EXPANSIVE MULTIPARAMETER ACTIONS AND MEAN DIMENSION

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Abstract. Mañé (1979) proved that if a compact metric space admits an expansive homeomorphism then it is finite dimensional. We generalize this theorem to multiparameter actions. The generalization involves mean dimension theory, which counts “averaged dimension” of a dynamical system. We prove that if $T : \mathbb{Z}^k \times X \to X$ is expansive and if $R : \mathbb{Z}^{k-1} \times X \to X$ commutes with $T$ then $R$ has finite mean dimension. When $k = 1$, this statement reduces to Mañé’s theorem. We also study several related issues, especially the connection with entropy theory.

1. Introduction

1.1. Main results. Let $(X, d)$ be a compact metric space. A homeomorphism $T : X \to X$ is said to be expansive if there exists $c > 0$ such that any distinct two points $x$ and $y$ in $X$ satisfy

$$\sup_{n \in \mathbb{Z}} d(T^n x, T^n y) > c.$$ 

Hyperbolic dynamics provides many examples of expansive maps [Bow75, Chapter 3]: A diffeomorphism is expansive on hyperbolic sets. Motivated by the work of Bowen [Bow70] on hyperbolic minimal sets, Mañé [Ma79] investigated the topological dimension of a compact metric space admitting an expansive homeomorphism:

**Theorem 1.1** (Mañé, 1979). Let $T : X \to X$ be an expansive homeomorphism. Then $X$ is finite dimensional.

Therefore infinite dimensional spaces (e.g. the infinite dimensional cube $[0, 1]^\mathbb{N}$) cannot admit expansive homeomorphisms. This is a rather surprising result because the definition of expansiveness seems to have nothing to do with dimension theory. Fathi [Fa89, Corollaries 5.4 and 5.5] revisited this phenomena from the viewpoint of entropy theory and he proved:

**Theorem 1.2** (Fathi, 1989). Let $T : X \to X$ be an expansive homeomorphism. If the topological entropy $h_{\text{top}}(T)$ is zero, then $X$ is zero dimensional.
The main purpose of this paper is to generalize the above theorems of Mañé and Fathi to multiparameter actions (i.e. $\mathbb{Z}^k$-actions). A continuous action $T : \mathbb{Z}^k \times X \to X$ on a compact metric space $(X, d)$ is said to be expansive if there exists $c > 0$ such that any two distinct points $x$ and $y$ in $X$ satisfy $\sup_{u \in \mathbb{Z}^k} d(T^u x, T^u y) > c$. At first sight, it looks nonsense to study $\mathbb{Z}^k$-versions of Theorems 1.1 and 1.2 because we can easily find examples which seemingly deny this direction:

Example 1.3. 

(1) Let $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the standard two dimensional torus and $h : T^2 \to T^2$ a hyperbolic toral automorphism, e.g. $h = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. $h$ is an expansive homeomorphism. Consider the two-sided infinite product $X := (T^2)^{\mathbb{Z}}$ (with the product topology) and let $\sigma : X \to X$ be the shift. Define $h_{\mathbb{Z}} : X \to X$ by $h_{\mathbb{Z}}(x_n)_{n \in \mathbb{Z}} = (h(x_n))_{n \in \mathbb{Z}}$. Then $\sigma$ and $h_{\mathbb{Z}}$ generate an expansive $\mathbb{Z}^2$-action on $X$ although $X$ is infinite dimensional (Shi–Zhou [SZ05, Proposition 4.2]). See also Example 1.7 below for a different kind of expansive actions on infinite dimensional spaces.

(2) Let $\text{id} : T^2 \to T^2$ be the identity map. Then $\text{id}$ and the hyperbolic toral automorphism $h : T^2 \to T^2$ generate an expansive $\mathbb{Z}^2$-action on $T^2$ with zero topological entropy although $T^2$ is positive dimensional. With a bit more effort, we can construct an expansive $\mathbb{Z}^2$-action of zero topological entropy on an infinite dimensional space. (Let $D = 0$ in Example 1.10 below.)

The above examples show that we cannot naively generalize Theorems 1.1 and 1.2 to $\mathbb{Z}^k$-actions. We have to change our viewpoint.

It turns out that mean dimension theory provides a reasonable framework for the problem. This is a topological invariant of dynamical systems introduced by Gromov [Gro99], which counts the number of variables per iterate to describe a point in a dynamical system. We denote by $\text{mdim}(X, T)$ the mean dimension of a continuous action $T : \mathbb{Z}^k \times X \to X$. We will review its definition in §2. It is known that (if $k \geq 1$) mean dimension is zero for all finite dimensional systems and finite topological entropy systems. The $\mathbb{Z}^k$-shift on $([0, 1]^D)^{\mathbb{Z}^k}$ has mean dimension $D$ and the $\mathbb{Z}^k$-shift on $([0, 1]^N)^{\mathbb{Z}^k}$ has infinite mean dimension.

Mean dimension has applications to topological dynamics [LW00, Lin99, Gut11, GLT16, GT, GQT]. As an illustration, we review one result [GQT, Main Theorem 1] which originated in the work of Lindenstrauss [Lin99, Theorem 5.1]: If a free minimal $\mathbb{Z}^k$-action $(X, T)$ satisfies $\text{mdim}(X, T) < D/2$ then we can equivariantly embed it in the $\mathbb{Z}^k$-shift on $([0, 1]^D)^{\mathbb{Z}^k}$. The value $D/2$ here is optimal. This shows in particular that a free minimal $\mathbb{Z}^k$-action $(X, T)$ has finite mean dimension if and only if one can equivariantly embed it in the $\mathbb{Z}^k$-shift on $([0, 1]^D)^{\mathbb{Z}^k}$, for some finite $D$.

The following is our first main result.
Theorem 1.4. Let $T : \mathbb{Z}^k \times X \to X$ be an expansive action on a compact metric space $X$, and let $R : \mathbb{Z}^{k-1} \times X \to X$ be a continuous action that commutes with $T$. Namely, $T^v \circ R^u = R^u \circ T^v$ for all $v \in \mathbb{Z}^k$ and $u \in \mathbb{Z}^{k-1}$. Then

$$\text{mdim} (X, R) < \infty.$$ 

For a subgroup $A \subset \mathbb{Z}^k$ we denote by $T|_A : A \times X \to X$ the restriction of $T$ to $A$. Letting $R = T|_A$ with rank $A = k - 1$ we have the following special case:

Corollary 1.5. Let $T : \mathbb{Z}^k \times X \to X$ be an expansive action on a compact metric space $X$. Then for any subgroup $A \subset \mathbb{Z}^k$ with rank $A = k - 1$

$$\text{mdim} (X, T|_A) < \infty.$$ 

Namely, the restriction of $T$ to any rank $(k - 1)$ subgroup has finite mean dimension.

Remark 1.6. Here are several remarks on the theorem:

- When $k = 1$, an action $R : \mathbb{Z} \times X \to X$ is the trivial action, and the mean dimension $\text{mdim} (X, R)$ is equal to the topological dimension $\dim X$. Thus the statement of Theorem 1.4 and also of Corollary 1.5 reduce to the original theorem of Mañé (Theorem 1.1) in this case.

- In Example 1.3 (1), the mean dimension of $\sigma$ is two and the mean dimension of $h_{\mathbb{Z}}$ is zero.

- Expansive actions always have finite topological entropy. So the mean dimension of an expansive action $T : \mathbb{Z}^k \times X \to X$ itself is zero since finite topological entropy systems are zero mean dimensional. This is trivial and uninteresting. The point of Corollary 1.5 is that it provides a nontrivial information of actions of infinite index subgroups. Our viewpoint is summarized by the following correspondence:

Expansiveness of $\mathbb{Z}^k$-actions $\longleftrightarrow$ mean dimension of $\mathbb{Z}^{k-1}$-actions.

Example 1.7. Let $T = \mathbb{R}/\mathbb{Z}$ be the circle and consider the infinite product $T^{\mathbb{Z}^2}$ indexed by $\mathbb{Z}^2$. Let $\sigma$ be the $\mathbb{Z}^2$-shift on it. We define a $\sigma$-invariant closed subset $X$ of $T^{\mathbb{Z}^2}$ by

$$X = \left\{ (x_{mn})_{(m,n) \in \mathbb{Z}^2} \in T^{\mathbb{Z}^2} \mid \forall (m,n) \in \mathbb{Z}^2 : 3x_{mn} + x_{m+1,n} + x_{m,n+1} = 0 \right\}.$$ 

Then $(X, \sigma)$ is expansive. (This fact can be directly and easily checked. See Schmidt [Sch90] for a more general and systematic study of this kind of examples.) We can check that for any rank one subgroup $A \subset \mathbb{Z}^2$ the mean dimension $\text{mdim} (X, \sigma|_A)$ is positive and finite.

We will provide two proofs of Theorem 1.4. The first proof (given in §3) has the advantage that it is elementary and self-contained. It uses only the definitions of mean dimension. The second proof (given in §4) requires more machineries, in particular Lindenstrauss–Weiss’ theory of metric mean dimension [LW00]. The advantage of the
second approach is that it also provides the following generalization of Fathi’s theorem (Theorem 1.2). This is our second main result:

**Theorem 1.8.** Let \( T : Z^k \times X \to X \) be an expansive action on a compact metric space \( X \) and let \( R : Z^{k-1} \times X \to X \) be a continuous action that commutes with \( T \). If the topological entropy of \( T \) is zero, then

\[
\text{mdim} (X, R) = 0.
\]

**Corollary 1.9.** Let \( T : Z^k \times X \to X \) be an expansive action on a compact metric space \( X \). If the topological entropy of \( T \) is zero, then for any subgroup \( A \subset Z^k \) with \( \text{rank} A = k - 1 \)

\[
\text{mdim} (X, T|_A) = 0.
\]

We would like to note that the expansiveness is an essential assumption in the statement of Theorem 1.8 (i.e. the zero entropy of \( T \) alone does not imply \( \text{mdim}(X, R) = 0 \)). For example, the identity map \( \text{id} \) and the \( Z^k \)-shift \( \sigma \) on \([0, 1]^{Z^k} \) generate a (non-expansive) zero entropy \( Z^{k+1} \)-action on \([0, 1]^{Z^k} \) although the mean dimension \( \text{mdim}\left([0, 1]^{Z^k}, \sigma\right) \) is positive (equal to one).

**Example 1.10.** Let \( h : T^r \to T^r \) be a hyperbolic toral automorphism as in Example 1.3 (1), where \( r = 2 \). Let \( \Lambda \) be a subset of \( Z \). The **upper Banach density** of \( \Lambda \) is given by

\[
D := \lim_{N \to \infty} \sup_{n \in Z} \frac{|\Lambda \cap [n, n+N)|}{N} \in [0, 1].
\]

This limit exists because \( \sup_{n \in Z} |\Lambda \cap [n, n+N)| \) is a subadditive function in \( N \). Let \( A \subset T^2 \) be a non-empty closed \( h \)-invariant subset such that the topological entropy \( h_{top}(A, h) \) of the restriction of \( h \) on \( A \) is zero. For example we can choose \( A = \{ \text{the fixed point of} \ h \} \). From Fathi’s theorem (Theorem 1.2), \( A \) is zero dimensional. Let \( Y_0 \) be the set of points \( x = (x_n)_{n \in Z} \) in \( X = (T^r)^Z \) satisfying

\[
\forall n \in Z \setminus \Lambda : \quad x_n \in A.
\]

This is \( h_Z \)-invariant but not \( \sigma \)-invariant. We define \( Y \subset X \) by

\[
Y = \bigcup_{n \in Z} \sigma^n (Y_0).
\]

\( Y \) is invariant under both \( h_Z \) and \( \sigma \). So they generate an expansive \( Z^2 \)-action on \( Y \). The topological entropy of this \( Z^2 \)-action is given by

\[
h_{\text{top}}(Y, \sigma, h_Z) = h_{\text{top}}(h) D.
\]

Here \( h_{\text{top}}(h) \) is the topological entropy of \( h : T^r \to T^r \). The above formula can be verified by a direct computation, or by using the following well-known formula (see [Pa12, Lemma 3.1]):

\[
h_{\text{top}}(Y, \sigma, h_Z) = \lim_{n \to \infty} \frac{1}{N} h_{\text{top}} (\pi_N(Y), h_N),
\]
where \( \pi_N : Y \to (\mathbb{T}^r)^{\{1,\ldots,N\}} \) is the obvious projection map and \( h_N : \pi_N(Y) \to \pi_N(Y) \) is defined by applying \( h \) on each coordinate.

On the other hand the mean dimension of \( \sigma \) on \( Y \) is given by

\[
\text{mdim}(Y, \sigma) = rD.
\]

This formula relies on the fact that \( \text{Widim}_\varepsilon(\mathbb{T}^m, \ell^\infty) = m \) for sufficiently small \( \varepsilon \) independent of \( m \), where \( \ell^\infty \) is the metric on \( \mathbb{T}^m \) that comes from \( \| \cdot \|_\infty \) norm on \( \mathbb{R}^m \) (see Lemma 5.1 below).

We see that in this case \( \text{mdim}(Y, \sigma) \) becomes zero exactly when \( h_{\text{top}}(Y, \sigma, h_Z) \) is zero. This behavior is (of course) compatible with the statement of Theorem 1.8. We also would like to remark that if \( A \) is an infinite set then the topological entropy of \( \sigma \) on \( Y \) is always infinite regardless of the value of \( D \). Thus it is not the topological entropy \( h_{\text{top}}(Y, \sigma) \) but the mean dimension \( \text{mdim}(Y, \sigma) \) that reflects the circumstances properly.

1.2. Jointly expansive automorphisms of \( Z^k \)-actions. Here we discuss the materials in §1.1 from the viewpoint of automorphisms of dynamical systems. An advantage of this approach is that we can also apply it to general amenable group actions. (See §1.4 below.)

Definition 1.11. Let \( T : Z^k \times X \to X \) be a continuous action (not necessarily expansive) on a compact metric space \( X \).

(1) A homeomorphism \( f : X \to X \) is called an automorphism of \((X, T)\) if it commutes with the \( T \)-action: \( T^u \circ f = f \circ T^u \) for all \( u \in Z^k \).

(2) An automorphism \( f \) of \((X, T)\) is said to be jointly expansive if \( f \) and \( T \) generate an expansive \( Z^{k+1} \)-action.

Example 1.12. The following are examples of existence/non-existence of jointly expansive automorphisms:

(1) Example 1.3 (1) shows that the shift \( \sigma \) on \((\mathbb{T}^2)^Z\) admits a jointly expansive automorphism \( h_Z \).

(2) The \( Z^k \)-shift \( \sigma \) on \([0,1]^Z\) does not admit a jointly expansive automorphism: If \( f : [0,1]^Z \to [0,1]^Z \) is a jointly expansive automorphism, then it yields an expansive homeomorphism on a fixed point set \( \text{Fix}(\sigma) \) of \( \sigma \). But \( \text{Fix}(\sigma) \) is homeomorphic to the unit interval \([0,1]\), which does not admit an expansive homeomorphism (cf. [KH95, Proposition 1.1.6]).

Example 1.12 (2) shows that the set of periodic points are (sometimes) obstructions to the existence of jointly expansive automorphisms. But if a system is free (i.e. it has no periodic points), then we cannot use this obstruction. Mean dimension provides another obstruction:

Corollary 1.13. If a \( Z^k \)-action \((X, T)\) admits a jointly expansive automorphism, then \( \text{mdim}(X, T) \) is finite.
This is an immediate corollary of Theorem 1.4. By using the method of Lindenstrauss–Weiss [LW00, Proposition 3.5], we can easily construct plenty of examples of free (and, moreover, minimal) $\mathbb{Z}^k$-actions of infinite mean dimension. Such systems do not admit jointly expansive automorphisms although they have no periodic points.

Of course, in general, neither periodic points nor mean dimension provide a sufficient criterion for the existence of a jointly expansive automorphism. For example, an irrational rotation on the circle does not admit a jointly expansive automorphism\footnote{A circle homeomorphism commuting with an irrational rotation must be a rotation. It is proved in [SZ05, Theorem 3.1] that the circle does not admit an expansive $\mathbb{Z}^k$-action for any $k \geq 1.$} although it is free and zero mean dimensional.

1.3. **On expansive and minimal $\mathbb{Z}^k$-actions and mean dimension of lower rank subgroups.** As we briefly noted in the beginning, the original motivation of Mañé came from Bowen’s work [Bow70]. Bowen [Bow70] proved that hyperbolic minimal sets of a diffeomorphism are always zero dimensional. Mañé [Ma79] generalized this to a more abstract setting (see also Artigue [Ar15] for a recent new proof):

**Theorem 1.14** (Mañé, 1979). *If $f : X \to X$ is an expansive and minimal homeomorphism on a compact metric space $X$, then $X$ is zero dimensional. Here $f$ is said to be minimal if the orbit $\{f^n x\}_{n \in \mathbb{Z}}$ is dense in $X$ for every $x \in X.$*

Contrary to Theorems 1.1 and 1.2, we do not currently have an appropriate multiparameter version of Theorem 1.14. The following results preclude some seemingly plausible generalizations:

**Proposition 1.15.** *There exists a positive mean dimensional $\mathbb{Z}$-action $(X, T)$ admitting a jointly expansive and minimal automorphism $f : X \to X.$*

**Proposition 1.16.** *There exists a minimal and expansive $\mathbb{Z}^2$-action with the property that for every line $L \subset \mathbb{R}^2$ the directional mean dimension of the action with respect to $L$ is positive.*

*Directional mean dimension* is a mean dimension analogue of *directional entropy* (Milnor [Mi88] and Boyle–Lind [BL97]) that was suggested recently by Lind. It counts the averaged dimension of $(X, T)$ along the $L$ direction. We will define it in §6.1. Propositions 1.15 and 1.16 both show that a for $k > 1$, a rank $(k - 1)$-subaction of a minimal and expansive $\mathbb{Z}^k$-action need not have zero mean dimension, in contrast to the case $k = 1$. Furthermore, proposition 1.15 shows that this can happen even when an single element of $\mathbb{Z}^2$ acts minimally, and proposition 1.16 shows in particular that a minimal and expansive $\mathbb{Z}^2$-action can have positive mean dimension for *every* element of $\mathbb{Z}^2$.

We will prove proposition 1.15 in §5 and proposition 1.16 in §6. The question remains:

**Problem 1.17.** Is there a reasonable generalization of Theorem 1.14 to $\mathbb{Z}^k$-actions?
1.4. Noncommutative versions. We can consider generalizations of §1.1 and §1.2 to noncommutative group actions. Let \((X, d)\) be a compact metric space.

**Polynomial growth groups:** Let \(G\) and \(H\) be finitely generated groups of polynomial growth. We denote by \(\text{deg}(G)\) and \(\text{deg}(H)\) the degrees of the polynomial growth of \(G\) and \(H\) respectively (e.g. \(\text{deg}(\mathbb{Z}^k) = k\)).

**Theorem 1.18.** Let \(T : G \times X \to X\) and \(R : H \times X \to X\) be continuous actions which commute with each other. Suppose \(T\) is expansive, namely there exists \(c > 0\) such that any distinct \(x, y \in X\) satisfy \(\sup_{g \in G} d(T^g x, T^g y) > c\).

1. Suppose \(\text{deg}(G) = \text{deg}(H) + 1\). Then:
   - (a) The mean dimension \(\text{mdim}(X, R)\) is finite.
   - (b) If the topological entropy of \(T\) is zero then \(\text{mdim}(X, R) = 0\).

2. Suppose \(\text{deg}(G) = \text{deg}(H)\). Then:
   - (a) The topological entropy of \(R\) is finite.
   - (b) If the topological entropy of \(T\) is zero then the topological entropy of \(R\) is also zero.

3. Suppose \(\text{deg}(G) < \text{deg}(H)\). Then the topological entropy of \(R\) is zero.

The case (1) is the most nontrivial case with respect to the viewpoint of mean dimension theory. The mean dimension of \(R\) is zero in the cases (2) and (3) because finite topological entropy systems are zero mean dimensional. Indeed, as we saw at the end of Example 1.10, finiteness of topological entropy is a strictly stronger condition than zero mean dimensionality.

**Remark 1.19.** The case (b) of (2) and the case (3) above were already proved by Shereshevsky [She96].

**Amenable groups:** Amenable groups may have exponential growth. So we cannot apply the framework of Theorem 1.18 to general amenable groups. However the formulation in §1.2 using automorphisms can be naturally generalized to amenable group actions. Let \(G\) be a finitely generated amenable group and \(T : G \times X \to X\) a continuous action. A homeomorphism \(f : X \to X\) is called an automorphism of \((X, T)\) if it commutes with the \(T\)-action. An automorphism \(f\) is said to be jointly expansive if there exists \(c > 0\) satisfying \(\sup_{n \in \mathbb{Z}, g \in G} d(f^n T^g x, f^n T^g y) > c\) for any two distinct \(x, y \in X\).

**Theorem 1.20.** Suppose a \(G\)-action \((X, T)\) admits a jointly expansive automorphism \(f\). Then:

1. The mean dimension \(\text{mdim}(X, T)\) is finite.

2. If the topological entropy of the \(G \times \mathbb{Z}\)-action generated by \(T\) and \(f\) is zero, then the mean dimension \(\text{mdim}(X, T)\) is zero.
The proofs of Theorems 1.18 and 1.20 are straightforward generalizations of the proofs of Theorems 1.4 and 1.8. But we have not so far found any interesting phenomena specific to the noncommutative case. So the main body of the paper concentrates on the case of $\mathbb{Z}^k$-actions and we omit the detailed explanations of the noncommutative case. We believe that experienced readers will not find any difficulty to extend the arguments of §3 and §4 to noncommutative group actions.

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2. Mean dimension

Here we review basics of mean dimension. Readers can find (much) more information in [Gro99, LW00, Lin99].

Let $(X, d)$ be a compact metric space. Let $U = \{U_i\}_{i \in I}$ be an open cover of $X$. We define $\text{mesh}(U, d)$ as the supremum of $\text{diam}(U_i)$ over $U_i \in U$. We define the order $\text{ord}(U)$ as the maximum integer $n \geq 0$ such that there exist pairwise distinct $i_0, i_1, \ldots, i_n \in I$ satisfying $U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_n} \neq \emptyset$. An open cover $V = \{V_j\}_{j \in J}$ of $X$ is called a refinement of $U$ if for every $j \in J$ there exists $i \in I$ satisfying $V_j \subset U_i$. We define the degree $D(U)$ as the minimum of $\text{ord}(V)$ over all refinements $V$ of $U$.

For two open covers $U = \{U_i\}_{i \in I}$ and $V = \{V_j\}_{j \in J}$ of $X$ we define a new open cover $U \vee V$ by

$$U \vee V = \{U_i \cap V_j | i \in I, j \in J\}.$$

We can check that [LW00, Corollary 2.5]

$$D(U \vee V) \leq D(U) + D(V). \tag{2.1}$$

Suppose $\mathbb{Z}^k$ continuously acts on $X$ by $T : \mathbb{Z}^k \times X \to X$. We define the mean dimension $\text{mdim}(X, T)$ by

$$\text{mdim}(X, T) = \sup_{U: \text{open cover of } X} \left( \lim_{N \to \infty} \frac{D\left( \bigvee_{u \in [-N,N]^k \cap \mathbb{Z}^k} T^{-u}U \right)}{(2N + 1)^k} \right). \tag{2.2}$$

This limit exists because of the subadditivity (2.1). The mean dimension is a topological invariant of $(X, T)$.
The above formulation (2.2) was introduced by [LW00]. Another formulation (closer to the original definition of [Gro99]) will be also useful later (§3): For $\varepsilon > 0$ we define the $\varepsilon$-width dimension $\text{Widim}_\varepsilon(X, d)$ as the minimum of $\text{ord}(U)$ over all open covers $U$ of $X$ satisfying $\text{mesh}(U, d) \leq \varepsilon$. Given an action $T : \mathbb{Z}^k \times X \to X$ and a subset $\Omega \subset \mathbb{R}^k$, the distance $d^T_{\Omega}$ on $X$ is defined by

$$d^T_{\Omega}(x, y) = \sup_{u \in \Omega \cap \mathbb{Z}^k} d(T^u x, T^u y).$$

When there is no ambiguity about the action $T$ we will write $d_{\Omega} = d^T_{\Omega}$.

Then $\text{mdim}(X, T)$ is given by

$$(2.3) \text{mdim}(X, T) = \lim_{\varepsilon \to 0} \left( \lim_{N \to \infty} \frac{\text{Widim}_\varepsilon \left( X, d^T\left[ -N, N \right]^k \right)}{(2N + 1)^k} \right).$$

The equivalence of (2.2) and (2.3) easily follows from the consideration on the Lebesgue number. The above definitions are all we need in §3. So readers may skip the rest of this section and directly go to §3.

Next we introduce metric mean dimension [LW00]. This will be used in §4. Let $\varepsilon > 0$ and $(X, d)$ a compact metric space. We define $\#(X, d, \varepsilon)$ as the minimum cardinality $|U|$ of open covers $U$ of $X$ satisfying $\text{mesh}(U, d) < \varepsilon$. Let $T : \mathbb{Z}^k \times X \to X$ be a continuous action. We define the entropy $S(X, T, d, \varepsilon)$ at the scale $\varepsilon > 0$ by

$$(2.4) S(X, T, d, \varepsilon) := \lim_{N \to \infty} \frac{\log \# \left( X, d^T\left[ -N, N \right]^k, \varepsilon \right)}{(2N + 1)^k} = \inf_{N \geq 1} \log \frac{\# \left( X, d^T\left[ -N, N \right]^k, \varepsilon \right)}{(2N + 1)^k}.$$  

The second equality follows from a standard deviation argument. The topological entropy $h_{\text{top}}(T)$ is given by

$$(2.5) h_{\text{top}}(T) = \lim_{\varepsilon \to 0} S(X, T, d, \varepsilon).$$

We define the upper/lower metric mean dimensions $\underline{\text{mdim}}(X, T, d)$ and $\overline{\text{mdim}}(X, T, d)$ by

$$\underline{\text{mdim}}(X, T, d) = \limsup_{\varepsilon \to 0} \frac{S(X, T, d, \varepsilon)}{\log(1/\varepsilon)}, \quad \overline{\text{mdim}}(X, T, d) = \liminf_{\varepsilon \to 0} \frac{S(X, T, d, \varepsilon)}{\log(1/\varepsilon)}.$$  

The following is a fundamental theorem [LW00, Theorem 4.2].

**Theorem 2.1** (Lindenstrauss–Weiss, 2000).

$$\text{mdim}(X, T) \leq \underline{\text{mdim}}(X, T, d) \leq \overline{\text{mdim}}(X, T, d).$$
3. First proof of Theorem 1.4

Here we prove Theorem 1.4 by adapting Mañé’s method [Ma79, pp. 318-319] to the settings of mean dimension theory. Throughout this section we assume that \((X, d)\) is a compact metric space with an expansive action \(T : \mathbb{Z}^k \times X \to X\), and that \(R : \mathbb{Z}^{k-1} \times X \to X\) is another action that commutes with \(T\). We choose \(c > 0\) such that any two distinct points \(x, y \in X\) satisfy

\[
\sup_{u \in \mathbb{Z}^k} d(T^u x, T^u y) > 2c. \quad (3.1)
\]

**Lemma 3.1.** There exists \(\delta > 0\) such that if \(N \geq 1\) and \(x, y \in X\) satisfy

\[
c \leq d_{[-N,N]^k}^T(x, y) \leq 2c,
\]

then \(d_{\partial[-N,N]^k}^T(x, y) > \delta\). Here \(\partial[-N,N]^k\) is the boundary of \([-N,N]^k\), i.e. it is given by

\[
\bigcup_{i=1}^k \{ x \in [-N,N]^k | x_i \in \{-N,N\} \}.
\]

**Proof.** Suppose the statement is false: There exist \(N_n \geq 1\) and \(x_n, y_n \in X\) \((n \geq 1)\) satisfying

\[
c \leq d_{[-N_n,N_n]^k}^T(x_n, y_n) \leq 2c, \quad \lim_{n \to \infty} d_{\partial[-N_n,N_n]^k}^T(x_n, y_n) = 0.
\]

Then \(N_n \to \infty\) as \(n \to \infty\) and there exists \(a_n \in [-N_n,N_n]^k\) satisfying \(d(T^{a_n} x_n, T^{a_n} y_n) \geq c\). It follows from \(\lim_{n \to \infty} d_{\partial[-N_n,N_n]^k}^T(x_n, y_n) = 0\) that the distance between \(a_n\) and \(\partial[-N_n,N_n]^k\) goes to infinity as \(n \to \infty\). We can assume that \(T^{a_n} x_n \to x\) and \(T^{a_n} y_n \to y\) by choosing subsequences (if necessary). Then \(d(x, y) \geq c\) and \(\sup_{u \in \mathbb{Z}^k} d(T^u x, T^u y) \leq 2c\). This contradicts (3.1). \(\square\)

**Lemma 3.2.** For any \(\varepsilon > 0\) there exists \(m = m(\varepsilon) > 0\) such that if \(x, y \in X\) satisfy

\[
d_{[-m,m]^k}^T(x, y) \leq 2c,
\]

then \(d(x, y) < \varepsilon\).

**Proof.** Suppose the statement is false: There exist \(\varepsilon > 0\) and \(x_n, y_n \in X\) \((n \geq 1)\) satisfying \(d_{[-n,n]^k}^T(x_n, y_n) \leq 2c\) and \(d(x_n, y_n) \geq \varepsilon\). Choose subsequences \(\{x_n\}\) and \(\{y_n\}\) converging to some \(x\) and \(y\) respectively. Then \(\sup_{u \in \mathbb{Z}^k} d(T^u x, T^u y) \leq 2c\) and \(d(x, y) \geq \varepsilon\), which contradicts (3.1). \(\square\)

**Proposition 3.3.**

\[
\limsup_{N \to \infty} \text{Widim}_{2c} \left( X, d_{[-N,N]^k}^T \right) < \infty.
\]
Proof. Let $\delta > 0$ be the constant introduced in Lemma 3.1. Choose an open cover $\mathcal{U} = \{U_1, \ldots, U_L\}$ of $X$ with $\text{mesh}(\mathcal{U}, d) < \delta$. For $N \geq 1$ we consider the open cover

$$\bigvee_{u \in \partial[-N,N]^k} T^{-u}\mathcal{U}.$$  

It follows from (2.1) in §2 that

$$\mathcal{D}\left(\bigvee_{u \in \partial[-N,N]^k} T^{-u}\mathcal{U}\right) \leq |Z^k \cap \partial[-N,N]^k| \cdot \mathcal{D}(\mathcal{U}) \leq 2^k(2N + 1)^{k-1}L.$$  

Thus there exists a refinement $\mathcal{V}_N$ of (3.2) satisfying $\text{ord}(\mathcal{V}_N) \leq 2^k(2N + 1)^{k-1}L$.

Take $V \in \mathcal{V}_N$. We define an equivalence relation on $V$ as follows: For $x, y \in V$ we write $x \sim_V y$ if there exists a finite sequence $x_0, x_1, \ldots, x_n$ in $V$ satisfying

$$x_0 = x, \quad x_n = y, \quad \forall 0 \leq i < n : d^T_{[-N,N]^k}(x_i, x_{i+1}) < c.$$  

Let $V = V_1 \cup \cdots \cup V_{a(V)}$ be the decomposition into the equivalence classes. Set $\mathcal{W}_N = \{ V_i | V \in \mathcal{V}_N, 1 \leq i \leq a(V) \}$. That is, $\mathcal{W}_N$ is obtained from $\mathcal{V}_N$ by breaking its elements into “c-approximately connected components” with respect to the metric $d^T_{[-N,N]^k}$. This is an open cover of $X$ with

$$\text{ord}(\mathcal{W}_N) = \text{ord}(\mathcal{V}_N) \leq 2^k(2N + 1)^{k-1}L.$$  

Claim 3.4.

$$\text{mesh}(\mathcal{W}_N, d^T_{[-N,N]^k}) \leq 2c.$$  

Proof. Suppose the statement is false: There exist $V \in \mathcal{V}_N$ and $x, y \in V$ satisfying $x \sim_V y$ and $d^T_{[-N,N]^k}(x, y) > 2c$. It follows from the definition of $\sim_V$ that we can find $x_0, \ldots, x_n$ in $V$ satisfying (3.3). Then some $x_i$ must satisfy $c \leq d^T_{[-N,N]^k}(x_i, x_{i+1}) \leq 2c$. Since $\mathcal{V}_N$ is a refinement of (3.2), it also satisfies

$$d^T_{\partial[-N,N]^k}(x, x_i) \leq \text{mesh}(\mathcal{U}, d) < \delta.$$  

This contradicts Lemma 3.1. \hfill $\Box$

From Claim 3.4 and (3.4)

$$\text{Widim}_{2c}\left( X, d^T_{[-N,N]^k} \right) \leq \text{ord}(\mathcal{W}_N) \leq 2^k(2N + 1)^{k-1}L.$$  

Thus we get

$$\limsup_{N \to \infty} \frac{\text{Widim}_{2c}\left( X, d^T_{[-N,N]^k} \right)}{N^{k-1}} \leq 2^{2k-1}L.$$  

The following lemma enables us to control the $R$-action by the information of the $T$-action. This is contained in Shereshevsky [She96, Lemma 2.2].
Lemma 3.5. There exists $K > 0$ such that for every $N > 0$ the following holds: If $x, y \in X$ satisfy

$$d^T_{[-K,N,K]}(x, y) \leq 2c,$$

then

$$d^R_{[-N,N]}(x, y) \leq 2c.$$

Proof. There exists $\varepsilon > 0$ such that $d(R^u x, R^u y) \leq 2c$ for all $u \in \{-1, 0, 1\}^{k-1}$ whenever $d(x, y) < \varepsilon$. By Lemma 3.2 there exists $K > 0$ so that $d^T_{[-K,K]}(x, y) \leq 2c$ implies $d(x, y) < \varepsilon$. For $A \times B \subset \mathbb{R}^{k-1} \times \mathbb{R}^k$ write

$$d^{(R,T)}_{A \times B}(x, y) = \max_{u \in A \cap \mathbb{Z}^{k-1}, v \in B \cap \mathbb{Z}^k} d(R^u(T^v(x)), R^u(T^v(y))).$$

Suppose $d^T_{[-K,N,K]}(x, y) \leq 2c$. Then it follows by induction on $0 \leq j \leq N$ that

$$d^{(R,T)}_{[-j,j]^{k-1} \times [-K(N-j),K(N-j)]}(x, y) \leq 2c.$$

The claim of the lemma now follows by setting $j = N$. □

Proof of Theorem 1.4. The mean dimension $\text{mdim}(X, R)$ is given by

$$\lim_{\varepsilon \to 0} \left( \lim_{N \to \infty} \frac{\text{Widim}_\varepsilon(X, d^R_{[-N,N]^{k-1}})}{(2N+1)^{k-1}} \right).$$

By Lemma 3.2 for every $\varepsilon > 0$ there exists $m = m(\varepsilon) > 0$ so that

$$d^{(R,T)}_{[-N,N]^{k-1} \times [-m,m]}(x, y) \leq 2c \implies d^R_{[-N,N]}(x, y) < \varepsilon.$$

By Lemma 3.5

$$d^T_{[-K,N-m,K+N+m]}(x, y) \leq 2c \implies d^{(R,T)}_{[-N,N]^{k-1} \times [-m,m]}(x, y) \leq 2c.$$

Hence

$$\text{Widim}_\varepsilon(X, d^R_{[-N,N]^{k-1}}) \leq \text{Widim}_{2c}(X, d^T_{[-K,N-m,K+N+m]}).$$

Noting $m = m(\varepsilon)$ is independent of $N$

$$\lim_{N \to \infty} \frac{\text{Widim}_\varepsilon(X, d^R_{[-N,N]^{k-1}})}{(2N+1)^{k-1}} \leq \frac{(K/2)^{k-1}}{N^{k-1}} \limsup_{N \to \infty} \frac{\text{Widim}_{2c}(X, d^T_{[-N,N]^{k}})}{N^{k-1}}.$$

By Proposition 3.3 the right-hand side is finite and independent of $\varepsilon$. Thus $\text{mdim}(X, R)$ is finite □

Remark 3.6. A similar argument shows that there exists $C < \infty$ such that every rank $(k-1)$ subgroup $A \subset \mathbb{Z}^k$ satisfies

$$\text{mdim}(X, T|_A) \leq C \lim_{N \to \infty} \frac{N^{k-1}}{|A \cap [-N,N]^k|}.$$
4. Second proof of Theorem 1.4 and the proof of Theorem 1.8

Here we prove Theorems 1.4 and 1.8 by adapting Fathi’s method [Fa89, Section 5] to our settings.

4.1. Frink’s metrization theorem. Here we review a classical theorem of Frink [Fr37].

**Theorem 4.1** (Frink, 1937). Let $X$ be a set and let $\rho$ a nonnegative function on $X \times X$ satisfying

1. $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$.
2. $\rho(x, y) = 0$ if and only if $x = y$.
3. $\rho(x, z) \leq 2 \max(\rho(x, y), \rho(y, z))$ for all $x, y, z \in X$.

Then there exists a distance function $D$ on $X$ satisfying

$$\frac{\rho(x, y)}{4} \leq D(x, y) \leq \rho(x, y), \quad (\forall x, y \in X).$$

Here “distance function” means that it satisfies the above (1), (2) and the triangle inequality.

**Proof.** We explain Frink’s nice proof [Fr37, pp. 134-135] for readers’ convenience.

**Claim 4.2.** For $x_0, x_1, \ldots, x_n \in X$ with $n \geq 2$

$$\rho(x_0, x_n) \leq 2\rho(x_0, x_1) + 4 \sum_{i=1}^{n-2} \rho(x_i, x_{i+1}) + 2\rho(x_{n-1}, x_n).$$

When $n = 2$, the right-hand side is just $2\rho(x_0, x_1) + 2\rho(x_1, x_2)$.

Assuming this claim for the moment, we prove the theorem. Let $x, y \in X$. We define $D(x, y)$ as the infimum of $\sum_{i=0}^{n-1} \rho(x_i, x_{i+1})$ over all $x_0, x_1, \ldots, x_n \in X$ with $x_0 = x$ and $x_n = y$. It immediately follows that $D$ satisfies $D(x, y) = D(y, x), D(x, y) \leq \rho(x, y)$ and the triangle inequality. The inequality $\rho(x, y)/4 \leq D(x, y)$ follows from Claim 4.2. Then $D$ satisfies (2) and becomes a distance function.

Now we start the proof of Claim 4.2. The proof is the induction on $n$. The statement is true for $n = 2$ by the condition (3). Let $N \geq 3$ and suppose the statement is true for $n \leq N - 1$. Let $x_0, \ldots, x_N \in X$. We can assume $x_0 \neq x_N$. We define $m \in [1, N]$ as the minimum integer satisfying $\rho(x_0, x_m) > \rho(x_m, x_N)$. If $m = 1$ then the statement follows from (3) because

$$\rho(x_0, x_N) \leq 2\max(\rho(x_0, x_1), \rho(x_1, x_N)) = 2\rho(x_0, x_1).$$

If $m = N$ then $\rho(x_0, x_{N-1}) \leq \rho(x_{N-1}, x_N)$ and hence the statement follows from

$$\rho(x_0, x_N) \leq 2\max(\rho(x_0, x_{N-1}), \rho(x_{N-1}, x_N)) = 2\rho(x_{N-1}, x_N).$$

So we assume $2 \leq m \leq N - 1$. This implies

$$\rho(x_0, x_{m-1}) \leq \rho(x_{m-1}, x_N), \quad \rho(x_0, x_m) > \rho(x_m, x_N).$$
Hence by (3)
\[ \rho(x_0, x_N) \leq 2 \max (\rho(x_0, x_m), \rho(x_m, x_N)) = 2\rho(x_0, x_m), \]
\[ \rho(x_0, x_N) \leq 2 \max (\rho(x_0, x_{m-1}), \rho(x_{m-1}, x_N)) = 2\rho(x_{m-1}, x_N). \]
By adding these two inequalities, we get
\[ \rho(x_0, x_N) \leq \rho(x_0, x_m) + \rho(x_{m-1}, x_N). \]
By the induction hypothesis,
\[ \rho(x_0, x_m) \leq 2\rho(x_0, x_1) + 4 \sum_{i=1}^{m-2} \rho(x_i, x_{i+1}) + 2\rho(x_{m-1}, x_m), \]
\[ \rho(x_{m-1}, x_N) \leq 2\rho(x_{m-1}, x_m) + 4 \sum_{i=m}^{N-2} \rho(x_i, x_{i+1}) + 2\rho(x_{N-1}, x_N). \]
By adding these two inequalities, we get the statement of the claim. \( \Box \)

4.2. Proofs of Theorems 1.4 and 1.8. Throughout this subsection we assume that 
\((X, d)\) is a compact metric space with an expansive action \(T : \mathbb{Z}^k \times X \to X\), and that 
\(R : \mathbb{Z}^{k-1} \times X \to X\) commutes with \(T\). We choose \(c > 0\) such that any two distinct points 
\(x, y \in X\) satisfy
\[ \sup_{u \in \mathbb{Z}^k} d(T^u x, T^u y) > c. \]
Fix an integer \(l > 0\) such that if \(x, y \in X\) satisfy \(d(x, y) \geq c/2\) then there exists \(u \in \mathbb{Z}^k\) 
with \(|u| \leq l\) satisfying \(d(T^u x, T^u y) \geq c\). Fix \(\alpha > 1\) with \(\alpha^l < 2\).

Let \(x, y \in X\). We define
\[ n(x, y) = \min \{ n \geq 0 \mid \exists u \in \mathbb{Z}^k : |u| \leq n \text{ and } d(T^u x, T^u y) \geq c \}. \]
If \(x = y\) then we set \(n(x, y) = \infty\). We set \(\rho(x, y) = \alpha^{-n(x,y)}\).

Lemma 4.3. The function \(\rho\) satisfies:

1. \(\rho(x, y) = \rho(y, x)\).
2. \(\rho(x, y) = 0\) if and only if \(x = y\).
3. \(\rho(x, z) \leq 2 \max(\rho(x, y), \rho(y, z))\).
4. If \(x_n \to x\) and \(y_n \to y\) in \(X\) as \(n \to \infty\) then
   \[ \limsup_{n \to \infty} \rho(x_n, y_n) \leq \rho(x, y). \]
5. \(\rho\) is compatible with the topology of \(X\). Namely, the balls (with respect to \(\rho\))
   \[ B_r(x, \rho) = \{ y \in X \mid \rho(x, y) < r \} \quad (x \in X, r > 0) \]
   form an open base of the topology of \(X\).
Thus $\rho_n, N$. Then for any $v$ \(|v| \leq l\) with $d(T^{u+v}x, T^{u+v}y) \geq c$. Then $n(x, y) \leq m + l$ and hence

$$\rho(x, y) = \alpha^{-n(x, y)} \geq \alpha^{-m(\alpha - l)} > \frac{\rho(x, z)}{2},$$

where we used $\alpha' < 2$.

(4) It is straightforward to check $\liminf_{n \to \infty} n(x_n, y_n) \geq n(x, y)$.

(5) It follows from the property (4) above that the balls $B_r(x, \rho)$ are open (with respect to $d$). Expansiveness implies that for any $x \in X$ and $R > 0$ there exists $r > 0$ satisfying $B_r(x, \rho) \subset B_R(x, d)$. Then the statement can be easily checked.

By the properties (1), (2), (3) of Lemma 4.3 and Frink’s metrization theorem (Theorem 4.1) we can find a distance function $D$ on $X$ satisfying

$$\frac{\rho(x, y)}{4} \leq D(x, y) \leq \rho(x, y).$$

By the property (5) of Lemma 4.3, the distance $D$ is compatible with the topology of $X$.

**Lemma 4.4.** If $n \geq 1$ and $x, y \in X$ satisfy $\max_{|u| < n} D(T^u x, T^u y) < 1/(4\alpha)$ then $D(x, y) < \alpha^{-n}$.

**Proof.** It follows $\max_{|u| < n} \rho(T^u x, T^u y) < 1/\alpha$ and hence $\min_{|u| < n} n(T^u x, T^u y) > 1$. This implies $n(x, y) > n$ and $D(x, y) \leq \rho(x, y) < \alpha^{-n}$. \(\square\)

Let $x \neq y$ be two points in $X$. There exists $v \in \mathbb{Z}^k$ with $d(T^v x, T^v y) \geq c$ and hence $\rho(T^v x, T^v y) = 1$. Thus

$$\sup_{u \in \mathbb{Z}^k} D(T^u x, T^u y) \geq \frac{1}{4} > \frac{1}{4\alpha}.$$ 

By the same argument as in Lemma 3.5, we can prove that there exists $K \geq 1$ such that for any $N \geq 1$

$$D^T_{[-K, N, K]^k}(x, y) < \frac{1}{4\alpha} \implies D^R_{[-N, N]^{k-1}}(x, y) < \frac{1}{4\alpha}.$$

Then for any $n, N \geq 1$

$$D^T_{[-KN-n, KN+n]^k}(x, y) < \frac{1}{4\alpha} \implies D^{(R, T)}_{[-N, N]^{k-1} \times [-n, n]^k}(x, y) < \frac{1}{4\alpha} \implies D^R_{[-N, N]^{k-1}}(x, y) < \alpha^{-n} \quad \text{(by Lemma 4.4)}.$$

Thus

$$(4.1) \quad \# \left(X, D^R_{[-N, N]^{k-1}, \alpha^{-n}}\right) \leq \# \left(X, D^T_{[-KN-n, KN+n]^k}, 1/(4\alpha)\right).$$
Proof of Theorems 1.4 and 1.8. We will prove

\[
\text{mdim}(X, R, D) \leq 2(K + 1)^k \frac{h_{\text{top}}(T)}{\log \alpha}.
\]

This shows Theorems 1.4 and 1.8 because \(h_{\text{top}}(T) < \infty\) for every expansive action and \(\text{mdim}(X, R) \leq \text{mdim}(X, R, D)\) by Theorem 2.1.

Let \(\delta > 0\) be arbitrary. It follows from the definition of the topological entropy that there exists \(M > 0\) such that for any \(N \geq M\)

\[
\frac{1}{(2N + 1)^k} \log \left(\# \left( X, D_{[-(K+1)N, (K+1)N]^k}^T, 1/(4\alpha) \right) \right) < (K + 1)^k (h_{\text{top}}(T) + \delta).
\]

Let \(0 < \varepsilon < \alpha^{-M}\). Choose \(N \geq M\) with \(\alpha^{-N} < \varepsilon \leq \alpha^{-N+1}\). By using (4.1) with \(n = N\)

\[
\# \left( X, D_{[-N, N]^k}^R, \varepsilon \right) \leq \# \left( X, D_{[-N, N]^k}^R, \alpha^{-N} \right) \leq \# \left( X, D_{[-K^N, K^N]^k}^T, 1/(4\alpha) \right).
\]

Thus

\[
\log \left(\# \left( X, D_{[-N, N]^k}^R, \varepsilon \right) \right) < (2N + 1)^k (K + 1)^k (h_{\text{top}}(T) + \delta).
\]

By the second equality of (2.4)

\[
S(X, R, D, \varepsilon) < (2N + 1)(K + 1)^k (h_{\text{top}}(T) + \delta).
\]

From \(\varepsilon \leq \alpha^{-N+1}\),

\[
N - 1 \leq \frac{\log(1/\varepsilon)}{\log \alpha}.
\]

Thus \(S(X, R, D, \varepsilon)\) is bounded by

\[
\left( \frac{2 \log(1/\varepsilon)}{\log \alpha} + 3 \right) (K + 1)^k (h_{\text{top}}(T) + \delta).
\]

So we get

\[
\text{mdim}(X, R, D) = \limsup_{\varepsilon \to 0} \frac{S(X, R, D, \varepsilon)}{\log(1/\varepsilon)} \leq 2(K + 1)^k \frac{h_{\text{top}}(T) + \delta}{\log \alpha}.
\]

Since \(\delta > 0\) is arbitrary, this proves (4.2).

4.3. Remark on entropy and metric mean dimension. The idea of the previous subsection is roughly summarized by the following correspondence:

Topological entropy of \(\mathbb{Z}^k\)-actions \(\leftrightarrow\) Metric mean dimension of \(\mathbb{Z}^{k-1}\)-actions.

We will give one remark on this correspondence. Let \((X, T)\) be a \(\mathbb{Z}^k\)-action (not necessarily expansive) and let \(d\) be a metric that generates the topology of \(X\) such that
**Proposition 4.5.** Under the above circumstances,

\[ h_{\text{top}}(T, f) \leq \log^+ L \cdot \text{mdim}(X, T, d), \]

where the left-hand side is the topological entropy of the \( \mathbb{Z}^k \times \mathbb{Z}_{\geq 0} \)-action generated by \( T \) and \( f \).

**Proof.** For \( \Omega \subset \mathbb{R}^k \times \mathbb{R}_{\geq 0} \) we define a distance \( d_\Omega \) on \( X \) by

\[ d_\Omega(x, y) = \sup_{(u, n) \in \Omega \cap (\mathbb{Z}^k \times \mathbb{Z}_{\geq 0})} d(T^u \circ f^n(x), T^u \circ f^n(y)). \]

Take \( K > \max(1, L) \). Take \( \varepsilon_0 > 0 \) such that if \( d(x, y) < \varepsilon_0 \) then \( d(f(x), f(y)) < Kd(x, y) \).

If \( U \subset X \) satisfies \( \text{diam} \left( (U, d([-N,N]^k \times \{0\}) < \varepsilon/K^n \right) \) for some \( n, N > 0 \) and \( 0 < \varepsilon < \varepsilon_0 \) then \( \text{diam} \left( (U, d([-N,N]^k \times [0,n]) \right) < \varepsilon \).

Hence for \( 0 < \varepsilon < \varepsilon_0 \)

\[ \#(X, d([-N,N]^k \times [0,n]), \varepsilon) \leq \#(X, d([-N,N]^k \times \{0\}), \varepsilon/K^n). \]

We choose positive numbers \( \varepsilon_1 > \varepsilon_2 > \cdots \to 0 \) satisfying

\[ \lim_{i \to \infty} \frac{S(X, T, d, \varepsilon_i)}{\log(1/\varepsilon_i)} = \text{mdim}(X, T, d). \]

Fix \( 0 < \varepsilon < \varepsilon_0 \). We choose integers \( n_i \to \infty \) satisfying \( \varepsilon_i K^{n_i} \leq \varepsilon < \varepsilon_i K^{n_i-1} \). Then for every \( i \geq 1 \)

\[ \#(X, d([-N,N]^k \times [0,n_i]), \varepsilon) \leq \#(X, d([-N,N]^k \times [0,n_i]), \varepsilon_i K^{n_i}) \leq \#(X, d([-N,N]^k \times \{0\}), \varepsilon_i). \]

From \( \varepsilon < \varepsilon_i K^{n_i-1} \),

\[ n_i > \frac{\log(1/\varepsilon_i) + \log \varepsilon + \log K}{\log K}. \]

It follows that

\[ \lim_{N \to \infty} \frac{\log \#(X, d([-N,N]^k \times [0,n_i]), \varepsilon)}{(n_i + 1)(2N + 1)^k} \leq \frac{S(X, T, d, \varepsilon_i)}{\log(1/\varepsilon_i)} \cdot \frac{\log K \cdot \log(1/\varepsilon_i)}{\log(1/\varepsilon_i) + \log \varepsilon + 2 \log K}. \]

Letting \( i \to \infty \)

\[ S(X, (T, f), d, \varepsilon) \leq \text{mdim}(X, T, d) \cdot \log K. \]

Since \( 0 < \varepsilon < \varepsilon_0 \) and \( K > \max(1, L) \) are arbitrary, this proves the statement. \( \square \)

When \( k = 0 \), Proposition 4.5 is just a standard relation between topological entropy and box dimension ([Fa89, Theorem 5.6], [KH95, Theorem 3.2.9]).

---

\(^3\)Namely \( f \) is a continuous map (not necessarily invertible) from \( X \) to \( X \) satisfying \( f \circ T^u = T^u \circ f \) for all \( u \in \mathbb{Z}^k \).
Example 4.6. Let $M$ be a compact $C^1$-manifold (of finite dimension). Consider the $\mathbb{Z}^k$-shift $\sigma$ on $M^{\mathbb{Z}^k}$. Let $A \subset \mathbb{Z}^k$ be a finite set and $F : M^A \to M$ a $C^1$-map. We define an endomorphism $f : M^{\mathbb{Z}^k} \to M^{\mathbb{Z}^k}$ of $\sigma$ by the smooth local rule $F$:

$$f ((x_u)_{u \in \mathbb{Z}^k}) = (F((x_v)_{v \in u+A}))_{u \in \mathbb{Z}^k}.$$ 

Then the topological entropy $h_{\text{top}}(\sigma, f)$ is finite: Take some distance $d$ on $M$ which comes from a Riemmanian metric and define a distance $D$ on $M^{\mathbb{Z}^k}$ by

$$D(x, y) = \sum_{u \in \mathbb{Z}^k} 2^{-|u|} d(x_u, y_u).$$ 

The metric mean dimension $\text{mdim}(M^{\mathbb{Z}^k}, \sigma, D)$ is equal to $\dim M$ (and hence finite) and the map $f$ is Lipschitz with respect to $D$.

5. Proof of Proposition 1.15

Here we prove Proposition 1.15. Our construction is based on Lindenstrauss–Weiss [LW00, Proposition 3.5].

5.1. Width dimension. Let $M = \mathbb{R}^2 / \mathbb{Z}^2$ with the standard flat distance $d$. (The diameter of $M$ is $1/\sqrt{2}$.) We denote by $\ell^\infty$ the sup-distance on the product space $M^n$. For $x = (x_0, \ldots, x_{n-1}), y = (y_0, \ldots, y_{n-1}) \in M^n$

$$\ell^\infty(x, y) = \max_{0 \leq i \leq n-1} d(x_i, y_i).$$

Lemma 5.1. For $0 < \varepsilon < 1/2$

$$\text{Widim}_\varepsilon(M^n, \ell^\infty) = 2n.$$ 

Proof. There exists a distance-nondecreasing continuous map from $([0, 1/2]^{2n}, \text{sup-distance})$ to $(M^n, \ell^\infty)$. Hence

$$\text{Widim}_\varepsilon(M^n, \ell^\infty) \geq \text{Widim}_\varepsilon([0, 1/2]^{2n}, \text{sup-distance}).$$

The right-hand side is equal to $2n$ for $\varepsilon < 1/2$ by [LW00, Lemma 3.2].

5.2. Proof of Proposition 1.15. Let $h : M \to M$ be a hyperbolic toral automorphism. An important fact for us is that periodic points of $h$ are dense in $M$. We define $h_n : M^n \to M^n$ by $h_n(x_0, \ldots, x_{n-1}) = (h(x_0), \ldots, h(x_{n-1}))$. Consider the two-sided infinite product $M^\mathbb{Z}$ and let $\sigma : M^\mathbb{Z} \to M^\mathbb{Z}$ be the shift: $\sigma(x)_n = x_{n+1}$. Define $h_\mathbb{Z} : M^\mathbb{Z} \to M^\mathbb{Z}$ by $h_\mathbb{Z}(x) = (h(x_n))_{n \in \mathbb{Z}}$. The transformations $\sigma$ and $h_\mathbb{Z}$ generate an expansive $\mathbb{Z}^2$-action on $M^\mathbb{Z}$. We will construct an appropriate subsystem $X$. We define a distance $D$ on $M^\mathbb{Z}$ by

$$D(x, y) = \sum_{n \in \mathbb{Z}} 2^{-|n|} d(x_n, y_n).$$ 

For $x = (x_n)_{n \in \mathbb{Z}} \in M^\mathbb{Z}$ and integers $a < b$ we denote

$$x^b_a = (x_a, x_{a+1}, \ldots, x_b).$$
Let \( A \subset M^n \) be a \( h_n \)-invariant closed subset. We define a \( \mathbb{Z}^2 \)-invariant closed subset \( X(A) \subset M^Z \) as the set of \( x \) satisfying

\[
\exists l \in \mathbb{Z} : \forall m \in \mathbb{Z} : \quad x_{l+(m+1)n-1}^{l+mn} \in A.
\]

In particular if \( A = M \) then \( X(A) = M^Z \).

We inductively define positive integers \( L_n \) and closed \( h_{3^n L_0 \cdots L_n} \)-invariant sets \( A_n \subset M^{3^n L_0 \cdots L_n} \) such that

\[
\text{(5.1) periodic points of } h_{3^n L_0 \cdots L_n} \text{ are dense in } A_n.
\]

First we set \( L_0 = 1 \) and \( A_0 = M \). Suppose we have already defined \( L_n \) and \( A_n \). It follows from (5.1) that there exists a finite set \( B_n = \{ y^{(1)}, \ldots, y^{(b_n)} \} \) in \( A_n^3 \subset M^{3^n L_0 \cdots L_n} \) such that

- \( B_n \) is \((1/n)\)-dense in \( A_n^4 \) with respect to the distance \( \ell^\infty \), namely, for every \( x \in A_n^3 \) there exists \( y \in B_n \) with \( \ell^\infty(x, y) < 1/n \).
- \( B_n \) is \( h_{3^{n+1} L_0 \cdots L_n} \)-invariant. In particular every point \( y^{(i)} \) is \( h_{3^{n+1} L_0 \cdots L_n} \)-periodic.

We choose \( L_{n+1} \) sufficiently larger than \( b_n \). (Indeed \( L_{n+1} > 2^{n+1} b_n \) is enough. But the detail of the choice is not important.) We define closed (but not necessarily invariant) set \( C_n \subset (A_n^3)^{L_{n+1}} \) as the set of points \( x = (x_0, \ldots, x_{L_{n+1}-1}) \), \( x_i \in A_n^3 \), satisfying

\[
x_{L_{n+1}-i} = y^{(i)} \quad (\forall 1 \leq i \leq b_n).
\]

We define a closed invariant set \( A_{n+1} \subset (A_n^3)^{L_{n+1}} \) by

\[
A_n = \bigcup_{m=0}^{\infty} h_{3^{n+1} L_0 \cdots L_{n+1}}^{m} (C_n), \quad \text{(this becomes a finite union)}.
\]

Periodic points of \( h_{3^{n+1} L_0 \cdots L_{n+1}} \) are dense in \( A_{n+1} \). So we can continue the induction.

The closed \( \mathbb{Z}^2 \)-invariant sets \( X(A_n) \) form a decreasing sequence:

\[
M^Z = X(A_0) \supset X(A_1) \supset X(A_2) \supset \ldots.
\]

We set \( X = \bigcap_{n=0}^{\infty} X(A_n) \). This is a closed \( \mathbb{Z}^2 \)-invariant set of \( M^Z \).

**Claim 5.2.** For any \( x \in X(A_n) \) and any \( y \in X(A_{n+1}) \) there exists \( p \in \mathbb{Z} \) satisfying

\[
D(x, \sigma^p y) < \frac{3}{n} + 2^{1-3^n L_0 \cdots L_n}.
\]

Since the right-hand side goes to zero as \( n \to \infty \), this shows that \( (X, \sigma) \) is minimal.

**Proof.** Let \( x = (x_m)_{m \in \mathbb{Z}} \in X(A_n) \) \( (x_m \in M) \). There exists \( l \in \mathbb{Z} \) such that

\[
\forall m \in \mathbb{Z} : \quad x_{l+(m+1)n-1}^{l+mn} \in A_n.
\]

We can assume \(-2 \cdot 3^n L_0 \cdots L_n < l \leq -3^n L_0 \cdots L_n \). Since \( B_n \) is \((1/n)\)-dense in \( A_n^3 \), we can find \( y^{(i)} \in B_n \) which is \((1/n)\)-close to \( x_{l+(m+1)n-1}^{l+mn} \) with respect to the sup-distance \( \ell^\infty \).
From the definition of $A_{n+1}$, any point $y \in X (A_{n+1})$ “contains” $y^{(i)}$ somewhere, namely there exists $q \in \mathbb{Z}$ with $y_{q+3^n L_0 \cdots L_{n-1}} = y^{(i)}$. Then

$$D(x, \sigma^q y) \leq \sum_{m=0}^{l+3^n L_0 \cdots L_{n-1}-1} 2^{-|m|} \varepsilon^{x^{(l+3^n L_0 \cdots L_{n-1}), y^{(i)}}} + \sum_{m<l \text{ or } m \geq l+3^n L_0 \cdots L_n} 2^{-|m|} \leq \frac{3}{n} + \sum_{|m|>3^n L_0 \cdots L_n} 2^{-|m|} = \frac{3}{n} + 2^{-3^n L_0 \cdots L_n}.

□

**Claim 5.3.** The mean dimension $\text{mdim}(X, \sigma \circ h_Z)$ is positive.

**Proof.** For $N \geq 1$ we define a distance $D_N$ on $X$ by

$$D_N(x, y) = \max_{0 \leq m < N} D((\sigma \circ h_Z)^m x, (\sigma \circ h_Z)^m y).$$

The mean dimension $\text{mdim}(X, \sigma \circ h_Z)$ is given by

$$\text{mdim}(X, \sigma \circ h_Z) = \lim_{\varepsilon \to 0} \left( \lim_{N \to \infty} \frac{\text{Widim}_\varepsilon(X, D_N)}{N} \right).$$

We inductively define $I_n \subset \{0, 1, 2, \ldots, 3^n L_0 \cdots L_n - 1\}$ by $I_0 = \{0\}$ and

$$I_{n+1} = \bigcup_{m=0}^{3L_{n+1}-3b_n-1} (m3^n L_0 \cdots L_n + I_n).$$

Roughly, $I_n$ is the positions of “free variables” of $A_n \subset M^{3^n L_0 \cdots L_n}$. We have $|I_{n+1}| = (3L_{n+1} - 3b_n)|I_n|$. Hence for $n \geq 1$

$$|I_n| = (3L_n - 3b_{n-1})(3L_{n-1} - 3b_{n-2}) \cdots (3L_1 - 3b_0).$$

Choose a point $z \in X$ satisfying $z_{m3^n L_0 \cdots L_n} \in A_n$ for all $m \in \mathbb{Z}$ and $n \geq 0$. We define a continuous map $f : M^{I_n} \to X$ by

$$f(x)_m = \begin{cases} h^{-m}(x_m) & (m \in I_n) \\ z_m & (m \notin I_n). \end{cases}$$

For $x, y \in M^{I_n}$

$$\ell^\infty(x, y) \leq D_{3^n L_0 \cdots L_n} (f(x), f(y)).$$

Then for $0 < \varepsilon < 1/2$

$$\text{Widim}_\varepsilon(X, D_{3^n L_0 \cdots L_n}) \geq \text{Widim}_\varepsilon(M^{I_n}, \ell^\infty) = 2|I_n| \quad \text{(Lemma 5.1)}.$$

So

$$\lim_{n \to \infty} \text{Widim}_\varepsilon(X, D_{3^n L_0 \cdots L_n}) \geq \lim_{n \to \infty} \frac{2|I_n|}{3^n L_0 \cdots L_n} = 2 \prod_{n=1}^{\infty} \left( 1 - \frac{b_{n-1}}{L_n} \right).$$

Since we chose $L_n$ sufficiently larger than $b_{n-1}$, the right-hand side is positive. □
Proof of Proposition 1.15. Set \( T = \sigma \circ h_Z \) and \( f = \sigma \). Then \( (X, T) \) is positive mean dimensional (Claim 5.3) and has a jointly expansive and minimal (Claim 5.2) automorphism \( f \). This proves the proposition. \( \square \)

Remark 5.4. Here are some remarks on the construction:

1. The transformation \( f = \sigma \) is also positive mean dimensional on \( X \).
2. We can slightly modify the above construction so that \( T = \sigma \circ h_Z \) also becomes minimal on \( X \). The modified construction is roughly as follows: We choose \( L_{n+1} \) sufficiently larger than \( b_n^2 \) and define \( C_n \subset (A_n^3)^{L_{n+1}} \) as the set of points \( x = (x_0, \ldots, x_{L_{n+1}-1}) \), \( x_i \in A_n^3 \), satisfying
   \[
   x_{L_{n+1}-ib_n}^{L_{n+1}-1-(i-1)b_n} = (y_i^{(i)}, \ldots, y_i^{(i)}) \quad (\forall 1 \leq i \leq b_n).
   \]
   We define \( A_{n+1} \) and \( X \) as before.
3. The action of \( h_Z \) on \( X \) is zero mean dimensional. So the above \( X \) does not provide the proof of Proposition 1.16. We will prove it by modifying the above construction. A basic idea is as follows: We consider \( X \times X \) with the \( \mathbb{Z}^2 \)-action defined by
   \[
   (m, n) \cdot (x, y) = (\sigma^m h_n^m(x), \sigma^{m+n} h_n^m(y)).
   \]
   Then we can check that every directional mean dimension of this action is positive. But (5.2) is not minimal (or, at least, we cannot prove its minimality). We need to modify the construction so that it becomes minimal. In other words the first and second factor of (5.2) should be disjoint. This is the main task of §6.2.

6. Directional mean dimension and the proof of Proposition 1.16

6.1. Directional mean dimension. Here we introduce the notion of directional mean dimension by mimicking the definition of directional entropy [Mi88, BL97]. We recommend readers to review the definitions in §2. The concept of directional mean dimensional was suggested to us by Doug Lind.

Let \( (X, d) \) be a compact metric space and \( T : \mathbb{Z}^2 \times X \to X \) a continuous action. Let \( L \subset \mathbb{R}^2 \) be a line. Let \( r > 1/\sqrt{2} \) and set
\[
B_r(L) = \{ x \in \mathbb{R}^2 | \exists y \in L : |x - y| < r \}.
\]
We define the directional mean dimension \( \text{mdim}(X, T, L) \) by
\[
\text{mdim}(X, T, L) = \lim_{\epsilon \to 0} \left( \liminf_{N \to \infty} \frac{\text{Widim}_\epsilon \left( X, d_{B_r(L) \cap (-N,N)^2} \right)}{\text{Length} \left( L \cap (-N,N)^2 \right)} \right).
\]

Remark 6.1. The following properties can be easily checked:

1. The value of \( \text{mdim}(X, T, L) \) is independent of \( r > 1/\sqrt{2} \) and the choice of the distance \( d \) (compatible with the underlying topology).
(2) If $L$ and $L'$ are two parallel lines in $\mathbb{R}^2$ then $\text{mdim}(X, T, L) = \text{mdim}(X, T, L')$. So it is enough to consider lines passing through the origin.

(3) If $L = \mathbb{R}u$ for some $u \in \mathbb{Z}^2$ then

$$\text{mdim}(X, T, L) = |u| \text{mdim}(X, T|_{\mathbb{Z}u}).$$

Here $\text{mdim}(X, T|_{\mathbb{Z}u})$ is the mean dimension of the restriction of $T$ on the subgroup $\mathbb{Z}u \subset \mathbb{Z}^2$.

6.2. Proof of Proposition 1.16. Here we prove Proposition 1.16 by modifying the construction in §5.2. The argument is a bit more technical. We recommend readers to check Remark 5.4 (3).

We continue to use the notations introduced in §5.2. We briefly recall them: $M = \mathbb{R}^2/\mathbb{Z}^2$ with a hyperbolic toral automorphism $h$. The two-sided infinite product $M^\mathbb{Z}$ admit the shift $\sigma$ and $h_\mathbb{Z}$ (the component-wise action of $h$), which generate an expansive $\mathbb{Z}^2$-action. For a closed and $h_\mathbb{Z}$-invariant subset $A \subset M^n$ we defined the $\mathbb{Z}^2$-invariant closed subset $X(A) \subset M^\mathbb{Z}$. The torus $M$ has the standard flat distance $d$ and we defined the distance $D$ on $M^\mathbb{Z}$ by $D(x, y) = \sum 2^{-|n|}d(x_n, y_n)$.

We will inductively define positive integers $L_n$ and closed $h_{3^n L_0 \cdots L_n}$-invariant subsets $A_n$ and $A'_n$ in $M^{3^n L_0 \cdots L_n}$ such that periodic points of $h_{3^n L_0 \cdots L_n}$ are dense both in $A_n$ and $A'_n$. First we set $L_0 = 1$ and $A_0 = A'_0 = M$. Suppose we have already defined $L_n$, $A_n$ and $A'_n$. There exist $h_{3^{n+1} L_0 \cdots L_n}$-invariant subsets $B_n = \{y^{(1)}, \ldots, y^{(b_n)}\} \subset A_n^3$ and $B'_n = \{z^{(1)}, \ldots, z^{(b_n)}\} \subset (A'_n)^3$ such that $B_n$ and $B'_n$ are $(1/n)$-dense in $A_n^3$ and $(A'_n)^3$ respectively with respect to the distance $\ell_\infty$ on $M^{3^{n+1} L_0 \cdots L_n}$. We choose $a_n \geq b_n$ which is a period of all $y^{(i)}$ and $z^{(i)}$ (for simplicity of the notation we set $H = h_{3^n L_0 \cdots L_n}$): $H^{a_n} (y^{(i)}) = y^{(i)}$, $H^{a_n} (z^{(i)}) = z^{(i)}$.

We choose $L_{n+1}$ sufficiently larger than $a_n^2 b_n$. We define closed subsets $C_n \subset (A_n^3)^{L_{n+1}}$ and $C'_n \subset ((A'_n)^3)^{L_{n+1}}$ as follows: A point $x = (x_0, \ldots, x_{L_{n+1}-1})$, $x_i \in A_n^3$, belongs to $C_n$ if for all $1 \leq i \leq b_n$

$$x_{L_{n+1} - 1 - (i-1)a_n^2}^{L_{n+1} - 1 - (i-1)a_n^2} = \underbrace{(y^{(i)}, \ldots, y^{(i)})}_{a_n^2}.$$

A point $x = (x_0, \ldots, x_{L_{n+1}-1})$, $x_i \in (A'_n)^3$, belongs to $C'_n$ if for all $1 \leq i \leq b_n$

$$x_{L_{n+1} - 1 - (i-1)a_n^2}^{L_{n+1} - 1 - (i-1)a_n^2} = \underbrace{(z^{(i)}, \ldots, z^{(i)})}_{a_n^2}, \underbrace{H(z^{(i)}), \ldots, H(z^{(i)})}_{a_n}, \underbrace{H^{a_n-1}(z^{(i)}), \ldots, H^{a_n-1}(z^{(i)})}_{a_n}.$$

We define $A_{n+1}$ and $A'_{n+1}$ by

$$A_{n+1} = \bigcup_{m=0}^{\infty} h_{3^{n+1} L_0 \cdots L_{n+1}}^m (C_n), \quad A'_{n+1} = \bigcup_{m=0}^{\infty} h_{3^{n+1} L_0 \cdots L_{n+1}}^m (C'_n).$$
Claim 6.2. For any \( p \) there exist integers \( Z \) positive. We define a distance \( D \) on \( M^Z \times M^Z \) by

\[ T_1(x, y) = (\sigma(x), \sigma(y)), \quad T_2(x, y) = (h_Z(x), \sigma \circ h_Z(y)). \]

We will prove that the expansive \( Z^2 \)-action \((T_1, T_2)\) on \( X \times X'\) satisfies the statement of Proposition 1.16, namely it is minimal and its every directional mean dimension is positive. We define a distance \( D \) on \( M^Z \times M^Z \) by

\[ D((x, y), (z, w)) = \max(D(x, z), D(y, w)) \quad (x, y, z, w \in M^Z). \]

**Claim 6.2.** For any \((x, y) \in X(A_n) \times X(A_n')\) and \((z, w) \in X(A_n+1) \times X(A_n'+1)\) there exist integers \( p \) and \( q \) satisfying

\[ D((x, y), T_1^p \cdot T_2^q(z, w)) < \frac{3}{n} + 2^{1-3^p L_0 \cdots L_n}. \]

The right-hand side goes to zero as \( n \to \infty \). Therefore the \( Z^2 \)-action \((X \times X', (T_1, T_2))\) is minimal.

**Proof.** Let \( x = (x_m)_{m \in \mathbb{Z}}, \ y = (y_m)_{m \in \mathbb{Z}}, \ z = (z_m)_{m \in \mathbb{Z}} \) and \( w = (w_m)_{m \in \mathbb{Z}} \) with \( x_m, y_m, z_m, w_m \in M \). There exist integers \( l_1 \) and \( l_2 \) in \((-2 \cdot 3^p L_0 \cdots L_n, -3^p L_0 \cdots L_n)\) such that

\[ \forall m \in \mathbb{Z}: \ x_{l_1+m3^p L_0 \cdots L_n} - 1 \in A_n, \quad y_{l_2+m3^p L_0 \cdots L_n} - 1 \in A_n'. \]

Since \( B_n \) and \( B_n' \) are \((1/n)\)-dense in \( A_n^3 \) and \((A_n')^3\) respectively, we can find some points in \( B_n \) and \( B_n' \) (say, \( y^{(i)} \) and \( z^{(j)} \)) which are \((1/n)\)-close to \( x_{l_1+3^p+1 L_0 \cdots L_n} - 1 \) and \( y_{l_2+3^p+1 L_0 \cdots L_n} - 1 \) respectively.

By applying \( T_1 \) and \( T_2 \) to \((z, w)\) in an appropriate number of times, we can assume that there exist integers \( s_1 \) and \( s_2 \) such that

\[ z_0^{3^p+1 L_0 \cdots L_n a_n^2} - 1 = (H^{s_1}(y^{(i)}), \ldots, H^{s_1}(y^{(i)})), \]

\[ w_{l_2-l_1}^{l_2-l_1+3^p+1 L_0 \cdots L_n a_n^2} - 1 = (H^{s_2}(z^{(j)}), \ldots, H^{s_2}(z^{(j)}), H^{s_2+1}(z^{(j)}), \ldots, H^{s_2+(a_n-1)}(z^{(j)})), \]

By applying \( T_1 \) and \( T_2 \) to \((z, w)\) in an appropriate number of times again, we can assume

\[ z_0^{3^p+1 L_0 \cdots L_n-1} = y^{(i)}, \quad w_{l_2-l_1}^{l_2-l_1+3^p+1 L_0 \cdots L_n-1} = z^{(j)}. \]

Finally, by applying \( T_1^{-l_1} \) to \((z, w)\), we can assume

\[ z_{l_1}^{l_1+3^p+1 L_0 \cdots L_n-1} = y^{(i)}, \quad w_{l_2}^{l_2+3^p+1 L_0 \cdots L_n-1} = z^{(j)}. \]
Then
\[
D(x, z) < \sum_{m=l_1}^{l_1+3^n+1L_0\cdots L_{n-1}} 2^{-|m|} \frac{1}{n} + \sum_{m<l_1 \text{ or } m \geq l_1+3^n+1L_0\cdots L_n} 2^{-|m|},
\]
\[
D(y, w) < \sum_{m=l_2}^{l_2+3^n+1L_0\cdots L_{n-1}} 2^{-|m|} \frac{1}{n} + \sum_{m<l_2 \text{ or } m \geq l_2+3^n+1L_0\cdots L_n} 2^{-|m|}.
\]
Both are bounded by
\[
\frac{3}{n} + \sum_{|m|>3^nL_0\cdots L_n} 2^{-|m|} = \frac{3}{n} + 2^{1-3^nL_0\cdots L_n}.
\]

The proof of Proposition 1.16 is completed by the next claim.

**Claim 6.3.** For any line \(L \subseteq \mathbb{R}^2\) the directional mean dimension \(\text{mdim} (X \times X', (T_1, T_2), L)\) is positive.

**Proof.** As we remarked in Remark 6.1 (2), we can assume that \(L\) passes through the origin. For a subset \(\Omega \subseteq \mathbb{R}^2\) we define a distance \(D_\Omega\) on \(X \times X'\) by
\[
D_\Omega ((x, y), (z, w)) = \sup_{(m, n) \in \Omega \cap \mathbb{Z}^2} D(T_1^mT_2^n(x, y), T_1^mT_2^n(z, w)) = \sup_{(m, n) \in \Omega \cap \mathbb{Z}^2} \max (D(\sigma^m h^n_2(x), \sigma^m h^n_2(z)), D(\sigma^{m+n} h^n_2(y), \sigma^{m+n} h^n_2(w))).
\]
The directional mean dimension \(\text{mdim} (X \times X', (T_1, T_2), L)\) is given by
\[
\lim_{\varepsilon \to 0} \left( \lim_{N \to \infty} \frac{\text{Widim}_\varepsilon (X \times X', D_{B_1(L) \cap (-N,N)^2})}{\text{Length}(L \cap (-N, N)^2)} \right).
\]
This is proportional to
\[
(6.1) \quad \lim_{\varepsilon \to 0} \left( \lim_{N \to \infty} \frac{\text{Widim}_\varepsilon (X \times X', D_{B_1(L) \cap (-N,N)^2})}{N} \right).
\]
We will prove that (6.1) is positive for any \(L\).

We inductively define \(I_n \subset \{0, 1, 2, \ldots, 3^nL_0\cdots L_{n-1}\}\) by \(I_0 = \{0\}\) and
\[
I_{n+1} = \bigcup_{m=0}^{3L_{n+1} - 3a^2_n b_n - 1} (m3^n L_0 \cdots L_n + I_n).
\]
\(I_n\) is the positions of the free variables of \(A_n\). It follows that \(|I_{n+1}| = (3L_{n+1} - 3a^2_n b_n) |I_n|\) and hence
\[
\frac{|I_n|}{3^n L_0 \cdots L_n} = \left(1 - \frac{a^2_{n-1} b_{n-1}}{L_n}\right) \left(1 - \frac{a^2_{n-2} b_{n-2}}{L_{n-1}}\right) \cdots \left(1 - \frac{a^2_0 b_0}{L_1}\right).
\]
Since we chose \(L_{n+1}\) sufficiently larger than \(a^2_n b_n\), we can assume that for all \(n \geq 0\)
\[
(6.2) \quad \frac{|I_n|}{3^n L_0 \cdots L_n} > \frac{1}{2}.
\]
\{I_n\}_{n=0}^{\infty} forms an increasing sequence. We define $I \subset \mathbb{Z}$ as the union of all $I_n$.

**Subclaim 6.4.** For any $t \geq 1$ we have $|(0, t) \cap I| > t/4$.

**Proof.** Let $n \geq 0$ be the integer satisfying $3^n L_0 \cdots L_n \leq t < 3^{n+1} L_0 \cdots L_{n+1}$.

**Case 1:** $t < 2 \cdot 3^n L_0 \cdots L_n$. Then by (6.2)
\[
\frac{|(0, t) \cap I|}{t} \geq \frac{|I_n|}{t} > \frac{|I_n|}{2 \cdot 3^n L_0 \cdots L_n} > \frac{1}{4}.
\]

**Case 2:** $2 \cdot 3^n L_0 \cdots L_n \leq t \leq (3L_{n+1} - 3a_n^2 b_n)3^n L_0 \cdots L_n$. Take the integer $m$ satisfying $m3^n L_0 \cdots L_n \leq t < (m+1)3^n L_0 \cdots L_n$. Then
\[
m \geq \frac{t}{3^n L_0 \cdots L_n} - 1 \geq \frac{t}{2 \cdot 3^n L_0 \cdots L_n}.
\]
\[
\frac{|(0, t) \cap I|}{t} \geq m|I_n| \geq \frac{|I_n|}{m3^n L_0 \cdots L_n} \geq \frac{1}{4}.
\]

**Case 3:** $(3L_{n+1} - 3a_n^2 b_n)3^n L_0 \cdots L_n < t < 3^{n+1} L_0 \cdots L_{n+1}$. Then
\[
\frac{|(0, t) \cap I|}{t} = \frac{|I_{n+1}|}{t} > \frac{|I_{n+1}|}{3^{n+1} L_0 \cdots L_{n+1}} > \frac{1}{2}.
\]

\[\square\]

Choose points $z \in X$ and $w \in X'$ so that $z^{(m+1)3^n L_0 \cdots L_n} \in A_n$ and $w^{(m+1)3^n L_0 \cdots L_n} \in A'_n$ for all $m \in \mathbb{Z}$ and $n \geq 0$.

**Case 1.** Suppose $L = \{(t, \alpha t) | t \in \mathbb{R}\}$ with some $\alpha$. We assume $\alpha \geq 0$. (The case $\alpha < 0$ is the same.) We set $\beta = \max(1, \alpha)$. (We recommend readers to assume that $\alpha \geq 1$ and hence $\beta = \alpha$. This is a more important case.) Let $N \geq \beta$ be an integer. Notice that $(m, [\alpha m]) \in B_1(L) \cap (-N, N)^2$ for $m \in [0, N/\beta)$. By Subclaim 6.4, $|I \cap [0, N/\beta)| > N/(4\beta)$. We define a continuous map $f : M^{I \cap [0, N/\beta)} \to X \times X'$ by
\[
f(x)_m = \begin{cases} (h^{-[\alpha m]}(x_m), w_m) & (m \in I \cap [0, N/\beta)) \\ (z_m, w_m) & (m \notin I \cap [0, N/\beta)). \end{cases}
\]

Then for any $x, y \in M^{I \cap [0, N/\beta)}$
\[
\ell^{\infty}(x, y) \leq D_{B_1(L) \cap (-N, N)^2}(f(x), f(y)).
\]

It follows from Lemma 5.1 that for $0 < \varepsilon < 1/2$
\[
\text{Widim}_e \left( X \times X', D_{B_1(L) \cap (-N, N)^2} \right) \geq \text{Widim}_e \left( M^{I \cap [0, N/\beta)}, \ell^{\infty} \right) = 2|I \cap [0, N/\beta)| > \frac{N}{2\beta}.
\]
This shows that (6.1) is larger than or equal to $1/(2\beta)$.

**Case 2.** Suppose $L = \{(0, t) | t \in \mathbb{R}\}$. This case is essentially the same with Claim 5.3 because of the form
\[
T_2^m \ast (x) = (\ast, \sigma^m h_Z^m(x)).
\]
But we provide the proof for the completeness. Let $N$ be a natural number. We define a continuous map $f : M^{I \cap [0, N)} \to X \times X'$ by

$$f(x)_m = \begin{cases} (z_m, h^{-m}(x_m)) & (m \in I \cap [0, N)) \\ (z_m, w_m) & (m \notin I \cap [0, N)) \end{cases}$$

Then for any $x, y \in M^{I \cap [0, N)}$

$$\ell^\infty(x, y) \leq D_{B_1(L) \cap (-N,N)^2} (f(x), f(y)).$$

From Lemma 5.1, for $0 < \varepsilon < 1/2$

$$\operatorname{Widim}_\varepsilon \left( X \times X', D_{B_1(L) \cap (-N,N)^2} \right) \geq \operatorname{Widim}_\varepsilon \left( M^{I \cap [0, N)}, \ell^\infty \right) = 2|I \cap [0, N)| > \frac{N}{2}.$$ 

Hence (6.1) is larger than or equal to $1/2$. 

\[\square\]

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