HYBRID MIXED DISCONTINUOUS GALERKIN FINITE ELEMENT METHOD FOR INCOMPRESSIBLE WORMHOLE PROPAGATION PROBLEM

JIANSONG ZHANG, YUN YU, JIANG ZHU, YUE YU AND RONG QIN

ABSTRACT. Wormhole propagation plays a very important role in the product enhancement of oil and gas reservoir. A new combined hybrid mixed finite element method is proposed to solve incompressible wormhole propagation problem with discontinuous Galerkin finite element procedure, in which, the new hybrid mixed finite element algorithm is established for concentration equation, while the discontinuous Galerkin finite element method is considered for concentration equation, and then the porosity function is computed straightforwardly by the approximate value of the concentration. This new combined method can keep local mass balance, meantime it also keeps the boundedness of the porosity. The convergence of the proposed method is analyzed and the optimal error estimate is derived. Finally, numerical examples are presented to verify the validity of the algorithm and the correctness of the theoretical results.

1. INTRODUCTION

The acid treatment of carbonate reservoirs is a widely practiced oil and gas well stimulation technique. In fact, when acids are injected into oil production wells, chemical reactions cause the dissolution of the material near the wellbore to result into flow channels. Such flow channels look like worm holes that they are usually called wormholes. Because of its important role in the product enhancement of oil and gas reservoir, the wormhole propagations have been a topic of key interest for research during recent decades. The theoretical researches on numerical methods for these problems have extensive practicability and important significance.

Here, we will construct a new combined numerical procedure to solve the incompressible wormhole propagation problem which is usually described by the following nonlinear partial differential equations (see [123]):

\[
\begin{aligned}
&\frac{\partial \phi}{\partial t} = \frac{\alpha k_c a_v (c_f - c_s)}{\rho_s}, \quad x \in \Omega, \quad 0 \leq t \leq T, \\
&\frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{u} = f, \quad x \in \Omega, \quad 0 \leq t \leq T, \\
&\mathbf{u} = \frac{-k(\phi)}{\mu} \nabla p, \quad x \in \Omega, \quad 0 \leq t \leq T, \\
&\frac{\partial (\phi c_f)}{\partial t} + \nabla \cdot (\mathbf{uc}_f) = \nabla \cdot (\phi \mathbf{D}(\mathbf{u}) \nabla c_f) + k_c a_v (c_s - c_f) + f_I c_I - f_P c_f,
\end{aligned}
\]

and the corresponding initial-boundary conditions are considered as follows:

\[
\begin{aligned}
&\phi(x, 0) = \phi_0(x), \quad c_f(x, 0) = c_{f0}(x), \quad x \in \Omega, \\
&\mathbf{u} \cdot \mathbf{n} = 0, \quad (\phi \mathbf{D}(\mathbf{u}) \nabla c_f - \mathbf{f} c_f) \cdot \mathbf{n} = 0, \quad x \in \partial \Omega, \quad 0 \leq t \leq T,
\end{aligned}
\]

where \( \Omega \subset \mathbb{R}^d (d = 2, 3) \) denotes a bounded polygonal/polyhedral domain; \( \alpha \) is the dissolving constant of the acid; \( \rho_s \) is the density of the rock; \( \mathbf{n} \) is the unit outward normal vector to \( \partial \Omega \); the functions \( p \) and \( \mathbf{u} \) denote the pressure and Darcy velocity; \( \phi \) and \( k \) are the porosity and permeability of rocks, and \( \mu \) is the viscosity of fluid; \( a_v \) is the interfacial area available for reaction; \( f, f_I \) and \( f_P \) are the external volumetric flow rate, the injection flow rate and the production flow rate, respectively; \( c_f, c_s \) and \( c_f \) are the concentration of acid in the fluid phase, the fluid-solid interface and the injected flow, respectively; Diffusion coefficient \( \mathbf{D}(\mathbf{u}) = \phi [d_m I + |\mathbf{u}|(d_f E(\mathbf{u}) + d_t E^T(\mathbf{u}))] \) comes from two aspects: small molecule diffusion of oil field scale problem, and speed-related diffusion in petroleum engineering, where the matrix \( E(\mathbf{u}) = (u_i u_j / |\mathbf{u}|^2)_{d \times d} \) and

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Corresponding author: Jiansong Zhang.

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Assume that the parameters \( \mu, k_v, k_s, \alpha, \rho_s \) are positive constants, and that \( \phi_0, \frac{k(\phi)}{\mu} \) and \( f(\cdot, t) \) are bounded as follows:

\[
0 < a_s \leq \frac{k(\phi)}{\mu} \leq a_s^*, \quad 0 < \phi_0 < 1, \quad |f(\cdot, t)| \leq C,
\]
where $a_\ast$, $a^*$ and $C$ are some positive constants. And we also assume that the diffusion coefficient $D(u)$ satisfies the uniformly positive definiteness and Lipschitz continuousness

$$D(u)\nabla c \cdot \nabla c \geq D_\ast |\nabla c|^2$$

and

$$\|D(u) - D(v)\|_{[L^2]^d} \leq D^* \|u - v\|_{[L^2]^d},$$

where $D^*$ and $D_\ast$ are two positive constants independent of $u$ and $v$ and $c$.

**Assumption 1.2.** Assume that the solution $(p, u, c_f)$ of the system $(1)$ has the regularities as follows:

(a) $p \in L^2(0, T; H^{k+2}(\Omega))$, $\phi_0 \leq \phi \leq C_1 < 1$,

(b) $u \in L^\infty(0, T; H^{k+1}(\Omega)) \cap L^\infty(0, T; L^\infty(\Omega))$,

(c) $c_f \in H^1(0, T; H^{k+1}(\Omega)) \cap L^\infty(0, T; W^{1,\infty}(\Omega))$,

(d) $\frac{\partial c_f}{\partial t} \in L^2(0, T; H^{k+1}(\Omega)) \cap L^\infty(0, T; L^\infty(\Omega))$.

Moreover, we only consider the homogeneous boundary condition case in this article for simplification. For nonhomogeneous boundary value problem, we can use some simple technique to transform it into homogeneous boundary value problem, so our method proposed later is still valid.

2. The formulation of HMDG method

In order to illustrate our method, we first give a uniform regular partition of $\Omega$, denoted by $T_h = \{K_1, K_2, ..., K_n\}$. We denote $E_h = \bigcup_{K \in T_h} \{e \in \partial K\}$ to be the set of all cell edges and $E_h^i = E_h \cap \partial \Omega$ as all the interior ones. Furthermore, let $h_e = \text{diam}(e)$ for all $e \in E_h$. Introduce the following piecewise Sobolev spaces associated with $T_h$

$$H^s(T_h) = \{v \in L^2(\Omega) : v|_K \in H^s(K), K \in T_h\}, \quad s \geq 0.$$

We also define the following spaces on $E_h$:

$$L^2(E_h) = \{v \in L^2(e) : v e \in E_h\}, \quad L^2(E_h^i) = \{v \in L^2(e) : v e \in E_h^i\}.$$

For $e \in E_h^i$, denote by $n_e$ a fixed unit normal direction. For $e \in \partial \Omega$, $n_e = n$. We define averages $\{\cdot\}$ and jumps $[\cdot]$

$$\{v\} = \frac{1}{2}[(v|_{K_1})|_e + (v|_{K_2})|_e], \quad [v] = (v|_{K_1})|_e - (v|_{K_2})|_e, \quad \text{on} \ e \in E_h^i.$$

In particular, if $e \in \partial \Omega$, $\{v\} = v|_e$, $[v] = v|_e$. Meantime, we define inner products as follows:

$$\langle \cdot, \cdot \rangle_{T_h} = \sum_{K \in T_h} \langle \cdot, \cdot \rangle_K, \quad \langle \cdot, \cdot \rangle_{E_h} = \sum_{e \in E_h} \langle \cdot, \cdot \rangle_e, \quad \langle \cdot, \cdot \rangle_{E_h^i} = \sum_{e \in E_h^i} \langle \cdot, \cdot \rangle_e,$$

and the norms $\|\cdot\|_{T_h} = \sqrt{\langle \cdot, \cdot \rangle_{T_h}}$ and $|\cdot|_{\partial T_h} = \sqrt{\langle \cdot, \cdot \rangle_{E_h}}$.

Introduce the discrete approximate spaces denoted by $\Psi_h, \Lambda_h, \Pi_h$ and $\Sigma_h$ as follows:

$$\Psi_h = \{v \in H^k(T_h) : v|_K \in P_k(K), K \in T_h\},$$

$$\Lambda_h = \{v \in L^2(T_h) : v|_K \in P_k(K), K \in T_h\},$$

$$\Pi_h = \{v \in [H^k(T_h)]^d : v|_K \in RT_k(K), K \in T_h\},$$

$$\Sigma_h = \{v \in L^2(E_h) : v|_e \in P_k(e), e \in E_h\},$$

where $P_k(K), P_k(e)$ are the spaces of polynomial functions of degree at most $k$ for each $K \in T_h$ and each $e \in E_h$, respectively, $RT_k(K) = [P_k(K)]^d \oplus xP_k(K)$ denotes the Raviart-Thomas mixed finite element space as in [9, 27, 28].

Set

$$\kappa = \frac{k_ck_a0}{(k_c + k_s)(1 - \phi_0)}.$$


Using (5) and (4), we can rewrite (1) in the following equivalent form:

\[
\frac{\partial \phi}{\partial t} = \frac{\alpha \kappa}{\rho_s}(1 - \phi) c_f,
\]

(8) \[\frac{\alpha \kappa}{\rho_s}(1 - \phi) c_f + \nabla \cdot \mathbf{u} = f, \quad \mathbf{u} = -\frac{k(\phi)}{\mu} \nabla p, \]

Next, we will formulate our method for wormhole propagation.

For the discretization of the porosity, we consider the similar technique as in [3]. The discrete porosity is point-wise defined and can be stated as follows:

(9) \[\frac{\partial \phi_h}{\partial t} = \frac{\alpha \kappa}{\rho_s}(1 - \phi_h)\bar{c}_h, \]

where \(\bar{c}_h = \max(0, \min(c_h, 1))\), \(c_h\) is a given approximation of the concentration \(c_f\).

2.1. Hybrid mixed finite element scheme for pressure and velocity. In this subsection, we give the hybrid mixed finite element (HMFE) method for pressure and velocity, which can be written as below:

**Algorithm 1.** For given approximate values of \(\phi_h\) and \(c_h\), seek \((p_h, \mathbf{u}_h, \lambda_h) \in \Lambda_h \times \Pi_h \times \Sigma_h\) such that

\[
\sum_{K \in \mathcal{T}_h} \left( \frac{\alpha \kappa}{\rho_s}(1 - \phi_h)\bar{c}_h, v_h \right)_K + \sum_{K \in \mathcal{T}_h} (\nabla \cdot \mathbf{u}_h, v_h)_K = \sum_{K \in \mathcal{T}_h} (f, v_h)_K, \quad \forall v_h \in \Lambda_h,
\]

(10) \[\sum_{K \in \mathcal{T}_h} (a(\phi_h)\mathbf{u}_h, \omega_h)_K - \sum_{K \in \mathcal{T}_h} (p_h, \nabla \cdot \omega_h)_K + \sum_{e \in \mathcal{E}_h} (\lambda_h, [\omega_h] \cdot n_e)_e = 0, \quad \forall \omega_h \in \Pi_h,
\]

\[\sum_{e \in \mathcal{E}_h} (\mu_h, [\mathbf{u}_h] \cdot n_e)_e = 0, \quad \forall \mu_h \in \Sigma_h,
\]

where \(a(\phi_h) = \frac{\alpha \kappa}{\rho_s}\) and \(\bar{c}_h = \max(0, \min(c_h, 1))\).

Define the bilinear form:

\[
B_{\phi}(\mathbf{u}_h; \phi_h, v_h) := \left( \mathbf{u}_h, \nabla v_h \right)_{\mathcal{T}_h} + (\nabla p_h, \omega_h)_{\mathcal{T}_h} + (a(\phi_h)\mathbf{u}_h, \omega_h)_{\mathcal{T}_h} + \left( \lambda_h - p_h, [\omega_h] \cdot n_e \right)_{\mathcal{E}_h} + \left( [\mathbf{u}_h] \cdot n_e, \mu_h - v_h \right)_{\mathcal{E}_h},
\]

We can rewrite (10) into the following equivalent form:

**Algorithm 2 (HMFE Algorithm).** For given \(\phi_h\) and \(c_h\), find \((\mathbf{u}_h, p_h, \lambda_h) \in \Pi_h \times \Lambda_h \times \Sigma_h\) such that

\[
B_{\phi}(c_h; \phi_h, v_h) + B_{\mathbf{u}}((\mathbf{u}_h, p_h, \lambda_h), (\omega_h, v_h, \mu_h)) = -(f, v_h)_{\mathcal{T}_h},
\]

(11) \[\forall (\omega_h, v_h, \mu_h) \in \Pi_h \times \Lambda_h \times \Sigma_h.
\]

2.2. DGFE method for the concentration. Due to the flexibility of the discontinuous Galerkin finite element method in constructing feasible local-shape function spaces and the advantage in capturing non-smooth or oscillatory solutions effectively, we consider it to be applied for the concentration.

Define the bilinear form

\[
B_{\mathbf{c}}(c_h, z_h) := \sum_{K \in \mathcal{T}_h} \int_K (\phi_h D(\mathbf{u}_h) \nabla c_h - \mathbf{u}_h c_h) \nabla z_h \, dx - \sum_{e \in \mathcal{E}_h} \int_e \{(\phi_h D(\mathbf{u}_h) \nabla z_h - \mathbf{u}_h z_h) \cdot n_e\} [c_h] ds
\]

\[ - \sum_{e \in \mathcal{E}_h} \int_e \{(\phi_h D(\mathbf{u}_h) \nabla c_h - \mathbf{u}_h c_h) \cdot n_e\} [z_h] ds + J_0^\gamma(c_h, z_h),
\]

where \(J_0^\gamma(c_h, z_h)\) denotes the penalty term defined by

\[
J_0^\gamma(c_h, z_h) = \sum_{e \in \mathcal{E}_h} \int_e \frac{\gamma}{h_e^\gamma} [c_h] [z_h],
\]

where \(\gamma\) is called penalty parameter and bounded below by a large enough constant, and \(\beta\) denotes some positive constant.

Now we can reach the SIPDG method for the concentration equation.
Proof. First, we prove that the solution of \((8)\) also solves \((13)\). For this, let \(z\) be an element in \(\Psi_h\). We multiply the third equation of \((8)\) by \(z\) and integrate on one element \(K\):

\[
\int_K \frac{\partial \phi}{\partial t} z \, dx + \int_K \phi \mathbf{D}(\mathbf{u}) \nabla c_f - \mathbf{u} c_f \cdot \nabla z \, dx - \int_{\partial K} \phi \mathbf{D}(\mathbf{u}) \nabla c_f - \mathbf{u} c_f \cdot n_K z \, ds
\]

\[
= \int_K (f_I c_f - f_P c_f - \kappa (1 - \phi) c_f) z \, dx.
\]

Summing it over all elements and using \((2)\), we observe that

\[
\sum_{K \in T_h} \int_{\partial K} (\phi \mathbf{D}(\mathbf{u}) \nabla c_f - \mathbf{u} c_f) \cdot n_e z \, ds = \sum_{e \in \mathcal{E}_h} \int_e (\phi \mathbf{D}(\mathbf{u}) \nabla c_f - \mathbf{u} c_f) \cdot n_e z \, ds.
\]

By the regularities of the solution \(\phi, u,\) and \(c\), we have

\[
(a) \quad (\phi \mathbf{D}(\mathbf{u}) \nabla c_f - \mathbf{u} c_f) \cdot n_e z \, ds = (\phi \mathbf{D}(\mathbf{u}) \nabla c_f - \mathbf{u} c_f) \cdot n_e[z] \, ds,
\]

\[
(b) \quad [c_f] = 0.
\]

Therefore, we obtain the second equation of the scheme \((13)\). Conversely, take \(z \in H^1(\Omega)\) and \(c_f \in H^1(\Omega) \cap \Psi_h\). Then \((13)\) reduces to

\[
\sum_{K \in T_h} \int_K \frac{\partial \phi}{\partial t} + \nabla (\mathbf{u} \cdot c_f) = \nabla (\phi \mathbf{D}(\mathbf{u}) \nabla c_f) + f_I c_f - f_P c_f - \kappa (1 - \phi) c_f, \quad \text{in } K.
\]

Finally, let \(K_1\) and \(K_2\) to be two adjacent elements, and \(e = \partial K_1 \cap \partial K_2\). Take \(z \in C_0^\infty(K_1 \cup K_2)\) and extend it by zero over the rest of the domain. Integrating by parts in the second equation of \((8)\), we can get

\[
\left(\frac{\partial \phi}{\partial t}, z\right)_{K_1 \cup K_2} + (\phi \mathbf{D}(\mathbf{u}) \nabla c_f - \mathbf{u} c_f, \nabla z)_{K_1 \cup K_2} - (\phi \mathbf{D}(\mathbf{u}) \nabla c_f - \mathbf{u} c_f, n_e[z], z)_{e}
\]

\[
= (f_I c_f - f_P c_f - \kappa (1 - \phi) c_f, z)_{K_1 \cup K_2}.
\]
On the other hand, (13) reduces to
\[
(\frac{\partial(\phi c_f)}{\partial t}, z)_{K_1 \cup K_2} + (\phi D(u)\nabla c_f - uc_f, \nabla z)_{K_1 \cup K_2} = (f_I c_f - f_p c_f - \kappa(1 - \phi) c_f, z)_{K_1 \cup K_2}.
\]
Hence, we have
\[
(\phi D(u)\nabla c_f - uc_f, z)_{K_1 \cup K_2} = 0, \quad \forall z \in C_0^\infty(K_1 \cup K_2).
\]
Since this holds for all \( e \), it implies that \((\phi D(u)\nabla c_f - uc_f)\cdot n_e = 0 \) on \( \partial \Omega \) and \( \nabla \cdot (\phi D(u)\nabla c_f - uc_f) \in L^2(\Omega) \), hence we have
\[
(\frac{\partial(\phi c_f)}{\partial t}) + \nabla \cdot (uc_f - \phi D(u)\nabla c_f) = f_I c_f - f_p c_f - \kappa(1 - \phi) c_f.
\]
\[\square\]

We can easily show that the discrete solution \( \phi_h \) of \( \phi \) satisfies the following boundedness.

**Theorem 2.2** (The boundedness of porosity). For any time \( t \in (0, T) \), the approximate porosity \( \phi_h \) is bounded, that is,
\[
(15) \quad \phi_0 \leq \phi_h \leq 1 - (1 - \phi_0)e^{-\eta t} < 1,
\]
\[
(16) \quad 0 \leq \partial_t \phi_h \leq \frac{\alpha \kappa}{\rho s},
\]
where \( \eta = \frac{\alpha \kappa}{\rho s} \) and \( \phi_0 > 0 \).

**Proof.** First, we can rewrite (9) as the following integral form
\[
\int_{\phi_0}^{\phi_h} \frac{1}{1 - \phi_h} d\phi = \int_0^t \eta \bar{c}_h d\tau,
\]
where \( \eta > 0 \). We deduce that
\[
\ln \frac{1 - \phi_h}{1 - \phi_0} = -\eta \bar{c}_h t.
\]
Notice that
\[
\phi_h = 1 - (1 - \phi_0)e^{-\eta \bar{c}_h t} \leq 1 - (1 - \phi_0)e^{-\eta t}.
\]
It is easily seen that the approximate value of the porosity increases with \( t \), and \( \phi_h = \phi_0 \) at \( t = 0 \), so we can get the estimate (15). The estimate (16) is reached by (15) and (9).
[\square]

For HMDG Algorithm, we have the following main convergence theorem.

**Theorem 2.3.** Under the assumptions (1.1) and (1.2), for \( t > 0 \), there is an a priori error estimate as follows:
\[
(17) \begin{cases}
(\text{a}) & \|\phi - \phi_h\|_{T_h} + \|u - u_h\|_{T_h} + \|c_f - c_h\|_{T_h} \leq Ch^s(\|c_f\|_{H^1(0, T; H^s(T_h))} + \|p\|_{L^2(0, T; H^{s+1}(T_h))}), \\
(\text{b}) & \|\nabla(p_h - \Pi_h p)\|_{T_h} + h^{-\frac{1}{2}} \lambda_h - p\|_{T_h} \leq Ch^s(\|c_f\|_{H^1(0, T; H^s(T_h))} + \|p\|_{L^2(0, T; H^{s+1}(T_h))}), \\
(\text{c}) & \|p - p_h\|_{T_h} \leq Ch^s(\|c_f\|_{H^1(0, T; H^s(T_h))} + \|p\|_{L^2(0, T; H^{s+1}(T_h))}),
\end{cases}
\]
where when \( d = 2 \), \( 1 \leq s \leq k + 1 \); when \( d = 3 \), \( 3/2 \leq s \leq k + 1 \).

3. Some important projections and lemmas

In this section, we will give some important projection operators and approximate properties, which is used to show the convergence theorem of our proposed method.

Firstly, we introduce the following norms with respect to the bilinear form \( B_u \):
\[\|\omega, v, \mu\|_B^2 := \|\omega\|_{T_h}^2 + \|\nabla v\|_{T_h}^2 + \frac{1}{h_e} |\mu - v|_{T_h}^2,\]
and
\[\|\omega, v, \mu\|_{B_{xu}}^2 := \|\omega, v, \mu\|_B^2 + h |\omega \cdot n_e|_{T_h}^2.\]
As in (11)(26), we can read the following stability and boundedness of the bilinear form \( B_u \).
Lemma 3.1 (Stability and Boundedness). Assume that $\phi$ and $\phi_h$ are fixed, for all $(u, p, \lambda) \in H^k(T_h) \times L^2(T_h) \times L^2(\partial T_h)$ and $(\omega_h, v_h, \mu_h) \in \Pi_h \times \Lambda_h \times \Sigma_h$, there holds

\[
(a) \quad |B_u((u, p, \lambda), (\omega_h, v_h, \mu_h))| \leq K^* \| (u, p, \lambda) \|_{B, \star} \| (\omega_h, v_h, \mu_h) \|_{B},
\]

\[
(b) \quad \sup_{(\omega_h, v_h, \mu_h) \in \Pi_h \times \Lambda_h \times \Sigma_h} \| B_u((u_h, p_h, \lambda_h), (\omega_h, v_h, \mu_h)) \|_{\omega_h, v_h, \mu_h} \geq K_s \| (u_h, p_h, \lambda_h) \|_{B},
\]

where $K_s$ and $K^*$ denote two positive constants independent of the mesh size $h$.

Introduce the local $L^2$-projection operators $\Pi_h$ and $\Pi_e$ as follows:

\[
(p - \Pi_h p, v_h)_K = 0, \quad \forall v_h \in P_k(K),
\]

\[
\langle \lambda - \Pi_e p, \mu_h \rangle_e = 0, \quad \forall \mu_h \in P_k(e),
\]

where $K \in T_h$, $e \in E_i^h$, $p \in L^2(K)$ and $\lambda \in L^2(e)$.

Lemma 3.2 (29). For the local $L^2$-projection operators $\Pi_h$ and $\Pi_e$, there exists the following approximate property

\[
\| p - \Pi_h p \|_K \leq Ch^s \| p \|_{s, K}, \quad 0 \leq s \leq k + 1,
\]

\[
\| \nabla(p - \Pi_h p) \|_K \leq Ch^s \| p \|_{s+1, K}, \quad 0 \leq s \leq k,
\]

\[
\| p - \Pi_h p \|_e + \| p - \Pi_e p \|_e \leq Ch^{s+\frac{1}{2}} \| p \|_{s+1, K}, \quad 0 \leq s \leq k.
\]

The classical Raviart-Thomas projection operator as in [9] is also used

\[
(u - \Pi^{RT} u, \omega_h)_K = 0, \quad \forall \omega_h \in [P_{k-1}(K)]^d,
\]

and

\[
\langle (u - \Pi^{RT} u) \cdot n_e, \mu_h \rangle_e = 0, \quad \forall \mu_h \in P_k(e), \quad e \in \partial K.
\]

We can reach the error estimate as in [9]:

Lemma 3.3. For the Raviart-Thomas interpolation $\Pi^{RT}$, the following estimate hold

\[
\| \nabla \cdot (u - \Pi^{RT} u) \|_K \leq Ch^s \| \nabla \cdot u \|_{s, K}, \quad 1 \leq s \leq k + 1,
\]

\[
\| u - \Pi^{RT} u \|_K + \frac{1}{2} \| u - \Pi^{RT} u \| \leq Ch^s \| u \|_{s, K}, \quad \frac{1}{2} \leq s \leq k + 1.
\]

Utilizing the above results element-wise, we can easily get the following error estimates.

Lemma 3.4. If $a(\phi)$ is bounded, there exists the following inequality

\[
\| (u - \Pi^{RT} u, p - \Pi_h p, \lambda - \Pi_e p) \|_{B, \star} \leq Ch^s | p |_{s+1, T_h}, \quad \frac{1}{2} < s \leq k.
\]

Remark 3.1. From [24], the following estimate holds: for any $1/2 \leq s \leq k + 1$,

\[
\| u - \Pi^{RT} u \|_K \leq Ch^s | p |_{s, K}.
\]

For the concentration, we introduce another projection operator $\Pi_s$ as follows:

\[
B_{c_s}(c_f - \Pi_s c_f, z_h) + (c_f - \Pi_s c_f, z_h) = 0, \quad \forall z_h \in \Psi_h,
\]

where $\delta$ should be some sufficient large constant.

As in [17], under the following inductive hypothesis

\[
\| u_h \|_{L^\infty} \leq C_u,
\]

where $C_u$ is a positive constant, we can reach the following estimates:

\[
\| c_f - \Pi_s c_f \|_{s, T_h} \leq Ch^s | c_f |_{s, T_h}, \quad 0 \leq s \leq k + 1,
\]

\[
\| \partial_t (c_f - \Pi_s c_f) \|_{s, T_h} \leq Ch^s (| c_f |_{s, T_h} + \| \partial_t (c_f) \|_{s, T_h}), \quad 0 \leq s \leq k + 1.
\]

The following trace inequalities will be also used to prove the convergence theorem (see Lemma 3.1 in [23]).

Lemma 3.5. For $\forall v \in H^1(K)$, the trace inequalities are shown below

\[
\| v \|_{0, e}^2 \leq C(h^{-1}_e \| v \|_{0, K}^2 + h_e \| v \|_{1, K}^2),
\]

\[
\| \nabla v \cdot n_e \|_{0, e}^2 \leq C(h^{-1}_e \| \nabla v \|_{0, K}^2 + h_e \| \nabla^2 v \|_{0, K}^2).
\]
4. THE PROOF OF CONVERGENCE THEOREM

Now, we can complete the proof of our convergence theorem [2, 3].

Proof. We firstly give the bound of \( \|c_h - c_f\|_{T_h} \). Set \( \xi_c = c_h - \Pi_s c_f, \xi_c = c_f - \Pi_s c_f \). Taking \( z_h = \xi_c \) in (26), we have

\[
\sum_{K \in T_h} \int_K \frac{\partial \phi_h \xi_c}{\partial t} dx + \sum_{K \in T_h} \int_K \phi_h \mathbf{D}(\mathbf{u}_h) \nabla \xi_c \cdot \nabla \xi_c dx \\
+ \sum_{K \in T_h} \int_K \mathbf{f}_p \xi_c^2 dx + J_0^\ast (\xi_c, \xi_c)
\]

\[
= \sum_{K \in T_h} \int_K \frac{\partial \phi_h \xi_c}{\partial t} dx + \sum_{K \in T_h} \int_K (\phi \mathbf{D}(\mathbf{u}) - \phi_h \mathbf{D}(\mathbf{u}_h)) \nabla c_f \cdot \nabla \xi_c dx \\
+ 2 \sum_{e \in \mathcal{E}_h} \int_e \{ \phi_h \mathbf{D}(\mathbf{u}_h) \nabla \xi_c \cdot n_e \} [\xi_c] ds + \sum_{K \in T_h} \int_K (\mathbf{u}_h - \mathbf{u}) c_f \nabla \xi_c dx
\]

(30)

\[
+ \sum_{e \in \mathcal{E}_h} \int_e \{ (\mathbf{u}_h - \mathbf{u}) \nabla c_f \cdot n_e \} [\xi_c] ds - 2 \sum_{e \in \mathcal{E}_h} \int_e \{ \mathbf{u}_h \xi_c \cdot n_e \} [\xi_c] ds
\]

\[
+ \sum_{K \in T_h} \int_K (f_P - \delta) \xi_c dx + \sum_{K \in T_h} \int_K \kappa [(1 - \phi)c_f - (1 - \phi_h)c_h] \xi_c dx
\]

\[
+ \sum_{K \in T_h} \int_K \frac{\partial (\phi - \phi_h)}{\partial t} c_f \xi_c dx + \sum_{K \in T_h} \int_K \frac{\partial c_f}{\partial t} (\phi - \phi_h) \xi_c dx
\]

\[= F_1 + F_2 + \cdots + F_{12}. \]

Now we estimate the terms on the right hand side of (30) one by one. Using (6) and Lemma 2.2, we can get the following result

\[ |F_1| + |F_2| + |F_4| + |F_6| + |F_9| + |F_{10}| + |F_{11}| + |F_{12}| \]

\[ \leq C \{ \|\xi_c\|_{T_h}^2 + \|\nabla \xi_c\|_{T_h}^2 + \frac{\|\partial \xi_c\|_{T_h}}{\partial t}^2 + \frac{\|\partial (\phi - \phi_h)\|_{T_h}}{\partial t}^2 \} + \|\phi - \phi_h\|_{T_h}^2 + \|\mathbf{u} - \mathbf{u}_h\|_{T_h}^2 + \varepsilon \|\nabla \xi_c\|_{T_h}^2. \]

For \( F_3 \), using (29) we have

\[ |F_3| \leq \varepsilon J_0^\ast (\xi_c, \xi_c) + C \frac{\sum_{e \in \mathcal{E}_h} \gamma^{-1} h_e \|\nabla \xi_c\|_{L^2(e)}^2}{\|\xi_c\|_{[\mathcal{E}]} \|\xi_c\|_{L(e)}} \]

\[ \leq \varepsilon J_0^\ast (\xi_c, \xi_c) + C_1 \gamma^{-1} \|\nabla \xi_c\|_{T_h}^2. \]

Next, we estimate \( F_5 \) with (6) and (29)

\[ |F_5| \leq \|\nabla c_f\|_{L^\infty} \sum_{e \in \mathcal{E}_h} \|\mathbf{D}(\mathbf{u}) - \mathbf{D}(\mathbf{u}_h)\|_{L^2(e)} \|\xi_c\|_{L(e)} \]

\[ \leq \varepsilon J_0^\ast (\xi_c, \xi_c) + C \frac{\sum_{e \in \mathcal{E}_h} \gamma^{-1} h_e \|\mathbf{u} - \mathbf{u}_h\|_{L(e)}^2}{\|\xi_c\|_{[\mathcal{E}]} \|\xi_c\|_{L(e)}} \]

\[ \leq \varepsilon J_0^\ast (\xi_c, \xi_c) + C \gamma^{-1} \|\mathbf{u} - \mathbf{u}_h\|_{T_h}^2. \]

Using the same technique as above, we can reach

\[ |F_7| \leq \varepsilon J_0^\ast (\xi_c, \xi_c) + C \frac{\sum_{e \in \mathcal{E}_h} \gamma^{-1} h_e \|\mathbf{u} - \mathbf{u}_h\|_{L(e)}^2}{\|\xi_c\|_{[\mathcal{E}]} \|\xi_c\|_{L(e)}} \]

\[ \leq \varepsilon J_0^\ast (\xi_c, \xi_c) + C \gamma^{-1} \|\mathbf{u} - \mathbf{u}_h\|_{T_h}^2, \]

\[ |F_8| \leq \varepsilon J_0^\ast (\xi_c, \xi_c) + \|\xi_c\|_{T_h}^2, \]
where $\gamma$ is large enough, $\varepsilon$ is small enough and they satisfy
\[
C_1 \gamma^{-1} \leq \frac{d^s_m}{4}, \quad \varepsilon \leq \min\left(\frac{1}{4}, \frac{d^s_m}{4}\right).
\]

Next, we deal with the first term on the left hand side of (30). Since $\frac{\partial \phi_h}{\partial t} \xi_c, \xi)_{T_h} \geq 0$, we can get
\[
\frac{\partial (\phi_h \xi_c, \xi)_{T_h}}{\partial t} \geq \frac{1}{2} \frac{\partial}{\partial t} (\phi_h \xi_c, \xi)_{T_h}.
\]
Substituting these estimates into (30), we have the inequality
\[
\| (\Pi RT u - u, \Pi f, \Pi p - p, \Pi_{p, p} - p)_{(\omega, v, \mu)} \| \leq \alpha (\Pi RT u - u, \omega, v, \mu)_{T_h}.
\]

According to the boundedness and stability of the bilinear form $B_u$, we have the estimate
\[
K_* \| (\Pi RT u - u, \Pi f, \Pi p - p, \Pi_{p, p} - p)_{(\omega, v, \mu)} \| \leq \sup_{(\omega, v, \mu)} \frac{B_u((\Pi RT u - u, \Pi f, \Pi p - p, \Pi_{p, p} - p)_{(\omega, v, \mu)})}{\| (\omega, v, \mu) \|}.
\]

Hence we get
\[
\| (\Pi RT u - u, \Pi f, \Pi p - p, \Pi_{p, p} - p)_{(\omega, v, \mu)} \| \leq C \| (\Pi RT u - u) \|_{T_h} + \| \phi - \phi_h \|_{T_h}.
\]

Using (33), we get the estimate
\[
\| (\Pi RT u - u, \Pi f, \Pi p - p, \Pi_{p, p} - p)_{(\omega, v, \mu)} \| \leq C \| (\Pi RT u - u) \|_{T_h} + \| \phi - \phi_h \|_{T_h}.
\]

So we can get that
\[
\frac{\partial (\phi - \phi_h)}{\partial t} \leq \frac{\alpha c}{\rho_*} \left[ \frac{\Delta}{(1 - \phi_h) c_f - c_h} + \frac{(\phi - \phi_h) c_f}{\| - e \|_{T_h}} \right].
\]

Substituting the above estimate into (31), and using (28), (34) and Gronwall’s inequality, we can get the following estimate
\[
\| c_f - c_h \|_{T_h} + \| \phi - \phi_h \|_{T_h} + \| u - u_h \|_{T_h} \leq C \int_0^t (\| \xi_c \|_{T_h} + \| \xi_e \|_{T_h}) d\tau
\]
Combined the above estimate with (34), we get the second inequality of (17).
It is easily seen that our estimates are derived under the induction hypothesis (27). Now, we check it. Note that
\[
\|u_h\|_{L^\infty} \leq \|u_h - \Pi^{RT}u\|_{L^\infty} + \|\Pi^{RT}u - u\|_{L^\infty} + \|u\|_{L^\infty} \\
\leq C h^{s-\frac{d}{2}} + \|u\|_{L^\infty} \leq C_u.
\]
Thus, the hypothesis (27) holds.

Using the similar technique as in [30], we know that
\[
\|\Pi h p - p_h\|_{T h} \leq C (1 + \|u\|_{L^\infty}) \|c_h - c_f\|_{T h}.
\]
Using Lemma 3.2 and (17)(a), we get (17)(c).

5. Numerical Examples

In this section, we will test the efficiency of our proposed method by some numerical examples. We firstly use HMFE method for the linear elliptic problem, and then we consider SIPDG method for the convection-diffusion equation. Next, we confirm the convergence rate of our combined method for the coupled problem. Finally, we apply the combined method to a “real” incompressible wormhole problem.

5.1. Convergence test of HMFE method. Here we will test the accuracy of the HMFE scheme. The HMFE method is considered for solving the elliptic problem with RT0 – P0, RT1 – P1 and RT2 – P2 elements. The exact solution is taken by \( p = \sin \pi x \sin \pi y \) in \([0, 1] \times [0, 1]\) and \( u = -\nabla p \), respectively. For different mesh size \( h = 1/8, 1/16, 1/32, 1/64 \), a convergence study is presented. The \( L^2 \)-norm errors and convergence accuracies are shown in Tables 5.1–5.3. As seen in these tables, the optimal convergence rates for pressure and velocity are evaluated.

| Table 5.1. Numerical results for \( p \) and \( u \) with RT0 – P0. |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( h \)         | \( \|p - p_h\|_{L^2} \) rates | \( \|u - u_h\|_{L^2} \) rates |
| 1/8             | 7.1830e-02 *     | 2.4473e-02 *     |
| 1/16            | 3.5977e-02 0.9975| 1.2508e-02 0.9684|
| 1/32            | 1.7992e-02 0.9997| 6.2691e-03 0.9965|
| 1/64            | 8.9969e-03 0.9985| 3.1355e-04 1.0005|

| Table 5.2. Numerical results for \( p \) and \( u \) with RT1 – P1. |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( h \)         | \( \|p - p_h\|_{L^2} \) rates | \( \|u - u_h\|_{L^2} \) rates |
| 1/8             | 2.7875e-02 *     | 1.1113e-02 *     |
| 1/16            | 7.1654e-03 1.9599| 2.8341e-03 1.9713|
| 1/32            | 1.8225e-03 1.9752| 7.1763e-04 1.9816|
| 1/64            | 4.6070e-04 1.9840| 1.8099e-04 1.9874|

| Table 5.3. Numerical results for \( p \) and \( u \) with RT2 – P2. |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( h \)         | \( \|p - p_h\|_{L^2} \) rates | \( \|u - u_h\|_{L^2} \) rates |
| 1/8             | 4.4473e-04 *     | 2.5267e-02 *     |
| 1/16            | 5.8189e-05 2.9341| 2.9358e-03 3.1054|
| 1/32            | 7.3831e-06 2.9785| 3.3422e-04 3.1349|
| 1/64            | 9.2786e-07 2.9922| 3.9131e-05 3.0944|
5.2. Convergence test of SIPDG method. Here we first test the convergent accuracy of the SIPDG method for convection-diffusion equation

$$\frac{\partial c_f}{\partial t} + \nabla \cdot (uc_f - D\nabla c_f) = f$$

with homogeneous and nonhomogeneous boundary value conditions. For this purpose, we take the two different exact solutions respectively as

$$c_f = e^{-t} \sin \pi x \sin \pi y, \quad \text{and} \quad c_f = e^{-y^2 - x - t}, \quad (x, y) \in [0, 1] \times [0, 1].$$

The velocity function $u = [-y, x]$ and the diffusion coefficient $D = 1.0$. The initial-boundary conditions and the right hand side term can be computed by the exact solutions. For the practical computation, the first-order Euler backward difference scheme in time is used and $L^2$-projection of the initial condition is also used. Setting $T = 1.0$ and time size $\Delta t = 1e^{-3}$, for different mesh size, we give some numerical results with $P_1$ discontinuous finite element space in Table 5.4. These numerical results show that SIPDG method has the optimal convergence rates in $L^2$-norm for both homogeneous and nonhomogeneous boundary conditions.

**Table 5.4.** Numerical results with $P_1$ element for homogeneous and nonhomogeneous boundary cases.

| $h$  | $L^2$ error homogenous rates | $L^2$ error nonhomogenous rates |
|------|-----------------------------|-------------------------------|
| 1/8  | 1.1599e-00 *                | 2.2602e-02 *                  |
| 1/16 | 3.0175e-01 1.9425           | 5.8884e-03 1.9405             |
| 1/32 | 7.6512e-02 1.9796           | 1.5668e-03 1.9100             |
| 1/64 | 1.9192e-02 1.9952           | 4.1542e-04 1.9152             |

In addition, we also consider our method for the porosity and the concentration. Initial-boundary conditions can be given by the exact solutions

$$c_f(x, y, t) = \frac{2e^2}{2e^2 + 4Dt} \exp\left\{ -\frac{(x \cos 4t + y \sin 4t + 0.2)^2 + (-x \sin 4t + y \cos 4t)^2}{2e^2 + 4Dt} \right\},$$

$$\phi(x, y, t) = 0.5 + 0.4 \sin(x + t) \sin(y + t), \quad \text{in} \quad \Omega = [0, 1] \times [0, 1].$$

The other parameters are taken as:

$$c_I = k_c = k_s = a_0 = \frac{\alpha}{\rho_s} = 1, \quad D = 0.1, \quad \epsilon = 0.1.$$  

Here we still use the first-order backward Euler scheme in time, and take time step $\Delta t = 1e^{-3}$. The computational results at $T = 1.0$ are shown as in Tables 5.5 and 5.6 with the uniform triangular meshes $h = 1/8, 1/16, 1/32, 1/64, 1/128$. From these tables, we can get the optimal convergence rates in $L^2$-norm with $P1$ and $P2$ discontinuous elements.

**Table 5.5.** Numerical results with $P1$ element for $c_f$ and $\phi$.

| $h$  | $\|c_f - c_h\|_{L^2}$ rates | $\|\phi - \phi_h\|_{L^2}$ rates |
|------|-----------------------------|-------------------------------|
| 1/8  | 7.0710e-02 *                | 8.2931e-02 *                  |
| 1/16 | 1.7289e-02 2.0320           | 2.4605e-02 1.7529             |
| 1/32 | 4.2388e-03 2.0281           | 6.7051e-03 1.8756             |
| 1/64 | 1.0472e-03 2.0171           | 1.7132e-03 1.9686             |
| 1/128| 2.6026e-04 2.0085           | 4.3005e-04 1.9941             |

**Table 5.6.** Numerical results with $P2$ element for $c_f$ and $\phi$.

| $h$  | $\|c_f - c_h\|_{L^2}$ rates | $\|\phi - \phi_h\|_{L^2}$ rates |
|------|-----------------------------|-------------------------------|
| 1/8  | 5.2783e-03 *                | 2.7432e-02 *                  |
| 1/16 | 6.3176e-04 3.0626           | 4.0521e-03 2.7591             |
| 1/32 | 7.5796e-05 3.0592           | 5.3296e-04 2.9266             |
| 1/64 | 9.1162e-06 3.6314           | 6.7731e-05 2.9761             |
| 1/128| 1.0039e-06 3.1828           | 8.0221e-06 3.0429             |
5.3. **Convergence test of the combined method.** In this experiment, we will show the convergence of our combined method. Here the analytic solution in $\Omega = [0, 1] \times [0, 1]$ is given as in [2]

\[
\begin{align*}
p(x, y, t) &= t \cos \pi x \cos \pi y, \\
c_f(x, y, t) &= tx^2(1-x)^2y^2(1-y)^2, \\
\phi(x, y, t) &= 1 - e^{-\frac{1}{16}t^2x^2(1-x)^2y^2(1-y)^2}e^{x+y+1-(x+y+1)}.
\end{align*}
\]

The parameters are taken as $D = 10^{-2}I$, $k_0 = 1$, $a_0 = 0.5$, $\rho_s = 10$, $\alpha = 1$, $k_c = k_s = 1$, $\mu = f_I = 1$, where $I$ is an identity matrix. And choosing $T = 1.0$ and time step $\Delta t = h^2$, we give some numerical results with $RT1 - P1$ element and $P1$ discontinuous element in Tables 5.7 and 5.8. We can easily find that our combined method is of second-order accuracy in $L^2$-norm, which is coincided with our theoretical analysis.

**Table 5.7.** Numerical results for $c_f$ and $\phi$ with $P1$ element.

| $h$   | $\|c_f - c_h\|_{L^2}$ rates | $\|\phi - \phi_h\|_{L^2}$ rates |
|-------|------------------------------|---------------------------------|
| 1/8   | 1.1109e-03 *                 | 2.4572e-02 *                    |
| 1/16  | 2.9657e-04 1.9053            | 6.8963e-03 1.8331               |
| 1/32  | 7.6954e-05 1.9463            | 1.7291e-03 1.9958               |
| 1/64  | 1.9001e-05 2.0181            | 4.2112e-04 2.0377               |
| 1/128 | 4.4123e-06 2.1065            | 1.0021e-04 2.0712               |

**Table 5.8.** Numerical results for $u$ and $p$ with $RT1 - P1$ element.

| $h$   | $\|u - u_h\|_{L^2}$ rates | $\|p - p_h\|_{L^2}$ rates |
|-------|----------------------------|--------------------------|
| 1/8   | 2.4932e-03 *               | 6.2173e-03 *             |
| 1/16  | 6.2776e-04 1.9897          | 1.7321e-03 1.8438        |
| 1/32  | 1.7290e-04 1.8603          | 3.9021e-04 2.1502        |
| 1/64  | 4.2003e-05 2.0414          | 9.7001e-05 2.0082        |
| 1/128 | 1.0010e-05 2.0691          | 2.5231e-05 1.9428        |

5.4. **Simulation for a “real” incompressible wormhole propagation.** In this experiment, a 0.2-meter computational domain is considered, and the first-order Euler backward time discretization is used. We set a singular area on the middle of the left boundary with space size to be 0.01-meter and time size to be $1e - 4$ to observe the phenomenon of wormhole propagation. The initial values and the parameters in the porous medium are taken as in Table 5.9. Initial concentration of acid and initial porosity of rock in this domain are set to be $c_0 = 0$ and $\phi_0 = 0.2$, respectively. The top and bottom boundaries of the domain are impermeable.
TABLE 5.9. The properties of acid flow and porous medium.

| Properties                        | Value                  |
|-----------------------------------|------------------------|
| the viscosity of fluid ($\mu$)    | $1 \text{ Pa} \cdot \text{s}$ |
| the injection flow rate ($f_I$)   | 4.5                    |
| the production flow rate ($f_P$)  | 2.5                    |
| the dispersion tensor ($D$)       | 0.01                   |
| the local mass-transfer coefficient ($k_c$) | 1 m/s                     |
| the density of the rock ($\rho_s$) | 2000 kg/m$^2$            |
| the dissolving constant of the acid ($\alpha$) | 0.1 kg/mole             |
| the kinetic constant for reaction ($k_s$) | 10 m/s                  |
| the initial interfacial area available for reaction ($a_0$) | 0.2 m$^{-1}$            |

The numerical results of the concentration and porosity at different time are shown in Figures 5.1 and 5.2. From these figures, we can observe $c_f, \phi \in [0, 1]$ and the phenomenon of wormhole propagation, which shows the effectiveness of the combined method.

![Porosity of rock at different time steps](image1.png)

**FIGURE 5.1.** Porosity of rock at the different time steps.

![Concentration of acid at different time steps](image2.png)

**FIGURE 5.2.** Concentration of acid at the different time steps.

REFERENCES

[1] C. Zhao, Physical and chemical dissolution front instability in porous media. Cham, Switzerland: Springer, 2014.
[2] X. Li, H. Rui, Characteristic block-centered finite difference method for simulating incompressible wormhole propagation, Comput. Math. Appl. 73 (2017) 2171-2190.
[3] J. Kou, S. Sun, Y. Wu, Mixed finite element-based fully conservative methods for simulating wormhole propagation, Comput. Methods Appl. Mech. Engrg. 298 (2016) 279-302.
[4] Y. Wu, A. Salama, S. Sun, Parallel simulation of wormhole propagation with the Darcy-Brinkman-Forchheimer framework, Comput. Geotech. 69 (2015) 564-577.
[5] X. Li, H. Rui, Block-centered finite difference method for simulating compressible wormhole propagation, J. Sci. Comput. 74 (2018) 1115-1145.
[6] J. Zhang, X. Shen, H. Guo, H. Fu, H. Han, Characteristic splitting mixed finite element analysis of compressible wormhole propagation, Appl. Numer. Math. 147 (2020) 66-87.
[7] H. Guo, L. Tian, Z. Xu, Y. Yang, N. Qi, High-order local discontinuous Galerkin method for simulating wormhole propagation, J. Comput. Appl. Math. 350 (2019) 247-261.
[8] D.N. Arnold, F. Brezzi, Mixed and nonconforming finite element methods: implementation, postprocessing and error estimates, ESAIM: Math. Model. Numer. Anal. 19 (1985) 7-32.
[9] F. Brezzi, M. Fortin, Mixed and Hybrid Finite Element Methods, Springer, New York, 1991.
[10] B. Cockburn, J. Gopalakrishnan, R. Lazarov, Unified hybridization of discontinuous Galerkin, mixed and conforming Galerkin methods for second order elliptic problems, SIAM J. Numer. Anal. 47 (2009) 1319-1365.

[11] H. Egger, J. Schoberl, A hybrid mixed discontinuous Galerkin finite-element method for convection-diffusion problems, IMA J. Numer. Anal. 30 (2010) 1206-1234.

[12] J. Zhang, H. Han, H. Guo, X. Shen, A combined hybrid mixed element method for incompressible miscible displacement problem with local discontinuous Galerkin procedure, Numer. Methods Part. D. E. 36 (2020) 1629-647.

[13] J. Zhu, H. Vargas, Robust and efficient mixed hybrid discontinuous finite element methods for elliptic interface problems, Int. J. Numer. Anal. Mod. 16 (2019) 72-86.

[14] L. Bevilacqua, R. Feijoo, L.F. Rojas M, A variational principle for the Laplace operator with application in the torsion of composite rods, Int. J. Solids Struct. 10 (1974) 1091-1102.

[15] B. Fraeijs de Veubeke, Displacement and equilibrium models in the finite element method, in: O.C. Zienkiewicz, G. Holister (Eds.), Stress Analysis, John Wiley and Sons, New York, 1965.

[16] W.H. Reed, T.R. Hill, Triangular mesh methods for the neutron transport equation, Tech. Report No. LA-UR-73-479, Los Alamos Scientific Laboratory, Los Alamos, New Mexico, 1973.

[17] D.N. Arnold, An interior penalty finite element method with discontinuous element, SIAM J. Numer. Anal. 19 (1982) 742-760.

[18] B. Rivieva, M.F. Wheeler, Discontinuous Galerkin methods for flow and transport problem in porous media, Commun. Numer. Methods Eng. 18 (2002) 63-68.

[19] R. Zhang, X. Yu, J. Zhu, A.F.D. Loula, Direct discontinuous Galerkin method for nonlinear reaction-diffusion systems in pattern formation, Appl. Math. Model. 38 (2014) 1612-1621.

[20] J. Zhu, X. Yu, A.F.D. Loula, Mixed discontinuous Galerkin analysis of thermally nonlinear coupled problem, Comput. Methods Appl. Mech. Engrg. 200 (2011) 1479-1489.

[21] X. Li, J. Zhu, R. Zhang, S. Cao, A combined discontinuous Galerkin method for the dipolar Bose-Einstein condensation, J. Comput. Phys. 275 (2014) 363-376.

[22] S. Sun, B. Riviera, M.F. Wheeler, A combined mixed and discontinuous Galerkin methods for coupled flow and reactive transport problems, Appl. Numer. Math. 52 (2005) 273-298.

[23] S. Sun, M.F. Wheeler, Discontinuous Galerkin methods for coupled flow and reactive transport problems, Appl. Numer. Math. 52 (2005) 273-298.

[24] M.R. Cui, A combined mixed and discontinuous Galerkin method for compressible miscible displacement problem in porous media, J. Comput. Appl. Math. 198 (2007) 19-34.

[25] J. Zhang, J. Zhu, R. Zhang, D. Yang, A.F.D. Loula, A combined discontinuous Galerkin finite element method for miscible displacement problem, J. Comput. Appl. Math. 309 (2017) 44-55.

[26] J.C. Nedelec, Mixed finite element in R3, Numer. Math. 35 (1980) 105-335.

[27] S.C. Brenner, L.R. Scott, The Mathematical Theory of Finite Element Methods, Springer, New York, 2002.

[28] J. Douglas Jr., R.E. Ewing, M.F. Wheeler, Approximation of the pressure by a mixed method in the simulation of miscible displacement, RAIRo Anal. Numer. 17 (1983) 17-33.

JIANSONG ZHANG: COLLEGE OF SCIENCE, CHINA UNIVERSITY OF PETROLEUM, QINGDAO 266580, CHINA
Email address: jszhang@upc.edu.cn

YUN YU: COLLEGE OF SCIENCE, CHINA UNIVERSITY OF PETROLEUM, QINGDAO 266580, CHINA
Email address: yuyun19970321@163.com

JIANG ZHU: LABORATÓRIO NACIONAL DE COMPUTAÇÃO CIENTÍFICA, MCTI, AVENIDA GETÚLIO VARGAS 333, 25651-075 PETRÓPOLIS, RJ, BRAZIL
Email address: jiang@lncc.br

YUE YU: COLLEGE OF SCIENCE, CHINA UNIVERSITY OF PETROLEUM, QINGDAO 266580, CHINA
Email address: m18766215811@163.com

RONG QIN: COLLEGE OF SCIENCE, CHINA UNIVERSITY OF PETROLEUM, QINGDAO 266580, CHINA
Email address: qr-920diana@126.com