Asymptotics of the Fourier and Laplace transforms in weighted spaces of analytic functions *

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In memory of Evsei Dyn’kin (1949–1999)

Abstract

We study the asymptotics near the origin of the Fourier transform in weighted Hardy spaces of analytic functions in the upper half-plane, and of the Laplace transform in weighted spaces of entire functions of zero exponential type.

These results are applied to two closely related problems posed by E. Dyn’kin: we find the asymptotics of the depth of zero in non-quasianalytic Denjoy-Carleman classes, and of the exact Levinson-Sjöberg majorant.

1 Main results

Let \( H^p \), \( 1 \leq p \leq \infty \), be the Hardy spaces of analytic functions in the upper half-plane \( \mathbb{C}_+ \), and let \( W \) be a non-vanishing analytic function in \( \mathbb{C}_+ \). The weighted Hardy spaces \( H^p(W) \) are defined as follows:

\[
H^p(W) = \{ f : f \text{ is analytic in } \mathbb{C}_+, \ W f \in H^p \},
\]

\[
\| f \|_{H^p(W)} \overset{def}{=} \| W f \|_{H^p}.
\]

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Throughout this paper, we assume that the weight function $W$ is an outer function, that is

$$W(z) = \exp\left\{ \frac{1}{\pi i} \int_{\mathbb{R}} \left[ \frac{1}{t-z} - \frac{t}{t^2+1} \right] \varphi(t) \, dt \right\}. \quad (1.1)$$

About the logarithmic weight $\varphi$ we always assume that

(i) $\varphi$ is non-negative and even on $\mathbb{R}$;
(ii) $\varphi$ increases on $(0, \infty)$ and

$$\lim_{t \to \infty} \frac{\varphi(t)}{\log t} = \infty;$$

(iii)

$$\int_{\mathbb{R}} \varphi(t) \frac{t}{t^2+1} \, dt < \infty;$$

(iv) at least one of the following two conditions holds true:

(iv-a) the function $\tau \mapsto \varphi(e^\tau)$ is convex;
(iv-b) the function $\varphi(t)$ is concave on $(0, \infty)$ and

$$\lim_{t \to \infty} t \varphi'(t + 0) = \infty.$$ 

(In the case (iv-a), the latter limit is also infinite due to condition (ii)).

Below, we tacitly assume that $\varphi$ is a $C^2$-function, and $\varphi(0) = 0$; however these assumptions can be easily dropped.

Functions from the space $H^p(W)$ rapidly decrease along the real axis, and we can define the (inverse) Fourier transform

$$(\mathcal{F}^{-1} f)(s) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-isx} \, dx.$$

Our assumptions guarantee that $\mathcal{F}^{-1} f$ is a $C^\infty$-function vanishing for $s \leq 0$ (the integration can be shifted to the line $\mathbb{R} + iy$ for an arbitrarily large positive $y$). The question we are interested in is:

- how deep is the zero of the function $(\mathcal{F}^{-1} f)(s)$ at $s = 0$?
We measure its depth by
\[ \rho_{p,W}(s) \overset{\text{def}}{=} \sup_{||f||_{H^p(W)} \leq 1} |(\mathcal{F}^{-1}f)(s)|, \]
and we compare this quantity with the (upper) Legendre transform
\[ Q(s) \overset{\text{def}}{=} \sup_{y>0} \log |W(iy)| - ys = \sup_{y>0} \left[ \frac{2y}{\pi} \int_0^\infty \frac{\varphi(t)dt}{t^2 + y^2} - ys \right]. \quad (1.2) \]
After a change of variables, one can express this function in another form. Let \( q(y) = \log |W(iy)| \). In our assumptions (i)–(iv), the function \( q(y) \) is a concave increasing function, and \( q'(y) \) steadily tends to zero when \( y \) goes to infinity (see Corollary 3.2 below). Then
\[ Q(q'(y)) = q(y) - yq'(y) = \frac{4y^3}{\pi} \int_0^\infty \frac{\varphi(t)dt}{(t^2 + y^2)^2}. \quad (1.3) \]
Loosely speaking, the function \( Q(q'(y)) \) has the same growth as \( \varphi(y) \); more precisely, one can check using (1.3) that
\[ \frac{4}{3\pi} \varphi(y) \leq Q(q'(y)) \leq \frac{16}{3\pi} y^3 \int_y^\infty \frac{\varphi(t)}{t^4} dt. \]

**Theorem 1.1** Let the logarithmic weight \( \varphi \) satisfy conditions (i)-(iv), and let
\[ (v) \quad \lim_{t \to \infty} \frac{t \varphi'(t)}{t^3 \int_t^\infty \frac{\varphi(\xi)}{\xi^2} d\xi} = +\infty. \]

Then for \( s \to 0 \)
\[ \log \rho_{p,W}(s) = -(1 + o(1))Q(s), \quad (1.4) \]
and
\[ \left( \mathcal{F}^{-1} \frac{1}{W} \right)(s) = (1 + o(1)) \sqrt{Q''(s)} e^{-Q(s)}. \quad (1.5) \]
Furthermore, if additionally, \( \varphi \) has a positive lower order, i.e
\[ (vi) \quad \liminf_{t \to \infty} \frac{\log \varphi(t)}{\log t} > 0, \]
then for \( s \to 0 \)
\[ \log \rho_{p,W}(s) = -Q(s) + O(\log Q(s)). \quad (1.6) \]
Condition (v) is not very restrictive since generally speaking $t\varphi'(t)$ has the same order of growth as the expression
\[ t^3 \int_t^\infty \frac{\varphi(s)}{s^4} ds. \]
For example, if $\varphi$ is concave (or, less restrictively, the function $\varphi'(t)/t$ does not increase), and if the function $\varphi(t) \log^{-3} t$ steadily increases with $t$ to $+\infty$, then by a straightforward inspection condition (v) holds. We do not know whether condition (v) is really needed for asymptotics (1.4) and (1.6).

In a special case, when the weight function is an entire function of genus zero with zeros on the negative imaginary semi-axes:

\[ W(z) = \prod_k \left( 1 + \frac{z}{ia_k} \right), \quad \sum_k \frac{1}{a_k} < \infty, \]

asymptotics (1.4), (1.3) and (1.4) hold without any additional restrictions. This can be deduced from a result of Hirschman and Widder [16, Chapter V, §3].

A dual problem to the one considered above is:

- to estimate the growth of the Laplace transform

\[ (\mathcal{L}f)(\zeta) = \int_0^\infty f(x)e^{-\zeta x} dx, \]

of an entire function $f$ of zero exponential type, when $\zeta$ approaches the origin.

Indeed,
\[ t^3 \int_t^\infty \frac{\varphi'((\xi/t)^2)}{\xi^4} d\xi = \frac{1}{3} \left\{ \varphi(t) + t^3 \int_t^\infty \frac{\varphi'((\xi/t)^2)}{\xi^3} d\xi \right\} \leq \frac{1}{3} \left\{ \varphi(t) + t\varphi'(t) \right\}. \]

Since the function $\varphi(t) \log^{-3} t$ increases, we have
\[ t\varphi'(t) \geq \frac{3\varphi(t)}{\log t}, \]
and
\[ \frac{t\varphi'(t)}{\varphi^{2/3}(t)} \geq \frac{3\varphi^{1/3}(t)}{\log t} \uparrow \infty. \]

This yields condition (v)
Let $W$ be an outer function (1.1). Define the space $B(W)$ of entire functions:

$$B(W) = \{ f : f \text{ is entire}, \quad f/W, f^*/W \in H^\infty \},$$

where $f^*(z) = \overline{f(\bar{z})}$,

$$||f||_{B(W)} \overset{def}{=} \sup_{t \in \mathbb{R}} \frac{|f(t)|}{|W(t)|}.$$

Since the function $W$ is outer, $B(W)$ consists of entire functions of zero exponential type. The Laplace transform $(\mathcal{L}f)(\zeta)$ is analytic in the right half-plane $\text{Re} \zeta > 0$, and since $f$ has zero exponential type, we can turn the integration line in any direction and thus obtain that $\mathcal{L}f$ is analytic in $\mathbb{C} \setminus \{0\}$ and vanishes at infinity. Set

$$\lambda_W(s) \overset{def}{=} \sup_{||f||_{B(W)} \leq 1} \max_{|\zeta| = s} |(\mathcal{L}f)(s)|.$$

**Theorem 1.2** Let the logarithmic weight $\varphi$ satisfy conditions (i)-(iii), (iv-a) and (iv-b). Then for $s \to 0$

$$\log \lambda_W(s) = (1 + o(1))Q(s). \quad (1.7)$$

If additionally condition (vi) holds, then

$$\log \lambda_W(s) = Q(s) + O(\log Q(s)). \quad (1.8)$$

We do not know whether condition (iv-a) is needed for asymptotics (1.7).

It is possible to introduce the scale of de Branges-type spaces $B^p(W)$, $1 \leq p \leq \infty$. In all these spaces the asymptotic relations (1.7) and (1.8) continue to hold.

The proofs are based on the classical Laplace method of asymptotic evaluation of integrals. Let us mention briefly another situation when the technique developed here could be useful. Let us drop condition (iii) and consider the so-called quasianalytic weights $\varphi(t)$ such that

$$\int_{\mathbb{R}} \frac{\varphi(t)}{t^2 + 1} \, dt = \infty,$$
but
\[ \int_{\mathbb{R}} \frac{\varphi(t)}{1 + |t|^3} dt < \infty. \]

Then one can define an analytic weight function \( W(z) \) using the modified Schwartz integral:
\[ W(z) = \exp \left\{ \frac{1}{\pi i} \int_{\mathbb{R}} \frac{(tz + 1)^2 \varphi(t)}{(t^2 + 1)^2(t - z)} dt \right\}, \]
and look at the asymptotics of the Fourier transform
\[ \left( \mathcal{F} \frac{1}{W} \right)(s) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{e^{isx}}{W(x)} dx, \quad s \to \infty, \]
and of the Laplace transform
\[ \left( \mathcal{L} \frac{1}{W} \right)(s) = \int_0^\infty \frac{e^{sy}}{W(iy)} dy, \]
when \( s \to \infty \) in \( \mathbb{C} \). These functions are closely related to the study of primary ideals in weighted convolution algebras on \( \mathbb{R} \) with quasianalytic weights (cf. Borichev’s work [4] and references therein).

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2 Some applications

2.1 Depth of zero in non-quasianalytic classes

Let \( C\{M_n\} \) be a non-quasianalytic Denjoy-Carleman class of \( C^\infty \)-functions on \( \mathbb{R} \) such that
\[ \sup_{\mathbb{R}} |g^{(n)}| \leq M_n, \quad n = 0, 1, 2, \ldots. \]

We are interested in

- the asymptotic behaviour of the quantity
\[ \delta_{\{M_n\}}(s) = \sup \left\{ |g(s)| : g \in C\{M_n\}, \quad g^{(n)}(0) = 0, \quad n = 0, 1, 2, \ldots \right\}, \]
when \( s \to 0 \).
This problem was raised and treated by Th. Bang in the paper [1]; a related problem in quasianalytic classes was considered earlier by Carleman [7, pp. 24–27]. Bang observed that if the sequence \( \{M_n\} \) grows sufficiently slowly (but still non-quasianalytically), then the estimate obtained from the Taylor formula

\[
\delta(M_n)(s) \leq \inf_{n \geq 0} \frac{M_n s^n}{n!}
\]  

leads to a very crude bound which does not capture the transition from non-quasianalyticity to quasianalyticity. For example, if

\[
M_n = n!(\log n)^{n(1+\beta)}, \quad \beta > 0,
\]

then (2.1) yields

\[
\log \frac{1}{\delta(M_n)(s)} \geq e^{cs^{-1/(\beta+1)}}.
\]

This bound does not blow up when \( \beta \to 0 \) as it should. The bound obtained by Bang by an ingenious and elementary method is

\[
\log \frac{1}{\delta(M_n)(s)} \geq e^{cs^{-1/\beta}}.
\]

The problem of finding the asymptotic behaviour of \( \delta(M_n)(s) \) was raised anew by Dyn'kin [13] in the seventies at a time when he was not aware of Bang’s work. A lower bound for \( \delta(M_n)(s) \) follows from the construction of V. P. Gurarie [15] and Beurling [2]. The upper bound for \( \delta(M_n)(s) \) was obtained by Volberg [25] and Dyn’kin [14] by different methods. Borichev has a related result in [3]. These results provide good bounds for \( |\log \delta(M_n)(s)| \) but are far from being the asymptotics. For example, in the case (2.2), the best they could achieve is

\[
 cs^{-1/\beta} \leq \log |\log \delta(M_n)(s)| \leq Cs^{-1/\beta}, \quad s \leq s_0.
\]

The discrepancy between the upper and lower bounds worsens if the class \( C\{M_n\} \) is “less” non-quasianalytic.

Here, using Theorem 1.1 we obtain a much sharper result. Let

\[
T(r) = \sup_{n \geq 0} \frac{r^n}{M_n}
\]
be A. Ostrowski’s function. Then the function
\[ \varphi(t) = \log T(|t|) = \sup_{n \geq 0} [n \log |t| - \log M_n] \]
(2.3)
automatically satisfies conditions (i)-(iii) and (iv-a).

**Lemma 2.1** Let \( W \) be the weight function defined by (1.1) with \( \varphi \) given by (2.3). Then
\[ \sqrt{2\pi} \rho_{1,W}(s) \leq \delta_{\{M_n\}}(s) \leq \frac{e}{\sqrt{2\pi}} s \rho_{\infty,W}(s). \]
(2.4)

**Proof of Lemma 2.1:** We repeat Carleman’s argument [7] with minor variations. Let \( g \in C\{M_n\} \), and \( g^{(n)}(0) = 0, \ n \in \mathbb{Z}_+ \). Without loss of generality, we assume that \( g(s) \equiv 0 \) for \( s \leq 0 \), and consider the Fourier transform
\[ (\mathcal{F}g)(z) = \frac{1}{\sqrt{2\pi}} \int_0^\infty g(s) e^{isz} \, ds \]
analytic in the upper half-plane. Then
\[ (\mathcal{F}g)(z) = \frac{(-1)^n}{(iz)^n \sqrt{2\pi}} \int_0^\infty g^{(n)}(s) e^{isz} \, ds, \]
whence
\[ |(\mathcal{F}g)(z)| \leq \frac{1}{\sqrt{2\pi} \gamma T(|z|)}, \quad z \in \mathbb{C}_+, \]
or
\[ ||(\mathcal{F}g)(z+i\tau)||_{\mathcal{H}^\infty(W)} \leq \frac{1}{\sqrt{2\pi} \tau}, \quad \tau > 0, \]
where the weight \( W \) is defined by (2.3). Then
\[ g(s) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (\mathcal{F}g)(x+i\tau) e^{-is(x+i\tau)} \, dx, \]
and
\[ |g(s)| \leq \frac{e^{\pi \tau}}{\sqrt{2\pi} \tau} \rho_{\infty,W}(s). \]
Choosing here the optimal value \( \tau = 1/s \), we obtain the second half of (2.4).
Now, let \( f \in H^1(W) \), and \( ||f||_{H^1(W)} \leq 1 \). Then \((\mathcal{F}^{-1}f)(s)\) vanishes for \( s \leq 0 \), and
\[
| (\mathcal{F}^{-1}f)^{(n)}(s) | \leq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{|x|^n}{T(|x|)} |f(x)|T(|x|) \, dx \leq \frac{M_n}{\sqrt{2\pi}}.
\]
Therefore, \( \sqrt{2\pi} \mathcal{F}^{-1}f \in C\{M_n\} \) and the first inequality in (2.4) follows. \( \square \)

Combining this lemma with Theorem 1.1, we get

**Theorem 2.1** If the logarithm of Ostrowski’s function \( \varphi(r) = \log T(r) \) satisfies condition (v), then
\[
\log \delta_{\{M_n\}}(s) = -(1 + o(1))Q(s), \quad s \to 0,
\]
where \( Q(s) \) is the upper Legendre transform (1.2).

If \( \varphi(r) \) also satisfies condition (vi), then a sharper asymptotic relation
\[
\log \delta_{\{M_n\}}(s) = -Q(s) + O(\log Q(s)), \quad s \to 0,
\]
holds.

### 2.2 Distance from the polynomials to an imaginary exponent in weighted spaces.

Our second application pertains to the weighted polynomial approximation.

Let \( T(r) = e^{\varphi(r)} \), where \( \varphi \) satisfies assumptions (i)–(iv), and let \( C^0_T \) be the weighted space of continuous functions \( h \) on \( \mathbb{R} \) such that
\[
\lim_{t \to \infty} \frac{h(t)}{T(|t|)} = 0,
\]
and
\[
||h||_{C^0_T} = \sup_{t \in \mathbb{R}} \frac{|h(t)|}{T(|t|)}.
\]

Let \( X = \text{Clos}_{C^0_T} \mathcal{P} \) be the closure of algebraic polynomials \( \mathcal{P} \) in \( C^0_T \).

Due to condition (iii), the polynomials are not dense in \( C^0_T \) and

\[
\text{Clos}_{C^0_T} \mathcal{P} \subset \mathcal{E}_0 \cap C^0_T, \quad (2.5)
\]
where \( \mathcal{E}_0 \) is a space of all entire functions of zero exponential type (cf. [18, Section VI]). It has been known for a long time that under condition (iv-a) plus some mild regularity of \( \varphi \) there is always an equality sign in (2.5) (Khachatrian, Koosis, Levinson-McKean). Recently, Borichev proved in [6] that this always holds whenever \( \varphi \) satisfies conditions (i)-(iii) and (iv-a). We shall not use this result.

Let \( e_s(t) = e^{ist} \). Here, we are interested in

- **the asymptotics of the quantity**

\[
 d_T(s) \overset{\text{def}}{=} \text{dist}_{C^0_T}(X, e_s) = \text{dist}_{C^0_T}(\mathcal{P}, e_s),
\]

when \( s \to 0 \).

A well-known duality argument links this question to the previous one:

**Lemma 2.2** Let \( W \) be an outer function (1.1) with the boundary values \( |W(t)| = T(|t|), t \in \mathbb{R} \). Then

\[
 \sqrt{2\pi} \rho_{1,W}(s) \leq d_T(s) \leq \frac{e}{\sqrt{2\pi}} s \rho_{\infty,W}(s).
\]

**Proof of Lemma 2.2:** We identify the dual space \((C^0_T)^*\) with the space of complex-valued measures \( \mu \) on \( \mathbb{R} \) such that

\[
 ||\mu||_T \overset{\text{def}}{=} \int_{\mathbb{R}} T(|t|) \, d\mu(t) < \infty.
\]

Then by the Hahn-Banach theorem

\[
 d_T(s) = \sup_{\mu \in \mathcal{P}^\perp, ||\mu||_T \leq 1} |\mu(e_s)| = \frac{\sqrt{2\pi}}{\sqrt{2\pi}} \sup_{\mu \in \mathcal{P}^\perp, ||\mu||_T \leq 1} |(\mathcal{F}\mu)(s)|,
\]

where

\[
 (\mathcal{F}\mu)(s) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ist} \, d\mu(t)
\]

is the Fourier transform of the measure \( \mu \), and \( \mathcal{P}^\perp \subset (C^0_T)^* \) is the annihilator of the polynomials. Clearly, \( (\mathcal{F}\mu)(s) \) is a \( C^\infty \)-function on \( \mathbb{R} \) such that

\[
 (\mathcal{F}\mu)^{(n)}(0) = \frac{\sqrt{2\pi}}{\sqrt{2\pi}} \int_{\mathbb{R}} t^n \, d\mu(t) = 0, \quad n \in \mathbb{Z}_+.
\]
and
\[ |(F\mu)^{(n)}(s)| = \left| \frac{1}{\sqrt{2\pi}} \int_\mathbb{R} (it)^n e^{ist} d\mu(t) \right| \]
\[ \leq \frac{1}{\sqrt{2\pi}} \int_\mathbb{R} |t|^n |d\mu(t)| \]
\[ = \frac{1}{\sqrt{2\pi}} \int_\mathbb{R} \frac{|t|^n}{T(|t|)} |d\mu(t)| \]
\[ \leq \frac{1}{\sqrt{2\pi}} \sup_{r \geq 0} \frac{r^n}{T(r)} =: \frac{1}{\sqrt{2\pi}} M_n \]

( the function \( T(r) \) is Ostrowski’s function for the sequence \( M_n \) defined in the last line). Juxtaposing this estimate with (2.6) and using the Lemma 2.1, we obtain
\[ d_T(s) \leq \delta_{\{M_n\}}(s) \leq \frac{e}{\sqrt{2\pi}} \rho_{\infty,W}(s). \]

In order to get the lower bound for \( d_T(s) \), we define the measure \( d\mu(t) = f(t) dt \), where \( f \in H^1(W) \) and \( ||f||_{H^1(W)} \leq 1 \). Then \( ||\mu||_T = ||f||_{H^1(W)} \leq 1 \). Since \( F^{-1}f \) always has a zero of infinite order at the origin, this measure annihilates the polynomials. Therefore, by (2.6)
\[ d_T(s) \geq \sup_{||f||_{H^1(W)} \leq 1} |(F^{-1}f)(s)| = \sqrt{2\pi} \rho_{1,W}(s), \]
completing the proof. \( \square \)

Combining this lemma with Theorem 1.1, we obtain

**Theorem 2.2** Let the function \( \varphi = \log T \) satisfy condition (v). Then
\[ \log d_T(s) = -(1 + o(1))Q(s), \quad s \to 0, \]
where \( Q \) is the upper Legendre transform (1.3). If \( \varphi \) also satisfies condition (vi), then
\[ \log d_T(s) = -Q(s) + O(\log Q(s)), \quad s \to 0. \]

The same asymptotics holds in the weighted \( L^p \)-spaces on \( \mathbb{R} \) under the same assumptions on the weight \( T \).
2.3 Exact Levinson-Sjöberg majorant

Let \( S = \{ \zeta = \xi + i\eta : |\xi| < 1, |\eta| < 1 \} \). The Carleman-Levinson-Sjöberg theorem says that the set of analytic functions in \( S \) such that

\[
|F(\zeta)| \leq \mathcal{M}(|\xi|)
\]

is locally uniformly bounded in \( S \) provided that the majorant \( \mathcal{M}(\xi) \) does not increase on \((0,1)\), and

\[
\int_0 \log \log \mathcal{M}(\xi) \, d\xi < \infty.
\]

In this form the theorem was proved independently by Levinson \[20\] and Sjöberg \[24\]. However a decade before Carleman proved an equivalent result in \[8\]. It is easy to see that the result persists for analytic functions in the punctured square \( S^* = S \setminus \{0\} \) which satisfy conditions \((2.7)\) and \((2.8)\); i.e. the corresponding family is locally uniformly bounded in \( S^* \).

In \[13\], \[14\], E. Dyn’kin raised the question of

- finding the growth of the exact majorant

\[
\mathcal{M}^*(s) = \sup_{F} \max_{|\zeta|=s} |F(\zeta)|,
\]

where the supremum is taken over all analytic functions \( F \) in the \( S^* = S \setminus \{0\} \) satisfying conditions \((2.7)\), \((2.8)\).

An upper bound for \( \mathcal{M}^*(s) \) can be obtained using Domar’s approach \[9\], \[11\] (cf.\[18\], Section VII D7)). A duality argument developed by the first-named author in \[21\] shows that the Carleman-Levinson-Sjöberg theorem is equivalent to the Denjoy-Carleman quasianalyticity theorem. Later, this fact was re-discovered by Dyn’kin in \[12\]. This leads to a lower bound for \( \mathcal{M}^*(s) \) in terms of \( \delta_{\{M_n\}}(s) \) with a properly chosen non-quasianalytic sequence \( \{M_n\} \).

The upper and lower bounds obtained in this way are not tight and there is a gap between them. See the discussion and summary of known results in the survey papers \[14\] and \[22\], pp. 69–71].

Here, we shall not use these results, but directly exploiting Theorem 1.2 we find an asymptotics for \( \log \mathcal{M}^*(s) \) when \( s \to 0 \).

Let

\[
\varphi(r) \overset{\text{def}}{=} \inf_{\xi > 0} [\log \mathcal{M}(\xi) + r\xi]
\]

(2.9)
be the (lower) Legendre transform of $\log M(\xi)$. We assume that the majorant $\mathcal{M}$ grows sufficiently fast: for each $N < \infty$

$$\lim_{\xi \to 0} \xi^N \mathcal{M}(\xi) = \infty.$$  \hfill (2.10)

Then the logarithmic weight function $\varphi(r)$ automatically satisfies conditions (i)–(iii) and (iv-b): condition (iii) follows from (2.8), see [18, Section VI IID2], and (2.10) yields condition (ii).

To ensure condition (iv-a) for the function $\varphi$, we assume that the functions $s \mapsto \log M(e^{-s})$ and $\xi \mapsto \log M(\xi)$ are convex. (2.11)

Then the function $\varphi(e^{\tau})$ is convex. Indeed, set $m(\xi) = \log M(\xi)$ and assume without loss of generality that $m$ is a $C^2$-function. Let $\xi = \xi_r$ be the unique solution of the equation $m'(\xi) = -r$. Then

$$r\varphi''(r) + \varphi'(r) = \frac{-m'(\xi)}{-m''(\xi)} + \xi = \frac{m'(\xi) + \xi m''(\xi)}{m''(\xi)} \geq 0,$$

which yields convexity of the function $\tau \mapsto \varphi(e^{\tau})$.

**Lemma 2.3**: Let the majorant $\mathcal{M}(\xi)$ satisfy conditions (2.8) and (2.10). Then

$$\lambda W^*(s) \leq \mathcal{M}^*(s) \leq C \lambda_W(s), \quad s \leq 1/2,$$

where $W$ is the outer function (1.1) constructed by the logarithmic weight $\varphi$ from (2.3), $W^*(z) = W(z)/(z + i)^2$, and $\lambda_W$ is the same as in the Theorem 1.2.

**Proof of Lemma 2.3**: Let $\gamma$ be a simple closed curve in $S^*$ which surrounds the origin, and let $\Omega_o$ and $\Omega_i$ be the outer and inner components of $\overline{C} \setminus \gamma$, i.e. $\Omega_o$ contains infinity, and $\Omega_i$ contains the origin. If $F$ is analytic in $S^*$, then

$$\frac{1}{2\pi i} \int_{\gamma} \frac{F(w)w dw}{\zeta(w - \zeta)} = \begin{cases} F_o(\zeta), & \zeta \in \Omega_o, \\ F_i(\zeta), & \zeta \in \Omega_i, \end{cases}$$

where $F_o$ is analytic in $\overline{C} \setminus \{0\}$ and decays at infinity at least as $O(|\zeta|^{-2})$, $F_i$ is analytic in $S^*$ and has at most a simple pole at the origin, and

$$F(\zeta) = F_i(\zeta) - F_o(\zeta), \quad \zeta \in S^*.$$
By the Carleman-Levinson-Sjöberg theorem,

$$\max_\gamma |F(w)| \leq C_{\gamma, M},$$

where $C_{\gamma, M}$ depends on the function $M$ and on $\text{dist}(\gamma, \partial S^*)$ but is independent of $F$. Therefore,

$$|F_i(\zeta)| \leq \frac{C}{|\zeta|}, \quad |\zeta| \leq \frac{1}{2}.$$

Now, for $|\zeta| \leq 1/2$

$$|F_o(\zeta)| \leq |F(\zeta)| + |F_i(\zeta)| \leq M(|\zeta|) + \frac{C}{|\zeta|} \leq C M(|\zeta|)$$

(the constants $C = C_M$ may vary from line to line). For $|\zeta| \geq 1/2$, we have

$$|F_o(\zeta)| \leq \frac{C}{|\zeta|^2}.$$

So that,

$$|F_o(\zeta)| \leq \frac{C M(|\zeta|)}{1 + \eta^2}, \quad \zeta \in \mathbb{C} \setminus \{0\}$$

(we put here $M(\xi) = M(1 - 0)$ for $\xi \geq 1$).

Since $F_o$ is analytic at $\mathbb{C} \setminus \{0\}$ and vanishes at infinity, it is a Laplace transform of an entire function $f$ of zero exponential type:

$$F_o(\zeta) = \int_0^\infty f(x) e^{-\zeta x} dx, \quad \xi = \text{Re} \zeta > 0.$$ 

By the inversion formula

$$f(x) e^{-\xi x} = \frac{1}{2\pi} \int_\mathbb{R} e^{i\eta x} F_o(\xi + i\eta) d\eta, \quad x > 0, \quad \xi > 0,$$

whence

$$|f(x)| \leq C \inf_{\xi > 0} [M(\xi) e^{\xi x}] = C e^{\varphi(x)}, \quad x > 0.$$

The same estimate holds for $x < 0$. Therefore, $f \in B(W)$, and for $s \leq 1/2$

$$\max_{|\zeta| = s} |F(\zeta)| \leq \max_{|\zeta| = s} |F_o(\zeta)| + \max_{|\zeta| = s} |F_i(\zeta)| \leq C \left( \lambda_{\infty, W}(s) + \frac{1}{s} \right) \leq C \lambda_{\infty, W}(s),$$
proving the upper bound for $\mathcal{M}^*(s)$.

To get the lower bound, we take $f \in B(W^*)$. Then the Laplace transform $L f$ is analytic in $\mathbb{C} \setminus \{0\}$, and for $\xi > 0$

$$|(Lf)(\xi + i\eta)| \leq \int_0^\infty e^{-\xi x + \varphi(x)} \frac{|f(x)|(1 + x^2)}{e^{\varphi(x)}} \frac{dx}{1 + x^2}$$

$$\leq \pi ||f||_{B(W^*)} \exp\left\{\sup_{x > 0} [\varphi(x) - \xi x]\right\} = \pi ||f||_{B(W^*)} \mathcal{M}(|\xi|).$$

The same bound holds for $\xi < 0$. Therefore, $\mathcal{M}^*(s) \geq \pi \lambda_{W^*}(s)$ completing the proof of the Lemma. \qed

Set $Q^*(s) = \sup_{y > 0} [\log |W^*(iy)| - ys]$. It can be easily seen that

$$Q(s) - o(Q(s)) \leq Q^*(s) \leq Q(s).$$

Indeed, if the maximum in (1.2) is attained at a point $y_s$ (this point is unique and tends to $\infty$ as $s$ approaches the origin), then

$$Q(s) \geq Q^*(s) \geq \log |W(iy_s)| - y_s s - 2 \log(1 + y_s)$$

$$= Q(s) - 2 \log(1 + y_s) = Q(s) - o(Q(s)),$$

(cf. (1.22) below). Applying Theorem 1.2, we obtain the asymptotic formula for $\log \mathcal{M}^*(s)$:

**Theorem 2.3** Let the majorant $\mathcal{M}(\xi)$ satisfy conditions (2.8), (2.10) and (2.11). Then for $s \to 0$

$$\log \mathcal{M}^*(s) = (1 + o(1))Q(s),$$

where $Q(s)$ is the upper Legendre transform (1.2).

As above, under an additional assumption, the remainder can be improved to $O(\log Q(s))$.

### 3 Preliminaries

In this section, we collect various elementary estimates which will be used in the proofs of Theorems 1.1 and 1.2.
3.1 The logarithmic weights

Here, we establish several simple facts about the logarithmic weight functions \( \varphi(t) \) satisfying conditions (i)–(iv), namely:

\[
\lim_{t \to \infty} \frac{\varphi(t)}{t} = 0, \quad (3.1)
\]
\[
\lim_{t \to \infty} \varphi'(t) = 0, \quad (3.2)
\]
\[
\int_1^\infty \frac{\varphi'(t)}{t} \, dt < \infty, \quad (3.3)
\]
\[
\int_1^\infty |\varphi''(t)| \, dt < \infty. \quad (3.4)
\]

First, let us prove relation (3.2), then (3.1) follows by integration. Assume, for example, that (iv-a) holds but (3.2) fails, and set \( \Phi(\tau) = \varphi(e^\tau) \). Then for a sequence \( \tau_j \uparrow +\infty \), \( \Phi'(\tau_j)e^{-\tau_j} \geq c > 0 \), and

\[
\Phi(\tau_j + \xi) = \Phi(\tau_j) + \int_{\tau_j}^{\tau_j + \xi} \Phi'(\tau) \, d\tau \geq c\xi e^{\tau_j}.
\]

Assuming without loss of generality that the intervals \([\tau_j, \tau_j + 1]\) are disjoint, we arrive at the contradiction:

\[
\infty > \int_1^\infty \Phi(\tau)e^{-\tau} \, d\tau \geq \sum_{j \geq 1} \int_{\tau_j}^{\tau_j + 1} \Phi(\tau)e^{-\tau} \, d\tau \geq c \sum_{j \geq 1} \int_0^1 \xi e^{-\xi} \, d\xi = +\infty.
\]

The second case, when condition (iv-b) holds, is even simpler: since the derivative \( \varphi' \) decreases, \( \varphi'(t) \geq ct \) for all \( t \geq 0 \), and therefore, \( \varphi(t) \geq ct \) which again contradicts condition (iii).

Now, let us prove relation (3.3): for each \( A > 1 \)

\[
\int_1^A \frac{\varphi(t)}{t^2} \, dt = \varphi(1) - \frac{\varphi(A)}{A} + \int_1^A \frac{\varphi'(t)}{t} \, dt.
\]
Letting $A \to \infty$ and using (3.1), we obtain (3.3).

In order to obtain (3.4), first observe that due to (3.2) the integral
\[
\int_{1}^{\infty} \phi''(t) dt \tag{3.5}
\]
is convergent. If condition (iv-b) holds, then the integrand in (3.5) is non-positive and there is nothing to prove. Furthermore, convergence of the integrals (3.3) and (3.5) yields convergence of
\[
\int_{1}^{\infty} \frac{t^2 \phi''(t) + t \phi'(t)}{t^2} dt.
\]
If condition (iv-a) holds, then the last integrand is non-negative, which yields the absolute convergence of the integral (3.4).

### 3.2 Differentiating the Poisson integral

Let
\[
u(z) = \log |W(z)| = \frac{\text{Im } z}{\pi} \int_{\mathbb{R}} \frac{\phi(t) dt}{|t - z|^2}, \quad \text{Im } z > 0,
\]
and
\[
q(y) = u(iy) = \log |W(iy)|.
\]
In this section we collect rather straightforward estimates of the derivatives of these two functions which will be used later.

**Lemma 3.1** Let $\phi$ be a $C^2$-function satisfying conditions (i)–(iv). Then
\[
u_x'(x, y) = \frac{4xy}{\pi} \int_{0}^{\infty} \frac{t\phi'(t) dt}{[(t-x)^2 + y^2] [(t+x)^2 + y^2]}, \tag{3.6}
\]
\[
q''(y) = -\frac{4y}{\pi} \int_{0}^{\infty} \frac{t\phi'(t) dt}{(t^2 + y^2)^2} \tag{3.7}
\]
\[
= \frac{2}{\pi y} \int_{0}^{\infty} \frac{t^2 \phi''(t)}{t^2 + y^2} dt, \tag{3.8}
\]
and
\[
u_{\theta\theta}(r, \theta) = -\frac{r \sin \theta}{\pi} \int_{\mathbb{R}} \frac{t^2 \phi''(t) + t \phi'(t)}{|t - re^{i\theta}|^2} dt. \tag{3.9}
\]
Proof: For \( x \geq 0 \) we have

\[
  u(x, y) = \frac{y}{\pi} \int_{0}^{\infty} \varphi(t) \left[ \frac{1}{(t-x)^2 + y^2} + \frac{1}{(t+x)^2 + y^2} \right] dt
\]

\[
  = -\frac{1}{\pi} \int_{0}^{\infty} \varphi'(t) \left[ \arctan \frac{t-x}{y} + \arctan \frac{t+x}{y} \right] dt. \quad (3.10)
\]

Integration by parts is justified since for \( t \to \infty \)

\[
  \arctan \frac{t-x}{y} + \arctan \frac{t+x}{y} = \arctan \frac{2y}{x} \left( \frac{1}{t^2} \right) + O \left( \frac{1}{t^2} \right).
\]

Differentiating (3.10) under the integral sign, we obtain

\[
  u'_x(x, y) = \frac{y}{\pi} \int_{0}^{\infty} \varphi'(t) \left[ \frac{1}{(t-x)^2 + y^2} - \frac{1}{(t+x)^2 + y^2} \right] dt
\]

\[
  = \frac{4xy}{\pi} \int_{0}^{\infty} \frac{t \varphi'(t) dt}{[(t-x)^2 + y^2][(t+x)^2 + y^2]},
\]

proving (3.6).

In order to get (3.7) we differentiate (3.6) by \( x \) and recall that \( u''_{yy} = -u''_{xx} \). Now, since the RHS of (3.7) equals

\[
  \frac{2}{\pi y} \int_{0}^{\infty} \varphi'(t) d \left( \frac{t^2}{t^2 + y^2} \right)
\]

relation (3.8) follows by integration by parts.

Next, we compute \( ru'_r(r, \theta) \) starting again with relation (3.10) rewritten in the polar coordinate and then differentiating it with respect to \( r \). We have

\[
  u(r, \theta) = -\frac{1}{\pi} \int_{0}^{\infty} \varphi'(t) \left[ \arctan \left( \frac{t}{r \sin \theta} - \cot \theta \right) + \arctan \left( \frac{t}{r \sin \theta} + \cot \theta \right) \right] dt,
\]

and

\[
  \left[ \ldots \right]'_r = -t \sin \theta \left[ \frac{1}{|t - re^{i\theta}|^2} + \frac{1}{|t + re^{i\theta}|^2} \right];
\]

therefore

\[
  ru'_r(r, \theta) = \frac{r \sin \theta}{\pi} \int_{\mathbb{R}} \frac{t \varphi'(t) dt}{|t - re^{i\theta}|^2}. \quad (3.11)
\]
Differentiation under the integral sign is justified by (3.3).

Iterating this procedure once more (and using (3.4) for justification of the differentiation), we obtain (3.9):

\[-u''_{\theta\theta} = r \left( ru'_r(r, \theta) \right)'_r = \frac{r \sin \theta}{\pi} \int_{\mathbb{R}} \frac{t^2 \varphi''(t) + t \varphi'(t)}{|t - re^{i\theta}|^2} \, dt. \quad (3.12)\]

This completes the proof. \(\Box\)

Observe several immediate corollaries:

**Corollary 3.1** The function \(x \mapsto u(x, y)\) increases with \(x\) on \([0, \infty)\).

**Corollary 3.2** The function \(q'(y)\) decreases (that is, \(q(y)\) is concave), and \(\lim_{y \to \infty} q'(y) = 0\).

\(^2\)Here is another argument suggested by I. Ostrovskii. To prove (3.6), observe that

\[
 u'_x(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \varphi(t) \frac{1}{(x-t)^2 + y^2} \, dt \\
 = -\frac{y}{\pi} \int_{-\infty}^{\infty} \varphi(t) \frac{1}{t^2 + y^2} \, dt \\
 = \frac{y}{\pi} \int_{-\infty}^{\infty} \varphi'(t) \frac{dt}{(x-t)^2 + y^2}.
\]

Relations (3.7) and (3.8) follow at once, as above.

To prove (3.12), we make a change of variable in the Poisson formula for \(u(z)\): set \(z = e^\zeta\), \(t = e^\tau\) for \(t > 0\) and \(t = -e^\tau\) for \(t < 0\). We obtain the Poisson formula for a strip:

\[
 u(e^\zeta) = \frac{\sin \theta}{2\pi} \int_{-\infty}^{\infty} \frac{\varphi(e^\tau)}{\cosh(\xi - \tau) - \cos \theta + \cos \theta} \, d\tau \\
 + \frac{\sin \theta}{2\pi} \int_{-\infty}^{\infty} \frac{\varphi(-e^\tau)}{\cosh(\xi - \tau) + \cos \theta} \, d\tau,
\]

\(\zeta = \xi + i\theta, \quad -\infty < \xi < \infty, \quad 0 < \theta < \pi\).

Differentiate the last formula twice by \(\xi\). Obviously, the second derivatives of the integrands by \(\xi\) can be replaced by their second derivatives by \(\tau\). Then integrating twice by parts we obtain

\[
 [u(e^\zeta)]''_{\zeta\zeta} = \frac{\sin \theta}{2\pi} \int_{-\infty}^{\infty} \frac{[\varphi(e^\tau)]''_{\tau\tau} \, d\tau}{\cosh(\xi - \tau) - \cos \theta + \cos \theta} + \frac{\sin \theta}{2\pi} \int_{-\infty}^{\infty} \frac{[\varphi(-e^\tau)]''_{\tau\tau} \, d\tau}{\cosh(\xi - \tau) + \cos \theta}.
\]

Returning to the original variables, we get (3.12) and hence (3.9). \(\Box\)
**Corollary 3.3** If \( \varphi \) satisfies conditions (i)–(iii) and (iv-a), then the function \( \theta \mapsto u(r, \theta) \) is concave on \([0, \pi]\) and

\[
u \left( r, \frac{\pi}{2} \right) = \max_{0 \leq \theta \leq \pi} u(r, \theta).
\]

**Corollary 3.4** If \( \varphi \) satisfies conditions (i)–(iii) and (iv-b), then the function \( q''(y) \) increases to 0 as \( y \to \infty \).

In the rest of this subsection, we shall estimate the partial derivatives of the function \( u \).

**Lemma 3.2** If the function \( \varphi(t) \) satisfies conditions (i)–(iv), then

\[
\frac{\varphi'(y)}{3 \pi y} \leq |q''(y)| \leq \frac{24}{\pi y} \int_{y}^{\infty} \frac{\varphi(t)}{t^2} \, dt. \tag{3.13}
\]

In particular,

\[
\lim_{y \to \infty} |q''(y)| = 0,
\]

and

\[
\lim_{y \to \infty} y|q''(y)|^{1/2} = \infty. \tag{3.14}
\]

**Proof:** We start with the lower bound in (3.13). First, assume that condition (iv-a) holds. Then

\[
|q''(y)| \geq \frac{4y}{\pi} \int_{y}^{\infty} \frac{t \varphi'(t)}{4t^4} \, dt \overset{(iv-a)}{\geq} \frac{y}{\pi} \cdot y \varphi'(y) \int_{y}^{\infty} \frac{dt}{t^4} = \frac{\varphi'(y)}{3 \pi y}. \tag{3.17}
\]

\footnote{A. Borichev brought our attention to the fact that the result ceases to hold if assumption (iv-a) is replaced by (iv-b). Indeed, let \( \varphi(t) = \min(t, R) \). Then

\[
u(iR) = \frac{2R}{\pi} \int_{0}^{R} \frac{t \, dt}{t^2 + R^2} + \frac{2R}{\pi} \int_{R}^{\infty} \frac{R \, dt}{t^2 + R^2}
\]

\[
< \frac{2R}{\pi} \left( \frac{1}{R^2} \cdot \frac{R^2}{2} + R \cdot \frac{1}{R} \right) = \frac{3R}{4\pi} < \frac{R}{4} = \frac{u(\pm R)}{4}.
\]

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If condition (iv-b) holds, then we have even a slightly better bound:

\[ |q''(y)| \geq \frac{4y}{\pi} \int_0^y \frac{t \varphi'(t)}{4y^4} \, dt \geq \frac{\varphi'(y)}{\pi y^3} \int_0^y t \, dt = \frac{\varphi'(y)}{2\pi y}. \]

Now, we prove the upper bound in (3.13) (this does not need condition (iv)):

\[ |q''(y)| \leq \frac{4y}{\pi} \left\{ \int_0^y \frac{t \varphi'(t)}{4y^4} \, dt + \int_y^\infty \frac{t \varphi'(t)}{t^4} \, dt \right\} \leq \frac{4y}{\pi} \left\{ \frac{\varphi(y)}{y^3} + \int_y^\infty \frac{\varphi(t)}{t^4} \, dt \right\} \leq \frac{24y}{\pi} \int_y^\infty \frac{\varphi(t)}{t^4} \, dt \leq \frac{24}{\pi y} \int_y^\infty \frac{\varphi(t)}{t^2} \, dt, \]

completing the proof. □

**Lemma 3.3** The following estimates hold:

\[ u'_x(x,y) \geq \frac{x}{6} |q''(y)|, \quad 0 \leq x \leq y; \]  

(3.15)  

\[ u'_x(x,y) \geq \frac{2}{5\pi} \min_{t \in [x/2,3x/2]} \varphi'(t), \quad x \geq y; \]  

(3.16)  

and  

\[ u'_x(x,y) \geq \frac{10}{x}, \quad x \geq 8|q''(y)|^{-1/2}, \quad y \geq y_0. \]  

(3.17)  

**Proof:** First, let \( 0 \leq x \leq y \). Then for \( t \geq 0 \)

\[ (t-x)^2 + y^2 \leq 2(t^2 + y^2), \quad \text{and} \quad (t+x)^2 + y^2 \leq 3(t^2 + y^2). \]

Hence  

\[ u'_x(x,y) \geq \frac{2xy}{3\pi} \int_0^\infty \frac{t \varphi'(t) \, dt}{(t^2 + y^2)^2} \geq \frac{3}{6} \frac{x}{6} |q''(y)|, \]

proving (3.15).

Now, let \( x \geq y \) and \( |t-x| \leq \frac{y}{2} \). Then

\[ (t-x)^2 + y^2 \leq \frac{5y^2}{4}, \quad \text{and} \quad (t+x)^2 + y^2 \leq \frac{15x^2}{2}. \]
Therefore,
\[
\begin{align*}
\frac{1}{4 x y} \int_{|t-x| \leq y/2} t \varphi'(t) \, dt & \cdot \frac{8}{75} \\
\geq & \frac{1}{5 \pi x y} \min_{|t-x| \leq y/2} \varphi'(t) \left[ \left( x + \frac{y}{2} \right)^2 - \left( x - \frac{y}{2} \right)^2 \right] \\
\geq & \frac{2}{5 \pi} \min_{x/2 \leq t \leq 3x/2} \varphi'(t),
\end{align*}
\]
proving (3.16).

At last, let \(8|q''(y)|^{-1/2} \leq x\). If \(x \leq y\), then
\[
u'(x, y) \geq \frac{x}{6} |q''(y)| \geq \frac{10}{x},
\]
and if \(x \geq y\), then
\[
u'(x, y) \geq \frac{2}{5 \pi} \min_{t \in [x/2, 3x/2]} \frac{t \varphi'(t)}{t} \geq \frac{4}{15 \pi x} \min_{t \geq x/2} t \varphi'(t) \geq \frac{10}{x},
\]
provided that \(y \geq y_0\), where \(y_0\) is large enough. This completes the proof of the lemma. \(\square\)

### 3.3 The Legendre transform \(Q(s)\)

Here, for the convenience of references, we collect several elementary facts about the behaviour of the function \(Q(s) = \sup_{y > 0}[q(y) - sy]\).

Let \(y_s\) be a unique solution of the equation
\[
q'(y) = s. \tag{3.18}
\]

Due to the Corollary 3.2, \(y_s \uparrow + \infty\) when \(s \to 0\). Then \(Q(s) = q(y_s) - sy_s\). Differentiating this equation, we obtain
\[
Q'(s) = q'(y_s) \frac{dy_s}{ds} - y_s - s \frac{dy_s}{ds} \tag{3.19}
\]
Then differentiating (3.18) by \(s\), and using (3.18), we get
\[
1 = q''(y_s) \frac{dy_s}{ds} = -q''(y_s)Q''(s),
\]
so that
\[ Q''(s) = -\frac{1}{q''(y_s)}. \]  
(3.20)

Now,
\[ Q(s) = q(y_s) - y_s q'(y_s) = \int_0^{y_s} \xi |q''(\xi)| d\xi \geq \frac{\varphi(y_s)}{3\pi}. \]  
(3.21)

Next, for \( s \leq s_0 \),
\[
0 \leq \log |Q'(s)| = \log y_s \overset{(ii)}{=} o(\varphi(y_s)) \overset{(3.21)}{=} o(Q(s)),
\]  
(3.22)

and
\[
Q''(s) = \frac{1}{|q''(y_s)|} \overset{\text{(13)}}{\leq} \frac{3\pi y_s}{\varphi'(y_s)} \overset{(iv)}{\leq} Cy_s^2.
\]

Hence,
\[
0 \leq \log Q''(s) = O(\log y_s) = o(\varphi(y_s)) = o(Q(s)), \quad s \to 0.
\]  
(3.23)

If \( \varphi \) has a positive lower order (condition (vi)), then the estimates are much better:
\[
\log y_s = O(\log \varphi(y_s)) = O(\log Q(s)),
\]
whence
\[
0 \leq \log |Q'(s)| \leq O(\log Q(s))
\]  
(3.24)

and
\[
0 \leq \log Q''(s) \leq O(\log Q(s))
\]  
(3.25)

for \( s \to 0 \).

4 Proof of Theorem 1.1

4.1 The upper bound for \( |(\mathcal{F}^{-1}f)(s)| \)

Lemma 4.1 Let the logarithmic weight satisfy conditions (i)-(iv). Then for \( s \leq s_0 \)
\[
\rho_{p,W}(s) \leq C \frac{\sqrt{Q''(s)} e^{-Q(s)}}{|Q'(s)|^{1/p}}.
\]
Proof: Let \( f \in H^p(W) \), and \( \|f\|_{H^p(W)} \leq 1 \). Then by a well-known estimate
\[
|f(z)| \leq \frac{|W(z)|}{(\pi \text{Im} z)^{1/p}}, \quad z \in \mathbb{C}_+
\] (see e.g. [7]). We estimate the Fourier transform \((\mathcal{F}^{-1}f)(s)\).

By Cauchy’s theorem, for each \( y > 0 \) and \( s > 0 \),
\[
\mathcal{F}^{-1}f(s) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-is(x+iy)} f(x + iy) \, dx.
\]

Making use of the bound (4.1) and, as above, denoting by \( u = \log |W| \) the Poisson integral of the function \( \varphi \), \( q(y) = u(0, y) \), we have
\[
|(\mathcal{F}^{-1}f)(s)| \leq \frac{e^{sy}}{\sqrt{2\pi} (\pi y)^{1/p}} \int_{\mathbb{R}} e^{-[u(x,y) - u(0,y)]} \, dx.
\]

We choose here \( y = y_s \), where \( y_s \) is a unique solution of the equation (3.18). Then according to the definition (1.2) of the upper Legendre transform \( Q \)
\[
|(\mathcal{F}^{-1}f)(s)| \leq \frac{e^{Q(s)}}{\sqrt{2\pi} (\pi y_s)^{1/p}} \int_{\mathbb{R}} e^{-[u(x,y_s) - u(0,y_s)]} \, dx.
\] (4.2)

For \( x \leq y \),
\[
u(x, y) - u(0, y) = \int_0^x u'_x(\xi, y) \, d\xi \geq \frac{1}{6}|q''(y)| \int_0^x \xi \, d\xi = \frac{|q''(y)|}{12} x^2,
\]
and therefore
\[
\int_{|x| \leq y_s} e^{-[u(x,y_s) - u(0,y_s)]} \, dx \leq \int_{\mathbb{R}} \exp \left[ -\frac{1}{12} |q''(y_s)| x^2 \right] \, dx
\]
\[
= \frac{\sqrt{12\pi}}{|q''(y_s)|}.
\] (4.3)

Next, for \( x \geq y, y \geq y_0 \), we have \( 8|q''(y)|^{-1/2} \leq y \), and
\[
\nu(x, y) - u(0, y) \geq \int_{8|q''(y)|^{-1/2}}^x u'_x(\xi, y) \, d\xi \geq 10 \log \left( \frac{x}{8|q''(y)|^{-1/2}} \right).
\]
Therefore,
\[
\int_{|x| \geq y_s} e^{-[u(x,y_s) - u(0,y_s)]} \, dx \leq 2 \int_{y_s}^{\infty} \left[ \frac{8|q''(y_s)|^{-1/2}}{x} \right]^{10} \, dx
\]
\[
= \frac{2 \cdot 8^{10}}{9} \frac{|q''(y_s)|^{-1/2}}{[y_s|q''(y_s)|^{1/2}]^9}
\]
\[
\leq |q''(y_s)|^{-1/2}, \quad (3.14)
\]
if \( s \leq s_0 \).

Combining estimates (4.2)–(4.4), we obtain
\[
|(F^{-1}f)(s)| \leq \frac{Ce^{-Q(s)}}{y_s^{1/p}|q''(y_s)|^{1/2}}. \quad (4.5)
\]
It remains to plug in relations (3.19) and (3.20) in (4.3). \( \Box \)

### 4.2 Asymptotics of \((F^{-1} \frac{1}{W})(s)\)

**Lemma 4.2** Let the logarithmic weight \( \varphi \) satisfy conditions (i)–(v), and let
\[
f(z) = \frac{1}{(1 - iz)^{2/p}W(z)}, \quad 1 \leq p \leq \infty,
\]
where the branch of the function \((1 - iz)^{2/p}\) is positive when \( z = iy, \, y > 0 \).
Then
\[
(F^{-1}f)(s) = (1 + o(1)) \frac{\sqrt{Q''(s)} e^{-Q(s)}}{Q'(s)^{2/p}}.
\]
In particular, if \( p = \infty \), we get the asymptotic relation (1.15).

**Proof:** Set
\[
h(z) = \log W(z) = \frac{1}{\pi i} \int_{\mathbb{R}} \left[ \frac{1}{t - z} - \frac{t}{t^2 + 1} \right] \varphi(t) \, dt.
\]
Then, applying Cauchy’s theorem, we get
\[
(F^{-1}f)(s) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{e^{-is(x+iy) - h(x+iy)}}{(1 + y - ix)^{2/p}} \, dx
\]
\[
= \frac{e^{sy - q(y)}}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{e^{-isx - [h(x+iy) - h(iy)]}}{(1 + y - ix)^{2/p}} \, dx
\]
since \( h(iy) = q(y) \). As in the previous section, we choose here \( y = y_s \) (see (3.18)) and split the integral into the main part

\[
I(y) \overset{\text{def}}{=} \int_{|x| \leq \omega(y)|q''(y)|^{-1/2}} e^{-i sx - [h(x+iy) - h(iy)]} \frac{1}{(1 + y - ix)^{2/p}} \, dx
\]

and the tail

\[
J(y) \overset{\text{def}}{=} \int_{|x| > \omega(y)|q''(y)|^{-1/2}} e^{-i sx - [h(x+iy) - h(iy)]} \frac{1}{(1 + y - ix)^{2/p}} \, dx,
\]

where

\[
\lim_{y \to \infty} \omega(y) = \infty. \tag{4.6}
\]

Later, we impose other restrictions on the function \( \omega(y) \).

First, we estimate the tail \( J(y) \). We have

\[
|J(y)| \leq 2 \int_{\omega(y)|q''(y)|^{-1/2}} e^{-[u(x,y) - u(0,y)]} \frac{1}{(1 + y)^{2/p}} \, dx.
\]

If \( s \leq s_0 \) and \( y = y_s \), then \( \omega(y) \geq 8 \), so that in the latter integral \( x \geq 8|q''(y)|^{-1/2} \), and we can use estimate (3.17):

\[
u(x, y) - u(0, y) \geq \int_{8|q''(y)|^{-1/2}}^{x} \frac{10}{\xi} \, d\xi = \log \left[ \frac{x}{8|q''(y)|^{-1/2}} \right]^{10}.
\]

Then

\[
|J(y)| \leq 2 \int_{\omega(y)|q''(y)|^{-1/2}} \left[ \frac{8|q''(y)|^{-1/2}}{x} \right]^{10} \frac{dx}{(1 + y)^{2/p}}
\]

\[
\leq \frac{2 \cdot 8^{10}}{9} \frac{|q''(y)|^{-5}}{[\omega(y)|q''(y)|^{-1/2}]^9 (1 + y)^{2/p}} \overset{[L.4]}{=} o(1) \frac{1}{|q''(y)|^{1/2} (1 + y)^{2/p}}, \quad y \to \infty.
\]

Now, we consider the main integral

\[
I(y) = \int_{|x| \leq \omega(y)|q''(y)|^{-1/2}} \exp \left\{-i sx - h'(iy)x - h''(iy)\frac{x^2}{2} - \sum_{k=3}^{\infty} \frac{h^{(k)}(iy)}{k!} x^k \right\} \frac{dx}{(1 + y - ix)^{2/p}}.
\]

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Because of the choice of $y = y_s$,

$$-is - h'(iy) = -is + iu_y'(0, y) = -i(s - q'(y)) = 0.$$ 

So taking into account that $h''(iy) = -u''_{yy}(0, y) = -q''(y)$, we obtain

$$I(y) = \int_{|x| \leq \omega(y)|q''(y)|^{-1/2}} \frac{\exp \left\{ -|q''(y)|^{1/2} - \sum_{k=3}^{\infty} \frac{h^{(k)}(iy)}{k!} x^k \right\}}{(1 + y - ix)^{2/p}} \, dx$$

$$= \frac{1 + o(1)}{y^{2/p}|q''(y)|^{1/2}} \int_{|\xi| \leq \omega(y)} \exp \left\{ -\frac{\xi^2}{2} \right\}$$

$$- \sum_{k=3}^{\infty} \frac{h^{(k)}(iy)}{k!} \frac{\xi^k}{|q''(y)|^{k/2}} \right\} d\xi ,$$

assuming that

$$\omega(y)|q''(y)|^{-1/2} = o(y) , \quad y \to \infty$$

(4.8)

(due to (3.14) this assumption does not contradict to (4.6)).

Now, we shall show that

$$\sup_{|\xi| \leq \omega(y)} \left| \sum_{k\geq 3} \frac{h^{(k)}(iy)}{k!} \frac{\xi^k}{|q''(y)|^{k/2}} \right| = o(1) , \quad y \to \infty .$$

(4.9)

For this, we need estimates of $h^{(k)}(iy)/k!$. We have

$$\left| \frac{h^{(k)}(iy)}{k!} \right| = \left| \frac{1}{\pi} \int_{\mathbb{R}} \frac{\varphi(t)}{(t - iy)^{k+1}} \, dt \right|$$

$$\leq \frac{2}{\pi} \int_{0}^{\infty} \frac{\varphi(t)}{(t^2 + y^2)^{(k+1)/2}} \, dt$$

$$\leq \frac{2}{\pi} \left\{ \frac{1}{y^{k+1}} \int_{0}^{y} \varphi(t) \, dt + \int_{y}^{\infty} \frac{\varphi(t)}{t^{k+1}} \, dt \right\}$$

Then the LHS of (4.9) is

$$\leq \frac{2}{\pi} \left\{ \frac{1}{y} \int_{0}^{y} \varphi(t) \, dt \sum_{k\geq 3} \left( \frac{\omega(y)}{|q''(y)|^{1/2}} \right)^k + \int_{y}^{\infty} \frac{\varphi(t)}{t} \sum_{k\geq 3} \left( \frac{\omega(y)}{t|q''(y)|^{1/2}} \right)^k \, dt \right\}$$

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\[
\frac{4}{\pi} \left( \frac{\omega(y)}{|q'''(y)|^{1/2}} \right)^3 \left\{ \frac{1}{y^4} \int_0^y \varphi(t) \, dt + \int_y^\infty \frac{\varphi(t)}{t^4} \, dt \right\} \\
\leq \frac{16}{3\pi} \left( \frac{\omega(y)}{|q'''(y)|^{1/2}} \right)^3 \int_y^\infty \frac{\varphi(t)}{t^4} \, dt \\
\leq \text{const}^3 \frac{y^3 \int_y^\infty \frac{\varphi(t)}{t^4} \, dt}{(y\varphi'(y))^{3/2}}. \tag{4.10}
\]

In the next to the last inequality, we used the estimate
\[
\frac{1}{y^4} \int_0^y \varphi(t) \, dt \leq \frac{\varphi(y)}{y^3} \leq \frac{1}{3} \int_y^\infty \frac{\varphi(y)}{t^4} \, dt \leq \frac{1}{3} \int_y^\infty \frac{\varphi(t)}{t^4} \, dt.
\]

By virtue of condition (v), we can choose \(\omega(y)\) increasing to infinity with \(y\) and such that the RHS of (4.10) is \(o(1)\). This proves (1.9).

Now, making use of (1.9), we continue the estimate (1.1):
\[
I(y) = \frac{1 + o(1)}{y^{2/p} |q'''(y)|^{1/2}} \int_{|\xi| \leq \omega(y)} e^{-\xi^2/2} \, d\xi = \frac{\sqrt{2\pi} + o(1)}{y^{2/p} |q'''(y)|^{1/2}},
\]

completing the proof. \(\square\)

### 4.3 Conclusion of the proof of Theorem 1.1

Juxtaposing lemmas 4.1 and 4.2, we get
\[
-Q(s) + \frac{1}{2} \log Q''(s) - \frac{2}{p} \log |Q'(s)| - O(1) \\
\leq \log \rho_p, W(s) \leq -Q(s) + \frac{1}{2} \log Q''(s) - \frac{1}{p} \log |Q'(s)| + O(1).
\]

Now, estimates (3.22) and (3.23) give us (1.4). When condition (vi) holds, estimates (3.24) and (3.23) yield (1.6). \(\square\)

### 5 Proof of Theorem 1.2

As above, we prove separately an upper and a lower bound for \(\lambda_W(s)\) which together immediately yield the theorem.
5.1 A version of the Laplace asymptotic estimate

Here, we give a version of the Laplace asymptotic estimate needed for the proof of Theorem 1.2. This result is a minor modification of the well-known (cf. [23, §17]), however, for the reader’s convenience we bring it with the proof.

**Theorem 5.1** Let

\[ N(s) = \int_1^\infty y^a e^{-sy+q(y)} \, dy, \]  

(5.1)

where \( a \in \mathbb{R}, \) and \( q \) is a \( C^3(0, \infty) \) concave function such that

(a) \( q \) steadily increases to \(+\infty\) with \( y; \)

(b) \( q' \) steadily decreases to 0 as \( y \) tends to \(+\infty;\)

(c) \( q'' \) steadily increases to 0 as \( y \) tends to \(+\infty;\)

(d) \[ \lim_{y \to \infty} y^2 |q''(y)| = +\infty; \]

(e) \[ \lim_{y \to \infty} \frac{|q''(y)|^{3/2}}{q'''(y)} = +\infty. \]

Then for \( s \to 0 \)

\[ N(s) = (1 + o(1)) |Q'(s)|^a \sqrt{2\pi Q''(s)} e^{Q(s)}, \]  

(5.2)

where

\[ Q(s) = \sup_{y > 0} [q(y) - sy] \]

is the upper Legendre transform of \( q. \)

The proof is based on two auxiliary lemmas.

**Lemma 5.1** In assumptions of Theorem 5.1, there is a function \( \eta(y) \leq y/2, \) \( 1 \leq y < \infty, \) such that

\[ \lim_{y \to \infty} \eta(y) \sqrt{|q''(y)|} = \infty, \]  

(5.3)
\[ \lim_{y \to \infty} \frac{\eta(y)}{y} = 0, \quad (5.4) \]

and

\[ \lim_{t \to \infty} \sup_{t - \eta(t) \leq y \leq t + \eta(t)} \left| \frac{q''(y)}{q''(t)} - 1 \right| = 0. \quad (5.5) \]

**Proof of Lemma 5.1:** Let \( y > t \). Then for some \( \xi \in (t, y) \)

\[ \left| \frac{q''(y)}{q''(t)} - 1 \right| = (y - t) \frac{q'''(\xi)}{|q''(t)|} \]

\[ \leq (y - t) \sqrt{|q''(t)|} \frac{q'''(\xi)}{|q''(\xi)|^{3/2}} \]

\[ \leq (y - t) \frac{\sqrt{|q''(t)|}}{\gamma(t)}, \]

where

\[ \gamma(t) \overset{\text{def}}{=} \inf_{\xi \geq t} \frac{|q''(\xi)|^{3/2}}{|q'''(\xi)|}, \quad \gamma(t) \uparrow \infty. \]

Let \( \gamma_1(t) \) be a minorant of \( \gamma(t) \) such that \( \gamma_1(t) \uparrow \infty \), and \( \gamma_1(t) = o(1) t^2 |q''(t)| \), for \( t \to \infty \). Defining

\[ \eta_1(t) \overset{\text{def}}{=} \min \left( \sqrt{\frac{\gamma_1(t)}{|q''(t)|}}, \frac{t}{4} \right), \]

we obtain a one-sided version of (5.5):

\[ \sup_{t \leq y \leq t + \eta_1(t)} \left| \frac{q''(y)}{q''(t)} - 1 \right| \leq \eta_1(t) \frac{\sqrt{|q''(t)|}}{\gamma(t)} \leq \frac{1}{\sqrt{\gamma(t)}} \to 0, \]

together with

\[ \frac{\eta_1(t)}{t} \leq \sqrt{\frac{\gamma_1(t)}{t^2 |q''(t)|}} = o(1), \]

and

\[ \eta_1(t) \sqrt{|q''(t)|} = \min \left( \sqrt{\gamma_1(t)}, \frac{1}{4} t |q''(t)| \right) \to \infty, \]

when \( t \to \infty \).
Set $\eta(t) = \eta_1(\tau_t)$, where $\tau_t \overset{df}{=} \inf\{ \xi : \xi + \eta_1(\xi) \geq t \}$. Then $\frac{2}{3}t \leq \tau_t \leq t$, and (5.4) follows. Since the function $|q''(t)|$ decreases, relation (5.3) follows as well. At last, $\tau_t + \eta_1(\tau_t) = t$, so that

$$\sup_{t - \eta_1(\tau_t) \leq y \leq t} \frac{|q''(y)|}{|q''(t)|} - 1 = \frac{|q''(\tau_t)|}{|q''(\tau_t + \eta_1(\tau_t))|} - 1 = o(1), \quad t \to \infty$$

by the choice of the function $\eta_1(t)$. □

The next lemma is useful for estimation of the tails.

**Lemma 5.2** Let $\psi \in C^2(a, \infty)$ be an increasing convex function. Then

$$\int_a^\infty e^{-\psi(x)} \, dx < \frac{e^{-\psi(a)}}{\psi'(a)}.$$

**Proof of the Lemma 5.2:** Since $\psi$ is convex and increasing,

$$\psi(x) - \psi(a) \geq (x - a)\psi'(a),$$

so that

$$\int_a^\infty e^{-\psi(x)} \, dx \leq \int_a^\infty e^{-\psi(a) - (x-a)\psi'(a)} \, dx = \frac{e^{-\psi(a)}}{\psi'(a)}.$$ 

Done! □

**Proof of the Theorem 5.1** Let $t = t_s$ be a unique solution of the equation

$$q'(t) = s, \quad s \leq s_0 = q'(+0).$$

Then we split the integral $N(s)$ into three parts:

$$N(s) = \left( \int_{t = t_s}^{t + \eta} + \int_{t - \eta}^{t + \eta} + \int_{t + \eta}^{\infty} \right) y^a e^{-q(s) + q(y)} \, dy = \sum_{k=1}^{3} I_k(s),$$

where $t = t_s$, $\eta = \eta(t_s)$ is a function from Lemma 5.1. The asymptotics (5.4) is defined by the integral $I_2(s)$; the other two integrals give the remainder.

Let us start with the principal term:

$$I_2(s) = e^{q(t) - st} \int_{t - \eta}^{t + \eta} y^a e^{q(y) - q(t) - (y-t)q'(t)} \, dy$$
\begin{equation}
= (1 + o(1)) t^a e^{g(t) - st} \int_{t-\eta}^{t+\eta} \exp \left\{ -\frac{1}{2} (y - t)^2 |q''(\xi(y))| \right\} \, dy
\end{equation}

\begin{equation}
(1 + o(1)) |Q'(s)|^a e^{Q(s)} \int_{t-\eta}^{t+\eta} \exp \left\{ -\frac{1}{2} (y - t)^2 |q''(t)| \right\} \, dy
\end{equation}

\begin{equation}
= (1 + o(1)) |Q'(s)|^a e^{Q(s)} \sqrt{2\pi Q''(s)} \tag{5.6}
\end{equation}

for \( s \to 0 \).

For estimates of the integrals \( I_1 \) and \( I_3 \) we use Lemma 5.2. First, we observe that the function \( y \mapsto sy - q(y) - a \log y \) is convex. Therefore, the Lemma 5.2 is applicable:

\begin{equation}
\int_{t+\eta}^{\infty} y^a e^{-sy+q(y)} \, dy \leq (t + \eta)^a \frac{e^{-(t+\eta)s + q(t+\eta)}}{s - q(t + \eta) - a(t + \eta) - 1}
\end{equation}

\begin{equation}
= (1 + o(1)) t^a e^{q(t) - st} \frac{e^{q(t+\eta) - q(t) - \eta q'(t)}}{q'(t) - q'(t + \eta) + O(1/t)}
\end{equation}

\begin{equation}
= (1 + o(1)) |Q'(s)|^a e^{Q(s)} \frac{e^{1/2 - \eta^2 |q''(t)|}}{\eta |q''(t)| (1 + o(1)) + O(1/t)}
\end{equation}

\begin{equation}
\leq (1 + o(1)) |Q'(s)|^a \sqrt{Q''(s)} e^{Q(s)} \frac{e^{1/2 - \eta^2 |q''(t)|}}{\eta |q''(t)|^{1/2} + O(\frac{1}{t |q''(t)|^{1/2}})}
\end{equation}

\begin{equation}
= o(1) |Q'(s)|^a \sqrt{Q''(s)} e^{Q(s)} \quad s \to 0 . \tag{5.7}
\end{equation}

The same estimate holds for the second integral:

\begin{equation}
I_2(s) < (t - \eta)^a \frac{e^{-(t-\eta)s + q(t-\eta)}}{q'(t - \eta) - s + O(1/t)} = o(1) |Q'(s)|^a \sqrt{Q''(s)} e^{Q(s)} . \tag{5.8}
\end{equation}

Collecting estimates (5.6)–(5.8), we finally obtain (5.2). \( \square \)

**5.2 The upper bound for \( \lambda_W(s) \)**

**Lemma 5.3** Let \( \varphi(t) \) satisfy conditions (i)–(iii), (iv-a), and (iv-b). Then for \( s \to 0 \)

\begin{equation}
\lambda_W(s) \leq (1 + o(1)) \sqrt{2\pi Q''(s)} e^{Q(s)} . \tag{5.9}
\end{equation}
\textbf{Proof:} Let $\|f\|_{B(W)} \leq 1$. Then

$$|f(z)|, |f(\overline{z})| \leq |W(z)|, \quad z \in \mathbb{C}_+.$$ 

Rotating the integration line in the Laplace transform, and then using the Corollary 3.3, we get

$$\left| (\mathcal{L}f)(se^{i\psi}) \right| = \left| \int_0^\infty f(re^{-i\psi})e^{-sr} \, dr \right|$$

$$\leq \int_0^\infty \exp \left\{ \log |W(re^{-i\psi})| - sr \right\} \, dr$$

$$\leq \int_0^\infty \exp \left\{ \log |W(ir)| - sr \right\} \, dr$$

$$\leq \int_1^\infty \exp \left\{ \log |W(ir)| - sr \right\} \, dr + O(1), \quad s \to 0.$$

Now, we check that the function $q(y) = \log |W(iy)|$ meets conditions (a)--(e) of Theorem 5.1, and then apply this theorem with $a = 0$. Conditions (a) and (b) follow from Corollary 3.2, (c) follows from Corollary 3.4, and condition (d) follows from the lower bound in the estimate (3.13) and (iv-b). In order to check condition (e), we estimate from above the third derivative $q'''$ using relation (3.8). Differentiating (3.8) once (this is permitted due to (3.4)), we get

$$q'''(y) = \frac{2}{\pi y^2} \int_0^\infty \frac{t^2[-\varphi''(t)]}{t^2 + y^2} \, dt + \frac{4}{\pi} \int_0^\infty \frac{t^2[-\varphi''(t)]}{(t^2 + y^2)^2} \, dt$$

$$\leq \frac{6}{\pi y^2} \int_0^\infty \frac{t^2[-\varphi''(t)]}{t^2 + y^2} \, dt = \frac{3|q''(y)|}{y},$$

and therefore

$$\frac{q'''(y)}{|q''(y)|^{3/2}} \leq \frac{3}{y|q''(y)|^{1/2}} \leq o(1), \quad y \to \infty.$$

Thus, applying the Theorem 5.1, we get

$$\left| (\mathcal{L}F)(se^{i\psi}) \right| \leq (1 + o(1))\sqrt{2\pi Q''(s)} e^{Q(s)}, \quad s \to 0.$$

This proves the upper bound (5.9). $\square$
5.3 The lower bound for $\lambda_W(s)$

Lemma 5.4 Let $\varphi(t)$ satisfy conditions (i)–(iii), (iv-a), and (iv-b). Then there is an entire function $E \in B(W)$ such that for some $N < \infty$

$$|(LE)(-is)| \geq (1 + o(1)) \left|Q'(s)\right|^{-N} \sqrt{Q''(s)}e^{Q(s)}, \quad s \to 0. \quad (5.10)$$

First, applying a result of Y. Domar [10, Lemma 4] (more precisely, we use his intermediate estimate (19)), we find an even entire function

$$G(z) = \sum_{n \geq 0} a_{2n} z^{2n}, \quad a_{2n} \geq 0,$$

such that

$$O < c_1|x|^{-2N} \leq G(x)e^{-2\varphi(x)} \leq c_2 < \infty, \quad |x| \geq 1.$$ 

The constants $c_1$ and $c_2$ depend on the function $\varphi$ and are independent of $x$. Since

$$M(r, G) \left(\overset{\text{def}}{=} \max_{|z| = r} |G(z)|\right) = G(r),$$

the function $G$ has zero exponential type, and moreover, belongs to the convergence class:

$$\int_{1}^{\infty} \log M(r, G) \frac{dr}{r^2} < \infty.$$

Applying the Krein-Akhiezer factorization theorem [19, Appendix V], we factorize

$$G(z) = E(z) \overline{E(\overline{z})},$$

where $E$ is an entire function of zero exponential type with zeroes in the lower half-plane. Since $G(x) = |E(x)|^2$,

$$c_1|x|^{-N} \leq |E(x)|e^{-\varphi(x)} \leq c_2, \quad |x| \geq 1. \quad (5.11)$$

In particular, $E/W$ is bounded on the real axis. Since $E$ has zero exponential type and $W$ is outer, we can apply the Phragmén-Lindelöf principle to the functions $E/W$, $E^*/W$, and conclude that these functions are in $H^\infty$. Hence $E \in B(W)$.

Without loss of generality, we assume that $E(0) = G(0) = 1$, so that $E$ is a canonical product of genus zero with the zero set symmetric with respect
to the imaginary axis (since $G$ is even). Thus $E(iy) \geq 0$ for $y \geq 0$. Applying again the Phragmén-Lindelöf principle to the function $E/W$ in $\mathbb{C}_+$ and using estimate (5.11), we get

$$E(iy) \geq c_1 y^{-N}|W(iy)|, \quad y \geq 1.$$  

(5.12)

Therefore, for $s \to 0$,

$$|(\mathcal{L}E)(-is)| = \int_0^\infty E(iy)e^{-sy}dy \stackrel{(5.13)}{=} c \int_1^\infty y^{-N}e^{\log|W(iy)|-sy}dy - O(1).$$

Above, in the proof of the previous lemma, we already checked that the function $q(y) = \log|W(iy)|$ satisfies all assumptions of Theorem 5.1. Applying this result, we obtain the estimate (5.10). □

**Remark** If the logarithmic weight $\varphi$ satisfies an additional assumption

(vii)

$$\liminf_{\tau \to \infty} \frac{d^2 \varphi(e^\tau)}{d\tau^2} > 0,$$

then instead of the result of Domar we may use a recent result of Borichev [5] and construct an entire function $G$ of genus zero with non-negative Taylor coefficients and such that

$$0 < c_1 \leq G(x)e^{-2\varphi(x)} \leq c_2 < \infty, \quad x \in \mathbb{R}.$$  

Repeating verbatim the rest of the argument, we obtain an entire function $E \in B(W)$ such that

$$|(\mathcal{L}E)(-is)| \leq (1 + o(1))\sqrt{Q''(s)}e^{Q(s)},$$

whence

$$0 < a \leq \frac{\lambda W(s)}{\sqrt{Q''(s)}e^{Q(s)}} \leq b < \infty.$$  

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