The existence of a measure-preserving bijection from a
unit square to a unit segment

Cong-Dan Pham
Duy Tan University
congdan.pham@gmail.com
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Abstract

In this paper, we prove the existence of a measure-preserving bijection from unit square to unit segment. This bijection is also called the probability isomorphism between two probability spaces. Then we give a new proof of the existence of the independent random variables on Borel probability space \([0, 1], \mathcal{B}([0, 1]), \mathbb{L})\) that their distribution functions are the given distribution functions.

1 Introduction

In classical analysis theory, there are strange functions from a unit segment to a unit square. A bijection is called a function of type Cantor and a continuous surjection is called a function of type Peano (see [2]). In this paper we study the question of the existence of a bijection between two the Borel probability spaces \(([0, 1], \mathcal{B}([0, 1]), \mathbb{L})\) and \(([0, 1]^2, \mathcal{B}([0, 1]^2), \mathbb{L})\) (maybe difference a set with null measure), where \(\mathbb{L}\) is the Lebesgue measure. The difference a set with null measure means that there exist two set \(B \subseteq [0, 1]^2, K \subseteq [0, 1]\) with \(\mathbb{L}(B) = \mathbb{L}(K) = 1\) and the bijection maps from \(B\) to \(K\). This bijection satisfies that it and its reverse function are measure-preserving functions. We also call this type bijection being the bi-measure-preserving function or probability isomorphism. Our main result reads then as follows:

Theorem 1.1. There exists a probability isomorphism \(f\) between two Borel probability spaces the unit square and the unit segment.

This result is applied to give a new proof about the existence of the independent random variables on Borel probability space \(([0, 1], \mathcal{B}([0, 1]), \mathbb{L})\) that their distribution functions are given distribution functions. In classical probability theory, we know that, for a finite number of the given distribution functions, there exists a family of independent random variables on \(([0, 1], \mathcal{B}([0, 1]), \mathbb{L})\) such that theirs distributions are the given distributions. This result is proved by using Rademacher functions (see for instance [3], [1]).
2 Proof of Theorem 1.1

To prove Theorem, we need some lemmas as follows:

**Lemma 2.1.** Let \( \varphi : \Omega \to (E, \mathcal{M}) \) where \( \Omega, E \) are two space, \( \mathcal{M} \) is a \( \sigma \)-algebra generated by a collection of sets \( \mathcal{C} \). Set \( f^{-1}(\mathcal{M}) := \{ f^{-1}(B) : B \in \mathcal{M} \} \). Then \( f^{-1}(\mathcal{M}) \) is a \( \sigma \)-algebra generated by \( f^{-1}(\mathcal{C}) \).

One proved this lemma in somewhere in measure theory. Now, we present an important theorem to apply in the next part:

**Theorem 2.2** (probability isomorphism). Let two probability spaces \((\Omega, \mathcal{F}, \mathbb{P}), (E, \mathcal{A}, \mathbb{Q})\) and a bijection \( \varphi : \Omega \to E \) such that:

\( \mathcal{F}, \mathcal{A} \) are respectively generated by two collection of sets \( \mathcal{E}, \mathcal{M} \) i.e. \( \mathcal{F} = \sigma(\mathcal{E}), \mathcal{A} = \sigma(\mathcal{M}) \) and \( f(\mathcal{E}) = \{ f(A) : \forall A \in \mathcal{E} \} \subset \mathcal{A} \); \( f^{-1}(\mathcal{M}) = \{ f^{-1}(B) : \forall B \in \mathcal{M} \} \subset \mathcal{F} \) and \( \mathbb{P}(f^{-1}(B)) = \mathbb{Q}(B) \forall B \in \mathcal{M} \).

Then \( f, f^{-1} \) are measurable (\( f \) is bi-measurable) and if \( \mathcal{M} \) is closed by intersection so \( f, f^{-1} \) are measure-preserving (\( f \) is called bi-measure-preserving or probability isomorphism.)

**Proof.** By Lemma 2.1 then \( f^{-1}(A) \) is \( \sigma \)-algebra generated by \( f^{-1}(\mathcal{M}) \), moreover \( f^{-1}(\mathcal{M}) \subset \mathcal{F} \), this implies \( f^{-1}(A) \subset \mathcal{F} \). It is similar to have that \( f(\mathcal{E}) \subset \mathcal{A} \). Therefore \( f, f^{-1} \) are measurable and there exist a bijection \( \varphi : \mathcal{F} \to \mathcal{A} \) such that \( \varphi(A) := f(A) \). Now, consider \( \mathcal{L} = \{ B \in \mathcal{A} : \mathbb{P}(f^{-1}(B)) = \mathbb{Q}(B) \} \). By assumption, \( \mathcal{M} \subset \mathcal{L} \). We prove that \( \mathcal{L} \) is a \( \sigma \)-additive class. In fact, \( f^{-1}(\emptyset) = \emptyset \) and \( f^{-1}(\Omega) = \Omega \) then \( \emptyset, \Omega \in \mathcal{L} \). Consider \( A \subset B \) and \( A, B \in \mathcal{L} \), we have \( \mathbb{P}(f^{-1}(B \setminus A)) = \mathbb{P}(f^{-1}(B) \setminus f^{-1}(A)) = \mathbb{P}(f^{-1}(B)) - \mathbb{P}(f^{-1}(A)) = \mathbb{Q}(B) - \mathbb{Q}(A) \). Therefore \( B \setminus A \in \mathcal{L} \). It remains to prove the \( \sigma \)-additivity. Let \( \{ B_n \}_{n \geq 1} \) are pairwise disjoint. So we get

\[
\mathbb{P}[f^{-1}(\bigcup_{n \geq 1} B_n)] = \mathbb{P}[\bigcup_{n \geq 1} f^{-1}(B_n)] = \sum_{n \geq 1} \mathbb{P}[f^{-1}(B_n)] = \sum_{n \geq 1} \mathbb{Q}(B_n) = \mathbb{Q}(\bigcup_{n \geq 1} B_n)
\]

This equation implies that \( \bigcup_{n \geq 1} B_n \in \mathcal{L} \). Therefore, \( \mathcal{L} \) is \( \sigma \)-additive class concluding \( \mathcal{M} \).

Now, we return to the proof of Theorem 1.1. Denote \( \Box = [0, 1]^2, \Delta = [0, 1] \). We will show that there exist a Borel set \( B \) on \( \Box \) such that \( L(B) = 1 \) and a bijection \( f : B \to \Delta \) such that \( f, f^{-1} \) are Borel measurable, measure-preserving. We know that any Borel set of \( \mathbb{R}^2, \mathbb{R} \) has a positive measure then it has continuum cardinality. We will construct the bijection \( f \) and the set \( B \) as follows:

Step 1: The square is divided into 4 smaller squares that are numerated as figure above. The unit segment is divided uniformly into 4 segments and they also numerated.

Next step we redivide uniformly every smaller squares and small segments into 4 parts. We continue this process until infinite steps and numerate the squares and segments by 1, 2, 3, 4 at every step. Consider 4 apexes of the square by coordinates: \((0, 0); (0, 1); (1, 0); (1, 1)\). Set \( M := \{(x, y) \in \Box : x \text{ or } y = \frac{a}{2^n}, 0 \leq a \leq 2^n, a \in \mathbb{N}\} \) and \( \Box^* = \Box \setminus M \). \( M \) is a Borel set.
and has the Lebesgue measure $\mathbb{L}(M) = 0$. Similarly, we put a Descartes system to have two endpoints of the unit segment $(0, 0)$; $(1, 0)$. Set $C = \{x \in \triangle : x = \frac{a}{4^n}, a \in \mathbb{N}, n \in \mathbb{N}^+\}$. $C$ is denumerable and $\mathbb{L}(C) = 0$. Denote $\bigtriangleup_{n}^{j_1j_2 \ldots j_m}$, $\triangle_{n}^{j_1j_2 \ldots j_m}$ be respectively the closed square, close segment obtained at the $n + th$ step and $j_k$ is order number of the square, segment at $k - th$ step contains it ($j_k \in \{1, 4\}$). Set $\star\bigtriangleup_{n}^{j_1j_2 \ldots j_m} := \bigtriangleup_{n}^{j_1j_2 \ldots j_m} \setminus M$, $\star\triangle_{n}^{j_1j_2 \ldots j_m} := \triangle_{n}^{j_1j_2 \ldots j_m} \setminus C$.

We construct a sequence of bijections $\{f_n\}_{n \geq 1}$, $f_n : \bigtriangleup^* \to \triangle^*$, satisfying $f(\star\bigtriangleup_{n}^{j_1j_2 \ldots j_m}) = \star\triangle_{n}^{j_1j_2 \ldots j_m}$, we can chose a bijection like that because two sets have the same continuum cardinality. With this construction, we see that for all $m > n$:

$$|f_m(x) - f_n(x)| < \frac{1}{4^n}. $$

Indeed, let $x \in \star\bigtriangleup_{n}^{j_1j_2 \ldots j_m}$ and $y \in \star\bigtriangleup_{n}^{j_1j_2 \ldots j_m}$ with $j_N \neq j'_N$. If $|j_N - j'_N| > 1$ then $\triangle_{n}^{j_1j_2 \ldots j_N}$, $\triangle_{n}^{j_1j_2 \ldots j'_N}$ are two disjoint segments for all $n \geq N$. Moreover, $f_n(x) \in \star\bigtriangleup_{n}^{j_1j_2 \ldots j_m}$, $f_n(y) \in \star\bigtriangleup_{n}^{j_1j_2 \ldots j'_N}$, take $n$ tend to infinity we get $f(x) \in \star\bigtriangleup_{n}^{j_1j_2 \ldots j_N}$, $f(y) \in \star\bigtriangleup_{n}^{j_1j_2 \ldots j'_N}$, it implies $f(x) \neq f(y)$. This is contradictory with the assumption $f(x) = f(y)$ so $|j_N - j'_N| = 1$. Hence we have three cases $(j_N, j'_N) \in \{(1, 2); (2, 3); (3, 4)\}$ (suppose $j_N < j'_N$). Since $\lim f_n(x) = \lim f_n(y) = f(x) = f(y)$, $f_n(x)$, $f_n(y)$ always belong to two segments side by side at the step $n$ and $f_n(x)$ always belong to segments numerated by 4 and $f_n(y)$ always belong to segments numerated by 1. It means that $f_n(x) \in \star\bigtriangleup_{n}^{j_1j_2 \ldots j_N}$, $f_n(y) \in \star\bigtriangleup_{n}^{j_1j_2 \ldots j'_N}$. By the definition of $f_n$, it implies $x \in \bigtriangleup_{n}^{j_1j_2 \ldots j_N}$, $y \in \bigtriangleup_{n}^{j_1j_2 \ldots j'_N}$.

Consider three case:

$$\begin{align*}
j_N = 1, j'_N = 2 & \text{ then } x = A, y = B \\
j_N = 2, j'_N = 3 & \text{ then } x = O, y = O \\
j_N = 3, j'_N = 4 & \text{ then } x = C, y = A
\end{align*}$$

All three cases are contradictory because $x \not\in M$, $y \not\in M$. Therefore, the assumption of the proof by contradiction is wrong then $f$ is injective. Now, we prove $\triangle^* \subset f(\bigtriangleup^*)$. Indeed, let $y \in \triangle^*$, at the step $n$ suppose that $y \in \star\triangle_{n}^{j_1j_2 \ldots j_m}$. This implies $f_n^{-1}(y) \in \star\bigtriangleup_{n}^{j_1j_2 \ldots j_m}$. The notation "0" indicate the interior of a set. Because $\bigtriangleup_{n}^{j_1j_2 \ldots j_m} \subset \bigtriangleup_{n}^{j_1j_2 \ldots j_m}$ then there exist a sequence $\{n_k\}$ such that $f^{-1}_n(y) \in \bigtriangleup_{n_k}^{j_1j_2 \ldots j_m}$ and $\bigtriangleup_{n_k}^{j_1j_2 \ldots j_m} \subset \bigtriangleup_{n_k}^{j_1j_2 \ldots j_m}$. Hence,

$$\bigcap_{k=1}^{\infty} \bigtriangleup_{n_k}^{j_1j_2 \ldots j_m} = \bigcap_{k=1}^{\infty} \bigtriangleup_{n_k}^{0j_1j_2 \ldots j_m} = \bigcap_{k=1}^{\infty} \bigtriangleup_{n_k}^{j_1j_2 \ldots j_m} = \{x\}.$$ 

The last equality is followed by the sequence of closed squares has the lengths of the sides.
Since, and Borel measure-preserving. Now we prove for all Lebesgue sets. Firstly, let us prove $L$ is Borel measurable. The family $F$: $\forall \triangle \in F$ is a bijection from a subset $\triangle = \Delta_k$. Indeed, let a set $\Delta$ that there is also a probability isomorphism from the unit square to the unit segment such that it maps bijectively from a subset $\Delta_k$ to $\triangle$. The image of $\Delta_k$ is a probability isomorphism between two Lebesgue spaces $(\text{Lebesgue } \sigma)$-algebra $B(\Delta)$ where $\mathcal{P}(M')$ is the family of all subsets of $M'$. The family $\{\emptyset, \triangle_n \in \mathcal{P}(C)\}$ is the generating collection of $B(\Delta)$. Remark that the two generating collections are closed by the intersection. The image of $\square_n \triangle_n$ by $f$ is $\triangle_n$ and maybe add a denumerable or finite number of points of $C$. So $f(\square_n \triangle_n) \in B(\Delta)$.

$$\mathbb{L}(f(\square_n \triangle_n)) = \mathbb{L}(\triangle_n) = \frac{1}{4^n}$$

The image of $\triangle_n$ by $f^{-1}$ is $\square_n \triangle_n$ and maybe subtract a denumerable or finite number of points of $f^{-1}(C)$ then $f^{-1}(\triangle_n) \in B(\Delta)$. Since $f$ satisfies Theorem 2.2 then $f, f^{-1}$ are bijection, measurable and measure-preserving i.e. $f$ is a probability isomorphism.

There are some consequences as follows:

**Corollary 2.3.** There exist a bijection from the unit square to the unit segment such that it is a probability isomorphism between two Lebesgue spaces (Lebesgue $\sigma$-algebra).

**Proof.** We proved the existence of probability isomorphism from $\square$ to $\triangle$ with the difference null set i.e. the bijection from $B$ to $\triangle$. It is still open for the question of the existence a bijection from $\square$ to $\triangle$, Borel measurable and is measure preserving. Now we will prove that there is also a probability isomorphism from $\square$ to $\triangle$ which is Lebesgue measurable. Indeed, let a set $Y \subset \triangle$ such that the cardinality of $Y$ is continuum and $\mathbb{L}(Y) = 0$. Set $X := f^{-1}(Y)$. Take a arbitrary bijection from $(\square \setminus B) \cup X$ to $Y$ (a bijection between two continuum sets), combine with the bijection $f|_{B \setminus X} : B \setminus X \rightarrow \triangle \setminus Y$ we get the bijection $F : \square \rightarrow \triangle$. For simplicity, we still denote $F$ by $f$. We proved $f$ is Borel measurable and Borel measure-preserving. Now we prove for all Lebesgue sets. Firstly, let us prove $\mathbb{L}(X) = 0$. For all $\varepsilon > 0$, there exist an open set $G$ such that $Y \subset G$ and $\mathbb{L}(G) < \varepsilon$. This implies $X$ is Lebesgue measurable and $\mathbb{L}(X) = 0$. For every Lebesgue measurable set $H$ of $\triangle$ we can write $H = H_1 \cup H_2$ where $H_1$ is Borel set, $\mathbb{L}(H_2) = 0$ and they are disjoint. So $f^{-1}(H) = f^{-1}(H_1) \cup f^{-1}(H_2)$ is Lebesgue measurable, $\mathbb{L}(f^{-1}(H)) = \mathbb{L}(f^{-1}(H_1)) = \mathbb{L}(H_1) = \mathbb{L}(H)$. So $f$ is a probability isomorphism between two Lebesgue probability spaces $\square$ and $\triangle$.

**Corollary 2.4.** There exists a probability isomorphism between two Lebesgue spaces $[0, 1]^m$ and $[0, 1]^n$, where $m, n$ are positive integers.

Now, we give an application of Theorem 1.1. We recover the existence of random variables on the probability space $\triangle$ associate the given distribution functions.
Corollary 2.5. Let $F_1, F_2, ..., F_n$ be the distribution functions and $(\triangle, \mathcal{B}(\triangle), \mathbb{L})$ be the Borel probability space. Then there exits $n$ independent random variables $X_1, ..., X_n : \triangle \rightarrow \mathbb{R}$ such that their distribution functions are $F_{X_1} = F_1, ..., F_{X_n} = F_n$.

Proof. This result is proved by using Ramdemacher functions,( see [3],[1]). Let us explain how do we prove it. Let a probability isomorphism $f : \triangle \rightarrow \mathbb{R}^n$. Let $X_i : B^n \rightarrow \mathbb{R}$ define by $X_i(x_1, ..., x_i, ..., x_n) = \sup\{t \in \mathbb{R} : F_i(t) < x_i\}$. It is clear that $\{X_i\}_{i=1}^n$ are independents and one proved that they have respectively the distribution functions $F_1, ..., F_n$. Set $X_i = X_i \circ f$, then $\{X_i\}$ are independents and their distribution functions:

$$F_{X_i}(x) = \mathbb{L}(X_i^{-1}(-\infty, x]) = \mathbb{L}(f^{-1} \circ X_i^{-1}(-\infty, x]) = \mathbb{L}(X_i^{-1}(-\infty, x]) = F_i(x).$$

3 Open question

It is still open for the existence of a probability isomorphism between two Borel probability spaces $([0, 1], \mathcal{B}([0, 1]), \mathbb{L})$ and $([0, 1]^2, \mathcal{B}([0, 1]^2), \mathbb{L})$ such that it has not the difference of a null measure set. \hfill \qed

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