ALGEBRAIC MODELS OF LOCAL PERIOD MAPS AND YUKAWA COUPLING

RUGGERO BANDIERA AND MARCO MANETTI

ABSTRACT. We describe some $L_\infty$ model for the local period map of a compact Kähler manifold. Application includes the study of deformations with associated variation of Hodge structure constrained by certain closed strata of the Grassmannian of the de Rham cohomology. As a byproduct we obtain an interpretation in the framework of deformation theory of the Yukawa coupling.

INTRODUCTION

For a better understanding of the content of this paper it is preferable to begin with a heuristic and somewhat imprecise description. Let $X$ be a compact Kähler manifold of dimension $n$; the Hodge filtration on the de Rham complex of $X$ determines a filtration of graded vector spaces

$$0 \subset F^0 H^*(X, \mathbb{C}) \subset F^1 H^*(X, \mathbb{C}) \subset \cdots \subset F^n H^*(X, \mathbb{C}) = H^*(X, \mathbb{C}) = \bigoplus_i H^i(X, \mathbb{C})$$

and therefore a sequence of elements in the total Grassmannian Grass$(H^*(X, \mathbb{C}))$, i.e., in the (reducible) variety of all graded vector subspaces of the de Rham cohomology. We recall that Grass$(H^*(X, \mathbb{C}))$ is smooth and the Zariski tangent space at a point $A^*$ is

$$T_{A^*} \text{Grass}(H^*(X, \mathbb{C})) = \prod_{i=0}^n \text{Hom}(A^i, H^i(X, \mathbb{C})).$$

Let $X \to (U, 0)$ be a deformation of $X$, with $U$ sufficiently small and contractible; then topologically $X$ is the product $U \times X$, in particular for every $u \in U$ there exists a natural isomorphism $H^*(X_u, \mathbb{C}) = H^*(X_0, \mathbb{C})$, induced by the Gauss-Manin connection, and it is well known that the local $i$th period map

$$\mathcal{P}^i : U \to \text{Grass}(H^*(X, \mathbb{C})), \quad \mathcal{P}^i(u) = F^i H^*(X_u, \mathbb{C}),$$

is holomorphic, cf. [35, Thm. 10.9]. Since Grassmannians admits natural stratifications, e.g. via Schubert varieties, it is natural to consider deformations whose period map is constrained into a certain closed strata. For instance, given $X \to (U, 0)$ as above one can consider the determinantal loci

$$(0.1) \quad \Sigma_{i,j,p,q}(U) = \left\{ u \in U \mid \text{rank} \left( F^i H^p(X_u, \mathbb{C}) \to H^p(X_0, \mathbb{C}) \right) \leq q \right\}.$$

Using Griffiths’ description of the differential of the period map, it is straightforward to calculate the Zariski tangent space of $\Sigma_{i,j,p,q}(U)$ in terms of the Kodaira-Spencer map $T_0 U \to H^1(X, \Theta_X)$; transversality implies in particular that $T_0 \Sigma_{i,j,p,q}(U) = T_0 U$ for every $i > j$. On the other side, if $j < i \leq p < n + j$ it is natural to expect $\Sigma_{i,j,p,q}(U)$ a closed analytic proper subset of $U$, provided $q$ sufficiently small and $X \to (U, 0)$ sufficiently general. In conclusion, the determinantal loci $\Sigma_{i,j,p,q}(U)$ are generally singular and a study of them requires a deep understanding of higher derivatives of the local period maps.
In this paper, we restrict our attention to the subsets
\[ Y_U = \sum_{n,1,n,0}(U) = \cap_p \sum_{n,1,p,0}(U) \]
although we think that ideas and methods developed here can be applied in a more or less straightforward way to more general situations. This choice is also motivated to a better understanding of the Yukawa coupling, see e.g. [18, p. 36], [5, p. 132], from the point of view of deformation theory.

It is now a matter of fact that higher derivatives of holomorphic maps have lots on common with Massey product structures: this is not surprising since the latter may be thought as higher derivatives of Kuranishi maps. The quoting of Schlessinger and Stasheff [30] “Massey product structures can be very helpful, though they are in general described in a form that is unsatisfactory” is valid also in relation to higher derivatives. It is nowadays well understood that Massey product structures should be replaced by DG-Lie algebras and $L_\infty$-algebras, following the general principles exposed by Deligne and Drinfeld in their famous letters [8, 12]; in the same circle of ideas, every morphism of deformation theories is induced by a morphism in the homotopy category of DG-Lie algebras, which admits a representative as an $L_\infty$ morphism, unique up to homotopy equivalence.

In our situation, an $L_\infty$ representative for the local universal period map has been described in [15], where universal means that we consider the variation of the Hodge over the semiuniversal deformation $X \to B$ of $X$. Using this description it is not difficult to write down a set equations for $Y_B$ and prove that, at least when $B$ is smooth, its tangent cone is defined by homogeneous polynomials of degree $\geq n$, and the ones of degree $n$ are precisely those in the Yukawa linear system.

However the situation is not yet satisfactory since the above approach does not give any deformation theoretic interpretation of $Y_B$; in other words it is not clear under what extent $Y_B$ is the local moduli space for some deformation problem. The natural starting point is that $Y_B$ fits into a pull-back diagram

\[
\begin{array}{ccc}
Y_B & \rightarrow & \text{Grass}(F^1H^*(X, \mathbb{C})) \\
\downarrow & & \downarrow \\
B & \rightarrow & \text{Grass}(H^*(X, \mathbb{C}))
\end{array}
\]

and the attempt to replicate this picture in the category of $L_\infty$ algebras it is not easy at it seems at a first sight, since the $L_\infty$ morphism representing $P^n$ is defined up to homotopy and fiber products of $L_\infty$-morphisms are not defined in general. The most natural solution to both problems is to consider homotopy fiber products. This is possible since DG-Lie algebras and $L_\infty$-algebras are pointed categories of fibrant objects, although the usual procedure for the construction of homotopy fiber products gives models which are in general very far from being minimal and therefore with Maurer-Cartan equation of difficult geometric interpretation.

The main and probably most difficult part of this paper is devoted to the construction of some small, and in some cases minimal, models for the homotopy pull-back relative to the $L_\infty$ version of the above diagram (0.2).

It is well known that deformations of $X$ are controlled by the Kodaira-Spencer DG-Lie algebra $KS_X$, defined as the Dolbeault’s resolution of the holomorphic tangent sheaf, equipped with the natural bracket and the opposite of the Dolbeault’s differential. Let’s fix a Kähler metric on $X$ and denote by $H^p,q_X$ the space of harmonic $(p,q)$-forms. If

\[
U = \oplus_q H^{n,q}, \quad V = \oplus_{p<n} \oplus_q H^{p,q}, \quad W = \oplus_{0<p<n} \oplus_q H^{p,q},
\]

then the graded vector spaces $\text{Hom}^*(U, V)[-1]$ and $\text{Hom}^*(U, W)[-1]$, considered as DG-Lie algebras with trivial bracket and trivial differential, control the Grassmann functors of embedded deformations of $F^0H^*(X, \mathbb{C})$ inside $H^*(X, \mathbb{C})$ and $F^1H^*(X, \mathbb{C})$ respectively. The first main, and
computationally the hardest, result of this paper is the complete description of an $L_\infty$ morphism

$$P^n: KS_X \to \text{Hom}^*(U, V)[-1]$$

representing the $n$th period map (Theorem 7.8); unfortunately such a description involves Green’s operators in its Taylor coefficients of order $\geq 2$. At this point we can consider the homotopy pull-back diagram

$$\begin{array}{ccc} Y\text{u}_X & \longrightarrow & \text{Hom}^*(U, W)[-1] \\ f \downarrow & & \downarrow \\ KS_X & \overset{P^n}{\longrightarrow} & \text{Hom}^*(U, V)[-1] \end{array}$$

keeping in mind that $Y\text{u}_X$ is defined up to isomorphism in the homotopy category and then its $L_\infty$ model is determined only up to quasi-isomorphism. For simplicity of notation we shall refer to $Y\text{u}_X$ as the Yukawa algebra of $X$.

Our second main result is the description of an $L_\infty$ model for the Yukawa algebra in which the underlying complex is $KS_X \times \text{Hom}^*(U, V/W)[-2]$ and $f$ is the projection on the first factor (Theorem 8.2): this is done by using the theory of derived brackets and $L_\infty$ extensions. We shall see that at the level of deformation functors the induced map $f: \text{Def}_{Y\text{u}_X} \to \text{Def}_{KS_X}$ is generally not injective, since its fibre is pro-represented by the vector space $\text{Hom}(H^{i,0}_X, H^{0,-n}) \oplus \text{Hom}(H_X^{1,0}, H^{0,n})$, while the image $f(\text{Def}_{Y\text{u}_X}) \subset \text{Def}_{KS_X}$ is pro-represented by the previously defined locus $Y_B$. The philosophical interpretation of this fact is that, as frequently happen in deformation theory, the “geometric” definition

$$(0.3) \quad Y_B = \left\{ u \in U \mid \text{rank} \left( F^n H^n(X_u, \mathbb{C}) \to \frac{H^n(X_0, \mathbb{C})}{F^1 H^n(X_0, \mathbb{C})} \right) = 0 \right\}.$$ 

requires the additional framing of a homotopy between 0 and the map $F^n H^n(X_u, \mathbb{C}) \to \frac{H^n(X_0, \mathbb{C})}{F^1 H^n(X_0, \mathbb{C})}$ in order to be considered a genuine, i.e., not artificially truncated, deformation problem.

The last result of this paper is the proof that if $X$ is a K3 surface, then $Y\text{u}_X$ is formal; this implies in particular that, denoting by $X \to (B, 0)$ its universal deformation, there exists an isomorphism of germs $i: (B, 0) \to (H^1(X, \Theta_X), 0)$ such that $i(Y_B)$ is the quadratic cone defined by the Yukawa coupling.

1. Review of holomorphic Cartan homotopy formulas and Kähler identities

Unless otherwise specified every vector space is considered over the field of complex numbers. Given a complex manifold $X$ let’s denote by $(A_X^\omega, d = \partial + \overline{\partial})$ its de Rham complex; later on we shall also consider its subcomplexes,

$$A_X^{\geq p, *} = \bigoplus_{i \geq p} A_X^{i,*}, \quad A_X^{\geq p, *} = \bigoplus_{i > p} A_X^{i,*},$$

together the quotient complexes

$$A_X^{\leq p, *} = A_X^{\leq p-1, *}, \quad A_X^{\leq p, *} = A_X^{\leq p, *}, \quad A_X^{\geq p, *} = A_X^{\geq p, *}, \quad A_X^{\geq p, *} = A_X^{\geq p, *}.$$ 

Denote by $(A_X^0(\Theta_X), \overline{\partial})$ the Dolbeault complex of the holomorphic tangent sheaf. For notational simplicity, unless otherwise specified, we shall denote by the same symbol $[-, -]$ both the usual bracket on $A_X^0(\Theta_X)$ and the graded commutator in the space $\text{Hom}^*(A_X^\omega, A_X^\omega)$. The contraction map $\Theta_X \otimes \Omega^{-1}_X \to \Omega^{-1}_X$ extends in the obvious way to map of degree $-1$:

$$i: A^0_X(\Theta_X) \to \text{Hom}^*(A_X^\omega, A_X^\omega), \quad \xi \mapsto i_\xi, \quad i_\xi(\omega) = \xi \omega,$$
which satisfies the “holomorphic Cartan homotopy formulas”, see e.g. [15]:
\[ l: A_X^{0,*}(\Theta_X) \to \text{Hom}^*(A_X^{0,*}, A_X^{0,*}), \quad \xi \mapsto l_\xi = [\partial, i_\xi], \]

The morphism of degree 0 satisfies the equalities \( \iota_{i_\eta} = -[\overline{\partial}, l_\eta] \) and \( l_{[\eta, \mu]} = [\eta, l_\mu] \). A straightforward computation in local coordinates shows that every \( \xi \in A_X^{0,p}(\Theta_X) \) can be recovered from the linear map \( l_\xi: A_X^{0,0} \to A_X^{0,p} \); in particular both \( i \) and \( l \) are injective maps. We shall refer to \( l_\xi \) as the holomorphic Lie derivative of \( \xi \).

For every \( \xi \in A_X^{0,1}(\Theta_X) \), the operator \( i_\xi \) is nilpotent of degree 0 and then it makes sense do consider its exponential
\[ e^{i_\xi}: A_X^{0,*} \to A_X^{0,*}. \]
Since \([i_\eta, i_\mu] = 0\) we always have \( e^{i_\xi} e^{i_\eta} = e^{i_{\xi+i_\eta}} = e^{i_{\xi+n}} \).

**Lemma 1.1.** For an element \( \xi \in A_X^{0,1}(\Theta_X) \) the following are equivalent:

1. \( \overline{\partial} + l_\xi = \frac{1}{2} [\overline{\partial}, l_\xi] \), i.e., \( \xi \) is integrable (in the sense of Newlander-Nirenberg);
2. \( e^{-i_\xi} de^{i_\xi} = d + l_\xi \);
3. \( (d + l_\xi)^2 = 0 \);
4. \( (\overline{\partial} + l_\xi)^2 = 0 \).

**Proof.** The implications (2) \( \Rightarrow \) (3) \( \Rightarrow \) (4) are completely trivial. For the equivalence (1) \( \Leftrightarrow \) (4) it is sufficient to write
\[
(\overline{\partial} + l_\xi)^2 = \frac{1}{2} [\overline{\partial}, l_\xi] + \frac{1}{2} [l_\xi, l_\xi] = l_\xi + \overline{\partial} l_\xi + \frac{1}{2} [l_\xi, l_\xi] = l_\xi + i_\xi \iota_{\overline{\partial}} l_\xi + \frac{1}{2} l_\xi l_\xi,
\]
and keep in mind the injectivity of \( l \). At last, for the implication (1) \( \Rightarrow \) (2) we have
\[
[-i_\xi, d] = [d, i_\xi] = [\partial, i_\xi] + [\overline{\partial}, i_\xi] = i_\xi + \iota_{\overline{\partial}} i_\xi, \quad [-i_\xi, i_\xi] = [-i_\xi, \iota_{\overline{\partial}} i_\xi] = 0,
\]
and therefore
\[
e^{-i_\xi} de^{i_\xi} = d + \sum_{n \geq 0} \frac{[-i_\xi, -]_n}{(n+1)!} ([i_\xi, d])
\]

In the above setup, let \( n \) be the dimension of \( X \). For every \( 0 \leq p \leq n \) and for every \( \xi \in A_X^{0,1}(\Theta_X) \) denote
\[ F_\xi^p = e^{i_\xi}(A_X^{0,p-,*}) \subset A_X^{0,*}. \]

In particular for \( \xi = 0 \) we recover the usual Hodge filtration of the de Rham complex. It follows from Lemma 1.1 that \( F_\xi^p \) is a subcomplex of \( A_X^{0,*} \) whenever \( \xi \) is integrable and
\[ e^{i_\xi}: (F_\xi^p, d + l_\xi) \to (F_\xi^p, d). \]

\[ □ \]
is an isomorphism of complexes.

**Lemma 1.2.** Let \( \xi, \eta \in A_X^{0,1}(\Theta_X) \) be two integrable sections: then the image of

\[
H^*(F_\xi, d) \to H^*(X, \mathbb{C}) = H^*(A_X^{\ast, \ast}, d)
\]

is contained in the image of

\[
H^*(F_\eta, d) \to H^*(X, \mathbb{C}) = H^*(A_X^{\ast, \ast}, d)
\]

if and only if for every \( x \in A_X^{n,0} \) such that \( (\overline{\partial} + I_\xi)x = 0 \) there exists \( y \in A_X^{0,n-1} \) such that

\[
\frac{i^n}{n!} x = (\overline{\partial} + I_\eta)y.
\]

**Proof.** Let \( e^{i\xi}x \) be an element of \( F_\xi^n \), with \( x \in A_X^{n,\ast} \); we have already proved that \( de^{i\xi}x = 0 \) if and only if \( (\overline{\partial} + I_\xi)x = 0 \). Then the cohomology class of \( e^{i\xi}x \) belongs to the image of \( H^*(F_\xi, d) \to H^*(X, \mathbb{C}) \) if and only if there exists \( x \in A_X^{\ast, \ast} \) such that \( e^{i\xi}x - dz \in F_\eta^1 \) or equivalently if and only if

\[
e^{-i\eta}(e^{i\xi}x - dz) = e^{i\xi}x - (d + I_\eta)e^{-i\eta}z \in F_\eta^1 = A_X^{n,0}.
\]

When \( x \in A_X^{n,0} \) the above equation is verified for \( z = 0 \), while for \( x \in A_X^{n,0} \) the above equation admits a solution if and only if

\[
\frac{i^n}{n!} x = (\overline{\partial} + I_\eta)y
\]

where \( y \) is the component of \( e^{-i\eta}z \) of type \((0, n-1)\). \( \square \)

Assume now that \( X \) is a compact Kähler manifold, i.e., a compact complex manifold equipped with Kähler metric, then the Laplacians associated to the operators \( d, \partial, \overline{\partial} \) satisfy the well known equalities [38, p. 44]:

\[
\Delta_{\overline{\partial}} = \Delta_{\partial} = \frac{1}{2}\Delta_d
\]

and therefore they determine the same spaces of harmonic forms

\[
H_{X}^{p,q} = \ker(\Delta : A_X^{p,q} \to A_X^{p,q}), \quad \Delta = \Delta_{\overline{\partial}}, \Delta_{\partial}, \Delta_d.
\]

Denoting by \( \iota : H_{X}^{p,q} \to A_X^{p,q} \) the inclusion, by \( \pi : A_X^{p,q} \to H_{X}^{p,q} \) the harmonic projection and by \( G_{\overline{\partial}} \) the Green operator for \( \Delta_{\overline{\partial}} \) we have the equalities [38, p. 66]:

\[
\begin{align*}
\partial \iota &= \pi \partial = I_{\overline{\partial}}, \\
\overline{\partial} \iota &= \pi \overline{\partial} = \pi G_{\overline{\partial}} = G_{\overline{\partial}} I_{\overline{\partial}} = 0, \\
\Delta_{\overline{\partial}} G_{\overline{\partial}} &= G_{\overline{\partial}} \Delta_{\overline{\partial}} = Id - \pi
\end{align*}
\]

and by Hodge theory the inclusions \( \iota : (H_{X}^{p,\ast}, 0) \to (A_X^{p,\ast}, \overline{\partial}) \) is a quasi-isomorphism of differential graded vector spaces.

Moreover the Kähler identities gives in particular the following commuting relations in the graded Lie algebra \( \text{Hom}^{\ast}(A_X^{\ast, \ast}, A_X^{\ast, \ast}) \) [38, pp. 44, 45 and 67):

\[
\begin{align*}
\Delta_{\overline{\partial}} &= [\overline{\partial}, \overline{\partial}], \\
[\partial, \overline{\partial}] &= [\partial, \Delta_{\overline{\partial}}] = [\partial, G_{\overline{\partial}}] = [\overline{\partial}, G_{\overline{\partial}}] = [\overline{\partial}, \overline{\partial}] = [\overline{\partial}, \overline{\partial}] = 0.
\end{align*}
\]

Alongside to the excellent Weil’s book we also refer to [25, 35, 39] as additional resources for the above formulas.

**Definition 1.3.** Given a compact Kähler manifold \( X \), the operator

\[
h = -\overline{\partial} G_{\overline{\partial}} = -G_{\overline{\partial}} \overline{\partial} \in \text{Hom}^{-1}(A_X^{\ast, \ast}, A_X^{\ast, \ast})
\]

will be called the \( \overline{\partial} \)-propagator.
It is straightforward to verify that the $\overline{\partial}$-propagator satisfies the following identities
\[ [\overline{\partial}, h] = \overline{\partial} h + h \overline{\partial} = i\pi - Id, \quad h_1 = \pi h = h^2 = 0, \quad [\partial, h] = 0, \quad [h \overline{\partial}, \overline{\partial}] = [h, \overline{\partial} \overline{\partial}] = \partial . \]

It is worth to point out that the above equalities give a simple and short proof of the equality $\overline{\partial} \partial(A_X^{p,q}) = \ker \overline{\partial} \cap \partial(A_X^{p,q})$, and more precisely that a $\partial$-exact element $x = \partial \alpha$ is $\overline{\partial}$-closed if and only if $x = \overline{\partial} \partial h \alpha$: in fact we can write
\[ x = \partial \alpha = [h, \overline{\partial}] \alpha = h \overline{\partial} \partial \alpha - \overline{\partial} h \alpha = \overline{\partial} \partial h \alpha - h \overline{\partial} x . \]

Similarly the degeneration of the Hodge to de Rham spectral sequence can be proved as a simple consequence of the formulas $[\partial, h \overline{\partial}] = 0$, $[h \overline{\partial}, \overline{\partial}] = \partial$, since they easily imply the equality
\[ e^{h \overline{\partial}} \overline{\partial} e^{-h \overline{\partial}} = (Id + h \overline{\partial})\overline{\partial}(Id - h \overline{\partial}) = \partial + \overline{\partial} . \]

According to the terminology introduced by Eilenberg and Mac Lane [13], cf. also [20, 28], we may express the equalities $\overline{\partial} h + h \overline{\partial} = i\pi - Id$, $h_1 = \pi h = h^2 = 0$, by saying that for every $k$ the diagram
\[ (H^k_X, 0) \xrightarrow{1} (A^{k\ast}_X, \overline{\partial}) \]

is a contraction of complexes.

2. Infinitesimal deformations and variations of Hodge structures

According to the standard terminology adopted in deformation theory, by an infinitesimal deformation we mean a deformation over a local Artin ring; when the Artin ring has square zero maximal ideal we shall speak of first order deformations. Let $\text{Art}$ be the category of local Artin $\mathbb{C}$-algebras with residue field $\mathbb{C}$; unless otherwise specified, for every $B \in \text{Art}$ we shall denote by $m_B$ its maximal ideal.

Given a complex manifold $X$, let’s denote by $A^{p,q}_X$ the sheaf of smooth differentiable forms of type $(p, q)$ on $X$. For a local Artin algebra $B$ and a section $\xi \in A^{0,1}_X(\Theta_X) \otimes m_B$, one can consider the sheaf of $B$-modules on $X$:
\[ \mathcal{O}_{X,\xi} = \ker(\overline{\partial} + l_\xi : A^{0,0}_X \otimes B \rightarrow A^{0,1}_X \otimes B) . \]

It is well known that $\mathcal{O}_{X,\xi}$ is a sheaf of flat $B$-modules and $\mathcal{O}_{X,\xi} \otimes_B \mathbb{C} = \mathcal{O}_X$ if and only if $(\overline{\partial} + l_\xi)^2 = 0$. Thus, when $\xi$ is integrable the local ringed space $X_\xi = (X, \mathcal{O}_{X,\xi})$ is a deformation of $X$ over $\text{Spec}(B)$ and every deformation is obtained, up to isomorphism, in this way. Finally, two integrable sections $\eta, \xi \in A^{0,1}_X(\Theta_X) \otimes m_B$ give isomorphic deformations if and only if there exists $a \in A^{0,0}_X(\Theta_X) \otimes m_B$ such that $e^{a}(\overline{\partial} + l_\eta) = (\overline{\partial} + l_\xi)e^{a}$: for a complete and detailed proof of the above assertions see e.g. [21].

**Proposition 2.1.** In the notation above, for every integrable section $\xi \in A^{0,1}_X(\Theta_X) \otimes m_B$ and every positive integer $p$ let $F_p^\xi \subset A^{p,\ast}_X \otimes B$ be the ideal sheaf generated by the image of the multiplication map
\[ \bigwedge^p d\mathcal{O}_{X,\xi} \rightarrow \bigoplus_{i+j=p} A^{i,j}_X \otimes B . \]

Then
\[ F_p^\xi = e^{\xi} \left( A^{p,\ast}_X \otimes B \right) = \Gamma(X, F_p^\xi) \subset A^{p,\ast}_X \otimes B . \]

In particular the subcomplexes $F_p^\xi$ represent the variation of the Hodge filtration along the infinitesimal deformation $X_\xi$. 

Proof. This is worth to recall that the filtration $F^n_p$ depends by the section $\xi$ and we shall see later that it can be recovered from the deformation $\hat{X}_\xi$ only up to automorphisms of the de Rham complex inducing the identity in cohomology. □

When $X$ is compact Kähler, and $\xi \in A^X_0(\Theta_X) \otimes m_B$ is integrable, a useful description of the cohomology of the complex $(A^X_\ast \otimes B, \overline{\partial} + l_\xi)$ in terms of the $\overline{\partial}$-propagator can be obtained by homological perturbation theory.

**Lemma 2.2.** Let $X$ be compact Kähler with $\overline{\partial}$-propagator $h = -\overline{\partial} G_{\overline{\partial}}$ and denote by $H^\ast X_\ast \subset A^X_\ast$ the subcomplex of harmonic forms. If $B \in \text{Art}$ and $\xi \in A^X_0(\Theta_X) \otimes m_B$ is integrable, then the map

$$t_\xi = (Id - \partial h)^{-1} = \sum_{n \geq 0} (h l_\xi)^n: (H^\ast X_\ast \otimes B, 0) \to (A^\ast X_\ast \otimes B, \overline{\partial} + l_\xi)$$

is a homotopy equivalence of complexes of $B$-modules with homotopy inverse

$$\pi_\xi = \pi(Id - l_\xi h)^{-1} = \sum_{n \geq 0} \pi(l_\xi h)^n: (A^\ast X_\ast \otimes B, \overline{\partial} + l_\xi) \to (H^\ast X_\ast \otimes B, 0).$$

In particular the cohomology groups of $(A^\ast X_\ast \otimes B, \overline{\partial} + l_\xi)$ are free $B$-modules.

Proof. In view of the formulas $[\overline{\partial}, h] = \overline{\partial} h + h \overline{\partial} = \pi - Id$, $h_1 = \pi h = h^2 = 0$, the ordinary perturbation lemma [20, 28] tells that

$$t_\xi: (H^\ast X_\ast \otimes B, \partial) \to (A^\ast X_\ast \otimes B, \overline{\partial} + l_\xi), \quad \delta_\xi = \sum_{n \geq 0} \pi(l_\xi h)^n l_{\xi}^0,$$

is a homotopy equivalence of complexes of $B$-modules with homotopy inverse $\pi_\xi$. It is now sufficient to observe that, since $\partial \pi = \pi \partial = [\partial, h] = [\partial, l_\xi] = 0$, we have $l_\xi h \partial = \partial l_\xi h$ and then

$$\delta_\xi = \sum_{n \geq 0} \pi(l_\xi h)^n l_{\xi}^0 = \sum_{n \geq 0} \pi(l_\xi h)^n \partial l_{\xi} = \sum_{n \geq 0} \pi \partial(l_\xi h)^n l_{\xi} = 0.$$

Notice that $\pi_\xi t_\xi$ is the identity; moreover the ordinary perturbation lemma also says that

$$(\overline{\partial} + l_\xi) h_\xi + h_\xi (\overline{\partial} + l_\xi) = \pi_\xi t_\xi - Id, \quad \text{where} \quad h_\xi = \sum_{n \geq 0} h(l_\xi h)^n.$$

Since $\partial l_\xi = \sum_{n \geq 0} \partial(h l_\xi)^n = \sum_{n \geq 0} (h l_\xi)^n \partial h = 0$, the Lemma 2.2 also gives another proof of the fact that $H^\ast (F^p_{\xi+1}) \to H^\ast (F^p_{\xi})$ is split injective for every $p$. In fact $(d + l_\xi) t_\xi = 0$ and the conclusion follows by a straightforward chasing in the diagram of complexes

$$0 \to (A^p_{\ast X} \otimes B, d + l_\xi) \to (A^p_{\ast X} \otimes B, d + l_\xi) \to (A^p_{\ast X} \otimes B, \overline{\partial} + l_\xi) \to 0$$

$$\xrightarrow{\rho_\xi} \xrightarrow{t_\xi} \xrightarrow{\rho_\xi} \xrightarrow{\rho_\xi} \xrightarrow{\rho_\xi}$$

$$\xrightarrow{\rho_\xi} (F^p_{\xi}, d) \xrightarrow{\rho_\xi} (F^p_{\xi+1}, d) \xrightarrow{\rho_\xi} (H^p_{\ast X}, 0)$$

in which the upper row is exact and every vertical arrow is a quasi-isomorphism.

The next proposition shows that the results of Lemma 2.2 hold in a stronger form when the de Rham complex is replaced by the subcomplex of $\partial$-closed forms.

**Proposition 2.3.** If $X$ is compact Kähler and $\xi \in A^0_{\ast X}(\Theta_X) \otimes m_B$ is integrable, then the map

$$(Id - l_\xi): (\ker \partial \otimes B, \overline{\partial} + l_\xi) \to (\ker \partial \otimes B, \overline{\partial})$$
is an isomorphism of complexes. In particular, if \( \dim X = n \), then the map
\[
\sum_{i \geq 0} (h_\xi)^i : (A^{n+}_X \otimes B, \overline{\partial}) \to (A^{n+}_X \otimes B, \overline{\partial} + l_\xi)
\]
is an isomorphism of complexes.

Proof. We first notice that, since \( \partial h_\xi = h_\xi \overline{\partial} \), the above maps make sense and are isomorphisms of graded vector spaces. We have
\[
\overline{\partial}(l d - h_\xi) - (l d - h_\xi)(\overline{\partial} + l_\xi) = h_\xi \overline{\partial} + h_\xi \overline{\partial}^2 - \overline{\partial} h_\xi - l_\xi
\]
\[
= h_\xi \pi_\xi - h_\xi \overline{\partial} + \frac{h}{2} [\xi, l_\xi] - \overline{\partial} h_\xi - l_\xi
\]
and, whenever \( \partial x = 0 \) we get \( \pi l_\xi x = \pi \partial \xi x = 0 \). \( \square \)

Putting together Lemma 1.2 and Lemma 2.2 we obtain immediately the following result:

**Theorem 2.4.** Let \( X \) be a compact Kähler manifold of dimension \( n \) with \( \overline{\partial} \)-propagator \( h \) and let \( \xi, \eta \in A^{0,1}_X(\Theta_X) \otimes m_B \) be two integrable sections: then
\[
H^*(F^{\eta}_\xi) \subset H^*(F^{\eta}_h) \subset H^*(X, \mathbb{C}) \otimes B
\]
if and only if the map
\[
\psi : H^{n,0}_X \otimes B \to H^{0,n}_X \otimes B,
\]
\[
\psi(x) = \sum_{i,j \geq 0} \pi(l_\xi h)^i j \xi^{\eta}_\xi - (h_\xi)^j i = \pi \eta^\theta_i \xi^{\eta}_\xi
\]
is trivial.

**Corollary 2.5.** Let \( X \) be a compact Kähler manifold of dimension \( n \) with \( \overline{\partial} \)-propagator \( h \) and \( \xi \in A^{0,1}_X(\Theta_X) \otimes m_B \) an integrable section: then
\[
H^*(F^{\eta}_\xi) \subset H^*(A^{0,+}_X \otimes B)
\]
if and only if the map
\[
\psi : H^{n,0}_X \otimes B \to H^{0,n}_X \otimes B,
\]
\[
\psi(x) = \sum_{j \geq 0} \pi \xi^{\eta}_\xi (h_\xi)^j i = \pi \xi^{\eta}_\xi
\]
is trivial.

### 3. Review of \( A_\infty \) algebras and \( L_\infty \) algebras

From now on we assume that the reader is familiar with the notion of \( A_\infty \) and \( L_\infty \)-algebras; this section is devoted to fix notations and recall the basic results that we need in the sequel of this paper.

Given a graded vector space \( V = \oplus V^i \) we shall denote by \( |x| \) the degree of a homogeneous element; given an integer \( n \) we shall denote by \( V[n] \) the same space with the degrees shifted by \( n \), i.e., \( V[n]^i = V[i+n] \) for every \( i \); finally we shall denote by \( s : V[n+1] \to V[n] \) the tautological linear isomorphism of degree +1.

For every graded vector space \( V \) we shall denote by \( \text{Hoch}(V) \) the graded Lie algebra of coderivations of the tensor coalgebra \( T(V) = \oplus_{k \geq 0} V \otimes^k \). Corestriction induces an isomorphism of graded vector spaces
\[
\text{Hoch}(V) \to \prod_{k \geq 0} \text{Hom}^*(V \otimes^k, V), \quad Q \mapsto pq = (q_0, q_1, \ldots, q_k, \ldots),
\]
where we denote by $p: T(V) \to V$ the natural projection. We call the $q_k: V^\otimes k \to V$ the Taylor coefficients of $Q$. We shall denote by $Q^1_k$ the composition $V^\otimes k \to T(V) \xrightarrow{\mathcal{Q}} T(V) \to V^\otimes j$, where the first arrow is the inclusion and the last one is the projection. A coderivation $Q$ is determined by its Taylor coefficients according to $Q^1_k = q_0: V^\otimes 0 = K \to V$, $Q^1_k = 0$ for $j \neq 1$, $Q^0_k = 0$ for all $k$,

$$Q^j_k(v_1 \otimes \cdots \otimes v_k) = \sum_{i=1}^j (-1)^{\sum_{<i} |Q|} q_{k-j+1}((v_1 \otimes \cdots \otimes v_{k-j}) \otimes \cdots \otimes v_k)$$

for $1 \leq j \leq k + 1$ and finally $Q^j_k = 0$ for $j > k + 1$. Given a morphism of graded coaugmented coalgebras $F: T(V) \to T(W)$, we shall similarly denote by $f_k: V^\otimes k \to W$ the components of the corestriction $T(V) \xrightarrow{\mathcal{F}} T(W) \xrightarrow{\mathcal{G}} W$ and call them the Taylor coefficients of $F$: the morphism $F$ is determined by its Taylor coefficients according to (where again we denote by $F^1_k$ the composition $V^\otimes k \to T(V) \xrightarrow{\mathcal{F}} T(W) \to V^\otimes j$)

$$F^j_k(v_1 \otimes \cdots \otimes v_k) = \sum_{i_1+\cdots+i_j=k} f_{i_1}(v_1 \otimes \cdots \otimes v_{i_1}) \otimes \cdots \otimes f_{i_j}(v_{k-i_j+1} \otimes \cdots \otimes v_k)$$

for $1 \leq j \leq k$, where the sum is taken over all ordered partitions $i_1+\cdots+i_j = k$ with $i_1, \ldots, i_j \geq 1$, and finally $F^j_k = 0$ for $j > k$.

**Definition 3.1.** An $A_{\infty}[1]$ algebra structure on a graded space $V$ is a DG-coalgebra structure on $T(V)$ vanishing at 1, that is, a degree one coderivation $Q \in \text{Hoch}(V)$ such that $[Q,Q] = 0$ and $q_0 = 0$. An $A_{\infty}[1]$ morphism between $A_{\infty}[1]$ algebras $(V,Q)$ and $(W,R)$ is a morphism of DG-coalgebras $F = (f_1, \ldots, f_k, \ldots) : (V,Q) \to (W,R)$. A morphism $F$ is called strict if $f_k = 0$ for every $k \geq 2$.

**Remark 3.2.** It is well known that the above definition is equivalent to the condition that the higher products on the suspension $V[-1]$, induced by the Taylor coefficients $q_k$ via the inverse décalage isomorphism $\text{Hom}^*(V^\otimes k, V) \to \text{Hom}^*(V[-1]^\otimes k, V[-1])[1-k]$, define a strong homotopy associative algebra structure, also known as an $A_{\infty}$ algebra structure, on $V[-1]$. In particular, given a differential graded associative algebra $(A,d,\cdot)$ there is an induced $A_{\infty}[1]$ algebra structure on $A[1]$ with Taylor coefficients $q_1(s^{-1}a) = -s^{-1}da, q_2(s^{-1}a_1 \otimes s^{-1}a_2) = (-1)^{|a_1|} s^{-1}(a_1 \cdot a_2)$ and $q_k = 0$ for $k \geq 3$.

We shall denote by $CE(V)$ the graded Lie algebra of coderivations of the symmetric coalgebra $S(V) = \otimes_{k \geq 0} V^\otimes k$; again corestriction induces an isomorphism of graded vector spaces $CE(V) \to \text{Hom}^*(S(V), V)$ and we call the components $q_k: V^\otimes k \to V$ of $Q \in CE(V)$ under corestriction the Taylor coefficients of $Q$: they determine $Q$ according to $Q^1_k = q_0, Q^1_k = 0$ for $j \neq 1$, $Q^0_k = 0$ for all $k$,

$$Q^j_k(v_1 \otimes \cdots \otimes v_k) = \sum_{\sigma \in S(k-j+1,j-1)} \varepsilon(\sigma) v_{\sigma(k)} \otimes \cdots \otimes v_{\sigma(k-j+1)} \otimes \cdots \otimes v_{\sigma(1)}$$

where we denote by $S(p,q)$ the set of $(p,q)$ unshuffles and by $\varepsilon(\sigma) = \varepsilon(\sigma; v_1, \ldots, v_k)$ the Koszul sign. A morphism of graded coaugmented coalgebras $F: S(V) \to S(W)$ is determined by its Taylor coefficients $f_k: V^\otimes k \to W$ according to $F^0_k(1) = 1, F^1_k = F^0_k = 0$ for $j,k \geq 1$,

$$F^j_k(v_1 \otimes \cdots \otimes v_k) = \frac{1}{j!} \sum_{i_1+\cdots+i_j=k} \sum_{\sigma \in S(1,\ldots,i_j)} \varepsilon(\sigma) f_{i_1}(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(i_1)}) \otimes \cdots \otimes f_{i_j}(v_{\sigma(k-i_j+1)} \otimes \cdots \otimes v_{\sigma(k)})$$

for $1 \leq j \leq k$ and $F^j_k = 0$ for $j > k$. 
**Definition 3.3.** An $L_{\infty}[1]$ algebra structure on a graded vector space $V$ is a DG-coalgebra structure on $S(V)$ vanishing at 1. An $L_{\infty}$ morphism between $L_{\infty}[1]$ algebras is a morphism of DG-coalgebras $F = (f_1, \ldots, f_k, \ldots) : (V, Q) \to (W, R)$: it is called strict if $f_k = 0$ for every $k \geq 2$.

**Remark 3.4.** The above definition is equivalent to say that the higher brackets on $V[-1]$ induced by the inverse décalage isomorphism $Hom^*(V \otimes^k, V) \to Hom^*(V[-1] \wedge^k, V[-1])[1-k]$ define a strong homotopy Lie algebra structure, also known as an $L_{\infty}$ algebra structure, on $V[-1]$; cf. [24]. In particular, given a DG-Lie algebra $(L, d, [\cdot, \cdot])$ there is an induced $L_{\infty}[1]$ algebra structure on $L[1]$ with Taylor coefficients $q_1(s^{-1}l) = -s^{-1}dl$, $q_2(s^{-1}l_1 \circ s^{-1}l_2) = (-1)^{|l_1|} s^{-1}|l_1, l_2|$ and $q_k = 0$ for $k \geq 3$.

**Remark 3.5.** There is a symmetrization functor $sym : A_{\infty}[1] \to L_{\infty}[1]$ from the category of $A_{\infty}[1]$ algebras to that of $L_{\infty}[1]$ algebras [24], generalizing the classical construction from the category of differential graded associative algebras: an $A_{\infty}[1]$ algebra $(V, q_1, \ldots, q_k, \ldots)$ is mapped to $(V, sym(q_1), \ldots, sym(q_k), \ldots)$, and sending an $A_{\infty}[1]$ morphism $F = (f_1, \ldots, f_k, \ldots) : (V, Q) \to (W, R)$ to the $L_{\infty}[1]$ morphism $sym(F) = (sym(f_1), \ldots, sym(f_k), \ldots) : (V, sym(Q)) \to (W, sym(R))$, $sym(f_k)(v_1 \circ \cdots \circ v_k) = \sum_{\sigma \in S_k} \varepsilon(\sigma)f_k(v_{\sigma(1)} \circ \cdots \circ v_{\sigma(k)})$.

**Definition 3.6.** Given an $A_{\infty}[1]$ (resp.: $L_{\infty}[1]$) morphism $F = (f_1, \ldots, f_k, \ldots) : (V, Q) \to (W, R)$ then $f_1; (V, q_1) \to (W, r_1)$ is a morphism of complexes: we shall say that $F$ is a quasi-isomorphism, or equivalently a weak equivalence, if such is $f_1$.

Most of our computations will be based on the following important and well known homotopy transfer theorem: for a proof we refer to [14, 19] and references therein.

**Theorem 3.7.** Given an $A_{\infty}[1]$ (resp.: $L_{\infty}[1]$) algebra $(V, q_1, \ldots, q_k, \ldots)$, a differential graded vector space $(W, r_1)$ and a diagram

$$(W, r_1) \xrightarrow{f_1} (V, q_1)$$

where $f_1, q_1$ are morphisms of cochain complexes and $K$ is a homotopy such that $g_1 f_1 = Id$, $q_1 K + K q_1 = f_1 g_1 - Id$, then there is an induced $A_{\infty}[1]$ (resp.: $L_{\infty}[1]$) structure $(W, r_1, \ldots, r_k, \ldots)$ on $W$ and an $A_{\infty}[1]$ (resp.: $L_{\infty}[1]$) quasi-isomorphism $F = (f_1, \ldots, f_k, \ldots) : W \to V$, defined by the recursions

$$f_k = \sum_{j=2}^{k} K q_j F_k^j, \quad \text{for } k \geq 2,$$

$$r_k = \sum_{j=2}^{k} q_1 q_j F_k^j, \quad \text{for } k \geq 2.$$  

Notice that $F_k^j$ only depends on $f_1, \ldots, f_{k-j+1}$. Moreover there exists an $A_{\infty}[1]$ (resp.: $L_{\infty}[1]$) quasi-isomorphism $G = (g_1, \ldots, g_k, \ldots) : V \to W$ such that $GF$ is the identity. In the $A_{\infty}[1]$ case the Taylor coefficients of the morphism $G$ may be defined by the recursive formulas

$$g_k = \sum_{j=1}^{k-1} g_j Q_k^j K_k,$$
where $K_k := \sum_{i=0}^{k-1} i! \cdot K \otimes (f_1 g_1)^{k-i+1} : V^{\otimes k} \to V^{\otimes k}$.

Remark 3.8. It is useful to notice that homotopy transfer is compatible in the obvious sense with the symmetrization functor $\text{sym} : A_{\infty}[1] \to L_{\infty}[1]$ of Remark 3.5. In the $L_{\infty}[1]$ case it is also possible to give an explicit description of the quasi-isomorphism $G$, although via recursive formulas much more complicated than in the $A_{\infty}[1]$ case, cf. [2, 11].

We have already defined the category $\text{Art}$ of local Artin $\mathbb{C}$-algebras with residue field $\mathbb{C}$; in addition we shall denote by $\text{Set}$ the category of sets and by $\text{Grp}$ the category of groups.

For a Lie algebra $L^0$, its exponential functor is

$$\exp_{L^0} : \text{Art} \to \text{Grp}, \quad \exp_{L^0}(B) = \exp(L^0 \otimes m_B).$$

Given a DG-Lie algebra $L$, its Maurer-Cartan functor is

$$\text{MC}_L : \text{Art} \to \text{Set}, \quad \text{MC}_L(B) = \{ x \in L^1 \otimes m_B \mid dx + \frac{1}{2}[x, x] = 0 \}.$$

The (left) gauge action of $\exp_{L^0}$ on $\text{MC}_L$ may be written as

$$e^a \ast x = x + \sum_{n \geq 0} \frac{[a, \ldots, a]^n}{(n + 1)!} [(a, x) - da]$$

and the corresponding quotient is called the deformation functor associated to $L$:

$$\text{Def}_L = \frac{\text{MC}_L}{\exp_{L^0}}.$$

The basic theorem of deformation theory asserts that every quasi-isomorphism $L \to M$ of DG-Lie algebras induces an isomorphism of functors $\text{Def}_L \to \text{Def}_M$. In one of its simplest interpretations, the basic theorem of derived deformation theory asserts that in characteristic 0 every “deformation problem” is described by the functor $\text{Def}_L$ for a suitable DG-Lie algebra $L$ determined up to homotopy; similarly every “morphism of deformation theories” is induced by a morphism in the homotopy category of DG-Lie algebras: here enters in the game also $L_{\infty}$-algebras as a fundamental tool, every morphism in the homotopy category of DG-Lie algebras can be represented by a direct $L_{\infty}$ morphism.

The construction of the functor $\text{Def}_L$ extends, although in a non trivial way, also to $L_{\infty}[1]$ algebras. First notice that if $V = (V, q_1, q_2, \ldots)$ is an $L_{\infty}[1]$ algebra and $(C, d)$ is a commutative DG-algebra, then the natural extensions

$$q_1 : V \otimes C \to V \otimes C, \quad q_1(v \otimes c) = q_1(v) \otimes c + (-1)^{|v|} v \otimes dc,$$

$$q_2 : (V \otimes C)^{\otimes 2} \to V \otimes C, \quad q_2(v_1 \otimes c_1, v_2 \otimes c_2) = (-1)^{|v_1| |v_2|} q_2(v_1, v_2) \otimes c_1 c_2,$$

$$q_k : (V \otimes C)^{\otimes k} \to V \otimes C, \quad q_k(v_1 \otimes \cdots \otimes v_k) \otimes c_1 \cdots c_k,$$

gives an $L_{\infty}[1]$ algebra structure on $V \otimes C$. Then the Maurer-Cartan functor is defined as

$$\text{MC}_V : \text{Art} \to \text{Set}, \quad \text{MC}_V(B) = \left\{ x \in V^0 \otimes m_B \mid \sum_{n \geq 0} \frac{1}{n!} q_n(x^{\otimes n}) \right\},$$

while the gauge action is replaced my the homotopy equivalence: two Maurer-Cartan elements $x, y \in \text{MC}_V(B)$ are said to be homotopy equivalent if there exists $z(t) \in \text{MC}_V[t, dt] (B)$ such that $z(0) = x$ and $z(1) = y$; here $V[t, dt] := V \otimes \mathbb{C}[t, dt]$ and $\mathbb{C}[t, dt]$ is the de DG-algebra of polynomial differential forms on the affine line. The functor $\text{Def}_V$ is defined as the quotient of $\text{MC}_V$ by homotopy equivalence.

For a DG-Lie algebra we recover the same construction as above and every $L_{\infty}$-morphism $V \to W$ induces two natural transformations $\text{MC}_V \to \text{MC}_W$, $\text{Def}_V \to \text{Def}_W$, cf. [25, 30].
4. The Fiorenza-Manetti model for the local period maps

The starting point of this paper are the results of [15], some of them we have already used in Proposition 2.1. Apart from the technicalities, the main contribution of [15] is the fact that the “correct” period domain for a compact Kähler manifold $X$ is not a subset of the product of the Grassmannians $\prod_i \text{Grass}(H^i(X, \mathbb{C}))$ but the Grassmannian of the entire de Rham complex $(A^\infty_X, d)$, which is a completely different object in the framework of derived geometry and carries a richer algebro-geometric structure, cf. [9, 10].

Given a vector space $V$ and a vector subspace $F \subset V$, the Grassmann functor of the pair $(V, F)$ is

$$G_{V,F} : \text{Art} \rightarrow \text{Set}, \quad G_{V,F}(B) = \{ F \mid F \subset V \otimes B \text{ flat } B\text{-submodule}, \ F \otimes_B \mathbb{K} = F \}.$$  

If $V$ is finite dimensional, the tautological bundle on the Grassmannian is a universal family for the above functor and then $G_{V,F}(B)$ is identified with the set of morphism of pointed schemes $(\text{Spec}(B), 0) \rightarrow (\text{Grass}(V), F)$.

If $V$ is a complex of vector spaces and $F \subset V$ is a subcomplex we can define the functor $G_{V,F}$ as

$$G_{V,F}(B) = \left\{ \begin{array}{l}
\text{subcomplexes of flat } B\text{-modules } F \subset V \otimes B \text{ such that } F \otimes_B \mathbb{K} = F \\
\text{on } V \text{ and inducing the identity in cohomology}
\end{array} \right\} \cup \left\{ \begin{array}{l}
\text{B-linear automorphisms of the complex } V \otimes B \text{ lifting the identity}
\end{array} \right\}.$$

The main reason of taking quotient for the automorphisms which are homotopy equivalent to the identity is that in this way the functor $G_{V,F}$ is homotopy invariant; this means that every morphism of pairs $\alpha : (V, F) \rightarrow (W, H)$ such that both $\alpha : V \rightarrow W$ and $\alpha : F \rightarrow H$ are quasi-isomorphisms, induces an isomorphism of functors $G_{V,F} \cong G_{W,H}$. This fact is essentially proved in [15] (especially in the ArXiv versions); a more detailed study will appear in the forthcoming paper [16]. A first consequence of the homotopy invariance is that if $H^*(F) \rightarrow H^*(V)$ is injective, then the natural transformation

$$G_{V,F} \rightarrow G_{H^*(V), H^*(F)}, \quad F \mapsto H^*(F),$$

is well defined and is an isomorphism of functors: to see this it is sufficient to choose a set $H \subset V$ of harmonic representatives for the cohomology of $V$ such that $H \cap F \rightarrow F$ is a quasi-isomorphism and apply homotopy invariance to the morphism of pairs $(H, H \cap F) \rightarrow (V, F)$.

There exist several equivalent models of $L_\infty$ algebras governing the functor $G_{V,F}$; in this section we explain the one usually called Fiorenza-Manetti mapping cone, which is very convenient for the algebraic description of period maps. Another model, much more convenient for the goal of this paper, will be described in next sections. To this end we need to start by recalling some basic stuff about homotopy fibers of morphisms of differential graded Lie algebras.

Given a morphism of DG-Lie algebras $f : L \rightarrow M$ one can define the analog of Maurer-Cartan functor:

$$\text{MC}_f(A) = \left\{ (l, e^m) \in L^1 \otimes m_A \times \exp(M^0 \otimes m_A) \mid dl + \frac{1}{2}[l, l] = 0, \ e^m \ast f(l) = 0 \right\}.$$

The gauge action on $\text{MC}_L$ lifts to a (left) action of the group functor $\exp_{L^0} \times \exp_{dM^{-1}}$ on $\text{MC}_f$ by setting

$$(e^a, e^{du}) \ast (l, e^m) = (e^a \ast l, e^{du} e^m e^{-f(a)}), \quad (l, u) \in (L^0 \oplus M^{-1}) \otimes m_A.$$

**Lemma 4.1** ([27, Thm. 6.14]). Given a morphism of differential graded Lie algebras $f : L \rightarrow M$, the functor of Artin rings

$$\text{Def}_f = \frac{\text{MC}_f}{\exp_{L^0} \times \exp_{dM^{-1}}}.$$


is isomorphic to the deformation functor of the homotopy fiber of $f$. This means that every commutative diagram of differential graded Lie algebras

$$
\begin{array}{ccc}
L & \xrightarrow{\alpha} & P \\
\downarrow{f} & & \downarrow{1} \\
M & \xrightarrow{\beta} & Q
\end{array}
\begin{array}{c}
\xrightarrow{\ker(q)} \\
\xrightarrow{i}
\end{array}
\xrightarrow{\alpha}
\begin{array}{c}
\ker(q) \\
i
\end{array}
$$

with $q$ surjective, $\alpha, \beta$ quasi-isomorphisms and $i$ the inclusion, induces canonically two isomorphisms of functors

$$\text{Def}_f \xrightarrow{\cong} \text{Def}_q \xrightarrow{\cong} \text{Def}_n = \text{Def}_{\ker(q)}.$$ 

Notice that, when $f: L \rightarrow M$ is an injective morphism of DG-Lie algebras, the Maurer-Cartan functor admits the simpler description

$$\text{MC}_f(A) = \{ e^m \in \exp(M^0 \otimes m_A) \mid e^{-m} \cdot 0 \in f(L^1) \otimes m_A \}.$$ 

On the other side, when $M = 0$ the functors $\text{MC}_f$ and $\text{Def}_f$ reduces to the usual definitions of $\text{MC}_L$ and $\text{Def}_L$ respectively.

**Remark 4.2.** The importance of homotopy fibres in deformation theory has been clarified in several places, see e.g. [14, 16, 27], since they are the right object for the study of semitrivialized deformation problems.

We shall talk about semitrivialized deformation problems when we consider deformations of a geometric object together with a trivialization of the deformation of a specific part of it. For instance, in this class we have Grassmann functors and more generally embedded deformations of a subvariety $Z$ of a complex manifold $X$: such deformations (over a base $B$) can be considered as deformations $Z \subset X$ of the pairs $Z \subset X$ equipped with a trivialization $X \cong X \times B$.

Keeping in mind that deformations = solutions of Maurer-Cartan equation and trivializations = gauge equivalences, it becomes natural that, according to Lemma 4.1, every semitrivialized deformation problem is governed by a functor $\text{Def}_f$ for a suitable morphism of differential graded Lie algebras $f: L \rightarrow M$.

**Example 4.3.** Let $(V, d)$ be a complex of vector spaces and $F \subset V$ a subcomplex. Then we have an inclusion morphism of DG-Lie algebras $\chi: \text{End}^*(V; F) \rightarrow \text{End}^*(V) = \text{Hom}^*(V, V)$, where

$$\text{End}^*(V; F) = \{ f \in \text{End}^*(V) \mid f(F) \subset F \}.$$ 

Given $A \in \text{Art}$ and $m \in \text{Hom}^0(V, V) \otimes m_A$, from the formula $e^{-m} de^m = d + e^{-m} \cdot 0$ it follows immediately that that $e^m \in \text{MC}_\chi(A)$ if and only if $e^m(F \otimes A)$ is a subcomplex of $V \otimes A$. If $e^m, e^n \in \text{MC}_\chi(A)$ are gauge equivalent, i.e., if $e^n = e^{du} e^m e^{-\chi(a)}$ then $e^{du}: V \otimes A \rightarrow V \otimes A$ is a morphism of complexes homotopic to the identity and, since $e^n(F \otimes A) = e^{du} e^m(F \otimes A)$, the two maps

$$H^*(e^n(F \otimes A)) \rightarrow H^*(V \otimes A), \quad H^*(e^m(F \otimes A)) \rightarrow H^*(V \otimes A),$$

have the same image. Thus we have defined a natural transformation

$$\text{Def}_\chi \rightarrow G_{V,F}$$

which, by the results of [15] is an isomorphism of functors.

One of the main results of [14] is the concrete description of an $L_\infty$-algebra $C(f)$ representing the homotopy fiber of $f: L \rightarrow M$, together with a natural isomorphism of functors $\text{MC}_f \simeq \text{MC}_{C(f)}$. The underline complex is the mapping cocone of $f$ in the category of complexes, i.e., $C(f) = L \times M[-1]$ and therefore $C(f)[1] = L[1] \times M$ equipped with the differential

$$q_1(s^{-1}l, m) = (-s^{-1}dl, dm - f(l)).$$
The only non trivial contributions to the higher brackets $q_k : C(f)[1]^{\otimes k} \to C(f)[1]$, $k > 1$, are

\[ q_2 : L[1]^{\otimes 2} \to L[1], \quad q_{k+1} : L[1] \otimes M^{\otimes k} \to M, \quad k \geq 1, \]

defined by the formulas

\[
q_2(s^{-1}l_1 \otimes s^{-1}l_2) = (-1)^{|l_1||l_2|}s^{-1}[l_1, l_2],
\]

\[
q_{k+1}(s^{-1}m_1 \otimes \cdots \otimes m_k) = -\frac{B_k}{k!} \sum_{\sigma \in S_k} \varepsilon(\sigma)[\cdots [[f(l_{\sigma(1)}), m_{\sigma(2)}], \cdots, m_{\sigma(k)}],
\]

where $\varepsilon(\sigma)$ is the Koszul sign and $B_0, B_1, \ldots$ are the Bernoulli numbers:

\[
\frac{t}{e^t - 1} = \sum_{k \geq 0} \frac{B_k}{k!} t^k = 1 - \frac{t}{2} + \frac{t^2}{12} - \frac{t^4}{720} + \frac{t^6}{30240} - \frac{t^8}{1209600} + \cdots.
\]

**Lemma 4.4 ([14, Thm. 7.5]).** In the above notation, $C(f)$ is weak equivalent (as $L_\infty$-algebra) to the homotopy fibre of $f : L \to M$. For every $B \in \textbf{Art}$ we have

\[
(x, m) \in \text{MC}_{C(f)}(B) \iff (x, e^m) \in \text{MC}_{f}(B).
\]

Moreover $(x_1, m_1), (x_2, m_2) \in \text{MC}_{C(f)}(B)$ are homotopy equivalent if and only if $(x_1, e^{m_1}), (x_2, e^{m_2}) \in \text{MC}_{f}(B)$ are gauge equivalent; in other words the isomorphism (4.2) induces an isomorphism of functors $\text{Def}_{C(f)} = \text{Def}_f$.

For reader’s convenience we shall reproduce here the proof of (4.2). By definition, a pair $(x, m) \in C(f)[1]^{\otimes} \otimes m_B = (L^1 \times M^0) \otimes m_B$ satisfies the Maurer Cartan equation in $C(f)$ if and only if $\sum_{n>0} \frac{1}{n!} q_n((x, m)^{\otimes n}) = 0$. We have

\[
\sum_{n>0} \frac{1}{n!} q_n((x, m)^{\otimes n}) = -dx - \frac{1}{2}[x, x], \quad dm - f(x) + \sum_{n>0} \frac{1}{n!} q_{n+1}(x, m, \ldots, m)
\]

and, since

\[
\frac{1}{n!} q_{n+1}(x, m, \ldots, m) = -\frac{B_n}{n!}[-m, -]^{n}(f(x)) = -\frac{B_n}{n!}[-m, -]^{n}(f(x)),
\]

we have

\[
dm - f(x) + \sum_{n>0} \frac{1}{n!} q_{n+1}(x, m, \ldots, m) = dm - \frac{[-m, -]}{e^{[-m, -]} - 1}(f(x)).
\]

Applying the operator $\frac{e^{[-m, -]} - 1d}{[-m, -]}$ to both sides we obtain that

\[
dm - f(x) + \sum_{n>0} \frac{1}{n!} q_{n+1}(x, m, \ldots, m) = 0
\]

if and only if

\[
f(x) = \frac{e^{[-m, -]} - 1d}{[-m, -]}(dm) = e^{-m} \ast 0.
\]

For a complex manifold $X$, the functor $\text{Def}_X : \textbf{Art} \to \text{Set}$ of its infinitesimal deformations is controlled by the Kodaira-Spencer DG-Lie algebra

\[
KS_X = (A_X^*(\Theta_X), -\partial, [-, -]),
\]

while the deformations of its de Rham complex are controlled by the DG-Lie algebra $\text{End}^*(A_X^{*\cdot}) = \text{Hom}^*(A_X^{*\cdot}, A_X^{*\cdot})$ and the Hodge filtration gives a sequence of inclusions of DG-Lie subalgebra

\[
\text{End}^*(A_X^{*\cdot}; A_X^{p\cdot}) = \{ f \in \text{End}^*(A_X^{*\cdot}) \mid f(A_X^{p\cdot}) \subseteq A_X^{p\cdot} \} \xrightarrow{\chi_p} \text{End}^*(A_X^{*\cdot}).
\]
According to Example 4.3 and Lemma 4.4, the $L_{\infty}$ structure on $C(\chi_i)$ controls the homotopical deformations of the $i$th subcomplex of the Hodge filtration inside the de Rham complex: here homotopical means that two deformations are considered isomorphic if they differ by an automorphism of the de Rham complex which is homotopic to the identity.

It is therefore natural to expect that the universal $i$th period map is induced by an $L_{\infty}$-morphism from $KS_X$ to $C(\chi_i)$; in fact we have:

**Theorem 4.5.** In the above notation, the map

$$\mathcal{P}^i: KS_X \to C(\chi_i), \quad \xi \mapsto (l_\xi, i_\xi),$$

is a strict $L_{\infty}$-morphism inducing the $i$th period map.

**Proof.** For the proof we refer either to \[I\] or to the paper \[\[35\]\] in which the finite dimensional assumption can be easily removed. \[\square\]

5. **Semidirect products of $L_{\infty}[1]$ algebras and homotopy pull-backs**

Let $(L,q_1,\ldots,q_k,\ldots)$ be an $L_{\infty}[1]$ algebra and $I \subset L$ an $L_{\infty}[1]$ ideal, i.e., a graded subspace $I \subset L$ such that $q_k(I \otimes L^{\otimes k-1}) \subset I$ for every $k \geq 1$. Then there is a unique induced $L_{\infty}[1]$ algebra structure on the quotient $L/I$ such that the projection $L \to L/I$ is a strict morphism of $L_{\infty}[1]$ algebras.

According to \[7, 29\] the usual construction of semidirect products of Lie algebras extends to $L_{\infty}[1]$ algebras in the following way: let $(M,r_1,\ldots,r_n,\ldots)$ and $(L,q_1,\ldots,q_n,\ldots)$ be two $L_{\infty}[1]$ algebras; denote by $CE(I)$ the Chevalley-Eilenberg DG-Lie algebra of $I$ and let

$$\phi = (\phi_1,\ldots,\phi_k,\ldots): M \to CE(I)[1]$$

an $L_{\infty}[1]$ morphism. The semidirect product $I \rtimes_{\phi} M$ is the $L_{\infty}[1]$ structure on the graded vector space $I \times M$ given by the Taylor coefficients $\tilde{q}_k: (I \times M)^{\otimes k} \to I \times M$:

$$\tilde{q}_k(i_1 \otimes \cdots \otimes i_k) = (q_k(i_1 \otimes \cdots \otimes i_k),0),$$

$$\tilde{q}_j(m_1 \otimes \cdots \otimes m_j) = (s\phi_j(m_1 \otimes \cdots \otimes m_j)_0(1), r_j(m_1 \otimes \cdots \otimes m_j)),$$

$$\tilde{q}_{j+k}(m_1 \otimes \cdots \otimes m_j \otimes i_1 \otimes \cdots \otimes i_k) = (s\phi_j(m_1 \otimes \cdots \otimes m_j)_k(i_1 \otimes \cdots \otimes i_k),0),$$

where $s\phi_j$ is the composition $M^{\otimes j} \xrightarrow{\phi_j} CE(I)[1] \xrightarrow{s} CE(I)$. Assuming for a while that $\bar{\mathcal{Q}} = (\bar{q}_1,\ldots,\bar{q}_k,\ldots)$ is an $L_{\infty}[1]$ structure, then the natural inclusion $I \to I \rtimes_{\phi} M$ and the natural projection $I \rtimes_{\phi} M \to M$ are strict $L_{\infty}[1]$ morphisms.

**Theorem 5.1.** In the notation above, for every $L_{\infty}[1]$ morphism $\phi: M \to CE(I)[1]$, the semidirect product $I \rtimes_{\phi} M$ is an $L_{\infty}[1]$ algebra. Conversely every $L_{\infty}[1]$ structure on $I \times M$ such that the inclusion $I \to I \times M$ and the projection $I \times M \to M$ are strict $L_{\infty}[1]$ morphisms, is the semidirect product $I \rtimes_{\phi} M$ for a unique $L_{\infty}[1]$ morphism $\phi: M \to CE(I)[1]$.

**Proof.** For the proof we refer either to \[7, Prop. 3.5\] or to the paper \[29\] in which the finite dimensional assumption can be easily removed. \[\square\]
The above result can be used to give a construction of homotopy fiber products in the category of $L_\infty[1]$ algebras and $L_\infty[1]$ morphisms: we remark that this category is not complete, the problem being that in general the equalizer of two $L_\infty[1]$ morphisms may not exist. We follows essentially the argument used in Schuhmacher’s thesis [31]. Recall that, by definition, a morphism of $L_\infty[1]$ algebras $F = (f_1, f_2, \ldots) : L \to M$ is a fibration if its linear part $f_1$ is surjective.

**Lemma 5.2.** Every fibration $N \to M$ of $L_\infty[1]$ algebras admits a factorization $N \to \tilde{N} \to M$ where the first arrow is an $L_\infty[1]$ isomorphism and the second one is a strict fibration.

*Proof.* This is proved in full details [31, Lemma 1.5.4], for reader convenience we give here a sketch of proof. Denoting by $V$ and $W$ the underlying complexes of $N$ and $M$ respectively, and by $f_k : V^\otimes k \to W$, $k > 0$, the Taylor coefficients of the fibration, since $f_1$ is surjective there exist a sequence of maps $g_k : V^\otimes k \to V$, $k > 0$, such that $g_1 = Id$ and $f_1g_k = f_k$ for every $k$. These maps induce an isomorphism of graded coalgebras $G = (g_1, \ldots) : S(V) \to S(V)$ and we define $\tilde{N}$ as the unique $L_\infty[1]$ structure on $V$ such that $G : N \to \tilde{N}$ is an isomorphism of $L_\infty[1]$ algebras.

**Proposition 5.3.** Given the $L_\infty[1]$ algebras $L$, $M$, $N$, an $L_\infty[1]$ morphism $F : L \to M$ and a fibration $G : N \to M$, the fiber product $L \times_M N$ exists in the category of $L_\infty[1]$ algebras and $L_\infty[1]$ morphisms.

*Proof.* According to Lemma 5.2 it is not restrictive to assume that $G = g : N \to M$ is a strict fibration; thus the kernel $I := \text{Ker}(g)$ is an $L_\infty[1]$ ideal and therefore by Theorem 5.1 we have $N = I \times M$ for a well defined $L_\infty[1]$ morphism $\phi : M \to CE(I)[1]$. Considering the composite morphism $\psi = \phi \circ F : L \to CE(I)[1]$, we claim that the diagram

\[
\begin{array}{ccc}
I \times_M L & \overset{\tilde{F}}{\longrightarrow} & I \times_M M \\
\downarrow & & \downarrow \\
L & \overset{F}{\longrightarrow} & M
\end{array}
\]

is cartesian in the category of $L_\infty[1]$ algebras and $L_\infty[1]$ morphisms, where the vertical arrows are the projections (which are strict $L_\infty[1]$ morphisms) and $\tilde{F}$ is given in Taylor coefficients by

$$
\tilde{f}_1(l, i) = (f_1(l), i), \quad \tilde{f}_k((l_1, i_1) \otimes \cdots \otimes (l_k, i_k)) = (0, f_k(l_1 \otimes \cdots \otimes l_k)) \quad \text{for } k \geq 2
$$

it is straightforward to see that $\tilde{F}$ is an $L_\infty[1]$ morphism [31, Prop. 1.2.4]. Given a commutative diagram of $L_\infty[1]$ algebras

\[
\begin{array}{ccc}
X & \overset{H}{\longrightarrow} & I \times M \\
\downarrow & & \downarrow \\
K & \overset{F}{\longrightarrow} & M
\end{array}
\]

it is easy to see that the morphism of graded coalgebras $S(X) \to S(I \times_M L)$ given in Taylor coefficients by $(\eta_1h_1, k_1) : X^\otimes 2 \to I \times_M L$, where $\eta_1 : I \times_M M \to I$ is the projection, is the only one making the required diagram commutative, and again one should check that this is an $L_\infty[1]$ morphism. We leave to the reader to fill in the details of the easy computations.

We point out that Proposition 5.3 is the only non trivial step in the proof that the category of $L_\infty[1]$ algebras and $L_\infty[1]$ morphisms is a pointed category of fibrant objects, as defined in [4].

In particular, it makes sense the notion of homotopy fiber product $L \times_M N$ of a pair of $L_\infty[1]$ morphisms $F : L \to M$ and $G : N \to M$. Notice that it is properly defined only up to quasi-isomorphism and:
(1) if $G$ is a fibration we simply take the fiber product $L \times_M N$ as in Proposition 5.3; in particular, if $G$ is a strict fibration then $N = I \times_\phi M$ for some $\phi: M \rightarrow \text{CE}(I)[1]$ and therefore $I \times_\phi F L$ is a model for $L \times_M N$.

(2) if $G$ is not a fibration, consider any factorization $G: N \overset{i}{\rightarrow} \tilde{N} \overset{\tilde{\phi}}{\rightarrow} M$, with $i$ quasi-isomorphism and $\tilde{G}$ fibration: then $L \times_M \tilde{N}$ is a model for $L \times_M N$.

Since every $L_\infty[1]$ algebras $V$ admits the path object $V[t, dt] = V \otimes \mathbb{C}[t, dt]$, the Brown’s factorization lemma [4, p. 421] gives as canonical factorization of a morphism $G: N \rightarrow M$ the diagram

$$N \xrightarrow{m \mapsto (n, G(n))} \{(n, m(t)) \in N \times M[t, dt] \mid G(n) = m(1)\} \xrightarrow{m(t) \mapsto m(0)} M.$$

Unfortunately, for the application we have in mind, the above factorization is too big. However when $G: N \rightarrow M$ is an injective morphism of DG-Lie algebras, a smaller factorization can be obtained by using the theory of derived brackets. Choosing a graded vector subspace $A \subset M$ such that $M = A \oplus N$, we denote by $d$ the differential on $M$ and by $P: M \rightarrow A$ the projection with kernel $N$. Since the complex $(A, Pd)$ is a deformation retract of the (desuspended) mapping cocone $C(i)[1]$ (see (6.6) for explicit formulas), homotopy transfer gives an $L_\infty[1]$ algebra structure $\phi(d)$ on $A$ which is a model for the homotopy fiber of the inclusion $G$.

Moreover, according to [1, Rem. 5.9] (cf. also [3]), there exists a DG-Lie algebra morphism

$$\phi: (M, d, [-, -]) \rightarrow (\text{CE}(A), [\phi(d), -], [-, -])$$

such that the map $N \rightarrow A[-1] \times_\phi M$, $n \mapsto (0, n)$, is a strict quasi-isomorphism of $L_\infty[1]$ algebras and thus the projection $A[-1] \times_\phi M \rightarrow M$ is weakly equivalent to the inclusion $G: N \rightarrow M$. Thus, by Proposition 5.3, for any $L_\infty$ morphism $L \rightarrow M$ the $L_\infty$ algebra $L \times_\phi A[-1]$ is a model for the homotopy pullback $L \times_M N$.

Finally, explicit formulas for the $L_\infty[1]$ structure on $A$ and the morphism $\phi$ are given in [1] under the additional assumption that $A \subset M$ is a graded Lie subalgebra: if moreover $A$ is an abelian graded Lie subalgebra one recovers the $L_\infty[1]$ algebra structure on $A$ given by Voronov second construction of higher derived brackets [37]:

$$\phi(d)_1(a) = Pda, \quad \phi(d)_k(a_1 \odot \cdots \odot a_k) = P[\cdots [da_1, a_2] \cdots, a_k] \text{ for } k \geq 2,$$

and the morphism $\phi: M \rightarrow \text{CE}(A)$, $m \mapsto (\Phi(m)_0, \ldots, \Phi(m)_k, \ldots)$, of DG-Lie algebras given by Voronov first construction of higher derived brackets [36]:

$$\phi(m)_0(1) = Pm, \quad \phi(m)_k(a_1 \odot \cdots \odot a_k) = P[\cdots [m, a_1] \cdots, a_k] \text{ for } k \geq 1.$$

In particular if $A \subset M$ is an abelian graded Lie subalgebra, then $A[-1] \times_\phi M$ is exactly the the $L_\infty$ algebra defined in [37, Sec. 4, Thm. 2]. Summing together the results of this section we finally reach our goal, expressed by the following theorem:

**Theorem 5.4.** Let $(M, d, [-, -])$ be a DG-Lie algebra, $N \subset M$ a DG-Lie subalgebra and $A \subset M$ an abelian graded Lie subalgebra such that $M = N \oplus A$ as graded vector spaces: denote by $P: M \rightarrow A$ the projection with kernel $N$. Given another DG-Lie algebra $L$ and an $L_\infty[1]$ morphism $F = (f_1, \ldots, f_k, \ldots): L[1] \rightarrow M[1]$, the $L_\infty[1]$ algebra

$$(A \times_\phi F L[1], q_1, \ldots, q_k, \ldots),$$
where the brackets are
\[
q_1(a, s^{-1}x) = \left( P(da + sf_1(s^{-1}x)), s^{-1}dx \right),
\]
\[
q_2(s^{-1}x_1 \circ s^{-1}x_2) = \left( Psf_2(s^{-1}x_1 \circ s^{-1}x_2), (-1)^{|x_1|} s^{-1}[x_1, x_2] \right),
\]
\[
q_j(s^{-1}x_1 \circ \cdots \circ s^{-1}x_j) = \left( Psf_j(s^{-1}x_1 \circ \cdots \circ s^{-1}x_j), 0 \right) \quad j \geq 3,
\]
\[
q_k(a_1 \circ \cdots \circ a_k) = \left( P[\cdots [da_1, a_2] \cdots, a_k], 0 \right) \quad k \geq 2,
\]
\[
q_{j+k}(s^{-1}x_1 \circ \cdots \circ s^{-1}x_j \otimes a_1 \circ \cdots \circ a_k) = \left( P[\cdots [sf_j(s^{-1}x_1 \circ \cdots \circ s^{-1}x_j), a_1] \cdots, a_k], 0 \right) \quad j, k \geq 1,
\]
is a model for the homotopy fiber product \( L \times^h_M N \) (here \( sf_j \) is the composition \( L[1] \overset{j} \to M[1] \overset{s} \to M \)).

6. Models of the Homotopy Fiber

The aim of this section is to compare several \( A_\infty \) models of the homotopy fiber of an inclusion of DG associative algebras: we will apply these results in the next section to the case of the inclusion \( \chi_p \) from (4.3). As recalled in Section 4, for any morphism of DG Lie algebras \( f : L \to M \) we may put an \( L_\infty \) algebra structure on the mapping cone \( C(f) = L \times M[-1] \) which is a model for the homotopy fiber \( 0 \times^h_M L \) of \( f \). This \( L_\infty \) structure is induced via homotopy transfer from the following DG Lie algebra model for \( 0 \times^h_M L \)
\[
K_f := \{(l, m(t, dt)) \in L \times M[t, dt] \mid m(t, dt)_{|t=0} = 0, m(t, dt)_{|t=1} = f(l)\},
\]
on a certain natural contraction called Dupont’s contraction, cf. [6, 14]; we also refer to [22] for the explicit \( L_\infty \) algebra structure on the mapping cone \( C(f - g) \) which is a model for the homotopy equalizer of two morphisms \( f, g : L \to M \) of DG-Lie algebras.

For a morphism \( f : A \to B \) of DG associative algebras we may similarly put a DG associative algebra structure on \( K_f \) and via homotopy transfer along Dupont’s contraction an \( A_\infty \) algebra structure on the mapping cone \( C(f) = A \times B[-1] \). This \( A_\infty \) structure was essentially computed in [6, Prop. 19]: the differential is as usual \( q_1(s^{-1}a, b) = (-s^{-1}da, db - f(a)) \), the nontrivial contributions to the higher products \( q_k : C(f)^{[1]} \overset{k} \to C(f)[1] \) are \( q_2(s^{-1}a_1 \otimes s^{-1}a_2) = (-1)^{|a_1|} s^{-1}(a_1a_2) \) and
\[
(6.1) \quad q_{i+j+1}(b_1 \otimes \cdots \otimes b_1 \otimes s^{-1}a \otimes b_{i+1} \otimes \cdots \otimes b_{i+j}) = \frac{B_{i+j}}{i! j!} (-1)^{i+j+1} \sum_{k=1}^{\infty} [b_k] b_1 \cdots b_i f(a) b_{i+1} \cdots b_{i+j}, \quad \forall i, j \geq 1.
\]
It is clear, either by Remark 3.8 or by a direct computation (recall the well known formula \([a, -]^k(b) = \sum_{i=0}^{k} (-1)^{i+k} a^{k-i} b^i a^i\) and apply polarization, also recall \( B_{i+j} = (-1)^{i+j} B_{i+j} \) for \( i + j \geq 2 \)) that the symmetrized \( L_\infty \) algebra structure on \( C(f) \) coincides with Fiorenza-Manetti mapping cone of \( f_{\text{Lie}} : A_{\text{Lie}} \to B_{\text{Lie}} \).

In the DG associative case there is however a simpler DG associative algebra structure on \( C(f) \) given by the product \( \bullet : C(f)^{[2]} \to C(f) \),
\[
(6.2) \quad (a_1, sb_1) \bullet (a_2, sb_2) = (a_1a_2, s(b_1a_2)),
\]
which is also immediately seen to be a model for the homotopy fiber \( 0 \times^h_B A \). The first result of this section is the determination of explicit \( A_\infty \) isomorphisms between these two models of the homotopy fiber, roughly given by the exponential and logarithm on \( B \): this will be used to simplify the computations in Theorem 6.3.
**Definition 6.1.** We will call $C(f)$ with the DG algebra structure (6.2) the associative mapping cone of $f$ and denote it by $C(f)_{A_{s}}$, while we will call $C(f)$ with the $A_{\infty}$ algebra structure (6.1) the Fiorenza-Manetti mapping cone of $f$ and denote it by $C(f)_{\infty}$.

**Lemma 6.2.** Given a morphism $f: A \to B$ of DG associative algebras, there exist two isomorphisms of $A_{\infty}$ [1] algebras $E: C(f)_{\infty}[1] \to C(f)_{A_{s}}[1]$ and $L: C(f)_{A_{s}}[1] \to C(f)_{\infty}[1]$, the one inverse of the other, whose Taylor coefficients $e_{k}, l_{k}: C(f)[1]^{\otimes k} \to C(f)[1]$ are $e_{1} = l_{1} = \text{id}_{C(f)[1]}$ and, for $k \geq 2$,

$$e_{k}((s^{-1}a_{1}, b_{1}) \otimes \cdots \otimes (s^{-1}a_{k}, b_{k})) = \left(0, \frac{1}{k!}b_{1} \cdots b_{k}\right),$$

$$l_{k}((s^{-1}a_{1}, b_{1}) \otimes \cdots \otimes (s^{-1}a_{k}, b_{k})) = \left(0, \frac{(-1)^{k+1}}{k}b_{1} \cdots b_{k}\right).$$

**Proof.** We denote by $(C(f)_{A_{s}}[1], r_{1}, r_{2}, 0, \ldots, 0, \ldots)$ the $A_{\infty}$[1] structure on $C(f)_{A_{s}}[1]$. It is straightforward that $E$ and $L$ are inverse automorphisms of the tensor coalgebra over $C(f)[1]$, thus it suffices to show that $L$ is an $A_{\infty}$[1] morphism, that is, the relation

$$l_{k}R_{k}^{k} + l_{k-1}R_{k-1}^{k-1} = \sum_{i=1}^{k} q_{i}L_{k}, \quad k \geq 2,$$

(cf. Section 3 for notations). Looking at the explicit formulas (6.1) and (6.2) one quickly realizes that the only cases where the necessary relation (6.3) is not trivially satisfied are terms of type $b_{1} \otimes \cdots \otimes b_{i} \otimes s^{-1}a \otimes b_{i+1} \otimes \cdots \otimes b_{i+j}$: in this case the left hand side of (6.3) is easily computed and it is equal to

$$(l_{k}R_{k}^{k} + l_{k-1}R_{k-1}^{k-1})(s^{-1}a \otimes b_{1} \otimes \cdots \otimes b_{j}) = \frac{(-1)^{j+1}}{j+1}f(a)b_{1} \cdots b_{j}$$

when $i = 0$, and to

$$(l_{k}R_{k}^{k} + l_{k-1}R_{k-1}^{k-1})(b_{1} \otimes \cdots \otimes b_{i} \otimes s^{-1}a \otimes b_{i+1} \otimes \cdots \otimes b_{i+j}) =$$

$$= (-1)^{j} \frac{1}{j+1} \sum_{i \geq 1} q_{i} \frac{(-1)^{i+j+1}}{i+j+1} f(a)b_{i+1} \cdots b_{i+j}$$

when $i \geq 1$. We turn our attention to the rhs of (6.3): this becomes

$$\sum_{i=1}^{\infty} \sum_{j_{1}+\cdots+j_{p}=i} q_{p+q+1} \frac{(-1)^{i+j+1}}{i+1} b_{1} \cdots b_{i} \otimes \cdots \otimes s^{-1}a \otimes \cdots \otimes \frac{(-1)^{j+1}}{j+1} b_{i+j-} \cdots b_{i+j}.$$
Next we observe that \(e^{\log(1+w) - \log(1+z)} - 1 = \frac{1+w}{1+z} - 1 = \frac{w-z}{1+z}\), and therefore

\[
\varphi(\log(1+z), \log(1+w)) = -\frac{1+z}{w-z}(\log(1+w) - \log(1+z)) = \frac{1+z}{w-z} \sum_{k \geq 1} (-1)^k k (w-z)(w^{k-1} + zw^{k-2} + \cdots + z^{k-1}),
\]

from which we get \(C_{0,j} = \frac{(-1)^{j+1}}{j+1}\) and \(C_{i,j} = \frac{(-1)^{i+j+1}}{i+j+1} + \frac{(-1)^{i+j}}{i+j}\) for \(i \geq 1\). By comparison with (6.4) and (6.5) this concludes the proof of the lemma.

Next we consider the particular case when \(f = i: A \to B\) is an inclusion. Given a graded subspace \(C \subset B\) such that \(B = A \oplus C\) as graded spaces we can consider the following contraction, where we denote by \(P: B \to C\) the projection with kernel \(A\), by \(P^\perp = \text{id}_B - P\), by \(d\) the differential on \(B\) and by \(q_i\) the differential on \(C(i)[1]\) (notice that since \(d(A) \subset A\) we have \((Pd)^2 = P(d)^2 = 0\) and thus \(Pd\) is a differential on \(C\):)

\[
(C, Pd) \xrightarrow{f_i} (C[i][1], q_i),
\]

\[
f_i(c) = (s^{-1}P^\perp dc, c), \quad g_1(s^{-1}a, b) = Pb, \quad K(s^{-1}a, b) = (s^{-1}P^\perp b, 0).
\]

Via homotopy transfer from the Fiorenza-Manetti mapping cocone or the associative mapping cone there are induced \(A_\infty[1]\) algebra structures on \(C\) giving us two other models for the homotopy fiber of \(i\). In the particular case of interest to us we may choose a \(C\) as above such that moreover \(C\) is a square zero graded subalgebra of \(B\) (this is a DG associative analog of the setup to Voronov constructions of higher derived brackets \([36, 37]\)). In the following proposition we make explicit the homotopy transfer formulas under this additional hypothesis, first we introduce a notation: for an integer \(k \geq 1\) and an ordered partition \(k = i_1 + \cdots + i_j\) with \(j, i_1, \ldots, i_j \geq 1\), we define the degree zero map \(g^{i_1, \ldots, i_j}: C(i)^{\otimes k} \to C\) by

\[
g^{i_1, \ldots, i_j}((s^{-1}a_1, b_1) \otimes \cdots \otimes (s^{-1}a_k, b_k)) = P(b_1 \cdots b_i \cdot g_1(b_{i_1+1} \cdots b_{i_1+i_2}, \cdots P(\cdots P(b_{i_1+\cdots+i_{j-1}+1} \cdots b_k) \cdots ))),
\]

where the inner \(P_i\)s are inserted according to the partition \((i_1, \ldots, i_j)\). For instance:

\[
g^1(s^{-1}a, b) = g_1(s^{-1}a, b) = P(b), \quad g^2(s^{-1}a_1, b_1) @ (s^{-1}a_2, b_2) @ (s^{-1}a_3, b_3) = P(b_1, b_2, b_3), \quad g^{2,1,2}(s^{-1}a_1, b_1) @ \cdots @ (s^{-1}a_5, b_5) = P(b_1 \cdot b_2 \cdot P(b_3 \cdot P(b_4 \cdot b_5))).
\]

Theorem 6.3. Given a DG associative algebra \((B, d, \cdot, \cdot)\) together with a decomposition \(B = A \oplus C\) into the direct sum of a DG subalgebra \(A\) and a graded subspace \(C\) such that \(C \cdot C = 0\), there is an \(A_\infty[1]\) algebra structure \((C, Pd, \Phi(d)_2, 0, \ldots, 0, 0)\) on \(C\) given by the derived product

\[
\Phi(d)_2(c_1 \otimes c_2) = P(dc_1 \cdot c_2)
\]

together with \(A_\infty[1]\) quasi-isomorphisms \(F_{As}: C \to C(i)_{A_\infty}, G_{As}: C(i)_{As} \to C, F_{\infty}: C \to C(i)_{\infty}, G_{\infty}: C(i)_{\infty} \to C\) given by \(f_{As,1} = f_1 = f_{\infty,1}\),

\[
f_{As,2}(c_1 \otimes c_2) = s^{-1}P^\perp (dc_1 \cdot c_2), \quad f_{\infty,2}(c_1 \otimes c_2),
\]

\[
f_{As,k} = 0 = f_{\infty,k} \quad \text{for } k \geq 3.
\]

In terms of the linear maps (6.7) we have

\[
g_{As,k} = \sum_{i_1 + \cdots + i_j = k} (-1)^{k+j} g^{i_1, \ldots, i_j}, \quad g_{\infty,k} = \sum_{i_1 + \cdots + i_j = k} \frac{(-1)^{k+j}}{i_1! \cdots i_j!} g^{i_1, \ldots, i_j},
\]
The following diagrams are commutative, where $E$ and $L$ are the $A_\infty[1]$ isomorphisms described in Lemma 6.2:

$\begin{array}{ccc}
C(i)_{\infty} & \xrightarrow{E} & C(i)_{A_\infty} \\
\downarrow F_{\infty} & & \downarrow F_{A_\infty} \\
C & & C
\end{array}$

$\begin{array}{ccc}
C(i)_{\infty} & \xleftarrow{E} & C(i)_{A_\infty} \\
\downarrow G_{\infty} & & \downarrow G_{A_\infty} \\
C & & C
\end{array}$

Proof. To begin we will show that the given $A_\infty[1]$ structure on $C$ and $A_\infty[1]$ quasi-isomorphisms $F_{A_\infty} : C \to C(i)_{A_\infty}[1]$, $G_{A_\infty} : C(i)_{A_\infty}[1] \to C$ are the ones induced via homotopy transfer from $(C(i)_{A_\infty}, r_1, r_2, 0, \ldots, 0, \ldots)$ along the contraction $(6.6)$. We denote by $p_B : C(i)[1] \to B$ the projection, then

$$p_B r_1f_1^{(2)}(c_1 \otimes c_2) = p_B r_2((s^{-1}P^+ dc_1, c_1 \otimes (s^{-1}P^+ dc_2, c_2)) = (-1)^{(|c_1|+1)}c_1 \cdot P^+ dc_2 = dc_1 \cdot c_2,$$

since we have $c \cdot P^+ b = c \cdot b$ for all $c \in C$, $b \in B$, and $0 = d(c_1 \cdot c_2) = dc_1 \cdot c_2 + (-1)^{|c_1|}c_1 \cdot dc_2$ for all $c_1, c_2 \in C$. For the moment we denote by $(C, Pd, \phi_2, \ldots, \phi_k, \ldots)$ and $F = (f_1, \ldots, f_k, \ldots)$ the $A_\infty[1]$ structure on $C$ and $A_\infty[1]$ morphism $F : C \to C(i)_{A_\infty}[1]$ induced via homotopy transfer from $C(i)_{A_\infty}[1]$, by the above $\phi_2(c_1 \otimes c_2) = g_1 r_2 f_1^{(2)}(c_1 \otimes c_2) = P(d(c_1 \otimes c_2) = \Phi(d_2(c_1 \otimes c_2))$ and $f_2(c_1 \otimes c_2) = K r_2 f_1^{(2)}(c_1 \otimes c_2) = s^{-1}P^+(dc_1 \cdot c_2) = f_{A_\infty,2}(c_1 \otimes c_2)$. Next we see that

$$p_B r_2 F_2^{(2)}(c_1 \otimes c_2 \otimes c_3) = p_B r_2((s^{-1}P^+ dc_1 \cdot c_2) \otimes c_3 + c_1 \otimes s^{-1}P^+(dc_2 \cdot c_3))$$

$$= (-1)^{|c_1|+1}c_1 \cdot dc_2 \cdot c_3 = dc_1 \cdot c_2 \cdot c_3 = 0,$$

hence $\phi_3 = p_1 r_2 F_3^{(2)} = 0 = K r_2 F_3^{(2)} = f_3$. Assuming inductively that $\phi_j = 0 = f_j$ for all $3 \leq j < k$ it is straightforward to see that $p_B r_2 F_k^{(2)} = 0$ and thus also $\phi_k = 0 = f_k$. It remains to show that the $A_\infty[1]$ isomorphism $G = (g_1, \ldots, g_k, \ldots) : C(i)_{A_\infty}[1] \to C$ given by the homotopy transfer formulas is the same as $G_{A_\infty}$ in the claim of the proposition. Recall that $g_k$ is defined by the recursion

$$(6.8) \quad g_k((s^{-1}a_1, b_1) \otimes \cdots \otimes (s^{-1}a_k, b_k)) = g_{k-1}(P_k^{-1} K_k ((s^{-1}a_1, b_1) \otimes \cdots \otimes (s^{-1}a_k, b_k)),$$

where $K_k := \sum_{j=1}^k \text{id}_{(i_1(1)}^{(i_1(1]} \otimes \Phi^{-1}(0)(j-k-1) : C(i)[1]^{\otimes k} \to C(i)[1]^{\otimes k}$.

Starting with $g_{A_\infty,1} = g_1$ we assume inductively we have shown $g_{A_\infty,j} = g_j$ for $1 \leq j < k$. We claim that $g_{k-1}((s^{-1}a_1, b_1) \otimes \cdots \otimes (s^{-1}a_k, b_k)) = 0$ whenever $b_{k-1} \in C$: this is because there is a bijective correspondence between the set of ordered partitions $k = i_1 + \cdots + i_p$ of $k$ with $i_p = 1$ and those with $i_p > 1$ given by $(i_1, \ldots, i_{j+1}) \to (i_1, \ldots, i_j + 1)$. Moreover when $b_{k-1} \in C$ we have $g_{i_1, \ldots, i_k}((s^{-1}a_1, b_1) \otimes \cdots \otimes (s^{-1}a_{k-1}, b_{k-1})) = g_{i_1, \ldots, i_{j+1}}((s^{-1}a_1, b_1) \otimes \cdots \otimes (s^{-1}a_{k}, b_{k}))$ and the two appear with opposite signs in the summation for $g_{k-1} = g_{A_\infty, k-1}$. Clearly, we also have $g_{k-1}((s^{-1}a_1, b_1) \otimes \cdots \otimes (s^{-1}a_k, b_k)) = 0$ whenever $b_j = 0$ for some $1 \leq j \leq k-1$, putting these two fact together we see that (6.8) reduces to:

$$g_k((s^{-1}a_1, b_1) \otimes \cdots \otimes (s^{-1}a_k, b_k)) =$$

$$= (-1)^{|b_{k-1}|}g_{k-1}((s^{-1}a_1, b_1) \otimes \cdots \otimes r_2 ((s^{-1}a_{k-2}, b_{k-1}) \otimes (s^{-1}P^+ b_k, 0)))$$

$$= g_{k-1}((s^{-1}a_1, b_1) \otimes \cdots \otimes (s^{-1}a_{k-2}, b_{k-2}) \otimes (\cdots, b_{k-1} \cdot P b_k))$$

$$= \sum_{i_1+\cdots+i_j=k-1} (-1)^{k+j}g_{i_1, \ldots, i_j}((s^{-1}a_1, b_1) \otimes \cdots \otimes (s^{-1}a_{k}, b_{k} - b_{k-1} \cdot P b_k))$$

$$= \sum_{i_1+\cdots+i_j=k-1} ((-1)^{k+j}g_{i_1, \ldots, i_j} + (-1)^{k+j+1}g_{i_1, \ldots, i_j+1})\ ((s^{-1}a_1, b_1) \otimes \cdots \otimes (s^{-1}a_{k}, b_{k}$$

$$= \sum_{i_1+\cdots+i_j=k} (-1)^{k+j}g_{i_1, \ldots, i_j}((s^{-1}a_1, b_1) \otimes \cdots \otimes (s^{-1}a_{k}, b_{k}))$$

$$= g_{A_\infty,k}((s^{-1}a_1, b_1) \otimes \cdots \otimes (s^{-1}a_{k}, b_{k})).$$
This concludes the first part of the proof.

At this point we could similarly prove that \((C, Pd, \Phi(d)_2, 0, \ldots, 0)\), \(F_\infty\) and \(G_\infty\) are the \(A_\infty[1]\) structure and \(A_\infty[1]\) quasi-isomorphisms induced via homotopy transfer from \(C(i)_\infty[1]\) along the contraction (6.6): this is true but the computation is a little harder and we won’t actually need this fact, as we only need to be able to compare the various models. Instead, we observe that the claim of the theorem follows once we show \(F_\infty = LF_A\) and \(G_\infty = G_A E\). The first identity is easy and left to the reader; for the second we have

\[
\sum_{i=1}^{k} g_{A,i} E_k^i ((s^{-1} a_1, b_1) \otimes \cdots \otimes (s^{-1} a_k, b_k)) =
\]

\[
= \sum_{j_1+\cdots+j_1=k} \sum_{i_1+\cdots+i_1=j_1} (-1)^{i+j} g^{i_1,\ldots,i_j} \left( \left( \cdots, \frac{1}{j_1!} b_1 \cdots b_{j_1} \right) \otimes \cdots \otimes \left( \frac{1}{j_k!} b_{k-j_k+1} \cdots b_k \right) \right)
\]

\[
= \sum_{i_1+\cdots+i_j=k} \left( \sum_{h_1+\cdots+h_{p_1}=i_1} \sum_{\ldots} (-1)^{i_1+p_1} \frac{1}{h_1! \cdots h_{p_1}!} g^{i_1,\ldots,i_j} ((s^{-1} a_1, b_1) \otimes \cdots \otimes (s^{-1} a_k, b_k)) \right)
\]

and by the equality

\[
\sum_{i \geq 1} \left( \sum_{h_1+\cdots+h_{p_1}=i_1} (-1)^{p+1} \frac{1}{h_1! \cdots h_{p_1}!} \right) t^i = \sum_{p \geq 1} \left( \sum_{h \geq 1} (-1)^{h+1} \frac{1}{h!} t^h \right)^p
\]

\[
= \sum_{p \geq 1} (1 - e^{-t})^p = \frac{1 - e^{-t}}{e^{-t}} = e^t - 1 = \sum_{i \geq 1} \frac{1}{i!} t^i
\]

we get

\[
\sum_{h_1+\cdots+h_{p_1}=i_1} (-1)^{i_1+p_1} \frac{1}{h_1! \cdots h_{p_1}!} \frac{1}{h_1! \cdots h_{p_1}!} = (-1)^{i_1+\cdots+i_j} \frac{1}{i_1! \cdots i_j!}
\]

\[
= (-1)^{j+k} \frac{1}{i_1! \cdots i_j!}
\]

\[
\square
\]

Remark 6.4. In general homotopy transfer from Fiorenza-Manetti mapping cocone or the associative one will induce different \(A_\infty[1]\) structures on \(C\): the fact that they are the same in the case of the previous proposition follows from the hypothesis \(C^C = 0\). Moreover, we remark that we only used this hypothesis in the computation of the \(A_\infty[1]\) structure on \(C\) and the \(A_\infty[1]\) morphisms \(F_A\), \(F_\infty\) and never in the computation of \(G_A\), \(G_\infty\).

7. Algebraic models of formal period maps

We recall, from [15], the definition and the basic properties of Cartan homotopies.

Definition 7.1. A Cartan homotopy between two DG Lie algebras \(L\) and \(M\) is a linear map \(i: L \to M, x \mapsto i_x\), of degree \(-1\) such that the following formal Cartan identities are satisfied:

\[
[i_x, i_y] = 0, \quad [i_x, I_y] = i_{[x, y]}, \quad \forall x, y \in L,
\]
where \( l : L \to M \), \( x \mapsto l_x \), is the degree zero map, called the boundary of \( i \), defined by \( l_x := d_M t_x + i_{d_L} x \). Then it is easy to prove that \( l \) is a morphism of graded Lie algebras: moreover, there is proven in [15, Cor. 3.7] that when \( l \) factors through the inclusion \( i : N \to M \) of a DG-Lie subalgebra then \( L \to C(i) \), \( x \mapsto (l_x, s_i x) \), is a strict morphism of \( L_\infty \) algebras.

**Example 7.2.** Let \( X \) be a complex manifold of dimension \( n \), \( L = KS_X \) the Kodaira-Spencer DG Lie algebra and \( M = \text{End}(A_X^{\bullet,*}) \) the DG Lie algebra of endomorphisms of the de Rham complex, then the holomorphic Cartan formulas (1.1) show that \( i : KS_X \to \text{End}(A_X^{\bullet,*}) , \xi \mapsto \xi_i \), is a Cartan homotopy in the above sense. Moreover, \( \xi_i \) is the holomorphic Lie derivative with respect to \( \xi \) and thus it preserves the Hodge filtration on \( A_X^{\bullet,*} \); in particular, for all \( 0 \leq i \leq n \) the boundary of \( i \) factors through the inclusion \( \chi_i : \text{End}(A_X^{\bullet,*}, A_X^{\bullet+1,*}) \to \text{End}(A_X^{\bullet,*}) \). The induced strict \( L_\infty \) morphism \( P^i : KS_X \to C(\chi_i) \) is the algebraic model of the \( i \)th period map from Theorem 4.5.

The following definition, cf. [16], generalizes the previous example.

**Definition 7.3.** A formal period data is the data of a DG space \((V,d)\) a DG subspace \( W \subset V \), a DG-Lie algebra \( L \) and a Cartan homotopy \( i : L \to \text{End}(V) \) such that the boundary \( l \) factors through the inclusion \( i : \text{End}(V) \to \text{End}(V) \), where \( \text{End}(V) := \{ f \in \text{End}(V) \text{ s.t. } f(W) \subset W \} \). We call the strict \( L_\infty \) morphism \( L \to C(i) \), \( x \mapsto (l_x, s_i x) \), the formal period map associated to the formal period data. A split formal period data is a formal period data together with the choice of a graded subspace \( A \subset V \) such that \( V = W \oplus A \) as graded spaces.

Given a split formal period data as above we denote by \( P : V \to A \) the projection with kernel \( W \) and by \( P^\perp = id_V - P \); then \( \tilde{P} : \text{End}(V) \to \text{End}(V) : f \mapsto P f P^\perp \) is a projection with kernel \( \text{End}(W) \) and image which we may and will identify with \( \text{Hom}^*(W,A) \). The decomposition \( \text{End}(V) = \text{End}(V,W) \oplus \text{Hom}^*(W,A) \) satisfies the hypotheses of Theorem 6.3, thus there is a DG associative algebra structure on \( \text{Hom}^*(W,A)[-1] \) with differential \( \delta(fg) = -s(P df f P^\perp) \) and product \( sf \circ sg = s(fdg) \); the associated DG Lie algebra is a model for the homotopy fiber of the inclusion \( i \). The symmetrization of the morphism \( G_\infty \) of Theorem 6.3 gives an \( L_\infty[1] \) quasi-isomorphism from Fiorenza-Manetti mapping cocone \( C(i)[1] \) to \( \text{Hom}^*(W,A) \) and the composition of this morphism and the formal period map is the following \( L_\infty[1] \) morphism:

\[
\Pi_\infty = (\pi_1, \ldots, \pi_k, \ldots) : L[1] \to \text{Hom}^*(W,A),
\]

\[
(7.1)
\]

\[
(7.2)
\pi_k(s^{-1}x_1 \circ \cdots \circ s^{-1}x_k) = \\
\sum_{\sigma \in S_k} \epsilon(\sigma) \sum_{i_1 + \cdots + i_k = k} \frac{(-1)^{k+j}}{i_1! \cdots i_j!} \pi_{i_\sigma(1)} \cdots \pi_{i_\sigma(i_j)} \cdots \pi_{i_\sigma(k_j+1)} \cdots \pi_{i_\sigma(k)} P^\perp
\]

We shall call the map (7.2) the split formal period map associated to the split formal period data.

**Example 7.4.** The formal period data in Example 7.2 splits canonically \( A_X^{\bullet,*} = A_X^{\bullet+1,*} \oplus A_X^{\bullet,*} \). We want to compute the associated split formal period map \( \Pi_\infty : KS_X[1] \to \text{Hom}^*(A_X^{\bullet+1,*}, A_X^{\bullet,*}) \). To apply the previous formula for \( \pi_k(s^{-1}x_1 \circ \cdots \circ s^{-1}x_k) \) we consider separately the various components \( \pi_k(s^{-1}x_1 \circ \cdots \circ s^{-1}x_k) : A_X^{\bullet+1,k+*} \to A_X^{\bullet+k,*} \), where \( 0 \leq j < \min(k, n - i+1) \) (obviously \( \pi_k(s^{-1}x_1 \circ \cdots \circ s^{-1}x_k) \) vanishes when \( j \) is not in this range),

\[
\pi_k(s^{-1}x_1 \circ \cdots \circ s^{-1}x_k) = \\
\sum_{\sigma \in S_k} \epsilon(\sigma) \left( \sum_{i_1 + \cdots + i_k = k, i_j > j} \frac{(-1)^{h+k}}{i_1! \cdots i_k!} \pi_{i_\sigma(1)} \cdots \pi_{i_\sigma(i_j)} \cdots \pi_{i_\sigma(k_j+1)} \cdots \pi_{i_\sigma(k)} \right)
\]

\[
= k! \frac{(-1)^{k+1}}{k!} + \sum_{i_h = j+1}^{k-1} \frac{(-1)^{h+1}}{i_h!} \sum_{i_1 + \cdots + i_{k-1} = k-i_h} \frac{(-1)^{h-1+1}}{i_1! \cdots i_{k-1}!} \pi_{i_\sigma(1)} \cdots \pi_{i_\sigma(k)}
\]
and then, according to (6.9),

\[
\pi_{k}(s^{-1}\xi_{1} \oplus \cdots \oplus s^{-1}\xi_{k}) = \left( -k! \sum_{i_{h}=j+1}^{k} \frac{(-1)^{i_{h}}}{i_{h}!(k-i_{h})!} \right) \hat{i}_{\xi_{1}} \cdots \hat{i}_{\xi_{k}} \\
= \left( \sum_{h=0}^{j} (-1)^{h} \left( \begin{array}{c} k \\ h \end{array} \right) \right) \hat{i}_{\xi_{1}} \cdots \hat{i}_{\xi_{k}}.
\]

For \( i = n \), the computation of Example 7.4 gives:

**Theorem 7.5.** Let \( X \) be a complex manifold of dimension \( n \). Then the \( L_{\infty}[1] \) morphism

\[
\Pi_{\infty} = (\pi_{1}, \ldots, \pi_{k}, \ldots): KSX[1] \to \text{Hom}^{*}(A_{X}^{n,+}, A_{X}^{n,-}),
\]

with Taylor coefficients

\[
\pi_{k}(s^{-1}\xi_{1} \oplus \cdots \oplus s^{-1}\xi_{k}) = \hat{i}_{\xi_{1}} \cdots \hat{i}_{\xi_{k}}: A_{X}^{n,+} \to A_{X}^{n-k,+}, \quad 0 < k \leq n,
\]

is an algebraic model of the \( n \)th period map.

Suppose now that \( X \) is a compact Kähler manifold; as above we denote by \( H_{\mathcal{K}}^{X} \) the space of harmonic forms, by \( \iota: H_{\mathcal{K}}^{X} \to A_{X}^{n,+} \) and \( \pi: A_{X}^{n,+} \to H_{\mathcal{K}}^{X} \) the inclusion and the harmonic projection and by \( h = -\overline{\partial} \overline{\partial}^{*} \) the \( \overline{\partial} \)operator (Definition 1.3). As usual we denote by \( P: A_{X}^{n,+} \to A_{X}^{n,-} \) the projection with kernel \( A_{X}^{n,+} \). We have the following contraction

\[
(\text{Hom}^{*}(H_{\mathcal{K}}^{X}, H_{\mathcal{K}}^{X}), 0) \xrightarrow{i} \left( \text{Hom}^{*}(A_{X}^{n,+}, A_{X}^{n,-}), \delta \right), \quad \delta(f) = Pd\!f - (\!1)^{|f|}f\!d,
\]

\[
i(f) = \iota f, \quad g_{i}(f) = \pi f, \quad K(f) = hf + (\!1)^{|f|}f\!d.
\]

In fact

\[
(\delta K + K\delta)(f) = \delta(hf + (\!1)^{|f|}f\!d) + K(Pd\!f - (\!1)^{|f|}f\!d)
\]

\[
= Pd\!f + (\!1)^{|f|}(hf + (\!1)^{|f|}f\!d)d + hPd\!f - (\!1)^{|f|}hf\!d + \pi f\!d
\]

\[
= Pd\!f + hPd\!f + \pi f\!d(h\!d + \!d\!h).
\]

Next we notice that \( dh + hd = \pi - \text{id} \) and \( [P, h] = 0 \), thus \( Ph\!d + hP\!d\!f = P(\pi - \text{id})\!f = (\pi - \text{id})\!f \) and

\[
(\delta K + K\delta)(f) = (\pi - \text{id})\!f + \pi f(\pi - \text{id}) = \pi f\!\pi - \pi = ig_{i}(f) - f.
\]

The side conditions \( g_{i}K = K^{2} = Ki = 0 \) are also easily verified.

**Proposition 7.6.** The homotopy transfer along the contraction (7.3) induces: the trivial \( L_{\infty}[1] \) structure on \( \text{Hom}^{*}(H_{\mathcal{K}}^{X}, H_{\mathcal{K}}^{X}) \), the \( L_{\infty}[1] \) quasi-isomorphism

\[
G = (g_{1}, \ldots, g_{n}, \ldots): \text{Hom}^{*}(A_{X}^{n,+}, A_{X}^{n,-}) \to \text{Hom}^{*}(H_{\mathcal{K}}^{X}, H_{\mathcal{K}}^{X}),
\]

\[
g_{k}(f_{i} \odot \cdots \odot f_{j}) = \sum_{\sigma \in S_{k}} \varepsilon(\sigma)\!f_{\sigma(1)}h\partial f_{\sigma(2)}h\partial \cdots h\partial f_{\sigma(k)}h,
\]

and the strict \( L_{\infty}[1] \) quasi-isomorphism \( i: \text{Hom}^{*}(H_{\mathcal{K}}^{X}, H_{\mathcal{K}}^{X}) \to \text{Hom}^{*}(A_{X}^{n,+}, A_{X}^{n,-}) \).

**Remark 7.7.** One should notice that in the above formula the \( f_{j} \in \text{Hom}^{*}(A_{X}^{n,+}, A_{X}^{n,-}) \) are identified with endomorphisms \( f_{j} \in \text{End}(A_{X}^{n,+}) \) such that \( \text{Im}(f_{j}) \subset A_{X}^{n,+} \subset \text{Ker}(f_{j}) \). The same remark applies to the following computations.
Proof. In the computation of the homotopy transfer we take advantage of the fact that the \(L^\infty[1]\) structure on \(\text{Hom}^*(A^\infty_X, A^\leq n_X^*)\) is the symmetrized of the \(A^\infty[1]\) structure \((q_1, q_2, 0, \ldots, 0, \ldots)\),

\[
q_2(f_1 \otimes f_2) = (-1)^{|f_1|+1}f_1df_2 = (-1)^{|f_1|+1}f_1d\!
\]

where the last equality follows by the previous remark. For the moment we denote the \(A^\infty[1]\) structure induced on \(\text{Hom}^*(H^p_X, H^q_X)\) via homotopy transfer along (7.3) by \((0, r_2, \ldots, r_k, \ldots)\), and we denote by \((i, i_2, \ldots, i_k, \ldots): \text{Hom}^*(H^p_X, H^q_X) \to \text{Hom}^*(A^\infty_X, A^\leq n_X)\) the induced \(A^\infty[1]\) quasi-isomorphism. Since \(dt = \pi d = 0\) we see that \(q_2^{i\otimes 2} = 0\), and thus \(r_2 = g_1q_2^{i\otimes 2} = 0 = Kq_2^{i\otimes 2} = i_2\). Since \(g_0 = 0\) for \(k \geq 3\) a straightforward induction shows that \(i_k = r_k = 0\) for all \(k \geq 2\). It remains to prove that the induced \(A^\infty[1]\) quasi-isomorphism

\[
G = (g_1, \ldots, g_k, \ldots): \text{Hom}^*(A^\infty_X, A^\leq n_X^*) \to \text{Hom}^*(H^p_X, H^q_X)
\]

is given by

\[
g_k(f_1 \otimes \cdots \otimes f_k) = \pi f_1\!d\!f_2\!h\!d\!\cdots\!h\!d\!f_k t,
\]

that is, that the \(g_k\) satisfy the recursive relation

\[
\tag{7.4}
g_k(f_1 \otimes \cdots \otimes f_k) = g_{k-1}Q^{k-1}_k(K_1 \otimes \cdots \otimes f_k),
\]

where as usual \(K_1 = \sum_{j=0}^{k-1} 1 \otimes \cdots \otimes K \otimes (i_1 \otimes k-j-1)\). We first consider the case \(k = 2\): as for all \(f_1, f_2 \in \text{Hom}^*(A^\infty_X, A^\leq n_X^*)\) we have \(q_2(f_1 \otimes \pi f_2) = 0\) it follows that \(Q^2_1 K_2(f_1 \otimes f_2) = (-1)^{|f_1|} q_2(f_1 \otimes K(f_2))\) and thus

\[
g_2(f_1 \otimes f_2) = (-1)^{|f_1|} g_1 q_2(f_1 \otimes (h f_2 + (-1)^{|f_1|} \pi f_2 h)) = -g_1(f_1 \!d\! f_2) = \pi f_1\!d\!f_2 t.
\]

We suppose inductively that \(g_1, \ldots, g_{k-1}\) are of the desired form: in particular for all \(f \in \text{Hom}^*(A^\infty_X, A^\leq n_X^*)\) we have \(g_{k-1}(\cdots \otimes \pi f) = g_{k-1}(\cdots \otimes K(f)) = 0\) and thus the right hand side of (7.4) becomes

\[
g_{k-1}Q^{k-1}_k K_1 (f_1 \otimes \cdots \otimes f_k) = (-1)^{|f_{k-1}|} g_{k-1} (f_1 \otimes \cdots \otimes f_{k-2} \otimes q_2(f_{k-1} \otimes K(f_k)))
\]

\[
= g_{k-1}(f_1 \otimes \cdots \otimes f_{k-2} \otimes \langle f_{k-1}h f_k \rangle) = \pi f_1\!d\!f_2\!h\!d\!\cdots\!h\!d\!f_k t.
\]

\(
\square
\)

We are now able to present a third algebraic model of the \(n\)th period map with the nice property that the target \(L^\infty\) algebra is minimal (in fact, it has the trivial \(L^\infty\) structure).

**Theorem 7.8.** The \(L^\infty[1]\) morphism \(P^\infty = (p_1, \ldots, p_k, \ldots): K S X [1] \to \text{Hom}^*(H^{n,n}_X, H^{n,n}_X)\), from the Kadaira-Spencer DG-Lie algebra to \(\text{Hom}^*(H^{n,n}_X, H^{<n,n}_X)\) with the trivial \(L^\infty[1]\) structure, given in Taylor coefficients by

\[
p_k(s^{-1}_1 \xi_1 \otimes \cdots \otimes s^{-1}_k \xi_k) = \sum_{j=1}^k \varepsilon (\sigma) \pi \varepsilon \xi_{s(1)} \varepsilon \xi_{s(2)} \cdots \varepsilon \xi_{s(k)} h \!d\! \xi_{s(j+1)} h \!d\! \xi_{s(j+2)} \cdots \!h\! \xi_{s(k)},
\]

is an algebraic model of the \(n\)th period map.

**Proof.** We have to show that the map \(P^\infty\) defined in (7.5) is the composition of the model of the \(n\)th period map \(\Pi^\infty: K S X [1] \to \text{Hom}^*(A^{\infty,n}_X, A^{\leq n,n}_X)\), \(p_k(s^{-1}_1 \xi_1 \otimes \cdots \otimes s^{-1}_k \xi_k) = i_\xi_1 \cdots i_\xi_k\), of Theorem 7.5 and the \(L^\infty[1]\) quasi-isomorphism \(G: \text{Hom}^*(A^{\infty,n}_X, A^{\leq n,n}_X) \to \text{Hom}^*(H^{n,n}_X, H^{<n,n}_X)\) of Proposition 7.6.
First of all we notice that $g_k(f_1 \circ \cdots \circ f_k) = 0$ whenever at least two entries $f_i$, $f_j$ belong to the subspace $\text{Hom}^*(A_X^{n,*}, A_X^{n-1,*}) \subset \text{Hom}^*(A_X^{n,*}, A_X^{<n,*})$. Thus the composition $GI\Pi_\infty$ is given by

$$
\sum_{k=1}^{N} \frac{1}{k!} \sum_{\sigma} \sum_{\xi_1, \ldots, \xi_k} \varepsilon(\sigma) g(\xi_{\sigma(1)} \cdots \xi_{\sigma(k)}) = 
= \sum_{j=1}^{n} \sum_{\xi_1, \ldots, \xi_k} \varepsilon(\sigma) k_{j-1} \xi_{\sigma(j)} \circ \cdots \circ \xi_{\sigma(k)}
= \sum_{j=1}^{n} \sum_{\xi_1, \ldots, \xi_k} \varepsilon(\sigma) p_{1} \xi_{\sigma(j)} \circ \cdots \circ \xi_{\sigma(k)}
= \sum_{j=1}^{n} \sum_{\xi_1, \ldots, \xi_k} \varepsilon(\sigma) p_{1} \xi_{\sigma(j)} \circ \cdots \circ \xi_{\sigma(k)}
$$

To justify the last passage we claim that $h \partial \xi_1 \circ \cdots \circ \xi_n = h \xi_1 \circ \cdots \circ \xi_n$ for all $k \geq 1$ and $\xi_1, \ldots, \xi_k \in KS_X$. In fact, since $\partial h = \partial \partial h = 0$, we have

$$
h \xi_1 \circ \cdots \circ \xi_n = h \xi_1 \circ \cdots \circ \xi_n + h \partial \xi_1 \circ \cdots \circ \xi_n = \cdots
$$

$\square$

Remark 7.9. Taking the composition of the morphism $P_\infty$ of Theorem 7.8 with the natural projection $\text{Hom}^*(H_X^{<n,*}, H_X^{<n-1,*}) \rightarrow \text{Hom}^*(H_X^{<n,*}, H_X^{<n-2,*})$ we recover the $L_\infty[1]$ morphism introduced in [26] for proving that the obstructions to deformations of $X$ are annihilated by the map

$$
i: H^2(X, \Theta_X) \rightarrow \prod_i \text{Hom}(H^i(X, \Omega_X^2), H^{i+2}(X, \Omega_X^{n-1})).$$

Taking the composition of the morphism $P_\infty$ of Theorem 7.8 with the natural projection $\text{Hom}^*(H_X^{<n,*}, H_X^{<n-1,*}) \rightarrow \text{Hom}^*(H_X^{<n,*}, H_X^{<n-2,*})$ we get the $L_\infty[1]$ morphism

$$
\Phi = (\phi_1, \ldots, \phi_k, \ldots): KS_X[1] \rightarrow \text{Hom}^*(H_X^{<n,*}, H_X^{<n-2,*}),
$$

where $\phi_k = 0$ for $k < n$ and

$$
\phi_k(s^{-1} \xi_1 \circ \cdots \circ s^{-1} \xi_n) = \sum_{\sigma \in S(n,1,\ldots,1)} \varepsilon(\sigma) p_{1} \xi_{\sigma(n)} \circ \cdots \circ \xi_{\sigma(1)}
$$

for $k \geq n$, which is the (formal, pointed) analog of the Yukawa potential function (see e.g. [34]) for an arbitrary compact Kähler manifold.

8. $L_\infty$ MODELS FOR YUKAWA ALGEBRAS

Let $X$ be a Kähler manifold of dimension $n \geq 2$ with a fixed Kähler metric. In Theorem 7.8 we constructed an $L_\infty[1]$ model

$$
\text{Hom}^*(H_X^{<n,*}, H_X^{<n-2,*}) \rightarrow KS_X[1] \rightarrow \text{Hom}^*(H_X^{<n-*}, H_X^{<n-2,*})
$$
of the geometric diagram of pointed moduli spaces

\[(\text{Grass}(F^1 H^* (X, \mathbb{C})), F^n H^* (X, \mathbb{C})) \xrightarrow{\mathcal{P}^n} (\text{Grass}(H^* (X, \mathbb{C})), F^n H^* (X, \mathbb{C}))\]

where \(B\) is the base of the semiuniversal deformation of \(X\) and \(\mathcal{P}^n\) the \(n\)th local period map. As explained in the introduction, if we want to replicate the geometric diagram

\[Y_B \rightarrow (\text{Grass}(F^1 H^* (X, \mathbb{C})), F^n H^* (X, \mathbb{C})) \xrightarrow{\mathcal{P}^n} (\text{Grass}(H^* (X, \mathbb{C})), F^n H^* (X, \mathbb{C}))\]

in the category of \(L_\infty\) algebras, the only thing that makes sense (according to the general philosophy of derived deformation theory) is to replace the above cartesian diagram of complex singularities by a homotopy cartesian diagram of \(L_\infty\) algebras

\[\text{Yu}_{X[1]} \xrightarrow{\text{Hom}^* (H^n_X, H^0_X H^*_X < n)} KS_{X[1]} \xrightarrow{\mathcal{P}_\infty} \text{Hom}^* (H^n_X, H^*_X H^*_X < n)\]

**Definition 8.1.** We shall denote a homotopy fiber product of the previous diagram of \(L_\infty\) algebras by \(\text{Yu}_{X[1]}\) (thus, strictly speaking, \(\text{Yu}_{X[1]}\) denotes an object in the homotopy category of \(L_\infty\) algebras). In particular the associated deformation functor \(\text{Def}_{\text{Yu}_{X[1]}} : \text{Art} \rightarrow \text{Set}\) is well defined.

By a straightforward application of Theorem 5.4 we obtain the following concrete \(L_\infty[1]\) model of \(\text{Yu}_{X[1]}\).

**Theorem 8.2.** The \(L_\infty[1]\) algebra \((KS_{X[1]} \times \text{Hom}^* (H^n_X, H^0_X H^*_X < n), \{q_1, \ldots, q_k, \ldots\})\), where the nontrivial brackets are

\[q_1 (s^{-1} \xi, sf) = (s^{-1} s^{-1} \xi_1, 0),\]

\[q_2 (s^{-1} \xi_1 \circ s^{-1} \xi_2) = \begin{cases} \{((-1)^{s^{-1} \xi_1} s^{-1} \xi_1, s \pi \xi_1 \cdot \xi_2 t) & \text{if } n = 2, \\ (-1)^{s^{-1} \xi_1} s^{-1} \xi_1 \cdot \xi_2 t) & \text{if } n > 2, \end{cases}\]

\[q_k = 0 \text{ for } 2 < k < n \text{ and finally }\]

\[q_n (s^{-1} \xi_1 \circ \cdots \circ s^{-1} \xi_{n+k}) = \sum_{\sigma \in S(n, 1, \ldots, 1)} \varepsilon (\sigma) s \pi \xi_{\sigma (1)} \cdots \xi_{\sigma (n)} h l \xi_{\sigma (n+1)} h \cdots h l \xi_{\sigma (n+k)} t,\]

is a homotopy fiber product of the diagram

\[\text{Hom}^* (H^n_X, H^0_X H^*_X < n) \xrightarrow{\mathcal{P}_\infty} \text{Hom}^* (H^n_X, H^*_X H^*_X < n)\]

and thus a model of \(\text{Yu}_{X[1]}\).
Fixing the above choice of a model of Yu$_X[1]$, the Maurer-Cartan functor becomes
\[
\text{MCYu}_X[1] : \text{Art} \to \text{Set}, \quad B \mapsto \text{MCYu}_X[1](B) = \left\{ (s^{-1}_q, sf) \in (\text{Yu}_X[1] \otimes \text{m}_B)^0 \ \big| \ \overline{\partial}_q = \frac{1}{2} \partial, \sum_{k \geq 0} \pi \left( \frac{v_k}{n!} \right) (\epsilon_q)_k = \pi \left( \frac{v_k}{n!} \right) s = 0 \right\}
\]
thus recovering the equation of Corollary 2.5.

The previous model of Yu$_X \to KS_X$ has the advantage to be a fibration of $L_\infty$ algebras with minimal fiber. On the other hand it has the disadvantage that the formulas for the brackets involve Green’s operator which is almost never explicitly known. We conclude this section by giving a second, more treatable, model of Yu$_X[1]$. We replace our algebraic model of the period map by the weakly equivalent one from Theorem 7.5, then we apply again Theorem 5.4 to obtain an explicit model for the homotopy fiber product of
\[
\text{Hom}^\ast(A^n_X, A^{0<\cdots<n}_X)
\]
\[
KS_X[1] \xrightarrow{\Pi_{\infty}} \text{Hom}^\ast(A^n_X, A^{0<\cdots<n}_X)
\]
Recall that the DG-Lie algebra structure on $(\text{Hom}^\ast(A^n_X, A^{0<\cdots<n}_X)[-1], \delta, [-, -])$ is given by
\[
\delta(sf) = s(-[\overline{\partial}, f] - P\partial f), \quad [sf_1, sf_2] = s (f_1 \partial f_2 - (-1)^{|f_1|(|f_2|+1)} f_2 \partial f_1),
\]
where $P : A^n_X \to A^{n,n}_X$ is the projection with kernel $A^{n,n}_X$, and we identify every element $f \in \text{Hom}^\ast(A^n_X, A^{0\cdots<n}_X)$ with its image in $\text{End}(A^n_X)$; in particular $\text{Im}(f) \subseteq A^{n,n}_X \subseteq \text{Ker}(f)$. For degree reasons we have $[sf_1, sf_2] = 0$ whenever $f_1, f_2 \in \text{Hom}^\ast(A^n_X, A^{0\cdots<n}_X)$ and therefore the decomposition
\[
\text{Hom}^\ast(A^n_X, A^{0\cdots<n}_X)[-1] = \text{Hom}^\ast(A^n_X, A^{0\cdots<n}_X)[-1] \oplus \text{Hom}^\ast(A^n_X, A^{0\cdots<n}_X)[-1]
\]
satisfies the hypotheses of Voronov construction of higher derived brackets [36, 37].

**Lemma 8.3.** The $L_\infty[1]$ algebra structure on $\text{Hom}^\ast(A^n_X, A^{0\cdots<n}_X)[-1]$ induced via higher derived brackets is abelian with differential $q_1(sf) = -s[\overline{\partial}, f]$.

**Proof.** We claim that $[\delta(sf_1), sf_2] = 0$ in the DG-Lie algebra $\text{Hom}^\ast(A^n_X, A^{0\cdots<n}_X)[-1]$ whenever $f_1, f_2 \in \text{Hom}^\ast(A^n_X, A^{0\cdots<n}_X)$, which obviously proves the lemma. This is clear by degree reasons if $n \geq 3$, while for $n = 2$ we have
\[
[\delta(sf_1), sf_2] = s(-[\overline{\partial}, f_1] - \partial f_1), sf_2 = (-1)^{|f_1|(|f_2|+1)} s(f_2 \partial^2 f_1) = 0.
\]

**Theorem 8.4.** The $L_\infty[1]$ algebra $(KS_X[1] \times \text{Hom}^\ast(A^n_X, A^{0\cdots<n}_X)[-1], q_1, \ldots, q_n, \ldots)$, where the only nontrivial brackets are $q_1(s^{-1}_q, sf) = (s^{-1}_q [\overline{\partial}, f])$
\[
q_2(s^{-1}_q \xi_1 \circ s^{-1}_q \xi_2) = \begin{cases} (1)^{|\xi_1| s^{-1}_q \xi_2} & \text{if } n = 2, \\ (1)^{|\xi_1| s^{-1}_q \xi_2} & \text{if } n > 2, \end{cases}
\]
\[
q_2(sf \circ s^{-1}_q \xi) = (-1)^{|f| s(f \xi)),
\]
\[
q_n(s^{-1}_q \xi_1 \circ \cdots \circ s^{-1}_q \xi_n) = s(\xi_1 \cdots \xi_n),
\]
is a homotopy fiber product of the diagram

\[ \text{Hom}^* (A_X^{\alpha}, A_X^{0 < \alpha < n}) \]

\[ \xrightarrow{\Pi_n} \text{Hom}^* (A_X^{\alpha}, A_X^{< n}) \]

and a model for \( Y_X [1] \).

**Proof.** We have to show that the \( L_\infty [1] \) algebra structure in the statement is the one given by the formulas of Theorem 5.4. This is a consequence of the fact that nested brackets of the form \( [\cdots [s(\xi_1, \cdots, \xi_k), s(f_1, \cdots, s(f_k)] \), where \( f_1, \ldots, f_k \in \text{Hom}^* (A_X^{\alpha}, A_X^{0}) \) and the brackets are computed in the DG-Lie algebra \( \text{Hom}^* (A_X^{\alpha}, A_X^{< n})[-1] \), are trivial for obvious degree reasons except in the case \( j = k = 1 \), where

\[ s(\xi, sf) = (-1)^{|s|}|f| s(f \partial \xi) \in \text{Hom}^* (A_X^{\alpha}, A_X^{0})[-1], \]

as \( l_\xi | A_X^{\alpha} = \partial \xi | A_X^{\alpha} \). The restriction of the \( L_\infty [1] \) structure to the fiber \( \text{Hom}^* (A_X^{\alpha}, A_X^{0})[-1] \) is homotopy abelian by the previous lemma. Finally, it is clear by degree reason that the only bracket \( q_k (s^{-1} \xi_1 \circ \cdots \circ s^{-1} \xi_k) \) with a nontrivial component along \( \text{Hom}^* (A_X^{\alpha}, A_X^{0})[-1] \) is \( q_n \). \( \square \)

### 9. Formality of Yukawa algebras for K3 surfaces

When \( X \) is a compact Kähler manifold with trivial canonical bundle, we get a consistent simplification of our formulas. If \( n = \text{dim} \, X \), let \( \Omega \in \mathcal{H}^n(X, K_X) \) be a holomorphic volume form. The Bogomolov-Tian-Todorov theorem says that the Kodaira-Spencer algebra \( KS_X \) is homotopy abelian, where, according to [17, p. 357] (cf. also [25, Section 7.3]), an explicit homotopy equivalence is given by the pair of quasi-isomorphisms of DG-Lie algebras

\[ KS_X \xleftarrow{\alpha} L = \{ \xi \in KS_X \mid l_\xi (\Omega) = 0 \} \xrightarrow{\beta} M = \frac{L}{I}, \]

where \( I \) is the differential Lie ideal

\[ I = \{ \xi \in KS_X \mid i_\xi (\Omega) \in \partial (A^{n-2}) \}. \]

More precisely, the Koszul-Tian-Todorov lemma ([23, Prop. 2.3], [32, Lemma 3.1], [33, Lemma 1.2.4]) tell us that both \( L, M \) are DG-Lie algebras and the bracket on \( M \) is trivial, while the \( \partial \bar{\partial} \)-lemma implies that both arrows are quasi-isomorphisms and the differential on \( M \) is trivial.

Taking the pull-back of the fibration of \( L_\infty [1] \)-algebras

\[ KS_X [1] \times \text{Hom}(H_X^{\alpha}, H_X^{0})[-1] \rightarrow KS_X [1] \]

via the strict quasi-isomorphism \( \alpha \) we get a strict quasi-isomorphism

\[ L \times \text{Hom}(H_X^{\alpha}, H_X^{0})[-1] \rightarrow KS_X [1] \times \text{Hom}(H_X^{\alpha}, H_X^{0})[-1] \]

in which every Taylor coefficient of degree > \( n \) in \( L \times \text{Hom}(H_X^{\alpha}, H_X^{0})[-1] \) vanishes. More precisely the nontrivial brackets are \( q_1 (s^{-1} \xi, sf) = (s^{-1} \partial \xi, 0) \),

\[ q_2 (s^{-1} \xi_1 \circ s^{-1} \xi_2) = \begin{cases} ((-1)^{|s|} s^{-1} [\xi_1, \xi_2], s \pi i \xi_1 i \xi_2) & \text{if } n = 2, \\ ((-1)^{|s|} s^{-1} [\xi_1, \xi_2], 0) & \text{if } n > 2, \end{cases} \]

\( q_k = 0 \) for \( 2 < k < n \) and finally

\[ q_n (s^{-1} \xi_1 \circ \cdots \circ s^{-1} \xi_n) = s \pi i \xi_1 \cdots i \xi_n. \]
Lemma 9.1. In the above notation, if $X$ is a K3 surface then, for every $\xi \in L$ and every $\eta \in I$ we have

$$\pi i_\xi i_\eta = \pi i_\eta i_\xi = 0.$$  

Proof. Notice first that $i_\xi i_\eta = \pm i_\eta i_\xi$ and then it is sufficient to prove the first equality $\pi i_\xi i_\eta = 0$. Since $H^3_X = H^{3,0}_X = 0$ the lemma is trivially verified when $|\xi| + |\eta| \neq 0, 2$.

If $|\xi| + |\eta| = 2$, by Serre duality it is sufficient to show that

$$\int_X \pi i_\xi i_\eta \Omega \wedge \Omega = 0.$$  

Since $\Omega$ is $\overline{\partial}$-closed and $\pi i_\xi i_\eta(\Omega) - i_\xi i_\eta(\Omega)$ is $\partial$-exact we have

$$\int_X \pi i_\xi i_\eta (\Omega) \wedge \Omega = \int_X i_\xi i_\eta (\Omega) \wedge \Omega = - \int_X i_\eta (\Omega) \wedge i_\xi (\Omega),$$

where the second equality follows from the fact that $i_\xi$ is a derivation of degree 0 of the de Rham algebra. The last integral is trivial since by assumption $i_\xi(\Omega)$ is $\partial$-closed and $i_\eta(\Omega)$ is $\partial$-exact.

If $|\xi| + |\eta| = 0$, again by Serre duality it is sufficient to show that

$$\int_X \pi i_\xi i_\eta (\Omega) \wedge \Omega = 0.$$  

Since $i_\rho(\Omega) = 0$ for every $\rho$, the same argument used in the previous case implies that

$$\int_X \pi i_\xi i_\eta (\Omega \wedge \Omega) = \int_X i_\xi i_\eta (\Omega \wedge \Omega) = - \int_X i_\eta (\Omega \wedge \Omega) \wedge i_\xi (\Omega)$$

and the last integral vanishes since $i_\eta(\Omega \wedge \Omega) = i_\eta(\Omega) \wedge \Omega$ is $\partial$-exact and $i_\xi(\Omega \wedge \Omega)$ is $\partial$-closed.

Since $\Omega \wedge \Omega$ is a scalar multiple of the Kähler volume form, we have $\overline{\partial} (\Omega \wedge \Omega) = 0$ and, since $\pi i_\xi i_\eta(\Omega) - i_\xi i_\eta(\Omega)$ is $\overline{\partial}$-exact, we have

$$\int_X \pi i_\xi i_\eta (\Omega) \wedge \Omega \wedge \Omega = \int_X \pi i_\xi i_\eta (\Omega) \wedge \Omega \wedge \Omega$$

and the same argument as above shows that the last integral vanishes.  

Theorem 9.2. Let $X$ be a K3 surface. The the $L_\infty$ algebra $Yu_X$ is formal. More precisely $Yu_X$ is quasi-isomorphic to the graded Lie algebra

$$H^*(X, \Theta_X) \times Hom^*(H^3_X, H^*_{\mathcal{O}_X}), \quad [(\xi, u), (\eta, v)] = (-1)^{|\xi|} i_\xi i_\eta.$$  

Proof. By the previous lemma the subspace $I[1] \subset L[1] \times Hom(\Omega^{1,0}_X, \Omega^{0,1}_X)[-1]$ is an $L_\infty[1]$ ideal and then the projection

$$L[1] \times Hom(\Omega^{1,0}_X, \Omega^{0,1}_X)[-1] \rightarrow \frac{L[1]}{I[1]} \times Hom(\Omega^{1,0}_X, \Omega^{0,1}_X)[-1]$$

is a strict quasi-isomorphism. Since $Hom(\mathcal{O}^{1,0}_X, \mathcal{O}^{0,1}_X)[-1] = Hom(H^{1,0}_X, H^{0,1}_X)$ it is sufficient to rewrite the induced $L_\infty[1]$ structure on the quotient via the natural isomorphisms

$$\frac{L[1]}{I[1]} \simeq H^*(X, \Theta_X)[1], \quad H^{1,0}_X \simeq H^*(X, \Omega^2_X), \quad H^{0,1}_X \simeq H^*(X, \mathcal{O}_X)$$

and then apply the décalage functor.  

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Università degli studi di Roma La Sapienza,
Dipartimento di Matematica “Guido Castelnuovo”,
P.le Aldo Moro 5, I-00185 Roma, Italy.
E-mail address: bandiera@mat.uniroma1.it

E-mail address: manetti@mat.uniroma1.it
URL: www.mat.uniroma1.it/people/manetti/