Tighter bound of Sketched Generalized Matrix Approximation

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Abstract

Generalized matrix approximation plays a fundamental role in many machine learning problems, such as CUR decomposition, kernel approximation, and matrix low rank approximation. Especially with today’s applications involved in larger and larger dataset, more and more efficient generalized matrix approximation algorithms become a crucially important research issue. In this paper, we find new sketching techniques to reduce the size of the original data matrix to develop new matrix approximation algorithms. Our results derive a much tighter bound for the approximation than previous works: we obtain a \((1 + \epsilon)\) approximation ratio with small sketched dimensions which implies a more efficient generalized matrix approximation.

1. Introduction

Matrix manipulations are the basis of modern data analysis. As the datasets becomes larger and larger, it is much more difficult to perform exact matrix multiplication, inversion, and decomposition. Consequently, matrix approximation techniques have been extensively studied, including approximate matrix multiplication [5, 6, 10] and low-rank matrix approximation [1, 2, 4, 13].

In this paper we are concerned with the generalized matrix approximation problem [16, 8, 15]:

\[
\min_X \| A - MXN \|_F,
\]

where \(A \in \mathbb{R}^{m \times n}, M \in \mathbb{R}^{m \times c} \) and \(N \in \mathbb{R}^{r \times n} \). It is well known that the solution is \(X^* = M^\dagger AN^\dagger \). It costs \(O(mz(A) \cdot \min(c, r) + mc^2 + nr^2) \) time to solve the generalized matrix approximation exactly to get \(X^* \). Generalized matrix approximation takes an important role in solving some machine learning problems such as the CUR decomposition [17, 9, 18], modified Nyström method [17], and distributed PCA [12, 3]. It is also a key research topic in numerical linear algebra [16, 8].

When \(N \) is the identity matrix, the generalized matrix approximation degenerates to the ordinary least squares regression. To solve the least squares regression \(\hat{X} = \text{argmin}_X \| MX - A \|_F \) more efficiently when \(c \ll m \), recent studies suggest multiplying a sketching matrix \(S \in \mathbb{R}^{s_c \times m} \), where \(s_c = O(c/\epsilon) \), to get a sketched least squares regression \(\hat{X} = \text{argmin}_X \| S(MX - A) \|_F \). The studies [4, 7, 14] also prove that the reduced least squares regression obtains a \((1 + \epsilon)\) relative error bound, which is, \(\| MX - A \|_F \leq (1 + \epsilon)\| MX - A \|_F \).
Let **s** be the **m** x **n** identity matrix. Given a matrix **A** = \([a_{ij}]\) \in \mathbb{R}^{m \times n} of rank \(\rho\), its SVD is given as \(\mathbf{A} = \mathbf{U} \Sigma \mathbf{V}^T = \mathbf{U}_k \Sigma_k \mathbf{V}_k^T + \mathbf{U}_{\rho\setminus k} \Sigma_{\rho\setminus k} \mathbf{V}_{\rho\setminus k}^T\), where \(\mathbf{U}_k\) and \(\mathbf{U}_{\rho\setminus k}\) contain the left singular vectors of \(\mathbf{A}\), \(\mathbf{V}_k\) and \(\mathbf{V}_{\rho\setminus k}\) contain the right singular vectors of \(\mathbf{A}\), and \(\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_\rho)\) with \(\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_\rho > 0\) are the nonzero singular values of \(\mathbf{A}\). Accordingly, \(\|\mathbf{A}\|_F = \left(\sum_{i,j} a_{ij}^2\right)^{1/2} = \left(\sum_i \sigma_i^2\right)^{1/2}\) is the Frobenius norm of \(\mathbf{A}\) and \(\|\mathbf{A}\|_2 \equiv \sigma_1\) is the spectral norm.

Additionally, \(\mathbf{A}^\dagger \equiv \mathbf{V} \Sigma^{-1} \mathbf{U}^T \in \mathbb{R}^{n \times m}\) is the Moore-Penrose pseudoinverse of \(\mathbf{A}\), which is unique.

It is easy to verify that \(\text{rank}(\mathbf{A}^\dagger) = \text{rank}(\mathbf{A}) = \rho\). Moreover, for all \(i = 1, \ldots, \rho\), \(\sigma_i(\mathbf{A}^\dagger) = 1/\sigma_{\rho - i + 1}(\mathbf{A})\). If \(\mathbf{A}\) is of full row rank, then \(\mathbf{A} \mathbf{A}^\dagger = \mathbf{I}_m\). Also, if \(\mathbf{A}\) is of full column rank, then \(\mathbf{A}^\dagger \mathbf{A} = \mathbf{I}_n\). When \(m = n\), \(\text{tr}(\mathbf{A}) = \sum_{i=1}^n a_{ii}\) is the trace of \(\mathbf{A}\).

It is well known that \(\mathbf{A}_k \equiv \mathbf{U}_k \Sigma_k \mathbf{V}_k^T\) is the minimizer of both \(\min \|\mathbf{A} - \mathbf{X}\|_F\) and \(\min \|\mathbf{A} - \mathbf{X}\|_2\) over all matrices \(\mathbf{X} \in \mathbb{R}^{m \times n}\) of rank at most \(k \leq \rho\). Thus, \(\mathbf{A}_k\) is called the best rank-\(k\) approximation of \(\mathbf{A}\). Let \(\text{mz}(\mathbf{A})\) denote the number of nonzero entries of \(\mathbf{A}\).
2.2 Subspace Embedding

Oblivious subspace embedding is an important sketching tool in randomized numerical linear algebra. By oblivious subspace embedding, a matrix can be projected to a much lower dimensional subspace, which leads to much faster matrix operations.

**Definition 1** ([19]) Given $\varepsilon > 0$ and $\delta > 0$, let $\Pi$ be a distribution on $s \times m$ real matrices, where $s$ relies on $m$, $d$, $\varepsilon$ and $\delta$. Suppose that with probability at least $1 - \delta$, for any fixed $m \times d$ matrix $A$, a matrix $S$ drawn from distribution $\Pi$ is a $(1 + \varepsilon)$ $\ell_2$-subspace embedding for $A$, that is, for all $x \in \mathbb{R}^d$, $\|SAx\|_2^2 = (1 \pm \varepsilon)\|Ax\|_2^2$ with probability $1 - \delta$. Then we call $\Pi$ an $(\varepsilon, \delta)$-oblivious $\ell_2$-subspace embedding.

For the sake of conciseness, the $(\varepsilon, \delta)$-oblivious $\ell_2$-subspace embedding is referred to as an $\varepsilon$-subspace embedding. Now we list some important subspace embedding matrices and their properties which will be used in this paper.

**Definition 2** (leverage-score sketching matrix) Let $V \in \mathbb{R}^{n \times k}$ be column orthonormal basis for $A \in \mathbb{R}^{n \times k}$ with $n > k$, and $v_{i,*}$ denote the $i$-th row of $V$. Let $\ell_i = \|v_{i,*}\|_F^2/k$ and $r$ be an integer with $1 \leq r \leq n$. Then the $\ell_i$’s are leverage scores for $A$. Construct a sampling matrix $\Omega \in \mathbb{R}^{n \times r}$ and a rescaling matrix $D \in \mathbb{R}^{r \times r}$ as follows. For every $j = 1, \ldots, r$, independently and with replacement, pick an index $i$ from the set $\{1, 2, \ldots, n\}$ with probability $\ell_i$ and set $\Omega_{ij} = 1$ and $D_{jj} = 1/\sqrt{\ell_i}$. The leverage-score sketching matrix $S$ for $A$ is then defined as $S = \Omega D$.

**Theorem 3** ([19]) Given $A \in \mathbb{R}^{m \times d}$ of full column rank, assume $S \in \mathbb{R}^{s \times m}$ is an $\varepsilon$-subspace embedding matrix for $A$. If $S$ is a sparse subspace embedding matrix, then $s = O(d^2\varepsilon^{-2})$ is sufficient. If $S$ is an $s \times m$ matrix of i.i.d. normal random variables with variance $1/s$, then $s = \Theta(d\varepsilon^{-2})$ is sufficient. For a leverage-score sketching matrix $S$, $s = O(d\log d\varepsilon^{-2})$ is needed.

**Theorem 4** ([4, 3]) For $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times k}$, there is an $s = \Theta(\varepsilon^{-2})$, so that for an $s \times m$ sparse embedding matrix $S$ or an $s \times n$ matrix $S$ of i.i.d. normal random variables with variance $1/s$, or an $s \times m$ leverage-score sketching matrix for $A$ under the condition that $A$ has orthonormal columns, then it holds that

$$\|A^TSA^T - A^TB\|_F^2 < \varepsilon^2\|A\|_F^2\|B\|_F^2$$

with probability at least $1 - \delta$ for any fixed $\delta > 0$.

Other types of sketching matrices like Subsampled Randomized Hadamard Transformation and detailed properties of sketching matrices and subspace embedding matrices can be found in the survey [19].

3. Main Result

We first give the conditions that subspace embedding matrices should satisfy for a sketched generalized matrix approximation to achieve a $(1 + \varepsilon)$ error bound. The detailed conditions are depicted in Theorem 5.

**Theorem 5** Given that $A \in \mathbb{R}^{m \times n}$, $M \in \mathbb{R}^{m \times c}$ and $N \in \mathbb{R}^{r \times n}$, assume that $S_M \in \mathbb{R}^{s_c \times m}$ is a subspace embedding matrix for $M$ with error parameter $\epsilon_0 = 1/2$, and $S_M$ also makes Eqn. (1) hold with error parameter $\sqrt{\epsilon/c}$. Similarly, $S_N \in \mathbb{R}^{r \times n}$ is a subspace embedding matrix for $N^T$ with error parameter $\epsilon_0 = 1/2$ and $S_N$ also makes Eqn. (1) hold with error parameter $\sqrt{\epsilon/r}$. Let

$$X^* = M^TAN^T = \arg\min_X \|A - MXN\|_F$$

(2)
and

\[ \hat{X} = (S_M M)^{\dagger} S_M A S_N^T (N S_N^T)^{\dagger}. \]  

(3)

Then we have

\[ \|A - MXN\|_F \leq (1 + \epsilon)\|A - M\hat{X} N\|_F. \]

The conditions required in Theorem 5 are all satisfied by oblivious embedding matrices, including sparse embedding matrices, gaussian matrices, subsampled randomized Hadamard matrices [19], as well as their combinations. Now we present Theorem 6, which shows that an \( \text{nnz}(A) \) generalized matrix approximation solver can be achieved and the corresponding sketched dimensions are \( s_c = \mathcal{O}(c/\epsilon) \) and \( s_r = \mathcal{O}(r/\epsilon) \), respectively.

**Theorem 6** We are given \( A \in \mathbb{R}^{m \times n}, M \in \mathbb{R}^{m \times c} \) and \( N \in \mathbb{R}^{r \times n} \). Assume that \( \Pi_M \) is a \( t \times m \) sparse embedding matrix and \( G_M \) is an \( s_c \times t \) matrix of i.i.d. normal random variables with variance \( 1/s_c \), where \( t = \mathcal{O}(c/\epsilon + c^2) \) and \( s_c = \mathcal{O}(c/\epsilon) \). Construct \( S_M = G_M \Pi_M \). Similarly, \( \Pi_N \) is a \( t' \times n \) sparse embedding matrix and \( G_N \) is a \( s_r \times t' \) matrix of i.i.d. normal random variables with variance \( 1/s_r \), where \( t' = \mathcal{O}(r/\epsilon + r^2) \) and \( s_r = \mathcal{O}(r/\epsilon) \). Then we construct \( S_N = G_N \Pi_N \). If \( X^* \) and \( \hat{X} \) are defined as (2) and (3) respectively, then we have

\[ \|A - MXN\|_F \leq (1 + \epsilon)\|A - M\hat{X} N\|_F, \]

with high probability and \( \hat{X} \) can be constructed with the computational complexity of

\[ O(\text{nnz}(A) + mc + nr + c^2r/\epsilon^3 + cr^2/\epsilon^3 + c^3r/\epsilon^2 + c^3r/\epsilon^2 + cr^3/\epsilon^2 + c^2r^3/\epsilon + c^3r^2/\epsilon + c^4/\epsilon + r^4/\epsilon). \]

(4)

**Proof** Let \( \Pi_M \) and \( G_M \) be \( \epsilon_0 \)-subspace embedding matrices for \( c \)-subspace with \( \epsilon_0 = 1/4 \), which needs \( t = \mathcal{O}(c^2) \) and \( s_c = \mathcal{O}(c) \). By Lemma 9, we have \( G_M \Pi_M \) is a \( 1/2 \)-subspace embedding matrix for \( M \). By Lemma 11 and Theorem 4, \( t = \mathcal{O}(c/\epsilon) \) and \( s_c = \mathcal{O}(c/\epsilon) \), and \( S_M = G_M \Pi_M \) makes Eqn. (1) hold with error parameter \( \sqrt{c/\epsilon} \). The similar result holds for \( \Pi_N, G_N \) and \( S_N \). Thus, \( S_M \) and \( S_N \) satisfy all conditions in Theorem 5. By Theorem 5, we obtain the result.

For computation complexity, it needs \( O(\text{nnz}(A) + c^2r/\epsilon^3 + cr^2/\epsilon^3 + c^3r/\epsilon^2 + c^3r/\epsilon^2 + cr^3/\epsilon^2 + \min(2c^2r^3/\epsilon, c^3r^2/\epsilon)) \) to compute \( S_M A S_N^T \). Computing \( (S_M M)^{\dagger} \) and \( (N S_N^T)^{\dagger} \) requires \( O(mc + c^2r/\epsilon^3 + c^3r/\epsilon^3) \) and \( O(nr + c^2r/\epsilon^3 + cr^2/\epsilon^3 + c^3r/\epsilon^2) \) respectively. Matrix multiplications of \( (S_M M)^{\dagger} S_M A S_N^T (N S_N^T)^{\dagger} \) need \( O(\min(2c^2r^3/\epsilon^2 + cr^2/\epsilon^2 + c^3r^2/\epsilon^2 + c^3r^2/\epsilon^2 + cr^3/\epsilon^2 + \min(c^2r^3/\epsilon, c^3r^2/\epsilon)) \) and \( c^4/\epsilon + r^4/\epsilon \).

The lemmas mentioned in the proof of Theorem 6 are given in Appendix A.

Leverage-score sketching matrices are significant in randomized numerical linear algebra. Using leverage-score sketching matrices, we can achieve faster sketched generalized matrix approximation than \( O(\text{nnz}(A)) \). It just needs \( O(\text{nnz}(M) + \text{nnz}(N)) \) arithmetic operations comparing with \( O(\text{nnz}(A)) \) in Theorem 6.

**Theorem 7** We are given \( A \in \mathbb{R}^{m \times n}, M \in \mathbb{R}^{m \times c} \) and \( N \in \mathbb{R}^{r \times n} \). Let \( S_M \in \mathbb{R}^{s_c \times m} \) be the leverage-score sketching matrix for \( M \) with \( s_c = \mathcal{O}(c/\epsilon + c\log c) \). Similarly, \( S_N \in \mathbb{R}^{s_r \times n} \) is the leverage-score sketching matrix for \( N^T \) with \( s_r = \mathcal{O}(r/\epsilon + r\log r) \). If \( X^* \) and \( \hat{X} \) are defined as (2) and (3) respectively, then we have

\[ \|A - MXN\|_F \leq (1 + \epsilon)\|A - M\hat{X} N\|_F \]

with high probability. And \( \hat{X} \) can be constructed with the computational complexity of

\[ O(\text{nnz}(M) + \text{nnz}(N) + (c^2r + cr^2)/\epsilon^2 + (c^2r + cr^2)\log(c\epsilon)/\epsilon \]

\[ + (c^3 + r^3)/\epsilon + (c^2r + cr^2)\log c \log r + c^3\log c + r^3\log r) \]
Proof By Theorem 3 and 4, it is easy to check that $S_M$ and $S_N$ satisfy the conditions in Theorem 5. Hence, the result holds by Theorem 5.

As for computational cost, it costs $O(nnz(M))$ and $O(nnz(N))$ time to compute leverage scores of $M$ and $N^T$ respectively [11]. And it takes $O(cr/\epsilon^2 + cr \log (cr)/\epsilon + \log c \log r)$ to compute $S_M AS_N^T$. And it requires $O(c^3/\epsilon + r^3/\epsilon + c^3 \log c + r^3 \log r)$ time to compute $(S_M M)^\dagger$ and $(S_N N^T)^\dagger$. It costs $O((c^3 r + cr^2)/\epsilon^2 + (c^3 r + cr^2) \log (cr)/\epsilon + (c^2 r + cr^2) \log c \log r)$ computational operations to achieve the multiplication of $(S_M M)^\dagger S_M AS_N^T (NS_N^R)^\dagger$. Hence, the total cost of constructing $\hat{X}$ is

$$O(nnz(M) + nnz(N) + (c^2 r + cr^2)/\epsilon^2 + (c^2 r + cr^2) \log (cr)/\epsilon + (c^3 r + cr^2) \log c \log r + c^3 \log c + r^3 \log r)$$

Now we consider the symmetric case where $A$ is symmetric and $N = M^T$, $X^*$ constructed as Eqn. (2) is a symmetric matrix. Note that $X$ is asymmetric in most cases since $S_M$ and $S_N$ are chosen independently. However, we can construct $X = (X + X^T)/2$ which is symmetric and can still keep relative error bound.

Corollary 8 Let $A \in \mathbb{R}^{m \times m}$ be a symmetric matrix. $M$ is an $m \times c$ matrix. $S_1$ and $S_2$ are the subspace embedding matrices for $M$ with error parameter $\epsilon_0 = 1/2$, and they also satisfy Eqn. (1) with error parameter $\sqrt{\epsilon}/c$. Let

$$X^* = M^T A (M^T)^\dagger = \arg\min_{X} \|A - XMX^T\|_F$$

and

$$\hat{X} = (S_M M)^\dagger S_M AS_N^T (NS_N^T)^\dagger.$$  

Construct a symmetric matrix $\tilde{X}$ as

$$\tilde{X} = (X + \hat{X})/2.$$  

Then we have

$$\|A - MX^T M^T\|_F \leq (1 + \epsilon)\|A - MX^* M^T\|_F.$$  

4. Proof of Main Theorem

In this section, we give the detailed proof of our main theorem, i.e., Theorem 5. Proof of Theorem 5 We define

$$A^+ \triangleq A - MX^* N.$$  

We let the condensed SVDs of $M$ and $N$ be respectively

$$M = X_M \Sigma_M V_M^T$$  

and

$$N = X_N \Sigma_N V_N^T.$$  

We have

$$MXN = M (S_M M)^\dagger S_M AS_N^T (NS_N^T)^\dagger N$$  

$$= U_M \Sigma_M V_M^T (S_M U_M \Sigma_M V_M^T)^\dagger (S_M AS_N^T (U_N \Sigma_N V_N^T S_N^T)^\dagger U_N \Sigma_N V_N^T)$$  

$$= U_M \Sigma_M V_M^T (S_M U_M)^\dagger (S_M AS_N^T (V_N S_N^T)^\dagger (U_N \Sigma_N)^\dagger U_N \Sigma_N V_N^T$$  

$$= U_M (S_M U_M)^\dagger S_M AS_N^T (V_N S_N^T)^\dagger V_N^T.$$  

where (5) is because $S_M U_M$ is of full column rank and $\Sigma_M V_M^T$ is of full row rank.

We define $Z^*$ by

$$MX^* N = U_M (\Sigma_M V_M X^* U_N \Sigma_N) V_N^T \equiv U_M Z^* V_N^T.$$  

.$$
Similarly, we define \( \hat{Z} \) by

\[
\text{M}XN = U_M \hat{Z} V_N^T.
\]

Then, we have that

\[
(S_M U_M)^T (S_M U_M) \hat{Z} (V_N^T S_N^T) (V_N^T S_N^T)^T
= (S_M U_M)^T S_M A^+ S_N^T (V_N^T S_N^T)^T
= (S_M U_M)^T S_M (A^+ + U_M Z^* V_N^T) S_N^T (V_N^T S_N^T)^T.
\]

Hence, we have

\[
(S_M U_M)^T S_M A^+ S_N^T (V_N^T S_N^T)^T = (S_M U_M)^T S_M U_M (\hat{Z} - Z^*) V_N^T S_N^T (V_N^T S_N^T)^T
= (S_M U_M)^T S_M U_M ZV_N^T S_N^T (V_N^T S_N^T)^T,
\]

where we define \( Z = \hat{Z} - Z^* \).

Since \( S_M U_M \) is of full column rank and \( s_c \geq c \geq \rho_c \), where \( \rho_c \) is the rank of \( M \), we have that \( (S_M U_M)^T S_M U_M \) is nonsingular. Similarly, \( V_N^T S_N^T (V_N^T S_N^T)^T \) is nonsingular. We obtain

\[
Z = [(S_M M)^T S_M U_M]^{-1} [(S_M U_M)^T S_M A^+ S_N (V_N^T S_N^T)^T] [V_N^T S_N^T (V_N^T S_N^T)^T]^{-1}
\]

and thus

\[
\| Z \|_F \leq \| (S_M U_M)^T S_M U_M \|_F^{-1} \| [V_N^T S_N^T (V_N^T S_N^T)^T]^{-1} \|_F \| [(S_M U_M)^T S_M A^+ S_N (V_N^T S_N^T)^T] \|_F.
\]

We can expand \( \| A - MXN \|_F^2 \) as follows:

\[
\| A - MXN \|_F^2
= \| A - MX^* N + MX^* N - MXN \|_F^2
= \| A - MX^* N \|_F^2 + \| MX^* N - MXN \|_F^2 + 2 \text{tr}[(A - MX^* N)^T (MX^* N - MXN)]
= \| A - MX^* N \|_F^2 + \| U_M ZV_N^T \|_F^2,
\]

where (6) is because

\[
\text{tr}[(A - MX^* N)^T (MX^* N - MXN)]
= \text{tr}[(I_m - MM^T) A + (MM^T) A (I_n - N^T N)] M (X^* - X) N
= \text{tr}[A^T (I_m - MM^T) M (X^* - X) N] + \text{tr}[N (I_n - N^T N) A^T MM^T M (X^* - X)]
= 0.
\]

Now we need to bound \( \| Z \|_F \). First, we express \( A^+ \) as follow

\[
A^+ = A - MX^* N = U_M^*(U_M^*)^T A + U_M U_M^T A V_N^T (V_N^T)^T.
\]

Then, we have

\[
\| [(S_M U_M)^T S_M A^+ S_N^T (V_N^T S_N^T)^T] \|_F
\leq \| (S_M U_M)^T S_M U_M (U_M^*)^T A S_N^T (V_N^T S_N^T)^T \|_F + \| (S_M U_M)^T S_M U_M U_M^T A V_N^T (V_N^T)^T S_N^T (V_N^T S_N^T)^T \|_F.
\]
Since $U_M^T U_M(U_M^T)^T A S_N^T (V_N^T S_N^T)^T = 0$, and by Theorem 4, we have

$$\|(S_M U_M)^T S_M U_M(U_M^T)^T A S_N^T (V_N^T S_N^T)^T\|_F = \|U_M^T S_M U_M(U_M^T)^T A S_N^T (V_N^T S_N^T)^T - U_M^T U_M(U_M^T)^T A S_N^T (V_N^T S_N^T)^T\|_F \\
\leq \frac{\sqrt{c}}{\sqrt{r}} \|U_M^T(U_M^T)^T A\|_F \|V_N\|_F = \sqrt{c} \|U_M^T(U_M^T)^T A\|_F$$

Therefore, we obtain

$$\|U_M^T(U_M^T)^T A S_N^T S_N V_N\|_F \leq (1 + \sqrt{c}) \|U_M^T(U_M^T)^T A\|_F.$$  \hfill (8)

Now, we get

$$\|(S_M U_M)^T S_M U_M(U_M^T)^T A S_N^T (V_N^T S_N^T)^T\|_F \leq 2\sqrt{c} \|U_M^T(U_M^T)^T A\|_F.$$  \hfill (9)

For $\|(S_M U_M)^T S_M U_M U_M^T A V_N^T (V_N^T S_N^T)^T\|_F$, we have,

$$\|(S_M U_M)^T S_M U_M U_M^T A V_N^T (V_N^T S_N^T)^T\|_F \leq \|(S_M U_M)^T S_M U_M U_M^T A V_N^T (V_N^T S_N^T)^T\|_F \\
\leq \|(1 + 0.5) A V_N^T (V_N^T S_N^T)^T\|_F$$

where $\|(S_M U_M)^T S_M U_M U_M^T A V_N^T (V_N^T S_N^T)^T\|_F \leq 2\sqrt{c} \|A\|_F.$

Thus, we have

$$\|(S_M U_M)^T S_M U_M(U_M^T)^T A S_N^T (V_N^T S_N^T)^T\|_F \leq 2\sqrt{c} \|U_M^T(U_M^T)^T A\|_F + 2\sqrt{c} \|A V_N^T (V_N^T S_N^T)^T\|_F$$

where last inequality follow from the fact that $\|U_M^T(U_M^T)^T A\|_F \leq \|A - M X^* N\|_F$ and $\|A V_N^T (V_N^T S_N^T)^T\|_F \leq \|A - M X^* N\|_F$. Thus, we have,

$$Z \leq \sigma_{\min}^2(S_M U_M) \sigma_{\min}^2(S_N V_N) \cdot \|(S_M U_M)^T S_M A^* S_N^T (V_N^T S_N^T)^T\|_F$$

where the second inequality is because subspace embedding property of $S_M$ and $S_N$. It holds that

$$\sigma_{\min}^2(S_M U_M) \leq \frac{1}{(1 - 0.5)^2} = 2,$$
and
\[ \sigma_{\min}^{-2}(S_N V_N) \leq \frac{1}{(1 - 0.5)\sigma_{\min}^2(V_N)} = 2. \]
Finally, we reach that
\[ \|A - MXN\|_F^2 = \|A - MX^*N\|_F^2 + \|MX^*N - MX^*N\|_F^2 \]
\[ = \|A - MX^*N\|_F^2 + \|U_M ZV_N^T\|_F^2 \]
\[ \leq \|A - MX^*N\|_F^2 + 256\epsilon \|A - MX^*N\|_F^2 \]
\[ = (1 + 256\epsilon)\|A - MX^*N\|_F^2. \]
By rescaling \( \epsilon \), we get the result.

5. Conclusion
In this paper we have studied fast generalized matrix approximation using sketching techniques. We have given a tighter bound of reduced dimensions \( s_c \) and \( s_r \) to reach a \( (1 + \epsilon) \) error bound and obtained an \( \mathcal{O}(\text{nnz}(M) + \text{nnz}(N)) \) generalized matrix approximation.

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Appendix A. Important Lemmas

Lemma 9 \( G \in \mathbb{R}^{\ell \times t} \) and \( \Pi \in \mathbb{R}^{t \times m} \) are \( \epsilon \)-subspace embedding matrices for \( k \)-subspace. \( A \in \mathbb{R}^{m \times k} \) has at most rank \( k \). Then, for any vector \( x \in \mathbb{R}^k \), it holds that
\[
\|G\Pi Ax\|_2^2 = (1 \pm 2\epsilon)\|Ax\|_2^2.
\]

Proof Since \( G \in \mathbb{R}^{\ell \times n} \) and \( \Pi \in \mathbb{R}^{n \times m} \) are \( \epsilon \)-subspace embedding matrix for \( k \)-subspace, we have
\[
\|G\Pi Ax\|_2^2 = (1 \pm 2\epsilon)\|\Pi Ax\|_2^2 = (1 \pm 2\epsilon)^2\|Ax\|_2^2 = (1 \pm 2\epsilon)\|Ax\|_2^2,
\]
where the last equality omit the high order \( \epsilon^2 \).

Lemma 10 ([3, 4]) Given \( A \in \mathbb{R}^{m \times n} \), there is an \( s = \Omega(\epsilon^{-2}) \) so that for an \( s \times m \) sparse embedding matrix \( S \) or an \( s \times m \) matrix \( S \) of i.i.d. normal random variables with variance \( 1/s \), then with high probability,
\[
\|SA\|_F^2 = (1 \pm \epsilon)\|A\|_F^2.
\]

Lemma 11 If \( G \in \mathbb{R}^{\ell \times t} \) and \( \Pi \in \mathbb{R}^{t \times m} \) can be used to approximate matrix products with error parameter \( \epsilon \) i.e. Equation (1) holding, besides \( \Pi \) can keep the Frobenius norm of matrix with error parameter \( \epsilon_0 = 1 \), i.e Equation (11) holding with error parameter \( 1 \), then given \( A \in \mathbb{R}^{m \times n} \) and \( B \in \mathbb{R}^{m \times k} \), we have
\[
\|A^T\Pi^T G^T \Pi B - A^TB\|_F \leq 5\epsilon\|A\|_F\|B\|_F.
\]

Proof Since \( G \in \mathbb{R}^{\ell \times n} \) and \( \Pi \in \mathbb{R}^{n \times m} \) can be used to approximate matrix products, we have
\[
\|A^T\Pi^T G^T \Pi B - A^TB\|_F = \|A^T\Pi^T G^T \Pi B - A^T\Pi^T \Pi B + A^T\Pi^T \Pi B - A^TB\|_F
\]
\[
\leq \|A^T\Pi^T G^T \Pi B - A^T\Pi^T \Pi B\|_F + \|A^T\Pi^T \Pi B - A^TB\|_F
\]
\[
\leq \epsilon(\|\Pi A\|_F\|\Pi B\|_F + \|A\|_F\|B\|_F)
\]
\[
\leq 5\epsilon\|A\|_F\|B\|_F,
\]
where the last inequality is because \( \|\Pi A\|_F \leq (1 + 1)\|A\|_F \) and \( \|\Pi B\|_F \leq (1 + 1)\|B\|_F \).