A predictive approach to Bayesian forecasting

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Abstract: Given a sequence $X = (X_1, X_2, \ldots)$ of random observations, a Bayesian forecaster aims to predict $X_{n+1}$ based on $(X_1, \ldots, X_n)$ for each $n \geq 0$. To this end, she only needs to select a collection $\sigma = (\sigma_0, \sigma_1, \ldots)$, called “strategy” in what follows, where $\sigma_0(\cdot) = P(X_1 \in \cdot)$ is the marginal distribution of $X_1$ and $\sigma_n(\cdot) = P(X_{n+1} \in \cdot \mid X_1, \ldots, X_n)$ the $n$-th predictive distribution. Because of the Ionescu-Tulcea theorem, $\sigma$ can be assigned directly, without passing through the usual prior/posterior scheme. One main advantage is that no prior probability is to be selected. In a nutshell, this is the non-standard approach to Bayesian predictive inference. A concise review of the latter is provided in this paper. We try to put such an approach in the right framework, to make clear a few misunderstandings, and to provide a unifying view. Some recent results are discussed as well. In addition, some new strategies are introduced and the corresponding distribution of the data sequence $X$ is determined. The strategies concern generalized Polya urns, random change points, covariates and stationary sequences.

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1. Introduction

To give a general definition of Bayesian inference is obviously hard. Nevertheless, a tentative definition may be useful. Let us call $O$ the object of inference. Roughly speaking, $O$ denotes whatever we ignore but would like to know. For instance, $O$ could be a parameter (finite or infinite dimensional), a set of future observations, an unknown probability distribution, the effect of some action, or something else. According to us, the distinguishing feature of the Bayesian approach is to regard $O$ as the realization of a random element, and not as an
unknown but fixed constant. As a consequence, the main goal of any Bayesian inferential procedure is to determine the conditional distribution of $O$ given the available information.

Note that, unless $O$ itself is a parameter, no other parameter is necessarily involved.

Prediction of unknown observable quantities is a fundamental part of statistics. Initially, it was probably the most prevalent form of statistical inference. The wind changed at the beginning of the 20th century when statisticians’ attention shifted to other issues, such as parametric estimation and testing; see e.g. [32]. Nowadays, prediction is back in the limelight again, and plays a role in modern topics including machine learning and data mining; see e.g. [16], [17], [24], [39].

This paper deals with prediction of future observations, based on the past ones, from the Bayesian point of view. Precisely, we focus on a sequence $X = (X_1, X_2, \ldots)$ of random observations and, at each time $n$, we aim to predict $X_{n+1}$ based on $(X_1, \ldots, X_n)$. Hence, for each $n$, the object of inference is $O = X_{n+1}$, the available information is $(X_1, \ldots, X_n)$, and the target is the \emph{predictive distribution} $P(X_{n+1} \in \cdot \mid X_1, \ldots, X_n)$. We point out that, apart from technicalities, most of our considerations could be generalized to the case where $O$ is an arbitrary (measurable) function of the future observations, say $O = f(X_{n+1}, X_{n+2}, \ldots)$.

This case is recently object of increasing attention; see e.g. [26] and [36].

No parameter $\theta$ plays a role at this stage. The forecaster may involve some $\theta$, if she thinks it helps, but she is not interested in $\theta$ as such. To involve $\theta$ means to model the probability distribution of $X$ as depending on $\theta$, and then to exploit this fact to calculate the predictive distributions $P(X_{n+1} \in \cdot \mid X_1, \ldots, X_n)$.

To better address our prediction problem, it is convenient to introduce the notion of \emph{strategy}. Let $(S, \mathcal{B})$ be a measurable space, with $S$ to be viewed as the set where the observations $X_n$ take values. Following Dubins and Savage [23], a strategy is a sequence $\sigma = (\sigma_0, \sigma_1, \sigma_2, \ldots)$ such that

- $\sigma_0$ and $\sigma_n(x)$ are probability measures on $\mathcal{B}$ for all $n \geq 1$ and $x \in S^n$;
- The map $x \mapsto \sigma_n(x, A)$ is $\mathcal{B}^n$-measurable for fixed $n \geq 1$ and $A \in \mathcal{B}$.

Here, $\sigma_0$ should be regarded as the marginal distribution of $X_1$ and $\sigma_n(x)$ as the conditional distribution of $X_{n+1}$ given that $(X_1, \ldots, X_n) = x$. Moreover,
\[ \sigma_n(x, A) \] denotes the value taken at \( A \) by the probability measure \( \sigma_n(x) \). We also note that strategies are often called prediction rules in the framework of species sampling sequences; see [50, p. 251].

Strategies are a natural tool to frame a prediction problem from the Bayesian standpoint. In fact, a strategy \( \sigma \) can be regarded as the collection of all predictive distributions (including the marginal distribution of \( X_1 \)) in the sense that

\[ \sigma_n(x, \cdot) = P(X_{n+1} \in \cdot \mid (X_1, \ldots, X_n) = x) \]

for all \( n \geq 0 \) and \( x \in S^n \). Thus, in a sense, everything a Bayesian forecaster has to do is to select a strategy \( \sigma \). Obviously, the problem is how to do it. A related problem is whether, in order to choose \( \sigma \), involving a parameter \( \theta \) is convenient or not.

An important special case is exchangeability. In fact, if \( X \) is assumed to be exchangeable, there is natural way to involve a parameter \( \theta \). To see this, take the parameter space \( \Theta \) as

\( \Theta = \{ \text{all probability measures on } B \} \).

Moreover, for each \( \theta \in \Theta \), denote by \( P_\theta \) a probability measure which makes \( X \) i.i.d. with common distribution \( \theta \), i.e.,

\[ P_\theta(X_1 \in A_1, \ldots, X_n \in A_n) = \prod_{i=1}^{n} \theta(A_i) \text{ for all } n \geq 1 \text{ and } A_1, \ldots, A_n \in B. \]

Then, under mild conditions on \( (S, B) \), de Finetti’s theorem yields

\[ P(X \in \cdot) = \int_\Theta P_\theta(X \in \cdot) \pi(d\theta) \]

for some (unique) prior probability \( \pi \) on \( \Theta \). Thus, conditionally on \( \theta \in \Theta \), the observations are i.i.d. with common distribution \( \theta \). This suggests calculating the strategy \( \sigma \) as follows.

(i) Select a prior \( \pi \) on \( \Theta \);
(ii) For each \( n \geq 1 \) and \( x \in S^n \), evaluate the posterior of \( \theta \) given \( x \), namely, the conditional distribution of \( \theta \) given that \( (X_1, \ldots, X_n) = x \);
(iii) Calculate \( \sigma \) as

\[ \sigma_n(x, A) = \int_\Theta \theta(A) \pi_n(d\theta \mid x) \text{ for all } A \in B, \]

where \( \pi_n(\cdot \mid x) \) is the posterior and \( \pi_0(\cdot \mid x) \) is meant as \( \pi_0(\cdot \mid x) = \pi \).

Steps (i)-(ii)-(iii) are familiar in a Bayesian framework. Henceforth, the forecaster is said to follow the standard approach (SA) if she selects \( \sigma \) via (i)-(ii)-(iii).
1.1. Non-standard approach to Bayesian predictive inference

There is another approach to Bayesian predictive inference, which is quite recurrent in the Bayesian literature and recently gained increasing attention. To fix ideas, we call it the non-standard approach (NSA). Obviously, the words “standard” and “non-standard” have no intrinsic meaning. They are used only to distinguish one approach from the other and to point out that the first is much more popular than the second.

According to NSA, the forecaster just selects her strategy $\sigma$. Merely, for each $n \geq 0$, she selects the predictive $\sigma_n$ without passing through the prior/posterior scheme described above. Among others, NSA is supported by de Finetti, Savage, Dubins and more recently by Diaconis and Regazzini. NSA is also strictly connected to Dawid’s prequential approach and to Pitman’s treatment of species sampling sequences. A non-exhaustive list of references is: [4], [8], [9], [15], [18], [19], [20], [21], [22], [23], [26], [28], [29], [31], [36], [37], [40], [44], [50], [51], [52].

The theoretical foundation of NSA is the Ionescu-Tulcea theorem; see e.g. [42, p. 159]. To state such a theorem, for notational simplicity, the $X_n$ are taken to be the coordinate random variables on $(S^\infty, B^\infty)$.

**Theorem 1. (Ionescu-Tulcea).** For any strategy $\sigma$, there is a unique probability measure $P_\sigma$ on $(S^\infty, B^\infty)$ such that

$$ P_\sigma(X_1 \in \cdot) = \sigma_0(\cdot) \quad \text{and} \quad P_\sigma(X_{n+1} \in \cdot \mid (X_1, \ldots, X_n) = x) = \sigma_n(x, \cdot) \quad (1) $$

for all $n \geq 1$ and $P_\sigma$-almost all $x \in S^n$.

In view of Theorem 1, to make predictions on $X$, the forecaster is free to select an arbitrary strategy $\sigma$. In fact, for any $\sigma$, there is a (unique) probability distribution for the sequence $X$, denoted above by $P_\sigma$, whose predictives $P_\sigma(X_{n+1} \in \cdot \mid X_1, \ldots, X_n)$ agree with $\sigma$ in the sense of equation (1).

The strengths and weaknesses of SA versus NSA are discussed in [8]. To summarize, SA is not motivated by prediction alone. The main goal of SA is to make inference on other features of the data distribution (typically some parameters) and in this case the posterior $\pi_n(\cdot \mid x)$ is fundamental. However, to assess a prior $\pi$ is not an easy task. In addition, once $\pi$ is selected, to evaluate the posterior $\pi_n(\cdot \mid x)$ is quite difficult as well. Frequently, $\pi_n(\cdot \mid x)$ cannot be written in closed form but only approximated numerically. In short, SA is a cornerstone of Bayesian inference, but, when prediction is the main target, it is actually quite involved.

In turn, NSA has essentially four merits. First, to avoid an explicit choice of the prior $\pi$ is practically useful. Indeed, when prediction is the main target, why select $\pi$ explicitly? Rather than wondering about $\pi$, it seems reasonable to reflect on how the next observation $X_{n+1}$ is affected by $(X_1, \ldots, X_n)$. Needless to say, circumventing $\pi$ is especially useful in the nonparametric framework. Second,
the data sequence $X$ is not forced to satisfy any distributional assumption. In particular, $X$ may fail to be exchangeable. Third, NSA requires the assignment of probabilities on observable facts only. The value of $X_{n+1}$ is actually observable, while $\pi$ and $\pi_n$ (being probabilities on $\Theta$) do not necessarily deal with observable facts. Fourth, the strategy $\sigma$ may be assigned stepwise. At each time $n$, the forecaster has observed $x = (x_1, \ldots, x_n) \in S^n$ and has already selected $\sigma_0, \sigma_1(x_1), \ldots, \sigma_{n-1}(x_1, \ldots, x_{n-1})$. Then, to predict $X_{n+1}$, she is still free to select $\sigma_n(x)$ as she wants. No choice of $\sigma_n(x)$ is precluded. According to us, this is consistent with the Bayesian view, where the observed data are fixed and one should condition on them. A similar point of view is highlighted in [26].

A basic remark is that both SA and NSA completely determine the probability distribution of $X$. In particular, if $X$ is exchangeable, selecting a strategy $\sigma$ via NSA uniquely determines the prior $\pi$. An intriguing line of research is in fact to identify the prior corresponding to a given $\sigma$; see e.g. [10], [21], [22], [28].

Finally, we note that several prediction procedures arising in non-necessarily Bayesian frameworks, such as Machine Learning and Data Mining, seem to be closer to NSA than SA. See e.g. [16], [17], [24], [39].

1.2. Characterizations

Recall that, for any strategy $\sigma$, there is a unique probability measure $P_\sigma$ on $(S^\infty, B^\infty)$ satisfying condition (1).

As noted above, when applying NSA, the data sequence $X$ is not required any distributional assumption. Nevertheless, in a specific problem, the forecaster may impose some conditions on $X$. For instance, she may wish $X$ to be exchangeable, or stationary, or Markov, or a martingale, and so on. In these cases, $\sigma$ is subjected to some constraints. For instance, if $X$ is required to be exchangeable, $\sigma$ should be such that $P_\sigma$ is exchangeable. Hence, those strategies $\sigma$ which make $P_\sigma$ exchangeable should be characterized.

More generally, fix any collection $C$ of probability measures on $(S^\infty, B^\infty)$ and suppose the data distribution is required to belong to $C$. Then, NSA gives rise to the following problem:

**Problem (\*)**: Characterize those strategies $\sigma$ such that $P_\sigma \in C$.

Sometimes, Problem (\*) is trivial (Markov, martingales) but sometimes it is not (stationarity, exchangeability). To illustrate, we mention three examples.

In the exchangeable case, Problem (\*) has a nice solution [28, Th. 3.1] but the conditions on $\sigma$ are quite hard to check in real problems. This is one of the reasons why NSA has not been developed for exchangeable data. Another (related) reason is the lack of constructive procedures for determining $\sigma$. 
A condition weaker than exchangeability, discussed in Section 3, is $P_\sigma$ conditionally identically distributed (c.i.d.). Even in this case, Problem (*) has been solved [6, Th. 3.1] but this time the conditions on $\sigma$ are simpler to verify. The class of admissible strategies is larger and includes several meaningful elements which cannot be used if $X$ is required to be exchangeable. As a consequence, NSA works quite well for c.i.d. data; see [8]-[9].

The stationary case is more involved. In fact, to our knowledge, there is no general characterization of the strategies $\sigma$ which make $P_\sigma$ stationary. However, such a characterization is available in some meaningful special cases (e.g. when $P_\sigma$ is also required to be Markov); see Section 4.

Finally, Problem (*) becomes simpler in the following (important) special case.

**Dominated strategies:** Let $\lambda$ be a $\sigma$-finite measure on $(S, B)$. Say that a strategy $\sigma$ is dominated by $\lambda$ if each $\sigma_n(x)$ admits a density $f_n(\cdot \mid x)$ with respect to $\lambda$, namely,

$$
\sigma_0(dy) = f_0(y) \lambda(dy) \quad \text{and} \quad \sigma_n(x, dy) = f_n(y \mid x) \lambda(dy)
$$

for all $n \geq 1$ and $x \in S^n$. Here, $f_0 : S \to \mathbb{R}^+$ and $f_n : S \times S^n \to \mathbb{R}^+$ are non-negative measurable functions.

For instance, if $S = \mathbb{R}$ and $\sigma_n(x)$ is a non-degenerate normal distribution for all $n$ and $x$, then $\sigma$ is dominated by $\lambda = \text{Lebesgue measure}$. Or else, if $S$ is countable, any strategy is dominated by $\lambda = \text{counting measure}$. Instead, if $S$ is uncountable, a non-dominated strategy is $\sigma_n(x_1, \ldots, x_n) = \delta_{x_n}$ where $\delta_{x_n}$ denotes the unit mass at the point $x_n$. Another non-dominated strategy is the empirical measure $\sigma_n(x_1, \ldots, x_n) = (1/n) \sum_{i=1}^n \delta_{x_i}$.

In a sense, dominated strategies play an analogous role to the usual dominated models in parametric statistical inference. The main advantage is that one can use the conditional density $f_n(\cdot \mid x)$ instead of the conditional measure $\sigma_n(x)$. A related advantage is that, if one fixes $\lambda$ and restricts to strategies dominated by $\lambda$, Problem (*) becomes simpler.

### 1.3. Content of this paper and further notation

This is a review paper on NSA which also includes some (minor) new results. Essentially, we aim to achieve three goals. First, we try to put NSA in the right framework, to provide a unifying view, and to make clear a few misunderstandings. This has been done in the Introduction. Second, in Section 2 and Subsection 3.1, we report some known results. Third, we provide some new strategies and we prove a few related results. The strategies, introduced by means of examples, deal with generalized Polya urns, random change points, covariates and stationary sequences. The results consist in determining the distribution of the data.
sequence $X$ under such strategies. To our knowledge, Examples 7, 9, 12, 14 and Theorems 8, 11, 13 are actually new, while Theorem 6 makes precise a claim contained in [26]. Moreover, as far as we know, Section 4 is the first attempt to develop NSA for stationary data. It provides a brief discussion of Problem (*) and introduces two large classes of stationary sequences.

Three dependence forms for $X$ are taken into account, namely, exchangeability, stationarity and conditional identity in distribution. Exchangeability is a typical assumption in Bayesian prediction problems. Stationarity is often neglected by Bayesian forecasters while it is a routine condition in the classical treatment of time series. Thus, it seems reasonable to consider stationarity from the Bayesian point of view as well. Conditional identity in distribution is not very popular but it is quite suitable for NSA; see Section 3. Finally, these three conditions are strictly connected among them. In fact,

$$X \text{ is exchangeable } \iff \ X \text{ is stationary and c.i.d.}$$

See [5] and [43].

The paper is organized in three sections, each concerned with a specific assumption on $X$, plus a final section of open problems. All the proofs are gathered in a final appendix.

We close this Introduction with some further notations.

As usual, $\delta_u$ is the unit mass at the point $u$. For each $x \in S^n$, where $n$ is a positive integer or $n = \infty$, we denote by $x_i$ the $i$-th coordinate of $x$. Moreover, we take $X$ to be the sequence of coordinate random variables on $S^\infty$, namely,

$$X_i(x) = x_i \quad \text{for all } i \geq 1 \text{ and } x \in S^\infty.$$ 

From now on, we fix a strategy $\sigma$ and we assume

$$X \sim P_\nu.$$ 

We write $\nu$ instead of $\sigma_0$ (i.e., we let $\sigma_0 = \nu$). Hence, $\nu$ is a probability measure on $B$ to be regarded as the distribution of $X_1$ under the strategy $\sigma$. Finally, to avoid technicalities, $S$ is assumed to be a Borel subset of a Polish space and $B$ the Borel $\sigma$-field on $S$.

2. Exchangeable data

A permutation of $S^n$ is a map $\phi : S^n \to S^n$ of the form

$$\phi(x) = (x_{j_1}, \ldots, x_{j_n}) \quad \text{for all } x \in S^n$$
where \((j_1, \ldots, j_n)\) is a fixed permutation of \((1, \ldots, n)\). A sequence \(Y = (Y_1, Y_2, \ldots)\) of random variables is *exchangeable* if
\[
\phi(Y_1, \ldots, Y_n) \sim (Y_1, \ldots, Y_n) \quad \text{for all } n \geq 2 \text{ and all permutations } \phi \text{ of } S^n.
\]

As noted in Subsection 1.2, if \(X\) is required to be exchangeable, to apply NSA is usually hard. But there are a few exceptions and two of them are discussed in this section. To this end, we first recall that \(X\) is a Dirichlet sequence (or a Polya sequence) if
\[
\sigma_n(x) = \frac{c \nu + \sum_{i=1}^{n} \delta_{x_i}}{n + c} \quad \text{for all } n \geq 0 \text{ and } x \in S^n,
\]
where \(c > 0\) is a constant, \(\nu\) a probability measure on \(\mathcal{B}\), and \(\sigma_0(x) = \nu\); see [11] and [25]. The role of Dirichlet sequences is actually huge in various frameworks, including Bayesian nonparametrics, population genetics, ecology, combinatorics and number theory; see e.g. [33], [41], [50], [51], [52]. From our point of view, however, two facts are to be stressed. First, a Dirichlet sequence is exchangeable. Second, being defined through its predictive distributions, a Dirichlet sequence is a natural candidate for NSA.

### 2.1. Species sampling sequences

For \(n \geq 1\) and \(x = (x_1, \ldots, x_n) \in S^n\), denote by \(k_n = k_n(x)\) the number of distinct values in the vector \(x\) and by \(x_1^*, \ldots, x_{k_n}^*\) such distinct values (in the order that they appear). Say that \(X\) is a *species sampling sequence* if it is exchangeable, \(\sigma_0 = \nu\) is non-atomic, and
\[
\sigma_n(x) = \sum_{j=1}^{k_n} p_{j,n}(x) \delta_{x_j^*} + q_n(x) \nu \quad \text{for all } n \geq 1 \text{ and } x \in S^n
\]
where the \(p_{j,n}\) are non-negative measurable functions on \(S^n\) and \(q_n = 1 - \sum_{j=1}^{k_n} p_{j,n}\). Under this strategy, quoting from [38, p. 253], \(X\) can be regarded as: “the sequence of species of individuals in a process of sequential random sampling from some hypothetical infinite population of individuals of various species. The species of the first individual to be observed is assigned a random tag \(X_1 = X_1^*\) distributed according to \(\nu\). Given the tags \(X_1, \ldots, X_n\) of the first \(n\) individuals observed, it is supposed that the next individual is one of the \(j\)-th species observed so far with probability \(p_{j,n}\), and one of a new species with probability \(q_n\).”

A nice consequence of the definition is that \(p_{j,n}(x)\) depends on \(x\) only through the vector \((N_{1,n}, \ldots, N_{k_{n,n}}, \ldots)\), where
\[
N_{j,n} = N_{j,n}(x) = \text{card}\left\{ k : 1 \leq k \leq n, x_k = x_j^* \right\}
\]
is the number of times that \( x_j^* \) appears in the vector \( x \); see [38] and [50].

The most popular example of species sampling sequence is probably the two-parameter Poisson-Dirichlet, introduced by Pitman in [49], which corresponds to the weights

\[
p_{j,n}(x) = \frac{N_{j,n} - b}{n + c} \quad \text{and} \quad q_n(x) = \frac{b k_n + c}{n + c}
\]

where \( b \) and \( c \) are constants such that: either (i) \( 0 \leq b < 1 \) and \( c > -b \) or (ii) \( b < 0 \) and \( c = -m b \) for some integer \( m \geq 2 \). In this model, if \( L \) denotes the number of distinct values appearing in the sequence \( X \), one obtains \( L \overset{a.s.}{=} \infty \) under (i) and \( L \overset{a.s.}{=} m \) under (ii). Note also that \( X \) reduces to a Dirichlet sequence in the special case \( b = 0 \).

Another example, due to [35], is

\[
p_{j,n}(x) = \frac{(N_{j,n} + 1)(n - k_n + b)}{n^2 + bn + c} \quad \text{and} \quad q_n(x) = \frac{k_n^2 - bk_n + c}{n^2 + bn + c}
\]

where \( b > 0 \) and \( c \) is such that \( k^2 + bk + c > 0 \) for all integers \( k > 0 \). This time, unlike the two-parameter Poisson-Dirichlet, \( L \) is a finite but non-degenerate random variable.

In general, to obtain a species sampling sequence, the forecaster needs to select \( \nu \) and the weights \( p_{j,n} \). While the choice of \( \nu \) is free (apart from non-atomicity) the \( p_{j,n} \) are subjected to the constraint that \( X \) should be exchangeable. (Incidentally, the choice of \( p_{j,n} \) is a good example of the difficulty of applying NSA when \( X \) is required to be exchangeable). The usual method to select \( p_{j,n} \) involves exchangeable random partitions. Let \( \mathbb{N} = \{1, 2, \ldots\} \) and let \( \Pi \) be a random partition of \( \mathbb{N} \). For each \( n \geq 1 \), call \( \Pi_n \) the restriction of \( \Pi \) to \( \{1, \ldots, n\} \), namely, the random partition of \( \{1, \ldots, n\} \) whose elements are of the form \( \{1, \ldots, n\} \cap A \) for some \( A \in \Pi \). Say that \( \Pi \) is exchangeable if

\[
\varphi(\Pi_n) \sim \Pi_n
\]

for all \( n \geq 1 \) and all permutations \( \varphi \) of \( \{1, \ldots, n\} \), where \( \varphi(\Pi_n) \) denotes the random partition \( \varphi(\Pi_n) = \{ \varphi(B) : B \in \Pi_n \} \). For instance, given any sequence \( Y = (Y_1, Y_2, \ldots) \) of random variables, define \( \Pi \) to be the random partition of \( \mathbb{N} \) induced by the equivalence relation \( i \sim j \iff Y_i = Y_j \). Then, \( \Pi \) is exchangeable provided \( Y \) is exchangeable. Now, the weights \( p_{j,n} \) of a species sampling sequence correspond, in a canonical way, to the probability law of an exchangeable partition; see [49]-[50]. Hence, choosing the \( p_{j,n} \) essentially amounts to choosing an exchangeable partition. We stop here for a detailed discussion of exchangeable partitions is beyond the scopes of this paper. The interested reader is referred to [34], [35], [49], [49], [52] and references therein.
A last remark is that the definition of species sampling sequences can be generalized. In particular, non-atomicity of $\nu$ can be dropped (as in [3] and [13]) and exchangeability can be replaced by some weaker condition (as in [1] and [2]).

### 2.2. Kernel based Dirichlet sequences

In [10], to generalize Dirichlet sequences while preserving their main properties, a class of strategies has been introduced. Among other things, such strategies make $X$ exchangeable.

A kernel $\alpha$ on $(S, B)$ is a collection

$$\alpha = \{\alpha(\cdot \mid x) : x \in S\}$$

such that $\alpha(\cdot \mid x)$ is a probability measure on $B$, for each $x \in S$, and the map $x \mapsto \alpha(A \mid x)$ is measurable for each $A \in B$. Sometimes, to make the notation easier, we will write $\alpha_x$ instead of $\alpha(\cdot \mid x)$. A straightforward example of kernel is $\alpha_x = \delta_x$ for each $x \in S$.

Fix a probability measure $\nu$ on $B$, a constant $c > 0$, a kernel $\alpha$ on $(S, B)$, and define the strategy

$$\sigma_n(x) = \frac{c \nu + \sum_{i=1}^{n} \alpha_{x_i}}{n + c}$$

for all $n \geq 0$ and $x \in S^n$. (2)

Clearly, $X$ reduces to a Dirichlet sequence if $\alpha = \delta$. In this case, we also say that $X$ is a classical Dirichlet sequence.

If $\alpha$ is an arbitrary kernel, $X$ may fail to be exchangeable. However, an useful sufficient condition for exchangeability is available. In fact, $X$ is exchangeable if $\alpha$ agrees with the conditional distribution for $\nu$ given some sub-$\sigma$-field $\mathcal{G} \subset B$. For instance, if $\mathcal{G} = B$, then $\alpha = \delta$ and $X$ is a classical Dirichlet sequence. At the opposite extreme, if $\mathcal{G}$ is the trivial $\sigma$-field, then $\alpha_x = \nu$ for all $x \in S$ and $X$ is i.i.d. with common distribution $\nu$. In general, for fixed $\nu$ and $c$, a strategy $\sigma$ which makes $X$ exchangeable can be associated with any sub-$\sigma$-field $\mathcal{G} \subset B$. It suffices to take $\alpha$ as the conditional distribution for $\nu$ given $\mathcal{G}$.

**Example 2. (Countable partitions).** Let $\mathcal{H}$ be a (non random) countable partition of $S$ such that $H \in \mathcal{B}$ and $\nu(H) > 0$ for all $H \in \mathcal{H}$. For $x \in S$, denote by $H_x$ the only $H \in \mathcal{H}$ such that $x \in H$. The conditional distribution for $\nu$ given the sub-$\sigma$-field generated by $\mathcal{H}$ is

$$\alpha(\cdot \mid x) = \sum_{H \in \mathcal{H}} 1_H(x) \nu(\cdot \mid H) = \nu(\cdot \mid H_x)$$

for all $x \in S$.

Hence, $X$ is exchangeable whenever

$$\sigma_n(x) = \frac{c \nu + \sum_{i=1}^{n} \nu(\cdot \mid H_{x_i})}{n + c}$$

for all $n \geq 0$ and $x \in S^n$.

Some remarks on the above strategy $\sigma$ are in order.
• \(\sigma\) may be reasonable when the basic information provided by each observation \(x_i\) is \(H_{x_i}\), namely, the element of the partition \(\mathcal{H}\) including \(x_i\).

• If \(S\) is countable, each sub-\(\sigma\)-field \(\mathcal{G} \subset \mathcal{B}\) is generated by a partition \(\mathcal{H}\) of \(S\). Hence, \(\alpha\) is necessarily as above.

• \(\sigma_n(x)\) is absolutely continuous with respect to \(\nu\) for all \(n\) and \(x\). This is a striking difference with classical Dirichlet sequences. To make an example, call \(\sigma^*\) the strategy obtained by \(\sigma\) replacing \(\alpha\) with \(\delta\). Under \(\sigma^*\), \(X\) is a classical Dirichlet sequence. Moreover, suppose \(\nu\) is nonatomic and define the set \(B(x) = \{x_1, \ldots, x_n\}\) for each \(x = (x_1, \ldots, x_n) \in S^n\). Since \(\nu\) is nonatomic and \(B(x)\) is finite,

\[
P_{\sigma}(X_{n+1} = X_i \text{ for some } i \leq n \mid (X_1, \ldots, X_n) = x) = \sigma_n(x, B(x)) = 0.
\]

On the other hand, since \(\delta_{x_i}(B(x)) = 1\) for each \(i = 1, \ldots, n,
\]

\[
P_{\sigma^*}(X_{n+1} = X_i \text{ for some } i \leq n \mid (X_1, \ldots, X_n) = x) = \sigma^*_n(x, B(x)) = n/(n + c).
\]

As a consequence, one obtains

\[
P_{\sigma}(\text{all the observations are distinct}) = 1 \quad \text{and} \quad P_{\sigma^*}(\text{all the observations are distinct}) = 0.
\]

• \(\sigma\) can be generalized replacing \(\alpha\) with

\[
\beta(\cdot \mid x) = 1_A(x) \delta_x + 1_{A^c}(x) \nu(\cdot \mid A^c \cap H_x),
\]

where \(A \in \mathcal{B}\) is a suitable set. Note that \(\beta\) reduces to \(\alpha\) if \(A = \emptyset\). Roughly speaking, \(\beta\) is reasonable in those problems where there is a set \(A\) such that \(x_i\) is informative about the future observations only if \(x_i \in A\). Otherwise, if \(x_i \not\in A\), the only relevant information provided by \(x_i\) is \(H_{x_i}\). As a trivial example, take

\[
S = \mathbb{R}, \quad \mathcal{H} = \{(-\infty, 0), \{0\}, (0, \infty)\}, \quad A = [-u, u]
\]

for some \(u > 0\). Then, \(\beta\) is reasonable if \(x_i\) is informative only if \(|x_i| \leq u\). Otherwise, if \(|x_i| > u\), the only meaningful information provided by \(x_i\) is its sign.

**Example 3. (Polya urns).** Some Polya urns are covered by Example 2. It follows that, for such urns, the sequence \(X\) of observed colors is exchangeable. To our knowledge, this fact was previously unknown.

As an example, consider sequential draws from an urn and denote by \(X_n\) the color of the ball extracted at time \(n \geq 1\). At time \(n = 0\), the urn contains \(m_j\) balls of color \(j\) where \(j \in \{1, \ldots, k\}\). Define

\[
S = \{1, \ldots, k\}, \quad m = \sum_{j=1}^k m_j \quad \text{and} \quad \nu \{ j \} = \frac{m_j}{m} \quad \text{for each } j \in S.
\]
The sampling scheme is as follows. Fix a partition $H$ of $S$ and define

$$m_j^* = m \nu \{\{j\} \mid H_j\} = \frac{m m_j}{\sum_{i \in H_j} m_i}.$$  

For each $n \geq 1$, one obtains $X_n \in H$ for some unique $H \in H$. In this case (i.e., if $X_n \in H$) the extracted ball is replaced together with $m_j^*$ more balls of color $j$ for each $j \in H$. In other terms, if the observed color belongs to $H$, each color in $H$ is reinforced (and not only the observed color). In particular, after each draw, $m$ new balls are added to the urn. Hence, denoting by $\sigma$ the strategy of Example 2 with $c = 1$, one obtains

$$P(X_{n+1} = j \mid (X_1, \ldots, X_n) = x) = \frac{m_j + \sum_{i=1}^n 1_{H_j}(x_i) m_j^*}{m + m n} = \frac{\nu \{j\} + \sum_{i=1}^n 1_{H_j}(x_i) \nu \{\{j\} \mid H_j\}}{1 + n} = \frac{c \nu \{j\} + \sum_{i=1}^n \nu \{\{j\} \mid H_{x_i}\}}{c + n} = \sigma_n(x)\{j\}.$$

If $\sigma$ is the strategy (2), in addition to exchangeability, $X$ satisfies various other properties of classical Dirichlet sequences. We refer to [10] for details. Here, we just note that the prior $\pi$ and the posterior $\pi_n$ can be explicitly determined. In particular, up to replacing $\delta$ with $\alpha$, the Sethuraman’s representation of $\pi$ (see [53]) is still true. Precisely, $\pi$ is the probability distribution of a random probability measure $\mu$ of the form

$$\mu(\cdot) = \sum_j V_j \alpha(\cdot \mid Z_j)$$

where:

- $(Z_j)$ and $(V_j)$ are independent sequences of random variables;
- $(Z_j)$ is i.i.d. with common distribution $\nu$;
- $V_j = U_j \prod_{i=1}^{j-1} (1 - U_i)$ for all $j \geq 1$, where $(U_i)$ is i.i.d. with common distribution beta(1, $c$). Namely, $(V_j)$ has the stick breaking distribution with parameter $c$.

3. Conditionally identically distributed data

A sequence $Y = (Y_1, Y_2, \ldots)$ of random variables is conditionally identically distributed (c.i.d.) if

$$Y_2 \sim Y_1 \quad \text{and} \quad P(Y_2 \in \cdot \mid Y_1, \ldots, Y_n) = P(Y_{n+1} \in \cdot \mid Y_1, \ldots, Y_n) \quad \text{a.s.}$$

for all $k > n \geq 1$. Such a condition is weaker than exchangeability. Indeed, $Y$ is exchangeable if and only if it is stationary and c.i.d.; see [5] and [43]. We also note that a c.i.d. sequence is identically distributed.
C.i.d. sequences have been introduced in [5] and [43] and then investigated or applied in various papers; see e.g. [1], [2], [6], [7], [8], [9], [14], [26], [27], [30] and references therein.

There are reasons for taking c.i.d. data into account in Bayesian prediction problems. In a sense, c.i.d. sequences have been actually introduced having prediction in mind. If \( X \) is c.i.d., at each time \( n \), the future observations \( (X_k : k > n) \) are identically distributed given the past, and this is reasonable in several prediction problems. Examples arise in clinical trials, generalized Polya urns, species sampling models, survival analysis and disease surveillance; see [1], [2], [5], [14], [26], [27], [31]. A further reason for assuming \( X \) c.i.d. is the asymptotic theory. In fact, the asymptotics of c.i.d. sequences is very close to that of exchangeable ones. As a consequence, a meaningful part of the usual Bayesian machinery can be developed under the sole assumption that \( X \) is c.i.d.; see [26]. Finally, it is not hard to characterize the strategies which make \( X \) c.i.d. Hence, unlike the exchangeable case, NSA can be easily implemented for c.i.d. data. A number of interesting strategies, which cannot be used if \( X \) is required to be exchangeable, become available if \( X \) is only asked to be c.i.d.; see e.g. [8] and [9].

In this section, following [8]-[9], NSA is applied to c.i.d. data. We first report some known strategies (Subsection 3.1) and we then introduce two new strategies which make \( X \) c.i.d. (Subsection 3.2).

### 3.1. Fast recursive update of predictive distributions

A possible property of a strategy \( \sigma \) is

\[
\sigma_{n+1}(x, y) \text{ is a function of } \sigma_n(x) \text{ and } y \tag{3}
\]

for all \( n \geq 0, x \in S^n \) and \( y \in S \). Here, \( y \) denotes the \((n+1)\)-th observation and \n
\[(x, y) = (x_1, \ldots, x_n, y).\]

Condition (3), introduced in [37], is motivated by applications. It means that the predictive \( \sigma_{n+1}(x, y) \) is just a recursive update of the previous predictive \( \sigma_n(x) \) and the last observation \( y \).

For each \( n \geq 0 \), let \( q_n : S^n \to [0, 1] \) be a measurable function (with \( q_0 \) constant) and \( \alpha_n \) a kernel on \( (S, B) \). Define a strategy \( \sigma \) through the recursive equations

\[
\sigma_0 = \nu \quad \text{and} \quad \sigma_{n+1}(x, y) = q_n(x) \sigma_n(x) + (1 - q_n(x)) \alpha_n(\cdot | y) \tag{4}
\]

for all \( n \geq 0, x \in S^n \) and \( y \in S \). Such a \( \sigma \) satisfies condition (3). In fact, each predictive \( \sigma_{n+1}(x, y) \) is a convex combination of the previous predictive \( \sigma_n(x) \) and the kernel \( \alpha_n(\cdot | y) \) which depends only on \( y \). The obvious interpretation is
that, at time \( n + 1 \), after observing \((x, y)\), the next observation is drawn from \( \sigma_n(x) \) with probability \( q_n(x) \) and from \( \alpha_n(\cdot \mid y) \) with probability \( 1 - q_n(x) \).

The strategies satisfying equation (4) are investigated in [9]. Under such strategies, \( X \) is usually not exchangeable but it is c.i.d. under some conditions on the kernels \( \alpha_n \). Precisely, \( X \) is c.i.d. if \( \alpha_n(\cdot \mid y) = \delta_y \) for all \( y \in S \) (just take \( G_n = B \) for all \( n \geq 0 \)).

**Example 4. (Finer countable partitions).** For each \( n \geq 0 \), let \( \mathcal{H}_n \) be a countable partition of \( S \) such that \( H \in B \) and \( \nu(H) > 0 \) for all \( H \in \mathcal{H}_n \). Suppose that \( \mathcal{H}_{n+1} \) is finer than \( \mathcal{H}_n \) for all \( n \geq 0 \). Define \( \sigma \) through equation (4) with

\[
\alpha_n(\cdot \mid y) = \sum_{H \in \mathcal{H}_n} 1_H(y) \nu(\cdot \mid H) = \nu(\cdot \mid H_n^y)
\]

where \( H_n^y \) denotes the only \( H \in \mathcal{H}_n \) such that \( y \in H \). The kernel \( \alpha_n \) is the conditional distribution for \( \nu \) given \( G_n \), where \( G_n \) is the \( \sigma \)-field generated by \( \mathcal{H}_n \). Since \( \mathcal{H}_{n+1} \) is finer than \( \mathcal{H}_n \), one obtains \( G_n \subset G_{n+1} \). Hence, \( X \) is c.i.d. Note also that the \( \mathcal{H}_n \) could be chosen such that

\[
\{y\} = \bigcap_n H_n^y \quad \text{for all } y \in S.
\]

In this case, as \( n \to \infty \), the partitions \( \mathcal{H}_n \) shrink to the partition of \( S \) in the singletons.

For instance, consider the strategy of Example 2 and suppose the forecaster wants to replace the fixed partition \( \mathcal{H} \) with a sequence \( \mathcal{H}_n \) of finer partitions. This is possible at the price of having \( X \) c.i.d. instead of exchangeable. In fact, with \( q_n = \frac{n + c}{n + 1 + c} \), one obtains

\[
\sigma_n(x) = \frac{c \nu + \sum_{i=1}^n \alpha_{i-1}(\cdot \mid x_i) + \sum_{i=1}^n \nu(\cdot \mid H_x^{i-1})}{n + c}.
\]

Or else, given a constant \( q \in (0, 1) \), consider the strategy

\[
\sigma^*_n(x) = q^n \nu + (1 - q) \sum_{i=1}^n q^{n-i} \delta_x.
\]

To make predictions through \( \sigma^* \) corresponds to exponential smoothing and may be reasonable when the forecaster has only vague opinions about the data. Moreover, \( P_{\sigma^*} \) is c.i.d.; see [8, Ex. 7]. The strategy \( \sigma \) could be used to decrease
the impact of the observed data while preserving the c.i.d. condition. In fact, with \( q_n = q \) for all \( n \geq 0 \), one obtains

\[
\sigma_n(x) = q^n\nu + (1 - q)\sum_{i=1}^{n} q^{n-i}\nu(H_{x_i}^{-1}).
\]

It is worth noting that equation (4) reminds Newton’s algorithm [47]-[48]. Such an algorithm provides in fact a further example of a strategy satisfying (4). More precisely, the algorithm aims to estimate the prior distribution rather than to make predictions. However, it may be reinterpreted as a predictive rule. In other terms, Newton’s algorithm corresponds to a strategy \( \sigma \) and such a \( \sigma \) meets equation (4) for a suitable choice of \( q_n \) and \( \alpha_n \); see e.g. [31, p. 1095]. Moreover, as shown in [31], \( \sigma \) makes \( X \) c.i.d.

We next turn to a strategy introduced in [37]. Once again, under this strategy, the data are c.i.d. but not necessarily exchangeable.

Example 5. (Hahn, Martin and Walker; Copulas). In this example, \( S = \mathbb{R} \) and “density function” means “density function with respect to Lebesgue measure”. A bivariate copula is a distribution function on \( \mathbb{R}^2 \) whose marginals are uniform on \((0, 1)\). The density function of a bivariate copula, provided it exists, is said to be a copula density.

In [37], in order to realize condition (3), the following updating rule is introduced. Fix a density \( f_0 \) and a sequence \( c_1, c_2, \ldots \) of bivariate copula densities. For the sake of simplicity, we assume \( f_0 > 0 \) and \( c_n > 0 \) for all \( n \geq 1 \). For \( n = 0 \), define

\[
\sigma_0(dz) = f_0(z) \, dz
\]

and call \( F_0 \) the distribution function corresponding to \( \sigma_0 \). Then, for each \( y \in \mathbb{R} \), define

\[
\sigma_1(y, dz) = f_1(z \mid y) \, dz \quad \text{where} \quad f_1(z \mid y) = c_1\{F_0(z), F_0(y)\} f_0(z).
\]

In general, for each \( n \geq 0 \) and \( x \in \mathbb{R}^n \), suppose \( \sigma_n(x) \) has been defined and denote by \( f_n(\cdot \mid x) \) and \( F_n(\cdot \mid x) \) the density and the distribution function of \( \sigma_n(x) \). Then, one can define

\[
\sigma_{n+1}(x, y, dz) = f_{n+1}(z \mid x, y) \, dz \quad \text{for all} \ y \in \mathbb{R}, \tag{5}
\]

where

\[
f_{n+1}(z \mid x, y) = c_{n+1}\{F_n(z \mid x), F_n(y \mid x)\} f_n(z \mid x). \tag{6}
\]

Equations (5)-(6) define a strategy \( \sigma \) dominated by Lebesgue measure.
In [37] (but not here) the $c_n$ are also required to be symmetric. Furthermore, in [37], equation (6) is not necessarily viewed as a method for obtaining a strategy but is *deduced* as a consequence of exchangeability. From our point of view, instead, equations (5)-(6) define a strategy $\sigma$ which we call HMW’s strategy.

Under HMW’s strategy, $X$ is not necessarily exchangeable, even if the $c_n$ are symmetric and $c_n \to 1$ (in some sense) as $n \to \infty$. To see this, recall that $X$ is i.i.d. if and only if it is exchangeable and $X_1$ is independent of $X_2$. In turn, $X_1$ is independent of $X_2$ if $c_1$ is the independence copula density (i.e., $c_1(u,v) = 1$ for all $(u,v) \in [0,1]^2$). Therefore, $X$ fails to be exchangeable whenever $c_1$ is the independence copula density and $c_2 \neq c_1$. However, as noted in [26], $X$ turns out to be c.i.d.

**Theorem 6.** If $\sigma$ is HMW’s strategy, then $X$ is c.i.d.

A proof of Theorem 6 is provided in the Appendix. We note that, for Theorem 6 to hold, the positivity assumption on $f_0$ and $c_n$ may be dropped and the $c_n$ can be taken to be conditional copula densities; see Remark 16.

3.2. Further examples

Suppose that, according to the forecaster, the data should be exchangeable until a random time and then go on so as to form a c.i.d. sequence. This idea is realized in the next example.

**Example 7. (Change points).** A predictable stopping time is a function $T$ on $S^\infty$, with values in \{2, 3, \ldots, \infty\}, satisfying

$$\{T = n+1\} = \{(X_1, \ldots, X_n) \in A_n\}$$

for some set $A_n \in B^n$. (7)

Basically, condition (7) means that the event $\{T = n+1\}$ depends only on $(X_1, \ldots, X_n)$. Similarly, $\{T \leq n+1\} = \bigcup_{j=2}^{n+1}\{T = j\}$ depends only on $(X_1, \ldots, X_n)$. As a consequence, for all $x \in S^n$ and $y \in S$, the indicators of $\{T \leq n+1\}$ and $\{T > n+1\}$ depend on $x$ but not on $y$.

Fix a predictable stopping time $T$ and a strategy $\beta = (\beta_0, \beta_1, \ldots)$ which makes $X$ exchangeable. Moreover, as in Subsection 3.1, fix the measurable functions $q_n : S^n \to [0,1]$. Then, define $\sigma_0 = \beta_0$, $\sigma_1 = \beta_1$, and

$$\sigma_{n+1}(x,y) = 1\{T > n+1\}(x) \beta_{n+1}(x,y) + 1\{T \leq n+1\}(x) \left\{q_n(x) \sigma_n(x) + (1 - q_n(x)) \delta_y\right\}$$

for all $n \geq 1$, $x \in S^n$ and $y \in S$. In the Appendix, it is shown that:
Theorem 8. The above strategy $\sigma$ makes $X$ c.i.d. Moreover, if
\[ A_n \text{ is invariant under permutations of } S^n \text{ for all } n \geq 1 \]
(where $A_n$ is the set involved in condition (7)) then $(X_1, \ldots, X_n)$ is exchangeable conditionally on $T > n$, in the sense that
\[ P_\sigma(\phi(X_1, \ldots, X_n) \in \cdot \mid T > n) = P_\sigma((X_1, \ldots, X_n) \in \cdot \mid T > n) \]
for all $n$ such that $P_\sigma(T > n) > 0$ and all permutations $\phi$ of $S^n$.

Theorem 8 is still valid if $\sigma$ is defined differently at the times subsequent to $T$. For instance, given a countable partition $\mathcal{H}$ of $S$, the conclusions of Theorem 8 are true even if
\[ \sigma_{n+1}(x, y) = q_n(x) \sigma_n(x) + (1 - q_n(x)) \sigma_n(x, \cdot \mid H_y) \]
for all $x \in S^n$ and $y \in S$ such that $T \leq n + 1$ and $\sigma_n(x, H_y) > 0$. Here, $\sigma_n(x, \cdot \mid H_y)$ denotes the probability measure
\[ \sigma_n(x, A \mid H_y) = \frac{\sigma_n(x, A \cap H_y)}{\sigma_n(x, H_y)} \]
for all $A \in \mathcal{B}$.

Censored survival times are a possible application of $\sigma$. Suppose that $S = \{0, 1\} \times (0, \infty)$ and the $i$-th observation is a pair $x_i = (j_i, t_i)$ where $t_i$ is the survival time of item $i$, or the time when item $i$ leaves the trial, according to whether $j_i = 1$ or $j_i = 0$. In this framework, $T - 1$ could be the first time when a fixed number $k$ of survival times is observed, namely,
\[ T = 1 + \inf\{n : \sum_{i=1}^{n} j_i = k\} \quad \text{with the usual convention } \inf \emptyset = \infty. \]

Finally, the strategy $\beta$ could be as in Subsection 2.2. In fact, classical Dirichlet sequences are a quite popular model to describe censored survival times but have the drawback of ties. This drawback is decreased if $\beta$ is of the form
\[ \beta_n(x) = c \nu + \sum_{i=1}^{n} \alpha_{x_i} \]
and the kernel $\alpha$ is such that $\alpha_u$ is nonatomic for all $u \in S$.

So far, the $n$-th predictive distribution has been meant as the conditional distribution of $X_{n+1}$ given $(X_1, \ldots, X_n)$. But the information available at time $n$ is often larger than $(X_1, \ldots, X_n)$. To model this situation, we consider a sequence of pairs
\[ X_n = (Y_n, Z_n) \]
where \( Y_n \) is the object of interest while \( Z_n \) is a covariate. At each time \( n \), the forecaster aims to predict \( Y_{n+1} \) based on \( (Y_1, Z_1), \ldots, (Y_n, Z_n) \). She is not interested in \( Z_{n+1} \) as such, but \( Z_1, \ldots, Z_n \) can not be neglected since they are informative on \( Y_{n+1} \). Moreover, she wants \( Y = (Y_1, Y_2, \ldots) \) to be c.i.d. and \( Z = (Z_1, Z_2, \ldots) \) unconstrained as much as possible. One solution could be a strategy which makes \( X \) c.i.d. However, if \( X \) is c.i.d., both \( Y \) and \( Z \) are marginally c.i.d., and having \( Z \) c.i.d. may be unwelcome. In the next example, \( Y \) is c.i.d. but \( Z \) is not. In addition, \( Y \) satisfies a condition stronger than the c.i.d. one, that is,

\[
Y_2 \sim Y_1 \quad \text{and} \quad P(Y_k \in \cdot \mid X_1, \ldots, X_n) = P(Y_{n+1} \in \cdot \mid X_1, \ldots, X_n) \quad \text{a.s. (8)}
\]

for all \( k > n \geq 1 \); see [5].

**Example 9. (Covariates).** Let \( S = \mathbb{R}^2 \) and

\[
0 < b_0 < b_1 < b_2 < \ldots, \quad \sup_n b_n \leq 1,
\]

a bounded strictly increasing sequence of real numbers. Take \( \sigma_0 \) as the probability distribution of \( (U + V, V) \) where

\[
U \text{ independent of } V, \quad U \sim \mathcal{N}(0, b_1), \quad V \sim \mathcal{N}(0, 1 - b_1).
\]

Similarly, for each \( n \geq 1 \) and

\[
x = (x_1, \ldots, x_n) = ((y_1, z_1), \ldots, (y_n, z_n)),
\]

take \( \sigma_n(x) \) as the probability distribution of \( (U_n(x) + V_n(x), V_n(x)) \) where

\[
U_n(x) \text{ independent of } V_n(x), \quad U_n(x) \sim \mathcal{N}(y_n - z_n, b_{n+1} - b_n), \quad V_n(x) \sim \mathcal{N}(0, 1 - b_{n+1}).
\]

Then, \( Z \) is not c.i.d. while \( Y \) satisfies condition (8). Furthermore, arguing as in [9, Sect. 4], the normal distribution could be replaced by any symmetric stable law.

To see that \( Z \) is not c.i.d., just note that \( Z \) fails to be identically distributed. To prove condition (8), take a collection \( \{T_n, W_n : n \geq 1\} \) of independent standard normal random variables and define the sequence \( X^* = (X_1^*, X_2^*, \ldots) \), where

\[
X_n^* = (Y_n^*, Z_n^*), \quad Y_n^* = \sum_{j=1}^n \sqrt{b_j - b_{j-1}} T_j + \sqrt{1 - b_n} W_n, \quad Z_n^* = \sqrt{1 - b_n} W_n.
\]

It is not hard to verify that \( X^* \sim X \). Hence, it suffices to prove (8) with \( X^* \) in the place of \( X \), and this can be done arguing as in [5, Ex. 1.2]. We omit the explicit calculations.
4. Stationary data

A sequence \( Y = (Y_1, Y_2, \ldots) \) of random variables is *stationary* if

\[
(Y_2, \ldots, Y_{n+1}) \sim (Y_1, \ldots, Y_n) \quad \text{for all } n \geq 1.
\]

In the non-Bayesian approaches to predictive inference, stationarity is a classical assumption. In a Bayesian framework, instead, stationarity seems to be less popular. In particular, to our knowledge, there is no systematic treatment of NSA for the stationary case. This section aims to fill this gap and begins an investigation of NSA when the data sequence \( X \) is required to be stationary.

After some general remarks on Problem (*), two large classes of stationary sequences are introduced. This is just a preliminary step, however, and much more work is to be done.

If \( X \) is required to be stationary, for NSA to apply, the strategies which make \( X \) stationary should be characterized. Hence, one comes across Problem (*) with \( C \) the class of stationary probability measures on \((S^\infty, B^\infty)\). This version of Problem (*) is quite hard and we are not aware of any general solution; see e.g. [12], [46] and references therein. Fortunately, however, Problem (*) is simple (or even trivial) in a few special cases. As an example, a strategy \( \sigma \) makes \( X \) a stationary (first order) Markov chain if and only if

\[
\int \sigma_1(x, \cdot) \sigma_0(dx) = \sigma_0(\cdot) \quad \text{and} \quad \sigma_n(x) = \sigma_1(x_{n})
\]

for all \( n \geq 1 \) and \( P_\sigma \)-almost all \( x \in S^n \). Even if obvious, this fact has an useful practical consequence. If the data are required to be stationary and Markov, to make Bayesian predictive inference via NSA is actually straightforward.

Another remark is that, unlike the exchangeable case, a finite dimensional stationary random vector can be always extended to an (infinite) stationary sequence. To formalize this fact, we first recall that the probability distribution of the random vector \((X_1, \ldots, X_n)\) is completely determined by \( \sigma_0, \sigma_1, \ldots, \sigma_{n-1} \).

**Lemma 10.** Fix \( n \geq 1 \), select \( \sigma_0, \sigma_1, \ldots, \sigma_{n-1} \) and define

\[
\sigma_j(u, x) = \sigma_{n-1}(x) \quad \text{for all } j > n - 1, u \in S^{j-n+1} \text{ and } x \in S^{n-1}.
\]

If \((X_2, \ldots, X_n) \sim (X_1, \ldots, X_{n-1})\), then \( X \) is stationary.

Lemma 10 is probably well known, but again we do not know of any explicit reference. Anyway, the proof is straightforward. It suffices to note that, under the strategy of Lemma 10, \( X_{j+1} \) is conditionally independent of \((X_1, \ldots, X_{j-n+1})\) given \((X_{j-n+2}, \ldots, X_j)\).

A last remark is that Problem (*) admits an obvious solution for dominated strategies. In this case, incidentally, Problem (*) can be easily solved even for exchangeable data.
Theorem 11. Let $\lambda$ be a $\sigma$-finite measure on $(S, \mathcal{B})$ and $\sigma$ a strategy dominated by $\lambda$, say

$$\sigma_0(dy) = f_0(y) \lambda(dy) \quad \text{and} \quad \sigma_n(x, dy) = f_n(y \mid x) \lambda(dy)$$

for all $n \geq 1$ and $x \in S^n$. Define

$$g_n(x) = f_0(x_1) f_1(x_2 \mid x_1) \cdots f_{n-1}(x_n \mid x_1, \ldots, x_{n-1})$$

for all $n \geq 1$ and $x \in S^n$. Then,

- $P_\sigma$ is stationary if and only if
  $$g_n(x) = \int g_{n+1}(u, x) \lambda(du)$$
  for all $n \geq 1$ and $P_\sigma$-almost all $x \in S^n$.

- $P_\sigma$ is exchangeable if and only if
  $$g_n(\phi(x)) = g_n(x)$$
  for all $n \geq 2$, all permutations $\phi$ of $S^n$ and $P_\sigma$-almost all $x \in S^n$.

The proof of Theorem 11 is given in the Appendix.

We finally give two examples. In both, $X$ is a stationary Markov sequence, possibly of order greater than 1.

Example 12. (Generalized autoregressive sequences). Let $S = \mathbb{R}$. Fix a probability measure $\mu$ on $\mathcal{B}$ and a measurable function $f : \mathbb{R} \to \mathbb{R}$. Define

$$\sigma_1(x, A) = P(f(x) + U \in A) \quad \text{for all } x \in \mathbb{R} \text{ and } A \in \mathcal{B},$$

where $U$ is a real random variable such that $U \sim \mu$. Suppose now that

$$\int \sigma_1(x, A) \nu(dx) = \nu(A), \quad A \in \mathcal{B}, \quad (9)$$

for some probability measure $\nu$ on $\mathcal{B}$. Then, $X$ is a stationary Markov chain provided

$$\sigma_0 = \nu \quad \text{and} \quad \sigma_n(x) = \sigma_1(x_n) \quad \text{for all } n \geq 2 \text{ and } x \in \mathbb{R}^n.$$  

Note that $Y \sim P_\sigma$ for any sequence $Y = (Y_1, Y_2, \ldots)$ such that

$$Y_1 \sim \nu \quad \text{and} \quad Y_n = f(Y_{n-1}) + U_n \text{ for } n \geq 2,$$

where $(U_n : n \geq 2)$ is i.i.d., independent of $Y_1$, and $U_2 \sim \mu$. Thus, $\mu$ can be regarded as the distribution of the “errors” $U_n$ and $\nu$ as the marginal distribution.
of the observations $Y_n$. For instance, the usual Gaussian (first order) autoregressive processes correspond to $f(x) = cx$, $\mu = \mathcal{N}(0, b)$ and $\nu = \mathcal{N}(0, b/(1 - c^2))$, where $c \in (-1, 1)$ and $b > 0$ are constants.

To make the above argument concrete, the following problem is to be solved: For fixed $f$ and $\mu$, give conditions for the existence of $\nu$ satisfying equation (9). More importantly, give an explicit formula for $\nu$ provided it exists. We next focus on this problem in the (meaningful) special case where $\mu$ is a symmetric stable law.

Let $\gamma \in (0, 2]$ be a constant and $Z$ a real random variable with characteristic function
\[
E\{\exp(itZ)\} = \exp\left(-\frac{|t|\gamma}{2}\right) \quad \text{for all } t \in \mathbb{R}.
\]
(The exponent $\gamma$ is usually denoted by $\alpha$, but this notation cannot be adopted in this paper since $\alpha$ denotes a kernel). For $a \in \mathbb{R}$ and $b > 0$, denote by $S(a, b)$ the probability distribution of $a + b^{1/\gamma}Z$, namely
\[
S(a, b; A) = P\left(a + b^{1/\gamma}Z \in A\right) \quad \text{for all } A \in \mathcal{B}.
\]
The probability measure $S(a, b)$ is said to be a symmetric stable law with exponent $\gamma$. Note that $S(a, b) = \mathcal{N}(a, b)$ if $\gamma = 2$ and $S(a, b) = \mathcal{C}(a, b)$ if $\gamma = 1$, where $\mathcal{C}(a, b)$ is the Cauchy distribution with density $f(x) = \frac{2b}{\pi(1 + b^2(x-a)^2)}$ (the standard Cauchy distribution corresponds to $a = 0$ and $b = 2$).

**Theorem 13.** Let $c \in (-1, 1)$ be a constant. If $\mu = S(a, b)$ and $f(x) = -a + cx$, then equation (9) is satisfied by
\[
\nu = S\left(0, \frac{b}{1 - |c|^\gamma}\right).
\]
By Theorem 13, which is proved in the Appendix, one obtains (first order) stationary autoregressive processes with any symmetric stable marginal distribution.

**Example 14.** (Markov sequences of arbitrary order). Let $\lambda$ be a $\sigma$-finite measure on $(S, \mathcal{B})$. Fix $n \geq 2$ and a measurable function $h$ on $S^n$ such that $h > 0$ and $\int h \, d\lambda^n = 1$. Given $h$, define a further function $g$ via cyclic permutations of $h$, namely
\[
g(x) = \frac{1}{n} \left\{h(x_1, \ldots, x_n) + h(x_2, \ldots, x_n, x_1) + \ldots + h(x_n, x_1, \ldots, x_{n-1})\right\} \quad \text{for all } x \in S^n.
\]
Such a $g$ is still a density with respect to $\lambda^n$ (since $\int g \, d\lambda^n = 1$) and satisfies
\[
g(x, y) = g(y, x) \quad \text{for all } x \in S^{n-1} \text{ and } y \in S.
\]
Next, define
\[ f_0(x) = \int g(x, v) \lambda^{n-1}(dv) \quad \text{for all } x \in S, \]
\[ f_{n-1}(x_n \mid x_1, \ldots, x_{n-1}) = \frac{g(x)}{\int g(x_1, \ldots, x_{n-1}, v) \lambda(dv)} \quad \text{for all } x \in S^n, \]
and
\[ f_{j-1}(x_j \mid x_1, \ldots, x_{j-1}) = \frac{\int g(x, v) \lambda^{n-j}(dv)}{\int g(x_1, \ldots, x_{j-1}, v) \lambda^{n-j+1}(dv)} \]
for all \( 2 \leq j \leq n-1 \) and \( x \in S^j \). Finally, define a strategy \( \sigma \) dominated by \( \lambda \) as
\[ \sigma_0(dz) = f_0(z) \lambda(dz), \quad \sigma_j(x, dz) = f_j(z \mid x) \lambda(dz) \quad \text{for } 1 \leq j \leq n-1, \ x \in S^j, \]
and \( \sigma_j(u, x) = \sigma_{n-1}(x) \) for all \( j > n-1, \ u \in S^{j-n+1} \) and \( x \in S^{n-1} \).

Under \( \sigma \), a density of \( (X_1, \ldots, X_n) \) is given by \( g \). By equation (10),
\[ \int g(v, x) \lambda(dv) = \int g(x, v) \lambda(dv) \quad \text{for all } x \in S^{n-1} \]
and this in turn implies
\[ (X_2, \ldots, X_n) \sim (X_1, \ldots, X_{n-1}). \]

Therefore, \( X \) is stationary because of Lemma 10. Note also that \( X \) is a Markov sequence of order \( n-1 \).

5. Concluding remarks and open problems

As noted in the Introduction, when prediction is the main target, NSA has various advantages with respect to SA. To make NSA a concrete tool, however, some further work is to be done. We close this paper with a brief list of open problems and possible hints for future research.

- In most applications, the available information strictly includes the values taken by the past observations. Hence, the topic addressed in Example 9 should be developed.

- Similarly, Section 4 should be expanded. It would be nice to have a general solution of Problem (*) for both the stationary and the stationary-ergodic cases. Further examples of stationary sequences (possibly, non Markovian) would be welcome as well.

- Obviously, NSA could be investigated under other distributional assumptions, in addition to exchangeability, stationarity and conditional identity in distribution.
• A question, related to Example 5, is: Under what conditions \( X \) is exchangeable when \( \sigma \) is HMW’s strategy?

• While probably hard, the problem raised in Example 12 looks intriguing. In Theorem 13, such a problem has been addressed when \( \mu \) is a symmetric stable law and \( f \) has a special form. What happens if \( \mu \) and \( f \) are both arbitrary?

• A question, raised by an anonymous referee, is whether NSA can be connected with the decision theory interpretation of Bayesian statistics and its relation to frequentist notions, such as the complete class theorems. Among other things, the referee claims: “In an introductory Bayes class, it is very nice to point out that a Bayes estimator minimizes Bayes risk in the case that the analyst’s prior is the same as “nature’s”. Can this basic result be translated or generalized into the NSA notion of strategies in any interesting way?”

• A last general issue is as follows. According to us, a Bayesian forecaster should take her responsibility and make the resulting choices. That is to say, she has to select a prior \( \pi \), if she adopts SA, or a strategy \( \sigma \) if she prefers NSA. However, in case of NSA, she could also take an empirical Bayes point of view. Hence, every unknown quantity involved in her strategy could be estimated based on the available data. This is admissible but there is a technical issue. To illustrate, let us take \( \mu = \mathcal{N}(0, b) \) and \( f(x) = c x \) in Example 12, and suppose that \( b \) and \( c \) are estimated instead of crudely assigned. Let \( \hat{b}_n(x) \) and \( \hat{c}_n(x) \) be the estimates of \( b \) and \( c \) based on \( x = (x_1, \ldots, x_n) \). Also, let \( \hat{\sigma}_n(x) \) be the probability measure obtained by \( \sigma_n(x) \) replacing \( b \) and \( c \) with \( \hat{b}_n(x) \) and \( \hat{c}_n(x) \). If stationarity is required, \( \hat{b}_n(x) \) and \( \hat{c}_n(x) \) should be such that the strategy \( \hat{\sigma} = (\hat{\sigma}_0, \hat{\sigma}_1, \ldots) \) makes \( X \) stationary. This is a completely open problem.

Appendix

This appendix contains the proofs of some claims scattered throughout the text. We will need the following characterization of c.i.d. sequences in terms of strategies.

**Theorem 15.** *(Theorem 3.1 of [6]).* Let \( \sigma \) be a strategy. Then, \( P_\sigma \) is c.i.d. if and only if

\[
\sigma_n(x, A) = \int \sigma_{n+1}(x, y, A) \sigma_n(x, dy)
\]

for all \( n \geq 0 \), all \( A \in \mathcal{B} \) and \( P_\sigma \)-almost all \( x \in S^n \).
Proof of Theorem 6. In this proof, “density function” stands for “density function with respect to Lebesgue measure”. We first recall a well known fact.

Let $C$ be a bivariate copula and $F_1$, $F_2$ distribution functions on $\mathbb{R}$. Suppose that $C$, $F_1$ and $F_2$ all have densities, say $c$, $f_1$ and $f_2$, respectively. Then,

$$F(x, y) = C\{F_1(x), F_2(y)\}$$

is a distribution function on $\mathbb{R}^2$ and

$$f(x, y) = c\{F_1(x), F_2(y)\} f_1(x) f_2(y)$$

is a density of $F$. Therefore, for all $y \in \mathbb{R}$ with $f_2(y) > 0$, one obtains

$$\int c\{F_1(x), F_2(y)\} f_1(x) \, dx = \int \frac{f(x, y)}{f_2(y)} \, dx = 1.$$

We next show that equations (5)-(6) actually define a strategy $\sigma$. Fix a density $f_0 > 0$ and a sequence $c_1, c_2, \ldots$ of strictly positive bivariate copula densities. For each $y \in \mathbb{R}$,

$$\int f_1(z \mid y) \, dz = \int c_1\{F_0(z), F_0(y)\} f_0(z) \, dz = 1 \quad \text{since } f_0(y) > 0.$$

Moreover, $f_1(z \mid y) > 0$ for all $z$ due to $f_0 > 0$ and $c_1 > 0$. Next, suppose that $f_n(\cdot \mid x)$ is a strictly positive density for some $n \geq 1$ and $x \in \mathbb{R}^n$. Then, for all $y \in \mathbb{R}$,

$$\int f_{n+1}(z \mid x, y) \, dz = \int c_{n+1}\{F_n(z \mid x), F_n(y \mid x)\} f_n(z \mid x) \, dz = 1 \quad \text{since } f_n(y \mid x) > 0.$$

Furthermore, $f_{n+1}(z \mid x, y) > 0$ for all $z$ since $f_n(\cdot \mid x) > 0$ and $c_{n+1} > 0$. By induction, this proves that $f_n(\cdot \mid x)$ is a density for all $n \geq 1$ and $x \in \mathbb{R}^n$. Therefore, equations (5)-(6) define a strategy $\sigma$ (called HMW’s strategy in Example 5).

Finally, we prove that $P_\sigma$ is c.i.d. if $\sigma$ is HMW’s strategy. By Theorem 15, it suffices to prove condition (11). In turn, since $\sigma$ is dominated by Lebesgue measure, condition (11) reduces to

$$f_n(z \mid x) = \int f_{n+1}(z \mid x, y) f_n(y \mid x) \, dy$$

for all $n \geq 0$, almost all $z \in \mathbb{R}$ and $P_\sigma$-almost all $x \in \mathbb{R}^n$. Such a condition follows directly from the definition of $\sigma$. In fact, for all $n \geq 0$ an $x \in \mathbb{R}^n$, one obtains

$$\int f_{n+1}(z \mid x, y) f_n(y \mid x) \, dy = \int c_{n+1}\{F_n(z \mid x), F_n(y \mid x)\} f_n(z \mid x) f_n(y \mid x) \, dy = f_n(z \mid x)$$

for almost all $z$. This concludes the proof. \qed
Remark 16. HMW’s strategy $\sigma$ has been defined under the assumption that $f_0 > 0$ and $c_n > 0$ for all $n \geq 1$. Such an assumption is superfluous and has been made only to avoid annoying complications in the definition of $\sigma$. Similarly, $X$ is c.i.d. even if the $c_n$ are conditional copulas, in the sense that they are allowed to depend on past data. Precisely, for each $n \geq 1$ and $x \in \mathbb{R}^n$, fix a bivariate copula density $c_{n+1}(\cdot \mid x)$. Then, the proof Theorem 6 still applies if equation (6) is rewritten as

$$f_{n+1}(z \mid x, y) = c_{n+1}\{F_n(z \mid x), F_n(y \mid x) \mid x\} f_n(z \mid x).$$

Proof of Theorem 8. We show that $X$ is c.i.d. via Theorem 15. Fix $A \in \mathcal{B}$ and $n \geq 0$. Since $P_\beta$ is exchangeable (and thus c.i.d.) Theorem 15 yields

$$\beta_n(x, A) = \int \beta_{n+1}(x, y, A) \beta_n(x, dy)$$

for $P_\beta$-almost all $x \in S^n$. Hence, up to changing $\beta$ on a $P_\beta$-null set, equation (12) can be assumed to hold for all $x \in S^n$. If $n = 0$,

$$\int \sigma_1(y, A) \sigma_0(dy) = \int \beta_1(y, A) \beta_0(dy) = \beta_0(A) = \sigma_0(A)$$

where the first equality is because $\sigma_0 = \beta_0$ and $\sigma_1 = \beta_1$ while the second follows from (12). Next, suppose $n \geq 1$ and take $x \in S^n$ and $y \in S$. By assumption, the events $\{T > n + 1\}$ and $\{T \leq n + 1\}$ depend on $x$ but not on $y$. If $T > n + 1$, one obtains $\sigma_{n+1}(x, y) = \beta_{n+1}(x, y)$ and $\sigma_n(x) = \beta_n(x)$. Hence, equation (12) implies again

$$\int \sigma_{n+1}(x, y, A) \sigma_n(x, dy) = \int \beta_{n+1}(x, y, A) \beta_n(x, dy) = \beta_n(x, A) = \sigma_n(x, A).$$

Similarly, if $T \leq n + 1$,

$$\int \sigma_{n+1}(x, y, A) \sigma_n(x, dy) = \int \{q_n(x) \sigma_n(x, A) + (1 - q_n(x)) \delta_y(A)\} \sigma_n(x, dy)$$

$$= q_n(x) \sigma_n(x, A) + (1 - q_n(x)) \int \delta_y(A) \sigma_n(x, dy) = \sigma_n(x, A).$$

In view of Theorem 15, this proves that $X$ is c.i.d.

Finally, suppose that $A_n$ is invariant under permutations of $S^n$ for each $n \geq 1$. We have to show that $(X_1, \ldots, X_n)$ is exchangeable conditionally on $T > n$. Fix $n$, a set $C \in \mathcal{B}^n$, and a permutation $\phi$ of $S^n$. For each $j \geq n$, it is easily seen that

$$P_\sigma(T = j + 1, \phi(X_1, \ldots, X_n) \in C) = P_\beta(T = j + 1, \phi(X_1, \ldots, X_n) \in C).$$
Therefore,

\[ P_\sigma \left( T = j + 1, \, \phi(X_1, \ldots, X_n) \in C \right) = P_\beta \left( T = j + 1, \, \phi(X_1, \ldots, X_n) \in C \right) \]

\[ = P_\beta \left( (X_1, \ldots, X_j) \in A_j, \, \phi(X_1, \ldots, X_n) \in C \right) \]

\[ = P_\beta \left( (X_1, \ldots, X_j) \in A_j, \, (X_1, \ldots, X_n) \in C \right) \]

where the last equality is because \( P_\beta \) is exchangeable and \( A_j \) is invariant under permutations of \( S^j \). In turn, this implies

\[ P_\sigma \left( T > n, \, \phi(X_1, \ldots, X_n) \in C \right) = \sum_{j \geq n} P_\sigma \left( T = j + 1, \, \phi(X_1, \ldots, X_n) \in C \right) \]

\[ = \sum_{j \geq n} P_\beta \left( T = j + 1, \, (X_1, \ldots, X_n) \in C \right) \]

\[ = \sum_{j \geq n} P_\beta \left( T = j + 1, \, (X_1, \ldots, X_n) \in C \right) \]

\[ = P_\sigma \left( T > n, \, (X_1, \ldots, X_n) \in C \right) \]

This concludes the proof. \( \square \)

**Proof of Theorem 11.** Just note that \( g_n \) is a density of \((X_1, \ldots, X_n)\) with respect to \( \lambda^n \), where \( \lambda^n = \lambda \times \ldots \times \lambda \) is the \( n \)-fold product of \( \lambda \). Therefore, Theorem 11 follows from the very definitions of stationarity and exchangeability, after noting that \( \int g_{n+1}(u, \cdot) \lambda(du) \) is a density of \((X_2, \ldots, X_{n+1})\) with respect to \( \lambda^n \). \( \square \)

**Proof of Theorem 13.** We first recall that

\[ \int \mathcal{S}(x, b; A) \mathcal{S}(0, r; dx) = \mathcal{S}(0, b + r; A) \quad \text{for all } A \in \mathcal{B} \text{ and } b, r > 0. \]

This can be checked by a direct calculation. For a proof, we refer to the Claim of [9, Th. 10]. Having noted this fact, define

\[ \mu = \mathcal{S}(a, b), \quad f(x) = -a + cx, \quad \nu = \mathcal{S} \left( 0, \frac{b}{1 - |c|} \right), \]

and denote by \( Z \) a real random variable such that \( Z \sim \mathcal{S}(0, 1) \). Define also

\[ r = \frac{b |c|^\gamma}{1 - |c|^\gamma}, \quad h(x) = cx, \]
and call $\nu^*$ the probability distribution of $h$ under $\nu$. On noting that
\[ a + b^{1/\gamma} Z \sim \mu \quad \text{and} \quad \nu^* = S(0, r), \]
one obtains
\[
\int \sigma_1(x, A) \nu(dx) = \int P(f(x) + a + b^{1/\gamma} Z \in A) \nu(dx) = \int P(h(x) + b^{1/\gamma} Z \in A) \nu(dx)
\]
\[
= \int P(x + b^{1/\gamma} Z \in A) \nu^*(dx) = \int S(x, b; A) S(0, r; dx)
\]
\[
= S(0, b + r; A) = S \left(0, \frac{b}{1 - |c\gamma|}; A \right) = \nu(A).
\]
Therefore, equation (9) holds. \hfill \Box

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