Strong maximum principle for a sublinear elliptic problem at resonance

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Abstract. We examine the semilinear resonant problem

$$-\Delta u = \lambda_1 u + \lambda g(u) \quad \text{in } \Omega, \quad u \geq 0 \quad \text{in } \Omega, \quad u_{|\partial \Omega} = 0,$$

where $\Omega \subset \mathbb{R}^N$ is a smooth, bounded domain, $\lambda_1$ is the first eigenvalue of $-\Delta$ in $\Omega$, $\lambda > 0$. Inspired by a previous result in literature involving power-type nonlinearities, we consider here a generic sublinear term $g$ and single out conditions to ensure: the existence of solutions for all $\lambda > 0$; the validity of the strong maximum principle for sufficiently small $\lambda$. The proof rests upon variational arguments.

Keywords: resonant problem, existence, maximum principle.

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded domain of class $C^2$, and let $\lambda_1$ be the first eigenvalue of $-\Delta$ in $\Omega$ with Dirichlet boundary conditions. The issue of the existence of solutions of the problem

$$\begin{cases} 
-\Delta u = \lambda_1 u + u^{s-1} - \mu u^{r-1} & \text{in } \Omega \\
u \geq 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}$$

(1.1)

$s \in (1,2)$, $r \in (1,s)$, and $\mu > 0$, has been the subject of study of the recent [3]. As a distinctive feature, the right-hand side term $f(t) := \lambda_1 t + t^{s-1} - \mu t^{r-1}$ in (1.1) is not locally Lipschitz near 0, and moreover satisfies the sign property

$$f^{-1}((-\infty,0]) \supseteq (0,a], \quad \text{for some } a > 0.$$

As a result, from the celebrated paper [13] (see also [8]), it is known that the strong maximum principle may fail to be valid in this context. By adopting minimax and perturbation
techniques, the author of [3] showed instead that such a principle does hold as long as the perturbation parameter is chosen sufficiently large. More precisely, the main results in [3] state that problem (1.1) has non-zero solutions for the entire positive range of $\mu$; positive solutions for $\mu$ large enough.

The fact that, after a rescaling, (1.1) can be turned into the problem

$$\begin{cases}
-\Delta u = \lambda_1 u + \lambda (u^s - 1 - u^r - 1) & \text{in } \Omega \\
u \geq 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}$$

(1.2)

for a suitable $\lambda > 0$, raises the natural question whether, as explicitly expressed in [3, Remark 2.4], the same results mentioned above continue to hold when the powers in (1.2) are replaced by a generic nonlinear term $g$. And, if it is so, it would be interesting of course to identify some “minimal” structure conditions on $g$ for the validity of such results. In the present paper we address these questions and consider the problem

$$\begin{cases}
-\Delta u = \lambda_1 u + \lambda g(u) & \text{in } \Omega \\
u \geq 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}$$

(P$_\lambda$)

where $g : [0, +\infty) \to \mathbb{R}$ is continuous, $g(0) = 0$, and obeys the following conditions:

$(g_1)$ there exists $q \in (1, 2)$ such that $k_1 := \sup_{t > 0} \frac{|g(t)|}{1 + t^{q-1}} < +\infty$;

$(g_2)$ $\lim_{t \to 0^+} \frac{g(t)}{t} = -\infty$;

$(g_3)$ $\liminf_{t \to +\infty} G(t) > 0$;

$(g_4)$ $\lim_{t \to +\infty} (g(t)t - 2G(t)) = -\infty$,

where, as usual,

$$G(t) := \int_0^t g(s)ds, \quad \text{for all } t \geq 0.$$

Problems like (P$_\lambda$) are being investigated since Landesman and Lazer’s pioneering work [9], in which sufficient conditions, based on the interaction between the nonlinearity and the spectrum of the linear operator, were given for them to have a solution. Noteworthy contributions following that work can be found in [2, 5, 12] and also in [6, 7, 10, 11, 14] (see the related references as well) in which several classes of elliptic problems at resonance are investigated via variational and topological methods.

Coming back to (P$_\lambda$), our approach develops along the same line of reasoning as [3]. We prove initially that (P$_\lambda$) has at least a non-zero solution for all $\lambda > 0$. This is accomplished by considering a sequence of problems near resonance whose solutions are shown to converge to a solution of the original problem. In this regard, assumption $(g_4)$ comes into play to prove the boundedness of the sequence of approximating solutions. Then, by exploiting the classical decomposition of $H^1_0(\Omega)$ into the first eigenspace and its orthogonal complement, we show
for sufficiently small $\lambda$, the set of solutions to $(P_{\lambda})$ is contained in the interior of the positive cone of $C^1_0(\overline{\Omega})$. It still remains an open question to investigate the uniqueness of positive solutions to $(P_{\lambda})$ (in the one-dimensional case and for power-nonlinearities it has instead been established in [4]), as well as the existence of non-zero solutions compactly supported in $\Omega$, in the spirit of [8].

Our main results, Theorems 2.3 and 2.4, are stated and proved in the coming section. Before going on, we arrange some notation and the variational framework for $(P_{\lambda})$. We set

$$\|u\| := \left(\int_{\Omega} |\nabla u|^2 \, dx\right)^{\frac{1}{2}},$$

for all $u \in H^1_0(\Omega)$, and denote by $\|\cdot\|_p$, $p \in [1, +\infty)$, the classical $L^p$-norm on $\Omega$. We also set

$$c_p := \sup_{u \in H^1_0(\Omega) \setminus \{0\}} \frac{\|u\|}{\|u\|_p}$$

for each $p \geq 1$, with $p \leq \frac{2N}{N-2}$ if $N \geq 3$, and denote by $\phi_1$ the positive eigenfunction associated with $\lambda_1$ and normalized with respect to $\|\cdot\|_\infty$. We recall that the first two eigenvalues $\lambda_1, \lambda_2$ of $-\Delta$ in $\Omega$ admit the variational characterization

$$\lambda_1 = \inf_{u \in H^1_0(\Omega) \setminus \{0\}} \frac{\|u\|^2}{\|u\|_2^2}, \quad \lambda_2 = \inf_{u \in \text{span}\{\phi_1\}^\perp \setminus \{0\}} \frac{\|u\|^2}{\|u\|_2^2}.$$

Given a set $E \subset \mathbb{R}^N$, its Lebesgue measure will be denoted by the symbol $|E|$. Throughout this paper, the symbols $C, C_1, C_2, \ldots$ represent generic positive constants whose exact value may change from occurrence to occurrence.

For all $\lambda > 0$, we denote by $I_{\lambda} : H^1_0(\Omega) \to \mathbb{R}$ the energy functional associated with $(P_{\lambda})$,

$$I_{\lambda}(u) := \frac{1}{2} \|u\|^2 - \frac{\lambda_1}{2} \|u\|_2^2 - \lambda \int_{\Omega} G(u_+) \, dx,$$

for all $u \in H^1_0(\Omega)$, where $u_+ = \max\{u, 0\}$. By a weak solution to $(P_{\lambda})$ we mean any $u \in C^0(\overline{\Omega}) \cap H^1_0(\Omega)$ verifying

$$\int_{\Omega} (\nabla u \nabla v - \lambda_1 u v - \lambda g(u) v) \, dx = 0, \quad \text{for all } v \in H^1_0(\Omega).$$

## 2 Results

As already mentioned, we start by considering a sequence of approximating problems.

**Lemma 2.1.** For each $\lambda > 0$, there exists $\tilde{n} \in \mathbb{N}$ such that the problem

$$\begin{cases}
-\Delta u = \left(\lambda_1 - \frac{1}{n}\right) u + \lambda g(u) & \text{in } \Omega \\
u \geq 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
$$

admits a non-zero weak solution $u_n$, with positive energy, for all $n \geq \tilde{n}$. 

Proof. Fix $\lambda > 0$ and let $n \in \mathbb{N}$ with $n > \frac{1}{\lambda_1}$. Let us first show that the energy functional $I_n : H^1_0(\Omega) \to \mathbb{R}$ corresponding to $(P_n)$,

$$I_n(u) := I_\lambda(u) + \frac{1}{2n} \|u_+\|_2^2 = \frac{1}{2} \|u\|^2 - \frac{1}{2} \left(\lambda_1 - \frac{1}{n}\right) \|u_+\|_2^2 - \lambda \int_\Omega G(u_+) dx,$$  \tag{2.1}

for all $u \in H^1_0(\Omega)$, has the mountain pass geometry for sufficiently large $n \in \mathbb{N}$.

Fix $k \in (2, 2^*)$ and set

$$M := \frac{k}{2} \sup_{t > 0} \frac{\lambda_1 t^2 + 2 \lambda G(t)}{t^k}.$$

By $(g_1)$ and $(g_2)$ one has $0 < M < +\infty$ and $\frac{1}{2} t^2 + \lambda G(t) \leq \frac{M}{k} t^k$, for all $t > 0$. Then, defining

$$R := \left(M c_1^k\right)^{\frac{1}{k}},$$

we easily obtain

$$\inf_{u \in S_R} I_n(u) \geq \inf_{\|u\| = R} \left(\frac{1}{2} \|u\|^2 - \frac{M}{k} \|u\|^k\right) \geq \inf_{\|u\| = R} \left(\frac{1}{2} \|u\|^2 - \frac{M c_1^k}{k} \|u\|^k\right) = \left(\frac{1}{2} - \frac{1}{k}\right) R^2 > 0,$$  \tag{2.2}

for any $n \in \mathbb{N}$, where $S_R := \{u \in H^1_0(\Omega) : \|u\| = R\}$.

Now, let us show that there exist $u_1 \in H^1_0(\Omega)$, with $\|u_1\| > R$, and $\bar{n} \in \mathbb{N}$, such that $I_n(u_1) < 0$ for all $n \geq \bar{n}$. Owing to $(g_3)$, there exist $L, b > 0$ such that

$$G(t) \geq L, \quad \text{for all } t \geq b.$$

If we denote by

$$E_\gamma := \{x \in \Omega : \phi_1(x) < \gamma\},$$

with $\gamma > 0$, then there exists $\gamma_1 > 0$ such that

$$L > \frac{k_1(bq + b^\theta)|E_\gamma|}{q(|\Omega| - |E_\gamma|)}, \quad \text{for all } \gamma \in (0, \gamma_1).$$  \tag{2.3}

Fix $\bar{\gamma} \in \mathbb{R}$ satisfying

$$0 < \bar{\gamma} < \min\left\{\gamma_1, \frac{b}{R}\right\}.$$

Since the function $\psi(t) := q\bar{\gamma} t + \bar{\gamma}^\theta t^\theta$ is continuous in $(0, +\infty)$ and $\psi\left(\frac{b}{\bar{\gamma}}\right) = bq + b^\theta$, thanks to (2.3), there exists $\bar{t} > \frac{b}{\bar{\gamma}}$ such that

$$L > \frac{k_1(q\bar{\gamma}\bar{t} + \bar{\gamma}^\theta \bar{t}^\theta)|E_{\bar{\gamma}}|}{q(|\Omega| - |E_{\bar{\gamma}}|)}.$$  \tag{2.4}
With the aid of (g₁) and (2.4) we then obtain

\[
\int_{\Omega} G(\tilde{I} \phi_1) dx = \int_{E_\\gamma} G(\tilde{I} \phi_1) dx + \int_{\{\phi_1 \geq \gamma\}} G(\tilde{I} \phi_1) dx \\
\geq -k_1 \int_{E_\\gamma} \left( \tilde{I} \phi_1 + \frac{(\tilde{I} \phi_1)^q}{q} \right) dx + \int_{\{\phi_1 \geq \gamma\}} G(\tilde{I} \phi_1) dx \\
\geq -k_1 \left( \tilde{I}_\\gamma + \frac{1}{q} \phi_1^q \right) |E_\\gamma| + L(|\Omega| - |E_\\gamma|) \\
> 0.
\]

As a result, there exists \( \bar{n} \in \mathbb{N} \), with \( \bar{n} > \frac{1}{\lambda_1} \), such that

\[
I_n(\tilde{I} \phi_1) = \frac{r^2}{2n} \|\phi_1\|^2 - \lambda \int_{\Omega} G(\tilde{I} \phi_1) dx < 0
\]

for all \( n \geq \bar{n} \). Therefore, the functional \( I_n \) satisfies the geometric conditions required by the mountain pass theorem for all \( n \geq \bar{n} \).

Moreover, by (g₁) and Sobolev embeddings, one has

\[
I_n(u) \geq \frac{1}{2n\lambda_1} \|u\|^2 - \lambda k_1 \left( \int_{\Omega} |u| dx + \frac{1}{q} \int_{\Omega} |u|^q dx \right) \\
\geq \frac{1}{2n\lambda_1} \|u\|^2 - \lambda c_1 k_1 \|u\| - \frac{\lambda c_1 k_1}{q} \|u\|^q,
\]

and thus \( I_n(u) \to +\infty \) as \( \|u\| \to +\infty \). This fact, in addition to standard arguments (see for instance Example 38.25 of [15]), ensures that \( I_n \) satisfies the Palais–Smale condition. Then, by invoking the classical mountain pass theorem, \( I_n \) admits a critical point \( u_n \in H^1_0(\Omega) \setminus \{0\} \) for all \( n \geq \bar{n} \), and, by (2.2), one also has

\[
I_n(u_n) = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_n(\gamma(t)) \geq \left( \frac{1}{2} - \frac{1}{k} \right) R^2,
\]

where \( \Gamma := \{ \gamma \in C^0([0,1], H^1_0(\Omega)) : \gamma(0) = 0, \gamma(1) = u_1 \} \). This concludes the proof.

**Lemma 2.2.** Let \( \lambda > 0 \), \( \bar{n} \in \mathbb{N} \) and let \( u_n \), with \( n \geq \bar{n} \), be as in Lemma 2.1. Then, the sequence \( \{u_n\}_{n \geq \bar{n}} \) is bounded in \( H^1_0(\Omega) \).

**Proof.** Let \( n \in \mathbb{N} \), \( n \geq \bar{n} \). By standard regularity theory, \( u_n \in C^{1,\alpha}(\overline{\Omega}) \), for some \( \alpha \in (0,1) \). For any \( \bar{n} \in \mathbb{N} \), \( n \geq \bar{n} \) there exist, uniquely determined, \( t_n \in \mathbb{R} \) and \( w_n \in \text{span}\{\phi_1\}^\perp \) such that

\[
u_n = t_n \phi_1 + w_n.
\]

It is straightforward to verify that \( w_n \in C^{1,\alpha}(\overline{\Omega}) \) is a weak solution to

\[
\begin{cases}
-\Delta u = \left( \lambda_1 - \frac{1}{n} \right) u + \lambda g(t_n \phi_1 + u) - \frac{t_n}{n} \phi_1 & \text{in } \Omega \\
u \geq 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

(2.6)
and therefore, also by \((g_1)\), one has
\[
\|w_n\|^2 \leq \left( \frac{\lambda_1 - \frac{1}{n}}{\lambda_2} \right) \|w_n\|^2 + \lambda \int_{\Omega} g(t_n \phi_1 + w_n) w_n \, dx \\
\leq \left( \frac{\lambda_1 - \frac{1}{n}}{\lambda_2} \right) \|w_n\|^2 + \lambda k_1 \|w_n\|_1 + \lambda k_1 t_n^{q-1} \|\phi_1\|_{\infty}^{q-1} \|w_n\|_1 + \lambda k_1 \|w_n\|_q^q.
\]
(2.7)

From (2.7), it follows that
\[
\|w_n\| \leq C \left( (1 + t_n^{q-1}) + \|w_n\|^{q-1} \right),
\]
(2.8)

for some \(C > 0\). We claim that the sequence \(\{t_n\}_{n \geq 0}\) is bounded in \(\mathbb{R}\). Arguing by contradiction, assume that, up to a subsequence, \(t_n \to +\infty\) as \(n \to +\infty\). Without loss of generality, we can assume that \(t_n \geq 1\) for all \(n \geq n\) and, since
\[
y^{q-1} \leq C_1 + \frac{1}{2C} \leq C_1 t_n^{q-1} + \frac{1}{2C} y, \quad \text{for all} \quad y > 0,
\]
from (2.8) we deduce
\[
\|w_n\| \leq 2C t_n^{q-1} + C \|w_n\|^{q-1} \leq 2C t_n^{q-1} + CC_1 t_n^{q-1} + \frac{1}{2} \|w_n\|,
\]
and then
\[
\|w_n\| \leq C_2 t_n^{q-1}.
\]

Therefore, fixing \(p > \max \left\{ \frac{N}{q}, \frac{q}{q-1} \right\} \), we obtain
\[
\|w_n\|_\infty \leq C_3 \left( \|w_n\|_p + \|g(t_n \phi_1 + w_n)\|_p + \frac{t_n}{n} \|\phi_1\|_p \right) \\
\leq C_4 \left( \|w_n\|_\infty^{\frac{p-1}{q}} \|w_n\|_1 + 1 + t_n^{q-1} + \|w_n\|_{\infty}^{q-1} - \|w_n\|_q^q + \frac{t_n}{n} \right) \\
\leq C_5 \left( \|w_n\|_\infty^{\frac{p-1}{q}} t_n^{q-1} + t_n^{q-1} + \|w_n\|_{\infty}^{q-1} - \|w_n\|_q^q \frac{q(q-1)}{p(q-1)} + \frac{t_n}{n} \right).
\]

Dividing the first and the last side of the previous inequality by \(t_n\) and bearing in mind that \(y^m \leq 1 + y\), for all \(m \in [0,1]\) and \(y > 0\), we get
\[
\left\| \frac{w_n}{t_n} \right\|_\infty \leq C_5 \left( \left\| \frac{w_n}{t_n} \right\|_\infty^{\frac{p-1}{q}} t_n^{q-2} + t_n^{q-2} + \left\| \frac{w_n}{t_n} \right\|_{\infty}^{q-1} - \frac{q(q-1)}{p(q-1)} + \left(1 + \frac{t_n}{n}\right) \right) \\
\leq C_5 \left( t_n^{q-2} + \left(1 + \frac{t_n}{n}\right) \right) \left(1 + \left\| \frac{w_n}{t_n} \right\|_\infty + \frac{1}{n}\right) \\
\leq C_5 \left( t_n^{q-2} + 2 t_n^{q-2} \left(1 + \left\| \frac{w_n}{t_n} \right\|_\infty + \frac{1}{n}\right) \right).
\]

It follows that
\[
\left(1 - 2C_5 t_n^{q-2}\right) \left\| \frac{w_n}{t_n} \right\|_\infty \leq 3C_5 t_n^{q-2} + \frac{C_5}{n},
\]
and, as a consequence,
\[
\lim_{n \to +\infty} \left\| \frac{w_n}{t_n} \right\|_\infty = 0.
\]
i.e.,

\[ \frac{u_n}{t_n} \to \phi_1 \quad \text{uniformly in } \overline{\Omega}. \]

So, fixing \( \gamma \in (0, \|\phi_1\|_{\infty}) \), we can find \( E \subset \Omega \), with \( |E| > 0 \), and \( \bar{n} \in \mathbb{N} \), \( \bar{n} \geq \tilde{n} \), such that

\[ u_n(x) \geq \gamma t_n, \quad \text{for all } n \geq \tilde{n} \quad \text{and} \quad x \in E. \]

At this point, set

\[ \delta := \sup_{t > 0} (g(t)t - 2G(t)) \in [0, +\infty), \]

and let \( \bar{\ell} > 0 \) such that

\[ g(t) \leq \frac{(\delta + 1)|\Omega|}{|E|}, \quad \text{for all } t \geq \bar{\ell}, \]

and \( n^* \geq \tilde{n} \) such that \( t_n \geq \frac{\bar{\ell}}{\gamma} \) for all \( n \geq n^* \). Then, for all \( n \geq n^* \), taking also (2.5) into account, we obtain

\[ 0 < \int_{\Omega} (g(u_n)u_n - 2G(u_n))dx \]

\[ = \int_{\Omega \setminus E} (g(u_n)u_n - 2G(u_n))dx + \int_E (g(u_n)u_n - 2G(u_n))dx \]

\[ \leq \delta|\Omega| - (\delta + 1)|\Omega| < 0, \]

a contradiction. Therefore, the sequence \( \{t_n\}_{n \geq \bar{n}} \) is bounded in \( \mathbb{R} \) and (2.8) yields the boundedness of \( \{w_n\}_{n \geq \bar{n}} \) in \( H^1_0(\Omega) \), as well. As a consequence, we get the boundedness of \( \{u_n\}_{n \geq \bar{n}} \) in \( H^1_0(\Omega) \), as desired.

Collecting the results of the previous lemmas, it is now easy to derive our first existence result.

**Theorem 2.3.** For all \( \lambda > 0 \), problem \( (P_\lambda) \) has at least one non-zero solution.

**Proof.** Let \( \{u_n\} \) be the sequence of solutions to \( (P_n) \) in Lemma 2.1. By Lemma 2.2 there exists \( u^* \in H^1_0(\Omega) \) such that, up to a subsequence,

\[ u_n \to u^* \quad \text{in } H^1_0(\Omega), \quad u_n \to u^* \quad \text{in } L^p(\Omega), \quad \text{for all } p \in [1, 2^*). \]

Fixing \( v \in H^1_0(\Omega) \) and taking the limit as \( n \to +\infty \) in the identity \( I_n'(u_n)(v) = 0 \), we get \( I_n'(u^*)(v) = 0 \), i.e. \( u^* \) is a weak solution to \( (P_\lambda) \). To justify that \( u^* \neq 0 \), observe that, by (2.5) one has

\[ 0 < \left( \frac{1}{2} - \frac{1}{k} \right) R^2 \]

\[ \leq \lambda \int_{\Omega} (g(u_n)u_n dx - 2G(u_n))dx \]

\[ \leq \lambda k_1 \left( \|u_n\|_1 + \|u_n\|_q^q \right) + 2\lambda k_1 \left( \|u_n\|_1 + \frac{1}{q} \|u_n\|_q^q \right), \]

and so, letting \( n \to +\infty \), the conclusion is achieved.
We now show that, when $\lambda$ approaches zero, every non-zero solution to $(P_{\lambda})$ is actually positive. To this aim, for all $\lambda > 0$, set

$$S_{\lambda} := \{ u \in H^1_0(\Omega) \setminus \{0\} : u \text{ is a solution to } (P_{\lambda}) \},$$

and denote by $P$ the interior of the positive cone of $C^1_0(\Omega)$, i.e.

$$P := \left\{ u \in C^1_0(\Omega) : u > 0 \text{ in } \Omega, \frac{\partial u}{\partial \nu} < 0 \text{ on } \partial \Omega \right\},$$

$v$ being the unit outer normal to $\partial \Omega$. Our second result reads as follows:

**Theorem 2.4.** There exists $\Lambda^* > 0$ such that for each $\lambda \in (0, \Lambda^*)$, $S_{\lambda} \subset P$.

**Proof.** We first observe that, by the regularity theory of elliptic equations, for all $\lambda > 0$ and $u_{\lambda} \in S_{\lambda}$, one has $u_{\lambda} \in C^{1,\alpha}(\Omega)$, for some $\alpha \in (0, 1)$.

If $u_{\lambda} \in S_{\lambda}$, it is straightforward to check that $v_{\lambda} := \lambda^{-1} u_{\lambda}$ is a solution to the problem

$$\left\{ \begin{array}{ll}
-\Delta u = \lambda_1 u + g(\lambda u) & \text{in } \Omega \\
u \geq 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{array} \right. \quad (\tilde{P}_{\lambda})$$

clearly equivalent to $(P_{\lambda})$. Note that $(g_2)$ ensures the existence of some $a > 0$ such that $g(t) < 0$ for all $t \in (0, a)$, and moreover it must hold

$$\|v_{\lambda}\|_{\infty} \geq \frac{a}{\lambda}, \quad (2.9)$$

otherwise we would get $g(u_{\lambda}) < 0$ in $\Omega \setminus u_{\lambda}^{-1}(0)$, and so

$$\|u_{\lambda}\|^2 - \lambda_1 \|u_{\lambda}\|^2 = \lambda \int_{\Omega} g(u_{\lambda}) u_{\lambda} dx < 0,$$

against the definition of $\lambda_1$. From now on, we will then focus on $(\tilde{P}_{\lambda})$. We split the proof in several steps.

**Step 1.** We show that there exist two constants $C^*, \Lambda_0 > 0$ such that, for any $\lambda \in (0, \Lambda_0]$ and for any $v_{\lambda} \in S_{\lambda}$,

$$\|v_{\lambda}\| \geq \frac{C^*}{\lambda}. \quad (2.10)$$

Fix $\beta > \max\{ \frac{N}{2}, \frac{1}{q-1} \}$. By [1, Theorem 8.2] and the embedding $W^{2,\beta}(\Omega) \hookrightarrow C^1(\overline{\Omega})$, one has $v_{\lambda} \in W^{2,\beta}(\Omega)$ and there exists a constant $C_0 > 0$, independent of $\lambda$, such that

$$\|v_{\lambda}\|_{C^1(\overline{\Omega})} \leq C_0 \left( (\lambda_1 + 1) \|v_{\lambda}\|_{\beta} + \|g(\lambda v_{\lambda})\|_{\beta} \right). \quad (2.11)$$

So, by $(g_1)$ and Hölder’s inequality, we get

$$\int_{\Omega} |g(\lambda v_{\lambda})|^\beta dx \leq k_1^\beta \int_{\Omega} \left( 1 + (\lambda v_{\lambda})^{q-1} \right)^\beta dx \leq 2^{\beta-1} k_1^\beta \left( |\Omega| + \lambda^{\beta(q-1)} \|v_{\lambda}\|_{\infty}^{\beta(q-1)-1} \|v_{\lambda}\|_{1} \right),$$
and therefore

\[
\|v_\lambda\|_\infty \leq C_0 \left( (\lambda_1 + 1) \|v_\lambda\|_\infty^{\frac{\delta - 1}{\delta}} \|v_\lambda\|_1^{\frac{1}{\delta}} + 2 \frac{\beta - 1}{\beta} k_1 \left( \|\Omega\|^{\frac{1}{\beta}} + \lambda^{q-1} \|v_\lambda\|_\infty^{\frac{q-1}{\beta}} \|v_\lambda\|_1^{\frac{1}{\beta}} \right) \right).
\]

Now, dividing by \(\|v_\lambda\|_\infty^{\frac{\delta - 1}{\delta}}\) both sides of the previous inequality and taking (2.9) into account, we obtain,

\[
\left( \frac{a}{\lambda} \right)^{\frac{1}{\beta}} \leq \|v_\lambda\|_\infty^{\frac{1}{\beta}} \leq C_1 \left( \|v_\lambda\|_1^{\frac{1}{\beta}} + \|v_\lambda\|_\infty^{\frac{1 - \delta}{\beta}} + \lambda^{q-1} \|v_\lambda\|_\infty^{q-2} \|v_\lambda\|_1^{\frac{1}{\beta}} \right)
\]

\[
\leq C_1 \left( \|v_\lambda\|_1^{\frac{1}{\beta}} + a^{1 - \delta} \lambda^{\frac{1 - \delta}{\beta}} + a^{q-2} \lambda \|v_\lambda\|_1^{\frac{1}{\beta}} \right)
\]

(2.12)

\[
\leq C_2 \left( (1 + \lambda) \|v_\lambda\|_1^{\frac{1}{\beta}} + \lambda^{\frac{\delta - 1}{\beta}} \right).
\]

Now, if \(0 < \lambda \leq \min\{1, a(2C_2)^{-\beta}\} := \Lambda_0\), one has

\[
\|v_\lambda\|_1^{\frac{1}{\beta}} \geq \frac{1}{2C_2} \left( \frac{a}{\lambda} \right)^{\frac{1}{\beta}} - \frac{1}{2} \geq \frac{1}{4C_2} \left( \frac{a}{\lambda} \right)^{\frac{1}{\beta}}
\]

and hence (2.10) is fulfilled with \(C^* = a(4C_2)^{-\beta}\). Since of course \(\|v_\lambda\| \to +\infty\) as \(\lambda \to 0^+\), by (2.12) we can determine \(C_3 > 0\) and \(\Lambda_1 \in (0, \Lambda_0]\) such that \(\|v_\lambda\| \geq 1\) and

\[
\|v_\lambda\|_\infty \leq C_3 \|v_\lambda\| \tag{2.13}
\]

for any \(\lambda \in (0, \Lambda_1]\). For the rest of the proof, we assume \(\lambda \in (0, \Lambda_1]\).

**Step 2.** We now show that, writing \(v_\lambda\) as

\[
v_\lambda = t_\lambda \phi_1 + w_\lambda,
\]

with \(t_\lambda \in \mathbb{R}\) and \(w_\lambda \in \text{span}\{\phi_1\}^\perp\), then it holds

\[
\|w_\lambda\|_{C^0(\Omega)} \leq \tilde{C} \|v_\lambda\|_1^{\frac{1}{\beta}}, \tag{2.14}
\]

for some \(\tilde{C} > 0\). By the same arguments as [3], it is easily seen that \(t_\lambda > 0\) and that \(w_\lambda\) is a weak solution to

\[
\begin{cases}
-\Delta u = \lambda_1 u + g(\lambda v_\lambda) & \text{in } \Omega \\
 u = 0 & \text{on } \partial \Omega.
\end{cases} \tag{2.15}
\]

The relation \(I_a'(v_\lambda)(\phi_1) = 0\) and the definition of \(\phi_1\) imply that

\[
\int_\Omega \nabla v_\lambda \nabla \phi_1 dx - \lambda_1 \int_\Omega v_\lambda \phi_1 dx - \int_\Omega g(\lambda v_\lambda) \phi_1 dx = - \int_\Omega g(\lambda v_\lambda) \phi_1 dx = 0,
\]

and therefore

\[
\int_\Omega g(\lambda v_\lambda) w_\lambda dx = \int_\Omega g(\lambda v_\lambda)(v_\lambda - t_\lambda \phi_1) dx = \int_\Omega g(\lambda v_\lambda) v_\lambda dx.
\]
So, we get
\[
\|w_\lambda\|^2 = \lambda_1 \|w_\lambda\|^2 + \int_\Omega g(\lambda v_\lambda) w_\lambda \, dx \\
\leq \frac{\lambda_1}{\lambda_2} \|w_\lambda\|^2 + \int_\Omega g(\lambda v_\lambda) v_\lambda \, dx \\
\leq \frac{\lambda_1}{\lambda_2} \|w_\lambda\|^2 + k_1 \left( \|v_\lambda\|_1 + \lambda^{q-1} \|v_\lambda\|_q^q \right) \\
\leq \frac{\lambda_1}{\lambda_2} \|w_\lambda\|^2 + C_4 \|v_\lambda\|^q,
\]
from which we deduce the estimate
\[
\|w_\lambda\|^2 \leq C_5 \|v_\lambda\|^q
\] (2.16)
being \(C_5 = \frac{\lambda_1 C_4}{\lambda_2 \lambda_1}\). By applying the same arguments as before to the function \(w_\lambda\) and bearing in mind also (2.13) and (2.16), we obtain
\[
\|w_\lambda\|_{C^1(\Omega)} \leq C_6 \left( (\lambda_1 + 1) \|w_\lambda\|_\beta + \|g(\lambda v_\lambda)\|_\beta \right) \\
\leq C_6 \left( (\lambda_1 + 1) \|w_\lambda\|_{C^1(\Omega)} \|v_\lambda\|_1^\frac{\gamma}{\alpha} + 2 \frac{\beta_1}{\gamma} k_1 \left( |\Omega|^\frac{1}{\gamma} + \lambda^{q-1} \|v_\lambda\|_\infty^{q-1-\frac{1}{\gamma}} \|v_\lambda\|_1^{\frac{1}{\gamma}} \right) \right) \\
\leq C_7 \left( \|w_\lambda\|_{C^1(\Omega)} \|v_\lambda\|_1^{\frac{q}{\gamma}} + 1 + \lambda^{q-1} \|v_\lambda\|^{q-1} \right) \\
\leq C_7 \left( \|w_\lambda\|_{C^1(\Omega)} \|v_\lambda\|_1^{\frac{q}{\gamma}} + 2 \|v_\lambda\|^{q-1} \right).
\]
So, either
\[
\|w_\lambda\|_{C^1(\Omega)} \leq 2 C_7 \|w_\lambda\|_{C^1(\Omega)} \|v_\lambda\|_1^{\frac{q}{\gamma}}
\] or
\[
\|w_\lambda\|_{C^1(\Omega)} \leq 4 C_7 \|v_\lambda\|^{q-1}.
\]
In any case, we get
\[
\|w_\lambda\|_{C^1(\Omega)} \leq \tilde{C} \|v_\lambda\|^{\frac{q}{2}}
\] (2.17)
where \(\tilde{C} = 4 C_7\), as desired.

Step 3 (conclusion). Taking (2.10) and (2.16) into account, for \(0 < \lambda \leq \min\{1, \Lambda_0, \Lambda_1, \Lambda_2\}\), where \(\Lambda_2 := \left(\frac{1}{2C_5}\right)^{\frac{1}{q-2}} C^*\), we obtain
\[
t_\lambda^2 \geq \frac{\|v_\lambda\|^2 - C_5 \|v_\lambda\|_q^q}{\|\phi_1\|^2} \geq \frac{\|v_\lambda\|^2}{\|\phi_1\|^2} \left( 1 - \frac{C_5 C^*^{q-2}}{\lambda^{q-2}} \right) \geq \frac{\|v_\lambda\|^2}{2 \|\phi_1\|^2} = C_8 \|v_\lambda\|^2,
\] (2.18)
where \(C_8 = \frac{1}{2\|\phi_1\|^2}\). For this range of \(\lambda\), in view of (2.17), we then obtain
\[
\left\| t_\lambda^{-1} v_\lambda - \phi_1 \right\|_{C^1(\Omega)} = t_\lambda^{-1} \|w_\lambda\|_{C^1(\Omega)} \leq \tilde{C} C_8^{-\frac{1}{2}} \|v_\lambda\|^{\frac{q}{2}-1} \leq C_9 \lambda^{1-\frac{q}{2}}
\]
with \(C_9 = \tilde{C} C_8^{-\frac{1}{2}} C^*^{\frac{1}{2}-1}\). Since \(\phi_1 \in \mathcal{P}\) and \(\mathcal{P}\) is an open subset of \(C^1(\Omega)\), there exists \(\delta > 0\) such that
\[
\{ u \in C^1(\Omega) : \|u - \phi_1\|_{C^1(\Omega)} < \delta \} \subset \mathcal{P}.
\]
So, setting \(\Lambda_3 := \left(\frac{C_9}{\delta}\right)^{\frac{1}{q-2}}\), for all \(0 < \lambda \leq \min\{1, \Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3\} := \Lambda^*\), one has \(t_\lambda^{-1} v_\lambda \in \mathcal{P}\) and hence \(v_\lambda \in \mathcal{P}\). This concludes the proof. \(\square\)
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