ALMOST NON-DEGENERATE FUNCTIONS AND A ZARISKI PAIR OF LINKS

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Abstract. Let \( f(z) \) be an analytic function defined in the neighborhood of the origin of \( \mathbb{C}^n \) which have some Newton degenerate faces. We generalize the Varchenko formula for the zeta function of the Milnor fibration of a Newton non-degenerate function \( f \) to this case. As an application, we give an example of a pair of hypersurfaces with the same Newton boundary and the same zeta function with different tangent cones.

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1. INTRODUCTION

Consider an analytic function \( f(z) = \sum \nu a_{\nu} z^\nu \) of \( n \) variables defined in a neighborhood of the origin of \( \mathbb{C}^n \). Assume that we are given a good resolution \( \hat{\pi} : X \to \mathbb{C}^n \) of the function \( f \) and let \( E_1, \ldots, E_s \) be the exceptional divisors of \( \hat{\pi} \), that is \( \hat{\pi}^* f^{-1}(V) = \hat{V} \cup \bigcup_{i=1}^s E_j \) where \( \hat{V} \) is the strict transform of the hypersurface \( V = f^{-1}(0) \). Consider the open dense subset \( E''_j = E_j \cap \hat{\pi}^{-1}(0) \setminus \hat{V} \cup_{i \neq j} E_i \). Let \( m_j \) be the multiplicity of \( \hat{\pi}^* f \) along \( E_j \). By A’Campo [1], the zeta function of the Milnor monodromy at the origin is given as

\[
(AC') \quad \zeta(t) = \prod_{j=1}^s (1 - t^{m_j})^{-\chi(E''_j)}.
\]
Suppose that \( f(z) \) is Newton non-degenerate. Then using a toric modification \( \tilde{\pi} : X \to \mathbb{C}^n \) which is admissible with the dual Newton diagram \( \Gamma^*(f) \), the zeta function can be computed combinatorially (Varchenko, [15]). More precisely the zeta function is given as

\[
(V) \quad \zeta(t) = \prod_I \zeta_I(t), \quad \zeta_I(t) = \prod_{Q \in \mathcal{P}_I} (1 - t^{d(Q; f^I)})^{-\chi(Q)}
\]

where \( f^I \) is the restriction of \( f \) to the coordinate subspace \( \mathbb{C}^I \) and \( d(Q, f^I) \) is the minimal value of the monomials in \( f^I \) with respect to the weight vector \( Q \). \( I \subset \{1, \ldots, n\} \) moves so that \( f^I \) is not zero. \( \mathcal{P}_I \) is the set of primitive integer weight vectors of the coordinate subspace \( \mathbb{C}^I \) which correspond to the maximal faces of \( \Gamma(f^I) \). For \( I = \{1, \ldots, n\} \), we simply denote \( \mathcal{P} \). The number \( \chi(Q) \) in (V) is defined as follows.

\[
\chi(Q) = (-1)^{|I|-1} n^{|I|} \text{Vol}_k C(\Delta(Q; f^I), O_I)/d(Q; f^I).
\]

Here \( C(\Delta(Q; f^I), O_I) \) is the cone over \( \Delta(Q; f^I) \) with a vertex at the origin \( O_I \) of \( \mathbb{C}^I \) and \( \text{Vol}_k \) is the \( k \)-dimensional Euclidean volume. See [15, 12] or Chapter 3 of [10] for details. The formula (V) does not require any explicit regular subdivision of \( \Gamma^*(f) \) and is convenient for computing the zeta function. However, to compute the intersection numbers of exceptional divisors, we need to use an explicit toric resolution. A’Campo’s formula using a toric modification is convenient in such a situation, as it express the geometry more directly.

The purpose of this paper is to generalize Varchenko’s formula for certain functions which have some Newton degenerate faces. In §2, we recall the basic definitions about good resolutions, the Newton boundary and the non-degeneracy and admissible toric modifications with respect to the dual Newton diagram. In §3, we introduce the class of almost non-degenerate functions and give the first main result on the zeta function (Theorem 8). As an application of Theorem 8, we give a Zariski pair of links. Namely we give two hypersurfaces of dimension 2 with the same zeta function whose tangent cones gives a Zariski pair in \( \mathbb{P}^2 \) in §4 (Theorem 12, Theorem 14).

Remark 1. There is a canonical projection \( \pi_I : \mathbb{Z}^n \to \mathbb{Z}^I \) associated with the projection \( \pi^I : \mathbb{R}^n \to \mathbb{R}^I \) but \( P_I = \pi_I(P) \) is not necessarily primitive for \( P \in \mathcal{P} \). In the formula (V), \( Q \in \mathcal{P}_I \) is not necessarily a vertex of \( \Sigma^* \) and in general \( Q \) is not in the image \( \pi_I(\mathcal{P}) \) but we use the information \( Q \in \mathcal{P}_I \) for the calculation. See the proof of Theorem (5.3) ([10]). On the other hand, using an admissible toric modification, (AC) is restated as

\[
(AC') \quad \zeta(t) = \prod_{P \in V^+} (1 - t^{d(P)})^{-\tilde{E}(P)''}
\]

where \( V^+ \) is the set of strictly positive vertices of \( \Sigma^* \), which do not necessarily correspond to the maximal dimensional faces. In general, \( \mathcal{P} \subset V^+ \). Thus we need only the information of the vertices in \( V^+ \) but we do not use lower faces \( f^I \).
In this paper we use the notations:

\[ C^I = \{ z \in \mathbb{C}^n \mid z_j = 0 \text{ for } j \notin I \}, \quad f^I = f|_{C^I}, \]
\[ C^{\ast I} = \{ z \in \mathbb{C}^n \mid z_j = 0 \iff j \notin I \}. \]

In particular, we write simply \( \mathbb{C}^n \) and \( \mathbb{C}^{\ast n} \) for \( I = \{1, \ldots, n\} \).

2. Preliminaries

2.1. A good resolution of a function. Let \( f \) be an analytic function defined in a neighborhood \( U \) of the origin of \( \mathbb{C}^n \). Let \( X \) be a complex manifold of dimension \( n \) and \( \tilde{\pi} : X \to U \) is a proper holomorphic function. \( \tilde{\pi} : X \to U \) is called a good resolution of \( f \) if it satisfies the following:

1. \( \tilde{\pi} \) is biholomorphic on the restriction to \( X \setminus \tilde{\pi}^{-1}(V) \to U \setminus V, V = f^{-1}(0) \).

Assume that the divisor \( (\tilde{\pi}^* f) \) is given as \( \tilde{V} + \sum_{i=1}^k m_i E_i \) where \( \tilde{V} \) is the strict transform of \( V = f^{-1}(0) \) and \( m_i \) is the multiplicity of \( \tilde{\pi}^* f \) along \( E_i \). Let \( \tilde{V}_i, i = 1, \ldots, m \) be the irreducible components of \( \tilde{V} \).

2. Each irreducible component \( \tilde{V}_i \) and the divisors \( E_1, \ldots, E_k \) are non-singular and \( \tilde{V} \cup \cup_{i=1}^k E_i \) has, at most, ordinary normal crossing singularities. Namely take \( p \in \tilde{\pi}^{-1}(0) \) and let \( \tilde{I} \subset \{1, \ldots, k + m\} \) be the set of \( i \) such that \( p \in E_i \). Then \( |I| \leq n \) and there is an analytic coordinate chart \( (U_p, (v_1, \ldots, v_n)) \) in a small neighborhood \( U_p \) of \( p \) and an injective mapping \( \tilde{\xi} : I \to \{1, \ldots, n\} \) so that \( E_i = \{ v_{\xi(i)} = 0 \} \) for \( i \in I \). Here we put \( E_{k+i} = \tilde{V}_i \) for simplicity.

2.2. The Newton boundary and the dual Newton diagram. Let \( M \) be the space of monomials of the fixed coordinate variables \( z_1, \ldots, z_n \) of \( \mathbb{C}^n \) and let \( N \) be the space of weights of the variables \( z_1, \ldots, z_n \). We identify the monomial \( z^\nu = z_1^{\nu_1} \cdots z_n^{\nu_n} \) and the integral point \( \nu = (\nu_1, \ldots, \nu_n) \in \mathbb{R}^n \). A weight \( P \) is also identified with the column vector \( \ell(p_1, \ldots, p_n) \in \mathbb{R}^n \) where \( p_i = \deg_P(z_i) \) and we call \( P \) a weight vector. Let \( f(z) = \sum_{\nu} a_\nu z^\nu \) be a given holomorphic function defined by a convergent series. The Newton polygon \( \Gamma^+(f) \) with respect to the given coordinates \( z = (z_1, \ldots, z_n) \) is the convex hull of the union \( \cup_{\nu, a_\nu \neq 0} \{ \nu + \mathbb{R}_{\geq 0}^n \} \) and the Newton boundary \( \Gamma(f) \) is defined by the union of compact faces of \( \Gamma^+(f) \). An integral point \( \nu = (\nu_1, \ldots, \nu_n) \in \Gamma^+(f) \) corresponds to the monomial \( z^\nu = z_1^{\nu_1} \cdots z_n^{\nu_n} \) and we consider \( \Gamma^+(f), \Gamma(f) \) as subspaces of \( M^+_{\mathbb{R}} \) where \( M^+_{\mathbb{R}} = M \otimes \mathbb{R} \) and we identify \( M^+_{\mathbb{R}} \) with \( \mathbb{R}_{\geq 0}^n \). Similarly we identify \( N_{\mathbb{R}} := N \otimes \mathbb{R} \) with \( \mathbb{R}^n \). For a positive weight vector \( P = \ell(p_1, \ldots, p_n) \), we consider the canonical linear function \( \ell_P \) on \( \Gamma^+(f) \) which is defined by \( \ell_P(\nu) = \sum_{i=1}^n \nu_i p_i \). This is nothing but the degree mapping \( \deg_P \nu = \sum_{i=1}^n p_i \nu_i \). The minimal value of \( \ell_P \) is denoted by \( d(P; f) \). Put \( \Delta(P; f) := \{ \nu \in \Gamma^+(f) \mid \ell_P(\nu) = d(P) \} \). We will use the simplified notations \( d(P) \) and \( \Delta(P) \) if any ambiguity seems unlikely.
In general, $\Delta(P)$ is a face of $\Gamma^+(f)$ and $\Delta(P) \subset \Gamma(f)$ if $P$ is \textit{strictly positive} (i.e., $p_i > 0, \forall i$). A face $\Delta \subset \Gamma(f)$ with $\dim \Delta = n - 1$, there is a unique strictly positive primitive integer vector $P$ such that $\Delta(P) = \Delta$. (Recall that $P = t(p_1, \ldots, p_n)$ is primitive if $\gcd \{p_1, \ldots, p_n\} = 1$.) The partial sum $\sum_{\nu \in \Delta} a_{\nu}z^\nu$ is called the \textit{face function} of weight $P$ and we denote it as $f_P$ or $f_\Delta$. It is a polynomial if $P$ is strictly positive. Two weight vectors $P, Q$ are equivalent if and only if $\Delta(P) = \Delta(Q)$ and this equivalent relation gives a conical subdivision of the positive weight vectors $\mathbb{N}^+ \subset \mathbb{R}^n$ (i.e., $\overline{\sigma}_* P$). We say, $f$ is \textit{Newton non-degenerate} on a face $\Delta$ of $\Gamma(f)$ if $f_\Delta : \mathbb{C}^n \rightarrow \mathbb{C}$ has no critical points. $f$ is \textit{Newton non-degenerate} if it is non-degenerate on every face $\Delta \subset \Gamma(f)$ of any dimension. The closure of an equivalent class can be irredundantly expressed as

$$\text{Cone}(P_1, \ldots, P_k) := \left\{ \sum \lambda_i P_i \mid \lambda_i \geq 0 \right\}$$

where $P_1, \ldots, P_k$ are chosen to be primitive integer vectors. That is, $k$ is minimal among any possible such expressions. A cone $\sigma = \text{Cone}(P_1, \ldots, P_k)$ is \textit{simplicial} if $\dim \sigma = k$ and $\sigma$ is \textit{regular} if $P_1, \ldots, P_k$ are primitive integer vectors which can be extended to a basis of the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$.

Recall that $f$ is \textit{convenient} if $\Gamma(f)$ touches with every coordinate axis. We say $f$ is \textit{pseudo-convenient} if $f$ is written as $f(z) = z^{\nu_0} f'(z)$ where $f'$ is convenient and $\nu_0$ is a positive integer vector.

\subsection{2.3. Toric modification.}

A regular simplicial cone subdivision $\Sigma^*$ of the space of positive weight vectors $\mathbb{N}^+ \subset \mathbb{R}^n$ is \textit{admissible with the dual Newton diagram} $\Gamma^*(f)$ if $\Sigma^*$ is a subdivision of $\Gamma^*(f)$. For such a regular simplicial cone subdivision, we associate a modification $\hat{\pi} : X \rightarrow \mathbb{C}^n$ as follows: let $\mathcal{S}$ be the set of $n$-dimensional cones in $\Sigma^*$. For each $\sigma = \text{Cone}(P_1, \ldots, P_n) \in \mathcal{S}$, we identify $\sigma$ with the unimodular matrix:

$$\sigma = \begin{pmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & \ddots & \vdots \\ p_{n1} & \cdots & p_{nn} \end{pmatrix}$$

with $P_j = t(p_{1j}, \ldots, p_{nj})$. To each $\sigma \in \mathcal{S}$, we associate an affine coordinate chart $(\mathbb{C}^n_{\sigma}, u_{\sigma})$ with $u_{\sigma} = (u_{\sigma 1}, \ldots, u_{\sigma n})$. The modification $\hat{\pi}$ is defined as follows. For each $\sigma \in \mathcal{S}$, we associate a birational mapping $\hat{\pi}_\sigma : \mathbb{C}^n_{\sigma} \rightarrow \mathbb{C}^n$ by $z_i = u_{\sigma i 1}^{p_{1i}} \cdots u_{\sigma i n}^{p_{ni}}$ for $i = 1, \ldots, n$ and $X$ is the complex manifold obtained by gluing $\mathbb{C}^n_{\sigma}$ and $\mathbb{C}^n_{\tau}$ by $\hat{\pi}_\sigma^{-1} \circ \hat{\pi}_\tau : \mathbb{C}^n_{\sigma} \rightarrow \mathbb{C}^n_{\tau}$ where it is well-defined. This defines the modification $\hat{\pi} : X \rightarrow \mathbb{C}^n$ which is proper and the restriction $\hat{\pi}$ to the torus $\mathbb{C}^n_{\sigma} \subset \mathbb{C}^n_{\sigma}$ is an isomorphism onto the torus $\mathbb{C}^n_{\sigma} = (\mathbb{C}^*)^n$ in the base space. If $\sigma = \text{Cone}(P_1, \ldots, P_n)$ and $\tau = \text{Cone}(Q_1, \ldots, Q_n)$ have a same vertex $Q_1 = P_1$, the hyperplane $u_{\sigma 1} = 0$ glues canonically with the hyperplane $\{u_1 = 0\}$. Thus any vertex $P$ of $\Sigma^*$, gluing the hyperplanes on every such toric coordinates with $P_1 = P$, defines a divisor in $X$, and we denote this divisor by $\hat{E}(P)$. If $P$ is strictly positive, $\hat{E}(P)$ is a compact
divisor and \( \hat{\pi}(\hat{E}(P)) = \{O\} \). Recall that a vertex of \( \Sigma^* \) is a primitive integer generator of a 1-dimensional cone of \( \Sigma^* \). Note that if \( f \) is pseudo-convenient, there exists a regular simplicial subdivision \( \Sigma^* \) whose vertices are strictly positive except for the canonical weight vectors \( e_i = t^i(0, \ldots, 1, \ldots, 0), i = 1, \ldots, n \). For simplicity we assume hereafter that \( f(z) \) has a convenient or pseudo-convenient Newton boundary and the vertices of \( \Sigma^* \) are strictly positive except the canonical weight vectors \( e_i, i = 1, \ldots, n \). Recall \( \hat{E}(e_i) \) is bijectively mapped onto \( \{ z_i = 0 \} \). Let \( \nu^+ \) be the set of strictly positive vertices of \( \Sigma^* \). Then the exceptional divisors of \( \hat{\pi} : X \to \mathbb{C}^n \) corresponds bijectively to the vertices of \( \nu^+ \). The pull-back \( \hat{\pi}^*f \) of \( f \) is expressed in the toric chart \( \mathbb{C}^n_\sigma \) with \( \sigma = \text{Cone}(P_1, \ldots, P_n) \) as follows:

\[
\hat{\pi}^*f(u_\sigma) = (\prod_{i=1}^n u_{\sigma,i}^{d(P_i)}) \tilde{f}(u_\sigma)
\]

and \( \tilde{f}(u_\sigma) \) is the defining function of the strict transform \( \tilde{V} \) of \( V \). The intersection \( E(P) := \tilde{V} \cap \hat{E}(P) \) is defined in \( \hat{E}(P) \) by \( \tilde{f}(0, u_{\sigma 2}, \ldots, u_{\sigma n}) = 0 \). \( E(P) \) is an exceptional divisor of the restriction \( \pi := \hat{\pi}|_\tilde{V} : \tilde{V} \to V \). We recall that \( \pi^{-1}(O) \) is the union of \( E(P) \) such that \( P \in \nu^+ \) and \( \Delta(P) \geq 1 \). Two exceptional divisors \( \hat{E}(P) \) and \( \hat{E}(Q) \) intersect if and only if there is a \( \sigma \in S \) such that \( \sigma = \text{Cone}(P, Q, P_3, \ldots, P_n) \). However for \( E(P) \cap E(Q) \) to be non-empty, besides the existence of such a \( \sigma \), it is also necessary that \( \dim \Delta(P) \cap \Delta(Q) \geq 1 \). See Proposition (1.3.2), in Chapter II ([10]).

If \( f \) is Newton non-degenerate, any admissible toric modification \( \hat{\pi} : X \to \mathbb{C}^n \) gives a good resolution of \( f \) and by (AC) the zeta function is written as

\[
(1) \quad \zeta(t) = \prod_{P \in \nu^+} (1 - t^{d(P)})^{-\chi(\hat{E}(P))''}
\]

\[
(2) \quad \hat{E}(P)'' = \hat{E}(P) \setminus (\tilde{V} \cup_{Q \in \nu^+, Q \neq P} \hat{E}(Q))
\]

where \( \tilde{V} \) is the strict transform of \( V = f^{-1}(0) \) on \( X \). Let \( \hat{E}(P)^{st} := \hat{E}(P) \cap \mathbb{C}^s \) and \( E(P)^{st} = E(P) \cap \mathbb{C}^s \). Then we use the toric decomposition \( \hat{E}(P) = \cup_I \hat{E}(P)^{st} \) and the equality

\[
\chi(E(P)^{st}) = \chi(Q) = (-1)^{|I|-1} |I| ! \text{Vol}_I \mathcal{C}(\Delta(Q) \cap \mathbb{R}^I, 0)/d(Q)
\]

for the computation of the Euler characteristic \( \chi(\hat{E}(P)'') \) where \( Q \in N^I \) is chosen to be a primitive integer vector such that \( \Delta(Q) = \Delta(P) \cap \mathbb{R}^I \) (See [5] [14]). This formula says that the Euler characteristics of \( E(P)^{st} \) is zero if \( \dim \Delta(P) \cap \mathbb{R}^I < |I| - 1 \).

3. Almost non-degenerate functions

Consider a function \( f(z) = \sum a_{\sigma} z^\sigma \) which is expanded in the Taylor series and let \( \Gamma(f) \) be the Newton boundary. Let \( \hat{\pi} : X \to \mathbb{C}^n \) be a toric modification with respect to \( \Sigma^* \) which is a simplicial regular subdivision of
the dual Newton diagram $\Gamma^*(f)$. Let $\mathcal{M}$ be the set of maximal dimensional faces of $\Gamma(f)$ and let $\mathcal{M}_0$ be the subset of $\mathcal{M}$ so that for $\Delta \in \mathcal{M}$, $f_{\Delta} : \mathbb{C}^* \to \mathbb{C}$ is degenerate if and only if $\Delta \in \mathcal{M}_0$. For $P$ which corresponds to a maximal face $\Delta \in \mathcal{M}$, we denote the exceptional divisor by $\hat{E}(P)$ which corresponds to $P$. We say that $f$ is an almost non-degenerate function if it satisfies the following conditions.

(A1) For any face $\Delta$ of $\Gamma(f)$ with either $\Delta \in \mathcal{M} \setminus \mathcal{M}_0$ or $\dim \Delta \leq n - 2$, $f$ is Newton non-degenerate on $\Delta$. For $\Delta \in \mathcal{M}_0$, $f_{\Delta} : \mathbb{C}^* \to \mathbb{C}$ has a finite number of 1-dimensional critical loci which are $\mathbb{C}^*$-orbits through the origin.

Here we recall that $f_{\Delta}(z)$ is a weighted homogeneous polynomial with respect to the weight vector $P$ and there is an associated $\mathbb{C}^*$-action defined by $t \circ (z_1, \ldots, z_n) = (t^{p_1} z_1, \ldots, t^{p_n} z_n)$, $t \in \mathbb{C}^*$ where $\Delta(P) = \Delta$ and $P = (p_1, \ldots, p_n)$. Critical points loci of $f_{\Delta}$ are stable under this action.

Let $\sigma = \text{Cone}(P_1, \ldots, P_n)$ be a simplicial cone in $\Sigma^*$ such that $\Delta(P_1) = \Delta \in \mathcal{M}_0$. Let $u_\sigma = (u_{\sigma 1}, \ldots, u_{\sigma n})$ be the corresponding toric coordinate chart. The strict transform $\hat{V}$ of $V(f)$ is defined by $\hat{f}(u_\sigma) = 0$ where $\hat{f}$ is defined by the equality:

$$\hat{\pi}^* f = \left( \prod_{i=1}^{n} u_{\sigma,i}^{d_i(P_1)} \right) \hat{f}(u_\sigma) = 0$$

and $E(P_1) \subset \hat{E}_0$ is defined by $\{ u_\sigma | u_{\sigma 1} = 0, g_{\Delta}(u_{\sigma 2}, \ldots, u_{\sigma n}) = 0 \}$ where $g_{\Delta}(u_{\sigma 2}, \ldots, u_{\sigma n}) := \hat{f}(0, u_{\sigma 2}, \ldots, u_{\sigma n})$. The assumption (A1) implies $E(P_1)$ has a finite number of isolated singular points. Let $S(\Delta)$ be the set of the singular points of $E(P_1)$. Take any $q \in S(\Delta)$ and assume $q = (0, \beta_2, \ldots, \beta_n)$ in $\mathbb{C}_a^*$. An admissible coordinate chart at $q$ is an analytic coordinate chart $(U_q, w)$, $w = (w_1, \ldots, w_n)$ centered at $q$ where $U_q$ is an open neighborhood of $q$ and $(w_2, \ldots, w_n)$ is an analytic coordinate change of $(u_{\sigma 2}, \ldots, u_{\sigma n})$, but we do not change $u_{\sigma 1}$ and we always assume $w_1 = u_{\sigma 1}$. (In many cases, we can take $w_i = u_{\sigma,i} - \beta_i$, $i = 2, \ldots, n$.) As $w_1 = u_{\sigma 1}$, $w_1 = 0$ is the defining function of $\hat{E}(P_1)$. As a second condition, we assume

(A2) For any $\Delta \in \mathcal{M}_0$ and $q \in S(\Delta)$, there exists an admissible coordinate $(U_q, w)$ centered at $q$ such that $\hat{\pi}^* f(w)$ is Newton non-degenerate and pseudo-convenient with respect to this coordinates $(U_q, w)$.

Remark 2. The conditions (A1), (A2) can be generalized for maximal faces of $f^I$. The definition is similar.

3.1. First modification of $f$. We assume that $f(z)$ is an almost non-degenerate function and we take an admissible toric modification as in the previous section. We consider the tubular Milnor fibration

$$\text{(*) } f : U(\varepsilon, \delta)^* \to D_\delta^0, U(\varepsilon, \delta) = \{ z | 0 < |f(z)| \leq \delta, \|z\| \leq \varepsilon \}, \delta \ll \varepsilon.$$ 

This fibration is isomorphically lifted on $X$ so that $\hat{f} : \hat{U}(\varepsilon, \delta)^* \to D_\delta^0$ is equivalent to the Milnor fibration (\text{(*)}). Here $\hat{U}(\varepsilon, \delta)^* := \hat{\pi}^{-1}(U(\varepsilon, \delta))$ and
\( \hat{f} = f \circ \hat{\pi} \). This fibration can be decomposed as the union of the fibrations along \( \hat{E}(P) \) for \( P \in V^+ \) and local Milnor fibrations of \( \hat{f} \) at \( q \in S(\Delta) \), \( \Delta \in \mathcal{M}_0 \). In Figure 1, brown circles are the spheres of radius \( \varepsilon' \) and \( \varepsilon'' \) where \( \varepsilon'' < \varepsilon' \) and \( \varepsilon' - \varepsilon'' \) is sufficiently small. At each \( q \in S(\Delta) \), we take a small ball \( B_{\varepsilon'}(q) \) and we consider the local Milnor fibration of the function \( \hat{f}(w) = \hat{\pi}^* f(w) \) at \( q \):

\[
\hat{f} : U_q(\varepsilon', \delta)^* \to D_\delta^*,
\]

where

\[
U_q(\varepsilon', \delta)^* = \{ w \in U_q \mid 0 < |\hat{f}(w)| \leq \delta, \|w\| \leq \varepsilon' \}, \delta \ll \min\{\varepsilon', \varepsilon\}.
\]

We assume \( \delta \) is small enough so that we can use the same \( \delta \) in (\Ast) for the local Milnor fibrations at \( q \). This means that for any \( \eta \neq 0, |\eta| \leq \delta \), the level hypersurface \( \hat{f}^{-1}(\eta) \) intersects transversely with the sphere \( S_{2\varepsilon'}^{2n-1}(q) \).

Here we assume that \( w \) is an admissible coordinate at \( q \). Let us consider the decomposition of of the total space of the lifted Milnor fibration \( \hat{U}(\varepsilon, \delta)^* \),

\[
\hat{U}(\varepsilon, \delta)^* = \hat{U}(\varepsilon, \delta)^* \cup \left( \bigcup_{q \in \Delta, \Delta \in \mathcal{M}_0} U_q(\varepsilon', \delta) \right)
\]

where \( \hat{U}(\varepsilon, \delta)^* := U(\varepsilon, \delta)^* \setminus \left( \bigcup_{q \in \Delta, \Delta \in \mathcal{M}_0} U_q(\varepsilon''', \delta) \right) \) and we consider the corresponding decomposition of the Milnor fibration. We take \( \varepsilon''' \) a bit smaller than \( \varepsilon' \). Assume \( \{q_1, \ldots, q_m\} = \{ q \in S(\Delta) \mid \Delta \in M_0 \} \).
Then the zeta function of $f$ is given as the product
\begin{equation}
\zeta(t) = \zeta'(t) \prod_{q \in S(\Delta)} \zeta_q(t)
\end{equation}
where each product factor $\zeta_q(t)$ can be computed by Varchenko’s formula (V). Then we have

**Lemma 3.** Assume that $f(z)$ is an almost non-degenerate function as above. Then the zeta function of $f$ is given as the product
\begin{equation}
\zeta'(t) = \prod_{I \subseteq \{1, \ldots, n\}} \zeta_I(t) \prod_{P \in \mathcal{P} \setminus \mathcal{P}_0} (1 - t^{d(P)})^{-\chi(P)} \prod_{P \in \mathcal{P}_0} (1 - t^{d(P)})^{-\chi(P)+(-1)^{n-1} \sum_{q \in S(\Delta(P))} \mu_q}.
\end{equation}

**Proof.** The factor $\zeta_I(t)$ and $(1 - t^{d(P)})^{-\chi(P)}$ in the first line of (4) are the same as in (V). The equality (4) will be proved [3, 4] where $\mu_q$ is the Milnor number of the hypersurface $E(P)$ in $\mathcal{E}(P)$ at $q$. The proof of the assertion [3] is essentially the same as the proof of Theorem (5.2), Chapter I, [10].

For the proof, we use the following Sublemma [4] and Proposition [5].

**Sublemma 4** (Lemma (5.3), [10]). Let $U \subset \pi^{-1}(U(\varepsilon, \delta))$ and suppose that there is a manifold $M$ and a submersion $p : U \rightarrow M$ so that $p \times \tilde{f} : U \rightarrow M \times D^*_\delta$ is a locally trivial fibration. Its restriction to $p^{-1}(m)$, $\tilde{f} : p^{-1}(m) \rightarrow D^*_\delta$, with $m \in M$ is also a fibration. Let $\zeta(t)$ and $\zeta_M^+(t)$ be the respective zeta functions of the fibrations $\tilde{f} : U \rightarrow D^*_\delta$ and $\hat{f} : p^{-1}(m) \rightarrow D^*_\delta$. Then we have the equality: $\zeta(t) = (\zeta^+_M(t))^{\chi(M)}$.

The assertion is trivial when $p : U \rightarrow M$ is a trivial fibration. Then we apply Mayer-Vietoris argument. $\zeta_M^+(t)$ is called the normal zeta function along $M$.

**Proposition 5** (Proposition (2.8), [10]). Let $U = U_1 \cup U_2$ be an open covering of the fibration $p : U \rightarrow D^*_\delta$ where the restriction of $p$ to $U_1, U_2$ and
$U_{12} := U_1 \cap U_2$ is also fibration. Consider four fibrations. Let $F, F_1, F_2, F_{12}$ be the respective fibers and let $\zeta, \zeta_1(t), \zeta_2(t), \zeta_{12}$ be their zeta functions. Then

$$\chi(F) = \chi(F_1) + \chi(F_2) - \chi(F_{12}), \quad \zeta(t) = \zeta_1(t)\zeta_2(t)\zeta_{12}(t)^{-1}.$$ 

The assertion follows easily from the Mayer-Vietoris argument.

We apply Proposition 5 to the union $\hat{U}_{i+1}(\varepsilon, \delta) = \hat{U}_i(\varepsilon, \delta) \cup U_{q_{i+1}}(\varepsilon', \delta)$. Let $W_{i+1} = U_{q_{i+1}}(\varepsilon', \delta) \cap U_{q_{i+1}}(\varepsilon'', \delta) = \hat{U}_i(\varepsilon, \delta) \cap U_{q_{i+1}}(\varepsilon', \delta)$. The proof of the inductive assertion (5) follows from the next assertion.

**Assertion 6.** The contribution to the zeta function from $W_{i+1} = \hat{U}_i(\varepsilon, \delta) \cap U_{q_{i+1}}(\varepsilon', \delta)$ is trivial.

Assuming Assertion (6), Lemma 3 follows by the inductive argument.

**Proof of Assertion (6).** We take $\delta \ll \varepsilon, \varepsilon'$. Assume that $q_{i+1} \in \hat{E}(P), \, P \in P_0$ and put

$$\hat{E}(P)_{i+1}' = \hat{E}(P) \cap (B_{r'}(q_{i+1}) \setminus B_{r''}(q_{i+1})), \quad E(P)_{i+1}' = E(P) \cap (B_{r'}(q_{i+1}) \setminus B_{r''}(q_{i+1}))$$

where $B_r(q_{i+1})$ is the ball of radius $r$ with the center $q_{i+1}$. $\hat{E}(P)_{i+1}'$ and $E(P)_{i+1}'$ are non-singular and homotopically equivalent to $S^{2n-3}$ and to the link $K_{r'} := g_{P'}^{-1}(0) \cap S^{2n-3}_r$ respectively. Note that in the toric coordinate chart $\sigma = \text{Cone}(P_1, \ldots, P_n)$ with $P = P_1$, $\hat{f}$ is written as $u_{\sigma_1}^{d_1}(P) f(u_{\sigma})$ and $g_{P}(u_{\sigma}) = \hat{f}(0, u_{\sigma'})$ and $u_{\sigma'} = (u_{\sigma_2}, \ldots, u_{\sigma_n})$ and $g_{P} = 0$ defines the hypersurface $E(P)$. Note that $u_{\sigma_1}, g_{P}(u_{\sigma})$ can be considered as a part of an analytic coordinate chart at any $q_{i+1}' \in E(P)_{i+1}'$. Take a small tubular neighborhood $U_{\gamma}$ in $X$ of $E(P)_{i+1}'$ of radius $\gamma$ with some distance function from $E(P_i)$. $U_{\gamma}$ is the set of points whose distance to $E(P)_{i+1}'$ is less than $\gamma$. Consider the union $W_{i+1}' = W_{i+1}' \cup U_{\gamma}'$ where $W_{i+1}' = \hat{W}_{i+1} \setminus U_{\gamma/2}$ and $U_{\gamma}' = U_{\gamma} \cap \hat{U}(\varepsilon, \delta)$. We can identify $W_{i+1}'$ as a tubular neighborhood of $\hat{E}(P)_{i+1}'$ in $X$ over $E(P) \setminus U_{\gamma}$ and choose compatible projections $p_1 : W_{i+1}' \to \hat{E}(P)_{i+1}'$ and $p_2 : U_{\gamma} \to E(P)_{i+1}'$. Here compatible means $p_2 \circ p_1 = p_1$ wherever both sides are defined. (When we take a point $q_{i+1}' \in E(P)_{i+1}'$ and a small neighborhood of $q_{i+1}'$, so that $(u_{\sigma_1}, g_{P}, u_{\sigma_3}, \ldots, u_{\sigma_n})$ are coordinates, we can assume that $p_1 : (u_{\sigma_1}, g_{P}, u_{\sigma_3}, \ldots, u_{\sigma_n}) \mapsto (0, g_{P}, u_{\sigma_3}, \ldots, u_{\sigma_n})$ and $p_2 : (u_{\sigma_1}, g_{P}, u_{\sigma_3}, \ldots, u_{\sigma_n}) \mapsto (0, 0, u_{\sigma_3}, \ldots, u_{\sigma_n})$.) First we consider $\hat{f} \times p_1 : W_{i+1}' \to D_{\delta} \times (\hat{E}(P) \setminus U_{\gamma/2})$. The restriction of the Milnor fibration over a fiber of $p_1$ is a cyclic covering corresponding to $u_{\sigma_1}^{d_1}(P)$. Thus the normal zeta function along $\hat{E}(P)_{i+1}'$ is $(1 - t^{d_1}(P))$. As $E(P)_{i+1}' \setminus U_{\gamma/2}$ is homotopic to the complement of the link $K_{r'}$ in $S^{2n-3}_r$, the Euler characteristic is zero and the zeta function of $\hat{f} : W_{i+1}' \to D_{\delta}^*$ is trivial by Sublemma 4. Next, we consider $\hat{f} \times p_2 : U_{\gamma}' \to D_{\delta}^* \times E(P)_{i+1}'$. The normal zeta function along $E(P)_{i+1}'$ is 1, as it corresponds to the function of two variables $u_{\sigma_1}^{d_1}(P) g_{P} (u_{\sigma})$.
(3.7.3), Chapter I, [10]. Thus combining the argument, the zeta functions of the restrictions of the Milnor fibration to \( W_{i+1}' \), to \( U_i' \) and to \( W_{i+1}' \cap U_i' \) are all trivial.

**Remark 7.** The assertion (3) in Lemma 3 is true without assuming the non-degeneracy of \( \tilde{f} \) at \( q \). In that case, \( \zeta_q(t) \) must be computed without using the formula (V). We need a practical resolution information of \( \tilde{f} \) at each singular point \( q \in S(\Delta) \). If \( \tilde{f} \) is non-degenerate at \( q \) with respect to an admissible coordinate \( w \), we only need the information of \( \Gamma(\tilde{f}, w) \). Then we apply (V).

### 3.2. Zeta function \( \zeta'(t) \) and the proof of the assertion (4) in Lemma 3

Recall that \( \zeta'(t) \) is the zeta function for the first toric modification, outside of the singular points \( \{ q \in S(\Delta), \Delta \in \mathcal{M}_0 \} \). Applying A'Campo's formula and Varchenko's description, we get a formula for the zeta function \( \zeta'(t) \) which is given as follows.

For \( \Delta \in \mathcal{M} \setminus \mathcal{M}_0 \), the calculation is the same as in the proof of Theorem (5.3), [10]. Assume that \( \Delta \in \mathcal{M}_0 \). Let \( P \) be the primitive weight vector which corresponds to \( \Delta \) and take a toric chart \( \mathcal{C}_\sigma^\Delta \). \( \sigma = \text{Cone}(P_1, \ldots, P_n) \) with \( P = P_1 \) and let \( u_\sigma = (u_{\sigma 1}, \ldots, u_{\sigma n}) \) the toric coordinate of this chart. Then as in the previous section, the exceptional divisor \( \tilde{E}(P) \) is defined by \( u_{\sigma 1} = 0 \) and \( E(P) := \tilde{E}(P) \cap \tilde{V} \) is defined in \( \tilde{E}(P) \) by \( g_P(u_{\sigma 2}, \ldots, u_{\sigma n}) = 0 \) where

\[
\tilde{\pi}^* f(u_\sigma) = \left( \prod_{i=1}^n u_{\sigma,i}^{d(P_i; f)} \right) \tilde{f}(u_\sigma)
\]

\[
g_P(u_{\sigma 2}, \ldots, u_{\sigma n}) := \tilde{f}(0, u_{\sigma 2}, \ldots, u_{\sigma n}).
\]

Let \( \mu_q \) be the Milnor number of \( (g_P, q) \) as a germ of a hypersurface at \( q \in \tilde{E}(P) \). We take a small ball \( B_q(\varepsilon) \) centered at \( q \) for \( q \in S(\Delta) \). Let us consider a small perturbation family \( f_s(z) \), \( |s| \leq 1 \) of the coefficients of \( f \) so that \( f = f_0, \Gamma(f_s) = \Gamma(f) \) for any \( s \) and \( f_s \) is non-degenerate for \( s \neq 0 \). More precisely, we need only move a bit the coefficients of \( f_\Delta, \Delta \in \mathcal{M}_0 \). The same toric modification \( \tilde{\pi} : X \to \mathbb{C}^n \) gives a good resolution of the family \( f_s \) for any \( s \neq 0 \). Let \( \zeta^{(s)}(t) \) be the zeta function of \( f_s, s \neq 0 \). Let \( \tilde{V}_0 = \tilde{V} \) and \( \tilde{V}_s \) be the strict transform of \( f^{-1}(0) \) and \( f_s^{-1}(0) \) respectively. Let \( \tilde{E}(P)_s \) be the corresponding factor of the exceptional divisor of \( \tilde{E}(P) \) which appears in the product expression of \( \zeta^{(s)}(t) \) in the formula (AC) of A'Campo and (V) of Varchenko and let \( E(P)_s \) and \( E(P)_0 \) (= \( \tilde{E}(P) \) the intersections of \( \tilde{E}(P) \) and the strict transforms \( \tilde{V}_s \) of \( f_s = 0 \) and \( \tilde{V}_0 \) of \( f_0 = 0 \) respectively. Note that \( \chi(E(P)) = \chi(E(P)_s) + (-1)^{n-1} \sum_{q \in S(\Delta)} \mu_q \). Therefore

\[
\chi(\tilde{E}(P) \setminus (\tilde{V}_0 \cup_{Q \neq P} \tilde{E}(Q))) = \chi(\tilde{E}(P) \setminus (\tilde{V}_s \cup_{Q \neq P} \tilde{E}(Q)) - (-1)^{n-1} \sum_{q \in S(\Delta)} \mu_q.
\]
Thus the factor of zeta function \( \zeta'(t) \) coming from the divisor \( \hat{E}(P) \) is changed from
\[
(1 - t^{d(P)})^{-\chi(\hat{E}(P)'')} \quad \text{to} \quad (1 - t^{d(P)})^{-\chi(\hat{E}(P)'')} + (-1)^{n-1} \sum_{q \in S(\Delta)} \mu_q.
\]

For \( \Delta \in \mathcal{M}_0 \), take \( P \in \mathcal{P}_0 \) with \( \Delta(P) = \Delta \) and we put
\[
\zeta_\Delta(t) := \prod_{q \in S(\Delta)} \zeta_q(t),
\]
\[
\zeta^{cr}_\Delta(t) := (1 - t^{d(P)}) \left( -1 \right)^{n-1} \sum_{q \in S(\Delta)} \mu_q
\]
\[
\zeta^{cr}(t) := \prod_{\Delta \in \mathcal{M}_0} \zeta^{cr}_\Delta(t)
\]

Using the above notations, we now obtain the equality:
\[
\zeta'(t) = \prod_{I \subseteq \{1, \ldots, n\}} \zeta_I(t) \prod_{P \in \mathcal{P} \setminus \mathcal{P}_0} (1 - t^{d(P)})^{-\chi(P)}
\times \prod_{P \in \mathcal{P}_0} (1 - t^{d(P)})^{-\chi(\hat{E}(P)'')} + (-1)^{n-1} \sum_{q \in S(\Delta)} \mu_q
\]
\[
= \zeta^{(s)}(t) \zeta^{cr}(t).
\]

This completes the proof of (4) in Lemma 3.

3.3. Second modifications. For \( \Delta \in \mathcal{M}_0 \) and \( q \in S(\Delta) \), we choose admissible coordinates \( w = (w_1, \ldots, w_n) \) centered at \( q \) and take an admissible regular simplicial subdivision \( \Sigma^*_q \) of \( \Gamma^*(\hat{f}; w) \). As \( \hat{f}(w) \) is pseudo-convenient, we assume that (\( \sharp \)) the vertices of \( \Sigma^*_q \) are strictly positive except \( e_1, \ldots, e_n \).

Then we take the toric modification \( \hat{\omega}_q : Y_q \to X \) with respect to \( \Sigma^*_q \). Taking the toric modification at each \( q \in S(\Delta) \), \( \Delta \in \mathcal{M}_0 \), let \( \hat{\omega} : Y \to X \) is the union of the toric modification. Here \( Y \) is the canonical gluing of the union of \( Y_q, q \in S(\Delta), \Delta \in \mathcal{M}_0 \). The composition
\[
\Pi : Y \overset{\hat{\omega}}{\longrightarrow} X \overset{\hat{f}}{\longrightarrow} \mathbb{C}^n
\]
gives a good resolution of \( f \). The exceptional divisors of \( \Pi \) are all compact under the assumption (\( \sharp \)). The zeta function \( \zeta_q(t) \) is described by (AC) or (V). Thus by Lemma 3 we have the following generalization of Varchenko’s formula:

**Theorem 8.** The zeta function of \( f \) is given by
\[
\zeta(t) = \zeta^{(s)}(t) \zeta^{cr}(t) \prod_{\Delta \in \mathcal{M}_0} \zeta_\Delta(t).
\]

where \( \zeta^{(s)}(t) \) is the zeta function of the Newton non-degenerate function \( f_s \) with \( \Gamma(f_s) = \Gamma(f) \).
3.4. Examples. Example 1. Consider \( f = (x - y)^2 + y^3 \). \( f \) has one face with the weight vector \( P = (1, 1) \). First toric modification \( \tilde{\pi} : X \rightarrow \mathbb{C}^2 \) is the ordinary blowing up. Take the chart \((U_1, (u_{\sigma_1}, u_{\sigma_2}))\) with \( \tilde{\pi}(u_{\sigma_1}, u_{\sigma_2}) = (u_{\sigma_1}u_{\sigma_2}, u_{\sigma_1}) \). The pull back of \( f \) is given as

\[
\tilde{\pi}^* f(u_{\sigma_1}, u_{\sigma_2}) = u_{\sigma_1}^2 ((u_{\sigma_2} - 1)^2 + u_{\sigma_1}).
\]

Exceptional divisor is given by \( u_{\sigma_1} = 0 \). Changing coordinate \( w_1 = u_{\sigma_1}, w_2 = u_{\sigma_1} - 1, \tilde{\pi}^* f = w_1^2 (w_2^2 + w_1) \) and we see that \( \mu_q = 1 \). The Newton boundary \( \Gamma(\tilde{\pi}^* f, (w_1, w_2)) \) has one face with weight \( Q = t^4(1, 1) \) and \( d(Q) = 6 \). Thus by (V),

\[
\zeta_q(t) = (1 - t^6)(1 - t^3)^{-1}
\]

where the factor \((1 - t^3)^{-1}\) comes from the vertex \( w_3^3 \). \( \zeta(t)' = \zeta(s)(t)\zeta^{cr}(t) \), \( \zeta(s)(t) = 1 \) and \( \zeta^{cr}(t) = (1 - t^2)^{-1} \) and \( \zeta(t) = (1 - t^6)(1 - t^2)^{-1}(1 - t^3)^{-1} \).

Example 2. Consider \( f = z^3 + y^3 + z^3 - 3xyz + z^4 \). Note that \( x^3 + y^3 + z^3 - 3xyz = 0 \) consists of three planes \( x + y + z = 0, x + \omega y + \omega^2 z = 0, x + \omega^2 y + \omega z = 0 \) with \( \omega = \exp(2\pi i / 3) \). The dual Newton diagram has a single strictly positive vertex \( P = t^4(1, 1, 1) \) and it is already regular simplicial cone subdivision. The corresponding toric modification is nothing but the ordinary blowing up. Here an ordinary blowing up is a toric modification with one strictly positive vertex \( P = t^4(1, 1, 1) \). It has 3 canonical toric charts. After one blowing up, we work in the toric coordinate chart \( \mathbb{C}^3_{\sigma}, u_\sigma = (u_{\sigma_1}, u_{\sigma_2}, u_{\sigma_3}) \) where \( \sigma = \text{Cone}(P, e_1, e_2), P = t^4(1, 1, 1) \) and \( x = u_{\sigma_1}u_{\sigma_2}, y = u_{\sigma_1}u_{\sigma_3}, z = u_{\sigma_1} \). We have

\[
\tilde{\pi}^* f = u_{\sigma_1}^2 \tilde{f}(u_\sigma),
\]

\[
\tilde{f}(u_\sigma) = (u_{\sigma_2} + u_{\sigma_3} + 1)(u_{\sigma_2} + \omega u_{\sigma_3} + \omega^2)(u_{\sigma_2} + \omega^2 u_{\sigma_3} + \omega) + u_{\sigma_1}
\]

and \( E(P) \) is defined by \( u_{\sigma_1} = 0, \tilde{f}(0, u_{\sigma_2}, u_{\sigma_3}) = 0 \) which consists of three \( \mathbb{P}^1 \) and three singular points are the intersection points of the lines which are \( q_0 := (0, 1, 1), q_1 = (0, \omega, \omega^2) \) and \( q_2 = (0, \omega^2, \omega) \). For example, at \( q_0 \), taking the coordinate \( w_1 = u_{\sigma_1}, w_2 = u_{\sigma_2} - 1, w_3 = u_{\sigma_3} - 1, \tilde{\pi}^* f \) is written as

\[
\tilde{\pi}^* f = u_{\sigma_1}^3(u_{\sigma_3}^3 + u_{\sigma_2}^3 - 3u_{\sigma_3}u_{\sigma_2} + u_{\sigma_1} + 1)
\]

\[
= w_1^3(w_3^3 + w_2^3 + 3w_3^2 - 3w_3w_2 + 3w_2^2 + w_1)
\]

By the symmetry of the equation, the singularities of \( E(P) \) are isomorphic at any \( q_i \). They are \( A_1 \) singularity and \( \mu_{q_i} = 1 \). Thus \( \zeta(s)(t) = (1 - t^3)^{-3}, \zeta^{cr}(t) = (1 - t^3)^3 \) and \( \zeta(t)' = 1 \). \( \zeta(t) \) is given as \((1 - t^4)^{-1} \). Thus we get

**Assertion 9.** \( \zeta(t) = (1 - t^4)^{-3} \) and \( \mu = 11 \).

The second assertion follows from \( -\deg \zeta = 1 + \mu = 12 \).

The above calculation can be generalized for \( f_n = x^3 + y^3 + z^3 - 3xyz + z^n, \ n \geq 4 \).

**Assertion 10.** The zeta function of \( f_n \) is given by \((1 - t^n)^{-3} \) and \( \mu(f_n) = 3n - 1 \).
In fact, after one blowing up, \( \tilde{V} \) has three singularities which are defined (up to isomorphism) by

\[
\hat{\pi}^* f = w_1^3 (u_{w_2}^3 + u_{w_3}^3 - 3u_{w_2}u_{w_3} + w_{1^3}^n + 1) \\
= w_1^3((u_3 + 1)^3 + (w_2 + 1)^3 - 3(w_3 + 1)(w_2 + 1) + 1 + w_1^{n-3}) \\
= w_1^3(3w_3^2 + 3w_2^2 - 3w_3w_2 + w_1^{n-3}).
\]

Example 3. Let \( f_d(x, y, z) \) be an irreducible convenient homogeneous polynomial which defines a projective curve of degree \( d \) with \( k \leq \frac{(n-1)(n-2)}{2} \) nodes. See [9] for an example of a maximal nodal curve. We consider \( f = f_d + x^{d+1} \). As an affine polynomial, \( f_d \) has a single maximal dimensional face with weight vector \( P \), and the strict transform \( \tilde{f} \) is defined by each point, \( f \). We first consider the zeta function of \( f \). To compute the zeta function of \( f \), we have

\[
\hat{\pi}^* f(u_{w_1}, u_{w_2}, u_{w_3}) = u_{w_1}^d (\hat{f}_d(1, u_{w_2}, u_{w_3}) + u_{w_1})
\]

and the strict transform \( \tilde{V} \) is defined by \( \hat{f}_d(1, u_{w_2}, u_{w_3}) + u_{w_1} = 0 \), \( \hat{E}_0 \) is defined by \( u_{w_1} = 0 \) and \( \tilde{V} \cap \hat{E}_0 \) has \( k \) nodal singularities at \( \{q_1, \ldots, q_k\} \). At each point, \( f(1, u_{w_2}, u_{w_3}) \) is written as

\[
\hat{f}_d(1, u_{w_1}, u_{w_2}) = Q(w_2, w_3) + R(w_2, w_3)
\]

where \( w \) is the admissible coordinate at \( q \) and \( Q(u_{w_2}, u_{w_3}) \) is a non-degenerate quadratic form, and we can assume \( Q = w_2^2 + w_3^2 \). The last term \( R(w_2, w_3) \) is a polynomial of degree greater than or equal to 3.

We first consider the zeta function of \( f_d \). Thus \( \hat{\pi}^* f_d(w) \) is equivalent to \( w_1^d(w_2^2 + w_3^2 + R(w_2, w_3)) \) which is non-degenerate and its zeta function is trivial at \( q_i \). Thus by Theorem 8, the zeta function of \( f_d \) is given as \( \zeta_{f_d}(t) = (1 - t^d)^{-d^2+3d-3+k} \). Recall \( \zeta_{f_d}(t) \) is equal to the product \( P_0(t)^{-1}P_1(t)P_2(t)^{-1} \) where \( P_j(t) \) is the characteristic polynomial of the monodromy action \( h_{s_j} : H_j(F; \mathbb{Q}) \to H_j(F; \mathbb{Q}) \) and \( F = f_d^{-1}(1) \) is the Milnor fiber of \( f_d \). By [3] (see also [2]), \( \pi_1(\mathbb{P}^2 \setminus \{f_d = 0\}) = \mathbb{Z}/d\mathbb{Z} \) and the canonical mapping \( p : F \to \mathbb{P}^2 \setminus \{f_d = 0\} \) is \( d \)-cyclic covering. This implies \( F \) is simply connected and thus \( P_1(t) = 1 \) and we get

\[
\zeta_{f_d}(t) = (1 - t^d)^{-d^2+3d-3+k}, \quad P_2(t) = (1 - t^d)^{d^2-3d+3-k}(1 - t)^{-1}.
\]

The polynomial \( f_d \) gives an example of a function which has one dimensional singularities but still the Milnor fiber is 1-connected.

To compute the zeta function of \( f_d \), we need to take one more blowing up at each singular point \( q_i = (\alpha_i, \beta_i) \), \( i = 1, \ldots, k \). We can choose admissible local coordinates \( w_i = (w_{i1}, w_{i2}, w_{i3}) \) with \( w_{i1} = u_{w_1} \) and \( (w_{i2}, w_{i3}) \) is a linear change of \( (u_{w_2} - \alpha_i, u_{w_3} - \beta_i) \) so that

\[
\hat{\pi}^* f = w_{i1}^d (w_{i2}^2 + w_{i3}^2 + w_{i1} + \text{(higher terms)}).
\]
where (higher terms) contains only variables \( w_{i2} \) and \( w_{i3} \). Thus the local zeta function is \( \zeta_q(t) = (1 - t^{d+1})^{-1} \) and we get

\[
\zeta(t) = (1 - t^d)^{-d+3d-3+k(1 - t^{d+1})^{-k}}, \quad \mu(f) = (d - 1)^3 + k.
\]

The last equality is generalized in Theorem 18 in §5.

4. APPLICATION: A ZARISKI PAIR OF LINKS

It is well-known that there exists a pair of projective curves \{C, C'\} of degree 6 (so called a Zariski pair) with 6 cusps whose complements have different topologies (Zariski [16]). The first curve \( C \) is sextic of torus type. A typical one is defined as follows:

\[
C := \{(x, y, z) \in \mathbb{P}^2 \mid f_6(x, y, z) = 0\}
\]

\[
f_6 = f_3^3 + f_2^2, \quad f_2 = x^2 + y^2 + z^2, \quad f_3 = x^3 + y^3 + z^3.
\]

Another curve \( C' \) is a sextic with 6 cusps such that there exists no conic which passes through these 6 points. We use the sextic which is given in [11]. For our purpose, we took the change of coordinates \((x, y) \mapsto (x + 1/2, y + 2)\). This change of coordinates is simply to put the singular locus off the coordinate planes.

\[
C' : \quad g_6(x, y, z) = 0
\]

(6)

\[
g_6 = -\frac{215}{64} z^6 + \frac{51}{16} x z^5 + \frac{63}{16} x^2 z^4 - \frac{3}{2} x^3 z^3 - \frac{9}{4} x^4 z^2
\]

\[+ 3 x^5 z + x^6 - \frac{41}{4} y z^5 + 8 x y z^4 + 10 x^2 y z^3 - 4 x^4 y z - \frac{571}{48} y^2 z^4
\]

\[+ \frac{22}{3} x y^2 z^3 + 27 x^2 y^2 z^2 - x^4 y^2 - \frac{190}{27} y^3 z^3 + \frac{8}{3} x y^3 z^2
\]

\[+ \frac{8}{3} x^2 y^3 z - \frac{85}{36} y^4 z^2 + \frac{x y^4 z}{3} + \frac{x^2 y^4}{3} - \frac{4}{9} y^5 z - \frac{y^6}{27}
\]

To make the singularity of \( f_6^{-1}(0) \) and \( g_6^{-1}(0) \) to be isolated at the origin, we put

\[
f = f_6 + z^7, \quad g = g_6 + z^7
\]

and we consider the corresponding hypersurface \( f^{-1}(0) \) and \( g^{-1}(0) \) at the origin. We will show that they have the same Newton boundary, the same zeta function (thus the same Milnor number too), the same dual resolution graph and thus their links \( M_f \) and \( M_g \) are diffeomorphic where \( M_f = V(f) \cap S^{2n-1}_{\varepsilon} \) and \( M_g = V(g) \cap S^{2n-1}_{\varepsilon} \) and \( \varepsilon \) is small enough. Their tangent cones gives a Zariski pair in \( \mathbb{P}^2 \). We call such a pair of links of hypersurfaces \( V(f), V(g) \) a Zariski pair of links.
4.0.1. Torus type sextic as the tangent cone. We consider first \( V = f^{-1}(0) \).

Consider the polynomials:

\[
\begin{align*}
    f_2 &= x^2 + y^2 + z^2, \\
    f_3 &= x^3 + y^3 + z^3, \\
    f_6 &= f_2^3 + f_3^2, \\
    f &= f_6 + z^7.
\end{align*}
\]

We first take an ordinary blowing up \( \hat{\pi} : X \to \mathbb{C}^3 \) and we take the chart \((U, \mathbf{u}_\sigma)\), \( \mathbf{u}_\sigma = (u_{\sigma 1}, u_{\sigma 2}, u_{\sigma 3}) \) with \( \hat{\pi}(\mathbf{u}_\sigma) = (u_{\sigma 1} u_{\sigma 3}, u_{\sigma 2} u_{\sigma 3}, u_{\sigma 3}) \). Denote the exceptional divisor by \( \hat{E}_0 \) which is diffeomorphic to \( \mathbb{P}^2 \) and it is defined by \( u_{\sigma 3} = 0 \) in \( U \). \( E_0 = \hat{E}_0 \cap \hat{V} \) is the exceptional divisor of the restriction of \( \hat{\pi} \) to \( \hat{V} \), corresponding to the strict transform of \( f_6 = 0 \). More precisely we have:

\[
\begin{align*}
    \hat{\pi}^* f &= u_{\sigma 1}^6 (f_6 + u_{\sigma 3}), \\
    \hat{f}_6 &= f_2^3 + f_3^2, \\
    \hat{f}_2 &= u_{\sigma 1}^2 + u_{\sigma 2}^2 + 1, \\
    \hat{f}_3 &= u_{\sigma 1}^3 + u_{\sigma 2}^3 + 1.
\end{align*}
\]

Let \( \hat{V} \) be the strict transform of \( V \). 6 singular points, say \( \rho_1, \ldots, \rho_6 \), of \( E_0 \) are located at the intersection of curves \( \hat{f}_2 = \hat{f}_3 = 0 \) in \( E_0 \) and they are \( A_2 \) singularities. On each intersection, in \( E_0 \) the curves \( \hat{f}_2 = 0 \) and \( \hat{f}_3 = 0 \) are non-singular and intersect transversely. As \( V(\hat{f}_2), V(\hat{f}_3), E_0 = V(u_{\sigma 3}) \) intersect transversely at \( \rho_i \), we can take \( w_1 = \hat{f}_2, w_2 = \hat{f}_3, w_3 = u_{\sigma 3} \) as analytic coordinates in a small neighborhood \( U_i \) of \( \rho_i \) and \((U_i \cap E_0, (w_1, w_2))\) is a local coordinate chart of \( E_0 \). Then the pull-back is written as

\[
    \hat{\pi}^* f = w_{\sigma 3}^6 (w_1^3 + w_2^2 + w_3).
\]

The dual Newton diagram \( \Gamma^*(\hat{\pi}^* f(\mathbf{w})) \) is as Figure 2 and we take a regular subdivision \( \Sigma^*_\rho_i \) as in Figure 2. The bullets in Figure 2 are the vertices of

\[
\begin{align*}
P &= \ell(2, 3, 6), \\
T &= \ell(1, 2, 3), \\
S &= \ell(1, 1, 2), \\
R &= \ell(1, 1, 1)
\end{align*}
\]

Figure 2. Dual Newton diagram
\( \Gamma^*(\hat{\pi}^*f) \) and by adding three vertices \( S, T, R \), we get a regular simplicial subdivision \( \Sigma_{\rho_i}^* \). We take the associated toric modification \( \hat{\omega}_i : Y_i \to U_i \). It has four exceptional divisors \( \hat{E}(P), \hat{E}(T), \hat{E}(S), \hat{E}(R) \). The strict transform \( \hat{V}_i \) of \( V \) is smooth. The restriction \( \hat{\omega}_i : \hat{V}_i \to V \) is a resolution of \( V \) at \( \rho_i \) and it has three exceptional divisors \( E(P) = \hat{E}(P) \cap \hat{V}_i, E(T) = \hat{E}(T) \cap \hat{V}_i \) and \( E(S) = \hat{E}(S) \cap \hat{V}_i \). On the other hand, \( \hat{E}(R) \cap \hat{V}_i \) is empty as \( \Delta(R) = \left\{ (0, 0, 7) \right\} \). (See for example Proposition (3.7), page 131, \[10\].) \( E(e_1), E(e_2) \) are strict transforms of the conic \( f_2 = 0 \) and the cubic \( f_3 = 0 \). \( E(e_3) \) is (the pull-back of) the exceptional divisor \( E_0 \). To distinguish divisors over other singular points \( \rho_i, 1 \leq i \leq 6 \), we denote them by \( E(P)_i, E(T)_i, E(S)_i \). We do the same toric modification at \( \rho_1, ..., \rho_6 \) and let \( \hat{\omega} : Y \to X \) be the union of these modifications and let \( \hat{\Pi} = \hat{\pi} \circ \hat{\omega} : Y \to X \to \mathbb{C}^n \) the composition of the modifications. The exceptional divisors of \( \hat{\Pi} \) is given as

\[
D := \hat{\Pi}^{-1}(0) \cap \hat{V} = E_0 + \sum_{i=1}^{6} (E(P)_i + E(T)_i + E(S)_i).
\]

Note that \( E(P)_i, E(T)_i, E(S)_i \) are isomorphic to \( \mathbb{P}^1 \). See for example, Lemma (6.4), p.158, \[10\]. By abuse of notation, we denote \( \hat{\omega}^*(E_0) \) by \( E_0 \). As for \( E_0 \), we assert:

**Assertion 11.** \( E_0 \) has genus 4.

**Proof.** The assertion follows from the Euler characteristic calculation, \( \chi(E_0) = -18 + 6 \cdot 2 = -6 \). Here \(-18\) is the Euler characteristic of the smooth sextic and 12 is the defect from 6 cusps. \( \square \)

The link of \( V \) is diffeomorphic to the boundary of the tubular neighborhood of the total exceptional divisor \( D \). To study geometry further, we consider the divisor defined by \( \hat{\Pi}^*f_2 = 0 \) and compute the intersection numbers. We use the property \( (\hat{\Pi}^*f_2) \cdot C = 0 \) for any compact divisor (see for example Theorem 2.6 of \[6\]). We use three toric charts \( \sigma = \text{Cone}(P, T, e_3), \tau = \text{Cone}(P, S, e_3) \) and \( \xi = \text{Cone}(S, e_1, e_3) \) with respective toric coordinates \( (u_{\sigma_1}, u_{\sigma_2}, u_{\sigma_3}), (u_{\tau_1}, u_{\tau_2}, u_{\tau_3}) \) and \( (u_{\xi_1}, u_{\xi_2}, u_{\xi_3}) \). By an easy computation, we get

\[
\hat{\Pi}^*f_2 = u_{\sigma_3}^2(u_{\sigma_1}^2 + u_{\sigma_2}^2 + 1) = u_{\sigma_1}^4 u_{\tau_2} u_{\tau_3}^2 = u_{\tau_1}^4 u_{\tau_2} u_{\tau_3}^2 = u_{\xi_1}^5 u_{\xi_2} u_{\xi_3}^2
\]

Thus we have

\[
(\hat{\Pi}^*f_2) = 2E_0 + \hat{V}_2 + \sum_{i=1}^{6} (14E(P)_i + 7E(T)_i + 5E(S)_i).
\]

Here \( \hat{V}_2 \) is the intersection of the strict transforms of \( \hat{V}(f_2) \cap \hat{V}(f) \). Note that \( \hat{V}_2 \) intersects only with \( E(S)_i, 1 \leq i \leq 6 \). Let \( I_f \) be the 19 \( \times \) 19 intersection matrix of the exceptional divisors. Thus we conclude

\[
(7) \quad E(P)_i^2 = -1, \quad E(T)_i^2 = -2, \quad E(S)_i^2 = -3, \quad E_0^2 = -42, \quad \det I_f = 6.
\]
The total configuration graph is one vertex $E_0$ at the center and 6 branches, $\Gamma_1, \ldots, \Gamma_6$, are is joined to the center divisor $E_0$ through $E(P)_i$ for $i = 1, \ldots, 6$.

$$\Gamma_i : \ E(S)_i \longrightarrow E(P)_i \longrightarrow E(T)_i$$

See Figure 4 for the whole resolution graph.

4.0.2. **Non Torus type sextic as the tangent cone.** We consider $V(g)$ where $g = g_6 + z^7$ and $g_6$ is given in (6). First we take a blowing up at the origin. Using the same coordinates $(u_{\sigma_1}, u_{\sigma_2}, u_{\sigma_3})$,

\[
\hat{\pi}^* g = u_{\sigma_3}^6 (\hat{g}_6(u_{\sigma_1}, u_{\sigma_2}) + u_{\sigma_3})
\]

\[
\hat{g}_6(u_{\sigma_1}, u_{\sigma_2}) = 1728 u_{\sigma_1}^6 + 5184 u_{\sigma_2}^5 + (-1728 u_{\sigma_2}^2 - 6912 u_{\sigma_2} - 3888) u_{\sigma_1}^4
\]

\[-2592 u_{\sigma_2}^3 + (576 u_{\sigma_2}^4 + 4608 u_{\sigma_2}^3 + 13536 u_{\sigma_2}^2 + 17280 u_{\sigma_2} + 6804) u_{\sigma_1}^2
\]

\[+ (576 u_{\sigma_2}^4 + 4608 u_{\sigma_2}^3 + 12672 u_{\sigma_2}^2 + 13824 u_{\sigma_2} + 5508) u_{\sigma_1} - 64 u_{\sigma_2}^6
\]

\[-768 u_{\sigma_2}^5 - 4080 u_{\sigma_2}^4 - 12160 u_{\sigma_2}^3 - 20556 u_{\sigma_2}^2 - 17712 u_{\sigma_2} - 5805
\]

The graph of $\hat{g}_6 = 0$ is given in Figure 3. The singular points are

\[\rho_1 = \left(\frac{1}{2}, -1, 0\right), \rho_2 = \left(\frac{1}{2}, -3, 0\right), \rho_3 = \left(-\frac{1}{2}, -1, 0\right),\]

\[\rho_4 = \left(-\frac{1}{2}, -3, 0\right), \rho_5 = \left(-\frac{1}{2} + \frac{\sqrt{6}}{3}, -\frac{1}{2}, 0\right), \rho_6 = \left(-\frac{1}{2} - \frac{\sqrt{6}}{3}, -\frac{1}{2}, 0\right).
\]

and the tangent cones are vertical for $\{\rho_1, \rho_2\}$ and horizontal for $\{\rho_3, \rho_4, \rho_5, \rho_6\}$. This implies the elementary choice of coordinates are admissible. The resolutions are similar for $\rho_1, \rho_2$ and also the resolutions for $\rho_3, \ldots, \rho_6$ are similar. So we will see two resolutions at $\rho_1$ and $\rho_3$.

We use the pull back of $x$ for the calculation of the intersection numbers. As none of $\rho_i$ is on $\{u_{\sigma_1} = 0\} \subset \C^3_\sigma$ and $\hat{\pi}^* x = u_{\sigma_1} u_{\sigma_3}$, it is enough to compute the pullback of $u_{\sigma_3}$ to compute the divisor $(\hat{\pi}^* x)$.

(1) First we consider the resolution at $\rho_1 = \left(\frac{1}{2}, -1, 0\right)$. Taking the coordinate $w_1 = u_{\sigma_1} - 1/2$, $w_2 = u_{\sigma_2} + 1$, $w_3 = u_{\sigma_3}$, we have

\[
\hat{\pi}^* g = \hat{w}_3^6 (\hat{g}_6(w_1, w_2) + w_3),
\]

\[
\hat{g}_6(w_1, w_2) = -512 w_1^3 + 5184 w_2^2 + \text{(higher terms)},
\]

\[
\hat{\pi}^* x = (w_1 + 1/2) w_3.
\]

Thus the Newton boundary of $\hat{\pi}^* g$ is the same as $\Gamma^* (\hat{\pi}^* f(w))$ in the previous section (i.e., $\Gamma((w_3^6 (-512 w_1^3 + 5184 w_2^2)))$ and we can take the exact same regular simplicial subdivision $\Sigma^*$). Taking the toric modification $\hat{\omega}_1 : Y_1 \to X$ with respect to $\Sigma^*$,

\[
\hat{\omega}_1^* (w_3) = u_{\sigma_1}^3 u_{\sigma_2} u_{\sigma_3}, \quad \text{in } \C^3_\sigma, \quad \sigma = \text{Cone}(P, T, e_3)
\]

\[
= u_{\tau_1}^3 u_{\tau_2}^2 u_{e_3}, \quad \text{in } \C^3_\tau, \quad \tau = \text{Cone}(P, S, e_3)
\]

\[
= u_{\xi_1}^2 u_{\xi_3}, \quad \text{in } \C^3_\xi, \quad \xi = \text{Cone}(S, e_1, e_3).
\]
Thus \( \hat{\omega}_1^* w_3 \) = 6E\( (P) \)1 + 3E\( (T) \)1 + 2E\( (S) \)1 + E0 and

\[(8) \quad (\hat{\omega}_1^* \hat{\pi}^* x) = 6E\( (P) \)1 + 3E\( (T) \)1 + 2E\( (S) \)1 + E0 + (u\( \sigma_1 \) = 0) \text{ in } Y_1.\]

The same equality for \( \hat{\omega}_2 : Y_2 \rightarrow X \).

(2) Consider the resolution at \( \rho_3 = (-\frac{1}{2}, -1, 0) \). Taking the coordinate \( w_1 = u_{\sigma 1} + 1/2, w_2 = u_{\sigma 2} + 1, w_3 = u_{\sigma 3} \), we have

\[
\hat{\pi}^* g = w_3^6(\bar{g}_6 + w_3), \\
\bar{g}_6 = -3456w_1^3 - 2304w_2^2 + \text{(higher terms)}, \\
\hat{\pi}^* u_{\sigma 3} = (w_1 - 1/2)w_3, u_{\sigma 3} = w_3.
\]

We take the toric modification with respect to the same \( \Sigma_{\rho}^*, \hat{\omega}_3 : Y_3 \rightarrow X \) and we get

\[(9) \quad (\hat{\omega}_3^* \hat{\pi}^* x) = 6E\( (P) \)3 + 3E\( (T) \)3 + 2E\( (S) \)3 + E0 + (u\( \sigma_1 \) = 0) \text{ in } Y_3.\]

Note that this is the same equality with \( \text{(8)} \). Taking the same toric modification \( \hat{\omega}_j : Y_j \rightarrow X \) at all \( \rho_j \) and we take the union of these 6 toric modifications to obtain the resolution of all 6 singular points \( \hat{\omega} : Y \rightarrow X \) and let \( \hat{\Pi} : Y \rightarrow \mathbb{C}^3 \) be the composition \( \hat{\pi} \circ \hat{\omega} \). We observe that the configuration of the exceptional divisors of \( V(g) \) by \( \Pi : \tilde{V} \rightarrow V = V(g), \Pi = \hat{\Pi}|_{\tilde{V}} \)
is the same with that of the resolution of $V(f)$. Note that

$$( \Pi^* x ) = E_0 + (u_{\sigma_1} = 0) + \sum_{i=1}^{6} (6E(P)_i + 3E(T)_i + 2E(S)_i),$$

Note also that the intersection number $(u_{\sigma_1} = 0) \cdot E_0$ is 6. This follows from the observation that $\{u_{\sigma_1} = 0\} \cap E_0$ corresponds to the 6 roots of $\tilde{g}(0,\sigma_2) = 0$. Using the above equality we get

$$(10) \quad E(P)_i^2 = -1, \ E(T)_i^2 = -2, \ E(S)_i^2 = -3, \ E_0^2 = -42.$$  

These intersection numbers are the same with that of the resolution of $V(f)$. We can also check the genus of $E_0$ is 4. Let $I_g$ be the $19 \times 19$ intersection matrix of the exceptional divisors. The intersection matrix $I_f$ and $I_g$ are the same. The calculation of $\zeta(t)$ and $\zeta_{\rho_i}(t)$ is also exactly same. Thus we conclude

**Theorem 12.** The functions $f$ and $g$ have the same zeta function:

$$\zeta(t) = (1 - t^6)^{-9}, \quad \zeta_{\rho_i}(t) = \frac{(1 - t^{21})(1 - t^{14})}{(1 - t^{12})(1 - t^7)}$$

and the Milnor number is 137. They have the same resolution graph with 19 exceptional divisors and their self-intersection numbers are given as $E(P)_i^2 = -1, E(T)_i^2 = -2, E(S)_i^2 = -3$ and $E_0^2 = -42$. The determinant of the intersection matrix is $-6$.

Note that $137 = (6 - 1)^3 + 6 \times 2$ and $5^3$ is the Milnor number of non-degenerate homogeneous polynomial of degree 6. This observation is generalized in Theorem 18 in §5. For the calculation of $\zeta(t)$, we use the equalities $\zeta^{(s)}(t) = (1 - t^d)^{-21}$ and $\zeta^{(r)}(t) = (1 - t^6)^{12}$.

**Remark 13.** Consider a $(p, q)$-torus curve $f_{pq} = 0$ ([13]) and its isolation function $h$:

$$f_p := x^p + y^p + z^p, \quad f_q := x^q + y^q + z^q, \quad \gcd(p, q) = 1$$

$$h = f_{pq} + z^{pq+1}, \quad f_{pq} = f_p^q + f_q^p.$$  

The calculation of the zeta function is completely parallel for $f$. We take first a blowing up $\hat{\pi} : X \to \mathbb{C}^3$ and we take the chart $(U, u_{\sigma} = (u_{\sigma_1}, u_{\sigma_2}, u_{\sigma_3}))$ with $\hat{\pi}(u_{\sigma}) = (u_{\sigma_1}u_{\sigma_3}, u_{\sigma_2}u_{\sigma_3}, u_{\sigma_3})$. Denote the exceptional divisor by $E_0$ which is $\mathbb{P}^2$ and it is defined by $u_{\sigma_3} = 0$ in $U$. $E_0 = E_0 \cap V$ is the exceptional divisor of the restriction to $V(h)$. $E_0$ is equal to the projective curve defined by $f_{pq} = 0$ in $\mathbb{P}^2$ and it has $pq$ singularities $\rho_i, i = 1, \ldots, pq$ which are
\[(p, q)\text{-cusps: } x^p + y^q = 0. \] After one ordinary blowing up \( \hat{\pi} : X \to \mathbb{C}^3 \), we have
\[
\hat{\pi}^* h = u^d_{\sigma3} (f_p^p + f_q^q + u_{\sigma3})
\]
\[
\tilde{f}_p = u^p_{\sigma1} + u^p_{\sigma2} + 1, \quad \tilde{f}_q = u^q_{\sigma1} + u^q_{\sigma2} + 1
\]
and the zeta function of \( h \) can be computed in the exact same way as follows.
\[
\zeta'(t) = (1 - t^{pq})^{-pq^2 + 3pq - 3 + pq(p-1)(q-1)},
\]
\[
\zeta_{\rho_i}(t) = \frac{(1 - t^{p(pq+1)})(1 - t^{q(pq+1)})}{(1 - t^{pq(pq+1)})(1 - t^{pq+1})}, \quad i = 1, \ldots, pq
\]
\[
\zeta(t) = \frac{1}{(1 - t^{pq})pq^2 - 3pq + 3pq(p-1)(q-1)} \left( \frac{(1 - t^{pq+1})(1 - t^{q(pq+1)})}{(1 - t^{pq(pq+1)})(1 - t^{pq+1})} \right)^{pq}
\]
and \( \mu(h) = (pq - 1)^3 + pq(p - 1)(q - 1) \).

4.0.3. Computation of \( H_1(M_f) \) and \( H_1(M_g) \). Consider compact Riemann surfaces \( E_1, \ldots, E_\ell \) embedded in a complex manifold of dimension 2 which intersect either transversely at a point or does not intersect and no three \( E_i, E_j, E_k \) intersect. Consider the graph \( \Gamma \) with vertices \( v_1, \ldots, v_\ell \) which correspond to \( E_1, \ldots, E_\ell \). We assume that \( \Gamma \) is a tree graph. Let \( s_{ij} \) be the intersection number \( E_i \cdot E_j \) and let \( s_{ii} \) be the self-intersection number of \( E_i \). Take a small tubular neighborhood \( N(E_i) \) and we assume that they intersect transversely and \( N(E_i) \cap N(E_j) \) is diffeomorphic to \( D^2 \times D^2 \) if \( s_{ij} = 1 \). Put \( N(\Gamma) = \bigcup_{i=1}^\ell N(E_i) \) and put \( M(\Gamma) \) be the boundary of \( N(\Gamma) \).
$M(\Gamma)$ is a 3-manifold with corners. Assume the $E_j$ is $\mathbb{P}^1$ for each $j$. Taking suitable generators $g_j$ of $\pi_1(M(\Gamma))$ represented by a fiber of $\partial N(E_j)$, the fundamental group is generated by $g_1, \ldots, g_\ell$ with the relations

$$[g_i, g_j] = e, \text{ if } s_{ij} = 1 \text{ and } g_1^{s_{ij}} \cdots g_\ell^{s_{ij}} \text{ for } 1 \leq j \leq \ell.$$ 

This is shown in [8]. Now we compute the homology of $M_f$ and $M_g$ identifying them as the graph manifolds. Take the tubular neighborhood $N(E(P)_{i})$, $N(E(T)_{i})$ and $N(E(S)_{i})$ for $1 \leq i \leq 6$ and $N(E_0)$ sufficiently small and let $N = N(E_0) \cup \bigcup_{i=1}^{6} (N(E(P)_{i}) \cup N(E(T)_{i}) \cup N(E(S)_{i}))$. Let $N(\Gamma)'$ be the boundary of the subgraph $\Gamma_i$, cut off $D_i^2 \times S^1$ at the intersection of $N(P_i)$ and $N(E_0)$. Note that $\partial N(\Gamma)' = \partial D_i \times S^1$. We compute $H_1(\partial N(\Gamma)')$ following the recipe of [8, 4]. Take 1-cycles $p_i, t_i, s_i$ represented by the respective fiber of the boundary of the tubular neighborhoods of $\partial N(E(P)_{i})$, $\partial N(E(T)_{i})$, $\partial N(E(S)_{i})$ and $e_0$ the fiber of $N(E_0)$. Then $H_1(N(\Gamma)')$ is generated by $e_0, p_i, t_i, s_i$ which satisfies the relations:

$$(11) \quad p_i - 2t_i = 0, \ p_i - 3s_i = 0, \ e_0 - p_i + t_i + s_i = 0, \ 1 \leq i \leq 6.$$ 

See page 10, [8]. Thus we can solve this as

$$(12) \quad p_i = 6e_0, \ t_i = 3e_0, \ s_i = 2e_0.$$ 

Let $D_1, \ldots, D_6$ be small disks on $E_0$ obtained as the intersection of $E_0$ and $N(E(P)_{i})$, $i = 1, \ldots, 6$. Take two disks $D, D'$ such that $D \subset D'$ and $D' \setminus D$ include 6 small disks $D_1, \ldots, D_6$. The restriction of $N(E_0)$ over $E_0 \setminus D$ is trivial and diffeomorphic to $(E_0 \setminus \text{Int}(D)) \times S^1$ and under the gluing with $D \times S^1$, $\partial D \times \{\ast\}$ is homologous to $-42e_0$ where $e_0$ is represented by $\{\ast\} \times S^1 \subset (E_0 \setminus \text{Int}(D)) \times S^1$. Take the generators $a_1, b_1, \ldots, a_4, b_4$ of $\pi_1(E_0)$ in $E_0 \setminus D'$ so that $\partial D' - 1 = [a_1, b_1] \cdots [a_4, b_4]$. As the boundary of the region $D' \setminus (D \cup \bigcup_{i=1}^{6} D_i)$ is $\partial D' - \partial D - \sum_{i=1}^{6} \partial D_i$, we get the relation:

$$(R_0) \quad -(p_1 + \cdots + p_6) - \partial D = 0 \text{ i.e. } 6e_0 = 0 \text{ by (12)},$$ 

where the boundary of disks are oriented counterclockwise and the homology class of $[a_1, b_1] \cdots [a_4, b_4]$ is zero. See Figure 5. The relation $(R_0)$ is the same as the assertion on p. 10, in [8].

Thus we have the following:

**Theorem 14.** The links of the surfaces $V(f)$ and $V(g)$ are diffeomorphic and their first homology group is given as:

$$H_1(M_f) = H_1(M_g) = \mathbb{Z}^8 \oplus \mathbb{Z}/6\mathbb{Z}.$$ 

The order of the torsion parts come from the absolute values of the determinant of the intersection matrix.

**Problem 15.** The homeomorphism of $M_f$ and $M_g$ comes from the graph manifold structure. Does this homeomorphism extend to a homeomorphism of the sphere $S^5$?
5. Sift formula for Milnor number

5.1. Non-degenerate polynomial. Let \( h(y) = \sum_{i=1}^{k} a_i y^{\nu_i} \) be a polynomial of \( m \) variables \( y_1, \ldots, y_m \). The Newton polygon \( \Delta(h) \) is defined by the convex hull of \( \{\nu_1, \ldots, \nu_k\} \) in \( \mathbb{R}^m \). We assume here that \( a_i \neq 0 \) for any \( 1 \leq i \leq k \). For a face \( \Xi \subset \Delta(h) \) (\( \Xi \) can be \( \Delta(h) \) itself), \( h_{\Xi}(y) \) is defined by the sum \( \sum_{\nu \in \Xi} a_{\nu} y^{\nu} \). \( h \) is called Newton non-degenerate as a polynomial if \( h_{\Xi} : \mathbb{C}^m \to \mathbb{C} \) has no critical points for any \( \Xi \subset \Delta(h) \). If \( f(z) = \sum_{\nu} a_{\nu} z^{\nu} \) is a Newton non-degenerate function germ, any face function \( f_{\Xi}(z) \) for a strictly positive weight vector is non-degenerate as a polynomial.

Lemma 16 (Kouchnirenko [5], Oka [14]). Assume that \( h(y) \) is Newton non-degenerate as a polynomial and let \( V(h)^* = \{y \in \mathbb{C}^m \mid h(y) = 0\} \). Then the Euler characteristic is given as
\[
\chi(V(h)^*) = (-1)^{m-1} m! \text{Vol}_m \Delta(h).
\]
In particular, if \( \dim \Delta(h) < m \), \( \chi(V(h)^*) = 0 \).

Corollary 17. Let \( h(y) \) be a non-degenerate polynomial and consider a polynomial of \( m + 1 \) variables \( \hat{h}(y,w) := h(y) + w \). Then \( \hat{h} \) is also non-degenerate and the following equality holds.
\[
m! \text{Vol}_m(\Delta(h)) = (m + 1)! \text{Vol}_{m+1}(\Delta(\hat{h})) \quad \text{and} \quad \chi(V(\hat{h})^*) = -\chi(V(h)^*).
\]

5.2. Shift of Milnor number. Consider a convenient homogeneous polynomial \( f_d(z) \) of degree \( d \) which defines a projective hypersurface \( V \) with \( s \) isolated singular points \( \rho_1, \ldots, \rho_s \). We assume the singular points are in...
the projective chart \( \{ z_n \neq 0 \} \). We assume that for each \( \rho_i \), there exists a local coordinates \((U_i, w)\) so that the local equation of \( V \cap U_{\rho_i} \) is a Newton non-degenerate function \( f_i(w) \). Put \( \mu_i \) the Milnor number of \( f_i \) at \( \rho_i \) and put \( \mu_{\text{tot}} := \sum_{i=1}^{s} \mu_i \). Consider the modified function \( f(z) = f_d(z) + z_{d+1} \) which makes \( V(f) \) has an isolated singularity at the origin. Then we have:

**Theorem 18** (Shift formula of Milnor number). Assume that \( f_d(z) \) be as above. Then \( f(z) = f_d(z) + z_{d+1} \) is an almost Newton non-degenerate function and the Milnor number \( \mu(f) \) is given as \( \mu(f) = (d-1)^n + \mu_{\text{tot}} \).

**Proof.** Note that \((d-1)^n \) is the Milnor number of a homogeneous polynomial of degree \( d \). The zeta function \( \zeta(t) \) of \( f \) is given by Theorem 8 as

\[
\zeta(t) = \zeta_d(t)(1 - t^d)^{-1} \mu_{\text{tot}} \prod_{i=1}^{s} \zeta_{\rho_i}(t) \tag{13}
\]

where \( \zeta_d(t) \) is the zeta function of a convenient homogeneous polynomial of degree \( d \) with an isolated singularity at the origin of \( \mathbb{C}^n \) and \( \zeta_{\rho_i}(t) \) is the zeta function of the \( \hat{\pi}^*f \) at \( \rho_i \). First we take an ordinary blowing up \( \hat{\pi} : X \to \mathbb{C}^n \) at the origin and take the chart \( U_n \) with coordinates \((u_1, \ldots, u_n)\) so that \( \pi(u) = z \) with \( z_i = u_1u_n, i \leq n-1 \) and \( z_n = u_n \). Put \( u' = (u_1, \ldots, u_{n-1}) \). The exceptional divisor is defined by \( u_n = 0 \). Consider the pull-back of the functions:

\[
\hat{\pi}^*f_d(u) = u_n f_d(u'), \quad \hat{\pi}^*f(u) = u_n f(u'), \quad \hat{\pi}^*f(u) := \hat{f_d}(u') + u_n. \tag{14, 15}
\]

For simplicity, we use the notations \( \hat{f_d}(u) \) and \( \hat{f}(u) \) for \( \hat{\pi}^*f_d(u) \) and \( \hat{\pi}^*f(u) \). Note that \( u' \) can be considered as the projective coordinates of \( \{ u_n \neq 0 \} \cap \mathbb{P}^{n-1} \) and \( \hat{f_d}(u') \) is the defining polynomial of \( V(\hat{f_d}) \cap \mathbb{P}^{n-1} \) and \( \hat{f}(u) \) is the defining polynomial of the strict transform of \( V(f) \). Take a singular point \( \rho_i \) and choose a local coordinates \( w' = (w_1, \ldots, w_{n-1}) \) of the exceptional divisor \( \{ u_n = 0 \} \cong \mathbb{C}^{n-1} \) at \( \rho_i \) so that \( \hat{f_d}(w') \) is non-degenerate with respect to this coordinates. The zeta function \( \zeta_i(t) \) of \( \hat{f_d}(w') \) at \( \rho_i \) is given by Varchenko formula as

\[
\zeta_i(t) = \prod_I \zeta_I(t)
\]

\[
\zeta_I(t) = \prod_{Q \in \pi_I(\hat{f_d})} (1 - t^{d(Q, \hat{f_d})} - \chi(Q)),
\]

\[
\chi(Q) = (-1)^{|I|}|I|! \text{Vol}_I(\text{Cone}(\Delta(Q))/d(Q, \hat{f_d})
\]

where \( \text{Vol}_I(\text{Cone}(\Delta(Q))) \) is the volume of the cone of the linear subspace defined by \( Q \).
where \( \mathcal{P}_I(\vec{f}_d) \) is the primitive weight vectors corresponding to the maximal faces of \( \Gamma(\vec{f}_d^I(w')) \) with \( I \subset \{1, \ldots, n-1\} \). As a corollary, we have

\[
(16) -1 + (-1)^{n-1} \mu_i = \deg \zeta_i(t) = \sum_I \sum_{Q \in \mathcal{P}_I(\vec{f}_d)} (-1)^{|I|!|I!\text{Vol}_I|\text{Cone}(\Delta(Q, \vec{f}_d^I))} = \sum_I \sum_{Q \in \mathcal{P}_I(\vec{f}_d)} -d(Q, \vec{f}_d^I) \chi(Q).
\]

Similarly we define \( \mathcal{P}_I(\hat{f}) \) the set of weight vectors corresponding to maximal faces of \( \Gamma(\hat{f}^I(w)) \) where \( w = (w', w_n) \) and \( w_n = u_n \). Observe that the dual Newton diagram \( \Gamma^*(\hat{f}(w)) \) is equal to the dual Newton diagram of the reduced function \( \hat{f}(w) \), as \( \hat{f} \) is pseudo-convenient. Also observe that \( \hat{f}^I \) is not identically zero if and only if \( n \in I \). It is clear that \( \hat{f}(w) \) is non-degenerate with respect to this coordinates as \( \hat{f}_d(w') \) is non-degenerate by the assumption. Thus \( f(z) \) is an almost non-degenerate function. \( \hat{f}(w) \) is also pseudo-convenient. Thus we can resolve the singularity by a toric modification which is biholomorphic outside of the origin. This shows, in particular, \( f \) has an isolated singularity at the origin. For a strictly positive vertex \( P = \{p_1, \ldots, p_{n-1}\} \in \mathcal{P}(\vec{f}_d(w')) \) of \( \Gamma^*(\vec{f}_d) \), we put \( \hat{P} = \{p_1, \ldots, p_{n-1}, p_n\} \) with \( p_n = d(P, f_d) \). For \( I \subset \{1, \ldots, n-1\} \) and we put \( \hat{I} = I \cup \{n\} \). \( P \) is a weight vector of \( w \) and by the definition of \( \hat{P} \), \( \hat{f}(w') \) is non-degenerate by the assumption. Thus \( f(z) \) is an almost non-degenerate function. \( \hat{f}(w) \) is also pseudo-convenient. Thus we can resolve the singularity by a toric modification which is biholomorphic outside of the origin. This shows, in particular, \( f \) has an isolated singularity at the origin.

\[
(17) \quad \hat{f}_Q^I(w) = (\vec{f}_d^I(w') + w_n)w_n^d, \quad d(\hat{Q}, \hat{f}^I) = d(Q, \vec{f}_d^I)(1 + d).
\]

The zeta function \( \zeta_{\rho_i}(t) \) of \( \hat{f}(w) \) at \( \rho_i \) is given as

\[
(18) \quad \zeta_{\rho_i}(t) = (1 - t^{d+1})^{-1} \prod_{Q \in \mathcal{P}_I(\vec{f}_d)} (1 - t^{d(\hat{Q}, \hat{f}^I)}) - \chi(\hat{Q}).
\]

Here the term \( (1 - t^{d+1})^{-1} \) comes from \( \{n\} \), the single monomial \( w_n^{d+1} \) which is the only case, not corresponding to any factor of \( \zeta_i(t) \). We assert:

**Assertion 19.** \( \chi(Q) = -\chi(\hat{Q}) \) for \( Q \in \mathcal{P}_I(\vec{f}_d) \).

As the argument is completely parallel for any \( I \), we assume that \( I = \{1, \ldots, n-1\} \). We consider a regular simplicial cone subdivision \( \Sigma^*_I \) of \( \Gamma^*(\vec{f}_d) \) and put \( \Sigma^*_i \) the join of \( \Sigma^*_I \) and \( e_n \). It gives an admissible regular simplicial subdivision of \( \Gamma^*(\hat{f}) \). Take the corresponding toric modification \( \omega : X \to \mathbb{C}^{n-1} \) and \( \hat{\omega} : Y \to \mathbb{C}^n \) and take a maximal simplex \( \sigma = \text{Cone}(P_1, \ldots, P_{n-1}) \) such that \( Q = P_1 \) and put \( \hat{\sigma} = \text{Cone}(P_1, \ldots, P_{n-1}, e_n) \). In the coordinate chart \( \mathbb{C}^{n-1}_{\sigma} \) and \( \mathbb{C}^n_{\hat{\sigma}} \) with coordinates \( u^I = (u_{\sigma, 1}, \ldots, u_{\sigma, n-1}) \) and
\( \mathbf{u}_\sigma = (\mathbf{u}'_\sigma, u_{\sigma n}) \), we have

\[
\omega^* \bar{f}_dQ(\mathbf{u}'_\sigma) = \left( \prod_{i=1}^{n-1} u_{\sigma i}^d(P_i, \bar{f}_d) \right) \bar{f}_dQ(\mathbf{u}'_\sigma), \quad \mathbf{u}'_\sigma = (u_{\sigma 2}, \ldots, u_{\sigma n-1}),
\]

\[
\hat{\omega}^* \hat{f}_Q(\mathbf{u}_\sigma) = \left( \prod_{i=1}^{n-1} u_{\sigma i}^d(P_i, \hat{f}_d) \right) u_{\sigma n}^d(\bar{f}_dQ(\mathbf{u}'_\sigma) + u_{\sigma n}).
\]

Here \( \bar{f}_dQ(\mathbf{u}'_\sigma) \) and \( \bar{f}_dQ(\mathbf{u}'_\sigma) + u_{\sigma n} \) are the defining polynomials of \( E(Q) \) and \( E(\hat{Q}) \). The polynomial \( \bar{f}_dQ \) is non-degenerate as a polynomial by the non-degeneracy assumption of \( \bar{f}_d(w') \). Thus we have

\[
\chi(Q) = (-1)^{n-2}(n-1)! \text{Vol}_{n-1} \text{Cone}(\Delta(Q, \bar{f}_d))/d(Q, \bar{f}_d) = (-1)^{n-2}(n-2)! \text{Vol}_{n-2} \Delta(\bar{f}_dQ(\mathbf{u}'_\sigma)),
\]

\[
\chi(\hat{Q}) = (-1)^{n-1}n! \text{Vol}_{\hat{Q}} \text{Cone}(\Delta(\hat{Q}, \hat{f}))/d(\hat{Q}, \hat{f}) = (-1)^{n-1}(n-1)! \text{Vol}_{n-1} \Delta(\bar{f}_dQ(\mathbf{u}'_\sigma) + u_{\sigma n}) = -\chi(Q) \text{ by Corollary [17]}
\]

This proves the Assertion [19]. Now we are ready to show the assertion for Milnor number. By the above argument, we have

\[
-1 + (-1)^{n-1} \mu_i = \sum I \sum_{Q \in \mathcal{P}_I} -d(Q, \bar{f}_d)\chi(Q),
\]

\[
\deg \zeta_{\rho_i}(t) = -(1 + d) + \sum I \sum_{Q \in \mathcal{P}_I} -d(Q, \bar{f}_d)(1 + d)\chi(\hat{Q}) = -(1 + d) + \sum I \sum_{Q \in \mathcal{P}_I} d(Q, \bar{f}_d)(1 + d)\chi(Q) = -(1 + d) - (d + 1)(-1 + (-1)^{n-1} \mu_i) = (-1)^n(1 + d)\mu_i.
\]

Thus we get

\[
-1 + (-1)^n \mu(f) = \deg \zeta(t) = -1 + (-1)^n \mu_d + (-1)^{n-1} d \mu_{\text{tot}} + \sum_{i=1}^{s} \deg \zeta_{\rho_i}(t) = -1 + (-1)^n(\mu_d + \mu_{\text{tot}}).
\]

This completes the proof of Theorem [16].

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