COMPLETELY DEGENERATE EQUILIBRIA
OF THE KURAMOTO MODEL ON NETWORKS

DAVIDE SCLOSA

Abstract. Kuramoto Networks contain non-hyperbolic equilibria whose stability is sometimes difficult to determine. We consider the extreme case in which all Jacobian eigenvalues are zero. In this case linearizing the system at the equilibrium leads to a Jacobian matrix which is zero in every entry. We call these equilibria completely degenerate. We prove that they exist for certain intrinsic frequencies if and only if the underlying graph is bipartite, and that they do not exist for generic intrinsic frequencies. In the case of zero intrinsic frequencies, we prove that they exist if and only if the graph has an Euler circuit such that the number of steps between any two visits at the same vertex is a multiple of 4. The simplest example is the cycle graph with 4 vertices. We prove that graphs with this property exist for every number of vertices \( N \geq 6 \) and that they become asymptotically rare for \( N \) large. Regarding stability, we prove that for any choice of intrinsic frequencies, any coupling strength and any graph with at least one edge, completely degenerate equilibria are not Lyapunov stable. As a corollary, we obtain that stable equilibria in Kuramoto Networks must have at least one strictly negative eigenvalue.

1. Motivation

Kuramoto Networks are widespread in neuroscience, biology, chemistry and engineering, as a natural framework for modeling synchronization and pattern formation [2, 3, 10, 11]. Changing the underlying graph, even just removing one edge, can lead to a dramatically different dynamical behavior. Therefore, great effort has been put into understanding dynamics on different classes of graphs: complete graphs [12, 19], cycles [4, 21, 17], bipartite graphs [20], planar graphs [8], dense graphs [14, 15, 16, 19, 22], sparse graphs [18], 3-regular graphs [9], trees [7, 13], and stars [5]. Here, rather than focusing on a particular type of graph, we focus on a particular type of equilibrium. We believe that degenerate equilibria in Kuramoto Networks are always caused by some special combinatorial properties of the underlying graph, which can be exploited to determine stability when linearization cannot. The results in this paper support this belief.

Let \( G \) be a graph with vertices \( 1, \ldots, N \) and adjacency matrix \((a_{jk})_{i,j} \). To every vertex \( k \) we associated a phase \( \theta_k \) in the 1-dimensional \( T = \mathbb{R}/2\pi\mathbb{Z} \), and a constant intrinsic frequency \( \omega_k \in \mathbb{R} \). The connection between vertices is modulated by a coupling strength \( K > 0 \). We call Kuramoto Network on \( G \) the coupled dynamical system

\[
\dot{\theta}_k = \omega_k + K \sum_{j=1}^{N} a_{jk} \sin(\theta_j - \theta_k), \quad k = 1, \ldots, N. \tag{1}
\]

The equilibria of (1) are never isolated, due to the presence of phase-shift symmetry, and in particular they are never hyperbolic. Some equilibria are hyperbolic up to symmetry: a typical example is the fully synchronized state, in which all the vertices
Figure 1. Completely degenerate equilibria on the square and on the hypercube. Blue, green, red and yellow denote 0, $\pi/2$, $\pi$ and $3\pi/2$ respectively.

share the same phase. In this paper we are interested in equilibria that are non-hyperbolic up to symmetry.

We say that an equilibrium is **completely degenerate** if the Jacobian matrix, obtained by linearizing the vector field at the equilibrium, is zero in every entry. As we will see, the Jacobian matrices obtained by linearizing (1) are symmetric. Therefore, they are zero in every entry if and only if all the eigenvalues are zero.

Completely degenerate equilibria are interesting. First, they are a source of counterexamples. For instance, they show that Taylor’s inequalities [19, Lemma 2.1] are not sufficient to guarantee stability. Second, they are equilibria with critical edges (see [9] for the definition), for which little is known. Third, they are closely related to Eulerian graphs, a class of graphs not yet analyzed in Kuramoto Networks literature. Fourth, as their stability cannot be determined by linear stability analysis, they require alternative techniques: in this paper we will combine results on real-analytic gradient systems with graph-theoretical arguments.

2. Zero Intrinsic Frequencies

2.1. **Classification.** Let us begin by analyzing completely degenerate equilibria in the case of zero intrinsic frequencies, that is $\omega_1 = \cdots = \omega_N = 0$. Up to rescaling time we can suppose $K = 1$, obtaining

$$\dot{\theta}_k = \sum_{j=1}^{N} a_{jk} \sin(\theta_j - \theta_k), \quad k = 1, \ldots, N. \tag{2}$$

Let $F$ denote the vector field induced by (2). Let $DF(\theta)$ denote the differential of the vector field at $\theta$. For every $j, k \in \{1, \ldots, N\}$ we have

$$DF(\theta)_{jk} = \begin{cases} a_{jk} \cos(\theta_j - \theta_k) & \text{if } j \neq k, \\ -\sum_{j} a_{jk} \cos(\theta_j - \theta_k) & \text{if } j = k. \end{cases} \tag{3}$$

If (2) contains a completely degenerate equilibrium, we say that the graph $G$ admits completely degenerate equilibria.

**Lemma 1.** A point $\theta \in \mathbb{T}^N$ is a completely degenerate equilibrium if and only if, for every vertex $k$, half of its neighbors have phase $\theta_k + \pi/2$ and the other half $\theta_k - \pi/2$. 


Proof. From (2) and (3) it follows that a point $\theta$ is a completely degenerate equilibrium if and only if
\[ \sum_j a_{jk} \sin(\theta_j - \theta_k) = 0 \] (4)
for every vertex $k$ and
\[ \cos(\theta_j - \theta_k) = 0 \] (5)
for every edge $jk$. These equations are satisfied if and only if for every edge $jk$ we have $\sin(\theta_j - \theta_k) = \pm 1$ and for every vertex half the sines are equal to $+1$, the other half to $-1$. □

Let us recall some definitions from graph theory. A cycle is a closed graph walk without repeated vertices. A circuit is a closed graph walk without repeated edges. An Euler circuit is a circuit that uses all the graph edges. If an Euler circuit exists, the graph is called Eulerian. Notice that every cycle is a circuit.

The following theorem characterizes graphs admitting completely degenerate equilibria as well as the set of completely degenerate equilibria given a graph. Notice that, since distinct connected components have independent dynamics, it is enough to consider connected graphs.

**Theorem 2.** A connected graph $G$ admits a completely degenerate equilibrium if and only if there is an Euler circuit such that the number of steps between any two visits at the same vertex is a multiple of $4$. Every completely degenerate equilibrium is obtained by fixing such an Eulerian circuit, choosing the phase of one vertex arbitrarily and increasing the phase of the next vertex by $\pi/2$ at each step of the circuit.

**Proof.** Suppose that $G$ admits completely degenerate equilibria. Fix a completely degenerate equilibrium $\theta$. We say that a circuit has the property $P$ if at each step the phase increases by $\pi/2$. By Lemma 1, it follows that every edge of $G$ is contained in a circuit satisfying $P$. Moreover, it follows that if we remove from $G$ the edges of a circuit satisfying $P$, then $\theta$ is still a completely degenerate equilibrium. Therefore, the set of edges of $G$ is the union of edge-disjoint circuits satisfying $P$. If two such circuits intersect in a vertex, their union can be walked in a way that makes it a circuit satisfying $P$. Since $G$ is connected, we conclude that there is a circuit satisfying $P$ containing every edge, that is, an Euler circuit.

Conversely, suppose that $G$ contains an Euler circuit satisfying $P$. Fix such a circuit. Choose the phase of one vertex arbitrarily and increase the phase of the next vertex by $\pi/2$ at each step of the Euler circuit. Since the circuit visits every vertex and satisfies $P$, this process defines a phase $\theta_k$ at each vertex $k$ in a consistent way. By Lemma 1, it follows that $\theta$ is a completely degenerate equilibrium. In particular, this proves that $G$ admits completely degenerate equilibria. □

**Example 3.** The cycle graph with $4$ vertices admits completely degenerate equilibria. An explicit example is given in Figure 1. This graph admits two completely degenerate equilibria up to phase-shift symmetry, one for every orientation of the cycle.

**Example 4.** The graph formed by the vertices and edges of the $4$-dimensional hypercube admits completely degenerate equilibria. An explicit example is given in Figure 1. More generally, any even-dimensional hypercube admits completely degenerate equilibria.

There are infinitely many graphs admitting completely degenerate equilibria:
Proposition 5. For every $N \geq 6$ there is a connected graph on $N$ vertices admitting completely degenerate equilibria.

Proof. Explicit examples with $N = 6, 7, 8$ vertices are given in Figure 2. Given a completely degenerate equilibrium on a connected graph with $N$ vertices, we can obtain a completely degenerate equilibrium on a connected graph with $N + 3$ vertices by gluing a 4-cycle to the original graph. For example, the graph with 7 vertices of Figure 2 is obtained by gluing two 4-cycles together. \qed

Although infinite in number, graphs admitting completely degenerate equilibria are asymptotically rare:

Proposition 6. The probability that a graph chosen uniformly at random among the graphs with $N$ vertices admits completely degenerate equilibria goes to 0 as $N$ goes to infinity.

Proof. By Lemma 1 a graph admitting completely degenerate equilibria is triangle-free. Let us prove that triangle-free graphs have asymptotic probability 0. Let $G$ be a graph on $N$ vertices. Partition the vertices into $\lfloor N/3 \rfloor$ subsets of size 3, and possibly a subset of smaller size. There are 8 distinct graphs on 3 vertices, and 7 of these are not triangles. Therefore, the probability that $G$ is triangle-free is at most $(7/8)^{\lfloor N/3 \rfloor}$. In particular, the probability goes to 0 as $N$ goes to infinity. \qed

Remark 7. From Lemma 1 it follows that if $G$ admits completely degenerate equilibria then $G$ contains no cycles of odd length. In particular $G$ is bipartite. That is, bipartiteness is necessary for the existence completely degenerate equilibria in the case of zero intrinsic frequencies. Theorem 2 implies that it is not sufficient. In a later section (Theorem 10) we will show that bipartiteness is necessary and sufficient for the existence of completely degenerate equilibria for some (not necessarily all zero) intrinsic frequencies.

2.2. Stability. Stability of completely degenerate equilibria cannot be determined by linearization. The goal of this section is proving that completely degenerate equilibria are never Lyapunov stable. In particular, they are never asymptotically stable.

It is well known \cite{15, 21} that (2) is a gradient system with respect to the energy function

$$E(\theta) = \sum_{jk \in e(G)} a_{jk} (1 - \cos(\theta_j - \theta_k)).$$

Here $e(G)$ denotes the set of edges of $G$ and each edge $jk$ is counted exactly once in the sum. For every point $\theta \in T^N$ the identity $F(\theta) = -DE(\theta)$ holds. Intuitively, this means that trajectories always evolve in the direction in which $E$ decreases maximally.
Intuitively, the energy decreases over time until a stationary point (or equilibrium) is reached. The connection between stability of and minimality is, however, subtle. There are smooth energy functions with Lyapunov stable equilibria that are not local minimizers and local minimizers that are not Lyapunov stable \cite{1}. However, if the energy function is real-analytic, then the Lyapunov stable equilibria are exactly the local minimizers of the energy \cite{1}. Since (6) is real-analytic, this result applies to our case.

Since distinct connected components have independent dynamics, it is enough to consider connected graphs. Moreover, we assume that the graph has at least one edge. Notice that if $G$ has no edges then there is no dynamics and every point is a (completely degenerate) Lyapunov stable equilibrium.

**Theorem 8.** For every connected graph with at least one edge, completely degenerate equilibria are not Lyapunov stable.

**Proof.** Let $\theta \in \mathbb{T}^N$ be a completely degenerate equilibrium. We will show that $\theta$ is a saddle point of the energy function (6), that is, a point that is neither a local minimizer nor a local maximizer. It follows that $\theta$ is not Lyapunov-stable.

Fix an Euler circuit as in Theorem 2 and let $j, k$ be two consecutive vertices in the circuit. We have $\theta_k = \theta_j + \pi/2$. For every $x \in \mathbb{R}$ let $\theta^x \in \mathbb{T}^N$ be defined as

\[
\begin{aligned}
\theta^x_j &= \theta_j + x, \\
\theta^x_k &= \theta_k - x, \\
\theta^x_h &= \theta_h, & h \notin \{j, k\}.
\end{aligned}
\]

Our goal is proving that $E(\theta) - E(\theta^x)$ changes sign in every neighborhood of $x = 0$. By \[(5)\] and \[(6)\] it follows that

\[
E(\theta) - E(\theta^x) = \sum_{pq \in E(G)} \cos(\theta^x_p - \theta^x_q).
\]

Let us compute this sum by following the circuit. The edge entering $j$ before $jk$, the edge $jk$, and the edge leaving $k$ after $jk$ amount to

\[
\cos(\pi/2 + x) + \cos(\pi/2 - 2x) + \cos(\pi/2 - x) = \sin(2x) - 2\sin(x).
\]

We claim the other terms in the sum are either zero or they cancel out. If an edge is neither adjacent to $j$ nor $k$, then $\theta^x_p = \theta_p$ and $\theta^x_q = \theta_q$ and therefore $\cos(\theta^x_p - \theta^x_q) = 0$.

Every visit to $j$ that is not the one preceding $jk$ amounts to

\[
\cos(\pi/2 + x) + \cos(\pi/2 - x) = -\sin(x) + \sin(x) = 0.
\]

Similarly any other visits to $k$ that is not the one involving $jk$ amounts to 0. Therefore

\[
E(\theta) - E(\theta^x) = \sin(2x) - 2\sin(x).
\]

Since this function changes sign in every neighborhood of $x = 0$, the point $\theta$ is a saddle of the energy. \qed

3. **Non-Zero Intrinsic Frequencies**

3.1. **Classification.** In the previous section we assumed the intrinsic frequencies to be all equal to 0. Let us return to the general case

\[
\dot{\theta}_k = \omega_k + K \sum_{j=1}^N a_{jk} \sin(\theta_j - \theta_k).
\]
Notice that linearizing (2) gives the same Jacobian matrix as the zero-frequency case (3). It follows that a point \( \theta \in \mathbb{T}^N \) is a completely degenerate equilibrium of (7) if and only if
\[
\sum_j a_{jk} \sin(\theta_j - \theta_k) = -\frac{\omega_k}{K}
\] (8)
for every vertex \( k \) and
\[
\cos(\theta_j - \theta_k) = 0
\] (9)
for every edge \( jk \). In particular we obtain:

**Proposition 9.** For a generic choice of intrinsic frequencies \( \omega_1, \ldots, \omega_N \) there are no graphs admitting completely degenerate equilibria.

**Proof.** If equation (8) and equation (9) hold then \( \omega_k/K \) is an integer for every \( k \). Therefore, if there is a graph admitting completely degenerate equilibria then the intrinsic frequencies are all elements of the lattice \( K\mathbb{Z} \).

We showed (Proposition 9) that for generic intrinsic frequencies there are no graphs admitting completely degenerate equilibria. On the other hand, we know that for zero intrinsic frequencies there are infinitely many graphs admitting completely degenerate equilibria (Proposition 5). This suggests asking which graphs admit completely degenerate equilibria for some intrinsic frequencies:

**Theorem 10.** For every graph \( G \) the following facts are equivalent:
(i) There are coupling strength \( K \) and intrinsic frequencies \( \omega_1, \ldots, \omega_N \) such that \( G \) admits completely degenerate equilibria;
(ii) \( G \) is bipartite.

**Proof.** Let \( \theta \) be a completely degenerate equilibrium. By (9) the phase differences of adjacent vertices are \( \pm \pi/2 \). In particular \( G \) contains no cycle of odd length. This is equivalent to bipartiteness.

Conversely, suppose that \( G \) is bipartite. Let the phases of one part be all equal to 0 and of the other part all equal to \( \pi/2 \). Then (9) is satisfied. Now choose any \( K > 0 \) and for every vertex \( k \) define
\[
\omega_k = -K \sum_j a_{jk} \sin(\theta_j - \theta_k).
\]
Then (8) is also satisfied and \( \theta \) is a completely degenerate equilibrium.

3.2. **Stability.** As we will see, the argument used to determine instability in the case of zero intrinsic frequencies can be adapted to the general case. There is, however, a technical difference: while (2) is a gradient system on \( \mathbb{T}^N \), in general (7) is only a gradient system locally, in a neighborhood of the equilibrium. This is explained in more details in the following proof.

**Theorem 11.** For every connected graph with at least one edge, every coupling strength \( K > 1 \) and every intrinsic frequencies \( \omega_1, \ldots, \omega_N \in \mathbb{R} \), the completely degenerate equilibria are not Lyapunov stable.

**Proof.** Up to rescaling time we can suppose \( K = 1 \). Let \( \theta \in \mathbb{T}^N \) be a completely degenerate equilibrium. By (8) and (9) the neighbors of a vertex \( k \) have phases \( \theta_k + \pi/2 \) or \( \theta_k - \pi/2 \). If, for every vertex \( k \), half of its neighbors have phase \( \theta_k + \pi/2 \) and the other half \( \theta_k - \pi/2 \), then from equation (8) it follows that \( \omega_k = 0 \) for every \( k \) and Theorem 8 applies.
Otherwise, there is a vertex $k$ such that $k$ has $d^+$ neighbors with phase $\theta_k + \pi/2$ and $d^-$ neighbors with phase $\theta_k - \pi/2$, where $d^+ \neq d^-$. For every $x \in \mathbb{R}$ let $\theta^x \in \mathbb{T}^N$ be defined as
\[
\begin{align*}
\theta^x_k &= \theta_k + x, \\
\theta^x_j &= \theta_j, 
\end{align*}
\]
In a neighborhood of $\theta$ the system (7) is gradient with energy function
\[
E(\theta) = -\sum_k \omega_k \theta_k + \sum_{jk \in E(G)} (1 - \cos(\theta_j - \theta_k)).
\]
Notice that the terms $\omega_k \theta_k$ are only well-defined locally since there are no injective continuous maps from $\mathbb{T}$ to $\mathbb{R}$. This is not an obstacle for the proof: all we need is the function $x \mapsto E(\theta^x)$ to be defined in some neighborhood of $x = 0$. We have
\[
E(\theta) - E(\theta^x) = \omega_k x + \sum_j a_{jk} (\cos(\theta_j - \theta_k - x) - \cos(\theta_j - \theta_k))
\]
\[
= \omega_k x + d^+ \cos(\pi/2 - x) + d^- \cos(-\pi/2 - x)
\]
\[
= \omega_k x + (d^+ - d^-) \sin(x).
\]
From equation (8) and equation (9) it follows that $d^+ - d^- = -\omega_k$. Therefore
\[
E(\theta) - E(\theta^x) = (d^+ - d^-)(-x + \sin(x)).
\]
This function changes sign in every neighborhood of $x = 0$. We conclude that $\theta$ is not Lyapunov stable.

As a corollary we obtain the following general result:

**Corollary 12.** For every graph with at least one edge, every coupling strength and every intrinsic frequencies, the Lyapunov stable equilibria of (7) have at least one strictly negative eigenvalue.

**Proof.** Without loss of generality, it is enough to consider connected graphs. Let $\theta$ be a Lyapunov stable equilibria. By the Center Manifold Theorem the Jacobian eigenvalues at $\theta$ are negative or zero. By Theorem 11 they are not all zero.

Corollary 12 shows that the presence of strictly negative eigenvalues, although not sufficient, is necessary for stability. Therefore, albeit linear stability analysis is not enough to determine the set of stable equilibria, it helps by restricting the search.

**Remark 13.** The proof of Theorem 11 actually shows something more: completely degenerate equilibria are saddles of the energy. In particular, it follows that they are unstable in both forward and backward time, or if $K$ is chosen to be negative.

**References**

[1] P. A. Absil and K. Kurdyka, *On the stable equilibrium points of gradient systems*, Systems and Control Letters, 55 (2006), pp. 573–577.

[2] J. A. Acebrón, L. L. Bonilla, C. J. P. Vicente, F. Ritort, and R. Spigler, *The kuramoto model: A simple paradigm for synchronization phenomena*, Reviews of modern physics, 77 (2005), p. 137.

[3] A. Arenas, A. Díaz-Guilera, J. Kurths, Y. Moreno, and C. Zhou, *Synchronization in complex networks*, Physics Reports, 469 (2008), pp. 93–153.

[4] E. Canale and P. Monzon, *Global properties of kuramoto bidirectionally coupled oscillators in a ring structure*, in 2009 IEEE Control Applications,(CCA) & Intelligent Control,(ISIC), IEEE, 2009, pp. 183–188.

[5] Z. Chen, Y. Zou, S. Guan, Z. Liu, and J. Kurths, *Fully solvable lower dimensional dynamics of Cartesian product of Kuramoto models*, New Journal of Physics, 21 (2019).

[6] F. R. Chung, *Spectral graph theory*, cbms reg, in Conf. Ser. Math, vol. 92, 1997, p. 11241131.
A. H. Dekker and R. Taylor, Synchronization properties of trees in the kuramoto model, SIAM Journal on Applied Dynamical Systems, 12 (2013), pp. 596–617.

R. Delabays, T. Coletta, and P. Jacquod, Multistability of phase-locking in equal-frequency kuramoto models on planar graphs, Journal of Mathematical Physics, 58 (2017), p. 032703.

L. DeVille and B. Ermentrout, Phase-locked patterns of the kuramoto model on 3-regular graphs, Chaos, 26 (2016), pp. 1–11.

F. Dörfler and F. Bullo, Synchronization in complex networks of phase oscillators: A survey, Automatica, 50 (2014), pp. 1539–1564.

F. Dörfler, M. Chertkov, and F. Bullo, Synchronization in complex oscillator networks and smart grids, Proceedings of the National Academy of Sciences, 110 (2013), pp. 2005–2010.

A. Jadbabaie, N. Motee, and M. Barahona, On the stability of the kuramoto model of coupled nonlinear oscillators, in Proceedings of the 2004 American Control Conference, vol. 5, IEEE, 2004, pp. 4296–4301.

M. Jafarian, X. Yi, M. Pirani, H. Sandberg, and K. H. Johansson, Synchronization of kuramoto oscillators in a bidirectional frequency-dependent tree network, in 2018 IEEE Conference on Decision and Control (CDC), IEEE, 2018, pp. 4505–4510.

M. Kassabov, S. H. Strogatz, and A. Townsend, Sufficiently dense Kuramoto networks are globally synchronizing, Chaos: An Interdisciplinary Journal of Nonlinear Science, 31 (2021), p. 073135.

S. Ling, R. Xu, and A. S. Bandeira, On the landscape of synchronization networks: A perspective from nonconvex optimization, SIAM Journal on Optimization, 29 (2019), pp. 1879–1907.

J. Lu and S. Steinerberger, Synchronization of kuramoto oscillators in dense networks, Nonlinearity, 33 (2020), p. 5905.

T. K. Roy and A. Lahiri, Synchronized oscillations on a kuramoto ring and their entrainment under periodic driving, Chaos, Solitons & Fractals, 45 (2012), pp. 888–898.

Y. Sokolov and G. B. Ermentrout, When is sync globally stable in sparse networks of identical kuramoto oscillators?, Physica A: Statistical Mechanics and its Applications, 533 (2019), p. 122070.

R. Taylor, There is no non-zero stable fixed point for dense networks in the homogeneous kuramoto model, Journal of Physics A: Mathematical and Theoretical, 45 (2012), p. 055102.

M. Verwoerd and O. Mason, On computing the critical coupling coefficient for the kuramoto model on a complete bipartite graph, SIAM Journal on Applied Dynamical Systems, 8 (2009), pp. 417–453.

D. A. Wiley, S. H. Strogatz, and M. Girvan, The size of the sync basin, Chaos: An Interdisciplinary Journal of Nonlinear Science, 16 (2006), p. 015103.

R. Yoneida, T. Tatsukawa, and J. N. Teramae, The lower bound of the network connectivity guaranteeing in-phase synchronization, Chaos, 31 (2021).