Exactly solvable chain of interacting electrons with correlated hopping and pairing

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A generalization of the Mattis-Nam model (J. Math. Phys., 13 (1972), 1185), which takes into account a correlated hopping and pairing of electrons, is proposed, its exact solution is obtained. In the framework of the model the stability of the zero energy Majorana fermions localized at the boundaries is studied in the chain in which electrons interact through both the one-site Hubbard interaction \( U \) and the correlated hopping and pairing \( t \). The ground-state phase diagram of the model is calculated in the coordinates \( t - U \), the region of existence of topological states is determined. It is shown that low-energy excitations destroy bonds between fermions in the chain, leading to a dielectric state.

INTRODUCTION

Behavior of electrons interacting via short-range interaction is described in the framework of the Hubbard model, exact solution of which has been obtained in one dimension in [1] (see also [2]). The Hubbard model with correlated hopping on a chain has been proposed and solved exactly in [3][5]. Mattic and Nam (MN) proposed modification of the Hubbard model for interacting electrons forming pairs, and solved it exactly in special point [6] (better known as the Kitaev point [7]). In contrast to traditional Hubbard model [1], the MN model describes topological states of interacting electrons [7–9], quantum topological phase transition between topological trivial and nontrivial phases. In this context, it is interesting to discuss a new model, which is a modification [3][6], the exact solution of which takes place for arbitrary one-site interaction, correlated hopping and pairing.

The realization of topological states in real systems is determined by the stability of the topological phase in the presence of interaction and disorder. Accounting for the defective structure of the material does not presuppose a special complexity, since nonlinearity in the system, as a rule, does not arise [10][13]. The interaction between electrons destroys the topological state of the system, the task is how to approach the solution of the key problem. Low-dimensional quantum models can be solved exactly at certain points, corresponding to defined values of the parameters. Using the example of a chain of interacting electrons, MN determined the region of stability of topological states with an arbitrary value of the interaction between electrons [6].

In the paper, we considered an extended modification of the MN model, taking into account also the correlated hopping and pairing of electrons. Using the MN approach, we will show, that the model has exact solution for arbitrary values of one-site interaction and correlated hopping and pairing. On the phase diagram, topological trivial and nontrivial phases are separated by the lines of quantum topological phase transitions. Spinless fermions move in a static \( Z_2 \) field, which is uniform in the ground state, similar to the Kitaev model [14]. ‘Defect’ in the \( Z_2 \) configuration breaks two bonds between electron and its nearest-neighbors, forming an isolated state on the site. Such type of excitations leads to transition to insulator state of the chain.

THE MODEL HAMILTONIAN

The Hamiltonian of the model is the sum of two terms, the first of which is determined in accordance with the MN model, the second takes into account the correlated hopping and pairing term within this model \( \mathcal{H} = \mathcal{H}_{MH} + \mathcal{H}_{ch} \)

\[
\mathcal{H}_{MN} = -\sum_{j=1}^{N-1} \sum_{\sigma = \uparrow, \downarrow} (c_{j,\sigma}^\dagger c_{j,\sigma} + c_{j,\sigma} c_{j+1,\sigma})(c_{j+1,\sigma}^\dagger - c_{j+1,\sigma}) + U \sum_{j=1}^{N} X_j,
\]

\[
\mathcal{H}_{ch} = -t \sum_{j=1}^{N-1} \sum_{\sigma = \uparrow, \downarrow} X_j (c_{j,\sigma}^\dagger + c_{j,\sigma})(c_{j+1,\sigma}^\dagger + c_{j+1,\sigma})X_{j+1},
\]

where \( c_{j,\sigma}^\dagger, c_{j,\sigma} (\sigma = \uparrow, \downarrow) \) are the fermion operators determined on a lattice site \( j, U \) is the value of the one-site Hubbard interaction determined by the operator \( X_j = (n_{j,\uparrow} - \frac{1}{2})(n_{j,\downarrow} - \frac{1}{2}) \), here \( n_{j,\sigma} = c_{j,\sigma}^\dagger c_{j,\sigma} \). \( t \) defines the correlated hopping and pairing of fermions.

Using the Jordan-Wigner transformation and the MN transformation of the spin operators via new set of the spin-\( \frac{1}{2} \) matrices \( P_j^\alpha \) and \( J_j^\alpha \) [6], we redefine the total Hamiltonian in the following form

\[
\mathcal{H} = -\sum_{j=1}^{N-1} (t J_j^+ J_{j+1}^- + 4 J_j^x J_{j+1}^x)(4 P_j^z P_{j+1}^z + 1)
\]

\[+ \frac{1}{2} U \sum_{j=1}^{N} J_j^z.
\]

The \( P_j^z \) operators commute with the total Hamiltonian, they are the integrals of the motion. The ground
state energy of the spin chain corresponds to a nontrivial Hamiltonian \( \mathcal{H} \) with \( P_j^x P_{j+1}^x = \frac{1}{2} \), for even number of sites in the chain the ground state is twice degenerated. The Hamiltonian \( \mathcal{H} \) describes the XY-Heisenberg spin-\( \frac{1}{2} \) chain in a magnetic field \( \mathcal{H} = -2 \sum_{j=1}^{N-1} (J_j^x J_{j+1}^x + 4J_j^z J_{j+1}^z) + \frac{1}{2}U \sum_{j=1}^{N} J_j^z \). Redetermine the Hamiltonian \( \mathcal{H} \) in the operators of spinless fermions \( a_j^\dagger \) and \( a_j \)

\[
\mathcal{H} = \sum_{j=1}^{N-1} \left[ \left( 2 + \frac{1}{2}t \right) (a_{j+1}^\dagger a_j + a_j^\dagger a_{j+1}) + \left( 2 - \frac{1}{2}t \right)(a_{j+1}^\dagger a_{j+1} + a_{j+1} a_j) \right] + \frac{1}{2}U \sum_{j=1}^{N} \left( a_j^\dagger a_j - \frac{1}{2} \right) \tag{4}
\]

THE GROUND-STATE PHASE DIAGRAM

The total Hamiltonian with the on-site Hubbard interaction and correlated hopping and pairing is mapped to a
noninteracting model of spinless fermions, namely to the XY-Heisenberg spin-$\frac{1}{2}$ chain [4]. The spectrum of spinless fermions is symmetric with respect to zero energy, equal to

$$\epsilon(k) = \pm \sqrt{\left[\frac{U}{4} - \left(2 + \frac{t}{2}\right) \cos k\right]^2 + \left(2 - \frac{t}{2}\right)^2 \sin^2 k},$$

(5)

where $k$ is the wave vector of fermion excitations along the chain.

At the point of the topological phase transition the gap in the excitation spectrum $\Delta$ disappears, the point of the phase transition separates trivial and nontrivial topological phases. The gaps in the spectrum disappear at $t = -4 \pm \frac{|U|}{2}$ (for arbitrary $U$) and at an additional point $t = 4$ for $|U| < 16$. At $t = -4$ for $U = 0$ (see in figure 1(h)) and $t = 4$ for $U < 16$ (see in figure 1(h)-c) the phase state of the system is not changed at the points of the phase transitions. At $U = 0$ the point of the spectrum $t = -4$ is twice degenerated (see in figure 1(h)), for $|U| < 16$ the point $t = 4$ is also twice degenerated (see in figure 1(h),c)). The one-site Hubbard interaction $U \neq 0$ and $|U| > 16$ removes the degeneration of energy at the points $t = -4$ (see in figure 1(h)-c) and $t = 4$ (see in figure 1(h)) and the phase transition points separate the trivial and nontrivial topological phases.

At $U = 0$ the topological state, which is characterized by the zero energy Majorana fermions, localized at the boundaries [7], is realized for an arbitrary $t$. The behavior of the wave function of zero energy Majorana fermions is shown in figure 2. The gaps in the spectrum vanish at $t = \pm 4$ (see in figure 2), these points are twice degenerated, as a result, the phase states are not changed at the points of the phase transitions. When accounting for the Hubbard interaction, the spectrum degeneracy is lifted and it splits. The region of existence of the trivial topological phase is determined by the value of the splitting (see in figure 1(h),c)). At $|U| \geq 16$ the fermion spectrum is gapless between two points in the coordinates $t$ (see in figure 2). The zero energy Majorana fermions are realized at $|U| < 8 + 2t$ or $|U| > -8 - 2t$ (we don't fix the signum of $t$ now). As we noted above, these points are the points of the topological phase transitions, they determine the region of stability of the phases, $|U| = 8 + 2t$ are the lines of the topological phase transitions on the ground state phase diagram, which separate trivial and nontrivial topological states of the chain with strong interaction between electrons (see in figure 3). In topological trivial phase zero energy Majorana fermions are absent (see in figure 2). The one-site Hubbard interaction kills the zero energy Majorana fermions at $|U| > 8 + 2t$, it limits the ambitions of topological phase in the chain with strong interaction between electrons. The correlated hopping and pairing considered in the model Hamiltonian expands the region of existence of the topological phase when it has the same sign with the single-particle hopping integral, and decreases it when they have different signs (see in figure 3). Near the point $t = -4$ the topological state is not stable with respect to small fluctuations of the one-site Hubbard interaction, small fluctuations of $U$ open the gap in the fermion spectrum, stabilizing the topological trivial state in the chain.

**INSULATOR STATE AS A RESULT OF EXCITATIONS IN THE CHAIN**

At half filling the Fermi energy is equal to zero, electrons are paired in the pairs which form condensate. Consider excitations in topological state, as a example the case $U = 2$, $t = 1$ is shown in figure 4. In this case the energy of isolated electron, equal to $\frac{t}{2}$, lies into the gap. Hight energy subband corresponds to one-particle excitations of electrons. As we note in previous section the ground state of the chain corresponds to uniform configurations of $P_{jz}$ operators, which are integrals of motion $P_{jz} = \frac{1}{2}$ or $P_{jz} = -\frac{1}{2}$. We show, that in the chain with periodic boundary conditions, one 'P$^-$-defect' at $l$-site $(P_{lj}^z \neq P_{lj}^z)$ leads to an isolator state of the chain. According to [3] one 'P$^-$-defect' forms an fermion, isolated on $l$-site, as a result, we obtain the chain with open boundary conditions. One electron pair decays into one electron with energy $\frac{U}{2}$, localized at $l$-site, and zero energy Majorana fermions localized at the boundaries $l-1$ and $l+1$. This state, which has a minimum excitation energy $\frac{U}{2} < \frac{3}{2}$, corresponds to the breaking of bonds between electrons in a chain (between $l-1$ and $l+1$ sites), triggers isolator state of the chain.

**CONCLUSION**

We have considered the exact solution of a two-parameter family of 1D models of electrons interacting via the one-site Hubbard repulsion and correlated hopping and pairing. The model Hamiltonian reduces to non-interacting spinless fermions hopping and pairing in the background of static $Z_2$ field configurations, the
ground state corresponds to uniform configurations. It is shown that the excitations corresponding to the defect in the static field configuration leads to breaking of the bonds between the electrons, induces the dielectric state of the chain. The ground-state phase diagram includes topological trivial and nontrivial phases. Criteria of realization topological state, which characterized by zero energy Majorana fermions localized at the boundaries are calculated for arbitrary values of one-site Hubbard interaction and integral of correlated hopping and pairing.

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