PATTERN FORMATION OF A COUPLED TWO-CELL
SCHNakenberg MODEL

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Abstract. In this paper, we study a coupled two-cell Schnakenberg model
with homogenous Neumann boundary condition, i.e.,

\[
\begin{aligned}
-d_1 \Delta u &= a - u + u^2 v + c(w - u), & \text{in } \Omega, \\
-d_2 \Delta v &= b - u^2 v, & \text{in } \Omega, \\
-d_1 \Delta w &= a - w + w^2 z + c(u - w), & \text{in } \Omega, \\
-d_2 \Delta z &= b - w^2 z, & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = \frac{\partial z}{\partial \nu} &= 0, & \text{on } \partial \Omega.
\end{aligned}
\]

We give a priori estimate to the positive solution. Meanwhile, we obtain the
non-existence and existence of positive non-constant solution as parameters
d_1, d_2, a and b changes.

1. Introduction. The Schnakenberg model [12] is a well-known autocatalytic
chemical reaction model with limit cycle behavior, which was introduced by
Schnakenberg in 1979. The trimolecular reactions between two chemical products
X, Y and two chemical sources A, B are described by the following equations:

\[
A \rightleftharpoons X, \quad B \rightarrow Y, \quad 2X + Y \rightarrow 3X.
\]  

(1)

Using the law of mass action, one can obtain a system of reaction-diffusion equations
for the concentrations u, v of the chemical products X, Y which describes the
reactions in (1). The non-dimensional form of the equations is

\[
\begin{aligned}
&u_t - d_1 \Delta u = a - u + u^2 v, \\
v_t - d_2 \Delta v = b - u^2 v.
\end{aligned}
\]

(2)

In the above equations, d_1, d_2 are the diffusion coefficients of the chemicals X, Y,
and a, b are the concentrations of A, B, respectively. It is also assumed that A, B are
in abundance so a and b are kept constant. Model (2) has been studied by various
researchers from both analytical and numerical points of view (see [2, 9, 11]). Non-
existence, existence of positive solutions and Hopf bifurcation of the steady state

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equation have been considered recently in [6, 16]. The Turing patterns and spike solutions are shown in [4, 15].

The study of two-cell model of two coupled components is a substantial progress from one-cell model of two-component reaction-diffusion system [3, 5]. Coupled cells with diffusion-reaction and mutual mass exchange are often used to describe the processes in living cells and tissues, or in distributed chemical reactions [13, 14]. The incentive for studying these problems is to investigate the self-oscillation produced by the reactions in systems of finite sizes. In fact, the study of two coupled components are much more complicated than that of two-component reaction-diffusion system [18]. The spatial patterns of a coupled two-cell reaction diffusion model with autocatalytic have been considered by utilizing the bifurcation theory and degree theory in [1, 19, 20].

In this paper, we mainly consider a coupled Schnakenberg model with homogeneous Neumann boundary condition, i.e.,

\[
\begin{align*}
\frac{du}{dt} &= d_1 \Delta u + a - u + u^2 v + c(w - u), & \text{in } \Omega \times (0, \infty), \\
\frac{dv}{dt} &= d_2 \Delta v + b - u^2 v, & \text{in } \Omega \times (0, \infty), \\
\frac{dw}{dt} &= d_1 \Delta w + a - w + w^2 z + c(u - w), & \text{in } \Omega \times (0, \infty), \\
\frac{dz}{dt} &= d_2 \Delta z + b - w^2 z, & \text{in } \Omega \times (0, \infty), \\
\frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0, & \text{on } \partial \Omega \times (0, \infty),
\end{align*}
\]

(3)

where \(a, \ b, \ c, \ d_1 \) and \(d_2\) are positive constants, \(\Omega \subset \mathbb{R}^N (N \geq 1)\) is a bounded domain with smooth boundary \(\partial \Omega\) and \(\nu\) is the outward unit normal vector on \(\partial \Omega\). The homogeneous Neumann boundary condition indicates that the system (3) is self-contained with zero-flux.

The rest of the paper is organized as follows. In Section 2, we establish a priori estimates for the positive solutions of the system (4). In Section 3 and 4, we consider the non-existence and existence to the positive non-constant solutions of the system (4).

2. A priori estimates. In this section, we give a priori estimates for the solutions of a steady state system (4) which is consistent with system (3):

\[
\begin{align*}
-d_1 \Delta u &= a - u + u^2 v + c(w - u), & \text{in } \Omega, \\
-d_2 \Delta v &= b - u^2 v, & \text{in } \Omega, \\
-d_1 \Delta w &= a - w + w^2 z + c(u - w), & \text{in } \Omega, \\
-d_2 \Delta z &= b - w^2 z, & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = \frac{\partial z}{\partial \nu} &= 0, & \text{on } \partial \Omega.
\end{align*}
\]

(4)

We now recall the following result, due to Lou and Ni (see [7, 8]).

**Lemma 2.1.** Let \(g \in C^1(\Omega \times \mathbb{R})\).

(i) If \(w \in C^2(\Omega) \times C^1(\Omega)\) satisfies

\[
\begin{align*}
\Delta w + g(x, w) &\geq 0, & \text{in } \Omega, \\
\frac{\partial w}{\partial \nu} &\leq 0, & \text{on } \partial \Omega,
\end{align*}
\]

and \(w(x_0) = \max_{\Omega} w\), then \(g(x_0, w(x_0)) \geq 0\).
(ii) If \( w \in C^2(\Omega) \times C^1(\overline{\Omega}) \) satisfies
\[
\begin{align*}
\Delta w + g(x,w) & \leq 0, \quad \text{in } \Omega, \\
\frac{\partial w}{\partial \nu} & \geq 0, \quad \text{on } \partial \Omega,
\end{align*}
\]
and \( w(x_0) = \min_{\overline{\Omega}} w \), then \( g(x_0,w(x_0)) \leq 0 \).

**Theorem 2.2.** Any solution \((u,v,w,z)\) of (4) satisfies
\[
a \leq u \leq 2(a + b) + \frac{2d_2b}{d_1a^2},
\]
\[
b[2(a + b) + \frac{2d_2b}{d_1a^2}]^{-2} \leq v \leq ba^{-2},
\]
\[
a \leq w \leq 2(a + b) + \frac{2d_2b}{d_1a^2},
\]
\[
b[2(a + b) + \frac{2d_2b}{d_1a^2}]^{-2} \leq z \leq ba^{-2}.
\]

**Proof.** By adding the first equation of (4) to the third one, we have
\[
-d_1 \Delta (u + w) = 2a - (u + w) + u^2v + w^2z \geq 2a - (u + w),
\]
that is, \( u + w \) satisfies
\[
\begin{align*}
2a - (u + w) + d_1 \Delta (u + w) & \leq 0, \quad \text{in } \Omega, \\
\frac{\partial (u + w)}{\partial \nu} & = 0, \quad \text{on } \partial \Omega.
\end{align*}
\]
Let \( x_0 \) be the minimum of \( u + w \). Lemma 2.1 implies that \( 2a - (u + w)(x_0) \leq 0 \), i.e., \( (u + w)(x_0) \geq 2a \), which yields \( u + w \geq 2a, \forall x \in \overline{\Omega} \). With the first equation of (4) and \( w \geq 2a - u \), we get
\[
-d_1 \Delta u = a - u + u^2v + c(w - u)
\]
\[
\geq a - u + u^2v + c[2a - u - u] \\
\geq a + 2ac - (1 + 2c)u,
\]
it means \( u \) satisfies
\[
\begin{align*}
d_1 \Delta u + a(1 + 2c) - (1 + 2c)u & \leq 0, \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} & = 0, \quad \text{on } \partial \Omega.
\end{align*}
\]
We define \( x_1 \) by the minimum of \( u \). Applying Lemma 2.1 again, it follows that \( a(1 + 2c) - (1 + 2c)u(x_1) \leq 0 \), i.e., \( u \geq a \). Since \( v \) satisfies
\[
\begin{align*}
-d_2 \Delta v = b - u^2v, \quad \text{in } \Omega, \\
\frac{\partial v}{\partial \nu} & = 0, \quad \text{on } \partial \Omega,
\end{align*}
\]
then denote the maximum of \( v \) by \( x_2 \). By Lemma 2.1, it is concluded that \( b - u(x_2)^2v(x_2) \geq 0 \), therefore, \( v(x) \leq v(x_2) \leq \frac{b}{u(x_2)^2} \leq \frac{b}{a^2}, \forall x \in \overline{\Omega} \). Similarly, we have \( w \geq a \) and \( z \leq \frac{b}{a^2} \).

Adding the equations in (4) all together, we obtain
\[
-\Delta(d_1u + d_2v + d_1w + d_2z) = 2(a + b) - (u + w).
\]
Setting $Q = d_1(u + w) + d_2(v + z)$, it is easy to see that
\[
\begin{cases}
-\Delta Q = 2(a + b) - (u + w), & \text{in } \Omega, \\
\frac{\partial Q}{\partial \nu} = 0, & \text{on } \partial \Omega.
\end{cases}
\]
Assume $x_3$ is the maximum of $Q$. Lemma 2.1 indicates $2(a + b) - (u + w)(x_3) \geq 0$, i.e., $2(a + b) \geq (u + w)(x_3)$, which yields,
\[d_1(u + w) \leq Q(x) \leq Q(x_3) \leq 2d_1(a + b) + \frac{2d_2b}{a^2},\]
and thus
\[u \leq 2(a + b) + \frac{2d_2b}{d_1a^2}, \quad w \leq 2(a + b) + \frac{2d_2b}{d_1a^2}.
\]
Also by Lemma 2.1 if $x_4$ is a minimum of $v$, it follows that $b - u^2(x_4)v(x_4) \leq 0$,
which implies
\[v(x) \geq v(x_4) \geq \frac{b}{u^2(x_4)} \geq b[2(a + b) + \frac{2d_2b}{d_1a^2}]^{-2}.
\]
Similarly, by the fourth equation of (4), we have $w(x) \geq b[2(a + b) + \frac{2d_2b}{d_1a^2}]^{-2}$. This
concludes our proof. \qedhere

**Proposition 1.** Let $a, B, D_1, D_2 > 0$ be fixed. Then, there exist two positive
constants $C_1, C_2 > 0$ depending on $a, B, D_1, D_2$, such that for all
\[0 < b < B, \quad d_1 > D_1, \quad 0 < d_2 < D_2,
\]
any solution $(u, v, w, z)$ of (4) satisfies
\[C_1 < \min_{x \in \Omega} \{u(x), v(x), w(x), z(x)\} \leq \max_{x \in \Omega} \{u(x), v(x), w(x), z(x)\} < C_2.
\]

Furthermore, by the standard elliptic arguments and Theorem 2.2 we can obtain:

**Proposition 2.** Let $a, B, D_1, D_2 > 0$ be fixed. Then, for any positive integer $k \geq 1$,
there exists a positive constant $C > 0$ depending on $a, B, D_1, D_2, k, N, \Omega$, such that for all
\[0 < b < B, \quad d_1 > D_1, \quad 0 < d_2 < D_2,
\]
any solution $(u, v, w, z)$ of (4) satisfies
\[|u|_k + |v|_k + |w|_k + |z|_k \leq C,
\]
where $| \cdot |_k$ denotes the norm of $C^k(\Omega)$. 

3. **Non-existence of non-constant solutions.** In this section, we show some
results for non-existence of positive non-constant solution of (4) as $d_1$ is sufficiently
large or $b$ is sufficiently small.

**Theorem 3.1.** (i) Let $a, b, d_2 > 0$ be fixed. Then, there exists a positive constant
$D > 0$ depending on $a, b, d_2$, such that for all $d_1 > D$, (4) has no non-constant
solution, i.e., the only solution of (4) is $(a + b, \frac{b}{(a + b)^2}, a + b, \frac{b}{(a + b)^2})$.

(ii) Let $a, d_1, d_2 > 0$ be fixed. Then, there exists a positive constant $B > 0$ depending
on $a, d_1, d_2$, such that for all $0 < b < B$, (4) has no non-constant solution,
i.e., the only solution of (4) is $(a, 0, a, 0)$.

First, we give a lemma, which is essential in the proof of Theorem 3.1.
Lemma 3.2. (i) Let $a, b, d_2 > 0$ be fixed and let $\{\delta_n\} \subset (0, \infty)$ be a sequence such that $\delta_n \to \infty$ as $n \to \infty$. If $(u_n, v_n, w_n, z_n)$ is a solution of (4) with $d_1 = \delta_n$, then

$$(u_n, v_n, w_n, z_n) \to (a + b, \frac{b}{(a + b)^2}, a + b, \frac{b}{(a + b)^2}) \text{ in } C^2(\Omega) \text{ as } n \to \infty.$$  

(ii) Let $a, d_1, d_2 > 0$ be fixed and let $\{\gamma_n\} \subset (0, \infty)$ be a sequence such that $\gamma_n \to 0$ as $n \to \infty$. If $(u_n, v_n, w_n, z_n)$ is a solution of (4) with $b = \gamma_n$, then

$$(u_n, v_n, w_n, z_n) \to (a, 0, a, 0) \text{ in } C^2(\Omega) \text{ as } n \to \infty.$$  

Proof. (i) By Proposition 1 and 2 for any positive integer $k$, there exist two positive constants $C_1$ and $C_2$ depending on $a$, $b$, $d_2$ and $k$, such that

$$|u_n|_k + |v_n|_k + |w_n|_k + |z_n|_k \leq C_1 \text{ and } \min_{x \in \Omega} \{u_n, v_n, w_n, z_n\} \geq C_2.$$  

Choosing $k > 2$, since the embedding $(C^k(\Omega))^4 \hookrightarrow (C^2(\Omega))^4$ is compact, there exists a subsequence of $\{u_n, v_n, w_n, z_n\}$, still denoted by itself, and positive functions $u, v, w, z \in C^2(\Omega)$, such that

$$(u_n, v_n, w_n, z_n) \to (u, v, w, z) \text{ in } C^2(\Omega) \text{ as } n \to \infty.$$  

Since $d_1 = \delta_n \to \infty$, using the first equation of (4), it follows that $u \equiv \xi > 0$ (is a constant), $w \equiv \eta > 0$ (is a constant), and $(\xi, v, \eta, z)$ satisfies

$$\begin{cases}
-d_2 \Delta v = b - \xi^2 v, \\
\frac{\partial v}{\partial \nu} = 0, \\
\int_\Omega a - \xi + \xi^2 v + c(\eta - \xi)dx = 0
\end{cases} \quad \text{and} \quad \begin{cases}
-d_2 \Delta z = b - \eta^2 z, \\
\frac{\partial z}{\partial \nu} = 0, \\
\int_\Omega a - \eta + \eta^2 z + c(\xi - \eta)dx = 0.
\end{cases}$$  

(5)

Multiplying both sides of the first equation in (5) by $b - \xi^2 v$, and integrating over $\Omega$, we have

$$\int_\Omega (-d_2 \Delta v)(b - \xi^2 v)dx = \int_\Omega (b - \xi^2 v)^2dx,$$

that is,

$$0 \leq \frac{d_2}{\xi^2} \int_\Omega |D(b - \xi^2 v)|^2dx = - \int_\Omega (b - \xi^2 v)^2dx \leq 0,$$

which yields $D(b - \xi^2 v) = 0$. Integrating the first equation of (5) over $\Omega$, we obtain

$$\int_\Omega (b - \xi^2 v)dx = \int_\Omega -d_2 \Delta vdx = 0.$$  

Hence $v = \frac{b}{\xi^2}$. Inserting $v = \frac{b}{\xi^2}$ into the third equation of (5), we get

$$\int_\Omega a + b - \xi + c(\eta - \xi)dx = 0.$$  

(6)

Similarly, we have

$$\int_\Omega a + b - \eta + c(\xi - \eta)dx = 0.$$  

(7)

Adding (6) to (7), thus

$$\int_\Omega 2(a + b) - (\xi + \eta)dx = 0,$$
it means $\xi + \eta = 2(a + b)$. Inserting $\eta = 2(a + b) - \xi$ into (6), it implies
\[
\int_\Omega (2c + 1)(a + b) - (2c + 1)\xi dx = 0,
\]
therefore, $\xi = a + b$. Furthermore, with $\xi + \eta = 2(a + b)$ it follows $\eta = a + b$. Since $v = \frac{b}{\xi^2}$ and $z = \frac{b}{\eta^2}$, we get $v = \frac{b}{(a + b)^2}$ and $z = \frac{b}{(a + b)^2}$.

(ii) The proof is similar to that of (i). We get $(u_n, v_n, w_n, z_n) \to (u, v, w, z)$, $n \to \infty$ in $C^2(\Omega)$ as $b = \gamma_n \to 0$, and $(u, v, w, z)$ satisfies
\[
\begin{aligned}
\begin{cases}
-d_1\Delta u = a - u + u^2v + c(w - u), & \text{in } \Omega, \\
-d_2\Delta v = -u^2v, & \text{in } \Omega, \\
-d_1\Delta w = a - w + w^2z + c(u - w), & \text{in } \Omega, \\
-d_2\Delta z = -w^2z, & \text{in } \Omega, \\
\partial u \partial \nu = \partial v \partial \nu = \partial w \partial \nu = \partial z \partial \nu = 0, & \text{on } \partial \Omega.
\end{cases}
\end{aligned}
\]
Adding the equations in (8) all together, it yeilds
\[
\begin{aligned}
\begin{cases}
-\Delta(d_1u + d_2v + d_1w + d_2z) = 2a - (u + w), & \text{in } \Omega, \\
\partial(d_1u + d_2v + d_1w + d_2z) \partial \nu = 0, & \text{on } \partial \Omega,
\end{cases}
\end{aligned}
\]
that is, $u + w = 2a$. Adding the first two equations of (8) and using $w = 2a - u$, we have
\[
\begin{aligned}
\begin{cases}
-\Delta(d_1u + d_2v) + (1 + 2c)u = (1 + 2c)a, & \text{in } \Omega, \\
\partial(d_1u + d_2v) \partial \nu = 0, & \text{on } \partial \Omega,
\end{cases}
\end{aligned}
\]
it means $u = a, w = a$. Inserting $u = a, w = a$ into the second and fourth equation of (8) respectively, therefore,
\[
\begin{aligned}
\begin{cases}
-d_2\Delta v = -a^2v, & \text{in } \Omega, \\
\partial v \partial \nu = 0, & \text{on } \partial \Omega,
\end{cases}
\end{aligned}
\quad \text{and} \quad
\begin{aligned}
\begin{cases}
-d_2\Delta z = -a^2z, & \text{in } \Omega, \\
\partial z \partial \nu = 0, & \text{on } \partial \Omega,
\end{cases}
\end{aligned}
\]
so $v = z = 0$. \qed

Now, we can use the technique in [10] to prove Theorem 3.1.

Proof. (i) Let $u = \xi + \tilde{u}$, $w = \eta + \tilde{w}$, where $\int_\Omega \tilde{u} = 0$, $\int_\Omega \tilde{w} = 0$ and $\xi, \eta \in \mathbb{R}^+$, then discussing the solution of (4) is equivalent to solving the equation
\[
\begin{aligned}
\begin{cases}
\Delta \tilde{u} + \rho P[a - (\xi + \tilde{u}) + (\xi + \tilde{u})^2v + c(\eta - \xi) + c(\tilde{w} - \tilde{u})] = 0, & \text{in } \Omega, \\
\int_\Omega [a - (\xi + \tilde{u}) + (\xi + \tilde{u})^2v + c(\eta - \xi) + c(\tilde{w} - \tilde{u})]dx = 0, & \text{in } \Omega, \\
\Delta \tilde{v} + d_2^{-1}[b - (\xi + \tilde{u})^2v] = 0, & \text{in } \Omega, \\
\Delta \tilde{w} + \rho P[a - (\eta + \tilde{w}) + (\eta + \tilde{w})^2z + c(\xi - \eta) + c(\tilde{u} - \tilde{w})] = 0, & \text{in } \Omega, \\
\int_\Omega [a - (\eta + \tilde{w}) + (\eta + \tilde{w})^2z + c(\xi - \eta) + c(\tilde{u} - \tilde{w})]dx = 0, & \text{in } \Omega, \\
\Delta \tilde{z} + d_2^{-1}[b - (\eta + \tilde{w})^2z] = 0, & \text{in } \Omega, \\
\partial \tilde{u} \partial \nu = \partial \tilde{v} \partial \nu = \partial \tilde{w} \partial \nu = \partial \tilde{z} \partial \nu = 0, & \text{on } \partial \Omega,
\end{cases}
\end{aligned}
\]
where \( \rho = d_1^{-1} \) and \( \mathcal{P} \phi = \phi - \frac{1}{|\Omega|} \int_\Omega \phi(x)dx \), i.e., \( \mathcal{P} : L^2(\Omega) \rightarrow L^2_0(\Omega) = \{ g \in L^2(\Omega) \colon \int_\Omega g(x)dx = 0 \} \) is a projective operator. Obviously \( U_0 \equiv (\hat{u}, \xi, v, \hat{w}, \eta, z) = (0, a + b, \frac{b}{(a + b)^2}, 0, a + b, \frac{b}{(a + b)^2}) \) is a solution of (9). Next, it is enough to prove that \( \rho > 0 \) is sufficiently small, then \( U_0 \) is the unique solution of (9). Define

\[
F(\rho, \hat{u}, \xi, v, \hat{w}, \eta, z) = (f_1, f_2, f_3, f_4, f_5, f_6)^T:
\]

\[
\mathbb{R}^+ \times ((L^2_0(\Omega) \cap W^{2,2}_\nu(\Omega)) \times \mathbb{R}^+ \times W^{2,2}_\nu(\Omega))^2 \rightarrow (L^2_0(\Omega) \times \mathbb{R} \times L^2(\Omega))^2
\]

where

\[
\begin{align*}
f_1(\rho, \hat{u}, \xi, v, \hat{w}, \eta, z) &= \Delta \hat{u} + \rho \mathcal{P}[a - (\xi + \hat{u}) + (\xi + \hat{u})^2 v + c(\eta - \xi) + c(\hat{w} - \hat{u})], \\
f_2(\rho, \hat{u}, \xi, v, \hat{w}, \eta, z) &= \int_\Omega a - (\xi + \hat{u}) + (\xi + \hat{u})^2 v + c(\eta - \xi) + c(\hat{w} - \hat{u})dx, \\
f_3(\rho, \hat{u}, \xi, v, \hat{w}, \eta, z) &= \Delta \hat{v} + d_2^{-1}[b - (\xi + \hat{u})^2 v], \\
f_4(\rho, \hat{u}, \xi, v, \hat{w}, \eta, z) &= \Delta \hat{w} + \rho \mathcal{P}[a - (\eta + \hat{w}) + (\eta + \hat{w})^2 z + c(\xi - \eta) + c(\hat{u} - \hat{w})], \\
f_5(\rho, \hat{u}, \xi, v, \hat{w}, \eta, z) &= \int_\Omega a - (\eta + \hat{w}) + (\eta + \hat{w})^2 z + c(\xi - \eta) + c(\hat{u} - \hat{w})dx, \\
f_6(\rho, \hat{u}, \xi, v, \hat{w}, \eta, z) &= \Delta z + d_2^{-1}[b - (\eta + \hat{w})^2 z],
\end{align*}
\]

where \( W^{2,2}_\nu(\Omega) = \{ g \in W^{2,2}(\Omega) \mid \frac{\partial g}{\partial \nu} = 0 \text{ on } \partial \Omega \} \).

Apparently, finding the solution of (9) is equivalent to solving \( F(\rho, \hat{u}, \xi, v, \hat{w}, \eta, z) = 0 \). Moreover, system (9) has a unique solution \( (\hat{u}, \xi, v, \hat{w}, \eta, z) = U_0 \) with \( \rho = 0 \).

Denote \( X_0 := (0, 0, a + b, \frac{b}{(a + b)^2}, 0, a + b, \frac{b}{(a + b)^2}) \), it can be easily calculated that

\[
D_{(\hat{u}, \xi, v, \hat{w}, \eta, z)} F(X_0) = \begin{pmatrix}
\Delta \hat{u} \\
\int_\Omega \frac{b - a}{a + b} \hat{u} + \frac{b - a}{a + b} \xi + (a + b)^2 \hat{v} + c(\hat{w} - \hat{u})dx \\
\Delta \hat{v} + d_2^{-1}(a + b)^2 \hat{v} + d_2^{-1} \frac{2b}{a + b} (-\hat{u} - \xi) \\
\int_\Omega \frac{b - a}{a + b} \hat{w} + \frac{b - a}{a + b} \eta + (a + b)^2 \hat{z} + c(\hat{u} - \hat{w})dx \\
\Delta \hat{z} - d_2^{-1}(a + b)^2 \hat{z} + d_2^{-1} \frac{2b}{a + b} (-\hat{w} - \hat{\eta})
\end{pmatrix}.
\]

Since \( \Delta : L^2_0(\Omega) \cap W^{2,2}_\nu(\Omega) \rightarrow L^2_0(\Omega) \) is invertible, so \( D_{(\hat{u}, \xi, v, \hat{w}, \eta, z)} F(X_0) \) is invertible too if and only if \( (10) \) is invertible. Let
Let \( \lambda \) influence of the size of the region \( \Omega \) on the pattern formation of (4). Let \( \tilde{\Omega} = \Omega \), then system (4) can be written as follows

\[
\begin{align*}
\tilde{\Delta} \tilde{v} + d_2^{-1}(-\frac{2b}{a+b} \tilde{\eta} - (a+b)^2 \tilde{\eta}) = 0 \\
\tilde{\Delta} \tilde{\eta} + d_2^{-1}(-\frac{2b}{a+b} \tilde{\hat{\eta}} - (a+b)^2 \tilde{\hat{\eta}}) = 0 \quad \text{in } \tilde{\Omega}.
\end{align*}
\]

one can verify that \( \mathcal{L} : (\mathbb{R} \times W^{2,2}_\nu(\Omega))^2 \to (\mathbb{R} \times L^2(\Omega))^2 \) is invertible, and \( D_{(\tilde{u},\tilde{v},\tilde{w},\tilde{\eta},\tilde{z})} F(X_0) \) is surjective. By the Implicit Function Theorem, there exist positive constants \( \rho_0 \) and \( \delta_0 \) such that for each \( \rho \in [0,\rho_0) \), \( U_0 \) is the unique solution of \( F(\rho, \tilde{u}, \tilde{v}, \tilde{w}, \tilde{\eta}, \tilde{z}) = 0 \) in \( B_{\delta_0} \), where \( B_{\delta_0}(U_0) \) is a ball in \((L_0(\Omega) \cap W^{2,2}_\nu(\Omega)) \times \mathbb{R} \times W^{2,2}_\nu(\Omega))^2 \) of radius \( \delta_0 \) centered at \( U_0 \). For \( \rho > 0 \) is sufficiently small, let \((\tilde{u}_\rho, \tilde{v}_\rho, \tilde{w}_\rho, \tilde{\eta}_\rho, \tilde{z}_\rho)\) be any solution of (9), Lemma 3.2(i) implies \((\tilde{u}_\rho, \tilde{v}_\rho, \tilde{w}_\rho, \tilde{\eta}_\rho, \tilde{z}_\rho)\) be any solution of \((11)\) as \( d_1 \) is sufficiently large.

(ii) The proof is similar to that of (i), we construct the operator \( F \) as follows

\[
F(b, u, v, w, z) = \begin{pmatrix}
d_1 \Delta u + a - u + u^2 v + c(w - u) \\
d_2 \Delta v + b - u^2 v \\
d_1 \Delta w + a - w + w^2 z + c(u - w) \\
d_2 \Delta z + b - w^2 z
\end{pmatrix},
\]

then

\[
F(b, u, v, w, z) : \mathbb{R} \times (W^{2,2}_\nu)^4 \to (L^2(\Omega))^4,
\]

it is also easy to verify that \( D_{(b,u,v,w,z)} F(0, a, 0, a, 0) \) is bijection. Thus, by Lemma 3.2(ii) and Implicit Function Theorem, we complete the proof.

\[\square\]

4. Existence of non-constant solution. In this section, we consider the existence of non-constant positive solutions to the system \((4)\). Now, we study the influence of the size of the region \( \Omega \) on the pattern formation of \((4)\). Let \( \tilde{\Omega} = \Omega \), \( \tilde{\Omega} \), \( \tilde{\Omega} \), \( \tilde{\Omega} \), \( \tilde{\Omega} \), \( \tilde{\Omega} \), \( \tilde{\Omega} \), \( \tilde{\Omega} \), \( \tilde{\Omega} \), \( \tilde{\Omega} \), \( \tilde{\Omega} \), then system \((12)\) can be written as follows

\[
\begin{align*}
-\frac{d_1}{l^2} \Delta \hat{u} &= a - \hat{u} + \hat{u}^2 \hat{v} + c(\hat{w} - \hat{u}), & \text{in } \tilde{\Omega}, \\
-\frac{d_2}{l^2} \Delta \hat{v} &= b - \hat{u}^2 \hat{v}, & \text{in } \tilde{\Omega}, \\
-\frac{d_1}{l^2} \Delta \hat{w} &= a - \hat{w} + \hat{w}^2 \hat{z} + c(\hat{u} - \hat{w}), & \text{in } \tilde{\Omega}, \\
-\frac{d_2}{l^2} \Delta \hat{z} &= b - \hat{w}^2 \hat{z}, & \text{in } \tilde{\Omega}, \\
\frac{\partial \hat{u}}{\partial \nu} = \frac{\partial \hat{v}}{\partial \nu} = \frac{\partial \hat{w}}{\partial \nu} = \frac{\partial \hat{z}}{\partial \nu} = 0, & \text{on } \partial \tilde{\Omega}.
\end{align*}
\]

Let \( \lambda = \frac{l^2}{d_2} \) and \( \theta = \frac{d_1}{d_2} \) writing \( \hat{u}, \hat{v}, \hat{w}, \hat{z} \) and \( \tilde{\Omega} \) instead of \( u, v, w, z \) and \( \Omega \), then
Clearly, $\lambda$ is the measure of the size of the domain. Noting the relations $\lambda = \frac{l^2}{d_2}$ and $\theta = \frac{d_1}{d_2}$ between $\lambda, \theta, d_1$ and $d_2$, we obtain the following corollary from (i) of Theorem 3.1.

**Corollary 1.** Let $\theta, a, b > 0$ be fixed. Then there exists $\hat{\lambda} > 0$, depending on $a$, $b$, $\theta$ and $\Omega$, such that the problem (14) has no non-constant solution for $0 < \lambda \leq \hat{\lambda}$, i.e., the only solution of (14) is $(a + b, \frac{b}{(a + b)^2}; a + b, \frac{b}{(a + b)^2})$.

Next, we consider the existence of non-constant positive solutions to system (13). Denote $\bar{u} = (u, v, w, z)$ and $u^* = (a + b, \frac{b}{(a + b)^2}; a + b, \frac{b}{(a + b)^2})$, define

$$
X = \{u \in (C^1(\overline{\Omega}))^4 | \partial_u u = \partial_u v = \partial_u w = \partial_u z = 0 \text{ on } \partial \Omega \},
$$

$$
X^+ = \{u \in X | u, v, w, z > 0 \text{ on } \Omega \},
$$

$$
B(C) = \{u \in X | C^{-1} < u, v, w, z < C \text{ on } \overline{\Omega}, C > 0 \},
$$

$$
G(\bar{u}) = \begin{pmatrix}
\lambda \theta^{-1}(a - u + u^2v + c(w - u)) \\
\lambda(b - u^2v) \\
\lambda \theta^{-1}(a - w + w^2z + c(u - w)) \\
\lambda(b - w^2z)
\end{pmatrix},
$$

and

$$
A(\lambda, \theta) = D_\bar{u}G(u^*)
$$

$$
= \begin{pmatrix}
\lambda \theta^{-1}(\frac{2b}{a + b} - 1 - c) & \lambda \theta^{-1}(a + b)^2 & \lambda \theta^{-1}c & 0 \\
\lambda \theta^{-1}c & -(a + b)^2 \lambda & 0 & 0 \\
0 & \lambda \theta^{-1}(\frac{2b}{a + b} - 1 - c) & \lambda \theta^{-1}(a + b)^2 \\
0 & 0 & \frac{a + b}{a + b} & -(a + b)^2 \lambda
\end{pmatrix}.
$$

Then system (13) can be written as

$$
\begin{cases}
-\Delta \bar{u} = G(\bar{u}), & \text{in } \Omega, \\
\frac{\partial \bar{u}}{\partial \nu} = 0, & \text{on } \partial \Omega,
\end{cases}
$$

and $\bar{u}$ is a positive solution of (14) if and only if $F(\bar{u}) = 0$, where $F(\bar{u}) = \bar{u} - (I - \Delta)^{-1}(G(\bar{u}) + \bar{u}), \bar{u} \in X^+$. Since $F(\cdot)$ is a compact operator for any $B = B(C)$, the Leray-Schauder degree $deg(F(\cdot), 0, B)$ is well defined if $F(\bar{u}) \neq 0, \bar{u} \in \partial B$. Due to

$$
D_\bar{u}F(u^*) = I - (I - \Delta)^{-1}(A + I),
$$
and if \( D_0 F(\vec{u}_\gamma^*) \) is invertible, it follows that

\[
\text{index}(F, \vec{u}_\gamma^*) = (-1)^\beta,
\]

where \( \beta \) is the number of negative eigenvalue of \( D_0 F(\vec{u}_\gamma^*) \). For any positive integer \( i \geq 0 \), \( X_i \) is invariant under \( D_0 F(\vec{u}_\gamma^*) \) and \( \gamma_i \) is an eigenvalue of \( D_0 F(\vec{u}_\gamma^*) \) on \( X_i \) if and only if \( \gamma_i \) is an eigenvalue of the matrix \( \frac{1}{1 + \mu_i} (\mu_i I - A) \). Therefore, \( D_0 F(\vec{u}_\gamma^*) \) is invertible if and only if the matrix \( \mu_i I - A \) is non-singular for any all \( i \geq 0 \). Denote

\[
H(\mu) = \det(\mu I - A),
\]

if \( H(\mu_i) \neq 0 \), then the number of negative eigenvalue of \( D_0 F(\vec{u}_\gamma^*) \) on \( X_i \) is odd if and only if \( H(\mu_i) < 0 \). Let \( m(\mu_i) \) be the multiplicity of \( \mu_i \), so we can get the following proposition (see [17], Theorem 7.3.9):

**Proposition 3.** Suppose that the matrix \( \mu_i I - A \) is non-singular for all \( i \geq 0 \). Then

\[
\text{index}(F, \vec{u}_\gamma^*) = (-1)^\beta, \quad \text{where} \quad \beta = \sum_{i \geq 0, H(\mu_i) < 0} m(\mu_i).
\]

It is easy to calculate that

\[
H(\mu) = \det(\mu I - A)
\]

\[
= \begin{vmatrix}
\mu - \lambda \theta^{-1} \left( -\frac{2b}{a + b} - 1 - c \right) & -\lambda \theta^{-1} (a + b)^2 & -\lambda \theta^{-1} c & 0 \\
\frac{a + b}{\lambda} & \mu + \lambda (a + b)^2 & 0 & 0 \\
-\lambda \theta^{-1} c & 0 & \mu - \lambda \theta^{-1} \left( -\frac{2b}{a + b} - 1 - c \right) & -\lambda \theta^{-1} (a + b)^2 \\
0 & 0 & \frac{a + b}{\lambda} & \mu + \lambda (a + b)^2
\end{vmatrix}
\]

\[
= [(\mu + \lambda (a + b)^2)(\mu - \lambda \theta^{-1} (\frac{2b}{a + b} - 1)) + 2\lambda^2 \theta^{-1} b(a + b)]
\]

\[
\cdot [(\mu + \lambda (a + b)^2)(\mu - \lambda \theta^{-1} (\frac{2b}{a + b} - 1 - 2c)) + 2\lambda^2 \theta^{-1} b(a + b)].
\]

So if

\[
\frac{2b}{a + b} > [(2c + 1) \frac{1}{2} + (a + b) \theta \frac{1}{2}]^2,
\]

\( H(\mu) = 0 \) has four different positive roots given by

\[
\mu_1(\lambda, \theta) = \frac{\lambda}{2\theta} \left( \frac{2b}{a + b} - 1 - (a + b)^2 \theta - \sqrt{\Delta_1} \right),
\]

\[
\mu_2(\lambda, \theta) = \frac{\lambda}{2\theta} \left( \frac{2b}{a + b} - 2c - 1 - (a + b)^2 \theta - \sqrt{\Delta_2} \right),
\]

\[
\mu_3(\lambda, \theta) = \frac{\lambda}{2\theta} \left( \frac{2b}{a + b} - 2c - 1 - (a + b)^2 \theta + \sqrt{\Delta_2} \right),
\]

\[
\mu_4(\lambda, \theta) = \frac{\lambda}{2\theta} \left( \frac{2b}{a + b} - 1 - (a + b)^2 \theta + \sqrt{\Delta_1} \right),
\]

respectively, where \( \Delta_1(\theta) := \left[ \frac{2b}{a + b} - 1 - (a + b)^2 \theta \right]^2 - 4\theta(a + b)^2 \) and \( \Delta_2(\theta) := \left[ \frac{2b}{a + b} - 1 - 2c - (a + b)^2 \theta \right]^2 - 4(2c + 1) \theta(a + b)^2 \). That is, \( \mu_1(\lambda, \theta), \mu_2(\lambda, \theta), \mu_3(\lambda, \theta) \) and \( \mu_4(\lambda, \theta) \) are four different eigenvalues of \( A(\lambda, \theta) \). Furthermore, \( H(\mu) < 0 \) if and only if \( \mu \in (\mu_1^*, \mu_2^*) \cup (\mu_3^*, \mu_4^*) \), because that \( H(\mu) = (\mu - \mu_1^*)(\mu - \mu_2^*)(\mu - \mu_3^*)(\mu - \mu_4^*) \) and \( 0 < \mu_1^* < \mu_2^* < \mu_3^* < \mu_4^* \).
Theorem 4.1. Suppose that $\frac{2b}{a+b} > [(2c+1)\frac{1}{2} + (a+b)\theta^2]^2$ holds, and there exists 0 ≤ i < j < h < l such that 

(i) $\mu_i^*(\lambda, \theta) \in (\mu_i, \mu_{i+1})$, $\mu_j^*(\lambda, \theta) \in (\mu_j, \mu_{j+1})$, $\mu_k^*(\lambda, \theta) \in (\mu_k, \mu_{k+1})$ and $\mu^*_h(\lambda, \theta) \in (\mu_{h}, \mu_{h+1})$,

(ii) $\sum_{k=i+1}^h m(\mu_k) + \sum_{k=h+1}^l m(\mu_k)$ is odd.

Then (13) has at least one non-constant solution.

Proof. Let $\bar{\lambda}$ be sufficiently small, such that $\mu_i^*(\bar{\lambda}, \theta)$, $\mu_j^*(\bar{\lambda}, \theta)$, $\mu_k^*(\bar{\lambda}, \theta)$ and $\mu^*_h(\bar{\lambda}, \theta) < \mu_1$. Meanwhile, by Corollary 1, (13) has no non-constant solution. For 0 ≤ t ≤ 1, define 

$$G(\bar{u}, t) = \left( \begin{array}{c} (I - \Delta)^{-1}(t\lambda + (1 - t)\bar{\lambda})\theta^{-1}(a + u^2v + c(w - u)) \\ (I - \Delta)^{-1}(t\lambda + (1 - t)\bar{\lambda})(b - u^2v + v) \\ (I - \Delta)^{-1}(t\lambda + (1 - t)\bar{\lambda})\theta^{-1}(a + w^2z + c(u - w)) \\ (I - \Delta)^{-1}(t\lambda + (1 - t)\bar{\lambda})(b - u^2z + z) \end{array} \right).$$

Proposition 1 implies that there exist positive constants $C_1 = C_1(\alpha, \beta, \theta, \bar{\lambda}) > 0$ and $C_2 = C_2(\alpha, \beta, \theta, \lambda)$ such that $E \subset (C^1(\Omega))^4 | \bar{u} < u, v, w, z < C_2$.

From $G(\bar{u}, t) : E \times [0, 1] \to (C^1(\Omega))^4$ is compact, the degree $\text{deg}(I - G(\bar{u}, t), \mathcal{E}, 0)$ is well defined. Due to the choice of $\bar{\lambda}$ and the Proposition 3 we have 

$$\text{deg}(I - G(\bar{u}, 0), \mathcal{E}, 0) = \text{index}(G(\bar{u}, 0), \bar{u}_*^*) = 1. \quad (16)$$

If system (13) has no solution, then by Proposition 3 we get 

$$\text{deg}(I - G(\bar{u}, 1), \mathcal{E}, 0) = \text{index}(G(\bar{u}, 1), \bar{u}_*) = (-1)^{\sum_{k=i+1}^h m(\mu_k) + \sum_{k=h+1}^l m(\mu_k)} = -1. \quad (17)$$

Applying the homotopy invariance of degree, we obtain 

$$1 = \text{deg}(I - G(\bar{u}, 0), \mathcal{E}, 0) = \text{deg}(I - G(\bar{u}, 1), \mathcal{E}, 0) = -1. \quad (18)$$

A contradiction. Therefore, system (13) has at least one non-constant solution. \qed

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