A Large-Deviation Upperbound on Directed Last-Passage Percolation Growth Rate Based on Entropy of Direction Vector

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Abstract

This short note provides a large-deviation-based upper bound on the growth rate of directed last passage percolation (LPP) using the entropy of the normalized direction vector.

1 Growth Rate in LPP

Consider the $d$-dimensional integer lattice $\mathbb{Z}^d$. For each point $z \in \mathbb{Z}^d$ we associate a random variable $X(z)$, with finite expectation $\mu := E[X(z)]$. The stochastic process $X(z)$ is i.i.d. across $z$.

A directed path from the origin to $z$ is a finite sequence of elements of $\mathbb{Z}^d$ such that the difference of two consecutive elements is a unit vector that has all zero entries except one location. For example, for $d = 2$, these constitute up-right paths on $\mathbb{Z}^2$. Let $\Pi(z)$ be the set of all directed paths from the origin to $z$. The last passage time from the origin to $z$ is defined as

$$T(z) = \max_{\pi \in \Pi(z)} \sum_{v \in \pi} X(v),$$

which is the random variable given by the weight of the directed path $\pi$ from the origin to $z$ with the maximum sum. Despite the terminology “last passage time”, we allow the random variables $X(z)$ to be negative. The terminology is a carry-over from first-passage percolation time where often the random variables are required to be positive, even though that will not be necessary herein.

We will be interested in the function

$$g(x) = \sup_{n \in \mathbb{N}} \frac{1}{n} E[T(\lfloor nx \rfloor)]$$

where the floor function $\lfloor \cdot \rfloor$ is interpreted elementwise. From $\Pi$ we have the following properties for $g(x)$.
\[ \lim_{n \to \infty} \frac{1}{n} T(\lfloor nx \rfloor) = g(x), \text{ almost surely.} \]
\[ g(\alpha x) = \alpha g(x) \text{ for } \alpha > 0 \]
\[ g(x) \text{ is a symmetric function} \]
\[ g(x) \text{ is super-additive: } g(x) + g(y) \leq g(x + y) \text{ for } x, y \in \mathbb{R}^d_+ \]
\[ g(x) \text{ is concave on } \mathbb{R}^d_+ \]

1.1 Schur-Concavity of \( g(x) \)

Since \( g(x) \) is homogeneous and \( x \in \mathbb{R}^d_+ \) with nonnegative entries, to fully define \( g(x) \) it suffices to consider \( x \) on the probability simplex \( P := \{ p \in \mathbb{R}^d_+: \| p \|_1 = 1 \} \). Let \( e_i \) be the \( d \)-dimensional unit vector in the \( i \)th direction. Since \( \Pi(e_i) \) only contains a single path, \( T(e_i) \) does not involve a maximization, and \( g(e_i) = \mu \) due to the law of large numbers. Hence, when \( x \) is on the probability simplex, \( g(x) \) is minimized if \( x = e_i \), a unit vector concentrated along the \( i \)th dimension.

In fact, since \( g(x) \) is concave and symmetric, it is also Schur-concave \[2\]. Loosely speaking, this means that \( g(x) \) is smaller on the probability simplex when \( x \) is concentrated along a few directions only. To state this more precisely, we recall the definition of majorization. Let \( x [\underline{i}] \) be a sorting of the elements of \( x \) so that \( x[1] \geq x[2] \ldots \geq x[d] \). We denote the majorization of \( x \) by \( y \) as \( x \preceq y \) defined as

\[ \sum_{i=1}^k x[i] \leq \sum_{i=1}^k y[i] \quad k = 1, \ldots, d - 1 , \]

and equality holds in \( \text{(3)} \) when \( k = d \). Schur-concavity of \( g(x) \) means that \( x \preceq y \) implies \( g(y) \leq g(x) \).

1.2 Upperbound on \( g(x) \)

We now derive a large-deviation-based upper bound on \( g(x) \) which depends on the entropy of the normalized direction vector defined by \( x \). This bound was derived in \[3\] for \( d = 2 \), where the connection with entropy was not provided.

Since the entries of \( x \) are nonnegative, and \( g(x) \) is homogeneous, we will focus on upperbounding \( g(p) \) where \( p \) is a probability vector that will have the interpretation of the empirical distribution of the fraction of total number of steps taken in each dimension. The upperbound for \( g(x) \) will require the large-deviation rate function of the random variable \( X(z) \) defined as

\[ I(x) := \sup_{\nu > 0} \left( x\nu - \log(M(\nu)) \right) , \]

where \( M(\nu) := E[\exp(\nu X(z))] \). Note that \( I(\cdot) \) is monotonically increasing for arguments greater than \( \mu \). The upperbound is now given in the following theorem:

**Theorem:** Let \( p \in \mathcal{P} \) be a probability vector with rational entries, and assume \( E[\exp(\nu X(z))] \) exists for some \( \nu \) in the neighborhood of the origin. Then

\[ g(p) \leq I^{-1}_\mu \left( H(p) \right) \]
where
\[ I^{-1}_+ (\beta) := \inf_{\theta} \{ \beta < I(\theta), \text{ and } \theta > \mu \}, \] (6)
is the inverse of the monotonic portion of the rate function above the mean, and \( H(p) \) is the entropy function of a \( d \) dimensional probability vector [I].

**Proof:** We will show that
\[
\lim_{l \to \infty} P \left[ \frac{1}{l} T(lp) > \alpha \right] = 0 \quad (7)
\]
exponentially, whenever \( \alpha > I^{-1}_+ (H(p)) \).

Since \( p \) has rational entries, there exists an integer \( m \) such that \( x := mp \in \mathbb{Z}^d \), so that \( \|x\|_1 = m \). Setting \( l = mn \) we can write
\[
\frac{1}{l} T(lp) = \frac{1}{mn} T(nx) = \max_{\pi \in \Pi(nx)} \frac{1}{mn} \sum_{v \in \pi} X(v). \quad (8)
\]
The number of paths in \( \Pi(nx) \) can be counted and upperbounded using [II pp. 351]
\[
|\Pi(nx)| = \left( \frac{nm}{nmp_1 \ldots nmp_d} \right) \leq \exp(nmH(p)), \quad (9)
\]
and each path \( \pi \) in \( \Pi(nx) \) involves \( \|nx\|_1 = nm \) random variables. We have
\[
P \left[ \frac{1}{l} T(lp) > \alpha \right] = P \left[ \bigcup_{\pi \in \Pi(nx)} \left( \frac{1}{mn} \sum_{v \in \pi} X(v) \right) > \alpha \right] \quad (10)
\]
\[
\leq \exp(nmH(p))P \left[ \left( \frac{1}{mn} \sum_{v \in \pi} X(v) \right) > \alpha \right] \quad (11)
\]
\[
\leq \exp(nmH(p)) \exp(-nmI(\alpha)) \quad (12)
\]
\[
\leq \exp(-l(I(\alpha) - H(p))) \quad (13)
\]
where (10) can be written as a union due to the max operator, (11) is due to the union bound and (9), (12) is due to the Chernoff bound, and (13) is obtained by substituting \( l = nm \), and the monotonicity of \( I(\cdot) \) for arguments above \( \mu \). This proves that the probability in question decays exponentially in \( l \), whenever \( \alpha > I^{-1}_+ (H(p)) \).

Note that if one is interested in bounding \( g(x) \) with \( \|x\|_1 \neq 1 \), one can use the homogeneity of \( g(x) \). If \( p \) has irrational entries, a rational approximation along with continuity arguments will show that the bound in the Theorem is still valid. It is well-known that the entropy function is Schur-concave in the probability vector. Due to the monotonically increasing nature of the rate function and its inverse, \( I^{-1}_+ (H(p)) \) is also Schur-concave, just like the quantity \( g(p) \) it is bounding in (9). Hence, one appealing feature of the upperbound is that it retains the Schur-concavity of the quantity it is bounding.
References

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