On the Phase Space Partition in High Energy Collisions†‡

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In high energy hadron-hadron and \(e^+e^-\) collisions, to isolate a part of the phase space in multihadron final states is necessary for exploring the underlying dynamics. It is shown that the partition of phase space according to the value of rapidity, popularly used in hadron-hadron collisions, is inappropriate for the study of \(e^+e^-\) collisions. The proper way in the latter case is to identify the visible jets and take them as objects for detailed study, forming an extended phase space. The value \(d_{\text{cut}}^{(0)}\) of the distance-measure for the identification of visible jets is determined. A new variable \(r\) is introduced to further partition the phase space inside jets, which possesses a very good anomalous scaling property, showing that the \(r\)-distribution inside jets is a self-similar fractal.

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The dynamics of multi-hadron final states in high energy collisions is an idea laboratory for the study of the perturbative and nonperturbative properties of the basic theory of strong interaction — QuantumChromo Dynamics QCD. This study, starting from the sixties of last century, has a long history. The multiplicity as well as the phase space volume have increased for orders of magnitude since then. It becomes necessary to take a part of the phase space for detailed analysis instead of to merely study it as a whole.

In hadron-hadron collisions, already in the 1970s the ISR experiments started to study the “central rapidity region” \(|y| < Y_{\text{cut}}^{(0)}\), where the rapidity \(y\) is defined as

\[
y = \frac{1}{2} \ln \frac{E + p_\parallel}{E - p_\parallel},
\]

with \(E\) the energy and \(p_\parallel\) the longitudinal momentum of the particle in the c.m. frame. \(Y_{\text{cut}}^{(0)}\) is a specially chosen cut smaller than the maximum rapidity reached in the collisions. The longitudinal direction is taken to be the direction of the momenta of the two incident particles.

Later, the successful operation of the \(p\-p\) collider SP\(\Phi\)S at CERN raised the c.m. energy \(\sqrt{s}\) for an order of magnitude. It enabled the UA5 \(^{[2]}\) and other collaborations to carry on their study in various rapidity windows.

More recently, this approach has been extended to \(e^+e^-\) collisions. For example, DELPHI \(^{[3]}\) and OPAL \(^{[4]}\) Collaborations have studied the intermittency and correlations in hadronic \(Z^0\) decay in a restricted rapidity region \(|y| < 2\), with the thrust or sphericity axis taken as the longitudinal direction.

However, the dynamics of \(e^+e^-\) collisions is different from that of hadron-hadron collisions. In hadron-hadron collisions the central rapidity region has a special physical meaning. Particle production occurs mainly in this region. The regions outside of the central one are the fragmentation regions, where the leading particle effect is present. On the other hand, in \(e^+e^-\) collisions the final-state system comes from a point source (virtual photon or \(Z^0\)). There is no leading particle effect, or, if some similar effect exists in the individual jets, it is not directed along the thrust (or sphericity) axis in multi-jet events, see e.g. Fig.1 and Fig.2. In this case, to study the “central rapidity region” with thrust or (sphericity) axis as longitudinal direction is doubtful.

In 1994 J. D. Bjorken \(^{[5]}\) proposed to isolate the gluon jets for the study of the crucial features of the underlying dynamics and put forward the idea of “extended phase-space”, cf. figure 1 of Ref.\(^{[6]}\). In this approach gluon jets produced in \(e^+e^-\) collisions are first identified and then taken out as a part of the extended phase-space. Further partition of phase space is done inside the jets.

Different ways of cutting phase-space, as e.g. those done in Ref.\(^{[7]}\) and those proposed in Ref.\(^{[8]}\), will apparently result in different physics. What is the proper way for the partition of phase-space? How to realize it? are urgent problems not only for the study of \(e^+e^-\) collisions but also for the study of the ultra-high energy hadron-hadron and/or relativistic heavy ion collisions at Fermilab-TEVATRON, Brookhaven-RHIC and CERN-LHC, where a vast amount of jets are being or will be produced.

In this letter we will try to give an answer to these questions. For illustration, we will use LUND Monte Carlo JETSET7.4 to generate a sample of 1.5 million \(e^+e^-\) collision events at \(\sqrt{s} = 91.2\) GeV.

Let us first analyse the usefulness of “central rapidity region” in \(e^+e^-\) collisions. As already pointed out, the final state hadrons in these collisions are produced from a point source. There is no leading particle effect, or in multi-jet events such effect is not along the “longitudinal” (thrust or sphericity) direction. In these collisions there are 2-jet, 3-jet, 4-jet, ... events. A cut in rapidity (with the thrust or sphericity axis as ”longitudinal”) will have very different effect on different-number-of-jet events. For example, in a 2-jet event a \(y\) cut will cut out the most energetic particles in both jets; while in a 3-jet event a \(y\) cut will only cut out the most energetic particles of one jet but retain almost all the particles of the other two jets. Two typical examples are given in Fig.1,

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where the dotted lines denote the rapidity cut \( |y| < 2 \). It cuts out a large part of the left-hand-side jet but retains almost all the particles of the other two jets. The results from such an asymmetric cut is hard to be interpreted.

Physically, a more appropriate cut is not to select a group of particles with rapidity in some range but to isolate the individual jets. The corresponding cut condition is then \( r < R_{\text{cut}} \) instead of \( y < Y_{\text{cut}} \), where \[ r = \sqrt{(\eta - \eta_{\text{jet}})^2 + (\varphi - \varphi_{\text{jet}})^2}, \] (2)

\( \eta \) and \( \varphi \) are the pseudo-rapidity and azimuthal angle of a final state particle in a jet, respectively, while \( \eta_{\text{jet}} \) and \( \varphi_{\text{jet}} \) are those of the jet. \( R_{\text{cut}} \) is a number around unity. For the dashed circles in Fig.1 this number has been chosen to be \( R_{\text{cut}} = 0.7 \). Note that in Eq.(2) we have used pseudo-rapidity \( \eta \) defined as

\[ \eta = -\ln \tan(\theta/2) \] (3)

instead of the rapidity defined in Eq.(1). At high energy, \( \eta \approx y \). For the study of phase space partition, \( \eta-\varphi \) is a natural choice, which are directly connected with the corresponding variables in a space-time description \[ \# \]. Note also that, when a jet has its axis (the total momentum of the particles inside it) nearly collinear with the thrust (or sphericity) axis, \( \eta_{\text{jet}} \) and \( \varphi_{\text{jet}} \) are almost all the particles of the other two jets. The results from such an asymmetric cut is hard to be interpreted.

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In this approach the basic questions are: How to identify the jets? How to partition the phase space further after isolating out the individual jets? Let us try to answer these questions.

The identification of jets from the final-state multiparticle system can be done using some algorithm, e.g. the

\[ (\eta_i)_ij = \sqrt{d_{ij}} \cdot \sqrt{s}. \] (5)

In Eq.(4) \( E_i \) and \( E_j \) are the energies of the two jets (or two particles), \( \theta_{ij} \) is the angle between them and \( s \) is the square of c.m. energy of the event. Note that we have used the notation \( d_{ij} \) instead of \( y_{ij} \), currently used in the literature, to avoid confusion with the rapidity cut. Jets or particles with \( d_{ij} \leq d_{\text{cut}} \) are combined into a single jet. This procedure is repeated until all pairs of jets \( i \) and \( j \) satisfy \( d_{ij} > d_{\text{cut}} \).

In principle, \( d_{\text{cut}} \) could be chosen arbitrarily. The resulting “jets” are, however, very different for different values of \( d_{\text{cut}} \). For our present purpose, we want to cut the phase-space according to jets, so the “jets” should be directly observable, cf. Fig’s.1 and 2. Such directly observable jets will be referred to as “visible jets”.

It has been shown in Ref. \[ \# \] that the scale for visible jets is \( 5 \leq k_t \leq 10 \) GeV. For the case of \( \sqrt{s} = 91.2 \) GeV, which we are taking as example, this range of \( k_t \) corresponds to \( 10^{-2.5} \leq d_{\text{cut}} \leq 10^{-1.9} \).

In order to be more apparent, we plot in Fig’s.2 (a)-(c) the view in momentum space of the same event, shown in Fig.1(b) as lego-plot. The number of jets \( N_{\text{jet}} \) versus log \( d_{\text{cut}} \) in this event is given in Fig.2(d). It can be seen from the figure that if we were using a value of \( d_{\text{cut}} \) as small as \( 10^{-4} \) (or \( 10^{-5} \)) then we would conclude that this is a 10-jet (or 5-jet) event; while if we used a value of \( d_{\text{cut}} \) as
greater than $10^{-1}$ then we would take this event as a 2-jet one. Although all these values of $d_{\text{cut}}$ are equally possible in principle, it is very unlikely that there are 10 (or 5) jets or only 2 jets in this event, cf. Fig’s.1 and 2. For our present purpose of constructing extended phase space through plotting sub-lego plots of individual jets, this event should certainly be considered as a 3-jet event. It can be seen from Fig.2(d) that a value of $d_{\text{cut}}$ about equal to or a little smaller than $10^{-2}$ will combine the wrongly identified five jets into three. This is the $d_{\text{cut}}$ value corresponding to visible jets.

A further check of this statement could be obtained from the rapidity distributions of 2-jet events for four different values of $d_{\text{cut}}$, shown in Fig.3 as histograms. In the same figures, the full triangles (open circles) are the distributions for those events in which the bigger jet is at the right (left). It can be seen from the figures that for log $d_{\text{cut}}$, the full triangles manifest a shoulder at about $y = 1.8$, showing that at this value of $d_{\text{cut}}$ a third jet starts to appear inside the big jet. This confirms the assertion that $d_{\text{cut}} = -2.5 \sim 1.9$ is the appropriate distance-measure for identifying the visible jets in $e^+e^−$ collisions at $\sqrt{s} = 91.2$ GeV. For convenience, we will in the following refer to this measure as $d_{\text{cut}}^{(0)}$ and use a value $d_{\text{cut}}^{(0)} = 10^{-2.2}$.

Now let us turn to the second question: How to partition the phase space further inside the identified jets?

The first step is to take out the identified jets and draw sub-lego plots for them, forming an extended phase space together with the mother lego plot. In these sub-lego plots the jet axes are taken as the center to define new variables

$$\eta' = \eta - \eta_{\text{jet}}, \quad \varphi' = \varphi - \varphi_{\text{jet}}.$$  \hspace{1cm} (6)

As example, the sub-lego plots of the two identified jets in Fig.1(b) are shown in Fig.4.

The pair of variables $(\eta_{\text{jet}}, \varphi_{\text{jet}})$ determine the direction of jet axis. A circular symmetry is expected to exist around this point in the $\eta-\varphi$ plane. Therefore, a natural variable for the description of the phase space region inside a jet is the distance $r$ in $\eta-\varphi$ plane from the particle inside the jet to the jet axis $(\eta_{\text{jet}}, \varphi_{\text{jet}})$, cf. Eq.(2).

The distribution of $r$ is shown in Fig.5(a). It has a peak at about $r = 0.3$ and tends to zero when $r$ increases. Choosing a certain value of $R_0$, e.g. $R_0 = 1.5$, shown as solid circle in Fig’s.4, a phase space region can be defined as $0 < r \leq R_0$. A partition of this region using different values of $r$ results in a series of rings, cf. the dashed circles in Fig’s.4. The distance $\delta r$ between the neighboring rings is the scale of the partition.

It is interesting to check whether there is a scale independence or anomalous scaling in the 1-dimensional $r$ space, or in other words, whether the $r$-distribution is a self-similar fractal. This can be done using the standard technique [11].

The first step is to change the variable $r$ into the corresponding cumulant variable $r_c$ defined as [12]

$$r_c = \int_0^r \rho(r)dr / \int_0^{R_0} \rho(r)dr$$  \hspace{1cm} (7)

to eliminate the influence of nonflat distribution. The resulting distribution of $r_c$ is shown in Fig.5(b).

The second step is to divide the region $0 < r \leq R_0$ ($0 < r_c \leq 1$) into $M$ bins and define the $q$th order normalized factorial moments NFM as [13]

$$F_q(M) = \frac{1}{M} \sum_{m=1}^{M} \frac{\langle n_m(n_m-1)\cdots(n_m-q+1) \rangle}{(n_m)^q},$$  \hspace{1cm} (8)

where $n_m$ is the number of particles in the $m$th bin, $\langle \cdots \rangle$ denotes the average over the sample of identified jets.
The advantage of NFM is that it can eliminate the statistical fluctuations provided the latter is of the Poissonian type \[13\]. The anomalous scaling of NFM

\[ F_q(M) \propto M^{\theta_q} \quad M \to \infty \quad (9) \]

as the increasing of the partition number \( M \) or as the decreasing of the phase space scale \( \delta r = R_0/M \), will signal the existence of fractal property — self-similarity.

In Fig.6 is shown the log \( F_q \) versus log \( M \) plot in the region \( 0 < r \leq \pi \quad [14] \) for \( q = 2, 3, 4, 5 \). It can be seen that very good scaling is obtained for \( M = 2 - 40 \). This astonishing phenomenon is closely related to the color coherence — angular ordering in parton fragmentation and is worthwhile further investigation, both experimentally and theoretically.

In this letter the proper way for the phase space partition in high energy e\(^+\)e\(^-\) collisions is analysed in some detail. It is shown that the partition of phase space according to the value of rapidity, popularly used in hadron-hadron collisions, is inappropriate for the study of e\(^+\)e\(^-\) collisions. The proper way in the latter case is to identify the visible jets and take them as objects for detailed study, forming an extended phase space. The value of the distance-parameter \( d_{\text{cut}} \) of the DURHAM algorithm, which is able to identify the visible jets, is discussed in detail and is found to be about \( d_{\text{cut}}^{(0)} = 10^{-2.2} \) (at c.m. energy \( \sqrt{s} = 91.2 \text{ GeV} \)).

A new variable \( r \) is introduced to further partition the phase space inside jets, which possesses a very good anomalous scaling property, showing that the \( r \)-distribution inside jets is a self-similar fractal. This spectacular phenomenon, being connected with the underlying dynamics, is worthwhile further theoretical and experimental investigations.

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