Martingale-driven approximations of singular stochastic PDEs

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K. Matetski
Columbia University, Email: matetski@math.columbia.edu

Abstract
We define multiple stochastic integrals with respect to càdlàg martingales and prove moment bounds and chaos expansions, which allow to work with them in a way similar to Wiener stochastic integrals. In combination with the discretization framework [EH17], our results give a tool for proving convergence of interacting particle systems to stochastic PDEs using regularity structures. As examples, we prove convergence of martingale-driven discretizations of the 3-dimensional stochastic quantization equation and the KPZ equation.

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1 Introduction

We are interested in singular stochastic PDEs, which can be written in the following form:

\[ \mathcal{L} u = F(u, \nabla u, \xi) , \]  

(1.1)

where \( \mathcal{L} \) is a linear operator (e.g. \( \mathcal{L} = \partial_t - \Delta \)), \( \xi \) is an irregular random noise (typically a Gaussian white noise) and \( F \) is a local non-linearity. In the case, when the problem (1.1) is locally subcritical, a notion of solution was provided in the framework of regularity structures by Hairer \cite{Hairer2014}. Alternative approaches to singular stochastic PDEs were developed by Gubinelli, Imkeller and Perkowski \cite{Gubinelli2015} and by Kupiainen \cite{Kupiainen2016}. Since such equations are ill-posed in the classical sense, one has to renormalize them by subtracting ‘infinite constants’. A general theory or renormalization has been built by Bruned, Hairer and Zambotti \cite{Bruned2016}.

Equations of the type (1.1) usually arise as scaling limits of microscopic models in statistical mechanics, e.g. the KPZ equation is the limit of weakly asymmetric growing interfaces \cite{Baik1997}, the \( \Phi^4_3 \) equation describes macroscopic behaviour of the ferromagnetic Ising model \cite{Glockner1999, Molchan2017, Schutz2018} at criticality, and random walks in random environments can converge to the parabolic Anderson model \cite{Koenig2016}. This is a motivation to consider spatial discretizations of a general type

\[ \mathcal{L}_\varepsilon u^\varepsilon = F^\varepsilon(u^\varepsilon, \nabla u^\varepsilon, \xi^\varepsilon) , \]  

(1.2)

when the scaling parameter \( \varepsilon > 0 \) tends to zero. In the case, when the solution of (1.1) is a function in time, a framework for spatial discretizations (1.2) on dyadic grids was developed in \cite{Hairer2018}; and a completely general discretization framework was introduced in \cite{Eguchi2017}. One of the main steps, when proving convergence of such approximations using regularity structures, is obtaining moment bounds and convergence of a ‘model’, which is given by a lift of the driving noise \( \xi^\varepsilon \) to a much more complicated object. In the case of a Gaussian noise, such bounds can be obtained using calculus of multiple Wiener integrals \cite{Nualart2006}, which includes Wick’s lemma, Nelson’s estimate and chaos expansions (see \cite{Hairer2014} in the continuous case, and \cite{Hairer2018} in the discrete case). The latter procedure was automated by Chandra and Hairer \cite{Chandra2016}, a unification of which with the general renormalization was done in \cite{Bruned2017}. Unfortunately, all the above mentioned examples, coming from statistical mechanics, don’t fit into this framework, because all of them are Markov processes, which implies that the driving noise \( \xi^\varepsilon \) is typically a càdlàg martingale or its time derivative.

When working with the KPZ equation \cite{Kardar1986}, which is an example of equations (1.1), one can avoid using regularity structures (or its alternatives). More precisely, the Hopf-Cole transformation turns the KPZ equation into the stochas-
tic heat equation with multiplicative noise, which is defined using the standard stochastic calculus. Discrete versions of this transformation proved to be useful when studying convergence of particular interacting particle systems [BG97, DT16, CT17, CST16, CS16, CGST18]. Unfortunately, such non-linear transformations are very sensitive to perturbations of models, which was for example the reason for having a strict limitation on jump ranges in the article by Dembo and Tsai [DT16]. Another approach to the KPZ equation is to write it as a martingale problem, which was proposed by Gonçalves and Jara [GJ14], and developed by Gubinelli and Jara [GJ13], and Gubinelli and Perkowski [GP18]. A restriction of this approach is that it works only in equilibrium.

In [HS17] the authors considered the KPZ equation driven by a sufficiently nice stationary, non-Gaussian noise. Using regularity structures and the general definition of Wick product, they proved convergence of a renormalized solution to the solution of the standard KPZ equation. This approach cannot be applied to the models of our interest, because the driving martingales are usually non-stationary and have slow decorrelation, which violates the assumptions of [HS17].

To avoid these limitations, one can work directly with singular stochastic PDEs, using regularity structures. For this, one needs to embed càdlàg martingales into the theory of regularity structures, which is a goal of this article. More precisely, we define multiple stochastic integrals with respect to càdlàg martingales and prove moment bounds and chaos expansions, which allow to work with them in a way similar to Wiener stochastic integrals. In particular, this allows to prove moment bounds and convergence of canonical lifts of càdlàg martingales, similarly to how it was done for a Gaussian noise.

Combining our results with the discretization framework [EH17], we get a powerful tool to prove convergences of general approximations (1.2). As examples, we prove convergence of martingale-driven discretizations of the stochastic quantization equation in 3D and the KPZ equation. These discretizations are not describing some particular interacting particle systems, but rather serve as a demonstration of our results. An application to a general class of weakly asymmetric exclusion processes is the content of the upcoming article [MQ].

1.1 Structure of the article

In Section 2 we define multiple stochastic integrals with respect to càdlàg martingales. Section 3 is devoted to a generalization of the bracket processes. In the consecutive Sections 4 and 5 we prove a ‘chaos expansion’ of products and moment bounds for stochastic integrals. In Section 6 we analyse multiple stochastic integrals with kernels given by generalized convolutions, which are typical objects in the theory of regularity structures. In Section 7 we prove convergence of martingale-driven discretizations of the stochastic quantization equation and the KPZ equation. Finally, Appendix A lists the properties of martingales which are
used in the article.

1.2 Notation

In this article we work in $\mathbb{R}^{d+1}$, for $d \geq 1$, and consider the coordinates as time-space $(t,x)$ with $t \in \mathbb{R}$ and $x \in \mathbb{R}^d$. We work with the parabolic scaling $s = (2,1,\ldots,1)$ of $\mathbb{R}^{d+1}$, for which we define the norm
\[
\|z\|_s := \max\{\sqrt{|t|}, |x_1|, \ldots, |x_d|\},
\]
where $z = (t,x_1,\ldots,x_d)$. Moreover, we set $|s| := 2 + d$, and for a multiindex $k = (k_0,\ldots,k_d) \in \mathbb{N}^{d+1}$ we use the notation $|k|_s := 2k_0 + \sum_{i=1}^d k_i$ (the set of natural numbers $\mathbb{N}$ includes 0).

For a function $\varphi : \mathbb{R}^{d+1} \to \mathbb{R}$, for a scaling parameter $\lambda \in (0,1]$ and for two points $z = (t,x), \bar{z} = (\bar{t},\bar{x}) \in \mathbb{R}^{d+1}$, we use the following notation for a rescaled and recenter function:
\[
\varphi_\lambda^s(z) := \lambda^{-|s|}\varphi(\lambda^{-2}(\bar{t} - t), \lambda^{-1}(\bar{x} - x)).
\]

For $\varepsilon \in (0,1]$ we define the grid $\Lambda_\varepsilon := \varepsilon \mathbb{Z}$ and the domain $D_\varepsilon := \mathbb{R} \times \Lambda_\varepsilon^d$. We equip the space $\ell^2(\Lambda_\varepsilon^d)$ with the standard norm, approximating $\|\cdot\|_{\ell^2}$ as $\varepsilon \to 0$. We refer to Appendix A for relevant definitions and properties of càdlàg martingales.

In estimates we will often use ‘$\lesssim$’, which means that the bound ‘$\leq$’ holds with a constant multiplier, independent of relevant quantities.

2 Multiple stochastic integrals

In this section we define multiple stochastic integrals with respect to càdlàg martingales with the spatial dimension $d \geq 1$. To this end, for $\varepsilon > 0$ we define the grid $\Lambda_\varepsilon := \varepsilon \mathbb{Z}$ and the time-space domain $D_\varepsilon := \mathbb{R} \times \Lambda_\varepsilon^d$. We equip the space $\ell^2(\Lambda_\varepsilon^d)$ with the standard norm, approximating $\|\cdot\|_{\ell^2}$ as $\varepsilon \to 0$. We refer to Appendix A for relevant definitions and properties of càdlàg martingales.

We consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ which satisfies the ‘usual conditions’ (i.e. completeness and right-continuity [JS03, Def. I.1.3]), and a collection of square-integrable càdlàg martingales $(M_\varepsilon^{k-}(x))_{t \geq 0}$ on this space, parametrized by $x \in \Lambda_\varepsilon^d$ and $k \in \mathcal{K} := \{1, \ldots, r\}$. The value of $r$ is the number of different martingales which we consider, i.e. we would like to consider
r-dimensional driving noise in (1.2). The required properties of these martingales are listed in the following assumption:

**Assumption 2.1** We assume the càdlàg square-integrable martingales \( M_t^{\varepsilon,k}(x) \) to have the following properties:

- For every \( k \) and \( x \), the process \( t \mapsto M_t^{\varepsilon,k}(x) \) is of finite total variation, and starting at zero, i.e. \( M_0^{\varepsilon,k}(x) = 0 \).
- There exists an adapted (in \( t \)) random function \( C_{\varepsilon}^k(t, \cdot) : \Lambda_{\varepsilon}^d \to \mathbb{R} \) such that
  \[
  \langle M_t^{\varepsilon,k}(x), M_t^{\varepsilon,\ell}(y) \rangle_t = \delta_{x,y} \delta_{k,\ell} \int_0^t C_{\varepsilon}^k(s, x) \, ds ,
  \]
  where \( \delta_{x,y} := \varepsilon^{-d} \delta_{x,y} \) is a rescaled Kronecker’s delta function.
- The function \( C_{\varepsilon}^k(z) \) is a.s. bounded uniformly in \( z \) and \( \varepsilon \). Moreover, for every test function \( \varphi \), every fixed \( t > 0 \) and some constants \( \alpha_k > 0 \), one has the following limit in distribution
  \[
  \lim_{\varepsilon \to 0} \int_0^t (t \varepsilon C_{\varepsilon}^k(s, \cdot))(\varphi) = \alpha_k t \int_{\mathbb{R}^d} \varphi(x) dx .
  \]
- One has the identity \( |\Delta_t M_t^{\varepsilon,k}(x)| = \varepsilon^{1-d/2} \) a.s., where \( \Delta_t M_t^{\varepsilon,k}(x) = M_t^{\varepsilon,k}(x) - M_{t-}^{\varepsilon,k}(x) \) is a jump at time \( t \).
- For \( t > 0 \), the jump \( \Delta_t M_t^{\varepsilon,k}(x) \) is a.s. non-zero for at most one point \( x \) and at most one value \( k \in \mathbb{R} \).
- The number of jumps of \( t \mapsto M_t^{\varepsilon,k}(x) \) over the time interval \([0, \varepsilon^2 T]\) has bounded moments, uniformly in \( \varepsilon \) and locally uniformly in \( T \).

Let \( C^\alpha(\mathbb{R}^d) \), with \( \alpha < 0 \), be the Besov space of distributions defined in [Hai14, Eq. 3.2]. For a function \( u^\varepsilon \) on \( \Lambda_{\varepsilon}^d \), we say that it converges to \( u \in C^\alpha(\mathbb{R}^d) \), if the following limit holds:

\[
\lim_{\varepsilon \to 0} \sup_{\varphi, x, \lambda \in (0,1]} (\lambda \vee \varepsilon)^{-\alpha} |(t \varepsilon u^\varepsilon)(\varphi^\lambda_x) - u(\varphi^\lambda_x)| = 0 ,
\]

where the supremum is taken over all points \( x \in \Lambda_{\varepsilon}^d \), and all test functions \( \varphi \), bounded uniformly by 1 together with all its derivatives of order up to \( \lceil -\alpha \rceil \). The following result proves that such martingales weakly converge to a cylindrical Wiener process [DPZ92].

**Lemma 2.2** If Assumption 2.1 is satisfied, then for every \( T > 0 \) and \( \alpha < -d/2 \) the martingales \( (M_t^{\varepsilon,k} : k \in \mathbb{R})_{t \geq 0} \) weakly converge in the Skorokhod topology \( \mathcal{D}([0, T], C^\alpha_{\varepsilon}(\mathbb{R}^d)^{\otimes r}) \) to an \( r \)-dimensional cylindrical Wiener process with variances \( \alpha_k \).
Proof. Let us take a test function $\varphi$ and consider martingales $t \mapsto (\varepsilon M_{t,k}^\varepsilon)(\varphi)$. Then the jump can be bounded by $|\Delta (\varepsilon M_{t,k}^\varepsilon)(\varphi)| \leq \varepsilon^{1+d/2} \|\varphi\|_\infty \to 0$, and the Aldous’ criterion \cite{Ald78} implies that $(\varepsilon M_{t,k}^\varepsilon)(\varphi)$ are tight in $\mathcal{D}([0, T], \mathbb{R})$ and every limiting point is in $C([0, T], \mathbb{R})$. Moreover, the assumption (2.2) and uniform boundedness of $C_\varepsilon$ imply the $L^1$ limit

$$
\lim_{\varepsilon \to 0} \langle (\varepsilon M_{t,k}^\varepsilon)(\varphi), (\varepsilon M_{t,k}^\varepsilon)(\varphi) \rangle_t = \delta_k, \ell \alpha_k \|\varphi\|_{L^2}^2,
$$

combining which with the Lévy characterization theorem we conclude that the limit of $M_{t,k}^\varepsilon$ in the Skorokhod topology $\mathcal{D}([0, T], \mathcal{S}(\mathbb{R}^d)^{\otimes r})$ is an $r$-dimensional cylindrical Wiener process with variances $\alpha_k$.

In order to prove convergence in $C_\varepsilon^0$, it is sufficient to show tightness of $M_{t,k}^\varepsilon$. Taking $\lambda \in (0, 1]$ and a rescaled test function $\varphi_\lambda^\varepsilon$, and repeating the argument in the proof of Theorem 2.3 with minor modifications, we obtain

$$
\left( \mathbb{E} \left[ \sup_{t \in [0, T]} |(\varepsilon M_{t,k}^\varepsilon)(\varphi_\lambda^\varepsilon)|^p \right] \right)^{1/p} \lesssim T \|\varphi_\lambda^\varepsilon\|_{\ell^2(\mathcal{N}_k^d)} + \varepsilon^{1+d/2} \|\varphi_\lambda^\varepsilon\|_\infty \lesssim (\lambda \lor \varepsilon)^{-d/2},
$$

which gives the required tightness in $C_\varepsilon^0$ with $\alpha < -d/2$. \hfill \Box

Our aim is to define space-time convolutions of $M_{t,k}^\varepsilon$ with sufficiently regular kernels, and to derive their moment bounds. We start with extending the noise to the whole line $\mathbb{R}$ in time in the following way: we take an independent copy $\tilde{M}_{t,k}^\varepsilon$ of $M_{t,k}^\varepsilon$, and define the two-sided martingale

$$
M_{t,k}^\varepsilon(x) := \begin{cases} M_{t,k}^\varepsilon(x) & \text{if } t \geq 0, \\ \tilde{M}_{t,k}^\varepsilon(x) & \text{if } t < 0. \end{cases}
$$

Furthermore, we introduce the Hilbert space $\mathcal{H}_\varepsilon := L^2(\mathbb{R}) \otimes \ell^2(\mathcal{N}_k^d)$ and its subset $\mathcal{H}_\varepsilon \subset \mathcal{H}_\varepsilon$ of continuous, compactly supported functions. Often we will work with the set $[n] := \{1, \cdots, n\}$ for $n \in \mathbb{N}$, and for a subset $n \subset [n]$, when there is no ambiguity, we will write $n^*$ for $[n] \setminus n$. Finally, for any such subset $n$, a function $f \in \mathcal{H}_\varepsilon^{\otimes n}$ and a vector of points $z = (z_1, \ldots, z_n) \in \mathcal{D}_\varepsilon^{\otimes n}$, we denote the vector $z_n := (z_i : i \in n)$ and the function $f_n(z_n; z_{n^*}) := f(z)$, where we use the natural order of elements in $n$.

With this notation at hand, we define for $f \in \mathcal{H}_\varepsilon$, $k \in \mathfrak{K}$ and $t \geq 0$ the integral

$$
\mathcal{I}_k^\varepsilon(f)_t := \varepsilon^d \sum_{x \in \mathcal{N}_k^d} \int_{-t}^{t} f(i, \bar{x}) \, dM_{t,k}^\varepsilon(\bar{x}),
$$

in the Itô sense. In order to simplify our notation, we will write the expression on the right-hand side of (2.4a) as $\int_{\mathcal{D}_{\varepsilon,t}} f(\bar{z}) \, dM_{t,k}^\varepsilon(\bar{z})$, where $\mathcal{D}_{\varepsilon,t} := [-t, t] \times \mathcal{N}_k^d$. 


Furthermore, for every $n \geq 2$, every function $f \in \mathcal{H}_x^{\otimes n}$ and every vector $\mathcal{R} = (k_1, \ldots, k_n) \in \mathbb{R}^n$, we define the multiple stochastic integral in the Itô sense recursively by

\[
\mathcal{I}_\mathcal{R}^\varepsilon(f)_t := \sum_{i=1}^n \int_{D_{x,t}} \mathcal{I}_{1 \setminus i}^\varepsilon(f(i))(\bar{z})_{i-} \, dM^{\varepsilon,k_i}(\bar{z}) ,
\]

where $\mathcal{R} \setminus i$ is the vector obtained from $\mathcal{R}$ after removing the $i$-th entry, $s^-$ mean the left limit at $s$, and $f(i)$ is defined above as $f_n$ with $n = (i)$. We note that due to compact support of the functions all the processes $t \mapsto \mathcal{I}_\mathcal{R}^\varepsilon(f)_t$ are well-defined. Moreover, all of them are càdlàg square-integrable martingales. In case, when we integrate over the whole time line $\mathbb{R}$, we will simply write $\mathcal{I}_\mathcal{R}^\varepsilon(f) := \mathcal{I}_\mathcal{R}^\varepsilon(f)_{+\infty}$.

We define a new domain $D_{x,t} := \Lambda_{x,t} \times \Lambda^d_{\varepsilon,t}$, which corresponds to space-time discrete grid, and introduce the norm $\| \cdot \|_{L^2}$ on the Hilbert space $\mathcal{H}_x^{\otimes n}$ by $\|f\|_{L^2}^2 := \int_{D_{x,t}} |f(z)|^2 \, dz$. Then, similarly to the multiple Wiener integrals [Nua06, Ch. 1], we have the following moment bounds:

**Proposition 2.3** Let martingales $M_{t}^{\varepsilon,k}(x)$ satisfy Assumption 2.1. Then for every (non-random) function $f \in \mathcal{H}_x^{\otimes n}$, every $p \geq 1$ and every vector $\mathcal{R} \in \mathbb{R}^n$, there is a constant $C$, depending on $p$ and support of $f$, such that

\[
E \left[ \sup_{t \geq 0} |\mathcal{I}_\mathcal{R}^\varepsilon(f)_t|^{p} \right]^{1/p} \leq C \|f\|_{L^2} .
\]

**Proof.** One can see that if we bound moments of each of the integrals in (2.4b) separately, then the Jensen and Minkowski inequalities will give us a bound on the moments of $\mathcal{I}_\mathcal{R}^\varepsilon(f)_t$. Therefore, it is sufficient to bound only the integral

\[
\mathcal{J}_\mathcal{R}^\varepsilon(f)_t := \int_{D_{x,t}} \mathcal{I}_{\mathcal{R} \setminus i}^\varepsilon(f(i))(\bar{z})_{i-} \, dM^{\varepsilon,k}(\bar{z}) ,
\]

for $t \geq 0$ for some $\mathcal{R} \in \mathbb{R}^{n-1}$ and $k \in \mathcal{R}$, where we postulate $\mathcal{I}_\mathcal{R}^\varepsilon(f(1))(\bar{z})_s = f(\bar{z})$ for any $s \geq 0$.

In order to bound $\mathcal{J}_\mathcal{R}^\varepsilon(f)$, we will use the Burkholder-Davis-Gundy inequality (A.2). Since the jump $\Delta_s M^{\varepsilon,k}(x)$ is non-zero for at most one point $x$ and $M^{\varepsilon,k}$ is of bounded total variation, we get almost surely the identity for the quadratic variation (see Appendix A for the definition)

\[
[\mathcal{J}_\mathcal{R}^\varepsilon(f), \mathcal{J}_\mathcal{R}^\varepsilon(f)]_t = \sum_{0 < s \leq t} (\Delta_s \mathcal{J}_\mathcal{R}^\varepsilon(f))^2
\]

\[
= \varepsilon^{2d} \sum_{-t \leq s \leq t} \sum_{y \in \mathcal{N}_t} \left( I_{\mathcal{R}}(f_\varepsilon(s,y))_{s-} \right)^2 \left( \Delta_s M^{\varepsilon,k}(y) \right)^2 ,
\]
where for negative $s$ the jump is defined as $\Delta_s M^{\varepsilon,k}(y) := M_{s}^{\varepsilon,k}(y) - M_{s^-}^{\varepsilon,k}(y)$ with $s^+$ meaning the right limit at $s$. Let us split $[-t, t]$ into subintervals of length $\varepsilon^2$, i.e. let $t_i := [\varepsilon^{-2} i]$ and $I_i^\varepsilon := [\varepsilon^2 i, \varepsilon^2 (i + 1)]$ for integer values of $i$ such that $|i| < t_i$, and let $I_{-t_i}^\varepsilon := [-t_i, -\varepsilon^2 t_i]$. Let furthermore $n_{i,y}^{\varepsilon,k}$ be the number of jumps of $t \mapsto M_{t_i}^{\varepsilon,k}(y)$ during the time interval $I_i^\varepsilon$. Then Assumption 2.1 yields the estimate

$$
\sum_{s \in I_i^\varepsilon} (\Delta_s M^{\varepsilon,k}(y))^2 \leq \varepsilon^{2 - d} n_{i,y}^{\varepsilon,k}.
$$

Thus, applying the Jensen inequality for $p \geq 2$ we obtain the bound

$$
E \left[ \mathcal{J}^\varepsilon(h), \mathcal{J}^\varepsilon(f) \right]^{2/p} \lesssim \varepsilon^{2(d/2 + 1)} \sum_{i = -t_i}^{t_i} \sum_{y \in \mathcal{N}} \sup_{s \in I_i^\varepsilon} E \left[ \sup_{r \in I_i^\varepsilon} |\mathcal{I}^\varepsilon_r(f_{\{s\}}(s, y))|_{r-} \right]^{2p} (n_{i,y}^{\varepsilon,k})^p \right]^{1/p},
$$

where the proportionality constant depends on $p$ and support of $f$ (we can take the maximal value of $t$, which is the support radius of $f$).

In the case $n = 1$ we use the identity $\mathcal{I}^\varepsilon_r(f_{\{1\}}(s, y))_{|r-} = f(s, y)$ and the fact that $E[(n_{i,y}^{\varepsilon,k})^p]^{1/p}$ is bounded uniformly in $\varepsilon$ (by Assumption 2.1) to get

$$
E \left[ \mathcal{J}^\varepsilon(h), \mathcal{J}^\varepsilon(f) \right]^{2/p} \lesssim \varepsilon^{2(1 + d/2)} \sum_{i = -t_i}^{t_i} \sum_{y \in \mathcal{N}} \sup_{s \in I_i^\varepsilon} |f(s, y)|^2 \lesssim \|f\|^2_{L^2}.
$$

Thus, combining the Burkholder-Davis-Gundy inequality (A.2) with the above bound, for $n = 1$ and $p \geq 4$ we obtain

$$
E \left[ \sup_{t \geq 0} |\mathcal{J}^\varepsilon_r(f)|^p \right]^{1/p} \lesssim \|f\|_{L^2} ,
$$

which is the required result (2.5). The same bound for $1 \leq p \leq 3$ follows from the Hölder inequality.

In the case $n \geq 2$, we use the Jensen and Cauchy-Schwarz inequalities in (2.6), moments bounds for $n_{i,y}^{\varepsilon,k}$ and the fact that $f$ is compactly supported to obtain

$$
E \left[ \mathcal{J}^\varepsilon(h), \mathcal{J}^\varepsilon(f) \right]^{2/p} \lesssim \varepsilon^{2(1 + d/2)} \sum_{i = -t_i}^{t_i} \sum_{y \in \mathcal{N}} \sup_{s \in I_i^\varepsilon} E \left[ \sup_{r \geq 0} \mathcal{I}^\varepsilon_r(f_{\{s\}}(s, y))_{r}^{4p} \right]^{1/2p}
\lesssim \int_{D_{1,2,\varepsilon}} E \left[ \sup_{r \geq 0} \mathcal{I}^\varepsilon_r(f_{\{s\}}(\bar{z}))_{r}^{4p} \right]^{1/2p} d\bar{z}.
$$
Combining now the Burkholder-Davis-Gundy inequality (A.2) with the bound (2.8), we conclude

\[ \mathbb{E} \left[ \sup_{t \geq 0} |\mathcal{J}_R^c(f)_{t_r}|^p \right]^{2/p} \lesssim \int_{D_{x,e}} \mathbb{E} \left[ \sup_{r \geq 0} |\mathcal{J}_R^c(f_r)(\tilde{z})|^{4p} \right]^{1/2p} d\tilde{z} \]

for \( p \geq 4 \), and the required bound follows by induction over \( n \). As before, the bound for \( 1 \leq p \leq 3 \) follows from the Hölder inequality. \( \square \)

3 Generalized bracket processes

We need to extend the definitions of the covariation processes, which we defined in Appendix A. More precisely, we define the following limit in probability:

\[
[M^\varepsilon; R]_t(x) := \varepsilon^{d(n-1)} \lim_{m \to \infty} \sum_{i=0}^{m-1} \prod_{j=1}^{n} (M_{\varepsilon,k_j}^{x,t_i}(x) - M_{\varepsilon,k_j}^{x,t_{i+1}}(x)), \tag{3.1}
\]

where \( \mathcal{R} = (k_1, \ldots, k_n) \in \mathbb{R}^n \) and where \( 0 = t_0 < t_1 < \cdots < t_m = |t| \) is a partition of the interval \([0, |t|]\), whose diameter vanishes as \( m \to \infty \). Under Assumption 2.1 these processes are well-defined [Myk94, Prop. 1]. One can see that in the cases \( n = 1 \) and \( n = 2 \) we respectively have \( [M^\varepsilon; R]_t(x) = M_{t}^{\varepsilon,k_1}(x) \) and \( [M^\varepsilon; R]_t(x) = \varepsilon^{d}[M_{t}^{\varepsilon,k_1}(x), M_{t}^{\varepsilon,k_2}(x)]_t \), where the latter is the quadratic covariation defined in Appendix A. Using Assumption 2.1, we can prove the following properties of these bracket processes:

**Lemma 3.1** If the martingales \( M^\varepsilon \) satisfy Assumption 2.1, then \( [M^\varepsilon; R]_t(x) = 0 \) if \( |\mathcal{R}| \geq 2 \) and not all components of \( \mathcal{R} \) are equal. Moreover, for \( \mathcal{R} = k^{1:m} \) one has

\[
[M^\varepsilon; R]_t(x) = \begin{cases} 
\varepsilon^{(n-1)|s|/2} M_{t}^{\varepsilon,k}(x), & \text{if } n \text{ is odd}, \\
\varepsilon^{n|s|/2-2} [M_{t}^{\varepsilon,k}(x); M_{t}^{\varepsilon,k}(x)]_t, & \text{if } n \text{ is even},
\end{cases}
\]

where \( |s| = d + 2 \), and the quadratic covariation is given by

\[
[M_{t}^{\varepsilon,k}(x); M_{t}^{\varepsilon,k}(x)]_t = \sum_{0 < s \leq |t|} (\Delta_{s}M_{t}^{\varepsilon,k}(x))^2.
\]

In particular, these identities imply that for two vectors \( \mathcal{R} \) and \( \mathcal{L} \) one has

\[
\varepsilon^{d}[M^\varepsilon; \mathcal{R}(x), M^\varepsilon; \mathcal{L}(x)]_t = [M^\varepsilon; \mathcal{R} \sqcup \mathcal{L}]_t(x), \tag{3.2}
\]

where \( \sqcup \) is the concatenation operation of two vectors.
Proof. For \(|\mathcal{R}| = n \geq 2\), Assumption 2.1 and [Myk94, Prop. 1] yield

\[
[M^\varepsilon; \mathcal{R}]_t(x) = \varepsilon d(n-1) \sum_{0<s \leq |t|} \prod_{i=1}^{n} \Delta_{\text{sgn}(t)s} M^{\varepsilon,k_i}(x),
\]

and all the statements of the lemma follow from Assumption 2.1.

We define a generalization of predictable quadratic covariation as the compensator of \([M^\varepsilon; \mathcal{R}]_t(x)\) (see [JS03, Sec. I.3b]), i.e. an adapted process \(t \mapsto \langle M^\varepsilon; \mathcal{R} \rangle_t(x)\) of finite total variation, such that for every fixed \(x\) the following process is a martingale:

\[
t \mapsto N^\varepsilon(t, x) := [M^\varepsilon; \mathcal{R}]_t(x) - \langle M^\varepsilon; \mathcal{R} \rangle_t(x).
\]

Then we have the following result:

**Lemma 3.2** If Assumption 2.1 is satisfied, then the compensator of \([M^\varepsilon; \mathcal{R}]_t(x)\) is given by \(\langle M^\varepsilon; \mathcal{R} \rangle_t(x) = 0\), if \(|\mathcal{R}|\) is odd or if \(|\mathcal{R}| \geq 2\) such that not all components of \(\mathcal{R}\) are equal. In the case when \(|\mathcal{R}|\) is even with equal components, the process \(t \mapsto \langle M^\varepsilon; \mathcal{R} \rangle_t(x)\) is differentiable in \(t\) and satisfies the following bound:

\[
\left| \frac{d}{dt} \langle M^\varepsilon; \mathcal{R} \rangle_t(x) \right| \leq C \varepsilon \left( |\mathcal{R}| / 2 - 1 \right) |s|,
\]

uniformly in \(t\), \(x\) and \(\varepsilon\).

**Proof.** Since \([M^\varepsilon; \mathcal{R}]_t(x) = 0\) if \(|\mathcal{R}| \geq 2\) and not all components of \(\mathcal{R}\) are equal, its compensator is 0. In the case, when all components of \(\mathcal{R}\) are equal the statements follows from Lemma 3.1, Assumption 2.1.

In the case \(|\mathcal{R}| = 2\), the process \(\langle M^\varepsilon; \mathcal{R} \rangle_t(x)\) is the predictable quadratic covariation \(\varepsilon d(\langle M^{\varepsilon,k_1}(x), M^{\varepsilon,k_2}(x) \rangle_t,\mathcal{R})\), defined in Appendix A. In general, the following limit holds in probability:

\[
\langle M^\varepsilon; \mathcal{R} \rangle_t(x) = \varepsilon d(n-1) \lim_{m \to \infty} \sum_{i=0}^{m-1} \mathbb{E} \left[ \prod_{j=1}^{n} (M^{\varepsilon,k}_{\text{sgn}(t)j+1}(x) - M^{\varepsilon,k}_{\text{sgn}(t)j}(x)) \right] \mathcal{F}_t,
\]

where \(0 = t_0 < t_1 < \cdots < t_m = |t|\) is a partition of the interval \([0, |t|]\) with vanishing diameter as \(m \to \infty\), and where \(\mathcal{F}_t\) is the underlying filtration. It can be proved using [Myk94, Eq. 6.7], the definition (3.1) and Assumption 2.1, by approximating continuous-time càdlàg martingales by martingales with discrete time.
4 Products of multiple stochastic integrals

The aim of this section is to prove an analogue of the Wiener chaos expansion [Nua06, Prop. 1.1.3] for the integrals $I^\varepsilon_K$, which can involve also products of more than two integrals.

4.1 A motivating example

Before giving technical definitions, we prefer to explain them on a simple example. Let us take two functions $f : \mathbb{R}^{2(d+1)} \to \mathbb{R}$ and $g : \mathbb{R}^{3(d+1)} \to \mathbb{R}$, and let $I^2(f)$ and $I^3(g)$ be two stochastic Wiener integrals with respect to a Wiener process [Nua06]. Then the result [Nua06, Prop. 1.1.3] (which is also called the Wick lemma) gives a Wiener chaos expansion for the product:

$$I^2(f)I^3(g) = I^5(f \otimes g) + I^3(f \otimes _2 g) + I^1(f \otimes _4 g),$$

for $f \otimes g : \mathbb{R}^{5(d+1)} \to \mathbb{R}$ and the function $f \otimes _2 g : \mathbb{R}^{3(d+1)} \to \mathbb{R}$ is given by

$$(f \otimes _2 g)(z_1, z_2, z_3) = \int f(\tilde{z}, z_1)g(\tilde{z}, z_2, z_3)d\tilde{z} + \cdots$$

In other words, to define $f \otimes _2 g$, we chose one variable of the function $f$ and one variable of $g$, we ‘contract’ these variables and integrate out. After that we sum over all possible choices of the two variables. Similarly, the function $f \otimes _4 g : \mathbb{R}^{d+1} \to \mathbb{R}$ is built by ‘contracting’ any pair of variables of $f$ with any pair of $g$, and can be written as

$$(f \otimes _4 g)(z) = \int f(\tilde{z}_1, \tilde{z}_2)g(z, \tilde{z}_1, \tilde{z}_2)d\tilde{z}_1d\tilde{z}_2 + \cdots$$

It is convenient to use diagrams to represent such functions and contractions. More precisely, let us draw ‘$\bigcirc$’ for a variable in $\mathbb{R}^{d+1}$, and let us draw the functions as $f = \bigcirc \bigcirc \bigcirc \bigcirc$ and $g = \bigcirc \bigcirc \bigcirc \bigcirc$. When we contract two variables, we replace them by ‘$\bigcirc \bigcirc$’ and connect them by an arrow ‘$\rightarrow$’. Then the functions in the chaos expansion (4.1) are given by the diagrams $f \otimes g = \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc$ , as well as

$$f \otimes _2 g = \bigcirc \bigcirc \bigcirc \bigcirc + \bigcirc \bigcirc \bigcirc \bigcirc + \cdots,$$

$$f \otimes _4 g = \bigcirc \bigcirc \bigcirc \bigcirc + \bigcirc \bigcirc \bigcirc \bigcirc + \cdots.$$

In the case, when we consider stochastic integrals $I^\varepsilon_K$ with respect to martingales, we have to consider contractions not only of two variables, but also of three and more. For example, the chaos expansion (4.1) in this case will also contain an integral of second order with the kernel given by

$$f \otimes _3 g = \bigcirc \bigcirc \bigcirc \bigcirc + \cdots,$$
where we contract three variables (see also [HS17], where such contractions appear as cumulants). Moreover, the variables should be integrated out with respect to one of the bracket processes defined in (3.1), which makes the kernel $f \otimes_3 g$ random and creates problems with adaptedness in the definition of a stochastic integral.

In the following section, we denote by $\mathbb{V}_0$ the set of all vertices in such diagrams. We call a component each set of vertices, which are contracted and integrated out together, and denote by $E_\gamma$ the set of all components. Moreover, we allow every component to be integrated out either with respect to a bracket process (3.1), or a martingale $N^\varepsilon$ defined in (3.4). The set of all components integrated out with respect to martingales $N^\varepsilon$ is denoted by $F_\gamma \subset E_\gamma$. Thus, a contraction is a triplet $\gamma = (\mathbb{V}_0, E_\gamma, F_\gamma)$.

### 4.2 Stochastic integrals and contractions

Let us have a non-empty finite set of vertices $\mathbb{V}_0$ (we use this notation to be consistent with Section 6) with assigned labels $\ell(v) \in \mathbb{R}$ for $v \in \mathbb{V}_0$. We define a contraction $\gamma = (\mathbb{V}_0, E_\gamma, F_\gamma)$ to be a hypergraph with vertices $\mathbb{V}_0$ and edges $E_\gamma$, so that each edge is a non-empty subset of vertices from $\mathbb{V}_0$. Here, $F_\gamma$ is a subset of $E_\gamma$, which contains all components of odd cardinalities (but not necessarily only them). In what follows, the edges will be called components (to avoid confusion with edges of graphs in Section 6). We denote by $\Gamma(\mathbb{V}_0)$ the set of all contractions $\gamma$, and for our future convenience we assume that $E_\gamma$ always contains isolated vertices as components of cardinality one. Moreover, it will be convenient to fix any orders of the vertices in $\mathbb{V}_0$ and of components. Then for $\bar{\mathbb{V}} \subset \mathbb{V}_0$ we write $\ell(\bar{\mathbb{V}})$ for the vector of labels $(\ell(v) : v \in \bar{\mathbb{V}})$ with respect to this order.

Let us have a set of vertices $\mathbb{V}_0$, such that $|\mathbb{V}_0| = n \geq 1$, a function $f \in \mathcal{H}^{\otimes n}_\varepsilon$ and a contraction $\gamma \in \Gamma(\mathbb{V}_0)$. If $\gamma$ has only one component $E_\gamma = \{\bar{R}\}$ with $\bar{R} \in \mathbb{R}^n$, then we define the integral

$$I^\varepsilon_\gamma(f)_t := \begin{cases} \int_{D_{\varepsilon,t}} f(\tilde{z}^{\otimes n}) \, d[M^\varepsilon; \bar{R}](\tilde{z}) , & \text{if } F_\gamma \neq E_\gamma , \\ \int_{D_{\varepsilon,t}} f(\tilde{z}^{\otimes n}) \, dN^\varepsilon_{\bar{R}}(\tilde{z}) , & \text{if } F_\gamma = E_\gamma , \end{cases} \tag{4.2a}$$

where we use the martingales (3.4) and where $\tilde{z}^{\otimes n}$ is the vector of length $n$ with each entry equal $\tilde{z}$. For $\gamma$ with more than one component, we define the integral recursively

$$I^\varepsilon_\gamma(f)_t := \sum_{e \in E_\gamma \setminus F_\gamma} \int_{D_{\varepsilon,t}} I^\varepsilon_{\gamma\setminus e}(f_e(\tilde{z}^{\otimes |e|})_{|\bar{e}|} - d[M^\varepsilon; \ell(e)](\tilde{z})

+ \sum_{e \in F_\gamma} \int_{D_{\varepsilon,t}} I^\varepsilon_{\gamma\setminus e}(f_e(\tilde{z}^{\otimes |e|})_{|\bar{e}|} - dN^\varepsilon_{\ell(e)}(\tilde{z}) , \tag{4.2b}$$
where the contraction $\gamma \setminus e$ is obtained from $\gamma$ after removing the vertices contained in the component $e$.

It follows from our definitions that the integrating process in all integrals in (4.2) are adapted, and Lemma 5.1 implies that all of them are $L^2$-bounded. Thus, the integrals with respect to the martingales $N^e_\gamma$ and $N_{\ell(e)}$ are well-defined as usual stochastic integrals. Furthermore, Assumption 2.1 yields that the integrals with respect to the brackets $[M^e; \mathcal{R}]$ and $[M^e; \ell(e)]$ can be a.s. written as finite sums over jump-times, which makes them well-defined.

In the case $E_\gamma = \mathcal{V}_o$, the definitions (2.4) and (4.2) coincide, but in general the process $t \mapsto \mathcal{I}_\gamma(f)_t$ is not a martingale, since the bracket process $[M^e; \mathcal{R}]_t(x)$ is not a martingale for even $|\mathcal{R}|$. One can see that the set $F_\gamma$ contains the components which correspond to the integrals with respect to the compensated processes (3.4), while the other integrals are with respect to the bracket process.

**Remark 4.1** We will use the integrals (4.2) also with respect to cylindrical Wiener processes. More precisely, in the case $\varepsilon = 0$ the domain $D_0$ equals $\mathbb{R}^{d+1}$ and we denote $W^k := M^{0,k}$, for $k \in \mathbb{N}$, to be independent cylindrical Wiener processes. Then we have $[W; \mathcal{R}]_t = (W; \mathcal{R})_t = 0$ if $|\mathcal{R}| \geq 3$, and we denote the respective multiple stochastic integral by $\mathcal{I}_\gamma(f)_t$.

### 4.3 Chaos expansions of products

Now, we will describe an analogue of the Wiener chaos expansion of a product of multiple stochastic integrals (2.4). More precisely, for integer values $n \geq 1$ and $m_i \geq 1$, and sets $\mathcal{R}^{(i)} = (k^{(i)}_1, \ldots, k^{(i)}_{m_i}) \in \mathcal{R}^{m_i}$ with $1 \leq i \leq n$, we would like to consider the product of the integrals $\mathcal{I}_{\mathcal{R}^{(i)}}$. For this we define the set $\mathcal{V}_o$ which contains pairs $(i, j)$, with $1 \leq i \leq n$ and $1 \leq j \leq m_i$. Furthermore, we define the set of contractions $\Gamma = \Gamma(\mathcal{R}^{(1)}, \ldots, \mathcal{R}^{(n)}) \subset \Gamma(\mathcal{V}_o)$ with vertices from $\mathcal{V}_o$, so that if $\gamma \in \Gamma$, then each component in $E_\gamma$ is a set of vertices $(i, j) \in \mathcal{V}_o$ with different values $i$, and $F_\gamma$ contains only components of odd cardinalities. For each vertex $v = (i, j) \in \mathcal{V}_o$ we assign the label $\ell(v) = k^{(i)}_j$. Then we have the following result:

**Proposition 4.2** For any integers $n \geq 1$, $m_i \geq 1$, sets $\mathcal{R}^{(i)} \subset \mathcal{R}^{m_i}$, with $1 \leq i \leq n$, (non-random) functions $f_i \in \mathcal{H}^{m_i}$ and the set of contractions $\Gamma = \Gamma(\mathcal{R}^{(1)}, \ldots, \mathcal{R}^{(n)})$, one has the expansion

$$
\prod_{i=1}^n \mathcal{I}_{\mathcal{R}^{(i)}}(f_i)_t = \sum_{\gamma \in \Gamma} \mathcal{I}_{\mathcal{R}} \left( \bigotimes_{i=1}^n f_i \right)_t .
$$

**Proof.** In the proof we consider sets $\mathcal{R}^{(i)} = (k^{(i)}_1, \ldots, k^{(i)}_{m_i}) \in \mathcal{R}^{m_i}$ with $1 \leq i \leq n$, and we denote $\Gamma = \Gamma(\mathcal{R}^{(1)}, \ldots, \mathcal{R}^{(n)})$ and $\tilde{\Gamma} = \Gamma(\mathcal{R}^{(1)}, \ldots, \mathcal{R}^{(n-1)})$. We start by...
proving the following identity, for \( n = 2 \) and \( \bar{\gamma} \in \bar{\Gamma} \),

\[
\mathcal{I}^z_{\bar{\gamma}}(f_1)_t \mathcal{I}^z_{\bar{\gamma}\mathbb{R}^{(2)}}(f_2)_t = \sum_{\gamma \in \Gamma(\bar{\gamma}, \mathbb{R}^{(2)})} \mathcal{I}^z_{\gamma}(f_1 \otimes f_2)_t ,
\]

(4.4)

where \( \Gamma(\bar{\gamma}, \mathbb{R}^{(2)}) \) contains those contractions \( \gamma \in \Gamma \), for which every component from \( E_\gamma \) contains components from \( E_\bar{\gamma} \) with vertices from \( \mathbb{R}^{(2)} \). Using the definitions (A.1), (2.4) and (4.2), and the identities (3.3) and (3.2) we can write

\[
\mathcal{I}^z_{\gamma}(f_1)_t \mathcal{I}^z_{\bar{\gamma}\mathbb{R}^{(2)}}(f_2)_t = \int_0^t \mathcal{I}^z_{\gamma}(f_1)_s - d\mathcal{I}^z_{\bar{\gamma}\mathbb{R}^{(2)}}(f_2)_s + \int_0^t \mathcal{I}^z_{\bar{\gamma}\mathbb{R}^{(2)}}(f_2)_s - d\mathcal{I}^z_{\gamma}(f_1)_s
\]

\[
+ \sum_{e \in E_\bar{\gamma}} \sum_{j=1}^{m_2} \int_{D_{e,t}} \mathcal{I}^z_{\bar{\gamma} \gamma}( (f_1)_e(\bar{\zeta})|_\gamma) - \mathcal{I}^z_{\bar{\gamma}\mathbb{R}^{(2)}}( (f_2)_j(\bar{\zeta}))|_\gamma - d[M^e; \ell(e) \cup k^{(2)}_j](\bar{\zeta}) .
\]

In the case \( m_1 = m_2 = 1 \), the last term in the above expression equals \( \mathcal{I}^z_{\gamma}(\mathbb{R}^{(2)} f_1)_t \), where \( \gamma \in \Gamma(\mathbb{R}^{(1)}, \mathbb{R}^{(2)}) \) is the only contraction with \( E_\gamma \neq \{ \emptyset \} \). The two stochastic integrals in (4.5) obviously give the term \( \mathcal{I}^z_{\gamma}(\mathbb{R}^{(2)} f_1)_t \), for the contraction \( \gamma \in \Gamma(\mathbb{R}^{(1)}, \mathbb{R}^{(2)}) \) such that \( E_\gamma = \{ \emptyset \} \).

Now, we will prove that the claim (4.4) holds for \( m_1, m_2 \geq 1 \). In order to proceed by induction, we assume that it holds for all pairs \( \bar{m}_1, \bar{m}_2 \geq 1 \), such that \( \bar{m}_1 \leq m_1 \) and \( \bar{m}_2 \leq m_2 \), where at least one of the inequalities is strict. Furthermore, we define the sets of vertices \( V_\bar{\gamma} := \{(i, j) : 1 \leq j \leq m_2 \} \subset \emptyset \).

Then we can write the first two integrals in (4.5) as

\[
\int_0^t \mathcal{I}^z_{\gamma}(f_1)_s - d\mathcal{I}^z_{\bar{\gamma}\mathbb{R}^{(2)}}(f_2)_s + \int_0^t \mathcal{I}^z_{\bar{\gamma}\mathbb{R}^{(2)}}(f_2)_s - d\mathcal{I}^z_{\gamma}(f_1)_s
\]

\[
= \sum_{j=1}^{m_2} \int_{D_{e,t}} \mathcal{I}^z_{\gamma}(f_1)|_{\gamma} - \mathcal{I}^z_{\bar{\gamma}\mathbb{R}^{(2)}}(f_2)|_{\gamma} - d[M^{(e, \mathbb{R}^{(2)}/j)}](\bar{\zeta})
\]

\[
+ \sum_{e \in E_\bar{\gamma}} \sum_{j=1}^{m_2} \int_{D_{e,t}} \mathcal{I}^z_{\bar{\gamma} \gamma}( (f_1)_e(\bar{\zeta})|_\gamma) - \mathcal{I}^z_{\bar{\gamma}\mathbb{R}^{(2)}}( (f_2)_j(\bar{\zeta}))|_\gamma - d[M^e; \ell(e)\cup k^{(2)}_j](\bar{\zeta}) .
\]

(4.6)

Applying the induction hypothesis to the first integral in (4.6), we can write it as

\[
\sum_{j=1}^{m_2} \sum_{\gamma \in \Gamma(\bar{\gamma}, \mathbb{R}^{(2)})} \int_{D_{e,t}} \mathcal{I}^z_{\gamma}(f_1 \otimes (f_2)_j(\gamma))|_\gamma - d[M^{(e, \mathbb{R}^{(2)/j})}\gamma](\bar{\zeta})
\]

\[
= \sum_{\gamma \in \Gamma(\bar{\gamma}, \mathbb{R}^{(2)})} \sum_{e \in E_\bar{\gamma} : e \cap V_1 = \emptyset} \int_{D_{e,t}} \mathcal{I}^z_{\bar{\gamma} \gamma}( (f_1 \otimes f_2)_e(\bar{\zeta})|_\gamma) - d[M^e; \ell(e)\gamma](\bar{\zeta}) ,
\]

(4.7)

where we have changed the order of summation. Applying furthermore the induction hypothesis to the product of two integrals in the last term in (4.6), we...
obtain
\[
\sum_{e \in E_\varepsilon} \sum_{\gamma \in \Gamma(\varepsilon, \gamma)} \int_{D_{\varepsilon, t}} \mathcal{I}_\gamma^e((f_1)_\varepsilon(\varepsilon[|e|]) \otimes f_2)|_{\tilde{\varepsilon}} \ d[M^e; \mathbf{e}(\varepsilon)](\tilde{\varepsilon}) \quad (4.8)
\]
\[
= \sum_{\gamma \in \Gamma(\varepsilon, \gamma)} \sum_{e \in E_\varepsilon: e \cap V_1 = \emptyset} \int_{D_{\varepsilon, t}} \mathcal{I}_\gamma^e((f_1 \otimes f_2)_\varepsilon(\varepsilon[|e|]))|_{\tilde{\varepsilon}} \ d[M^e; \mathbf{e}(\varepsilon)](\tilde{\varepsilon}) ,
\]
where again we changed the order of summation. Finally, we apply the induction hypothesis to the product of two integrals in the last term in (4.5) to obtain
\[
\sum_{e \in E_\varepsilon} \sum_{j=1}^{m_2} \sum_{e \in \Gamma(\varepsilon, \gamma, e_2) \setminus \gamma} \int_{D_{\varepsilon, t}} \mathcal{I}_\gamma^e((f_1)_\varepsilon(\varepsilon[|e|]) \otimes (f_2)_{j\varepsilon}(\varepsilon))|_{\tilde{\varepsilon}} \ d[M^e; \mathbf{e}(\varepsilon)](\tilde{\varepsilon})
\]
\[
= \sum_{\gamma \in \Gamma(\varepsilon, \gamma)} \sum_{e \in E_\varepsilon: e \cap V_1 = \emptyset} \int_{D_{\varepsilon, t}} \mathcal{I}_\gamma^e((f_1 \otimes f_2)_\varepsilon(\varepsilon[|e|]))|_{\tilde{\varepsilon}} \ d[M^e; \mathbf{e}(\varepsilon)](\tilde{\varepsilon}) ,
\]
where we changed the summation variables. Combining the identities (4.7), (4.8) and (4.9), we obtain exactly the right-hand side of (4.4).

Now we turn to the proof of the identity (4.3). In the case \(n = 2\) it follows immediately from (4.4) with \(\bar{\gamma}\) containing only the isolated vertices. Let us assume that (4.3) holds for \(n - 1\) and will prove it for \(n \geq 3\). Denoting \(\bar{f} = \otimes_{i=1}^{n-1} f_i\), and using the induction hypothesis for the first \(n - 1\) integrals, we can write
\[
\prod_{i=1}^{n} \mathcal{I}_{\bar{\gamma}^{(n)}}^e(f_i)_{t_i} = \sum_{\gamma \in \bar{\Gamma}} \mathcal{I}_\gamma^e(\bar{f})_{t_i} \mathcal{I}_{\bar{\gamma}^{(n)}}^e(f_n)_{t_i} .
\]
Applying furthermore (4.4), after the respective relabeling of vertices, we obtain
\[
\prod_{i=1}^{n} \mathcal{I}_{\bar{\gamma}^{(n)}}^e(f_i)_{t_i} = \sum_{\gamma \in \bar{\Gamma}} \sum_{\gamma \in \Gamma(\bar{\gamma}^{(n)})} \mathcal{I}_\gamma^e(\bar{f} \otimes f_n)_{t_i} = \sum_{\gamma \in \bar{\Gamma}} \mathcal{I}_\gamma^e(\bar{f} \otimes f_n)_{t_i} ,
\]
which is exactly the required identity (4.3).

5 Moment bounds for multiple stochastic integrals

Our next aim is to get moment bounds for the integrals (4.2). As in Section 2, we will use the domain \(D_{\varepsilon, \varepsilon} := \Lambda_{\varepsilon^2} \times \Lambda_{\varepsilon}^d\), and for a contraction \(\gamma \in \Gamma(V_0)\) we define
the following norm on $f \in \mathcal{H}_\varepsilon^{\{V_0\}}$: if $E_\gamma = \{V_0\}$, i.e. when $E_\gamma$ has only one component, we set

$$
\|f\|_{\mathcal{H}_\varepsilon;\gamma} := \begin{cases} 
1_{\{E_\gamma \neq F_\gamma\}} & \varepsilon |\{V_0\}|^{2-1/\varepsilon} \int_{D_\varepsilon} |f(\bar{z}^{\{V_0\}})| \, d\bar{z} \\
+ \varepsilon |\{V_0\}|/2 \left( \int_{D_\varepsilon} |f(\bar{z}^{\{V_0\}})|^2 \, d\bar{z} \right)^{1/2},
\end{cases} 
$$

(5.1a)

where as before, $\bar{z}^{\{V_0\}}$ is the vector of length $n$ with each entry equal $\bar{z}$. In the case $|E_\gamma| \geq 2$, we pick a component $e \in E_\gamma$ and define recursively

$$
\|f\|_{\mathcal{H}_\varepsilon;\gamma} := 1_{\{e \not\in F_\gamma\}} \varepsilon |\{e\}|^{2-1/\varepsilon} \int_{D_\varepsilon} \|f_e(\bar{z}^{\{e\}})\|_{\mathcal{H}_\varepsilon;\gamma \setminus e} \, d\bar{z} \\
+ \varepsilon |\{e\}|/2 \left( \int_{D_\varepsilon} \|f_e(\bar{z}^{\{e\}})\|^2_{\mathcal{H}_\varepsilon;\gamma \setminus e} \, d\bar{z} \right)^{1/2},
$$

(5.1b)

where the contraction $\gamma \setminus e$ is obtained from $\gamma$ after removing the vertices contained in $e$. Then we have the following moment bounds:

**Lemma 5.1** Let martingales $M^{\varepsilon,k}$ satisfy Assumption 2.1. Then for every contraction $\gamma \in \Gamma(\mathcal{V}_0)$, every (non-random) function $f \in \mathcal{H}_\varepsilon^{\{V_0\}}$ and every $p \geq 1$, there is a constant $C$, depending on $p$ and support of $f$, such that

$$
\mathbb{E} \left[ \sup_{t \geq 0} |\mathcal{I}_\gamma(\ell_t)|^p \right]^{1/p} \leq C \|f\|_{\mathcal{H}_\varepsilon;\gamma} .
$$

(5.2)

**Proof.** We will prove the bound (5.2) by induction over the number of components in $\gamma$. We first consider the case when $\gamma$ has only one component $e = V_0$. In this case we write

$$
\mathcal{I}_\gamma(\ell_t) = 1_{\{e \not\in F_\gamma\}} \int_{D_\varepsilon} f(\bar{z}^{\{e\}}) \, dM^\varepsilon(e) + \int_{D_\varepsilon} f(\bar{z}^{\{e\}}) \, dN^\varepsilon_{(e)}(\bar{z}) ,
$$

where the process $N^\varepsilon$ has been defined in (3.4). We denote the two terms on the right-hand side by $\mathcal{J}_\gamma^{\varepsilon,1}(\ell_t)$ and $\mathcal{J}_\gamma^{\varepsilon,2}(\ell_t)$, respectively. The first one vanishes if $E_\gamma = F_\gamma$, and otherwise we apply Lemma 3.2 to get

$$
\mathbb{E} \left[ \sup_{t \geq 0} |\mathcal{J}_\gamma^{\varepsilon,1}(\ell_t)|^p \right]^{1/p} \lesssim \varepsilon |\{e\}|^{2-1/\varepsilon} \int_{D_\varepsilon} |f(\bar{z}^{\{e\}})| \, d\bar{z} .
$$

For the second integral we use the Burkholder-Davis-Gundy inequality (A.2) and the identity (3.2):

$$
\mathbb{E} \left[ \sup_{t \geq 0} |\mathcal{J}_\gamma^{\varepsilon,2}(\ell_t)|^p \right] \lesssim \mathbb{E} \left[ \left( \int_{D_\varepsilon} f(\bar{z}^{\{e\}})^2 \, d[\ell^\varepsilon(\bar{z})] \right)^{p/2} \right] .
$$
\[ \lesssim \varepsilon^{p(|e|-1)|\delta|/2} \left( \int_{D_{1/2},e} |f(\tilde{\zeta}|\cdot|e|)|^2 \, d\tilde{\zeta} \right)^{p/2}, \]

where the last bound is proved similarly to (2.6). Combining the last two estimates, we conclude
\[ \mathbb{E} \left[ \sup_{t \geq 0} |\mathcal{I}_\gamma^e(f)_t|^p \right]^{1/p} \lesssim \|f\|_{\mathcal{R}_{\epsilon,\gamma}}, \]

which is the required bound (6.25) in the case when \( \gamma \) has only one component.

In the case when \( \gamma \) has more than one component, we bound each term in the sum (4.2b) separately. Then the required bound will follow from the Jensen and Minkowski inequalities. To this end, we pick an component \( e \in E_\gamma \) and write for brevity
\[ \mathcal{I}_{\gamma,e}^e (f)_t := 1_{\{e \notin F_\gamma\}} \int_{D_{\epsilon,\ell}} \mathcal{I}_{\gamma,e}^e (f_e(\tilde{\zeta}|\cdot|e|))_{\gamma} \cdot d\langle M^e; \mathcal{L}(e) \rangle (\tilde{\zeta}) + \int_{D_{\epsilon,\ell}} \mathcal{I}_{\gamma,e}^e (f_e(\tilde{\zeta}|\cdot|e|))_{\gamma} \cdot dN_{\mathcal{L}(e)} (\tilde{\zeta}) . \]

We denote the last two integrals by \( \mathcal{J}_{\gamma,e}^{3,e}(f)_t \) and \( \mathcal{J}_{\gamma,e}^{4,e}(f)_t \) respectively. Similarly to above, we use the Hölder inequality and Lemma 3.2 to get
\[ \mathbb{E} \left[ \sup_{t \geq 0} |\mathcal{J}_{\gamma,e}^{3,e}(f)_t|^p \right]^{1/p} \lesssim 1_{\{e \notin F_\gamma\}} \varepsilon^{(|e|/2-1)|\delta|} \int_{D_{\epsilon,\ell}} \mathbb{E} \left[ \sup_{r \geq 0} |\mathcal{I}_{\gamma,e}^e (f_e(\tilde{\zeta}|\cdot|e|))_{\gamma} |^p \right]^{1/p} d\tilde{\zeta} . \]

For the last term in (5.3) we use the Burkholder-Davis-Gundy inequality (A.2) and the identity (3.2):
\[ \mathbb{E} \left[ \sup_{t \geq 0} |\mathcal{J}_{\gamma,e}^{4,e}(f)_t|^p \right]^{1/p} \lesssim \mathbb{E} \left[ \left( \int_{D_{\epsilon,\ell}} |\mathcal{I}_{\gamma,e}^e (f_e(\tilde{\zeta}|\cdot|e|))_{\gamma} |^2 \cdot d\langle M^e; \mathcal{L}(e) \rangle (\tilde{\zeta}) \right)^{p/2} \right]^{1/p} \]
\[ \lesssim \varepsilon^{p(|e|-1)|\delta|/2} \left( \int_{D_{1/2,e}} \mathbb{E} \left[ \sup_{r \geq 0} |\mathcal{I}_{\gamma,e}^e (f_e(\tilde{\zeta}|\cdot|e|))_{\gamma} |^4 \right]^{1/2} d\tilde{\zeta} \right)^{p/2}, \]

where the last bound follows similarly to (2.6). Combining the last two estimates, we obtain
\[ \mathbb{E} \left[ \sup_{t \geq 0} |\mathcal{I}_{\gamma,e}^e (f)_t|^p \right]^{1/p} \lesssim 1_{\{e \notin F_\gamma\}} \varepsilon^{(|e|/2-1)|\delta|} \int_{D_{\epsilon,\ell}} \mathbb{E} \left[ \sup_{r \geq 0} |\mathcal{I}_{\gamma,e}^e (f_e(\tilde{\zeta}|\cdot|e|))_{\gamma} |^p \right]^{1/p} d\tilde{\zeta} + \varepsilon^{p(|e|-1)|\delta|/2} \left( \int_{D_{1/2,e}} \mathbb{E} \left[ \sup_{r \geq 0} |\mathcal{I}_{\gamma,e}^e (f_e(\tilde{\zeta}|\cdot|e|))_{\gamma} |^4 \right]^{1/2} d\tilde{\zeta} \right)^{1/2}. \]

The required bound (5.2) follows now by induction for the integral \( \mathcal{I}_{\gamma,e}^e \).
6 Kernels in the form of generalized convolutions

In this section we obtain moment bounds for the multiple stochastic integrals (4.2), when the function \( f \) is of a very specific form, similar to a generalized convolution introduced in \[HQ15, \text{App. A}\]. This type of kernels appears in canonical lifts of random noises in the theory of regularity structures. As before, we work in the space \( \mathbb{R}^{d+1} \) with \( d \geq 1 \) and the parabolic scaling \( s = (2, 1, \ldots, 1) \).

Following the idea of \[HQ15, \text{App. A}\], it will be convenient to describe the kernels using labeled graphs. More precisely, we consider a finite directed multigraph \( G = (V, E) \) with a set of vertices \( V \) and with edges \( e \in E \) labelled by pairs \((a_e, r_e) \in \mathbb{R}_+ \times \mathbb{Z}\), so that multiple edges of the same direction can connect two vertices. Furthermore, we assume that the graph is weakly connected and loopless, i.e. every vertex has either an outgoing or incoming edge, and there are no edges from a vertex to itself. We require \( G \) to contain a distinguished vertex \( \star \in V \), connected by outgoing edges with a non-empty set of vertices \( V^\uparrow_\star \subset V \setminus \{\star\} \).

We also allow \( \star \) to have incoming edges (which by the above assumptions cannot come from \( V^\uparrow_\star \)). Finally, we assume that the graph contains a non-empty set \( V_\circ \) of distinguished vertices different from \( \star \) and \( V^\uparrow_\star \), which have only outgoing edges.

In the following figure we provide an example of such labeled graph \( \mathcal{G} \), where we omit labels and use various decorations for nodes and edges.

![Labeled Graph](image)

We define the set \( V_\circ := V \setminus \{\star\} \) and for a directed edge \( e \in E \) we write \( e_\pm \) for the two vertices so that \( e = (e_-, e_+) \) is directed from \( e_- \) to \( e_+ \). We make the following assumption concerning labels of the edges:

**Assumption 6.1** The described graph has the following properties: no edge incoming to \( \star \) may have \( r_e > 0 \); every edge \( e \) connecting two elements from \( V^\uparrow_\star \cup \{\star\} \) has \( r_e = 0 \); at most one edge with \( r_e < 0 \) may be incident to the same vertex; if there is more than one edge connecting two vertices \( e_- \) and \( e_+ \), at most one can have nonzero value \( r_e \), in which case \( r_e > 0 \).

For a multigraph \( \mathcal{G} = (V, E) \) and a contraction \( \gamma \in \Gamma(V_\circ) \), introduced in Section 4, we define the simple (containing no multiedges) graph \( \mathcal{G}_\gamma = (\bar{V}, \bar{E}) \), with labels \((\bar{a}_e, \bar{r}_e)\), in the following way: the set of vertices \( \bar{V} \subset V \) is obtained from \( V \) by identifying those vertices from \( V_\circ \) which belong to the same component...
in \( \gamma \). We denote this ‘identification’ by a surjective map \( i_\gamma : V \to \hat{V} \). In particular, \( i_\gamma \) maps isolated vertices to themselves. We denote \( \hat{V}_0 \) to be the image of \( V_0 \) under the map \( i_\gamma \). Then we define the set of edges \( \hat{E} \) on \( \hat{V} \) to contain \((i_\gamma \otimes i_\gamma) e = (i_\gamma(e_\gamma), i_\gamma(e_\gamma^+)) \) for all \( e = (e_\gamma, e_\gamma^+) \in E_\gamma \), with the label \((a_e, r_e)\). Finally, the unique edge \( e \in \hat{E} \) from \( e_- \) to \( e_+ \) is obtained by contracting all edges \( \hat{e} \) from \( e_- \) to \( e_+ \) in \( \hat{E}_\gamma \), and the label \((\hat{a}_e, \hat{r}_e)\) of \( e \) is the sum of the labels of all such parallel edges \( \hat{e} \).

It follows from Assumption 6.1 that the values \( \hat{r}_e \) is either 0 (if all edges \( \hat{e} \in \hat{E} \) connecting \( e_- \) to \( e_+ \) have \( r_{\hat{e}} = 0 \)), or coincides with the only non-zero value \( r_{\hat{e}} \), for \( \hat{e} \in \hat{E} \) connecting \( e_- \) to \( e_+ \). In what follows, we will call \( G_\gamma \) a contracted graph corresponding to \( G \) and \( \gamma \).

For a subset \( \hat{V} \subset \hat{V} \) we define the outgoing edges \( \hat{E}_+^1(\hat{V}) := \{ e \in \hat{E}_\gamma : e_+ \in \hat{V} \} \), incoming edges \( \hat{E}_-^1(\hat{V}) := \{ e \in \hat{E}_\gamma : e_- \in \hat{V} \} \), internal edges \( \hat{E}_0(\hat{V}) := \{ e \in \hat{E}_\gamma : e_\gamma \in \hat{V} \} \), and incident edges \( \hat{E}(\hat{V}) := \{ e \in \hat{E}_\gamma : e_\gamma \in \hat{V} \text{ or } e_\gamma^+ \in \hat{V} \} \). Finally, we define the sets \( \hat{E}_+^1(\hat{V}) := \{ e \in \hat{E}(\hat{V}) : r_e > 0 \} \), \( \hat{E}_+^1 := \hat{E}_+^1 \cap \hat{E}_+^1 \) and \( \hat{E}_+^0 := \hat{E}_+^0 \cap \hat{E}_+^0 \). Then require the contracted graph to satisfy Assumption A.1 in [HQ15]:

**Assumption 6.2** A graph \( G = (V, E) \) and a contraction \( \gamma \in \Gamma(V_0) \) are such that the contracted graph \( G_\gamma = (\hat{V}, \hat{E}) \) has the following properties:

1. For any edge \( e \in \hat{E} \) we have \( \hat{a}_e + (\hat{r}_e \land 0) < |s| \).
2. For every subset \( \hat{V} \subset \hat{V}_\gamma \) of cardinality at least 3 one has
   \[
   \sum_{e \in \hat{E}(\hat{V})} \hat{a}_e < (|\hat{V}| - 1)|s|.
   \]
3. For every subset \( \hat{V} \subset \hat{V}_\gamma \) containing \( \star \) of cardinality at least 2 one has
   \[
   \sum_{e \in \hat{E}(\hat{V})} \hat{a}_e + \sum_{e \in \hat{E}_+^1(\hat{V})} (\hat{a}_e + \hat{r}_e - 1) - \sum_{e \in \hat{E}_-^1(\hat{V})} \hat{r}_e < (|\hat{V}| - 1)|s|.
   \]
4. For every non-empty subset \( \hat{V} \subset \hat{V}_\gamma \) one has
   \[
   \sum_{e \in \hat{E}(\hat{V}) \setminus \hat{E}_+^1(\hat{V})} \hat{a}_e + \sum_{e \in \hat{E}_+^1(\hat{V})} \hat{r}_e - \sum_{e \in \hat{E}_+^1(\hat{V})} (\hat{r}_e - 1) > |\hat{V}| |s|.
   \]

For \( \varepsilon > 0 \) and for every vertex \( v \notin \hat{V}_0 \cup \{ \star \} \) we assume to be given a translation invariant measure \( \mu_v^\varepsilon \) on \( R^{d+1} \), which is supported in a ball of radius proportional to \( \varepsilon \) and has finite total variation. Furthermore, for every \( e \in E \) we consider a smooth kernel \( K_e^\varepsilon : R^{d+1} \to R \), which is supported in the ball of radius 1 around the origin, and for any \( q > 0 \) the following quantity is bounded uniformly in \( \varepsilon > 0 \):

\[
\| K_e^\varepsilon \|_{\alpha, q} := \sup_{\| z \| \leq 1} \sup_{\| \varepsilon \| \leq 1} \| z \|^\alpha \varepsilon^{\alpha + |k|} |D^k K_e^\varepsilon(\varepsilon z)| . \tag{6.1}
\]
Moreover, for edges $e$ with $r_e < 0$ let us be given a collection of constants $\{I_{e,k}^\varepsilon : k \in \mathbb{N}^{d+1}, |k|_s < |r_e| \}$, which will be used in our definition of renormalized kernels.

### 6.1 Bounds on generalized convolutions

The constant $r_e$, assigned to an edge $e \in \mathbb{E}$, describes a renormalization of the singularity, which we define in the following way. If $r_e < 0$, then for a test function $\varphi$ on $\mathbb{R}^{d+1} \times \mathbb{R}^{d+1}$ we define the expansion

$$
(T_{r_e} \varphi)(z_{e_-}, z_{e_+}) := \varphi(z_{e_-}, z_{e_+}) - \sum_{|k| < |r_e|} \frac{(z_{e_+} - z_{e_-})^k}{k!} D^k_2 \varphi(z_{e_-}, z_{e_+}),
$$

where $D^k_2$ is the multi-derivative in the second argument. Furthermore, we associate to $K^\varepsilon_{e}$ the distributional ‘kernel’

$$
\hat{K}^\varepsilon_{e}(\varphi) := \int_{\mathbb{R}^{d+1}} \int_{\mathbb{R}^{d+1}} K^\varepsilon_{e}(z_{e_+} - z_{e_-}) (T_{r_e} \varphi)(z_{e_-}, z_{e_+}) \mu^\varepsilon_{e_-}(dz_{e_-}) \mu^\varepsilon_{e_+}(dz_{e_+})
$$

$$
+ \sum_{|k| < |r_e|} \frac{I_{e,k}^\varepsilon}{k!} \int_{\mathbb{R}^{d+1}} \varphi(z_{e_-}, z_{e_+}) \mu^\varepsilon_{e_-}(dz_{e_-}),
$$

for any test function $\varphi$, where $I_{e,k}^\varepsilon$ are the fixed constants introduced in the previous section. In the case $r_e \geq 0$, we define the renormalized kernel

$$
\hat{K}^\varepsilon_{e}(z_{e_-}, z_{e_+}) = K^\varepsilon_{e}(z_{e_+} - z_{e_-}) - \sum_{|k| < r_e} \frac{z^k_e}{k!} D^k K^\varepsilon_{e}(-z_{e_-}).
$$

Finally, for a smooth test function $\varphi$ on $\mathbb{R}^{d+1}$ we define its rescaled version $\varphi^\lambda(z) := \varphi^\lambda_0(z)$ as in (1.3) and the generalized convolution

$$
K^\varepsilon_{G}^{\lambda}(z^\varphi) := \int_{(\mathbb{R}^{d+1})^\varepsilon} \prod_{e \in \mathbb{E}} \hat{K}^\varepsilon_{e}(z_{e_-}, z_{e_+}) \prod_{v \in \mathbb{V}_e} \varphi^\lambda(z_v) \hat{\mu}^\varepsilon_{\mathbb{V}, \varphi}(dz),
$$

where $z = (z_v \in \mathbb{R}^{d+1} : v \in \mathbb{V}_e), z^\varphi = (z^\varphi_v \in \mathbb{R}^{d+1} : v \in \mathbb{V}_e)$ and the measure

$$
\hat{\mu}^\varepsilon_{\mathbb{V}, \varphi}(dz) := \prod_{v \in \mathbb{V}_\varphi \setminus \mathbb{V}_0} \mu^\varepsilon_{e}(dz_v) \prod_{w \in \mathbb{V}_0} \delta_{z_w - z^\varphi_w} dz_w,
$$

with the Dirac delta-function $\delta_z$. Since the kernels $K^\varepsilon_{e}$ are smooth, our assumptions guarantee that the generalized convolution (6.5) is well-defined.
Remark 6.3 The kernel (6.5) can be defined also in the case $\varepsilon = 0$. More precisely, we take $\mu_\varepsilon := \mu_0^\varepsilon$ to be the Lebesgue measures on $\mathbb{R}^{d+1}$, and we assume that the kernels $K_\varepsilon := K_0^\varepsilon : \mathbb{R}^{d+1} \setminus \{0\} \rightarrow \mathbb{R}$ have the bounded quantities (6.1) with $\varepsilon = 0$. We prove in Theorem 6.4 below that the kernel $K_\varepsilon^\gamma$ is well-defined by (6.5), with respect to these kernels and measures.

We have the following bound on multiple stochastic integrals, a proof of which we provide in the following sections:

**Theorem 6.4** Let $G = (V, E)$ be a graph with labels $\{a_e, r_e\}_{e \in E}$ satisfying Assumption 6.1 and let $\gamma \in \Gamma(V_0)$ be a contraction (as defined in Section 4) such that the contracted graph $G_{\gamma} = (\hat{V}, \hat{E})$ satisfies Assumption 6.2. Let furthermore $\mathcal{I}_\gamma$ be multiple stochastic integrals defined in (4.2) with respect to càdlàg martingales satisfying Assumption 2.1. Then, for every $p \geq 1$, there exists a constant $C$, depending only on $p$, the graph $G_{\gamma} = (\hat{V}, \hat{E})$ and supports of the measures $\mu_\varepsilon^\gamma$ and the kernels $K_\varepsilon^\gamma$, such that the bound

$$\left( \mathbb{E}\left[ \sup_{t \geq 0} |\mathcal{I}_\gamma^\lambda(K_{G}^\gamma e)_{t}\right]^p \right)^{1/p} \leq C \varepsilon^{\beta_\gamma}(\lambda \vee \varepsilon)^{a_\gamma} \prod_{e \in E} \|K_e^\gamma\|_{a_e, q} \prod_{v \in V_e \setminus V_0} \|\mu_v\|_{TV}$$

(6.7)

holds uniformly in $\lambda \in (0, 1]$ for some value $q \geq 0$ determined by the graph, where

$$\alpha_\gamma := (|V_e \setminus \hat{V}_e| - |F_\gamma|/2)|s| - \sum_{e \in E} a_e, \quad \beta_\gamma := (|V_0| - 2|E_\gamma| + |F_\gamma|)|s|/2.$$  

(6.8)

Let moreover the following limits hold as $\varepsilon \rightarrow 0$: the kernel $K_\varepsilon^\gamma$ converges pointwise to a kernel $K_\varepsilon$ on $\mathbb{R}^{d+1} \setminus \{0\}$ for each $e \in E$ and the latter satisfies (6.1) with $\varepsilon = 0$; the measure $\mu_\varepsilon^\gamma$ converges in total variation to the Lebesgue measure $\mu_{\varepsilon}$ on $\mathbb{R}^{d+1}$ for each $v \notin V_0 \cup \{\star\}$. Then the multiple integrals $\mathcal{I}_\gamma^\lambda(K_{G}^\varepsilon e)$ converge in the sense of (6.7) to the multiple Wiener integral $\mathcal{I}_\gamma^\lambda$ (see Lemma 2.2 and Remark 6.3), and in particular the latter is well-defined.

If we would like to consider a recentered test function $\varphi_\varepsilon^\lambda(z) := \varphi_\lambda(z - \bar{z})$, we need to shift respectively all the variables in the generalized convolution:

$$K_{G;\varepsilon}^\lambda(z^\varepsilon) := \int_{(\mathbb{R}^{d+1})^{V_e}} \prod_{e \in E} \hat{K}_e^\varepsilon(z_{-} - \bar{z}, z_{+} - \bar{z}) \prod_{v \in V_e} \varphi_\varepsilon^\lambda(z_v) \mu_v^\varepsilon(\bar{z}) d\bar{z},$$

(6.9)

for which we have the following result:

**Corollary 6.5** Under the assumptions of Theorem 6.4, the bound (6.7) and the convergence result hold for the multiple integrals $\mathcal{I}_\gamma^\lambda(K_{G;\varepsilon}^\lambda e)$. 
Proof. One can see that $\mathcal{K}^\lambda_{\mathbb{G},z}(\mathbf{z}) = \mathcal{K}^\lambda_{\mathbb{G}}(\mathbf{z}^0 - \mathbf{z}^{1|\mathcal{V}_0})$, where as before $\mathbf{z}^{1|\mathcal{V}_0}$ is the vector of length $n$ with all entries equal $\bar{z}$. This yields $\mathcal{I}_r(\mathcal{K}^\lambda_{\mathbb{G},z}) = \mathcal{I}_r(\hat{\mathcal{K}}^\lambda_{\mathbb{G}})$, where $\mathcal{I}_r$ are the multiple integrals with respect to the shifted martingales $\hat{\mathcal{M}}^r(z) := \mathcal{M}^r(z + \bar{z})$. Since the latter satisfy Assumption 2.1 (up to a shift), the same results hold for the multiple stochastic integrals.

From (6.1) one can see, that multiplication of a kernel $K^\varepsilon_\gamma$ by $\varepsilon^\delta$, with $\delta > 0$, ‘improves’ its singularity by $\delta$. That’s why a positive power of $\varepsilon$ in (6.7) can compensate violation of Assumption 6.2. More precisely, let us change the labels of the graph as follows:

$$\hat{a}_e := \tilde{a}_e + \begin{cases} 0, & \text{if } e_\gamma \notin \hat{\mathcal{V}}_0, \\ (1 - |i^{-1}_\gamma(e_-)|/2)|s|, & \text{if } e_\gamma \in \hat{\mathcal{V}}_0 \setminus i_\gamma(F_\gamma), \\ (1 - |i^{-1}_\gamma(e_-)|)|s|/2, & \text{if } e_\gamma \in i_\gamma(F_\gamma), \end{cases}$$

(6.10)

where the map $i_\gamma$ has been defined above, and we recall that by assumption all components of odd cardinalities are contained in $F_\gamma$. Then we have the following result:

**Corollary 6.6** In the settings of Theorem 6.4, let the contracted graph $\mathbb{G}_\gamma = (\hat{\mathcal{V}}, \hat{\mathcal{E}})$ be such that the labels $(\hat{a}_e, \hat{r}_e)$ satisfy Assumption 6.2. Then one has the bound

$$\left( \mathbb{E} \left[ \sup_{t \geq 0} |\mathcal{I}_r(\mathcal{K}^{\lambda}_\mathbb{G},t)|^p \right] \right)^{1/p} \leq C\varepsilon^\kappa(\lambda \vee \varepsilon)\hat{\alpha}_\gamma - \kappa \prod_{e_\gamma \in \hat{\mathcal{E}}} ||K^\varepsilon_\gamma||_{tc} \prod_{v \in \mathcal{V}_0 \setminus \mathcal{V}_0} ||\mu^\varepsilon||_{TV},$$

(6.11)

for any $\kappa \in [0, (|\mathcal{V}_0| - 2|F_\gamma| + |F_\gamma|)|s|/2]$, such that there is an edge $e \in \hat{\mathcal{E}}$ with $e_\gamma \in \hat{\mathcal{V}}_0$ and Assumption 6.2 holds if we replace $\hat{a}_e$ by $\bar{a}_e + \kappa$. The constant $\hat{\alpha}_\gamma$ equals $\alpha_\gamma + \beta_\gamma$ and can be written as

$$\hat{\alpha}_\gamma := (|\mathcal{V}_0| \setminus (|\mathcal{V}_0| \cup |\mathcal{V}_0|)) + |\mathcal{V}_0|/2 + |F_\gamma|/2)|s| - \sum_{e \in \hat{\mathcal{E}}} a_e.$$  

(6.12)

Moreover, a respective analogue of Corollary 6.5 for recentered test functions holds.

**Proof.** We define the kernels $\hat{K}^\varepsilon_\gamma := \varepsilon^\deltaK^\varepsilon_\gamma$ with ‘improved’ singularities, where

$$\delta_e := \begin{cases} 0, & \text{if } e_\gamma \notin \hat{\mathcal{V}}_0, \\ (|i^{-1}_\gamma(e_-)|/2 - 1)|s|, & \text{if } e_\gamma \in \hat{\mathcal{V}}_0 \setminus i_\gamma(F_\gamma), \\ (|i^{-1}_\gamma(e_-)| - 1)|s|/2, & \text{if } e_\gamma \in i_\gamma(F_\gamma). \end{cases}$$
Then the new kernels $\hat{K}^e_\gamma$ satisfy (6.1) with the values $\tilde{a}_e$, defined in (6.10). Let $\hat{K}^\lambda_\gamma$ be the generalized convolution defined as in (6.5) using the kernels $\hat{K}^e_\gamma$. Then we can write $I_\gamma(\hat{K}^\lambda_\gamma) = e^{-\beta} I_\gamma(\hat{K}^\lambda_\gamma)$, and Theorem 6.4 yields the bound (6.11) with $\kappa = 0$. In order to have this bound with $\kappa > 0$, we take an edge $e \in \hat{\mathbb{E}}$, such that $e_+ \in \hat{\mathbb{V}}$, $\delta_e > 0$ and Assumption 6.2 holds for the labels $\hat{a}_e$ by $\hat{a}_e + \kappa$. Then we simply decrease the value of $\delta_e$ by $\kappa \in (0, \delta_e)$ in the argument above. □

6.2 Multiscale decomposition of the generalized convolution

Our aim is to use a decomposition of the kernels $K^e_\gamma$ in the generalized convolution (6.5). To this end, we make the graph $G_\gamma$ complete, by adding into the set of edges $\hat{\mathbb{E}}$ those $e \in \hat{\mathbb{V}}^2$ which are not already connected by edges in $\hat{\mathbb{E}}$. We can chose any direction of $e$, and to all these new edges we assign the labels $\hat{a}_e = \hat{r}_e = 0$ and the kernels $\hat{K}^e \equiv 1$, so that (6.5) is unaffected. With abuse of notation we refer to this enhanced graph $G_\gamma = (\hat{\mathbb{V}}, \hat{\mathbb{E}})$. Furthermore, for $v \in \hat{\mathbb{V}}$ and the edge $e = (\star, v)$, we view the test function $\varphi^\lambda(z_v)$ as a new kernel $\hat{K}^e(z_+, z_v)$, where $z_+ = 0$, and where we recall $\hat{a}_e = \hat{r}_e = 0$.

We start with providing the following decomposition of the kernels, which can be proved in the same way as [HQ15, Lem. A.4], taking into account that we consider singularities of the kernels only up to the scale $\varepsilon$ and that we integrate with respect to measures $\mu^e_\varepsilon$:

**Lemma 6.7** For every $e \in \hat{\mathbb{E}}$, the kernel $\hat{K}^e_\varepsilon$ has the following properties:

1. one can write $\hat{K}^e_\varepsilon(z) = \sum_{n=0}^{N} K^e_\varepsilon^n(z)$, where $N := \lfloor \log_2 \varepsilon \rfloor$;
2. the function $K^e_\varepsilon^n(z)$ is supported in $2^{-(n+2)} \leq \|z\|_s \leq 2^{-n}$;
3. for any $q \geq 0$ large enough and for some $C > 0$, independent of $\varepsilon$, one has
   \[
   \|D^{k} K^e_\varepsilon^n(z)\| \leq C 2^n (a_+ + k) \|K^e_\varepsilon^n\|_a, q,
   \]
   uniformly in $z$, $|k|_s \leq q$ and $n \geq 0$;
4. if $r_+ \leq 0$, then $\int_{\mathbb{R}^d+1} D^{k} K^e_\varepsilon^n(z) \mu^e_\varepsilon(z) dz = 0$, for all $n > 0$ and $|k|_s < |r_+|$;
5. if $r_+ \leq 0$, then the kernel (6.3) can be written as
   \[
   \hat{K}^e_\varepsilon(\varphi) = \sum_{n=0}^{N} \int K^e_\varepsilon^n(z_+ - z_0) \varphi(z_0, z_+) \mu^e_\varepsilon(z_0) \mu^e_\varepsilon(dz_0) \mu^e_\varepsilon(dz_+),
   \]
   for every test function $\varphi$ on $\mathbb{R}^{d+1} \times \mathbb{R}^{d+1}$.

Our next aim is to decompose the kernels in (6.9) into sums of localized functions. To this end, using Lemma 6.7 we write $K^\varepsilon(z) = \sum_{n \geq 0} K^\varepsilon_+^n(z)$, where $K^\varepsilon_+^n \equiv 0$ for $n > N$. Then, for $\mathbf{n} \in \mathbb{N}^3$ and $e \in \hat{\mathbb{E}}$ such that $r_+ \leq 0$, we define the kernel

\[
\hat{K}^\varepsilon_+^\mathbf{n}(z, \hat{z}) := \begin{cases} 
K^\varepsilon_+^k(z - \hat{z}), & \text{if } \mathbf{n} = (k, 0, 0), \\
0, & \text{if } \mathbf{n} \neq (k, 0, 0).
\end{cases}
\]
In the case \( r_e > 0 \), we take any smooth functions \( \psi^{(n)}(z) \), such that \( \psi^{(n)}(z) \) is supported in \( 2^{-(n+2)} \leq \|z\|_s \leq 2^{-n} \), scales as \( 2^{-n} \) and satisfies \( \sum_{n \geq 0} \psi^{(n)}(z) = 1 \) for all \( z \). Then for \( n = (k, p, m) \) we set

\[
\hat{K}^{\varepsilon,n}(z, \tilde{z}) := \psi^{(k)}(\tilde{z} - z)\psi^{(p)}(z)\psi^{(m)}(\tilde{z})\hat{K}_e^\varepsilon(z, \tilde{z}) , \tag{6.14}
\]

where the kernel \( \hat{K}_e^\varepsilon \) has been defined in (6.4). For a function \( n : E \to N^3 \) and a point \( z = \{z_v : v \in V\} \), we define

\[
\hat{K}^{\varepsilon,n}(z) := \prod_{v \in E} \hat{K}^{\varepsilon,n}(z_{e_-}, z_{e_+}) . \tag{6.15}
\]

For \( \lambda \in (0, 1] \), let \( N_{\lambda} := \{n : E \to N^3 : 2^{-|n_{\ast,v}|} \leq \lambda, v \in V_\ast\} \), where it follows from the assumption \( r_{(\ast,v)} = 0 \) that only the first entry of \( n_{(\ast,v)} \) can be non-zero. Since the functions \( \psi^{(n)} \) sum up to 1 and since we consider the test function \( \varphi^\lambda \) as a kernel, one can rewrite the generalized convolution (6.5) as

\[
K_{\gamma}^{\lambda,\varepsilon}(z^\ast) := \sum_{n \in N_{\lambda}} \int_{(R^{d+1})^V} \hat{K}^{\varepsilon,n}(z) \mu_{\gamma,V,E}^{\ast}(dz) , \tag{6.16}
\]

where we set the measure \( \mu_{\gamma,V,E}^{\ast}(dz) \) on \( z^\ast \in (R^{d+1})^V \) to be \( \mu_{\gamma,V,E}^{\ast}(d\tilde{z}) \delta_{z^\ast,dz^\ast} \) where \( z^\ast = \tilde{z} \sqcup z_\ast, \tilde{z} \in (R^{d+1})^V, \) and \( \delta_{z^\ast} \) is the Dirac delta function.

Since we are interested in evaluating the integrals \( \mathcal{I}_\gamma^\varepsilon(K_{\gamma}^{\lambda,\varepsilon}) \), we can exploit the fact that the integration variables \( z_v \) in the kernel (6.16), for vertices \( v \) belonging to the same component of \( \gamma \), are equal. More precisely, for a function \( n \) from \( N_{\lambda,\gamma} := \{n : \tilde{E} \to N^3 : 2^{-|n_{\ast,v}|} \leq \lambda, v \in V_\ast\} \) and a point \( z = \{z_v : v \in V\} \), we define the kernel \( \hat{K}^{\varepsilon,n}(z) \) as in (6.15), but with the product over \( \tilde{E} \). Furthermore, we define the measure \( \hat{\mu}_{\gamma,V,E}^{\ast}(dz) \) on \( (R^{d+1})^V \) by

\[
\hat{\mu}_{\gamma,V,E}^{\ast}(dz) := \prod_{w \in V \setminus V_0} \mu_{\gamma}^{\ast}(dz_w) \prod_{w \in V_0} \delta_{z_w - z_{w^\ast}} dz_w ,
\]

where \( w_\ast \) is the first element (with respect to the chosen order of vertices) in \( i^{-1}_\gamma(w) \), and the map \( i_\gamma \) has been introduced in the beginning of this section. Then we define the kernel

\[
K_{\gamma}^{\varepsilon,n}(z^\ast) := \int_{(R^{d+1})^V} \hat{K}^{\varepsilon,n}(z) \hat{\mu}_{\gamma,V,E}^{\ast}(dz) , \tag{6.17}
\]

and write the multiple stochastic integral in the following way:

\[
\mathcal{I}_\gamma^\varepsilon(K_{\gamma}^{\lambda,\varepsilon})_t = \sum_{n \in N_{\lambda,\gamma}} \mathcal{I}_\gamma^\varepsilon(K_{\gamma}^{\varepsilon,n})_t .
\]
Using this expansion and applying Minkowski’s inequality, we obtain the bound
\[
\left( \mathbb{E} \left[ \sup_{t \geq 0} \mathcal{I}_t^\gamma (\mathcal{K}_{G}^{\lambda, \gamma}) | t |^p \right] \right)^{1/p} \lesssim \sum_{n \in \mathcal{N}_{\lambda, \gamma}} \left( \mathbb{E} \left[ \sup_{t \geq 0} \mathcal{I}_t^\gamma (\mathcal{K}_{G}^{\lambda, \gamma, n}) | t |^p \right] \right)^{1/p}.
\] (6.18)

Now, bounding a multiple integral of the generalized convolution are boiled down to bounding integrals in (6.18) and a smart summation over the functions \( n \in \mathcal{N}_{\lambda, \gamma} \). This is what we do in the next sections, where, following the idea of [HQ15, Sec. A.2], we use a multiscale clustering in the summation over \( n \).

### 6.3 Multiscale clustering

We associate to every point \( z \in (\mathbb{R}^{d+1})^\mathbb{N} \) a rooted labelled binary tree \((T, \ell)\), such that \( \|z_v - z_w\|_s \sim 2^{-\ell_v \wedge \ell_w} \), where \( v \wedge w \) is the closest common ancestor of \( v \) and \( w \). Moreover, the labels \( \ell \) satisfy \( \ell_v \geq \ell_{\omega} \) whenever \( \nu \geq \omega \), where \( \nu \geq \omega \) means that \( \omega \) belongs to the shortest path from \( \nu \) to the root of the tree \( T \) (see [HQ15, Sec. A.2] for construction of such tree). Given a set of vertices \( \hat{\mathcal{V}} \), denote by \( \mathcal{T}(\hat{\mathcal{V}}) \) the set of rooted labelled binary trees \((T, \ell)\) as above, which have \( \hat{\mathcal{V}} \) as their set of leaves. Denote furthermore by \( \mathcal{T}_\gamma(\hat{\mathcal{V}}) \), the subset of those labelled trees in \( \mathcal{T}(\hat{\mathcal{V}}) \) with the property that \( 2^{-\ell_{v \wedge w}} \leq \lambda \) for any two leaves \( v, w \in \hat{\mathcal{V}} \cup \{ \star \} \).

Our next aim is to write summation in (6.18) over such labeled trees \((T, \ell)\) and then over those functions \( n \) which are close in some sense to the labeling \( \ell \). To this end, for \( c := \log |\mathcal{V}| + 2 \), we define the set \( \mathcal{N}_{\gamma} (T, \ell) \) to consist of all functions \( n : \hat{\mathcal{E}} \to \mathbb{N}^{3} \) such that for every edge \( e = (v, w) \) with \( r_e \leq 0 \), one has \( n_e = (k, 0, 0) \) with \( |k - \ell_{v \wedge w}| \leq c \), and for every edge \( e = (v, w) \) with \( r_e > 0 \), one has \( n_e = (k, p, m) \) with \( |k - \ell_{v \wedge w}| \leq c \), \( |p - \ell_{v \wedge w}| \leq c \), and \( |m - \ell_{v \wedge w}| \leq c \). Then (6.18) and [HQ15, Lem. A.9] yield
\[
\left( \mathbb{E} \left[ \sup_{t \geq 0} \mathcal{I}_t^\gamma (\mathcal{K}_{G}^{\lambda, \gamma, n}) | t |^p \right] \right)^{1/p} \lesssim \sum_{(T, \ell) \in \mathcal{T}_\gamma(\hat{\mathcal{V}})} \sum_{n \in \mathcal{N}_{\gamma}(T, \ell)} \left( \mathbb{E} \left[ \sup_{t \geq 0} \mathcal{I}_t^\gamma (\mathcal{K}_{G}^{\lambda, \gamma, n}) | t |^p \right] \right)^{1/p}.
\] (6.19)

Moreover, we notice that if there is an \( e \in \hat{\mathcal{E}} \) such that at least one entry of \( n_e \) is greater than \( N \) (recall that \( N := -\lfloor \log_2 \varepsilon \rfloor \)), then the kernel \( \mathcal{K}_{G}^{\lambda, \gamma, n} \) vanishes, since \( \mathcal{K}_{e, \mathcal{E}} \equiv 0 \), by assumption which we made above (6.13). That’s why the two sums in (6.19) can be restricted to \((T, \ell) \in \mathcal{T}_\gamma(\hat{\mathcal{V}}) \) and \( n \in \mathcal{N}_{\gamma}(T, \ell) \), where \( \mathcal{T}_\gamma(\hat{\mathcal{V}}) \) contains those labeled trees \((T, \ell) \in \mathcal{T}_\gamma(\hat{\mathcal{V}}) \) whose labels satisfy \( \ell_v \leq N + c \), for all \( v \), and \( \mathcal{N}_{\gamma}(T, \ell) \) contains the functions from \( \mathcal{N}_{\gamma}(T, \ell) \) with values in \( \{0, \ldots, N\}^3 \). Our aim is now to bound the expression in (6.19).

### 6.4 Bounds on multiple integrals

In this section we are going to bound the integrals appearing on the right-hand side of (6.19). We start with modifying the kernels in these integrals, taking into account negative renormalization.
Given a labelled tree \((T, \ell) \in \mathcal{T}(\hat{V})\), we denote by \(\mathcal{D}(T, \ell)\) the subset of \((\mathbb{R}^{d+1})^\hat{V}\) such that \(\|z_v - z_w\|_s \leq |\hat{V}|2^{-\ell v \wedge w}\) for all vertices \(v, w \in \hat{V}\). Furthermore, in exactly the same way as in [HQ15, Sec. A.5], we can build new kernels \(\hat{K}^{\varepsilon, \mathbf{n}}(\mathbf{z})\) such that \(\text{supp}(\hat{K}^{\varepsilon, \mathbf{n}}) \subset \mathcal{D}(T, \ell)\) and which satisfy
\[
\int_{(\mathbb{R}^{d+1})^\hat{V}} \hat{K}^{\varepsilon, \mathbf{n}}(\mathbf{z}) \mu^{\varepsilon, \mathbf{x}^*}_V(d\mathbf{z}) = \int_{(\mathbb{R}^{d+1})^\hat{V}} \hat{K}^{\varepsilon, \mathbf{n}}(\mathbf{z}) \mu^{\varepsilon, \mathbf{x}^*}_V(d\mathbf{z}) .
\]
(6.20)

The only difference with the argument in [HQ15, Sec. A.5] consists in our integration with respect to measures \(\mu^e_r\). One can see that Lemma 6.7.4 is the only property required to compensate this difference.

Using these kernels and the comments below (6.19), the latter can be written as
\[
\left( E\left[\sup_{t \geq 0} |I_t^r(K^{\lambda \varepsilon}_G, \ell)|^p\right]\right)^{1/p} \lesssim \sum_{(T, \ell) \in \mathcal{T}(\hat{V})} \sum_{\mathbf{n} \in \mathcal{N}_e(T, \ell)} \left( E\left[\sup_{t \geq 0} |I_t^r(K^{\varepsilon, \mathbf{n}}_G, \ell)|^p\right]\right)^{1/p},
\]
(6.21)

with the new kernels
\[
\hat{K}^{\varepsilon, \mathbf{n}}(\mathbf{z}_e) := \int_{\mathcal{D}(T, \ell)} \hat{K}^{\varepsilon, \mathbf{n}}(\mathbf{z}) \mu^{\varepsilon, \mathbf{x}^*}_V(d\mathbf{z}) .
\]
(6.22)

Moreover, [HQ15, Lem. A.16] combined with Lemma 6.7 gives the bound
\[
\|\hat{K}^{\varepsilon, \mathbf{n}}\|_\infty \lesssim \prod_{v \in T^0} 2^{-\ell_v(\hat{\eta}_e - |s|)} \prod_{e \in E} \|K^e_r\|_{\text{TV}} \prod_{v \in V^\setminus V_0} \|\mu^e_r\|_{\text{TV}},
\]
(6.23)

uniformly over all \(\mathbf{n} \in \mathcal{N}_e(T, \ell)\), where \(T^0\) denotes the set of interior nodes of the tree \(T\), \(\hat{\eta}_e := |s| + \sum_{e \in \hat{E}} \hat{\eta}_e(v)\) and
\[
\begin{align*}
\hat{\eta}_e(v) := & -\hat{a}_e 1_{v _e}(v) + \hat{r}_e (1_{v _e \wedge e}(v) - 1_{v _e}(v)) 1_{v _e > 0, e _e \wedge * > e _e} \\
& + (1 - \hat{r}_e - \hat{a}_e)(1_{v _e \wedge e}(v) - 1_{v _e}(v)) 1_{v _e > 0, e _e \wedge * > e _e} \\
& - \hat{r}_e 1_{e \in A'}(1_{v _e}(v) - 1_{v _e}(v)) ,
\end{align*}
\]
(6.24)

with \(1_{v _e}(v) = 1\) and \(1_{v _e}(v) = 0\) for \(w \neq v\). Here, \(e _e = e _e \wedge e _e \in T\) for an edge \(e = (e _e, e _e) \in \hat{E}\), the set \(A' \subset \hat{E}\) contains those edges \(e = (e _e, e _e)\) which have the label \(r_e < 0\), and for which any two vertices \(\{u, v\}\) satisfying \(u \wedge v = e _e\) coincide with \(e _e, e _e\). The interior node \(e _e \in T^0\) is the element of the form \(w \wedge e _e\) with \(w \not\in e \) which is furthest from the root. The following result provides a bound for each term on the right-hand side of (6.21).
Lemma 6.8 For any labeled tree $(T, \ell) \in T_\lambda(\tilde{\mathcal{V}})$ and any $p \geq 1$ there is a constant $C$ such that for every $n \in \mathcal{N}_\gamma(T, \ell)$ one has the bound
\[
\left( \mathbb{E} \left[ \sup_{\ell \geq 0} |\mathcal{I}_\gamma^\varepsilon(\tilde{K}^\varepsilon_{\mathcal{G}, \gamma})|_1^p \right] \right)^{1/p} \leq C \varepsilon^{\beta_\gamma} \prod_{v \in T^0} 2^{-\ell_v|s|} \|\tilde{K}^\varepsilon_{\mathcal{G}, \gamma}\|_\infty ,
\] (6.25)
where $\beta_\gamma := (|V_0| - 2|E_\gamma| + |F_\gamma|)|s|/2$ and the kernel has been defined in (6.22).

Proof. Applying Lemma 5.1, we obtain the following bound:
\[
\left( \mathbb{E} \left[ \sup_{\ell \geq 0} |\mathcal{I}_\gamma^\varepsilon(\tilde{K}^\varepsilon_{\mathcal{G}, \gamma})|_1^p \right] \right)^{1/p} \lesssim \|\tilde{K}^\varepsilon_{\mathcal{G}, \gamma}\|_{\mathcal{H}^{\gamma}},
\]
where the norm on the right-hand side is defined in (5.1), and in order to get (6.25) we need to prove the bound
\[
\|\tilde{K}^\varepsilon_{\mathcal{G}, \gamma}\|_{\mathcal{H}^{\gamma}} \lesssim \varepsilon^{\beta_\gamma} \prod_{v \in T^0} 2^{-\ell_v|s|} \|\tilde{K}^\varepsilon_{\mathcal{G}, \gamma}\|_\infty .
\] (6.26)

We are going to prove this claim by induction over the number of components in the contraction $\gamma$. Let us first consider the case when $\gamma$ has only one component, i.e. $E_\gamma = \{V_0\}$. Then we use the simple bound on (5.1a) to get
\[
\|\tilde{K}^\varepsilon_{\mathcal{G}, \gamma}\|_{\mathcal{H}^{\gamma}} \lesssim (1_{\{E_\gamma \neq \emptyset\}} \varepsilon^{(|V_0|/2 - 1)|s|} + \varepsilon^{(|V_0| - 1)|s|/2}) \sup_{z} |\bar{K}^\varepsilon_{\mathcal{G}}(\tilde{z}^{\downarrow} V_0)| .
\]
Furthermore, the definition (6.22) and properties of the measures $\mu_\varepsilon$ yield
\[
\sup_{z} |\bar{K}^\varepsilon_{\mathcal{G}}(\tilde{z}^{\downarrow} V_0)| \lesssim |A_{\varepsilon, \gamma}|_{\tilde{\mathcal{V}}} \|\tilde{K}^\varepsilon_{\mathcal{G}}\|_\infty ,
\]
where we write $| \cdot |_{\alpha}$ for the $(\alpha|s|)$-dimensional Hausdorff measure, and the set $A_{\varepsilon, \gamma} \subset (\mathbb{R}^{d+1})^{\tilde{\mathcal{V}}}$ contains all points $z$ satisfying $|z_v - z_w|_s \leq C 2^{-\ell_v w} v \in T^0$ for all vertices $v, w \in \tilde{\mathcal{V}}$. Here, we use the fact that $2^{-\ell_v w} \geq \varepsilon$, which is a consequence of the assumption $(T, \ell) \in T_\lambda(\mathcal{V})$. For an interior node $v \in T^0$, let us choose $v_\pm \in \tilde{\mathcal{V}}$ so that $v_- \wedge v_+ = v$. Then one can see that the following inclusion holds:
\[
A_{\varepsilon, \gamma} \subset \left\{ z \in (\mathbb{R}^{d+1})^{\tilde{\mathcal{V}}} : |z_{v_-} - z_{v_+}|_s \leq C 2^{-\ell_v}, \forall v \in T^0 \right\} .
\]
Computing the Hausdorff measure of the right-hand side, we obtain the bound $|A_{\varepsilon, \gamma}|_{\tilde{\mathcal{V}}} \lesssim \prod_{v \in T^0} 2^{-\ell_v|s|}$. Thus, have the following bound
\[
\|\tilde{K}^\varepsilon_{\mathcal{G}, \gamma}\|_{\mathcal{H}^{\gamma}} \lesssim (1_{\{E_\gamma \neq \emptyset\}} \varepsilon^{(|V_0|/2 - 1)|s|} + \varepsilon^{(|V_0| - 1)|s|/2}) \prod_{v \in T^0} 2^{-\ell_v|s|} \|\tilde{K}^\varepsilon_{\mathcal{G}}\|_\infty .
\]
which is exactly (6.26) with $|E_\gamma| = 1$ and $|F_\gamma| \in \{0, 1\}$.

Now we turn to the case when the contraction $\gamma$ has more than one components, i.e. $|E_\gamma| \geq 2$. We pick a component $\bar{e} \in E_\gamma$ and write $\bar{v} \in \bar{V}_\delta$ for the vertex corresponding to $\bar{e}$, i.e. $i_\gamma(\bar{e}) = \{\bar{v}\}$. We denote $\bar{\gamma} = \gamma \setminus \bar{e}$, and write $G_{\bar{\gamma}}$ for the graph obtained from $G_\gamma$ after removing the vertex $\bar{v}$ with all adjacent edges. Then the labeled tree $(T, \bar{\ell})$ reduces (after removing the nodes $\bar{v}$ and $\bar{v}^\uparrow$) to a labeled tree $(\bar{T}, \bar{\ell})$ on the vertices of $G_{\bar{\gamma}}$. Furthermore, we introduce the kernel $\hat{\mathcal{K}}_{G_{\bar{\gamma}}, \bar{z}}^\varepsilon$ which is obtained from $\hat{\mathcal{K}}_{G, z}^\varepsilon$ after fixing the value $z_\delta = \bar{z}$ for the vertex $\bar{v}$. Then, by induction hypothesis, the kernel $\hat{\mathcal{K}}_{G_{\bar{\gamma}}, \bar{z}}^\varepsilon$ satisfies (6.26) with the graph $G_{\bar{\gamma}}$:

$$
\| \hat{\mathcal{K}}_{G_{\bar{\gamma}}, \bar{z}}^\varepsilon \|_{\mathcal{H}_{\bar{v}; \bar{\gamma}}} \lesssim e^{\beta_{\bar{v}}} \prod_{v \in T^\circ} 2^{-\ell_v|\bar{v}|} \| \hat{\mathcal{K}}_{\bar{z}}^\varepsilon \|_{\infty}, \tag{6.27}
$$

where the kernel $\hat{\mathcal{K}}_{G_{\bar{\gamma}}, \bar{z}}^\varepsilon$ is obtained from $\hat{\mathcal{K}}_{G, z}^\varepsilon$ after fixing the value $\bar{z}$ for the vertex $\bar{v}$. We use the definition (5.1b) to get

$$
\| \hat{\mathcal{K}}_{G, z}^\varepsilon \|_{\mathcal{H}_{v; \gamma}} \lesssim (1_{\{e \notin F_\gamma\}} e^{(|\bar{\ell}|/2 - 1)|\bar{v}|} + e^{(|\bar{\ell}| - 1)|\bar{v}|/2}) \sup_{\bar{z}} \| \hat{\mathcal{K}}_{G_{\bar{\gamma}}, \bar{z}}^\varepsilon \|_{\mathcal{H}_{\bar{v}; \bar{\gamma}}}.
$$

Combining this with the assumption (6.27) we obtain

$$
\| \hat{\mathcal{K}}_{G_{\bar{\gamma}}, \bar{z}}^\varepsilon \|_{\mathcal{H}_{\bar{v}; \bar{\gamma}}} \lesssim (1_{\{e \notin F_\gamma\}} e^{(|\bar{\ell}|/2 - 1)|\bar{v}|} + e^{(|\bar{\ell}| - 1)|\bar{v}|/2}) \sup_{\bar{z}} \| \hat{\mathcal{K}}_{G_{\bar{\gamma}}, \bar{z}}^\varepsilon \|_{\infty} \prod_{v \in T^\circ} 2^{-\ell_v|\bar{v}|} \| \hat{\mathcal{K}}_{\bar{z}}^\varepsilon \|_{\infty}, \tag{6.28}
$$

From $T^\circ = \bar{T}^\circ \cup \{\bar{v}^\uparrow\}$ we conclude that the following identity holds:

$$
2^{-\ell_v|\bar{v}|} \prod_{v \in T^\circ} 2^{-\ell_v|\bar{v}|} = \prod_{v \in T^\circ} 2^{-\ell_v|\bar{v}|}.
$$

Furthermore, in the case $\bar{e} \notin F_\gamma$, we have $|E_\gamma| = |E_{\bar{\gamma}}| + 1$ and $|F_\gamma| = |F_{\bar{\gamma}}|$, which yields $\beta_{\bar{\gamma}} = \beta_{\gamma} + (|\bar{\ell}|/2 - 1)|\bar{v}|$. In the case $\bar{e} \in F_\gamma$, we have $|E_\gamma| = |E_{\bar{\gamma}}| + 1$ and $|F_\gamma| = |F_{\bar{\gamma}}| + 1$, which implies $\beta_{\bar{\gamma}} = \beta_{\gamma} + (|\bar{\ell}| - 1)|\bar{v}|/2$. Combining all these identities together, we conclude that (6.28) is exactly the required bound (6.26).

\[ \square \]

### 6.5 Proof of Theorem 6.4

Using the results from the previous sections, we are ready to prove Theorem 6.4. Combining Lemma 6.8 and the bounds (6.21), we get the following moment bound:

$$
\left( \mathbb{E} \left[ \sup_{\ell \geq 0} |T_{\gamma}^\varepsilon (\mathcal{K}_{G, z}^\varepsilon)_{\ell}|^p \right] \right)^{1/p} \lesssim e^{\beta_{\gamma}} \sum_{(T, \ell) \in T^\varepsilon(\bar{V})} \prod_{v \in T^\circ} 2^{-\ell_v|\bar{v}|}, \tag{6.29}
$$
where \( \tilde{\eta} \) is defined in (6.24), and where the proportionality constant is a multiple of \( \prod_{v \in E} \| K^v_x(\gamma_{e,v}) \|_{TV} \). Then the bound (6.7) follows from (6.29) and [HQ15, Lem. A.10]. Indeed, we obtain

\[
\left( \mathfrak{E} \left[ \sup_{t \geq 0} | I_\gamma^x(\mathcal{K}^{\lambda,\varepsilon}_G) |^p \right] \right)^{1/p} \lesssim \varepsilon^{\beta_1} (\lambda \vee \varepsilon)^{\tilde{\alpha} |\eta|} ,
\]

where \( |\eta| := \sum_{v \in T_0} \hat{\eta}_v \). As at the end of the proof of [HQ15, Lem. A.19], we get \( |\eta| = |s| |\hat{\mathcal{V}}_s \setminus \hat{\mathcal{V}}_r^s| - \sum_{e \in E} \hat{a}_e \), which finishes the proof of the bound (6.7).

Now, we turn to the proof of the convergence result. To this end, we separate the scaling of the martingales and measures with the scaling of the kernels. More precisely, we take \( \varepsilon, \tilde{\varepsilon} > 0 \) and define the kernel \( \mathcal{K}^{\lambda,\varepsilon,\tilde{\varepsilon}}_G \) as in (6.5), but with measures \( \mu^\varepsilon \) and with kernels \( K^x \). Then the limit \( \lim_{\varepsilon \to 0} \mathcal{I}_\gamma^x(\mathcal{K}_G^\lambda,\varepsilon) = \mathcal{I}_\gamma(\mathcal{K}_G^\lambda) \) will follow from the limits \( \lim_{\varepsilon \to 0} \mathcal{I}_\gamma^x(\mathcal{K}_G^\lambda,\varepsilon) = \mathcal{I}_\gamma(\mathcal{K}_G^\lambda) \), where in the last case we use \( \tilde{\varepsilon} \) on the right-hand side of (6.7). For every fixed \( 0 < \varepsilon < \tilde{\varepsilon} \), the kernel \( \mathcal{K}^{\lambda,\varepsilon,\tilde{\varepsilon}}_G \) is smooth and compactly supported. Hence, the limit \( \lim_{\varepsilon \to 0} \mathcal{I}_\gamma^x(\mathcal{K}_G^\lambda,\varepsilon) = \mathcal{I}_\gamma(\mathcal{K}_G^\lambda) \) follows from Lemma 2.2 and multilinearity of the integral in \( M^e \). Furthermore, the bound (6.7) implies that the generalized convolution is multilinear in the measures \( \mu^\varepsilon \). Hence, for every fixed \( \bar{\varepsilon} > 0 \), we can take the limits of the measures \( \mu^\varepsilon \) one by one, to obtain \( \lim_{\varepsilon \to 0} \mathcal{I}_\gamma^x(\mathcal{K}_G^\lambda,\varepsilon) = \mathcal{I}_\gamma(\mathcal{K}_G^\lambda) \). Similarly, the limit \( \lim_{\varepsilon \to 0} \mathcal{I}_\gamma(\mathcal{K}_G^\lambda) = \mathcal{I}_\gamma(\mathcal{K}_G^\lambda) \) follows from multilinearity of the generalized convolution in the kernels \( K^x \).

7 Applications to particular stochastic PDEs

In this section we consider specific approximations of the two SPDEs: the stochastic quantization equation in 3D and the KPZ equation, with driving noises given by martingales satisfying Assumption 2.1. For simplicity, we consider only one martingale (i.e. \( r = 1 \)), so that we don’t need to modify the regularity structures from [Hai14] and [FH14] for the equations. We prove convergence of such approximations to the solutions of the respective continuous equations.

7.1 The stochastic quantization equation in 3D

The stochastic quantization equation on the three dimensional torus \( T^3 := (\mathbb{R}/\mathbb{Z})^3 \) can be formally described by the equation

\[
\partial_t \Phi = \Delta \Phi + (\infty - a) \Phi - \lambda \Phi^3 + \xi , \quad \Phi(0, \cdot) = \Phi_0(\cdot), \tag{7.1}
\]

where \( \Delta \) is the Laplace operator on \( T^3 \), \( a \) and \( \lambda \) are two fixed constants, \( \Phi_0 \) is an initial state, and \( \xi \) is the space-time white noise on \( L^2(\mathbb{R} \times T^3) \). The reason for the appearance of \( \infty \) is that solutions are random Schwartz distributions (this
is already the case for the linear equation [DPZ14], so that their third power is undefined in the classical sense. The equation (7.1) in this case is interpreted as the limit of solutions to the equations obtained by mollifying the noise $\xi$ and replacing $\infty$ by a diverging constant. It was shown in [Hai14] that this limit exists and is independent of the choice of the mollifier.

We consider the following martingale-driven discretizations of (7.1):

$$\frac{d}{dt}\Phi^\varepsilon = \Delta_\varepsilon \Phi^\varepsilon + (C^\varepsilon - a)\Phi^\varepsilon - \lambda (\Phi^\varepsilon)^3 + \xi^\varepsilon, \quad \Phi^\varepsilon(0) = \Phi^\varepsilon_0(\cdot),$$

(7.2)
on a uniform discretization $T^3_\varepsilon$ of the torus $T^3$ with mesh size $\varepsilon > 0$, with an initial state $\Phi^\varepsilon_0$, and where $\Delta_\varepsilon$ is a discretization of the Laplacian, with the properties:

1. $\Delta_\varepsilon P(x) = \Delta P(x)$, for any polynomial $P$ with $\deg(P) \leq 2$;
2. one can write $\Delta_\varepsilon \varphi(x) = \frac{1}{\varepsilon^2} \int_{\mathbb{R}^3} \varphi(x + \varepsilon y) \mu(dy)$, for every function $\varphi \in C(\mathbb{R}^3)$ and a measure $\mu$, concentrated on a finite neighbourhood of the origin on $\mathbb{Z}^3$;
3. the Fourier transform of $\mu$ vanishes only on $\mathbb{Z}^3$;

For example the nearest-neighbour approximation of the Laplacian satisfies these assumptions, see [HM18, Ex. 5.2]. The renormalization constants $C^\varepsilon$ diverge as $\varepsilon \to 0$, and the driving noise is given by $\xi^\varepsilon = \partial_t M^\varepsilon$ where $M^\varepsilon$ is a càdlàg martingale, satisfying Assumption 2.1, with $r = 1$ and with $c^\varepsilon \equiv 1$. For example, the martingales $M^\varepsilon_t(x)$ can be independent compensated Poissonian processes with rates 1. Our convergence result is the following:

**Theorem 7.1** Let for some $\eta \in (-\frac{2}{3}, -\frac{1}{2})$ one has $\Phi_0 \in C^9(\mathbb{T}^3)$ a.s., and let $\Phi^\varepsilon_0$ be a random function on $\mathbb{T}^3$ satisfying $\lim_{\varepsilon \to 0} \|\Phi_0; \Phi^\varepsilon_0\|_{C^\eta} = 0$ a.s. Let furthermore the driving martingales $M^\varepsilon$ and the discrete Laplacian $\Delta_\varepsilon$ have the properties stated above. Then for every $\alpha < -\frac{1}{2}$ and every $\eta < \eta \wedge \alpha$ there is a sequence of renormalisation constants $C^\varepsilon \sim \varepsilon^{-1}$ such that the limit $\lim_{\varepsilon \to 0} \|\Phi; \Phi^\varepsilon\|_{C^\eta} = 0$ holds in probability.

**Proof.** We prefer not to duplicate here definitions from the theory of regularity structures. Instead, we refer to the original manuscript [Hai14] (as well as [FH14, Hai15]) and the discretization framework [EH17].

In order to describe equation (7.1), we use the regularity structure $\mathcal{S} = (\mathcal{A}, \mathcal{T}, \mathcal{G})$ defined in [Hai14, Sec. 9.4]. Let furthermore $G$ be the Greens function of the operator $\partial_t - \Delta$, which can be written as $G = K + R$, where $K$ is its singular part and $R$ is smooth and compactly supported [Hai14, Lem. 5.5]. Unfortunately, we cannot write the discrete Green’s function $G^\varepsilon$ of $\partial_t - \Delta_\varepsilon$, since it has a jump at $t = 0$. Instead, we can separate this singular part of the kernel from the rest. More precisely, using [HM18, Lem. 5.4], we can write the Green’s function of $\partial_t - \Delta_\varepsilon$ as $G^\varepsilon = K^\varepsilon + K^\varepsilon + R^\varepsilon$, where $K^\varepsilon$ is supported in a box of size $c\varepsilon$, ...
for some \(c > 0\), satisfies \(|D^k K^\varepsilon(z)| \lesssim \varepsilon^{-3-|k|}|s|\) for \(t \neq 0\), and is discontinuous at \(t = 0\). The function \(\tilde{R}^\varepsilon\) is \(C^3\), compactly supported and converges to \(R\) in \(C^3\). The function \(K^\varepsilon\) is \(C^3\), compactly supported, satisfies (6.1) with \(a = 3\), and converges to \(K\) pointwise outside of the origin.

For an admissible model \((\Pi, \Gamma)\) on \(\mathcal{T}\), the abstract version of equation (7.1) is

\[
u = \mathcal{P}_\tau 1_+(-au - \lambda u^3 + \Xi) + G\Phi_0\, ,
\]

see [Hai14, Eq. 9.13] for the definitions of the involved operators. Next, we will reformulate the discrete equation (7.2) in a similar way.

The canonical lift \((\hat{\Pi}^\varepsilon, \tilde{\Gamma}^\varepsilon)\) of the noise \(\xi^\varepsilon\) is defined as in [Hai14], where we use discrete objects instead of continuous: \(\hat{K}^\varepsilon\) instead of \(K\), and summation over the grid \(\Lambda^3\) instead of integration over \(\mathbb{R}^3\) in space. The respective renormalized model \((\Pi^\varepsilon, \hat{\Gamma}^\varepsilon)\) is constructed in [Hai14, Sec. 9.2], using two renormalization constants \(C^\varepsilon_1\) and \(C^\varepsilon_2\), which we define in (7.5) and (7.6) below.

We need to show that the pair \((\Pi^\varepsilon, \hat{\Gamma}^\varepsilon)\) is an admissible discrete model in the sense of [EH17, Def. 2.5] and Example 2 on p.8 in [EH17]. Moreover, it weakly converges to the continuous model \((\Pi, \Gamma)\) constructed in [Hai14]. By an argument as in the proof if [Hai14, Thm. 10.7], it is sufficient to prove that for every \(\tau \in \mathcal{T}\) such that \(|\tau| < 0\), every \(p \geq 1\) and every test function \(\varphi\), bounded uniformly by 1 together with its derivatives, one has the bound

\[
E[\nu \cdot (\tilde{\Pi}^\varepsilon_2 \tau)(\varphi^\lambda_\varepsilon)]^p \leq C(\lambda \vee \varepsilon)^p|\tau|^{\kappa},
\]

uniformly in \(z \in D_\varepsilon\) and \(\lambda \in (0, 1]\), where the constant \(C\) can depend on \(p\) and where \(|\tau|\) is the homogeneity of the symbol \(\tau\). Convergence of models follows from the following limit

\[
\lim_{\varepsilon \to 0} \sup_{z, \lambda, \varphi} (\lambda \vee \varepsilon)^p|\tau|^{\kappa} E[(\nu \cdot (\tilde{\Pi}^\varepsilon_2 \tau) - \Pi_\varepsilon \tau)(\varphi^\lambda)]^p = 0 ,
\]

for some \(\kappa > 0\). These bounds and limits are proved in Section 7.1.1.

The reconstruction operator is defined by \((\tilde{R}^\varepsilon H^\varepsilon)(z) := (\hat{\Pi}^\varepsilon H^\varepsilon(z))(z)\), for all modeled distributions \(H^\varepsilon : \mathcal{D}_\varepsilon \to \mathcal{T}\), not containing the abstract noise \(\Xi\). Its action on the noise \(\Xi\) is defined by \(\nu(\tilde{R}^\varepsilon \Xi)(\varphi) := \int_{D_\varepsilon} \varphi(z) dM^\varepsilon(z)\), for any test function \(\varphi\), where we use the extension \(\nu\) from (1.4). Next, using the discrete kernels we define the integration operator on modeled distributions: \(P^\varepsilon \Xi := \hat{K}^\varepsilon + (\tilde{R}^\varepsilon + \hat{K}^\varepsilon)\tilde{R}^\varepsilon\), where the two operators \(\hat{K}^\varepsilon\) and \(\tilde{R}^\varepsilon\) are defined in [Hai14, Eq. 5.15, Eq. 7.7], and \(\hat{K}^\varepsilon\) maps discrete function into abstract Taylor expansions, the same as \(R^\varepsilon\). Finally, we write \(G^\varepsilon \Phi_0^\varepsilon\) for the harmonic extension as in [Hai14, Lem 7.5].

Then, for a function \(u^\varepsilon : \mathcal{D}_\varepsilon \to \mathcal{T}\) we define the fixed point equation

\[
u^\varepsilon = P^\varepsilon \Xi \left(-au^\varepsilon - \lambda(u^\varepsilon)^3 + \Xi\right) + G^\varepsilon \Phi_0^\varepsilon .
\]
Using the definitions of the involved operators, one can see that if \( u^\varepsilon \) solves this equation, then \( \hat{R}^\varepsilon u^\varepsilon = \Phi^\varepsilon \), where the latter is a classical solution of (7.2), with the renormalization constant \( C^\varepsilon := 3C_1^\varepsilon - 9C_2^\varepsilon \sim \varepsilon \). Now, the statement of this theorem follows from [EH17, Thm. 6.4] and convergence of the discrete models.

7.1.1 Bounds on the discrete models

It is convenient to use graphical notation to represent random variables and integrals, where nodes represent variables and arrows represent kernels. The node ‘\( \circ \)’ with \( n \geq 1 \) outgoing arrows represents a variable in \( D_\varepsilon \) which is integrated out with respect to the martingale \( N^\varepsilon \), defined in (3.4), such that \( |\hat{\mathcal{R}}| = n \). The node ‘\( \bullet \)’ will represent integration with respect to the bracket \( [M^\varepsilon; \hat{\mathcal{K}}] \). By the node ‘\( \bullet \)’ we will denote an integrated out variable in \( D_\varepsilon \). A vertex ‘\( \bullet \)’ labeled with \( z \) represents the basis point \( z \in D_\varepsilon \). The arrow ‘\( \rightarrow \)’ represents the discrete kernel \( K^\varepsilon \), and we will write two lapels \( (a_e, r_e) \) on this arrow, which correspond to the labels on graphs in Section 6. More precisely, since the kernel \( K^\varepsilon \) satisfies the bound (6.1) with \( a = 3 \), we will draw ‘\( 3,0 \)’. Finally, ‘\( \rightarrow \)’ represents a test function \( \varphi^\lambda \).

In the bounds on the renormalized model \( \hat{\Pi}^\varepsilon \), we will use Theorem 6.4 and its corollary. To this end, we will use Proposition 4.2 to express the map \( \iota^\varepsilon (\hat{\Pi}^\varepsilon z_\tau) (\varphi^\lambda_z) \) as a sum of integrals with kernels, given by generalized convolutions (6.5).

As it was stated in the previous section, we need to prove bound only for the symbols \( \tau \in \mathcal{T} \) such that \( |\tau| < 0 \). The list if these symbols is \( \Xi, \Psi, \Psi^2, \Psi^3, \Psi^2 X_i, \mathcal{I}(\Psi^3) \Psi, \mathcal{I}(\Psi^3) \Psi^2, \mathcal{I}(\Psi^3) \Psi^3 \) (see the proof of [Hai14, Thm. 10.22]). Below, we consider these symbols one by one.

**The symbol \( \tau = \Xi \).** The definition of the canonical lift yields

\[
\iota^\varepsilon (\hat{\Pi}^\varepsilon \Xi) (\varphi^\lambda_z) = \int_{D_\varepsilon} \varphi^\lambda_z (z) dM^\varepsilon (z) .
\]

Lemma 2.2 now gives the required bounds (7.3) with \( |\Xi| = -5/2 - \kappa \), where \( \kappa > 0 \), where \((\Pi, \Gamma)\) is the continuous model, constructed in [Hai14, Thm. 10.22]:

\[
(\Pi z_\Xi) (\varphi^\lambda_z) = \int_{\mathbb{R}^4} \varphi^\lambda_z (z) dW (z) ,
\]

and \( W \) is a Wiener process.

**The symbol \( \tau = \Psi \).** We can represent the map \( \hat{\Pi}^\varepsilon \tau \) diagrammatically as

\[
\iota^\varepsilon (\hat{\Pi}^\varepsilon \tau) (\varphi^\lambda_z) = \circ \rightarrow \varphi^\lambda_z .
\]

This diagram is the stochastic integral \( I^\varepsilon(f) = I^\varepsilon(f)_\infty \) as defined in (4.2a), where the contraction \( \gamma \) is defined on the set of vertices \( \mathcal{V}_o \) such that \( |\mathcal{V}_o| = |E_\circ| = \)
The kernel \( f \) is in this case the generalized convolution \( K_{\xi}^{\lambda,\epsilon} \), as in (6.9), given by

\[
K_{\xi}^{\lambda,\epsilon}(\xi) = \int_{D_{\epsilon}} \varphi_{\lambda}(\tilde{z}) K^{\epsilon}(\tilde{z} - \xi) d\tilde{z}.
\]

One can check that Assumption 6.2 holds for this diagram. Applying Corollary 6.6 and recalling that \( |\tau| = -1/2 - \kappa \), one obtains the bound

\[
\left( E|\iota_{\epsilon}(\hat{\Pi}_{\xi}^{\epsilon}\tau)(\varphi^{3}_{\lambda})|^{p} \right)^{1/p} \lesssim (\lambda \vee \epsilon)^{|r| + \kappa},
\]

which is exactly our claim for the symbol \( \Psi \). The limiting model \( \Pi_{\xi} \) is obtained by replacing the discrete kernel and the martingale in \( \hat{\Pi}_{\xi}^{\epsilon} \tau \) by the continuous kernel and the Wiener process respectively. In what follows, we prefer not to specify contractions \( \gamma \) every time, as it will be clear from diagrams. Moreover, it will be always clear how the limiting model \( \Pi_{\xi} \) is defined, which we also prefer not to specify every time.

The symbols \( \tau = \Psi^{2} \) and \( \tilde{\tau} = \Psi^{2}X_{t} \). Applying Proposition 4.2 and taking into account the renormalization, the map \( \hat{\Pi}_{\xi}^{\epsilon} \tau \) can be represented by the diagrams

\[
i_{\epsilon}(\hat{\Pi}_{\xi}^{\epsilon}\tau)(\varphi^{3}_{\lambda}) = \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} + \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} - C^{\epsilon}_{1} = \begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} + \begin{array}{c} \text{Diagram 7} \\ \text{Diagram 8} \end{array},
\]

where the renormalization constant is given by

\[
C^{\epsilon}_{1} := \int_{D_{\epsilon}} K^{\epsilon}(\tilde{z})K^{\epsilon}(\tilde{z}) d\tilde{z} \sim \epsilon^{-1},
\]

where we use the assumption that the predictable covariation process of the martingales is \( C^{\epsilon} \equiv 1 \). Recalling, that the vertices on the diagrams belong to \( F_{\gamma} \), and applying Corollary 6.6 to these two diagrams, one obtains the bounds (7.3) with \( |\tau| = -1 - 2\kappa \). The required bound on the symbol \( \tilde{\tau} = \Psi^{2}X_{t} \) follows in a straightforward way from the definition \( (\hat{\Pi}_{\xi}^{\epsilon}\tilde{\tau})(\tilde{z}) = (\hat{\Pi}_{\xi}^{\epsilon}\tau)(\tilde{z})(\tilde{x} - x) \), where \( z = (t, x) \) and \( \tilde{z} = (\tilde{t}, \tilde{x}) \).

The symbol \( \tau = \Psi^{3} \). A diagrammatic representation of the map \( \hat{\Pi}_{\xi}^{\epsilon} \tau \) is

\[
i_{\epsilon}(\hat{\Pi}_{\xi}^{\epsilon}\tau)(\varphi^{3}_{\lambda}) = \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} + 3 \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} + \begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array},
\]

where we have used renormalization. Then Corollary 6.6 gives the required bounds (7.3), where \( |\tau| = -3/2 - 3\kappa \).
The symbol $\tau = \mathcal{I}(\Psi^3)\Psi$. The diagrams for the map $\hat{\Pi}_\varepsilon \tau$ are

$$
\iota_\varepsilon(\hat{\Pi}_\varepsilon \tau)(\varphi_3) = 3 + 3 \xi + 3 \xi + 3 \xi + 3 \xi,
$$

where the arrow ‘$\rightarrow$’ represents the renormalized kernel $(z_1, z_2) \mapsto K_\varepsilon(z_2 - z_1) - K_\varepsilon(z - z_1)$. Multiplying as before some of the kernels by powers of $\varepsilon$, we can apply Corollary 6.6 and get the bounds (7.3) with $|\tau| = -4K$.

The symbol $\tau = \mathcal{I}(\Psi^2)\Psi^2$. We can represent the map $\hat{\Pi}_\varepsilon \tau$ diagrammatically:

$$
\iota_\varepsilon(\hat{\Pi}_\varepsilon \tau)(\varphi_3) = 3 + 3 \xi + 2 \xi + 2 \xi + 2 \xi + 2 \xi - C_2^\varepsilon.
$$

All the diagrams except the third one satisfy Assumption 6.2. In order to make the third diagram satisfy Assumption 6.2.2, we need to chose an appropriate renormalization constant. We chose the renormalization constant as follows:

$$
C_2^\varepsilon = 2 < 3,0,3,0 > ,
$$

and after that the third diagram can be rewritten as

$$
2 < 3,0,3,0 > - C_2^\varepsilon = 2 < 3,0,3,0 > + 2 < 3,0,3,0 > + 2 < 3,0,3,0 >.
$$
which now satisfies the required assumptions. Applying Corollary 6.6 and recalling that $|\tau| = -4\kappa$, one obtains the bounds (7.3).

**The symbol** $\tau = I(\Psi^3)\Psi^2$. Taking into account renormalization, the diagrams for the map $\hat{\Pi}_x^\epsilon \tau$ are the following:

\[
\ell_\epsilon(\hat{\Pi}_x^\epsilon \tau)(\varphi_{x}^\epsilon) = \sum_{\text{diagrams}} \text{Diagrammatics}
\]

Recalling that the constant $C_2^\epsilon$ is (7.6), the last two diagrams can be written (up to the multiplier 6) as

\[
\sum_{\text{diagrams}} \text{Diagrammatics}
\]

where we use negative renormalization in the last diagram. The fact that

\[
\text{satisfies (6.1) with } a = 5 \text{ is a consequence of [Hai14, Lem. 10.14]. Now, all diagrams satisfy Assumption 6.2, and applying Corollary 6.6 one obtains the bounds (7.3) with } |\tau| = -1 - 5\kappa.
\]
7.2 The KPZ equation

The KPZ equation is a canonical model in the class of one-dimensional stochastic growth processes [KPZ86], and is formally given by

\[ \partial_t h = \Delta h + \lambda((\partial_x h)^2 - \infty) + \xi, \]

on the torus \( T \) with an initial condition \( h_0(\cdot) \) at time \( t = 0 \). As in (7.1), the non-linearity is undefined in the classical sense and renormalization required to define a solution. The discrete approximation of the KPZ equation we consider to be

\[ \frac{d}{dt} h^\varepsilon = \Delta h^\varepsilon + \lambda((\nabla h^\varepsilon)^2 - C^\varepsilon) + \xi^\varepsilon, \]

where \( \nabla \) is a discrete derivative satisfying \( \lim_{\varepsilon \to 0} \| \nabla \varphi - \varphi' \|_\infty = 0 \) for every \( \varphi \in C^1 \), and the discrete Laplacian \( \Delta \) and the noise \( \xi^\varepsilon \) are as in (7.2), but in one dimension. Then our convergence result is the following:

**Theorem 7.2** In the described context, let \( h_0 \in C^\eta(T) \) almost surely, for some \( \eta > 0 \), and let \( h^\varepsilon_0 \) be a random function on \( T^\varepsilon \) satisfying almost surely \( \lim_{\varepsilon \to 0} \| h_0 - h^\varepsilon_0 \|_{C^\eta} = 0 \). Then for every \( \alpha < \frac{1}{2} \) there is a sequence of renormalisation constants \( C^\varepsilon \sim \varepsilon^{-1} \) in (7.8) such that, for every \( \bar{\eta} < \eta \wedge \alpha \) one has the limit in probability \( \lim_{\varepsilon \to 0} \| h - h^\varepsilon \|_{C^{\bar{\eta}}} = 0 \).

**Proof.** The regularity structure \( \mathcal{J} \) for the equation (7.7) and a canonical lift \( (\Pi^\varepsilon, \Gamma^\varepsilon) \) were defined in [HQ15] (in a more general setting). The renormalized model \( (\hat{\Pi}^\varepsilon, \hat{\Gamma}^\varepsilon) \) is defined in the same article, using the renormalization constant (7.9). The rest of the proof goes along the lines of Theorem 7.1, and we only need to prove the bounds (7.3), where \( (\Pi, \Gamma) \) is a continuous model constructed in [HQ15]. These bounds are proved in Section 7.3 below.

\[ \square \]

7.3 Moment bounds for discrete models

The list of symbols with negative homogeneities is \( \Xi, \Psi^2, \Psi \mathcal{T}(\Psi), \Psi \mathcal{T}(\Psi^2), (\mathcal{T}(\Psi^2))^2, \Psi \mathcal{T}(\Psi \mathcal{T}(\Psi^2)) \), where this time the we use the notation \( \Psi := \mathcal{T}(\Xi) \). We will consider these symbols one by one and prove the bounds (7.3). For this, we will make use of the graphical notation from Section 7.1.1, using the fact that the kernel \( \nabla K^\varepsilon \) satisfies (6.1) with \( a = 2 \).

The symbol \( \Xi \) is treated in exactly the same way as in Section 7.1.1, but now \( |\Xi| = -3/2 - \kappa \) with \( \kappa > 0 \).

**The symbol** \( \tau = \Psi^2 \). Using our graphical notation, we can represent the map \( \iota_{\varepsilon}(\hat{\Pi}^\varepsilon \tau)(\varphi^\varepsilon) \) diagrammatically as

\[ \iota_{\varepsilon}(\hat{\Pi}^\varepsilon \tau)(\varphi^\varepsilon) = \frac{\circ 2_0 2_0}{2_0 2_0} + \frac{\circ 2_0 2_0}{2_0 2_0} - C^\varepsilon_1 = \frac{\circ 2_0 2_0}{2_0 2_0} + \frac{\circ 2_0 2_0}{2_0 2_0} , \]
where we use the renormalization constant

\[
C_\varepsilon := \int_{D_\varepsilon} \nabla_\varepsilon K^\varepsilon(\bar{z}) \nabla_\varepsilon K^\varepsilon(\bar{z}) \, d\bar{z} \sim \varepsilon^{-1}.
\] (7.9)

One can check that Assumption 6.2 holds in this case. Thus, applying Corollary 6.6 and recalling that \( |\tau| = -1 - 2\kappa \), one obtains the bounds (7.3).

**The symbol \( \tau = \Psi \mathcal{I}'(\Psi) \).** Using the definition of the renormalized model and Proposition 4.2, we obtain for the symbol \( \tau \) the following identity:

\[
\iota_\varepsilon(\hat{\Pi}^\varepsilon \tau)(\varphi_2^\lambda) = \begin{array}{c}
\text{Diagram 1}\n\end{array} + \begin{array}{c}
\text{Diagram 2}\n\end{array}.
\] (7.10)

The last diagram doesn’t satisfy Assumption 6.2 and one needs to write it in a different way. More precisely, we notice that due to symmetry of the kernels the following identity holds:

\[
\begin{array}{c}
\text{Diagram 3}\n\end{array} = 0.
\]

Hence, the last diagram in (7.10) can be written as

\[
\begin{array}{c}
\text{Diagram 4}\n\end{array} = \begin{array}{c}
\text{Diagram 5}\n\end{array} + \begin{array}{c}
\text{Diagram 6}\n\end{array},
\]

and now Assumption 6.2 is satisfied. Corollary 6.6 yields the bounds (7.3) with \( |\tau| = -2\kappa \).

**The symbol \( \tau = \Psi \mathcal{I}'(\Psi^2) \).** Applying Proposition 4.2, we get the identity

\[
\iota_\varepsilon(\hat{\Pi}^\varepsilon \tau)(\varphi_2^\lambda) = \begin{array}{c}
\text{Diagram 7}\n\end{array} + \begin{array}{c}
\text{Diagram 8}\n\end{array} + \begin{array}{c}
\text{Diagram 9}\n\end{array} + \begin{array}{c}
\text{Diagram 10}\n\end{array}.
\]

At this stage, we cannot apply Corollary 6.6 to the third diagram, because the triangular loop does not satisfy Assumption 6.2.2. This problem can be resolved by renormalizing this loop. More precisely, we rewrite this diagram in the following way:

\[
\begin{array}{c}
\text{Diagram 11}\n\end{array} = \begin{array}{c}
\text{Diagram 12}\n\end{array} + \begin{array}{c}
\text{Diagram 13}\n\end{array},
\] (7.11)
where the first diagram on the right-hand side now satisfies Assumption 6.2. Next, we introduce the kernel

\[
\begin{align*}
&= \\
&= \\
&= \\
&= \
\end{align*}
\]

which satisfies the bound (6.1) with \( a = 3 \) (this follows from [Hai14, Thm. 10.7]). Thus, the second term in (7.11) can be written as

\[
\begin{align*}
&= \\
&= 
\end{align*}
\]

where the two diagrams satisfy Assumption 6.2, and the constant \( c_\epsilon \) is given by

\[
(7.12)
\]

Using bounds on kernels [Hai14, Thm. 10.7], we can replace discrete derivatives in this diagram by continuous and write

\[
c_\epsilon = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \nabla K^\varepsilon(z_1) \nabla K^\varepsilon(z_1 - z_2) \nabla K^\varepsilon(-z_2) \, dz_1 \, dz_2 + a_\epsilon 
\]

with a remainder \( a_\epsilon \to 0 \) as \( \varepsilon \to 0 \). Finally, due to antisymmetry of the kernels, the double integrals equals 0, which yields \( c_\epsilon \to 0 \) as \( \varepsilon \to 0 \). Applying Corollary 6.6 to all these diagrams separately, we conclude that the bounds (7.3) hold with \( |\tau| = -\frac{1}{2} - 3\kappa \).

The symbol \( \tau = (\mathcal{I}'(\Psi^2))^2 \). Using the definition of the renormalized model and Proposition 4.2 we obtain

\[
(7.13)
\]

All these diagrams, except the last one, satisfy Assumption 6.2. The last diagrams required renormalization, for which we define the constant

\[
C^\varepsilon_2 = 4
\]
Then the last two terms in (7.13) can be written in the following way:

\[ 4 \Psi_{2,0}(\Psi_{2,0}(\Psi_{2,0})) - C_c^e = 4 \Psi_{2,0}(\Psi_{2,0}(\Psi_{2,0})) + 4 \Psi_{2,0}(\Psi_{2,0}(\Psi_{2,0})) , \]

and these diagrams satisfy Assumption 6.2. Thus, Corollary 6.6 yields the bounds (7.3) with \( |\tau| = -\frac{1}{2} - 4\kappa \), as required.

The symbol \( \tau = \Psi' \Psi \Psi' \Psi^2 \). For the symbol \( \tau \) we use Proposition 4.2 to obtain the identity

\[ \iota_c(\hat{\Pi}_c(\varphi^2)) = \Psi_{2,0}(\Psi_{2,0}(\Psi_{2,0})) + 2 \Psi_{2,0}(\Psi_{2,0}(\Psi_{2,0})) + \Psi_{2,0}(\Psi_{2,0}(\Psi_{2,0})) + \Psi_{2,0}(\Psi_{2,0}(\Psi_{2,0})) + 2 \Psi_{2,0}(\Psi_{2,0}(\Psi_{2,0})) + 2 \Psi_{2,0}(\Psi_{2,0}(\Psi_{2,0})) - C_c^e , \]

and below we treat one by one the last four diagrams, which don’t satisfy Assumption 6.2. In the fourth to the last diagram, we separate the renormalized kernel and then treat the triangle subgraph in the same way as in (7.11):
where the constant $c_\varepsilon$ has been defined in (7.12). Now, these diagrams satisfy Assumption 6.2. The third to the last diagram in (7.14) is treated in exactly the same way. Similarly, to the second to the last diagram we obtain:

$$ z_{2,0} = z_{3,-1} + c_\varepsilon z_{2,0}, $$

which satisfy Assumption 6.2. The last diagram in (7.14) need renormalization. To this end, we define the constant $C_3^{\varepsilon}$ as

$$ C_3^{\varepsilon} = 2 z_{2,0} - C_3^{\varepsilon}. $$

Then the renormalized last diagram in (7.14) can be written as

$$ 2 z_{2,0} - C_3^{\varepsilon} = 2 z_{2,0} + 2, $$

and all these diagrams satisfy Assumption 6.2. Applying Corollary 6.6 to all the diagrams described above, we obtain the bounds (7.3) with $|\tau| = -4\kappa$.

### Appendix A Properties of càdlàg martingales

Following [JS03, Ch. I.4], we list those properties of martingales which are used in the article. Let $(M_t)_{t\geq 0}$ and $(N_t)_{t\geq 0}$ be two càdlàg square-integrable martingales on the same filtered probability space. Their predictable quadratic covariation $\langle M, N \rangle_t$ is the unique adapted process with bounded total variation, such that $M_tN_t - \langle M, N \rangle_t$ is a martingale. The quadratic covariation $[M, N]_t$ is defined by

$$ [M, N]_t := M_tN_t - M_0N_0 - \int_0^t M_s - dN_s - \int_0^t N_s - dM_s, \quad (A.1) $$
\[ M_s^- := \lim_{r \uparrow s} M_r \]

is the left limit of \( M \) at time \( s \). Another way to define these quadratic covariations is the following: if \( 0 = t_0 \leq \cdots \leq t_n = t \) is a partition with diameter \( \max_i (t_{i+1} - t_i) \) tending to zero as \( n \to \infty \), then \([M, N]_t\)

is equal to the limit in probability of the sums \( \sum_{i=0}^{n-1} (M_{t_{i+1}} - M_{t_i})(N_{t_{i+1}} - N_{t_i}) \) as \( n \to \infty \) (see \([JS03, Thm. I.4.47]\)), and \( \langle M, N \rangle_t \)

is the probability limit of the sums \( \sum_{i=0}^{n-1} \mathbb{E}[(M_{t_{i+1}} - M_{t_i})(N_{t_{i+1}} - N_{t_i}) | \mathcal{F}_{t_i}] \), where \( \mathcal{F}_{t_i} \) is the underlying filtration \([JS03, Prop. I.4.50]\).

The difference \([M, N]_t - \langle M, N \rangle_t\)

of the two covariation processes is always a martingale \([JS03, Thm. I.4.52]\), meaning that \( \langle M, N \rangle_t \)

is a compensator of \([M, N]_t\). In particular, if the martingales are continuous then the two covariation processes coincide. Let \( \Delta_s M := M_s - M_s^- \) be the jump of \( M \) at time \( s \). If the martingales are of bounded total variation, then one has the identity \([M, N]_t = \sum_{0 < s \leq t} (\Delta_s M)(\Delta_s N)\), where the sum runs over the jump times on the interval \((0, t]\), see \([JS03, Thm. I.4.52]\).

Moment bounds for martingales are usually proved using the Burkholder-Davis-Gundy inequality, which can be found for example in \([Pro05, Ch. IV.4.48]\).

**Lemma A.1** Let \( M \) be a càdlàg square-integrable martingale. Then for every \( p \geq 1 \) there exists a constant \( C_p \), such that for every \( t \geq 0 \) one has

\[
\mathbb{E} \left[ \sup_{s \in [0,t]} |M_s - M_0|^p \right] \leq C_p \mathbb{E} \left[ [M,M]_t^{p/2} \right].
\] (A.2)

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