y_0 = j') \mathbb{E}(T|y_0 = j')
= \frac{1}{2n} \left[ \mathbb{E}(T|y_0 = 0) + \mathbb{E}(T|y_0 = 1') + \mathbb{E}(T|y_0 = n/2) + \mathbb{E}(T|y_0 = (n/2 + 1)') \right]
+ \frac{2}{2n} \left[ \sum_{i=1}^{\frac{n}{2}-1} \mathbb{E}(T|y_0 = i) + \sum_{j=2}^{\frac{n}{2}} \mathbb{E}(T|y_0 = j') \right]
\]

**Case 1.1** For \( \frac{n}{2} \) even

\[
\mathbb{E}(T) = \frac{1}{2n} \left[ \mathbb{E}(T|y_0 = 0) + \mathbb{E}(T|y_0 = 1') + \mathbb{E}(T|y_0 = n/2) + \mathbb{E}(T|y_0 = (n/2 + 1)') \right]
+ \frac{2}{2n} \left[ \sum_{i=1}^{\frac{n}{2}-1} \mathbb{E}(T|y_0 = i) + \sum_{j=2}^{\frac{n}{2}} \mathbb{E}(T|y_0 = j') \right]
= \frac{1}{2n} \left[ \frac{\frac{n}{2} - 5}{3} + \frac{8}{3} \left( \frac{1}{4} \right)^{\frac{\frac{n}{2} + 1}{2}} + \frac{n}{2} + 2 \right]
+ \frac{2}{2n} \left[ \sum_{i=1}^{\frac{n}{2}-1} \mathbb{E}(T|y_0 = i) + \sum_{j=2}^{\frac{n}{2}} \mathbb{E}(T|y_0 = j') \right]
\]
By using geometric series, we get

\[
\frac{2}{2n} \left[ \sum_{i=1}^{2^{-1}} \mathbb{E}(T|y_0 = i) \right] = \frac{2}{2n} \left[ \sum_{i=1}^{2^{-1}} \mathbb{E}(T|y_0 = 2i - 1) + \sum_{i=1}^{2^{-1}} \mathbb{E}(T|y_0 = 2i) \right]
\]

\[
= \frac{2}{2n} \left[ \sum_{i=1}^{2^{-1}} \left( 2i - \frac{2}{3} - \frac{4}{3} \left( \frac{1}{4} \right)^i \right) + \sum_{i=1}^{2^{-1}} \left( 2i + \frac{4}{3} - \frac{4}{3} \left( \frac{1}{4} \right)^i \right) \right]
\]

\[
= \left( \frac{n}{16} + \frac{1}{12} - \frac{4}{9n} + \frac{4}{9n} \left( \frac{1}{4} \right)^{\frac{2}{2}} \right)
\]

\[
+ \left( \frac{n}{16} + \frac{1}{12} - \frac{16}{9n} + \frac{16}{9n} \left( \frac{1}{4} \right)^{\frac{2}{2}} \right)
\]

\[
= \frac{n}{8} + \frac{1}{6} - \frac{20}{9n} + \frac{20}{9n} \left( \frac{1}{4} \right)^{\frac{2}{2}}
\]

\[
\frac{2}{2n} \left[ \sum_{j=2}^{2} \mathbb{E}(T|y_0 = j') \right] = \frac{2}{2n} \left[ \sum_{j=1}^{2} \mathbb{E}(T|y_0 = (2j)') + \sum_{j=1}^{2} \mathbb{E}(T|y_0 = (2j + 1)') \right]
\]

\[
= \frac{2}{2n} \left[ \sum_{j=1}^{2} \left( 2j + \frac{1}{3} + \frac{8}{3} \left( \frac{1}{4} \right)^j \right) \right]
\]

\[
+ \sum_{j=1}^{2} \left( 2j + 2 - \frac{5}{3} + \frac{8}{3} \left( \frac{1}{4} \right)^{j+1} \right)
\]

\[
= \left( \frac{n}{16} + \frac{1}{3} + \frac{8}{9n} - \frac{8}{9n} \left( \frac{1}{4} \right)^{\frac{2}{2}} \right)
\]
Therefore, the drunk capture time is

\[
dct(C_n \square P_2) \leq \mathbb{E}(T) = \left[ \frac{1}{2} + \frac{4}{3n} - \frac{1}{3n} \left( \frac{1}{4} \right)^{\frac{q}{2}} \right] + \frac{2}{2n} \left[ \sum_{i=1}^{q-1} \mathbb{E}(T|y_0 = i) + \sum_{j=2}^{q} \mathbb{E}(T|y_0 = j') \right]
\]

\[
= \left[ \frac{1}{2} + \frac{4}{3n} - \frac{1}{3n} \left( \frac{1}{4} \right)^{\frac{q}{2}} \right] + \left[ \frac{n}{8} + \frac{1}{6} - \frac{20}{9n} + \frac{20}{9n} \left( \frac{1}{4} \right)^{\frac{q}{2}} \right]
\]

\[
= \frac{n}{4} + \frac{5}{6} - \frac{1}{9n} + \frac{1}{9n} \left( \frac{1}{4} \right)^{\frac{q}{2}}
\]

**Case 1.2** For \( \frac{q}{2} \) odd

\[
\mathbb{E}(T) = \frac{1}{2n} \left[ \mathbb{E}(T|y_0 = 0) + \mathbb{E}(T|y_0 = 1') + \mathbb{E}(T|y_0 = n/2) + \mathbb{E}(T|y_0 = (n/2 + 1)') \right]
\]

\[
+ \frac{2}{2n} \left[ \sum_{i=1}^{q-1} \mathbb{E}(T|y_0 = i) + \sum_{j=2}^{q} \mathbb{E}(T|y_0 = j') \right]
\]

\[
= \frac{1}{2n} \left[ 0 + 1 + \frac{1}{3} + \frac{8}{3} \left( \frac{1}{4} \right)^{\frac{q+1}{2}} \right] + \frac{n}{2} + 1 - \frac{2}{3} - \frac{4}{3} \left( \frac{1}{4} \right)^{\frac{q+1}{2}} + n + 1
\]

\[
+ \frac{2}{2n} \left[ \sum_{i=1}^{q-1} \mathbb{E}(T|y_0 = i) + \sum_{j=2}^{q} \mathbb{E}(T|y_0 = j') \right]
\]

\[
= \left[ \frac{1}{2} + \frac{4}{3n} + \frac{1}{3n} \left( \frac{1}{4} \right)^{\frac{q}{2}} \right] + \frac{2}{2n} \left[ \sum_{i=1}^{q-1} \mathbb{E}(T|y_0 = i) + \sum_{j=2}^{q} \mathbb{E}(T|y_0 = j') \right]
\]

By using geometric series, we get

\[
\frac{2}{2n} \left[ \sum_{i=1}^{q-1} \mathbb{E}(T|y_0 = i) \right] = \frac{2}{2n} \left[ \sum_{i=1}^{q-\frac{1}{2}} \mathbb{E}(T|y_0 = 2i - 1) + \sum_{i=1}^{q-\frac{1}{2}} \mathbb{E}(T|y_0 = 2i) \right]
\]

\[
= \frac{2}{2n} \left[ \sum_{i=1}^{q-\frac{1}{2}} \left( 2i - \frac{2}{3} - \frac{4}{3} \left( \frac{1}{4} \right)^i \right) + \sum_{i=1}^{q-\frac{1}{2}} \left( 2i + \frac{4}{3} - \frac{4}{3} \left( \frac{1}{4} \right)^i \right) \right]
\]
\[
\begin{align*}
&= \left( \frac{n}{16} - \frac{1}{6} - \frac{13}{36n} + \frac{8}{9n} \left( \frac{1}{4} \right)^{\frac{n}{2}} \right) \\
&\quad + \left( \frac{n}{16} + \frac{1}{3} - \frac{49}{36n} + \frac{8}{9n} \left( \frac{1}{4} \right)^{\frac{n}{2}} \right) \\
&= \frac{n}{8} + \frac{1}{6} - \frac{31}{18n} + \frac{16}{9n} \left( \frac{1}{4} \right)^{\frac{n}{2}}
\end{align*}
\]

\[
\frac{2}{2n} \left[ \sum_{j=2}^{\frac{n}{2}} \mathbb{E}(T | y_0 = j) \right] = \frac{2}{2n} \left[ \sum_{j=1}^{\frac{n}{2}-1} \mathbb{E}(T | y_0 = (2j)) + \sum_{j=1}^{\frac{n}{2}-1} \mathbb{E}(T | y_0 = (2j+1)) \right]
\]

\[
= \frac{2}{2n} \left[ \sum_{j=1}^{\frac{n}{2}-1} \left( 2j + \frac{1}{3} + \frac{8}{3} \left( \frac{1}{4} \right)^{j} \right) \right]
\]

\[
\quad + \sum_{j=1}^{\frac{n}{2}-1} \left( 2j + 2 - \frac{5}{3} + \frac{8}{3} \left( \frac{1}{4} \right)^{j+1} \right)
\]

\[
= \left( \frac{n}{16} + \frac{1}{12} + \frac{17}{36n} - \frac{16}{9n} \left( \frac{1}{4} \right)^{\frac{n}{4}} \right)
\]

\[
+ \left( \frac{n}{16} + \frac{1}{12} - \frac{7}{36n} - \frac{4}{9n} \left( \frac{1}{4} \right)^{\frac{n}{4}} \right)
\]

\[
= \frac{n}{8} + \frac{1}{6} + \frac{10}{36n} - \frac{20}{9n} \left( \frac{1}{4} \right)^{\frac{n}{4}}
\]

Therefore, the drunk capture time is

\[
dct(C_n \square P_2) \leq \mathbb{E}(T) = \left[ \frac{1}{2} + \frac{4}{3n} + \frac{1}{3n} \left( \frac{1}{4} \right)^{\frac{n}{4}} \right]
\]

\[
+ \frac{2}{2n} \left[ \sum_{i=1}^{\frac{n}{2}-1} \mathbb{E}(T | y_0 = i) + \sum_{j=2}^{\frac{n}{2}} \mathbb{E}(T | y_0 = j) \right]
\]

\[
= \left[ \frac{1}{2} + \frac{4}{3n} + \frac{1}{3n} \left( \frac{1}{4} \right)^{\frac{n}{4}} \right] + \left[ \frac{n}{8} + \frac{1}{6} - \frac{31}{18n} + \frac{16}{9n} \left( \frac{1}{4} \right)^{\frac{n}{4}} \right]
\]

\[
+ \left[ \frac{n}{8} + \frac{1}{6} + \frac{10}{36n} - \frac{20}{9n} \left( \frac{1}{4} \right)^{\frac{n}{4}} \right]
\]

\[
= \frac{n}{4} + \frac{5}{6} - \frac{1}{9n} - \frac{1}{9n} \left( \frac{1}{4} \right)^{\frac{n}{4}}
\]
Case 2 For $n$ odd (Fig. 4)

\[
\mathbb{E}(T) = \sum_{i=0}^{n-1} \mathbb{P}(y_0 = i) \mathbb{E}(T | y_0 = i) + \sum_{j=1}^{n} \mathbb{P}(y_0 = j') \mathbb{E}(T | y_0 = j')
\]

\[
= \frac{1}{2n} \left[ \mathbb{E}(T | y_0 = 0) + \mathbb{E}(T | y_0 = 1') \right] + \frac{2}{2n} \left[ \sum_{i=1}^{\frac{n-1}{2}} \mathbb{E}(T | y_0 = i) + \sum_{j=2}^{\frac{n+1}{2}} \mathbb{E}(T | y_0 = j') \right]
\]

\[
= \frac{1}{2n} + \frac{2}{2n} \left[ \sum_{i=1}^{\frac{n-1}{2}} \mathbb{E}(T | y_0 = i) + \sum_{j=2}^{\frac{n+1}{2}} \mathbb{E}(T | y_0 = j') \right]
\]

Case 2.1 For $\frac{n-1}{2}$ even

By using geometric series, we get

\[
\frac{2}{2n} \left[ \sum_{i=1}^{\frac{n-1}{2}} \mathbb{E}(T | y_0 = i) \right] = \frac{2}{2n} \left[ \sum_{i=1}^{\frac{n-1}{4}} \mathbb{E}(T | y_0 = 2i - 1) + \sum_{i=1}^{\frac{n-1}{4}} \mathbb{E}(T | y_0 = 2i) \right]
\]

\[
= \frac{2}{2n} \left[ \sum_{i=1}^{\frac{n-1}{4}} \left( 2i - 2 - \frac{4}{3} \left( \frac{1}{4} \right)^i \right) + \sum_{i=1}^{\frac{n-1}{4}} \left( 2i + \frac{4}{3} - \frac{4}{3} \left( \frac{1}{4} \right)^i \right) \right]
\]

\[
= \frac{n}{16} - \frac{1}{24} - \frac{67}{144n} + \frac{4\sqrt{2}}{9n} \left( \frac{1}{4} \right) \frac{n}{4}
\]

\[
+ \frac{n}{16} + \frac{11}{24} - \frac{139}{144n} + \frac{4\sqrt{2}}{9n} \left( \frac{1}{4} \right) \frac{n}{4}
\]

\[
= \frac{n}{8} + \frac{5}{12} - \frac{206}{144n} + \frac{8\sqrt{2}}{9n} \left( \frac{1}{4} \right) \frac{n}{4}
\]

\[
\frac{2}{2n} \left[ \sum_{j=2}^{\frac{n+1}{2}} \mathbb{E}(T | y_0 = j') \right] = \frac{2}{2n} \left[ \sum_{j=1}^{\frac{n-1}{4}} \mathbb{E}(T | y_0 = (2j)') + \sum_{j=1}^{\frac{n-1}{4}} \mathbb{E}(T | y_0 = (2j+1)') \right]
\]
\[
\begin{align*}
\frac{2}{2n} & \left[ \sum_{j=1}^{\frac{n-1}{4}} \left( 2j + \frac{1}{3} + \frac{8}{3} \left( \frac{1}{4} \right)^j \right) \\
+ \sum_{j=1}^{\frac{n-1}{4}} \left( 2j + 2 - \frac{5}{3} + \frac{8}{3} \left( \frac{1}{4} \right)^{j+1} \right) \right] \\
= & \frac{n}{16} + \frac{5}{24} + \frac{89}{144n} - \frac{8\sqrt{2}}{9n} \left( \frac{1}{4} \right)^{\frac{n}{4}} \\
+ & \frac{n}{16} + \frac{5}{24} - \frac{7}{144n} - \frac{2\sqrt{2}}{9n} \left( \frac{1}{4} \right)^{\frac{n}{4}} \\
= & \frac{n}{8} + \frac{5}{12} + \frac{82}{144n} - \frac{10\sqrt{2}}{9n} \left( \frac{1}{4} \right)^{\frac{n}{4}} \\
\end{align*}
\]

Therefore, the drunk capture time is

\[
\text{dct}(C_n \square P_2) \leq \mathbb{E}(T) = \frac{1}{2n} + \frac{2}{2n} \left[ \sum_{i=1}^{\frac{n-1}{4}} \mathbb{E}(T|y_0 = i) + \sum_{j=2}^{\frac{n}{4}} \mathbb{E}(T|y_0 = j^+) \right] \\
= \frac{1}{2n} + \left[ \frac{n}{8} + \frac{5}{12} - \frac{206}{144n} + \frac{8\sqrt{2}}{9n} \left( \frac{1}{4} \right)^{\frac{n}{4}} \right] \\
+ \left[ \frac{n}{8} + \frac{5}{12} + \frac{82}{144n} - \frac{10\sqrt{2}}{9n} \left( \frac{1}{4} \right)^{\frac{n}{4}} \right] \\
= \frac{n}{4} + \frac{5}{6} - \frac{13}{36n} - \frac{2\sqrt{2}}{9n} \left( \frac{1}{4} \right)^{\frac{n}{4}} \\
\]

**Case 2.2** For \( \frac{n-1}{2} \) odd

There is some special initial position of robber which is \( R = \left( \frac{n+1}{2} \right)^{\frac{n-1}{2}} \) and \( R = \left( \frac{n+3}{2} \right)^{\frac{n-1}{2}} \). The expected capture time \( \mathbb{E}\left(T|y_0 = \left( \frac{n+1}{2} \right)^{\frac{n-1}{2}}\right) = \mathbb{E}\left(T|y_0 = \left( \frac{n+3}{2} \right)^{\frac{n-1}{2}}\right) \) can be calculated by the following equation

\[
\mathbb{E}\left(T|y_0 = \left( \frac{n+1}{2} \right)^{\frac{n-1}{2}}\right) = \frac{3 + \mathbb{E}\left(T|y_0 = \left( \frac{n-1}{2}\right)^{\frac{n-1}{2}}\right) + \mathbb{E}\left(T|y_0 = \frac{n+1}{2}\right)}{2} \\
= \frac{n}{2} + 2 - \frac{7}{6} + \frac{4}{6} \left( \frac{1}{4} \right)^{\frac{n+1}{4}} \\
\]

By using geometric series, we get

\[
\frac{2}{2n} \left[ \sum_{i=1}^{\frac{n-1}{4}} \mathbb{E}(T|y_0 = i) \right] = \frac{2}{2n} \left[ \sum_{i=1}^{\frac{n+1}{4}} \mathbb{E}(T|y_0 = 2i - 1) + \sum_{i=1}^{\frac{n+3}{4}} \mathbb{E}(T|y_0 = 2i) \right] \\
= \frac{2}{2n} \left[ \sum_{i=1}^{\frac{n+1}{4}} \left( 2i - \frac{2}{3} - \frac{4}{3} \left( \frac{1}{4} \right)^i \right) + \sum_{i=1}^{\frac{n+3}{4}} \left( 2i + \frac{4}{3} - \frac{4}{3} \left( \frac{1}{4} \right)^i \right) \right] \\
\]
\[
= \frac{n}{16} + \frac{5}{24} - \frac{43}{144n} + \frac{2\sqrt{2}}{9n} \left( \frac{1}{4} \right)^{\frac{q}{4}} \\
+ \frac{n}{16} + \frac{5}{24} - \frac{235}{144n} + \frac{8\sqrt{2}}{9n} \left( \frac{1}{4} \right)^{\frac{q}{4}} \\
= \frac{n}{8} + \frac{5}{12} - \frac{139}{72n} + \frac{10\sqrt{2}}{9n} \left( \frac{1}{4} \right)^{\frac{q}{4}}
\]

\[
\frac{2}{2n} \left[ \sum_{j=2}^{\frac{n+1}{4}} \mathbb{E}(T | y_0 = j') \right] = \frac{2}{2n} \left[ \sum_{j=1}^{\frac{n+1}{4}} \mathbb{E}(T | y_0 = (2j)'') + \sum_{j=1}^{\frac{n-3}{4}} \mathbb{E}(T | y_0 = (2j + 1)') \right] \\
= \frac{2}{2n} \left[ \sum_{j=1}^{\frac{n+1}{4}} \left( 2j + 1 + \frac{8}{3} \left( \frac{1}{4} \right)^{\frac{j}{4}} \right) \\
+ \sum_{j=1}^{\frac{n+1}{4}} \left( 2j + 2 - \frac{5}{3} + \frac{8}{3} \left( \frac{1}{4} \right)^{j+1} \right) \right] \\
+ \frac{2}{2n} \left[ \sum_{j=1}^{\frac{n+1}{4}} \left( 2j + 2 - \frac{7}{6} + \frac{4}{6} \left( \frac{1}{4} \right)^{\frac{j}{4}} \right) \right] \\
= \frac{n}{16} + \frac{11}{24} + \frac{185}{144n} - \frac{13\sqrt{2}}{9n} \left( \frac{1}{4} \right)^{\frac{q}{4}} \\
+ \frac{n}{16} - \frac{1}{24} - \frac{31}{144n} - \frac{4\sqrt{2}}{9n} \left( \frac{1}{4} \right)^{\frac{q}{4}} \\
= \frac{n}{8} + \frac{5}{12} + \frac{77}{72n} - \frac{17\sqrt{2}}{9n} \left( \frac{1}{4} \right)^{\frac{q}{4}}
\]

Therefore, the drunk capture time is

\[
dct(C_n \Box P_2) \leq \mathbb{E}(T) = \frac{1}{2n} + \frac{2}{2n} \left[ \sum_{i=1}^{\frac{n-1}{4}} \mathbb{E}(T | y_0 = i) + \sum_{j=1}^{\frac{n}{4}} \mathbb{E}(T | y_0 = j') \right] \\
= \frac{1}{2n} + \left[ \frac{n}{8} + \frac{5}{12} - \frac{139}{72n} + \frac{10\sqrt{2}}{9n} \left( \frac{1}{4} \right)^{\frac{q}{4}} \right] \\
+ \left[ \frac{n}{8} + \frac{5}{12} + \frac{77}{72n} - \frac{17\sqrt{2}}{9n} \left( \frac{1}{4} \right)^{\frac{q}{4}} \right] \\
= \frac{n}{4} + \frac{5}{6} - \frac{13}{36n} - \frac{7\sqrt{2}}{9n} \left( \frac{1}{4} \right)^{\frac{q}{4}}
\]

**Proof of Theorem 5** In this section, we establish an upper bound of an expected capture time on grid graphs, \( P_n \Box P_m \) where \( n \leq m \). As mentioned earlier, we apply the method from Komarov and Winkler [13] together with finding an appropriate vertex to land the cop. In their classic paper, the method consists of 2 progresses which are Gross and Fine, each of
which has 2 stages. However, to prove Theorem 5 it suffices to reduce first progress to only one stage. Hence, in the following, we may introduce the modified version for this paper, and if the readers would like to see the original version, please see in [13].

The first stage is in the part of Gross Progress which the cop aims to terminate at the initial position of drunk, the vertex $y_0$. That is, the cop does not want to chase the drunk directly but she is quite sure that, once she arrives at the initial place of the drunk, he is still around. This is the idea of the first stage. After this, it will be in the part of Fine Progress which, at Stage 2, cop moves 4 steps at Turns $i, i + 1, i + 2$ and $i + 3$ on a shortest path to the position of drunk at Turn $i$. She, the cop, will re-target every 4 steps on a shortest path of each turn until she is close enough to the drunk so that she can chase him directly without spending too many moves; this is Stage 3.

First, we may give results concerning the number of vertices at distance $k$ from a given vertex on grid graphs which are needed in the first progress.

**Observation 1** For any vertex $(a, b)$ of a grid graph, the number of vertices having distance $k$ from $(a, b)$ is at most $4k$.

By Observation 1, we can find an upper bound of the number of vertices that have distance from $l$ to $q$ from any given vertex.

**Observation 2** For a vertex $(a, b)$ of a grid graph, the number of vertices that have distance from $l$ to $q$ from $(a, b)$ is at most $4q^2$.

**Proof** By Observation 1, we have that the number of vertices that have distance from $l$ to $q$ of $(a, b)$ is $4(l + (l + 1) + \cdots + q) = 4(q(q + 1)/2 - (l - 1)/2) \leq 4q^2$ and this completes the proof. □

Further, for a grid graph $P_n \Box P_m$, we may assume that the cop always locates at the middle of graphs that is the vertex $(\lceil n/2 \rceil, \lceil m/2 \rceil)$ to make her far from all vertices at most $\lfloor n/2 \rfloor + \lfloor m/2 \rfloor$ moves. By a simple counting, we obtain the following observation concerning the number of vertices at distance $k$ from the vertex $(\lceil n/2 \rceil, \lceil m/2 \rceil)$ of grid graphs $P_n \Box P_m$ when $n \leq m$.

**Observation 3** Let $G$ be a grid graph $P_n \Box P_m$ when $n \leq m$, and let $s = \lfloor n/2 \rfloor$, $p = \lfloor m/2 \rfloor$. For an integer $0 \leq k \leq s + p$, we let $l(k)$ be as follows:

$$l(k) = \begin{cases} 1 & \text{if } k = 0, \\ 4k & \text{if } 0 < k < s, \\ 4s - 1 & \text{if } k = s, \\ 4s & \text{if } s < k < p, \\ 4s - 1 & \text{if } k = p, \\ 4(s + p - k) & \text{if } p < k < s + p, \\ 1 & \text{if } k = s + p. \end{cases}$$

Further, we let $l_s(k)$ and $l_p(k)$ as follows:

$$l_s(k) = \begin{cases} 1 & \text{if } k = s, \\ 2 & \text{if } s < k < s + p, \\ 1 & \text{if } k = s + p. \end{cases}$$
\[ l_p(k) = \begin{cases} 1 & \text{if } k = p, \\ 2 & \text{if } p < k < s + p, \\ 1 & \text{if } k = s + p. \end{cases} \]

Then for the number of vertices having distance \( k \) from the vertex \( ([n/2], [m/2]) \), \( l^*(k) \), define to be

\[ l^*(k) = \begin{cases} l(k) & \text{if } n = 2s \text{ and } m = 2p, \\ l(k) + l_p(k) & \text{if } n = 2s \text{ and } m = 2p + 1, \\ l(k) + l_s(k) & \text{if } n = 2s + 1 \text{ and } m = 2p, \\ l(k) + l_s(k) + l_p(k) & \text{if } n = 2s + 1 \text{ and } m = 2p + 1. \end{cases} \]

Note that we have defined that a vertex of grid graph is denoted by an ordered pair; however, for the sake of convenience, we may let a small alphabet denote a vertex of grid graph too. Hence, we recall from Section 1 that \( x_0 \) and \( y_0 \) are the initial positions that the cop and the drunk land when the game starts, respectively. That is \( x_0 = ([n/2], [m/2]) \). Our first result in this section shows that drunk usually commits a crime roughly \((n + m)/4\) steps away from the cop, the proof of which applies Observation 3. Hence, we might omit the proof.

**Theorem 10** For a grid graph \( P_n \square P_m \), if the cop lands at the vertex \( x_0 = ([n/2], [m/2]) \), the expected value of distance between the initial position of cop and drunk is

\[ E(d(x_0, y_0)) = \begin{cases} s + p \quad & \text{if } n = 2s \text{ and } m = 2p, \\ s + p + \frac{s}{4s+2} + \frac{p}{4p+2} & \text{if } n = 2s + 1 \text{ and } m = 2p + 1, \\ s + p + \frac{s}{4s+2} & \text{if } n = 2s + 1 \text{ and } m = 2p. \end{cases} \]

By Theorem 10, we may let \( t = E(d(x_0, y_0)) \). That is

\[ t = \begin{cases} s + p \quad & \text{if } n = 2s \text{ and } m = 2p, \\ s + p + \frac{s}{4s+2} + \frac{p}{4p+2} & \text{if } n = 2s + 1 \text{ and } m = 2p + 1, \\ s + p + \frac{s}{4s+2} & \text{if } n = 2s + 1 \text{ and } m = 2p. \end{cases} \]

Note that \( t \) will be the expected value of the distance between cop and drunk when the game starts.

In the following, for \( 1 \leq i \leq 3 \), we let

- \( T_i \) be the number of moves that cop spends at Stage \( i \) and
- \( D_i \) be the distance between cop and drunk at the end of Stage \( i \).

Remind that, at the first stage, cop goes to the crime scene. By Theorem 10, we have that

\[ E(T_i) = E(d(x_0, y_0)) = t. \]

It is worth noting that, after the drunk lands his first position, \( t \) is fixed. We let \( q \) be the maximum distance of any vertex from \( x_0 \), that is \( q = \lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor \). By the famous...
Lemma 1 Let $y_0, y_1, \ldots, y_t$ be a random walk with $t$ moves on a grid graph. Then, $\mathbb{E}(d(y_0, y_t)) \leq 1 + \sqrt{t} \sqrt{1 + 2 \ln(4\sqrt{2}q^3)}$.

Proof We let $p_t(y_0, y)$ be the probability that a random walk starting from $y_0$ stops at $y$ when we moves exactly $t$ steps. Recall that all vertices of a grid graph have degree either 2, 3 or 4. Hence $\deg_G(y)/\deg_G(x) \leq 2$ for any pair vertices $x, y$ of grid graph. We may let $c = \sqrt{\frac{1 + 2 \ln(4\sqrt{2}q^3)}{\sqrt{t}}}$ where $c = \sqrt{\frac{1 + 2 \ln(4\sqrt{2}q^3)}{\sqrt{t}}}$ By Varopoulos–Carne’s bound and Observation 2, we have that

$$\mathbb{P}(d(y_0, y_t) \geq c\sqrt{t}) = \sum_{y : d(y_0, y) \geq c\sqrt{t}} p_t(y_0, y)$$

$$\leq \sum_{y : d(y_0, y) \geq c\sqrt{t}} \sqrt{e} \sqrt[4]{\frac{\deg_G(y)}{\deg_G(y_0)}} \exp\left(-\frac{d(y_0, y)^2}{2t}\right)$$

$$\leq \sum_{y : d(y_0, y) \geq c\sqrt{t}} \sqrt{\frac{c\sqrt{2}}{2t}} \exp\left(-\frac{(c\sqrt{t})^2}{2t}\right)$$

$$\leq 4q^2 \sqrt{2e} \frac{1-q^2}{2t} \frac{1+2\ln(4\sqrt{2}q^3)}{2t}$$

$$= 4\sqrt{2q} e^{\frac{1-1-2\ln(4\sqrt{2}q^3)}{2t}}$$

$$\leq 1/q.$$ 

Since $t$ is fixed, it follows that

$$\mathbb{E}(d(y_0, y_t)|d(y_0, y_t) < c\sqrt{t}) < c\sqrt{t}.$$ 

Clearly, $\mathbb{E}(d(y_0, y_t)|d(y_0, y_t) \geq c\sqrt{t}) \leq t < q$ because $y_0$ and $y_t$ are $t$ steps apart and $t < q$ by their definitions. Hence, 

$$\mathbb{E}(d(y_0, y_t)) = \mathbb{E}(d(y_0, y_t)|d(y_0, y_t) < c\sqrt{t}) \mathbb{P}(d(y_0, y_t) < c\sqrt{t})$$

$$+ \mathbb{E}(d(y_0, y_t)|d(y_0, y_t) \geq c\sqrt{t}) \mathbb{P}(d(y_0, y_t) \geq c\sqrt{t})$$

$$\leq c\sqrt{t}(1 + q(1/q)) = 1 + \sqrt{t} \sqrt{1 + 2 \ln(4\sqrt{2}q^3)}$$

and this completes the proof.

Observation 4 For any random walk $z_0, \ldots, z_4$ with 4 steps on grid graph $G$, we have that $\mathbb{P}(d(z_0, z_4) < 4) \geq 8/9.$
Proof Since $\mathbb{P}(d(z_0, z_4) = 4) + \mathbb{P}(d(z_0, z_4) < 4) = 1$, it suffices to show that $\mathbb{P}(d(z_0, z_4) = 4) \leq 1/9$. For $1 \leq i \leq 4$, we may let $D_i$ be the set of all vertices that have distance $i$ from $z_0$. For any vertex $y \in D_4$, we have that $|N_{D_4}(y)| = 1$ or $|N_{D_4}(y)| = 2$ because $G$ is grid graph.

When $|N_{D_4}(y)| = 1$. Clearly, $\mathbb{P}(d(z_0, z_4) = 4) \leq 1/54$ where the equality holds when $z_0$ is a corner and $y$ is on the boundary. When $|N_{D_4}(y)| = 2$, we may let $\{u_1, u_2\} = N_{D_4}(y)$. Since $G$ is a grid graph, $|N_{D_2}(u_1) \cap N_{D_2}(u_2)| = 1$. Further one of $N_{D_2}(u_1) \setminus N_{D_2}(u_2)$ and $N_{D_2}(u_2) \setminus N_{D_2}(u_1)$ is non-empty. If one of which is empty, $\mathbb{P}(d(z_0, z_4) = 4)$ is maximized at $10/108$ when $G$ is $P_2 \Box P_4$ and $z_0$ is at $(1, 1)$ and $z_4$ is at $(2, 4)$. If both are non-empty, $\mathbb{P}(d(z_0, z_4) = 4)$ is maximized at $1/9$ when $G$ is $P_3 \Box P_3$ and $z_0$ is at $(1, 1)$ and $z_4$ is at $(3, 3)$. This completes the proof.

Now, we are in Stage 2 which is the first stage of Fine Progress. We may recall that, at the beginning of this stage, cop is at $y_0$ and drunk is at $y_I$. As we mentioned before, at Step 4$k$ for $0 \leq k$, the cop will find a shortest path $P^k$ to drunk and moves along $P^k$ with 4 steps before re-targeting at Step 4$(k + 1)$ to find a shortest path $P^{k+1}$ between her and drunk and moves along $P^{k+1}$. She keeps doing this until, after drunk moves, the distance between her and drunk is at most 3. Every time that cop re-targets, drunk walks randomly 4 steps. By Observation 4, the distance between them decreases at least one with probability at least 8/9. Hence, we let $w_k$ be a $\{0, 1\}$-random variables which $\mathbb{P}(w_k = 1) = 8/9$ for all $k \in \mathcal{N}$, that is $w_k = 1$ when the distance decreases and, otherwise, $w_k = 0$. Further, we let $S_k = w_1 + \cdots + w_k$ and let $\{w_k\}_{k \in \mathcal{N}}$ be the random process with the rule that the process is ended at time $\tau$ if $S_\tau = D_1 - 3$, that is when the distance between cop and drunk decreases $D_1 - 3$ yielding that two of them are at most 3 apart and the next move belongs to cop. By Wald’s identity [21], we have that

$$
\mathbb{E}(S_\tau) = \mathbb{E}(\tau)\mathbb{E}(w_k) = \mathbb{E}(\tau) \left( 1 \cdot \frac{8}{9} + 0 \cdot \frac{1}{9} \right) = \frac{8}{9} \mathbb{E}(\tau).
$$

Since $\mathbb{E}(S_\tau) = \mathbb{E}(D_1) - 3$, we have that $\mathbb{E}(\tau) = 9/8(\mathbb{E}(D_1) - 3)$. Because cop spends 4 moves in every turn before re-targeting, we have

$$
\mathbb{E}(T_2) = 4\mathbb{E}(\tau) = 9/2(\mathbb{E}(D_1) - 3) < 9/2(\sqrt{1 + 2 \ln(4\sqrt{2}q^3)} - 2).
$$

Finally, at Stage 3, the cop starts chasing the drunk directly and it is the cop’s turn when this stage begins. Once the drunk moves backward to the cop, he will be under arrest by the next move of the cop. We may let $\mathbb{P}(r \rightarrow c)$ be the probability that the drunk moves backward to the cop and $\mathbb{P}(cr \rightarrow)$ be the probability that the drunk moves away from the cop. Clearly, $\mathbb{P}(r \rightarrow c) \leq 1$. As $G$ is a grid graph, $\mathbb{P}(cr \rightarrow) \leq 3/4$. Hence,

$$
\mathbb{E}(T_3) = \sum_{k=2}^{\infty} k \mathbb{P}(cr \rightarrow)^{k-2} \mathbb{P}(r \rightarrow c)
\leq \sum_{k=2}^{\infty} k (3/4)^{k-2} (1)
= 2 + 3 \left( \frac{3}{4} \right)^1 + 4 \left( \frac{3}{4} \right)^2 + \cdots
= \frac{4}{3} \left( 2 \left( \frac{3}{4} \right) + 3 \left( \frac{3}{4} \right)^2 + 4 \left( \frac{3}{4} \right)^3 \cdots \right)
$$

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We will find $dct$. Further recall that $M_i = M - m_i$ and $x = \frac{(m_1 - 1)(m_2 - 1)}{M_1 M_2}$. We assume that $i$ is the partite set with $m_i$ vertices. To maximize the probability that the drunk will be in the different partite set of the cop in each turn, she chooses her initial position on set $A_1$. Further, in each turn, she moves alternately between $A_1$ and $A_2$ until the drunk makes mistake by moving to a vertex which is adjacent to the cop, a vertex on the different partite set that the cops is currently in. Observe that she is in $A_1$ in Turn $2l$ and in $A_2$ in Turn $2l + 1$ for integer $l \geq 0$. We will find $dct(K_{m_1, m_2, \ldots, m_k})$ by the following equation.

$$dct(K_{m_1, m_2, \ldots, m_k}) = \mathbb{E}(T) = \sum_{n=1}^{\infty} n \mathbb{P}(T = i).$$

First, we will find the probability that the cop catches the drunk at Turn $i$. Hence, we distinguished 2 cases (1) the cop catches the drunk and (2) the drunk randomly moves to the cop.

**Case 1** The cop catches the drunk.

In this case, the cop moves $i$ turns but the drunk moves only $i - 1$ turn. We let $\mathbb{P}_1(T = i)$ be the probability that the cop catches the drunk at Turn $i$ in this case. When $i = 1$, the drunk lands his first position to any partite set which is not $A_1$. That is $\mathbb{P}_1(T = 1) = \frac{M_1}{M}$. We may assume that $i > 1$.

When $i$ is odd, the drunk initially lands in $A_1$ but different vertex of the cop with the probability $\frac{m_2 - 1}{M_1}$. Next, we may let $1 \leq j \leq i - 2$. When $j$ is odd, the drunk is in $A_1$ at Turn $j - 1$ and moves in $A_2$ to a different vertex of the cop at Turn $j$ with the probability $\frac{m_2 - 1}{M_1}$. When $j$ is even, the drunk is in $A_2$ at Turn $j - 1$ and moves in $A_1$ to a different vertex of the cop at Turn $j$ with the probability $\frac{m_1 - 1}{M_2}$. Finally, at Turn $i - 2$, the drunk is in $A_1$ and moves to any partite set apart from $A_2$ with the probability $\frac{M_1 - 1}{M_2}$. Hence, $\mathbb{P}_1(T = i) = \left(\frac{m_1 - 1}{M_1}\right)^{i - 1} \left(\frac{m_2 - 1}{M_2}\right)^{i - 2} \left(\frac{M_1 - 1}{M_2}\right)$. Similarly, when $i$ is even, we have that $\mathbb{P}_1(T = i) = \left(\frac{m_1 - 1}{M_1}\right)^{i - 1} \left(\frac{m_2 - 1}{M_2}\right)^{i - 2} \left(\frac{M_1 - 1}{M_2}\right)$. Therefore,

$$\mathbb{P}_1(T = i) = \begin{cases} \frac{M_1}{M}, & i = 1 \\ \left(\frac{m_1 - 1}{M_1}\right)^{i - 1} \left(\frac{m_2 - 1}{M_2}\right)^{i - 2} \left(\frac{M_1 - 1}{M_2}\right), & i \text{ is odd and } i \neq 1 \\ \left(\frac{m_1 - 1}{M_1}\right)^{i - 1} \left(\frac{m_2 - 1}{M_2}\right)^{i - 2} \left(\frac{M_1 - 1}{M_2}\right), & i \text{ is even} \end{cases}$$

**Case 2** The drunk randomly moves to the cop.
In this case, both the cop and the drunk move $i$ turns. By similar arguments as Case 1, we have that
\[ P_2(T = i) = \begin{cases} \frac{1}{M}, & i = 0 \\ \left( \frac{m_1-1}{M_1} \right) \left( \frac{m_1-1}{M_2} \right)^{i-1} ; & i \text{ is odd} \\ \left( \frac{m_1-1}{M_1} \right) \left( \frac{m_1-1}{M_2} \right)^{i-1} \left( \frac{m_1-1}{M_1} \right)^{\frac{i}{2}} ; & i \text{ is even and } i \neq 0 \end{cases} \]

Hence, the probability that the cop catches the drunk at Turn $i$ is $P(T = i) = P_1(T = i) + P_2(T = i)$. Therefore,
\[
det(K_{m_1,m_2,...,m_k}) = E(T) = \sum_{i=1}^{\infty} i P(T = i) = \sum_{i=1}^{\infty} i P_1(T = i) + \sum_{i=1}^{\infty} i P_2(T = i)
\]

From the probability $P_1$ and $P_2$, we get
\[
\sum_{i=1}^{\infty} i P_1(T = i) = \frac{m_1 - 1}{MM_1 (1-x)} \left[ \frac{2M_{1,2} (M - 1)}{M_2 (1-x)} + M_{1,2} \frac{m_2 - 1}{M - 2} \right] + \frac{M_1}{M}
\]

and
\[
\sum_{i=1}^{\infty} i P_2(T = i) = \frac{m_1 - 1}{MM_1 (1-x)} \left[ \frac{2(M - 1) (m_2 - 1)}{M_1 M_2 (1-x)} + 1 \right]
\]

Hence, the drunk capture time of $K_{m_1,m_2,...,m_k}$ is
\[
det(K_{m_1,m_2,...,m_k}) = \frac{M_1}{M} + \frac{m_1 - 1}{MM_1 (1-x)} \left[ \frac{2(M - 1)}{M_2 (1-x)} \left( M_{1,2} + \frac{m_2 - 1}{M_1} \right) + M_{1,2} \left( \frac{m_2 - 1}{M_2} \right) + 1 \right]
\]

\[ \square \]

**Proof of Theorem 7** Recall that $m_1 = m_2 = \cdots = m_l = n$ for $2 \leq l \leq k$ and $m_j > n$ for all $j > l$. We let $A_j$ be the partite set having $m_j$ vertices. Since the cop plays optimally, she lands her initial position on a smallest partite set that has $n$ vertices. Assume without loss of generality that the cop lands on $A_1$ and moves alternately between $A_1$ and $A_2$ until the drunk randomly moves to a vertex joining to her position. We will find $det(K_{m_1,m_2,...,m_k})$ by the following equation.

\[ det(K_{m_1,m_2,...,m_k}) = E(T) = \sum_{i=1}^{\infty} i P(T = i). \]

First, we will find the probability that the cop catches the drunk at Turn $i$. We distinguished 2 cases (1) the cop catches the drunk and (2) the drunk randomly moves to the cop.

**Case 1** The cop catches the drunk.

In this case, the cop moves $i$ turns but the drunk moves only $i - 1$ turn. We let $P_1(T = i)$ be the probability that the cop catches the drunk at Turn $i$ in this case. If $i = 1$, the drunk lands in any set which is not $A_1$ and the cop catches him in the first turn. Thus, $P_1(T = 1) = \frac{M_1 - n}{M}$. 

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If \( i \geq 2 \), the drunk lands his first position in \( A_1 \) but different vertex of the cop with the probability \( \frac{n-1}{M} \). Then, for \( 1 \leq j \leq i - 2 \), the drunk moves to the same partite set of the cop at Turn \( j \) with the probability \( \frac{n-1}{M-n} \). Finally, in Turn \( i - 1 \), he moves to a partite set which is neither \( A_1 \) nor \( A_2 \) with the probability \( \frac{M-2n}{M-n} \). Thus, \( P_1(T = i) = (\frac{n-1}{M})(\frac{M-2n}{M-n})(\frac{n-1}{M-n})^{i-2} \) when \( i \geq 2 \). Therefore,

\[
P_1(T = i) = \begin{cases} \frac{M-n}{M}, & i = 1 \\ \left(\frac{n-1}{M}\right) \left(\frac{M-2n}{M-n}\right) \left(\frac{n-1}{M-n}\right)^{i-2}, & i \geq 2. \end{cases}
\]

**Case 2** drunk moves to cops.

In this case, both the cop and the drunk move \( i \) turns. By the similar arguments as Case 1, we have that

\[
P_2(T = i) = \begin{cases} \frac{1}{M}, & i = 0 \\ \left(\frac{n-1}{M}\right) \left(\frac{1}{M-n}\right) \left(\frac{n-1}{M-n}\right)^{i-1}, & i \geq 1. \end{cases}
\]

Hence, the probability that the cop catches the drunk at \( i^{th} \) turn is \( P(T = i) = P_1(T = i) + P_2(T = i) \). Therefore,

\[
dct(K_{m_1,m_2,...,m_k}) = E(T) = \sum_{i=1}^{\infty} i P(T = i) = \sum_{i=1}^{\infty} i P_1(T = i) + \sum_{i=1}^{\infty} i P_2(T = i)
\]

\[
= \frac{M-n}{M} + \left(\frac{n-1}{M}\right) \left(\frac{M-2n}{M-n}\right) \sum_{i=0}^{\infty} (i+2) \left(\frac{n-1}{M-n}\right)^i
\]

\[
+ \left(\frac{n-1}{M}\right) \left(\frac{1}{M-n}\right) \sum_{i=0}^{\infty} i \left(\frac{n-1}{M-n}\right)^{i-1}
\]

\[
= \frac{M-n}{M} + \left(\frac{n-1}{M}\right) \left(\frac{M-2n}{M-n}\right) \left(\frac{2M-3n+1}{M-n}\right) \left(\frac{M-n}{M-2n+1}\right)^2
\]

\[
+ \left(\frac{n-1}{M}\right) \left(\frac{1}{M-n}\right) \left(\frac{M-n}{M-2n+1}\right)^2
\]

\[
= \frac{M^2 - M - Mn + 2n - n^2}{M(M-2n+1)}.
\]

\[\square\]

**Proof of Theorem 8** Recall that \( m \leq n \). We may let \( A_1 \) and \( A_2 \) be the partite sets whose number of vertices are \( m \) and \( n \), respectively. Since the cop plays optimally, she lands in \( A_1 \). The drunk randomly locates (1) in \( A_2 \) with probability \( n/(m+n) \), and he is captured in the first move of the cop, or (2) in \( A_1 \) but different vertex of the cop with the probability \( (m-1)/(m+n) \), or (3) the same vertex of the cop with the probability \( 1/(m+n) \). Clearly,
Case (3) yields the capture time 0 and Case (1) yields the capture time 1. By similar arguments as the proof of Theorems 6 and 7, we have that

$$\mathbb{P}(T = i) = \begin{cases} 
\left( \frac{m-1}{m+n} \right) \left( \frac{m-1}{m} \right)^{i-1} \left( \frac{n-1}{n} \right)^{\frac{i}{2}} \left( \frac{1}{n} \right); & i \text{ is odd} \\
\left( \frac{m-1}{m+n} \right) \left( \frac{m-1}{m} \right)^{i-2} \left( \frac{n-1}{n} \right)^{i} \left( \frac{1}{m} \right); & i \text{ is even and } i \neq 0 
\end{cases}$$

Thus,

$$dct(K_{m,n}) = \mathbb{E}(T) = \sum_{i=0}^{\infty} i \mathbb{P}(T = i)$$

$$= 0 \left( \frac{1}{m+n} \right) + 1 \left( \frac{n}{m+n} \right) + 1 \left( \frac{m-1}{m+n} \right) \left( \frac{1}{n} \right)$$

$$+ 2 \left( \frac{m-1}{m+n} \right) \left( \frac{n-1}{n} \right) \left( \frac{1}{m} \right) + \cdots$$

$$= \sum_{i=1}^{\infty} (2i - 1) \left( \frac{m-1}{m+n} \right) \left( \frac{m-1}{m} \right)^{i-1} \left( \frac{n-1}{n} \right)^{i-1} \left( \frac{1}{n} \right)$$

$$+ \sum_{i=1}^{\infty} (2i) \left( \frac{m-1}{m+n} \right) \left( \frac{m-1}{m} \right)^{i-1} \left( \frac{n-1}{n} \right)^{i} \left( \frac{1}{m} \right) + \left( \frac{n}{m+n} \right)$$

By geometric series formulas and its differentiation

$$= 2 \left( \frac{m-1}{m+n} \right) \left( \frac{1}{n} \right) \left( \frac{1}{1 - \left( \frac{m-1}{m} \right) \left( \frac{n-1}{n} \right)^{\frac{i}{2}}} \right) - \left( \frac{m-1}{m+n} \right) \left( \frac{1}{n} \right) \left( \frac{1}{1 - \left( \frac{m-1}{m} \right) \left( \frac{n-1}{n} \right)^{\frac{i}{2}}} \right)$$

$$+ 2 \left( \frac{m-1}{m+n} \right) \left( \frac{n-1}{n} \right) \left( \frac{1}{m} \right) \left( \frac{1}{1 - \left( \frac{m-1}{m} \right) \left( \frac{n-1}{n} \right)^{\frac{i}{2}}} \right) + \left( \frac{n}{m+n} \right)$$

$$= \left( \frac{m-1}{m+n} \right) \left( \frac{1}{n} \right) \left( \frac{mn}{m+n-1} \right) \left( \frac{m+n-1}{m} \right) \left( \frac{2mn}{m+n-1} - 1 \right) + \frac{n}{m+n}$$

$$= \left( \frac{m-1}{m+n} \right) \left( \frac{m}{m+n-1} \right) (2n-1) + \frac{n}{m+n}$$

$$= \frac{2m^2n - m^2 + n^2 - mn - n + m}{(m+n-1)(m+n)}.$$ 

\[\square\]

**Proof of Theorem 9** Let $D$ be the distance between the cop and the drunk initial positions. Since $C_n$ is symmetric, we have that $D \leq \lfloor \frac{n}{2} \rfloor$. The cop has two ways on a cycle to chase the drunk. As she plays optimal, she chases the drunk on a shortest path between them. Further, if the game is played on $C_n$ when $n$ is even and the cop and the drunk land their initial positions on the opposite vertices, then the cop chases the drunk in the clockwise direction. We have that

$$dct(C_n) = \mathbb{E}(T) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \mathbb{E}(T \mid D = i) \mathbb{P}(D = i).$$

In the following, we will find $\mathbb{E}(T \mid D = i)$. 

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We first consider the case when $i$ is even. In this case, the last move of the drunk is Step $i$ by moving backward to the cop position. For an integer $1 \leq k \leq n/4$, we let $D = 2k$. Hence, the minimum capture time is $k$ when the drunk moves all $k$ steps backward to the cop. For a positive integer $r \geq k$, we will show that the probability that the cop captures the drunk at turn $r$ is $\binom{r-1}{k-1}(\frac{1}{2})^r$. When $r = k$, it is easy to show that the probability is $1/2^r = \binom{k-1}{\frac{1}{2}}^r$. Hence, we may assume that $r \geq k + 1$. Clearly, there are two possibilities of the drunk in each move which are forward (the drunk move one step away from the cop) and backward (the drunk move one step back to the cop) as the two directions between the cop to the drunk become different after the cop’s first move. Hence, there are $(\frac{1}{2})^r$ possible moves in the sample space. Now, we count the number of random moves that the drunk is caught at turn $r$ in his move. It is not difficult to see that the number of random moves is equal to the number of permutations of $b$ (backward) and $f$ (forward) with multiplicities $k$ and $r-k-1$, respectively, which one of $b$ is fixed to be the last position. There are $\binom{k-1}{r-1}$ such permutations. Hence, the probability that the cop captures the drunk at Turn $r$ is $\binom{k-1}{r-1}(\frac{1}{2})^r$.

By applying the identity $\frac{1}{(1-x)^{k+1}} = \sum_{j=0}^{\infty} \binom{j+k}{k} x^j$, we have

$$
\mathbb{E}(T \mid D = 2k) = \sum_{r=k}^{\infty} r \binom{r-1}{k-1} \left(\frac{1}{2}\right)^r
= 2k \sum_{r=k}^{\infty} \frac{r!}{k!(r-k)!} \left(\frac{1}{2}\right)^r
= 2k \sum_{j=0}^{\infty} \binom{j+k}{k} \left(\frac{1}{2}\right)^j
= \frac{2k}{2^{k+1}} \frac{1}{1 - 1/2}^{k+1} = 2k.
$$

Now, we consider the case when $i$ is odd. In this case, last move of drunk is Step $i - 1$ by moving backward and cops captures drunk at Step $i$ in his turn. For an integer $0 \leq k \leq n/4$, we let $D = 2k + 1$. Hence, minimum capture time is $k + 1$ when drunk moves backward to cop. For a positive integer $r \geq k + 1$, we will show that the probability that cop catches drunk at turn $r$ is $\binom{r-2}{k-1}(\frac{1}{2})^{r-1}$. Recall that in this case the drunk’s last move is on turn $r - 1$. Similarly, the number of random moves is equal to the number of permutations of $b$ and $f$ with multiplicities $k$ and $r-k-1$, respectively, which one of $b$ is fixed to be the last position. There are $\binom{k-1}{r-2}$ such permutations. Hence, probability that cop captures drunk at Turn $r$ is $\binom{k-1}{r-2}(\frac{1}{2})^{r-1}$. Hence,

$$
\mathbb{E}(T \mid D = 2k + 1) = \sum_{r=k+1}^{\infty} r \binom{r-2}{k-1} \left(\frac{1}{2}\right)^{r-1} = 2k + 1.
$$

From the two cases, we have $\mathbb{E}(T \mid D = i) = i$. Therefore, when $n$ is even, we have

$$
dct(C_n) = \mathbb{E}(T) = \sum_{i=0}^{\frac{n}{2}} \mathbb{E}(T \mid D = i) \mathbb{P}(D = i)
= \frac{1}{n} \mathbb{E}(T \mid D = 0) + \frac{2}{n} \sum_{i=1}^{\frac{n}{2}-1} \mathbb{E}(T \mid D = i) + \frac{1}{n} \mathbb{E} \left( T \mid D = \frac{n}{2} \right).
$$
\begin{align*}
  &= 0 + \frac{2}{n} \left[ 1 + 2 + \cdots + \frac{n}{2} - 1 \right] + \frac{1}{2} \\
  &= \frac{n}{4}.
\end{align*}

When \( n \) is odd, we have

\[
dct(C_n) = \mathbb{E}(T) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \mathbb{E}(T|D = i) \mathbb{P}(D = i)
\]

\[
= \frac{1}{n} \mathbb{E}(T|D = 0) + \frac{2}{n} \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \mathbb{E}(T|D = i)
\]

\[
= 0 + \frac{2}{n} \left[ 1 + 2 + \cdots + \frac{n-1}{2} \right]
\]

\[
= \frac{n}{4} - \frac{1}{4n}.
\]

\[\square\]

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