Acyclic Edge Coloring of 2-Degenerate Graphs

Manu Basavaraju and L. Sunil Chandran

COMPUTER SCIENCE AND AUTOMATION DEPARTMENT
INDIAN INSTITUTE OF SCIENCE
BANGALORE-560012, INDIA
E-mail: manu@csa.iisc.ernet.in; sunil@csa.iisc.ernet.in

Received December 6, 2008; Revised January 19, 2010

Published online 16 December 2010 in Wiley Online Library (wileyonlinelibrary.com).
DOI 10.1002/jgt.20559

Abstract: An acyclic edge coloring of a graph is a proper edge coloring such that there are no bichromatic cycles. The acyclic chromatic index of a graph is the minimum number $k$ such that there is an acyclic edge coloring using $k$ colors and is denoted by $a'(G)$. A graph is called 2-degenerate if any of its induced subgraph has a vertex of degree at most 2. The class of 2-degenerate graphs properly contains series-parallel graphs, outerplanar graphs, non-regular subcubic graphs, planar graphs of girth at least 6 and circle graphs of girth at least 5 as subclasses. It was conjectured by Alon, Sudakov and Zaks (and much earlier by Fiamcik) that $a'(G) \leq \Delta + 2$, where $\Delta = \Delta(G)$ denotes the maximum degree of the graph. We prove the conjecture for 2-degenerate graphs. In fact we prove a stronger bound: we prove that if $G$ is a 2-degenerate graph with maximum degree $\Delta$, then $a'(G) \leq \Delta + 1$. © 2010 Wiley Periodicals, Inc. J Graph Theory 69: 1–27, 2012

Keywords: acyclic edge coloring; acyclic edge chromatic number; 2-degenerate graphs; series-parallel graphs; outer planar graphs

1. INTRODUCTION

All graphs considered in this article are finite and simple. A proper edge coloring of $G=(V,E)$ is a map $c:E \rightarrow C$ (where $C$ is the set of available colors) with $c(e) \neq c(f)$ for
any adjacent edges $e, f$. The minimum number of colors needed to properly color the edges of $G$ is called the chromatic index of $G$ and is denoted by $\chi'(G)$. A proper edge coloring $c$ is called acyclic if there are no bichromatic cycles in the graph. In other words, an edge coloring is acyclic if the union of any two color classes induces a set of paths (i.e., linear forest) in $G$. The acyclic edge chromatic number (also called acyclic chromatic index), denoted by $\chi'_e(G)$, is the minimum number of colors required to acyclically edge color $G$. The concept of acyclic coloring of a graph was introduced by Grünbaum [15].

The acyclic chromatic index and its vertex analogue can be used to bound other parameters like oriented chromatic number and star chromatic number of a graph, both of which have many practical applications, for example, in wavelength routing in optical networks [5, 16]. Let $\Delta = \Delta(G)$ denote the maximum degree of a vertex in graph $G$. By Vizing’s theorem, we have $\Delta \leq \chi'(G) \leq \Delta + 1$ (see [11] for proof).

Since any acyclic edge coloring is also proper, we have $\chi'_e(G) \geq \chi'(G) \geq \Delta$.

It has been conjectured by Alon et al. [3] (and much earlier by Fiamcik [12]) that $\chi'_e(G) \leq \Delta + 2$ for any $G$. Using probabilistic arguments Alon et al. [2] proved that $\chi'_e(G) \leq 60\Delta$. The best known result up to now for arbitrary graph is by Molloy and Reed [17] who showed that $\chi'_e(G) \leq 16\Delta$. Muthu et al. [18] proved that $\chi'_e(G) \leq 4.52\Delta$ for graphs $G$ of girth at least 220 ($Girth$ is the length of a shortest cycle in a graph).

Though the best known upper bound for general case is far from the conjectured $\Delta + 2$, the conjecture has been shown to be true for some special classes of graphs. Alon et al. [3] proved that there exists a constant $k$ such that $\chi'_e(G) \leq \Delta + 2$ for any graph $G$ whose girth is at least $k\Delta \log \Delta$. They also proved that $\chi'_e(G) \leq \Delta + 2$ for almost all $\Delta$-regular graphs. This result was improved by Nešetřil and Wormald [22] who showed that for a random $\Delta$-regular graph, $\chi'_e(G) \leq \Delta + 1$. Muthu, Narayanan and Subramanian proved the conjecture for grid-like graphs [19]. In fact they gave a better bound of $\Delta + 1$ for these classes of graphs. From Burnstein’s [9] result it follows that the conjecture is true for subcubic graphs. Skulrattanakulchai [23] gave a polynomial time algorithm to color a subcubic graph using $\Delta + 2 = 5$ colors.

Determining $\chi'_e(G)$ is a hard problem both from the theoretical and algorithmic points of view. Even for the simple and highly structured class of complete graphs, the value of $\chi'_e(G)$ is still not determined exactly. It has also been shown by Alon and Zaks [4] that determining whether $\chi'_e(G) \leq 3$ is NP-complete for an arbitrary graph $G$. The vertex version of this problem has also been extensively studied [15, 9, 8]. A generalization of the acyclic edge chromatic number has been studied: The $r$-acyclic edge chromatic number $\chi'_r(G)$ is the minimum number of colors required to color the edges of the graph $G$ such that every cycle $C$ of $G$ has at least $\min\{|C|, r\}$ colors [13, 14].

**Our Result:** We prove the conjecture for 2-degenerate graphs. A graph $G$ is called $k$-degenerate if any induced subgraph of $G$ has a vertex of degree at most $k$. For example, planar graphs are 5-degenerate, forests are 1-degenerate. The earliest result on acyclic edge coloring of 2-degenerate graphs was by Card and Roditty [10], where they proved that $\chi'_e(G) \leq \Delta + k - 1$, where $k$ is the maximum edge-connectivity, defined as $k = \max_{u,v \in V(G)} \lambda(u,v)$, where $\lambda(u,v)$ is the edge-connectivity of the pair $u,v$. Note that here $k$ can be as high as $\Delta$. Muthu et al. [20] proved that $\chi'_e(G) \leq \Delta + 1$ for outerplanar graphs which are a subclass of 2-degenerate graphs and posed the problem of proving the conjecture for 2-degenerate graphs as an open problem. In fact, they have informed us that very recently they have also derived an upper bound of $\Delta + 1$ for series–parallel...
Let $G$ be a 2-degenerate graph with maximum degree $\Delta$, then $a'(G) \leq \Delta + 1$.

Our result is tight since there are 2-degenerate graphs which require $\Delta + 1$ colors (e.g., cycle, non-regular subcubic graphs, etc.). Most of the work in this field has been non-constructive, using probabilistic methods. In contrast, our proof is constructive (e.g., cycle, non-regular subcubic graphs, etc.). Most of the work in this field has been non-constructive, using probabilistic methods. In contrast, our proof is constructive.

**Remark.** It may be noted that though $a'(G) \leq \Delta + 1$ for 2-degenerate graphs, it is not so in general. In fact every $\Delta$-regular graph on $2n$ vertices with $\Delta > n$ requires at least $\Delta + 2$ colors to be acyclically edge colored [7].

## 2. PRELIMINARIES

Let $G = (V, E)$ be a simple, finite and connected 2-degenerate graph of $n$ vertices and $m$ edges. Let $x \in V$. Then $N_G(x)$ will denote the neighbors of $x$ in $G$. For an edge $e \in E$, $G - e$ will denote the graph obtained by deletion of the edge $e$. For $x, y \in V$, when $e = (x, y) = xy$, we may use $G - \{xy\}$ instead of $G - e$. Let $c: E \to \{1, 2, \ldots, k\}$ be an acyclic edge coloring of $G$. For an edge $e \in E$, $c(e)$ will denote the color given to $e$ with respect to the coloring $c$. For $x, y \in V$, when $e = (x, y) = xy$ we may use $c(x, y)$ instead of $c(e)$. For $S \subseteq V$, we denote the induced subgraph on $S$ by $G[S]$.

**Partial Coloring:** Let $H$ be a subgraph of $G$. Then an acyclic edge coloring $c'$ of $H$ is also a partial coloring of $G$. Note that $H$ can be $G$ itself. Thus, a coloring $c$ of $G$ itself can be considered a partial coloring. A coloring $c$ of $G$ is said to be a proper partial coloring if $c$ is proper. A proper partial coloring $c$ is called acyclic if there are no bichromatic cycles in the graph. Sometimes we also use the word valid coloring instead of acyclic coloring. Note that with respect to a partial coloring $c$, $c(e)$ may not be defined for an edge $e$. So, whenever we use $c(e)$, we are considering an edge $e$ for which $c(e)$ is defined, though we may not always explicitly mention it.

Let $c$ be a partial coloring of $G$. We denote the set of colors in the partial coloring $c$ by $C = \{1, 2, \ldots, \Delta + 1\}$. For any vertex $u \in V(G)$, we define $F_u(c) = \{c(u, z) : z \in N_G(u)\}$. For an edge $ab \in E$, we define $S_{ab}(c) = F_b(c) - \{c(a, b)\}$. Note that $S_{ab}(c)$ need not be the same as $S_{ba}(c)$. We will abbreviate the notation to $F_u$ and $S_{ab}$ when the coloring $c$ is understood from the context.

To prove the main result, we plan to use contradiction. Let $G$ be the minimum counter example for the statement in Theorem 1. Let $G = (V, E)$ be a graph on $m$ edges where $m \geq 1$. We will remove an edge $e$ from $G$ and get a graph $G' = (V, E')$. By the minimality of $G$, the graph $G'$ will have an acyclic edge coloring $c: E' \to \{1, 2, \ldots, \Delta + 1\}$. Our
intention will be to extend the coloring \( c \) of \( G' \) to \( G \) by assigning an appropriate color for the edge \( e \), thereby contradicting the assumption that \( G \) is a minimum counter example.

The following definitions arise out of our attempt to understand what may prevent us from extending a partial acyclic coloring of \( G - e \) to \( G \).

**Maximal Bichromatic Path:** Consider the subgraph induced by any two colors \( \alpha \) and \( \beta \) with respect to any proper coloring \( c \). The \((\alpha, \beta)\)-subgraph consists of even cycles, bichromatic paths of length at least two, isolated edges, and isolated vertices. Now when we say maximal bichromatic path, we only concentrate on bichromatic paths of length at least two, ignoring the even bichromatic cycles, isolated edges, and isolated vertices. Thus, an \((\alpha, \beta)\) maximal bichromatic path with respect to a proper coloring \( c \) of \( G \) is a path-component of the \((\alpha, \beta)\)-subgraph that consists of at least two edges. An \((\alpha, \beta, a, b)\) maximal bichromatic path is an \((\alpha, \beta)\) maximal bichromatic path which starts at the vertex \( a \) with an edge colored \( \alpha \) and ends at \( b \). We emphasize that the edge of the \((\alpha, \beta, a, b)\) maximal bichromatic path incident on vertex \( a \) is colored \( \alpha \) and the edge incident on vertex \( b \) can be colored either \( \alpha \) or \( \beta \). Thus, the notations \((\alpha, \beta, a, b)\) and \((\alpha, \beta, b, a)\) have different meanings. Also note that any maximal bichromatic path will have at least two edges. The following fact is obvious from the definition of acyclic edge coloring:

**Fact 1.** Given a pair of colors \( \alpha \) and \( \beta \) of a proper coloring \( c \) of \( G \), there can be at most one maximal \((\alpha, \beta)\) bichromatic path containing a particular vertex \( v \), with respect to \( c \).

A color \( \alpha \neq c(e) \) is a candidate for an edge \( e \) in \( G \) with respect to a partial coloring \( c \) of \( G \) if none of the adjacent edges of \( e \) are colored \( \alpha \). A candidate color \( \alpha \) is valid for an edge \( e \) if assigning the color \( \alpha \) to \( e \) does not result in any bichromatic cycle in \( G \).

Let \( e = (a, b) \) be an edge in \( G \). Note that any color \( \beta \notin F_a \cup F_b \) is a candidate color for the edge \( ab \) in \( G \) with respect to the partial coloring \( c \) of \( G \). But \( \beta \) may not be valid. What may be the reason? It is clear that color \( \beta \) is not valid if and only if there exists \( \alpha \neq \beta \) such that a \((\alpha, \beta)\) bichromatic cycle gets formed if we assign color \( \beta \) to the edge \( e \). In other words, if and only if, with respect to coloring \( c \) of \( G \) there existed a \((\alpha, \beta, a, b)\) maximal bichromatic path with \( \alpha \) being the color given to the first and last edge of this path. Such paths play an important role in our proof. We call them critical paths. It is formally defined below:

**Critical Path:** Let \( ab \in E \) and \( c \) be a partial coloring of \( G \). Then a \((\alpha, \beta, a, b)\) maximal bichromatic path which starts out from the vertex \( a \) via an edge colored \( \alpha \) and ends at the vertex \( b \) via an edge colored \( \alpha \) is called an \((\alpha, \beta, ab)\) critical path. Note that any critical path will be of odd length. Moreover, the smallest length possible is three.

**Remark.** When we define critical paths, we always keep the graph \( G \) in mind even though we are dealing with coloring of a subgraph. Thus when we say \( ab \) is an edge, its an edge in \( G \).

An obvious strategy to extend a valid partial coloring \( c \) of \( G \) would be to try to assign one of the candidate colors to an uncolored edge \( e \). The condition that a candidate color being not valid for the edge \( e \) is captured in the following fact.

**Fact 2.** Let \( c \) be a partial coloring of \( G \). A candidate color \( \beta \) is not valid for the edge \( e = (a, b) \) if and only if \( \exists \alpha \in S_{ab} \cap S_{ba} \) such that there is a \((\alpha, \beta, ab)\) critical path in \( G \) with respect to the coloring \( c \).
If all the candidate colors turn out to be invalid, we try to slightly modify the partial coloring \( c \) in such a way that with respect to the modified coloring, one of the candidate colors becomes valid. An obvious way to modify is to recolor an edge so that some critical paths are broken and a candidate color becomes valid. Sometimes, we resort to a slightly more sophisticated strategy to modify the coloring namely color exchange defined below:

Color Exchange: Let \( c \) be a partial coloring of \( G \). Let \( u, i, j \in V(G) \) and \( ui, uj \in E(G) \). We define Color Exchange with respect to the edge \( ui \) and \( uj \), as the modification of the current partial coloring \( c \) by exchanging the colors of the edges \( ui \) and \( uj \) to get a partial coloring \( c' \), i.e., \( c'(u,i) = c(u,j), c'(u,j) = c(u,i) \) and \( c'(e) = c(e) \) for all other edges \( e \) in \( G \). The color exchange with respect to the edges \( ui \) and \( uj \) is said to be proper if the coloring obtained after the exchange is proper. The color exchange with respect to the edges \( ui \) and \( uj \) is valid if and only if the coloring obtained after the exchange is acyclic. The following fact is obvious:

**Fact 3.** Let \( c' \) be the partial coloring obtained from a valid partial coloring \( c \) by the color exchange with respect to the edges \( ui \) and \( uj \). Then the partial coloring \( c' \) will be proper if and only if \( c(u,i) \notin S_{uj} \) and \( c(u,j) \notin S_{ui} \).

The color exchange is useful in breaking some critical paths as is clear from the following lemma:

**Lemma 1.** Let \( c \) be a partial coloring of \( G \) and \( u, i, j, a, b \in V(G) \), \( ui, uj, ab \in E \). Also let \( \{ \lambda, \xi \} \in C \) such that \( \{ \lambda, \xi \} \cap \{ c(u,i), c(u,j) \} \neq \emptyset \) and \( \{ i,j \} \cap \{ a,b \} = \emptyset \). Suppose there exists an \( (\lambda, \xi, ab) \) critical path that contains vertex \( u \), with respect to a valid partial coloring \( c \) of \( G \). Let \( c' \) be the partial coloring obtained from \( c \) by the color exchange with respect to the edges \( ui \) and \( uj \). If \( c' \) is proper, then there will not be any \( (\lambda, \xi, ab) \) critical path in \( G \) with respect to the partial coloring \( c' \).

**Proof.** Firstly, \( \{ \lambda, \xi \} \neq \{ c(u,i), c(u,j) \} \). This is because, if there is a \( (\lambda, \xi, ab) \) critical path that contains vertex \( u \), with respect to a valid partial coloring \( c \) of \( G \), then it has to contain the edge \( ui \) and \( uj \). Since \( i \notin \{ a,b \} \), vertex \( i \) is an internal vertex of the critical path which implies that both the colors \( \lambda \) and \( \xi \) (that is \( c(u,i) \) and \( c(u,j) \)) are present at vertex \( i \). That means \( c(u,j) \in S_{ui} \) and this contradicts Fact 3, since we are assuming that the color exchange is proper. Thus \( \{ \lambda, \xi \} \neq \{ c(u,i), c(u,j) \} \).

Now let \( P \) be the \( (\lambda, \xi, ab) \) critical path with respect to the coloring \( c \). Without loss of generality assume that \( \gamma = c(u,i) \in \{ \lambda, \xi \} \). Since vertex \( u \) is contained in path \( P \), by the maximality of the path \( P \), it should contain the edge \( ui \) since \( c(u,i) = \gamma \in \{ \lambda, \xi \} \). Let us assume without loss of generality that path \( P \) starts at vertex \( a \) and reaches vertex \( i \) before it reaches vertex \( u \). Now after the color exchange with respect to the edges \( ui \) and \( uj \), i.e., with respect to the coloring \( c' \), there will not be any edge adjacent to vertex \( i \) that is colored \( \gamma \). So if any \( (\lambda, \xi) \) maximal bichromatic path starts at vertex \( a \), then it has to end at vertex \( i \). Since \( i \neq b \), by Fact 1 we infer that the \( (\lambda, \xi, ab) \) critical path does not exist.

**3. PROOF OF THEOREM 1**

**Proof.** We prove the theorem by way of contradiction. Let \( G \) be a 2-degenerate graph with \( n \) vertices and \( m \) edges which is a minimum counter example for the theorem.
properties which in turn will lead to a contradiction. Example in mind, we will show that any partial coloring $c$ of $G$ should satisfy certain properties which in turn will lead to a contradiction.

Selection of the Primary Pivot: Let $W_0 = \{z \in V(G) | \text{degree}_G(z) = 2\}$. Since $G$ is 2-degenerate $W_0 \neq \emptyset$. We may assume that $V - W_0 \neq \emptyset$ because otherwise, $G$ is a cycle and it is easy to see that it is $\Delta + 1 = 3$ acyclically edge colorable. Thus $\Delta(G) \geq 3$. Let $G' = G[V - W_0]$ and $W_1 = \{z \in V(G') | \text{degree}_{G'}(z) \leq 2\}$. By the definition of 2-degeneracy there exists at least one vertex of degree at most 2 in $G'$ and thus $W_1 \neq \emptyset$.

Let $V' = V(G')$. If $V' - W_1 \neq \emptyset$, then there exists at least one vertex of degree at most 2 in $G'[V' - W_1]$. Let $G'' = G[V' - W_1]$ and $W_2 = \{z \in V(G'') | \text{degree}_{G''}(z) \leq 2\}$. Let $q \in W_2$. Clearly $N_G(q) \cap W_1 \neq \emptyset$ and let $x \in N_G(q) \cap W_1$. On the other hand if $V' - W_1 = \emptyset$, then let $x \in W_1$. We call $x$ the Primary Pivot, since $x$ plays an important role in our proof. Let $N'_G(x) = N_G(x) \cap W_0$ and $N''_G(x) = N_G(x) - N'_G(x)$. Since $x \in W_1$, it is easy to see that $|N''_G(x)| \leq 2$ and $|N'_G(x)| \geq 1$.

Let $N''(x) = \{y_1, y_2, \ldots, y_7\}$. Also $\forall y_i$, let $N(y_i) = \{x, y'_i\}$ (see Fig. 1). $\forall y_i$, let $G_i$ denote the graph obtained by removing the edge $(x, y_i)$ from the graph $G$. Let $N'_i(x) = N'_G(x) - \{y_i\}$ and $N''_i(x) = N_G(x) - N'_G(x)$. By the minimum choice of $G$, graph $G_i$ is $\Delta + 1$ acyclically edge colorable. Let $c_i$ be a valid coloring of $G_i$ and thus a partial coloring of $G$. We denote the set of colors by $C = \{1, 2, \ldots, \Delta + 1\}$.

Comment: Note that the figures given in this article are only for providing visual aid for the reader. They do not capture all possible configurations.

A. Properties of Any Valid Coloring $c_i$ of $G_i$

Let $F_x(c_i) = \{c_i(x, z) | z \in N_G(x)\}$. Let $F'_x(c_i) = \{c_i(x, z) | z \in N'_G(x)\}$ and $F''_x(c_i) = \{c_i(x, z) | z \in N''_G(x)\}$. Note that $F_x(c_i)$ is the disjoint union of $F'_x(c_i)$ and $F''_x(c_i)$ and also $|F''_x(c_i)| \leq 2$.

**Lemma 2.** With respect to any valid coloring $c_i$ of $G_i$, $c_i(y_i, y'_i) \in F''_x(c_i)$.

*Journal of Graph Theory* DOI 10.1002/jgt
Lemma 4. With respect to any valid coloring \( c_i \) of \( G_i \), let \( F \) be a valid coloring of \( G_i \) such that \( c_i((y_i, y'_i)) \in F_x(c_i) \). Otherwise all the candidate colors are valid for the edge \( xy_i \), since any cycle involving the edge \( xy_i \) will contain the edge \( y_iy'_i \) as well as an edge incident on \( x \) in \( G_i \) and thus the cycle will have at least 3 colors. Suppose \( c_i((y_i, y'_i)) \in F_x(c_i) \). Clearly we have \( |F_x(c_i) \cup \{c_i((y_i, y'_i))\}| \leq \Delta - 1 \). Thus, there are at least two candidate colors for the edge \( xy_i \). Let \( y_j \in N^c_{G_i}(x) \) be the vertex such that \( c_i((y_j, y'_j)) = c_i((x, y_j)) \). When we color edge \( xy_i \), there is a possibility of a bichromatic cycle only if we assign \( c_i((y_j, y'_j)) \) to the edge \( xy_i \) since \( \text{degree}_{G_i}(y_j) = 2 \). But since we have at least two candidate colors for edge \( xy_i \), this situation can easily be avoided. We infer that \( c_i((y_j, y'_j)) \notin F_x(c_i) \).

\[ \square \]

Lemma 5. With respect to any valid coloring \( c_i \) of \( G_i \), \( |F_x^*(c_i)| = 2 \).

\[ \text{Proof.} \] Suppose not. Then \( |F_x^*(c_i)| \leq 1 \). Since \( |F_x(c_i)| \leq \Delta \), we have at least one candidate color for the edge \( y_iy'_i \). Let \( c_i \) be the valid coloring obtained by recoloring the edge \( y_iy'_i \) with a candidate color. By Lemma 2, we have \( c_i((y_j, y'_j)) \notin F_x(c_i) \). Clearly since \( |F_x^*(c_i)| \leq 1 \) and \( c_i((y_j, y'_j)) \neq c_i((y_i, y'_i)) \), we can infer that \( c_i((y_j, y'_j)) \notin F_x^*(c_i) = F_x(c_i) \), a contradiction to Lemma 2.

An immediate consequence of Lemma 3 is that \( |N^c_{G_i}(x)| = 2 \). Moreover by the way we have selected vertex \( x \), at least one of them should belong to \( W_1 \cup W_2 \). We make the following assumption:

Assumption 1. With respect to any valid coloring \( c_i \) of \( G_i \), without loss of generality let \( F_x^*(c_i) = \{1, 2\} \) and \( N^c_{G_i}(x) = \{q, q'\} \). Thus \( \{c_i(x, q), c_i(x, q')\} = \{1, 2\} \). Also without loss of generality we assume that \( q \in W_2 \cup W_1 \) (see Fig. 1).

\[ \text{Lemma 4.} \] With respect to any valid coloring \( c_i \) of \( G_i \), colors \( 1, 2 \notin S_{y_iy'_i} \).

\[ \text{Proof.} \] Since \( |F_x^*(c_i)| \leq \Delta \), we have at least one candidate color \( \gamma \neq c_i((y_i, y'_i)) \) for the edge \( y_iy'_i \). Note that \( \gamma \) is valid for the edge \( y_iy'_i \) in \( G_i \) since \( y_i \) is a pendant vertex in \( G_i \). Let \( c_i' \) be the valid coloring obtained by recoloring the edge \( y_iy'_i \) with \( \gamma \). Now since \( c_i \) and \( c_i' \) are valid, by Lemma 2, we have \( \{c_i((y_i, y'_i)), c_i'((y_i, y'_i))\} = F_x^*(c_i) = \{1, 2\} \) (by Assumption 1). Since \( c_i((y_i, y'_i)) \notin S_{y_iy'_i} \) and \( c_i'((y_i, y'_i)) \notin S_{y_iy'_i} \), we have \( 1, 2 \notin S_{y_iy'_i} \).

Let \( C' = C - \{1, 2\} \). For each color \( \gamma \in C' \), we define a graph \( G_{i,\gamma} \) as follows:

\[
G_{i,\gamma} = \begin{cases} 
G_i & \text{if } \gamma \in C' - F_x^*(c_i) \\
G_i - xy_a, \text{ where } c_i(x, y_a) = \gamma & \text{if } \gamma \in F_x^*(c_i)
\end{cases}
\]

Also let \( c_{i,\gamma} \) be the valid coloring of \( G_{i,\gamma} \) derived from \( c_i \) of \( G_i \), that is by discarding the color of the edge \( xy_a \), where \( y_a \) is the vertex such that \( c_i(x, y_a) = \gamma \). Also if \( c_{i,\gamma} \) is a valid coloring of \( G_{i,\gamma} \), then \( c_{i,\gamma} \) is said to be derivable from \( c_i \) if we can extend the coloring \( c_{i,\gamma} \) of \( G_{i,\gamma} \) to the coloring \( c_i \) of \( G_i \). Also note that even though we define these graphs \( G_{i,\gamma} \), we always have the original graph in mind when using definitions like critical paths, which are defined with respect to an edge in the graph.

\[ \text{Lemma 5.} \] Let \( c_i \) be any valid coloring of \( G_i \). With respect to coloring \( c_{i,\gamma} \) of \( G_{i,\gamma} \), \( \forall \gamma \in C' - F_x^*(c_i), \exists (\mu, \gamma, xy_i) \) critical path, where \( \mu = c_i((y_i, y'_i)) \).

\[ \text{Journal of Graph Theory DOI 10.1002/jgt} \]
Proof. Recall that when $\gamma \in C' - F'_x(c_i)$, we have $G_{i, i} = G_i$ and hence $c_{i, i} = c_i$. Suppose if there is no $(\mu, \gamma, xy_j)$ critical path, then by Fact 2 color $\gamma$ is valid for the edge $xy_i$. Thus we get a valid coloring of $G$, a contradiction.

Lemma 6. Let $c_i$ be any valid coloring of $G_i$. With respect to coloring $c_{i, i}$ of $G_{i, i}$, $\forall \gamma \in C' - F'_x(c_i)$, $\exists (v, \gamma, x, y'_i)$ maximal bichromatic path, where $\{v\} = \{1, 2\} - \{c_i(y_i, y'_i)\}$.

Proof. Recall that when $\gamma \in C' - F'_x(c_i)$, we have $G_{i, i} = G_i$ and hence $c_{i, i} = c_i$. Suppose there is no $(v, \gamma, x, y'_i)$ maximal bichromatic path, where $\gamma \in C' - F'_x(c_i)$.

Recall that $F_x(c_i)$ is valid for the edge $xy_i$. Now recolor the edge $y_i y'_i$ with color $v$ to get a valid coloring $c'_i$ of $G_i$. Since by our assumption that there is no $(v, \gamma, x, y'_i)$ maximal bichromatic path with respect to $c_{i, i} = c_i$, there cannot be any $(v, \gamma, xy_j)$ critical path with respect to the coloring $c'_i$, a contradiction to Lemma 5 (Note that the color $\mu$ discussed in Lemma 5 and assumption is same as $v = c'_i(y_i, y'_i)$ in $c'_i$).

Assumption 2. Since $|F_x(c_i)| \leq \Delta - 1$, we have $|C - F_x(c_i)| \geq 2$. Since $C - F_x(c_i) = C' - F'_x(c_i)$, we have $|C' - F'_x(c_i)| \geq 2$. Thus, $\deg_G(y'_i) \geq 3$ and hence $\deg_G(y'_i) \geq 3$.

Let $\alpha, \beta \in C' - F'_x(c_i)$.

Lemma 7. Let $c_i$ be any valid coloring of $G_i$. With respect to coloring $c_{i, i}$ of $G_{i, i}$, $\forall \gamma \in F'_x(c_i)$, $\exists (\mu, \gamma, xy_i)$ critical path, where $\mu = c_i(y_i, y'_i)$.

Proof. Let $c_i(x, y_j) = \gamma$, where $\gamma \in F'_x(c_i)$. Suppose if there is no $(\mu, \gamma, xy_j)$ critical path, then by Fact 2 color $\gamma$ is valid for the edge $xy_i$ with respect to the coloring $c_{i, i}$.

Color the edge $xy_i$ with color $\gamma$ to get a valid coloring $d$ of $G - \{xy_j\}$. Now we will show that we can extend the coloring $d$ of $G - \{xy_j\}$ to a valid coloring of the graph $G$ by giving a valid color for the edge $xy_j$, leading to a contradiction of our assumption that $G$ was a minimum counter example. We claim the following:

Claim 1. With respect to the coloring $d$, either color $\alpha$ or $\beta$ is valid for the edge $xy_j$ (Recall that $\alpha, \beta \in C' - F'_x(c_i)$ by Assumption 2).

Proof. Without loss of generality, let $d(y_j, y'_i) = \eta$. Note that $\eta \neq \gamma = c_i(x, y_j)$. Now if,

1. $\eta \notin F_x(c_i)$.

In view of Assumption 2, $\alpha, \beta \notin F_x(c_i)$. Noting that $\eta$ cannot be equal to both $\alpha$ and $\beta$, without loss of generality, let $\eta \neq \alpha$. Then color the edge $(x, y_j)$ with color $\alpha$ to get a proper coloring $d'$. If a bichromatic cycle gets formed, then it should contain the edge $xy_j$ and also involve both the colors $\eta$ and $\alpha$ since $\deg_G(y_j) = 2$. But since $\eta \notin F_x(c_i)$, such a bichromatic cycle is not possible. Thus, the coloring $d'$ is valid.

2. $\eta \in \{1, 2\} = \{\mu, v\} = F'_x(c_i)$. Recolor the edge $xy_j$ with color $\alpha$ to get a coloring $d'$. We claim that the coloring $d'$ is valid. This is because if it is not valid, then there has to be a $(x, \eta)$ bichromatic cycle containing the edge $xy_j$ with respect to $d'$. This implies that there has to be a $(\eta, \alpha, xy_j)$ critical path with respect to the coloring $d$ and hence with respect to the coloring $c_{i, i}$ (Note that the coloring $d$ is obtained from $c_{i, i}$ just by giving the color $\gamma$ to the edge $xy_i$ and $\eta$, $\gamma \neq \alpha, \beta$).

If $\eta = \mu$, this means that there was a $(\eta = \mu, x, xy_j)$ critical path with respect to $c_{i, i}$. But this is not possible by Fact 1, since there is already a $(\mu, x, xy_i)$ critical path with respect to $c_{i, i}$ (by Lemma 5) and $y_i \neq y_j$. 

Journal of Graph Theory DOI 10.1002/jgt
Lemma 9. Let $c_i$ be any valid coloring of $G_i$. But this is not possible by Fact 1 since there is already a $(v, z, x, y'_j)$ maximal bichromatic path with respect to $c_i$. By Lemma 6 and $y'_j \neq y_j$ since by Assumption 2, $\text{deg}_G(y'_j) = 3$. Thus, there cannot be any bichromatic cycles with respect to the coloring $d'$. Thus, the coloring $d'$ is valid.

3. $\eta \in F'_i(c_i)$. Let $y_k \in N'_G(x)$ be such that $d(x, y_k) = \eta$. With respect to colors $\{z, \beta\}$, without loss of generality let $d(y_k, y'_k) = \eta$. Now recolor the edge $xy_j$ with color $\beta$ to get a coloring $d'$. Now if a bichromatic cycle gets formed, then it should contain the edge $xy_j$ and also involve both the colors $\eta$ and $\beta$. Thus, the bichromatic cycle should contain the edge $xy_k$. Since $\text{deg}_G(y_k) = 2$, the bichromatic cycle should contain the edge $y_ky'_k$. But by our assumption, $c_i(y_k, y'_k) \neq \beta$, a contradiction. Thus, the coloring $d'$ is valid.

Hence, either color $z$ or $\beta$ is valid for the edge $xy_j$.

Thus we have a valid coloring (i.e., $d'$) for the graph $G$, a contradiction.

Lemma 8. Let $c_i$ be any valid coloring of $G_i$. With respect to coloring $c_{i, \gamma}$ of $G_{i, \gamma}$, $\forall \gamma \in F'_i(c_i), \exists (v, \gamma, x, y'_j)$ maximal bichromatic path, where $\{v\} = \{1, 2\} - \{c_i(y_i, y'_j)\}$.

Proof. Suppose if there is no $(v, \gamma, x, y'_j)$ maximal bichromatic path, where $\gamma \in F'_i(c_i)$, then by Lemma 4, color $v$ is a candidate for the edge $yi y'_j$. Now recolor the edge $yi y'_j$ with color $\gamma$ to get a valid coloring $c'_{i, \gamma}$ of $G_i$. Since by our assumption that there is no $(v, \gamma, x, y'_j)$ maximal bichromatic path with respect to $c_{i, \gamma}$, there cannot be any $(v, \gamma, x, y'_j)$ critical path with respect to the coloring $c'_{i, \gamma}$, a contradiction to Lemma 7 (Note that the color $\mu$ discussed in Lemma 7 and assumption is same as $v=c'_{i, \gamma}(y_i, y'_j)$ in $c'_{i, \gamma}$).

Critical Path Property: In the rest of the article, we will have to repeatedly use the properties (namely the presence of $(\mu, \gamma, x, y_j)$ critical path in $G_{i, \gamma}$, where $\mu = c_{i, \gamma}(y_i, y'_j)$) described by Lemmas 5 and 7. Therefore, we will name these properties as the Critical Path Property of the graph $G_{i, \gamma}$.

If $c_i$ is any valid coloring of $G_i$, then in $G_{i, \gamma}$, $\forall \gamma \in C$, by Critical Path Property (i.e., Lemmas 5 or 7) there exists a $(\mu, \gamma, x, y_j)$ critical path and by Lemmas 6 and 8 there exists a $(v, \gamma, x, y'_j)$ maximal bichromatic path, where $\mu = c_i(y_i, y'_j)$ and $\{v\} = F'_i(c_i) - \{\mu\}$. Recall that $|S_{ab}| \leq \Delta - 1$ for any $ab \in E$. As an immediate consequence, we have

$$S_{xy} = S_{xy'} = S_{y_jy_j'} = C - \{1, 2\} = C'. \tag{1}$$

In view of (1), we have

$$|S_{xy}| = |S_{xy'}| = |S_{y_jy_j'}| = |C'| = \Delta - 1. \tag{2}$$

Lemma 9. Let $c_i$ be any valid coloring of $G_i$. Let $\mu = c_i(y_i, y'_j) \in \{1, 2\}$. Also let $y_j \in N_G(x) - \{y_i\}$. Then $\forall \gamma \in C'$, the $(\mu, \gamma, x, y_j)$ critical path in $G_{i, \gamma}$ does not contain the vertex $y_j$.

Proof. Suppose there exists a $(\mu, \gamma, x, y'_j)$ critical path that contains the vertex $y_j$, then $y_j$ cannot be an end vertex as $y_j \neq y_j$. Thus, $y_j$ is an internal vertex. Now since $\text{deg}_G(y_j) = 2$, the $(\mu, \gamma, x, y_j)$ critical path should contain the edge $xy_j$ as well. But
the \((\mu,\gamma,xy_i)\) critical path ends at vertex \(x\) with color \(\mu\) which implies \(c_i(x,y_j) = \mu\), a contradiction since \(c_i(x,y_j) \notin \{1,2\} = \{\mu,v\}\).

**Lemma 10.** Let \(c_i\) be any valid coloring of \(G_i\) and let \(u \in \{q,q'\}\). Let \(\mu = c_i(y_i,y'_i) = c_i(x,u) \in \{1,2\}\) and \(v = \{1,2\} - \{\mu\}\). Then \(\forall \gamma \in C'\), the \((\mu,\gamma,xy_i)\) critical path in \(G_i,\gamma\) has length at least five.

**Proof.** Suppose not. Then the \((\mu,\gamma,xy_i)\) critical path has length three which implies that the vertices in the critical path are \(x, u, y'_i, y_i\) in that order. Thus \(\mu \in F_u(c_i)\) and \(F_u(c_i) = S_{xu} \cup \{\mu\}\). Now change the color of the edge \(y'_i y_i\) to \(v\). It is proper since by 1, we have \(\{1,2\} = \{\mu,v\} \notin S_{y'_iy_i}\). It is valid since \(y_i\) is a pendant vertex in \(G_i,\gamma\). Now in view of Critical Path Property (i.e., Lemmas 5 or 7) there has to be a \((v,\gamma,xy_i)\) critical path that passes through the vertex \(y'_i\) with respect to this new coloring. Since \(c_i,\gamma(u,y'_i) = \gamma\), this \((v,\gamma,xy_i)\) critical path should contain vertex \(u\) as an internal vertex, which implies that color \(v \in F_u(c_i)\). Recalling that \(F_u(c_i) = S_{xu} \cup \{\mu\}\), we have \(v \in S_{xu}\), a contradiction in view of (1). Thus, the \((\mu,\gamma,xy_i)\) critical path has length at least five with respect to the coloring \(c_i,\gamma\) of \(G_i,\gamma\). \(\blacksquare\)

**B. The Structure of the Minimum Counter Example in the Vicinity of the Primary Pivot, \(x\)**

**Lemma 11.** The minimum counter example \(G\) satisfies the following properties:

\(\forall u, v \in N_G(x), (u,v) \notin E(G)\).

\(\forall y_i \in N'_G(x) \text{ and } \forall v \in N_G(x) - \{y_i\}, \text{ we have } (v,y'_i) \notin E(G)\).

**Proof.** To prove (a) we consider the following cases:

**Case 1.1.** \(u,v \in N'_G(x)\).

Let \(u = y_k\) and \(v = y_j\). Now if \(u \in N_G(v)\), then \(u = y'_j\). Recalling that \(\Delta(G) \geq 3\), in view of (2), we have \(ddeg_G(u) = ddeg_G(y'_j) \geq 5\). But \(ddeg_G(u) = ddeg_G(y_k) = 2\), a contradiction.

**Case 1.2.** \(u,v \in N''_G(x)\).

Then we need to show that \(q' \notin N_G(q)\). To see this, consider the coloring \(c_i\) of graph \(G_i\). We know that \(\{c_i(x,q),c_i(x,q')\} = \{\mu,v\}\). Without loss of generality, let \(c_i(x,q) = c_i(y_i,y'_i) = \mu\). Note that by (2), we have \(S_{xq} = C'\). If \(q' \in N_G(q)\), then \(c_i(q,q') \in C'\). Let \(c_i(q,q') = \gamma \notin \{\mu,v\}\). Now in \(G_i,\gamma\), the \((\mu,\gamma)\) maximal bichromatic path that starts at vertex \(x\) contains only edges \(xq\) and \(qq'\) since \(\mu \notin F'_G(c_i)\) (by (2)). Thus by Fact 1, there cannot be a \((\mu,\gamma,xy_i)\) critical path in \(G_i,\gamma\), a contradiction to Critical Path Property (i.e., Lemmas 5 or 7). Thus \(q' \notin N_G(q)\).

**Case 1.3.** \(u \in N''_G(x)\) and \(v \in N'_G(x)\).

Let \(v = y_i\). Then we have to show that \(y'_i \notin N''_G(x) = \{q,q'\}\). To see this, consider the coloring \(c_i\) of graph \(G_i\). Recall that \(\{c_i(x,q),c_i(x,q')\} = \{\mu,v\}\). Without loss of generality let \(c_i(x,q) = c_i(y_i,y'_i) = \mu\). Now if \(y'_i = q\), then we have \(c(q,y_i) = c(q,y'_i,y_i) = \mu\), a contradiction since \(c(x,q) = \mu\). On the other hand if \(y'_i = q'\), then \(c(q',y_i) = c(y'_i,y_i) = \mu\). This means that \(\mu \in S_{q'q}\), a contradiction in view of (1). Thus \(y'_i \neq q,q'\).

Thus \(\forall u,v \in N_G(x)\), we have \((u,v) \notin E(G)\).

To prove (b), we consider the following cases:

**Case 2.1.** \(v \in N'_G(x)\).
Let \( v = y_j \in N'_G(x) \). If \((v, y'_j) = (y_j, y'_j) \in E(G)\), then \( y'_i = y'_j \). Consider the coloring \( c_j \) of graph \( G_j \). Let \( c_j(y'_j, y'_i) = \mu \). Recall that by (2), we have \( S_{y'_j} = C' \). If \( y'_i = y'_j \), then \( c_j(y'_j, y'_i) = \gamma \). Now in \( G_{i,j} \), the \((\mu, \gamma)\) maximal bichromatic path that starts at vertex \( y_i \) contains only edges \( y_j y'_j, y'_j y_i \) and thus ends at vertex \( y_i \) since \( \mu \notin F_{y_j}(c_j) \).

This is because \( N_{G_{i,j}}(y_i) = \{y'_j, x\} \) and we have \( c_j(y'_j, y_i) = \gamma \) and \( c_j(x, y_i) \neq \mu \) (since by Assumption 1, \( \mu \in c_j(x, q), c_j(x, q') \)). Thus by Fact 1, there cannot be a \((\mu, \gamma, xy)\) critical path in \( G_{i,j} \), a contradiction to Critical Path Property (i.e., Lemmas 5 or 7). Thus \( y'_i \neq y'_j \).

**Case 2.2.** \( v \in N'_G(x) = \{q, q'\} \).

Then we have to show that \( y'_i \notin N_G(q) \cup N_G(q') \). To see this, consider the coloring \( c_i \) of graph \( G_i \). Recall that \( \{c_i(x, q), c_i(x, q')\} = \{\mu, v\} \). Without loss of generality, let \( c_i(x, q) = c_i(y_i, y'_i) = \mu \). Suppose \( y'_i \in N_G(q) \), then we have \( c(y'_i, q) \in S_{xy} \). Thus by (1), we have \( c_i(y'_i, q) \neq v \). Now there exists a \((\mu, c_i(y'_i, q) \neq v, xy)\) critical path of length 3, a contradiction to Lemma 10. Now if \( y'_i \in N_G(q') \), then we recolor the edge \( y_j y'_i \) with color \( v \) to get a valid coloring \( c'_j \). Now there exists a \((\mu, c_i(y'_i, q'), xy)\) critical path of length 3, a contradiction to Lemma 10. Thus \( y'_i \notin N_G(q) \cup N_G(q') \).

Thus \( \forall y_i \in N'_G(x) \) and \( \forall v \in N_G(x) \) \(-\{y_i\} \), we have \((v, y'_i) \notin E(G)\).

**C. Modification of Valid Coloring \( c_1 \) of \( G_1 \) to Get Valid Coloring \( c_j \) of \( G_j \)**

**Assumption 3.** Let \( c_1 \) be a valid coloring of \( G_1 \) and without loss of generality let \( c_1(x, q) = 1, c_1(x, q') = 2, \) and \( c_1(y_1, y'_1) = \mu = 1 \).

**Remark.** In view of Assumption 3, the Critical Path Property with respect to the coloring \( c_1 \) of \( G_1 \) reads as follows: With respect to the coloring \( c_1, \gamma \), there exists a \((1, \gamma, xy_1)\) critical path, for all \( \gamma \in C' \).

Let \( f_1 \) be the coloring of \( G_1 \) obtained from \( c_1 \) by exchanging the colors of the edges \( xq \) and \( xq' \). Also for \( \gamma \in C' \), we define the coloring \( f_1, \gamma \) as the coloring obtained from \( c_1, \gamma \) by exchanging the colors with respect to the edges \( xq \) and \( xq' \). Note that \( f_1, \gamma \) can be obtained from \( f_1 \) just by discarding the \( \gamma \) colored edge incident on vertex \( x \) for \( \gamma \in F_{xy_1}(f_1) \).

**Claim 2.** The coloring \( f_1 \) is proper but is not valid.

**Proof.** The coloring \( f_1 \) is proper since in view of (1), \( 2 \notin S_{xq} \) and \( 1 \notin S_{xq'} \). Suppose the coloring \( f_1 \) is valid. Let \( \gamma \) be a candidate color for the edge \( xy_1 \). Clearly \( \gamma \in C - F_{xy_1}(f_1) \).

Now since \( f_1 \) is proper, taking \( u = x, i = q, j = q', ab = xy_1, \lambda = 1, \) and \( \xi = \gamma \), Lemma 1 can be applied. There existed a \((1, \gamma, xy_1)\) critical path with respect to coloring \( c_1 \). By Lemma 1, we infer that there cannot be any \((1, \gamma, xy_1)\) critical path with respect to the coloring \( f_1 \). Thus by Fact 2, candidate color \( \gamma \) is valid for the edge \( xy_1 \). Thus we have obtained a valid coloring for the minimum counter example \( G \), a contradiction.

By Claim 2, there exist bichromatic cycles with respect to the coloring \( f_1 \). It is clear that each bichromatic cycle with respect to \( f_1 \) has to contain either the edge \( xq \) or \( xq' \) since we have changed only the colors of the edges \( xq \) and \( xq' \) to get the coloring \( f_1 \) from \( c_1 \). Thus, each such bichromatic cycle should be either a \((1, \gamma)\) bichromatic cycle or a \((2, \gamma)\) bichromatic cycle. Note that each of these bichromatic cycles should pass through the vertex \( x \). Moreover, observe that there cannot be any \((1, 2)\) bichromatic
cycle since color $1 \notin S_{xq}$ with respect to $f_1$ in view of (1). Thus $\gamma \in F'_1(f_1)$. From this we infer that $|F'_1(f_1)| \geq 1$. Recalling Assumption 2, we have $|C - F_1(f_1)| \geq 2$. It follows that $|C'| \geq 3$. Thus we have,

$$\Delta(G) \geq \text{degree}_{G_1}(q) \geq |S_{xy}| + 1 \geq |C'| + 1 \geq 4.$$  (3)

Let

$$C_1 = C_1(f_1) = \{ \gamma \in F'_1(c_1) \mid \exists (1, \gamma) \text{ bichromatic cycle with respect to coloring } f_1 \}.$$ 

$$C_2 = C_2(f_1) = \{ \gamma \in F'_1(c_1) \mid \exists (2, \gamma) \text{ bichromatic cycle with respect to coloring } f_1 \}.$$ 

Note that from the discussion above, any bichromatic cycle with respect to the coloring $f_1$ contains a vertex $y_i \in N'_{G_1}(x)$. But degree$_{G_1}(y_i) = 2$ and therefore $|S_{xy}| = 1$. Thus, $S_{xy}$ contains exactly one of the color 1 or 2. Thus with a fixed color $\gamma \in C_1 \cup C_2$ there exists exactly one of $(1, \gamma)$ or $(2, \gamma)$ bichromatic cycle, which implies that the sets $C_1$ and $C_2$ cannot have any element in common (see Fig. 2). Thus, we have

$$C_1 \cap C_2 = \emptyset.$$  (4)

Recall that in view of Critical Path Property (i.e., Lemmas 5 or 7), for a coloring $c_1, \gamma$ of $G_{1, \gamma}, \forall \gamma \in C'$, there exists a $(1, \gamma, xy_1)$ critical path. With respect to the new coloring $f_1, \gamma$, since the colors of only edges $xq$ and $xq'$ are changed, this path starts from $y_1$ and reaches the vertex $q$. But since color 1 is not present at vertex $q$ with respect to the coloring $f_1, \gamma$, the bichromatic path ends at vertex $q$. Thus, the $(1, \gamma, xy_1)$ critical path with respect to coloring $c_{1, \gamma}$ gets curtailed to a $(\gamma, 1, q, y_1)$ maximal bichromatic path with respect to $f_1, \gamma$. Also note that in view of Lemma 10, the length of this $(\gamma, 1, q, y_1)$ maximal bichromatic path is at least four. This is true for the coloring $f_1$ also i.e., there exists a $(\gamma, 1, q, y_1)$ maximal bichromatic path with respect to $f_1$. To see this observe that $f_1$ is obtained from $f_1, \gamma$ by putting back the edge $xy_a$, where $c_1(x, y_a) = \gamma$. This cannot alter the $(\gamma, 1, q, y_1)$ maximal bichromatic path since $x$ does not belong to this path and also $y_a \neq y_1$. Also in view of Lemma 9, none of the above maximal bichromatic paths contain vertex $y_j, \forall y_j \in N'_{G}(x) - \{y_1\}$. Thus, the coloring $f_1$ satisfies the following property which we name as Property A:

**Property A.** A partial coloring of $G$ is said to satisfy Property A iff $\forall \gamma \in C - \{1, 2\}$, there exists a $(\gamma, 1, q, y_1)$ maximal bichromatic path of length at least four. Moreover,
none of the above maximal bichromatic paths contain vertex $x$ or vertex $y_i$, where $y_i \in N_G'(x) - \{y_1\}$.

**Claim 3.** There exists a proper coloring $f_1'$ obtained from $f_1$ such that $\forall i \in \{1, 2\}$, $|C_i| \leq 1$, where $C_i = C_i(f_1')$. Moreover $f_1'$ satisfies Property A.

**Proof.** If $|C_1| \leq 1$ and $|C_2| \leq 1$, then let $f_1' = f_1$. If $|C_1| \leq 1$, then let $f_1'' = f_1$. Otherwise if $|C_1| \geq 2$, then let $C_1 = \{y_{i_0}, y_{i_1}, \ldots, y_{i_{k-1}}\}$ and also let $y_{i_j}$ be the vertex such that $f_1(x, y_{i_j}) = y_{i_j}$, $\forall j \in \{0, 1, 2, \ldots, k-1\}$ (see Fig. 2). Now let the coloring $f_1''$ be defined as $f_1'(x, y_{i_j}) = y_{i_j}$, where $l = j + 1(\text{mod } k)$, $\forall j \in \{0, 1, 2, \ldots, k-1\}$ and $f_1''(e) = f_1(e)$ for all other edges. (Note that we have only shifted the colors of the edges $x y_{i_0}, x y_{i_1}, \ldots, x y_{i_{k-1}}$ circularly. We call this procedure deranging of colors.)

Note that we are changing only the colors of the edges $x y_{i_j}$ for $j = 0, 1, 2, \ldots, k-1$. Also we are using only the colors $y_{i_j} \in C_1$ for recoloring. Since with respect to the coloring $f_1$, a $(1, \gamma_{i_j})$ bichromatic cycle passed through $y_{i_j}$ and $\text{degree}_{G_1}(y_{i_j}) = 2$, we have $S_{xy_{i_j}} = \{1\}$. Thus, the coloring $f_1''$ is proper.

Since for all $y_{i_j}, 0 \leq j \leq k - 1$ we have $S_{xy_{i_j}} = \{1\}$ with respect to the coloring $f_1''$, it is clear that any new bichromatic cycle created (in the process of getting $f_1''$ from $f_1'$) has to be a $(1, \gamma)$ bichromatic cycle, where $\gamma \in C_1$.

We claim that the coloring $f_1''$ does not have any $(1, \gamma)$ bichromatic cycle for $\gamma \in C_1$. To see this consider a $\gamma \in C_1$, say $\gamma_{i_1}$. There existed a $(1, \gamma_{i_1})$ bichromatic cycle with respect to $f_1$. It contained the edge $x y_{i_1}$. Now with respect to $f_1''$, edge $x y_{i_1}$ is colored with color $\gamma_{i_1}$. Thus, the $(1, \gamma_{i_1})$ maximal bichromatic path which contains the vertex $x$ has one end at vertex $y_{i_1}$ since color $\gamma_{i_1}$ is not present at the vertex $y_{i_1}$ with respect to $f_1''$. Thus, $(1, \gamma_{i_1})$ bichromatic cycle cannot exist with respect to the coloring $f_1''$. This argument works for all $\gamma \in C_1$ and thus for any color $\gamma \in C_1$, there is no $(1, \gamma)$ bichromatic cycle with respect to $f_1''$.

If $|C_2| \leq 1$, then $f_1' = f_1''$. Otherwise if $|C_2| \geq 2$, by performing similar recoloring (now starting with $f_1''$) as we did to get rid of the $(1, \gamma)$ bichromatic cycles, we can get a coloring $f_1'''$ without any $(2, \gamma)$ bichromatic cycle. Now let $f_1'' = f_1'''$. Thus, we get a coloring $f_1'$ from $f_1$ which has $|C_1| \leq 1$ and $|C_2| \leq 1$.

Note that we are changing only the colors of the edges $x y_{i_1}$, for $y_{i_1} \in N_G'(x)$. But the coloring $f_1$ satisfied Property A and hence none of the $(\gamma, 1, q, y_{i_1})$ maximal bichromatic paths, $\forall \gamma \in C - \{1, 2\}$, contained the vertex $y_{i_1}$ or $x$. Thus, these bichromatic paths have not been altered (i.e., neither broken nor extended) by the recoloring to get $f_1'$ from $f_1$. Thus, the coloring $f_1'$ satisfies Property A.

**Observation 1.** Note that the color of the edge $y_1 y_1'$ is unaltered in $f_1'$, i.e., $f_1'(y_1, y_1') = f_1(y_1, y_1') = c_1(y_1, y_1') = 1$. Also only the colors of certain edges incident on the vertex $y_1$, where $y_1 \in N_G'(x) - \{y_1\}$, are modified when we obtained $f_1'$ starting from $c_1$. (This information is required later in the proof.)

It is easy to see that $f_1'$ is proper but not valid. It is not valid because, if it is valid then since $f_1'$ satisfies Property A, there are $(\gamma, 1, q, y_{i_1})$ maximal bichromatic paths, $\forall \gamma \in C - \{1, 2\}$. Thus by Fact 1, for any $\theta \in C - F_\theta(f_1')$, there cannot be a $(1, \theta, x y_{i_1})$ critical path. Thus by Fact 2, color $\theta$ is valid for the edge $x y_{i_1}$. Thus we have a valid coloring for the graph $G$, a contradiction. Thus, $f_1'$ is not valid. It implies that at least one of $C_1$ or $C_2$ is non-empty. In the next lemma, we further refine the proper coloring $f_1'$.
Lemma 12. There exists a proper coloring \( h_1 \) of \( G_1 \) obtained from \( f_1' \) such that there is at most one bichromatic cycle. Moreover, \( h_1 \) satisfies Property A.

**Proof.** By Claim 3, we have \( |C_1| \leq 1 \) and \( |C_2| \leq 1 \). If exactly one of \( C_1, C_2 \) is singleton, then let \( h_1 = f_1' \). Otherwise we have \( |C_1| = 1 \) and \( |C_2| = 1 \).

Assumption 4. Without loss of generality let \( C_1 = \{ \gamma \} \) and \( C_2 = \{ \emptyset \} \). Let \( f_1'(x,y_j) = \gamma \) and \( f_1'(x,y_k) = \emptyset \). Thus \( f_1'(y_j,y_j') = 1 \) and \( f_1'(y_k,y_k') = 2 \), since there are \((1,\gamma)\) and \((2,\emptyset)\) bichromatic cycles passing through the vertex \( x \).

Claim 4. Color \( 2 \notin S_{y_jy_j'} \).

**Proof.** Suppose not, then \( 2 \in S_{y_jy_j'} \). Since there is a \((1,\gamma)\) bichromatic cycle passing through \( y_j', \) the colors 1 and \( \gamma \) are present at \( y_j' \). It follows that there exists \( \eta \in C - \{1,2,\gamma\} \) missing at \( y_j' \). Now recolor edge \( y_jy_j' \) with color \( \eta \) to get a coloring \( f_1'' \). If the color \( \eta \) is valid for the edge \( y_jy_j' \), then let \( h_1 = f_1'' \) and we are done as the situation reduces to having only one bichromatic cycle (i.e., \( |C_2| = 1 \) and \( |C_1| = 0 \)). If the color \( \eta \) is not valid for the edge \( y_jy_j' \), then there has to be a \((\gamma,\eta)\) bichromatic cycle that passes through vertex \( x \). Let \( f_1'(x,y_i) = \eta \). Since \( deg_G(y_i) = 2 \), we have \( S_{xy_i} = \{f_1''(y_i,y_i'), \emptyset\} = \{\gamma\} \). Recall that by Assumption 2, \( x \in C' - F'(c_1) \) and thus \( x \in C' - F'(f_1') \). Clearly \( x \neq \eta \). Recolor the edge \( xy_i \) with color \( x \) to get a coloring \( f_1''' \). Note that the color \( x \) is valid for the edge \( xy_j \) because if there is a \((x,\eta)\) bichromatic cycle, then it implies that \( S_{xy_i} = \{\eta\} \). But we know that \( S_{xy_i} = \{\gamma\} \), a contradiction. Thus, let \( h_1 = f_1''' \) and the situation reduces to having only one bichromatic cycle (i.e., \( |C_2| = 1 \) and \( |C_1| = 0 \)).

In view of Claim 4, color 2 is a candidate for the edge \( y_jy_j' \). Recolor edge \( y_jy_j' \) with color 2 to get a coloring \( f_1''' \). If the color 2 is valid for the edge \( y_jy_j' \), then let \( h_1 = f_1''' \) and the situation reduces to having only one bichromatic cycle (i.e., \( |C_2| = 1 \) and \( |C_1| = 0 \)). If the color 2 is not valid for the edge \( y_jy_j' \), then there has to be a \((\gamma,2)\) bichromatic cycle created due to the recoloring, thereby reducing the situation to \( |C_2| = 2 \) and \( |C_1| = 0 \). Now we can recolor the graph using the procedure similar to that in the proof of Claim 3 (i.e., derangement of colors in \( C_2 \)) to get a valid coloring \( h_1 \) without any bichromatic cycles.

The coloring \( f_1' \) satisfied Property A and hence none of the \((\gamma,1,q,y_1)\) maximal bichromatic paths , \( \forall y \in C - \{1,2\} \), contained the vertex \( y_j \). Thus, none of the \((\gamma,1,q,y_1)\) maximal bichromatic paths will be *broken* or *curtailed* in the process of getting \( h_1 \) from \( f_1' \). This is because we are changing only the colors of the edges incident on the vertex \( y_j \) or \( y_k \) and if a \((\gamma,1,q,y_1)\) maximal bichromatic path gets *broken* or *curtailed*, it means that the vertex \( y_j \) or \( y_k \) was contained in those maximal bichromatic path, a contradiction to Property A of \( f_1' \) since \( y_j \in N'(x) - \{y_1\} \). On the other hand, if any of these paths gets extended, then vertex \( y_j \in \{y_1,q\} \). But in view of Lemma 11 (part \( a) \) this is not possible. Thus, the \((\gamma,1,q,y_1)\) maximal bichromatic paths have not been extended. Thus, these bichromatic paths have not been altered by the recolorings to get \( h_1 \) from \( f_1' \). Thus, the coloring \( h_1 \) satisfies Property A.

Observation 2. Note that the color of the edge \( y_1y_1' \) is unaltered in \( h_1 \), i.e., \( h_1(y_1,y_1') = f_1'(y_1,y_1') = 1 \) (by Observation 1). Also only the colors of certain edges incident on the vertex \( y_j \), where \( y_i \in N'(x) - \{y_1\} \) are modified.
It is easy to see that \( h_1 \) is proper but not valid. It is not valid because if it is valid then since \( h_1 \) satisfies Property A, there are \((\gamma, 1, q, y_1)\) maximal bichromatic paths, \(\forall \gamma \in C - \{1, 2\}\). Thus by Fact 1, for any \( \theta \in C - F_x(h_1) \), there cannot be a \((1, \theta, x_1)\) critical path. Thus by Fact 2, color \( \theta \) is valid for the edge \( xy_1 \). Thus we have a valid coloring for the graph \( G \), a contradiction. Thus \( h_1 \) is not valid. Then in view of Lemma 12, we make the following assumption:

**Assumption 5.** Without loss of generality, let the only bichromatic cycle in the coloring \( h_1 \) of \( G_1 \) pass through the vertex \( y_j, j \neq 1 \). Also let \( h_1(x, y_j) = \rho \).

We get a coloring \( c_j \) of \( G_j \) from \( h_1 \) of \( G_1 \) by:

1. Removing the edge \( xy_j \).
2. Adding the edge \( xy_1 \) and coloring it with the color \( h_1(x, y_j) = \rho \).

Note that the coloring \( c_j \) is proper since \( \rho \neq c_j(y_1', y_j') = h_1(y_1, y_j') = 1 \) (by Observation 2) and \( \rho \notin S_{y_1}(c_j) \) (by the definition of \( c_j \)). Note that by removing the edge \( xy_j \), we have broken the only bichromatic cycle that existed with respect to \( h_1 \). The coloring \( c_j \) is valid because if there is a bichromatic cycle in \( G_j \) with respect to \( c_j \), then it should contain the edge \( xy_1 \) and thus it should be a \((1, \rho)\) bichromatic cycle since \( c_j(x, y_1) = h_1(x, y_j) = \rho \) and \( c_j(y_1, y_j') = 1 \). The \((\rho, 1, q, y_1)\) maximal bichromatic path with respect to \( h_1 \) is still a bichromatic path with respect to \( c_j \). And since no edge incident to \( q \) is recolored, there is a \((\rho, 1)\) maximal bichromatic path that starts at \( q \) and contains vertex \( y_1 \). This clearly implies that there cannot be a \((\rho, 1)\) bichromatic cycle containing vertex \( y_1 \). It follows that the coloring \( c_j \) of \( G_j \) is acyclic. Therefore, all the Lemmas in previous sections are applicable to the coloring \( c_j \) also.

Now we may assume that \( c_j(y_j, y_j') = 2 \) because if \( c_j(y_j, y_j') = 1 \), then we can change the color of the edge \( y_j y_j' \) to 2 without altering the validity of the coloring since \( y_j \) is a pendant vertex in \( G_j \). Thus, we make the following assumption:

**Assumption 6.** Without loss of generality, let \( c_j(y_j, y_j') = 2 \). Also recall that \( c_j(x, q) = 2 \) and \( c_j(x, q') = 1 \).

**Remark.** In view of Assumption 6, the Critical Path Property with respect to the coloring \( c_j \) of \( G_j \) reads as follows: With respect to the coloring \( c_j, \gamma \), there exists a \((2, \gamma, xy_j)\) critical path, for all \( \gamma \in C' \). The reader may contrast the Critical Path Property of \( c_j \) with that of \( c_1 \) (see Remark after Assumption 3). This correspondence is very important for the proof.

**Observation 3.** Note that \( c_j(x, q) = 2, c_j(x, q') = 1, c_j(y_1, y_j') = 1, c_j(y_j, y_j') = 2, \) and \( c_j(x, y_1) = \rho \notin \{1, 2\} \). Also if \( e \) is an edge such that none of its end points is \( x \) or \( y_i \), where \( y_i \in \mathcal{N}_G'(x) \), we have \( c_j(e) = c_1(e) \).

**Lemma 13.** Coloring \( c_{j, \gamma} \) of \( G_{j, \gamma} \) satisfies Property A.

**Proof.** We consider the following cases:

**Case 1.** \( \gamma \in C' - \{\rho\} \).

Recall that the coloring \( h_1 \) satisfied Property A. In getting \( c_j \) from \( h_1 \), we have only colored the edge \( xy_1 \) with color \( \rho \) and have discarded the edge \( xy_j \). Thus \( \forall \gamma \in C' - \{\rho\} \), there exists a \((\gamma, 1, q, y_1)\) maximal bichromatic path in \( c_j \) also. Noting that by Property A,
the maximal bichromatic path does not contain vertex $x$ or $y_i$, where $\forall y_i \in N'_G(x) - \{y_1\}$, we infer that even in $G_{j;\gamma}$ the $(\gamma,1,q,y_1)$ maximal bichromatic path is unaltered.

**Case 2.** $\gamma = \rho$.

Then $G_{j,\rho}$ is the graph obtained by removing the edge $xy_1$ from $G_j$ since $c_j(x,y_1) = \rho$. Recall that with respect to the coloring $h_1$, we have a $(\rho,1,q,y_1)$ maximal bichromatic path. Removal of edge $xy_1$ from $G_j$ cannot alter this path since $h_1$ satisfies Property A and thus edge $xy_1$ is not in the path. Now the graph obtained is nothing but the graph $G_{j,\rho}$ with respect to the coloring $c_{j,\rho}$. Thus, $G_{j,\rho}$ satisfies Property A. $\blacksquare$

**Property B.** Let $c_{1,\eta}$ be a partial coloring of $G_{1,\eta}$, for $\eta \in C - \{1,2\}$. Then $c_{1,\eta}$ is said to satisfy Property B if $\forall \gamma \in C - \{1,2\}$, there exists a $(\gamma,2)$ maximal bichromatic path which starts at vertex $q$ and involves the vertex $y'_j$. Also the length of the segment of this bichromatic path between the vertices $q$ and $y'_j$ is at least three. Moreover in none of the above maximal bichromatic paths the segment between the vertices $q$ and $y'_j$ contains vertex $x$ or vertex $y_i$, where $y_i \in N'_G(x)$.

**Lemma 14.** Coloring $c_{1,\eta}$ of $G_{1,\eta}$ satisfies Property B, for $\eta \in C - \{1,2\}$.

**Proof.** By Critical Path Property (i.e., Lemmas 5 or 7) and Lemma 10, $\forall \gamma \in C'$, there exists a $(2,\gamma,x,y_j)$ critical path of length at least five in $G_{j,\gamma}$. Also by Lemma 9, these critical paths do not contain vertex $y_i$, $\forall y_i \in N'_G(x) - \{y_1\}$. Recall that we obtained $c_j$ from $c_1$ by a series of recolorings. How will the above-mentioned critical paths change if we undo all these recolorings and get back $c_1$? Note that in the process of obtaining coloring $c_j$ from $c_1$, we have only changed the colors incident on the vertices $y_i$, where $y_i \in N'_G(x)$ and have exchanged the colors of the edges $xq$ and $xq'$ (by Observation 3). Thus, only the colors of edge $xq$ and possibly edge $y_jy'_j$ of these critical paths will get modified when we undo the recolorings. The reader may recall that the first step in getting $c_j$ from $c_1$ was to exchange the colors of edges $xq$ and $xq'$. It follows that with respect to a coloring $c_{1,\eta}$, there exists a $(\gamma,2)$ maximal bichromatic path which starts at vertex $q$ and involves the vertex $y'_j$. It also follows that the length of the segment of the bichromatic path between the vertices $q$ and $y'_j$ is at least three. Moreover, it is easy to see that in none of the above maximal bichromatic paths the segment between the vertices $q$ and $y'_j$ contains vertex $x$ or vertex $y_i$, where $y_i \in N'_G(x)$. $\blacksquare$

**D. Selection of Secondary Pivot $p$ and Properties of $c_1$ and $c_j$ in the Vicinity of $p$**

Let $N'_G(q) = N_G(q) \cap (W_1 \cup W_0)$ and $N''_G(q) = N_G(q) - N'_G(q)$. Since $q \in W_2 \cup W_1$ (see Assumption 1), it is easy to see that $|N''_G(q)| \leq 2$. Now recall that in view of (2) $\deg_G(q) = \Delta$ and by (3), $\Delta \geq 4$. Thus, we have $|N''_G(q)| \geq 2$.

Let $p \in N''_G(q)$ be such that $p \neq x$. In the rest of the proof, this vertex $p$ will play a central role. Therefore, we name it as the **Secondary Pivot**. Let $c_1(q,p) = \eta$. Note that $\eta \in C'$ by (1). Thus by Critical Path Property (i.e., Lemmas 5 or 7), there exists a $(1,\eta,x,y_1)$ critical path with respect to the coloring $c_{1,\eta}$ that passes through the vertex $p$ and clearly $qp$ is the second edge of this critical path. Recalling that this critical path has length at least five (by Lemma 10), we can infer that $p \neq y_1$ and $\deg_G(p) \geq 2$. Now since $p \in W_1 \cup W_0$, there is at most one neighbor of $p$ other than $q$ which is not in $W_0$. If such a vertex exists, let it be $p'$. Otherwise clearly $(N_G(p) \cap W_0) \neq \emptyset$ and let
It is interesting to note that where.

Lemma 17. From part (a) of Lemma 11.

Remark. In view of Lemma 17, degree\(G \geq 3\). Therefore \(p \notin W_0\). It follows that \(p \in W_1\).

It is interesting to note that \(p\) could have been selected as the Primary Pivot instead of \(x\). The reader may want to reread the procedure for selecting the primary pivot given.
Claim 5. Know by definition of $e$.

The minimum counter example $G$ satisfies the following properties:

(a) \( \forall u, v \in N_G(p), (u, v) \notin E(G) \).

(b) \( \forall z_i \in N_G(p) - \{q, p'\} \) and \( \forall v \in N_G(p) \), we have \( (v, z'_i) \notin E(G) \).

This lemma is not explicitly used in the proof, but we believe that this information will help the reader to visualize the situation better.

In view of Lemma 17, let $e_1$ and $e_2$ be the edges incident on $p$ such that $c_{1, \eta}(e_1) = 1$ and $c_{1, \eta}(e_2) = 2$. Then we claim the following:

Lemma 19. $c_{j, \eta}(e_1) = 1$ and $c_{j, \eta}(e_2) = 2$.

Proof. Recall that by Observation 3, only the edges incident on vertices $x$ or $y_i$, where $y_i \in N'_G(x)$, are altered while obtaining coloring $c_j$ from $c_1$. Let $e_1 = (p, z_i)$ and $e_2 = (p, z_i)$. Now to show that $c_{j, \eta}(e_1) = 1$ and $c_{j, \eta}(e_2) = 2$, it is enough to verify that $p, z_i, z'_i \notin \{x\} \cup N'_G(x)$. But this true by Lemma 15.

Lemma 20. $c_{1, \eta}(p, p') \in \{1, 2\}$ (In other words, one of the edge $e_1$ or $e_2$ is $pp'$. By Lemma 19, this also implies that $c_{j, \eta}(p, p') = c_{1, \eta}(p, p') \in \{1, 2\}$.)

Proof. Suppose not. Then $e_1 \neq pp'$ and $e_2 \neq pp'$. Without loss of generality, let $e_1 = (p, z_1)$ and $e_2 = (p, z_2)$. Thus $c_{1, \eta}(p, z_1) = 1$ and $c_{1, \eta}(p, z_2) = 2$. By Lemma 10 there exists a $(1, \eta, xy_1)$ critical path of length at least five with respect to $c_{1, \eta}$. This implies that $c_{1, \eta}(z_1, z'_1) = \eta$. Now by Lemma 14, with respect to $c_{1, \eta}$ there exists a $(\eta, 2)$ maximal bichromatic path which starts at vertex $q$ and contains vertex $y_j'$. Moreover, the segment of this bichromatic path between the vertices $q$ and $y_j'$ is of length at least three with respect to $c_{1, \eta}$. Since $p, z_1$ is only the second edge of this path, we can infer that $c_{1, \eta}(z_2, z'_2) = \eta$.

Now with respect to the coloring $c_{1, \eta}$, we exchange the colors of the edges $p, z_1$ and $p, z_2$ to get a coloring $c'_{1, \eta}$.

Claim 5. Coloring $c'_{1, \eta}$ is valid.

Proof. Note that $c'_{1, \eta}$ is proper since $c_{1, \eta}(z_1, z'_1) = \eta$ and $c_{1, \eta}(z_2, z'_2) = \eta$. Now the coloring $c'_{1, \eta}$ is valid because otherwise there has to be a $(\eta, 1)$ or $(\eta, 2)$ bichromatic cycle since only the colors of the edges $p, z_1$ and $p, z_2$ are altered. Thus, such a bichromatic cycle has to contain the edge $qp$ since $c'_{1, \eta}(q, p) = \eta$. From (1), we can infer that color 2 $\notin F_{\eta}(c'_{1, \eta})$. But if there exists a bichromatic cycle with respect to the coloring $c'_{1, \eta}$, it has to contain vertex $q$. From this we can infer that it has to be a $(\eta, 1)$ bichromatic cycle. This means that the cycle has to contain the vertex $x$ since $c'_{1, \eta}(x, q) = 1$. But we know by definition of $c_{1, \eta}$ that $\eta \notin F_{\eta}(c_{1, \eta}) = F_{\eta}(c'_{1, \eta})$. Thus, there does not exist a $(\eta, 1)$ bichromatic cycle with respect to the coloring $c'_{1, \eta}$. We conclude that the coloring $c'_{1, \eta}$ of $G_{1, \eta}$ is valid. □
Claim 6. With respect to the partial coloring \( c'_{1,\eta} \), there does not exist any \((1, \eta, xy_1)\) critical path.

Proof. Now since \( c'_{1,\eta} \) is proper, taking \( u=p, i=z_1, j=z_2, ab=xy_1, \lambda=1, \xi=\eta \), and noting that \( \{x, y_1\} \cap \{z_1, z_2\} = \emptyset \) (by Lemma 15), Lemma 1 can be applied. There existed a \((1, \eta, xy_1)\) critical path containing vertex \( p \) in coloring \( c_{1,\eta} \). By Lemma 1, we infer that there cannot be any \((1, \eta, xy_1)\) critical path in the coloring \( c'_{1,\eta} \).

Claim 7. There exists a valid coloring \( c'_1 \) of \( G_1 \) such that the coloring \( c'_{1,\eta} \) of \( G_{1,\eta} \) is derivable from \( c'_1 \).

Proof. It is enough to show that we can extend the coloring \( c'_{1,\eta} \) of \( G_{1,\eta} \) to a valid coloring \( c'_1 \) of \( G_1 \). If \( \eta \in C - F_x(c_1) \), then by definition \( G_{1,\eta} = G_1 \) and thus \( c'_1 = c'_{1,\eta} \). Otherwise, let \( y_k \in N'_G(x) \) be the vertex such that \( c_1(x, y_k) = \eta \). Note that \( k \neq 1 \). Recall that \( c'_{1,\eta} \) is obtained by discarding the color on the edge \( xy_k \). Thus, it is enough to extend the coloring \( c'_{1,\eta} \) to \( c'_1 \) by assigning an appropriate color to the edge \( xy_k \).

Note that there exists a \((1, x, xy_1)\) critical path with respect to \( c_{1,\eta} \), for \( z \in C - F_x(c_1) \) (by Lemma 5). Clearly \( z \neq \eta \). We claim that the \((1, x, xy_1)\) critical path exists even with respect to \( c'_{1,\eta} \). To see this, note that we have changed the colors of only edges \( p_1z_1 \) and \( p_2z_2 \) to get \( c'_{1,\eta} \) from \( c_{1,\eta} \). Note that by this exchange the \((1, x, xy_1)\) critical path cannot be extended since \( p_1, p_2 \notin \{x, y_1\} \) (by Lemma 15). Now if the \((1, x, xy_1)\) critical path gets altered, it means that this critical path contained the edge \( p_1z_1 \) (recall that \( c_{1,\eta}(p_1, z_1) = 1 \)) and hence \( c_{1,\eta}(z_1, z'_1) = \eta \). But we know that \( c_{1,\eta}(z_1, z'_1) = \eta \), a contradiction. Thus we have,

With respect to the partial coloring \( c'_{1,\eta} \), there exists a \((1, x, xy_1)\) critical path,

for \( z \notin F_x(c'_1) \) and \( z \neq \eta \). (5)

Now color the edge \( xy_k \) with color \( \eta \) to get a coloring \( d_1 \) of \( G_1 \). If \( d_1 \) is valid we are done and \( c'_1 = d_1 \). If it is not valid, then there has to be a bichromatic cycle containing the color \( \eta \). Note that the colorings \( d_1 \) and \( c_1 \) differ only due to the exchange of colors of edges \( p_1z_1 \) and \( p_2z_2 \). Thus, it has to contain one of the edges \( p_1z_1 \) or \( p_2z_2 \). Therefore, it has to be either a \((\eta, 1)\) or \((\eta, 2)\) bichromatic cycle since \( d_1(p_1, z_1) = 2, d_1(p_2, z_2) = 1 \). This also means that the bichromatic cycle has to contain the vertex \( q \), since \( d(p, q) = \eta \). Thus, the bichromatic cycle has to be a \((\eta, 1)\) bichromatic cycle since \( 2 \notin F_q(d_1) \). This means that \( d_1(y_k, y'_k) = 1 \). Now recolour the edge \( xy_k \) with color \( z \) to get a coloring \( d'_1 \) of \( G_1 \). If \( d'_1 \) is valid, we are done and \( c'_1 = d'_1 \). If it is not valid then there has to be a \((z, 1)\) bichromatic cycle containing the vertex \( x \), implying that there existed a \((1, z, xy_k)\) critical path with respect to the coloring \( d_1 \) and hence with respect to the coloring \( c'_{1,\eta} \). But in view of (5), there already exists a \((1, z, xy_1)\) critical path and by Fact 1, \((1, z, xy_k)\) critical path is not possible, a contradiction. Thus, the coloring \( d'_1 \) is valid and let \( c'_1 = d'_1 \).

Thus there exists a valid coloring \( c'_1 \) of \( G_1 \) such that the coloring \( c'_{1,\eta} \) of \( G_{1,\eta} \) is derivable from \( c'_1 \).

Now in view of Claims 6 and 7 there does not exist any \((1, \eta, xy_1)\) critical path with respect to the coloring \( c'_{1,\eta} \) of \( G_{1,\eta} \), a contradiction to Critical Path Property (i.e., Lemmas 5 or 7).

Journal of Graph Theory DOI 10.1002/jgt
We conclude that \( c_{1,\eta}(p, p') \in \{1, 2\}. \)

Assumption 7. In view of Lemmas 17, 19, and 20, let \( z_1 \) be the vertex such that \( \{c_{1,\eta}(p, z_1)\} = \{1, 2\} - \{c_{1,\eta}(p, p')\} \). It follows that \( \{c_{j,\eta}(p, z_1)\} = \{1, 2\} - \{c_{j,\eta}(p, p')\} \) and \( \{e_1, e_2\} = \{pp', pz_1\} \).

**Observation 4.**

(a) If \( c_{1,\eta}(p, p') = c_{1,\eta}(p, p') = 2 \), we have by Assumption 7 that \( c_{1,\eta}(p, z_1) = c_{j,\eta}(p, z_1) = 1 \). Thus with respect to the partial coloring \( c_{1,\eta} \), there exists a \((1, \eta, xy_1)\) critical path of length at least five which contains the vertex \( z_1 \). It follows that \( c_{1,\eta}(z_1, z_1') = \eta \) since \( z_1 z_1' \) is just the fourth edge of this \((1, \eta, xy_1)\) critical path.

(b) If \( c_{1,\eta}(p, p') = c_{1,\eta}(p, p') = 1 \), we have by Assumption 7 that \( c_{1,\eta}(p, z_1) = c_{j,\eta}(p, z_1) = 2 \). Thus with respect to the partial coloring \( c_{j,\eta} \), there exists a \((2, \eta, xy_1)\) critical path of length at least five which contains the vertex \( z_1 \). It follows that \( c_{j,\eta}(z_1, z_1') = \eta \) since \( z_1 z_1' \) is just the fourth edge of this \((2, \eta, xy_1)\) critical path.

**Local Recolorings:** If a partial coloring \( h \) of \( G \) is obtained from a partial coloring \( c \) of \( G \) by recoloring only certain edges incident on the vertices belonging to \( N_G(p) - \{p', q\} = \{z_1, z_2, \ldots, z_k\} \) and also possibly the edge \( pp' \), then \( h \) is said to be obtained from \( c \) by local recolorings.

The concept of local recolorings turns out to be crucial for the rest of the proof. The following lemma provides the main tool in this respect.

**Lemma 21.**

(a) Let \( c_{1,\eta}(p, p') = c_{1,\eta}(p, p') = 2 \). Also let \( h_{1,\eta} \) be any valid coloring obtained from \( c_{1,\eta} \) by recoloring only certain edges incident on the vertices belonging to \( N_G(p) - \{p', q\} = \{z_1, z_2, \ldots, z_k\} \) and also possibly the edge \( pp' \) (i.e., by only local recolorings). Then there exists a valid coloring \( h_1 \) of \( G_1 \) such that the valid coloring \( h_{1,\eta} \) of \( G_{1,\eta} \) is derivable from \( h_1 \).

(b) Let \( c_{1,\eta}(p, p') = c_{1,\eta}(p, p') = 1 \). Also let \( f_{j,\eta} \) be any valid coloring obtained from \( c_{j,\eta} \) by recoloring only certain edges incident on the vertices belonging to \( N_G(p) - \{p', q\} = \{z_1, z_2, \ldots, z_k\} \) and also possibly the edge \( pp' \) (i.e., by only local recolorings). Then there exists a valid coloring \( f_j \) of \( G_j \) such that the valid coloring \( f_{j,\eta} \) of \( G_{j,\eta} \) is derivable from \( f_j \).

**Proof.**

(a) Recall that \( \eta \neq 1, 2 \). If \( \eta \notin F_x(c_1) \), then \( c_{1,\eta} = c_1 \). In this case, we take \( h_1 = c_1 \). Otherwise if \( \eta \in F_x'(c_1) \), let \( xy_k \) be the edge in \( G_1 \) such that \( c_1(x, y_k) = \eta \). Note that \( k \neq 1 \). It is enough to show that we can extend the valid coloring \( h_{1,\eta} \) of \( G_{1,\eta} \) to a valid coloring \( h_1 \) of \( G_1 \) by assigning an appropriate color to the edge \( xy_k \). (Reader may note that neither \( pp' \) nor any edge incident on the vertices in \( \{z_1, z_2, \ldots, z_k\} \) can be the edge \( xy_k \) since \( x \notin (p, p', z_1, z_2, \ldots, z_k, z_1', z_2', \ldots, z_k') \) due to Lemma 15.)

Now assign color \( \eta \) to the edge \( xy_k \) to get a coloring \( d \). If the coloring \( d \) is valid, we are done and we have \( h_1 = d \). If it is not valid, then there has to be a bichromatic cycle created in \( G_1 \) with respect to the coloring \( d \). The cycle has
to be a \((\eta, \theta)\) bichromatic cycle, where \(d(y_k, y'_k) = \theta\). Moreover we can infer that 
\(\theta \in F_x(d)\). If \(\theta \neq d(p, p') = 2\), then let \(d' = d\). Otherwise we have \(\theta = d(p, p') = 2\). Now there exists a color \(\omega \neq 2, \eta\) that is a candidate for the edge \(y_ky'_k\). Recolor the edge \(y_ky'_k\) using color \(\omega\) to get a coloring \(d'\) of \(G_1\). Now if \(d'\) is a valid coloring, then we are done and we have \(h_1 = d'\). If it is not valid, then \(d'(y_k, y'_k) \neq 2\). Let 
\(d'(y_k, y'_k) = \beta \neq 2\). Moreover with respect to the coloring \(d'\), there should be a 
\((\eta, \beta)\) bichromatic cycle. Also let \(z \neq \eta \in C - F_x(c_1) = C - F_x(d')\). Now if, 
\((1)\) \(\beta = 1\).

**Claim 8.** None of the \((1, \gamma, xy_1)\) critical paths, where \(\gamma \neq \eta \in C - F_x(c_1)\) 
are altered in the process of getting the coloring \(h_1, \eta\) from \(c_1, \eta\).

**Proof.** Recall that only the edges incident on vertices \(z_i\), where \(z_i \in N_G(p) - \{p', q\}\), and edge \(pp'\) are possibly recolored to get the coloring \(h_1, \eta\) of 
\(G_1\). Note that by these recolorings, the \((1, \gamma, xy_1)\) critical path cannot 
be extended since \(x, y_1 \notin \{p, p', z_1, z_2, \ldots, z_k, \ldots, z'\}\) due to Lemma 15. 
Now if any \((1, \gamma, xy_1)\) critical paths are altered, then they have to contain the 
above-mentioned edges. Note that none of the vertices in \(\{z_1, z_2, \ldots, z_k\}\) or 
vertex \(p'\) can be the end vertices \(x\) or \(y_1\) and hence any critical path containing 
\(z_i\) or \(p'\) should also contain the vertex \(p\) since \(deg_{G_1, \eta}(z_i) = 2\).

We can infer that with respect to the coloring \(c_1, \eta\), the \((1, \gamma, xy_1)\) critical path 
passes through the vertex \(p\). It follows that this critical path has to contain 
the edge \(cz_1\) since \(c_1, \eta(p, z_1) = 1\) (from part (a) of Observation 4). Now since 
\(z_1 \in W_0\) (i.e., \(deg_{G_1, \eta}(z_1) = 2\)), this implies that \(c_1, \eta(z_1, \ldots, \gamma, z_1') = \eta\), 
a contradiction since from part (a) of Observation 4, we know that \(c_1, \eta(z_1, \ldots, \eta) = \eta\). Thus 
there cannot be any \((1, \gamma, xy_1)\) critical path containing the edges incident on 
vertices \(z_i\), where \(z_i \in N_G(p) - \{p', q\}\) and edge \(pp'\). Thus none of the \((1, \gamma, xy_1)\) 
critical paths, where \(\gamma \in C - F_x(c_1), \gamma \neq \eta\), are altered.

Since \(d'\) is not valid, there has to be a \((\eta, 1)\) bichromatic cycle that passes 
through the vertex \(x\). Now recolor the edge \(xy_k\) with color \(z\) to get a coloring \(d''\). 
Now if still there is a bichromatic cycle, then it should contain the edge 
\(xy_k\) and hence the edge \(y_ky'_k\). Therefore, it is a \((x, 1)\) bichromatic cycle. This 
implies by Fact 2 that there existed a \((1, \gamma, xy_k)\) critical path with respect to 
the coloring \(d''\) and hence with respect to the coloring \(h_1, \eta\). But in view of 
Claim 8, there exists a \((1, \gamma, xy_1)\) critical path with respect to the coloring \(h_1, \eta\), 
a contradiction in view of Fact 1. Thus, the coloring \(d''\) is valid.

\((2)\) \(\beta \neq 1\). This implies that \(\beta \neq \eta \in F_{x'}(d'')\). Let \(y_i \in N_G(x)\) be such that \(d'(x, y_i) = \beta\). 
Thus \(d'(y, y'_i) = \eta\). Now recolor the edge \(xy_k\) with color \(z \in C - F_x(d'')\) to get 
a coloring \(d''\). Note that \(x \neq \eta\) since \(\eta \notin C - F_x(d'')\). Now if still there is a 
bichromatic cycle, then it should contain the edge \(xy_k\) and hence the edge \(y_ky'_k\). 
Therefore, it is a \((x, \beta)\) bichromatic cycle. Thus, the bichromatic cycle should 
contain the edge \(xy_i\). Since \(deg_{G_1}(y_i) = 2\), the bichromatic cycle 
should contain the edge \(y_iy'_i\). But by our assumption, \(d''(y_i, y'_i) = d'(y_i, y'_i) = \eta \neq x\), a contradiction. Thus, the coloring \(d''\) is valid.

Now let \(h_1 = d''\). Thus, we get a valid coloring of \(G_1\) from \(h_1, \eta\).

(b) The proof of this is similar to that of part (a) with \(G_j, c_j, y_j\) taking the roles of 
\(G_1, c_1, y_1\), respectively, and the colors 1 and 2 exchanging their roles. ■
Lemma 22.

(a) If $c_{1,\eta}(p,p') = c_{j,\eta}(p,p') = 2$, then with respect to the coloring $c_{1,\eta}$, $2 \notin S_{z_1z'_1}$.

(Recall that by Assumption 7, $\{c_{1,\eta}(p,z_1)\} = \{c_{j,\eta}(p,z_1)\} = \{1, 2\} - \{c_{1,\eta}(p,p')\} = \{1\}$.)

(b) If $c_{j,\eta}(p,p') = c_{1,\eta}(p,p') = 1$, then with respect to the coloring $c_{j,\eta}$, $1 \notin S_{z_1z'_1}$.

(Recall that by Assumption 7, $\{c_{j,\eta}(p,z_1)\} = \{c_{1,\eta}(p,z_1)\} = \{1, 2\} - \{c_{j,\eta}(p,p')\} = \{2\}$.)

Proof.

(a) Suppose not. That is $2 \in S_{z_1z'_1}$. Note that by part (a) of Observation 4, we have $c_{j,\eta}(p,z_1) = 1$ and $c_{j,\eta}(z_1,z'_1) = \eta$. Therefore, there exists some $\theta \notin \{1, 2, \eta\}$ missing in $S_{z_1z'_1}$. Now recolor edge $z_1z'_1$ with color $\theta$ to get a coloring $c'_{1,\eta}$. If the coloring $c'_{1,\eta}$ is valid, then let $c''_{1,\eta} = c'_{1,\eta}$. Otherwise, a bichromatic cycle gets formed by the recoloring. Since $c'_{1,\eta}(p,z_1) = 1$, it has to be a $(1, \theta)$ bichromatic cycle and it passes through the vertex $p$. Thus there exists $z_1 \in N_G(p) - \{q,p'\}$ such that $c'_{1,\eta}(p,z_1) = \theta$ and $c''_{1,\eta}(z_1,z'_1) = 1$.

Now there exists a color $\mu \notin \{1,\theta,2,\eta\}$ missing at $p$. Recolor the edge $p\bar{z}_1$ with color $\mu$ to get a coloring $c'''_{1,\eta}$. This clearly breaks the $(1, \theta)$ bichromatic cycle that existed with respect to $c'_{1,\eta}$. But if a new bichromatic cycle gets formed with respect to $c'''_{1,\eta}$, then it has to contain vertex $z_1$ and therefore the edge $z_1z'_1$, implying that it has to be a $(\mu,\theta)$ bichromatic cycle since $c'''_{1,\eta}(z_1,z'_1) = \theta$. This cycle passes through the vertex $p$ and hence passes through the vertex $z_1$ since $c'''_{1,\eta}(p,z) = \theta$, implying that $c'''_{1,\eta}(z_1,z'_1) = \mu$, a contradiction since $c'''_{1,\eta}(z_1,z'_1) = 1$.

Thus, the coloring $c''_{1,\eta}$ is valid. Note that we have possibly changed the colors of the edges $p\bar{z}_1$ and $z_1z'_1$ to get $c'''_{1,\eta}$ from $c_{1,\eta}$ (i.e., only local recolorings are done). Therefore, by part (a) of Lemma 21 we infer that there exists a coloring $c''_{1,\eta}$ of $G_1$ such that $c''_{1,\eta}$ is derivable from $c'_{1,\eta}$. It follows from Critical Path Property (i.e., Lemmas 5 or 7) that there exists a $(1,\eta,xy_1)$ critical path with respect to the coloring $c''_{1,\eta}$. On the other hand, recall that with respect to $c_{1,\eta}$ there existed a $(1,\eta,xy_1)$ critical path passing through $p\bar{z}_1$ and $z_1z'_1$ (by part (a) of Observation 4). But while getting $c''_{1,\eta}$ from $c_{1,\eta}$ we have indeed changed the color of at least one of the edges $p\bar{z}_1$ or $z_1z'_1$ using a color other than 1 and $\eta$. It follows that the $(1,\eta)$ maximal bichromatic path, which contains the vertex $x$, ends at either vertex $p$ or $z_1$. Noting that $p,z_1 \neq y_1$, we infer by Fact 1 that there cannot be a $(1,\eta,xy_1)$ critical path with respect to the coloring $c''_{1,\eta}$, a contradiction.

(b) The proof of this is similar to that of part (a) with $G_{j,\eta}$, $c_{j,\eta}$, $y_j$ taking the roles of $G_{1,\eta}$, $c_{1,\eta}$ and $y_1$, respectively, and the colors 1 and 2 exchanging their roles.

E. Getting a Valid Coloring That Contradicts the Critical Path Property Either From $c_1$ or From $c_j$

In this section, we will get the final contradiction in the following way: If $c_{1,\eta}(p,p') = c_{j,\eta}(p,p') = 1$, then we will show that we can get a coloring $c'_j$ from $c_j$ that contradicts
the Critical Path Property. Otherwise if \( c_1,\eta(p,p') = c_j,\eta(p,p') = 2 \), then we will show that we can get a coloring \( c_1' \) from \( c_1 \) that contradicts the Critical Path Property.

The two colorings \( c_1 \) and \( c_j \) are very similar and hence we will only describe the way we get \( c_1' \) from \( c_1 \). The same arguments can be imitated easily for \( c_j \) by keeping the following correspondences in mind.

1. Vertex \( y_1 \) has same role as vertex \( y_j \).
2. Colors 1 and 2 exchange their roles.
3. \((1, \gamma, xy_1)\) critical path has the same role as \((2, \gamma, xy_j)\) critical path, for \( \gamma \in C' \).
   The Critical Path Property of \( c_1 \) corresponds to that of \( c_j \) (see Remarks after Assumptions 3 and 6).
4. Part (a) of Lemmas 21 and 22 applies to coloring \( c_1 \) while part (b) applies to coloring \( c_j \) in a corresponding way.
5. Lemma 14 has the same role as Lemma 13.

We make the following assumption:

**Assumption 8.** Let \( c_1,\eta(p,p') = c_j,\eta(p,p') = 2 \).

**Observation 5.** In view of Assumption 8, from Observation 4 there exists a \((1, \eta, xy_1)\) critical path which contains the vertex \( z_1 \) with respect to the partial coloring \( c_1,\eta \). Moreover, this path is of length at least five. It follows that \( c_1,\eta(p,z_1) = 1 \) and \( c_1,\eta(z_1,z_1') = \eta \). The first five vertices of the path are \( x, q, p, z_1, z_1' \). Then clearly \( z_1' \neq y_1 \) and hence is not a pendant vertex in \( G_1,\eta \). Thus, we have \( S_{z_1,z_1'} \neq \emptyset \) and \( 1 \in S_{z_1,z_1'} \).

Getting a valid coloring \( d_1 \) of \( G_1,\eta - \{pz_1\} \) from \( c_1,\eta \) by only local recolorings

In view of Lemma 22 and since \( c_1,\eta(p,z_1) = 1 \), the color 2 is a candidate for the edge \( z_1z_1' \). We get a valid coloring \( d_1 \) of \( G_1,\eta - \{pz_1\} \) from \( c_1,\eta \) by removing the edge \( pz_1 \) and recoloring the edge \( z_1z_1' \) by the color 2. Note that \( d_1 \) is valid since \( z_1 \) is a pendant vertex in \( G_1,\eta - \{pz_1\} \). Moreover, we have broken the \((1, \eta, xy_1)\) critical path. Hence, we have

\[
\text{With respect to the partial coloring } d_1, \text{ there does not exists any } (1, \eta, xy_1) \text{ critical path.} \tag{6}
\]

**Lemma 23.** With respect to the partial coloring \( d_1 \) of \( G_1,\eta \), \( \forall \gamma \in C - F_p(d_1) \), there exists a \((2, \gamma, pz_1)\) critical path. Since each of these critical paths has to contain the edge \( pp' \), we can infer that \( C - F_p(d_1) \subseteq S_{pp'} \).

**Proof.** Suppose not. Then there exists a color \( \gamma \in C - F_p(d_1) \) such that there is no \((2, \gamma, pz_1)\) critical path. By Fact 2 color \( \gamma \) is valid for the edge \( pz_1 \). Thus, we get a valid coloring \( d_1' \) of \( G_1,\eta \) by coloring the edge \( pz_1 \) with color \( \gamma \).

Note that we have possibly changed the colors of the edges \( pz_1 \) and \( z_1z_1' \) to get \( d_1' \) from \( c_1,\eta \) (i.e., only local recolorings are done). Therefore by part (a) of Lemma 21 we infer that there exists a valid coloring of \( G_1 \) from which \( d_1' \) can be derived. It follows from Critical Path Property (i.e., Lemmas 5 or 7) that there exists a \((1, \eta, xy_1)\) critical path with respect to the coloring \( d_1' \). On the other hand, recall that with respect to \( c_1,\eta \) there existed a \((1, \eta, xy_1)\) critical path passing through \( pz_1 \) and \( z_1z_1' \) (by Observation 5). But while getting \( d_1' \) from \( c_1,\eta \) we have indeed changed the color of the edges \( z_1z_1' \) using...
the color $2 \notin \{1, \eta\}$. It follows that the $(1, \eta)$ maximal bichromatic path which contains the vertex $x$ ends at either vertex $p$ or $z_1$. Noting that $p, z_1 \neq y_1$, we infer that there cannot be a $(1, \eta, xy_1)$ critical path with respect to the coloring $d'_1$, a contradiction. ■

Note that with respect to $G_{1, \eta} \setminus \{pz_1\}$, $|F_p(d_1)| \leq \Delta - 1$ and therefore $|C - F_p(d_1)| \geq 2$. But we know that color $1 \notin F_p(d_1)$. Since $|C - F_p(d_1)| \geq 2$, there exists a color $\mu \neq 1 \in C - F_p(d_1)$. Note that $\mu \neq 2$ also. The following observation is obvious in view of Claim 23:

**Observation 6.** With respect to the partial coloring $d_1$ of $G_{1, \eta}$, $1, \mu \notin F_p(d_1)$ and there exist $(2, 1, pz_1)$ and $(2, \mu, pz_1)$ critical paths.

**Selection of a special color $\theta$.** Since $|F_p'(d_1)| \leq \Delta$, there exists a color $\theta$ missing at vertex $p'$. By Lemma 23, $\theta \notin C - F_p(d_1) \subseteq F_p'(d_1)$. Thus $\theta \in F_p(d_1)$. Clearly $\theta \neq 2$ since $2 \in F_p'$ and $\theta \neq 1, \mu$ because $1, \mu \notin F_p(d_1)$ and hence by Lemma 23 we have $1, \mu \in S_{pp}(d_1)$. Further $\theta \neq \eta$. This is because by Lemma 14, the $(\eta, 2)$ maximal bichromatic path starts at vertex $q$ and contains the vertex $y_j'$. Clearly, the first three vertices of this path are $q, p, p'$. Recall that the length of the segment of this path between vertices $q$ and $y_j'$ is at least three. Therefore $\eta \in S_{pp}(d_1)$. Now without loss of generality let $d_1(p, z_2) = \theta(\neq 1, \eta, \mu, 2)$. Note that $z_2$ is a vertex different from $z_1$.

Note that with respect to the coloring $c_{1, \eta}$, the $(1, \eta, xy_1)$ critical path passes through the vertex $z_1$ (by (5)). This critical path cannot contain the vertex $z_2$. This is because if $z_2$ is an internal vertex of this critical path, then the edge $pz_2$ should be contained in the path, a contradiction since $c_{1, \eta}(p, z_2) = \theta \neq 1, \eta$. On the other hand, if $z_2$ is an end vertex then it implies that $z_2 \in \{x, y_1\}$, a contradiction in view of Lemma 15. Thus, vertex $z_2$ is not contained in the $(1, \eta, xy_1)$ critical path. While getting the coloring $d_1$ from $c_{1, \eta}$, this path was broken due to the recoloring of $z_1 z_1'$ and $pz_1$. It follows that the $(1, \eta)$ maximal bichromatic path that starts at vertex $y_1$ does not contain vertex $z_2$. Thus we can infer that

**Observation 7.** With respect to the coloring $d_1$, there cannot exist a $(1, \eta, y_1, z_2)$ maximal bichromatic path.

**Getting a valid coloring $d_2$ of $G_{1, \eta} \setminus \{pz_2\}$ from $d_1$ of $G_{1, \eta} \setminus \{pz_1\}$ by only local recolorings**

We get a coloring $d''_1$ of $G_{1, \eta} \setminus \{pz_1, pz_2\}$ from $d_1$ by discarding the edge $pz_2$. Note that the partial coloring $d''_1$ of $G_{1, \eta}$ is valid.

Now recolor the edge $pz_1$ with coloring the special color $\theta$ to get a coloring $d_2$ of $G_{1, \eta} \setminus \{pz_2\}$. Note that the color $\theta$ is a candidate for the edge $pz_1$ with respect to the coloring $d''_1$ since $d''_1(z_1, z'_1) = 2$ and $\theta \notin F_p(d''_1)$, since we have removed the edge $pz_2$. (Recall that $d''_1(p, z_2) = \theta$.) We claim that $d_2$ is valid also. Clearly if there is any bichromatic cycle created, then it has to be a $(\theta, 2)$ bichromatic cycle since $d_2(z_1, z'_1) = 2$. Now this bichromatic cycle has to pass through vertex $p'$ since $d_2(p, p') = 2$. But by the definition of color $\theta$, it was not present at vertex $p'$. Thus, there cannot be a $(\theta, 2)$ bichromatic cycle. It follows that the partial coloring $d_2$ of $G_{1, \eta} \setminus \{pz_2\}$ is valid. Recall that by (6) that there exists no $(1, \eta, xy_1)$ critical path with respect to $d_1$. Note that to get $d_2$ from $d_1$, we just assigned $\theta(\neq 1, 2, \eta, \mu)$ to the edge $pz_1$ and removed the edge $pz_2$. Thus, there is no chance of $(1, \eta, xy_1)$ critical path getting created with respect to $d_2$. 

*Journal of Graph Theory DOI 10.1002/jgt*
Hence we have

With respect to the partial coloring \( d_2 \), there does not exists any

\[(1, \eta, xy_1) \] critical path.

(7)

Getting a valid coloring \( c'_{1, \eta} \) of \( G_{1, \eta} \) from \( d_2 \) of \( G_{1, \eta} - \{pz_2\} \) by only local recolorings

Now we will show that we can give a valid color for the edge \( pz_2 \) to get a valid coloring for the graph \( G_{1, \eta} \). We claim the following:

**Lemma 24.** With respect to the coloring \( d_2 \) at least one of the colors 1, \( \mu \) is valid for the edge \( pz_2 \). (Recall that by Observation 6, \( 1, \mu \notin F_p(d_1) \) and therefore \( 1, \mu \notin F_p(d_2) \).)

**Proof.** Let \( d_2(z_2, z'_2) = \sigma \). Now if,

1. \( \sigma = 2 \). Recolor the edge \( pz_2 \) using color 1 to get a coloring \( d_3 \). The coloring \( d_3 \) is valid because if a bichromatic cycle gets formed it has to be (1, 2) bichromatic cycle containing the vertex \( p \) implying that there was a (2,1,\( pz_2 \)) critical path with respect to \( d_2 \). But by Observation 6, there was a (2,1,\( pz_1 \)) critical path with respect to the coloring \( d_1 \) and hence with respect to the coloring \( d_2 \). (Note that to get \( d_2 \) from \( d_1 \), we just assigned \( d_1(p,z_2) = \theta \) (\( \neq 1,2,\eta,\mu \)) to edge \( pz_1 \) and removed the edge \( pz_2 \). Thus, the (2,1,\( pz_1 \)) critical path is not altered during this recoloring.) Thus in view of Fact 1, there cannot be any (2,1,\( pz_2 \)) critical path with respect to \( d_2 \) since \( z_1 \neq z_2 \), a contradiction. Thus, the coloring \( d_3 \) is valid.

2. \( \sigma \in \{1, \mu\} \). Recolor the edge \( pz_2 \) using color \( \{1, \mu\} - \{\sigma\} \) to get a coloring \( d_3 \). The coloring \( d_3 \) will be valid because if a bichromatic cycle gets formed, it has to be (1,\( \mu \)) bichromatic cycle containing the vertex \( p \). But since color \( \sigma \in \{1, \mu\} \) is not present at vertex \( p \), such a bichromatic cycle is not possible.

3. \( \sigma \notin \{1,2,\mu\} \). Recolor the edge \( pz_2 \) using color 1 to get a coloring \( d'_2 \). If the coloring \( d'_2 \) is valid, then let \( d_3 = d'_2 \). Otherwise if the coloring \( d'_2 \) is not valid, then there has to be a (\( \sigma, 1 \)) bichromatic cycle. Now let \( d'_2(p,z_j) = \sigma \). Then the bichromatic cycle passes through the vertex \( z_j \) and hence \( d'_2(z_j, z'_j) = 1 \), since \( deg_{G}(z_j) = 2 \). Now we recolor edge \( pz_2 \) with color \( \mu \) to get a coloring \( d_3 \). If there is a bichromatic cycle formed with respect to the coloring \( d_3 \), then it has to be a (\( \mu, \sigma \)) bichromatic cycle and hence it passes through the vertex \( z_j \). But color \( \mu \) is not present at \( z_j \) since \( d'_2(z_j, z'_j) = 1 \). Thus, there cannot be any (\( \mu, \sigma \)) bichromatic cycle. Hence, the coloring \( d_3 \) is valid.

Thus, either color 1 or \( \mu \) is valid for the edge \( pz_2 \).

To get the coloring \( d_3 \) from \( d_2 \) we have only given a valid color for the edge \( pz_2 \) and have not altered the color of any other edge (i.e., only local recolorings are done). Recall that \( d_2 \) does not have any \( (1, \eta, xy_1) \) critical path (by (7)). Note that \( d_3(q, p) = 1 \) and \( d_3(q, p) = \eta \). If we give color \( \mu \neq 1, \eta \) to the edge \( pz_2 \), there is no chance of a \( (1, \eta, xy_1) \) critical path getting formed in \( d_3 \). On the other hand, by giving color 1 to the edge \( pz_2 \) if a \( (1, \eta, xy_1) \) critical path gets formed, then it means that there exists a \( (1, \eta, xy_1, z_2) \) maximal bichromatic path with respect to \( d_2 \) and hence with respect to \( d_1 \). But by Observation 7 such a bichromatic path does not exist. Now let \( c'_{1, \eta} = d_3 \). Thus we have,

With respect to the valid coloring \( c'_{1, \eta} \) of \( G_{1, \eta} \), there does not exists any

\[(1, \eta, xy_1) \] critical path.

(8)
In getting $c'_{1,\eta}$ from $c_{1,\eta}$ we have done only local recolorings and thus by Lemma 21 $c'_{1,\eta}$ can be derived from some valid coloring $c'_1$ of $G_1$. Note that we have not changed the color of the edge $y_1y'_1$ while getting $c'_{1,\eta}$ from $c_{1,\eta}$ since $y_1 \notin \{p,p',z_1,\ldots,z_k,z'_1,\ldots,z'_k\}$ (by Lemma 15). Thus $c'_{1,\eta}(y_1,y'_1)=1$. It follows that the Critical Path Property of $c'_{1,\eta}$ is the same as Critical Path Property of $c_{1,\eta}$. This implies that there exists a $(1,\eta,xy_1)$ critical path with respect to the coloring $c_{1,\eta}$, a contradiction in view of (8).

This completes the proof.

REFERENCES

[1] A. A. Ageev, Every circle graph of girth at least 5 is 3-colorable, Discrete Math 195 (1999), 229–233.
[2] N. Alon, C. J. H. McDiarmid, and B. A. Reed, Acyclic coloring of graphs, Random Struct Alg 2 (1991), 343–365.
[3] N. Alon, B. Sudakov, and A. Zaks, Acyclic edge-colorings of graphs, J Graph Theory 37 (2001), 157–167.
[4] N. Alon and A. Zaks, Algorithmic aspects of acyclic edge colorings, Algorithmica 32 (2002), 611–614.
[5] D. Amar, A. Raspaud, and O. Togni, All to all wavelength routing in all-optical compounded networks, Discrete Math 235 (2001), 353–363.
[6] M. Basavaraju and S. Chandran, Acyclic edge coloring of subcubic graphs, Discrete Math 308 (2008), 6650–6653.
[7] M. Basavaraju, S. Chandran, and M. Kummini, d-regular graphs of acyclic chromatic index at least $d+2$, J Graph Theory 63 (2010), 226–230.
[8] O. V. Borodin, Acyclic colorings of planar graphs, Discrete Math 25 (1979), 211–236.
[9] M. I. Burnstein, Every 4-valent graph has an acyclic five-coloring, Soobsč Akad Nauk Gruzin SSR 93 (1979), 21–24.
[10] Y. Card and Y. Roditty, Acyclic edge-colorings of sparse graphs, Appl Math Lett 7 (1994), 63–67.
[11] R. Diestel, Graph Theory, 2nd edn, 173, Springer, New York, 2000.
[12] J. Fiamcik, The acyclic chromatic class of a graph, Math Slovaca 28 (1978), 139–145 (Russian).
[13] S. Gerke and M. Raemy, Generalised acyclic edge colourings of graphs with large girth, Discrete Math 307 (2007), 1668–1671.
[14] C. Greenhill and O. Pikhurko, Bounds on the generalised acyclic chromatic numbers of bounded degree graphs, Graphs Combin 21 (2005), 407–419.
[15] B. Grünbaum, Acyclic colorings of planar graphs, Israel J Math 14 (1973), 390–408.
[16] A. Kostochka, E. Sopena, and X. Zhu, Acyclic and oriented chromatic numbers of graphs, J Graph Theory 24 (1997), 331–340.
[17] M. Molloy and B. Reed, Further algorithmic aspects of Lovász local lemma, Proceedings of the 30th Annual ACM Symposium on Theory of Computing, Dallas, Texas, United States, 1998, pp. 524–529.
[18] R. Muthu, N. Narayanan, and C. R. Subramanian, Improved bounds on acyclic edge coloring, Elect Notes Discrete Math 19 (2005), 171–177.
[19] R. Muthu, N. Narayanan, and C. R. Subramanian, Optimal acyclic edge coloring of grid like graphs, Proceedings of the 12th International Conference, COCOON, Lecture Notes in Computer Science 4112, Springer, Berlin, 2006, pp. 360–367.
[20] R. Muthu, N. Narayanan, and C. R. Subramanian, Acyclic edge colouring of outerplanar graphs, Algorithmic Aspects in Information and Management, Lecture Notes in Computer Science 4508, Springer, Berlin, 2007, pp. 144–152.
[21] R. Muthu, N. Narayanan, and C. R. Subramanian, Acyclic edge colouring of partial 2-trees, submitted, 2008, available at http://www.imsc.res.in/narayan/p2t.pdf
[22] J. Něsetřil and N. C. Wormald, The acyclic edge chromatic number of a random d-regular graph is $d+1$, J Graph Theory 49 (2005), 69–74.
[23] S. Skulrattankulchai, Acyclic colorings of subcubic graphs, Inf Process Lett 92 (2004), 161–167.