Abstract

GANs excel at learning high dimensional distributions, but they can update generator parameters in directions that do not correspond to the steepest descent direction of the objective. Prominent examples of problematic update directions include those used in both Goodfellow’s original GAN and the WGAN-GP. To formally describe an optimal update direction, we introduce a theoretical framework which allows the derivation of requirements on both the divergence and corresponding method for determining an update direction, with these requirements guaranteeing unbiased mini-batch updates in the direction of steepest descent.

We propose a novel divergence which approximates the Wasserstein distance while regularizing the critic’s first order information. Together with an accompanying update direction, this divergence fulfills the requirements for unbiased steepest descent updates. We verify our method, the First Order GAN, with image generation on CelebA, LSUN and CIFAR-10 and set a new state of the art on the One Billion Word language generation task. Code to reproduce experiments is available.

1. Introduction

Generative adversarial networks (GANs) (Goodfellow et al., 2014) excel at learning generative models of complex distributions, such as images (Radford et al., 2016; Ledig et al., 2017), textures (Jetchev et al., 2016; Bergmann et al., 2017; Jetchev et al., 2017), and even texts (Gulrajani et al., 2017; Heusel et al., 2017).

GANs learn a generative model \( G \) that maps samples from multivariate random noise into a high dimensional space. The goal of GAN training is to update \( G \) such that the generative model approximates a target probability distribution. In order to determine how close the generated and target distributions are, a class of divergences, the so-called adversarial divergences was defined and explored by (Liu et al., 2017). This class is broad enough to encompass most popular GAN methods such as the original GAN (Goodfellow et al., 2014), \( f \)-GANs (Nowozin et al., 2016), moment matching networks (Li et al., 2015), Wasserstein GANs (Arjovsky et al., 2017) and the tractable version thereof, the WGAN-GP (Gulrajani et al., 2017).

GANs learn a generative model with distribution \( Q \) by minimizing an objective function \( \tau(\mathbb{P} || \mathbb{Q}) \) measuring the similarity between target and generated distributions \( \mathbb{P} \) and \( \mathbb{Q} \). In most GAN settings, the objective function to be minimized is an adversarial divergence (Liu et al., 2017), where a critic function is learned that distinguishes between target and generated data. For example, in the classic GAN (Goodfellow et al., 2014) the critic \( f \) classifies data as real or generated, and the generator \( G \) is encouraged to generate samples that \( f \) will classify as real.

Unfortunately in GAN training, the generated distribution often fails to converge to the target distribution. Many popular GAN methods are unsuccessful with toy examples, for example failing to generate all modes of a mixture of Gaussians (Srivastava et al., 2017; Metz et al., 2017) or failing to learn the distribution of data on a one-dimensional line in a high dimensional space (Fedus et al., 2017). In these situations, updates to the generator don’t significantly reduce the divergence between generated and target distributions; if there always was a significant reduction in the divergence then the generated distribution would converge to the target.

The key to successful neural network training lies in the ability to efficiently obtain unbiased estimates of the gradients of a network’s parameters with respect to some loss. With GANs, this idea can be applied to the generative setting. There, the generator \( G \) is parameterized by some values \( \theta \in \mathbb{R}^m \). If an unbiased estimate of the gradient of the divergence between target and generated distributions with respect to \( \theta \) can be obtained during mini-batch learning, then SGD can be applied to learn \( G \).

In GAN learning, intuition would dictate updating the generated distribution by moving \( \theta \) in the direction of steepest descent \(- \nabla_\theta \tau(\mathbb{P} || \mathbb{Q}_\theta)\). Unfortunately, \(- \nabla_\theta \tau(\mathbb{P} || \mathbb{Q}_\theta)\) is generally intractable, therefore \( \theta \) is updated according to...
a tractable method; in most cases a critic \( f \) is learned and the gradient of the expected critic value \( \nabla_{\theta} \mathbb{E}_Q q_\theta [f] \) is used as the update direction for \( \theta \). Usually, this update direction and the direction of steepest descent \( -\nabla_{\theta} \mathbb{E}_Q (\mathcal{P} || \mathcal{Q}_\theta) \), don’t coincide and therefore learning isn’t optimal. As we see later, popular methods such as WGAN-GP (Gulrajani et al., 2017) are affected by this issue.

Therefore we set out to answer a simple but fundamental question: Is there an adversarial divergence and corresponding method that produces unbiased estimates of the direction of steepest descent in a mini-batch setting?

In this paper, under reasonable assumptions, we identify a path to such an adversarial divergence and accompanying update method. Similar to the WGAN-GP, this divergence also penalizes a critic’s gradients, and thereby ensures that the critic’s first order information can be used directly to obtain an update direction in the direction of steepest descent.

This program places four requirements on the adversarial divergence and the accompanying update rule for calculating the update direction that haven’t to the best of our knowledge been formulated together. This paper will give rigorous definitions of these requirements, but for now we suffice with intuitive and informal definitions:

A. the divergence used must decrease as the target and generated distributions approach each other. For example, if we define the trivial distance between two probability distribution to be 0 if the distributions are equal, and 1 otherwise, i.e.
\[
\tau_{\text{trivial}}(\mathcal{P} || \mathcal{Q}) := \begin{cases} 0 & \mathcal{P} = \mathcal{Q} \\ 1 & \text{otherwise} \end{cases}
\]
then even as \( Q \) gets close to \( P \), \( \tau_{\text{trivial}}(\mathcal{P} || \mathcal{Q}) \) doesn’t change. Without this requirement, \( \nabla_{\theta} \tau(\mathcal{P} || \mathcal{Q}_\theta) = 0 \) and every direction is a “direction of steepest descent,”

B. critic learning must be tractable,

C. the gradient \( \nabla_{\theta} \tau(\mathcal{P} || \mathcal{Q}_\theta) \) and the result of an update rule must be well defined,

D. the optimal critic enables an update which is an estimate of \( -\nabla_{\theta} \tau(\mathcal{P} || \mathcal{Q}_\theta) \).

In order to formalize these requirements, we define in Section 2 the notions of adversarial divergences and optimal critics. In Section 3 we will apply the adversarial divergence paradigm and begin to formalize the requirements above and better understand existing GAN methods. The last requirement is defined precisely in Section 4 where we explore criteria for an update rule guaranteeing a low variance unbiased estimate of the true gradient \( -\nabla_{\theta} \tau(\mathcal{P} || \mathcal{Q}_\theta) \).

After stating these conditions, we devote Section 5 to defining a divergence, the Penalized Wasserstein Divergence that fulfills the first two basic requirements. In this setting, a critic is learned, that similarly to the WGAN-GP critic, pushes real and generated data as far apart as possible while being penalized if the critic violates a Lipschitz condition.

As we will discover, an optimal critic for the Penalized Wasserstein Divergence between two distributions need not be unique. In fact, this divergence only specifies the values that the optimal critic assumes on the supports of generated and target distributions. Therefore, for many distributions, multiple critics with different gradients on the support of the generated distribution can all be optimal.

We apply this insight in Section 6 and add a gradient penalty to define the First Order Penalized Wasserstein Divergence. This divergence enforces not just correct values for the critic, but also ensures that the critic’s gradient, its first order information, assumes values that allow for an easy formulation of an update rule. Together, this divergence and update rule fulfill all four requirements.

We hope that this gradient penalty trick will be applied to other popular GAN methods and ensure that they too return better generator updates. Indeed, (Fedus et al., 2017) improves existing GAN methods by adding a gradient penalty.

Finally in Section 7, the effectiveness of our method is demonstrated by generating images and texts.

### 2. Notation, Definitions and Assumptions

In (Liu et al., 2017) an adversarial divergence is defined:

**Definition 1** (Adversarial Divergence). Let \( X \) be a topological space, \( C(X^2) \) the set of all continuous real valued functions over the Cartesian product of \( X \) with itself and set \( \mathcal{G} \subseteq C(X^2) \), \( \mathcal{G} \neq \emptyset \). An adversarial divergence \( \tau(\cdot || \cdot) \) over \( X \) is a function
\[
\mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{+\infty\} \\
(\mathcal{P}, \mathcal{Q}) \mapsto \tau(\mathcal{P} || \mathcal{Q}) = \sup_{g \in \mathcal{G}} \mathbb{E}_{\mathcal{P} \otimes \mathcal{Q}}[g].
\]

The function class \( \mathcal{G} \subseteq C(X^2) \) must be carefully selected if \( \tau(\cdot || \cdot) \) is to be reasonable. For example, if \( \mathcal{G} = C(X^2) \) then the divergence between two Dirac distributions \( \tau(\delta_0 || \delta_1) = \infty \), and if \( \mathcal{G} = \{0\} \), i.e. \( \mathcal{G} \) contains only the constant function which assumes zero everywhere, then \( \tau(\cdot || \cdot) = 0 \).

Many existing GAN procedures can be formulated as an adversarial divergence. For example, setting
\[
\mathcal{G} = \{x, y \mapsto \log(u(x)) + \log(1 - u(y)) \mid u \in \mathcal{V}\} \\
\mathcal{V} = (0, 1)^X \cap C(X)^1
\]
\(1(0, 1)^X \) denotes all functions mapping \( X \) to \((0, 1)\).
results in $\tau_G(\mathbb{P}\|\mathbb{Q}) = \sup_{f \in \mathcal{F}} \mathbb{E}_{\mathbb{P}\otimes\mathbb{Q}}[g]$, the divergence in Goodfellow’s original GAN (Goodfellow et al., 2014). See (Liu et al., 2017) for further examples.

For convenience, we’ll restrict ourselves to analyzing a special case of the adversarial divergence (similar to Theorem 4 of (Liu et al., 2017)), and use the notation:

**Definition 2 (Critic Based Adversarial Divergence).** Let $X$ be a topological space, $\mathcal{F} \subseteq C(X)$, $\mathcal{F} \neq \emptyset$. Further let $f \in \mathcal{F}$, $m_f : X \times X \to \mathbb{R}$, $m_f : (x,y) \mapsto m_1(f(x)) - m_2(f(y))$ and $r_f \in C(X^2)$. Then define
\[
\tau : \mathcal{P}(X) \times \mathcal{P}(X) \times \mathcal{F} \to \mathbb{R} \cup \{+\infty\}
\]
\[
(\mathbb{P}, \mathbb{Q}, f) \mapsto \tau(\mathbb{P}\|\mathbb{Q}; f) = \mathbb{E}_{\mathbb{P}\otimes\mathbb{Q}}[m_f - r_f]
\]
and set $\tau(\mathbb{P}\|\mathbb{Q}) = \sup_{f \in \mathcal{F}} \tau(\mathbb{P}\|\mathbb{Q}; f)$.

For example, the $\tau_G$ from above can be equivalently defined by setting $\mathcal{F} = (0,1)^X \cap C(X)$, $m_1(x) = \log(x)$, $m_2(x) = \log(1-x)$ and $r_f = 0$. Then
\[
\tau_G(\mathbb{P}\|\mathbb{Q}) = \sup_{f \in \mathcal{F}} \mathbb{E}_{\mathbb{P}}[m_f - r_f]
\]
is a critic based adversarial divergence.

An example with a non-zero $r_f$ is the WGAN-GP (Gulrajani et al., 2017), which is a critic based adversarial divergence when $\mathcal{F} = C^1(X)$, the set of all differentiable real functions on $X$, $m_1(x) = m_2(x) = x, \lambda > 0$ and
\[
r_f(x, y) = \lambda \mathbb{E}_{\alpha \sim U([0,1])}[\|\nabla_x f(z)\|_{1+\alpha} - 1]^2.
\]
Then the WGAN-GP divergence $\tau_I(\mathbb{P}\|\mathbb{Q})$ is:
\[
\tau_I(\mathbb{P}\|\mathbb{Q}) = \sup_{f \in \mathcal{F}} \tau_I(\mathbb{P}\|\mathbb{Q}; f) = \sup_{f \in \mathcal{F}} \mathbb{E}_{\mathbb{P}\otimes\mathbb{Q}}[m_f - r_f].
\]

While Definition 1 is more general, Definition 2 is more in line with most GAN models. In most GAN settings, a critic in the simpler $C(X)$ space is learned that separates real and generated data while reducing some penalty term $r_f$ which depends on both real and generated data. For this reason, we use exclusively the notation from Definition 2.

One desirable property of an adversarial divergence is that $\tau(\mathbb{P}^*\|\mathbb{P})$ obtains its infimum if and only if $\mathbb{P}^* = \mathbb{P}$, leading to the following definition adapted from (Liu et al., 2017):

**Definition 3 (Strict adversarial divergence).** Let $\tau$ be an adversarial divergence over a topological space $X$. $\tau$ is called a strict adversarial divergence if for any $\mathbb{P}, \mathbb{P}^* \in \mathcal{P}(X)$,
\[
\tau(\mathbb{P}^*\|\mathbb{P}) = \inf_{\mathbb{P} \in \mathcal{P}(X)} \tau(\mathbb{P}^*\|\mathbb{P}') \Rightarrow \mathbb{P}^* = \mathbb{P}
\]
In order to analyze GANs that minimize a critic based adversarial divergence, we introduce the set of optimal critics.

**Definition 4 (Optimal Critic, $\text{OC}_\tau(\mathbb{P}\|\mathbb{Q})$).** Let $\tau$ be a critic based adversarial divergence over a topological space $X$ and $\mathbb{P}, \mathbb{Q} \in \mathcal{P}(X)$, $\mathcal{F} \subseteq C(X)$, $\mathcal{F} \neq \emptyset$. Define $\text{OC}_\tau(\mathbb{P}\|\mathbb{Q})$ to be the set of critics in $\mathcal{F}$ that maximize $\tau(\mathbb{P}\|\mathbb{Q})$. That is
\[
\text{OC}_\tau(\mathbb{P}\|\mathbb{Q}) := \{ f \in \mathcal{F} | \tau(\mathbb{P}\|\mathbb{Q}; f) = \tau(\mathbb{P}\|\mathbb{Q}) \}.
\]

Note that $\text{OC}_\tau(\mathbb{P}\|\mathbb{Q}) = \emptyset$ is possible, (Arjovsky & Bottou, 2017). In this paper, we will always assume that if $\text{OC}_\tau(\mathbb{P}\|\mathbb{Q}) \neq \emptyset$, then an optimal critic $f^* \in \text{OC}_\tau(\mathbb{P}\|\mathbb{Q})$ is known. Although is an unrealistic assumption, see (Bińkowski et al., 2018), it is a good starting point for a rigorous GAN analysis. We hope further works can extend our insights to more realistic cases of approximate critics.

Finally, we assume that generated data is distributed according to a probability distribution $\mathbb{Q}_{\theta}$ parameterized by $\theta \in \Theta \subseteq \mathbb{R}^m$ satisfying the mild regularity Assumption 1. Furthermore, we assume that $\mathbb{P}$ and $\mathbb{Q}$ both have compact and disjoint support in Assumption 2. Although we conjecture that weaker assumptions can be made, we decide for the stronger assumptions to simplify the proofs.

**Assumption 1 (Adapted from (Arjovsky et al., 2017)).** Let $\Theta \subseteq \mathbb{R}^m$. We say $\mathbb{Q}_{\theta} \in \mathcal{P}(X)$, $\theta \in \Theta$ satisfies Assumption 1 if there is a locally Lipschitz function $g : \Theta \times \mathbb{R}^d \to X$ which is differentiable in the first argument and a distribution $\mathbb{Z}$ with bounded support in $\mathbb{R}^d$ such that for all $\theta \in \Theta$ it holds $\mathbb{Q}_{\theta} \sim g(\theta, z)$ where $z \sim \mathbb{Z}$.

**Assumption 2 (Compact and Disjoint Distributions).** Using $\Theta \subseteq \mathbb{R}^m$ from Assumption 1, we say that $\mathbb{P}$ and $(\mathbb{Q}_{\theta})_{\theta \in \Theta}$ satisfies Assumption 2 if for all $\theta \in \Theta$ it holds that the supports of $\mathbb{P}$ and $\mathbb{Q}_{\theta}$ are compact and disjoint.

3. Requirements Derived From Related Work

With the concept of an Adversarial Divergence now formally defined, we can investigate existing GAN methods from an Adversarial Divergence minimization standpoint. During the last few years, weaknesses in existing GAN frameworks have been highlighted and new frameworks have been proposed to mitigate or eliminate these weaknesses. In this section we’ll trace this history and formalize requirements for adversarial divergences and optimal updates.

Although using two competing neural networks for unsupervised learning isn’t a new concept (Schmidhuber, 1992), recent interest in the field started when (Goodfellow et al., 2014) generated images with the divergence $\tau_G$ defined in Eq. 2. However, (Arjovsky & Bottou, 2017) shows if $\mathbb{P}, \mathbb{Q}_{\theta}$ have compact disjoint support then $\nabla_{\theta} \tau_G(\mathbb{P}\|\mathbb{Q}_{\theta}) = 0$, preventing the use of gradient based learning methods.

In response to this impediment, the Wasserstein GAN was proposed in (Arjovsky et al., 2017) with the divergence:
\[
\tau_W(\mathbb{P}\|\mathbb{Q}) = \sup_{\|f\|_{L^1} \leq 1} \mathbb{E}_{x \sim \mathbb{P}}[f(x)] - \mathbb{E}_{x' \sim \mathbb{Q}}[f(x')]
\]
where \( \|f\|_L \) is the Lipschitz constant of \( f \). The following example shows the advantage of \( \tau_W \). Consider a series of Dirac measures \((\delta_\frac{1}{n})_{n>0}\). Then \( \tau_W(\delta_0|\delta_\frac{1}{n}) = \frac{1}{n} \) while \( \tau_G(\delta_0|\delta_\frac{1}{n}) = 1 \). As \( \delta_\frac{1}{n} \) approaches \( \delta_0 \), the Wasserstein divergence decreases while \( \tau_G(\delta_0|\delta_\frac{1}{n}) \) remains constant. This issue is explored in (Liu et al., 2017) by creating a weak ordering, the so-called, strength of divergences. A divergence \( \tau_1 \) is said to be stronger than \( \tau_2 \) if for any sequence of probability measures \((P_n)_{n \in \mathbb{N}}\) and any target probability measure \( P \), the convergence \( \tau_1(P^*\|P_n) \xrightarrow{n \to \infty} \inf_{P \in \mathcal{P}(X)} \tau_1(P^*\|P) \) implies \( \tau_2(P^*\|P_n) \xrightarrow{n \to \infty} \inf_{P \in \mathcal{P}(X)} \tau_2(P^*\|P) \). The divergences \( \tau_1 \) and \( \tau_2 \) are equivalent if \( \tau_1 \) is stronger than \( \tau_2 \) and \( \tau_2 \) is stronger than \( \tau_1 \). The Wasserstein distance \( \tau_W \) is the weakest divergence in the class of strict adversarial divergences (Liu et al., 2017), leading to the following requirement:  

**Requirement 1 (Equivalence to \( \tau_W \)).** An adversarial divergence \( \tau \) is said to fulfill Requirement 1 if \( \tau \) is a strict adversarial divergence which is weaker than \( \tau_W \). The issue of the zero gradients was side stepped in (Goodfellow et al., 2014) with non-convex constraints on the parameters. Thus, finding an optimal Lipschitz continuous critic is an ideal. (Petzka et al., 2018) showed that an optimal critic for \( \tau_I \) has undefined gradients on the support of the generated distribution \( Q_\theta \). Thus, the update direction \( \nabla_\theta E_{Q_\theta}[f^\ast] \) is undefined; even if a direction was chosen from the subgradient field (meaning the update direction is defined but random) the update direction won’t generally point in the direction of steepest gradient descent. This naturally leads to the next requirement:  

**Requirement 2 (Convex Admissible Critic Parameter Set).** Assume \( \tau \) is a critic based adversarial divergence where critics are chosen from a set \( \mathcal{F} \). Assume further that in training, a parameterization \( \vartheta \in \mathbb{R}^C \) of the critic function \( f_\vartheta \) is learned. The critic based adversarial divergence \( \tau \) is said to fulfill requirement 2 if the set of admissible parameters \( \{ \vartheta \in \mathbb{R}^C \ | \ f_\vartheta \in \mathcal{F} \} \) is convex.

It was reasoned in (Gulrajani et al., 2017) that since a Wasserstein critic must have gradients of norm at most 1 everywhere, a reasonable strategy would be to transform the constrained optimization into an unconstrained optimization problem by penalizing the divergence when a critic has non-unit gradients. With this strategy, the so-called Improved Wasserstein GAN or WGAN-GP divergence defined in Eq. 3 is obtained. The generator parameters are updated by training an optimal critic \( f^\ast \) and updating with \( \nabla_\theta E_{Q_\theta}[\hat f^\ast] \). Although this method has impressive experimental results, it is not yet ideal. (Petzka et al., 2018) showed that an optimal critic for \( \tau_I \) has undefined gradients on the support of the generated distribution \( Q_\theta \). Thus, the update direction \( \nabla_\theta E_{Q_\theta}[f^\ast] \) is undefined; even if a direction was chosen from the subgradient field (meaning the update direction is defined but random) the update direction won’t generally point in the direction of steepest gradient descent. This naturally leads to the next requirement:  

**Requirement 3 (Well Defined Update Rule).** An update rule is said to fulfill Requirement 3 on a target distribution \( \mathbb{P} \) and a family of generated distributions \( (Q_\theta)_{(\theta \in \Theta)} \) if for every \( \theta \in \Theta \) the update rule at \( \mathbb{P} \) and \( Q_\theta \) is well defined.

Note that kernel methods such as (Dziugaite et al., 2015) and (Li et al., 2015) provide exciting theoretical guarantees and may well fulfill all four requirements. Since these guarantees come at a cost in scalability, we won’t analyze them further.

### 4. Correct Update Rule Requirement

In the previous section, we stated a bare minimum requirement for an update rule (namely that it is well defined). In this section, we’ll go further and explore criteria for a “good” update rule. For example in Lemma 8 in Section A of Appendix, it is shown that there exists a target \( \mathbb{P} \) and a family |\( \Theta \) |\( \tau_1 \) |\( \tau_2 \) |\( \tau_I \) |\( \tau_W \) |\( \tau_G \) |
|---|---|---|---|---|---|
|GAN |no |✓ |✓ |✓ |no |
|WGAN |✓ |✓ |✓ |✓ |
|WGAN-GP |✓ |✓ |no |no |
|WGAN-LP |✓ |✓ |no |no |
|DRAGAN |✓ |✓ |✓ |no |
|PWGAN |✓ |✓ |no |no |
|FOGAN |✓ |✓ |✓ |

Table 1. Comparing existing GAN methods with regard to the four Requirements formulated in this paper. The methods compared are the classic GAN (Goodfellow et al., 2014), WGAN (Arjovsky et al., 2017), WGAN-GP (Gulrajani et al., 2017), WGAN-LP (Petzka et al., 2018), DRAGAN (Kodali et al., 2017), PWGAN (our method) and FOGAN (our method).
of generated distributions \((Q_\theta)_{\theta \in \Theta}\) fulfilling Assumptions 1 and 2 such that for the optimal critic \(f_{\theta_0} \in \text{OC}_\gamma (P, Q_{\theta_0})\) there is no \(\gamma \in \mathbb{R}\) so that

\[
\nabla_\theta \tau_f (P \parallel Q_\theta)|_{\theta_0} = \gamma \nabla_\theta E_{Q_\theta}[f_{\theta_0}]|_{\theta_0}
\]

for all \(\theta_0 \in \Theta\) if all terms are well defined. Thus, the update rule used in the WGAN-GP setting, although well defined for this specific \(P\) and \(Q_{\theta_0}\), isn’t guaranteed to move \(\theta\) in the direction of steepest descent. In fact, (Mandel et al., 2018) shows that the WGAN-GP does not converge for specific classes of distributions. Therefore, the question arises why such well defined update rule also moves \(\theta\) in the direction of steepest descent?

The most obvious candidate for an update rule is simply use the direction \(-\nabla_\theta \tau (P \parallel Q_\theta)\), but since in the adversarial divergence setting \(\tau (P \parallel Q_\theta)\) is the supremum over a set of infinitely many possible critics, calculating \(-\nabla_\theta \tau (P \parallel Q_\theta)\) directly is generally intractable.

One strategy to address this issue is to use an envelope theorem (Milgrom & Segal, 2002). Assuming all terms are well defined, then for every optimal critic \(f^* \in \text{OC}_\gamma (P, Q_{\theta_0})\) it holds \(\nabla_\theta \tau (P \parallel Q_\theta)|_{\theta_0} = \nabla_\theta \tau (P \parallel Q_\theta; f^*)|_{\theta_0}\). This strategy is outlined in detail in (Aronov et al., 2017) when proving the Wasserstein GAN update rule, and explored in the context of the classic GAN divergence \(\tau_G\) in (Arjovsky & Bottou, 2017).

Yet in many GAN settings, (Goodfellow et al., 2014; Arjovsky et al., 2017; Salimans et al., 2016; Petzka et al., 2018), the update rule is to train an optimal critic \(f^*\) and then take a step in the direction of \(-\nabla_\theta E_{Q_\theta}[f^*]\). In the critic based adversarial divergence setting (Definition 2), a direct result of Eq. 1 together with Theorem 1 from (Milgrom & Segal, 2002) is that for every \(f^* \in \text{OC}_\gamma (P, Q_{\theta_0})\)

\[
\nabla_\theta \tau (P \parallel Q_\theta)|_{\theta_0} = \nabla_\theta \tau (P \parallel Q_\theta; f^*)
\]

when all terms are well defined. Thus, the update direction \(\nabla_\theta E_{Q_\theta}[f^*]\) only points in the direction of steepest descent for special choices of \(m_2\) and \(r_f\). One such example is the Wasserstein GAN where \(m_2(x) = x\) and \(r_f = 0\).

Most popular GAN methods don’t employ functions \(m_2\) and \(r_f\) such that the update direction \(\nabla_\theta E_{Q_\theta}[f^*]\) points in the direction of steepest descent. For example, with the classic GAN, \(m_2(x) = \log(1 - x)\) and \(r_f = 0\), so the update direction \(\nabla_\theta E_{Q_\theta}[f^*]\) clearly is not oriented in the direction of steepest descent \(\nabla_\theta E_{Q_\theta}[\log(1 - f^*)]\). The WGAN-GP is similar, since as we see in Lemma 8 in Appendix, Section A, \(\nabla_\theta E_{P \otimes Q_\theta}[r_f]\) is not generally a multiple of \(\nabla_\theta E_{Q_\theta}[f^*]\).

The question arises why this direction is used instead of directly calculating the direction of steepest descent? Using the correct update rule in Eq. 4 above involves estimating \(\nabla_\theta E_{P \otimes Q_\theta}[r_f]\), which requires sampling from both \(P\) and \(Q_\theta\). GAN learning happens in mini-batches, therefore \(\nabla_\theta E_{P \otimes Q_\theta}[r_f]\) isn’t calculated directly, but estimated based on samples which can lead to variance in the estimate.

To analyze this issue, we use the notation from (Bellemare et al., 2017) where \(X_m := X_1, X_2, \ldots, X_m\) are samples from \(P\) and the empirical distribution \(\hat{P}_m\) is defined by \(\hat{P}_m := \frac{1}{m} \sum_{i=1}^{m} X_i\). Further let \(\nabla X_m \sim \theta\) be the element-wise variance. Now with mini-batch learning we get

\[
\nabla X_m \sim \theta \left[ \nabla_\theta E_{P \otimes Q_\theta}[m_f - r_f] \right]_{\theta_0} = \nabla X_m \sim \theta \left[ \nabla_\theta (E_{P \otimes Q_\theta}[m_2(f^*)] - E_{Q_\theta}[m_2(f^*)]) - E_{P \otimes Q_\theta}[r_f] \right]_{\theta_0} = \nabla X_m \sim \theta \left[ \nabla_\theta E_{P \otimes Q_\theta}[r_f] \right]_{\theta_0}.
\]

Therefore, estimation of \(\nabla_\theta E_{P \otimes Q_\theta}[r_f]\) is an extra source of variance.

Our solution to both these problems chooses the critic based adversarial divergence \(\tau\) in such a way that there exists a \(\gamma \in \mathbb{R}\) so that for all optimal critics \(f^* \in \text{OC}_\gamma (P, Q_{\theta_0})\) it holds

\[
\nabla_\theta E_{P \otimes Q_\theta}[r_f]|_{\theta_0} \approx \gamma \nabla_\theta E_{Q_\theta}[m_2(f^*)]|_{\theta_0}.
\] (5)

In Theorem 2 we see conditions on \(P, Q_\theta\) such that equality holds. Now using Eq. 5 we see that

\[
\nabla_\theta \tau (P \parallel Q_\theta)|_{\theta_0} = \nabla_\theta (E_{Q_\theta}[m_2(f^*)] + E_{P \otimes Q_\theta}[r_f])|_{\theta_0} \approx \nabla_\theta (E_{Q_\theta}[m_2(f^*)] + \gamma E_{Q_\theta}[m_2(f^*)])|_{\theta_0} = -(1 + \gamma) \nabla_\theta E_{Q_\theta}[m_2(f^*)]|_{\theta_0}
\]

making \((1 + \gamma) \nabla_\theta E_{Q_\theta}[m_2(f^*)]\) a low variance update approximation of the direction of steepest descent.

We’re then able to have the best of both worlds. On the one hand, when \(r_f\) serves as a penalty term, training of a critic neural network can happen in an unconstrained optimization fashion like with the WGAN-GP. At the same time, the direction of steepest descent can be approximated by calculating \(\nabla_\theta E_{Q_\theta}[m_2(f^*)]\), and as in the Wasserstein GAN we get reliable gradient update steps.

With this motivation, Eq. 5 forms the basis of our final requirement:

**Requirement 4 (Low Variance Update Rule).** An adversarial divergence \(\tau\) is said to fulfill requirement 4 if \(\tau\) is a critic based adversarial divergence and for every optimal critic \(f^* \in \text{OC}_\gamma (P, Q_{\theta_0})\) fulfills Eq. 5.

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Because the first expectation doesn’t depend on \(\theta\), \(\nabla_\theta E_{Q_\theta}[m_1(f^*)] = 0\). In the same way, because the second expectation doesn’t depend on the mini-batch \(X_m\) sampled, \(\nabla X_m \sim \theta [E_{Q_\theta}[m_2(f^*)]] = 0\).
It should be noted that the WGAN-GP achieves impressive experimental results; we conjecture that in many cases \( \nabla_\theta \mathbb{E}_{Q_\theta} [f^*] \) close enough to the true direction of steepest descent. Nevertheless, as the experiments in Section 7 show, our gradient estimates lead to better convergence in a challenging language modeling task.

5. Penalized Wasserstein Divergence

We now attempt to find an adversarial divergence that fulfills all four requirements. We start by formulating an adversarial divergence \( \tau_P \) and a corresponding update rule that can be shown to comply with Requirements 1 and 2. Subsequently in Section 6, \( \tau_P \) will be refined to make its update rule practical and conform to all four requirements.

The divergence \( \tau_P \) is inspired by the Wasserstein distance, there for an optimal critic between two Dirac distributions \( f^* \in \Omega \tau(\delta_a, \delta_b) \) it holds \( f(a) - f(b) = |a - b| \). Now if we look at

\[
\tau_{\text{simple}}(\delta_a \| \delta_b) := \sup_{f \in \mathcal{F}} f(a) - f(b) - \frac{(f(a) - f(b))^2}{|a - b|} \quad (6)
\]

it’s easy to calculate that \( \tau_{\text{simple}}(\delta_a \| \delta_b) = \frac{1}{2} |a - b| \), which is the same up to a constant (in this simple setting) as the Wasserstein distance, without being a constrained optimization problem. See Figure 1 for an example.

This has another intuitive explanation. Because Eq. 6 can be reformulated as

\[
\tau_{\text{simple}}(\delta_a \| \delta_b) = \sup_{f \in \mathcal{F}} f(a) - f(b) - |a - b| \left( \frac{f(a) - f(b)}{|a - b|} \right)^2
\]

which is a tug of war between the objective \( f(a) - f(b) \) and the squared Lipschitz penalty \( \frac{f(a) - f(b)}{|a - b|} \) weighted by \( |a - b| \). This \( |a - b| \) term is important (and missing from (Gulrajani et al., 2017), (Petzka et al., 2018)) because otherwise the slope of the optimal critic between \( a \) and \( b \) will depend on \( |a - b| \).

The penalized Wasserstein divergence \( \tau_P \) is a straightforward adaptation of \( \tau_{\text{simple}} \) to the simple dimensional case.

**Definition 5** (Penalized Wasserstein Divergence). Assume \( X \subseteq \mathbb{R}^n \) and \( \mathbb{P}, Q \in \mathcal{P}(X) \) are probability measures over \( X \), \( \lambda > 0 \) and \( \mathcal{F} = C^1(X) \). Set

\[
\tau_P(\mathbb{P} \| \mathbb{Q}; f) := \mathbb{E}_{x \sim \mathbb{P}} [f(x)] - \mathbb{E}_{x' \sim \mathbb{Q}} [f(x')] - \lambda \mathbb{E}_{x \sim \mathbb{P}, x' \sim \mathbb{Q}} \left[ \frac{(f(x) - f(x'))^2}{\|x - x'\|} \right].
\]

Define the penalized Wasserstein divergence as

\[
\tau_P(\mathbb{P} \| \mathbb{Q}) = \sup_{f \in \mathcal{F}} \tau_P(\mathbb{P} \| \mathbb{Q}; f).
\]

This divergence is updated by picking an optimal critic \( f^* \in \Omega \tau_P(\mathbb{P}, \mathbb{Q}_\theta) \) and taking a step in the direction of \( \nabla_\theta \mathbb{E}_{Q_\theta} [f^*] \).

This formulation is similar to the WGAN-GP (Gulrajani et al., 2017), restated here in Eq. 3.

**Theorem 1**. Assume \( X \subseteq \mathbb{R}^n \), and \( \mathbb{P}, \mathbb{Q}_\theta \in \mathcal{P}(X) \) are probability measures over \( X \) fulfilling Assumptions 1 and 2. Then for every \( \theta_0 \in \Theta \) the Penalized Wasserstein Divergence with it’s corresponding update direction fulfills Requirements 1 and 2.

Further, there exists an optimal critic \( f^* \in \Omega \tau_P(\mathbb{P}, \mathbb{Q}_\theta) \) that fulfills Eq. 5 and thus Requirement 4.

**Proof**. See Appendix, Section A.

Note that this theorem isn’t unique to \( \tau_P \). For example, for the penalty in Eq. 8 of (Petzka et al., 2018) we conjecture that a similar result can be shown. The divergence \( \tau_P \) is still very useful because, as will be shown in the next section, \( \tau_P \) can be modified slightly to obtain a new critic \( \tau_F \), where every optimal critic fulfills Requirements 1 to 4.

Since \( \tau_P \) only constrains the value of a critic on the supports of \( \mathbb{P} \) and \( \mathbb{Q}_\theta \), many different critics are optimal, and in general \( \nabla_\theta \mathbb{E}_{Q_\theta} [f^*] \) depends on the optimal critic choice and is thus not well defined. With this, Requirements 3 and 4 are not fulfilled. See Figure 1 for a simple example.

In theory, \( \tau_P \)'s critic could be trained with a modified sampling procedure so that \( \nabla_\theta \mathbb{E}_{Q_\theta} [f^*] \) is well defined and Eq. 5 holds, as is done in both (Kodali et al., 2017) and (Unterthiner et al., 2018). By using a method similar to (Bishop et al., 1998), one can minimize the divergence \( \tau_P(\mathbb{P}, \mathbb{Q}_\theta) \) where \( \mathbb{Q}_\theta \) is data equal to \( x' + \epsilon \) where \( x' \) is sampled from \( \mathbb{Q}_0 \) and \( \epsilon \) is some zero-mean uniform distributed noise. In this way the support of \( \mathbb{Q}_\theta \) lives in the full space \( X \) and not the submanifold \( \text{supp}(\mathbb{Q}_\theta) \). Unfortunately, while this method works in theory, the number of samples required for accurate gradient estimates scales with the dimensionality of the underlying space \( X \), not with the dimensionality of data or generated submanifolds \( \text{supp}(\mathbb{P}) \) or \( \text{supp}(\mathbb{Q}_\theta) \).

In response, we propose the First Order Penalized Wasserstein Divergence.

6. First Order Penalized Wasserstein Divergence

As was seen in the last section, since \( \tau_P \) only constrains the value of optimal critics on the supports of \( \mathbb{P} \) and \( \mathbb{Q}_\theta \), the gradient \( \nabla_\theta \mathbb{E}_{Q_\theta} [f^*] \) is not well defined. A natural method to refine \( \tau_P \) to achieve a well defined gradient is to enforce two things:
This divergence is updated by picking an optimal critic $f^* \in \text{OC}_{\tau_F}(\mathbb{P}, \mathbb{Q}_{\theta_0})$ and taking a step in the direction of $\nabla_{\theta} \mathbb{E}_{\mathbb{Q}_\theta}[f^*] |_{\theta_0}$.

In order to define a GAN from the First Order Penalized Wasserstein Divergence, we must define a slight modification of the generated distribution $\mathbb{Q}_\theta$ to obtain $\mathbb{Q}_\theta'$. Similar to the WGAN-GP setting, samples from $\mathbb{Q}_\theta'$ are obtained by $x' - \alpha(x' - x)$ where $x \sim \mathbb{P}$ and $x' \sim \mathbb{Q}_\theta$. The difference is that $\alpha \sim \mathcal{U}([0, \varepsilon])$, with $\varepsilon$ chosen small, making $\mathbb{Q}_\theta$ and $\mathbb{Q}_\theta'$ quite similar. Therefore updates to $\theta$ that reduce $\tau_F(\mathbb{P} \Vert \mathbb{Q}_0')$ also reduce $\tau_F(\mathbb{P} \Vert \mathbb{Q}_\theta)$.

Conveniently, as is shown in Lemma 5 in Appendix, Section A, any optimal critic for the First Order Penalized Wasserstein divergence is also an optimal critic for the Penalized Wasserstein Divergence. The key advantage to the First Order Penalized Wasserstein Divergence is that for any $\mathbb{P}$, $\mathbb{Q}_0$ fulfilling Assumptions 1 and 2, $\tau_F(\mathbb{P} \Vert \mathbb{Q}_0')$ with its corresponding update rule $\nabla_{\theta} \mathbb{E}_{\mathbb{Q}_\theta}[f^*]$ on the slightly modified probability distribution $\mathbb{Q}_\theta'$ fulfills requirements 3 and 4.

**Theorem 2.** Assume $X \subseteq \mathbb{R}^n$, and $\mathbb{P}, \mathbb{Q}_0 \in \mathcal{P}(X)$ are probability measures over $X$ fulfilling Assumptions 1 and 2 and $\mathbb{Q}_0'$ is $\mathbb{Q}_0$ modified using the method above. Then for every $\theta_0 \in \Theta$ there exists at least one optimal critic $f^* \in \text{OC}_{\tau_F}(\mathbb{P}, \mathbb{Q}_{\theta_0}')$ and $\tau_F$ combined with update direction $\nabla_{\theta} \mathbb{E}_{\mathbb{Q}_\theta}[f^*] |_{\theta_0}$ fulfills Requirements 1 to 4. If $\mathbb{P}, \mathbb{Q}_\theta'$ are such that $\forall x, x' \in \text{supp}(\mathbb{P}), \text{supp}(\mathbb{Q}_\theta')$ it holds $f^*(x) - f^*(x') = c \Vert x - x' \Vert$ for some constant $c$, then equality holds for Eq. 5.

**Proof.** See Appendix, Section A

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**7. Experimental Results**

**7.1. Image Generation**

We begin by testing the FOGAN on the CelebA image generation task (Liu et al., 2015), training a generative model with the DCGAN architecture (Radford et al., 2016) and...
obtaining Fréchet Inception Distance (FID) scores (Heusel et al., 2017) competitive with state of the art methods without doing a tuning parameter search. Similarly, we show competitive results on LSUN (Yu et al., 2015) and CIFAR-10 (Krizhevsky & Hinton, 2009). See Table 2, Appendix B.1 and released code.

7.2. One Billion Word

Finally, we use the First Order Penalized Wasserstein Divergence to train a character level generative language model on the One Billion Word Benchmark (Chelba et al., 2013). In this setting, a 1D CNN deterministically transforms a latent vector into a $32 \times C$ matrix, where $C$ is the number of possible characters. A softmax nonlinearity is applied to this output, and given to the critic. Real data is one-hot encoding of 32 character texts sampled from the true data.

We conjecture this is an especially difficult task for GANs, since data in the target distribution lies in just a few corners of the $32 \times C$ dimensional unit hypercube. As the generator is updated, it must push mass from one corner to another, passing through the interior of the hypercube far from any real data. Methods other than Coulomb GAN (Unterthiner et al., 2018) WGAN-GP (Gulrajani et al., 2017; Heusel et al., 2017) and the Sobolev GAN (Mroueh et al., 2018) have not been shown to be successful on this task.

We use the same setup as in both (Gulrajani et al., 2017; Heusel et al., 2017) with two differences. First, we train to minimize our divergence from Definition 6 with parameters $\lambda = 0.1$ and $\mu = 1.0$ instead of the WGAN-GP divergence. Second, we use batch normalization in the generator, both for training our FOGAN method and the benchmark WGAN-GP; we do this because batch normalization improved performance and stability of both models.

As with (Gulrajani et al., 2017; Heusel et al., 2017) we use the Jensen-Shannon-divergence (JSD) between $n$-grams from the model and the real world distribution as an evaluation metric. The JSD is estimated by sampling a finite number of 32 character vectors, and comparing the distributions of the $n$-grams from said samples and true data. This estimation is biased; smaller samples result in larger JSD estimations. A Bayes limit results from this bias; even when samples are drawn from real world data and compared with real world data, small sample sizes result in large JSD estimations. In order to detect performance difference when training with the FOGAN and WGAN-GP, a low Bayes limit is necessary. Thus, to compare the methods, we sampled 6400 32 character vectors in contrast with the 640 vectors sampled in past works. Therefore, the JSD values in those papers are higher than the results here.

For our experiments we trained both models for 500,000 iterations in 5 independent runs, estimating the JSD between 6-grams of generated and real world data every 2000 training steps, see Figure 2. The results are even more impressive when aligned with wall-clock time. Since in WGAN-GP training an extra point between real and generated distributions must be sampled, it is slower than the FOGAN training; see Figure 2 and observe the significant $(2\sigma)$ drop in estimated JSD.

\[\text{Table 2. Comparison of different GAN methods for image and text generation. We measure performance with respect to the FID on the image datasets and JSD between } n \text{-grams for text generation.}\]

| Task       | BEGAN | DCGAN | Coulomb | WGAN-GP | FOGAN |
|------------|-------|-------|---------|---------|-------|
| CelebA     | 28.5  | 12.5  | 9.3     | 4.2     | 6.0   |
| LSUN       | 112   | 57.5  | 31.2    | 9.5     | 11.4  |
| CIFAR-10   | -     | -     | 27.3    | 24.8    | 27.4  |
| 4-gram     | -     | -     | .220 ± .006 | .226 ± .006 |
| 6-gram     | -     | -     | .573 ± .009 | .556 ± .004 |

\[\text{Figure 2. Five training runs of both WGAN-GP and FOGAN, with the average of all runs plotted in bold and the } 2\sigma \text{ error margins denoted by shaded regions. For easy visualization, we plot the moving average of the last three } n \text{-gram JSD estimations. The first two plots both show training w.r.t. number of training iterations; the second plot starts at iteration } 50. \text{ The last plot show training with respect to wall-clock time, starting after } 6 \text{ hours of training.}\]
Acknowledgements

This work was supported by Zalando SE with Research Agreement 01/2016.

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Yu, F., Zhang, Y., Song, S., Seff, A., and Xiao, J. Lsun: Construction of a large-scale image dataset using deep learning with humans in the loop. *arXiv preprint arXiv:1506.03365*, 2015.
A. Proof of Things

Proof of Theorem 1. The proof of this theorem is split into smaller lemmas that are proven individually.

- That $\tau_p$ is a strict adversarial divergence which is equivalent to $\tau_w$ is proven in Lemma 4, thus showing that $\tau_p$ fulfills Requirement 1.
- $\tau_p$ fulfills Requirement 2 by design.
- The existence of an optimal critic in $OC_{\tau_p}(P, Q)$ follows directly from Lemma 3.
- That there exists a critic $\gamma \in OC_{\tau_p}(P, Q)$ that fulfills Eq. 5 is because Lemma 3 ensures that a continuous differentiable $f^*$ exists in $OC_{\tau_p}(P, Q)$ which fulfills Eq. 9. Because Eq. 9 holds for $\gamma \in C(X)$, the same reasoning as the end of the proof of Lemma 7 can be used to show Requirement 4.

We prepare by showing a few basic lemmas used in the remaining proofs

Lemma 1 (concavity of $\tau_p(\|P\|Q; \cdot$)). The mapping $C^1(X) \to \mathbb{R}, f \mapsto \tau_p(\|P\|Q; f)$ is concave.

Proof. The concavity of $f \mapsto \mathbb{E}_{x \sim P}[f(x)] - \mathbb{E}_{x' \sim Q}[f(x')]$ is trivial. Now consider $\gamma \in (0, 1)$, then

$$\mathbb{E}_{x \sim P, x' \sim Q} \left[ \frac{\gamma(f(x) - f(x')) + (1 - \gamma)(f(x) - f(x'))^2}{\|x - x'\|} \right]$$

$$\leq \mathbb{E}_{x \sim P, x' \sim Q} \left[ \frac{\gamma(f(x) - f(x'))^2 + (1 - \gamma)(f(x) - f(x'))^2}{\|x - x'\|} \right]$$

$$= \gamma \mathbb{E}_{x \sim P, x' \sim Q} \left[ \frac{(f(x) - f(x'))^2}{\|x - x'\|} \right] + (1 - \gamma) \mathbb{E}_{x \sim P, x' \sim Q} \left[ \frac{(f(x) - f(x'))^2}{\|x - x'\|} \right],$$

thus showing concavity of $\tau_p(\|P\|Q; \cdot$).

Lemma 2 (necessary and sufficient condition for maximum). Assume $P, Q \in P(X)$ fulfill assumptions 1 and 2. Then for any $f \in OC_{\tau_p}(P, Q)$ it must hold that

$$P_{x' \sim Q} \left( \mathbb{E}_{x \sim P} \left[ \frac{f(x) - f(x')}{\|x - x'\|} \right] = \frac{1}{2\lambda} \right) = 1$$

and

$$P_{x \sim P} \left( \mathbb{E}_{x' \sim Q} \left[ \frac{f(x) - f(x')}{\|x - x'\|} \right] = \frac{1}{2\lambda} \right) = 1.$$}

Further, if $f \in C^1(X)$ and fulfills Eq. 7 and 8, then $f \in OC_{\tau_p}(P, Q)$.

Proof. Since in Lemma 1 it was shown that the the mapping $f \mapsto \tau_p(\|P\|Q, f)$ is concave, $f \in OC_{\tau_p}(P, Q)$ if and only if $f \in C^1(X)$ and $f$ is a local maximum of $\tau_p(\|P\|Q; \cdot$). This is equivalent to saying that all $u_1, u_2 \in C^1(X)$ with $\text{supp}(u_1) \cap \text{supp}(Q) = \emptyset$ and $\text{supp}(u_2) \cap \text{supp}(P) = \emptyset$ it holds

$$\nabla_{(\varepsilon, \rho)} \left[ \mathbb{E}_P[f + \varepsilon u_1] - \mathbb{E}_Q[f + \rho u_2] - \lambda \mathbb{E}_{x \sim P, x' \sim Q} \left[ \frac{((f + \varepsilon u_1)(x) - (f + \rho u_2)(x'))^2}{\|x - x'\|^2} \right] \right]_{\varepsilon = 0, \rho = 0} = 0$$

which holds if and only if

$$\mathbb{E}_{x \sim P} \left[ u_1(x) \left( 1 - 2\lambda \mathbb{E}_{x' \sim Q} \left[ \frac{(f(x) - f(x'))}{\|x - x'\|} \right] \right) \right] = 0$$

and

$$\mathbb{E}_{x' \sim Q} \left[ u_2(x') \left( 1 - 2\lambda \mathbb{E}_{x \sim P} \left[ \frac{(f(x) - f(x'))}{\|x - x'\|} \right] \right) \right] = 0$$

proving that Eq. 7 and 8 are necessary and sufficient.
Lemma 3. Let $P, Q \in \mathcal{P}(X)$ be probability measures fulfilling Assumptions 1 and 2. Define an open subset of $X$, $\Omega \subseteq X$, such that $\text{supp}(Q) \subseteq \Omega$ and $\inf_{x \in \text{supp}(P), x' \in \Omega} \|x - x'\| > 0$. Then there exists a $f \in \mathcal{F} = C^1(X)$ such that

$$\forall x' \in \Omega: \quad E_{x \sim P} \left[ \frac{f(x) - f(x')}{\|x - x'\|} \right] = \frac{1}{2\lambda}$$

and

$$\forall x \in \text{supp}(P): \quad E_{x' \sim Q} \left[ \frac{f(x) - f(x')}{\|x - x'\|} \right] = \frac{1}{2\lambda}$$

Proof. Since $\tau_P(P\|Q; f) = \tau(P\|Q; f + c)$ for any $c \in \mathbb{R}$ and is only affected by values of $f$ on $\text{supp}(P) \cup \Omega$ we first start by considering

$$\mathcal{F} = \left\{ f \in C^1(\text{supp}(P) \cup \Omega) \mid E_{x \sim P, x' \sim Q} \left[ \frac{f(x')}{\|x - x'\|} \right] = 0 \right\}$$

Observe that Eq. 9 holds if

$$x' \in \Omega: \quad f(x') = \frac{E_{x \sim P} \left[ \frac{f(x)}{\|x - x'\|} \right] - \frac{1}{2\lambda}}{E_{x \sim P} \left[ \frac{1}{\|x - x'\|} \right]}$$

and similarly for Eq. 10

$$\forall x \in \text{supp}(P): \quad f(x) = \frac{E_{x' \sim Q} \left[ \frac{f(x')}{\|x - x'\|} \right] + \frac{1}{2\lambda}}{E_{x' \sim Q} \left[ \frac{1}{\|x - x'\|} \right]}$$

Now it’s clear that if the mapping $T: \mathcal{F} \to \mathcal{F}$ defined by

$$T(f)(x) := \begin{cases} \frac{E_{x' \sim Q} \left[ f(x') \right]}{E_{x \sim P} \left[ \frac{1}{\|x - x'\|} \right]} & x \in \text{supp}(P) \\ \frac{E_{x' \sim Q} \left[ f(x') \right]}{E_{x \sim P} \left[ \frac{1}{\|x - x'\|} \right]} & x \in \Omega \end{cases}$$

admit a fix point $f^* \in \mathcal{F}$, i.e. $T(f^*) = f^*$, then $f^*$ is a solution to Eq. 9 and 10, and with that a solution to Eq. 7 and 8 and $\tau_P(P\|Q; f^*) = \tau_P(P\|Q)$.

Define the mapping $S: \mathcal{F} \to \mathcal{F}$ by

$$S(f)(x) = \frac{f(x)}{2\lambda E_{\tilde{x} \sim P, x' \sim Q} \left[ \frac{f(x) - f(x')}{\|\tilde{x} - x'\|} \right]}$$

Then

$$E_{\tilde{x} \sim P, x' \sim Q} \left[ S(f)(\tilde{x}) - S(f)(x') \right] = \frac{1}{2\lambda}$$

and

$$S(S(f))(x) = \frac{S(f)(x)}{2\lambda}$$

making $S$ a projection. By the same reasoning, if $E_{\tilde{x} \sim P, x' \sim Q} \left[ \frac{f(\tilde{x}) - f(x')}{\|\tilde{x} - x'\|} \right] = \frac{1}{2\lambda}$ then $f$ is a fix-point of $S$, i.e. $S(f) = f$.

Assume $f$ is such a function, then by definition of $T$ in Eq. 11

$$E_{\tilde{x} \sim P, x' \sim Q} \left[ \frac{T(f)(\tilde{x}) - T(x)(x')}{\|\tilde{x} - x'\|} \right] = E_{\tilde{x} \sim P} \left[ E_{x' \sim Q} \left[ \frac{T(f)(\tilde{x})}{\|\tilde{x} - x'\|} \right] - E_{x' \sim Q} \left[ E_{\tilde{x} \sim P} \left[ \frac{T(f)(x')}{\|\tilde{x} - x'\|} \right] \right] \right]$$

$$= E_{\tilde{x} \sim P} \left[ E_{x' \sim Q} \left[ \frac{f(x')}{\|\tilde{x} - x'\|} \right] + \frac{1}{2\lambda} \right] - E_{x' \sim Q} \left[ E_{\tilde{x} \sim P} \left[ \frac{f(\tilde{x})}{\|\tilde{x} - x'\|} \right] - \frac{1}{2\lambda} \right]$$

$$= -E_{\tilde{x} \sim P, x' \sim Q} \left[ \frac{f(\tilde{x}) - f(x')}{\|\tilde{x} - x'\|} \right] + 2 \frac{1}{2\lambda}$$

$$= \frac{1}{2\lambda}$$
Therefore, \(S(T(S(f))) = T(S(f))\). We can define \(S(F) = \{S(f) \mid f \in F\}\) and see that \(T : S(F) \rightarrow S(F)\). Further, since \(S(\cdot)\) only multiplies with a scalar, \(S(F) \subseteq F\).

Let \(f_1, f_2 \in S(F)\). From Eq. 12 we get

\[
\mathbb{E}_{x \sim P, x' \sim Q} \left[ \frac{f_1(x') - f_2(x')}{\|x - x'\|} \right] = \mathbb{E}_{x \sim P, x' \sim Q} \left[ \frac{f_1(x) - f_2(x)}{\|x - x'\|} \right].
\]

Now since for every \(f \in F\) it holds by design that \(\mathbb{E}_{x \sim P, x' \sim Q} \left[ \frac{f(x')}{\|x - x'\|} \right] = 0\) and since \(S(F) \subseteq F\) we see that \(f_1, f_2 \in S(F)\) that

\[
\mathbb{E}_{x \sim P, x' \sim Q} \left[ f_1(x') - f_2(x') \right] = \mathbb{E}_{x \sim P, x' \sim Q} \left[ f_1(x) - f_2(x) \right] = 0.
\]

Using this with the continuity of \(f_1, f_2\), there must exist \(x_1 \in \text{supp}(P)\) with

\[
\mathbb{E}_{x' \sim Q} \left[ f_1(x') - f_2(x') \right] = 0.
\]

With this (and compactness of our domain), \(Q\) must have mass in both positive and negative regions of \(f_1 - f_2\) and exists a constant \(p < 1\) such that for all \(f_1, f_2 \in S(F)\) it holds

\[
\sup_{x \in \text{supp}(P)} \mathbb{E}_{x' \sim Q} \left[ \frac{f_1(x') - f_2(x')}{\|x - x'\|} \right] \leq p \sup_{x \in \text{supp}(P)} \mathbb{E}_{x' \sim Q} \left[ \frac{1}{\|x - x'\|} \right] \sup_{x' \in \Omega} |f_1(x') - f_2(x')|.
\]

(13)

To show the existence of a fix-point for \(T\) in the Banach Space \((\bar{F}, \|\cdot\|_{\infty})\) we use the Banach fixed-point theorem to show that \(T\) has a fixed point in the metric space \((S(F), ||\cdot||_\infty)\) (remember that \(T : S(F) \rightarrow S(F)\) and \(S(F) \subseteq F\)). If \(f_1, f_2 \in S(F)\) then

\[
\sup_{x \in \text{supp}(P)} |T(f_1)(x) - T(f_2)(x)| = \sup_{x \in \text{supp}(P)} \mathbb{E}_{x' \sim Q} \left[ \frac{f_1(x') - f_2(x')}{\|x - x'\|} \right] \leq p \sup_{x' \in \text{supp}(Q)} |f_1(x') - f_2(x')| \text{ using Eq. 13}
\]

The same trick can be used to find some some \(q < 1\) and show

\[
\sup_{x' \in \Omega} |T(f_1)(x') - T(f_2)(x')| \leq q \sup_{x \in \text{supp}(P)} |f_1(x) - f_2(x)|
\]

thereby showing

\[
\|T(f_1) - T(f_2)\|_{\infty} < \max(p,q)\|f_1 - f_2\|_{\infty}
\]

The Banach fix-point theorem then delivers the existence of a fix-point \(f^* \in S(F)\) for \(T\).

Finally, we can use the Tietze extension theorem to extend \(f^*\) to all of \(X\), thus finding a fix point for \(T\) in \(C^1(X)\) and proving the lemma.

**Lemma 4.** \(\tau_p\) is a strict adversarial divergence and \(\tau_p\) and \(\tau_W\) are equivalent.

**Proof.** Let \(P, Q \in \mathcal{P}(X)\) be two probability measures fulfilling Assumptions 1 and 2 with \(P \neq Q\). It’s shown in (Sriperumbudur et al., 2010) that \(\mu = \tau_W(P, Q) > 0\), meaning there exists a function \(f \in C(X), \|f\|_L \leq 1\) such that

\[
\mathbb{E}_P[f] - \mathbb{E}_Q[f] = \mu > 0.
\]

The Stone–Weierstrass theorem tells us that there exists a \(f' \in C_{\infty}(X)\) such that \(\|f - f'\|_{\infty} \leq \frac{\mu}{4}\) and thus \(\mathbb{E}_P[f'] - \mathbb{E}_Q[f'] \geq \frac{\mu}{2}\). Now consider the function \(\varepsilon f'\) with \(\varepsilon > 0\), it’s clear that

\[
\tau_p(P||Q) \geq \tau_p(P||Q; \varepsilon f') = \varepsilon (\mathbb{E}_P[f'] - \mathbb{E}_Q[f']) - \varepsilon^2 \lambda \mathbb{E}_{x \sim P, x' \sim Q} \left[ (f'(x) - f'(x'))^2 \right] \geq \frac{\mu}{2}.
\]
and so for a sufficiently small $\varepsilon > 0$ we’ll get $\tau_P(\|P\|Q; \varepsilon f') > 0$ meaning $\tau_P(\|P\|Q) > 0$ and $\tau_P$ is a strict adversarial divergence.

To show equivalence, we note that

$$\tau_P(\|P\|Q) \leq \sup_{m \in C(\mathcal{X}^2)} \mathbb{E}_{x \sim P, x' \sim Q}[m(x, x') \left(1 - \frac{\lambda m(x, x')}{\|x - x'\|}\right)]$$

therefore for any optimum it must hold $m(x, x') \leq \frac{\|x - x'\|}{2\lambda}$, and thus (similar to Lemma 2) any optimal solution will be Lipschitz continuous with a the Lipschitz constant independent of $P, Q$. Thus $\tau_W(\|P\|Q) \geq \gamma \tau_P(\|P\|Q)$ for $\gamma > 0$, from which we directly get equivalence.

**Proof of Theorem 2.** We start by applying Lemma 5 giving us

- $\text{OC}_{\tau_F}(P, Q'_{\theta_0}) \neq \emptyset$.
- For any $P, Q \in \mathcal{P}(X)$ fulfilling Assumptions 1 and 2, it holds that $\tau_F(\|P\|Q) = \tau_P(\|P\|Q)$, meaning $\tau_F$ is like $\tau_P$ a strict adversarial divergence which is equivalent to $\tau_W$, showing Requirement 1.
- $\tau_F$ fulfills Requirement 2 by design.
- Every $f^* \in \text{OC}_{\tau_F}(P, Q'_{\theta_0})$ is in $\text{OC}_{\tau_P}(P, Q'_{\theta_0}) \subseteq C^1(X)$, therefore $f^*$ the gradient $\nabla_{\theta} \mathbb{E}_{Q_{\theta}}[f^*]_{\theta_0}$ exists. Further Lemma 7 shows that the update rule $\nabla_{\theta} \mathbb{E}_{Q_{\theta}}[f^*]_{\theta_0}$ is unique, thus showing Requirement 3.
- Lemma 7 gives us every $f^* \in \text{OC}_{\tau_F}(P, Q'_{\theta_0})$ with the corresponding update rule fulfills Requirement 4, thus proving Theorem 2.

Before we can show this theorem, we must prove a few interesting lemmas about $\tau_F$. The following lemma is quite powerful; since $\tau_P(\|P\|Q) = \tau_F(\|P\|Q)$ and $\text{OC}_{\tau_P}(P, Q) \subseteq \text{OC}_{\tau_F}(P, Q)$ any property that’s proven for $\tau_P$ automatically holds for $\tau_F$.

**Lemma 5.** If let $X \subseteq \mathbb{R}^n$ and $P, Q \in \mathcal{P}(X)$ be probability measures fulfilling Assumptions 1 and 2. Then

1. there exists $f^* \in \text{OC}_{\tau_F}(P, Q)$ so that $\tau_F(\|P\|Q; f^*) = \tau_P(\|P\|Q; f^*)$.
2. $\tau_F(\|P\|Q) = \tau_P(\|P\|Q)$.
3. $\emptyset \neq \text{OC}_{\tau_F}(P, Q)$.
4. $\text{OC}_{\tau_F}(P, Q) \subseteq \text{OC}_{\tau_P}(P, Q)$.

Claim (4) is especially helpful, now anything that has been proven for all $f^* \in \text{OC}_{\tau_F}(P, Q)$ automatically holds for all $f^* \in \text{OC}_{\tau_P}(P, Q)$

**Proof.** For convenience define

$$G(P, Q; f) := \mathbb{E}_{x' \sim Q}[\left\|\nabla_x f(x)\right\|_x^2 - \left\|\frac{\mathbb{E}_{\tilde{x} \sim P}[\tilde{x} - x'] f(\tilde{x}) - f(x')}{\left\|x' - \tilde{x}\right\|_x^2}\right\|^2]$$

($G$ is for gradient penalty) and note that

$$\tau_F(\|P\|Q; f) = \tau_P(\|P\|Q; f) - G(P, Q; f) \geq 0$$

Therefore it’s clear that $\tau_F(\|P\|Q) \leq \tau_P(\|P\|Q)$.
Claim (1). Let $\Omega \subseteq X$ be an open set such that $\text{supp}(Q) \subseteq \Omega$ and $\Omega \cap \text{supp}(P) = \emptyset$. Then Lemma 3 tells us there is a $f \in OC_{\tau_P}(P, Q)$ (and thus $f \in C^1(X)$) such that

$$\forall x' \in \Omega : \mathbb{E}_{\tilde{x} \sim P} \left[ \frac{f(\tilde{x}) - f(x')}{\|\tilde{x} - x'\|} \right] = \frac{1}{2\lambda}$$

and thus, because $\text{supp}(Q) \subseteq \Omega$ open and $f \in C^1(X)$,

$$\forall x' \in \text{supp}(Q) : \left. \nabla_x \mathbb{E}_{\tilde{x} \sim P} \left[ \frac{f(\tilde{x}) - f(x)}{\|\tilde{x} - x\|} \right] \right|_{x'} = 0$$

Now taking the gradients with respect to $x'$ gives us

$$\nabla_x \mathbb{E}_{\tilde{x} \sim P} \left[ \frac{f(\tilde{x}) - f(x)}{\|\tilde{x} - x\|} \right] = -\nabla_x f(x)|_{x'} \mathbb{E}_{\tilde{x} \sim P} \left[ \frac{1}{\|\tilde{x} - x\|} \right] + \mathbb{E}_{\tilde{x} \sim P} \left[ \frac{(\tilde{x} - x') f(\tilde{x}) - f(x')}{\|\tilde{x} - x'\|^3} \right]$$

meaning

$$\forall x' \in \text{supp}(Q) : \nabla_x f(x)|_{x'} = \frac{\mathbb{E}_{\tilde{x} \sim P} \left[ (\tilde{x} - x') f(\tilde{x}) - f(x') \right]}{\mathbb{E}_{\tilde{x} \sim P} \left[ \frac{1}{\|\tilde{x} - x\|} \right]}$$

thus $G(P, Q; f) = 0$, showing the claim.

Claims (2) and (3). The claims are a direct result of claim (1); for every $P, Q \in \mathcal{P}(X)$ there exists a $f^* \in OC_{\tau_P}(P, Q)$ such that $G(P, Q; f^*) = 0$. Therefore

$$\tau_P(P||Q) \geq \tau_F(P||Q) \geq \tau_F(P||Q; f^*) = \tau_P(P||Q; f^*) = \tau_P(P||Q)$$

thus showing both $\tau_P(P||Q) = \tau_F(P||Q)$ and $f^* \in OC_{\tau_P}(P||Q)$.

Claim (4). This claim is a direct result of claim (2); since $\tau_F(P||Q) = \tau_F(P||Q)$, that means that if $f^* \in OC_{\tau_F}(P||Q)$, then

$$\tau_F(P||Q) = \tau_F(P||Q; f^*) = \tau_F(P||Q; f^*) - G(P, Q; f) \leq \tau_P(P||Q; f^*) \leq \tau_P(P||Q) = \tau_F(P||Q)$$

thus $\tau_P(P||Q; f^*) = \tau_F(P||Q)$ and $f^* \in OC_{\tau_P}(P||Q)$.

\[\square\]

Lemma 6. For every $f^* \in OC_{\tau_P}(P, Q')$ it holds

$$\forall x' \in \text{supp}(Q') : \left. \nabla_x \mathbb{E}_{\tilde{x} \sim P} \left[ \frac{f^*(\tilde{x}) - f^*(x)}{\|\tilde{x} - x\|} \right] \right|_{x'} = 0$$

Proof. Set

$$v = \frac{\mathbb{E}_{\tilde{x} \sim P}[(\tilde{x} - x') \frac{f^*(\tilde{x}) - f^*(x')}{\|\tilde{x} - x'\|^3}]}{\mathbb{E}_{\tilde{x} \sim P}[(\tilde{x} - x') \frac{f^*(\tilde{x}) - f^*(x')}{\|\tilde{x} - x'\|^3}]}$$

and note that due to construction of $Q'$ and $v, v$ is such that for almost all $x' \in \text{supp}(Q')$ there exists an $a \neq 0$ where for all $\varepsilon \in [0, |a|]$ it holds $x' + \varepsilon \text{sign}(a) v \in \text{supp}(Q')$.

Since $f^* \in C^1(X)$ it holds

$$\left. \frac{d}{d\varepsilon} f^*(x' + \varepsilon v) \right|_{\varepsilon = 0} = \langle v, \nabla_x f^*(x') \rangle.$$
Using Eq. 7 we see,
\[
E_{x \sim P} \left[ f^*(x) - f^*(x' + \varepsilon v) \right]
= \varepsilon \left\langle v, \nabla_x E_{x \sim P} \left[ f^*(x) - f^*(\hat{x}) \right] \right\rangle_{x'} + O(\varepsilon^2)
\]
\[
= \varepsilon \left\langle v, \begin{bmatrix} f^*(x) - f^*(\hat{x}) \end{bmatrix} \right\rangle_{x'} + O(\varepsilon^2)
\]
\[
= \frac{1}{\sqrt{x}} E_{x \sim P} \left[ \left\langle \hat{x} - x', f^*(\hat{x}) - f^*(x') \right\rangle \right]
= \frac{1}{\sqrt{x}} E_{x \sim P} \left[ \left\langle \hat{x} - x', f^*(\hat{x}) - f^*(x') \right\rangle \right]
\]
which means
\[
0 = \frac{d}{d\varepsilon} \left. E_{x \sim P} \left[ f^*(x) - f^*(x' + \varepsilon v) \right] \right|_{\varepsilon=0}
= -\frac{d}{d\varepsilon} f^*(x' + \varepsilon v)_{|\varepsilon=0} E_{x \sim P} \left[ \frac{1}{\|x - x'\|^2} \right] - E_{x \sim P} \left[ \left\langle v, x' - x \right\rangle f^*(x) - f^*(x') \right].
\]
Therefore,
\[
\frac{d}{d\varepsilon} f^*(x' + \varepsilon v)_{|\varepsilon=0} = \left\langle v, \nabla_x f^*(x) \right\rangle_{x'}
\]
\[
= \frac{E_{\hat{x} \sim P} \left[ \left\langle \hat{x} - x', f^*(\hat{x}) - f^*(x') \right\rangle \right]}{E_{\hat{x} \sim P} \left[ 1/\|\hat{x} - x'\| \right]}
\]
\[
= \frac{\left\langle v, E_{\hat{x} \sim P} \left[ (\hat{x} - x') f^*(\hat{x}) - f^*(x') \right] \right\rangle}{E_{\hat{x} \sim P} \left[ 1/\|\hat{x} - x'\| \right]}
\]
\[
= \frac{E_{\hat{x} \sim P} \left[ (\hat{x} - x') f^*(\hat{x}) - f^*(x') \right]}{E_{\hat{x} \sim P} \left[ 1/\|\hat{x} - x'\| \right]}
\]
Now from the proof of Lemma 5 claim (4), we know that since \(G(P, Q; f^*) = 0\) we get
\[
\|\nabla_x f^*(x)\|_{x'} = \frac{E_{\hat{x} \sim P} \left[ (\hat{x} - x') f^*(\hat{x}) - f^*(x') \right]}{E_{\hat{x} \sim P} \left[ 1/\|\hat{x} - x'\| \right]}
= \left\langle v, \nabla_x f^*(x) \right\rangle_{x'}
\]
and since for \(x \neq 0\) and \(\|w\| = 1\) it holds \(\langle w, x \rangle = \|x\| \iff \|w\| = x\) we discover \(\nabla_x f^*(x) = v\|\nabla_x f^*(x)\|_{x'}\) and thus
\[
\nabla_x f^*(x)_{x'} = v\|\nabla_x f^*(x)\|_{x'} = \frac{E_{\hat{x} \sim P} \left[ (\hat{x} - x') f^*(\hat{x}) - f^*(x') \right]}{E_{\hat{x} \sim P} \left[ 1/\|\hat{x} - x'\| \right]}
\]
and with
\[
\nabla_x f^*(x)_{x'} E_{\hat{x} \sim P} \left[ 1/\|\hat{x} - x'\| \right] = E_{\hat{x} \sim P} \left[ (\hat{x} - x') f^*(\hat{x}) - f^*(x') \right].
\]
Plugging this into Eq. 14 gives us
\[ \forall x' \in \text{supp}(Q') : \nabla_x E_{\tilde{x} \sim P} \left[ \frac{f^*(\tilde{x}) - f^*(x)}{\|\tilde{x} - x\|} \right]_{x'} = 0 \]

**Lemma 7.** Let \( P \) and \((Q_\theta)_{\theta \in \Theta}\) in \( P(X) \) and fulfill Assumptions 1 and 2, further let \((Q'_\theta)_{\theta \in \Theta}\) be as defined in introduction to Theorem 2, then for any \( f^* \in \text{OC}_{\tau_F}(P, Q'_\theta) \)
\[ \nabla_{\theta} \tau_F(P\|Q'_\theta) \approx -\frac{1}{2} \nabla_{\theta} E_{x' \sim Q'_\theta} [f^*(x')] \]

thus \( f^* \) fulfills Eq. 5 and \( \tau_F \) fulfills Requirement 4. Further, if \( P, Q_\theta \) are such that there exists an \( f \) with \( f(x) - f(x') = \|x - x'\| \) for all \( x \in \text{supp}(P) \) and \( x' \in \text{supp}(Q) \) then
\[ \nabla_{\theta} \tau_F(P\|Q_\theta) = -\frac{1}{2} \nabla_{\theta} E_{x' \sim Q_\theta} [f^*(x')] \]

**Proof.** Start off by noting that for some \( f^* \in \text{OC}_{\tau_F}(P, Q_\theta) \), Theorem 1 from (Milgrom & Segal, 2002) gives us
\[ \nabla_{\theta} \tau_F(P\|Q'_\theta) |_{\theta_0} = \nabla_{\theta} \tau_F(P\|Q'_\theta; f^*) |_{\theta_0} \]

Further, since for \( f^* \in \text{OC}_{\tau_F}(P, Q_\theta) \) it holds
\[ \|\nabla_x f^*(x)\|_{x'} = \frac{\|E_{\tilde{x} \sim P} (\tilde{x} - x') f^*(\tilde{x}) - f^*(x')\|}{E_{\tilde{x} \sim P} (1/\|\tilde{x} - x\|)} \]

the gradient of the gradient penalty part is zero, i.e.
\[ \nabla_{\theta} E_{x' \sim P, x' \sim Q_\theta} \left[ \|\nabla_x f^*(x)\|_{x'} - \frac{\|E_{\tilde{x} \sim P} (\tilde{x} - x') f^*(\tilde{x}) - f^*(x')\|}{E_{\tilde{x} \sim P} (1/\|\tilde{x} - x\|)} \right]^2 = 0. \]

One last point needs to be made before the main equation, which is for \( x \in \text{supp}(P) \)
\[ \nabla_{\theta} E_{x' \sim Q'_\theta} \left[ f^*(x) - f^*(x') \right]_{\|x - x'\|} \approx 0. \]

This is from the motivation of the penalized Wasserstein GAN where for an optimal critic it should hold that \( f^*(x) - f^*(x') \) is close to \( c\|x - x'\| \) for some constant \( c \). Note that if \( P \) and \( Q_\theta \) are such that \( f^*(x) - f^*(x') = c\|x - x'\| \) is possible everywhere, then this term is equal to zero.

\[ \nabla_{\theta} \tau_F(P\|Q_\theta) |_{\theta_0} = \nabla_{\theta} E_{P \| Q'_\theta} [(f^*(x) - f^*(x'))(1 - \lambda f^*(x) - f^*(x'))] |_{\theta_0}. \]

Since \( Q_\theta \) fulfills Assumption 1, \( Q_\theta \sim g(\theta, z) \) where \( g \) is differentiable in the first argument and \( z \sim Z \) (\( Z \) was defined in Assumption 1). Therefore if we set \( g_\theta(\cdot) = g(\theta, \cdot) \) we get
\[ \nabla_{\theta} \tau_F(P\|Q_\theta) |_{\theta_0} = \nabla_{\theta} E_{\tilde{x}, \tilde{z} \sim P, z \sim Z, \alpha \sim U([0, 1])} \left[ (f^*(x) - f^*(\alpha \tilde{x} + (1 - \alpha) g_\theta(z))) \left( 1 - \lambda f^*(x) - f^*(\alpha \tilde{x} + (1 - \alpha) g_\theta(z)) \right) \right] |_{\theta_0} \]
\[ = -E_{\tilde{x}, \tilde{z} \sim P, z \sim Z, \alpha \sim U([0, 1])} \nabla_{\theta} f^*(\alpha \tilde{x} + (1 - \alpha) g_\theta(z)) |_{\theta_0} \left( 1 - \lambda f^*(x) - f^*(\alpha \tilde{x} + (1 - \alpha) g_\theta(z)) \right) \]
\[ - \lambda E_{\tilde{x}, \tilde{z} \sim P, z \sim Z, \alpha \sim U([0, 1])} \left[ (f^*(x) - f^*(\alpha \tilde{x} + (1 - \alpha) g_\theta(z))) \nabla_{\theta} \left( f^*(x) - f^*(\alpha \tilde{x} + (1 - \alpha) g_\theta(z)) \right) \right] |_{\theta_0}. \]
Now if we look at the 17 term of the equation, we see that it’s equal to:

\[- \mathbb{E}_{\tilde{x} \sim \mathbb{P}, z, \alpha \sim \mathcal{U}([0, 1])} \left[ \nabla_\theta f^*(\alpha \tilde{x} + (1 - \alpha)g_\theta(z)) \right] \mathbb{E}_{\tilde{x} \sim \mathbb{P}} \left[ \mathbb{E}_{x' \sim \mathbb{Q}_\theta} \left[ f^*(x') \right] \right] \]

and term 18 of the equation is equal to

\[- \frac{1}{2} \nabla_\theta \mathbb{E}_{x' \sim \mathbb{Q}_\theta} \left[ f^*(x') \right] \theta_0 \]

thus showing

\[\nabla_\theta \tau_{\mathcal{P}}(\mathbb{P} \| \mathbb{Q}_\theta) \theta_0 \approx - \frac{1}{2} \nabla_\theta \mathbb{E}_{x' \sim \mathbb{Q}_\theta} \left[ f^*(x') \right] \theta_0 \]

Lemma 8. Let \( \tau_1 \) be the WGAN-GP divergence defined in Eq. 3, let the target distribution be the Dirac distribution \( \delta_0 \) and the family of generated distributions be the uniform distributions \( \mathcal{U}([0, \theta]) \) with \( \theta > 0 \). Then there is no \( C \in \mathbb{R} \) that fulfills Eq. 5 for all \( \theta > 0 \).

Proof. For convenience, we’ll restrict ourselves to the \( \lambda = 1 \) case, for \( \lambda \neq 1 \) the proof is similar. Assume that \( f \in \text{OC}_{\tau_1}(\delta_0, \mathcal{U}([0, \theta])) \) and \( f(0) = 0 \). Since \( f \) is an optimal critic, for any function \( u \in C^1(\mathcal{X}) \) and any \( \varepsilon \in \mathbb{R} \) it holds \( \tau_1(\delta_0 \| \mathcal{U}([0, \theta]); f) \geq \tau_1(\delta_0 \| \mathcal{U}([0, \theta]); f + \varepsilon u) \). Therefore \( \varepsilon = 0 \) is a maximum of the continuously differentiable function \( \varepsilon \mapsto \tau_1(\delta_0 \| \mathcal{U}([0, \theta]); f + \varepsilon u) \) and \( \frac{d}{d\varepsilon} \tau_1(\delta_0 \| \mathcal{U}([0, \theta]); f + \varepsilon u) \bigg|_{\varepsilon = 0} = 0 \). Therefore

\[\frac{d}{d\varepsilon} \tau_1(\delta_0 \| \mathcal{U}([0, \theta]); f + \varepsilon u) \bigg|_{\varepsilon = 0} = - \int_0^\theta u(t) dt - \int_0^\theta \frac{2}{t} \int_0^t u'(x) (f'(x) + 1) dx dt = 0\]

multiplying by -1 and deriving with respect to \( \theta \) gives us

\[u(\theta) + \frac{2}{\theta} \int_0^\theta u'(x) (f'(x) + 1) dx = 0.\]

Since we already made the assumption that \( f(0) = 0 \) and since \( \tau_1(\mathbb{P} \| \mathbb{Q}; f) = \tau_1(\mathbb{P} \| \mathbb{Q}; f + c) \) for any constant \( c \), we can assume that \( u(0) = 0 \). This gives us \( u(\theta) = \int_0^\theta u'(x) dx \) and thus

\[\int_0^\theta u'(x) dx + \frac{2}{\theta} \int_0^\theta u'(x) (f'(x) + 1) dx = \frac{2}{\theta} \int_0^\theta u'(x) \left( \frac{\theta}{2} + f'(x) + 1 \right) dx.\]

Therefore, for the optimal critic it holds \( f'(x) = -\left( \frac{\theta}{2} + 1 \right) \), and since \( f(0) = 0 \) the optimal critic is \( f(x) = -\left( \frac{\theta}{2} + 1 \right) x \).

Now

\[\frac{d}{d\theta} \mathbb{E}_{\mathcal{U}([0, \theta])}[f] = - \frac{d}{d\theta} \int_0^\theta \left( \frac{\theta}{2} + 1 \right) x dx = - \left( \frac{\theta}{2} + 1 \right) \theta \]

and

\[\frac{d}{d\theta} \mathbb{E}_{\delta_0 \& \mathcal{U}([0, \theta])}[\tau_1] = \frac{d}{d\theta} \frac{1}{\theta} \int_0^\theta \left( \frac{\theta}{2} \right)^2 dx = \frac{d}{d\theta} \frac{\theta^2}{4} = \frac{\theta}{2}.\]

Therefore there exists no \( \gamma \in \mathbb{R} \) such that Eq. 5 holds for every distribution in the WGAN-GP context.
B. Experiments

B.1. CelebA

The parameters used for CelebA training were:

```python
'batch_size': 64,
'betal': 0.5,
'c_dim': 3,
'calculate_slope': True,
'checkpoint_dir': 'logs/1127_220919_.0001_.0001/checkpoints',
'checkpoint_name': None,
'counter_start': 0,
'data_path': 'celeba_cropped/',
'dataset': 'celebA',
'discriminator_batch_norm': False,
'epoch': 81,
'fid_batch_size': 100,
'fid_eval_steps': 5000,
'fid_n_samples': 50000,
'fid_sample_batchsize': 1000,
'fid_verbose': True,
'gan_method': 'penalized_wgan',
'gradient_penalty': 1.0,
'incept_path': 'inception-2015-12-05/classify_image_graph_def.pb',
'input_fname_pattern': '*.jpg',
'input_height': 64,
'input_width': None,
'is_crop': False,
'is_train': True,
'learning_rate_d': 0.0001,
'learning_rate_g': 0.0005,
'lipschitz_penalty': 0.5,
'load_checkpoint': False,
'log_dir': 'logs/0208_191248_.0001_.0005/logs',
'lr_decay_rate_d': 1.0,
'lr_decay_rate_g': 1.0,
'num_discriminator_updates': 1,
'optimize_penalty': False,
'output_height': 64,
'output_width': None,
'sample_dir': 'logs/0208_191248_.0001_.0005/samples',
'stats_path': 'stats/fid_stats_celeba.npz',
'train_size': inf,
'visualize': False
```

The learned networks (both generator and critic) are then fine-tuned with learning rates divided by 10. Samples from the trained model can be viewed in figure 3.
Figure 3. Images from a First Order GAN after training on CelebA data set.
B.2. CIFAR-10

The parameters used for CIFAR-10 training were:

```
BATCH_SIZE: 64
BETA1_D: 0.0
BETA1_G: 0.0
BETA2_D: 0.9
BETA2_G: 0.9
BN_D: True
BN_G: True
CHECKPOINT_STEP: 5000
CRITIC_ITERS: 1
DATASET: cifar10
DATA_DIR: /data/cifar10/
DIM: 32
D_LR: 0.0003
FID_BATCH_SIZE: 200
FID_EVAL_SIZE: 50000
FID_SAMPLE_BATCH_SIZE: 1000
FID_STEP: 5000
GRADIENT_PENALTY: 1.0
G_LR: 0.0001
INCEPTION_DIR: /data/inception-2015-12-05
ITERS: 500000
ITER_START: 0
LAMBDA: 10
LIPSCHITZ_PENALTY: 0.5
LOAD_CHECKPOINT: False
LOG_DIR: logs/
MODE: fogan
N_GPUS: 1
OUTPUT_DIM: 3072
OUTPUT_STEP: 200
SAMPLES_DIR: /samples
SAVE_SAMPLES_STEP: 200
STAT_FILE: /stats/fid_stats_cifar10_train.npz
TBOARD_DIR: /logs
TTUR: True
```

The learned networks (both generator and critic) are then fine-tuned with learning rates divided by 10. Samples from the trained model can be viewed in figure 4.
Figure 4. Images from a First Order GAN after training on CIFAR-10 data set.
B.3. LSUN

The parameters used for LSUN Bedrooms training were:

```python
BATCH_SIZE: 64
BETA1_D: 0.0
BETA1_G: 0.0
BETA2_D: 0.9
BETA2_G: 0.9
BN_D: True
BN_G: True
CHECKPOINT_STEP: 4000
CRITIC_ITERS: 1
DATASET: lsun
DATA_DIR: /data/lsun
DIM: 64
D_LR: 0.0003
FID_BATCH_SIZE: 200
FID_EVAL_SIZE: 50000
FID_SAMPLE_BATCH_SIZE: 1000
FID_STEP: 4000
GRADIENT_PENALTY: 1.0
G_LR: 0.0001
INCEPTION_DIR: /data/inception-2015-12-05
ITERS: 500000
ITER_START: 0
LAMBDA: 10
LIPSCHITZ_PENALTY: 0.5
LOAD_CHECKPOINT: False
LOG_DIR: /logs
MODE: fogan
N_GPUS: 1
OUTPUT_DIM: 12288
OUTPUT_STEP: 200
SAMPLES_DIR: /samples
SAVE_SAMPLES_STEP: 200
STAT_FILE: /stats/fid_stats_lsun.npz
TBOARD_DIR: /logs
TTUR: True
```

The learned networks (both generator and critic) are then fine-tuned with learning rates divided by 10. Samples from the trained model can be viewed in figure 5.
Figure 5. Images from a First Order GAN after training on LSUN data set.
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Beacker it this that that W
Though ’s lunge plans wignsper c
He says : WalaMurka in the moroe
Dry Hall Siting tven the concer
There are court phinchs hasffort
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Figure 6. Samples generated by First Order GAN trained on the One Billion Word benchmark with FOGAN (left) the original TTUR method (right).

B.4. Billion Word

The parameters used for the Billion Word training were one run with the following settings, followed by a second run using initialized with the best saved model from the first run and learning rates divided by 10. Samples from our method and the WGAN-GP baseline can be found in figure 6.

`activation_d': 'relu',
'batch_norm_d': False,
'batch_norm_g': True,
'batch_size': 64,
'checkpoint_dir': 'logs/checkpoints/0201_181559_0.000300_0.000100',
'critic_iters': 1,
'data_path': '1-billion-word-language-modeling-benchmark-r13output',
'dim': 512,
'gan_divergence': 'FOGAN',
'gradient_penalty': 1.0,
'is_train': True,
'iterations': 500000,
'jsd_test_interval': 2000,
'learning_rate_d': 0.0003,
'learning_rate_g': 0.0001,
'lipschitz_penalty': 0.1,
'load_checkpoint_dir': 'False',
'log_dir': 'logs/tboard/0201_181559_0.000300_0.000100',
'max_n_examples': 10000000,
'n_ngrams': 6,
'num_sample_batches': 100,
'print_interval': 100,
'sample_dir': 'logs/samples/0201_181559_0.000300_0.000100',
'seq_len': 32,
'squared_divergence': False,
'use_fast_lang_model': True