FUNCTIONS CONTINUOUS ON CURVES IN O-MINIMAL STRUCTURES

JANAK RAMAKRISHNAN

Abstract. We give necessary and sufficient conditions on a non-oscillatory curve in an o-minimal structure such that, for any bounded definable function, there exists a definable closed set containing an initial segment of the curve on which the function is continuous. This question is translated into one on types: What are the conditions on an n-type such that, for any bounded definable function, there is a definable closed set containing the type on which the function is continuous. We introduce two concepts related to o-minimal types: that of scale, which measures the “density” of a smaller model inside a larger one at some point, and that of a decreasing type, which allows us to manipulate types more easily than before. We formalize the notion of scale mentioned in [MS94] and refine the characterization there of definable types in o-minimal theories. Then we join it with the notion of decreasing type to achieve our main result. A decreasing sequence has the property that the map taking initial segments of the sequence to the T-convex subrings generated by them preserves inclusion.

1. Introduction

The study of o-minimal structures often encounters functions that are not first-order definable in such structures. Such functions may be definable in an o-minimal expansion of the original structure or lie in a Hardy field extension of the field of germs of definable functions. In this article, we examine non-oscillatory curves in an o-minimal structure – curves that are not definable in the structure, but are “well-behaved,” in that their component functions do not oscillate with respect to the definable functions. For example, \( \langle t, e^t \rangle \) is non-oscillatory in \((\mathbb{R}, +, \cdot, <, 0, 1)\), despite not being a definable curve, since \( e^t \) does not oscillate with respect to any rational function.

We will focus on a question which arose from trying to generalize Theorem 7.1 of [Mal74], on the existence of a formal solution to a differential equation implying the existence of a \( C^\infty \) solution with Taylor series the formal solution. The question is as follows: given a bounded definable function and a non-oscillatory curve, when is the curve contained in a definable closed set on which the function is continuous? The answer is not “always”, as shown by the curve \( \langle t, -1/\ln t \rangle \) near 0 and the function \( \min(1, y/x) \) in the structure \((\mathbb{R}, +, \cdot, <, 0, 1)\) (see Corollary 2.10).

Date: February 26, 2009.

2000 Mathematics Subject Classification. Primary 03C64; Secondary 26B05, 12J15.

Thanks to Thomas Scanlon, my thesis advisor, for discussing this problem with me, and for all his valuable help. The central question of this paper was brought to my attention by Patrick Speissegger. I would like to thank him and Leo Harrington for reading and commenting on various versions of the results.
To answer this question, we use an elementary observation — that any non-oscillatory curve in \( n \) dimensions has associated to it a complete \( n \)-type — to turn the question into one about types, namely: when is a type contained in a definable closed set on which the function is continuous?

The way that such a closed set can fail to exist is that, in some sense, the type lies in a “gap” between two regions on which the function takes very different values, but which share a common boundary point that is infinitesimally close to the type. The right way to formalize these notions of “gap” and “region” comes from [MS94]’s concept of “scale” (our terminology). However, such concepts turn out to be insufficient in the absence of certain guarantees on the order of the variables of the type. To that end, this article introduces the notion of a “decreasing type,” which simplifies the use of scale. We are then equipped to state and prove our main theorem:

**Theorem A.** Let \( T \) be such that all principal types are interdefinable (e.g., \( T \) expands the theory of an ordered field). Let \( M \models T \), and let \( A \subseteq M \). Let \( p \) be a decreasing \( n \)-type over \( A \). Let \( \bar{c} = \langle c_1, \ldots, c_n \rangle \models p \), and let \( \text{tp}(c_1/M) \) be principal. Then the following statements are equivalent:

1. For every \( A \)-definable bounded \( n \)-ary function, \( F \), defined on \( \bar{c} \), there is an \( A \)-definable set, \( C \), such that \( \bar{c} \in C \), \( F \) is continuous on \( C \), and \( F \upharpoonright C \) extends continuously to \( \text{cl}(C) \).
2. For \( i = 1, \ldots, n \), \( \text{tp}(c_i/Ac_{<i}) \) is algebraic, principal, or out of scale on \( A \).

Using our correspondence between types and curves, we obtain our desired result:

**Theorem B.** Let \( T \) be such that all principal types are interdefinable, let \( M \models T \), and let \( \bar{\gamma}(t) = \langle \gamma_1(t), \ldots, \gamma_n(t) \rangle \) be a (not necessarily definable) non-oscillatory curve in \( M^n \). Then the following statements are equivalent:

1. \( \bar{\gamma} \) can be reordered so that \( \text{tp}(\bar{\gamma}/M) \) is decreasing, and \( \text{tp}(\gamma_i/M\bar{\gamma}_{<i}) \) is algebraic, principal, or out of scale on \( M \).
2. For any bounded \( M \)-definable \( n \)-ary function, \( F \), continuous on \( \gamma([0, s)) \) for some \( s > 0 \), there is an \( M \)-definable subset of \( M^n \), \( C \), such that \( F \upharpoonright C \) continuously extends to \( \text{cl}(C) \) and \( \bar{\gamma}([0, t)) \subseteq \text{cl}(C) \), for some \( t > 0 \).

The structure of the paper is as follows. In Section 2 we state the question of the paper and reduce it from one about curves to one about types. In Section 3 we obtain some basic results on \( o \)-minimal types, following [Mar86]. In Section 4 we define scale, give some results of [MS94], and obtain some new ones. In Section 5 we give the notion of a decreasing type. Finally, in Section 6 we prove Theorem A and get Theorem B as a corollary.

Throughout, we fix an \( o \)-minimal theory, \( T \), and structure, \( M \). All structures are assumed to be embedded in a monster model, \( \mathfrak{C} \), in which lie all elements and sets. “Definable” means “definable with parameters in \( M \)”. Tuples (of elements or functions) will be indicated by a bar above the symbol. Subscripts, like \( x_i \), indicate the \( i \)th coordinate of \( \bar{x} \), and \( x_{<i} \) is the tuple \( \langle x_1, \ldots, x_{i-1} \rangle \). Similarly for \( x_{\leq i} \), \( x_{> i} \), and \( x_{\geq i} \). We will do the same for a function with image in \( M^n \), like \( \gamma_i \) to mean the \( i \)th component of \( \bar{\gamma} \). We let \( \pi_{<i} \) denote projection onto the first \( i - 1 \) coordinates.

A “curve” is a continuous (though not necessarily definable) map from \([0, 1) \cap M\) to \( M^n \) for some \( n \). We denote the topological closure of a set \( A \) by \( \text{cl}(A) \). If \( A \) is a set, \( \text{Pr}(A) \) is the prime model containing \( A \). If \( N \) is a model and \( A \) is a set, \( N(A) \) denotes the prime model containing \( NA \).
2. Exploring the question

In this section, we assume that $T$ expands the theory of a real closed field. Although a stronger assumption than necessary, it simplifies the discussion. We will see in Section 3 what the correct assumption should be.

Given any definable curve in $M^n$, $\tilde{\gamma}$, and any bounded definable $n$-ary function, $F$, such that $F$ is defined on $\tilde{\gamma}((0, s))$ for some $s \in (0, 1)$, we have the existence of $\lim_{t \to 0} F(\tilde{\gamma}(t))$ in $M$, by o-minimality. But we can say more. Closely related results which imply the following lemma were proved in [FKMS08]. However, the proof is quite direct and illuminates our strategy in the future, so we present it here.

**Lemma 2.1.** Let $\tilde{\gamma}$ be a definable curve in $M^n$ and let $F$ be a definable bounded $n$-ary function with $F$ defined on $\tilde{\gamma}((0, a))$ for some $a \in (0, 1)$. If $F(\tilde{\gamma}(0)) = \lim_{t \to 0} F(\tilde{\gamma}(t))$, then there is a definable closed set, $D$, containing $\tilde{\gamma}((0, s))$ for some $s \in (0, 1)$, such that $F$ is continuous.

**Proof.** We may definably transform $\tilde{\gamma}$ and assume that $\gamma_1$ is either $t$ or $-t$. We may also choose $s'$ such that $\tilde{\gamma}((0, s'))$ lies entirely within some cell, $C'$, on which $F$ is continuous. An easy induction argument shows that we may take $C'$ to be open. Let the definition of $C'$ be

$$\left\{ \bar{x} \in \mathbb{C}^n \mid \bigwedge_{i \leq n} x_i \in (f_i(\bar{x}_{<i}), g_i(\bar{x}_{<i})) \right\},$$

where, for $i \leq n$, $f_i$, $g_i$ are $i-1$-ary functions, with $f_i < g_i$, defined on the domain

$$\left\{ \bar{x} \in \mathbb{C}^{i-1} \mid \bigwedge_{j < i} x_j \in (f_j(\bar{x}_{<j}), g_j(\bar{x}_{<j})) \right\}.$$

(In future, we will just say that $f_1, g_1, \ldots, f_n, g_n$ is the cell-definition of $C'$.) First, we wish to ensure that no points in the closure other than $\gamma(0)$ will present a problem. We do this by shrinking the set so that the closure lies inside $C'$ as much as possible. Define $C''$ by

$$\left\{ \bar{x} \in C' \mid 0 < x_1 < s'/2, \bigwedge_{1 \leq i \leq n} x_i \in \left( \frac{f_i(\bar{x}_{<i}) + \gamma_i(x_1)}{2}, \frac{g_i(\bar{x}_{<i}) + \gamma_i(x_1)}{2} \right) \right\}.$$

By construction, $\text{cl}(C'') \setminus \{0\} \times \mathbb{C}^{n-1} \subseteq C'$. Thus, $F$ is continuous on $\text{cl}(C'') \setminus \{0\} \times \mathbb{C}^{n-1}$. We have $\tilde{\gamma}((0, s'/2)) \subseteq C''$.

Now, to ensure that taking limits of $F$ as we approach $\gamma(0)$ will yield consistent results, we can define

$$D := \{ \bar{x} \in \text{cl}(C'') \mid x_1 \in [0, s'], |F(\bar{x}) - F(\tilde{\gamma}(x_1))| \leq x_1 \}.$$

It is clear that $D$ is closed and contains $\tilde{\gamma}((0, s'/2))$, and that $F$ is continuous on $D \setminus \{0\} \times \mathbb{C}^{n-1}$. However, we can see that $F$ is actually continuous on $D$ using the sup norm, $\| \cdot \|$: take any point $\bar{a} \in D$ with first coordinate $0$. By definition of $D$, $F(\bar{a}) = \lim_{t \to 0} F(\tilde{\gamma}(t))$. For any $\varepsilon$, choose $\delta$ such that $\delta < \varepsilon/2$ and $|F(\tilde{\gamma}(t)) - F(\tilde{\gamma}(0))| < \varepsilon/2$ for $t \in (0, \delta)$. Then any $\bar{b} \in D$ with $\|\bar{b} - \bar{a}\| < \delta$ must have $b_1 < \delta$, and so

$$|F(\bar{b}) - F(\bar{a})| \leq |F(\bar{b}) - F(\tilde{\gamma}(b_1))| + |F(\tilde{\gamma}(b_1)) - F(\tilde{\gamma}(0))| \leq b_1 + \varepsilon/2 < \varepsilon.$$

Thus, $D$ is as desired.
We now consider the case where \( \bar{\gamma} \) is not definable, but still has initial segment in \( M^* \). Of course, we wish at a minimum for \( \lim_{t \to 0^+} F(\bar{\gamma}(t)) \) to exist. This is assured if \( \bar{\gamma} \) is definable in an o-minimal expansion of \( M \), or belongs to a Hardy field extending the field of germs of definable functions of \( M \), but there is a more general property.

**Definition 2.2.** (LMS03) Let \( \bar{\gamma} \) be a (not necessarily definable) curve. Say that \( \bar{\gamma} \) is non-oscillatory if, for each definable function \( f \) from \( M^{m+1} \) to \( M \), there exists \( t_f > 0 \) such that either \( f(t, \bar{\gamma}(t)) = 0 \) for all \( t \in (0, t_f) \) or \( f(t, \bar{\gamma}(t)) \neq 0 \) for all \( t \in t_f \).

We now show that being non-oscillatory is precisely what we need for limits to exist.

**Lemma 2.3.** \( \bar{\gamma} \) is non-oscillatory iff \( \lim_{t \to 0^+} F(\bar{\gamma}(t)) \) exists in \( \mathcal{C} \) for any definable bounded \( F \) (note that the limit may not be unique).

**Proof.** For the forward direction, suppose that, for some \( a \in (0, 1) \), for any \( c, s \in (0, a) \), we know that there exists \( s^-(c) \in (0, c) \) such that \( F(\bar{\gamma}(v)) < F(\bar{\gamma}(c)) \) for all \( v \in (0, s^-(c)) \). Then we can take \( \inf \{ F(\bar{\gamma}(t)) \mid t \in (0, a) \} \) to be the limit, which exists in \( \mathcal{C} \) (though not uniquely). Similarly, if we can find \( s^+(c) \) with \( F(\bar{\gamma}(v)) > F(c) \) for \( v \in (0, s^+(c)) \), we take the sup. We are left with the case in which, for any \( t \in (0, 1) \) and \( s \in (0, t) \), we can find \( v_1, v_2 \in (0, s) \) with \( F(\bar{\gamma}(v_1)) < F(\bar{\gamma}(t)) \) and \( F(\bar{\gamma}(v_2)) > F(\bar{\gamma}(t)) \). Fix \( c \) such that \( F \) is continuous on \( \bar{\gamma}((0, c]) \). By continuity of \( F(\bar{\gamma}(t)) \), we must then have that \( \bar{\gamma} \) enters and leaves the definable (since \( F(\bar{\gamma}(c)) \) is consistent) infinitely many times. After decomposing \( A \) into finitely many cells, we know that \( \bar{\gamma} \) enters and leaves some cell infinitely many times, which is easily seen to contradict the definition of non-oscillatory.

For the reverse direction, suppose that \( \bar{\gamma} \) is not non-oscillatory. Then there is some function, \( f \), such that there are arbitrarily small values of \( t \) with \( f(t, \bar{\gamma}(t)) \) equal to and not equal to 0. Let \( F = f \).

We can now ask the same question of non-oscillatory curves as we did of definable curves – given a bounded definable function whose value at one endpoint agrees with the limit on the curve, is there a closed set containing the curve on which the function is continuous?

It is helpful here to examine the behavior of a non-oscillatory curve in the structure \( M \) more closely.

**Definition 2.4.** Let \( \bar{\gamma} = \langle \gamma_1, \ldots, \gamma_k \rangle \) be a non-oscillatory curve. Let

\[
\tp(\bar{\gamma}/M) := \lim_{t \to 0^+} \tp(\bar{\gamma}(t)/M) = \{ \varphi(\bar{x}) \mid \exists s \forall t \in (0, s) \varphi(\bar{\gamma}(t)) \}.
\]

**Lemma 2.5.** \( \tp(\bar{\gamma}/M) \) is a complete, consistent type.

**Proof.** It is clear that it is consistent, so it remains to show completeness. Consider any formula, \( \varphi(\bar{x}) \). By cell decomposition, \( \varphi \) is equivalent to a disjunction of cell definitions, say \( \bigvee_{i=1}^m \bar{x} \in C_i \). We may assume by induction on \( k \) that \( \exists x_k \varphi(\bar{x}) \) is determined by \( \tp(\bar{\gamma}/M) \). If it is not in \( \tp(\bar{\gamma}/M) \), then clearly \( \varphi \) is not either, so we may assume that it is. Since \( \exists x_k \varphi(\bar{x}) \) defines the set \( \bigvee_{i=1}^m \pi_{<k}(C_{i}) \), we must have that \( \bar{\gamma}_{<k}(t) \) lies in the projection of the cells \( \pi_{<k}(C_{i_1}), \ldots, \pi_{<k}(C_{i_r}) \), for \( t \in (0, s) \), some positive number \( s \), and \( i_1, \ldots, i_r \leq m \). Let the \( k \)th coordinate cell definition of \( C_{i_j} \) be given by \( (f^{i_j}, g^{i_j}) \). If the ordering of \( \gamma_k \) in the set \( \{ F(\bar{\gamma}_{<k}(i)), g^{i_j}(\bar{\gamma}_{<k}) \mid j = 1, \ldots, r \} \) is determined, then we are done. But \( \bar{\gamma} \) is non-oscillatory, which is sufficient. \( \square \)
Lemma 2.6. For any definable set, $C$, $\text{tp}(\bar{\gamma}/M) \vdash \bar{x} \in C$ iff $\bar{\gamma}((0, s)) \in C$ for some $s > 0$.

Proof. By the definition of $\text{tp}(\bar{\gamma}/M)$. \hfill \Box

Lemma 2.7. Let $\bar{\gamma}$ be a non-oscillatory curve in $M^n$. The following conditions are equivalent.

1. For any bounded definable function, $F$, there exists a definable set, $C$, such that $F \upharpoonright C$ extends continuously to $\text{cl}(C)$ and $\text{cl}(C)$ contains an initial segment of $\bar{\gamma}$.

2. For any bounded definable function, $F$, there is a definable $C$ containing $\text{tp}(\bar{\gamma})$ such that $F \upharpoonright C$ continuously extends to $\text{cl}(C)$.

Proof. Trivial, using Lemma 2.6. \hfill \Box

We can then ask our question about types. But it is not hard to construct an example where no appropriate set exists.

Example 2.8. Let $M = (\mathbb{R}, +, \cdot, <)$, the reals as an ordered field. Let $p(x_1, x_2)$ be the type generated by the formulas $x_1 > 0$, $x_1 < a$, $0 < x_2 < x_1$, $x_2 < ax_1$, and $ax_1^q < x_2$, for $a \in \mathbb{R}_+$, $q \in \mathbb{Q}_{>1}$. Let $F(x_1, x_2)$ be the function $\min(y/x, 1)$ on the open first quadrant, and 0 everywhere else.

Claim 2.9. $p$ is consistent and complete, and if $D$ is any definable set containing the realizations of $p$, then $F \upharpoonright D$ does not extend continuously to $\text{cl}(D)$.

Proof. We leave verification of $p$'s consistency and completeness as routine. For the last statement, take a cell decomposition of $D$, say $C_1, \ldots, C_k$. WLOG, assume that $C_1 \in p$. Note that $C_1$ must be open. Let the cell definition of $C_1$ be given by $f_1, g_1, f_2, g_2$, where $f_1$ and $g_1$ are constants. Note that $f_1 \leq 0$. By Theorem 4.6 of [Mil94], $f_2(x_1)$ and $g_2(x_1)$ asymptotically approach rational powers as $x_1$ goes to 0. Since $p$ requires that $x_2$ is greater than $x_1^d$ for any rational $d > 1$, $g_2(x_1)$ must approach a rational power of $x_1$ with exponent at most 1. Similarly, since $p$ requires that $x_2$ is less than $ax_1$ for any positive $a \in \mathbb{R}$, $f_2(x_1)$ must approach a rational power of $x_1$ with exponent greater than 1. But then $F(x_1, f_2(x_1))$ and $F(x_1, g_2(x_1))$ have different limits as $x_1$ goes to 0. Since the sets $\{x \mid x_2 = f_2(x_1)\}$ and $\{x \mid x_2 = g_2(x_1)\}$ are in the closure of $C_1$, they are in $\text{cl}(D)$, so it is impossible for $F \upharpoonright D$ to extend continuously to 0 in $\text{cl}(D)$. \hfill \Box

Corollary 2.10. With $M$ and $F$ as above, if $\bar{\gamma}$ is the curve $(t, -t/\ln t)$, there is no closed set containing an initial segment of $\bar{\gamma}$ on which $F$ is continuous.

Proof. $\text{tp}(\bar{\gamma}/M)$ is $p$ in Example 2.8. \hfill \Box

We may ask, then, for necessary and sufficient conditions on $p$, an $n$-type in an o-minimal structure, so that, for any $F$, a bounded definable function continuous on $p$, there is a closed definable set, $C$, containing $p$ such that $F$ is continuous on $C$. In order to characterize such types, we will need to extend a classification of o-minimal types developed by [Mar86] and [MS94].
3. O-minimal types

Before we begin to present any new machinery, we will need to state some basic results that follow from [Mar86]. We use here the results of [Mar86] but follow some of the terminology of [Tre05]: the definable 1-types are called “principal”. To each principal type over a set $A$ is associated a unique element of $\text{dcl}(A) \cup \{\pm \infty\}$, $a$, which is “closest” to. We say that a principal type is “principal above/below/near $a$.” The results of [Mar86] will be used freely – the reader is referred there for background. We first present some results easily derived from the definitions.

Lemma 3.1. Let $c_1, c_2$ be principal over $A$, near $\beta_1, \beta_2 \in \text{dcl}(A) \cup \{\pm \infty\}$ respectively. If $c_1$ is non-principal over $c_2A$, then there is some $A$-definable function $f(x)$ such that $\lim_{x \to \beta_2^+} f(x) = \beta_2$ and $c_2$ lies between $f(c_1)$ and $\beta_2$.

Proof. We assume that $c_1, c_2$ are above $\beta_1, \beta_2$, respectively – the proof is similar for the other possibilities. Since $c_1$ is non-principal over $c_2A$, there is some $A$-definable $g$ such that $\beta_1 < g(c_2) < c_1$. Since $g(c_2)$ cannot be in $\text{dcl}(A)$, we must have $\lim_{x \to \beta_2^+} g(x) = \beta_1$: if not, there is some $A$-definable interval above $\beta_2$ where $\beta_1 < a < g(x)$, for a fixed $a \in \text{dcl}(A)$, which is impossible, since any $A$-definable interval above $\beta_2$ contains $c_2$, and $\beta_1 < g(c_2) < c_1 < a$ for every $a > \beta_1 \in \text{dcl}(A)$.

Thus, $\lim_{x \to \beta_2^+} g(x) = \beta_1$, and as well, $g$ is increasing in a definable neighborhood of $\beta_2$ – else we could find an element of $\text{dcl}(A)$ between $\beta_1$ and $g(c_2)$. Let $f(x) = g^{-1}(x)$. Then $f(c_1) > c_2$, and moreover $\lim_{x \to \beta_2^+} f(x) = \beta_2$. \qed

Lemma 3.2. Let $S'$ be a definable set in $\mathcal{C}^{m+n}$. Let $S = \{\bar{x} \mid \exists a \in \pi_{\leq m}(S)(\bar{x} \in \text{cl}(|\bar{a}| \times S_\bar{a}))\}$. Then there is a partition of $\mathcal{C}^m$ into definable subsets $A_1, \ldots, A_k$ such that $\text{cl}(S') \cap (A_i \times \mathcal{C}^n) = S \cap (A_i \times \mathcal{C}^n)$, for $i = 1, \ldots, k$. In other words, the fiber of the closure is the closure of the fiber.

Proof. $S$ and $\text{cl}(S')$ satisfy the conditions of Corollary 2.3, Chapter 6, of [vdD98], with $A = \mathcal{C}^n$, so we can find $A_1, \ldots, A_k$ such that $S \cap (A_i \times \mathcal{C}^n)$ is closed in $\text{cl}(S') \cap (A_i \times \mathcal{C}^n)$, which implies that the two sets are equal, for each $i = 1, \ldots, k$. \qed

Lemma 3.3. Let $A$ be a set. If, for any elements $b, c$, the principal types above $b$ and near $\infty$ are interdefinable (over $Ab$), the principal types above $b$ and below $c$ are interdefinable (over $A_b$), and the principal types near $\pm\infty$ are interdefinable (over $A$), then, for any $B \supseteq A$, $\text{dcl}(B)$ is dense without endpoints.

Proof. To show that $\text{dcl}(B)$ is dense without endpoints, it suffices to show that $\text{dcl}(B)$ is nonempty, and, given a point, $b$, there are points $b^-, b^+ \in \text{dcl}(Ab)$ with $b^- < b < b^+$, and, given $b < c$, there is $d \in (b, c) \cap \text{dcl}(Abc)$. The argument for all three is the same – namely, we take an interval, and show that the map between the principal type above the left-hand endpoint and the principal type below the right-hand endpoint yields a point in the interval definable from $A$ and the endpoints. We apply the principal type above the left-hand endpoint and the principal type below the right-hand endpoint yields a point in the interval definable from $A$ and the endpoints. We apply this to the interval $(-\infty, \infty)$ to show $\text{dcl}(B)$ is nonempty, to the intervals $(b, \infty)$ and $(-\infty, b)$ to get $b^+$ and $b^-$, respectively, and to $(b, c)$ to get $d$. So let $(\alpha, \beta)$ be an interval, with $\alpha, \beta \in \mathcal{C} \cup \{\pm \infty\}$. By hypothesis, there is an $A$-definable function, $f$, such that $\lim_{x \to \alpha^+} f(x, \alpha, \beta) = \beta$. (If $\alpha$ or $\beta$ is $\pm \infty$, it will not be a parameter of $f$.) Then $f(-, \alpha, \beta)$ is necessarily decreasing on an interval with left endpoint $\alpha$. If $f$ stops decreasing at some point between $\alpha$ and $\beta$, then that point is $A\alpha\beta$-definable. If $f$ does not stop decreasing, then it has a definable infimum.
existence of Skolem functions, there is an $A$ that was first defined in [MS94], although not formally named. We will say that all principal types are interdefinable over $A$ if the premise of Lemma 3.3 holds. We will say that all principal types are interdefinable in $T$ if, for any $A$, all principal types are interdefinable over $A$.

It is easy to see that if all principal types are interdefinable, then the theory, $T$, has Skolem functions, by the same argument as in [vdD98], Chapter 6, Proposition 1.2. Note that, by compactness, if all principal types are interdefinable over $A$, then the principal type above 0 is mapped by $A$-definable functions to the principal types above (below) any $a$, uniformly, via some functions $h^+(x, y)$, where $h^+(x, a)$ maps the principal type above 0 to the principal type above (below) $a$.

**Lemma 3.5.** Let $A$ be a set. If, for any elements $a, b$, the principal type above $a$ (over $Aab$) is interdefinable with the principal type above $b$, then all elements are non-trivial in the terminology of [PS98], and thus a definable group chunk exists around each one.

**Proof.** By interdefinability, there is an $A$-definable function $f$, with $f(x, b, a)$ mapping an interval above $a$ to an interval above $b$. It is clear that $f(x, b, a)$ must be increasing. If we let $b$ vary, then it is also clear that $f(c, -, a)$ must be increasing, for some $c$ sufficiently close to $a$. Then $f(-, -, a)$ witnesses the non-triviality of $a$. □

**Lemma 3.6.** Let all principal types be interdefinable over a set, $A$. If $a \in \text{cl}(X) \setminus X$, where $X$ is an $A$-definable subset of $\mathbb{C}^n$, then there is an $A\bar{a}$-definable continuous injective map $\bar{\gamma} : (0, s) \to X$, for some $s > 0$, such that $\lim_{t \to 0} \bar{\gamma}(t) = \bar{a}$.

**Proof.** The proof is based on [vdD98], Chapter 6, Corollary 1.5, although the proof there assumes that $T$ expands the theory of an ordered group.

Since $a \in \text{cl}(X)$, any open set containing $a$ intersects $X$. As noted above, there exist $A$-definable functions, $h^+(x, y)$, such that $h^+(x, a_i)$ maps an interval above 0 to an interval above (below) $a_i$. Restrict to an $A\bar{a}$-definable interval above 0 on which all $h^+(\epsilon, a_i)$ are continuous. Then, for each $\epsilon$ in this interval, the box bounded by $h^+(\epsilon, a_i)$ is open, and thus must contain some point of $X$. By the existence of Skolem functions, there is an $A\bar{a}$-definable function, $\bar{\gamma}$, such that $\bar{\gamma}(\epsilon)$ is such a point in $x$. Restricting the interval to a smaller neighborhood above 0 so that $\bar{\gamma}$ is continuous and injective, we are done. □

4. Scale and definable $n$-types

The notion of a “region” that we referred to earlier is closely related to a concept that was first defined in [MS94], although not formally named.

**Definition 4.1.** Let $p = \text{tp}(a/B)$ be non-principal, with $A \subset B$. Let $M = \text{dcl}(A)$, $N = \text{dcl}(B)$. Let $p$ be all out of scale on $A$ iff for every $k$ and every $B$-definable $k$-ary function $f$, $f(M^k)$ is neither cofinal nor coinitial at $a$ in $N$. Let $p$ be $k$-in scale on $A$ iff for some $k$ and some $B$-definable $k$-ary function $f$, $f(M^k)$ is cofinal and coinitial at $a$. Let $p$ be $k$-near scale on $A$ iff it is not $k$-in scale for any $k$, and, for some $k$ and some $B$-definable $k$-ary function $f$, $f(M^k)$ is cofinal (or coinitial)
at \( a \). We say in scale to denote “1-in scale”, and near scale for “1-near scale”, and out of scale to denote “not 1-in scale or 1-near scale”.

\[ \text{M94} \] obtains the following theorems.

**Theorem 4.2.** (\[MS94\], Theorem 2.1) Let \( p \in S_n(M) \). Then \( p \) is definable iff for any \( \bar{c} \) realizing \( p \), \( M(\bar{c}) \) realizes only principal types over \( M \).

**Theorem 4.3.** \[MS94\] Let \( p \) be an \( n \)-type over a structure \( M \). Let \( \bar{c} \) be a realization of \( p \). Then \( p \) is definable iff for each \( i \leq n \), \( tp(c_i/M\bar{c}_{<i}) \) is principal, all out of scale on \( M \), or \( k \)-near scale on \( M \), for some \( k \).

We can tighten the notion of scale here, which allows us to refine \[MS94\]’s results slightly.

**Lemma 4.4.** Let \( M \prec N \), suppose that \( N \) realizes only principal types in \( M \), and let \( \bar{c} \in N \) be a tuple. Suppose that \( f(\bar{c}, M^n) \) is cofinal and/or coinitial at \( d \) in \( N \), for \( f \) an \( M \)-definable function and \( d \) non-principal over \( N \). Then there is a unary \( N \)-definable function, \( g \), such that \( g(M) \) is cofinal (coinitial) at \( d \) in \( N \) iff \( f(\bar{c}, M^n) \) is.

**Proof.** We go by induction on \( n \). The case \( n = 1 \) is trivial. Note that, since \( N \) realizes only principal types in \( M \), by Theorem 4.2 \( \bar{c} \) is definable over \( M \). If \( f(\bar{c}, M^n) \) is cofinal and coinitial at \( d \) in \( N \), then we are in the conditions of Lemma 2.8 of \[MS94\]. But this implies that \( M \) is not Dedekind complete in \( M(\bar{c}, d) \). Thus, there must be some \( M \bar{c} \)-definable function, \( g \), such that \( g(d) \) is non-principal in \( M \) – that is, since \( N \) realizes only principal types in \( M \), \( g^{-1}(M) \) is cofinal and coinitial at \( d \) in \( N \), so we are done.

Now we show that if \( f(\bar{c}, M^n) \) is only cofinal (coinitial) at \( d \) in \( N \), then we can find the appropriate \( g \). WLOG, we assume \( f(\bar{c}, M^n) \) is cofinal at \( d \) in \( N \). Note that, by Lemma 2.7 of \[MS94\], we know that \( tp(\bar{c}, d/M) \) is \( M \)-definable. Let \( f(\bar{x}) = f(\bar{c}, \bar{x}) \).

We can find an \( M \bar{c} \)-definable cell, \( C \subseteq \mathcal{C} \), such that \( f(C \cap M^n) \) is cofinal at \( d \) in \( N \) and \( f(C) < d \). Since \( tp(\bar{c}, d/M) \) is \( M \)-definable, \( C \) can be taken to be \( M \)-definable. Consider \( \sup(f(x, C_x)) \). For each \( a \in N \), \( \sup(f(a, C_a)) \in N \), so \( \sup(f(a, C_a)) < d \). Let \( g(x) = \sup(f(x, C_x)) \). Since \( d > \sup(\{g(a) \mid a \in M\}) \geq f(C) \), we know that \( g(M) \) is cofinal at \( d \) in \( N \).

By Lemma 4.4. “\( k \)-near scale” and “\( k \)-in scale” are equivalent to “near scale,” and “in scale” when we are working over a definable extension.

**Corollary 4.5.** Let \( p \) be an \( n \)-type over \( M \) with \( \bar{c} \) a realization. \( p \) is definable iff \( tp(c_i/M\bar{c}_{<i}) \) is principal or out of scale or \( k \)-near scale on \( M \) for \( i \leq n \).

**Proof.** For the forward direction, if \( p \) is definable, then for each \( i \leq n \), \( tp(c_i/M\bar{c}_{<i}) \) is principal or all out of scale or \( k \)-near scale on \( M \) for some \( k \). But since \( c_{<i} \) is definable over \( M \), the latter two properties are equivalent to out of scale and near scale, respectively.

The reverse direction proceeds by induction on the indices of \( \bar{c} \). If \( p \) is not definable, then at some least coordinate, \( i \), there is an \( M \)-definable function, \( f \), such that \( f(\bar{c}_{<i}, M^i) \) is cofinal and coinitial in \( M(\bar{c}_{<i}) \) at \( c_i \). But by Lemma 4.4 since \( tp(\bar{c}_{<i}/M) \) is definable by induction, there is some \( M \)-definable \( g \) such that \( g(\bar{c}, M) \) is cofinal and coinitial in \( M(\bar{c}_{<i}) \) at \( c_i \), so \( tp(c_i/M\bar{c}_{<i}) \) is not principal, out of scale, or near scale on \( M \).
5. Decreasing types

Given an n-type, the ordering of the variables can affect the type of each variable over the preceding ones. Consider the type of $(\epsilon, \epsilon')$ over $M = (\mathbb{R}, +, \cdot, <)$, where $1 \gg \epsilon \gg \epsilon' > 0$. We have that $\text{tp}(\epsilon/M)$ and $\text{tp}(\epsilon'/Me)$ are principal. However, if we consider the elements in reverse order, $\text{tp}(\epsilon'/M)$ is still principal, but now $\text{tp}(\epsilon/Me')$ is non-principal. We wish to fix a class of orderings of $p$'s coordinates that will provide some predictability.

Convention 5.1. We will assume from now on that all principal types are interdefinable in $T$, except where otherwise noted. By Lemma 5.3, this implies that there is at least one $\emptyset$-definable element. As well, by Lemma 5.6, around this $\emptyset$-definable element is a $\emptyset$-definable group chunk, with a $\emptyset$-definable identity element. Let “0” denote some such $\emptyset$-definable element such that there exists a $\emptyset$-definable group chunk containing it, and in which it is the identity element.

We begin by defining a partial ordering that we will use henceforth.

Definition 5.2. Let $A$ be a set. Let $a \prec_A b$ iff there exists $a' \in \text{dcl}(aA)$ such that $a' > 0$, and $(0, a') \cap \text{dcl}(bA) = \emptyset$. Let $a \sim_A b$ iff $a \not\prec_A b$ and $b \not\prec_A a$. Finally, let $a \preceq_A b$ iff $a \sim_A b$ or $a \prec_A b$.

Lemma 5.3. $\sim_A$ is an equivalence relation, and $\prec_A$ totally orders the $\sim_A$-classes.

Proof. It is trivial to see that $\sim_A$ is an equivalence relation – transitivity is true because coinitiality (near 0) is transitive. Similarly, $\prec_A$ totally orders the $\sim_A$-classes because “coinitiality” totally orders sets, up to coinitiality equivalence. □

Definition 5.4. Given a base set, $A$, and a tuple, $\bar{c} = \langle c_1, \ldots, c_n \rangle$, let $c_j \prec_i c_k$, for $i \leq j, k$, iff $c_j \sim_{A \bar{c}_i} c_k$. Given a type, $p(x_1, \ldots, x_n)$, let $x_j \prec_i x_k$ iff, for some realization $\bar{c}$ of $p$, $c_j \prec_i c_k$.

Note that “some realization” is equivalent to “every realization”.

Lemma 5.5. Let $p$ be an n-type over a set $A$. Then there exists a re-ordering of the variables of $p$ such that, in the new ordering, $x_i \preceq_i x_j$, for $1 \leq i < j$.

Proof. We reorder $p$ in stages. At stage $i$, having determined $\bar{x}_{<i}$, there is at least one maximal element in the order $\prec_i$ among the remaining $x_j$. Set any such maximal element to be $x_i$. □

Definition 5.6. If the variables of $p$ satisfy the conclusion of Lemma 5.5, we say that $p$ is decreasing. For $i$ an index in the variables of $p$, let $Q(i)$ denote the greatest index at most $i$ such that $\text{tp}(c_{Q(i)}/\bar{c}_{<Q(i)}A)$ is principal, and 0 if such index does not exist.

We note here a connection between decreasing sequences and the $T$-convex subrings of $\mathbb{R}$. If $\bar{c}$ is a decreasing sequence over $A$, then for $1 \leq j < k \leq n$, the convex hull of $\Pr(A\bar{c}_{<j})$ is a $T$-convex subring contained in the convex hull of $\Pr(A\bar{c}_{<k})$, with equality if and only if $Q(k) \leq j$. The connections between decreasing sequences and $T$-convex subrings and valuations, as well as results on infinite sequences, will be presented in a future paper. In this work, we are only concerned with developing the tools necessary to answer our original question on curves.

Lemma 5.7. Let $p$ be a decreasing type over a set $A$, let $\bar{c} \models p$, and let $k$ be an index such that $\text{tp}(c_k/A\bar{c}_{<k})$ is principal. Then for $i \geq k$, $\text{tp}(c_i/A\bar{c}_{<k}A)$ is principal.
Proof. Note that, since $c_k \geq_k c_i$ (by definition of “decreasing”), we know that $\text{dcl}(c_i, A \bar{c}_{<k})$ is coinitial above $0$ in $\text{dcl}(A \bar{c}_{<k})$. Since $c_k$ is principal over $A \bar{c}_{<k}$, there is some $d \in \text{dcl}(A \bar{c}_{<k})$, principal above $0$ over $A \bar{c}_{<k}$. By coinitiality, there is some $d' \in \text{dcl}(c_i, A \bar{c}_{<k})$, with $0 < d' < d$, but then $d'$ witnesses that $c_i$ is principal over $A \bar{c}_{<k}$.

6. Good bounds and $i$-closures

We will be helped in answering our question by some technical results and lemmas concerning the closures of sets. We are interested in picking out the elements on the boundaries of definable sets that will cause problems with continuity – those will be the $i$-closures. We can assure continuity by bounding the various values a function takes by a function that goes to $0$ as it approaches an $i$-closure point. As a technical point, we also develop an approximation to the distance between two points in the case that our structure does not expand an ordered group.

Condition 6.1. We will be working under the following assumptions for the rest of this section. Let $p$ be a decreasing independent $n$-type over a set $A$, $\bar{c}$ a realization of $p$, $i$ an index in $p$’s coordinates, and $k = Q(i) > 0$. As well, we assume that $\text{tp}(c_j/\bar{c}_{<k} A)$ is principal above $\beta_j(\bar{c}_{<k}) \neq \pm \infty$ for $j \geq k$. Let $\bar{\beta} = (\beta_k, \ldots, \beta_n)$.

Note that if Condition 6.1 is true for some $\bar{c}$, it is true for any $\bar{c}_0 \models p$, and thus can also be thought of as a condition just on $p$, $A$, $i$, and $k$. As well, note that, for any $p$ a decreasing type over $A$ and $c \models p$ with $k = Q(i)$ for some coordinate $i$ and $j \geq k$, $\text{tp}(c_j/\bar{c}_{<k} A)$ is principal by Lemma 5.7.

Lemma 6.2. If $p$, $A$, $\bar{c}$, $i$, and $k$ satisfy Condition 6.1, then there is an $A$-definable set, $C^0$, containing $\bar{c}$ such that, for every $\bar{a} \in \pi_{<k-1}(C^0)$, $\text{cl}(C^0)$ contains a unique point, $\bar{d}$, with $\bar{d}_{<k} = \langle \bar{a}, \beta_k(\bar{a}) \rangle$. Moreover, for each $\bar{a}$ (and in particular for $\bar{c}_{<k}$), this point is independent of choice of $C^0$ – in fact, it is $\langle \bar{a}, \bar{\beta}(\bar{a}) \rangle$.

Proof. By Lemma 3.1 for each $j > k$ there is some $A$-definable $k$-ary function, $h_j$, such that

$$c_j < h_j(\bar{c}_{<k}),$$

and

$$\lim_{x \to \beta_j(\bar{c}_{<k})} h_j(\bar{c}_{<k}, x) = \beta_j(\bar{c}_{<k}).$$

Let $C$ be an $A$-definable set containing $\bar{c}$ such that $\bar{\beta}$ is continuous on $C$, $h_j > \beta_j(\bar{c}_{<k})$ for $j \geq k$ (possible since $h_j(\bar{c}_{<k}) > \beta_j(\bar{c}_{<k})$), and (1) holds on all of $C$ (possible since it holds for $\bar{c}$ – note that the limit statement is first-order). Let

$$B = \{ \bar{x} \mid x_j > \beta_j(\bar{x}_{<k}) \land x_j < h_j(\bar{x}_{<k}), \text{for } j > k \}.$$ 

Let $C' = C \cap B$. Note that, since $\bar{c} \in C$, $\bar{c} \in B$, we know $C'$ is non-empty. Now, by Lemma 3.2 we can decompose $C'$ into definable sets, $C_0, \ldots, C_r$, on each of which, for any $\bar{a} \in \pi_{<k-1}(C'^0)$, $\text{cl}(C^0) = \text{cl}(C^0)^{\bar{a}}$ – the closure of a fiber is the fiber of the closure. WLOG, let $C^0$ be the cell containing $\bar{c}$.

Let $\bar{a} \in \pi_{<k-1}(C'^0)$. Let $D = \{ \bar{a} \} \times C^0_0$. Let $\bar{d} \in \text{cl}(C'^0_0)$, with $\bar{d}_{<k} = \langle \bar{a}, \beta_k(\bar{a}) \rangle$. Note that this implies $d \in \text{cl}(D)$. We want to show that $\bar{d} = \langle \bar{a}, \bar{\beta}(\bar{a}) \rangle$. Let $\bar{\gamma}(t)$ be an $A\bar{a}$-definable curve in $D$, with $\bar{\gamma}(0) = \bar{d}$, with existence guaranteed by Lemma 3.6. Then, for $j > k$,

$$d_j \geq \lim_{t \to 0^+} \beta_j(\bar{\gamma}(t)_{<k}) = \beta_j(\bar{a}).$$
Similarly,
\[ d_j \leq \lim_{t \to 0^+} h_j(\gamma(t)) = \lim_{y \to \beta_j(y)^+} h_j(\bar{a}, y) = \beta_j(\bar{a}). \]
Thus, \( \bar{d} = (\bar{a}, \bar{\beta}(\bar{a})) \).

**Definition 6.3.** Assume Condition 6.1 holds. Then, for any tuple \( \bar{a} \) with length at least \( k - 1 \), such that \( \bar{a}_{\leq k} \in \pi_{\leq k-1}(C^0) \), let
\[ \text{icl}_p(i, \bar{a}) = (\bar{a}_{\leq k}, \bar{\beta}(\bar{a}_{\leq k})). \]
When \( p \) is clear from context, we may omit it, writing simply \( \text{icl}(i, \bar{a}) \), and also referring to this as the \( i \)-closure of \( \bar{a} \).

Note that \( \text{icl}(i, \bar{a}) \) is an \( A\bar{a}_{\leq k} \)-definable point.

**Lemma 6.4.** Assume Condition 6.1 holds. If \( \text{tp}(c_i/A\bar{c}_{<i}) \) is non-principal, then \( \text{icl}(i, \bar{x}) = \text{icl}(i - 1, \bar{x}) \).

**Proof.** If \( \text{tp}(c_i/A\bar{c}_{<i}) \) is non-principal, then, by definition, \( Q(i) \neq i \), so \( Q(i) \leq i - 1 \). Since now the conditions on \( Q(i) \) and \( Q(i - 1) \) are identical, \( Q(i) = Q(i - 1) \), and so
\[ \text{icl}(i, \bar{x}) = \langle \bar{x}_{Q(i)}, \beta_{Q(i)}(\bar{x}_{Q(i)}), \ldots, \beta_n(\bar{x}_{Q(i)}) \rangle = \text{icl}(i - 1, \bar{x}) \).
\]

**Definition 6.5.** Assume Condition 6.1 holds. Let \( f \) be an \( i \)-ary \( A \)-definable function such that, for some \( A \)-definable \( C \) with \( \bar{c} \in C \), \( f \) is continuous on \( C \) (as a function of the first \( i \) coordinates), non-negative, and moreover \( f \) is continuous on \( C \cup \{ \text{icl}(i, \bar{x}) \mid \bar{x} \in C \} \), with \( f(\text{icl}(i, \bar{x})) = 0 \), for all \( x \in C \). Then we call \( f \) a good bound at \( i \).

The set of good bounds at \( i \) (for a given \( p \)) forms a vector space over \( A \).

**Condition 6.6.** We now extend Condition 6.1. Let \( C \) be an \( A \)-definable set with \( \bar{c} \in C \), with the following properties. For any \( \bar{a} \in C \), \( h^+(x, \beta_k(\bar{a}_{\leq k})) \), the map taking an interval above 0 to an interval above \( \beta_k(\bar{a}_{\leq k}) \), has \( a_k \) in its image. For any element \( d \), denote the inverse to \( h^+(x, d) \) by \( h_d \). For any \( \bar{a} \in C \), \( h_{\beta_k(\bar{a}_{\leq k})}(a_k) \) lies in the group chunk around 0.

Note that, given any \( C \ni \bar{c} \), we may shrink \( C \) to a smaller \( A \)-definable set to satisfy Condition 6.6.

**Definition 6.7.** Let \( c < a, b \) be elements with \( \hat{h}_c(b), \hat{h}_c(a) \) in the group chunk around 0. Define \( |b - a| \) to be \( |\hat{h}_c(b) - \hat{h}_c(a)| \), with subtraction and absolute value in the sense of the group chunk around 0.

Note that, since \( 0 < \hat{h}_c(b), \hat{h}_c(a) \), the subtraction operation is well-defined.

**Definition 6.8.** For any \( j \in \mathbb{N}^+ \), let
\[ m_i(\bar{x}_{\leq i}) = |x_{Q(i)} - \beta_{Q(i)}(\bar{x}_{Q(i)})|. \]
If Condition 6.6 holds, then \( m_i \) is a good bound at \( i \), with domain \( C \).

**Lemma 6.9.** Assume Condition 6.1 holds. If \( f \) is a good bound at \( i \) then there exists \( f' \) such that \( f' \geq f \) on some definable set containing \( \bar{c} \), and \( f' \) is a good bound at \( i - 1 \).

**Proof.** Note that there must be a principal type at or before the \( i - 1 \) coordinate in \( p \), otherwise the good bound requirement is vacuous.
Case 1: $tp(c_i/\bar{c}_{<i}A)$ is principal. By the definition of a good bound, there is some $A$-definable $C$ such that $f$ is continuous and non-negative on $C$, and $f$ extends to $cl(C)$ such that $f(icl(i, \bar{x})) = 0$ for all $\bar{x} \in C$. Since $c_{i-1}$ is principal near some $A\bar{c}_{<Q(i-1)}$-definable element, assume $c_{i-1}$ is principal above (WLOG) $\beta_{i-1}(\bar{c}_{<Q(i-1)})$, where $\beta_{i-1}$ is $A$-definable (note that $\beta_{i-1}$ is not part of the original sequence of functions, $\beta$). We may restrict $C$ so that $icl(i-1, \bar{x}) \notin C$, since we can take $C$ to have lower boundary at least $\beta_{i-1}(\bar{c}_{<Q(i-1)})$ at the $i-1$st coordinate. Note that, on $C$, $icl(i-1, \bar{x}) \neq icl(i, \bar{x})$, since $icl(i-1, \bar{x})_{i-1} = \beta_{i-1}(\bar{x}_{<Q(i-1)}) < x_{i-1} = icl(i, \bar{x})_{i-1}$. Since $f$ is a good bound at $i$, we know that $f(icl(i, \bar{x})) = 0$ for $\bar{x} \in C$, and therefore $f(icl(i, \bar{x})) < m_{i-1}(icl(i, \bar{x}))$. Note that $Q(i) = i$. Then for each $\bar{x}$, there is some $A$-definable $h(\bar{x}_{<i})$ such that, if $x_i \in (\beta_i(\bar{x}_{<i}), h(\bar{x}_{<i}))$, $f(\bar{x}) < m_{i-1}(\bar{x})$. Restrict $C$ to have upper boundary at most $h$ on the $i$th coordinate. Then, on our new $C$, $m_{i-1} > f$, and $m_{i-1}$ is a good bound at $i-1$.

Case 2: $tp(c_i/\bar{c}_{<i}A)$ is non-principal. There is a closed $A\bar{c}_{<i}$-definable interval, $J(\bar{c}_{<i})$, about $c_i$ on which $f(\bar{c}_{<i}, -)$ is continuous. Thus, for $\bar{x}_{<i}$ in some $A$-definable set containing $\bar{c}$, say $\bar{C}$, there is $J(\bar{x}_{<i})$, a closed $A\bar{x}_{<i}$-definable interval such that $f(\bar{x}_{<i}, -)$ is continuous. Let $C' = \{\bar{x} \in \bar{C} | x_i \in J(\bar{x}_{<i})\}$, an $A$-definable set.

We can then let $f'(\bar{x}_{<i}) = sup\{f(\bar{x}_{<i}) | x_i \in J(\bar{x}_{<i})\}$. Clearly, $f(\bar{x}_{<i}) \leq f'(\bar{x})$. We must also show that $f'$ is a good bound at $i-1$ - that is, $f'$ can be extended continuously to icl($i-1, \bar{x}$) by $0$, for $\bar{x} \in C'$. Since $f$ extends to icl($i, \bar{x}$) = icl($i-1, \bar{x}$) by $0$, we know that, for any definable curve in $C$ with limit point icl($i, \bar{x}$), $f(\bar{x}_{<i})$ goes to $0$ on the curve. This implies that $sup\{f(\bar{x}_{<i}) | x_i \in J(\bar{x}_{<i})\}$ also goes to $0$ on the curve - suppose not. Then there is a curve, $\gamma$, and $\epsilon > 0$ such that, for each $t > 0$, there is an $x_i \in J(\gamma(t)_{<i})$ such that $f(\gamma(t)_{<i}, x_i) > \epsilon$. We can definably choose $x_i$ as a function of $t$ (and $\epsilon$), thus yielding a new curve, $\gamma'$, with $f(\gamma'(t))$ not going to $0$, contradiction.

\[ \square \]

7. Main result

We are now ready to prove our main theorem. We restate it, since all terms have finally been defined.

Theorem 7.1. Let all principal types be interdefinable over $T$. Let $A$ be a set. Let $p$ be a decreasing $n$-type over $A$. Let $\bar{c} = \langle c_1, \ldots, c_n \rangle \models p$, with $tp(c_1/A)$ principal. The following statements are equivalent.

1. For every $A$-definable function, $F$, with domain including $\bar{c}$, there is an $A$-definable set, $C$, such that $\bar{c} \in C$, $F$ is continuous on $C$, and $F \restriction C$ extends continuously to $cl(C)$.

2. For $i = 1, \ldots, n$, $tp(c_i/A\bar{c}_{<i})$ is algebraic, principal, or out of scale on $A$.

Remark 7.2. While, on its face, the theorem only applies to types which, when reordered, have first element principal, other types can be transformed in the following manner. Let $q$ be a decreasing $n$-type and $\bar{c}$ a realization, such that, for $i \leq k \geq 1$, $tp(c_i/A\bar{c}_{<i})$ is non-principal. Then we may replace $A$ by $A\bar{c}_{<k}$ and apply the theorem. Suppose that a counterexample function, $F(\bar{c}_{<k}, \bar{x})$, can be constructed, with $F(\bar{y})$ an $A$-definable function. Then it is not hard to see that, taking $F$ as an $n$-ary function, it will be a counterexample for the full $\bar{c}$. Conversely, if for any $F(\bar{c}_{<k}, \bar{x})$ the desired set, $C_{\bar{c}_{<k}}$, does exist, $A$-definable with parameters $\bar{c}_{<k}$, then we can again take $F$ to be $n$-ary, and replace $C$ by the set $\{d \times C_d | d \in D\}$, for some sufficiently small $A$-definable $D$ containing $\bar{c}_{<k}$.
We first prove the “if” of Theorem 7.1.

**Proposition 7.3.** Let all principal types be interdefinable over $T$. Let $A$ be a set and let $p$ be a decreasing $n$-type over $A$. Let $\bar{c} = \langle c_1, \ldots, c_n \rangle$ realize $p$. Suppose that, for $i = 1, \ldots, n$, $\text{tp}(c_i/\bar{c}_{\leq i-1})$ is algebraic, principal, or out of scale on $A$. Then, for any $F$ a bounded $A$-definable function on $\mathcal{E}^n$, there is an $A$-definable set, $C$, such that $F$ is continuous on $C$, $F$ can be continuously extended to $\text{cl}(C)$, and $\bar{c} \in \text{cl}(C)$.

**Proof.** We will go by induction on $n$, although we will also have an additional “inner” induction. Note that, by absorbing $A$ into our language, we may assume that $A = \emptyset$. Let $P$ be the prime model of $T$.

**Regularizing principal types.** Let the coordinates at which $\text{tp}(c_i/\bar{c}_{\leq i})$ is principal be $i_1, \ldots, i_l$, and let $I = \{i_1, \ldots, i_l\}$. Let the function of $\bar{c}_{< i}$ over which $c_i$ is principal be $\alpha_i$.

**Claim 7.4.** It is sufficient to prove Proposition 7.3 in the case where no $c_i$ is principal near $\pm \infty$ over $\emptyset$.

**Proof.** Let $I^+_\pm = \{i \in I \mid \text{tp}(c_i) = \infty^-\}$ and $I^-_\pm = \{i \in I \mid \text{tp}(c_i) = -\infty^+\}$. Let $q^\pm$ be the maps taking the principal type near $\pm \infty$ to the principal type above $0$. Then we can consider the map $\xi(x_1, \ldots, x_n) = \langle y_1, \ldots, y_n \rangle$, with $y_i = q^+(x_i)$ if $i \in I^+_\infty$, $y_i = q^-(x_i)$ if $i \in I^-\infty$, and $x_i$ otherwise.

We assume that there is a $\emptyset$-definable set $C'$ containing $\xi(\bar{c})$ such that $F \circ \xi^{-1}$ is continuous on $C'$, and extends continuously as $F \circ \xi^{-1}$ to $\text{cl}(C')$, and prove Proposition 7.3 in this case, giving the claim. Let $I$ be an interval above $0$ such that $q^\pm$ are continuous and monotone on $I$. Then we can form a cell, $B$, from the cartesian product of $\mathcal{E}$ and $I$, with $I$ appearing at coordinates for which $i \in I^\pm_\infty$. $\xi^{-1}$ is continuous on $B$. We may assume that, if $d \in \text{cl}(B)$ and $d_{\bar{i}} \neq 0$ for any $i = 1, \ldots, n$, then $d \in \text{Im}(\xi)$, by restricting the upper boundary functions of $B$. We may assume that $C' \subseteq B$ by intersecting $C'$ with $B$, since for $i \in I^\pm_\infty$, $c_i$ is principal above $0$.

Then, if we let $B' = \xi^{-1}(B)$, $\xi$ is a homeomorphism on $B'$. We restrict $\xi$ to have domain $B'$. Note that, since $q^\pm$ are continuous and monotone on $I$, if we consider inf$(q^+(I))$ and sup$(q^-(I))$, we obtain bounds, $r^+$ and $r^-$, such that, if $\bar{a}$ is a point in $\mathcal{E}^n$, with $a_i > r_+$ for $i \in I^+_\infty$ and $a_i < r_-$ for $i \in I^-\infty$, then $\bar{a} \in B'$.

Let $C'' = \text{cl}(C') \cap \text{dom}(\xi^{-1})$. Let $C = \xi^{-1}(C'')$. Note that, since $\xi$ is a homeomorphism on $B'$, and $\text{cl}(C') \cap \text{Im}(\xi)$ is closed in $\text{Im}(\xi)$, $C$ is closed in $\text{dom}(\xi)$. We want to show that $C$ is actually closed. Since we have bounded $C''$ away from the right endpoint of $I$ on each coordinate $i \in I^\pm_\infty$, we have necessarily bounded $C$ away from $r^-$ and $r^+$ on those coordinates. Thus, it follows that $C$ is closed.

We claim that $F$ has a continuous extension, $\mathcal{F}$, on $C$, defined by $\mathcal{F}(\bar{x}) = (F \circ \xi^{-1} \circ \xi)(\bar{x})$.

To prove $\mathcal{F}$ is continuous, take $D$ a closed subset of $\mathcal{E}$; we wish to show that $\mathcal{F}^{-1}(D) \cap C$ is closed. Since $\xi$ is a homeomorphism on a set containing $C$ and $\mathcal{F}$ is continuous, $\mathcal{F}^{-1}(D) \cap C$ is closed.

\*The use of van den Dries’ result on fiberwise-continuous functions is based on the proof in [Spe08].
\( F^{-1}(D) \subseteq C \), this is equivalent to asking if \( \xi(F^{-1}(D)) \) is closed. We have
\[
(\xi \circ F^{-1})(D) = \xi \circ (F \circ \xi^{-1} \circ \xi^{-1})(D) = \xi \circ \xi^{-1} \circ F \circ \xi^{-1}(D) = F \circ \xi^{-1}(D),
\]
which is closed by continuity of \( F \circ \xi \). Thus, \( F \) is continuous on \( C \), proving Proposition 7.3. \( \square \)

Thus, we may assume that no \( \alpha_i \) is \( \pm \infty \).

**Claim 7.5.** It is sufficient to prove Proposition 7.3 in the case where each \( c_i \) is principal above \( \alpha_i \).

**Proof.** This can be proved by the same method. Using interdefinability of principal types, for each \( i \in I \) with \( c_i \) principal below \( \alpha_i \), we can apply the map taking the noncut below \( \alpha_i \) to the noncut above \( \alpha_i \), and take the identity on all other coordinates. \( \square \)

**Making \( F \) and \( \bar{c} \) \( n \)-dimensional.**

**Claim 7.6.** It suffices to prove Proposition 7.3 in the case that \( F \) is non-constant in each coordinate in a neighborhood of \( \bar{c} \).

**Proof.** Suppose that \( F \) is constant in the \( i \)th coordinate at \( \bar{c} \). Then we may take \( D \) to be a \( \emptyset \)-definable set containing \( \bar{c} \) on which \( F \) is continuous and constant in the \( i \)th coordinate. Since no \( c_i \) is principal near \( \pm \infty \), we may assume that \( D \) is bounded. Let \( \pi(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \). Then let \( D' = \pi(D) \).

Note that \( \pi(\bar{c}) \in D' \). For \( \bar{d} \in D' \), let \( \delta(\bar{d}) \) denote an arbitrary element of \( \pi^{-1}(\bar{d}) \).

We can take \( \delta \) to be \( \emptyset \)-definable. Then define
\[
F'(\bar{d}) = F(\delta(\bar{d})),
\]
which is well-defined by our assumption that \( F \) is constant on the \( i \)th coordinate in \( D \), and thus on \( \pi^{-1}(\bar{d}) \). Then, by induction, we may find a subset of \( D' \) on which \( F' \) is continuous, and such that \( F' \) extends continuously to the closure of \( D' \). We continue to call this extension \( F' \). We may take \( D' \) to be a cell and replace \( D \) by \( \pi^{-1}(D') \cap D \). Then by 1.7 of Chapter 6 of [vdD98], \( \pi(\text{cl}(D)) = \text{cl}(\pi(D)) = \text{cl}(D') \).

We now show directly that \( F \) extends continuously to \( \text{cl}(D) \), with the extension, which we continue to call \( F \), defined by \( F(\bar{x}) = F'(\pi(\bar{x})) \).

Let \( \bar{x} \in \text{cl}(D) \), and \( \epsilon > 0 \). We can find an open \( B \) around \( \pi(\bar{x}) \) such that \( |F'(\bar{y}) - F'(\pi(\bar{x}))| < \epsilon \), for \( \bar{y} \in B \).

Thus, \( |F(\bar{z}) - F'(\bar{x})| < \epsilon \), for \( \bar{z} \in \pi^{-1}(B) \), but since \( \pi \) is continuous, \( \pi^{-1}(B) \) is an open set containing \( \bar{x} \), and thus we have found an open set containing \( \bar{x} \) such that \( |F(\bar{z}) - F(\bar{x})| < \epsilon \) for \( z \) in this set, and thus \( F \) is continuous on \( \text{cl}(D) \). \( \square \)

Therefore, we may assume that \( F \) is non-constant in each coordinate near \( \bar{c} \).

We can partition \( C \) into \( \emptyset \)-definable cells on which \( F \) is continuous and monotonic in each coordinate. The closure of (at least) one of these cells must contain \( \bar{c} \). Let \( D \) be a cell of lowest dimension on which \( F \) is continuous, monotonic in each coordinate and whose closure contains \( \bar{c} \).

**Claim 7.7.** It suffices to prove Proposition 7.3 in the case that \( D \) is open.

**Proof.** We suppose that \( D \) is not open and then show Proposition 7.3. Using the \( p_D \) defined in [vdD98], Chapter 3, 2.7, we can homeomorphically map \( D \) to \( p_D(D) \), with \( p_D(D) \subseteq \mathbb{C}^m \), for \( m < n \). Note that \( p_D \) is still a homeomorphism on \( \text{cl}(D) \). Then, by induction, if \( F' = F \circ p_D^{-1} \), we can find \( \emptyset \)-definable \( C' \) such that
$p_D(c) \in C'$, $F'$ is continuous on $C'$, and $F'$ extends continuously as $\overline{F}$ to $\text{cl}(C')$. Let $C = p_D^{-1}(C')$. Note that $\text{cl}(C) = p_D^{-1}(\text{cl}(C'))$, by [vdD98, Chapter 6, 1.7] again. Let $\overline{F}(x) = \overline{F} \circ p_D$. Note that, on $C$, $\overline{F} = F$. Now we show $\overline{F}$ is continuous. Let $E \subseteq M$ be closed. Then $\overline{F}^{-1}(E) \cap \text{cl}(C)$ is closed iff $p_D(\overline{F}^{-1}(E) \cap C)$ is closed. As in the proof of Claim 7.4, this set can be written as $p_D(\overline{F}^{-1}(E)) \cap C'$. Continuing to follow Claim 7.4, we can write $p_D(\overline{F}^{-1}(E)) \cap C' = \overline{F}^{-1}(E) \cap C'$, which is closed by continuity of $\overline{F}$ on $C'$. Thus, $\overline{F}$ is continuous on $\text{cl}(C)$.

**Inner Induction.** Let $f_i, g_i$ be the $\emptyset$-definable lower and upper bounding functions in the construction of $D$ as a cell. We now construct new $\emptyset$-definable bounding functions to replace these, starting at $i = n$ and going down to $i = 1$. It is easy to see that Condition 6.4 holds for $p$ and $c$ here. Moreover, we can definable shrink $D$ so that Condition 6.6 also holds. For any $\bar{x}$ with $\bar{x}_{\leq i} \in \pi_{\leq i}(D)$, let $E_\bar{x}^i = \{\bar{x}_{\leq i}\} \times D_{\bar{x}_{\leq i}}$. We refine $F$ as we go, contracting its domain, but expanding the points on the closure of the domain onto which it extends continuously.

We have two induction statements at stage $i$:

1. For any $\bar{x} \in \pi_{\leq i}(D)$, $F(\bar{x}, -)$ is continuous on $E_\bar{x}^i$, and extends continuously to $\text{cl}(E_\bar{x}^i)$.

2. There is a $\emptyset$-definable $G(\bar{x}_{\leq i})$, a good bound at $i$, such that for any $\bar{a} = \langle a_1, \ldots, a_n \rangle$ and $\bar{a}' = \langle a_1, \ldots, a_i, a_{i+1}, \ldots, a_n \rangle$ with $\bar{a}, \bar{a}' \in D$, $|F(\bar{a}) - F(\bar{a}')| \leq G(\bar{a}_{\leq i})$.

**Remark 7.8.** Note that we may not have a global addition, and so on its face, the absolute value expression above makes no sense. However, we can reduce to the situation in Definition 6.7 by restricting our set $D$ as follows. Since $\bar{c}$ is definable by Corollary 4.5, we know that $F(\bar{c})$ must be principal over $P$. Let $F_0$ be the point near which it is principal (note that, since $F$ is bounded, $F_0 \in P$). We may assume WLOG that $F(\bar{c}) > F_0$. Then we may assume that $F(D) > F_0$. Moreover, since $F(\bar{c})$ fulfills the condition on $a$ in Definition 6.7 with $c = F_0$, we may assume that every element of $F(D)$ does too (in fact, that every element in $\text{cl}(F(D))$ does as well). Thus, the expression above is well-defined, in the sense of Definition 6.7.

**Claim 7.9.** We may refine $D$ so that, for $\bar{x} \in \pi_{\leq i}(D)$, $F$ is continuous on $E_{\bar{x}}^{i-1}$ and extends continuously to $\text{cl}(E_{\bar{x}}^{i-1}) \cap \{\bar{y} \mid \bar{y}_{\leq i} \in \pi_{\leq i}(D)\}$.

**Proof.** Let $\overline{F}$ be $F$ with its domain extended onto $\text{cl}(E_{\bar{x}}^i)$ for each $\bar{x} \in \pi_{\leq i}(D)$. By (11) and Corollary 2.4 of Chapter 6 of [vdD98], for any $\bar{x} = \langle x_1, \ldots, x_{i-1} \rangle \in \pi_{\leq i-1}(D)$, we can partition $(f_i(\bar{x}_{\leq i}), g_i(\bar{x}_{\leq i}))$ into intervals $I_1(\bar{x}), \ldots, I_r(\bar{x})(\bar{x})$ (and their endpoints) so that $\overline{F}$ is continuous on

$$\{\bar{y} \in \text{cl}(D) \mid \bar{y}_{\leq i} = \bar{x} \wedge y_i \in I_j(\bar{x})\},$$

for $1 \leq j \leq r$. Then we can find a $\emptyset$-definable open set $U \subseteq \mathcal{C}^i$ containing $\bar{c}_{\leq i}$ such that $\text{r}(\bar{x})$ is constant on $U$, and we denote this constant value by $r$. Let $I_j(\bar{c}_{\leq i})$ be given by $(h_j(\bar{c}_{\leq i}), h_{j+1}(\bar{c}_{\leq i}))$, for some $\emptyset$-definable $h_j$, $j = 1, \ldots, r$, with $h_1 = f_i$, and $h_{r+1} = g_i$, with $h_j$ $\emptyset$-definable, for $j = 1, \ldots, r$. Then we may further assume that, on $U$, $I_j(\bar{x}) = (h_{j}(\bar{x}), h_{j+1}(\bar{x}))$. Replace $D$ by $D \cap \{\bar{y} \mid \bar{y}_{\leq i} \in U\}$, and replace $f_i, g_i$ by $h_j, h_{j+1}$, respectively, for the $j$ such that $h_j(\bar{c}_{\leq i}) < c_i < h_{j+1}(\bar{c}_{\leq i})$. Furthermore, we can assume that, for any $\bar{a} \in \pi_{\leq i}(D)$, we have $\text{cl}(E_{\bar{a}}^{i-1}) = \{\bar{a}\} \times
\[ \text{cl}(D_a) = \{a\} \times \text{cl}(D)_a, \text{ by Lemma 3.2} \] Thus, \( F \) is continuous on
\[
\{y \in \text{cl}(D) \mid y <_i x, y <_i E\} = \text{cl}(E^{i-1}_x) \cap \{y \mid y <_i E \in \pi <_i (D)\}.
\]

With the claim, all that remains to show (11) for \( i - 1 \) is to consider points in \( \text{cl}(E^{i-1}_x) \setminus \{y \mid y <_i E \in \pi <_i (D)\} \) - points with \( i \)th coordinate equal to \( f_i(x) \) or \( g_i(x) \).

We must now consider two cases - when \( c \) is principal over \( c <_i \). when and when it is non-principal. The algebraic case is done by openness of \( D \).

**Case 1:** \( \text{tp}(c_i/c_{<i}M) \) is principal. We may assume that \( f_i \leq \alpha_i \), since this is true at \( c <_i \), and so we may actually assume that \( f_i = \alpha_i \). If we replace \( g_i \) by a definable function lying between \( f_i \) and \( g_i \) (guaranteed by Lemma 3.3 - \( (g_i + f_i)/2 \) in the case that \( T \) expands the theory of a group) we guarantee that, for \( x <_i E \), \( F \) is continuous on the set
\[
\{y \in \text{cl}(E^{i}_x) \mid f_i(y) < y \leq g_i(y)\}.
\]

Thus, it only remains to show that \( F \) extends continuously onto the points where \( y_i = f_i(y) = \alpha_i(y) \). But by Lemma 3.2 if we are given \( x \) as above, we can restrict \( D \) further so there is only one such point - \( \text{icl}(i, x) \) (note that \( Q(i) = i \)). We have \( \text{icl}(i, c) \in \text{cl}(E^{i}_x) \). By Lemma 3.6, we can find a \( c <_i \)-definable curve, \( \gamma(t, c) \), such that \( \gamma(0, c) = \text{icl}(i, c) \), and \( \gamma(t, c) \in E^{i}_x \), for \( t > 0 \). We may then assume that, for any \( y <_i E \), \( \gamma(t, y) \in E^{i}_y \) is a curve in \( E^{i}_y \) with \( \gamma(0, y) = \text{icl}(i, y) \). Since \( F \) is bounded and continuous, \( \lim_{t \to 0} F(\gamma(t, y)) \) exists, for each \( y \in D^i \). Let \( \gamma^1(t, y_1, \ldots, y_{i-1}) \), \( \gamma^2(t, y_1, \ldots, y_{i-1}) \) be \( \theta \)-definable curves in \( E^{i}_y \) with limit at \( t = 0 \) of \( \text{icl}(i, y) \). Fix \( a \in \pi_{i-1}(D) \). Let \( r_j = \lim_{t \to 0^+} F(\gamma^j(t, a)) \). Let \( \epsilon \) be any positive element. By (12), there exists a \( \theta \)-definable \( G \), a good bound at \( i \), such that \( |F(y) - F(y')| < G(y) \), for \( y, y' \in D \) with \( a = y <_i y' = y' \). Since \( G \) is a good bound at \( i \), we can choose \( s_1, s_2 > 0 \) such that, for \( t \in (0, s_1) \), \( G(\gamma(t, a)) \leq \epsilon/3 \) and \( |F(\gamma(t, a)) - F(\gamma(t, a))| \). Let \( r_1, r_2 = r_1, \) and so \( F \) extends continuously to \( \text{icl}(i, x) \), satisfying (11) for \( i - 1 \).

We must also satisfy condition (12) for \( i - 1 \). Let \( G' \) be the good bound at \( i - 1 \) with \( G' \geq G \) guaranteed from Lemma 3.9 (we may restrict \( D \) so that \( D \) is the appropriate domain for \( G' \)). Restrict \( D \) so that the image of \( G'(D) \) under \( h_0 \) contains only elements that can be multiplied by \( 4 \). Let \( \tilde{\gamma} \) be the curve from above. Restrict its domain (possibly further restricting \( D \)) so that \( \tilde{\gamma} \) is monotonic in the \( i \)th coordinate. Then
\[
S(\bar{x}, z) = \sup \{y : \exists t (\gamma(t, x) = y \land |F(\gamma(t, x)) - F(\text{icl}(i, x))| < z)\}
\]
is a function that is decreasing in \( z \) for every \( x \). Now replace our \( i \)th coordinate boundary function, \( g_i \), with \( \min(g_i(x), S(\bar{x}, G'(\bar{x})) \). We have then guaranteed that applying \( F \) to any point on \( \tilde{\gamma} \) will yield a value differing little from \( F \) applied to the \( i \)-closure point.
Given $\bar{y}, \bar{y}' \in D$ with $\bar{y}_{<i} = \bar{y}'_{<i}$, we can find $t, t'$ with $\bar{\gamma}(t, \bar{y}_{<i})_{<i} = \bar{y}_{<i}$ and $\bar{\gamma}(t', \bar{y}_{<i})_{<i} = \bar{y}'_{<i}$. Then

$$4G'(\bar{y}_{<i}) \geq G(\bar{y}_{<i}) + G(\bar{y}'_{<i}) + |F(\bar{\gamma}(t, \bar{y}_{<i})) - F(\text{icl}(i, \bar{y}_{<i}))| + |F(\bar{\gamma}(t', \bar{y}_{<i})) - F(\text{icl}(i, \bar{y}_{<i}))| \geq |F(\bar{y}) - F(\bar{\gamma}(t, \bar{y}_{<i}))| + |F(\bar{y}') - F(\bar{\gamma}(t', \bar{y}_{<i}))| + |F(\bar{\gamma}(t', \bar{y}_{<i})) - F(\bar{\gamma}(t, \bar{y}_{<i}))| \geq |F(\bar{y}) - F(\bar{y}')|.$$ 

Note that all addition is well-defined. Thus, since $4G'$ is a good bound at $i - 1$, we have satisfied (12) for $i - 1$.

**Case 2:** $\text{tp}(c_i/\bar{c}_{<i})$ is non-principal. Condition (11) for $i - 1$ is easily satisfied, because we can choose $f_i$ and $g_i$ such that $\langle \bar{x}, f_i(\bar{x}) \rangle$ and $\langle \bar{x}, g_i(\bar{x}) \rangle$ are in the interior of $D$, for $\bar{x} \in \pi_{<i}(D)$. Thus, we know that $F$ is continuous on $\text{cl}(E_{x_i}^{-1})$.

Define $\mu(\bar{x}) = \sup\{F(\bar{y}) \mid \bar{y}_{<i} = \bar{x}_{<i}\}$. The function $\mu$ will play a similar role to the curve $\bar{\gamma}$ that was used in the principal case. For $\bar{x} \in D$, note that $|\mu(\bar{x}_{<i}) - F(\bar{x})| \leq G(\bar{x}_{<i})$, for some $G$ a good bound at $i$, by (12) for $i$. If we can bound $|\mu(\bar{x}_{<i}) - \mu(\bar{x}'_{<i})|$ by some good bound at $i - 1$, where $\bar{x}_{<i} = \bar{x}'_{<i}$, we will be done.

**WLOG,** assume that $F$ is increasing in the $i$th coordinate. Now, consider $\mu_{\bar{c}_{<i}}$. Let $k = Q(i)$ (from Definition 5.6). We know that $k > 0$, since if $k = 0$, then $\text{Pr}(\bar{c}_{<i})$ would contain no principal types.

By hypothesis, $\mu_{\bar{c}_{<i}}^{-1}(P)$ is neither cofinal nor coinitial at $c_i$. We can thus replace $f_i$ and $g_i$ by $\emptyset$-definable functions $f'_i$ and $g'_i$ such that, for $y_i \in \{f_i(\bar{c}_{<i}), g_i(\bar{c}_{<i})\}$, we have $\mu(\bar{c}_{<i}, y_i) \notin P$, and thus, $\text{tp}(\mu(\bar{c}_{<i}, y_i)/P) = \text{tp}(\mu(\bar{c}_{<i}, y'_i)/P)$, for any $y_i, y'_i \in \{f_i(\bar{c}_{<i}), g_i(\bar{c}_{<i})\}$, since for two elements to have different types over $P$, there must be an element of $P$ between them. By induction (on $n$), we may assume that $f_i$ and $g_i$ extend continuously to $\text{cl}(\pi_{<i}(D))$ (it is trivial to check that they can be taken to be bounded), so we can assume that $f_i(\text{icl}(i, \bar{x}))$ and $g_i(\text{icl}(i, \bar{x}))$ are defined.

We may also require that $f_i(\text{icl}(i, \bar{x})) = g_i(\text{icl}(i, \bar{x}))$, for any $\bar{x} \in \pi_{<i}(D)$. First, we restrict $g_i(\bar{x}_{<k})$ so that (1) it satisfies Definition (3) with $c = \alpha_k(\bar{x}_{<k})$; (2) $|g_k(\bar{x}_{<k}) - \alpha_k(\bar{x}_{<k})|$ can be multiplied by $2$; and (3) $2|g_k(\bar{x}_{<k}) - \alpha_k(\bar{x}_{<k})|$ is in the domain of $h^+(\cdot, f_i(\text{icl}(i, \bar{x})))$. We may restrict $D$ so that the same three conditions hold for $f_i(\bar{x})$, with $c = f_i(\text{icl}(i, \bar{x}))$. Then replace $g_i$ by

$$\min(g_i(\bar{x}), h^+(x_k - \alpha_k(\bar{x}_{<k}))) + |f_i(\bar{x}) - f_i(\text{icl}(i, \bar{x}))|.$$ 

The second expression goes to $f_i(\text{icl}(i, \bar{x}))$ as $x_k$ goes to $\alpha_k(\bar{x}_{<k})$, since both absolute values go to $0$.

**Claim 7.10.** If $b, b' \in [f_i(\bar{c}_{<i}), g_i(\bar{c}_{<i})]$, then $\text{tp}(|\mu(\bar{c}_{<i}, b) - \mu(\bar{c}_{<i}, b')|/P)$ is principal near $0$.

**Proof.** Note that, since $\mu$ is a bounded function (since $F$ is), it cannot be the case that $\mu(\bar{c}_{<i}, b)$ is principal near $\pm \infty$ over $P$. By Corollary (3) since $\text{tp}(c_j/\bar{c}_{<j})$ is principal or out of scale on $P$ for every $j \leq i$, we have that $\text{tp}(\bar{c}_{<i}/P)$ is definable, and hence $P(\bar{c}_{<i})$ realizes only principal types over $P$, so $\mu(\bar{c}_{<i}, b)$ is principal. Then $|\mu(\bar{c}_{<i}, b) - \mu(\bar{c}_{<i}, b')|$ is principal near $0$, since two elements principal over the same element cannot be separated by a non-infinitesimal amount. 

---

*This is trivial if we have a global addition $-g_i(\bar{x}) = f_i(\bar{x}) + x_k - \alpha_k(\bar{x}_{<k})$. 


Thus, 
\[ \mu(\xi_{<i}) = \sup\{|\mu(\xi_{<i}, x_i) - \mu(\xi_{<i}, x'_i)| : x_i, x'_i \in [f_i(\xi_{<i}), g_i(\xi_{<i})]| \]

is principal near 0 over \( P \). Note that \( \mu \) is \( \emptyset \)-definable.

By induction (on \( n \)), we know that \( \mu \) is continuous on the closure of some \( \emptyset \)-definable set \( C \) containing \( \xi_{<i} \). For any \( \bar{x} \in C \), \( \mu(\text{icl}(i, \bar{x})) = 0 \), since \( f_i(\text{icl}(i, \bar{x})) = g_i(\text{icl}(i, \bar{x})) \). Thus, \( \mu \) is a good bound at \( i - 1 \), by definition. Let \( G' \) be the good bound at \( i - 1 \) bounding \( G \) guaranteed by Lemma \ref{lem:good_bound}. We may restrict \( D \) so that \( G'(D) \) and \( \mu(D) \) are both multiply by 4. Since \( \mu(\xi_{<i}) \geq |\mu(\xi_{<i}) - \mu(\xi'_{<i})| \) when \( \xi_{<i} = \xi'_{<i} \), we can now satisfy (12) for \( i - 1 \): given \( x, x' \) as in Condition (12) for \( i - 1 \),

\[ |F(x) - F(x')| \leq |F(x) - \mu(\xi_{<i})| + |F(x') - \mu(\xi'_{<i})| + |\mu(\xi_{<i}) - \mu(\xi'_{<i})| \leq 2G'(\xi_{<i}) + 2\mu(\xi_{<i}), \]

and thus we are done.

This concludes the proof of the “if” direction. We now do the “only if” part of Theorem \ref{thm:principal}

**Proposition 7.11.** Let \( p \) be a decreasing \( n \)-type over a set \( A \), and \( \bar{c} = \langle c_1, \ldots, c_n \rangle \) a tuple realizing \( p \), such that, for some \( i \), \( \text{tp}(c_i/\xi_{<i}A) \) is in scale or near scale on \( A \) (a fortiori on \( A\bar{c}_{\text{Q}(i)} \)), and \( \text{tp}(c_i/A) \) is principal. Then there exists a bounded \( A \)-definable function, \( F \), such that, for any \( A \)-definable set containing \( \bar{c} \), \( F \) is not continuous on the closure of the set.

**Proof.** As before, we may assume \( A = \emptyset \). Let \( P \) be the prime model, i.e. \( P = \text{dcl}(\emptyset) \).

We will construct a \( \emptyset \)-definable \( i \)-ary function, extending it to be constant on the last \( n - 1 \) coordinates, so we may assume that \( i = n \). Let \( k = Q(n) \). Note that \( k > 0 \). By hypothesis, there is some \( \xi_{<n} \)-definable function, \( f_{\xi_{<n}} \), such that \( f_{\xi_{<n}}(P) \) is cofinal or coinitial at \( c_n \) in \( \text{dcl}(\bar{c}_{<n}) \). WLOG, assume it is coinitial. Define \( F(x_1, \ldots, x_n) = f_{\xi_{<n}}^{-1}(x_n) \). Suppose that \( C \) is a \( \emptyset \)-definable set containing \( \bar{c} \). Using Lemma \ref{lem:principal}, we replace \( C \) by a \( \emptyset \)-definable subset such that \( \text{cl}(C) \) contains exactly one point with first \( k \) coordinates \( \langle \xi_{<k}, \alpha(\xi_{<k}) \rangle \), where \( \alpha \) is the \( \emptyset \)-definable function near which \( c_k \) is principal. We may further assume that \( C \) is a cell. Let \( g_n \) be the function bounding the \( n \)-th coordinate of \( C \) from above. Since \( f_{\xi_{<n}}(P) \) is coinitial at \( c_n \) in \( \text{dcl}(\bar{c}_{<n}) \), there is some element, \( r, P \), such that \( c_n < f_{\xi_{<n}}(r) < g_n(\bar{c}_{<n}) \).

By coinitiality, we can then find \( r' \in P \) with \( c_n < f_{\xi_{<n}}(r') < f_{\xi_{<n}}(r) \).

Note that, since \( F(\xi_{<n}, f_{\xi_{<n}}(r)) = r \), and \( F(\xi_{<n}, f_{\xi_{<n}}(r')) = r' \), with \( r, r' \in P \), we must have \( \emptyset \)-definable sets \( D_1 = \{ \bar{x} \in D^i \mid F(\bar{x}) = r \} \) and \( D_2 = \{ \bar{x} \in D^i \mid F(\bar{x}) = r' \} \).

Again by Lemma \ref{lem:principal}, we may possibly shrink \( D_1 \) and \( D_2 \), keeping \( \xi_{<k} \in \pi_{<k}(D_1), \pi_{<k}(D_2) \), and then assume that for each set \( \text{cl}(D_1) \) and \( \text{cl}(D_2) \), there is a unique point in the set with first \( k \) coordinates \( \langle \xi_{<k}, \alpha_k(\xi_{<k}) \rangle \). But since both \( \text{cl}(D_1) \) and \( \text{cl}(D_2) \) are subsets of \( \text{cl}(C) \), and \( \text{cl}(C) \) has a unique such point, there is a common point in \( \text{cl}(D_1) \) and \( \text{cl}(D_2) \). Since \( F = r \) on \( D_1 \), and \( F = r' \) on \( D_2 \), \( F \) cannot be extended continuously to this common point.

We are now ready to prove Theorem \ref{thm:no_continuous} as a corollary of Theorem \ref{thm:principal}

**Theorem 7.12.** Let \( M \) be an \( o \)-minimal structure, with \( T \) such that all principal types are interdefinable, and let \( \gamma(t) = \langle \gamma_1(t), \ldots, \gamma_n(t) \rangle \) be a (not necessarily
definable) non-oscillatory curve in \( M^n \). Then the following two statements are equivalent:

1. \( \bar{\gamma} \) can be reordered so that \( \text{tp}(\bar{\gamma}/M) \) is decreasing, and \( \text{tp}(\gamma_i/M\bar{\gamma}_{<i}) \) is algebraic, principal, or out of scale on \( M \).
2. For any bounded \( M \)-definable function, \( F \), continuous on \( \bar{\gamma}([0, s)) \) for some \( s > 0 \), there is an \( M \)-definable subset of \( M^n \), \( C \), such that \( F \mid C \) extends continuously to \( \text{cl}(C) \), and \( \bar{\gamma}([0, t)) \subseteq \text{cl}(C) \), for some \( t > 0 \).

Proof. Note that the first non-algebraic coordinate in \( \text{tp}(\bar{\gamma}/M) \) must be principal, since otherwise \( \bar{\gamma}(0) \) would not exist in \( M^n \).

For the forward direction, let \( p = \text{tp}(\bar{\gamma}/M) \), which is well-defined by Lemma 2.5.

Then, since (1) holds, \( p \) satisfies the conditions of Theorem A with \( A = M \), and so we can find the definable set, \( C' \), guaranteed by Theorem A with \( F \mid C' \) extending continuously to \( \text{cl}(C') \). Since \( \bar{\gamma}(0) \) must lie in the closure of any set containing \( p \), we are done.

Inversely, if (1) does not hold, then reorder \( \bar{\gamma} \) so that \( \text{tp}(\bar{\gamma}/M) \) is decreasing. Thus, the failure of (1) guarantees an in scale or near scale type. Then Theorem A gives us an \( F \) that does not extend continuously to the closure of any definable set containing \( p \). Since any definable set containing an initial segment of \( \bar{\gamma} \) must contain \( p \), we are done.

\[\square\]

REFERENCES

[FKMS08] Harvey Friedman, Krysztof Kurdyka, Chris Miller, and Patrick Speissegger, *Expansions of the real field by open sets: definability versus interpretability*, Preprint, November 2008.

[LMS03] Jean-Marie Lion, Chris Miller, and Patrick Speissegger, *Differential equations over polynomially bounded o-minimal structures*, Proc. Amer. Math. Soc. 131 (2003), no. 1, 175–183.

[Mal74] Bernard Malgrange, *Sur les points singuliers des équations différentielles*, Enseign. Math. 20 (1974), no. 2, 147–176.

[Mar86] David Marker, *Omitting types in o-minimal theories*, J. Symbolic Logic 51 (1986), no. 1, 63–74.

[Mil94] Chris Miller, *Expansion of the real field with power functions*, Ann. Pure Appl. Logic 68 (1994), no. 1, 79–94.

[MS94] David Marker and Charles Steinhorn, *Definable types in o-minimal theories*, J. Symbolic Logic 59 (1994), no. 1, 185–197.

[PS98] Ya’akov Peterzil and Sergei Starchenko, *A trichotomy theorem for o-minimal structures*, Proc. London Math. Soc. (3) 77 (1998), no. 3, 481–523.

[Spe08] Patrick Speissegger, *Continuity near a point*, Unpublished Note, 2008.

[Tre05] Marcus Tressl, *Model completeness of o-minimal structures expanded by dedekind cuts*, J. Symbolic Logic 70 (2005), no. 1, 29–60.

[vdD98] Lou van den Dries, *Tame topology and o-minimal structures*, Cambridge University Press, 1998.

[vdDL95] Lou van den Dries and Adam Lewenberg, *T-convexity and tame extensions*, J. Symbolic Logic 60 (1995), no. 1, 74–102.

\[\text{E-mail address: janak@math.univ-lyon1.fr}\]