Error analysis of a demodulation procedure for multicarrier signals with slowly-varying carriers

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Abstract—We propose a procedure to demodulate analog signals encoded by a multicarrier modulator, with slowly-varying carrier shapes. We prove that the asymptotic demodulation error can be made arbitrarily small. The intended application is the “sensorless” control of AC electric motors at or near standstill, through the decoding of the PWM-induced current ripple.

Index Terms—analog demodulation, multicarrier signals, slowly-varying carriers, multiple access methods, PWM injection

I. INTRODUCTION

We consider a composite signal \( y(t) \) of the form

\[
y(t) := \sum_{i=1}^{n} z_i(t) s_i(t, \frac{t}{\varepsilon}) + d(t, \frac{t}{\varepsilon}) + O(\varepsilon^5),
\]

where the \( z_i \)'s are (known) 1-periodic functions in the second variable; \( \varepsilon \) being a (known) “small” parameter, the \( z_i \)'s can be seen as rapidly oscillating carriers with slowly varying shapes modulating the (unknown) \( z_i \)'s. The function \( d \) is a disturbance, also 1-periodic in the second variable, about which little is known except that for each \( t \) the support of \( d(t, \cdot) \) is contained in a “well-behaved” known subset \( D_t \) of \([0, 1)\). In other words, on each period of the carriers, part of the signal \( y \) is garbled and considered useless. Finally, the \( O(\varepsilon^5) \) term corresponds to “small” disturbances, where \( O \) denotes the (uniform) “big O” symbol of analysis, i.e.

\[
f(t, \varepsilon) = O(\varepsilon^5) \text{ if } \|f(t, \varepsilon)\| \leq K \varepsilon^5 \text{ for some } K \text{ independent of } t \text{ and } \varepsilon.
\]

The objective is to recover by an implementable causal process the unknown \( z_i \)'s with an accuracy of up to \( O(\varepsilon^5) \) from the known \( y \) and \( s_i \)'s, provided the \( s_i \)'s and \( z_i \)'s satisfy some suitable regularity assumptions.

The motivation for this problem is the following. When operating an AC electric motor through a PWM inverter with period \( \varepsilon \), an analysis based on the theory of averaging reveals that the currents in the motor have the form

\[
y(t) = y_a(t) + \varepsilon y_v(t) s(t, \frac{t}{\varepsilon}) + O(\varepsilon^2) + d(t, \frac{t}{\varepsilon}),
\]

which is a particular instance of (1) with \( z_1 := y_a \), \( z_2 := \varepsilon y_v \), \( s_1 := 1 \), and \( s_2 := s \), where \( s \) is determined by the PWM process [1], [2]. The \( O(\varepsilon^2) \) term corresponds to a small higher-order ripple which can be ignored. The disturbance \( d \) consists of short spikes appearing at each PWM commutation, due to stray capacitances in the power electronics. A typical (synthetic) signal \( y \) is shown in Fig. 4, see also [2, Fig. 9] for experimental data. In “sensorless” industrial drives, these currents are the only measurements, and controlling the motor at or near standstill with this sole information is a difficult problem for several theoretical and technological reasons. A way to achieve this is to extract \( y_a \) and \( \varepsilon y_v \) from the modulated currents \( y \); a suitable processing of \( y_v \), then gives access to the motor angular position [2], which is instrumental in controlling the motor. It is therefore very important to ensure the demodulation error is at most \( O(\varepsilon^2) \).

The demodulation procedure proposed in this paper, essentially consisting of multiplications by known signals followed by low-pass filters, is reminiscent of various schemes in communication theory and signal processing. Nevertheless, nothing really close seems to exist in the literature, let alone a quantitative analysis of the demodulation error:

- it is of course a generalization of coherent demodulation in quadrature carrier multiplexing, with more than two carriers not restricted to sine and cosine, see e.g. [3, section 4.4]; but even in this simple case, no analysis of the error is usually performed, the challenges being more on carrier reconstruction
- it somewhat looks like synchronous decorrelating detection in Code-Division Multiple Access communication systems, where the \( s_i \)'s would play the roles of the signature waveforms and the \( z_i \)'s the role of the symbols, see e.g. [4, section 5.1]; but the encoded signals being there digital, the issues and analysis are very different
- it is also akin to multicarrier reception, with or without multiple access, see e.g [5, section 12.2] and [6, section 2.2]; but once again that field is exclusively concerned with digital encoded signals
- finally, it bears some resemblance for its filtering part with the interpolation/compensation filters used in \( \Delta \Sigma \) analog-to-digital converters, see e.g. [7, chapter 14].

The paper extends the previous work [8] in two ways that are paramount for the intended application: on the one hand, it considers carriers with slowly-varying shapes, which makes the error analysis much more difficult; on the other hand, the procedure is not restricted to “orthogonal” demodulation, hence can directly handle the disturbance \( d \) without ad-hoc prefiltering as in [2].

The paper runs as follows: in section II, we collect notations
and definitions, in particular the $A_k$ regularity property; in section III we state and prove the main result; in section IV we illustrate this result and confirm the error estimates with numerical experiments.

II. NOTATIONS AND DEFINITIONS

We collect here definitions used throughout the paper. The most important notion is the $A_k$ regularity property introduced in proposition 1, which is needed in lemma 4 to repeatedly integrate by parts; this property, which is paramount for handling carriers with slowly-varying shapes, is trivially satisfied for fixed-shape carriers as in [8].

Let $g(t, \sigma)$ be a function of two variables; informally speaking, $t$ represents the slow timescale and $\sigma$ the fast timescale. We will often use the convenient notation $g_x(t) := g(t, \frac{1}{n} t)$. The function $g$ is $1$-periodic in the second variable if $g(t, \sigma+1) = g(t, \sigma)$ for all $t$. Its mean in the second variable is the function $\bar{g}(t) := \int_0^1 g(t, \sigma) d\sigma$. For brevity, we will usually omit the phrase “in the second variable”. If $g$ is $1$-periodic with zero mean, any of its primitives (in the second variable) is also $1$-periodic, in particular its zero-mean primitive $g^{(-1)}(t, \sigma) := \int_0^\sigma g(t, \tau) d\tau - \int_0^1 \int_0^\sigma g(t, \tau) d\tau d\sigma$. Likewise, $g^{(k-1)}$ denotes the zero-mean primitive of $g^{(k)}$.

We say $g$ is Lipschitz if $\|g(t_1, \sigma) - g(t_2, \sigma)\| \leq L\|t_1 - t_2\|$ for some $L$ independent of $t_1$, $t_2$ and $\sigma$. Finally, we introduce the $A_k$ regularity property.

Definition 1 ($A_k$ property). Let $g(t, \sigma)$ be $1$-periodic with zero mean. It is said to be $A_k$, $k \geq 1$, if $g^{(-k)}$ is $k-1$ times differentiable in the first variable, with bounded derivatives at all orders, and $\partial_{t}^{k-1} g^{(-k)}$ Lipschitz.

A typical $A_k$ function encountered in practice is $g(t, \sigma) = \text{sign}(u(t) - \sigma) - 2u(t) + 1$ where $\sigma := \text{mod} 1$; $u(t) \in (0,1)$ represents the PWM duty cycle and is assumed $k-1$ times differentiable, with bounded derivatives at all orders, and $u^{(k-1)}$ Lipschitz.

It is easy to show that if on the one hand $g(t, \sigma)$ is $A_k$, and on the other hand $z(t)$ is $k-1$ times differentiable, with bounded derivatives at all orders, and $z^{(k-1)}$ Lipschitz, then the product $zg$ is also $A_k$.

III. THE DEMODULATION PROCEDURE

The demodulation procedure for an error of order $e^k$ consists of multiplications by a suitable demodulating basis $R := (r_1, \ldots, r_n)^T$, followed by a bank of low-pass finite impulse response filters with kernel $\pi_k$; see section III-A for a discussion of how to select $R$. The kernel $\pi_k$ is a “compensated” $k$-times iterated moving average, namely a suitable linear combination of shifted instances of $\pi_k$, where the kernel $\pi_k$ is defined recursively by $\pi_1 := \frac{1}{2} \mathbb{I}_{[0,1]}$ and $\pi_k := 2^{-k} \pi_{k-1} + \pi_k$, see e.g. [9, chapter 6.7] for explicit expressions. For instance for $k = 3$, the linear combination is $\pi_3(t) := \frac{17}{8} \pi(t) - 5 \pi(t - e) + \frac{3}{4} \pi(t - 2e)$, see section III-B for more details.

Fig. 1 illustrates the whole demodulation procedure:

- $y(t)$ is multiplied by $R^T(t)$, and filtered by $\tilde{\pi}_k$; the result, $\tilde{\pi}_k \ast (yR^T(t))$, turns out to be $Z^T(t)\pi(t) + O(e^k)$, where $Z := (z_1, \ldots, z_n)^T$ is the vector signal to recover
- the modulating basis $S := (s_1, \ldots, s_n)^T$ is also multiplied by $R^T(t)$, and filtered by $\pi_k$; the result, $(\tilde{\pi}_k \ast (SR^T(t)))$, turns out to be $\pi^T(t) + O(e^k)$
- finally, $(\tilde{\pi}_k \ast (yR^T(t)))$ is multiplied by the inverse of the matrix $(\tilde{\pi}_k \ast (sR^T(t)))^{-1}$; the result, $(\tilde{\pi}_k \ast (yR^T(t)) \times (\tilde{\pi}_k \ast (sR^T(t)))^{-1}(t)$, is as desired $Z^T(t) + O(e^k)$.

As pointed out in the introduction, this demodulation scheme is at first sight not completely surprising. What is much less obvious is that the overall demodulation error is indeed of order $e^k$.

A. Main result

We assume that the $s_i$’s are independent outside the subset $D_1$ containing the support of the disturbance $d(t, \cdot)$, i.e. that the $s_i$’s defined by $s_i(t, \sigma) := (1 - D_1)(\sigma) s(t, \sigma)$ are linearly independent. We can thus choose the demodulating basis $R := (r_1, \ldots, r_n)^T$ such that $R d = 0$ and $\pi T R$ is invertible, where $S := (s_1, \ldots, s_n)^T$ is the modulating basis; one simple choice is for instance $R(t, \sigma) := (1 - D_1(\sigma)) S(t, \sigma)$. A delicate point is to select $R$ also such that $\pi^{-1} R^{-1}$ is $A_k$, provided of course that $D_1$ is “well-behaved” (for instance a finite union of intervals with sufficiently regular moving bounds). For simplicity, we just assume this is the case (and check it a posteriori in the numerical experiments of section IV).

Finally, we assume the $z_i$’s are $k-1$ times differentiable, with bounded derivatives at all orders, and $z_i^{(k-1)}$ Lipschitz, so that $z_i(\pi - \pi T^{-1} \pi)$ is also $A_k$.

Theorem 1. $Z := (z_1, \ldots, z_n)^T$ can be recovered to order $e^k$ from $y$ by the causal process $P_k$ defined by $P_k[\pi](t) := (\tilde{\pi}_k \ast (yR^T(t))) \times (\tilde{\pi}_k \ast (sR^T(t)))^{-1}(t)$.

In other words, $Z^T(t) = P_k[\pi](t) + O(e^k)$.

B. Proof of theorem 1

Rewriting (1) as

$$y(t) = Z^T(t) S(t, \frac{1}{e}) + d(t, \frac{1}{e}) + O(e^k),$$
right-multiplying by $R_k^T$ and convolving with $K_k$ yields
\[
(\tilde{K}_k \ast (yR_k^T))(t) = (\tilde{K}_k \ast (Z^T S_k R_k^T))(t) + O(\varepsilon^k)
\]
\[
= \left[ K_k \ast (Z^T (S_k R_k^{-} - S^R)) \right](t)
+ \left( K_k \ast (Z^T S^R) \right)(t) + O(\varepsilon^k)
\]
\[
= (\tilde{K}_k \ast (Z^T S^R))(t) + O(\varepsilon^k);
\]
to obtain the last line, we have applied Lemma 4 with $g(t, \sigma) := Z^T(t) (S(t, \sigma) R^T(t, \sigma) - S^R(t))$, which is by construction zero-mean and $A_k$. The result obviously holds also if $K_k(t)$ is replaced by the shifted kernel $\tau_T K_k(t) := K_k(t - T)$.

On the other hand, [8, Theorem 1] asserts that a $C^k$-function $\varphi$ with bounded $\varphi^{(k)}$ is left unchanged to order $\varepsilon^k$ by a suitable linear combination $\tilde{K}_k$ of the shifted kernels $\tau_T K_k$, $i = 0, \ldots, k - 1$, i.e. $(\tilde{K}_k \ast \varphi)(t) = \varphi(t) + O(\varepsilon^k)$. For instance,
\[
\tilde{K}_1(t) := K_1(t)
\]
\[
\tilde{K}_2(t) := 2K_2(t) - K_2(t - \varepsilon),
\]
\[
\tilde{K}_3(t) := \frac{17}{2} K_3(t) - 5 K_3(t - \varepsilon) + \frac{2}{3} K_3(t - 2 \varepsilon).
\]
Actually, we must slightly extend the result to the case where $\varphi$ is $k - 1$ times differentiable with $\varphi^{(k-1)}$ Lipschitz, which we omit by lack of space. A consequence,
\[
(Z^T(t) S^R)(t) = Z^T(t) S^R(t) + O(\varepsilon^k).
\]
Since $S^R(t)$ is invertible, $Z(t)$ can be recovered to order $\varepsilon^k$.

To make the process truly implementable in practice, notice $S^R(t)$ can be computed to order $\varepsilon^k$ by
\[
\tilde{K}_k \ast (yR_k^T)(t) = Z^T S^R(t) + O(\varepsilon^k),
\]
which is an instance of the previous equation with $Z(t) = I_n$.

In conclusion, $Z(t)$ is recovered to order $\varepsilon^k$ by
\[
P_k[y](t) := (\tilde{K}_k \ast (yR_k^T))(t) \times (\tilde{K}_k \ast (S_k R_k^T))^{-1}(t)
= Z^T(t) + O(\varepsilon^k),
\]
where the process $P_k$ is causal since the kernel $\tilde{K}_k$ is supported on $[0, 2\varepsilon] \subset \mathbb{R}^+$. 

C. Technical lemmas

This section is quite technical and can be skipped without disturbing the flow of ideas. Its goal is to establish Lemma 4, which is instrumental in the proof of Theorem 1. Lemma 4 relies on Lemma 3, which itself relies on Lemma 2. Lemmas 4 and 3 are in some sense properties of the convolution kernel $K_k$, whereas Lemma 2 extends to our context a classical result of finite-differences calculus. Notice the use of the $A_k$ property when integrating by parts in Lemma 4, which is the main trick to extend the ideas of [8] to slowly-moving carriers.

Define the $k^{th}$-order backward difference $\Delta_k g_{\varepsilon}$ of the function $g_{\varepsilon}(t) := g(t, \frac{\varepsilon}{1+\varepsilon})$ by
\[
(\Delta_k g_{\varepsilon})(t) := \sum_{i=0}^{k} (-1)^i \binom{k}{i} g_{\varepsilon}(t - i\varepsilon).
\]
On the other hand, recall that $K_k$ is $k - 1$ times differentiable, with compact support for all the derivatives. As for $K_k^{(k)}$, it can be defined in the distributional sense, and is a linear combination of Dirac delta functions, and in particular also has compact support; for instance, $K_1^{(1)} = \frac{1}{\varepsilon}(\delta_0 - \frac{\varepsilon}{\varepsilon})$.

Lemma 2. Let $g(t, \sigma)$ be $1$-periodic, and $k - 1$ times differentiable in the first variable with $\partial_1^{(k-1)} g$ Lipschitz. Then $(\Delta_k g_{\varepsilon})(t) = O(\varepsilon^k)$.

Proof. By the Lipschitz form of Taylor’s formula [10, (2.1)],
\[
g(t + \mu, \sigma) = \sum_{j=0}^{k-1} \frac{\mu^j}{j!} \partial_1^j g(t, \sigma) + \mu^k \rho(\mu, \sigma),
\]
where the remainder $\rho$ is $O(1)$ since it satisfies
\[
\mu^j \rho(\mu, \sigma) = \frac{1}{(k-2)!} \int_0^1 (1-\tau)^{k-2} \times (\partial_1^{(k-1)} g(t + \mu \tau, \sigma) - \partial_1^{(k-1)} g(t, \sigma)) d\tau.
\]
Applying this to $g(t - i\varepsilon, \frac{t-i\varepsilon}{\varepsilon}) = g(t - i\varepsilon, \frac{t}{\varepsilon})$ since $g$ is $1$-periodic yields
\[
(\Delta_k g_{\varepsilon})(t) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} g(t - i\varepsilon, \frac{t-i\varepsilon}{\varepsilon}) + O(\varepsilon^k)
\]
\[
= \sum_{j=0}^{k-1} \left( -\frac{\varepsilon}{\varepsilon} \right)^j \partial_1^j g(t, \frac{t}{\varepsilon}) + O(\varepsilon^k)
\]
As $\sum_{i=0}^{k} (-1)^i \binom{k}{i} j^j = 0$, see [11, Cor. 2], this gives the desired result. \hfill \Box

Lemma 3. Let $g(t, \sigma)$ be $1$-periodic, and $k - 1$ times differentiable in the first variable with $\partial_1^{(k-1)} g$ Lipschitz. Then $(K_k^{(k)} + g_{\varepsilon})(t) = O(1)$.

Proof. We first prove by induction that $K_k^{(k)} \ast g_{\varepsilon} = \frac{1}{\varepsilon^k} \Delta_k g_{\varepsilon}$. Indeed, $K_1^{(1)} \ast g_{\varepsilon} = \frac{1}{\varepsilon}(\delta_0 - \frac{\varepsilon}{\varepsilon})$ and $g_{\varepsilon} = \frac{1}{\varepsilon} \Delta_1 g_{\varepsilon}$. Assuming the property holds at rank $k$,
\[
K_{k+1}^{(k+1)} \ast g_{\varepsilon} = (K_{k+1} \ast K_1^{(1)})^{(k+1)} \ast g_{\varepsilon}
\]
\[
= K_{k}^{(k)} \ast K_1^{(1)} \ast g_{\varepsilon}
\]
\[
= \frac{1}{\varepsilon^k} \Delta_k (K_1^{(1)} \ast g_{\varepsilon})
\]
\[
= \frac{1}{\varepsilon^k} \Delta_k (\frac{\Delta_1 g_{\varepsilon}}{\varepsilon})
\]
\[
= \frac{1}{\varepsilon^{k+1}} \Delta_{k+1} g_{\varepsilon}.
\]
To obtain the second line, we have repeatedly used $(T \ast S)' = (T' \ast S) = T \ast S'$.

Applying Lemma 2, we eventually find $(K_k^{(k)} \ast g_{\varepsilon})(t) = \frac{1}{\varepsilon^k} \Delta_k g_{\varepsilon}(t) = \frac{1}{\varepsilon^k} O(\varepsilon^k) = O(1)$. \hfill \Box

Lemma 4. Let $g$ be $A_k$. Then $(K_k \ast g_{\varepsilon})(t) = O(\varepsilon^k)$
Proof. We sketch the proof for \( k = 3 \), the general result following by induction. Notice that by assumption \( g^{(-3)} \) is twice differentiable in the first variable with \( \partial_1^2 g^{(-3)} \) Lipschitz, which will be used each time Lemma 3 is invoked.

We first prove \( \left( K_3'' \ast (g^{(-2)}) \right)(t) = O(\varepsilon) \). Starting from
\[
\left( (g^{(-3)})_x \right)' = (\partial_1 g^{(-3)})_x + \frac{1}{\varepsilon} (\partial_2 g^{(-3)})_x
\]
we find after convolving with \( K_3'' \) and integrating by parts
\[
K_3'' \ast (g^{(-2)})_x = \varepsilon K_3''' \ast (g^{(-3)})_x - \varepsilon K_3'' \ast (\partial_1 g^{(-3)})_x,
\]
the boundary terms vanish since \( K_3'' \) has compact support. The first term is \( O(\varepsilon) \) by Lemma 3. Using \( K_3'' = (K_1 \ast K_2)'' = K_1 \ast K_2 '' \), the second term reads \( \varepsilon K_3 \ast (K_3'' \ast (\partial_1 g^{(-3)})_x) \), hence is also \( O(\varepsilon) \) by Lemma 3. The sum of the two terms is therefore also \( O(\varepsilon) \).

We next prove \( \left( K_3 \ast (g^{(-1)}) \right)(t) = O(\varepsilon^2) \). Indeed, using successively
\[
(g^{(-1)})_x = \varepsilon (g^{(-2)})_x - \varepsilon (\partial_1 g^{(-2)})_x
\]
\[
(\partial_1 g^{(-2)})_x = \varepsilon (\partial_2 g^{(-3)})_x - \varepsilon (\partial_1^2 g^{(-3)})_x.
\]
Convolving with \( K_3 \) and integrating by parts,
\[
K_3 \ast (g^{(-1)})_x = \varepsilon K_3'' \ast (g^{(-2)})_x - \varepsilon K_3'' \ast (\partial_1 g^{(-3)})_x
+ \varepsilon^2 K_3' \ast (\partial_1^2 g^{(-3)})_x,
\]
the boundary terms vanish since \( K_3 ' \) has a bounded support. We already know the first two terms are \( O(\varepsilon^2) \). Using \( K_3' = (K_2 \ast K_1)' = K_2 \ast K_1 ' \), the last term reads \( \varepsilon^2 K_2 \ast \left( K_3 ' \ast (\partial_1^2 g^{(-3)})_x \right) \), hence is also \( O(\varepsilon^2) \) by lemma 3. The sum of the three terms is therefore also \( O(\varepsilon^2) \).

We finally prove \( \left( K_3 \ast g_x \right)(t) = O(\varepsilon^3) \). Indeed, using successively
\[
g_x = \varepsilon (g^{(-1)})_x - \varepsilon (\partial_1 g^{(-1)})_x
\]
\[
(\partial_1 g^{(-1)})_x = \varepsilon (\partial_2 g^{(-2)})_x - \varepsilon (\partial_1^2 g^{(-2)})_x
\]
\[
(\partial_1^2 g^{(-2)})_x = \varepsilon (\partial_1^3 g^{(-3)})_x - \varepsilon (\partial_1^3 g^{(-3)})_x,
\]
we find
\[
g_x = \varepsilon (g^{(-1)})_x - \varepsilon^2 (\partial_1 g^{(-2)})_x
+ \varepsilon^3 (\partial_1^2 g^{(-3)})_x - \varepsilon^3 (\partial_1^3 g^{(-3)})_x.
\]
Convolving with \( K_3 \) and integrating by parts,
\[
K_3 \ast g_x = \varepsilon K_3'' \ast (g^{(-1)})_x - \varepsilon^2 K_3' \ast (\partial_1 g^{(-2)})_x
+ \varepsilon^3 K_3' \ast (\partial_1^2 g^{(-3)})_x - \varepsilon^3 K_3 \ast (\partial_1^3 g^{(-3)})_x
\]
the boundary terms vanish since \( K_3 \) has a bounded support. We already know the first and third terms are \( O(\varepsilon^3) \). The second term reads
\[
\varepsilon^2 K_3' \ast (\partial_1 g^{(-2)})_x = \varepsilon^3 K_3' \ast (\partial_1 g^{(-3)})_x - \varepsilon^3 K_3' \ast (\partial_1^2 g^{(-3)})_x
\]
\[
= \varepsilon^3 K_3 \ast (K_3'' \ast (\partial_1 g^{(-3)})_x)
- \varepsilon^3 K_3' \ast (\partial_1^2 g^{(-3)})_x,
\]
and is also \( O(\varepsilon^3) \) by using Lemma 3 twice. Finally, by Young’s convolution inequality, the fourth term satisfies
\[
\| \varepsilon^3 K_3 \ast (\partial_1^3 g^{(-3)})_x \|_\infty \leq \varepsilon^3 \| K_3 \|_1 \| \partial_1^3 g^{(-3)} \|_\infty,
\]

hence is also \( O(\varepsilon^3) \); notice \( \partial_1^3 g^{(-3)} \) is bounded by assumption, and so is \( K_3 \). The sum of the four terms is therefore also \( O(\varepsilon^3) \), which concludes the proof.

\[
\text{Figure 2. Encoded signals } z_1(t), z_2(t), z_3(t).
\]

\[
\text{Figure 3. Carriers } s_1(t), s_2(t), s_3(t) \text{ (zoom).}
\]

\[
\text{Figure 4. Composite signal } y(t) \text{ (zoom).}
\]

\[
\text{IV. Numerical experiments}
\]

We illustrate the error analysis of Theorem 1 with numerical experiments for \( k = 1, 2, 3 \). Consider the composite signal \( y \) defined on \([0, 5]\) by
\[
y(t) = z_1(t)s_1(t, \frac{t}{5}) + z_2(t)s_2(t, \frac{t}{5}) + z_3(t)s_3(t, \frac{t}{5}) + d(t, \frac{t}{5}),
\]
with encoded signals \( z_1, z_2, z_3 \) (see Fig. 2)
\[
\begin{align*}
z_1(t) &= 2 \sin(t) - 1.5 \sin\left(\frac{t}{4}\right) \\
z_2(t) &= \cos(t) - 1.2 \sin\left(\frac{t}{4}\right) \\
z_3(t) &= 1.4 \cos\left(\frac{t}{4}\right)^2;
\end{align*}
\]
and carriers \( s_1, s_2, s_3 \) (see Fig. 3)
\[
\begin{align*}
s_1(t, \sigma) &= 1 \\
s_2(t, \sigma) &= \text{sign}\left(\frac{1}{20} + \sigma - 0.5\right) \\
s_3(t, \sigma) &= \begin{cases} 
\cos(t) + \sigma & \sigma \leq 0.5 \\
\cos(t) + 1 - \sigma & \sigma \geq 0.5,
\end{cases}
\]
where \( \sigma := \sigma \mod 1 \). The support of the disturbance \( d \) is
\[
D_{s} := \left[ f(t) - \frac{1}{20}, f(t) + \frac{1}{20} \right] \cup \left[ g(t) - \frac{1}{20}, g(t) + \frac{1}{20} \right],
\]
with \( f(t) := \frac{1}{4} (1 + \sin(t)) \) and \( g(t) := \frac{1}{4} (1 + \cos(t)) \); hence, on a window of length \( \varepsilon \) between 10% (when the two intervals coincide) and 20% (when the two intervals are disjoint) of the signal is corrupted. Fig 4 displays the resulting signal \( y \), with the spikes caused by \( d \) clearly visible.

We select the simplest demodulating basis that is zero on \( D_s \), namely \( R(t, \sigma) := (1 - \mathbb{1}_{D_s}(\sigma)) S(t, \sigma) \); tedious but routine computations show \( SR^T \) is invertible by plotting its condition number \( \kappa \), see Fig. 6: indeed, \( SR^T(t) \) is always well-conditioned, except during the filter initialization.

We focus on the recovery of \( z_2 \), since it is modulated by the least regular carrier. We consider the error \( e_k(t) := z_2(t) - P_k^2[y](t) \), where \( P_k^2[y] \) denotes the second component of \( P_k[y] \). For \( \varepsilon \) fixed, the error decreases as anticipated with \( k \), see Fig 5. To study the asymptotic behavior as a function of \( \varepsilon \), we consider the \( L_2 \)-error \( \| e_k \| := \left( \int_{-1}^{1} (e_k(t))^2 dt \right)^{1/2} \); the first second of data is discarded to ensure the filters are well initialized. As anticipated, the plots in log scale are straight lines with slopes equal to the orders of the estimates, see Fig. 7.

V. CONCLUSION

We have proposed a demodulation procedure to recover analog signals encoded by multiple carriers with slowly-varying shapes. Though the procedure is not completely surprising at first sight, proving that the overall demodulation error is arbitrarily small is not obvious. Arguably, the framework is somewhat peculiar, which explains why no similar work seems to exist in the literature. Nevertheless, the result is exactly what we need for the application we have in mind, namely the “sensorless” control of AC electric motors at or near standstill. In this application, the composite signal \( y \) to be decoded is the (vector) current in the motor, the motor itself acting as a multicarrier modulator when fed by a PWM inverter; a suitable processing of the demodulated signal then yields the rotor angle, which is needed to accurately control the motor.

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