Estimation of a motion equations system of a potential two-dimensional in a water flow plan to dimensionless form

Anatoly Kondratenko¹ and Marija Alexandrova²

¹ Timiryzev Russian State Agrarian University Moscow Agricultural Akademy
Moscow Russia
² Platov South-Russian State Polytechnic University (NPI) Novocherkassk Russia

E-mail: e_masha@mail.ru

Abstract. In terms of stationary open water flow, the boundary conditions in the flow free spreading problem are reduced to a dimensionless form by the various coordinates and flow parameters’ transformations, including I.A. Sherenkov’s transformations, which bring the boundary value problem to a dimensionless form. It was found that the equations system itself can be reduced to a universal dimensionless form, but the boundary problem cannot be reduced, since the boundary conditions both at the flow outlet from a free-flow pipe and at the flow infinity downstream are not reduced to a universal dimensionless form. It is concluded that it is impossible to solve the boundary value problem once and then use this solution under any boundary conditions. It was also revealed that the problem solution depends on a dimensionless parameter - the Froude criterion at the flow outlet from the pipe. This proves that it is possible to build a universal graph, a universal method for solving the problem with Froude numbers at the flow outlet from the pipe more than unity or close to unity. But increasing the Froude number, it is necessary to build a series of graphs, and it is better to create a single theory, an algorithm for solving this problem.

1. Introduction
The work was written for a critical assessment of the existing methods for solving the problem and substantiating the topic of postgraduate student dissertation work. The fact that the calculation error with an increase in the Froude number when using one graph of I.A. Sherenkov, according to the geometry mismatch of the extreme streamline can reach 50% and more, is also proved experimentally. However, this does not depreciate all the advantages of the previous works carried out by the Russian researchers (I.A. Sherenkov et al.), who proposed a fair, easy-to-use method for calculating the water flows free spreading parameters (although calculated approximately in terms of adequacy to the real flow), but for the first time in the world science practice it is not the semi-analytical (graph + formulas) method. The presented work is the continuation of the research previously carried out by the authors [1-5].

Obtaining a system of equations for the water flow motion in a dimensionless form is necessary to identify the parameters (dimensionless criteria) affecting the process and to identify the flow features. The task is relevant both in theoretical and practical terms, because the motion equations and the results obtained on their basis are necessary for the practice of building the hydraulic structures’ (HS) culverts under highways and railways [6].

We will take the following system of the fluid flow differential equations as a basis [7]:
where $V_x, V_y$ are the projection of the local velocity vector on the coordinate axis OX, OY;

$h$ is the local flow depth;

$g$ is the gravity acceleration.

The system (1) describes a stationary flow of an open two-dimensional in plan water flow during its flow along a horizontal smooth channel [7, 8].

Let us consider various options for reducing the system of equations (1) to a dimensionless form.

2. Option one

Let us introduce the following dimensionless coordinates and flow parameters:

$$\bar{y} = \frac{y}{b} ; \quad \bar{x} = \frac{x}{b} ; \quad \bar{h} = \frac{h}{h_0} ; \quad \bar{V}_x = \frac{V_x}{V_0} ; \quad \bar{V}_y = \frac{V_y}{V_0},$$

(2)

where $b$ is a characteristic linear flow size;

$V_0, h_0$ are flow velocity and depth at some characteristic point.

After passing in equations (1) to dimensionless quantities $\bar{x}, \bar{y}, \bar{h}, \bar{V}_x, \bar{V}_y$, we get the following system:

$$
F \bar{t}_0 \left[ V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} + g \frac{\partial h}{\partial x} \right] = 0;
$$

$$
F \bar{t}_0 \left[ V_x \frac{\partial V_y}{\partial x} + V_y \frac{\partial V_y}{\partial y} + g \frac{\partial h}{\partial y} \right] = 0;
$$

$$
\frac{\partial (V_x h)}{\partial x} + \frac{\partial (V_y h)}{\partial y} = 0,
$$

(3)

where $F \bar{t}_0 = \frac{V_0^2}{gh_0}$ denotes the Froude number at the flow characteristic point.

In the case of the potential flow additional condition, it is necessary to add the equation to the system of equations (1) [9]:

\[ \square \]
\[ W = \frac{\|V_x\|_y - \|V_y\|_x}{\|x\|_y} = 0. \]  

(4)

Or in a dimensionless form:

\[ \frac{V_0}{b} \frac{\|\bar{V}_x\|_{\bar{y}} - \|\bar{V}_y\|_{\bar{x}}}{\|\bar{x}\|_{\bar{y}}} = 0. \]  

(5)

Comparing the equations (4) and (5) it becomes obvious, that their form coincides, since \( \frac{V_0}{b} \neq 0 \).

For example, let us consider the solution to the boundary value problem of a potential turbulent flow free spreading [10-12]. To solve this problem, we proceed from the equations (3), (5) and the boundary conditions in the dimensionless form:

\[ \bar{h} = 1; \quad -\frac{1}{2} J \bar{y} J \frac{1}{2}; \quad \bar{V}_x = 1. \]  

(6)

For this task \( h = h_0, V_x = V_0 \) are the parameters of the flow outgoing from a rectangular pipe with width \( b \) in a free water flow.

As a characteristic point, it is possible to select a point at the flow spill from the pipe with the parameters \( h_0, V_0 \).

Let us choose a coordinate system (OXY) in terms of flow: OX – longitudinal flow axis, OY – completes the system to the right coordinate system.

With this choice of coordinates, in the case of flow symmetry about the axis OX the dimensionless boundary conditions are valid (6).

Analyzing the equations (3), (5) and the dimensionless boundary conditions (6), we can state that the nature of the turbulent flow free spreading will depend on the dimensionless Froude criterion \( F_0 \), because, although the equation (5), the boundary conditions (6) and the third equation in the system (3) do not depend on \( F_0 \) (their form is universal), the first and second system equations (3) depend on \( F_0 \).

Intermediate terminal

Since under the transformation (2) [8, 13], it is impossible to formulate the boundary value problem in a dimensionless form, independent of any criteria, it is impossible to reduce the solution of the problem to the use of a universal schedule, i.e., to compose it once and then use it to recalculate the flow parameters.

Let us consider the question: is it possible to choose such an affine similarity transformation [14] in which the system of equations with the boundary conditions for the free flow spreading problem would be reduced to a universal dimensionless form? Universal means not depending on any criteria.

3. Option two

Let us choose a similarity transformation in the form:

\[ \frac{\bar{y}}{b} = \frac{y}{h_0}; \quad \frac{\bar{x}}{b} = \frac{x}{h_0}; \quad \bar{h} = \frac{h}{h_0}; \quad \bar{V}_x = \frac{V_x}{K_x}; \quad \bar{V}_y = \frac{V_y}{K_y}. \]  

(7)

Using the equality (7), the system of equations (1) is reduced to the form:
For the system (8) to be reduced to a dimensionless form, we require the satisfying of the condition:

\[ K_x = K_y = \sqrt{gh_0}. \]  

(9)

Then the system (8) is reduced to the form:

\[
\begin{align*}
K_x \frac{\| \nabla \bar{V}_x \|_{x}}{\| \nabla \bar{V}_y \|_{y}} + K_y \frac{\| \nabla \bar{V}_y \|_{y}}{\| \nabla \bar{V}_x \|_{x}} + \frac{\| \nabla \bar{h} \|_{y}}{\| \nabla \bar{h} \|_{x}} &= 0; \\
K_x \frac{\| \nabla (\bar{V}_x \bar{h}) \|_{x}}{\| \nabla \bar{h} \|_{y}} + K_y \frac{\| \nabla (\bar{V}_y \bar{h}) \|_{y}}{\| \nabla \bar{h} \|_{x}} &= 0. 
\end{align*}
\]

(10)

The flow potentiality condition (4) is reduced to the form:

\[
\frac{\| \nabla V_x \|_{y}}{\| \nabla V_y \|_{x}} \quad \frac{\| \nabla \bar{V}_x \|_{y}}{\| \nabla \bar{V}_y \|_{x}} = K_x b \frac{\| \nabla \bar{y} \|_{y}}{\| \nabla \bar{y} \|_{x}} = 0.
\]

(11)

or

\[
\frac{\| \nabla \bar{V}_x \|_{y}}{\| \nabla \bar{V}_y \|_{x}} = 0.
\]

(12)

The system of equations (10) and the equation (12) are reduced to a dimensionless form.

The boundary conditions for specifying the parameters’ calculation of the free flow spreading:

\[
\begin{align*}
\bar{y} &= \frac{y}{b}; \quad \bar{x} = \frac{x}{b}; \quad \bar{h} = \frac{h}{h_0}; \quad \bar{V}_x = \frac{V_x}{\sqrt{gh_0}}; \quad \bar{V}_y = \frac{V_y}{\sqrt{gh_0}}. 
\end{align*}
\]

(13)
- \( \frac{b}{2} J \ y J \ \frac{b}{2} \); \( x = 0; \ h = h_0; \ V_x = V_0; \ V_y = 0. \)

(14)

At the flow spill from the pipe, the boundary conditions in a dimensionless form will be as follows:

- \( \frac{1}{2} J \ y J \ \frac{1}{2} \); \( \bar{x} = 0; \ \bar{h} = 1; \ \bar{V}_x = \frac{V_0}{\sqrt{gh_0}} = \sqrt{F_{r_0}} ; \ \bar{V}_y = 0. \)

(15)

The condition \( \bar{V}_x = \sqrt{F_{r_0}} \) is not universal, it depends on \( F_{r_0} \).

Therefore, when transforming (13), it is impossible to construct a universal method for calculating the flow parameters. It is necessary to frame a calculation for each value \( F_{r_0} \).

Let us now show that from the graph of I.A. Sherenkov [15, 16] in the coordinates

\[ \bar{y} = \frac{y}{b}; \ \bar{x} = \frac{x}{b\sqrt{F_{r_0}}} \]

(16)

it is possible to obtain the results that are adequate to the real (or model) only when the Froude numbers at the flow spill from the pipe supplying water are close to unity, i.e.:

\[ F_{r_0} \approx 1. \]

(17)

With the increasing parameter \( F_{r_0} \), the calculated parameters’ mismatch using the graph of I.A. Sherenkov will increase in comparison with the field and experimental data.

According to the flow motion equations transformation to the dimensionless parameters I.A. Sherenkov proceeded from the conditions:

\[ \bar{y} = \frac{y}{b}; \ \bar{x} = \frac{x}{b\sqrt{F_{r_0}}}; \ \bar{h} = \frac{h}{h_0}; \ \bar{V}_x = \frac{V_x}{V_0K_x}; \ \bar{V}_y = \frac{V_y}{V_0K_y}. \]

(18)

In this case, the system of equations (1) is transformed to the following dimensionless form:

\[ F_{r_0}K_x^2\bar{V}_x\frac{\|\bar{V}_x\}}{\|\bar{x}\}} + F_{r_0}^{3/2}K_xK_y\bar{V}_y\frac{\|\bar{V}_y\}}{\|\bar{y}\}} \frac{\|\bar{h}\}}{\|\bar{x}\}} + \frac{\|\bar{h}\}}{\|\bar{x}\}} = 0; \]

\[ F_{r_0}^{3/2}K_xK_y\bar{V}_x\frac{\|\bar{V}_y\}}{\|\bar{x}\}} + F_{r_0}K_y^2\bar{V}_y\frac{\|\bar{V}_y\}}{\|\bar{y}\}} \frac{\|\bar{h}\}}{\|\bar{y}\}} = 0; \]

\[ \frac{K_x}{\sqrt{F_{r_0}}} \frac{\|\bar{V}_x\bar{h}\}}{\|\bar{x}\}} + K_y \frac{\|\bar{V}_y\bar{h}\}}{\|\bar{y}\}} = 0. \]

(19)
The system of equations (19) at $K_x = \frac{1}{\sqrt{F_{r_0}^2}}$, $K_y = \frac{1}{F_{r_0}}$ will take a universal, dimensionless form:

\[
\begin{align*}
V_x \frac{\nabla V_x}{\nabla x} + V_y \frac{\nabla V_y}{\nabla y} + \frac{\nabla h}{\nabla x} &= 0; \\
V_x \frac{\nabla V_y}{\nabla x} + V_y \frac{\nabla V_y}{\nabla y} + \frac{\nabla h}{\nabla y} &= 0; \\
\frac{\nabla (V_x h)}{\nabla x} + \frac{\nabla (V_y h)}{\nabla y} &= 0.
\end{align*}
\]

(20)

The boundary conditions (14) taking into account the transformation (18), will take the following dimensionless form:

\[-\frac{1}{2} \frac{\partial h}{\partial y} \frac{\partial y}{\partial j} \frac{1}{2}; \quad \bar{x} = 0; \quad \bar{h} = 1; \quad \bar{V_x} = \sqrt{F_{r_0}^2}; \quad \bar{V_y} = 0.
\]

(21)

It follows from (21) that the boundary conditions supplementing the system (20) to the possibility of uniquely determining the solution to the boundary value problem of a potential stationary open water flow free spreading during its unpressurized outflow from a rectangular pipe into a wide smooth horizontal channel do not have a universal form. Therefore, only one I.A. Sherenkov’s graph might not be enough for the calculation, but it is possible to use a series of graphs for different Froude numbers. When $F_{r_0}$ is close to one, it is possible to use the graph of I.A. Sherenkov.

4. Conclusions

When solving various boundary problems along water flows, it is possible without solving the problem to reduce the system of motion and boundary conditions equations to dimensionless form and answer the questions:

– what criteria does the solution of the problem depend on?

– is it possible to reduce the solution of the problem to a universal dimensionless form, and, consequently, is it possible to solve it once, and then use only the recalculation of the results under various boundary conditions?

I.A. Sherenkov’s graph is given in various monographs [7, 15, 16], in reference manuals on hydraulics [6], in dissertations [17]. It was used in the hydraulic structure construction [6].

However, the theoretical studies in this work and the experimental studies carried out in the laboratory of NSRI by Tsyvin M.N., Kolchenko O.L., Tkachenko N.I., show the need to revise the solution of the problem at different Froude numbers at the flow spill from the pipe.

The discrepancies between the results on the geometry of the extreme streamline by different methods and the experiments results are shown in figure 1.
5. General Conclusions

The results on the two-dimensional spreading in terms of water flows are necessary in the design of hydraulic structures, as they are widely known in the technical literature.

Without solving the boundary problem, we can say that the search for the methods for solving the problem of a potential water flow free spreading should be continued.

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Figure 1. Taken from the monograph [10]: 1- Theoretical data, 2- Experimental data, 3- According to G.A. Lilitsky.
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