Trace anomalies and the string-inspired definition of quantum-mechanical path integrals in curved space.

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Abstract

We consider quantum-mechanical path integrals for non-linear sigma models on a circle defined by the string-inspired method of Strassler, where one considers periodic quantum fluctuations about a center-of-mass coordinate. In this approach one finds incorrect answers for the local trace anomalies of the corresponding \(n\)-dimensional field theories in curved space. The quantum field theory approach to the quantum-mechanical path-integral, where quantum fluctuations are not periodic but vanish at the endpoints, yields the correct answers. We explain these results by a detailed analysis of general coordinate invariance in both methods. Both approaches can be derived from the same operator expression and the integrated trace anomalies in both schemes agree. In the string-inspired method the integrands are not invariant under general coordinate transformations and one is therefore not permitted to use Riemann normal coordinates.

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1 Introduction

One-dimensional (quantum-mechanical) path integrals in curved space (non-linear sigma models) have been used to compute the chiral and trace anomalies of n-dimensional quantum field theories coupled to external gravitation and Yang-Mills fields. In addition, quantum-mechanical path integrals in flat space have proven useful to calculate one- and higher-loop Green’s functions for field theories. In both cases one considers the evaluation of the partition function of the one-dimensional theory on a finite time interval $[-\beta_1, \beta_2]$.

For the evaluation of anomalies the quantum-mechanical path integral has been defined, in analogy to quantum field theory (QFT), by time slicing and requiring that all paths attain the same values at $-\beta_1$ and $\beta_2$ for bosonic point particles (or opposite values for fermionic point particles). One decomposes the paths (“fields”) $x^i(t)$ into a background solution of the free theory satisfying the boundary conditions at $-\beta_1$ and $\beta_2$ and quantum deviations vanishing at these points. Compactifying the interval to a circle (as seems natural for evaluating the trace, which yields the partition function), such paths will in general possess a kink-discontinuity because their derivatives need not be continuous at the point where the endpoints come together. The kinetic operator in the space of functions which vanish at the endpoints has no zero modes, and is readily inverted. To compute the partition function one finally integrates over all possible boundary values.

On the other hand, for the evaluation of Green’s functions in flat space, a string-inspired approach to the path integral has been used. Here the periodicity of the partition function is reflected in the fact that bosonic paths are decomposed into periodic fluctuations about a constant center-of-mass mode (and fermionic paths are expanded into antiperiodic fluctuations). The propagator for these deviations is constructed by inverting the kinetic operator in the space of periodic functions orthogonal to the constant modes, which are zero modes of the one-dimensional Laplacian, and only at the very last does one integrate over the center-of-mass coordinate.

In this letter we consider quantum-mechanical path integrals in curved space using the string-inspired definition of path integrals; we will use the calculation of trace anomalies to illustrate our points. Both definitions can be derived from the same operator expression by time-slicing. At first sight one would thus expect that this method should give the same results as the QFT approach. In fact it has been observed that for linear sigma models (flat-space) the target space Green’s functions in both methods differ by total derivatives and we have earlier shown why chiral anomalies are independent of the approach one uses. We find that the results for local trace anomalies differ, due to a subtlety having to do with general coordinate invariance (Einstein invariance) in curved space. Namely, in the QFT approach the Riemann summands $\sqrt{g(x_0)} \langle x_0 | \exp(-\frac{\bar{\hbar}}{\hbar} \hat{H})| x_0 \rangle d^n x_0$ of the (discretized) partition function $Z(\beta) = \text{Tr} \exp(-\frac{\bar{\hbar}}{\hbar} \hat{H})$ are Einstein invariant, whereas in the string-inspired

\footnote{In the case of chiral anomalies, the fermionic point particles attain the same values at $-\beta_1$, and $\beta_2$.}
approach the corresponding expressions are not Einstein invariant (even if $\hat{H}$ is Einstein invariant, i.e., even if $\hat{H}$ commutes with the generator of general coordinate transformations). Consequently, one cannot simplify the calculations in the string-inspired approach by using normal coordinates. Only the integrated expressions in this approach are general coordinate invariant. In the QFT approach, on the other hand, one obtains directly the correct local trace anomaly, and the invariance of the summands allows normal coordinates to be used. Since both approaches are derived from the same expression, this indicates that again the integrands computed with either method should differ by a total derivative. We present explicit two-loop and three-loop calculations which support our arguments, and our conclusion is that the easiest way to compute quantum-mechanical amplitudes in a curved space-time background is to follow the QFT approach rather than the string-inspired method.

2 Two approaches to the path integral

The basic object to compute is the partition function: the trace of the transition element $\langle z | \exp(-\beta / \hbar \hat{H}) | y \rangle$. Here $\hat{H}(\hat{x}^i, \hat{p}_i)$ is an arbitrary but fixed Hamiltonian operator with $i = 1, \ldots, n$. For definiteness we shall choose an Einstein invariant $\hat{H}$, which means that $\hat{H}$ commutes with the generator $\hat{G}(\xi^i(\hat{x}))$ for general coordinate transformations with parameters $\xi^i(\hat{x})$. The Hamiltonian we consider is given by

$$\hat{H} = \frac{1}{2} g^{-1/4}(\hat{x}) \hat{p}_i g^{ij}(\hat{x}) g^{1/2}(\hat{x}) \hat{p}_j g^{-1/4}(\hat{x}) ; \quad g(\hat{x}) = \det g_{ij}(\hat{x}) . \quad (2.1)$$

In the transition element one inserts $N-1$ complete sets of $\hat{x}$-eigenstates and $N$-sets of $\hat{p}$-eigenstates, rewrites the Hamiltonian in Weyl-ordered form\(^2\) replaces the $\hat{x}$ and $\hat{p}$-operators by their $c$-number eigenvalues and integrates over all $p$'s. The factors $\det(g_{ij}(x_k + x_{k-1}/2))$ produced by the integration over the $p$'s are exponentiated by anticommuting Lee-Yang ghosts $b^i$, $c^i$ and a commuting Lee-Yang ghost $a^i$\(^2\). Since we are interested in taking traces, we set $z = y$.

From this point on the QFT approach and the string-inspired approach go separate ways. In the QFT approach one expands the $z^i = x_0^i, x_1^i, \ldots, x_{N-1}^i, x_N^i = y^i$ into a constant background part $x_0^i = z^i = y^i$ satisfying the free field equations of the free field part of the action $\frac{1}{2} g_{ij}(x_0) \dot{x}_i \dot{x}_j + \frac{1}{2} g_{ij}(x_0) (x_k^i - x_{k-1}^i)^2 (x_k^j - x_{k-1}^j) \sim \frac{1}{2} g_{ij}(x_0) \dot{x}_k^i \dot{x}_k^j$ (where $\epsilon = \beta / N$), and quantum deviations $q_k^i$ which vanish at the endpoints $k = 0$ and $k = N$. The $q_k^i$ are expanded into eigenfunctions that satisfy these boundary conditions

$$q_k^i = \sum_{m=1}^{N-1} r_m^i \sqrt{\frac{2}{N}} \sin \left( \frac{km\pi}{N} \right) ; \quad k = 1, \ldots, N - 1 ; \quad i = 1, \ldots, n . \quad (2.2)$$

\(^2\)It has been shown in [9] that also in curved space one may replace the Weyl-ordered exponential by the exponential of the Weyl-ordered Hamiltonian, because the difference vanishes for $N \to \infty$. This is due to the fact that the propagators $\langle p_i(t_1) p_j(t_2) \rangle$ and $\langle p_i(t_1) q_j(t_2) \rangle$ are not singular in $\epsilon \equiv \beta / N$. With $\hat{H}$ in Weyl-ordered form one may use the midpoint rule to evaluate arbitrary functions of the operators $\hat{x}$ and $\hat{p}$ in terms of the eigenvalues $x$ and $p$. Weyl-ordering of $[2,3]$ leads to well-known extra noncovariant terms of order $\hbar^2$ which are crucial for the general coordinate invariance of the transition element.
By coupling $\frac{1}{2}(q_k^i + q_{k-1}^i)$ and $(q_k^i - q_{k-1}^i)/\epsilon$ to external sources, one obtains the discretized propagators in closed form, and by taking the limit $N \to \infty$ one reads off the correct Feynman rules [3] (the rules how to evaluate integrals over products of distributions $\theta(\sigma - \tau)$ and $\delta(\sigma - \tau)$ as they appear in Feynman graphs). The upshot is that $\delta(\sigma - \tau)$ is to be considered a Kronecker delta function and that $\theta(0) = 1/2$, so that for example (we define $t = \beta \tau$ for convenience)

$$\int_{-1}^{0} d\sigma \int_{-1}^{0} d\tau \delta(\sigma - \tau) \theta(\sigma - \tau) \theta(\sigma - \tau) = \frac{1}{4} = \int_{-1}^{0} d\sigma \int_{-1}^{0} d\tau \delta(\sigma - \tau) \theta(\sigma - \tau) \theta(\tau - \sigma) .$$

(2.3)

After a careful detailed analysis [3] one finds that

$$\text{Tr} \langle z | \exp(-\frac{\beta}{\hbar} \hat{H}) | y \rangle = \int d^n x_0 \sqrt{g(x_0)} \frac{1}{(2\pi \beta \hbar)^{n/2}} \langle x_0 | \exp(-\frac{i}{\hbar} S_{\text{int}}) | x_0 \rangle ,$$

(2.4)

where

$$- \frac{1}{\hbar} S_{\text{int}} = - \frac{1}{2\beta \hbar} \int_{-1}^{0} d\tau [g_{ij}(x_0 + q) - g_{ij}(x_0)] (\dot{q}^i \dot{q}^j + b^i c^j + a^i d^j) - \beta \hbar \int_{-1}^{0} d\tau \left[ R(x_0 + q) + g^{ij} \Gamma_m^{im} \Gamma_j^n m(x_0 + q) \right] .$$

(2.5)

The integral $\int d^n x_0 \sqrt{g(x_0)}$ on the r.h.s. of (2.4) comes from taking the trace and the factor $(2\pi \beta \hbar)^{-n/2}$ coincides with the Feynman measure. The terms with $R$ and $\Gamma$ are created by Weyl ordering eq. (2.1). The continuum limits of the propagators to be used for the perturbative evaluation of the last factor are

$$\langle q^i(\sigma) \dot{q}^i(\tau) \rangle_{x_0} = -\beta \hbar g^{ij}(x_0) \Delta^{QFT}_{ij}(\sigma, \tau) ,$$

$$\langle b^i(\sigma) c^j(\tau) \rangle_{x_0} = -2\beta \hbar g^{ij}(x_0) \delta(\sigma - \tau) = -2 \langle a^i(\sigma) a^j(\tau) \rangle_{x_0}$$

(2.6)

with

$$\Delta^{QFT}(\sigma, \tau) = \sigma(\sigma + 1) \theta(\sigma - \tau) + \tau(\sigma + 1) \theta(\tau - \sigma) .$$

(2.7)

Note that $\langle \dot{q}^i(\sigma) \dot{q}^i(\tau) \rangle \sim \Delta^{QFT}_{ij}(\sigma, \tau) = 1 - \delta(\sigma - \tau)$ is singular for $\sigma \to \tau$ but that this singularity cancels in the sum $\langle \dot{q}^i \dot{q}^j \rangle + \langle b^i c^j \rangle + \langle a^i d^j \rangle$. At higher loops the $a^i, b^i, c^j$ ghosts are crucial to remove all ultraviolet ($\sigma - \tau \to 0$) singularities and as a result all integrals are finite (as they should be in quantum mechanics, despite the double derivative interactions in $\frac{1}{2} g_{ij}(x) \dot{x}^i \dot{x}^j$). Since we work on the finite interval $[-\beta, 0]$ (or $[-1, 0]$ for $\tau = t/\beta$) there are no infrared singularities.

The string-inspired approach to the partition function starts from the same discretized expression of the transition element as an integral over the complete sets $d^n x_1, ..., d^n x_{N-1}$, but one then includes the trace integration over $x_0 = x_N$ and treats all points $x_0, x_1, ..., x_{N-1}$ on equal footing. One thus considers the same object

$$\text{Tr} \langle z | \exp(-\frac{\beta}{\hbar} \hat{H}) | y \rangle = \int d^n x_0 \sqrt{g(x_0)} \langle x_0 | \exp(-\frac{i}{\hbar} \hat{H}) | x_0 \rangle ,$$

(2.8)

\footnote{Our conventions for the Riemann and Ricci tensor are that $R_{ijk}^l = \partial_i \Gamma_{jk}^l + \Gamma_{im}^j \Gamma_m^l - (i \leftrightarrow j)$ and $R_{ik} = R_{ijk}^j$.}
but now one expands the \( n \times N \) integration variables \( x_k^i \) into periodic functions which are eigenfunctions of the free action

\[
\frac{2}{\sqrt{N}} \text{cos} \left( \frac{2kp\pi}{N} \right) r_{p0}^{ic} + \sum_{p=1}^{N/2-1} \frac{2}{\sqrt{N}} \text{sin} \left( \frac{2kp\pi}{N} \right) r_{p}^{is} ; \quad k = 1, \ldots, N . \tag{2.9}
\]

(We take \( N \) to be even.) In the discretized expression for (2.3), only factors of the metric \( g_{ij} \) at midpoints occur, and no explicit \( g_{ij}(x_0) \) survive. We decompose \( x_k^i \) into a center of mass part \( x_c^i = \frac{2}{\sqrt{N}} r_{0}^{ic} \) and fluctuations \( q_k^i \) around \( x_c^i \). Coupling the \( q_k^i \) to the same external sources as before, but substituting then (2.9) one finds after considerable tedious but straightforward algebra the propagators for the string-inspired method in closed discretized form [4]. They contain discretized theta- and delta-functions with the same properties as before; in particular (2.3) holds again. The continuum propagators are now given by (2.6) with

\[
\Delta^{SI}(\sigma, \tau) = \frac{1}{2} (\sigma - \tau) \epsilon(\sigma - \tau) - \frac{1}{2} (\sigma - \tau)^2 - \frac{1}{12} . \tag{2.10}
\]

and \( x_0 \) replaced by \( x_c \). As expected, this propagator is translation invariant, and satisfies \( \partial^2 \Delta^{SI}(\sigma - \tau) = \delta(\sigma - \tau) - 1 \) which is the Dirac delta-function in the space of functions \( q(\tau) \) orthogonal to the constant. The factor \(-\frac{1}{12}\) ensures that \( \int_0^1 \Delta^{SI}(\sigma, \tau) d\tau = 0 \), which should be satisfied as \( \langle q(\sigma) \int_0^1 q(\tau) d\tau \rangle = 0 \).

The expression for the trace in (2.4) is then formally the same as obtained from (2.3), namely

\[
\int d^nx_0 \sqrt{g(x_0)} \langle x_0 | \exp(-\beta \hat{H}) | x_0 \rangle = \int d^nx_c \sqrt{g(x_c)} \frac{1}{(2\pi\hbar)^{n/2}} \langle e^{-\frac{1}{\hbar} S_{int}} \rangle_{x_c} , \tag{2.11}
\]

where the right-hand-side is now evaluated using \( \Delta^{SI}(\sigma, \tau) \). In particular \( S_{int} \) is the same as (2.3) but with \( x_k^i \) replaced by \( x_c^i \). However, the integrand \( \langle \exp(-\frac{1}{\hbar} S_{int}) \rangle_{x_c} \) cannot be written as a matrix element with well-defined position bras and kets; rather it is a function which can be computed using (2.10) and (2.3). In particular the \( x_c \) in \( \sqrt{g(x_c)} \) is not a true position like the \( x_0 \) in \( \sqrt{g(x_0)} \), which is an eigenvalue of the \( \hat{x} \)-operator. It is this fact which is of crucial importance for the issues of general covariance. Starting with an Einstein invariant Hamiltonian our final answer for the partition function will be general coordinate invariant. In the QFT case the integrands \( \langle z | \exp(-\frac{2}{\hbar} \hat{H}) | y \rangle \) and the tracing measure \( \int d^nx_0 \sqrt{g(x_0)} \) are separately Einstein invariant by construction. In the string-inspired case there is no reason why the integrands on the right-hand-side of (2.11) should be Einstein invariant, and explicit calculations show that they are not Einstein invariant. Only the integrated expression for the partition function is guaranteed to be covariant (and is the same in both approaches since in both cases one integrates over all \( n \times N \) variables).

We have now defined the path integral according to the QFT method and the string-inspired method. Both have the same non-covariant order \( \hbar^2 \) counterterm needed for the Einstein invariance of the final results, and both have the same vertices because both are based on time-slicing. The difference rests solely in the order
of the integration over \( d^n x_1, d^n x_2, ..., d^n x_{N-1}, d^n x_N = d^n x_0 \): in the QFT approach we first integrate over intermediate \( x_1^i, ..., x_{N-1}^i \) using \( (2.7) \) and then integrate over \( \sqrt{g(x_0)}d^n x_0 \), whereas in the string-inspired method one first integrates over the fluctuations about the center-of-mass \( x_c^i = 1/N \sum x_k^i \) using \( (2.10) \), and then over \( \sqrt{g(x_c)}d^n x_c \). Note that the number of eigenfunctions into which quantum fluctuations are expanded is the same in both cases (sines and cosines of the double angle vs. only sines of the single angle). However, if we view the integration region in both cases as a circle, we have a kink in the paths in the first case but not in the second case.

3 Comparison of approaches: Trace anomalies in \( n = 2 \) and \( n = 4 \) dimensions

The anomalies of \( n \)-dimensional quantum field-theories can be written as

\[
A_n = \lim_{\beta \to 0} \text{Tr} J \exp(-\frac{\beta}{\hbar} \hat{R}) ,
\]

(3.1)

where \( J \) is the Jacobian \( \partial \delta \phi(x)/\partial \phi(y) \) of the fields \( \phi(x) \) transforming under the symmetry with \( \delta \phi(x) \) and \( \hat{R} \) is a regulator. For consistent anomalies the form of \( \hat{R} \) is unique \([10]\) and for local trace (Weyl) anomalies for real scalar fields one finds that \( \hat{R} \) is equal to the Hamiltonian \( (2.1) \) with an improvement term. The QFT approach then yields \([2]\)

\[
A_{NW} = \lim_{\beta \to 0} \text{Tr} \frac{1}{4}(2-n)\sigma(\hat{x}) \exp(-\frac{\beta}{\hbar} \hat{H})
\]

(3.2)

where \( \sigma(x) \) is the Weyl parameter, normalized to \( \delta_W g_{ij} = \sigma(x)g_{ij}(x) \) and

\[
\hat{H} = \frac{1}{2} g^{-1/4}(\hat{x}) \hat{p}_i g^{1/2}(\hat{x}) g^{ij}(\hat{x}) \hat{p}_j g^{-1/4}(\hat{x}) - \frac{1}{2} \hbar^2 \xi \hat{R}(\hat{x}) .
\]

(3.3)

The term with \( R(\hat{x}) \) is the improvement term and \( \xi = \frac{1}{4}(n-2)/(n-1) \). The trace in \([3,2]\) can be written as a path integral. For the QFT approach one finds then that the local trace anomaly is proportional to \( \int d^n x_0 \sqrt{g(x_0)} \sigma(x_0) \exp(-\frac{1}{\hbar} S_{int}) \). Naively, one might expect a similar expression for the trace anomaly in the string-inspired method, proportional to \( \int d^n x_c \sqrt{g(x_c)} \sigma(x_c) \exp(-\frac{1}{\hbar} S_{int}) \). One should, however, take care in defining the expectation value of a local operator in the string-inspired approach. The naive equivalence with the QFT approach by taking the local operator at the point \( x_c \) is incorrect since at a discretized intermediate stage

\[
\int \frac{1}{N} \sum_{k=1}^{N} \sigma(x_k) \exp(-\frac{1}{\hbar} S) \neq \int \sigma(x_c = \frac{1}{N} \sum_{k=1}^{N} x_k) \exp(-\frac{1}{\hbar} S)
\]

(3.4)

where \( \int \) denotes \( \prod_{i=1}^{n} \prod_{k=1}^{N} \int \sqrt{g(x_k)} dx_k \) and \( S \) is the discretized action. Using cyclicity of the trace, \([3,2]\) leads to the left-hand-side of \((3.4)\), not to the right-hand-side.
Unambiguous is the global trace anomaly where $\sigma(\hat{x})$ is a constant. In that case the anomaly is just proportional to the partition function and this is the case we shall consider.

For $n = 2$ we need all (connected and disconnected) two-loop graphs on the worldline\footnote{The $\beta$ independent terms in (2.4) come from $1 + \frac{1}{2}$ loops (one-loop graphs are independent of $\beta$, two-loop graphs are proportional to $\beta$, etc.).} If one uses normal coordinates, only graphs with the topology of the number 8 and the counterterm contribute. The local anomaly is then proportional to

$$
\langle e^{-\frac{\hbar}{\kappa} S_{\text{int}}} \rangle_{x_0}, n=2 = -\frac{\beta \hbar}{6} R(x) \int_{-1}^{0} d\tau \Delta(\tau, \tau) \{ \Delta^*(\tau, \tau) + \delta(0) - \Delta(\tau, \tau) \Delta(\tau, \tau) \}
$$

$$
-\frac{\beta \hbar}{8} R(x)
.$$  

The singularities with $\delta(0)$ are well-defined in the discretized expressions and cancel in both approaches. Following the QFT approach, and substituting (2.7) in the above, one obtains the correct answer $-\frac{1}{12} R$. In the string-inspired method, substituting (2.10), the second term vanishes due to the fact that $\Delta^S(\tau, \tau) = 0$ and the final result is $-\frac{1}{9} R$, which is incorrect.

The results for the calculation in general coordinates are summarized in figure 1, where we introduce the notation $\partial^2 g = g^{ij} g^{kl} \partial_{ik} \partial_{lj} g_{ij}, \partial^i \partial^j g_{ij} = g^{ik} g^{jl} \partial_{ik} \partial_{lj} g_{ij}, \partial_m g = g^{ij} \partial_m g_{ij}, g_m = g^{mn} \partial_n g_{mn}, g_m = g^{mn} g_n, (\partial_i g_{jk})^2 = g^{ij} g^{kl} \partial_i g_{jk} \partial_j g_{kl}$ and similarly for $\partial_i g_{jk} \partial_j g_{ik}$. In the first line in figure 1 we find the contribution due to expanding $\exp \left[ -\frac{1}{2\kappa} \int_{-1}^{0} \frac{1}{2} g^{ij} \partial_{ik} \partial^i_{lj} g_{ij}(x_0) \partial_k g_{ij}(x_0) \left\{ q^i q^j + b^i b^j + a^i a^j \right\} d\tau \right]$ to first order and taking the contractions $\langle q^i q^j + b^i b^j + a^i a^j \rangle$ and $\langle g^i g^j \rangle$. In the QFT approach one finds $-\frac{\beta \hbar}{4} \partial^2 g(-\frac{1}{12})$ while in the string approach one obtains $-\frac{\beta \hbar}{4} \partial^2 g(-\frac{1}{12})$. In the last line one finds the contribution from the order $\hbar^2$ terms in the action which were produced by Weyl ordering.

Adding all contributions, we find for $\langle \exp \left( -\frac{\hbar}{\kappa} S_{\text{int}} \right) \rangle_{x_0}$ (proportional to the local Weyl anomaly of a real scalar field in $n = 2$ dimensions) in the QFT approach an Einstein invariant result, which of course equals that of the normal coordinate calculation.

$$
\langle e^{-\frac{\hbar}{\kappa} S_{\text{int}}} \rangle_{x_0}, n=2 = \left( -\frac{1}{4} \partial^2 g \right) \left( -\frac{1}{6} \right) + \left( -\frac{1}{2} \partial^i \partial^j g_{ij} \right) \left( \frac{1}{12} \right)
$$

$$
+ \left( \frac{1}{8} \partial^m g \partial_m g \right) \left( \frac{1}{12} \right) + \left( -\frac{1}{2} \partial^m g g_m \right) \left( \frac{1}{12} \right)
$$

$$
+ \left( \frac{1}{2} g^m g_m \right) \left( -\frac{1}{12} \right) + \left( -\frac{1}{4} (\partial_i g_{jk})^2 \right) \left( \frac{1}{4} \right)
$$

$$
+ \left( -\frac{1}{2} \partial_i g_{jk} \partial_j g_{ik} \right) \left( -\frac{1}{6} \right) + \left( -\frac{1}{8} R - \frac{1}{8} g^{ij} \Gamma_i \Gamma_j \right)
$$

$$
= -\frac{1}{12} R
.$$  

(We have written each term as a product of a tensor structure times the result of integrations over expressions depending on $\Delta(\sigma, \tau)$. Using normal coordinates one
Figure 1: The results for the trace anomaly for a real scalar field in $n = 2$ dimensions. Dotted lines denote ghosts. For each graph, the result in either the QFT or string-inspired (SI) approach is obtained by taking the product of the tensor structure and the number in the appropriate column. The tensor structure is the same in both cases as the vertices are the same. Because $\Delta_{\text{SI}}(\tau, \tau) = 0$ in the string-inspired approach, there are fewer nonvanishing contributions than in the QFT approach. However, the sum of all contributions is general coordinate invariant in the QFT but not in the SI approach.

Hence the integrands in both methods differ. Yet the integrated expressions (3.6) and (3.7) should be the same, as we explained in the previous section. By integrating these expressions with $\int d^n x_0 \sqrt{g(x_0)}$ and $\int d^n x_c \sqrt{g(x_c)}$, respectively, and using partial integration we may replace terms with double derivatives of the metric by products of terms with single derivatives. This yields under the integral sign the “equalities”

$$\sqrt{g} \partial^2 g \leftrightarrow \sqrt{g} \left( -\frac{1}{2} \partial^m g \partial_m g + (\partial_i g_{jk})^2 + g^m \partial_m g \right).$$
\[
\sqrt{g} \nabla^i \nabla^j g_{ij} \iff \sqrt{g} \left( -\frac{1}{2} \partial^m gg_m + \partial_i g_{jk} \partial_j g_{ik} + g^m g_m \right). \tag{3.8}
\]

One then indeed finds that the integrated Weyl anomalies agree
\[
\int dx_0 \sqrt{g(x_0)} \langle e^{-\frac{1}{\hbar}S_{int}} \rangle_{x_0}, n=2 = \int dx_c \sqrt{g(x_c)} \langle e^{-\frac{1}{\hbar}S_{int}} \rangle_{x_c}, n=2. \tag{3.9}
\]

The case \( n = 2 \) is rather special as the answer is proportional to \( \sqrt{g} R^{(n=2)} \) which is itself a total derivative. Let us therefore consider the trace anomaly for a real scalar field in \( n = 4 \) dimensions and compute it using normal coordinates. This will explicitly demonstrate that one gets incorrect results from the string-inspired method. As we explained before, this is due to the fact that one can only choose normal coordinates at one point, but since the integrands (functions of \( x_c \)) are not Einstein scalars, one cannot use normal coordinates at each point \( x_c \).

In \( n = 4 \) the improvement term in (3.3) is nonvanishing and effectively converts the term \( \frac{K^2}{8} R \) in (2.5) into \( \frac{K^2}{24} R \). The action yields the following vertices in normal coordinates
\[
-\frac{1}{\hbar} S_{int} = -\frac{1}{2\beta \hbar} \int_{-1}^{0} d\tau \left[ -\frac{1}{3} R_{iklj} q^k q^l - \frac{1}{6} D_m R_{iklj} g^m q^k q^l 
- \left\{ \frac{1}{20} D_m D_n R_{iklj} - \frac{2}{45} R_{iktm} R_{jl} \right\} q^m q^n q^k q^l + \ldots \right](q^i q^j + b^i c^j + a^i a^j)
- \beta \hbar \int_{-1}^{0} d\tau \left( \frac{1}{24} R(x_0 + q) + \frac{1}{8} g^{ij} \Gamma_{ik} \Gamma_{jl} (x_0 + q) \right). \tag{3.10}
\]

The \( \beta \)-independent term (which yields the trace anomaly) now arises from three-loop diagrams. Since there are no three-point vertices in normal coordinates, the five-point vertices do not contribute. The four- and six-point vertices yield the graphs in figure 2. In the one-but-last line of figure 2 one finds the contribution from the order \( \hbar^2 \) vertex \( \frac{1}{24} R(x_0 + q) + \frac{1}{8} g^{ij} \Gamma_{ik} \Gamma_{jl} (x_0 + q) \) due to expanding it to second order in \( q^i \) and contracting the two \( q^i \)'s to an equal time loop. Note that it would have been incorrect to omit the \( \Gamma \Gamma \) term altogether by arguing that it will not contribute in normal coordinates. In the last line one finds the contribution from the disconnected graphs; they are proportional to the square of the \( n = 2 \) result, but with the improvement term their contribution vanishes in the QFT approach. This is coincidental as for the trace anomaly in \( n = 6 \) dimensions the contribution from disconnected graphs does not cancel. (It is given by the products of lower dimensional trace anomalies, namely \( \left( -\frac{1}{12} R + \frac{1}{10} R \right) \frac{1}{720} (R^2_{mn} - R^2_{mn} - \frac{1}{5} D^2 R) + \frac{1}{6} (-\frac{1}{12} R + \frac{1}{10} R)^3 \) where the terms \( \frac{1}{10} R \) and \( -\frac{1}{5} D^2 R \) come from the improvement term.)

Adding all contributions, we obtain from the QFT approach
\[
\langle e^{-\frac{1}{\hbar}S_{int}} \rangle_{x_0}, n=4 = \frac{(\beta \hbar)^2}{72} \left[ \left( -\frac{1}{6} R_{mn}^2 \right) + \left( \frac{3}{20} D^2 R + \frac{1}{10} R_{mn}^2 + \frac{1}{15} R_{mn}^2 \right) \right.
+ \left. \left( -\frac{1}{4} R_{mn}^2 \right) + \left( -\frac{1}{4} D^2 R + \frac{1}{4} R_{mn}^2 \right) \right]
= \frac{(\beta \hbar)^2}{720} \left[ R_{mn}^2 - R_{mn}^2 - D^2 R \right]. \tag{3.11}
\]
Graph:  

\[ QFT: \begin{align*}
&-\frac{1}{6} R^2_{mn} \\
&-\frac{1}{4} R^2_{mnpq} \\
&\frac{3}{20} D^2 R + \frac{1}{10} R^2_{mnpq} + \frac{1}{40} R^2_{mn} - \frac{1}{8} D^2 R + \frac{1}{8} R^2_{mnpq}
\end{align*} \]

\[ SI: \begin{align*}
&-\frac{7}{180} R^2_{mn} \\
&-\frac{1}{60} R^2_{mnpq} \\
&\frac{1}{8} D^2 R + \frac{1}{8} R^2_{mnpq}
\end{align*} \]

\[ SL: \begin{align*}
&0 \\
&\frac{1}{2} \left( \frac{1}{36} R \right)^2
\end{align*} \]

Figure 2: The results for the trace anomaly for a real scalar field in \( n = 4 \) dimensions, calculated in Riemann normal coordinates. Only the topology of the graphs is indicated and all terms should be multiplied by an overall factor \((\beta \bar{h})^2/72\) to obtain the results in (3.11) and (3.12). In the string-inspired approach the fact that \( \Delta_{SI}(\tau, \tau) = 0 \) again leads to great simplifications but in this approach the integrands are not Einstein invariant and the use of normal coordinates is therefore illegal.

which is the correct result \[8\]. In the string inspired result assuming (incorrectly) that one may use normal coordinates in the integrands, we find instead

\[
\langle e^{-\frac{1}{\hbar}S_{int}} \rangle_{x_c, n=4} = \frac{(\beta \bar{h})^2}{72} \left[ \left( -\frac{7}{180} R^2_{mn} \right) + \left( \frac{1}{40} D^2 R + \frac{1}{60} R^2_{mnpq} + \frac{1}{90} R^2_{mn} \right) \\
+ \left( -\frac{1}{60} R^2_{mnpq} \right) + \left( -\frac{1}{8} D^2 R + \frac{1}{8} R^2_{mnpq} \right) + \frac{1}{36} R^2 \right]
\]

\[
= \frac{(\beta \bar{h})^2}{72} \left[ \frac{1}{8} R^2_{mnpq} - \frac{1}{36} R^2_{mn} - \frac{1}{10} D^2 R + \frac{1}{36} R^2 \right]. \quad (3.12)
\]

The \( D^2 R \) terms come out the same, for which we have no explanation at hand, but the other terms are different. This demonstrates that using normal coordinates in the string-inspired approach yields incorrect results for the local trace anomaly.

4 Conclusion

We have considered path integrals of quantum-mechanical non-linear sigma models. The operator expression for the partition function can be evaluated in two different ways: the QFT approach (where paths vanish at the endpoints) and the string-inspired approach (where paths are manifestly periodic). At the discretized level,
where all expressions are well-defined, the difference is that in the QFT approach one first integrates over the intermediate points \( q_k = x_k - x_0 \) for \( k = 1, \ldots, N \) and then over the endpoint \( x_0 = x_N \), while in the string-inspired approach one first integrates over the deviations \( q_k = x_k - x_c \) and then over the center-of-mass \( x_c = \frac{1}{N} \sum x_k \). As the starting point is the same expression for both path integrals they should yield the same answers.

As a test we considered local trace anomalies, which are proportional to
\[
\text{Tr} \sigma(\hat{x}) \exp(-\frac{\beta}{\hbar} \mathcal{R})
\]
with \( \mathcal{R} \) a regulator equal to the Hamiltonian \( \hat{H} \) with an improvement term. This yields the correct local trace anomalies using the QFT approach. The string-inspired approach to path integrals does not reproduce the correct local trace anomaly from
\[
\int d^n x_c \sqrt{g(x_c)} \sigma(x_c) \langle \exp(-\frac{1}{\hbar} S_{\text{int}}) \rangle_{x_c}
\]
We understood this by going back to the discretized level in which case \( \sigma(x_c) \) equals \( \sigma(\frac{1}{N} \sum x_k) \) and not \( \frac{1}{N} \sum \sigma(x_k) \). Interpreting \( \int d^n x_c \sqrt{g(x_c)} \sigma(x_c) \langle \exp(-\frac{1}{\hbar} S_{\text{int}}) \rangle_{x_c} \) as a regularization scheme for \( \text{Tr} \sigma(\hat{x}) \), our result shows that different regularization schemes (for example the QFT approach and the SI approach) lead to different local trace anomalies. Since trace anomalies are not topological, this result could have been expected.

The integrated or global anomaly, for which \( \sigma(\hat{x}) \) is constant, is proportional to the partition function. The answers for this case are the same, as they should, but a second subtlety was discovered in the process. Namely, the QFT approach has the additional benefit that the transition element \( \langle x_0 | \exp(-\frac{\beta}{\hbar} \hat{H}) | x_0 \rangle \) is itself general covariant and one may use normal coordinates in evaluating Feynman diagrams. The string-inspired approach uses the cyclic symmetry of the partition function, which causes many diagrams to vanish trivially but the integrand \( \langle \exp(-\frac{1}{\hbar} S_{\text{int}}) \rangle_{x_c} \) is not covariant and the benefits of cyclicity are offset by the fact that one is not allowed to use normal coordinates.

Our final conclusions are twofold. First, one cannot use the string-inspired path integral to construct regulators for local trace anomalies which keep Einstein invariance at the quantum level. Second, one can use the string-inspired method to evaluate partition functions, but one cannot use normal coordinates and the integrands differ from those in the QFT approach by total derivatives.

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References

[1] L. Alvarez-Gaumé and E. Witten, Nucl. Phys. B234 (1984) 269.

[2] F. Bastianelli, Nucl. Phys. B376 (1992) 113.

F. Bastianelli and P. van Nieuwenhuizen, Nucl. Phys. B389 (1993) 53.

[3] M. Strassler, Nucl. Phys. B385 (1992) 145.
[4] C. Schubert, *The worldline path integral approach to the Bern-Kosower formalism*, Humboldt-U preprint PRINT-97-273.

[5] J. de Boer, B. Peeters, K. Skenderis, and P. van Nieuwenhuizen, Nucl. Phys. B459 (1996) 631 and B446 (1995) 211.

[6] A. Hatzinikitas, K. Schalm, P. van Nieuwenhuizen, Nucl. Phys. B518 (1998) 424.

[7] B.S. DeWitt, *Supermanifolds*, second edition, Cambridge University Press (1992).

[8] F. Bastianelli, K. Schalm, and P. van Nieuwenhuizen, Phys. Rev. D 58 (1998) 044002.

[9] J. de Boer, B. Peeters, K. Skenderis, and P. van Nieuwenhuizen, Strings ’95 and 1995 Leuven Workshop on Gauge theories, Applied supersymmetry, and Quantum gravity.

[10] A. Diaz, W. Troost, A. Van Proeyen and P. van Nieuwenhuizen, Int. J. Mod. Phys. A4 (1989) 3959.