LARGE TIME BEHAVIOR OF A CONSERVED PHASE-FIELD SYSTEM

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Abstract. We investigate the large time behavior of a conserved phase-field system that describes the phase separation in a material with viscosity effects. We prove a well-posedness result, the existence of the global attractor and its upper semicontinuity, when the heat capacity tends to zero. Then we prove the existence of inertial manifolds in one space dimension, and for the case of a rectangular domain in two space dimension. We also construct robust families of exponential attractors that converge in the sense of upper and lower semicontinuity to those of the viscous Cahn-Hilliard equation. Continuity properties of the intersection of the inertial manifolds with bounded absorbing sets are also proven. This work extends and improves some recent results proven by A. Bonfoh for both the conserved and non-conserved phase-field systems.

1. Introduction. The conserved phase-field system that takes into account the effects of viscosity in a material occupying either a bounded domain $\Omega$ of $\mathbb{R}^d$, with smooth boundary, or $\Omega = \Pi_{i=1}^d(0, L_i)$, $L_i > 0$, for $d \leq 3$, reads (cf. [7, 13, 17])

\[
\begin{align*}
\tau \phi_t - \Delta (\delta \phi_t - \Delta \phi + g(\phi) - u) &= 0, \\
\epsilon u_t + \phi_t - \Delta u &= 0,
\end{align*}
\]

where $\tau > 0$ is a relaxation time, $\delta \geq 0$ is the viscosity parameter, $\epsilon \geq 0$ is the heat capacity, $\phi$ is the order parameter, $u$ is the absolute temperature and $g = G'$ with $G$ a double-well potential. These systems of equations describe phase transition processes such as melting or solidification. If $\epsilon = 0$, then System (1) reduces to the viscous Cahn-Hilliard equation:

\[
\begin{align*}
\tau \phi_t - \Delta (\delta \phi_t - \Delta \phi + g(\phi) - u) &= 0, \\
\phi_t - \Delta u &= 0,
\end{align*}
\]

which can be written in an equation for the single unknown, namely

\[
(1 + \tau) \phi_t - \Delta (\delta \phi_t - \Delta \phi + g(\phi)) = 0,
\]

(cf. [20]; cf. also [10, 12]).

In [13], the author proved a well-posedness result for a problem in 3d including Problem (1), with irregular potentials such as logarithmic functions, and subject
to dynamic boundary conditions. Global and exponential attractors and also some stability results were proven in [16, 17] for a 3d conserved phase-field system with viscosity and memory terms and subject to Neumann boundary conditions.

System (1) with $\delta = 0$ were considered in [1, 5, 6, 14]. In [5, 6], the authors considered the problem, subject to Neumann boundary conditions, and with an arbitrary polynomial $g(\phi)$ when $d = 1, 2$, and $g(\phi) = \frac{1}{4}(\phi^3 - \phi)$ when $d = 3$. Introducing the change of variable $v = \epsilon u + \phi$, they showed the existence of the global and exponential attractors for the problem in the variable $(\phi, v)$. In [14], A. Miranville considered the problem subject to Dirichlet boundary conditions, for $d = 2$ or $3$ and for a large class of functions $G(\phi)$ including polynomials of any arbitrary odd degree with a strictly positive leading coefficient. He proved the existence of exponential attractors. More recently, in [1], A. Bonfoh proved the existence of the global and exponential attractors and also inertial manifolds. He also proved some convergence properties of the dynamics to the one of the Cahn-Hilliard equation as $\epsilon$ goes to zero. Note that the exponential attractors attract bounded sets of a closed subspace of the space $H^1(\Omega) \times L^2(\Omega)$ in [1] while this space is $H^2_0(\Omega) \times H^1_0(\Omega) \times H^1_0(\Omega)$ in [14] and $H^2(\Omega) \times H^2(\Omega)$ in [5, 6].

Our aim in this paper is to extend and improve the analysis carried out in [1] to System (1) but with $\delta > 0$ and subject to the boundary conditions either of Neumann or periodic type. More precisely, we prove the existence of the global and exponential attractors and also inertial manifolds. Then we compare the dynamics of (1) with the one of the viscous Cahn-Hilliard equation. Let us now mention an earlier work of A. Bonfoh [2] where a similar study was done on a non-conserved phase-field system having the viscous Cahn-Hilliard equation as singular limit. The present paper also aims to improve some methods and results of [2]. Here, we give a direct proof of the existence of inertial manifolds that differs from the method used in [2] (inspired by [8]) that was based on introducing the change of variable $w = 2\epsilon^{-1/2} \phi + \sqrt{\epsilon} u$ and an auxiliary problem in the variable $(\phi, w)$. Also, the continuity properties do not require smoothness of exponential attractors and absorbing sets as previously needed in [1, 2].

This paper is organized as follows. In Section 2, we set the problem. In Sections 3 and 4 we derive a priori estimates and we demonstrate the well-posedness of the problem and the existence of the global attractor which is upper semicontinuous at $\epsilon = 0$. Then, in Section 5, we prove the existence of inertial manifolds in one space dimension, and for the case of a rectangular domain in two space dimension. In the final Section 6, we construct exponential attractors that are continuous at $\epsilon = 0$ in a metric independent of $\epsilon$. Continuity properties of intersection of the inertial manifolds with bounded sets are also examined.

2. Functional setting. If $W$ is a Sobolev-type space, then we set

$$\dot{W} = \{ \varphi \in W, m(\varphi) = 0 \},$$

where

$$m(\varphi) = \frac{1}{|\Omega|} \int_{\Omega} \varphi(x) dx.$$

Moreover, we set

$$\tilde{\varphi} = \varphi - m(\varphi).$$

We denote by $W'$ the dual space of $W$. 
System (1) is subject to the boundary conditions either of Neumann or periodic type
\[ \partial_n \phi|_{\partial \Omega} = \partial_n \Delta \phi|_{\partial \Omega} = \partial_n u|_{\partial \Omega} = 0, \] (4)
(the symbol \( \partial_n \) denotes the outward normal derivative) if \( \Omega \) is a bounded domain of \( \mathbb{R}^d \), with smooth boundary \( \partial \Omega \), or
\[ \begin{cases} u|_{x_i = 0} = u|_{x_i = L_i}, & u|_{x_i = 0} = u|_{x_i = L_i}, \quad i = 1, \ldots, d, \\ \phi|_{x_i = 0} = \phi|_{x_i = L_i}, & \phi|_{x_i = 0} = \phi|_{x_i = L_i}, \quad i = 1, \ldots, d, \end{cases} \] (5)
for \( \phi \) and the derivatives of \( \phi \) of order \( \leq 3 \),
if \( \Omega = \Pi_{i=1}^d (0, L_i) \).

Let us define the linear unbounded operator, with domain \( \mathcal{D}(N) \),
\[ N = -\Delta : \mathcal{D}(N) \rightarrow \dot{L}^2(\Omega), \]
with
\[ \mathcal{D}(N) = \begin{cases} \{ \phi \in H^2(\Omega), \partial_n \phi|_{\partial \Omega} = 0 \}, & \text{in case of (4)}, \\ H^2_{per}(\Omega), & \text{in case of (5)}, \end{cases} \]
which is self-adjoint and nonnegative. If \( N \) is restricted to \( \mathcal{D}(N) \cap \dot{L}^2(\Omega) \), then it turns to be positive with compact inverse \( N^{-1} \). Moreover, one can define the powers \( N^r \) of \( N \) for \( r \in \mathbb{R} \) (cf. [23] at page 57). The spaces \( V_r = \mathcal{D}(N^{r/2}) \) are Hilbert spaces. In particular, \( V_r = (H^1(\Omega))' \) or \( (H^1_{per}(\Omega))' \), \( V_0 = L^2(\Omega) \), \( V_1 = H^1(\Omega) \) or \( H^1_{per}(\Omega) \). The injection \( V_{r_1} \hookrightarrow V_{r_2} \) is compact whenever \( r_1 > r_2 \). We denote by \( \| \cdot \| \) and \( (\cdot, \cdot) \) the usual norm and scalar product in \( L^2(\Omega) \) (and also in \( L^2(\Omega)^d \)). When \( r \) is positive, \( V_r \) is a subspace of \( H^r(\Omega) \) and
\[ \| \varphi \|_r = \left( \| N^{r/2} \varphi \|^2 + |m(\varphi)|^2 \right)^{1/2} \]
is a norm on \( V_r \) which is equivalent to the usual \( H^r(\Omega) \)–norm; we endow \( V'_r \) with the norm
\[ \| \varphi \|_{-r} = \left( \| N^{-r/2} \varphi \|^2 + |m(\varphi)|^2 \right)^{1/2}. \]
Note that, for \( k \in \mathbb{N} \) and any \( \varphi \in V_k \), we have
\[ \| N^{k/2} \varphi \| = \| \Delta^{k/2} \varphi \| \]
when \( k \geq 2 \) is even, and
\[ \| N^{k/2} \varphi \| = \| \nabla N^{(k-1)/2} \varphi \| = \| \nabla \Delta^{(k-1)/2} \varphi \| \]
when \( k \geq 1 \) is odd.

We consider the problem
\[ \begin{cases} \tau \phi_t + N(\delta \phi_t + N \phi + g(\phi) - u) = 0, \\ \epsilon u_t + \phi_t + Nu = 0, \\ \phi|_{t=0} = \phi_0, \quad u|_{t=0} = u_0, \end{cases} \] (6)
where $\epsilon \in (0,1]$ and $\delta > 0$. We denote the function $G(s) = \int_0^s g(\varsigma) d\varsigma$ and we assume that $g \in C^2(\mathbb{R})$ and the following conditions hold (cf., e.g., [4]):

$$G(s) \geq -C_1, \quad C_1 \geq 0, \quad \forall s \in \mathbb{R},$$

$$\forall \gamma \in \mathbb{R}, \quad \exists C_2(\gamma) > 0, C_3(\gamma) \geq 0 \text{ such that}$$

$$(s - \gamma)g(s) - C_2G(s) \geq -C_3, \quad \forall s \in \mathbb{R},$$

(where $C_2, C_3$ are bounded when $\gamma$ is bounded)

$$g'(s) \geq -C_4, \quad C_4 \geq 0, \quad \forall s \in \mathbb{R},$$

$$|g''(s)| \leq C_5 \left(|s|^p + 1\right), \quad C_5 > 0, \quad \forall s \in \mathbb{R},$$

where $p > 0$ is arbitrary when $d = 1, 2$ and $p \in [0,3]$ when $d = 3$. For instance, $g(s) = s^3 - s$ satisfies $(7)$-$(10)$. However, we note that, in one space dimension, no growth assumption on $g$ is needed.

The space $\overline{X}$ denotes the closure of a metric space $X \subset Y$ in the topology of the complete metric space $Y$. Furthermore, there exist two positive constants $C_\alpha, C_\gamma$ such that

$$\|\varphi\|_{-1} \leq C_\alpha \|\tilde{\varphi}\| \leq C_\gamma \|\nabla \varphi\|, \quad \forall \varphi \in V_1.$$ 

For every $r \geq 0$, we endow the Hilbert space

$$U_r = \mathcal{D}(N^{r/2}) \times \mathcal{D}(N^{(r-1)/2}),$$

with the norm

$$\|(\varphi, \psi)\|_{U_r} = \left(\|(I + N)^{r/2}\varphi\|^2 + \|(I + N)^{(r-1)/2}\psi\|^2\right)^{1/2}.$$ 

Note that $\|(I + N)^{r/2}\|^2$ is a norm on $V_r$ which is equivalent to $\|\cdot\|_r$. Sometimes, we will use the equivalent norm

$$\|(\varphi, \psi)\|_{U_r, \epsilon} = \left(\|(I + N)^{r/2}\varphi\|^2 + \epsilon\|(I + N)^{(r-1)/2}\psi\|^2\right)^{1/2}.$$ 

The Hausdorff semi-distance with respect to the metric of $E$ is defined as:

$$\text{dist}_E(A, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\|_E,$$

whereas the symmetric Hausdorff distance between $A$ and $B$ is

$$\text{dist}_{E}^{\text{sym}}(A, B) = \max\{\text{dist}_E(A, B), \text{dist}_E(B, A)\}.$$ 

Then, we set

$$K_\alpha = \{\varphi \in L^2(\Omega), \ |m(\varphi)| \leq \alpha\},$$

$$K_{\alpha, \sigma} = \{(\varphi, \psi) \in U_1, \ |m(\varphi)| \leq \alpha, \ |m(\psi)| \leq \sigma\},$$

for some $\alpha \geq 0$ and $\sigma \geq 0$.

We multiply $(6)_1$ and $(6)_2$ by 1 and we integrate over $\Omega$, and we find

$$\frac{d}{dt} m(\phi) = 0$$

and

$$\frac{d}{dt} [m(\phi) + cm(u)] = 0,$$

respectively, so that

$$m(\phi(t)) = m(\phi_0), \quad \forall t \geq 0,$$
and
\[ m(u(t)) = m(u_0), \quad \forall t \geq 0. \]

3. **A priori estimates.** We multiply (6) by \( N^{-1}\phi_t \) and (6) by \( u \) and we integrate over \( \Omega \), respectively. Summing the resulting equations, we obtain
\[ \frac{1}{2} \frac{d}{dt} \left( \epsilon \| u \|^2 + \| \nabla \phi \|^2 + 2 \int_{\Omega} G(\phi) \right) + \| \nabla u \|^2 + \| \phi_t \|^2 + 2 \delta \| \phi_t \|^2 = 0. \tag{11} \]
We multiply (6) by \( N^{-1}\phi_t \) and we integrate over \( \Omega \), and we obtain, owing to (8),
\[ \frac{d}{dt} \left( \tau \| \phi_t \|^2 + \delta \| \phi_t \|^2 \right) + \| \phi_t \|^2 + c_1 \int_{\Omega} G(\phi) \leq c_1 \| u \|^2 + c_2, \tag{12} \]
where \( c_1 = c(m(\phi_0)) \) and \( c_2 = c(m(\phi_0)) \), but we will omit the dependence of constants with respect to \( m(\phi_0) \). We multiply (6) by \( \phi_t \) and we integrate over \( \Omega \), and we obtain, owing to (9),
\[ \frac{d}{dt} \left( \tau \| \phi_t \|^2 + \delta \| \phi_t \|^2 \right) + \| \phi_t \|^2 + c_1 \int_{\Omega} G(\phi) \leq c_1 \| u \|^2 + c_2, \tag{13} \]
Summing (11), \( \varpi_1(12) \) and \( \varpi_2(13) \), with appropriate choices of \( \varpi_1, \varpi_2 > 0 \), we deduce
\[ \frac{d}{dt} E(t) + c \left( \| u \|^2 + \| \phi \|^2 + 2 \int_{\Omega} G(\phi) + \| \phi_t \|^2 + \| \phi_t \|^2 + 2 \delta \| \phi_t \|^2 \right) \leq c', \tag{14} \]
where \( c' = c(m(\psi_0), m(u_0)) \), and
\[ E(t) = \epsilon \| u \|^2 + \| \nabla \phi \|^2 + 2 \int_{\Omega} G(\phi) \]
\[ + \varpi_1 (\tau \| \phi_t \|^2 + \delta \| \phi_t \|^2) + \varpi_2 (\tau \| \phi_t \|^2 + \delta \| \nabla \phi \|^2). \]
There exist \( c_0, c_1, c_2 \geq 0 \), independent of \( \epsilon \), such that
\[ \| \phi \|^2 + \| u \|^2 + \int_{\Omega} G(\phi) \geq c_0 E(t), \]
and
\[ c_1 \left( \| (\phi(t), u(t)) \|^2_{H^1} - 1 \right) \leq E(t) \leq c_2 \left( \epsilon \| u(t) \|^2 + \| \phi(t) \|^2_{H^1} + 1 \right), \]
since
\[ |G(\phi)| \leq c \| \phi \|^{p+2} + 1 \]
\[ + \| \nabla \phi \|^2 + \| \phi_t \|^2, \]
due to (8) and (10). Thus, we deduce from (14) that
\[ \frac{d}{dt} E(t) + c_0 E(t) + c_1 \left( \| u \|^2 + \| \phi \|^2 + \| \phi_t \|^2 + \| \phi_t \|^2 + \delta \| \phi_t \|^2 \right) \leq c. \tag{15} \]
Now, we multiply (6) by \( \phi_t \) and \( N\phi \) and we integrate over \( \Omega \), respectively, and we obtain
\[ \frac{1}{2} \frac{d}{dt} \| N\phi \|^2 + \| \phi_t \|^2 + \delta \| \nabla \phi \|^2 + (g'(\phi) \nabla \phi, \nabla \phi) - (\nabla u, \nabla \phi) = 0, \tag{16} \]
and
\[ \frac{1}{2} \frac{d}{dt} (\tau \| \nabla \phi \|^2 + \delta \| N\phi \|^2) + \| \nabla \phi \|^2 + (g'(\phi) \nabla \phi, \nabla N\phi) - (\nabla u, \nabla N\phi) = 0. \tag{17} \]
We multiply (6) by \( Nu \) and we integrate over \( \Omega \), and we obtain
\[ \frac{c}{2} \frac{d}{dt} \| \nabla u \|^2 + \| Nu \|^2 + (\nabla \phi_t, \nabla u) = 0. \tag{18} \]
When $d = 1$, we have
\[
|g'(\phi)\nabla \phi, \nabla \phi_t| \leq c\|g'(\phi)\|_{L^\infty(\Omega)} \|\nabla \phi\| \|\nabla \phi_t\|
\leq c\|g'(\phi)\|_{L^\infty(\Omega)} \|\phi\|_1 \|\phi_t\|_1.
\]

When $d = 2$, we have
\[
|g'(\phi)\nabla \phi, \nabla \phi_t| \leq c\left( \|\phi\|_{L_t^{p+1}(\Omega)}^{p+1} + 1 \right) \|\nabla \phi\|_{L^p(\Omega)^2} \|\nabla \phi_t\|
\leq c\left( \|\phi\|_{L_t^{p+1}}^{p+1} + 1 \right) \|\phi\|_2 \|\phi_t\|_1.
\]

When $d = 3$, using Agmon’s inequality, we have
\[
|g'(\phi)\nabla \phi, \nabla \phi_t| \leq c\|\phi\|_{L^2(\Omega)}^2 \|\nabla \phi\|_{L^6(\Omega)^2} \|\nabla \phi_t\| + c\|\nabla \phi\| \|\nabla \phi_t\|
\leq c\left( \|\phi\|^3_2 + \|\phi\|_1 \right) \|\phi_t\|_1.
\]

The term $|g'(\phi)\nabla \phi, \nabla N\phi|$ is treated in the same way. Summing (16), (17) and (18), we deduce
\[
\frac{d}{dt} (\epsilon \|\nabla u\|^2 + \|N\phi\|^2 + \tau \|\nabla \phi\|^2 + \delta \|N\phi\|^2) + \|\phi\|^2_2 + \|u\|^2_3 + 2\tau \|\phi_t\|^2 + \delta \|\nabla \phi_t\|^2
\leq M_1(t) (\|\phi\|^2_2 + 1) \|\phi\|_2^2,
\]
where
\[
M_1(t) = \begin{cases} 
  c\|g'(\phi)\|_{L_t^{p+1}(\Omega)}^2, & \text{if } d = 1, \\
  c\left( \|\phi\|_{L_t^{p+2}}^{2p+2} + 1 \right), & \text{if } d = 2, \\
  c\left( \|\phi\|_{L_t^6}^6 + 1 \right), & \text{if } d = 3.
\end{cases}
\]

4. Well-posedness and the global attractor. We start this section by proving a well-posedness result.

**Theorem 4.1.** We assume that (7)-(10) hold. If $(\phi_0, u_0) \in \mathcal{U}_1$, then (6) possesses a unique solution $(\phi, u)$ such that
\[
(\phi, u) \in C([0, T]; \mathcal{U}_1) \cap L^2(0, T; \mathcal{U}_2), \quad m(\phi(t)) = m(\phi_0), \quad m(u(t)) = m(u_0),
\]
for any $T > 0$. Moreover, if $(\phi_0, u_0) \in \mathcal{U}_2$, then
\[
(\phi, u) \in C([0, T]; \mathcal{U}_2) \cap L^2(0, T; \mathcal{U}_3).
\]

**Proof.** (i) Existence: The existence follows from standard arguments, using Galerkin approximations and then passing to the limit (see for instance [23]). If $(\phi_0, u_0) \in \mathcal{U}_1$, then the approximate solutions $(\phi_m, u_m)$ are bounded independently of $m$ (cf. (15)), and using weak compactness, we find a subsequence still denoted by $(\phi_m, u_m)$ and a pair of functions $(\phi, u)$ such that $\phi \in L^\infty(0, T; V_1) \cap L^2(0, T; V_2)$, $u \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; V_1)$, and
\[
\phi_m \to \phi \text{ in } L^2(0, T; V_1) \text{ strongly and a.e. in } \Omega \times (0, T),
\phi_m \to \phi \text{ in } L^\infty(0, T; V_1) \text{ weakly} - \star,
\phi_m \to \phi \text{ in } L^2(0, T; V_2) \text{ weakly},
\quad \phi_m \to \phi \text{ in } L^\infty(0, T; L^2(\Omega)) \text{ weakly} - \star,
\quad u_m \to u \text{ in } L^\infty(0, T; L^2(\Omega)) \text{ weakly} - \star,
\quad u_m \to u \text{ in } L^2(0, T; V_1) \text{ weakly}.
\]

Since $g$ is continuous, we can pass to the limit as $m \to \infty$ in the approximate problem, and $(\phi, u)$ is solution to (6). From classical compactness theorems, it
follows that $\phi$ is weakly continuous from $[0, T]$ into $V_1$, and $u$ is weakly continuous from $[0, T]$ into $L^2(\Omega)$. Using (11), we can see that the real function $t \rightarrow \epsilon ||u||^2 + ||\nabla \phi||^2$ is continuous on $[0, T]$. We can conclude that $\phi$ is strongly continuous from $[0, T]$ into $V_1$, and $u$ is strongly continuous from $[0, T]$ into $L^2(\Omega)$. If $(\phi_0, u_0) \in U_2$, we can proceed like in part (i) to show the existence of a pair of functions $(\phi, u)$ solution to (6) such that $\phi \in C([0, T]; V_2) \cap L^2(0, T; V_3)$ and $u \in C([0, T]; V_1) \cap L^2(0, T; V_2)$. (ii) Uniqueness: Let $(\phi_1, u_1)$ and $(\phi_2, u_2)$ be two solutions of (6). Setting $\phi = \phi_1 - \phi_2$ and $u = u_1 - u_2$, we have $\phi(0) = 0$, $u(0) = 0$, $m(\phi(t)) = 0$, $m(u(t)) = 0$, $\forall t \geq 0$, and $(\phi, u)$ satisfies the equations

$$\tau \phi_t + N(\delta \phi_t + N\phi + g(\phi_1) - g(\phi_2) - u) = 0, \quad (20)$$

$$\epsilon \phi_t + \phi_t + Nu = 0. \quad (21)$$

We multiply (20) by $N^{-1} \phi_t$, and (21) by $u$ and we integrate over $\Omega$, respectively, and we obtain

$$\frac{1}{2} \frac{d}{dt} ||\phi||^2 + \tau ||\phi||^2_1 + \delta ||\phi||^2 + (g(\phi_1) - g(\phi_2), \phi_t) - (u, \phi_t) = 0, \quad (22)$$

and

$$\frac{\epsilon}{2} \frac{d}{dt} u^2 + ||u||^2_1 + (\phi_t, u) = 0. \quad (23)$$

When $d = 1$, we have

$$||g(\phi_1) - g(\phi_2)|| \leq \sup_{\theta \in [0, 1]} ||g'(\theta \phi_1 + (1 - \theta) \phi_2)||_{L^\infty(\Omega)} ||\phi||.$$  

When $d = 2$, we have

$$||g(\phi_1) - g(\phi_2)|| \leq c \left( ||\phi_1||_{L^{p+1}}^2 + ||\phi_2||_{L^{p+1}}^2 + 1 \right) ||\phi||_{L^p(\Omega)}$$

$$\leq c \left( ||\phi_1||_{L^p}^2 + ||\phi_2||_{L^p}^2 + 1 \right) ||\phi||_{L^p(\Omega)}.$$  

When $d = 3$, we have

$$||g(\phi_1) - g(\phi_2)|| \leq c \left( 1 + ||\phi_1||_{L^{p+1}}^2 + ||\phi_2||_{L^{p+1}}^2 + ||\phi_1||_{L^p}^2 + ||\phi_2||_{L^p}^2 \right) ||\phi||_{L^p(\Omega)}$$

$$\leq c \left( 1 + ||\phi_1||^2 + ||\phi_2||^2 \right) \left( 1 + ||\phi_1||^2 + ||\phi_2||^2 \right) ||\phi||_1.$$  

Summing (22) and (23), we deduce

$$\frac{d}{dt} \left( ||\phi||^2 + \epsilon ||u||^2 \right) + ||u||^2_1 + 2\tau ||\phi_t||^2_1 + \delta ||\phi_t||^2 \leq M_2(t)||\phi||^2_1, \quad (24)$$

where

$$M_2(t) = \begin{cases} c \left( \sup_{\theta \in [0, 1]} ||g'(\theta \phi_1 + (1 - \theta) \phi_2)||_{L^\infty(\Omega)} \right)^2, & \text{if } d = 1, \\
 c \left( ||\phi_1||_{L^p}^{2p+2} + ||\phi_2||_{L^p}^{2p+2} + 1 \right), & \text{if } d = 2, \\
 c \left( 1 + ||\phi_1||^2 + ||\phi_2||^2 \right) \left( 1 + ||\phi_1||^2 + ||\phi_2||^2 \right), & \text{if } d = 3. \end{cases}$$

Applying the Gronwall’s lemma to (24), we deduce that

$$||(\phi(t), u(t))||^2_{U_{t,s}} \leq c e^{\int_t^s M_2(s) ds} ||(\phi(0), u(0))||^2_{U_{0,s}}, \quad \forall t \geq 0,$$

hence the result.

\[\Box\]
Theorem 4.3. The global attractor \( A^\alpha_{\epsilon} \) is upper semicontinuous at \( \epsilon = 0 \), that is,

\[
\lim_{\epsilon \to 0} \text{dist}_{t \in [0, \infty)} (A^\alpha_{\epsilon}, A^\alpha) = 0.
\]
Proof. Let \((\phi_0, u_0) \in A_\epsilon^{\alpha, \sigma}\) (we have \(A_\epsilon^{\alpha, \sigma} \subset B_2\); cf. Sect. 6). Thus, owing to the definition of the global attractor \(A_\epsilon\) and also (97),
\[
\text{dist}_{U_\epsilon}(S(t)\phi_0, L_{m(u_0)}S(t)\phi_0, (A_\epsilon)^\sigma) \to 0, \quad \text{when } t \to +\infty. \tag{27}
\]
If \(\eta > 0\), then we can show, on account of (70) and (27), that there exist \((\phi, \zeta)\) belonging to \((A_\epsilon)^\sigma\), \(\xi\eta > 0\) and \(\epsilon_\eta\) (all depending only on \(\eta\)) such that
\[
\|S_\epsilon(\xi\eta)(\phi_0, u_0) - (\phi, \zeta)\|_{U_\epsilon} \leq \eta. \tag{28}
\]
The upper semicontinuity (26) follows from (28) and the invariance property
\[
S_\epsilon(\xi\eta)A_\epsilon^{\alpha, \sigma} = A_\epsilon^{\alpha, \sigma}
\]
(cf. [2]).

5. Inertial manifolds. In this section only, we take \(d = 1\) or \(2\), and we assume
\(\Omega = \Pi_{i=1}^d (0, L_i)\) and \(L_1/L_2\) is a rational number. In order to prove the existence of an inertial manifold for Problem (6)-(10), we introduce the ”prepared problem”:
\[
\begin{cases}
\tau \phi_t + N(\delta \phi_t + N \phi + g(\phi) - u) = 0, \\
\epsilon u_t + \phi_t + Nu = 0,
\end{cases} \tag{29}
\]
where
\[
g(\phi) = \theta \left( \frac{\|A\phi\|}{r_d} \right) g(\phi), \tag{30}
\]
\(r_d\) is the radius of the absorbing set \(B_d \subset K_{\alpha, \sigma} \cap U_d\) for \(S_\epsilon(t)|_{K_{\alpha, \sigma}}, d = 1, 2,\)
\[
A = \begin{cases}
(I + N)^{1/2}, & \text{if } d = 1, \\
I + N, & \text{if } d = 2,
\end{cases}
\]
and \(\theta : \mathbb{R}^+ \to [0, 1]\) is a \(C^\infty\) function such that \(\theta(s)\) is equal to 1 when \(0 \leq s \leq 1,\)
and is equal to 0 when \(s > 2,\) and \(|\theta'(s)| \leq 2, \forall s \geq 0\). Then we write (29) in the following form:
\[
U_t + AU + G(U) = 0, \tag{31}
\]
where \(U = (\phi, u),\)
\[
G(U) = \left( (\tau I + \delta N)^{-1} N g(\phi), -\frac{1}{\epsilon}(\tau I + \delta N)^{-1} N g(\phi) \right),
\]
and
\[
A = \begin{pmatrix}
(\tau I + \delta N)^{-1} N^2 & -N (\tau I + \delta N)^{-1} N + \frac{1}{\epsilon} N \\
\frac{1}{\epsilon}(\tau I + \delta N)^{-1} N^2 & 1 - (\tau I + \delta N)^{-1} N^2 + \frac{1}{\epsilon} N
\end{pmatrix}.
\]
The operator \(A : U_3 \to U_4\) is non self-adjoint, positive and has a discrete spectrum
\[
\mu_k^\pm = \frac{\lambda_k}{2\epsilon(\tau + \delta \lambda_k)} \left( 1 + \tau + (\delta + \epsilon)\lambda_k \pm \sqrt{(1 + \tau + (\delta + \epsilon)\lambda_k)^2 - 4\epsilon\lambda_k(\tau + \delta \lambda_k)} \right),
\]
for \(k = 0, 1, 2, \ldots\) and corresponding eigenfunctions
\[
U_k^\pm = (e_k, -\mu_k^\pm e_k),
\]
where
\[
\hat{\mu}_k^\pm = \frac{1}{2\epsilon} \left( 1 + \tau + (\delta - \epsilon)\lambda_k \pm \sqrt{(1 + \tau + (\delta + \epsilon)\lambda_k)^2 - 4\epsilon\lambda_k(\tau + \delta \lambda_k)} \right),
\]
and
\[
\hat{\mu}_k^\pm = \frac{1}{2\epsilon} \left( 1 + \tau + (\delta - \epsilon)\lambda_k \pm \sqrt{(1 + \tau + (\delta + \epsilon)\lambda_k)^2 - 4\epsilon\lambda_k(\tau + \delta \lambda_k)} \right),
\]
and
\{\lambda_k\} are the eigenvalues of \(N\) ordered in an increasing sequence and \(\{e_k\}\) are the corresponding eigenfunctions which is an orthogonal basis of \(L^2(\Omega)\). These eigenvalues and eigenfunctions have the form:

\[
\lambda = \begin{cases} 
\pi^2 \frac{k_1^2}{L_1^2}, & \text{if } d = 1, \\
\pi^2 \left( \frac{k_1^2}{L_1^2} + \frac{k_2^2}{L_2^2} \right), & \text{if } d = 2,
\end{cases}
\]

and

\[
e(x) = \begin{cases} 
\sqrt{\frac{2}{L_1}} \cos \frac{\pi k_1 x_1}{L_1}, & \text{if } d = 1, \\
\sqrt{\frac{4}{L_1 L_2}} \cos \frac{\pi k_1 x_1}{L_1} \cos \frac{\pi k_2 x_2}{L_2}, & \text{if } d = 2,
\end{cases}
\]

for \(k_1, k_2 = 0, 1, 2, \ldots\).

Let \(c_1 > 0\). There exists \(n\) such that \(\lambda_n \geq 1\) and

\[
\lambda_{n+1} - \lambda_n > \max \{4c_1(\tau + 1), 4c_1\delta\}. \tag{32}
\]

Indeed, if \(d = 1\), then this is immediate; and if \(d = 2\), then the result is due to I. Richards (see [21]), since \(L_1/L_2\) is rational. Of course, the latter estimate implies that

\[
\lambda_{n+1}^2 - \lambda_n^2 > \max \{4c_1(\tau + 1), 4c_1\delta\}(\lambda_n + \lambda_{n+1}).
\]

Denote

\[
D_k = \epsilon^2 \lambda_k^2 + 2\epsilon(1 - \tau - \delta \lambda_k)\lambda_k + (1 + \tau + \delta \lambda_k)^2,
\]

\[
\Delta_k = \frac{\lambda_{k+1} \sqrt{D_{k+1}}}{\tau + \delta \lambda_{k+1}} - \frac{\lambda_k \sqrt{D_k}}{\tau + \delta \lambda_k},
\]

\[
f_k = \frac{\lambda_{k+1}^2}{\tau + \delta \lambda_{k+1}} - \frac{\lambda_k^2}{\tau + \delta \lambda_k} - 2c_1,
\]

\[
g_k^\pm = \frac{(1 + \tau + \delta \lambda_{k+1})\lambda_{k+1}}{\tau + \delta \lambda_{k+1}} \pm \frac{(1 + \tau + \delta \lambda_k)\lambda_k}{\tau + \delta \lambda_k},
\]

\[
h_k^\pm = \frac{\lambda_{k+1}^2}{(\tau + \delta \lambda_{k+1})^2} \pm \frac{\lambda_k^2}{(\tau + \delta \lambda_k)^2},
\]

\[
i_k^\pm = \frac{(1 - \tau - \delta \lambda_{k+1})\lambda_{k+1}^3}{(\tau + \delta \lambda_{k+1})^2} \pm \frac{(1 - \tau - \delta \lambda_k)\lambda_k^3}{(\tau + \delta \lambda_k)^2},
\]

\[
j_k^\pm = \frac{(1 + \tau + \delta \lambda_{k+1})^2 \lambda_{k+1}^2}{(\tau + \delta \lambda_{k+1})^2} \pm \frac{(1 + \tau + \delta \lambda_k)^2 \lambda_k^2}{(\tau + \delta \lambda_k)^2}.
\]

Let us now prove the following results.

**Lemma 5.1.** Provided that \(n\) is large enough for (32) to hold, there exists \(c(n)\) suitably small such that the following inequalities are satisfied:

(i)

\[
f_n \geq 0.
\]

(ii)

\[
\Delta_n \geq 0,
\]

for every \(\epsilon \in (0, c(n))\).

(iii)

\[
f_n^4 - 2f_n^2h_n^+ + (h_n^-)^2 > 0.
\]
(iv) \[ i_n^+ g_n^- + f_n (g_n^-)^2 - (i_n^+ g_n^- + j_n^+ f_n)^2 < 0. \]

(v) \[ \epsilon^2 h_n^+ + 2 \epsilon i_n^+ + j_n^+ - (\epsilon f_n + g_n^-)^2 > 0, \]

for every \( \epsilon \in (0, \epsilon(n)] \).

**Proof.** (i) The inequality \( f_n \geq 0 \) is equivalent to
\[ \tau \left[ \lambda_n^2 + \lambda_n^2 \right] - 2 c_1 \delta (\lambda_n + \lambda_{n+1}) - 2 c_1 \tau + \delta \lambda_n (\lambda_n + 1) \geq 0, \]
which holds true, whenever \((32)\) is satisfied.

(ii) The inequality \( \Delta_n \geq 0 \) is equivalent to
\[ \tau \left( \lambda_{n+1} \sqrt{D_{n+1}} - \lambda_n \sqrt{D_n} \right) + \delta \lambda_n \lambda_{n+1} \left( \sqrt{D_{n+1}} - \sqrt{D_n} \right) \geq 0. \]

Now, the term \( \lambda_{n+1} \sqrt{D_{n+1}} - \lambda_n \sqrt{D_n} \) and \( \lambda_n^2 D_{n+1} - \lambda_n^2 D_n \) have the same sign, and
\[ \lambda_n^2 D_{n+1} - \lambda_n^2 D_n = \delta^2 \left( \lambda_{n+1}^2 - \lambda_n^2 \right) \left( 1 + \tau \right)^2 \left( \lambda_n^2 - \lambda_{n+1}^2 \right) + 2 \delta (1 + \tau) \left( \lambda_{n+1}^3 - \lambda_n^3 \right) - 2 \delta \epsilon (\lambda_{n+1}^2 - \lambda_n^2) + \epsilon^2 \left( \lambda_{n+1}^2 - \lambda_n^2 \right) + 2 \epsilon (1 - \tau) \left( \lambda_{n+1}^3 - \lambda_n^3 \right). \]

(33)

Similarly, the term \( \sqrt{D_{n+1}} - \sqrt{D_n} \) and \( D_{n+1} - D_n \) have the same sign, and
\[ D_{n+1} - D_n = \delta^2 \left( \lambda_{n+1}^2 - \lambda_n^2 \right) + 2 \delta (1 + \tau) \left( \lambda_n^2 - \lambda_{n+1}^2 \right) - 2 \delta \epsilon (\lambda_{n+1}^2 - \lambda_n^2) + \epsilon^2 \left( \lambda_{n+1}^2 - \lambda_n^2 \right) + 2 \epsilon (1 - \tau) \left( \lambda_{n+1}^3 - \lambda_n^3 \right). \]

(34)

It is clear that both quantities \((33)\) and \((34)\) are positive for every \( \epsilon \in (0, \epsilon_1(n)] \), for some \( \epsilon_1(n) > 0 \), hence the result.

(iii) The quadratic equation \( x^2 - 2 h_n^+ x + (h_n^-)^2 = 0 \) has two positive real roots
\[ x^\pm = \left( \frac{\lambda_{n+1}^2 + \lambda_n^2}{\tau + \delta \lambda_{n+1}} \right) \pm \frac{\sqrt{\left( \lambda_{n+1}^2 + \lambda_n^2 \right)^2 - 4 (\lambda_{n+1}^2 - \lambda_n^2)}}{2 \tau + 2 \delta \lambda_n} \],

and we have \( f_n^2 < x^- \leq x^+ \), hence (iii).

(iv) A computation shows that
\[ \frac{i_n^+ g_n^- - i_n^+ g_n^-}{2 \lambda_n \lambda_{n+1}} \]
\[ \times \left[ \frac{(1 + \tau + \delta \lambda_n)(1 + \tau - \delta \lambda_{n+1}) \lambda_{n+1}^2}{\tau + \delta \lambda_{n+1}} - \frac{(1 - \tau - \delta \lambda_n)(1 + \tau + \delta \lambda_{n+1}) \lambda_n^2}{\tau + \delta \lambda_n} \right] \]

(35)

and
\[ f_n \left[ (g_n^-)^2 - j_n^+ \right] = -2 \lambda_n \lambda_{n+1} \frac{(1 + \tau + \delta \lambda_n)(1 + \tau + \delta \lambda_{n+1})}{(\tau + \delta \lambda_n)(\tau + \delta \lambda_{n+1})} \left( \frac{\lambda_{n+1}^2}{\tau + \delta \lambda_{n+1}} - \frac{\lambda_n^2}{\tau + \delta \lambda_n} - 2 c_1 \right). \]

(36)

Summing (35) with (36), we deduce that (iv) is equivalent to
\[ \frac{\lambda_{n+1}^2}{1 + \tau + \delta \lambda_{n+1}} - \frac{\lambda_n^2}{1 + \tau + \delta \lambda_n} - c_1 > 0, \]
which holds whenever (32) is satisfied (refer to the proof of (i)).

(v) The inequality
\[ \epsilon^2 h_n^+ + 2\epsilon i_n^+ + j_n^- - (\epsilon f_n + g_n^-)^2 > 0 \]
is equivalent to
\[ \epsilon^2 (h_n^+ - f_n^2) + 2\epsilon (i_n^+ - f_n g_n^-) + j_n^- - (g_n^-)^2 > 0. \]

A computation shows that
\[ h_n^+ - f_n^2 = \frac{2\lambda_n^2 \lambda_{n+1}^2}{(\tau + \delta\lambda_n)(\tau + \delta\lambda_{n+1})} + 4c_1 \left( \frac{\lambda_{n+1}^2}{\tau + \delta\lambda_{n+1}} - \frac{\lambda_n^2}{\tau + \delta\lambda_n} - c_1 \right) > 0, \]
and
\[ j_n^- - (g_n^-)^2 = \frac{2(1 + \tau + \delta\lambda_n)(1 + \tau + \delta\lambda_{n+1})\lambda_n \lambda_{n+1}}{(\tau + \delta\lambda_n)(\tau + \delta\lambda_{n+1})} > 0. \]

If \( i_n^+ - f_n g_n^- \geq 0 \), then (v) holds for all \( \epsilon \in (0,1) \). But, if \( i_n^+ - f_n g_n^- \leq 0 \), then (v) holds for at least every \( \epsilon \) such that
\[ 0 < \epsilon \leq \epsilon_2(n) = \frac{j_n^- - (g_n^-)^2}{2(f_n g_n^- - i_n^+)}. \]

The proof of the lemma is completed, by setting \( \epsilon(n) = \min\{1, \epsilon_1(n), \epsilon_2(n)\} \).

We have the following result.

**Proposition 1.** For every \( c_1 > 0 \), there exists \( n \) (independent of \( \epsilon \)) and a \( \bar{\epsilon}(n) > 0 \) such that, for every \( \epsilon \in (0, \bar{\epsilon}(n)) \), the spectral gap condition
\[ \mu_{n+1}^- - \mu_n^- > c_1, \] \hspace{1cm} (37)
holds.

**Proof.** It is clear that \( g_k^- \) is positive for every \( k \). We have
\[ \mu_{n+1}^- - \mu_n^- = \frac{1}{2\epsilon} (\epsilon f_n + g_n^- - \Delta_n) + c_1. \]

Thanks to (i) and (ii) of Lemma 5.1, the inequality
\[ \epsilon f_n + g_n^- - \Delta_n > 0 \]
is equivalent to
\[ (\epsilon f_n + g_n^-)^2 > \Delta_n^2. \] \hspace{1cm} (38)

We set \( \Delta_n = A - B \), with
\[ A = \frac{\lambda_{n+1} \sqrt{D_{n+1}}}{\tau + \delta\lambda_{n+1}} \] and \[ B = \frac{\lambda_n \sqrt{D_n}}{\tau + \delta\lambda_n}, \]
so that, (38) is exactly the following
\[ (\epsilon f_n + g_n^-)^2 > A^2 + B^2 - 2AB. \] \hspace{1cm} (39)

We observe that
\[ A^2 + B^2 = \epsilon^2 h_n^+ + 2\epsilon i_n^+ + j_n^- \] and \[ A^2 - B^2 = \epsilon^2 h_n^- + 2\epsilon i_n^- + j_n^- \].

Now, on account of (v) of Lemma 5.1, inequality (39), in turn, is equivalent to
\[ 4A^4 B^4 > \left[(A^2 + B^2)^2 - (\epsilon f_n + g_n^-)^2\right]^2, \]
which can be rewritten as
\[(A^2 - B^2)^2 - 2(A^2 + B^2)^2 (\epsilon f_n + g_n)^2 + (\epsilon f_n + g_n)^4 < 0,\]
that is,
\[\left(\epsilon^2 h_n^{-2} + 2\epsilon i_n^{-2} + j_n^{-2}\right) - 2\left(\epsilon^2 h_n^{+2} + 2\epsilon i_n^{+2} + j_n^{+2}\right) (\epsilon f_n + g_n)^2 + (\epsilon f_n + g_n)^4 < 0.\]

The inequality (40), in turn, can be rewritten as
\[\epsilon^4 \left[h_n^{-4} + f_n^{-4} - 2f_n^{+4}h_n^{+4}\right] + 4\epsilon^3 \left[h_n^{-4}i_n^{-4} + f_n^{-4}g_n^{-4} - (h_n^{+4}f_n^-g_n^{-4} + i_n^{+4}f_n^{-4})\right] + 2\epsilon^2 \left[2(i_n^{-4} - h_n^{-4}j_n^{-4} + 3f_n^{-4}(g_n) - (h_n^{+4}(g_n)^2 + 4i_n^{+4}(f_n) - j_n^{+4}(f_n))\right] + 4g_n^{-4} \left[i_n^{+4}(g_n)^2 - (i_n^{+4}(g_n) + j_n^{+4}(f_n))\right] < 0.\]

Noting that \(j_n^{-} = g_n^+ g_n^{-}\), we compute that
\[(j_n^{-2} + (g_n)^4 - 2g_n^{+2}(g_n)^2 = 0,\]
and we find that (41) is equivalent to
\[\epsilon^3 \left[(h_n)^2 + f_n^{+2} - 2f_n^{-2}h_n^{-2}\right] + 4\epsilon^2 \left[h_n^{-4}i_n^{-4} + f_n^{-4}g_n^{-4} - (h_n^{+4}f_n^-g_n^{-4} + i_n^{+4}f_n^{-4})\right] + 2\epsilon \left[2(i_n^{-4} - h_n^{-4}j_n^{-4} + 3f_n^{-4}(g_n) - (h_n^{+4}(g_n)^2 + 4i_n^{+4}(f_n) - j_n^{+4}(f_n))\right] + 4g_n^{-4} \left[i_n^{+4}(g_n)^2 - (i_n^{+4}(g_n) + j_n^{+4}(f_n))\right] < 0.\]

On account of (i), (iii) and (iv) of Lemma 5.1, the inequality (42) holds for every \(\epsilon \in (0, \epsilon_3(n))\), for some \(\epsilon_3(n) > 0\). Finally, it results from (42), (ii) and (v) of Lemma 5.1 that (37) holds for every \(\epsilon \in (0, \hat{\epsilon}(n))\), where \(\hat{\epsilon}(n) = \min\{\epsilon(n), \epsilon_3(n)\}\).  

We now prove the existence of an inertial manifold for Problem (6).

**Theorem 5.2.** Let (7)-(10) hold. We assume \(\Omega = \Pi_{i=1}^{d}(0, L_i)\), \(d \leq 2\) and \(L_1/L_2\) is rational if \(d = 2\). Then, there exists \(\hat{\epsilon}(n)\) such that, for every \(\epsilon \in (0, \hat{\epsilon}(n))\), System (6) has an inertial manifold \(\mathcal{M}_{\alpha, \sigma}^{0, \sigma} \subset K_{\alpha, \sigma} \cap \mathcal{U}_d\), that is,

(i) \(\mathcal{M}_{\alpha, \sigma}^{0, \sigma}\) is a finite-dimensional Lipschitz manifold in \(K_{\alpha, \sigma} \cap \mathcal{U}_d\);
(ii) \(\mathcal{M}_{\alpha, \sigma}^{0, \sigma}\) is positively invariant under \(S_t\), that is, \(S_t(\mathcal{M}_{\alpha, \sigma}^{0, \sigma}) \subset \mathcal{M}_{\alpha, \sigma}^{0, \sigma}, \forall t \geq 0\);
(iii) \(\mathcal{M}_{\alpha, \sigma}^{0, \sigma}\) is exponentially attracting, that is, there exists a constant \(c_0\) such that, for every \((\phi_0, u_0) \in K_{\alpha, \sigma} \cap \mathcal{U}_d\), there is a constant \(c_1(\phi_0, u_0) > 0\) such that
\[\mathrm{dist}(S_t(\phi_0, u_0), \mathcal{M}_{\alpha, \sigma}^{0, \sigma}) \leq c_1 e^{-c_0 t}, \forall t \geq 0,\]
where \(\mathrm{dist}\) is taken in the norm \(\|\Gamma, \cdot\|\), with \(\|\cdot\|\) and \(\Gamma\) defined by (47) and by (50), respectively.

**Proof.** We set
\[X_n = \text{span}\left\{U_k^\pm, \quad k = 0, 1, ..., n\right\}, \quad Y_n = \text{span}\left\{U_k^\pm, \quad k = n + 1, n + 2, ..., \right\}, \quad X_{n1} = \text{span}\{U_i^-, U_m^-, \mu_i^-, \mu_m^+ \in \sigma_1\}, \quad X_{n2} = \text{span}\{U_i^+, \mu_i^- \leq \mu_i^+ < \mu_i^+\},\]
with \(\sigma_1 = \{\mu_i^-, \mu_m^+, \max\{\mu_i^-\}, \mu_m^+\} \leq \mu_0^+\),
\[\sigma_2 = \{\mu_i^+, \mu_m^+, \mu_i^- \leq \mu_0^- \leq \min\{\mu_i^+, \mu_m^+\}\}.
\]
We introduce the scalar product $⟨.,.⟩$ in $\mathcal{U}_1$ (inspired by [24]) defined by
\[
⟨U,V⟩ = Ψ_1(P_{X_n}U,P_{X_n}V) + Ψ_2(P_{Y_n}U,P_{Y_n}V),
\]
where $P_{X_n}$ and $P_{Y_n}$ are, respectively, the projections from $\mathcal{U}_1$ onto $X_n$ and $Y_n$ and the functions $Ψ_1 : X_n \times X_n \to \mathbb{R}$ and $Ψ_2 : Y_n \times Y_n \to \mathbb{R}$ are defined by
\[
Ψ_1(U,V) = (1 + τ)(u,y) + (1 + δ − ε)(∇u,∇y) + ε(u,z) + ε(y,v), \quad (43)
\]
\[
Ψ_2(U,V) = (1 + τ)(u,y) + τ(∇u,∇y) + ε(u,z) + ε(y,v) + ε(v,z), \quad (44)
\]
with $U = (u,v)$, $V = (y,z)$ in $X_n$ and $Y_n$, respectively. Indeed, we have
\[
Ψ_1(U,U) = (1 + τ)∥u∥^2 + (1 + δ − ε)∥∇u∥^2 + 2ε(u,v) + ε∥v∥^2
\]
\[
\geq (1 + τ)∥u∥^2 + (1 + δ − ε)∥∇u∥^2 - 2ε∥u∥∥v∥ + ε∥v∥^2
\]
\[
= (1 + τ - 2ε)∥u∥^2 + (1 + δ − ε)∥∇u∥^2 + \frac{ε}{2}∥v∥^2, \quad \forall U ∈ X_n,
\]
\[
Ψ_2(U,U) = (1 + τ)∥u∥^2 + τ∥∇u∥^2 + 2ε(u,v) + ε∥v∥^2
\]
\[
\geq (1 + τ)∥u∥^2 + τ∥∇u∥^2 - 2ε∥u∥∥v∥ + ε∥v∥^2
\]
\[
= (1 + τ - 2ε)∥u∥^2 + τ∥∇u∥^2 + \frac{ε}{2}∥v∥^2, \quad \forall U ∈ Y_n.
\]
Thus, for $U_i^− ∈ X_{n_1}$ and $U_i^+ ∈ X_{n_2}$, noting that $(ε_i,ε_k) = δ_{ik},$
\[
\tilde{μ}_i^+ - \tilde{μ}_i^- = \frac{1}{ε}((δ - ε)λ_i + 1 + τ), \quad \tilde{μ}_i^+ \tilde{μ}_i^- = -\frac{λ_i}{ε},
\]
we have that
\[
⟨U_i^−,U_i^+⟩ = (1 + τ) + (1 + δ − ε)λ_i - ε(\tilde{μ}_i^+ + \tilde{μ}_i^-) + ε\tilde{μ}_i^− \tilde{μ}_i^+ = 0.
\]
As a consequence, $X_{n_1}$ is orthogonal to $X_{n_2}$ and to $Y_n$, and the decomposition $K_{α,σ} = X_{n_1} ⊕ X_{n_2} ⊕ Y_n$ is orthogonal with respect to the scalar product $⟨.,.⟩$ and we set $\mathcal{U}_1^1 = X_{n_1}$ and $\mathcal{U}_1^{1+} = X_{n_2} ⊕ Y_n$. Let $\mathcal{P}$ and $\mathcal{Q}$ be the unique orthogonal projections onto $\mathcal{U}_1^1$ and $\mathcal{U}_1^{1+}$, respectively. We now define the norm
\[
∥U∥ = ⟨U,U⟩^{1/2}.
\]
From (45) and (46), we can deduce that there exists $c$ independent of $ε$ such that
\[
∥U∥ ≥ c∥ϕ₁, \quad \forall U = (ϕ,u) ∈ \mathcal{U}_1.
\]
Now, we note that $g$, $g'$ and $g''$ are bounded continuous functions on $K_{α} ∩ D(Λ)$, and there exists a constant $c > 0$ such that
\[
∥Αg(ϕ)∥ ≤ c, \quad \forall ϕ ∈ K_{α} ∩ D(Λ),
\]
\[
∥Αg'(ϕ)v∥ ≤ c∥Λv||, \quad \forall ϕ,v ∈ K_{α} ∩ D(Λ),
\]
hence, there exists $c > 0$ such that
\[
∥Αg(ϕ) - Αg(φ)∥ ≤ c∥Λ(ϕ - φ)||, \quad ∀ϕ,φ ∈ K_{α} ∩ D(Λ),
\]
(cf. [23, Lemmas 2.1 and 2.2]). Thus, $G(U) : \mathcal{U}_d → \mathcal{U}_d$ is globally Lipschitz continuous, that is, there exists $c > 0$, independent of $ε$, such that
\[
∥ΓG(U)∥ ≤ c, \quad ∀U ∈ K_{α,σ} ∩ \mathcal{U}_d,
\]
\[
∥ΓG(U) - ΓG(V)∥ ≤ c∥Γ(U - V)||, \quad ∀U,V ∈ K_{α,σ} ∩ \mathcal{U}_d.
\]
where
\[
\Gamma = \begin{cases} 
I, & \text{if } d = 1, \\
(I + N)^{1/2}, & \text{if } d = 2.
\end{cases}
\] (50)
Moreover, there exist \(C_1, C_2 > 0\), independent of \(\epsilon\), such that
\[
\|Q e^{s\Delta}Q\|_{L^2(\Omega \mathcal{U}_d)} \leq C_1 e^{s\mu_{n+1}}, \quad s < 0,
\]
\[
\|P e^{-sA\mathcal{P}}\|_{L^2(\mathcal{P}\mathcal{U}_d)} \leq C_2 e^{-s\mu_n}, \quad s \leq 0,
\]
for every \(\epsilon \in (0,1]\). It follows from the existence theorem of inertial manifolds [23, Chap. 9, Theorem 2.1] (cf. also [22]) that the semigroup \(\tilde{S}_\epsilon(t)\) generated by Equation (31) admits an inertial manifold \(\mathcal{M}^\alpha_{\epsilon,\sigma}\) in \(K_{\alpha,\sigma} \cap \mathcal{U}_d\). More precisely, there exists a Lipschitz mapping \(\Phi_{\epsilon,\sigma}: K_{\alpha,\sigma} \cap \mathcal{P}\mathcal{U}_d \to \mathcal{Q}\mathcal{U}_d\) such that the graph of \(\Phi_{\epsilon,\sigma}\) coincides with that of \(\tilde{S}_\epsilon(t)\) on \(\mathcal{M}^\alpha_{\epsilon,\sigma}\). Hence, \(\mathcal{M}^\alpha_{\epsilon,\sigma}\) is also the desired inertial manifold for \(S_\epsilon(t)\), that is, \(\mathcal{M}^\alpha_{\epsilon,\sigma} = \mathcal{M}^\alpha_{\epsilon,\sigma}\).

Remark 1. The norm \(\|\cdot\|\) is equivalent to \(\|\cdot\|_{\mathcal{U}_d}\) on \(\mathcal{U}_d\), in the sense that, there exist \(c_1, c_2 > 0\) independent of \(\epsilon\), such that
\[
c_1 \|U\|_{\mathcal{U}_d} \leq \|U\| \leq c_2 \|U\|_{\mathcal{U}_d}.
\]
This follows from (45) and (46). We thus take this opportunity to correct (4.10) and (4.11) in [1] by replacing \(\epsilon^2(v,z)\) with \(\epsilon(v,z)\) and adding \((1 - \epsilon)(\nabla u, \nabla y)\) to (4.10).

6. Exponential attractors.

6.1. Estimates of the difference of two solutions. Firstly, we estimate the difference of two solutions of (6).

Proposition 2. There exist \(c, c' > 0\) independent of \(\epsilon\) such that
\[
\|S_\epsilon(t)z_1 - S_\epsilon(t)z_2\|_{\mathcal{U}_d}^2 \leq c(t^{-1} + 1) e^{c't} \|z_1 - z_2\|_{\mathcal{U}_d}^2, \quad \forall t > 0,
\] (51)
for any \(z_i = (\phi_{0i}, u_{0i}) \in \mathcal{B}_2\), \(i = 1, 2\), and any \(\epsilon \in (0,1]\).

Proof. We consider two solutions \((\phi_1, u_1)\) and \((\phi_2, u_2)\) of (6) with initial conditions
\[
\phi_i|_{t=0} = \phi_{0i}, \quad u_i|_{t=0} = u_{0i},
\]
such that \((\phi_{0i}, u_{0i}) \in \mathcal{B}_2\), \(i = 1, 2\). We set \(\phi = \phi_1 - \phi_2\), \(u = u_1 - u_2\), \(\tilde{\phi}_0 = \phi_{01} - \phi_{02}\) and \(\tilde{u}_0 = u_{01} - u_{02}\). There exists \(c > 0\), independent of \(\epsilon\), such that
\[
\|\phi(t)\|^2_2 + \epsilon \|u(t)\|^2_2 \leq c, \quad \forall t \geq 0.
\] (52)
The pair \((\phi, u)\) satisfies the problem
\[
\tau \phi_t + N(\phi + g(\phi) - g(\phi_2) - u) = 0,
\] (53)
\[
eu_t + \phi + Nu = 0,
\] (54)
\[
\phi|_{t=0} = \tilde{\phi}_0, \quad u|_{t=0} = \tilde{u}_0.
\] (55)
We multiply (53) and (54) by \( N^{-1} \phi_t \) and by \( u \), respectively, we integrate over \( \Omega \), and we obtain
\[
\frac{1}{2} \frac{d}{dt} \| \nabla \phi \|^2 + \tau \| \phi_t \|^2 + \delta \| \phi_t \|^2 + (g(\phi_1) - g(\phi_2)) - (u, \phi_t) = 0,
\]
and
\[
\frac{\epsilon}{2} \frac{d}{dt} \| u \|^2 + \| \nabla u \|^2 + (\phi_t, u) = 0.
\]
Summing (56) and (57), we deduce
\[
\frac{d}{dt} (\epsilon \| u \|^2 + \| \nabla \phi \|^2) + 2\| \nabla u \|^2 + 2\tau \| \phi_t \|^2 + \delta \| \phi_t \|^2 \leq c \| \phi \|^2_t,
\]
hence
\[
\| (\phi(t), u(t)) \|^2_{\mathcal{U}_{t, t}} + \int_0^t (\| u(s) \|^2 + \| \phi_t(s) \|^2) \, ds \leq c \| (\phi_0, u_0) \|^2_{\mathcal{U}_{t, t}} e^{\epsilon t}, \quad \forall t \geq 0,
\]
where \( c \) and \( c' \) are independent of \( \epsilon \). We can also deduce from (53) and (58) that
\[
\int_0^t \| \phi(s) \|^2_{\mathcal{U}_{t, t}} \, ds \leq c \| (\phi_0, u_0) \|^2_{\mathcal{U}_{t, t}} e^{\epsilon t}, \quad \forall t \geq 0.
\]
Now, we multiply (53) and (54) by \( \phi_t \) and \( Nu \), respectively, we integrate over \( \Omega \), and we find
\[
\frac{1}{2} \frac{d}{dt} \| Nu \|^2 + \tau \| \phi_t \|^2 + \delta \| \phi_t \|^2 + (\nabla (g(\phi_1) - g(\phi_2)), \nabla \phi_t) - (\nabla u, \nabla \phi_t) = 0,
\]
and
\[
\frac{\epsilon}{2} \frac{d}{dt} \| \nabla u \|^2 + \| Nu \|^2 + (\phi_t, \nabla u) = 0.
\]
Summing (60) and (61), we deduce
\[
\frac{d}{dt} (\epsilon \| \nabla u \|^2 + \| Nu \|^2) + c \left( \| Nu \|^2 + \| \phi_t \|^2 \right) \leq c \| \phi \|^2_{t}.
\]
We multiply (62) by \( t \) and obtain
\[
\frac{d}{dt} \left( ct \| \nabla u \|^2 + t \| Nu \|^2 \right) + ct \left( \| Nu \|^2 + \| \phi_t \|^2 \right) \leq c(t+1) \left( ct \| u \|^2 + \| \phi \|^2 \right).
\]
Integrating (63) between 0 and \( t \), and using (58) and (59), we deduce the result. \( \square \)

There exists \( t^* > 0 \) such that \( S_{\epsilon}(t) \mathcal{B}_k \subset \mathcal{B}_k \) for all \( t \geq t^* \). From now on, we set \( \mathcal{B}_k = S_{\epsilon}(t^*) \mathcal{B}_k \), and we will always assume that \( t^* \geq 1 \). Then we have that
\[
\mathcal{B}_k \subset \{ (\varphi, \psi) \in K_{\alpha, \sigma} \cap \mathcal{U}_k, \| (\varphi, \psi) \|^2_{\mathcal{U}_{t, t}} \leq r_k \}.
\]
Note that \( \mathcal{B}_k \) is a bounded absorbing set for \( S_{\epsilon}(t) |_{K_{\alpha, \sigma}} \) as well.

Now, we prove the following

**Proposition 3.** There exists \( c > 0 \), independent of \( \epsilon \), such that
\[
\| z \|^2_{\mathcal{U}_k} \leq c,
\]
for any \( z \in \mathcal{B}_k \) and any \( \epsilon \in (0, 1) \).
Proof. Multiplying (6)_2 by \( Nu \) and integrating over \( \Omega \), we obtain
\[
\epsilon \frac{d}{dt} \| \nabla u \|^2 + \| Nu \|^2 + (\phi_t, Nu) = 0. \tag{65}
\]
From (6)_1, we deduce that
\[
\phi_t = - (\tau I + \delta N)^{-1} N (N \phi + g(\phi) - u). \tag{66}
\]
Substituting (66) into (65), we find
\[
\epsilon \frac{d}{dt} \| \nabla u \|^2 + \| Nu \|^2 + \| N(\tau I + \delta N)^{-1/2} u \|^2
\]
\[
= \left( (\tau I + \delta N)^{-1} N \phi, Nu \right) + \left( (\tau I + \delta N)^{-1} N g(\phi), Nu \right). \tag{67}
\]
We have
\[
\| g(\phi(t)) \|_{L^\infty(\Omega)} \leq c,
\]
since \( \| \phi(t) \|_2 \leq c, \forall t \geq 0 \), and we deduce from (67) that
\[
\epsilon \frac{d}{dt} \| \nabla u \|^2 + c \| \nabla u \|^2 \leq c. \tag{68}
\]
We first multiply (68) by \( e^{c t/\epsilon} \), then we integrate between \( s \) and \( t + 1 \), for any \( s \leq t + 1 \). This yields
\[
\epsilon \| \nabla u(t + 1) \|^2 e^{c(t+1)/\epsilon} \leq c \| \nabla u(s) \|^2 e^{c s/\epsilon} + c \epsilon \left( e^{c(t+1)/\epsilon} - e^{c s/\epsilon} \right). \tag{69}
\]
Integrating now (69) between \( t \) and \( t + 1 \) with respect to \( s \), we deduce
\[
\| u(t) \|^2 \leq c, \forall t \geq 1,
\]
since \( |m(u_0)| \leq \sigma \), hence the result. \( \square \)

From now on, we will assume that
\[
\mathcal{B} = \{ \varphi \in \mathcal{D}(N^{k/2}), \| (I + N)^{k/2} \varphi \| \leq r_k \}
\]
is an absorbing set of \( S(t) \) on \( K_n \), where \( r_k > 0 \) are the same as in \( \mathcal{B}_k, k = 1, 2 \).

We now show the following estimate.

**Proposition 4.** There exist \( t_* > 0, c > 0 \) and \( c' > 0 \) (all independent of \( \epsilon \)) such that
\[
\| S_\epsilon(t)(\phi_0, u_0) - (S(t)\phi_0, L_{m(u_0)} S(t)\phi_0) \|_{L^\infty_t} \leq c \sqrt{\epsilon} e^{c' t}, \forall t \geq t_*, \tag{70}
\]
for any \( (\phi_0, u_0) \in \bar{\mathcal{B}}_2 \), and any \( \epsilon \in (0, 1] \).

**Proof.** Let us take \( (\phi_0, u_0) \in \bar{\mathcal{B}}_2 \). We set
\[
(\phi^\epsilon(t), u^\epsilon(t)) = S_\epsilon(t)(\phi_0, u_0), \quad (\phi(t), u(t)) = (S(t)\phi_0, L_{m(u_0)} S(t)\phi_0).
\]
On account of (52) and (64), there exist \( c > 0 \) such that
\[
\| \phi^\epsilon(t) \|^2 + \| u^\epsilon(t) \|^2 \leq c, \forall t \geq 0, \tag{71}
\]
\[
\| \phi(t) \|_2 \leq c, \forall t \geq 0. \tag{72}
\]
We now set \( P = \phi^\epsilon - \phi \) and \( R = u^\epsilon - u \) and they satisfy the following problem:
\[
\tau P_t + N(\delta P_t + N P + g(\phi^\epsilon) - g(\phi) - R) = 0, \tag{73}
\]
\[
c R_t + P_t + N R = -c u_t, \tag{74}
\]
\[
P|_{t=0} = 0, \quad R|_{t=0} = u_0 - L_{m(u_0)} \phi_0. \tag{75}
\]
We also have
\[ m(P(t)) = m(R(t)) = m(u_t(t)) = 0, \quad \forall t \geq 0. \]
We multiply (73) and (74) by \( N^{-1}P_t \) and \( R \), we integrate over \( \Omega \), and we obtain
\[ \frac{1}{2} \frac{d}{dt} \| P \|^2 + \tau \| P_t \|^2 + \| R \|^{2} + \delta \| P_t \|^{2} - \langle R, P_t \rangle + (g(\phi') - g(\phi), P_t) = 0, \tag{76} \]
and
\[ \frac{\epsilon}{2} \frac{d}{dt} \| R \|^2 + \| R \|^{2} + (P_t, R) = -\epsilon (u_t, R), \tag{77} \]
respectively. Summing (76) and (77) and noting that
\[ \| g(\phi') - g(\phi) \| \leq c \| P \|_1, \tag{78} \]
we deduce that
\[ \frac{d}{dt} (\| P \|^2 + \epsilon \| R \|^2) + 2 \| P_t \|^2 + \| R \|^{2} \leq c_1 \| P \|^2 + \epsilon^2 \| u_t \|^{2}. \tag{79} \]
We will show that
\[ \| u_t(t) \|^2 \leq ce^{ct}, \quad \forall t \geq 0. \tag{80} \]
We then apply the Gronwall’s lemma to (79), and we find
\[ \| P(t) \|^2 + \epsilon \| R(t) \|^2 \leq c \left( \epsilon \| u_0 - L_{m(u_0)} \phi_0 \|^2 + \epsilon \right) e^{ct}, \quad \forall t \geq 0, \tag{81} \]
due to (80). Integrating now (79) between 0 and \( t \), we obtain
\[ \int_0^t (\| P_s \|^2 + \| R_s \|^2) \, ds \leq c \left( \epsilon \| u_0 - L_{m(u_0)} \phi_0 \|^2 + \epsilon \right) e^{ct}, \quad \forall t \geq 0, \tag{82} \]
due to (80) and (81). From (73), we deduce that
\[ P_t = -(\tau I + \delta N)^{-1}N (NP + g(\phi') - g(\phi) - R). \tag{83} \]
Substituting (83) into (77), we find
\[ \frac{\epsilon}{2} \frac{d}{dt} \| R \|^2 + \| R \|^{2} + \| \nabla (\tau I + \delta N)^{-1/2} R \|^2
\]
\[ = \langle \nabla (\tau I + \delta N)^{-1}NP, \nabla R \rangle + \langle \nabla (\tau I + \delta N)^{-1}[g(\phi') - g(\phi)], \nabla R \rangle
\]
\[ - \epsilon (N^{-1/2}u_t, N^{1/2}R). \tag{84} \]
There exist \( C_1, C_2 > 0 \) such that
\[ C_1 \| \tilde{q} \| \leq \| \nabla (\tau I + \delta N)^{-1/2} q \|, \quad \forall q \in L^2(\Omega), \]
and
\[ \| (\tau I + \delta N)^{-\alpha} N^\alpha q \| \leq C_2 \| q \|, \quad \forall q \in H(N^\alpha), \]
for any \( \alpha > 0 \). We deduce from (84) and (78) that
\[ \frac{\epsilon}{2} \frac{d}{dt} \| R \|^2 + \| R \|^{2} + C_1 \| R \|^2 \leq c \| P \|^2 + \alpha \epsilon^2 \| u_t \|^{2}. \]
Also, due to (80) and (81), we have
\[ \frac{d}{dt} \left( \epsilon \| R \|^2 e^{ct} / \epsilon \right) \leq \epsilon \| R \|^2 e^{ct} / \epsilon + c \epsilon e^{ct} \left( \epsilon \| u_0 - L_{m(u_0)} \phi_0 \|^2 + \epsilon \right) e^{ct}. \tag{85} \]
We multiply (85) by \( t \) and we deduce that
\[ \frac{d}{dt} \left( \epsilon \| R \|^2 e^{ct} / \epsilon \right) \leq \epsilon \| R \|^2 e^{ct} + cte^{ct} \left( \epsilon \| u_0 - L_{m(u_0)} \phi_0 \|^2 + \epsilon \right) e^{ct}. \tag{86} \]
Integrating (86) between 0 and t, we find
\[ ct\|R(t)\|^2 \leq c \int_0^t \|R(s)\|^2 ds + cct \left( \epsilon \|u_0 - L_{m(u_0)}\phi_0\|^2 + \epsilon \right) e^c t, \]
so that, due to (82),
\[ \epsilon \|R(t)\|^2 \leq cc \left( \epsilon \|u_0 - L_{m(u_0)}\phi_0\|^2 + \epsilon \right) t^{-1} e^c t, \quad \forall t > 0, \]
hence
\[ \epsilon \|R(\sqrt{t})\|^2 \leq c\sqrt{t} \left( \epsilon \|u_0 - L_{m(u_0)}\phi_0\|^2 + \epsilon \right). \tag{87} \]
Like Estimate (81), we can show that
\[ \|P(t)\|^2 + \epsilon \|R(t)\|^2 \leq c \left( \epsilon \|u_0 - L_{m(u_0)}\phi_0\|^2 + \epsilon \right) e^c t, \quad \forall t \geq 0. \]
On the other hand, we have (cf. again (82))
\begin{align*}
\|P(t)\|^2 & \leq c\|P(t)\| \|P(t)\|_2 \\
& \leq c\|P(t)\|_2 \int_0^t \|P_i(s)\| ds \\
& \leq c \left( \epsilon \|u_0 - L_{m(u_0)}\phi_0\|^2 + \epsilon \right) \sqrt{t} e^c t, \quad \forall t \geq 0,
\end{align*}
so that
\[ \|P(\sqrt{t})\|^2 \leq c\sqrt{t} \left( \epsilon \|u_0 - L_{m(u_0)}\phi_0\|^2 + \epsilon \right). \tag{88} \]
We now apply the Gronwall’s lemma to (79) between \( \sqrt{t} \) and \( t + \sqrt{t} \). We find
\[ \left( \|P\|^2 + \epsilon \|R\|^2 \right) (t + \sqrt{t}) \leq c \left[ \left( \|P\|^2 + \epsilon \|R\|^2 \right) \left( \sqrt{t} + \sqrt{t} \right) \right] e^c t, \quad \forall t \geq 0. \tag{89} \]
Thanks to (87) and (88), from (89) it follows that
\[ \left( \|P\|^2 + \epsilon \|R\|^2 \right) (t + \sqrt{t}) \leq c\sqrt{t} \left( \epsilon \|u_0 - L_{m(u_0)}\phi_0\|^2 + \epsilon \right) e^c t, \quad \forall t \geq 0. \tag{90} \]
Again apply the Gronwall’s lemma to (79) between \( s \) and \( t \), we obtain the following estimate
\[ \|P(t)\|^2 + \epsilon \|R(t)\|^2 \leq c \left( \|P(s)\|^2 + \epsilon \|R(s)\|^2 + \epsilon^2 \right) e^c t, \]
for any given \( s \geq 0 \) and any \( t > s \). Let \( t_0 > 0 \), independent of \( \epsilon \), be such that \( t_0 > \sqrt{t} \). This latter estimate, with \( s = \sqrt{t} \), combined with (90) gives
\[ \|P(t)\|^2 + \epsilon \|R(t)\|^2 \leq c\sqrt{t} \left( \epsilon \|u_0 - L_{m(u_0)}\phi_0\|^2 + \epsilon \right) e^c t, \quad \forall t > \sqrt{t}. \tag{91} \]
Finally, estimate (71) and (91) yield the result. \( \square \)

**Proof of (80).** Firstly, we observe that \( v = \phi_t \) is solution to the problem
\begin{align*}
(1 + \tau)v_t + N(\delta v_t + Nv + g'(\phi)v) = 0, \tag{92} \\
v|_{t=0} = I\phi_0, \tag{93}
\end{align*}
where \( \phi(t) = S(t)\phi_0 \),
\[ I\phi_0 = -[(1 + \tau)I + \delta N]^{-1}N(N\phi_0 + g(\phi_0)), \]
and, \( w = u_t \) satisfies
\[ w(t) = [(1 + \tau)I + \delta N]^{-1}(Nv(t) + g'(\phi(t))v(t)), \quad \forall t \geq 0. \]
We have
\[ \|g'(\phi(t))\|_{L^\infty((1))} \leq c, \quad \forall t \geq 0, \tag{94} \]
due to (72), and therefore,
\[ \|g'(\phi(t))v(t)\| \leq c \|v(t)\|, \quad \forall t \geq 0. \]
We multiply (92) by \( N^{-1}v \) and integrate over \( \Omega \), and we deduce
\[ \frac{d}{dt} \left[ (1 + \tau)\|v\|_{E_1}^2 + \delta \|v\|_{V_1}^2 \right] + \|\nabla v\|^2 \leq c \|v\|^2, \]
and then
\[ \|v(t)\|^2 \leq ce^{ct}, \quad \forall t \geq 0. \]
Hence the result. \( \square \)

6.2. A robust family of exponential attractors. We now give sufficient conditions ensuring the existence of uniform exponential attractors that are continuous with respect to \( \epsilon \) (cf. [3, Theorem 5.1]; cf. also [2, 11, 15]). More precisely, we have the following theorem.

Theorem 6.1 ([3]). Let \( E^1, E^2, V^1, V^2, W^1, W^2 \) be Banach spaces such that \( W^i \subseteq V^i \subseteq E^i, i = 1, 2 \). Set \( E_\epsilon = E^1 \times E^2, V_\epsilon = V^1 \times V^2, W_\epsilon = W^1 \times W^2 \) and endow them with the following norms
\[ \|(p, q)\|_{E_\epsilon} = (\|p\|_{E^1}^2 + \epsilon \|q\|_{E^2}^2)^{1/2}, \]
\[ \|(p, q)\|_{V_\epsilon} = (\|p\|_{V^1}^2 + \epsilon \|q\|_{V^2}^2)^{1/2}, \]
\[ \|(p, q)\|_{W_\epsilon} = (\|p\|_{W^1}^2 + \epsilon \|q\|_{W^2}^2)^{1/2}, \]
respectively, where \( \epsilon \in [0, 1] \), with the convention that \( E_0 = E^1, V_0 = V^1, \) and \( W_0 = W^1 \). Let \( B_\epsilon(r) \) denote a closed ball in \( W_\epsilon \) of radius \( r > 0 \) and centered at zero. Consider a one-parameter family of strongly continuous semigroups \( \{S_\epsilon(t)\}_\epsilon \) acting on the phase-space \( E_\epsilon \), for each \( \epsilon \in [0, 1] \). Then assume that there exist \( \alpha, \beta, \gamma, \vartheta \in (0, 1], \kappa \in (0, \frac{1}{2}), \epsilon_j \geq 0, \) and \( \varrho > 0 \) (all independent of \( \epsilon \)) such that, setting \( B_\epsilon = B_\epsilon(\varrho) \), the following conditions hold:

1. There exists a map \( L : B_0 \to V^2 \) which is Hölder continuous of exponent \( \alpha \). Here \( B_0 \) is endowed with the metric topology of \( E^1 \).
2. There exists \( t^*_\epsilon > 0 \), independent of \( \epsilon \), such that
\[ S_\epsilon(t)B_\epsilon \subseteq B_\epsilon, \quad \forall t \geq t^*_\epsilon, \]
and \( B_\epsilon \) is uniformly bounded (with respect to \( \epsilon \)) in the \( E_1 \)-norm. Moreover, setting \( S_\epsilon(t^*_\epsilon) = S_\epsilon \), the map \( S_\epsilon \) satisfies, for every \( z_1, z_2 \in B_\epsilon \),
\[ S_\epsilon z_1 - S_\epsilon z_2 = L_\epsilon(z_1, z_2) + K_\epsilon(z_1, z_2), \]
where
\[ \|L_\epsilon z_1 - L_\epsilon z_2\|_{E_\epsilon} \leq \kappa \|z_1 - z_2\|_{E_\epsilon}, \]
\[ \|K_\epsilon z_1 - K_\epsilon z_2\|_{V_\epsilon} \leq \epsilon_j \|z_1 - z_2\|_{E_\epsilon}. \]
3. For any \( z \in B_\epsilon \), there hold
\[ \|S_\epsilon^m z - L_\epsilon S_0^m \Pi_\epsilon z\|_{E_1} \leq \epsilon_j m^\beta, \quad \forall m \in \mathbb{N}, \]
\[ \|S_\epsilon(t)z - L_\epsilon S_0(t)\Pi_\epsilon z\|_{E_1} \leq \epsilon_j t^{\epsilon_j}, \quad \forall t \in [t^*_\epsilon, 2t^*_\epsilon]. \]

Here the "lifting" map \( L_\epsilon : B_0 \to E_\epsilon \) is defined by
\[ L_\epsilon x = \begin{cases} (x, Lx), & \text{if } \epsilon > 0, \\ x, & \text{if } \epsilon = 0, \end{cases} \]
Moreover, there exist $$\Pi_\epsilon : B_\epsilon \to B_0$$ is the projection onto the first component when $$\epsilon > 0$$, and the identity map otherwise.

4. The map $$z \mapsto S_\epsilon(t)z$$ is Lipschitz continuous on $$B_\epsilon$$ endowed with the metric topology of $$E_\epsilon$$, with a Lipschitz constant independent of $$\epsilon$$ and $$t \in [t^*, 2t^*]$$.

5. The map

$$(t, z) \mapsto S_\epsilon(t)z : [t^*, 2t^*] \times B_\epsilon \to B_\epsilon$$

is Hölder continuous of exponent $$\theta$$, where $$B_\epsilon$$ is endowed with the metric topology of $$E_\epsilon$$.

Then there exists a family of exponential attractors $$E_\epsilon$$ on $$B_\epsilon = \overline{E}_\epsilon$$ with the following properties:

(i) $$E_\epsilon$$ attracts $$B_\epsilon$$ with an exponential rate which is uniform with respect to $$\epsilon$$, that is,

$$\text{dist}_{E_\epsilon}(S_\epsilon(t)B_\epsilon, E_\epsilon) \leq M_1 e^{-\omega t}, \ \forall t \geq 0,$$

for some $$M_1 > 0$$ and some $$\omega > 0$$.

(ii) The fractal dimension of $$E_\epsilon$$ is uniformly bounded with respect to $$\epsilon$$, that is,

$$\text{dim}_{E_\epsilon}(E_\epsilon) \leq M_2.$$

(iii) The family $$E_\epsilon$$ is Hölder continuous with respect to $$\epsilon$$, that is, there exist a positive constant $$M_3$$ and $$\tau \in (0, \frac{1}{2}]$$ such that

$$\text{dist}_{E_\epsilon}^\text{sym}(E_\epsilon, L_c E_0) \leq M_3 \epsilon^\tau,$$

for all $$0 < \epsilon \leq 1$$. In addition, there exist a positive constant $$M_4$$ and $$\sigma \in (0, \frac{1}{2}]$$ such that

$$\text{dist}_{E_\epsilon}(E_\epsilon, L_c E_0) \leq M_4 \epsilon^\sigma,$$

for all $$0 < \epsilon \leq 1$$, and

$$\lim_{\epsilon \to 0} \text{dist}_{E_\epsilon}(L_c E_0, E_\epsilon) = 0.$$

Here $$\omega$$, $$\tau$$, $$\sigma$$ and $$M_j$$ are independent of $$\epsilon$$, and they can be computed explicitly.

We prove the following result.

**Theorem 6.2.** For every $$\epsilon \in (0, 1]$$, the semigroup $$S_\epsilon(t)$$ possesses an exponential attractor $$E^{\alpha, \sigma}_\epsilon$$ in $$K_{\alpha, \sigma}$$, that is,

(i) $$E^{\alpha, \sigma}_\epsilon$$ is compact and positively invariant, that is, $$S_\epsilon(t)E^{\alpha, \sigma}_\epsilon \subset E^{\alpha, \sigma}_\epsilon, \ \forall t \geq 0$$;

(ii) the fractal dimension of $$E^{\alpha, \sigma}_\epsilon$$ is finite;

(iii) there exists a constant $$c_0 > 0$$ such that, for every bounded subset $$B \subset K_{\alpha, \sigma}$$, there exists a constant $$c_1(B) > 0$$ such that $$c_1$$ and $$c_2$$ are independent of $$\epsilon$$

$$\text{dist}_{K_{\alpha, \sigma}}(S_\epsilon(t)B, E^{\alpha, \sigma}_\epsilon) \leq c_1 e^{-c_0 t}, \ \forall t \geq 0.$$

Moreover, there exist $$0 < \omega_1 \leq \frac{1}{2}$$ and $$M_1 > 0$$ (all independent of $$\epsilon$$) such that

$$\text{dist}_{K_{\alpha, \sigma}}(E^{\alpha, \sigma}_\epsilon, (K^{\alpha})^\sigma) \leq M_1 \epsilon^{\omega_1},$$

for any $$\epsilon \in (0, 1]$$, and

$$\lim_{\epsilon \to 0} \text{dist}_{K_{\alpha, \sigma}}((K^{\alpha})^\sigma, E^{\alpha, \sigma}_\epsilon) = 0,$$

where $$E^{\alpha}_\epsilon$$ is an exponential attractor for the semigroup $$S(t)$$ on $$K_{\alpha}$$.
Proof. We first observe that the semigroup $S(t)$ has an exponential attractor $\mathcal{E}^\alpha$ on $K_\alpha$ (see [4]). Let us now prove the theorem. On account of Theorem 6.1, we let $E_\epsilon = \mathcal{U}_1$, $V_\epsilon = W_\epsilon = \mathcal{U}_2$, $B_\epsilon = \overline{\mathcal{B}_2}$ and we check all the assumptions 1-5. To verify Assumption 1, we show that there exists a constant $c$ such that

$$
\|L_\beta \phi_1 - L_\beta \phi_2\| = \|(1 + \tau)I + \delta N\|^{-1}(N(\phi_1 - \phi_2) + g(\phi_1) - g(\phi_2))
\leq c(\|\phi_1 - \phi_2\| + \|g(\phi_1) - g(\phi_2)\|)
\leq c(\|\phi_1 - \phi_2\|_1),
$$

(97)

for any $\phi_1$ and $\phi_2$ in $\mathcal{B}$. Assumption 2 is satisfied by Propositions 2 and 3. Assumption 3 is given by Proposition 4. We choose $t^*$ such that (70) is satisfied. Thus, we are left to check Assumptions 4 and 5. Indeed, defining

$$(\phi_i(t), u_i(t)) = S_\epsilon(t)z_{0i},$$

with $z_{0i} = (\phi_{0i}, u_{0i}) \in \overline{\mathcal{B}_2}$, $i = 1, 2$, and $t \in [t^*, 2t^*]$, we obtain

$$
\|S_\epsilon(t)z_{01} - S_\epsilon(t')z_{02}\|_{\mathcal{U}_{t',t}}
\leq \|S_\epsilon(t)z_{01} - S_\epsilon(t')z_{01}\|_{\mathcal{U}_{t',t}} + \|S_\epsilon(t')z_{01} - S_\epsilon(t')z_{02}\|_{\mathcal{U}_{t',t}}, \forall t, t' \in [t^*, 2t^*].
$$

On the one hand, we have

$$
\int_0^t \|\phi(t)\|^2 ds + \int_0^t \|u(t)\|^2 ds \leq c(t + 1), \forall t \geq 0.
$$

(98)

Therefore,

$$
\|S_\epsilon(t)z_{01} - S_\epsilon(t')z_{01}\|_{\mathcal{U}_{t',t}} \leq c \left( \|\phi(t) - \phi(t')\|_1 + \sqrt{c}\|u(t) - u(t')\| \right)
\leq c \int_t^{t'} \|\phi(s)\|_1 ds + c \int_t^{t'} \sqrt{c}\|u(s)\| ds
\leq c(t^*)|t' - t|^{1/2}.
$$

On the other hand, it follows from (51) that

$$
\|S_\epsilon(t')z_{01} - S_\epsilon(t')z_{02}\|_{\mathcal{U}_{t',t}} \leq c(t^*)|z_{01} - z_{02}|_{\mathcal{U}_{t',t}}, \forall t' > 0.
$$

Hence, we conclude with

$$
\|S_\epsilon(t)z_{01} - S_\epsilon(t')z_{02}\|_{\mathcal{U}_{t',t}} \leq c(t^*)(|t' - t|^{1/2} + |z_{01} - z_{02}|_{\mathcal{U}_{t',t}}).
$$

This shows the existence of exponential attractors on $\overline{\mathcal{B}_2}$ that satisfy (95) and (96). Then, like in [12], we can extend the basin of attraction to the whole phase-space $\mathcal{U}_1$ by using the transitivity property of the exponential attraction. \hfill \Box

Proof of (98). We deduce from (66) and (71) that

$$
\|\phi_i(t)\|^2 \leq c, \forall t \geq 0,
$$

(99)

and

$$
\int_0^t \|\phi(t)\|^2 ds \leq c \int_0^t \left( \|\phi(s)\|^2 + \|u(s)\|^2 + 1 \right) ds, \forall t \geq 0.
$$

(100)

Integrating (19) between 0 and $t$, we deduce that

$$
\int_0^t \left( \|\phi(s)\|^2 + \|u(s)\|^2 \right) ds \leq c(t + 1), \forall t \geq 0.
$$

(101)

The estimate on $\phi_i$ follows from (100) and (101).
To prove the remaining estimate on \( u_t \), we multiply (6) by \( u_t \), and we integrate over \( \Omega \), and we find
\[
\frac{1}{2} \frac{d}{dt} \|\nabla u\|^2 + \epsilon \|u_t\|^2 + (\phi_t, u_t) = 0.
\] (102)
Substituting (66) into (102), we find
\[
\frac{1}{2} \frac{d}{dt} \|\nabla u\|^2 + \|\nabla (\tau I + \delta N)^{-1/2} u\|^2 - (N^2 (\tau I + \delta N)^{-1} \phi, u)
\]
\[= -(N(\tau I + \delta N)^{-1} \phi_t, Nu) - (N(\tau I + \delta N)^{-1} \phi_t g' \phi), u). \] (103)
Using (99) and (94), we deduce from (103) that
\[
\frac{1}{2} \frac{d}{dt} \|\nabla u\|^2 + \|\nabla (\tau I + \delta N)^{-1/2} u\|^2 - (N(\tau I + \delta N)^{-1} \phi, u)
\]
\[= -(N(\tau I + \delta N)^{-1} \phi_t, Nu) - (N(\tau I + \delta N)^{-1} \phi_t g' \phi), u). \] (104)
Integrating (104) between 0 and \( t \), and using (71) and (101), the result follows. □

6.3. Continuity of inertial manifolds. We now want to prove some stability properties of the inertial manifolds. Firstly, we recall that the semigroup \( S(t) \) possesses an inertial manifold \( M \) on \( K_\alpha \) (see [10, 19]). More precisely, there exists a Lipschitz mapping \( \Phi^\alpha : PK_\alpha \cap D(\Lambda) \to QD(\Lambda) \) such that the graph of \( \Phi^\alpha \) defines an inertial manifold
\[
M^\alpha = \{ p + \Phi^\alpha(p), \; \; p \in PK_\alpha \cap D(\Lambda) \},
\]
for the unperturbed "prepared problem":
\[
(1 + \tau) \phi_t + N(\delta \phi_t + N\phi + g(\phi)) = 0,
\]
where \( g \) is defined by (30). Here \( P \) is the unique orthogonal projection in \( D(\Lambda) \) onto the space spanned by \( \{ e_0(x), e_1(x), \ldots, e_n(x) \} \) (cf. Sect. 5), and \( Q = I - P \).

For any arbitrary \( R > 0 \), we define the following bounded sets
\[
M^\alpha_{c,R} = \{ (\phi, u), \; (\phi, u) = p + \Phi^\alpha(p), \; \| p \| \leq R \},
\]
\[
M^\alpha_R = \{ \phi, \; \phi = p + \Phi^\alpha(p), \; \| p \| \leq R \},
\]
\[
M^\alpha_{c,R} = \{ \phi \in M^\alpha_R, \; m(\phi) = \mu \},
\]
and
\[
(M^\alpha_{c,R}^\mu)^{\sigma,\beta} = \{ (\varphi, L_{\beta} \varphi), \; \varphi \in M^\alpha_{c,R}^\mu \},
\]
which are intersection of the inertial manifolds \( M^\alpha_{c,R} \) proven in Theorem 5.2 and \( M^\alpha \) with bounded sets (we recall that \( L_{\beta} \) is defined by (25)).

We prove the following results.

**Theorem 6.3.** Let the assumptions of Theorem 5.2 hold and let us assume that \( M^\alpha_{c,R} \subset \overline{B}_2 \). Then
\[
\lim_{\epsilon \to 0} \text{dist}_{C^k}(M^\alpha_{\epsilon,R}^\sigma, (M^\alpha)^\sigma) = 0.
\] (105)
In addition, let \( \mu \in [-\alpha, \alpha] \), \( \beta \in [-\sigma, \sigma] \). If we assume that \( M^\alpha_{R} \subset B \), then
\[
\lim_{\epsilon \to 0} \text{dist}_{C^k}((M^\alpha_{\epsilon,R}^\mu)^{\sigma,\beta}, M^\alpha_{c,R}^\sigma) = 0.
\] (106)
Proof. Exactly like (26), we can show that, if \( \eta > 0 \), then there exist \( t_\eta > 0 \) and \( \epsilon_\eta \) (all depending only on \( \eta \)) such that

\[
\forall \epsilon \leq \epsilon_\eta, \quad \text{dist}_{\mathcal{U}_\epsilon}(S_\epsilon(t_\eta)) \leq \eta.
\]  

(107)

If we assume now that \( \mathcal{M}_\epsilon^{\alpha,\sigma} \subset \bar{B}_2 \) then (105) follows from (107), the invariance property

\[
\bar{S}_\epsilon(t)\mathcal{M}_\epsilon^{\alpha,\sigma} = \mathcal{M}_\epsilon^{\alpha,\sigma}, \quad \forall t \in \mathbb{R},
\]

and the fact that \( \bar{S}_\epsilon(t) \) and \( S_\epsilon(t) \) coincide on \( \bar{B}_2 \).

In order to show (106), we need an intermediate result. That is, there exists \( M_2 = M_2(n) > 0 \), independent of \( \epsilon \), such that

\[
\text{dist}_{\mathcal{U}_\epsilon}((\mathcal{M}_\epsilon^{\alpha,\mu})^{\sigma,\beta}, \mathcal{M}_\epsilon^{\alpha,\sigma}) \leq M_2 \sqrt{\epsilon},
\]

(108)

for every \( \epsilon \in (0, \epsilon(n)] \).

Let us now prove (106). Let \( (\phi_0, L_\beta \phi_0) \in (\mathcal{M}_\epsilon^{\alpha,\mu})^{\sigma,\beta} \). There exists a sequence \( (\phi_\epsilon, v_\epsilon)_{\epsilon > 0} \) of solutions to Problem (31) such that \( (\phi_\epsilon, v_\epsilon) \in \mathcal{M}_\epsilon^{\alpha,\sigma} \) and converges to \( (\phi_0, L_\beta \phi_0) \), in the norm \( \| \cdot \|_{\mathcal{U}_\epsilon} \), as \( \epsilon \) goes to zero, due to (108). Since \( \mathcal{M}_\epsilon^{\alpha,\sigma} \subset \bar{B}_2 \), \( \mathcal{M}_\epsilon^{\alpha,\sigma} \) is uniformly (in \( \epsilon \)) bounded in the norm \( \| \cdot \|_{\mathcal{U}_\epsilon} \) (cf. (64)), there exists a subsequence which we still denote by \( (\phi_\epsilon, v_\epsilon)_{\epsilon > 0} \) and which converges to some \( (\phi, \xi) \in (\mathcal{M}_\alpha)^\sigma \), in the norm \( \| \cdot \|_{\mathcal{U}_\epsilon} \), due to (105). Clearly, \( \phi = \phi_0 \) and \( \xi = L_\beta \phi_0 \), and this limit is independent of the subsequence chosen. Consequently, the whole sequence \( (\phi_\epsilon, v_\epsilon)_{\epsilon > 0} \) converges to \( (\phi_0, L_\beta \phi_0) \), hence the result.

Proof of (108). Firstly, from Remark 1 we can deduce that, there exist \( c_1 \) and \( c_2 \) (independent of \( \epsilon \)) such that

\[
c_1 \| U \|_{\mathcal{U}_\epsilon} \leq \| U \| \leq c_2 \| U \|_{\mathcal{U}_\epsilon}, \quad \forall U \in \mathcal{U}_\epsilon.
\]

(109)

For any \( \eta > 0 \), we denote by \( C_\eta((\infty, 0]; \mathcal{U}_\epsilon) \) the following Banach space (cf., e.g., [8, 18])

\[
C_\eta((\infty, 0]; \mathcal{U}_\epsilon) = \{ f : f : (\infty, 0] \to \mathcal{U}_\epsilon \text{ continuous and sup}_{t \leq 0} e^{\eta t} \| \Gamma f(t) \| < \infty \},
\]

with norm

\[
\| f \|_{C_\eta((\infty, 0]; \mathcal{U}_\epsilon)} = \sup_{t \leq 0} e^{\eta t} \| \Gamma f(t) \|.
\]

Now let us prove (108). We consider an element \( U_0 = (\phi_0, L_\beta \phi_0) \) of \( (\mathcal{M}_R^{\alpha,\mu})^{\sigma,\beta} \). Since \( \phi_0 \in \mathcal{M}_R^{\alpha,\mu} \), there exists a complete trajectory \( (\phi(t))_{t \in \mathbb{R}} \) lying in \( \mathcal{M}_\alpha \) such that the pair \( (\phi, u) \) is solution of the unperturbed problem, written in the form:

\[
\begin{align*}
\tau \phi_t + N(\delta \phi_t + N \phi + g(\phi) - u) &= 0, \\
\epsilon u_t + \phi_t + Nu &= \epsilon u_t, \\
\phi(0) &= \phi_0, \quad u(0) = u_0, \quad m(\phi_0) = \mu, \quad m(u_0) = \beta,
\end{align*}
\]

(110)

(the term \( \epsilon u_t \) was added to both sides of (2) to obtain (110)). Thus, for all time \( t \in \mathbb{R} \), the function \( U(t) = (\phi(t), u(t)) \) satisfies the non autonomous initial value problem:

\[
\begin{align*}
U_t + AU + G(U) &= F(t), \\
U(0) &= (\phi_0, L_\beta \phi_0),
\end{align*}
\]

(111)

where

\[
F(t) = (0, u_t(t))
\]

(we refer the reader to (29) and (31)).
We will prove that, there exists $M = M(n) > 0$, independent of $\epsilon$, such that
\[
\|\Gamma u_t(s)\| \leq Me^{-\gamma_n s}, \quad \forall s \leq 0,
\] (112)
and, for every $\epsilon \in (0, \tilde{c}(n)]$, there holds
\[
\mu_n^- < \gamma_n < \mu_{n+1}^-,
\] (113)
where
\[
\gamma_n = \frac{\lambda_n^2}{1 + \tau + \delta \lambda_n}.
\]
We now choose $\eta = \eta(n)$ (independent of $\epsilon$) such that
\[
\mu_n^- < \gamma_n \leq \eta < \mu_{n+1}^-.
\] (114)
Let $(U_\epsilon(t))_{t \in \mathbb{R}}$ be a complete trajectory lying in $\mathcal{M}^{\alpha,\sigma}$ and solution to (31). The function $z(t) = U(t) - U_\epsilon(t)$ is defined for all time $t \in \mathbb{R}$ and satisfies the problem
\[
\begin{align*}
&\left\{ z_t + A z + G(U) - G(U_\epsilon) = F(t), \\
&z(0) = U_0 - U_\epsilon(0).
\right.
\end{align*}
\] (115)
We write $z(t) = p(t) + q(t)$, where $p = \mathcal{P}z$ and $q = \mathcal{Q}z$; and we have that
\[
z(t) = e^{-A P t} p(0) + \int_0^t e^{-A P (t-s)} P \{G(U_\epsilon(s)) - G(U(s)) + F(s)\} \, ds
\] + \int_{-\infty}^t e^{-A Q (t-s)} Q \{G(U_\epsilon(s)) - G(U(s)) + F(s)\} \, ds,
\] (116)
(cf., e.g., [8, 9, 23]). Since $\mathcal{P} U_d$ is a finite-dimensional subspace of $U_d$, we can choose $U_\epsilon(0)$ such that $p(0) = 0$. On account of (126), and the fact that $\|A \Phi(t)\| \leq c, \forall t \in \mathbb{R}$, we can deduce that $\|A \Phi(t)\| \leq ce^{-\gamma_n t}, \forall t \leq 0$, and then $\|\Gamma U(t)\| \leq ce^{-\gamma_n t}, \forall t \leq 0$. Similarly, we can show that $\|\Gamma U_\epsilon(t)\| \leq ce^{-\mu_n^- t}, \forall t \leq 0$. Hence, $z \in C_n((\alpha, \sigma); U_d)$, due to (114). Thus, we deduce from (116) and (49) that
\[
\|z\|_{C_n((\alpha, \sigma); U_d)} = \sup_{t \leq 0} e^{\eta t} \left\{ \int_0^t e^{-A P (t-s)} P \{G(U_\epsilon(s)) - G(U(s)) + F(s)\} ds \right. \\
+ \left. \int_{-\infty}^t e^{-A Q (t-s)} Q \{G(U_\epsilon(s)) - G(U(s)) + F(s)\} ds \right\}
\leq c_1 \sup_{t \leq 0} \left\{ e^{\eta t} \int_0^t e^{(s-t)\mu_n^-} ds + e^{\eta t} \int_{-\infty}^t e^{(s-t)\mu_{n+1}^-} ds \right\} \|z\|_{C_n((\alpha, \sigma); U_d)}
+ c_2 \sup_{t \leq 0} \left\{ e^{\eta t} \int_0^t e^{(s-t)\mu_n^-} \|\Gamma u_t(s)\| ds + e^{\eta t} \int_{-\infty}^t e^{(s-t)\mu_{n+1}^-} \|\Gamma u_t(s)\| ds \right\}.
\] (117)
We infer from (117), (112), (113) and (114) that
\[
\|z\|_{C_n((\alpha, \sigma); U_d)} \leq c_1 \sup_{t \leq 0} \left\{ \frac{1}{\mu_n} e^{(\eta - \mu_n^-) t} [1 - e^{t\mu_n^-}] + \frac{1}{\mu_{n+1}^-} e^{\eta t} \right\} \|z\|_{C_n((\alpha, \sigma); U_d)}
+ c_2 \sup_{t \leq 0} \left\{ \frac{1}{\gamma_n - \mu_n} [e^{(\eta - \gamma_n) t} - e^{(\eta - \mu_n^-) t}] + \frac{1}{\mu_{n+1}^- - \gamma_n} e^{(\eta - \gamma_n) t} \right\},
\]
that is,
\[ \|z\|_{C_n((-\infty,0];U_t)} \leq c_1 \left( \frac{1}{\mu_n} + \frac{1}{\mu_{n+1}} \right) \|z\|_{C_n((-\infty,0];U_t)} + c_2 \sqrt{\epsilon} \left( \frac{1}{\gamma_n - \mu_n} + \frac{1}{\mu_{n+1} - \gamma_n} \right). \]  
(118)

We then deduce from (118), with an appropriate choice of \( n \), that
\[ \|z\|_{C_n((-\infty,0];U_t)} \leq \frac{1}{2} \|z\|_{C_n((-\infty,0];U_t)} + M \sqrt{\epsilon}, \]
that is,
\[ \|z\|_{C_n((-\infty,0];U_t)} \leq M \sqrt{\epsilon}. \]  
(119)

Observing that
\[ \|\Gamma z(0)\| \leq \|z\|_{C_n((-\infty,0];U_t)}, \]
we deduce from (119) that
\[ \|\Gamma z(0)\| \leq M \sqrt{\epsilon}, \]
that is,
\[ \|\Gamma(\phi_0, L_{B} \phi_0) - \Gamma U_{0}(0)\| \leq M \sqrt{\epsilon}. \]  
(120)

Estimates (120) and (109) imply the lower semicontinuity estimate (108). \( \square \)

**Proof of (112).** Because \( \phi_0 \in \mathfrak{M}^\mu_R \), the function \( \phi(t) \) is in the form
\[ \phi(t) = p(t) + \Phi^\alpha(p(t)), \quad m(\phi(t)) = \mu, \quad \forall t \in \mathbb{R}, \]
where \( p(t) \) satisfies \( p \in C(\mathbb{R}, PK_{\alpha} \cap \mathcal{D}(\Lambda)) \) and is the unique solution to the problem
\[ p_t + \Lambda p + P \Phi^\alpha(p) = 0, \]  
(121)
\[ p|_{t=0} = p_0, \]  
(122)
with \( \|p_0\| \leq R \). Here \( H(\phi) = B^{-1} N g(\phi) \),
\[ B = (1 + \tau) I + \delta N \quad \text{and} \quad A = B^{-1} N^2 : \mathcal{D}(N) \to L^2(\Omega). \]

Since \( g \in C^2 \), it is known that \( \Phi^\alpha \in C^2 \) and
\[ \|\Lambda(\Phi^\alpha)'(p)\| \leq 1, \quad \forall p \in PK_{\alpha} \cap \mathcal{D}(\Lambda), \]
(cf. [9]). We have
\[ \phi_t(t) = p_t(t) + (\Phi^\alpha)'(p(t)) p_t(t) \]
and
\[ \|\Lambda \phi_t(t)\| \leq \|\Lambda p_t(t)\| + \|\Lambda((\Phi^\alpha)'(p(t)) p_t(t))\| \leq c \|\Lambda p(t)\|. \]  
(123)

From (121), we can also deduce that
\[ \|\Lambda p_t(t)\| \leq \|A \Lambda p(t)\| + \|\Lambda H(p + \Phi^\alpha(p))\| \leq \gamma_n \|\Lambda p(t)\| + c. \]  
(124)

We take the \( L^2 \)-scalar product of (121) with \( \Lambda^2 p \) and we obtain
\[ \frac{1}{2} \frac{d}{dt} \|\Lambda p\|^2 + (A \Lambda p, \Lambda p) + (P \Lambda H(p + \Phi^\alpha(p)), \Lambda p) = 0, \]

hence
\[ \frac{1}{2} \frac{d}{dt} \|\Lambda p\|^2 + (A \Lambda p, \Lambda p) \leq c \|\Lambda p\|. \]

We then deduce that
\[ -\|\Lambda p\| \frac{d}{dt} \|\Lambda p\| \leq \gamma_n \|\Lambda p\|^2 + c \|\Lambda p\|. \]
and, therefore,
\[ -\frac{d}{dt} \|Ap\| \leq \gamma_n \|Ap\| + c. \tag{125} \]
We now apply the Gronwall lemma to (125) between \( t \) and 0, \( t \leq 0 \), and we find
\[ \|Ap(t)\| \leq c \|Ap_0\| e^{-\gamma_n t}, \quad \forall t \leq 0. \tag{126} \]
Finally, we remind that
\[ u_t(t) = B^{-1}(N\phi_t(t) + g'(\phi(t))\phi_t(t)), \quad \forall t \in \mathbb{R}, \]
and we can deduce that
\[ \|Au_0(t)\| \leq c \|A\phi_0\| e^{-\gamma_n t} + c \|g'(\phi(t))\phi_t(t)\| \leq c \|A\phi_t(t)\| \quad \forall t \in \mathbb{R}. \tag{127} \]
Estimate (112) follows from (123), (124), (126) and (127).

**Proof of (113).** We have
\[ \mu_{n+1}^- - \gamma_n = \frac{S_n + \epsilon T_n - (1 + \tau + \delta \lambda_n)\lambda_{n+1} + \lambda_{n+1}^\perp n\perp n\perp D_{n+1}}{2\epsilon(\tau + \delta \lambda_n)(1 + \tau + \delta \lambda_n)}, \]
where
\[ S_n = (1 + \tau + \delta \lambda_n)(1 + \tau + \delta \lambda_{n+1})\lambda_{n+1} \]
and
\[ T_n = (1 + \tau + \delta \lambda_n)\lambda_{n+1}^2 - 2\lambda_n^2(\tau + \delta \lambda_{n+1}). \]
When \( \epsilon \in (0, \epsilon(n)] \), we can see that the sign of \( S_n + \epsilon T_n - (1 + \tau + \delta \lambda_n)\lambda_{n+1} + \lambda_{n+1}^\perp n\perp n\perp D_{n+1} \) is the same as that of \( (S_n + \epsilon T_n)^2 - (1 + \tau + \delta \lambda_n)^2\lambda_{n+1}^2 D_{n+1}, \) and
\[ (S_n + \epsilon T_n)^2 - (1 + \tau + \delta \lambda_n)^2\lambda_{n+1}^2 D_{n+1} \]
\[ = 2\epsilon(1 + \tau + \delta \lambda_n)\lambda_{n+1}[T_n(1 + \tau + \delta \lambda_{n+1}) - (1 + \tau + \delta \lambda_n)(1 - \tau - \delta \lambda_{n+1})\lambda_{n+1}^2] \]
\[ + \epsilon^2[T_n^2 - (1 + \tau + \delta \lambda_n)^2\lambda_{n+1}^2]. \]
Now, a simple computation shows that
\[ T_n(1 + \tau + \delta \lambda_{n+1}) - (1 + \tau + \delta \lambda_n)(1 - \tau - \delta \lambda_{n+1})\lambda_{n+1}^2 \]
\[ = 2\tau(1 + \tau)(\lambda_{n+1}^2 - \lambda_n^2) + 2\delta \lambda_n\lambda_{n+1}^2(\lambda_{n+1}^2 - \lambda_n) \]
\[ + 2\delta \lambda_{n+1}[\tau \lambda_n(\lambda_{n+1}^2 - \lambda_n) + (1 + \tau)(\lambda_{n+1}^2 - \lambda_n^2)], \]
which is positive, and
\[ T_n^2 - (1 + \tau + \delta \lambda_n)^2\lambda_{n+1}^4 \]
\[ = -4(\tau \lambda_n^2 + \delta \lambda_{n+1}^2)(\lambda_{n+1}^2 + \tau(\lambda_{n+1}^2 - \lambda_n^2) + \delta \lambda_n\lambda_{n+1}(\lambda_{n+1}^2 - \lambda_n)), \]
which is negative. Thus, \( \mu_{n+1}^- - \gamma_n \) is positive, for every \( \epsilon \in (0, \epsilon(n)]. \)

Let us now show the other inequality. We have
\[ \mu_n^- - \gamma_n = \frac{Q_n + \epsilon R_n - (1 + \tau + \delta \lambda_n)\lambda_n + \lambda_{n+1}^\perp n\perp n\perp D_{n+1}}{2\epsilon(\tau + \delta \lambda_n)(1 + \tau + \delta \lambda_n)}, \]
where
\[ Q_n = (1 + \tau + \delta \lambda_n)^2\lambda_n \]
and
\[ R_n = (1 - \tau - \delta \lambda_n)^2\lambda_n. \]
When \( \epsilon \in (0, \tilde{\epsilon}(n)] \), we can see that the sign of \( Q_n + \epsilon R_n - (1 + \tau + \delta \lambda_n) \lambda_n \sqrt{D_n} \) is the same as that of \( (Q_n + \epsilon R_n)^2 - (1 + \tau + \delta \lambda_n)^2 \lambda_n^2 D_n \), and
\[
(Q_n + \epsilon R_n)^2 - (1 + \tau + \delta \lambda_n)^2 \lambda_n^2 D_n = -4 \epsilon^2 (\tau + \delta \lambda_n) \lambda_n^4,
\]
which is negative. The proof of (113) is completed.

**Remark 2.** All the continuity properties of the global and exponential attractors and inertial manifolds proven in [1] relied on an estimate that was valid on an absorbing set \( B_8 \) in \( U_8 \) and that required the condition \( g \in C^8(\mathbb{R}) \). This estimate can be proven on an absorbing set \( B_4 \) in \( U_4 \) and under a weaker condition \( g \in C^4(\mathbb{R}) \) at most, adapting the proof of (70). In addition, we give here all the details of the proofs of Theorems 6.2 and 6.3 compared to [1] where only sketches of the proofs were given.

**Remark 3.** In [2], we analyzed the non-conserved phase-field system
\[
\begin{aligned}
\tau \phi_t - \Delta \phi + g(\phi) &= u, \\
\epsilon u_t + \phi_t - \Delta u &= 0,
\end{aligned}
\]
and subject to Dirichlet boundary conditions. To prove continuity properties of the global and exponential attractors and inertial manifolds we required the condition \( g \in C^3(\mathbb{R}) \) and an absorbing set \( B_3 \) in \( U_3 \). Using the methods of the present paper, this condition could be weaken to \( g \in C^2(\mathbb{R}) \) and we would only need to work with an absorbing set \( B_2 \) in \( U_2 \). Also, an explicit proof of the spectral gap condition (8.8) in [2] was not given. Here we give a detailed proof of the spectral gap condition (cf. (37)). In addition, we prove the existence of inertial manifolds for Problem 6 in the variable \((\phi, u)\) directly, and not via an auxiliary problem in the variable \((\phi, w)\) as mentioned in the introduction.

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