Equivalence theorem
and dynamical symmetry breaking

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Abstract

The equivalence theorem, between the longitudinal gauge bosons and the states eaten up by them in the process of symmetry breaking, is shown to be valid in a class of models where the details of dynamical symmetry breaking makes it obscure.

In a recent paper [1], Donoghue and Tandean have made an intriguing point regarding the validity of the equivalence theorem [2, 3, 4, 5, 6, 7] in a class of gauge models which exhibit dynamical symmetry breaking. As a paradigm, they considered the pedagogical model [8, 9] where the electroweak gauge symmetry is dynamically broken by quark condensates which are also responsible for the chiral symmetry breaking:

\[ \langle u_L u_R \rangle = \langle d_L d_R \rangle \neq 0. \]

(1)

The pions, \( \pi^\pm \pi^0 \), which would have been the Goldstone bosons for the chiral symmetry breaking \( SU(2)_L \times SU(2)_R \rightarrow SU(2)_V \), are eaten up by the electroweak gauge bosons in this model in absence of any fundamental Higgs bosons. Donoghue and Tandean [1] then calculate the amplitude of the process \( e^+ e^- \rightarrow Z \gamma \) at energies larger than \( M_Z \) mediated by the triangle diagram. Both quarks and leptons can appear in the triangle, and from the condition of vanishing of gauge anomalies, they obtain that the amplitude vanishes. Since \( \pi^0 \) constitutes the longitudinal part of the \( Z \) in this model, they then compare it with the process \( e^+ e^- \rightarrow \pi^0 \gamma \), mediated through the triangle diagram. However, since the pion consists of quarks but no leptons, only quarks circulate in the loop now, and therefore the amplitude does not vanish. This, they claim, is a violation of the equivalence theorem. The purpose of this article is to examine this claim.

In Higgs models of symmetry breaking, one can calculate the couplings of the unphysical Higgs bosons (which are eaten up by the gauge bosons) from those of the gauge bosons without having to make any assumption about the Higgs content of the model. This can be done by demanding that, if one uses the \( R_\xi \) gauges to calculate any amplitude, the \( \xi \)-dependent poles must cancel [10, 11].
Thus, for two fermions $a$ and $b$ which are represented by their field operators $\psi_a$ and $\psi_b$, if the coupling to any gauge boson $V$ is given by

$$\bar{\psi}_a \gamma^\mu (G_{ab} + G'_{ab} \gamma_5) \psi_b V_\mu ,$$

this requirement demands that the corresponding unphysical Higgs boson, $S$, will have the coupling

$$\frac{1}{M_V} \bar{\psi}_a \left[ (m_a - m_b)G_{ab} + (m_a + m_b)G'_{ab} \gamma_5 \right] \psi_b S ,$$

where $M_V$ is the mass of the gauge boson $V$ after symmetry breaking. From this requirement alone, one can verify the equivalence theorem for any amplitude.

However, the equivalence theorem is more deep-rooted than the Higgs mechanism of symmetry breaking for the following reason. In any gauge symmetry breaking, some gauge bosons obtain masses $M_V$ whose exact values depend on some parameters of the theory. For any $M_V \neq 0$, the longitudinal components of the massive gauge bosons are physical states. The “Nambu-Goldstone” modes, the states absorbed by the gauge bosons, are unphysical. On the other hand, for $M_V = 0$, the Nambu-Goldstone modes are physical states but the longitudinal components are not, since the symmetry is not broken. The equivalence theorem then merely states that all observables are continuous in the limit $M_V \to 0$. In other words, in that limit, the amplitudes with any process with longitudinal component of a vector boson is the same (apart from a phase maybe) with the amplitudes for the corresponding processes where the longitudinal gauge bosons are replaced by the states that are eaten up by them in the process of symmetry breaking. Stated this way, the equivalence theorem seems to be a statement of continuity of certain parameters of the theory, and hence is expected to be valid for any model with symmetry breaking. In light of this, the claim of Donoghue and Tandean [1] is indeed surprising.

To examine their claim, let us use their paradigm of QCD condensates breaking the electroweak gauge symmetry [8, 9], and for the sake of definiteness, let us talk about processes involving $Z$ bosons only. Obviously, one can make similar arguments for processes involving $W$ bosons. Since the $Z$-boson does not provide any flavor changing neutral current with the standard model fermions, the indices $a$ and $b$ in Eq. (2) have to be equal, and we denote the couplings $G$ and $G'$ in this case with a single index. From Eq. (3) now, we see that proving the equivalence theorem is tantamount to proving that the coupling of the $\pi^0$ to the fermion field $\psi_a$ is given by

$$\frac{2m_a}{M_Z} G'_a \bar{\psi}_a \gamma_5 \psi_a \pi^0 .$$

If the fermion $a$ in question is the electron, for example, it might naively seem that such a coupling with the pion cannot exist since the pion wave function does not have any electron. However, this is not true, as can be seen from the diagram of Fig. [1]. Two important points need to be made before we calculate this coupling. First, the intermediate line can only be $Z$. The diagram with intermediate photon line cannot contribute since the photon couplings are vectorial, whereas the pion couples only to the axial vector current through the relation

$$\left\langle 0 \left| Q \gamma^\mu \gamma_5 \frac{1}{2} Q \right| \pi^J(q) \right\rangle = f_\pi q^\mu \delta^{IJ} .$$
where $|0\rangle$ is the hadronic vacuum, $I, J$ indices run over the adjoint representation of the isospin symmetry SU(2)$_V$, and

$$Q \equiv \begin{pmatrix} u \\ d \end{pmatrix}. \quad (6)$$

Second, the diagram must be calculated in the unitary gauge where unphysical degrees of freedom cannot appear in the intermediate state. Otherwise, the pions can appear even as intermediate states and it will be impossible to calculate the diagram. In the unitary gauge, the propagator of the gauge boson is given by

$$-iD^{\mu \nu}(q) = -i\left(g^{\mu \nu} - q^\mu q^\nu/M_Z^2\right)q^2 - M_Z^2. \quad (7)$$

Thus the effective interaction between the fermions and the pion derived from this diagram is given by

$$iL_{\text{eff}} = \bar{u}_a(p') \gamma^\mu \left(G_a + G'_a \gamma_5\right) u_a(p) \cdot [-iD^{\mu \nu}(q)] \cdot i \langle 0 | J^{(Z)}_{\nu} | \pi^0(q) \rangle, \quad (8)$$

where $J^{(Z)}_{\nu}$ is the current that couples to the $Z$ boson:

$$J^{(Z)}_{\nu} = -\frac{g}{4\cos \theta_W \pi} \left(\gamma_{\nu} \gamma_5 u - \bar{d} \gamma_\nu \gamma_5 d + \cdots\right), \quad (9)$$

where $g$ is the weak SU(2) gauge coupling, $\sin \theta_W = e/g$, and the dots signify vector currents as well as currents of fermions other than the up and the down quarks which are of no interest for us. From Eqs. (5), (7) and (9), we obtain

$$D^{\mu \nu}(q) \left\langle 0 \left| J^{(Z)}_{\nu} \right| \pi^0(q) \right\rangle = \frac{q^\mu}{M_Z}, \quad (10)$$

using the relations for the gauge boson masses obtained in this model, viz.,

$$M_W = M_Z \cos \theta_W = \frac{1}{2} g f_\pi. \quad (11)$$

Putting Eq. (10) in Eq. (8), it is straightforward to show that the amplitude is

$$\frac{2m_a}{M_Z} G'_a \bar{u}_a(p') \gamma_5 u_a(p), \quad (12)$$



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1The up-quark field $u$ (in italics) is not to be confused with the positive energy spinor $u$ (in boldface).
where we have used the spinor definitions

\[ \not{p}u(p) = mu(p), \quad \bar{u}(p)\not{p} = m\bar{u}(p). \tag{13} \]

Obviously, Eq. (12) is equivalent to the interaction of Eq. (4). Since this coupling is obtained in this model, it is now easy to verify that the equivalence theorem is valid for any amplitude, as we argued before.

The skeptic in us may wonder whether our proof is valid for processes where \( Z \) or \( \pi^0 \) couples to internal fermion lines, given that we have used the on-shell condition for the spinors, Eq. (13), to derive Eq. (12).\footnote{In fact, one can ask the same question about Higgs models of symmetry breaking since Eq. (13) is used to derive Eq. (3) as well.} We put to rest such doubts by explicitly calculating the process \( e^+e^- \rightarrow Z\gamma \), which is the process calculated by Donoghue and Tandean [1]. However, we note that the triangle diagrams considered by them are not the lowest order diagrams for this process. There are tree diagrams, given in Fig. 2, which contribute. The amplitude for this diagram, \( A_Z \), can be written as

\[ iA_Z = ie\varepsilon^\mu(k)\epsilon^\nu(p)\not{\nabla}(p')\Gamma_{\mu\nu}u(p), \tag{14} \]

where \( \varepsilon \) and \( \epsilon \) represent the polarization vectors of the \( Z \) and the photon respectively, and

\[ \Gamma_{\mu\nu} = \gamma^\nu \frac{\not{p} - \not{q} + m_e^2}{(p - q)^2 - m_e^2} \gamma^\mu (G_e + G_e'\gamma_5) + \gamma^\mu (G_e + G_e'\gamma_5) \frac{\not{p} - \not{k} + m_e}{(p - k)^2 - m_e^2} \gamma^\nu. \tag{15} \]

The diagrams for \( e^+e^- \rightarrow \pi^0\gamma \), on the other hand, are obtained if the \( Z \)-boson lines of Fig. 2 couple to the pion wavefunction in the manner shown in Fig. 3. The amplitude of such diagrams is given by

\[ iA_\pi = i\epsilon \varepsilon^\mu(k)\not{\nabla}(p')\Gamma_{\mu\nu}u(p) \cdot [ -iD(\rho)(q) ] \cdot i \langle 0 \mid J^{(\pi)}_{\rho} \mid \pi^0(q) \rangle = i\epsilon \varepsilon^\mu(k)\not{\nabla}(p')\Gamma_{\mu\nu}u(p) \frac{q_\nu}{M_Z}, \tag{16} \]

using Eq. (10) in the last step. In general, the amplitudes in Eqs. (14) and (13) are not equal, even in magnitude. However, if we consider a longitudinal polarized \( Z \) boson in Eq. (14) whose 4-momentum is given by \( (E, \kappa\hat{n}) \) for some unit 3-vector \( \hat{n} \), the polarization vector will be given by \( \varepsilon^\mu_{\text{long}}(q) \equiv (\kappa, E\hat{n})/M_Z \). In the limit \( M_Z \rightarrow 0 \) or equivalently \( E/M_Z \rightarrow \infty \), since \( \kappa \approx E \), we obtain

\[ e^\pm(p) \rightarrow Z(q) \]

\[ 1 \]

\[ e^\pm(p') \rightarrow \gamma(k) \]

\[ \gamma(k) \]

\[ e^\mp(p - q) \]

\[ e^\pm(p - k) \]

\[ Z(q) \]

\[ \gamma(k) \]

\[ e^\pm(p') \]

\[ Z(q) \]

\[ e^\pm(p - k) \]
\[
\begin{align*}
\text{\begin{picture}(10,10)
\put(0,0){Z} & \put(2,0){\ldots} & \put(3,0){\ldots} & \put(4,0){\ldots} & \put(5,0){\ldots} & \put(6,0){\ldots} & \put(7,0){\ldots} & \put(8,0){\ldots} & \put(9,0){\ldots} \end{picture}}
+ \begin{picture}(10,10)
\put(0,0){Z} & \put(2,0){\ldots} & \put(3,0){\ldots} & \put(4,0){\ldots} & \put(5,0){\ldots} & \put(6,0){\ldots} & \put(7,0){\ldots} & \put(8,0){\ldots} & \put(9,0){\ldots} \end{picture}
\begin{picture}(10,10)
\put(0,0){\Pi^0} & \put(2,0){\ldots} & \put(3,0){\ldots} & \put(4,0){\ldots} & \put(5,0){\ldots} & \put(6,0){\ldots} & \put(7,0){\ldots} & \put(8,0){\ldots} & \put(9,0){\ldots} \end{picture}
+ \begin{picture}(10,10)
\put(0,0){Z} & \put(2,0){\ldots} & \put(3,0){\ldots} & \put(4,0){\ldots} & \put(5,0){\ldots} & \put(6,0){\ldots} & \put(7,0){\ldots} & \put(8,0){\ldots} & \put(9,0){\ldots} \end{picture}
\begin{picture}(10,10)
\put(0,0){\Pi^0} & \put(2,0){\ldots} & \put(3,0){\ldots} & \put(4,0){\ldots} & \put(5,0){\ldots} & \put(6,0){\ldots} & \put(7,0){\ldots} & \put(8,0){\ldots} & \put(9,0){\ldots} \end{picture}
\begin{picture}(10,10)
\put(0,0){\Pi^0} & \put(2,0){\ldots} & \put(3,0){\ldots} & \put(4,0){\ldots} & \put(5,0){\ldots} & \put(6,0){\ldots} & \put(7,0){\ldots} & \put(8,0){\ldots} & \put(9,0){\ldots} \end{picture}
+ \ldots
\end{align*}
\]

Figure 3: Diagrams responsible for the mass of the \(Z\)-boson.

\(\varepsilon_{\text{long}}^\mu(q) = q^\mu/M_Z\). Thus, in this limit, the amplitudes for \(e^+e^- \rightarrow Z_{\text{long}}\gamma\) and \(e^+e^- \rightarrow \pi^0\gamma\) are indeed equal, as seen from Eqs. (14) and (16). This is the verification of the equivalence theorem to this order.

It is now easy to see that the proof can be extended to any process, e.g., with any number of \(Z\)-bosons in the initial and final states. If we have a diagram with an external \(Z_{\text{long}}\) line, we obtain a factor \(\varepsilon_{\text{long}}^\mu(q)\) in the amplitude. On the other hand, if we let the \(Z\)-boson to couple to the \(\pi^0\) wavefunction, we will obtain the factors \([-iD^{\mu\rho}(q)]\cdot i \langle 0 | J_{\rho}^{(Z)}(Z) | \pi^0(q) \rangle\), coming from the \(Z\) propagator and the matrix element involving the pion wavefunction. However, as shown in Eq. (14), this equals \(q^\mu/M_Z\), which is \(\varepsilon_{\text{long}}^\mu(q)\) in the limit \(M_Z \rightarrow 0\). Thus, equivalence theorem is valid for any process.

And in fact, it is also easy to see that the proof can be easily extended to any model of dynamical symmetry breaking. In general, let us denote the spin-0 state eaten up by the \(Z\)-boson by \(\Pi^0\). Since the \(Z\) mass is generated by the series of diagrams given in Fig. 3, it is easy to see that one requires

\[
\langle 0 | J_{\nu}^{(Z)} | \Pi^0(q) \rangle = -M_Z q_{\nu},
\]

no matter how \(M_Z\) is related to the parameters of the unbroken theory\(^3\). This is all one needs to verify Eq. (14), and thereby the equivalence theorem.

Earlier, we said that the equivalence theorem is the statement of continuity of physical observables in the limit \(M_V \rightarrow 0\). Since this limit can be realized as \(q \rightarrow 0\), it is obvious that one expects the equivalence at only the lowest non-trivial order in the gauge coupling constant \(\gamma\) to all orders in other couplings in the model. And we have already proved the theorem to this order for the process \(e^+e^- \rightarrow Z_{\text{long}}\gamma\). The diagram discussed by Donoghue and Tandean \(\Pi\) is higher order in gauge coupling constant and hence is not relevant for the validity of the equivalence theorem for the process \(e^+e^- \rightarrow Z\gamma\).

One can of course consider some other process for which tree diagrams do not exist \(\Pi\). Take, for example, the process \(\nu\bar{\nu} \rightarrow Z_{\text{long}}\gamma\). Here, even the non-triangle diagrams are fourth order in gauge coupling constants, and so is the triangle-mediated diagram shown in Fig. 4. To examine the validity of the equivalence theorem for this process, we will have to get into a detailed analysis of the triangle part of the diagram. This has been done in earlier papers \(\Pi, \Pi, \Pi\) in a different context. We mainly follow the notation and analysis of Hikasa \(\Pi\) in what follows.

The triangle part gives a \(Z\gamma\pi\) effective coupling. Since only the vector part of the \(Z\) coupling matters, we can consider it as a \(\gamma^*\gamma\pi\) coupling apart from some irrelevant differences in the coupling constants, \(\gamma^*\) being a photon which is not necessarily on shell. The matrix element for \(\pi^0 \rightarrow \gamma^*\gamma\)

\(^3\)Actually, the right side can have an arbitrary phase, which will appear as an overall phase of all couplings of \(\Pi^0\). But this phase does not affect any physics, including the equivalence theorem.
transition is given by

$$\left\langle \gamma^*(K)\gamma(k)|\pi^0(q)\right\rangle = \lim_{q^2 \to m_{\pi}^2} \left( m_{\pi}^2 - q^2 \right) \left\langle \gamma^*(K)\gamma(k)|\phi_\pi|0\right\rangle.$$ (18)

where $\phi_\pi$ is the interpolating field for the pion. We now employ the PCAC relation:

$$\partial_\mu J^\mu_5 = f_\pi m_{\pi}^2 \phi_{\pi} + \frac{e^2}{16\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu},$$ (19)

where $J^\mu_5$ is the axial vector current of the quarks, and $F_{\mu\nu}$ is the electromagnetic field strength tensor. Using this, we can rewrite the matrix element of Eq. (18) as

$$\left\langle \gamma^*(K)\gamma(k)|\pi^0(q)\right\rangle = \left\{ \frac{m_{\pi}^2 - q^2}{f_\pi m_{\pi}^2} \left\{ -e^2 \left\langle 0|T(\partial_\mu J^\mu_5 J^\alpha_5)|0\right\rangle - \frac{e^2}{4\pi^2} [kK]_{\alpha\beta} \right\} \right\}$$

$$= e^2 \left\{ \frac{m_{\pi}^2 - q^2}{f_\pi m_{\pi}^2} \left\{ q^\mu T_{\mu\alpha\beta} - \frac{1}{4\pi^2} [kK]_{\alpha\beta} \right\} \right\},$$ (20)

where $T_{\mu\alpha\beta}$ is the 3-point function with the axial current and two vector currents, and

$$[kK]_{\alpha\beta} \equiv \varepsilon_{\alpha\beta\lambda\rho} k^\lambda K^\rho.$$ (21)

Now, from the requirements of vector current conservation and Lorentz invariance, one can write down the most general form for $T_{\mu\alpha\beta}$ as follows:

$$T_{\mu\alpha\beta} = \left\{ k^2 \varepsilon_{\mu\alpha\beta\rho} K^\rho + k_\alpha [kK]_{\mu\beta} \right\} F_1$$
$$+ \left\{ K^2 \varepsilon_{\mu\alpha\beta\rho} K^\rho + K_\beta [kK]_{\mu\alpha} \right\} F_2$$
$$+ (k + K)_\mu [kK]_{\alpha\beta} F_3 + (k - K)_\mu [kK]_{\alpha\beta} F_4,$$ (22)

where $F_i (i = 1 \cdots 4)$ are form factors. Thus,

$$q^\mu T_{\mu\alpha\beta} = \left\{ k^2 F_1 - K^2 F_2 - q^2 F_3 + (k^2 - K^2) F_4 \right\} [kK]_{\alpha\beta}.$$ (23)

4Note that $F_4 = 0$ from Bose symmetry. But we do not impose Bose symmetry at this level, since we want to use our results in the case where the two vector particles are the photon and the Z.
We are obviously interested in the case $k^2 = 0$ since one photon is on shell, and $q^2 = 0$ since the pion is a Goldstone boson in this model. Thus, only the form factors $F_2$ and $F_4$ are relevant, and from the general formulas given by Hikasa [15], we obtain

$$F_2 = \frac{1}{2\pi^2} \int_0^1 dz \int_0^{1-z} d'z' \frac{zz'}{m^2 - zz'K^2},$$  \hspace{1cm} (24)

$$F_4 = 0,$$  \hspace{1cm} (25)

where $m$ is the mass of the fermions in the loop. Now, the fermions in the loop are obviously the $u$ and the $d$ quarks. Since the pion mass is zero, and $m^2_\pi \propto m_u + m_d$, the up and the down quarks must be massless in this model. Thus, putting $m = 0$ in Eq. (24), we obtain $F_2 = -(4\pi^2K^2)^{-1}$, so that from Eqs. (23) and (20), we find that the amplitude of the triangular loop actually vanishes in this case.

Two comments should be made here. First, the vanishing of this amplitude has nothing to do with the decay $\pi^0 \rightarrow \gamma\gamma$ in the real world where the pion has a small mass [13, 14, 15]. For that case, in the soft pion limit one is interested in the limit $k^2 = K^2 = 0$ and $q^2$ small, so that the form factor $F_3$ is important in that case. Moreover, since $F_3 = (48\pi^2m^2)^{-1}$, Eq. (23) shows that the term $q^\mu T_{\mu\alpha\beta}$ vanishes in Eq. (20) for $q^2 = 0$, which is the famous Sutherland-Veltman theorem [16, 17].

Second, in the present limit this implies that the contribution of the diagram in Fig. 4 is actually zero no matter which fermion-antifermion pair is considered at the outer lines. This is in complete agreement with the equivalence theorem. The result can be, and has been [13], obtained by using a $\sigma$-model which incorporates the anomalous contributions in a straightforward way.

We can summarize as follows. In models exhibiting dynamical symmetry breaking, verification of the equivalence theorem may not be obvious, but is nevertheless possible. And we believe that the equivalence theorem is always valid because it is based only on the requirement that physical observables are continuous in the values of certain parameters of the theory.

### Note added:
After the paper was submitted for publication, I was made aware of a paper by Zhang [18] which also addresses the issue of the Equivalence theorem in models where symmetry is broken by fermion condensates. The conclusions of that paper is similar to the present paper.

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