LP APPROACH TO EXPONENTIAL STABILIZATION OF
SINGULAR LINEAR POSITIVE TIME-DELAY SYSTEMS VIA
MEMORY STATE FEEDBACK

NGUYEN H. SAU
Department of Mathematics, Electric Power University
235 Hoang Quoc Viet Road, Hanoi, Vietnam

VU N. PHAT
Institute of Mathematics, VAST
18 Hoang Quoc Viet Road, Hanoi, Vietnam

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Abstract. This paper deals with the exponential stabilization problem by
means of memory state feedback controller for linear singular positive systems
with delay. By using system decomposition approach, singular systems the-
ory and Lyapunov function method, we obtain new delay-dependent sufficient
conditions for designing such controllers. The conditions are given in terms
of standard linear programming (LP) problems, which can be solved by LP
optimal toolbox. A numerical example is given to illustrate the effectiveness
of the proposed method.

1. Introduction. Over the past few decades, singular systems (also referred to
as descriptor systems, semi-state systems, implicit systems, differential-algebraic
systems, or generalized state-space systems) have attracted much attention due to
the comprehensive applications in control theory [4, 5, 12, 19]. The analysis and the
controller design for such systems have received considerable attention in the past
decades (see e.g. [2, 8, 20, 22]). However, physical systems in the real world involve
variables that have nonnegative sign, say, population levels, electrical circuits, power
systems, absolute temperature, and so on. Such systems are referred to as positive
systems (see e.g. [9, 11, 25]), where their output states are nonnegative provided the
initial conditions are nonnegative. Due to widespread applications, it is necessary
to investigate the stability and control problem for positive systems. Example
applications include irrigation networks, highway and air-traffic flows, job-balancing
in computer clusters, and chemical networks, to name just a few. The states of
positive systems are confined within a “cone” located in the positive orthant rather
than in the wholes pace, which makes their analysis and synthesis a challenging and
interesting job (see e.g. [3, 26]). It is worth noting that a lot of well-known results
for normal linear systems cannot be simply applied to positive systems, since the
states of positive systems are located in the positive orthant rather than in linear
spaces. Among the large number of research results obtained for singular linear
positive systems, much attention has been devoted to the stability analysis of such systems. A successful application to exponential stability of positive systems has been proposed in [10, 13, 18, 27], however, they did not consider the singularity case.

The problem of observer and control design via LP approach for singular delay systems has been considered in [7, 14, 17], but the positivity case was not considered. Note that the LP approach may have a numerical advantage versus the LMI approach since the existing LMI softwares cannot handle large size problems and are not numerically stable. It should be noted that due to the singularity of derivative matrix and the non-negativity of variables in positive singular systems, much of the developed theory for such systems is still not up to a quantitative level. This feature makes the analysis and synthesis of singular linear positive systems a challenging task [3, 23]. Moreover, stability and control problems of singular positive systems with time delay have been extensively studied due to the fact that the singular positive time-delay system better describes physical systems than state-space systems. By using state-space decomposition approach and linear matrix inequality technique, a solution to stability problem for singular linear positive systems was given in [16, 24, 27], unfortunately, the conditions are obtained there under the regularity and impulse-free assumptions and the design feedback controller was not considered. To the best of our knowledge, very few results on stability and control for singular linear positive systems with time delay have been reported in the literature.

In this paper, we provide a treatment based on linear programming for positivity and stabilization of singular linear positive systems with delay. The particular property of singular positive systems that we uncover in this paper (and which renders structured memory state feedback control design convex) has consequences for general plants and it is also applicable to pairs of systems and supply rates that can be transformed into the problem studied here via a change of control variables. Our main propose is to design a memory state feedback controller which guarantees the exponential stability of the closed-loop system. The significant contribution of this paper lies in two aspects. First, new delay-dependent sufficient conditions are provided, which determine the regularity, impulse-free and positivity of the system. Second, based on the structural decomposition of system matrices and the singular value decomposition method, we establish new sufficient conditions for designing memory state feedback controllers to stabilizes the closed-loop system. The derived conditions are described in a linear programming form and do not impose any restriction on the matrix dynamics of the governed system, so they can be easier solved by a standard linear programming problem than by the LMI method.

The remainder of this paper is organized as follows. In Section 2, necessary preliminaries and some propositions are provided for the proof of the main result. Section 3 proposes the delay-depending sufficient conditions for designing memory state feedback controllers of exponential stabilization problem. The paper ends with a conclusion and cited references.

**Nomenclatures:** $x \in \mathbb{R}^n$ is called nonnegative (positive) if all ist entries are nonnegative (positive). $x \in \mathbb{R}^n : \|x\| = \max_{1 \leq i \leq n} |x_i|$. $\mathbb{R}_{0,+}^n (\mathbb{R}_+^n)$ denotes the space of all nonnegative (positive) vectors in $\mathbb{R}^n$. $\mathbb{R}^{m \times n}$ denotes the set of all real $(m \times n)$ matrices. $I_n$ denotes the identity matrix in $\mathbb{R}^{n \times n}$. A matrix $B \in \mathbb{R}^{n \times n}$ is called Metzler if all its off diagonal elements are non-negative. $B \succeq 0 (B > 0)$ means all its
entries are nonnegative (positive). $A \succeq B (A \succ B)$ means $A - B \succeq 0 (A - B \succ 0)$. $\mathcal{C}([-h, 0], \mathbb{R}^n)$ denotes the space of all continuous functions defined on $[-h, 0]$.

2. Preliminaries. Consider the following singular linear system with delay

$$
\begin{aligned}
&E\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + B u(t), \quad t \geq 0, \\
x(t) = \varphi(t), \quad t \in [-h, 0],
\end{aligned}
$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control vector, $A_0, A_1 \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, and the matrix $E \in \mathbb{R}^{n \times n}$ is singular and $\text{rank} E = r < n$. The initial condition $\varphi(t) \in \mathcal{C}([-h, 0], \mathbb{R}^n)$, $h > 0$.

**Definition 2.1.** ([5]): (i) The pair $(E, A_0)$ is said to be regular if $\text{det}(sE - A_0)$ is not identically zero. (ii) The pair $(E, A_0)$ is said to be impulse-free if $\text{deg}(\text{det}(sE - A_0)) = \text{rank} E$. (iii) The singular delay system (1) $(u(t) = 0)$ is said to be regular and impulse-free if the pair $(E, A_0)$ is regular and impulse free.

**Definition 2.2.** ([9]): The singular unforced system (1) $(u(t) = 0)$ is said to be positive if for any initial condition $\varphi: [-h, 0] \to \mathbb{R}^n_{0+}$, the state trajectory satisfies $x(t) \geq 0$ for all $t \geq 0$.

**Definition 2.3.** Given $\alpha > 0$. The singular unforced system (1) is said to be $\alpha-$stable if it is regular, impulse-free and there exist a positive number $N > 0$ such that the solution $x(t, \varphi)$ satisfies

$$
\|x(t, \varphi)\| \leq Ne^{-\alpha t}\|\varphi\|, \quad \forall t \geq 0.
$$

One way to solve the stabilization problem of singular systems with delay is to design memoryless controllers (see, e.g. [8, 20, 22]). The memoryless controllers are the control laws of the form $u(t) = Kx(t)$ or of more general controllers with memory that include, nevertheless, an instantaneous feedback term $u(t) = Kx(t) + \sum_{i=1}^{m} K_i x(t-h_i)$. Although the memoryless controllers in the mentioned papers are easy to implement, it was pointed out in [15], that they tend to be more conservative when the time delay is small. In fact, information on the size of the delay is often available in many processes. Hence, by using this information and employing a feedback of the past control history as well as the current state, we may expect to achieve an improved performance. Therefore, in this paper, as in [13] we will design the memory static feedback controller for the exponential stabilization of singular linear positive systems with delay.

**Definition 2.4.** Given $\alpha > 0$. The singular delay system (1) is said to be $\alpha-$stabilizable if there exists a memory feedback control $u(t) = K x(t) + F x(t-h)$, $K, F \in \mathbb{R}^{m \times n}$ such that the closed-loop system

$$
\begin{aligned}
&E\dot{x}(t) = (A_0 + BK)x(t) + (A_1 + BF)x(t-h), \quad t \geq 0, \\
x(t) = \varphi(t), \quad t \in [-h, 0],
\end{aligned}
$$

is positive and $\alpha-$stable.

Since $\text{rank} E = r < n$, it is known ([5]) that there are two nonsingular matrices $P, Q$ such that $PEQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$. Let us denote

$$
PEQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} := \tilde{E}, \quad PA_0Q = \begin{bmatrix} A_{01} & A_{02} \\ A_{03} & A_{04} \end{bmatrix} := \tilde{A}_0,
$$
Proposition 1. Assume that the system (1) is regular and impulse-free, written in the form (2) with $\det(A_{04}) \neq 0$. The system (2) is positive if and only if $A_{04}$ is Hurwitz, $A_0$ is Metzler and $\bar{A}_1 \succeq 0$, $\bar{B} \succeq 0$.

Applying the memory feedback control
\[ u(t) = K_1 y_1(t) + K_2 y_2(t) + F_1 y_1(t-h) + F_2 y_2(t-h), \]
where $K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$, $F = \begin{bmatrix} F_1 & F_2 \end{bmatrix}$, $K_1, F_1 \in \mathbb{R}^{m \times r}$, $K_2, F_2 \in \mathbb{R}^{m \times (n-r)}$, the system (2) is reduced to the closed-loop system
\[
\begin{aligned}
\dot{y}_1(t) &= A_{01} y_1(t) + A_{02} y_2(t) + A_{11} y_1(t-h) + A_{12} y_2(t-h) + B_1 u(t), \\
y_1(t) &= \phi_1(t), \quad t \in [-h, 0], \\
0 &= A_{03} y_1(t) + A_{04} y_2(t) + A_{13} y_1(t-h) + A_{14} y_2(t-h) + B_2 u(t), \\
y_2(t) &= \phi_2(t), \quad t \in [-h, 0],
\end{aligned}
\]
where $Q^{-1} \varphi(t) = [\phi_1(t), \phi_2(t)]$.

Proposition 2. Assume that $Q \succeq 0$. If the system (2) is $\alpha-$stabilizable by the feedback control (3), then the system (1) is $\alpha-$stabilizable by the feedback control $u(t) = KQ^{-1}x(t) + FQ^{-1}x(t-h)$.

Proof. Systems (4) is regular, impulse-free if and only if $\det(A_{04} + B_2 K_2) \neq 0$ ([12]). Setting
\[
\begin{aligned}
P &= \begin{bmatrix} I_r & -(A_{02} + B_1 K_2)(A_{04} + B_2 K_2)^{-1} \\ 0 & I_{n-r} \end{bmatrix}, \\
\overline{Q} &= Q \begin{bmatrix} I_r & 0 \\ -(A_{04} + B_2 K_2)^{-1}(A_{03} + B_2 K_1) & (A_{04} + B_2 K_2)^{-1} \end{bmatrix}.
\end{aligned}
\]
It is easy to verify that
\[
P E \overline{Q} = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix}, \quad P(A_0 + BKQ^{-1}) \overline{Q} = \begin{bmatrix} A_0 & 0 \\ 0 & I_{n-r} \end{bmatrix}.
\]
\[
\begin{aligned}
P(sE - (A_0 + BKQ^{-1})) \overline{Q} &= sPE \overline{Q} - P(A_0 + BKQ^{-1}) \overline{Q} = \begin{bmatrix} sI_r - \hat{A}_0 & 0 \\ 0 & -I_{n-r} \end{bmatrix}, \\
\end{aligned}
\]
where $\hat{A}_0 = (A_{01} + B_1 K_1) - (A_{02} + B_1 K_2)(A_{04} + B_2 K_2)^{-1}(A_{03} + B_2 K_1)$. Therefore
\[
\begin{aligned}
\det(sE - (A_0 + BKQ^{-1})) &= \det(P^{-1} P(sE - (A_0 + BKQ^{-1})) \overline{Q} \overline{Q}^{-1}) \\
&= (-1)^{n-r} \det(P^{-1} \det(sI_r - \hat{A}_0) \det(\overline{Q}^{-1}).
\end{aligned}
\]
Moreover, noting that \( det(sI - \hat{A}_0) = \sum_{k=1}^{r} a_k s^k, a_r = 1, \) and
\[
det(P^{-1}) \neq 0, \ det(Q^{-1}) \neq 0,
\]
due to the singularity of \( P \) and \( Q \), the polynomial \( det(sE - (A_0 + BKQ^{-1})) \) is not identically zero and
\[
\text{deg}(det(sE - (A_0 + BKQ^{-1}))) = r = \text{rank}E,
\]
which implies that the closed-loop system of (1) is regular and impulse-free. Suppose that system (4) is positive, i.e. \( y(t) \succeq 0, t \geq 0, \) we have \( x(t) = Qy(t) \succeq 0, t \geq 0 \) because \( Q \succeq 0 \) hence the closed-loop system of (1) is positive. On the other hand, if \( \|y(t)\| \leq Me^{-\alpha t} \|\phi\|, \forall t \geq 0, \) then
\[
\|x(t)\| \leq \|Q\|\|y(t)\| \leq Ne^{-\alpha t} \|\phi\|, \forall t \geq 0.
\]
\[\square\]

**Proposition 3.** ([6]) Let \( A \) be a Metzler matrix. Then the following conditions are equivalent.

1) \( A \) is Hurwitz.

2) There exists \( \gamma \in \mathbb{R}^n \) such that \( \gamma \succ 0 \) and \( A\gamma \prec 0. \)

3) There exists \( \lambda \in \mathbb{R}^n \) such that \( \lambda \succ 0 \) and \( \lambda^T A \prec 0. \)

4) The matrix \( A \) is nonsingular and satisfies \( A^{-1} \preceq 0. \)

**Proposition 4.** Given a number \( \nu > 0, \) a matrix \( Y \succeq 0, \) and \( X \) is Metzler and Hurwitz, if for some vector \( \lambda \in \mathbb{R}_+^m \) satisfying
\[
\lambda^T (X + e^\nu Y) \prec 0,
\]
then there exist a vector \( \eta \in \mathbb{R}_+^m \) and a number \( \rho \in (0, 1) \) such that
\[
\eta^T (X^{-1}) Y \preceq \rho \eta^T.
\]

**Proof.** From (5) and \( \lambda \succ 0, \ Y \succeq 0, \ e^\nu > 1 \) it follows that
\[
\lambda^T (X + Y) \prec 0.
\]
Since \( X + Y \) is Metzler, using Proposition 3 and (7), we can find a vector \( \eta_1 \in \mathbb{R}_+^m \) satisfying
\[
(X + Y)\eta_1 \prec 0.
\]
Moreover, since \( X \) is Hurwitz and Metzler, using Proposition 3 again we get \( -X^{-1} \succeq 0. \) Now, pre-multiplying both sides of equation (8) with the nonsingular matrix \( -X^{-1} \succeq 0, \) we have
\[
\left( -X^{-1} Y - I_m \right) \eta_1 \prec 0,
\]
which derives by Proposition 3 that there exists \( \eta \in \mathbb{R}_+^m \) satisfying
\[
\eta^T \left( -X^{-1} Y - I_m \right) \prec 0.
\]
We now will show that there exists a number \( \rho \in (0,1) \) such that
\[-\eta^T X^{-1} Y \leq \rho \eta^T.\] (11)
Indeed, defining \( \eta^T := (\xi_1, \xi_2, ..., \xi_m) > 0 \) and \( q := -\eta^T X^{-1} Y = (q_1, q_2, ..., q_m) \geq 0. \)
If \( q_i = 0, \forall i = 1, 2, ..., m \), then for any \( \rho \in (0,1) \) we get (11). If \( q \neq 0 \), then setting \( \rho := \max_{i=1,2,...,m} \left\{ \frac{q_i}{\xi_i} \right\} \). The inequality (10) implies \( q < \eta^T \), i.e. \( q_i < \xi_i, \forall i = 1, 2, ..., m \), hence \( \rho = \max_{i=1,2,...,m} \left\{ \frac{q_i}{\xi_i} \right\} \in (0,1) \). Moreover, since \( q_i \leq \rho \xi_i \), for all \( i = 1, 2, ..., m \), we obtain \( q \leq \rho \eta^T \), which implies the condition (11).

3. Main result. This section explores the exponential stabilization of the linear singular positive system with delay given in (1). The design of memory state feedback controller is given in terms of sufficient conditions by solving Linear Programming Problem. The memory state feedback controller (3) guarantees the corresponding closed-loop system (4) to be positive, regular and impulse-free and exponentially stable. For the brief expression of the proof we denote \( \hat{A}_0 = [a_{ij}^{(0)}]_{n \times n}, \hat{A}_1 = [a_{ij}^{(1)}]_{n \times n}, b_i^T \) denotes the \( i \)-th row of \( \hat{B} \).

**Theorem 3.1.** Given \( \alpha > 0 \), the system (1) is \( \alpha \)– stabilizable if the following linear programming problem is feasible in the variables \( \beta = (\hat{\beta}_1, \hat{\beta}_2, ..., \hat{\beta}_n) \in \mathbb{R}^n_+ \), \( k_j, f_j \in \mathbb{R}^n, j = 1, 2, ..., n : \)
\[
\begin{align*}
  a_{ij}^{(0)} \hat{\beta}_j + b_i^T k_j & \geq 0, \quad i, j = 1, ..., n; \ i \neq j, \\
  a_{ij}^{(1)} \hat{\beta}_j + b_i^T f_j & \geq 0, \quad i, j = 1, ..., n,
\end{align*}
\]
(12)
\[
(\alpha \hat{E} + \hat{A}_0 + \hat{A}_1 e^{\alpha h}) \beta + \hat{B} \left( \sum_{i=1}^n k_i + e^{\alpha h} \sum_{j=1}^n f_i \right) < 0.
\]
(13)
Moreover, the stabilizing control matrices are given by
\[
  K = \begin{bmatrix} k_1 & k_2 & \cdots & k_n \\
  \hat{\beta}_1 & \hat{\beta}_2 & \cdots & \hat{\beta}_n \end{bmatrix}, \quad F = \begin{bmatrix} f_1 & f_2 & \cdots & f_n \\
  \hat{\beta}_1 & \hat{\beta}_2 & \cdots & \hat{\beta}_n \end{bmatrix}.
\]

**Proof.** Based on the Proposition 2 it suffices to prove the \( \alpha \)-stabilizability of the system (4). Our proof is divided into two parts. In the first part we show the system (4) is regular, impulse-free and positive. In the second part we prove that the system (4) is exponential stable.

1. **Positivity, regularity and impulse-free** For this we rewrite the system (4) in the form
\[
\begin{align*}
  \overline{E} y(t) = \overline{A}_0 y(t) + \overline{A}_1 y(t-h), & \quad t \geq 0, \\
  y(t) = \phi(t), & \quad t \in [-h, 0],
\end{align*}
\]
(14)
where \( \overline{E} = \begin{bmatrix} I_r & 0 \\
  0 & 0 \end{bmatrix} \), and
\[
\overline{A}_0 = \begin{bmatrix} A_{01} + B_1 K_1 & A_{02} + B_1 K_2 \\
  A_{03} + B_2 K_1 & A_{04} + B_2 K_2 \end{bmatrix}, \quad \overline{A}_1 = \begin{bmatrix} A_{11} + B_1 F_1 & A_{12} + B_1 F_2 \\
  A_{13} + B_2 F_1 & A_{14} + B_2 F_2 \end{bmatrix}.
\]
Note that the condition \( a_{ij}^{(0)} \hat{\beta}_j + b_i^T k_j \geq 0, \quad i, j = 1, ..., n; \ i \neq j, \) is equivalent to \( a_{ij}^{(0)} + b_i^T \frac{k_j}{\hat{\beta}_j} \geq 0, \quad i, j = 1, ..., n; \ i \neq j, \) and hence \( \hat{A}_0 + \hat{B} K \) is Metzler. Moreover,
we have
\[ \overline{A}_0 = \begin{bmatrix} A_{01} + B_1K_1 & A_{02} + B_1K_2 \\ A_{03} + B_2K_1 & A_{04} + B_2K_2 \end{bmatrix} = \begin{bmatrix} A_{01} & A_{02} \\ A_{03} & A_{04} \end{bmatrix} + \begin{bmatrix} B_1K_1 & B_1K_2 \\ B_2K_1 & B_2K_2 \end{bmatrix} = \tilde{A}_0 + \tilde{B}K, \]
which gives \( \overline{A}_0 \) is Metzler. Similarly,
\[ a_{ij}^{(1)} \beta_j + b_j^T f_j \geq 0 \iff a_{ij}^{(1)} + b_j^T \beta_j \geq 0, \]
which gives \( \tilde{A}_1 + \tilde{B}F \succeq 0 \). Since
\[ \overline{A}_1 = \begin{bmatrix} A_{11} + B_1F_1 & A_{12} + B_1F_2 \\ A_{13} + B_2F_1 & A_{14} + B_2F_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{13} & A_{14} \end{bmatrix} + \begin{bmatrix} B_1F_1 & B_1F_2 \\ B_2F_1 & B_2F_2 \end{bmatrix} = \tilde{A}_1 + \tilde{B}F, \]
we get \( \overline{A}_1 \succeq 0 \). Setting
\[ \sum_{i=1}^{n} k_i = \left[ \frac{k_1}{\beta_1} \frac{k_2}{\beta_2} \ldots \frac{k_n}{\beta_n} \right] \beta = K\beta, \quad \sum_{i=1}^{n} f_i = \left[ \frac{f_1}{\beta_1} \frac{f_2}{\beta_2} \ldots \frac{f_n}{\beta_n} \right] \beta = F\beta, \]
and using condition (13) we have
\[ (\alpha \bar{E} + \tilde{A}_0 + \tilde{A}_1 e^{\alpha h}) \beta + \tilde{B} \left( \sum_{i=1}^{n} k_i + e^{\alpha h} \sum_{j=1}^{n} f_i \right) = (\alpha \bar{E} + \tilde{A}_0 + \tilde{A}_1 e^{\alpha h}) \beta \]
\[ + \tilde{B}(K + e^{\alpha h} F) \beta = (\alpha \bar{E} + \tilde{A}_0 + \tilde{A}_1 e^{\alpha h} + \tilde{B}(K + e^{\alpha h} F)) \beta \prec 0 \]  
\[ (15) \]
\[ \alpha \bar{E} + \overline{A}_0 + \overline{A}_1 e^{\alpha h} = \alpha \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} A_{01} + B_1K_1 & A_{02} + B_1K_2 \\ A_{03} + B_2K_1 & A_{04} + B_2K_2 \end{bmatrix} 
+ e^{\alpha h} \begin{bmatrix} A_{11} + B_1F_1 & A_{12} + B_1F_2 \\ A_{13} + B_2F_1 & A_{14} + B_2F_2 \end{bmatrix} 
= \alpha \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} A_{01} & A_{02} \\ A_{03} & A_{04} \end{bmatrix} + e^{\alpha h} \begin{bmatrix} A_{11} & A_{12} \\ A_{13} & A_{14} \end{bmatrix} + \begin{bmatrix} B_1K_1 & B_1K_2 \\ B_2K_1 & B_2K_2 \end{bmatrix} 
+ e^{\alpha h} \begin{bmatrix} B_1F_1 & B_1F_2 \\ B_2F_1 & B_2F_2 \end{bmatrix} 
= \alpha \bar{E} + \tilde{A}_0 + \tilde{A}_1 e^{\alpha h} + \tilde{B}(K + e^{\alpha h} F). \]  
\[ (16) \]
From (15) and (16) it follows that
\[ (\alpha \bar{E} + \overline{A}_0 + \overline{A}_1 e^{\alpha h}) \beta \prec 0. \]  
\[ (17) \]
On the other hand, from \( (\alpha \bar{E} + \overline{A}_1 e^{\alpha h}) \beta \succeq 0 \) and (17) it follows that
\[ \overline{A}_0 \beta \prec 0. \]  
\[ (18) \]
Decomposing \( \beta := \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{R}_+^{n}, \quad v_1 \in \mathbb{R}_+^{r}, \quad v_2 \in \mathbb{R}_+^{n-r}, \) we derive from (18) that
\[ \overline{A}_0 \beta = \begin{bmatrix} A_{01} + B_1K_1 & A_{02} + B_1K_2 \\ A_{03} + B_2K_1 & A_{04} + B_2K_2 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \prec 0, \]  
\[ (19) \]
which gives
\[ (A_{03} + B_2K_1)v_1 + (A_{04} + B_2K_2)v_2 \prec 0. \]  
\[ (20) \]
Since $\overline{A}_0$ is Metzler matrix, $v_1 \in \mathbb{R}_+^r$, we have $(A_{04} + B_2K_1)v_1 \geq 0$, and hence from (20), we obtain
\[(A_{04} + B_2K_2)v_2 < 0.\] (21)

Remark that the matrix $A_{04} + B_2K_2$ is Metzler, using Proposition 3 and (21) we obtain that the matrix $A_{04} + B_2K_2$ is Hurwitz, and $\text{det}(A_{04} + B_2K_2) \neq 0$, which implies that the system (4) is regular, impulse-free. Moreover, since $\overline{A}_1 \geq 0$, $\overline{A}_0$ is Metzler and $A_{04} + B_2K_2$ is Hurwitz, $\text{det}(A_{04} + B_2K_2) \neq 0$, using Proposition 1 we get that the system (4) is positive.

2. Exponential stability

Since system (4) is a linear differential-algebraic equation, we need to prove the exponential stability of each solution $y_1(t), y_2(t)$.

2.1. Exponential stability of $y_1(t)$.

We first note that the matrix $\alpha E + \overline{A}_0 + \overline{A}_1 e^{\alpha h}$ is Metzler satisfying (17), using Proposition 3, there exist $\lambda \in \mathbb{R}_+^n$ such that
\[\lambda^T [\alpha E + \overline{A}_0 + \overline{A}_1 e^{\alpha h}] < 0.\] (22)

Consider the following nonnegative function
\[V(t,y) = e^\alpha \lambda^T Ey(t) + \int_{t-h}^t e^\alpha(s+h) \lambda^T \overline{A}_1 y(s) ds.\] (23)

Taking the derivative in $t$ along the solution we have
\[\dot{V}(t,y) = \alpha \lambda^T e^\alpha Ey(t) + \lambda^T e^\alpha Ey(t) + \lambda^T \overline{A}_1 e^{\alpha(t+h)} y(t) - \lambda^T \overline{A}_1 e^{\alpha t} y(t-h)\]
\[= \alpha \lambda^T e^\alpha Ey(t) + \lambda^T e^\alpha \overline{A}_0 y(t) + \lambda^T e^{\alpha(t+h)} \overline{A}_1 y(t)\]
\[= e^\alpha \lambda^T [\alpha E + \overline{A}_0 + \overline{A}_1 e^{\alpha h}] y(t).\] (24)

From the condition (22) it follows that
\[\dot{V}(t,y) \leq 0, \quad t \geq 0.\] (25)

Integrating both sides of (25) from 0 to $t$ leads to
\[V(t,y(t)) \leq V(0,y(0)) = \lambda^T \overline{A}_0 e^{\alpha(s+h)} y(s) ds \leq \gamma \| \phi \|.\] (26)

where $\gamma = n \| \lambda \| + nhe^{\alpha h} \| \overline{A}_1 \|$, $\overline{A}_1 = \overline{A}_1 + BF$. On the other hand, we have
\[V(t,y(t)) \geq \lambda^T e^\alpha Ey(t) \geq \Lambda e^\alpha \| y_1(t) \|,\] (27)

where $\Lambda = \min_{i=1,2,...,n} \lambda_i$. Combining (26) with (27) yields
\[\| y_1(t) \| \leq \gamma e^{-\alpha t} \| \phi \| := \nu \| \phi \| e^{-\alpha t}, \forall t \geq 0.\] (28)

2.2. Exponential stability of $y_2(t)$. In this subsection we will prove the second component solution $y_2(t)$ of the system is also exponentially stable with the same
Further, note that which gives

\[
\lambda^T [aE + \overline{A}_0 + \overline{A}_1 e^{\alpha h}] = \begin{bmatrix} \lambda_1^T & \lambda_2^T \end{bmatrix} \begin{bmatrix} \lambda_1 & \lambda_2 \\ \alpha_r & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} A_{01} + B_1 K_1 & A_{02} + B_1 K_2 \\ A_{03} + B_2 K_1 & A_{04} + B_2 K_2 \end{bmatrix} + e^{\alpha h} \begin{bmatrix} A_{11} + B_1 F_1 & A_{12} + B_1 F_2 \\ A_{13} + B_2 F_1 & A_{14} + B_2 F_2 \end{bmatrix}
\]

\[
= \begin{bmatrix} \lambda_1^T & \lambda_2^T \end{bmatrix} \mathcal{M},
\]

(29)

where

\[
\mathcal{M} = \begin{bmatrix} \alpha I_r + A_{01} + B_1 K_1 + e^{\alpha h} (A_{11} + B_1 F_1) & A_{02} + B_1 K_2 + e^{\alpha h} (A_{12} + B_1 F_2) \\ A_{03} + B_2 K_1 + e^{\alpha h} (A_{13} + B_2 F_1) & A_{04} + B_2 K_2 + e^{\alpha h} (A_{14} + B_2 F_2) \end{bmatrix}.
\]

Using (22) and (29) we have

\[
[\lambda_1^T \ \lambda_2^T] \mathcal{M} \prec 0,
\]

(30)

and hence

\[
\lambda_1^T (A_{02} + B_1 K_2 + e^{\alpha h} (A_{12} + B_1 F_2)) + \lambda_2^T (A_{04} + B_2 K_2 + e^{\alpha h} (A_{14} + B_2 F_2)) \prec 0.
\]

(31)

Since \( \overline{A}_1 \succeq 0, \lambda_1 \succeq 0, \) and \( \overline{A}_0 \) is Metzler, we derive that \( \lambda_1^T (A_{02} + B_1 K_2 + e^{\alpha h} (A_{12} + B_1 F_2)) \succeq 0 \) and from (31) it follows that

\[
\lambda_2^T (A_{04} + B_2 K_2 + e^{\alpha h} (A_{14} + B_2 F_2)) \prec 0.
\]

Further, note that \( A_{04} + B_2 K_2 \) is Metzler and Hurwitz, \( (A_{14} + B_2 F_2) \succeq 0 \) due to \( \overline{A}_1 \succeq 0, \) and \( \alpha h > 0. \) Using Proposition 4 we can find a number \( \rho \in (0, 1) \) and vector \( \eta \in \mathbb{R}^n_{++} \) such that

\[
- \eta^T (A_{04} + B_2 K_2)^{-1} (A_{14} + B_2 F_2) \leq \rho \eta^T.
\]

(32)

Let us denote

\[
p(t) = -(A_{04} + B_2 K_2)^{-1} \left( (A_{03} + B_2 K_1) y_1(t) + (A_{13} + B_2 F_1) y_1(t - h) \right).
\]

If \( t > h \) then

\[
\|y_1(t - h)\| \leq \frac{\gamma}{\lambda} e^{-\alpha (t - h)} \| \phi \| \leq \frac{\gamma}{\lambda} e^{\alpha h} \| \phi \| e^{-\alpha t}, \forall t > h.
\]

(33)

If \( t \in [0, h] \) then

\[
\|y_1(t - h)\| = \| \phi_1 \| \leq \| \phi \| e^{-\alpha (t - h)} \leq e^{\alpha h} \| \phi \| e^{-\alpha t},
\]

which gives

\[
\|y_1(t - h)\| \leq \rho e^{\alpha h} \| \phi \| e^{-\alpha t}, \forall t \in [0, h].
\]

(34)

From (33), (34) it follows that

\[
\|y_1(t - h)\| \leq \nu e^{\alpha h} \| \phi \| e^{-\alpha t}, \forall t \geq 0.
\]

(35)
By the notation of vector function \( p(t) \), we obtain from (28), (35) that

\[
\eta^T p(t) = -\eta^T (A_{04} + B_2 K_2)^{-1} \left( (A_{03} + B_2 K_1) y_1(t) + (A_{13} + B_2 F_1) y_1(t - h) \right)
\]

\[
\leq r \| \eta^T (A_{04} + B_2 K_2)^{-1} (A_{03} + B_2 K_1) \| \| y_1(t) \| 
+ r \| \eta^T (A_{04} + B_2 K_2)^{-1} (A_{13} + B_2 F_1) \| \| y_1(t - h) \|
\leq \nu_1 \| \phi \| e^{-\alpha t}, \ \forall t \geq 0.
\]  

(36)

where

\[
\nu_1 = r \nu e^{\alpha h} \left( \| \eta^T (A_{04} + B_2 K_2)^{-1} (A_{03} + B_2 K_1) \| 
+ \| \eta^T (A_{04} + B_2 K_2)^{-1} (A_{13} + B_2 F_1) \| \right).
\]

Moreover, from the second equation of (4) we have

\[
y_2(t) = - (A_{04} + B_2 K_2)^{-1} \left( (A_{03} + B_2 K_1) y_1(t) + (A_{13} + B_2 F_1) y_1(t - h) 
+ (A_{14} + B_2 F_2) y_2(t - h) \right)
= - (A_{04} + B_2 K_2)^{-1} (A_{14} + B_2 F_2) y_2(t - h) + p(t).
\]

Therefore

\[
\eta^T y_2(t) = -\eta^T (A_{04} + B_2 K_2)^{-1} (A_{14} + B_2 F_2) y_2(t - h) + \eta^T p(t). \quad (37)
\]

Combining the derived conditions (32), (36) and (37) gives

\[
\eta^T y_2(t) \leq \rho \eta^T y_2(t - h) + \eta^T p(t)
\leq \rho \eta^T y_2(t - h) + \nu_1 \| \phi \| e^{-\alpha t}, \ \forall t \geq 0.
\]  

(38)

Let \( N = \max\{ (n - r) \| \eta \| e^{\alpha h}, \nu_1 e^{\alpha h} \} \). If \( t \in [0, h] \) then \( t - h \in [-h, 0] \) and we have

\[
\| y_2(t - h) \| = \| \phi_2 \| \leq \| \phi \| \leq \| \phi \| e^{-\alpha(t - h)} \leq e^{\alpha h} \| \phi \| e^{-\alpha t},
\]

which implies

\[
\| y_2(t - h) \| \leq e^{\alpha h} \| \phi \| e^{-\alpha t}, \ \forall t \in [0, h]. \quad (39)
\]

Hence from (38) and (39) we obtain that

\[
\eta^T y_2(t) \leq \rho (n - r) \| \eta \| \| y_2(t - h) \| + \nu_1 \| \phi \| e^{-\alpha t}
\leq \rho N \| \phi \| e^{-\alpha t} + N \| \phi \| e^{-\alpha t}
\leq N (1 + \rho) \| \phi \| e^{-\alpha t}, \ \ t \in [0, h].
\]  

(40)

If \( t \in [h, 2h] \) then \( t - h \in [0, h] \). From (38) and (40) we have

\[
\eta^T y_2(t) \leq \rho \left( N (1 + \rho) \| \phi \| e^{-\alpha t} \right) + \nu_1 \| \phi \| e^{-\alpha t} \leq N (1 + \rho + \rho^2) \| \phi \| e^{-\alpha t}.
\]  

(41)

Similarly, for the case \( t \in [(k - 1)h, kh] \), we have

\[
\eta^T y_2(t) \leq N \left( \rho + \rho + \ldots + \rho^k \right) \| \phi \| e^{-\alpha t}.
\]
Thus, when \( t \in [kh, (k+1)h], t-h \in [(k-1)h, kh] \), using the inductive supposition and (38), we have
\[
\eta^T y_2(t) \leq \rho \left( N (1 + \rho + \ldots + \rho^k) \|\phi\| e^{-\alpha t} \right) + \nu_1 \|\phi\| e^{-\alpha t} \\
\leq N \left( 1 + \rho + \ldots + \rho^{k+1} \right) \|\phi\| e^{-\alpha t}.
\]
Since \( \rho \in (0, 1) \), and by induction, we get
\[
\eta^T y_2(t) \leq N \|\phi\| e^{-\alpha t} (1 + \rho + \ldots + \rho^k + \ldots) \leq \frac{N}{1-\rho} \|\phi\| e^{-\alpha t}.
\]
Hence
\[
\|y_2(t)\| \leq \frac{N}{(1-\rho)\eta_{\min}} \|\phi\| e^{-\alpha t},
\]
where \( \eta_{\min} = \min_{1 \leq i \leq n-r} |\xi_i|; \eta = (\xi_1, \xi_2, \ldots, \xi_{n-r}) \in \mathbb{R}^{n-r}_+ \). From (28) and (42) we finally obtain
\[
\|y(t)\| \leq M \|\phi\| e^{-\alpha t}, \forall t \geq 0.
\]

**Remark 1.** Theorem 3.1 gives sufficient conditions for the positivity and stabilization of the system (1). The conditions (12), (13) are described by a system of linear scalar inequalities (a linear programming form) w.r.t. \( \beta, k_j, f_j, j = 1, 2, \ldots, n \), and do not impose any restriction on the matrix dynamics of the governed system, so these conditions can be easier solved by a standard linear programming problem than by the LMI method. Since there exist powerful LP softwares (as Cplex) that can solve efficiently very large size problems, the LP approach is more simple and can have a legitimate numerical advantage in comparison to the LMI method [1, 3, 13, 18, 21, 26].

**Remark 2.** It is worth noting that the stability analysis of linear positive time-delay system has been studied in [10, 13], but the results were limited to linear normal regular systems and the method used there can not be applied to singular systems. Also, singular value approach to positivity and stability of system (1) was proposed in [16], however the conditions are obtained under the regularity and impulse-free assumptions and the design feedback controller was not considered.

**Remark 3.** The following procedure for constructing the memory state feedback controllers can be applied.

**Step 1.** Define nonsingular matrices \( P, Q \) with \( Q \succeq 0 \), then transform the original system \((E, A_0, A_1, B)\) into the equivalent system \((\tilde{E}, \tilde{A}_0, \tilde{A}_1, \tilde{B})\).

**Step 2.** Determine vectors \( \beta \in \mathbb{R}^n_+, k_j, f_j \in \mathbb{R}^m, j = 1, 2, \ldots, n \), satisfying all the conditions (12) -(13) by solving linear programming optimal toolbox [1].

**Step 3.** Construct the feedback control matrices \( K, F \) given by
\[
K = \begin{bmatrix} k_1 & k_2 & \ldots & k_n \\ \beta_1 & \beta_2 & \ldots & \beta_n \end{bmatrix}, \quad F = \begin{bmatrix} f_1 & f_2 & \ldots & f_n \\ \beta_1 & \beta_2 & \ldots & \beta_n \end{bmatrix}.
\]

**Step 4.** Define feedback control function \( u(t) = KQ^{-1}x(t) + FQ^{-1}x(t-h) \).
4. Numerical example. Consider the system (1), where

\[
E = \begin{bmatrix}
1 & 0 & 2 \\
0 & 0 & 0 \\
2 & 0 & 2 \\
\end{bmatrix}, \quad A_0 = \begin{bmatrix}
-6 & 1 & 2 \\
0 & 0 & 0 \\
-16 & 1 & 2 \\
\end{bmatrix}, \\
A_1 = \begin{bmatrix}
5 & -20 & 2 \\
-1 & 1 & 2 \\
6 & -20 & 2 \\
\end{bmatrix}, \quad B = \begin{bmatrix}
2 & 0 \\
0 & 1 \\
2 & 0 \\
\end{bmatrix}, \quad h = 2.85.
\]

We can find two nonsingular matrices

\[
P = \begin{bmatrix}
-1 & 0 & 1 \\
1 & 0 & -1/2 \\
0 & 1 & 0 \\
\end{bmatrix}, \quad Q = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0 \\
\end{bmatrix},
\]

such that \( E, A_0, A_1, B \) are partitioned accordingly as

\[
\tilde{E} = PEQ = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}, \quad \tilde{A}_0 = PA_0Q = \begin{bmatrix}
-10 & 0 & 0 \\
2 & 1 & 1/2 \\
0 & 0 & 0 \\
\end{bmatrix},
\]

\[
\tilde{A}_1 = PA_1Q = \begin{bmatrix}
1 & 0 & 0 \\
2 & 1 & -10 \\
1 & 2 & -1 \\
\end{bmatrix}, \quad \tilde{B} = \begin{bmatrix}
0 & 0 \\
1 & 0 \\
0 & 1 \\
\end{bmatrix}.
\]

Therefore,

\[
A_{01} = \begin{bmatrix}
-10 & 0 \\
2 & 1 \\
\end{bmatrix}, \quad A_{02} = \begin{bmatrix}
0 & 0 \\
1/2 & 0 \\
\end{bmatrix}, \quad A_{03} = \begin{bmatrix}
0 & 0 \\
\end{bmatrix}, \quad A_{04} = \begin{bmatrix}
0 & 0 \\
\end{bmatrix}, \quad B_1 = \begin{bmatrix}
0 & 0 \\
1 & 0 \\
\end{bmatrix}, \quad A_{11} = \begin{bmatrix}
1 & 0 \\
2 & 1 \\
\end{bmatrix}, \quad A_{12} = \begin{bmatrix}
0 & 0 \\
-10 & 0 \\
\end{bmatrix}, \quad A_{13} = \begin{bmatrix}
1 & 2 \\
\end{bmatrix}, \quad A_{14} = \begin{bmatrix}
-1 \\
\end{bmatrix}, \quad B_2 = \begin{bmatrix}
0 & 1 \\
\end{bmatrix}.
\]

Figure 1. State response of the closed-loop system
For $\alpha = 0.5$ we solve the LP problem (12)-(13) and obtain the following solutions:

$$k_1 = (-20, 1), \quad k_2 = (-100, 0), \quad k_3 = (-10, -100),$$

$$f_1 = (1, 1), \quad f_2 = (1, 1), \quad f_3 = (200, 20), \quad \beta = (10, 1, 20).$$

Therefore, the control gain matrices are given by

$$K = \begin{bmatrix} -2 & -100 & -0.5 \\ 0.1 & 0 & -5 \end{bmatrix}, \quad F = \begin{bmatrix} 0.1 & 1 & 10 \\ 0.1 & 1 & 1 \end{bmatrix}.$$ 

In this example, direct calculation shows that $\det(sE - A_0) \equiv 0$, for all $s \in \mathbb{C}$, then we have the open-loop system is not regular. However, by Theorem 3.1, the system is 0.5-stabilizable under memory feedback controllers defined by

$$\begin{cases} u_1(t) = -2x_1(t) + 0.1x_1(t - 2.85) - 0.5x_2(t) + 10x_2(t - 2.85) - 100x_3(t) + x_3(t - 2.85), \\ u_2(t) = 0.1x_1(t) + 0.1x_1(t - 2.85) - 5x_2(t) + x_2(t - 2.85) + x_3(t - 2.85). \end{cases}$$

Figure 1 shows the trajectories of $x_1(t), x_2(t)$ and $x_3(t)$ of the closed-loop system with the initial condition $\varphi(t) = [9t^2, \frac{1}{30}(39t^2 + 99(t - 2.85)^2), (t + 2.85)^2], \quad t \in [-2.85, 0].$

5. Conclusions. In this paper, we have studied problem of exponential stabilization for singular linear positive systems with delay. Based on the system decomposition approach, singular systems theory and Lyapunov function methods, we have provided delay-dependent sufficient conditions for designing memory state feedback controllers of exponential stabilization problem via linear programming. The results on positivity and stability analysis have been obtained in this paper for the systems with constant delay. We have a strong prospect that the results can be extended to the linear singular positive systems with time-varying delays.

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*E-mail address: ngu yenhuusau87@gmail.com*

*E-mail address: vnphat@math.ac.vn*