Abstract—Among the intelligent safety technologies for road vehicles, active suspensions controlled by embedded computing elements for preventing rollover have received a lot of attention. The existing models for synthesizing and allocating forces in such suspensions are conservatively based on the constraint that no wheels lift off the ground. However, in practice, smart/active suspensions are more necessary in the situation where the wheels have just lifted off the ground. The difficulty in computing control in the last situation is that the problem requires satisfying disjunctive constraints on the dynamics. To the authors’ knowledge, no efficient solution method is available for the simulation of dynamics with disjunctive constraints and thus hardware realizable and accurate force allocation in an active suspension tends to be a difficulty. In this work we give an algorithm for and simulate numerical solutions of the force allocation problem as an optimal control problem constrained by dynamics with disjunctive constraints. In particular we study the allocation and synthesis of time-dependent active suspension forces in terms of sensor output data in order to stabilize the roll motion of the road vehicle. An equivalent constraint in the form of a convex combination (hull) is proposed to satisfy the disjunctive constraints. The validated numerical simulations show that it is possible to allocate and synthesize control forces at the active suspensions from sensor output data such that the forces stabilize the roll moment of the vehicle with its wheels just lifted off the ground during arbitrary fish-hook maneuvers.

I. INTRODUCTION

Traditionally forces allocated in intelligent suspension systems of a road vehicle to prevent its rollover during a severe maneuver have been based on the conservative constraint that none of its wheels has lifted off the road surface (see, for example, models in [?] and [?]; see [?] for a detailed recent literature review which we omit here to avoid repetition). However, this condition may not be practical in an actual situation where the vehicle does a severe fish-hook maneuver to avoid a sudden obstacle around a corner or on a relatively tight curve, when there is no sufficient space and time left for slowing down. Such maneuvers are well known to produce large yaw rates that induce rollover by lifting the wheels on one side of the vehicle off the road surface. The more severe problem of stabilization when wheels have lifted off yields disjunctive constraints (i.e., either sufficient anti-roll moment to the left or to the right, depending on which wheels are lifted off, must be available) on the vehicle dynamics. It then becomes necessary to solve a disjunctively constrained dynamical optimization problem to obtain the stabilizing forces that must be synthesized in the suspensions. In this work we formulate and solve a disjunctively constrained dynamical optimization problem and in the process, find forces in the suspensions that would assist in preventing the rollover of a vehicle in the more severe situation of wheels just lifting off the ground.

Disjunctively constrained dynamics has not been studied often, although general disjunctive programming with nonlinear algebraic constraints have received sufficient attention in recent years. A review of methods of handling disjunctive constraints in non-linear optimization problems where constraints do not include dynamics, can be found in [?] and in Part II of [?].

In computation schemes for collision avoidance (e.g., intelligent transportation system involving many vehicles) [?], [?], [?], [?], disjunctive constraints on the dynamics are converted into more conservative negated conjunctive constraints on the critical section in which collision is to be avoided and the schemes use large numbers and Kronecker deltas. A drawback of this approach (called “big-M constraints” in [?]) method is that the computational relaxation affected is often weak [?] resulting in failure of the disjunctive program. In the rollover prevention problem, it is more safety critical to satisfy the disjunctive constraints tightly.

Thus a general but efficient approach must be developed for computing the force allocation in suspensions correctly in the context of the rollover prevention problem. A convex hull (outlined in [?] and in Part II of [?]) of the the functions which enter the disjunctive constraints is used in this work to compute the effective force allocations in the active suspensions of the vehicle.

The paper is organized as follows. At first the mathematical model of a road vehicle undergoing a severe maneuver along with different constraints is described, followed by a method to handle the disjunctive constraints that allow the wheels of the vehicle being lifted off the road surface. Section III-C1 describes the discretization of the disjunctively constrained dynamics. Section III-C2 outlines the direct transcription method of solving the resulting optimal control problem with disjunctive dynamics as constraint. Comparison of the disjunctive dynamics approach with the more conservative conjunctive constraints is illustrated with numerical results in section V. The rollover preventive forces with wheels lifted off are computed as sensor adapted controls in section V to show that it is possible to synthesize the desired controls in terms of yaw rate using the present disjunctive dynamics model.
II. MAIN RESULTS

The main results of this work are: (a) disjunctive dynamics constrained model of rollover of a road vehicle undergoing a severe maneuver in which wheels on one side are allowed to be lifted off the ground and (b) computation, using this model, to find the control forces that should be synthesized in an active suspension system in order to stabilize the roll of the vehicle even when wheels on one side have just lifted off the ground. The control forces are shown to be obtained in terms of the yaw rate sensor output.

The maneuver indicated by figure 2 is a fish-hook maneuver in which wheels are allowed to be lifted off the ground and the control forces in the active suspension are generated based on yaw rate sensor output to stabilize the vehicle. It is found that the synthesized control forces make the vertical reactions on the right wheels zero during the course of the maneuver in which wheels lift off, and the resulting roll moment is negative, indicating that the rollover tendency of the vehicle with wheels lifted-off is neutralized. The numerical solutions show that satisfying the disjunctive constraints is key to computing the anti-roll controls.

III. MATHEMATICAL AND COMPUTATIONAL PRELIMINARIES

In this section we layout the mathematical and computational tools and approach that are the bases of our vehicle rollover model with disjunctively constrained dynamics and subsequent force synthesis.

A. Dynamic Optimization Problem

We approach the present control synthesis problem in the following dynamic optimization form:

\[
\min \int_{t_0}^{t_f} L(x, u, t) \, dt \quad \text{(objective function)} \tag{1a}
\]

subject to

\[
\dot{x} = f(x, u, t) \quad \text{(dynamics)} \tag{1b}
\]

\[
0 \leq \phi(x, t) \quad \text{(inequality path constraints)} \tag{1c}
\]

\[
0 = \varphi(x, t) \quad \text{(equality path constraints)} \tag{1d}
\]

\[
\sum_{i=1}^{m_d} (0 \leq \varphi_i) \quad \text{(inclusive disjunction on constraints)} \tag{1e}
\]

\[
b_{upper} \geq \begin{pmatrix} x \\ u \end{pmatrix} \geq b_{lower} \quad \text{(bounds)} \tag{1f}
\]

\[
x(0), u(0) \text{ are consistent initial conditions} \tag{1g}
\]

where \(x \in \mathbb{R}^m\) denotes the differential state variables, \(f : \mathbb{R}^m \times \mathbb{R}^{q_x} \times \mathbb{R} \to \mathbb{R}^m\), \(u \in \mathbb{R}^{q_u}\) is the vector of controls and algebraic state variables; \(\phi : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^{q_d}\) is the vector of inequality path constraints; \(\varphi_i : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^{q_r}\), \(i = 1 \cdots m_d\) are the constraints over which disjunction is taken, \(\sum_{i=1}^{m_d} (0 \leq \varphi_i) \leq 0\) is the vector of inequality path constraints and \(t \in [t_0, t_f] \subset \mathbb{R}\) with \(t_f > t_0\). In the above \(q_x, q_d, q_r\) are such that no more than \(q_x\) constraints are active at any given time in the simulation interval.

B. Disjunctive Constraints as Convex Hull

Disjunctive constraints can be incorporated as convex constraints. This is equivalent to representing the disjunction as a convex hull of the constraints entering the disjunction. An inclusive disjunction or inclusive logical or over the functions \(f_i : \mathbb{R}^n \to \mathbb{R}\), \(i \in \{1, \cdots, m\}\) can be represented as follows:

\[
\begin{aligned}
\left\{ f_i(x) \leq 0 \right\} &\iff \sum_{i=1}^{m_d} \lambda_i f_i(x) \leq 0, \quad \text{such that} \\
\sum_{i=1}^{m_d} \lambda_i = 1, \lambda_i \geq 0 &\forall i \in \{1, \cdots, m_d\} \tag{2}
\end{aligned}
\]

From the construction of (2) it is clear that \(\{\lambda_i \sum_{i=1}^{m_d} \lambda_i = 1, \lambda_i \geq 0 \forall i \in \{1, \cdots, m_d\}\}\) can be found if at least one of the \(f_i\)'s is non-positive. If \(f_i(x) \geq 0, \forall i \in \{1, \cdots, m_d\}\), i.e., all of the functions, over which the inclusive disjunction is specified, are positive, then no \(\lambda_i\)'s can be found and the inclusive disjunction is correctly indicated as infeasible.

It follows that exclusive disjunction or exclusive-or between two constraints, i.e., \((f_1 \leq 0) \oplus (f_2 \leq 0)\) can be represented by the following constraints: \(\lambda_1 f_1 + \lambda_2 f_2 \leq 0, \pi_1 f_1 + \pi_2 f_2 \geq 0, \lambda_1 + \lambda_2 = 1, \pi_1 + \pi_2 = 1, \pi_1 \geq 0, \lambda_1 \geq 0, i = 1, 2, \lambda_2 = 1 - \lambda_1\).

In the context of the constrained dynamics, the inclusive disjunction on constraints at some \(t \in I \subseteq [t_0, t_f]\) implies that at least one of the constraints (over which disjunction is taken) is satisfied. Consider \(f_1(t, X(t))\) and \(f_2(t, X(t))\) to be two constraint functions. In each sub-interval \(I \in [t_0, t_f]\) if \(f_1(t, X(t)) > 0\) then \(f_2(t, X(t)) \leq 0\) must hold and vice versa. The inclusive disjunctive constraint is also satisfied, when, as appropriate, both \(f_1\) and \(f_2\) are non-positive. We implement the above by satisfying \(g(t, X(t)) := \lambda(t) f_1(t, X(t)) + (1 - \lambda(t)) f_2(t, X(t)) \leq 0\) such that \(\lambda(t) \in [0, 1]\).

**Theorem III.1.** Consider the inclusive disjunctive constraint at a \(t \in [t_0, t_f]\): \((f_1(t, X(t)) \leq 0) \lor (f_2(t, X(t)) \leq 0\). If and only if \(X(t)\) satisfies \(g(t, X(t)) := \lambda(t) f_1(t, X(t)) + (1 - \lambda(t)) f_2(t, X(t)) \leq 0\) for some \(\lambda(t) \in [0, 1]\), the inclusive disjunction is satisfied.

**Proof:** Suppose at least one of \((f_1(t, X(t)) \leq 0)\) and \((f_2(t, X(t)) \leq 0)\) is true at \(t \in [t_0, t_f]\). By construction, we can choose \(\lambda(t) \in [0, 1]\) for \(t \in [t_0, t_f]\) so that \(g(t, X(t)) \leq 0\). A \(\lambda(t)\) cannot be found only when both \(f_1\) and \(f_2\) are positive. The converse is as follows. Let \(g(t, X(t)) \leq 0\) hold for some \(\lambda(t) \in [0, 1]\). Since the sum of two positive reals can not be negative, \(g(t, X(t)) \leq 0\) implies that

\[
\begin{aligned}
\text{either } \lambda(t) f_1(t, X(t)) \leq 0 \text{ or } (1 - \lambda(t)) f_2(t, X(t)) \leq 0, \text{ or,} \\
\text{both } f_1(t, X(t)) \leq 0 \text{ and } f_2(t, X(t)) \leq 0.
\end{aligned}
\]

Hence the claim follows.

We note that the variable \(\lambda(t)\) used to handle the inclusive disjunctive constraints does not have a unique solution. However, from the computational point of view \(\lambda(t)\) is treated as an algebraic variable (in the context of a differential-algebraic equation model of the vehicle rollover dynamics). Equations (14a) and (14b) in section IV-B are based on this approach.
C. Numerical Algorithm for Solving the Dynamic Optimization Problem

In this section we give a step-wise algorithm for solving the dynamic optimization formulation (1) of the control problem.

Step 1 The ODE (converted to canonical first order) describing the dynamics over the entire time interval of the maneuver is discretized.

Step 2 As described in section III-B, the convex hull equivalent of the disjunctive constraints is appended to the discretized dynamics.

Step 3 The finite dimensional dynamic optimization problem thus formed is solved by the direct transcription method described in section III-C2 using a nonlinear programming solver.

1) The \( \alpha \)-Method Discretization of the Dynamics: For definiteness, we use the \( \alpha \)-method to discretize the dynamics in our numerical method (cf. [7], [8] for first order DAEs, [9] for second order DAEs). The method has the property of producing regularized (reduced condition number) constraint Jacobians and its DAE discretization is unconditionally stable (cf. [9] for mathematical theory and computational properties when used in a direct transcription). These two properties are useful because of the stiffness the disjunctive constraints produce by the switching action inherent in the disjunction. The method discretizes the first order initial value problem \( \dot{x} = f(x, t), g(x, u, t) = 0, x(t_0) = x_0 \) as

\[
x_{n+1} = x_n + \left(1 - \frac{\beta}{\gamma}\right) h_n f(x_n, t_n) + \frac{\beta}{\gamma} h_n f(x_{n+1}, t_{n+1}) + \left(\frac{1}{2} - \frac{\beta}{\gamma}\right) h^2_n a_n
\]

where \( h_n = t_{n+1} - t_n \) is the time step size, and \( a_0 \) is either given or calculated by \( a_0 = \frac{d}{dt} \) at \( t = 0 \). The parameters \( \gamma \) and \( \beta \) are computed as

\[
\gamma = \frac{2}{\rho + 1} - \frac{1}{2}
\]

\[
\beta = \frac{1}{(\rho + 1)^2}
\]

where \( \rho \in [0, 1) \) is a user-selected variable.

2) Direct Transcription Method: Consider the following optimal control problem.

Minimize \( J = \int_{t_0}^{t_f} L(x, u, t) \) subject to

\[
\dot{x} = f(x, u, t), x(t_0) = x_0, \quad g(x, u, t) = 0, \quad h(x, u, t) \leq 0 \]

where the initial value \( u(t_0) = u_0 \) may or may not be given. This is an infinite-dimensional continuous problem over \([t_0, t_f]\).

We approximate the problem by a finite-dimensional version by discretizing the dynamics over \([t_0, t_f]\) partitioned as \( t_0 < t_1 < \cdots < t_N = t_f \). The objective function \( J \) is approximated by the trapezoidal rule whereas the dynamics is discretized by the \( \alpha \)-method and the constraints are required to be satisfied at each grid point. We assume that \( u(t_0) = : u_0 \) is either known or can be computed such that \( x_0, u_0 \) are consistent with the equality and inequality constraints (6c) and (6d). The finite dimensional discretized problem is then written as

\[
\min \sum_{i=0}^{i=N-1} \frac{t_{i+1} - t_i}{2} \times (L(x_i, u_i, t_i) + L(x_{i+1}, u_{i+1}, t_{i+1}))
\]

subject to, for \( n = 0, 1, \cdots, N - 1 \) and \( h_n = t_{n+1} - t_n \),

\[
x_{n+1} = x_n + \left(1 - \frac{\beta}{\gamma}\right) h_n f(x_n, t_n) + \frac{\beta}{\gamma} h_n f(x_{n+1}, t_{n+1}) + \left(\frac{1}{2} - \frac{\beta}{\gamma}\right) h^2_n a_n
\]

\[
a_{n+1} = \frac{1}{h_n \gamma} (f(x_{n+1}, t_{n+1}) - f(x_n, t_n)) + \left(1 - \frac{1}{\gamma}\right) \gamma
\]

\[0 = g(x_{n+1}, u_{n+1}, t_{n+1}) \]

\[0 \leq h(x_{n+1}, u_{n+1}, t_{n+1}) \]

\[x(t_0) = x_0; \ 0 = g(x_0, u_0, t_0); \ 0 \geq h(x_0, u_0, t_0) \]

in which \( a_0 \) is also computed when not known. The problem (7a) (7f) is solved by a nonlinear programming (NLP) solver, such as, a sequential quadratic programming (SQP) method. For a more detailed description of the method, the reader is referred to [9].

IV. MATHEMATICAL MODEL OF THE ROLL STABILIZATION

The vehicle dynamics model (cf. [9] for description) is described as a constraint in an optimal control problem. The model, with reference to figures [1] and [2] and with reference to the parameters described in the Appendix, is given as

Minimize \( J = \int_{t_0}^{t_f} \left\{ [X(t) - \overline{X}(t)]^2 + [Y(t) - \overline{Y}(t)]^2 \right\} dt \)

subject to the following equations of motion:

\[
M \ddot{X} = \sum_{i=1}^{4} \mu F_{Yi} \sin(\theta_Z + \delta_i)
\]

\[
M \ddot{Y} = \sum_{i=1}^{4} -\mu F_{Yi} \cos(\theta_Z + \delta_i)
\]

\[
I_{ZZ} \ddot{\theta}_Z = \sum_{i=1}^{4} (-\mu F_{Yi} \cos \delta_i r_{Xi} - \mu F_{Yi} \sin \delta_i r_{Yi})
\]

\[
M \ddot{Z} = \sum_{i=1}^{4} F_{Zi} - Mg
\]

\[
I_{XX} \ddot{\theta}_X = (F_{Z1} - F_{Z2} + F_{Z3} - F_{Z4}) T
\]

\[
+ (F_{Z1} + F_{Z2} + F_{Z3} + F_{Z4}) Z \tan \theta_X
\]

\[
+ MZ (Y \cos \theta_Z - \dot{X} \sin \theta_Z)
\]

to which we shall append the roll stabilization specific constraints in section [V-B]. Other constraints are bounds from
A. Forces and parameters

The wheel reaction forces $F_{Zi}$, $i = 1, \ldots, 4$ are defined by the following formulae.

\[
\begin{align*}
F_{Z1} &= F_l + \frac{b}{2(a + b)} Mg + K \left( Z_0 - \left( Z + \frac{T}{2} \theta_X \right) \right) - C \left( Z + \frac{T}{2} \theta_X \right) \tag{9a} \\
F_{Z2} &= F_r + \frac{b}{2(a + b)} Mg + K \left( Z_0 - \left( Z - \frac{T}{2} \theta_X \right) \right) - C \left( Z - \frac{T}{2} \theta_X \right) \tag{9b} \\
F_{Z3} &= F_l + \frac{a}{2(a + b)} Mg + K \left( Z_0 - \left( Z + \frac{T}{2} \theta_X \right) \right) - C \left( Z + \frac{T}{2} \theta_X \right) \tag{9c} \\
F_{Z4} &= F_r + \frac{a}{2(a + b)} Mg + K \left( Z_0 - \left( Z - \frac{T}{2} \theta_X \right) \right) - C \left( Z - \frac{T}{2} \theta_X \right) \tag{9d}
\end{align*}
\]

The tire forces $F_{Yi}$, $i = 1, \ldots, 4$ are defined as

\[
F_{Yi} = \begin{cases} 
0 & \text{if } F_{Zi} \leq 0 \\
D \sin \left( C_T \arctan \left( B \phi \right) \right) & \text{otherwise}
\end{cases} \tag{10}
\]

where the parameters used in the definition of $F_{Yi}$ are calculated using the following formulae.

\[
\begin{align*}
\phi &= (1 - E)(\alpha + \Delta S_h) \\
&\quad + (E/B) \arctan(B(\alpha + \Delta S_h)) \tag{11a} \\
D &= a_1 F_{Zi}^2 + a_2 F_{Zi}' \tag{11b} \\
B &= \frac{a_3 \sin \left( a_4 \arctan \left( a_5 F_{Zi}' \right) \right)}{C_T D} \tag{11c} \\
E &= a_6 F_{Zi}^2 + a_7 F_{Zi}' + a_8 \tag{11d} \\
\alpha &= \frac{180}{\Pi} \left\{ -\delta_i - \arctan \left( \frac{\hat{X} \sin \theta_Z - \hat{Y} \cos \theta_Z - r_X \dot{\theta}_X}{\hat{X} \cos \theta_Z + \hat{Y} \sin \theta_Z - r_Y \dot{\theta}_Z} \right) \right\} \tag{11e} \\
F_{Zi}' &= \frac{F_{Zi}}{1000} \tag{11f}
\end{align*}
\]

A detailed description of the above tire model formulae is given in [2] (also see [2]). The constant values used for numerical computations in this work are given in the Appendix.

B. Constraints on the Dynamics

1) Suspension travel limits:

\[
\begin{align*}
Z_{\min} &\leq Z + \frac{T}{2} \theta_X \leq Z_{\max} \tag{12a} \\
Z_{\min} &\leq Z - \frac{T}{2} \theta_X \leq Z_{\max}. \tag{12b}
\end{align*}
\]

2) Controlling force limits:

\[
\begin{align*}
-F_{\max} &\leq F_l \leq F_{\max} \tag{13a} \\
-F_{\max} &\leq F_r \leq F_{\max}. \tag{13b}
\end{align*}
\]

3) Anti-roll moment constraints (Inclusive Disjunctive Constraints):

\[
\begin{align*}
\left( -F_{Z1} - F_{Z3} \leq 0 \right) \lor \\
\left( \hat{Y} \cos \theta_Z - \hat{X} \sin \theta_Z - \frac{T}{2Z} \leq 0 \right) \tag{14a}
\end{align*}
\]

and

\[
\begin{align*}
\left( -F_{Z2} - F_{Z4} \leq 0 \right) \lor \\
\left( -\hat{Y} \cos \theta_Z - \hat{X} \sin \theta_Z + \frac{T}{2Z} \leq 0 \right). \tag{14b}
\end{align*}
\]

The disjunctive anti-roll constraints (14) are treated in the existing literature (e.g., see [2], [2], [2]) in the more conservative (conjunctive) form:

\[
-F_{Z1} - F_{Z3} \leq 0 \text{ and } -F_{Z2} - F_{Z4} \leq 0 \tag{15}
\]

which imply that the wheels do not lift off the ground. This treatment of constraints fails to generate any controller intervention when the wheels have lifted off. Further, it also does not check whether the vehicle roll has become unstable (see rollover index analysis in section VI) and needs controller intervention at all.
and disjunctive constraints (i.e., wheels lifting off) are employed. A. Numerical Solutions with Disjunctive Constraints

N with \( \alpha \) an NLP solver. It is seen that the \( \alpha \) solved by the direct transcription method (cf. section III-C2) with backward Euler discretization, fail to converge to a solution \((\ldots)\). Discretizations that do not affect a regularization, such as \( \text{IN NEWTON} \), which is partitioned into \( N \) equally spaced grid points with the initial guess \( \ldots \). Various combinations of the constraints and of the initial guesses for the control profile are experimented with for obtaining the numerical solutions. The \text{fmincon} \) function of the MATLAB\textsuperscript{\textcopyright}, a sequential quadratic programming (SQP) routine, is used as an NLP solver. It is seen that the \( \alpha \)-method discretization with \( N \geq 121 \) uniform time steps captures the required time resolution of the stiff dynamics of the maneuvers and produces convergent numerical solutions.

A. Numerical Solutions with Disjunctive Constraints

V. Computation of the Control Forces

The switching of the disjunctive constraints, as expected, introduces stiffness in the dynamics resulting in a possibly large condition number in the constraint Jacobian of the NLP solver. The \( \alpha \)-method when used as the discretization method in a transcription scheme can affect regularization of the Jacobian \((\ldots)\). Discretizations that do not affect a regularization, such as the backward Euler discretization, fail to converge to a solution in the NLP problem.

Hence, the optimal control problem with its dynamics and disjunctive constraints as described in section [V] is numerically solved by the direct transcription method (cf. section [III-C2]) with \( \alpha \)-method discretization over the maneuver simulation interval \([t_0, t_f]\), which is partitioned into \( N \) equally spaced grid points. The step size for the \( \alpha \)-method is then \( h = \frac{t_f - t_0}{N - 1} \). Various combinations of the constraints and of the initial guesses for the control profile are experimented with for obtaining the numerical solutions. The \text{fmincon} \) function of the MATLAB\textsuperscript{\textcopyright}, a sequential quadratic programming (SQP) routine, is used as an NLP solver. It is seen that the \( \alpha \)-method discretization with \( N \geq 121 \) uniform time steps captures the required time resolution of the stiff dynamics of the maneuvers and produces convergent numerical solutions.

A severe fishhook maneuver with steering input as in figure 23 and disjunctive constraints (i.e., wheels lifting off) are employed. Figures 3 and 4 show the controls obtained from the simulation of the disjunctively constrained dynamic optimization problem with a partition of \( N = 151 \) grid points with the initial guess values of \( F_l \) and \( F_r \) being zero (i.e., inactive). Figure 4 shows the evolution of the control forces needed to stabilize the vehicle. Using initial guess of control variables for direct transcription as \( F_l(t) = F_r(t) = 1000 \) instead of zeros and a uniform grid with \( N = 121 \) points, the dynamic optimization yields the results shown in figures 6 and 7. Figure 6 shows that the inclusive disjunctive constraints are satisfied. The time profile of the variables \( X, Y, \theta_X \) is similar to that in figure 5 while \( Z \) and \( \theta_X \) shows minor differences with the zero initial guess. However, the obtained control forces are very different from the ones in figure 4 implying that there are possibly several local optima. However, for the realization on a active suspension
system, it is desirable to have a single time profile for the control forces insensitive to the perturbation of initial guesses and grid size in the optimizer. This can be accomplished using a set of anti-symmetric control forces, i.e., $F_l = -F_r$. With this additional constraint $F_l + F_r = 0$, the solutions are obtained with different initial guesses as before. The controls then become almost insensitive to the perturbation of initial guesses and of grid sizes, as the anti-symmetric control force constraint resolves the problem of the local optima. Figure 8 shows the control forces, satisfying the anti-symmetric force constraints and computed with the initial guess $F_r(t) = -F_l(t) = 1000$ using $N = 151$ grid points, while figure 9 shows the time profile of $Z$ and $\theta_X$.

VI. ROLLOVER INDEX AND EFFICACY OF DISJUNCTIVE CONSTRAINTS

To underscore the effectiveness of the disjunctive constraints approach over the conservative constraints, we use the concept of rollover index [7]. The rollover index $R$ is defined as

$$R = \frac{(F_{Z2} + F_{Z4}) - (F_{Z1} + F_{Z3})}{(F_{Z2} + F_{Z4}) + (F_{Z1} + F_{Z3})}.$$  

The wheel lift-off starts occurring at $|R| = 1$. $|R| < 1$ indicates no lift-off of the wheels and hence no rollover. On the other hand $|R| > 1$ indicates lift-off of the wheel. But not all wheel lift-off causes rollover. This is exactly where the present disjunctive constraint [14] approach becomes useful in computing anti-rollover forces in the suspensions, i.e., a correct negative roll moment can stabilize the vehicle even when the wheels have lifted off the ground. The following comparison of absolute values of rollover index over the simulation interval shows that the disjunctive approach is a more realistic approach.

Let us consider the vehicle control problem with the following computation for both the disjunctive constraint and the conventional conservative approaches. The transcription is done with $N = 151$ equally spaced grid points along with the initial guess for controls as $F_l = F_r = 0$. The solution plots for the disjunctive constraints are given in figure 8; these indicate the absence of rollover. An anti-rollover solution with conservative constraints (no wheels are allowed to lift-off) is computed, as shown in figure 9. Rollover index from both the simulations are plotted in figure 10. Figure 10 shows that the disjunctive constraints allow the wheels to lift-off and yet the vehicle gets stabilized by the computed controls. The same figure compares the disjunctive constraint approach to the conventional conservative approach which does not cover the more severe situation of lift off of the wheels. Thus, whenever there is room for stabilization and prevention of rollover even when the wheels have lifted off during a severe maneuver, the disjunctive constraint approach provides an effective way to compute the control forces in active suspensions.

Figure 11 shows the rollover index obtained from the computation by the transcription method to find anti-symmetric controls (see figure 8) for the disjunctive dynamics. In this case too, the vehicle is stabilized with anti-roll moment induced by the anti-symmetric control forces at the suspensions even though wheels are allowed to be lifted off (corresponding to $|R| > 1$).

VII. SYNTHESIS OF CONTROL FORCES IN TERMS OF SENSOR DATA OUTPUT

In order to develop an effective control system, it is necessary to investigate if the control forces can be represented as a linear combination of sensible parameters so that the sensor output data from the system can be used to synthesize the control forces in the active suspensions. Specifically, we seek to determine the (local) optimum values (in the neighborhood some values that are useful and attainable from the engineering point of view) of the coefficients $\varphi$’s in the formula

$$F_l = \varphi_1 \theta_X + \varphi_2 \dot{\theta}_X + \varphi_3 \dot{\theta}_Z + \varphi_4 (Z - Z_0) + \varphi_5 \dot{Z}$$

along with the constraint $F_r = -F_l$. For synthesis of the control force, we find the dominating terms in the above formula to determine which sensor parameters are critical and determine the
force. The linear combination of the dominant terms which best approximates \( F_l \) is determined. The control output is synthesized from these terms weighted by the respective coefficients. The weights computed for a range of maneuvers similar to that in our numerical computation are stored in a look up table on the embedded computing element of the controller and fed forward into the active suspension. Using \( \dot{\phi} \) to denote the vector \([\varphi_1 \varphi_2 \varphi_3 \varphi_4 \varphi_5]\), a local optimum is obtained with the initial guess \( \dot{\varphi} = 0 \) and \( N = 151 \) grid points to yield \( \dot{\varphi} = [-916.5607 -2102.4 -4799.4 3.8244 \times 10^{-4} -0.0078] \). The time-variation of \( F_l \) is shown in figure 13. Also, the magnitudes of the individual terms that make up the equation \( F_l = \varphi_1 \dot{\theta}_X + \varphi_2 \dot{\theta}_X + \varphi_3 \dot{\theta}_Z + \varphi_4 (Z - Z_0) + \varphi_5 \dot{Z} \) is shown in figure 13. It is apparent that the term \( \varphi_3 \dot{\theta}_Z \) is the most dominating term and approximates the total control force the best. This means, applying a reaction force proportionate to the rate of yaw stabilizes the roll over tendency of the vehicle undergoing fishhook maneuver with wheels lifting off the ground. In another numerical experiment with different initial guess (active forces) we obtain \( F_l \) as shown in figure 14 and this closely matches the one in figure 13 showing the robustness of the approximation, i.e., insensitivity to perturbation of initial guesses in the optimizer. The coefficient \( \varphi_3 \) is found in this case to be \(-4796.2\) which is comparable to its value from the previous example, with an initial guess that the control forces are inactive. This shows an effective way to synthesize the control forces that can vary linearly in proportion to the sensed parameter \( \dot{\theta}_Z \) and stabilize the vehicle in spite of the wheels lifting off the ground.

A. Disjunctive Constraints vis-a-vis Conservative Constraints

We compare solutions with disjunctive constraints with those corresponding to the conservative constraints [15] defined in section IV-B. Choosing the same initial guesses and the step-size as for the solution shown in figure 8 the control forces obtained for the conservative case are shown in figure 15. As before we model

![Fig. 11. Comparison of the magnitude of the rollover index over time. The present model using disjunctive constraints show stabilization even after wheels have lifted off (rollover index magnitude greater than one) while the existing conservative approach does not allow wheels to lift off.](image1)

![Fig. 12. Magnitude of the rollover index over time with anti-symmetric control forces and disjunctive constraints. The present model using disjunctive constraints show stabilization even after wheels have lifted off (rollover index magnitude greater than one).](image2)

![Fig. 13. Control variable \( F_l \) for Disjunctive Constraints along with its Approximation](image3)

![Fig. 14. Control variable \( F_l \) expressed as \( F_l = \varphi_3 \dot{\theta}_Z \) for Disjunctive Constraints](image4)

![Fig. 15. Anti-Symmetric Controls with Conservative Constraints; \( N = 151 \)](image5)
the control forces as a linear combination of sensible parameters and find the (local) optimum values of the coefficients $\varphi$’s in the formula $F_l = \varphi_1 \dot{\theta} X + \varphi_2 \dot{\theta} X + \varphi_3 \dot{\theta} Z + \varphi_4 (Z - Z_0) + \varphi_5 \dot{Z}$ along with the constraint $F_r = -F_l$. Figure 16 shows the control forces $F_l$ and $F_r$ obtained with the initial guess $\varphi_i = 0$, $\forall i = 1, \ldots, 5$ and with $N = 151$ grid points, while figure 17 shows the satisfaction of the conservative constraints. The (local) optimal value of the coefficients $\varphi$’s in the above linear combination formula of $F_l$ is found to be

$$\varphi = \begin{bmatrix} -800.8541 & -1556.5 & -4763.9 & 0 & 6.0974 \times 10^{-4} \end{bmatrix}.$$ 

In figure 18 we find that $\varphi_3 \dot{\theta} Z$ is the dominating term approximating $F_l$ the closest. Hence, as before, $F_l$ is approximated by $\varphi_3 \dot{\theta} Z$ alone. The control force $F_l$ is re-calculated setting $F_l = \varphi_3 \dot{\theta} Z$ and $\varphi_5 = -4761.2$ is obtained. This is shown in figure 19.

It is apparent that with conservative constraints, the force requirements are comparable to that when the inclusive disjunctive constraints are used, although the latter ones correctly stabilize the roll in spite of the wheels being lifted off. Thus, control forces as linear functions of $\theta Z$ (sensed yaw rate data) while satisfying the disjunctive constraints (i.e., with wheels lifting off) is realizable within the existing hardware systems but would ensure safety under more severe maneuvering conditions.

VIII. VALIDATION USING ARBITRARILY STEERING MANEUVER WITH THE SYNTHESIZED CONTROLS AS INPUT

In the preceding sections we found that in case of the anti-symmetric controls, $F_l$ (and hence $F_r$) can be taken as a linear function of the yaw rate, i.e., $\dot{\theta} Z$. The same synthesized controls are then used against an arbitrary steering input given in figure 24 (cf. [?]) and we check whether the disjunctive constraints are satisfied.

1) Inclusive disjunctive constraints, anti-symmetric controls: $F_l$ is approximated by the linear formula $F_l = \varphi_3 \dot{\theta} Z$ and $\varphi_3 = -4796.2$ as computed in the previous section. The simulation of the dynamics with $F_l = -4796.2 \dot{\theta} Z$ and $F_r = -F_l$ yields the plots given in figure 20 which in turn shows that the disjunctive constraints are satisfied.

2) Conservative constraints, anti-symmetric controls: The synthesized control in this case is $F_l = \varphi_3 \dot{\theta} Z$, $\varphi_3 = -4761.2$. The plots in the same figure 20 verify that the constraints are satisfied when the synthesized force is used as the control input.

A. Robustness Check Against More Arbitrary Steering Inputs

We use the steering inputs in figures 25 and 26. It may be observed that although the steering input rates and values are higher the controls synthesized with the steering input of figure 25 still works as the control is primarily dependent on the yaw rates which remain considerable in these severe maneuvers. The disjunctive constraints are satisfied in case of both the inputs. We can see that in figure 21 the first plot shows that at about 1 s, for a short while only one constraint is satisfied, indicating that the control forces are providing the anti-roll moment. Although the input in figure 26 is a more severe maneuver, the plots in figure...
found that the disjunctive constraints enable us to stabilize vehicles with wheels lifted off, whenever there is a room for doing so, using control forces comparable to that required for the existing conservative approach of not allowing the wheels to lift off. This increases safety under severe maneuver over the conservative approach which is limited to wheels not lifting off the ground. Finally we arrived at a simple linear formula proportional to sensor output data enabling the synthesis of the anti-rollover control forces. Future work may be directed toward obtaining smoother independent control forces in the suspensions. Other objective functions based on control effort and handling comfort in place of (8a) need to be explored. Additionally, path constraints such as \( X(t) = P_1(t) \) and \( Y(t) = P_2(t) \) where \( P_1(t) \) and \( P_2(t) \) may be prescribed as known paths in time for increased smoothness and handling comfort. However, the high index of the resulting differential-algebraic equation model may be a potential computational difficulty that would need finding effective discretization and optimization strategies.

**APPENDIX: PARAMETERS USED IN COMPUTATIONS**

We list below the simulation data and values of model parameters used in the numerical computations.

**Time Interval:** \( t_0 = 0 \) s and \( t_f = 1.5 \) s.

**Initial Conditions**
- \( X(0) = 0, \dot{X}(0) = \dot{X}_0 = \frac{400}{g} \) m/s
- \( Y(0) = 0, \dot{Y}(0) = 0 \)
- \( Z(0) = Z_0, \dot{Z}(0) = 0 \)
- \( \theta_X(0) = 0, \dot{\theta}_X(0) = 0 \)
- \( \theta_Z(0) = 0, \dot{\theta}_Z(0) = 0 \)

**Parameter Values Used in the Model**
- \( M = 1400 \) kg (vehicle mass) \( T = 1.5 \) m (track width)
- \( K = 30000 \) kg/s\(^2\) (suspension stiffness)
- \( C = 4000 \) kg/s (suspension damping)
- \( I_{XX} = 1300 \) kgm\(^2\) (roll moment of inertia)
- \( I_{ZZ} = 4000 \) kgm\(^2\) (yaw moment of inertia)
- \( h = 0.7 \) m (height of center of gravity (CG))
- \( a = 1.4 \) m (longitudinal distance of front axle from CG)
- \( b = 1.5 \) m (longitudinal distance of rear axle from CG)
- \( g = 9.8 \) m/s\(^2\) (acceleration due to gravity)
- \( \mu = 1.3 \) (friction coefficient)
- \( r_{X1} = r_{X2} = a \)
- \( r_{X3} = r_{X4} = -b \)
- \( r_{Y1} = r_{Y3} = -r_{Y2} = -r_{Y4} = \frac{T}{2} \)
- \( \delta_3 = \delta_4 = 0 \) (steering angle of rear wheels)
- \( \delta_1 = \delta_2 = \delta(t) \) (steering angle of front wheels, see figures 23 and 26)
- \( Z_{min} = 0.5 \) m (minimum height of suspension mount point)
- \( Z_{max} = 0.9 \) m (maximum height of suspension mount point)
- \( F_{max} = 10000 \) N (maximum controlling force limit)

**Constants Used in Tire Force Calculation**
- \( C_T = 1.30, \Delta S_h = 0 \)
- \( a_1 = -22.1, a_2 = 1011, a_3 = 1078 \)
- \( a_4 = 1.82, a_5 = 0.208, a_6 = 0 \)
- \( a_7 = -0.354, a_8 = 0.707 \)
Fig. 23. Fishhook steering function $\delta_i(t)$, $i = 1, 2$ in degrees

Fig. 24. Arbitrary steering function $\delta_i(t)$, $i = 1, 2$ in degrees. Although similar to the input with which control was synthesized, the steering rate is faster.

Fig. 25. Arbitrary steering function $\delta_i(t)$, $i = 1, 2$ in degrees. This is a more severe maneuver.

Fig. 26. Arbitrary steering function $\delta_i(t)$, $i = 1, 2$ in degrees. Steering input rates are much faster and the maneuver is even more severe than that in figure

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