COMPACT MODULI OF K3 SURFACES
WITH A NONSYMPLECTIC AUTOMORPHISM

VALERY ALEXEEV, PHILIP ENGEL, AND CHANGHO HAN

ABSTRACT. We construct a modular compactification via stable slc pairs for the moduli spaces of K3 surfaces with a nonsymplectic group of automorphisms under the assumption that some combination of the fixed loci of automorphisms defines an effective big divisor, and prove that it is semitoroidal.

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1. INTRODUCTION

Let $X$ be a smooth K3 surface over the complex numbers. An automorphism $\sigma$ of $X$ is called non-symplectic if it has finite order $n > 1$ and $\sigma^*(\omega_X) = \zeta_n \omega_X$, where $\omega_X \in H^{2,0}(X)$ is a nonzero 2-form and $\zeta_n$ is a primitive $n$th root of identity. By changing the generator of the cyclic group $\mu_n$ we can and will assume that $\zeta_n = \exp(2\pi i/n)$. It is well known that a K3 surface admitting such an automorphism is projective. The possibilities for the order are the numbers $n$ whose Euler function satisfies $\varphi(n) \leq 20$ with the single exception $n \neq 60$, see [MO98, Thm. 3].

In this paper we study compactification of moduli spaces of pairs $(X, \sigma)$. But to begin with, the automorphism group $\text{Aut}(X, \sigma)$, i.e. those automorphisms of $X$ commuting with $\sigma$, may be infinite. To fix this, we will usually additionally assume:

$(\exists g \geq 2)$ The fixed locus $\text{Fix}(\sigma)$ contains a curve $C_1$ of genus $g \geq 2$.

By looking at the $\mu_n$-action on the tangent space of any fixed point, it is easy to see that $\text{Fix}(\sigma)$ is a disjoint union of several smooth curves and points. The Hodge index theorem implies at most one of the fixed curves has genus $g \geq 2$. One could instead have one or two fixed curves of genus $g = 1$. All other fixed curves are isomorphic to $\mathbb{P}^1$.

Under the $(\exists g \geq 2)$ assumption, the group $\text{Aut}(X, \sigma)$ is finite. The opposite is almost true. For example let $n = 2$, i.e. $\sigma$ is an involution. Then $\sigma^*$ fixes the

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Neron-Severi lattice $S_X \subset H^2(X, \mathbb{Z})$ and acts as multiplication by $(-1)$ on the lattice $T_X = S_X^\perp$ of transcendental cycles. In this case $\text{Aut}(X, \sigma) = \text{Aut}(X)$.

Deformation classes of such K3 surfaces $(X, \sigma)$ are classified by the primitive 2-elementary hyperbolic sublattices $S \subset L_{K3}$. By Nikulin [Nik79b] there are 75 cases, uniquely determined by certain invariants $(g, k, \delta)$. Among them 51 satisfy $(\exists g \geq 2)$. The only case when $|\text{Aut}(X)| < \infty$ but $(\exists g \geq 2)$ is not satisfied is $(g, k, \delta) = (1, 9, 1)$ which is the one-dimensional mirror family to K3 surfaces of degree 2. In the case $(g, k, \delta) = (2, 1, 0)$ one has $|\text{Aut}(X)| = \infty$ but the set $\text{Fix}(\sigma)$ consists of two elliptic curves, so $(\exists g \geq 2)$ does not hold.

Since the moduli stack of smooth quasipolarized K3 surfaces is notoriously non-separated, so is usually the moduli stack of smooth K3s with a nonsymplectic automorphism. For a fixed isometry $\rho \in O(L_{K3})$ of order $n$, there exists the moduli stack and moduli space of smooth K3 surfaces “of type $\rho$”: those pairs $(X, \sigma)$ where the action of $\sigma^*$ on $H^2(X, \mathbb{Z})$ can be modeled by $\rho$. We construct them in Section 2. The maximal separated quotient of $F_\rho$ is $(\mathbb{D}_\rho \setminus \Delta_\rho)/\Gamma_\rho$, where $\mathbb{D}_\rho$ is a symmetric Hermitian domain of type IV if $n = 2$ or a complex ball if $n > 2$, $\Gamma_\rho$ is an arithmetic group, and $\Delta_\rho \subset \mathbb{D}_\rho$ is the discriminant locus.

Under the assumption $(\exists g \geq 2)$, the space $F^{\text{ade}}_\rho := (\mathbb{D}_\rho \setminus \Delta_\rho)/\Gamma_\rho$ is the coarse moduli space for the K3 surfaces $\overline{X}$ with $ADE$ singularities, obtained from the smooth K3 surfaces $X$ by contracting the $(-2)$-curves perpendicular to the component $C_1$ with $g \geq 2$ in $\text{Fix}(\sigma)$. The stack of such $ADE$ K3 surfaces is separated.

The main goal of this paper is to construct a functorial, geometrically meaningful compactification of the moduli space $F^{\text{ade}}_\rho$, under the assumption $(\exists g \geq 2)$. Let $R = C_1, \varphi_{|mR}: X \to \overline{X}$ be the contraction as above and $\overline{R}$ be the image of $R$. Then for any $\epsilon > 0$ the pair $(\overline{X}, \epsilon \overline{R})$ is a stable pair with semi log canonical singularities. Then the theory of KSBA moduli spaces (see [Kol21] for the general case or [AET19, ABE20] for the much easier special case needed here) gives a moduli compactification $\overline{F}^{\text{slc}}_\rho$ to a space of stable pairs with automorphism.

Our main Theorem 3.24 says that $\overline{F}^{\text{slc}}_\rho$ is a semitoroidal compactification of $\mathbb{D}_\rho/\Gamma_\rho$. This class of compactifications was introduced by Looijenga [Loo03b] as a common generalization of Baily-Borel and toroidal compactifications. As a corollary, the family of $ADE$ K3 surfaces with an automorphism extends along the inclusion $(\mathbb{D}_\rho \setminus \Delta_\rho)/\Gamma_\rho \hookrightarrow \mathbb{D}_\rho/\Gamma_\rho$.

The proof applies a modified form of one of the main theorems of [AE21] about so-called recognizable divisors. The $g \geq 2$ component of the fixed locus is a canonical choice of a polarizing divisor. We prove that this divisor is recognizable.

As we point out in Section 5, the results also extend to the more general situation of a symmetry group $G \subset \text{Aut} X$ which is not purely symplectic.

The cases $n = 2, 3, 4, 6$ are of the most interest for compactifications. If $n \neq 2, 3, 4, 6$ then the space $\mathbb{D}_\rho/\Gamma_\rho$ is already compact, see [Mat16] or Corollary 3.14.

K3 surfaces with an involution were classified by Nikulin in [Nik79b]. K3s with a non-symplectic automorphism of prime order $p \geq 3$ were classified by Artebani, Sarti, and Taki in [AS08, AST11]. The case $n = 4$ was treated by Artebani-Sarti in [AS15] and the case $n = 6$ by Dillies in [Dil09, Dil12].

We note two cases where our KSBA, semitoroidal compactification $\overline{F}^{\text{slc}}_\rho$ is computed in complete detail: Alexeev-Engel-Thompson [AET19] for the case of K3
surfaces of degree 2, generically double covers of \( \mathbb{P}^2 \), and a forthcoming work Deopurkar-Han [DH22] which treats a 9-dimensional component in the moduli for \( n = 3 \).

The paper is organized as follows. In Section 2 we set up the general theory of the moduli of K3 surfaces with a non-symplectic automorphisms. In Section 3 we define the stable pair compactifications and prove the main Theorem 3.24. In Section 4 we relate K3 surfaces with nonsymplectic automorphisms with their quotients \( Y = X/\mu_n \), and the compactification \( \mathcal{F}_\rho^{slc} \) with the KSBA compactification of the moduli spaces of log del Pezzo pairs \( (Y, \frac{n-1+e}{n} B) \).

In Section 5 we extend the results in two different ways: to K3 surfaces with a finite group of symmetries \( G \subset \text{Aut} X \) which is not purely symplectic, and to more general polarizing divisors associated with such a group action.

Throughout, we work over the field of complex numbers.

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2. Moduli of K3s with a Nonsymplectic Automorphism

2A. Notations. A lattice is a free abelian group with an integral-valued symmetric bilinear form. Let \( L = H^\oplus 3 \oplus E^\oplus 2 \) be a fixed copy of the even unimodular lattice of signature \((3,19)\), where \( H = \Pi_{11} \) corresponds to the bilinear form \( b(x,y) = xy \) and \( E_8 \) is the standard negative definite even lattice of rank 8. For any smooth K3 surface \( X \) the cohomology lattice \( H^2(X,\mathbb{Z}) \) is isometric to \( L \).

Denote by \( S = S_X \) the Neron-Severi lattice \( \text{Pic}(X) = \text{NS}(X) \). By the Lefschetz (1,1)-theorem, it equals \( (H^2)^1 \cap H^2(X,\mathbb{Z}) \subset H^2(X,\mathbb{C}) \). We have \( H^{2,0}(X) = \mathbb{C}\omega_X \) for some nowhere vanishing holomorphic two-form \( \omega_X \). If \( X \) is projective, then \( S_X \) is nondegenerate of signature \((1,r_X - 1)\). In this case, its orthogonal complement \( T_X = (S_X)^\perp \subset H^2(X,\mathbb{Z}) \) is the transcendental lattice, of signature \((2,20 - r_X)\).

The Kähler cone \( K_X \subset H^{1,1}(X,\mathbb{R}) \) is the set of classes of Kähler forms on \( X \); it is an open convex cone.

**Theorem 2.1** (Torelli Theorem for K3 surfaces, [PSS71]). The isomorphisms \( \sigma: X' \rightarrow X \) are in bijection with the isometries \( \sigma^*: H^2(X,\mathbb{Z}) \rightarrow H^2(X',\mathbb{Z}) \) satisfying the conditions \( \sigma^*(H^{2,0}(X)) = H^{2,0}(X') \) and \( \sigma^*(K_X) = K_{X'} \).

For any lattice \( H, \) a root is a vector \( \delta \in H \) with \( \delta^2 = -2 \). The set of all roots is denoted by \( H_{-2} \). The Weyl group \( W(H) \) is the group generated by reflections \( v \mapsto v + (v, \delta)\delta \) for \( \delta \in H_{-2} \). It is a normal subgroup of the isometry group \( O(H) \).

2B. Moduli of marked unpolarized K3s. The basic reference here is [ast85]. Let \( X \) be a K3 surface. A marking is an isometry \( \phi: H^2(X,\mathbb{Z}) \rightarrow L \). Let

\[ D = \mathbb{P}\{ x \in L_C \mid x \cdot x = 0, \ x \cdot \bar{x} > 0 \}, \ \text{dim } D = 20. \]

There exists a fine moduli space \( \mathcal{M} \) of marked K3 surfaces and a period map \( \pi: \mathcal{M} \rightarrow D, \ (X,\phi) \mapsto \phi(H^{2,0}(X)) \in \mathbb{P}(L_C) \). \( \mathcal{M} \) is a non-Hausdorff 20-dimensional complex manifold with two isomorphic connected components interchanged by negating \( \phi \). The period map is étale and surjective.

For a period point \( x \in D \), the vector space \( (\mathbb{C} x \oplus \mathbb{C} \bar{x}) \cap L_R \subset L_C \) is positive definite of rank 2 and its orthogonal complement \( x^\perp \cap L_R \) has signature \((1,19)\). Let

\[ \{ v \in x^\perp \cap L_R \mid v^2 > 0 \} = P_x \cup (-P_x) \]
be the two connected components of the set of positive square vectors. Then the fiber \( \pi^{-1}(x) \) is identified with the set of connected components \( C \) of

\[(1) \quad (P_x \cup (-P_x)) \setminus \cup_{\delta} \delta^\perp \text{ for } \delta \in (x^\perp \cap L)^{-2}.\]

Namely, an open chamber \( C \) is identified with the Kähler cone \( \mathcal{K}_X \) of the corresponding marked K3 surface \( X \) via the marking \( \phi \). The connected components are permuted by the reflections and \( \pm \text{id} \), and \( \pi^{-1}(x) \) is a torsor under the group \( \mathbb{Z}_2 \times W_x \), where \( W_x = W(x^+ \cap L) \). Since \( x^+ \cap L_\mathbb{R} \) is hyperbolic, the group and the fiber \( \pi^{-1}(x) \) may be infinite. For a general point \( x \in \mathcal{D} \), the lattice \( x^+ \cap L \) has no roots and the fiber \( \pi^{-1}(x) \) consists of two points, one in each connected component of \( \mathcal{M} \).

2C. Moduli of \( \rho \)-marked and \( \rho \)-markable K3 surfaces with automorphisms.

Fix \( \rho \in O(L) \) an isometry of order \( n > 1 \) and consider a K3 surface \( X \) with a non-symplectic automorphism \( \sigma \) of order \( n \).

**Definition 2.2.** A \( \rho \)-marking of \((X, \sigma)\) is an isometry \( \phi : H^2(X, \mathbb{Z}) \rightarrow L \) such that \( \sigma^* = \phi^{-1} \circ \rho \circ \phi \). We say that \((X, \sigma)\) is \( \rho \)-markable if it admits a \( \rho \)-marking.

A family of \( \rho \)-marked surfaces is a smooth morphism \( f : (X, \sigma_B) \rightarrow B \) with an automorphism \( \sigma_B : X \rightarrow X \) over \( B \), together with an isomorphism of local systems \( \phi_S : R^1 f_* \mathbb{Z} \rightarrow L \otimes \mathbb{Z}_B \) such that every fiber is a K3 surface with a \( \rho \)-marking. A family \( f : (X, \sigma_B) \rightarrow B \) is \( \rho \)-markable if such an isomorphism exists locally in complex-analytic topology on \( B \).

We define the moduli stacks \( \mathcal{M}_\rho \) of \( \rho \)-marked, resp. \( \mathcal{F}_\rho \) of \( \rho \)-markable K3 by taking \( \mathcal{M}_\rho(B) \), resp. \( \mathcal{F}_\rho(B) \) to be the groupoids of such families over base \( B \).

**Definition 2.3.** Define \( \mathcal{L}_{C}^\rho \) to be the eigenspace \( x \in L_C \) such that \( \rho(x) = \zeta_n x \), and the subdomain \( \mathcal{D}_\rho = \mathbb{P}(\mathcal{L}_{C}^\rho) \cap \mathbb{D} \subset \mathbb{D} \). Define \( \Gamma_\rho \subset O(L) \) as the group of changes-of-marking: \( \Gamma_\rho := \{ \gamma \in O(L) \mid \gamma \circ \rho = \rho \circ \gamma \} \).

**Definition 2.4.** Let the generic transcendental lattice \( T_\rho := \mathcal{L}_{C}^\rho \cap L \) be the intersection of \( L \) with the sum of all primitive eigenspaces of \( \rho \), and let the generic Picard lattice be \( S_\rho = (T_\rho)^{\perp} \). Let \( L^G = \text{Fix}(\rho) \subset S_\rho \) be classes in \( L \) fixed by \( \rho \). (Here, we use \( G = \langle \rho \rangle \simeq \mathbb{Z}_n \) to avoid confusing notation, as \( L^G \) would be.)

Note that the \( \zeta_n \)-eigenspaces \( \mathcal{L}_{C}^\rho \) and \( T_{\rho, C}^\rho \) coincide, and that for any K3 surface with a \( \rho \)-marking the two fixed sublattices \( \phi : S^G_X = H^2(X, \mathbb{Z})^G \cong L^G \) are identified.

For there to exist a \( \rho \)-markable algebraic K3 surface, the signature of \( T_\rho \) must be \((2, \ell)\) for some \( \ell \), as there is necessarily a vector of positive norm fixed by \( \sigma^* \) (the sum of a \( \sigma^* \)-orbit of an ample class). The converse is also true.

When \( n = 2 \), we have that \( \mathcal{D}_\rho \subset \mathbb{P}(T_{\rho, C}) \) is (two copies of) the Type IV domain associated to the lattice \( T_\rho \). When \( n \geq 3 \), the condition that \( x \cdot y = 0 \) is vacuous on \( \mathcal{D}_\rho \) because \( x \cdot y = 0 \) for eigenvectors \( x, y \) of \( \rho \) with non-conjugate eigenvalue. Thus,

\[ \mathcal{D}_\rho = \mathbb{P}\{x \in T_{\rho, C}^\rho \mid x \cdot \bar{x} > 0\} \]

is a complex ball, a Type I domain. The Hermitian form \( x \cdot \bar{y} \) on \( T_{\rho, C}^\rho \) necessarily has signature \((1, \ell)\) for some \( \ell \) for there to exist a \( \rho \)-markable K3 surface.

**Definition 2.5.** The discriminant locus is \( \Delta_\rho := (\cup_{\delta} \delta^\perp) \cap \mathcal{D}_\rho \) ranging over all roots \( \delta \) in \((L^G)^{\perp} \).
Lemma 2.6. Let \( \rho \in O(L) \) be an isometry of order \( n > 1 \). Then

1. A marking \( \phi : H^2(X, \mathbb{Z}) \to L \) defines a \( \rho \)-marking, i.e. defines an automorphism \( \sigma \) such that \( \sigma^* = \phi^{-1} \circ \rho \circ \phi \) iff the period \( x = \pi((X, \phi)) \) lies in \( \mathbb{D}_\rho \setminus \Delta_\rho \) and there exists an ample line bundle \( \mathcal{L}_h \) on \( X \) with \( h = \phi(\mathcal{L}_h) \in L^G \).

2. For a point \( x \in \mathbb{D}_\rho \setminus \Delta_\rho \) the set of \( \rho \)-marked K3s with this period is a torsor over the group \( \Gamma_\rho \cap (\mathbb{Z}_2 \times W_x) \).

Proof. Because the action is nonsymplectic, \( \rho(x) = \zeta_n x \neq x \). For any \( h \in L^G \) one has \( \rho(h) = h \), which implies that \( hx = 0 \). Thus, \( L^G \perp x \) and \( S^G_X \cong L^G \).

Clearly, one must have \( x \in \mathbb{D}_\rho \). By the Torelli theorem, automorphism \( a = \phi^{-1} \circ \rho \circ \phi \) of \( H^2(X, \mathbb{Z}) \) is induced by an automorphism \( \sigma \) of \( X \) iff it sends the Kähler cone \( K_X \) to itself. By averaging, this is equivalent to having an \( a \)-invariant Kähler class \( \mathcal{L}_h \in K_X \cap H^2(X, \mathbb{Z}) \). And since \( L^G \perp x \), one has \( \mathcal{L}_h \perp \omega_X \), so \( \mathcal{L}_h \in S_X \) and \( \mathcal{L}_h \) is an ample line bundle. This proves (1).

If \( x \perp \delta \) for some root \( \delta \in (L^G)^\perp \) then \( \mathcal{L}_\delta = \phi^{-1}(\delta) \in \text{Pic}(X) \) and either \( \mathcal{L}_\delta \) or \( \mathcal{L}_\delta^{-1} \) is effective. Then for \( \mathcal{L}_h \) as in part (1) one has both \( \mathcal{L}_h \), \( \mathcal{L}_\delta = 0 \) because \( h \perp \delta \) and \( \mathcal{L}_h \cap \mathcal{L}_\delta \neq 0 \) because \( \mathcal{L}_h \) is ample. Contradiction.

On the other hand, let \( x \in \mathbb{D}_\rho \setminus \Delta_\rho \). Then \( L^G \not\subset \cup \delta \delta^\perp \) for \( \delta \in (x \perp \perp \cap L^G) \). Thus, there exists a chamber \( \mathcal{C} \in \dim_{\rho} \setminus \cup \delta \delta^\perp \) such that \( \mathcal{C} \cap L^G \neq \emptyset \). Let \( (X, \phi) \) be the K3 surface corresponding to this chamber. Then there exists \( h \in \mathcal{C} \cap L^G \) and by part (1) the marking \( \phi \) is a \( \rho \)-marking.

Any surface with the same period \( x \) is isomorphic to \( X \), but with a marking \( \phi' = g \circ \phi \) for some \( g \in \mathbb{Z}_2 \times W_x \). Then one has both \( \sigma^* = \phi^{-1} \circ \rho \circ \phi \) and \( \sigma^* = (\phi')^{-1} \circ \rho \circ \phi' \) iff \( g \in \Gamma_\rho \). This proves (2). \( \square \)

Lemma 2.7. There exists a fine moduli space \( \mathcal{M}_\rho \) of \( \rho \)-marked K3 surfaces with a non-symplectic automorphism. \( \mathcal{M}_\rho \) an open subset of \( \pi^{-1}(\mathbb{D}_\rho \setminus \Delta_\rho) \).

Proof. The points of \( \mathcal{M} \) are chambers \( \mathcal{C} \) in Equation (1) over \( x \in \mathbb{D}_\rho \setminus \Delta_\rho \). As in the proof of Lemma 2.6, one has \( \mathcal{C} \in \mathcal{M}_\rho \) iff \( \mathcal{C} \cap L^G \neq \emptyset \). This is an open condition. \( \square \)

The restriction of \( \pi : \mathcal{M} \to \mathbb{D} \) gives the period map \( \pi_\rho : \mathcal{M}_\rho \to \mathbb{D}_\rho \setminus \Delta_\rho \). The general fiber of \( \pi_\rho \) is a torsor over \( \Gamma_\rho \cap (\mathbb{Z}_2 \times W(S_\rho)) \). Thus, \( \mathcal{M}_\rho \) is not separated iff there exists \( x \in \mathbb{D}_\rho \setminus \Delta_\rho \) such that \( \Gamma_\rho \cap W_x \supset \Gamma_\rho \cap W(S_\rho) \). This indeed happens:

Example 2.8. Consider the 9-dimensional family of \( \mu_3 \)-covers of \( \mathbb{P}^1 \times \mathbb{P}^1 \) branched in a curve \( B \) of bidegree \( (3, 3) \), studied by Kondo [Kon02]. In this case,

\[
S_\rho = L^G = (\text{Pic}(\mathbb{P}^1 \times \mathbb{P}^1))(3) = H(3) \quad \text{and} \quad T_\rho = (L^G)^\perp = H \oplus H(3) \oplus E_8^2.
\]

Let \( \overline{Y} \) be a degeneration of the quadric \( \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3 \) to a quadratic cone and \( \overline{X} \to \overline{Y} \) be the \( \mu_3 \)-cover branched in a curve \( B \in |\mathcal{O}_{\overline{Y}}(3)| \) not passing through the apex. Let \( Y = \mathbb{P}_2 \) and \( X \) be the minimal resolutions of \( \overline{Y} \) and \( \overline{X} \). The \( \mathbb{P}^1 \)-fibration on \( Y \) gives an elliptic fibration on \( X \), and the preimage of the \( (-2) \)-section of \( Y \) is a union of three disjoint \( (-2) \)-sections \( e, e', e'' \) on \( X \), interchanged by the automorphism \( \sigma \). The invariant sublattice \( S'_x = (\text{Pic}(\mathbb{F}_2))(3) = H(3) \) is generated by \( f \) and \( f' = f + \sum_{i=0}^2 \sigma^i e \).

The only \( (-2) \)-curves on \( X \) are \( \sigma^i e \) and they do not lie in \( (S'_x)^\perp \). Thus, once we fix a marking \( \phi \), the period \( x \) of \( X \) will be in \( \mathbb{D}_\rho \setminus \Delta_\rho \). The reflections \( w_i \) in the roots \( \rho' \phi(e) \) commute. Their product \( w = w_0 w_1 w_2 \) is non-trivial: on \( L^G \) it acts as the reflection that interchanges \( \phi(f) \) and \( \phi(f') \). It is easy to check that \( w \in \Gamma_\rho \). So \( \Gamma_\rho \cap W_x \neq 1 \) and \( W(L^G) = 1 \).
Thus, the map $\mathcal{M}_\rho \to \mathbb{D}_\rho \setminus \Delta_\rho$ is not separated in this case. Locally it looks like the “double-headed snake” $\mathbb{A}^1 \cup_{\mathbb{A}^1 \setminus 0} \mathbb{A}^1 \to \mathbb{A}^1$ times $\mathbb{A}^8$. Here is another way to see the same. The positive cone $P$ in $H(3)_\mathbb{R}$ is the unique Weyl chamber for the Weyl group $W(H(3)) = 1$; its rays are $\phi(f)$ and $\phi(f')$. The hyperplane $\phi(e)^\perp$ cuts it in half. The intersections of the Weyl chambers $C \subset P_x \setminus \delta^\perp$ of Equation 1 with $P$ are either halves of $P$.

**Theorem 2.9.** The moduli stack $F_\rho$ of $\rho$-markable K3 surfaces with a non-symplectic automorphism is the quotient $F_\rho = \mathcal{M}_\rho/\Gamma_\rho$. Its coarse moduli space admits a bijective period map to $(\mathbb{D}_\rho \setminus \Delta_\rho)/\Gamma_\rho$, and the coarse moduli space of the separated quotient $F_\rho^{\text{sep}}$ is $(\mathbb{D}_\rho \setminus \Delta_\rho)/\Gamma_\rho$. The generic stabilizer is the group

$$K_\rho := \ker(\Gamma_\rho \to \text{Aut}(\mathbb{D}_\rho))/\Gamma_\rho \cap (\mathbb{Z}_2 \times W(S_\rho))$$

**Proof.** The statement is immediate from the definitions and the above two Lemmas by quotienting the period map $\pi_\rho$. The points of $\pi_\rho^{-1}(x)$ are permuted by $\Gamma_\rho$, thus they are identified in the $\Gamma_\rho$-quotient. They are also identified in the separated quotient.

For $\rho$ to correspond to any K3 surface with a nonsymplectic automorphism, $S_\rho$ must have signature $(1, r - 1)$ for some $r$, and for $T_\rho$ to have signature $(20 - r)$. The action of $\Gamma_\rho$ on the Type IV domain $\mathbb{D}(T_\rho)$ factors through $O(T_\rho)$ and is therefore properly discontinuous. Thus, the action of $\Gamma_\rho$ on $\mathbb{D}_\rho$ is properly discontinuous, and so $(\mathbb{D}_\rho \setminus \Delta_\rho)/\Gamma_\rho$ is makes sense as a complex-analytic space. (It is also quasiprojective by Baily-Borel.)

The last statement follows from Lemma 2.6(2) by noting that for a generic $x \in \mathbb{D}_\rho \setminus \Delta_\rho$ one has $x^\perp \cap L = S_\rho$. \hfill \Box

**Remark 2.10.** The proof of part (1) of Lemma 2.6 and of Theorem 2.9 follow the arguments of Dolgachev-Kondo [DK07, Thms. 11.2, 11.3]. Sections 10 and 11 of [DK07] contain a construction of the moduli space of K3 surfaces with a non-symplectic automorphism that is based on moduli of lattice polarized K3s. But it uses [Dol96, Thm. 3.1] which unfortunately is false, as was noted in [AE21] and as Example 2.8 also shows. For this reason, we decided to give an alternative construction.

**Remark 2.11.** Even though the map to $(\mathbb{D}_\rho \setminus \Delta_\rho)/\Gamma_\rho$ in Theorem 2.9 is bijective, the coarse moduli space of $F_\rho$ is a non-separated algebraic space when $\mathcal{M}_\rho$ is not separated. This is very similar to the algebraic space obtained by dividing a “two-headed snake” $\mathbb{A}^1 \cup_{\mathbb{A}^1 \setminus 0} \mathbb{A}^1$ by the involution $z \to -z$ exchanging the heads. The quotient is a non-separated algebraic space with a bijection to $\mathbb{A}^1 = \mathbb{A}^1/\pm$.

We note that the separated quotient $F_\rho^{\text{sep}}$ is a stack $[\mathbb{D}_\rho \setminus \Delta_\rho : W \Gamma_\rho]$ which can be locally constructed near $x \in \mathbb{D}_\rho \setminus \Delta_\rho$ by first taking a coarse quotient by the normal subgroup $\Gamma_\rho \cap (\mathbb{Z}_2 \times W_x) \leq \text{Stab}_x(\Gamma_\rho)$ and then taking the stack quotient by $\text{Stab}_x(\Gamma_\rho)/\Gamma_\rho \cap (\mathbb{Z}_2 \times W_x)$. See [AE21, Rem. 2.36].

**Proposition 2.12.** Suppose $\sigma \in \text{Aut}(X)$ fixes a curve $R$ of genus at least 2, i.e. the assumption $(\exists g \geq 2)$ holds. Then $\text{Aut}(X, \sigma)$ is finite.

**Proof.** Let $h \in \text{Aut}(X, \sigma)$ be an automorphism of $X$ satisfying $h \circ \sigma = \sigma \circ h$. Then $h$ permutes the fixed components of $\sigma$. Since there is at most one component $R$ of genus $g \geq 2$, we conclude $h(R) = R$. Hence $h \in \text{Aut}(X, \mathcal{O}(R))$, a finite group. \hfill \Box
Note that generic stabilizer $K_\rho$ from Theorem 2.9 is never the trivial group, as $\rho \in K_\rho$ is a nontrivial element. As this is the automorphism group of a generic element $(X, \sigma) \in F_\rho$, if $g \geq 2$ holds then $K_\rho$ is finite by Proposition 2.12.

Example 2.13. Consider the double cover $\pi: X \to \mathbb{P}^2$ branched over a smooth sextic $B$. There is a non-symplectic involution $\sigma$ switching the two sheets of $X$, acting on $H^2(X, \mathbb{Z})$ by fixing $h = c_1(\pi^*\mathcal{O}(1))$ and negating $h^\perp$. Choosing a model $\rho$ for the action of $\sigma^*$ on cohomology, we have that $S_\rho = \langle 2 \rangle$ and $T_\rho = (-2) \oplus H^\perp \oplus E_8^{\oplus 2}$ are the $(+1)$- and $(-1)$-eigenspaces, respectively.

The divisor $\Delta_\rho/\Gamma_\rho \subset \mathbb{P}^1/\Gamma_\rho = F_2$ has two irreducible components corresponding to $\Gamma_\rho$-orbits of roots $\delta \in (T_\rho)^\perp$. Such an orbit is uniquely determined by the divisibility (1 or 2) of $\delta \in T_\rho^\perp$. The case where the divisibility is 2 corresponds to when $B$ acquires a node. Then there is an involution $\sigma$ on the minimal resolution of the double cover $X \to \overline{X} \to \mathbb{P}^2$, but $\sigma^*(\delta) = \delta$, $\sigma^*(h) = h$ and the $(+1, -1)$-eigenspaces of $\sigma^*$ have dimensions $(2, 20)$. Thus, no $\rho$-marking can be extended over a family $\mathcal{X} \to C$ with central fiber $X$ and general fiber as above.

When the divisibility of $\delta$ is 1, $\mathbb{P}^2$ degenerates to $\mathbb{P}^1_4 = \mathbb{P}(1, 1, 4)$ and the minimal resolution of the double cover $X \to \overline{X} \to \mathbb{P}^1_4$ is an elliptic K3 surface with $\sigma$ the elliptic involution. Again the eigenspaces have dimension profile $(2, 20)$ and so $(X, \sigma)$ is not $\rho$-markable for the $\rho$ as above.

3. Stable pair compactifications

3A. Complete moduli of stable slc pairs. We refer the reader to [ABE20, Sec. 2B] and [AE21, Sec. 7D] for a detailed discussion of stable K3 surface pairs and their compactified moduli. Briefly:

Definition 3.1. In our context, a stable slc surface pair is a pair $(S, \epsilon D)$, where

1. $S$ is a connected, reduced, projective Gorenstein surface $S$ with $\omega_S \simeq \mathcal{O}_S$ which has semi log canonical singularities.

2. $D$ is an effective ample Cartier divisor on $S$ that does not contain any log canonical centers of $S$.

Then for sufficiently small rational number $\epsilon > 0$ the pair $(S, \epsilon D)$ is stable, meaning:

1. it has semi log canonical singularities, and
2. the $\mathbb{Q}$-Cartier divisor $K_S + \epsilon D$ is ample.

“Sufficiently small” works in families: for a fixed $D^2$ there exists $\epsilon_0$ so that if a pair $(S, \epsilon D)$ is stable in the above definition for some $\epsilon$ then it is stable for any $0 < \epsilon \leq \epsilon_0$.

The main application to K3 surfaces is an observation that for any K3 surface $\overline{X}$ with ADE singularities and an effective ample divisor $\overline{R}$, the pair $(\overline{X}, \epsilon \overline{R})$ is stable. Indeed, $\omega_{\overline{X}} \simeq \mathcal{O}_{\overline{X}}$, the surface $\overline{X}$ has canonical singularities—which is much better than semi log canonical—and there are no log centers.

As usual, let $F_{2d}$ denote the moduli space of polarized K3 surfaces $(\overline{X}, \overline{L})$ with ADE singularities and ample primitive line bundle $\overline{L}$ of degree $\overline{L}^2 = 2d$, and $P_{2d, m} \to F_{2d}$ denote the moduli space of pairs $(\overline{X}, \epsilon \overline{R})$ with an effective divisor $\overline{R} \in |m \overline{L}|$. Then the main result for K3 surfaces is the following:

Theorem 3.2. (1) For the stable pairs as above there exists an algebraic Deligne-Mumford moduli stack $\mathcal{M}_{\text{slc}}$, with a coarse moduli space $\mathcal{M}_{\text{slc}}$. 
(2) The closure $P_{2d,m}^{\text{sep}}$ of $P_{2d,m}$ in $M^{\text{slc}}$ is projective and provides a compactification of $P_{2d,m}$ to a moduli space of stable slc pairs.

To apply this result to a compactification of $F_{\rho}^{\text{sep}}$ one needs to choose, in a canonical manner, a big and nef divisor on the generic $(X, \sigma) \in F_{\rho}$.

**Definition 3.3.** A canonical choice of polarizing divisor is an algebraically varying big and nef divisor $R$ defined over a Zariski dense subset $U \subset F_{\rho}$ of the moduli space of $\rho$-markable K3 surfaces.

3B. Stable pair compactification of $F_{\rho}^{\text{sep}}$. We apply Theorem 3.2 to construct a stable pair compactification in the present context as follows.

Suppose that for each surface $(X, \sigma) \in F_{\rho}$ assumption $(\exists g \geq 2)$ holds, i.e. the fixed locus Fix($\sigma$) contains a component $C_1$ of genus $g \geq 2$, as well as possibly several smooth rational curves $C_i$ and some isolated points. In fact, it suffices that a single $(X, \sigma) \in F_{\rho}$ satisfies assumption $(\exists g \geq 2)$ because the genus of $C_1$ is constant in a family of smooth K3 surfaces with non-symplectic automorphism. So $R = C_1$ gives a canonical choice of polarizing divisor for all of $U = F_{\rho}$.

Let $\pi : X \to \overline{X}$ be the contraction to an ADE K3 surface such that the divisor $\overline{R} := \pi(C_1)$ is ample; it has degree $\overline{R}^2 = 2g(C_1) - 2 > 0$. It provides us with an ample divisor on $\overline{X}$. If $\mathcal{O}(\overline{R}) = \overline{L}$ for a primitive $\overline{L}$ then the pair $(\overline{X}, \mathcal{O}(\overline{R}))$ is a point of $F_{2d,m}$, and the pair $(\overline{X}, \epsilon \overline{R})$ is a point of $P_{2d,m}$.

**Definition 3.4.** We define the map $\psi : F_{\rho} \to P_{2d,m}$ as follows. Pointwise, it sends $(X, \sigma)$ to $(\overline{X}, \epsilon \overline{R})$. In every flat family $f : X \to S$ of K3 surfaces with automorphism, the sheaf $\mathcal{O}_X(\overline{R})$ is relatively big and nef. Since $R^iL^d = 0$ for $i > 0$, $d > 0$, it gives a contraction to a flat family $\overline{f} : (\overline{X}, \overline{R}) \to S$. This induces the map on moduli.

**Lemma 3.5.** The map $\psi : F_{\rho} \to P_{2d,m}$ defined above induces an injective map $F_{\rho}^{\text{sep}} \to \text{im}(\psi)$.

**Proof.** The map $\psi$ factors through the separated quotient of $F_{\rho}$ because $P_{2d,m}$ is separated. Now suppose there is an isomorphism of pairs $\overline{f} : (X_1, R_1) \to (X_2, R_2)$ inducing an isomorphism of the minimal resolutions $f : (X_1, R_1) \to (X_2, R_2)$. Consider the morphism $\varphi = \sigma_1^{-1}f^{-1}\sigma_2f$. Then $\varphi$ is a symplectic automorphism of $X_1$ fixing the curve $R_1$ pointwise. Since $\varphi$ preserves $\mathcal{O}_{X_1}(R_1)$, it has finite order. By [Nik79a] the fixed set of a finite order symplectic K3 automorphism is finite. Thus, $\varphi = \text{id}$ and $f$ preserves the group action. So, $(X, \sigma)$ is uniquely determined by $(\overline{X}, \overline{R})$. \hfill $\Box$

**Remark 3.6.** $F_{\rho}^{\text{sep}}$ itself has a moduli interpretation: It is the moduli space $F_{\rho}^{\text{slc}}$ of ADE K3 surfaces $(\overline{X}, \overline{\sigma})$ with automorphism, for which Fix$(\overline{\sigma})$ is ample, and for which the minimal resolution $(X, \sigma) \to (\overline{X}, \overline{\sigma})$ is $\rho$-markable.

**Definition 3.7.** Let $Z = \text{im}(\psi)$ and let $\overline{Z}$ be its closure in $P_{2d,m}^{\text{slc}}$, with reduced scheme structure. The stable pair compactification

$$F_{\rho}^{\text{sep}} = F_{\rho}^{\text{slc}} \hookrightarrow F_{\rho}^{\text{slc}}$$

is defined as the normalization of $\overline{Z}$.

In particular, $F_{\rho}^{\text{slc}}$ is normal by definition. Points correspond to the pairs $(\overline{X}, \epsilon \overline{R})$, possibly degenerate, with some finite data.
3C. Kulikov degenerations of K3 surfaces. A basic tool in the study of degenerations of K3 surfaces is Kulikov models. We use them in the argument below, so we briefly recall the definition.

Let $(C,0)$ denote the germ of a smooth curve at a point $0 \in C$ and let $C^* = C \setminus 0$. Let $X^* \to C^*$ be a family of algebraic K3 surfaces.

**Definition 3.8.** A Kulikov model $X \to (C,0)$ is an extension of $X^* \to C^*$ for which $X$ is a smooth algebraic space, $K_X \sim_C 0$, and $X_0$ has reduced normal crossings.

We say the $X$ is Type I, II, or III, respectively, depending on whether $X_0$ is smooth, has double curves but no triple points, or has triple points, respectively. We call the central fiber $X_0$ of such a family a Kulikov surface.

A key result on the degenerations of K3 surfaces is the theorem of Kulikov [Kul77] and Persson-Pinkham [PP81]:

**Theorem 3.9.** Let $Y^* \to C^*$ be a family of algebraic K3 surfaces. Then there is a finite base change $(C',0) \to (C,0)$ and a sequence of birational modifications of the pull back $Y' \to X$ such that $X$ has smooth total space, $K_X \sim_{C'} 0$, and $X_0$ has reduced normal crossings.

We recall some fundamental results about Kulikov models. The primary reference is [FS86]. Let $T : H^2(X_t, \mathbb{Z}) \to H^2(X_t, \mathbb{Z})$ denote the Picard-Lefschetz transformation associated to an oriented simple loop in $C^*$ enclosing 0. Since $X_0$ is reduced normal crossings, $T$ is unipotent. Let

$$N := \log T = (T - I) - \frac{1}{2}(T - I)^2 + \cdots$$

be the logarithm of the monodromy.

**Theorem 3.10.** [FS86][Fri84] Let $X \to (C,0)$ be a Kulikov model. We have that

- if $X$ is Type I, then $N = 0$,
- if $X$ is Type II, then $N^2 = 0$ but $N \neq 0$,
- if $X$ is Type III, then $N^3 = 0$ but $N^2 \neq 0$.

The logarithm of monodromy is integral, and of the form $Nx = (x \cdot \lambda)\delta - (x \cdot \delta)\lambda$ for $\delta \in H^2(X_t, \mathbb{Z})$ a primitive isotropic vector, and $\lambda \in \delta^\perp/\delta$ satisfying

$$\lambda^2 = \#\{\text{triple points of } X_0\}.$$  

When $\lambda^2 = 0$, its imprimitivity is the number of double curves of $X_0$.

Thus, the Types I, II, III of Kulikov model are distinguished by the behavior of the monodromy invariant $\lambda$: either $\lambda = 0$, $\lambda^2 = 0$ but $\lambda \neq 0$, or $\lambda^2 \neq 0$ respectively.

**Definition 3.11.** Let $J \subset H^2(X_t, \mathbb{Z})$ denote the primitive isotropic lattice $\mathbb{Z}\delta$ in Type III or the saturation of $\mathbb{Z}\delta \oplus \mathbb{Z}\lambda$ in Type II.

3D. Baily-Borel compactification. Let $N$ be a lattice of signature $(2,\ell)$, together with an isometry $\rho \in O(N)$ of finite order $n$, such that all eigenvalues of $\rho$ on $N_C$ are primitive $n$th roots of unity, and $N_C^{\perp_n}$ contains a vector $x$ of positive Hermitian norm $x \cdot \bar{x}$. This is the situation which arises for a non-symplectic automorphism of an algebraic K3 surface, with $N = T_\rho$. Then we have a Type IV domain

$$D_N = \mathbb{P}\{x \in N_C \mid x \cdot x = 0, \ x \cdot \bar{x} > 0\}$$

For $n = 2$ one has $D_\rho = D_N$. For $n > 2$ one has a Type I subdomain of $D_N$.

$$D_\rho = \mathbb{P}\{x \in N_C^{\perp_n} \mid x \cdot \bar{x} > 0\}$$
Thus, Baily-Borel \([BB66]\). For Type IV domains \(\mathbb{D}_N\) and \(\mathbb{D}_\rho\) embed into their compact duals \(\mathbb{D}_N^\circ, \mathbb{D}_\rho^\circ\), which are defined by dropping the condition that \(x \cdot \bar{x} > 0\). Define \(\mathbb{D}_N \subset \mathbb{D}_N^\circ, \mathbb{D}_\rho \subset \mathbb{D}_\rho^\circ\) as their topological closures. One has a well known description of the rational boundary components of \(\mathbb{D}_N\), see e.g. see [Loo03b].

**Definition 3.12.** A rational boundary component of \(\mathbb{D}_N\) is an analytic subset \(B_J \subset \mathbb{D}_N\) of the form:

1. \((\mathbb{P}J_C \setminus \mathbb{P}J_R) \cap \mathbb{D}_N\) for \(\text{rk} J = 2\) a primitive isotropic sublattice of \(N\),
2. \(\mathbb{P}J_C \cap \mathbb{D}_N\) for \(\text{rk} J = 1\) a primitive isotropic sublattice of \(N\).

The rational boundary components of \(\mathbb{D}_\rho\) are intersections of \(B'_J = B_J \cap \mathbb{D}_\rho\).

One defines the rational closure of \(\mathbb{D}_\rho\) to be \(\mathbb{D}_\rho^{bb} := \mathbb{D}_\rho \cup B'_J\), topologized via a horoball topology at the boundary. Then the Baily-Borel compactification of \(\mathbb{D}_\rho/\Gamma\) is (at least topologically) \(\mathbb{D}_\rho/\Gamma^{bb} := \mathbb{D}_\rho^{bb} / \Gamma\).

The space \(\mathbb{D}_\rho/\Gamma^{bb}\) was shown to have the structure of a projective variety by Baily-Borel [BB66]. For Type IV domains \(\mathbb{D}_N\) and \(\mathbb{D}_\rho\) if \(n = 2\), the boundary components (1) are isomorphic to \(\mathbb{H} \sqcup (-\mathbb{H})\) and the boundary components (2) are points. For \(n > 2\), the boundary components of the Type I domain \(\mathbb{D}_\rho\) are points. If \(\text{rk} J = 2\) then a point \([x] \in B_J\) corresponds to the elliptic curve \(E_x = J_C(J + \mathbb{C}x)\).

**Lemma 3.13.** In the case \(n > 2\), for each boundary component \(B'_J\) we necessarily have \(\text{rk} J = 2\) and \(n \in \{3,4,6\}\), and \(x \in B'_J\) corresponds to the elliptic curve with \(j(E_x) = 0\) if \(n = 3\) or 6, and with \(j(E_x) = 1728\) if \(n = 4\).

**Proof.** If \(B'_J\) is boundary component of \(\mathbb{D}_\rho\) then \(N^*_C \cap J_C \neq 0\). Since \(J\) is defined over \(\mathbb{Z}\) and \(\zeta_n \notin \mathbb{R}\), then \(N^*_C \cap J_C \neq 0\) as well. This implies that \(\text{rk} J = 2\) and 

\[J_C = J^*_C \oplus J^*_C.\]

Thus, \(\rho(J_C) = J_C\), implying that \(\rho(J) = J\). Therefore \(\rho|_J \in \text{GL}(J) \cong \text{GL}_2(\mathbb{Z})\) necessarily has order \(n\). Thus, \(n \in \{3,4,6\}\). For a point \([x] \in B'_J\) one has \(x \in N^*_C\) and \(\mu_n \subset \text{Aut}(E_x)\). This determines \(E_x\). \(\square\)

**Corollary 3.14.** If \(n \neq 2,3,4,6\) then the rational closure of \(\mathbb{D}_\rho\) is simply \(\mathbb{D}_\rho\) itself. So \(\mathbb{D}_\rho/\Gamma\) is already compact.

The following is a well-known consequence of Schmid’s nilpotent orbit theorem:

**Proposition 3.15.** Let \(X^* \rightarrow C^*\) be a degeneration of a \(\rho\)-markable K3 surfaces over a punctured analytic disk \(C^*\). A lift of the period mapping \(\overline{C}^* \cong \mathbb{H} \rightarrow \mathbb{D}_\rho\) approaches the Baily-Borel cusp \(B_J\) as \(\text{Im}(\tau) \rightarrow \infty\), where \(J\) is the monodromy lattice in \(H^2(X_0, \mathbb{Z})\), cf. Definition 3.11. When \(\text{rk} (J) = 2\), the limiting point \(x \in B_J\) corresponds to an elliptic curve \(E_x\) isomorphic to any double curve of the central fiber \(X_0\) of a Kulikov model \(X \rightarrow C\).

**Corollary 3.16.** If \(n \neq 2,3,4,6\), any degeneration of \((X,\sigma) \in F_\rho\) has Type I. If \(n \in \{3,4,6\}\), any degeneration of \((X,\sigma) \in F_\rho\) has Type I or II.

The last statement was also proved by Matsumoto [Mat16] using different techniques. His proof also holds in some prime characteristics.
3E. Semitoroidal compactifications. Semitoroidal compactifications of arithmetic quotients $\mathbb{D}/\Gamma$ for type IV Hermitian symmetric domains $\mathbb{D}$ were defined by Looijenga [Loo03b] (where they were called “semitoric”). They simultaneously generalize toroidal and Baily-Borel compactifications of $\mathbb{D}/\Gamma$. The case of the complex ball $\mathbb{D}$ (a type I symmetric Hermitian domain) is comparatively trivial. The semitoroidal compactifications in this case are implicit in [Loo03a, Loo03b]. We quickly overview the construction in both cases now.

**Definition 3.17.** A $\Gamma$-admissible semifan $\mathfrak{F}$ consists of the following data:

When $n = 2$, it is a convex, rational, locally polyhedral decomposition $\mathfrak{F}_J$ of the rational closure $C^+(J^+/J)$ of the positive norm vectors, for all rank 1 primitive isotropic sublattices $J \subset N$, such that:

1. $\{\mathfrak{F}_J\}_{J \subset N}$ is $\Gamma$-invariant. In particular, a fixed $\mathfrak{F}_J$ is invariant under the natural action of $\text{Stab}_J(\Gamma)$ on $C^+(J^+/J)$.
2. A compatibility condition of the $\{\mathfrak{F}_J\}_{J \subset N}$ along any primitive isotropic lattice $J' \subset N$ of rank 2 holds, see Definition 3.18.

When $n > 2$, the data is much simpler: It consists, for each primitive isotropic sublattice $J \subset N$ satisfying $J \cap N^C \neq \emptyset$, of a primitive sublattice $\mathfrak{F}_J \subset J^+/J$ such that the collection $\{\mathfrak{F}_J\}$ is $\Gamma$-invariant.

**Definition 3.18.** Let $J' \subset N$ be primitive isotropic of rank 2. We say that the collection $\{\mathfrak{F}_J\}_{J \subset N}$ is compatible along $J'$ if, given any primitive sublattice $J \subset J'$ of rank 1, the kernel of the hyperplanes of $\mathfrak{F}_J$ containing $J'/J$, when intersected with $(J')^+/J \subset J^+/J$ and then descended to $(J')^+/J'$, cut out a fixed sublattice $\mathfrak{F}_{J'} \subset (J')^+/J'$ which is independent of $J$.

In both the $n = 2$ and $n > 2$ cases, we use the same notation $\mathfrak{F} := \{\mathfrak{F}_J\}_{J \subset N}$ even though $J$ ranges over rank 1 isotropic sublattices when $n = 2$ and ranges over rank 2 isotropic sublattices when $n > 2$.

In the Type IV case, Looijenga constructs a compactification $\mathbb{D}/\Gamma \rightarrow \overline{\mathbb{D}/\Gamma}^\mathfrak{F}$ for any $\Gamma$-admissible semifan $\mathfrak{F}$, so consider the Type I case. By Lemma 3.13 we may restrict to $n \in \{3, 4, 6\}$. There is a $\mathbb{Z}[\zeta_n]$-lattice

$$Q := (N \otimes \mathbb{Z}[\zeta_n])^{\zeta_n} \subset N^{\zeta_n} = Q_C$$

on which Hermitian form $x \cdot \bar{y}$ defines a $\mathbb{Z}[\zeta_n]$-valued Hermitian pairing of signature $(1, \ell)$ for some $\ell$. Any element of $\tilde{\Gamma}_\rho$ (in particular, any element of $\Gamma$) preserves $Q$ and the Hermitian form on it. The converse also holds. Thus $\Gamma \subset U(Q)$ is a finite index subgroup of the group of unitary isometries of $Q$ and $\Gamma_R = U(Q_C) = U(1, \ell)$.

The boundary components $B_J = \mathbb{P}(J^{\zeta_n})$ are then projectivizations of the isotropic $\mathbb{Z}[\zeta_n]$-lines $K \subset Q$. Here $K_C = J_C^{\zeta_n}$.

Choose a generator $k \in K$. Then any $x \in \mathbb{D}_\rho \subset \mathbb{P}Q_C$ has a unique representative $x \in Q_C$ for which $k \cdot x = 1$. This realizes $\mathbb{D}_\rho$ as a generalized tube domain in the affine hyperplane $V_k := \{k \cdot x = 1\} \subset Q_C$.

Let $U_K \subset \text{Stab}_K(\Gamma)$ be the unipotent subgroup (i.e. $U_K$ acts on $K$, $K^\perp/K$, and $Q/K^\perp$ by the identity). Then $U_K$ acts on $V_k$ by translations. Choosing some isotropic $k' \in Q_C$ for which $k' \cdot k = 1$, any element $x \in V_k$ can be written uniquely as $x = k' + x_0 + cxk$ for some $x_0 \in \{k, k'\}^\perp$ and $c \in \mathbb{C}$. The image of $\mathbb{D}_\rho$ is exactly those $x$ satisfying $2\text{Re}(c) > -x_0 \cdot \bar{x}_0$. 
The fibration $D_\rho \to K_C^\perp/K_C$ sending $x \mapsto x_0 \mod K_C$ is a fibration of right half-planes. The action of $U_K$ fibers over the action of a translation subgroup $U_K \subset K_\perp/K$ on $K_C^\perp/K_C$ and thus, there is a fibration

$$D_\rho/U_K \to (K_C^\perp/K_C)/U_K =: A_K$$

over an abelian variety. The fibers are quotients of the right half-planes with coordinate $c$ by a discrete, purely imaginary, translation group isomorphic to $\mathbb{Z}$. This realizes $D_\rho/U_K$ is a punctured holomorphic disc bundle over $A_K$.

**Definition 3.19.** $D_\rho/U_K$ is the first partial quotient associated to the Baily-Borel cusp $K$. The extension of this punctured disc bundle to a disc bundle $D_\rho/\Gamma$ can $\to A_K$ for a given $K$ is called the toroidal extension at the cusp $K$.

We will identify the divisor at infinity, i.e. the zero section of the disc bundle, with $A_K$ itself.

**Construction 3.20.** The toroidal compactification of $D_\rho/\Gamma$ is constructed as follows: Let $\Gamma_K$ be the finite group defined by the exact sequence

$$0 \to U_K \to \text{Stab}_K(\Gamma) \to \Gamma_K \to 0.$$ 

For each cusp $K$, quotient the toroidal extension $V_K := D_\rho/\Gamma_K \supset D_\rho/\Gamma \cap (1 + F_K^*)$. A well-known theorem states that there exists a horoball neighborhood $P_K^\perp \subset N_K \subset D_{bb}$ such that $(N_K \setminus P_K^\perp)/\Gamma_K$ injects. Thus, we can glue a neighborhood of the boundary $A_K/\Gamma_K \subset V_K$ to $D_\rho/\Gamma$, ranging over all $\Gamma$-orbits of cusps $K$. The result is the toroidal compactification $D_\rho/\Gamma_{tor}$.

The boundary divisors of $D_\rho/\Gamma_{tor}$ are in bijection with $\Gamma$-orbits of isotropic $\mathbb{Z}[\zeta_n]$-lines $K \subset \mathbb{Q}$ and the boundary divisor is isomorphic to $A_K/\Gamma_K$, where $\Gamma_K$ acts by a subgroup of the finite group $U(K_\perp/K)$. There is a morphism

$$\overline{D_\rho/\Gamma_{tor}} \to \overline{D_\rho/\Gamma_{bb}}$$

which contracts each boundary divisor to a point. As such, the normal bundle of the boundary divisor is anti-ample. Passing to a finite index subgroup $\Gamma_0 \subset \Gamma$, we can assume that $\Gamma_K$ is trivial for all cusps $K$ and the anti-ampleness still holds. This proves that the normal bundle to $A_K \subset \overline{D_\rho/\Gamma_{can}}$ in the first partial quotient is anti-ample.

Using [Gra62] one shows that a divisor in a smooth analytic space, isomorphic to an abelian variety and with anti-ample normal bundle, can be contracted along any abelian subvariety. In particular, for any sub-$\mathbb{Z}[\zeta_n]$-lattice $\mathfrak{f}_K \subset K_\perp/K$, there is a contraction

$$\overline{D_\rho/\Gamma_{can}} \to \overline{D_\rho/\Gamma_{\mathfrak{f}K}}$$

which is an isomorphism away from the boundary divisor and contracts exactly the translates of the abelian subvariety $\text{im}(\mathfrak{f}_K) \subset A_K$.

To construct the semitoroidal compactification $\overline{D_\rho/\Gamma_{\mathfrak{f}K}}$, we wish to glue, at each cusp $K$, a punctured analytic open neighborhood of the boundary of $\overline{D_\rho/\Gamma_{K_{\mathfrak{f}K}}} \to A_K$ to $\overline{D_\rho/\Gamma}$. This is only possible if the action of $\Gamma_K$ on $\overline{D_\rho/\Gamma_{can}}$ descends along the above contraction. The condition in Definition 3.17 ensures that the collection
admits a birational automorphism which is the action of the automorphism \( \sigma \)

Theorem 3.23. A feature of the construction is that one can pull back a semifan \( \mathfrak{F} \) for a Type IV domain to any Type I subdomain, and there will be a morphism between the corresponding semitoric compactifications.

3F. Recognizable divisors. We recall the main new concept “recognizability” introduced in [AE21]. We slightly modify the definition as necessary for moduli spaces of K3 surfaces with \( \rho \)-markable automorphism:

Definition 3.22. A canonical choice of polarizing divisor \( R \) for \( U \subset F_{\rho} \) is recognizable if for every Kulikov surface \( X_0 \) of Type I, II, or III which smooths to some \( \rho \)-markable K3 surface, there is a divisor \( R_0 \subset X_0 \) such that on any smoothing into \( \rho \)-markable K3 surfaces \( X \to (C, 0) \) with \( C^* \subset U \), the divisor \( R_0 \) is, up to the action of \( \text{Aut}^{0}(X_0) \), the flat limit of \( R_t \) for \( t \neq 0 \in C^* \).

We use the term “smoothing” to mean specifically a Kulikov model \( X \to (C, 0) \).

Roughly, Definition 3.22 amounts to saying that the canonical choice \( R \) can also be made on any Kulikov surface, including smooth K3s.

Theorem 3.23. If \( R \) is recognizable, then \( F_{\rho}^{\text{slc}} \) is semitoroidal compactification of \( F_{\rho} \) for a unique semifan \( \mathfrak{F}_R \).

Proof. The proof when \( n = 2 \) is essentially the same as [AE21, Thm. 1.2]. So we restrict our attention to the Type I case \( n > 2 \), which is ultimately much simpler anyways. First, we show that \( F_{\rho}^{\text{slc}} \) contains \( D_{\rho}/\Gamma_{\rho} \).

Let \( M_{\rho}^* \) be the closure of the moduli space of \( \rho \)-marked K3 surfaces \( M_{\rho} \) in the space of all marked K3 surfaces \( M \) and let \( F_{\rho}^* = M_{\rho}^*/\Gamma_{\rho} \) be the quotient. Given any smooth K3 surface \( X_0 \in F_{\rho}^* \), the recognizability implies that the universal family \((X^*, R^*) \to U \) extends over \( F_{\rho}^* \) by the same argument as [AE21, Prop. 6.3]. Thus, the argument of Lemma 3.5 shows that there is a morphism \((F_{\rho}^*)^{\text{sep}} = D_{\rho}/\Gamma_{\rho} \to P_{2d,m}\) and so we may as well have constructed \( F_{\rho}^{\text{slc}} \) by taking the normalization of the closure of the image of \( D_{\rho}/\Gamma_{\rho} \), which is notably already normal. This completes the proof when \( n \neq 3, 4, 6 \).

So let \( \mathbb{P}K_{\mathbb{C}} \) be a Baily-Borel cusp of \( D_{\rho} \) when \( n \in \{3, 4, 6\} \). We observe that the closure of \( D_{\rho}/U_K \) in the toroidal extension \( D(J) \subset D(J)^{\lambda} \) of the “universal” first partial quotient for unpolarized K3 surfaces, cf. [AE21, Def. 4.18], is simply the first partial quotient \( D_{\rho}/U_K^{\text{can}} \). [AE21, Prop. 4.16] shows that \( D(J) \) embeds into a family of affine lines over \( J^{\perp}/J \otimes_{\mathbb{Z}} \tilde{E} \) where \( \tilde{E} \) is the universal elliptic curve over \( \mathbb{H} \cup (-\mathbb{H}) \) and \( D(J)^{\lambda} \) is its closure in a projective line bundle. The space \( D_{\rho}/U_K \) sits inside this affine line bundle as the inverse image of

\[
K^{\perp} \text{ in } Q/K \otimes_{\mathbb{Z}[\zeta_n]} E \subset J^{\perp}/J \otimes_{\mathbb{Z}} \tilde{E}
\]

where \( E \) is the elliptic curve admitting an action of \( \zeta_n \) (note that \( K = J \) but with the additional structure of a \( \mathbb{Z}[\zeta_n] \)-lattice).

Thus we may restrict a Type II \( \lambda \)-family, cf. [AE21, Def. 5.34], to a family

\[
\mathcal{X} \to D_{\rho}/U_K^{\text{can}}
\]

of Kulikov surfaces of Types I + II. We call \( \mathcal{X} \) a \( K \)-family. Note that any \( K \)-family admits a birational automorphism which is the action of the automorphism \( \sigma \) on the restriction of \( \mathcal{X} \) to \( (D_{\rho} \setminus \Delta_{\rho})/U_K \).
The arguments in [AE21, Secs. 6,8], leading up to the proof of Theorem 1.2 of loc. cit. now all apply to $K$-families $\mathcal{X}$, showing that there is a sandwich of normal compactifications
\[
\overline{D_\rho}/\Gamma_\rho \dashv \overline{F}_\rho^{\text{slc}} \to \overline{D_\rho}/\Gamma_\rho^{\text{bb}}.
\]
Using that the normal image of an abelian variety is an abelian variety (a similar argument is used in [AE21, Thm. 7.18]), we conclude that there must exist a $\Gamma_\rho$-admissible semifan $\mathcal{G}_R$ for which $\overline{F}_\rho^{\text{slc}} = \overline{D_\rho}/\Gamma_\rho^{\mathcal{G}_R}$. $\square$

3G. The main theorem.

**Theorem 3.24.** Under the assumption $(\exists g \geq 2)$, $R = C_1$ is recognizable for $F_\rho$. The stable pair compactification $\overline{F}_\rho^{\text{slc}}$ is a semitoroidal compactification of $D_\rho/\Gamma_\rho$.

**Proof.** By Theorem 3.23, the second statement follows from the first. Let $(X, R) \to (C, 0)$ be a Kulikov model with a flat family of divisors $R \subset X$ for which

1. there is an automorphism $\sigma$ on $X^* \to C^*$ making $(X_t, \sigma_t) \in F_\rho$ for $t \neq 0$,
2. $R_t \subset \text{Fix}(\sigma_t)$ is the fixed component of genus at least 2 for $t \neq 0$, and
3. $R_0 = \lim_{t \to 0} R_t$.

By [AE21, Prop. 6.12], it suffices to show that if we make a one-parameter deformation the smoothing of $X_0$ into $F_\rho$ that keeps $X_0$ constant, the limiting curve $R_0$ does not deform, up to $\text{Aut}^0(X_0)$.

The automorphism $\sigma$ on the generic fiber of any smoothing defines a birational automorphism of $X$. Any two Kulikov models are related by an automorphism followed by a sequence of Atiyah flops of types 0, I, II along curves in $X_0$ which are either $(−2)$-curves or $(−1)$-curves on component(s) of $X_0$. As such, there are only countably many curves in $X_0$ along which it is possible to make an Atiyah flop, and this continues to be the case after a flop is made. Thus, up to conjugation by $\text{Aut}^0(X_0)$, there are only countably many possibilities for the birational automorphism $\sigma_0 := |x_0|: X_0 \to X_0$.

Hence if $X_0 \mapsto X$ and $X_0 \mapsto \tilde{X}$ are smoothings into $F_\rho$ as above, we have $\sigma_0 = \psi \circ \sigma_0 \circ \psi^{-1}$ for some $\psi \in \text{Aut}^0(X_0)$.

Let $\{A_j\}$ be the countable set of curves in $X_0$ along which $\sigma_0$ can be indeterminate. Any such curve $A_j$ is $\text{Aut}^0(X_0)$-invariant. Let $A = \cup_j A_j$ be their union. Clearly, the limit divisor $R_0$ is contained in the union of $A \cup S$ where $S$ is the closure of the fixed locus of $\sigma_0$ in its locus of determinacy. Similarly, $R_0$ is contained in $A \cup \tilde{S}$ and $\sigma_0(P) = P$ if and only if $\sigma_0(\psi(P)) = \psi(P)$. Since the smoothing $\tilde{X}$ is a deformation of the smoothing $X$ and the limiting divisor of $R$ varies continuously, we conclude that $\tilde{R}_0 = \psi(R_0)$ and therefore $R$ is recognizable. $\square$

**Proposition 3.25.** Any element $(\mathcal{X}, e\mathcal{R}) \in \overline{F}_\rho^{\text{slc}}$ has an automorphism $\pi \in \text{Aut}(\mathcal{X})$.

Furthermore, $\overline{\mathcal{R}} = \text{Fix}(\pi)$ and $\pi^*$ acts on $H^0(\mathcal{X}, \omega_{\mathcal{X}}) \cong \mathbb{C}$ by multiplication by $\zeta_n$.

**Proof.** As noted in Remark 3.6, any point in $\overline{F}_\rho^{\text{sep}} = (\overline{D_\rho} \setminus \Delta_\rho)/\Gamma_\rho$ corresponds to a pair $(\mathcal{X}, \pi)$ of an ADE K3 surface with automorphism, for which $\overline{\mathcal{R}} = \text{Fix}(\pi)$ is ample and the minimal resolution is $\rho$-markable. Then any boundary point $(\mathcal{X}_0, e\mathcal{R}_0) \in \overline{F}_\rho^{\text{slc}}$ is a stable limit of such ADE K3 surface pairs $f: (\mathcal{X}, e\mathcal{R}) \to C$.

Since $\overline{\mathcal{R}_0}$ is $\pi$-invariant and the canonical model is unique, $\mathcal{X}$ admits an automorphism $\pi$ whose fixed locus contains $\overline{\mathcal{R}_0}$. In fact, $\text{Fix}(\pi) = \overline{\mathcal{R}_0}$. $\text{Fix}(\pi)$ is a
Cartier divisor, and thus forms a flat family of divisors containing $R$. But $\text{Fix}(\sigma_0)$ already contains the flat limit $R_0$. The statement about $\omega_{\mathcal{X}/C}$ follows from the fact that $f^*\omega_{\mathcal{X}/C}$ is invertible (by Base Change and Cohomology, since $R^1f^*\omega_{\mathcal{X}/C} = 0$) and $\sigma_t^*$ acts by $\zeta_n$ on the generic fiber of this line bundle.

\section{Moduli of Quotient Surfaces}

We refer the reader to [Kol13] for the notions appearing in the following definitions. The pair $(Y, \Delta)$ is called demi-normal if $X$ satisfies Serre’s $S_2$ condition, has double normal crossing singularities in codimension 1, and $\Delta = \sum d_iD_i$ is an effective Weil $\mathbb{Q}$-divisor with $0 < d_i \leq 1$ not containing any components of the double crossing locus of $Y$.

The following is [Kol13, Prop. 2.50(4)], using our adopted notations.

\begin{proposition}
\label{prop:etaleleftrightarrowlocal}
Étale locally, there is a one-to-one correspondence between
\begin{enumerate}[(a)]
\item Local demi-normal pairs $(y \in Y, \frac{n-1}{n}B)$ of index $n$, i.e. such that the divisor $nK_Y + (n-1)B$ is Cartier.
\item Local demi-normal pairs $(\bar{y} \in \bar{Y})$ such that $K_{\bar{Y}}$ is Cartier, with a $\mu_n$-action that is free on a dense open subset, and such that the induced action on $\omega_{\bar{Y}} \otimes C(\bar{y})$ is faithful.
\end{enumerate}
Moreover, the pair $(Y, \frac{n-1}{n}B)$ is slc iff so is $\bar{Y}$.
\end{proposition}

The variety $\bar{Y}$ is called the local index-1 cover of the pair $(Y, \frac{n-1}{n}B)$. [Kol13, Sec. 2] also gives a global construction.

\begin{theorem}
\label{thm:index1cover}
Let $(\mathcal{X}, \epsilon R) \in \mathcal{F}_{\rho}^{\text{slc}}$ and let $\pi : \mathcal{X} \to Y = \mathcal{X}/\mu_n$ be the quotient map with the branch divisor $B = f(R)$. Then
\begin{enumerate}[(1)]
\item $nK_Y + (n-1)B \sim 0$,
\item $B$ and $-K_Y$ are ample $\mathbb{Q}$-Cartier divisors,
\item the pair $(Y, \frac{n-1+\epsilon}{n}B)$ is stable for any rational $0 < \epsilon \ll 1$, i.e. it has slc singularities and the $\mathbb{Q}$-divisor $K_Y + \frac{n-1+\epsilon}{n}B$ is ample.
\end{enumerate}
Vice versa, for a pair $(Y, B)$ satisfying the above conditions, its index-1 cover $\mathcal{X}$ with the ramification divisor $\mathcal{R}$ satisfies:
\begin{enumerate}[(1)]
\item $K_{\mathcal{X}} \sim 0$ and the $\mu_n$-action on $\mathcal{X}$ is non-symplectic,
\item $\mathcal{R}$ is $\mathbb{Q}$-Cartier,
\item the pair $(\mathcal{X}, \epsilon \mathcal{R})$ is stable for any rational $0 < \epsilon \ll 1$.
\end{enumerate}
\end{theorem}

\begin{proof}
Follows from the above Proposition \ref{prop:etaleleftrightarrowlocal} and the formulas
\[\pi^*(B) = n\mathcal{R}, \quad \pi^*(K_Y + \frac{n-1+\epsilon}{n}B) = K_{\mathcal{X}} + \epsilon \mathcal{R}.\]
\end{proof}

\begin{corollary}
The coarse moduli space $\mathcal{F}_{\rho}^{\text{slc}}$ coincides with the normalization of the KSBA compactification of the irreducible component in the moduli space of the log canonical pairs $(Y, \frac{n-1+\epsilon}{n}B)$ of log del Pezzo surfaces $Y$ with $(n-1)B \in |-nK_Y|$ in which a generic surface is a quotient of a K3 surface with a non-symplectic automorphism of type $\rho$. The stack for the former is a $\mu_n$-gerbe over the stack for the latter.

For the proof, we note that a small deformation of a K3 surface is a K3 surface.
Example 4.4. The KSBA compactification moduli of K3 surfaces of degree 2 for the ramification divisor $R$ constructed in [AET19] is equivalent to the Hacking’s compactification [Hac04] of the moduli space of pairs $(\mathbb{P}^2, \frac{1+\epsilon}{2} B_6)$ of plane sextic curves.

5. Extensions

The results of this paper are easily extended to the case of a nonsymplectic action by an arbitrary finite group $G$ and to more general divisors defined by group actions. Most of the changes amount to introducing more cumbersome notations.

5A. A general nonsymplectic group of automorphisms.

Definition 5.1. Let $X$ be a smooth K3 surface and $\sigma: G \subset \text{Aut} X$ be a finite symmetry group. The action of $G$ on $H^{2,0}(X) = \mathbb{C}\omega_X$ gives the exact sequence

$$0 \rightarrow G_0 \rightarrow G \xrightarrow{\mu_n} \mu_n \rightarrow \mathbb{C}^*. $$

One says that $G$ is nonsymplectic (or not purely symplectic) if $G \neq G_0$, i.e. $\alpha \neq 1$.

We now extend the results of Section 2 directly to this more general setting.

Definition 5.2. Fix a finite subgroup $\rho: G \rightarrow O(L)$ and a nontrivial character $\chi: G \rightarrow \mathbb{C}^*$. Let $(X, \sigma: G \rightarrow \text{Aut} X)$ be a K3 surface with a non-symplectic automorphism group.

A $(\rho, \chi)$-marking of $(X, \sigma)$ is an isometry $\phi: H^2(X, \mathbb{Z}) \rightarrow L$ such that for any $g \in G$ one has $\phi \circ \sigma(g)^* = \rho(g) \circ \phi$ and such that the character $\alpha: G \rightarrow \mathbb{C}^*$ induced by $\sigma$ coincides with $\chi$. We say that $(X, \sigma)$ is $\rho$-markable if it admits a $\rho$-marking.

A family of $(\rho, \chi)$-marked K3 surfaces is a smooth family $f: (\mathcal{X}, \sigma_B, \phi_B) \rightarrow B$ with a group of automorphisms $\sigma_B: G \rightarrow \text{Aut}(\mathcal{X}/B)$ and with a marking $\phi_B: R^2 f_* \mathbb{Z} \rightarrow L \otimes \mathbb{Z}_B$ such that every fiber is a $(\rho, \chi)$-marked K3 surface.

A family of smooth $\rho$-markable K3 surfaces is a family $f: (\mathcal{X}, \sigma_B) \rightarrow B$ of K3 surfaces with a group of automorphisms over base $B$ which admits a $\rho$-marking locally on $B$.

We define the moduli stacks $\mathcal{M}_{\rho, \chi}$ of $(\rho, \chi)$-marked, resp. $F_{\rho, \chi}$ of $(\rho, \chi)$-markable K3 by taking $\mathcal{M}_{\rho, \chi}(B)$, resp. $F_{\rho, \chi}(B)$ to be the groupoids of such families over $B$.

Definition 5.3. Define the vector space

$$L_{\rho, \chi} = \{x \in L \mathbb{C} \mid \rho(g)(x) = \chi(g)x\}$$

to be the intersection of the eigenspaces for the individual $g \in G$, and the period domain as

$$D_{\rho, \chi} = \mathbb{P}\{x \in L_{\rho, \chi} \mid x \cdot \bar{x} > 0\}$$

The second condition is redundant if there exists $g \in G$ with $\chi(g) > 2$. Thus, $D_{\rho}$ is a type IV domain if $|\chi(G)| = 2$ and a complex ball, a type I domain if $|\chi(G)| > 2$.

The discriminant locus is $\Delta_{\rho} := \cup_{\delta} \delta^2 \cap \Delta_{\rho}$ ranging over all roots $\delta$ in $(L^G)^\perp$, where $L^G = \{a \in L \mid \rho(g)(a) = a\}$ is the sublattice of $L$ fixed by $G$.

Definition 5.4. The group of changes-of-marking is

$$\Gamma_{\rho} := \{\gamma \in O(L) \mid \gamma \circ \rho = \rho \circ \gamma\}.$$

Then the direct analogue of Lemma 2.6 and Theorem 2.9 is

Theorem 5.5. For a fixed finite group $\rho: G \rightarrow O(L)$ with a nontrivial character $\chi: G \rightarrow \mathbb{C}^*$:
(1) There exists a fine moduli space $\mathcal{M}_{\rho,\chi}$ of $(\rho,\chi)$-marked K3 surfaces $(X,\sigma,\phi)$. It admits an étale period map $\pi_{\rho}: \mathcal{M}_{\rho,\chi} \to \mathbb{D}_{\rho,\chi} \setminus \Delta_{\rho}$. The fiber $\pi_{\rho}^{-1}(x)$ over a point $x \in \mathbb{D}_{\rho,\chi} \setminus \Delta_{\rho}$ is a torsor over $\Gamma_{\rho} \cap (\mathbb{Z}_2 \cap W_x)$.

(2) The moduli stack of $p$-markable K3 surfaces $(X,\sigma)$ is obtained as a quotient of $F_{\rho,\chi}$ by $\Gamma_{\rho}$. On the level of coarse moduli spaces it admits a bijective map to $(\mathbb{D}_{\rho,\chi} \setminus \Delta_{\rho})/\Gamma_{\rho}$.

Proof. If the group $G$ does not act purely symplectically, i.e. there exists $g \in G$ with $\rho(g)(x) \neq x$ then $L^G \perp x$ and $S^G_0 \simeq L^G$. The rest of the proof of Lemma 2.6 works the same for any finite group. And the proof of Theorem 2.9 goes through verbatim. □

5B. More general polarizing divisors. With a more general group action, there are more choices for the polarizing divisors. For a generic K3 surface $X$ with a period $x \in \mathbb{D}_{\rho,\chi} \setminus \Delta_{\rho}$ we can consider any combination $\sum b_i B_i$ of curves $B_i$ which are either fixed by some element $g \in G$ or are some of the $(-2)$-curves corresponding to the roots in the generic Picard lattice $(L_{\rho,\chi}^C)^{\perp} \cap L$ that generically gives a big and nef divisor on $X$. Theorem 3.24 extends immediately to this situation with the same proof.

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Email address: valery@uga.edu

Email address: philip.engel@uga.edu

Email address: changho.han@uga.edu

Department of Mathematics, University of Georgia, Athens GA 30602, USA