HAUSDORFF DIMENSION OF THE SECOND GRIGORCHUK GROUP

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Abstract. We show that the Hausdorff dimension of the closure of the second Grigorchuk group is $43/128$. Furthermore we establish that the second Grigorchuk group is super strongly fractal and that its automorphism group equals its normaliser in the full automorphism group of the tree.

1. Introduction

Let $T$ be the $d$-adic rooted tree, for $d \geq 2$. The first Grigorchuk group [8], more commonly known as the Grigorchuk group, was the earliest example of a regular branch group. The class of regular branch groups consists of subgroups of $Aut(T)$ that mimic key properties of $Aut(T)$; see Section 2 for precise definitions. The second Grigorchuk group was introduced in the same paper [8], and shares several interesting properties with the first Grigorchuk group, such as being infinite, periodic, finitely generated, just infinite, as well as possessing the congruence subgroup property, and having finitely many maximal subgroups only of finite index, see [10][11].

Regular branch groups, especially the two Grigorchuk groups, have been studied extensively from various aspects over the past twenty years; see [3] for a good introduction. In this note, we are interested in the Hausdorff dimension of regular branch groups acting on the 4-adic rooted tree $T$. Let

$$\Gamma = \lim_{\leftarrow n \in \mathbb{N}} C_4 \wr \ldots \wr C_4$$

be the group of all 4-adic automorphisms, which is contained in a Sylow pro-2 subgroup of $Aut(T)$. For $G \leq \Gamma$, the Hausdorff dimension of the closure of $G$ in $\Gamma$ is given by

$$hdim_{\Gamma}(\overline{G}) = \lim_{n \to \infty} \frac{\log |G : St_G(n)|}{\log |\Gamma : St_\Gamma(n)|} \in [0, 1],$$

where $\lim$ represents the lower limit, and for a subgroup $H \leq \Gamma$, the normal subgroup $St_H(n)$ is the $n$th level stabiliser of $H$ (also called the $n$th principal congruence subgroup of $H$); we refer to Section 2 for further details. The Hausdorff dimension of $\overline{G}$ is a measure of how dense $\overline{G}$ is in $\Gamma$. This was first applied by Abercrombie [1], Barnea and Shalev [2] in the more general setting of profinite groups.

Key results concerning the Hausdorff dimension of Grigorchuk-type groups were established in [5–7,12]. Although it is well known that the closure of the first Grigorchuk group has Hausdorff dimension 5/8, the computation of the Hausdorff dimension of the second Grigorchuk group does not appear to be recorded anywhere in the literature. Here, we close this gap.

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**Theorem A.** Let $G$ be the second Grigorchuk group acting on the 4-adic rooted tree. Then

(i) the orders of the congruence quotients of $G$ are given by

$$\log_4 |G : St_G(n)| = \begin{cases} 
1 & \text{if } n = 1, \\
3 & \text{if } n = 2, \\
\frac{1}{3}(86 \cdot 4^n - 4) & \text{if } n > 2.
\end{cases}$$

(ii) the Hausdorff dimension of the closure of $G$ in $\Gamma$ is

$$\text{hdim}_\Gamma(G) = \frac{43}{128}.$$ 

We also prove the following, which is of independent interest.

**Proposition B.** Let $G$ be the second Grigorchuk group. Then the following hold:

(i) the group $G$ is super strongly fractal;

(ii) the group $\text{Aut}(G)$ equals the normaliser of $G$ in $\text{Aut}(T)$.

**Organisation.** Section 2 of this paper consists of background material for regular branch groups. Section 3 contains properties of the second Grigorchuk group $G$, and includes the proof of Proposition B. Theorem A is proved in Section 4.

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2. Preliminaries

For $d \geq 2$, let $T$ be the $d$-adic rooted tree, meaning all vertices have $d$ children. Using the alphabet $X = \{1, \ldots, d\}$, the vertices $u_\omega$ of $T$ are labelled bijectively by elements $\omega$ of the free monoid $X^*$ as follows: the root of $T$ is labelled by the empty word $\emptyset$, and for each word $\omega \in X^*$ and letter $x \in X$ there is an edge connecting $u_\omega$ to $u_\omega x$. We say that $u_\omega$ precedes $u_\lambda$ whenever $\omega$ is a prefix of $\lambda$.

A natural length function on $X^*$ is defined as follows: the words $\omega$ of length $|\omega| = n$, representing vertices $u_\omega$ that are at distance $n$ from the root, are the $n$th level vertices and form the $n$th layer of the tree. The elements of the boundary $\partial T$ correspond naturally to infinite rooted paths.

Denote by $T_u$ the full rooted subtree of $T$ that has its root at a vertex $u$ and includes all vertices succeeding $u$. For any two vertices $u = u_\omega$ and $v = u_\lambda$, the map $u_\omega \tau \mapsto u_\lambda \tau$, induced by replacing the prefix $\omega$ by $\lambda$, yields an isomorphism between the subtrees $T_u$ and $T_v$.

Every automorphism of $T$ fixes the root and the orbits of $\text{Aut}(T)$ on the vertices of the tree $T$ are precisely its layers. For $f \in \text{Aut}(T)$, the image of a vertex $u$ under $f$ is denoted by $u^f$. Observe that $f$ induces a faithful action on the monoid $X^*$ such that $(u_\omega)^f = u_{\omega^f}$.

For $\omega \in X^*$ and $x \in X$ we have $(\omega x)^f = \omega^f x'$ where $x' \in X$ is uniquely determined by $\omega$ and $f$. This induces a permutation $f(\omega)$ of $X$ so that

$$(\omega x)^f = \omega^f x'^f(\omega), \quad \text{and hence} \quad (u_\omega x)^f = u_{\omega^f x'^f(\omega)}.$$ 

The automorphism $f$ is **rooted** if $f(\omega) = 1$ for $\omega \neq \emptyset$. It is **directed**, with directing path $\ell \in \partial T$, if the support $\{\omega \mid f(\omega) \neq 1\}$ of its labelling is infinite and marks only vertices at distance 1 from the set of vertices corresponding to the path $\ell$. 
2.1. Subgroups of $\text{Aut}(T)$. Let $G \leq \text{Aut}(T)$. For $n \in \mathbb{N}$, the $n$th level stabiliser $\text{St}_G(n) = \bigcap_{\omega = n} \text{St}_G(u_\omega)$ is the subgroup consisting of automorphisms that fix all vertices at level $n$. Denoting by $T_n$ the finite subtree of $T$ on vertices up to level $n$, we see that $\text{St}_G(n)$ is equal to the kernel of the induced action of $G$ on $T_n$.

The full automorphism group $\text{Aut}(T)$ is a profinite group:

$$\text{Aut}(T) = \varprojlim_n \text{Aut}(T_n)$$

The topology of $\text{Aut}(T)$ is defined by the open subgroups $\text{St}_{\text{Aut}(T)}(n)$, $n \in \mathbb{N}$. A subgroup $G$ of $\text{Aut}(T)$ has the congruence subgroup property if for every subgroup $H$ of finite index in $G$, there exists some $n$ such that $\text{St}_G(n) \subseteq H$.

Each $g \in \text{St}_{\text{Aut}(T)}(n)$ can be described completely in terms of its restrictions to the subtrees rooted at vertices at level $n$. Indeed, there is a natural isomorphism

$$\psi_n : \text{St}_{\text{Aut}(T)}(n) \rightarrow \prod_{\omega = n} \text{Aut}(T_{u_\omega}) \cong \text{Aut}(T) \times d^n \times \text{Aut}(T).$$

For ease of notation, we write $\psi = \psi_1$.

Let $\omega \in X^*$ be of length $n$. We further define

$$\varphi_\omega : \text{St}_{\text{Aut}(T)}(n) \rightarrow \text{Aut}(T_{u_\omega}) \cong \text{Aut}(T)$$

to be the natural restriction of $\psi_n$.

A group $G \leq \text{Aut}(T)$ is said to be self-similar if the images under $\varphi_\omega$ and $\psi_n$ are contained in $G$ and $G \times d^n \times G$, respectively.

For $G \leq \text{Aut}(T)$, the vertex stabiliser $\text{St}_G(u)$ is the subgroup consisting of elements in $G$ that fix the vertex $u$.

We say that the group $G$ is fractal if $\varphi_\omega(\text{St}_G(u_\omega)) = G$ for every $\omega \in X^*$, after the natural identification of subtrees. Furthermore we say that the group $G$ is strongly fractal if $\varphi_x(\text{St}_G(1)) = G$ for every $x \in X$, and we say that the group $G$ is super strongly fractal if, for each $n \in \mathbb{N}$, we have $\varphi_\omega(\text{St}_G(n)) = G$ for every word $\omega \in X^*$ of length $n$. For more information on these definitions and examples of groups satisfying these properties, see [11].

We recall that a spherically transitive self-similar group $G$ with $1 \neq K \leq \text{St}_G(1)$ such that $K \times d^\omega \times K \subseteq \psi(K)$ and $|G : K| < \infty$, is said to be regular branch over $K$.

3. Some properties of the second Grigorchuk group

Let $T$ be the 4-adic rooted tree. The second Grigorchuk group $\mathcal{G} \leq \text{Aut}(T)$ is generated by two automorphisms $a$ and $b$, where $a$ is the rooted automorphism corresponding to the cycle $(1234)$, and $b \in \text{St}_G(1)$ is recursively defined by $\psi(b) = (a, 1, a, b)$. The group $\mathcal{G}$ is periodic [9], and from [10] Proof of Lem. 2.1], it follows that $\mathcal{G}$ is regular branch over $\gamma_3(\mathcal{G})$ and furthermore

$$\psi(\gamma_3(\text{St}_G(1))) = \gamma_3(\mathcal{G}) \times \gamma_3(\mathcal{G}) \times \gamma_3(\mathcal{G}) \times \gamma_3(\mathcal{G}).$$

Also we have $\text{St}_G(3) \leq \gamma_3(\mathcal{G})$; see [10] Lem. 3.2]. Note that $\text{St}_G(2) \not\leq \gamma_3(\mathcal{G})$, since

$$\psi(b(b^{-2})^{-1}) = (1, b^{-1}, 1, b) \in \text{St}_G(2),$$

but $b(b^{-2})^{-1} = [b^{-1}, a^2] \not\subseteq \gamma_3(\mathcal{G})$ by [10] Lem. 2.1].

**Lemma 3.1.** We have $|G : \gamma_3(\mathcal{G})| = d^3$ and $|\text{St}_G(1) : \gamma_3(\text{St}_G(1))| = d^7$.

**Proof.** The first statement follows from [10] Lem. 1.2 and 2.1]. For the second statement, note that

$$\text{St}_G(1) / \text{St}_G(1)' = \langle b, b^a, b^{a^2}, b^{a^3} \rangle \text{St}_G(1)' \cong C_4 \times C_4 \times C_4 \times C_4,$$
and
\[ \text{St}_G(1)/\gamma_3(\text{St}_G(1)) = \langle [b, b^a], [b, b^a]^a, [b, b^a^3] \rangle \gamma_3(\text{St}_G(1)) \]
\[ \cong C_4 \times C_4 \times C_4. \]

Thus
\[ |\text{St}_G(1) : \gamma_3(\text{St}_G(1))| = |\text{St}_G(1) : \text{St}_G(1)'| \cdot |\text{St}_G(1)' : \gamma_3(\text{St}_G(1))'| = 4^7. \]

The above result has the following application.

**Lemma 3.2.** For \( G \) the second Grigorchuk group, we have \( \psi^{-1}([b^{-1}, a^2], 1, 1, 1) \notin G \).

**Proof.** For simplicity, we write \( x = \psi^{-1}([b^{-1}, a^2], 1, 1, 1) \). If \( x \) were in \( G \), then clearly \( x \) would be in \( \text{St}_G(1)' \).

Recall that \([b^{-1}, a^2] \notin \gamma_3(G)\). Hence \( x \notin \gamma_3(\text{St}_G(1)) \). Therefore it suffices to show that the image of \( x \) modulo \( \gamma_3(\text{St}_G(1)) \) does not lie in the finite abelian group of exponent 4 below:
\[ \text{St}_G(1)'/\gamma_3(\text{St}_G(1)) = \langle [b, b^a], [b, b^a]^a, [b, b^a^3] \rangle \gamma_3(\text{St}_G(1)) \]
\[ \cong \langle ([a, b], 1, 1, [a, b]), ([b, a], [a, b], 1, 1, [a, b], [b, a]) \rangle \psi(\gamma_3(\text{St}_G(1))). \]

From the above presentation, it is now clear that \( x \notin \text{St}_G(1)' \), and hence \( x \notin G \), as required. \( \square \)

3.1. **Partial weights.** In this subsection, we introduce partial weights in order to give a complete description of the elements in \( \text{St}_G(2) \) and \( \text{St}_G(3) \), which we will need in Section 3.

The definitions and the notation follow [6].

We write \( b_0 = b, b_1 = b^a, b_2 = b^a^2, \) and \( b_3 = b^a^3 \). Recall that \( \text{St}_G(1) = \langle b_0, b_1, b_2, b_3 \rangle \). Hence for \( g \in \text{St}_G(1) \), we can write \( g \) as a word in \( b_0, b_1, b_2, b_3 \), that is,
\[ g = \omega(b_0, b_1, b_2, b_3), \]
where \( \omega = \omega(x_0, x_1, x_2, x_3) \) is a group word in four variables \( x_0, x_1, x_2, x_3 \).

For \( \omega \) a group word in the variables \( x_0, x_1, x_2, x_3 \),

(i) the partial weight of \( \omega \) with respect to \( x_i \), for \( 0 \leq i \leq 3 \), is the sum of the exponents of \( x_i \) in \( \omega \), reduced modulo 4, and

(ii) the partial weight of \( \omega \) is the sum of all its partial weights, reduced modulo 4.

Let \( g = \omega(b_0, b_1, b_2, b_3) \) be an element of \( \text{St}_G(1) \). For \( 0 \leq i \leq 3 \), write \( r_i \) for the partial weight of \( \omega \) with respect to \( x_i \). Then
\[ \psi(g) = (a^{r_0 + r_2} \omega_1(b_0, b_1, b_2, b_3), a^{r_1 + r_3} \omega_2(b_0, b_1, b_2, b_3), a^{r_0 + r_2} \omega_3(b_0, b_1, b_2, b_3), a^{r_1 + r_3} \omega_0(b_0, b_1, b_2, b_3)) \]
\[ (1) \]
where \( \omega_i \) is a word of total weight \( r_i \), for each \( 0 \leq i \leq 3 \).

**Theorem 3.3.** Let \( G \) be the second Grigorchuk group, and let \( g \in \text{St}_G(1) \). Then the partial and total weights are the same for all representations of \( g \) as a word in \( b_0, b_1, b_2, b_3 \).

**Proof.** We proceed as in [6] Proof of Thm. 2.8]. First we note that it suffices to prove that, if \( \omega \) is a word such that \( \omega(b_0, b_1, b_2, b_3) = 1 \), then the total weight of \( \omega \) is zero, and all partial weights \( r_0, r_1, r_2, r_3 \) of \( \omega \) are also zero.

We observe from the expression for \( \psi(g) \) above that
\[ r_0 + r_2 = r_1 + r_3 = 0, \]
which proves that the total weight of \( \omega \) is zero.
Additionally, since \( \omega(b_0, b_1, b_2, b_3) = 1 \), in the above expression for \( \psi(g) \), we must have \( \omega_i(b_0, b_1, b_2, b_3) = 1 \) for all \( 0 \leq i \leq 3 \). Since the total weight of such a \( \omega_i \), which is \( r_i \), has just been proved to be zero, the result follows.

Let \( g \in St_G(1) \). The partial weight of \( g \) with respect to \( b_i \), for \( 0 \leq i \leq 3 \), and the total weight of \( g \), as the corresponding weights for any word representing \( g \). For \( g \) with partial weights \( r_i \) with respect to \( b_i \) for \( 0 \leq i \leq 3 \), we further refer to \( (r_0, r_1, r_2, r_3) \in (\mathbb{Z}/4\mathbb{Z})^4 \) as the weight vector of \( g \).

The following is key.

**Theorem 3.4.** Let \( G \) be the second Grigorchuk group and suppose \( g \in St_G(1) \) has weight vector \( (r_0, r_1, r_2, r_3) \). Then

(i) we have \( g \in St_G(2) \) if and only if \( r_0 + r_2 = r_1 + r_3 = 0 \);

(ii) if \( g \in St_G(3) \) then \( r_0 = r_1 = r_2 = r_3 = 0 \).

**Proof.**

(i) This is clear from (1).

(ii) Suppose that \( g \in St_G(3) \). From (1), it follows that \( \omega_i(b_0, b_1, b_2, b_3) \in St_G(2) \) for all \( 0 \leq i \leq 3 \). Since the total weight for \( \omega_i(b_0, b_1, b_2, b_3) \) is \( r_i \), the result now follows from part (i).

**Theorem 3.5.** Let \( G \) be the second Grigorchuk group. Then

\[ |St_G(2) : St_G(3)| = 2^{11} \]

**Proof.** Let \( g \in St_G(1) \) have weight vector \( (r_0, r_1, r_2, r_3) \). From Theorem 3.4 (i), it follows that there are exactly \( 4^3 \) possibilities for \( (r_0, r_1, r_2, r_3) \), for \( g \) to be in \( St_G(2) \). Using the notation of [6], Proof of Thm. 3.5, we denote the possibilities by

\[ r^{(i)} = (r_0^{(i)}, r_1^{(i)}, r_2^{(i)}, r_3^{(i)}) \]

for each \( 1 \leq i \leq 16 \). For convenience, we write \( r^{(i)} = (0, 0, 0, 0) \).

We proceed as in [6], Proof of Thm. 3.5. First note that each solution \( r^{(i)} \) determines a subset \( R^{(i)} \) of \( St_G(2) \), consisting of all the elements whose weight vector is \( r^{(i)} \). For the natural map \( \pi : G \to G/ St_G(3) \), we set \( S^{(i)} = \pi(R^{(i)}) \). From the previous paragraph, we have

\[ St_G(2)/St_G(3) = \bigcup_{i=1}^{16} S^{(i)} \]

The theorem follows once we establish the following:

(a) For \( i \neq j \in \{1, \ldots, 16\} \), the sets \( S^{(i)} \) and \( S^{(j)} \) are disjoint.

(b) We have \( |S^{(i)}| = 2^7 \) for all \( i \in \{1, \ldots, 16\} \).

To prove (a), we consider two elements \( g \in R^{(i)} \) and \( h \in R^{(j)} \) with the same image under \( \pi \). Then \( gh^{-1} \in St_G(3) \) and so the weight vector of \( gh^{-1} \) is the zero vector, by Theorem 3.4 (ii). Since the weight vector defines a homomorphism from \( St_G(1) \) to \( (\mathbb{Z}/4\mathbb{Z})^4 \), we deduce that \( S^{(i)} = S^{(j)} \), and thus \( i = j \), as required.

For part (b), we begin by observing that \( S^{(i)} \) is non-empty for each \( 1 \leq i \leq 16 \). Also, if \( h_i \in S^{(i)} \) then \( S^{(i)} = h_i S^{(1)} \). Therefore \( |S^{(i)}| = |S^{(1)}| \), and hence it suffices to prove that \( |S^{(1)}| = 2^7 \).

Let \( g \in St_G(2) \). Then \( g \in R^{(1)} \) if and only if every component of \( \psi(g) \) has zero total weight. Since \( G' \) consists of all the elements of \( St_G(1) \) whose total weight is equal to 0, it follows that \( g \in R^{(1)} \) if and only if

\[ \psi(g) \in G' \times G' \times G' \times G' \]

and thus

\[ R^{(1)} = G \cap \psi^{-1}(G' \times G' \times G' \times G') \].
As in [G] Proof of Thm. 3.7] (note that [G] Thm. 2.14] holds with \( p \) replaced by 4), we deduce that

\[
|G' \times G' \times G' \times G' : \psi(R^{(1)})| = 4 \quad \text{and} \quad R^{(1)} = St_G(1').
\]

Now, as in [G] Thm. 2.4(i)], we have \( |St_G(1) : St_G(2)| = 4^2 \). Thus

\[
|G' \times G' \times G' \times G' : St_G(2) \times St_G(2) \times St_G(2) \times St_G(2)| = 4^4,
\]

and it follows that \( |S^{(1)}| = 2^7 \) once we establish that

\[
|St_G(2) \times St_G(2) \times St_G(2) \times St_G(2) : \psi(St_G(3))| = 2.
\]

As \( St_G(3) = St_G(1) \cap \psi^{-1}(St_G(2) \times St_G(2) \times St_G(2) \times St_G(2)) \), it suffices to show that

\[
|\psi^{-1}(St_G(2) \times St_G(2) \times St_G(2) \times St_G(2)) \cdot St_G(1) = \psi^{-1}([a^{-1}, a^2], 1, 1, 1)) St_G(1)|.
\]

By Theorem [3.3] we have \( St_G(2) = \langle bb_2^{-1}, b_1 b_3^{-1} \rangle_G = \langle bb_2^{-1} \rangle_G = \langle [b^{-1}, a^2] \rangle_G \). Since \( \langle b^{-1}, a^2 \rangle \in \gamma_3(G) \subseteq St_G(1) \), by Lemma [3.2] it is enough to prove the following equality.

\[
\frac{\psi^{-1}(St_G(2) \times St_G(2) \times St_G(2) \times St_G(2)) \cdot St_G(1)}{St_G(1)} = \frac{\psi^{-1}([b^{-1}, a^2]) St_G(1)}{St_G(1)}.
\]

Note that the expression above is clear from the fact that

\[
\psi^{-1}(([b^{-1}, a^2], 1, 1)) = \psi^{-1}([b^{-1}, a^2], 1, 1, 1)) \cdot \psi^{-1}([b, b^{-1}, 1, 1, 1, 1), a]
\]

\[
= \psi^{-1}([b^{-1}, a^2], 1, 1, 1)) \cdot \psi^{-1}((b, b^{-1}, b b^{-1}, 1, 1))
\]

and

\[
(b, b^{-1}, b, b^{-1}, 1, 1) = ([a^2, b^{-1}], [b^{-1}, a^2], 1, 1) \in \gamma_3(G) \leq St_G(1).
\]

Hence we are done. \( \Box \)

### 3.2. Further properties.

We end this section with a few results of independent interest.

**Lemma 3.6.** The second Grigorchuk group \( G \) is super strongly fractal.

*Proof.* Observe that \( G \) is level-transitive, that is, transitive on every layer of the tree. Also, the group \( G \) is fractal since \( \psi(b) = (a^1, a, b) \) and \( \psi(b^a) = (b^a, a, a) \). From [14] Lem 2.5], \( G \) is also strongly fractal.

We first show that \( \varphi_{ux}(St_G(2)) = G \) for \( u, x \in X \). As seen in the proof of Theorem [3.5] we have \( |G : St_G(2)| = 4^3 \). Observe that \( St_G(1) \leq St_G(2) \), hence it follows from Lemma [3.1] and the remark above it, that

\[
\frac{St_G(2)}{St_G(1)'} = \frac{\langle b(b^a)^{-1}, b^a(b^a)^{-1} \rangle St_G(1)'}{St_G(1)'}.
\]

We deduce that, for \( u \in X \),

\[
\varphi_u(St_G(2)) = \langle b \rangle G'.
\]

As \( \psi(b) = (a^1, a, b) \) and \( b^a = b[b, a] \), it follows that \( \varphi_{ux}(St_G(2)) = G \), as required.

Now let \( n \in \mathbb{N} \) and write \( K_n = \psi_n^{-1}(\gamma_3(G) \times \mathbb{X}^n \times \gamma_3(G)) \). Since \( G \) is regular branch over \( \gamma_3(G) \), we have

\[
\psi_n(K_n) = \gamma_3(G) \times \mathbb{X}^n \times \gamma_3(G) \subseteq (St_G(1) \times \mathbb{X}^n \times St_G(1)) \cap \psi_n(St_G(n)) = \psi_n(St_G(n + 1)).
\]

From the equation

\[
\psi([b, a, a]) = (b^{-1}ab^{-1}a, a^{-2}b, a^2, a^{-1}ba^{-1}),
\]

we deduce that \( \varphi_x(\gamma_3(G)) \subseteq \langle a^2, b \rangle G' \neq G \), for all \( x \in X \), however from

\[
\psi([b, a, a^2]) = (b^{-1}, a^{-1}ba, b, a^{-1}ba) \in \psi(St_G(2)),
\]

we deduce that \( \varphi_{ux}(\gamma_3(G)) = G \) for all \( u, x, \in X \).
Write $N = \langle [b, a, a^2] \rangle^G$, and $M_n = \psi_n^{-1}(N \times 4^n \times N)$. Then
\[
\psi_n(M_n) \subseteq (\text{St}_G(2) \times 4^n \times \text{St}_G(2)) \cap \psi_n(\text{St}_G(n)) = \psi_n(\text{St}_G(n+2)).
\]

Hence for any $\omega \in X^*$ of length $n$, we have $N = \varphi_\omega(M_n) \subseteq \varphi_\omega(\text{St}_G(n+2))$. It remains to show that if we take $\omega \in X^*$ of length $n+2$, we have $\varphi_\omega(\text{St}_G(n+2)) = G$, where $n \in \mathbb{N}$. But for any $u, x \in X$, and any $\nu \in X^*$ of length $n$, we have $G = \varphi_\nu x(\omega \in X^*$, we have $\varphi_\omega(\text{St}_G(n+2)) = G$, as required. 

In what follows, we prove that the second Grigorchuk group is saturated. We recall that a group $G \leq \text{Aut}(T)$ is said to be saturated if for any $n \in \mathbb{N}$ there exists a subgroup $H_n \leq \text{St}_G(n)$ that is characteristic in $G$ and level-transitive on every $n$th level subtree. Examples of saturated groups acting on rooted trees are, among others, the first Grigorchuk group \[9\], the Basilica $p$-groups \[1\], and the branch multi-EGS groups \[13\].

**Theorem 3.7.** The second Grigorchuk group $G$ is saturated.

**Proof.** We want to prove that, for any $n \in \mathbb{N}$, there exists a subgroup $H_n \leq \text{St}_G(n)$ characteristic in $G$ and level-transitive on every subtree of the $n$th level. To this end, define the subgroups $H_n \leq \text{St}_G(n)$ inductively as follows: $H_0 = G$, and $H_{n+1} = H'_n$.

Then $H_1$ contains $\psi([a, b]) = (b^{-1}a, a^{-1}, a, b^{-1}a)$ and $\psi([a, b]^2) = (a, b^{-1}, b^{-1}, a, a^{-1})$. We deduce that $\varphi_1(H_1) = G$, and thus $\varphi_x(H_1) = G$ for any $x \in X$. Hence $H_1$ acts level-transitively on all subtrees rooted at a level one vertex. By induction it follows that the restriction of $H_n$ to subtrees rooted at an $n$th level vertex contains $G$ and thus the required level-transitively of $H_n$ follows. \[\square\]

As a straightforward application of \[9\] Thm. 7.5] since $G$ is saturated, we obtain the following.

**Corollary 3.8.** The automorphism group $\text{Aut}(G)$ of the second Grigorchuk group coincides with the normaliser of $G$ in $\text{Aut}(T)$, that is, $\text{Aut}(G) = N_{\text{Aut}(T)}(G)$.

4. Hausdorff dimension of $G$.

This section is devoted to determine Hausdorff dimension of $G$.

**Theorem 4.1.** Let $G$ be the second Grigorchuk group acting on the 4-adic rooted tree. Then

(i) the orders of the congruence quotients of $G$ are given by
\[
\log_4 |G : \text{St}_G(n)| = \begin{cases} 1 & \text{if } n = 1, \\ 3 & \text{if } n = 2, \\ \frac{1}{3}(86 \cdot 4^{n-4} + 4) & \text{if } n > 2. \end{cases}
\]

(ii) the Hausdorff dimension of the closure of $G$ in $\Gamma$ is
\[
\text{hdim}_\Gamma(\overline{G}) = 43/128.
\]

**Proof.** (i) For convenience, we write $G_n = G/\text{St}_G(n)$. It is easy to see that $|G_1| = 4$. For $n = 2$, as stated in the proof of Theorem 3.5 we have $|G_2| = 4^3$. For $n = 3$, the result follows from Theorem 3.5.
Now we let \( n \geq 4 \). As \( \text{St}_G(3) \leq \gamma_3(G) \) and \( G \) is regular branch over \( \gamma_3(G) \), we have
\[
|G_n| = |G : \text{St}_G(1)| \cdot |\text{St}_G(1) : \text{St}_G(n)|
\]
\[
= 4 \cdot \frac{|G \times \cdot \cdot \cdot \times G : \text{St}_G(n-1) \times \cdot \cdot \cdot \times \text{St}_G(n-1)|}{|G \times \cdot \cdot \cdot \times G : \psi(\text{St}_G(1))|}
\]
\[
= 4 \cdot \frac{|\gamma_3(G)|^2}{|\text{St}_G(1) : \gamma_3(\text{St}_G(1))|}
\]
\[
= 4^5,
\]
Since
\[
|G \times \cdot \cdot \cdot \times G : \psi(\text{St}_G(1))| = \frac{|G \times \cdot \cdot \cdot \times G : \gamma_3(G) \times \cdot \cdot \cdot \times \gamma_3(G)|}{|\psi(\text{St}_G(1)) : \gamma_3(\text{St}_G(1))|}
\]
\[
= \frac{|G : \gamma_3(G)|^4}{|\text{St}_G(1) : \gamma_3(\text{St}_G(1))|}
\]
\[
= 4^5,
\]
we have
\[
\log_4 |G_n| = -4 + 4 \log_4 |G_{n-1}|,
\]
and the result follows by induction.

This follows immediately from (i). Indeed, if \( n > 2 \), we have
\[
\text{hdim}_f(G) = \lim_{n \to \infty} \frac{\log_4 |G_n|}{\log_4 |\Gamma : \text{St}_G(n)|}
\]
\[
= \lim_{n \to \infty} \frac{4^3}{(86 \cdot 4^{n-4} + 4) \cdot 4^n}
\]
\[
= \frac{86}{4^4} = \frac{43}{128},
\]
as required. \( \square \)

References

[1] A.G. Abercrombie, Subgroups and subrings of profinite rings, Math. Proc. Camb. Phil. Soc. 116 (2) (1994), 209–222.
[2] Y. Barnea and A. Shalev, Hausdorff dimension, pro-p groups, and Kac-Moody algebras, Trans. Amer. Math. Soc. 349 (1997), 5073–5091.
[3] L. Bartholdi, R.I. Grigorchuk and Z. Šunić, Handbook of algebra 3, North-Holland, Amsterdam, 2003.
[4] E. Di Domenico, M. Noce and A. Thillaisundaram, Generalized Basilica groups, in preparation.
[5] G.A. Fernández-Alcober and A. Zugadi-Reizabal, Spinal groups: semidirect product decompositions and Hausdorff dimension, J. Group Theory 14 (2011), 491–519.
[6] G.A. Fernández-Alcober and A. Zugadi-Reizabal, GGS-groups: order of congruence quotients and Hausdorff dimension, Trans. Amer. Math. Soc. 366 (2014), 1993–2007.
[7] G.A. Fernández-Alcober, Ş. Gül and A. Thillaisundaram, The congruence quotients of multi-EGS groups, in preparation.
[8] R.I. Grigorchuk, On Burnside’s problem on periodic groups, Funktsional. Anal. i Prilozhen 14 (1980), 53–54.
[9] Y. Lavreniuk and V. Nekrashevych, Rigidity of branch groups acting on rooted trees, Geom. Dedicata 89 (2002), 159–179.
[10] E.L. Pervova, Profinite topologies in just infinite branch groups, preprint 2002-154 of the Max Planck Institute for Mathematics, Bonn, Germany.
[11] E.L. Pervova, Maximal subgroups of non-locally finite p-groups, preprint 2002-158 of the Max Planck Institute for Mathematics, Bonn, Germany.
[12] Z. Šunić, Hausdorff dimension in a family of self-similar groups, Geom. Dedicata 124 (2007), 213–236.
[13] A. Thillaisundaram and J. Uria-Albizuri, The profinite completion of multi-EGS groups, arXiv preprint: 1910.03399.
[14] J. Uria-Albizuri, On the concept of fractality for groups of automorphisms of a regular rooted tree, in Reports@SCM, 2, pp. 33–44 (2016).
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