On non-Abelian group of generalized covariant Hamilton system

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Abstract

This paper considers the non-abelian Lie algebra of Lie groups, by using the structure constant to construct some new quantities, the GCHS defined by GSPB is for the covariant evolution equation that consists of two parts, TGHS and S dynamics. Meanwhile, the generalized force field given by the GCHS in terms of the momentum applies to deduce some results.

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1 Preliminaries

1.1 Non-Abelian group, Lie algebra

A non-abelian group, sometimes called a non-commutative group, is a group \((G, \ast)\) in which there exists at least one pair of elements \(a\) and \(b\) of \(G\), such that \(a \ast b \neq b \ast a\). This class of groups contrasts with the abelian groups. Both discrete groups and continuous groups may be non-abelian.
A Lie algebra is a vector space $\mathfrak{g}$ together with a non-associative, alternating bilinear map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}; (x, y) \mapsto [x, y]$, called the Lie bracket, satisfying the Jacobi identity.

For any associative algebra $B$ with multiplication $\ast$, one can construct a Lie algebra $L(B)$. As a vector space, $L(B)$ is the same as $B$. The Lie bracket of two elements of $L(B)$ is defined to be their commutator in $B$:

$$[a, b] = a \ast b - b \ast a$$

The associativity of the multiplication $\ast$ in $B$ implies the Jacobi identity of the commutator in $L(B)$.

**Definition 1.** A Lie algebra is a vector space $\mathfrak{g}$ over some field $F$ together with a binary operation $[\cdot, \cdot]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the Lie bracket that satisfies the following axioms:

(i) **Bilinearity,**

$$[ax + by, z] = a[x, z] + b[y, z], \quad [z, ax + by] = a[z, x] + b[z, y]$$

for all scalars $a, b \in F$ and all elements $x, y, z \in \mathfrak{g}$.

(ii) **Alternativity,** $[x, x] = 0$, for all $x \in \mathfrak{g}$.

(iii) **The Jacobi identity,** $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$, for all $x, y, z \in \mathfrak{g}$.

(iv) **Anticommutativity,** $[x, y] = -[y, x]$, for all elements $x, y \in \mathfrak{g}$.

Most of the interesting Lie groups are non-abelian, and these play an important role in gauge theory.

**Lemma 1.** Let $X_1, \ldots, X_n$ be a basis of Lie algebra $\mathcal{G}$ of Lie groups, $[X_i, X_j] \in \mathcal{G}$, hence there are constants $c_{ij}^k \in C^\infty(U, \mathbb{R})$ such that

$$[X_i, X_j] = c_{ij}^k X_k$$

satisfying the following properties

1. $c_{ij}^k + c_{ki}^j = 0$
2. $c_{ij}^r c_{rk}^s + c_{jk}^r c_{ri}^s + c_{ki}^r c_{rj}^s = 0$

Suppose that $\frac{\partial}{\partial x^i}, \ldots, \frac{\partial}{\partial x^n}$ is the basis of Lie algebra $\mathcal{G}$ on the Euclidean space $\mathbb{R}^n$, then it yields $[X_i, X_j] = 0$, where $X_i = \frac{\partial}{\partial x^i}$, subsequently, structure constant $c_{ij}^k \in C^\infty(U, \mathbb{R})$ of Lie groups hold $c_{ij}^k = 0$. As explained, the GCHS on generalized Poisson manifold $(P, S, \{, , \})$ is only for the non-Euclidean space.

In the GSPB, we will consider that $[X_i, X_j] \neq 0$ for the basis $X_i \in \mathcal{G}$. On the basis of lemma 1, we can obtain $[X_i, X_j] s = c_{ij}^r A_r$, where $A_r$ is structural derivative. Hence,

$$L_{ij} = [X_i, X_j] + [X_i, X_j] s = c_{ij}^r D_r$$

The algebraic definition for the Lie derivative of a tensor field follows from the following theorems.
Theorem 1. Let $M$ be $n$-dimensional $C^\infty$ manifold, $X \in C^\infty(TM)$, let

$$L_X : C^\infty(\otimes^r,sTM) \to C^\infty(\otimes^r,sTM)$$

that is, $\theta \mapsto L_X \theta$ satisfy

1. $L_X f = X f, f \in C^\infty(M, \mathbb{R}) = C^\infty(\otimes^0,0TM)$,
2. $L_X Y = [X, Y], Y \in C^\infty(TM) = C^\infty(\otimes^1,0TM)$

1.2 Generalized covariant Hamilton systems and generalized force field

In order to better understand the concept of the GCHS for the subsequent discussions, in this section, we briefly retrospect some basic concepts of entire framework of GCHS [1]. We begin with the generalized Poisson bracket (GPB) that is defined as the bilinear operation [1–4, 6–9]

$$\{f, g\}_{GPB} = \nabla^T f J \nabla g = J_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_i}$$

where structural matrix $J$ satisfies antisymmetric $J_{ij} = \{x_i, x_j\}_{GPB} = -J_{ji}$. The GCHS in terms of the coordinates has generalized the GHS to the more general form

$$\frac{dx}{dt} = \dot{x} + w x = WH(x), \quad x \in \mathbb{R}^m$$

and greatly opened up a new insight to study the Hamiltonian system with extra structure. The GCHS includes the TGHS

$$\dot{x} = \frac{dx}{dt} = J(x) DH(x)$$

and S dynamics

$$w = \hat{S} H = \{1, H\} \in C^\infty(M, \mathbb{R})$$

The GCHS is defined by the GSPB [1]

$$\{f, g\} = \{f, g\}_{GPB} + G(s, f, g)$$

for $f, g \in C^\infty(M, \mathbb{R})$, and geometric bracket is

$$G(s, f, g) = f\{s, g\}_{GPB} - g\{s, f\}_{GPB}$$

Based on the GCHS and the GSPB, we obtained the covariant evolution as a general dynamical system $\frac{dx}{dt} = \{f, H\}$.

Conservative force, a particle of mass $m$ moves under the influence of a conservative force derived from the gradient $\nabla$ of a scalar potential, the force is given by $F = -\nabla V(r)$, where $V$ is potential energy. According to momentum theorem $\frac{dp}{dt} = F$, we can obtain the following corollary
Corollary 1. The generalized force field $F$ on the generalized Poisson manifold $(P, S, \{\cdot, \cdot\})$ is shown as

$$F = \frac{d}{dt}p = -DH$$

Its components is $F_k = \dot{p}_k = -D_kH$, where $D = \nabla + A$, $A = \nabla s$.

Hence the GCHS can be naturally rewritten in the form

$$\{f, H\} = J_{ij}D_i f D_j H$$

in which $F_i = -D_i H$ is the components of force $F$. Based on the GSPB, theoretically, we have the conservation of energy given by

$$\{H, H\} = J_{ij}D_i H D_j H = J_{ij} F_i F_j = 0$$

and the TCHS is rewritten as

$$\dot{x}_k = J_{kj} D_j H = -J_{kj} F_j = J_{jk} F_j$$

This expression implies the connection between the generalized force field and the velocity.

1.3 Generalized structural Poisson bracket

Definition 2. [1] Let a vector field $X_f = J_{ij} \partial_i f \partial_j \in T_p M$ be given on manifolds, so there is a transformation given by

$$X_f \rightarrow X_f^M = X_f + fX_s$$

for all $X_f, X_s = J_{ij} A_i \partial_j \in T_p M$, $\hat{S} = X_s$ is called structural operator.

Analytic expression of the GSPB

$$\{f, g\} = D^T f JDg = J_{ij}D_i f D_j g \in C^\infty (M, \mathbb{R})$$

with the vector operator $D = \nabla + A$,

$$\{f, g\} = D^T f JDg = \{f, g\}_{GPB} + f\{s, g\}_{GPB} - g\{s, f\}_{GPB}$$

for all $f, g \in C^\infty (M, \mathbb{R})$.

Theorem 2. [1] The generalized structural Poisson bracket of two functions $f, g \in C^\infty (M, \mathbb{R})$ is shown as

$$\{f, g\} = \{f, g\}_{GPB} + G(s, f, g)$$

and geometric bracket is

$$G(s, f, g) = f\{s, g\}_{GPB} - g\{s, f\}_{GPB}$$

where $\{f, g\} = - \{g, f\}$ is skew-symmetric.
Theorem 3. For all \( f, g, h \in C^\infty(M, \mathbb{R}) \), \( \lambda, \mu \in \mathbb{R} \), the GSPB has the following important properties

1. Symmetry: \( \{ f, g \} = -\{ g, f \} \).
2. Bilinearity: \( \{ \lambda f + \mu g, h \} = \lambda \{ f, h \} + \mu \{ g, h \} \).
3. GJI: \( \{ f, \{ g, h \} \} + \{ g, \{ h, f \} \} + \{ h, \{ f, g \} \} = 0 \).
4. Generalized Leibnitz identity: \( \{ fg, h \} = \{ fg, h \}_{GPB} + G(s, fg, h) \).
5. Non degeneracy: if for all \( F \), \( \{ f, g \} = 0 \), then \( \{ f, g \}_{GPB} = G(s, g, f) \).

With the properties, we can define a general form to extend the Poisson manifolds.

Definition 3. The generalized structural Poisson bracket on the smooth manifold \( M \) is an operation on the smooth function space \( C^\infty(M) \), for \( F, G \in C^\infty(M) \), it exists \( \{ F, G \}_{GPB} \in C^\infty(M) \), the operation satisfies the condition 1–4, \( (M, S, \{ \}, \) is correspondingly called generalized Poisson manifolds.

1.4 TGHS, S-dynamics, GCHS

Theorem 4 (TGHS, S-dynamics, GCHS). The TGHS, S-dynamics, GCHS can be respectively formulated as

**TGHS:** \( \frac{d}{dt} f = \dot{f} = \{ f, H \}_{GPB} - H\{ s, f \}_{GPB} \).

**S-dynamics:** \( \frac{d}{dt} s = w = \{ s, H \}_{GPB} = \{ 1, H \} \).

**GCHS:** \( \frac{D}{dt} f = \{ f, H \} = \{ f, H \}_{GPB} + G(s; f, H) \).

2 Some results based on GCHS

The commutation relation of operator \( D_i = \partial_i + A_i \) is

\[
[D_i, D_j] = F_{ij} = [\partial_i, \partial_j] + \partial_i A_j - \partial_j A_i
\]

Generally, \( [\partial_i, \partial_j] = 0 \), we naturally lead to the result \( F_{ij} = [D_i, D_j] = 0 \) by using \( A_j = \partial_j s \).

As lemma shown, \( [X_i, X_j] = c_{ij}^k X_k \), by using the structure constant \( c_{ij}^k \), we let \( v^j c_{ij}^k = w_i^j \) be denoted.

Claim 1. On the non-abelian Lie groups, two identities hold \( L_{ij} = c_{ij}^k D_k \), \( q_k = w_i^k D_k \), where \( v^j c_{ij}^k = w_i^j \).

Subsequently, it leads to

\[
t_k = q_k H = w_i^k D_j H = -w_i^j F_j
\]
and
\[ L_{ij}H = c_{ij}^k D_k H = -c_{ij}^k F_k = c_{ji}^k F_k \]

Plugging the non-Abel condition into the corollary 1, then it gives the equation
\[ u_{jk} = D_j F_k - D_k F_j = H F_{kj} = 0 \]

and S dynamics can be reshown as \( w = J_{ij} \partial_i s F_j \), therefore, we can obtain the below corollary

**Corollary 2.** Based on lemma 1 and corollary 1, one has
\[ t_k = q_k H = w^l_k D_l H = -w^l_k F_l \]
\[ L_{ij} H = c_{ij}^k F_k \]
\[ u_{jk} = F_{kj} H = 0 \]

Within the framework of the GCHS, there are two second order of the GCHS with respect to time and coordinates \( \frac{D^2 x_k}{dt^2}, D_i D_j \frac{D x_k}{dt} \), the former one is about the acceleration, meanwhile the latter is about curvature. \( D_k \frac{D x_k}{dt} \) is also the considerable factors.

**Theorem 5.** Let vector field \( v = v_i \frac{\partial}{\partial x^i} \in T_p M \) be on the generalized Poisson manifold \((P, S, \{\cdot, \cdot\})\), then dot product between \( D = \nabla + A \) and the GCHS is given by
\[ D \cdot \frac{D x}{dt} = D \cdot v + (x \cdot D + 1) w \]

**Proof.** Based on the equation (1), let \( D_l \) act on the component expression of GCHS and obtain
\[ D_l \frac{D x_k}{dt} = D_l x_k + D_l (w x_k) = D_l x_k + x_k D_l w + w \delta_{lk} \]
\[ = \partial_l x_k + A_l x_k + x_k \partial_l w + x_k A_l w + w \delta_{lk} \] (2)

If \( l = k \) holds for the (2), then the consequence will be accordingly obtained
\[ D_k \frac{D x_k}{dt} = D \cdot v + (x \cdot D + 1) w \] (3)

where the divergence of vector \( v \) is \( \nabla \cdot v = \partial_k x_k \).

Apparently, this equation of theorem 4 simultaneously contains two kinetic quantities \( v, w \).

According to theorem 1, we have
\[ L_{X_s} f = X_s f \]
\[ L_{X_s} X_f = [X_s, X_f] \]
by using structural operator \( X_s \).

**Corollary 3.** The S dynamics can be rewritten as
\[ w = L_{X_s} H = X_s H = \hat{S} H \]
2.1 Quadratic invariance

Corollary 4. There is an invariance on the generalized Poisson manifold \((P, S, \{\cdot, \cdot\})\) such that

\[
\frac{D^2}{dt^2} f = \ddot{f} + 2w \dot{f} + \beta f
\]

for all \(f \in C^\infty (M, \mathbb{R})\), where \(\frac{df}{dt} = \dot{f}, \frac{d^2 f}{dt^2} = \ddot{f}\), and \(\beta = w^2 + \frac{d}{dt}w\).

Thusly, the second order covariant derivative of time \(\frac{D^2}{dt^2} = \frac{d^2}{dt^2} + 2w \frac{d}{dt}\) is invariant mathematical structure on the generalized Poisson manifold. In order to prove the practical effect of corollary 4, we can put it to the covariant canonical Hamilton’s equations,

\[
\overset{\circ}{F}_k = \frac{Dp_k}{dt} = -D_kH + p_kw
\] (4)

Then we act the CTO on covariant force \(F_k\) which leads to the consequence

\[
\frac{D}{dt} \frac{Dp_k}{dt} = -\frac{D}{dt}D_kH + \frac{D}{dt}(p_kw)
\]

\[
= -\frac{d}{dt}D_kH - wD_kH + \frac{d}{dt}(p_kw) + p_kw^2
\]

\[
= \frac{d^2}{dt^2}p_k + 2w \frac{d}{dt}p_k + p_k\beta
\]

Obviously, we can draw a conclusion that second order form of covariant canonical Hamilton’s equations with respect to time \(t\) holds invariance.

\[
\left\{\begin{array}{l}
a_i = \frac{\tau^2 x_i}{at^2} = \frac{d^2}{dt^2}x_i + 2w \frac{d}{dt}x_i + x_i\beta \\
\frac{\tau^2 p_k}{at^2} = \frac{d^2}{dt^2}p_k + 2w \frac{d}{dt}p_k + p_k\beta
\end{array}\right.
\]

Hence we can regard acceleration \(a_i = \frac{\tau^2 x_i}{at^2} = \frac{d^2}{dt^2}x_i + 2w \frac{d}{dt}x_i + x_i\beta\) as general geodesic equation.

Theorem 6. Let covariant force \(F_k = \frac{Dp_k}{dt}\) be given on the \((P, S, \{\cdot, \cdot\})\), then

\[f_{kj} = D_j F_k - D_k F_j = c_{kj}S_H = c_{kj}w\]

Proof. The covariant force is shown by equation (4), then acting the covariant derivative \(D_j\) on it, we derive the equation

\[
D_j F_k = D_j \frac{Dp_k}{dt} = -D_jD_kH + D_j(p_kw)
\]

\[
= -D_jD_kH + wD_jp_k + p_kD_jw - A_jp_kw
\]

Similarly, we have \(D_k F_j = -D_kD_jH + p_jD_kw + wD_kp_j - A_kp_jw\), we can derive the difference by subtracting the two equations above

\[f_{kj} = D_j F_k - D_k F_j\]
\[ F_{kj}H + w (\xi_{jk} - \xi_{kj}) + (p_k D_j w - p_j D_k w) \]
\[ = F_{kj}H + (p_k \partial_j - p_j \partial_k) w + w (\xi_{jk} - \xi_{kj} + A_j p_k - A_k p_j) \]
\[ = F_{kj}H + (p_k \partial_j - p_j \partial_k + \xi_{jk} - \xi_{kj} + A_j p_k - A_k p_j) w \]
\[ = F_{kj}H + (p_k \beta_j - p_j \beta_k + \xi_{jk} - \xi_{kj} + A_j p_k - A_k p_j) \hat{S}H \]
\[ = c_{kj} \hat{S}H \]

where S-dynamics is \( w = \hat{S}H \), and
\[ c_{kj} = p_k \partial_j - p_j \partial_k + \xi_{jk} - \xi_{kj} + A_j p_k - A_k p_j \]

As a result, it gets
\[ f_{kj} = c_{kj} w \]

Since then the corollary 2 has been used for the proof.

**References**

[1] G Wang. A study of generalized covariant Hamilton systems on generalized Poisson manifold[J]. arXiv:1710.10597v7.

[2] A. Weinstein. The Local Structure of Possion Manifold [J]. Diff. Geom. 1983, 18: 523-557.

[3] Marsden J E, Ratiu T S. Introduction to mechanics and symmetry: a basic exposition of classical mechanical systems [M]. Springer Science , Business Media, 2013, 61-85,157-165.

[4] Jibin Li, Xiaohua Zhao, Zhengrong Liu. Generalized Hamiltonian systems theory and Applications (Second Edition) [M]. series of modern mathematical foundations, Science Press, 2007.

[5] Jost, Jürgen. Riemannian geometry and geometric analysis. Springer Science, Business Media, 2008, 17-20.

[6] Stephen Wiggins. Introduction to Applied Nonlinear Dynamical Systems and Chaos [M]. Springer, 2003, 197-225.

[7] Weinstein A. Lectures on symplectic manifolds [M]. American Mathematical Soc., 1977.15-20.

[8] Da Silva A C, Da Salva A C. Lectures on symplectic geometry [M]. Berlin: Springer, 2001.

[9] Meyer K, Hall G, Offin D. Introduction to Hamiltonian dynamical systems and the N-body problem [M]. Springer Science, Business Media, 2008.