PALEY-WIENER DESCRIPTION OF $K$-SPHERICAL BESOV SPACES ON THE HEISENBERG GROUP

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Abstract. We characterize the Besov spaces associated to the Gelfand pairs on the Heisenberg group. The characterization is given in terms of bandlimited wavelet coefficients where the bandlimitedness is introduced using spherical Fourier transform. To obtain these results we develop an approach to the characterization of Besov spaces in abstract Hilbert spaces through compactly supported admissible functions.

1. Introduction

One of the main themes in Analysis is correlation between frequency content of a function and its smoothness. On the classical level the frequency is understood in terms of Fourier transform (or Fourier series) and smoothness is described in terms of Sobolev, Lipschitz, and Besov norms.

For these notions it was well understood (see [1], [2], [28], [41]) that there exists a perfect balance between rate of approximation by bandlimited functions (by trigonometric polynomials) and smoothness described by Besov norms.

A form of a harmonic analysis which holds this balance true in general Hilbert spaces and manifolds was recently developed in [36]-[38] and [20]. In this papers Sobolev and Besov spaces on manifolds were associated with elliptic Laplace-Beltrami operators on manifolds.

We note that harmonic analysis associated with Besov norms, approximations by bandlimited vectors and $K$-functions was considered in [29]-[34], [22].

New developments in this direction were recently published in [8]-[14], [20], [36]. In [14] the authors characterize inhomogeneous Besov spaces on stratified Lie groups using some functional calculus and the spectral theory for the sub-Laplacian operators on these groups.

We base our work on the observation (see Definition 1.1 below) that every time one has some kind of Fourier transform, the functions that are compactly supported on the “frequency side are natural generalizations of the classical bandlimited (Paley-Wiener) functions. In the introduction we formulate main results obtained in our paper. The exact definitions of all notions are given in the text.

In the introduction we formulate main results obtained in our paper. The exact definitions of all notions are given in the text.

We start with a self-adjoint positive definite operator $\Delta$ in a Hilbert space $\mathcal{H}$ and consider its positive root $D = \Delta^{1/2}$. The domain $\mathcal{D}_s$, $s \in \mathbb{R}$, of the operator $D^s$, $s \in \mathbb{R}$, plays the role of the Sobolev space. In what follows, the notation $\| \cdot \|$ means $\| \cdot \|_{\mathcal{H}}$. We define the following graph norm for the Sobolev spaces $\mathcal{D}_r$, as domain of the operator $\Delta^{r/2}$:

$$\|f\|_r = \|f\| + \|\Delta^{r/2}f\|.$$
The inhomogeneous Besov space $B^α_{2,q} = B^α_{2,q}(\Delta)$ was introduced as an interpolation space between the Hilbert space $\mathcal{H}$ and Sobolev space $D_{\alpha/2}$ where $\alpha$ can be any natural number such that $0 < \alpha < r, 1 \leq q < \infty$, or $0 \leq \alpha \leq r, q = \infty$ ([2], [4], [22], [23], [41]). It is known that the Besov space can be characterized as space of all functions in $\mathcal{H}$ whose Besov norm can be described in terms of moduli of continuity in terms of wave semigroup $e^{itD}$ ([2], [4], [22], [36], [38], [41]).

The notion of the Paley-Wiener spaces for the abstract Hilbert space $\mathcal{H}$ associated to the positive self-adjoint operator $D$ is given as below. According to the spectral theory [3], there exists a unitary operator $F_D$ from $\mathcal{H}$ onto a Hilbert space $X$, where $X = \int X(\lambda)dm(\lambda)$ is a direct integral of Hilbert spaces $X(\lambda)$ and is the space of all $m$-measurable functions $\lambda \mapsto x(\lambda) \in X(\lambda)$, for which the norm

$$\|x\|_X = \left(\int_0^\infty \|x(\lambda)\|^2_{X(\lambda)}dm(\lambda)\right)^{1/2}$$

is finite. The unitary operator $F_D$ transforms domain of $D^k, k \in \mathbb{N}$, onto $X_k = \{x \in X|\lambda^kx \in X\}$ with norm

$$\|x(\lambda)\|_{X_k} = \left(\int_0^\infty \lambda^{2k}\|x(\lambda)\|^2_{X(\lambda)}dm(\lambda)\right)^{1/2}. \quad (1)$$

Besides $F_D(D^k f) = \lambda^k(F_D f)$, if $f$ belongs to the domain of $D^k$.

The following definition can be found in [31] and [35].

**Definition 1.1.** Let $D$ be the same as above. Then we say a vector $f$ from $\mathcal{H}$ belongs to the Paley-Wiener space $PW_\omega = PW_\omega(D)$ if the support of the spectral Fourier transform $F_D f$ belongs to $[0, \omega]$. For a vector $f \in PW_\omega$ the notation $\omega_f$ will be used for a positive number such that $[0, \omega_f]$ is the smallest interval which contains the support of the spectral Fourier transform $F_D f$. We call the vectors in Paley-Wiener spaces bandlimited functions.

The goal of this article is to realize the above notions for the Heisenberg group and describe the Besov norms for the group in terms of bandlimited and admissible (wavelet) functions.

Let $\mathbb{H}_n \cong \mathbb{R}^n \times \mathbb{R}$ denote the $n$-dimensional Heisenberg group and $K$ is a Lie compact subgroup of $U(n)$, the group of unitary automorphisms on $\mathbb{H}_n$. We let the space $L^2_K(\mathbb{H}_n)$ denote the space of all functions $f$ in $L^2(\mathbb{H}_n)$ which are $K$-invariant: $f(kw) = f(w), \forall k \in K, \forall w \in \mathbb{H}_n$.

For functions in $L^2_K(\mathbb{H}_n)$ one can introduce the $K$-spherical transform $F$. It is known that $F$ is a unitary operator from $L^2_K(\mathbb{H}_n)$ onto a certain $L^2$ space of functions defined on $\mathbb{R}^* \times \mathbb{N}^n, \mathbb{R}^* = \mathbb{R}/\{0\}$. We say a function $f \in L^2_K(\mathbb{H}_n)$ is a Paley-Wiener function if $F(f)$ has compact support in $\mathbb{R}^* \times \mathbb{N}^n$.

In what follows the $l_w$ and $\delta_α$ are standard translation and dilation operators on $\mathbb{H}_n$. We call $\Psi \in L^2_K(\mathbb{H}_n)$ a wavelet if the measurable coefficient map $W_{f,\Psi} : (w,\alpha) \mapsto \langle f, l_w \delta_\alpha \Psi \rangle$ is square integrable.

Our main result is Corollary 4.9 as following:

**Main result.** Let $(K, \mathbb{H}_n)$ be a Heisenberg Gelfand pair, i.e., $L^1_K(\mathbb{H}_n)$ is a commutative algebra with convolution operator. Then there exists a bandlimited wavelet $\Psi \in L^2_K(\mathbb{H}_n)$ such that for any $f \in L^2_K(\mathbb{H}_n)$ the following holds true. For any $f \in L^2_K(\mathbb{H}_n)$
\[ \|f\|_{B_{2,q}^\alpha} \asymp \|f\| + \left( \sum_{j \geq 0} \left( 2^{-j((n+1)-\alpha/q)} \| \mathcal{F}(f) A_{2^j} \mathcal{F}(\Psi) \| \right)^q \right)^{1/q} \]

where \( \delta_{2^j} \Psi(w) = 2^{j(n+1)} \Psi(2^j w) \) for all \( w \in \mathbb{H}_n \), and for \( a > 0 \), \( A_a \) is a unitary dilation.

The above equivalency is understood in this sense that \( f \) is a \( K \)-spherical Besov function if and only if the sum is finite and two norms are equivalent. We will use the following main technical lemmas for the proof of our main results.

**Lemma 1.2.** Let \( \mathcal{H} \) be a general Hilbert space. Then for any \( f \in \mathcal{H} \) there exists a sequence of bandlimited functions \( f_j := f_j(f) \in PW_{2^{j+1}} \) and \( g \in PW_1 \), and a sequence of operators \( S_j : \mathcal{H} \to PW_{2^{j+1}} \) and \( S : \mathcal{H} \to PW_1 \) such that in \( \mathcal{H} \)

\[ f = S(g) + \sum_{j=0}^{\infty} S_j(f_j). \]

**Lemma 1.3.** Let \( \mathcal{H} \) be a general Hilbert space and \( f \in \mathcal{H} \). Then there exist a sequence of bandlimited functions \( f_j := f_j(f) \in PW_{2^{j+1}} \) and \( g \in PW_1 \) such that for \( \alpha > 0 \), \( 1 \leq q < \infty \), the Besov norm is equivalent to

\[ \|g\| + \left( \sum_{j=0}^{\infty} \left( 2^{j\alpha} \|f_j\| \right)^q \right)^{1/q} \quad \text{and,} \quad \|f\| + \left( \sum_{j=0}^{\infty} \left( 2^{j\alpha} \|f_j\| \right)^q \right)^{1/q}, \]

and the equivalency of norms also holds for \( q = \infty \) with the standard convention.

The existence of bandlimited atoms \( f_j(f) \) was already proved on manifolds and general Hilbert spaces in [37, 38]. However, in the present paper we develop this result further. Namely, we give a “constructive” description of such atoms \( f_j(f) \) which are “infinitely smooth” on the space and enjoy “all vanishing moments” property. This implies, as usual, that our atoms have perfect localization on the frequency side and a reasonable localization on the space (Heisenberg group).

The outline of this paper is as following. After introducing some preliminaries and notations we prove the main Lemmas 1.2 and 1.3 in Section 3 and hence we establish an equivalent Besov norm on general Hilbert spaces in terms of “admissible” functions. In Section 4 we introduce the Gelfand pairs and \( K \)-spherical (Gelfand) transform on the Heisenberg group associated to the sub-Laplacian on this group. We conclude this article with the proofs of our main results in Theorem 4.7 and Corollary 4.9.

### 2. Preliminaries and Notations

We introduce Besov spaces on \( \mathcal{H} \), \( B_{2,q}^\alpha = B_{2,q}^\alpha(\Delta), \alpha > 0, 1 \leq q \leq \infty \), using Peetre’s interpolation \( K \)-functions (\([2, 4, 22, 23, 41]\)). That is

\[ B_{2,q}^\alpha = \left( \mathcal{H}, \mathcal{D}_{r/2} \right)^K_{\alpha/r, q}, \]

where \( r \) can be any natural number such that \( 0 < \alpha < r \) for \( 1 \leq q < \infty \), or \( 0 \leq \alpha \leq r \) for \( q = \infty \).
For any \( f \in \mathcal{H} \), let \( K(\cdot, f, \mathcal{H}, D_{r/2}) \) be the Peetre’s \( K \)-functional for the pair \((\mathcal{H}, D_{r/2})\) given on \( \mathbb{R}^+ \) by

\[
K(t, f, \mathcal{H}, D_{r/2}) = \inf_{g \in D_{r/2}} (\|f - g\| + t\|g\|_r), \quad t > 0, \ f \in \mathcal{H}.
\]

We introduce the following functionals

\[
\Phi_{\theta, q}^\varepsilon(\varphi(t)) = \left( \int_0^\varepsilon (t^{-\theta} \varphi(t))^q \frac{dt}{t} \right)^{1/q}, \ 1 < \theta < \infty, 1 \leq q < \infty,
\]

and

\[
\Phi_{\theta, \infty}^\varepsilon(\varphi(t)) = \sup_{0 \leq t \leq \varepsilon} t^{-\theta}(\varphi(t)), \ q = \infty.
\]

The spaces \((\mathcal{H}, D_{r/2})_{\theta, q}^K, \ 0 < \theta < 1, 1 \leq q \leq \infty,\) are the sets of abstract functions in \( \mathcal{H} \) for which the following norm

\[
\|f\| + \Phi_{\theta, q}^\varepsilon(K(t, f, \mathcal{H}, D_{r/2})) \quad f \in \mathcal{H}
\]

is finite. The fact that \( H^r \subset \mathcal{H} \) implies that for any \( \varepsilon > 0 \) the following two norms are equivalent.

\[
\|f\| + \Phi_{\theta, q}^\varepsilon(K(t, f, \mathcal{H}, D_{r/2})) \asymp \Phi_{\theta, \infty}^\varepsilon(K(t, f, \mathcal{H}, D_{r/2})) \quad f \in \mathcal{H}.
\]

It is known result in \([2, 4, 36]\) that this Besov norm can be described in terms of a modulus of continuity constructed in terms of the wave semigroup \( e^{itD} \).

**Theorem 2.1.** Let \( \alpha < r \in \mathbb{N} \). The norm of the Besov space \( B_{2,q} \) on \( \mathcal{H} \) is equivalent to

\[
\|f\| + \left( \int_0^1 (s^{-\alpha} \Omega_r(s, f))^q \frac{ds}{s} \right)^{1/q}
\]

for \( 1 \leq q < \infty \) and equivalent to

\[
\|f\| + \sup_{0 < s < 1} (s^{-\alpha} \Omega_r(s, f)) \frac{ds}{s}
\]

for \( q = \infty \), where modulus of continuity is introduced as

\[
\Omega_r(s, f) = \sup_{0 < r \leq s} \| (I - e^{ir\sqrt{\Delta}})^r f \|.
\]

**2.1. Functional Calculus.** By the spectral theory, suppose that \( \Delta \) has the unique spectral resolution or decomposition \( P \) of the identity

\[
\Delta = \int_0^\infty \xi dP_\xi.
\]

\( dP \) is a projection-valued measure concentrated on the spectrum of \( \Delta \), \( \sigma(\Delta) = (0, \infty) \), with orthogonal projections \( P_\xi \) on \( \mathcal{H} \) with \( P_{\{0\}}(\mathcal{H}) = 0 \) and \( P_{\sigma(\Delta)} = I \). Therefore by the spectral theory, for any \( f \in D(\Delta) \) and \( g \in \mathcal{H} \)

\[
\langle \Delta f, g \rangle = \int_0^\infty \xi d(\xi f, g),
\]
and

$$\mathcal{D}(\Delta) = \left\{ f \in \mathcal{H} : \|\Delta f\|^2 := \int_0^\infty \xi^2 d(P_\xi f, f) < \infty \right\}.$$ 

For $\beta$ a bounded Borel function on $\sigma(\Delta)$, we define the commutative integral operator $\beta(\Delta)$ by

$$\beta(\Delta) := \int_0^\infty \beta(\xi) dP_\xi;$$

by the spectral theory this is a bounded operator with domain

$$\mathcal{D}(\beta(\Delta)) = \left\{ f \in \mathcal{H} : \|\beta(\Delta) f\|^2 := \int_0^\infty |\beta(\xi)|^2 d(P_\xi f, f) < \infty \right\}.$$ 

The operator norm is $\|\beta(\Delta)\| = \|\beta\|_\infty$ and the following hold:

(a) $\beta \gamma(\Delta) = \beta(\Delta) \gamma(\Delta) = \gamma(\Delta) \beta(\Delta)$
(b) $\beta(\Delta)^* = \bar{\beta}(\Delta)$
(c) for any $f \in \mathcal{D}(\beta(\Delta))$ and $g \in \mathcal{H}$

$$\langle \beta(\Delta) f, g \rangle = \int_0^\infty \beta(\xi) d(P_\xi f, g).$$

$\beta$ is real-valued, then the operator $\beta(\Delta)$ is self-adjoint by (b). The operator is positive definite if $\beta$ takes its values in $\mathbb{R}^+$. Throughout this paper, by $A \asymp B$ we shall mean that $A, B$ are positive numbers and that there are positive constants $c_1, c_2$ such that $c_1 A \leq B \leq c_2 A$. Similarly, we say $A \preceq B$ if there exists $c > 0$ such that $A \leq cB$.

In a complete analogy to the Fourier transform of a function on $\mathbb{R}$, we shall use $\hat{\psi}$ for the functions given on the spectrum of $\Delta$.

### 3. Proof of Lemma 1.2 and Lemma 1.3

**Proof of Lemma 1.2.** Suppose $\hat{\phi}, \hat{\psi} \in L^\infty(0, \infty)$ with supp$(\hat{\phi}) \subseteq [0, 1]$ and supp$(\hat{\psi}) \subseteq [1/2, 2]$ for which the following resolution of the identity holds:

$$|\hat{\phi}(\xi)|^2 + \sum_{j \in \mathbb{Z}^+} |\hat{\psi}_j(\xi)|^2 = 1 \quad \forall \xi \in \mathbb{R}^+ \quad (6)$$

where $\hat{\psi}_j(\xi) := \hat{\psi}(2^{-2j}\xi)$. Thus supp$(\hat{\psi}_j) \subseteq [2^{j-1}, 2^{j+1}]$.

Applying the spectral theory for $\xi$, the following version of Calderón decomposition, in complete analogy to the Euclidean setting, holds:

$$\hat{\phi}(\Delta)^* \hat{\phi}(\Delta) f + \sum_{j \in \mathbb{Z}^+} \hat{\psi}_j(\Delta)^* \hat{\psi}_j(\Delta) f = f \quad \forall f \in \mathcal{H} \quad (7)$$

where the series converges in $\mathcal{H}$. Now define $S_j := \hat{\psi}_j(\Delta)$ and $f_j := S_j(f)$. Then the functions $f_j$ are in $PW_{2j+1}$ and the assertions of the lemma hold. \qed
For the proof of Lemma 3.3 we need the following two technical lemmas.

Lemma 3.1. For any $\tau > 0$ and natural number $r$, in the operator norm
\[
\| (I - e^{i\sqrt{\Delta}})^r \hat{\phi}(\Delta) \| \leq \tau^r, \text{ and }
\| (I - e^{i\sqrt{\Delta}})^r \hat{\psi}_j(\Delta) \| \leq \tau^r 2^{(j+1)r/2} \quad j \in \mathbb{Z}^+.
\]

Proof. We prove the inequality for $\hat{\psi}_j$ and the proof for $\hat{\phi}$ follows in an analogy way. Since $\text{supp} \hat{\psi}_j \subseteq [2^{j-1}, 2^{j+1}]$,
\[
\| (I - e^{i\sqrt{\Delta}})^r \hat{\psi}_j(\Delta) \| = \sup_{2^{j-1} \leq \xi \leq 2^{j+1}} \| (I - e^{i\sqrt{\Delta}})^r \hat{\psi}_j(\xi) \| 
\leq \sup_{2^{j-1} \leq \xi \leq 2^{j+1}} \| (1 - e^{i\sqrt{\Delta}})^r \|
\leq \tau^r 2^{(j+1)r/2}.
\]

Note that to pass from (10) to (8) we use $|\hat{\psi}_j(\xi)| \leq 1$ which is driven from (6). And, to pass from (9) to (12) we use the Fundamental Theorem of Calculus for the function $h(x) = e^{i\sqrt{x}}$ on $[0, \tau]$. \hfill \Box

Lemma 3.2. Let $r \in \mathbb{N}$, $m, k \in \mathbb{R}$ such that $k + m \geq 0$ and $k < 0$. Let $f \in \mathcal{H}$. Define $w_j = 2^{kj}$ and $c_j = 2^{mj}$ for $j \in \mathbb{Z}$. Then for any $1 \leq \bar{q} < \infty$
\[
\Omega_r(s, f)^{\bar{q}} \leq s^{\bar{q}r} \sum_{j=-1}^{\infty} \left( 2^{j\bar{q}r/2} w_j c_j \| \hat{\psi}_j(\Delta)f \|^\bar{q} \right) \quad \forall \ s \in (0, 1].
\]

Proof. Take $\hat{\psi}_{-1} := \hat{\phi}$ and let $f \in B_{2,q}^\alpha$. By Applying the decomposition (7) to $f$, for any $\tau \leq s$ we have
\[
\| (I - e^{i\sqrt{\Delta}})^r f \| = \sum_{j=-1}^{\infty} \| (I - e^{i\sqrt{\Delta}})^r \hat{\psi}_j(\Delta)f \| \leq \sum_{j=-1}^{\infty} \| (I - e^{i\sqrt{\Delta}})^r \hat{\psi}_j(\Delta)f \| \hat{\psi}_j(\Delta)f \|.
\]

Taking supremum over $\tau$ in (11) yields
\[
\sup_{\tau \leq s} \| (I - e^{i\sqrt{\Delta}})^r f \| \leq \sup_{\tau \leq s} \sum_{j=-1}^{\infty} \| (I - e^{i\sqrt{\Delta}})^r \hat{\psi}_j(\Delta)f \| \leq \sum_{j=-1}^{\infty} \| (I - e^{i\sqrt{\Delta}})^r \hat{\psi}_j(\Delta)f \|.
\]

Therefore for $\bar{q} > 1$
\[
\left( \sup_{\tau \leq s} \| (I - e^{i\sqrt{\Delta}})^r f \| \right)^{\bar{q}} \leq \left( \sum_{j=-1}^{\infty} w_j c_j \sup_{\tau \leq s} \| (I - e^{i\sqrt{\Delta}})^r \hat{\psi}_j(\Delta)f \| \right)^{\bar{q}}. \quad (12)
\]
The Hölder inequality for the series in (12) and for the pair \((\tilde{q}, \frac{q}{q-1})\) with the weights \(w_j\) yields the following estimations up to some constants independent of \(f\):

\[
\begin{align*}
(12) \lesssim \left( \sum_{j=-1}^{\tilde{q}-1} w_j \right)^{\frac{\tilde{q}}{q}} & \sum_{j=-1}^{\tilde{q}} w_j c_j^q \sup_{\tau \leq s} \| (I - e^{ir\sqrt{\Delta}})^{r} \hat{\psi}_j(\Delta)^{r} \hat{\psi}_j(\Delta) f \|^{\tilde{q}} \\
\lesssim & \sum_{j=-1}^{\tilde{q}} w_j c_j^q \sup_{\tau \leq s} \| (I - e^{ir\sqrt{\Delta}})^{r} \hat{\psi}_j(\Delta)^{r} \hat{\psi}_j(\Delta) f \|^{\tilde{q}} \\
\lesssim & \sum_{j=-1}^{\tilde{q}} w_j c_j^q \| \hat{\psi}_j(\Delta) f \|^{\tilde{q}} \sup_{\tau \leq s} \| (I - e^{ir\sqrt{\Delta}})^{r} \hat{\psi}_j(\Delta)^{r} \|^{\tilde{q}} \\
\lesssim & \sum_{j=-1}^{\tilde{q}} w_j c_j^q \| \hat{\psi}_j(\Delta) f \|^{\tilde{q}} \sup_{\tau \leq s} (\tau r 2^{(j+1)r/2})^{\tilde{q}} \\
\lesssim & s^{\tilde{q}r} \sum_{j=-1}^{\tilde{q}} 2^{j\tilde{q}/2} w_j c_j^q \| \hat{\psi}_j(\Delta) f \|^{\tilde{q}}.
\end{align*}
\]

We note that to pass from (13) to (14) we applied Lemma 3.1 for \(\tilde{\psi}\). Interfering the preceding estimations in (12) we achieve the result. The assertion for \(q = 1\) is obtained with a similar argument.

Proof of Lemma 1.3. Let \(f_j\) be as above. We prove that for these functions the equivalency (2) of the Lemma 1.3 hold. We shall prove this in two parts.

Part I. For any \(f \in \mathcal{H}\), if \(\{2^{j\alpha} f_j\}_j \in l^q(\mathbb{Z}^+, \mathcal{H})\), then \(f \in B_{2,q}^{\alpha}\) and

\[
\|f\| + \left( \int_0^1 (s^{-\alpha} \Omega_r(f,s))^{q} ds/s \right)^{1/q} \lesssim \|f\| + \left( \sum_{j \in \mathbb{Z}^+} (2^{j\alpha} \|f_j\|)^q \right)^{1/q}.
\]

And, for \(q = \infty\) the result holds.

Proof of Part I. Let \(1 \leq q < \infty\) and take \(\tilde{q} = q\) in Lemma 3.2. Let \(r \leq 2\alpha\) and \(k\) and \(m\) satisfy the inequality \(k + mq \leq q(\alpha - r/2)\). (In fact, there exists a large class of pairs \((k,m)\) such that \(k + m \geq 0\) and simultaneously satisfy the inequality.) By Lemma 3.2

\[
\left( \sup_{\tau \leq s} \| (I - e^{ir\sqrt{\Delta}})^{r} f \| \right)^{q} \lesssim s^{qr} \sum_{j=-1}^{\tilde{q}} 2^{j\tilde{q}/2} w_j c_j^q \| \hat{\psi}_j(\Delta) f \|^{\tilde{q}}.
\]

By integrating the both sides of (16) on \([0,1]\) with respect to the measure \(s^{-\alpha q} ds/s\) we get

\[
\int_0^1 s^{-\alpha q} \left( \sup_{\tau \leq s} \| (I - e^{ir\sqrt{\Delta}})^{r} f \| \right)^{q} ds/s \lesssim \left( \int_0^1 s^{r(\alpha - q)} ds/s \right) \sum_{j=-1}^{\tilde{q}} 2^{j\tilde{q}/2} w_j c_j^q \| \hat{\psi}_j(\Delta) f \|^{\tilde{q}}
\]

\[
\quad = \frac{1}{q(r - \alpha)} \sum_{j=-1}^{\tilde{q}} 2^{j\tilde{q}/2} w_j c_j^q \| \hat{\psi}_j(\Delta) f \|^{\tilde{q}}.
\]
Hence
\[
\left( \int_0^1 \left( s^{-\alpha} \sup_{\tau \leq s} \left\| \left( I - e^{i\tau \Delta} \right)^{r} f \right\| \right)^q ds/s \right)^{1/q} \leq \left( \sum_{j=-1} 2^{jqr/2} w_j c_j^q \left\| \hat{\psi}_j(\Delta) f \right\|^q \right)^{1/q} \tag{17}
\]

To complete the proof of part I, define
\[
A_j := \begin{cases} 
2^{jr/2} w_j^{1/q} c_j \| \hat{\psi}_j(\Delta) f \|^q & \text{if } j = -1 \\
0 & \text{if } j \geq 0 ,
\end{cases}
\]
and
\[
B_j := \begin{cases} 
0 & \text{if } j = -1 \\
2^{jr/2} w_j^{1/q} c_j \| \hat{\psi}_j(\Delta) f \|^q & \text{if } j \geq 0 .
\end{cases}
\]

We rewrite the right hand side of (17) as follows.
\[
\left( \sum_{j=-1} 2^{jqr/2} w_j c_j^q \left\| \hat{\psi}_j(\Delta) f \right\|^q \right)^{1/q} = \left( \sum_{j=-1} |A_j + B_j|^q \right)^{1/q} \leq \left( \sum_{j=-1} |A_j|^q \right)^{1/q} + \left( \sum_{j=-1} |B_j|^q \right)^{1/q} \tag{18}
\]

Substituting back \( A_j \) and \( B_j \) in above and using \( 2^{-r/2} w_{-1}^{1/q} c_{-1} \leq 1 \), and \( 2^{jqr/2} w_j c_j^q \leq 2^{jq} \) we get
\[
\left( \sum_{j=0} 2^{jqr/2} w_j c_j^q \left\| \hat{\psi}_j(\Delta) f \right\|^q \right)^{1/q} \leq 2^{r/2} w_{-1}^{1/q} c_{-1} \| \hat{\psi}_{-1}(\Delta) f \| + \left( \sum_{j=0} 2^{jqr/2} w_j c_j^q \left\| \hat{\psi}_j(\Delta) f \right\|^q \right)^{1/q} \leq \| f \| + \left( \sum_{j=0} 2^{jqr/2} w_j c_j^q \left\| \hat{\psi}_j(\Delta) f \right\|^q \right)^{1/q} \leq \| f \| \left( \sum_{j=0} \left\| \hat{\psi}_j(\Delta) f \right\| \right)^{1/q} = \| f \| B_2^{q} .
\]

This completes the proof of \( I \) for \( 1 \leq q < \infty \). By Lemma 3.2 for \( \tilde{q} = 1 \) we have
\[
\Omega_r(s, f) \leq s^r \sum_{j=-1} 2^{jr/2} w_j c_j \| \hat{\psi}_j(\Delta) f \|. \tag{19}
\]
By multiplying both sides of (19) by \( s^{-\alpha} \) and taking the supremum over \( 0 < s < 1 \) we get

\[
\sup_{0 < s < 1} s^{-\alpha} \Omega_r(s, f) \leq \left( \sup_{0 < s < 1} s^{r-\alpha} \right) \sum_{j=-1}^{\infty} 2^{jr/2} w_j c_j \| \hat{\psi}_j(\Delta) f \| \quad (r - \alpha > 0)
\]

\[
\leq \sum_{j=-1}^{\infty} 2^{jr/2} w_j c_j \| \hat{\psi}_j(\Delta) f \|
\]

\[
\leq \left( \sup_{j \geq -1} 2^{j\alpha} \| \hat{\psi}_j(\Delta) f \| \right) \left( \sum_{j=-1}^{\infty} 2^{-j(r-\alpha/2)} w_j c_j \right)
\]

Recall that here \( \bar{q} = q = 1 \) and \( r - \alpha > 0 \). With these restrictions and \( k + m \leq \alpha - r/2 \) the sum \( \sum_{j=-1}^{\infty} 2^{-j(r-\alpha/2)} w_j c_j \) is finite. Therefore

\[
\sup_{0 < s < 1} s^{-\alpha} \Omega_r(s, f) \leq \sup_{j \geq -1} 2^{j\alpha} \| \hat{\psi}_j(\Delta) f \| = \sup_{j \geq 0} 2^{j\alpha} \| \hat{\psi}_j(\Delta) f \| + \| f \| .
\]

This completes the proof of Part I for \( q = \infty \).

**Part II.** For any \( f \in \mathcal{H} \)

\[
\| f \| + \left( \sum_{j \in \mathbb{Z}^+} (2^{j\alpha} \| \hat{\psi}_j(\Delta) f \|)^q \right)^{1/q} \leq \| f \| + \left( \int_0^1 (s^{-\alpha} \Omega_r(f, s))^{q} ds/s \right)^{1/q} \quad (20)
\]

And, for \( q = \infty \) the statement also holds true.

**Proof of Part II.** Put \( c = \int_0^1 s^{-\alpha} |1 - e^{is/\sqrt{t}}|^{4r} ds \). Then \( c > 0 \) and without lose of generality we assume that \( c = 1 \). Let \( 1/4 \leq t \leq 1 \). By substituting \( s \mapsto 2s\sqrt{t} \) in the above integral we get

\[
1 = \int_0^{1/\sqrt{t}} \frac{1}{(2s\sqrt{t})^{-\alpha} |1 - e^{is\sqrt{t}}|^{4r} \sqrt{t} ds}.
\]

Since \( 1/\sqrt{t} \leq 2 \) and \( \sqrt{t} \leq 1 \), the followings hold for any \( M > 0 \) up to some constants independent of \( t \).

\[
1 = \int_0^{1/\sqrt{t}} \frac{1}{(2s\sqrt{t})^{-\alpha} |1 - e^{is\sqrt{t}}|^{4r} \sqrt{t} ds} \leq \int_0^1 (s\sqrt{t})^{-\alpha} |1 - e^{is\sqrt{t}}|^{4r} ds \leq \int_0^1 s^{-\alpha} (\sqrt{t})^{-(\alpha+M)} |1 - e^{is\sqrt{t}}|^{4r} ds.
\]

Now define \( \xi(t) := \int_0^1 s^{-\alpha} (\sqrt{t})^{-(\alpha+M)} |1 - e^{is\sqrt{t}}|^{4r} ds \) on \( 1/4 \leq t \leq 1 \) and zero elsewhere. The map \( \xi \) is bounded and \( \xi(t) \geq 1 \). By the spectral theory for \( \Delta \) the operator \( \xi(\Delta) \) is bounded on \( \mathcal{H} \) and we obtain the following in the weak sense.

\[
I \leq \int_0^1 s^{-\alpha} (\sqrt{\Delta})^{-(\alpha+M)} |1 - e^{i\sqrt{\Delta}}|^{4r} ds \quad (21)
\]

where \( I \) is the identity operator on the Hilbert space \( \mathcal{H} \). Therefore for any \( g \in \mathcal{H} \)

\[
\| g \|^2 \leq \int_0^1 s^{-\alpha} ((\sqrt{\Delta})^{-(\alpha+M)/2} (1 - e^{i\sqrt{\Delta}})^{2r} g \|^{2} ds . \quad (22)
\]

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Take \( g := \hat{\psi}_j(\Delta)f \), \( j \geq 0 \). An application of (21) and Lemma 3.1 gives
\[
\|
\hat{\psi}_j(\Delta)f\|_2 \leq \int_0^1 s^{-\alpha} \|(\sqrt{\Delta})^{-(\alpha+M)/2}(1 - e^{i s \sqrt{\Delta}})^{2r} \hat{\psi}_j(\Delta)f\|_2^2 \, ds \\
\leq \int_0^1 s^{-\alpha} \|(1 - e^{i s \sqrt{\Delta}})^{2r} \hat{\psi}_j(\Delta)f\|_2^2 \, ds \\
\leq \int_0^1 s^{-\alpha} \|s^{-\alpha} \Omega_r(s,f)\|^2 \, ds \\
= 2^{-j(\alpha+M-2r)/2} \int_0^1 (s^{-\alpha} \Omega_r(s,f))^2 s^{2r+\alpha} \, ds \\
= 2^{-j(\alpha+M-2r)/2} \|s \mapsto s^{-\alpha} \Omega_r(s,f)\|^2_{L^2((0,1], dm(s))} \\
\tag{23}
\]
with \( dm(s) = s^{1+r+\alpha} \). Take \( F(s) := s^{-\alpha} \Omega_r(s,f) \) and \( G(s) := 1, 0 < s \leq 1 \). The inverse of the Hölder inequality for \( p = 2 \) implies that
\[
\|F\|_{L^2} \|G\|_{L^{-1}} \leq \|FG\|_{L^1},
\]
equivalently,
\[
\|F\|_{L^2} \leq \|G\|_{L^1} \|FG\|_{L^1}.
\]
This translates to
\[
\int_0^1 (s^{-\alpha} \Omega_r(s,f))^2 s^{r+\alpha} \, ds \leq \left( \int_0^1 s^{2r+\alpha} \, ds \right)^2 \left( \int_0^1 s^{-\alpha} \Omega_r(s,f)^2 s^{2r+\alpha} \, ds \right)^2 \\
= c \left( \int_0^1 s^{-\alpha} \Omega_r(s,f)^2 s^{2r+\alpha} \, ds \right)^2
\]
for \( c = c(\alpha, r) = (2r + \alpha + 1)^{-2} \). By interfering this in (23) we get
\[
\|\hat{\psi}_j(\Delta)f\|^2 \leq 2^{-j(\alpha+M-2r)/2} \|s \mapsto s^{-\alpha} \Omega_r(s,f)\|^2_{L^2((0,1], dm(s))},
\]
or equivalently,
\[
\|\hat{\psi}_j(\Delta)f\|^2 \leq 2^{-j(\alpha+M-2r)/4} \|s \mapsto s^{-\alpha} \Omega_r(s,f)\|^2_{L^1((0,1], dm(s))}.
\]
Therefore for \( 1 < q < \infty \) and \( q' = \frac{q}{q-1} \)
\[
\|\hat{\psi}_j(\Delta)f\| \leq c \, 2^{-j(\alpha+M-2r)/4} \|s^{-\alpha} \Omega_r(s,f)\|_{L^1((0,1], dm(s))} \\
\leq c' \, 2^{-j(\alpha+M-2r)/4} \|s^{-\alpha} \Omega_r(s,f)\|_{L^1((0,1], dm(s))} \|1\|_{L^{q'}((0,1], dm(s))} \\
\leq c' \, 2^{-j(\alpha+M-2r)/4} \|s^{-\alpha} \Omega_r(s,f)\|_{L^{q'}((0,1], dm(s))} \\
\tag{25}
\]
with \( c' = c'(r, \alpha) = (r + \alpha + 1)^{-2+1/q'} \). By \( 0 < s < 1 \), (24) leads to
\[
\|\hat{\psi}_j(\Delta)f\| \leq c' \, 2^{-j(\alpha+M-2r)/4} \|s^{-\alpha} \Omega_r(s,f)\|_{L^{q'}((0,1], ds/s)}
\]
and hence
\[ \| \hat{\psi}_j(\Delta)f \|^q \leq c 2^{-j(q(\alpha + M - 2r)/4)} \left( \int_0^1 (s^{-\alpha}\Omega_r(s,f))^q ds/s \right). \]

Therefore from above, for \( M > 3\alpha + 2r \) we have

\[ \sum_{j \geq 0} 2^{j\alpha q} \| \hat{\psi}_j(\Delta)f \|^q \leq \left( \sum_{j \geq 0} 2^{-jq(M-3\alpha-2r)/4} \right) \left( \int_0^1 (s^{-\alpha}\Omega_r(s,f))^q ds/s \right) = c \left( \int_0^1 (s^{-\alpha}\Omega_r(s,f))^q ds/s \right). \]

Or equivalently,

\[ \left( \sum_{j \geq 0} 2^{j\alpha q} \| \hat{\psi}_j(\Delta)f \|^q \right)^{1/q} \leq \left( \int_0^1 (s^{-\alpha}\Omega_r(s,f))^q s^r ds/s \right)^{1/q}. \]

This completes the proof of Part II for \( 1 < q < \infty \). Proof for \( q = 1 \) can be obtained from the preceding calculations. To prove the inequality (20) for case \( q = \infty \), recall that in (24)

\[ \| \hat{\psi}_j(\Delta)f \| \leq 2^{-j(\alpha + M - 2r)/4} \int_0^1 s^{-\alpha}\Omega_r(s,f) s^{2r+\alpha} ds \]

\[ \leq 2^{-j(\alpha + M - 2r)/4} \sup_{0 < s < 1} (s^{-\alpha}\Omega_r(s,f)) \]

Therefore for any \( M > 3\alpha + 2r \)

\[ \sup_{j \geq 0} 2^{j\alpha} \| \hat{\psi}_j(\Delta)f \| \leq \sup_{j \geq 0} (2^{-j(M-3\alpha-2r)/4}) \sup_{0 < s < 1} (s^{-\alpha}\Omega_r(s,f)) \leq \sup_{0 < s < 1} (s^{-\alpha}\Omega_r(s,f)), \]

and this completes the proof for \( q = \infty \). \( \square \)

4. The Heisenberg Group

Let \( \mathbb{H}_n \) denote the \((2n + 1)\)-dimensional Heisenberg group \( \mathbb{H}_n \) identified by \( \mathbb{C}^n \times \mathbb{R} \). The multiplication is given by

\[ (z,t) \cdot (z',t') = \left( z + z', t + t - \frac{1}{2} Im((z,z')) \right) \quad (z,t), (z',t') \in \mathbb{H}_n, \]

with the identity \( e = (0,0) \) and \( (z,t)^{-1} = (-z,-t) \).

Any \( a > 0 \) defines an automorphism of \( \mathbb{H} \) defined by

\[ a(z,t) = (az, a^2t) \quad \forall (z,t) \in \mathbb{H}_n \]

(28)

where \( az = (a z_1, \ldots, a z_n) \). We fix the Haar measure \( dv \) on the Heisenberg group which is the Lebesgue measure \( dzdt \) on \( \mathbb{C}^n \times \mathbb{R} \). For each \( a > 0 \), the unitary dilation operator \( D_a \) on \( L^2(\mathbb{H}_n) \) is given by

\[ \delta_a f(z,t) = a^{-(n+1)} f(a^{-1}z, a^{-2}t) \quad \forall f \in L^2(\mathbb{H}_n), \]

(29)

and for any \( \omega \in \mathbb{H} \), the left translation operator \( T_\omega \) is given by

\[ l_\omega f(v) = f(\omega^{-1}v) \quad \forall v \in \mathbb{H}. \]

Define \( \tilde{f}(z,t) = \tilde{f}(-z,-t) \). We say \( f \) is self-adjoint if \( \tilde{f} = f \). It is easy to show that \( \delta_a f = \hat{\delta_a f} \). The \textit{quasiregular representation} \( \pi \) of the semidirect product \( G := \mathbb{H}_n \rtimes (0,\infty) \)
acts on $L^2(\mathbb{H}_n)$ by the dilation and translation operators, as follows. For any $f \in L^2(\mathbb{H})$ and $(\omega, a) \in G$ and $v \in \mathbb{H}_n$

$$(\pi(\omega, a)f)(v) := l_\omega \delta_a f(v) = a^{-(n+1)} f(a^{-1}(\omega^{-1}v)).$$

For $a > 0$ define $f_a(z, t) = a^{-2(n+1)} f(a^{-1}z, a^{-2}t)$. Thus $\delta_a f = a^{n+1} f_a$ and $\tilde{f}_a = \tilde{f}_a$.

We let $\mathcal{S}(\mathbb{H}_n)$ denote the space of Schwartz functions on $\mathbb{H}_n$. By definition $\mathcal{S}(\mathbb{H}_n) = \mathcal{S}(\mathbb{R}^{2n+1})$. Provided that the integral exists, for any functions $f$ and $g$ on $\mathbb{H}_n$, the convolution of $f$ and $g$ is defined by

$$f \ast g(\omega) = \int_{\mathbb{H}_n} f(\nu)g(\nu^{-1}\omega) \, d\nu.$$

We fix the basis $\frac{\partial}{\partial t}$ and $Z_j, \bar{Z}_j$, $j = 1, \ldots, n$ for the Lie algebra of $\mathbb{H}_n$ where

$$Z_j = 2 \frac{\partial}{\partial \bar{Z}_j} + i \frac{z_j}{2} \frac{\partial}{\partial t}, \quad \bar{Z}_j = 2 \frac{\partial}{\partial z_j} - i \frac{\bar{z}_j}{2} \frac{\partial}{\partial t}$$

and $\frac{\partial}{\partial z_j}$ and $\frac{\partial}{\partial \bar{Z}_j}$ are the standard derivations on $\mathbb{C}$ and $\frac{\partial}{\partial t}$ is the derivation operator in direction $\mathbb{R}$. These operators generate the algebra of left-invariant differential operators on $\mathbb{H}_n$. Associated to this basis, the Heisenberg sub-Laplacian is defined by

$$\Delta := -\frac{1}{2} \sum_j (Z_j \bar{Z}_j + Z_j \bar{Z}_j).$$

$\Delta$ is self-adjoint and positive definite.

For general introduction to the Heisenberg group and its representations we refer to the papers of Geller [15, 16].

4.1. Gelfand pairs associated to the Heisenberg group. Let $U(n)$ denote the group of $n \times n$ unitary matrices. This group is a maximal compact and connected Lie subgroup of the automorphisms group $\text{Aut}(\mathbb{H}_n)$ and

$$\sigma(z, t) = (\sigma z, t) \quad \forall \sigma \in U(n), \ (z, t) \in \mathbb{H}_n,$$

for any $z \in \mathbb{C}^n$ and $t \in \mathbb{R}$. These automorphisms are called rotations and are usually denoted by $R_\sigma$, instead. All compact connected subgroups of $\text{Aut}(\mathbb{H}_n)$ can be obtained by conjugating $U(n)$ by an automorphism.

Suppose $K \subseteq U(n)$ is a Lie compact subgroup acting on $\mathbb{H}_n$. A function $f$ on $\mathbb{H}_n$ is called $K$-invariant if for any $k \in K$ and $\omega \in \mathbb{H}_n$, $f(k\omega) = f(\omega)$. If we let $L^p_K(\mathbb{H}_n)$ denote the $K$-invariant subspace of $L^p(\mathbb{H}_n)$, then for $K = \{I\}$ we have $L^p_K(\mathbb{H}_n) = L^p(\mathbb{H}_n)$ where $I$ is the identity operator. For $K = U(n)$, the space $L^p_K(\mathbb{H}_n)$ contains all rotation invariant elements in $L^p(\mathbb{H}_n)$.

For a subgroup $K$, the pair $(K, \mathbb{H}_n)$ is called Gelfand pair associated to the Heisenberg group, or simply Gelfand pair, if the space of measurable $K$-invariant and integrable functions $L^1_K(\mathbb{H}_n)$ is a commutative algebra with respect to the convolution operator, i.e., $f \ast g = g \ast f$. It was known that $L^1_K(\mathbb{H}_n)$ is a commutative algebra for $K = U(n)$, and thus $(U(n), \mathbb{H}_n)$ is a Gelfand pair [6].
4.2. **K-spherical transform.** A smooth $K$-invariant function $\phi : \mathbb{H}_n \to \mathbb{C}$ is called $K$-spherical associated to the Gelfand pair $(K, \mathbb{H}_n)$ if $\phi(e) = 1$ and $\phi$ is joint eigenfunction for all differential operators on $\mathbb{H}_n$ which are invariant under the action of $K$ and $\mathbb{H}_n$. Equivalently, the $K$-spherical functions are homomorphisms of the commutative algebra $L^1_K(\mathbb{H}_n)$. The general theory of $K$-spherical functions for Gelfand pairs $(K, \mathbb{H}_n)$ was studied by Benson et al. at [5, 6] and [39].

The set of $K$-spherical functions associated to the pair $(K, \mathbb{H}_n)$ is identified by the space $\mathbb{R}^* \times \mathbb{N}^n$ and can be explicitly computed for concrete examples of $K$ ([6]). The space $\mathbb{R}^* \times \mathbb{N}^n$ is called Gelfand space. Let $\phi_{\lambda, m}$ denote the $K$-spherical function associated to $(\lambda, m) \in \mathbb{R}^* \times \mathbb{N}^n$. Then for some polynomial $q_m$ on $\mathbb{C}^n$ with $|q_m(z)| \leq |q_m(0)| = 1,$

$$\phi_{\lambda, m}(z, t) = e^{i\lambda t} e^{-|\lambda| |z|^2/4} q_m(\sqrt{|\lambda|} z) \quad \forall z \in \mathbb{C}^n, \ t \in \mathbb{R}.$$  

The $K$-spherical functions $\phi_{\lambda, m}$ are eigenfunctions of the sub-Laplacian operator $\Delta$ with eigenvalues given by

$$\Delta(\phi_{\lambda, m}) = |\lambda|(2|m| + n)\phi_{\lambda, m},$$

where $|m| = |m_1| + \cdots + |m_n|$.

**Definition.** ($K$-spherical transform associated to $\Delta$) The $K$-spherical transform $\mathcal{F} := \mathcal{F}_K$ of a function $f \in L^1_K(\mathbb{H}_n)$ at the character $(\lambda, m) \in \mathbb{R}^* \times \mathbb{N}^n$ is defined by

$$\mathcal{F}(f)(\lambda, m) := \int_{\mathbb{H}_n} f(z,t)\phi_{\lambda, m}(z,t)dzdt,$$  

(30)

where $dzdt$ is the Haar measure for $\mathbb{H}_n$ and the integral is well-defined.

**Lemma 4.1.** For any $f$ and $g$ in $L^1_K \cap L^2_K(\mathbb{H}_n)$

$$\mathcal{F}(f \ast g)(\lambda, m) = \mathcal{F}(f)(\lambda, m)\mathcal{F}(g)(\lambda, m) \quad a.e. \ (\lambda, m).$$

And, by density of $L^1_K \cap L^2_K(\mathbb{H}_n)$ in $L^2_K(\mathbb{H}_n)$ the assertion also holds for all $f$ and $g$ in $L^2_K(\mathbb{H}_n)$.

**Proof.** This is straightforward from (30). \hfill \Box

The definition (30) implies that the spherical Fourier transform $\mathcal{F}(f)$ is bounded with $\|\mathcal{F}(f)\|_{\infty} \leq \|f\|_1$ and lies in $C_0(\mathbb{R}^* \times \mathbb{N}^n)$, the space of continuous functions with fast decay at infinity. (We say a function $F : \mathbb{N}^n \to \mathbb{C}$ has fast decay at infinity if for any sequence $\{a_m\}$ of complex numbers with $|a_m| \to 0$ as $|m| \to \infty$,

$$\lim_{|m| \to \infty} |a_m F(m)| < \infty.$$  

The Godement’s Plancherel theory for the Gelfand pairs (see [21]) guaranties the existence of a unique positive Borel measure $d\mu$ on $\mathbb{R}^* \times \mathbb{N}^n$ for which for all continuous functions $f \in L^1_K(\mathbb{H}_n) \cap L^2_K(\mathbb{H}_n)$

$$\int_{\mathbb{H}_n} |f(z,t)|^2dzdt = \int_{\mathbb{R}^*} \sum_{n \in \mathbb{N}^n} |\mathcal{F}(f)(\lambda, m)|^2d\mu.$$  

(31)
The Godement-Plancherel measure \( d\mu \) on the Gelfand space \( \mathbb{R}^* \times \mathbb{N}^n \) is given explicitly by

\[
\int_{\mathbb{R}^* \times \mathbb{N}^n} F(\lambda, m) d\mu(\lambda, m) = (2\pi)^{-(n+1)} \int_{\mathbb{R}^*} \sum_{n^m} w_m F(\lambda, m) |\lambda|^n d\lambda
\]

for some positive constant weights \( w_m \) dependent on degree of the polynomials \( g_m \) and \( d\lambda \) is the Lebesgue measure on \( \mathbb{R} \). (For the proof of this see e.g. [5] and [42].)

The inversion formula for a continuous function \( f \in L^1_K(\mathbb{H}_n) \cap L^2_K(\mathbb{H}_n) \) with integrable \( K \)-spherical transform \( F(f) \) is given by

\[
f(z, t) = (2\pi)^{-(n+1)} \int_{\mathbb{R}^* \times \mathbb{N}^n} w_m F(f)(\lambda, m) \phi_{\lambda, m}(z, t) |\lambda|^n d\lambda. \tag{32}
\]

The transform \( F \) extends uniquely to a unitary operator between \( L^2_K(\mathbb{H}_n) \) and the Hilbert space \( L^2(\mathbb{R}^* \times \mathbb{N}^n, d\mu) \). We let \( F \) denote this unitary operator. Therefore the inversion formula (32) also holds for all \( f \in L^2_K(\mathbb{H}_n) \) in the weak sense.

Let \( f \in L^2_K(\mathbb{H}_n) \) and \( k \) be a natural number. Then \( f \) lies in the domain of \( \Delta^k \) if and only if the measurable map \( (\lambda, m) \mapsto |\lambda|^k(2|m| + n)^k F(f)(\lambda, m) \) is in \( L^2(\mathbb{R}^* \times \mathbb{N}^n, d\mu) \). This indicates that the domain \( D(\Delta^k) \) is transferred by the unitary operator \( F \) onto the subspace of functions \( f \in L^2(\mathbb{H}_n) \) for which

\[
\int_{\mathbb{R}^* \times \mathbb{N}^n} |\lambda|^{2k}(2|m| + n)^{2k} |F(f)(\lambda, m)|^2 d\mu(\lambda, m) < \infty.
\]

We let \( \Delta^k \) denote the closure of \( \Delta^k \) on \( L^2_K(\mathbb{H}_n) \). The preceding also shows that this operator is uniquely equivalent to a multiplication operator \( M_k \) acting on \( L^2(\mathbb{R}^* \times \mathbb{N}^n, d\mu) \) through \( F \) where for any \( F \in L^2(\mathbb{R}^* \times \mathbb{N}^n, d\mu) \)

\[
M_k(F) : (\lambda, m) \mapsto |\lambda|^k(2|m| + n)^k F(\lambda, m).
\]

From above we can conclude that if \( \beta \in L^\infty(0, \infty) \), then for any \( f \in L^2_K(\mathbb{H}_n) \)

\[
F(\beta(\Delta) f)(\lambda, m) = \beta(|\lambda|(2|m| + n)) F(f)(\lambda, m) \quad \text{a.e. (\lambda, m)} \tag{33}
\]

It is known by the spectral theory that for any bounded \( \beta \in L^\infty(0, \infty) \) the operator \( \beta(\Delta) \) is a bounded and kernel operator on \( L^2 \) with a kernel in \( L^2 \). The following constructive lemma shows how to obtain this kernel using the \( K \)-spherical Fourier transform for the Gelfand pairs \( (K, \mathbb{H}_n) \).

**Lemma 4.2.** Let \( \beta \in L^\infty(0, \infty) \). For \( a > 0 \) define \( \beta^a(\xi) = \beta(a\xi) \). Then the integral operator \( \beta^a(\Delta) \) is a convolution operator and there exists \( B \in L^2_K(\mathbb{H}_n) \) such that for any \( f \in L^2_K(\mathbb{H}_n) \)

\[
\beta^a(\Delta) f = f * B_{|\sigma} \tag{34}
\]

where \( B_{|\sigma}(\omega) = a^{-(n+1)} B(a^{-1/2} \omega) \). If \( \beta \) is real-valued, then \( \tilde{B} = B \).

**Proof.** Without loss of generality we prove that \( \beta^a(\Delta) f = f * B_a \). Let \( \alpha : \mathbb{R}^* \times \mathbb{N}^n \mapsto (0, \infty) \) be the measurable map defined by \( \alpha(\lambda, m) = |\lambda|(2|m| + n) \). Since \( \beta \) is bounded, the map \( \beta \circ \alpha : (\lambda, m) \mapsto \beta(|\lambda|(2|m| + n)) \) is bounded, measurable, and due to the finiteness of the measure \( d\mu \) it lies in \( L^2(\mathbb{R}^* \times \mathbb{N}^n, d\mu) \). Let \( B \in L^2_K(\mathbb{H}_n) \) denote the spherical Fourier inverse of \( \beta \circ \alpha \). Therefore for a.e. \( (\lambda, m) \)

\[
F(B)(\lambda, m) = \beta(|\lambda|(2|m| + n)).
\]
By the definition of dilation operator and the preceding result, for any \( a > 0 \) we have
\[
\mathcal{F}(B_a)(\lambda, m) = \beta(a^2|\lambda(2|m| + n)) = \beta(a^2(|\lambda(2|m| + n)));
\] (34)
thus
\[
\mathcal{F}^{-1}(\beta a^2 \circ \alpha) = B_a.
\]
To complete the proof of the lemma, we need to show that the operator \( \beta a^2(\Delta) \) is a convolution operator and for any \( f \in L^2_K(\mathbb{H}_n) \)
\[
\beta a^2(\Delta)f = f * B_a
\] (35)
in \( L^2 \)-norm. If we replace \( \beta \) by \( \beta a^2 \) in (33), for almost every \((\lambda, m)\) we get
\[
\mathcal{F}(\beta a^2(\Delta)f)(\lambda, m) = \beta a^2(|\lambda(2|m| + n)) \mathcal{F}(f)(\lambda, m).
\]
And, by (34)
\[
\mathcal{F}(B_a)(\lambda, m) = \beta a^2(|\lambda(2|m| + n)).
\]
Using this and Lemma 4.1 we arrive at
\[
\mathcal{F}(\beta a^2(\Delta)f)(\lambda, m) = \mathcal{F}(B_a)(\lambda, m) \mathcal{F}(f)(\lambda, m) = \mathcal{F}(f \ast B_a)(\lambda, m) \quad \text{a.e. } (\lambda, m).
\]
The inverse of spherical Fourier transform concludes that \( \beta a^2(\Delta)f = f \ast B_a \) in \( L^2 \) norm. The fact that \( \tilde{B} = B \) for the real-valued function \( \beta \) is an immediate application of the spectral theory.

\[ \square \]

**Theorem 4.3.** ([24]) Let \( \beta \in \mathcal{S}(\mathbb{R}^+) \). Then \( B \), the kernel of operator \( \beta(\Delta) \), is in \( \mathcal{S}(\mathbb{H}_n) \) and
\[
\beta(\Delta)f = f \ast B \quad \forall \, f \in L^2.
\] (36)

**Notation.** In the sequel we shall call \( B \) the distribution kernel for \( \beta \). Note that in our situation the kernel \( B \) is the inverse of \( K \)-spherical transform of the map \( \beta \circ \alpha \) (see above for the definition of \( \alpha \)). And, from now on, if \( \hat{\psi} \in L^\infty(0, \infty) \), then \( \Psi \in L^2_K(\mathbb{H}_n) \) denotes the distribution kernel of the operator \( \hat{\psi}(\Delta) \).

### 4.3. Wavelets for \( L^2_K(\mathbb{H}_n) \).

The existence of wavelets in \( L^2 \) was proved by Liu-Peng [26] for the Heisenberg group, by Führ [13] (Corollary 5.28) for general homogeneous groups, and by Currey [12] for nilpotent Lie groups. The construction of Shannon wavelets using multiresolution analysis method was presented for the Heisenberg group in [27].

In contrast to those works, this article does not use any representation theory. The first wavelet systems on stratified Lie groups possessing a lattice were constructed by Lemarié [25], by suitably adapting concepts from spline theory. More recent construction of both continuous and discrete wavelet systems were based on the spectral theory of the sub-Laplacian for stratified Lie groups [17]. Their wavelets are Schwartz and enjoy all vanishing moments, or compactly supported with arbitrary many vanishing moments. Using the spectral theory methods, construction of smooth wavelets with more significant properties were introduced on compact and smooth manifolds in [18] and [19]. In this article, the construction of continuous and bandlimited wavelets and the classification of Besov spaces in terms of these wavelets.
are based on the spectral theory techniques. The definition of a wavelet on $\mathbb{H}_n$ is given as following.

We say $\varphi \in L^2_K(\mathbb{H}_n)$ is a continuous wavelet if for any $f \in L^2_K(\mathbb{H}_n)$ the isometry

$$\|f\|^2 = c_\varphi \int_{\mathbb{H}_n} \int_0^\infty |\langle f, l_w \delta_a \varphi \rangle|^2 d\mu(a, w)$$

holds for some constant $c = c_\varphi > 0$, where $d\mu(a, w) = a^{-(2n+3)} du dw$ is the left Haar-measure for the product group $\mathbb{H}_n \times (0, \infty)$. Associated to a wavelet $\varphi$ and a function $f \in L^2_K$, we define the coefficient map $W_{f, \varphi} : \mathbb{H}_n \times (0, \infty) \ni (w, a) \mapsto \langle f, l_w \delta_a \varphi \rangle$ and call $\langle f, l_w \delta_a \varphi \rangle$ the continuous wavelet coefficient of $f$ at position $w$ and scale $a$. The isometry (37) implies the following inversion formula

$$f = c_\varphi^{-1} \int_{\mathbb{H}_n} \int_0^\infty \langle f, l_w \delta_a \varphi \rangle l_w \delta_a \varphi d\mu(a, w),$$

where the integral is understood in the weak sense. (38) can simply be interpreted as reconstruction of $f$ through its wavelet coefficients $\langle f, l_w \delta_a \varphi \rangle$ associated to the wavelet $\varphi$.

We say a measurable function $\nu : [0, \infty) \to \mathbb{R}$ is admissible if $\int_0^\infty |\nu(t)|^2 dt/t < \infty$.

Theorem 4.4. Let $\alpha$ be the measurable map defined in Lemma 4.2. Let $\nu : [0, \infty) \to \mathbb{R}$ be a measurable function and $\nu \circ \alpha$ is in $L^2(\mathbb{R}^* \times \mathbb{N}^n)$. Put $\varphi := F^{-1}(\nu \circ \alpha)$. Then $\varphi$ is a continuous wavelet if and only if $\nu$ is admissible. In this case $c_\varphi = \frac{1}{2} \int_0^\infty |\nu(t)|^2 dt/t$.

Proof. Put $Q := 2n + 2$. Then for any $f \in L^2_K(\mathbb{H}_n)$

$$\int_0^\infty \int_{\mathbb{H}_n} |\langle f, l_w \delta_a \varphi \rangle|^2 a^{-(Q+1)} du dw = \int_0^\infty \int_{\mathbb{H}_n} |f \ast \delta_a \varphi(w)|^2 a^{-(Q+1)} du dw$$

$$= \int_0^\infty \|f \ast \delta_a \varphi\|^2 a^{-(Q+1)} du$$

(39)

By the Plancherel theorem and Lemma 4.1

$$\int_0^\infty \|\mathcal{F}(f \ast \delta_a \varphi)\|^2 d\mu(a)$$

$$= \int_0^\infty \int_{\mathbb{R}^n} \sum_m |\mathcal{F}(f)(\lambda, m)|^2 |\mathcal{F}(\delta_a \varphi)(\lambda, m)|^2 d\mu(\lambda, m) a^{-(Q+1)} da$$

$$= \int_0^\infty \int_{\mathbb{R}^n} \sum_m |\mathcal{F}(f)(\lambda, m)|^2 |(\nu \circ \alpha)(\lambda, m)|^2 d\mu(\lambda, m) a^{-1} da$$

$$= \frac{1}{2} \int_0^\infty \int_{\mathbb{R}^n} \sum_m |\mathcal{F}(f)(\lambda, m)|^2 |(\nu \circ \alpha)(\lambda, m)|^2 d\mu(\lambda, m) dt/t$$

$$= \frac{1}{2} \int_{\mathbb{R}^n} \sum_m |\mathcal{F}(f)(\lambda, m)|^2 \left( \int_0^\infty |(\nu \circ \alpha)(\lambda, m)|^2 dt/t \right) d\mu(\lambda, m).$$

By dilation invariant property of the measure $dt/t$, the inner integral is

$$\int_0^\infty |(\nu \circ \alpha)(\lambda, m)|^2 dt/t = \int_0^{\infty} |\nu(t)|^2 dt/t.$$
Interfering this in the previous calculations, we can immediately conclude that \( \varphi \) is a wavelet if and only if \( \int_{0}^{\infty} |\nu(t)|^2 \, dt/t < \infty \).

\[ \square \]

**Corollary 4.5.** Let \( \nu \in L^\infty(0, \infty) \) and \( \nu \circ \alpha \in L^2(\mathbb{R}^* \times \mathbb{N}^n) \). Then \( \varphi = F^{-1}(\nu \circ \alpha) \) is a wavelet if one of the followings holds:

(a) \( \nu(0) = 0 \)

(b) \( \nu \) is with support away from zero.

(c) \( \nu \) is given by \( \nu(\xi) = \xi^k \nu_0(\xi) \) where \( 0 \neq \nu_0 \in \mathcal{S}(\mathbb{R}^+) \).

### 4.4. Coefficient characterization of Besov norms.

Let \( \hat{\psi} \) and \( \hat{\psi}_j \) be the same bounded and real-valued measurable functions on \( \mathbb{R}^+ \) introduced in the proof of Lemma 4.2 in Section 3. If we let \( \Psi \) and \( \Psi_j \) denote the (distribution) kernel of the operators \( \hat{\psi}(\Delta) \) and \( \hat{\psi}_j(\Delta) \), respectively, (see the definition followed by Lemma 4.2) then we have

**Lemma 4.6.** The following hold:

(a) \( \Psi \) is a wavelet

(b) \( \Psi \in D(\Delta^k) \) for any natural number \( k \), and

(c) \( \Psi_j = 2^{-j(n+1)} \delta_{2^{-j}} \).

**Proof.** (a) and (b) are trivial by Corollary 4.5 and the bandlimitedness of \( \Psi \). The proof of (c) is deduced from the proof of Lemma 4.2 and the spectral theory.

Let \( Y = L^2_{\alpha,q}(\mathbb{H}_n \times \mathbb{Z}^+) \), \( \alpha > 0 \), \( 1 \leq q < \infty \), denote the space of measurable functions \( F \) on \( \mathbb{H}_n \times \mathbb{Z}^+ \) for which

\[
\|F\|_Y := \left( \sum_{j \geq 0} 2^{-j(q(n+1)-\alpha)} \left( \int_{\mathbb{H}_n} |F(w, j)|^q \, dw \right)^{q/2} \right)^{1/q} < \infty,
\]

and with standard definition for \( q = \infty \). \( \| \cdot \|_Y \) defines a complete norm for \( Y \). For any wavelet \( \Psi \) and any function \( f \) we shall call the map \( \mathbb{H}_n \times \mathbb{N}^n \ni (w, j) \mapsto \langle f, T_w D_{2^{-j}} \Psi \rangle \) the wavelet coefficient map and denote it by \( W_{f,\Psi} \).

**Theorem 4.7.** [Main Theorem] There exists a bandlimited wavelet \( \Psi \in L^2_K(\mathbb{H}_n) \) such that \( f \in L^2_K(\mathbb{H}_n) \) is in Besov space \( B^\alpha_{2,q} \) if and only if its wavelet coefficient map \( W_{f,\Psi} \) belongs to the Banach space \( Y = L^2_{\alpha,q} \). And,

\[
\|f\|_{B^\alpha_{2,q}} \leq \|f\| + \left( \sum_{j \geq 0} 2^{-j(q(n+1)-\alpha)} \|f * \delta_{2^{-j}} \Psi^*\|^q \right)^{1/q}
\]

**Proof.** Let \( \hat{\phi}, \hat{\psi}, \hat{\psi}_j \) be the same functions introduced in Section 3 for which the equation (6) holds. Let \( \Phi \) denote the kernel of operator \( \hat{\phi}(\Delta) \). Then by Lemma 4.2 we have

\[
\hat{\phi}(\Delta)f = f * \Phi, \quad \hat{\psi}_j(\Delta)f = f * \Psi_j \quad \forall \quad f \in L^2_K(\mathbb{H}_n).
\]

(40)

This translates the Besov norm to

\[
\|f\|_{B^\alpha_{2,q}} \leq \|f \Phi\| + \left( \sum_{j \geq 0} 2^{j\alpha} \|f \Psi^j\|^q \right)^{1/q} \leq \|f\| + \left( \sum_{j \geq 0} 2^{j\alpha} \|f \Psi^j\|^q \right)^{1/q}
\]
This is same as to say that
\[
\|f\|_{B_{2,q}^\alpha} \asymp \|f\| + \left( \sum_{j \geq 0} 2^{j\alpha} \left( \int_{\mathbb{H}_n} |\langle f, l^j w \Psi \rangle|^2 dw \right)^{q/2} \right)^{1/q}.
\]

By Lemma 4.6 we have \(\Psi^j = 2^{-j(n+1)} \delta_{2^{-j} \Psi}\). Therefore
\[
\|f\|_{B_{2,q}^\alpha} \asymp \|f\| + \left( \sum_{j \geq 0} 2^{-j(q(n+1)-\alpha)} \left( \int_{\mathbb{H}_n} |\langle f, l^j w \delta_{2^{-j} \Psi} \rangle|^2 dw \right)^{q/2} \right)^{1/q}
\]
and this completes the proof of the theorem. \(\square\)

**Lemma 4.8.** For given \(f \in L_K^2(\mathbb{H}_n)\) and \(a > 0\),
\[
\mathcal{F}(\delta_a f)(\lambda, m) = a^{(n+1)} \mathcal{F}(f)(a^2 \lambda, m) \quad \text{for a.e. } (\lambda, m).
\]

**Proof.** The proof is straightforward by the definition of dilation \(\delta_a\) and the \(K\)-spherical Fourier transform. \(\square\)

Based on above lemma, for any \(a > 0\) we define the dilation \(A_a\) of function \(F\) on the Gelfand space \(\mathbb{R}^* \times \mathbb{N}^0\) by
\[
A_a F(\lambda, m) = a^{-(n+1)/2} F(a^{-1} \lambda, m).
\]
Therefore the characterization of \(K\)-spherical Besov norms in terms of Gelfand transform of a wavelet follows.

**Corollary 4.9.** Let \(\alpha > 0\) and \(\Psi\) be the same as in Theorem 4.7. Then for any \(f \in L_K^2(\mathbb{H}_n)\)
\[
\|f\|_{B_{2,q}^\alpha} \asymp \|f\| + \left( \sum_{j \geq 0} (2^{-j(n+1)-\alpha/q} \|\mathcal{F}(f) A_{2^j} \mathcal{F}(\Psi)\|^q)^{1/q} \right)
\]

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