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Properties of the Global Total k-Domination Number

Frank A. Hernández Mira 1, Ernesto Parra Inza 2, José M. Sigarreta Almira 3,* and Nodari Vakhania 2

1 Regional Development Sciences Center, Autonomous University of Guerrero, Los Pinos s/n, Suburb El Roble, Acapulco, Guerrero 39070, Mexico; fmira890@gmail.com
2 Science Research Center, Autonomous University of Morelos, Cuernavaca 62209, Mexico; eparrainza@gmail.com (E.P.I.); nodari@uaem.mx (N.V.)
3 Faculty of Mathematics, Autonomous University of Guerrero, Carlos E. Adame 5, Col. La Garita, Acapulco, Guerrero 39070, Mexico
* Correspondence: josemariasigarretaalmira@hotmail.com

Abstract: A nonempty subset \( D \subseteq V \) of vertices of a graph \( G = (V, E) \) is a dominating set if every vertex of this graph is adjacent to at least one vertex from this set except the vertices which belong to this set itself. \( D \subseteq V \) is a total \( k \)-dominating set if there are at least \( k \) vertices in set \( D \) adjacent to every vertex \( v \in V \), and it is a global total \( k \)-dominating set if \( D \) is a total \( k \)-dominating set of both \( G \) and \( \overline{G} \). The global total \( k \)-domination number of \( G \), denoted by \( \gamma_t^k(G) \), is the minimum cardinality of a global total \( k \)-dominating set of \( G \), GTkD-set. Here we derive upper and lower bounds of \( \gamma_t^k(G) \), and develop a method that generates a GTkD-set from a GT\((k−1)\)-set for the successively increasing values of \( k \). Based on this method, we establish a relationship between \( \gamma_t^k(G) \) and \( \gamma_t^k(G) \), which, in turn, provides another upper bound on \( \gamma_t^k(G) \).

Keywords: global total domination; total \( k \)-domination number

1. Introduction

We start by introducing the basic notation. Suppose we are given a simple graph \( G = (V, E) \) with \( |V| = n \) (\( n \) is called the order of graph \( G \)) and \( |E| = m \) (\( m \) is called the size of graph \( G \)). Given \( D \subseteq V \) (\( D \neq \emptyset \)) and vertex \( v \in V \), let \( N_D(v) \) be the set of all vertices from set \( D \), adjacent to vertex \( v \) (also called the neighbors of vertex \( v \) from set \( D \)); we will use \( \overline{N}_D(v) \) for the set of vertices in set \( D \) which are not neighbors of vertex \( v \) \( (\overline{N}_D(v) = \overline{N}_D(v) \cup \{v\}) \). We let \( N_D[v] = N_D(v) \cup \{v\} \), and we call \( \overline{d}_D(v) = |N_D(v)| \) the degree of vertex \( v \) in set \( D \). We denote by \( \overline{\delta}_D(v) \) the cardinality of set \( \overline{N}_D(v) \) \( (\overline{\delta}_D(v) = |\overline{N}_D(v)|) \). We will use more compact notation \( N(v) \), \( N[v] \), \( \delta(v) \), \( \overline{N}(v) \) and \( \overline{N}[v] \) instead of \( N_G(v) \), \( N_G[v] \), \( \delta_G(v) \), \( \overline{N}_G(v) \) and \( \overline{N}_G[v] \), respectively. When this will cause no confusion. The minimum (the maximum, respectively) degree in graph \( G \) is traditionally denoted by \( \delta \) (\( \Delta \), respectively). \( G[S] \) and \( \overline{G} \), respectively, will stand for the subgraph of graph \( G \) induced by \( S \subseteq V \) and the complement of graph \( G \), respectively.

Let \( X \) and \( Y \) be subsets of set \( V \). We denote by \( E(X, Y) \) the set of all the edges in graph \( G \) joining a vertex \( x \in X \) with a vertex \( y \in Y \). Let \( u \) and \( v \) be vertices from set \( V \). Then the distance between these two vertices \( d(u, v) \) is the length (the number of edges) of a minimum \( u − v \)-path. The length of the longest \( u − v \) path, for any \( u \) and \( v \), is called the diameter of graph \( G \), denoted by \( diam(G) \). The girth of graph \( G \) is the length of the shortest cycle in that graph and is denoted by \( g(G) \).

Let \( D \subseteq V \) be a nonempty subset of set \( V \). Then \( D \) is called a total \( k \)-dominating set for graph \( G \) if there are at least \( k \) vertices in set \( D \) adjacent to every vertex \( v \in V \) (we will also say that vertex \( v \) is totally \( k \)-dominated by set \( D \)). The cardinality of a total \( k \)-dominating set in graph \( G \) with the minimum cardinality is called the total \( k \)-domination number of graph \( G \) and is denoted by \( \gamma_t^k(G) \). We will refer to a total \( k \)-dominating set with cardinality \( \gamma_t^k(G) \) as a \( \gamma_t^k(G) \)-set. A total 1-dominating set is normally referred to as a total dominating set,
and the total 1-domination number is referred to as the total domination number, denoted by $\gamma_t(G)$. We refer the reader to [1–9] for more detail on these definitions.

Given again a non-empty set $D \subseteq V$, $D$ is called a global total $k$-dominating set of graph $G$ (GTKD set for short) if $D$ is a total $k$-dominating set of both graphs $G$ and $\overline{G}$. The global total $k$-domination number of $G$, denoted by $\gamma_{gt}^k(G)$, is the cardinality of a global total $k$-dominating set with the minimum cardinality. A global total $k$-dominating set of cardinality $\gamma_{gt}^k(G)$ will be referred to as a $\gamma_{gt}^k(G)$-set. Again, if $k = 1$, a global total 1-dominating set is a global total dominating set (see [10,11]).

As it is well-known and also easily be seen, $2k + 1 \leq \gamma_{gt}^k(G) \leq n$, for any graph $G$ with order $n$. Here we shall exclusively deal with the connected graphs due to a known fact that if $G_1, G_2, \ldots, G_r$ ($r \geq 2$) are the connected components in graph $G$, then

$$\gamma_{gt}^k(G) = \sum_{i=1}^{r} \gamma_{kt}(G_i)$$

(see [12]).

The main goal of this paper is to complete the current study of global total $k$-domination number in graphs. First, we give upper and lower bounds on $\gamma_{gt}^k(G)$, and then we develop a method that generates a GT$kD$-set from a GT$((k-1))D$-set for the successively increasing values of $k$. Based on this method, we establish a relationship between $\gamma_{gt}^k(G)$ and $\gamma_{kt}(G)$, which, in turn, provides another upper bound on $\gamma_{gt}^k(G)$.

The rest of the paper is organized as follows. In the next section, we present known results and give some remarks. In Sections 3 and 4, we derive upper and lower bounds, respectively, for global total $k$-domination number. In the Section 5, we provide our method that obtains a global total $(k+1)$-dominating set from a global total $k$-dominating set.

2. Relations between $\gamma_{gt}^k(G)$ and $\gamma_{kt}(G)$

Clearly, the definition of a GT$kD$ set gives us an implicit lower bound for the parameter $\gamma_{kt}(G)$:

Observation 1. Let $G$ be a graph; then $\gamma_{kt}(G) \geq \max\{\gamma_{kt}(G), \gamma_{kt}(\overline{G})\}$.

The above lower bound is not necessarily attainable, as we illustrate in the following figure: we depict graph $G$ and its complement $\overline{G}$, and the corresponding minimum total 2-dominating set in both graphs (black vertices); see Figure 1.

![Figure 1](image_url)

Figure 1. Graph $G$ and its complement $\overline{G}$, which satisfy $\gamma_{2t}(G) = 5$, $\gamma_{2t}(\overline{G}) = 5$ and $\gamma_{2t}^5(G) = 6$.

The following proposition was proved in [12].

Proposition 1. Let $G$ be a graph,

(i) If $\gamma_{kt}(G) > \Delta(G) + k$, then $\gamma_{kt}^5(G) = \gamma_{kt}(G)$.

(ii) If $\gamma_{kt}(G) \leq \Delta(G) + k$, then $\gamma_{kt}^5(G) \leq \Delta(G) + k + 1$. 

Corollary 1. Let $G$ be a graph with maximum degree $\Delta$. Then, $\gamma^k_{\text{gt}}(G) \leq \max\{\gamma_k(G), \Delta + k + 1\}$.

Proposition 2. Let $G$ be a graph with order $n$ and maximum degree $\Delta$. If $n > \frac{\Delta(\Delta+k)}{k}$, then $\gamma^k_{\text{gt}}(G) = \gamma_k(G)$.

Proof. If $n > \frac{\Delta(\Delta+k)}{k}$, then $\Delta + k < \frac{kn}{\Delta} \leq \gamma_k(G)$; consequently, $\Delta + k + 1 \leq \gamma_k(G)$. By Corollary 1 we have $\gamma^k_{\text{gt}}(G) = \gamma_k(G)$.

Theorem 1. For any graph $G$, $\gamma^k_{\text{gt}}(G) = \gamma_k(G)$ if and only if there exists a minimum total $k$-dominating set $D$ such that any subset $D'$ of $D$ with $|D| - k + 1$ vertices is not included in any star in the graph—that is, and only if there is not a vertex $v \in V$ such that $D' \subseteq N[v]$.

Proof. Let $D$ be a minimum total $k$-dominating set which is also a global total $k$-dominating set, and let $D'$ be a subset of $D$ with cardinality $|D| - k + 1$. If there exists a vertex $v \in V$ such that $D' \subseteq N[v]$, then $v \in D'$ and it is adjacent to $|D| - k$ vertices in $D'$, so $v$ has less than $k$ non-adjacent vertices in $D$, or $v \notin D'$, and it is adjacent to $|D| - k + 1$ vertices in $D'$, so $v$ has less than $k$ non-adjacent vertices in $D$. In both cases we have a contradiction with the fact that $D$ is a global total $k$-dominating set.

On the other hand, we take a minimum total $k$-dominating set $D$ such that for any subset $D'$ of $D$ with $|D| - k + 1$ vertices and every vertex $v \in V$, we have $D' \not\subseteq N[v]$. Then, for any vertex $v \in D$ we have $|N(v)| < |D| - k$, so $v$ has, at least, $k$ non-neighbors in $D$. If $v \in V \setminus D$ we have $|N(v)| < |D| - k + 1$, so $v$ has, at least, $k$ non-neighbors in $D$. Therefore, $D$ is a global total $k$-dominating set.

3. Upper Bounds for the Global Total $k$-Domination Number

In this section, we obtain some upper bounds for the global total $k$-domination number in a graph. Bermudo et al. in [12] showed a characterization when the global total $k$-domination number is equal to the order of the graph, but we give here that characterization in a more specific way. To do that, in the following proposition we give a condition to guarantee that the global total $k$-domination number is less than $n$.

Proposition 3. Let $G$ be a graph with order $n$, minimum degree $\delta$ and maximum degree $\Delta$. If $k < \min\{\delta, n - \Delta - 1\}$, then $\gamma^k_{\text{gt}}(G) \leq n - 1$.

Proof. Let us see that, for any $v \in V$, the set $D = V \setminus \{v\}$ is a GT$k\text{D}$ set of $G$. We have that $\delta_D(v) = \delta(v) \geq \delta > k$ and $\delta_D(v) = n - 1 - \delta(v) \geq n - 1 - \Delta > k$. For every $u \in D$ we have $\delta_D(u) \geq \delta(u) - 1 \geq \delta - 1 \geq k$ and $\delta_D(u) \geq n - 1 - \delta(u) - 1 \geq n - 2 - \Delta \geq k$. Therefore, $D$ is a GT$k\text{D}$ set of $G$.

Proposition 3 is not an equivalence, as we can see if we consider a triangle and we add a leaf to every vertex of the triangle. In such a case $\gamma^k_{\text{gt}}(G) \leq n - 1 = 5$ and $k = 1 = \min\{\delta, n - \Delta - 1\}$.

Now, in order to present the characterization of all graphs having a global total $k$-domination number equal to the number of vertices, we need to define the following set. Given a graph $G$ and an integer $i$, let $T_i(G) = \{v \in V(G) : \delta(v) = i\}$ (i.e., the set of vertices in graph $G$ with the degree $i$).

Theorem 2. Given graph $G$ with order $n$ and the minimum and the maximum degrees $\delta$ and $\Delta$, $\gamma^k_{\text{gt}}(G) = n$ if and only if one of the conditions (a)–(c) below hold

(a) $k = \delta < n - \Delta - 1$ and $V = \bigcup_{v \in T_\delta(G)} N(v)$.

(b) $k = n - \Delta - 1 < \delta$ and $V = \bigcup_{w \in T_\delta(G)} (V \setminus N[w])$.
(c) \( k = \delta = n - \Delta - 1 \) and \( V = \left( \bigcup_{v \in T_\delta(G)} N(v) \right) \cup \left( \bigcup_{w \in T_\Delta(G)} (V \setminus N[w]) \right) \).

**Proof.** (a) If \( k = \delta < n - \Delta - 1 \) and \( V = \bigcup_{v \in T_\delta(G)} N(v) \), we consider \( D = V \setminus \{u\} \) for any \( u \in V \). We note that there exists \( v \in N(u) \) such that \( \delta(v) = k \); this implies that \( \delta_D(v) < k \). Thus, \( D \) is not a GTkD set of \( G \). Hence, \( \gamma^k_{\delta}(G) = n \).

(b) If \( k = n - \Delta - 1 < \delta \) and \( V = \bigcup_{w \in T_\Delta(G)} (V \setminus N[w]) \), for any \( u \in V \) there exists \( w \in V \) such that \( \delta(w) = \Delta \) and \( u \notin N[w] \). If we consider \( D = V \setminus \{u\} \), then \( \delta_D(w) \leq n - \Delta - 2 < k \); thus, \( D \) is not a GTkD set of \( G \). Therefore, \( \gamma^k_{\Delta}(G) = n \).

(c) If \( k = \delta = n - \Delta - 1 \) and \( V = \left( \bigcup_{v \in T_\delta(G)} N(v) \right) \cup \left( \bigcup_{w \in T_\Delta(G)} (V \setminus N[w]) \right) \), using (a) or (b), we obtain that \( V \setminus \{u\} \) is not a GTkD set of \( G \), for any \( u \in V \). Consequently, \( \gamma^k_{\delta}(G) = n \).

Finally, if we assume that \( \gamma^k_{\delta}(G) = n \), by Proposition 3 we have that \( k \notin \{\delta, n - \Delta - 1\} \). For every vertex \( v \in V \), we note that \( D = V \setminus \{v\} \) is not a GTkD set of \( G \), so there exists \( u \in D \) such that \( \delta_D(u) < k \) or \( \delta_D(u) < k \). If \( k = \delta < n - \Delta - 1 \), since \( \delta_D(u) \geq n - 2 - \delta(u) \geq n - 2 - \Delta \geq k \), then we have that \( \delta_D(u) < k \); this implies that \( u \in T_\delta(G) \) and \( v \in N(u) \). If \( k = n - \Delta - 1 < \delta \), since \( \delta_D(u) \geq \delta(u) - 1 \geq \delta - 1 \geq k \), then we have that \( n - 2 - \delta(u) \leq \delta_D(u) < k = n - \Delta - 1 \); that is, \( n - 2 - \delta(u) = \delta_D(u) = n - \Delta - 2 \), so \( u \in T_\Delta(G) \) and \( v \in V \setminus N[u] \). If \( k = \delta = n - \Delta - 1 \), since \( \delta_D(u) < k \) or \( \delta_D(u) < k \), we have that \( u \in T_\delta(G) \) and \( v \in N(u) \), or \( u \in T_\Delta(G) \) and \( v \in V \setminus N[u] \).

The following corollary was directly obtained from Theorem 2.

**Corollary 2.** Let \( G \) be a graph with minimum degree \( \delta \), maximum degree \( \Delta \) and order \( n \neq \Delta + \delta + 1 \). Then \( \gamma^k_{\delta}(G) = n \) if and only if one of the following condition holds:

(a) \( k = \delta < n - \Delta - 1 \) and \( \gamma^k_{\delta}(G) = n \).

(b) \( k = n - \Delta - 1 < \delta \) and \( \gamma^k_{\delta}(G) = n \).

**Corollary 3.** Let \( G \) be a graph of order \( n \), minimum degree \( \delta \) and maximum degree \( \Delta \). If one of the following conditions holds:

(a) \( k = \delta < n - \Delta - 1 \) and \( |T_\delta(G)| \geq n - \delta \).

(b) \( k = n - \Delta - 1 < \delta \) and \( |T_\Delta(G)| \geq \Delta + 1 \).

(c) \( k = \delta = n - \Delta - 1 \) and \( |T_\delta(G)| \geq n - \delta \) or \( |T_\Delta(G)| \geq \Delta + 1 \),

then \( \gamma^k_{\delta}(G) = n \).

**Proof.** Since \( \gamma^k_{\delta}(G) = \gamma^k_{\delta}(\overline{G}) \), \( \overline{G} = n - \delta - 1 \), \( T_\Delta(\overline{G}) = T_\delta(G) \) and \( V \setminus N[w] = N(w) \), it is enough to check that \( |T_\Delta(\overline{G})| \geq \Delta + 1 \) implies \( V = \bigcup_{w \in T_\Delta(G)} (V \setminus N[w]) \). However, for any vertex \( v \in V \), if \( |T_\Delta(G)| \geq \Delta + 1 \), then there exists a vertex \( w \in T_\Delta(G) \) which is not a neighbor of \( v \), so \( v \in \bigcup_{w \in T_\Delta(G)} (V \setminus N[w]) \).

It was proved in [12] that \( \gamma^k_{\delta}(G) \leq \min \{ \gamma^k_{\delta}(G) + \Delta, \gamma^k_{\delta}(G) + \gamma^k_{\Delta}(\overline{G}) \} \). It would be convenient to characterize the graphs \( G \) such that \( \gamma^k_{\delta}(G) = \gamma^k_{\delta}(G) + \Delta \), and the graphs \( G \) such that \( \gamma^k_{\delta}(G) = \gamma^k_{\delta}(G) + \gamma^k_{\Delta}(\overline{G}) \). On the other hand, the invariants of a graph are important when characterizing them; below we use some of them such as diameter and girth. The following proofs use the ideas showed in [11].
Theorem 3. If $G$ is a graph such that $\text{diam}(G) \geq 5$, every total $k$-dominating set is a $G\text{TkD}$ set of $G$.

Proof. Let $D$ be a total $k$-dominating set and $u, v \in V$ such that $d(u, v) \geq 5$. Since $\delta_D(u) \geq k$ and $\delta_D(v) \geq k$, there exist $\{u_1, \ldots, u_k\} \subseteq D \cap N(u)$ and $\{v_1, \ldots, v_k\} \subseteq D \cap N(v)$. For any vertex $w \in V$ we know that $\delta_D(w) \geq k$. If $u_i \in N(w)$ for some $i \in \{1, \ldots, k\}$, then $w \notin \bigcup_{i=1}^{k} N[v_i]$; that means, $\delta_D(w) \geq k$. Therefore, $D$ is a $G\text{TkD}$ set of $G$. \hfill \Box

Corollary 4. If $G$ is a graph such that $\text{diam}(G) \geq 5$, then $\gamma_{G\text{kl}}(G) = \gamma_{G\text{kl}}(G)$.

According to the idea given in [11], we obtain the following result.

Proposition 4. If $G$ is a graph such that $\text{diam}(G) = 4$ and there exist $\{u, v_1, \ldots, v_k\} \subseteq V$ such that $\text{dist}(u, v_j) = 4$ for every $j \in \{1, \ldots, k\}$, then $\gamma_{G\text{kl}}^5(G) \leq \gamma_{G\text{kl}}(G) + k$.

Proof. Let $D$ be a minimum total $k$-dominating set of a graph; then there exists the vertex set $\{u_1, \ldots, u_k\} \subseteq D$ such that $\{u_1, \ldots, u_k\} \subseteq N(u)$. For any vertex $w \in V$ and $i \in \{1, \ldots, k\}$, $w$ cannot be adjacent to both $u_i$ and $v_i$, so $D \cup \{v_1, \ldots, v_k\}$ is a global total $k$-dominating set. \hfill \Box

In Figure 2 we can see an example where the equality in Proposition 4 for $k = 2$ is attained. Taking into account that any neighbor of a vertex of degree 2 must belong to any total 2-dominating set (grey vertices), we show in that figure the minimum total 2-dominating set (b) and the minimum global total 2-dominating set (c).

![Figure 2](image-url)

Figure 2. (a) Grey vertices are neighbors of vertices of degree 2. (b) Minimum total 2-dominating set and (c) minimum global total 2-dominating set.

For a graph $G$, we let $\delta^*(G) = \min\{\delta(G), \delta(\overline{G})\}$.

Proposition 5. Let $G$ be a graph of order $n$ and minimum degree $\delta$; then $\gamma_{G\text{kl}}^\delta(G) \leq n - \delta^*(G) + k$.

Proof. Let us see that every set $D \subseteq V$ such that $|D| \geq n - \delta^*(G) + k$ is a global total $k$-dominating set. Since $|D| \geq n - \delta + k$, every vertex $v$ satisfies $\delta_{V \setminus D}(v) \leq \delta - k$, $\delta_D(v) \geq k$. Since $|D| \geq n - \delta + k$, every vertex $v$ satisfies $\delta_{V \setminus D}(v) \leq \delta - k$, so $\bar{\delta}_D(v) \geq k$. \hfill \Box

4. Lower Bounds for the Global Total $k$-Domination Number

We know that any graph $G$ satisfies $\gamma_{G\text{kl}}^\delta(G) \geq 2k + 1$, and a characterization for graphs satisfying the equality was given in [12]. Additionally, in that work the authors showed the following inequality.

Remark 1. Let $G$ be a graph with order $n$, minimum degree $\delta$ and maximum degree $\Delta$. Then,

$$\gamma_{G\text{kl}}^\delta(G) \geq \max\left\{\frac{kn}{\Delta}, \frac{kn}{n - \delta - 1}\right\}.$$

For example, the lower bound given above can be reached in the graph shown in Figure 3.
Theorem 4. Let $G$ be a graph of order $n$, maximum degree $\Delta$ and size $m$. Then

$$\gamma_{kl}^e(G) \geq \frac{2m + n(2k - \Delta) + (2k + 1)^2}{n + 2k}.$$  

Proof. Let $D$ be a $\gamma_{kl}(G)$-set. Since every vertex in $V \setminus D$ cannot have more that $|D| - k$ neighbors in $D$, we have $E(D, V \setminus D) \leq (n - |D|)(|D| - k)$, so

$$m = E(D, D) + E(D, V \setminus D) + E(V \setminus D, V \setminus D) \leq \frac{|D|\Delta(G) - E(D, V \setminus D) + E(D, V \setminus D) + (\Delta - k)(n - |D|)}{2}$$

$$\leq \frac{|D|\Delta + (n - |D|)(|D| - k) + (\Delta - k)(n - |D|)}{2}$$

$$= \frac{|D|\Delta + (n - |D|)(|D| - 2k + \Delta)}{2}$$

$$= \frac{-|D|^2 + (n + 2k)|D| + n\Delta - 2kn}{2},$$

which implies that

$$(2k + 1)^2 + 2m \leq |D|^2 + 2m \leq (n + 2k)|D| + n\Delta - 2kn,$$

then

$$|D| \geq \frac{2m + n(2k - \Delta) + (2k + 1)^2}{n + 2k}.$$  

\[\square\]

Theorem 5. Let $G$ be a graph with order $n$, maximum degree $\Delta$ and size $m$. Then,

$$\gamma_{kl}^e(G) \geq \frac{2m + n(\Delta - 2k)}{n + k - \Delta}.$$  

Proof. We suppose that $D$ is a $\gamma_{kl}(G)$-set and $|D| \geq 2r + 1$ for some $r \geq 2$, and $|D| \geq 2k + 2$. Since $D$ is minimal, for any vertex $v_1 \in D$ there exists a vertex $w_{v_1}$ such that one of the following conditions holds.

1) $w_{v_1} \in D$, $v_1 \in N(w_{v_1})$ and $\delta_D(w_{v_1}) = k$,
2) $w_{v_1} \in D$, $v_1 \notin N(w_{v_1})$ and $\delta_D(w_{v_1}) = |D| - k - 1$,
3) $w_{v_1} \in V \setminus D$, $v_1 \in N(w_{v_1})$ and $\delta_D(w_{v_1}) = k$,
4) $w_{v_1} \in V \setminus D$, $v_1 \notin N(w_{v_1})$ and $\delta_D(w_{v_1}) = |D| - k$.

Now, in cases (1) and (3), we take $v_2 \in D \setminus N(w_{v_1})$, and in cases (2) and (4), we take $v_2 \in D \cap N(w_{v_1})$, and we know that there exists a vertex $w_{v_2} \neq w_{v_1}$ such that one of the above conditions holds. Since $|D| \geq 2r + 1$ we can obtain $w_{v_1}, \ldots, w_{v_r}$ vertices satisfying
one of the conditions above. We suppose that there exist \(i, j, s\) and \(r - j - s\) vertices satisfying (1), (2), (3) and (4), respectively. Then,

\[
E(D, D) \leq \frac{ik + (j - i)(|D| - k - 1) + (|D| - j)(|D| - k - 1)}{2} \\
= \frac{ik - i(|D| - k - 1) + |D|(|D| - k - 1)}{2} \\
= \frac{i(2k - |D| + 1) + |D|(|D| - k - 1)}{2},
\]

\[
E(D, V \setminus D) \leq \frac{sk + (r - j - s)(|D| - k) + (n - |D| - r + j)(|D| - k)}{2} \\
= \frac{sk - s(|D| - k) + (n - |D|)(|D| - k)}{2} \\
= \frac{(n - |D|)(|D| - k) + s(2k - |D|)}{2},
\]

and

\[
E(V \setminus D, V \setminus D) \leq \frac{s(\Delta - k) + (r - j - s)(\Delta - |D| + k)}{2} \\
+ \frac{(n - |D| - r + j)(\Delta - k)}{2} \\
= \frac{s(\Delta - k) + (r - j - s)(\Delta - k - |D| + 2k)}{2} \\
+ \frac{(n - |D| - r + j)(\Delta - k)}{2} \\
= \frac{(\Delta - k)(n - |D|) + (r - j - s)(2k - |D|)}{2},
\]

therefore,

\[
m \leq E(D, D) + E(D, V \setminus D) + E(V \setminus D, V \setminus D) \\
\leq \frac{i(2k - |D| + 1) + |D|(|D| - k - 1)}{2} + \frac{(n - |D|)(|D| - k) + s(2k - |D|)}{2} \\
+ \frac{(\Delta - k)(n - |D|) + (r - j - s)(2k - |D|)}{2} \\
= \frac{i(2k - |D| + 1) + |D|(|D| - k - 1)}{2} \\
+ \frac{(n - |D|)(|D| - 2k + \Delta) + (r - j)(2k - |D|)}{2} \\
= \frac{|D|(n + k - \Delta) + n(2k + \Delta) + (i + r - j)(2k - |D|) + i}{2} \\
\leq \frac{|D|(n + k - \Delta) + n(2k + \Delta)}{2}.
\]

then

\[
|D| \geq \frac{2m + n(\Delta - 2k)}{n + k - \Delta}.
\]

\[
\square
\]

Let us see another lower bound using the algebraic connectivity. Given a graph \(G\), its adjacency matrix \(A\) and the diagonal matrix \(D\) whose entries are the degrees of all vertices in the graph, the Laplacian matrix is defined as \(L = A - D\). The algebraic connectivity of \(G\), denoted by \(\mu\) is the second smallest eigenvalue of the Laplacian matrix.
The algebraic connectivity of \( G = (V, E) \) with order \( n \) satisfies the following equality given by Fielder [13].

\[
\mu = 2n \min \left\{ \sum_{v \in V} \sum_{j \in E} (w_i - w_j)^2 : w \neq a \text{ for } a \in \mathbb{R} \right\},
\]

where \( j = (1, 1, \ldots, 1) \) and \( w \in \mathbb{R}^n \).

**Theorem 6.** Let \( G \) be a graph with order \( n \) and algebraic connectivity \( \mu \). Then,

\[
\gamma_{kl}(G) \geq \frac{kn}{n - \mu}.
\]

**Proof.** Let \( D \) be a \( \gamma_{kl}(G) \)-set. It can be found that if we take

\[
w = \begin{cases} 
1 & \text{if } v \in D \\
0 & \text{if } v \not\in D
\end{cases}
\]

in the set given above, since \( \mu \) is the minimum, we have

\[
\mu \leq n \sum_{v \in D} \delta_D(v) \leq n(n - |D|)(|D| - k) = \frac{n(|D| - k)}{|D|};
\]

therefore, \( |D| \geq \frac{kn}{n - \mu} \).

**Theorem 7.** Let \( G \) be a graph of order \( n \) and maximum degree \( \Delta \). If \( k \geq \min \left\{ \frac{\Delta}{2}, \frac{n - \Delta - 1}{2} \right\} \), then

\[
\gamma_{kl}(G) \geq \sqrt{\frac{4kn + 4k + 1}{2}}.
\]

**Proof.** Let \( D \) be a \( \gamma_{kl}(G) \)-set. For every \( v \in D \), if we suppose that \( k \geq \frac{\Delta}{2} \), we have \( \delta_D(v) \geq \delta_D(v) \), then

\[
|D|(|D| - k - 1) \geq \sum_{v \in D} \delta_D(v) \geq \sum_{v \in D} \delta_D(v) \geq (n - |D|)|k,
\]

which implies that \( |D|^2 - |D| \geq kn \), or equivalently, that \( \left( |D| - \frac{1}{2} \right)^2 \geq kn + \frac{1}{4} \); that is, \( |D| \geq \sqrt{kn + 1 - 1} \).

If \( \frac{n - \Delta - 1}{2} \leq k < \frac{\Delta}{2} \), since \( \gamma_{kl}(G) = \gamma_{kl}(\overline{G}) \) and \( \overline{A} = n - \delta - 1 \), we can obtain the same result.

The lower bound given in Theorem 7 is attained, for instance, in the graph given in Figure 4.

![Figure 4](image_url)
In graph theory, it is common to analyze graphs obtained by some transformation from an originally given graph. An example of such a transformation is the elimination of one or more edges of the graph. Given a graph \( G \), it is natural to think about what happens if you add or delete edges on the graph. We note that removing an edge in \( G \) is equivalent to adding an edge to graph \( \overline{G} \). Therefore, it suffices to study just one of these cases.

**Proposition 6.** Let \( G \) be a graph with order \( n \), minimum degree \( \delta \) and maximum degree \( \Delta \), and let \( k < \min\{\delta, n - \Delta - 1\} \). Then the following inequalities are satisfied (for an edge \( e \)):

\[
\gamma_{kt}^G(G - e) \leq \gamma_{kt}^G(G) + 2, \\
\gamma_{kt}^G(G + e) \leq \gamma_{kt}^G(G) + 2.
\]

**Proof.** Let \( G \) be a graph and \( D \) be a \( \gamma_{kt}^G \)-set, and we consider \( e \in E \). Notice that \( e \in E(V \setminus D, V \setminus D) \), \( e \in E(D, V \setminus D) \) or \( e \in E(D, D) \); we will divide the proof into three cases and we denote \( G' = G - e \).

**Case 1:** If \( e \in E(V \setminus D, V \setminus D) \). Note that every vertex in \( V(G') \) has at least \( k \) neighbors and \( k \) non-neighbors in \( D \). Therefore, \( \gamma_{kt}^G(G') \leq |D| = \gamma_{kt}^G(G) < \gamma_{kt}^G(G) + 2 \).

**Case 2:** If \( e \in E(D, V \setminus D) \). Let \( e = uv \), where \( u \in D \) and \( v \in V \setminus D \). We note that for every \( v \in V(G) - \{u\} \), \( \delta_D(v) \geq k \) and \( \delta_D(v) \geq k \). On the other hand, note that \( \delta_D(v) > k \) in \( G' \), and if \( \delta_D(v) \geq k \) in \( G' \), then \( \gamma_{kt}^G(G') \leq |D| = \gamma_{kt}^G(G) < \gamma_{kt}^G(G) + 2 \). Now, \( \delta_D(v) = k - 1 \) in \( G' \), then there exists \( w \in V(G') \setminus D \) such that \( w \in N_{G'}(v) \). Therefore, \( D \cup \{w\} \) is a GTD set of \( G' \), so \( \gamma_{kt}^G(G') \leq |D \cup \{w\}| = \gamma_{kt}^G(G) + 1 < \gamma_{kt}^G(G) + 2 \).

**Case 3:** If \( e \in E(D, D) \). Let \( e = uv \) where \( u, v \in D \). We note that for every \( v \in V(G) - \{u, v\} \), \( \delta_D(v) \geq k \) and \( \delta_D(v) \geq k \). In the worst case \( \delta_D(u) < k \) and \( \delta_D(v) < k \), the others cases are solved as the above; there exists \( w, p \in V(G') \setminus D \) such that \( w \in N_{G'}(u) \) and \( p \in N_{G'}(v) \). Now, if \( w = p \) then \( D \cup \{w\} \) is a GTD set of \( G' \) and \( \gamma_{kt}^G(G') \leq |D \cup \{w\}| = \gamma_{kt}^G(G) + 1 < \gamma_{kt}^G(G) + 2 \); otherwise, \( w \neq p \) and then \( D \cup \{w, p\} \) is a GTD set of \( G' \); hence \( \gamma_{kt}^G(G') \leq |D \cup \{w, p\}| = \gamma_{kt}^G(G) + 2 \).

Thus, the first inequality is satisfied: \( \gamma_{kt}^G(G - e) \leq \gamma_{kt}^G(G) + 2 \). Now, as we say above for this problem, removing an edge in \( G \) is analogous to adding an edge in \( \overline{G} \). Since \( G - e \) and \( G + e \) are complementary graphs and it is known that \( \gamma_{kt}^G(G) = \gamma_{kt}^G(G) \), it is verified that \( \gamma_{kt}^G(G - e) = \gamma_{kt}^G(G + e) \). Hence, by the first inequality \( \gamma_{kt}^G(G + e) = \gamma_{kt}^G(G - e) \leq \gamma_{kt}^G(G) + 2 = \gamma_{kt}^G(G) + 2 \). So, \( \gamma_{kt}^G(G + e) \leq \gamma_{kt}^G(G) + 2 \). \( \square \)

Let \( S \) be a subset of \( V \) such that the maximum degree of the subgraph induced by the vertices from set \( S \) is no more than \( k - 1 \). Then set \( S \) will be referred to as a \( k \)-independent set of vertices. The cardinality of a \( k \)-independent set of the maximum cardinality will be referred to as the \( k \)-independence number in graph \( G \) and will be denoted by \( \beta_k(G) \). The lower \( k \)-independence number \( i_k(G) \) is the minimum cardinality of a maximal \( k \)-independent set in graph \( G \).

**Proposition 7.** Let \( D \) be a global total \( k \)-dominating set in \( G \) and let \( V \setminus D \) be a maximum \((\Delta - k)\)-independent. Then,

\[
n - \beta_{\Delta - k}(G) \leq |D| \leq \min\{n - \gamma(G), n - i_{\Delta - k}(G)\}.
\]

**Proof.** Since \( V \setminus D \) is a maximal \((\Delta - k)\)-independent set, \( V \setminus D \) is a dominating set; thus, \( n - |D| \geq \gamma(G) \). Moreover, \( i_{\Delta - k}(G) \leq n - |D| \leq \beta_{\Delta - k}(G) \). \( \square \)

5. Deriving Upper Bounds for \( \gamma_{(k+1)t}^G(G) \) from \( \gamma_{kt}^G(G) \)

It is intuitively clear that the greater \( k \) is, the more difficult is to find a global total \( k \)-dominating set of graph \( G = (V,E) \) with the minimum cardinality. In particular, the following relationship is easy to see: \( \gamma_{kt}^G(G) \leq \gamma_{2t}^G(G) \leq \gamma_{3t}^G(G) \leq \ldots \leq \gamma_{kt}^G(G) \), for every
which, in turn, provides upper bounds for $\gamma_{(k+1)D}(G)$, which is not an easy task. In this next section we develop a method that generates a GT$(k + 1)D$ set from a GT$D$, based on which we establish a relationship between minimum cardinality GT$D$ and GT$(k + 1)D$ sets—more precisely, between $\gamma^D_{(k+1)}(G)$ and $\gamma^k_{(k+1)}(G)$, which, in turn, provides upper bounds for $\gamma_{(k+1)D}(G)$.

We first need to introduce some necessary definitions. Given $D \subseteq V$, a subset of the set of vertices $V$, let $N(D)$ be the set of vertices from $V \setminus D$ having at least one neighbor in $D$; that is, $N(D) = \{x \in V \setminus D \mid \exists y \in D \text{ such that } x \in N_G(y)\}$. Similarly, we denote by $N(D)$ the set of vertices from $V \setminus D$ having at least one non-neighbor in $D$.

Now let $A$ and $B$ be subsets of $V$. We will say that a subset $D \subseteq A$ is a relative dominating set of $B$ from set $A$ if for every $x \in B$ there exists at least one vertex $v \in D$ such that $v \in N(x)$ or $v \in B$. Correspondingly, we call the minimum cardinality of such a relative dominating set the relative domination number of set $B$ from set $A$ and denote it by $\gamma'(A, B)$. We abbreviate by $\gamma'(A, B)$-set a relative dominating set of $B$ from set $A$ of cardinality $\gamma'(A, B)$.

Finally, $\gamma'(A, B)$ is the relative domination number of $B$ from set $A$ in graph $G$ and $\gamma'(A, B)$-set is a relative dominating set of $B$ from set $A$ in graph $G$ with cardinality $\gamma'(A, B)$; see an example in Figure $5$.

**Lemma 1.** Let $G$ be a graph with $\text{diam}(G) = 2$ and $g(G) = 4$, and let $S$ be an induced subgraph isomorphic to $C_4$. Let $B = V \setminus (N(S) \cup S)$ and $A = N(B)$. Then $\gamma^D_{(k+1)}(G) \leq \gamma'(A, B) + 4$.

**Proof.** Let $D'$ be a $\gamma'(A, B)$-set, $D = S \cup D'$ and $v \in V$. Note that since $\text{diam}(G) = 2$, $D' \subseteq A \subseteq N(S)$. Thus, we can see that $v \in N(S)$, $v \in B$ or $v \in S$. If $v \in N(S)$, then it has at least one neighbor in $S$ and hence also in $D$. On the other hand, if $v \in B$, then $v$ must have at least one neighbor in $D'$ and hence also in $D$. If $v \in S$, then $v$ has at least one neighbor in $S$, and hence also in $D$. Therefore, $D$ is a total 1-dominating set of $G$.

If $v \in S$, then there exists one non-neighbor vertex of $v$ in $S$, and hence also in $D$. If $v \in B$, then the four vertices in $S$ are non-neighbors of $v$, and hence vertex $v$ has at least one non-neighbor in set $D$. If $v \in N(S)$, since $g(G) = 4$, $v$ has at most two neighborhoods in $S$; thus, it has at least two non-neighbors in $S$ and hence also in $D$. Therefore, $D$ is a global 1-dominating set of $G$. Finally, $D$ is a global total 1-dominating set of $G$, so $\gamma^D_{(k+1)}(G) \leq \gamma'(A, B) + |S| = \gamma'(A, B) + 4$. $\square$

![Figure 5](image-url)  
*Figure 5.* In the depicted graph $G$, the set $S$ is formed by the white vertices, set $A$ is formed by the black vertices and set $B$ is formed by the gray vertices. Note that $\gamma'(A, B) = 2$ (the set $\{u, v\}$ is a $\gamma'(A, B)$-set) and $\gamma^D_{(k+1)}(G) = 6$.

**Corollary 5.** Let $G$ be a graph with $\text{diam}(G) = 2$ and $g(G) = 4$; let $S$ be an induced subgraph isomorphic to $C_4$, $B = V \setminus (N(S) \cup S)$ and $A = N(B)$. Then the following conditions hold:

- If $B = \emptyset$, then $\gamma^D_{(k+1)}(G) = 4$.
- Since $\gamma'(A, B) \leq |B|$, $\gamma^D_{(k+1)}(G) \leq |B| + 4$. 

Let $k$ be a positive integer with $1 \leq k < \min\{\delta, n - \Delta - 1\}$, and $D$ be a $\gamma_{k \delta}^G(G)$-set for graph $G$. Below we define special sets of vertices that will be used in future derivations.

- $H = V(G) \setminus D$.
- $Z = \{x \in H \mid \delta_D(x) \geq k + 1 \text{ and } \delta_D(x) \geq k + 1\}$ are all vertices in $H$ which are global total $(k + 1)$-dominated.
- $X = T_k(G[D])$ are all vertices in $D$ with only $k$ neighbors.
- $Y = T_{|D| - k - 1}(G[D])$ are all vertices in $D$ with only $k$ non-neighbors.
- $X' = N(X) \cap H$ are all the vertices in $H$ which have at least one neighbor in set $X$.
- $N = \gamma'(X', X)$-set, a relative dominating set of $X$ from set $X'$.
- $Y' = \mathcal{N}(Y) \cap H$ are all the vertices in set $H$ which have at least one non-neighbor in set $Y$.
- $R = \gamma'(Y', Y)$-set, a relative dominating set of $X$ from set $X'$ in $\overline{G}$.
- $P = H \setminus Z$ are all the vertices in $H$ which are not yet global total $(k + 1)$-dominated.
- $M = \gamma'(H, P)$-set $\cup \gamma'(H, P)$-set;
- $S = D \cup N \cup R \cup M$.

Now we show that the set $S$ obtained as above is a global total $(k + 1)$-dominating set given a $\gamma_{k \delta}^G(G)$-set $D$.

**Theorem 8.** Let $G$ be a graph and $D$ be an arbitrary $\gamma_{k \delta}^G(G)$-set. Then the set $S$ obtained as above is a global total $(k + 1)$-dominating set of graph $G$.

**Proof.** Let $D$ be an arbitrary $\gamma_{k \delta}^G(G)$-set, $H = V \setminus D$, $Z = \{x \in H \mid \delta_D(x) \geq k + 1 \text{ and } \delta_D(x) \geq k + 1\}$, $X = T_k(G[D])$ and $Y = T_{|D| - k - 1}(G[D])$. Further, let $P = H \setminus Z$, $E$ be a $\gamma'(H, P)$-set, $F$ be a $\gamma'(H, P)$-set and $M = E \cup F$ (all these sets being constructed as above specified). If $X = \emptyset$ and $Y = \emptyset$, then every vertex from $D \cup Z$ has at least $k + 1$ adjacent and $k + 1$ non-adjacent vertices in set $D$. Besides, note that every vertex $v \in P$ has at least $k + 1$ adjacent and $k + 1$ non-adjacent vertices in set $D \cup M$. Additionally, since $V = D \cup Z \cup P, D \cup M$ is a global total $(k + 1)$-dominating set of graph $G$.

Assume now that $X \neq \emptyset$ and $Y \neq \emptyset$, and let $X' = N(X) \cap H$ and $N$ be a $\gamma'(X', X)$-set (notice that by the construction of the set $X'$, there always exists the set $N$). Observe that every vertex from set $D \cup Z$ has at least $k + 1$ adjacent and $k + 1$ non-adjacent vertices in set $D \cup M$. Besides, every vertex $v \in P$ has at least $k + 1$ adjacent and $k + 1$ non-adjacent vertices in set $D \cup M$. Since $V = D \cup Z \cup P, D \cup N \cup M$ is a global total $(k + 1)$-dominating set of $G$.

The case $X = \emptyset$ and $Y \neq \emptyset$ is analogous to the above case. We obtain that $D \cup R \cup M$ is a global total $(k + 1)$-dominating set of $G$, where $Y' = \overline{N}(Y) \cap H$ and $R$ is a $\gamma'(Y', Y)$-set.

Finally, assume that $X \neq \emptyset$ and $Y \neq \emptyset$. Let $X' = N(X) \cap H, Y' = \overline{N}(Y) \cap H, N$ be a $\gamma'(X', X)$-set and $R$ be a $\gamma'(Y', Y)$-set. Using a similar arguments as above, we again obtain that $S$ is a global total $(k + 1)$-dominating set of graph $G$. 

In the next proposition we derive an upper bound on the cardinality of the global total $(k + 1)$-domination number. In the same lemma, we give a necessary condition when the global total $(k + 1)$-domination number is equal to the total $(k + 1)$-domination number.

**Proposition 8.** Let $G$ be a graph with $\delta \geq k$ and $D$ be a $\gamma_{k \delta}^G(G)$-set. Then the following conditions hold:

(a) $\gamma_{(k + 1)\delta}^G(G) \leq \gamma_{k \delta}^G(G) + |N \cup R \cup M|$.

(b) If $|N \cup M| > \Delta + k - \gamma_{k \delta}^G(G)$, then $\gamma_{(k + 1)\delta}^G(G) = \gamma_{(k + 1)\delta}^G(G)$.

**Proof.** (a) By Theorem 8, $S$ is a global total $(k + 1)$-dominating set of $G$; hence, the bound trivially holds.
(b) Recall that \( |S| = \gamma^k_{\delta k}(G) + |N \cup R \cup M| \). Additionally, it is easy to see that \( S \setminus R \) is a total \((k+1)\)-dominating set of \( G \). In [12] it is proved that if \( \gamma^k_{\delta k}(G) > \Delta + k \), then \( \gamma^k_{\delta k}(G) = \gamma^k_{\delta k}(G) \) (see Proposition 2.10). Hence, if \( |S| \geq \gamma^k_{\delta k}(G) + |N \cup M| \geq \gamma^{k+1}_{\delta k}(G) > \Delta + k + 1 \), then \( \gamma^k_{\delta k}(G) = \gamma^{k+1}_{\delta k}(G) \). Hence, if \( |N \cup M| = \Delta + k + 1 - \gamma^k_{\delta k}(G) \) then \( \gamma^k_{\delta k}(G) = \gamma^{k+1}_{\delta k}(G) \). \( \square \)

Using the definition of the above introduced sets and Theorem 8 and Proposition 8, we can obtain a global total \( k \)-domination set for any \( k = 2, \ldots, \min\{\delta, n - \Delta - 1\} \). As a side-result, we also obtain the corresponding upper bounds to a global total \( k \)-domination number. Finally, we note that this procedure provides a global total \( k \)-dominating set of minimum cardinality, \( 2 \leq k \leq \min\{\delta, n - \Delta - 1\} \), for some graphs; see Figure 6.

![Figure 6](image_url)

**Figure 6.** A graph \( G \) with \( \gamma^5_{11}(G) = 4, \gamma^7_{22}(G) = 6 \) and \( \gamma^5_{33}(G) = 8 \). Note that if \( D = \{v_1, v_2, v_3, v_4\} \) a \( \gamma^5_{11}(G) \)-set, then \( S = \{v_1, v_2, v_3, v_4, v_5, v_7\} \) which is a \( \gamma^7_{22}(G) \)-set. Likewise, from \( S \) we construct \( S' = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\} \) which is a \( \gamma^5_{33}(G) \)-set.

6. Conclusions

We studied the global total \( k \)-domination number in general graphs. In particular, we presented new upper and lower bounds using the algebraic connectivity in graphs. We also established a relationship between the global total \( k \)-domination numbers of the originally given graph \( G \) and the transformed ones. Then we derived an explicit relationship between a \( \gamma^k_{\delta k}(G) \)-set and a \( \gamma^k_{\delta(k+1)}(G) \)-set, which allowed us to obtain another upper bound for the global total \( k \)-domination number in a recurrent fashion, starting from \( k = 1 \). We gave an example of a graph \( G \) for which a \( \gamma^k_{\delta k}(G) \)-set, for every \( k = 2, \ldots, \min\{\delta, n - \Delta - 1\} \) is provided. For future work, the global total \( k \)-domination number could be studied on unitary operations in graphs, such as edge subdivision, edge contraction, path contraction and removal of a vertex. It would be a challenging task to adopt the proposed method as such and also extend it for a wider class of graphs.

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