A short remark on the nonlinear oscillator with a damping term

Shaowen Yao¹ and Zhibo Cheng¹,²

Abstract
A nonlinear oscillator with a damping term can model many nonlinear vibration problems. This short remark insights into its physical understanding by the variational principle, which is established by the semi-inverse method. The dissipative energy involved in the variational formulation can be explained by the two-scale thermodynamics. Taylor series method is used to solve its frequency-amplitude relation.

Keywords
Semi-inverse method, variational principle, dissipative energy, Taylor series method, He’s frequency formulation

Introduction
In our previous publication, we studied a nonlinear oscillator with a damping term; however, we found a typo in equation (1), which was wrongly written as

\[ \ddot{u} + \frac{ku}{1 + cu} + bu = 0, \quad u(0) = A, \quad \dot{u}(0) = 0 \] (1)

It should be corrected as

\[ \ddot{u} + \frac{ku}{1 + cu^2} + bu = 0, \quad u(0) = A, \quad \dot{u}(0) = 0 \] (2)

As this nonlinear oscillator has been caught an immediate attention, see some accepted papers in this journal with DOI numbers: 10.1177/1461348419847307, 10.1177/1461348419851931 and 10.1177/1461348418812327, the correction to equation (1) is very much needed, and its physical explanation will be also given in this short remark.

Physical insight into equation (2) and its variational principle
Nonlinear vibration is extremely important in engineering from nanoscale attachment to form a nanofiber membrane to building’s anti-seismic design.³ Equation (2) can be written as

\[ \ddot{u} + cu^2 + (k + b + bcu^2)u = 0 \] (3)

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Generally, a nonlinear oscillator with a damping term admits no variational principle; however, the variational formulation for equation (3) can be obtained by the semi-inverse method. When \( b = 0 \), its variational formulation is

\[
J(u) = \int_0^T \left\{ \frac{1}{2} \dot{u}^2 - \frac{1}{2} ku^2 + \frac{c}{12} \dot{u}^4 \right\} dt
\]

(4)

where \( \frac{1}{2} \dot{u}^2 \) is the kinetic energy, \( \frac{1}{2} ku^2 \) is the potential energy, and \( \frac{c}{12} \dot{u}^4 \) is the dissipative energy, which should be negative. The dissipative term involved in equation (2) can be explained by the two-scale thermodynamics. We consider a spring vibrating in a fractal space, for example, in water, when \( c = 0 \), it is a linear spring, but it ignores the effects of water molecule’s size and distribution on its oscillatory property. When we study the vibration in the fractal space on a smaller scale, saying a molecular scale, the dissipative energy should be a function of the square of its velocity (\( \dot{u}^2 \)). Considering the dissipative energy in the fractal space is irrelevant to the direction of the velocity, so it can be expressed \( \frac{c}{12} \dot{u}^4 \), where \( c \) is related to the fractal dimensions of the fractal space.

Now, we consider the case of \( b \neq 0 \). The term \( bc\dot{u}^2 \) can be considered as a viscous force

\[
F = -bc\dot{u}^2 u
\]

(5)

The work done by the viscous force can be written as

\[
\delta w = -\int_0^T bc\dot{u}^2 u \dot{u} dt
\]

(6)

The variational formulation can be written in the form

\[
\delta J(u) = \int_0^T \left\{ \frac{1}{2} \dot{u}^2 - \frac{1}{2} (k + b)u^2 + \frac{c}{12} \dot{u}^4 \right\} dt + \delta w = \int_0^T \left\{ \frac{1}{2} \dot{u}^2 - \frac{1}{2} (k + b)u^2 + \frac{c}{12} \dot{u}^4 - \frac{1}{2} bc\dot{u}^2 u^2 \right\} dt
\]

(7)

where \( \dot{u} \) is considered as a constrained function, \( \delta \dot{u} = 0 \). The concept of the constrained function is widely used in the variational iteration method to identify approximately the Lagrange multiplier involved in the iteration algorithm. We obtain the following approximate variational principle

\[
J(u) = \int_0^T \left\{ \frac{1}{2} \dot{u}^2 - \frac{1}{2} (k + b)u^2 + \frac{c}{12} \dot{u}^4 - \frac{1}{2} bc\dot{u}^2 u^2 \right\} dt
\]

(8)

**Taylor series method**

Hereby, we introduce He’s frequency formulation by Taylor series. He’s frequency formulation and its various modifications have been proved to be extremely simple but remarkably accurate. For simplicity, we consider the case when \( b = 0 \). In view of the initial conditions and by differentiating equation (3) and setting \( t = 0 \) in the resultant equations, we have

\[
\ddot{u}(0) = -kA
\]

(9)

\[
\ddot{u}(0) = 0
\]

(10)

\[
\ddot{u}(0) = 2ck^2A^2 + k^2A
\]

(11)
Its Taylor series solution is

\[ u(t) = u(0) + \dot{u}(0)t + \frac{1}{2!}\ddot{u}(0)t^2 + \frac{1}{3!}\dddot{u}(0)t^3 + \frac{1}{4!}\ddddot{u}(0)t^4 = A - \frac{1}{2}kAr^2 + \frac{1}{24}[2ck^3A^3 + k^2A]t^4 \]  

(12)

Setting

\[ u(t) \bigg|_{t=\frac{\pi}{2\omega}} = 0 \]  

(13)

we obtain the frequency-amplitude relationship

\[ A - \frac{1}{2}kA \left( \frac{\pi}{2\omega} \right)^2 + \frac{1}{24}[2ck^3A^3 + k^2A] \left( \frac{\pi}{2\omega} \right)^4 = 0 \]  

(14)

Similarly, for the case when \( b \neq 0 \), we have

\[ \dddot{u}(0) = -(k+b)A \]  

(15)

\[ \dddot{u}(0) = 0 \]  

(16)

\[ \dddot{u}(0) = 2c(k+b)^3A^3 + (k+b)^2A - 2bc(k+b)^2A^3 \]  

(17)

The fourth-order series solution is

\[
\begin{align*}
    u(t) &= u(0) + \dot{u}(0)t + \frac{1}{2!}\ddot{u}(0)t^2 + \frac{1}{3!}\dddot{u}(0)t^3 + \frac{1}{4!}\ddddot{u}(0)t^4 \\
    &= A - \frac{1}{2}(k+b)Ar^2 + \frac{1}{24}\left[2c(k+b)^3A^3 + (k+b)^2A - 2bc(k+b)^2A^3\right]t^4 
\end{align*}
\]  

(18)

The frequency-amplitude relationship is obtained

\[ A - \frac{1}{2}(k+b)A \left( \frac{\pi}{2\omega} \right)^2 + \frac{1}{24}\left[2c(k+b)^3A^3 + (k+b)^2A - 2bc(k+b)^2A^3\right] \left( \frac{\pi}{2\omega} \right)^4 = 0 \]  

(19)

**Conclusion**

In this paper, the semi-inverse method\(^{3-11}\) is adopted to establish a variational formulation for a nonlinear oscillator with a damping term. The established variational formulation can give a good physical insight into equation (2) in an energy view, and the damping term \((\frac{k}{\pi^2\omega^2})\) involved in equation (2) can be explained as the dissipative term. Though equation (2) can be solved by various analytical methods; here, its frequency-amplitude relationship is obtained approximately by Taylor series method.\(^{36}\) Two special cases are discussed for \( b = 0 \) and \( b \neq 0 \) for easy understanding.

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