Optimal Finite Homogeneous Sphere Approximation

Omer Lavi

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Abstract
The two-dimensional sphere can’t be approximated by finite homogeneous spaces. We describe the optimal approximation and its distance from the sphere. We compare this distance to the distance achieved by all Platonic and Archimedean solids.

Keywords
Finite approximation · Coxeter groups · Platonic solids · Archimedean solid

Mathematics Subject Classification
20F55 · 20H15 · 20B25

1 Introduction

Tsachik Gelander and Itai Benjamini asked me what is the optimal approximation of the sphere by a finite homogeneous space (see also [5, Rem. 1.5]). It was known that the sphere cannot be approximated arbitrarily well by finite homogeneous spaces, but not which (if any) finite homogeneous space gives the best approximation of it.

We call a subset $X \subseteq \mathbb{R}^n$ intrinsically homogeneous if there is a subgroup $G \leq \text{Isom}(\mathbb{R}^n)$ which preserves it (i.e., $G(X) = X$) such that the induced action $G \curvearrowright X$ is transitive. Observe that if $X$ is finite then, up to conjugation, $G$ must be contained in $O_n(\mathbb{R})$ (the subgroup of $n \times n$ orthogonal matrices). Denote by $d_H$ the Hausdorff metric in $\mathbb{R}^n$ (see below). A metric space $A \subseteq \mathbb{R}^n$ is called approximable if, for every $\varepsilon > 0$, there is a finite, intrinsically homogeneous subset $X \subseteq \mathbb{R}^n$ such that $d_H(X, A) < \varepsilon$. Alternately, a finite intrinsically homogeneous subset $X \subseteq \mathbb{R}^n$ is called an optimal approximation of a subset $A \subseteq \mathbb{R}^n$ if

$$d_H(X, A) \leq d_H(Y, A)$$

1 Weizmann Institute of Science, Mizra, Israel
for every other finite intrinsically homogeneous subset $Y \subseteq \mathbb{R}^n$. In this case, $d_H(X, A)$ is called the \textit{approximation distance} of $A$.

\textbf{Example 1.1} The $n$-dimensional torus is approximable, by a sequence of regular, transitive graphs with bounded geometry (see [3]).

Gelander showed that this is the only example: a compact manifold can be approximated by finite homogeneous metric spaces if and only if it is a torus [5, Cor. 1.3]).

Our main result is as follows:

\textbf{Main Theorem 1.2} The optimal approximation of the sphere $\mathbb{S}^2$ is a subset of size 120, which are the orbit of the Chebyshev center of the fundamental domain of the group $H_3$ (see below). Its faces consist of squares, regular hexagons and regular decagons. The approximation distance, calculated up to four digits, is 0.3208.

\textbf{Remark 1.3} Note that this subset does not consist of an Archimedean solid. We will compute below the Hausdorff distance from the sphere to each Platonic and Archimedean solid.

\textbf{Remark 1.4} Observe that we restrict our attention to intrinsically homogeneous spaces and that we use the Hausdorff distance inside $\mathbb{R}^3$, rather than the more abstract concept of \textit{Gromov–Hausdorff distance}. It is an interesting question whether an optimal approximation for the sphere exist in this sense, and what it is if it does. Note that such a space might not be embeddable in $\mathbb{R}^3$.

\textbf{Notation 1.5} Henceforth, by “homogeneous” we shall mean intrinsically homogeneous, since we restrict our attention to such subsets.

\subsection*{1.1 The Hausdorff Distance}

Recall the definition of the \textit{Hausdorff distance} between sets.

\textbf{Definition 1.6} Let $(X, d)$ be a metric space. Let $A, B$ be two subsets in $X$. We define the Hausdorff distance $d_H(A, B)$ as

$$d_H(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\}.$$  

Informally, two sets are \textit{close} in the Hausdorff distance if every point of either set is close to a point in the other. Alternatively it measure how much one should \textit{inflate} each set so that it will contain the other. The Hausdorff distance is widely used both in pure and applied mathematics. It is also used in fields such as computer vision, image processing, computer graphics, pattern recognition and more (see for example [7, 10]).

In the case of measuring how close a set is to the sphere or equivalently, how \textit{round} is a
set one can adopt also a definition that takes into account the ratio between the square of the volume of the set and the square of its surface area (normalized appropriately). This definition of roundness was given by Pólya in [9]. Given a finite homogeneous set, one can measure the roundness of its convex hull to the sphere in Pólya’s sense. It is natural then to ask if a best approximation exist and if so what it is. Aravind addressed this problem in [1], where he computed the \textit{roundness} (in Pólya’s sense) of all Platonic and Archimedean solids and found the best approximation among them. It is interesting to mention that the best approximation among Archimedean solids in Pólya’s sense (namely the Snub Dodecahedron) is different than the one we found (see Table 3). Another way of measuring roundness of a set is the ratio between the inscribed radius and circumscribed radius (see [8]). Huybers described how one can get better approximation to the sphere in that sense however many of his examples are not homogeneous. As is explained by both Aravind and Huybers, the task of finding the \textit{roundest polytope} attracts a lot of attention as it is related to soccer. Scientific papers as well as commercial effort was made in this direction. (See for example patents issued by Nike and others referenced in [8].)

**Archimedean Solids**

The mentioned above Archimedean and Platonic solids give us examples of special homogeneous spaces. The five Platonic solids, \textit{tetrahedron}, \textit{cube}, \textit{octahedron}, \textit{dodecahedron}, and \textit{icosahedron} were studied extensively already by the ancient Greeks. It is believed that Theaetetus was the first one to discover all five and to give the first proof that no other convex regular polyhedra exist. In addition to their beauty they appear in surprisingly many places in nature. Maybe the most famous are crystals but also in the structure of other minerals, viruses and more. As their name suggests the Archimedean solids were also studied by the ancient Greeks, it is believed that Archimedes already counted them all (for an account on this subject see [6]). Independently, almost two millennia later Kepler rediscovered them. Archimedean solids also appears in many places in nature. Many of them for example appear in the structure of fullerenes, others in quasicrystals, the medicine industry, chemistry, physics and more.

Although Platonic and Archimedean solids were studied and well understood years before people ever started to study groups, it is very natural to construct them using group theory and more precisely the theory of discrete subgroups of $O_3(\mathbb{R})$. This was observed by Coxeter and others (see Sect. 2). It can be shown that the vertices of these solids lie on the orbit of special points, namely points that are stabilized by certain subgroups of $O_3(\mathbb{R})$. The theory of discrete reflection groups was developed in the early 20th century. It combines ideas from combinatorics and group theory and gives a complete description of such groups. Today we call these groups Coxeter groups, after the man who gave such a big contribution to the understanding of them. In the interplay between understanding groups through their combinatorics and geometry, this paper try to do the way back. Our main idea is to use the well-developed theory of Coxeter groups to study finite homogeneous sets. The idea is to study finite homogeneous sets by means of their isometry groups, the Coxeter groups. We will describe the classification of Coxeter groups based on his paper [4]. We will divide these groups
into two families. The first is an infinite family of almost cyclic groups (i.e., these groups have an index two cyclic subgroup) and the second consist of three groups that have a far more interesting action on $\mathbb{R}^3$. Next we will construct a fundamental domain to each group. A fundamental domain related to a group $G$ acting on the sphere is a set on the sphere which has the property that its (finitely many) copies under the action of $G$ cover the sphere. It will turn out that in the search of optimal sphere approximation we can restrict to orbits of the center of fundamental domains (see Corollary 2.18). This will give us three candidates for optimal approximation (i.e., the three centers related to three Coxeter groups). The optimal approximation will be the best among these three.

2 Finite Reflection Groups

In this section we give a short summary of the results we will need from the theory of finite Coxeter groups.

If $R \in O_3(\mathbb{R})$ is a reflection through the plane $\mathcal{R} \subseteq \mathbb{R}^3$, then $\mathcal{R}$ is exactly the set of fixed points of $R$. If $R_1, R_2 \in O_3(\mathbb{R})$ are two reflections through the planes $\mathcal{R}_1, \mathcal{R}_2 \subseteq \mathbb{R}^3$, then the composition $R_1 \circ R_2$ (which we will denote hereafter simply as $R_1R_2$) is a rotation around the axis $\mathcal{R}_1 \cap \mathcal{R}_2$ (which is a line).

Suppose $R_1, R_2, R_3 \in O_3(\mathbb{R})$ are three reflections, reflecting through the planes $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$ respectively. Suppose the angle between $\mathcal{R}_1$ and $\mathcal{R}_2$ is $\pi/n$, the angle between $\mathcal{R}_2$ and $\mathcal{R}_3$ is $\pi/m$, and the angle between $\mathcal{R}_3$ and $\mathcal{R}_1$ is $\pi/2$. Then it is a result of Coxeter that the group $\langle R_1, R_2, R_3 \rangle$ is finite, and admits the following presentation:

$$\langle R_1, R_2, R_3 \rangle \cong \langle a, b, c : a^2 = b^2 = c^2 = (ab)^n = (bc)^m = (ca)^2 = \text{id} \rangle.$$ 

Moreover, the subgroup $\langle R_1, R_2, R_3 \rangle \leq O_3(\mathbb{R})$ is determined up to conjugation by these angles. We denote it by $[n, m]$. We moreover denote

$$S_1 = R_1R_2, \quad S_2 = R_2R_3, \quad S_3 = R_3R_1,$$

so that $S_1$ is a rotation around the axis $\mathcal{R}_1 \cap \mathcal{R}_2$ by an angle of $2\pi/n$, $S_2$ is a rotation around the axis $\mathcal{R}_2 \cap \mathcal{R}_3$ by an angle of $2\pi/m$ and $S_3$ is a rotation around the axis $\mathcal{R}_1 \cap \mathcal{R}_3$ by an angle of $\pi$. We have $S_3 = S_1S_2$, and the subgroup $\langle S_1, S_2 \rangle$ is of index two inside $[n, m]$.

Let $n \in \mathbb{N}$. The group $[2, n]$ admits a subgroup of index two which is cyclic of order $n$; it consists of rotations, which all have the same axis.

**Theorem 2.1** (Coxeter) Let $G \leq O_3(\mathbb{R})$ be a finite subgroup. Then $G$ is contained in one of the following groups: $[2, n]$ (for some $n$), $[3, 3]$, $[3, 4]$, or $[3, 5]$.

**Notation 2.2** We will call the groups $[3, 3]$, $[3, 4]$, and $[3, 5]$ the tetrahedral group (and write $A_3$), octahedral group (or $B_3$), and icosahedral group ($H_3$) respectively.
2.1 Orbits of Coxeter Groups

Recall that when a group $G$ acts on a set $X$, the orbit of a point $x \in X$ under this action is denoted by $Gx$. Obviously, if $X \subseteq \mathbb{R}^3$ is a finite homogeneous subset, then $X = Gx$ for some $x \in \mathbb{R}^3$ and some finite $G < O_3(\mathbb{R})$. Therefore, it is enough to check the Hausdorff distances of subsets of this form.

If $G \leq [2, n]$, then the rotations in $G$ have only one axis of rotation. The orbit of a point under the action of $G$ is contained in at most two latitudes. The Hausdorff distance from the sphere would be lowest if they are at an angle of $\pi/4$ from the equator (possibly on a sphere of a shorter radius). In that case, the distance from any point $x$ in the orbit to the north pole $P$ is $d(x, P) = \sqrt{2}/2$. Therefore, the Hausdorff distance between this orbit and the sphere is at least $\sqrt{2}/2 > 0.7$. This means this family is irrelevant to our quest.

We are mainly interested in groups that have more than one axis for their rotations, namely the $[3, n]$ family. Set $V = \mathbb{R}^3$. The volume enclosed by the reflecting planes is then a fundamental domain for the action of the group on $\mathbb{R}^3$. In the sequel we will denote this fundamental domain by $B$. For the action on the unit sphere we have then a fundamental domain which is an intersection of $B$ and the sphere. We will denote this domain by $B'$. Since the group is generated by three reflections, the fundamental domain is a spherical triangle. If $G = \langle R_1, R_2, R_3 \rangle$ and we set $G_1 = \langle R_1, R_2 \rangle$, $G_2 = \langle R_2, R_3 \rangle$, $G_3 = \langle R_3, R_1 \rangle$, then a choice of points in $V^{G_1} \cap S^2$, $V^{G_2} \cap S^2$, $V^{G_3} \cap S^2$ gives us the vertices of this triangle and a choice of respective segments in $V^{R_i} \cap S^2$ gives the sides of this triangle. We will therefore denote the groups $G_i$ as vertex groups and the groups $R_i$ as edge groups. Note that every group element in $G$ is conjugated to an element in a vertex groups.

Given a group $G$ we would like to find the point whose orbit is closest to the sphere among all other $G$ orbits.

**Definition 2.3** Given a group $G < O_3(\mathbb{R})$ we say that $x \in \mathbb{R}^3$ is an optimal $G$ sphere approximation if

$$d_H(Gx, S^2) = \inf_{y \in \mathbb{R}^3} d_H(Gy, S^2).$$

The search for optimal approximation can be limited to fundamental domain:

**Remark 2.4** If $B \subseteq \mathbb{R}^3$ is a fundamental domain for the action of $G$ then $x \in B$ is optimal $G$ approximation if and only if

$$d_H(Gx, S^2) = \inf_{y \in B} d_H(Gy, S^2).$$

We can define a distance between groups and the sphere.

**Definition 2.5** If $G$ is a group the we define the distance between $G$ and the sphere as

$$d_H(G, S^2) = \inf_{y \in \mathbb{R}^3} d_H(Gy, S^2).$$
Remark 2.6 Note that if $B$ is a fundamental domain and $B' = B \cap S^2$ then

$$d_{\text{H}}(G, S^2) = \inf_{x \in B} \sup_{y \in B'} d(Gx, y).$$

The next result will provide a candidate for optimal approximation. The following lemma is well known and its proof is straightforward (see for example [2, Lem. 2.2.7]).

Lemma 2.7 (lemma on the center) Let $V$ be a real or complex Hilbert space, and let $A \subset V$ be a non-empty bounded set of $V$. Among all closed balls in $V$ containing $A$, there exist a unique one with minimum radius.

Definition 2.8 The center of the unique closed ball with minimal radius containing $A$ in the previous lemma is called Chebyshev center (or circumcenter) of $A$.

Example 2.9 Let $A$ be an Euclidean triangle. In this case finding the Chebyshev center is an easy task. For obtuse or right triangle the center is just at the middle of the long side. Otherwise it is at the intersection of the perpendicular bisectors of the sides of the triangle.

Remark 2.10 Since the fundamental domain $B'$ is not convex the Chebyshev center is not contained in $B'$. Usually it will not be on the unit sphere $S^2$ but inside the ball of radius one.

2.2 Fundamental Domain of Coxeter Groups

We describe now the fundamental domains of the three-dimensional Coxeter groups, $A_3$, $B_3$, and $H_3$. Our task then is to find the three reflecting planes. As the group $O_3(\mathbb{R})$ is acting transitively on two-dimensional subspaces, we can assume without any loss of generality that one of the planes, which we denote by $\mathcal{R}_1$, is $XY$, i.e., that it is defined by the normal vector $\mathcal{R}_1^\perp = (0, 0, 1)$. Note that $O_2(\mathbb{R})$ is embedded naturally in $O_3(\mathbb{R})$ as the stabilizer of $\mathcal{R}_1$, and acts transitively on the collection of one-dimensional subspaces. We can assume then that $\mathcal{R}_3$ is the plane $XZ$. It is perpendicular to $\mathcal{R}_1$ and its normal is $\mathcal{R}_3^\perp = (0, 1, 0)$. (We number this plane by $\mathcal{R}_3$ to be consistent with the notation of the Coxeter groups.) The last space, $\mathcal{R}_2$, whose unit normal we denote by $\mathcal{R}_2^\perp$, intersects the other two with angles of $\pi/3$ and $\pi/m$ respectively. Set $\mathcal{R}_2^\perp = (x, y, z)$, then

$$\langle \mathcal{R}_1^\perp, \mathcal{R}_2^\perp \rangle = z = \cos \frac{\pi}{3} = \frac{1}{2}, \quad \langle \mathcal{R}_2^\perp, \mathcal{R}_3^\perp \rangle = y = \cos \frac{\pi}{m}.$$

Next, since we restrict our attention to unit vectors we can compute $x$:

$$1 = \langle \mathcal{R}_2^\perp, \mathcal{R}_2^\perp \rangle = x^2 + \cos^2 \frac{\pi}{m} + \cos^2 \frac{\pi}{3}.$$

We can calculate now spherical triangles for the three groups.
The tetrahedral group $A_3$: The group $A_3 = [3, 3]$ is generated by three reflections whose angles are $\pi/3$, $\pi/3$, and $\pi/2$. As explained above, there is a fundamental domain described by the following three unit normal vectors:

1. $R_1^\perp = (0, 0, 1)$,
2. $R_2^\perp = (1/\sqrt{2}, 1/2, 1/2)$,
3. $R_3^\perp = (0, 1, 0)$.

Now we wish to construct a spherical triangle. The intersection $R_1 \cap R_3$ is the $X$ axis, we denote the first vertex $z = (1, 0, 0)$. Next $R_1 \cap R_2 = \{(t, -2t/\sqrt{2}, 0) : t \in \mathbb{R}\}$. Therefore we set $x = (1/\sqrt{3}, -2/\sqrt{6}, 0)$. Finally by symmetry, $y = (1/\sqrt{3}, 0, -2/\sqrt{6})$.

The octahedral group $B_3$: The group $B_3 = [3, 4]$ is generated by three reflections whose angles are $\pi/3$, $\pi/4$, and $\pi/2$. As explained above, there is a fundamental domain described by the following three unit normal vectors:

1. $R_1^\perp = (0, 0, 1)$,
2. $R_2^\perp = (1/\sqrt{2}, 1/\sqrt{2}, 1/2)$,
3. $R_3^\perp = (0, 1, 0)$

Now we wish to construct a spherical triangle. The intersection $R_1 \cap R_3$ is the $X$ axis, we denote the first vertex $z = (1, 0, 0)$. Next $R_1 \cap R_2 = \{(t, -t\sqrt{2}/2, 0) : t \in \mathbb{R}\}$. Therefore we set $x = (\sqrt{2}/\sqrt{3}, -1/\sqrt{3}, 0)$. Finally $R_2 \cap R_3 = \{(t, 0, -t) : t \in \mathbb{R}\}$. Therefore we set $y = (1/\sqrt{2}, 0, -1/\sqrt{2})$. (See Fig. 2, $z$ is represented by the red dot, $y$ is the blue dot and the red triangle represents $x$.)

The icosahedral group $H_3$: The group $H_3 = [3, 5]$ is generated by three reflections whose angles are $\pi/3$, $\pi/5$, and $\pi/2$. As explained above, there is a fundamental domain described by the following three unit normal vectors (Fig. 1):

1. $R_1^\perp = (0, 0, 1)$,
2. $R_2^\perp = (\sqrt{3}/4 - \cos^2(\pi/5), \cos(\pi/5), 1/2)$,
3. $R_3^\perp = (0, 1, 0)$.
Now we wish to construct a spherical triangle. The intersection \( \mathcal{R}_1 \cap \mathcal{R}_3 \) is the \( X \) axis, we denote the first vertex \( z = (1, 0, 0) \). Next:

\[
\mathcal{R}_1 \cap \mathcal{R}_2 = \left\{ \left( t \cos \frac{\pi}{5}, -t \sqrt{\frac{3}{4} - \cos^2 \frac{\pi}{5}}, 0 \right) : t \in \mathbb{R} \right\}.
\]

Therefore we set

\[
x = \frac{2}{\sqrt{3}} \left( \cos \frac{\pi}{5}, -\sqrt{\frac{3}{4} - \cos^2 \frac{\pi}{5}}, 0 \right).
\]

Finally,

\[
\mathcal{R}_2 \cap \mathcal{R}_3 = \left\{ \left( \frac{t}{2}, 0, -t \sqrt{\frac{3}{4} - \cos^2 \frac{\pi}{5}} \right) : t \in \mathbb{R} \right\},
\]

so set

\[
y = \frac{1}{\sin (\pi/5)} \left( \frac{1}{2}, 0, -\sqrt{\frac{3}{4} - \cos^2 \frac{\pi}{5}} \right).
\]

(See Fig. 2, \( z \) is represented by the red dot, \( y \) is the blue dot and the red triangle represents \( x \).)

We summarize the above in Table 1 (for later use we calculated also the midpoints of every side):

| Group \((t, s)\) | Chebyshev center | \(d_H(G, S^2)\) |
|------------------|------------------|------------------|
| \(A_3\) \((0.3255, 0.0872)\) | \((0.6511, -0.3370, -0.3370)\) | 0.5907 |
| \(B_3\) \((0.3929, 0.0397)\) | \((0.7858, -0.2498, -0.3255)\) | 0.4628 |
| \(H_3\) \((0.4485, 0.0153)\) | \((0.8971, -0.1655, -0.2548)\) | 0.3208 |

The groups names are according to the Coxeter notation, \((t, s)\) is the solution to (2) and \(d_H(G, S^2) = d(c, x) = d(c, y) = d(c, z)\).
|       | $A_3$                      | $B_3$                      | $H_3$                      |
|-------|---------------------------|---------------------------|---------------------------|
| $z$   | $(1, 0, 0)$               | $(1, 0, 0)$               | $(1, 0, 0)$               |
| $x$   | $\left( \frac{1}{\sqrt{3}}, -\frac{2}{\sqrt{6}}, 0 \right)$ | $\left( \frac{\sqrt{3}}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, 0 \right)$ | $\frac{2}{\sqrt{3}} \left( \cos \frac{\pi}{5}, -\sqrt{\frac{3}{4} - \cos^2 \frac{\pi}{5}}, 0 \right)$ |
| $y$   | $\left( \frac{1}{\sqrt{3}}, 0, -\frac{2}{\sqrt{6}} \right)$ | $\left( \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \right)$ | $\frac{1}{\sin \left( \frac{\pi}{5} \right)} \left( \frac{1}{2}, 0, -\sqrt{\frac{3}{4} - \cos^2 \frac{\pi}{5}} \right)$ |
| $m_1$ | $\left( \frac{1 + \sqrt{3}}{2}, -\frac{1}{\sqrt{6}}, 0 \right)$ | $\left( \frac{\sqrt{3} + \sqrt{3}}{2 \sqrt{3}}, -\frac{1}{2 \sqrt{3}}, 0 \right)$ | $\frac{1}{\sqrt{3}} \left( \cos \frac{\pi}{5} + \frac{\sqrt{3}}{2}, -\sqrt{\frac{3}{4} - \cos^2 \frac{\pi}{5}}, 0 \right)$ |
| $m_2$ | $\left( \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}} \right)$ | $\left( \frac{2 + \sqrt{3}}{2 \sqrt{6}}, -\frac{1}{2 \sqrt{3}}, -\frac{1}{2 \sqrt{2}} \right)$ | $\frac{\sqrt{3/4 - \cos^2 \left( \frac{\pi}{5} \right)}}{\sqrt{3}} \left( \frac{\sqrt{3 \sin \left( \frac{\pi}{5} \right) + 4 \cos \left( \frac{\pi}{5} \right)}}{4 \sqrt{3/4 - \cos^2 \left( \frac{\pi}{5} \right)}}, -1, -\frac{\sqrt{3}}{2 \sin \left( \frac{\pi}{5} \right)} \right)$ |
| $m_3$ | $\left( \frac{1 + \sqrt{3}}{2 \sqrt{3}}, 0, -\frac{1}{\sqrt{6}} \right)$ | $\left( \frac{1 + \sqrt{3}}{2 \sqrt{2}}, 0, -\frac{1}{2 \sqrt{2}} \right)$ | $\frac{1}{2 \sin \left( \frac{\pi}{5} \right)} \left( \frac{1}{2}, \sin \frac{\pi}{5}, 0, -\sqrt{\frac{3}{4} - \cos^2 \frac{\pi}{5}} \right)$ |
2.3 The Optimal Sphere Approximation of a Group

We will now show that for a fixed $n$, the Coxeter group $[3,n]$ has an optimal sphere approximation. Note that if two points $x, y \in \mathbb{R}^3$ are in the same side of a plane $R$, then reflecting $x$ about this plane will enlarge the distance from $y$. The next few lemmas will show that this phenomena happens for all the elements of $G$. Fix $n \in \{3, 4, 5\}$ and set $G = [3,n]$ (Table 2).

**Lemma 2.11** Let $R$ be a reflection about the plane $R$. Let $R^\perp$ be a normal to $R$, and suppose that $x, y \in \mathbb{R}^3$ are two points with

$$\langle x, R^\perp \rangle \cdot \langle y, R^\perp \rangle \geq 0,$$

then $d(x, y) \leq d(Rx, y)$.

**Proof.** Note that for two points $x, y \in \mathbb{R}^3$, the distance $d(x, y)^2$ is given by

$$d(x, y)^2 = \|x - y\|^2 = \langle x - y, x - y \rangle = \|x\|^2 + \|y\|^2 - 2 \langle x, y \rangle.$$

Denote by $P_{R}x$ as the orthogonal projection of $x$ on $R$ and $P_{R^\perp}x$, as the orthogonal projection of $x$ on $R^\perp$. Then

$$\langle P_{R^\perp}x, P_{R^\perp}y \rangle \geq 0.$$ It follows that

$$d(Rx, y)^2 = \|Rx\|^2 + \|y\|^2 - 2 \langle Rx, y \rangle$$

$$= \|x\|^2 + \|y\|^2 - 2 [\langle RP_{R}x, P_{R}y \rangle + \langle RP_{R^\perp}x, P_{R^\perp}y \rangle]$$

$$= \|x\|^2 + \|y\|^2 - 2 [\langle P_{R}x, P_{R}y \rangle - \langle P_{R^\perp}x, P_{R^\perp}y \rangle]$$

$$= \|x - y\|^2 + 2 \langle P_{R^\perp}x, P_{R^\perp}y \rangle \geq d(x, y)^2. \quad \square$$

**Lemma 2.12** Let $R_1, R_2$ be two reflection with angle $\angle (R_1, R_2) = \varphi \leq \pi/2$ and let $x$ be a point with $\angle (x, R_i) \leq \varphi$ (for $i = 1, 2$). Then

$$\langle x, R_1^\perp \rangle \cdot \langle R_2x, R_1^\perp \rangle \geq 0.$$

**Proof** Indeed, if $x \in R_1$ then $\langle x, R_1^\perp \rangle = 0$. Otherwise $0 < \angle (R_2x, R_1) < \pi$ hence for every point $p$ in the segment $[x, R_2x]$, we have $0 < \angle (p, R_1) < \pi$, so the segment $[x, R_2x]$ has no $R_1$ fixed point. If we had

$$\langle x, R_1^\perp \rangle \cdot \langle R_2x, R_1^\perp \rangle < 0$$

then by the intermediate value theorem we had a point $p \in [x, R_2x]$ that was fixed by $R_1$. \square

We now show that rotations also enlarge distances.
Lemma 2.13 Let $R_1, R_2$ be two reflection with angle $\angle (R_1, R_2) = \varphi \leq \pi/2$ and let $x, y$ by two points with $\angle (x, R_i) \leq \varphi$ and $\angle (y, R_i) \leq \varphi$ (for $i = 1, 2$). Set $S = R_1 R_2$, then

$$d(Sx, y) \geq d(x, y).$$

**Proof.** Indeed,

$$d(Sx, y) = d(R_i R_j x, y) = d(R_j x, R_i y).$$

Now $\langle R_i y, R_j^\perp \rangle \langle x, R_j^\perp \rangle \geq 0$ by Lemma 2.12. It follows from Lemma 2.11 (applied twice) that

$$d(Sx, y) = d(R_j x, R_i y) \geq d(x, R_i y) \geq d(x, y).$$

Tsachik Gelander suggested the following lemma which makes proofs much easier.

Lemma 2.14 Let $x, y \in B$ two points in the fundamental domain, and $g \in G$ be any element. Then for every generating reflection $R_i$,

$$\langle gx, R_i^\perp \rangle \cdot \langle gy, R_i^\perp \rangle \geq 0.$$

**Proof** Assume first that $x, y$ are inner points. Suppose towards contradiction that $\langle gx, R_i^\perp \rangle \cdot \langle gy, R_i^\perp \rangle \leq 0$. Since inner product is a continuous function there is a point $z \in [gx, gy]$ with $\langle z, R_i^\perp \rangle = 0$. Then $g^{-1}z$ is in $B$ and is fixed by $R_i g$ which is a contradiction since $B$ is a fundamental domain. If one (or both) points are at the boundary then the lemma follows by the continuity of the inner product.

**Corollary 2.15** Let $x, y \in B$ and $g$ be any element of $G$ then $d(R_i gx, gy) \geq d(x, y)$.

**Proof** Follows from Lemmas 2.14 and 2.11.

Lemma 2.16 Let $R_1, R_2$ be two reflections whose angle is $\pi/n$ for some $n \in \{2, 3, 4, 5\}$. Let $H$ be the vertex group generated by $R_1, R_2$. Let $B$ the fundamental domain for the action of $H$, enclosed by the planes associated with $R_1, R_2$. Then for any $x, y \in B$ and $g \in H$

$$d(x, y) \leq d(gx, y).$$

**Proof** Let $R_i, i = 1, 2$, denote the reflecting planes of $R_i$ respectively and let $U = R_1 \cap R_2$. Let $H'$ be the group generated by $S = R_1 R_2$. Then $H'$ is a cyclic group of rotations about the axis $U$. Note that $[H : H'] = 2$, and that $|H| = n$.

Assume first that $g \in H'$, so $g = S^k$ for some $1 \leq k \leq n$. If $k \leq n/2$ then $g$ is a rotation at angle $k \pi /n$. It is a product of $R_1$ and an image of $R_2, R'$. Note that since $k \leq n/2$, $\angle (R_1, R') \leq \pi/2$ we have $d(x, y) \leq d(gx, y)$ by Lemma 2.13. The other case is similar. If $k > n/2$, then $g = S^{-(n-k)}$ and $g$ is a rotation which is a product of
Let $g \in H \setminus H'$ then $g = S^k R_1$ for some element $k$. We distinguish between two cases.

- Assume first that $k = 2l$. Note that $\frac{\pi}{2} \leq \angle (R_2, R') \leq \pi$. Therefore,

  $$d(gx, y) = d(R_1S^{-l}x, S^{-l}y).$$

  The result follows from Corollary 2.15 applied with $g = S^{-l}$.

- Now assume that $k = 2l + 1$, then

  $$d(gx, y) = d(R_2R_1S^lx, R_1S^ly).$$

  The result follows from Corollary 2.15 applied with $g = R_1S^l$. \hfill $\square$

Recall that for a given set of points $A \subset X$ in a metric space $X$, we can define Dirichlet domains related to $A$. The points in $A$ are called seeds. Every point $x \in X$ is labeled by the seed that is closest to $x$ (in $A$). This partition of $X$ is called a Voronoi diagram and every cell is called a Dirichlet domain. We now show that the fundamental domain is a Dirichlet domain related to the orbit of any point in $B$, and the set of copies of $B$ gives a tessellation of $\mathbb{R}^3$ which is a Voronoi diagram.

**Proposition 2.17** Let $g \in G$ be any element. Then, for every $x, y \in B$,

$$d(x, y) \leq d(gx, y).$$

Equivalently, $x$ is the nearest point to $y$ in $Gx$.

**Proof** If $g \in G$ is any element then $g = h^{-1}g'h$ for some $g'$ in one of the vertex groups. If $g' \in G'$ is a rotation then by Lemma 2.13

$$d(gx, y) = d(g'hx, hy) \geq d(x, y).$$

Otherwise as in Lemma 2.16, $g$ is conjugated to some $R_i$ and the lemma follows from Corollary 2.15. \hfill $\square$

**Corollary 2.18** If $c$ is the Chebyshev center then $Gc$ is an optimal $G$ sphere approximation.

**Proof** Recall that an optimal $G$ sphere approximation is such that

$$d_H(Gc, S^2) = \inf_{x \in B} \sup_{y \in B'} d_H(Gx, y).$$

By Proposition 2.17,

$$d_H(Gx, y) = d(x, y).$$
So

\[ d_H(Gc, S^2) = \inf_{x \in B} \sup_{y \in B'} d(x, y). \]

On the other hand, for every \( x \),

\[ \sup_{y \in B'} d(x, y) \]

is the radius of the smallest ball centered at \( x \) and containing \( B' \) and the proof follows.

\[ \square \]

### 2.4 The Chebyshev Center

Let \( G \) be one of the Coxeter groups, \( A_3, B_3, \) and \( H_3 \). Let \( B' \) be the spherical triangle which is a fundamental domain for the action of \( G \) on the sphere. Our goal now is to show that the minimal ball containing its vertices contains \( B' \). In light of Example 2.9 this will make the task of finding the center fairly easy.

**Lemma 2.19** Let \( G \) be one of the groups, \( A_3, B_3, \) and \( H_3 \). Let \( B \) and \( B' \) be its fundamental domains for the action on \( \mathbb{R}^3 \) and the sphere, respectively. Set \( \{x, y, z\} \) as the vertices of the spherical triangle defining \( B' \). Denote \( c \) as the Chebyshev center of the triple \( \{x, y, z\} \). Then \( c \) is also the Chebyshev center of \( B' \).

**Proof** Clearly it is enough to show that for any two vertices, say \( x, y \in V^{R1} \), and any point \( p \) in the spherical segment \( [x, y]^s \) related to the segment \( [x, y] \),

\[ d(c, p) \leq d(c, x). \]

Denote the orthogonal projection of \( c \) on \( V^{R1} \) by \( c' \). For any point \( q \in V^{R1} \) the Pythagorean theorem implies

\[ d^2(q, c) = d^2(q, c') + d^2(c, c'). \]

It is therefore enough to show that

\[ d(p, c') \leq d(x, c'). \]

Note that it follows from Table 2.1 that for all three groups and all segments \( [a, b] \) of the spherical triangles we have

\[ \|m_i\| \geq \|c\| \geq \|c'\|, \]

in particular, we can assume without any loss of generality that \( \angle([0, p], [0, c']) \leq \pi/2 \) (otherwise argue for \( y \)), so we obtain

\[ 0 \leq \angle([0, p], [0, c']) \leq \angle([0, x], [0, c']) \leq \frac{\pi}{2}. \quad (1) \]
Set $a = \|c'\|$. It follows from the theorem of cosine with respect to the triangle $x, 0, c'$ that

$$d^2(x, c') = 1 + a^2 - 2a \cos (\langle [x, 0], [0, c'] \rangle)$$

Implementing the same theorem for the triangle $p, 0, c'$ we get

$$d^2(p, c') = 1 + a^2 - 2a \cos (\langle [p, 0], [0, c'] \rangle).$$

The result then follows from (1).

## 2.5 The Chebyshev Center Related to Each of the Groups

Now we find the Chebyshev centers of all three fundamental domains. The center whose distance to any (all) vertex of the spherical triangle is the shortest among all three groups, will give the optimal finite homogeneous approximation. To this end, we find the perpendicular bisectors of each of the sides in the three cases. Denote $u = x - y$ and $v = x - z$. Then the perpendicular bisectors are in the affine subspace spanned by $u, v$. Let $V$ be the space spanned by $u, v$, then an orthogonal vector to $u$ in $V$ is of the form $w_1 = u + av$,

$$\langle u + av, u \rangle = 0.$$ 

Therefore, 

$$a = -\|u\| \langle u, v \rangle.$$ 

Therefore the line that pass through the midpoint $m_1 = (x + y)/2$ is 

$$l_1 = \{m_1 + tw_1 : t \in \mathbb{R}\} = \left\{m_1 + t \left(u - \frac{\|u\|}{\langle u, v \rangle}v\right) : t \in \mathbb{R}\right\}.$$ 

Similarly, the line that pass through $m_2 = (x + z)/2$ is 

$$l_2 = \{m_2 + tw_2 : t \in \mathbb{R}\} = \left\{m_2 + t \left(v - \frac{\|v\|}{\langle u, v \rangle}u\right) : t \in \mathbb{R}\right\}.$$ 

Both line intersects at the point which satisfies 

$$m_1 + t \left(u - \frac{\|u\|}{\langle u, v \rangle}v\right) = m_2 + s \left(v - \frac{\|v\|}{\langle u, v \rangle}u\right),$$

or, equivalently,
Table 3  Hausdorff distances from each of the Platonic and Archimedean solids to the sphere

| Solid                  | Generating point | Isometry group | Distance to the sphere |
|------------------------|------------------|----------------|------------------------|
| **Platonic solids**    |                  |                |                        |
| Tetrahedron            | $x/3, y/3$       | $A_3$          | 0.9428                 |
| Octahedron             | 0.5774 $y$       | $B_3$          | 0.8165                 |
| Cube                   | 0.5774 $x$       | $B_3$          | 0.8165                 |
| Icosahedron            | 0.7947 $y$       | $H_3$          | 0.6071                 |
| Dodecahedron           | 0.7947 $x$       | $H_3$          | 0.6071                 |
| **Archimedean solids** |                  |                |                        |
| Truncated tetrahedron  | 0.5774 $m_1$, 0.5774 $m_3$ | $A_3$ | 0.8586                 |
| Cuboctahedron          | 0.8660 $m_2$     | $A_3$          | 0.7071                 |
| Truncated cube         | 0.7071 $m_1$     | $B_3$          | 0.7388                 |
| Truncated octahedron   | $m_3$            | $B_3$          | 0.678                  |
| Rhombicuboctahedron    | 0.9659 $m_2$     | $B_3$          | 0.5140                 |
| Truncated cuboctahedron| 0.9516 $(x + y + z)/3$ | $B_3$ | 0.5248                 |
| Snub cube              | 0.9516 $(x + y + z)/3$ | $B_3$ | 0.5248                 |
| Icosidodecahedron      | 0.8507 $z$       | $H_3$          | 0.5257                 |
| Truncated dodecahedron | 0.8507 $m_1$     | $H_3$          | 0.5479                 |
| Truncated icosahedron  | $m_3$            | $H_3$          | 0.443                  |
| **Rhombicosidodecahedron** | 0.9945 $m_2$       | $H_3$          | 0.3354                 |
| Truncated icosidodecahedron | 0.9727 $(x + y + z)/3$ | $H_3$ | 0.3773                 |
| Snub dodecahedron      | 0.9727 $(x + y + z)/3$ | $H_3$ | 0.3773                 |

The best approximation of the sphere among these is the rhombicosidodecahedron

\[ m_1 - m_2 = s \left( v - \frac{\|v\|}{\langle u, v \rangle} u \right) - t \left( u - \frac{\|u\|}{\langle u, v \rangle} v \right). \]  

(2)

We get a set of three linear non homogeneous equations which by the way constructed has a unique solution. We solved the equations for each group using matlab®. The distance \(d_H(G, S^2)\) will be then \(d(c, x)\) where \(c\) is the Chebyshev center. We summarize the results in Table 3.

**Example 2.20** We give now some examples of some known polyhedrons.

**The octahedron and the cube** The octahedron is the orbit of \(y\) under the octahedral group \(B_3\). Note that \(d(x, y) = 0.9194\) (clearly \(d(y, z)\) is smaller). The Hausdorff distance from the orbit of \(y\) to the two-dimensional sphere is 0.9194. For every \(t > 0\) the orbit of \(ty\) will also be an octahedron and all octahedrons arise in this way. One can easily compute that the optimal \(t\) is \(t = 0.5774\). Then the distance is \(d_H(G(ty), S^2) = 0.8165\). The cube is its dual. It is the orbit of \(x\) (and any \(tx\)) and has the same Hausdorff distance from the sphere.
The icosahedron and the dodecahedron
Both Platonic solids are homogeneous spaces of $H_3$. The first is the orbit of $y$. Its stabilizer is therefore $G_2 = \langle R_2, R_3 \rangle$. This group has ten elements hence the orbit has twelve vertices and its faces are all regular triangles. The second is its dual. It is the orbit of $x$. It has twenty vertices and its faces are all regular pentagons. The distance $d(0.7947x, y) = 0.6071$ and this is also the Hausdorff distance of both polyhedrons from the sphere.

The truncated icosahedron
The truncated icosahedron for example is the polyhedron whose vertices are the $H_3$ orbit of the point that lies in the middle of $[y, z]$, which we denote by $m_3$. The reflection $R_3$ is its stabilizer and the action of $G_2 = \langle R_2, R_3 \rangle$ has five points forming a regular pentagon. On the other hand the orbit group $G_1 = \langle R_1, R_2 \rangle$ has six points which generate a regular hexagon. This homogeneous space is very familiar as it was the one skeleton of the official soccer ball for many years. The distance $d(m_3, x) = 0.443$ and this is also the Hausdorff distance to the two-dimensional sphere. It is interesting to note that the orbit of $0.9945m_2$—the middle of $[x, y]$—is of Hausdorff distance 0.3354. The orbit is the vertices set of the Archimedean solid, rhombicosidodecahedron. Indeed, $m_2$ is very close to the Chebyshev center (we had $s = 0.0153$) therefore its orbit is very close to be optimal $G$ sphere approximation.

In Table 3 we list all thirteen Archimedean and five Platonic solids and their distance from the sphere. It is pleasant to mention here the work of Shaked Bader and Sapir Freizeit. They calculated these distances in an exercise assigned to them by Tsachik Gelander. Note that if a solid is the orbit of some point $p$ then for every $t > 0$, the orbit of $tp$ will give the same solid scaled differently. We performed grid search with Matlab to get best approximation.

To conclude, the optimal finite homogeneous approximation of the $S^2$ sphere is a space with 120 elements which are the orbit of the Chebyshev center of the fundamental domain of the group $H_3$. Like the truncated icosidodecahedron its faces are squares, regular hexagons and regular decagons.

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