Irreducible antifield BRST-anti-BRST formalism for reducible gauge theories

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Abstract
In this paper we develop an irreducible antifield BRST-anti-BRST formalism for reducible gauge theories.
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1 Introduction

The most powerful manifestly covariant quantization method for gauge theories was proved to be the antifield BRST formalism [1]–[3]. An other approach of the same kind is represented by the antifield BRST-anti-BRST formalism. The BRST-anti-BRST method was differently implemented at the Hamiltonian [6]–[10] and Lagrangian [10]–[23] levels. Although it does not play such an important role like the BRST symmetry itself, the BRST-anti-BRST formulation is a helpful background for the geometrical (superfield) description of the BRST transformation, the investigation of the perturbative renormalizability of the Yang-Mills models, a consistent approach of anomalies, as well as for the correct understanding of the non-minimal sector involved with the BRST quantization [24]–[30]. The antifield BRST-anti-BRST symmetry can

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be implemented in the context of irreducible, as well as of reducible gauge
theories. However, in the reducible case the BRST-anti-BRST ghost and
antifield spectra are more involved precisely due to the fact that the gauge
generators are no longer independent. In view of this, the derivation of the
solution to the master equation corresponding to the reducible case is more
difficult than in the irreducible one.

This paper investigates the possibility of quantizing reducible gauge the-
ories by employing the BRST-anti-BRST prescriptions for irreducible sys-
tems. Our main result consists in proving that a large class of reducible
gauge systems can be covariantly quantized accordingly the irreducible antifield
BRST-anti-BRST manner. Our treatment is based on the fact that the
antifield BRST-anti-BRST symmetry for a given gauge theory exists sim-
ply provided that the corresponding antifield BRST symmetry exists. In
this light, we enforce the following steps: (i) we transform the initial re-
ducible gauge theory into an irreducible one in a manner that allows the
replacement of the BRST quantization of the reducible system by that of the
irreducible theory, and (ii) we quantize the irreducible gauge theory within
the antifield BRST-anti-BRST framework. Step (i) results in the possibil-
ity to derive an irreducible BRST symmetry associated with the reducible
theory. This makes legitimate the application of the irreducible antifield
BRST-anti-BRST machinery to the irreducible theory associated with the
original reducible system.

Our paper is organized in five sections. Section 2 realizes a brief review on
the main ingredients of the irreducible antifield BRST-anti-BRST construc-
tion. In Section 3 we derive an irreducible theory associated with the starting
reducible one and show that it is permissible from the BRST point of view
to replace the quantization of the reducible system by that of the irreducible
theory. We subsequently develop the antifield BRST-anti-BRST quantiza-
tion of the irreducible theory associated with the reducible system. Section
4 illustrates the theoretical part of the paper for the Freedman-Townsend
model and for a model with abelian three-form gauge fields. Section 5 ends
the paper with some conclusions.
2 Main ideas of the irreducible antifield BRST-anti-BRST construction

In this section we present a summary of basic elements required at the construction of the BRST-anti-BRST symmetry corresponding to an irreducible gauge theory. We start from an arbitrary action (local functional) depending on the fields $\Phi^i$

$$S^L_0[\Phi^i] = \int d^Dx L(\Phi^i(x), \partial_{\mu_1} \Phi^i(x), \cdots, \partial_{\mu_s} \cdots \partial_{\mu_s} \Phi^i(x)),$$  \hspace{1cm} (1)

which is assumed invariant under the gauge transformations (written in the De Witt condensed notations)

$$\delta_\epsilon \Phi^i = R^i_\alpha \epsilon^\alpha \hspace{0.5cm} (\Leftrightarrow \delta_\epsilon \Phi^i(x) = \int d^Dy R^i_\alpha (x, y) \epsilon^\alpha (y)).$$  \hspace{1cm} (2)

For definiteness we consider the bosonic case, but the analysis can be straightforwardly extended to fermions modulo introducing some appropriate phases. We suppose that $R^i_\alpha$ form an irreducible generating set, i.e., these functions are independent and complete. The completeness of the gauge generators induces that

$$R^j_\alpha \frac{\delta R^i_\beta}{\delta \Phi^j} - R^j_\beta \frac{\delta R^i_\alpha}{\delta \Phi^j} \approx C^{\gamma}_{\alpha \beta} R^i_\gamma,$$  \hspace{1cm} (3)

where $C^{\gamma}_{\alpha \beta}$ may involve the fields, and the weak equality ‘$\approx$’ means an equality valid when the field equations hold.

The fundamental scope of the antifield BRST-anti-BRST formalism is to construct two differentials defining an algebra of the type

$$s^2_1 = 0 = s^2_2, \ s_1 s_2 + s_2 s_1 = 0,$$  \hspace{1cm} (4)

where $s_1$ and $s_2$ are respectively called the BRST and anti-BRST operators, and must be such their cohomology at degree zero is given by the classical observables of (1) (gauge invariant functions defined on the stationary surface of field equations $\Sigma : \delta S^L_0 = 0$). From (1) and the zeroth order cohomological requirement it is clear that

$$s = s_1 + s_2,$$  \hspace{1cm} (5)

is also nilpotent, and, moreover, describes a BRST symmetry associated with a redundant description of the gauge symmetries inferred by duplicating the
gauge generators. Conversely, any such redundant BRST symmetry of (1) that splits like in (5) implies the BRST-anti-BRST algebra (4) for its separate pieces plus the isomorphism between the zero degree cohomologies of the individual parts and the classical observables of (1). These two aspects lead to the fact that one can replace (4) by the unique relation

\[ s^2 = 0, \]

provided \( s \) can be made to split as in (6). In other words, one can simply follow the standard BRST rules for first-stage reducible gauge systems in order to construct \( s \) (and therefore \( s_1 \) and \( s_2 \), once the split is shown to hold). In order to handle appropriately the two pieces of \( s \) it is necessary to introduce a bidegree that distangles between them. It is called ghost bidegree or bighost number, is denoted by \( \text{bigh} = (gh_1, gh_2) \), and is defined through

\[ \text{bigh}(s_1) = (1, 0), \text{bigh}(s_2) = (0, 1), \]

such that the resulting degree, called ghost number and denoted by \( gh \), will be \( gh = gh_1 + gh_2 \).

In order to perform a proper construction of \( s \) (and consequently of \( s_1 \) and \( s_2 \)), it is necessary to duplicate the gauge generators \( R^i_\alpha \) and to introduce the corresponding reducibility functions

\[ \left( \begin{array}{c} \delta^\alpha_{\beta} \\ -\delta^\alpha_{\beta} \end{array} \right). \]

Following the standard BRST receipt, the ghost spectrum will contain the ghosts \( \eta^\alpha_1, \eta^\alpha_2 \) and the ghosts of ghosts \( \pi^\alpha \), respectively associated with the new gauge generators \( (R^i_\alpha, \bar{R}^i_\alpha) \) and the reducibility functions (8). The bighost number and Grassmann parity (\( \epsilon \)) of the original fields and ghost spectrum variables are defined by setting

\[ \text{bigh}(\Phi^i) = (0, 0), \text{bigh}(\eta^\alpha_1) = (1, 0), \text{bigh}(\eta^\alpha_2) = (0, 1), \text{bigh}(\pi^\alpha) = (1, 1), \]

\[ \epsilon(\Phi^i) = 0, \epsilon(\eta^\alpha_1) = \epsilon(\eta^\alpha_2) = 1, \epsilon(\pi^\alpha) = 0, \]

such that \( gh(\eta^\alpha_1) = gh(\eta^\alpha_2) = 1 \) and \( gh(\pi^\alpha) = 2 \). From now on we include the ghost bidegree of an object by means of an additional superscript such that if \( F \) has \( \text{bigh}(F) = (a, b) \), we simply write \( ^{a,b} F \). For further purpose, we
generically call the original fields, ghosts and ghosts of ghosts by ‘fields’, and
denote them like
\[
\Phi^A = \begin{pmatrix}
\Phi^i, \\ \eta_1^\alpha, \\ \eta_2^\alpha, \\ \pi^\alpha
\end{pmatrix}.
\] (11)

The longitudinal exterior derivative associated with the redundant descrip-
tion of the gauge symmetries, \( D \), is defined on the stationary surface \( \Sigma \) accordingly the usual BRST prescriptions as
\[
D \ (0,0) \Phi^i = R^i_\alpha \left( \eta_1^\alpha + \eta_2^\alpha \right),
\] (12)
\[
D \ (1,0) \eta_1^\alpha = \pi^\alpha + \frac{1}{2} C^\alpha_{\beta \gamma} \eta_1^\beta \left( \eta_1^\gamma + \eta_2^\gamma \right),
\] (13)
\[
D \ (0,1) \eta_2^\alpha = -\pi^\alpha + \frac{1}{2} C^\alpha_{\beta \gamma} \eta_2^\beta \left( \eta_1^\gamma + \eta_2^\gamma \right),
\] (14)
\[
D \ (1,1) \pi^\alpha = \frac{1}{2} C^\alpha_{\beta \gamma} \pi^\beta \left( \eta_1^\gamma + \eta_2^\gamma \right),
\] (15)
such that \( D^2 \approx 0 \). Thus, the total differential \( D \) splits as \( D = D_1 + D_2 \), with \( \text{bigh}(D_1) = (1,0) \) and \( \text{bigh}(D_2) = (0,1) \). The action of \( D_1 \) and \( D_2 \) can be immediately inferred from (12–15) by identifying the components accordingly the ghost bidegree. The weak nilpotency of \( D \) yields \( D_1^2 \approx 0 \approx D_2^2 \) and \( D_1D_2 + D_2D_1 \approx 0 \).

It is well-known that in the standard BRST formalism there is one antibracket with ghost number one. Here it is necessary to construct two antibrackets in order to make the antibracket structure compatible with the ghost bidegrees of the ghost spectrum while preserving the symmetry between the two degrees \( gh_1 \) and \( gh_2 \). We denote the two antibrackets by \( (,)_1 \) and \( (,)_2 \) requiring that they possess the biggest host number \( (1,0) \), respectively, \( (0,1) \), and introduce a pair of antifields \( (\Phi^*_A(1), \Phi^*_A(2)) \) respectively conjugated to a field \( \Phi^A \) in the first and second antibracket. The characteristics of the antifields are as follows
\[
\text{bigh} \left( \Phi^*_A(1) \right) = \left( -gh_1(\Phi^A) - 1, -gh_2(\Phi^A) \right),
\] (16)
\[ bigh(\Phi_A^{*(2)}) = (-gh_1(\Phi^A), -gh_2(\Phi^A) - 1), \quad (17) \]
\[ \epsilon(\Phi_A^{*(1)}) = \epsilon(\Phi_A^{*(2)}) = \epsilon(\Phi^A) + 1 \mod 2, \quad (18) \]

and the fundamental antibrackets are defined by
\[ (\Phi^A, \Phi_B^{*(1)})_1 = (\Phi^A, \Phi_B^{*(2)})_2 = \delta^A_B, (\Phi^A, \Phi_B^{*(1)})_2 = (\Phi^A, \Phi_B^{*(2)})_1 = 0. \quad (19) \]

However, there appear two problems with the definition of the Koszul-Tate operator along the standard line of the antifield procedure, namely, it fails to be nilpotent and there are non-trivial co-cycles at positive resolution degrees. These matters are due in principle to the fact that at the Lagrangian level the duplication of the gauge symmetries is not accompanied by a duplication of the field equations defining the stationary surface. Therefore, the complete reducible description of gauge orbits does not define a complete redundant description of the stationary surface. These points prevent the direct construction of a Koszul-Tate differential \( \delta \) that splits into two pieces generating a biresolution of \( C^\infty(\Sigma) \) in the usual manner. The difficulties mentioned above can be surpassed by adding further variables \( \Phi_A \) called ‘bar variables’, with the features
\[ bigh(\Phi_A) = (-gh_1\Phi^A - 1, -gh_2\Phi^A - 1), \quad \epsilon(\Phi_A) = \epsilon(\Phi^A), \quad (20) \]

and by modifying the action of \( \delta \) as the sum between a canonical (BRST-like component) \( \delta_{\text{can}} \) and a non-canonical part expressed via an operator \( V \) that acts (only on the bar variables) through
\[ V = (\Phi_A^{*(2)} - \Phi_A^{*(1)}) \frac{\delta}{\delta \Phi_A}, \quad (21) \]

such that
\[ \delta = \delta_{\text{can}} + V, \quad (22) \]

with \( gh(\delta) = gh(\delta_{\text{can}}) = gh(V) = 1. \) Putting together the generators \( \Phi_A^{*(1)}, \Phi_A^{*(2)} \) and \( \Phi_A \) of \( \delta \), we find the BRST-anti-BRST antifield spectrum for an original irreducible gauge theory under the form
\[ \Phi_A^{*(1)} = \begin{pmatrix}
(-1,0)^{(1)} & (-2,0)^{(1)} & (-1,-1)^{(12)} & (-2,-1)^{(1)} \\
\Phi^i & \eta^\alpha & \eta^\alpha & \pi^\alpha
\end{pmatrix}, \quad (23) \]
\[ \Phi^{(2)}_A = \begin{pmatrix} (0,-1)^{\ast (2)} & (-1,-1)^{\ast (21)} & (0,-2)^{\ast (22)} & (-1,-2)^{\ast (2)} \\ \Phi^i & \eta^-_\alpha & \eta^-_\alpha & \pi^-_\alpha \end{pmatrix}, \]  

(24)

\[ \bar{\Phi}_A = \begin{pmatrix} (-1,-1) & (-2,-1) & (-1,-2) & (-2,-2) \\ \bar{\Phi}^i & \bar{\eta}^-_{1\alpha} & \bar{\eta}^-_{2\alpha} & \bar{\pi}^-_\alpha \end{pmatrix}. \]  

(25)

The correct actions of the Koszul-Tate operator \( \delta \) associated with the redundant description of the gauge symmetries read as

\[ \delta \Phi^A = 0, \]  

(26)

\[ \delta (-1,0)^{\ast (1)} \Phi^i = \delta (0,-1)^{\ast (2)} \Phi^i = -\delta S_L^{\Phi^i}, \]  

(27)

\[ \delta (-2,0)^{\ast (11)} \eta^-_\alpha = \delta (-1,-1)^{\ast (21)} \eta^-_\alpha = \Phi^i R^i_\alpha, \]  

(28)

\[ \delta (-1,-1)^{\ast (12)} \eta^-_\alpha = \delta (0,-2)^{\ast (22)} \eta^-_\alpha = \Phi^i R^i_\alpha, \]  

(29)

\[ \delta (-2,-1)^{\ast (1)} \eta^-_\alpha = \delta (-1,-2)^{\ast (2)} \eta^-_\alpha = -\left( (0,-1)^{\ast (22)} \eta^-_\alpha + (0,-1)^{\ast (2)} \eta^-_\alpha + (0,-1)^{\ast (21)} \Phi^i R^i_\alpha \right), \]  

(30)

\[ \delta \bar{\Phi}_A = \Phi^{(2)}_A - \bar{\Phi}^{(1)}_A. \]  

(31)

Under these considerations, the total Koszul-Tate differential splits as \( \delta = \delta_1 + \delta_2 \), with \( \text{bigh}(\delta_1) = (1, 0), \text{bigh}(\delta_2) = (0, 1) \), the concrete actions of the two Koszul-Tate components being deduced on account of identifying the expressions from (26–31) with respect to the resolution bidegree (also called antighost bidegree or biantighost number). The prior definitions restore both the nilpotency and the acyclicity of \( \delta \) at positive resolution degrees, and, moreover, induce the required relations

\[ \delta_1^2 = 0 = \delta_2^2, \delta_1 \delta_2 + \delta_2 \delta_1 = 0. \]  

(32)

In addition, it is easy to check that while \( \delta \) realizes a resolution of the algebra \( C^\infty(\Sigma) \), \( (\delta_1, \delta_2) \) perform a biresolution of the same algebra. The resolution bidegree \( \text{bires} = (\text{res}_1, \text{res}_2) \) of the BRST and anti-BRST Koszul-Tate components are expressed as expected by \( \text{bires}(\delta_1) = (-1, 0), \text{bires}(\delta_2) = (0, -1) \), while the resolution degree of an object with the resolution bidegree
\((res_1, res_2)\) is obtained via \(res = res_1 + res_2\). The resolution bidegrees of all the ‘fields’ are \((0, 0)\), and those of the antifields \((23, 25)\) can be computed by \(\text{bires} \left( \text{antifield} \right) = (-gh_1, -gh_2)\). Now, it is easy to see that \(\delta_1\) and \(\delta_2\) can be decomposed like \(\delta\) in \((22)\), i.e.,

\[
\delta_1 = \delta\text{can} + V_1, \quad \delta_2 = \delta\text{can} + V_2,
\]

with \(\text{bires}(\delta\text{can}) = (−1, 0) = \text{bires}(V_1), \text{bires}(\delta\text{can}) = (0, −1) = \text{bires}(V_2)\), where

\[
V_1 = \Phi^*_{iA} \frac{\delta}{\delta \Phi_A}, \quad V_2 = -\Phi^*_{iA} \frac{\delta}{\delta \Phi_A}.
\]

In the standard antifield approach the implementation of the BRST symmetry is accomplished by means of a ghost number zero bosonic anticanonical generator that is the solution of the master equation and leads to the effective action. In the antifield BRST-anti-BRST approach there is also a single bosonic generator \(S\) of ghost bidegree \((0, 0)\) (and thus of ghost number zero) that implements this symmetry through

\[
sF = (F, S) + VF,
\]

for any \(F\) depending on the ‘fields’ and antifields, where the antibracket \((,\)\) is defined by

\[
(,) = (,)_1 + (,)_2 .
\]

The nilpotency of \(s\) implies at the antibracket level the classical master equation of the BRST-anti-BRST formalism

\[
\frac{1}{2} (S, S) + VS = 0,
\]

whose generator is subject to the boundary conditions

\[
S = S_0^L, \quad S = \Phi^*_{iA} \frac{\partial}{\partial \Phi_A} \eta^\alpha_1 + \Phi^*_{iA} \frac{\partial}{\partial \Phi_A} \eta^\alpha_2 ,
\]

\[
S = \begin{pmatrix}
(0, -1)^*(21) & (0, -1)^*(12) & (0, 1)^*(1, 1) \\
\eta^\alpha_1 - \eta^\alpha_2 & \Phi^*_{iA} R^i_\alpha & \pi^\alpha_1 + \text{“more”}
\end{pmatrix},
\]

and to the properties \(\epsilon(S) = 0, \text{bigh}(S) = (0, 0)\). Actually, it was shown \([10]\) that the solution to the master equation \((37)\) satisfying the required
conditions exists, which further yields that we can decompose \( s \) precisely into two pieces \( s_1 \) and \( s_2 \) of ghosts bidegrees \( (7) \) that act like

\[
s_a F = (F, S)_a + V_a F, \quad a = 1, 2,
\]

where \((.,.)_a\) and \(V_a\) are defined by \((19)\) and \((34)\). The master equation \((37)\) is in fact equivalent with two equations, namely,

\[
\frac{1}{2} (S, S)_1 + V_1 S = 0, \quad \frac{1}{2} (S, S)_2 + V_2 S = 0.
\]

However, the results obtained are not entirely convenient as the generator of the Lagrangian BRST-anti-BRST symmetry is neither BRST, nor anti-BRST invariant. This matter will be solved within the gauge-fixing process.

In order to appropriately fix the gauge, it is necessary to forget for the moment about the second antibracket, and to add some new fields in the theory, denoted by \( \mu^A_{(1)} \) and \( \rho^A_{(1)} \), which are respectively conjugated in the first antibracket with \( \Phi_A \) and \( \Phi^*_A \)

\[
\begin{align*}
(\mu^A_{(1)}, \Phi_B) &= \delta^A_B, \\
(\Phi^*_A, \rho^B_{(1)}) &= \delta^B_A.
\end{align*}
\]

The properties of the new variables read as

\[
\begin{align*}
\text{bigh} \left( \mu^A_{(1)} \right) &= (-g h_1 \Phi_A - 1, -g h_2 \Phi_A), \quad \epsilon \left( \mu^A_{(1)} \right) = \epsilon \left( \Phi_A \right) + 1 \text{ mod } 2, \\
\text{bigh} \left( \rho^A_{(1)} \right) &= (-g h_1 \Phi^*_A - 1, -g h_2 \Phi^*_A), \quad \epsilon \left( \rho^A_{(1)} \right) = \epsilon \left( \Phi^*_A \right) + 1 \text{ mod } 2.
\end{align*}
\]

The generator of the BRST-anti-BRST symmetry is extended on the new variables by means of the relation

\[
S_1 = S + \Phi^*_A \mu^A_{(1)},
\]

such that the master equation \((37)\) will be equivalent to \((S_1, S_1)_1 = 0\), which has the well-known form of the master equation from the standard antifield BRST method. Applying now the usual BRST gauge-fixing process, namely, choosing a gauge-fixing fermion that involves only the variables playing role of fields \( \psi = \psi \left[ \Phi^A, \mu^A_{(1)}, \Phi^*_A \right] \), we can eliminate the variables playing role
of antifields from the theory and find as usually the corresponding gauge-fixed action, $S_{1\psi}$. An alternative way of fixing the gauge is to take a bosonic functional $F$ that depends only on the ‘fields’ $\Phi^A$ through

$$\psi = \mu^A_{(1)} \frac{\delta F}{\delta \Phi^A},$$

(47)

which yields

$$\Phi^A = \frac{\delta F}{\delta \Phi^A}, \quad \Phi^{* (1)} = \mu^A_{(1)} \frac{\delta^2 F}{\delta \Phi^B \delta \Phi^A}, \quad \rho^A_{(1)} = 0.$$

(48)

Introducing some Lagrange multipliers $\mu^A_{(2)}$ and $\lambda^A$ that implement the gauge conditions (48), we obtain that the usual path integral

$$Z_\psi = \int D\Phi^A D\mu^A_{(1)} D\Phi^{* (2)} A \exp iS_{1\psi},$$

(49)

becomes

$$Z_F = \int D\Phi^A D\mu^A_{(1)} D\Phi^{* (1)} A D\mu^A_{(2)} D\Phi^{* (2)} A D\lambda^A D\bar{\Phi}^A \exp iS_F,$$

(50)

with the effective action $S_F$ given by

$$S_F = S + \Phi^{* (2)} A \mu^A_{(1)} - \Phi^{* (1)} A \mu^A_{(2)} + \mu^A_{(1)} \frac{\delta^2 F}{\delta \Phi^B \delta \Phi^A} \mu^B_{(2)} + \left( \Phi^A - \frac{\delta F}{\delta \Phi^A} \right) \lambda^A.$$

(51)

We remark that if one integrates in (50) over the multipliers $(\mu^A_{(2)}, \lambda^A)$, and also over $(\Phi^{* (1)} A, \bar{\Phi}^A)$, one re-obtains (49) for the choice (47) of $\psi$. One can show by direct computation that the effective action is both BRST and anti-BRST invariant.

### 3 Irreducible antifield BRST-anti-BRST quantization of reducible gauge theories

The basic purpose of this section is to show how reducible gauge theories can be quantized along the irreducible antifield BRST-anti-BRST formalism. This task can be accomplished by: (i) building an irreducible theory
equivalent to the original reducible one in a way that allows us to substitute the BRST quantization of the redundant system by that of the irreducible one, and (ii) quantizing the resulting irreducible system within the BRST-anti-BRST framework. The legitimacy of (ii) is implied by (i) and also by the fact that the BRST-anti-BRST symmetry for a given theory exists provided the standard BRST symmetry for that theory can be enforced.

### 3.1 Irreducible theories associated with reducible ones

Our starting point is a gauge invariant Lagrangian action

\[ S_0[\Phi^\alpha_0] = \int d^Dx L_0(\Phi^\alpha_0, \partial_\mu \Phi^\alpha_0, \cdots, \partial_\mu \cdots \partial_\mu \Phi^\alpha_0), \]  

subject to the gauge transformations

\[ \delta_\epsilon \Phi^\alpha_0 = Z^\alpha_0_{\alpha_1} \epsilon^{\alpha_1}, \quad \alpha_0 = 1, \cdots, M_0, \quad \alpha_1 = 1, \cdots, M_1, \]

which are assumed to be \( L \)-stage reducible

\[ Z^\alpha_0_{\alpha_1} Z^\alpha_1_{\alpha_2} = C^\alpha_0_{\alpha_2} \frac{\delta S_0}{\delta \Phi^\beta_0}, \quad \alpha_2 = 1, \cdots, M_2, \]

\[ Z^\alpha_1_{\alpha_2} Z^\alpha_2_{\alpha_3} = C^\alpha_1_{\alpha_3} \frac{\delta S_0}{\delta \Phi^\beta_0}, \quad \alpha_3 = 1, \cdots, M_3, \]

\[ \vdots \]

\[ Z^\alpha_{L-2} \alpha_{L-1} Z^\alpha_{L-1} \alpha_L = C^\alpha_{\alpha_L} \frac{\delta S_0}{\delta \Phi^\beta_0}, \quad \alpha_L = 1, \cdots, M_L, \]

\[ Z^\alpha_{L-1} \alpha_L Z^\alpha_{L+1} = C^\alpha_{\alpha_{L+1}} \frac{\delta S_0}{\delta \Phi^\beta_0}, \quad \alpha_{L+1} = 1, \cdots, M_{L+1}, \]

where \( L \) is supposed finite. For the sake of notational simplicity we take the original fields to be bosonic, but the analysis can be extended to fermions modulo adding some appropriate sign factors. It is understood that the functions \( Z^\alpha_{\alpha_0} \) and \( (Z^\alpha_{\alpha_{k+1}})_{k=1,\cdots,L} \) form a complete set of gauge generators, respectively, reducibility functions.

Initially, we construct an irreducible theory associated with the reducible one. In this respect, from (54-57) we remark that

\[ \text{rank}(Z^\alpha_{\alpha_k}) \approx \sum_{i=k}^{L+1} (-1)^{k+i} M_i, \quad k = 1, \cdots, L + 1, \]
where the weak equality ‘≈’ means an equality valid on the stationary surface of field equations \( \frac{\delta S_0}{\delta \Phi^{\alpha_0}} = 0 \). Let \( \left( A^\beta_{\beta_{k-1}} \right)_{k=1, \cdots, L+1} \) be some matrices that may involve the fields \( \Phi^{\alpha_0} \), taken such that

\[
\text{rank} \left( D^\beta_{\alpha_k} \right) \approx \sum_{i=k}^{L+1} (-)^{k+i} M_i, \ k = 1, \cdots, L + 1,
\]

where

\[
D^\beta_{\alpha_k} = A^\beta_{\alpha_{k-1}} Z^{\alpha_{k-1}}, \ k = 1, \cdots, L + 1.
\]

In particular we can take \( A^\beta_{\beta_{k-1}} = \left( Z^{\beta_{k-1}} \right)^T \), with \( \left( Z^{\beta_{k-1}} \right)^T \) the transposed of \( Z^{\beta_{k-1}} \). From (60) it follows directly that

\[
D^\beta_{\alpha_k} Z^{\alpha_{k+1}} \approx 0, \ k = 1, \cdots, L.
\]

Relations (61) allow us to represent \( D^\beta_{\alpha_k} \) under the form

\[
D^\beta_{\alpha_k} = \delta^\beta_{\alpha_k} - Z^\beta_{\alpha_k + 1} A_{\alpha_k}^{\alpha_{k+1}}, \ k = 1, \cdots, L + 1.
\]

Throughout the paper we work with the conventions

\[
f^{\alpha_k} = 0 \text{ if } k < 0 \text{ or } k > L + 1.
\]

It is easy to see that (62) satisfy (61). With these observations at hand, we pass to the concrete construction of an irreducible theory associated with the starting reducible one. In view of this, we add the fields \( \Phi^{\alpha_2k}, k = 1, \ldots, a \), and the gauge parameters \( \epsilon^{\alpha_2k+1}, k = 1, \ldots, b \), corresponding to every reducibility relation (54–57) with even, respectively, odd free indices, where

\[
a = \begin{cases} \frac{L}{2}, & \text{for } L \text{ even}, \\ \frac{L+1}{2}, & \text{for } L \text{ odd}, \end{cases} \quad b = \begin{cases} \frac{L}{2}, & \text{for } L \text{ even}, \\ \frac{L+1}{2}, & \text{for } L \text{ odd}. \end{cases}
\]

Under these considerations, we associate the theory described by the action

\[
S_0 \left[ \Phi^{\alpha_0}, \Phi^{\alpha_2k} \right] = S_0 \left[ \Phi^{\alpha_0} \right],
\]

and subject to the gauge transformations

\[
\delta_\epsilon \Phi^{\alpha_0} = Z^{\alpha_0}_{\alpha_1} \epsilon^{\alpha_1},
\]
\[ \delta \Phi^{\alpha_2} = A_{\alpha_1}^{\alpha_2} \epsilon^{\alpha_1} + Z_{\alpha_3}^{\alpha_2} \epsilon^{\alpha_3} , \]  

\[ \vdots \]

\[ \delta \epsilon \Phi^{\alpha_{2k}} = A_{\alpha_{2k-1}}^{\alpha_{2k}} \epsilon^{\alpha_{2k-1}} + Z_{\alpha_{2k+1}}^{\alpha_{2k}} \epsilon^{\alpha_{2k+1}} , \]  

\[ \vdots \]

\[ \delta \epsilon \Phi^{\alpha_{2a}} = \begin{cases} 
A_{\alpha_{L-1}}^{\alpha_{L}} \epsilon^{\alpha_{L-1}} + Z_{\alpha_{L+1}}^{\alpha_{L}} \epsilon^{\alpha_{L+1}}, & \text{for } L \text{ even}, \\
A_{\alpha_{L+1}}^{\alpha_{L+1}} \epsilon^{\alpha_{L}}, & \text{for } L \text{ odd}, 
\end{cases} \]  

with the starting reducible system. In (67–69) \( A_{\alpha_{2k}} \) are some matrices that satisfy (59). It is obvious that the transformations (66–69) leave the action (55) invariant. From (53) we observe that the weak equality associated with the new system coincides with that corresponding to the original theory because the field equations of the supplementary fields are trivial.

Now, it is easy to show that the gauge transformations (66–69) are irreducible. If we take

\[ \epsilon^{\alpha_{2k+1}} = Z_{\alpha_{2k+2}}^{\alpha_{2k+1}} \theta^{\alpha_{2k+2}} , \]  

with arbitrary \( \theta \)'s, the transformations (60–69) become

\[ \delta \theta \Phi^{\alpha_0} \approx 0 , \]  

\[ \delta \theta \Phi^{\alpha_2} \approx D_{\beta_2}^{\alpha_2} \theta_{\beta_2} , \]  

\[ \vdots \]

\[ \delta \theta \Phi^{\alpha_{2k}} \approx D_{\beta_{2k}}^{\alpha_{2k}} \theta_{\beta_{2k}} , \]  

\[ \vdots \]

\[ \delta \theta \Phi^{\alpha_{2a}} \approx \begin{cases} 
D_{\alpha_{L}}^{\alpha_{L}} \theta_{\beta_{L}}, & \text{for } L \text{ even}, \\
D_{\alpha_{L+1}}^{\alpha_{L+1}} \theta_{\beta_{L+1}}, & \text{for } L \text{ odd}, 
\end{cases} \]  

Using (61) and the completeness of the reducibility functions, it follows that

\[ \delta \theta \Phi^{\alpha_{2k}} \approx 0, \ k = 0, \ldots, a, \]  

if and only if

\[ \theta_{\beta_{2k}} \approx Z_{\beta_{2k+1}}^{\beta_{2k}} \lambda_{\beta_{2k+1}}, \ k = 1, \ldots, a, \]  

for some arbitrary functions \( \lambda_{\beta_{2k+1}} \). Inserting (75) in (70), it results that

\[ \delta \epsilon \Phi^{\alpha_{2k}} \approx 0 \iff \epsilon^{\alpha_{2k+1}} \approx 0 , \]  

13
so the gauge transformations with the parameters (70) are trivial. This establishes the irreducibility of the gauge transformations (66–69). The prior construction of the irreducible gauge transformations does not guarantee their completeness. In fact, it is impossible to prove in general the completeness of the irreducible gauge transformations (66–69) as it depends on the choice of the matrices $A_{\alpha k-1}^{\alpha k}$ and also on the original reducibility matrices. This is why in the sequel we assume the completeness of (66–69). This feature is required by the possibility to construct a weakly nilpotent longitudinal exterior derivative along the gauge orbits connected with the irreducible theory.

At this point we investigate whether it is legitimate or not to replace the BRST quantization of the reducible theory by that of the irreducible system constructed previously. Initially, we construct the BRST symmetry associated with the irreducible system. First, we derive the Koszul-Tate differential in a way that ensures its acyclicity. The minimal antifield spectrum includes the fermionic antifields

$$\left( \Phi^*_{\alpha 0}, \Phi^*_{\alpha 2k} \right), \quad k = 1, \ldots, a, \quad (77)$$

with antighost number one, and the bosonic antifields

$$\eta^*_{\alpha 2(k+1)}, \quad k = 0, \ldots, b, \quad (78)$$

with antighost number two. The irreducible Koszul-Tate operator, $\delta$, acts on its generators through

$$\delta \Phi_{\alpha 0} = 0, \quad \delta \Phi_{\alpha 2k} = 0, \quad k = 1, \ldots, a, \quad (79)$$

$$\delta \Phi^*_{\alpha 0} = -\frac{\delta S_0}{\delta \Phi_{\alpha 0}}, \quad (80)$$

$$\delta \Phi^*_{\alpha 2k} = -\frac{\delta S_0}{\delta \Phi_{\alpha 2k}} \equiv 0, \quad k = 1, \ldots, a, \quad (81)$$

$$\delta \eta^*_{\alpha 2(k+1)} = \Phi^*_{\alpha 2k} Z^\alpha_{\alpha 2k+1} + \Phi^*_{\alpha 2k+2} A^\alpha_{\alpha 2k+1}, \quad k = 0, \ldots, b, \quad (82)$$

such that $\delta$ is clearly nilpotent, $\delta^2 = 0$. Relations (80–81) give rise to the co-cycles $\Phi^*_{\alpha 0} Z_{\alpha 1}$ and $\Phi^*_{\alpha 2k}$, with $k = 1, \ldots, a$, while (82) lead to some combinations of these co-cycles which are $\delta$-exact. However, this does not ensure the separate $\delta$-exactness of $\Phi^*_{\alpha 0} Z_{\alpha 1}$ and $\Phi^*_{\alpha 2k}$. Let us prove that these co-cycles are indeed $\delta$-exact. For definiteness we expose the case $L$ even, the other
situation being solved in the same fashion. Multiplying \((82)\) for \(k = L/2 - 1\) by \(Z^{\alpha_{L-1}}_{\beta L}\) and using \((62)\) we find

\[
\Phi^*_{\beta L} = \delta \gamma_{\beta L},
\]  

(83)

with

\[
\gamma_{\beta L} = \left(\eta^*_{\alpha_{L-1}}Z^{\alpha_{L-1}}_{\beta L} + \alpha_2 \Phi^*_{\alpha L-2} \Phi^*_{\alpha_0} C^{\alpha L-2\alpha_0}_{\beta L} + \eta^*_{\alpha_{L+1}} A^{\alpha_{L+1}}_{\beta L}\right),
\]  

(84)

where \(\alpha_2 = 1/2\) for \(L = 2\), and \(\alpha_2 = 1\) otherwise. Then, \((82)\) for \(k = L/2 - 1\) takes the form

\[
\delta \left(\eta^*_{\alpha_{L-1}} - A^{\beta L}_{\alpha_{L-1}} \gamma_{\beta L}\right) = \Phi^*_{\alpha_{L-2}} Z^{\alpha_{L-2}}_{\beta_{L-2}}.
\]  

(85)

Now, we multiply the relation corresponding to \((82)\) for \(k = L/2 - 2\) by \(Z^{\alpha_{L-3}}_{\beta_{L-2}}\), and consequently obtain

\[
\delta \left(\eta^*_{\alpha_{L-3}} Z^{\alpha_{L-3}}_{\beta_{L-2}} + \alpha_4 \Phi^*_{\alpha L-4} \Phi^*_{\alpha_0} C^{\alpha L-4\alpha_0}_{\beta_{L-2}}\right) = \Phi^*_{\alpha_{L-2}} D^{\alpha_{L-2}}_{\beta_{L-2}},
\]  

(86)

where \(\alpha_4 = 1/2\) for \(L = 4\), and \(\alpha_4 = 1\) otherwise. Replacing \((82)\) in \((86)\) and employing \((85)\) it results that \(\Phi^*_{\beta L-2}\) is also \(\delta\)-exact. Reprising the same procedure for each level we infer

\[
\Phi^*_{\alpha_0} Z^{\alpha_0}_{\alpha_{1}} = \delta \gamma_{\alpha_1}, \quad \Phi^*_{\beta_{2k}} = \delta \gamma_{\beta_{2k}}, \quad k = 1, \ldots, a,
\]  

(87)

with

\[
\gamma_{\beta_{2k}} = \left(Z^{\beta_{2k-1}}_{\beta_{2k}} \eta^*_{\beta_{2k-1}} + \alpha_{L-2k+2} C^{\beta_{2k-2\alpha_0}}_{\beta_{2k}} \Phi^*_{\beta_{2k-2}} \Phi^*_{\alpha_0} + A^{\beta_{2k+1}}_{\beta_{2k}} \left(\eta^*_{\beta_{2k+1}} - A^{\beta_{2k+2}}_{\beta_{2k+1}} \gamma_{\beta_{2k+2}}\right)\right),
\]  

(88)

\[
\gamma_{\alpha_{1}} = \eta^*_{\alpha_{1}} - A^{\alpha_2}_{\alpha_1} \gamma_{\alpha_{2}},
\]  

(89)

and \(\alpha_{L-2k+2} = 1\) for \(L \neq 2k - 2\), respectively, \(\alpha_{L-2k+2} = 1/2\) for \(L = 2k - 2\). The last relations enforce the triviality of the above mentioned co-cycles at antighost number one. Moreover, there are no non trivial cocycles at resolution degrees greater than one due to the irreducibility of the gauge transformations \((56-59)\), hence the irreducible Koszul-Tate differential is acyclic.

The construction of the longitudinal exterior differential along the gauge orbits, \(d\), follows the general irreducible BRST line \([5]\), the hypothesis of completeness on the irreducible gauge transformations ensuring the weak
nilpotency of $d$ without introducing any ghosts of ghosts. Under these considerations, the homological perturbation theory guarantees the existence of the irreducible BRST symmetry, $s_I$. We observe that the two theories display the same classical observables as the fields $(\Phi^{\alpha_2})_{k=1,...,a}$ are not effectively involved with the action (65) (of the irreducible system), being therefore purely gauge. Consequently, the observables of the irreducible theory do not depend on these fields and check the equations

$$\frac{\delta F}{\delta \Phi^{\alpha_0}} Z^{\alpha_0}_{\alpha_2} \approx 0,$$

which are nothing but the equations that must be verified by the observables of the reducible theory. As the observables of the irreducible and reducible theories coincide, the zeroth order cohomological groups corresponding to the irreducible and reducible formulations are also equal

$$H^0(s_I) = H^0(s_R),$$

with $s_R$ denoting the reducible BRST symmetry. Hence, the irreducible and reducible theories are equivalent from the BRST point of view, i.e., from the point of view of the basic equations describing this formalism

$$s^2 = 0, \ H^0(s) = \{\text{physical observables}\}.$$  

The last conclusion ensures that we can substitute the BRST quantization of the reducible theory by that of the irreducible system derived at the beginning of this section. Taking into account that both the existence and construction of the antifield BRST-anti-BRST symmetry are essentially based on the existence of the standard Lagrangian BRST symmetry, it follows that we can safely replace the BRST-anti-BRST quantization of the original reducible theory with that of the irreducible system. This is the concern of the next subsection.

### 3.2 Irreducible antifield BRST-anti-BRST quantization

In this subsection we perform the BRST-anti-BRST quantization of the irreducible theory constructed in the previous subsection, which is described by the action (65) and is subject to the irreducible gauge transformations (66–69). The initial field spectrum and gauge parameters are respectively
given by \((\Phi^{\alpha_{2k}})_{k=0,\ldots,a}\) and \((\epsilon^{\alpha_{2k+1}})_{k=0,\ldots,b}\), hence we can make the following analogies between \(2)\) and our system

\[
\Phi^i \leftrightarrow (\Phi^{\alpha_{2k}})_{k=0,\ldots,a}, \quad \epsilon^\alpha \leftrightarrow (\epsilon^{\alpha_{2k+1}})_{k=0,\ldots,b},
\]

(93)

where \(R^i_\alpha\) is expressed in our case by a matrix containing \((a + 1) \times (b + 1)\) blocks of elements \(Z^{\alpha_{2k}}_{\alpha_{2k+1}}\) and \(A^{\alpha_{2k}}_{\alpha_{2k+1}}\) structured accordingly (66–69). In agreement with the discussion from Section 2, the field, ghost and antifield spectra (see (11) and (23–25)) are organized as

\[
\Phi^4 = \begin{pmatrix}
(0,0)^{\alpha_{2k}}, (1,0)^{\alpha_{2k+1}}, (0,1)^{\alpha_{2k+1}}, (1,1)^{\alpha_{2k+1}}
\end{pmatrix},
\]

(94)

\[
\Phi^*_A^{(1)} = \begin{pmatrix}
(1,0)^{(1)}, (0,1)^{(11)}, (1,1)^{(12)}, (2,1)^{(1)}
\end{pmatrix},
\]

(95)

\[
\Phi^*_A^{(2)} = \begin{pmatrix}
(0,1)^{(2)}, (1,1)^{(21)}, (1,2)^{(22)}, (2,2)^{(2)}
\end{pmatrix},
\]

(96)

\[
\Phi_A = \begin{pmatrix}
(1,1), (1,2), (2,2)
\end{pmatrix},
\]

(97)

where the superscript indicates the highest number.

The definitions of \(\delta_1\) and \(\delta_2\) (see (23 31)) on the generators from the BRST-anti-BRST complex are expressed by

\[
\delta_1 (0,0)^A \Phi = \delta_2 (0,0)^A \Phi = 0,
\]

(98)

\[
\delta_1 (1,0)^{(1)} \Phi^{\alpha_{2k}} = \delta_2 (0,1)^{(2)} \Phi^{\alpha_{2k}} = -\frac{\delta S_0}{\delta \Phi^{\alpha_{2k}}},
\]

(99)

\[
\delta_1 (1,2)^{(1)} \Phi^{\alpha_{2k}} = \delta_2 (0,1)^{(2)} \Phi^{\alpha_{2k}} = -\frac{\delta S_0}{\delta \Phi^{\alpha_{2k}}}, \quad k \geq 1,
\]

(100)

\[
\delta_1 (-2,0)^{(11)} \eta^{\alpha_{2k+1}} = \delta_2 (1,1)^{(21)} \eta^{\alpha_{2k+1}} = \eta \Phi^{\alpha_{2k}} Z^{\alpha_{2k}}_{\alpha_{2k+1}} + \Phi^{\alpha_{2k}} A^{\alpha_{2k+1}}_{\alpha_{2k+2}}, \quad k \geq 0,
\]

(101)

\[
\delta_1 (-1,1)^{(12)} \eta^{\alpha_{2k+1}} = \delta_2 (0,2)^{(22)} \eta^{\alpha_{2k+1}} = \eta \Phi^{\alpha_{2k}} Z^{\alpha_{2k}}_{\alpha_{2k+1}} + \Phi^{\alpha_{2k}} A^{\alpha_{2k+1}}_{\alpha_{2k+2}}, \quad k \geq 0,
\]

(102)
As we have already discussed in the previous subsection, there appear some problems connected with the existence of some apparently non trivial co-cycles for the Koszul-Tate operator in the context of the antifield BRST formulation (see formulas \((80-81)\)). The same problem is present within the BRST-anti-BRST approach, namely, the co-cycles 
\[
\delta_1 (\Phi_{\alpha_0} Z^\alpha_{\alpha_1})_{k \geq 1}, \quad (\Phi_{\alpha_2k})_{k \geq 1}
\]

and 
\[
(\Phi_{\alpha_0} Z^\alpha_{\alpha_1}), \quad (\Phi_{\alpha_2k})_{k \geq 1}
\]

(obtained from \(\Phi_{\alpha_0}^*\)) multiplied by \(Z^\alpha_{\alpha_1}\) and \((100)\) are both \(\delta_1\)- and \(\delta_2\)-closed. However, the first two sets of co-cycles are killed in the homology of \(\delta_2\), and the same is for the other two sets, but in the homology of \(\delta_1\) (see \((103)\)). In fact, all the co-cycles \(\Phi_{\alpha_0}^*\) and \(\Phi_{\alpha_2k}\) are dropped out from the homology of \(\delta_1\), respectively, \(\delta_2\) by means of the bar variables (see \((103, 108)\). In consequence, the only dangerous co-cycles are represented by 
\[
(\Phi_{\alpha_0} Z^\alpha_{\alpha_1}), \quad (\Phi_{\alpha_2k})_{k \geq 1}
\]

in the homology of \(\delta_1\), and 
\[
(\Phi_{\alpha_0} Z^\alpha_{\alpha_1}), \quad (\Phi_{\alpha_2k})_{k \geq 1}
\]

in the homology of \(\delta_2\). In order to perform a proper construction of the Koszul-Tate bicomplex it is necessary to investigate their exactness. The proof showing the \(\delta_1\)-, respectively, \(\delta_2\)-exactness of
the above invoked co-cycles will be done below for definiteness in the case $L$ even, but the opposite situation can be solved in a similar fashion.

We start from the last two relations (101) with respect to $\delta_1$ in the hypothesis $L$ even

$$
\delta_1 \eta^{\ast(11)}_{\alpha L-1} = \Phi_{\alpha L-2} Z^{\alpha L-2}_{\alpha L-1} + \Phi_{\alpha L} A_{\alpha L-1}, \tag{109}
$$

$$
\delta_1 \eta^{\ast(11)}_{\alpha L+1} = \Phi_{\alpha L} Z^{\alpha L}_{\alpha L+1}. \tag{110}
$$

If we multiply (110) by $A^{\alpha L+1}_{\beta L}$, (109) by $Z^{\alpha L-1}_{\beta L}$ and sum the resulting relations, we find

$$
\delta_1 \left( \eta^{\ast(11)}_{\alpha L-1} Z^{\alpha L-1}_{\beta L} + \eta^{\ast(11)}_{\alpha L+1} A^{\alpha L+1}_{\beta L} \right) = \Phi^{\ast(1)}_{\alpha L-2} Z^{\alpha L-2}_{\alpha L-1} Z^{\alpha L}_{\beta L} + \Phi^{\ast(1)}_{\alpha L} A^{\alpha L+1}_{\alpha L+1}. \tag{111}
$$

Taking now into account (56), (62) for $k = L$, (99) with respect to $\delta_1$ and also (100) for $k = \frac{L}{2} - 1$ (and with respect to $\delta_1$), it follows that

$$
\Phi^{\ast(1)}_{\beta L} = \delta_1 \gamma^{\ast(1)} \beta L, \tag{112}
$$

where

$$
\gamma^{\ast(1)} \beta L = \eta^{\ast(11)}_{\alpha L-1} Z^{\alpha L-1}_{\beta L} + \eta^{\ast(11)}_{\alpha L+1} A^{\alpha L+1}_{\beta L}, \tag{113}
$$

with

$$
a_2 = \begin{cases} 1, & \text{if } L \neq 2, \\ \frac{1}{2}, & \text{if } L = 2. \end{cases} \tag{114}
$$

Inserting the relations (112) in (109), we have that

$$
\Phi^{\ast(1)}_{\alpha L-2} Z^{\alpha L-2}_{\alpha L-1} = \delta_1 \left( \eta^{\ast(11)}_{\alpha L-1} - \gamma^{\ast(1)} \alpha L A^{\alpha L}_{\alpha L-1} \right). \tag{115}
$$
Next, we pass to the definitions (101) for 
\[ k = \frac{L}{2} - 2 \] (with respect to \( \delta_1 \))
\[
\delta_1 \left( \frac{-2,0}{\eta} \right)_{L=3}^{(11)} = \Phi_{\alpha_{L-4}} Z_{\alpha_{L-3}}^{\alpha_{L-4}} + \Phi_{\alpha_{L-2}} A_{\alpha_{L-3}}^{\alpha_{L-2}},
\] (116)
and multiply these relations by \( Z_{\beta_{L-2}}^{\alpha_{L-3}} \), arriving at
\[
\delta_1 \left( \frac{-2,0}{\eta} \right)_{L=3}^{(11)} = \Phi_{\alpha_{L-4}} Z_{\alpha_{L-3}}^{\alpha_{L-4}} Z_{\beta_{L-2}}^{\alpha_{L-3}} + \Phi_{\alpha_{L-2}} A_{\alpha_{L-3}}^{\alpha_{L-2}} Z_{\beta_{L-2}}^{\alpha_{L-3}},
\] (117)
If we take into consideration the reducibility relations (54–57) for \( Z_{\alpha_{L-3}}^{\alpha_{L-4}} Z_{\beta_{L-2}}^{\alpha_{L-3}} \),
(39) with respect to \( \delta_1 \) and (62) for \( k = L - 2 \), we are led to
\[
\delta_1 \left( \frac{-2,0}{\eta} \right)_{L=3}^{(11)} + a_4 \left( \frac{-1,0}{\Phi_{\alpha_{L-4}}} \right)_{L=3}^{(1)} C_{\beta_{L-2}}^{\alpha_{L-4}} \beta_0 + \Phi_{\alpha_{L-2}} A_{\alpha_{L-3}}^{\alpha_{L-2}} Z_{\beta_{L-2}}^{\alpha_{L-3}},
\] (118)
where
\[
a_4 = \begin{cases} 
1, & \text{if } L \neq 4, \\
\frac{1}{2}, & \text{if } L = 4.
\end{cases}
\] (119)
At this moment we employ (113) and derive
\[
\left( \frac{-1,0}{\Phi_{\beta_{L-2}}} \right) = \delta_1 \left( \frac{-2,0}{\eta} \right)_{\beta_{L-2}}^{(1)},
\] (120)
with
\[
\left( \frac{-2,0}{\eta} \right)_{\beta_{L-2}}^{(11)} = \left( \frac{-2,0}{\eta} \right)_{L=3}^{(11)} Z_{\alpha_{L-3}}^{\alpha_{L-4}} Z_{\beta_{L-2}}^{\alpha_{L-3}} + a_4 \left( \frac{-1,0}{\Phi_{\alpha_{L-4}}} \right)_{L=3}^{(1)} C_{\beta_{L-2}}^{\alpha_{L-4}} \beta_0 + \left( \frac{-2,0}{\Phi_{\gamma}} \right)_{L=3}^{(11)} \left( \frac{-2,0}{\eta}_{\alpha_{L-1}} - \gamma_{\alpha L} A_{\alpha_{L-1}}^{\alpha_{L-1}} \right) A_{\beta_{L-2}}^{\alpha_{L-1}}.
\] (121)
Substituting (120) in (116) we additionally infer
\[
\left( \frac{-1,0}{\Phi_{\alpha_{L-4}}} \right)_{L=3}^{(11)} Z_{\alpha_{L-3}}^{\alpha_{L-4}} = \delta_1 \left( \frac{-2,0}{\eta} \right)_{L=3}^{(11)} \left( \frac{-2,0}{\beta_{L-2}} \right)_{L=3}^{(1)} A_{\alpha_{L-3}}^{\alpha_{L-2}}.
\] (122)
Reprising the same treatment on the other definitions (101) with respect to $\delta_1$ we consequently deduce that all the antifields $\left(\begin{array}{c} (-1,0)^{(1)} \\
\Phi_{\alpha_{L-2k}} \end{array} \right)_{k=0,\ldots,a-1\equiv \frac{L}{2}-1}$ are $\delta_1$-exact, i.e.,

$$\frac{(-1,0)^{(1)}}{\Phi_{\beta_{L-2k}}} = \delta_1 \frac{(-2,0)}{\gamma_{\beta_{L-2k}}}$$

with

$$\frac{(-2,0)}{\gamma_{\beta_{L-2k}}} = \frac{(-2,0)^{(11)}}{\eta_{\alpha_{L-2k-1}}} Z^{\alpha_{L-2k-1}}_{\beta_{L-2k}} + a_{2k+2} \frac{(-1,0)^{(1)}}{\Phi_{\alpha_{L-2k-2}}} C^{\alpha_{L-2k-2}\beta_0}_{\beta_{L-2k}} \frac{(-1,0)^{(1)}}{\Phi_{\beta_0}} \right) A^{\alpha_{L-2k+1}}_{\beta_{L-2k}}$$

where

$$a_{2k+2} = \begin{cases} 1, & L \neq 2k + 2, \\ \frac{1}{2}, & L = 2k + 2. \end{cases}$$

The definition of $\delta_1$ acting on $\frac{(-2,0)^{(11)}}{\eta_{\alpha_{1}}}$

$$\delta_1 \frac{(-2,0)^{(11)}}{\eta_{\alpha_{1}}} = \frac{(-1,0)^{(1)}}{\Phi_{\alpha_{0}}} Z^{\alpha_{0}}_{\alpha_{1}} + \frac{(-1,0)^{(1)}}{\Phi_{\alpha_{2}}} A^{\alpha_{2}}_{\alpha_{1}},$$

together with (123) for $k = \frac{L}{2} - 1$ yield

$$\frac{(-1,0)^{(1)}}{\Phi_{\alpha_{0}}} Z^{\alpha_{0}}_{\alpha_{1}} = \delta_1 \left( \frac{(-2,0)^{(11)}}{\eta_{\alpha_{1}}} - \frac{(-2,0)^{(1)}}{A^{\alpha_{2}}_{\alpha_{1}}} \right).$$

In this way we managed to show that all the dangerous $\delta_1$-co-cycles $\left(\begin{array}{c} (-1,0)^{(1)} \\
\Phi_{\alpha_{2k}} \end{array} \right)_{k \geq 1}$ and $\frac{(-1,0)^{(1)}}{\Phi_{\alpha_{0}}} Z^{\alpha_{0}}_{\alpha_{1}}$ are indeed $\delta_1$-exact. Following a similar line we can prove that the $\delta_2$ co-cycles $\left(\begin{array}{c} (0,-1)^{(2)} \\
\Phi_{\alpha_{2k}} \end{array} \right)_{k \geq 1}$ and $\frac{(-1,0)^{(1)}}{\Phi_{\alpha_{0}}} Z^{\alpha_{0}}_{\alpha_{1}}$ are $\delta_2$-exact

$$\frac{(0,-1)^{(2)}}{\Phi_{\alpha_{L-2k}}} = \delta_2 \frac{(0,-2)}{\gamma_{\alpha_{L-2k}}} \right)_{k = 0, \ldots, a - 1 \equiv \frac{L}{2} - 1,}$$
by the equations \( \delta S \) \( \eta \). All these results lead to the conclusion that the Koszul-Tate component is prevented precisely by the irreducibility of the gauge transformations \( \gamma \). The appearance of non-trivial co-cycles at higher order resolution bidegree (1, 0) or (0, 1) are trivial, and, on the other hand, the appearance of non-trivial co-cycles at resolution bidegree (0, 2) furnishes a correct biresolution of \( \Sigma' \), where \( \Sigma' \) is defined by the equations \( \delta S_{\Phi} = 0 \), such that it is permissible to approach the antifield BRST-anti-BRST quantization of the irreducible gauge theory along the general lines exposed in Section 2.

At this stage we construct the longitudinal exterior derivatives along the gauge orbits, \( D_1 \) and \( D_2 \). In view of this we need to know the coefficients appearing at the commutators among the irreducible gauge transformations \( \alpha \). As underlined in the previous subsection, it is reasonable to assume the completeness of the irreducible gauge generators. This assumption leads to the next general relations expressing the on-shell closedness of the irreducible generators

\[
(0, -1)^{(2)} \Phi \frac{\delta Z^{a_0}}{\delta \Phi^{a_0}} = \delta_1 \left( (0, -2)^{(2)} \eta \frac{\delta A_{\alpha_1}}{\delta \Phi_{\alpha_1}} - (0, -2)^{(2)} \eta' \frac{\delta A_{\alpha_2}}{\delta \Phi_{\alpha_2}} \right),
\]

where

\[
(0, -2)^{(2)} \gamma' = \eta \frac{\delta A_{\alpha_2}}{\delta \Phi_{\alpha_2}} Z^{a_0}_{\alpha_1} + a_{2k+2} \left( (0, -1)^{(2)} \Phi \frac{\delta A_{\alpha_2}}{\delta \Phi_{\alpha_2}} C_{\alpha_{L-2k-2}}^{\beta_0} (0, -1)^{(2)} \right) + (0, -2)^{(2)} \gamma \frac{\delta A_{\alpha_2}}{\delta \Phi_{\alpha_2}} A_{\alpha_{L-2k+1}}^{\alpha_0} \Phi_{\beta_0}^{\alpha_0} (0, -1)^{(2)} \Phi_{\beta_0}^{\alpha_0}.
\]

In consequence, there are no non-trivial co-cycles of \( \delta_1 \) and \( \delta_2 \) at positive resolution bidegrees because on the one hand we established that all the co-cycles at resolution bidegree (1, 0) or (0, 1) are trivial, and, on the other hand, the appearance of non-trivial co-cycles at higher order resolution bidegrees is prevented precisely by the irreducibility of the gauge transformations \( \Sigma' \). All these results lead to the conclusion that the Koszul-Tate components \( \delta_1 \) and \( \delta_2 \) furnish a correct biresolution of \( C^\infty (\Sigma') \), where \( \Sigma' \) is defined by the equations \( \delta S_{\Phi} = 0 \), such that it is permissible to approach the antifield BRST-anti-BRST quantization of the irreducible gauge theory along the general lines exposed in Section 2.
where the coefficients involved with the right-hand sides of the prior formulas may depend in principle on the fields \( \Phi^{a_0} \).

The definitions of \( D_1 \) and \( D_2 \) acting on the generators \((134)\) are constructed with the help of \((12 \, 13) \) and \((133 \, 134)\). In the sequel we omit the superscript for notational simplicity. Related to the ghost number zero fields, these definitions take the concrete form

\[
D_1 \Phi^{a_0} = Z^{a_0}_{\alpha_1} \eta_1^{\alpha_1}, \quad D_2 \Phi^{a_0} = Z^{a_0}_{\alpha_1} \eta_2^{\alpha_1},
\]

\[
D_1 \Phi^{a_2k} = Z^{a_2k}_{\alpha_2k+1} \eta_1^{\alpha_2k+1} + A^{a_2k}_{\alpha_2k-1} \eta_1^{\alpha_2k-1}, \quad k = 1, \ldots, a,
\]

\[
D_2 \Phi^{a_2k} = Z^{a_2k}_{\alpha_2k+1} \eta_2^{\alpha_2k+1} + A^{a_2k}_{\alpha_2k-1} \eta_2^{\alpha_2k-1}, \quad k = 1, \ldots, a.
\]

The actions of \( D_1 \) and \( D_2 \) on the ghosts are given by

\[
D_1 \eta_1^{a_1} = \frac{1}{2} C^{a_1}_{\beta_1 \gamma_1} \eta_1^{\beta_1} \eta_1^{\gamma_1} + C^{a_1}_{\beta_1 \gamma_2} \eta_1^{\beta_1} \eta_2^{\gamma_2},
\]

\[
D_1 \eta_2^{a_1} = C^{a_2k+1}_{\beta_1 \gamma_1} \eta_1^{\beta_1} \eta_1^{\gamma_2k+1} + C^{a_2k+1}_{\beta_1 \gamma_2} \eta_1^{\beta_1} \eta_1^{\gamma_2k+1} + \bar{a}_{2k+1} C^{a_2k+1}_{\beta_1 \gamma_2} \eta_1^{\beta_1} \eta_1^{\gamma_2k+1}, \quad k = 1, \ldots, b,
\]

\[
D_2 \eta_1^{a_1} = -\pi^{a_1} + \frac{1}{2} C^{a_1}_{\beta_1 \gamma_1} \eta_1^{\beta_1} \eta_1^{\gamma_2} + \frac{1}{2} C^{a_1}_{\beta_1 \gamma_2} \left( \eta_2^{\beta_1} \eta_3^{\gamma_2} + \eta_1^{\beta_1} \eta_2^{\gamma_2} \right),
\]

\[
D_2 \eta_2^{a_1} = -\pi^{a_2k+1} + \frac{1}{2} C^{a_2k+1}_{\beta_1 \gamma_1} \left( \eta_1^{\beta_1} \eta_1^{\gamma_2k+1} + \eta_1^{\beta_1} \eta_2^{\gamma_2k+1} \right) + \frac{1}{2} \bar{a}_{2k+1} C^{a_2k+1}_{\beta_1 \gamma_2} \left( \eta_1^{\beta_1} \eta_2^{\gamma_2k+1} + \eta_1^{\beta_1} \eta_2^{\gamma_2k+1} \right), \quad k = 1, \ldots, b,
\]
Finally, we have the following actions with respect to the ghosts of ghosts

\[ D_2 \beta_2^{(k)} = \frac{1}{2} C_{\beta_1 \gamma_1}^{\alpha_1} \gamma_2 \eta_2^{(k)} + C_{\beta_1 \gamma_3}^{\alpha_1} \beta_2^{(k)} \eta_2, \quad (144) \]

\[ D_2 \beta_2^{(k+1)} = C_{\beta_1 \gamma_2}^{\alpha_2k+1} \eta_2^{(k+1)} + C_{\beta_1 \gamma_3}^{\alpha_2k+1} \beta_2 \eta_2^{(k+1)}, \quad k = 1, \ldots, b. \quad (145) \]

Finally, we have the following actions with respect to the ghosts of ghosts

\[ D_1 \pi^{\alpha_1} = \frac{1}{2} C_{\beta_1 \gamma_1}^{\alpha_1} \pi^{(k)} \eta_1^{(k)} + \frac{1}{2} C_{\beta_1 \gamma_3}^{\alpha_1} \left( \pi^{(k)} \eta_1^{(k)} - \eta_1^{(k)} \pi^{(k)} \right), \quad (146) \]

\[ D_1 \pi^{(k+1)} = \frac{1}{2} C_{\beta_1 \gamma_2}^{\alpha_2k+1} \left( \pi^{(k+1)} \eta_1^{(k+1)} - \eta_1^{(k+1)} \pi^{(k+1)} \right) + \frac{1}{2} C_{\beta_1 \gamma_2}^{\alpha_2k+1} \left( \pi^{(k+1)} \eta_1^{(k+1)} - \eta_1^{(k+1)} \pi^{(k+1)} \right) + \frac{1}{2} \bar{a}_{2k+1} C_{\beta_1 \gamma_2}^{\alpha_2k+1} \left( \pi^{(k+1)} \eta_1^{(k+1)} - \eta_1^{(k+1)} \pi^{(k+1)} \right), \quad k = 1, \ldots, b. \quad (147) \]

\[ D_2 \pi^{\alpha_1} = \frac{1}{2} C_{\beta_1 \gamma_1}^{\alpha_1} \pi \eta_2^{(k+1)} + \frac{1}{2} C_{\beta_1 \gamma_3}^{\alpha_1} \left( \pi \eta_2^{(k+1)} - \eta_2^{(k+1)} \pi \right), \quad (148) \]

\[ D_2 \pi^{(k+1)} = \frac{1}{2} C_{\beta_1 \gamma_2}^{\alpha_2k+1} \left( \pi \eta_2^{(k+1)} - \eta_2^{(k+1)} \pi \right) + \frac{1}{2} C_{\beta_1 \gamma_2}^{\alpha_2k+1} \left( \pi \eta_2^{(k+1)} - \eta_2^{(k+1)} \pi \right) + \frac{1}{2} \bar{a}_{2k+1} C_{\beta_1 \gamma_2}^{\alpha_2k+1} \left( \pi \eta_2^{(k+1)} - \eta_2^{(k+1)} \pi \right), \quad k = 1, \ldots, b. \quad (149) \]

In the above, we used the notation

\[ \bar{a}_{2k+1} = \begin{cases} \frac{1}{2}, & \text{for } k = 1, \\ 1, & \text{for } k \neq 1. \end{cases} \quad (150) \]

Taking into account the above definitions of \( \delta_1, \delta_2, D_1 \) and \( D_2 \), the first two pieces of the solution to the master equation \((37)\) for our irreducible gauge theory read as

\[ S_0 = S_0 [\Phi^0], \quad (151) \]
\[ S = \Phi_{\alpha_0}^* Z^{\alpha_0}_{\alpha_1} \eta_1^{\alpha_1} + \sum_{k=1}^{a} \left( \Phi_{\alpha_2k}^* Z^{\alpha_{2k}}_{\alpha_{2k+1}} \eta_1^{\alpha_{2k+1}} + \Phi_{\alpha_2k}^* A_{\alpha_{2k-1}} \eta_1^{\alpha_{2k-1}} \right) + \Phi_{\alpha_0}^* Z^{\alpha_0}_{\alpha_1} \eta_2^{\alpha_1} + \sum_{k=1}^{a} \left( \Phi_{\alpha_2k}^* Z^{\alpha_{2k}}_{\alpha_{2k+1}} \eta_2^{\alpha_{2k+1}} + \Phi_{\alpha_2k}^* A_{\alpha_{2k-1}} \eta_2^{\alpha_{2k-1}} \right). \]  

The third piece from \( S \) contains, apart from the usual terms exposed in Section 2, some supplementary terms that take into account the more complete definitions (138–149), and is expressed by

\[ S = \sum_{k=0}^{b} \left( \eta_0^{(2)}(1) - \eta_0^{(12)} + \Phi_{\alpha_2k} Z^{\alpha_{2k}}_{\alpha_{2k+1}} + \Phi_{\alpha_2k+2} A_{\alpha_{2k+1}} \right) \pi^{\alpha_{2k+1}} + \]

\[ \eta_{\alpha_1}^{(11)} \left( \frac{1}{2} C_{\alpha_1 \gamma_1} \eta_1^{\beta_1 \gamma_1} + C_{\alpha_1 \gamma_3} \eta_1^{\beta_1 \gamma_3} \right) + \]

\[ \sum_{k=1}^{b} \eta_0^{(11)} \left( C_{\alpha_{2k+1} \gamma_1} \eta_1^{\beta_1 \gamma_1} + C_{\alpha_{2k+1} \gamma_3} \eta_1^{\beta_1 \gamma_3} \right) \]

\[ C_{\alpha_{2k+1} \beta_1 \gamma_2} \eta_1^{\beta_1 \gamma_2} + \alpha_{2k+1} C_{\alpha_{2k+1} \beta_1 \gamma_2} \eta_1^{\beta_1 \gamma_2} \]

\[ \left( \eta_0^{(11)} + \eta_0^{(12)} \right) \left( \frac{1}{2} C_{\alpha_1 \gamma_1} \eta_1^{\beta_1 \gamma_1} + \frac{1}{2} C_{\alpha_1 \gamma_3} \eta_1^{\beta_1 \gamma_3} \right) \]

\[ \sum_{k=1}^{b} \left( \eta_0^{(2)}(1) + \eta_0^{(12)} \right) \left( \frac{1}{2} C_{\alpha_{2k+1} \beta_1 \gamma_2} \eta_1^{\beta_1 \gamma_2} + \eta_1^{\beta_1 \gamma_2} \right) + \]

\[ \frac{1}{2} C_{\alpha_{2k+1} \beta_1 \gamma_2} \left( \eta_1^{\beta_1 \gamma_1} + \eta_1^{\beta_1 \gamma_2} \right) + \]

\[ \frac{1}{2} C_{\alpha_{2k+1} \beta_1 \gamma_2} \left( \eta_1^{\beta_1 \gamma_1} + \eta_1^{\beta_1 \gamma_2} \right) \]

\[ \sum_{k=1}^{b} \eta_{\alpha_{2k+1}} \left( C_{\alpha_{2k+1} \beta_1 \gamma_2} \eta_1^{\beta_1 \gamma_2} + C_{\alpha_{2k+1} \beta_1 \gamma_2} \eta_1^{\beta_1 \gamma_2} \right) + \]

\[ \frac{1}{2} C_{\alpha_{2k+1} \beta_1 \gamma_2} \left( \eta_1^{\beta_1 \gamma_1} + \eta_1^{\beta_1 \gamma_2} \right) \]

\[ \eta_{\alpha_1}^{(22)} \left( \frac{1}{2} C_{\alpha_1 \beta_1 \gamma_2} \eta_2^{\beta_1 \gamma_2} + C_{\alpha_1 \beta_1 \gamma_2} \eta_2^{\beta_1 \gamma_2} \right) \]
\[ \frac{1}{2} C^{\alpha_{2k+1}}_{\beta_1 \gamma_{2k+3}} (\pi^{\beta_1} \eta_1^{\gamma_{2k+3}} - \eta_1^{\beta_1} \pi^{\gamma_{2k+3}}) + \]

\[ \frac{1}{2} a_{2k+1} C^{\alpha_{2k+1}}_{\beta_1 \gamma_{2k+1}} (\pi^{\beta_1} \eta_1^{\gamma_{2k+1}} - \eta_1^{\beta_1} \pi^{\gamma_{2k+1}}) + \]

\[ \pi^{*}_1 \left( \frac{1}{2} C^{\alpha_1}_{\beta_1 \gamma_1} \pi^{\beta_1} \eta_2^{\gamma_1} + \frac{1}{2} C^{\alpha_1}_{\beta_1 \gamma_3} (\pi^{\beta_1} \eta_3^{\gamma_3} - \eta_3^{\beta_1} \pi^{\gamma_3}) \right) + \]

\[ \sum_{k=1}^{b} \pi^{*}_{\alpha_{2k+1}} \left( \frac{1}{2} C^{\alpha_{2k+1}}_{\beta_1 \gamma_{2k+1}} (\pi^{\beta_1} \eta_2^{\gamma_{2k+1}} - \eta_2^{\beta_1} \pi^{\gamma_{2k+1}}) + \right. \]

\[ \left. \frac{1}{2} C^{\alpha_{2k+1}}_{\beta_1 \gamma_{2k+3}} (\pi^{\beta_1} \eta_2^{\gamma_{2k+3}} - \eta_2^{\beta_1} \pi^{\gamma_{2k+3}}) + \right. \]

\[ \frac{1}{2} a_{2k+1} C^{\alpha_{2k+1}}_{\beta_1 \gamma_{2k-1}} (\pi^{\beta_1} \eta_2^{\gamma_{2k-1}} - \eta_2^{\beta_1} \pi^{\gamma_{2k-1}}) \right) \ldots . \] (153)

The remaining terms from \( S \), as well as the higher-order pieces of the solution to the master equation can be derived by means of projecting the master equation in the antifield BRST-anti-BRST formalism on increasing biresolution degrees.

The gauge-fixing process goes as explained in Section 2, and requires the supplementary variables

\[ \mu^{(1)}_{*} = \begin{pmatrix} (0,1)^{(\Phi)_{\alpha_{2k}}} & (1,1)^{(\eta_1)_{\alpha_{2k+1}}} & (0,2)^{(\eta_2)_{\alpha_{2k+1}}} & (1,2)^{(\pi)_{\alpha_{2k+1}}} \\ \mu \end{pmatrix}. \] (154)

With the help of these new fields, we pass to the solution of the master equation (in the first antibracket)

\[ S_1 = S + \int d^D x \left( \sum_{k=0}^{a} \Phi^{*}_{\alpha_{2k}} \mu^{(\Phi)_{\alpha_{2k}}} + \right. \]

\[ \left. \sum_{k=0}^{b} \left( \eta^{*}_{\alpha_{2k+1}} \mu^{(\eta_1)_{\alpha_{2k+1}}} + \eta^{*}_{\alpha_{2k+1}} \mu^{(\eta_2)_{\alpha_{2k+1}}} + \pi^{*}_{\alpha_{2k+1}} \mu^{(\pi)_{\alpha_{2k+1}}} \right) \right). \] (155)

The gauge-fixed action results as in Section 2 by choosing an appropriate gauge-fixing boson. A possible gauge-fixing boson is expressed by

\[ F = \frac{1}{2} \sum_{k=0}^{a} \int d^D x \left( \Phi_{\alpha_{2k}} K_{\alpha_{2k} \beta_{2k}} \Phi^{\beta_{2k}} \right). \] (156)
where $K_{\alpha_{2k}\beta_{2k}}$ stand for some symmetric field-independent invertible matrices playing the role of metric tensors with respect to the field indices. Eliminating the bar variables and the antifields conjugated with the ‘fields’ in the first antibracket from (155) on behalf of (156) (see (48)), and further all $\Phi^{*2}(2)$ and $\mu^{A}_{(1)}$ on their equations of motion, we consequently arrive at the gauge-fixed action

$$S_{1F} = \int d^{D}x \left( \sum_{k=0}^{b} (\Phi^{2k}_{\alpha} K_{\beta_{2k} \alpha_{2k}} Z_{\alpha_{2k+1}}^{\alpha_{2k+1}} + \Phi^{2k+2}_{\beta} K_{\alpha_{2k+2} \beta_{2k+2}} A_{\alpha_{2k+1}}^{\alpha_{2k+2}}) \pi^{\alpha_{2k+1}} - (Z_{\alpha_{1}}^{\alpha_{0}} \eta^{\alpha_{1}}_{2}) K_{\alpha_{0} \beta_{0}} (Z_{\beta_{1}}^{\beta_{0}} \eta^{\beta_{1}}_{1}) - \sum_{k=1}^{a} (A_{\alpha_{2k-1}}^{\alpha_{2k-1}} \eta^{\alpha_{2k-1}}_{2} + Z_{\alpha_{2k+1}}^{\alpha_{2k+1}} \eta^{\alpha_{2k+1}}_{2}) K_{\alpha_{2k} \beta_{2k}} \times (A_{\beta_{2k-1}}^{\beta_{2k-1}} \eta^{\beta_{2k-1}}_{1} + Z_{\beta_{2k+1}}^{\beta_{2k+1}} \eta^{\beta_{2k+1}}_{1}) + S_{0}[\Phi^{\alpha_{0}}] + \ldots \right).$$

If one eventually needs to enforce some Gaussian averages with respect to $\pi^{\alpha_{2k+1}}$, then it is necessary to add to $F$ some terms quadratic in the ghosts $\eta^{\alpha_{2k+1}}_{1}$ and $\eta^{\alpha_{2k+1}}_{2}$. In this way, our irreducible treatment for reducible gauge theories is completely elucidated.

### 4 Examples

In this section we apply the general theory on two interesting models of field theory, namely, the Freedman-Townsend model and an example involving abelian three-form gauge fields.

#### 4.1 The Freedman-Townsend model

We start with the Lagrangian action

$$S_{0}^{L} \left[ B_{\mu\nu}^{a}, A_{\mu}^{a} \right] = \frac{1}{2} \int d^{4}x \left( -B_{\mu\nu}^{a} F_{\mu\nu}^{a} + A_{\mu}^{a} A_{\mu}^{a} \right),$$

where $B_{\mu\nu}^{a}$ stands for an antisymmetric tensor field, and the field strength, $F_{\mu\nu}^{a}$, is defined by

$$F_{\mu\nu}^{a} = \partial_{\mu} A_{\nu}^{a} - \partial_{\nu} A_{\mu}^{a} - f_{bc}^{a} A_{\mu}^{b} A_{\nu}^{c}. \quad (159)$$

Action (158) is invariant under the first-stage reducible gauge transformations

$$\delta_{\epsilon} B_{\mu\nu}^{a} = \epsilon_{\mu\nu\lambda\rho} \left( D^{\lambda} \right)^{a}_{b} \epsilon^{b}, \quad \delta_{\epsilon} A_{\mu}^{a} = 0,$$

where $D^{\lambda}$ is the covariant derivative.

4 Examples

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#### 4.1 The Freedman-Townsend model

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$$\delta_{\epsilon} B_{\mu\nu}^{a} = \epsilon_{\mu\nu\lambda\rho} \left( D^{\lambda} \right)^{a}_{b} \epsilon^{b}, \quad \delta_{\epsilon} A_{\mu}^{a} = 0,$$
with
\[
(D^\lambda)^a_b = \delta^a_b \partial^\lambda + f^a_{bc} A^\lambda c.
\] (161)

The field equations deriving from (158) read as
\[
\frac{\delta S_0^L}{\delta B^a_{\mu\nu}} \equiv -\frac{1}{2} F^\mu_{\nu a} = 0, \quad \frac{\delta S_0^L}{\delta A^a_\mu} \equiv A^a_\mu + (D^\lambda)^a_b B^b_{\lambda\mu} = 0.
\] (162)

The non-vanishing gauge generators of (160)
\[
(Z_{\mu\nu\rho})^a_b = \varepsilon_{\mu\nu\lambda\rho} (D^\lambda)^a_b,
\] (163)
admit the first-order on-shell reducibility relations
\[
(Z_{\mu\nu\rho})^a_b (Z^\rho)^b_c = -\frac{1}{2} \varepsilon_{\mu\nu\lambda\rho} f^a_{cd} \frac{\delta S_0^L}{\delta B_{\lambda\rho d}},
\] (164)
where the first-stage reducibility functions are expressed by
\[
(Z^\rho)^b_c = (D^\rho)^b_c.
\] (165)

To every reducibility relation (164) we attach a scalar field, \( \varphi^{a2} \equiv \varphi^c \), subject to the gauge transformations
\[
\delta_\epsilon \varphi^{a2} = A^{a2}_{\alpha_1} \epsilon^{\alpha_1},
\] (166)
where \( A^{a2}_{\alpha_1} \) is such that \( A^{\beta_2}_{\alpha_1} Z_{\alpha_2}^{\alpha_1} \) is invertible. For example, we take
\[
A^{\beta_2}_{\alpha_1} = -\delta^a_b \partial^\mu \delta^4 (x - y),
\] (167)
hence
\[
\delta_\epsilon \varphi^a = \partial^\mu \epsilon^a_\mu.
\] (168)

The field, ghost, and antifield spectra of the antifield BRST-anti-BRST background are respectively given by
\[
\begin{pmatrix}
(0,0)^a_B, (0,0)^a_A, (0,0)^a_{\varphi}, (0,0)^a_{\eta_1}, (0,0)^a_{\eta_2}, (0,0)^a_{\pi}
\end{pmatrix},
\] (169)
\[
\begin{pmatrix}
(-1,0)^{(1)}_{\mu\nu} B^a, (-1,0)^{(1)}_{\mu} A^a, (-1,0)^{(1)}_{\mu} \varphi^a, (-1,0)^{(1)}_{\mu} \eta_{1a}, (-1,0)^{(1)}_{\mu} \eta_{2a}, (-1,0)^{(1)}_{\mu} \pi_{\nu a}
\end{pmatrix},
\] (170)
Choosing the gauge-fixing boson of the type $(156)$ plus an appropriate term and work with the solution 

\[ S_1 = S_0^L + \int d^4 x \left( \varepsilon_{\mu\nu\rho\sigma} \left( B_{a}^{(1)\mu\nu}(D^{\lambda})_{b}^{a} \eta_{\lambda}^{\rho\sigma} + B_{a}^{(2)\mu\nu}(D^{\lambda})_{b}^{a} \eta_{\lambda}^{\rho\sigma} \right) \right) + \]

\[ \right. \left( \frac{(0,-1)^{(2)\mu\nu}}{B_a}, \frac{(0,-1)^{(2)\mu}}{A_a}, \frac{(0,-1)^{(2)\nu}}{\varphi_a} \right) \left( \eta_{\nu a}, \eta_{\mu a}, \pi_{\nu a}, \pi_{\mu a} \right) \left( \frac{(0,-1)^{(2)\mu}}{B_a}, \frac{(0,-1)^{(2)\mu}}{A_a}, \frac{(0,-1)^{(2)\nu}}{\varphi_a} \right) \left( \eta_{\mu a}, \eta_{\nu a}, \pi_{\mu a}, \pi_{\nu a} \right) \right). \] 

(171)

(172)

In the following we discard the superscript for the sake of notation simplicity. The solution of the master equation associated with the irreducible formalism takes the form

\[ S = S_0^L + \int d^4 x \left( \varepsilon_{\mu\nu\rho\sigma} \left( B_{a}^{(1)\mu\nu}(D^{\lambda})_{b}^{a} \eta_{\lambda}^{\rho\sigma} + B_{a}^{(2)\mu\nu}(D^{\lambda})_{b}^{a} \eta_{\lambda}^{\rho\sigma} \right) \right) + \]

\[ \left( \frac{(0,1)^{(2)\mu\nu}}{(0,1)^{(2)\mu}} \frac{(0,1)^{(2)\nu}}{(0,1)^{(2)}} \frac{(0,2)^{(2)\mu\nu}}{(0,2)^{(2)}} \frac{(0,2)^{(2)}}{(0,2)^{(2)}} \frac{(1,1)^{(2)\mu\nu}}{(1,1)^{(2)\mu}} \frac{(1,1)^{(2)\nu}}{(1,2)^{(2)\mu\nu}} \frac{(1,1)^{(2)}}{(1,2)^{(2)}} \right) \]

(173)

(174)

and work with the solution

\[ S_1 = S_0^L + \int d^4 x \left( B_{a}^{(2)\mu\nu}(D^{\lambda})_{b}^{a} \eta_{\lambda}^{\rho\sigma} + B_{a}^{(2)\mu\nu}(D^{\lambda})_{b}^{a} \eta_{\lambda}^{\rho\sigma} \right) + \]

\[ \left( \frac{(0,2)^{(2)\mu\nu}}{(0,1)^{(2)}} \frac{(0,2)^{(2)\mu}}{(0,1)^{(2)}} \frac{(0,2)^{(2)\nu}}{(0,1)^{(2)}} \frac{(0,2)^{(2)}}{(0,1)^{(2)}} \right). \] 

(175)

Choosing the gauge-fixing boson of the type $(156)$ plus an appropriate term that leads to a Gaussian average, namely,

\[ F = \int d^4 x \left( -\frac{1}{4} B_{\mu\nu} B_{a}^{\mu\nu} + \frac{1}{2} \varphi_a \varphi_a + \eta_{2\nu a} \eta_{\lambda}^{\rho\sigma} \right), \] 

(176)

eliminating the antifields with the index (1) and the bar variables in the standard way (with the help of (176)) from $S_1$, and subsequently eliminating the antifields bearing the index (2) and the $\mu(1)$’s on their equations of motion, we finally reach the gauge-fixed action

\[ S_{1,F} = S_0^L + \int d^4 x \left( -\frac{1}{2} \left( (D^{\lambda})_{b}^{d} \eta_{2\rho\sigma} \right) \left( (D^{\lambda})_{b}^{a} \eta_{\rho\sigma} \right) \right) - \]

\[ \left. \left( \frac{1}{2} \varepsilon_{\mu\nu\lambda\rho} (D^{\mu})_{a}^{b} B_{b}^{\lambda a} - \partial_{\mu} \varphi_a - \pi_{\rho a} \right) \pi_{b}^{a} \right). \] 

(177)

Action (177) possesses no gauge invariances. This ends our irreducible antifield BRST-anti-BRST treatment of the Freedman-Townsend model.
4.2 A model with three-form gauge fields

Here, we start with the Lagrangian action

\[
S^L_0[A_{\mu\nu\lambda}] = \int d^7x \left( \frac{2\alpha^2 (3!)^2 4!}{M^2} F_{\mu\nu\lambda\rho} F^{\mu\nu\lambda\rho} + \alpha \varepsilon_{\mu\nu\lambda\rho\sigma\beta\gamma} F^{\mu\nu\lambda\rho} A^{\sigma\beta\gamma} \right), \tag{178}
\]

where \( A_{\mu\nu\lambda} \) denote abelian three-form gauge fields, \( \varepsilon_{\mu\nu\lambda\rho\sigma\beta\gamma} \) stands for the completely antisymmetric symbol in seven dimensions, \( \alpha \) and \( M \) are some constants, and the field strength is defined by

\[
F_{\mu\nu\lambda\rho} = \partial_\mu A_{\nu\lambda\rho} - \partial_\nu A_{\mu\lambda\rho} + \partial_\lambda A_{\rho\mu\nu} - \partial_\rho A_{\lambda\mu\nu} \equiv \partial_{[\mu} A_{\nu\lambda\rho]}, \tag{179}
\]

Action (178) is invariant under the gauge transformations

\[
\delta_{\epsilon} A^{\mu\nu\lambda} = \partial^{[\mu} \epsilon^{\nu\lambda]}, \tag{180}
\]

where the gauge generators are of the form

\[
Z^{\mu\nu\lambda}_{\beta\gamma} = \frac{1}{2} \partial^{[\mu} \delta^{\nu]} \delta^{\lambda]}_{\beta\gamma}. \tag{181}
\]

The above gauge generators are second stage reducible, with the reducibility relations

\[
Z^{\mu\nu\lambda}_{\beta\gamma} Z^{\beta\gamma}_{\rho} = 0, \tag{182}
\]

\[
Z^{\beta\gamma}_{\rho} Z^{\rho} = 0, \tag{183}
\]

where the first, respectively, second order reducibility functions are expressed by

\[
Z^{\beta\gamma}_{\rho} = \partial^{[\beta} \delta^{\gamma]}_{\rho]}, \tag{184}
\]

\[
Z^{\rho} = \partial^{\rho}. \tag{185}
\]

The role of the indices \( \alpha_0, \alpha_1, \) and \( \alpha_2 \) is played in our case by \( \mu\nu\lambda, \beta\gamma, \) respectively, \( \rho, \) while \( \alpha_3 \) is one-valued and is omitted for notational simplicity. In agreement with Section 3, we add the fields \( A^{\alpha_2} \equiv A^\rho, \) the gauge parameters \( \epsilon^{\alpha_3} \equiv \epsilon, \) and impose the gauge transformations of the new fields like

\[
\delta_{\epsilon} A^\rho = \partial_\lambda \epsilon^{\lambda\rho} + \partial^\rho \epsilon. \tag{186}
\]
It is clear that the gauge transformations (180) and (186) form a complete and irreducible set. The field, ghost and antifield spectra are organized as

$$\left( (0,0)^{\mu \nu \lambda}, (0,0)^{\mu}, (1,0)^{\mu \nu}, (1,0), (0,1)^{\mu \nu}, (0,1), (1,1)^{\mu \nu}, (1,1) \right),$$

(187)

$$\left( (-1,0)^{s(1)} A_{\mu \nu \lambda}, A_{\mu}, \eta_{\mu \nu}, \eta \right),$$

(188)

$$\left( (0,-1)^{s(2)} A_{\mu \nu \lambda}, A_{\mu}, \eta_{\mu \nu}, \eta \right),$$

(189)

$$\left( (-1,-1) A_{\mu \nu \lambda}, \eta_{\mu \nu}, \eta \right).$$

(190)

The solution of the master equation (37) reads as

$$S = S_0^L + \int d^7x \left( A_{\mu \lambda}^{s(1)} \partial^{\mu} \eta_{\lambda}^{1} + A_{\mu \lambda}^{s(2)} \partial^{\mu} \eta_{\lambda}^{2} + A_{\rho}^{s(1)} \left( \partial_{\rho} \eta_{\lambda}^{1} + \partial^{\rho} \eta_{\lambda} \right) + A_{\rho}^{s(2)} \left( \partial_{\rho} \eta_{\lambda}^{2} + \partial^{\rho} \eta_{\lambda} \right) + \left( \eta_{\mu \nu}^{s(21)} - \eta_{\mu \nu}^{s(12)} \right) \pi + \tilde{A}_{\mu \lambda} \partial^{\mu} \pi_{\lambda} + \tilde{A}_{\rho} \left( \partial_{\rho} \pi_{\lambda} + \partial^{\rho} \pi \right) \right).$$

(191)

In order to fix the gauge, it is necessary to introduce the ‘fields’

$$\left( (0,1)^{(A)} A_{\mu \lambda}, (0,1)^{(A)} \mu, (1,1)^{(\eta_{1}) \mu \nu}, (1,1)^{(\eta_{2}) \mu \nu} \right),$$

(192)
and to work with the solution

\[ S_1 = S + \int d^7 x \left( A^{(2)}_{\mu \nu \lambda} \mu^{(1)}(A) A^{(2)}_{\mu \nu} \mu^{(1)}(A) + \eta^{(2)}_{\mu \nu} \mu^{(1)}(\eta) \eta^{(2)}_{\mu \nu} \mu^{(1)}(\eta) + \eta^{(2)}_{\mu \nu} \mu^{(1)}(\eta) \eta^{(2)}_{\mu \nu} \mu^{(1)}(\eta) + \eta^{(2)}_{\mu \nu} \mu^{(1)}(\eta) \eta^{(2)}_{\mu \nu} \mu^{(1)}(\eta) + \pi^{(2)}_{\mu \nu} \mu^{(1)}(\pi) \pi^{(2)}_{\mu \nu} \mu^{(1)}(\pi) \right). \]  

We choose the gauge-fixing boson also of the form (156), i.e.,

\[ F = -\int d^7 x \left( \frac{1}{6} A_{\mu \nu \lambda} A^{\mu \nu \lambda} + \frac{1}{2} A_{\mu} A^{\mu} \right), \]  

and consequently derive the gauge-fixed action (after elimination of some auxiliary ‘fields’ on their equations of motion)

\[ S_{1F} = S_0^L + \int d^7 x \left( -\eta_{2\mu \nu} \Box \eta_{1\mu \nu} - \eta_{2} \Box \eta_{1} + \pi_{\mu \nu} \left( \partial_{\lambda} A^{\lambda \mu \nu} + \frac{1}{2} \partial^{[\mu} A^{\nu]i} \right) + \pi \partial_{\mu} A^{\mu} \right). \]  

It is clear that the gauge-fixed action displays a propagating character. The same line can be applied if one adds to the action (178) any interaction terms that are invariant also under the gauge transformations (180).

### 5 Conclusion

To conclude with, in this paper we develop a method that allows the application of the irreducible antifield BRST-anti-BRST quantization to a large class of reducible gauge theories. The crucial point of our procedure is expressed by the replacement of the starting reducible system by an equivalent irreducible one, such that we can substitute the antifield BRST-anti-BRST quantization of the reducible theory with that of the corresponding irreducible system. The quantization of the irreducible system follows the standard rules of the irreducible antifield BRST-anti-BRST method, the acyclicity of the Koszul-Tate bicomplex being ensured. In due course we emphasize a possible class of gauge-fixing bosons which is relevant in the context of our procedure. Finally, we show how our mechanism can be applied to practical solutions on two models of interest.
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