The orientation of small ice crystals in cold clouds determines the reflection of light from the sun. This orientation results from the torque exerted by the fluid on the settling crystals. Here, we compute the torque acting on a small spheroid in a uniform flow by solving numerically the Navier-Stokes equations, and we compare our results with recent predictions [Dabade et al. (2015)], derived for small particle Reynolds numbers \( \text{Re} \ll 1 \). We find that the angular dependence of the torque predicted by the theory remains qualitatively correct even when the Reynolds numbers is as high as \( \text{Re} \sim 10 \). The theory describes qualitatively very well how the magnitude of the torque depends on the aspect ratio of the spheroid, for oblate and prolate particles. At larger Reynolds numbers the flow past spheroids acquires a more complicated structure, resulting in systematic deviations from the theoretical predictions. Overall, our numerical results provide an important justification of recent theoretical approaches to predict the statistics of orientation of ice-crystals settling in a turbulent flow.
I. INTRODUCTION

How does a spheroidal particle settle in a quiescent fluid? When the settling velocity is small enough, so that the fluid motion induced by the particle can be described by the Stokes approximation [38, 39], the particle settles at an arbitrary constant orientation equal to its initial orientation. But since the initial particle orientation is marginally stable, any small perturbation must affect the particle orientation. For example, for very small particles, upon which thermal noise plays a significant role, Brownian torques induce random orientation. In addition, slight breaking of the fore-aft symmetry of the particle [40–42] gives rise to a torque causing the particle to settle at a steady angle determined by particle shape, independent of its initial orientation. These torques, induced either by thermal fluctuations or by specific fore-aft asymmetry of the particle, compete with the inertial torque arising from convective inertial corrections to the Stokes approximation. A heavy particle settling steadily in a fluid experiences an undisturbed uniform mean flow corresponding to the negative settling velocity. This mean flow exerts a convective inertial torque on the particle. Its effect depends upon the particle Reynolds number

$$Re = \frac{Ua_{\text{max}}}{\nu}.$$  

Here $U$ is the settling speed of the particle, $\nu$ is the kinematic viscosity of the fluid, and $a_{\text{max}}$ measures the maximal linear size of the particle – the half length of a rod or the radius of a disk. For small $Re$, the convective inertial torque turns the spheroid so that it settles with its broad side first. Brenner & Cox [43] calculated the torque by perturbation theory in $Re$, for nearly spherical particles in a uniform flow. A technically important point is that the convective-inertia torque induced by the flow results from a singular perturbation of the Stokes equation, so that straightforward perturbation theory in $Re$ fails even at very small values of $Re$. Using asymptotic matching methods [44], Khayat & Cox [40] obtained the convective-inertia torque in the slender-body limit, complementing the earlier results for nearly spherical particles. More recently, Dabade et al. [45] used the reciprocal theorem to calculate this torque for spheroids of arbitrary aspect ratio – disks and rods – to linear order in $Re$.

Several earlier numerical studies have been devoted to a determination of the torque acting on spheroids in a uniform flow. Hölzer and Sommerfeld [46] used a lattice-Boltzmann method (LBM) to compute the steady-flow torque on non-spherical particles of different shapes, amongst others for a prolate spheroid ($\lambda = 3/2$) at different angles of inclination to the flow. Ouchene et al. [47, 48] used a commercial Navier-Stokes solver to resolve the flow field around prolate spheroids with aspect ratios $\lambda$ ranging from 5/4 to 32/1. Zastawny et al. [49] considered both prolate ($\lambda = 5/4$ and 5/2) and oblate ($\lambda = 1/5$) spheroids by means of an immersed boundary method and Sanjeevi et al. [50] used a LBM approach to compute the flow field around a prolate ($\lambda = 5/2$) and an oblate ($\lambda = 2/5$) spheroid at various angles of inclination. Zastawny et al. [49], Ouchene et al. [48] and Sanjeevi et al. [50] proposed semi-empirical correlation formulae for their torque data, in the form of explicit functions of inclination angle and Reynolds number. These earlier studies provide important insight for several spheroid shapes. However, they give the torque only for certain shapes, particle inclination to the flow, and particle Reynolds number. For slender fibres more is known. Shin et al. performed numerical simulations, and their Fig. 5 shows that the Khayat & Cox theory works well for slender fibers up to Reynolds numbers of the order of $\sim 10$.

Our goal here is to validate the small-Re-model [40, 43, 45] for spheroids of different aspect ratios in a steady homogeneous flow, and to determine how the torque changes as the Reynolds number increases. To answer this question, we solved numerically the Navier-Stokes equations past a spheroid at rest, in a uniform and steady flow, as schematically illustrated in Fig. 1 (left) at several values of the Reynolds number, $Re$, and of the particle shape (aspect ratio of the spheroid), in the case of small platelets and of small columns.

FIG. 1. Left: prolate spheroid with symmetry axis $n$ in a uniform flow with velocity $u = -U\hat{e}_x$. The angle $\varphi$ between $n$ and the $\hat{e}_x$-axis is called tilt angle. Right: shape factor $F(\lambda)$ determining the torque $\tau_z$, Eq. (7), red solid line. Also shown are the slender-body asymptote [3], black dotted line, as well as the near-spherical asymptote [4], black dashed line.
II. METHOD

Much of the literature on viscous and convective torques on small non-spherical particles in a flow uses spheroids as model shapes because the resistance tensors that determine the motion of the particle in the fluid are known \[39\], and because fore-aft and rotational symmetry lead to a comparatively simple angular dynamics. In the following we consider spheroidal particles. Similarities and differences between the angular dynamics of spheroids and crystals with discrete rotation and reflection symmetry was discussed by Fries et al. \[31\].

We denote the symmetry axis of the spheroidal particle by \( n \). The length of the symmetry axis is \( 2a_\parallel \), and the diameter of the spheroid is \( 2a_\perp \). The aspect ratio of the spheroid is defined as \( \lambda = a_\parallel / a_\perp \). Oblate particles (platelets) have \( \lambda < 1 \), while prolate particles (columns) have \( \lambda > 1 \). The Reynolds number defined in Eq. (1) is based upon \( a_{\max} = \max\{a_\parallel, a_\perp\} \). We consider a small spheroidal particle at a fixed position in a steady homogeneous flow with velocity \( \mathbf{u} \), the flow a small ice crystal experiences as it settles through quiescent air with settling velocity \(-\mathbf{u}\). For a prolate spheroid the setup is shown in Fig. 1(a). The tilt angle \( \varphi \) is defined as the angle between the particle-symmetry vector \( n \) and \(-\mathbf{u}\), for prolate as well as for oblate spheroids. For fore-aft symmetric particles, it is sufficient to consider angles \( \varphi \) in the interval \([0, \pi/2]\).

We computed the torque upon the particle by numerical solution of the full three-dimensional Navier-Stokes equations for incompressible flow, using a solver of the incompressible, three-dimensional Navier-Stokes equations, MGLET \[52\]. This code was recently used to document the computational challenges of calculating forces and torques upon rods in uniform flows \[53\]. The method is briefly described in appendix A. The simulations of Ref. \[53\] give precise results for the inertial torque for a rod of aspect ratio \( \lambda = 6 \) in a uniform flow, at different Reynolds numbers. In the following we show results for the convective-inertial torque for different aspect ratios, not only for rods but also for disks. To quantify the convective-inertial effect for particles of different sizes and shapes we keep the Reynolds number \( \text{Re} \) based on the maximal particle dimension, \( a_{\max} \), constant as we vary particle shape.

The small-Re theory \[40, 43, 45\] says that the inertial torque on a small spheroid in a uniform flow is of the form

\[
\tau^{(\text{Re})} = F(\lambda)\rho U^2 a_{\max}^3 (n \cdot \hat{u})(n \wedge \hat{u})
\]

(2)

to linear order in \( \text{Re} = U a_{\max}/\nu \). Here and in Eq. (2), \( U = |\mathbf{u}| \), and \( \hat{u} = \mathbf{u}/U \). The shape factor \( F(\lambda) \) was computed in Ref. \[44\], and it is shown in Fig. 1(b). Also shown is the slender-body limit derived by Khayat and Cox \[40\],

\[
F(\lambda) \sim -5\pi/[3(\log \lambda)^2],
\]

(3)
as well as the near-spherical expansion \[45\]

\[
F(\lambda) \sim \mp811\pi\varepsilon/560
\]

(4)

for small eccentricity \( \varepsilon \). Here the eccentricity parameter is defined by \( \lambda = 1 + \varepsilon \) for prolate particles, and \( \lambda = (1 - \varepsilon)^{-1} \) for oblate particles. Up to a relative error of order \( 10^{-3} \) in the numerical prefactor Eq. (1) agrees with the result of Cox \[43\] for nearly spherical particles, as mentioned by the authors of Ref. \[45\].

In the following we assume without loss of generality that gravity points along the \( \hat{e}_z \)-axis, and that the symmetry vector \( n \) lies in the \( \hat{e}_z - \hat{e}_y \)-plane (Fig. 1). Then the torque points along the \( \hat{e}_z \)-axis, \( \mathbf{\tau} = \tau_z \hat{e}_z \), where \( \hat{e}_z = \hat{e}_x \wedge \hat{e}_y \). In this case Eq. (2) implies that the torque depends on its tilt angle \( \varphi \) as

\[
\tau_z^{(\text{Re})} = -\frac{1}{2} F(\lambda) \rho U^2 a_{\max}^3 \sin 2\varphi .
\]

(5)
The torque \( \tau_z \) is zero both when \( \varphi = 0 \), corresponding to \( \mathbf{\hat{n}} \) parallel to \( \mathbf{\hat{u}} \), or when \( \varphi = \pi/2 \), when \( \mathbf{\hat{n}} \) and \( \mathbf{\hat{u}} \) are perpendicular to each other. This is a consequence of the symmetry of the problem. The \( \sin(2\varphi) \)-dependence in Eq. (5) is the simplest possible angular dependence that respects these constraints. This is exactly the result of the small-Re perturbation theory at order \( \text{Re} \). We see that the angular dependence is symmetric around the tilt angle \( \varphi = \pi/4 \). The sign of the torque (5) is such that \( \varphi = \pi/2 \) is stable for prolate particles (rods), whereas \( \varphi = 0 \) is stable for oblate particles (disks). In the next Section we summarise our numerical results, that show how the torque varies as a function of particle shape, Reynolds number, and tilt angle.

III. RESULTS

We de-dimensionalise the torque as

\[
\tau_z' = \frac{\tau_z}{\rho U^2 a_{\max}^3} .
\]

(6)
Our initial configuration is symmetric w.r.t. reflection in the $x$-$y$ plane [Fig. 1(left)]. We have checked that the flow remains symmetric and steady for all simulations described in this article, for Reynolds numbers up to $Re = 30$.

Fig. 2 shows our simulation results for the dimensionless torque for prolate and oblate spheroids (appendix B), compared with the small-Re theory [5], which reads in dimensionless form:

$$\tau'_z = -\frac{1}{2}F(\lambda) \sin 2\varphi.$$  \hspace{1cm} (7)

This theory is shown as a thick solid line. Panel (a) contains the results for a prolate spheroid with $\lambda = 6$ as a function of tilt angle, for different particle Reynolds numbers (symbols). Filled symbols correspond to data from Table 5 in Ref. [53]. Thin solid lines are fits to the theoretically predicted angular dependence, proportional to $\sin 2\varphi$. (b) Same for an oblate spheroid with $\lambda = 1/6$. (c) Maximal torque $\tau'_z$ (evaluated at $\varphi = 45^\circ$) as a function of aspect ratio $\lambda$, for different particle Reynolds numbers. Symbols show simulation results, the black solid line is $\frac{1}{2}F(\lambda)$. (d) Dependence of $\delta \tau_z = |\tau_z - \tau_z^{(Re)}|$ upon Reynolds number, for $\varphi = 45^\circ$, $\lambda = 6$ (filled symbols), $\lambda = 1/6$ (empty symbols). Here $\tau_z^{(Re)}$ is given by Eq. (5). The dashed line represents $\sqrt{Re}$.

Our results for $\lambda = 6$ and $Re = 0.3$ (red, □), $Re = 3$ (green, ◼), and $Re = 30$ (blue, △). Filled symbols correspond to data from Ref. [53]. The theory (7) is shown as a solid black line. Coloured lines are fits of the angular dependence in Eq. (7), proportional to $\sin 2\varphi$. The results are qualitatively similar to those obtained for $\lambda = 6$, but there are two important differences. First, we have no data points for $Re = 0.3$. The smallest Reynolds-number simulations are very costly because one must use a large domain size at the same time as a small spatial mesh [53]. This is particularly challenging for disks because a finer mesh is needed to resolve the flow in the vicinity of the strongly curved periphery of flat disks. Second, for disks the $\varphi$-dependence develops an asymmetry around $\varphi = 45^\circ$ at larger values of $Re$. Panel (c) shows the torque at $\varphi = 45^\circ$ as a function of particle aspect ratio in comparison with Eq. (7). We infer that the theory describes the shape dependence of the inertial torque well, quantitatively at $Re = 0.3$, and qualitatively at larger $Re$. Panel (d) shows the relative difference between the numerical results and Eq. (7) as a function of Reynolds number, for $\lambda = 1/6$ and 6. The deviations decrease quite slowly as $Re$ decreases at the Reynolds numbers for which we have numerical data. Since we lack data at small values of $Re$, it is difficult to infer the order of the next term in the expansion in $Re$. The next order could be $Re^{3/2}$, giving rise to a $Re^{1/2}$-correction to $\tau_z^{(Re)}$ (dashed line), but there could also be logarithmic corrections, as in the small-Re theory for the drag coefficient [54].

Fig. 3 quantifies the asymmetry of the $\varphi$-dependence of the torque around $\varphi = 45^\circ$ that develops for disks at larger Reynolds numbers. What is the origin of this asymmetry? To understand the mechanism, we visualise the fluid-velocity field around a disk with aspect ratio $\lambda = 1/3$ at $\varphi = 30^\circ$ and $60^\circ$ in Fig. 4. We observe that the streamlines closely follow the surface of the spheroid in panels (a) and (b). This reflects that flow remains attached to the surface of this oblate spheroid at small Reynolds numbers. At $Re = 30$, by contrast, the flow separates as the
FIG. 3. Torque on a disk as a function of Reynolds number. Results for spheroids of two different shapes are shown: \( \lambda = 1/2 \) (circles), \( \lambda = 1/3 \) (squares). Empty symbols correspond to tilt angle \( \varphi = 30 \), and full symbols to \( \varphi = 60 \). At \( \text{Re} = 3 \) and 5, empty and full symbols lie on top of each other.

FIG. 4. Streamlines of the flow around a disk in the \( x-y \) plane at \( z = 0 \), for \( \text{Re} = 3 \) and \( \lambda = 1/3 \). The tilt angle is (a) \( \varphi = 30^\circ \) and (b) \( \varphi = 60^\circ \); (c) and (d) show the same but for \( \text{Re} = 30 \).

Oblate spheroid meets the flow with its broad side, resulting in quite different flow patterns for \( \varphi = 30^\circ \) and \( 60^\circ \). This certainly contributes to the asymmetry of the torque.
IV. DISCUSSION AND CONCLUSIONS

We performed numerical simulations determining the torque on oblate and prolate spheroids that settle steadily in a quiescent fluid. Our results show that the small-Re theory [40, 43, 45] works reasonably well for small Reynolds numbers. The shape dependence remains qualitatively correct for the largest Reynolds numbers we have considered, Re= 30. But in general the torque is smaller than the small-Re theory (2) predicts. For example, Fig. (2b) shows that the maximal Re = 30-torque on a disk is smaller than the small-Re prediction by about a factor of two. What does this imply for the angular dynamics of ice platelets settling in turbulent clouds? When particle inertia is negligible, theory suggests [55-57] that the variance of the tilt angle is inversely proportional to the maximal value of the torque squared. This means that at Re = 30 the standard deviation of the tilt angle is larger than predicted by the theory, by a factor of two.

The small-Re theory for the torque exhibits a symmetry around $\varphi = 45^\circ$. For prolate particles our numerical simulations exhibit this symmetry quite accurately even at the largest Re we simulated. For disks, by contrast, this symmetry is clearly broken at Re= 30 (and this may imply that the maximum of the torque is not precisely at $\varphi = 45^\circ$). We computed the disturbance flow and saw that the flow detaches from the particle when it faces the flow with its broadest side first, causing the torque asymmetry. It is likely that this asymmetry in the $\varphi$-dependence is a precursor of a bifurcation, as the Reynolds number increases. Indeed, experiments show that there is a transition for a disk. It settles with its broad side down at small Re, but exhibits other kinds of periodic or chaotic lateral and angular dynamics at larger Re, due to interactions between the disk and the induced vortex street. A bifurcation to periodic angular dynamics occurs at Re ~ 100 [55-57]. Unsteady dynamics and symmetry breaking occur also for rods, as the simulations summarised in Ref. [60] show.

In the present work, and in particular when solving numerically the Navier-Stokes equations, we assumed that the settling particle experiences a homogeneous flow. If by contrast the fluid is in motion, then fluid-velocity gradients give rise to additional torques. In the Stokes approximation these torques were first calculated by Jeffery [61], and they compete with the torque due to fluid inertia. As shown in Refs. [62, 63], the magnitude of the fluid-inertia torque in a linear flow depends upon the shear Reynolds number $Re_s = a_{max}^8/\nu$ where the shear rate $s$ is an estimate of the magnitude of the fluid-velocity gradients. When $Re \gg \sqrt{Re_s}$, then the shear-induced inertial torque is negligible compared to inertial corrections due to the slip velocity, at least for steady flows [64]. Lopez & Guazzelli [65] measured the angular dynamics of rods settling in a steady vortex flow. Their model takes into account the torque induced by fluid inertia for fibers in the slender body limit [40], but neglects shear-induced torques, and it qualitatively explains their experimental results.

How the alignment of spheroids settling through a fluid is affected by unsteady fluid motion is not known in general. Yet the extent to which turbulence destroys alignment of settling particles has important consequences in the atmospheric sciences, where reflection of polarised light reveal small orientation fluctuations of small ice crystals [66, 67] settling in turbulent clouds [68]. A model for this effect [55-57, 65, 69, 70] assumes that the fluid torque on a settling crystal can be approximated by the superposition of the Jeffery torque due to the turbulent fluid-velocity gradients, and the small-Re expression for the convective inertial torque, Eq. [5]. The model predicts that the settling particles tend to orient so as to maximize their drag. Typical ice-crystal sizes in clouds range from 150 $\mu$m to 1 mm. Given the settling speed of such crystals in air (see Table 10.3a in Ref. [66]), this corresponds to Reynolds numbers of the order of 2 to 15. The model analysed in Refs. [55, 57, 65, 69, 70] nevertheless uses the small-Re expression for the torque, Eq. [5], for two reasons. Firstly, higher Re-corrections are not known except in the slender-body limit [40]. Secondly, the empirical correlations for the torque have been verified only for certain Reynolds numbers, particle shapes, and inclinations of the particle to the flow, as mentioned in the Introduction. The results reported here support one of the assumptions that enter the theory [55-57, 65, 69, 70] for the angular dynamics of ice platelets settling in turbulent clouds [the shape and angular dependence of the steady inertial torque, Eq. [5]].

Experiments at large Re (Re ~ 1000) compare the trajectories and velocities of platelets settling in a quiescent fluid, with those settling in a turbulent background flow [71]. The authors find that the background turbulence has a significant effect upon the settling dynamics. This is expected because the fluid-velocity gradients give rise to Jeffery torques, as mentioned above. In addition, fluctuations in the translational slip velocity appear to have a profound effect on the orientation of the settling particle [57]. But fluid-velocity gradients must also affect the form of the convective inertial torque, discussed here for a quiescent fluid. This effect is not taken into account in the model used in Refs. [55, 57, 65, 69, 70]. To validate the model, it would be of interest to conduct experiments at smaller Reynolds numbers, so that one can compare and contrast with the predictions of Refs. [57], for example. We intend to run fully resolved simulations of particles settling in turbulence in order to justify and refine the model. But this remains a challenge for the future.
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Appendix A: Description of simulations

MGLET is a finite-volume code that directly solves the full time-dependent three-dimensional Navier-Stokes equations for incompressible fluids. The computational domain is discretised on a multi-level staggered Cartesian mesh with cubic grid cells. A third-order explicit low-storage Runge-Kutta scheme [72] is used for the time dependence. Stone’s strongly implicit procedure [73] is applied for pressure correction in each time step. To represent the curved particle surface in the Cartesian mesh, MGLET uses a direct-forcing immersed boundary method, representing no-slip and impermeable boundary condition at the particvle surface. The code has been extensively validated for various flows in a wide Reynolds number range, among which [53, 60, 74] are in the low Re-regime and most relevant to the present study.

All simulations in the present study used the largest practically possible computational domain ($205a_{\text{min}} \times 205a_{\text{min}} \times 205a_{\text{min}}$) with $a_{\text{min}} = \min\{a_{\parallel}, a_{\perp}\}$, see Ref. [53]. The minimum grid cell size is 0.02$a_{\text{min}}$. The relatively fine mesh and large computational domain lead to big mesh size (40–50 million grid cells), and the explicit time-evolution scheme leads to very small time step size when Re is very low. These challenges are discussed in Ref. [53].

We define the Reynolds number using $a_{\text{max}} = \max\{a_{\parallel}, a_{\perp}\}$, Eq. (1). The authors of Ref. [53] define the Reynolds number (Re$_D$ in their notation) in terms of the short-axis length $D$, equal to 2$a_{\text{min}}$. For the aspect ratio $\lambda = 6$ studied in Ref. [53] we have $Re = \frac{1}{2}Re_D = 3Re_D$. To determine the effect of particle shape, we varied the aspect ratio keeping $a_{\text{min}}$ constant. It follows that the relation between $Re_D$ and Re defined in Eq. (1) is

$$Re = \frac{1}{2}Re_D \left\{ \begin{array}{ll}
\lambda & \text{for } \lambda > 1, \\
\lambda^{-1} & \text{for } \lambda < 1.
\end{array} \right.$$  \hspace{1cm} (A1)

The authors of Ref. [53] also define a second Reynolds number, Re$_p$ in their notation, in terms of the sphere-equivalent diameter $d_0 = 2a_0$. Since the volume of the spheroid is $\frac{4}{3}\pi a_{\parallel}a_{\perp}^{2}$, we have that $a_0 = (a_{\parallel}a_{\perp}^{2})^{1/3} = \lambda^{1/3}a_{\perp} = \lambda^{-2/3}a_{\parallel}$. The authors of Ref. [53] de-dimensionalise the torque by dividing by $\frac{1}{2}\rho U^2 \pi d_0^3$. To compare with their results for $\lambda = 6$ we use

$$d_0 = 2a_0 = 2a_{\text{max}} \left\{ \begin{array}{ll}
\lambda^{-2/3} & \text{for } \lambda > 1, \\
\lambda^{1/3} & \text{for } \lambda < 1.
\end{array} \right.$$  \hspace{1cm} (A2)

The ratio of normalisation factors is

$$\frac{\frac{1}{2}\rho U^2 \pi d_0^3}{\rho U^2 a_{\text{max}}^3} = \frac{\pi}{2} \left\{ \begin{array}{ll}
\lambda^{-2} & \text{for } \lambda > 1, \\
\lambda & \text{for } \lambda < 1.
\end{array} \right.$$  \hspace{1cm} (A3)
### Appendix B: Summary of simulation results

TABLE I. Numerical results (MGLET) for torque $\tau' = \tau_s/(\rho U^2 a_{max})$ upon a spheroid in a uniform flow, as a function of tilt angle $\varphi$, Reynolds number $Re$, and particle aspect ratio $\lambda$.

| $\lambda$ | $\varphi$ [deg] | 15  | 30  | 45  | 60  | 75  |
|---|---|---|---|---|---|---|
| 6  | 0.112 | 0.196 | 0.226 | 0.199 | 0.114 |
| 3  | 0.340 |
| 2  | 0.393 |
| $\frac{1}{3}$ | -0.707 |
| $\frac{1}{6}$ | -0.853 |

| $\lambda$ | $\varphi$ [deg] | 15  | 30  | 45  | 60  | 75  |
|---|---|---|---|---|---|---|
| 6  | 0.076 | 0.133 | 0.159 | 0.135 | 0.078 |
| 3  | 0.120 | 0.211 | 0.244 | 0.213 | 0.122 |
| 2  | 0.145 | 0.251 | 0.291 | 0.255 | 0.145 |
| $\frac{1}{3}$ | -0.283 | -0.487 | -0.558 | -0.487 | -0.275 |
| $\frac{1}{6}$ | -0.340 | -0.586 | -0.681 | -0.586 | -0.335 |
| $\frac{1}{6}$ | -0.340 | -0.628 | -0.746 | -0.649 | -0.369 |

| $\lambda$ | $\varphi$ [deg] | 15  | 30  | 45  | 60  | 75  |
|---|---|---|---|---|---|---|
| 6  | 0.033 | 0.057 | 0.065 | 0.057 | 0.033 |
| 3  | 0.068 | 0.117 | 0.136 | 0.119 | 0.068 |
| 2  | 0.094 | 0.161 | -0.185 | 0.161 | 0.090 |
| $\frac{1}{3}$ | -0.220 | -0.369 | -0.416 | -0.353 | -0.196 |
| $\frac{1}{6}$ | -0.267 | -0.450 | -0.497 | -0.408 | -0.225 |
| $\frac{1}{6}$ | -0.293 | -0.503 | -0.547 | -0.432 | -0.233 |