ABSENCE OF CARTAN SUBALGEBRAS FOR RIGHT-ANGLED HECKE VON NEUMANN ALGEBRAS

MARTIJN CASPERS

Abstract. For a right-angled Coxeter system \((W, S)\) and \(q > 0\), let \(M_q\) be the associated Hecke von Neumann algebra, which is generated by self-adjoint operators \(T_s, s \in S\) satisfying the Hecke relation \((\sqrt{q} T_s - q)(\sqrt{q} T_s + 1) = 0\) as well as suitable commutation relations. Under the assumption that \((W, S)\) is reduced and \(|S| \geq 3\) it was proved by Garncarek [Gar15] that \(M_q\) is a factor (of type \(\text{II}_1\)) for a range \(q \in [\rho, \rho^{-1}]\) and otherwise \(M_q\) is the direct sum of a \(\text{II}_1\)-factor and \(\mathbb{C}\).

In this paper we prove (under the same natural conditions as Garncarek) that \(M_q\) is non-injective, that it has the weak\(^*\)-completely contractive approximation property and that it has the Haagerup property. In the hyperbolic factorial case \(M_q\) is a strongly solid algebra and consequently \(M_q\) cannot have a Cartan subalgebra. In the general case \(M_q\) need not be strongly solid, but still we prove that \(M_q\) does not possess a Cartan subalgebra.

1. Introduction

Hecke algebras are one-parameter deformations of group algebras of a Coxeter group. They were the fundament for the theory of quantum groups [Jim86, Kas95] and have remarkable applications in the theory of knot invariants [Jon85] as was shown by V. Jones. A wide range of applications of Coxeter groups and their Hecke deformations can be found in [Dav08]. In [Dym06] (see also [Dav08, Section 19]) Dymara introduced the von Neumann algebras generated by Hecke algebras. Many important results were then obtained (see also [DDJB07]) for these Hecke von Neumann algebras. This gave for example insight in the cohomology of associated buildings and its Betti numbers. In this paper we investigate the approximation properties of Hecke von Neumann algebras as well as their Cartan subalgebras (here we mean the notion of a Cartan subalgebra in the von Neumann algebraic sense which we recall in Section 6 and not the Lie algebraic notion).

Let us recall the following definition. Let \(q > 0\) and let \(W\) be a right-angled Coxeter group with generating set \(S\) (see Section 2). The associated Hecke algebra is a \(*\)-algebra generated by \(T_s, s \in S\) which satisfies the relation:

\[
(\sqrt{q} T_s - q)(\sqrt{q} T_s + 1) = 0, \quad T_s^* = T_s \quad \text{and} \quad T_s T_t = T_t T_s,
\]

for \(s, t \in S\) with \(st = ts\). Hecke algebras carry a canonical faithful tracial vector state (the vacuum state) and therefore generate a von Neumann algebra \(M_q\) under its GNS construction. It was recently proved by Garncarek [Gar15] that if \((W, S)\) is reduced (see Section 2) and \(|S| \geq 3\), the von Neumann algebra \(M_q\) is a factor in case \(q \in [\rho, \rho^{-1}]\) where \(\rho\) is the radius of convergence of the fundamental power series (2.2). If \(q \notin [\rho, \rho^{-1}]\) then \(M_q\) is the direct sum of a \(\text{II}_1\) factor and \(\mathbb{C}\). For

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more general coxeter groups/Hecke algebras (not necessarily being right angled, or for multi-parameters \( q \)) this result is unknown. It deserves to be emphasized that this in particular shows that the isomorphism class of \( M_q \) depends on \( q \); an observation that was already made in the final remarks [Dav08, Section 19].

The first aim of this paper is to determine approximation properties of \( M_q \) (assuming the same natural conditions as Garnarek). We first show that \( M_q \) is a non-injective von Neumann algebra and therefore falls outside Connes’ classification of hyperfinite factors [Con76]. Secondly we show that \( M_q \) has the weak-* completely contractive approximation property (wk-\(*\) CCAP). This means that there exists a net of completely contractive finite rank maps on \( M_q \) that converges to the identity in the point \( \sigma \)-weak topology. In case \( q = 1 \) the algebra \( M_q \) is the group von Neumann algebra of a right-angled Coxeter group. In this case the result was known. For instance the CCAP follows from Reckwerdt’s result [Rec15] and non-injectivity follows easily from identifying a copy of the free group inside \( W \). In this context we also mention the parallel results for \( q \)-Gaussian algebras with \(-1 < q < 1\): factoriality by Ricard [Ric05], non-injectivity by Nou [Nou04] and the completely contractive approximation property by Avsec [Avs11]. We owe some to the ideas of [Nou04] (and [BoSp94]) for the proof of non-injectivity. Nou also uses a Khintchine inequality though our proof is more in the spirit of [RiXu06].

Another important result concerning the approximation properties of operator algebras was obtained by Houdayer and Ricard [HoRi11] who settled the approximation properties of free Araki-Woods factors. For our Hecke von Neumann algebra \( M_q \) we summarize:

**Theorem A.** Let \( q > 0 \).

1. Let \((W, S)\) be a reduced right-angled Coxeter system with \(|S| \geq 3\). Then \( M_q \) is non-injective.

2. For a general right-angled Coxeter system \((W, S)\) the associated Hecke von Neumann algebra \( M_q \) has the wk-* CCAP and the Haagerup property.

Obviously non-injectivity and the wk-* CCAP of Theorem A have different proofs. However the two proofs each borrow some ideas from [RiXu06] where Ricard and Xu proved that weak amenability with constant 1 is preserved by taking free products of discrete groups. In order to prove non-injectivity we first obtain a Khintchine inequality for Hecke algebras. We show that this Khintchine inequality leads to a contradiction in case \( M_q \) were to be injective. For the wk-* CCAP we first obtain cb-estimates for radial multipliers and then use estimates of word length projections (see Proposition 5.12) going back to Haagerup [Haa78].

Our second aim is the study of Cartan subalgebras of the Hecke von Neumann algebra \( M_q \). Recall that a Cartan subalgebra of a \( \text{II}_1 \)-factor is by definition a maximal abelian subalgebra whose normalizer generates the \( \text{II}_1 \)-factor itself. Cartan subalgebras arise typically in crossed products of free ergodic probability measure preserving actions of discrete groups on a probability measure space. In fact Cartan subalgebras always come from some orbit equivalence class in the following sense: for a separable \( \text{II}_1 \) factor \( \mathcal{M} \) any Cartan subalgebra \( \mathcal{A} \subseteq \mathcal{M} \) gives rise to a standard probability measure space \( X \) and an orbit equivalence class \( \mathcal{R} \) with cocycle \( \sigma \) such that \( (\mathcal{A} \subseteq \mathcal{M}) \simeq (\ell^\infty(X) \subseteq \mathcal{R}(X, \sigma)) \). We refer to [FeMo77] for details. Cartan subalgebras can be used to obtain further rigidity results, see [PoVa14], [IPV13] for two very prominent illustrations of this: the first showing that (suitable) actions of
free groups on probability spaces remember the number of generators of the group; the second showing that certain group von Neumann algebras completely remember the group ($W^*$-superrigidity). These results typically rely on the uniqueness of a Cartan subalgebra. Other applications can be found in prime factorization theorems [OzPo04, HoIs15] in which the absence of Cartan algebras plays a crucial role.

In [Vo96] Voiculescu was the first one to find factors (namely free group factors) that do not have a Cartan subalgebra. His proof relies on estimates for the free entropy dimension of the normalizer of an injective von Neumann algebra. Using a different approach Ozawa and Popa [OzPo10] were also able to find classes of von Neumann algebras that do not have a Cartan subalgebra (including the free group factors). These results generalized to a larger class of class of groups. In particular in [PoVa14] Popa and Vaes (see also Chifan-Sinclair [ChSi13]) proved absence of Cartan subalgebras for group factors of bi-exact groups that have the CBAP. Isono [Iso15] then put the results from [PoVa14] into a general von Neumann framework in order to prove absence of Cartan subalgebras for free orthogonal quantum groups. Isono proved that factors with the wk-∗ CBAP that satisfy condition (AO)$^+$ are strongly solid, meaning that the normalizer of an injective von Neumann algebra is injective again. Using this strong solidity result by Isono/Popa–Vaes we are able to prove the following.

**Theorem B.** Let $q \in [\rho, \rho^{-1}]$ with $\rho$ as in Theorem 2.2. Let $(W, S)$ be a reduced right-angled Coxeter system with $|S| \geq 3$. Assume that $W$ is hyperbolic. Then the associated Hecke von Neumann algebra $\mathcal{M}_q$ is strongly solid.

In turn as $\mathcal{M}_q$ is non-injective by Theorem A we are able to derive the result announced in the title of this paper for the hyperbolic case.

**Corollary C.** Let $q \in [\rho, \rho^{-1}]$ with $\rho$ as in Theorem 2.2. For a reduced right-angled hyperbolic Coxeter system $(W, S)$ with $|S| \geq 3$ the associated Hecke von Neumann algebra $\mathcal{M}_q$ does not have a Cartan subalgebra.

General right-angled Hecke von Neumann algebras are not strongly solid, see Remark 6.6. Still we can prove that they do not possess a Cartan subalgebra. We do this by showing that if $\mathcal{M}_q$ were to have a Cartan subalgebra then each of the three alternatives in [Vae14, Theorem A] fails to be true, which leads to a contradiction. It deserves to be noticed that the proof of Theorem D uses Theorem A as well.

**Theorem D.** Let $q > 0$. For a reduced right-angled Coxeter system $(W, S)$ with $|S| \geq 3$ the associated Hecke von Neumann algebra $\mathcal{M}_q$ does not have a Cartan subalgebra.

**Structure.** In Section 2 we introduce Hecke von Neumann algebras and some basic algebraic properties. Lemma 2.7 is absolutely crucial as each of the results in this paper rely in their own way on this decomposition lemma. In Section 3 we obtain universal properties of Hecke von Neumann algebras. In Section 4 we prove that $\mathcal{M}_q$ is non-injective. In Section 5 we find approximation properties of $\mathcal{M}_q$ and conclude Theorem A. Section 6 proves the strong solidity result of Theorem B from which Corollary C shall easily follow. Finally Section 7 proves absence of Cartan subalgebras in the general case.
Convention. Let $X$ be a set and let $A, B \subseteq X$. We will briefly write $A \setminus B$ for $A \setminus (A \cap B)$.

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2. Notation and preliminaries

Standard result on operator spaces can be found in [EfRu00], [Pis02]. Standard references for von Neumann algebras are [StZs75] and [Tak79]. Recall that $ucp$ stands for unital completely positive.

2.1. Coxeter groups. A Coxeter group $W$ is a group that is freely generated by a finite set $S$ subject to relations

$$(st)^{m(s,t)} = 1,$$

for some constant $m(s,t) \in \{1, 2, \ldots, \infty\}$ with $m(s,t) = m(t,s) \geq 2$, $s \neq t$ and $m(s,s) = 1$. The constant $m(s,t) = \infty$ means that no relation is imposed, so that $s, t$ are free variables. The Coxeter group $W$ is called right-angled if either $m(s,t) = 2$ or $m(s,t) = \infty$ for all $s, t \in S, s \neq t$ and this is the only case we need in this paper. Therefore we assume from now on that $W$ is a right-angled Coxeter group with generating set $S$. The pair $(W, S)$ is also called a Coxeter system.

Let $w \in W$ and suppose that $w = w_1 \ldots w_n$ with $w_i \in S$. The representing expression $w_1 \ldots w_n$ is called reduced if whenever also $w = w_1' \ldots w_m'$ with $w_i' \in S$ then $n \leq m$, i.e. the expression is of minimal length. In that case we will write $|w| = n$. Reduced expressions are not necessarily unique (only if $m(s,t) = \infty$ whenever $s \neq t$), but for each $w \in W$ we may pick a reduced expression which we shall call minimal.

Convention: For $w \in W$ we shall write $w_i$ for the minimal representative $w = w_1 \ldots w_n$.

To the pair $(W, S)$ we associate a graph $\Gamma$ with vertex set $VT = S$ and edge set $ET = \{(s,t) \mid m(s,t) = 2\}$. A subgraph $\Gamma_0$ of $\Gamma$ is called full if the following property holds: $\forall s, t \in VT_0$ with $(s,t) \in ET$ we have $(s,t) \in ET_0$. A clique in $\Gamma$ is a full subgraph in which every two vertices share an edge. We let $\text{Cliq}(\Gamma)$ denote the set of cliques in $\Gamma$. To keep the notation consistent with the literature the empty graph is in $\text{Cliq}(\Gamma)$ by convention (in this paper we shall sometimes exclude the empty graph from $\text{Cliq}(\Gamma)$ explicitly or treat it as a special case to keep some of the arguments more transparent). We let $\text{Cliq}(\Gamma, l)$ be the set of cliques with $l$ vertices.

Definition 2.1. A Coxeter system $(W, S)$ is called reduced if the complement of $\Gamma$ is connected.

2.2. Hecke von Neumann algebras. Let $(W, S)$ be a right-angled Coxeter system. Let $q > 0$. By [Dav08 Proposition 19.1.1] there exists a unique unital
*-algebra $\mathbb{C}_q(\Gamma)$ generated by a basis $\{\tilde{T}_w \mid w \in W\}$ satisfying the following relations. For every $s \in S$ and $w \in W$ we have:

$$
\tilde{T}_s \tilde{T}_w = \begin{cases} 
\tilde{T}_{sw} & \text{if } |sw| > |w|, \\
q \tilde{T}_{sw} + (1 - q) \tilde{T}_w & \text{otherwise}, 
\end{cases}
$$

We define normalized elements $T_w = q^{-|w|/2} \tilde{T}_w$. Then for $w \in W$ and $s \in S$,

$$
T_s T_w = \begin{cases} 
T_{sw} & \text{if } |sw| > |w|, \\
T_{sw} + pT_w & \text{otherwise}, 
\end{cases}
$$

where

$$
p = \frac{q - 1}{\sqrt{q}}.
$$

There is a natural positive linear tracial map $\tau$ on $\mathbb{C}_q(\Gamma)$ satisfying $\tau(T_w) = 0$, $w \neq 1$ and $\tau(1) = 1$. Let $L^2(\mathcal{M}_q)$ be the Hilbert space given by the closure of $\mathbb{C}_q(\Gamma)$ with respect to $(x, y) = \tau(y^* x)$ and let $\mathcal{M}_q$ be the von Neumann algebra generated by $\mathbb{C}_q(\Gamma)$ acting on $L^2(\mathcal{M}_q)$. $\tau$ extends to a state on $\mathcal{M}_q$ and $L^2(\mathcal{M}_q)$ is its GNS space with cyclic vector $\Omega := T_e$. $\mathcal{M}_q$ is called the Hecke von Neumann algebra at parameter $q$ associated to the right-angled Coxeter system $(W, S)$.

**Theorem 2.2** (see [Gar14]). Let $(W, S)$ be a reduced right-angled Coxeter system and suppose that $|S| \geq 3$. Let $\rho$ be the radius of convergence of the fundamental power series:

$$
\sum_{k=0}^{\infty} |\{w \in W \mid |w| = k\}| z^k.
$$

For every $q \in [\rho, \rho^{-1}]$ the von Neumann algebra $\mathcal{M}_q$ is a factor. For $q > 0$ not in $[\rho, \rho^{-1}]$ the von Neumann algebra $\mathcal{M}_q$ is the direct sum of a factor and $\mathbb{C}$.

As $\mathcal{M}_q$ possesses a normal faithful tracial state the factors appearing in Theorem 2.2 are of type $\Pi_1$.

For the analysis of $\mathcal{M}_q$ we shall in fact need $\mathcal{M}_1$ which is the group von Neumann algebra of the Coxeter group $W$. It can be represented on $L^2(\mathcal{M}_q)$. Indeed, let $T_w^{(1)}$ denote the generators of $\mathcal{M}_1$ as in (2.1) and let $T_w$ be the generators of $\mathcal{M}_q$. Set the unitary map,

$$
U : L^2(\mathcal{M}_1) \to L^2(\mathcal{M}_q) : T_w^{(1)} \Omega \to T_w \Omega.
$$

In this paper we shall always assume that $\mathcal{M}_1$ is represented on $L^2(\mathcal{M}_q)$ by the identification $\mathcal{M}_1 \to B(L^2(\mathcal{M}_q)) : x \mapsto U x U^*$. Note that this way

$$
T_w^{(1)}(T_w^{(1)} \Omega) = T_{vw} \Omega.
$$

For $w \in W$ we shall write $P_w$ for the projection of $L^2(\mathcal{M}_q)$ onto the closure of the space spanned linearly by $\{T_v \Omega \mid |w^{-1} v| = |v| - |w|\}$ (see Remark 2.3 below). For $\Gamma_0 \in \text{Clq}(\Gamma)$ we shall write $P_{v\Gamma_0}$ for $P_w$ where $w \in W$ is the product of all vertex elements of $\Gamma_0$ and $|V \Gamma_0|$ for the number of elements in $V \Gamma_0$. Similarly we shall write $P_{v\Gamma_0}$ for $P_w$ where $w \in W$ is the product of all vertex elements of $\Gamma_0$. 

Remark 2.3 (Creation and annihilation arguments). Note that for \( w, v \in W \) saying that \(|w^{-1} v| = |v| - |w|\) just means that the start of \( v \) contains the word \( w \). Throughout the paper we say that \( s \in W \) acts by means of a creation operator on \( v \in W \) if \(|sv| = |v| + 1\). It acts as an annihilation operator if \(|sv| = |v| - 1\). Note that as \( W \) is right-angled we cannot have \(|sv| = |v|\). For \( v, w \in W \) we may always decompose \( w = w' w'' \) such that \(|w| = |w'| + |w''|, |w''v| = |v| - |w''|\) and \(|wv| = |v| - |w''| + |w'|\). That is \( w \) first acts by means of annihilations of the letters of \( w'' \) and then \( w' \) acts as a creation operator on \( w''v \). We will use such arguments without further reference.

The following Lemma 2.7 together with Lemma 2.4 say that \( T_w \) decomposes in terms of a sum of operators that first act by annihilation (this is \( T_w^{(1)} \)) then a diagonal action (this is the projection \( P_{uVT_0} \)) and finally by creation (this is \( T_w^{(1)} \)). This decomposition is crucial for each of our main results.

Definition 2.4. Let \( w \in W \). Let \( A_w \) be the set of triples \((w', \Gamma_0, w'')\) with \( w', w'' \in W \) and \( \Gamma_0 \in \text{Clq}(\Gamma) \) such that: (1) \( w = w'VT_0w'' \), (2) \(|w| = |w'| + |VT_0| + |w''|\), (3) if \( s \in S \) commutes with \( VT_0 \) then \(|sw'| > |w'|\) (that is, letters commuting with \( VT_0 \) cannot occur at the end of \( w' \) but if they are there they should occur at the start of \( w'' \) instead).

Lemma 2.5. For \((w', \Gamma_0, w'') \in A_w\) there exist \( u, u', u'' \in W \) such that

\[
T_w^{(1)} P_{uVT_0} T_w^{(1)} = T_u^{(1)} P_{uVT_0} T_u^{(1)},
\]

and moreover if \( s \in W \) is such that \(|u's| < |u'|\) then \(|su''| > |u''|\). We may assume that \( u' = w'u^{-1} \) and \( u'' = uw'' \).

Proof. Let \( u \in W \) be the (unique) element of maximal length such that \(|w'u^{-1}| = |w'| - |u|\) and \(|uw''| = |w''| - |u|\). Set \( u' = w'u^{-1} \) and \( u'' = uw'' \). It then remains to prove (2.4) as the rest of the properties are obvious or follow by maximality of \( u \). For \( v \in W \) such that \(|VT_0w''v| = |w''v| - |VT_0|\) we have,

\[
T_u^{(1)} P_{uVT_0} T_u^{(1)} (T_\Gamma) = T_u^{(1)} P_{uVT_0} (T_{uw''\gamma} \Omega) = T_u^{(1)} (T_{uw''\gamma})
\]

and

\[
=T_{w''\gamma} (T_{w''\gamma}) = T_u^{(1)} P_{VT_0} (T_{w''\gamma}) = T_{w''\gamma} (T_{w''\gamma}) = T_u^{(1)} P_{VT_0} (T_{w''\gamma}) = T_{w''\gamma} (T_{w''\gamma}) = T_{w''\gamma} (T_{w''\gamma}) = T_u^{(1)} (T_{w''\gamma}) = T_u^{(1)} (0) = 0.
\]

Remark 2.6. In Lemma 2.5 the property that \(|u's| < |u'|\) implies that \(|su''| > |u''|\) is equivalent to \(|u'u''| = |u'| + |u''|\). The words \( u' \) and \( u'' \) in Lemma 2.4 are not unique: in case \(|su''| = |u''| - 1\) and \( s \) commutes with \( VT_0 \) then we may replace \((u', u'')\) by \((u's, u'u'')\).

Lemma 2.7. We have,

\[
T_w = \sum_{(w', \Gamma_0, w'') \in A_w} p_{VT_0}^{(1)} T_w^{(1)} P_{VT_0} T_w^{(1)},
\]
where $A_w$ is given in Definition 2.4.

**Proof.** The proof proceeds by induction on the length of $w$. If $|w| = 1$ then $T_w = T_w^{(1)} + P_w$ by (2.1). Now suppose that (2.5) holds for all $w \in W$ with $|w| = n$. Let $v \in W$ be such that $|v| = n + 1$. Decompose $v = sw$, $|w| = n, s \in S$. Then,

$$T_v = T_s T_w$$

(2.6)

$$= \left( T_s^{(1)} + pP_s \right) \left( \sum_{(w', \Gamma_0, w'') \in A_w} p^{\#VT_0} T_w^{(1)} P_{VT_0} T_w^{(1)} \right)
- T_s w^{(1)} + \sum_{(w', \Gamma_0, w'') \in A_w} \left( p^{\#VT_0} T_{s w'} P_{VT_0} T_{w''}^{(1)} + p^{\#VT_0 + 1} P_s T_w^{(1)} P_{VT_0} T_w^{(1)} \right).$$

Now we need to make the following observations.

1. If $sw' = ws'$ then $P_s T_w^{(1)} = T_w^{(1)} P_s$. So in that case,

$$P_s T_w^{(1)} P_{VT_0} T_{w'}^{(1)} = T_w^{(1)} P_s P_{VT_0} T_{w'}^{(1)}.$$ 

Moreover $P_s P_{VT_0}$ equals $P_{s VT_0}$ in case $s$ commutes with all elements of $VT_0$ and it equals 0 otherwise.

2. In case $sw' \neq w's$ we claim that $P_s T_w^{(1)} P_{VT_0} T_w^{(1)} = 0$. To see this, rewrite $P_s T_w^{(1)} P_{VT_0} T_{w'}^{(1)} = P_s T_u^{(1)} P_{VT_0} T_{u'}^{(1)}$ with $u, u', u''$ as in Lemma (2.5). As $sw' \neq w's$ we have $su' \neq su'$ and/or $su \neq us$ (because $w' = uu'$ with $|w'| = |u'| + |u|$, c.f. Lemma (2.5).

(a) Assume $su' \neq u's$. For $v \in W$ with $T_u^{(1)} \Omega$ in the range of $P_{u VT_0}$,

$$P_s T_u^{(1)} P_{u VT_0} T_{u'}^{(1)} (T_v \Omega) = P_s T_u^{(1)} T_v \Omega.$$

Furthermore, the assertions of Lemma (2.5) imply $|u'v VT_0| = |u'| + |u VT_0|$ and therefore (recalling that $T_u^{(1)} \Omega$ is in the range of $P_{u VT_0}$) we get that $|u'v| = |u'| + |u|$ which implies (because $su' \neq u's$ and $u'vu'$ starts with all letters of $u'$) that (2.7) is 0. For $v \in W$ with $T_{u'}^{(1)} \Omega$ not in the range of $P_{u VT_0}$ we have $T_{u'}^{(1)} P_{u VT_0} T_{u'}^{(1)} (T_v \Omega) = 0$.

We conclude in all we conclude $P_s T_u^{(1)} P_{u VT_0} T_{u'}^{(1)} = 0$.

(b) Assume $su' = u's$ but $su \neq us$. Then $P_s T_u^{(1)} P_u = T_{u'}^{(1)} P_s P_u = 0$.

So in all, (2.6) gives,

$$T_v = T_s w^{(1)} + \sum_{(w', \Gamma_0, w'') \in A_w} p^{\#VT_0} T_w^{(1)} P_{VT_0} T_w^{(1)}
+ \sum_{(w', \Gamma_0, w'') \in A_w, s w' = w's} p^{\#VT_0 + 1} T_w^{(1)} P_{VT_0} T_w^{(1)},$$

and in turn an identification of all summands shows that the latter expression equals,

$$\sum_{(v', \Gamma_0, v'') \in A_w} p^{\#VT_0} T_{v'}^{(1)} P_{VT_0} T_{v'}^{(1)}.$$

This concludes the proof. □
2.3. Group von Neumann algebras. Let $G$ be a locally compact group with left regular representation $s \mapsto \lambda_s$, and group von Neumann algebra $L(G) = \{ \lambda_s | s \in G \}^\prime$. We let $A(G)$ be the Fourier algebra consisting of functions $\varphi(s) = \langle \lambda_s \xi, \eta \rangle, \xi, \eta \in L^2(G)$. There is a pairing between $A(G)$ and $L(G)$ which is given by $\langle \varphi, \lambda(f) \rangle = \int_G f(s) \varphi(s) ds$ which turns $A(G)$ into an operator space that is completely isometrically identified with $L(G)_\pi$. We let $M_{CB}A(G)$ be the space of completely bounded Fourier multipliers of $A(G)$. For $m \in M_{CB}A(G)$ we let $T_m : L(G) \to L(G)$ be the normal completely bounded map determined by $\lambda(f) \mapsto \lambda(mf)$. The following theorem is due to Bozejko and Fendler [BoFe84] (see also [JNR09 Theorem 4.5]).

**Theorem 2.8.** Let $m \in M_{CB}A(G)$. There exists a unique normal completely bounded map $M_m : B(L^2(G)) \to B(L^2(G))$ that is an $L^\infty(G)$-bimodule homomorphism and such that $M_m$ restricts to $T_m : \lambda(f) \mapsto \lambda(mf)$ on $L(G)$. Moreover, $\| M_m \|_{CB} = \| T_m \|_{CB} = \| m \|_{M_{CB}A(G)}$.

3. Universal property and conditional expectations

In this section we establish some standard universal properties for subalgebras of $\mathcal{M}_q$.

**Theorem 3.1.** Let $q > 0$ put $p = (q - 1)/\sqrt{q}$ and let $(W,S)$ be a right-angled Coxeter system with associated Hecke von Neumann algebra $(\mathcal{M}_q, \tau)$. Suppose that $(N,\tau_N)$ is a von Neumann algebra with GNS faithful state $\tau_N$ that is generated by self-adjoint operators $R_s, s \in S$ that satisfy the relations $R_s R_t = R_t R_s$ whenever $m(s,t) = 2$, $R_s^2 = 1 + p R_s, s \in S$ and further $\tau_N(R_{w_1} \cdots R_{w_n}) = 0$ for every non-empty reduced word $w = w_1 \cdots w_n$ in $W$. Then there exists a unique normal *-homomorphism $\pi : \mathcal{M}_q \to N$ such that $\pi(T_s) = R_s$. Moreover $\tau_N \circ \pi = \tau$.

**Proof.** The proof is routine, c.f. [CaPi15 Proposition 2.12]. We sketch it here. Let $(L^2(N), \pi_N, \eta)$ be a GNS construction for $(N,\tau_N)$. As $\tau_N$ is GNS faithful we may assume that $N$ is represented on $L^2(N)$ via $\pi_N$. We define a linear map $V : L^2(\mathcal{M}_q) \to L^2(N)$ by $V \Omega = \eta$ and

$$V(T_w \Omega) = R_w \eta,$$

where $w \in W$,

and $R_w := R_{w_1} \cdots R_{w_n}$. One checks that $V$ is isometric by showing that $\{ R_w \eta \mid w \in W \}$ is an orthonormal system\footnote{The proof goes as follows. We may find unique coefficients $c_w$ such that $T_w \cdots T_{w_1} T_{w_1} \cdots T_{w_n} = \sum_{\nu \in W} c_w T_{\nu}$. We have $c_\emptyset = 1$ if $w = w'$ and $c_\emptyset = 0$ if $w \neq w'$ by comparing the trace of both sides of this expression. In fact the coefficients $c_w$ may be found by using the commutation relations for $T_s$ and the Hecke relation $T_s^2 = 1 + p T_s$ to `reduce' the left hand side of this expression. As the same relations hold for the operators $R_s$ (by assumption of the lemma) we also get $R_s \cdots R_{w_1} R_{w_1} \cdots R_{w_n} = \sum_{\nu \in W} c_w R_{\nu}$. So, $$(R_w \eta, R_{w'} \eta) = \tau_N(R_{w'} R_w) = \tau_N(R_{w_1} \cdots R_{w_1} R_{w_1} \cdots R_{w_n}) = \tau_N \left( \sum_{\nu \in W} c_w R_{\nu} \right) = c_\emptyset.$$ This proves that indeed $V$ is isometric.} Putting $\pi(\cdot) = V(\cdot)V^*$ concludes the lemma. As $V \Omega = \eta$ we get $\tau_N \circ \pi = \tau$. $\Box$
Proof. Theorem 3.1 implies that set, \( q > 0 \) preserves normal conditional expectation value, c.f. [Tak03, Theorem IX.4.2].

We shall say that \((\tilde{W}, \tilde{S})\) is a Coxeter subsystem of \((W, S)\) if \( \tilde{S} \subseteq S \) and \( \tilde{m}(s, t) = m(s, t) \) for all \( s, t \in \tilde{S} \). Here \( \tilde{m} \) is the function on \( \tilde{S} \times \tilde{S} \) that determines the commutation relations for \( \tilde{W} \), c.f. Section 2.1.

Corollary 3.3. Let \( q > 0 \). Let \((\tilde{W}, \tilde{S})\) be a Coxeter subsystem of a right-angled Coxeter system \((W, S)\). Let \( \tilde{M}_q \) and \( M_q \) be their respective Hecke von Neumann algebras. There exists a trace preserving normal conditional expectation value, c.f. [Tak03, Theorem IX.4.2]. □

Consider the Hecke von Neumann algebra \( M_q \) for the case that \( S \) is a one-point set, \( q > 0 \) and \( p = \frac{q-1}{\sqrt{q}} \). In that case we have \( W = \{e, s\} \) and \( L^2(M_q) \) has a canonical basis \( \Omega \) and \( T_s\Omega \). With respect to this basis \( T_s \) takes the form \( \begin{pmatrix} 0 & 1 \\ 1 & p \end{pmatrix} \) and one sees (using for example the relation \( T_s^2 = 1 + pT_s \)) that \( M_q = \text{CId}_2 \oplus CT_s \), i.e. it is two dimensional. The following corollary uses the graph product, for which we refer to [CaFi15]. It is a generalization of the free product by adding a commutation relation to vertex algebras that share an edge; the free product is then given by a graph product over a graph with no edges.

Corollary 3.4. Let \((W, S)\) be an arbitrary right-angled Coxeter system and let \( q > 0 \). Let \( \Gamma \) be the graph associated to \((W, S)\) as before. For \( s \in S \) let \( M_q(s) \) be the 2-dimensional Hecke von Neumann subalgebra corresponding to the one-point set \( \{s\} \). Then we have a graph product decomposition \( M_q = *_{s \in \Gamma} M_q(s) \).

Proof. Let \( T_s \in M_q \), \( s \in S \) be the operators as introduced in Section 2.2. Let \( \bar{T}_s \), \( s \in S \) be the operator \( T_s \) but then considered in the algebra \( M_q(s) \) which in turn is contained in \( *_{s \in \Gamma} M_q(s) \) with conditional expectation. Now the map \( T_s \mapsto \bar{T}_s \) determines an isomorphism by Theorem 3.1 and the universal property of the graph product given by [CaFi15, Proposition 2.12]. □

Remark 3.5. Note that Corollary 3.4 allows us sometimes to use general results of graph product von Neumann algebras to study \( M_q \), see [CaFi15]. Note however that properties as strong solidity, injectivity, et cetera have not been studied beyond the case of free products. Moreover for many stability results for free products one needs to impose assumptions on the product algebras which are not satisfied by \( M_q(s) \) (such as being diffuse or being a II_1-factor, see [CaFi15, Corollary 2.29] for example). This makes the balance between general graph product results and applications to \( M_q \) rather delicate.

4. Non-injectivity of \( M_q \)

Recall that a von Neumann algebra \( M \subseteq B(\mathcal{H}) \) is called injective if there exists a (not necessarily normal) conditional expectation \( \mathcal{E} : B(\mathcal{H}) \to M \). This means that \( \mathcal{E} \) is a completely positive linear map which satisfies \( \forall x \in M : \mathcal{E}(x) = x \).
We prove that $\mathcal{M}_q$ is non-injective. The proof is based on a Khintchine type inequality. For the sake of presentation we shall first prove non-injectivity in the free case, meaning that $m(s, t) = \infty$ whenever $s \neq t$. The proof is conceptually the same as the general case but the notation simplifies quite a lot, making the proof much more accessible.

4.1. **Non-injectivity of $\mathcal{M}_q$: the free case.** In this subsection assume that $m(s, t) = \infty$ for all $s, t \in S, s \neq t$. In particular this means that in Lemma 2.7 every clique appearing in the sum has only 1 vertex or is empty. We proceed now as in [RiXu06, Section 2]. Define the following two linear subspaces of $\mathcal{B}(L^2(\mathcal{M}_q))$:

\begin{align}
L_1 & := \text{span}\left\{ P_s T_s^{(1)} P_s^\perp \mid s \in S \right\}, \\
K_1 & := \text{span}\left\{ P_s^\perp T_s^{(1)} P_s \mid s \in S \right\}.
\end{align}

(4.1)

Note that as $s^2 = e$ in fact $P_s T_s^{(1)} P_s^\perp = T_s^{(1)} P_s^\perp$ and $P_s^\perp T_s^{(1)} P_s = T_s^{(1)} P_s$. $L_1$ is a space of creation operators and $K_1$ is a space of annihilation operators. $L_1$ and $K_1$ are not contained in $\mathcal{M}_q$ but live in the ambient space $\mathcal{B}(L^2(\mathcal{M}_q))$. The following Lemma 4.1 is a special case of [RiXu06, Lemma 2.3 and Corollary 2.4].

**Lemma 4.1.** We have complete isometric identifications,

\begin{align}
L_1 & \simeq (\mathbb{C}^\# S)_{\text{column}} : T_s^{(1)} P_s^\perp \mapsto e_s, \\
K_1 & \simeq (\mathbb{C}^\# S)_{\text{row}} : T_s^{(1)} P_s \mapsto e_s,
\end{align}

where the lower scripts indicate the operator space structure of a column and row Hilbert space with orthonormal basis $e_s, s \in S$.

We define $\Sigma_1 = \text{span}\{ T_s \mid s \in S \}$ and subsequently:

$$\Sigma_d = \text{span}\{T_{w_1} \otimes \ldots \otimes T_{w_d} \mid w \in W\},$$

which is contained in the $d$-fold algebraic tensor copy of $\Sigma_1$. There exists a canonical map $\sigma_d : \Sigma_d \to \mathcal{B}(L^2(\mathcal{M}_q)) : T_{w_1} \otimes \ldots \otimes T_{w_d} \mapsto T_w$. For $s \in V$ we let $A_s := \mathbb{C} P_s$, i.e. a 1-dimensional operator space. Using $\otimes_h$ for the Haagerup tensor product we set $L_k = (L_1)^{\otimes_h k}, K_k = (K_1)^{\otimes_h k}$ and

$$X_d = \left( \bigoplus_{k=0}^d L_k \otimes_h K_{d-k} \right) \bigoplus \left( \bigoplus_{s \in S} \bigoplus_{k=0}^{d-1} L_k \otimes_h A_s \otimes_h K_{d-k-1} \right).$$

Here the sums are understood as $\ell^\infty$-direct sums of operator spaces. The multiplication maps $L_k \otimes_h K_{d-k} \to \mathcal{B}(L^2(\mathcal{M}_q))$ and $L_k \otimes_h A_s \otimes_h K_{d-k-1} \to \mathcal{B}(L^2(\mathcal{M}_q))$ are completely contractive by the very definition of the Haagerup tensor product. Extending linearly to $X_d$ gives a map $\Pi_d : X_d \to \mathcal{B}(L^2(\mathcal{M}_q))$ with

$$\|\Pi_d\|_{CB} \leq (d + 1) + d \# S.\tag{4.2}$$

In fact the tensor amplification

$$\langle \iota \otimes \Pi_d \rangle : \mathcal{M}_q \otimes_{\text{min}} X_d \to \mathcal{M}_q \otimes_{\text{min}} \mathcal{B}(L^2(\mathcal{M}_q)) \tag{4.3}$$

has complete bound majorized by \((d+1) + d \# S\). Define a mapping \(j_d : \Sigma_d \to X_d\),
\[
T_{w_1} \otimes \ldots \otimes T_{w_d} \mapsto \\
\left( \bigoplus_{k=0}^{d} T_{w_1} P_{w_1}^\perp \otimes \ldots \otimes T_{w_k} P_{w_k}^\perp \otimes T_{w_k+1} P_{w_k+1}^\perp \otimes \ldots \otimes T_{w_d} P_{w_d} \right) \\
\bigoplus \left( \bigoplus_{s \in S} \bigoplus_{k=0}^{d-1} T_{w_1} P_{w_1}^\perp \otimes \ldots \otimes T_{w_k} P_{w_k}^\perp \otimes P_s \otimes T_{w_k+2} P_{w_k+2}^\perp \otimes \ldots \otimes T_{w_d} P_{w_d} \right),
\]
and extend linearly.

**Lemma 4.2.** We have \(\sigma_d = \Pi_d \circ j_d\).

**Proof.** The lemma follows if we could prove the following equalities, the first one being Lemma 2.7
\[
(4.5)\]
\[
T_w = (P_{w_1} + P_{w_2}) T_{w_1}^{(1)} (P_{w_1} + P_{w_2}) \ldots (P_{w_d} + P_{w_d}) T_{w_d}^{(1)} T_{w_d}^{(1)} (P_{w_d} + P_{w_d}) \\
+ p \sum_{k=0}^{d-1} (P_{w_1} + P_{w_2}) T_{w_1}^{(1)} (P_{w_1} + P_{w_2}) \ldots (P_{w_k} + P_{w_k}) T_{w_k}^{(1)} (P_{w_k} + P_{w_k}) \times \\
P_{w_{k+1}} T_{w_{k+2}}^{(1)} (P_{w_{k+2}} + P_{w_{k+2}}) \ldots (P_{w_d} + P_{w_d}) T_{w_d}^{(1)} (P_{w_d} + P_{w_d}) \\
= \sum_{k=0}^{d-1} (T_{w_1} P_{w_1}^\perp) \ldots (T_{w_k} P_{w_k}^\perp) (T_{w_{k+1}} P_{w_{k+1}}^\perp) \ldots (T_{w_d} P_{w_d}) \\
+ \sum_{k=0}^{d-1} (T_{w_1} P_{w_1}^\perp) \ldots (T_{w_k} P_{w_k}^\perp) P_{w_{k+1}} (T_{w_{k+2}} P_{w_{k+2}}^\perp) \ldots (T_{w_d} P_{w_d}).
\]
The proof is a creation/annihilation argument as in [RiXn06, Fact 2.6]. First note that from the fact that \(w_i^2 = 1\) we obtain that \(T_{w_i}^{(1)} P_{w_i} = P_{w_i}^\perp T_{w_i}^{(1)} P_{w_i} = P_{w_i}^\perp T_{w_i}^{(1)} P_{w_i}\) and by taking adjoints \(P_{w_i} T_{w_i}^{(1)} = P_{w_i} T_{w_i}^{(1)} P_{w_i} = T_{w_i}^{(1)} P_{w_i}\). Therefore also \(P_{w_i} T_{w_i}^{(1)} P_{w_i} = P_{w_i} T_{w_i}^{(1)} T_{w_i}^{(1)} P_{w_i} = 0\). Next note that \(P_{w_i} P_{w_{i+1}} = 0\) since in this subsection we assume that \(w_i\) and \(w_{i+1}\) do not commute. Using these considerations we see that in the left hand side expression of \((4.5)\) all terms are zero except for the ones that remain on the right hand side of \((4.5)\). Indeed consider a product \(\prod_{i=1}^{d} T_{w_i}^{(1)} Q_{w_i}\) with \(Q_{w_i} = P_{w_i}\) or \(Q_{w_i} = P_{w_i}^\perp\). If a factor \(T_{w_{i+1}}^{(1)} P_{w_{i+1}}^\perp\) occurs then we must have \(Q_{w_i} = P_{w_i}^\perp\) or this product is zero. This shows that the only non-zero summands in the first term on the left side of \((4.5)\) are the ones appearing in the first summation on the right side. Similarly the second summations on each side of \((4.5)\) may be identified.

**Remark 4.3.** The inequality \((4.2)\) can be interpreted as a Khintchine inequality. It is also possible (in the free case at least) to obtain the reverse Khintchine inequality \(\|j_d(x)\| \leq \|\sigma_d(x)\|\) following (almost exactly) the proof of [RiXn06, Theorem 2.5]. The reverse inequality will not be used in this paper.

**Lemma 4.4.** Let \(A\) be a C*-algebra and let \(M = (M_{i,j})_{1 \leq i \leq k, 1 \leq j \leq l} \in M_{k,l}(A)\) be a \(k \times l\)-matrix with entries \(M_{i,j} \in A\). Assume \(k > l\). Suppose that if \(M_{i,j}\) and \(M_{i,j'}\)
are non-zero then \( j = j' \). Then,

\[
\|M\|_{M_k(A)} \leq \sqrt{\sum_{i,j} M_{i,j}^* M_{i,j}}.
\]

**Proof.** The assumption that if \( M_{i,j} \) and \( M_{i,j'} \) are non-zero then \( j = j' \) just means that after possibly permuting the standard basis vectors we may represent the transpose of \( M \) by means of the following matrix,

\[
\begin{pmatrix}
M_{1,1} & \ldots & M_{1,n_1} & 0 & \ldots & 0 & \ldots & 0 \\
0 & \ldots & 0 & M_{2,n_1+1} & \ldots & M_{2,n_2} & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & 0 & \ldots & 0 & \ldots & M_{k,n_{k-1}+1} & \ldots & M_{k,l}
\end{pmatrix}.
\]

Then \( \|M\|_{M_k(A)} = \max_i \sqrt{\sum_{j=1}^{n_i} M_{i,j}^* M_{i,j}} \), which yields the lemma. \( \square \)

**Theorem 4.5.** Assume that \( |S| \geq 3 \) and \( \forall s,t \in S, \sigma(s,t) = \infty \). The Hecke von Neumann algebra \( M_q \) is not injective.

**Proof.** We may assume that \( S \) is finite because of Corollary 3.3 and the fact that it is enough to prove the existence of an expected non-injective subalgebra. Let \( M_q^{op} \) be the opposite algebra of \( M_q \). We may identify \( M_q^{op} \) with \( J M_q J \) through the map \( y \mapsto J y^* J \) and put \( \bar{g} = J y J \) where \( J \) is the (anti-linear) Tomita-Takesaki modular conjugation, c.f. \[Tak03\]. If \( M_q \) were to be injective then we would have for all choices \( x_i, y_i \in M_q \), \( 1 \leq i \leq m \) that

\[
(4.6) \quad \| \sum_{i=1}^m x_i \otimes \bar{y}_i \|_{M_q \otimes M_q^{op}} \geq \| \tau( \sum_{i=1}^m x_i y_i^* ) \|,
\]

c.f. \[Was77\] Corollary 2, \[Con76\] Theorem 5.1] (in conjunction with Theorem 2.2 so that we are in the \( \Pi_1 \)-case). We show that this contradicts (4.2). Fix \( d \in \mathbb{N} \) and find a (finite) sequence \( w^{(i)} \in W \) with \( |w^{(i)}| = 2d \) that satisfies the property that \( i = j \) whenever

\[
(4.7) \quad w_1^{(i)} \ldots w_d^{(i)} = w_1^{(j)} \ldots w_d^{(j)} \text{ or } w_{d+1}^{(i)} \ldots w_{2d}^{(i)} = w_{d+1}^{(j)} \ldots w_{2d}^{(j)}.
\]

One can choose such a sequence of length equal to at least \( 2^d - 1 \). (Indeed let \( s, t, r \in S \) be three different generators. There are exactly \( 2^d - 1 \) reduced words \( w_1 \ldots w_{d-1} \) with \( w_i \in \{ s, t, r \} \). Call these words \( A \). Also there are exactly \( 2^d - 1 \) words \( w_{d+2} \ldots w_{2d} \) with \( w_i \in \{ s, t, r \} \). Call these words \( B \). Take some bijection \( \varphi : A \rightarrow B \) and consider the words \( w \varphi(w) \). This results in \( 2^d - 1 \) words with property (4.7); in fact the only thing that matters for the proof is that the length of such a sequence is exponential in \( d \). (4.6) then reads

\[
(4.8) \quad \| \sum_{i=1}^{2^d - 1} T_{w^{(i)}} \otimes \tau(\bar{w}^{(i)}) \|_{M_q \otimes M_q^{op}} \geq 2^d - 1.
\]

On the other hand, the Khintchine inequality \[1.2\] with length \( 2d \) gives

\[
(4.9) \quad ((2d + 1) + 2d |S|) A_d \geq \| \sum_{i=1}^{2d - 1} T_{w^{(i)}} \otimes \tau(\bar{w}^{(i)}) \|_{M_q \otimes M_q^{op}},
\]
where \( A_d \) is the maximum over the norms of each of the following expressions:

\[
T_{w_1} P_{w_1} \otimes \cdots \otimes T_{w_k} P_{w_k} \otimes T_{w_{k+1}} P_{w_{k+1}} \otimes \cdots \otimes T_{w_{2d}} P_{w_{2d}} \otimes T_{w_l},
\]
with \( 0 \leq k \leq 2d \) and

\[
\sum_{i=1}^{2d-1} T_{w_1} P_{w_1} \otimes \cdots \otimes T_{w_k} P_{w_k} \otimes P_{w_{k+1}} \otimes T_{w_{k+1}} P_{w_{k+2}} \otimes \cdots \otimes T_{w_{2d}} P_{w_{2d}} \otimes T_{w_l},
\]
with \( 0 \leq k \leq 2d-1, s \in S \). Here these expressions are identified in respectively

\[
(L_k\otimes_h K_{2d-k}) \otimes_{\min} \mathcal{M}_q^\text{op} \simeq \mathcal{B}(\mathcal{C}(\#S)^k, \mathcal{C}(\#S)^{2d-k}) \otimes_{\min} \mathcal{M}_q^\text{op},
\]
with \( 0 \leq k \leq 2d \) and

\[
(L_k\otimes_h A_{w_k} \otimes_h K_{2d-k-1}) \otimes_{\min} \mathcal{M}_q^\text{op} \simeq \mathcal{B}(\mathcal{C}(\#S)^k, \mathcal{C}(\#S)^{2d-k-1}) \otimes_{\min} A_{w_k} \otimes_{\min} \mathcal{M}_q^\text{op},
\]
with \( 0 \leq k \leq 2d \). The isomorphism (4.12) is given by

\[
T_{s_{k+1}} P_{s_{k+1}} \otimes \cdots \otimes T_{s_k} P_{s_k} \otimes P_{s_{k+1}} \otimes T_{s_{k+1}} P_{s_{k+2}} \otimes \cdots \otimes T_{s_{2d}} P_{s_{2d}} \otimes T,\]
and therefore Condition [4.7] and Lemma [4.3] show that the norm of (4.10) can be upper estimated by:

\[
\| \sum_{i=1}^{2d-1} T_{w(i)} T_{w(i)} \|^{\frac{1}{2}} \leq 2^{(d-1)/2} \max_i \| T_{w(i)} T_{w(i)} \|^{\frac{1}{2}} = 2^{(d-1)/2} \max_i \| T_{w(i)} \|.
\]

The expression can be upper estimated by \( 2^{(d-1)/2} (1 + (d + 1)p) \), c.f. Lemma [2.7].

A similar argument shows that we may upper estimate the norm of (4.11) with \( 2^{(d-1)/2} (1 + (d + 1)p) \). Indeed the isomorphism (4.13) is given by

\[
T_{s_{k+1}} P_{s_{k+1}} \otimes \cdots \otimes T_{s_k} P_{s_k} \otimes P_{s_{k+1}} \otimes T_{s_{k+1}} P_{s_{k+2}} \otimes \cdots \otimes T_{s_{2d}} P_{s_{2d}} \otimes T,\]
and therefore Condition [4.7] and Lemma [4.3] we see that (4.11) can be upper estimated with

\[
\| \sum_{i=1}^{2d-1} T_{w(i)} T_{w(i)} \|^{\frac{1}{2}} \leq 2^{(d-1)/2} \max_i \| T_{w(i)} \| \leq 2^{(d-1)/2} (1 + (d + 1)p).
\]

So in all:

\[
\| \sum_{i=1}^{2d-1} T_{w(i)} T_{w(i)} \| \mathcal{M}_q \otimes \mathcal{M}_q^\text{op} \leq (1 + (d + 1)p)2^{(d-1)/2}((2d + 1) + 2d \#S).
\]

Combining this with (4.18) shows that for every \( d \) we must have:

\[
2^{d-1} \leq 2^{(d-1)/2} (1 + (d + 1)p)((2d + 1) + 2d \#S).
\]
As $2^{d-1}$ grows faster than $2^{(d-1)/2}p(d)$ for any polynomial $p$ this leads to a contradiction and we conclude that $\mathcal{M}_q$ cannot be injective.

\[\mathbf{Corollary 4.6.\ Let (W, S) be a right angled Coxeter system with associated Hecke von Neumann algebra } \mathcal{M}_q. \text{ Let } (\tilde{W}, \tilde{S}) \text{ be a Coxeter subsystem of } (W, S) \text{ that is free, i.e. } \tilde{m}(s, t) = \infty \text{ for all } s, t \in \tilde{S}. \text{ Then } \mathcal{M}_q \text{ is non-injective.}\]

\[\text{Proof.} \text{ This follows as the Hecke von Neumann algebra } \tilde{\mathcal{M}}_q \text{ of } (\tilde{W}, \tilde{S}) \text{ is an expected subalgebra of } \mathcal{M}_q, \text{ c.f. Corollary 4.5. If } \mathcal{M}_q \text{ were to be injective then so would } \tilde{\mathcal{M}}_q \text{ which contradicts Theorem 4.5.}\]

\[\mathbf{Remark 4.7.} \text{ Following the argument of Corollary 4.6 we would be able to prove non-injectivity of a general reduced Coxeter system } (W, S) \text{ with } |S| \geq 3 \text{ if we could prove this for the case that } |S| = 3, \text{ say } S = \{r, s, t\}, \text{ and } m(r, s) = \infty, m(s, t) = \infty \text{ and } m(r, t) = 2. \text{ Though that this special case suffices, we derive nevertheless a general Khintchine inequality in the next section as this involves the same modifications.}\]

4.2. Non-injectivity of $\mathcal{M}_q$: the general case. We now assume again that $(W, S)$ is a general Coxeter system of a right angled Coxeter group, i.e. $\forall s, t \in S$ we have that $m(s, t)$ equals either 2 or $\infty$. Non-injectivity of $\mathcal{M}_q$ follows essentially by the same argument as in the free case. We only need to treat the Khintchine inequality with more care. Therefore we introduce some additional terminology. Firstly, for $s \in S$ we set

\[
\text{Link}(s) = \{t \in S \mid m(s, t) = 2\},
\]

so these are all vertices in $\Gamma$ that have distance exactly 1 to $s$. For a subset $X \subseteq V\Gamma$ we set $\text{Link}(X) = \bigcap_{s \in X} \text{Link}(s)$. We sometimes regard $\text{Link}(X)$ as a full subgraph of $\Gamma$. We let $\Sigma_d$ be span$\{T_{w_1} \otimes \ldots \otimes T_{w_d} \mid w \in W\}$ which is contained in the algebraic tensor product $\Sigma_1^{\otimes d}$ and $\Sigma_1$ the linear space spanned by $T_s, s \in S$. Let $(W_f, S)$ be the free Coxeter system which is determined by the same generating set $S$ but with relations $m(f, s, t) = \infty, s \neq t$. Let $\mathcal{M}_q^f$ be the free Hecke von Neumann algebra and $L^2(\mathcal{M}_q^f)$ its GNS space. We define the spaces $L_1$ and $K_1$ exactly as in (4.11) but with respect to the Coxeter system $(W_f, S)$. In particular Lemma 4.7 remains valid. Lemma 4.1 is not valid for the system $(W, S)$ which is the reason we need to introduce an extra intertwining argument in this section. Then set $L_k = (L_1)^{\otimes k}, K_k = (K_1)^{\otimes k}$. Let $\text{Cliq}(\Gamma, l)$ be the set of cliques in $\Gamma$ with $l$ vertices. For $\Gamma_0 \in \text{Cliq}(\Gamma, l)$ we let $\text{Comm}(\Gamma_0)$ be the set of all pairs $(\Gamma_1, \Gamma_2) \in \text{Clq}(\text{Link}(\Gamma_0)) \times \text{Clq}(\text{Link}(\Gamma_0))$ such that $V\Gamma_1 \cap V\Gamma_2 = \emptyset$. Let

\[
(4.14) \quad X_d = \bigoplus_{l=0}^{d} \bigoplus_{\Gamma_0 \in \text{Clq}(\Gamma, l)} \bigoplus_{(\Gamma_1, \Gamma_2) \in \text{Comm}(\Gamma_0)} K_{d-k-l},
\]

where $A_{\Gamma_0} = \mathbb{C}P_{\text{VT}_{\Gamma_0}}^{l} \subseteq B(L^2(\mathcal{M}_q^f))$; here $P_{\text{VT}_{\Gamma_0}}^{l}$ is the projection onto the vectors $T_\Omega \in L^2(\mathcal{M}_q^f)$ with $\Omega$ starting with letters $\text{VT}_0$ ordered in minimal order (see the Convention in Section 2).

Parallel to the free case we shall define a mapping $j_d : \Sigma_d \to X_d$. Let $0 \leq l \leq d, 0 \leq k \leq d - l$ and let $\Gamma_0 \in \text{Clq}(\Gamma, l), (\Gamma_1, \Gamma_2) \in \text{Comm}(\Gamma_0)$. The image of
Lemma 4.8. Let \( \sigma \) be the zero vector. \( \Gamma \) does not exist then the summand is understood as 0.

Now (4.5) becomes as follows. The proof is comparable to [CaFi15, Proposition 4.6].

**Lemma 4.8.** Let \( \mathbf{w} \in W \). Let \( A_w(k, \Gamma_0) \) be the set of pairs \((\mathbf{w}', \mathbf{w}'')\) with \( \mathbf{w} = \mathbf{w}' \Gamma_0 \mathbf{w}''\), \( |\mathbf{w}'| = k \) and \( |\mathbf{w}| = |\mathbf{w}'| + |\Gamma_0| + |\mathbf{w}''| \). For each \( \Gamma_1 \in \text{Cliq}(\text{Link}(\Gamma_0)) \) there exists at most one \((\mathbf{w}', \mathbf{w}'')\) in \( A_w(k, \Gamma_0) \) with \((\ast)\) for all \( s \in \Gamma_1 \), \( |\mathbf{w}' s| = |\mathbf{w}'| - 1 \), for all \( s \in \text{Link}(\Gamma_0) \setminus \Gamma_1 \), \( |\mathbf{w}' s| = |\mathbf{w}'| + 1 \).

**Proof.** Suppose that \((\mathbf{w}', \mathbf{w}'')\) \(\in A_w(k, \Gamma_0)\) and \((\mathbf{w}', \mathbf{w}'')\) \(\in A_w(k, \Gamma_0)\) both satisfy the property \((\ast)\) of the lemma. We have \( \mathbf{w} = \mathbf{w}_1' \Gamma_0 \mathbf{w}_1'' = \mathbf{w}_2' \Gamma_0 \mathbf{w}_2'' \) with \( |\mathbf{w}_1'| = |\mathbf{w}_2'| = k \). The only way \( \mathbf{w}_1' \) and \( \mathbf{w}_2' \) could possibly differ is if one of the end letters of \( \mathbf{w}_1' \) is exchanged with one of the start letters of \( \mathbf{w}_2'' \). Suppose that for some \( t \in S \) we have \( |\mathbf{w}_1'| = |\mathbf{w}_2'| - 1 \) but \( |\mathbf{w}_2'| = |\mathbf{w}_2'| + 1 \). Then we must have \( |\mathbf{w}_1'| = |\mathbf{w}_1'| + 1 \) but \( |\mathbf{w}_2'| = |\mathbf{w}_2'| - 1 \) and moreover \( t \) commutes with \( \Gamma_0 \), i.e. \( t \in \text{Link}(\Gamma_0) \). If \( t \in \Gamma_1 \) then \( |\mathbf{w}_1'| = |\mathbf{w}_2'| + 1 \) cannot hold by \((\ast)\) and if \( t \notin \Gamma_0 \) then \( |\mathbf{w}_1'| = |\mathbf{w}_2'| - 1 \) cannot hold, also by \((\ast)\). I.e. we get a contradiction. So we must have \( \mathbf{w}_1' = \mathbf{w}_2' \) and hence also \( \mathbf{w}_1'' = \mathbf{w}_2'' \).

Now (4.15) becomes as follows. The proof is comparable to [CaFi15, Proposition 4.6].

**Lemma 4.9.** For every \( T_{w_1} \otimes \ldots \otimes T_{w_d} \in \Sigma_d \) we have,

\[
T_{w_1} \otimes \ldots \otimes T_{w_d} = \sum_{i=0}^{d-1} \sum_{\Gamma_0 \in \text{Cliq}(\Gamma_1) \setminus \Gamma_0} \sum_{\Gamma_1 \in \text{Comm}(\Gamma_0)} p^{\# \Gamma_0} T_{w_1}^{T_1(1)} P_{w_{\sigma(1)}}^{\perp} \ldots T_{w_{\sigma(i)}}^{T_1(1)} P_{w_{\sigma(i)}}^{\perp} \ldots T_{w_{\sigma(k)}}^{T_1(1)} P_{w_{\sigma(k)}}^{\perp} \ldots T_{w_{\sigma(d)}}^{T_1(1)} P_{w_{\sigma(d)}}^{\perp} \Gamma_0 \times P_{\Gamma_0} T_{w_2}^{T_1(1)} P_{w_{\sigma(2)}}^{\perp},
\]

where \( \sigma \) (changing over the summation) is as in (11) above this lemma. If such \( \sigma \) does not exist then the summand is understood as 0.

**Proof.** We first note that we may decompose,

\[
T_{w_1} \otimes \ldots \otimes T_{w_d} = (P_{w_1} + P_{w_1}^{\perp}) T_{w_1} (P_{w_2} + P_{w_2}^{\perp}) \ldots (P_{w_d} + P_{w_d}^{\perp}) T_{w_d} (P_{w_d} + P_{w_d}^{\perp})
\]
Therefore consider an expression of the form:

\[(4.18) \quad Q^{(1)}_{w_1} T_{w_1} Q^{(2)}_{w_2} \cdots Q^{(1)}_{w_d} T_{w_d} Q^{(2)}_{w_d}, \]

where \(Q^{(j)}_{w_i}\) equals either \(P_{w_i}\) or \(P^\perp_{w_i}\). Throughout the proof we shall assume that \((4.18)\) is non-zero. The following claims show that after possibly interchanging commuting factors in the expression \((4.18)\) we may assume that \((4.18)\) is of a specific form.

**Claim 1.** The expression \((4.18)\) is after possibly interchanging commuting letters in \(w_1 \ldots w_d\) of the form:

\[(4.19) \quad Q^{(1)}_{w_1} T_{w_1} Q^{(2)}_{w_2} \cdots Q^{(1)}_{w_s} T_{w_s} Q^{(2)}_{w_s}(P^\perp_{w_{s+1}} T_{w_{s+1}} P_{w_{s+1}}) \cdots (P^\perp_{w_{d}} T_{w_{d}} P_{w_{d}}). \]

Moreover, the tail of annihilation operators is maximal in the sense that if for some \(i \leq s\) we have \(Q^{(2)}_{w_i} = P_{w_i}\) then \(Q^{(1)}_{w_i} = P^\perp_{w_i}\).

**Proof of Claim 1.** Suppose that we are given an expression as in \((4.19)\). Suppose that for some \(i < s\) we have \(Q^{(1)}_{w_i} = P^\perp_{w_i}\) and \(Q^{(2)}_{w_i} = P_{w_i}\). Then we need to show that \(w_i\) commutes with \(w_{i+1} \ldots w_s\). To do so we may suppose the index \(i\) was chosen maximal (but still smaller than \(s\)). Suppose that \(w_i\) and \(w_{i+1} \ldots w_s\) do not commute and let \(w_k\) be the first letter in \(w_{i+1} \ldots w_s\) that does not commute with \(w_i\). Our choice of \(i\) yields that \(Q^{(1)}_{w_i} = P_{w_k}\) (indeed if \(Q^{(1)}_{w_i}\) were to be \(P^\perp_{w_i}\) then \((4.19)\) is 0 in case \(Q^{(2)}_{w_k} = P^\perp_{w_k}\) and in case \(Q^{(2)}_{w_k} = P_{w_k}\) then \(i\) was not maximal). But then \((4.19)\) contains a factor \(P_{w_i}P_{w_k} = 0\) which means that \((4.19)\) would be zero which in turn is a contradiction.

**Claim 2.** The expression \((4.18)\) is after possibly interchanging commuting letters in \(w_1 \ldots w_d\) of the form:

\[(4.20) \quad Q^{(1)}_{w_1} T_{w_1} Q^{(2)}_{w_2} \cdots Q^{(1)}_{w_s} T_{w_s} Q^{(2)}_{w_s}(P^\perp_{w_{s+1}} T_{w_{s+1}} P_{w_{s+1}}) \cdots (P^\perp_{w_{d}} T_{w_{d}} P_{w_{d}}). \]

Moreover, the tail of annihilation and diagonal operators is maximal in the sense that if for some \(i \leq r\) we have \(Q^{(1)}_{w_i} = P_{w_i}\) then \(Q^{(2)}_{w_i} = P^\perp_{w_i}\).

**Proof of Claim 2.** Suppose that we are given a (non-zero) expression as in \((4.20)\). Suppose that for some \(i < r\) we have \(Q^{(1)}_{w_i} = P_{w_i}\) and \(Q^{(2)}_{w_i} = P^\perp_{w_i}\). Then we need to show that \(w_i\) commutes with \(w_{i+1} \ldots w_r\). To do so we may suppose the index \(i < r\) was chosen maximal. Suppose that \(w_i\) and \(w_{i+1} \ldots w_r\) do not commute and let \(w_k\) be the first letter in \(w_{i+1} \ldots w_r\) that does not commute with \(w_i\). Our (maximal) choice of \(i\) yields that \(Q^{(1)}_{w_i} = P_{w_k}\) and \(Q^{(2)}_{w_i} = P^\perp_{w_k}\). But then \((4.20)\) contains a factor \(P_{w_i}P_{w_k} = 0\) which means that \((4.20)\) would be zero. As this is a contradiction the claim follows.

**Claim 3.** The expression \((4.18)\) is after possibly interchanging commuting letters in \(w_1 \ldots w_d\) of the form:

\[(4.21) \quad (P_{w_1} T_{w_1} P^\perp_{w_1}) \cdots (P_{w_s} T_{w_s} P^\perp_{w_s})(P^\perp_{w_{s+1}} T_{w_{s+1}} P_{w_{s+1}}) \cdots (P^\perp_{w_{d}} T_{w_{d}} P_{w_{d}}). \]

Moreover \(w_{r+1} \ldots w_s\) forms a clique.
Proof of Claim 3. This is obvious now from Claim 2 and the fact that $P_{w_i}^+ T_w T_{w_i}^+ P_{w_i}^+ = 0$. As $P_{w_i} P_{w_j}$ is non-zero only if $w_i$ and $w_j$ commute we must have that $w_{r+1} \ldots w_s$ forms a clique.

Remainder of the proof. Note that as $T_w = T_w^{(1)} + p P_w$ we have that (4.18) after possibly interchanging commuting letters in $w_1 \ldots w_d$ equals:

$$p^{|\Gamma_0|} (T_w^{(1)} P_{w_1}^+) \ldots (T_w^{(1)} P_{w_r}^+) P_{\Gamma_0} (T_w^{(1)} P_{w_{r+1}}^+) \ldots (T_w^{(1)} P_{w_d}^+),$$

where $\Gamma_0$ is the clique comprising the letters $w_{r+1} \ldots w_r$ as in Claim 3. Therefore the non-zero terms on the right hand side of (4.17) all occur in the summation (4.16). Note that the ‘possible commutations’ in each of the claims do not affect the permutation $\sigma$ in (4.16). That the summands of (4.16) are in 1–1 correspondence with the non-zero terms of (4.17) follows by Lemma 4.10. \hfill \square

We define

$$\Pi_d : j_d(\Sigma_d) \to B(L^2(\mathcal{M}_q)) : j_d(x) \mapsto \sigma_d(x).$$

As $\Sigma_d$ is finite dimensional this map is completely bounded and by definition $\sigma_d = \Pi_d \circ j_d$. It remains to obtain control over the complete bound of $\Pi_d$ in terms of $d$. This is done by means of the following intertwining lemma.

Lemma 4.10. Suppose that $\Gamma$ is finite. $\Pi_d$ defined in (4.22) has complete bounded that is majorized by $Cd^2$ for a constant $C$ that is independent of $d$.

Proof. The proof is an intertwining argument between product maps associated to the general and to the free case. Let us make this precise. Let $L^2(\mathcal{M}_q^f)$ be the GNS space of the Hecke algebra $\mathcal{M}_q^f$ generated by $(W_f,S)$ where again $W_f$ is the ‘free’ Coxeter group with generating set $S$ and relations $\forall s,t \in S, m_f(s,t) = \infty$. Let $0 \leq l \leq d, 0 \leq k \leq d - l$ and let $\Gamma_0 \in \text{Clq}(\Gamma, l), (\Gamma_1, \Gamma_2) \in \text{Comm}(\Gamma_0)$. Let,

$$\Pi_{d,k,l,\Gamma_0,\Gamma_1,\Gamma_2} : L_k \otimes_h A_{\Gamma_0} \otimes_h K_{d-k-1} \to B(L^2(\mathcal{M}_q^f))$$

be the product map. This map is completely bounded as follows from the definition of the Haagerup tensor product.

Definition of the intertwining maps. We define two unitary maps. Note that the second map only differs from the first one at the place we put an exclamation mark.

- We define the intertwining map,

$$Q_{k,l,\Gamma_0,\Gamma_1} : L^2(\mathcal{M}_q) \to L^2(\mathcal{M}_q^f),$$

by sending a vector $T_w \Omega$ with $|w| = d$ to $T_{w_{\sigma(1)} \ldots w_{\sigma(d)}} \Omega$ where $\sigma$ is the permutation defined in (11–4) after (4.15). Moreover we assume that this $\sigma$ is chosen such that each of the expressions $w_{\sigma(1)} \ldots w_{\sigma(k)}$, $w_{\sigma(k+1)} \ldots w_{\sigma(k+l)}$ and $w_{\sigma(k+l+1)} \ldots w_{\sigma(d)}$ are minimal (which uniquely determines $Q_{k,l,\Gamma_0,\Gamma_1}(T_w \Omega)$). If such $\sigma$ does not exist then $Q_{k,l,\Gamma_0,\Gamma_1}(T_w \Omega)$ is understood as the zero vector.

- We define the intertwining map,

$$R_{k,l,\Gamma_0,\Gamma_1} : L^2(\mathcal{M}_q) \to L^2(\mathcal{M}_q^f),$$

by sending a vector $T_w \Omega$ with $|w| = d$ to $T_{w_{\sigma(1)} \ldots w_{\sigma(d)}} \Omega$ where $\sigma$ is the permutation defined in (11–4) after (4.15). Moreover we assume that this $\sigma$ is chosen such that each of the expressions $w_{\sigma(1)} \ldots w_{\sigma(k)}$, $w_{\sigma(k+1)} \ldots w_{\sigma(k+l)}$ and $w_{\sigma(k+l+1)} \ldots w_{\sigma(d)}$ are minimal (which uniquely determines $R_{k,l,\Gamma_0,\Gamma_1}(T_w \Omega)$). If such $\sigma$ does not exist then $R_{k,l,\Gamma_0,\Gamma_1}(T_w \Omega)$ is understood as the zero vector.
by sending a vector $T_w \Omega$ with $|w| = d$ to $T^f_{w_{\sigma(1)} \ldots w_{\sigma(d)}} \Omega^f$ where $\sigma$ is the permutation defined in (4.14–4.45) after expression (4.15). Moreover we assume that this $\sigma$ is chosen such that each of the expressions $w_{\sigma(1)} \ldots w_{\sigma(k)}$ (!), $w_{\sigma(k+1)} \ldots w_{\sigma(k+l)}$ and $w_{\sigma(k+l+1)} \ldots w_{\sigma(d)}$ are minimal (which uniquely determines $R_{k,l,\Gamma_0,\Gamma_1}(T_w \Omega)$). If such $\sigma$ does not exist then $R_{k,l,\Gamma_0,\Gamma_1}(T_w \Omega)$ is understood as the zero vector.

Claim. Let $x = T_{w_1} \otimes \ldots \otimes T_{w_d} \in \Sigma_d$ and let $x_{d,k,l,\Gamma_0,\Gamma_1,\Gamma_2}$ with $0 \leq k \leq d - l$, $\Gamma_0 \in \text{Cliq}(\Gamma, l)$ and $(\Gamma_1, \Gamma_2) \in \text{Comm}(\Gamma_0)$ be the corresponding summands of $j_d(x)$ in $X_d$. We have,

$$(4.27) \quad R_{k,l,\Gamma_0,\Gamma_1} \Pi^f_{d,k,l,\Gamma_0,\Gamma_1,\Gamma_2}(x_{d,k,l,\Gamma_0,\Gamma_1,\Gamma_2})Q_{d-l-k,l,\Gamma_0,\Gamma_2} = (T^f_{w_{\sigma(1)}} P^f_{w_{\sigma(1)}}) \ldots (T^f_{w_{\sigma(k)}} P^f_{w_{\sigma(k)}}) P_{\Gamma_0} (T^f_{w_{\sigma(k+1)}} P_{w_{\sigma(k+l+1)}}) \ldots (T^f_{w_{\sigma(d)}} P_{w_{\sigma(d)}}),$$

where $\sigma$ is defined as in (1) – (6) and the right hand side should be understood as 0 if such $\sigma$ does not exist.

Proof of the Claim. Note that both sides of (4.27) equal 0 if $\sigma$ as in the statement of the claim does not exist, c.f. the definition of $j_d$. So we assume that this is not the case. Take an elementary tensor product $T_{w_1} \otimes \ldots \otimes T_{w_d} \in \Sigma_d$. As both sides of (4.27) change in the same way under interchanging $w_i$ and $w_{i+1}$ in case $m(w_i, w_{i+1}) = 2$, we may assume that the tensor product $T_{w_1} \otimes \ldots \otimes T_{w_d}$ is ordered in such a way that the permutation $\sigma$ on the right hand side of (4.27) is trivial.

Now take a vector $T_v \Omega$ and set,

$$Q_{d-l-k,l,\Gamma_0,\Gamma_2}(T_v \Omega) = T^f_v \Omega,$$

with $v'_i = v_{\sigma(i)}$ and $\sigma$ as in the definition of $Q_{d-l-k,l,\Gamma_0,\Gamma_2}$ so that $v'_{d-k-l} \ldots v'_l$, $v'_{d-k-l} \ldots v'_d$ and $v'_{d-k+l} \ldots v'_n$ are minimal. If such $\sigma$ does not exist then $Q_{d-l-k,l,\Gamma_0,\Gamma_2}(T_v \Omega) = 0$. We have,

$$(4.28) \quad \Pi^f_{d,k,l,\Gamma_0,\Gamma_1,\Gamma_2}(x_{d,k,l,\Gamma_0,\Gamma_1,\Gamma_2}) = (T^f_{w_1} P^f_{w_1}) \ldots (T^f_{w_d} P^f_{w_d}) P_{\Gamma_0} (T^f_{w_{k+l+1}} P_{w_{k+l+1}}) \ldots (T^f_{w_{d}} P_{w_{d}}).$$

And therefore, if $Q_{d-l-k,l,\Gamma_0,\Gamma_2}(T_v \Omega)$ is non-zero,

$$(4.29) \quad \Pi^f_{d,k,l,\Gamma_0,\Gamma_1,\Gamma_2}(x_{d,k,l,\Gamma_0,\Gamma_1,\Gamma_2}) Q_{d-l-k,l,\Gamma_0,\Gamma_2}(T_v \Omega) = (T^f_{w_1} \Omega, T^f_{w_d} \Omega) \ldots (T^f_{v'_{d-k-l}} \Omega, T^f_{v'_{d-k+l}} \Omega) T^f_{v'_{d-k-l} \ldots v'_{d-k+l}} \Omega.$$

On the other hand consider an expression

$$\quad (T^f_{w_k} P_{w_k}) \ldots (T^f_{w_{k+l+1}} P_{w_{k+l+1}}) P_{\Gamma_0} \times (T^f_{w_k} P_{w_k}) T_v \Omega.$$

Because $w_{k+l+1} \ldots w_d$ starts with $\Gamma_2$ (since we assumed that (4.27) is non-zero) this expression can only be non-zero if there exists $v'$ as defined above. As we assumed that $v'_{d-k-l} \ldots v'_l$ and $w_{k+l+1} \ldots w_d$ are minimal the inner products match
up as in (4.30) possibly being 0 so that we get,

\[ (4.29) = (T_{w_1}^{1}) P_{w_1}^{\perp} \cdots (T_{w_k}^{1}) P_{w_k}^{\perp} P_{T_0} \]

\[ \times (T_{w_{k+1}}^{1}) P_{w_{k+1}}^{\perp} \cdots (T_{w_d}^{1}) P_{w_d}^{\perp} T_{\Omega}^{\perp} \]

\[ = (T_{v_1}^{1} \Omega, T_{w_1}^{1} \Omega) \cdots (T_{v_1}^{d-k-1} \Omega, T_{w_{k+1}}^{d-k+1} \Omega) \]

\[ T_{w_1}, \ldots, w_k T_{v_1}^{d-k} \cdots v_1^{d-k+1} \cdots \Omega \]

Clearly the image of (4.30) under \( R_{k,I,T,0,1} \) equals (4.28). This concludes the claim.

Remainder of the proof. From Lemma 4.9 we see that \( \Pi_{q} \) is given by the direct sum of the maps \( R_{k,I,T,0,1}^{q}, \Pi_{d,k,I,T,0,1}^{q}, \Pi_{d,k,I,T,0,1}^{q} (\gamma) \mathcal{Q}_{d+1-k,I,T,0,1}^{q} \), which are defined on the corresponding summands of \( X_{d} \). As each of these summands is completely contractive and as we assumed that \( \Gamma \) is finite there are \( C d^{2} \) summands for some constant \( C \) independent of \( d \), we see that \( \Pi_{q} \) is completely bounded with complete bound majorized by \( C d^{2} \).

\[ \square \]

**Theorem 4.11.** Let \( q > 0 \) and let \( (W,S) \) be a reduced right-angled Coxeter system with \( |S| \geq 3 \). The Hecke von Neumann algebra \( M_{q} \) is not injective.

**Proof.** Following Remark 4.7 it suffices to consider the case \( |S| = 3 \), say \( S = \{ r, s, t \} \), with commutation relations \( m(r,s) = \infty, m(s,t) = \infty, m(r,t) = 2 \). The proof of this case is now a mutatis mutandis copy of the proof of Theorem 4.5.

Note that at the beginning of the proof we had to justify that there existed a sequence of words \( w^{(1)} \) of length \( 2^{d-1} \) such that (4.7) holds. For the proof it only matters that the maximal possible length of such a sequence is exponential in \( d \). We claim that in the current case there is such a sequence of length at least \( 2^{d}(d-1) \) in case \( d \) is odd. Indeed, for \( d \) odd we may consider words \( a_{1}sa_{3}sa_{5} \cdots a_{d-2}sa_{d} \) with \( a_{i} \in \{ r, t \} \). Such words form a set \( A \) of size \( 2^{d(d-1)} \). Similarly the words \( ts a_{1}sa_{3}sa_{5} \cdots sa_{d-2}a_{d} \) with \( a_{i} \in \{ r, t \} \) form a set \( B \) of size \( 2^{d(d-1)} \). Letting again \( \varphi : A \to B \) be a bijection and considering the words \( w, w' \in A \) shows that there is a sequence satisfying (4.7) of length at least \( 2^{d(d-1)} \).

Next note that the Khintchine inequality applied in (4.30) gets replaced by Lemma 4.10. The rest of the proof of Theorem 4.5 changes mutatis mutandis to the Khintchine decomposition (4.10).

\[ \square \]

**5. Completely contractive approximation property and Haagerup property**

We show that for a right angled Coxeter system \( (W,S) \) the Hecke von Neumann algebra \( M_{q} \) has the wk*-CCAP, see Definition 5.13. The proof follows a – by now standard – strategy of Haagerup [Haa78] by considering radial multipliers first and then showing that word length cut-downs have a complete bound that is at most polynomial in the word length. As a direct consequence of our radial multipliers we find that \( M_{q} \) also has the Haagerup property.

**5.1. Radial multipliers.** We first observe the following.

**Proposition 5.1.** Let \( (W,S) \) be a right-angled Coxeter group with Hecke von Neumann algebra \( M_{q}, q > 0 \). For every \( 0 < r < 1 \) there exist a normal unital completely positive map \( \Phi_{r} : M_{q} \to M_{q} \) that is determined by \( \Phi_{r}(T_{w}) = r^{|w|}T_{w} \).
Proof. As in Corollary 3.4 we identify $M_q$ with the graph product $*_{s \in V_T}(M_q(s), \tau_s)$ where $\tau_s$ is the tracial state on $M_q(s)$. Consider the map $\Phi_{r,s} : M_q(s) \to M_q(s)$ determined by $1 \mapsto 1, T_s \mapsto rT_s$. This map is unital and completely positive: indeed consider matrices
\[
A := \begin{pmatrix} \sqrt{1-r} & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & \sqrt{1-r} \\ 0 & 0 \end{pmatrix}, C = \begin{pmatrix} \sqrt{r} & 0 \\ 0 & \sqrt{r} \end{pmatrix}.
\]
Then $\Phi_{r,s}$ agrees with $x \mapsto A^*xA + B^*xB + C^*xC$ as before Corollary 3.4 we already noted that $T_s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Furthermore $\Phi_{r,s}$ preserves the trace $\tau_s$ as $\tau_s$ is the vector state associated with $(1,0)^t$. Therefore we may apply [Cart13 Proposition 2.30] and obtain the graph product ucp map $\Phi_r := *_{s \in V_T} \Phi_{r,s}$ which proves the proposition. 

5.2. Creation/annihilation arguments. Here we present some combinatorial arguments that we need in Section 5.3. We have chosen to separate them from the proofs of Section 5.3 so that the reader could skip them at first sight.

We introduce the following notation. Let $x, w \in W$. We shall write $w \leq x$ for saying that $|w^{-1}x| = |x| - |w|$. Then $w < x$ is defined naturally. So $w \leq x$ means that $w$ is obtained from $x$ by cutting off a tail. An element $v \in W$ is called a clique word in case its letters form a clique. For $\Lambda$ a clique in $W$ and $v \in W$ we define $v(2,0)$ as the maximal clique $\Gamma_0$ such that $|vV\Gamma_0| = |v| - |V\Gamma_0|$. Then we set the decomposition $v = v(1,\Lambda)v(2,\Lambda)$ with $|v| = |v(1,\Lambda)| + |v(2,\Lambda)|$ and $v(2,\Lambda) = v(2,0)\Lambda$ (which uniquely determines $v(1,\Lambda)$). For $g \leq x$ we let $\Lambda_{g,x}$ be $(x^{-1}g)(2,\emptyset)$. In other words $\Lambda_{g,x}$ is the maximal clique that appears at the start of $g^{-1}x$. We let $C(g,x)$ be the collection of $w \in W$ with $g \leq w \leq g\Lambda_{g,x}$. Note that $C(g,x)$ contains at least $g$ and $g\Lambda_{g,x}$ (and the latter elements can be equal). We write $C(g,\emptyset)$ for $\cup_{g \leq x}C(g,x)$.

Example 5.2. Consider the Coxeter system $(W,S)$ with $S = \{r,s,t\}$ in which $m(r,s) = 2$ and $m(r,t) = m(s,t) = \infty$. Consider $v = trs$. Then $v(1,\emptyset) = t, v(2,\emptyset) = rs, v(1,r) = tr$ and $v(2,r) = s$. Also $\Lambda_{t,tru} = \{t, tr, ts, trs\}$.

Lemma 5.3. Let $x, w \in W$. Let $w = w'w''$ be the decomposition with $|w| = |w'| + |w''|$ such that $|w''x| = |x| - |w''|$ and $|wx| = |x| - |w'| + |w''|$. Take $(w'')^{-1} \leq g \leq x$. Then, for $v \in C(g,x)$,

\[
(vw)(2, (wg)(2,0) \setminus g(2,0)) = v(2, g(2,0) \setminus (wg)(2,0))
\]

and

\[
|(vw)(1, (wg)(2,0) \setminus g(2,0))| = |v(1, g(2,0) \setminus (wg)(2,0))| - |w''| + |w'|.
\]

Proof. Let $v \in C(g,x)$. The clique $v(2, \emptyset)$ consists of the clique $g^{-1}v$ plus all letters in $g(2,\emptyset)$ that commute with $g^{-1}v$. Therefore $v(2, g(2,0) \setminus (wg)(2,0))$ is the clique consisting of $g^{-1}v$ plus all letters in $(wg)(2,0) \cap g(2,\emptyset)$ that commute with $g^{-1}v$. On the other hand $(vw)(2,\emptyset)$ consists of the clique $g^{-1}v$ together with all letters in $(wg)(2,0)$ that commute with $g^{-1}v$. Then

Suppose that $\Gamma_0$ and $\Gamma_1$ are cliques such that both $|vV\Gamma_i| = |v| - |V\Gamma_i|$ then the letters $V\Gamma_0$ and $V\Gamma_1$ must commute. So the union $\Gamma_2 = \Gamma_0 \cup \Gamma_1$ is a clique with $|vV\Gamma_2| = |v| - |V\Gamma_2|$. 

\end{proof}
equals $g^{-1}v$ together with all elements in $(wg)(2,\emptyset) \cap g(2,\emptyset)$ that commute with $g^{-1}v$. So we conclude (5.1). Therefore,
\[(wv)(1,(wg)(2,\emptyset)\setminus g(2,\emptyset)) |\] 
\[= |wv| - |(wv)(2,(wg)(2,\emptyset)\setminus g(2,\emptyset))| \]
\[(5.3) = |v| - |w''| + |w| - |v(2,g(2,\emptyset))(wg)(2,\emptyset)| \]
\[= |v(1,g(2,\emptyset)\setminus wg(2,\emptyset))| - |w''| + |w|. \]

\[\square\]

**Lemma 5.4.** Let $x, w \in W$ and decompose $w = w'w''$ such that $|w| = |w'| + |w''|$ and $|w'x| = |x| - |w''|$ and $|w x| = |x| - |w''| + |w|$. Let $(w'')^{-1} \leq g \leq x$. Then:

1. $g(2,\emptyset)(wg)(2,\emptyset) = g(2,\emptyset)(w''g)(2,\emptyset)$;
2. For $v \in C(g, x)$ we have
\[v(2,v(2,\emptyset)(w''v)(2,\emptyset)) = v(2,g(2,\emptyset)(w''g)(2,\emptyset)). \]

**Proof.** (1) Because $(w'')^{-1} \leq g \leq x$ we also have $|w''g| = |g| - |w''|$ and $|wg| = |g| - |w''| + |w|$. So $w'$ creates letters in $w''g$ so that $g(2,\emptyset)(wg)(2,\emptyset) = g(2,\emptyset)(w''g)(2,\emptyset)$.

(2) Let $A$ be the set of letters in $g(2,\emptyset)$ that commute with $g^{-1}v$. The clique $v(2,\emptyset)$ consists of $g^{-1}v \cup A$. This means that $v(2,v(2,\emptyset)(w''v)(2,\emptyset))$ consists of $g^{-1}v \cup A$ intersected with $(w''v)(2,\emptyset)$. The intersection of $(w''v)(2,\emptyset)$ with $g^{-1}v$ is $g^{-1}v \cup A \cap (w''v)(2,\emptyset) = g^{-1}v \cup (A \cap (w''v)(2,\emptyset))$. On the other hand $v(2,g(2,\emptyset)(w''g)(2,\emptyset))$ equals $g^{-1}v \cup (A \cap (w''g)(2,\emptyset))$ and as $g(2,\emptyset) \cap (w''g)(2,\emptyset) = g(2,\emptyset) \cap (w''v)(2,\emptyset)$ clearly $(A \cap (w''v)(2,\emptyset)) = (A \cap (w''g)(2,\emptyset))$. This proves (5.4).

\[\square\]

Although Coxeter groups generally do not have polynomial growth (and neither they are hyperbolic) we still have the polynomial estimate of the following Lemma 5.5. We do not believe that the degree of the polynomial bound we obtain in Lemma 5.5 is optimal, but it suffices for our purposes and it admits a short proof.

**Lemma 5.5.** Let $W$ be a right-angled Coxeter group with finite graph $\Gamma$. Let $x \in W$. For $a \in \mathbb{N}$ define
\[\kappa_x(a) = \# \{ w \leq x \mid |w| = a \}. \]
Then $\kappa_x(a) \leq Ca^{(V\Gamma)^{-2}}$. Moreover, the constant $C$ can be taken uniform in $x$.

**Proof.** To do the proof we shall actually count a more refined number. We write $\Lambda \leq \Gamma$ for saying that $\Lambda$ is a complete subgraph of $\Gamma$. We say that $w$ is a ($\leq \Lambda$)-word if its letters (in reduced form) are all in $V\Lambda$ (they do not need to exhaust all of $V\Lambda$); we say that $w$ is a $\Lambda$-word if its letters are exactly $V\Lambda$. Then define
\[\kappa^\Lambda_x(a) = \# \{ v \leq x \mid |v| = a \text{ and } v \text{ is a ($\leq \Lambda$)-word} \}. \]
Let $c$ and $k_0$ be constants such that for $0 \leq a \leq 1$ we have for all $\emptyset \neq \Lambda \leq \Gamma$ we have $\kappa^\Lambda_x(a) \leq c(a + k_0)^{|V\Lambda|-2}$ and further for all $a \in \mathbb{N}$ and all non-empty complete subgraphs $\Lambda$ of $\Gamma$ we have $2^{|V\Lambda|}ca \leq (a + k_0)^2$. We prove by induction on $a$ that for all $\emptyset \neq \Lambda \leq \Gamma$ we have $\kappa^\Lambda_x(a) \leq c(a + k_0)^{|V\Lambda|-2}$. In order to do so pick some fixed $w < x$ with $|w| = a$ and $w$ a $\Lambda$-word. Now if $v < x$ with $|v| = a$ then let $v_0$ be an element of maximal length such that both $v_0 \leq v$ and $v_0 \leq w$ (we leave in the middle if $v_0$ is unique). Let $s \in W$ be a letter that appears at the start of $v_0^{-1}w$. As $v_0$ has maximal length $s$ cannot appear at the start of $v_0^{-1}v$. As $v$ and
are both below \( x \) the letter \( s \) must commute with \( v_0^{-1} v \) and this is only true if \( s \) commutes with all the letters occuring in \( v_0^{-1} v \) (this is the normal form theorem [Gre07, Theorem 3.9]). So if \( v_0^{-1} w \) is a \( \Lambda \)-word then \( v_0^{-1} v \) is a \( \text{Link}(\Lambda) \)-word (recall \( \text{Link}(\Lambda) = \cap_{s \in V \Lambda} \text{Link}(s) \)); in fact it must be a \( (\text{Link}(\Lambda) \cap \Lambda) \)-word as we only deal with words with letters in \( \Lambda \). Moreover \( v_0^{-1} v \) must appear at the start of \( w^{-1} x \). So every word in the set we count in (5.5) is obtained from \( w^{-1} x \) with words with letters in \( \Lambda \). Moreover \( v_\omega \) must commute with \( v \) (this is \( \text{Link}(\Lambda) \cap \Lambda \)-word (recall [Gre07, Theorem 3.9]). So if \( \omega \) is a \( \Lambda \)-word then \( \omega \) is a \( \Lambda \)-word. Therefore we get,

\[
\kappa^\Lambda_{\omega}(a) \leq \sum_{\Lambda' \leq \Lambda} \sum_{v < w} \kappa^{\text{Link}(\Lambda')}_{w^{-1} x}(|v^{-1} w|).
\]

\( v^{-1} w \) is a \( \Lambda' \)-word.

Note that the number of \( \omega \in W \) with \( \omega < v_0^{-1} v \), \( |\omega| = l \) and \( v \) a \( \Lambda' \)-word is smaller than or equal to \( \kappa_{w^{-1} x}^{\Lambda'}(|v^{-1} w| - l) \). In case \( l = 0 \) we have \( \kappa_{w^{-1} x}^{\Lambda'}(|v^{-1} w| - l) = 1 \) (elementary) and in case \( l > 0 \) we can apply our induction hypothesis to get \( \kappa_{w^{-1} x}^{\Lambda'}(|v^{-1} w| - l) \leq c(a - l + k_0)^{|V\Lambda'| - 2} \). Therefore we get,

\[
\kappa^\Lambda_{\omega}(a) \leq \sum_{\Lambda' < \Lambda} \sum_{v < w} c(a + k_0)^{|\text{Link}(\Lambda') \cap V\Lambda| - 2}.
\]

\( v^{-1} w \) is a \( \Lambda' \)-word

\[
\leq \sum_{\Lambda' < \Lambda} \sum_{l=0}^a c^2(a - l + k_0)^{|V\Lambda'| - 2}(a + k_0)^{|\text{Link}(\Lambda') \cap V\Lambda| - 2} - 2^{|V\Gamma|} 2(a + 1)(a + k_0)^{|V\Lambda'| - 2}(a + k_0)^{|\text{Link}(\Lambda') \cap V\Lambda| - 2} \leq c(a + k_0)^{|V\Lambda| - 2}.
\]

The last line follows from the choice of \( c \) and \( k_0 \).

5.3. Word length projections. The aim of this section is to prove that \( \mathcal{T}_w \mapsto \delta(|w| \leq n) \mathcal{T}_w \) gives a complete bounded multiplier of \( \mathcal{M}_q \) with complete bound growing at most polynomially in \( n \). Firstly we simplify notation a little bit.

Remark 5.6. Note that we may identify \( \ell^2(W) \) with basis \( \delta_x, x \in W \) with \( L^2(\mathcal{M}_q) \) with basis \( \mathcal{T}_x \Omega \). This way \( \ell^2(W) \) acts on \( L^2(\mathcal{M}_q) \) by means of the left regular representation.

We borrow the following construction from [Oza08]. We let \( B_f(W) \) be the set of finite subsets of \( W \). For \( A \in B_f(W) \) we define \( \tilde{\xi}_A^\pm \) to be the vectors in \( \ell^2(B_f(W)) \) given by

\[
\tilde{\xi}_A^\pm(\omega) = \begin{cases} 1 & \text{if } \omega \subseteq A, \\ 0 & \text{otherwise,} \end{cases} \quad \tilde{\xi}_A^-(\omega) = \begin{cases} (-1)^{|\omega|} & \text{if } \omega \subseteq A, \\ 0 & \text{otherwise,} \end{cases}
\]

Using the binomial formula (see Lemma 4 of [Oza08]), we have \( \|\tilde{\xi}_A^\pm\|^2 = 2^{|A|} \) and

\[
\langle \tilde{\xi}_A^\pm, \tilde{\xi}_B^\pm \rangle = \begin{cases} 0 & A \cap B \neq \emptyset, \\ 1 & \text{otherwise.} \end{cases}
\]

We let

\[
R = \text{span} \{ P_w \mid w \in W \}.
\]
Let $Q_w$ be the operator

$$Q_w \delta_x = \delta(w = x) \delta_x,$$

i.e. $Q_w$ is the Dirac delta function at $w$ seen as a multiplication operator.

**Lemma 5.7.** For $w \in W$ we have

$$Q_w = \sum_{v \in C(w,+)} (-1)^{|w^{-1} v|} P_v.$$

**Proof.** Firstly, $Q_w(w) = 1 = P_w(w) = (\sum_{v \in C(w,+)} (-1)^{|w^{-1} v|} P_v)(w)$. Let $x \in W$.

If $w \not\leq x$ we get $Q_w(x) = 0 = (\sum_{v \in C(w,+)} (-1)^{|w^{-1} v|} P_v)(x)$. In case $w < x$ we find

$$Q_w(x) = \sum_{v \in C(w,x)} (-1)^{|w^{-1} v|} P_v,$$

and this expression equals 0 by the binomial formula. Indeed, let $A_{w,x}$ be the maximal clique appearing at the start of $w^{-1} x$ (see Section 5.2). The number of words smaller than $A_{w,x}$ of length $l$ is $|A_{w,x}|$ choose $l$. So (5.7) equals,

$$\sum_{l=0}^{|A_{w,x}|} \sum_{v \in C(w,x), |w^{-1} v| = l} (-1)^{|w^{-1} v|} = \sum_{l=0}^{|A_{w,x}|} \binom{|A_{w,x}|}{l} (-1)^{|w^{-1} v|} = 0.$$

This concludes the lemma. \qed

Now let $A_q$ be the $*$-algebra generated by the operators $T_w, w \in W$. So $M_q$ is the $\sigma$-weak closure of $A_q$. We define

$$\Psi_{\leq n} : A_q \to M_q : T_w \mapsto \begin{cases} T_w & |w| \leq n, \\ 0 & \text{otherwise}. \end{cases}$$

We also set $\Psi_n = \Psi_{\leq n} - \Psi_{\leq (n-1)}$. The crucial part which we need to prove is that $\Psi_{\leq n}$ is completely bounded with a complete bound that can be upper estimated in $n$ polynomially. In order to do so we first introduce 3 auxiliary maps.

**Auxiliary map 1.** Recall that $M_1$ is just the group von Neumann algebra of the right-angled Coxeter group $W$. For $k \in \mathbb{N}$ define the multiplier $A_1 \to A_1$,

$$\rho_k(T^{(1)}_w) = \delta(|w| = k) T^{(1)}_w.$$

This map is completely bounded as the range is finite dimensional. We may extend $\rho_k$ to a $\sigma$-weakly continuous map $M_1 \to M_1$ (for convenience of the reader we provided details of this extension trick through double duality in Theorem 5.14). By the Bozejko-Fendler Theorem 2.8 we may extend $\rho_k$ uniquely to a $\sigma$-weakly continuous $\ell^\infty(W)$-bimodule map $\mathcal{B}(\ell^2(W)) \to \mathcal{B}(\ell^2(W))$ with the same completely bounded norm. Using Lemma 2.7 we see that

$$\Psi_{\leq n} = \sum_{k=0}^n \rho_k \circ \Psi_{\leq n}.$$

We emphasize at this point that in our proofs we shall not need a growth estimate for $\|\rho_k\|_{CB}$ in terms of $k$. It is known however by [Rec15] that $\|\rho_k\|_{CB}$ admits a polynomial bound in $k$. In the hyperbolic case this would already follow from [Oza08, Theorem 1 (2)].

Only in the hyperbolic case it is known by [Oza08, Theorem 1 (2)] that this map is completely bounded and moreover $\|\rho_k\|_{CB} \leq C(k + 1)$ for some constant $C$ independent of $k$. 
Auxiliary map 2. Let $\mathbb{T}$ be the unit circle in $\mathbb{C}$. For $z \in \mathbb{T}$ we define a unitary map,

$$A_z : \ell^2(W) \to \ell^2(W) : \delta_w \mapsto z^{|w|}\delta_w.$$ 

We set for $i \in \mathbb{Z}$,

$$\Phi_i : \mathcal{B}(\ell^2(W)) \to \mathcal{B}(\ell^2(W)) : x \mapsto \int_{\mathbb{T}} z^{-i}A_z^*xA_zdz,$$

where the measure is the normalized Lebesgue measure on $\mathbb{T}$. Intuitively $\Phi_i$ cuts out the operators that create $i$ more letters than it annihilates (where a negative creation is an annihilation). Using Lemma 2.7 we see that

$$\Psi_{\leq n} = \sum_{i=-n}^{n} \Phi_i \circ \Psi_{\leq n}.$$

Auxiliary map 3. Assume that $\Gamma$ is finite. For $a \in \mathbb{N}$ we define Stinespring dilations,

$$U^+_a : \ell^2(W) \to \ell^2(W) \otimes \ell^2(W) \otimes \ell^2(W) \otimes \ell^2(B_f(W)),$$

by mapping $\delta_x$ to (see Section 5.2 for notation),

$$\sum_{g \leq x} \sum_{\Lambda \leq g(2,0)} \beta^+_{g,x,\Lambda,a} \delta_g \otimes \delta_{g^{-1}x} \otimes \delta_{g(2,\Lambda)} \otimes \xi^\Lambda_\Lambda.$$

Here,

$$\beta^+_{g,x,\Lambda,a} = \sum_{v \in C(g,x)} (-1)^{|g^{-1}v|} F_{\Lambda,a}(v),$$

where $F_{\Lambda,a}(v) = 1$ if

$$2|v(1,\Lambda)| + |v(2,\Lambda)| \leq a,$$

and else $F_{\Lambda,a}(v) = 0$. We let $\beta^-_{g,x,\Lambda,a} = 1$ if $\beta^+_{g,x,\Lambda,a} \neq 0$ and $\beta^-_{g,x,\Lambda,a} = 0$ otherwise. Then set,

$$\sigma_{a,b}(x) = (U^-_b)^*(x \otimes 1 \otimes 1 \otimes 1)U^+_a.$$ 

The map $U^\pm_a$ is bounded with polynomial bound in $a$ by the following lemma.

Lemma 5.8. If $\Gamma$ is finite, the map $U^\pm_a$ is bounded. Moreover, there exists a polynomial $P$ such that $||U^\pm_a|| \leq P(a)$.

Proof. It follows from comparing the first two tensor legs in the definition of $U^\pm_a$ that the images of $\delta_x, x \in W$ are orthogonal vectors. Therefore it suffices to show that $\sup_{x \in W} ||U^\pm_a \delta_x||$ is bounded polynomially. Now let $C = \sum_{\Lambda \in \text{Cliq}(\Gamma)} 2^{|\Lambda|} |\beta^\pm_{g,x,\Lambda,a}|$. Then

$$||U^\pm_a \delta_x|| = \sum_{g \leq x} \sum_{\Lambda \leq g(2,0)} \beta^\pm_{g,x,\Lambda,a} \delta_g \otimes \delta_{g^{-1}x} \otimes \delta_{g(2,\Lambda)} \otimes \xi^\Lambda_\Lambda \leq \sum_{g \leq x} \sum_{\Lambda \leq g(2,0)} |\beta^\pm_{g,x,\Lambda,a}| 2^{|\Lambda|}.$$

In case

$$a \leq 2|g(1,\Lambda)| + |g(2,\Lambda)|,$$

we define Stinespring dilations,

$$U^+_a : \ell^2(W) \to \ell^2(W) \otimes \ell^2(W) \otimes \ell^2(W) \otimes \ell^2(B_f(W)),$$
then $\beta_{g,x,\Lambda,a}^\pm = 0$ by definition. \eqref{beta} will certainly hold when $a \leq |g|$. Let $M$ be the maximum length of a clique in $\text{Cliqu}(\Gamma)$. Then if
\begin{equation}
2|g(1,\Lambda)| + |g(2,\Lambda)| \leq a - 2M - 1,
\end{equation}
we find that $\beta_{g,x,\Lambda,a}^\pm = 0$ by the binomial formula as for every $v \in C(g,x)$ we have $F_{\Lambda,a}(v) = 1$. \eqref{beta} will certainly hold if $2|g| \leq a - 2M - 1$. So \eqref{5.3} can be estimated by $C$ times the number of $g \leq x$ with
\[
\frac{1}{2}(a - 2M - 1) \leq |g| \leq a.
\]
But the number such $g$’s grows polynomially in $a$, c.f. Lemma 5.5.

\begin{lemma}
Let $x \in W$. Let $u',u'' \in W$ be such that $|u''x| = |x| - |u''|$, $|u'u''x| = |x| - |u'| + |u'|$. Let $v \in W$ be such that $(u'')^{-1} \leq v \leq x$. Then,
\begin{equation}
\sum_{v \leq g \leq x} \beta_{g,x,g(2,\emptyset) \setminus (u'u''g)(2,\emptyset),a}^+ - \beta_{g,x,g(2,\emptyset) \setminus (u'u''g)(2,\emptyset),a}^-
\end{equation}
\[
\begin{cases}
1 & \text{in case } 2|v(1,v(2,\emptyset) \setminus (u'u''v)(2,\emptyset))| + |v(2,v(2,\emptyset) \setminus (u'u''v)(2,\emptyset))| \leq a, \\
0 & \text{otherwise}.
\end{cases}
\]
\end{lemma}

\begin{proof}
By Equations \eqref{5.1} and \eqref{5.2} for $v \leq g$ we get,
\[
\beta_{g,x,g(2,\emptyset) \setminus (u'u''g)(2,\emptyset),a}^+
\begin{aligned}
&= \sum_{w \in C(g,x)} (-1)^{|g^{-1}w|} F_{g(2,\emptyset) \setminus (u'u''g)(2,\emptyset),a}(w) \\
&= \sum_{w \in C(u'u''g,u'u''x)} (-1)^{|g^{-1}w|} F_{(u'u''g)(2,\emptyset) \setminus g(2,\emptyset),a-2|u'|+2|u'|}(w) \\
&= \beta_{g,x,g(2,\emptyset) \setminus (u'u''g)(2,\emptyset),a}^-
\end{aligned}
\]
Therefore also,
\[
\beta_{g,x,g(2,\emptyset) \setminus (u'u''g)(2,\emptyset),a}^-
\begin{aligned}
&= \beta_{u'u''g,u'u''x,(u'u''g)(2,\emptyset) \setminus g(2,\emptyset),a-2|u'|+2|u'|}^-
\end{aligned}
\]
We therefore have that the left hand side of \eqref{5.4} equals,
\[
\sum_{v \leq g \leq x} \beta_{g,x,g(2,\emptyset) \setminus (u'u''g)(2,\emptyset),a}^+ - \beta_{g,x,g(2,\emptyset) \setminus (u'u''g)(2,\emptyset),a}^-
\]
To compute this sum, recall that $\mathcal{R}$ was defined in \eqref{5.0}, and define the mapping
\[
\kappa_a : \mathcal{R} \to \mathcal{R} : P_w \mapsto F_{w(2,\emptyset) \setminus (u'u''w)(2,\emptyset),a}(w)P_w.
\]
Then, using Lemma 5.7, the definition of $\kappa_a$, Lemma 5.4 and the definition \eqref{5.9},
\[
\kappa_a(Q_g)(x) = \kappa_a \left( \sum_{w \in C(g,x)} (-1)^{|g^{-1}w|} P_w(x) \right) = \sum_{w \in C(g,x)} (-1)^{|g^{-1}w|} F_{w(2,\emptyset) \setminus (u'u''w)(2,\emptyset),a}(w) = \beta_{g,x,g(2,\emptyset) \setminus (u'u''g)(2,\emptyset),a}^+.
\]
Lemma 5.10. Assume that $\Gamma$ is finite so that (5.10) is defined boundedly. We have for $n \in \mathbb{N}$ that $\Psi \leq n = \sum_{i=-n}^{n} \sigma_{n-i,n+i} \circ \Phi_{i} \circ \Psi \leq n$.

Proof. Let $T_{w} \in M_{q}$ with $|w| \leq n$. We need to show that,

$$T_{w} = \sum_{k=0}^{n} \sum_{i=-n}^{n} \sigma_{n-i,n+i} \circ \Phi_{i} \circ \rho_{k}(T_{w}).$$

We split by Lemma 2.7

$$T_{w} = \sum_{(w',r_{0},w'')} T_{w'}^{(1)} P_{VT_{0}} T_{w''}^{(1)},$$

and show that $\sum_{k=0}^{n} \sum_{i=-n}^{n} \sigma_{n-i,n+i} \circ \Phi_{i} \circ \rho_{k}$ applied to each of these summands acts as the identity. Let us consider a summand $T_{w'}^{(1)} P_{VT_{0}} T_{w''}^{(1)}$ with $(w',r_{0},w'') \in A_{w}$.

Let $u, u', u''$ be as in Lemma 5.10 so that $T_{w'}^{(1)} P_{VT_{0}} T_{w''}^{(1)} = T_{u}^{(1)} P_{u VT_{0}} T_{u}^{(1)}$. We have

$$\rho_{k}(T_{u}^{(1)} P_{u VT_{0}} T_{u}^{(1)}) = \begin{cases} T_{u}^{(1)} P_{u VT_{0}} T_{u}^{(1)} & \text{if } k = |u'| + |u''|, \\ 0 & \text{otherwise.} \end{cases}$$

So the only non-zero summand is $k = |u'| + |u''|$ so that it remains to show that for $x, y \in W$,

$$(5.15) \langle \sum_{i=-n}^{n} \sigma_{n-i,n+i} \circ \Phi_{i}(T_{u}^{(1)} P_{u VT_{0}} T_{u}^{(1)}) \delta_{x}, \delta_{y} \rangle = \langle T_{u}^{(1)} P_{u VT_{0}} T_{u}^{(1)}, \delta_{x}, \delta_{y} \rangle.$$

If the right hand side is non-zero then we must have $y = u' u'' x$. Furthermore, recall that there is a choice for $u', u''$ and we may choose them (depending on $x$) such that $|u'' x| = |x| - |u''|$ and $|u' u'' x| = |x| - |u'| + |u''|$. After making this choice the right hand side is non-zero in case $(u'')^{-1} u V T_{0} \leq x$, in which case the expression equals 1.

Now consider the left hand side of (5.15),

$$\langle (\Phi(T_{u}^{(1)} P_{u VT_{0}} T_{u}^{(1)}) \otimes 1 \otimes 1) U^{+}_{n-i} \delta_{x}, U^{+}_{n-i} \delta_{y} \rangle$$

$$= \left( \sum_{g \leq x, A \leq g(2,0)} \sum_{h \leq y, A' \leq h(2,0)} \sum_{i=-n}^{n} \beta_{g, x, A, n+i}^{+} \Phi_{i}(T_{u}^{(1)} P_{u VT_{0}} T_{u}^{(1)}) \delta_{g} \otimes \delta_{g^{-1} x} \otimes \delta_{g(2, A)} \otimes \xi_{A}^{+},

(5.16) \right.$$
to be non-zero. Taking into account $\Phi$, we see that (5.10) is non-zero only if $i = |u''| - |u'|$.

Next we note that by comparing the last two tensor legs, if a summand in (5.10) is non-zero then we have $g(2, \Lambda) = h(2, \Lambda')$ and $\Lambda \cap \Lambda' = \emptyset$. Recall that $h = u'u''g$. But then $\Lambda$ must equal the letters in $g(2, \emptyset)$ that are not any more in $(u'u''g)(2, \emptyset)$ and $\Lambda'$ must equal the letters in $(u'u''g)(2, \emptyset)$ that are not anymore in $g(2, \emptyset)$. This precisely means that $\Lambda = g(2, \emptyset)(u'u''g)(2, \emptyset)$ and $\Lambda' = (u'u''g)\setminus g(2, \emptyset)$.

In all, we find that

$$\sum_{g\leq x} \sum_{\Lambda\leq g(2, \emptyset)} \beta_{g, x, \Lambda, n-1} T_{u'} T_{u''} T_{u''} = \sum_{h\leq y} \sum_{\Lambda'\leq h(2, \emptyset)} \beta_{h, y, \Lambda', n+i} \delta_h \otimes \delta_{h(2, \Lambda')} \otimes \tilde{\xi}_h,$$

We claim that this expression is 1 by verifying Lemma 5.9. Indeed set $w := (u'')^{-1}uV_0$. First suppose that $u$ is the empty word. Then

$w(2, w(2, \emptyset)(u'u''w)(2, \emptyset)) = V_0$.

and so

$w(1, w(2, \emptyset)(u'u''w)(2, \emptyset)) = (u'')^{-1}$.

If $u$ is not the empty word, then let $s \in W$ be a final letter of $u$ (i.e. $|us| = |u| - 1$). Then $s$ cannot commute with $V_0$ as this would violate the equation $T_{u'} T_{u''} T_{u''} = T_{w'} P_{V_0} T_{w''}$. Therefore again,

$w(2, w(2, \emptyset)(u'u''w)(2, \emptyset)) = w(2, \emptyset) = V_0$.

and so

$w(1, w(2, \emptyset)(u'u''w)(2, \emptyset)) = (u'')^{-1}u$.

Further our constructions give that $|u'| = \frac{k+i}{2}$ and $2|u| + |V_0| = |w| - |u'| - |u''| = |w| - k$. So we have,

$2|w(1, w(2, \emptyset)(u'u''w)(2, \emptyset))| + |w(2, w(2, \emptyset)(u'u''w)(2, \emptyset))|$

$= 2(|u'')^{-1}| + 2|u| + |V_0| = 2\frac{k+i}{2} + (|w| - k)$

$= |w| - i \leq n - i$,

so that by Lemma 5.9 we see that (5.10) is 1. So we conclude that (5.15) holds.

\[\Box\]

**Lemma 5.11.** Assume that $\Gamma$ is finite so that (5.10) is defined boundedly. We have for $n \in \mathbb{N}, -n \leq i \leq n$:

$$\sigma_{n-i, n+i} \circ \Phi_i \circ \Psi_{\leq n} = \sigma_{n-i, n+i} \circ \Phi_i.$$

**Proof:** The proof pretty much parallels the proof of Lemma 5.10. We need to show that the right hand side applied to $T_w$ with $|w| > n$ equals 0. Therefore we may look at the summands $T_{w'} T_{V_0} T_{w''}$ with $(w', \Gamma_0, w'') \in A_w$ which can be further
decomposed as $T_u^{(1)}P_uVT_0T_u^{(1)}$ with $u, u', u''$ as in Lemma 5.10. It suffices then to show that for all choices of $k$ the following expression is 0:

$$\langle \sigma_{n-i,n+i} \circ \Phi_i \circ \rho_k(T_u^{(1)}P_uVT_0T_u^{(1)})\delta_x, \delta_y \rangle.$$  

(5.18)

Firstly, this expression is 0 in case $|u'| + |u''| \neq k$. So assume $|u'| + |u''| = k$. Then,

$$\langle \sigma_{n-i,n+i} \circ \Phi_i(T_u^{(1)}P_uVT_0T_u^{(1)})\delta_x, \delta_y \rangle.$$  

(5.18)

As in the proof of Lemma 5.10 the expression (5.18) equals 0 unless $u'u'x = y$ or $(u')^{-1}uVT_0 \leq x$ with $u''$, $u'$ chosen in such a way that $|u'x| = |x| - |u''|$ and $|u'u'x| = |x| - |u''| + |u'|$. In that case $i = |u'| - |u''|$. As in (5.16),

$$\langle (T_u^{(1)}P_uVT_0T_u^{(1)} \otimes 1 \otimes 1)U_{n-i}^+\delta_x, U_{n+i}^-\delta_y \rangle$$

(5.18) = \sum_{(u')^{-1}uVT_0 \leq x} \beta_{g,x,g(2,0)}(u'u'g(2,0),n-i|u'u'\delta_x, (u'u'g(2,0))g(2,0),n+i).$$

As for $w := (u'')^{-1}uVT_0$ we have again by the same reasoning as in/before (5.17) that,

$$2|w(1, w(2, 0)\setminus (u'u'w(2, 0))| + |w(2, w(2, 0)\setminus (u'u'w(2, 0))| = |w| - i > n - i.$$

The expression (5.19) is zero by Lemma 5.9.

**Proposition 5.12.** We have $\|\Psi_{\leq n}\|_{CB} \leq P(n)$ for some polynomial $P$.

**Proof.** By Lemmas 5.10 and 5.11 we have,

$$\Psi_{\leq n} = \sum_{i=-n}^{n} \sigma_{n-i,n+i} \circ \Phi_i \circ \Psi_{\leq n}$$

$$= \sum_{i=-n}^{n} \sigma_{n-i,n+i} \circ \Phi_i,$$

and the right hand side is completely bounded with polynomial bound in $n$; indeed the bound of $\sigma_{n-i,n+i}$ is polynomial in $n$ by its very definition and Lemma 5.8.

**Definition 5.13.** A von Neumann algebra $\mathcal{M}$ has the weak-* completely bounded approximation property (wk-* CBAP) if there exists a net of normal finite rank maps $\Phi_i : \mathcal{M} \to \mathcal{M}$ such that $\Phi_i(x) \to x$ in the $\sigma$-weak topology and moreover $\sup_i \|\Phi_i\|_{CB} < \infty$. If the maps $\Phi_i$ can be chosen such that $\limsup_i \|\Phi_i\|_{CB} \leq 1$ then $\mathcal{M}$ is said to have the weak-* completely contractive approximation property (wk-* CCAP).

**Theorem 5.14.** Let $(W, S)$ be a right angled Coxeter system and let $q > 0$. The Hecke von Neumann algebra $\mathcal{M}_q$ has the wk-* CCAP.

**Proof.** By an inductive limit argument and Corollary 3.3 we may assume that $\Gamma$ is finite. The proof goes back to Haagerup [Han78]. Consider the completely bounded map $\Psi_{\leq n} \circ \Phi_r : \mathcal{A}_q \to \mathcal{M}_q$. Clearly as $n \to \infty$ and $r \not\to 1$ this map converges to the identity in the point $\sigma$-weak topology. Let $\epsilon > 0$. We have,

$$\|\Psi_{\leq n} \circ \Phi_r\|_{CB} \leq \|(\Psi_{\leq n} - \text{Id}) \circ \Phi_r\|_{CB} + \|\Phi_r\|_{CB} \leq \sum_{i=n+1}^{\infty} r^n \|\Psi_{\leq n}\|_{CB} + \|\Phi_r\|_{CB},$$
which shows using Proposition 5.12 and Proposition 5.1 that we may let \( r \not\to 1 \) and
then choose \( n := n_r \) converging to \( \infty \) such that \( \| \Psi_{\leq n_r} \circ \Phi_r \|_{CB} \leq 1 + \epsilon \) for some constant.

The map \( \Phi_r \) is normal. Also \( \Psi_{\leq n} \) is normal by a standard argument: indeed
using duality and Kaplansky’s density theorem one sees that \( \Psi_n \) maps \( L^1(M_q) \to \)
\( L^1(M_q) \) boundedly. Then taking the dual of this map yields that \( \Psi_n : M_q \to M_q \)
is a normal map. We may extend \( \Psi_{\leq n} \circ \Phi_r \) to a normal map \( M_q \to M_q \).
Then using a standard approximation argument yields the result.

\[ \Box \]

Remark 5.15. Recall that a von Neumann algebra \( M \) with normal faithful tracial
state \( \tau \) has the Haagerup property if there exists a net \( \Phi_i \) of \( \tau \)-preserving ucp maps
\( M \to M \) such that \( T_i : x\Omega_\tau \mapsto \Phi_i(x)\Omega_\tau \) is compact and converges to \( 1 \) strongly.
Proposition 5.1 directly shows that \( M_q \) has the Haagerup property in case \( \Gamma \) is
finite (to obtain compactness) and then the general case follows by an inductive
limit argument using the conditional expectations from Corollary 3.3.

Remark 5.16. In case our right-angled Coxeter group is free (i.e. \( m(s, t) = \infty \)
for all \( s \neq t \)) it is possible to adapt the arguments of \[ \text{RiXu06} \] in order to obtain word length cut downs with polynomial bound. This argument – purely based on book keeping of creations/annihilations – seems unrepairable in the general case. In case \( q = 1 \) for a general right-angled Coxeter group word length cut-downs were obtained in \[ \text{Rec15} \] by using actions on CAT(0)-spaces. The connection with the general Hecke case is unclear.

6. Strong solidity in the hyperbolic case

We prove that in the factorial case (see Theorem 2.2) \( M_q \) is a strongly solid von Neumann algebra in case the Coxeter group is hyperbolic.

6.1. Preliminaries on strongly solid algebras. The normalizer of a von Neumann subalgebra \( P \) of \( M \) is defined as \( \{ u \in U(M) \mid uPu^* = P \} \). We define \( \text{Nor}_P(M) \) as the von Neumann algebra generated by the normalizer of \( P \) in \( M \). A von Neumann algebra is called diffuse if it does not contain minimal projections.

Definition 6.1. A finite von Neumann algebra \( M \) is strongly solid if for any diffuse
injective von Neumann subalgebra \( P \subseteq M \) the von Neumann algebra \( \text{Nor}_M(P) \) is
again injective.

In \[ \text{OzPo10} \] Ozawa and Popa proved that free group factors are strongly solid and consequently they could prove that these are \( \text{II}_1 \) factors that have no Cartan subalgebras (as was proved by Voiculescu \[ \text{Voi96} \] earlier on by a completely different method). A general source for strongly solid von Neumann algebras are group von Neumann algebras of groups that have the weak+-completely bounded approximation property and are bi-exact (see \[ \text{ChSi13}, \text{CSU13}, \text{PoVa14} \]; we also refer to these sources for the definition of bi-exactness). The following definition and subsequent theorem were then introduced and proved in \[ \text{Iso15} \]. For standard forms of von Neumann algebras we refer to \[ \text{Tak03} \].

Definition 6.2. Let \( M \subseteq B(H) \) be a von Neumann algebra represented on the
standard Hilbert space \( H \) with modular conjugation \( J \). We say that \( M \) satisfies
condition \((AO)^+\) if there exists a unital \( \text{C}^* \)-subalgebra \( A \subseteq M \) that is \( \sigma \)-weakly
dense in \( M \) and which satisfies the following two conditions:
(1) $A$ is locally reflexive.
(2) There exists a ucp map $\theta : A \otimes_{\min} JAJ \to B(\mathcal{H})$ such that $\theta(a \otimes b) - ab$ is a compact operator on $\mathcal{H}$.

**Theorem 6.3** ([Iso15]). Let $\mathcal{M}$ be a II$_1$-factor with separable predual. Suppose that $\mathcal{M}$ satisfies condition $(AO)^+$ and has the weak-$*$ completely bounded approximation property. Then $\mathcal{M}$ is strongly solid.

A maximal abelian von Neumann subalgebra $P \subseteq \mathcal{M}$ of a II$_1$ factor $\mathcal{M}$ is called a Cartan subalgebra if $\text{Nor}_{\mathcal{M}}(P) = \mathcal{M}$. It is then obvious that if $\mathcal{M}$ is a non-injective strongly solid II$_1$-factor, then $\mathcal{M}$ cannot contain a Cartan subalgebra. Therefore we will now prove that the Hecke von Neumann algebra $\mathcal{M}_q$ in the factorial, hyperbolic case satisfies condition $(AO)^+$.

6.2. **Crossed products.** Let $A$ be a C*-algebra that is represented on a Hilbert space $\mathcal{H}$. Let $\alpha : G \curvearrowright A$ be a continuous action of a locally compact group $G$ on $A$. The reduced crossed product $A \rtimes_r G$ is the C*-algebra of operators acting on $\mathcal{H} \otimes \ell^2(G)$ generated by operators

$$u_g := \sum_{h \in G} 1 \otimes e_{gh,h}, \quad g \in G, \quad \text{and} \quad \pi(x) := \sum_{h \in G} h^{-1} \cdot x \otimes e_{h,h}, \quad x \in A.$$  

Here the convergence of the sums should be understood in the strong topology. There is also a universal crossed product $A \rtimes_u G$ for which we refer to [BrOz08] (in the case we need it, it turns out to equal the reduced crossed product).

6.3. **Gromov boundary and condition $(AO)^+$.** Let again $(W, S)$ be a Coxeter system which we assume to be hyperbolic (see [BrOz08] Section 5.3). Let $\Lambda$ be the associated Cayley tree. A geodesic ray starting at a point $w \in \Lambda$ is a sequence $(w, wv_1, wv_1v_2, \ldots)$ such that $|wv_1 \ldots v_n| = |w| + n$. We typically write $\omega = (\omega(0), \omega(1), \ldots)$ for a geodesic ray. Let $\delta W$ be the Gromov boundary of $W$ which is the collection of all geodesic rays starting at the identity of $W$ modulo the equivalence $\omega_1 \simeq \omega_2$ iff $\lim_{x,y \to \infty} \text{dist}(\omega_1(x), \omega_2(y)) = 0$. $W \cup \delta W$ may be topologized as in [BrOz08] Section 5.3.

Let $W \curvearrowright W$ be the action by means of left translation. The action extends continuously to $W \cup \delta W$ and then restricts to an action $W \curvearrowright \delta W$. We may pull back this action to obtain $W \curvearrowright C(\delta W)$. As in this section we assumed that $W$ is a hyperbolic group the action $W \curvearrowright \delta W$ is well-known to be amenable [BrOz08] which implies that $C(\delta W) \rtimes_u W = C(\delta W) \rtimes_r W$ and furthermore this crossed product is a nuclear C*-algebra. Let $f \in C(\delta W)$, let $\tilde{f}_1, \tilde{f}_2 \in C(W \cup \delta W)$ be two continuous extensions of $f$ and let $f_1$ and $f_2$ be their respective restrictions to $W$. Then $f_1 - f_2 \in C_0(W)$. That is, the multiplication map $f_1 - f_2$ acting on $\ell^2(W)$ determines a compact operator. So the assignment $f \mapsto f_1$ is a well-defined *-homomorphism $C(\delta W) \to B(\ell^2(W))/K$ where $K$ are the compact operators on $\ell^2(W)$. It is easy to check that this map is $W$-equivariant and thus we obtain a *-homomorphism:

$$\pi_1 : C(\delta W) \rtimes_u W \to B(\ell^2(W))/K.$$  

Let again $W \curvearrowright W$ be the action by means of left translation which may be pulled back to obtain an action $W \curvearrowright \ell^\infty(W)$. Let

$$\rho : \ell^\infty(W) \rtimes_r W \to B(\ell^2(W))$$  

be the $\sigma$-weakly continuous $*$-isomorphism determined by $\rho : u_w \mapsto T_w^{(1)}$ and $\rho : \pi(x) \mapsto x$ (see [Vac01, Theorem 5.3]). In fact $\rho$ is an injective map (this follows immediately from [Com11, Theorem 2.1] as the operator $G$ in this theorem equals the multiplicative unitary/structure operator [Tak03, p. 68]). Let $C_\infty(W)$ be the $C^*$-algebra generated by the projections $P_w, w \in W$. Take $f \in C_\infty(W)$ and let $\tilde{f}$ be the continuous extension of $f$ to $W \cup \delta W$. The map $f \mapsto \tilde{f}|_{\delta W}$ determines a $*$-homomorphism $\sigma : C_\infty(W) \to C(\delta W)$ that is $W$-equivariant. Therefore it extends to the crossed product map

$$\sigma \times_r \text{Id} : C_\infty(W) \times_r W \to C(\delta W) \times_r W.$$ 

**Theorem 6.4.** Let $(W, S)$ be a right-angled hyperbolic Coxeter group and let $q \in [\rho, \rho^{-1}]$, see Theorem [2.2]. The von Neumann algebra $M_q$ satisfies condition (AO)$^\ast$.

**Proof.** We let $A_q$ be the unital $C^*$-subalgebra of $M_q$ generated by operators $T_w, w \in W$. It is easy to see that $A_q$ is preserved by the multipliers that we constructed in order to prove that $M_q$ had the wk-$*$ CBAP, see Section [5] (indeed these were compositions of radial multipliers – see Proposition [6.1] and word length projections – see Proposition [5.12]). Therefore $A_q$ has the CBAP, hence by the remarks before [HaKr94, Theorem 2.2] it is exact. Therefore $A_q$ is locally reflexive [BrOz08, Chapter 18].

It remains to prove condition (2) of Definition [6.2]. By Lemma [2.7] we see that $A_q$ is contained in the $C^*$-subalgebra of $B(\ell^2(W))$ generated by the operators $P_w, T_w^{(1)}$ with $w \in W$. So $\rho^{-1}(A_q)$ is contained in $C_\infty(W) \times_r W$ and therefore we may set

$$\gamma : A_q \to C(\delta W) \times_r W \quad \text{as} \quad \gamma = (\sigma \times_r \text{Id}) \circ \rho^{-1}.$$ 

The mapping $\pi_2 : J A_q J \to B(\ell^2(W))/K : b \mapsto b$ is a $*$-homomorphism and its image commutes with the image of $\pi_1$ of (6.2) (as was argued in [HiGm04, Lemma 6.2.8]). By definition of the maximal tensor product there exists a $*$-homomorphism:

$$(\pi_1 \otimes \pi_2) : (C(\delta W) \times_r W) \otimes_{\text{max}} J A_q J \to B(\ell^2(W))/K : a \otimes J b J \mapsto \pi_1(a) J b J.$$ 

We may now consider the following composition of $*$-homomorphisms:

$$(6.3) \quad A_q \otimes_{\text{min}} J A_q J \xrightarrow{\pi_1 \otimes \pi_2} (C(\delta W) \times_r W) \otimes_{\text{min}} J A_q J \xrightarrow{\gamma \otimes \text{Id}} B(\ell^2(W))/K \xrightarrow{\pi_1 \otimes \pi_2} (C(\delta W) \times_u W) \otimes_{\text{max}} J A_q J.$$ 

By construction this map is given by:

$$(6.4) \quad a \otimes J b J \mapsto a J b J + K, \quad \text{where} \quad a, b \in A_q.$$ 

The map $\pi_1$ is nuclear because we already observed that $C(\delta W) \times_u W$ is nuclear. Also $\pi_2$ is nuclear as it equals $J(\cdot) J \circ \pi_1 \circ \gamma \circ J(\cdot) J$. Therefore $\pi_1 \otimes \pi_2 : (C(\delta W) \times_r W) \otimes_{\text{min}} J A_q J \to B(\ell^2(W))/K$ in diagram (6.3) is nuclear and we may apply the Choi-Effros lifting theorem [ChEf76] in order to obtain a ucp lift $\theta : (C(\delta W) \times_r W) \otimes_{\text{min}} J A_q J \to B(\ell^2(W))$. Then $\theta \circ (\gamma \otimes \text{Id})$ together with (6.4) witness the result.

**Corollary 6.5.** Let $(W, S)$ be a reduced hyperbolic Coxeter system with $|S| \geq 3$ and $q \in [\rho, \rho^{-1}]$. Then the Hecke von Neumann algebra $M_q$ has no Cartan subalgebra.
Proof. This is a consequence of Theorem [6.3] together with Theorems [1.11] [5.13] and [6.4]

Remark 6.6. In case \( W \) is not hyperbolic, it is not necessarily true that the group von Neumann algebra \( M_1 \) is strongly solid. The easiest case is when \( \Gamma \) is \( K_{2,3} \); the complete bipartite graph with 2+3 vertices. Then the graph product \( W = *_{K_{2,3}}Z_2 = (Z_2 * Z_2) * (Z_2 * Z_2 * Z_2) \) contains a copy of \( Z \times F_2 \). Then \( M_1 \) cannot be strongly solid as it contains the group von Neumann algebra of \( Z \times F_2 \). Note that \( K_{2,3} \) is not an irreducible graph but the same argument applies if one adds one point with no edges to \( K_{2,3} \).

7. Absence of Cartan subalgebras

As we saw in Remark 6.6 the absence of Cartan subalgebras for general right-angled Hecke von Neumann algebras cannot be proved through strong solidity. In this section we obtain absence of Cartan subalgebras through an ‘unscrewing argument’ combined with a theorem by Vaes [Vae14, Theorem A] (see also [Ioa15] for related results). We need some terminology first.

Definition 7.1. Let \( N, P \subseteq M \) be finite von Neumann algebras. We say that \( N \) is injective (or amenable) relative to \( P \) if there is a completely positive map \( \Phi \) from the basic construction \( \langle M, e_P \rangle \) onto \( N \) such that \( \Phi|_M \) is the conditional expectation of \( M \) onto \( N \). Here \( e_P \) is the Jones projection, i.e. the conditional expectation of \( M \) to \( P \) on the \( L^2 \)-level.

The following Theorem 7.2 uses Popa’s intertwining by bi-modules technique. For us it suffices that for finite (separable) von Neumann algebras \( N, P \subseteq M \) we say that \( N \prec_M P \) if there exists no sequence of unitaries \( w_k \) in \( N \) such that for all \( x, y \in M \) we have \( \|S_P(xw_ky)\|_2 \to 0 \). The following theorem is a somewhat less general version of [Vae14, Theorem A].

Theorem 7.2. Let \( N_i, i = 1, 2 \) be von Neumann algebras with a common von Neumann subalgebra \( B \). Let \( N = N_1 \star_B N_2 \) be the amalgamated free product. Let \( A \subseteq N \) be a von Neumann subalgebra that is injective relative to one of the \( N_i, i = 1, 2 \). Then at least one of the following statements holds true: (1) \( A \prec_N B \), (2) There exists \( i \) such that \( \text{Nor}_N(A) \prec_N N_i \), (3) \( \text{Nor}_N(A) \) is injective relative to \( B \).

Recall that for a graph \( \Gamma \) and \( v \in V \) we have \( \text{Link}(v) = \{ w \in V \Gamma \mid (v, w) \in E \Gamma \} \) and \( \text{Star}(v) = \text{Link}(v) \cup \{ v \} \).

Lemma 7.3. Every reduced graph \( \Gamma \) with \( |V \Gamma| \geq 3 \) contains a vertex \( v \in V \) such that \( V \Gamma - \text{Star}(v) \) contains at least two points.

Proof. Pick some random point \( v \in V \). We cannot have \( \text{Star}(v) = V \Gamma \) because then \( \Gamma \) would not be reduced. So there is at least one point \( w \in V \Gamma - \text{Star}(v) \). If there is another point in \( V \Gamma - \text{Star}(v) \) then we are done, so we assume that \( w \) is the only point in \( V \Gamma - \text{Star}(v) \). This implies that \( \text{Link}(v) \) is non-empty. \( \text{Star}(w) \) does not contain \( v \) as \( w \notin \text{Star}(v) \). Also there must be at least one point \( u \in \text{Link}(v) \) (which was non-empty!) that is not connected to \( w \) because if this is not the case then every two elements in \( \text{Link}(v) \) and \( \{ v, w \} \) would be connected so that \( \Gamma \) is not reduced. In all we proved that \( w \) has the property that \( V \Gamma - \text{Star}(w) \) contains at least two elements, namely \( v \) and \( u \).
**Theorem 7.4.** Let \((W, S)\) be a reduced right-angled Coxeter group with \(|S| \geq 3\). Let \(q > 0\). The Hecke-von Neumann algebra \(M_q\) does not have a Cartan subalgebra.

**Proof.** Let \(\Gamma = (VT, ET)\) be the graph of \((W, S)\). By Corollary 3.3 we get a graph product decomposition \(M_q = \ast_{s \in VT} M_q(s)\) with \(M_q(s)\) the Hecke-von Neumann algebra associated with the Coxeter subsystem generated by just \(s\) (so it is 2-dimensional by Section 3). Choose \(r \in VT\) such that \(VT - \text{Star}(r)\) contains at least 2 elements, c.f. Lemma 7.3. Put \(N_1 = \ast_{s \in \text{Star}(r)} M_q(s)\), \(N_2 = \ast_{s \in VT - \text{Star}(r)} M_q(s)\) and \(B = \ast_{s \in \text{Link}(r)} M_q(s)\). Simply write \(M\) for \(M_q\). By the unscrewing theorem [CaFi15, Theorem 2.26] we get:

\[
M = N_1 \ast_B N_2.
\]

Now suppose that \(A \subseteq M\) is a Cartan subalgebra. We are going to derive a contradiction by showing that any of the three alternatives of Theorem 7.2 is absurd.

**Claim 1:** We cannot have \(\text{Nor}_M(A) \prec_M N_i\) for either \(i = 1, 2\).

**Proof of the claim.** As \(A\) is assumed to be Cartan we need to prove that \(M \not\prec_M N_i\). Let \(t \in VT - \text{Star}(r)\). Then the subalgebra of \(M\) generated by \(M_q(r)\) and \(M_q(t)\) is \(M_q(r) \ast M_q(t)\). Take unitaries \(u \in M_q(r)\) and \(v \in M_q(t)\) with trace 0. Put \(w_k = (uv)^k\) which then is a unitary in \(M_q(r) \ast M_q(t)\) with trace 0.

We need to show that for all \(x, y \in M\) we have \(\|E_N(xw_k y)\|_2 \to 0\). Recall that \(M_q(s)^\circ\) is the space of elements \(z \in M_q(s)\) with \(E_B(z) = 0\). By a density argument we may and will assume that \(x = x_1 \ldots x_k\) and \(y = y_1 \ldots y_l\) are reduced operators with \(x_i, y_i \in M_q(s)^\circ\) for some \(s\) (see [CaFi15, Definition 2.10] for the definition of a reduced operator). If \(x_{m+1}, \ldots, x_k \in M_q(r)^\circ \cup M_q(t)^\circ\) then decompose \(x' = x_1 \ldots x_m\) and \(a = x_{m+1} \ldots x_k\). We may assume that this decomposition is taken in such a way that the length of \(a\) is maximal, in other words: the end of the expression \(x'\) has (after possible commutations) no factors \(x_i\) that come from \(M_q(r)^\circ\) and \(M_q(t)^\circ\). We take a similar decomposition for \(y\). We may write \(y = y' b\) with \(y' = y_{n+1} \ldots y_l\) and \(b = y_1 \ldots y_n\) with \(y_i, 1 \leq i \leq n\) elements of either \(M_q(r)^\circ\) and \(M_q(t)^\circ\). Again we may assume that this decomposition is maximal meaning that (after possible commutations) the expression \(y'\) does not have factors at the start that come from either \(M_q(r)^\circ\) or \(M_q(t)^\circ\).

Now write \(xw_k y = x'(aw_k b)y'\). For \(k\) big (in fact \(k \geq m + n + 1\) suffices) we get that \(aw_k b\) is not contained in \(N_i\) for neither \(i = 1, 2\). Indeed \(a\) and \(b\) can never cancel all the occurrences of \(u\) and \(v\) in \(w_k = (uv)^k\) so that \(aw_k b \in M_q(r) \ast M_q(t) \ominus (M_q(r) \cup M_q(t))\). So \(xw_k y = x'(aw_k b)y' \not\in N_i\) for either \(i = 1, 2\). Therefore \(\|E_N(xw_k y)\|_2 \to 0\) as \(k \to \infty\).

**Claim 2:** We do not have \(A \prec_M B\).

**Proof of the claim.** If \(A \prec_M B\) then we certainly have \(A \prec_M N_i\) for both \(i = 1, 2\). But then by [Ion15, Lemma 9.4] and the fact that the inclusion \(N_i \subseteq M\) is mixing (see the proof of Corollary C, part 1, in [Vae14, Section 7]; also we refer to [Ion15] and [Vae14] for the definition of mixing) we get that also \(\text{Nor}_M(A) \prec_M N_i, i = 1, 2\). However this is impossible by Claim 1.

**Claim 3:** \(M\) is not relative injective with respect to \(B\).
Proof of the claim. Recall our choice of \( r \in V T \) at the start of the proof. Let \( t_1, t_2 \) be two different points in \( V T - \text{Star}(r) \). Let \( \Lambda \) be the full subgraph of \( G \) with vertex set \( \{r, t_1, t_2\} \). Let \( \mathcal{N} = s_{t \in \Lambda} M_q(s) \). Note that \( \mathcal{N} \cap \mathcal{B} = \mathbb{C} \). Suppose that \( \mathcal{M} \) to be relative injective with respect to \( \mathcal{B} \). Then there exists a conditional expectation \( \Phi : (\mathcal{M}, e_B) \to \mathcal{M} \). We shall prove that this implies that \( \mathcal{N} \) is injective. Consider \( L^2(\mathcal{M}) \). It contains the closed subspace \( \mathcal{K} := \text{span} \{nb\Omega, n \in \mathcal{N}, b \in B\} \). Let \( \{n_i\}_{i \in I} \) be elements in \( \mathcal{N} \) such that \( \{n_i\Omega\}_{i \in I} \) forms an orthonormal basis in \( L^2(\mathcal{N}) \). Then \( p = \sum_{i \in I} n_ie_Bn_i^* \) is the projection of \( L^2(\mathcal{M}) \) onto \( \mathcal{K} \). As \( \mathcal{K} \) is an invariant subspace for \( \mathcal{N} \) we have \( p \in \mathcal{N} \cap (\mathcal{M}, e_B) \).

Now consider \( \Psi : p(\mathcal{M}, e_B)p \to \Phi(p)^{-1}\Phi(p) \to \mathcal{K} \to \mathcal{N} \). This is a completely positive map and for \( n \in \mathcal{N} \) we have \( \Phi(p)^{-1}(\mathcal{E}_N \circ \Phi)(np) = \Phi(p)^{-1}(\mathcal{E}_N \circ \Phi)(np) = n\Phi(p)^{-1}(\mathcal{E}_N \circ \Phi)(np) = n \). So that \( \Psi \) is a conditional expectation for the inclusion \( \mathcal{N} \to p(\mathcal{M}, e_B)p : n \to npn \). We may identify \( \mathcal{F} \) with \( L^2(\mathcal{N}) \otimes L^2(\mathcal{B}) \) canonically by \( nb\Omega \to n\Omega \otimes b\Omega \). This way \( e_B \) acting on \( \mathcal{K} \) corresponds to \( p\Omega \otimes \text{Id}_{L^2(\mathcal{B})} \) with \( p\Omega \) the orthogonal projection onto \( \Omega \in L^2(\mathcal{N}) \). It follows that the algebra generated by \( p\mathcal{N}p \) and \( e_B \) equals \( \mathcal{B}(L^2(\mathcal{N})) \otimes \text{Id}_{L^2(\mathcal{B})} \). Hence we may restrict \( \Psi \) to the latter algebra to obtain a conditional expectation for the inclusion \( \mathcal{N} \to \mathcal{B}(L^2(\mathcal{N})) \otimes \text{Id}_{L^2(\mathcal{B})} : n \to npn \). But \( \mathcal{B}(L^2(\mathcal{N})) \otimes \text{Id}_{L^2(\mathcal{B})} \) is a type I algebra. So \( \mathcal{N} \) is the range of a conditional expectation on a type I algebra and hence it is injective. This contradicts Theorem A/Corollary 1.11 (by our choice of the vertices \( r, t_1 \) and \( t_2 \)) and we may conclude the claim.

Remainder of the proof. Now Theorem 7.2 implies that either (1) \( \text{Nor}_\mathcal{M}(A) \prec \mathcal{N} \) for either \( i = 1 \) or \( i = 2 \); (2) \( \mathcal{A} \prec \mathcal{B} \); (3) \( \mathcal{M} \) is injective relative to \( \mathcal{B} \). The three claims above rule out all of these possibilities showing that \( \mathcal{M} \) does not possess a Cartan subalgebra.

\[ \square \]

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Martijn Caspers

Utrecht University, Budapestlaan 6, 3584 CD Utrecht, The Netherlands

*E-mail address:* m.p.t.caspers@uu.nl