LOWER BOUNDS FOR POSSIBLE BLOW–UP SOLUTIONS FOR THE NAVIER–STOKES EQUATIONS REVISITED.

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ABSTRACT. In this paper we give optimal lower bounds for the blow-up rate of the $H^s(\mathbb{T}^3)$-norm, $\frac{1}{2} < s < \frac{5}{2}$, of a putative singular solution of the Navier–Stokes equation.

1. Introduction

In [1] the authors, based on ideas presented by Robinson, Sadowski, Silva in [5], showed an almost optimal lower bound for the blow–up rate of solutions of the Navier–Stokes equations with periodic boundary conditions on a bounded maximal interval of existence $(0, T)$, $T < \infty$, when this solution belongs to $\dot{H}^{\frac{1}{2}}(\mathbb{T}^3) \cap \dot{H}^{\frac{5}{2}}(\mathbb{T}^3)$. To be more precise, it was shown that a regular solution of the Navier–Stokes equation whose maximal interval of existence (or regularity) is $(0, T)$, must satisfy

$$\|u(t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{T}^3)} \geq \frac{c}{\sqrt{(T-t)|\log(T-t)|}},$$

for a constant $c > 0$. In this paper we go a little further and give a proof of the expected optimal lower blow–up rate. Namely, we prove the following estimate on the blow–up rate of putative singular solutions to the Navier–Stokes equations:

$$\frac{C}{t^{\frac{1}{2}}} \leq \|u(T-t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{T}^3)}, \quad C > 0.$$  

The proof of this result requires a detailed inspection of the bounds on the non-linear term of the Navier–Stokes equations found in [5], and the application of an interpolation technique inspired by the method used by Hardy to prove Carlson’s inequality (see [2]).

The lower blow–up rates for putative singular solutions to the Navier–Stokes equations can be interpreted as a regularity criterion for solutions of the equation (as they give a lower bound on the size of the maximal interval of existence). These blow–up estimates were first stated for the $L^p$ spaces, $p > 3$, without proof by Leray in his remarkable paper [4], and proved by Giga in [3] via semigroup theory. In this paper, we rather follow the elementary proof on homogeneous Sobolev spaces given by Robinson, Sadowski and Silva for their blow–up estimates.

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The next statement is essentially the same given in [5]. The main difference is that we show a proof which includes the case when the solution belongs to $\dot{H}^{s}((T^3) \cap \dot{H}^{s+1}(T^3))$.

**Theorem 2.1.** Let $u(x,t) = (u_1, u_2, u_3)$ be a solution Navier–Stokes equations whose maximum interval of existence is $(0,T)$, $0 < T < \infty$, and such that $u \in C((0,T), \dot{H}^{s}(T^3) \cap \dot{H}^{s+1}(T^3))$, with $\frac{1}{2} < s < \frac{5}{2}$. Then the following estimate holds

$$\frac{C_s}{t^{\frac{s}{2}-\frac{1}{2}}} \leq \|u(T-s)\|_{\dot{H}^{s}(T^3)}.$$  

Proof. First, we must recall the energy inequality found in [5]:

$$\frac{d}{dt} \left( \|u(t)\|^2 \right) + 4\pi^2 \|u(t)\|_{s+1} \leq C_\varepsilon \left( \sum_k |\hat{a}_k| |k|^r \right) \|u(t)\|_s \|u\|_{s+1-r},$$

with $0 \leq r \leq 1$. Here, we use $\|v\|_s$ to denote the norm of $v$ in the homogeneous Sobolev spaces $\dot{H}^{s}(T^3)$. Now we pick $r = \frac{1}{2} \left( s - \frac{1}{2} \right)$, and apply the interpolation technique employed by Hardy in his proof of Carlson’s inequality (see [2]), to the first factor on the right hand side of (2), to obtain:

$$\sum_k |\hat{a}_k| |k|^{\frac{s}{2}-\frac{1}{2}} = \sum_k |\hat{a}_k| |k|^{\frac{s}{2}-\frac{1}{2}} \frac{\sqrt{a|k|^{s+\frac{1}{2}} + b|k|^{s+\frac{1}{2}}}}{\sqrt{a|k|^{s+\frac{1}{2}} + b|k|^{s+\frac{1}{2}}}}$$

$$\leq \left( a \|u\|_s^2 + b \|u\|_{s+1}^2 \right)^\frac{1}{2} \left( \sum_k \frac{1}{a|k|^{s+\frac{1}{2}} + b|k|^{s+\frac{1}{2}}} \right)^\frac{1}{2}$$

$$\leq \left( a \|u\|_s^2 + b \|u\|_{s+1}^2 \right)^\frac{1}{2} \left( \frac{4\pi}{\sqrt{ab}} \left( \frac{\sqrt{a}}{\sqrt{b}} \int_0^\infty \frac{y^{\frac{s}{2}-\frac{1}{2}}}{1 + y^2} \, dy \right) \right)^\frac{1}{2},$$

if we choose $a = \|u(t)\|_s^2$ and $b = \|u(t)\|_{s+1}^2$ then the energy inequality (2) becomes

$$\frac{d}{dt} \left( \|u(t)\|^2 \right) + 4\pi^2 \|u(t)\|_{s+1}^2 \leq C_\varepsilon \|u(t)\|_{s+\frac{s}{2}} \|u(t)\|^{\frac{s}{s+1}} \|u(t)\|^{\frac{s}{s+1}}.$$  

Now, observe that $\frac{s}{2} + \frac{5}{4} = \left( \frac{s}{2} - \frac{1}{4} \right) s + \left( \frac{5}{4} - \frac{s}{2} \right) (s+1)$, so by interpolation between homogeneous Sobolev spaces, we get

$$\|u\|_{s+\frac{s}{2}} \leq \|u\|_{s}^{\frac{s}{s+1}} \|u\|_{s+1}^{\frac{s}{s+1}}.$$  

Therefore, from inequality (3) we obtain

$$\frac{d}{dt} \left( \|u(t)\|^2 \right) + 4\pi^2 \|u(t)\|_{s+1}^2 \leq C_\varepsilon \|u(t)\|_{s}^{s+\frac{s}{2}} \|u(t)\|^{\frac{s}{s+1}}.$$
It is time to use Young’s inequality \( ab \leq \frac{a^p}{p} + \frac{b^q}{q} \), with the choice
\[
p = \frac{2(s+\frac{1}{2})}{s-\frac{1}{2}} \quad \text{and} \quad q = \frac{2}{s-\frac{1}{2}}.
\]
We thus get
\[
\frac{1}{2} \frac{d}{dt}(\|u(t)\|_s^2) \leq c_s(\|u(t)\|_s^2)^{\left(1+\frac{1}{s-\frac{1}{2}}\right)}.
\]
Finally, by integrating between \( T-t \) and \( T \) the previous estimate, inequality (1) follows.

**Remark 1.** Theorem 2.1 is also valid when we consider the case of the whole space, i.e., for solutions \( u(t) \in \dot{H}^{\frac{3}{2}}(\mathbb{R}^3) \cap \dot{H}^{\frac{5}{2}}(\mathbb{R}^3) \), this because all the calculations leading to its proof are valid on \( \mathbb{R}^3 \) if we change sums by integrals.

The previous proof gives us also an lower bound on size on the maximal interval of existence. Indeed, the following result holds.

**Corollary 2.1.** Let \( u(x,t) \) be a solution of the Navier–Stokes equations with initial condition \( u_0(x) \in \dot{H}^{s}(\mathbb{T}^3) \), \( \frac{1}{2} < s < \frac{5}{2} \), and let \( T > 0 \) be the minimum time for blow–up. Then

\[
\frac{K_s}{(\|u_0\|_s)^{\frac{1}{2(s-\frac{1}{2})}}} \leq T.
\]

3. **Conclusions**

Theorem 2.1 includes the optimal lower bound for blow–up rates when \( u \in \dot{H}^{\frac{3}{2}}(\mathbb{T}^3) \cap \dot{H}^{\frac{5}{2}}(\mathbb{T}^3) \); this particular case was missing in the proof given in [5], and in [11] a non optimal bound was proved. These bounds raise the following question: If there exists some \( C > 0 \) such that \( \|u(T-t)\|_s \leq Ct^{-\frac{1}{2}(s-\frac{1}{2})} \), does \( \|u(T-t)\|_s \) blow–up? Furthermore, a lower blow-up rate for \( u \in \dot{H}^{\frac{3}{2}}(\mathbb{T}^3) \), for putative blow–up solutions to the Navier–Stokes equations, is yet unknown.

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