Learning Decentralized Linear Quadratic Regulators
with $\sqrt{T}$ Regret

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Abstract

We propose an online learning algorithm that adaptively designs decentralized linear quadratic regulators when the system model is unknown a priori and new data samples from a single system trajectory become progressively available. The algorithm uses a disturbance-feedback representation of state-feedback controllers coupled with online convex optimization with memory and delayed feedback. Under the assumption that the system is stable or given a known stabilizing controller, we show that our controller enjoys an expected regret that scales as $\sqrt{T}$ with the time horizon $T$ for the case of partially nested information pattern. For more general information patterns, the optimal controller is unknown even if the system model is known. In this case, the regret of our controller is shown with respect to a linear sub-optimal controller. We validate our theoretical findings using numerical experiments.

1 Introduction

A fundamental challenge in decentralized control is that a local controller at a subsystem of the networked system may have access only to a subset of the global state information (e.g., [38, 25]). Control design under specified information patterns has been widely studied (e.g., [21, 36, 26, 17, 39]). Finding an optimal controller under general information constraints is NP-hard (e.g., [43, 4]). Much work has been done to identify special information constraints [21, 25, 26, 35, 36] that yield tractable optimal solutions. In particular, the partially nested information constraint [21, 25, 26] assumes that the information propagates at least as fast as dynamics in networked systems consisting of multiple interconnected subsystems [21].

However, almost all existing work assumes the knowledge of the system model when designing the control policy. In this paper, we study controller design for decentralized Linear Quadratic Regulator (LQR) when the system model is unknown. In this problem, the controller needs to be designed in an online manner utilizing data samples that become available from a single system trajectory. We propose a model-based learning algorithm that regulates the system while learning the system matrices (e.g., [12, 42, 9]). As is standard in online learning, we measure the performance of our controller design algorithm using the notion of regret (e.g., [7]), which compares the cost incurred by the controller designed using an online algorithm to that incurred by the optimal decentralized controller. Perhaps surprisingly, despite the existence of the information constraint, our regret bound matches with the one provided in [1, 11] for centralized LQR, which has also been shown to be the best regret guarantee that can be achieved by any online learning algorithms (up to logarithmic factors in the time horizon $T$ and other problem constants) [40, 8, 51].

Related Work

For control design in centralized LQR, many approaches have now appeared in the literature. Offline learning algorithms solve the case when data samples from the system trajectories can be collected offline before the

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control policy is designed (e.g., [13, 50]). The metric of interest here is sample complexity which relates the gap between the performances of the proposed control policy and the optimal control policy to the number of data samples from the system trajectories that are collected offline. Online learning algorithms solve the case when data samples from a single system trajectory become available in an online manner and the control policy needs to be designed simultaneously (e.g., [1, 11, 41, 8, 10, 24, 33]). The metric of interest here is the regret of the adaptive approach. These approaches can also be classified into model-based learning algorithms in which a system model is first estimated from data samples from the system trajectories and then a control policy is designed based on the estimated system model (e.g., [8, 37, 12, 31, 44]), or model-free learning algorithms in which a control policy is directly obtained from the system trajectories (e.g., [16, 30, 6, 20, 48, 32]) using zeroth-order optimization methods (e.g., [19]). We focus on model-based online learning of LQR controllers, but in a decentralized control problem.

As compared to the centralized case, there are only a few results available for solving decentralized linear quadratic control problems with information constraints and unknown system models. In [18], the authors assumed a quadratic invariance condition on the information constraint in a decentralized output-feedback linear quadratic control problem over a finite horizon and proposed a model-free offline learning algorithm along with a sample complexity analysis. In [29], the authors proposed a model-free offline learning algorithm for multi-agent decentralized LQR over an infinite horizon, where each agent (i.e., controller) has access to a subset of the global state without delay. They showed that their (consensus-based) algorithm converges to a control policy that is a stationary point of the objective function in the LQR problem. In [15], the authors studied model-based offline learning for LQR with subspace constraints on the closed-loop responses, which may not lead to controllers that satisfy the desired information constraints (e.g., [49]). Finally, in [45], the authors considered model-based offline learning for decentralized LQR with a partially nested information constraint using certainty equivalence approach and analyzed the sample complexity of the learning algorithm.

Contributions
Our approach and contributions can be summarized as follows.

• We begin by assuming that the information pattern is partially nested, for which an optimal decentralized controller can be designed if the model is known [26]. In Section 2.4, we show that this optimal controller can be cast into a Disturbance-Feedback Controller (DFC) which has been utilized in learning centralized LQR [3, 41, 28]. Adapting the DFC structure to a decentralized controller requires more care since the resulting DFC needs to respect the prescribed information pattern.

• In Section 3, we present our online decentralized control algorithm, which first identifies a system model based on a single system trajectory, and then adaptively designs a control policy (with the DFC structure and the partially nested information pattern) based on the estimated system model. The control policy utilizes a novel Online Convex Optimization (OCO) algorithm with memory and delayed feedback.

• In Section 4, we prove that the expected regret of our online decentralized control algorithm scales as $\sqrt{T}$ under the assumption that the system is stable, where $T$ is the length of the time horizon. We first analyze the regret bounds for the general OCO algorithm with memory and delayed feedback (which works for general OCO problems and is of independent interest), and then specialize the results to our setting. Surprisingly, our regret bound matches with the one provided in [1, 11] for learning centralized LQR in terms of $T$, which has been shown to be the best regret guarantee (up to logarithmic factors in $T$ and other constants) [40, 8, 10]. Our regret result is in stark contrast to the sample complexity result in [45] which shows a degradation for decentralized LQR compared to the centralized case [31].

• In Section 5, we show that all of our results can be extended to a general information pattern and stabilizable systems. Since the optimal decentralized controller with a general information pattern is unknown even if the model is known, the regret analysis compares our design to a particular sub-optimal controller.

Notation and Terminology The sets of integers and real numbers are denoted as $\mathbb{Z}$ and $\mathbb{R}$, respectively. The set of integers (resp., real numbers) that are greater than or equal to $a \in \mathbb{R}$ is denoted as $\mathbb{Z}_{\geq a}$ (resp., $\mathbb{R}_{\geq a}$). The space of $m$-dimensional real vectors is denoted by $\mathbb{R}^m$ and the space of $m \times n$ real matrices is
denoted by $\mathbb{R}^{m \times n}$. For a matrix $P \in \mathbb{R}^{n \times n}$, let $P^T$, $\text{Tr}(P)$, and $\{\sigma_i(P) : i \in \{1, \ldots, n\}\}$ be its transpose, trace, and the set of singular values, respectively. Without loss of generality, let the singular values of $P$ be ordered as $\sigma_1(P) \geq \cdots \geq \sigma_n(P)$. Let $\|\cdot\|$ denote the Euclidean norm for a vector or spectral norm for a matrix, i.e., $\|P\| = \sqrt{\sigma_1(P)}$ for $P \in \mathbb{R}^{n \times n}$ and $\|x\| = \sqrt{x^T x}$ for $x \in \mathbb{R}^n$. Let $\|P\|_F = \sqrt{\text{Tr}(P^TP)}$ denote the Frobenius norm of $P \in \mathbb{R}^{n \times m}$. A positive semidefinite matrix $P$ is denoted by $P \succeq 0$, and $P \succeq Q$ if and only if $P - Q \succeq 0$. Let $S_{+}^n$ (resp., $S_{++}^n$) denote the set of $n \times n$ positive semidefinite (resp., positive definite) matrices. Let $I_n$ denote an $n \times n$ identity matrix; the subscript is omitted if the dimension can be inferred from the context. Given any integer $n \geq 1$, we define $[n] = \{1, \ldots, n\}$. The cardinality of a finite set $A$ is denoted by $|A|$. Let $\mathcal{N}(0, \Sigma)$ denote a Gaussian distribution with mean 0 and covariance $\Sigma \succeq 0$. For a vector $x$, let $\text{dim}(x)$ denote its dimension. Let $\sigma(\cdot)$ denote the sigma field generated by the corresponding random vectors. For $[n]$ and $P_i \in \mathbb{R}^{n \times m_i}$ for all $i \in [n]$, denote $[P_i]_{i \in [n]} = [P_1 \cdots \ P_n]$.

2 Problem Formulation and Preliminary Results

2.1 Decentralized LQR with Sparsity and Delay Constraints

Consider a networked system with $p \in \mathbb{Z}_{\geq 1}$ dynamically coupled linear-time-invariant (LTI) subsystems. Let $\mathcal{V} = [p]$ be a set that contains the indices of all the $p$ subsystems. Specifically, denoting the state, input and disturbance of subsystem $i \in [p]$ at time $t$ by $x_{t,i} \in \mathbb{R}^{n_i}$, $u_{t,i} \in \mathbb{R}^{m_i}$, and $w_{t,i} \in \mathbb{R}^{n_i}$, respectively, the dynamics of subsystem $i$ is given by

$$x_{t+1,i} = \left( \sum_{j \in \mathcal{N}_i} A_{ij}x_{t,j} + B_{ij}u_{t,j} \right) + w_{t,i}, \quad \forall i \in \mathcal{V},$$

where $\mathcal{N}_i \subseteq [p]$ denotes the set of subsystems whose states and inputs directly affect the state of subsystem $j$, $A_{ij} \in \mathbb{R}^{n_i \times n_j}$ and $B_{ij} \in \mathbb{R}^{n_i \times m_j}$ are given coupling matrices, and $w_{t,i} \sim \mathcal{N}(0, \Sigma_i)$ is a white Gaussian noise process for $t \in \mathbb{Z}_{\geq 0}$ with $\Sigma_i \in \mathbb{R}^{n_i \times n_i}$. We assume that $w_{t,i}$ and $w_{t,j}$ are independent for all $i, j \in \mathcal{V}$ with $i \neq j$ and for all $t_1, t_2 \in \mathbb{Z}_{\geq 0}$, which implies that $w_t \sim \mathcal{N}(0, \Sigma_t)$ is a white Gaussian noise process for $t \in \mathbb{Z}_{\geq 0}$. For simplicity, we assume throughout that $n_i \geq m_i$ for all $i \in \mathcal{V}$. We can rewrite Eq. (1) as

$$x_{t+1,i} = A_i x_{t,\mathcal{N}_i} + B_i u_{t,\mathcal{N}_i} + w_{t,i}, \quad \forall i \in \mathcal{V},$$

where $A_i \triangleq [A_{ij}]_{j \in \mathcal{N}_i}$, $B_i \triangleq [B_{ij}]_{j \in \mathcal{N}_i}$, $x_{t,\mathcal{N}_i} \triangleq [x_{t,j}]^T_{j \in \mathcal{N}_i}$, and $u_{t,\mathcal{N}_i} \triangleq [u_{t,j}]^T_{j \in \mathcal{N}_i}$, with $\mathcal{N}_i = \{j_1, \ldots, j_{|\mathcal{N}_i|}\}$. Furthermore, letting $n = \sum_{i \in \mathcal{V}} n_i$ and $m = \sum_{i \in \mathcal{V}} m_i$, and defining $x_t = [x_{t,i}]_{i \in \mathcal{V}}^T$, $u_t = [u_{t,i}]_{i \in \mathcal{V}}^T$, and $w_t = [w_{t,i}]_{i \in \mathcal{V}}^T$, we can write Eq. (1) into the following matrix form:

$$x_{t+1} = A x_t + B u_t + w_t,$$

where the $(i,j)$th block of $A \in \mathbb{R}^{n \times n}$ (resp., $B \in \mathbb{R}^{n \times m}$), i.e., $A_{ij}$ (resp., $B_{ij}$) satisfies $A_{ij} = 0$ (resp., $B_{ij} = 0$) if $j \notin \mathcal{N}_i$. Without loss of generality, we assume that $x_0 = 0$.

**Information Structure.** A key difficulty in decentralized control is that the control input design must satisfy the constraints of a prescribed information flow among the subsystems in $[p]$. We can use a directed graph $\mathcal{G}(\mathcal{V}, \mathcal{A})$ (with $\mathcal{V} = [p]$) to characterize the information flow under sparsity and delay constraints on the communication among the subsystems, where each node in $\mathcal{G}(\mathcal{V}, \mathcal{A})$ represents a subsystem in $[p]$, and we assume that $\mathcal{G}(\mathcal{V}, \mathcal{A})$ does not have self loops. Moreover, we associate any edge $(i, j) \in \mathcal{A}$ with a delay of either 0 or 1, further denoted as $i \xrightarrow{0} j$ or $i \xrightarrow{1} j$, respectively. Then, we define the delay matrix corresponding to $\mathcal{G}(\mathcal{V}, \mathcal{A})$ as $D \in \mathbb{R}^{p \times p}$ such that: (i) If $i \neq j$ and there is a directed path from $j$ to $i$ in $\mathcal{G}(\mathcal{V}, \mathcal{A})$, then $D_{ij}$ is equal to the sum of delays along the directed path from node $j$ to node $i$ with the smallest accumulative delay; (ii) If $i \neq j$ and there is no directed path from $j$ to $i$ in $\mathcal{G}(\mathcal{V}, \mathcal{A})$, then $D_{ij} = +\infty$; (iii) $D_{ii} = 0$ for all $i \in \mathcal{V}$. For the remainder of this paper, we focus on the scenario where the information (e.g., state information)

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1. The analysis can be extended to the case when $w_t(t)$ is assumed to be a zero-mean white Gaussian noise process with covariance $W \in S_{++}^n$. In that case, our analysis will depend on $\max_{i \in \mathcal{V}} \sigma_i(W_i)$ and $\min_{i \in \mathcal{V}} \sigma_i(W_i)$.

2. The framework described in this paper can also be used to handle more general delay values; see [26] for a detailed discussion.
corresponding to subsystem \( j \in \mathcal{V} \) can propagate to subsystem \( i \in \mathcal{V} \) with a delay of \( D_{ij} \) (in time), if and only if there exists a directed path from \( j \) to \( i \) with an accumulative delay of \( D_{ij} \). As argued in [26], we assume that there is no directed cycle with zero accumulative delay; otherwise, one can first collapse all the nodes in such a directed cycle into a single node, and equivalently consider the resulting directed graph in the framework described above.

Thus, considering any \( i \in \mathcal{V} \) and any \( t \in \mathbb{Z}_{\geq 0} \), the state information that is available to the controller corresponding to \( i \in \mathcal{V} \) is given by

\[
\mathcal{I}_{t,i} = \{ x_{k,j} : j \in \mathcal{V}_i, 0 \leq k \leq t - D_{ij} \},
\]

where \( \mathcal{V}_i \triangleq \{ j \in \mathcal{V} : D_{ij} \neq +\infty \} \). In the sequel, we refer to \( \mathcal{I}_{t,i} \) as the information set of controller \( i \in \mathcal{V} \) at time \( t \in \mathbb{Z}_{\geq 0} \). Note that \( \mathcal{I}_{t,i} \) contains the states of the subsystems in \( \mathcal{V} \) such that there is sufficient time for these state values to reach subsystem \( i \in \mathcal{V} \) at time \( t \in \mathbb{Z}_{\geq 0} \), in accordance with the sparsity and delay constraints described above. Now, based on the information set \( \mathcal{I}_{t,i} \), we define \( \pi_i(\mathcal{I}_{t,i}) \) to be the set that consists of all the policies \( u_{t,i} \) that map the states in \( \mathcal{I}_{t,i} \) to a control input at node \( i \).

Similarly to [26, 45, 46], we make the following assumption on the information structure associated with system (1). Later in Section 5, we will show how to extend our analysis to the setting when Assumption 1 does not hold.

**Assumption 1.** For any \( i \in \mathcal{V} \), it holds that \( D_{ij} \leq 1 \) for all \( j \in \mathcal{N}_i \), where \( \mathcal{N}_i \) is given in Eq. (1).

The above assumption ensures that the state of subsystem \( i \in \mathcal{V} \) is affected by the state and input of subsystem \( j \in \mathcal{V} \), if and only if there is a communication link with a delay of at most 1 from subsystem \( j \) to \( i \) in \( \mathcal{G}(\mathcal{V}, \mathcal{A}) \). This assumption ensures that the information structure associated with the system given in Eq. (1) is partially nested [21], which is a condition for tractability of decentralized (or distributed) control problems that is frequently used in the literature (e.g., [26, 38] and the references therein). This assumption is also satisfied in networked systems where information propagates at least as fast as dynamics. To illustrate the above arguments, we introduce Example 1.

![Figure 1: The directed graph of Example 1. Node \( i \in \mathcal{V} \) represents a subsystem with state \( x_i(t) \), a solid edge \((i, j) \in \mathcal{A}\) is labeled with the information propagation delay from \( i \) to \( j \), and the dotted edges represent the coupling of the dynamics among the nodes in \( \mathcal{V} \).](image)

**Example 1.** Consider a directed graph \( \mathcal{G}(\mathcal{V}, \mathcal{A}) \) given in Fig. 1, where \( \mathcal{V} = \{1, 2, 3\} \) and each directed edge is associated with a delay of 0 or 1. The corresponding LTI system is then given by

\[
\begin{bmatrix}
    x_1(t + 1) \\
    x_2(t + 1) \\
    x_3(t + 1)
\end{bmatrix} =
\begin{bmatrix}
    A_{11} & A_{12} & A_{13} \\
    A_{21} & A_{22} & A_{23} \\
    0 & A_{32} & A_{33}
\end{bmatrix}
\begin{bmatrix}
    x_1(t) \\
    x_2(t) \\
    x_3(t)
\end{bmatrix} +
\begin{bmatrix}
    B_{11} & B_{12} & B_{13} \\
    B_{21} & B_{22} & B_{23} \\
    0 & B_{32} & B_{33}
\end{bmatrix}
\begin{bmatrix}
    u_1(t) \\
    u_2(t) \\
    u_3(t)
\end{bmatrix}.
\]

(5)

**Decentralized LQR and Its Solution.** The decentralized LQR problem can then be posed as follows:

\[
\min_{u_{t,i} \forall i \in \mathcal{V}, t \in \mathbb{Z}_{\geq 0}} \lim_{T \to \infty} \mathbb{E} \left[ \frac{1}{T} \sum_{t=0}^{T-1} (x_t^\top Q x_t + u_t^\top R u_t) \right]
\]

s.t. \( x_{t+1} = Ax_t + Bu_t + w_t \),

\( u_{t,i} \in \pi_i(\mathcal{I}_{t,i}), \forall i \in \mathcal{V}, \forall t \in \mathbb{Z}_{\geq 0}, \)

with cost matrices \( Q \in \mathbb{S}^n \) and \( R \in \mathbb{S}^{m \times n} \), and the expectation taken with respect to \( w_t \) for all \( t \in \mathbb{Z}_{\geq 0} \).

Following the steps in, e.g., [26], for solving (6), we first construct an information graph \( \mathcal{P}(\mathcal{U}, \mathcal{H}) \). Considering any directed graph \( \mathcal{G}(\mathcal{V}, \mathcal{A}) \) with \( \mathcal{V} = \{p\} \), and the delay matrix \( D \in \mathbb{R}^{p \times p} \) as we described above,
let us first define $s_{k,j}$ to be the set of nodes in $G(V, A)$ that are reachable from node $j$ within $k$ time steps, i.e., $s_{k,j} = \{i \in V : D_{ij} \leq k\}$. The information graph $P(U, H)$ is then constructed as

$$
U = \{s_{k,j} : k \geq 0, j \in V\},
$$

$$
H = \{(s_{k,j}, s_{k+1,j}) : k \geq 0, j \in V\}.
$$

We see from (7) that each node $s \in U$ corresponds to a set of nodes from $V = [p]$ in the original directed graph $G(V, A)$. As discussed in [26], the nodes in $U$ specify a partition of the noise history in (3) (i.e., $\{w_t\}_{t \geq 0}$) with respect to the information constraint, and an edge $(s_{k,j}, s_{k+1,j}) \in H$ indicates that the set of noise terms corresponding to $s_{k,j}$ belongs to that corresponding to $s_{k+1,j}$. If there is an edge from $s$ to $r$ in $P(U, H)$, we also denote the edge as $s \to r$. Additionally, considering any $s_0,i \in U$, we write $w_i \to s_0,i$ to indicate that the noise $w_{t,i}$ is injected to node $i \in V$ for $t \in \mathbb{Z}_{\geq 0}$.

The information graph $P(U, H)$ constructed from the directed graph $G(V, A)$ in Fig. 1 is given in Fig. 2.

Figure 2: The information graph of Example 1. Each node in the information graph is a subset of the nodes in the directed graph given in Fig. 1.

Throughout this paper and without loss of generality, we assume that the elements in $V = [p]$ are ordered in an increasing manner, and that the elements in $s$ are also ordered in an increasing manner for all $s \in U$. Moreover, for any $X, Y \subseteq V$, we use $A_{XY}$ (or $A_{X,Y}$) to denote the submatrix of $A$ that corresponds to the nodes of the directed graph $G(V, A)$ contained in $X$ and $Y$. For example, $A_{\{1\}, \{1,2\}} = [A_{11} \ A_{12}]$. In the sequel, we will also use similar notations to denote submatrices of other matrices, including $B$, $Q$, $R$ and the identity matrix $I$. We will make the following standard assumption (see, e.g., [26]).

**Assumption 2.** For any $s \in U$ that has a self loop, the pair $(A_{ss}, B_{ss})$ is stabilizable and the pair $(A_{ss}, C_{ss})$ is detectable, where $C_{ss}$ is such that $Q_{ss} = C_{ss}^\top C_{ss}$.

Leveraging the partial nestedness from Assumption 1 and the information graph constructed above, [26] gives a closed-form expression for the optimal solution to (6).

**Lemma 1.** [26, Corollary 4] Suppose Assumptions 1-2 hold. Consider the problem given in (6), and let $P(U, H)$ be the associated information graph. Suppose Assumption 2 holds. For all $r \in U$, define matrices $P_r$ and $K_r$ recursively as

$$
K_r = -(R_{rr} + B_{sr}^\top P_r B_{sr})^{-1} B_{sr}^\top P_r A_{sr},
$$

$$
P_r = Q_{rr} + K_r^\top R_{rr} K_r + (A_{sr} + B_{sr} K_r)^\top P_s (A_{sr} + B_{sr} K_r),
$$

where for each $r \in U$, $s \in U$ is the unique node such that $r \to s$. In particular, for any $s \in U$ that has a self loop, the matrix $P_s$ is the unique positive semidefinite solution to the Riccati equation given by Eq. (9), and

\(^3\)Since we have assumed that there is no directed cycle with zero accumulative delay in $P(U, H)$, one can show that $w_{t,i}$ is the unique noise term such that $w_{t,i}$
the matrix $A_{ss} + B_{ss}K_s$ is stable. The optimal solution to (6) is then given by

$$\zeta_{t+1,s} = \sum_{r \rightarrow s} (A_{sr} + B_{sr}K_r)\zeta_{t,r} + \sum_{w_i \rightarrow s} I_s, i) w_t,i,$$

$$u^*_t,i = \sum_{r \rightarrow s} I_{(i),r}K_r\zeta_{t,r},$$

for all $t \in \mathbb{Z}_{\geq 0}$, where $\zeta_{t,s}$ is an internal state initialized with $\zeta_{0,s} = \sum_{w_i \rightarrow s} I_s, i) x_0,i = 0$ for all $s \in \mathcal{U}$. The corresponding optimal cost of (6), denoted as $J_*$, is given by

$$J_* = \sigma^2 \sum_{i \in \mathcal{V}} \text{Tr}(I_{(i),s}^2 P_s I_{(s),i}^2).$$

Note that Eqs. (8)-(9) need to be computed for all $s \in \mathcal{U}$ to obtain $u^*_t,i$, however, from [26, Proposition 1], $|\mathcal{U}| \leq p^2 - p + 1$, where $p$ is the number of nodes in $\mathcal{G}(\mathcal{V}, \mathcal{A})$.

### 2.2 Problem Considered in this Paper and Our Result

We consider the problem of online learning for decentralized LQR with the information structure described in Section 2.1 when the system model is unknown a priori. Following [38, 26, 45, 46], we assume that the cost matrices $Q, R$, the graph $\mathcal{G}(\mathcal{V}, \mathcal{A})$ and the delay matrix $D$ are known. For any $t \in \mathbb{Z}_{\geq 0}$ and any $i \in \mathcal{V}$, a decentralized online control algorithm first chooses $u^*_{t,i} \in \pi_i(\mathcal{L}, i)$ based only on $x^*_{0,i}, \ldots, x^*_{t,i}$ observed so far and then incurs a cost $c(x^*_{t,i}, u^*_{t,i})$, where $c(x,u) \triangleq x^\top Q x + u^\top R u$ and $x^*_{t,i}$ is the state of subsystem (3) when the inputs $u^*_{0,i}, \ldots, u^*_{t-1,i}$ designed by the algorithm were applied. We wish to find a decentralized online control algorithm that minimizes the regret defined as

$$\text{Regret} = \sum_{t=0}^{T-1} (c(x^*_{t,i}, u^*_{t,i}) - J_*),$$

which compares the cost incurred by the algorithm against the optimal cost $J_*$ of (6) (given in Lemma 1) over the horizon $T \in \mathbb{Z}_{\geq 1}$. The following standard result is standard, which will be useful in our later analysis.

**Lemma 2.** Consider $Q \in \mathbb{S}^n_+$ and $R \in \mathbb{S}^n$. The quadratic function $c(x,u) = x^\top Q x + u^\top R u$ is Lipschitz continuous such that

$$|c(x_1,u_1) - c(x_2,u_2)| \leq 2(R_x + R_u) \max\{\sigma_1(Q), \sigma_2(R)\} \norm{x_1 - x_2} + \norm{u_1 - u_2},$$

for all $x_1, x_2 \in \mathbb{R}^n$ and all $u_1, u_2 \in \mathbb{R}^m$ such that $\norm{x_1} \leq R_x, \norm{x_2} \leq R_x, \norm{u_1} \leq R_u, \norm{u_2} \leq R_u$.

**Main result (informal).** We design an algorithm for learning the decentralized LQR under the information constraint that achieves $\text{Regret} = \mathcal{O}(\sqrt{T})$, where $\mathcal{O}(\cdot)$ hides polynomial factors in $\log T$ and other problem parameters.

As we discussed in the introduction, $\mathcal{O}(\sqrt{T})$ matches with the regret lower bound for learning centralized LQR which naturally holds for the decentralized setting.

### 2.3 Summary of Main Symbols

To ease our presentation in the remainder of this paper, we summarize the main symbols used throughout. $i \in \mathcal{V}$ represents a subsystem from $\mathcal{V} = [p]$ and the corresponding controller of the subsystem. $\mathcal{P}(\mathcal{U}, \mathcal{H})$ is the information graph constructed by (7) from the directed graph $\mathcal{G}(\mathcal{V}, \mathcal{A})$ and $D_{\text{max}}$ is the maximum accumulative delay along any path in $\mathcal{G}(\mathcal{V}, \mathcal{A})$. $\mathcal{D}$ and $\mathcal{D}_0$ represent sets of DFC parameterized by some $M$. $u$, $x$, and $\zeta$, $w$ and $\eta$ refer to control input, system state, and internal state (see Lemma 1), disturbance and concatenate disturbance vector, respectively. Subscript $i \in \mathcal{V}$ (resp., $s \in \mathcal{U}$) denotes the node index in $\mathcal{V}$ (resp., $\mathcal{U}$). Superscript $M$ (resp., $\text{alg}$) refers to the quantities corresponding to a DFC (resp., the proposed algorithm). For example, $x^*_{t,i}$ represents the state vector of system (3) when the control inputs $u^*_{0,i}, \ldots, u^*_{t-1,i}$ were applied, and $x^*_{t,i}$ refers to the states in $x^*_{t,i} \in \mathbb{R}^n$ that correspond to subsystem $i \in \mathcal{V}$. $R_0$ typically denotes an upper bound on $\phi \in \{u, x, w\}$. Using * in a symbol means the symbol represents some optimal quantity, e.g., $J_*$ is the optimal cost of (6). Finally, $\widehat{\cdot}$ is used for estimated quantities, and $\varepsilon$ denotes estimation error.
2.4 Disturbance-Feedback Controller

For our analyses in the remaining of this paper, we cast the optimal control policy given by Lemma 1 into a Disturbance-Feedback Controller (DFC) introduced in, e.g., [47, 26, 41]). As we will see in later sections, it is crucial to leverage the DFC structure in our decentralized control policy design in order to achieve the desired $O(\sqrt{T})$ regret. To proceed, recall the information graph $\mathcal{P}(\mathcal{U}, \mathcal{H})$ constructed in (7). Let $\mathcal{L}$ denote the set of all the leaf nodes in $\mathcal{P}(\mathcal{U}, \mathcal{H})$, i.e.,

$$\mathcal{L} = \{ s_{0,i} \in \mathcal{U} : i \in \mathcal{V} \}. \quad (14)$$

For any $s \in \mathcal{U}$, we denote

$$\mathcal{L}_s = \{ v \in \mathcal{L} : v \leadsto s \}, \quad (15)$$

where we write $v \leadsto s$ if and only if there is a unique directed path from node $v$ to node $s$ in $\mathcal{P}(\mathcal{U}, \mathcal{H})$. In other words, $\mathcal{L}_s$ is the set of leaf nodes in $\mathcal{P}(\mathcal{U}, \mathcal{H})$ that can reach $s$. Moreover, for any $v, s \in \mathcal{U}$ such that $v \leadsto s$, let $l_{vs}$ denote the length of the unique directed path from $v$ to $s$ in $\mathcal{P}(\mathcal{U}, \mathcal{H})$; let $l_{vs} = 0$ if $v = s$. Furthermore, let $\mathcal{U}_1$ (resp., $\mathcal{U}_2$) be the set of all the nodes in $\mathcal{U}$ that have (resp., do not have) a self loop. As shown in [45], for any $s \in \mathcal{U}_1$, one can rewrite Eq. (10) as

$$\zeta_{t+1,s} = (A_{ss} + B_{ss}K_s)\zeta_{t,s} + \sum_{v \in \mathcal{L}_s} H_{v,s}I_{v,\{j_v\}}w_{t-l_{vs},j_v}, \quad (16)$$

with $w_{j_v} \rightarrow v$, where

$$H_{v,s} \equiv (A_{sr_1} + B_{sr_1}K_{r_1}) \cdots (A_{r_{l_{vs}}-1} + B_{r_{l_{vs}}-1}K_v), \quad (17)$$

with $H_{v,s} = I$ if $v = s$, $K_v$ is given by Eq. (8) for all $r \in \mathcal{U}$, and $v, r_{l_{vs}}-1, \ldots, r_1$ are the nodes along the directed path from $v$ to $s$ in $\mathcal{P}(\mathcal{U}, \mathcal{H})$. Note that we let $w_t = 0$ for all $t < 0$. Unrolling Eq. (16), and recalling that $\zeta_{0,s} = 0$ for all $s \in \mathcal{U}$, we have

$$\zeta_{t,s} = \sum_{k=0}^{t-1} (A_{ss} + B_{ss}K_s)^{t-(k+1)} \sum_{v \in \mathcal{L}_s} H_{v,s}I_{v,\{j_v\}}w_{k-l_{vs},j_v},$$

$$= \sum_{k=0}^{t-1} (A_{ss} + B_{ss}K_s)^{t-(k+1)} [H_{v,s}I_{v,\{j_v\}}]_{v \in \mathcal{L}_s} \eta_{k,s}, \quad (18)$$

where

$$\eta_{k,s} = [w_{k-l_{vs},j_v}^T]_{v \in \mathcal{L}_s}^T. \quad (19)$$

Similarly, for any $s \in \mathcal{U}_2$, one can rewrite Eq. (10) as

$$\zeta_{t,s} = \sum_{v \in \mathcal{L}_s} H_{v,s}I_{v,\{j_v\}}w_{t-1-l_{vs},j_v},$$

$$= [H_{v,s}I_{v,\{j_v\}}]_{v \in \mathcal{L}_s} \eta_{t-1,s}. \quad (20)$$

Using Eqs. (18) and (20), one can then show that $u_t^* = \sum_{s \in \mathcal{U}} I_{v,s}K_s\zeta_{t,s}$ given by Eq. (10) can be expressed as the following DFC:

$$u_t^* = \sum_{s \in \mathcal{U}} I_{v,s} \sum_{k=1}^{t} M_{s,k}^s \eta_{t-k,s}, \quad \forall t \in \mathbb{Z}_{\geq 0} \quad (21)$$

$$M_{s,k}^s = \begin{cases} K_s(A_{ss} + B_{ss}K_s)^{k-1} [H_{v,s}I_{v,\{j_v\}}]_{v \in \mathcal{L}_s} & \text{if } s \in \mathcal{U}_1 \text{ and } k \in [t], \\ K_s [H_{v,s}I_{v,\{j_v\}}]_{v \in \mathcal{L}_s} & \text{if } s \in \mathcal{U}_2 \text{ and } k = 1, \\ 0 & \text{if } s \in \mathcal{U}_2 \text{ and } k \in \{2, \ldots, t\}. \end{cases} \quad (22)$$

Note that $M_{s,k}^s \in \mathbb{R}^{n_x \times n_c}$ for all $s \in \mathcal{U}$ and all $k \in [t]$, where $n_s = \sum_{i \in \mathcal{U}_s} n_i$ and $n_{\mathcal{L}_s} = \sum_{v \in \mathcal{L}_s} n_v$. Hence, Eq. (21) gives a disturbance-feedback representation of the optimal control policy $u_t^*$. Note that $u_t^*$ given
in Eq. (21) depends on all the past disturbances \( w_0, \ldots, w_{t-1} \). To mitigate the dependency on all the past disturbances, we further introduce a (truncated) DFC defined as follows (e.g., [2]). We will later show in our theoretical analysis in Section 4 that leveraging the DFC given by Definition 1 in our algorithm design suffices to achieve the \( \sqrt{T} \)-regret.

**Definition 1.** A DFC parameterized by \( M = [M_s^k]_{k \in [h], s \in \mathcal{U}} \) is given by

\[
u_t^M = \sum_{s \in \mathcal{U}} \sum_{k=1}^h I_{s, s} M_s^k \eta_{t-k, s}, \quad \forall t \in \mathbb{Z}_{\geq 0},
\]  

where \( h \in \mathbb{Z}_{\geq 1} \), and \( \eta_{k, s} \) is given by Eq. (19) and satisfies \( \eta_{k, s} = 0 \) for all \( k < 0 \) and all \( s \in \mathcal{U} \). Moreover, let \( x_t^M \) be the state of system (3) when the input sequence \( u_0^M, \ldots, u_t^M \) is applied.

We assume that the following assumption holds for now and similar assumptions can be found in [28, 24]. Later in Section 5, we show that Assumption 3 can be relaxed.

**Assumption 3.** The system matrix \( A \in \mathbb{R}^{n \times n} \) is stable, and \( \|A^k\| \leq \kappa_0 \gamma_0^k \) for all \( k \in \mathbb{Z}_{\geq 0} \), where \( \kappa_0 \geq 1 \) and \( \rho(A) < \gamma_0 < 1 \).

Now, recall from Lemma 1 that for any \( s \in \mathcal{U}_1 \), the matrix \( A_{ss} + B_{ss} K_s \) is stable, where \( K_s \) is given by Eq. (8). We then have from the Gelfand formula (e.g., [22]) that for any \( s \in \mathcal{U}_1 \), there exist \( \kappa_s \in \mathbb{R}_{\geq 1} \) and \( \gamma_s \in \mathbb{R} \) with \( \rho(A_{ss} + B_{ss} K_s) < \gamma_s < 1 \) such that \( \|(A_{ss} + B_{ss} K_s)^k\| \leq \kappa_s \gamma_s^k \) for all \( k \in \mathbb{Z}_{\geq 0} \). To proceed, we denote

\[
\Gamma = \max \{ \| A \|, \| B \|, \max_{s \in \mathcal{U}_1} \| P_s \|, \max_{s \in \mathcal{U}_1} \| K_s \|, 1 \},
\gamma = \max_{s \in \mathcal{R}} \{ \max \gamma_s, \gamma_0 \}, \kappa = \{ \max \kappa_s, \kappa_0 \},
\]  

where \( \mathcal{R} \subseteq \mathcal{U} \) denotes the set of root nodes in \( \mathcal{U} \), and \( P_s \) is given by Eq. (9) for all \( s \in \mathcal{U} \). Similarly to the learning algorithms for LQR proposed in [9, 29, 45], we assume (upper bounds on) the parameters in (24) are known for our algorithm design. Moreover, we denote

\[
D_{\text{max}} = \max_{v, j \in \mathcal{V}} D_{ij},
\]  

where we write \( j \sim i \) if and only if there is a directed path from node \( j \) to node \( i \) in \( G(\mathcal{V}, \mathcal{A}) \), and recall that \( D_{ij} \) is the sum of delays along the directed path from \( j \) to \( i \) with the smallest accumulative delay. Note that \( D_{\text{max}} \ll p \) when the delay in the information flow among the nodes in \( \mathcal{V} = [p] \) is small. We then have the following result, which shows that the optimal DFC given by (21) belongs to a class of DFCs with bounded norm.

**Lemma 3.** For \( M_{s, s} \) given by (22), it holds that

\[
\| M_{s, s}^k \| \leq \kappa \gamma^{k-1} \rho^{2D_{\text{max}} + 1}, \quad \| M_{s, s}^k \|_{F} \leq \sqrt{n} \kappa \gamma^{k-1} \rho^{2D_{\text{max}} + 1},
\]

for all \( k \in [t] \) and all \( s \in \mathcal{U} \), where \( h \in \mathbb{Z}_{\geq 1} \).

**Proof.** First, we have from Eq. (22) that for any \( s \in \mathcal{U}_1 \) and any \( k \in [t] \),

\[
\| M_{s, s}^k \| \leq \| K_s \| \|(A_{ss} + B_{ss} K_s)^k\| \|[H_{v, s} I_{v, (j_s)}]\|_{v \in \mathcal{L}_s}
\leq \Gamma \kappa_s \gamma_s^{k-1} \sum_{v \in \mathcal{L}_s} \| H_{v, s} I_{v, (j_s)} \|
\leq \Gamma \kappa_s \gamma_s^{k-1} \mathcal{L}_s \rho^{2D_{\text{max}}} \leq \kappa \gamma^{k-1} \rho^{2D_{\text{max}} + 1},
\]
where $\mathcal{L}_s = \{v \in \mathcal{L} : v \leadsto s\}$ is the set of leaf nodes in $\mathcal{P}(\mathcal{U}, \mathcal{H})$ that can reach $s$, and the third inequality follows from the facts that $|\mathcal{L}_s| \leq |\mathcal{V}| = p$ and $\|H_{v,s}\| \leq \Gamma_{2\max}$ for all $v, s \in \mathcal{U}$. Similarly, we have from (22) that for any $s \in \mathcal{U}$ and any $k \in [t]$, 
$$
\|M_s[k]\| \leq \kappa s^{-1} p \Gamma_{2\max}. \tag{25}
$$

Finally, for any $s \in \mathcal{U}$ and any $k \in [t]$, we have from the above arguments that
$$
\|M_s[k]\|_{2} \leq \sqrt{\min\{n_s, n_{\mathcal{L}_s}\}} \|M_s[k]\|_{F} \leq \sqrt{n_s \kappa_s^{-1}} p \Gamma_{2\max} + 1,
$$
where $M_s[k] \in \mathbb{R}^{n_s \times n_{\mathcal{L}_s}}$, and we use the fact that $n_s \leq n_{\mathcal{L}_s}$ with $n_s \leq n$.

**Candidate DFCs.** Based on Definition 1 and Lemma 3, we define a class of DFCs parameterized by $M = [M_s[k]]_{k \in [n], s \in \mathcal{U}}$:
$$
\mathcal{D} = \left\{ M = [M_s[k]]_{k \in [n], s \in \mathcal{U}} : \|M_s[k]\|_{F} \leq 2 \sqrt{n_s \kappa_s^{-1}} p \Gamma_{2\max} + 1 \right\}. \tag{26}
$$

Our decentralized control policy design is then based on the DFCs from $\mathcal{D}$. Consider any $M \in \mathcal{D}$ and any $i \in \mathcal{V}$, the corresponding control input $u_{t,i}^M$ (i.e., the $i$th element of $u_t^M$ given by Definition 1) needs to be determined based on the state information in $\mathcal{T}_{t,i}$ defined in Eq. (4). In particular, the past disturbances that are required to compute $u_{t,i}^M$ may be determined exactly via Eq. (3), using the state information in $\mathcal{T}_{t,i}$ and the system matrices (e.g., [26, 45]). Since we consider the scenario with unknown system matrices $A$ and $B$, $u_t^M$ given by Definition 1 cannot be directly implemented, which together with the information constraints create the major challenge when designing and analyzing our decentralized online control algorithm in the next section.

## 3 Algorithm Design

The decentralized online control algorithm that we propose contains two phases. The first phase is a pure exploration phase dedicated to identifying the system matrices $A$ and $B$. The second phase leverages an Online Convex Optimization (OCO) algorithm to find the decentralized control policy.

### 3.1 Phase I: System Identification

**Algorithm 1 Least Squares Estimation of $A$ and $B$**

**Input:** parameter $\lambda \in \mathbb{R}_{>0}$, time horizon $N$, graph $G(\mathcal{V}, \mathcal{A})$ with delay matrix $D$

1. for $t = 0, \ldots, N-1$ and for each $i \in \mathcal{V}$ in parallel do
   1. Play $u_{t,i}^{\text{alg, i.d.}} \sim \mathcal{N}(0, \sigma_u^2 I_{m_i})$
   2. Obtain $\Phi_N$ using (28)
   3. Extract $\hat{A}$ and $\hat{B}$ from $\Phi_N$
   4. $A \leftarrow \hat{A}$, $B \leftarrow \hat{B}$
   5. Set $\hat{A}_{ij} = 0$ and $\hat{B}_{ij} = 0$ if $D_{ij} = \infty$
   6. Return $\hat{A}$ and $\hat{B}$

During the first phase, the algorithm uses a least squares method to obtain estimates of $A$ and $B$, denoted as $\hat{A}$ and $\hat{B}$, respectively, using a single system trajectory consisting of the control input sequence $\{u_0^{\text{alg}}, \ldots, u_{N-1}^{\text{alg}}\}$ and the corresponding system state sequence $\{x_0^{\text{alg}}, \ldots, x_N^{\text{alg}}\}$, where $N \in \mathbb{Z}_{\geq 1}$. Here, the inputs $u_0^{\text{alg}}, \ldots, u_{N-1}^{\text{alg}}$ are drawn independently from a Gaussian distribution $\mathcal{N}(0, \sigma_u^2 I_{m})$, where $\sigma_u \in \mathbb{R}_{>0}$.

In other words, we have $u_t^{\text{alg, i.d.}} \sim \mathcal{N}(0, \sigma_u^2 I_{m})$ for all $t \in \{0, \ldots, N-1\}$. Moreover, we assume that the input $u_t$ and the disturbance $w_t$ are independent for all $t \in \{0, \ldots, N-1\}$. Specifically, denote
$$
\Phi = \begin{bmatrix} A & B \end{bmatrix}, \; z_t = \begin{bmatrix} x_t^{\text{alg}} \; u_t^{\text{alg}} \end{bmatrix}^T, \tag{27}
$$
where $\Phi \in \mathbb{R}^{n \times (n+m)}$ and $z_t \in \mathbb{R}^{n+m}$. Given the sequences $\{z_0, \ldots, z_{N-1}\}$ and $\{x^\alg_1, \ldots, x^\alg_N\}$, the algorithm uses a regularized least squares method to obtain an estimate of $\Phi$, denoted as $\hat{\Phi}_N$, i.e.,

$$
\hat{\Phi}_N = \arg\min_{Y \in \mathbb{R}^{n \times (n+m)}} \left\{ \lambda \|Y\|^2_F + \sum_{t=0}^{N-1} \|x^\alg_{t+1} - Y z_t\|^2 \right\},
$$

(28)

where $\lambda \in \mathbb{R}_{>0}$ is the regularization parameter. The least squares method is summarized in Algorithm 1. Note that the last step in Algorithm 1 sets certain elements in $\hat{A}$ and $\hat{B}$ (extracted from $\hat{\Phi}_N$) to be zero, according to the directed graph $G(V, A)$ and the corresponding delay matrix $D$ as we described in Section 2.1. Algorithm 1 returns the final estimates of $\hat{A}$ and $\hat{B}$ as $\hat{A}$ and $\hat{B}$, respectively. Denote

$$
\Delta = \Phi - \hat{\Phi}_N, \quad \Delta_N = \Phi - \hat{\Phi}_N,
$$

(29)

where $\hat{\Phi}_N = [\hat{A} \quad \hat{B}]$, with $\hat{A}$ and $\hat{B}$ returned by Algorithm 1. To obtain $\hat{A}$ and $\hat{B}$, we set certain entries in $\hat{A}$ and $\hat{B}$ obtained from (28) to zero according to the delay matrix $D$ that is assumed to be known. In fact, there exist system identification methods (for sparse system identification) that return $\hat{A}$ and $\hat{B}$ with the same sparsity pattern as $A$ and $B$, under extra assumptions on $A$ and $B$ (e.g., [14]). However, the extra assumptions on $A$ and $B$ can be restrictive and hard to check in practice.

**Gaussian Inputs.** Algorithm 1 needs to use inputs drawn from a Gaussian distribution, which is typical in algorithms for learning (centralized) LQR (e.g., [11, 9, 41]). As we will see later, using the Gaussian inputs is crucial to provide upper bounds on the estimation error terms $\Delta$ and $\Delta_N$. We leave relaxing the Gaussian input assumption to future work.

**Requirement on Global Knowledge.** Algorithm 1 considers the scenario where each $i \in V$ plays the control $u^\alg_{i,t} \sim \mathcal{N}(0, \sigma^2_{i,t} I_{m_i})$ in parallel for $t = 0, \ldots, N-1$, which implies $u^\alg_{i,t} \sim \mathcal{N}(0, \sigma^2_{i,t} I_{m_i})$, and sends the local $x^\alg_{i,t}$ and $\sigma^\alg_{i,t}$ to a central agent for $t = 0, \ldots, N-1$. The central agent obtains $\hat{\Phi}_N$ using (28) based on a single system trajectory $\{(x^\alg_0, u^\alg_0), \ldots, (x^\alg_{N-1}, u^\alg_{N-1})\}$, and then sends the estimates $\hat{A}, \hat{B}$ back to each $i \in V$. Nonetheless, Algorithm 1 can be implemented without violating the information constraints given by Eq. (4), since $u^\alg_{i,t} \sim \mathcal{N}(0, \sigma^2_{i,t} I_{m_i})$ is not a function of the states in the information set defined in Eq. (4) for any $t \in \{0, \ldots, N-1\}$. After receiving the global $\hat{A}$ and $\hat{B}$, each $i \in V$ enters the second phase of the algorithm and computes $u^\alg_{i,t}$ in a fully decentralized manner (without access to the centralized agent). Even when the system model is known, existing algorithms for decentralized controller design typically require the global knowledge of the system model (e.g., [26, 25, 35, 36]). Relaxing this requirement is an interesting avenue for future work.

### 3.2 Phase II: Decentralized Online Control

The second phase leverages a general OCO algorithm which we introduce first.

#### 3.2.1 OCO with Memory and Delayed Feedback

We solve a general OCO problem with memory and delayed feedback, which is of independent interest. At each time step $t \geq \tau$, a decision maker first chooses $x_t \in \mathcal{W} \subseteq \mathbb{R}^d$ and incurs a cost $F_t(x_{t-\tau}, \ldots, x_t)$, and the function $F_{t-\tau}()$ is revealed to the decision maker, with $x_k \in \mathcal{W}$ arbitrarily if $k < 0$. Different from classic OCO frameworks, but unifying the ones in [3] and [27], the incurred cost at time step $t$ has a memory that depends on the choices $x_{t-\tau}, \ldots, x_t$, and the function $F_{t-\tau}()$ is revealed at time step $t$. The learner wishes to minimize its regret $\sum_{t=\tau}^{T-1} (F_t(x_{t-\tau}, \ldots, x_t) - f_t(x^*))$ for some $x^* \in \mathcal{W}$, where $f_t(x) = F_t(x, \ldots, x)$ for $x \in \mathcal{W}$. For this, we propose Algorithm 2 based on an online projected gradient descent scheme, where $x_{t+1}$ is updated from $x_t$ using the delayed gradient $g_{t-\tau}$ from $\nabla f_{t-\tau}(x_{t-\tau})$ with an additive error $\epsilon_{t-\tau} \in \mathbb{R}$. This error is useful when the gradient may not be exactly evaluated as in our decentralized LQR design.

We analyze the regret of Algorithm 2 under the following assumptions.

**Assumption 4.** For any $t \in \{0, \ldots, T-1\}$, the cost function $F_t : \mathcal{W}^{h+1} \rightarrow \mathbb{R}$ is $L_c$-coordinatewise-Lipschitz.$^4$

\[ F_t() \text{ is } L_c\text{-coordinatewise-Lipschitz if } |F_t(x_0, \ldots, x_j, \ldots, x_h) - F_t(x_0, \ldots, \tilde{x}_j, \ldots, x_h)| \leq L_c \|x_j - \tilde{x}_j\|, \text{ for all } j \in \{0, \ldots, h\} \text{ and all } x_j, \tilde{x}_j \in \mathcal{W}. \]
Algorithm 2 OCO with memory and delayed feedback

Input: time horizon $T$, step size $\eta_t \forall t \in \{\tau, \ldots, T + \tau\}$, delay $\tau \in \mathbb{Z}_{\geq 0}$, feasible set $\mathcal{W}$

1: Initialize $x_0 = \cdots = x_{\tau - 1} \in \mathcal{W}$ arbitrarily
2: for $t = \tau, \ldots, T - 1 + \tau$ do
3: Obtain $g_{t-\tau} = \nabla f_{t-\tau}(x_{t-\tau}) + \varepsilon_{t-\tau}$
4: Update $x_{t+1} = \Pi_{\mathcal{W}}(x_t - \eta_t g_{t-\tau})$

Assumption 5. Let $k \in \mathbb{Z}_{\geq 0}$ with $\tau < k < T$ and $\tau \in \mathbb{Z}_{\geq 0}$, and let $\{\mathcal{F}_t\}_{t \geq 0}$ be a filtration. The induced unary function $f_t : \mathcal{W} \to \mathbb{R}$ is $L_f$-Lipschitz with $\max_{x \in \mathcal{W}} \| \nabla^2 f_t(x) \|_2 \leq \beta$ for all $t \in \{0, \ldots, T - 1\}$, and $f_{t:k}(x) \triangleq \mathbb{E}[f_t(x)|\mathcal{F}_{t-1}]$ is $\alpha$-strongly convex $\forall t \in \{k, \ldots, T - 1\}$.\footnote{$f_t(.)$ is $L_f$-Lipschitz if $|f_t(x_1) - f_t(x_2)| \leq L_f \|x_1 - x_2\|$ for all $x_1, x_2 \in \mathcal{W}$. $f_t(.)$ is $\alpha$-strongly convex if and only if $\nabla^2 f_t(x) \succeq \alpha I_d$ for all $x \in \mathcal{W}$ (e.g., [5]).}

Assumption 6. The set $\mathcal{W} \subseteq \mathbb{R}^d$ is assumed to be convex and satisfies that $\text{Diam}(\mathcal{W}) \triangleq \sup_{x,y \in \mathcal{W}} \|x - y\| \leq G$. Suppose $\|g_t\| \leq L_g$ for all $t \in \{0, \ldots, T - 1\}$.

The following results are proved in Appendix A.

Lemma 4. Let Assumptions 4-6 hold. Set $\eta_t = \frac{3}{\alpha T}$ for $t \in \{\tau, \ldots, T + \tau\}$ in Algorithm 2 and define $\varepsilon_t \triangleq \nabla f_{t-\tau}(x_{t-\tau}) - \nabla f_{t-k}(x_{t-\tau})$. Then,

$$
\sum_{t=k}^{T-1} (f_{t,k}(x_t) - f_{t:k}(x_*)) \leq -\frac{\alpha}{6} \sum_{t=0}^{T-1} \|x_t - x_*\|^2 + \frac{2G^2(3k + 5\tau + 3)}{6} + \frac{3L_g^2(1 + \tau)}{\alpha} \log T + \sum_{t=k}^{T-1} \left( 3 - \varepsilon_t^T (x_t - x_*) \right), \forall x_* \in \mathcal{W}.
$$

Lemma 5. Let Assumptions 4-6 hold, and set $\eta_t = \frac{3}{\alpha T}$ for $t \in \{\tau, \ldots, T + \tau\}$ in Algorithm 2. With $X_t(x_*) \triangleq (f_t(x_*) - f_{t-k}(x_*)) - (f_t(x_t) - f_{t-k}(x_t)) + \nabla (f_t - f_{t-k})(x_t-k)^\top (x_{t-k} - x_*)$,

$$
\sum_{t=k}^{T-1} (f_t(x_t) - f_t(x_*)) \leq -\frac{\alpha}{6} \sum_{t=0}^{T-1} \|x_t - x_*\|^2 + \frac{2G^2(3k + 5\tau + 3)}{6} + \sum_{t=k}^{T-1} \left( \frac{3}{2\alpha} \varepsilon_t^2 + \frac{3L_g}{\alpha} (L_g(1 + \tau) + (4\beta G + 8L_f)k) \log T - \sum_{t=k}^{T-1} X_t(x_*) \right)
$$

Proposition 1. Let Assumptions 4-6 hold. Let $k > h$ and $\eta_t = \frac{3}{\alpha T}$ for $t \in \{\tau, \ldots, T + \tau\}$ in Algorithm 2. Then, for any $\delta > 0$, with probability at least $1 - \delta$:

$$
\sum_{t=k}^{T-1} (F_{t,h}(x_*), \ldots, x_t) - f_t(x_*)) \leq -\frac{\alpha}{12} \sum_{t=0}^{T-1} \|x_t - x_*\|^2 + O(1) \left( \sum_{t=k}^{T-1} \frac{\|\varepsilon_t\|^2}{\alpha} + \frac{k \delta L_f^2}{\alpha} \log \frac{T(1 + \log_+(\alpha G^2))}{\delta} \right) + \frac{\alpha G^2 k + \frac{L_g}{\alpha} (L_g + \beta G + L_f)k + L_c h^2}{T} \log T, \forall x_* \in \mathcal{W},
$$

where $O(1)$ denotes a universal constant, and $\log_+(x) \triangleq \log(\max\{1, x\}) \forall x \in \mathbb{R}_{>0}$.

3.2.2 Decentralized Control Policy Design

After obtaining $\hat{A}, \hat{B}$ from the system identification phase, Algorithm 3 computes $u^\text{alg}_{k,i}$ for each $i \in V$ in a fully decentralized manner (lines 2-16). Algorithm 3 chooses a DFC $u^\text{alg}_{k}(\text{from the convex set } \mathcal{D}$ of DFCs
given by Eq. (26)) parameterized by \( M_t = [M_t[k]]_{k \in [h], s \in \mathcal{T}} \) for all \( t \in \{N, \ldots, T-1\} \), based on estimates of the true disturbance \( w_t \) in Eq. (3). Specifically, for any \( j \in \mathcal{V} \), let \( \hat{w}_{t,j} \) be an estimate of the disturbance \( w_{t,j} \) in Eq. (2) obtained as

\[
\hat{w}_{t,j} = \begin{cases} 0, & \text{if } t \leq N - 1, \\
 x_{t+1,j}^{\text{alg}} - \hat{A}_j x_{t,N_j}^{\text{alg}} - \hat{B}_j u_{t,N_j}^{\text{alg}}, & \text{if } t \geq N,
\end{cases}
\]  

(33)

where we replace \( A_j \) and \( B_j \) in Eq. (2) with the estimates \( \hat{A}_j \) and \( \hat{B}_j \) obtained from Algorithm 1, respectively, and \( x_{t,N_j}^{\text{alg}} = \left[ x_{t,j_1}^{\text{alg}} \right]_{j_1 \in N_j} \) and \( u_{t,N_j}^{\text{alg}} = \left[ u_{t,j_1}^{\text{alg}} \right]_{j_1 \in N_j} \) with \( N_j \) given in Eq. (3). Denote

\[
\hat{\eta}_{k,s} = \left[ \hat{w}_{k-1,t,v,j}^{\text{alg}} \right]_{v \in \mathcal{L}_s},
\]  

(34)

with \( w_{j,v} \rightarrow v \) and \( \mathcal{L}_s \) given by Eq. (15).

**The OCO Subroutine in Algorithm 3.** In line 14 of Algorithm 3, the parameter \( M_t = [M_t[k]]_{k \in [h], s \in \mathcal{T}} \in \mathcal{D} \) of the DFC is updated using the OCO subroutine from Algorithm 2, where we denote \( M_{t,s} = [M_t[k]]_{k \in [h]} \). In line 14 of Algorithm 3, \( \Pi_{\mathcal{D}}(\cdot) \) denotes the projection onto the set \( \mathcal{D} \), which yields efficient implementations (e.g., [34, 41]). Formally, we introduce the following definitions.

**Definition 2. (Counterfactual Dynamics and Cost)** Let \( M_k \in \mathcal{D} \) for all \( k \geq 0 \), where \( \mathcal{D} \) is given by Eq. (26). First, for any \( k \geq 0 \), define

\[
u_k(M_k | \hat{w}_0:k-1) = \sum_{s \in \mathcal{L}} \sum_{k'=1}^h I_{V,s} M_{k,k'}^{k'} \hat{\eta}_{k'-s},
\]  

(35)

where \( \hat{w}_{0:t-1} \) denotes the sequence \( \hat{w}_0, \ldots, \hat{w}_{t-1} \). Let \( u_k(M_k | \hat{w}_0:k-1) \) denote the \( i \)th element of \( \nu_k(M_k | \hat{w}_0:k-1) \). Next, for any \( t \geq h \), define

\[
x_t(M_{t-h:t-1} | \hat{\Phi}, \hat{w}_{0:t-1}) = \sum_{k=t-h}^{t-1} \hat{A}^{-(k+1)} (\hat{w}_k + \hat{B} u_k(M_k | \hat{w}_0:k-1)),
\]  

(36)

\[
F_t(M_{t-h:t} | \hat{\Phi}, \hat{w}_{0:t-1}) = c \left( x_t(M_{t-h:t} | \hat{\Phi}, \hat{w}_{0:t-1}), u_t(M_t | \hat{w}_{0:t-1}) \right),
\]  

(37)

where \( \hat{\Phi} = \begin{bmatrix} \hat{A} & \hat{B} \end{bmatrix} \). Finally, for any \( t \geq h \), define

\[
x_t(M | \hat{\Phi}, \hat{w}_{0:t-1}) = x_t(M_{t-h:t-1} | \hat{\Phi}, \hat{w}_{0:t-1}),
\]  

(38)

\[
f_t(M | \hat{\Phi}, \hat{w}_{0:t-1}) = f_t(M_{t-h:t} | \hat{\Phi}, \hat{w}_{0:t-1}),
\]  

(39)

if \( M_{t-h} = \cdots = M_k = M \) with \( M \in \mathcal{D} \). Similarly, define \( u_k(M_k | \hat{w}_0:k-1) \), \( x_t(M_{t-h:t} | \hat{\Phi}, \hat{w}_{0:t-1}) \), \( F_t(M_{t-h:t} | \hat{\Phi}, \hat{w}_{0:t-1}) \), \( x_t(M | \hat{\Phi}, \hat{w}_{0:t-1}) \), and \( f_t(M | \hat{\Phi}, \hat{w}_{0:t-1}) \).

Since the cost \( F_t : \mathcal{D}^{h+1} \rightarrow \mathbb{R} \) is defined using the estimated system \( \hat{A}, \hat{B} \) rather than the real system \( A, B \), we refer to \( F_t(\cdot) \) as the counterfactual cost. Next, we define a cost corresponding to the true system dynamics \( x_t^{\text{alg}} = A x_t^{\text{alg}} + B u_t^{\text{alg}} + w_t \).

**Definition 3. (Prediction Cost)** Let \( M_k \in \mathcal{D} \) for all \( k \geq 0 \), where \( \mathcal{D} \) is given by Eq. (26). For any \( t \geq h \), define

\[
x_t^{\text{pred}}(M_{t-h:t-1}) = \sum_{k=t-h}^{t-1} A^{-(k+1)} (w_k + B u_k(M_k | \hat{w}_0:k-1)),
\]  

\[
F_t^{\text{pred}}(M_{t-h:t}) = c \left( x_t^{\text{pred}}(M_{t-h:t-1}), u_t(M_t | \hat{w}_{0:t-1}) \right),
\]  

where \( u_k(M_k | \hat{w}_0:k-1) \) is defined in Definition 2. Also define the terms \( x_t^{\text{pred}}(M) \) and \( f_t^{\text{pred}}(M) \) analogous to \( x_t(M | \hat{\Phi}, \hat{w}_{0:t-1}) \) and \( f_t(M | \hat{\Phi}, \hat{w}_{0:t-1}) \) in Definition 2.
Algorithm 3 Decentralized online control algorithm

Input: parameters $\lambda, N, R_{\cdots}, R_u, v$, cost matrices $Q$ and $R$, directed graph $G(V,A)$ with delay matrix $D$, time horizon length $T$, step sizes $\eta_t$ for $t \in \{N, \ldots, T + D_{\text{max}} - 1\}$

1: Use Algorithm 1 to obtain $\hat{A}$ and $\hat{B}$
2: For any $i \in V$, initialize $K_{i,1} = \bar{K}_{i,1}$ and $K_{i,2} = \bar{K}_{i,2}$
3: for $t = N, \ldots, N + D_{\text{max}} - 1$ do
   4:    Set $M_i = 0$ and play $u_{t+1, j}^{\text{alg}} = u_{t,i}(M_i | \hat{w}_{0:t-1}) = 0$
5:   for $t = N + D_{\text{max}}, \ldots, T + D_{\text{max}} - 1$ and for each $i \in V$ in parallel do
6:      for $s \in \mathcal{L}(T_i)$ do
7:          Find $u_s$ s.t. $j \in V$ and $s_0,j = s$
8:          Obtain $\hat{w}_{t-D_{ij}-1,j}$ from Eq. (33)
9:          $K_{i,1} \leftarrow K_{i,1} \cup \{\hat{w}_{t-D_{ij}-1,j}\}$
10:         if $\|x_t^{\text{alg}}\| > R_s$ or $\|u_{t,i}(M_i | \hat{w}_{0:t-1})\| > R_u$ then
11:            Play $u_{t, i}^{\text{alg}} = 0$ until $t = T + D_{\text{max}} - 1$
12:       end if
13:     end for
14:      $M_{t+1,s} \leftarrow \Pi_D (M_{t,s} - \eta_t \frac{\partial f_{t-D_{\text{max}}}(M_{t-D_{\text{max}}}, \hat{\Phi}, \hat{\psi}_{0:t-D_{\text{max}}})}{\partial M_{t-D_{\text{max}}}})$
15:     $K_{i,2} \leftarrow K_{i,2} \cup \{M_{t+1,s}\} \setminus \{M_{t-D_{\text{max}}-1,s}\}$
16:     $K_{i,1} \leftarrow K_{i,1} \setminus \{\hat{w}_{t-2D_{\text{max}}-2h,j} : s \in \mathcal{L}(T_i), w_j \rightarrow s\}$

---

$x_t^{\text{pred}}(M_{t-h:t-1})$ is a prediction of the true state $x_t^{\text{alg}}$ based on the past control inputs and disturbances from a length-$h$ window. Thus, we refer to the corresponding cost $f_t^{\text{pred}}(\cdot)$ as the prediction cost.

Based on the above arguments, line 14 in Algorithm 3 can be viewed as applying the OCO subroutine given by Algorithm 2 to the function sequence $F_t^{\text{pred}} : D^{h+1} \rightarrow \mathbb{R}$ and $f_t^{\text{pred}} : D \rightarrow \mathbb{R}$ for $t \in \{N + D_{\text{max}}, \ldots, T + D_{\text{max}} - 1\}$, where $F_t^{\text{pred}}(\cdot)$ and $f_t^{\text{pred}}(\cdot)$ are defined in Definition 3, $D_{\text{max}}$ is defined in Eq. (24). $N$ is an input parameter to Algorithm 3, and we set the delay $\tau \in \mathbb{Z}_{\geq 0}$ to be $\tau = D_{\text{max}}$ in the OCO subroutine.

Furthermore, we let the delayed gradient $g_{t-\tau}$ in the OCO subroutine to be $g_{t-\tau} = \nabla f_{t-\tau}(M_{t-\tau}, \hat{\Phi}, \hat{\psi}_{0:t-\tau-1})$ and thus we have

$$
\epsilon_{t-\tau} = g_{t-\tau} - \nabla f_{t-\tau}(x_{t-\tau}) = \nabla f_{t-\tau}(M_{t-\tau}, \hat{\Phi}, \hat{\psi}_{0:t-\tau-1}) - \nabla f_t^{\text{pred}}(M_{t-\tau}),
$$

where $f_{t-\tau}(\hat{\Phi}, \hat{\psi}_{0:t-\tau-1})$ is defined in Definition 2. We will later show in Section 4.3 that our analysis developed in Section 3.2.1 for the OCO with memory and delayed feedback can be specialized to our setting.

The Decentralized Structure of Algorithm 3. To see that Algorithm 3 is decentralized, for any $i \in V$, let $T_i$ denote the set of disconnected directed trees in $\mathcal{P}(U,H)$ such that the root node of any tree in $T_i$ contains $i$. Reloading the notation, also let $T_i$ denote the set of nodes of all the trees in $T_i$. Moreover, denote

$$
\mathcal{L}(T_i) = T_i \cap \mathcal{L},
$$

where $\mathcal{L}$ is defined in Eq. (14), i.e., $\mathcal{L}(T_i)$ is the set of leaf nodes of all the trees in $T_i$. For example, in Fig. 1, $T_1 = \{\{1, 2, 3\}, \{1, 2\}, \{3\}\}$, and $\mathcal{L}(T_1) = \{\{1\}, \{1, 2\}, \{3\}\}$. For any $i \in V$, Algorithm 3 maintains sets $K_{i,1}$ and $K_{i,2}$ in its current memory, which are initialized as $K_{i,1} = \bar{K}_{i,1}$ and $K_{i,2} = \bar{K}_{i,2}$ with

$$
\bar{K}_{i,1} = \{\hat{w}_{k,j} : k \in \{N' - 2D_{\text{max}} - 2h, \ldots, N' - D_{ij} - 2\}, s \in \mathcal{L}(T_i), j \in V, s_0,j = s\},
$$

$$
\bar{K}_{i,2} = \{M_{k,s} = [M_{k,s}^{(k)}]_{k' \in [h]} : k \in \{N' - D_{\text{max}} - 1, \ldots, N'\}, s \in T_i\},
$$

where $N' = N + D_{\text{max}}$, and we let $\hat{w}_{k,j} = 0$ for all $\hat{w}_{k,j} \in \bar{K}_{i,1}$, $M_{k,s} = 0$ for all $M_{k,s} \in \bar{K}_{i,2}$, and $M_t = 0$ for all $t < N$. For any $i \in V$ and any $t \geq N + D_{\text{max}}$, $K_{i,1}$ and $K_{i,2}$ can be obtained based on the information set $\mathcal{T}_{i,t}$ defined in Eq. (4), which implies that $u_{t, i}^{\text{alg}}$ can be computed from $\mathcal{T}_{i,t}$. Finally, note that the number of iterations of the for loop in lines 6-9 (resp., lines 13-15) in Algorithm 3 is bounded by $|\mathcal{L}(T_i)| \leq p$ (resp., $|\mathcal{T}_i| \leq |\mathcal{M}| \leq p^2 - p + 1$).
Remark 1. For any $s, r \in \mathcal{L}(T)$, let $j_1, j_2 \in \mathcal{V}$ be such that $s_{0,j_1} = s$ and $s_{0,j_2} = r$. In Algorithm 3, we assume without loss of generality that the elements in $\mathcal{L}(T)$ are already ordered such that if $D_{i,j_1} > D_{i,j_2}$, then $s$ comes before $r$ in $\mathcal{L}(T)$. We then let the for loop in lines 7-14 in Algorithm 3 iterate over the elements in $\mathcal{L}(T)$ according to the above order. For example, considering node 2 in the directed graph $\mathcal{G}(\mathcal{V}, \mathcal{A})$ in Example 1, we see from Figs. 1-2 that $\mathcal{L}(T_2) = \{(1), (1, 2), (3)\}$, where $s_{0,1} = \{1\}$, $s_{0,2} = \{1, 2\}$ and $s_{0,3} = \{3\}$. Since $D_{21} = 1, D_{22} = 0$ and $D_{23} = 1$, we let the elements in $\mathcal{L}(T_2)$ be ordered such that $\mathcal{L}(T_2) = \{(1), \{3\}, \{1, 2\}\}$.

3.3 Results for Algorithm 3

First, we show that the decentralized control policy given by Algorithm 3 satisfies the required information constraint described in Section 2.1. We will make the following assumption on the cost matrices $Q, R$.

Assumption 7. Let $\psi \in \mathbb{Z}_{\geq 2}$ be the number of strongly connected components in the directed graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, and let $\mathcal{V} = \cup_{l \in [\psi]} \mathcal{V}_l$, where $\mathcal{V}_l \subseteq \mathcal{V}$ is the set of nodes in the $l$th strongly connected component. For any $l_1, l_2 \in [\psi]$, any $i \in \mathcal{V}_{l_1}$ and any $j \in \mathcal{V}_{l_2}$, it holds that $(i, j) \notin \mathcal{A}$ and $(j, i) \notin \mathcal{A}$. Moreover, the cost matrices $Q \in \mathbb{S}^n_+$ and $R \in \mathbb{S}^m_+$ have a block structure according to the directed graph $\mathcal{G}(\mathcal{V}, \mathcal{A})$ such that $Q_{\mathcal{V}_l \mathcal{V}_l} = 0$ and $R_{\mathcal{V}_l \mathcal{V}_l} = 0$ for all $l \in [\psi]$, where $\mathcal{V}_l^c = \mathcal{V} \setminus \mathcal{V}_l$.

Supposing the directed graph $\mathcal{G}(\mathcal{V}, \mathcal{A})$ is strongly connected, one can check that Assumption 7 holds and $Q$ and $R$ need not possess the block structure.

Proposition 2. Suppose Assumption 7 holds and any controller $i \in \mathcal{V}$ at any time step $t \in \{N + D_{\text{max}}, \ldots, T + D_{\text{max}} - 1\}$ has access to the states in $\mathcal{I}_{t,i}$ defined as

$$\mathcal{I}_{t,i} = \left\{ x_{k,j}^{\text{alg}} : j \in \mathcal{V}, D_{ij} < \infty, k \in \{t - D_{\text{max}} - 1, \ldots, t - D_{ij}\} \right\} \subseteq \mathcal{I}_{t,i},$$

where $\mathcal{I}_{t,i}$ is defined in Eq. (4). For any $i \in \mathcal{V}$ and any $t \in \{N + D_{\text{max}}, \ldots, T + D_{\text{max}} - 1\}$, the sets $\mathcal{K}_{t,1}$ and $\mathcal{K}_{t,2}$ at the beginning of iteration $t$ of the for loop in lines 5-16 of the algorithm satisfy

$$\mathcal{K}_{t,1} = \left\{ \hat{w}_{k,s} : k \in \{t - 2D_{\text{max}} - 2h, \ldots, t - D_{ij} - 2\}, s \in \mathcal{L}(T_t), j \in \mathcal{V}, s_{0,j} = s \right\},$$

$$\mathcal{K}_{t,2} = \left\{ M_{k,s} = (M_{k,s}^{[k']})_{k' \in [s]} : k \in \{t - D_{\text{max}} - 1, \ldots, t\}, s \in \mathcal{I}_t \right\},$$

and the control input $u_{t,3}^{\text{alg}}$ in line 12 can be determined based on $\mathcal{K}_{t,1}$ and $\mathcal{K}_{t,2}$ after line 9 and before line 16 in iteration $t$ of the for loop in lines 5-16 of the algorithm.

Proposition 2 proved in Appendix B shows that for any $i \in \mathcal{V}$, the sets $\mathcal{K}_{t,1}$ and $\mathcal{K}_{t,2}$ can be recursively updated in the memory of Algorithm 3 based on $\mathcal{I}_{t,i} \subseteq \mathcal{I}_{t,i}$, such that $u_{t,j}^{\text{alg}}$ can be determined from the current $\mathcal{K}_{t,1}$ and $\mathcal{K}_{t,2}$ for all $t \in \{N + D_{\text{max}}, \ldots, T + D_{\text{max}} - 1\}$. Thus, Proposition 2 precisely shows that $u_t^{\text{alg}}$ for any $i \in \mathcal{V}$ designed by Algorithm 3 satisfies the required decentralized information constraints and can be implemented in a fully decentralized manner based only on the local state information. Noting that $|\mathcal{L}(T_t)| \leq p$ and $|\mathcal{T}_t| \leq |\mathcal{U}| \leq p^2 - p + 1$ for all $i \in \mathcal{U}$ [26], one can show that the size of the memory $\mathcal{K}_{t,1}$ and $\mathcal{K}_{t,2}$ satisfies that $|\mathcal{K}_{t,1}| \leq (2D_{\text{max}} + 2h - 1)p$ and $|\mathcal{K}_{t,2}| \leq (D_{\text{max}} + 2)q$ for all $i \in \mathcal{U}$. Algorithm 3 is also significantly different from the offline learning algorithm for decentralized LQR proposed in [45] (see our discussions below for more details). The proof of Proposition 2 then deviates from the proof of [45, Proposition 2]. Next, we upper bound the regret of Algorithm 3.

Theorem 1. Suppose Assumptions 1-2 and 7 hold. There exist input parameters $\lambda, N, R_t, R_u$ and step sizes $\eta_t$ for all $t \in \{N + D_{\text{max}} + T + D_{\text{max}} - 1\}$ for Algorithm 3 such that $\mathbb{E}[\text{Regret}] = \mathcal{O}(\sqrt{T})$, for all $T \geq N + 3h + D_{\text{max}}$ with $N = \max\{\Theta(1), \sqrt{T}\}$ and $h = \max\{4D_{\text{max}} + 4, \frac{4}{\eta_0}\log T\}$, where let $\mathcal{O}(\cdot)$ and $\Theta(\cdot)$ compress polynomial factors in $\log T$ and other problem parameters.

Theorem 1 shows that Algorithm 3 achieves $\sqrt{T}$-regret under properly chosen input parameters; the proof is in the next section. Below, we provide some remarks.

Optimality of the Regret. Despite the information constraints, our regret bound for learning decentralized LQR matches with the regret bound for learning centralized LQR [1, 11, 41] in terms of $T$, which has been shown to be the best regret that can be achieved (up to polynomial factors in $\log T$ and some other
constants) [40, 8, 10]. Particularly, it was shown that the regret of any learning algorithm for centralized LQR (with stable $A$ matrix) is lower bounded by $\Omega(\sqrt{T})$ [9].

**Necessity of DFC and OCO and Comparison to [45].** An offline learning algorithm based on certainty equivalence approach for decentralized LQR with partially nested information structure was proposed in [45]. It requires that the system restart to the initial state after identifying $A$ and $B$ from the system trajectory and cannot be directly applied to our online setting, where restarting the system to the initial state is not allowed. Moreover, [45] only provides a $\tilde{O}(1/\sqrt{N})$ sample complexity result (with $\tilde{O}()$ hiding polynomial factors in $\log N$) for the offline algorithm, where $N$ is the length of the system trajectory for identifying the system matrices. As argued in [9, 31], ignoring the restart issue, an offline algorithm using a certainty equivalence approach for learning LQR with $\tilde{O}(1/\sqrt{N})$ sample complexity can only be translated into an online algorithm with $\tilde{O}(T^{2/3})$ regret. Instead, we leverage the DFC structure and the OCO subroutine in our control policy design to provide an online algorithm with $\tilde{O}(\sqrt{T})$ regret. Thus, the result in Theorem 1 stands in stark contrast to the results in [45], since there is a performance gap between the offline learning algorithm proposed for decentralized LQR in [45] and that proposed for centralized LQR in [31].

**Expected Regret.** Both expected regret and high probability regret have been proved for online learning algorithms for centralized LQR (e.g. [9, 11]). We leave investigating whether the regret bound $\tilde{O}(\sqrt{T})$ also holds with a high probability (instead of only in expectation) as future work. Note that the online learning algorithms (for centralized LQR) that yield high probability sublinear regret (e.g., [11]) typically incur potentially unbounded regret under a failure event with small probability. In contrast, we upper bound the expected regret of our algorithm on the failure event using the condition in line 10 of the algorithm for

\[ R_{\text{FC}}(\theta) = \max_{\theta} \tilde{R}(\theta) \]





4 Regret Analysis: Proof of Theorem 1

Throughout this proof, we assume that Assumptions 1-3 and 7 hold. We first decompose the regret of Algorithm 3 defined in Eq. (13) as

\[
\text{Regret} = \sum_{t=0}^{N_0-1} c(x_t^{\text{alg}}, u_t^{\text{alg}}) + \sum_{t=N_0}^{T-1} c(x_t^{\text{alg}}, u_t^{\text{alg}}) - \sum_{t=N_0}^{T-1} R_t^{\text{pred}}(M_{t-h:t}) \\
+ \sum_{t=N_0}^{T-1} R_t^{\text{pred}}(M_{t-h:t}) - \sum_{t=N_0}^{T-1} f_t^{\text{pred}}(M_{\text{apx}}) + \sum_{t=N_0}^{T-1} R_t^{\text{pred}}(M_{\text{apx}}) - \inf_{M \in D_0} \sum_{t=N_0}^{T-1} f_t(M|\Phi, w_{0:t-1}) \\
+ \inf_{M \in D_0} \sum_{t=N_0}^{T-1} f_t(M|\Phi, w_{0:t-1}) - \inf_{M \in D_0} \sum_{t=N_0}^{T-1} c(x_t^M, u_t^M) + \inf_{M \in D_0} \sum_{t=N_0}^{T-1} c(x_t^M, u_t^M) - TJ^* \tag{46}
\]

where $N_0 = N + D_{\text{max}} + 3h$, $M_{\text{apx}} \in D$ will be specified later with $D$ given by Eq. (26), $J^*$ is the optimal cost to (6) given by Eq. (12), and $D_0$ is given by\(^6\)

\[
D_0 = \left\{ M = (M[k]_{s \leq t})_{k \in [\frac{T}{4}], s \leq t} : \|M[k]_{s \leq t}\|_F \leq \sqrt{\eta \kappa} \Gamma^{2D_{\text{max}}+1} \right\}. \tag{47}
\]

\(^6\)To ease our presentation, we assume that $\frac{h}{4} \in \mathbb{Z}_{\geq 1}$.
To be more specific, first note that the result in Proposition 3 shows that the estimation error of the least squares approach used in the first phase of Algorithm 3 satisfies \( \| \Delta_N \| = \tilde{O}(1/\sqrt{N}) \) (on the event \( \mathcal{E} \) defined in Eq. (50)), where \( \Delta_N \) is defined in (29) and \( N \) is the length of the system trajectory used for system identification. Thus, choosing \( N \geq \sqrt{T} \) yields that \( \| \Delta_N \| \leq \bar{\epsilon} \), where \( \bar{\epsilon} \) satisfies \( \bar{\epsilon} \leq \tilde{O}(1/T^{1/4}) \). By the choice of \( N \), we show that \( R_0 \) contributes \( \tilde{O}(\sqrt{T}) \) to \( \text{Regret} \). In the second phase, Algorithm 3 uses the OCO with memory and delayed feedback subroutine to adaptively choose the control input \( u^\text{alg}_t \) (while satisfying the information constraints). Choosing \( M_{\text{gpx}} \in \mathcal{D} \) carefully, we show that \( R_2 \) and \( R_3 \) together contribute \( \tilde{O}(\bar{\epsilon}^2T) \) to \( \text{Regret} \). Furthermore, based on the choice of \( h \), we show that \( R_1 \) and \( R_4 \) together contribute \( \tilde{O}(1) \) to \( \text{Regret} \), and \( R_5 \) contributes \( \tilde{O}(\sqrt{T}) + TJ_* \) to \( \text{Regret} \). Putting the above arguments together will give \( \mathbb{E}[\text{Regret}] = \tilde{O}(\sqrt{T}) \).

- **Major challenges in the proof.** We summarize the major differences between our proof and those in the related literature that use DFC and OCO for online centralized LQR.

First, [41] studies learning centralized LQR, and proposes an online algorithm that leverages the OCO with memory subroutine from [3]. Since we study learning the decentralized LQR (with sparsity and delayed constraints), we propose a general OCO algorithm with both memory and delayed feedback, and use it as a subroutine in Algorithm 3. We prove the regret of the general OCO algorithm with memory and delayed feedback, and specialize to the learning decentralized LQR setting (Section 4.3).

Second, unlike [3, 41] that considers bounded noise, we assume that the noise to system (3) is Gaussian. Therefore, we need to first conduct our analysis on an event under which the Gaussian noise is bounded (Section 4.1), and then bound the (expected) regret of Algorithm 3 when the Gaussian noise is unbounded (Section 4.4).

Third, unlike [3, 41] that consider high probability regret and set the finite-horizon LQR cost as the benchmark, we prove the expected regret and set the averaged expected infinite-horizon LQR cost as the benchmark. Therefore, the regret decomposition in Eq. (46) is different from those in [3, 41].

### 4.1 Properties under a Good Probabilistic Events

Before we upper bound the terms in Eq. (46), we introduce several probabilistic events, on which we will prove that several favorable properties hold. To simplify the notations in the sequel, we denote

\[
R_w = \sigma_w \sqrt{10n \log 2T},
\]

\[
\sigma = \min\{\sigma_w, \sigma_u\}, \quad \bar{\sigma} = \max\{\sigma_w, \sigma_u\},
\]

Define

\[
\mathcal{E}_w = \left\{ \max_{0 \leq t \leq T-1} \| w_t \| \leq R_w \right\},
\]

\[
\mathcal{E}_u = \left\{ \max_{0 \leq t \leq N-1} \| u^\text{alg}_t \| \leq \sigma_u \sqrt{5m \log 4NT} \right\},
\]

\[
\mathcal{E}_\Phi = \left\{ \text{Tr}(\Delta_N^T V_N \Delta_N) \leq 4\sigma_N^2 n \log \left( 4nT \frac{\text{det}(V_N)}{\text{det}(\lambda I_{n+m})} \right) + 2\lambda^2 \| \Phi \|^2 T \right\},
\]

\[
\mathcal{E}_z = \left\{ \sum_{t=0}^{N-1} z_t z_t^\top \geq \frac{(N-1)\bar{\sigma}^2}{40} I \right\},
\]

where \( N, T \) are the input parameters to Algorithm 3, and \( \Delta_N \) and \( z_t \) are define in (29) and (27), respectively. Denoting

\[
\mathcal{E} = \mathcal{E}_w \cap \mathcal{E}_u \cap \mathcal{E}_\Phi \cap \mathcal{E}_z,
\]

we have the following result that shows \( \mathcal{E} \) holds with a high probability.

**Lemma 6.** [45, Lemma 4] For any \( N \geq 200(n + m) \log 48T \), it holds that \( \mathbb{P}(\mathcal{E}) \geq 1 - 1/T \).
Recalling $\hat{\Delta}_N$ and $\Delta_N$ defined in (29), we then have the following result proved in Appendix C, which relates the estimation error of $\hat{A}$ and $\hat{B}$ to that of $A$ and $B$.

**Lemma 7.** Suppose $\|\hat{\Delta}_N\| \leq \varepsilon$ with $\varepsilon \in \mathbb{R}_{>0}$. Then, $\|\Delta_N\| \leq \sqrt{\varepsilon}$, where $\psi \in \mathbb{Z}_{\geq 1}$ is the number of strongly connected components in the directed graph $\mathcal{G}(V, A)$.

The following result adapts [45, Proposition 1] to prove an upper bound on $\|\hat{\Delta}_N\|$ and then invokes Lemma 7; the proof can be found in Appendix C.

**Proposition 3.** Denote

$$z_b = \frac{5\kappa}{1-\gamma} \sqrt{2(\Gamma^2 m + m + n) \log 2T},$$

and

$$\bar{\varepsilon} = \min \left\{ \sqrt{\frac{480\psi}{T} \sigma_w^2 (n + m) \left( \log(T + \frac{z_b^2}{\gamma})T \right)^2}, \varepsilon_0 \right\}.$$  \hspace{1cm} (51)

with $\varepsilon_0 = \frac{(1-\gamma)\sigma_w}{\sqrt{\Gamma \sigma_w^2 m \max + 1} \sqrt{T}}$. Let the input parameters to Algorithm 3 satisfy that $T \geq N$ and $\lambda = \sigma_w^2$ with

$$N = \left[ \max \left\{ \frac{480\psi \sigma_w^2 (n + m) \left( \log(T + \frac{z_b^2}{\gamma})T \right)^2}{\sigma_w^2 \bar{\varepsilon}} \right\} \right],$$  \hspace{1cm} (52)

where $\kappa, \gamma, \Gamma$ are given in (24), $D_{\max}$ is given in Eq. (25), and $\psi$ is described in Lemma 7. Then, on the event $\mathcal{E}$ defined in Eq. (50), it holds that $\|A - \hat{A}\| \leq \bar{\varepsilon}$ and $\|B - \hat{B}\| \leq \bar{\varepsilon}$.

Proposition 3 shows that $\bar{\varepsilon} = \mathcal{O}(1/T^{1/4})$ by choosing the length $N$ of the system identification phase sufficiently large. Since this phase uses the control $u_t \overset{i.i.d.}{\sim} N(0, \sigma_w^2 I_m)$ for $t = 0, \ldots, N$, Assumption 3 ensures $\|x_t^{\text{alg}}\| = \mathcal{O}(\sqrt{T})$ for $t = 0, \ldots, N - 1$ on the event $\mathcal{E}$. From now on, we set $N$ as Eq. (52) in Algorithm 3. Note from our choices of $h$ and $N$ that $N \geq h$. Moreover, we have the following results proved in Appendix C.

**Lemma 8.** Suppose the event defined in Eq. (50) holds. Let $x_t^*$ be the state of system (3) when we use the optimal control policy $u_0^*, \ldots, u_{n-1}^*$ given by Eq. (21). Then,

$$\|u_t^*\| \leq \frac{pq\kappa \Gamma^2 D_{\max} + 1 + q R_w}{1-\gamma}, \quad \text{and} \quad \|x_t^*\| \leq \frac{pq\kappa \Gamma^2 D_{\max} + 1 + q R_w}{1-\gamma},$$  \hspace{1cm} (53)

for all $t \in \mathbb{Z}_{\geq 0}$. Letting $M^{[k]} = M^{[k]}_t$ in Definition 1, for all $s \in \mathcal{U}$ and all $k \in [g]$, it holds that $M = [M^{[k]}_{t}]_{k \in [g], s \in \mathcal{U}}$ satisfies $M \in \mathcal{D}$, and that

$$\|u^M_t\| \leq \frac{pq\kappa \Gamma^2 D_{\max} + 1 + q R_w}{1-\gamma}, \quad \text{and} \quad \|x^M_t\| \leq \frac{2pq\kappa \Gamma^2 D_{\max} + 1 + q R_w}{(1-\gamma)^2},$$  \hspace{1cm} (54)

for all $t \in \mathbb{Z}_{\geq 0}$, where $\mathcal{D}$ is defined in Eq. (26). Moreover, it holds that

$$c(x^M_t, u^M_t) - c(x_t^*, u_t^*) \leq \frac{12\bar{\sigma}p^2q^2\kappa^4 \Gamma^4 (D_{\max} + 1) R_w^2}{(1-\gamma)^4},$$  \hspace{1cm} (55)

for all $t \in \mathbb{Z}_{\geq 0}$.

**Lemma 9.** On the event $\mathcal{E}$ defined in Eq. (50), $\|v^{\text{alg}}_t\| \leq R_u$, $\|x^{\text{alg}}_t\| \leq R_x$, $\|x^{\text{pred}}_t(M_{t-h:t-1})\| \leq R_x$, $\|\hat{w}_t\| \leq R_w$, and $\|\hat{w}_t - w_t\| \leq \Delta R_w \bar{\varepsilon}$, for all $t \in \{N, \ldots, T - 1\}$, where

$$R_u = 4qR_u h \sqrt{n \kappa^2 \Gamma^2 D_{\max}},$$

$$R_x = \frac{\bar{\sigma} \sqrt{20(m + n)\Gamma^2}}{1-\gamma} + \frac{(\Gamma R_u + R_w)\kappa}{1-\gamma},$$

$$R_w = \left( \frac{\Gamma \kappa}{1-\gamma} + 1 \right) \bar{\varepsilon} R_u + R_w + \frac{\bar{\varepsilon} \kappa}{1-\gamma} (\kappa \bar{\sigma} \sqrt{20(m + n)\Gamma} + R_w),$$

$$\Delta R_w = \left( \frac{\Gamma \kappa}{1-\gamma} + 1 \right) R_u + \frac{\kappa}{1-\gamma} (\kappa \bar{\sigma} \sqrt{20(m + n)\Gamma} + R_w).$$  \hspace{1cm} (59)
Lemma 8 shows that under the event $\mathcal{E}$, the DFC given in Definition 1 can be used to approximate the optimal control policy $u_t^*$ given by Eq. (21) with the approximation error (in terms of the resulting cost) decaying exponentially in $h$. Lemma 9 provides bounds on the norm of the state, input, estimated disturbance and its estimation error in Algorithm 3 when the event $\mathcal{E}$ holds. We now set $R_u$ and $R_w$ in Algorithm 3 as Eqs. (56) and (57), respectively. Proposition 3 and Lemma 9 yield that the conditions in line 10 in Algorithm 3 are not satisfied for any $i \in \mathcal{V}$ on the event $\mathcal{E}$. The above results will be useful to bound $R_0, \ldots, R_5$ in the sequel.

### 4.2 Upper Bounds on $R_0, R_1, R_4$ and $R_5$

The following results are proved in Appendix D.

**Lemma 10.** On the event $\mathcal{E}$ defined in Eq. (50), it holds that $R_0 \leq \left( \sigma_1(Q)R_x^2 + \sigma_1(R)\frac{R_u^2 \sigma_w^2}{\sigma_w^2} \right) N_0$.

**Lemma 11.** On the event $\mathcal{E}$ defined in Eq. (50), it holds that $R_1 \leq 2R_x \sigma_1(Q) \Gamma R_u + R_u \frac{\gamma^h}{(1-\gamma)^2} T$.

**Lemma 12.** On the event $\mathcal{E}$ defined in (50), it holds that

$$R_4 \leq 8\sigma_1(Q)q^2 nh^2 \kappa^2 \gamma^2 \Gamma T \max_{i,j} R_w^2 \frac{\gamma^h}{(1-\gamma)^2} T.$$  

**Lemma 13.** $E[\{\mathcal{E}\} R_5] \leq 128p^2 q e^{4 T} \max_{i,j} R_w^2 \frac{\gamma^h}{(1-\gamma)^2} T + T J_*$.

Note that the upper bounds shown in Lemma 10 naturally hold for the expected values $E[\{\mathcal{E}\} R_0]$, $E[\{\mathcal{E}\} R_1]$ and $E[\{\mathcal{E}\} R_4]$. $R_0$ corresponds to the system identification phase in Algorithm 3. Based on the choice of the system identification phase length $N$ and Lemma 9, we show in Lemma 10 that $E[\{\mathcal{E}\} R_0] = \mathcal{O}(\sqrt{T})$, where recall that $N_0 = N + D_{\max} + 3h$. $R_1$ (resp., $R_4$) corresponds to the error when approximating the true cost by the prediction cost (resp., counterfactual cost), and the proof of the upper bounds on $R_1, R_4$ in Lemma 10 uses the results in Lemma 9. Recalling $0 < \gamma < 1$, one can choose $h$ to be sufficiently large as $h = \mathcal{O}(\log T)$ such that $\gamma^h T = \mathcal{O}(1)$, which implies via Lemma 10 that $E[\{\mathcal{E}\} (R_1 + R_4)] = \mathcal{O}(1)$. Finally, $R_5$ corresponds to the error when approximating the optimal decentralized control policy given by Lemma 1 with the (truncated) DFC and we show that $E[\{\mathcal{E}\} R_5] = \mathcal{O}(\sqrt{T}) + T J_*$. 

**Remark 2.** We upper bound $E[\{\mathcal{E}\} R_5]$ rather than $\{\mathcal{E}\} R_5$ in Lemma 13. The reason is that by taking the expectation $E[.]$ (with respect to the disturbances $w_t$), we can leverage the assumption made before that the disturbances $w_{t,i}$ and $w_{t,j}$ are independent for different $i, j \in \mathcal{V}$ of system (1) so that $E[w_{t,i} w_{t,j}] = 0$ for $i \neq j$. We then use Lemma 8 to relate the optimal finite-horizon decentralized LQR cost achieved by the DFC to the optimal expected averaged infinite-horizon decentralized LQR cost, and get the upper bound $E[\{\mathcal{E}\} R_5]$ in Lemma 13.

### 4.3 Upper Bound on $R_2$ and $R_3$

$R_2$ and $R_3$ are due to the OCO subroutine given by Algorithm 2 with memory and delayed feedback. To upper bound $R_2, R_3$, we will apply Proposition 1 shown for the general OCO algorithm. Recall that the OCO subroutine in Algorithm 3 is applied to the function sequence $F^\text{pred}_t : \mathcal{D}^{h+1} \to \mathbb{R}$ and $f^\text{pred}_t : \mathcal{D} \to \mathbb{R}$ for $t \in \{N + D_{\max} - 1 \} \cup \{N + D_{\max} , \ldots , T + D_{\max} - 1 \}$, where $F^\text{pred}_t(\cdot)$ and $f^\text{pred}_t(\cdot)$ are defined in Definition 3 and $\mathcal{D}$ defined in Eq. (26) is convex. We first show that Assumptions 4-6 hold in our problem setting. We need to define the following.

**Definition 4.** Let $(\mathcal{F}_t)_{t \geq N}$ be a filtration with $\mathcal{F}_t = \sigma(w_N, \ldots, w_t)$ for all $t \geq N$. For any $t \geq N + k_f$ with $k_f \in \mathbb{Z}_{\geq 0}$ and any $M \in \mathcal{D}$, define $f^\text{pred}_{t:k_f}(M) := E[f^\text{pred}_{t}(M) | \mathcal{F}_{t-k_f}]$.

Note that in order to leverage the result in Proposition 1, we first view the function $f^\text{alg}_t(\cdot)$ as a function of $\text{Vec}(M)$, where $\text{Vec}(M)$ denotes the vector representation of $M = [M^k_j]_{k \in \mathbb{N}, j \in \mathbb{N}}$. We then show in Lemma 14...
that $f_t^{\text{pred}}(\cdot)$ is Lipschitz with respect to $\text{Vec}(M)$.

In Lemmas 15–17, we assume for notational simplicity that $M$ is already written in its vector form $\text{Vec}(M)$. The proofs of Lemmas 14–17 are included in Appendix E.

**Lemma 14.** Suppose the event $\mathcal{E}$ defined in Eq. (50) holds. For any $t \in \{N, \ldots, T-1\}$, $f_t^{\text{pred}} : \mathcal{D} \to \mathbb{R}$ is $L'_f$-Lipschitz and $f_t^{\text{pred}} : \mathcal{D}^{h+1} \to \mathbb{R}$ is $L'_c$-coordinatewise-Lipschitz, where

$$L'_f = L'_c = 2(R_u + R_x) \max\{\sigma_1(Q), \sigma_1(R)\} \frac{(1 - \gamma) \sqrt{qhR_{\bar{w}}}}{1 - \gamma}.$$ \hfill (60)

**Lemma 15.** On the event $\mathcal{E}$ defined in Eq. (50), it holds that $\|\nabla^2 f_t^{\text{pred}}(M)\| \leq \beta'$ for all $t \geq \{N, \ldots, T-1\}$ and all $M \in \mathcal{D}$, where

$$\beta' = \frac{2qhR_{\bar{w}}^2 (\sigma_1(Q)(1 - \gamma)^2 + \sigma_1(R)k^2)}{(1 - \gamma)^2}.$$ \hfill (51)

**Lemma 16.** On the event $\mathcal{E}$ defined in Eq. (50), it holds that

$$\left\| \frac{\partial f_t(M)}{\partial M} - \frac{\partial f_t^{\text{pred}}(M)}{\partial M} \right\| \leq \Delta L_f \bar{e},$$ \hfill (61)

for all $t \in \{N, \ldots, T-1\}$ and all $M \in \mathcal{D}$, where

$$\Delta L_f = \frac{34\sigma_1(Q)\sqrt{qhR_{\bar{w}}r^2}}{(1 - \gamma)^2} \left((R_{\bar{w}} + \Gamma R_u) \frac{34k^2}{(1 - \gamma)^2} + R_x \right),$$ \hfill (62)

with $\Gamma = \Gamma + 1$.

**Lemma 17.** Suppose the event $\mathcal{E}$ defined in Eq. (50) holds, and let $k_f \geq D_{\text{max}} + h + 1$, where $D_{\text{max}}$ is defined in Eq. (25). Then, for any $t \geq N + k_f$, the function $f_{t+k_f}(\cdot)$ given by Definition 4 is $\frac{\sigma_m(R)^2}{2}$-strongly convex.

Recalling the expression of $O_2$ given in Eq. (46) and leveraging Lemmas 14–17, we can specialize Proposition 1 shown for the general OCO with memory subroutine to the following upper bound on $O_2$.

**Lemma 18.** Let $\eta_t = \frac{3}{d'f_t}$ for all $t \in \{N + D_{\text{max}}, \ldots, T + D_{\text{max}} - 1\}$ in Algorithm 3, where $\alpha' \triangleq \frac{\sigma_m(R)^2}{2}$. Then, for any $M_{\text{apx}} \in \mathcal{D}$ and any $\delta > 0$, the following holds with probability at least $1 - \delta$:

$$R_2 \leq -\frac{\alpha'}{12} \sum_{t=N}^{T-1} \|\text{Vec}(M_t) - \text{Vec}(M_{\text{apx}})\|^2$$

$$+ O(1) \left( \sum_{t=N_0}^{T-1} \frac{\Delta L_f^2}{\alpha'} \|\epsilon\|^2 + \frac{k_f d' L_f^2}{\alpha'} \log \left( \frac{T (1 + \log \frac{(\alpha' G'^2)}{\delta})}{\delta} \right) + \alpha' G'^2 (k_f + D_{\text{max}}) + \frac{L_{\text{apx}}}{\alpha'} (L'_f D_{\text{max}} + (\beta' G' + L'_f)k_f + L'_f h^2) \log T \right),$$ \hfill (63)

where $M_t$ is chosen by Algorithm 3 for all $t \in \{N, \ldots, T-1\}$, $\alpha'$, $\beta'$, $L_f'$, $L_c'$, $\Delta L_f$ are given in Lemmas 14–17, $\bar{e}$ is given in Eq. (51), $L_f' = L_f' + \Delta L_f$, $k_f = 3h + D_{\text{max}} + 1$, $d' = \dim(\text{Vec}(M)) = h \sum_{s \in \mathcal{U}} n_s n_L$, and $G' = 4\sqrt{qhm}\Gamma^{2D_{\text{max}} + 1}$.

The upper bound on $R_2$ in Lemma 18 holds for all $M_{\text{apx}} \in \mathcal{D}$. To upper bound $R_3$, we choose a particular $M_{\text{apx}} \in \mathcal{D}$ and upper bound the difference between $f_t^{\text{pred}}(M_{\text{apx}})$ and $f_t(M_{\star}(\Phi, u_{0:t-1}))$, where $M_{\star} \in \arg\inf_{M \in \mathcal{D}_0} \sum_{t=N_0}^{T-1} f_t(M_{\Phi, u_{0:t-1}})$ with $\mathcal{D}_0$ defined in Eq. (47) and $f_t(\cdot, \Phi, u_{0:t-1})$ given by Definition 2. We begin with the following intermediate lemma; the proof is included in Appendix F.

\footnote{Note that $\|P\|_F^2 = \|\text{Vec}(P)\|^2$ for all $P \in \mathbb{R}^{n \times n}$, which explains why we define $\mathcal{D}$ in Eq. (26) based on $\|\cdot\|_F^2$. To be more precise, projecting $M = [M^k_{\star}]_{k \in \mathcal{U}, k \in [h]}$ onto the set $\mathcal{D}$ is equivalent to projecting $\text{Vec}(M)$ onto a Euclidean ball in $\mathbb{R}^{\dim(\text{Vec}(M))}$.}
Lemma 19. Suppose the event $\mathcal{E}$ defined in Eq. (50) holds. Let
\begin{equation}
\hat{M}_t \in \arg \inf_{M \in \mathcal{D}_0} \sum_{t=N_0}^{T-1} f_t(M|\Phi, w_{0:t-1}), \tag{64}
\end{equation}
where $\mathcal{D}_0$ is defined in Eq. (47), and $f_t(M|\Phi, w_{0:t-1})$ is given by Definition 2. Then, for any $t \in \{N_0, \ldots, T-1\}$ and any $M_{\text{apx}} \in \mathcal{D},$
\begin{equation}
 f_t^{\text{pred}}(M_{\text{apx}}) - f_t(\hat{M}_t|\Phi, w_{0:t-1}) \leq 2(R_w\sigma_1(R) + R_x\sigma_1(Q)) \frac{\Gamma \kappa}{1 - \gamma} \max_{k \in \{t-h, \ldots, t\}} \| u_k(M_{\text{apx}}|\hat{w}_{0:k-1}) - u_k(\hat{M}_t|w_{0:k-1}) \|. \tag{65}
\end{equation}

Next, the following lemma finds an $M_{\text{apx}}$ that depends on $\hat{M}_t$ such that the difference between $\hat{M}_t$ and $M_{\text{apx}}$ can be related to the difference between $M_t$ and $M_{\text{apx}}$. The proof of Lemma 20 is provided in Appendix F.

Lemma 20. Suppose the event $\mathcal{E}$ defined in Eq. (50) holds. Then, there exists $M_{\text{apx}} \in \mathcal{D}$ such that for any $k \in \{N + 2h, \ldots, T-1\}$ and any $\mu \in \mathbb{R}_{>0}$,
\begin{equation}
\| u_k(\hat{M}_t|w_{0:k-1}) - u_k(M_{\text{apx}}|\hat{w}_{0:k-1}) \| \leq \bar{c}(\Gamma R_u + R_w) p^2 q h \sqrt{n} \frac{\kappa^2 \Gamma^2 D_{\max}^2 + 1}{4(1 - \gamma)} + \frac{p^2 q^2 h^2}{8\mu} \sqrt{n} \kappa \Gamma^2 D_{\max} + 1 (\kappa + 1) \varepsilon^2 T + \frac{p^5 q^3 n^2}{8} h^5 \kappa^4 \Gamma^6 D_{\max} + 1 (\kappa + 1) R_w \varepsilon^2 T + qh^2 R_w \mu \sum_{t=N}^{T-1} (\| \text{vec}(M_t) - \text{vec}(M_{\text{apx}}) \|)^2. \tag{66}
\end{equation}
where $M_t$ is chosen by Algorithm 3 for all $t \in \{N, \ldots, T-1\}$.

Combining Lemmas 19-20 yields the following upper bound on $R_3$; the proof is included in Appendix F.

Lemma 21. Suppose the event $\mathcal{E}$ defined in Eq. (50) holds. Then, there exists $M_{\text{apx}} \in \mathcal{D}$ such that for any $\mu \in \mathbb{R}_{>0},$
\begin{equation}
R_3 \leq (R_w\sigma_1(R) + R_x\sigma_1(Q)) \frac{2\Gamma \kappa}{1 - \gamma} \left( \bar{c}(\Gamma R_u + R_w) p^2 q h \sqrt{n} \frac{\kappa^2 \Gamma^2 D_{\max}^2 + 1}{4(1 - \gamma)} T + \frac{p^2 q^2 h^2}{8\mu} \sqrt{n} \kappa \Gamma^2 D_{\max} + 1 (\kappa + 1) \varepsilon^2 T + \frac{p^5 q^3 n^2}{8} h^5 \kappa^4 \Gamma^6 D_{\max} + 1 (\kappa + 1) R_w \varepsilon^2 T + qh^2 R_w \mu \sum_{t=N}^{T-1} (\| \text{vec}(M_t) - \text{vec}(M_{\text{apx}}) \|)^2 \right). \tag{67}
\end{equation}

4.4 Upper bound on Regret

We are now in place to upper bound $\mathbb{E}[\text{Regret}]$, where $\text{Regret}$ is defined in Eq. (13). First, we recall that Lemma 18 provides a high probability upper bound on $R_2$. Hence, we define $\mathcal{E}_{R_2}$ to be the event on which the upper provided in Lemma 18 holds with probability at least $1 - 1/T$, i.e., we let $\delta = 1/T$ in (63). We then have from Lemma 18 that on the event $\mathcal{E} \cap \mathcal{E}_{R_2},$
\begin{equation}
R_2 \leq -\alpha' \frac{1}{12} \sum_{t=N}^{T-1} (\| \text{vec}(M_t) - \text{vec}(M_{\text{apx}}) \|)^2 + \mathcal{O}(1) \left( \sum_{t=N_0}^{T-1} \frac{\Delta L_f^2}{\alpha'} \| \varepsilon \|^2 + \frac{k_f d' L_f^2}{\alpha'} \log \left( T + \log_+ (\alpha' G^2) \right) + \alpha' G^2 (k_f + D_{\max}) + \frac{L_f'}{\alpha'} (L_f' D_{\max} + (\beta' G' + L_f') k_f + L_f' h^2 \log T) \right). \tag{68}
\end{equation}
Next, we have from Lemma 21 that on the event $\mathcal{E} \cap \mathcal{E}_{R_2}$, the following holds for any $\mu \in \mathbb{R}_{>0}$:
\begin{equation}
R_3 \leq (R_w\sigma_1(R) + R_x\sigma_1(Q)) \frac{2\Gamma \kappa}{1 - \gamma} \left( \bar{c}(\Gamma R_u + R_w) p^2 q h \sqrt{n} \frac{\kappa^2 \Gamma^2 D_{\max}^2 + 1}{4(1 - \gamma)} T + \frac{p^2 q^2 h^2}{8\mu} \sqrt{n} \kappa \Gamma^2 D_{\max} + 1 (\kappa + 1) \varepsilon^2 T + \frac{p^5 q^3 n^2}{8} h^5 \kappa^4 \Gamma^6 D_{\max} + 1 (\kappa + 1) R_w \varepsilon^2 T + qh^2 R_w \mu \sum_{t=N}^{T-1} (\| \text{vec}(M_t) - \text{vec}(M_{\text{apx}}) \|)^2 \right). \tag{69}
\end{equation}
Thus, setting \( \mu \) properly and summing up the above two inequalities, we cancel the \( \sum_{i=1}^{T-1} \| \text{Vec}(M_k) - \text{Vec}(M_{\text{alg}}) \|^2 \) term. Moreover, since \( h \geq \frac{4}{1+\gamma} \log T \) by our choice of \( h \), one can show that \( \gamma^{h/4} \leq \frac{1}{T^{1/4}} \). Noting from Eq. (51) that \( \bar{c} \leq \hat{O}(\frac{1}{T^{1/4}}) \), we can combine the above arguments together and obtain that

\[
E[\mathbf{1}\{\mathcal{E} \cap \mathcal{E}_{R_2}\}(R_2 + R_3)] = \hat{O}(\sqrt{T}).
\]  

Recalling from Lemma 6 that \( P(\mathcal{E}) \geq 1 - 1/T \), we have from the union bound that \( P(\mathcal{E} \cap \mathcal{E}_{R_2}) \geq 1 - 2/T \). Similarly, combining the results in Lemmas 10, 11, 12 and 13, we obtain that

\[
E[\mathbf{1}\{\mathcal{E} \cap \mathcal{E}_{R_2}\}(R_0 + R_1 + R_4 + R_5)] = \hat{O}(\sqrt{T}) + TJ_\ast.
\]  

It then follows from Eqs. (68)-(69) that

\[
E\left[\mathbf{1}\{\mathcal{E} \cap \mathcal{E}_{R_2}\} \sum_{i=0}^{T-1} c(x_t^{\text{alg}}, u_t^{\text{alg}})\right] = E\left[\mathbf{1}\{\mathcal{E} \cap \mathcal{E}_{R_2}\} \sum_{i=0}^{5} R_i\right] = \hat{O}(\sqrt{T}) + T J_\ast.
\]  

(70)

Finally, in order to prove \( \mathbb{E}[\text{Regret}] = \hat{O}(\sqrt{T}) \), it remains to upper bound \( \mathbb{E}\left[\mathbf{1}\{\mathcal{E} \cap \mathcal{E}_{R_2}\} \sum_{t=0}^{T-1} c(x_t^{\text{alg}}, u_t^{\text{alg}})\right] \). We prove the following result in Appendix G.

**Lemma 22.** It holds that

\[
\mathbb{E}\left[\mathbf{1}\{\mathcal{E} \cap \mathcal{E}_{R_2}\} \sum_{t=0}^{T-1} c(x_t^{\text{alg}}, u_t^{\text{alg}})\right] = \hat{O}(1).
\]  

(71)

Combining the above arguments complete the proof of Theorem 1.

5 Extensions

5.1 General Information Structure

We show that our analyses and results can be extended to system (1) with general information structure even when Assumption 1 does not hold. To this end, let us consider the following decentralized LQR problem which is the finite-horizon counterpart of (6):

\[
\min_{u_0^\ast, \ldots, u_{T-1}^\ast} \mathbb{E}\left[\frac{1}{T} \sum_{t=0}^{T-1} (x_t^\top Q x_t + u_t^\top R u_t)\right]
\]

s.t. \( x_{t+1} = Ax_t + Bu_t + w_t \),

\[
u_t^M = \sum_{s \in U} I_{\mathcal{V},s} \sum_{k=1}^{h} M_s[k] y_{t-k,s} \quad \forall i \in \mathcal{V}, \forall t \in \{0, \ldots, T - 1\},
\]

\[M = [M_s[k]]_{k \in [h], s \in U} \in D_0',
\]

where \( D_0' = \{ M = [M_s[k]]_{k \in [h], s \in U} : \| M_s[k] \|_F \leq \kappa' \} \) with \( \kappa' \in \mathbb{R}_{>0} \) and \( h \in \mathbb{Z}_{\geq 1} \) to be parameters that one can choose, and we denote the optimal solution to (72) as \( J' \). We note from our arguments in Section 2.4 that given the system matrices \( A \) and \( B \), (72) in fact minimizes over a class of linear state-feedback controller \( u_0^M, \ldots, u_{T-1}^M \) with \( u_t^M \in \pi(\mathcal{I}_t,i) \) for all \( i \in \mathcal{V} \) and all \( t \in \{0, \ldots, T - 1\} \). Since Assumption 1 does not hold, Lemma 1 cannot be applied to ensure that the class of linear controller is optimal to the decentralized LQR problem. However, the class of linear controller considered in (72) already includes any linear state-feedback controller (that depends on the states that are at most \( h \) time steps in the past) under the information constraint. Now, we apply Algorithm 3 to (72) and compute \( u_t^M \) in (72) as \( u_t^{\text{alg}} \) given by the algorithm, since the system matrices \( A \) and \( B \) are unavailable to obtain \( y_{t-1-k,s} \) for \( u_t^M \). We then consider the following regret of Algorithm 3:

\[
\text{Regret}' = \sum_{t=0}^{T-1} (c(x_t^{\text{alg}}, u_t^{\text{alg}}) - J'_t).
\]

(73)
Following similar arguments to those in the proof of Theorem 1, one can decompose $\text{Regret}'$ as
\[
\text{Regret}' = \sum_{i=0}^{5} R'_i - T J'_*,
\]
where $R'_0, \ldots, R'_5$ are defined similarly to $R_0, \ldots, R_5$ in the decomposition in Eq. (46) with $D_0$ and $D$ replaced by $D'_0$ and $D'$, respectively, where $D' \triangleq \{M = [M_{i,k}]_{k \in [n], s \in U} : \|M_{s,k}\|_F \leq 2k\}'$. Moreover, following similar arguments to those in the proof of Lemma 13, one can show that $\mathbb{E}[\{E\} R_5] \leq T J'$. Thus, one can obtain similar results for $\text{Regret}'$ to those in Theorem 1 with $\mathbb{E}[\text{Regret}'] = \mathcal{O}(\sqrt{T})$.

5.2 Stabilizable Systems

Assumption 3 can be relaxed to assuming $(A, B)$ is stabilizable with a priori known stabilizing $K$, as in works on centralized LQR [31, 11, 9, 40]. With a known $K \in \mathbb{R}^{m \times n}$ such that $A + BK$ is stable and $u_t = K x_t$ satisfies the information constraint, we can consider $u_t = K x_t + u'_t$ with $u'_{t,i} \in \pi_i(L_{t,i})$ for all $i \in V$ in (6), and optimize over $u'_0, u'_1, \ldots$. Note that $u_t = K x_t + u'_t$ also satisfies the information constraint. Replacing $A$ with $\bar{A} \triangleq A + BK$, one can check that our analyses and results follow verbatim. Finally, using the techniques from [10, 46], one can add a preliminary phase in Algorithm 3 to obtain a stabilizing controller from system trajectory, which incurs an extra regret of $2\mathcal{O}(n)$ to the algorithm.

6 Numerical Results

For the system in Example 1, we set $\sigma_w = 1$ and $Q = R = I_3$, and generate the system matrices $A$ and $B$ randomly with each nonzero entry chosen independently from a normal distribution. In Fig. 3, we plot $\text{Regret}$ defined in Eq. (13) when Algorithm 3 is used and $N = \sqrt{T}$. The results are averaged over 10 independent experiments to approximate $\mathbb{E}[\text{Regret}]$, and the shaded areas in Fig. 3 display quartiles. We see from Fig. 3 that when $T$ increases, $\text{Regret}/T$ decreases, $\text{Regret}/\sqrt{T}$ slowly increases and $\text{Regret}/(\sqrt{T} \log T)$ remains unchanged (after $T \geq 300$). Thus, Fig. 3 matches with the regret bound $\mathbb{E}[\text{Regret}] = \mathcal{O}(\sqrt{T} \log T)$ in Theorem 1 and also show that the bound is tight for certain problem instances.

![Figure 3: Results for $\text{Regret}$ when Algorithm 3 is applied to Example 1.](image)

Next, we compare our approach with the one in [45] based on the certainty equivalence (CE) approach. We consider the same instance of Example 1 as that constructed above. The results in Table 1(a) are averaged over 5 independent experiments. As we argued in Section 3.3, there is no regret guarantee of the algorithm in [45] since it is an offline algorithm; thus, we can see that Algorithm 3 achieves better performance.

We further compare the performances of Algorithm 3 under different information patterns. Specifically, we use the information propagation pattern in Example 1 given in the directed graph $G(V, A)$ in Fig. 1 as the benchmark (i.e., Info pattern 1). We obtain Info pattern 2 by removing the (solid) directed edge from node 1 to node 2 in Fig. 1, and Info pattern 3 by further removing the direct edge from node 2 to node 3. We keep all the other problem parameters the same as Example 1. Neither of the Info patterns 2 or 3 is partially nested, so there is no closed-form solution to (6) or (72) as that given by Lemma 1. Hence, we compute the cost $\sum_{t=1}^{T} c(x_t^{\text{alg}}, u_t^{\text{alg}})$ of Algorithm 3 directly and obtain the results in Table 1(b) which
are averaged over 5 independent experiments. In general, Info pattern 1 (resp., Info pattern 3) tends to have the lowest (resp., highest) cost among the three for $t = 100, \ldots, 1000$, possibly because graph $G(V, A)$ with more edges implies that the controller at $i \in V$ becomes more informative as it receives more state information from other subsystems in $V$.

| Regret ($\times 10^3$) | 100 | 200 | 300 | 400 | 500 | 600 | 700 | 800 | 900 | 1000 |
|------------------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|
| Algorithm 3            | 2.27| 3.08| 4.13| 4.83| 5.89| 5.65| 6.59| 7.97| 7.32| 8.66 |
| CE                     | 3.16| 4.13| 4.99| 5.92| 6.80| 7.83| 8.62| 9.51| 10.86| 11.10 |

(a) Compare Regret of different algorithms under different $T$'s.

| Cost ($\times 10^4$) | 100 | 200 | 300 | 400 | 500 | 600 | 700 | 800 | 900 | 1000 |
|---------------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|
| Info pattern 1      | 0.95| 1.01| 1.73| 4.81| 4.40| 4.41| 5.21| 8.04| 7.91| 6.66 |
| Info pattern 2      | 0.81| 1.36| 1.63| 4.33| 4.17| 3.60| 4.69| 8.46| 7.79| 8.03 |
| Info pattern 3      | 1.28| 1.77| 2.89| 4.12| 4.45| 3.59| 4.73| 8.90| 7.80| 7.53 |

(b) Compare costs of Algorithm 3 for different info patterns under different $T$'s.

Table 1: Comparisons of different algorithms and information patterns.

7 Conclusion

We considered the problem of learning decentralized linear quadratic regulator under an information constraint on the control policy, with unknown system models. We proposed a model-based online learning algorithm that adaptively designs a control policy when new data samples from a single system trajectory become available. Our algorithm design was built upon a disturbance-feedback representation of state-feedback controllers, and an online convex optimization with memory and delayed feedback subroutine. We showed that our online algorithm yields a controller that satisfies the desired information constraint and yields a $\sqrt{T}$ expected regret. We validated our theoretical results using numerical simulations.

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we let
where the inequality follows from the fact that
the distance between two vectors cannot increase when
projecting onto a convex set. It follows that
First, let us consider any
A.1 Proof of Lemma 4

The Appendix is organized as follows. In Appendix A, we proof the results for the general OCO with memory
and delayed feedback setting and characterize the regret of Algorithm 2. In Appendix B, we prove the
first main result summarized in Section 3.3 (i.e., Proposition 2), which shows that Algorithm 3 can be
implemented in a fully decentralized manner while satisfying the information constraints given by (4) (during
its decentralized online control phase). Appendices C-G contain all the proofs that lead to the final regret
guarantee in Theorem 1. Appendix H contain some helper lemmas used in our proofs.

A Proofs Pertaining to OCO with Memory and Delayed Feedback

A.1 Proof of Lemma 4
First, let us consider any \( x_\ast \in \mathcal{W} \) and any \( t \in \{k, \ldots, T - 1\} \). We know from the strong convexity of \( f_{t,k}(\cdot) \) that (e.g., [5])
\[
2(f_{t,k}(x_t) - f_{t,k}(x_\ast)) \leq 2\nabla f_{t,k}(x_t)\top (x_t - x_\ast) - \alpha \|x_t - x_\ast\|^2.
\]
(75)
Since \( x_{t+1} = \Pi_{\mathcal{W}}(x_t - \eta_t g_{t,\tau}) \), we have
\[
\|x_{t+1} - x_t\|^2 \leq \|x_t - \eta_t g_{t,\tau} - x_\ast\|^2
= \|x_t - x_\ast\|^2 + \eta_t^2 \|g_{t,\tau}\|^2 - 2\eta_t g_{t,\tau}\top (x_t - x_\ast) - 2\eta_t g_{t,\tau}(x_t - x_\ast),
\]
where the inequality follows from the fact that the distance between two vectors cannot increase when
projecting onto a convex set. It follows that
\[
-2g_{t,\tau}\top (x_t - x_\ast) \geq \frac{\|x_{t+1} - x_\ast\|^2 - \|x_t - x_\ast\|^2}{\eta_t} - \eta_t \|g_{t,\tau}\|^2 + 2g_{t,\tau}\top (x_t - x_\ast).
\]
(76)
Moreover, noting that \( g_{t,\tau} = \nabla f_{t-k,\tau}(x_{t-k}) + \varepsilon_{t,\tau} \), we have
\[
-g_{t,\tau}\top (x_t - x_\ast) = -(\nabla f_{t-k,\tau}(x_{t-k}) + \varepsilon_{t,\tau})\top (x_t - x_\ast) - \varepsilon_{t,\tau}\top (x_t - x_\ast)
\leq -(\nabla f_{t-k,\tau}(x_{t-k}) + \varepsilon_{t,\tau})\top (x_t - x_\ast) + \alpha \mu \|x_t - x_\ast\|^2 + \frac{1}{2\alpha \mu} \|\varepsilon_{t,\tau}\|^2,
\]
(77)
where the inequality follows from the fact that \( \alpha \mu \leq \frac{a^2}{2\mu} + \frac{\mu}{2} b^2 \) for any \( a, b \in \mathbb{R} \) and any \( \mu \in \mathbb{R}_{>0} \). To proceed,
we let \( 0 < \mu < 1 \). Combining (76)-(77), and recalling Assumption 6, we obtain
\[
2\nabla f_{t-k,\tau}(x_{t-k})\top (x_t - x_\ast) \leq \frac{\|x_t - x_\ast\|^2 - \|x_{t+1} - x_\ast\|^2}{\eta_t} + \eta_t L_0^2 + \alpha \mu \|x_t - x_\ast\|^2 + \frac{1}{\alpha \mu} \|\varepsilon_{t,\tau}\|^2
- 2\varepsilon_{t,\tau}\top (x_t - x_\ast) - 2g_{t,\tau}\top (x_t - x_\ast),
\]
(78)
where we note that
\[-2g^\top_{t-\tau}(x_t - x_{t-\tau}) = 2\|g_t-\tau\|\|x_t - x_{t-1} + \cdots + x_{t-\tau+1} - x_t\|\]
\[\leq 2\|g_t-\tau\|\sum_{j=1}^{\tau} \eta_{t-j}\|g_{t-j-\tau}\| \leq 2L_g^2 \sum_{j=1}^{\tau} \eta_{t-j},\]
where we let \(g_{t-j-\tau} = 0\) if \(t-j-\tau < 0\). Now, we see from (75) and (78) that
\[
\sum_{t=k}^{T-1} (f_{t,k}(x_t) - f_{t,k}(x_*)) \leq -\frac{\alpha}{2} \sum_{t=k}^{T-1} \|x_t - x_*\|^2 + \sum_{t=k+\tau}^{T-1+\tau} \nabla f_{t-\tau,k}(x_{t-\tau})^\top (x_{t-\tau} - x_*) \leq \frac{1}{2} \sum_{t=k+\tau}^{T-1+\tau} \left( \frac{1}{\eta_{t+1}} - \frac{1}{\eta_t} - \alpha(1-\mu) \right) \|x_{t+1} - x_*\|^2 + \frac{1}{2} \sum_{t=T-1}^{T-1+\tau} \left( \frac{1}{\eta_{t+1}} - \frac{1}{\eta_t} \right) \|x_{t+1} - x_*\|^2
\[+ \sum_{t=k+\tau}^{T-1+\tau} \left( \frac{L_g^2 \eta_t}{2} + \frac{1}{2\alpha \mu} \|x_{t+\tau} - x_*\|^2 - \varepsilon_{t-\tau}(x_{t-\tau} - x_*) + L_g^2 \sum_{j=1}^{\tau} \eta_{t-j} + \frac{\|x_{t-\tau} - x_*\|^2}{2\eta_{t+\tau}}. \quad (79)\]
Setting \(\eta_t = \frac{1}{\alpha t}\) for all \(t \in \{\tau, \ldots, T+\tau\}\) and \(\mu = \frac{1}{3}\), one can show that the following hold:
\[\frac{1}{\eta_{t+1}} - \frac{1}{\eta_t} - \alpha(1-\mu) = -\frac{\alpha}{3} \sum_{t=T-1}^{T-1+\tau} \left( \frac{1}{\eta_{t+1}} - \frac{1}{\eta_t} \right) \|x_{t+1} - x_*\|^2 \leq \frac{\alpha(\tau+1)}{3} G^2,\]
\[\sum_{t=k+\tau}^{T-1} L_g^2 \eta_t \leq \sum_{t=k}^{T-1} L_g^2 \eta_t \leq \frac{3L_g^2}{\alpha} \log T, \quad \|x_{t+\tau} - x_*\|^2 \leq \frac{\alpha G^2 (k+\tau)}{3}\]
\[L_g^2 \sum_{t=k+\tau}^{T-1+\tau} \sum_{j=1}^{\tau} \eta_{t-j} \leq \frac{3L_g^2 \tau}{\alpha} \sum_{t=k+\tau}^{T-1+\tau} \frac{1}{t-\tau} \leq \frac{3L_g^2 \tau}{\alpha} \log T.\]
Meanwhile, we have
\[\frac{\alpha}{6} \sum_{t=k+\tau+1}^{T-1} \|x_t - x_*\|^2 \leq \frac{\alpha G^2 (k+\tau+1)}{6} - \frac{\alpha}{6} \sum_{t=0}^{T-1} \|x_t - x_*\|^2.\]
It then follows from (79) that (30) holds.  

\[\Box\]

### A.2 Proof of Lemma 5

Using similar arguments to those for the proof of [41, Lemma K.2] one can show that
\[
\sum_{t=k}^{T-1} (f_t(x_t) - f_t(x_*)) - \sum_{t=k}^{T-1} \varepsilon_t^\top (x_t - x_*) \leq \sum_{t=k}^{T-1} (f_{t,k}(x_t) - f_{t,k}(x_*)) \]
\[+ \sum_{t=k}^{T-1} X_t(x_*) + \sum_{t=k}^{T-1} (2\beta G + 4L_f) \|x_t - x_{t-k}\|.\]
Moreover, we have
\[
\sum_{t=k}^{T-1} (2\beta G + 4L_f)\|x_t - x_{t-k}\| = (2\beta G + 4L_f) \sum_{t=k}^{T-1} \|x_t - x_{t-1} + \cdots + x_{t-k+1} - x_{t-k}\|
\leq (2\beta G + 4L_f) \sum_{t=k}^{T-1} k \sum_{j=1}^{T-1} \|g_{t-j}\| \leq (4\beta G + 8L_f) T g \sum_{t=k+1}^{T} \frac{3k}{\alpha(t-k)} \leq (4\beta G + 8L_f) 3L_g k \log T,
\]
where we let \(\eta_0 = \frac{3}{T}\), \(\eta_t = \frac{3}{T'}\) if \(t' \geq 1\), and \(g_{t-j-\tau} = 0\) if \(t - j - \tau < 0\). Combining the above arguments with Lemma 4, we conclude that \((31)\) holds. \(\blacksquare\)

### A.3 Proof of Proposition 1

First, let us consider any \(\delta > 0\) and any \(x_\ast \in W\). Using similar arguments to those for [41, Lemma K.3], one can show that the following holds with probability at least \(1 - \delta\):
\[
- \sum_{t=k}^{T-1} X_t(x_\ast) - \frac{\alpha}{12} \sum_{t=0}^{T-1} \|x_t - x_\ast\|^2 \leq \mathcal{O}(1) \frac{kL_f^2}{\alpha} \log \left(\frac{k(1 + \log_+ (\alpha T G^2))}{\delta}\right),
\]
where \(\log_+ (x) = \log(\max\{1, x\})\). It then follows from Lemma 5 that the following holds with probability at least \(1 - \delta\):
\[
\sum_{t=k}^{T-1} (f_t(x_t) - f_t(x_\ast)) \leq - \frac{\alpha}{12} \sum_{t=0}^{T-1} \|x_t - x_\ast\|^2 + \mathcal{O}(1) \left(\alpha G^2 (k + \tau) + \frac{L_g}{\alpha} (L_g \tau + (\beta G + L_f)k) \log T \right.
\]
\[\left. + \sum_{t=k}^{T-1} \frac{1}{\alpha} \|\tilde{\epsilon}_t\|^2 + \frac{kL_f^2}{\alpha} \log \left(\frac{T(1 + \log_+ (\alpha G^2))}{\delta}\right)\right),
\]
where we use the facts that \(k \leq T\) and \(\log_+ (\alpha T G^2) \leq T \log_+ (\alpha G^2)\). Next, following [41, Claim K.4] and [41, Claim K.5], one can further show that the following hold with probability at least \(1 - \delta\):
\[
\sum_{t=k}^{T-1} (f_t(x_t) - f_t(x_\ast)) \leq - \frac{\alpha}{12} \sum_{t=0}^{T-1} \|x_t - x_\ast\|^2 + \mathcal{O}(1) \left(\sum_{t=k}^{T-1} \frac{1}{\alpha} \|\tilde{\epsilon}_t\|^2 + \frac{kL_f^2}{\alpha} \log \left(\frac{T(1 + \log_+ (\alpha G^2))}{\delta}\right) + \alpha G^2 (k + \tau) + \frac{L_g}{\alpha} (L_g \tau + (\beta G + L_f)k) \log T\right), \quad \forall x_\ast \in W. \quad (80)
\]

Now, following the proof of [2, Theorem 4.6], we have that for any \(x_\ast \in W\),
\[
\sum_{t=k}^{T-1} (F_t(x_t, \ldots, x_{t-h}) - f_t(x_\ast)) = \sum_{t=k}^{T-1} (f_t(x_t) - f_t(x_\ast)) + \sum_{t=k}^{T-1} (F_t(x_t, \ldots, x_{t-h}) - f_t(x_t)). \quad (81)
\]
The second term on the right-hand side of the above equation can be bounded as
\[
\sum_{t=k}^{T-1} (F_t(x_t, \ldots, x_{t-h}) - f_t(x_t)) \leq L_c \sum_{t=k}^{T-1} \sum_{i=1}^{h} \|x_t - x_{t-i}\|
\]
\[
\leq L_c \sum_{t=k}^{T-1} \sum_{j=1}^{h} \eta_{t-j}\|g_{t-j}\|
\]
\[
\leq L_c L_g \sum_{t=k}^{T-1} \sum_{i=1}^{h} \frac{3i}{\alpha(t-i)}
\]
\[
\leq L_c L_g \frac{3}{\alpha(t-h)} \leq \frac{3L_c L_g h^2}{\alpha} \log T,
\]
where we let \(\eta_t = \frac{3}{\alpha t}\) for all \(t' \geq 1\), and \(g_{t-j} = 0\) if \(t-j-t' < 0\). Combining (80)-(82) together completes the proof of the proposition.

B Proof of Proposition 2

Let us consider any \(i \in \mathcal{V}\). To prove part (a), we use an induction on \(t = N + D_{\text{max}} + \ldots, T + D_{\text{max}} - 1\). For the base case \(t = N\), we see directly from line 2 in Algorithm 3 and Eq. (42) that \(K_{i,1}\) (resp., \(K_{i,2}\)) satisfies Eq. (44) (resp., Eq. (45)) at the beginning of iteration \(t = N\) of the for loop in lines 7-18 of the algorithm. For the inductive step, consider any \(t \in \{N + D_{\text{max}} + \ldots, T + D_{\text{max}} - 1\}\) and suppose \(K_{i,1}\) (resp., \(K_{i,2}\)) satisfies Eq. (44) (resp., Eq. (45)) at the beginning of iteration \(t\) of the for loop in lines 7-18 of the algorithm.

First, let us consider \(K_{i,1}\), and consider any \(s \in \mathcal{L}(\mathcal{T}_i)\) with \(j \in \mathcal{V}\) and \(s_{0,j} = s\) in the for loop in lines 8-11 of the algorithm, where \(\mathcal{L}(\mathcal{T}_i)\) is defined in Eq. (41). We aim to show that \(\hat{w}_{t-D_{ij} - 1,j}\) can be determined, using Eq. (33) and the current \(K_{i,1}\) and \(K_{i,2}\). We see from Eq. (33) that in order to determine \(\hat{w}_{t-D_{ij} - 1,j}\), we need to know \(x_{t-D_{ij},j}^{\text{alg}}\), \(x_{t-D_{ij} - 1,j}^{\text{alg}}\), and \(u_{t-D_{ij} - 1,j}^{\text{alg}}\) for all \(j \in \mathcal{N}_j\). Note that \(D_{ij} \leq D_{ij} + 1\) for all \(j \in \mathcal{N}_j\), which implies that \(t - D_{\text{max}} - 1 \leq t - D_{ij} - 1 \leq t - D_{ij}\). Thus, we have that \(x_{t-D_{ij},j}^{\text{alg}} \in \mathcal{T}_{t,i}\) and \(x_{t-D_{ij} - 1,j}^{\text{alg}} \in \mathcal{T}_{t,i}\) for all \(j \in \mathcal{N}_j\), where \(\mathcal{T}_{t,i}\) is defined in Eq. (43). Then we focus on showing that \(u_{t-D_{ij} - 1,j}^{\text{alg}}\) for all \(j \in \mathcal{N}_j\) can be determined based on the current \(K_{i,1}\) and \(K_{i,2}\). Considering any \(j \in \mathcal{N}_j\), one can show via line 12 of the algorithm and Eq. (35) that
\[
u_{t-D_{ij} - 1,j}^{\text{alg}} = \sum_{r \geq j} I_{(j,v)}^h \sum_{k=1}^{h} M_{t}^{[k]} \hat{w}_{t-D_{ij} - 1,k,v},
\]
where \(\hat{w}_{t-D_{ij} - 1,k,v} = \left[\hat{w}_{t-D_{ij} - 1,k,v}^v\right]_{v \in \mathcal{V}}\). Now, let us consider any \(r \geq j\). Recalling the definition of \(\mathcal{P}(\mathcal{U}, \mathcal{H})\) in Eq. (7), and noting that \(j_0 \in v, j_1 \in r, v - r,\) one can show that
\[
D_{j_1j_0} \leq l_{vr} \leq D_{\text{max}}.
\]
To proceed, we split our arguments into two cases: \(D_{ij} \leq D_{ij} + D_{ij} \geq D_{ij} + 1\). Supposing \(D_{ij} \leq D_{ij}\), we have
\[
t - 2D_{\text{max}} - 2h \leq t - D_{ij} - 1 - k - l_{vs}
\]
\[
t - D_{ij} - 2 \geq t - D_{ij} - 1 - k - l_{vs},
\]
for all \(k \in [h]\). From the induction hypothesis, we know that \(\hat{w}_{k',j_0} \in K_{i,1}\) for all \(k' \in \{t - 2D_{\text{max}} - 2h, \ldots, t - D_{ij} - 2\}\), which implies via (84) that \(\hat{w}_{t-D_{ij} - 1,k,v}\) for all \(k \in [h]\) can be determined based on the current \(K_{i,1}\). Moreover, noting that \(j_1 \in r, j_1 \in \mathcal{N}_j\), one can show that \(r \in \mathcal{T}_i\). Since \(M_{t-D_{ij} - 1,r} \in K_{i,2}\) for all \(r \in \mathcal{T}_i\), we then have from the above arguments that \(u_{t-D_{ij} - 1,j}^{\text{alg}}\) can be determined based on the current \(K_{i,1}\) and \(K_{i,2}\). Next, suppose that \(D_{ij} \geq D_{ij} + 1\). Noting that \(j_0 \rightarrow j_1 \rightarrow i\), i.e., there exists a directed path from node \(j_0\) to node \(i\) that goes through \(j_1\) in \(\mathcal{G}(\mathcal{V}, \mathcal{A})\), we have
\[
D_{ij} \leq D_{ij} + D_{ij} \leq D_{ij} + D_{ji} + D_{jij_0},
\]
where \( D_{jj_1} \in \{0,1\} \). Combining (83) and (85), we have
\[
\begin{align*}
t - 2D_{\text{max}} - 2h & \leq t - D_{ij} - 1 - k - l_{uv} \\
t - D_{ij} - 1 & \geq t - D_{ij} - 1 - k - l_{uv},
\end{align*}
\] (86)

for all \( k \in [h] \). Recalling from Remark 1 that we have assumed without loss of generality that the for loop in lines 8-11 of Algorithm 3 iterate over the elements in \( \mathcal{L}(T_i) \) according to a certain order of the elements in \( \mathcal{L}(T_i) \). We then have from the fact \( D_{ij} \geq D_{ij} + 1 \) that when considering \( s \in \mathcal{L}(T_i) \) (with \( j \in V \) and \( s_{0,j} = s \)) in the for loop in lines 8-11 of Algorithm 3, the element \( s_{0,j} \in \mathcal{L}(T_i) \) has already been considered by the algorithm, i.e., \( \hat{w}_{k',j} \) for all \( k' \in \{t - 2D_{\text{max}} - 2h, \ldots, t - D_{ij} - 1\} \) are in the current \( K_{i,1} \). It then follows from (86) similar arguments to those above that \( u_{t-D_{ij}-1,j} \) can be determined based on the current \( K_{i,1} \) and \( K_{i,2} \). This completes the inductive step of the proof that \( K_{i,1} \) satisfies Eq. (44) at the beginning of any iteration \( t \in \{N + D_{\text{max}}, \ldots, T + D_{\text{max}} - 1\} \) of the for loop in lines 7-18 of Algorithm 3.

Next, let us consider \( K_{i,2} \), and consider any \( s \in T_i \) in the for loop in lines 15-17 of the algorithm. Note that \( j \sim i \) for all \( j \in s \). To simplify the notations in this proof, we denote
\[
 f_{t_D}(M_i) = f_{t_D}(M_{t_D} | \Phi, \hat{w}_{t-D_{D}-1}),
\]
Moreover, we see from Definition 2 that
\[
 f_{t_D}(M_i) = c(x_{t_D}, u_{t_D}),
\]
where, for notational simplicity, we denote
\[
x_{t_D} = \sum_{k=t_D-h}^{t_D-1} \hat{A}_{t_D}^{-1}(k) (\hat{w}_k + \hat{B} u_k),
\]
\[
 u_k = u_k (M_{t_D} | \Phi, \hat{w}_{0:k-1}) = \sum_{s \in \mathcal{T}} \sum_{k'=1}^{h} I_{V,s} M_{t_D,s} [k'] \hat{v}_{k-k',s},
\] (87)

for all \( k \in \{t_D - h, \ldots, t_D - 1\} \), where \( \hat{v}_{k-k',r} = [\hat{w}_{t-D_{D}-1}]^T \). Similarly, one can also show that
\[
 u_{k,j} = \sum_{t \not\sim j} I_{(j),r} \sum_{k'} M_{t_D,r} [k'] \hat{v}_{k-k',r},
\] (88)

for all \( k \in \{t_D - h, \ldots, t_D - 1\} \) and all \( j \in V \). We will then show that \( M_{t_D} \), i.e., \( \partial f_{t_D}(M_{t_D}) \), can be determined based on the current \( K_{i,1} \) and \( K_{i,2} \). In other words, viewing \( f_{t_D}(M_{t_D}) \) as a function of \( M_{t_D} \), we will show that for any term in \( f_{t_D}(M_{t_D}) \) that contains \( M_{t_D,s} \), the coefficients of that term can be determined based on the current \( K_{i,1} \) and \( K_{i,2} \). To proceed, we notice from Assumption 7 that
\[
c(x_{t_D}, u_{t_D}) = \sum_{l \in [v]} (x_{t_D,j}^T Q_{V,l} x_{t_D,j} + u_{t_D,l}^T R_{V,l} u_{t_D,l}),
\]
where \( x_{t_D,j} = [x_{t_D,j}^T]_{j \in V} \) and \( u_{t_D,l} = [u_{t_D,l,j}]_{j \in V} \). Let us consider any \( l \in [\psi] \). One can show using Eq. (88) that the term \( u_{t_D,l}^T R_{V,l} u_{t_D,l} \) contains \( M_{t_D,s} \) if and only if there exists \( j \in s \in T_i \) such that \( j \in V_l \). Supposing that \( l \in [\psi] \) satisfies this condition, one can show that either \( i \rightarrow j \) or \( j \rightarrow i \) holds in \( \mathcal{G}(V,A) \), which implies via Assumption 7 that \( i \in V_l \). Viewing the term \( u_{t_D,l}^T R_{V,l} u_{t_D,l} \) as a function of \( M_{t_D,s} \), we then aim to show that the coefficients of this term can be determined based on the current \( K_{i,1} \) and \( K_{i,2} \). To this end, considering any \( j_1 \in V_l \), we see from Eq. (88) that
\[
u_{t_D,j_1} = \sum_{r \not\sim j_1} I_{(j_1),r} \sum_{k=1}^h M_{t_D,r} [k] \hat{v}_{t-D_{D}-k,r},
\]
where \( \hat{\eta}_{tD-k,r} = [w_{tD-k-l,v}]_{l \in \mathcal{L}} \). Since \( i \in \mathcal{V}_t \) as we argued above, it holds that \( j \sim i \) in \( \mathcal{G}(\mathcal{V}, \mathcal{A}) \), which implies that \( r \in \mathcal{T}_i \) for all \( r \ni j_i \). It then follows from Eq. (45) that \( M_{t,r} \in \mathcal{K}_{i,2} \) for all \( r \ni j_i \). Moreover, noting that that \( l_{v} \geq D_{\text{max}} \), we then have from Eq. (44) that \( \hat{\eta}_{tD-k,r} \) for all \( k \in [h] \) and all \( r \ni j_i \) can be determined based on the current \( \mathcal{K}_{i,1} \). It follows that the coefficients in the term \( u_{tD,\mathcal{V}_t}^\top R_{\mathcal{V}_t,\mathcal{V}_t} u_{tD,\mathcal{V}_t} \) can be determined based on the current \( \mathcal{K}_{i,1} \) and \( \mathcal{K}_{i,2} \).

Now, let us consider any \( l \in [\psi] \) with the corresponding term \( x_{tD,\mathcal{V}_t}^\top R_{\mathcal{V}_t,\mathcal{V}_t} x_{tD,\mathcal{V}_t} \). Recall from Algorithm 1 that \( \mathcal{A} \) and \( \hat{\mathcal{B}} \) satisfy that for any \( i,j \in \mathcal{V}, A_{ij} = 0 \) and \( B_{ij} = 0 \) if and only if \( D_{ij} = \infty \), where \( D_{ij} < \infty \) if and only if there is a directed path from node \( j \) to node \( i \) in \( \mathcal{G}(\mathcal{V}, \mathcal{A}) \). One can also show that for any \( i,j \in \mathcal{V} \) and any \( k \in \{tD-h, \ldots, tD-1 \} \), \( (A^{tD-(k+1)})_{ij} = 0 \) and \( (\hat{A}^{tD-(k+1)}\hat{B})_{ij} = 0 \) if and only if \( D_{ij} = \infty \). It then follows from Eq. (87) and Assumption 7 that

\[
x_{tD,\mathcal{V}_t} = \sum_{k=tD-h}^{tD-1} \left( (A^{tD-(k+1)})_{\mathcal{V}_t,\mathcal{V}_t} \hat{w}_{k,\mathcal{V}_t} + (\hat{A}^{tD-(k+1)}\hat{B})_{ij} u_{k,\mathcal{V}_t} \right).
\]  

One can show via Eq. (89) that for any \( l \in [\psi] \), the term \( x_{tD,\mathcal{V}_t}^\top R_{\mathcal{V}_t,\mathcal{V}_t} x_{tD,\mathcal{V}_t} \) contains \( M_{t,s} \), if and only if there exists \( j \in s \) such that \( j \in \mathcal{V}_t \). Supposing that \( l \in [\psi] \) satisfies this condition, we have from similar arguments to those above that \( i \in \mathcal{V}_t \). Viewing the term \( x_{tD,\mathcal{V}_t}^\top R_{\mathcal{V}_t,\mathcal{V}_t} x_{tD,\mathcal{V}_t} \) as a function of \( M_{t,s} \) we aim to show that the coefficients of this term can be determined based on the current \( \mathcal{K}_{i,1} \) and \( \mathcal{K}_{i,2} \). To this end, let us consider any \( j_1 \in \mathcal{V}_t \) and any \( k \in \{tD-h, \ldots, tD-1 \} \). Since \( i \in \mathcal{V}_t \) as we argued above, it holds that \( j_1 \sim i \) in \( \mathcal{G}(\mathcal{V}, \mathcal{A}) \), which also implies that \( r \in \mathcal{T}_i \) for all \( r \ni j_1 \). It follows from Eq. (44) that \( u_{k,j_1} \) can be determined based on the current \( \mathcal{K}_{i,1} \). Moreover, recalling that \( u_{k,j_1} = \sum_{r \ni j_1} I_{\{j_1\},r} \sum_{k'=1}^{h} M_{t,r}^{[k]} \hat{\eta}_{k-k',r} \), one can show via similar arguments to those above that \( M_{t,s} \in \mathcal{K}_{i,2} \) for all \( r \ni j_1 \), and \( \hat{\eta}_{k-k',r} \) for all \( k' \in [h] \) and all \( r \ni j_1 \) can be determined based on the current \( \mathcal{K}_{i,1} \). It now follows from Eq. (89) that the coefficients of the term \( x_{tD,\mathcal{V}_t}^\top R_{\mathcal{V}_t,\mathcal{V}_t} x_{tD,\mathcal{V}_t} \) can be determined based on the current \( \mathcal{K}_{i,1} \) and \( \mathcal{K}_{i,2} \). This completes the inductive step of the proof that \( \mathcal{K}_{i,2} \) satisfies Eq. (44) at the beginning of any iteration \( t \in \{N + D_{\text{max}}, \ldots, T + D_{\text{max}} - 1 \} \) of the for loop in lines 7-18 of Algorithm 3.

We then prove part (b). It suffices to show that for any \( t \in \{N + D_{\text{max}}, \ldots, T + D_{\text{max}} - 1 \} \) and any \( i \in \mathcal{V} \), \( u_{t,s}^{\text{alg}} \) can be determined based on the current \( \mathcal{K}_{i,1} \) and \( \mathcal{K}_{i,2} \) after line 9 and before line 16 in iteration \( t \) of the algorithm, where we note that the current \( \mathcal{K}_{i,1} \) is given by

\[
\mathcal{K}_{i,1} = \{ \hat{w}_{k,j} : k \in \{t - 2D_{\text{max}} - 2h, \ldots, t - D_{ij} - 1 \}, s \in \mathcal{L}(\mathcal{T}_i), j \in \mathcal{V}, s_{0,j} = s \}.
\]

Note again from Eq. (35) that

\[
u_{t,i}^{\text{alg}} = \sum_{r \ni i} I_{\{i\},r} \sum_{k=1}^{h} M_{t,r}^{[k]} \hat{\eta}_{k-k,r},
\]

where \( \hat{\eta}_{k-k,r} = [\hat{w}_{tD-l,v}]_{l \in \mathcal{L},v}^{\top} \). Considering \( r \ni i \) and any \( j_v \) with \( w_{j_v} \rightarrow v \) and \( v \in \mathcal{L}_r \), we have

\[
t - 2D_{\text{max}} - 2h \leq t - k - l_{v},
\]

\[
t - D_{ij,v} - 1 \geq t - k - l_{v},
\]

for all \( k \in [h] \). Since \( i \in r \in \mathcal{T}_i \), we know that \( j_v \sim i \). It then follows from Eq. (44) that \( \hat{\eta}_{k-k,r} \) for all \( k \in [h] \) and all \( r \ni i \) can be determined based on the current \( \mathcal{K}_{i,1} \). Also noting from Eq. (45) that \( M_{t,r} \in \mathcal{K}_{i,2} \) for all \( r \in \mathcal{T}_i \), we conclude that \( u_{t,i}^{\text{alg}} \) can be determined based on the current \( \mathcal{K}_{i,1} \) and \( \mathcal{K}_{i,2} \) .

### C Proofs Omitted in Section 4.1

#### C.1 Proof of Lemma 7

Considering the directed graph \( \mathcal{G}(\mathcal{V}, \mathcal{A}) \), let \( \mathcal{V} = \cup_{i \in [\psi]} \mathcal{V}_i \), where \( \mathcal{V}_i \subseteq \mathcal{V} \) denotes the set of nodes in the \( i \)th strongly connected component of \( \mathcal{G}(\mathcal{V}, \mathcal{A}) \). For any \( i \in [\psi] \), let \( \hat{\mathcal{V}}_i \) be the set of nodes in \( \mathcal{G}(\mathcal{V}, \mathcal{A}) \) that
can reach any node in \( \mathcal{V} \), via a directed path in \( \mathcal{G}(\mathcal{V}, A) \), and let \( \hat{V}_i^c = \mathcal{V} \setminus \hat{V}_i \). Denoting \( \Delta A = A - \hat{A} \) and \( \Delta B = B - \hat{B} \), we see from line 6 in Algorithm 1 that \( [\Delta A_{V_i, \hat{V}_i} \ \Delta B_{V_i, \hat{V}_i}] \) is a submatrix of \( \tilde{\Delta}_N \). One can then show that \( \| [\Delta A_{V_i, \hat{V}_i} \ \Delta B_{V_i, \hat{V}_i}] \| \leq \| \tilde{\Delta}_N \| \). Moreover, we see from line 6 in Algorithm 1 that \( \hat{A}_{V_i, \hat{V}_i} = 0 \) and \( \hat{B}_{V_i, \hat{V}_i} = 0 \). Also noting from Eq. (3) that \( \Delta A_{V_i, \hat{V}_i} = 0 \) and \( B_{V_i, \hat{V}_i} = 0 \), one can then show via Eqs. (18)-(20) that

\[
\| \Delta A_{V_i, \hat{V}_i} \| = \| [\Delta A_{V_i, \hat{V}_i} \ \Delta B_{V_i, \hat{V}_i}] \|. \]

Now, assuming without loss of generality that the nodes in \( \mathcal{V} \) are ordered such that \( \Phi_N = [\tilde{\Phi}_V]_{\{i \in [\psi]\}}^1 \) we obtain from the above arguments that

\[
\Delta \leq \sqrt{\sum_{i \in [\psi]} \| [\Delta A_{V_i} \ \Delta B_{V_i}] \|^2} \leq \sqrt{\psi}. \]

**C.2 Proof of Proposition 3**

First, following the arguments for [45, Proposition 1], one can show that under the choice of \( N \) given by Eq. (52),

\[
\| \tilde{\Delta}_N \|^2 \leq \frac{160}{N \sigma^2} \left( 2n \sigma^2 (n + m) \left( \log(N + z_b^2/\lambda)T \right) + \lambda n \Gamma^2 \right). \]

We then have from Lemma 7 that

\[
\| \Delta_N \|^2 \leq \frac{160 \psi}{N \sigma^2} \left( 2n \sigma^2 (n + m) \left( \log(N + z_b^2/\lambda)T \right) + \lambda n \Gamma^2 \right)
\leq \frac{480 \psi}{N \sigma^2} n \sigma^2 (n + m) \left( \log(T + z_b^2/\lambda)T \right) \Gamma^2, \tag{90} \]

where \( \Delta_N \) is defined in (29). Plugging Eq. (52) into the above inequality and noting that \( \| \tilde{\Delta}_N \| \leq \| \Delta(N) \|^2 \) and \( \| \tilde{\Delta}_N \| \leq \| \Delta(N) \|^2 \), one can show that \( \| \tilde{\Delta}_N \| \leq \epsilon \) and \( \| \tilde{\Delta}_N \| \leq \epsilon \).

**C.3 Proof of Lemma 8**

Consider any \( t \in \mathbb{Z}_{\geq 0} \). We see from Eq. (21) that

\[
\| u_t \| \leq \sum_{s \in I_t} \sum_{k=1}^t \| M_{k,1}[k] \| \| \eta_{t-k,s} \| \leq q \kappa \int \Gamma^{2D_{\max} + 1} R_w \sum_{k=0}^{t-1} \gamma^k \leq \frac{qq \kappa \Gamma^{2D_{\max} + 1} R_w}{1 - \gamma}. \]

Next, we know from [25, Lemma 14] that \( x_t^* \) satisfies \( x_t^* = \sum_{s \in I_t} f_{V,s} \zeta_{t,s} \) for all \( t \in \mathbb{Z}_{\geq 0} \), where \( \zeta_{t,s} \) is given by Eq. (10). Using similar arguments to those above, one can then show via Eqs. (18)-(20) that

\[
\| x_t^* \| \leq \sum_{s \in I_t} \| \zeta_{t,s} \| \leq \frac{qq \kappa \Gamma^{2D_{\max} + 1} R_w}{1 - \gamma}. \]

Similarly, we obtain from Eq. (23) that

\[
\| u_t^M \| \leq \frac{qq \kappa \Gamma^{2D_{\max} + 1} R_w}{1 - \gamma}. \]

From Eq. (3), we know that

\[
x_t^M = \sum_{k=0}^{t-1} A^{t-(k+1)}(Bu_k^M + w_k), \]

32
which implies that
\[
\|x_t^M\| \leq \sum_{k=0}^{t-1} \|A^{t-(k+1)}\| \|Bu_k^M + w_k\|
\]
\[
\leq \left( \Gamma \frac{pqk\Gamma^2D_{\text{max}}+1R_w}{1-\gamma} + R_w \right) \sum_{k=0}^{t-1} \gamma^k
\]
\[
\leq \frac{2pqk^2\Gamma^2D_{\text{max}}+2R_w}{(1-\gamma)^2}.
\]
Note that $M \in \mathcal{D}$ trivially holds. Now, we have from the above arguments and Lemma 2 that
\[
c(x_t^M, u_t^M) - c(x_t^*, u_t^*) \leq 2\sigma \left( \frac{pqk\Gamma^2D_{\text{max}}+1R_w}{1-\gamma} + \frac{2pqk^2\Gamma^2D_{\text{max}}+2R_w}{(1-\gamma)^2} \right) (\|x_t^M - x_t^*\| + \|u_t^M - u_t^*\|)
\]
\[
\leq 6\sigma \frac{pqk^2\Gamma^2D_{\text{max}}+2R_w}{(1-\gamma)^2} (\|x_t^M - x_t^*\| + \|u_t^M - u_t^*\|).
\]
Moreover, we have
\[
\|u_t^M - u_t^*\| = \left\| \sum_{s \in \mathcal{U}} \sum_{k=h}^{t} I_{s,t} M_{s,k,t-k,s} \right\|
\]
\[
\leq R_w \kappa \Gamma D_{\text{max}} + \sum_{k=h}^{t-1} \gamma^k
\]
\[
\leq \frac{\kappa \Gamma D_{\text{max}}}{1-\gamma} R_w.
\]
From Eq. (3), we have
\[
\|x_t^M - x_t^*\| = \left\| \sum_{k=0}^{t-1} A^{t-(k+1)} B (u_k^M - u_k^*) \right\|
\]
\[
\leq \frac{\kappa \Gamma D_{\text{max}}+2\gamma^h R_w}{1-\gamma} \sum_{k=0}^{t-1} \gamma^k
\]
\[
\leq \frac{\kappa^2 \Gamma D_{\text{max}}+2\gamma^h R_w}{(1-\gamma)^2}.
\]
Combining the above inequalities completes the proof of the lemma. ■

C.4 Proof of Lemma 9

First, we see from Assumption 3 and (49) that
\[
\|x_N^\text{alg}\| = \left\| \sum_{k=0}^{N-1} A^k (Bu_N^\text{alg} + w_{N-k-1}) \right\|
\]
\[
\leq \left( \sigma_u \sqrt{5m \log 4NT} + \sigma_w \sqrt{10n \log 2T} \right) \Gamma \kappa \sum_{k=0}^{N-1} \gamma^k
\]
\[
\leq \sigma \sqrt{10(\sqrt{n} + \sqrt{m})} \Gamma \kappa \frac{\sqrt{20(m+n)\Gamma \kappa}}{1-\gamma}.
\]
Next, we recall from Eqs. (3) and (33) that for any \( t \in \{N, \ldots, T-1\} \), \( w_t = x_{t+1}^{alg} - Ax_t^{alg} - Bu_t^{alg} \) and \( \hat{w}_t = x_{t+1}^{alg} - \hat{A}x_t^{alg} - \hat{B}u_t^{alg} \), respectively. It follows that for any \( t \in \{N, \ldots, T-1\} \),

\[
\begin{align*}
    w_t - \hat{w}_t &= (\hat{A} - A)x_t^{alg} + (\hat{B} - B)u_t^{alg} \\
    &= (\hat{A} - A)\left( A^{t-N}x_N^{alg} + \sum_{k=0}^{t-N-1} A^k(Bu_{t-k-1}^{alg} + w_{t-k-1}) \right) + (\hat{B} - B)u_t^{alg}.
\end{align*}
\]

Let us denote \( R_{ut} = \max_{k \in \{N, \ldots, t\}} \|u_k^{alg}\| \) for all \( t \in \{N, \ldots, T-1\} \), where we note that \( R_{u_N} = 0 \) (from line 4 in Algorithm 3) and that \( R_{u_t} \geq R_{u_{t-1}} \). Now, consider any \( t \in \{N, \ldots, T-1\} \). Recalling from Proposition 3 that \( \|\hat{A} - A\| \leq \bar{\varepsilon} \) and \( \|\bar{B} - B\| \leq \bar{\varepsilon} \), we have

\[
\begin{align*}
    \|w_t - \hat{w}_t\| &\leq \bar{\varepsilon}\|A^{t-N}\|\|x_N^{alg}\| + \bar{\varepsilon}\sum_{k=0}^{t-N-1} A^k(Bu_{t-k-1}^{alg} + w_{t-k-1}) + \bar{\varepsilon}\|u_t^{alg}\| \\
    &\leq \bar{\varepsilon}\kappa \gamma^{t-N} \frac{2\sqrt{10m\Gamma\kappa}}{1 - \gamma} + \bar{\varepsilon}(\Gamma R_{u_{t-1}} + R_w)\kappa \sum_{k=0}^{t-1} \gamma^k + \bar{\varepsilon} R_{ut} \\
    &\leq \left( \frac{\Gamma \kappa}{1 - \gamma} + 1 \right) \bar{\varepsilon} R_{ut} + \frac{\bar{\varepsilon} \kappa}{1 - \gamma} (\kappa \sigma \sqrt{20(m + n)\Gamma} + R_w), \quad (91)
\end{align*}
\]

which implies that

\[
\|\hat{w}_t\| \leq \|w_t - w_{t-1}\| + \|w_t\| \leq \left( \frac{\Gamma \kappa}{1 - \gamma} + 1 \right) \bar{\varepsilon} R_{ut} + R_w + \frac{\bar{\varepsilon} \kappa}{1 - \gamma} (\kappa \sigma \sqrt{20(m + n)\Gamma} + R_w). \quad (92)
\]

Recall from Eq. (35) that \( u_t^{alg} = \sum_{s \in \mathcal{U}} \sum_{k=0}^{h} I_{v, s} M_{t,s}^{[k]} \hat{\eta}_{t-k, s} \), where \( \hat{\eta}_{t-k, s} = \left[ \hat{w}_{t-k-l v, s, j, v} \right]_{v \in \mathcal{L}_s}^T \), with \( w_{j,v} \rightarrow v \), for all \( k \in [h] \) and all \( s \in \mathcal{U} \). Noting that \( |\mathcal{L}_s| \leq p \), we have

\[
\begin{align*}
    \|\hat{\eta}_{t-k, s}\| &= \left\| \left[ \hat{w}_{t-k-l v, s, j, v} \right]_{v \in \mathcal{L}_s}^T \right\| \leq \sum_{v \in \mathcal{L}_s} \|\hat{w}_{t-k-l v, s, j, v}\| \leq p \|\hat{w}_{t-k-l v, s}\| \\
    &\leq p \left( \frac{\Gamma \kappa}{1 - \gamma} + 1 \right) \bar{\varepsilon} R_{ut} + p R_w + \frac{p \bar{\varepsilon} \kappa}{1 - \gamma} (\kappa \sigma \sqrt{20(m + n)\Gamma} + R_w).
\end{align*}
\]

Noting from the definition of Algorithm 3 that \( M_t \in \mathcal{D} \) for all \( t \in \{N, \ldots, T-1\} \), we then have

\[
\begin{align*}
    R_{ut} &\leq 2 \left( pq \left( \frac{\Gamma \kappa}{1 - \gamma} + 1 \right) \bar{\varepsilon} R_{u_{t-1}} + pq R_w + \frac{pq \bar{\varepsilon} \kappa}{1 - \gamma} (\kappa \sigma \sqrt{20(m + n)\Gamma} + R_w) \right) h \sqrt{n k p t^{2D_{max}+1}} \\
    &\leq \frac{1}{4} R_{u_{t-1}} + 3 pq R_w h \sqrt{n k p t^{2D_{max}}} (\sqrt{2D_{max}})^{T-2} - N \sum_{k=0}^{T-2-N} \frac{1}{4^k}, \quad (93)
\end{align*}
\]

where the first inequality follows from the definition of \( \mathcal{D} \) given by Eq. (26), and the second inequality follows from the choice of \( \bar{\varepsilon} \) in Eq. (51). Unrolling (93) with \( R_{u_N} = 0 \) yields

\[
\begin{align*}
    R_{u_{T-1}} &\leq \frac{1}{4^{T-1-N}} R_{u_N} + 3 q R_w h \sqrt{n k p t^{2D_{max}}} \sum_{k=0}^{T-2-N} \frac{1}{4^k} \\
    &\leq 4 q R_w h \sqrt{n k p t^{2D_{max}}},
\end{align*}
\]

which implies that \( \|u_t^{alg}\| \leq R_u \) for all \( t \in \{N, \ldots, T-1\} \). Recalling (91)-(92) yields \( \|\hat{w}_t - w_t\| \leq \Delta R_w \bar{\varepsilon} \) and \( \|\hat{w}_t\| \leq R_w \), respectively, for all \( t \in \{N, \ldots, T-1\} \).
To proceed, we note that
\[
\|x_t^{\text{alg}}\| = \left\| A^{t-N} x_N^{\text{alg}} + \sum_{k=0}^{t-N-1} A^k (Bu_{t-k-1}^{\text{alg}} + w_{t-k-1}) \right\|
\]
\[
\leq \kappa \sigma \sqrt{\frac{20(m+n)\Gamma}{1-\gamma}} + (\Gamma R_u + R_w) \kappa \sum_{k=0}^{t-1} \gamma^k
\]
\[
\leq \frac{\sigma \sqrt{20(m+n)\Gamma^2}}{1-\gamma} + \frac{(\Gamma R_u + R_w)\kappa}{1-\gamma} = R_x,
\]
for all \( t \in \{N, \ldots, T-1\} \). Similarly, we have from Definition 3 that
\[
\|x_t^{\text{pred}}(M_{t-h:t-1})\| = \left\| \sum_{k=t-h}^{t-1} A^{-(k+1)}(Bu_k(M_k|\hat{\Phi}, \hat{w}_{0:k-1}) + w_k) \right\|
\]
\[
\leq (\Gamma R_u + R_w) \kappa \sum_{k=0}^{h-1} \gamma^k
\]
\[
\leq \frac{(\Gamma R_u + R_w)\kappa}{1-\gamma} \leq R_x,
\]
for all \( t \in \{N, \ldots, T-1\} \), where we use the fact that \( u_k(M_k|\hat{\Phi}, \hat{w}_{0:k-1}) = u_t^{\text{alg}} \) if \( k \geq N \), and \( u_k(M_k|\hat{\Phi}, \hat{w}_{0:k-1}) = 0 \) if \( k < N \).

\section{D Proofs for Upper Bounds on \( R_0, R_1, R_4 \) and \( R_5 \)}

\subsection*{D.1 Proof of Lemma 10}

Suppose the event \( \mathcal{E} \) defined in Eq. (50) holds. Considering any \( t \in \{N, \ldots, N_0-1\} \), we know from Lemma 9 that \( \|u_t^{\text{alg}}\| \leq R_u \) and \( \|x_t^{\text{alg}}\| \leq R_x \) for all \( t \in \{N, \ldots, N_0-1\} \). Next, consider any \( t \in \{0, \ldots, N-1\} \). We have from (49) that \( \|u_t^{\text{alg}}\| \leq \sigma_u \sqrt{m \log 4NT} \). Moreover, under Assumption 3, we see that
\[
\|x_t^{\text{alg}}\| = \left\| \sum_{k=0}^{t-1} A^k (Bu_{t-k-1}^{\text{alg}} + w_{t-k-1}) \right\|
\]
\[
\leq (\sigma_u \sqrt{m \log 4NT} + \sigma_w \sqrt{10n \log 2T}) \frac{\Gamma}{1-\gamma} \sum_{k=0}^{t-1} \gamma^k
\]
\[
\leq \frac{\sigma u \sqrt{20(m+n)\Gamma}}{1-\gamma} \leq \frac{\sigma}{1-\gamma} \leq R_x.
\]

It then follows that
\[
c(x_t^{\text{alg}}, u_t^{\text{alg}}) = x_t^{\text{alg}}^\top Q x_t^{\text{alg}} + u_t^{\text{alg}}^\top R u_t^{\text{alg}}
\]
\[
\leq \sigma_1(Q) R_x^2 + \sigma_1(R) \frac{R^2 \sigma_u^2}{\sigma_w^2},
\]
for all \( t \in \{0, \ldots, N_0-1\} \).
D.2 Proof of Lemma 11

Suppose the event \( \mathcal{E} \) holds, and consider any \( t \in \{N_0, \ldots, T-1\} \). Noting from Lemma 9 that \( \|x_t^{\text{alg}}\| \leq R_x \) and \( \|x_t^{\text{pred}}\| \leq R_x \), we have from Definition 3 that
\[
c(x_t^{\text{alg}}, u_t) - F_{\text{pred}}^{\text{pred}}(M_{t-h:t}) = x_t^{\text{alg}} \top Q x_t^{\text{alg}} - x_t^{\text{pred}} \top Q x_t^{\text{pred}} \leq 2R_x \sigma_1(Q) \|x_t^{\text{alg}} - x_t^{\text{pred}}\|
\]
\[
\leq 2R_x \sigma_1(Q) \left\| \sum_{k=h}^{t-1} A^k (B u_{t-k-1} + w_{t-k-1}) \right\|
\]
\[
\leq 2R_x \sigma_1(Q) (\Gamma R_u + R_w) \frac{\kappa^h}{1 - \gamma},
\]
where the first inequality follows from similar arguments to those for Lemma 2.

D.3 Proof of Lemma 12

Consider any \( M \in \mathcal{D}_0 \). First, recalling Eq. (23) and Definition 2, we see that \( u_t^M = u_t(M|w_{0:t-1}) = \sum_{s \in \mathcal{U}} \sum_{k=1}^{h} I_{v,s} M_s[k] \eta_{t-k, s} \), for all \( t \geq 0 \), where
\[
\|u_t^M\| \leq \sum_{s \in \mathcal{U}} \sum_{k=1}^{h} \|M_s[k]\| \|\eta_{t-k, s}\|
\]
\[
\leq \sqrt{n h q \kappa \Gamma^{2D_{\max}+1}} R_w.
\]

Considering any \( t \in \{N_0, \ldots, T-1\} \), we then see that
\[
f_t(M|\Phi, w_{0:t-1}) - c(x_t^M, u_t^M) = x_t(M|\Phi, w_{0:t-1}) \top Q x_t(M|\Phi, w_{0:t-1}) - x_t^M \top Q x_t^M,
\]
where
\[
x_t(M|\Phi, w_{0:t-1}) = \sum_{k=t-h}^{t-1} A^{t-(k+1)} (w_k + B u_k(M|w_{0:k-1}))
\]
\[
x_t^M = \sum_{k=0}^{t-1} A^{t-(k+1)} (w_k + B u_k^M).
\]

Moreover, we have
\[
\|x_t(M|\Phi, w_{0:t-1})\| = \left\| \sum_{k=t-h}^{t-1} A^{t-(k+1)} (w_k + B u_k(M|w_{0:k-1})) \right\|
\]
\[
\leq \left( 1 + \sqrt{n h q \kappa \Gamma^{2D_{\max}+2}} \right) R_w \|\sum_{k=0}^{h-1} A^k\|
\]
\[
\leq 2 \sqrt{n h q \kappa \Gamma^{2D_{\max}+2}} \frac{R_w \kappa}{1 - \gamma},
\]

Similarly, we have
\[
\|x_t^M\| \leq 2 \sqrt{n h q \kappa \Gamma^{2D_{\max}+2}} \frac{R_w \kappa}{1 - \gamma}.
\]

It then follows from similar arguments to those for Lemma 2 that
\[
f_t(M|\Phi, w_{0:t-1}) - c(x_t^M, u_t^M) \leq 4 \sigma_1(Q) \sqrt{n h q \kappa \Gamma^{2D_{\max}+2}} \frac{R_w \kappa}{1 - \gamma} \|x_t(M|\Phi, w_{0:t-1}) - x_t^M\|,
\]

36
where
\[ \|x_t(M|\Phi, w_{0:t-1}) - x_t^M\| = \left\| \sum_{k=0}^{t-h-1} A^{t-(k+1)}(u_k + Bu_k^M) \right\| \]
\[ \leq 2q\sqrt{nh}\kappa p^2\Gamma^2 D_{\max} + 2 R_w \sum_{k=h}^{t-1} \|A^k\| \]
\[ \leq 2q\sqrt{nh}\kappa p^2\Gamma^2 D_{\max} + 2 R_w \kappa^2 \gamma^h \frac{1}{1-\gamma}, \]
which implies that
\[ f_t(M|\Phi, w_{0:t-1}) - c(x_t^M, u_t^M) \leq 8\sigma_1(Q)q^2 nh^2 \kappa^2 p^2 \Gamma^4 D_{\max} + 4 R_w^2 \kappa^2 \gamma^h \frac{1}{1-\gamma^2}. \]
Thus, denoting \( M_{**} \in \arg \inf_{M \in D_0} \sum_{t=0}^{T-1} c(x_t^M, u_t^M) \), we have from the above arguments that
\[ R_4 \leq \sum_{t=0}^{T-1} \left(f_t(M_{**}|\Phi, w_{0:t-1}) - c(x_t^{M_{**}}, u_t^{M_{**}})\right) \]
\[ \leq 8\sigma_1(Q)q^2 nh^2 \kappa^2 p^2 \Gamma^4 D_{\max} + 4 R_w^2 \kappa^2 \gamma^h \frac{1}{1-\gamma^2} T. \]

\section*{D.4 Proof of Lemma 13}
Let \( M_s[k] = M_{*s}^k \) for all \( s \in \mathcal{U} \) and all \( k \in [h/4] \), where \( M_{*s}^k \) is given by Eq. (22). We know from Lemma 8 that \( M \in D_0 \), where \( M = [M_s^k]_{s \in \mathcal{U}, k \in [h/4]} \). It follows that
\[ R_5 \leq \sum_{t=0}^{T-1} c(x_t^M, u_t^M) \leq \sum_{t=0}^{T-1} c(x_t^M, u_t^M). \]
Using similar arguments to those for Lemma 8, one can show that on the event \( \mathcal{E} \),
\[ c(x_t^M, u_t^M) - c(x_t^*, u_t^*) \leq \frac{12\sigma_2 p^2 q h^2 \gamma^4 \Gamma^4 D_{\max} + 2 R_w^2 \gamma^h}{1-\gamma^2}; \]
for all \( t \in \mathbb{Z} \) and \( s \in \mathcal{U} \), where \( c(x_t^*, u_t^*) = x_t^{**\top} Q x_t^* + u_t^{**\top} R u_t^* \), and \( x_t^* \) is the state corresponding to the optimal control policy \( u_t^* \) given by Eq. (21).

Now, considering the internal state \( \zeta_{t,s} \) given in Eq. (10) for all \( t \in \mathbb{Z} \) and all \( s \in \mathcal{U} \), we recall from [26, Lemma 14] that for any \( t \in \mathbb{Z} \), the following hold: (a) \( \mathbb{E} \mathbb{E}_{\mathcal{T}_s} = 0 \), for all \( s \in \mathcal{U} \); (b) \( x_t^s = \sum_{s \in \mathcal{U}} I_{v_i} \zeta_{t,s} \); (c) \( \zeta_{t,s_1} \) and \( \zeta_{t,s_2} \) are independent for all \( s_1, s_2 \in \mathcal{U} \) with \( s_1 \neq s_2 \). Based on these results, we can relate \( \sum_{t=0}^{T-1} c(x_t^*, u_t^*) \) to \( J_s \). Specifically, considering any \( t \in \{0, \ldots, T-2\} \), we first recall from Eq. (10) that \( \zeta_{t+1,s} = \sum_{r \rightarrow s} (A_{sr} + B_{sr} K_r) \zeta_{t,r} + \sum_{i \rightarrow t} I_{s,i} w_{t,i} \) for all \( s \in \mathcal{U} \), where \( K_r \) is given by Eq. (8). For any \( s \in \mathcal{U} \), we have
\[ \mathbb{E}_{\mathcal{S}_{t+1,s}} \mathbb{P}_s \zeta_{t+1,s} = \mathbb{E}_{\mathcal{S}_t} \left[ \sum_{r \rightarrow s} (A_{sr} + B_{sr} K_r) \zeta_{t,r} + \sum_{i \rightarrow t} I_{s,i} w_{t,i} \right] \]
\[ = \mathbb{E} \left[ \sum_{r \rightarrow s} (A_{sr} + B_{sr} K_r) \mathbb{P}_s (A_{sr} + B_{sr} K_r) \zeta_{t,r} + \sum_{i \rightarrow t} I_{s,i} w_{t,i} \right], \]
where \( \mathbb{P}_s \) is given by Eq. (9). Moreover, for any \( s \in \mathcal{U} \), we have from Eq. (9) that
\[ \mathbb{E} \left[ \sum_{r \rightarrow s} \zeta_{t,r} P_r \zeta_{t,r} \right] = \mathbb{E} \left[ \sum_{r \rightarrow s} (Q_{rr} + K_r^{\top} R_{rr} K_r) \zeta_{t,r} \right] + \mathbb{E} \left[ \sum_{r \rightarrow s} \zeta_{t,r} (A_{sr} + B_{sr} K_r) \mathbb{P}_s (A_{sr} + B_{sr} K_r) \zeta_{t,r} \right] \]
\[ = \mathbb{E} \left[ \sum_{r \rightarrow s} (Q_{rr} + K_r^{\top} R_{rr} K_r) \zeta_{t,r} \right] + \mathbb{E} \left[ \zeta_{t+1,s} \mathbb{P}_s \zeta_{t+1,s} \right] - \mathbb{E} \left[ \sum_{i \rightarrow t} w_{t,i} I_{s,i} \mathbb{P}_s I_{s,i} w_{t,i} \right]. \]
Summing over all $s \in \mathcal{U}$ yields the following:
\[
E\left[\sum_{s \in \mathcal{U}} \sum_{t \rightarrow s} \zeta_{t,s}^T P_s \zeta_{t,r}\right] - E\left[\sum_{s \in \mathcal{U}} \zeta_{t+1,s}^T P_s \zeta_{t+1,r}\right] = E\left[\sum_{s \in \mathcal{U}} \zeta_{t,s}^T (Q_{t,s} + K_{t,s}^T R_{t,s} K_{t,s}) \zeta_{t,s}\right] - E\left[\sum_{s \in \mathcal{U}} \sum_{w \rightarrow s} w_{t,i}^s I_{(i),s}^s P_s I_{(i),s}^s w_{t,i}\right],
\]
which implies that
\[
E\left[\sum_{s \in \mathcal{U}} \zeta_{t,s}^T P_s \zeta_{t,s}\right] - E\left[\sum_{s \in \mathcal{U}} \zeta_{t+1,s}^T P_s \zeta_{t+1,s}\right] = E\left[\sum_{s \in \mathcal{U}} \zeta_{t,s}^T (Q_{s,s} + K_{s,s}^T R_{s,s} K_{s,s}) \zeta_{t,s}\right] - J^*_s
\]
where we use the definition of $J^*_s$ given by Eq. (12), and the fact that the information graph $\mathcal{P}(\mathcal{U}, \mathcal{H})$ has a tree structure as shown in [26, 45]. Noting from Lemma 1 that $x_t^* = \sum_{s \in \mathcal{U}} I_{v,s} \zeta_{t,s}$ and $u_t^* = \sum_{s \in \mathcal{U}} I_{v,s} K_{s,s} \zeta_{t,s}$ for all $t \in \{0, \ldots, T\}$, one can then show via Eq. (94) that
\[
E\left[\sum_{s \in \mathcal{U}} \zeta_{t,s}^T P_s \zeta_{t,s}\right] - E\left[\sum_{s \in \mathcal{U}} \zeta_{t+1,s}^T P_s \zeta_{t+1,s}\right] = E\left[x_t^*^T Q x_t^* + u_t^T R u_t^*\right] - J^*_s,
\]
for all $t \in \{0, \ldots, T - 1\}$. It then follows that
\[
E\left[\sum_{t=0}^{T-1} (x_t^*^T Q x_t^* + u_t^T R u_t^*)\right] = TJ^*_s + E\left[\sum_{t=0}^{T-1} (\zeta_{t,s}^T P_s \zeta_{t,s} - \zeta_{t+1,s}^T P_s \zeta_{t+1,s})\right]
\]
\[
\leq TJ^*_s + E\left[\sum_{s \in \mathcal{U}} \zeta_{0,s}^T P_s \zeta_{0,s}\right] = TJ^*_s,
\]
where we use the fact that $\zeta_{0,s} = 0$ as we assumed previously.

Putting the above arguments together, we have
\[
E[1 \{E\} R_5] \leq E\left[1 \{E\} \sum_{t=0}^{T-1} (c(x_t^M, u_t^M) - c(x_t^*, u_t^*))\right] + E\left[1 \{E\} \sum_{t=0}^{T-1} c(x_t^*, u_t^*)\right]
\]
\[
\leq \frac{12p^2 \kappa \Gamma \Gamma_{\max} + 2R_2^2}{(1 - \gamma)^4} T + E\left[\sum_{t=0}^{T-1} c(x_t^*, u_t^*)\right]
\]
\[
\leq \frac{12p^2 \kappa \Gamma \Gamma_{\max} + 2R_2^2}{(1 - \gamma)^4} T + TJ^*_s.
\]

\section{Proofs Pertaining to Upper Bounding $R_2$}

\subsection{Proof of Lemma 14}

Suppose the event $E$ holds, and consider any $t \in \{N, \ldots, T - 1\}$. For any $M \in D$, where $D$ is given by Eq. (26), we have from Lemma 9 that $\|u_t(M|\hat{w}_{0:t-1})\| \leq R_u$ and $\|x_t^\text{pred}(M)\| \leq R_x$. Now, consider any $M, \hat{M} \in D$, we have from Lemma 2 and Definition 3 that
\[
|f_t^\text{pred}(M) - f_t^\text{pred}(\hat{M})| \leq 2(R_u + R_x) \max \{\sigma_1(Q), \sigma_1(R)\}
\times (\|x_t^\text{pred}(M - \hat{M})\| + \|u_t(M - \hat{M}|\hat{w}_{0:t-1})\|).
\]
Recalling Definition 2, we have
\[
\|u_t(M - \hat{M}|\hat{w}_{0:t-1})\| = \left\|\sum_{s \in \mathcal{U}} \sum_{k=1}^h I_{v,s}(M_s[k] - \hat{M}_s[k]) \hat{\eta}_{t-k,s}\right\|.
\]
where we note from Lemma 9 that \(\|\hat{\eta}_{t-k,s}\| = \|\begin{bmatrix} \hat{a}_{t-k-l, u, j_v}^{\top} \end{bmatrix}_{v \in \mathcal{E}_v}\| \leq R_{\hat{a}}\). One can write the right-hand side of the above equation into a matrix form and obtain
\[
\left\| u_t(M - \hat{M} | \hat{w}_{0:t-1}) \right\| = \| \Delta M \| \eta_t, \\
\]where \(\Delta M = \left[ I_{\mathcal{V}, s}(M_s[k] - \hat{M}_s[k]) \right]_{s \in \mathcal{U}, k \in [k]}\) and \(\eta_t = \left[ \eta_{t-k,s}^{\top} \right]_{s \in \mathcal{U}, k \in [k]}\). One can also show that \(\|\eta_t\| \leq \sqrt{q} h R_{\hat{a}}\).
It then follows that
\[
\left\| u_t(M - \hat{M} | \hat{w}_{0:t-1}) \right\| \leq \sqrt{q} h R_{\hat{a}} \| \Delta M \| \\
\leq \sqrt{q} h R_{\hat{a}} \| \text{Vec}(M - \hat{M}) \|, \
(96)
\]
where \(\text{Vec}(M - \hat{M})\) denotes the vector representation of \((M - \hat{M})\). Now, we have from Definition 3 that
\[
\| x_{t}^{\text{pred}}(M - \hat{M} | \hat{\Phi}, \hat{w}_{0:t-1}) \| = \left\| \sum_{k=t-h}^{t-1} A^{t-(k+1)} B \left( u_k(M - \hat{M} | \hat{w}_{0:k-1}) \right) \right\| \\
\leq \Gamma \sqrt{q} h R_{\hat{a}} k \sum_{k=0}^{h-1} \gamma_k \| \text{Vec}(M - \hat{M}) \| \\
\leq \frac{\Gamma \sqrt{q} h R_{\hat{a}} k}{1 - \gamma} \| \text{Vec}(M - \hat{M}) \|. \
(97)
\]
Going back to (95), we conclude that \(f_t^{\text{alg}}(\cdot)\) is \(L_t'\)-Lipschitz. Now, noting the definition of \(F_t^{\text{alg}}(\cdot)\) from Definition 3, one can use similar arguments to those above and show that \(F_t^{\text{alg}}(\cdot)\) is \(L_t'\)-Lipschitz.\(\blacksquare\)

### E.2 Proof of Lemma 15

Suppose the event \(\mathcal{E}\) holds, and consider any \(t \in \{N, \ldots, T-1\}\) and any \(M \in \mathcal{D}\). For notational simplicity in this proof, we denote \(\tilde{x}_t(M) = x_t^{\text{pred}}(M)\) and \(\tilde{u}_t(M) = u_t(M | \hat{w}_{0:t-1})\), where \(x_t^{\text{pred}}(M)\) and \(u_t(M | \hat{w}_{0:t-1})\) are given by Definitions 3 and 2, respectively. Moreover, we assume that \(M = (M_s[k])_{k \in [k], s \in \mathcal{U}}\) is already written in its vector form \(\text{Vec}(M)\), i.e., we let \(M = \text{Vec}(M)\). We then have
\[
f_t^{\text{pred}}(M) = \tilde{x}_t(M)^{\top} Q \tilde{x}_t(M) + \tilde{u}_t(M)^{\top} R \tilde{u}_t(M).
\]
Taking the derivative, we obtain
\[
\frac{\partial f_t^{\text{pred}}(M)}{\partial M} = 2 \tilde{x}_t(M)^{\top} Q \frac{\partial \tilde{x}_t(M)}{\partial M} + 2 \tilde{u}_t(M)^{\top} R \frac{\partial \tilde{u}_t(M)}{\partial M}.
\]
Further taking the derivative, we obtain
\[
\nabla^2 f_t^{\text{pred}}(M) = 2 \frac{\partial \tilde{x}_t(M)^{\top}}{\partial M} Q \frac{\partial \tilde{x}_t(M)}{\partial M} + 2 \frac{\partial \tilde{u}_t(M)^{\top}}{\partial M} R \frac{\partial \tilde{u}_t(M)}{\partial M}, 
(98)
\]
where we use the fact that \(\tilde{x}_t(M)\) and \(\tilde{u}_t(M)\) are linear functions in \(M\). Moreover, we know from (96)-(97) in the proof of Lemma 14 that \(\tilde{u}_t(\cdot)\) is \(\sqrt{q} h R_{\hat{a}}\)-Lipschitz and \(\tilde{x}_t(\cdot)\) is \(\frac{\Gamma \sqrt{q} h R_{\hat{a}} k}{1 - \gamma}\)-Lipschitz, which implies that \(\| \frac{\partial \tilde{x}_t(M)}{\partial M} \| \leq \sqrt{q} h R_{\hat{a}}\) and \(\| \frac{\partial \tilde{u}_t(M)}{\partial M} \| \leq \frac{\Gamma \sqrt{q} h R_{\hat{a}} k}{1 - \gamma}\) (e.g., [5]). Combining the above arguments, we complete the proof of the lemma.\(\blacksquare\)

### E.3 Proof of Lemma 16

Suppose the event \(\mathcal{E}\) holds, and consider any \(t \in \{N, \ldots, T-1\}\) and any \(M \in \mathcal{D}\). For notational simplicity in this proof, we denote \(\tilde{x}_t(M) = x_t^{\text{pred}}(M)\), \(\tilde{u}_t(M) = u_t(M | \hat{w}_{0:t-1})\), and \(\hat{x}_t(M) = x_t(M | \hat{\Phi}, \hat{w}_{0:t-1})\), where \(x_t^{\text{pred}}(M)\) is given by Definition 3, and \(u_t(M | \hat{w}_{0:t-1})\), \(x_t(M | \hat{\Phi}, \hat{w}_{0:t-1})\) are given by Definition 2. We then have
\[
f_t(M | \hat{\Phi}, \hat{w}_{0:t-1}) - f_t^{\text{pred}}(M) = \hat{x}_t(M)^{\top} Q \hat{x}_t(M) - \tilde{x}_t(M)^{\top} Q \tilde{x}_t(M).
\]
Taking the derivative, we obtain

\[
\frac{\partial f_1(M)}{\partial M} - \frac{\partial f_1^{red}(M)}{\partial M} = 2 \frac{\partial \tilde{x}_t(M)}{\partial M} Q \tilde{x}_t(M) - 2 \frac{\partial \tilde{x}_t(M)}{\partial M} Q \tilde{x}_t(M) = 2 \left( \frac{\partial \tilde{x}_t(M)}{\partial M} - \frac{\partial \tilde{x}_t(M)}{\partial M} \right) Q \tilde{x}_t(M) + 2 \frac{\partial \tilde{x}_t(M)}{\partial M} Q \tilde{x}_t(M) - 2 \frac{\partial \tilde{x}_t(M)}{\partial M} Q \tilde{x}_t(M) = 2 \left( \frac{\partial \tilde{x}_t(M)}{\partial M} - \frac{\partial \tilde{x}_t(M)}{\partial M} \right) Q \tilde{x}_t(M) + 2 \frac{\partial \tilde{x}_t(M)}{\partial M} Q \tilde{x}_t(M) - 2 \frac{\partial \tilde{x}_t(M)}{\partial M} Q \tilde{x}_t(M) \right),
\]

(99)

where we assume for notational simplicity that \( M = (M_k^t)_{k \in [n], t \in T} \) is already written in its vector form \( \text{Vec}(M) \), i.e., we let \( M = \text{Vec}(M) \).

First, we upper bound \( \| \tilde{x}_t(M) - \tilde{x}_t(M) \| \). We see from Definitions 2-3 that

\[
\| \tilde{x}_t(M) - \tilde{x}_t(M) \| = \left\| \sum_{k=t-h}^{t-1} \left( A^t \tilde{x}_t(M) - A^t \tilde{x}_t(M) \right) \right\|
\]

Note from Proposition 3 that \( \| A - \hat{A} \| \leq \bar{\varepsilon} \) and \( \| B - \hat{B} \| \leq \bar{\varepsilon} \), and note from Assumption 3 that \( \| A_k \| \leq \kappa \gamma_k \) for all \( k \geq 0 \). We then have from Lemma 24 that

\[
\| \hat{A}_k - A_k \| \leq k \kappa^2 (\kappa \bar{\varepsilon} + \frac{1}{2} \gamma) k^{-1} \bar{\varepsilon} \\
\leq k \kappa^2 \left( \frac{3 + \gamma}{4} \right)^{k-1} \bar{\varepsilon}, \forall k \geq 0,
\]

where we use the fact that \( \bar{\varepsilon} \leq \frac{1}{k \gamma} \). We also have

\[
\| \hat{A}_k B - A_k B \| \leq \| \hat{A}_k - A_k \| \| B \| + \| A_k \| \| \hat{B} \| + \| A_k \| \| \hat{B} - B \| \\
\leq k \kappa^2 \left( \frac{3 + \gamma}{4} \right)^{k-1} \bar{\varepsilon}(\Gamma + \bar{\varepsilon}) + k \gamma_k \bar{\varepsilon}.
\]

Now, noting that one can use similar arguments to those for Lemma 9 and show that \( \| \tilde{x}_k \| \leq R_\tilde{x} \) and \( \| \tilde{x}_t(M) \| \leq R_\tilde{x} \) for all \( k \geq 0 \), we have from the above arguments that

\[
\| \tilde{x}_t(M) - \tilde{x}_t(M) \| \leq R_\tilde{x} \sum_{k=0}^{h-1} \| \hat{A}_k - A_k \| \| \hat{B} - B \| \\
\leq \left( R_\tilde{x} + (\Gamma + \bar{\varepsilon}) R_\tilde{x} \right) \kappa^2 \bar{\varepsilon} \sum_{k=0}^{h-1} \left( \frac{3 + \gamma}{4} \right) k^{-1} + \kappa^2 \bar{\varepsilon} \sum_{k=0}^{h-1} \gamma_k^k \\
\leq \left( R_\tilde{x} + (\Gamma + \bar{\varepsilon}) R_\tilde{x} \right) \kappa^2 \bar{\varepsilon} \frac{16}{(1 - \gamma)^2} + \frac{R_\tilde{x} \kappa \bar{\varepsilon}}{1 - \gamma},
\]

where the third inequality follows from a standard formula for series. Moreover, one can use similar arguments for Lemma 9 and show that \( \| \tilde{x}_t(M) \| \leq R_x \), which further implies that

\[
\| \tilde{x}_t(M) \| \leq \| \tilde{x}_t(M) - \tilde{x}_t(M) \| + \| \tilde{x}_t(M) \| \\
\leq \left( R_\tilde{x} + (\Gamma + \bar{\varepsilon}) R_\tilde{x} \right) \kappa^2 \bar{\varepsilon} \frac{16}{(1 - \gamma)^2} + \frac{R_\tilde{x} \kappa \bar{\varepsilon}}{1 - \gamma} + R_x.
\]

Next, we upper bound \( \| \hat{\partial}_s \tilde{x}_t(M)/\partial M - \hat{\partial}_s \tilde{x}_t(M)/\partial M \| \). Noting from (96) in the proof of Lemma 14 that \( \hat{\tilde{x}}(\cdot) \) is \( \sqrt{q}R_\tilde{x} \)-Lipschitz, we have that \( \| \hat{\partial}_s \tilde{x}_t(M)/\partial M \| \leq \sqrt{q}R_\tilde{x} \) (e.g., [5]). It then follows from Definitions 2-3
that
\[ \left\| \frac{\partial \hat{x}_i(M)}{\partial M} - \frac{\partial \hat{x}_i(M)}{\partial M} \right\| = \left\| \sum_{k=t-h}^{t-1} (\hat{A}^{t-(k+1)} B - \hat{A}^{t-(k+1)} B) \frac{\partial \hat{u}_k}{\partial M} \right\| \]
\[ \leq \sqrt{h R_w} \sum_{k=0}^{h-1} \left\| \hat{A}^k B - \hat{A}^k B \right\| \]
\[ \leq \sqrt{h R_w} \sum_{k=0}^{h-1} (k \kappa^2 \left( \frac{3 + \gamma}{4} \right)^{k-1} \varepsilon (\Gamma + \varepsilon) + \kappa \gamma^k \varepsilon) \]
\[ \leq \sqrt{h R_w} \left( \kappa^2 \varepsilon (\Gamma + \varepsilon) \frac{16}{(1 - \gamma)^2} + \frac{\kappa \varepsilon}{1 - \gamma} \right). \]

Similarly, noting from (97) in the proof of Lemma 14 that \( \hat{x}_i(\cdot) \) is \( \frac{\sqrt{\varepsilon}}{1 - \gamma} \)-Lipschitz, which implies that \( \left\| \frac{\partial \hat{x}_i(M)}{\partial M} \right\| \leq \frac{\sqrt{\varepsilon}}{1 - \gamma}. \)

Finally, combining the above arguments together and noting that \( \varepsilon \leq 1 \) and \( \kappa \geq 1 \), one can show via (99) and algebraic manipulations that (61) holds. ■

### E.4 Proof of Lemma 17

Let us consider any \( t \in \{N + k_f, \ldots, T - 1\} \) and any \( M \in \mathcal{D} \). First, we recall from Definition 2 that
\[ u_t(M|\hat{w}_{0:t-1}) = \sum_{s \in \mathcal{U}} \sum_{k=1}^{h} \int_{\mathcal{V}} M_s[k] \hat{\eta}_{t-k,s}, \]
where \( \hat{\eta}_{t-k,s} = [\hat{w}^\top_{t-k-\ell \cdots, \ell} j_{\ell}]_{\ell \in \mathcal{L}_s} \). Writing the right-hand side of the above equation into a matrix form yields
\[ u_t(M|\hat{w}_{0:t-1}) = \text{Mat}(M) \hat{\eta}, \]
where \( \text{Mat}(M) = \left[ \int_{\mathcal{V}} M_s[k] \right]_{s \in \mathcal{U}, k \in [h]} \) and \( \hat{\eta} = [\hat{\eta}^\top_{t-k,s}]_{s \in \mathcal{U}, k \in [h]} \). Further writing \( M = [M_s[k]]_{s \in \mathcal{U}, k \in [h]} \) into its vector form \( \text{Vec}(M) \), one can show that
\[ u_t(M|\hat{w}_{0:t-1}) = \hat{\Lambda}_t M, \]
where we let \( M = \text{Vec}(M) \) for notational simplicity, and \( \hat{\Lambda}_t = \text{diag}((\hat{\eta}^\top_i)_{i \in [n]}) \) with \( \hat{\eta}^\top_i = \hat{\eta}_i \). Taking the derivative, we obtain
\[ \frac{\partial u_t(M|\hat{w}_{0:t-1})}{\partial M} = \hat{\Lambda}_t. \]

Next, we recall from Eq. (98) in the proof of Lemma 15 that
\[ \nabla^2 f_t^{\text{pred}}(M) = 2 \frac{\partial \hat{x}_i(M)}{\partial M} \nabla M + 2 \frac{\partial \hat{u}_i(M)}{\partial M} \nabla M \]
\[ \geq 2 \sigma_m(R) \frac{\partial \hat{u}_i(M)}{\partial M} \nabla M \frac{\partial \hat{u}_i(M)}{\partial M} \]
\[ = 2 \sigma_m(R) \hat{\Lambda}_t \hat{\Lambda}_t, \]
(100)
where we denote \( \hat{x}_i(M) = x_i^{\text{pred}}(M) \) and \( \hat{u}_i(M) = u_t(M|\hat{w}_{0:t-1}) \), with \( x_i^{\text{pred}}(M) \) and \( u_t(M|\hat{w}_{0:t-1}) \) given by Definitions 3 and 2, respectively. To proceed, let us consider any \( i \in [n] \) and any element of the vector \( \hat{\eta}^\top_i \), and denote the element as \( \hat{\omega} \in \mathbb{R} \). Meanwhile, let \( w \in \mathbb{R} \) denote the element of \( \hat{\eta} = [\hat{\eta}^\top_{t-k,s}]_{s \in \mathcal{U}, k \in [h]} \) that corresponds to \( \hat{\omega} \), where \( \hat{\eta}_{t-k,s} = [w^\top_{t-k-\ell \cdots, \ell} j_{\ell}]_{\ell \in \mathcal{L}_s} \). Note that \( \hat{\omega}^2 \) is a diagonal element of the matrix \( \hat{\Lambda}_t \hat{\Lambda}_t \).
One can check that \( w \) is an element of some vector \( w_{t-k-l_v,j_v} \sim \mathcal{N}(0, \sigma_w^2 M_{t-j_v}) \), where \( k \in [h], v \in \mathcal{L}_s \), which implies that \( w \sim \mathcal{N}(0, \sigma_w^2) \). Recalling from the definition of the information graph \( \mathcal{P}(\mathcal{U}, \mathcal{H}) \) given by (7) that \( l_{v,s} \leq D_{\text{max}} \) for all \( v, s \in \mathcal{U} \), where \( D_{\text{max}} \) is defined in Eq. (25), we see that \( t - k - l_{v,s} > t - k_f \). Combining the above arguments, we obtain

\[
E[\hat{w}^2 | F_{t-k_f}] \geq \frac{1}{2} E[w^2 | F_{t-k_f}] - \frac{1}{2} E[(\hat{w} - w)^2 | F_{t-k_f}],
\]

where the equality follows from the fact that \( F_{t-k_f} \) is generated by the stochastic sequence \( w_N, \ldots, w_{t-k_f} \).

Now, suppose the event \( E \) holds. We know from Lemma 9 that \( \| \hat{w}_t - w_t \| \leq \Delta R_w \bar{\varepsilon} \leq \sigma_w / 2 \) for all \( t \in \{N, \ldots, T - 1\} \). Recalling from Definition 4 that \( f_{t:k_f}^\text{pred}(M) = E[f_t^\text{pred}(M)|F_{t-k_f}] \), we then have from (100)-(101) that \( \nabla^2 f_t^\text{pred}(M) \geq \frac{\sigma_w (R) \sigma^2}{2} I_{d_m} \), where \( d_m \) is the dimension of \( M \) (i.e., \( \text{vec}(M) \)).

\[\Box\]

### F Proof Pertaining to Upper Bounding \( R_3 \)

#### F.1 Proof of Lemma 19

Suppose the event \( E \) holds and consider any \( t \in \{N, \ldots, T - 1\} \). First, recall from Definitions 2-3 that

\[
f_t^\text{pred}(M_{\text{apx}}) - f_t^\text{pred}(M_{\hat{\Phi}, w_{0:t-1}}) = x_t^\text{pred}(M_{\text{apx}})^\top Q x_t^\text{pred}(M_{\text{apx}}) - x_t(M_{\hat{\Phi}}, w_{0:t-1})^\top Q x_t(M_{\hat{\Phi}}, w_{0:t-1})
\]

\[
+ u_t(M_{\text{apx}}|\hat{w}_{0:t-1})^\top R u_t(M_{\text{apx}}|\hat{w}_{0:t-1}) - u_t(M_{\hat{\Phi}}|w_{0:t-1})^\top R u_t(M_{\hat{\Phi}}|w_{0:t-1}).
\]

Noting that \( M_{\text{apx}} \in \mathcal{D} \) and \( M_{\hat{\Phi}} \in \mathcal{D}_0 \), one can use similar arguments to those for Lemma 9 and show that \( \| x_t^\text{pred}(M_{\text{apx}}) \| \leq R_x, x_t(M_{\hat{\Phi}}|w_{0:t-1}) \leq R_x, u_t(M_{\text{apx}}|\hat{w}_{0:t-1}) \leq R_u, \) and \( u_t(M_{\hat{\Phi}}|w_{0:t-1}) \leq R_u \). Using similar arguments to those for Lemma 2, one can then show that

\[
f_t^\text{pred}(M_{\text{apx}}) - f_t^\text{pred}(M_{\hat{\Phi}, w_{0:t-1}}) \leq 2 R_u \sigma_1(R) \| u_t(M_{\text{apx}}|\hat{w}_{0:t-1}) - u_t(M_{\hat{\Phi}}|w_{0:t-1}) \|
\]

\[
+ 2 R_x \sigma_1(Q) \sum_{k=t-h}^{t-1} A^{t-(k+1)}B(u_k(M_{\text{apx}}|\hat{w}_{0:k-1}) - u_k(M_{\hat{\Phi}}|w_{0:k-1}))
\]

\[
\leq 2(R_u \sigma_1(R) + R_x \sigma_1(Q)) \Gamma_{k-\gamma} \frac{1}{\gamma} \max_{\gamma \in \{t-h, \ldots, t\}} \| u_k(M_{\text{apx}}|\hat{w}_{0:k-1}) - u_k(M_{\hat{\Phi}}|w_{0:k-1}) \|
\]

where the second inequality follows from Assumption 3.

\[\Box\]

#### F.2 Proof of Lemma 20

Suppose the event \( E \) holds. Consider any \( t \in \{N + h, \ldots, T - 1\} \). First, recall from Definition 2 that

\[
u_t(M_{\hat{\Phi}}|w_{0:t-1}) = \sum_{s \in \mathcal{L}_s} \sum_{k=t-h}^{t-1} I_{v,s} M_{v,s} \eta_{k,s}, \text{ where } \eta_{k,s} = [w_{k-l_v, j_v}]^\top v_{k-l_v} \rightarrow v \text{ and } \mathcal{L}_s \text{ given by Eq. (15)}.
\]

We can then write

\[
u_t(M_{\hat{\Phi}}|w_{0:t-1}) = \sum_{s \in \mathcal{L}_s} \sum_{k=t-h}^{t-1} I_{v,s} \hat{M}_{v,s} \eta_{k,s},
\]

\[
\text{where } \hat{\eta}_{k,s} = [\hat{w}_{k-l_v, j_v}]^\top v_{k-l_v} \rightarrow v \in \mathcal{L}_s \text{.}
\]

Denoting \( \hat{s} = \{j_v : w_{j_v} \rightarrow v, v \in \mathcal{L}_s\} \), we see from Eq. (3) that

\[
\eta_{k,s} = \sum_{v \in \mathcal{L}_s} I_{s,\{j_v\}} I_{\{j_v\}, v} (x_{k-l_v+1} - A x_{k-l_v} - B u_{k-l_v}^\text{alg}).
\]

Similarly, we see from (33) that

\[
\hat{\eta}_{k,s} = \sum_{v \in \mathcal{L}_s} I_{s,\{j_v\}} I_{\{j_v\}, v} (x_{k-l_v+1} - \hat{A} x_{k-l_v} - \hat{B} u_{k-l_v}^\text{alg}).
\]
For notational simplicity in the remaining of this proof, we denote
\[
I(k) = \{k - \frac{h}{4}, \ldots, k - 1\},
\]
\[
I^0_{v_s}(k) = \{0, \ldots, k - l_{v_s} - 1\},
\]
\[
I^1_{v_s}(k) = \{k - l_{v_s} - \frac{h}{4}, \ldots, k - l_{v_s} - 1\},
\]
\[
I^2_{v_s}(k) = \{0, \ldots, k - l_{v_s} - \frac{h}{4} - 1\},
\]
for all \(k \geq 0\). Moreover, denote \(\Delta A = \hat{A} - A\) and \(\Delta B = \hat{B} - B\). Now, noting that
\[
x_{k-l_{v_s}} = \sum_{k'=0}^{k-l_{v_s}-1} A^{k-l_{v_s}-(k'+1)}(Bu_{k'}^{\text{alg}} + w_{k'}),
\]
for all \(k \in I(t)\), we have from the above arguments that
\[
u_t(\hat{M}_s|w_{0:t-1}) = \sum_{s \in \mathcal{U}} \sum_{k \in I(t)} I_{v_s} \hat{M}^{[t-k]_{s,k}} \left( \eta_{k,s} + \sum_{v \in \mathcal{L}_s} I_{v_s(j,v)} I_{v_s(j,v),\nu}(\Delta A x_{k-l_{v_s}} + \Delta Bu_{k-l_{v_s}}^{\text{alg}}) \right)
\]
\[
= \sum_{s \in \mathcal{U}} \sum_{k \in I(t)} I_{v_s} \hat{M}^{[t-k]_{s,k}} \left( \eta_{k,s} + \sum_{v \in \mathcal{L}_s} I_{v_s(j,v)} I_{v_s(j,v),\nu} \right)
\]
\[
\times \left( \Delta A \left( \sum_{k' \in I_{v_s}(k)} A^{k-l_{v_s}-(k'+1)}(Bu_{k'}^{\text{alg}} + w_{k'}) \right) + \Delta Bu_{k-l_{v_s}}^{\text{alg}} \right)
\]
\[
= u_t^{tr} + u_t^{na},
\]
where
\[
u_t^{tr} = \sum_{s \in \mathcal{U}} \sum_{k \in I(t)} I_{v_s} \hat{M}^{[t-k]_{s,k}} \left( \eta_{k,s} + \sum_{v \in \mathcal{L}_s} I_{v_s(j,v)} I_{v_s(j,v),\nu} \right)
\]
\[
\times \left( \Delta A \left( \sum_{k' \in I_{v_s}(k)} A^{k-l_{v_s}-(k'+1)}(Bu_{k'}^{\text{alg}} + w_{k'}) \right) \right),
\]
\[
u_t^{na} = \sum_{s \in \mathcal{U}} \sum_{k \in I(t)} I_{v_s} \hat{M}^{[t-k]_{s,k}} \left( \eta_{k,s} + \sum_{v \in \mathcal{L}_s} I_{v_s(j,v)} I_{v_s(j,v),\nu} \right)
\]
\[
\times \left( \Delta A \left( \sum_{k' \in I_{v_s}(k)} A^{k-l_{v_s}-(k'+1)}(Bu_{k'}^{\text{alg}} + w_{k'}) \right) + \Delta Bu_{k-l_{v_s}}^{\text{alg}} \right).
\]

In the following, we analyze the terms \(u_t^{tr}\) and \(u_t^{na}\).

**Claim 1.** It holds that \(\|u_t^{tr}\| \leq \bar{\varepsilon}(\Gamma R_u + R_w)p^2 qh\sqrt{n}K^{2D_{\text{max}}+1} \gamma^{h/4} \frac{h}{4(1-\gamma)}\).

**Proof.** Recall from Proposition 3 that \(\|\Delta A\| \leq \bar{\varepsilon}\), and recall from Lemma 9 that \(u_k^{\text{alg}} \leq R_u\) and \(\|w_k\| \leq R_w\). Also note from Eq. (47) that \(\|\hat{M}^{[k]}_s\| \leq \sqrt{n}Kp\Gamma^{2D_{\text{max}}+1}\) for all \(k \in \left[\frac{h}{4}\right]\), and note from Assumption 3 that \(\|A^k\| \leq \kappa\gamma^k\) for all \(k \geq 0\). We then have
\[
\|u_t^{tr}\| \leq \sum_{s \in \mathcal{U}} \sum_{k \in I(t)} \|\hat{M}^{[t-k]_{s,k}}\| \sum_{v \in \mathcal{L}_s} \bar{\varepsilon}(\Gamma R_u + R_w) \sum_{k' \in I_{v_s}(k)} \|A^{k-l_{v_s}-(k'+1)}\|
\]
\[
\leq \bar{\varepsilon}(\Gamma R_u + R_w) \sum_{s \in \mathcal{U}} \sum_{v \in \mathcal{L}_s} \sum_{k \in I(t)} \|\hat{M}^{[t-k]_{s,k}}\| \frac{K^{h/4}}{1-\gamma}
\]
\[
\leq \bar{\varepsilon}(\Gamma R_u + R_w) \frac{K^{h/4}}{4(1-\gamma)} qh\sqrt{n}Kp\Gamma^{2D_{\text{max}}+1}
\]
\[
\leq \bar{\varepsilon}(\Gamma R_u + R_w)p^2 qh\sqrt{n}K^{2D_{\text{max}}+1} \gamma^{h/4} \frac{h}{4(1-\gamma)},
\]
where the third inequality uses the fact that \(|\mathcal{L}_s| \leq p\) for all \(s \in \mathcal{U}\).
Note from the definition of Algorithm 3 that for any $k \geq N$,

$$u_k^\text{alg} = \sum_{s \in \mathcal{L}} \sum_{k' = t - h}^{k-1} I_{\mathcal{V}_s} M_{k,s} \hat{\eta}_{k',s}$$

$$= \left( \sum_{s \in \mathcal{L}} \sum_{k' \in I(k)} I_{\mathcal{V}_s} \tilde{M}_{k,s}^r \hat{\eta}_{k,s} \right) + u_k(M_k - M_t) \hat{w}_{0,k-1}), \quad (103)$$

where we use fact that $\tilde{M}_{k,s}^r = 0$ for all $k' > \frac{h}{4}$. Denoting $\Delta w_k = w_k - \hat{w}_k$ for all $k \geq 0$, we can then write

$$u_t^\text{apx} = \sum_{s \in \mathcal{L}} \sum_{k \in I(t)} I_{\mathcal{V}_s} \tilde{M}_{k,s}^r \left( \hat{\eta}_{k,s} + \sum_{i \in \mathcal{L}_s} I_i \eta_{i,(j,s)} \gamma_{(j,s)} \right)$$

$$= \sum_{\mathcal{L}} \sum_{k \in I(t)} I_{\mathcal{V}_s} \tilde{M}_{k,s}^r \left( \hat{\eta}_{k,s} + \sum_{i \in \mathcal{L}_s} I_i \eta_{i,(j,s)} \gamma_{(j,s)} \right)$$

$$\times \left( \Delta A \left( \sum_{k' \in I(t)} A^{k-t_{v,s}-(k'+1)} (B \hat{u}_{k'} + \Delta u_{k'}) + \Delta B \hat{u}_{k'-t_{v,s}} \right) \right)$$

$$= \sum_{\mathcal{L}} \sum_{k \in I(t)} I_{\mathcal{V}_s} \tilde{M}_{k,s}^r \left( \hat{\eta}_{k,s} + \sum_{i \in \mathcal{L}_s} I_i \eta_{i,(j,s)} \gamma_{(j,s)} \right)$$

$$\times \left( \Delta A \left( \sum_{k' \in I(t)} A^{k-t_{v,s}-(k'+1)} (B \hat{u}_{k'} + \Delta u_{k'}) + \Delta B \hat{u}_{k'-t_{v,s}} \right) \right)$$

$$= u_t^{\text{apx}} + u_t^{\text{apx}}, \quad (104)$$

where

$$u_t^{\text{apx}} = \sum_{s \in \mathcal{L}} \sum_{k \in I(t)} I_{\mathcal{V}_s} \tilde{M}_{k,s}^r \left( \hat{\eta}_{k,s} + \sum_{i \in \mathcal{L}_s} I_i \eta_{i,(j,s)} \gamma_{(j,s)} \right)$$

$$\times \left( \Delta A \left( \sum_{k' \in I(t)} A^{k-t_{v,s}-(k'+1)} (B \hat{u}_{k'} + \Delta u_{k'}) + \Delta B \hat{u}_{k'-t_{v,s}} \right) \right)$$

Claim 2. There exists $M_{\text{apx}} \in \mathcal{D}$ with $\mathcal{D}$ given by Eq. (26) such that $u_t^{\text{apx}} = u_t(M_{\text{apx}}) \hat{w}_{0,t-1}$, and $\|M_{\text{apx},s} - M_{[k]}^r\| \leq \frac{q_2 h n^2 a_{\text{apx}}}{4} + 4 \mathcal{D}_{\text{max}} + 3 \bar{\varepsilon}$ for all $k \in [h]$ and all $s \in \mathcal{U}$.

Proof. Noting (103), we can flat write

$$u_t = \left( \sum_{s \in \mathcal{L}} \sum_{k \in I(t)} I_{\mathcal{V}_s} \tilde{M}_{k,s}^r \hat{\eta}_{k,s} \right) + u_t^{\text{apx}} + u_t^{\text{apx}} + u_t^{\text{apx}},$$

where

$$u_t^{\text{apx}} = \sum_{r,s \in \mathcal{L}} \sum_{v \in \mathcal{L}} \sum_{k \in I(t)} \sum_{k' \in I_{t,v}(k)} \sum_{k'' \in I(k')} \sum_{k''} I_{\mathcal{V}_s} \tilde{M}_{k,s}^r I_{(j,s)} I_{(j,s)} \gamma_{(j,s)} \Delta A A^{k-t_{v,s}-(k'+1)} \gamma_{(j,s)} \hat{\eta}_{(j,s)}(k''),$$

$$u_t^{\text{apx}} = \sum_{r,s \in \mathcal{L}} \sum_{v \in \mathcal{L}} \sum_{k \in I(t)} \sum_{k' \in I_{t,v}(k)} \sum_{k''} I_{\mathcal{V}_s} \tilde{M}_{k,s}^r I_{(j,s)} I_{(j,s)} \gamma_{(j,s)} \Delta A A^{k-t_{v,s}-(k'+1)} \gamma_{(j,s)} \hat{\eta}_{(j,s)}(k''),$$

$$u_t^{\text{apx}} = \sum_{r,s \in \mathcal{L}} \sum_{v \in \mathcal{L}} \sum_{k \in I(t)} \sum_{k' \in I_{t,v}(k)} \sum_{k''} I_{\mathcal{V}_s} \tilde{M}_{k,s}^r I_{(j,s)} I_{(j,s)} \gamma_{(j,s)} \Delta A A^{k-t_{v,s}-(k'+1)} \gamma_{(j,s)} \hat{\eta}_{(j,s)}(k''),$$

where $M_{w,r} = I_{n_v}$ if $r \in \mathcal{L}$, and $M_{w,r} = 0$ if $r \notin \mathcal{L}$, with $w_{j,s} \to r$ and $\mathcal{L}$ defined in Eq. (14). Noting from the choice of $h$ that $h \geq 4(\mathcal{D}_{\text{max}} + 1)$, one can check that for any $k \in I(t)$, any $k' \in I_{t,v}(k)$ and any $k'' \in I(k')$, it holds that $k \in [t - h, \ldots, t - 1]$. It then follows that there exists $(\tilde{M}_{\text{apx},s,k})_{k \in [h], s \in \mathcal{U}}$ such that

$$u_t^{\text{apx}} + u_t^{\text{apx}} + u_t^{\text{apx}} = \sum_{s \in \mathcal{L}} \sum_{k \in I(t-h)} I_{\mathcal{V}_s} \tilde{M}_{k,s}^r \hat{\eta}_{k,s}.$$
Moreover, noting from Proposition 3 that $\| \Delta A \| \leq \varepsilon$ and $\| \Delta B \| \leq \varepsilon$, and noting from Eq. (64) that $\| M_{t,s}^{[k]} \| \leq \sqrt{n} \kappa p \Gamma^{2D_{\text{max}}+1}$ for all $s \in \mathcal{U}$ and all $k \in [h/4]$, where $0 < \gamma < 1$, one can show via the above arguments that

$$\| \tilde{M}_{t,s}^{[k]} \| \leq 3 p \left( \frac{h}{4} \right) ^2 n \kappa^3 p^3 \Gamma^{4D_{\text{max}}+3} \varepsilon, \forall k \in \{ t-h, \ldots, t-1 \}.$$ 

It follows that $u_{t}^{\text{apx}} = \sum_{s \in \mathcal{U}} \sum_{k \in (t)} I_{t,s}^{[k]} \tilde{M}_{t,s}^{[k]} \tilde{\eta}_{k,s} + \sum_{s \in \mathcal{U}} \sum_{k=t-h}^{t-1} I_{t,s}^{[k]} \tilde{M}_{t,s}^{[k]} \tilde{\eta}_{k,s}$, which also implies that there exist $M_{t,s}^{[k]} = (M_{t,s}^{[k]})_{k \in [t], s \in \mathcal{U}}$ such that $u_{t}^{\text{apx}} = u_{t}^{\text{apx}}(M_{t,s}^{[k]} | \tilde{w}_{0,t-1}) = \sum_{s \in \mathcal{U}} \sum_{k=t-h}^{t-1} I_{t,s}^{[k]} M_{t,s}^{[k]} \tilde{\eta}_{k,s}$, where

$$M_{t,s}^{[k]} = \begin{cases} M_{t,s}^{[k]} + \tilde{M}_{t,s}^{[k]} & \text{if } k \in \{ t - \frac{h}{2}, \ldots, t-1 \}, \\ M_{t,s}^{[k]} & \text{if } k \in \{ t-h, \ldots, t-h-1 \}, \end{cases}$$

for all $s \in \mathcal{U}$. Noting the fact that $\tilde{M}_{t,s}^{[k]} = 0$ for all $k > \frac{h}{4}$, we have $M_{t,s}^{[k]} - \tilde{M}_{t,s}^{[k]} = \tilde{M}_{t,s}^{[k]}$ for all $k \in \{ t-h, \ldots, t-1 \}$ and all $s \in \mathcal{U}$. Hence, we have from the above arguments that

$$\| M_{t,s}^{[k]} - \tilde{M}_{t,s}^{[k]} \| = \| \tilde{M}_{t,s}^{[k]} \| \leq 3 p \left( \frac{h}{4} \right) ^2 n \kappa^3 p^3 \Gamma^{4D_{\text{max}}+3} \varepsilon \leq \sqrt{n} \kappa p \Gamma^{2D_{\text{max}}+1},$$

for all $k \in \{ t-h, \ldots, t-1 \}$ and all $s \in \mathcal{U}$, where the second inequality follows from the choice of $\varepsilon$ in Eq. (51). It follows that

$$\| M_{t,s}^{[k]} \| \leq \| \tilde{M}_{t,s}^{[k]} \| + \sqrt{n} \kappa p \Gamma^{2D_{\text{max}}+1} \leq 2 \sqrt{n} \kappa p \Gamma^{2D_{\text{max}}+1},$$

for all $k \in \{ t-h, \ldots, t-1 \}$ and all $s \in \mathcal{U}$, which implies that $M_{t,s}^{[k]} \in \mathcal{D}$. \hfill \blacksquare

**Claim 3.** For any $t_1 \in \{ N_0, \ldots, T - 1 \}$ and any $\mu \in \mathbb{R}_{>0}$, it holds that

$$\| u_{t_1}^{\text{er}} \| \leq \frac{5}{8} h^5 \kappa^4 \Gamma^{6D_{\text{max}}+4} (\kappa + 1) R \delta \varepsilon^2 + \frac{1}{8} \left( p^2 q \sqrt{n} h^2 \kappa \Gamma^{2D_{\text{max}}+1} (\kappa + 1) \varepsilon \right).$$

**Proof.** Consider any $t_1 \in \{ N + 2h, \ldots, T - 1 \}$. We first write

$$u_{t_1}^{\text{er}} = u_{t_1}^{\text{er}1} + u_{t_1}^{\text{er}2} + u_{t_1}^{\text{er}3},$$

where

$$u_{t_1}^{\text{er}1} = \sum_{s \in \mathcal{U}} \sum_{k \in (t_1)} \sum_{v \in \mathcal{L}} \sum_{k' \in \mathcal{L}_r(k)} I_{v,s} M_{v,s}^{[t_1-k]} I_{s,(v)} I_{(v)}, \nu \Delta A \kappa^{-l_{x}}, \Delta B \Delta \tilde{u}_{k'},$$

$$u_{t_1}^{\text{er}2} = \sum_{s \in \mathcal{U}} \sum_{k \in (t_1)} \sum_{v \in \mathcal{L}} \sum_{k' \in \mathcal{L}_r(k)} I_{v,s} M_{v,s}^{[t_1-k]} I_{s,(v)} I_{(v)}, \nu \Delta A \kappa^{-l_{x}}, \Delta B \Delta \tilde{u}_{k'},$$

$$u_{t_1}^{\text{er}3} = \sum_{s \in \mathcal{U}} \sum_{k \in (t_1)} \sum_{v \in \mathcal{L}} \sum_{k' \in \mathcal{L}_r(k)} I_{v,s} M_{v,s}^{[t_1-k]} I_{s,(v)} I_{(v)}, \nu \Delta B \Delta \tilde{u}_{k-l_{x}}.$$
where the second inequality also uses the fact from Lemma 9 that $\Delta w_k \leq R_d \varepsilon$ for all $k \in \{N, \ldots, T - 1\}$. It then follows that

$$\|u_k^{tr}\| \leq p^2 q \sqrt{n} h^2 2^{T-1} D_{max} + 1 \frac{16}{16} R_d \varepsilon^2 + p^2 q \sqrt{n} h^2 2^{T-1} D_{max} + 1 \frac{16}{16} \varepsilon (k \Gamma + 1) \max_{k \in \{t - h, \ldots, t - 1\}} \|u_k(M_k - \tilde{M}_s) | \hat{w}_{0; k-1}\|.$$

(106)

Moreover, we have that

$$\|u_k(M_k - \tilde{M}_s) | \hat{w}_{0; k-1}\| \leq \|u_k(M_k - M_{\text{apx}}) | \hat{w}_{0; k-1}\| + \|u_k(\tilde{M}_s - M_{\text{apx}}) | \hat{w}_{0; k-1}\|,$$

for all $k \in \{t_1 - h, \ldots, t_1 - 1\}$. Noting that $t_1 \in \{N + 2h, \ldots, T - 1\}$, we know that $k \in \{N + h, \ldots, T - 1\}$ for all $k \in \{t_1 - h, \ldots, T - 1\}$. Now, consider any $k \in \{t_1 - h, \ldots, t_1 - 1\}$. Using similar arguments to those for (96) in the proof of Lemma 14, we have that

$$\|u_k(M_{\text{apx}} - \tilde{M}_s) | \hat{w}_{0; k-1}\| \leq \sqrt{q h} R_{\hat{w}} \|\text{Mat}(M_{\text{apx}}) - \text{Mat}(\tilde{M}_s)\|,$$

where $\text{Mat}(M_{\text{apx}}) = [I_{V_s M_{\text{apx}}^{[k]}}]_{s \in \mathcal{U}, k' \in [h]}$ and $\text{Mat}(\tilde{M}_s) = [I_{V_s \tilde{M}_{\text{apx}}^{[k]}}]_{s \in \mathcal{U}, k' \in [h]}$. Recalling from Claim 2 that $\| M_{\text{apx}, s}^{[k]} - M_{\text{apx}, s}^{[k]} \| \leq \frac{q h n \kappa}{4} \Gamma^4 D_{max} + 3 \varepsilon$ for all $k \in [h]$ and all $s \in \mathcal{U}$, one can also show that

$$\|\text{Mat}(M_{\text{apx}}) - \text{Mat}(\tilde{M}_s)\| \leq \frac{\sqrt{q h}}{4} q h n \kappa^3 \Gamma^4 D_{max} + 3 \varepsilon,$$

which implies that

$$\|u_k(M_{\text{apx}} - \tilde{M}_s) | \hat{w}_{0; k-1}\| \leq \frac{3 q h n \kappa^3}{4} \Gamma^4 D_{max} + 3 R_{\hat{w}} \varepsilon.$$

Thus, we have from (106) that

$$\|u_k^{tr}\| \leq \frac{p^2 q h^2}{2} \Gamma^6 D_{max} + 4 (k \Gamma + 1) R_{\hat{w}} \varepsilon^2 + \frac{1}{8} \mu \|u_k(M_k - \tilde{M}_s) | \hat{w}_{0; k-1}\|^2 + \frac{\mu}{2} \max_{k \in \{t_1 - h, \ldots, t_1 - 1\}} \|u_k(M_k - M_{\text{apx}}) | \hat{w}_{0; k-1}\|^2,$$

where the inequality follows from the fact that $ab \leq \frac{a^2}{2\mu} + \frac{b^2}{2\mu}$ for any $a, b \in \mathbb{R}$ and any $\mu \in \mathbb{R}_{> 0}$.

Finally, putting the above arguments (particularly Eq. (102) and (104), and Claims 1-3) together, we have that for any $k \in \{N + 2h, \ldots, T - 1\}$,

$$\|u_k(\tilde{M}_s) | \hat{w}_{0; t - 1}\| - u_k(M_{\text{apx}}) | \hat{w}_{0; t - 1}\| = \|u_k^{tr} + u_k^{apx} + u_k^{tr} - u_k(\tilde{M}_s) | \hat{w}_{0; k-1}\| \leq \|u_k^{tr}\| + \|u_k^{apx}\| + \|u_k^{tr} - u_k(\tilde{M}_s) | \hat{w}_{0; k-1}\| = \|u_k^{tr}\| + \|u_k^{apx}\|,$$

which implies that (66) holds, completing the proof of the lemma.

**F.3 Proof of Lemma 21**

First, we have from Lemma 19 that

$$R_3 \leq 2(R_u \sigma_1(R) + R_e \sigma_1(Q)) \frac{\Gamma_k}{1 - \gamma} \sum_{t = N_0}^{T-1} \max_{k \in \{t - h, \ldots, t\}} \|u_k(M_{\text{apx}}) | \hat{w}_{0; k-1}\| - u_k(\tilde{M}_s) | \hat{w}_{0; k-1}\|.$$

46
Noting that \( N_0 = N + 3h + D_{\text{max}} \), we see that for any \( t \in \{N_0, \ldots, T - 1\} \) and any \( k \in \{t - h, \ldots, t\} \), \( k \in \{N + 2h, \ldots, T - 1\} \). It then follows from Lemma 20 that

\[
R_3 \leq \left( R_x \sigma_1(R) + R_x \sigma_1(Q) \right) \frac{2 \Gamma \kappa}{1 - \gamma} \left( \varepsilon(R_x + R_u) p^2 q \sqrt{\kappa^2 E_0^2 + 1} T \right) + \frac{p^2 q h^2}{8 \mu} \sqrt{k \Gamma^2 D_{\text{max}} + 1} \left( \kappa \Gamma + 1 \right) \varepsilon^2 T
+ \frac{p^5 q n^2 h^5 \kappa^4}{8} \Gamma^4 D_{\text{max}} + 4 \left( \kappa \Gamma + 1 \right) R_{\alpha} \varepsilon^2 T + \frac{q h^2 R_{\alpha}}{2} \sum_{t=0}^{T-1} \max_{k \in \{t-h, \ldots, t\}} \left\| u_t \left( M_{\text{apx}} \hat{w}_{t-1} \right) - u_k \left( \hat{M}_{t-1} \hat{w}_{t-1} \right) \right\|^2.
\]

To complete the proof of the lemma, we note that

\[
\sum_{t=0}^{T-1} \max_{k \in \{t-h, \ldots, t\}} \left\| u_t \left( M_{\text{apx}} \hat{w}_{0:t-1} \right) - u_t \left( \hat{M}_{t-1} \hat{w}_{0:t-1} \right) \right\|^2 \leq 2h \sum_{t=0}^{T-1} \left\| u_t \left( M_t - M_{\text{apx}} \hat{w}_{0:t-1} \right) \right\|^2 
\leq 2q h^2 R_{\alpha} \sum_{t=0}^{T-1} \left\| \text{Vec}(M_t) - \text{Vec}(M_{\text{apx}}) \right\|^2,
\]

where the second inequality follows from similar arguments to those for (96) in the proof of Lemma 14. ■

### G Proof Omitted in Section 4.4

First, we recall line 10 in Algorithm 3 and define \( N_a = \min_{i \in V} N_{a,i} \), where

\[
N_{a,i} \triangleq \min \left\{ t \geq N + D_{\text{max}} : \| x^a_{i} \| > R_x \text{ or } \| u_{i,t} (M_i \hat{w}_{0:t-1}) \| > R_u \right\}, \quad \forall i \in V.
\]

(107)

Noting from Lemma 9 that \( N_a \geq T \) on the event \( E \), and noting that \( P(E) \geq 1 - 1/T \), we see that \( P\left( \mathbb{1}\{N_a \leq T - 1\} \right) \leq 1/T \). We then have the following result; the proof is included in Appendix H.

**Lemma 23.** It holds that

\[
E \left[ \left\| x^a_{N_a} \right\|^2 \mathbb{1}\{N_a \leq T - 1\} \right] \leq \frac{2 \Gamma^2 p (R_x + R_u)^2}{T} + \frac{3 \kappa^2 (1 + \Gamma^2) \bar{\sigma}^2 n \log \frac{3T}{2}}{(1 - \gamma)T},
\]

(108)

\[
E \left[ \left\| u^a_{N_a} \right\|^2 \mathbb{1}\{N_a \leq T - 1\} \right] \leq 2 q h^2 \kappa^2 T \Gamma^4 D_{\text{max}} \left( \frac{4 (1 + \Gamma) \bar{c} ^2 p}{T} \left( R_x + R_u \right)^2 + \frac{3 \kappa^2 (1 + \Gamma^2) \bar{\gamma}^2 n \log \frac{3T}{2}}{(1 - \gamma)T} \right).
\]

(109)

To proceed, denoting \( \tilde{E} = (E \cap E_{R_2})^c \), we further have the following decomposition:

\[
E \left[ \mathbb{1}\{\tilde{E}\} \sum_{t=0}^{T-1} c(x^a_t, u^a_t) \right] = E \left[ \mathbb{1}\{\tilde{E}\} \sum_{t=0}^{N-1} c(x^a_t, u^a_t) \right] + E \left[ \mathbb{1}\{\tilde{E}\} \sum_{t=N}^{N+D_{\text{max}}-1} c(x^a_t, u^a_t) \right] + E \left[ \mathbb{1}\{\tilde{E}\} \sum_{t=N+D_{\text{max}}}^{T-1} c(x^a_t, u^a_t) \right].
\]

(110)

Now, for any \( t \in \mathbb{Z}_{\geq 0} \), we have

\[
\left\| x^a_t \right\| = \left\| \sum_{k=0}^{t-1} A^{t-(k+1)} (u_k + B u^a_k) \right\|
\leq \left( \max_{0 \leq k \leq t-1} \left\| u_k \right\| + \Gamma \max_{0 \leq k \leq t-1} \left\| u^a_k \right\| \right) \sum_{k=0}^{t-1} \left\| A^k \right\|
\leq \left( \max_{0 \leq k \leq t-1} \left\| w_k \right\| + \Gamma \max_{0 \leq k \leq t-1} \left\| u^a_k \right\| \right) \frac{\kappa}{1 - \gamma},
\]

(111)
which implies that
\[
\|x_t^{\text{alg}}\|^2 \leq \left( \max_{0 \leq k \leq t-1} \|w_k\|^2 + \Gamma^2 \max_{0 \leq k \leq t-1} \|u_k^{\text{alg}}\|^2 \right) \frac{2\kappa^2}{(1 - \gamma)^2}. \tag{112}
\]

Noting that \(\mathbb{P}(\mathcal{E}) \leq 2/T\), and recalling from Algorithm 3 that \(u_t^{\text{alg}} \sim_{\text{i.i.d.}} \mathcal{N}(0, \sigma_u^2 I_m)\) for all \(t \in \{0, \ldots, N - 1\}\), we obtain from (112) and Lemma 25 that
\[
\bar{R}_0 \leq \frac{20N\kappa^2}{(1 - \gamma)^2T} \log \frac{3T}{2} (\sigma_u^2 n + \sigma_a^2 m).
\tag{113}
\]

Similarly, recalling from Algorithm 3 that \(u_t^{\text{alg}} = 0\) for all \(t \in \{N, \ldots, N + D_{\max} - 1\}\), we have that
\[
\bar{R}_1 \leq \frac{20D_{\max}\kappa^2}{(1 - \gamma)^2T} \log \frac{3T}{2} (\sigma_u^2 n + \sigma_a^2 m).
\tag{114}
\]

Moreover, we see from line 10 of Algorithm 3 that \(\|x_t^{\text{alg}}\| \leq \sqrt{\bar{R}_x}\) and \(\|u_t^{\text{alg}}\| \leq \sqrt{\bar{R}_u}\) for all \(t \in \{N + D_{\max}, \ldots, N_a - 1\}\), which implies that
\[
\bar{R}_2 \leq \frac{2N_a}{T} (\bar{R}_x + \bar{R}_u). \tag{115}
\]

Furthermore, noting line 11 of Algorithm 3, we have that for any \(t \in \{N_a + 1, \ldots, T - 1\}\,
\[
\|x_t^{\text{alg}}\|^2 = \|A^{t - N_a} x_{N_a}^{\text{alg}} + \sum_{k=N_a}^{t-1} A^{t -(k+1)} (w_k + Bu_k^{\text{alg}})\|^2
\leq 3\|A^{t - N_a} x_{N_a}^{\text{alg}}\|^2 + 3 \left\| \sum_{k=N_a}^{t-1} A^{t -(k+1)} w_k \right\|^2 + 3\|A^{t -(N_a + 1)} u_{N_a}^{\text{alg}}\|^2
\leq 3(\kappa^2 (t - N_a))^2 \|x_{N_a}^{\text{alg}}\|^2 + \frac{3\kappa^2}{(1 - \gamma)^2} \max_{N_a \leq k \leq t - 1} \|w_k\| + 3(\kappa^2 (t - (N_a + 1)))^2 \|u_{N_a}^{\text{alg}}\|^2.
\tag{116}
\]

Denoting the right-hand sides of (108) and (109) as \(\hat{R}_x\) and \(\hat{R}_u\), respectively, and invoking Lemma 25, one can show that
\[
\bar{R}_3 \leq \sigma_1(Q) \left( \hat{R}_x + (T - N_a - 1) (4\kappa^2 \gamma^2 + 1) \hat{R}_x + 3\kappa^2 \gamma^2 \hat{R}_u \right). \tag{117}
\]

Combining (113)-(115) and (117) together, we prove Lemma 22.  \(\blacksquare\)

**H Auxiliary Proof and Lemmas**

**H.1 Proof of Lemma 23**

Based on the definition of Algorithm 3, we split our arguments for the proof of (108) into \(N_a = N + D_{\max}\) and \(N + D_{\max} < N_a \leq T - 1\). First, supposing \(N_a = N + D_{\max}\), we have
\[
\|x_{N_a}^{\text{alg}}\| = \left\| \sum_{t=0}^{N_a - 1} A^{N_a -(t+1)} (w_t + Bu_t^{\text{alg}}) \right\|
\leq \max_{0 \leq t \leq N_a - 1} \|w_t + Bu_t^{\text{alg}}\| \sum_{t=0}^{N_a - 1} \|A^{N_a -(t+1)}\|
\leq \left( \max_{0 \leq t \leq N_a - 1} \|w_t\| + \Gamma \max_{0 \leq t \leq N_a - 1} \|u_t^{\text{alg}}\| \right) \frac{\kappa}{1 - \gamma}
\leq \left( \max_{0 \leq t \leq T - 1} \|w_t\| + \Gamma \max_{0 \leq t \leq N_a - 1} \|u_t^{\text{alg}}\| \right) \frac{\kappa}{1 - \gamma},
\]
where the last inequality follows from the definition of Algorithm 3, and we note that $u_t^{\text{alg}} \sim i.i.d. N(0, \sigma_u^2 I)$.

Since $\mathbb{P}\{N_a \leq T - 1\} \leq 2/T$, we see from Lemma 25 that

$$\mathbb{E}\left[ \mathbb{I}\{N_a \leq T - 1\} \max_{0 \leq t \leq T - 1} \|w_t\|^2 \right] \leq \frac{10\sigma_u^2 n}{T} \log 3T,$$

$$\mathbb{E}\left[ \mathbb{I}\{N_a \leq N - 1\} \max_{0 \leq t \leq T - 1} \|u_t^{\text{alg}}\|^2 \right] \leq \frac{10\sigma_u^2 m}{T} \log 3N.$$

It then follows that

$$\mathbb{E}\left[ \|x_{N_a}^{\text{alg}}\|^2 \mathbb{I}\{N_a = N + D_{\text{max}}\} \right] \leq \frac{2\kappa^2}{1 - \gamma}\left( \mathbb{E}\left[ \mathbb{I}\{N_a \leq T - 1\} \max_{0 \leq t \leq T - 1} \|w_t\|^2 \right] + \Gamma^2 \mathbb{E}\left[ \mathbb{I}\{N_a \leq N - 1\} \max_{0 \leq t \leq T - 1} \|u_t^{\text{alg}}\|^2 \right] \right)$$

$$\leq \frac{20\kappa^2(1 + \Gamma^2)\sigma^2 n \log 3T}{(1 - \gamma)T}.$$  \hfill (118)

Next, suppose $N + D_{\text{max}} < N_a \leq T - 1$. Noting that $\|x_{t,i}^{\text{alg}}\| \leq R_x$ and $\|u_{t,i}^{\text{alg}}\| \leq R_u$ for all $t < N_a$ and all $i \in V$, we have

$$\|x_{N_a}^{\text{alg}}\| = \|A_{N_a} x_{N_a - 1}^{\text{alg}} + Bu_{N_a - 1}^{\text{alg}} + w_{N_a - 1}\|$$

$$\leq \Gamma \sqrt{p}(R_x + R_u) + \max_{0 \leq t \leq T - 1} \|w_t\|,$$

which implies that

$$\mathbb{E}\left[ \|x_{N_a}^{\text{alg}}\|^2 \mathbb{I}\{N + D_{\text{max}} < N_a \leq T - 1\} \right] \leq \frac{2\Gamma^2 p(R_x + R_u)^2}{T} + \frac{10\sigma_u^2 n}{T} \log 3T. \hfill (119)$$

Combining (118)-(119) together completes the proof of (108).

We then prove (109). Consider $N + D_{\text{max}} \leq N_a \leq T - 1$. We recall from Algorithm 3 that $u_{N_a}^{\text{alg}} = u_{N_a}(M_{N_a}, \hat{w}_{0:N_a - 1}) = \sum_{s \in \mathcal{U}} \sum_{k=1}^{h} I_{V,s} M_{N_a,s}^{[k]} \hat{\eta}_{N_a-k,s}$, where $\hat{\eta}_{N_a-k,s} = [\hat{\omega}^T_{N_a-k-l,v,j_v}]_{v \in L_s}$ with $\hat{\omega}_{N_a-k-l,v,j_v}$ given by (33). It follows that

$$\|u_{N_a}^{\text{alg}}\|^2 \leq \left( \sum_{s \in \mathcal{U}} \sum_{k=1}^{h} \|M_{N_a,s}^{[k]}\| \|\hat{\eta}_{N_a-k,s}\| \right)^2.$$  \hfill (120)

Moreover, consider any $s \in \mathcal{U}$, any $t \in \{N_a - k - l_v : k \in [h], v \in L_s\}$, and any $j_v$ with $w_{j_v} \to v$, we have from (33) that

$$\|\hat{\omega}_{t,j_v}\| \leq \|x_{t+i,j_v}^{\text{alg}}\| + \|\hat{A}_j\| \|x_{t+i,N_a}^{\text{alg}}\| + \|\hat{B}_j\| \|u_{t,N_a}^{\text{alg}}\|$$

$$\leq \|x_{t+i+1,j_v}^{\text{alg}}\| + (\Gamma + \tilde{\varepsilon}) \sqrt{p}(R_x + R_u). \hfill (121)$$

Now, suppose $t = N_a - 1$. We see from (121) that

$$\|\hat{\omega}_{t,j_v}\| \leq \|x_{t+1,j_v}^{\text{alg}}\| + (\Gamma + \tilde{\varepsilon}) \sqrt{p}(R_x + R_u),$$

which implies via (108) and the fact $\mathbb{P}(\mathbb{I}\{N_a \leq T - 1\}) \leq 1/T$ that

$$\mathbb{E}\left[ \|\hat{\omega}_{t,j_v}\|^2 \mathbb{I}\{N + D_{\text{max}} \leq N_a \leq T - 1\} \right] \leq \frac{2T^2 p + 2(\Gamma + \tilde{\varepsilon})^2 p}{T} (R_x + R_u)^2 + \frac{30\kappa^2(1 + \Gamma^2)\sigma^2 n \log 3T}{(1 - \gamma)T}. \hfill (122)$$

Supposing $t < N_a - 1$ and noting from Algorithm 3 that $\|x_{t+1}^{\text{alg}}\| \leq R_x$, we have from (121) that

$$\|\hat{\omega}_{t,j_v}\| \leq \|x_{t+1,j_v}^{\text{alg}}\| + (\Gamma + \tilde{\varepsilon}) \sqrt{p}(R_x + R_u)$$

$$\leq R_x + (\Gamma + \tilde{\varepsilon}) \sqrt{p}(R_x + R_u),$$

49
which implies that

$$
\mathbb{E} \left[ \| \hat{w}_{i,j} \|^2 1 \{ N + D_{\text{max}} \leq N_a \leq T - 1 \} \right] \leq \frac{2R_x^2}{T} + \frac{2(\Gamma + \bar{\varepsilon})^2 p(R_x + R_u)^2}{T^3}.
$$

(123)

Noting from Eq. (26) that $$\| M_{N_a,s}^{[k]} \| \leq 2\sqrt{n}p \kappa_M \gamma_M$$ for all $$s \in \mathcal{U}$$ and all $$k \in [h]$$, and recalling that $$\hat{\eta}_{N_a-k,s} = \left( \hat{w}_{N_a-k-l_v,j_v} \right)^T v \in \mathcal{L}_s$$ (with $$|\mathcal{L}_s| \leq p$$) for all $$v \in \mathcal{L}_s$$ and all $$s \in \mathcal{U}$$, one can plug (122)-(123) into (120) and show that (109) holds.

\[\blacksquare\]

\section*{H.2 Auxiliary Lemmas}

\textbf{Lemma 24.} \cite[Lemma 5]{31} Consider any matrix $$M \in \mathbb{R}^{n \times n}$$ and any matrix $$\Delta \in \mathbb{R}^{n \times n}$$. Let $$\kappa_M \in \mathbb{R}_{\geq 1}$$ and $$\gamma_M \in \mathbb{R}_{> 0}$$ be such that $$\gamma_M \geq \rho(M)$$, and $$\| M^k \| \leq \kappa_M \gamma_M$$ for all $$k \in \mathbb{Z}_{\geq 0}$$. Then, for all $$k \in \mathbb{Z}_{\geq 0}, \quad \| (M + \Delta)^k - M^k \| \leq k \kappa_M^2 (\kappa_M \| \Delta \| + \gamma_M)^{k-1} \| \Delta \|.

\textbf{Lemma 25.} Let $$\mathcal{E}$$ be an probabilistic event with $$\mathbb{P}(\mathcal{E}) \leq \delta$$, where $$0 < \delta < 1$$, and let $$w_t \sim_i \mathcal{N}(0, \sigma_w^2 I_n)$$ for all $$t \in \{0, \ldots, T-1\}$$. Then, for any $$T > 1$$, it holds that

$$
\mathbb{E} \left[ \max_{0 \leq t \leq T-1} \| w_t \|^2 \right] \leq 5\sigma_w^2 n \delta \log \frac{3T}{\sigma}.
$$

\textit{Proof.} The proof first relies on the following quadratic form of Gaussian random vector proved in \cite{23}:

$$
\mathbb{P} \left( \| Ax \|^2 > \text{Tr}(\Sigma) + 2\sqrt{\text{Tr}(\Sigma^2)z + 2\| \Sigma \| z} \right) \leq e^{-z} \quad \forall z \in \mathbb{R}_{\geq 0},
$$

where $$x \sim \mathcal{N}(0, I_n)$$, $$A \in \mathbb{R}^{m \times n}$$ and $$\Sigma = A^T A$$. Next, following \cite[Lemma 34]{9}, we know that

$$
\max_{0 \leq t \leq T-1} \| w_t \| \leq \sigma \sqrt{5n \log \frac{T}{\delta}}.
$$

The rest of the proof follows from \cite[Lemma 35]{9}.

\[\blacksquare\]