THE UNDERLYING DIGRAPHS
OF A COINED QUANTUM RANDOM WALK

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Abstract. We give a characterization of the line digraph of a regular
digraph. We make use of the characterization, to show that the un-
derlying digraph of a coined quantum random walk is a line digraph.
We remark the connection between line digraphs and in-split graphs in
symbolic dynamics. (MSC2000: 05C50, 81P68)

1. Introduction

In this paper, we give a characterization of the line digraph of a regular
digraph. We make use of the characterization, to show that the underlying
digraph of a coined quantum random walk is a line digraph.

The structure of the paper is the following. In Section 2, we recall the def-
definition of line digraph and survey some properties. In Section 3, we give the
characterization. In Section 4, we consider coined quantum random walks
and remark the connection between these objects and line digraphs. Finally,
we remark the connection between line digraphs and in-split graphs as de-
defined in symbolic dynamics. The reader familiar with the general properties
of line digraphs can skip Section 2.

2. Line digraphs

2.1. Definition. The notion of line digraph has been introduced by Harary
and Norman in 1960 [HN60]. A classic survey on line graphs and digraphs
is [HB78]; a recent one is [P95]. Line digraphs are used in the design and
analysis of interconnection networks (see e.g. [FYA84]) and as a tool in algo-

A directed graph, for short digraph, consists of a non-empty finite set of
elements called vertices and a finite set of ordered pairs of vertices called
arcs. Let us denote by $D = (V, A)$ a digraph with vertex-set $V (D)$ and
arc-set $A (D)$. In an arc $(v_i, v_j)$, $v_i$ and $v_j$ are called end-vertices of $(v_i, v_j)$;
$v_i$ tail and $v_j$ head of $(v_i, v_j)$. A digraph $D$ is an empty-graph if $A (D)$ is
the empty-set.

Definition 1 (Line digraph). The line digraph of a digraph $D$ is denoted
by $\overrightarrow{L}D$ and defined as follows: the vertex set of $\overrightarrow{L}D$ is $A (D)$ and, for every

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v_h, v_i, v_j, v_k \in V(D)$, $(v_h, v_i), (v_j, v_k) \in A(LD)$ if and only if $v_i = v_j$. The $k$-iterated line digraph is recursively defined by 

$$L^kD = L^{k-1}LD.$$ 

Now we need some standard terminology.

A dipath is a non-empty digraph $D$, where

$$V(D) = \{v_0, v_1, ..., v_k\}, \quad A(D) = \{(v_0, v_1), (v_1, v_2), ..., (v_{k-1}, v_k)\},$$

and, for every $v_i, v_j \in V(D), v_i \neq v_j$. A dipath is called dicycle if $v_0 = v_k$. A $k$-dipath ($k$-dicycle), denoted by $\overrightarrow{P}_n$ ($\overrightarrow{C}_n$), is a dipath (dicycle) on $n$ vertices. The undirected analogues of a dipath and a dicycle are called path and cycle, respectively.

A digraph $H$ is a subdigraph of a digraph $D$ if $V(H) \subseteq V(D)$, $A(H) \subseteq A(D)$ and every arc in $A(H)$ has both end-vertices in $V(H)$. If $V(H) = V(D)$, $H$ is said to be a spanning subdigraph of $D$. If every arc of $A(D)$ with both end-vertices in $V(H)$ is in $A(H)$, we say that $H$ is induced by the set $X = V(H)$ and $H$ is an induced subdigraph of $D$.

This is an obvious but important remark: the set of vertices $V(L^kD)$ can be seen as the set of the $k$-dipaths in $D$.

2.2. General properties. The theorems stated here are well-known; the proofs can be found in [HB78].

For every $S \subset V(D)$, let

$$N^{-}_D(S) = \{v_i : (v_i, v_j) \in A(D), v_j \in S\}$$

and

$$N^{+}_D(S) = \{v_j : (v_i, v_j) \in A(D), v_i \in S\}$$

be the in-neighbourhood and the out-neighbourhood of $S$, respectively. If the context is not equivocque we then write $N^{-}(S)$ ($N^{+}(S)$). The in-degree of $S$ is $d^{-}(S) = |N^{-}(S)|$; the out-degree $d^{+}(S) = |N^{+}(S)|$.

A vertex $v_i$ in a digraph $D$ is isolated if it is not in a dipath with another vertex of $D$.

A digraph $D$ is connected if, for every $v_i, v_j \in V(D)$, there is a dipath containing $v_i$ to $v_j$, or viceversa; $D$ is strongly-connected if, for every $v_i, v_j \in V(D)$, there is a dipath from $v_i$ to $v_j$ and from $v_j$ to $v_i$.

Theorem 1. Let $D$ be a digraph on $n$ vertices (none of which isolated) and $m$ arcs. Then:

(i)

$$|V(LD)| = m \quad \text{and} \quad |A(LD)| = \sum_{v_i \in V(D)} d^{+}(v_i) d^{-}(v_i);$$

(ii)

$$N^{-}((v_i, v_j)) = d^{-}(v_i) \quad \text{and} \quad N^{+}((v_i, v_j)) = d^{+}(v_j);$$

(iii) $L^kD \cong \overrightarrow{P}_{n-1}$ if and only if $D \cong \overrightarrow{P}_n$.
Theorem 2. Let $D$ be a digraph. Then

(i) $\overrightarrow{L}^n D$ is an empty-graph, for some $n$, if and only if $D$ has no dipaths;

(ii) if $D$ has two dicycles joined by a $k$-dipath (possibly $k = 1$), then

$$\lim_{n \to \infty} p_n = \infty,$$

where $p_n$ is the number of vertices of $\overrightarrow{L}^n D$;

(iii) if $D$ is strongly connected, and if $\overrightarrow{L}^n D \cong D$ for some $n$, then $\overrightarrow{L} D \cong D$, and $D$ is a dicycle.

A digraph $D$ is hamiltonian when $V(D) = V(H)$, where $H$ is a dicycle. A digraph $D$ is eulerian if it is connected and, for every $v_i \in V(D)$, $d^-(v_i) = d^+(v_i)$. A digraph $D$ is regular if, for every $v_i, v_j \in V(D)$, $d^-(v_i) = d^+(v_i) = d^-(v_j) = d^+(v_j)$.

Theorem 3. Let $D$ be a digraph. Then

(i) $\overrightarrow{L} D$ is strongly connected if and only if $D$ is strongly connected;

(ii) $\overrightarrow{L} D$ is eulerian if and only if, for every arc $(v_i, v_j) \in A(D)$, $d^-(v_i) = d^+(v_j)$;

(iii) $\overrightarrow{L} D$ is hamiltonian if and only if $D$ is eulerian.

A general partition of a set $S$ is a collection $\{S_i\}_{i \in I}$ of (possibly empty) subsets of $S$, such that

$$S = \bigcup_{i \in I} S_i, \quad \text{and} \quad S_i \cap S_j = \emptyset \quad \text{if} \quad i \neq j.$$

A digraph $D$ is said to be $F$-free if it does not contain any subdigraph isomorphic to $F$.

The adjacency matrix of a digraph $D$ on $n$ vertices, denoted by $M(D)$, is the $n \times n (0, 1)$-matrix with $ij$-th entry equal to 1 if $(i, j) \in A(D)$, and equal to 0, otherwise. Let $r_i(M)$ and $c_j(M)$ be respectively the $i$-th row and the $j$-th column of a matrix $M$. Let $\langle a, b \rangle$ the inner product of vectors $a$ and $b$.

Theorem 4. Let $D$ be a digraph. Then the following statements are equivalent:

(i) $D$ is a line digraph;

(ii) there exist two general partitions $\{A_i\}_{i \in I}$ and $\{B_i\}_{i \in I}$ of $V(D)$ such that, for each $i$ and $j$,

$$|A_j \cap B_i| \leq 1 - \delta_{i,j}, \quad \text{and such that} \quad A(D) = \bigcup_{i \in I} A_i \times B_i;$$

(iii) any two rows of $M(D)$ are identical or orthogonal, $M_{i,i} = 0$ for all $i$, and if $r_i(M) = r_j(M) \neq 0$ then $\langle c_i(M), c_j(M) \rangle = 0$;

(iv) any two columns of $M(D)$ are identical or orthogonal, $M_{i,i} = 0$ for all $i$, and if $c_i(M) = c_j(M) \neq 0$ then $\langle r_i(M), r_j(M) \rangle = 0$;
(v) \( D \) is \( D_3 \)- and \( D_4 \)-free, where

\[
D_3 = (\{1, 2, 3, 4\}, \{(1, 2), (1, 3), (4, 2)\}),
\]

and

\[
D_4 = (\{1, 2, 3\}, \{(1, 2), (3, 1), (3, 2)\}).
\]

Any digraph obtained from \( D_3 \) or \( D_4 \) by adding arcs is not an induced subdigraph of \( D \), with the exception of

\[
D'_3 = (\{1, 2, 3, 4\}, \{(1, 2), (1, 3), (4, 2), (4, 3)\})
\]

and

\[
D'_4 = (\{1, 2, 3\}, \{(1, 2), (3, 1), (3, 2), (1, 1)\}).
\]

**Theorem 5** ([P01]). Let \( D \) be a strongly connected digraph. If \( D \) is regular then all its iterated line digraphs are regular, thus eulerian, and hamiltonian.

### 2.3. Algebraic properties.

#### 2.3.1. Spectrum.

Let \( D \) be a digraph on \( n \) vertices. The set of eigenvalues of the adjacency matrix \( M(D) \) is denoted by \( sp(D) \), and called spectrum of \( D \) [CDH95]. Let \( I_n \) be the \( n \times n \) identity matrix.

**Theorem 6** (see [LZ83] or [R01]). Let \( D \) be a digraph. The characteristic polynomial of \( \mathcal{L}D \) is

\[
P(\mathcal{L}D, x) = x^{|A(D)|-|V(D)|} P(D, x),
\]

where

\[
P(D, x) = \det (xI_n - M(D))
\]

is the characteristic polynomial of \( D \).

Let \( \{0\}_n \) be a set of \( n \) zeros. By Theorem 6 since

\[
sp(K^+_d) = \{d\} \cup \{0\}_{d-1},
\]

we have the following corollary.

**Corollary 1.** \( sp(B(d,k)) = \{d\} \cup \{0\}_{d^k-1}. \)

#### 2.3.2. Moore-Penrose inverse.

A matrix \( M^+ \) is a Moore-Penrose inverse of a matrix \( M \) if

(i) \( MM^+M = M \),

(ii) \( M^+MM^+ = M^+ \),

(iii) \( (MM^+)^\top = MM^+ \),

(iv) \( (M^+M)^\top = M^+M \).

**Theorem 7** ([Sm81]). A square \((0,1)\)-matrix has a Moore-Penrose inverse if and only if it is the adjacency matrix of a line digraph.
2.3.3. Permanent. The permanent of a \((0,1)\)-matrix \(M\) is

\[
\text{per}(M) = \sum_{\pi \in S_n} \prod_{i=1}^{n} A_{i,\pi(i)},
\]

where \(S_n\) is the symmetric group on an \(n\)-set.

**Theorem 8** \([\text{KSW97}]\). Let \(D\) be a digraph. Then

\[
\text{per}\left( M \left( \overrightarrow{L} D \right) \right) > 0
\]

if and only if each connected component of \(D\) is eulerian.

2.4. Line digraphs and unitary matrices. Let \(M\) be a matrix over any field. The support of \(M\) is the \((0,1)\)-matrix with \(ij\)-th element equal to 1 if \(M_{i,j} \neq 0\), and equal to 0, otherwise. The digraph of \(M\) is the digraph whose adjacency matrix is the support of \(M\). If a digraph \(D\) is the digraph of a matrix \(M\) then we say that \(D\), or indistinctly \(M(D)\), supports \(M\).

**Theorem 9** \([\text{Se03}]\). Let \(D\) be a digraph. Then \(\overrightarrow{L} D\) is the digraph of a unitary matrix if and only if every connected component of \(D\) is eulerian.

3. A characterization of the line digraph of a regular digraph

In this section, we characterize the adjacency matrix of the line digraph of a regular digraph.

3.1. The characterization. Let \(D\) be a digraph. A 1-cycle factor of \(D\) is the disjoint union of directed cycles spanning \(D\). The adjacency matrix of a 1-cycle factor is a permutation matrix.

A \(k\)-factor \(F\) of \(D\) is a \(k\)-regular spanning subdigraph of \(D\). A \(k\)-factorization of \(D\) is a set \(\{F_1, F_2, \ldots, F_m\}\) of pairwise arc-disjoint \(k\)-factors of \(D\) covering \(A(D)\), that is \(A(D) = F_1 \cup F_2 \cup \ldots \cup F_m\). More generally, a factorization of \(D\) is a set of pairwise arc-disjoint factors of \(D\), possibly of different degrees, covering \(A(D)\).

The growth \(\Upsilon_D(F)\) of \(D\), introduced in \([\text{HS96}]\), is a digraph derived from a spanning subdigraph \(F\) of \(D\), by adding, for each vertex \(v_i \in F\),

\[
l = |N^+_D(v_i)| - |N^+_F(v_i)| \quad \text{vertices} \quad \{v_1, \ldots, v_l\}
\]

and

\[
l \quad \text{arcs} \quad (v_i, v_1), (v_i, v_2), \ldots, (v_i, v_l).
\]

**Proposition 1.** Let \(D\) be a \(k\)-regular digraph. Let \(\{F_1, F_2, \ldots, F_k\}\) be a \(1\)-factorization of \(D\). Then there is a labeling of \(\overrightarrow{L} D\), such that

\[
M \left( \overrightarrow{L} D \right) = \begin{bmatrix}
M(F_1) & M(F_2) & \cdots & M(F_k) \\
M(F_1) & M(F_2) & \cdots & M(F_k) \\
\vdots & \vdots & \ddots & \vdots \\
M(F_1) & M(F_2) & \cdots & M(F_k)
\end{bmatrix}.
\]
Proof. Since $D$ is $k$-regular, the adjacency matrix of $D$ is

$$M(D) = \sum_{j=1}^{k} M(F_j).$$

Label by the pair $(F_j, v_l)$ the arc $(v_i, v_l)$ of $F_j$. Construct $\Upsilon_D(F_j)$. Note that $F_j$ is a subdigraph of $\Upsilon_D(F_j)$. So, in $\Upsilon_D(F_j)$, we can keep the same labelling of $F_j$. In addition, we label by $v_m$, with $m \neq l$, the $m$-th of the $k-1$ vertices which are heads of the arcs incident to $v_i$ and that are not in $F_j$. Then label the arc $(v_i, v_m)$ with the pair $(F_m, v_m)$. Since each $F_j$ is the disjoint union of dicycles, and since the line digraph of a dicycle is a dicycle (see, e.g., [HBTS], Theorem 7.1), it follows that

$$\Upsilon_D(F_j) \cong \Upsilon_{\overrightarrow{L}D}(F_j).$$

Given the chosen labelling and an ordering of the vertices,

$$M\left(\Upsilon_{\overrightarrow{L}D}(F_j)\right) = \begin{bmatrix}
0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \cdots & \cdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0 \\
M(F_1) & M(F_2) & \cdots & M(F_j) & \cdots & M(F_k) \\
0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \cdots & \cdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0 
\end{bmatrix},$$

where $M(F_j)$ is the $(j,j)$-th block of $M\left(\Upsilon_{\overrightarrow{L}D}(F_j)\right)$. Since, the set

$$\{F_1, F_2, \ldots, F_k\}$$

is a 1-factorization of $D$ if and only if

$$\left\{\Upsilon_{\overrightarrow{L}D}(F_1), \Upsilon_{\overrightarrow{L}D}(F_2), \ldots, \Upsilon_{\overrightarrow{L}D}(F_k)\right\}$$

is a 1-factorization of $\overrightarrow{L}D$ [HS96], it follows that

$$M\left(\overrightarrow{L}D\right) = \sum_{j=1}^{k} M\left(\Upsilon_{\overrightarrow{L}D}(F_j)\right).$$

3.2. Example. Consider the digraph $D$ with adjacency matrix matrix

$$M(D) = \begin{bmatrix}
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 
\end{bmatrix}.$$
Note that $D$ is a 2-cube. Chose a 1-factorization of $D$, $\{F_1, F_2\}$. For example, we chose $\{F_1, F_2\}$ such that

$$M(F_1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad M(F_2) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

We obtain

$$M(\Upsilon_D(F_1)) = \begin{bmatrix} F_{1,1} & F_{1,2} & F_{1,3} & F_{1,4} & F_{2,1} & F_{2,2} & F_{2,3} & F_{2,4} \\ F_{1,1} & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ F_{1,2} & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ F_{1,3} & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ F_{1,4} & 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$M(\Upsilon_D(F_2)) = \begin{bmatrix} F_{1,1} & F_{1,2} & F_{1,3} & F_{1,4} & F_{2,1} & F_{2,2} & F_{2,3} & F_{2,4} \\ F_{1,1} & 0 & 0 & \cdots & \cdots & 0 \\ F_{1,2} & 0 & 0 & \cdots & \cdots & 0 \\ F_{1,3} & 0 & \vdots & \ddots & \vdots & \vdots \\ F_{1,4} & 0 & \cdots & \cdots & 0 & \cdots \\ F_{2,1} & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ F_{2,2} & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ F_{2,3} & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ F_{2,4} & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Now,

$$\Upsilon_D(F_1) \cong \Upsilon_{LD}(F_1) \quad \text{and} \quad \Upsilon_D(F_2) \cong \Upsilon_{LD}(F_2)$$

Since $\{\Upsilon_{LD}(F_1), \Upsilon_{LD}(F_2)\}$ is a factorization of $\overrightarrow{LD}$,

$$M(\overrightarrow{LD}) = M(\Upsilon_D(F_1)) + M(\Upsilon_D(F_2)) = \begin{bmatrix} M(F_1) & M(F_2) \\ M(F_1) & M(F_2) \end{bmatrix}.$$ 

4. Line digraphs and coined quantum random walks

In this section, we first recall the definition of coined quantum random walk, then, we show that the underlying digraph of a coined quantum random walk is a line digraph. Coined quantum random walks, and continuous-time quantum random walks, are surveyed in [K03].
4.1. Definition. Let $D$ be a $k$-regular digraph on $n$ vertices. Consider two quantum systems, to which are respectively assigned the Hilbert spaces $\mathcal{H}^k$ and $\mathcal{H}^n$, of respective dimensions $k$ and $n$. Label each ray of the standard basis of $\mathcal{H}^k$ by a 1-factor of $D$. Label each ray of the standard basis of $\mathcal{H}^n$ by the vertices of $D$. Let
\[
\{|F_j,v_i\rangle : 0 \leq j \leq k, 0 \leq i \leq n\}
\]
be the standard basis of $\mathcal{H}^{k\cdot n} = \mathcal{H}^k \otimes \mathcal{H}^n$. Let $|\psi_t\rangle \in \mathcal{H}^{k\cdot n}$ be the state of the system at time $t$. Let $\hat{C} : \mathcal{H}^k \rightarrow \mathcal{H}^k$ and $\hat{T} : \mathcal{H}^{k\cdot n} \rightarrow \mathcal{H}^{k\cdot n}$ be unitary operators, such that $\hat{C} : |F_j\rangle \rightarrow \sum_{j=1}^{k} \alpha_j |F_j\rangle$ and $\hat{T} : |F_j,v_i\rangle \rightarrow |F_j,v_l\rangle$, where $(v_i,v_l) \in F_j$.

The matrices $C$ and $T$, arising from these operators, are respectively called coin and shift. Define the unitary operator
\[
\hat{U} : |F_j,v_i\rangle \rightarrow \sum_{j=1,t:(v_i,v_l)\in F_j}^{k} \alpha_j |F_j,v_l\rangle.
\]

A coined quantum random walk on $D$ with coin $C$, induced by the transition matrix
\[
U = T \cdot (C \otimes I_n),
\]
is a sequence $\{X_t\}$ of random variables. The sequence starts at $X_0 = v_i$, fixed, or drawn from some initial distribution. At time $t$, the probability that $X_t = v_j$, conditioned on an initial state $|\psi_0\rangle$, is
\[
Pr_t(v_j|\psi_0) = \sum_{i} \langle \psi_t | P^{(i,j)} | \psi_t \rangle, \quad \text{where} \quad |\psi_t\rangle = U^t |\psi_0\rangle
\]
and $P^{(i,j)}$ is the projector onto the $k$-dimensional subspace spanned $|F_j,v_i\rangle$.

4.2. The underlying digraph of a coined quantum random walk. The underlying digraph of a random walk induced by a transition matrix $M$ is the digraph of $M$. In this sense, the underlying digraph of a coined quantum random walk induced by a unitary matrix $U$ is the digraph of $U$.

**Proposition 2.** The underlying digraph of a coined quantum random walk is a line digraph.

**Proof.** Let $D$ be a $k$-regular digraph on $n$ vertices. Let $\{F_1,F_2,...,F_k\}$ be a factorization of $D$. By Proposition 1,
\[
M \left( \mathcal{L}D \right) = (M (K^+_k) \otimes I_n) \cdot T,
\]
where
\[
T = \bigoplus_{j=1}^{k} M (F_j),
\]
Observe that

\[ M \left( T^T D \right)^T = T \cdot \left( M \left( K_k^+ \right) \otimes I_n \right). \]

If in this equation we replace \( M \left( K_k^+ \right) \) with a unitary matrix \( C \) of size \( k \), we obtain

\[ T \cdot \left( C \otimes I_n \right), \]

which is the transition matrix of a coined quantum random walk on \( D \).

### 4.3. Example: the cycle.

The following construction is described in [FFY92].

Let \( D = \text{Cay} \left( G, S \right) \) be a Cayley digraph and let \( S = \{ s_1, s_2, \ldots, s_d \} \). Let \( H = \{ \pi_{s_i} : s_i \in S \} \) be a set of permutations of \( S \). Let \( H \) act on \( S \) as a regular group: we take \( \pi_{s_1} \) as identity element and, for every \( s_i, s_j \in S \), we assume that there exists a unique \( s_k \), such that \( \pi_{s_k} (s_i) = s_j \). The set \( H \) can be used to define the following operation on \( S \):

\[ s_i \odot s_j = \pi_{s_i} (s_j). \]

Let \( S^* \subseteq S \). The elements \( s_k \in S^* \), such that, for all \( s_i, s_j \in S \),

\[ s_k \odot (s_i \odot s_j) = (s_k \odot s_i) \odot s_j, \]

form a group \(( S^*, \odot )\), whose identity element is \( s_1 \).

Let \( \times_{sd} \) be the symbol of the (external) semidirect product. If \( H \leq \text{Aut} \left( G \right) \) then \(( S, \odot ) \cong H \). In such a case, the digraph \( T D \) is a Cayley digraph of

\[ \Omega \cong G \times_{sd} ( S, \odot ), \]

with respect to the set of generators

\[ \{ (s_1, s_i) : s_i \in S \}. \]

Let \( C_n \) be the cyclic group of order \( n \). Take \( n \) odd. Assume that \( \mathbb{Z}_n \cong C_n \) is generated by the set \( S = \{ 1, n - 1 \} \). The Cayley digraph \( D = \text{Cay} \left( \mathbb{Z}_n, S \right) \) is a cycle of length \( n \). Let \( H = \{ \pi_1, \pi_{n-1} \} \). Let \( \pi_1 \) be the identity and,

\[ \pi_{n-1} \left( n - 1 \right) = 1 \quad \text{and} \quad \pi_{n-1} \left( 1 \right) = n - 1. \]

Then \( H \cong \mathbb{Z}_2 \). The element \( \pi_{n-1} \in \text{Aut} \left( \mathbb{Z}_n \right) \). Then \( T D \) is the Cayley digraph of the group

\[ \Omega \cong \mathbb{Z}_n \times_{sd} \mathbb{Z}_2 \cong D_n, \]

where

\[ D_n = \langle a, b : a^n = b^2 = e, a^{n-1} = bab \rangle \]

is the dihedral group of order \( 2n \), generated by its standard presentation.

By denoting a permutation in the standard cycle notation, we write

\[ g = (1\ 2\ 3\ \ldots\ n) \quad \text{and} \quad g^{n-1} = (1\ n\ n-1\ \ldots\ 2\ 1), \]

where \( g = 1 \) and \( g^{n-1} = n - 1 \). Then

\[ \pi_{n-1} \left( 1 \right) = \sigma \left( 1 \right) \sigma = n - 1, \quad \text{and} \quad \pi_1 \left( n - 1 \right) = \sigma \left( n - 1 \right) \sigma = 1, \]
where
\[
\sigma = (2^n)(3^n - 1) \cdots \left( \frac{n+1}{2} \frac{n+1}{2} + 1 \right).
\]
The fixed-point of \(\sigma\) is 1. Under the isomorphism \(\iota : \mathbb{Z}_n \times_{sd} \mathbb{Z}_2 \longrightarrow D_n\), the generators of \(D_n\) are
\[
\iota([g,g]) = g = a \quad \text{and} \quad \iota([g,g^{n-1}]) = g\sigma = b,
\]
where
\[
g\sigma = (1^n)(2^n - 1) \cdots \left( \frac{n-1}{2} \frac{n+1}{2} + 1 \right).
\]
The fixed-point of \(g\sigma\) is \(\frac{n+1}{2}\). The Cayley digraph \(\text{Cay}(D_n, \{a, b\})\) is the 1-skeleton of an \(n\)-gon prism. Let \(\rho_{\text{reg}}\) be the (right) regular permutation representation of \(\mathbb{Z}_n\). By Proposition 1,
\[
M(\text{Cay}(D_n, \{a, b\})) = \begin{bmatrix}
\rho_{\text{reg}}(g) & \rho_{\text{reg}}(g^{n-1}) \\
\rho_{\text{reg}}(g) & \rho_{\text{reg}}(g^{n-1})
\end{bmatrix}.
\]

4.4. Line digraphs and in-split graphs. The notion of split graph is fundamental in symbolic dynamics (see, e.g., [LM95]). Let \(D\) be a digraph. Then \(\overrightarrow{L}D\) is a special case of split graph of \(D\), namely an in-split graph. Here is the definition. For every \(v_i \in V(D)\), let
\[
N^-(v_i) = I(1, v_i) \uplus I(2, v_i) \uplus \cdots \uplus I(m(v_i), v_i),
\]
where \(m(v_i)\) is the number of classes in the partition of \(N^-(v_i)\). Let \(\mathcal{P}\) be a partition of \(A(D)\) as above. The in-split graph formed from \(D\) using \(\mathcal{P}\) is denoted by \(D[\mathcal{P}]\) and defined as follows:
\[
V(D[\mathcal{P}]) = \{I(j, v_i) : v_i \in V(S), 1 \leq j \leq m(v_i)\};
\]
the number of arcs from the vertex \(I(k, v_i)\) to the vertex \(I(j, v_i)\) is the number of arcs in \(D\) which belong to \(I(j, v_i)\) and have \(v_i\) as tail. If \(\mathcal{P}\) is the maximal partition (all its classes have cardinality 1) then
\[
D[\mathcal{P}] = \overrightarrow{L}D.
\]

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