Another presentation
of even orthogonal Steinberg groups

Egor Voronetsky *
Chebyshev Laboratory,
St. Petersburg State University,
14th Line V.O., 29B,
Saint Petersburg 199178 Russia

December 23, 2020

Abstract

We use the pro-group approach to prove that StO(2l, R) admits van der Kallen’s “another presentation” for any commutative ring R and l ≥ 3. Moreover, we construct an analog of ESD-transvections in even orthogonal Steinberg pro-groups under some assumptions on their parameters.

1 Introduction

In [7] W. van der Kallen proved that the linear Steinberg group St(n, R) over a commutative ring R is a central extension of the elementary linear group E(n, R) provided that n ≥ 4. More precisely, he showed that the Steinberg group admits a more invariant presentation and actually is a crossed module over the general linear group GL(n, R). This was generalized by M. Tulenbaev in [6] for linear groups over almost commutative rings R.

For symplectic groups the same result was proved by A. Lavrenov in [11]. He used essentially the same approach: there is another presentation of the symplectic Steinberg group StSp(2l, R) over a commutative ring R for l ≥ 3 such that it is obvious that this group is a central extension of the elementary symplectic group ESp(2l, R). Together with S. Sinchuk he also proved centrality of the corresponding K_2-functors for Chevalley groups of types D_l for l ≥ 3 and E_l in [2, 5] using a different method.

In [8] we reproved that St(n, R) is a crossed module over GL(n, R) using pro-groups. This more powerful method allowed to generalize the result for isotropic linear groups over almost commutative rings and for matrix linear groups over non-commutative rings with a local stable rank condition. The same result for

*The work was supported by the Theoretical Physics and Mathematics Advancement Foundation «BASIS».
odd unitary groups (including the Chevalley groups of type $A_l$, $B_l$, $C_l$, and $D_l$) was proved in [9]. Finally, together with Lavrenov and Sinchuk we generalized this result for all the remaining simple simply connected Chevalley groups of rank at least 3. For Chevalley groups of rank 2 there are counterexamples, see [10].

The pro-groups approach does not give any “another presentation” of the corresponding Steinberg group by itself. In this paper we prove the following:

**Theorem.** Let $R$ be a commutative ring and $l \leq 3$ be an integer. Then the even orthogonal Steinberg group $\text{StO}(2l, R)$ is isomorphic to the abstract group $\text{StO}^*(2l, R)$ generated by symbols $X^*(u, v)$, where $u, v \in R^{2l}$ are vectors, $u$ is a column of an elementary orthogonal matrix, and $u \perp v$. The relations on these symbols are

- $X^*(u, v + v') = X^*(u, v) X^*(u, v')$;
- $X^*(u, v) X^*(u', v') X^*(u, v)^{-1} = X^*(T(u, v)u', T(u, v)v')$, where $T(u, v)$ is the corresponding ESD-transvection;
- $X^*(u, vr) = X^*(v, -ur)$ if $v$ is also a column of an elementary orthogonal matrix;
- $X^*(u, ur) = 1$.

Actually, instead of columns of elementary orthogonal matrices it is possible to take columns of all orthogonal matrices. Using this variant of “another presentation”, it is obvious, that $\text{StO}(2l, R)$ is a crossed module over $\text{O}(2l, R)$.

During the proof we also give a general definition of ESD-transvections $X(u, v) \in \text{StO}(2l, R)$ for isotropic unimodular $u \in R^{2l}$ and $u \perp v$. These elements lift ordinary ESD-transvections $T(u, v) \in \text{SO}(2l, R)$ and satisfy various identities, but their existence for completely general $u$ is unclear.

The author wants to express his gratitude to Nikolai Vavilov, Sergey Sinchuk and Andrei Lavrenov for motivation and helpful discussions.

## 2 Orthogonal Steinberg pro-groups

We use the group-theoretical notation $gh = g h g^{-1}$ and $[g, h] = g h g^{-1} h^{-1}$. If a group $G$ acts on a group $H$ by automorphisms, we denote the action by $g h$.

Let $R$ be a commutative unital ring, $l \geq 3$ be an integer. We consider the free $R$-module $R^{2l}$ with the basis $e_{-l}, \ldots, e_{-1}, e_1, \ldots, e_l$. This module has the split quadratic form $q$ given by $q(v) = \sum_{i=1}^{l} q_i q_{-i}$. Let $\langle u, v \rangle = q(u + v) - q(u) - q(v)$ be the associated symmetric bilinear form. The even orthogonal group $\text{O}(2l, R)$ consists of $g \in \text{GL}(2l, R)$ such that $q(gv) = q(v)$ for all vectors $v$.

Consider any vectors $u, v \in R^{2l}$ such that $q(u) = \langle u, v \rangle = 0$. In this case the operator

$$T(u, v) : R^{2l} \to R^{2l}, w \mapsto w + u(v, w) - v(u, w) - uq(v)\langle u, w \rangle$$
is called an Eichel–Siegel–Dickson transvection (or an ESD-transvection) with parameters $u$ and $v$, see [4] for details in a more general context. The following lemma summarized the well-known properties of these operators, all of them may be checked by direct computations.

**Lemma 1.** Each ESD-transvection lies in the special orthogonal group $SO(2l, R)$. Moreover,

- $T(u, v)T(u, v') = T(u, v + v')$ for $q(u) = (u, v) = (u, v') = 0$;
- $T(ur, v) = T(u, vr)$ for $q(u) = (u, v) = 0$ and $r \in R$;
- $\delta T(u, v) = T(gu, gv\lambda^{-1})$ for $q(u) = (u, v) = 0$, $g \in GO(2l, R)$, and $q(gw) = \lambda q(w)$ for all $w \in R^{2l}$;
- $T(u, v) = T(v, -u)$ for $q(u) = q(v) = (u, v) = 0$;
- $T(u, ur) = 1$ for $q(u) = 0$ and $r \in R$.

Recall that elementary orthogonal transvections are the elements $t_{ij}(r) = T(e_i, e_{-j}r)$ for $i \neq \pm j$. Clearly, $T(e_i, v) = \prod_{j \neq \pm i} t_{ij}(v_{-j})$ for $(e_i, v) = 0$.

The orthogonal Steinberg group $StO(2l, R^\infty)$ is the abstract group generated by elements $x_{ij}(r)$ for $r \in R$ and $i \neq \pm j$. The relations on these elements are the following:

- $x_{ij}(r + r') = x_{ij}(r)x_{ij}(r')$;
- $x_{ij}(r) = x_{-j,-i}(-r)$;
- $[x_{ij}(r), x_{kl}(r')] = 1$ for $j \neq k \neq -i \neq -l \neq j$;
- $[x_{ij}(r), x_{j,-i}(r')] = 1$;
- $[x_{ij}(r), x_{jk}(r')] = x_{ik}(rr')$ for $i \neq \pm k$.

There is a group homomorphism $\text{st}: StO(2l, R^\infty) \to SO(2l, R^\infty), x_{ij}(r) \mapsto t_{ij}(r)$.

We would like to find analogs of ESD-transvections in the Steinberg group $StO(2l, R^\infty)$. In order to do so, we use results from [3, 8, 9]. Recall that there is a "forgetful" functor from the category of pro-groups $\text{Pro(Grp)}$ to the category of group objects in the category of pro-sets $\text{Pro(Grp)}$. This functor is actually fully faithful, so we identify pro-groups with the corresponding pro-sets, and similarly for non-unital pro-$R$-algebras. The projective limits in $\text{Pro(Grp)}$ are denoted by $\lim\text{Pro}$, and various pro-sets are labeled with an upper index $(\infty)$, such as $X^{(\infty)}$.

We also use the following convention from [3, 8, 9]: if a morphism between pro-sets is given by a first order term (possibly many-sorted), then we add the upper index $(\infty)$ for the formal variables. For example, $[g^{(\infty)}, h^{(\infty)}]$ denotes the commutator morphism $G^{(\infty)} \times G^{(\infty)} \to G^{(\infty)}$ for a pro-group $G^{(\infty)}$, and $r^{(\infty)}m^{(\infty)}$ denotes the product morphism $R^{(\infty)} \times M^{(\infty)} \to M^{(\infty)}$ for a pro-ring $R^{(\infty)}$ and its pro-module $M^{(\infty)}$. The domains of such variables are usually clear from the context.
For any $s \in R$ an $s$-homotope of $R$ is the commutative non-unital $R$-algebra $R^{(s)} = \{ r^{(s)} \mid r \in R \}$. The operations are given by

\[ r^{(s)} + r^{(s)} = (r + r')^{(s)}, \quad rr^{(s)} = (rr')^{(s)}, \quad r^{(s)}r^{(s)} = (srr')^{(s)}. \]

Note that in the definition of $\text{StO}(2l, R)$ we do not need the unit of $R$. Hence we define $\text{StO}^{(s)}(2l, R) = \text{StO}(2l, R^{(s)})$ by the same generators and relations, but with parameters in $R^{(s)}$. If $s, s' \in R$, then there is a homomorphism $R^{(s s')} \to R^{(s)}, r^{(s s')} \mapsto (s' r)^{(s)}$ of commutative non-unital $R$-algebras, and an obvious homomorphism $\text{StO}^{(s s')}(2l, R) \to \text{StO}^{(s)}(2l, R)$ of their Steinberg groups.

Fix a multiplicative subset $S \subseteq R$. The formal projective limit $R^{(\infty, S)} = \lim_{\longrightarrow} R^{(s)}$ is actually a commutative non-unital pro-$R$-algebra. Similarly, the Steinberg pro-group $\text{StO}^{(\infty, S)}(2l, R) = \lim_{\longrightarrow} \text{StO}^{(s)}(2l, R)$ is indeed a pro-group. If $S = R \setminus \mathfrak{p}$ for a prime ideal $\mathfrak{p}$, then we write $R^{(\infty, \mathfrak{p})}$ and $\text{StO}^{(\infty, \mathfrak{p})}(2l, R)$ instead of $R^{(\infty, S)}$ and $\text{StO}^{(\infty, S)}(2l, R)$. Similarly, if $S = \{1, f, f^2, \ldots\}$ for $f \in R$, then we write $f$ instead of $S$ in the indices. Finally, the constructions $R^{(\infty, S)}$ and $\text{StO}^{(\infty, S)}(2l, R)$ are contravariant on $S$, in the case $S = \{1\}$ we get $R$ and $\text{StO}(2l, R)$ up to a canonical isomorphism. There are well-defined morphisms $x_a : R^{(\infty, S)} \to \text{StO}^{(\infty, S)}(2l, R)$ of pro-groups, they generate the Steinberg pro-group in the categorical sense by results from \[3\] or \[9\].

**Lemma 2.** Let $S \subseteq R$ be a multiplicative subset. Consider the family $\pi_m : R^{(\infty, m)} \to R^{(\infty, S)}$ of morphisms of pro-groups (and pro-$R$-algebras), where $\mathfrak{m}$ runs over all maximal ideals of $R$ disjoint with $S$. Then this family generates $R^{(\infty, S)}$ in the following sense: for every pro-group $G^{(\infty)} \in \text{Pro(Grp)}$ the map

\[ \text{Pro(Grp)}(R, G^{(\infty)}) \to \prod_{\mathfrak{m}} \text{Pro(Grp)}(R^{(\infty, \mathfrak{m})}, G^{(\infty)}) \]

is injective.

**Proof.** If suffices to consider an ordinary group $G$. Consider group homomorphisms $f, g : R^{(s)} \to G$ for some $s \in S$ such that $f \circ \pi_m = g \circ \pi_m : R^{(\infty, m)} \to G$ for all maximal ideals $\mathfrak{m}$ disjoint with $S$. The set

\[ \mathfrak{a} = \{ r \in R \mid f|_{R^{(s)}} = g|_{R^{(s)}} \} \]

is an ideal of $R$. By assumption, it is not contained in any such $\mathfrak{m}$, so $S^{-1}\mathfrak{a} = S^{-1}R$. In other words, $s' \in \mathfrak{a}$ for some $s' \in S$. This means that $f|_{R^{(s s')}} = g|_{R^{(s s')}}$. \( \square \)

Take a multiplicative subset $S \subseteq R$. Each $\frac{a}{s} \in S^{-1}1$ gives an endomorphism of the pro-group $R^{(\infty, S)}$ by $\frac{a}{s^{n}} : b^{(s^{n}s')} \mapsto (ab)^{(s')}$, This endomorphism is denoted by the term $\frac{a}{s^{n}} r^{(\infty)}$ with a free variable $r^{(\infty)}$. By results from \[3\] or \[9\], for any multiplicative subset $S \subseteq R$ the local Steinberg group $\text{StO}(2l, S^{-1} R)$ acts on the corresponding Steinberg pro-group $\text{StO}^{(\infty, S)}(2l, R)$ by automorphisms. Note that this action is not given by a morphism $\text{StO}(2l, S^{-1} R) \times$
over, it acts transitively on the set of pairs (unimodular vectors (i.e. \(v\) isotropic vectors and \(u\) unimodular vectors).

We need the following well-known result:

To prove the second claim, take such a pair \((u, v)\). Since \(u\) is unimodular and isotropic, we may assume that \(u = e_l\). Then \(v_{-l} = 0, l \geq 2, \) and \(v - v_{i} e_l\) is a unimodular isotropic vector in \(R^{2l-2}\). All matrices from EO\((2l - 2, R)\) fix \(u, v_{i} e_l\) respectively, and the conclusion follows.

3 Transvections in orthogonal Steinberg groups

We need the following well-known result:

\[ \text{EO}(2l, R) = \text{Im}(\text{st}: \text{St}(2l, R) \to \text{SO}(2l, R)) \]

acts transitively on the set of isotropic unimodular vectors (i.e. \(v \in R^{2l}\) such that \(R = \sum_i R v_i\) and \(q(v) = 0\)). Moreover, it acts transitively on the set of pairs \((u, v)\), where \(u, v\) are orthogonal isotropic vectors and \(u \wedge v\) is unimodular in \(\Lambda^2(R^{2l})\).

Proof. Let \(v \in R^{2l}\) be an isotropic unimodular vector. If \(v_l\) or \(v_{-l}\) is invertible, then \(v' = T(e_1, ae_{\pm l})\) satisfy \(v'_i \in R^*\) for a suitable \(a \in R\). In this case let \(j = 1\).

Otherwise there is an index \(j \neq \pm l\) such that \(v_j \in R^*\), so we take \(v' = v\).

In any case, there is unique \(b \in R\) such that \(v'' = T(e_1, be_{-j})v'\) satisfies \(v''_i = 1\). Now let \(v''' = \prod_{i \neq j} T(e_i, -v'_i e_{-i})v''\), it still has the property \(v'''_i = 1\). Moreover, \(v'''_i = 0\) for \(i \neq \pm l\). Since \(v'''\) is isotropic, \(v''' = e_l\).

To prove the second claim, take such a pair \((u, v)\). Since \(u\) is unimodular and isotropic, we may assume that \(u = e_l\). Then \(v_{-l} = 0, l \geq 2, \) and \(v - v_{i} e_l\) is a unimodular isotropic vector in \(R^{2l-2}\). All matrices from EO\((2l - 2, R)\) fix \(u, v_{i} e_l\) respectively, and the conclusion follows.
hence by the above we may assume that $u = e_t$ and $v = e_{l-1} + \nu_{l}e_t$. It remains to apply the transvection $T(e_t, -\nu_{l}e_{l-1})$ to get the pair $(e_t, e_{l-1})$.  

Let $S \subseteq R$ be a multiplicative subset and $u \in S^{-1} R^{2l}$ be an isotropic unimodular vector. There is a well-defined split epimorphism $\langle u, v(\infty) \rangle = \sum_{i} u_i v^{(\infty)}_i : (R(\infty,S))^{2l} \to R^{(\infty,S)}$ of pro-groups, i.e. its kernel $\ker(\langle u, v(\infty) \rangle)$ has a direct complement isomorphic to $R^{(\infty,S)}$. If $g \in \text{GO}(2l, S^{-1} R)$, then $g$ maps the direct summand $M_u^{(\infty,S)}$ of $(R^{(\infty,S)})^{2l}$ to $M_u^{(\infty,S)}$. Our goal is to construct morphisms $X(u,v^{(\infty)}): M_u^{(\infty,S)} \to \text{StO}^{(\infty,S)}(2l,R)$ of pro-groups satisfying various natural conditions.

**Lemma 4.** Let $\mathfrak{m}$ be a maximal ideal of $R$. Then for all isotropic unimodular vectors $u \in R^{2l}_\mathfrak{m}$ there are unique morphisms $X(u,v^{(\infty)}): M_u^{(\infty,\mathfrak{m})} \to \text{StO}^{(\infty,\mathfrak{m})}(2l,R)$ of pro-groups such that

1. $9X(u,v^{(\infty)}) = X(gu,gv^{(\infty)})$ for $g \in \text{O}(2l,R_\mathfrak{m})$;
2. $X(u, ur^{(\infty)}) = 1$;
3. $X(u, vr^{(\infty)}) = X(u,v^{(\infty)}r)$ for $r \in R^{*}_\mathfrak{m}$;
4. $X(u, ur^{(\infty)}) = X(v,-ur^{(\infty)})$ for orthogonal isotropic $u,v \in R^{2l}_\mathfrak{m}$ such that $u \wedge v$ is unimodular;
5. $X(e_i,e_j r^{(\infty)}) = x_{i-j}(r^{(\infty)})$ for $i \neq \pm j$.

**Proof.** Note that $M_{e_t}^{(\infty,\mathfrak{m})} = \bigoplus_{i \neq -l} e_t R^{(\infty,\mathfrak{m})}$. Let

$$X(e_t, v^{(\infty)}) = \prod_{0 < i < l} (x_{e_t + e_i}(v^{(\infty)}_i) x_{e_t - e_i}(v^{(\infty)}_{-i})): M_{e_t}^{(\infty,\mathfrak{m})} \to \text{StO}^{(\infty,\mathfrak{m})}(2l,R),$$

this is the only possible choice satisfying the second and the last conditions.

The subgroup $\{g \in \text{Spin}(2l,R_\mathfrak{m}) \mid ge_t \in e_t R_\mathfrak{m}\}$ coincides with the standard maximal parabolic subgroup $P_1 \leq \text{Spin}(2l,R_\mathfrak{m})$. It is generated by the standard maximal torus of the spin group and the elementary transvections stabilizing $e_t$, since $R_\mathfrak{m}$ is local. It is easy to see that $X(e_i, v^{(\infty)}) = X(e_i, gv^{(\infty)} \frac{ge_t}{e_t})$ for every such a generator $g \in P_1$: the torus acts by roots on both sides (see formula (4.2) in [1]), and for the elementary transvections this follows directly from the definitions. Here $\frac{ge_t}{e_t}$ denotes the element of $R^{*}_\mathfrak{m}$ such that $ge_t = e_t \frac{ge_t}{e_t}$.

Now let $u \in R^{2l}_\mathfrak{m}$ be any isotropic unimodular vector. By lemma [2] there is $g \in \text{Spin}(2l,R_\mathfrak{m})$ such that $u = ge_t$. Let $X(u,v^{(\infty)}) = gX(e_t, v^{(\infty)})$, this morphism is independent on $g$ and satisfies the properties 1, 2, 5.

To prove the third property, note that there is $h \in \text{Spin}(2l,R)$ such that $he_t = e_t r$ and $h$ lies in the maximal torus. Hence

$$X(e_t r, v^{(\infty)}) = hX(e_t, h^{-1} v^{(\infty)}) = X(e_t, v^{(\infty)} \frac{h v^{(\infty)}}{e_t r}) = X(e_t, v^{(\infty)} r),$$

and the third property for other vectors $u$ follows from the definition of $X(u,v^{(\infty)})$.  


These morphisms also satisfy the following identities:

\[ S \text{ isotropic unimodular vector. Then there exists at most one morphism } \]

\[ X(e_i, e_{i-1} r^{(\infty)}) = x_{i,1-i}(r^{(\infty)}) = x_{i-1,-i}(-r^{(\infty)}) = X(e_{i-1}, -e_i r^{(\infty)}). \]

Now let us return to the general case.

**Theorem 1.** Let \( S \subseteq R \) be a multiplicative subset and \( u \in S^{-1}R^{2l} \) be an isotropic unimodular vector. Then there exists at most one morphism \( X(u, v^{(\infty)}): M_u(\infty,S) \to \text{StO}(\infty,S)(2l, R) \) of pro-groups such that for any maximal ideal \( m \) of \( R \) disjoint with \( S \) the following diagram with canonical vertical arrows is commutative:

\[
\begin{array}{ccc}
M_u(\infty,m) & \xrightarrow{X(u,v^{(\infty)})} & \text{StO}(\infty,m)(2l, R) \\
\downarrow & & \downarrow \\
M_u(\infty,S) & \xrightarrow{X(u,v^{(\infty)})} & \text{StO}(\infty,S)(2l, R).
\end{array}
\]

These morphisms also satisfy the following identities:

1. \( g X(u, v^{(\infty)}) = X(g u, g v^{(\infty)}) \) for \( g \in \text{StO}(2l, S^{-1}R) \) if the left hand side is defined;
2. \( X(u, ur^{(\infty)}) = 1 \) if the left hand side is defined;
3. \( X(ur, v^{(\infty)}) = X(u, v^{(\infty)} r) \) for \( r \in S^{-1}R^* \) if both sides are defined;
4. \( X(u, vr^{(\infty)}) = X(v, -ur^{(\infty)}) \) for orthogonal isotropic \( u, v \in S^{-1}R^{2l} \) such that \( u \wedge v \) is unimodular and both sides are defined;
5. \( X(e_i, e_{i-j} r^{(\infty)}) = x_{i,-j}(r^{(\infty)}) \) for \( i \neq \pm j \).

**Proof.** By lemma \( 2 \) the pro-group \( M_u(\infty,S) \) is generated by the canonical morphisms \( M_u(\infty,m) \to M_u(\infty,S) \) for all maximal ideals \( m \) of \( R \) disjoint with \( S \). Hence the morphisms \( X(u, v^{(\infty)}) \) are unique whenever they exist. Clearly, such a morphism exists and satisfies the last property for some basis vector \( u = e_i \). The set of isotropic unimodular vectors \( u \in S^{-1}R^{2l} \) such that \( X(u, v^{(\infty)}) \) exists is closed under the action of \( \text{StO}(2l, S^{-1}R) \) by extranaturality of the action. All other properties follow from lemmas \( 2 \) and \( 4 \). \( \square \)

**Theorem 2.** Let \( S \subseteq R \) be a multiplicative subset, \( u \in S^{-1}R^{2l} \) be an isotropic unimodular vector. The morphism \( X(u, v^{(\infty)}): M_u(\infty,S) \to \text{StO}(\infty,S)(2l, R) \) from theorem \( 1 \) exists in the following cases:

1. \( S = R \setminus m \) for a maximal ideal \( m \);
2. \( u \) lies in the orbit of \( e_l \) under the action of \( \text{EO}(2l, S^{-1}R) \);
3. \( S = \{1\} \) and \( u \) is a column of an orthogonal matrix.

7
In the last case \( X(u, v) = X(gu, gv) \in \mathrm{StO}(2l, R) \) for \( g \in O(2l, R) \), \( u \in O(2l, R)e_i \), and \( u \perp v \). Also \( X(u, v) \in \mathrm{StO}(2l, R) \) maps to \( T(u, v) \in O(2l, R) \).

Proof. The first case is lemma [4], the second is theorem [1]. If \( S = \{1\} \) and \( u = ge_i \) for some \( g \in O(2l, R) \), then let \( X(u, v) = X(e_i, g^{-1}v) \) for all vectors \( v \perp u \). This morphism satisfies the definition from theorem [1] by extranaturality of the action. The last claims follows from the uniqueness of \( X(u, v) \) and the definitions.

Finally, we prove that the even orthogonal Steinberg group admits “another presentation” in the sense of van der Kallen. Note that the third property of \( X(u, v) \) from theorem [1] is not needed.

**Theorem 3.** Let \( \mathrm{StO}^*(2l, R) \) be the abstract group generated by symbols \( X^*(u, v) \) for vectors \( u, v \in R^{2l} \) such that \( u \in O(2l, R)e_i \) and \( v \perp u \). The relations are the following:

- \( X^*(u + v', v) = X^*(u, v) X^*(u', v) \);
- \( X^*(u, v') X^*(u', v') X^*(u, v)^{-1} = X^*(T(u, v)u', T(u, v)v') \);
- \( X^*(u, v) \) if \( v \) also lies in \( O(2l, R)e_i \);
- \( X^*(u, vr) = 1 \).

Then the canonical morphism \( \mathrm{StO}(2l, R) \to \mathrm{StO}^*(2l, R) \), \( x_{ij}(r) \mapsto X^*(e_i, e_{-j}r) \) is an isomorphism, the preimage of \( X^*(u, v) \) is the element \( X(u, v) \) from theorem [1]. Also we may write everywhere \( \mathrm{Spin}(2l, R)e_i \) or \( \mathrm{EO}(2l, R)e_i \) instead of \( O(2l, R)e_i \).

Proof. Let \( F : \mathrm{StO}(2l, R) \to \mathrm{StO}^*(2l, R) \) be the homomorphism from the statement. By theorems [1] and [2], there is a homomorphism \( G : \mathrm{StO}^*(2l, R) \to \mathrm{StO}(2l, R) \), \( X^*(u, v) \mapsto X(u, v) \). Clearly, \( G \circ F = \text{id} \).

The group \( O(2l, R) \) (or \( \mathrm{Spin}(2l, R) \) or \( \mathrm{EO}(2l, R) \)) acts on \( \mathrm{StO}^*(2l, R) \) by \( X^*(u, v) = X^*(gu, gv) \). Clearly, \( \mathrm{StO}^*(2l, R) \) is generated by the image of \( F \) and its conjugates under this action (here we need that \( u \) lies in the orbit of \( e_i \)). Since \( \mathrm{StO}(2l, R) \) is perfect, it follows that \( \mathrm{StO}^*(2l, R) \) is perfect. The canonical homomorphism \( \mathrm{StO}^*(2l, R) \to O(2l, R), X^*(u, v) \mapsto T(u, v) \) has central kernel by the second relation, hence the kernel of \( G \) is also central.

Now \( G : \mathrm{StO}^*(2l, R) \to \mathrm{StO}(2l, R) \) is a split perfect central extension. It is well-known that such a homomorphism is necessarily an isomorphism, its splitting \( F \) is the inverse.

**References**

[1] A. Lavrenov. Another presentation for symplectic Steinberg groups. *J. Pure Appl. Alg.*, 219(9):3755–3780, 2015.
[2] A. Lavrenov and S. Sinchuk. On centrality of even orthogonal $K_2$. *J. Pure Appl. Alg.*, 221(5):1134–1145, 2017.

[3] A. Lavrenov, S. Sinchuk, and E. Voronetsky. Centrality of odd unitary $K_2$-functor, 2020. Preprint, arXiv:2009.03999.

[4] V. Petrov. Odd unitary groups. *J. Math. Sci.*, 130(3):4752–4766, 2005.

[5] S. Sinchuk. On centrality of $K_2$ for Chevalley groups of type $E_t$. *J. Pure Appl. Alg.*, 220(2):857–875, 2016.

[6] M.S. Tulenbaev. Schur multiplier of the group of elementary matrices of finite order. *J. Sov. Math.*, 17(4):2062–2067, 1981.

[7] W. van der Kallen. Another presentation for Steinberg groups. *Indag. Math.*, 80(4):304–312, 1977.

[8] E. Voronetsky. Centrality of $K_2$-functor revisited. *J. Pure Appl. Alg.*, 225(4), 2020. published electronically.

[9] E. Voronetsky. Centrality of odd unitary $K_2$-functor, 2020. Preprint, arXiv:2005.02926.

[10] M. Wendt. On homotopy invariance for homology of rank two groups. *J. Pure Appl. Alg.*, 216(10):2291, 2012.