COMPLETIONS OF HIGHER EQUIVARIANT $K$-THEORY

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Abstract. For a linear algebraic group $G$ acting on a smooth variety $X$ over an algebraically closed field $k$ of characteristic zero, we prove a version of non-abelian localization theorem for the rational higher equivariant $K$-theory of $X$. This is then used to establish a Riemann-Roch theorem for the completion of the rational higher equivariant K-theory at a maximal ideal of the representation ring of $G$.

1. Introduction

Ever since Grothendieck proved his very general Riemann-Roch theorem, the Riemann-Roch problem has by now become one of the most important tools in algebraic geometry for studying the sheaves and bundles on varieties and analytic spaces in terms of the various cohomology theories on the variety such as the motivic and the singular cohomology. The well known Riemann-Roch theorem of Baum, Fulton and MacPherson [1] states that the Grothendieck group of coherent sheaves on an algebraic variety has a natural isomorphism with the Chow groups of algebraic cycles on the variety, when these groups are taken with the rational coefficients. This result was later generalized by Bloch [2] when he founded the theory of higher Chow groups and showed that these higher Chow groups form a cohomology theory. Moreover, when considered with the rational coefficients, they completely describe the higher K-theory of coherent sheaves of a quasi-projective variety.

When a linear algebraic group $G$ acts on a variety $X$, it becomes very natural to ask if the Grothendieck group or more generally, the $K$-theory of the equivariant coherent sheaves has similar description in terms of the geometry of the variety. Such a description becomes important even from the point of view of the representation theory of the group and gives a more concrete and geometric way of understanding the representation theory of linear algebraic groups. However, it turns out, as is evident from the work of Edidin and Graham [5], that if at all there is one, such a description can not be as direct as in the non-equivariant case. The basic reason for this is that the equivariant (higher) Chow groups are always complete with respect to certain linear topology while this is certainly not the case with the equivariant K-theory.

A variety in this paper will mean a reduced, connected and separated scheme of finite type with an ample line bundle over a field $k$. This base field $k$ will be fixed throughout this paper and will be assumed to be algebraically closed and of characteristic zero (except in Section 2, where $k$ is arbitrary). Let $G$ be a connected
linear algebraic group over \( k \) acting on such a variety \( X \). Recall that this action on \( X \) is said to be linear if \( X \) admits a \( G \)-equivariant ample line bundle, a condition which is always satisfied if \( X \) is normal [18, Theorem 2.5]. All \( G \)-actions in this paper will be assumed to be linear. For \( i \geq 0 \), let \( K^G_i(X) \) (resp. \( G^G_i(X) \)) denote the \( i \)th homotopy group of the \( K \)-theory spectrum of \( G \)-equivariant vector bundles (resp. coherent sheaves) on \( X \). The \( G \)-equivariant higher Chow groups \( CH^G_j(X,i) \) of \( X \) were defined by Edidin and Graham (cf. [7], also see Section 5 for more detail) as the ordinary higher Chow groups of the quotient space \( X \times U \), where \( U \) is an open subscheme of a representation of \( G \) on which the action of \( G \) is free, and its complement is of sufficiently high codimension. The representation ring \( R(G) \) of \( G \) is simply the group \( K^G_0(k) \). The ring structure is given by the tensor product of representations. In this paper, all the (equivariant) \( K \)-groups and higher Chow groups will be considered with the rational coefficients, and its complement is of sufficiently high codimension. The representation ring \( R(G) \) of \( G \) is simply the group \( K^G_0(k) \). The ring structure is given by the tensor product of representations. In this paper, all the (equivariant) \( K \)-groups and higher Chow groups will be considered with the rational coefficients. In particular, the representation ring \( R(G) \) will actually mean \( R(G) \otimes \mathbb{Z} \mathbb{Q} \). For any \( \mathbb{Q} \)-algebra \( A \) and a \( \mathbb{Q} \)-vector space \( V \), the group \( V_A \) will mean the group \( V \otimes \mathbb{Q} A \). These notations will be very often employed for the equivariant \( K \)-groups and higher Chow groups.

In view of the completeness of the product of equivariant Chow groups as stated above, Edidin and Graham showed in loc. cit. that if \( G^G_0(X) \) is completed with respect to the maximal ideal of the the virtual rank zero representations in \( R(G) \), then this completion is isomorphic to the product of the equivariant Chow groups via a Riemann-Roch map. This was the first result of Riemann-Roch type in the equivariant \( K \)-theory. Subsequently in [8], Edidin and Graham studied the problem of describing the completion of \( G^G_0(X) \) with respect to any given maximal ideal of \( R(G) \) in terms of algebraic cycles, in the case when the base field is \( \mathbb{C} \) and all the Grothendieck groups, Chow groups and the representation rings are considered with the complex coefficients. To state their result in a concise form, recall from [6] that if \( G \) is a complex algebraic group, every maximal ideal \( m_\Psi \) of \( R_\mathbb{C}(G) \) is given as the ideal of the representations whose characters vanish at a unique semi-simple conjugacy class \( \Psi \) in \( G \).

**Theorem 1.1** (Edidin-Graham). Let \( G \) act on a complex algebraic space \( X \). For any maximal ideal \( m_\Psi \) of \( R_\mathbb{C}(G) \), let \( Z \) denote the centralizer of an element \( h \) of \( \Psi \) and let \( X^h \) be the fixed subspace of \( X \) for \( h \). Then there is a Riemann-Roch isomorphism

\[
\tau^\Psi_X : (G^G_0(X)_\mathbb{C})_{m_\Psi} \rightarrow \prod_{j=0}^{\infty} CH^G_j(X^h)_{\mathbb{C}}.
\]

Moreover, for smooth \( X \), this map is described in terms of Chern characters and Todd classes.

Edidin and Graham prove this result by extending Thomason’s localization theorem [19] for the algebraic \( K \)-theory for the action of diagonalizable groups to the case of general complex algebraic groups (cf. loc. cit., Theorem 4.3). This localization theorem implies in particular that in the situation of Theorem 1.1, the restriction map of the completions of the equivariant \( K \)-theory induces an
isomorphism

\[ i^! : (\hat{G}_i^G(X)_C)_{\mathfrak{m}_\psi} \to (\hat{G}_i^\mathbb{Z}(X^h)_C)_h, \]

where the term on the right is the completion of the equivariant \( K \)-groups with respect to the maximal ideal corresponding to the conjugacy class in \( \hat{Z} \) consisting of the single element \( h \).

The results of this paper were motivated by some of the questions raised in [8]. The first natural question is the formulation of the Edidin-Graham’s completion theorem for the rational \( K \)-theory rather than the complex \( K \)-theory of varieties with group actions. The second and the more important question is whether it is possible to describe the completion of the higher equivariant \( K \)-theory with respect to the various maximal ideals of \( R(G) \) in terms of the algebraic cycles on some subspaces. Our main purpose in this paper is to answer these questions for the rational equivariant higher \( K \)-theory of smooth varieties.

We fix some notations before we state our first result. For any finitely generated abelian group \( N \) and any field \( l \) of characteristic zero, let \( T = D_l(N) \) denote the unique split diagonalizable group over \( l \) whose character group is given by \( N \). \( D_l(N) \) is the spectrum of the group algebra \( l[N] \). It is well known that \( D_l(N) \) is a torus if and only if \( N \) has no torsion. There is also a canonical isomorphism of \( \mathbb{Q} \)-algebras \( \mathbb{Q}[N] \to R(T) \). By [19] Lemme 1.1, Proposition 1.2, every maximal ideal \( \mathfrak{m} \) of \( R(T) \) corresponds to a unique diagonalizable closed subgroup \( T_\mathfrak{m} = D_l(N/N_\mathfrak{m}) \) of \( T \) where \( N_\mathfrak{m} \) is the kernel of the natural map of abelian groups

\[ N \xrightarrow{\phi} (R(T)/\mathfrak{m})^* \text{ given by } n \mapsto [n] \text{ mod } \mathfrak{m}. \]

\( T_\mathfrak{m} \) is called the support of \( \mathfrak{m} \). Note that \( T_\mathfrak{m} \) is connected if and only if the image of \( \phi \) has no torsion. Let \( G \) be a connected linear algebraic group over \( k \) with a fixed maximal torus \( T = D_k(N) \) of rank \( r \). It follows from Proposition [2.1] below that the natural map \( R(G) \to R(T) \) of \( \mathbb{Q} \)-algebras is finite. In particular, every maximal ideal \( \mathfrak{m} \) of \( R(G) \) can be lifted to a (not necessarily unique) maximal ideal \( \mathfrak{m}^\phi \) of \( R(T) \). There is a distinguished maximal ideal \( I_G \) of \( R(G) \), which is the ideal of the rank zero virtual representations, i.e., \( I_G \) is the kernel of the rank map \( R(G) \to \mathbb{Q} \). It is called the augmentation ideal of \( R(G) \). It is easy to see that (cf. Corollary [2.10]) \( I_T \) is the only lift of \( I_G \) in \( R(T) \) and the support of \( I_T \) is the identity subgroup of \( T \).

Suppose now that \( G \) acts linearly on a smooth quasi-projective variety \( X \) over \( k \). It is then well known that the natural map \( K_*^G(X) \to G_*^G(X) \) is an isomorphism. Hence we shall not make any distinction between these groups for smooth varieties in this paper. It is also well known that \( K_*^G(X) \) and \( G_*^G(X) \) are naturally modules for the \( \mathbb{Q} \)-algebra \( K_*^G(X) \) and hence they are also modules for the \( \mathbb{Q} \)-algebra \( R(G) \). For any maximal ideal \( \mathfrak{m} \) of a commutative ring \( A \) and for any \( A \)-module \( M \), let \( \hat{M} \) denote the \( \mathfrak{m} \)-adic completion of \( M \). We fix a maximal ideal \( \mathfrak{m} \) of \( R(G) \) and choose its lift \( \hat{\mathfrak{m}} \) as a maximal ideal of \( R(T) \). Let \( T_\hat{\mathfrak{m}} \subset T \) denote the support of \( \hat{\mathfrak{m}} \) and let \( Z = Z_G(S) \) denote the centralizer of \( S = T_\hat{\mathfrak{m}} \otimes \mathbb{Q} k \) in \( G \) under the inclusion \( S \subset T \subset G \). Let \( \overline{\mathfrak{m}} \) be the restriction of \( \hat{\mathfrak{m}} \) to \( R(Z) \) under the natural restriction maps \( R(G) \to R(Z) \to R(T) \). Our first main result deals with the problem of
representing the completions $\hat{K}_i^G(X)_m$ of the equivariant higher $K$-theory of $X$ in terms of algebraic cycles. For such a variety $X$, put
\[ CH^*_G(X, i) = \bigoplus_{j \geq 0} CH^j_G(X, i) \otimes \mathbb{Q} \quad \text{for} \quad i \geq 0. \]

It is known [7] that the term on the right is an infinite sum in general. Put $S(G) = CH^*_G(k, 0)$. Then $S(G)$ is a graded $\mathbb{Q}$-algebra with $S(G)_0 = CH^0_G(k) = \mathbb{Q}$, where the graded ring structure is given by the intersection product on the equivariant Chow groups of the classifying space $BG$. Moreover, this intersection product and the pullback maps on the equivariant Chow groups make $CH^*_G(X, i)$ a graded module for the ring $S(G)$. Let $J_G$ denote the maximal ideal $\bigoplus_{j \geq 1} CH^j_G(k, 0)$ and let $\hat{CH}^*_G(X, i)$ denote the $J_G$-adic completion of the $S(G)$-module $CH^*_G(X, i)$. It was shown in [12, Theorem 1.2] that if $G$ acts on a smooth quasi-projective variety $X$, then there is a Chern character map
\[ (1.2) \quad ch^X : K^G_i(X)_{i_G} \rightarrow \hat{CH}^*_G(X, i) \]
which is an isomorphism. This result was proved for $i = 0$ by Edidin and Graham [5, Theorem 4.1]. This shows that the completion of the higher $K$-groups at the augmentation ideal of $R(G)$ can be represented by the equivariant higher Chow groups via Chern characters. Our purpose here is to give a similar representation of the completion of the rational equivariant higher $K$-groups at other maximal ideals of the representation ring. This in particular extends the result of [8, Theorem 5.5] to the higher equivariant $K$-theory of smooth varieties and hence answers a question of Edidin and Graham in loc. cit. The following result in particular shows that although it is impossible to prove the direct equivariant analogue of the Riemann-Roch Theorem of Bloch [2, Theorem 9.1] showing the isomorphism between the rational higher $K$-theory and higher Chow groups, it is indeed possible to prove such an isomorphism whenever the equivariant higher $K$-groups are completed at a given maximal ideal of the representation ring of the underlying group.

**Theorem 1.2.** Let the group $G$ act on a smooth quasi-projective variety $X$ and let $m$ be a given maximal ideal of $R(G)$. Assume that the support $T_m$ is a subtorus of the maximal torus $T$ of $G$. Let $L = R(T)/m$ be the residue field of $m$ and let $f : X_m \hookrightarrow X$ denote the inclusion of the fixed locus for the action of $T_m$ on $X$. Then for every $i \geq 0$, there is a Chern character map
\[ ch^X_m : K^G_i(X)_mL \rightarrow CH^*_Z(X_m, i)_L \]
which is an isomorphism.

As in the proof of the twisted Riemann-Roch theorem in loc. cit. for the Grothendieck group of equivariant coherent sheaves, we deduce the above result by first proving the following version of the non-abelian completion theorem for the rational equivariant $K$-theory. We first generalize the completion result of Edidin and Graham for the complex equivariant $K$-theory to the case of $K$-theory with coefficients in the base field $k$. This is then used to prove the following result for
the rational equivariant $K$-theory by an extension and descent argument. To state the result, we fix a maximal ideal $m$ of $R(G)$ with a lift $\tilde{m}$ as a maximal ideal of $R(T)$. Let $\tilde{m}$ be the restriction of $\tilde{m}$ to $R(Z)$ in the notations of Theorem 1.2. We have the following restriction map of various completions.

\[ \text{res}_m : \widehat{K}_G^i(X)_m \to \widehat{K}_Z^i(X)_m \to \widehat{K}_Z^i(X)_{\tilde{m}} \]

**Theorem 1.3.** Let $G$ act on a smooth quasi-projective variety $X$ and let $m$ be a given maximal ideal of $R(G)$. Assume that the support $T_e$ is a subtorus of the maximal torus $T$ of $G$ and let $f : X_e \hookrightarrow X$ denote the inclusion of the fixed locus for the action of $T_e$ on $X$. Then for every $i \geq 0$, all the horizontal arrows in the commutative diagram

\[ \begin{array}{ccc}
\tilde{f}_m^i : K^i_G(X) \otimes_{R(G)} \widehat{R(G)}_m & \to & K^i_Z(X) \otimes_{R(Z)} \widehat{R(Z)}_m \\
\text{res}_m & \downarrow & \text{res}_m \\
\tilde{f}_m^i & \to & K^i_Z(X)_{\tilde{m}} \otimes_{R(Z)} \widehat{R(Z)}_m
\end{array} \]

are isomorphisms of $\widehat{R(G)}_m$-modules and the vertical arrows are injective.

We now make some remarks about the above results. We first point out that our results are stated for the action of reductive groups. However, the general case can be easily deduced from this by choosing a Levi subgroup $L$ containing $T$, which always exists in characteristic zero as can be seen using the Lie algebras. The proofs are then completed by Proposition 3.7. We do not say more on this. The more important remark we want to make is that our version of non-abelian completion and Riemann-Roch theorems are proved for the completion of the rational equivariant $K$-theory at the maximal ideals of the representation ring. As the reader would observe, many of the intermediate results proved in this paper hold for all prime ideals of the representation ring although they have been stated only for the maximal ideals. We expect that the main results of this paper can be proved for all prime ideals using these intermediate results and some more analysis. The reader can also see that even though the non-abelian completion holds for the rational $K$-theory, one has to replace the field of rationals by the residue field of the given maximal ideal for proving the Riemann-Roch theorem. In fact, our proof of Theorem 1.2 (especially Lemma 5.1) would show that it may not be possible to avoid doing this if one wants to represent the equivariant higher $K$-theory by the algebraic cycles.

We end the introduction with a brief outline of the various sections of this paper. As in [S], our proof of the Riemann-Roch theorem for equivariant higher $K$-theory is based on the non-abelian completion theorem for the rational equivariant $K$-theory. In order to prove this, we devote our next section to the study of the geometric aspects of the variety defined by the representation ring of the underlying algebraic group $G$ defined over an arbitrary field $k$ (of any characteristic). This helps in particular to determine all the maximal ideals of $R(G)$ as well as $R_k(G)$. The results of this section do not depend on the nature of the field $k$ and hence can be helpful in pursuing some of the questions discussed above. Section 3 is
devoted to proving most of the preliminary results one needs to prove Theorem 1.3. We finally prove this theorem in Section 4 by first proving it for those groups whose derived groups are simply connected and then deducing the general case from this through some reduction steps. Section 5 is devoted to the proof of our Riemann-Roch theorem. We do this by first defining an automorphism of the higher equivariant K-theory which converts the completion with respect to a given maximal ideal to the completion with respect to the augmentation ideal of the representation ring. This is then combined with Theorem 1.3 and a version of Riemann-Roch theorem for the higher equivariant K-theory, proved recently in [12].

2. Geometry of the Representation Ring \( R_k(G) \)

In this section, \( k \) will denote any field with arbitrary characteristic. Let \( G \) be a linear algebraic group over a field \( k \). Recall that \( R_k(G) \) denotes the representation ring of \( G \) with coefficients in \( k \), i.e., \( R_k(G) = R(G) \otimes \mathbb{Z} k \). We establish the preliminary results about \( R_k(G) \) in this section. Our main purpose here is to identify this \( k \)-algebra as the invariant subalgebra of the coordinate ring of the group \( G \) for the adjoint action of \( G \) on itself. This is then used to describe the maximal ideals of \( R_k(G) \) in terms of the closed points of the group \( G \) itself. Let \( k \) be any given field. We begin with the following finiteness result about the maps between the integral representation rings. When \( k \) is the field of complex numbers, such a result was proved for the rings \( R_C(G) \) in [6] using the corresponding result of Segal [16] for the compact Lie groups.

**Proposition 2.1.** Let \( G \) be a connected linear algebraic group over \( k \) and let \( H \) be a closed connected subgroup of \( H \). Then the restriction map of integral representation rings \( R(G) \to R(H) \) is finite.

**Proof.** We first prove the proposition when \( k \) is algebraically closed and \( H \) is a maximal torus \( T \) of \( G \). If \( V \) is an irreducible representation of \( G \) and if the unipotent radical \( R_uG \) of \( G \) acts non-trivially on \( V \), then the invariant subspace of \( V \) for this action is a non-zero subspace (cf. [3, Corollary 10.5]) which is invariant under \( G \) as well since \( R_uG \) is normal in \( G \). This contradicts the irreducibility of \( V \). Hence \( R_uG \) must act trivially on \( V \). Since \( R(G) \) is a free abelian group on the set of irreducible representations of \( G \), we see that the map \( R(G) \to R(G/R_uG) \) is an isomorphism. Hence we can assume that \( G \) is reductive. Let \( W \) denote the Weyl group of \( G \). Then \( W \) acts naturally on \( T \) by inner automorphisms which induces an action of \( W \) on \( R(T) \). Moreover, [7, Théoreme 4] implies that and the natural map of the representation rings induces a ring isomorphism

\[
R(G) \xrightarrow{\cong} (R(T))^W,
\]

where the term on the right is the ring of invariants for the Weyl group action. Since the natural map \( \mathbb{Z}[N] \to R(T) \) is an isomorphism of rings (where \( N \) is the character group of \( T \)), we see that \( R(T) \) is a finite type algebra over \( \mathbb{Z} \). In particular, it is finite over \( (R(T))^W \) which is then also a finite type \( \mathbb{Z} \)-algebra (cf. [12, Lemma 4.7]) and hence noetherian. Now if \( H \) is a closed connected subgroup of \( G \), we choose a maximal torus \( T \)
of $G$ containing $S$. Then we get a commutative diagram of representation rings

$$
\begin{array}{ccc}
R(G) & \longrightarrow & R(T) \\
\downarrow & & \downarrow \\
R(H) & \longrightarrow & R(S)
\end{array}
$$

We have shown above that the horizontal arrows are finite and injective maps. Since $S$ is a subtorus of $T$, there is a decomposition $T = S \times S'$ and hence the map $R(T) \rightarrow R(S)$ is surjective. In particular, $R(S)$ is finite over $R(G)$. Since we have also shown above that $R(G)$ is noetherian, we conclude that $R(H)$ is finite over $R(G)$.

Finally, we consider the case when $k$ is not necessarily algebraically closed. We can embed $G$ as a closed subgroup of $\text{GL}_n$ over $k$ for some $n$ and then it suffices to show that $R(H)$ is finite over $R(\text{GL}_n)$. For any algebraic group $G$ over $k$, let $G_\overline{k}$ denote the base change to the algebraic closure of $k$. Then we get the following commutative diagram of representation rings.

$$
\begin{array}{ccc}
R(\text{GL}_n) & \longrightarrow & R(\text{GL}_{n,\overline{k}}) \\
\downarrow & & \downarrow \\
R(H) & \longrightarrow & R(H_{\overline{k}})
\end{array}
$$

By the Galois descent for the rational algebraic $K$-theory, it is easy to see that when tensored with the field of rational numbers, the terms on the left are the Galois invariants of the corresponding terms on the right (cf. [10, Lemma 8.4]). In particular, the kernels of the horizontal maps are torsion. On the other hand, a representation ring is the free abelian group on the set of irreducible representations and hence has no torsion. We conclude that $R(H)$ is a $R(G)$-submodule of $R(H_{\overline{k}})$. Since $R(\text{GL}_n)$ is known be noetherian, we only now need to show that $R(H_{\overline{k}})$ is finite over $R(\text{GL}_n)$. But this is immediate from the case of algebraically closed fields and the fact that $R(\text{GL}_n) \cong R(\text{GL}_{n,\overline{k}})$.

**Remark 2.2.** It is shown in [12] that for the groups $G$ and $H$ as in Proposition 2.1, map $S(G) \rightarrow S(H)$ of the equivariant Chow rings is finite with the rational coefficients. It is still not known if this holds with the integral coefficients as well.

We next state a completely known result about the semi-simple conjugacy classes in a connected reductive group.

**Lemma 2.3.** Let $T$ be a maximal torus of a connected reductive group $G$ over an algebraically closed field $k$. Let $N$ denote the normalizer of $T$ in $G$. Then any two elements of $T$ which are conjugate in $G$, are also conjugate by an element of $N$. In particular, the set of semi-simple conjugacy classes in $G$ is in natural bijection with the set $T/W$ of $W$-orbits in $T$, where $W$ is the Weyl group of $G$.

**Proof.** (Sketch) Suppose $s, t \in T$ are such that $t = hsh^{-1}$ in $G$. It is then easy to check directly that $hTh^{-1}$ is contained in $Z_G(t)$ and hence $T$ and $hTh^{-1}$ are maximal tori of the identity component of $Z_G(t)$. Hence they are conjugate by an
element \(g\) of \(Z_G(t)\) (cf. [3, Corollary 11.3]). But this implies that \(gh \in N\) and 
\((gh)s(gh)^{-1} = gtg^{-1} = t\).

For the second assertion, first note that every semi-simple element lies in a maximal torus and hence its conjugacy class is same as the conjugacy class of an element of \(T\). Moreover, the above assertion shows that two such conjugacy classes are distinct if and only if they are not conjugate by an element of \(N\). On the other hand, \(W = N/T\) acts on \(T\) by conjugation. This implies that \(G\)-conjugacy classes of two elements of \(T\) are distinct if and only if they are in distinct \(W\)-orbits. □

Let \(G\) be a connected reductive group over a field \(k\). For any affine \(k\)-variety \(X\), let \(k[X]\) denote the coordinate ring of \(X\). Let \(\text{Sch}_k\) denote the category of finite type \(k\)-schemes. Note then that there is a natural isomorphism of \(k\)-algebras

\[
(2.2) \quad k[X] \xrightarrow{\cong} \text{Hom}_{\text{Sch}_k}(X, A^1_k).
\]

Recall that the algebraic group \(G\) is given by the data of \(k\)-morphisms

\[
G \times_k G \xrightarrow{\mu} G, \quad G \xrightarrow{\iota} G, \quad \text{Spec}(k) \xrightarrow{e} G,
\]

which define the multiplication, inverse and the identity section respectively. This data is equivalent to the following data of morphisms of the Hopf algebras over \(k\).

\[
k[G] \xrightarrow{\text{id}} k[G] \otimes_k k[G], \quad k[G] \xrightarrow{\iota} k[G], \quad k[G] \xrightarrow{e} k.
\]

Let \(G \xrightarrow{\Delta} G \times G\) denote the diagonal map and let \(G \xrightarrow{(\mu, \text{id})} G \times G\) denote the map which switches the coordinates. The adjoint action of \(G\) on itself by the composite map ‘ad’ given by

\[
(2.3) \quad G \times G \xrightarrow{(\Delta, \text{id})} G \times G \times G \xrightarrow{(\iota, \text{sw})} G \times G \times G \xrightarrow{(\mu, \text{id})} G \times G \xrightarrow{\mu} G.
\]

In terms of the dual action of Hopf algebras, we denote it by

\[
(2.4) \quad k[G] \xrightarrow{\text{ad}} k[G] \otimes_k k[G].
\]

It is clear from the definition that when \(k\) is algebraically closed, the action is indeed the classical adjoint action \(ghg^{-1}\) of \(G\) on itself. To show that (2.3) indeed defines an action, one needs to check that the following diagrams of the maps of Hopf algebras commute.

\[
\begin{array}{c}
k[G] \xrightarrow{\text{ad}} k[G] \otimes_k k[G] \\
\downarrow \text{id} \otimes \text{ad} & \text{id} \otimes \text{ad} \downarrow \\
k[G] \otimes_k k[G] \xrightarrow{\text{id} \otimes \text{ad}} k[G] \otimes_k k[G] \otimes_k k[G]
\end{array}
\]

\[
\begin{array}{c}
k[G] \xrightarrow{\text{ad}} k[G] \otimes_k k[G] \\
\downarrow \text{id} & \text{id} \downarrow \\
k[G] \otimes_k k[G] \xrightarrow{\text{id}} k[G] \otimes_k k[G]
\end{array}
\]

However, the injectivity of the natural map \(\text{Hom}_{\text{Sch}_k}(X, Y) \rightarrow \text{Hom}_{\text{Sch}_k}(X_T, Y_T)\) (which can be checked locally) implies that it is enough to check the commutativity
when \( k \) is algebraically closed. But this is a straightforward checking since the adjoint action in this case is just the conjugation.

We recall from [9, Section 1] that an action of \( G \) on an affine \( k \)-scheme \( X \) is equivalent to the map of \( k \)-algebras
\[
\hat{\alpha} : k[X] \rightarrow k[G] \otimes_k k[X]
\]
which satisfies the commutativity of certain diagram of maps of \( k \)-algebras similar to the above. We refer the reader to loc. cit., Section 1 for more detail.

**Definition 2.4.** We shall say that an element \( f \) of \( k[X] \) is \( G \)-invariant for this action if \( \hat{\alpha}(f) = 1 \otimes f \).

It is easy to check that the set of \( G \)-invariant elements in \( k[X] \) is a \( k \)-sub-algebra and will be denoted by \( k[X]^G \). This is the coordinate ring of the universal geometric quotient of \( X \) for the action of \( G \) on \( X \). In particular, if \( k \) is algebraically closed, then \( k[X]^G \) is the invariant elements in \( k[X] = \text{Hom}_{sch_k}(X, \mathbb{A}^1_k) \) for the action of \( G(k) \) on \( k[X] \) given by
\[
(2.6) \quad G \times k[X] \rightarrow k[X], \quad (g, f) \mapsto f^g,
\]
where \( f^g(x) = f \circ \alpha((g, x)) \).

We now consider the adjoint action of \( G \) on itself as described in 2.3. In this case, we shall often write \( k[X]^G \) as \( C[G] \). In order to understand the geometry of the representation ring of \( G \), our first step is to define a \( k \)-algebra map from \( R_k(G) \rightarrow C[G] \) which has some interesting properties. So let \((V, \rho)\) be an \( n \)-dimensional representation of \( G \) given by the morphism of \( k \)-algebraic groups \( G \rightarrow GL(V) \). Let \( \chi : GL(V) \subset \text{End}(V) \rightarrow \mathbb{A}^1_k \) denote the character morphism described algebraically by the composite map
\[
k[t] \xrightarrow{\hat{\chi}} k[X_{ij}] \hookrightarrow k[X_{ij}, 1/\det],
\]
\[
\hat{\chi}(t) = \sum_{i=0}^{n} X_{ii}.
\]
Composing this map with \( \rho \), we get the character map
\[
(2.7) \quad k[t] \xrightarrow{\hat{\chi}_\rho} k[G].
\]
It is easy to check from above that if \((V_1, \rho_1)\) and \((V_2, \rho_2)\) are two representations of \( G \), then \( \hat{\chi}_{\rho_1 \oplus \rho_2}(t) = \hat{\chi}_{\rho_1}(t) + \hat{\chi}_{\rho_2}(t) \) and \( \hat{\chi}_{\rho_1 \otimes \rho_2}(t) = \hat{\chi}_{\rho_1}(t) \cdot \hat{\chi}_{\rho_2}(t) \). In other words, the assignment \((V, \rho) \mapsto \hat{\chi}_\rho(t)\) induces a \( k \)-algebra morphism \( R_k(G) \xrightarrow{\phi_G} k[G] \).

**Proposition 2.5.** Let \( G \) be a connected reductive group over a field \( k \) such that it admits a maximal torus \( T \) which is split over \( k \). Then the map \( \phi_G \) induces an isomorphism of \( k \)-algebras
\[
(2.8) \quad \phi_G : R_k(G) \rightarrow C[G].
\]
Proof. We first have to show that the image of $\phi_G$ is contained in $C[G]$. For this, it suffices to show that for any representation $(V, \rho)$, one has $\hat{ad} \circ \hat{\chi}_\rho = 1 \otimes \hat{\chi}_\rho$. Since the map $k[G] \otimes _k k[G] \to \overline{k}[G] \otimes _\overline{k} k[G]$ is injective, it suffices to show that $\hat{ad}_\overline{k} \circ \hat{\chi}_\rho = 1 \otimes \hat{\chi}_\rho$. Thus we can assume that $k$ is algebraically closed. In this case, we can use [2, 4] and the discussion following [2, 4] to reduce to showing that $\hat{\chi}_\rho(ghg^{-1}) = \chi_\rho(h)$. But this follows immediately from the standard properties of the trace of matrices.

To prove the isomorphism of $\phi_G$, note that as $G$ is $k$-split, it is uniquely described by a root system over $k$ and hence its Weyl group $W$ is a constant finite group which does not depend on the base change of $G$ by any field extension of $k$. Moreover, $W$ acts on $T$ and hence on $R_k(T)$ and $C[T] = k[T]$ such that the map $\phi_T$ is $W$-equivariant. Thus we have the following commutative diagram.

\[
\begin{array}{ccc}
R_k(G) & \xrightarrow{\phi_G} & C[G] \\
\eta_R & & \eta_C \\
R_k(T)^W & \xrightarrow{\phi_T^W} & C[T]^W
\end{array}
\]

Since $T$ is a split torus, $\phi_T^W$ is an isomorphism. The left vertical map is an isomorphism by [17, Théorème 4]. This implies that the composite map $\eta_C \circ \phi_G$ is an isomorphism. Thus we only need to show that $\eta_C$ is injective.

We first observe that the isomorphism $k[G]_k \to \overline{k}[G_{\overline{k}}]$ induces a map $C[G]_k \to C[G_{\overline{k}}]$ which is injective since the map $C[G]_k \to k[G]_k$ is so. Now the commutative diagram

\[
\begin{array}{ccc}
C[G] & \xrightarrow{} & C[G]_k \\
\downarrow & & \downarrow \\
C[T] & \xrightarrow{} & C[T_{\overline{k}}]
\end{array}
\]

reduces the problem to showing that the right vertical map is injective. Hence we can assume that $k$ is algebraically closed.

We have seen above that in this case, $C[G]$ is same as the those functions on $G$ which are invariant under the adjoint action of $G(k)$ on $k[G]$ given by conjugation. Suppose now that $f$ and $f'$ are two functions in $C[G]$ such that they define the same elements of $C[T]^W$. That is, $f$ and $f'$ define same function on the $T/W$. We conclude from Lemma 2.3 that they take same value on any given semi-simple element of $G$. Since the set of semi-simple elements contains a dense open subset in $G$ (cf. [3, Theorem 11.10]), we see that $f$ and $f'$ are same. This proves the desired injectivity of $\eta_C$. \qed

**Corollary 2.6.** Let $G$ be a connected reductive group over an algebraically closed field $k$. Then $R_k(G)$ is naturally isomorphic to the $k$-algebra $C[G]$ of class functions on $G$. In particular, $C[G]$ is a finite type $k$-algebra and hence noetherian.
Proof. We have already seen above that in this case, \( C[G] \) is same as the algebra of class functions, i.e., those functions on \( G \) which take a constant value on a conjugacy class. The corollary now follows directly from Proposition 2.5. For the last part, we only need to show that \( R_k(G) \) has the desired property. But this follows from 2.1 and \([12\text{, Lemma 4.7}]\). □

Corollary 2.7. Let \( G \) be a connected split reductive group of rank \( r \) over a field \( k \). Then \( C[G] \) has a \( k \)-basis consisting of the characters of irreducible representations of \( G \). If \( G \) is simply connected, then \( C[G] \) is isomorphic to the polynomial algebra \( k[\chi_1, \cdots, \chi_r] \) in the characters of the irreducible representations with highest fundamental weights.

Proof. The first part follows from Proposition 2.5 since \( R_k(G) \) has the desired property. The second part also follows from this proposition and Chevalley’s theorem \([4]\) that \( R_k(G) \) has the desired form if \( G \) is simply connected. □

In the rest of this section, we shall assume \( k \) to algebraically closed (of any characteristic). Let \( G \) be a connected linear algebraic group and let \( \Psi = C_G(g) \) be a semi-simple conjugacy class. Let \( m_{\Psi} \) denote the kernel of the \( k \)-algebra map

\[
R_k(G) \xrightarrow{\chi(g)} k.
\]

Since \( \chi(g) \) takes the value one at the trivial character, we see that \( m_{\Psi} \) is a maximal ideal of \( R_k(G) \) with the residue field \( k \). We deduce some further consequences of Proposition 2.5.

Corollary 2.8 \((6)\). Assume \( G \) is connected and reductive. Then a conjugacy class \( \Psi \) in \( G \) is closed if and only if it is semi-simple.

Proof. It is known that the conjugacy class of a semi-simple element is always closed (cf. \([3\text{, Theorem 9.2}]\)). The proof of the converse follows exactly along the same line as in the proof of \([6\text{, Proposition 2.4}]\), once we use our Corollary 2.6 and observe that Mumford’s result \([9\text{, Chapter 1, Corollary 1.2}]\) holds over any algebraically closed field. □

Corollary 2.9 \((6)\). Let \( G \) be as in Corollary 2.8. Then the correspondence \( \Psi \mapsto m_{\Psi} \) gives a bijection between the set of semi-simple conjugacy classes in \( G \) and the maximal ideals of \( R_k(G) \). If the characteristic of \( k \) is zero, the same conclusion holds for any connected linear algebraic group.

Proof. The identification of the maximal ideals of \( R_k(G) \) is same as identification of the closed points of \( \text{Spec}(R_k(G)) \) and the latter is isomorphic to \( \text{Spec}(C[G]) = T/W \) by Proposition 2.5. On the other hand, the closed points of \( T/W \) are same as its \( k \)-rational points which are identified with the semi-simple conjugacy classes of \( G \) by Lemma 2.3. If \( k \) has characteristic zero, then \( G \) has a Levi subgroup \( L \) and one has \( R_k(G) \cong R_k(L) \) and the argument of loc. cit., Proposition 2.5 goes through. □

Corollary 2.10 \((6)\). Let \( G \hookrightarrow H \) be a closed embedding of connected reductive groups and let \( \Psi = C_G(h) \) be a semi-simple conjugacy class in \( H \). Then \( R_k(G)_{m_{\Psi}} \) is a semi-local ring with maximal ideals \( \{m_{\Psi_1}, \cdots, m_{\Psi_r}\} \) where \( \Psi_1 \prod \cdots \prod \Psi_r = \Psi \cap G \).
is the disjoint union of the semi-simple classes in $G$. If the characteristic of $k$ is zero, then the same conclusion holds for all connected linear algebraic groups.

Proof. The fact that $R_k(G)\mathfrak{m}_n$ is a semi-local ring follows from Proposition 2.1. The rest of the argument is same as in the proof of loc. cit., Proposition 2.6 in view of Proposition 2.5 and Corollary 2.9 above. □

3. Preliminary Results

For the rest of this paper, our ground field $k$ will be assumed to be algebraically closed and of characteristic zero. This field will be fixed from now on. Following the approach of [6], we shall first prove the non-abelian completion theorem for the rational equivariant higher $K$-theory for those groups whose commutators are simply connected. In this section, we collect all the preliminary results needed in this direction.

For a finitely generated abelian group $N$ and a field $l$ of characteristic zero, let $T = D_l(N)$ denote the split diagonalizable group over $l$ with the character group $N$. We shall sometime also write $T_l$ to emphasize the base field. Recall from [19] Proposition 1.2] that for every maximal ideal $\mathfrak{m}$ of the group algebra $l[N] = l[T]$, there is a unique diagonalizable closed subgroup $T_\mathfrak{m}$ of $T$ such that $\mathfrak{m}$ is the inverse image of a maximal ideal of $l[T]$; and $T_\mathfrak{m}$ is the smallest closed subgroup with this property. $T_\mathfrak{m}$ is called the support of $\mathfrak{m}$ and its coordinate ring is the group algebra $l[N/N_\mathfrak{m}]$, where

$$N_\mathfrak{m} = \{ n \in N | 1 - [n] \in \mathfrak{m} \},$$

which is same as the kernel of the map $N \xrightarrow{\phi} (l[N]/\mathfrak{m})^*$ given by $n \mapsto [n]$ modulo $\mathfrak{m}$. We also recall that if $X$ is a variety over a field $l$, then for any field extensions $l \subset l_1 \subset l_2$, there is a natural inclusion of sets $X(l_1) \hookrightarrow X(l_2)$ of rational points of $X$. This map is $x \in X(l_1) \mapsto \overline{x}$, where $\overline{x}$ is represented by the diagonal map $\text{Spec}(l_2) \to X \times_k \text{Spec}(l_2)$ and the projection to the first factor is $\text{spec}(l_2) \to X$ whose image is the point $x$. The following lemma follows easily from Thomason’s theorem in loc. cit..

Lemma 3.1. Let $T = D_Q(N)$ be a split torus and let $t$ be a closed point of $T$ defined by a maximal ideal $\mathfrak{m}$ of $Q[T]$ with residue field $l$. Let $L$ be an algebraically closed field containing $l$ and let $t_L \in T(L)$ denote the image of the point $x$ under the inclusion $T(l) \subset T(L)$. Then

$$T_\mathfrak{m} \times_Q \text{Spec}(L) \xrightarrow{\sim} S,$$

where $S \subset T_L$ is the closure of the cyclic subgroup generated by $t_L$.

Proof. Let $\mathfrak{m}_L$ be the maximal ideal of $L[N]$ defining the closed point $t_L$ of $T_L$. Since every closed subgroup of $T_L$ is diagonalizable, we see from above that the support $T_\mathfrak{m}_L$ is the smallest closed subgroup of $T_L$ containing $t_L$ and hence is same as $S$. Thus we need to show that $T_\mathfrak{m} \otimes_Q L \xrightarrow{\sim} T_\mathfrak{m}_L$.

We see from 3.1 above that $T_\mathfrak{m} = \text{Spec} \left( Q \left[ N/N_\mathfrak{m} \right] \right)$, where $N_\mathfrak{m} = \text{Ker} \left( N \to l^* \right)$. Similarly we have $T_\mathfrak{m}_L = \text{Spec} \left( L \left[ N/N_{\mathfrak{m}_L} \right] \right)$, where $N_{\mathfrak{m}_L} = \text{Ker} \left( N \to L^* \right)$. Thus it
suffices to show that $N_m = N_{m_e}$. But this follows immediately from the commutative diagram

\[(3.2)\]

\[
\begin{array}{ccc}
N & \xleftarrow{(Q[N])^*} & l^* \xrightarrow{\cdot \langle} L^*
\end{array}
\]

\[
\begin{array}{ccc}
(L[N])^* & \xrightarrow{(L \otimes L)^*} & L^*
\end{array}
\]

**Proposition 3.2.** Let $G$ be a connected reductive group over $k$ and let $T$ be a maximal torus of $G$. Let $m$ be a maximal ideal of $R(G)$ and let $\tilde{m}$ be a maximal ideal of $R(T)$ whose inverse image in $R(G)$ is $m$. Assume that $T_{\tilde{m}}$ is connected and is contained in the center of $G$. Then $\tilde{m}$ is the unique maximal ideal of $R(T)$ which contracts to $m$.

**Proof.** We first remark that there is always at least one maximal ideal of $R(T)$ whose inverse image in $R(G)$ is $m$, since the map $f : \text{Spec}(R(T)) \to \text{Spec}(R(G))$ is finite and dominant by Proposition 2.1 and hence surjective. To prove the proposition, let $t$ be the closed point of $X = \text{Spec}(R(T))$ defined by $\tilde{m}$ and let $x = f(t)$ be the closed point of $Y = \text{Spec}(R(G))$ defined by $m$.

Suppose $t'$ is another closed point of $X$ such that $x = f(t')$ and consider the base change map $f_k : X_k \to Y_k$. Note that $X_k$ is same as $T$ and $Y_k = T/W$, where $W$ is the Weyl group of $G$. Let $t_k$ and $t'_k$ be the images of $t$ and $t'$ respectively under the inclusion $X(l) \subset X(k)$, where $l = \mathbb{Q}(t, t')$. It is then easy to see that $f_k(t_k) = y_k = f(t'_k)$. It suffices to show that $t_k = t'_k$. However, since $k$ is algebraically closed, we can apply Corollary 2.10 to find that $f_k^{-1}(y_k) = \{m_{s_1}, \ldots, m_{s_r}\}$, where $C_G(t_k) \cap T = C_T(s_1) \coprod \cdots \coprod C_T(s_r)$ is the disjoint union of the conjugacy classes in $T$. Thus it suffices to show that $t_k \in Z(G)$. Since $S = T_{\tilde{m}}$ is the closure of the cyclic subgroup of $T$ generated by $t_k$ by Lemma 3.1 and since $S \subset Z(G)$ by our assumption, we only need to see that $Z_G(S) = Z_G(t_k)$. But this follows easily by observing that for any $g \in Z_G(t_k)$, the subgroup $Z_G(g)$ is closed.

We fix the following notations once and for all before proceeding further. Let $G$ be a connected reductive group over $k$ as above and let $T = D_k(N)$ be a maximal torus of $G$. Let $f : X = \text{Spec}(R(T)) \to Y = \text{Spec}(R(G))$ denote the natural map of $\mathbb{Q}$-schemes. We fix a maximal ideal $m$ of $R(G)$ and let $\tilde{m}$ be a maximal ideal of $R(T)$ such that $R(G) \cap \tilde{m} = m$. Let $t$ (resp. $x$) denote the closed point of $X$ (resp. $Y$) defined by $\tilde{m}$ (resp. $m$). Thus we have $f(t) = x$. Put $\delta = f^{-1}(x) = \{t_1, \ldots, t_r\}$ with $t_1 = t$. Assume that the support $T_{\tilde{m}}$ of $\tilde{m}$ is connected. Then $S = T_{\tilde{m}}$ is a subtorus of $T$ and the following property of its centralizer $Z = Z_G(S)$ in $G$ is well known.

**Lemma 3.3.** $Z = Z_G(S)$ is connected and reductive.

**Proof.** See [B] Corollary 11.12 and Section 13.17, Corollary 2].

\[\square\]
Let $W$ and $W'$ be the Weyl groups of $G$ and $Z$ respectively with respect to the maximal torus $T$. Then $W$ naturally acts on $T$ and hence on $X = \text{Spec } (R(T))$.

**Proposition 3.4.** For the action of the Weyl group $W$ on $X$ as above, one has $Wm = (\text{orbit of } t) = \delta$ and the stabilizer of $t$ is the subgroup $W'$.

*Proof.* Since the map $f : X \to Y$ is the quotient map for the action of the finite group $W$ on $X$ by Proposition 2.1, $Y$ is a universal geometric quotient of $X$. In particular, we have $f^{-1}(x) = f^{-1}(f(t)) = Wt = Wm$. This proves the first part. To prove the second part, we first observe that $Z$ is a connected reductive group by Lemma 3.3 and hence we can apply Proposition 3.2 to see that $t$ is the only closed point of $X$ which lies over $f'(t)$ under the quotient map $f' : X \to Y' = \text{Spec } (R(Z)) = X/W'$. Now we can use the first part of the proposition with $G$ replaced by $Z$ to conclude that $W' \subset \text{St}_W(t)$. Thus we need to show the reverse inclusion to finish the proof. So let $w \in \text{St}_W(t)$. Then $w$ also stabilizes $t_k \in T_{\text{max}} = S \subset T$. Since $W = N_G(T)/T$, we can lift $w$ to an element $\overline{w}$ of $G$ and then this lift acts on $T$ by conjugation. In particular, $w \in \text{St}_W(t_k)$ implies that $\overline{w} \in Z_G(t_k)$ and the latter is same as $Z_G(S) = Z$ as we have seen above. Hence $w \in N_Z(T)/T = W'$. This proves the proposition. \qed

In the next few paragraphs, we recollect some basic facts about the fundamental groups of linear algebraic groups which are relevant to us in this paper. Our main interest is Proposition 3.6 below which will be needed in the next section. Let $G$ be a connected reductive group over $k$ with a fixed maximal torus $T$ and the Weyl group $W$. Let $X^*(T)$ and $X_*(T)$ denote the groups of characters and cocharacters of $T$ respectively. Recall that since $T$ is a split torus, the group $G$ is uniquely described by its root system $(G, T, \Phi)$. The set $\Phi$ is the set of non-zero elements $\alpha \in X^*(T)$ such that for the action of $T$ on the Lie algebra $g$ of $G$ via the adjoint representation, the subspace

$$g_\alpha = \{ v \in g \mid tv = \alpha(t)v \forall t \in T \}$$

is not zero. It is clear from this that $\Phi$ is a finite set. If we put $V = X^*(T) \otimes_\mathbb{Z} \mathbb{R}$, then recall that $V$ has a $W$-invariant inner product $\langle , \rangle$ and this makes $(V, \Phi)$ an abstract root system. If $\Phi^\vee \subset V^*$ is set of all elements $\alpha^\vee$ such that $\langle \alpha^\vee, v \rangle = \frac{2}{\langle \alpha, \alpha \rangle} \alpha, v \rangle$ for $\alpha \in \Phi$, then $(V^*, \Phi^\vee)$ is called the dual root system. Furthermore, there is a perfect pairing $X^*(T) \times X_*(T) \to \mathbb{Z}$ given by $(f, g) \mapsto \langle f, g \rangle = f \circ g \in X_*(\mathbb{G}_m) = \mathbb{Z}$, and this identifies $X_*(T) \otimes_\mathbb{Z} \mathbb{R}$ with $V^*$ and the integrality condition of the root system implies that the subset

$$\Lambda(\Phi^\vee) = \{ v \in V^* \mid \langle \alpha^\vee, v \rangle \in \mathbb{Z} \}$$

is a lattice in $V^*$ and is in fact contained in $X_*(T)$.

The inclusion $T \to G$ defines a natural map $X_*(T) = \pi_1(T) \to \pi_1(G)$ which induces an isomorphism (cf. [11, Section 31.8])

$$X_*(T) \cong \frac{\Lambda(\Phi^\vee)}{\pi_1(G)},$$

(3.3)
where \((G', T', \Phi')\) is the root system of the derived subgroup \(G' = (G, G')\) of \(G\). It is easy to see that in case \(G\) is a complex reductive group, then \(\pi_1(G)\) coincides with the topological fundamental group of \(G\). The second part of the following result was proved in [8, Lemma 2.5] for the complex algebraic groups by purely topological means.

**Lemma 3.5.** Let \(G\) be as above. Then \(\pi_1(G)\) is a finitely generated abelian group and it is finite if and only if \(G\) is semi-simple. Furthermore, the derived subgroup \(G'\) of \(G\) is simply connected if and only if \(\pi_1(G)\) is torsion-free.

**Proof.** The first part of the lemma follows directly from 3.3 above once we observe that the rank of \(\Lambda(\Phi'^\vee)\) is same as the semi-simple rank of \(G\). To prove the second part, let \(T' = G' \cap T\). Then it is well known that \(T'\) is a maximal torus of \(G'\) and \(T / T'\) is also a torus. In particular, \(\frac{X_\ast(T)}{X_\ast(T')} \cong X_\ast(T / T')\) is a free abelian group. The lemma now follows immediately from the short exact sequence

\[
0 \rightarrow \frac{X_\ast(T')}{\Lambda(\Phi'^\vee)} \rightarrow \frac{X_\ast(T)}{\Lambda(\Phi'^\vee)} \rightarrow \frac{X_\ast(T)}{X_\ast(T')} \rightarrow 0. 
\]

The following result about the fundamental groups of the centralizers of the sub-tori of \(G\) will play a crucial role in the next section to prove the non-abelian completion theorem in a special case.

**Proposition 3.6.** Let \(G\) be as in Lemma 3.5 and assume that the derived subgroup \(G'\) of \(G\) is simply connected. Then for any torus \(S \subset G\) in \(G\), the derived subgroup \(Z'\) of the centralizer \(Z = Z_G(S)\) is also simply connected. In particular, the commutators of the Levi subgroups of any parabolic subgroup of \(G\) are simply connected.

**Proof.** Using Lemma 3.5, it is enough to show that \(\pi_1(Z)\) is torsion-free. Let \(T\) be a maximal torus of \(G\) containing \(S\) and let \((G, T, \Phi)\) be the associated root system. It is then well known (cf. [3, Section 11.17]) that there is a subset \(I\) of \(\Phi\) such that \(S\) is the identity component of the subgroup \(\bigcap_{\alpha \in I} \text{Ker}(\alpha)\) of \(T\) and hence \(Z = Z_G(S)\) is the Levi subgroup of a parabolic subgroup of \(P\) containing \(T\) (cf. [3, Proposition 14.18]). It is easy to check in this case that \(Z' = Z \cap G'\). Let \(\Phi'_I \subset \Phi'\) be the subset consisting of the linear combination of elements of \(I\). Then \(X_\ast(S)\) has an orthogonal complement \((X_\ast(S))'\) inside \(X_\ast(T)\) with respect to the inner product as described above and so has \(\Lambda(\Phi'^\vee)_I\) inside \(\Lambda(\Phi'^\vee)\). In particular, the quotient \(\pi_1(Z) = \frac{X_\ast(T)}{\Lambda(\Phi'^\vee)}\) is a direct summand of \(\frac{X_\ast(T)}{\Lambda(\Phi'^\vee)} = \pi_1(G)\) and the latter group is torsion-free. The second part of the proposition follows from the first because the Levi subgroup of any parabolic subgroup of \(G\) is isomorphic to the centralizer of the subtorus \(S\) of \(T\) as above (cf. loc. cit., Corollary 14.19).

We end this section with the following reduction step which can be used to prove the main results of this paper for non-reductive groups.
Proposition 3.7. Let $G$ be a connected linear algebraic group and let $L$ be a Levi subgroup of $G$. Suppose $G$ acts on a smooth quasi-projective variety $X$. Then the restriction map $K^G_*(X) \to K^L_*(X)$ is an isomorphism. In particular, $R(G) \cong R(L)$.

Proof. Let $U = R_u G$ denote the unipotent radical of $G$. Morita isomorphism (cf. [20, Theorem 1.10]) implies that there is an isomorphism $K^G_*(G, G/L \times X) \cong K^L_*(X)$. Since $X$ is a $G$-space, there is a $G$-equivariant isomorphism $G \times X \cong G/L \times X$. Thus we get an isomorphism

$$(3.5) \quad K^*_G(X) \cong K^*_G(G, G/L \times X) \cong K^*_G(U \times X).$$

Now, as $U$ is a split unipotent group, there is a composition series of closed subgroups of $G$ such that each successive quotient is isomorphic to the additive group $\mathbb{G}_a$. Using a descending induction and homotopy invariance, one gets an isomorphism $K^*_G(U \times X) \cong K^*_G(X)$. \hfill $\square$

4. Non-abelian Completion Theorem

Let $G$ be a connected reductive group over $k$ with a maximal torus $T$. Let $m$ be a maximal ideal of $R(G)$ and let $\overline{m}$ be a maximal ideal of $R(T)$ whose inverse image in $R(G)$ is $m$. Let $S = T_{\overline{m}} \subset T$ be the support of $\overline{m}$ and let $Z = Z_G(S)$ denote the centralizer of $S$ in $G$. Put $\overline{m} = \overline{m} \cap R(Z)$. Let $X$ be a smooth quasi-projective variety with a $G$-action and let $f : X^{\overline{m}} \hookrightarrow X$ be the fixed locus for the action of $S$ on $X$. It is known that (cf. [19, Proposition 3.1]) that $X^{\overline{m}}$ is smooth. Let $d$ be the codimension of $X^{\overline{m}}$ in $X$. Let $\mathcal{N}$ denote the conormal bundle of $X^{\overline{m}}$ in $X$ and put $\lambda_1(\mathcal{N}) = [\mathcal{O}_{X^{\overline{m}}}] - [\Lambda^1 \mathcal{N}] + \cdots + [\Lambda^d \mathcal{N}]$ as a class in $K^G_0(X^{\overline{m}})$. Recall from [13] that there are restriction maps of $R(G)_m$-modules

$$(4.1) \quad res_m : \widehat{K^G_*(X)} \to \widehat{K^G_*(X)^{\overline{m}}} \to \widehat{K^G_*(X)^{\overline{m}}},$$

and $$(4.2) \quad f^* : \widehat{K^G_*(X)^{\overline{m}}} \to \widehat{K^G_*(X^{\overline{m}})}.$$ 

We begin with the following extension of the localization result [19, Lemme 3.2] for the action of diagonalizable groups to the case of arbitrary linear algebraic groups.

Lemma 4.1. Let $G$ be a connected reductive group acting on a smooth quasi-projective variety $X$ as above. Let $\overline{m}$ be a maximal ideal of $R(T)$ with $m = \overline{m} \cap R(G)$. Assume that the support $T_{\overline{m}}$ is connected and is contained in the center of $G$. Then the pullback map

$$(4.3) \quad f^* : K^G_i(X)_m \to K^G_i(X^{\overline{m}})_m$$

is an isomorphism. In particular, the map

$$(4.4) \quad f^* : K^G_i(X) \otimes_{R(G)} \widehat{R(G)}_m \to K^G_i(X^{\overline{m}}) \otimes_{R(G)} \widehat{R(G)}_m$$

is also an isomorphism.
Proof. Since $X$ and $X^\hat{m}$ are smooth, we have the push forward map $f_* : K^G_i(X^\hat{m}) \to K^G_i(X)$ such that the composite map $f^* \circ f_*$ is multiplication with $\alpha = \lambda_{-1}(N)$ by the self-intersection formula (cf. [21, Theorem 2.1]). The map $f_*$ is an isomorphism after localization at $m$ by [19, Théorème 2.2]. Hence it suffices to show that $\alpha$ acts as a unit on $K^G_i(X^\hat{m})_m$.

We have seen in the proof of Proposition 3.7 that there are natural maps of $R(G)$-modules

$$K^G_i\left(X^\hat{m}\right) \xrightarrow{p^i} K^T_i\left(X^\hat{m}\right) \xrightarrow{\cong} K^G_*(G, G/T \times X^\hat{m}) \xrightarrow{p_n} K^G_i\left(X^\hat{m}\right)$$

such that the composite is the identity map (cf. [21, Section 1.6]). Hence it suffices to show that $\alpha$ acts as a unit on $K^T_i\left(X^\hat{m}\right)_m$. Since $S = T_m \subset Z(G)$, we can apply Proposition 3.2 to conclude that $\hat{m}$ is the unique maximal ideal of $R(T)$ which contracts to $m$ in $R(G)$. In particular, we have $K^T_i\left(X^\hat{m}\right)_m \cong K^T_i\left(X^\hat{m}\right)_{\hat{m}}$. Thus we need to show that $\beta = p^i(\alpha)$ acts as a unit on $K^T_i\left(X^\hat{m}\right)_{\hat{m}}$. But this follows from [19, Lemme 3.2] since $\beta$ is the class of $\lambda_{-1}(N)$ in $K^T_i\left(X^\hat{m}\right)$.

Finally, the isomorphism of (4.3) is proved by observing that the left side of (4.3) is same as $K^G_i(X) \otimes_{R(G)} R(G)_m$ (cf. [13, Theorem 4.4]) and similarly the right side. Moreover, (4.3) is an isomorphism of $R(G)_m$-modules and hence remains an isomorphism when tensored with the $R(G)_m$-module $\widehat{R(G)}_m$. \hfill \Box

Proposition 4.2. Let $G$ be a connected reductive group acting on a smooth quasi-projective variety $X$ as above. Assume that the derived subgroup $G'$ of $G$ is simply connected. Let $\widehat{m}$ be a maximal ideal of $R(T)$ with $m = \widehat{m} \cap \overline{R(G)}$. Assume that the support $T_{\widehat{m}}$ is connected. Then the map

$$\widehat{K_i^G(X)}_m \xrightarrow{\text{res}_m} \widehat{K_i^Z(X)}_{\widehat{m}}$$

is an isomorphism of $\widehat{R(G)}_m$-modules.

Proof. Let $W$ and $W'$ be the Weyl groups of $G$ and $Z$ respectively. We have the maps of $\mathbb{Q}$-algebras

$$R(G) \xrightarrow{f_G^Z} R(Z) \xrightarrow{f_T^Z} R(T)$$

which are all finite by Proposition 2.1. Put $\overline{m} = \widehat{m} \cap R(Z)$ and let $f_Z^{-1}(\overline{m}) = \{m_1, \cdots, m_s\}$ with $m_1 = \overline{m}$ and $f_T^{-1}(m_i) = \{m_{i1}, \cdots, m_{is_i}\}$ with $m_{i1} = \widehat{m}$. Since $f_T^{-1}(m_1) = \widehat{m} = m_{11}$ by Proposition 3.2 we can write

$$f_Z^{-1}(\overline{m}) = \{\widehat{m}\} \cup \left( \bigcup_{i \geq 2} \bigcup_{j=1}^{s_i} \{m_{ij}\} \right).$$
In particular, the natural maps
\[(4.7) \quad \hat{K}^T_i(X)_m \longrightarrow \prod_{i \geq 2, j \geq 1} K^T_i(X)_{m_{ij}} \bigoplus \hat{K}^T_i(X)_{\bar{m}} \]
are isomorphisms by [8, Lemma 3.1].

Since \( S = T_{\bar{m}k} \) is connected and \( G' \) is simply connected, we see from Lemma 3.3 and Proposition 3.6 that \( Z \) is a connected reductive subgroup of \( G \) of the same rank with simply connected commutator subgroup. Hence the natural maps
\[(4.8) \quad \hat{K}^T_i(X)_{\bar{m}} \longrightarrow \hat{K}^T_i(X)_{\bar{m}} \]
isomorphisms by [14, Proposition 8]. Taking the invariants for the Weyl groups both sides, we get isomorphisms
\[(4.9) \quad K^G_i(X) \otimes_{R(G)} R(T) \xrightarrow{\theta^G} K^T_i(X), \quad K^Z_i(X) \otimes_{R(Z)} R(T) \xrightarrow{\theta^Z} K^T_i(X) \]
isomorphisms by [13, Proposition 8]. Taking the \( m \)-adic completions and using [8, Lemma 3.2], we get isomorphisms of \( \hat{R}(G)_m \)-modules
\[(4.10) \quad \hat{K}^G_i(X)_m \xrightarrow{\sim} (\hat{K}^T_i(X)_m)^W \quad \text{and} \quad \hat{K}^Z_i(X)_{\bar{m}} \xrightarrow{\sim} (\hat{K}^T_i(X)_{\bar{m}})^W' \]
For the action of \( W \) on \( \hat{K}^T_i(X)_m \) as in 4.7 and for \( i, j \geq 1 \), put
\[ W_{ij} = \{ w \in W | wK^T_i(X)_{m_{ij}} = K^T_i(X)_{m_{ij}} \}. \]
Then Proposition 3.4 implies that \( W_{11} = W' \) and \( f^G_{T}(\bar{m}) \) is the orbit of \( \bar{m} \) for the action of \( W \). Using the fact that \( \bar{m}_{11} = \bar{m} \), we conclude using [8, Lemma 3.3] that the natural map
\[(4.12) \quad (\hat{K}^T_i(X)_{\bar{m}})^W' \rightarrow (\hat{K}^G_i(X)_m)^W \]
is an isomorphism. Moreover, 4.8 and 4.11 show that the term on the left is same as \( \hat{K}^Z_i(X)_{\bar{m}} \). We conclude using 4.11 again that the map \( \hat{K}^G_i(X)_m \rightarrow \hat{K}^Z_i(X)_{\bar{m}} \) is an isomorphism of \( \hat{R}(G)_m \)-modules. \( \square \)

Our final goal in this section is to deduce Theorem 1.3 from Proposition 4.2 using the change of groups argument of [8]. Accordingly, we first establish the following steps necessary for the purpose. Let \( G \) be a connected and reductive group over \( k \) with a fixed maximal torus \( T \).

**Lemma 4.3.** For a given maximal ideal \( \mathfrak{m} \) of \( R(G) \), we can embed \( G \) as a closed subgroup of a group \( \hat{H} \) such that \( H \) is product of general linear groups and \( \mathfrak{m} \) is the only maximal ideal lying over \( \mathfrak{m} \cap R(H) \) under the natural map \( R(H) \rightarrow R(G) \).
Proof. Put $X = \text{Spec}(R(G))$ and let $x$ denote the closed point of $X$ defined by the maximal ideal $\mathfrak{m}$. Let $L$ denote the residue field of $x$ and let $x_k$ be the image of $x$ under the inclusion $X(L) \subset X(k)$. Then by Corollaries 2.6 and 2.9 we have $R_k(G) \twoheadrightarrow \mathbb{C}[G]$ and $x_k$ is given by a unique conjugacy class $\mathfrak{m}_\Psi$ of a semi-simple element $h$ of $G$. Hence we can use [6, Proposition 2.8] to embed $G \hookrightarrow H = \prod_i \text{GL}_{n_i}$ as a closed subgroup such that $\Psi_H(h) \cap G = \Psi_G(h)$. Put $Y = \text{Spec}(R(H))$ and let $f : X \to Y$ be the induced map, which is finite by Proposition 2.1. Putting $y = f(x)$, we see by Corollary 2.10 that $x_k$ is the only closed point lying over $y_k = f(x_k)$ under the finite map $X_k \to Y_k$. We conclude that $x$ is the only closed point of $X$ lying over $y$. □

Let us fix a maximal ideal $\mathfrak{m}$ of $R(G)$ and let $G \hookrightarrow H$ be an embedding as a closed subgroup as in Lemma 4.3. Let $\mathfrak{m}$ be a maximal ideal of $R(T)$ lying over $\mathfrak{m}$. Such an ideal always exists as follows from Proposition 2.4. Let $T_{\mathfrak{m}}$ be the support of $\mathfrak{m}$ and let $Z$ denote the centralizer of $S = T_{\mathfrak{m}} \otimes_k \mathbb{Q}$ in $G$. Let $T'$ be a maximal torus of $H$ containing $T$ and $Z'$ the centralizer of $S$ in $H$. It is then clear that $Z = Z' \cap G$. Put

$$\overline{\mathfrak{m}} = \mathfrak{m} \cap R(Z), \quad \mathfrak{m}' = \mathfrak{m} \cap R(H), \quad \mathfrak{m}' = \mathfrak{m} \cap R(T'), \quad \text{and} \quad \overline{\mathfrak{m}'} = \overline{\mathfrak{m}} \cap R(Z').$$

Note that the map $R(T') \to R(T)$ is surjective and $\mathfrak{m}'$ is the unique maximal ideal of $R(T')$ lying over $\mathfrak{m}$. Let $X$ be a smooth quasi-projective variety over $k$ with an action of $G$ and let $f : X^{\mathfrak{m}} \hookrightarrow X$ be the inclusion of the fixed locus for the action of $S$ on $X$. Put $X' = H \times^G X$. Then $H$ acts naturally on $X'$. Let $f' : X^{'\mathfrak{m}'} \hookrightarrow X'$ denote the inclusion of the fixed locus for the action of $S \subset T'$ on $X'$. Then $X'$ and $X^{'}\mathfrak{m}'$ are also smooth and $f'$ is a regular embedding. Since $S$ is contained in the center of $Z$, we see that $X^{'\mathfrak{m}'}$ is $Z'$-invariant. Thus there is a natural map $Z' \times X^{'\mathfrak{m}'} \to X^{'\mathfrak{m}'}$ which in fact descends to a map

$$\phi : Z' \times X^{'\mathfrak{m}'} \to X^{'\mathfrak{m}'}.$$

Lemma 4.4. The map $\phi$ is an isomorphism of $Z'$-spaces.

Proof. We first observe from [3, Proposition 8.18] that there is a dense open subset $U_1$ of $S$ such that $Z_H(S) = Z_H(h)$ and hence $Z_G(S) = Z_G(h)$ for all $h \in U_1$. Next we claim that there is a dense open subset $U_2$ of $S$ such that $X^S = X^h$ and $X^{'S} = X^{'h}$ for all $h \in U_2$.

Since $S$ is a torus and $X$ (and hence $X'$) has ample line bundles, there is a finite cover of $X$ and $X'$ by $S$-invariant affine open sets (cf. [18, Corollary 3.11]). Thus by choosing our desired open set as the finite intersection of open subsets for the element of the affine cover, we can reduce to the case when $X$ and $X'$ are affine. Then there is a linear representation $V$ of $S$ such that $X \hookrightarrow V$ is an $S$-equivariant closed embedding. Since $X^S = V^S \cap X$ and $X^h = V^h \cap X$, we can assume that $X = V$. Then we can write $V = V_0 \oplus V_1 \oplus \cdots \oplus V_r$ as a direct sum of $S$-invariant subspaces, where $S$ acts on $V_i$ by a character $\chi_i$ and $\chi_i \neq \chi_j$ for $i \neq j$. We can find a dense open subset $U_2 \subset S$ such that $\chi_i(h) \neq \chi_j(h)$ for $i \neq j$ and $h \in U_2$. 

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For such $h$, it is easy to see that $V_0 = V^S = V^h$. The same proof works also for $X'$. This proves our claim.

Taking $U = U_1 \cap U_2$, we see that $U$ is a dense open subset of $S$ such that for all $h \in U$, one has $Z_H(S) = Z_H(h), X^S = X^h$ and $X'^S = X'^h$. In particular, we get

$$Z' \times X^\tilde{m} \cong Z_H(h) \times X^h$$

The lemma now follows from this and [6, Lemma 3.5].

**Remark 4.5.** Although the above lemma has been stated for the maximal ideals of $R(G)$, it is easy to see that the proof works for all prime ideals as long as their supports are connected.

**Proposition 4.6.** Let the notations be as above and consider the commutative diagram

$$\begin{array}{ccc}
K^H_i(X')_{m'} & \longrightarrow & K^Z_i(X')_{m'} \\
\downarrow f^!_{m'} & & \downarrow f^!_{m'} \\
K^G_i(X)_{m} & \longrightarrow & K^Z_i(X)_{m}
\end{array}$$

The map $f^!_m$ is an isomorphism if $f^!_{m'}$ is so.

**Proof.** We consider the following commutative diagram.

$$\begin{array}{ccc}
K^H_i(X')_{m'} & \longrightarrow & K^Z_i(X')_{m'} \\
\downarrow f^!_{m'} & & \downarrow f^!_{m'} \\
K^Z_i(X'_{m'})_{m} & \longrightarrow & K^Z_i(X_{m})_{m}
\end{array}$$

The first horizontal arrow in the top row is isomorphism by the Morita isomorphism and the second one is an isomorphism by Proposition [1.1, Lemma 4.4] and [8, Lemma 3.1]. The left vertical arrow is an isomorphism by our assumption. The first horizontal arrow in the bottom row is an isomorphism by Lemma [4.1] and the Morita isomorphism. We conclude that the middle vertical arrow is an isomorphism. We now claim that $\overline{m}$ is the only maximal ideal of $R(Z)$ which contracts to $\overline{m}$ in $R(Z')$.

To see this, suppose $\overline{m}_1$ is another such maximal ideal of $R(Z)$. Since the map $R(Z) \to R(T)$ is finite and dominant by Proposition [2, 1] we can choose a maximal ideal $\overline{m}_1$ of $R(T)$ which contracts to $\overline{m}_1$ in $R(Z)$. Since the map $R(T') \to R(T)$ is surjective, $\tilde{m}_1$ is the unique maximal ideal of $R(T)$ lying over $\overline{m}_1 = \overline{m}_1 \cap R(T')$ of $R(T')$. Hence we get two distinct maximal ideals $\tilde{m}'$ and $\tilde{m}_1'$ of $R(T')$ whose intersection with $R(Z')$ is same as $\overline{m}' \cap R(Z') = \overline{m}$, which is a contradiction by Proposition [3, 2] as $S$ is contained in the center of $Z'$. This proves the claim. Using
the claim and loc. cit., Lemma 3.1, we see that the second horizontal arrow on the bottom row in \[\text{(4.14)}\] is an isomorphism. A diagram chase shows that the map \( f_\mathfrak{m}^! \) is also an isomorphism. □

**Proof of Theorem 1.3.** We first show the isomorphism of the horizontal arrows in \[\text{(1.4)}\]. The isomorphism of \( \tilde{f}^* \) and \( f^* \) follows from Lemma 4.1. Thus it suffices to show that the composite horizontal maps in the top and the bottom rows are isomorphisms. We embed \( G \) as a closed subgroup of \( H \) as in Lemma 4.3. We first show that the composite map \( f_\mathfrak{m}^! \) in the bottom row of \[\text{(1.4)}\] is an isomorphism. In view of Proposition 4.6, it suffices to show that \( f'_\mathfrak{m}^! \) is an isomorphism. Hence we can assume that the derived subgroup \( G' \) of \( G \) is simply connected. In this case, the conclusion follows from Lemma 4.1 and Proposition 4.2.

Now we prove the isomorphism of the composite map \( \tilde{f}_\mathfrak{m}^! \) in the top row of \[\text{(1.4)}\]. We first assume that the derived subgroup of \( G \) is simply connected. This implies from Lemma 3.3 and Proposition 3.6 that \( \mathcal{Z} \) also has the similar property. Hence the natural map

\[
(4.15) \quad K^G_i(X) \otimes_{R(G)} R(Z) \to K^Z_i(X)
\]

is an isomorphism by [8, Proposition 2.1]. Next we take \( X = \text{Spec}(k) \) in the bottom row of \[\text{(1.4)}\] to see that map \( \tilde{R}(G)_\mathfrak{m} \to \tilde{R}(Z)_\mathfrak{m} \) is an isomorphism of complete local rings. Tensoring this isomorphism with \( K^G_i(X) \), we get

\[
K^G_i(X) \otimes_{R(G)} \tilde{R}(G)_\mathfrak{m} \cong K^G_i(X) \otimes_{R(G)} \tilde{R}(Z)_\mathfrak{m} = (K^G_i(X) \otimes_{R(G)} R(Z)) \otimes_{R(Z)} \tilde{R}(Z)_\mathfrak{m} \cong K^Z_i(X) \otimes_{R(Z)} \tilde{R}(Z)_\mathfrak{m}
\]

where the vertical arrow is isomorphism by \[4.15\]. The proof in the simply connected case is now completed by Lemma 4.1.

We now prove the isomorphism of \( \tilde{f}^!_\mathfrak{m} \) in \[\text{(1.4)}\] in the general case. So let \( G \) be a connected and reductive group with a fixed maximal torus \( T \). Let \( \mathfrak{m} \) be the given maximal ideal of \( R(G) \). We embed \( G \) as a closed subgroup of \( H \) as in Lemma 4.3 and follow the same notations as above. We have seen in the proof of Proposition 4.6 that in this case, \( \mathfrak{m} \) is the only maximal ideal of \( R(Z) \) whose intersection with \( R(Z') \) is \( \mathfrak{m}' \). Since the maps of these representation rings are finite by Proposition 2.1 we see that \( \tilde{R}(G)_\mathfrak{m}' \cong \tilde{R}(G)_\mathfrak{m} \) and \( \tilde{R}(Z)_\mathfrak{m}' \cong \tilde{R}(Z)_\mathfrak{m} \). We
Since the derived subgroup of $H$ is simply connected by our choice of $H$, the isomorphism of the top horizontal arrow in (4.16) is already shown. The arrows $l_1$ and $r_1$ are the Morita isomorphisms. The isomorphism of $l_2$ and $r_2$ is obvious. Next we see from (2.1) that $R(G)$ is the ring of invariants for the Weyl group action on $R(T)$ and hence is noetherian (cf. [12, Lemma 4.7]). The isomorphism of $l_3$ and $r_3$ is now an immediate consequence Proposition 2.1 and [13, Theorem 8.7]. Finally, we have already concluded above that the arrows $l_4$ and $r_4$ are isomorphisms. A diagram chase shows that the bottom horizontal arrow is an isomorphism. This completes the proof of the isomorphism of the horizontal arrows in (1.4).

Since we have just observed that $R(G)$ is noetherian, the injectivity of the vertical arrows in (1.4) follows from the following elementary lemma.

**Lemma 4.7.** Let $A$ be a commutative noetherian ring and let $\hat{A}$ denote the completion of $A$ with respect to a given ideal $I$ in $A$. Then for any $A$-module $M$, the natural map $M \otimes_A \hat{A} \to \hat{M}$ is injective.

**Proof.** Since $A$ is noetherian, the lemma is obvious if $M$ is a finitely generated $A$-module (cf. [13, Theorem 8.7]). In general, we can find a direct system $\{M_\gamma\}_{\gamma \in \Gamma}$ of finitely generated submodules of $M$ such that $\lim_{\gamma} M_\gamma \cong M$. In particular, we
get \( \lim_{\gamma} M_\gamma \otimes_A \hat{\mathbb{A}} \xrightarrow{\cong} M \otimes_A \hat{\mathbb{A}} \). Now we consider the following commutative diagram.

\[
\begin{array}{ccc}
\lim_{\gamma} M_\gamma \otimes_A \hat{\mathbb{A}} & \xrightarrow{\cong} & M \otimes_A \hat{\mathbb{A}} \\
\downarrow & & \downarrow \\
\lim_{\gamma} \hat{M}_\gamma & \rightarrow & \hat{M}
\end{array}
\]

The left vertical arrow is an isomorphism since each \( M_\gamma \) is finitely generated. Since each \( M_\gamma \) is a submodule of \( M \) and since the completion functor is left exact, we see that \( \hat{M}_\gamma \hookrightarrow \hat{M} \) for each \( \gamma \) and hence \( \lim_{\gamma} \hat{M}_\gamma \hookrightarrow \hat{M} \). A diagram chase now shows that the right vertical arrow is injective. \( \square \)

5. Riemann-Roch for Higher Equivariant \( K \)-theory

We begin this section with a brief recall of the equivariant higher Chow groups defined in [7]. Let \( G \) a linear algebraic group over \( k \) acting on a quasi-projective variety \( X \). For any integer \( j \geq 0 \), let \( V \) be a representation of \( G \) and let \( U \) be a \( G \)-invariant open subset of \( V \) such that the codimension of the complement \( V \setminus U \) in \( V \) is larger than \( j \), and \( G \) acts freely on \( U \). It is easy to see that such representations of \( G \) always exist. Let \( X_G \) denote the quotient \( X \times U \) of the product \( X \times U \) by the diagonal action of \( G \), which is free. The equivariant higher Chow group \( CH^*_G(X, i) \) is defined as the homology group \( H_i(Z(X_G, \cdot)) \), where \( Z(X_G, \cdot) \) is the Bloch’s cycle complex of the variety \( X_G \) [2]. It is known (loc. cit.) that this definition of \( CH^*_G(X, i) \) is independent of the choice of the above representation. One should also observe that \( CH^*_G(X, i) \) may be nonzero for infinitely many values of \( j \), a crucial change from the non-equivariant higher Chow groups. We refer the reader to loc. cit. and [12] for further details.

Now we come back to the case of a connected reductive group \( G \) over \( k \). We fix a maximal torus \( T = D_k(N) \) of \( G \), where \( N \) is the character group of \( T \). Let \( \mathfrak{m} \) be a given maximal ideal of \( R(G) \) and let \( \tilde{\mathfrak{m}} \) be a maximal ideal of \( R(T) = \mathbb{Q}[N] \) lying over \( \mathfrak{m} \). Assume that the support \( T_{\tilde{\mathfrak{m}}} \) of \( \tilde{\mathfrak{m}} \) is connected. Until we begin the proof of Theorem 1.2 we also assume in this section that \( S = T_{\tilde{\mathfrak{m}}k} \) is contained in the center of \( G \). Let \( L = R(T)/\tilde{\mathfrak{m}} \) be the residue field of \( \tilde{\mathfrak{m}} \). Let \( x \) and \( y \) be the closed points of \( X = \text{Spec}(R(T)) \) and \( Y = \text{Spec}(R(G)) \) defined by \( \tilde{\mathfrak{m}} \) and \( \mathfrak{m} \) respectively so that \( f(x) = y \) under the finite map of finite type \( \mathbb{Q} \)-schemes \( f : X \to Y \).

Note that \( X = D_{\mathbb{Q}}(N) \) is a split torus over \( \mathbb{Q} \) such that \( X_k \xrightarrow{\cong} D_k(N) = T \). Hence the natural map \( X^*(X) \to X^*(T) \) is an isomorphism of \( \mathbb{Q} \)-algebras. In the same way, we see that if \( M \) is the character group of \( S \), then the natural map \( X^*(T_{\tilde{\mathfrak{m}}}) \to X^*(S) = \mathbb{Q}[M] \) is an isomorphism of \( \mathbb{Q} \)-algebras. We conclude that every character \( \chi \) of \( S \) is given by the morphism of group schemes \( \chi : T_{\tilde{\mathfrak{m}}} \to \mathbb{G}_m \) over \( \mathbb{Q} \). In particular, we get \( \chi(x) \in L^* \).
If $V$ is a representation of $G$, then $S$ acts on $V$ via characters and this decomposes $V$ uniquely as

\[(5.1) \quad V = \bigoplus_{\chi \in M} V_{\chi}, \]

where $S$ acts on $V_{\chi}$ by $hv = \chi(h)v$. Since $S \subset Z(G)$, we see that the above is a decomposition of $V$ as direct sum of $G$-representations parameterized by $M$. This defines an $L$-algebra automorphism of $R_L(G)$

\[(5.2) \quad t_{\bar{m}} = t_x : R_L(G) \to R_L(G) \]

\[t_{\bar{m}}([V]) = \sum_{\chi \in M} \chi(x)[V_{\chi}].\]

The following lemma follows easily from above.

**Lemma 5.1.** The automorphism $t_{\bar{m}}$ takes the point $x$ of $Y_L$ to the closed point defined by the augmentation ideal $I_G$ of $R_L(G)$.

**Proof.** We see from the above discussion that $t_{\bar{m}}$ acts compatibly on $T_{\bar{m}L}$ and $X_L$ such that the diagram

\[
\begin{array}{ccc}
T_{\bar{m}L} \to & X_L & \xrightarrow{f} Y_L \\
t_{\bar{m}} \downarrow & & \downarrow t_{\bar{m}} \\
T_{\bar{m}L} \to & X_L & \xrightarrow{f} Y_L
\end{array}
\]

commutes. Since the augmentation ideal of $R(T_{\bar{m}})$ contracts to the augmentation ideal of $R(G)$, the lemma now follows immediately by noting that $x_L \in T_{\bar{m}L}$ and by a direct checking in (5.2) that $t'_{\bar{m}}(x_L) = (1, \cdots, 1) \in T_{\bar{m}}(L)$. \qed

For a quasi-projective variety $X$ with a $G$-action, let $G^G(X)$ denote the $K$-theory spectrum of the abelian category of $G$-equivariant coherent sheaves on $X$. Assume that $S \subset G$ acts trivially on $X$. We wish to decompose this spectrum in terms of the characters of $S$. Let $\mathcal{C}$ denote the abelian category of $G$-equivariant coherent sheaves on $X$. Since $S$ acts trivially on $X$, it acts on each coherent sheaf fiberwise. For any character $\chi$ of $S$, let $\mathcal{C}_\chi$ denote the subcategory of $\mathcal{C}$ given by those sheaves on which $S$ acts by the character $\chi$. That is,

\[
\text{ob} \mathcal{C}_\chi = \left\{ \mathcal{F} \in \mathcal{C} | \text{satisfies condition} \right\}
\]

Note that since every open subset of $X$ is $S$-invariant, the above definition makes sense.

**Proposition 5.2.** Each $\mathcal{C}_\chi$ is an abelian full subcategory of $\mathcal{C}$ and the inclusion $\mathcal{C}_\chi \hookrightarrow \mathcal{C}$ induces a natural isomorphism of $K$-theory spaces

\[(5.3) \quad \prod_{\chi \in M} K(\mathcal{C}_\chi) \cong K^G(X).\]
It is easy to see that $C_{\chi}$ is a full additive subcategory. To see that it has kernels and cokernels, one needs to show that the kernel and cokernel of a map $F \to G$ in $C_{\chi}$ also lie in $C_{\chi}$, one can check it at each affine open subset of $X$, where it can be checked directly. To prove (5.4), note that $M = X^*(S)$ is a filtered category in a natural way by putting $\text{Hom}(\chi, \chi')$ to be the empty set unless $\chi = \chi'$ in which case, we take this to be the singleton set consisting of the identity map. Then $M$ becomes a discrete filtered category and there is a functor $F : M \to \text{Cat}$ given by $F(\chi) = C_{\chi}$. Now, since $S$ acts trivially on $X$, every $F \in C$ has a unique decomposition

$$F = \prod_{\chi \in M} F_{\chi}, \text{ where }$$

$$F_{\chi}(U) = \{ f \in F(U)|sf = \chi(s)f \forall s \in S \}.$$  

Furthermore, the condition $S \subset Z(G)$ shows that this is in fact a canonical decomposition of $F$ as a direct sum of $G$-equivariant coherent subsheaves parameterized by $M$. This immediately shows that

$$\lim_{\chi \in M} F(\chi) \cong \prod_{\chi \in M} C_{\chi} \cong C$$

is an equivalence. In particular, we get $\prod_{\chi \in M} BQC_{\chi} \cong BQC$. The proposition now follows from [15, Proposition 3].

We shall denote the space $K(C_{\chi})$ also by $K^\chi(X)$. We consider $L = \mathbb{Q}(x)$ as a discrete space and write $G^G(X)_L = G^G(X)\Lambda L$ and $K^\chi(X)_L = K^\chi(X)\Lambda L$ as the smash products. Then we have $\pi_i (G^G(X)_L) = G^G_i(X)\otimes_{\mathbb{Z}}L$ and $\pi_i (G^\chi(X)_L) = K^\chi_i(X)\otimes_{\mathbb{Z}}L$. This allows us to define an $L$-linear automorphism of $K^\chi_i(X)_L$

$$t_{\bar{m}}^\chi : K^\chi_i(X)_L \to K^\chi_i(X)_L$$

$$t_{\bar{m}}^\chi(\alpha) = \chi(x)\alpha.$$  

This makes sense since $\chi(x) \in L^*$. By Proposition (5.2), this induces an $L$-linear automorphism of $G^G_i(X)_L$

$$t_{\bar{m}}^G = \prod_{\chi \in M} t_{\bar{m}}^\chi : G^G_i(X)_L \to G^G_i(X)_L.$$  

We shall sometimes write $t_{\bar{m}}^G$ also as $t_{\bar{m}}^G$ if we need to indicate the underlying group. It is easy to see from (5.4) and (5.5) that the automorphism $t_{\bar{m}}$ coincides with the one defined in (5.2) when $X = \text{Spec}(k)$.

**Proposition 5.3.** For every $\alpha \in R(G)_L$ and $\beta \in G^G_i(X)_L$, one has

$$t_{\bar{m}}(\alpha \cdot \beta) = t_{\bar{m}}(\alpha) \cdot t_{\bar{m}}(\beta)$$

for the $R(G)_L$-module structure on $G^G_i(X)_L$ given by $R(G)_L \otimes_{\mathbb{Z}}G^G_i(X)_L \to G^G_i(X)_L$. 
Proof. We first assume that $G = T$ is a torus. Then there is a decomposition $T = S \times T'$ where $T' = T/S$. Since $S$ acts trivially on $X$, we have the isomorphism $G^T_i(X)_L \otimes L R(S)_L \overset{\cong}{\rightarrow} G^T_i(X)_L$ by Lemma 5.4 below. Moreover, since $x_k \in S$ and since the characters of $S$ restrict to the trivial character on $T'$, we see that $t^T_m$ acts on $G^T_i(X)_L$ by $t^T_m(\beta' \otimes \alpha') = \beta' \otimes t^S_m(\alpha')$. Writing $R(T) = R(T') \otimes R(S)$, $\alpha = \alpha_S \otimes \alpha_{T'}$ and $\beta = \beta_S \otimes \beta_{T'}$, we get

$$
t^T_m(\alpha \cdot \beta) = t^T_m((\alpha_{T'} \cdot \beta_{T'}) \otimes (\alpha_S \cdot \beta_S)) = \alpha_{T'} \cdot \beta_{T'} \otimes t^S_m(\alpha_S \cdot \beta_S)
$$

$$
= \alpha_{T'} \cdot \beta_{T'} \otimes t^S_m(\alpha_S) \cdot t^S_m(\beta_S) = (\alpha_{T'} \otimes t^S_m(\alpha_S)) \cdot (\beta_{T'} \otimes t^S_m(\beta_S))
$$

$$
= t^T_m(\alpha) \cdot t^T_m(\beta),
$$

where the first equality in the second row holds because $t^T_m$ is a ring automorphism of $R(S)_L$ (cf. Lemma 5.1). This proves the proposition in the case of a torus.

To prove the general case, let $p^i : G^G_i(X)_L \rightarrow G^T_i(X)_L$ be the restriction map. We have seen before that $p^i$ is a split injective $R(G)_L$-linear map (cf. [21, Section 1.6]). Moreover, as $x_k \in S$, it is clear from definition that $t^T_m$ acts compatibly on $G^G_i(X)$ and $G^T(X)$. Now we have

$$
p^i(t^G_m(\alpha \cdot \beta)) = t^T_m(p^i(\alpha \cdot \beta)) = t^T_m(p^i(\alpha) \cdot p^i(\beta))
$$

$$
= t^T_m(p^i(\alpha)) \cdot t^T_m(p^i(\beta)) = p^i \circ t^T_m(\alpha) \cdot p^i \circ t^T_m(\beta)
$$

$$
= p^i(t^T_m(\alpha) \cdot t^T_m(\beta)),
$$

where the first equality in the second row follows from the torus case proved above. Since $p^i$ is injective, we get the desired result in the general case.

The following is a generalization of [22, Lemma 5.6] where it was proved when $X$ is affine.

**Lemma 5.4.** Let $T$ be a torus acting on a normal quasi-projective variety $X$ and let $S \subset T$ be a subtorus acting trivially on $X$ so that $T$ acts on $X$ via $T' = T/S$. Then there is an isomorphism

$$
G^T_i(X) \otimes R(S) \overset{\cong}{\rightarrow} G^T_i(X).
$$

**Proof.** When $X$ is affine, this was proved in [22, Lemma 3.1]. In general, as $X$ is normal, it can be covered by finitely many $T$-invariant affine open subsets (cf. [18, Corollary 3.11]). Now the lemma is easily proved by an induction on the number of $T$-invariant open subsets covering $X$ and the Mayer-Vietoris long exact sequence for the $K$-theory of $T$-equivariant coherent sheaves. As is well known, this Mayer-Vietoris property is a direct consequence of the localization and the excision properties, both of which hold for the $K$-theory of equivariant coherent sheaves (cf. [23, Theorem 2.7, Corollary 2.4]).

**Corollary 5.5.** The automorphism $t^T_m$ of $G^G_i(X)_L$ takes the $R(G)_L$-submodule $m^jG^G_i(X)_L$ to $P^j_iG^G_i(X)_L$ for all $j \geq 0$. In particular, it induces an isomorphism

$$
\tilde{t}^T_m : G^G_i(X)_{m,L} \overset{\cong}{\rightarrow} G^G_i(X)_{tG,L}.
$$
Proof. This follows immediately from Proposition 5.3 since the action of $t_m$ on $R(G)_L$ is by a ring automorphism which takes the maximal ideal $m$ to the augmentation ideal $I_G$ by Lemma 5.1. □

Proof of Theorem 1.2: Since $X$ is smooth, we can replace $G^G_i(X)$ with $K^G_i(X)$ and similarly for $X^\bar{m}$. By Theorem 1.3, the natural map

$$\widehat{K^G_i(X)}_{m,L} \xrightarrow{f^m_{\bar{m}}} \widehat{K^Z_i(X^\bar{m})}_{m,L}$$

is an isomorphism of $\widehat{R(G)}_m$-modules. Combining this with Corollary 5.5, we have an isomorphism

$$(5.6) \quad \widehat{K^G_i(X)}_{m,L} \xrightarrow{\cong} \widehat{K^Z_i(X^\bar{m})}_{I_G,L}.$$ 

On the other hand, [12, Theorem 1.2] implies that there is a Chern character map

$$(5.7) \quad \widehat{K^Z_i(X^\bar{m})}_{I_G,L} \xrightarrow{\widehat{\text{ch}}} \widehat{CH^*_Z(X^\bar{m}, i)}_{L}$$

which is an isomorphism. The theorem now follows from 5.6 and 5.7 by taking $\widehat{\text{ch}}_{m}$ to be $\widehat{\text{ch}} \circ f^m_{\bar{m}}$. □

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