A parity violating $\phi^6$ soliton model

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Abstract

In the present paper we propose an analytically solvable $\phi^6$ model with non-trivial soliton solutions connecting the trivial vacua. The model does not respect parity symmetry and like $\phi^4$ theory has two minima. The soliton solutions and the consequent results are solved in terms of Lambert W function, i.e. the inverse function of $f(W) = W e^W$. We compare the solution with the kink of $\phi^4$ theory which preserves the parity symmetry. Moreover, we study the interaction of boson and fermion fields with the proposed soliton considering two types of couplings for the boson-soliton interaction as well as Yukawa coupling for the fermion-soliton interaction. The bound and continuum states of the fields in the presence the soliton are studied. Most results are derived analytically which renders the model a fertile ground for further study including parity breaking related phenomena.

1 Introduction

There is a small class of nonlinear differential equations with soliton solutions. A soliton is a stable solution with localized energy density. It arises as a result of the interaction between nonlinearity and dispersion with a sharpening nonlinear term counterbalancing the dispersive term. The competition between these two contributions shapes the structure of the soliton and provides its stability. In the language of topology, the soliton configuration has an associated conserved topological charge or winding number which protects it against decaying into a trivial configuration. Solitons are fascinating due to their mathematical properties. However, their usefulness extends far beyond that, reaching many areas of science. Particularly, they are subject of research in various areas of physics, including high energy physics, nonlinear optics and condensed matter physics [1–5]. Amongst the most known solitons are skyrmions and domain walls in magnetic materials [6–8], vortices in superconductors and fluids [9–12] as well as magnetic monopoles, Q-balls, cosmic strings and instantons in high energy physics [13–21]. Besides the theoretical applications of solitons, they play an increasingly important role in technology, e.g. in communications [22,24].

Due to the fact that in most systems in physics the solitons are not isolated objects, their interaction with other fields has been subject to intense research in the literature. Boson and Dirac fields interacting with a soliton are known to affect or even create many intriguing phenomena including vacuum polarization and Casimir effect [25,26], superconductivity and

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Bose-Einstein condensation \cite{27,28}, localization of fermions in the braneworld scenarios \cite{29}, charge and fermion number fractionalization \cite{30} as well as conducting polymers \cite{31}. Massless Dirac fermions behave as the quasiparticles in materials such as graphene and topological insulators \cite{32,33}.

Exactly solvable models are considered indispensable tools to explore the physics of the system and the symmetries behind it. In this paper we introduce a parity breaking model with analytical soliton solution. The potential includes powers up to sixth order in the scalar field $\phi$, where alongside the even powers there exist odd powers of $\phi$ in the potential, causing a parity asymmetry. In \cite{34} the authors considered a massless Dirac field interacting with a skyrmion-like planar defect in a system that does not respect the parity symmetry. They studied the fermion bound spectrum as well as the scattering of fermions from the localized topological structure and found a closed form for the scattering cross section for small fermion-skyrmion coupling. Parity or inversion symmetry breaking models with topological solutions are of importance in many areas of physics, for example in the context of superconductivity \cite{35–39}, fractional quantum Hall effect \cite{40}, mesoscopic electron transport \cite{41}, current of abnormal parity \cite{42}, heavy-ion collisions \cite{43}, nonlinear Schrödinger equations \cite{44} and hydrodynamics \cite{45}.

In this paper, we consider a parity-breaking model with two minima where besides finding the soliton solution and stability equation analytically, we study the interaction of the soliton with boson and fermion fields. The soliton solution and the rest of the results are given in terms of Lambert W function. We consider two types of interactions with the boson field and Yukawa interaction for the fermion field. The boson bound and scattering states as well as the fermion zero mode are expressed in closed analytical forms. In Sec. 2 we introduce the model and the corresponding topological solution as well as analyze the small oscillations of the soliton. In Sec. 3 we study the interaction of boson and fermion fields with the soliton of our model. Finally, in Sec. 4 we summarize the results of the current work. The details of the calculations are given in Appendices.

## 2 Model

We propose the theory described by the following Lagragian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi),$$

where the potential term is given by

$$V(\phi) = \frac{\lambda^2}{2} (1 - \phi^2)^2 (1 - \phi^2).$$

The potential presents two minima, $\phi_0 = \pm 1$, which allows one to obtain solitonic solutions interpolating between them. The potential is of sixth order. However, unlike classical $\phi^6$ theory, it has no parity symmetry since odd powers of $\phi$ are also included. An equivalent potential could be considered by mapping $\phi \rightarrow -\phi$, where the roles of the kink and antikink solutions are interchanged. The coupling $\lambda$ has mass dimension one which defines a reference mass scale in the system with respect to which we rescale all the parameters. Nevertheless, when deem necessary we write the mass dimension explicitly, as a function of $\lambda$.

Although Lagragian (1) yields a second order equation of motion, thanks to the BPS condition one can obtain an equivalent first order equation

$$\partial_x \phi - (1 - \phi^2) (1 - \phi) = 0,$$

where
where the field $\phi$ is static. Integrating the above equation we find

\[ \frac{1}{4} \left[ \log \left( \frac{\phi + 1}{\phi - 1} \right) - \frac{2}{\phi} \right] = x + C, \quad (4) \]

where $C$ is the integration constant. We choose the center of the soliton at $\phi(0) = 0$, implying $C = \frac{1}{4}(2 - i\pi)$. This equation can be solved in terms of Lambert $W$ function\(^1\) The details of the calculation are provided in Appendix A. The solution is

\[ \phi_s(x) = 1 - \frac{2}{1 + W[e^{1+4x}]} \quad (5) \]

The corresponding antikink solution can be obtained by changing $\phi \to -\phi$. Figure 1 shows the kink profiles for $\phi^4$ and our models. Notice that in the kink profile of our model the parity is explicitly broken.

![Figure 1](image.png)

**Figure 1:** (a) Soliton profile. (b) Energy density. The solid line (blue) and the dashed line (red) show the soliton in our model and the kink of $\phi^4$ theory, respectively.

Using the BPS condition, it is straightforward to calculate the energy of the soliton configuration, the so-called classical mass of the soliton,

\[ M_{cl} = \int_{-\infty}^{\infty} E(x) dx = \int_{-\infty}^{\infty} \left[ \frac{1}{2} \left( \frac{d\phi}{dx} \right)^2 + V(x) \right] dx = \int_{-1}^{1} (1 - \phi^2)^2 (1 - \phi)^2 \ d\phi = \frac{4}{3}(\lambda), \quad (6) \]

where the energy density $E(x)$ is shown in Fig. 1 for our model and $\phi^4$ kink. Interestingly, despite the difference in the energy density of the two models, the resulting mass is the same.

Having the profile of the soliton, it is relevant to analyze the small fluctuations of the boson field described by the linear stability equation

\[ \left[ -\partial_x^2 + U[\phi(x)] \right] \eta_n(x) = \omega_n^2 \eta_n(x), \quad (7) \]

\(^1\)For the properties of Lambert $W$ function check, e.g., [46].
with the stability potential

\[
U[\phi(x)]|_{\phi_s(x)} = \frac{d^2 V}{d\phi^2} |_{\phi_s(x)} = 16 \left( 1 - 8 W \left[ e^{1+4x} \right] + 6 \left( W \left[ e^{1+4x} \right] \right)^2 \right) \left( 1 + W \left[ e^{1+4x} \right] \right)^4,
\]

where \( \eta_n \)'s are the normal modes of the fluctuations around the static solution. Due to the translational symmetry of the system there exists a zero mode, \( \omega_0 = 0 \). It is possible to show that it is as follows

\[
\eta_0 = \partial_x \phi = \frac{8 W \left[ e^{1+4x} \right]}{(W \left[ e^{1+4x} \right] + 1)^3}.
\]

Knowing that

\[
\frac{d^2 V}{d\phi^2} |_{\phi=1} = 0, \quad \frac{d^2 V}{d\phi^2} |_{\phi=-1} = 16 (\lambda^2),
\]

and requiring \( \omega_n^2 \) to be non-negative, implies that the zero mode is the only discrete mode. Figure 2 shows the stability potential \( U(x) \) (panel (a)) as well as the zero mode \( \eta_0 \) (panel (b)). As one can see, the potential presents different limits at \( x \to \pm \infty \). Due to this fact, only waves whose energy exceeds \( U(-\infty) \) are permitted when travelling from the left. In contrast, incoming waves from the right are allowed for lower energies starting from 0, the value of \( U(\infty) \). In this case, oscillations with an energy smaller than \( U(-\infty) \) are totally reflected by the potential barrier. Until now we have been concerned with the soliton solutions in isolation. In what follows, we consider the interaction of the soliton of our model with other fields, including boson and fermion fields. We consider two different types of couplings responsible for the soliton-boson interaction, besides a Yukawa coupling between the soliton and a Dirac field. In all three cases, we consider the soliton as a background field.

![Figure 2](image-url)
3 Interaction with a scalar field

3.1 Model I

First, let us consider the interaction of a real massive scalar field $\chi$ with the soliton of our model in the following form

$$L = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) + \frac{1}{2} \partial_\mu \chi \partial^\mu \chi + \frac{1}{2} m^2 \chi^2 - g \phi \chi, \quad (11)$$

where $m$ is the mass of the field $\chi$ and $g$ is the scalar-soliton coupling constant. This interaction yields a non-homogeneous Klein-Gordon equation

$$(\Box - m^2) \chi = -g \phi. \quad (12)$$

Separating the time dependence as $\chi = \chi_s e^{-iEt}$, we find the equation

$$(\partial_x^2 + k^2) \chi_s = g \phi, \quad (13)$$

where we define $k^2 \equiv E^2 - m^2$. First consider the bound states, for which we have $k^2 < 0$.

The solution to the equation of motion, eq. (13), is

$$\chi_s(x) = Ae^{\kappa x} + Be^{-\kappa x} + \frac{g}{\kappa} \int_x^\infty \sinh [\kappa(x-y)] \phi(y) \, dy, \quad (14)$$

introducing $k^2 \equiv -\kappa^2$. The first two terms come from the solution of the homogeneous equation and the last one is a particular solution. Focusing only on the integral term in the above solution and performing the change of variables $u = x - y$ results in

$$-\frac{g}{\kappa} \int \sinh (\kappa u) \left(1 - \frac{2}{W[e^{1+4(x-u)}] + 1}\right) \, du = -\frac{g}{\kappa^2} + \frac{2g}{\kappa} \int \frac{\sinh (\kappa u)}{1 + W[e^{1+4(x-u)}]} \, du. \quad (15)$$

Two more changes of variables, $v = e^{1+4(x-u)}$ followed by $w = W[v]$, allow us to rewrite the integral in a form that can be directly solved

$$-\frac{g}{\kappa^2} - \frac{g}{4\kappa} \int \sinh \left[-\frac{\kappa}{4} \left(\ln(w) + w - 1 - 4x\right)\right] \frac{1}{w} \, dw$$

$$= -\frac{g}{\kappa^2} - \frac{g}{4\kappa} \left[-\frac{\kappa}{4} e^{-\frac{\kappa}{4}(1+4x)} \Gamma \left(\frac{\kappa}{4}, \frac{\kappa}{4} W[e^{1+4x}]\right)\right]$$

$$-\frac{\kappa}{4} e^{\frac{\kappa}{4}(1+4x)} \Gamma \left(-\frac{\kappa}{4}, \frac{\kappa}{4} W[e^{1+4x}]\right). \quad (16)$$

Therefore, the general solution takes the form

$$\chi_s(x) = Ae^{\kappa x} + Be^{-\kappa x} - \frac{g}{\kappa^2} - \frac{g}{4\kappa} \left[-\frac{\kappa}{4} e^{-\frac{\kappa}{4}x} e^{-\frac{\kappa}{4}x} \Gamma \left(\frac{\kappa}{4}, \frac{\kappa}{4} W[e^{1+4x}]\right)\right]$$

$$-\frac{\kappa}{4} e^{\frac{\kappa}{4}x} \Gamma \left(-\frac{\kappa}{4}, \frac{\kappa}{4} W[e^{1+4x}]\right). \quad (17)$$

To obtain a real $\chi_s$ one has to impose the restriction $\kappa = 4n$ where $n$ is an integer number. This means that $E^2 = m^2 - 16n^2$ and also $n < m/4(\lambda)$ following the fact that $E^2$ is non-negative. Looking at the limit of $\chi_s$ when $x \to +\infty$ it is easy to see that, for the solution to
be finite, $A$ should be null. At this limit, the last term in the above expression vanishes and therefore the solution converges to $-g/\kappa^2$. It remains to determine the value of $B$ which can be found by requiring the solution to be finite when $x \to -\infty$. Doing so, it can be shown that

$$B = \frac{g}{16n} (-n)^{-n} e^{-n} \Gamma (n).$$

(18)

The detailed calculations are provided in Appendix B. Figure 3 shows the bound states for three values of $n$, 1, 2 and 3. As it can be seen, at the limits $x \to \pm \infty$ the solution converges to $\mp g/\kappa^2 = \mp g/16n^2$. This result is expected considering the eq. (13) when $\phi(x \to \pm \infty) = \pm 1$. Notice that, interestingly, bound states are also solitons which means that the original soliton can trap another boson field in the form of a soliton configuration.

Now, let us look at the case $k^2 > 0$ which corresponds to the scattering states. The general solution for this equation is in the form

$$\chi_s(x) = Ae^{ikx} + Be^{-ikx} + \frac{g}{k} \int_x^{\infty} \sin[k(x - y)] \phi(y) \, dy,$$

(19)

where again the first two terms come from the solution of the homogeneous equation and the last one is a particular solution. Following the same series of change of variables the integral in the above expression changes to

$$-\frac{g}{2k} \int \sin \left[ -\frac{k}{4} (\ln(w) + w - 1 - 4x) \right] \frac{1}{w} \, dw$$

$$= -\frac{ig}{4k} \left[ \Gamma(-ik/4, ikw/4)(ik/4)^{ik/4} e^{i(1+4x)k/4} - c.c. \right],$$

(20)

where $c.c.$ stands for the complex conjugate. After some simplifications, the solution (19) takes the form

$$\chi_s(x) = Ae^{ikx} + Be^{-ikx} + \frac{g}{k^2} +$$

$$+ \frac{g}{2k} \text{Im} \left[ \Gamma(-ik/4, ik W[e^{1+4x}]/4) e^{i(1+4x+ln(k/4)+i\pi/2)k/4} \right].$$

(21)
To verify the result, one can look at the limits \( x \to \pm \infty \). At the limit \( x \to +\infty \), the last term in the above solution tends to zero and we recover the expected result using eq. (13) when \( \phi \to 1 \)

\[
\chi_s(x \to +\infty) = Ae^{ikx} + Be^{-ikx} + \frac{g}{k^2}.
\]

(22)

The same goes for the limit \( x \to -\infty \) where the eq. (21) tends to

\[
\chi_s(x \to -\infty) = Ae^{ikx} + Be^{-ikx} - \frac{g}{k^2} + \frac{g}{2k} \text{Im} \left[ \left( \frac{k}{4} \right)^{ik/4} e^{i(\pi+1)k/4} \Gamma \left( -\frac{i}{4} \right) e^{ikx} \right].
\]

(23)

The last term can be removed through a redefinition of the coefficients \( A \) and \( B \), which gives the expected result using eq. (13) when \( \phi \to -1 \).

3.2 Model II

Now we introduce a different type of coupling between the soliton field \( \phi \) and the scalar field \( \chi \). Consider the following Lagrangian

\[
\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) + \frac{1}{2} \partial_\mu \chi \partial^\mu \chi + \frac{1}{2} m^2 \chi^2 + g \phi \chi^2.
\]

(24)

where the coupling between the fields is analogous to a Yukawa interaction. This interaction yields the equation of motion

\[
\left( \Box - m^2 \right) \chi - 2g \phi \chi = 0.
\]

(25)

Considering \( \chi = \chi_s e^{-iEt} \) and rearranging the terms we arrive at

\[
(-\partial_x^2 - 2g \phi) \chi_s = k^2 \chi_s.
\]

(26)

Replacing the solitonic solution of our model in the above equation leads to

\[
\left( -\partial_x^2 + \frac{4g}{1 + W[e^{1+4x}]} \right) \chi_s = (k^2 + 2g) \chi_s,
\]

(27)

which has the formal structure of the Schrödinger equation with energy equal to \((k^2 + 2g)\). Figure (4) shows the form of the potential term in the above Schrödinger-like equation. In [17], the author solved a similar equation. To map our system to the quantum mechanical system solved in the aforementioned paper we need first to consider the change of variables \( 1 + 4x \to -y \) which results in

\[
\left( -\partial_y^2 + \frac{g/4}{1 + W[e^{-y}]} \right) \chi_s = \frac{1}{16} (k^2 + 2g) \chi_s.
\]

(28)

Now, the map between their system and ours is given by \( 2m/h^2 \to 1 \), \( V_0 \to g/4 \), \( E \to (k^2 + 2g)/16 \) and \( \sigma \to 1 \). As a result the solution to our system is in the following form

\[
\chi_s = z^{i\delta^{-}/2} e^{-i\delta^{+}z/2} \left( \frac{du(z)}{dz} - i \frac{\delta^{+} + \delta^{-}}{2} u(z) \right).
\]

(29)
with $z = W[e^{1+4x}]$, $\delta^\pm = \frac{1}{2} \sqrt{k^2 \pm 2g}$, $a = (\delta^+ + \delta^-)^2/(4\delta^+)$,

$$u = C_1 (i\delta^+ z)^{1-i\delta^-} \text{$_1$F$_1$(}1 + i(a - \delta^-); 2 - i\delta^-; i\delta^+ z) + C_2 U(i\delta^-; i\delta^+ z),$$

where $C_1$ and $C_2$ are constants and $_1$F$_1$ and $U$ are the Kummer and Tricomi confluent hypergeometric functions, respectively. Due to the form of the potential it is easy to see that the system does not have any bound state. The scattering states from the right and the left are shown in Fig. 5(a). Besides that, in the same figure one can see the scattering from the right where the energy is below the threshold required to surpass the barrier. In this case obviously the wave is totally reflected which matches the result in this figure. Figure 5(b) shows the reflection coefficient as a function of the momentum for the waves from the right and left. For the momentum associated to the energy below the barrier for the wave coming from the right the reflection coefficient is 1 as expected. Also, for both cases, waves coming from the right or left, the reflection coefficient goes to zero at high energies due to the fact that the wave do not see the barrier.

Figure 4: Potential energy considering $g = 0.1$.

Figure 5: (a) Boson field continuum states. (b) Reflection coefficient. In both cases, the solid line (blue) and the dashed line (red) show the graphs for scattering from the right and scattering from the left considering $g = 0.1$, respectively. The dot-dashed curve (green) shows the scattering from the right for the case where the energy is below the threshold required to surpass the barrier.
4 Interaction with a fermion field

Fermions can be coupled to the soliton in various ways. We introduce a fermion field $\psi$ coupled to the soliton through a Yukawa coupling in the following form

$$ L = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) + \bar{\psi} i \gamma^n \partial_\mu \psi - g \phi \bar{\psi} \psi, $$

(30)

where $g$ is a coupling constant. The equation of motion in the background of the soliton reads

$$ i \gamma^\mu \partial_\mu \psi - g \phi \psi = 0. $$

(31)

Writing the spinor field $\psi$ in components as $\psi = e^{-iEt} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ one can find the pair of equations

$$ E \psi_1 + \psi_2' - g \phi \psi_2 = 0, $$

$$ E \psi_2 - \psi_1' - g \phi \psi_1 = 0, $$

(32)

where the representation for the Dirac matrices is taken as $\gamma^0 = \sigma_1$, $\gamma^1 = i \sigma_3$ and $\gamma^5 = \sigma_2$. For the case of $\phi^4$ model, a zero energy bound state or zero mode, is known to exist which is also the case for our model. The zero mode is given by

$$ \psi(x) = N \begin{pmatrix} e^{-g f(x) e^{1+4x}} \\ 0 \end{pmatrix}, $$

(33)

where $N$ is the normalization constant. Since one of the components is null the soliton never receives backreaction from this state and the solution is exact [48]. Performing the above integration we can obtain an explicit solution to the state

$$ \psi_1 = N \exp \left\{ -g \int^x \left[ 1 - \frac{2}{1 + W[e^{1+4x}]} \right] dx' \right\}. $$

(34)

Performing the change of variables $y = \exp(1 + 4x)$ this becomes

$$ \psi_1 = N \exp \left\{ -g x + \frac{g}{2} \int^{e^{1+4x}} W'[y] \frac{W[y]}{W[y]} dy \right\}, $$

(35)

using the property of Lambert $W$ function

$$ W'[y] = \frac{W[y]}{y (1 + W[y])}. $$

(36)

Therefore, the wavefunction becomes

$$ \psi = N \begin{pmatrix} \exp \left\{ -g x + \frac{g}{2} \ln \left[ W[e^{1+4x}] \right] \right\} \\ 0 \end{pmatrix}, $$

(37)

$$ = N \begin{pmatrix} \left( W[e^{1+4x}] \right)^{\frac{g}{2}} e^{-gx} \\ 0 \end{pmatrix}, $$

(38)

with the normalization constant

$$ N = \sqrt{\frac{g/2}{\Gamma[g/2]} \frac{2 e^{-g/2}}{2 e^{-g/2}}}. $$

(39)
Details of the above calculation are given in appendix C. In Fig. (6) we show the fermionic zero mode for two different values of the coupling. The resulting parity asymmetry in the zero mode of our model is visible especially when the fermion-soliton coupling $g$ increases. Besides that, we solve the equations of motion in (32) for nonzero bound states numerically where the result for the upper and lower components of the first and second fermionic bound states is presented in Fig. (7). A close inspection of the figure reveals that the states do not respect parity. Moreover, we plot the bound and threshold energies as a function of bound state number as well as the fermion-soliton coupling $g$ in Fig. (8) where the system is solved numerically. It is not difficult to show that the system has energy-reflection symmetry which is given by $\gamma_1$ in our model. In Fig. (8) the symmetry manifests itself by the symmetric form of the spectrum around $E = 0$ line. For really small values of $g$ the only discrete mode is the zero mode, however increasing it gradually there appear more and more bound states. Besides the bound states one can explore the scattering ones considering energies above threshold in the equation of motion (32). We show the result for the upper and lower components of the fermionic scattering states for both cases, scattering from the left and right, in Fig. (9). Again, it is easy to see that the states do not respect parity symmetry.

5 Conclusion

In this work, we have designed a parity-breaking solitonic model where the potential is up to sixth order in scalar field $\phi$, with two minima. The soliton solutions connecting the two minima has been given in terms of the Lambert W function. Although the system lacks $Z_2$ symmetry, changing $\phi \rightarrow -\phi$ only swaps the role of the soliton and antisoliton solutions. We have found the soliton mass which is equal to the one for the kink of $\phi^4$ theory, despite a very different energy density. Studying the linear stability equation for the small perturbations around the static soliton solutions, we have concluded that the only discrete mode is the zero mode associated to the translational invariance, in contrast with the parity-symmetric $\phi^4$ model. Besides that, we have studied the interaction of the boson and fermion fields with the soliton considering two different types of interaction terms for the bosonic one and the Yukawa interaction for the fermionic one. The first interaction we have considered has
Figure 7: (a) First fermionic bound state for $g = 1$. (b) Second fermionic bound state for $g = 1$. The solid (blue) and dashed (red) curves show the upper and lower components, $\psi_1$ and $\psi_2$, respectively.

Figure 8: (a) Fermionic bound energy spectrum for $g = 2$. The dashed lines (red) show the threshold energies. (b) Fermionic bound energy spectrum for the first three bound states as a function of the coupling $g$. The dashed lines (red) show the threshold energies.

Figure 9: (a) Fermionic continuum states in the case of the scattering from the right. (b) Fermionic continuum states in the case of the scattering from the left. In both graphs $g = 1$ and $k = 0.5$ and also the solid curve (blue) and dashed (red) curve show the upper and lower components, $\psi_1$ and $\psi_2$, respectively.
led to a non-homogeneous Klein-Gordon equation with interesting results. For example, we have shown that the boson bound state due to the interaction with the soliton is also in the form of a defect which means that the soliton in our model traps the bosonic field in a kink configuration. Considering the second interaction term, which is a Yukawa-like interaction, we have shown that one can write the equation of motion in the form of a the Schrödinger equation. With a change of variables and mapping the parameters with the results obtained in [47], we have found the bound and continuum states analytically. We have also studied the scattering of the waves from the right and left as well as the reflection coefficient knowing the barrier shape potential term. We have shown that for the waves below the barrier the reflection coefficient is unity and for the waves with high energy compared to the height of the barrier the coefficient goes to zero, as expected. We have solved the system analytically for both types of interactions. This is not common and makes the model valuable for further study. Moreover, in both cases we have verified that the results match the expectations in the limiting cases where $\phi(x \to \pm \infty) \to \pm 1$. Finally, the interaction of the fermion field with the soliton has been considered. In this case we have been able to find the normalized fermion zero mode analytically. We have also obtained the nonzero bound energy spectrum as a function of bound state number as well as the fermion-soliton coupling $g$ numerically.

The system has energy reflection symmetry given by $\gamma^1$ resulting in symmetric bound energy spectrum. We have shown that for very small values of the coupling the only discrete mode is the zero mode, then increasing it gradually there appear a growing number of bound states. At the end, the scattering oscillating modes for the waves coming from the right and the left have been shown. In all three types of interaction it is clear that the bound and continuum states do not respect parity where the strength of the parity breaking is related to the value of the coupling.

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**References**

[1] Yakov M Shnir. *Topological and non-topological solitons in scalar field theories*. Cambridge University Press, 2018.

[2] Fatkhulla Abdullaev, Sergei Darmanyan, Pulat Khabibullaev, and J Engelbrecht. *Optical solitons*. Springer Publishing Company, Incorporated, 2014.

[3] Tanmay Vachaspati. *Kinks and domain walls: An introduction to classical and quantum solitons*. Cambridge University Press, 2006.

[4] Yuri S Kivshar and Govind Agrawal. *Optical solitons: from fibers to photonic crystals*. Academic press, 2003.

[5] Ramamurti Rajaraman. *Solitons and instantons*. North Holland, 1982.

[6] Albert Fert, Nicolas Reyren, and Vincent Cros. Magnetic skyrmions: advances in physics and potential applications. *Nature Reviews Materials*, 2(7):17031, 2017.
[7] Stuart SP Parkin, Masamitsu Hayashi, and Luc Thomas. Magnetic domain-wall racetrack memory. *Science*, 320(5873):190–194, 2008.

[8] T Koyama, D Chiba, K Ueda, K Kondou, H Tanigawa, S Fukami, T Suzuki, N Ohshima, N Ishiwata, Y Nakatani, et al. Observation of the intrinsic pinning of a magnetic domain wall in a ferromagnetic nanowire. *Nature materials*, 10(3):194, 2011.

[9] Alexander L Fetter and Pierre C Hohenberg. Theory of type ii superconductors. In *Superconductivity*, pages 817–923. Routledge, 2018.

[10] Dustin Kleckner and William TM Irvine. Creation and dynamics of knotted vortices. *Nature physics*, 9(4):253, 2013.

[11] Ophir M Auslaender, Lan Luan, Eric WJ Straver, Jennifer E Hoffman, Nicholas C Koshnick, Eli Zeldov, Douglas A Bonn, Ruixing Liang, Walter N Hardy, and Kathryn A Moler. Mechanics of individual isolated vortices in a cuprate superconductor. *Nature Physics*, 5(1):35, 2009.

[12] AA Abrikosov. Nobel lecture: Type-ii superconductors and the vortex lattice. *Reviews of modern physics*, 76(3):975, 2004.

[13] Maxim V Polyakov and Hyeon-Dong Son. Nucleon gravitational form factors from instantons: forces between quark and gluon subsystems. *Journal of High Energy Physics*, 2018(9):156, 2018.

[14] Christian Schneider, Greger Torgrimsson, and Ralf Schützhold. Discrete worldline instantons. *Physical Review D*, 98(8):085009, 2018.

[15] Csaba Csáki, Yuri Shirman, John Terning, and Michael Waterbury. Kaluza-klein monopoles and their zero modes. *Physical review letters*, 120(7):071603, 2018.

[16] DF Jackson Kimball, D Budker, J Eby, M Pospelov, Szymon Pustelny, Theo Scholtes, YV Stadnik, Antoine Weis, and A Wickenbrock. Searching for axion stars and q-balls with a terrestrial magnetometer network. *Physical Review D*, 97(4):043002, 2018.

[17] Mark Hindmarsh, Kari Rummukainen, and David J Weir. New solutions for non-abelian cosmic strings. *Physical review letters*, 117(25):251601, 2016.

[18] Thomas Schaefer. Instanton effects in qcd at high baryon density. *Physical Review D*, 65(9):094033, 2002.

[19] Alexander Vilenkin and E Paul S Shellard. *Cosmic strings and other topological defects*. Cambridge University Press, 2000.

[20] Alexander Kusenko and Mikhail Shaposhnikov. Supersymmetric q-balls as dark matter. *Physics Letters B*, 418(1-2):46–54, 1998.

[21] Gerardus t Hooft. Magnetic monopoles in unified theories. *Nucl. Phys. B*, 79(CERN-TH-1876):276–284, 1974.

[22] Mercedes Alcon-Camas, AE El-Taher, Hai Wang, Paul Harper, Vasileios Karakelis, James A Harrison, and J-D Ania-Castañón. Long-distance soliton transmission through ultralong fiber lasers. *Optics letters*, 34(20):3104–3106, 2009.
[23] Pablo Marin-Palomo, Juned N Kemal, Maxim Karpov, Arne Kordts, Joerg Pfeifle, Martin HP Pfeiffer, Philipp Trocha, Stefan Wolf, Victor Brasch, Miles H Anderson, et al. Microresonator-based solitons for massively parallel coherent optical communications. *Nature*, 546(7657):274, 2017.

[24] Hermann A Haus and William S Wong. Solitons in optical communications. *Reviews of modern physics*, 68(2):423, 1996.

[25] Azadeh Mohammadi and ER Bezerra de Mello. Finite temperature bosonic charge and current densities in compactified cosmic string spacetime. *Physical Review D*, 93(12):123521, 2016.

[26] Siamk Sadat Gousheh, Azadeh Mohammadi, and Leila Shahkarami. Casimir energy for a coupled fermion-kink system and its stability. *Physical Review D*, 87(4):045017, 2013.

[27] Gordon W Semenoff and Pasquale Sodano. Stretching the electron as far as it will go. *arXiv preprint cond-mat/0605147*, 2006.

[28] Stefan Burger, Kai Bongs, Stefanie Dettmer, Wolfgang Ertmer, Klaus Sengstock, Anna Sanpera, Gora V Shlyapnikov, and Maciej Lewenstein. Dark solitons in bose-einstein condensates. *Physical Review Letters*, 83(25):5198, 1999.

[29] Alejandra Melfo, Nelson Pantoja, and Jose David Tempo. Fermion localization on thick branes. *Physical Review D*, 73(4):044033, 2006.

[30] Roman Jackiw and Cláudio Rebbi. Solitons with fermion number 1/2. *Physical Review D*, 13(12):3398, 1976.

[31] W,P Su, JR Schrieffer, and Ao J Heeger. Solitons in polyacetylene. *Physical review letters*, 42(25):1698, 1979.

[32] AH Castro Neto, Francisco Guinea, Nuno MR Peres, Kostya S Novoselov, and Andre K Geim. The electronic properties of graphene. *Reviews of modern physics*, 81(1):109, 2009.

[33] Xiao-Liang Qi and Shou-Cheng Zhang. Topological insulators and superconductors. *Reviews of Modern Physics*, 83(4):1057, 2011.

[34] Dionisio Bazeia and Azadeh Mohammadi. Dirac field in the background of a planar defect. *Physics Letters B*, 779:420–424, 2018.

[35] Hikaru Watanabe and Youichi Yanase. Group-theoretical classification of multipole order: Emergent responses and candidate materials. *Physical Review B*, 98(24):245129, 2018.

[36] Jun Ishizuka and Youichi Yanase. Odd-parity multipole fluctuation and unconventional superconductivity in locally noncentrosymmetric crystal. *Physical Review B*, 98(22):224510, 2018.

[37] Jonathan Ruhman, Vladyslav Kozii, and Liang Fu. Odd-parity superconductivity near an inversion breaking quantum critical point in one dimension. *Physical review letters*, 118(22):227001, 2017.
A Calculation of the soliton solution

Performing the change of variables $\chi = \phi - 1$ in Eq. (3) we obtain

$$\log \left[ \frac{1 + 2/\chi}{e^{2/\chi}} \right] = 4x - i\pi + 2,$$

which leads to

$$(-1 - 2/\chi) e^{-1-2/\chi} = e^{4x+1}.$$  (41)

Recalling that the Lambert $W$ function is defined as the inverse function of

$$f(W) = We^W,$$  (42)

we have

\[\text{...}\]
the previous equation can be written as

\[-1 - 2/\chi = W[e^{4x+1}] \cdot (43)\]

Now solving for \(\chi\) and reintroducing \(\phi\) we have the final result

\[\phi = 1 - \frac{2}{1 + W[e^{4x+1}]} \cdot (44)\]

**B Integration constants of the bound states in model I**

We start with the general form of the solution for model I, eq. (17),

\[\chi_s(x) = Ae^{4nx} + Be^{-4nx} + \frac{g}{16n} \left[ f_n(x) - f_{-n}(x) - \frac{1}{n} \right] \cdot (45)\]

where

\[f_n(x) \equiv n^n e^{n(1+4x)} \Gamma \left( -n, n W \left[ e^{1+4x} \right] \right) \cdot (46)\]

Let us first look at the limit \(x \to +\infty\). At this limit, \(f_n(x)\) becomes

\[f_n(x \to +\infty) \approx n^n e^{n(1+4x)} \Gamma \left( -n, n \left[ 1 + 4x - \ln(1 + 4x) \right] \right) \]

\[\approx n^n e^{n(1+4x)} \left[ \frac{1}{4x} e^{-n(1+4x)} n^{-1-n} \right] = \frac{1}{4nx}. \cdot (47)\]

Therefore,

\[\lim_{x \to +\infty} f_n(x) = 0, \cdot (48)\]

and similarly

\[\lim_{x \to +\infty} f_{-n}(x) = 0. \cdot (49)\]

Since the \(A\) term diverges in this limit and there is no other term to compensate it, the constant \(A\) should be set to zero.

Now considering the limit \(x \to -\infty\) we can determine the remaining constant \(B\). Using the expansion of the Lambert function for small arguments we have

\[f_n(x \to -\infty) \approx n^n e^{n(1+4x)} \Gamma \left( -n, n e^{1+4x} \right) \]

\[\approx n^n e^{n(1+4x)} \frac{1}{n!} \left[ \frac{e^{n e^{1+4x}}}{(n e^{1+4x})^n} (n - 1)! + (-1)^n \Gamma \left( 0, n e^{1+4x} \right) \right] \]

\[\approx n^n e^{n(1+4x)} \left\{ \frac{e^{n e^{1+4x}}}{n (n e^{1+4x})^n} + \frac{(-1)^n}{n!} \left[ -\gamma - \ln \left( n e^{1+4x} \right) + n e^{1+4x} \right] \right\}. \cdot (50)\]

In the limit \(x \to -\infty\), we can ignore the second term and replace \(e^{n e^{1+4x}}\) with 1, which gives

\[\lim_{x \to -\infty} f_n(x) = \frac{1}{n}. \cdot (50)\]
We need to deal with $f_{-n}(x)$ differently. For this function, we have
\[
 f_{-n}(x \to -\infty) \approx (-n)^{-n} e^{-n(1+4x)} \Gamma (n, -n e^{1+4x}) \\
= (-n)^{-n} e^{-n(1+4x)} \left[ \Gamma (n) - \gamma (n, -n e^{1+4x}) \right],
\] (51)
where $\gamma (s, z)$ is the lower incomplete gamma function. Therefore, we obtain
\[
 f_{-n}(x \to -\infty) \approx (-n)^{-n} e^{-n(1+4x)} \left[ \Gamma (n) - \gamma (n, -n e^{1+4x}) \right]
\] (52)
\[
= (-n)^{-n} e^{-n} \Gamma (n) e^{-4nx} - \frac{1}{n}.
\] (53)

The first term diverges at $x \to -\infty$ and should be cancelled by the $B$ term in the full solution. As a result, eq. (45) becomes
\[
\chi_s (x) = \frac{g}{16n} \left[ f_n (x) - f_{-n} (x) - \frac{1}{n} - (-n)^{-n} e^{-n} \Gamma (n) e^{-4nx} \right].
\] (54)

### C Normalization of the fermionic zero mode

In the eq. (37), we can find the normalization factor in the following way
\[
\mathcal{N}^2 = 1 \int_{-\infty}^{\infty} \psi_1^* \psi_1 \, dx = 1 \int_{-\infty}^{\infty} e^{-2gx} \left( W \left[ e^{1+4x} \right] \right)^g \, dx.
\] (55)

Choosing the transformation $y = e^{1+4x}$, it results in
\[
\mathcal{N}^2 = 1 \int_0^{\infty} e^{-\frac{g}{2} \left[ \ln(y) - 1 \right]} \left( W [y] \right)^g \frac{dy}{4y} \\
= e^{-g/2} \int_0^{\infty} \frac{1}{4y^{1+g/2}} \left( W [y] \right)^g dy.
\] (56)

Now let’s consider the change of variables $w = W (y)$ (notice that $y = we^w$, by the very definition of the Lambert $W$ function). Therefore,
\[
\mathcal{N}^2 = 4e^{-g/2} \int_0^{\infty} \frac{w^g}{(we^w)^{1+g/2}} (1 + w) e^w \, dw,
\] (58)
which leads to
\[
\mathcal{N}^2 = \left( \frac{g}{2} \right)^{g/2} \frac{2e^{-g/2}}{\Gamma [g/2]}.
\] (59)

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