ON THE INTERSECTION FORM OF SURFACES

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ABSTRACT. Given a closed, oriented surface $M$, the algebraic intersection of
closed curves induces a symplectic form $\text{Int}(\cdot,\cdot)$ on the first homology group
of $M$. If $M$ is equipped with a Riemannian metric $g$, the first homology
group of $M$ inherits a norm, called the stable norm. We study the norm of
the bilinear form $\text{Int}(\cdot,\cdot)$, with respect to the stable norm.

1. Introduction

Let $M$ be a closed (i.e. compact, without boundary) manifold of dimension
two, different from the 2-sphere, equipped with an orientation 2-form $\Omega$. If
$\alpha$ and $\beta$ are two $C^1$ closed curves on $M$ which intersect transversally, we call
algebraic intersection of $\alpha$ and $\beta$ the number
\[
\sum \frac{\Omega(\dot{\alpha}_x, \dot{\beta}_x)}{|\Omega(\dot{\alpha}_x, \dot{\beta}_x)|},
\]
where
- $\dot{\alpha}_x$ denotes the tangent vector to $\alpha$ at $x$
- the sum is taken over all pairs of parameter values $(s,t)$ such that $\alpha(s) = \beta(t)$.

It is classical that this number only depends on the homology classes of $\alpha$
and $\beta$. We denote it by $\text{Int}([\alpha],[\beta])$. The map $\text{Int}(\cdot,\cdot)$ extends by linearity
to a symplectic (i.e bilinear, antisymmetric, nondegenerate) form on the first
homology $H_1(M,\mathbb{R})$ of $M$.

The central question in this paper is when $M$ is endowed with a Riemannian
metric $g$, how much can two curves of a given length intersect?

This amounts to evaluating the norm of the bilinear form $\text{Int}(\cdot,\cdot)$ with respect
to a certain norm on $H_1(M,\mathbb{R})$, called the stable norm. Informally speaking
the stable norm measures the size, relative to the metric $g$, of a homology or
cohomology class. Various equivalent definitions exist, see [1,7,9,13]. We shall
use that of [9]: for $x \in M$ and a vector $v \in T_xM$, we denote by $|v|$ its Riemannian
norm. The \textit{comass} of a differential one-form on $M$ is given by
\[
\text{comass}(\omega) = \sup \left\{ \frac{|\omega(v)|}{|v|} : x \in M, \ v \in T_xM, \ v \neq 0 \right\}.
\]

Equation (1) defines a norm on the space $\mathcal{F}_1(M)$ of smooth 1-forms on $M$.
We get a norm on the first cohomology of $M$ by taking the infimum of the

\textit{Date:} February 27, 2013.
comass over all smooth closed 1-forms in a given cohomology class:
\[ \forall c \in H^1(M, \mathbb{R}), \|c\|_s := \inf\{\text{comass}(\omega) : \omega \in \mathcal{F}_1(M), d\omega = 0, [\omega] = c\}. \]
The norm \(\|\cdot\|_s\) is called the stable norm on \(H^1(M, \mathbb{R})\). We denote in the same way the dual norm on \(H^1(M, \mathbb{R})\).

We say a homology class \(h \in H_1(M, \mathbb{R})\) is integer if \(h\) is the image in \(H_1(M, \mathbb{R})\) of an element of \(H_1(M, \mathbb{Z})\). When \(M\) is an orientable surface of genus \(s\), and the homology class \(h\) is integer, the stable norm has a nice expression, see \([1, 12, 13]\):
\[ \|h\|_s \text{ is the minimum of all sums } \sum |r_i| l_g(\gamma_i), \text{ where} \]
- the index \(i\) ranges over \(0, \ldots, s\)
- \(l_g\) denotes the length with respect to \(g\)
- the \(r_i\) are integer numbers
- the \(\gamma_i\) are pairwise disjoint simple closed geodesics
- \(h = \sum_{i=1}^n r_i [\gamma_i]\).

The norm of the bilinear form \(\text{Int}(\cdot, \cdot)\) with respect to the stable norm on \(H_1(M, \mathbb{R})\) is then defined as:
\[ K(M, g) := \sup \left\{ \frac{\|\text{Int}(h_1, h_2)\|}{\|h_1\|_s \|h_2\|_s} : h_1, h_2 \in H_1(M, \mathbb{R}) \setminus \{0\} \right\}. \]
Observe that in the above expression, the supremum is actually a maximum, since the function \(\|\text{Int}(h_1, h_2)\|/\|h_1\|_s \|h_2\|_s\) is zero-homogeneous, so it is actually defined on the projectivized of \(H_1(M, \mathbb{R})\), which is compact.

When there is no ambiguity on \(M\) and \(g\), we shall sometimes abbreviate the notation \(K(M, g)\) to \(K\).

While, from a geometrical standpoint, the stable norm is the most natural norm on \(H_1(M, \mathbb{R})\), from the complex analysis viewpoint, the most natural norm is the \(L^2\)-norm. For any differential one-form \(\omega\) and for \(x \in M\) we denote by \(\|\omega_x\|\) the norm, with respect to the metric \(g\), of the corresponding linear form on \(T_xM\). Then we define the \(L^2\)-norm of \(\omega\) by the formula
\[ (\|\omega\|_2)^2 := \int_M \|\omega_x\|^2 d\text{vol}(x), \]
where \(\text{vol}\) denotes the volume element of the metric \(g\). We define the \(L^2\)-norm of a cohomology class \(c\) as \(\inf(\|\omega\|_2)\), over all 1-forms \(\omega \in c\). It is a remarkable fact (see \([3]\)) that this infimum is actually a minimum, and is achieved by the unique harmonic 1-form in the cohomology class \(c\). The norm on \(H_1(M, \mathbb{R})\) dual to the \(L^2\)-norm on \(H^1(M, \mathbb{R})\) will also be called \(L^2\)-norm, and will be denoted by the same symbol.

The original motivation for this article was to compare the stable norm and the \(L^2\)-norm. This is done in Section 2.2 and our result is:

**Theorem 1.1.** Let
- \((M, g)\) be a closed, oriented surface equipped with a Riemannian metric
- \(\text{vol}(M, g)\) be the total volume of \((M, g)\).

Then for all \(h \in H_1(M, \mathbb{R})\), we have
\[ \frac{1}{\sqrt{\text{vol}(M, g)}} \|h\|_s \leq \|h\|_2 \leq K(M, g) \sqrt{\text{vol}(M, g)} \|h\|_s. \]
This theorem was originally proved as Equation (4.8) of [12], see also [3, 11]. In Section 2 we give a short and simple proof. The first inequality, which is a straightforward consequence of the Cauchy-Schwarz inequality, has been extended to higher dimensions in [15]. It is also used in [10, 14].

Now that we've been introduced to the number $K(M, g)$, we want to know more about it. A trivial, but nice observation, is that Theorem 1.1 entails

$$K(M, g) \geq \frac{1}{\text{vol}(M, g)}.$$  \hspace{1cm} (4)

The first question that comes to mind is

**Question 1.2.** Is the lower bound of Equation (4) optimal? If not, what is the best possible lower bound? Is it realized by some surfaces, and if so, how to characterize such surfaces?

Such as it is, Question 1.2 is readily answered by [3, 11], and the answer is that $\text{Vol}(M, g)K(M, g) = 1$ if and only if $M$ is the two-torus and the metric $g$ is flat. The "if" part may be checked by elementary calculations and we leave it as an exercise. The "only if" holds because, by [11], if the stable norm and the $L^2$-norm are proportional, then each harmonic 1-form has constant norm. Then Proposition 6.2 of [3] implies that $(M, g)$ is a flat torus.

This answer to Question 1.2 prompts new questions:

**Question 1.3.** If we fix a genus $s > 1$ for $M$, what is the optimal lower bound? Is it realized by some Riemannian metrics? If so, are those metrics of constant curvature?

Another obvious question is

**Question 1.4.** Does $K(M, g)$ have an upper bound involving known geometric quantities such as the length of a homological systole (the length of a shortest, non-separating closed geodesic)?

The best we can do about Questions 1.3, 1.4 is summed up in Corollary 3.5 which we restate here for the commodity of the reader:

**Theorem 1.5.** Let

- $M$ be a closed, oriented surface of genus $s \geq 1$
- $g$ be a Riemannian metric on $M$
- $D := \text{diam}(M, g)$ be the diameter of $(M, g)$
- $l_1 := l_1(M, g)$ be the length of a homological systole of $(M, g)$.

Then we have

$$\frac{1}{2l_1D} \leq K(M, g) \leq \frac{9}{l_1^2}.$$

In Section 4 we specialize to metrics of constant negative curvature, and we obtain the

**Theorem 1.6.** Let

- $M$ be a closed, oriented surface of genus $s > 1$
- $g$ be a Riemannian metric of constant curvature $-1$ on $M$
- $l_1$ be the length of a homological systole of $(M, g)$.
Then there exist positive numbers $A(s)$ and $B(s)$, which depend only on the genus $s$ of $M$, such that when $l_1$ is small enough,

$$\frac{A(s)}{l_1|\log(l_1)|} \leq K(M, g) \leq \frac{B(s)}{l_1|\log(l_1)|}. $$

It would be interesting to know if there is a more precise asymptotic estimate for the behaviour of $K(M, g)$ when $g$ tends to infinity in the moduli space $\mathcal{M}_s$ of surfaces of genus $s$ and curvature $-1$. At least we know that $K(M, g)$ does not have a maximum in $\mathcal{M}_s$, but the following question remains:

**Question 1.7.** Does $K(M, g)$ have a minimum when $(M, g)$ ranges over the moduli space $\mathcal{M}_s$ of surfaces of genus $s$? If so, which surfaces realize the minimum?

There is still an obvious question that we haven’t addressed:

**Question 1.8.** Given a surface $(M, g)$, by which homology classes is $K(M, g)$ realized, as the maximum in Equation (3)? When is it realized by (the homology classes of) simple closed geodesics?

In the case of flat tori, it can be checked by elementary calculations that for almost every flat torus (with respect to Lebesgue measure on the moduli space of flat tori), $K(M, g)$ is not realized by the homology classes of simple closed geodesics. In the case of surfaces of constant negative curvature, we propose the following conjecture, inspired by Theorem 10.7 of [16]:

**Conjecture 1.9.** For any $s > 1$, for almost every $(M, g)$ in $\mathcal{M}_s$, $K(M, g)$ is realized by the homology classes of simple closed geodesics.

2. Comparison between the stable norm and the $L^2$-norm

**2.1. Poincaré duality.** First let us recall some basic facts. Let $\omega$ and $\omega'$ be two closed 1-forms on $M$, and let $c$ and $c'$ be their respective cohomology classes. The wedge product $\omega' \wedge \omega$ is a 2–form on $M$, so there exists $\lambda$ in $\mathbb{R}$ such that

$$[\omega' \wedge \omega] = \lambda [\Omega],$$

where $\Omega$ is the volume form of $M$. The number $\lambda$ only depends on the cohomology classes $c$ and $c'$, we denote it $c' \wedge c$ for the sake of brevity.

Recall that the Poincaré duality $P$ is the map from $H^1(M, \mathbb{R})$ to $H_1(M, \mathbb{R})$ induced by the wedge product of 1-forms: for any $c, c' \in H^1(M, \mathbb{R})$, we have

$$(c', P(c)) = c' \wedge c,$$

where $(\cdot, \cdot)$ denotes the usual duality bracket between $H^1(M, \mathbb{R})$ and $H_1(M, \mathbb{R})$. By [8], p. 59, the Poincaré duality maps the wedge product in $H^1(M, \mathbb{R})$ to the intersection pairing in $H_1(M, \mathbb{R})$, that is, for any $c, c' \in H^1(M, \mathbb{R})$, we have

$$c' \wedge c = \text{Int} (P(c'), P(c)).$$

We shall need the following three lemmas:

**Lemma 2.1.** The Poincaré duality map $P : H^1(M, \mathbb{R}) \to H_1(M, \mathbb{R})$ is an isometry with respect to the $L^2$-norm.
Proof. Let $c$ be any cohomology class in $H^1(M, \mathbb{R})$. Recall that the $L^2$-norm of $c$ is $\sqrt{|c \wedge^* c|}$, where $^*$ is Hodge’s star operator. Then by the definition of the dual $L^2$-norm on $H^1(M, \mathbb{R})$, we have

$$\|P(c)\|_2 = \sup_{c' \neq 0} \frac{\langle c', P(c) \rangle}{\|c'\|_2} = \sup_{c' \neq 0} \frac{c' \wedge c}{\sqrt{|c' \wedge^* c'|}}$$

and by the Schwarz inequality, the supremum above is achieved for $c' =^* c$, so we have

$$\|P(c)\|_2 = \sqrt{|c \wedge^* c|} = \|c\|_2.$$ 

□

Lemma 2.2. For any $c \in H^1(M, \mathbb{R})$ and any $h \in H_1(M, \mathbb{R})$ we have

$$\langle c, h \rangle = \text{Int}(Pc, h).$$

Proof. Since the wedge product and the intersection pairing are Poincaré dual, we have \text{Int}(Pc, h) = c \wedge P^{-1}h, and by Equation (5) we have

$$c \wedge P^{-1}h = \langle c, P(P^{-1}h) \rangle = \langle c, h \rangle.$$ 

□

Lemma 2.3. The norm of the inverse Poincaré duality map from $H_1(M, \mathbb{R})$ to $H^1(M, \mathbb{R})$, with respect to the stable norm, is $K$.

Proof. The norm of the inverse Poincaré duality map from $H_1(M, \mathbb{R})$ to $H^1(M, \mathbb{R})$ is

$$\sup_{h \neq 0} \frac{\|P^{-1}h\|_s}{\|h\|_s} = \sup_{h \neq 0} \frac{\langle P^{-1}h, h' \rangle}{\|h\|_s \|h'\|_s}$$

and by Lemma 2.2

$$\sup_{h \neq 0} \frac{\|P^{-1}h\|_s}{\|h\|_s} = \sup_{h \neq 0, h' \neq 0} \frac{\text{Int}(h, h')}{\|h\|_s \|h'\|_s} = K.$$ 

□

2.2. Proof of Theorem 1.1. Let $\omega$ be a differential 1-form on $M$. We have

$$(\|\omega\|_2)^2 = \int_M \|\omega_x\|^2 d\text{vol}(x) \leq \int_M (\sup_{x \in M} \|\omega_x\|^2) d\text{vol}(x) = \text{vol}(M, g)(\text{comass}(\omega))^2.$$ 

Thus,

$$\|\omega\|_2 \leq \sqrt{\text{vol}(M, g) \text{comass}(\omega)}.$$ 

Taking the infimum of either member of Equation (6) over all closed 1-forms in the cohomology class $c := [\omega]$, we get

$$\|c\|_2 \leq \sqrt{\text{vol}(M, g)} \|c\|_s$$

and since

$$\forall h \in H_1(M, \mathbb{R}), \|h\|_s = \sup_{c \neq 0} \frac{\langle c, h \rangle}{\|c\|_s}$$

we get

$$\forall h \in H_1(M, \mathbb{R}), \|h\|_s \leq \sup_{c \neq 0} \sqrt{\text{vol}(M, g)} \frac{\langle c, h \rangle}{\|c\|_2} = \sqrt{\text{vol}(M, g)} \|h\|_2,$$
which is the first inequality of Theorem 1.1.

Now let us prove the second inequality. Equation (7) and Lemma 2.3 tell us that for any homology class \( h \), we have
\[
\|Ph\| \leq \sqrt{\text{vol}(M,g)} \|Ph\|_s \leq \sqrt{\text{vol}(M,g)} K \|h\|_s.
\]

Now Lemma 2.1 says that \( \|Ph\| = \|h\| \), which completes the proof. \( \square \)

3. More on \( K \)

We get a better understanding of \( K \) by noticing that in its definition we may restrict to simple closed geodesics:

**Lemma 3.1.** We have

\[
K = \sup \left\{ \frac{|\text{Int}(\alpha,\beta)|}{l_g(\alpha)l_g(\beta)} : \alpha, \beta \text{ are simple closed geodesics} \right\}.
\]

**Proof.** First let us point out that for any simple closed geodesics \( \alpha \) and \( \beta \), we have \( l_g(\alpha) \geq \|\alpha\|_s \) and \( l_g(\beta) \geq \|\beta\|_s \), so
\[
\frac{|\text{Int}(\alpha,\beta)|}{l_g(\alpha)l_g(\beta)} \leq \frac{|\text{Int}(\alpha),\beta)|}{\|\alpha\|_s \|\beta\|_s} \leq K,
\]

whence
\[
K \geq \sup \left\{ \frac{|\text{Int}(\alpha,\beta)|}{l_g(\alpha)l_g(\beta)} : \alpha, \beta \text{ are simple closed geodesics} \right\}.
\]

To establish the reverse inequality, first observe that in the definition of \( K \) we may restrict to integer homology classes:

\[
K = \sup \left\{ \frac{|\text{Int}(h_1, h_2)|}{\|h_1\|_s \|h_2\|_s} : h_1, h_2 \in H_1(M, \mathbb{Z}) \setminus \{0\} \right\}
\]

because rational homology classes are dense in \( H_1(M, \mathbb{R}) \). Now take

- two integer homology classes \( h_1 \) and \( h_2 \)
- simple closed geodesics \( \alpha_1, \ldots, \alpha_k \) and \( \beta_1, \ldots, \beta_p \)
- integers \( a_1, \ldots, a_k \) and \( b_1, \ldots, b_p \)

such that

- \( h_1 = a_1 [\alpha_1] + \ldots + a_k [\alpha_k] \)
- \( \|h_1\|_s = |a_1| l_g(\alpha_1) + \ldots + |a_k| l_g(\alpha_k) \)
- \( h_2 = b_1 [\beta_1] + \ldots + b_p [\beta_p] \)
- \( \|h_2\|_s = |b_1| l_g(\beta_1) + \ldots + |b_p| l_g(\beta_p) \).

Without loss of generality we may assume that
\[
\frac{|\text{Int}(\alpha_1, \beta_1)|}{l_g(\alpha_1)l_g(\beta_1)} = \max_{i,j} \frac{|\text{Int}(\alpha_i, \beta_j)|}{l_g(\alpha_i)l_g(\beta_j)}.
\]
Then
\[ |\text{Int}(h_1, h_2)| = \left| \sum_{i,j} a_ib_j \text{Int}(\alpha_i, \beta_j) \right| \leq \sum_{i,j} |a_i||b_j| |\text{Int}(\alpha_i, \beta_j)| \]
\[ \leq \sum_{i,j} |a_i||b_j| l_g(\alpha_i)l_g(\beta_j) \frac{|\text{Int}(\alpha_1, \beta_1)|}{l_g(\alpha_1)l_g(\beta_1)} \]
\[ = \|h_1\|_s\|h_2\|_s |\text{Int}(\alpha_1, \beta_1)|, \]
whence
\[ \frac{|\text{Int}(h_1, h_2)|}{\|h_1\|_s\|h_2\|_s} \leq \frac{|\text{Int}(\alpha_1, \beta_1)|}{l_g(\alpha_1)l_g(\beta_1)} \]
which proves that
\[ K \leq \sup \left\{ \frac{|\text{Int}(\alpha, \beta)|}{l_g(\alpha)l_g(\beta)} : \alpha, \beta \text{ are simple closed geodesics} \right\}, \]
and the lemma. 

Let \( \epsilon > 0 \) be a positive real number and let \( \alpha \) and \( \beta \) be two simple closed geodesics, such that
\[ K \leq \frac{|\text{Int}(\alpha, \beta)|}{l_g(\alpha)l_g(\beta)} + \epsilon. \]

Set \( N := |\text{Int}(\alpha, \beta)| \). Replacing, if necessary, \( \alpha \) and \( \beta \) by shorter curves whose algebraic intersection is \( N \), we may assume that \( \alpha \) and \( \beta \) minimize the product \( l_g(\alpha) \cdot l_g(\beta) \) among all pairs of curves whose algebraic intersection is \( N \). Then the following lemma tells us that all intersections of \( \alpha \) and \( \beta \) have the same sign, that is, the algebraic intersection of \( \alpha \) and \( \beta \) coincides with the number of their intersection points. Therefore in the definition of \( K \) we may restrict to simple closed geodesics, where \( \#\{\alpha \cap \beta\} = |\text{Int}(\alpha, \beta)| \).

**Lemma 3.2.** Let \( N \) be a positive integer, and let \( \alpha, \beta \) be simple closed geodesics such that \( |\text{Int}(\alpha, \beta)| = N \) and
\[ \frac{|\text{Int}(\alpha, \beta)|}{l_g(\alpha)} = \sup \left\{ \frac{N}{l_g(\gamma)} : \gamma \text{ simple closed geodesic }, |\text{Int}(\alpha, \gamma)| = N \right\}, \]
\[ \frac{|\text{Int}(\alpha, \beta)|}{l_g(\beta)} = \sup \left\{ \frac{N}{l_g(\gamma)} : \gamma \text{ simple closed geodesic }, |\text{Int}(\beta, \gamma)| = N \right\}. \]

Then
\[ N = |\text{Int}(\alpha, \beta)| = \#\{\alpha \cap \beta\}. \]

**Proof.** By contradiction: assume that there exists a geodesic arc \( \alpha_1 \) of \( \alpha \) with endpoints \( p_1 \) and \( p_2 \) on \( \beta \), such that the sign of the intersection at \( p_1 \) is different from the sign at \( p_2 \). Let \( \beta_1 \) be the geodesic arc on \( \beta \) connecting \( p_1 \) and \( p_2 \) traversing \( \beta \) in the positive sense and let \( \beta_2 \) the remaining part of \( \beta \) (see Fig. 1).

Now assume without loss of generality that \( l_g(\alpha_1) \leq l_g(\beta_1) \). We construct a new curve \( \beta' \) by connecting \( \beta_2 \) with \( \alpha_1 \). By homotoping \( \beta' \) away from \( \alpha \) with a small deformation, the intersection points \( p_1 \) and \( p_2 \) disappear. Now the
Proposition 3.3. Let \((M, g)\) be a closed, oriented Riemannian surface. Let \(l_1\) be the length of a homological systole \(\alpha_1\) and let \(D\) be the diameter of \((M, g)\). Then
\[
\frac{1}{l_1 \cdot 2D} \leq K.
\]

Proof. Let \(\alpha_2\) be a shortest closed geodesic such that \(|\text{Int}(\alpha_1, \alpha_2)| = 1\), and let \(l_2\) be its length. Then we have \(K \geq (l_1 l_2)^{-1}\). We shall prove, by contradiction, that \(l_2 \leq 2D\), which entails the proposition.

Assume \(l_2 > 2D\). Then there exist two points \(p_1\) and \(p_2\) on \(\alpha_2\) whose distance is not realized by a geodesic arc on \(\alpha_2\). Let \(\delta_0\) be a geodesic arc, which is not an arc of \(\alpha_2\), and realizes the distance between \(p_1\) and \(p_2\). The points \(p_1\) and \(p_2\) divide \(\alpha_2\) into two arcs, \(\delta_1\) and \(\delta_2\). Denote by \(\delta_0\) and \(\delta_2\) the curves we obtain by connecting the arcs with the same name. Both of these curves are strictly shorter than \(\alpha_2\). Now we distinguish two cases.

Case 1: \(\delta_0\) and \(\alpha_1\) do not have two consecutive (along \(\delta_0\)) intersections with the same sign (this includes the case when \(\delta_0\) and \(\alpha_1\) have one, or zero, intersection point). Then the algebraic intersection between \(\delta_0\) and \(\alpha_1\) has absolute value zero or one. Therefore one of the curves \(\delta_1\delta_0\) and \(\delta_2\delta_0\) has algebraic intersection \(\pm 1\) with \(\alpha_1\). Since it is shorter than \(\alpha_2\), this contradicts the minimality of \(\alpha_2\).

Case 2: \(\delta_0\) and \(\alpha_1\) have two intersection points \(p_3\) and \(p_4\), consecutive along \(\delta_0\), with the same sign. Let \(\alpha_3\) be the closed curve obtained by joining the arc of \(\delta_0\) between \(p_3\) and \(p_4\), with an arc of \(\alpha_1\), of length \(\leq l_1/2\) (see Fig. 2). Then
\[
l_g(\alpha_3) \leq l_g(\delta_0) + \frac{l_1}{2} \leq D + \frac{l_1}{2} < \frac{l_2}{2} + \frac{l_1}{2} \leq l_2.
\]

On the other hand, \(\alpha_3\) is homotopic to a closed curve which intersects \(\alpha_1\) exactly once. Therefore it is homotopic to a closed geodesic \(c\), such that \(|\text{Int}(c, \alpha_1)| = |\text{Int}(\alpha_3, \alpha_1)| = 1\) (see [6], proof of Theorem 17.3.1), which is

\[\]
shorter than $\alpha_2$, a contradiction. \qed

In the following we will obtain an upper bound on $K$. To this end we prove the following proposition.

**Proposition 3.4.** Let

- $(M, g)$ be a closed, oriented Riemannian surface
- $l_1$ be the length of a homological systole of $(M, g)$
- $\alpha, \beta$ be simple closed geodesics in $(M, g)$.

Then

$$\frac{|\text{Int}(\alpha, \beta)|}{l_g(\alpha)l_g(\beta)} \leq \frac{9}{l_1^2}. \quad (9)$$

**Proof.** Take a real number $r < l_1/2$. We cut $\alpha$ and $\beta$ into segments of length $r$ and at most one segment of smaller length. Let $n_\alpha$ and $n_\beta$ be the respective numbers of segments obtained. Let $I$ and $J$ be a pair of these segments in $\alpha$ and $\beta$ respectively. We shall prove, by contradiction, that the algebraic intersection of $I$ and $J$ is at most one. Assume to the contrary. Then there exist two intersection points $p$ and $q$ of $I$ and $J$, consecutive along $I$, such that the intersections of $I$ and $J$ at $p$ and $q$ have the same sign. Let $\gamma$ be the closed curve formed by subsegments of $I$ and $J$ glued at $p$ and $q$. Then $\gamma$ is homotopic to a curve $\gamma'$ which intersects $\alpha$ exactly once (see Fig. 3).

In particular $\gamma'$ is non-separating. On the other hand, the length of $\gamma'$ is $\leq 2r < l_1$, which contradicts the definition of $l_1$. We have proven the inequality

$$|\text{Int}(\alpha, \beta)| \leq n_\alpha n_\beta. \quad (10)$$

By construction

$$(n_\alpha - 1)r \leq l_g(\alpha) \leq n_\alpha r \quad \text{and} \quad (n_\beta - 1)r \leq l_g(\beta) \leq n_\beta r$$

so

$$\frac{n_\alpha n_\beta}{l_g(\alpha)l_g(\beta)} \leq \left( \frac{1}{r} + \frac{1}{l_g(\alpha)} \right) \left( \frac{1}{r} + \frac{1}{l_g(\beta)} \right) \leq \left( \frac{1}{r} + \frac{1}{l_1} \right)^2.$$

Substituting this into Equation (10), and since $r$ is arbitrarily close to $l_1/2$, we obtain the claim. \qed
Summarizing Equations (4, 8 and 9), we obtain the following bounds.

**Corollary 3.5.** Let

- \((M, g)\) be a closed, oriented Riemannian surface
- \(D := \text{diam}(M, g)\) be its diameter
- \(l_1\) be the length of a homological systole of \((M, g)\)
- \(V := \text{Vol}(M, g)\) be the total volume of \((M, g)\)
- \(K := K(M, g)\).

Then

\[
\frac{1}{V} \leq K \quad \text{and} \quad \frac{1}{2l_1D} \leq K \leq \frac{9}{l_1^2}.
\]

4. **Surfaces of constant negative curvature: statement of Theorem 4.2**

From now on we assume that the genus of \(M\) is \(\geq 2\) and the metric \(g\) has curvature \(-1\) everywhere. Recall that we denote by \(\mathcal{M}_s\) the moduli space of surfaces of genus \(s\), that is, the set of metrics of curvature \(-1\) on \(M\), modulo isometries.

Let \(\eta\) be a simple closed geodesic on \(M\). Let \(\omega_\eta\) be the supremum of all \(w\), such that the geodesic arcs of length \(w\) emanating perpendicularly from \(\eta\) are pairwise disjoint. A **collar** around \(\eta\) or **cylinder** of width \(w < \omega_\eta\), \(C_w(\eta)\), is defined by

\[
C_w(\eta) := \{p \in M \mid \text{dist}(p, \eta) < w\}.
\]

By [4], Theorem 4.3.2 we have:

**Theorem 4.1** (Collar theorem). Let \(M\) be a closed, oriented surface of genus \(s \geq 2\), endowed with a metric \(g\) of curvature \(-1\). Let \(\eta\) be a simple closed
geodesic in \((M, g)\). Then
\[
\omega_\eta \geq cl(l_\eta(\eta)) := \text{arsinh} \left( \frac{1}{\sinh \left( \frac{l_\eta(\eta)}{2} \right)} \right).
\]
If \(\delta\) is another simple closed geodesic that does not intersect \(\eta\), then \(C_{cl(l_\eta(\eta))}(\eta)\) and \(C_{cl(l_\delta(\delta))}(\delta)\) are disjoint.

The main result of this section is:

**Theorem 4.2.** Let
- \(M\) be a closed, oriented surface \(M\) of genus \(s \geq 2\)
- \(g\) be a metric of curvature \(-1\) on \(M\)
- \(\alpha_1\) be a homological systole of \((M, g)\)
- \(l_1 := l_g(\alpha_1)\) be its length.

We have
\[
\left( (s-1) \cdot l_1 \cdot (105s + 4 \text{arsinh} \left( \frac{4}{l_1} \right)) \right)^{-1} < K(M, g) \leq 144 + \frac{18(s-1)}{l_1 \cdot cl(l_1)}.
\]

**Remark 4.3.** Since \(\text{arsinh} (4/l_1)\) and \(cl(l_1)\) are equivalent to \(-\log(l_1)\), when \(l_1\) goes to zero, Theorem 4.2 implies Theorem 1.6.

This means that \(K(M, g)\) tends to infinity if and only if \(l_1\), the length of a homological systole of \((M, g)\) goes to zero. In particular \(K(M, g)\) is unbounded.

4.1. **Proof of the lower bound in Theorem 4.2.** By [5] there exist simple closed geodesics \(\beta_1 = \alpha_1\) and \(\beta_2, \ldots, \beta_{2s}\) such that
- \([\beta_1], \ldots, [\beta_{2s}]\) is a basis of \(H_1(M, \mathbb{R})\) as a vector space
- \(\text{Int}([\beta_{2i-1}], [\beta_{2i}]) = 1\) for all \(i = 1, \ldots s\)
- \(\forall i = 1, \ldots, 2s, l_g(\beta_i) < (s-1) \left( 105s + 4 \text{arsinh} \left( \frac{4}{l_1} \right) \right)\).

In particular
\[
l_2 = l_g(\beta_2) < (s-1) \left( 105s + 4 \text{arsinh} \left( \frac{4}{l_1} \right) \right),
\]
As \(\text{Int}([\alpha_1], [\beta_2]) = 1\), we get:
\[
K(M, g) \geq \frac{1}{l_1 l_2} > \frac{1}{l_1 (s-1) \left( 105s + 4 \text{arsinh} \left( \frac{4}{l_1} \right) \right)}.
\]

4.2. **Preliminaries to the proof of the upper bound in Theorem 4.2.** Let \((\alpha_i)_{i=1,\ldots,k}\) be the set of non-separating simple closed geodesics, such that
\[
l_g(\alpha_i) < \frac{1}{4} < 2 \text{arsinh}(1).
\]
It follows from [4], Theorem 4.1.1 that \(k \leq 3s - 3\). For \(i = 1, \ldots, k\) we set \(w_i := cl(l_g(\alpha_i)) - 1.3\),
\[
C_i := C_{cl(l_g(\alpha_i))}(\alpha_i) \quad \text{and} \quad B_i := C_{w_i}(\alpha_i) \subset C_i \quad (\text{see Fig. 4}).
\]
Let furthermore $\partial_1 B_i$ and $\partial_2 B_i$ be the connected boundary components of $B_i$ and let $b_i$ be a geodesic arc realizing the distance between these two boundaries. We will gather some useful facts about the geometry of these collars in the following:

![Figure 4](image_url)

**Figure 4.** An embedded collar $C_i = C_{cl(l_g(\alpha_i))}(\alpha_i)$ of a short non-separating simple closed geodesic $\alpha_i$.

- By the collar theorem the $(C_i)_{i=1,\ldots,k}$ are pairwise disjoint.
- For $x \leq 2 \arcsinh(1)$ we have: $cl(x)$ is a monotonically decreasing function and $x \leq 2cl(x)$.
- $l_g(\partial_1 B_i) = l_g(\alpha_i) \cdot \cosh(w_i) = l_g(\alpha_i) \cdot \cosh(cl(l_g(\alpha_i)) - 1.3)$.
- As $l_g(\alpha_i) < \frac{1}{4}$ it follows from this formula and the definition of $cl(l_g(\alpha_i))$ that
  \[ l_g(b_i) = 2 \cdot (cl(l_g(\alpha_i)) - 1.3) > 5l_g(\partial_1 B_i) \quad \text{and} \]
  \[ l_g(\partial_1 B_i) > \frac{1}{2} > 2l_g(\alpha_i). \quad \text{(see Fig. 5)} \]

Let $\epsilon > 0$ be a positive real number and let $\gamma$ and $\delta$ be two simple closed geodesics, such that

\[ K(M, g) \leq \frac{|\text{Int}([\gamma], [\delta])|}{l_g(\gamma) \cdot l_g(\delta)} + \epsilon. \]

Set $N := |\text{Int}([\gamma], [\delta])|$. Replacing, if necessary, $\gamma$ and $\delta$ by shorter curves whose algebraic intersection is $N$, we may assume that $\gamma$ and $\delta$ minimize the product $l_g(\gamma) \cdot l_g(\delta)$ among all pairs of curves whose algebraic intersection is $N$. Then Lemma 3.2 tells us that all intersections of $\gamma$ and $\delta$ have the same sign, that is, the algebraic intersection of $\gamma$ and $\delta$ coincides with the number of their intersection points. We will obtain our result by distinguishing two cases: either $\gamma$ or $\delta$ is one of the $(\alpha_i)_{i=1,\ldots,k}$, or neither $\gamma$ nor $\delta$ is one of the $(\alpha_i)_{i=1,\ldots,k}$.
5. Proof of Theorem 4.2., Case 1: Neither $\gamma$ nor $\delta$ is one of the $(\alpha_i)_{i=1,..,k}$

Set

$$M_1 = \bigcup_{i=1,..,k} B_i \quad \text{and} \quad M_2 = M \setminus \bigcup_{i=1,..,k} B_i.$$  

For $i \in \{1, 2\}$, let $N_i := \#\{ p \in M_i : p \in \{ \gamma \cap \delta \} \}$ be the number of intersection points of $\gamma$ and $\delta$ in $M_i$. We have:

$$K(M, g) - \epsilon \leq \frac{N_1 + N_2}{l_g(\gamma) \cdot l_g(\delta)} = \frac{N_1}{l_g(\gamma) \cdot l_g(\delta)} + \frac{N_2}{l_g(\gamma) \cdot l_g(\delta)} = K_1 + K_2,$$

where $K_1$ and $K_2$ are defined by

$$K_1 := \frac{N_1}{l_g(\gamma) \cdot l_g(\delta)} \quad \text{and} \quad K_2 := \frac{N_2}{l_g(\gamma) \cdot l_g(\delta)}.$$  

Here we use the fact that the algebraic intersection of $\gamma$ and $\delta$ coincides with the number of their intersection points. We will establish two independent upper bounds on $K_1$ and $K_2$ to prove our theorem.

5.1. Upper bound on $K_1$. To find an upper bound on $K_1$, we establish bounds on each $B_i$. Denote by

- $N_i^1 := \#\{ p \in B_i : p \in \{ \gamma \cap \delta \} \}$ the number of intersection points of $\gamma$ and $\delta$ in $B_i$
- $K_1^i := \frac{N_i^1}{l_g(\gamma) \cdot l_g(\delta)}$
It follows that

\[ K_1 = \frac{N_1}{l_g(\gamma) \cdot l_g(\delta)} = \sum_{i=1}^k \frac{N_i}{l_g(\gamma) \cdot l_g(\delta)} = \sum_{i=1}^k K_i. \]

Now fix an index \( i \) and let us work in the cylinder \( B_i \). Let

\[ (\gamma_j)_{j=1,\ldots,n_1} = \gamma \cap B_i \quad \text{and} \quad (\delta_l)_{l=1,\ldots,n_2} = \delta \cap B_i \]

be the disjoint union of geodesic arcs of \( \gamma \) and \( \delta \), respectively, which traverse \( B_i \). We have:

\[ K_1^i = \frac{N_i^i}{l_g(\gamma) \cdot l_g(\delta)} < \frac{\sum_{j,l} \# \{ \gamma_j \cap \delta_l \}}{l_g(\gamma \cap B_i) \cdot l_g(\delta \cap B_i)} = \frac{\sum_{j,l} \# \{ \gamma_j \cap \delta_l \}}{\sum_{j,l} l_g(\gamma_j) \cdot l_g(\delta_l)}. \]

Now we may assume without loss of generality that

\[ \frac{\# \{ \gamma_1 \cap \delta_1 \}}{l_g(\gamma_1) \cdot l_g(\delta_1)} = \max_{j,l} \frac{\# \{ \gamma_j \cap \delta_l \}}{l_g(\gamma_j) \cdot l_g(\delta_l)}. \]

It follows that

\[ K_1^i < \frac{\sum_{j,l} \# \{ \gamma_j \cap \delta_l \} l_g(\gamma_j) \cdot l_g(\delta_l)}{\sum_{j,l} l_g(\gamma_j) \cdot l_g(\delta_l)} = \frac{\# \{ \gamma_1 \cap \delta_1 \}}{l_g(\gamma_1) \cdot l_g(\delta_1)}. \]

We now determine an upper bound on the intersection number and a lower bound on the length of \( \gamma_1 \) and \( \delta_1 \). To this end we will define the winding number of an arc traversing \( B_i \) and prove two lemmas concerning the intersection of two such geodesic arcs. Then we will provide a lower bound on the length of an arc.

The next three lemmata will also be used in the proof of the upper bound on \( K_2 \), where we shall need to apply them to pairs of geodesic arcs, which are not necessarily formed by an arc of \( \gamma \) and an arc of \( \delta \) traversing a cylinder \( B_i \).

**Lemma 5.1.** Let \( c \) be a geodesic arc traversing a cylinder \( B_i \). Let \( d \neq c \) be another geodesic arc traversing \( B_i \) or let \( d \) be the simple closed geodesic \( \alpha_i \). Then \( c \) and \( d \) intersect under the same sign at any intersection point.

**Proof.** By contradiction: Assume that there exists a geodesic arc \( c^1 \) of \( c \) with endpoints \( p_1 \) and \( p_2 \) on \( d \), consecutive along \( d \), such that the sign of the intersection at \( p_1 \) is different from the sign at \( p_2 \). Let \( d^1 \) be the geodesic arc on \( d \) connecting \( p_1 \) and \( p_2 \) in \( B_i \). Consider a lift of \( c^1 \) to the hyperbolic plane. By abuse of notation we denote the lift of \( c^1 \) by the same symbol. We also denote the lifts of \( p_1 \) and \( p_2 \) on \( c^1 \) by the same symbols. Let \( d^1 \) be the lift of \( d^1 \) intersecting \( c_1 \) at \( p_1 \) in the hyperbolic plane. Now, due to the topology of the cylinder, \( p_2 \) lies also on \( d^1 \). Hence in the hyperbolic plane \( p_1 \) and \( p_2 \) are connected by the two different geodesic arcs \( c^1 \) and \( d^1 \). These arcs belong to two different geodesics passing through \( p_1 \) and \( p_2 \). But in the hyperbolic plane there can be only one geodesic through any two distinct points (see [H], Theorem...
Let $c$ be a geodesic arc traversing $B_i$. With respect to its fixed endpoints on $\partial B_i$, $c$ is in the homotopy class 
$$[c] = [b' \cdot a \cdot b''].$$ 
Here $b'$ and $b''$ are directed geodesic arcs that meet $\alpha_i$ perpendicularly on opposite sides of $\alpha_i$, and $a$ is a directed arc on $\alpha_i$. We define the orientation $\sigma(c)$ of $c$ with respect to $\alpha_i$ as

$$\sigma(c) := \begin{cases} +1 & \text{if the orientation of } a \text{ agrees with that of } \alpha_i \\ 0 & \text{if } a \text{ is a single point} \\ -1 & \text{if the orientation of } a \text{ disagrees with that of } \alpha_i, \end{cases}$$

and the winding number of $c$ as

$$\tilde{c} := \sigma(c) \frac{l_g(a)}{l_g(\alpha_i)} = \sigma(a) \frac{l_g(a)}{l_g(\alpha_i)}$$

so we have $\sigma(c) = \text{sgn}(\tilde{c})$, where $\text{sgn}(\cdot)$ denotes the sign function.

Note that due to Lemma 5.1 an arc $c$ has only one intersection point with $\alpha_i$. Denote by the floor function the mapping $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}$, defined by $\lfloor x \rfloor := \max\{z \in \mathbb{Z} : z \leq x\}$ for all $x \in \mathbb{R}$. The number of intersection points of two geodesic arcs traversing a cylinder $B_i$ and the sign of their intersections are related to the winding numbers of these arcs in the following way:

**Lemma 5.2.** Let $c$ and $d$ be two geodesic arcs traversing a cylinder $B_i$. Let $\tilde{c}$ and $\tilde{d}$ be the winding numbers of the arcs $c$ and $d$, respectively. If $c$ and $d$ intersect $\alpha_i$ under the same sign, then

$$|\tilde{d} - \tilde{c}| \leq \#\{c \cap d\} \leq |\tilde{d} - \tilde{c}| + 1.$$

Furthermore $\text{sgn}(\tilde{d} - \tilde{c})$ determines the sign of the intersection of $c$ and $d$ at any intersection point.

If $c$ and $d$ intersect $\alpha_i$ under different signs, then

$$|\tilde{d} + \tilde{c}| \leq \#\{c \cap d\} \leq |\tilde{d} + \tilde{c}| + 1.$$

Furthermore $\text{sgn}(\tilde{d} + \tilde{c})$ determines the sign of the intersection of $c$ and $d$ at any intersection point.

**Proof.** We first consider the case, where $c$ and $d$ intersect $\alpha_i$ from the same side.

**Case 1:** $\text{Int}(c, \alpha_i) = \text{Int}(d, \alpha_i)$

We first treat the case where $\tilde{c} = 0$.

By Lemma 5.1, $c$ and $d$ always intersect under the same sign at any intersection point. Hence we have

$$\#\{c \cap d\} = |\text{Int}(c, d)|.$$

If $\tilde{c} = 0$, then $c$ intersects $\alpha_i$ perpendicularly. Now $d$ winds at least $|\tilde{d}|$ times around $B_i$ and intersects $c$ at least $|\tilde{d}|$ times. There might be an additional
intersection point, but not more than one, that is,

\[ |\tilde{d}| \leq \#\{c \cap d\} = |\text{Int}(c, d)| \leq |\tilde{d}| + 1.\]

Furthermore, the sign of the intersection of \(c\) and \(d\) at any intersection point is determined by the orientation \(\sigma(d)\) of \(d\), that is,

\[ \sigma(d) = \text{sgn}(\tilde{d}) = \begin{cases} 
\text{sgn}(\text{Int}(c, d)) & \text{if } \text{Int}(c, \alpha_i) = \text{Int}(d, \alpha_i) = +1 \\
-\text{sgn}(\text{Int}(c, d)) & \text{if } \text{Int}(c, \alpha_i) = \text{Int}(d, \alpha_i) = -1.
\end{cases} \]

Therefore our lemma is true if \(\tilde{c} = 0\).

The intersection number of two curves only depends on the homotopy class with fixed endpoints of the curves. Let \(\eta\) and \(\mu\) be two curves with fixed endpoints and with homotopy classes \([\eta] = [c]\) and \([\mu] = [d]\). If \(\tilde{c} = 0\) we obtain the following result:

\begin{align*}
(14) \quad |\tilde{d}| & \leq |\text{Int}(\eta, \mu)| \leq |\tilde{d}| + 1 \quad \text{and} \\
(15) \quad \text{sgn}(\tilde{d}) = \begin{cases} 
\text{sgn}(\text{Int}(\eta, \mu)) & \text{if } \text{Int}(\eta, \alpha_i) = \text{Int}(\mu, \alpha_i) = +1 \\
-\text{sgn}(\text{Int}(\eta, \mu)) & \text{if } \text{Int}(\eta, \alpha_i) = \text{Int}(\mu, \alpha_i) = -1.
\end{cases}
\end{align*}

If \(\tilde{c} \neq 0\), we apply a Dehn twist to the cylinder, which we define in the following via Fermi coordinates.

We recall that \(w_i = \frac{t_i(b_i)}{2}\). The \textit{Fermi coordinates} with base point \(p_1 := \alpha_i(0)\) are an injective parametrization

\[ \psi : \mathbb{R} \mod \{x \mapsto x + l_3(\alpha_i)\} \times (-w_i, w_i) \to B_i, \psi : (t, s) \mapsto \psi(t, s), \]

such that

\begin{itemize}
\item \(\psi(0, 0) = \alpha_i(0) = p_1\) and \(\psi(t, 0) = \alpha_i(t)\), for all \(t\)
\item \(s \mapsto \psi(t, s)\) is an arc-length parametrization of an oriented geodesic arc \(b_i\) that intersects \(\alpha_i\) perpendicularly in \(\alpha_i(t)\)
\item \(\text{Int}(b_i, \alpha_i) = +1\).
\end{itemize}

Let \(z \in \mathbb{R}\) be a real number. A Dehn twist of order \(z\), \(\mathcal{D}_z : B_i \to B_i\) is defined in Fermi coordinates by \(\mathcal{D}_z(\psi(t, s)) := \psi\left(t + z\frac{w_i + s}{2w_i}, s\right)\).

Let

\[ \mathcal{D} := \mathcal{D}_{-\text{Int}(c, \alpha_i)\tilde{c}} \]

be the Dehn twist of order \(-\text{Int}(c, \alpha_i)\tilde{c}\). Then the winding number \(\tilde{c}'\) of the geodesic arc \(c'\) in the homotopy class (with fixed extremities) of \(\mathcal{D}(c)\) is \(0\). The winding number \(\tilde{d}'\) of the geodesic arc \(d'\) in the homotopy class of \(\mathcal{D}(d)\) is \(\tilde{d} - \tilde{c}\).

Since \(\mathcal{D}\) is isotopic to the identity, we have \(\text{Int}(c, d) = \text{Int}(\mathcal{D}(c), \mathcal{D}(d))\). By (14) we have

\[ |\tilde{d}'| \leq \#\{c' \cap d'\} = |\text{Int}(c', d')| \leq |\tilde{d}| + 1 \]

that is,

\[ |\tilde{d} - \tilde{c}'| \leq \#\{c \cap d\} = |\text{Int}(c, d)| \leq |\tilde{d} - \tilde{c}| + 1. \]

It follows furthermore from (15) that \(\text{sgn}(\tilde{d}') = \text{sgn}(\tilde{d} - \tilde{c})\) determines the sign of the intersection of \(d\) and \(c\) at any intersection point. This completes the proof in the case, where here \(c\) and \(d\) intersect \(\alpha_i\) from the same side.

We now consider the case, where \(c\) and \(d\) intersect \(\alpha_i\) from different sides.

\textbf{Case 2:} \(\text{Int}(c, \alpha_i) = -\text{Int}(d, \alpha_i)\)
In this case let \( c^{-1} \) be the geodesic that coincides pointwise with \( c \), but which traverses \( B_i \) in the opposite sense. Let \( \tilde{c}^{-1} \) be the winding number of the arc \( c^{-1} \). We have that

\[
\tilde{c}^{-1} = -\tilde{c}, \quad \text{therefore} \quad \tilde{d} - \tilde{c}^{-1} = \tilde{d} + \tilde{c}.
\]

As \( \text{Int}(c, d) = -\text{Int}(c^{-1}, d) \) we have that

\[
\#\{c \cap d\} = \#\{c^{-1} \cap d\} \quad \text{and} \quad \text{sgn}(\text{Int}(c, d)) = -\text{sgn}(\text{Int}(c^{-1}, d)).
\]

As \( \text{Int}(c^{-1}, \alpha_i) = \text{Int}(d, \alpha_i) \), we can apply the result from Case 1. Therefore the statement for Case 2 follows from Case 1. This completes the proof of Lemma 5.2.

We now give two lower bounds for the length of a geodesic arc traversing a cylinder \( B_i \).

**Lemma 5.3.** Let \( c \) be a geodesic arc traversing a cylinder \( B_i \) and let \( \tilde{c} \) be its winding number. We have:

\[
\log(c) \geq 2 \cdot (\log(l_g(\alpha_i)) - 1.3) \quad \text{and} \quad \log(c) \geq |\tilde{c}| \cdot \log(\alpha_i).
\]

**Proof.** We lift \( c \) to \( c^* \) in the hyperbolic plane (see Fig. 6). Let \( \alpha_i'' \) be the lift of \( \alpha_i \) and let \( b_i' \) be a lift of \( b_i \) (see Fig. 4). Let \( B_i' \) be a fundamental domain of \( B_i \), whose boundary is \( b_i' \). Denote by \( c^p \) the arc which we obtain from the orthogonal projection of \( c^* \) onto \( \alpha_i'' \) in the hyperbolic plane. Let \( q \) be the midpoint of \( c^* \). Here, due to the symmetry of the situation, the midpoint of \( c^* \) lies on \( \alpha_i'' \) and is also the midpoint of \( c^p \). Let \( T \) be a triangle with vertices \( q \), an endpoint of \( c^* \) and an endpoint of \( c^p \) (see Fig. 6).
It follows from the geometry of the hyperbolic right-angled triangle $T$ (see [1], p. 454) that
\[ \cosh\left(\frac{l_g(c^*)}{2}\right) = \cosh\left(\frac{l_g(b'_i)}{2}\right) \cdot \cosh\left(\frac{l_g(c^p)}{2}\right). \]
As $l_g(b_i) = l_g(b'_i)$ and $l_g(c) = l_g(c^*)$ we obtain from the above equation that
\[ \cosh\left(\frac{l_g(b_i)}{2}\right) = \cosh\left(\frac{l_g(b'_i)}{2}\right) \cdot \cosh\left(\frac{l_g(c^p)}{2}\right). \]
As \( \cosh \) is a strictly increasing function on \( \mathbb{R}^+ \) and as \( \cosh(0) = 1 \), it follows from the above equation that
\[ \cosh\left(\frac{l_g(b_i)}{2}\right) \geq \cosh\left(\frac{l_g(b'_i)}{2}\right) \text{ and } \cosh\left(\frac{l_g(c^p)}{2}\right) \geq \cosh\left(\frac{l_g(c^p)}{2}\right). \]
From these two inequalities we obtain again by the monotonicity of the \( \cosh \) function on \( \mathbb{R}^+ \) that
\[ l_g(c) \geq l_g(b_i) = 2 \cdot (cl(l_g(\alpha_i)) - 1.3) \text{ and } l_g(c) \geq l_g(c^p). \]
Here the first inequality is the first inequality of the lemma. It follows furthermore from the definition of the winding number that
\[ |\tilde{c}| \cdot l_g(\alpha_i) = l_g(c^p) \leq l_g(c), \]
which proves the second inequality in Lemma 5.3.

We will denote in the following the winding number of an arc $\gamma_j$ by $\tilde{c}_j$ and the winding number of an arc $\delta_i$ by $\tilde{d}_i$. Now let $m_1, m'_1 \in \mathbb{N}$ be the natural numbers such that
\[ m_1 = |\tilde{c}| + 1 \text{ and } m'_1 = |\tilde{d}| + 1. \]
We may assume without loss of generality that $m'_1 \leq m_1$. It follows from Lemma 5.2 that
\[ \#\{\gamma_1 \cap \delta_1\} \leq m'_1 + m_1 \leq 2m_1. \]
It follows from the definition of $m_1$ and Lemma 5.3 that
\begin{align*}
(16) l_g(\gamma_1) & \geq |\tilde{c}| \cdot l_g(\alpha_i) \geq (m_1 - 1)l_g(\alpha_i) \text{ and } l_g(\gamma_1) \geq l_g(b_i) > l_g(\alpha_i) \\
(17) l_g(\delta_1) & \geq l_g(b_i) = 2(cl(l_g(\alpha_i)) - 1.3).
\end{align*}
Here the inequality $l_g(b_i) > l_g(\alpha_i)$ in (16) follows by combining the two inequalities in (11). Using the above inequalities we obtain from inequality (13) for $m_1 \geq 2$:
\[ K_1^i \leq \frac{\#\{\gamma_1 \cap \delta_1\}}{l_g(\gamma_1) \cdot l_g(\delta_1)} \leq \frac{2m_1}{(m_1 - 1) \cdot l_g(\alpha_i) \cdot l_g(\delta_1)} \leq \frac{2}{l_g(\alpha_i) (cl(l_g(\alpha_i)) - 1.3)}.
\]
If $m_1 = 1$, we use the second inequality in (16) to derive the same upper bound.

Combining the estimates for the $K^i_j$ we obtain that
\[ K_1 \leq \sum_{i=1}^{k} K_1^i \leq \sum_{i=1}^{k} \frac{2}{l_g(\alpha_i) (cl(l_g(\alpha_i)) - 1.3)} < \sum_{i=1}^{k} \frac{6}{l_g(\alpha_i) \cdot cl(l_g(\alpha_i))}. \]
To prove the last inequality we have to show that
\[ cl(l_g(\alpha_i)) - 1.3 > \frac{cl(l_g(\alpha_i))}{3} \] that is, \[ cl(l_g(\alpha_i)) > \frac{3}{2} \cdot 1.3. \]
We obtain this result by combining the two inequalities \( 2 \cdot (cl(l_g(\alpha_i)) - 1.3) > 5l_g(\partial_1 B_i) \) and \( l_g(\partial_1 B_i) > \frac{1}{2} \) in (11).

As the function \( \frac{1}{x \cdot cl(x)} \) is monotonously decreasing in the interval \([0, 2 \text{arsinh}(1)]\) (see Fig. 7), we obtain from the above inequality for \( K_1 \) that
\[
(18) \quad K_1 \leq \sum_{i=1}^{k} \frac{6}{l_g(\alpha_i) \cdot cl(l_g(\alpha_i))} \leq \sum_{i=1}^{k} \frac{6}{l_g(\alpha_1) \cdot cl(l_g(\alpha_1))} \leq \frac{18s - 18}{l_g(\alpha_1) \cdot cl(l_g(\alpha_1))}.
\]
Here the last inequality follows from the fact that we have at most \( 3s - 3 \) cylinders \( B_i \).

5.2. Upper bound on \( K_2 \). We recall that
\[ K_2 = \frac{N_2}{l_g(\gamma) \cdot l_g(\delta)}, \quad \text{where} \quad N_2 = \# \{ p \in M_2 = M \setminus \bigcup_{i=1,...,k} B_i : p \in \{ \gamma \cap \delta \} \}.
\]

To obtain an upper bound on \( K_2 \) we now construct a comparison surface \( M' \) from \( M \) such that
\[
(19) \quad l_g(\alpha_1') > \frac{1}{4},
\]
where \( \alpha_1' \) is the shortest non-separating simple closed geodesic in \( M' \). Then we construct two comparison curves \( \gamma' \subset M' \) and \( \delta' \subset M' \), such that
\[
l_g(\gamma') \leq l_g(\gamma), \quad l_g(\delta') \leq l_g(\delta), \quad \text{and} \quad N_2 \leq |\text{Int}([\gamma'], [\delta'])|.
\]
Now for every $\epsilon > 0$, we can approximate our non-smooth surface $(M', g')$ with a smooth surface $(M_\epsilon, g_\epsilon)$ such that the distance function of $(M_\epsilon, g_\epsilon)$ is $\epsilon$-close to that of $(M', g')$. It follows from this remark and by applying Proposition 3.4 to $(M_\epsilon, g_\epsilon)$ that

$$K_2 = \frac{N_2}{l_g(\gamma) \cdot l_g(\delta)} \leq \frac{|\text{Int}([\gamma'], [\delta'])|}{l_g(\gamma') \cdot l_g(\delta')} \leq K(M', g') \leq \frac{9}{l_g(\alpha')^2} \leq 144. \quad (*)$$

5.2.1. **Construction of $(M', g')$.** We construct a surface $(M', g')$ with a singular Riemannian metric in the following way. We cut out all collar $B_i$ from $M$ and then reconnect the open ends. Here we identify the sides in the following way. For all $i \in \{1, \ldots, k\}$, let $J_i$ be an isotopy

$$J_i : B_i \times [0, 1] \to B_i,$$

such that $J_i(\cdot, 0) = \text{id}$ and $J_i(B_i, 1) = \partial B_i$.

Here $J_i$ should satisfy the following condition. For all $t \in [0, 1]$ and all $p_1 \in \partial_i B_i$

$$J_i(p_1, t) \in b_{p_1} \subset B_i,$$

where $b_{p_1}$ is a geodesic arc in $B_i$ with endpoint $p_1$ and that intersects $\alpha_i$ perpendicularly. We define

$$M' := M \setminus \bigcup_{i=1}^{k} B_i \mod \{J_i(p_1, 0) = J_i(p_1, 1), \text{ for all } p_1 \in \partial_i B_i, i \in \{1, \ldots, k\}\}.$$  

We call $\partial B_i$ the image of $\partial_i B_i$ in $M'$.

Now we have to show that the length of a non-separating simple closed curve $\eta$ in $M'$ is bigger than $\frac{1}{4}$. We distinguish two cases: either $\eta$ intersects $\bigcup_{i=1}^{k} \partial B_i$, or not.

Consider first the case, where $\eta$ does not intersect a $\partial B_i$ in $M'$. Now the $(\alpha_i)_{i=1, \ldots, k}$ are the non-separating simple closed geodesics in $M$, such that $l_g(\alpha_i) < \frac{1}{4}$. In $M$ all non-separating simple closed curves of length smaller than $\frac{1}{4}$ are contained in the union $\bigcup_{i=1}^{k} B_i$ of cylinders $B_i$. As $\eta \subset M$ does not intersect this set, we have that

$$l_g(\eta) > \frac{1}{4}.$$  

Any simple closed curve $\eta$ in $M'$ that intersects a $\partial B_i$ either intersects a boundary of $C_i \setminus B_i$ or is contained in $C_i \setminus B_i$. In the first case $\eta$ is longer than the distance between a boundary of $C_i$ and $\partial B_i$. In the second case $\eta$ is either contractible, or is freely homotopic to $\partial B_i$, which is the shortest curve in its free homotopy class in $M'$. As the $(B_i)_{i=1, \ldots, k}$ are chosen in a way such that for all $i \in \{1, \ldots, k\}$

$$\text{dist}(\partial_i C_i, \partial_i B_i) = 1.3 \text{ and } l_g(\partial B_i) > \frac{1}{2} \text{ (see (11))},$$

we have in any case that

$$l_g(\eta) \geq \frac{1}{2}.$$  

Summarizing these cases we obtain that the length of any non-separating simple closed curve in $M'$ is bigger than $\frac{1}{4}$. Therefore inequality (19) holds. This proves our upper bound on $K_2$ in (20).
5.2.2. Construction of $\gamma'$ and $\delta'$. First we construct two comparison curves $\gamma' \subset M'$ and $\delta' \subset M'$, such that

\[ l_g(\gamma') \leq l_g(\gamma), \quad l_g(\delta') \leq l_g(\delta) \quad \text{and} \quad N_2 \leq |\text{Int}(\gamma', [\delta'])| \]

To this end we will replace all arcs of $\gamma$ and $\delta$ traversing a cylinder $C_i$ with shorter arcs in $C_i \setminus B_i$. Proceeding this way with all $(C_i)_{i=1,..,k}$, we obtain the comparison curves $\gamma'$ and $\delta'$ from $\gamma$ and $\delta$, respectively.

Let in the following $C_i$ be a fixed cylinder. Before we present the construction, we will first gather some information about the way $\gamma$ and $\delta$ intersect in $B_i \subset C_i$. We recall that $(\gamma_j)_{j=1,..,n_1} = \gamma \cap B_i$ and $(\delta_l)_{l=1,..,n_2} = \delta \cap B_i$ are the arcs of $\gamma$ and $\delta$ traversing $B_i$. The following lemma shows that all arcs of $\gamma$ and all arcs of $\delta$ intersect $\alpha_i$ under the same sign. More precisely:

**Lemma 5.4.** Let

- $\gamma_m$ and $\gamma_j$ be two distinct arcs of $\gamma$ traversing $B_i$
- $\delta_k$ and $\delta_l$ be two distinct arcs of $\delta$ traversing $B_i$.

Then

\[ \text{Int}(\gamma_m, \alpha_i) = \text{Int}(\gamma_j, \alpha_i) \quad \text{and} \quad \text{Int}(\delta_k, \alpha_i) = \text{Int}(\delta_l, \alpha_i). \]

**Proof.** We will prove the statement by contradiction. Recall that we assume that $\gamma$ is the shortest simple closed geodesic, such that $|\text{Int}(\gamma, [\delta])| = N$. Consider the arcs $(\gamma_j)_{j=1,..,n_1}$ of $\gamma$ traversing $B_i$. We assume that there exist two such arcs that intersect $\alpha_i$ under a different sign. Let without loss of generality $\gamma_1$ and $\gamma_2$ be these two arcs. As $\gamma$ is a simple closed geodesic, it has no self-intersection and these two arcs can not intersect. We can therefore connect the endpoints of $\gamma_1$ and $\gamma_2$ on $\partial_1 B_i$ with an arc $\eta_1$ on $\partial_1 B_i$ and the endpoints of these two
arcs on $\partial B_i$ with an arc $\gamma_2$ on $\partial B_i$, such that together with the arcs $\gamma_1$ and $\gamma_2$ the arcs form a simple closed curve $\gamma$ (see Fig. 8). As $\gamma$ is contractible to a point, $[\eta] = 0$. Hence replacing $\gamma_1$ and $\gamma_2$ of $\gamma$ with directed arcs corresponding to $\eta_1$ and $\eta_2$, but with inverse direction, we can create a new curve $\tilde{\gamma}$, such that $[\gamma] = [\tilde{\gamma}]$. But it follows from the first inequality in Lemma 5.3 and (11) that

$$l_g(\gamma_j) \geq 2(c l_g(\alpha_i)) - 1.3 > l_g(\partial_1 B_i) \geq l_g(\eta_j) \quad \text{for} \quad j \in \{1, 2\}.$$ 

Hence $l_g(\tilde{\gamma}) < l_g(\gamma)$. A contradiction to the minimality of $l_g(\gamma)$. As the same proof applies to $\delta$, we obtain our lemma. \qed

**Lemma 5.5.** Let $\tilde{c}_j$ be the winding number of the arc $\gamma_j$ and $\tilde{d}_j$ be the winding number of the arc $\delta_j$. Then

$$|\tilde{c}_j - \tilde{c}_1| < 1 \quad \text{and} \quad |\tilde{d}_j - \tilde{d}_1| < 1.$$ 

**Proof.** We will prove the lemma for two arcs $\gamma_j$ and $\gamma_l$ of $\gamma$. If $|\tilde{c}_j - \tilde{c}_1| \geq 1$, then $|\tilde{c}_j - \tilde{c}_1| \geq 1$. It follows from Lemma 5.2 that $\#\{\gamma_j \cap \gamma_l\} \geq 1$ and $\gamma_j$ intersects $\gamma_l$. A contradiction to the fact that $\gamma$ is a simple closed geodesic and therefore has no self-intersection. \qed

Let $\gamma_{\text{min}}$ and $\delta_{\text{min}}$ be two arcs of $\gamma$ and $\delta$, respectively, with minimal absolute value of the winding number. Let $\tilde{c}_{\text{min}}$ and $\tilde{d}_{\text{min}}$ be the winding numbers of these arcs. We have

$$|\tilde{c}_{\text{min}}| = \min_j |\tilde{c}_j| \quad \text{and} \quad |\tilde{d}_{\text{min}}| = \min_l |\tilde{d}_l|.$$ 

We set furthermore

$$|\tilde{c}_{\text{min}}| = m_{\gamma} \quad \text{and} \quad |\tilde{d}_{\text{min}}| = m_{\delta}.$$ 

Let $c$ be a geodesic arc traversing $B_i$. With respect to its fixed endpoints on $\partial B_i$, $c$ is in the homotopy class $[c] = [b' \cdot a \cdot b'']$. Here $b'$ and $b''$ are directed geodesic arcs that meet $\alpha_i$ perpendicularly on opposite sides of $\alpha_i$, and $a$ is a directed arc on $\alpha_i$. We recall that $\sigma(c)$, the orientation of $c$, is defined by

$$\sigma(c) := \begin{cases} +1 & \text{if the orientation of } a \text{ agrees with that of } \alpha_i \\ 0 & \text{if } a \text{ is a single point} \\ -1 & \text{if the orientation of } a \text{ disagrees with that of } \alpha_i. \end{cases}$$

If $\tilde{c}$ is the winding number of the arc $c$, then $\sigma(c) = \text{sgn}(\tilde{c})$. We have:

**Lemma 5.6.** Let $\tilde{c}_j$ be the winding number of the arc $\gamma_j$ and $\tilde{d}_j$ be the winding number of the arc $\delta_j$.

If $|\tilde{c}_{\text{min}}| \geq 1$ then $\sigma(\gamma_j) = \sigma(\gamma_{\text{min}})$ for all $j \in \{1, \ldots, n_1\}$.

If $|\tilde{d}_{\text{min}}| \geq 1$ then $\sigma(\delta_l) = \sigma(\delta_{\text{min}})$ for all $l \in \{1, \ldots, n_2\}$.

Furthermore

$$|\tilde{c}_j - \sigma(\gamma_j)m_{\gamma}| < 2 \quad \text{and} \quad |\tilde{d}_j - \sigma(\delta_l)m_{\delta}| < 2.$$
Proof. We will prove the lemma for an arc $\gamma_j$ of $\gamma$. The statement about the orientation $\sigma(\gamma_j)$ of $\gamma_j$ follows from Lemma 5.5. It follows from the triangle inequality that
\[
|\tilde{c}_j - \sigma(\gamma_j)m_\gamma| = |\tilde{c}_j - \tilde{c}_{min} + \tilde{c}_{min} - \sigma(\gamma_j)m_\gamma| \leq |\tilde{c}_j - \tilde{c}_{min}| + |\tilde{c}_{min} - \sigma(\gamma_j)m_\gamma| < 1 + |\tilde{c}_{min} - \sigma(\gamma_j)m_\gamma|.
\]
Here the last inequality follows from Lemma 5.5. Now if $|\tilde{c}_{min}| < 1$ then $m_\gamma = 0$ and the inequality is true. If $|\tilde{c}_{min}| \geq 1$ then $\sigma(\gamma_j) = \sigma(\gamma_\gamma)$ and the inequality follows from the definition of $|\tilde{c}_{min}|$ and $m_\gamma$. \hfill \square

We now define the comparison curves $\gamma'$ and $\delta'$ of $M'$. Let $v$ be a geodesic arc of $\gamma$ or $\delta$ that traverses $B_i$ with endpoints $p_1 \in \partial_1 B_i$ and $p_2 \in \partial_2 B_i$ and let $\tilde{v}$ be its winding number. We first replace $v$ with the geodesic arc $v'$ with the same endpoints $p_1$ and $p_2$ on $\partial B_i$, such that its winding number $\tilde{v}'$ has the following value:
If $|\tilde{c}_{min}| \leq |\tilde{d}_{min}|$ then $m_\gamma \leq m_\delta$ and we set
\begin{equation}
\tilde{v}' = \begin{cases} 
\tilde{v} - \sigma(v)(\max\{m_\gamma - 1, 0\} + \max\{m_\delta - m_\gamma - 2, 0\}) & \text{if } v \subset \gamma, \\
\tilde{v} - \sigma(v)(\max\{m_\gamma - 1, 0\} + \max\{m_\delta - m_\gamma - 2, 0\}) & \text{if } v \subset \delta.
\end{cases}
\end{equation}
If $|\tilde{d}_{min}| \leq |\tilde{c}_{min}|$ then $m_\delta \leq m_\gamma$ and we set
\[
\tilde{v}' = \begin{cases} 
\tilde{v} - \sigma(v)(\max\{m_\delta - 1, 0\} + \max\{m_\gamma - m_\delta - 2, 0\}) & \text{if } v \subset \delta, \\
\tilde{v} - \sigma(v)(\max\{m_\delta - 1, 0\} + \max\{m_\gamma - m_\delta - 2, 0\}) & \text{if } v \subset \gamma.
\end{cases}
\]
Denote by $\gamma^*$ and $\delta^*$ the curves which we obtain this way from $\gamma$ and $\delta$. We call
\[
(\gamma_j^*)_{j=1,...,n_1} = \gamma^* \cap B_i \quad \text{and} \quad (\delta_i^*)_{i=1,...,n_2} = \delta^* \cap B_i
\]
the arcs of $\gamma^*$ and $\delta^*$ traversing $B_i$ and denote by $\tilde{c}_j^*$ and $\tilde{d}_i^*$ the winding number of the arc $\gamma_j^*$ and $\delta_i^*$, respectively.
Now let $v''$ be the arc on $\partial B_i \subset M'$, such that
\[
v'' = J_1(v', 1).
\]
Replacing all arcs $v$ of $\gamma$ and $\delta$ in all $(B_i)_{i=1,...,k}$ with corresponding arcs $v''$ in $M'$ we obtain $\gamma^*$ and $\delta^*$.

Claim 5.7. $l_g(\gamma') \leq l_g(\gamma)$, $l_g(\delta') \leq l_g(\delta)$ and $N_2 \leq \|\text{Int}([\gamma'], [\delta'])\|$.

Before launching into the proof of Claim 5.7 let us explain the idea a little bit.
On the one hand, we need the comparison curves $\gamma'$ and $\delta'$ to be shorter than the original curves $\gamma$ and $\delta$, respectively. So we ensure that the winding numbers of the arcs of $\gamma^*$ and the arcs of $\delta^*$ in $B_i$ are no greater than the winding numbers of the corresponding arcs of $\gamma$ and $\delta$. This is the reason for the $\max\{m_\gamma - 1, 0\}$ and $\max\{m_\delta - 1, 0\}$ in the definition.

On the other hand, to prove the statement about the intersection number: $N_2 \leq \|\text{Int}([\gamma'], [\delta'])\|$, what we need to do is to make sure that the intersections between the comparison curves $\gamma^*$ and $\delta^*$ have the same sign as those of the
original curves. Recall that by Lemma 3.2 \( \gamma \) and \( \delta \) intersect always under the same sign and have

\[
N_2 = \#\{p \in M_2 = M \setminus \bigcup_{i=1}^{k} B_i : p \in \{\gamma \cap \delta\}\}
\]

intersection points outside the union of the cylinders \((B_i)_{i=1,...,k}\). As \( \gamma \) and \( \gamma^* \) and \( \delta \) and \( \delta^* \) coincide in \( M_2 \), \( \gamma^* \) and \( \delta^* \) have at least \( N_2 \) intersection points and their sign of intersection at any intersection point in \( M_2 \) is the same. We will show that due to the \( \max\{m_\gamma - m_\delta - 2, 0\} \) in our definition there are no two consecutive intersections of \( \gamma^* \) and \( \delta^* \) with different sign. It follows that \( \#\{\gamma^* \cap \delta^*\} = |\text{Int}(\gamma^*, \delta^*)| \), whence \( N_2 \leq |\text{Int}(\gamma^*, \delta^*)| \).

As furthermore \( \gamma' \) and \( \delta' \) are the image of \( \gamma^* \) and \( \delta^* \), respectively, under a continuous deformation of the surface \( M \), it follows that

\[
\text{Int}(\gamma^*, \delta^*) = \text{Int}(\gamma', \delta').
\]

In total we obtain:

\[
N_2 \leq \#\{\gamma^* \cap \delta^*\} = |\text{Int}(\gamma^*, \delta^*)| = |\text{Int}(\gamma', \delta')|.
\]

Now let us prove Claim 5.7. To simplify our proof we may assume without loss of generality that

\[
|\tilde{c}_{\min}| \leq |\tilde{d}_{\min}|.
\]

We will first show:

\[
l_g(\gamma') \leq l_g(\gamma) \text{ and } l_g(\delta') \leq l_g(\delta).
\]

To this end we first show that \( |\tilde{c}_{j}^*| < 3 \) and \( |\tilde{d}_{j}^*| < 5 \).

Consider an arc \( \gamma_j^* \) of \( \gamma^* \). It follows from Equation (21) that

\[
\tilde{c}_j = \tilde{c}_j - \sigma(\gamma_j) \max\{m_\gamma - 1, 0\}.
\]

From which follows by the triangle inequality that

\[
|\tilde{c}_j^*| = |\tilde{c}_j - \sigma(\gamma_j)(m_\gamma - m_\Gamma + \max\{m_\gamma - 1, 0\})| \leq |\tilde{c}_j - \sigma(\gamma_j)m_\gamma| + |\sigma(\gamma_j)| \cdot |m_\gamma - \max\{m_\gamma - 1, 0\}| < 2 + 1 = 3.
\]

Here the last inequality follows from Lemma 5.6 and the fact that \( |x - \max\{x - 1, 0\}| \leq 1 \) for all \( x \in \mathbb{R}_+ \).

Now let \( \delta_j^* \) be an arc of \( \delta^* \). It follows from Equation (21) that

\[
\tilde{d}_j = \tilde{d}_j - \sigma(\delta_j)(\max\{m_\gamma - 1, 0\} + \max\{m_\delta - m_\gamma - 2, 0\}).
\]

We distinguish two cases, \( m_\delta - m_\gamma \leq 2 \) and \( m_\delta - m_\gamma > 2 \).

i) \( m_\delta - m_\gamma \leq 2 \Rightarrow \max\{m_\delta - m_\gamma - 2, 0\} = 0 \)
We obtain from the triangle inequality and Lemma 5.6 that
\[
|\tilde{d}_i^*| = |\tilde{d}_i - \sigma(\delta_i)\max\{m_\gamma - 1, 0\}| = \\
|\tilde{d}_i - \sigma(\delta_i)(m_\delta - m_\gamma + \max\{m_\gamma - 1, 0\})| \leq \\
|\tilde{d}_i - \sigma(\delta_i)m_\delta| + |\sigma(\delta_i)|\cdot|m_\delta - \max\{m_\gamma - 1, 0\}| < \\
2 + |m_\delta - \max\{m_\gamma - 1, 0\}|.
\]
Applying again the triangle inequality we have that
\[
|\tilde{d}_i^*| < 2 + |m_\delta - m_\gamma + m_\gamma - \max\{m_\gamma - 1, 0\}| < \\
2 + |m_\delta - m_\gamma| + |m_\gamma - \max\{m_\gamma - 1, 0\}| < 2 + 2 + 1 = 5.
\]
Here the last inequality follows from the hypothesis and the fact that the function $|x - \max\{x - 1, 0\}| \leq 1$ for all $x \in \mathbb{R}_+$. Hence $|\tilde{d}_i^*| < 5$.

ii) $m_\delta - m_\gamma > 2 \Rightarrow \max\{(m_\delta - m_\gamma - 2), 0\} = m_\delta - m_\gamma - 2$

It follows from Equation (21) that in this case
\[
\tilde{d}_i^* = \tilde{d}_i - \sigma(\delta_i)(\max\{m_\gamma - 1, 0\} + m_\delta - m_\gamma - 2).
\]
In this case we apply the triangle inequality twice to $|\tilde{d}_i^*|$ and obtain
\[
|\tilde{d}_i^*| = |\tilde{d}_i - \sigma(\delta_i)(\max\{m_\gamma - 1, 0\} + m_\delta - m_\gamma - 2)| \leq \\
|\tilde{d}_i - \sigma(\delta_i)m_\delta| + |\sigma(\delta_i)|\cdot|m_\gamma - \max\{m_\gamma - 1, 0\}| + |2| < 5.
\]
Here the last inequality follows from Lemma 5.6 and the fact that the function $|x - \max\{x - 1, 0\}| \leq 1$ for all $x \in \mathbb{R}_+$. Hence in any case $|\tilde{v}_j^*| < 3$ and $|\tilde{d}_j^*| < 5$.

Let $v$ be an arc of $\gamma$ or $\delta$ traversing $B_i$. Let $v'$ be the replacement arc of $v$ according to Equation (21). The deformation $J_{i}$ collapses the cylinder $B_i$ onto $\partial B_i$. Here the arc $v'$ of $\gamma^*$ or $\delta^*$ is deformed into an arc $v''$ of $\gamma'$ or $\delta'$ in $M'$. If $\tilde{v}'$ is the winding number of an arc $v'$, then $v''$ winds $|\tilde{v}'| \times$ times around $\partial B_i$. Hence, as $|\tilde{v}'| < 5$, we have that
\[
l_g(v'') < 5l_g(\partial_1 B_i).
\]
As $v$ traverses $B_i$ we conclude with Lemma 5.3 that
\[
l_g(v) > 2(\alpha((\alpha_i)) - 1.3) > 5l_g(\partial_1 B_i) > l_g(v'').
\]
Here the second inequality follows from the first part of inequality (11). Hence our replacement arc $v''$ of $\gamma'$ or $\delta'$ is always shorter than the arc $v$ from $\gamma$ or $\delta$. This proves the first part of Claim 5.7. Now we prove the second part, that is,
\[
N_2 \leq |\text{Int}([\gamma'], [\delta'])|.
\]
This amounts to showing that there are no two consecutive intersections of $\gamma^*$ and $\delta^*$ with different sign. For this it suffices to show that the sign of the intersection between $\gamma^*$ and $\delta^*$ at any point $p$ inside a cylinder $B_i$ is always equal to $\text{sgn}(|\text{Int}(\gamma, \delta)|)$. We will therefore show that all comparison arcs $\gamma_j^*$ and $\delta_i^*$ intersect under the same sign as $\gamma_j$ and $\delta_i$ in a cylinder $B_i$. 


Therefore we have to treat two cases. Either both $\gamma$ and $\delta$ intersect $\alpha_i$ from the same side or $\gamma$ and $\delta$ intersect $\alpha_i$ from different sides. Here we will deduce the result of the second case from the result of the first case.

Case I: $\text{Int}(\gamma, \alpha_i) = \text{Int}(\delta, \alpha_i)$

By Lemma 5.2 it is sufficient to show that the sign of the difference of the winding numbers does not change, that is,

$$\text{sgn}(d_i^* - c_j^*) = \text{sgn}(\tilde{d}_i - \tilde{c}_j).$$

Recall that we assume that $|\tilde{c}_{\min}| \leq |\tilde{d}_{\min}|$. In this case we have

$$\max\{m_{\gamma} - 1, 0\} \leq m_\gamma \leq m_\delta \quad \text{and} \quad \max\{m_\delta - 1, 0\} \leq m_\delta.$$

We will prove our statement depending on whether $\tilde{c}_j$ and $\tilde{d}_i$ have different sign or are equal to zero or whether they have the same sign or are equal to zero.

a) $(\tilde{c}_j \leq 0$ and $\tilde{d}_i \geq 0)$ or $(\tilde{c}_j \geq 0$ and $\tilde{d}_i \leq 0)$

We assume without loss of generality that $\tilde{c}_j \leq 0$ and $\tilde{d}_i \geq 0$. We have that

$$\tilde{d}_i - \tilde{c}_j \geq 0.$$

As $\tilde{c}_j \leq 0$ and $\tilde{d}_i \geq 0$ it follows that $\tilde{d}_i - \tilde{c}_j = 0$ if and only if $\tilde{d}_i = \tilde{c}_j = 0$. This case implies that $|\tilde{c}_{\min}| = |\tilde{d}_{\min}| = 0$. It follows from Equation (21) that $d_i^* = \tilde{d}_i$ and $c_j^* = \tilde{c}_j$. Hence

$$\tilde{d}_i - \tilde{c}_j = d_i^* - c_j^* = 0.$$

and (22) holds.

Conversely if $\tilde{d}_i = \tilde{c}_j = 0$ does not hold then either $\tilde{d}_i$ or $\tilde{c}_j$ is different from zero. We may assume that $\tilde{d}_i > 0$ and we have that

$$\tilde{d}_i - \tilde{c}_j > 0 \text{ that is, } \text{sgn}(\tilde{d}_i - \tilde{c}_j) = +1.$$

It follows from Equation (21) and from inequality (23) that

$$c_j^* = \tilde{c}_j + \max\{m_{\gamma} - 1, 0\} \leq \tilde{c}_j + m_{\gamma} \leq 0.$$

Here the last inequality follows from the fact that $\tilde{c}_j \leq 0$ and $|\tilde{c}_j| \geq m_{\gamma}$. Furthermore

$$\tilde{d}_i^* = \tilde{d}_i - (\max\{m_{\gamma} - 1, 0\} + \max\{m_\delta - m_{\gamma} - 2, 0\}).$$

We now show that $\tilde{d}_i^* > 0$. As $c_j^* \leq 0$ it follows that

$$\tilde{d}_i^* - c_j^* > 0 \text{ that is, } \text{sgn}(\tilde{d}_i^* - c_j^*) = +1$$

and condition (22) follows with (24).

To show that $\tilde{d}_i^* > 0$ we distinguish the two subcases $m_\delta - m_{\gamma} \leq 2$ and $m_\delta - m_{\gamma} > 2$. 


i) $m_\delta - m_\gamma \leq 2 \Rightarrow \max\{(m_\delta - m_\gamma - 2), 0\} = 0$

Then as $m_\delta \geq m_\gamma$

$$\tilde{d}_l = \tilde{d}_l - \max\{m_\gamma - 1, 0\} \geq \tilde{d}_l - \max\{m_\delta - 1, 0\} > 0.$$ 

Here the last inequality follows by distinguishing the cases $m_\delta > 1$ and $m_\delta \leq 1$. In the latter case we use the fact that $\tilde{d}_l > 0$.

ii) $m_\delta - m_\gamma > 2 \Rightarrow \max\{(m_\delta - m_\gamma - 2), 0\} = m_\delta - m_\gamma - 2$

Then it follows with inequality (23):

$$\tilde{d}_l = \tilde{d}_l - \max\{m_\gamma - 1, 0\} - m_\delta + m_\gamma + 2 = (\tilde{d}_l - m_\delta) + (m_\gamma - \max\{m_\gamma - 1, 0\}) + 2 \geq 2 > 0.$$ 

b) $(\tilde{c}_j \leq 0 \text{ and } \tilde{d}_l \leq 0) \text{ or } (\tilde{c}_j \geq 0 \text{ and } \tilde{d}_l \geq 0)$

We assume without loss of generality that $\tilde{c}_j \geq 0$ and $\tilde{d}_l \geq 0$. We will further subdivide this case into the subcases $m_\delta - m_\gamma \leq 2$ and $m_\delta - m_\gamma > 2$.

i) $m_\delta - m_\gamma \leq 2$

In this case it follows from Equation (21) that

$$\tilde{c}_j^* = \tilde{c}_j - \max\{m_\gamma - 1, 0\} \text{ and } \tilde{d}_l^* = \tilde{d}_l - \max\{m_\gamma - 1, 0\}.$$ 

Hence

$$\tilde{d}_l^* - \tilde{c}_j^* = \tilde{d}_l - \tilde{c}_j \Rightarrow \text{sgn}(\tilde{d}_l^* - \tilde{c}_j^*) = \text{sgn}(\tilde{d}_l - \tilde{c}_j)$$ 

and the condition (22) is fulfilled.

ii) $m_\delta - m_\gamma > 2$

We have due to the definition of $m_\delta$ that

$$\tilde{d}_l - \tilde{c}_j \geq m_\delta - \tilde{c}_j = m_\delta - \tilde{c}_j + |\tilde{c}_{\min}| - |\tilde{c}_{\min}| + m_\gamma - m_\gamma = m_\delta - (\tilde{c}_j - |\tilde{c}_{\min}|) - (|\tilde{c}_{\min}| - m_\gamma) - m_\gamma.$$ 

We note that due to Lemma 5.5 $0 \leq \tilde{c}_j - |\tilde{c}_{\min}| < 1$. Furthermore due to the definition of $|\tilde{c}_{\min}|$ and $m_\gamma$ we have that $0 \leq |\tilde{c}_{\min}| - m_\gamma < 1$. Hence

$$\tilde{d}_l - \tilde{c}_j > m_\delta - 2 - m_\gamma > 0 \Rightarrow \text{sgn}(\tilde{d}_l - \tilde{c}_j) = +1.$$ 

Here the last inequality follows from our hypothesis. It follows from Equation (21) that

$$\tilde{c}_j^* = \tilde{c}_j - \max\{m_\gamma - 1, 0\} \text{ and }$$

$$\tilde{d}_l^* = \tilde{d}_l - \max\{m_\gamma - 1, 0\} - m_\delta + m_\gamma + 2.$$
Hence
\[ \tilde{d}_i^l - \tilde{c}_j^g = \tilde{d}_i - m_\delta + m_\gamma + 2 - \tilde{c}_j = 2 + (\tilde{d}_i - m_\delta) - (\tilde{c}_j - m_\gamma) > 2 + 0 - 2 > 0 \]
Here by the definition of \( m_\delta \), \( \tilde{d}_i - m_\delta \geq 0 \) and by Lemma 5.6 \( \tilde{c}_j - m_\gamma < 2 \).
Hence \( \text{sgn}(\tilde{d}_i^l - \tilde{c}_j^g) = +1 \).

Therefore
\[ \text{sgn}(\tilde{d}_i^l - \tilde{c}_j^g) = \text{sgn}(\tilde{d}_i - \tilde{c}_j) = +1 \]
and the condition (22) is fulfilled.
This proves that in any case condition (22) is fulfilled. This settles the claim in the case, where \( \gamma \) and \( \delta \) intersect \( \alpha_i \) from the same side.

Case II: \( \text{Int}(\gamma, \alpha_i) = - \text{Int}(\delta, \alpha_i) \)

If \( \gamma \) and \( \delta \) intersect \( \alpha_i \) from different sides, it is sufficient to show that
\[ \text{sgn}(\tilde{d}_i^l + \tilde{c}_j^g) = \text{sgn}(\tilde{d}_i + \tilde{c}_j). \]
This follows from Lemma 5.2.
In this case let \( \gamma^{-1} \) be the oriented geodesic that coincides pointwise with \( \gamma \), but which has opposite orientation. Let \( (\gamma_j^{-1})_{j=1,..,n_1} \) be the arcs of \( \gamma^{-1} \) traversing \( B_i \). Let \( \tilde{c}_j^{-1} \) be the winding number of the arc \( \gamma_j^{-1} \). We have that \( \tilde{c}_j^{-1} = -\tilde{c}_j \).
Applying Equation (21) to the arcs of \( \delta \) and \( \gamma^{-1} \), we obtain that
\[ \tilde{c}_j^{-1} = -\tilde{c}_j \quad \text{and} \quad \tilde{c}_j^{-1*} = -\tilde{c}_j^*. \]
Now \( \gamma^{-1} \) and \( \delta \) intersect \( \alpha_i \) under the same sign. Hence condition (22) from Case I is fulfilled, that is,
\[ \text{sgn}(\tilde{d}_i - \tilde{c}_j^{-1}) = \text{sgn}(\tilde{d}_i^* - \tilde{c}_j^{-1*}). \]
Combining the two previous equations we obtain:
\[ \text{sgn}(\tilde{d}_i - (-\tilde{c}_j)) = \text{sgn}(\tilde{d}_i - \tilde{c}_j^{-1}) = \text{sgn}(\tilde{d}_i^* - \tilde{c}_j^{-1*}) = \text{sgn}(\tilde{d}_i^* - (-\tilde{c}_j^*)) \]
and therefore condition (25) is fulfilled. This settles the claim in the case where \( \gamma \) and \( \delta \) intersect \( \alpha_i \) from different sides. Hence the remaining second part of Claim 5.7 is true.

This finishes the construction of the curves \( \gamma' \) and \( \delta' \). From this we get the upper bound on \( K_2 \) in Equation (20).

5.3. End of the proof of Case 1 of Theorem 4.2. Summarizing Case 1, we obtain from the inequalities (18) and (20) in (12) that
\[ K(M, g) - \epsilon \leq K_1 + K_2 \leq \frac{18s - 18}{l_g(\alpha_1)cl(l_g(\alpha_1))} + 144. \]
As \( \epsilon \) is arbitrarily small, we obtain the upper bound stated in Theorem 4.2 from this inequality.
6. Proof of Theorem 4.2., Case 2: Either $\gamma$ or $\delta$ is one of the $(\alpha_i)_{i=1,\ldots,k}$

We may suppose that $\gamma = \alpha_i$. In this case we have to verify the upper bound in Theorem 4.2 for

$$K(M, g) - \epsilon = \frac{|\mathrm{Int}(\alpha_i, \delta)|}{l_g(\alpha_i) \cdot l_g(\delta)} = \frac{N}{l_g(\alpha_i) \cdot l_g(\delta)}.$$ 

Now $\delta$ intersects $\alpha_i$ $N$ times. To this end it has to traverse $N$ times the collar $C_i$ of $\alpha_i$. Analogous to Lemma 5.3 we obtain from the length of the $N$ arcs $(\delta_j)_{j=1,\ldots,N}$ of $\delta$ traversing $C_i$:

$$l_g(\delta) > \sum_{j=1}^{N} l_g(\delta_j) \geq N \cdot 2cl(l_g(\alpha_i)).$$

Now from the monotonicity of the function $\frac{1}{x \cdot cl(x)}$ in the interval $(0, 2 \arcsinh(1)]$ (see Fig. 7) it follows that

$$K(M, g) - \epsilon \leq \frac{N}{2Nl_g(\alpha_i) \cdot cl(l_g(\alpha_i))} = \frac{1}{2l_g(\alpha_1) \cdot cl(l_g(\alpha_1))}.$$ 

Again we obtain our upper bound in Theorem 4.2 as $\epsilon$ is arbitrarily small. $\square$

Acknowledgement

While working on this article the second author has been supported by the Alexander von Humboldt foundation.

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