Entanglement is one of the key features of quantum world and any entanglement measure must satisfy some basic laws. Most important of them is the invariance of entanglement under local unitary operations. We show that this is no longer true with local $\mathcal{PT}$ symmetric unitary operations. If two parties share a maximally entangled state, then under local $\mathcal{PT}$ symmetric unitary evolution the entropy of entanglement for pure bipartite states does not remain invariant. Furthermore, we show that if one of the party has access to $\mathcal{PT}$-symmetric quantum world, then a maximally entangled state in usual quantum theory appears as a non-maximally entangled states for the other party. This we call as the “entanglement mismatch” effect which can lead to the violation of the no-signaling condition.

Quantum theory is arguably the most fundamental theory of Nature which has been tested over more than hundred years. Though, there have been several attempts to extend quantum theory, the basic tenets of the theory remained untouched so far. There have been non-linear generalizations of quantum theory [1–3], non-unitary modifications to the Schrödinger equations [4–7], complex extension of quantum theory [8, 9], and many more, to name a few. However, these generalizations are not free from importunate issues.

In conventional quantum theory the observables are represented by Hermitian operators and the evolution of a closed system is governed by unitary evolution. However, in recent years there have been considerable interests in quantum systems governed by non-Hermitian Hamiltonians [8–14]. It was discovered that there are class of non-Hermitian Hamiltonians which possesses real eigenvalues provided they respect $\mathcal{PT}$ symmetry and the symmetry is unbroken. In $\mathcal{PT}$-symmetric quantum mechanics the usual condition of Hermiticity of operators is replaced by the condition of $\mathcal{PT}$ invariance, where $\mathcal{C}$ stands for conjugation, $\mathcal{P}$ for parity and $\mathcal{T}$ for time reversal [9]. In standard quantum theory $\mathcal{CPT}$ symmetry and Hermiticity conditions are the same. The $\mathcal{CPT}$ invariance condition is a natural extension of Hermiticity condition that allows reality of observables and unitary dynamics. Using the operator $\mathcal{C}$, Bender et al [9] have introduced an inner product structure associated with $\mathcal{CPT}$ which can have positive definite norms for quantum states.

Entanglement is one of the weirdest feature of quantum mechanics. In the emerging field of quantum information theory entanglement plays a major role [15]. This is also a very useful resource in the sense that using entanglement one can do many things in the quantum world which are usually impossible in ordinary classical world. Some of these tasks include, but not limited to, quantum computing [16], quantum teleportation [17], quantum cryptography [18], remote state preparation [19], and quantum communication [20]. Usual discussions about quantum entanglement pertain to the realm of Hermitian quantum theory. However, recently the notion of entanglement for quantum systems described by $\mathcal{PT}$ symmetric Hamiltonians was introduced by the present author [21] and independently in Ref. [22].

Early formulation of $\mathcal{PT}$-symmetric quantum theory aimed to offer a genuine extension of usual quantum theory. Later, mathematical unitary equivalence has been shown between $\mathcal{PT}$ symmetric quantum theory and the usual quantum theory for single quantum systems [11]. However, entangled quantum systems may offer new insights into the nature of this (in)equivalence. Since the equivalence properties of entangled states are different under joint unitary and under local unitary transformations, it was conjectured by the present author that under local unitary transformations (or more generally under LOCC paradigm) equivalence between $\mathcal{PT}$ symmetric quantum theory and the usual quantum theory may not exists [21].

In addition, $\mathcal{PT}$ symmetric quantum theory offers new possibilities. It has been shown that faster time evolution [23] and state discrimination is possible with $\mathcal{PT}$ symmetric Hamiltonian [24]. Though, the faster time evolution with non-Hermitian Hamiltonian has been questioned in Ref. [25]. One assumption that is usually made is that one can describe a local subsystem by a $\mathcal{PT}$ symmetric Hamiltonian and it is possible to switch between $\mathcal{PT}$ symmetric world and the conventional quantum world. Similar assumptions [26, 27] have been recently scrutinized, and it has been shown that local $\mathcal{PT}$ symmetry acting on a composite system can lead to signaling [28].

Any entanglement measure must satisfy some basic laws. Most important of them is the non-increase of entanglement under local operations. Moreover, a stringent requirement is that it must be invariant under local unitary operations. In this paper, we show that this does not hold for local $\mathcal{PT}$ symmetric unitary operations. If Alice and Bob share a maximally entangled state then under local $\mathcal{PT}$ symmetric unitary evolution the entropy of entanglement for a pure bipartite state does not remain invariant. This proves our earlier conjecture that even though global $\mathcal{PT}$ symmetry is equivalent to conventional quantum theory, local $\mathcal{PT}$ symmetry can have inequivalent predictions. Because of the $\mathcal{CPT}$ inner product, orthogonal quantum states in ordinary quantum theory become non-orthogonal quantum states in non-Hermitian quantum theory. As a consequence, we will show that a maximally entangled state in ordinary theory can appear as a non-maximally entangled state to one observer if another observer has access to $\mathcal{PT}$ symmetric quantum world. This is a precursor for the violation of the no-signaling condition.

$\mathcal{PT}$ Symmetric Quantum Theory.– We will give the basic formalism that is necessary to develop the notion of entanglement in non-Hermitian quantum theory. In earlier formula-

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tion of $\mathcal{P}\mathcal{T}$-symmetric quantum theory, it turned out that $\mathcal{P}\mathcal{T}$-symmetric quantum theory admitted states which have negative norms. This had no clear interpretation. This was cured by introducing another operator $\mathcal{C}$ called as the conjugation operator \[8,9\]. This operator commutes with the Hamiltonian and the operator $\mathcal{P}\mathcal{T}$. Also, note that $\mathcal{C}^2 = I$, which implies that it has eigenvalues $\pm 1$.

Bender et al \[8,9\] have shown that non-Hermitian Hamiltonians can have real eigenvalues if it possess $\mathcal{P}\mathcal{T}$-symmetry, i.e., $[H, \mathcal{P}\mathcal{T}] = 0$ and the symmetry is unbroken (if all of the eigenfunctions of $H$ are simultaneous eigenfunction of the operator $\mathcal{P}\mathcal{T}$). Hamiltonians having unbroken $\mathcal{P}\mathcal{T}$ symmetry can define a unitary quantum theory. Unitarity can be shown by the fact that such Hamiltonians possess a new symmetry called conjugation $\mathcal{C}$ with $[\mathcal{C}, H] = 0$ and $[\mathcal{C}, \mathcal{P}\mathcal{T}] = 0$.

Quantum theory that deals with non-Hermitian Hamiltonians and respects $\mathcal{P}\mathcal{T}$ symmetry may be called non-Hermitian quantum theory. One can formalize the framework by stating the following postulates: (i) A quantum system is a three-tuple $(\mathcal{H}, H, \langle \cdot \rangle_{\mathcal{C}\mathcal{P}\mathcal{T}})$, where $\mathcal{H}$ is a physical Hilbert space with the $\mathcal{C}\mathcal{P}\mathcal{T}$ inner product $\langle \cdot \rangle_{\mathcal{C}\mathcal{P}\mathcal{T}}$ having a positive norm, and $H$ is the non-Hermitian Hamiltonian; (ii) The state of a system is a vector $|\psi\rangle$ in $\mathcal{H}$. For any two vectors the $\mathcal{C}\mathcal{P}\mathcal{T}$ inner product is defined as $\langle \psi |\phi\rangle_{\mathcal{C}\mathcal{P}\mathcal{T}} = \int dx |C\mathcal{P}\mathcal{T}\psi(x)\rangle \phi(x)\langle x|$, (iii) The time evolution of state vector is unitary with respect to $\mathcal{C}\mathcal{P}\mathcal{T}$ inner product, (iv) An observable can be a linear operator $O$; provided it is Hermitian with respect to the $\mathcal{C}\mathcal{P}\mathcal{T}$ inner product, i.e., $\langle O \cdot \rangle_{\mathcal{C}\mathcal{P}\mathcal{T}} = \langle O \cdot \rangle_{\mathcal{C}\mathcal{P}\mathcal{T}}$, (v) If we measure an observable $O$, then the eigenvalues are the possible outcomes, (vi) If measurement gives an eigenvalue $O_n$, the states makes a transition to the eigenstate $|\psi_n\rangle$ and the probability of obtaining the eigenvalues $O_n$ (say) in a state $|\psi\rangle$ is given by

$$p_n = \frac{\langle \psi |\psi_n\rangle_{\mathcal{C}\mathcal{P}\mathcal{T}}^2}{\| \psi \|_{\mathcal{C}\mathcal{P}\mathcal{T}} \| \psi_n \|_{\mathcal{C}\mathcal{P}\mathcal{T}}},$$

where $\| \psi \|_{\mathcal{C}\mathcal{P}\mathcal{T}} = \sqrt{\langle \psi |\psi\rangle_{\mathcal{C}\mathcal{P}\mathcal{T}}}$, and (vii) If we have two quantum systems $(\mathcal{H}_1, H_1, \langle \cdot \rangle_{\mathcal{C}\mathcal{P}\mathcal{T}})$ and $(\mathcal{H}_2, H_2, \langle \cdot \rangle_{\mathcal{C}\mathcal{P}\mathcal{T}})$, then the state of the combined system lives in a tensor product Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Entanglement for $\mathcal{P}\mathcal{T}$ symmetric qubits.— In $\mathcal{P}\mathcal{T}$-symmetric quantum mechanics if we store information in any two distinct orthogonal states of non-Hermitian Hamiltonian, then we call it as a $\mathcal{P}\mathcal{T}$-symmetric quantum bit or in short a $\mathcal{P}\mathcal{T}$qubit. In general a $\mathcal{P}\mathcal{T}$qubit is different from a qubit. In the limit of vanishing non-Hermiticity parameter, a $\mathcal{P}\mathcal{T}$qubit becomes a standard qubit.

In non-Hermitian quantum theory a general two-state system will be described by a $2 \times 2$ Hamiltonian which respects $\mathcal{C}\mathcal{P}\mathcal{T}$ symmetry. Following Bender et al \[8\], this Hamiltonian is given by

$$H = \begin{pmatrix} r & e^{i\theta} \\ t & re^{-i\theta} \end{pmatrix},$$

with $r, s, t, \text{ and } \theta$ all real numbers. It has eigenvalues $E_{\pm} = r \cos \theta \pm \sqrt{st - r^2 \sin^2 \theta}$. This Hamiltonian is non-Hermitian yet it has real eigenvalues whenever we have $st > r^2 \sin^2 \theta$. Also, $H$ is invariant under $\mathcal{C}\mathcal{P}\mathcal{T}$. Two distinct eigenstates of this Hamiltonian are given by

$$|\psi_\pm\rangle = \frac{1}{\sqrt{2\cos \alpha}} \begin{pmatrix} e^{i\alpha/2} \\ e^{-i\alpha/2} \end{pmatrix} |\psi_-\rangle = \frac{1}{\sqrt{2\cos \alpha}} \begin{pmatrix} e^{-i\alpha/2} \\ -e^{i\alpha/2} \end{pmatrix},$$

where $\alpha$ is defined through $\sin \alpha = \frac{r}{\sqrt{st}} \sin \theta$. With respect to the $\mathcal{C}\mathcal{P}\mathcal{T}$ inner product (which gives a positive definite inner product) we have $\langle \psi_\pm |\psi_\pm\rangle_{\mathcal{C}\mathcal{P}\mathcal{T}} = 1$ and $\langle \psi_\pm |\psi_\mp\rangle_{\mathcal{C}\mathcal{P}\mathcal{T}} = 0$.

The $\mathcal{C}\mathcal{P}\mathcal{T}$ inner product for any two states of $\mathcal{P}\mathcal{T}$qubit is given by

$$\langle \psi |\phi\rangle = [[(\mathcal{C}\mathcal{P}\mathcal{T})|\psi\rangle],\phi\rangle,$$

where $|\psi\rangle$ is the $\mathcal{C}\mathcal{P}\mathcal{T}$ conjugate of $|\psi\rangle$. In the 2-dimensional Hilbert space, the operator $\mathcal{C}$ is given by

$$\mathcal{C} = \frac{1}{\cos \alpha} \begin{pmatrix} i \sin \alpha & 1 \\ -i \sin \alpha & 1 \end{pmatrix}.$$

The operator $\mathcal{P}$ is unitary and is given by $\mathcal{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

The operator $\mathcal{T}$ is anti-unitary and its effect is to transform $x \rightarrow x, p \rightarrow -p$ and $i \rightarrow -i$.

Since the eigenstates $|\psi_\pm\rangle$ of the non-Hermitian Hamiltonian $H$ span the two-dimensional Hilbert space, one can encode one bit of information in these orthogonal states. An arbitrary state can be represented as superposition of these orthogonal states

$$|\Psi\rangle = \alpha |\psi_+\rangle + \beta |\psi_-\rangle = \alpha |0\rangle_{\mathcal{C}\mathcal{P}\mathcal{T}} + \beta |1\rangle_{\mathcal{C}\mathcal{P}\mathcal{T}}.$$

Thus, any arbitrary superposition of two orthogonal states of $\mathcal{P}\mathcal{T}$ invariant Hamiltonian will be called $\mathcal{P}\mathcal{T}$-quantum bit or $\mathcal{P}\mathcal{T}$qubit. In fact, any linear superposition of two orthogonal states of an observable $O$ in $\mathcal{P}\mathcal{T}$-symmetric quantum theory can represent a $\mathcal{P}\mathcal{T}$qubit.

In $\mathcal{P}\mathcal{T}$-symmetric quantum theory quantum entanglement can arise if we have more than one $\mathcal{P}\mathcal{T}$qubit. Now, suppose we have two quantum systems with non-Hermitian Hamiltonians $H_1$ and $H_2$, where

$$H_1 = \begin{pmatrix} re^{i\theta} & s \\ t & re^{-i\theta} \end{pmatrix}, H_2 = \begin{pmatrix} r'e^{i\theta'} & s' \\ t' & r'e^{-i\theta'} \end{pmatrix}.$$

Let $\{|\psi_\pm\rangle\} \in \mathcal{H}_1$ and $\{|\psi'_\pm\rangle\} \in \mathcal{H}_2$ are the eigenfunctions of the Hamiltonians $H_1$ and $H_2$, respectively. The state of the combined system will live in $\mathcal{H}_1 \otimes \mathcal{H}_2$ which is spanned by $\{|\psi_+\rangle \otimes |\psi'_+\rangle, |\psi_+\rangle \otimes |\psi'_-\rangle, |\psi_-\rangle \otimes |\psi'_+\rangle, |\psi_-\rangle \otimes |\psi'_-\rangle\}$. A general state of two $\mathcal{P}\mathcal{T}$qubits can be expanded using the joint basis in $\mathcal{H}_1 \otimes \mathcal{H}_2$ and it will be an entangled state.

The $\mathcal{C}\mathcal{P}\mathcal{T}$ inner products on the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ induce the inner product on $\mathcal{H}_1 \otimes \mathcal{H}_2$. For any two arbitrary vectors $|\Psi\rangle, |\Phi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$, we define the inner product between them as

$$\langle \Psi |\Phi\rangle_{\mathcal{C}\mathcal{P}\mathcal{T}} = [(\mathcal{C}\mathcal{P}\mathcal{T})|\Psi\rangle,|\Phi\rangle].$$
Using this inner product we can calculate relevant physical quantities for the composite system under consideration.

Now, we come to the central question: is it possible to describe a composite system where one part is described by a local PT symmetric Hamiltonian and the other part is by conventional quantum theory. We will show that this will lead to contradiction with a basic law of quantum entanglement.

Violation of Entanglement Invariance Under Local Unitary.– For any pure bipartite state $|\Psi\rangle_{AB}$ the entropy of any one of the reduced density matrix is a measure of entanglement [29]. It is given by

$$E(\Psi) = -\text{tr}_A(\rho_A \log \rho_A) - \text{tr}_B(\rho_B \log \rho_B),$$  \hspace{1cm} (9)

where $\rho_A = \text{tr}_B(|\Psi\rangle_{AB}\langle\Psi|)$ and $\rho_B = \text{tr}_A(|\Psi\rangle_{AB}\langle\Psi|)$. This measure of entanglement satisfies the following properties [30]: (i) $E(\Psi) = 0$ iff $|\Psi\rangle$ is separable, (ii) $E(\Psi)$ is invariant under local unitary transformations, i.e., $E(\Psi) = E(U_I \otimes V_I |\Psi\rangle)$, (iii) $E(\Psi)$ cannot increase under local operation and classical communications (LOCC) and (iv) the entanglement content of $n$ copies of $|\Psi\rangle$ is additive, i.e., $E(\Psi^\otimes n) = nE(\Psi)$.

Consider a situation where Alice and Bob share maximally entangled state as described by conventional quantum theory, i.e., a state of the form

$$|\Phi\rangle = \frac{1}{\sqrt{2}} (|0\rangle_0 |1\rangle_1 + |1\rangle_0 |0\rangle_1),$$  \hspace{1cm} (10)

where $|0\rangle, |1\rangle$ are the eigenstates of $\sigma_z$. Suppose Alice has a locally PT symmetric quantum system with a Hamiltonian $H$ and applies a local unitary $U(t) = \exp(-itH)$ ($h = 1$) to her subsystem. Then the composite system will evolve as

$$|\Phi(t)\rangle = \frac{1}{\sqrt{2}} (e^{-iHt}|0\rangle_0 |1\rangle_1 + e^{-iHt}|1\rangle_0 |0\rangle_1).$$  \hspace{1cm} (11)

Using the resolution of identity $\sum_{n=\pm} |\psi_n\rangle\langle\psi_n| = I$ for the eigenstates of the PT symmetric Hamiltonian, we can write the above time-evolved state as

$$|\Phi(t)\rangle = \frac{1}{\sqrt{2}} e^{-iE_{\pm}t}(|\phi_+\rangle_{\pm} + e^{-iE_{\pm}t}|\phi_-\rangle_{\pm}).$$  \hspace{1cm} (12)

where $|\phi_+\rangle = e^{(0)}_+ |0\rangle + e^{(1)}_+ |1\rangle$, $|\phi_-\rangle = e^{(0)}_- |0\rangle + e^{(1)}_- |1\rangle$, $e^{(0)}_+ = \langle \psi_+ | 0 \rangle$, $e^{(1)}_+ = \langle \psi_+ | 1 \rangle$, $e^{(0)}_- = \langle \psi_- | 0 \rangle$, and $e^{(1)}_- = \langle \psi_- | 1 \rangle$. Note that here $|\psi_{\pm}\rangle$ are CPT conjugates of $|\psi_{\pm}\rangle$.

One can simplify the above by noting that $e^{(0)}_+ = e^z$, $e^{(1)}_+ = e^c$, and $e^{(1)}_- = -e^z$, where $e^c = e^{\frac{i\phi}{\sqrt{2}}}$.

Suppose Bob is in the conventional quantum world, he will apply the rules of standard quantum theory. After the action of PT symmetric local unitary on Alice’s subsystem, the reduced state of Bob is given by (with normalization)

$$\rho_B = \frac{1}{N} e^{-i\phi} (2 + \cos(|E| - 2\phi) - \cos E|(|0\rangle_0 \langle 0|_0 - 2i \sin \phi (1 - \cos E|0\rangle_1 \langle 1|_0 + (2 + \cos(2\phi) - \cos E)|1\rangle_1 \langle 1|_1).

(13)

where $E = (E_+ - E_-)$ and $N = 4(\cos E \sin^2 \phi)$. The eigenvalues of the reduced density operator are simply $\lambda_\pm(t) = \frac{1}{2} \pm \frac{1}{2} N(t)$, where $N(t)$ is given by

$$N(t) = \frac{(7 - 8 \cos E_\phi + 2 \cos 2E_\phi + 2 \cos 2\phi \sin^2 E_\phi) \frac{1}{4} \sin \phi}{4(\cos E \sin^2 \phi)}.$$  \hspace{1cm} (14)

Therefore, the entanglement of the state $|\Psi(t)\rangle$ is given by $E(\Phi(t)) = -\sum_{\lambda_\pm} \ln \lambda_i(t)$, which is not unity. Thus, by a local PT symmetric unitary transformation, the maximally entangled state has changed to a non-maximally entangled state, there by showing that entanglement is not preserved under such local unitary. For simplicity, if we take $E_\phi = \pi/2$, then we can see that the reduced density matrix is given by

$$\rho_B = \frac{1}{2} \left( \begin{array}{cc} 1 & \sin \phi \cos \phi \\ -i \sin \phi & 1 - \sin \phi \cos \phi \end{array} \right)$$  \hspace{1cm} (15)

which depends only on the non-Hermiticity parameter $\phi$. The eigenvalues of the density matrix $\rho_B$ are given by $\lambda_\pm = \frac{1}{2} (1 \pm \sqrt{1 - \cos^2 \phi})$. The violation of conservation of entanglement under local PT symmetric unitary transformation holds for all values of the non-Hermitian parameter $\phi$. When $\phi = 0$, we see that entanglement is preserved under local unitary. This shows that under a PT symmetric local unitary transformation on one part, the entanglement as seen by an observer in the conventional quantum world is not preserved. A local PT symmetric unitary transformation acts like a global operation on the two entangled particles. Since this violates a basic conservation law of entanglement, most likely option may be that a local PT symmetric unitary transformation should not coexist with the conventional quantum theory.

One can see that the recent result that local PT symmetry violates the no-signaling condition [28] also follows from our result. Suppose that Alice want to send one classical bit using the shared maximally entangled state. If She wishes to send ‘0’, she decides to do nothing and if she wants to send ‘1’ she applies a local PT symmetric unitary. Accordingly, the state of Bob will be different: in the first case it is a random mixture $\frac{1}{2}$ and in the second case it is given by $\frac{1}{2}$. Thus, by a local PT symmetric unitary transformation, the state of Bob has changed from a random mixture to a non-degenerate density operator $\rho_B$. Since the local action by Alice changes the reduced density matrix for Bob who may be space-like separated, this can lead to signaling.

Entanglement mismatch and signaling.— Here, we will illustrate the effect of non-Hermiticity on the entanglement, which we call ‘entanglement mismatch’ in switching from usual quantum theory to PT symmetric quantum theory or vice versa. We will argue that this effect is already a signature of the violation of the no-signaling condition. First, note that the partial trace operation is indeed a quantum operation. Let us define the Kraus elements for the partial trace over the subsystem $A$ as $E_i = \langle \psi_i | \otimes I$, so that we have $E_i E_i^\dagger = |\psi_i⟩ \otimes I$ for some orthonormal basis $|\psi_i⟩ \in H_A$. This satisfies $\sum_i E_i^\dagger E_i = I$. If we have a composite state
\( \rho_{AB} \), then under this quantum operation we have \( \rho_{AB} \to \sum E_i \rho_{AB} E_i^\dagger = tr_A(\rho_{AB}) = \rho_B \). Moreover a local operation on the subsystem \( A \) cannot change the reduced state of the subsystem \( B \). However, we will show that such a local operation in \( \mathcal{PT} \) symmetric world can change the reduced state of the other subsystem which may be space-like separated.

Suppose that Alice and Bob share an entangled state \( |\Psi\rangle_{AB} = \sum_i \sqrt{\lambda_i} |\psi_i\rangle_A \otimes |\phi_i\rangle_B \), where \( \lambda_i \)'s are the Schmidt coefficients, and \( |\psi_i\rangle, |\phi_i\rangle \) are the Schmidt basis. If Alice and Bob both describe their particles using conventional quantum theory, then the state of Bob’s particle is \( \rho_B = \sum \lambda_i |\psi_i\rangle \langle \psi_i| \). The entanglement entropy is given by \( E(\Psi) = -\sum \lambda_i \log \lambda_i \). Now, we will show that if Alice is in a \( \mathcal{PT} \) symmetric quantum world, then the reduced states of the particle \( B \) will be different. Because the inner products in ordinary and \( \mathcal{PT} \)-symmetric quantum theory are different, the partial traces will give different results. For example, if Alice is in a \( \mathcal{PT} \)-symmetric quantum world, the reduced density matrix for the particle \( B \) will be

\[
\rho_B = \sum_{ij} \sqrt{\lambda_i \lambda_j} (CPT) |\psi_j\rangle \langle \psi_j| |\phi_i\rangle \langle \phi_i|. \tag{16}
\]

The above density matrix is not in the diagonal form because \( \langle \psi_j|\psi_i \rangle \neq \delta_{ij} \) in the usual sense. As a consequence, the entanglement content of a bipartite state depends on the observer’s world. This phenomenon we call as the ‘entanglement mismatch’ in switching from conventional quantum world to \( \mathcal{PT} \) symmetric quantum world.

To see this clearly, let us consider the situation where Alice and Bob share an entangled state of spin-singlet in ordinary quantum theory. If both observers are in the conventional quantum world, then they will agree that they share a maximally entangled state. However, if Alice is in a \( \mathcal{PT} \) symmetric quantum world, then the reduced density matrix for particle \( B \) is given by \( \rho_B = tr_A(|\Psi^\dagger\rangle \langle \Psi^\dagger|) = \frac{1}{2}(|0\rangle \langle 0|_{CPT} - |1\rangle \langle 1|_{CPT} + |1\rangle \langle 0|_{CPT} - |0\rangle \langle 1|_{CPT} \), where the \( CPT \) inner products are given by \( \langle 0|_{CPT} = \langle 1|_{CPT} = \frac{1}{\sqrt{2}} = \frac{\cos \alpha}{\sqrt{2}} \), \( |1\rangle_{CPT} = i \tan \alpha \) and \( -i \tan \alpha \). Using these, the reduced density matrix \( \rho_B \) is given by (after renormalization)

\[
\rho_B = \frac{1}{2} \begin{pmatrix} 1 & -i \sin \alpha \\ i \sin \alpha & 1 \end{pmatrix}. \tag{17}
\]

The eigenvalues of the density matrix \( \rho_B \) are now given by \( \lambda_{\pm} = \frac{1}{2}(1 \pm \cos \alpha) \). Therefore, the entanglement entropy is given by \( E(\Psi^\dagger) = -\frac{1}{2}(1 + \sin \alpha) \log \frac{1}{2}(1 + \sin \alpha) - \frac{1}{2}(1 - \sin \alpha) \log \frac{1}{2}(1 - \sin \alpha) \). In the Hermitian limit (\( \alpha = 0 \)), Alice and Bob will share an entangled state with \( E(\Psi^\dagger) = 1 \). This shows that if Alice is located in conventional quantum world then two distant parties will share one unit of entanglement, however if she is having access to \( \mathcal{PT} \) symmetric quantum world, the shared state will have less than one unit of entanglement. This is the “entanglement mismatch” effect which arise due to non-Hermiticity.

To see that the “entanglement mismatch” indeed leads to signaling, note that the state of Bob’s particle changes depending on the situation whether Alice is in conventional quantum world or in \( \mathcal{PT} \) symmetric quantum world. Since a local operation should not change the reduced state of a remote particle, this violates the no-signaling condition. One can also understand the signaling by saying that Alice carries out measurements in conventional computational basis \( \{0, 1\} \) in \( \mathcal{PT} \) symmetric basis \( \{\psi_+, \psi_-\} \). Depending on this choice, the reduced state of Bob will be different which can in principle be distinguishable.

Outlooks.– Quantum theory has phenomenal predictive power and still it continues to predict new effects. Can we say similar things for the complex extension of quantum theory governed by \( \mathcal{PT} \) symmetric Hamiltonians? To answer such questions, one must go beyond the single particle description of non-Hermitian Hamiltonian systems to composite systems. Indeed, we have shown that the notion of \( \mathcal{PT} \) qubit and entanglement for composite quantum systems described by \( \mathcal{PT} \) symmetric Hamiltonians can be introduced in a consistent manner. However, a pressing question is whether it is possible to describe a composite system where one part is described by a local \( \mathcal{PT} \) symmetric Hamiltonian and the other part is by conventional quantum theory, and whether it is possible to switch between two quantum worlds. We have shown that this, in fact, leads to a contradiction with a basic law of conservation of quantum entanglement. Specifically, we have shown that an observer having access to \( \mathcal{PT} \) symmetric local unitary, then that can change the entanglement content of the shared resource. Thus, one of the fundamental law of quantum entanglement is violated by local \( \mathcal{PT} \) symmetric unitary. We have shown how the entanglement property of quantum states also change if we switch from usual quantum world to non-Hermitian world. If one of the particle is in a \( \mathcal{PT} \) symmetric quantum world, then a maximally entangled state in the sense of ordinary quantum theory will appear as non-maximally entangled state for another observer. This shows that it is not enough to share a maximally entangled state between two distant parties, but they should know in which world they are located. This effect we call as the “entanglement mismatch” and this can be regarded as a signature of the violation of the no-signaling condition.

Therefore, one may accept that the complex extension of quantum theory with \( \mathcal{PT} \) symmetric Hamiltonian may be a genuine extension of quantum theory, as this can have inequivalent predictions for spatially separated systems. If this is true, then \( \mathcal{PT} \) symmetric Hamiltonians may be used as a powerful resource for quantum computation, quantum information and quantum communications. Other option is that local \( \mathcal{PT} \) symmetry may not coexist with conventional quantum theory and it may not be implementable in physics.

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