Resonance overlap and non-linear velocity spread in Hamiltonian beam-plasma systems

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In this paper, we analyze in some detail the properties of the beam-plasma instability with respect to both the morphology of the linear dispersion relation, and the non-linear behavior of the particle velocity spread. First, we investigate non-perturbative effects in the dispersion relation, characterizing the linear growth rates and the frequency shift with respect to the plasma frequency where the perturbative inverse Landau damping expression breaks down. Then, we discuss the behavior of the non-linear velocity spread as function of the linear growth rate. We introduce three basic criteria to estimate the non-linear velocity spread, and demonstrate that only the full change of the particle velocity profile is really predictive of resonance overlap. Finally, we discuss aspects of the mode saturation level in the case of a broad fluctuation spectrum and, by the help of an analytical toy model, we illuminate the mechanism responsible for higher saturation intensity with suitable overlapping resonances with respect to the case of single resonance with an isolated mode.

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I. INTRODUCTION

The beam-plasma interaction is one of the most interesting paradigms of plasma physics, not only for its direct implementation in laboratory physics (for instance in the physics of plasma accelerators), but also because it is isomorphic to the bump-on-tail problem. Moreover, the phenomenology of particle trapping in the Langmuir potential well has been proposed as reduced model for the behavior of fast ions interacting with Alfvén waves in fusion devices. The main features to be addressed in the beam-plasma interaction are the dispersion relation (characterizing the linear phase of instability) and the non-linear evolution of the mode, e.g., the saturation of fluctuating fields and particle trapping.

The beam-plasma problem can be cast as a N-body problem for both the thermal distribution of plasma electrons (assuming a neutralizing ion background) and the supra-thermal particles constituting the beam itself. However, in Ref. the problem was successfully reduced assuming the thermal plasma could be described as linear dielectric medium, in which the beam electrons interact with a single Langmuir mode. This approach has been generalized to the case of a warm electron beam (in which the initial velocity dispersion of the beam is significant) in Ref. and the implications of this paradigm for the transport features (convection and diffusion) of beam electrons have been analyzed in Ref. Examples of the practical use of this theoretical framework to interpret fast-ion transport in Tokamaks are in Ref.

Despite in Refs. the basic features of the interaction of a tenuous beam with a cold plasma have been extensively discussed, subtle issues remain open and are connected to fusion applications of the originally proposed paradigm. In particular, questions arise when multiple Langmuir modes are simultaneously considered. The linear growth rate of the beam-plasma instability is essentially fixed by the ratio between the beam and plasma density, while wave-particle power exchange, non-linear saturation and particle redistribution depend on the details of inverse Landau damping. From this features, it is possible to estimate the most relevant non-linear quantity, i.e., the mode saturation level, which is responsible, in turn, of the non-linear velocity spread produced in the beam distribution. In fact, near the resonance condition, when the particle velocity equals the Langmuir wave phase velocity, the distribution function of the supra-thermal electrons is modified over a velocity region whose amplitude depends on the fluctuating field intensity at saturation.

In this paper, we first investigate the details of the linear dispersion relation in the case of an initial Erfc distribution of the beam in the velocity space. In particular, we address non-perturbative effects and the break down of the perturbative inverse Landau damping expression where the ratio between the growth rate and the real frequency shift with respect to the plasma frequency becomes finite. We then address the more relevant question concerning the behavior of the non-linear particle velocity spread, and propose three different criteria to estimate the region of the velocity space involved in the wave particle power exchange. The first method relies on the calculation of the non-linear velocity spread as directly induced by the mode saturation level, i.e., as a quantity predicted by energy conservation in the isolated
wave-particle resonant interaction. The second criterion is based on a precise measure of the clump size in the particle phase-space at mode saturation, interpreted as the size of the region where the particle trapping in the wave electric field takes place. Finally, we introduce the possibility of measuring the non-linear velocity spread as the size of the velocity-space region over which the particle distribution function at saturation is modified with respect to the initial profile. We will show that this quantity differs from the clump size, which seems to determine only the “plateau” region of the distribution function. In fact, the distortion of the particle distribution function appears to be very important in characterizing the resonance overlapping region even beyond wave particle trapping; that is, further away than the clump size from the linear resonance condition. We will show that only the latter of the nonlinear velocity spread criteria introduced above is able to predict the interaction of two isolated resonances, as their distance in the velocity space is decreased. This evidence has the very important physical implication that also particles, which are not trapped by the wave near resonance \cite{15}, are relevant in the “active” overlap of different non-linear fluctuations, since the power transfer also involves those particles simultaneously feeling two (or multiple) electric fields. Through this mechanism, the resonant wave-particle power exchange can be enhanced. Our analysis has the relevant feature to determine the non-linear size of the resonance in a fully self-consistent scheme, where the distribution function is dynamically coupled to the spectral evolution. Such self-consistency allows to take into account the backreaction induced by the distribution function deformation of the fluctuation spectrum and this is a crucial point in identifying the real resonance region involved in the non-linear dynamics. In this respect, this analysis enters the long standing debate about the transition to stochasticity of adjacent resonances associated to the well-known Chirikov overlap criterion \cite{21, 21} and its deepening and upgradings \cite{22, 23}.

With the help of an analytical toy model, we further study the mechanism by which overlapping resonances yield enhanced fluctuation level at saturation with respect to the isolated resonance case. This behavior is a consequence of an efficient transfer of phase-space energy to the Langmuir modes. It takes place only in the presence of adjacent resonances that are weakly overlapped, and is a relevant physical process also for fast ion transport induced by Alfvén modes in Tokamaks (see, for instance, the recent analysis in Ref. \cite{24}).

The paper is articulated as follows. In Sec. II, the adopted equations for analyzing the the beam-plasma system are described and the basic simulation parameters are given. In Sec. III the linear dispersion relation is analyzed, emphasizing the non-perturbative behaviors and comparing theoretical predictions with numerical simulation results. In Sec. IV the particle trapping mechanism is discussed and the definitions of non-linear velocity spread are introduced. Trapping frequency, mode saturation level and non-linear velocity spread are also studied by means of numerical simulations as a function of the linear drive. In Sec. VI the issue of resonance overlap is addressed, discussing the features of the non-linear velocity spread. In Sec. VII the mode saturation level is analyzed by comparing the multi-mode simulations with simulation results for an isolated resonance. The interpretative analytical model is also introduced, in order to describe the enhanced energy transfer between particles and modes. Concluding remarks are given in Sec. VIII.

II. HAMILTONIAN DESCRIPTION OF THE BEAM-PLASMA INTERACTION

The beam-plasma system described in Refs. \cite{11, 12} deals with a fast electron beam injected into a 1D plasma, which is treated as a cold linear dielectric medium supporting longitudinal electrostatic Langmuir waves. The beam density $n_B$ is much smaller than that of the background plasma ($n_p$). Here, we adopt the Hamiltonian formulation of the problem described in Refs. \cite{14, 27} (and refs. therein), where the broad supra-thermal particle beam is discretized as superposition of $n \gg 1$ cold beams self-consistently evolving in the presence of $m$ modes at the plasma frequency, i.e., $\omega_j \approx \omega_p$ for $j = 1, ..., m$. This ensures that the dielectric function of the cold background plasma ($\epsilon = 1 - \omega_p^2/\omega^2$) is nearly vanishing \cite{11} and allows casting the Poisson equation for each plasma oscillation into the form of a simple evolution equation, while particle trajectories are solved from the Newton’s law \cite{12}. One single mode in the fluctuation spectrum is initially set in order to be resonantly exited by one of the $n$ cold beams (linearly unstable); i.e., with a wave number $k = \omega_p/v_r$, where $v_r$ is the initial velocity of the considered cold beam. The 1D cold plasma equilibrium is taken as a periodic slab of length $L$, and the position along the $x$ direction for each particle is labeled by $x_i$ while $N$ denotes the total particle number ($i = 1, ... N$). For the sake of convenience, beam particle positions are scaled as $\bar{x}_i = x_i(2\pi/L)$, and the Langmuir wave scalar potential $\varphi(x, t)$ is expressed in terms of the Fourier components $\phi_j(k_j, t)$. Introducing the parameter $\eta = n_B/n_p$, we use the normalization: $\tau = t\omega_p, \quad u_i = \bar{x}_i = \bar{x}_i(t) = v_j(2\pi/L)/\omega_p, \quad \ell_j = k_j(2\pi/L)^{-1}, \quad \phi_j = \varphi_j(2\pi/L)^2\omega_j^2 \phi_j, \quad \bar{\phi}_j = \phi_j e^{-\tau}$. The prime denotes derivative with respect $\tau$ and barred frequencies and growth rates associated to the beam-plasma instability are defined as $\bar{\omega} = \omega/\omega_p$ and $\bar{\gamma} = \gamma/\omega_p$, respectively.

The beam-plasma system is governed by the following set of equations in the laboratory frame

\begin{equation}
\bar{x}_i' = u_i, \quad u_i' = \sum_{j=1}^{m} (i \ell_j \bar{\phi}_j e^{i\ell_j \bar{x}_i} + \text{c.c.}), \\
\bar{\phi}_j' = -i\bar{\phi}_j + i\eta 2\pi \frac{N}{j} \sum_{i=1}^{N} e^{-i\ell_j \bar{x}_i}.
\end{equation}

(1)
We remark that resonance conditions rewrite $\ell_i u_{rj} = 1$ and $\ell_j$ is, thus, taken as the integer number corresponding to the best approximation of $1/u_{rj}$.

Since in the equations of motion (1) the dielectric is expanded near $\omega \simeq \omega_p$ [12, 27], the left hand side of the dispersion relation must be consistently rewritten as $\epsilon \simeq 2(\omega - 1)$. In order to study the instability features of a single resonant mode, we explicitly write $\bar{\omega} = \omega_0 + i \bar{\gamma}_L$, where $\omega_0$ contains a real frequency shift with respect to the Langmuir mode frequency $\omega_p$ and $\bar{\gamma}_L$ denotes the normalized linear growth rate. Using the proper normalizations, we can finally rewrite the dispersion relation for a given resonant velocity $u_r$ (corresponding to the selected resonant fluctuation with mode number $\ell = 1/u_r$) as

$$2(\omega_0 + i \bar{\gamma}_L - 1) - \frac{\eta u_r}{M} \int_{-\infty}^{+\infty} du \frac{\partial_u f_B(u)}{u/u_r - \omega_0 - i \bar{\gamma}_L} = 0, \quad (4)$$

where $M = \int_{-\infty}^{+\infty} du f_B(u)$.

Eq. (4) is numerically integrated by splitting its imaginary and real parts and by plotting the respective contour lines for assigned values of $\omega_0$ and $\bar{\gamma}_L$ in a given range corresponding to the most unstable mode. The intersection point of the two lines corresponds a solution of the dispersion relation. In Fig. 2, we show an example for 7 distinct modes and $\eta = 0.0007$, where each one is represented by a different color, and we also mark the various intersection points). The mode numbers are selected in order to resonate in different regions of the distribution function, $u_r = 1/\ell$. In particular, the resonant velocities vary from $u_r \simeq 0.0011$ to $u_r \simeq 0.0019$, with equal normalized velocity spacing.

It is worth stressing that the values of growth rates increase as the resonant velocity approaches the inflection point of $f_B(u)$ where the drive of the inverse Landau damping $\partial_u f_B$ is maximum. Furthermore, we see that the real part of the frequency starts to deviate from unity ($\bar{\omega}_0 = 1 \Rightarrow \omega = \omega_p$) on the right-hand flat profile of the distribution function. In fact, in that region the local value of $\partial_u f_B$ is essentially negligible, and a finite growth rate is indicative of the break-down of the perturbative Landau damping expression, which is necessarily
associated to a real frequency shift with respect to the plasma frequency.

Let us now compare the results obtained by analytical and semi analytical integration of the dispersion relation, respectively. By linearizing Eq. (4) and fixing \( \bar{\gamma}_L = 1 \), one can get the well known expression for the linear growth rates (denoted by the subscript \( \text{lin} \)) \cite{28}:

\[
\bar{\gamma}_{\text{lin}} - \frac{\pi \eta u_r^2}{2M} \partial_u f_B \bigg|_{u_r} = 0 . \tag{5}
\]

Meanwhile, the dispersion relation can be also analytically integrated in terms of the Faddeeva function \( w(x) \) and residue contributions \cite{29}. Since we easily get 
\[
\partial_u f_B = A \exp[-B^2(1 - D u)^2],
\]
Eq. (4) provides the following solution \cite{30}:

\[
2(\bar{\omega}_0 F + i \bar{\gamma}_F - 1) - \frac{\eta A u_r}{M} i \pi (2e^{-X} - w(X)) = 0 , \tag{6}
\]

with \( X = BC - BD(\bar{\omega}_0 F + i \bar{\gamma}_F) u_r \) (for the sake of completeness, in our specific case, we have: \( A = 2559.88, B = 5.89, C = 1.15 \) and \( D = 770.0 \)). Here, the subscript \( F \) denotes the values of \( \bar{\omega}_0 \) and \( \bar{\gamma}_L \) obtained by the analytical solution, Eq. (3), of the dispersion relation Eq. (4). The roots of Eq. (6) are obtained using the same contour lines method adopted for Eq. (4).

In Fig. 3 we plot the ratios between the growth rates \( \bar{\gamma} \) fitted from the time behavior of single mode simulations of the system \( \| \) and the obtained values of \( \bar{\gamma}_L, \bar{\gamma}_{\text{lin}} \) and \( \bar{\gamma}_F \) evaluated from Eqs. (4), (5) and (6), respectively, for \( \eta = 0.0007 \).

\begin{figure}[h]
  \centering
  \includegraphics[width=0.5\textwidth]{figure3.png}
  \caption{(Color online) Resonance velocity dependence of the ratios between the growth rate \( \bar{\gamma} \) and \( \bar{\gamma}_L, \bar{\gamma}_{\text{lin}} \) and \( \bar{\gamma}_F \) evaluated from Eqs. (4), (5) and (6), respectively, for \( \eta = 0.0007 \).}
\end{figure}

IV. ANALYSIS OF THE NON-LINEAR VELOCITY SPREAD

In this Section, we study the basic non-linear features of the system \( \| \) by means of numerical simulations. We recall that the dynamics of one isolated mode consists of an initial exponential growth of the mode followed by the non-linear saturation, where the particles are trapped and begin to slosh back and forth in the potential well of the wave. This makes the mode intensity oscillate and generates rotating clumps in the phase-space. A quadratic relation exists \cite{11, 12, 16} between the saturation level of the considered mode (dubbed \( |\phi|^2 \)) and the linear growth rate, i.e., \( |\phi|^2 = \alpha \bar{\gamma}_L^2 \) (with \( \alpha = \text{const.} \)). This relation holds only if the non-linear dynamics is not sensitive to the morphology of the distribution function \( \| \) \cite{19}; and all the analyses reported in the present study satisfy this condition.

A. Particle trapping and non-linear velocity spread

Assuming a single mode scheme, the approximation of the post-saturation dynamics by an instantaneous harmonic oscillator allows to identify the so called trapping (bounce) frequency \( \omega_B \) as

\[
\bar{\omega}_B = \sqrt{2F \phi^2} = \sqrt{2\alpha \bar{\gamma}_L} . \tag{7}
\]

Meanwhile, from energy conservation at saturation, one can estimate the non-linear velocity spread of the resonant particles, i.e., particles having velocity \( u_r = 1/\ell \). This quantity is clearly related to the (half) clump width
mentioned above and it is derived from the relation \(m(\Delta \phi_{NL})^2/2 = e|\phi(x,t)|^S\). Using the normalized variables, this definition of the nonlinear velocity spread can be cast as

\[
\frac{\Delta \tilde{u}_{NL}}{u_r} = 2L \sqrt{|\tilde{\phi}|^S} = \sqrt{2} \tilde{\omega}_B = 2L \sqrt{\alpha} \tilde{\gamma}_L . \tag{8}
\]

This estimate excludes effects of non strictly resonant particles. Thus, in order to get a satisfactory characterization of the non-linear dynamics, we introduce two further measures quantifying the non-linear velocity spread. First of all, we consider the clump width \(\Delta u^e_{NL}\) defined by measuring the largest instantaneous velocity of particles initialized at \(\tau = 0\) with \(u < u_r\); and, similarly, the smallest velocity of particles with \(u < u_r\) at \(\tau = 0\). This measure is performed during the temporal evolution of the system and \(\Delta u^e_{NL}\) is taken as the value at saturation time \(\tau_S\). Another way to define the particle non-linear velocity spread (in the following dubbed \(\Delta u^t_{NL}\)) is by a measure of the distortion of the distribution function at saturation, i.e.,

\[
|f_B(\tau_S) - f_B(0)|/f_B(0) \geq \varepsilon \quad \text{for} \quad |u - u_r| \leq \Delta u^t_{NL} \tag{9}
\]

with \(\varepsilon\) a small control parameter. In the presented analysis, this parameter is safely taken twice the numerical relative error (dependent on the velocity \(u\)) generated by the discretization procedure (smoothed histograms) with respect to the analytic expression of the distribution function, Eq. (2), at \(\tau = 0\). The main difference between the two methods consists in the dynamic role of trapped particles as part of the clump, while finite distortion of the distribution function (i.e., phase-space particle transport) is also due to un-trapped but nearly resonant particles [15]. The second estimate is expected to overestimate the clump size, since it includes effects at the edges of the plateau (flattened region of the distribution function, mainly coinciding with the clump size).

The aim of the following analysis is to study the behavior of the various definitions of nonlinear velocity spread, introduced above, as a function of the linear growth rate of the resonance. In particular, we analyze 5 distinct cases having different resonant velocities (namely \(u_r \simeq 0.0013, 0.0014, 0.0015, 0.0016, 0.0017\)). For each case, 10 simulations with different \(\eta\) values are studied (equipacing the \(\eta\) value from 0.00015 to 0.0025) in order to address the dependence of the non-linear velocity spread on the instability drive, \(\tilde{\gamma}_L\), which is proportional to \(\eta\).

For the sake of simplicity, we discuss in detail the case of \(u_r\) at the inflection point of \(f_B\) (case 3), and then we summarize the overall behavior of the fundamental quantities. First of all, in the upper panel of Fig. 4 we plot the clump width \(\Delta u^e_{NL}\) versus time and for the different values of \(\eta\) (different colors in the figure): as expected, the smaller \(\eta\), the smaller \(\Delta u^e_{NL}\) is. In fact, as the value of \(\eta\) is lowered, the instability drive becomes correspondingly weaker, and, in turn, the electric field amplitude at saturation and the clump width are decreased. We observe that, for a fixed \(\eta\) value, the clump width increases with time during the instability growth phase, until the saturation level is reached.

In the lower panel of Fig. 4, we illustrate for a fixed value of \(\eta\) the measure, \(\Delta u^t_{NL}\), of the region over which the distribution function is non-linearly distorted. Here, the existence of a plateau with smooth edges is clear. We will see how \(\Delta u^t_{NL}\) plays a crucial role in determining the resonance overlap criterion for multiple modes.

### B. Non-linear velocity spread versus linear growth rate

Let us now discuss the \(\tilde{\gamma}_L\) dependence of the quantities introduced above. In the upper panel of Fig. 5 (which, again, corresponds to the case \(u_r \simeq 0.0015\)), we show (in Log-Log scale) the quadratic behavior of the mode saturation amplitude with respect to \(\tilde{\gamma}_L\). In this specific case, we find \(|\psi|^S \simeq 1.2 \times 10^{-5} \tilde{\gamma}_L^2\) which can be shown to weakly depend on the model parameters. In the middle panel of Fig. 5, meanwhile, we plot \(\tilde{\omega}_B\) as a function
of $\bar{\gamma}_L$, confirming the linear scaling of Eq. (7). Finally, in the lower panel of Fig. 5 we we illustrate the behaviors of the various measures we introduced for characterizing the non-linear velocity spread. For the present case, we obtain $\Delta u_{NL}^f/u_r \simeq 8.95\bar{\gamma}_L$, $\Delta u_{NL}^c/u_r \simeq 6.45\bar{\gamma}_L$ and $\Delta \bar{u}_{NL}/u_r \simeq 4.62\bar{\gamma}_L$. We note that, consistent with the adopted definitions, $\Delta u_{NL}^f > \Delta u_{NL}^c > \Delta \bar{u}_{NL}$.

For the behavior of $|\tilde{\phi}|^2$ (i.e., $\alpha = 1.27 \pm 0.26 \times 10^{-5}$) and $\bar{\omega}_B$, our results are consistent with the original analysis in Refs. [12, 16], confirming the crucial role of wave-particle trapping for nonlinear mode saturation and resonance broadening. However, the study of the non-linear velocity spread suggests a more subtle physical interpretation of fluctuation induced particle transport in phase-space, based on the inequality $\Delta u_{NL}^f > \Delta u_{NL}^c > \Delta \bar{u}_{NL}$ noted above, which is consistent with the general description of wave-particle interactions of Ref. [13]. As we will see in the next Section, these different characterizations of the non-linear velocity spread have relevant implications for the mode-mode coupling associated to a warm beam. Finally, we emphasize that repeating the analysis above with different slope of the Erfc distribution, yields qualitatively comparable dependences of the non-linear velocity spread definitions. This suggests a universal character of the obtained behaviors and that the non-perturbative response of beam particles is, at the leading order, accounted for by the scaling of the non-linear velocity spread with $\bar{\gamma}_L$.

V. CHARACTERIZATION OF THE RESONANCE OVERLAP

Let now discuss how the results above can be applied to define the mode resonance overlap criterion. We set a system with $\eta = 0.00056$ and in which 3 distinct modes are excited in correspondence of different resonant velocities. Resonance overlap occurs when the phase-space regions associated to different resonances is mix, due to the non-linear velocity spread. In Fig. 6 we plot the values of the different definitions of velocity spread for each resonance (represented by a bullet): dotted lines indicate $\Delta \bar{u}_{NL}$, dashed lines are $\Delta u_{NL}^c$, while solid lines correspond to $\Delta u_{NL}^f$ (different colors denote different resonances). In Fig. 7 the system is evolved self-consistently for the 3 modes and it is compared with the single mode simulation results of each resonance (gray lines). As it can be argued from the plot, two resonances start to interact nearby the corresponding mode saturation, and the only quantity which properly predicts resonance overlap is $\Delta u_{NL}^f$. In fact, the two other estimates suggest that fluctuations should evolve as superposition of non-interacting modes. The remaining resonance (green curve) is isolated.
The analysis above is performed at fixed drive, namely η = 0.00056. By increasing the value of η, the resonances become increasingly more overlapped and interaction time become smaller. On the contrary, reducing the drive results into a progressive separation of the resonances, which eventually behave as isolated. We emphasize that the criterion for resonance overlap based on ∆uNL allows calculating the value of η below which resonances are isolated. For example, at η = 0.0005 no effective non-linear interplay should occur according to the criterion based on the ∆uNL. However, some residual non-linear interplay is found in numerical simulations, although, in this case, the overlap starts much later than single mode saturation time, where the analysis is not predictive because the measures of the non-linear velocity spread are taken for φ = φS.

VI. MODE AMPLITUDE AT SATURATION

The case analyzed in the previous Section corresponds to an intermediate situation between the limiting cases of isolated and strongly overlapping resonances. Fig.7 suggests that the mode saturation levels are larger in the multi mode simulation than in the single mode runs. In the following, we analyze the saturation levels as function of the resonance separation.

We initialize the beam particles using a distribution function fB(u) with a positive constant gradient and η = 0.002, i.e., we consider a linear initial profile in the velocity space. We run 7 simulations with two modes, fixing one of the resonances (namely at u1 = 0.00135) and sweeping the other one (dubbed u2) in order to span, with constant velocity increment, the resonance separations ∆uSEP ≡ u2 − u1 from 5 × 10−6 to 2 × 10−3. For each case, we compare numerical results with the evolutions obtained for isolated resonances.

In the presence of multiple modes and resonances, we remind that the total momentum and energy are conserved [12]. In particular, total conserved momentum can be written as

$$\mathcal{K}_P = \sum_j \frac{|\vec{\phi}_j|^2}{\ell_j} + \frac{2}{\eta N} \sum_i u_i . \quad (13)$$

Thus, for each case, we can measure the saturation level

![Figure 6: (Color online) Three resonance beam plasma system with η = 0.00056: the black line represents fB(u) of Eq. 4. Bullets denote the resonance velocities: u1 ≃ 0.0013 (green), u2 ≃ 0.0015 (yellow), u3 ≃ 0.0017 (blue). Different estimates of non-linear velocity spread are indicated with corresponding colors: u ± ∆uNL (dotted lines), u ± ∆uNL (dashed lines) and u ± ∆uNL (solid lines).]

![Figure 7: (Color online) Plot of the mode evolution of the 3 resonance model (colored lines) and the respective single mode simulations (gray lines). The color scheme is the same of Fig.4](nonetheless it is very close to overlap) and behaves as such at the time of single mode saturation, as predicted by the criterion based on ∆uNL. However, at later times, the synergistic non-linear interaction of the two overlapping resonances modify the dynamics and broaden the region affected by significant non-linear velocity spread and resonance overlap with respect to the criterion based on ∆uNL. As a consequence, this eventually allows synergistic interaction also with this isolated resonance.

The analysis above is performed at fixed drive, namely η = 0.00056. By increasing the value of η, the resonances become increasingly more overlapped and interaction time become smaller. On the contrary, reducing the drive results into a progressive separation of the resonances, which eventually behave as isolated. We emphasize that the criterion for resonance overlap based on ∆uNL allows calculating the value of η below which resonances are isolated. For example, at η = 0.0005 no effective non-linear interplay should occur according to

![Figure 8: (Color online) Saturation level ∑j |φ̂j|2/ℓj as a function of ∆uSEP = u2 − u1 for self-consistent simulations of two resonances (blue) compared with the saturation level obtained artificially superposing the evolution of the individual isolated resonances (yellow). The threshold for the onset for enhanced saturation level is the intersection point (dashed gray line) ∆uSEP ≃ 1.75 × 10−5.](as ∑j |φ̂j|2/ℓj. In Fig.8 we plot an interpolation of this)
quantity as function of $\Delta u_{SEP}$ for the self-consistent simulations of two modes (blue line) compared with the saturation level obtained artificially superposing the evolution of isolated resonances (yellow line). Note that for vanishing resonance separation, as expected, the self-consistent saturation level is half of the value obtained by artificial addition of single isolated modes. In fact, for coalescing resonances the modes become different realizations of the same fluctuating field and their saturation amplitude is reduced by a factor of two. It is immediate to see that a threshold value ($\Delta u_{SEP} \simeq 1.75 \times 10^{-5}$) emerges, below which the saturation level for the self-consistent evolution is lower than that obtained by artificial superposition of single isolated modes: this corresponds to the regime of strongly overlapped resonances. The opposite situation occurs above $\Delta u_{SEP}$ (clearly provided that $\Delta u_{SEP} \lesssim 2\Delta u_{NL}$, otherwise resonances are not overlapped and the modes evolve as single isolated fluctuations). In order to better illustrate such a behavior, in Fig.9 we plot the single and multi mode evolutions, for a case below (left-hand panel) and above (right-hand panel) threshold, respectively.

A. Interpretative model

When the multi mode saturation level is larger than for single isolated modes, it is evident that a more efficient process can tap energy from the particle phase-space. In the following, we propose a toy model to qualitatively describe this effect and the threshold condition introduced above.

For each resonance, respectively called $u_1$ and $u_2$, we assume to model the non-linear distorted distribution function at saturation by a flattening over a certain region; that is, as horizontal lines centered at the resonance position and extended over twice the non-linear velocity spread, respectively denoted as $\Delta u_1$ and $\Delta u_2$ for the two considered resonances. As described in the previous Section, resonance overlap occurs when the flattening regions intersect. We then describe the overlapping resonances as a single resonance having a new resonance velocity and non-linear velocity spread defined by

\[ u_r = u_1 + \Delta u_{SEP}/2 \ , \]
\[ \Delta u_r = \beta((u_2 + \Delta u_2) - (u_1 - \Delta u_1))/2 \ , \]

respectively, where $\Delta u_{SEP} = u_2 - u_1$ and $\beta = \mathcal{O}(1)$ is a control parameter for the model. This scheme is illustrated, for $\beta = 1$, in Fig.10 where we used realistic quantities (estimated from $\Delta u_{NL}$) for comparison with the simulation results described above.

The non-linear mode saturation corresponds to particle transfer from large to small velocities as indicated in the momentum conservation law Eq.(13). Thus, the relevant quantities to be analyzed are the areas of the triangles that are colored in the figure. Evolving the two modes as single isolated resonances, the saturation level scales as

\[ \tilde{A} = \frac{\tan \alpha}{2} ((\Delta u_1)^2 + (\Delta u_2)^2) \ , \]

i.e., as the sum of the areas built around the two resonances. Here $\alpha$ denotes the angular coefficient of the initial linear distribution. Meanwhile, the area of the triangle correspond to the new merged resonance can be evaluated by means of Eqs.(14) and (15) as

\[ A = \beta^2 \frac{\tan \alpha}{8} (\Delta u_{SEP} + \Delta u_1 + \Delta u_2)^2 \ . \]

Assuming to neglect the velocity dependence of $\Delta u_2$ (as $u_2$ is swept to change $\Delta u_{SEP}$), Fig.11 shows $\tilde{A}$ and $A$ (for $\beta = 1$) as functions of the resonance separation. For $\Delta u_{SEP} > \Delta u_1 + \Delta u_2$ there is no resonance overlap and the system response is consistent with two isolated resonances. For decreasing $\Delta u_{SEP}$ below the onset of the overlap (i.e., the system depicted in Fig.11), the saturation level for the merged resonance is larger than for the single isolated modes, i.e., $A > \tilde{A}$, or, more precisely, $A = 2\tilde{A}$ at the onset condition, suggesting a sudden transition in the non-linear dynamics due to the synergistic behavior of the interacting modes. In particular, it

![Figure 9](https://via.placeholder.com/150)

![Figure 10](https://via.placeholder.com/150)
Thus, we can identify a range for enhanced or synergistic interaction of overlapping resonances

\[ \Delta u_{SEP}^* = -\Delta u_1 - \Delta u_2 + 2 \frac{(\Delta u_1)^2 + (\Delta u_2)^2}{\beta^2}. \]  

We can further refine this range by setting a threshold.

\[ \Delta u_{SEP}^* < \Delta u_{SEP} < \Delta u_1 + \Delta u_2. \]  

Meanwhile, for \( \Delta u_{SEP} < \Delta u_{SEP}^* \), the two resonances are strongly overlapping and \( \bar{A} < A \), with \( A = \bar{A}/2 \). The limit of \( \Delta u_{SEP} = 0 \) is consistent with our numerical simulations.

### VII. CONCLUDING REMARKS

The analysis of this paper offers an interesting point of view for the further development of the bump-on-tail paradigm (to which the beam-plasma instability is isomorphic) as predictive models for the fast ion interaction with Alfvén eigenmodes. Quantitative prediction of resonance overlap and of enhanced saturation conditions and levels was the original motivation of this work, which has led us to conclude that the system evolution is strongly influenced by the global features of the distribution function both in the linear and non-linear phases.
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