Statistics of work distribution in periodically driven closed quantum systems

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We study the statistics of the work distribution $P(w)$ in a $d$–dimensional closed quantum system with linear dimension $L$ subjected to a periodic drive with frequency $\omega_0$. We show that after an integer number of periods of the drive, the corresponding rate function $I(w) = -\ln[P(w)]/L^d$ satisfies an universal lower bound $I(0) \geq n_d$ and has a zero at $w = Q$, where $n_d$ and $Q$ are the defect density and residual energy generated during the drive. We supplement our results by calculating $I(w)$ for a class of $d$-dimensional integrable models and show that it has oscillatory dependence on $\omega_0$ originating from Stuckelberg interference generated during multiple passage through intermediate quantum critical points or regions during the drive. We suggest experiments to test our theory.

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The study of non-equilibrium dynamics of closed quantum systems has gained tremendous momentum in recent years due to availability of experimental test bed in the form of ultracold atom systems [1–3]. Such cold atoms provide near perfect examples emulators model Hamiltonians such as the Ising or the Bose-Hubbard models [4–5]: in addition, they offer real time tunability of the parameters of the emulated Hamiltonians [6–8]. Consequently, they form perfect experimental platforms for studying non-equilibrium dynamical models. The initial focus of theoretical studies in these direction has been on sudden quench [9, 10] and ramp [11–14] protocols which take the system from one phase of the model to another through an intermediate quantum critical point. However, later studies have also focussed on periodic protocols which involve multiple passage through such critical points [15–17]; in particular it was shown that such a passage can lead to interesting phenomenon such as dynamic freezing [15, 16] and to novel steady states [17].

One of the quantities of interest in a many-body system driven out of equilibrium is its statistics of work distribution [18–23]. It is well-known that for thermodynamic systems in equilibrium, such a distribution follows the large deviation principle (LDP), namely $P(w) \sim \exp[-L^d I(w)]$, where $I(w)$ is the rate function characterizing the decay rate of $P(w)$ from its peak value which occurs at $w = \langle w \rangle$, where $\langle . \rangle$ is the average work done, and $L$ is the linear dimension of the $d$-dimensional quantum system. It was also shown that LDP is valid for a large class of classical non-equilibrium systems where the dynamics can be described by Markov processes [19]. More recently, the statistics of work distribution has been investigated in the context of closed quantum systems taken out of equilibrium by quench or ramp protocols [20, 21]. These works made general arguments about the properties of the rate functions of a generic quantum critical system, showed that they obeyed LDP, and computed it explicitly for a class of one-dimensional (1D) integrable models [20, 21] where the non-equilibrium dynamics involves single passage through an intermediate quantum critical point. To the best of our knowledge, such studies were never extended for periodic protocols beyond two-level systems [23]; in particular the effect of multiple passage through critical points due to the drive on $P(w)$ has not been studied.

In this work, we study the statistics of the work distribution $P(w)$ in a $d$–dimensional closed quantum system with linear dimension $L$ subjected to a periodic drive with frequency $\omega_0$. We provide formally exact expression of the moment generating function, $G(u) = \int dw P(w) \exp[-uw]$, for such a system after an integer number $(n)$ of periods. Using the expression of $G(u)$ and basic elements of large deviation theory, we show that the corresponding rate function $I(w) = -\ln[P(w)]/L^d$, for any closed quantum system after $n$ drive periods, satisfies an universal lower bound $I(0) \geq n_d$, where $n_d$ is the excitation (defect) density generated during the drive. We also show that for any periodic protocol $I(w)$ has a zero at $w = Q$, where $Q$ is the residual energy. We supplement our results by explicit calculation of $I(w)$ for a class of integrable models in $d$-dimensions. Specific examples of these models include the Ising and XY models in $d = 1$ and the Kitaev model in $d = 2$. We show that $I(w)$ has an non-monotonic dependence on the drive frequency $\omega_0$ which originates from the Stuckelberg interference generated during multiple passage of these systems through quantum critical points or lines during the drive. We suggest concrete experiments to test our theory. To the best of our knowledge, the existence of a universal lower bound for $I(0)$ linking it to an experimentally measurable quantity $n_d$ of a generic periodically driven closed quantum system has never been shown; also, the influence of quantum interference phenomenon on the statistics of work distribution of a closed quantum system has not been pointed out. Our work is expected to fill up these major gaps in the existing literature.

We begin by considering a generic time-dependent many-body Hamiltonian $H[\lambda(t)] \equiv H$ where the parameter $\lambda(t)$ has periodic time dependence with a characteristic frequency $\omega_0$ such that at $t = 0$, $\lambda(0) = \lambda_0$. Further,
\( \lambda(t) \) returns to its starting value \( \lambda_0 \) after \( n \) drive cycles at \( t = t_f = 2\pi n / \omega_0 \). We denote the instantaneous eigenstates and eigenenergies of \( H \) at \( t = 0, t_f \) by \( | \alpha \rangle \) and \( E_\alpha \) respectively; they obey \( H|\lambda_0\rangle|\alpha\rangle = E_\alpha|\alpha\rangle \). Assuming that the system is in the ground state \( (0) \) of \( H|\lambda_0\rangle \) at \( t = 0, P(w) \) can be written as

\[
P(w) = \sum_\alpha P_{|\alpha\rangle \rightarrow |\alpha\rangle} \delta(w - E_\alpha + E_0),
\]

where \( P_{|\alpha\rangle \rightarrow |\alpha\rangle} \) denotes the probability that the system reaches the state \( |\alpha\rangle \) at the end of \( n \) drive cycles starting from the ground state \( (0) \). The probability \( P_{|\alpha\rangle \rightarrow |\alpha\rangle} \) can be expressed in terms of the evolution operator \( S \) as

\[
P_{|\alpha\rangle \rightarrow |\alpha\rangle} = \langle \alpha | S(0) | \alpha \rangle^2, \quad S = T_t e^{\int_0^t \lambda(t') dt'}, \tag{2}
\]

where \( T_t \) denotes time ordering. We note that one can write \( S(0) = |\psi(t_f)\rangle = \sum_\alpha c_\alpha |\alpha\rangle \), where \( c_\alpha \) denotes the wavefunction overlap between \( |\psi(t_f)\rangle \) with the eigenstate \( |\alpha\rangle \); they satisfy \( \sum_\alpha |c_\alpha|^2 = 1 \). The excitation (defect) density \( n_d \) and the residual energy \( Q \) generated during the drive can be expressed in terms of \( c_\alpha \) as

\[
n_d = L^{-d} \sum_{\alpha \neq 0} |c_\alpha|^2, \quad Q = L^{-d} \sum_{\alpha \neq 0} (E_\alpha - E_0)|c_\alpha|^2. \tag{3}
\]

One can now compute the moment generating function of \( P(w) \), given by its Laplace transform, as \( \text{[20]} \)

\[
G(u) = \int dw \exp[-wu] P(w) = \sum_\alpha P_{|\alpha\rangle \rightarrow |\alpha\rangle} e^{-(E_\alpha - E_0)u}.
\]

Using Eq. 2 and the relations \( H(t_f) = H(0) \equiv H(\lambda_0) \) and \( S(0) = \sum_\alpha c_\alpha |\alpha\rangle \), one can express \( G(u) \) in terms of the eigenenergies \( E_\alpha \) and the wavefunction overlaps \( c_\alpha \):

\[
G(u) = |c_0|^2 + \sum_{\alpha \neq 0} |c_\alpha|^2 e^{-(E_\alpha - E_0)u}. \tag{4}
\]

Next we show that \( G(u) \) obtained in Eq. 4 satisfies LDP, namely, it can be expressed as \( G(u) = \exp[-L_d f(u)] \), where \( f(u) \) is a concave function of \( u \) \( \text{[18]} \). To do this, we express \( f(u) \), using Eq. 4 as

\[
f(u) = -L_d \ln \left[ |c_0|^2 + \sum_{\alpha \neq 0} |c_\alpha|^2 e^{-(E_\alpha - E_0)u} \right] \tag{5}
\]

To show that \( f(u) \) is a concave function, we observe that \( f(0) = 0 \) and \( f(\infty) = L_d \ln[1/|c_0|^2] \geq 0 \) \( \text{[24]} \). Moreover, we note that

\[
\partial_u f(u) = L_d \sum_{\alpha \neq 0} \frac{(E_\alpha - E_0)|c_\alpha|^2 e^{-(E_\alpha - E_0)u}}{|c_\alpha|^2} \geq 0 \tag{6}
\]

vanishes only at \( u = \infty \) since both \( E_\alpha - E_0 \) and \( |c_\alpha|^2 \) are positive definite for all \( \alpha \). Further \( \partial_u^2 f(u) < 0 \); thus for any generic closed quantum system subjected to a periodic drive, \( f(u) \), computed after \( n \) drive cycles, must be a concave function in the domain \( u \in (0, \infty) \). Consequently \( G(u) \) (and hence \( P(w) \)) obeys LDP.

The fact that \( G(u) \) obeys LDP can be utilized for computing the rate function \( I(w) = -L_d \ln[P(w)] \) by invoking the Gartner-Ellis theorem which links \( f(u) \) to the rate function \( I(w) \) \( \text{[13, 14, 22]} \)

\[
I(w) = f(u[w]) - wu[w], \quad \partial_u f(u)|_{u=u[w]} = w, \tag{7}
\]

We now use Eq. 7 to obtain the universal lower bound for \( I(0) \) as follows. From Eq. 7 we observe that \( I(0) = f(u[0]) \), where \( \partial_u f(u)|_{u=u[0]} = 0 \). From Eq. 6 we find that \( u[0] = \infty \); consequently, one obtains \( I(0) = f(\infty) = L_d \ln[1/|c_0|^2] \) \( \text{[24]} \). Using Eq. 8 one then obtains

\[
I(0) = L_d \ln \left( 1 - \sum_{\alpha \neq 0} |c_\alpha|^2 \right)^{-1} \geq L_d \sum_{\alpha \neq 0} |c_\alpha|^2 = n_d, \tag{8}
\]

which establishes the bound \( I(0) \geq n_d \). Note that the equality holds when \( n_d = 0 \) which occurs for perfect dynamic freezing \( \text{[13, 16]} \). Next, we relate the zero of \( I(w) \) to the residual energy generated during the drive. To this end, we first use Eqs. 3 and 6 to obtain \( Q = \partial_u f(u)|_{u=0} \). Using this observation, we note that Eq. 7 admits a solution \( u[w] = 0 \) for \( w = Q \) and that the corresponding value of the rate function is given by

\[
I(Q) = f(0) = 0. \tag{9}
\]

Eqs. 8 and 9 are the central results of this work. They relate the rate function of a periodically driven generic quantum system after an integer number of drive periods to physically measurable quantities \( n_d \) and \( Q \). These relations are universal in the sense that they hold irrespective of the system dimension, specific parameters of its Hamiltonian, and details of the periodic drive protocol. These details are encoded in \( c_0, \omega_0, \) and \( E_\alpha \) and thus affect \( n_d \) and \( Q \); however they do not alter Eqs. 8 and 9. Further, we note that Eq. 8 holds even when the system is in thermal or non-equilibrium ensemble at \( t = 0 \) and irrespective of applicability of LDP \( \text{[20]} \). We would also like to point out that for any drive protocol, the rate function vanishes at \( w = \langle w \rangle \) \( \text{[20, 21]} \); however, \( \langle w \rangle \) can not be related to \( Q \) or any other experimentally measurable quantity for such protocols. This equality of these two quantities stems from the periodicity of the drive which renders \( H_f = H_i \) after \( n \) drive cycles.

Next, we compute \( I(w) \) for a class of \( d \)-dimensional integrable models whose Hamiltonian are given by \( H_{\text{int}}(t) = \sum_k \psi_k^\dagger H_k(t) \psi_k \), where \( \psi_k^\dagger = (c_k^\dagger c_{-k}^\dagger) \) are Fermionic creation operators and \( H_k(t) \) is given by

\[
H_k(t) = \tau_3(\lambda_1(t) - b_k) + \tau_1 g_k. \tag{10}
\]

Here \( \tau_3 \) and \( \tau_1 \) denote usual Pauli matrices while \( b_k \) and \( g_k \) are general functions of momenta. It is well-known
From Eq. (12), we find that the model passes through a 2D Jordan-Wigner transformation \[ \lambda = \cos(\omega_0 t) \]
and compute \( I(w) \) for the model at end of one drive cycle.

The Hamiltonian of Ising model is given by
\[
H_{\text{Ising}} = -J \sum_{\langle ij \rangle} S_i^z S_j^z - h \sum_i S_i^x,
\]
where \( J \) is the nearest-neighbor coupling between the spins and \( h \) is the transverse field. It turns out that \( H_{\text{Ising}} \) reduces to \( H_{\text{int}} \) via Jordan-Wigner transformation \[ \text{[28]} \] with \( \lambda_0 = g = \hbar/J, b_k = \cos(k) \) and \( y_k = \sin(k) \).

The Kitaev model, describing half-integer spins on a 2D honeycomb lattice, has the Hamiltonian
\[
H' = \sum_{j+\ell=\text{even}} J_x S_{j,\ell}^x S_{j+1,\ell}^x + J_y S_{j,\ell}^y S_{j+1,\ell}^y + J_z S_{j,\ell}^z S_{j+1,\ell}^z,
\]
where \( J_{x,y,z} \) denote nearest-neighbor coupling between the spins and \( (j,\ell) \) describe 2D lattice coordinates. It is well-known that \( H' \) can also be mapped to \( H_{\text{int}} \) via a 2D Jordan-Wigner transformation \[ \text{[29]} \] with \( \lambda_0 = J_z, b_k = (J_x \cos(k \cdot M_1) + J_y \cos(k \cdot M_2)), \) and \( y_k = (J_z \sin(k \cdot M_1) - J_y \sin(k \cdot M_2)) \). Here \( M_{1,2} = (\sqrt{3}a/2, \pm(-3a)/2) \) denote the spanning vectors of the reciprocal lattice of the Kitaev model and \( a \) is the lattice spacing.

To obtain \( I(w) \) for \( H_{\text{int}}(t) \), we first note that the instantaneous eigenvalues of \( H_k(t) \) is given by
\[
E_k[\lambda(t)] = \pm \sqrt{(\lambda(t) - b_k)^2 + y_k^2},
\]
From Eq. (12) we find that the model passes through a critical point (line) in \( d = 1 \) (\( d = 2 \)) as shown schematically in Fig. 1. For \( d = 1 \), the critical point is reached twice for each cycle at \( t_1 = \omega_0^{-1} \arccos(b_{k_0}/\lambda_0 - 1) \) and \( t_2 = 2\pi/\omega_0 - t_1 \), where \( b_{k_0} = 0 \). For \( d = 2 \), the critical region is traversed for a time window \( t = t_i \) such that \( t_i = \omega_0^{-1} \arccos(b_{k_i}/\lambda_0 - 1) \) and \( t'_i = 2\pi/\omega_0 - t_i \), where \( k_i \) satisfies \( g_{k_i} = 0 \).

Let us consider the system in its ground state at \( t = 0 \) with \( \langle \psi_{k_0}^{|0\rangle} = u_{k_0}^{|0\rangle} + v_{k_0}^{|1\rangle} \), where
\[
\psi_{k_0}^{|0\rangle} = (1 + [-](2\lambda_0 - b_{k_0})/2E_k(\lambda_0))^{1/2}/\sqrt{2},
\]
and \( |0\rangle \) and \( |1\rangle \) denote the states \((1,0)\) and \((0,1)\) respectively for a given \( k \). The corresponding excited state is given by \( |\psi_{k_0}^{|1\rangle} = -v_{k_0}^{|0\rangle} + u_{k_0}^{|1\rangle} \). The state of the system at \( t = t_f \) is given by \( |\psi_{k_0}(t_f) = u_{k_0}^{|0\rangle} + v_{k_0}^{|1\rangle} \), where the expressions for \( u_k \) and \( v_k \) can be obtained by solving the Schrodinger equation \( i\hbar \partial_j |\psi_k(t) = H_k(t) |\psi_k(t) \rangle \).

Thus the wavefunction overlaps for any \( k \) is given by
\[
\alpha_k = (u_k^0 v_k + v_k^0 u_k), \quad \gamma_k = (u_k^0 v_k - v_k^0 u_k).\]

We note that one can express \( n_d \) and \( Q \) in terms of \( \gamma_k \) as \( n_d(Q) = \psi_{k_0}^{|1\rangle} = u_{k_0}^{|0\rangle} + v_{k_0}^{|1\rangle} \). Further, using Eqs. (4) and (5) one obtains, in terms of the wavefunction overlaps,
\[
f(u, \omega_0) = - \int \frac{d^d k}{(2\pi)^d} \ln \left[ |\alpha_k|^2 + |\gamma_k|^2 e^{-2E_k(\lambda_0)n} \right],
\]
where the integral is to be taken over the d-dimensional Brillouin zone. The corresponding rate function \( I(w, \omega_0) \) can be computed from Eq. (8) using Eq. (7) in particular, \( I(0, \omega_0) \) is given by
\[
I(0, \omega_0) = \int \frac{d^d k}{(2\pi)^d} \ln (1 - |\gamma_k|^2) \geq n_d(\omega_0),
\]
where we have used the expression of \( n_d(\omega_0) \). Eq. (15) and (16) along with Eq. (7) reduces the task of computing \( I(w) \) to computing \( \gamma_k \). This can be done exactly, albeit numerically, for both the Kitaev and the Ising models. The results of this numerics is shown In Figs. 2 and 3. In Fig. 2, we find that for both the Ising and the Kitaev models, the rate functions display a non-monotonic behavior as a function of \( \omega_0 \). Further in the left panels of Fig. 3 we show via explicit computation of \( I(0, \omega_0) \) (Eq. 16) and \( n_d(\omega_0) \), that the universal bound established in Eq. (8) is obeyed for both the models. We also compute the position of zeros of \( I(w, \omega_0) \) for several representative values of \( \omega_0 \) in the right panels of Fig. 3. These are then compared with the plot of \( Q(\omega_0) \); the plots show excellent match confirming Eq. (9).

Finally, we relate the non-monotonic behavior of \( I(w, \omega_0) \) to the Stuckelberg interference phenomenon.
this end, we focus on Ising model in $d = 1$. In this case, it is possible to obtain an approximate analytical expression of $\gamma_k$ using the adiabatic-impulse approximation \[30\]. Within this approximation, the excitation production occurs in the impulse region near the avoided level crossing for any given $k$; for the rest of the evolution, the system gathers a time dependent adiabatic phase factor. The details of the calculation of these phase factors and the probabilities of defect production during each passage through the critical point and the rate function at the end of $I(0)$ are identical to \[30\]. Within this approximation, the excitation production occurs in the impulse region near the avoided level crossing for any given $k$; for the rest of the evolution, the system gathers a time dependent adiabatic phase factor. The details of the calculation of these phase factors and the probabilities of defect production during each passage through the critical point and the rate function at the end of $I(0)$ are identical to \[30\].

\[
J(0) = \exp(-2\pi\delta_k), \quad \delta_k = \frac{g^2}{|d\lambda/dt|} t_{\text{cr}},
\]

\[
\phi_k = -\pi/4 + \delta_k(\ln(\delta_k - 1) + \text{Arg}(1 - i\delta_k))
\]

The expression for total defect formation probability, $|\gamma_k|^2$, after a full drive cycle is given in terms of these quantities as

\[
|\gamma_k|^2 = 4p_k(1 - p_k)\sin^2(\phi_k^{\text{st}})
\]

where $\phi_k^{\text{st}} = \xi_{2k} + \phi_k$ is the Stuckelberg phase originating from the interference of parts of the system wavefunction at ground and excited states during the second passage through the critical point, and $\xi_{2k} = \int_{t_1}^{t_2} 2E_k(t')dt'/\hbar$ is the phase acquired during passage between the crossings of the critical point at $t = t_1$ and $t_2$.\[30\][31].

From Eq. [18] we note that $|\gamma_k|^2 \sim 1$ provided $p_k \simeq 1/2$ and $\sin^2(\phi_k^{\text{st}}) \simeq \pi/2$ for the same $k$; otherwise it stays small. Since $p_k$ depends on $\omega_0$ through $\delta_k$ and $\sin^2(\phi_k^{\text{st}})$ is an oscillatory function of $\omega_0$, we expect periodic pattern of maxima and minima $|\gamma_k|^2$ as a function of $\omega_0$. A plot of $|\gamma_k|^2$ in the left and right panels of Fig. 4 as a function of $k$ for two representative values of $\omega_0 = 0.1$ and 0.12 demonstrates the above-mentioned effect for the Ising and the Kitaev models respectively. The left panel also shows a qualitative match between the analytical (Eq. [18]) and numerical (Eq. [14]) expressions of $|\gamma_k|^2$. It is then obvious from Eqs. [14] and [7] that a pattern of alternate maxima and minima as a function of $\omega_0$ shall also show up in $G(u, \omega_0)$ and $I(w, \omega_0)$ since they depend on momentum integral of $|\gamma_k|^2$. Thus we find that the peaks (dips) of $I(w, \omega_0)$ arises from constructive (destructive) interference of the ground and excited state wavefunctions; it constitutes a manifestation of quantum interference phenomenon in shaping the statistics of work distribution. The right panel shows analogous result for the Kitaev model showing qualitatively similar behavior.

The experimental verification of our work can be done by standard techniques of measuring characteristics function $G(iu)$ leading to construction of $P(w)$ by using single qubit interferometry \[32\][34]. Typically experiments involve spin 1/2 systems which are implemented using ion traps \[32\] or single two-level system \[33\] or nuclear spins of $\text{C}^{13}$ \[34\] and $P^{19}$ \[35\]. We suggest driving such systems using a periodic protocol with frequency $\omega_0$ for a time $T_0 = 2\pi/\omega_0$. Our theory predicts that $I(0) \geq n_d$ and $I(Q) = 0$ for such a drive. We note that $Q$ and $n_d$ can be separately measured. For a system of few spins, $n_d$ will be given by the number of spins deviating from their ground state configuration while $Q$ is given by energy of these spins; the corresponding quantities for a two-level system would involve occupation probability of the upper level ($n_d$) and the energy difference between the final and initial configurations ($Q$).

In conclusion, we have established that for a generic closed quantum system subjected to a periodic drive, the rate function at the end of $n$ drive cycles satisfies $I(0) \geq n_d$ and $I(Q) = 0$. These relations are universal and do not depend on system or protocol details as long the drive is periodic. We have also computed $I(w)$ for a
class of integrable models where the drive takes such systems through intermediate critical points (regions). We have shown that $I(w)$ is an oscillatory function of $\omega_0$, linked this behavior to the Stuckelberg interference phenomenon, and suggested experiments to test our theory.

**SUPPLEMENTARY MATERIAL: WORK DISTRIBUTION FOR ARBITRARY STARTING ENSEMBLE**

In this section, we provide a proof of the relation $I(0) \geq n_d$ for the case when the system starts at $t = 0$ from a thermal/non-equilibrium ensemble and without assuming the existence of LDP. We begin by assuming that the system is, at $t = 0$, in thermal/non-equilibrium ensemble where the probability of occupation of the state $|\alpha\rangle$ is given by $\rho_{0\alpha}$. For a thermal ensemble, for example, $\rho_{0\alpha} = \exp(-\beta E_\alpha)/Z$, where $E_\alpha$ are the eigenenergies of the system satisfying $H|\lambda_0\rangle = E_\alpha |\alpha\rangle$, $\beta = (k_BT)^{-1}$ is the inverse temperature, $k_B$ is the Boltzman constant, and $Z = \sum_\alpha \exp(-\beta E_\alpha)$ is the partition function. The precise form of the occupation probability $\rho_{0\alpha}$ shall be irrelevant in our derivation. We shall, however, $\sum_\alpha \rho_{0\alpha} = 1$ which is generically true for any ensemble.

The work distribution of such a system is given by a straightforward generalization of Eq. [1]. In contrast to the zero temperature case, where the system starts from the initial ground state $|0\rangle$ at $t = 0$, the system has a finite weight in all states $|\alpha\rangle$ and hence one has

$$P(w) = \sum_{\alpha\beta} P_{\alpha\rightarrow \beta} \delta(w - E_\beta + E_\alpha)$$

$$P_{\alpha\rightarrow \beta} = \rho_{0\beta}^{(\alpha)}|\langle \beta |S|\alpha\rangle|^2,$$  \hspace{1cm} (19)

where $P_{\alpha\rightarrow \beta} = \rho_{0\beta}$ is the probability that the system starts out in the state $|\alpha\rangle$ and $P_{\alpha\beta}$ is the conditional probability that it ends up in the state $|\beta\rangle$ after $n$ drive cycles having started from the state $|\alpha\rangle$. We note at the outset, since at the end of $n$ drive cycles $H_f = H_i = H|\lambda_0\rangle$, one can write $S|\alpha\rangle = \sum_\beta \Lambda_{\alpha\beta}|\beta\rangle$, where $\Lambda_{\alpha\beta}$ are the wavefunction overlap between the states $S|\alpha\rangle$ and $|\beta\rangle$. In this notation, $\Lambda_{\alpha\beta} = c_\beta$ which is used in the main text and for any $|\alpha\rangle$, $\sum_\beta |\Lambda_{\alpha\beta}|^2 = 1$. We also note that the excitation (defect) density $n_d$ of the system at the end of the periodic drive can be written as

$$n_d = L^{-d}(1 - \sum_\alpha \rho_{0\alpha}|\Lambda_{\alpha\alpha}|^2) = L^{-d} \sum_{\alpha\neq \beta} \rho_{0\alpha}|\Lambda_{\alpha\beta}|^2$$ \hspace{1cm} (20)

The characteristic function of the $\chi(u)$ of the driven system is defined as the Fourier transform of its work distribution $P(w)$. The expressions of $G(u)$ and $P(w)$ can be obtained using Eq. [19] as

$$\chi(u) = \int dw e^{-iuw}P(w) = \sum_{\alpha} \rho_{0\alpha}|\Lambda_{\alpha\alpha}|^2 e^{-iu(E_\beta - E_\alpha)}$$

$$= \sum_\alpha \rho_{0\alpha} \left(|\Lambda_{\alpha\alpha}|^2 + \sum_{\beta \neq \alpha} |\Lambda_{\alpha\beta}|^2 e^{-iu(E_\beta - E_\alpha)}\right)$$ \hspace{1cm} (21)

Using Eq. [21] one can obtain $P(w) = \int dw \exp[iuw]\chi(u)$. Then defining the rate function $I(w) = -L^{-d}\ln[P(w)]$, one can obtain, after some straightforward algebra,

$$I(0) = -L^{-d}\ln \left[\sum_\alpha \rho_{0\alpha}|\Lambda_{\alpha\alpha}|^2\right]$$

$$= -L^{-d}\ln \left[1 - \sum_{\beta \neq \alpha} \rho_{0\alpha}|\Lambda_{\alpha\beta}|^2\right]$$

$$\geq L^{-d}\sum_{\alpha \neq \beta} \rho_{0\alpha}|\Lambda_{\alpha\beta}|^2 = n_d$$ \hspace{1cm} (22)

which established the bound $I(0) \geq n_d$. As mentioned in the beginning of this section, we have not used the explicit expression of $\rho_{0\alpha}$ anywhere in this derivation. Further, we note that we have not assumed the existence of LDP in demonstrating the bound; thus our results holds for any non-equilibrium starting ensemble $\rho_{0\alpha}$ and for any finite-size system where LDP may not hold.

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[25] The most general form of the Gartner-Ellis theorem yields $I(u) = -\text{Inf}[uw - f(u)]$, where Inf stands for Infimum [15]. However, if $f'(u)$ exists, this definition of $I(u)$ coincides with the one given in Eq. 7.
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