A generalization of rotation of binary sequences and its applications to toggle dynamical systems

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Abstract

We study a simple generalization of the rotation (or circular shift) of the binary sequences. In particular, we show each orbit of this generalized rotation has a certain statistical symmetry. This generalized rotation naturally arises when we generalize the results of Joseph and Roby on a toggle dynamical system whose state space consists of independent sets on the path graphs.

Keywords: Combinatorics on words, Rotation, Toggle dynamical system.

1 Introduction and statement of the main result

The rotation (or circular shift) of the binary sequences is a fundamental operation which plays crucial roles in various kinds of computations. This paper introduces and studies a simple and natural generalization of the rotation of the binary sequences. Although the generalization is very simple, the generalized rotation exhibits very complicated orbit structures. Our main aim in this paper is to show each orbit of this generalized rotation has a certain statistical symmetry. Joseph and Roby answered affirmatively to the Propp’s question on a toggle dynamical system by reducing the toggle dynamical system to the dynamical system of the ordinary rotation. Our generalized rotation naturally arises when we generalize the Propp’s question. We will discuss this application later in Section 3.

Throughout this paper, $n$ denotes a positive integer, and $m$ denotes a positive integer not greater than $n$. For a word $w$ over a finite alphabet, $w_i$ denotes the $i$-th letter of $w$ and $w_{[i,j]}$ denotes the subword of the form $w_iw_{i+1} \cdots w_j$. The length $n$ of the word $w$ is denoted by $|w|$. For a finite alphabet $A$, the set of finite words over $A$ is denoted by $A^*$. The generalized rotation $\rho_m : \{0,1\}^n \rightarrow \{0,1\}^n$ is defined as follows: Let $w = w_0w_1w_2 \cdots w_{n-1} \in \{0,1\}^n$ be a word over $\{0,1\}$ of length $n$. Then,

$$\rho_m(w) = \begin{cases} w_{k+1}w_{k+2} \cdots w_{n-1}0 \underbrace{1\cdots 1}_m & \text{if there exists } k < m \text{ s.t. } w_{[0,k]} = \underbrace{1\cdots 1}_{k} 0, \\ w_mw_{m+1} \cdots w_{n-1} \underbrace{1\cdots 1} & \text{otherwise.} \end{cases}$$

It is clear that $\rho_1$ is the ordinary rotation and $\rho_m$ is a bijection. If $p$ is the smallest integer such that $\rho_m^p(w) = w$ then $p$ satisfies $p = \#\{\rho_m^k(w)| k \in \mathbb{Z}\}$. We call the sequence $w, \rho_m^1(w), \rho_m^2(w), \ldots, \rho_m^{p-1}(w)$ the $\rho_m$-orbit of $w$.

Example 1. When $m = 2$, we have

$$\rho_2(w) = \begin{cases} w_1w_2 \cdots w_n0 & w_0 = 0, \\ w_2w_3 \cdots w_n01 & w_0w_1 = 10, \\ w_2w_3 \cdots w_n11 & w_0w_1 = 11. \end{cases}$$

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The word $w = 101110 \in \{0,1\}^7$ have the $\rho_2$-orbit of length 12:

| $k$ | $\rho_2^k(w)$     |
|-----|-------------------|
| 0   | 1011110           |
| 1   | 1110011           |
| 2   | 1100111           |
| 3   | 0011111           |
| 4   | 0111110           |
| 5   | 1111100           |

| $k$ | $\rho_2^k(w)$     |
|-----|-------------------|
| 6   | 1110011           |
| 7   | 1001111           |
| 8   | 0111101           |
| 9   | 1110101           |
| 10  | 1011110           |
| 11  | 0101111           |

When $m = 3$, we have

$$\rho_3(w) = \begin{cases} w_1w_2\cdots w_n0 & w_0 = 0, \\ w_2w_3\cdots w_n01 & w_0w_1 = 10, \\ w_3w_4\cdots w_n011 & w_0w_1w_2 = 110, \\ w_3w_4\cdots w_n111 & w_0w_1w_2 = 111. \end{cases}$$

$w = 1011110$ has the $\rho_3$-orbit of $w$ of length 9:

| $k$ | $\rho_3^k(w)$     |
|-----|-------------------|
| 0   | 1011110           |
| 1   | 1111001           |
| 2   | 1001111           |
| 3   | 0111101           |
| 4   | 1111010           |
| 5   | 1010111           |
| 6   | 1011101           |
| 7   | 1110101           |
| 8   | 0101111           |

Our main aim in this paper is to show a statistical property of the cumulative sum

$$\rho_m^k(w)_0 + \rho_m^k(w)_1 + \cdots + \rho_m^k(w)_{j-1},$$

for $j = 1, 2, \ldots, n$ where $\rho_m^k(w)_i$ stands for the $i$-th letter (or bit) of the word $\rho^k(w)$. Here we recall the notation and definition of the multisets (see, e.g., [4]). A multiset is intuitively a set with repeated elements. More precisely, a multiset $M$ on a set $S$ is a pair $(S, \nu)$, where $\nu : S \to \mathbb{N}$ is the multiplicity function. For example $M = \{a, b, c\}$ is a multiset on $S = \{a, b, c\}$ with the multiplicity $\nu(a) = 1, \nu(b) = 2, \nu(c) = 3$. Let $a(w)$ be the number of the digits 1 in $w$. We define $L_m^{(j)}(w)$ as the multiset on $\{0, 1, \ldots, a(w)\}$ consisting of the left cumulative sums:

$$L_m^{(j)}(w) = \left\{ \sum_{i=0}^{j-1} \rho_m^k(w)_i \mid k = 0, 1, \ldots, p - 1 \right\}.$$
and \(R_m^{(j)}(w)\) the right cumulative sums:

\[
R_m^{(j)}(w) = \left\{ \sum_{i=0}^{j-1} \rho_k^i(w)_{n-i} \, \bigg| \, k = 0, 1, \ldots, p-1 \right\},
\]

for \(j = 1, 2, \ldots, n\), and, for convenience, we define \(L_m^{(0)} = R_m^{(0)} = \left\{ 0, 0, \ldots, 0 \right\}\), which consists of only 0s. In other words, we define

\[
\nu_{L_m^{(j)}}(s) = \# \left\{ k \in \{0, 1, \ldots, p-1\} \mid \sum_{i=0}^{j-1} \rho_k^i(w)_i = s \right\},
\]

and

\[
\nu_{R_m^{(j)}}(s) = \# \left\{ k \in \{0, 1, \ldots, p-1\} \mid \sum_{i=0}^{j-1} \rho_k^i(w)_{n-i-1} = s \right\}.
\]

We occasionally write simply \(L^{(j)}\) (resp. \(R^{(j)}\)) instead of \(L_m^{(j)}(w)\) (resp. \(R_m^{(j)}(w)\)) when the context is clear. Our main result in this paper is the following.

**Theorem 1.** Let \(w \in \{0, 1\}^n\) be a word over \(\{0, 1\}\) of length \(n\). Then, for \(j = 0, 1, \ldots, n-1\), as multisets

\[
L^{(j)} = R^{(j)},
\]

i.e., \(\nu_{L^{(j)}}(s) = \nu_{R^{(j)}}(s)\) for \(s \in \{0, 1, \ldots, a(w)\}\).

As an immediate corollary of this theorem, we have the following.

**Corollary 1.**

\[
\sum_{k=0}^{p-1} \rho_m^k(w)_{j} = \sum_{k=0}^{p-1} \rho_m^k(w)_{n-1-j}.
\]

**Example 2.** Let \(w = 1011110 \in \{0, 1\}^7\), same as in Example 1. Then we have the following tables of left cumulative sums and right cumulative sums.

| \(k\) \(\backslash\) \(j\) |  1   |  2   |  3   |  4   |  5   |  6   |  7   |
|-----------------|------|------|------|------|------|------|------|
| 0               | 1    | 1    | 2    | 3    | 4    | 5    | 5    |
| 1               | 1    | 1    | 2    | 3    | 4    | 5    | 5    |
| 2               | 1    | 1    | 1    | 2    | 3    | 4    | 5    |
| 3               | 0    | 1    | 2    | 3    | 4    | 4    | 5    |
| 4               | 1    | 2    | 3    | 4    | 4    | 5    | 5    |
| 5               | 1    | 1    | 2    | 2    | 3    | 4    | 5    |
| 6               | 1    | 1    | 2    | 3    | 4    | 4    | 5    |
| 7               | 1    | 2    | 3    | 3    | 4    | 4    | 5    |
| 8               | 0    | 1    | 1    | 2    | 3    | 4    | 5    |

| \(k\) \(\backslash\) \(j\) |  1   |  2   |  3   |  4   |  5   |  6   |  7   |
|-----------------|------|------|------|------|------|------|------|
| 0               | 0    | 0    | 1    | 2    | 3    | 4    | 5    |
| 1               | 1    | 2    | 3    | 4    | 4    | 5    | 5    |
| 2               | 1    | 1    | 2    | 2    | 3    | 4    | 5    |
| 3               | 1    | 1    | 2    | 3    | 4    | 4    | 5    |
| 4               | 1    | 2    | 3    | 3    | 4    | 4    | 5    |
| 5               | 0    | 1    | 1    | 2    | 3    | 4    | 5    |
| 6               | 1    | 1    | 1    | 2    | 3    | 4    | 5    |
| 7               | 1    | 2    | 3    | 3    | 4    | 4    | 5    |
| 8               | 1    | 1    | 1    | 2    | 3    | 4    | 5    |

Table 1: Left: table of left cumulative sums \(\sum_{i=0}^{j-1} \rho_m^k(w)_i\), Right: table of right cumulative sums \(\sum_{i=0}^{j-1} \rho_m^k(w)_{n-i-1}\)
We can summarize these tables by the frequency tables, that is, tables whose $j$-th column vector is

$$(\nu_{L(j)}(0), \nu_{L(j)}(1), \ldots, \nu_{L(j)}(5)) = (\nu_{R(j)}(0), \nu_{R(j)}(1), \ldots, \nu_{R(j)}(5)).$$

| $s \setminus j$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-----------------|---|---|---|---|---|---|---|---|
| 0               | 9 | 2 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1               | 0 | 7 | 6 | 2 | 0 | 0 | 0 | 0 |
| 2               | 0 | 0 | 3 | 4 | 3 | 0 | 0 | 0 |
| 3               | 0 | 0 | 0 | 3 | 4 | 3 | 0 | 0 |
| 4               | 0 | 0 | 0 | 0 | 2 | 6 | 7 | 0 |
| 5               | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 9 |

Figure 1: Left: Frequency table of cumulative sums $\sum_{i=0}^{j-1} \rho^k(1011110)_i$ for $j = 0, 1, 2, \ldots, 7$, i.e., table of $\nu_{L(j)}(s)$ and Right: its histogram representation for $j = 1, 2, \ldots, 6$.

Since the number $a(w)$ of digits 1 in $\rho^k(w)$ does not depend on $k$, we have

$$\rho^k_m(w_0) + \rho^k_m(w_1) + \cdots + \rho^k_m(w_{j-1}) = s \iff \rho^k_m(w_{n-1}) + \rho^k_m(w_{n-2}) + \cdots + \rho^k_m(w_j) = a(w) - s.$$  

Therefore, we have

$$\nu_{L(j)}(s) = \nu_{R(a-j)}(a(w) - s) = \nu_{L(a-j)}(a(w) - s).$$  

(3)

Figure 2: Histograms of the cumulative sums $\sum_{i=0}^{j-1} \rho^k_2(w)_i$, for $j = 25, 50, \ldots, 175$, where $w$ is a randomly chosen word of length 200 whose orbit length is 15010.
Example 3. Figure 2 shows the histogram representation of $L^{(j)}$ for a randomly chosen word $w$ of length 200, whose orbit length is 15010. Each of the 7 histograms in the figure shows the histogram of $L^{(j)}$ for $j = 25, 50, 75, \ldots, 175$, in which we can observe the relation (3).

Remark 1. By theorem 1 and the relation (3), if $n$ is even, then we have
\[ \nu_{L^{(n/2)}}(s) = \nu_{L^{(n/2)}}(a(w) - s), \]
where $a(w)$ is the number of digits 1 in $w$. In this sense, the frequency table of $L^{(n/2)}$ is symmetric for even $n$. Figure 3 shows the distributions of $L^{(800)}$ for two randomly chosen words $w$ of length 1600 whose orbit sizes are 5606832 and 5473312. The distribution seems to heavily depend on the choice of $w$. It is a fascinating problem to study the asymptotic behavior of large random sequences.

Remark 2. The reverse $R(w)$ of a word $w = w_1w_2\cdots w_n \in \{0, 1\}^n$ is defined by $R(w) = w_nw_{n-1}\cdots w_2w_1$. Then, it is clear that
\[ \rho^{-1} = R \circ \rho \circ R. \quad (4) \]
We remark that Theorem 1 can be easily shown if we assume the additional condition that the $\rho$-orbit of $w$ contains its reverse $R(w)$: Let $j$ be the smallest positive integer such that $R(w) = \rho^j(w)$ and let $O = \{\rho^k(w) \mid k \in \mathbb{Z}\}$. Then it is clear that $O = \{\rho^{-k}(w) \mid k \in \mathbb{Z}\}$ and $O = \{\rho^{j+k}(w) \mid k \in \mathbb{Z}\} = \{\rho^k \circ R(w) \mid k \in \mathbb{Z}\}$. From (4) and the fact $R^2$ is the identity map, we have $\rho^{-k} = R \circ \rho^k \circ R$. Therefore we have
\[ R(O) = \{R \circ \rho^k(w) \mid k \in \mathbb{Z}\} = \{R \circ \rho^k \circ R(w) \mid k \in \mathbb{Z}\} = \{\rho^{-k}(w) \mid k \in \mathbb{Z}\} = \{\rho^k(w) \mid k \in \mathbb{Z}\} = O, \]
which implies Theorem 1. However, there are words $w$ whose $\rho$-orbit does not contain $R(w)$. For example, the $\rho_2$-orbit of $w = 00100101$ is
\[ 00100101, 01001010, 10010100, 01010001, 10100010, 10001001, \]
which does not contain $R(w)$.
2 Proof of the main theorem

Let \( w = w_0w_1 \cdots w_{n-1} \in \{0, 1\}^n \) be a word of length \( n \). Then, the subword \( w_iw_{i+1} \cdots w_j \) of \( w \) is denoted by \( w_{[i,j]} \). Let \( m \) be a positive integer not greater than \( n \). We define an extension sequence of words \( w^{(0)}, w^{(1)}, w^{(2)}, \ldots \) as follows. We define \( w^{(0)} = w \), and for \( j \geq 0 \), \( w^{(j+1)} \) is obtained as an extension of \( w^{(j)} \) defined by

\[
  w^{(j+1)} = \begin{cases} 
    w^{(j)}01 \cdots 1 & \text{if there exists } k < m, \text{ s.t. } \rho^i(w)_{[0,k]} = 11 \cdots 10 \\
    w^{(j)}1 \cdots 1 & \text{otherwise}.
  \end{cases}
\]

Therefore \( w^{(k)} \) contains \( \rho_m^k(w) \) as its suffix. Let \( p \) be the size of \( \rho_m \)-orbit of \( w \). Then, \( p \) is the smallest non-negative integer such that \( w \) is a suffix of \( w^{(p)} \). We define \( \overline{w} = \overline{w_0w_1} \cdots \overline{w_{l-1}} \) to be the word obtained by removing the suffix \( w \) from \( w^{(p)} \). Thus, we can see \( \overline{w} \) as a compact representation of the \( \rho_m \)-orbit of \( w \). Let the sequence \( i_w = (i_0, i_1, \ldots, i_p) \) be defined by

\[
  i_k = |w^{(k)}| - n,
\]

where \( |w^{(k)}| \) is the length of the word \( w^{(k)} \). In other words, \( i_w = (i_0, i_1, \ldots, i_p) \) is the rising subsequence of \( (0, 1, \ldots, l) \) which satisfies

\[
  \rho_m^k(w) = w^{(p)}_{[i_k, i_{k+n-1}]},
\]

and therefore

\[
  w^{(p)}_{[i_k, i_{k+1}-1]} = \begin{cases} 
    11 \cdots 1 & i_{k+1} - i_k = m, \\
    \overline{11 \cdots 1 0} & \text{otherwise}.
  \end{cases}
\]

The main idea of the proof of Theorem 1 can be informally stated as follows. An element of the left hand side of (1) can be expressed in terms of \( \overline{w} \):

\[
  \sum_{i=0}^{j-1} \rho^k(w)_i = \sum_{i=0}^{j-1} \overline{w}_{ik+i},
\]

for \( j = 1, 2, \ldots, n \). A similar expression of elements of the right hand side of (1) can be obtained by using \( \rho^{-1} \) instead of \( \rho \). These expressions are used to prove the equality of these two multisets. The equality can be easily shown for \( j < m \), and the induction on \( j \) are used for \( j \geq m \).

**Example 4.** When \( m = 3 \) and \( w = 1011110 \), we have
Let \( l \) be the length of the word \( \overline{w} \) defined above. We divide the set \( I = \{0, 1, \ldots, l - 1\} \) of indices of \( \overline{w} \) into two disjoint subsets, \( I_0 = \{i \mid \overline{w}_i = 0\} \) and \( I_1 = \{i \mid \overline{w}_i = 1\} \). It is obvious that
\[
I = I_0 \cup I_1, \quad I_0 \cap I_1 = \emptyset.
\]
By (7), it is clear that \( I_0 \subset \{i_0 - 1, i_1 - 1, \ldots, i_{p-1} - 1\} \). We define
\[
I_T = I_1 \cap \{i_0 - 1, i_1 - 1, \ldots, i_{p-1} - 1\} = \left\{ i \in I \mid \overline{w}_{i-m+1, i} = 11 \ldots 1 \right\},
\]
and \( I_H = I_1 \setminus I_T \), where \( i - m + 1 \) is considered to be in \( I \) by taking modulo \( l \). Thus we have a decomposition, \( I = I_0 \cup I_H \cup I_T \). One of the most important properties of this decomposition is
\[
\{i_0, i_1, \ldots, i_{p-1}\} = \{k + 1 \mid k \in I_0 \cup I_T\},
\]
from which we obtain another expression of \( L_{m, k}^{(j)}(w) \):
\[
L_{m, k}^{(j)}(w) = \left\{ \sum_{i=0}^{j-1} \overline{w}_{k+i+1} \right\} \frac{1}{k \in I_0 \cup I_T}.
\]

**Example 5.** Let \( w = 1011110 \) and \( m = 3 \). Then, as we have shown in Example 4, \( \rho_3 \)-orbit of \( w \) is of length 9 and \( \overline{w} = 10111101111011110 \), which is indexed as follows:

\[
\begin{array}{cccccccccccccccc}
  i & i_0 & i_1 & i_2 & i_3 & i_4 & i_5 & i_6 & i_7 & i_8 \\
\hline
  \overline{w}_i & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0
\end{array}
\]

As we have also shown in Example 4, \( (i_0, i_1, \ldots, i_9) = (0, 2, 5, 7, 8, 11, 13, 15, 18, 19) \). Thus we have
\[
I_0 = \{1, 6, 7, 12, 14, 18\}, \quad I_H = \{0, 2, 3, 5, 8, 9, 11, 13, 15, 16\}, \quad I_T = \{4, 11, 17\}.
\]
Let \( j \) be a non-negative integer not greater than \( n \) and \( a, b \in \{0, H, T\} \). Then we define a multiset \( M_{(a,b)}^{(j)}(w) \) by

\[
M_{(a,b)}^{(j)}(w) = \left\{ \sum_{i=0}^{j-1} \overline{w}_{k+i} \mid k \in I_a, k + j - 1 \in I_b \right\}.
\]

We occasionally write simply \( M_{a,b}^{(j)} \) to express \( M_{m,(a,b)}^{(j)}(w) \) when \( m \) and \( w \) are fixed. Then, the left hand side \( L_m^{(j)}(w) \) of (11) has the following decomposition:

\[
L^{(j)} = \left( \bigcup_{b \in \{0,T,H\}} M_{0,b}^{(j+1)} \right) \cup \left( \bigcup_{b \in \{0,T,H\}} M_{T,b}^{(j+1)} - 1 \right),
\]

where \( M - 1 \) denotes the multiset \( \{m - 1 \mid m \in M\} \) for a multiset \( M \) of integers.

**Example 6.** Let \( w = 1011110 \) and \( m = 3 \), the same as the previous examples. Table 2 summarizes \( M_{m,(a,b)}^{(3)}(w) \) for \( w = 1011110 \) and \( m = 3 \). For instance, as we have seen in Example 5, \( I_0 = \{1,6,7,12,14,18\} \), and hence \((I_0 + 2) \cap I_0 = \{14,1\}\). Therefore we have \( M_{0,0,0}^{(3)} = \{\overline{w}_{12} + \overline{w}_{13} + \overline{w}_{14} = 1, \; \overline{w}_{18} + \overline{w}_{0} + \overline{w}_{1} = 1\} \). \( I_T = \{4,10,17\} \), \( I_H = \{0,2,3,5,8,9,11,13,15,16\} \) and hence \((I_T + 2) \cap I_H = \{0\}\). Therefore, we have \( M_{T,H}^{(3)} = \{\overline{w}_{17} + \overline{w}_{18} + \overline{w}_{0} = 2\} \). Table 2 shows that

\[
L^{(2)} = \left( \bigcup_{b \in \{0,T,H\}} M_{0,b}^{(3)} \right) \cup \left( \bigcup_{b \in \{0,T,H\}} M_{T,b}^{(3)} - 1 \right) = \{1,1\} \cup \{1,2,2,2\} \cup \{(2,2) - 1\} \cup \{(2) - 1\} = \{1,1,1,1,1,2,2,2\}.
\]

| \( a \setminus b \) | 0     | T     | H     |
|-----------------|------|------|------|
| 0               | \{1,1\} | \emptyset | \{1,2,2,2\} |
| T               | \{2,2\} | \emptyset | \{2\} |
| H               | \{1,2\} | \{3,3,3\} | \{2,2,2,3,3\} |

Table 2: Table of \( M_{3,(a,b)}^{(3)}(1011110) \).

To prove Theorem 1, we consider the *inverse* version of the above argument. Let \( w \in \{0,1\}^n \) be a word and \( p \) be the length of \( w \)’s \( \rho_m \)–orbit. We define a word \( \hat{w} = \hat{w}_1 \hat{w}_2 \cdots \hat{w}_{l-1} \) by removing the prefix \( w \) from \( w^{(p)} \). Then, we have

\[
\hat{w}_{[i_k-n,i_k]}^{(p)} = w_{[i_k,i_k+n-1]}^{(p)} = \rho_m^k(w).
\]

**Lemma 1.**

\[
\hat{w}_{[i_k,i_k+1-1]} = R(\overline{w}_{[i_k,i_k+1-1]}) = \begin{cases} 
11 \cdots 1 & i_{k+1} - i_k = m, \\
i_{k+1} - i_k & 011 \cdots 1, \text{ otherwise.}
\end{cases}
\]

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Proof. By (7) and
\[
\hat{w}_{[i_k,i_{k+1}-1]} = w_{[i_k+n,i_{k+1}+n]}^{(p)} = R \left( w_{[i_k,i_{k+1}-1]}^{(p)} \right) = R \left( w_{[i_k,i_{k+1}-1]} \right),
\]
where \(R(w)\) denotes the reverse of \(w\).

Example 7. Let \(w = 1011110\) and \(m = 3\). Then, as we have seen in Example 4, \(l = 19\), \(p = 9\), and \(w^{(p)} = 10111001111010111011110\), from which we remove the prefix \(w\) to obtain
\[
\hat{w} = 011110101110101110.
\]

Let \(l\) be the length of the word \(\hat{w}\) defined above, which is equal to the length of \(\overline{w}\). We divide the set \(I = \{0, 1, \ldots, l - 1\}\) of indices of \(\hat{w}\) into three disjoint subsets, \(\hat{I}_0, \hat{I}_T\) and \(\hat{I}_H\) defined as follows.
\[
\hat{I}_0 = \{i \mid \hat{w}_i = 0\}, \quad \hat{I}_1 = \{i \mid \hat{w}_i = 1\}.
\]

Then, by Lemma 1 it is clear that \(\hat{I}_0 \subset \{0, i_1, \ldots, i_{p-1}\}\). We subdivide \(\hat{I}_1\) into two disjoint subsets:
\[
\hat{I}_T = \hat{I}_1 \cap \{0, i_1, \ldots, i_{p-1}\}, \quad \hat{I}_H = \hat{I}_1 \setminus \hat{I}_T.
\]

One of the most important properties of this decomposition is \(\{0, i_1, \ldots, i_{p-1}\} = \hat{I}_0 \cup \hat{I}_T\), and therefore we obtain another expression of \(R_m^{(j)}(w)\):
\[
R_m^{(j)}(w) = \left\{ \sum_{i=0}^{j-1} \hat{w}_{k-i-1} \mid k \in I_0 \cup I_T \right\}.
\]

Example 8. Let \(w = 1011110\) and \(m = 3\). Then, as we have shown in Example 4, \(\hat{w} = 011110101110101110\) which has length \(l = 19\), we have
\[
\begin{array}{cccccccccccccccc}
i & i_0 & i_1 & i_2 & i_3 & i_4 & i_5 & i_6 & i_7 & i_8 \\
\hat{w}_i & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0
\end{array}
\]

Therefore \(\hat{I}_0 = \{0, 5, 7, 11, 13, 18\}\), \(\hat{I}_H = \{1, 3, 4, 6, 9, 10, 12, 14, 16, 17\}\), \(\hat{I}_T = \{2, 8, 15\}\).

Let \(j\) be a non-negative integer not greater than \(n\) and \(a, b \in \{0, T, H\}\). Then we define the multiset \(\mathcal{M}_{m,(a,b)}^{(j)}(w)\) by
\[
\mathcal{M}_{m,(a,b)}^{(j)}(w) = \left\{ \sum_{i=0}^{j-1} \hat{w}_{k-i} \mid k \in I_a, k - j + 1 \in I_b \right\}.
\]

We occasionally write simply \(\mathcal{M}_{a,b}^{(j)}\) to express \(\mathcal{M}_{m,(a,b)}^{(j)}(w)\) when \(m\) and \(w\) are fixed. Then, the right hand side \(R^{(j)}\) of (1) has the following decomposition:
\[
R^{(j)} = \bigcup_{b \in \{0,T,H\}} M_{0,b}^{(j+1)} \cup \bigcup_{b \in \{0,T,H\}} M_{T,b}^{(j+1)} - 1. \tag{9}
\]
By (8) and (9), to prove $L^{(j)} = R^{(j)}$, it suffices to show $M^{(j)}_{a,b} = \hat{M}^{(j)}_{a,b}$ for all $a, b \in \{0, T, H\}$.

| $a \setminus b$ | 0   | $T$ | $H$ |
|-----------------|-----|-----|-----|
| 0               | {1,1} | 0   | {1,2,2,2} |
| $T$             | {2,2} | 0   | {2} |
| $H$             | {1,2} | {3,3} | {2,2,2,3,3} |

Table 3: Table of $\hat{M}^{(3)}_{3,(a,b)}(1011110)$

**Example 9.** Let $w = 1011110$ and $m = 3$, the same as the previous examples. Table 3 summarizes $\hat{M}^{(3)}_{3}(w)$ for $w = 1011110$ and $m = 3$, which is same as Table 2 of $M^{(3)}_{a,b}(w)$ in Example 6. This coincidence is not trivial at all. For instance, as we have seen in Example 8, $\hat{I}_0 = \{0, 5, 7, 11, 13, 18\}$, and hence

$$(\hat{I}_0 - 2) \cap \hat{I}_0 = \{5, 11\}.$$ 

Therefore we have

$$\hat{M}^{(3)}_{T,H}(w) = \{\hat{w}_7 + \hat{w}_6 + \hat{w}_5 = 1, \ \hat{w}_{13} + \hat{w}_{12} + \hat{w}_{11} = 1\}.$$ 

$\hat{I}_T = \{2, 8, 15\}, \hat{I}_H = \{1, 3, 4, 6, 9, 10, 12, 14, 16, 17\}$ and hence

$$(\hat{I}_T - 2) \cap I_H = \{6\}.$$ 

Therefore, we have

$$\hat{M}^{(3)}_{T,H}(w) = \{\hat{w}_8 + \hat{w}_7 + \hat{w}_6 = 2\}.$$ 

Since $M^{(3)}_{a,b} = \hat{M}^{(3)}_{a,b}$ for all $a, b \in \{0, T, H\}$, we have $L^{(2)}_3(w) = R^{(2)}_3(w)$.

In the previous examples we have seen examples of $\overline{w}$ and $\hat{w}$ for $w = 1011110$:

| $\overline{w}_i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|-----------------|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|
| $\hat{w}_i$     | 0 | $H$ | $H$ | $T$ | $H$ | 0 | 0 | $H$ | $H$ | $H$ | 0 | 0 | $H$ | $H$ | $T$ | 0 |

As can be seen in these examples, the following two lemmas relating $\overline{w}$ and $\hat{w}$ hold.

**Lemma 2.** Let $w = w_0w_1\cdots w_{n-1} \in \{0, 1\}^n$ be a word of length $n$, and $\overline{w} = \overline{w_0w_1\cdots \overline{w}_{l-1}}$ and $\hat{w} = \hat{w}_0\hat{w}_1\cdots \hat{w}_{l-1}$ be as defined above. Then

$$\hat{w}_i = \overline{w}_{i+n},$$

where $i + n$ is considered to be in $\{0, 1, \ldots, l-1\}$ by taking modulo $l$. In other words, $\rho_1^n(\overline{w}) = \hat{w}$.

**Proof.** This is clear from the definition of $\overline{w}$ and $\hat{w}$. □
Lemma 3. The following maps are bijections.

\[ I_0 \ni i \mapsto i - n + 1 \in \hat{I}_0. \]  
(11)

\[ I_1 \ni i \mapsto i - n + 1 \in \hat{I}_1. \]  
(12)

\[ I_H \ni i \mapsto i + 1 \in \hat{I}_H. \]  
(13)

\[ I_T \ni i \mapsto i - m + 1 \in \hat{I}_T. \]  
(14)

Proof. Since \( \overline{w} \) is obtained from \( w^{(p)} \) by removing its suffix \( w \), and \( \hat{w} \) is obtained from the same sequence \( w^{(p)} \) by removing its prefix \( w \), we see that (11) and (12) are bijections. By (7) and Lemma 1, it is clear that (13) and (14) are bijections. \( \square \)

Now we start to prove

\[ M_{a,b}^{(j)} = \hat{M}_{a,b}^{(j)} \]  
(15)

for \( a, b \in \{0, T, H\} \) and \( j = 1, 2, \ldots, n \). We prove this by induction on \( j \). The proof is reminiscent of solving sudoku puzzles. When \( j = 1 \), (15) is clear since

\[ M_{a,b}^{(1)} = \hat{M}_{a,b}^{(1)} = \begin{cases} \emptyset & a \neq b, \\ \{0, 0, \ldots, 0\} & a = b = 0, \\ \{1, 1, \ldots, 1\} & a = b \in \{H, T\}. \end{cases} \]

We prove the cases where \( j = 2, \ldots, m \) and then prove for \( j > m \). First we prove some facts, Lemma 4 to 7 without assuming the induction hypothesis.

Lemma 4.

\[ M_{0,0}^{(j)} = \hat{M}_{0,0}^{(j)}, \quad M_{T,T}^{(j)} = \hat{M}_{T,T}^{(j)}. \]

Proof. This is clear from Lemma 3. \( \square \)

Lemma 5.

\[ \bigcup_{a,b \in I_H \cup I_T} M_{a,b}^{(j)} = \bigcup_{a,b \in I_H \cup I_T} \hat{M}_{a,b}^{(j)} \]

Proof. Since \( I_H \cup I_T = I_1 \) and \( \hat{I}_H \cup \hat{I}_T = \hat{I}_1 \), the lemma follows from the relation (12) between \( I_1 \) and \( \hat{I}_1 \) in Lemma 3. \( \square \)

Lemma 6.

\[ M_{0,0}^{(j)} \cup M_{0,T}^{(j)} \cup \left( M_{T,0}^{(j)} - 1 \right) \cup \left( M_{T,T}^{(j)} - 1 \right) = \hat{M}_{0,0}^{(j)} \cup \hat{M}_{0,T}^{(j)} \cup \left( \hat{M}_{T,0}^{(j)} - 1 \right) \cup \left( \hat{M}_{T,T}^{(j)} - 1 \right) \]  
(16)

Proof. Recall that \( i_s, i_{s+1}, \ldots, i_{s+1} - 2 \in I_H \) and \( i_{s+1} - 1 \in I_0 \cup I_T \) for every \( s \in \{0, 1, \ldots, p-1\} \). Since \( i_{s+1} - 1 \in I_0 \), i.e. \( \overline{w}_{i_{s+1}-1} = 0 \), if and only if \( \hat{w}_{i_s} = 0 \), we have

\[ \overline{w}_{i_s} + \overline{w}_{i_{s+1}} + \cdots + \overline{w}_{i_{s+1}-1} = \hat{w}_{i_s} + \hat{w}_{i_{s+1}} + \cdots + \hat{w}_{i_{s+1}-1} \]
Since $I_0 \cup I_T = I \setminus I_H$ and $\hat{I}_0 \cup \hat{I}_T = \hat{I} \setminus \hat{I}_H$, the bijection \((13)\) induces another bijection

$$I_0 \cup I_T \ni i \mapsto i + 1 \in \hat{I}_0 \cup \hat{I}_T.$$  

Therefore, $i, i + j - 1 \in I_0 \cup I_T$ if and only if $i + 1, i + j \in \hat{I}_0 \cup \hat{I}_T$. Hence, if $i, i + j - 1 \in I_0 \cup I_T$, then there are some $s, t \in \{0, 1, \ldots, p - 1\}$ such that $i + 1 = i_s$ and $i + j = i_t$. Therefore

$$\begin{align*}
\hat{w}_{i+1} + \hat{w}_{i+2} + \cdots + \hat{w}_{i+j-1} &= \hat{w}_{i+1} + \hat{w}_{i+2} + \cdots + \hat{w}_{i+j-1} \\
\end{align*}$$

\((17)\)

for all $i \in I_0 \cup I_T$ such that $i + j - 1 \in I_0 \cup I_T$. Each element of the multiset of the left (resp. right) hand side of \((16)\) is expressed as the left (resp. right) hand side of \((17)\) and the lemma follows.

\[ \square \]

**Lemma 7.**

$$M_{H,T}^{(j)} = \hat{M}_{H,T}^{(j)} \quad \text{for } j = 1, 2, \ldots, m.$$  

**Proof.** For $j \leq m$, $M_{H,T}^{(j)} = \{ j, j, \ldots, j \}$ which is clearly equal to $\hat{M}_{H,T}^{(j)} = \{ j, j, \ldots, j \}$.  

\[ \square \]

**Lemma 8.** Let $j$ be a positive integer and assume that $M_{a,b}^{(j-1)} = \hat{M}_{a,b}^{(j-1)}$ for $a, b \in \{0, T, H\}$. Then, for $a, b \in \{0, T, H\}$,

$$M_{a,0}^{(j)} \cup (M_{a,T}^{(j)} - 1) \cup (M_{a,H}^{(j)} - 1) = \hat{M}_{a,0}^{(j)} \cup (\hat{M}_{a,T}^{(j)} - 1) \cup (\hat{M}_{a,H}^{(j)} - 1),$$

\((18)\)

and

$$M_{0,b}^{(j)} \cup (M_{T,b}^{(j)} - 1) \cup (M_{H,b}^{(j)} - 1) = \hat{M}_{0,b}^{(j)} \cup (\hat{M}_{T,b}^{(j)} - 1) \cup (\hat{M}_{H,b}^{(j)} - 1).$$

\((19)\)

**Proof.** It is clear that

$$M_{0,b}^{(j-1)} \cup M_{T,b}^{(j-1)} \cup M_{H,b}^{(j-1)} = \{ w_i + w_{i+1} + \cdots + w_{i+j-2} \mid i + j - 2 \in I_b \},$$

and

$$M_{0,b}^{(j)} \cup M_{T,b}^{(j)} \cup M_{H,b}^{(j)} = \{ w_i + w_{i+1} + \cdots + w_{i+j-1} \mid i + j - 1 \in I_b \}. $$

Therefore, we have

$$M_{0,b}^{(j)} \cup (M_{T,b}^{(j)} - 1) \cup (M_{H,b}^{(j)} - 1) = M_{0,b}^{(j-1)} \cup M_{T,b}^{(j-1)} \cup M_{H,b}^{(j-1)},$$

and

$$\hat{M}_{0,b}^{(j)} \cup (\hat{M}_{T,b}^{(j)} - 1) \cup (\hat{M}_{H,b}^{(j)} - 1) = \hat{M}_{0,b}^{(j-1)} \cup \hat{M}_{T,b}^{(j-1)} \cup \hat{M}_{H,b}^{(j-1)}.$$  

By the assumption that $M_{a,b}^{(j-1)} = \hat{M}_{a,b}^{(j-1)}$ for $a, b \in \{0, T, H\}$, \((19)\) follows. The proof of \((18)\) is similar.  

\[ \square \]

**Proposition 1.** For $j = 1, 2, \ldots, m$ and $a, b \in \{0, T, H\}$,

$$M_{a,b}^{(j)} = \hat{M}_{a,b}^{(j)}.$$  

\[(20)\]
Proof. We have already shown this for \((a, b) = (0, 0)\) and \((T, T)\) in Lemma 4 and for \((a, b) = (H, T)\) in Lemma 9. Hence, by (19), we have \(M_{0, T}^{(j)} = M_{0, T}^{(j)}\). Then, we have \(M_{0, H}^{(j)} = M_{0, H}^{(j)}\) by (18), and \(M_{T, 0}^{(j)} = M_{T, 0}^{(j)}\) by Lemma 6. Then \(M_{H, 0}^{(j)} = M_{H, 0}^{(j)}\) follows from (19) and \(M_{T, H}^{(j)} = M_{T, H}^{(j)}\) from (18). \(M_{H, H}^{(j)} = M_{H, H}^{(j)}\) follows from (19). □

Lemma 9. Let \(j > m\) be a positive integer not greater than \(n\), and assume that \(M_{a, b}^{(j-m)} = \hat{M}_{a, b}^{(j-m)}\) for \(a, b \in \{0, T, H\}\). Then

\[
M_{H, T}^{(j)} = \hat{M}_{H, T}^{(j)}.
\]

Proof. Suppose that \(j > m + 1\) and \(i \in \{0, 1, \ldots, l - 1\}\). Then, by Lemma 3, \(i \in I_H\) and \(i + j - 1 \in I_T\) if and only if \(i + 1 \in \hat{I}_H\) and \(i + j - m \in \hat{I}_T\).

\[
\bar{w}_i + \cdots + \bar{w}_{i+j-1} = \begin{cases} 
\hat{w}_{i+1} + \cdots + \hat{w}_{i+j-m} + m & \text{there is a } d < m \text{ s.t. } i + d \in I_T, \\
\hat{w}_{i+1} + \cdots + \hat{w}_{i+j-m} + 1 & \text{otherwise.}
\end{cases}
\]

By the induction hypothesis,

\[
M_{T, H}^{(j-m)} = \hat{M}_{T, H}^{(j-m)}.
\]

Therefore, there is some bijections

\[
f : \left\{ k \in \hat{I}_H \mid k + j - m - 1 \in \hat{I}_T \right\} \rightarrow \left\{ k \in I_T \mid k + j - m - 1 \in I_H \right\},
\]

such that, for all \(k \in \left\{ k \in \hat{I}_H \mid k + j - m - 1 \in \hat{I}_T \right\},

\[
\hat{w}_k + \hat{w}_{k+1} + \cdots + \hat{w}_{k+j-m-1} = \bar{w}_{f(k)} + \bar{w}_{f(k)+1} + \cdots + \bar{w}_{f(k)+j-m-1}.
\]

Hence, \(i + 1 \in \hat{I}_H\) and \(i + j - m \in \hat{I}_T\), then

\[
\hat{w}_{i+1} + \hat{w}_{i+2} + \cdots + \hat{w}_{i+j-m} = \bar{w}_{i'+m-1} + \bar{w}_{i'+m} + \cdots + \bar{w}_{i'+j-2},
\]

where \(i' + m - 1 = f(i+1) \in I_T\). Then, by the relation (14) of Lemma 3, \(i' \in \hat{I}_T\) and

\[
\hat{w}_{i'} + \cdots + \hat{w}_{i'+j-1} = \begin{cases} 
\bar{w}_{i'+m-1} + \cdots + \bar{w}_{i'+j-2} + m & \text{there exists } d < m \text{ s.t. } i' + j - 2 + d \in I_T, \\
\bar{w}_{i'+m-1} + \cdots + \bar{w}_{i'+j-2} + m - 1 & \text{otherwise.}
\end{cases}
\]

We have shown \(M_{T, T}^{(j)} = \hat{M}_{T, T}^{(j)}\) for \(j = 1, 2, \ldots, n\) without using induction (Lemma 4). Therefore, we can choose the bijection \(f\) so that whenever \(i \in I_H\) and \(i + j - 1 \in I_T\), there exists \(d < m\) such that \(i + d \in I_T\) if and only if \(i' + j - 2 + d \in \hat{I}_T\). □

Proof of Theorem 1. Same as the proof of Proposition 1 □
3 Toggle dynamical system on $X_{N,m}$

Let $N$ be a positive integer and $m$ a positive integer not greater than $N$. Let $X_{N,m}$ denote the subset of $\{0,1\}^N$ defined by

$$X_{N,m} = \{ w = w_0 w_1 \cdots w_{N-1} \in \{0,1\}^N \mid w_i + w_{i+1} + \cdots + w_{i+m} \leq 1 \text{ for } i = 0,1,\ldots,N-m-1 \}.$$

In other words, $X_{N,m}$ is the language consisting of the words $w$ each of whose subwords of the form $w_{[i,i+m]}$ does not contain more than one 1's. We consider the dynamical system $(X_{N,m}, \varphi_m)$ with the state space $X_{N,m}$ and the transformation $\varphi_m : X_{N,m} \to X_{N,m}$ defined as follows: The toggle map $\tau_i : X_{N,m} \to X_{N,m}$ is defined by

$$\tau_i(w) = \begin{cases} w & w \neq 00100010 \cdots 10, \\ w_i \cdots w_{i-1}(1 - w_i)w_{i+1} \cdots w_{N-1} & w = 00100010 \cdots 10, \\ w_0w_1 \cdots (1 - w_i) \cdots w_{N-1} & w \in X_{N,m}, \\ w_0w_1 \cdots (1 - w_i) \cdots w_{N-1} & w \notin X_{N,m}, \end{cases}$$

and $\varphi_m = \tau_{N-1} \circ \tau_{N-2} \circ \cdots \circ \tau_0$. It is clear that every $\tau_i$ is a bijections from $X_{N,m}$ to itself, and so is $\varphi_m$. Therefore $\varphi_m$ decomposes $X_{N,m}$ into $\varphi_m$-orbits.

Joseph and Roby [3] studied the dynamical system $(X_{N,m}, \varphi_m)$ for $m = 1$ and showed some surprising properties. In particular, they showed the symmetry of the digit sum of orbits: When $m = 1$, for every $w \in X_{N,m}$ and $j \in \{0,1,\ldots,N-1\}$

$$\sum_{k=0}^{p-1} \varphi_m^k(w)_j = \sum_{k=0}^{p-1} \varphi_m^k(w)_{N-1-j},$$

where $p$ is the length of the $\varphi_m$-orbit of $w$ and $\varphi_m^k(w)_j$ is the $j$-th digit of the word $\varphi_m^k(w)$.

The key idea of the proof of (21) for $m = 1$ by Joseph and Roby [3] is the reduction of the toggle dynamical system to the rotation of the bit sequences by using the notion of the snakes. We show that (21) holds for general $m$ by reducing it to a dynamical system driven by the generalized rotations which has been discussed in previous sections.

Let $\{0,1\}^*$ denote the set of finite words over the alphabet $\{0,1\}$, and let $w = w_0 w_1 w_2 \cdots w_{|w|-1} \in \{0,1\}^*$ be a finite word. Then, we define $a(w) = \sum_{j=0}^{|w|-1} w_j$ and $b(w) = |w| - a(w)$, that is, $a(w)$ is the number of 1's in $w$ and $b(w)$ the number of 0's. Define the subset $Y_{n,m}$ of $\{0,1\}^*$ by

$$Y_{n,m} = \{ w \in \{0,1\}^* \mid a(w) + (m + 1)b(w) = n \}.$$  

Example 10. $Y_{3,3} = \{111\}$, $Y_{4,3} = \{1111,0\}$, $Y_{5,3} = \{11111,10,01\}$, and $Y_{6,3} = \{111111,110,101,011\}$.  

By modifying the argument using the notion of the snakes by Joseph and Roby [3], we construct a bijection between the orbits of $(X_{N+1,m}, \varphi_m)$ and those of $(Y_N, \rho_m)$.

Example 11. $(Y_{6,3}, \rho_3)$ has three orbits:

$$Y_{6,3} = \{111111\} \cup \{110,011\} \cup \{101\},$$

and so does $(X_{7,3}, \varphi_3)$:

$$X_{7,3} = \{100010,0100001,0010000,0001000,0000100\}$$

$$\cup \{1000100,0000010,1000001,0100000,0010001,0000000\} \cup \{1000000,0100010,0000001\}.$$

We construct an explicit bijection between these sets of orbits.
**Definition 1.** Let $S \in X_{N+1,m}$ be a word and $q$ be the length of $\varphi_m$-orbit of $S$. Then, define the orbit board $(S(i,j))_{0 \leq i < q, 0 \leq j \leq n}$ for the word $S$, by

$$S(i,j) = \varphi_m^i(w)_j.$$

**Table 4:** Orbit board of $10000000000100 \in X_{14,3}$. A snake consists of positions of 1’s which are marked by the circles.

**Lemma 10.**

1. When $S(i,j) = 1$ and $j \neq N-1$, either $S(i,j+m+1) = 1$ or $S(i+1,j+1) = 1$, and never both.

2. When $S(i,j) = 1$ and $j \neq 0$, either $S(i,j-m-1) = 1$ or $S(i-1,j-1) = 1$, and never both.

**Definition 2.** By Lemma 10, if $S(i,j) = 1$, then a sequence $s = ((i_0,j_0),(i_1,j_1),\ldots,(i_n,j_n))$ containing $(i,j)$ which has the following properties is uniquely determined.
1. \( j_0 = 0 \), and \( j_n = N \).

2. \( S(i_k, j_k) = 1 \) for \( k = 0, 1, \ldots, n \).

3. \((i_k, j_k) - (i_{k-1}, j_{k-1}) \in \{(1, 1), (0, m + 1)\} \) for \( k = 1, 2, \ldots, n \).

We call \( s \) the snake containing \((i, j)\). Since \( j_k - j_{k-1} \in \{1, m + 1\} \), we obtain an \( N - 1 \)'s composition \((j_1 - j_0)(j_2 - j_1) \cdots (j_n - j_{n-1})\) whose parts are in \( \{1, m + 1\} \). We call this composition the snake composition of \( s \).

Let \( c \in \{1, m + 1\}^* \) be a snake composition. Then we can transform \( c \) into a word \( \{0, 1\}^* \) by replacing \( m + 1 \) with \( 0 \), which we denote \( \tilde{c} \).

**Example 12.** Table 4 shows the orbit board of \( 10000000001000 \in X_{14,3} \). A snake consists of positions of 1's which are marked by the circles. There are 9 snakes in this orbit board. The snake compositions of these snakes are

\[
\begin{align*}
1411114, & \quad 1111411, 4111141, 1111414, 1414111, 1411141, 1114141, 4141111.
\end{align*}
\]

If we replace the digits 4 in the above compositions with 0, we obtain the \( \rho_3 \)-orbit of \( w = 1011110 \) which we have already seen in Example 1. We will explain this correspondence.

**Lemma 11.** Suppose \( S(i, j) = 1 \) and \( S(i + 2, j - d) = 1 \) with \( m \leq d \leq 2m \) and \( j \neq N - 1 \). Then, by Lemma 10 exactly one of \( S(i + 1, j + 1) = 1 \) and \( S(i, j + m + 1) = 1 \) occurs, for each of which we have

1. If \( S(i + 1, j + 1) = 1 \), then \( S(i + 3, j - d + 1) = 1 \).
2. If \( S(i, j + m + 1) = 1 \), then \( S(i + 2, j - d + m + 1) = 1 \).

**Proof.**

1. If \( S(i + 1, j + 1) = 1 \), then we have

\[
S(i + 2, j - k + 1) = S(i + 2, j - k + 2) = \cdots = S(i + 2, j + 1) = 0,
\]

and Lemma 10 implies \( S(i + 3, j - k + 1) = 1 \). (See Figure 4)

![Figure 4: Positions of 1's in Case 1](image)

2. If \( S(i, j + m + 1) = 1 \), then we have

\[
S(i + 1, j - k) = S(i + 1, j - k + 1) = \cdots S(i + 1, j + m + 1) = 0,
\]

and therefore \( S(i + 1, j - k + m) = 1 \).
Lemma 12. Suppose \( S(i, N - 1) = 1 \). Then, there exists a unique \( d \) such that \( m \leq d \leq 2m \) and \( S(i + 2, N - 1 - d) = 1 \), and

1. If \( d > m \), then we have

\[
S(i + 2, N - d + m) = S(i + 3, N - d + m + 1) = \cdots = S(i + 1 + d - m, N - 1) = 1.
\]

2. If \( d = m \), then we have

\[
S(i + 2, N - m - 1) = S(i + 3, N - m) = \cdots = S(i + 2 + m, N - 1) = 1.
\]

Proof. The uniqueness of such \( d \) is clear and \([1]\) and \([2]\) follows easily from the existence of such \( d \) and Lemma \([10]\). Therefore we show the existence: It is clear that

\[
S(i, N - 2) = S(i, N - 3) = \cdots = S(i, N - 1 - m) = 0.
\]

If \( S(i, N - 2 - m) = 1 \), then \( S(i + 1, N - 2 - 2m) = S(i + 1, N - 1 - 2m) = S(i + 1, N - 2m) = \cdots = S(i + 1, N - 1) = 0 \). Therefore, there exists exactly one \( d \) such that \( S(i + 2, N - 1 - d) = 1 \) and \( m \leq d \leq 2m \).
If $S(i, N - 2) = S(i, N - 3) = \cdots = S(i, N - 3 - 2m) = 0$, then we have exactly one $d$ such that $S(i + 1, N - 2 - d) = 1$ and $m \leq d \leq 2m$. Therefore, by Lemma 10 we have $S(i + 2, N - 1 - d) = 1$.

| 2m | 0 0 \cdots 0 0 1 |
|----|-------------------|
|    |                   |
|    | 1 1 \cdots 0 0 1 |
|    | m 1 \cdots 0 0 1 |
| i  | i+1 \cdots i+2   |

\[ \square \]

**Theorem 2.** Let $s$ be the snake containing $(i, 0)$ in the $\varphi_m$-orbit board of $S \in \{0, 1\}^*$. Let $c$ be the snake composition of $s$ and $i'$ be the least integer greater than $i$ for which $S(i', 0) = 1$.

Assume $c$ starts with $k$-repetition of 1, i.e., $11 \cdots 1$. Then,

1. If $k \geq m$, then $i' = i + m + 2$.
2. If $k < m$, then $i' = i + k + 2$.
3. Let $c$ be the snake composition of the snake containing $(i, 0)$ and let $c'$ be the snake composition of the snake containing $(i', 0)$. Then we have
   \[ \rho_m(\tilde{c}) = \tilde{c'}, \]  
   \[ (22) \]
   where $\tilde{c}$ is the word in $\{0, 1\}^*$ obtained from $c$ by replacing $m + 1$ with 0.

**Proof.** Without loss of generality, we can assume that $i = 0$.

If $k \geq m$, then
   \[ S(1, 1) = S(2, 2) = \cdots = S(m, m) = 1, \]
and hence $S(i, j) = 0$ for $i \leq m, 0 \leq j < i$. Therefore
   \[ S(m + 1, 0) = S(m + 1, 1) = \cdots = S(m + 1, m) = 0. \]

This implies $S(m + 2, 0) = 1$, which proves 1.

If $k < m$, then
   \[ S(1, 1) = S(2, 2) = \cdots = S(k, k) = S(k, k + m + 1) = 1. \]
and hence $S(k + 1, 0) = S(k + 1, 1) = \cdots = S(k + 1, k + m + 1) = 0$. This implies $S(k + 2, 0) = 1$, which proves 2.

We next prove 3. Let $s = ((0, j_0), (i_1, j_1), \ldots, (i_t, j_t))$. If $k < m$, then
   \[ S(k, k) = S(k, k + m + 1) = S(k + 2, 0) = 1, \]
and \((i_k, j_k) = (k, k)\) and \((i_{k+1}, j_{k+1}) = (k, k + m + 1)\). In particular, the snake composition \(c\) is expressed as

\[
c = (j_1 - j_0)(j_2 - j_1) \cdots (j_t - j_{t-1})
\]

\[
= 11 \cdots 1 \ (m + 1)(j_{k+2} - j_{k+1})(j_{k+3} - j_{k+2}) \cdots (j_t - j_{t-1}).
\]

Then, by Lemma 11 and 12 the snake composition \(c'\) of the snake \(s'\) starting from \((k + 2, 0)\) is

\[
(j_{k+2} - j_{k+1})(j_{k+3} - j_{k+2}) \cdots (j_t - j_{t-1})(m + 1) \underbrace{11 \cdots 1}_{k}.
\]

Therefore, (22) holds. If \(k \geq m\), then

\[
S(m, m) = S(m + 2, 0) = 1.
\]

By Lemma 11 and 12 the snake composition \(c'\) starting from \((m + 2, 0)\) is

\[
(j_m + 1 - j_m)(j_{m+2} - j_{m+1}) \cdots (j_t - j_{t-1}) \underbrace{11 \cdots 1}_{m}.
\]

\[
\square
\]

**Theorem 3.** For every \(S \in X_{N, m}\) and \(i \in \{0, 1, \ldots, n - 1\}\)

\[
\sum_{k=0}^{q-1} \varphi^t_{m}(S)_{i} = \sum_{k=0}^{q-1} \varphi^t_{m}(S)_{N-1-i},
\]

where \(q\) is the length of the \(\varphi_m\)-orbit of \(S\) and \(\varphi^t_{m}(S)_{i}\) is the \(i\)-th digit of the word \(\varphi^t_{m}(S)\).

**Proof.** Let \(s_0, s_1, \ldots, s_{p-1}\) be the snakes in the orbit board of \(S\) and \(c_0, c_1, \ldots, c_{p-1}\) be the corresponding snake compositions. Then, by Theorem 3 there exists a \(\{0, 1\}\)-composition \(w\) which satisfies

\[
c_0 = w, \ c_1 = \rho_m(w), \ldots, c_{p-1} = \rho_m^{p-1}(w).
\]

Then, \(\varphi_m^k(S)_{i} = 1\), if and only if \((k, i)\) is contained in a snake \(s \in \{s_0, s_1, \ldots, s_{p-1}\}\). Suppose that \(\varphi_m^t(S)_{i} = 1\) and \((t, i)\) is contained in a snake \(s_k\). Let the snake composition of \(s_k\) be \(c_k = a_0 a_1 \cdots a_{n-1}\). Then, we have some index \(j\) such that \(i = u_0 + u_1 + \cdots + u_{j-1}\), which is equal to

\[
(m + 1)j - m \left(\rho_m^k(w)_0 + \rho_m^k(w)_1 + \cdots + \rho_m^k(w)_{j-1}\right).
\]

This implies

\[
\rho_m^k(w)_0 + \rho_m^k(w)_1 + \cdots + \rho_m^k(w)_{j-1} = j - \frac{i - j}{m}.
\]

Therefore

\[
\sum_{k=0}^{q-1} \varphi^t_{m}(S)_{i} = \sum_{j=0}^{|w|-1} \# \left\{ k \left| \sum_{\nu=0}^{j-1} \rho_m^k(w)_{\nu} = j - \frac{i - j}{m} \right. \right\} = \sum_{j=0}^{|w|-1} \nu_i^{(l)}(w) \left( j - \frac{i - j}{m} \right).
\]
In the same manner, we can show
\[
\sum_{k=0}^{q-1} \varphi_m^t(S)_{N-1-i} = \sum_{j=0}^{\lfloor u \rfloor-1} \nu_R^{(j)}(u) \left( j - \frac{i - j}{m} \right).
\]
By Theorem 1 we are done.

**Example 13.** We can compute the bottom line, 9, 7, 3, 3, 4, 6, ..., 9 of the Table 4 from the table in Figure 1 as follows,

\[
9 = \nu_{L(0)} (0) + \nu_{L(3)} (4), \quad 7 = \nu_{L(1)} (1) + \nu_{L(4)} (5), \quad 3 = \nu_{L(2)} (2), \quad 3 = \nu_{L(3)} (3), \quad 4 = \nu_{L(1)} (0) + \nu_{L(4)} (4), \quad \text{and so forth.}
\]

4 Concluding remarks

Another generalization of [3] is studied in [2]. Our generalization of [3] can be considered as the toggle dynamical systems on more independent sets. It seems that results in this paper can be further generalized. For example, we can also consider less independent sets: Let \( Z_{N,m} \) be the set defined by

\[
Z_{N,m} = \left\{ z = z_0z_1 \cdots z_{N-1} \in \{0,1\}^N \mid z_i + z_{i+1} + \cdots + z_{i+m-1} < m \quad \text{for} \quad i = 0, 1, \ldots, N - m \right\}.
\]

In other words, \( Z_{N,m} \) is the language over \( \{0,1\} \) consisting of the words which do not contain \( m \) consecutive 1s as its subword. Numerical experiments suggests that the toggle dynamical system on \( Z_{N,m} \) has the same symmetric property as \( X_{N,m} \). However, we have not succeeded in finding the objects corresponding to the snakes on the orbit board of \( z \in Z_{N,m} \).

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