Numerical method for solving the initial value problem for binary systems

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Abstract.
In this article we describe a new numerical method for computing accurate initial data for binary compact objects such as black holes and neutron stars. Tests are performed for a simple model of the black hole fields and a preliminary application of the IWM formulation is presented.

1. Introduction
The two body problem in General Relativity, turned out to be an extremely difficult problem. Two black holes orbiting each other radiate gravitational waves, and radiation reaction drives the orbit to spiral in towards the merger. After the merger, a formed black hole continues to emit gravitational waves until the multipole deformations are dumped out. The merger of binary black holes is a highly non-linear phenomenon, and the numerical simulation is considered as the unique theoretical method to investigate it. Accurate inspiral to merger simulations are essential for constructing the waveform templates used in data analysis from the laser interferometric gravitational wave detectors. The ground based interferometers, such as Advanced LIGO may detect $\sim 10 M_\odot$ binary black holes inspiral within $z \sim 4$, while the space based interferometric detector such as LISA or DECIGO may detect the inspiral of $10^6 M_\odot$ supermassive binary black holes, and may discover the intermediate mass binary black holes in $\sim 10^3 M_\odot$.

In the simulation of binary systems, the spacetime is constructed as a time evolution of an initial spacelike slice; the Einstein equations written in 3+1 form, possibly coupled with matter fields, are integrated to evolve the initial data sets. The Einstein equations in 3+1 form contain not only the evolution equations but also the constraints on the data which must be satisfied on each slice including initial data itself. The problem raised then is to determine appropriate initial values for the binary inspiral simulations.
The Hamiltonian constraint can be solved by applying the Lichnerowicz decomposition, in which the 3-metric is decomposed into a conformal factor and a conformal 3-metric, thereby resulting to an elliptic equation for the conformal factor. The momentum constraints, on the other, are solved using the York decomposition of the tracefree part of the extrinsic curvature into its “longitudinal” part and a symmetric transverse-tracefree part. Depending on the definition of the vector field for this decomposition, and on how one specifies the conformal (background) 3-metric and other freely specifiable data, the momentum constraints may be solved analytically [1], or numerically which amounts in solving three dimensional vector elliptic equations (for detailed review see [2]). Two variations of the latter procedure is the conformal tranverse-tracefree (CTT) decomposition and the physical tranverse tracefree (PTT) decomposition. Difficulties are the arbitrariness of the freely specifiable data (choice of these data in order to specify realistic initial data), together with the choice for the lapse and the shift that control the motion of coordinates through spacetime.

In earlier days, black hole initial configurations were constructed with the assumption of time symmetry [3, 4, 5, 6]. These were largely motivated by the simple solution of the Hamiltonian constraint and the spatial trace of Einstein equations under the assumptions of static, asymptotically flat spacetime with a conformally flat 3-metric. Evolution of such data represented black hole collisions (from rest) and that was numerically computed by [7]. Time-asymmetric initial data in which black holes had arbitrary linear and angular momenta was constructed by [1]. The assumptions under this framework were asymptotic and conformal flatness, maximal slicing and the symmetric transverse-tracefree tensor of the CTT decomposition set to zero. That lead to a decoupling of the Hamiltonian from the momentum constraints which in turn had an analytic solution. Finally this solution was made inversion symmetric in order to have the nice properties of the Misner two-sheeted topology and because of the way it is constructed it is refered as the conformal imaging method. Generalizations to an arbitrary number of black holes was given by [8] and numerical simulations where performed by a number of investigators [9, 10, 11, 12].

During the late nineties new methods were incorporated into the black hole binary initial data construction. The first was the puncture method of Brandt and Brügmann [13] which combines the Brill-Lindquist data with the Bowen-York solution to the momentum constraints on a spatially conformal flat slice. A solution for the conformal factor is seeked in the form of Brill-Lindquist puncture data [5], plus a regular unknown function, which results into the Hamiltonian constraint being solved for this function. The second method called the conformal thin sandwich (CTS) decomposition formulated by York [14], which is closely related to the Isenberg-Wilson-Mathews formulation used for binary neutron stars computations [15, 16]. The difference with the CTT and PTT decompositions is that the CTS decomposition brings some of the coordinate dynamics, represented by the laspe and shift, back into the calculations. In fact the shift vector is now the unknown variable that is determined from the momentum constraints and the laspe (conformal lapse) is freely specifiable. The CTT, PTT, CTS decompositions were usually solving data on the conformal flat slice, while the possibility of non-conformally flat Kerr-Shild background data was explored by [17]. In most of the cases CTT, PTT, CTS produced data that contained unphysical gravitational radiation of the order of several percent of the total mass [18], but substantially smaller than the Kerr-Shild background data.

For constructing initial data of the binary neutron stars, the Isenberg-Wilson-Mathews formulation has been used. In this formulation, four constraints are solved for the conformal factor and the shift, while the spatial trace of the Einstein equation is solved for the lapse [15, 16, 19, 20]. This idea was applied succesfully to the binary black hole case by Gourgoulhon, Grandclément, and Bonazzola (GGB) [21, 22]. In their calculation, the orbital angular velocity
of the system is determined from an equality for the Komar and ADM mass (which reduces to the classical virial theorem at the Newtonian limit). Comparison with analytical results [23] showed that their approach agreed well with analytic calculations, especially in the location of the innermost stable circular orbit (ISCO). Also Cook [24], by a modification of the CTS decomposition in order to include the constant K equation for the calculation of the lapse, together with a novel set of apparent-horizon boundary conditions originated from [25, 26], produced similar initial data sets. This work was further refined by [27] and provided initial data for both single and binary black hole systems, under the assumption of conformal flat 3-geometry. On the same track of CTS decomposition with simpler boundary conditions, [28] provided initial data by taking the conformal 3-geometry to be a superposition of two Schwarzschild black holes in Kerr-Schild coordinates. Their quasicircular orbits were identified by imposing equality of the ADM mass of the binary and the Komar mass.

Other computations for binary black hole initial data include Baumgarte, [29], which was the first who applied the puncture method of Brandt and Brügmann to produce binary black hole data. Similar calculations were performed by [30]. The helical Killing vector idea was applied to the puncture method in [31] under the CTT decomposition, and the corresponding quasi-equilibrium binary black hole sequences were shown in [32]. There the ISCO was found to be close to the one found by Baumgarte. Similar puncture data were computed using the spectral code [33]. Finally a mixture of the CTS decomposition and puncture method was considered in [34] and used to construct data for boosted black holes in [35] and data for binary black holes [36].

In this article, we report a new numerical method suitable for computing accurate initial data sets of the binary black hole, black hole-neutron star, and binary neutron star systems. Initial data sets of these kinds are calculated from the Einstein equations written in the form of elliptic equations for metric components, the Poisson equations with non-linear sources. Our new Poisson solver patterns after the numerical method for computing rotating neutron stars, KEH method [37], and for binary neutron stars [38]. The KEH method uses Green’s formula to write the field equations in equivalent integral forms, and iteratively solve them on the spherical coordinate on which the Legendre expansion is applied to the Green’s function. The set of equations is discretized by the standard finite difference scheme.

We extend the method to handle binary configurations introducing three spherical coordinate patches. Two of them are centered at each hole and extend to a finite radius larger than the gravitational radius but small enough in order not to overlap. The third coordinate extends to asymptotics, and includes the other two coordinate patches near the origin. There are two important features of our new code: (1) The number of multipoles in the coordinate patch centered at the orbital center can be reduced to approximately ten since the size of patches for compact objects is extended to about half of the binary separation. (2) The data between those patches are communicated only at the boundary of those patches to minimize the amount of data to interpolate from one to the other. These novel features result to an efficient code even keeping a high resolution near the compact objects where the field is strong. The other advantages of the method may be that the coding is relatively simple, and that the iteration of the integral form converges robustly.

2. 3+1 formulation and helical symmetry
We assume the spacetime to be $\mathcal{M} = \mathbb{R} \times \Sigma$, and let $n^\mu$ be the future-pointing unit vector normal to the slices $t \times \Sigma$. The three metric $\gamma_{ij}(t)$ of each spatial slice $\Sigma_t$ is a restriction of the projection 4-tensor orthogonal to $n^\mu$, $\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$. In the chart $\{t, x^i\}$ the spacetime
metric is written
\[ ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt), \]
where \( \alpha \) and \( \beta^i \) are the lapse and the shift vector respectively. The Einstein equation can now be split into the constraint equations
\[ R - K_{ij}K^{ij} + K^2 = 16\pi \rho_H, \]  
\[ D_j(K^{ij} - \gamma^{ij}K) = 8\pi j^i, \]
and the evolution equations. The spatial trace of the Einstein equation combined with the Hamiltonian equation (2) gives
\[ \partial_t K = \alpha R - \Delta \alpha + \alpha K^2 + \beta^i D^i K + 4\pi \alpha (S - 3\rho_H), \]
where \( \rho_H := T_{\mu\nu}n^\mu n^\nu, j_i := -T_{\mu\nu}\gamma^{\mu\nu} n^i, \) and \( S := T_{\mu\nu}\gamma^{\mu\nu} \) are the source terms, \( D \) is covariant derivative associated with \( \gamma_{ij} \) and \( \Delta := D^i D_i \).

We assume quasi-circular state for the initial data of the inspiraling orbit. In the co-moving (rotating) frame, a helical vector, which is the time translation in the co-rotating frame is defined as
\[ k^\mu = t^\mu + \Omega \phi^\mu = (1, \Omega\phi^i). \]
where \( \Omega \) is a constant representing the orbital angular velocity. The stationarity is imposed assuming the helical vector \( k^\mu \) to be the Killing field; the spatial metric \( \gamma_{ij} \) and the extrinsic curvature \( K_{ij} \) satisfy
\[ \mathcal{L}_k \gamma_{ij} = 0 \quad \text{and} \quad \mathcal{L}_k K_{ij} = 0. \]

With a conformal transformation of the form \( \gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} \), and using the helical symmetry (6), the constraint equations (2), (3), and equation (4) become
\[ \tilde{\Delta} \psi = \frac{\psi^5}{8} \tilde{R} - \frac{\psi^5}{32\alpha^2} (\tilde{L}\omega)_{ij} (\tilde{L}\tilde{\omega})^{ij} + \frac{\psi^5}{12} K^2 \]
\[ \tilde{\Delta} \alpha = \psi^4 \left( \frac{1}{4\alpha} (\tilde{L}\omega)_{ij} (\tilde{L}\tilde{\omega})^{ij} + \frac{\alpha K^2}{3} + \tilde{\omega}^i D_i K \right) - \frac{2}{\psi} \tilde{D}_i \psi \tilde{D}^i \alpha \]
\[ \tilde{\Delta} \tilde{\omega}_i = -\frac{1}{3} \tilde{D}_i \tilde{D}_j \tilde{\omega}^j - \tilde{R}_{ij} \tilde{\omega}^i + \tilde{D}_i \ln \left( \frac{\alpha}{\psi^6} \right) (\tilde{L}\tilde{\omega})_{ij} + \frac{4\alpha}{3} \tilde{D}_i K \]
where \( \omega^i = \beta^i + \Omega \phi^i \) is the comoving shift and \( (\tilde{L}\omega)_{ij} = \tilde{D}_i \omega_j + \tilde{D}_j \omega_i - \frac{2}{3} \tilde{\gamma}_{ij} \tilde{D}_m \tilde{\omega}^m \) (all quantities with a bar above them refer to the conformal geometry \( \tilde{\gamma}_{ij} \)). Note that on the above equations we have taken the sources to be zero.

3. The KEH method and the Grid
All components of the field equations are written as elliptic equations, whose source term \( S \) may include non-linear terms in the field itself,
\[ \nabla^2 \Phi = S(\Phi, x), \]
where \( \Phi \) represents components of field on a slice \( \Sigma \). The flat Laplacian \( \nabla^2 \) is separated in the L.H.S. and inverted by a certain Poisson solver, and the non-linear equation (10) is solved iteratively. Our choice of the Poisson solver is to use Green’s formula,
\[ \Phi(x) = -\frac{1}{4\pi} \int_V G(x, x') S(x') d^3x' + \frac{1}{4\pi} \int_{\partial V} [G(x, x') \nabla^\alpha \Phi(x') - \Phi(x') \nabla^\alpha G(x, x')] dS'_a, \]
which is identical to the Poisson equation (10) for any function \( G(x, x') \) that satisfies \( \nabla^2 G(x, x') = -4\pi \delta(x - x') \); this is true even if the source \( S \) depends on the field \( \Phi \) non-linearly. Thereby, the elliptic equation with a non-linear source can be solved iteratively using Eq. (11). Hereafter we call this iteration the KEH iteration.

The function \( G(x, x') \) is expanded in terms of spherical harmonics

\[
G(x, x') = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} g_{\ell}^m(r, r') \frac{\ell!}{(\ell + m)!} P_{\ell}^m(\cos \theta) P_{\ell}^m(\cos \theta') \cos(m(\phi - \phi') ,
\]

where the coefficient \( \epsilon_m \) is one when \( m = 0 \) and two otherwise. For the radial part of the Green's function we chose one of the following

\[
g_{\ell}^{\text{NB}}(r, r') = \frac{r_{\ell}^r}{r_{\ell+1}^r} \quad \text{(13)}
\]

\[
g_{\ell}^{\text{DD}}(r, r') = \left[ 1 - \left( \frac{r_a}{r_b} \right)^{2\ell+1} \right]^{-1} \frac{r_a^\ell}{r_b^{\ell+1}} \left[ (r_a < r) \frac{\ell}{r_a} \left( r_a - \frac{r_a}{r_a} \right) + (r_b < r) \frac{\ell}{r_b} \left( r_b - \frac{r_b}{r_b} \right) \right] \quad \text{(14)}
\]

\[
g_{\ell}^{\text{ND}}(r, r') = \left[ 1 + \left( \frac{r_a}{r_b} \right)^{2\ell+1} \right]^{-1} \frac{r_a^\ell}{r_b^{\ell+1}} \left[ (r_a < r) \frac{\ell}{r_a} \left( r_a - \frac{r_a}{r_a} \right) + (r_b < r) \frac{\ell}{r_b} \left( r_b - \frac{r_b}{r_b} \right) \right] \times \left( \frac{r_a}{r_a} \right)^{\ell+1} - \left( \frac{r_b}{r_b} \right)^{\ell+1} \quad \text{(15)}
\]

where \( r_{\ell} := \text{sup}(r, r') \), and \( r_{\ell} := \text{inf}(r, r') \). The first gives the Green’s function for the Laplacian without boundary, the second gives the Green’s function for the Laplacian between two concentric spheres with radii \( r = r_a \) and \( r = r_b \) (\( r_a < r_b \)) and Dirichlet boundary conditions on both spheres \( g_{\ell}^{\text{DD}}(r, r_a) = g_{\ell}^{\text{DD}}(r, r_b) = 0 \), and the third choice gives the Green’s function between two concentric spheres with Neumann boundary condition \( \partial_r g_{\ell}^{\text{ND}}(r, r_a) = 0 \) at the inner sphere and Dirichlet boundary conditions at the outer sphere \( g_{\ell}^{\text{ND}}(r, r_b) = 0 \). An important requisit for the Green’s function \( G(x, x') \) to achieve successful convergence with the KEH iteration is that all multipole components of \( \nabla^m G(x, x') \) should not vanish at the boundary for solving Dirichlet problem, and similary \( G(x, x') \) should not vanish for Neumann.

We then introduce three spherical domains to describe the binary black hole/neutron star data; one of them is centered near the source (not necessarily the mass center of two black holes or neutron stars), and the other two are centered at each compact object. We call these coordinates the central coordinate system (CCS) and black hole coordinate system (BHCS), respectively. The computational domains are shown schematically in Fig. 1.

The boundaries \( I_1, I_2 \) are introduced to reduce the number of the Legendre expansion terms in the Green’s function (12) of the CCS. Taking radii of \( I_1 \) and \( I_2 \) large enough, contribution of higher multipoles in the CCS are included in the surface integrals over \( I_1 \) and \( I_2 \), and the small number of multipoles, typically ten, is enough for the volume integral of the source in the CCS. This allows to compute accurate solution without a high resolution in the CCS region, while one can increase the radial resolution near the black holes in the BHCS where the metric potentials may vary rapidly.

Between concentric spheres \( I_1 \) and \( S_{a_1} \) for the BHCS-1, and \( I_2 \) and \( S_{o_2} \) for the BHCS-2, we reserve overlapping regions, which appear shaded in Fig. 1. These interfaces \( I_i \) and \( S_{o_i} \) \( (i = 1 \text{ or } 2) \) are not physical boundaries; the boundary conditions of the fields are not prescribed there.
Instead, the value of the field on $I_i$ is calculated from the corresponding BHCS field, and the value of the field on $S_o$ is calculated from the field of the CCS in order to achieve communication of the three regions during the iterations. As a result, the field potentials become consistent throughout the BHCS and CCS, and satisfy the physical boundary conditions at the BH throat and at the asymptotics.

4. Numerical results
To test our elliptic solver we consider first a static asymptotically flat spacetime (time-symmetric initial data) of the form

$$ds^2 = -\alpha^2 dt^2 + \psi^4 f_{ij} dx^i dx^j,$$

where $f_{ij}$ is the flat metric. The Hamiltonian constraint and the spatial trace of the Einstein equation $G_{\alpha\beta\gamma}^{\alpha\beta} = 0$ give

$$\nabla^2 \psi = 0 \quad \text{and} \quad \nabla^2 (\alpha \psi) = 0,$$

respectively. These equations have solutions,

$$\psi = 1 + \frac{M}{2r} \quad \text{and} \quad \alpha \psi = 1 - \frac{M}{2r},$$

which correspond to the Schwarzschild solution with mass $M$ in isotropic coordinate, $\psi|_{r\to\infty} = 1$, and $\alpha|_{r\to\infty} = 1$. The solution can be regarded to be of Misner type where two identical spacelike sheets are connected by a throat of radius $r = M/2$. 

\textbf{Figure 1.} The computational domain. One central grid and two black hole grids with excised regions.
Since the above $\psi$ and $\psi\alpha$ are solutions of Laplace equation they can be easily generalized to represent two black holes by simple superposition (although now the resulted solution will no more be of Misner type),

$$
\psi = 1 + \frac{M_1}{2r_1} + \frac{M_2}{2r_2} \quad \text{and} \quad \alpha\psi = 1 - \frac{M_1}{2r_1} - \frac{M_2}{2r_2},
$$

(19)

where $r_1$ and $r_2$ are the distances from the center of the two throats which are taken to be at $r_1 = M_1/2$ and $r_2 = M_2/2$. We can impose either Dirichlet boundary condition, from the value of Eq. (19) at the throat, or Neumann boundary condition by taking the corresponding derivatives of $\psi$ and $\alpha\psi$ as for example

$$
\frac{\partial \psi}{\partial r_1} = \frac{M_1}{2r_1^2} - \frac{M_2}{2r_2^2} \frac{\partial r_2}{\partial r_1} \quad \text{at} \quad r_1 = \frac{M_1}{2},
$$

(20)

and 1 $\leftrightarrow$ 2 for the throat at $r_2 = M_2/2$. The converged solutions are computed by imposing either Dirichlet or Neumann boundary conditions and the accuracy of our Poisson solver is examined comparing these solutions to the analytic ones for the two throat case.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Fractional errors of the conformal factor $\psi$ along the x-axis.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{Fractional errors of the lapse $\alpha$ along the x-axis.}
\end{figure}

In Fig. 2, 3 we plot the fractional errors ($100|\text{Exact Value} - \text{Numerical Value}|/|\text{Exact Value}|$) of the conformal factor $\psi$ and the lapse $\alpha$ using the no boundary Green’s function $G^{\text{NB}}$ and imposing Neumann boundary conditions. The integral form involves solely the surface integrals of Eq.(11). Since 4th order accurate formulas are used for the numerical integrals and the derivatives that appear in the surface terms, the 4th order convergence of the numerical solution to the exact solution can be seen in Fig. 2 and 3. Starting from $\alpha = \psi = 1$, the solution converges after 15 iterations with a typical value of the convergence factor $c = 0.8$.

Next we apply our new numerical method to compute binary black hole initial data in the IWM formulation. Here the metric is as (1) with $\gamma_{ij} = \psi^4 f_{ij}$, $f_{ij}$ being the flat metric. The conformal factor $\psi$, shift $\beta^a$ and lapse $\alpha$ are solved from the Hamiltonian constraint, momentum constraints and the spatial trace of the Einstein’s equation, respectively, which are written as elliptic equations (7), (8), (9).
Dirichlet boundary conditions are given at the excision sphere \( r = r_a \) of BHCS to all variables \( \{\psi, \alpha, \beta^\nu\} \). For the boundary value of the conformal factor \( \psi_B \), a constant is chosen large enough to form horizons near the excision spheres. The boundary condition for the shift vector assigns a momentum and a spin to each hole.

5. Discussion
Convergence of the KEH iteration is robust and fast for the binary black hole cases. Typically, CPU time per iteration with a moderate resolution may be about 1 sec, and with the higher resolution less than 20 sec. This will enable us to do systematic calculations of various sequences of solutions with different mass and spin parameters of holes or different black hole boundary conditions [39]. Also, the coding of the program is less technical comparing, for example, with that of spectral methods. This is advantageous if we want to develop code with more sophisticated formulations as used in [40].

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