Robustness of quantum Fourier transform interferometry

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Boson Sampling photonic networks can be used to obtain Heisenberg limited measurements of optical phase gradients, by using quantum Fourier transform interferometry. Here, we use phase-space techniques and the complex P-distribution, to simulate a 100 qubit Fourier transform interferometer with additional random phases, to simulate decoherence. This is far larger than possible in conventional calculations of matrix permanents, which is the standard technique for such calculations. Our results show that this quantum metrology technique is robust against phase decoherence, and one can also measure lower order correlations without substantial degradation.

I. INTRODUCTION

Linear photonic networks employing a quantum computing technique called Boson Sampling, can be used to implement quantum Fourier transform (QFT) interferometers [1, 2]. These devices use high order quantum correlations to measure phase gradients with Heisenberg limited precision. There is an analytic formula known for this in ideal cases, yet the general performance of such photonic networks is difficult to compute. The photon counting output probability is determined by the square of a matrix permanent. Calculating this for a general unitary matrix is exponentially hard for large channel numbers. It falls into the complexity class called \#P, and is more difficult to evaluate than the well known NP complete and \#P hard problems.

To understand the performance of a real interferometer, one needs to go beyond the ideal case of a perfect quantum Fourier transform with a uniform gradient. Any real photonic interferometer has some fluctuations in the relative phases. In addition, even apart from losses, the object being measured will not have a completely uniform phase gradient. An important question, therefore, is whether the favorable scaling in the ideal interferometer can be translated to a real world case with phase noise. Questions like this also arise in the case of classical interferometry and grating design, and are answered by taking conventional Fourier transforms combined with Gaussian averaging.

The issue is not so easily resolved in the case of a large QFT network, as the calculation involves evaluating a large matrix permanent. Exact calculations of such permanents are notoriously difficult, and are generally not feasible for $M \times M$ QFT interferometers above $M = 50$ [3]. Here we analyse much larger QFT interferometers, with up to 100 channels, and calculate the effects of phase noise using phase-space techniques. Our results show that the QFT interferometer is in fact robust against phase noise, with a gradual degradation of the peak visibility as phase-noise increases, rather than a catastrophic failure when imperfections are introduced.

Importantly, our results far exceed previous size limits on such permanental calculations. In addition, this technique can represent any input state or output measurement. This is essential for understanding how this technology can be scaled to large sizes with imperfect sources [4], or used in different applications [5], or with imperfect detectors.

To illustrate this issue, we also calculate the QFT correlation peak for correlation functions of much less than optimal order. Such low-order correlations are another realistic requirement in a practical device. The maximal order correlations at large $M$ are likely to be heavily suppressed due to efficiency factors. Our results again show that the technique is quite surprisingly robust. Lowering the correlation order appears to have relatively little effect on the peak width.

II. COMPLEX P-REPRESENTATIONS

We use the complex P-representation [6] for the input and the output states of the Boson Sampling device. In this paper we only consider number states as an input, even though complex P-representations are capable of handling any input density matrix.

The complex P-representation is defined implicitly to produce the density matrix when integrated over a set of contours $C$ enclosing the origins of the complex planes corresponding to the variables $\alpha_k, \beta_k$:

$$\hat{\rho}^{\text{in}} = \iint_C P(\alpha, \beta) \hat{\Lambda}(\alpha, \beta) \, d\alpha \, d\beta.$$ 

Here the projector operator $\hat{\Lambda}$

$$\hat{\Lambda}(\alpha, \beta) = \frac{\langle \alpha| (\beta^+)^\|}{(\beta^+)|\alpha\rangle},$$

is defined in terms of $M$-mode un-normalised Bargmann-Glauber coherent states $|\alpha\rangle$, where

$$|\alpha\rangle = \prod_{k=1}^{M} \left[ \sum_{n_k} \alpha_k^{n_k} \sqrt{n_k!} |n_k\rangle \right].$$

There is more than one way to choose the contours $C$; in this work we use circular contours of radius $r$, where $r$ can
be varied. Later in the paper we show that the value of \( r \) strongly affects the effectiveness of the representation.

A \( P \) function defined this way exists for any density matrix (and, in general, is not unique) [6]. In this paper we consider the initial state to be a pure multimode state where each mode \( k \) is a number state with \( n_k \) bosons. Commonly in the Boson Sampling experiments one photon is sent in each of a subset \( \sigma \) of input modes, so \( n_k = 0, 1 \).

The input density matrix \( \hat{\rho}^{(\text{in})} \) is transformed to the output density matrix \( \hat{\rho}^{(\text{out})} \) when transmitted through a linear network. The network is described by a unitary matrix \( U \). Possible losses can be treated by adding modes to serve as noise channels, but in this paper we only consider the unitary case. The effect of the network matrix in the \( P \) representation is straightforward: it simply acts on the vectors of phase-space variables as \( \alpha^{(\text{out})}, \beta^{(\text{out})} = U \alpha, U^* \beta \) [7]. The resulting output density matrix is therefore

\[
\hat{\rho}^{(\text{out})} = \Re \mathcal{F} \int_C P(\alpha, \beta) \Lambda(T\alpha, T^*\beta) \, d\alpha d\beta.
\]

A typical observable in Boson Sampling experiments, and the one we will consider in this paper is a normally ordered correlation of output modes belonging to a set \( \sigma' \):

\[
\left\langle \prod_{k \in \sigma'} \hat{n}_k \right\rangle_Q = \mathcal{F} \int_C P(\alpha, \beta) \prod_{k \in \sigma'} \hat{n}_k^{(\text{out})}(\alpha, \beta) \, d\alpha d\beta
\equiv \left\langle \prod_{k \in \sigma'} n_k^{(\text{out})}(\alpha, \beta) \right\rangle_P.
\]

Here the output number variable \( n_k^{(\text{out})}(\alpha, \beta) \) is defined as

\[
n_k^{(\text{out})}(\alpha, \beta) = \alpha_k^{(\text{out})} \beta_k^{(\text{out})}.
\]

As the distribution is not unique (the choice of contours \( C \) being flexible), one can choose different representations of the input state. These lead to different strategies for using random sampling methods, as well as different sampling weights. In this work we use two kinds of representations tailored for different scenarios, described below: a continuous sampling method (VCP) and a discrete method (QCP). Which one is preferred depends on the correlation being calculated.

III. CONTINUOUS SAMPLING METHOD

We first consider a continuous analytic complex \( P \)-distribution for the initial state

\[
P(\alpha, \beta) = \prod_{k=1}^M P_k(\alpha_k, \beta_k),\]

where \( P_k \) is the single-mode distribution for the corresponding input channel \( k \):

\[
P_k(\alpha, \beta) = \begin{cases} \left(\frac{n_k}{2\pi}\right)^2 \frac{e^{n\beta}}{\delta(\alpha)\delta(\beta)} & \text{for } n_k > 0, \\ 1 & \text{for } n_k = 0, \end{cases}
\]

Here \( n_k \) is the number of photons in the input mode \( k \), and we simplified the distribution for the modes where \( n_k = 0 \) by shrinking the corresponding contours to the origin of the complex plane. For nonzero boson number inputs we use a circular contour of radius \( r \) and make a transition to polar coordinates, so that \( \alpha_k = r z_k = r \exp(i\phi_k^{(\alpha)}) \) and \( \beta_k = r \bar{z}_k = r \exp(-i\phi_k^{(\beta)}) \). The coherent modulus \( r \) is chosen to minimise the sampling error.

The phase variables can be understood intuitively on defining \( \phi_k = (\phi_k^{(\alpha)} + \phi_k^{(\beta)})/2 \) and \( \theta_k = \phi_k^{(\alpha)} - \phi_k^{(\beta)} \), where \( \phi_k \) is the classical phase, and \( \theta_k \) is a nonclassical phase which only exists when the quantum state has nonclassical features. In this case we get:

\[
\hat{\rho} = \prod_{k=1}^M J_k(\phi_k, \theta_k)(\alpha_k, \beta_k),
\]

where

\[
J_k = \left\{ \begin{array}{ll} \left(\frac{n_k}{2\pi}\right)^2 & \text{for } n_k > 0, \\ 1 & \text{for } n_k = 0, \end{array} \right.
\]

It is clearly possible to separate the real and imaginary parts of the exponential. We use random probabilistic sampling for the real part. We call the resulting distribution a circular von Mises complex-P distribution (VCP), since the weight around the contour has a von Mises probability distribution [8]. Once the random phase angle is randomly chosen, the imaginary part is included as an additional complex weight.

The sampled probability distribution for the subset of modes with nonzero inputs is:

\[
P_k(\phi_k, \theta_k) = \frac{1}{2\pi I_0(r^2)} \exp(r^2 \cos \theta_k), \quad n_k > 0,
\]

where \( I_0(x) \) is the modified Bessel function of the first kind of order 0, \( \theta_k \in [-\pi, \pi) \), and \( \phi_k \in [-\pi, \pi) \). This corresponds to sampling the variables separately as:

\[
\phi_k = \mathcal{U}(-\pi, \pi),
\]

\[
\theta_k = \mathcal{VM}(0, r^2),
\]

where \( \mathcal{U} \) is the uniform distribution, and \( \mathcal{VM} \) is the circular von Mises distribution [8]. For each sample we calculate the phase-space variables as \( \alpha_k = \)
\[ \Omega = \prod_{k=1}^{M} \left\{ \frac{(n_k!)^2 I_0(r^2)}{r^{2n_k}} \exp \left( i r^2 \sin \theta_k - n_k \theta_k \right) \right\}, \quad n_k > 0; \quad n_k = 0. \]

These are used to calculate any moment \( f(\alpha, \beta) \) in conjunction with the drawn samples of \( \alpha \) and \( \beta \) as

\[ \langle f(\alpha, \beta) \rangle = \frac{1}{L} \sum_{j=1}^{L} \Omega \left( \alpha^{(j)}, \beta^{(j)} \right) f \left( \alpha^{(j)}, \beta^{(j)} \right), \]

where \( L \) denotes the total number of samples, and \( \alpha^{(j)} \) and \( \beta^{(j)} \) is the \( j \)-th sample of a pair of random phase-space coordinates.

**IV. DISCRETE SAMPLING METHOD**

Suppose that the initial photon number is bounded, for example with a fixed input boson number. We now introduce a discrete sampling method, which we term the discrete or qudit complex P-representations (QCP) [9]. This construction is useful in the limit of discrete or qudit complex P-representations (QCP) [9].

The photon number phase-space variable, for an input of single bosons into a subset \( \sigma \) of size \( N \) of input modes

\[ n_k^{(\text{out})}(q, \bar{q}) = r^2 \left( \sum_{j \in \sigma} U_{kj} z_{\tilde{q}_j} \right) \left( \sum_{j' \in \sigma'} U_{kj'} \bar{z}_{\bar{q}_{j'}} \right)^*, \]

As a result, after including the complex P-function weights, an \( N \)-th order output correlation is

\[ \langle \prod_{k \in \sigma'} \hat{n}_k \rangle_Q = \left| \frac{1}{d^M} \sum_{q} \left( \prod_{i \in \sigma} z^{-q_i} \prod_{k \in \sigma'} \left( \sum_{j \in \sigma} U_{kj} z_{\tilde{q}_j} \right) \right) \right|^2. \]

Here perm indicates the matrix permanent. Note that if the order of the measured correlation is equal to the number of input bosons, the \( r \) factors cancel.

As expected [11], this is the square of the permanent of the sub-matrix of \( U \) with rows in \( \sigma' \) and columns in \( \sigma \), which we call \( U (\sigma', \sigma) \). After summation on the \( q \) indices, the only terms that survive involve products of distinct permutations of the matrix indices, which is the permanent. There are exponentially many terms involved at large \( N \). For reasons of efficiency, this is evaluated computationally by taking randomly chosen integers \( (q, \bar{q}) \), and averaging over many samples of these random phases.

In order to do that, we define a function

\[ p(\sigma, \sigma', q) = \prod_{i \in \sigma} z^{-q_i} \prod_{k \in \sigma'} \left( \sum_{j \in \sigma} U_{kj} z_{\tilde{q}_j} \right) \]

representing a single sample of the permanent for the submatrix defined by the set of inputs \( \sigma \) and the set of
outputs $\sigma'$, with $\mathbf{q}$ being a vector of random numbers. Then the permanent of the submatrix can be calculated probabilistically as

$$\text{perm} U (\sigma, \sigma') \approx \frac{1}{L} \sum_{j=1}^{L} p(\sigma, \sigma', \mathbf{q}^{(j)}) \equiv \langle p(\sigma, \sigma', \mathbf{q}) \rangle_{L}.$$ 

In order to avoid the bias in the calculation of the permanent-squared, we must use two independent sets of random variables to calculate the expectations of the permanent and permanent-conjugated, and multiply them together:

$$|\text{perm} [U (\sigma, \sigma')]|^2 \approx \Re \langle p(\sigma, \sigma', \mathbf{q}) \rangle_{L} \langle p(\sigma, \sigma', \mathbf{q}^\dagger) \rangle_{L}. \quad (3)$$

The noises variables factors into two terms which are independent but conjugate on average. The factored terms give independent estimates of the permanent and its conjugate, so their product is an unbiased estimate of the modulus squared. We take the real part to impose the constraint that the final result must be real.

V. QUANTUM FOURIER TRANSFORM INTERFEROMETRY

To test our methods for quantum enhanced technology we consider the quantum Fourier transform interferometer (QuFTI) experiment \cite{Motes}. This is a novel multi-mode interferometer that measures gradients of a phase-shift and corresponds to consider the case where the input state is one photon in each input mode. Here the permanent of the full matrix is evaluated, not the permanent-squared. We take the real part to impose the constraint that the final result must be real.

$$Q^{(\text{conj})} = \left| \text{perm} U^{(M)} \right|^2 = \prod_{j=1}^{M-1} \frac{2j(M-j) \cos(M\phi) + M^2 - 2jM + 2j^2}{M^2}. \quad (4)$$

Fig. 1 shows results obtained with our quantum software for the QuFTI. We confirm the interferometer conjecture with an unprecedented permanent size of 100 × 100. Exact methods would take trillions of years to compute this. In Fig. 1(a) we evaluate the error in the modulus square of the permanent using both the VCP and QCP representation with $d = 2$. The second panel, Fig. 1(b), shows the dependence of the error for the VCP method depending on the contour radius $r$, justifying the choice of $r = 0.1$.

In our simulations we include the effect of decoherence, using an independent phase noise term $\xi$ at each site with outputs $\sigma'$, with $\mathbf{q}$ being a vector of random numbers. Then the permanent of the submatrix can be calculated probabilistically as

$$\text{perm} U (\sigma, \sigma') \approx \frac{1}{L} \sum_{j=1}^{L} p(\sigma, \sigma', \mathbf{q}^{(j)}) \equiv \langle p(\sigma, \sigma', \mathbf{q}) \rangle_{L}.$$ 

In order to avoid the bias in the calculation of the permanent-squared, we must use two independent sets of random variables to calculate the expectations of the permanent and permanent-conjugated, and multiply them together:

$$|\text{perm} [U (\sigma, \sigma')]|^2 \approx \Re \langle p(\sigma, \sigma', \mathbf{q}) \rangle_{L} \langle p(\sigma, \sigma', \mathbf{q}^\dagger) \rangle_{L}. \quad (3)$$

The noises variables factors into two terms which are independent but conjugate on average. The factored terms give independent estimates of the permanent and its conjugate, so their product is an unbiased estimate of the modulus squared. We take the real part to impose the constraint that the final result must be real.
We also evaluate different order correlations. This is carried out since experimental detector inefficiencies may lead to greater count rates for lower order correlations. This is shown in figure 3(a), for $M = 30$, where we show three different order correlation functions, for $Q = \langle \hat{n}_1 \cdots \hat{n}_N \rangle$, with $N = 30$, $N = 25$ and $N = 20$. As the order is reduced the sampling error is clearly increasing, but the fringe width does not change dramatically, indicating that the method appears to be robust against lower order measurements.

Figure 3(b) gives the sampling error as a function of $r$. This demonstrates how the choice of integration contour can change the sampling error for these lower order correlation measurements. The results show that in this case, one should use a contour near $r = 1$. However, the exact choice of radius is important. For this order of correlation ($N = 25$), we find that for the lower-order correlations there is a finite optimal value of $r$, in this case near $r = 0.8$, where the error is the lowest.

VI. CONCLUSIONS

In conclusion, we develop two probabilistic methods based on complex $P$-distribution that can be used to find the expectation values of the correlations of the outputs of a Boson Sampling device. A more general one, VCP, can handle correlations of any order, while a more specialized QCP is limited to the correlations of maximum order (equal to the number of input photons), but has a much lower sampling error than VCP. These results are limited by sampling errors, even with the highly efficient methods we use. This approach also provides a link with contour integral methods for matrix permanents.

We use these methods to simulate the QuFTI experiment with up to 100 modes and test the conjecture by Motes et al for the expected output correlation value. In addition, we test the performance of the interferometer under noisy conditions (by adding noise to the measured phase gradient). Our results demonstrate the exceptional robustness of multi-photon quantum Fourier interferometry against decoherence.

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FIG. 3. (c) Correlations of different order for $M = 30$ obtained using the VCP method with 200 ensembles of $10^6$ samples. The VCP parameter was set to $r = 0.8$ to minimise the sampling error (see panel (b)). (b) Dependence of the error for the 25th order correlation as a function of $r$ for $M = 30$ and $\phi = 0.01$, obtained with the VCP method with 200 ensembles of $10^4$ samples.

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