ANOTHER PROOF OF WRIGHT’S INEQUALITIES

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Abstract. We present a short way of proving the inequalities obtained by Wright in [Journal of Graph Theory, 4: 393 – 407 (1980)] concerning the number of connected graphs with \(\ell\) edges more than vertices.

1. Preliminaries

For \(n \geq 0\) and \(-1 \leq \ell \leq \binom{n}{2} - n\), let \(c(n, n + \ell)\) be the number of connected graphs with \(n\) vertices and \(n + \ell\) edges. Quantifying \(c(n, n + \ell)\) represents one of the fundamental tasks in the theory of random graphs. It has been extensively studied since the Erdős-Rényi’s paper [3]. The generating functions associated to the numbers \(c(n, n + \ell)\) are due to Sir E. M. Wright in a series of papers including [11, 12]. He also obtained the asymptotic formula for \(c(n, n + \ell)\) for every \(\ell = o(n^{1/3})\). Using different methods, Bender, Canfield and McKay [1], Pittel and Wormald [8] and van der Hofstad and Spencer [9] were able to determine the asymptotic value of \(c(n, n + \ell)\) for all ranges of \(n\) and \(\ell\).

For \(\ell \geq -1\), let \(W_\ell\) be the exponential generating function (EGF, for short) of the family of connected graphs with \(n\) vertices and \(n + \ell\) edges. Thus, \(W_\ell(z) = \sum_{n=0}^{\infty} c(n, n + \ell) \frac{z^n}{n!}\). Let \(T(z)\) be the EGF of the Cayley’s rooted labeled trees. It is well known that \(T(z) = ze^{T(z)} = \sum_{n \geq 1} n^{n-1} \frac{z^n}{n!}\) (see for example [4, 5]). Among other results, Wright proved that the functions \(W_\ell(z)\), \(\ell \geq -1\), can be expressed in terms of \(T(z)\). Such results allowed penetrating and precise analysis when studying random graphs processes as it has been shown for example in the giant paper [5]. Throughout the rest of this note, all formal power series are univariate. Therefore, for sake of simplicity we will often omit the variable \(z\) so that \(T \equiv T(z)\), \(W_i \equiv W_i(z)\) and so on.

We need the following notations.

Definition. If \(A\) and \(B\) are two formal power series such that for all \(n \geq 0\) we have \([z^n] A(z) \leq [z^n] B(z)\) then we denote this relation \(A \preceq B\) or \(A(z) \preceq B(z)\).

The aim of this note is to provide an alternative and generating function based proof of the inequalities obtained by Sir Wright in [12] (in particular, he used numerous intermediate lemmas). More precisely, Wright obtained the following.
Theorem (Wright 1980). Let \( b_1 = \frac{5}{24} \) and \( c_1 = \frac{19}{24} \). Define recursively \( b_\ell \) and \( c_\ell \) by

\[
\begin{align*}
2(\ell + 1)b_{\ell+1} &= 3\ell(\ell + 1)b_\ell + 3 \sum_{t=1}^{\ell-1} t(\ell - t)b_{\ell-t}, \quad (\ell \geq 1) \\
2(3\ell + 2)c_{\ell+1} &= 8(\ell + 1)b_{\ell+1} + 3\ell b_\ell + (3\ell + 2)(3\ell - 1)c_\ell \\
&\quad + 6 \sum_{t=1}^{\ell-1} t(3\ell - 3t - 1)b_{\ell-t}, \quad (\ell \geq 1). 
\end{align*}
\]

Then, for all \( \ell \geq 1 \)

\[
\begin{align*}
\frac{b_\ell}{(1 - T(z))^{3\ell}} - \frac{c_\ell}{(1 - T(z))^{3\ell-1}} &\preceq W_\ell(z) \preceq \frac{b_\ell}{(1 - T(z))^{3\ell}}. 
\end{align*}
\]

(3) is known as Wright’s inequalities and such results has been extremely useful in the enumerative study of graphs as well as in the theory of random graphs [2, 5, 6, 7, 10].

Our proof of (3) is based upon two ingredients:

**Fact 1.** We know that the EGFs \( W_\ell \) satisfy

\[
\begin{align*}
W_0 &= -\frac{1}{2} \log (1 - T) - \frac{T^2}{2} - \frac{T^2}{4}, \\
(1 - T) \varphi_z W_{\ell+1} + (\ell + 1) W_{\ell+1} &= \left(\varphi_z^2 - \frac{3\varphi_z}{2} - \ell\right) W_\ell + \frac{1}{2} \sum_{k=0}^{\ell} (\varphi_z W_k)(\varphi_z W_{\ell-k}), \quad (\ell \geq 0), 
\end{align*}
\]

where \( T = T(z) \), \( W_k = W_k(z) \) and \( \varphi_z = z\frac{\partial}{\partial z} \) corresponds to marking a vertex (such combinatorial operator consists to choose a vertex among the others). For the combinatorial sense of (4), we refer the reader to [1, 5] or [11].

**Fact 2.** Let \( A \) and \( B \) be two formal power series and \( \ell \in \mathbb{N} \). If \((1 - T) \varphi_z A + (\ell + 1) A \preceq (1 - T) \varphi_z B + (\ell + 1) B \) then \( A \preceq B \).

To prove Fact 2, fix \( \ell \geq 0 \). We write

\[
B(z) - A(z) = \sum_{n=0}^{\infty} (b_n - a_n) \frac{z^n}{n!} \quad \text{and} \quad \forall n, c_n = b_n - a_n.
\]

Suppose that \((1 - T) \varphi_z A + (\ell + 1) A \preceq (1 - T) \varphi_z B + (\ell + 1) B \). We then have

\[
\begin{align*}
n! [z^n] ((1 - T(z)) \varphi_z (B(z) - A(z)) + (\ell + 1) (B(z) - A(z))) &= \\
(n + \ell + 1)c_n - \sum_{k=1}^{n} \binom{n}{k} k^{k-1} (n-k)c_{n-k} &\geq 0.
\end{align*}
\]

It is now easily seen that \( \forall n, c_n \geq 0 \). Therefore, \( A \preceq B \).

Our proof of (3) is divided into two parts each of each are given in the next Sections.
2. Proof of $W_\ell \preceq \frac{b_\ell}{(1-T)\ell}$

Define the family $(W_\ell)_{\ell \geq 0}$ as $W_0 = -\frac{1}{2} \log(1-T)$ and for $\ell \in \mathbb{N}^*$, $W_\ell = \frac{b_\ell}{(1-T)\ell}$. Observe that we have $W_0 \preceq W_0$ and $W_1 \preceq W_1$ has been proved in [12]. Now, we can proceed by induction. Suppose that for $2 \leq i \leq \ell$, $W_i \preceq W_i = \frac{b_i}{(1-T)i}$ and let us prove that $W_{\ell+1} \preceq W_{\ell+1} = \frac{b_{\ell+1}}{(1-T)^{\ell+1}}$. Simple calculations show that

$$\left(\frac{\partial^2 z - \partial z}{2}\right) W_\ell \preceq \frac{\partial^2 z}{2} W_\ell \preceq \frac{3\ell(3\ell + 2)}{2} \frac{b_\ell}{(1-T)^{3\ell+4}} - \frac{3\ell(3\ell + 2)}{2} \frac{b_\ell}{(1-T)^{3\ell+3}}\,,$$

(7)

$$\left(\partial_z W_0\right) \left(\partial_z W_\ell\right) \preceq \frac{3\ell b_\ell}{2} \frac{b_\ell}{(1-T)^{3\ell+4}} - \frac{3\ell b_\ell}{2} \frac{b_\ell}{(1-T)^{3\ell+3}}$$

and

$$\frac{1}{2} \sum_{p=1}^{\ell-1} \left(\partial_z W_p\right) \left(\partial_z W_{\ell-p}\right) \preceq \frac{1}{2} \sum_{p=1}^{\ell-1} 9p(\ell-p)b_pb_{\ell-p} \left(\frac{1}{(1-T)^{3\ell+4}} - \frac{1}{(1-T)^{3\ell+3}}\right)\,.$$  

(8)

Summing (7), (8), (9), using the recurrence (11) and the induction hypothesis, we find that

$$(1-T)\partial_z W_{\ell+1} + (\ell + 1)W_{\ell+1} \preceq \frac{3(\ell + 1)b_{\ell+1}}{(1-T)^{3\ell+4}} - \frac{3(\ell + 1)b_{\ell+1}}{(1-T)^{3\ell+3}}\,.$$  

(10)

Since

$$(1-T)\partial_z W_{\ell+1} + (\ell + 1)W_{\ell+1} = \frac{3(\ell + 1)b_{\ell+1}}{(1-T)^{3\ell+4}} - \frac{2(\ell + 1)b_{\ell+1}}{(1-T)^{3\ell+3}}\,.$$  

(11)

by Fact 2, we have $W_{\ell+1} \preceq W_{\ell+1}$.

3. Proof of $\frac{b_\ell}{(1-T)^{\ell}} - \frac{c_\ell}{(1-T)^{\ell+1}} \preceq W_\ell$

Define $W_0 = W_0$ and for $\ell \in \mathbb{N}^*$, $W_\ell = \frac{b_\ell}{(1-T)^{\ell}} - \frac{c_\ell}{(1-T)^{\ell+1}}$. As before, we shall proceed by induction. We have $W_0 \preceq W_0$ and

$$W_1 - W_1 = \frac{13}{12(1-T)} - \frac{1}{2} - \frac{T^2}{8} \geq \frac{13}{12} \left(\frac{1}{(1-T)} - T - 1\right) = \frac{13T^2}{12(1-T)} \geq 0\,.$$  

(12)

Suppose that for $2 \leq k \leq \ell$, $W_k = \frac{b_k}{(1-T)^{k}} - \frac{c_k}{(1-T)^{k+1}} \preceq W_k$. We have to prove that $W_{\ell+1} = \frac{b_{\ell+1}}{(1-T)^{\ell+1}} - \frac{c_{\ell+1}}{(1-T)^{\ell+2}} \preceq W_{\ell+1}$. For this purpose, define $\Psi_{\ell+1}$ as

$$\Psi_{\ell+1} = \left(\frac{\partial^2 z - 3\partial z}{2} - \ell\right) W_\ell + \left(\partial_z W_0\right) \left(\partial_z W_\ell\right) + \frac{1}{2} \sum_{k=1}^{\ell-1} \left(\partial_z W_k - \frac{(\ell - 1)c_k}{(1-T)^{3k}}\right) \left(\partial_z W_{\ell-k}\right) - \left(\frac{\alpha_\ell}{(1-T)^{3\ell+2}} + \frac{\beta_\ell}{(1-T)^{3\ell+1}} + \frac{\gamma_\ell}{(1-T)^{3\ell}} + \frac{\delta_\ell}{(1-T)^{3\ell-1}}\right)\,.$$  

(13)
where $\alpha_\ell$, $\beta_\ell$, $\gamma_\ell$ and $\delta_\ell$ are given by

$$\alpha_\ell = \frac{(7\ell + 4)c_{\ell+1}}{2} - 3(\ell + 1)b_{\ell+1} - \frac{3}{4}\ell b_\ell + \frac{3(\ell - 1)(3\ell + 4)}{4} c_\ell$$

(14)

$$+ \frac{1}{2} \sum_{t=1}^{\ell-1} (3t - 1)c_t (3\ell - 3t - 1)c_{\ell-t},$$

$$\beta_\ell = -\frac{(3\ell + 2)c_{\ell+1}}{2} + 2(\ell + 1)b_{\ell+1} - \frac{3}{4}\ell b_\ell - \frac{(3\ell - 1)(3\ell + 4)}{4} c_\ell$$

(15)

$$- \frac{1}{2} \sum_{t=1}^{\ell-1} (3t - 1)c_t (3\ell - 3t - 1)c_{\ell-t},$$

$$\gamma_\ell = \frac{\ell b_\ell}{2} + \frac{(3\ell - 1)c_\ell}{2} \quad \text{and} \quad \delta_\ell = -\frac{\ell - 1}{2} c_\ell.$$  

Rewriting the formal power series

$$\frac{(7\ell + 4)/2c_{\ell+1} - 3(\ell + 1)b_{\ell+1} - 3/4b_\ell}{(1-T)^{3\ell+2}} + \frac{(3\ell + 2)/2c_{\ell+1} - 2(\ell + 1)b_{\ell+1} + 3/4b_\ell}{(1-T)^{3\ell+1}}$$

(16)

$$= \Psi_\ell + \frac{2\ell b_\ell}{2(1-T)^{3\ell}} + \left(\frac{3\ell - 1}{2(1-T)^{3\ell}} - \frac{(\ell - 1)c_\ell}{(1-T)^{3\ell-1}}\right),$$

it is easily seen that if the quantity (coming from the denominators of the 2 first terms of the above equation)

$$2(\ell + 1)c_{\ell+1} - (\ell + 1)b_{\ell+1} - \frac{3}{2}\ell b_\ell \geq 0$$

(17)

then

$$\frac{\alpha_\ell}{(1-T)^{3\ell+2}} + \frac{\beta_\ell}{(1-T)^{3\ell+1}} + \frac{\gamma_\ell}{(1-T)^{3\ell}} + \frac{\delta_\ell}{(1-T)^{3\ell-1}} \geq 0. \quad \text{(We used } 1/(1-T)^a \geq 1/(1-T)^b \text{ if } a \geq b).$$

Using (11) and (12), after simple algebra we have (18). Therefore by construction, RHS of (11) $\geq \Psi_{\ell+1}$. After nice cancellations, it yields

$$\Psi_{\ell+1} = \frac{3(\ell + 1)b_{\ell+1}}{(1-T)^{3\ell+4}} - \frac{2(\ell + 1)b_{\ell+1} + (3\ell + 2)c_{\ell+1}}{(1-T)^{3\ell+3}} + \frac{(2\ell + 1)c_{\ell+1}}{(1-T)^{3\ell+2}}.$$  

(19)

Remarking that $(1-T)\partial_z W_{\ell+1} + (\ell + 1) W_{\ell+1} = \Psi_{\ell+1}$, we have completed the proof of $W_{\ell+1} \leq W_{\ell+1}$.

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