IMPROVED BOHR RADIUS FOR THE CLASS OF STARLIKE LOG-HARMONIC MAPPINGS

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ABSTRACT. Let \( H(D) \) be the linear space of analytic functions on the unit disk \( D = \{ z \in \mathbb{C} : |z| < 1 \} \) and let \( B = \{ w \in H(D) : |w(z)| < 1 \} \). The classical Bohr’s inequality states that if a power series \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) converges in \( D \) and \( |f(z)| < 1 \) for \( z \in D \), then
\[
\sum_{n=0}^{\infty} |a_n| r^n \leq 1 \quad \text{for} \quad r \leq \frac{1}{3}
\]
and the constant \( \frac{1}{3} \) is the best possible. The constant \( \frac{1}{3} \) is known as Bohr radius. A function \( f : D \to \mathbb{C} \) is said to be log-harmonic if there is a \( w \in B \) such that \( f \) is a non-constant solution of the non-linear elliptic partial differential equation
\[
\bar{f}z(z)/f(z) = w(z) f_z(z)/f(z).
\]
The class of log-harmonic mappings is denoted by \( S_{LH} \). The set of all starlike log-harmonic mapping is defined by
\[
ST_{LH} = \left\{ f \in S_{LH} : \frac{\partial}{\partial \theta} \text{Arg}(f(e^{i\theta})) = \text{Re} \left( \frac{zf_z - \bar{z}f_{\bar{z}}}{f} \right) > 0 \quad \text{in} \quad D \right\}.
\]
In this paper, we study several improved Bohr radius for the class \( ST_{LH}^0 \), a subclass of \( ST_{LH} \), consisting of functions \( f \in ST_{LH} \) which map the unit disk \( D \) onto a starlike domain (with respect to the origin).

1. INTRODUCTION AND PRELIMINARIES

A complex-valued function \( f \) in \( D \) is said to be harmonic if it satisfies the Laplace equation \( \Delta f = 4f_{z\bar{z}} = 0 \) in \( D \). Every harmonic function \( f \) in \( D \) has the unique canonical form \( f = h + \bar{g} \), where \( h \) and \( g \) are analytic in \( D \) with \( g(0) = 0 \). Every analytic function is a harmonic function. Let \( \mathcal{H} \) be the class of all complex-valued harmonic functions \( f = h + \bar{g} \) defined on \( D \), where \( h \) and \( g \) are analytic in \( D \) with the normalization \( h(0) = h'(0) - 1 = 0 \) and \( g(0) = 0 \). Here \( h \) is called analytic part and \( g \) is called co-analytic part of \( f \).

Harmonic mappings play the natural role in parameterizing minimal surfaces in the context of differential geometry. Planar harmonic mappings have application not only in the differential geometry but also in various field of engineering, physics, operations research and other intriguing aspects of applied mathematics. The theory of harmonic functions has been used to study and solve fluid flow problems [9].

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The theory of univalent harmonic functions having prominent geometric properties like starlikeness, convexity and close-to-convexity appear naturally while dealing with planner fluid dynamical problems. For instance, the fluid flow problem on a convex domain satisfying an interesting geometric property has been extensively studied by Aleman and Constantin [9]. With the help of geometric properties of harmonic mappings, Constantin and Martin [22] have obtained a complete solution of classifying all two dimensional fluid flows.

Let \( \mathcal{H}(\mathbb{D}) \) be the class of analytic functions \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) in the unit disk \( \mathbb{D} \). The origin of the Bohr phenomenon lies in the seminal work by Harald Bohr [21], which include the following result: Let \( f \in \mathcal{H}(\mathbb{D}) \) satisfies \( |f(z)| < 1 \) for all \( z \in \mathbb{D} \), then \( \sum_{n=0}^{\infty} |a_n||z|^n \leq 1 \) for all \( z \in \mathbb{D} \) with \( |z| = r \leq 1/3 \) and the constant 1/3 is the best possible. For \( f \in \mathcal{H}(\mathbb{D}) \), the majorant series is denoted by \( M_f(r) \) and is defined by \( M_f(r) = \sum_{n=0}^{\infty} |a_n||z|^n \). Bohr actually obtained the inequality \( M_f(r) \leq 1 \) for \( |z| \leq 1/6 \), but subsequently later, M. Riesz, I. Schur and F. Weiner, independently established this inequality for \( |z| \leq 1/3 \) and the constant 1/3 cannot be improved [39]. The constant \( r_0 = 1/3 \) is called the Bohr radius and the inequality \( M_f(r) \leq 1 \) is called Bohr inequality for bounded analytic functions in the unit disk \( \mathbb{D} \). Moreover, for the function \( \phi_a \) defined by

\[
\phi_a(z) = \frac{a - z}{1 - az}, \quad a \in [0, 1)
\]

it follows that \( M_{\phi_a}(r) > 1 \) if, and only if, \( r > 1/(1 + 2a) \), for which \( a \to 1 \) shows that 1/3 is optimal.

Using the Euclidian distance \( d \), the Bohr inequality for \( f \in \mathcal{H}(\mathbb{D}) \) can be written as

\[
(1.1) \quad d \left( \sum_{n=0}^{\infty} |a_n z^n|, |a_0| \right) = \sum_{n=1}^{\infty} |a_n z^n| \leq 1 - |f(0)| = d(f(0), \partial \mathbb{D}),
\]

where \( \partial \mathbb{D} \) is the boundary of the unit disk \( \mathbb{D} \).

Let \( \mathcal{M} \) be a class of analytic functions \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) which map the unit disk \( \mathbb{D} \) into a domain \( \Omega \subset \mathbb{C} \). We say the class \( \mathcal{M} \) satisfies Bohr phenomenon if there exists \( r^* \) such that \((1.1) \) holds for \( |z| = r \leq r^* \). The largest such \( r^* \) is called the Bohr radius for the class \( \mathcal{M} \).

In the recent years, studying Bohr inequalities become an interesting topic of research for the functions of one as well as several complex variables. The notion of Bohr inequality has been generalized to several complex variables (see [24, 37]), to planer harmonic mappings (see [23, 51]), to the solutions of elliptic partial differential equations (see [2, 12]), to elliptic equations (see [3]), to vector valued functions and operator valued functions (see [17, 18]), to analytic functions in norm linear spaces (see [14]) and in a more abstract setting (see [7]). In 1977, Boas and Khavinshon [20] extended the Bohr inequality to several complex variables by finding multidimensional Bohr radius. Bohr’s theorem attracted a greater interest after it was used by Dixon [24] in 1995 to characterize Banach algebras that satisfy von Neumann inequality. The generalization of Bohr's
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Theorem become now-a-days an active topic of research. In 2001, Aizenberg et al. [8], and in 2013 Aytuna and Djakov [15] studied the Bohr property of holomorphic functions while Paulsen et al. [32] extended the Bohr inequality to Banach algebra. The relevance between Banach theory and Bohr’s theorem was explored in [19, 23, 24].

Recently, Ali and Ng [11] have extended the classical Bohr inequality in the Poincare disk model of hyperbolic plane. Kayumov and Ponnusamy [31] have determined the Bohr radius for the class of analytic functions

\[ f(z) = z^m \sum_{k=0}^{\infty} a_k p^k, \]

\[ p \geq m \geq 0 \]

with \(|f(z)| \leq 1\). In 2018, Kayumov et al. [30] introduced the idea of \( p \)-Bohr radius for harmonic functions and obtained the \( p \)-Bohr radius for the class of odd harmonic functions. Kayumov et al. [30] have obtained the Bohr radius for the class of analytic Bloch functions and harmonic functions. Alkhaleefah et al. [13] have studied the Bohr radius for the class of quasi-subordinate functions which in particular gives the classical Bohr radius. Number of improved versions of the classical Bohr inequality have been proved in [32].

We now define Bohr radius in subordination and bounded harmonic classes. Let \( f \) and \( g \) be two analytic functions in the unit disk \( D \). We say that \( g \) is subordinate to \( f \) if there exists an analytic function \( \phi : D \rightarrow D \) with \( \phi(0) = 0 \) so that \( g = f \circ \phi \) and it is denoted by \( f \prec g \). If \( g \) is univalent and \( f(0) = g(0) \) then \( f(D) \subset g(D) \). We denote the class of all functions subordinate to a fixed function \( f \) by \( S(f) \) and \( f(D) = \Omega \). The class \( S(f) \) is said to have Bohr’s phenomenon if for any \( g(z) = \sum_{n=0}^{\infty} b_n z^n \in S(f) \) and \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) there is a \( r_0 \) in \((0, 1]\) such that

\[
\sum_{n=1}^{\infty} |b_n z^n| \leq d(f(0), \partial \Omega) \quad \text{for} \quad |z| < r_0.
\]

In 2010, it was established by Abu-Muhanna [3, Theorem] that the class \( S(f) \) has Bohr phenomenon when \( f \) is univalent in \( D \). In particular, the following interesting result was obtained.

**Theorem 1.1.** [3] If \( g(z) = \sum_{n=0}^{\infty} b_n z^n \in S(f) \) and \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) is univalent, then

\[
\sum_{n=1}^{\infty} |b_n z^n| \leq d(f(0), \partial \Omega) \quad \text{for} \quad |z| \leq r_0 = 3 - \sqrt{8} = 0.17157.
\]

Here \( r_0 \) is sharp for the Koebe function \( f_K(z) = z/(1 - z)^2 \).

In [3], Abu-Muhanna has proved the following lemma to find the lower bound of the distance \( d(f(0), \partial \Omega) \).

**Lemma 1.4.** [3] Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be an analytic univalent function from \( D \) onto a simply connected domain \( \Omega \). Then

\[
\frac{1}{4} |f'(0)| \leq d(f(0), \partial \Omega) \leq |f'(0)|.
\]

Next we discuss improved Bohr radius for starlike log-harmonic mappings. A function \( f : D \rightarrow \mathbb{C} \) is said to be log-harmonic if there is a \( w \in \mathcal{B} \) such that \( f \) is a
non-constant solution of the non-linear elliptic partial differential equation

\[(1.6)\quad \frac{f_z(z)}{\bar{f}(z)} = \frac{w(z)f_z(z)}{f(z)},\]

where the second dilation function \(w\) is such that \(|w(z)| < 1\) for all \(z \in \mathbb{D}\). The Jacobian

\[J_f = |f_z|^2 - |\bar{f}_z|^2 = |f_z|^2(1 - |w(z)|^2)\]

is positive, and therefore all the non-constant log-harmonic mappings are sense-preserving and open in \(\mathbb{D}\).

In 2013, Li et al. [34] proved a necessary and sufficient condition for a function to be log-p-harmonic and also studied local log-p-harmonic mappings. Mao et al. [38] have established Schwarz’ lemma for log-harmonic mappings, through which they proved two versions of the Landau’s theorem for these functions. In 2018, Liu and Ponnusamy [35] obtained the coefficient estimates and hence studied Bohr radius for log-harmonic mappings. Inner mapping radius by constructing a family of 1-slit log-harmonic mappings have been established in [35]. Several interesting properties have been established in [35] of log-harmonic mappings. In 2019, Liu and Ponnusamy [36] obtained the precise ranges of log-harmonic Koebe mapping, log-harmonic right half-plane mapping and log-harmonic two-slits mappings. Further, the coefficient estimates for univalent log-harmonic starlike mappings has been established in [36].

Let \(h_0\) and \(g_0\) be two functions defined by

\[(1.7)\quad h_0(z) = \frac{1}{1 - z} \exp\left(\frac{2z}{1 - z}\right) = \exp\left(\sum_{n=1}^{\infty} \left(2 + \frac{1}{n}\right) z^n\right)\]

\[(1.8)\quad g_0(z) = (1 - z) \exp\left(\frac{2z}{1 - z}\right) = \exp\left(\sum_{n=1}^{\infty} \left(2 - \frac{1}{n}\right) z^n\right) .\]

Then the function \(f_0\) defined by

\[(1.9)\quad f_0(z) = zh_0(z)g_0(z) = \frac{z(1 - \bar{z})}{1 - z} \exp\left(\Re\left(\frac{4z}{1 - z}\right)\right) \quad \text{for} \quad z \in \mathbb{D}\]

is the log-harmonic Koebe function.

In 2011, Duman [25] obtained the upper bound for \(|h(z)|\) and \(|g(z)|\). In 2016, Ali et al. [10, Theorem 2] established the sharp lower bounds and exhibited the corresponding extremal functions \(h_0\), \(g_0\) and \(f_0\). Ali et al. [10] extended the Bohr phenomenon to the context of starlike univalent log-harmonic mappings of the form

\[(1.10)\quad f(z) = zh(z)g(z) \quad \text{in} \quad \mathcal{S}T^1_{L,H},\]

and proved the following interesting result.
Theorem 1.2. \cite{10} Let $f$ be a function given by (1.10). Also, let $H(z) = zh(z)$ and $G(z) = zg(z)$. Then

$$
\begin{cases}
\frac{1}{2e} \leq d(0, \partial H(\mathbb{D})) \leq 1 \\
\frac{2}{e} \leq d(0, \partial G(\mathbb{D})) \leq 1 \\
\frac{1}{e^2} \leq d(0, \partial f(\mathbb{D})) \leq 1.
\end{cases}
$$

Equalities occur if, and only if, $h$, $g$ and $f$ are suitable rotation of $h_0$, $g_0$ and $f_0$.

In 1989, Abdulhadi and Hengartner \cite{2} established the sharp coefficient bounds for the function in the class $ST_{LH}^0$.

Theorem 1.3. \cite{2} Let $f$ be a function given by (1.10). Then

$$
|a_n| \leq 2 + \frac{1}{n} \quad \text{and} \quad |b_n| \leq 2 - \frac{1}{n} \quad \text{for all} \; n \geq 1.
$$

Equalities hold for rotation of the function $f_0$.

In 2016, Ali \textit{et al.} \cite{10} obtained Bohr radius for log-harmonic mappings of the class $ST_{LH}^0$.

Theorem 1.4. \cite{10} Let $f(z) = zh(z)g(z) \in ST_{LH}^0$ and $H(z) = zh(z)$ and $G(z) = zg(z)$. Then

(i) the inequality

$$
M_h(r) := |z| \exp \left( \sum_{n=1}^{\infty} |a_n||z|^n \right) \leq d(0, \partial H(\mathbb{D}))
$$

holds for $|z| \leq r_H \approx 0.1222$, where $r_H$ is the unique root in $(0, 1)$ of

$$
\frac{r}{1-r} \exp \left( \frac{2r}{1-r} \right) = \frac{1}{2e}.
$$

(ii) the inequality

$$
M_g(r) := |z| \exp \left( \sum_{n=1}^{\infty} |b_n||z|^n \right) \leq d(0, \partial G(\mathbb{D}))
$$

holds for $|z| \leq r_G \approx 0.3659$, where $r_G$ is the unique root in $(0, 1)$ of

$$
r(1-r) \exp \left( \frac{2r}{1-r} \right) = \frac{2}{e}.
$$

Both the radii are sharp and are attained by appropriate rotation of the functions $H_0(z) = zh_0(z)$ and $G_0(z) = zg_0(z)$. 

Theorem 1.5. \[10\] Let $f$ be a function given by (1.10). Then for any real $t$, the inequality
\[
|z| \exp \left( \sum_{n=1}^{\infty} |a_n + e^{it}b_n||z|^n \right) \leq d(0, \partial f(\mathbb{D}))
\]
holds for $|z| \leq r_f \approx 0.09078$, where $r_f$ is the unique root in $(0,1)$ of
\[
r \exp \left( \frac{4r}{1-r} \right) = \frac{1}{e^2}.
\]
The bound is sharp and is attained by suitable rotation of the log-harmonic Koebe function $f_0$.

Our another interest in this paper is to study Bohr radius for the class of analytic functions $f$ which map unit disk $\mathbb{D}$ into a concave-wedge domain. The concave-wedge domain is defined (see [4]) by
\[
W_\alpha = \left\{ w \in \mathbb{C} : |\arg w| < \frac{\alpha \pi}{2}, \ 1 \leq \alpha \leq 2 \right\}.
\]
It is known that the conformal mapping from $\mathbb{D}$ onto $W_\alpha$ is given by
\[
F_{\alpha,t}(z) = t \left( \frac{1 + z}{1 - z} \right)^\alpha = t \left( 1 + \sum_{n=1}^{\infty} A_n z^n \right) \quad \text{for } 1 \leq \alpha \leq 2 \text{ and } t > 0.
\]
(1.11)

It is easy to see that when $\alpha = 1$, the domain turns out to be a convex half-plane and when $\alpha = 2$ it gives a slit domain. Let $S_{W_\alpha}$ be the class of analytic functions $f$ which maps the unit disk $\mathbb{D}$ into the wedge domain $W_\alpha$.

In 2014, Abu-Muhana et al. [4] proved the following interesting result for functions in the class $S_{W_\alpha}$.

Theorem 1.6. \[4\] Let $\alpha \in [1,2]$. If $f(z) = a_0 + \sum_{n=1}^{\infty} a_n z^n \in S_{W_\alpha}$ with $a_0 > 0$, then the inequality
\[
\sum_{n=1}^{\infty} |a_n||z|^n \leq d(a_0, \partial W_\alpha)
\]
holds for $|z| \leq r_\alpha = (2^{1/\alpha} - 1)/(2^{1/\alpha} + 1)$. The function $f = F_{\alpha,a_0}$ in (1.11) shows that $r_\alpha$ is sharp.

The following lemma is useful to prove one of our main results for functions in the class $S_{W_\alpha}$.

Lemma 1.12. \[4\] Let $F_{\alpha,t}$ be given by (1.11), where $\alpha \in [1,2]$. Then $A_n > 0$ for all $n \geq 1$.

2. Main results

2.1. Bohr radius in subordination and bounded harmonic classes. It is natural to investigate the improved version of the Theorem [11]. We prove the following improved sharp Bohr radius for the class $S(f)$.
Theorem 2.1. Let $\beta \in [0, 1/4)$. If $g(z) = \sum_{n=0}^{\infty} b_n z^n \in \mathcal{S}(f)$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is univalent, then

\begin{equation}
\beta |f'(0)| + \sum_{n=0}^{\infty} |b_n z^n| \leq d(f(0), \partial \Omega) \quad \text{for } |z| \leq r_{\beta} = \frac{3 - 4\beta - \sqrt{8(1 - 2\beta)}}{1 - 4\beta}.
\end{equation}

The radius $r_{\beta}$ is sharp for the Koebe function $f_K(z) = z/(1-z)^2$.

Remark 2.1. In particular, when $\beta = 0$, the radius $r_\beta$ which has been proved in Theorem 2.1 coincides exactly with $r_0 = 3 - \sqrt{8} = 0.17159$ in Theorem 1.1. Further, in particular, we obtain $r_{\beta} = 5 - 2\sqrt{6} \approx 0.10102$ for $\beta = 1/8$, $r_{\beta} = 9 - 4\sqrt{5} \approx 0.05572$ for $\beta = 3/16$ and $r_{\beta} = 17 - 12\sqrt{2} \approx 0.02943$ for $\beta = 7/32$. In fact, we see that $\lim_{\beta \to (1/4)^-} r_{\beta} = 0$.

2.2. Improved Bohr radius for starlike log-harmonic mappings. It is known that if $f$ is a non-vanishing log-harmonic mapping then $f$ can be written as $f(z) = h(z)g(z)$ where $h$ and $g$ are analytic functions in $\mathbb{D}$. On the other hand, if $f$ vanishes at $z = 0$ but is not identically zero, then $f$ admits the following representation

\begin{equation}
f(z) = z^m |z|^{2m} h(z)g(z)
\end{equation}

where $m$ is a non-negative integer and $\text{Re } \beta > -1/2$, and $h$, $g$ are analytic functions in $\mathbb{D}$ with $g(0) = 1$ and $h(0) \neq 1$ (see [1]). The exponent $\beta$ in (2.2) depends only on $w(0)$ and it can be expresses as

$$\beta = \frac{w(0)}{1 - |w(0)|^2}.$$

Note that $f(0) \neq 0$ if, and only if, $m = 0$, and that a univalent log-harmonic mapping on $\mathbb{D}$ vanishes at the origin if, and only if, $m = 1$. Univalent log-harmonic mappings have been studied extensively by many researchers (see [10, 25, 29]). The class of log-harmonic mappings is denoted by $\mathcal{S}_{LH}$. Let $z |z|^{2\beta} h(z)g(z)$ be a log-harmonic univalent function. We say that $f$ is a starlike log-harmonic mapping if

\begin{equation}
\frac{\partial}{\partial \theta} \text{Arg}(f(e^{i\theta})) = \text{Re} \left( \frac{zf_\theta - \bar{z}f_{\bar{z}}}{f} \right) > 0 \quad \text{in } \mathbb{D}
\end{equation}

and we denote the set of all starlike log-harmonic functions by $\mathcal{S}_{LH}^L$. Let $\mathcal{S}_{LH}^{1,0}$ be a subclass of $\mathcal{S}_{LH}$, consisting of functions $f \in \mathcal{S}_{LH}$ which map the unit disk $\mathbb{D}$ onto a starlike domain (with respect to the origin).

Our main aim is to study Bohr radius for the class of sense-preserving starlike log-harmonic mappings in $\mathbb{D}$ of the form $f(z) = zh(z)g(z)$ with

$$h(z) = \exp \left( \sum_{n=1}^{\infty} a_n z^n \right) \quad \text{and} \quad g(z) = \exp \left( \sum_{n=1}^{\infty} b_n z^n \right),$$

where $h(z)$ and $g(z)$ may be called as analytic and co-analytic factors of the function $f(z)$.

We prove the following improved Bohr radius for functions in the class $\mathcal{S}_{LH}^{1,0}$.
Theorem 2.2. Let $f$ be a function given by (1.10). Then for any real $t$, the inequality
\[ |z| \exp \left( \sum_{n=1}^{\infty} \left| a_n + e^{it} b_n + \frac{n}{4n^2 - 1} a_n b_n \right| z^n \right) \leq d(0, \partial f(\mathbb{D})) \]
holds for $|z| \leq r_f \approx 0.08528$, where $r_f$ is the unique root in $(0,1)$ of
\[(2.4) \quad \frac{r}{1 - r} \exp \left( \frac{4r}{1 - r} \right) = \frac{1}{e^2} \text{ in } (0,1). \]

The radius $r_f$ is sharp and is attained by a suitable rotation of the log-harmonic Koebe function $f_0$ given by (1.9).

**Figure 1.** The radius $r_f \approx 0.08528$ is a root of (2.4) in $(0,1)$.

**Figure 2.** Image of unit disk $\mathbb{D}$ under the Koebe function $f(z) = \frac{z}{(1-z)^2}$ and log-harmonic Koebe function $f_0(z) = \frac{z(1-z)}{1-z} \exp \left( \text{Re} \left( \frac{1}{1-z} \right) \right)$.

Next we prove the sharp Bohr radius for the class $ST_{LH}^0$ in view of additional terms $|a_n|^2$ and $|b_n|^2$ in the series expansion of $h$ and $g$ respectively.
Figure 3. Image of unit disk $\mathbb{D}$ under the map $h_0(z) = \frac{1}{1-z} \exp\left(\frac{2z}{1-z}\right)$ and $g_0(z) = (1-z) \exp\left(\frac{2z}{1-z}\right)$.

Theorem 2.3. Let $f$ be a function given by (1.10) and $H(z) = zh(z)$ and $G(z) = zg(z)$. Then

(i) the inequality

$$|z| \exp\left(\sum_{n=1}^{\infty} \left|a_n\right| + \frac{n}{(2n+1)^2} \left|a_n\right|^2 \right) \leq d(0, \partial H(\mathbb{D}))$$

holds for $|z| \leq r_H \approx 0.09735$, where $r_H$ is the unique root of

$$r \left(1 - r\right)^2 \exp\left(\frac{2r}{1-r}\right) = \frac{1}{2e} \text{ in } (0, 1).$$

Figure 4. The radii $r_H \approx 0.09735$ and $r_G \approx 0.30539$ are roots of (2.5) and (2.6) respectively in $(0, 1)$. 

the inequality
\[ |z| \exp \left( \sum_{n=1}^{\infty} \left( |b_n| + \frac{n}{(2n-1)^2} |b_n|^2 \right) |z|^n \right) \leq d(0, \partial G(\mathbb{D})) \]
holds for \( |z| \leq r_G \approx 0.30539 \), where \( r_G \) is the unique root of
\[ r \exp \left( \frac{2r}{1-r} \right) = \frac{2}{e} \quad \text{in} \quad (0, 1). \]

Both the radii are sharp and are attained by appropriate rotation of \( H_0(z) = zh_0(z) \) and \( G_0(z) = zg_0(z) \).

We prove the next improved sharp Bohr radius for the class \( ST^0_{LH} \) adding \( |H(z)| \) and \( |G(z)| \) with \( M_h(r) \) and \( M_g(r) \) respectively.

**Theorem 2.4.** Let \( f \) be a function given by (1.10) and \( H(z) = zh(z) \) and \( G(z) = zg(z) \). Then

(i) the inequality
\[ |H(z)| + |z| \exp \left( \sum_{n=1}^{\infty} |a_n||z|^n \right) \leq d(0, \partial H(\mathbb{D})) \]
holds for \( |z| \leq r_H \approx 0.1073 \), where \( r_H \) is the unique root of
\[ r \left( \frac{2r}{1-r} - \log(1-r) + \frac{1}{1-r} \exp \left( \frac{2r}{1-r} \right) \right) = \frac{1}{2e} \quad \text{in} \quad (0, 1). \]

(ii) the inequality
\[ |G(z)| + |z| \exp \left( \sum_{n=1}^{\infty} |b_n||z|^n \right) \leq d(0, \partial G(\mathbb{D})) \]
holds for \( |z| \leq r_G \approx 0.3063 \), where \( r_G \) is the unique root of
\[ r \left( \frac{2r}{1-r} + \log(1-r) + (1-r) \exp \left( \frac{2r}{1-r} \right) \right) = \frac{2}{e} \quad \text{in} \quad (0, 1). \]

Both the radii are sharp and are attained by appropriate rotation of \( H_0(z) = zh_0(z) \) and \( G_0(z) = zg_0(z) \).

For any positive integer \( m \), considering \( |h(z)|^m \) and \( |g(z)|^m \), next we prove the improved sharp Bohr radius for the class \( ST^0_{LH} \).

**Theorem 2.5.** Let \( f \) be a function given by (1.10) and \( H(z) = zh(z) \) and \( G(z) = zg(z) \).

(i) If \( |h(z)| \leq 1 \), then for any \( m \in \mathbb{N} \), the inequality
\[ |z| \exp \left( |h(z)|^m + \sum_{n=1}^{\infty} |a_n||z|^n \right) \leq d(0, \partial H(\mathbb{D})) \]
Figure 5. The radii $r_H \approx 0.1073$ and $r_G \approx 0.3063$ are roots of (2.7) and (2.8) respectively in $(0, 1)$.

holds for $|z| \leq r_H \approx 0.0566$, where $r_H$ is the unique root of

$$re\frac{2r}{1-r} \exp\left(\frac{2r}{1-r}\right) = \frac{1}{2e} \quad \text{in} \quad (0, 1).$$

(ii) If $|g(z)| \leq 1$, then for any $m \in \mathbb{N}$, the inequality

$$|z| \exp\left(|g(z)|^m + \sum_{n=1}^{\infty} |b_n||z|^n\right) \leq d(0, \partial G(\mathbb{D}))$$

holds for $|z| \leq r_G \approx 0.1764$, where $r_G$ is the unique root of

$$re(1 - r) \exp\left(\frac{2r}{1-r}\right) = \frac{2}{e} \quad \text{in} \quad (0, 1).$$

Both the radii are sharp and are attained by a suitable rotation of $H_0(z) = zh_0(z)$ and $G_0(z) = zg_0(z)$.

Figure 6. The radii $r_H \approx 0.0566$ and $r_G \approx 0.1764$ are roots of (2.9) and (2.10) respectively in $(0, 1)$.

Remark 2.2. It is worth to notice in Theorem 2.5 that the Bohr radii $r_H$ and $r_G$ are independent of the choice of the positive integer $m$. 
We prove the improved sharp Bohr radius adding $|h(z) + g(z)|$ with the series $\sum_{n=1}^{\infty} |a_n + e^{it}b_n||z|^n$ for the class $ST_{LH}^0$.

**Theorem 2.6.** Let $f$ be a function given by (1.10) with $|h(z)| + |g(z)| \leq 1$. Then for any real $t$, the inequality

$$|z| \exp \left( |h(z) + g(z)| + \sum_{n=1}^{\infty} |a_n + e^{it}b_n||z|^n \right) \leq d(0, \partial f(D))$$

holds for $|z| \leq r_f \approx 0.04181$, where $r_f$ is the unique root of

$$er \exp \left( \frac{4r}{1-r} \right) = \frac{1}{e^2} \text{ in } (0,1).$$

The Bohr radius $r_f$ is sharp and is attained by suitable rotation of the log-harmonic Koebe function $f_0$.

Next we prove the improved sharp Bohr radius for the class $ST_{LH}^0$ adding $|f(z)|$.

**Theorem 2.7.** Let $f$ be a function given by (1.10) with $|h(z)| \leq 1$ and $|g(z)| \leq 1$. Then for any real $t$, the inequality

$$|f(z)| + |z| \exp \left( \sum_{n=1}^{\infty} |a_n + e^{it}b_n||z|^n \right) \leq d(0, \partial f(D))$$

holds for $|z| \leq r_f \approx 0.0592$, where $r_f$ is the unique root of

$$r \left( 1 + \exp \left( \frac{4r}{1-r} \right) \right) = \frac{1}{e^2} \text{ in } (0,1).$$

The Bohr radius $r_f$ is sharp for a suitable rotation of the log-harmonic Koebe function $f_0$.

![Figure 7](image.png)

**Figure 7.** The radius $r_f \approx 0.0592$ is the root of (2.12) in $(0,1)$. 
2.3. **Bohr radius for concave-wedge domain.** We prove the following result which is an improved version of Theorem 1.6.

**Theorem 2.8.** Let \( \alpha \in [1, 2] \) and \( \beta \in [0, 2) \). If \( f(z) = a_0 + \sum_{n=1}^{\infty} a_n z^n \in S_{W, \alpha} \) with \( a_0 > 0 \), then the inequality

\[
\frac{2\beta a_0}{\alpha \pi} |\arg f(z)| + \sum_{n=1}^{\infty} |a_n| |z|^n \leq d(a_0, \partial W, \alpha)
\]

holds for \( |z| \leq r_{\alpha, \beta} = ((2 - \beta)^{1/\alpha} - 1)/((2 - \beta)^{1/\alpha} + 1) \). The function \( f = F_{\alpha, a_0} \) in (1.11) shows that \( r_{\alpha, \beta} \) is sharp.

**Remark 2.3.** Since \( W_{\alpha} \) turns out to be a convex half-plane when \( \alpha = 1 \), it is evident that, for \( \alpha = 1 \) and \( \beta = 0 \), the radius \( r_{\alpha, \beta} \) coincides exactly with the Bohr radius 1/3.

![Figure 8](image-url)

**Figure 8.** Image of unit disk \( \mathbb{D} \) under the maps \( F_{1,1}(z) \), \( F_{1.5,20}(z) \) and \( F_{2,3}(z) \) respectively.

### 3. Proof of the Main Results

**Proof of Theorem 2.1.** Since \( g(z) = \sum_{n=0}^{\infty} b_n z^n \in S(f) \) and \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) are univalent, by the famous well-known de Brande's theorem [28, p.17], we have

\[
|b_n| \leq n |f'(0)|.
\]

Therefore, from (1.5) and (3.1), it is easy to see that

\[
|b_n| \leq 4n d(f(0), \partial \Omega).
\]
By a simple computation using (3.2), we obtain

\[ \beta |f'(0)| + \sum_{n=1}^{\infty} |b_n z^n| = \beta |f'(0)| + \sum_{n=1}^{\infty} |b_n| |z^n| \]

\[ \leq 4\beta d(f(0), \partial \Omega) + 4d(f(0), \partial \Omega) \sum_{n=1}^{\infty} nr^n \]

\[ = 4d(f(0), \partial \Omega) \left( \beta + \frac{r}{(1-r)^2} \right) \]

\[ \leq d(f(0), \partial \Omega) \]

if, and only if,

\[ 4 \left( \beta + \frac{r}{(1-r)^2} \right) \leq 1. \]

Therefore, (2.1) holds for

\[ |z| \leq r_0 = \frac{3 - 4\beta - \sqrt{8(1 - 2\beta)}}{1 - 4\beta}. \]

Since

\[ f_K(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} nz^n \text{ and } d(f(0), \partial \Omega) = \frac{1}{4}, \]

a simple computation shows that

\[ \beta |f'(0)| + \sum_{n=1}^{\infty} |b_n z^n| = \beta |f'_K(0)| + \sum_{n=1}^{\infty} |b_n|r_0^n \]

\[ = 4\beta d(f_K(0), \partial \Omega) + 4d(f_K(0), \partial \Omega) \sum_{n=1}^{\infty} nr_0^n \]

\[ = 4d(f_K(0), \partial \Omega) \left( \beta + \frac{r_0}{(1-r_0)^2} \right) \]

\[ = d(f(0), \partial \Omega). \]

This shows that the radius \( r_0 \) is the best possible. This completes the proof. \( \square \)

**Proof of Theorem 2.2.** Let \( f(z) = zh(z)g(z) \in \mathcal{S}T^0_{LH} \). Then in view of Theorem 1.3, we have the following coefficient bounds

\[ |a_n| \leq 2 + \frac{1}{n} \quad \text{and} \quad |b_n| \leq 2 - \frac{1}{n} \quad \text{for all } n \geq 1. \]

On the other hand, from Theorem 1.2, we have

\[ d(0, \partial f(\mathbb{D})) \geq \frac{1}{c^2}. \]
Therefore, a simple computation shows that
\[
\begin{align*}
\exp \left( \sum_{n=1}^{\infty} |a_n|r^n + \sum_{n=1}^{\infty} |b_n|r^n + \sum_{n=1}^{\infty} \frac{n}{4n^2 - 1} |a_n| |b_n|r^n \right) & \\
\leq \exp \left( \sum_{n=1}^{\infty} \left( 2 + \frac{1}{n} \right) r^n + \sum_{n=1}^{\infty} \left( 2 - \frac{1}{n} \right) r^n + \sum_{n=1}^{\infty} \frac{n}{4n^2 - 1} \left( 2 + \frac{1}{n} \right) \left( 2 - \frac{1}{n} \right) r^n \right) & \\
= \exp \left( 4 \sum_{n=1}^{\infty} r^n + \sum_{n=1}^{\infty} \frac{r^n}{n} \right) & \\
= \exp \left( \frac{4r}{1 - r} - \log(1 - r) \right) & \\
\leq d(0, \partial f(\mathbb{D}))
\end{align*}
\]
if, and only if,
\[
\exp \left( \frac{4r}{1 - r} - \log(1 - r) \right) \leq \frac{1}{e^2}
\]
which is equivalent to
\[
\frac{r}{1 - r} \exp \left( \frac{4r}{1 - r} \right) \leq \frac{1}{e^2}.
\]
The Bohr radius \(r_f\) is the unique root of the equation
\[
\frac{r}{1 - r} \exp \left( \frac{4r}{1 - r} \right) = \frac{1}{e^2}
\]
in \((0, 1)\), a simple computation shows that \(r_f \approx 0.08528\).

In order to show the sharpness of \(r_f\), let \(h_0, g_0\) and \(f_0\) be given by (1.7), (1.8) and (1.9) respectively. For these functions, it is easy to see that
\[
|a_n| = 2 + \frac{1}{n}, \quad |b_n| = 2 - \frac{1}{n} \quad \text{for all} \quad n \in \mathbb{N} \quad \text{and} \quad d(0, \partial f_0(\mathbb{D})) = \frac{1}{e^2}.
\]

A simple computation using (3.3) shows that
\[
\begin{align*}
\exp \left( \sum_{n=1}^{\infty} |a_n|r_f^n + \sum_{n=1}^{\infty} |b_n|r_f^n + \sum_{n=1}^{\infty} \frac{n}{4n^2 - 1} |a_n| |b_n|r_f^n \right) & \\
= \exp \left( \sum_{n=1}^{\infty} \left( 2 + \frac{1}{n} \right) r_f^n + \sum_{n=1}^{\infty} \left( 2 - \frac{1}{n} \right) r_f^n + \sum_{n=1}^{\infty} \frac{n}{4n^2 - 1} \left( 2 + \frac{1}{n} \right) \left( 2 - \frac{1}{n} \right) r_f^n \right) & \\
= \exp \left( 4 \sum_{n=1}^{\infty} r_f^n + \sum_{n=1}^{\infty} \frac{r_f^n}{n} \right) & \\
= \exp \left( \frac{4r_f}{1 - r_f} - \log(1 - r_f) \right) & \\
= \frac{1}{e^2} & \\
= d(0, \partial f_0(\mathbb{D})).
\end{align*}
\]
Therefore, the radius $r_f$ is the best possible. \qed

**Proof of Theorem 2.3.** Let $f(z) = zh(z)g(z) \in \mathcal{ST}^0_{LH}$, $H(z) = zh(z)$ and $G(z) = zg(z)$. (i) In view of Theorem 1.2 and Theorem 1.3, we have the following sharp coefficient estimates

$$|a_n| \leq 2 + \frac{1}{n} \quad \text{for all } n \geq 1 \quad \text{and } d(0, \partial H(\mathbb{D})) \geq \frac{1}{2e}$$

and equality holds for the function $h_0(z)$ and $H(z) = zh_0(z)$ respectively. A simple computation shows that

$$|z| \exp \left( \sum_{n=1}^{\infty} \left( |a_n| + \frac{n}{(2n+1)^2} |a_n|^2 \right) |z|^n \right) \leq r \exp \left( 2 \sum_{n=1}^{\infty} r^n + 2 \sum_{n=1}^{\infty} \frac{r^n}{n} \right)$$

$$= r \exp \left( \frac{2r}{1-r} - 2 \log(1-r) \right)$$

$$= \frac{r}{(1-r)^2} \exp \left( \frac{2r}{1-r} \right)$$

$$\leq d(0, \partial H(\mathbb{D}))$$

if, and only if,

$$\frac{r}{(1-r)^2} \exp \left( \frac{2r}{1-r} \right) \leq \frac{1}{2e}.$$ 

Therefore, the Bohr radius $r_H$ is the unique root of the equation

$$\frac{r}{(1-r)^2} \exp \left( \frac{2r}{1-r} \right) = \frac{1}{2e}$$

in $(0, 1)$, a computation shows that $r_H \approx 0.09735$.

To show the sharpness of the radius $r_H$, let $H_0(z) = zh_0(z)$, where $h_0$ is given by (1.7). It is easy to see that

$$|z| \exp \left( \sum_{n=1}^{\infty} \left( |a_n| + \frac{n}{(2n+1)^2} |a_n|^2 \right) |z|^n \right) = r_H \exp \left( 2 \sum_{n=1}^{\infty} r_H^n + 2 \sum_{n=1}^{\infty} \frac{r_H^n}{n} \right)$$

$$= r \exp \left( \frac{2r}{1-r_H} - 2 \log(1-r_H) \right)$$

$$= \frac{r_H}{(1-r_H)^2} \exp \left( \frac{2r_H}{1-r_H} \right)$$

$$= \frac{1}{2e}$$

$$= d(0, \partial H_0(\mathbb{D})).$$
Therefore, the radius $r_H$ is the best possible.

(ii) In view of Theorem 1.2 and Theorem 1.3, we have

$$|b_n| \leq 2 - \frac{1}{n} \quad \text{for all } n \geq 1 \quad \text{and} \quad d(0, \partial G(D)) \geq \frac{2}{e}.$$ 

Both the equalities hold for the function $g_0(z)$ and $G_0(z) = zg_0(z)$ respectively.

A simple computation shows that

$$|z| \exp \left( \sum_{n=1}^{\infty} \left( |b_n| + \frac{n}{(2n-1)^2} |b_n|^2 \right) |z|^n \right)$$

$$\leq r \exp \left( 2 \sum_{n=1}^{\infty} r^n \right)$$

$$= r \exp \left( \frac{2r}{1-r} \right)$$

$$\leq d(0, \partial G(D))$$

if, and only if,

$$r \exp \left( \frac{2r}{1-r} \right) \leq \frac{2}{e}.$$ 

Therefore, the Bohr radius $r_G$ is the unique root of the equation

$$r \exp \left( \frac{2r}{1-r} \right) = \frac{2}{e}$$

in $(0, 1)$ which can be computed as $r_G \approx 0.30539$.

To show the sharpness of the radius $r_G$, let $G_0(z) = zg_0(z)$, where $g_0$ is given by (1.7). It is easy to see that

$$|z| \exp \left( \sum_{n=1}^{\infty} \left( |b_n| + \frac{n}{(2n-1)^2} |b_n|^2 \right) |z|^n \right)$$

$$= r_G \exp \left( 2 \sum_{n=1}^{\infty} r^n_G \right)$$

$$= r_G \exp \left( \frac{2r_G}{1-r_G} \right)$$

$$= \frac{2}{e}$$

$$= d(0, \partial G_0(D)).$$

Therefore, the radius $r_G$ is the best possible. \hfill \Box

**Proof of Theorem 2.5.** Let $f(z) = zh(z)g(z) \in ST_{LH}^0$, $H(z) = zh(z)$ and $G(z) = zg(z)$.
(i) In view of Theorem 1.2 and Theorem 1.3 and using the fact that \(|h(z)| \leq 1\), for \(m \in \mathbb{N}\), we obtain
\[
|z| \exp \left( |h(z)|^m + \sum_{n=1}^{\infty} |a_n||z|^n \right) = |z| \exp(|h(z)|^m) \exp \left( \sum_{n=1}^{\infty} |a_n||z|^n \right)
\]
\[
\leq re \exp \left( \frac{2r}{1-r} - 2 \log(1-r) \right)
\]
\[
= \frac{re}{1-r} \exp \left( \frac{2r}{1-r} \right)
\]
\[
\leq d(0, \partial H(\mathbb{D}))
\]
if, and only if,
\[
\frac{re}{1-r} \exp \left( \frac{2r}{1-r} \right) \leq \frac{1}{2e}.
\]
Therefore, the Bohr radius \(r_H\) is the solution of
\[
\frac{er}{1-r} \exp \left( \frac{2r}{1-r} \right) = \frac{1}{2e}.
\]
It is easy to see that \(r_H \approx 0.0566\). The radius \(r_H\) is best possible and it can be shown by using the function \(H_0(z) = zh_0(z)\).

(ii) Since \(|g(z)| \leq 1\), in view of Theorem 1.2 and Theorem 1.3, by a simple computation, we obtain
\[
|z| \exp \left( |g(z)|^m + \sum_{n=1}^{\infty} |b_n||z|^n \right) = |z| \exp(|g(z)|^m) \exp \left( \sum_{n=1}^{\infty} |b_n||z|^n \right)
\]
\[
\leq re \exp \left( \sum_{n=1}^{\infty} \left( 2 - \frac{1}{n} \right) r^n \right)
\]
\[
= re \exp \left( \frac{2r}{1-r} + \log(1-r) \right)
\]
\[
= re(1-r) \exp \left( \frac{2r}{1-r} \right)
\]
\[
\leq d(0, \partial G(\mathbb{D}))
\]
if, and only if,
\[
re(1-r) \exp \left( \frac{2r}{1-r} \right) \leq \frac{2}{e}.
\]
Therefore, the Bohr radius \(r_G\) is the unique root of the equation
\[
re(1-r) \exp \left( \frac{2r}{1-r} \right) = \frac{2}{e}.
\]
A simple computation shows \(r_G \approx 0.1764\). The radius \(r_G\) is the best possible which can be shown by considering the function \(G_0(z) = zg_0(z)\), where \(g_0(z)\) is defined in (1.8). □
Proof of Theorem 2.6. In view of Theorem 1.3, we have the following sharp coefficient bounds

\[ |a_n| \leq 2 + \frac{1}{n} \quad \text{and} \quad |b_n| \leq 2 - \frac{1}{n} \quad \text{for all } n \geq 1, \]

which are attained by the function \( h_0 \) and \( g_0 \) respectively defined in (1.7) and (1.8). On the other hand, by Theorem 1.2, we have the sharp distance

\[ d(0, \partial f(\mathbb{D})) \geq \frac{1}{e^2} \]

which is attained by the function \( f_0 \) defined in (1.9).

Since \(|h(z)| + |g(z)| \leq 1\), we obtain

\[
|z| \exp \left( |h(z) + g(z)| + \sum_{n=1}^{\infty} |a_n||z|^n + \sum_{n=1}^{\infty} |b_n||z|^n \right) \\
= r \exp (|h(z)| + |g(z)|) \exp \left( \sum_{n=1}^{\infty} |a_n|r^n + \sum_{n=1}^{\infty} |b_n|r^n \right) \\
\leq er \exp \left( 4 \sum_{n=1}^{\infty} |a_n|r^n \right) \\
= er \exp \left( \frac{4r}{1 - r} \right) \\
\leq d(0, \partial f(\mathbb{D}))
\]

if, and only if,

\[ er \exp \left( \frac{4r}{1 - r} \right) \leq \frac{1}{e^2}. \]

Therefore, the Bohr radius \( r_f \) is the unique root of the equation

\[ er \exp \left( \frac{4r}{1 - r} \right) = \frac{1}{e^2} \]

in \((0, 1)\) which yields \( r_f \approx 0.04181 \). The sharpness of the radius \( r_f \) can be shown by considering a suitable rotation of the log-harmonic Koebe function \( f_0 \). This completes the proof. \( \Box \)
Proof of Theorem 2.7. Since $|h(z)| \leq 1$ and $|g(z)| \leq 1$, in view of Theorem 1.2 and Theorem 1.3, we obtain

$$|f(z)| + |z| \exp \left( \sum_{n=1}^{\infty} |a_n||z|^n + \sum_{n=1}^{\infty} |b_n||z|^n \right)$$

$$\leq r |h(z)||g(z)| + r \exp \left( \sum_{n=1}^{\infty} |a_n|r^n + \sum_{n=1}^{\infty} |b_n|r^n \right)$$

$$\leq r + r \exp \left( 4 \sum_{n=1}^{\infty} \frac{r^n}{r} \right)$$

$$= r + r \exp \left( \frac{4r}{1-r} \right)$$

$$\leq d(0, \partial f(\mathbb{D}))$$

if, and only if,

$$r \left( 1 + \exp \left( \frac{4r}{1-r} \right) \right) \leq \frac{1}{e^2}.$$ 

Thus, the Bohr radius $r_f$ is the unique root of the equation

$$r \left( 1 + \exp \left( \frac{4r}{1-r} \right) \right) = \frac{1}{e^2}$$

which gives $r_f \approx 0.0592$. The radius $r_f$ is sharp and can be shown by considering a suitable rotation of the log-harmonic Koebe function $f_0$. This completes the proof.

Proof of Theorem 2.4. By Theorem 1.2, we have

(3.4) \hspace{1cm} d(0, \partial H(\mathbb{D})) \geq \frac{1}{2e} \quad \text{and} \quad d(0, \partial G(\mathbb{D})) \geq \frac{2}{e}

and the sharp coefficient bounds

$$|a_n| \leq 2 + \frac{1}{n} \quad \text{and} \quad |b_n| \leq 2 - \frac{1}{n} \quad \text{for all} \ n \geq 1.$$ 

All the inequalities are attained by the extremal functions $H_0(z) = zh_0(z)$ and $G_0(z) = zg_0(z)$, where $h_0$ and $g_0$ are defined respectively in (1.7) and (1.8).

(i) Using Theorem 1.3 and (3.4), we obtain

$$|H(z)| + |z| \exp \left( \sum_{n=1}^{\infty} |a_n||z|^n \right) \leq |z| \left( \sum_{n=1}^{\infty} |a_n||z|^n \right) + |z| \exp \left( \sum_{n=1}^{\infty} |a_n||z|^n \right)$$

$$\leq r \left( \sum_{n=1}^{\infty} \left( 2 + \frac{1}{n} \right) r^n \right) + r \exp \left( \sum_{n=1}^{\infty} \left( 2 + \frac{1}{n} \right) r^n \right)$$

$$= r \left( \frac{2r}{1-r} - \log(1-r) \right) + r \exp \left( \frac{2r}{1-r} - \log(1-r) \right)$$

$$\leq d(0, \partial H(\mathbb{D}))$$
Improved Bohr radius for the class of starlike log-harmonic mappings

if, and only if,
\[ r \left( \frac{2r}{1 - r} - \log(1 - r) \right) + r \exp \left( \frac{2r}{1 - r} - \log(1 - r) \right) \leq \frac{1}{2e}. \]

Therefore, the Bohr radius \( r_H \) is the unique root of the equation
\[ r \left( \frac{2r}{1 - r} - \log(1 - r) \right) + r \exp \left( \frac{2r}{1 - r} - \log(1 - r) \right) = \frac{1}{2e} \]
in \((0, 1)\) which shows that \( r_H \approx 0.1073 \). In order to show the radius \( r_H \) is sharp, we consider the function \( H_0(z) = zh_0(z) \), where \( h_0 \) is defined in \([1.7]\). Therefore, a simple computation shows that
\[
|H_0(z)| + |z| \exp \left( \sum_{n=1}^{\infty} |a_n||z|^n \right) = \frac{1}{2e} = d(0, \partial H(\mathbb{D})).
\]
This shows that \( r_H \) is the best possible.

(ii) Using Theorem 1.3 and (3.4), we obtain
\[
|G(z)| + |z| \exp \left( \sum_{n=1}^{\infty} |b_n||z|^n \right) \leq |z| \left( \sum_{n=1}^{\infty} |b_n||z|^n \right) + |z| \exp \left( \sum_{n=1}^{\infty} |b_n||z|^n \right)
\leq r \left( \sum_{n=1}^{\infty} \left( 2 - \frac{1}{n} \right) r^n \right) + r \exp \left( \sum_{n=1}^{\infty} \left( 2 + \frac{1}{n} \right) r^n \right)
= r \left( \frac{2r}{1 - r} + \log(1 - r) \right) + r \exp \left( \frac{2r}{1 - r} + \log(1 - r) \right)
\leq d(0, \partial H(\mathbb{D})).
\]
if, and only if,
\[ r \left( \frac{2r}{1 - r} + \log(1 - r) \right) + r \exp \left( \frac{2r}{1 - r} + \log(1 - r) \right) \leq \frac{1}{2e}. \]
Therefore, the Bohr radius $r_G$ is the unique root of the equation
\[ r \left( \frac{2r}{1-r} - \log(1-r) \right) + r \exp \left( \frac{2r}{1-r} - \log(1-r) \right) = \frac{2}{e} \]
in $(0, 1)$ which yields $r_G \approx 0.3063$. To show the radius $r_G$ is best possible, we consider the function $G_0(z) = zg_0(z)$, where $g_0$ is defined in (1.8).

Thus, it is easy to see that
\[
|G_0(z)| + |z| \exp \left( \sum_{n=1}^{\infty} |b_n||z|^n \right) = |z| \left( \sum_{n=1}^{\infty} |b_n||z|^n \right) + |z| \exp \left( \sum_{n=1}^{\infty} |b_n||z|^n \right) = r_G \left( \sum_{n=1}^{\infty} \left( 2 - \frac{1}{n} \right) r_G^n \right) + r_G \exp \left( \sum_{n=1}^{\infty} \left( 2 + \frac{1}{n} \right) r_G^n \right) = r_G \left( \frac{2r_G}{1-r_G} + \log(1-r_G) \right) + r_G \exp \left( \frac{2r_G}{1-r_G} + \log(1-r_G) \right) = \frac{2}{e} = d(0, \partial G(D))
\]
which shows that $r_G$ is sharp. \qed

**Proof of Theorem 2.8.** Since $f \in S_{W_{\alpha}}$, we have $|\arg f(z)| \leq \pi \alpha/2$. Therefore, in view of Lemma 1.4, it is easy to see that
\[
\frac{2 \beta a_0}{\pi \alpha} |\arg f(z)| + \sum_{n=1}^{\infty} |a_n z^n| \leq \frac{2 \beta a_0}{\pi \alpha} \cdot \frac{\pi \alpha}{2} + a_0 \sum_{n=1}^{\infty} A_n r^n
\]
\[
= a_0 \left( \sum_{n=1}^{\infty} A_n r^n + \beta \right) = a_0 \left( \left( \frac{1+r}{1-r} \right)^{\alpha} - 1 + \beta \right) = d(a_0, \partial W_a) \left( \left( \frac{1+r}{1-r} \right)^{\alpha} - 1 + \beta \right) \leq d(a_0, \partial W_a)
\]
if, and only if,
\[
\left( \frac{1+r}{1-r} \right)^{\alpha} - 1 + \beta \leq 1.
\]
Therefore, the Bohr radius is the unique root of the equation
\[
\left( \frac{1+r}{1-r} \right)^{\alpha} - 1 + \beta = 1.
\]
A simple computation shows that \( r_{\alpha,\beta} = \frac{(2 - \beta)^{1/\alpha} - 1}{(2 - \beta)^{1/\alpha} + 1} \). By a suitable rotation of the function \( f = F_{\alpha,a_0} \) in (1.11), it can be shown that the radius \( r_{\alpha,\beta} \) is sharp. This completes the proof. \( \square \)

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