RIGIDITY THEOREM FOR COMPACT BACH-FLAT MANIFOLDS WITH
POSITIVE CONSTANT $\sigma_2$

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ABSTRACT. We prove that an $n(\geq 4)$-dimensional compact Bach-flat manifold with positive
constant $\sigma_2$ is an Einstein manifold, provided that its Weyl curvature satisfies a suitable
pinching condition.

1. Introduction

Let $(M^n, g)(n \geq 3)$ be an $n$-dimensional Riemannian manifold with the Riemannian
curvature tensor $Rm = \{R_{ijkl}\}$, the Weyl curvature tensor $W = \{W_{ijkl}\}$, the Ricci curvature
tensor $Ric = \{R_{ij}\}$ and the scalar curvature $R$. For any manifold of dimension $n \geq 4$, the
Bach tensor, introduced by Bach [1], is defined as

\begin{equation}
B_{ij} \equiv \frac{1}{n-3} \nabla^k \nabla^l W_{ikjl} + \frac{1}{n-2} R^k_{jkl},
\end{equation}

where $\nabla$ is the operator of covariant differentiation on $M^n$. Here and hereafter the Einstein
convention of summing over the repeated indices will be adopted. Recall that a metric $g$
is called Bach-flat and the manifold is called Bach-flat manifold if the Bach tensor vanishes. It is easy to see that $(M^n, g)(n \geq 4)$ is a Bach-flat manifold, if it is either a locally
conformally flat manifold, or an Einstein manifold.

The curvature pinching phenomenon plays an important role in global differential geom-
etry. Some isolation theorems of the Weyl curvature tensor of positive Ricci Einstein mani-
folds are given in [14, 16, 20], when its $L^2$-norm is small. Recently, two rigidity theorems
of the Weyl curvature tensor of positive Ricci Einstein manifolds are given in [4, 11, 12],
which improve results due to [14, 16, 20]. The second author and Xiao have studied comp-
act manifolds with harmonic curvature to obtain some rigidity results in [8, 9, 10]. Here
when a Riemannian manifold satisfies $\delta Rm = \{\nabla \delta R_{ijkl}\} = 0$, we call it a manifold with
harmonic curvature. Bach-flat manifolds have been studied by many authors. For any
complete Bach-flat manifold, Kim [17] has studied some rigidity phenomena and derived
that a complete Bach-flat manifold $M^4$ with nonnegative constant scalar curvature and pos-
itive Yamabe constant is an Einstein manifold if the $L^2$-norm of the trace-free Riemannian
curvature tensor $\tilde{Rm}$ is small enough. Later, Chu [7] improved Kim’s result and showed
that $M^4$ is in fact a space of constant curvature under the same assumptions. For a compact
Bach-flat manifold $M^4$ with the positive Yamabe constant, Chang et al. [5] proved that $M^4$
is conformal equivalent to the standard four-sphere provided that the $L^2$-norm of the Weyl
curvature tensor $W$ is small enough, and also showed that there is only finite diffeomor-
phism class with a bounded $L^2$-norm of $W$. Peng and the second author [13] showed that
the compact Bach-flat manifold with positive constant scalar curvature is spherical space

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form or Einstein manifold under some $L^p$ pinching conditions or some pointwise pinching conditions. For compact manifolds with the positive Yamabe constant, Chang et al. proved a sharp form of differentiable sphere theorem.

For a Riemannian manifold $(M^n,g)(n \geq 3)$, we denote by $\sigma_2(A_g)$ the 2nd-elementary symmetric function of the eigenvalues of the so-called Schouten tensor $A_g := Ric - \frac{\sigma_1}{2(n-1)}g$ with respect to $g$. Hence

$$\sigma_2(A_g) = \frac{1}{2} (trA_g)^2 - |A_g|^2 = \frac{1}{2} \left( \frac{(n-2)^2}{4n(n-1)}R^2 - |\hat{Ric}|^2 \right),$$

where $\hat{Ric} := Ric - \frac{\sigma_1}{n}g$ denotes the trace-free Ricci curvature tensor.

Our main result in this paper is the following:

**Theorem 1.1.** Let $(M^n,g)(n \geq 4)$ be an $n$-dimensional compact Bach-flat manifold with positive constant $\sigma_2(A_g)$. If

$$|W|^2 + \frac{n^2(n-4)}{2(n-2)}|\hat{Ric}|^2 < \frac{4n}{(n-2)^2}\sigma_2(A_g),$$

where $\hat{Ric} := Ric - \frac{\sigma_1}{n}g$ is the trace-free Ricci curvature tensor, then $M^n$ is an Einstein manifold.

**Remark 1.2.** The pinching condition of Theorem 1.1 is optimal. When $M^n = N^1 \times N^{n-1}(c)$, it is easy to compute that $\sigma_2(A_g) = \frac{(n-1)(n-2)}{8}c^2$ and $|\hat{Ric}|^2 = \frac{(n-1)(n-2)}{n}c^2$. In this case the equality in (1.3) holds.

**Corollary 1.3.** Let $(M^4,g)$ be a 4-dimensional compact Bach-flat manifold with positive scalar curvature and positive constant $\sigma_2(A_g)$. If

$$|W|^2 < 2\sigma_2(A_g),$$

then $M^4$ is isometric to a quotient of the round $\mathbb{S}^4$.

**Corollary 1.4.** Let $(M^4,g)$ be a 4-dimensional compact locally conformally flat manifold with positive scalar curvature and positive constant $\sigma_2(A_g)$. Then $M^4$ is isometric to a quotient of the round $\mathbb{S}^4$.

**Remark 1.5.** In [15], Hu-Li-Simon proved for a compact locally conformally flat manifold $(M^n,g)$ with constant non-zero $\sigma_2(A_g)$ for some $k \in \{2, 3, \ldots, n\}$, if the tensor $A_g$ is semi-positive definite, then $(M^n,g)$ is a space form of positive sectional curvature. We enhance this result when $n = 4$.

**Corollary 1.6.** Let $(M^n,g)(n \geq 5)$ be an $n$-dimensional compact locally conformally flat manifold with positive scalar curvature and positive constant $\sigma_2(A_g)$. If

$$|\hat{Ric}|^2 < \frac{1}{n(n-1)}R^2,$$

then $M^n$ is isometric to a quotient of the round $\mathbb{S}^n$.

**Remark 1.7.** In [10], Xiao and the second author proved that an $n$-dimensional compact locally conformally flat manifold $(M^n,g)(n \geq 4)$ with positive constant scalar curvature is isometric to a quotient of the round $\mathbb{S}^n$, if $\left( \int_{M^n} |\hat{Ric}|^2 \right)^{\frac{2}{n}} < \frac{1}{n(n-1)}Y(M^n,[g])$, where $Y(M^n,[g])$ denotes the Yamabe constant of $(M^n,g)$.

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2. Proof of Theorem

In what follows, we adopt, without further comment, the moving frame notation with respect to a chosen local orthonormal frame.

Let \((M^n, g)(n \geq 3)\) be an \(n\)-dimensional compact Riemannian manifold. Decomposing the Riemannian curvature tensor into irreducible components (see [2], Chapter 1, Section G) yields

\[
R_{ijkl} = W_{ijkl} + \frac{1}{n-2} \left( R_{ik} \delta_{jl} - R_{il} \delta_{jk} + R_{jl} \delta_{ik} - R_{jk} \delta_{il} \right) - \frac{R}{(n-1)(n-2)} \left( \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk} \right)
\]

\[
= W_{ijkl} + \frac{1}{n-2} \left( R_{ik} \delta_{jl} - \check{R}_{ik} \delta_{jl} + \check{R}_{il} \delta_{jk} - \check{R}_{jk} \delta_{il} \right) + \frac{R}{n(n-1)} \left( \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk} \right)
\]

\[
= W_{ijkl} + \frac{1}{n-2} \left( A_{ik} \delta_{jl} - A_{il} \delta_{jk} + A_{jl} \delta_{ik} - A_{jk} \delta_{il} \right),
\]

where \(R\) is the scalar curvature, \(R_{ijkl}, W_{ijkl}, R_{ij}, \check{R}_{ij}\) and \(A_{ij}\) denote the components of \(Rm\), the Weyl curvature tensor \(W\), the Ricci curvature tensor \(Ric\), the trace-free Ricci curvature tensor \(\check{Ric} = Ric - \frac{R}{n} g\) and the Schouten tensor \(A = Ric - \frac{R}{2(n-1)} g\), respectively.

Let \(e_1, \ldots, e_n\) be a local orthonormal frame field on \(M^n\), \(\omega_1, \ldots, \omega_n\) its dual coframe field, \(\phi = \sum_{i,j} \phi_{ij} \omega_i \otimes \omega_j\) be a symmetric (0,2)-type tensor defined on \(M^n\). By letting

\[
\phi_{ij,k} := \nabla_k \phi_{ij}, \phi_{ijkl} := \nabla_k \nabla_l \phi_{ij},
\]

where \(\nabla\) is the operator of covariant differentiation on \(M^n\), we have the following Ricci identities

\[
\phi_{ijkl} - \phi_{ij,kl} = \phi_{mj} R_{mikl} + \phi_{mk} R_{mjkl}.
\]

The norm of a \((0,4)\)-type tensor \(T\) is defined as

\[
|T|^2 = |T_{ijkl}|^2 = T_{ijkl} T^{ijkl}.
\]

By the second Bianchi identity

\[
R_{mik,j} + R_{mil,k} + R_{mij,kl} = 0,
\]

we have

\[
(2.3) \quad R_{ijk,k} - R_{ik,j} = R_{lk,j},
\]

and

\[
(2.4) \quad R_{ik,j} = \frac{1}{2} R_{ik}.
\]

Then

\[
(2.5) \quad \check{R}_{ijk,k} - \check{R}_{ik,j} = R_{lk,j} + \frac{R}{n} \delta_{ik} - \frac{R}{n} \delta_{ij},
\]

and

\[
(2.6) \quad \check{R}_{ik,j} = \frac{n-2}{2n} R_{ik}.
\]
By (2.3) and (2.4), we have
\[ W_{lik,lj} = \frac{1}{n-2} (R_{lk,l} \delta_{lj} - R_{lj,l} \delta_{lk} + R_{ij,jl} \delta_{lk} - R_{il,lj} \delta_{lk}) + \frac{R_{j}}{(n-1)(n-2)} (\delta_{lk} \delta_{lj} - \delta_{lj} \delta_{lk}) \]
\[ = R_{lk,lj} - \frac{1}{n-2} (R_{lk,l} \delta_{lj} - R_{lj,l} \delta_{lk} + R_{ij,jl} \delta_{lk} - R_{il,lj} \delta_{lk}) \]
\[ + \frac{R_{j}}{(n-1)(n-2)} \delta_{lj} - \frac{R_{j}}{(n-1)(n-2)} \delta_{lk} \]
\[ = R_{lk,lj} - \frac{1}{n-2} (R_{lk,l} \delta_{lj} - R_{lj,l} \delta_{lk} + R_{ij,jl} \delta_{lk} - R_{il,lj} \delta_{lk}) \]
\[ + \frac{R_{j}}{(n-1)(n-2)} \delta_{lj} - \frac{R_{j}}{(n-1)(n-2)} \delta_{lk} \]
\[ = \frac{n-3}{n-2} R_{lk,lj} + \frac{R_{j}}{2(n-1)(n-2)} (R_{lj,l} \delta_{lk} - R_{lk,lj} \delta_{lj}) \]
\[ = \frac{n-3}{n-2} R_{lk,lj} + \frac{R_{j}}{2(n-1)(n-2)} (R_{lj,l} \delta_{lk} - R_{lk,lj} \delta_{lj}). \]

If \( M^n \) is Bach-flat, i.e.,
\[ B_{ij} = \frac{1}{n-3} W_{lijk,k} + \frac{1}{n-2} R_{lk} W_{lijk} = 0, \]
then from the above we have
\[ R_{lk,lj} = \frac{1}{2(n-1)} (R_{lj,l} \delta_{lk} - R_{lk,lj} \delta_{lj} - R_{lk} W_{lijk}). \]

In order to prove Theorem 1.1, we need the following lemmas:

**Lemma 2.1.** Let \( (M^n, g)(n \geq 4) \) be an \( n \)-dimensional compact Bach-flat Riemannian manifold, then
\[ \frac{1}{2} |\nabla \tilde{Ric}|^2 = |\nabla \tilde{Ric}|^2 + \frac{n-2}{2(n-1)} \tilde{R}_{ij} R_{ij} - 2 \tilde{R}_{ij} \tilde{R}_{lk} W_{lijk} + \frac{n}{n-2} \tilde{R}_{ij} \tilde{R}_{lk} \tilde{R}_{ij} + \frac{R}{n-1} |\tilde{Ric}|^2. \]

**Proof.** By using (2.1), (2.5), (2.6), (2.7) and the Ricci identity, we get
\[ \frac{1}{2} |\nabla \tilde{Ric}|^2 = |\nabla \tilde{Ric}|^2 + \tilde{R}_{ij} \tilde{R}_{lk,j} \]
\[ = |\nabla \tilde{Ric}|^2 + \tilde{R}_{ij} (\tilde{R}_{ik,j} + \tilde{R}_{lk,j} + \frac{R_{j}}{n} \delta_{lj} - \frac{R_{j}}{n} \delta_{lk}) \]
\[ = |\nabla \tilde{Ric}|^2 + \tilde{R}_{ij} \tilde{R}_{ik,j} + \tilde{R}_{il,k} \frac{R_{j}}{n} \]
\[ = |\nabla \tilde{Ric}|^2 + \tilde{R}_{ij} (\tilde{R}_{ik,j} + \tilde{R}_{lk,j} + \tilde{R}_{ih,j} \tilde{R}_{lk,j}) + \tilde{R}_{j} \frac{R_{j}}{n} \]
\[ = |\nabla \tilde{Ric}|^2 + \frac{n-2}{2n} \tilde{R}_{ij} R_{ij} + \tilde{R}_{ij} \tilde{R}_{lk}[W_{lijk} + \frac{1}{n-2} (\tilde{R}_{hk,j} \delta_{lk} - \tilde{R}_{hk,l} \delta_{lj} + \tilde{R}_{hk,j} \delta_{lk} - \tilde{R}_{lk,j} \delta_{lj}) \]
\[ + \frac{R}{n(n-1)} (\delta_{lk} \delta_{lj} - \delta_{lj} \delta_{lk})] + \tilde{R}_{ij} \tilde{R}_{ih} (\tilde{R}_{ih,j} + \frac{R}{n} \delta_{lj}) + \tilde{R}_{j} \frac{R_{j}}{n} \]
\[ + \tilde{R}_{ij} \frac{1}{2(n-1)} (R_{lk,lj} - R_{lj,lk}) - \tilde{R}_{l} W_{lijk} \]
\[ = |\nabla \tilde{Ric}|^2 + \frac{n-2}{2(n-1)} \tilde{R}_{ij} R_{ij} - 2 \tilde{R}_{ij} \tilde{R}_{lk} W_{lijk} + \frac{n}{n-2} \tilde{R}_{ih,j} \tilde{R}_{ih} + \frac{R}{n-1} |\tilde{Ric}|^2. \]

This completes the proof of Lemma 2.1. \( \square \)
Lemma 2.2. Let \((M^n, g)(n \geq 4)\) be an \(n\)-dimensional compact Bach-flat Riemannian manifold with positive constant \(\sigma_2(A_g)\), then
\[
0 \geq \frac{n}{n-2} \int_{M^n} \hat{R}_{ijk} \hat{R}_{ijk} + \frac{1}{n-1} \int_{M^n} R \hat{Ric}^2 - 2 \int_{M^n} \hat{R}_{ijk} \hat{R}_{ikj}.
\]

Proof. We compute
\[
\begin{align*}
|\nabla A|^2 &= |\nabla Ric|^2 - \frac{1}{n-1} \left| \nabla R \right|^2 + \frac{\left| \nabla R \right|^2}{4(n-1)^2} n \\
&= |\nabla Ric|^2 - \frac{3n-4}{4(n-1)^2} \left| \nabla R \right|^2 \\
&= |\nabla \hat{Ric}|^2 + \frac{(n-2)^2}{4(n-1)^2} \left| \nabla R \right|^2,
\end{align*}
\]
and
\[
|\nabla tr A|^2 = \frac{(n-2)^2}{4(n-1)^2} \left| \nabla R \right|^2.
\]

Since \(\sigma_2(A_g)\) is a positive constant, the inequality of Kato type due to Hu-Li-Simon [15], Li [13] and Simon [19], i.e.,
\[
|\nabla A|^2 \geq |\nabla tr A|^2
\]
holds. From (2.9) and (2.10), (2.11) implies that
\[
|\nabla \hat{Ric}|^2 \geq \frac{(n-2)^2}{4(n-1)^2} \left| \nabla R \right|^2.
\]

Integrating (2.8) by parts on \(M^n\) and using (2.6) and (2.12) we have
\[
0 = \int_{M^n} |\nabla \hat{Ric}|^2 + \frac{n-2}{2(n-1)} \int_{M^n} \hat{R}_{ijk} \hat{R}_{ijk} - 2 \int_{M^n} \hat{R}_{ijk} \hat{R}_{ikj} + \frac{n-2}{n-1} \int_{M^n} \hat{R}_{ijk} \hat{R}_{ikj} + \frac{n-2}{n-1} \int_{M^n} \hat{R}_{ijk} \hat{R}_{ikj} + \int_{M^n} \frac{R}{n-1} |\hat{Ric}|^2
\]
\[
= \int_{M^n} |\nabla \hat{Ric}|^2 - \frac{n-2}{2(n-1)} \int_{M^n} \hat{R}_{ijk} \hat{R}_{ijk} - 2 \int_{M^n} \hat{R}_{ijk} \hat{R}_{ikj} + \frac{n-2}{n-1} \int_{M^n} \hat{R}_{ijk} \hat{R}_{ikj} + \frac{n-2}{n-1} \int_{M^n} \hat{R}_{ijk} \hat{R}_{ikj} + \int_{M^n} \frac{R}{n-1} |\hat{Ric}|^2
\]
\[
\geq -2 \int_{M^n} \hat{R}_{ijk} \hat{R}_{ikj} + \frac{n}{n-2} \int_{M^n} \hat{R}_{ijk} \hat{R}_{ikj} + \int_{M^n} \frac{R}{n-1} |\hat{Ric}|^2.
\]

This completes the proof of Lemma 2.2. \(\square\)

Lemma 2.3. Let \((M^n, g)(n \geq 4)\) be an \(n\)-dimensional Riemannian manifold, then
\[
\left| -W_{ijkl} \hat{R}_{ij} + \frac{n}{2(n-2)} \hat{R}_{ijkl} \hat{R}_{ikj} \right| \leq \sqrt{\frac{n-2}{2(n-1)}} \left| \hat{Ric} \right|^2 \left( |W|^2 + \frac{n}{2(n-2)} \left| \hat{Ric} \right|^2 \right)^{\frac{1}{2}}.
\]

Remark 2.4. We follow these proofs of Proposition 2.1 in [4] and Lemma 4.7 in [3] to prove this lemma which is proved in [13]. For completeness, we also write it out.

Proof. First of all we have
\[
(Ric \otimes g)_{ijkl} = \hat{R}_{ijk} g_{jl} - \hat{R}_{ijkl} g_{jl} + \hat{R}_{ijkl} g_{jl} = \hat{R}_{ijkl} g_{jl} - \hat{R}_{ijkl} g_{jl},
\]
\[
(Ric \otimes \hat{Ric})_{ijkl} = 2(\hat{R}_{ijkl} \hat{R}_{jkl} - \hat{R}_{ijkl} \hat{R}_{jkl}),
\]
where $\otimes$ denotes the Kulkarni-Nomizu product. An easy computation shows
\[
W_{ijkl} \check{R}_{ik} \check{R}_{jk} = \frac{1}{4} W_{ijkl} (\check{\text{Ric}} \otimes \check{\text{Ric}})_{ijkl},
\]
\[
\check{R}_{ij} \check{R}_{jk} \check{R}_{ik} = -\frac{1}{8} (\check{\text{Ric}} \otimes g)_{ijkl} (\check{\text{Ric}} \otimes \check{\text{Ric}})_{ijkl}.
\]
Hence we get the following equation
\[
(2.13) \quad W_{ijkl} \check{R}_{ik} \check{R}_{jk} + \frac{n}{2(n - 2)} \check{R}_{ijkl} \check{R}_{ik} \check{R}_{jk} = -\frac{1}{4} \left( W + \frac{n}{4(n - 2)} \check{\text{Ric}} \otimes g \right) (\check{\text{Ric}} \otimes \check{\text{Ric}})_{ijkl}.
\]
Since $\check{\text{Ric}} \otimes \check{\text{Ric}}$ has the same symmetries with the Riemannian curvature tensor, it can be orthogonally decomposed as
\[
\check{\text{Ric}} \otimes \check{\text{Ric}} = T + V' + U'.
\]
Here $T$ is totally trace-free, and
\[
V'_{ijkl} = -\frac{2}{n - 2} \left( \check{\text{Ric}}^2 \otimes g \right)_{ijkl} + \frac{2}{n(n - 2)} |\check{\text{Ric}}|^2 (g \otimes g)_{ijkl},
\]
\[
U'_{ijkl} = -\frac{1}{n(n - 1)} |\check{\text{Ric}}|^2 (g \otimes g)_{ijkl},
\]
where $\left( \check{\text{Ric}}^2 \right)_{ik} = \check{R}_{ip} \check{R}_{kp}$. Taking the squared norm we obtain
\[
|\check{\text{Ric}} \otimes \check{\text{Ric}}|^2 = 8|\check{\text{Ric}}|^4 - 8|\check{\text{Ric}}^2|^2,
\]
\[
|V'|^2 = \frac{16}{n - 2} |\check{\text{Ric}}|^2 - \frac{16}{n(n - 2)} |\check{\text{Ric}}|^4,
\]
\[
|U'|^2 = \frac{8}{n(n - 1)} |\check{\text{Ric}}|^4.
\]
In particular, one has
\[
(2.14) \quad |T|^2 + \frac{n}{2} |V'|^2 = |\check{\text{Ric}} \otimes \check{\text{Ric}}|^2 + \frac{n - 2}{2} |V'|^2 - |U'|^2 = \frac{8(n - 2)}{n - 1} |\check{\text{Ric}}|^4.
\]
We now estimate the right hand side of (2.13). Using (2.14), Cauchy-Schwarz inequality and the fact that $W$ and $T$ are totally trace-free we obtain
\[
\left| \left( W + \frac{n}{4(n - 2)} \check{\text{Ric}} \otimes g \right)_{ijkl} (\check{\text{Ric}} \otimes \check{\text{Ric}})_{ijkl} \right|^2 = \left| \left( W + \frac{n}{4(n - 2)} \check{\text{Ric}} \otimes g \right)_{ijkl} (T + V')_{ijkl} \right|^2 \leq \left( W + \frac{\sqrt{2}n}{4(n - 2)} \check{\text{Ric}} \otimes g \right)_{ijkl} \left( |T|^2 + \frac{n}{2} |V'|^2 \right) \leq \frac{8(n - 2)}{n - 1} |\check{\text{Ric}}|^4 \left( |W|^2 + \frac{n}{2(n - 2)} |\check{\text{Ric}}|^2 \right).
\]
This estimate together with (2.13) concludes this proof. \hfill \Box
Proof of Theorem 1.1. By (1.2), the pinching condition (1.3) in Theorem 1.1 is equivalent to

\[ |W|^2 + \frac{n}{2(n-2)} |\hat{\text{Ric}}|^2 < \frac{R^2}{2(n-1)(n-2)}. \]  

By Lemma 2.2 and Lemma 2.3 we obtain

\[ 0 \geq \int_M \left[ \frac{R}{n-1} - 2 \sqrt{\frac{n-2}{2(n-1)}} \left( |W|^2 + \frac{n}{2(n-2)} |\hat{\text{Ric}}|^2 \right) \right] |\hat{\text{Ric}}|^2. \]  

Combining (2.15) with (2.16), we get that \( \hat{\text{Ric}} = 0 \), i.e., \( M^n \) is an Einstein manifold. We finish the proof of Theorem 1.1. □

Proof of Corollary 1.3. When \( n = 4 \), the pinching condition (1.3) in Theorem 1.1 is reduced to (1.4) in Corollary 1.3. By Theorem 1.1, \( M^4 \) is Einstein. Thus by (1.2), (1.4) is equivalent to

\[ |W|^2 < \frac{R^2}{12}. \]  

By Theorem 1.8 in [8], we obtain that \( M^4 \) is isometric to a quotient of the round \( S^4 \). We finish the proof of Theorem 1.1. □

Proof of Corollary 1.4. According to \( \sigma_2(A_g) \) is a positive constant and the fact that a locally conformally flat manifold is also a Bach-flat manifold, \( (M^4, g) \) satisfies condition (1.4) in Corollary 1.3 then we know that \( M^4 \) is isometric to \( S^4 \). Hence we complete the proof of Corollary 1.4. □

Proof of Corollary 1.6. From (1.5) and (1.2), we can easily get

\[ |\hat{\text{Ric}}|^2 < \frac{8}{n(n-4)} \sigma_2(A_g). \]  

Since \( (M^n, g) \) is a compact locally conformally flat manifold, we have \( |W| = 0 \), then \( (M^n, g) \) satisfies condition (1.3) in Theorem 1.1. According to the fact that a locally conformally flat manifold is also a Bach-flat manifold, by use of Theorem 1.1 we know that \( M^n \) is an Einstein manifold. Then we can get the conclusion that \( (M^n, g) \) is isometric to a quotient of the round \( S^n \). Hence we complete the proof of Corollary 1.6. □

From the proofs of Lemma 2.2 and Theorem 1.1, we have

\[ \int_M |\nabla \hat{\text{Ric}}|^2 \geq \frac{(n-2)^2}{4n(n-1)} \int_M |\nabla R|^2, \]  

and

\[ |W|^2 + \frac{n}{2(n-2)} |\hat{\text{Ric}}|^2 < \frac{1}{2(n-2)(n-1)} R^2, \]  

then \( M^n \) is an Einstein manifold.

Remark 2.6. For Bach-flat manifolds with positive constant scalar curvature, (2.18) naturally holds. Proposition 2.5 improves Theorem 3 in [13].
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