Article

On Small Deviation Asymptotics in the $L_2$-Norm for Certain Gaussian Processes

Leonid Rozovsky

R&D Department, Saint-Petersburg State Chemical and Pharmaceutical University, 197376 Saint-Petersburg, Russia; info@pharminnotech.com

Abstract: The results obtained allow finding sharp small deviations in a Hilbert norm for centered Gaussian processes in the case where their covariances have a special form of the eigenvalues and allow us to describe small deviation asymptotics for certain Gaussian processes.

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MSC: 60G50; 60F99

1. Introduction and Results

Consider a centered Gaussian process $X(t)$, $0 \leq t \leq 1$, with covariance $G(t,s) = \mathbb{E}X(t)X(s)$, $t, s \in [0,1]$ such that $0 < \int_0^1 G(t,t) \, dt < \infty$. Set $||X(t)||_2^2 = \int X^2(t) \, dt$. Our aim is to obtain sharp asymptotics of the probability $\mathbb{P}(||X^2(t)||_2 < \varepsilon)$ as $\varepsilon \to 0$. Due to the well-known classical Karhunen-Loève expansion (see, for instance, [1]), the following equality in the distribution:

$$||X(t)||_2^2 = \int_0^1 X^2(t) \, dt = \sum_{n \geq 1} \lambda_n \tilde{\zeta}_n^2$$

takes place, where $\tilde{\zeta}_n$, $n \geq 1$, are standard independent Gaussian rv’s, while positive summable $\lambda_n$ are the eigenvalues of the integral equation:

$$\lambda f(s) = \int_0^s G(t,s) f(t) \, dt, \ 0 \leq s \leq 1.$$

Hence, the problem examined is equivalent to studying the asymptotic behavior of the probability $\mathbb{P}(\sum_{n \geq 1} \lambda_n \tilde{\zeta}_n^2 < \varepsilon)$ as $\varepsilon \to 0$ or, in a more general setting, the probability $\mathbb{P}(\sum_{n \geq 1} \lambda_n |\tilde{\zeta}_n|^p < \varepsilon)$, $p > 0$.

The problem of small deviations for the norms of Gaussian processes (including one of the simplest ones, the norm in $L_2$) has been studied quite intensively (see the bibliography in [2–4]). However, due to the fact that explicit formulas for $\lambda_n$ are known for a few concrete processes, sharp asymptotics often cannot be found. Nevertheless, this was done in [2] for an interesting and rather general case where the coefficients $\lambda_n$ behave as quotients of powers of two polynomials.

We present this nice result since it is the starting point for our research. To formulate it, we introduce some notation. For $p > 0$ and $V > 1$, set:

$$f(t) = \log \mathbb{E} \exp \{-t|\tilde{\zeta}_1|^p\}, \ t \geq 0; \ K(V, p) = -\int_0^\infty t^{-1/V} f'(t) \, dt. \quad (1)$$
For example, $K(V, 2) = 2^{(1-V)/V} \pi / \sin (\pi / V)$.

Let us define polynomials:

$$\begin{align*}
Q_r(x) &= x^r + A_{r-1}x^{r-1} + \cdots + A_0 = \prod_{i=1}^r (x + \theta_i), \\
P_q(x) &= x^q + B_{q-1}x^{q-1} + \cdots + B_0 = \prod_{i=1}^q (x + \phi_i),
\end{align*}$$

where $r, q \in \mathbb{Z}_+$, $A_i, B_i \in \mathbb{R}$, $\phi_i, \theta_i \in \mathbb{C}$. We assume that for $n = 1, 2, \ldots$, the polynomials $Q_r(n)$, $P_q(n)$ are strictly positive and that real constants $\mu, \nu$ satisfy the condition:

$$V = q\mu - r\nu > 1,$$

as well as that the sequence $a_n = Q_r(n)/P_q(n)$ does not increase.

**Proposition 1.** If:

$$\sigma = (\mu B_{q-1} - \nu A_{r-1}) / V > -1,$$

then:

$$\mathbf{P}\left( \sum_{n \geq 1} |a_n|^p < \varepsilon \right) \sim C \varepsilon^{1/p} \exp (-B \varepsilon^{-1/(p-1)})$$

as $\varepsilon \to +0$, where (see (1), (3), and (4)):

$$\begin{align*}
A &= \frac{p - (1 + 2\sigma) V}{2p(V - 1)}, \\
B &= (V - 1) \left( \frac{K}{V} \right)^{\nu / \mu}, \\
K &= K(V, p)
\end{align*}$$

and:

$$C = \frac{2^{\frac{-3p + 2V - 2\sigma}{4p}}}{\Gamma(1 + \frac{1}{p})^{\frac{1}{4p}} \sqrt{\Gamma(1 + \frac{1}{p}) \cdot \frac{\pi}{\Gamma(1 + \frac{1}{p})}}}.$$ 

Proposition 1 was proven in [2], Theorem 1. Note that Condition (4) and the assumption on the sequence $a_n$, which is non-increasing, which is used essentially in the proof, were not explicitly mentioned in the formulation of [2], Theorem 1.

The purpose of the present note is to extend Proposition 1 to a more general class of weights $a_n$ and to omit the constraint (4).

Let us formulate the results. For $n, m \in \mathbb{N}$, $\lambda_j \in \mathbb{R}$, and $z_j \in \mathbb{C}$ ($1 \leq j \leq m$), we set

$$V = \sum_{j=1}^m \lambda_j > 1, \quad a_n = \prod_{j=1}^m |n + z_j|^{-\lambda_j}.$$ 

Note that the weights $Q_r(n)/P_q(n)$ in Proposition 1 constitute a subcase of the weights $a_n$ above (one can derive this by taking $\lambda_j$ to be equal to $\mu$ or $-\nu$).

**Theorem 1.** Let the integer $k \geq 0$, $z_j \neq -1, -2, \ldots$, and:

$$\sigma := \frac{1}{V} \sum_{j=1}^m \lambda_j \Re z_j > -k - 1.$$ 

Then:
Then, for every $p > 0$:

$$
P\left( \sum_{n \geq 1} a_n |\xi_n|^p < \varepsilon \right) \sim \left( \prod_{j=1}^{m} |\Gamma(k + 1 + z_j)|^{-\lambda_j} \right)^{1/p} \left( \prod_{j=1}^{k} \sigma_j \right)^{-\frac{1}{p}} \cdot C \varepsilon^A \exp \left( -B \varepsilon^{-\frac{1}{p}} \right),
$$

(7)
as $\varepsilon \to +0$, with the same notation as in Proposition 1, except the “new” $V$ and $\sigma$.

Note that Equality (7) still holds true if we assume that $z_j \neq -k - 1, -k - 2, \ldots$ and replace the numbers $a_j$, $1 \leq j \leq k$, in (7) with arbitrary positive constants.

Note also that the minimal $k$ satisfying Condition (6) is equal to $\lfloor \max (-\sigma, 0) \rfloor$, where $\lfloor x \rfloor$ stands for the integer part of $x$.

2. Proofs

Let us prove the statement of Theorem 1 for $k = 0$. We follow the lines of the proof in [2]. Set $b_n = (n + \sigma)^{-V}$. Taking into account the equality $|n + z_j|^2 = n^2 + 2n\Re z_j + |z_j|^2$, it is easy to verify that:

$$
|1 - (b_n/a_n)| = O(\varepsilon^{-n^2}), \quad n \to \infty.
$$

Therefore,

$$
\sum_{n \geq 1} |1 - (b_n/a_n)| < \infty
$$

(8)
and hence, the product $\prod_{j=1}^{\infty} (b_n/a_n)$ converges. To calculate it, we use the well-known representation for the gamma function:

$$
\frac{1}{\Gamma(1 + z)} = e^{cz} \prod_{n=1}^{\infty} \left( 1 + \frac{z}{n} \right) e^{-z/n}, \quad z \in \mathbb{C},
$$

where $c$ is the Euler constant. According to this formula, we have:

$$
\prod_{n=1}^{\infty} \frac{n + z_j}{n + \sigma} e^{-(z_j - \sigma)/n} = e^{-c(z_j - \sigma)} \frac{\Gamma(1 + \sigma)}{\Gamma(1 + z_j)}.
$$

Taking into account that $\sum_{j=1}^{m} \lambda_j (\Re z_j - \sigma) = 0$, and $\sigma > -1$, we get:

$$
\left| \prod_{n=1}^{\infty} \prod_{j=1}^{m} \left( \frac{n + z_j}{n + \sigma} \right)^{\lambda_j} \right| = \left| \frac{\Gamma^V(1 + \sigma)}{\prod_{j=1}^{m} \Gamma^{\lambda_j}(1 + z_j)} \right|.
$$

Hence,

$$
\prod_{j=1}^{\infty} (b_n/a_n) = \Gamma^V(1 + \sigma) \prod_{j=1}^{m} |\Gamma(1 + z_j)|^{-\lambda_j}.
$$

Applying the comparison theorem ([5], Theorem 2.1) (which is possible due to Condition (8)) and [6], Lemma 1, we conclude that Theorem 1 holds for $k = 0$.

Now, let $k \geq 1$. Represent the probability on the left-hand side of (7) as $P(X_1 + X_2 < \varepsilon)$, where $X_1 = \sum_{n \geq 1} a_n |\xi_n|^p$, $X_2 = \sum_{n \geq 1} a_{n+k} |\xi_{n+k}|^p$, and let us denote $F_1(\cdot)$, $F_2(\cdot)$ the distribution functions of $X_1, X_2$, respectively.
It is easy to see that for any positive $a_n$:
\[
P(a_n | \xi_n| < \epsilon) \sim \sqrt{\frac{2}{\pi}} \left( \frac{\epsilon}{a_n} \right)^{\frac{1}{2}} \frac{1}{p}, \quad \epsilon \to +0.
\]

Hence, using [7], Corollary 1 and the just proven statement of Theorem 1 for $k = 0$, we obtain:
\[
F_1(\epsilon) \sim \left( \frac{\Gamma(1 + \frac{1}{p}) \sqrt{\frac{2}{\pi} \epsilon^{\frac{1}{2}}}}{\Gamma(1 + \frac{k}{p})} \right)^{\frac{k}{p}} \left( \prod_{j=1}^{k} a_j \right)^{-\frac{1}{p}}
\]
\[
F_2(\epsilon) \sim \left( \prod_{j=1}^{m} |\Gamma(k + 1 + z_j)|^{-\lambda_j} \right)^{1/p} \tilde{C} \epsilon^{\tilde{A}} \exp\left(-B \epsilon^{-\frac{1}{\tilde{A}}}ight)
\]  
(9)

as $\epsilon \to +0$, where $\tilde{C}$ and $\tilde{A}$ are defined similarly to $C$ and $A$ in Proposition 1 with $\sigma$ being replaced by $k + \sigma$.

From [7], Theorem 5, Equation (1.24), and Remark 3 and (9), it follows that:
\[
P(X_1 + X_2 < \epsilon) \sim \Gamma(1 + \alpha) F_2(\epsilon) F_1(1/|g_2'(\epsilon)|), \quad \epsilon \to +0,
\]  
(10)

where $\alpha = k/p$ and $g_2(\epsilon) = B \epsilon^{-\frac{1}{\tilde{A}}}$. The observation that the right-hand sides in (10) and (7) coincide finally proves Theorem 1.

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