Global Solution of the Electromagnetic Field-Particle System of Equations.

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In this paper we discuss global existence of the solution of the Maxwell and Newton system of equations, describing the interaction of a rigid charge distribution with the electromagnetic field it generates. A unique solution is proved to exist (for regular charge distributions) on suitable homogeneous and non-homogeneous Sobolev spaces, for the electromagnetic field, and on coordinate and velocity space for the charge; provided initial data belong to the subspace that satisfies the divergence part of Maxwell’s equations.

I. INTRODUCTION.

We are interested in the following system of equations: let \( e \in \mathbb{R} \), \( \varphi \) a sufficiently regular function; then the Maxwell-Newton equations are written in three spatial dimensions as

\[
\begin{aligned}
\partial_t B + \nabla \times E &= 0 \\
\partial_t E - \nabla \times B &= -j \\
\dot{\xi} &= v \\
\dot{v} &= e[(\varphi \ast E)(\xi) + v \times (\varphi \ast B)(\xi)]
\end{aligned}
\]

with

\[
j = ev\varphi(\xi - x), \quad \rho = e\varphi(\xi - x).
\]

This system can be used to describe motion of a non-relativistic rigid particle, with an extended charge distribution \( e \varphi \), interacting with its own electromagnetic field (in this case we could need some additional physical conditions, such as \( \int dx \varphi = 1 \), however these conditions are not necessary for the existence of a solution). So \( \xi, v \in \mathbb{R}^3 \) will be the position and velocity of the charge’s center of mass, \( E, B \) the electric and magnetic field vectors. We remark that in (M-N) charge is conserved, i.e.

\[
\partial_t \rho + \nabla \cdot j = 0.
\]
It is useful to construct the electromagnetic tensor $F^{\mu\nu}$:

$$F^{\mu\nu} = \begin{pmatrix}
0 & E_1 & E_2 & E_3 \\
-E_1 & 0 & B_3 & -B_2 \\
-E_2 & -B_3 & 0 & B_1 \\
-E_3 & B_2 & -B_1 & 0
\end{pmatrix}.$$ 

Therefore we make the following identifications:

$$E_j = \sum_{j=1}^{3} \delta_{ij} F_{0j}^{\mu},$$

$$B_j = \frac{1}{2} \sum_{k,l=1}^{3} \epsilon_{jkl} F^{kl};$$

where $\delta_{ij}$ is the Kronecker's delta and $\epsilon_{ijk}$ is the three-dimensional Levi-Civita symbol. From now on, we adopt the following notation: whenever an index is repeated twice, a summation over all possible values of such index is intended. Define $R_{kl}^{ij} = -\delta_{ik} \partial_k + \delta_{ij} \partial_l$ and $(R^*)_{kl}^{ij} = \delta_{il} \partial_k - \delta_{ik} \partial_l$, and let $\Omega = (RR^*)^{1/2}$, then

$$U(t) \equiv \begin{pmatrix}
\cos \Omega t \\
-R^* \sin \frac{\Omega t}{2}
\end{pmatrix} \left(\begin{array}{c}
\sin \frac{\Omega t}{2} \\
\cos(R^*R)^{1/2} t
\end{array}\right); 
\quad \text{(3)}$$

also, define

$$W(t) \equiv 1 + t \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \quad \text{(4)}$$

For the construction of $U(t)$ we have followed [8]. Then (MCN) can be rewritten as an integral equation: set

$$\mathbf{u}(t) = \begin{pmatrix} F_0(t) \\ F(t) \\ \xi(t) \\ v(t) \end{pmatrix}, \quad \text{(5)}$$

and let $\mathbf{u}(t_0) = \mathbf{u}(0)$ (we define the electric part as $F_0 = F^{0j}$, the magnetic part as $F = (F^{jk})_{j<k}$); also, let

$$j(t) = ev(t) \varphi(\xi(t) - x),$$

$$(f_{em}(t))_i = e \sum_{j=1}^{3} \delta_{ij} \left[ (\varphi \ast F^{0j}(t))(\xi(t)) + \sum_{k=1}^{3} v_k(t) (\varphi \ast F^{jk}(t))(\xi(t)) \right].$$
Then we can write the integral equation:

\[
\mathbf{u}(t) = \begin{pmatrix}
U(t-t_0) & 0 \\
0 & W(t-t_0)
\end{pmatrix} \mathbf{u}(0) + \int_{t_0}^{t} dt \begin{pmatrix}
U(t-\tau) & 0 \\
0 & W(t-\tau)
\end{pmatrix} \left( \begin{array}{c}
-j(\tau) \\
0 \\
0 \\
\mathcal{F}_{em}(\tau)
\end{array} \right) .
\] (M-N.i)

The second couple of Maxwell’s equations (namely \( \nabla \cdot E = \rho \) and \( \nabla \cdot B = 0 \)) have to be dealt with separately.

We are interested in solutions belonging to the following spaces: let \( \mathcal{X}_s \), for \( -\infty < s < 3/2 \), to be

\[
\mathcal{X}_s \equiv \left( \dot{H}^s(\mathbb{R}^3) \right)^3 \oplus \left( \dot{H}^s(\mathbb{R}^3) \right)^3 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3 .
\]

If \( \mathbf{u} \in \mathcal{X}_s \) has the form \( \mathbf{u}(t) \), then \( \mathcal{X}_s \) is a Hilbert space if equipped with the norm

\[
\| \mathbf{u} \|_{\mathcal{X}_s}^2 = \| F_0 \|_{(\dot{H}^s)^3}^2 + \| F \|_{(\dot{H}^s)^3}^2 + |\xi|^2 + |\xi|^2 ;
\]

where

\[
\| f \|_{(\dot{H}^s)^3}^2 = \sum_{j=1}^{3} \| \omega^s f_j \|_{L^2}^2 = \sum_{j=1}^{3} \int_{\mathbb{R}^3} dk |k|^{2s} |\hat{f}_j(k)|^2 ,
\]

with

\[
\omega = |\nabla| .
\]

The homogeneous Sobolev spaces \( \dot{H}^s(\mathbb{R}^d) \) are Hilbert spaces for all \( s < d/2 \) [see \( \mathbb{2} \)]. We will use as well the non-homogeneous Sobolev spaces \( H^r(\mathbb{R}^d) \), \( r \geq 0 \), complete with the norm

\[
\| f \|_{H^r}^2 = \int_{\mathbb{R}^d} dk (1 + |k|^2)^r |\hat{f}(k)|^2 .
\]

Let \( I \subseteq \mathbb{R} \); then we define

\[
\mathcal{X}_s(I) = C^0(I, \mathcal{X}_s) ;
\]

complete with the norm

\[
\| \mathbf{u}(\cdot) \|_{\mathcal{X}_s(I)} = \sup_{t \in I} \| \mathbf{u}(t) \|_{\mathcal{X}_s} .
\]

Our goal will be to prove that a unique global solution of (M-N) exists on \( \mathcal{X}_s(\mathbb{R}) \) whenever the initial datum belongs to (a subspace of) \( \mathcal{X}_s \) (theorem \( \mathbb{1} \); for all \( s < 3/2 \) and suitably regular \( \varphi \) (by the result for \( s = 0 \) global existence on non-homogenous Sobolev spaces is also proved, theorem \( \mathbb{2} \), \( \mathbb{2} \)).
Particles interacting with its electromagnetic field has been widely studied in physics. The study of a radiating point particle revealed the presence of divergencies. Therefore classically a radiating particle is always assumed to have extended charge distribution. The most used equations to describe such particle’s (and corresponding fields) motion are \([M-N]\) above, or its semi-relativistic counterpart, called Abraham model \([13]\) (also Maxwell-Lorentz equations in \([4]\)), see \([5]\) below. For a detailed discussion of their physical properties, historical background and applications the reader can consult any classical textbook on electromagnetism \([e.g. 14]\). On a mathematical standpoint, almost all results deal with the semi-relativistic system, and are quite recent. We mention an early work of Bambusi and Noja \([3]\) on the linearised problem; papers of Appel and Kiessling \([1]\), Kiessling \([12]\) on conservation laws and motion of a rotating extended charge. Concerning global existence of solutions, refer to Komech and Spohn \([13]\) and Bauer and Dürr \([4]\); the latter result has been developed further in Bauer et al. \([5, 6]\) to consider weighted \(L^2\) spaces. Imaykin et al. \([9, 11]\) have investigated soliton-type solutions and asymptotics. For a comprehensive review on the classical and quantum dynamics of particles and their radiation fields the reader may refer to the book by Spohn \([15]\).

The existence results \([4, 6, 13]\) are formulated for the semi-relativistic system \([8]\), but they should apply also to \([M-N]\): existence of a differentiable solution holds on suitable subspaces of \((L^2_w(\mathbb{R}^3))^3 \oplus (L^2_w(\mathbb{R}^3))^3 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3\), with \(w\) denoting an eventual weight on \(L^2\) spaces; in this paper a continuous global solution is proved to exist on a wider class of spaces: \((\dot{H}^s(\mathbb{R}^3))^3 \oplus (\dot{H}^s(\mathbb{R}^3))^3 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3, s < 3/2\) and \((H^r(\mathbb{R}^3))^3 \oplus (H^r(\mathbb{R}^3))^3 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3, r \geq 0\).

From a physical standpoint, it is a natural choice to consider the energy space \(\mathcal{X}_0(\mathbb{R})\) to solve \([M-N]\). Also, it is not necessary, in principle, to introduce new objects like the electromagnetic potentials \(\phi\) and \(A\). Nevertheless, there are plenty of situations where such potentials are either convenient or necessary: a simple example is in defining a Lagrangian or Hamiltonian function of the charge-electromagnetic field system. Once a gauge is fixed, investigating the regularity of the potentials \(\phi\) and \(A\) is equivalent to provide a solution to \([M-N]\) on a suitable space, often different form \(\mathcal{X}_0(\mathbb{R})\). In this context, homogeneous Sobolev spaces emerge. For example, consider the vector potential \(A\) in the Coulomb gauge. Then, given \(F^{ij}\), we have \(A_j = \omega^{-2} \sum_{i=1}^3 \partial_i F^{ij} \). So the requirement \(A \in (L^2(\mathbb{R}^3))^3\) is equivalent to \(B \in (\dot{H}^{-1}(\mathbb{R}^3))^3\). Another natural choice, if one wants to investigate the connection between the quantum and the classical theory, is to have \(A \in (\dot{H}^{1/2}(\mathbb{R}^3))^3\) and \(\partial_t A \in (\dot{H}^{-1/2}(\mathbb{R}^3))^3\) (because, roughly speaking, the classical correspondents of quantum creation/annihilation operators behave as \(\omega^{1/2}A, \omega^{-1/2}\partial_t A\) and are required to be square integrable). This is equivalent to solve \([M-N]\) with \(E \in (\dot{H}^{-1/2}(\mathbb{R}^3))^3\) and \(B \in (\dot{H}^{-1/2}(\mathbb{R}^3))^3\).
This led us to consider the existence of global solutions of \((M-N)\) on homogeneous Sobolev spaces, especially the ones with negative index \(s < 0\).

**Remark 1.** In formulating the equations, we have restricted to one charge for the sake of simplicity; results analogous to those stated in Theorems 1 and 2 should hold also in the case of \(n\) charges, even if they are subjected to mutual and external interactions, provided these interactions are regular enough.

The rest of the paper is organised as follows: in section II we summarise and discuss the results proved in this paper; in section III a local solution is constructed by means of Banach fixed point theorem; in section IV we prove uniqueness and construct the maximal solution; in section V we show that the maximal solution is defined for all \(t \in \mathbb{R}\); finally in section VI we discuss the divergence part of Maxwell’s equations.

### II. STATEMENT OF MAIN RESULTS.

In this section we summarise the results proved in the paper. This is done in Theorems 1 and 2.

We recall the Cauchy problem related to the Maxwell-Newton system of equations:

\[
\begin{align*}
\partial_t B + \nabla \times E &= 0 \\
\partial_t E - \nabla \times B &= -j \quad \hat{\xi} = v \\
E(t_0) &= E_0 \\
B(t_0) &= B_0
\end{align*}
\tag{6}
\]

\[
\begin{align*}
\nabla \cdot E &= \rho \\
\nabla \cdot B &= 0
\end{align*}
\tag{7}
\]

with \(j = ev\phi(\xi - x)\) and \(\rho = e\varphi(\xi - x)\). With an abuse of terminology, we will refer to the solutions of the Cauchy problem \((6), (7)\) simply as the solutions of Maxwell-Newton system.

**Theorem 1.** Let \(-\infty < s < 3/2\), \(\varphi\) a differentiable function of \(\mathbb{R}^3\) such that \(\|\varphi\|_Y < \infty\) (\(Y\) is \(H^{-s}, H^{-s+1}, H^s, H^{s+1}\)) and such that \((6)\) admits a solution on \((H^s(\mathbb{R}^3))^3 \oplus (H^s(\mathbb{R}^3))^3\). Furthermore let \(\xi_0, v_0 \in \mathbb{R}^3\) and \(E_0, B_0 \in (H^s(\mathbb{R}^3))^3\) satisfying \((7)\).

Then the Maxwell-Newton system \((M-N)\) admits a unique solution on \(C^0(\mathbb{R}, (H^s(\mathbb{R}^3))^3 \oplus (H^s(\mathbb{R}^3))^3 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3)\).
Corollary 1. Let the conditions of theorem 1 be satisfied. In addition, let \( \|\varphi\|_{H^{s-1}} < \infty \), \( E(0), B(0) \in (H^s(R^3))^3 \).

Then \( (M-N) \) admits a unique solution on \( C^0_0(\mathbb{R}, (H^s(R^3))^3 \oplus (H^{s-1}(R^3))^3 \oplus (\mathbb{R}^3 \oplus \mathbb{R}^3))^3 \). 

Theorem 2. Let \( r \geq 0 \), \( \varphi \) a differentiable function of \( R^3 \) such that \( \|\varphi\|_{H^r}, \|\varphi\|_{H^1} < \infty \) and such that \( (M-N) \) admits a solution on \( (H^r(R^3))^3 \oplus (H^r(R^3))^3 \). Furthermore let \( \xi(0), v(0) \in \mathbb{R}^3 \) and \( E(0), B(0) \in (H^r(R^3))^3 \) satisfying (7).

Then \( (M-N) \) admits a unique solution on \( C^0(\mathbb{R}, (H^r(R^3))^3 \oplus (H^r(R^3))^3 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3))^3 \). 

Proof of Theorem 2. Using Theorem 1 we prove existence of the unique solution of \( (M-N) \) on \( \mathcal{L}_0(\mathbb{R}) \). Then by Remark 3 it follows that if initial data are in \( (H^r(R^3))^3 \oplus (H^r(R^3))^3 \oplus \mathbb{R}^3 \) then also the solution is in the same space for all \( t \in \mathbb{R} \).

Remark 2. The methods and results of this paper should also apply to the semi-relativistic version of Maxwell-Newton system, called Abraham model:

\[
\begin{aligned}
\partial_t B + \nabla \times E &= 0 \\
\partial_t E - \nabla \times B &= -j \\
\dot{\xi} &= \frac{p}{\sqrt{1+p^2}} \\
\dot{p} &= e[(\varphi \ast E)(\xi) + \frac{p}{\sqrt{1+p^2}} \times (\varphi \ast B)(\xi)]
\end{aligned}
\]

with \( j = ep\varphi(\xi - x)/\sqrt{1+p^2} \) and \( \rho = e\varphi(\xi - x) \).

In Bauer and Dürr [4] an approach different to the one of this paper is taken; global existence for (8) is proved on a particular class of \( (L^2(R^3))^3 \oplus (L^2(R^3))^3 \oplus \mathbb{R}^3 \)-subspaces, namely \( D(B^n) \), \( n \geq 1 \) where \( B = (\nabla \times B, -\nabla \times E, 0, 0) \).
A. Rotating charge.

Let $m, I > 0$, and $\Omega(\cdot): \mathbb{R} \to \mathbb{R}^3$. Then global solution of the following Cauchy problem can be proved along the same guidelines as for theorems 1, 2

\[
\begin{align*}
\frac{\partial t}{\partial t} B + \nabla \times E &= 0 \\
\frac{\partial t}{\partial t} E - \nabla \times B &= -j \\
\dot{\xi} &= v \\
\dot{v} &= \frac{e}{m} \int dx \left[ E(x) + (v + \Omega \times (x - \xi)) \times B(x) \right] \varphi(\xi - x) \\
\dot{\Omega} &= \frac{1}{I} \int dx \left[ (x - \xi) \times [E(x) + (v + \Omega \times (x - \xi)) \times B(x)] \varphi(\xi - x) \right]
\end{align*}
\]

(9)

Then theorems 1 and 2 above can be reformulated for the system (9) as:

\[
\begin{align*}
E(t_0) = E_0 \\
B(t_0) = B_0
\end{align*}
\]

Define the following conditions:

**Condition 1.** $\varphi$ is a differentiable function of $\mathbb{R}^3$ such that $\forall i, i' = 1, 2, 3, \|\varphi(x)\|_Y, \|x_i \varphi(x)\|_Y, \|x_i x_i' \varphi(x)\|_Y < \infty$ when $Y = \dot{H}^{-s}, \dot{H}^{-s+1}, \dot{H}^s, \dot{H}^{s+1}$.

**Condition 2.** $\varphi$ is a differentiable function of $\mathbb{R}^3$ such that $\forall i, i' = 1, 2, 3, \|\varphi(x)\|_Y, \|x_i \varphi(x)\|_Y, \|x_i x_i' \varphi(x)\|_Y < \infty$ when $Y = H^r, H^1$.

Then theorems 1 and 2 above can be reformulated for the system (9) as:

**Theorem 1**. Let $-\infty < s < 3/2$, $\varphi$ satisfying Condition 1 and such that (7) admits a solution on $(\dot{H}^s(\mathbb{R}^3))^3 \oplus (\dot{H}^s(\mathbb{R}^3))^3$. Furthermore let $\xi_0(0), v(0), \Omega_0(0) \in \mathbb{R}^3$ and $E_0(0), B_0(0) \in (\dot{H}^s(\mathbb{R}^3))^3$ satisfying (7). Then Cauchy problem (9) admits a unique solution on $\mathcal{C}^0(\mathbb{R}, (\dot{H}^s(\mathbb{R}^3))^3 \oplus (\dot{H}^s(\mathbb{R}^3))^3 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3)$.

**Theorem 2**. Let $r \geq 0$, $\varphi$ satisfying Condition 2 and such that (7) admits a solution on $(H^r(\mathbb{R}^3))^3 \oplus (H^r(\mathbb{R}^3))^3$. Furthermore let $\xi_0(0), v(0), \Omega_0(0) \in \mathbb{R}^3$ and $E_0(0), B_0(0) \in (H^r(\mathbb{R}^3))^3$ satisfying (7). Then (9) admits a unique solution on $\mathcal{C}^0(\mathbb{R}, (H^r(\mathbb{R}^3))^3 \oplus (H^r(\mathbb{R}^3))^3 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3)$.
III. LOCAL SOLUTION.

In this section we construct a local solution of \([M-N_i]\) on \(\mathcal{X}_s(I)\), for a suitable interval \(I \subset \mathbb{R}\) containing \(t_0\); this is done in Proposition 2. We start our analysis summarizing the properties of \(U(t)\) and \(W(t)\), defined respectively in (3) and (4), we will use the most. This is done in the following Proposition:

Proposition 1. \(U(t)\) satisfies the following properties:

i. \(U(t)\) commutes with \(\omega^s\) for all \(s \in \mathbb{R}\), on suitable domains.

ii. \(U(t)\), \(t \in \mathbb{R}\), is a unitary one-parameter group on \((\dot{H}^s(\mathbb{R}^3))^3 \oplus (\dot{H}^s(\mathbb{R}^3))^3\), for \(s < 3/2\).

\(W(t)\) satisfies the following properties:

i. \(W(t)\) is differentiable in \(t\), and

\[
W(t) = \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}.
\]

ii. \(W(s)W(t) = W(s + t)\) for all \(s, t \in \mathbb{R}\).

A. Contraction mapping.

We want to prove local existence by means of Banach fixed-point theorem. In order to do that we need to define a strict contraction on a closed subspace of \(\mathcal{X}_s(I)\). Define

\[
A[t_0, \mathbf{u}(0)](\mathbf{u}(t)) \equiv \begin{pmatrix}
U(t - t_0) & 0 \\
0 & W(t - t_0)
\end{pmatrix} \mathbf{u}(0) + \int_{t_0}^t d\tau \begin{pmatrix}
U(t - \tau) & 0 \\
0 & W(t - \tau)
\end{pmatrix} \begin{pmatrix}
-j(\tau) \\
0 \\
f_{em}(\tau)
\end{pmatrix}.
\]

The following lemma is crucial for the analysis of the contraction map \(A[t_0, \mathbf{u}(0)]\):

Lemma 1. Let \(\mathbf{u}_1(1)\) and \(\mathbf{u}_2(2) \in \mathcal{X}_s(I)\), with \(I = [-T + t_0, t_0 + T]\) for some \(T > 0\); \(\varphi\) a differentiable
function such that \(\|\varphi\|_Y < \infty\), when \(Y\) is \(\hat{H}^{-s}\), \(\hat{H}^{-s+1}\), \(\hat{H}^{s}\) and \(\hat{H}^{s+1}\). Then, \(\forall s < 3/2\),

\[
\|A[t_0, u_0(0)](u_1) - A[t_0, u_0(0)](u_2)\|_{X_s(I)}^2 \leq |e|^2 T^2 \sup_{t \in I} \sup_{\alpha_i = 1, 2} \left\{ 2 \left( \|\varphi\|_{\hat{H}^{s+1}}^2 \|v(1) - v(2)\|(\|v(1) - v(2)\|)^2 + 3 \|\varphi\|_{\hat{H}^{s+1}}^2 \|v(1) - v(2)\|(\|v(1) - v(2)\|)^2 \right) + 2(1 + T^2) \left( \|\varphi\|_{\hat{H}^{s+1}}^2 \|F_0(v_2)\|(\|F_0(v_2)\|)^2 + 6 \|\varphi\|_{\hat{H}^{s+1}}^2 \|F_0(v_2)\|(\|F_0(v_2)\|)^2 \right) \right\}
\]

**Corollary 2.** If in addition to the conditions above: \(u_1(1)\) and \(u_2(2)\) \(B(I, \rho)\) (the ball of radius \(\rho\) of \(X_s(I)\)) for some \(\rho > 0\) and \(\max_y \|\varphi\|_Y = M\), then

\[
\|A[t_0, u_0(0)](u_1) - A[t_0, u_0(0)](u_2)\|_{X_s(I)} \leq T(T + 1)|e| \rho (4\rho^2 + 5) \|u_1 - u_2\|_{X_s(I)}.
\]

**Corollary 3.** Let \(u(0) \in B(\rho) \subset X_s\) (ball of radius \(\rho\) of \(X_s\)), \(u \in B(I, \rho_1) \subset X_s(I)\) and \(\max_y \|\varphi\|_Y = M\). Then

\[
\|A[t_0, u_0(0)](u)\|_{X_s(I)} \leq \sqrt{2}(T + 1)|e| + T(T + 1)|e| \rho_1 (4\rho^2 + 5).
\]

**Proof of Lemma** Consider the case \(t_0 < t\) (the other is perfectly analogous): by definition of \(A[t_0, u_0(0)]\) and Proposition \(2\) we write

\[
\|A[t_0, u_0(0)](u_1) - A[t_0, u_0(0)](u_2)\|_{X_s(I)}^2 \leq \sup_{t \in I} \|\varphi(\xi(1) - \tau)\|_{\hat{H}^{s}}^2 + \left( \|\varphi(\xi(1) - \tau)\|_{\hat{H}^{s}}^2 + (1 + |t - \tau|)^2 \right) \left( \|\varphi\|_{\hat{H}^{s}}^2 + \left( \|\varphi\|_{\hat{H}^{s}}^2 + 2(1 + |t - \tau|)^2 \right) \right)
\]

\[
\sum_{j=1}^{3} \left( \|\varphi(\xi(1) - \tau)\|_{\hat{H}^{s}}^2 + \left( \|\varphi(\xi(1) - \tau)\|_{\hat{H}^{s}}^2 + 2(1 + |t - \tau|)^2 \right) \right)
\]

\[
\equiv T^2 |e| \sup_{t \in I} \left\{ X_1(\tau) + 2(1 + |t - \tau|^2) \left[ X_2(\tau) + X_3(\tau) \right] \right\}.
\]

Consider now \(X_1(\tau)\), \(i = 1, 2, 3\) separately.

\[
X_1(\tau) \leq 2 \left( \|v(2) - v(1)\|_{\hat{H}^{s}}^2 + \|v(1)\|_{\hat{H}^{s}}^2 \right),
\]

Since \(\varphi\) is differentiable, we can write for some \(c \in [0, 1]\) (by the mean value theorem):

\[
\varphi(\xi(2) - x) - \varphi(\xi(1) - x) = \nabla \varphi((1 - c)\xi(2)(\tau) + c\xi(1)(\tau) - x) \cdot (\xi(2)(\tau) - \xi(1)(\tau)).
\]
Therefore we obtain

\[
X_1(\tau) \leq 2 \left( \|(v(1) - v(2))(\tau)\|^2 \|\varphi\|^2_{\dot{H}^s} + \sup_{\alpha=1,2} |v(\alpha)(\tau)|^2 \|\nabla \varphi((1-c)\xi_2)(\tau) + c\xi_1(\tau) - x\cdot (\xi_2(\tau) - \xi_1(\tau))\|^2_{\dot{H}^s} \right)
\]

\[
\leq 2 \left( \|(v(1) - v(2))(\tau)\|^2 \|\varphi\|^2_{\dot{H}^s} + 3 \|\xi_2(\tau) - \xi_1(\tau)\|^2 \sup_{\alpha=1,2} |v(\alpha)(\tau)|^2 \|\varphi((1-c)\xi_2(\tau) + c\xi_1(\tau) - \cdot)\|^2_{\dot{H}^{s+1}} \right); 
\]

in the last inequality we used the fact that, for all \(a, b \in \mathbb{R}^3\):

\[
\|\nabla \varphi(a - x) \cdot b\|^2_{\dot{H}^s} = \left\| \sum_{i=1}^{3} \partial_i(\omega^s \varphi(a - x)) b_i \right\|^2_{L^2} \leq 3 \sum_{i=1}^{3} |b_i|^2 \left\| \partial_i(\omega^s \varphi(a - x)) \right\|^2_{L^2}
\]

\[
\leq 3 \sum_{i=1}^{3} |b_i|^2 \langle \omega^s \varphi(a - x), -\Delta \omega^s \varphi(a - x) \rangle.
\]

Finally we obtain

\[
X_1(\tau) \leq 2 \left( \|(v(1) - v(2))(\tau)\|^2 \|\varphi\|^2_{\dot{H}^s} + 3 \|\xi_2(\tau) - \xi_1(\tau)\|^2 \sup_{\alpha=1,2} |v(\alpha)(\tau)|^2 \|\varphi\|^2_{\dot{H}^{s+1}} \right).
\]

A similar reasoning yields the following results for \(X_2(\tau)\) and \(X_3(\tau)\), using the fact that \(\|\varphi F\|_{L^\infty} \leq \|\omega^s \varphi\|_{L^2} \|\omega^s F\|_{L^2}^s:

\[
X_2(\tau) \leq 2 \left( \|\xi_1(\tau) - \xi_2(\tau)\|^2 \|\varphi\|^2_{\dot{H}^{s+1}} \sup_{\alpha=1,2} \|F(0)(\tau)\|^2_{(\dot{H}^s)^3} + \|\varphi\|^2_{\dot{H}^{s-\tau}} \|\varphi F(01) - F(02)(\tau)\|^2_{(\dot{H}^s)^3} \right),
\]

\[
X_3(\tau) \leq 6 \left( \|(v(1) - v(2))(\tau)\|^2 \|\varphi\|^2_{\dot{H}^{s+1}} \sup_{\alpha=1,2} \|F(0)(\tau)\|^2_{(\dot{H}^s)^3} + \|\varphi\|^2_{\dot{H}^{s-\tau}} \sup_{\alpha=1,2} |v(\alpha)(\tau)|^2 \|F(0)(\tau) - F(2)(\tau)\|^2_{(\dot{H}^s)^3} \right),
\]

\[
- \|F(2)(\tau)\|^2_{(\dot{H}^s)^3} + |(\xi_1(\tau) - \xi_2(\tau))|^2 \|\varphi\|^2_{\dot{H}^{s-\tau}} \sup_{\alpha,\beta=1,2} |v(\alpha)(\tau)|^2 \|F(\beta)(\tau)\|^2_{(\dot{H}^s)^3} \right).
\]

B. Existence of local solution.

We are now able to show that, if \(u(0) \in B(\rho) \subset \mathcal{S}_s\), then \(A[t_0, u(0)]\) is a strict contraction of \(B(I, 2\rho) \subset \mathcal{S}_s(I)\) for a suitable \(I\) that depends on \(\rho\). This fact proves that a local solution of \((M-N.1)\) exists on \(\mathcal{S}_s(I)\). The precise statement is contained in the following proposition:

**Proposition 2.** Let \(s < 3/2; \varphi\) a differentiable function such that \(\|\varphi\|_{Y} \leq M\), when \(Y\) is \(\dot{H}^{-s}, \dot{H}^{-s+1}, \dot{H}^s\) and \(\dot{H}^{s+1}\). Then for all \(\rho > 0\), \(\exists T(\rho) > 0\) such that, for all \(u(0) \in B(\rho) \subset \mathcal{S}_s\), Equation \((M-N.1)\) has a unique solution belonging to \(B(I, 2\rho)\) with \(I = [-T(\rho) + t_0, t_0 + T(\rho)]\).
Proof. Corollary 3 (with $\rho_1 = a\rho$) shows that for all $u_0 \in B(\rho)$, $A[t_0, u_0]$ maps $B(I, a\rho)$ into itself if $I = [-T + t_0, t_0 + T]$ with $0 < T \leq T_a, T_a(\rho)$ solution of $(T_a + 1)(\sqrt{2} + T_a |e| Ma(4a^2 \rho^2 + 5)) - a = 0$.

The last equation has at most one positive solution: the positive solution exists for all $a > \sqrt{2}$.

On the other hand, corollary 2 shows that $A[t_0, u_0]$ is a strict contraction on $B(I, 2\rho)$ for $I = [-T + t_0, t_0 + T], 0 < T < T_c$, with $T_c(2\rho)$ positive solution of $T_c(T_c + 1) |e| M(16\rho^2 + 5) = 1$.

Therefore defining $T(\rho) > 0$ as

$$T(\rho) = \min\{T_{a=2}(\rho), T_c(2\rho)/2\},$$

it follows that $A[t_0, u_0]$ is a strict contraction on $B(I, 2\rho)$ with $I = [-T(\rho) + t_0, t_0 + T(\rho)]$. By Banach’s fixed point theorem, the map $A[t_0, u_0]$ has then a unique fixed point on $B(I, 2\rho)$, solution of (M-N.i).

$$\Box$$

IV. UNIQUENESS, MAXIMAL SOLUTION.

In this section we prove that for all $s < 3/2$ the solution of (M-N.i) on $\mathcal{X}_s(I)$ is unique, provided it exists, for all $u_0 \in \mathcal{X}_s$ and $I \subseteq \mathbb{R}$. The uniqueness result yields the possibility to construct a maximal solution, and by Proposition 2 we establish the finite blowup alternative.

A. Uniqueness.

The uniqueness result is again based on Lemma 1, therefore the necessary conditions on $\varphi$ are the same.

**Proposition 3.** Let $s < 3/2; \varphi$ a differentiable function such that $\|\varphi\|_Y \leq M$, when $Y$ is $H^{-s}$, $H^{-s+1}, H^s$ and $H^{s+1}$. Suppose that Equation (M-N.i) has at least one solution on $\mathcal{X}_s(I), I \subseteq \mathbb{R}$ when $u_0 \in \mathcal{X}_s$. Then the solution is unique.

**Proof.** Suppose there are two solution corresponding to $u_0 \in \mathcal{X}_s$, namely $u_{(1)}(t)$ and $u_{(2)}(t)$. Define $u_- = u_{(1)} - u_{(2)}$. Then, following a reasoning analogous to the one for the proof of Lemma 1 we obtain

$$\|u_-(t)\|_{\mathcal{X}_s} \leq \int_{t_0}^{t} d\tau M(\tau)\|u_-(\tau)\|_{\mathcal{X}_s};$$
with
\[
M(\tau)^2 = 2|c|^2 M^2 \sup_{\alpha_i=1,2} \left\{ 1 + 3|v_{(\alpha_1)}(\tau)|^2 + 2(1 + |t - \tau|^2) \left[ 2\left( \|F_0(\alpha_2)(\tau)\|_{(H^s)^3}^2 \right) + 1 \right] + 6\left( \|F_{(\alpha_3)}(\tau)\|_{(H^s)^3}^2 + |v_{(\alpha_4)}(\tau)|^2 + |v_{(\alpha_5)}(\tau)|^2 \|F(\alpha_6)(\tau)\|_{(H^s)^3}^2 \right) \right\}.
\]

By Gronwall's Lemma, it follows that \( u_- = 0 \).

\[\square\]

B. Maximal Solution.

Using Propositions 2 and 3 we can construct the maximal solution of (M-N.i) on \( \mathscr{X}_s(\mathbb{R}) \). By maximal solution we mean that it is defined on the interval \( I_{\text{max}} = [-T_- + t_0, t_0 + T_+] \), for some \( T_-, T_+ > 0 \) and every solution on \( \mathscr{X}_s(I) \) is such that \( I \subseteq I_{\text{max}} \). In the next proposition we show that if either \( T_- \) or \( T_+ \) is finite, then the \( \mathscr{X}_s \) norm of the solution \( u(t) \) has to diverge when \( t \to T_- \) (or \( T_+ \)).

**Proposition 4.** Let \( s < 3/2 \), and \( T_- > 0 \) be such that \( I_{\text{max}} = [-T_- + t_0, t_0 + T_+] \) is the maximal interval where the solution of Equation (M-N.i) on \( \mathscr{X}_s(I_{\text{max}}) \) is defined, for \( u_0 \in \mathscr{X}_s \). Let \( u_{\text{max}} \) be such solution, then one of the following is true:

i. \( T_- < \infty \) and \( \|u_{\text{max}}(t)\|_{\mathscr{X}_s} \to \infty \) when \( t \to T_- + t_0 \);

ii. \( T_- = \infty \).

Equivalently:

i'. \( T_+ < \infty \) and \( \|u_{\text{max}}(t)\|_{\mathscr{X}_s} \to \infty \) when \( t \to t_0 + T_+ \);

ii'. \( T_+ = \infty \).

**Proof.** Assume there is a sequence \((t_i)_{i \in \mathbb{N}}\) and \( N > 0 \) such that \( t_i \to T_+ \) and \( \|u_{\text{max}}(t_i)\|_{\mathscr{X}_s} \leq N \) for all \( i \in \mathbb{N} \). Let \( t_k \) be such that \( t_k + T(N) > T_+ \), where \( T(N) \) is defined in Proposition 2. Starting from \( u_{\text{max}}(t_k) \) one can therefore extend, by Propositions 2 and 3, the solution \( u_{\text{max}}(t) \) to \( t = t_k + T(N) > T_+ \). This contradicts maximality. The proof for \( T_- \) is analogous.

\[\square\]
**V. GLOBAL EXISTENCE**

In this section we prove that $I_{\text{max}}$ defined in section IV.B is all $\mathbb{R}$. This is done by means of an energy-type estimate, given in Lemma 2. The result holds also when we substitute $\omega^s$ with $(1 - \Delta)^{s/2}$; as stated in remark 4.

We introduce the so-called interaction representation. Define

$$F \equiv \begin{pmatrix} F_0 \\ F \end{pmatrix}.$$  

Also, $\tilde{F}(t, t_0) = U^*(t - t_0)F(t)$. Therefore if $F(t)$ obeys the first part of (M-N.i), we have

$$\tilde{F}(t) = F(0) + \int_{t_0}^t d\tau U(t_0 - \tau) \begin{pmatrix} -j(\tau) \\ 0 \end{pmatrix}.$$  

Then for all $s < 3/2$, $\tilde{F}(t)$ is differentiable in $t$ on $(\dot{H}^s)^3 \oplus (\dot{H}^{-s})^3$, if $F(0) \in (\dot{H}^s)^3 \oplus (\dot{H}^{-s})^3$ (with $\varphi$ regular enough), and

$$\partial_t \tilde{F}(t) = U(t_0 - t) \begin{pmatrix} -j(t) \\ 0 \end{pmatrix}. \quad (10)$$

We also remark that $v(t)$ satisfying (M-N.i) is differentiable in $t$, and

$$\dot{v}(t) = f_{en}(t). \quad (11)$$

**Lemma 2.** Let $s < 3/2$, $\varphi$ such that $\|\varphi\|_{\dot{H}^{-s}}, \|\varphi\|_{\dot{H}^s} < \infty$. Then the following inequality hold, for $E$, $B$ and $v$ satisfying Equation (M-N.i):

$$\frac{1}{2} \left( v^2(t) + \int dx \left( (\omega^sE(t))^2 + (\omega^sB(t))^2 \right) \right) \leq \frac{1}{2} \left( v^2(t_0) + \int dx \left( (\omega^sE(t_0))^2 + (\omega^sB(t_0))^2 \right) \right) \exp \left\{ |e| |t - t_0| \left( 1 + 6 \|\varphi\|^2_{\dot{H}^s} + 3 \|\varphi\|^2_{\dot{H}^{-s}} \right) \right\}. \quad (M-N.s)$$

**Remark 3.** If $s = 0$ we can prove the conservation of energy

$$\frac{1}{2} \left( v^2(t) + \int dx \left( E^2(t) + B^2(t) \right) \right) = \frac{1}{2} \left( v^2(t_0) + \int dx \left( E^2(t_0) + B^2(t_0) \right) \right).$$

**Proof of Lemma 2** Define

$$M(t) = \frac{1}{2} \left( v^2(t) + \int dx \left( (\omega^sF_0(t))^2 + (\omega^sF(t))^2 \right) \right).$$

Then by Proposition 1 and (10):

$$\frac{dM(t)}{dt} = v\dot{v} + \frac{1}{2} \partial_t (\omega^s \tilde{F}, \omega^s \tilde{F}) = v\dot{v} + (\omega^s \tilde{F}, \omega^s U(t_0 - t) \begin{pmatrix} -j(t) \\ 0 \end{pmatrix}).$$
Proof of Proposition 5.

By Equation (M-N.i) and Lemma 2 we obtain:

\[
\frac{dM(t)}{dt} \leq |e| \left[ \frac{v^2}{2} + \frac{3}{2} \left( \| \varphi \ast E \|_2^2 + \| (\omega^2 \varphi) \ast E \|_2^2 + \| \varphi \ast (\omega^2 E) \|_2^2 \right) \right].
\]

Young’s inequality finally yields

\[
\frac{dM(t)}{dt} \leq |e| \left( 1 + 6 \| \varphi \|_{H^s}^2 + 3 \| \varphi \|_{\dot{H}^{s-\epsilon}}^2 \right) M(t).
\]

Apply Gronwall’s Lemma to obtain the sought result.

\[ \square \]

**Proposition 5.** Let \( s < 3/2 \), \( \varphi \) such that \( \| \varphi \|_{\dot{H}^{-s}} \), \( \varphi \|_{\dot{H}^s} < \infty \). Furthermore let \( u(0) \in X_s \) and \( u(t) \) a solution of Equation [M-N.a] on \( X_s(I) \) for some \( I \subseteq \mathbb{R} \). Then

\[
\| u(t) \|_{X_s}^2 \leq (1 + |t - t_0|^2)^{\exp \left\{ |e| |t - t_0| \left( 1 + 6 \| \varphi \|_{H^s}^2 + 3 \| \varphi \|_{\dot{H}^{s-\epsilon}}^2 \right) \right\}} \| u(0) \|_{X_s}^2.
\]

**Remark 4.** Since also \((1 - \Delta)^{s/2} \) commutes with \( U(t) \), the following statement also holds. Let \( r \geq 0 \), \( \varphi \) such that \( \| \varphi \|_{\dot{H}^r} < \infty \). Also, let \( \mathscr{P} = (H^r)^3 \oplus (H^r)^3 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3; u(0) \in \mathscr{P} \) and \( u(t) \) the solution of Equation [M-N.a] on \( \mathscr{P}(\mathbb{R}) \). Then

\[
\| u(t) \|_{\mathscr{P}}^2 \leq (1 + |t - t_0|^2)^{\exp \left\{ |e| |t - t_0| \left( 1 + 9 \| \varphi \|_{\dot{H}^r}^2 \right) \right\}} \| u(0) \|_{\mathscr{P}}^2.
\]

**Proof of Proposition 5.** By Equation [M-N.a] and Lemma 2 we obtain

\[
|\xi(t)|^2 \leq \int_{t_0}^t d\tau \varrho(\tau) \left[ v^2(t_0) + \int dx \left( (\omega^2 E(t_0))^2 + (\omega^s B(t_0))^2 \right) \right] \exp \left\{ |e| |t - t_0| \left( 1 + 6 \| \varphi \|_{H^s}^2 + 3 \| \varphi \|_{\dot{H}^{s-\epsilon}}^2 \right) \right\}.
\]

Hence the bound is proved using Lemma 2.

\[ \square \]

We are now able to prove that \( I_{\text{max}} \), defined on section [IV.B] is \( \mathbb{R} \):

**Proposition 6.** Let \( s < 3/2 \), \( \varphi \) a differentiable function such that \( \| \varphi \|_Y < \infty \), when \( Y \) is \( \dot{H}^{-s} \), \( \dot{H}^{-s+1} \), \( \dot{H}^s \) and \( \dot{H}^{s+1} \). Furthermore let \( u(0) \in X_s \) and \( u_{\text{max}} \in X_s(I_{\text{max}}) \) the corresponding maximal solution of Equation [M-N.a]. Then \( I_{\text{max}} = \mathbb{R} \).

**Proof.** By Proposition 5 we see that, for all \( s < 3/2 \), \( \| u_{\text{max}}(t) \|_{X_s} \) diverges if and only if \( t \to \pm \infty \). Then, by Proposition 4 \( I_{\text{max}} = \mathbb{R} \).

\[ \square \]
VI. THE SECOND COUPLE OF MAXWELL’S EQUATIONS.

In this section we analyse the divergence Maxwell’s equations, $\nabla \cdot E = \rho$, $\nabla \cdot B = 0$. We prove that any couple of tempered distributions that satisfies (6) (as distributions, i.e. acting on $\mathcal{S}(\mathbb{R}^3)$ functions), satisfies also (7), provided the initial data satisfy (7) themselves. Therefore the divergence part of Maxwell’s equations reduces to a constraint on the set of possible initial values of the electric and magnetic field.

Proposition 7. Let $E(0), B(0)$ satisfy (7); $E(t), B(t) \in \mathcal{S}'(\mathbb{R}^3)$ satisfy Cauchy problem (10). Also, let charge conservation (2) holds. Then $E(t), B(t)$ satisfy (7).

Remark 5. To apply this proposition, we need initial data of integral Equation (M-N.i) that satisfy (7). So given $\varphi$, we have to find vectors in $(\dot{H}^s)^3, s < 3/2$, that satisfy Equations (7). For example $\{B \in (\dot{H}^s)^3 : \nabla \cdot B = 0\}$ is a closed subspace of $(\dot{H}^s)^3$, whose orthogonal complement is $\{B \in (\dot{H}^s)^3 : \exists \Lambda \in \dot{H}^{s+1} \text{ such that } B = \nabla \Lambda\}$.

A thorough study of these equations is, however, beyond the scope of this paper, the interested reader should refer to Csató et al. [7, and references thereof contained]. Keep in mind that in general to fulfill (7) some regularity conditions on $\varphi$ may be necessary.

Proof of Proposition 7. Define the distributions $f(t), g(t) \in \mathcal{S}(\mathbb{R}^3)$ as

$$f(t) = \nabla \cdot E(t) - \rho(t);$$
$$g(t) = \nabla \cdot B(t).$$

Using the assumptions on initial data we see that $f(t_0) = g(t_0) = 0$. Furthermore, in the sense of distributions, using (10) we obtain:

$$\partial_t f(t) = \nabla \cdot \partial_t E(t) - \partial_t \rho(t) = \nabla \cdot \nabla \times B(t) - \nabla \cdot j(t) - \partial_t \rho(t);$$
$$\partial_t g(t) = \nabla \cdot \partial_t B(t) = -\nabla \cdot \nabla \times E(t).$$

Now since the divergence of a curl is equal to zero and using charge conservation (2):

$$\partial_t f(t) = 0;$$
$$\partial_t g(t) = 0.$$

Therefore $f(t) = g(t) = 0$ on $\mathcal{S}'(\mathbb{R}^3)$.

\qed
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