Fully Loop-Quantum-Cosmology-corrected propagation of gravitational waves during slow-roll inflation

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The cosmological primordial power spectrum is known to be one of the most promising observable to probe quantum gravity effects. In this article, we investigate how the tensor power spectrum is modified by loop quantum gravity corrections. The two most important quantum terms, holonomy and inverse-volume, are explicitly taken into account in a unified framework. The equation of propagation of gravitational waves is derived and solved for one set of parameters.

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I. INTRODUCTION

The inflationary scenario is currently the favored paradigm to describe the first stages of the evolution of the Universe (see, e.g., [1] for a recent review). Although still debated, it has received many experimental confirmations, including from the WMAP 5-year results [2], and solves most cosmological paradoxes.

On the other hand, a fully quantum theory of gravity is necessary to investigate situations where general relativity (GR) breaks down. The big bang is an example of such a situation where the backward evolution of a classical space-time comes to an end after a finite amount of time. Among the theories willing to reconcile the Einstein gravity with quantum mechanics, loop quantum gravity (LQG) is appealing as it is based on a nonperturbative quantization of 3-space geometry (see, e.g., [3] for an introduction). Loop quantum cosmology (LQC) is a finite, symmetry reduced model of LQG suitable for the study of the whole Universe as a physical system (see, e.g., [4]).

In this article, we consider the influence of LQC corrections to general relativity on the production and propagation of gravitational waves during inflation. We first derive the equation of propagation of gravitational waves with both holonomy and inverse-volume corrections. This equation is then reexpressed with the commonly used cosmological variables. It is finally solved for a specific set of parameters and the primordial power spectrum is derived. The aim of this work is to conclude our previous studies [5] and [6] where, respectively, only holonomy and only inverse-volume corrections were considered. By combining both terms, we show that the inverse-volume correction dominates over the holonomy one and dictates the overall shape of the tensor spectrum.

Quite a lot of work has already been devoted to gravitational waves in LQC [7]. Our approach assumes the background to be described by the standard slow-roll inflationary scenario whereas LQC corrections are taken into account to compute the propagation of tensor modes. This approach is heuristically justified (to decouple the physical effects) and intrinsically plausible (as, on the one hand, the LQC-driven superinflation can only be used to set the proper initial conditions to a standard inflationary stage if the horizon and flatness problems are both to be solved [8] and as, on the other hand, it seems that the quantum bounce can trigger a standard inflationary phase [9]). In addition, very few studies so far have taken into account both the holonomy and the inverse-volume corrections. This latter term is somehow more speculative than the former one as it was shown to exhibit a fiducial cell dependence (see, e.g., [10]). For the sake of completeness it is however obviously worth considering the fully corrected propagation of gravitational waves.

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II. EQUATION OF PROPAGATION FOR THE GRAVITON

The derivation of the equation of propagation of gravitational waves with both holonomy and inverse-volume corrections extensively uses the material developed in [11]: notations, conventions and framework of this work are the same and will not be explicitly restated. We begin by considering a Friedmann-Lemaître-Robertson-Walker universe with a spatial metric $g_{ab}$, which will be perturbed to account for gravitational waves. Hereafter, $N$ and $N^a$ are respectively the lapse function and the shift function. The metric components read as follows:

\begin{align*}
g_{00} &= -N^2 + q_{ab}N^aN^b = -a^2(\eta), \quad (1) \\
g_{0a} &= q_{ab}N^b = 0, \quad (2) \\
g_{ab} &= q_{ab} = a^2(\eta)(\delta_{ab} + h_{ab}). \quad (3)
\end{align*}

As usual in the formalism of LQC, we use the Ashtekar variables for an homogeneous and isotropic background: the connection $A^a_b$, and the triad density $\tilde{E}^a_i$. They can be written as a function of two other variables ($\tilde{k}, \tilde{p}$) as

\begin{align*}
\tilde{E}^a_i &= \tilde{p}\delta^a_i, \\
\tilde{A}^i_a &= \tilde{K}^i_a + \tilde{\Gamma}^i_a \\
\tilde{K}^i_a &= \tilde{k}\delta^i_a, \\
\tilde{N}^a &= 0, \quad \tilde{N} = \sqrt{\tilde{p}}. \quad (4)
\end{align*}

Hamilton-Jacobi equations will be used to determine the perturbed part of the Ashtekar variables. The Hamiltonian constraint reads as

\[ H[N] = \frac{1}{2\kappa} \int_{\Sigma} d^3x N |det E|^{-\frac{2}{3}} E^a_i E_b^i (\epsilon^{ijk} F_{ab}^i - 2(1 + \gamma^2)K^i_b K^j_a), \quad (5) \]

where $F_{ab} = \partial_a A^i_b - \partial_b A^i_a + \epsilon^{ijk} A^j_a A^k_b$ is the field strength. The Hamiltonian for a matter field $\Phi$ is given by

\[ H_{\text{matter}} = \int d^3x \left( \frac{1}{2} \frac{p_\Phi^2}{\sqrt{|det E^a_i|}} + \sqrt{|det E^a_i|} V(\Phi) \right). \quad (6) \]

With Eq. (5) and these Hamiltonians, the background is described by

\[ H^{\text{cond}}_{G}[\tilde{N}] = \frac{1}{2\kappa} \int_{\Sigma} d^3x \tilde{N} \left[ -6\sqrt{\tilde{p}}k^2 \right], \quad (7) \]

and

\[ H_{\text{matter}}[\tilde{N}] = \int_{\Sigma} d^3x \left( \frac{1}{2} \frac{p_\Phi^2}{\sqrt{3}} + \frac{p_\Phi^2}{2} \right) V(\Phi). \quad (8) \]

Perturbing the canonical variables (and going through the appropriate Poisson bracket) leads to:

\[ H_{G}[\tilde{N}] = \frac{1}{2\kappa} \int_{\Sigma} d^3x \tilde{N} \left[ -6\sqrt{\tilde{p}}k^2 - \frac{\tilde{k}^2}{2\sqrt{3}} (\delta E^c_j \delta E^d_k \delta^i_j \delta^j_i) + \sqrt{\tilde{p}} (\delta K^c_i \delta K^d_k \delta^i_j \delta^j_i) \\
- \frac{2\tilde{k}}{\sqrt{\tilde{p}}} (\delta E^c_j \delta K^d_k) - \frac{1}{\sqrt{\tilde{p}}}(\delta_{cd} \delta_{e}^{jk} E^d_j \delta^c_k \partial_c \partial_d \tilde{E}^i_k) \right], \quad (9) \]

where only the tensor perturbations (i.e. gravitational waves) are considered in $\delta E^a_i$.

This classical Hamiltonian is to be modified by quantum corrections. Because loop quantization is based on holonomies, i.e. exponentials of the connection rather than direct connection components, one needs to substitute in the gravitational sector

\[ \tilde{k} \rightarrow \sin(m\tilde{\mu}\gamma\tilde{k}) \quad (10) \]
where $\tilde{\mu}$ is a new parameter related to the action of the fundamental Hamiltonian on a lattice state. In addition, because of inverse powers of the densitized triad which, when quantized, becomes an operator with zero in the discrete part of its spectrum, the matter and gravitational Hamiltonians must be modified by introducing the function

$$\alpha(\tilde{p}, \delta E^a_i) = 1 + \lambda q^n = 1 + \lambda \left( \frac{l_{Pl}}{\tilde{p}} \right)^n.$$  \hfill (11)

At a semiclassical level, i.e. $q \ll 1$, the same parametric form of $\alpha$ can be used in both the matter Hamiltonian and the gravitational Hamiltonian. However, the two positive and real valued constants $\lambda$ and $n$ may differ from one sector to another. In the following, $(S, \lambda_s, s)$ and $(D, \lambda_d, d)$ will therefore denote $(\alpha, \lambda, n)$ for the gravitational sector and the matter sector respectively. With these two corrections, the Hamiltonians read

$$H_{G}^{\text{eff}}[\hat{N}] = \frac{1}{2\kappa} \int d^3x \hat{N} S(\tilde{p}) \left[ -6\sqrt{p} \left( \frac{\sin(\tilde{\mu}\gamma \tilde{k})}{\tilde{\mu}\gamma} \right)^2 \right],$$  \hfill (12)

$$H_{\text{matter}}[\hat{N}] = \int d^3x \left( \frac{1}{2} D(q) \frac{\tilde{p}^2}{\tilde{p}^2} + \tilde{p}^2 V(\Phi) \right),$$  \hfill (13)

with $H_{G}^{\text{eff}}$ the effective gravitational Hamiltonian describing the homogeneous background. The equations of motion for $(\hat{k}, \hat{p})$, i.e. the background equations, can be obtained in the Hamiltonian formalism

$$\hat{p}(p, H_{G}^{\text{eff}}[\hat{N}] + H_{\text{matter}}[\hat{N}]) ; \quad \hat{k}(k, H_{G}^{\text{eff}}[\hat{N}] + H_{\text{matter}}[\hat{N}]),$$  \hfill (14)

leading to:

$$\dot{\hat{p}} = 2 \cdot \tilde{p} \cdot S(\tilde{p}, \delta E) \cdot \left( \frac{\sin(2\tilde{\mu}\gamma \tilde{k})}{2\tilde{\mu}\gamma} \right),$$  \hfill (15)

$$\dot{\hat{k}} = \frac{\kappa}{3V_0} \frac{\partial H_{\text{matter}}}{\partial \tilde{p}} - \tilde{p} \frac{\partial S}{\partial \tilde{p}} \cdot \left( \frac{\sin(\tilde{\mu}\gamma \tilde{k})}{\tilde{\mu}\gamma} \right)^2 - \frac{S}{2} \left[ \left( \frac{\sin(\tilde{\mu}\gamma \tilde{k})}{\tilde{\mu}\gamma} \right)^2 + 2\tilde{p} \frac{\partial}{\partial \tilde{p}} \left( \frac{\sin(\tilde{\mu}\gamma \tilde{k})}{\tilde{\mu}\gamma} \right)^2 \right].$$  \hfill (16)

The same modification is applied to the perturbed gravitational Hamiltonian. Denoting $H_{G}^{\text{Phen}}[\hat{N}]$ the effective perturbed quantum-corrected gravitational Hamiltonian, it reads with both holonomy and inverse-volume corrections

$$H_{G}^{\text{Phen}}[\hat{N}] = \frac{1}{2\kappa} \int d^3x \hat{N} S(\tilde{p}, \delta E^a_i) \left[ -6\sqrt{p} \left( \frac{\sin(\tilde{\mu}\gamma \tilde{k})}{\tilde{\mu}\gamma} \right)^2 - \frac{1}{2\tilde{p}^{3/2}} \left( \frac{\sin(\tilde{\mu}\gamma \tilde{k})}{\tilde{\mu}\gamma} \right)^2 (\delta E^a_i \delta E^b_j \delta^i_j \delta^a_j) \right. + \frac{\sqrt{p}}{\tilde{p}^{3/2}} (\delta K^a_i \delta K^b_j \delta^a_j \delta^b_j) \left. - \frac{2}{\sqrt{p}} \left( \frac{\sin(2\tilde{\mu}\gamma \tilde{k})}{2\tilde{\mu}\gamma} \right) (\delta E^a_i \delta K^b_j) - \frac{1}{\tilde{p}^{3/2}} (\delta_{ce} \delta_{de} \delta E^c_j \delta^e_j \delta^c_j \delta^e_j) \right].$$  \hfill (17)

We now turn to the equation of motion of the graviton. The perturbed densitized triad is

$$\delta E^a_i = -\frac{1}{2} \tilde{p} h^a_i.$$  \hfill (18)

As has been done for the homogeneous canonical variables, it is possible to define the equation of motion for the perturbations:

$$\delta E^a_i = \{ \delta E^a_i, H_{G}^{\text{Phen}}[\hat{N}] + H_{\text{matter}}[\hat{N}] \} = \left\{ \delta K^a_i(x), \delta E^a_i(y) \right\} \frac{\delta}{\delta (\delta K^a_i)} (H_{G}^{\text{Phen}}[\hat{N}] + H_{\text{matter}}[\hat{N}]),$$

$$\delta \dot{K}^a_i = \{ \delta K^a_i, H_{G}^{\text{Phen}}[\hat{N}] + H_{\text{matter}}[\hat{N}] \} = \left\{ \delta K^a_i(x), \delta E^a_i(y) \right\} \frac{\delta}{\delta (\delta E^a_i)} (H_{G}^{\text{Phen}}[\hat{N}] + H_{\text{matter}}[\hat{N}]).$$

This leads to:

$$\delta \dot{E}^a_i = -\frac{1}{2} (\tilde{p} h^a_i + \tilde{p} h^a_i)$$  \hfill (19)

$$= -S(\tilde{p}, \delta E) \cdot \left[ \tilde{p} \cdot \delta K^c_i \cdot \delta^a_i \cdot \delta^b_i - \left( \frac{\sin(2\tilde{\mu}\gamma \tilde{k})}{2\tilde{\mu}\gamma} \right) \cdot \delta E^{a}_i \right].$$  \hfill (20)
By combining those equations and using the expression of \( \dot{\varphi} \), one obtains the expression of \( \delta K^i_a \) as a function of \( h^i_a \) and of \( \dot{h}^i_a \). The expression of \( \delta K^i_a \) is:

\[
\delta K^i_a = \frac{1}{2S} \delta h^i_a + \frac{1}{2} \left( \frac{\sin(2\bar{\mu}\gamma\bar{k})}{2\bar{\mu}\gamma} \right) h^i_a.
\]  

The equation of motion will lead to another derivative with respect to \( \eta \). The Hamilton-Jacobi equation for the perturbed connection can now be used to find the final equation of propagation for gravitational waves:

\[
\delta K^i_a = \frac{1}{2} \left[ \frac{\delta h^i_a}{S} \frac{\partial S}{\partial \eta} \cdot \delta h^i_a + \left( \frac{\sin(2\bar{\mu}\gamma\bar{k})}{2\bar{\mu}\gamma} \right) \delta h^i_a + h^i_a \frac{\partial}{\partial \eta} \left( \frac{\sin(2\bar{\mu}\gamma\bar{k})}{2\bar{\mu}\gamma} \right) \right]
\]

\[
= \frac{\delta K^i_a(x)}{\delta E^b_j} \left( H^{Phen}_G \frac{\delta}{\delta E^b_j} [\bar{N}] + H_{\text{matter}} [\bar{N}] \right).
\]

As

\[
\frac{\delta H^{Phen}_G}{\delta (\delta E^b_j)} = \frac{1}{2\kappa} \int \frac{\delta S}{\delta (\delta E^b_j)} \left[ \frac{\delta S}{\delta (\delta E^b_j)} \right] + \frac{1}{2\kappa} \int \frac{\delta S}{\delta (\delta E^b_j)} \frac{\partial S}{\partial \eta} \left[ \frac{\delta S}{\delta (\delta E^b_j)} \right] \frac{\sin(2\bar{\mu}\gamma\bar{k})}{2\bar{\mu}\gamma} \left( \delta E^c_i \cdot \delta^b_c \cdot \delta^b_j \right)
\]

\[
= \frac{1}{\sqrt{p}} \left( \frac{\sin(2\bar{\mu}\gamma\bar{k})}{2\bar{\mu}\gamma} \right) \delta K^i_a - \frac{2}{p^2} \left( \delta b_{b \cdot \delta^b_j} \cdot \delta^b_j \right) \left( \delta E^d_i \right) - \frac{\kappa}{\delta E^b_j} \delta H_{\text{matter}} [\bar{N}]
\]

where \([\ldots] \) stands for the term beginning with \([-6\sqrt{p} \left( \frac{\sin(\bar{\mu} \gamma \bar{k})}{\bar{\mu} \gamma} \right)^2 \ldots \) in \([23]\). one obtains (with \( \delta^f \frac{\partial}{\partial \epsilon} \frac{\partial (\delta E^d_i)}{\delta E^b_j} = \nabla^2 (\delta E^d_i) = -\frac{1}{2h^i_a} \cdot \nabla^2 h^i_a \))

\[
\frac{\delta H^{Phen}_G}{\delta (\delta E^b_j)} \frac{\delta}{\delta (\delta E^b_j)} \left[ \delta K^i_a \right] \left[ \delta E^b_j \right] = \frac{1}{2 \sqrt{p} \frac{\partial S}{\delta (\delta E^b_j)}} \left[ \frac{\delta S}{\delta (\delta E^b_j)} \right] \left[ \frac{\delta S}{\delta (\delta E^b_j)} \right]
\]

\[
= \frac{1}{2} \left[ \frac{\delta h^i_a}{S} \frac{\partial S}{\partial \eta} \cdot \delta h^i_a + \left( \frac{\sin(2\bar{\mu}\gamma\bar{k})}{2\bar{\mu}\gamma} \right) \delta h^i_a + h^i_a \frac{\partial}{\partial \eta} \left( \frac{\sin(2\bar{\mu}\gamma\bar{k})}{2\bar{\mu}\gamma} \right) \right] + \frac{\delta H_{\text{matter}} [\bar{N}]}{\delta (\delta E^b_j)}
\]

After quite a lot of algebra, the equation of motion of the graviton can be derived:

\[
\frac{1}{2} \left[ \frac{\delta h^i_a}{S} + S \frac{\sin(2\bar{\mu}\gamma\bar{k})}{2\bar{\mu}\gamma} \right] \delta h^i_a - \left( \frac{\sin(2\bar{\mu}\gamma\bar{k})}{2\bar{\mu}\gamma} \right) \delta h^i_a + h^i_a \frac{\partial}{\partial \eta} \left( \frac{\sin(2\bar{\mu}\gamma\bar{k})}{2\bar{\mu}\gamma} \right) \right] \right] = \kappa S \Pi^i_a,
\]

where

\[
T_Q = -2 \left( \frac{\bar{\mu} \frac{\partial \bar{\mu}}{\partial \bar{p}}} \right) \left( \frac{\sin(\bar{\mu} \gamma \bar{k})}{\bar{\mu} \gamma} \right)^4,
\]

\[
\Pi_{Q \delta} = \frac{1}{3V_0} \frac{\partial H_{\text{matter}}}{\partial \bar{p}} \left( \frac{\partial E^c_i \delta^b_c}{\partial \bar{p}} \cos(2\bar{\mu}\gamma\bar{k}) + \frac{\partial H_{\text{matter}}}{\delta (\delta E^b_j)} \right),
\]

\[
A^i_a = \frac{1}{2} \sqrt{p} \frac{\delta S}{\delta (\delta E^b_j)} \left[ \frac{\delta S}{\delta (\delta E^b_j)} \right] \left[ \frac{\delta S}{\delta (\delta E^b_j)} \right] - \bar{p} \frac{\partial S}{\partial \bar{p}} \cos(2\bar{\mu}\gamma\bar{k}) \left( \frac{\sin(\bar{\mu} \gamma \bar{k})}{\bar{\mu} \gamma} \right)^2 h^i_a.
\]

As usual, requiring an anomaly-free constraint algebra in the presence of quantum corrections requires \( A^i_a \) to vanish. It should be noticed that the inverse-volume correction is involved in each term, through the \( S \) and \( D \) factors, whereas the holonomy correction is only involved in the \( h^i_a \) term, in \( T_Q \) and in \( \Pi_{Q \delta} \).
It is worth studying a bit more into the details of this $\Pi_\text{od}$ source term as it seems to have been misunderstood in several works. In particular, it has often been either neglected or miscomputed. Without holonomy and inverse-volume correction, this term reads as

$$
\Pi_\text{od}^i = \left[ \frac{1}{3V_0} \frac{\partial H_{\text{matter}}}{\partial \bar{p}} \left( \frac{\delta E^c_i \delta \phi_i}{\bar{p}} \right) + \frac{\delta H_{\text{matter}}}{\delta(\delta E^i)} \right],
$$

(25)

with, in this case,

$$
E^a_i = \bar{p} \delta^a_i, \quad \delta E^a_i = -\frac{1}{2} \bar{p} h^a_i, \quad detE = \frac{1}{3!} e^{ijk} E^a_i E^b_j E^c_k.
$$

(26)

At the zeroth order in gravitational perturbation, one can show that

$$
\bar{H}_{\text{matter}} = \int_\Sigma d^3x \sqrt{\tilde{g}} \left( \frac{1}{2} \bar{p} \dot{\phi}^2 + \bar{p}^2 V(\phi) \right),
$$

(27)

and the nonlinear $H_{\text{matter}}$ is given by

$$
H_{\text{matter}} = \bar{H}_{\text{matter}} + \int_\Sigma d^3x \sqrt{\tilde{g}} \delta E^a_i \delta \phi_i \left( \frac{1}{2} \bar{p}^2 - V(\phi) \right),
$$

(28)

thus leading to

$$
\frac{\delta H_{\text{matter}}}{\delta(\delta E^a_i)} = \bar{N} \frac{\delta E^b_j \delta \phi_i}{\sqrt{\bar{p}}} \left( \frac{1}{2} \bar{p}^2 - V(\phi) \right).
$$

(29)

Restricting to the first order in perturbation, the derivative with respect to $\bar{p}$ can be evaluated and one finally obtains

$$
\frac{1}{3V_0} \frac{\partial H_{\text{matter}}}{\partial \bar{p}} \frac{\delta E^c_i \delta \phi_i}{\bar{p}} = -\frac{\delta H_{\text{matter}}}{\delta(\delta E^i)}.
$$

(30)

This easily establishes that classically $\Pi_\text{od}^i = 0$. However, when LQC corrections are taken into account the source term may not vanish anymore (because of the derivative of $D$ with respect to $\bar{p}$ for the inverse-volume correction and because of the cosine term for the holonomy one).

When only inverse-volume corrections are considered, the source term is still given by Eq. (25) but the matter Hamiltonian now reads

$$
H_{\text{matter}} = \bar{H}_{\text{matter}} + H_{\text{matter}}^{(\delta)}
$$

(31)

$$
= \int_\Sigma d^3x \sqrt{\tilde{g}} \left[ \left( D(\bar{p}, \delta E^a_i) \frac{1}{2} \bar{p} \dot{\phi}^2 + \bar{p}^2 V(\phi) \right) + \frac{1}{4\sqrt{\bar{p}}} \delta E^a_i \delta \phi_i \left( \frac{1}{2} \bar{p}^2 - V(\phi) \right) \right],
$$

which leads, at the leading order, to

$$
\frac{\delta H_{\text{matter}}}{\delta(\delta E^a_i)} = \bar{N} \left[ \frac{\delta E^b_j \delta \phi_i}{2 \sqrt{\bar{p}}} \left( \frac{1}{2} \bar{p}^2 - V(\phi) \right) + \frac{\delta D}{\bar{p}} \frac{\partial D}{\bar{p}} \frac{1}{2} \bar{p}^2 \right],
$$

(32)

and

$$
\frac{1}{3V_0} \frac{\partial H_{\text{matter}}}{\partial \bar{p}} \frac{\delta E^c_i \delta \phi_i}{\bar{p}} = \bar{N} \frac{1}{3} \left( \frac{\delta E^c_i \delta \phi_i}{\bar{p}} \right) \left[ -\frac{3}{4} \bar{p} \dot{\phi}^2 + \frac{3}{2} \bar{p} V(\phi) + \frac{\partial D}{\bar{p}} \frac{\partial D}{\bar{p}} \frac{1}{2} \bar{p}^2 \right].
$$

(33)

We finally obtain

$$
\Pi_{Q,\text{od}}^{(IV)} = \frac{1}{3V_0} \left( \frac{\delta E^c_i \delta \phi_i}{\bar{p}} \right) \frac{\partial H_{\text{matter}}}{\partial \bar{p}} + \frac{\delta H_{\text{matter}}}{\delta(\delta E^i)} = \frac{\bar{p}^2}{2p^2} \left[ \frac{1}{3} \left( \frac{\delta E^c_i \delta \phi_i}{\bar{p}} \right) \frac{\partial D}{\bar{p}} + \frac{\delta D}{\delta(\delta E^i)} \right].
$$

(34)
However, because of the anomaly-free condition (see Eq. (27) of [11]), this term is vanishing. This means that, at the leading order, $\Pi_{Q_a}^{(IV)} = 0$.

Considering now the holonomy correction alone, one can expand the cosine term in $\Pi_{Q_a}^{(i)}$ and show that

$$\Pi_{Q_a}^{(\text{holo})} = -2\bar{\mu}\bar{\gamma}\sin^2(\bar{\mu}\gamma\bar{k}) \frac{1}{3V_0} \frac{\partial \hat{H}_{\text{matter}}}{\partial \bar{p}} \left( \frac{\delta E_j^c \delta_j^a}{\bar{p}} \right),$$

as expected from the vanishing inverse-volume source term.

### III. SCHRÖDINGER EQUATION FOR THE FOURIER MODES

The energy density and the pressure of the cosmological fluid can be written as

$$\rho = \frac{1}{V_0\bar{p}^2} \frac{\delta H_{\text{matter}}}{\delta \bar{N}}, \quad p = -\frac{1}{NV_0} \frac{\delta H_{\text{matter}}}{\delta (\sqrt{\det E})},$$

With the Hamiltonian constraint, one obtains

$$0 = \frac{1}{2\kappa} \int_\Sigma d^3x S \left[ -6\sqrt{\bar{p}} \left( \frac{\sin(\bar{\mu}\gamma\bar{k})}{\bar{\mu}\gamma} \right)^2 + \frac{\delta H_{\text{matter}}}{\delta \bar{N}} \right],$$

which finally leads to

$$\rho = \frac{3}{\kappa} \bar{p} \left( \frac{\sin(\bar{\mu}\gamma\bar{k})}{\bar{\mu}\gamma} \right)^2.$$  (38)

Defining $\mathcal{H}$ as the Hubble parameter with respect to the conformal time ($\mathcal{H} = a^{-1} da(\eta)/d\eta$), we obtain the quantum Friedmann equations:

$$\mathcal{H}^2 = S^2 \left( \frac{\sin(2\bar{\mu}\gamma\bar{k})}{2\bar{\mu}\gamma} \right)^2 = S^2 \frac{\bar{p}}{3} \rho \left( 1 - \bar{\mu}^2 \gamma^2 \frac{\bar{p}}{3S \rho} \right),$$

which lead, with $\rho_c = 3/(\kappa \bar{\mu}^2 \gamma^2 \bar{p})$, to

$$\mathcal{H}^2 = a^2 \frac{\kappa}{3} \rho \left( S - \frac{\rho}{\rho_c} \right).$$

(39)

This equation, which has already been found in [12], includes all the LQC corrections and shows that the holonomy term, leading to the bounce, is the most important one as far as the background is concerned. This conclusion will be radically modified for perturbations.

The equation of motion for the graviton can now be reexpressed in terms of the commonly-used cosmological variables. By taking into account Eq. (38) and $\bar{\mu}^2\bar{p} = l_{PL}^2$, one obtains

$$S^2T_Q = -2 \left( \frac{\bar{p}}{\bar{\mu}} \frac{\partial \bar{\mu}}{\partial \bar{p}} \right) (S\bar{\mu}\gamma)^2 \left( \frac{\sin(\bar{\mu}\gamma\bar{k})}{\bar{\mu}\gamma} \right)^4 \frac{\kappa}{3} \frac{a^2}{\rho_c} \rho^2.$$  (41)

The multiplicative factor of $\bar{h}_a$ in Eq. (24) can be reexpressed as a function of the Hubble parameter

$$2S \left( \frac{\sin(2\bar{\mu}\gamma\bar{k})}{2\bar{\mu}\gamma} \right) \left( 1 - \frac{\bar{p}}{S} \frac{\partial S}{\partial \bar{p}} \right) = 2\mathcal{H} \left( 1 - \frac{1}{2} \frac{\dot{S}}{S} \right).$$

(42)
Finally, the source term can be explicitly computed
\[ \Pi^{i}_{Q_{a}} = \frac{h_{i}}{S \rho_{c}} \frac{\rho}{2} \left[ \rho - \frac{\Phi^{2}}{D(q) a^{2}} \left( 1 - \frac{1}{6} \frac{\dot{D}}{D} \frac{a}{\dot{a}} \right) \right]. \]

As in [11], we use the effective parametrization \( S = 1 + \lambda_{s}(q)^{-\xi} \) with \( q = (a/l_{PL})^{2} \). The equation of propagation can now be written as
\[ \ddot{h} + 2 \frac{\dot{a}}{a} \left( 1 - \frac{1}{2} \frac{\dot{\ln}(S)}{\ln(a)} \right) \dot{h} - (S^{2} \nabla^{2} + M^{2}(a)) h = 0, \tag{43} \]
with
\[ M^{2}(a) = \kappa \rho \frac{\rho_{c}}{a^{2}} \left( \frac{2}{3} \beta - \frac{\Phi^{2}}{D(q) a^{2}} \left( 1 - \frac{1}{6} \frac{\dot{D}}{D} \frac{a}{\dot{a}} \right) \right). \tag{44} \]

This can be usefully expressed as an equation for the spatial Fourier transform \( h_{k} \) of \( h \)
\[ \ddot{h}_{k} + 2 \frac{\dot{a}}{a} \left( 1 - \frac{1}{2} \frac{\dot{a} \dot{S}}{a S} \right) \dot{h}_{k} + (S^{2}k^{2} - M^{2}(a))h_{k} = 0. \tag{45} \]
The variables are changed according to \( \phi_{k} = h_{k}a/\sqrt{S} \), leading to a Schrödinger-like equation
\[ \ddot{\phi}_{k} + \left\{ S^{2}k^{2} - \left( \frac{\ddot{a}}{a} + M^{2}(a) - \frac{\dot{a} \dot{S}}{a S} + \frac{3}{4} \left( \frac{\dot{S}}{S} \right)^{2} - \frac{1}{2} \frac{\dot{S}}{S} \right) \right\} \phi_{k} = 0. \tag{46} \]

### IV. POWER SPECTRUM

The main question to address is to investigate if one correction, either holonomy or inverse-volume, dominates over the other as far as the production of gravitational waves during inflation is concerned. The system describing the dynamics is
\[ H^{2} = a^{2} \frac{\kappa}{3} \rho \left( S - \frac{\rho}{\rho_{c}} \right), \]
\[ 0 = \ddot{\Phi}_{k} + 2 \frac{\dot{a}}{a} \left( 1 - \frac{1}{2} \frac{\dot{a} \dot{\Phi}}{a \dot{\Phi}} \right) \dot{\Phi}_{k} + a^{2}DV,_{\Phi}(\Phi), \]
\[ 0 = \ddot{\phi}_{k} + \left\{ S^{2}k^{2} - \left( \frac{\ddot{a}}{a} + M^{2}(a) - \frac{\dot{a} \dot{S}}{a S} + \frac{3}{4} \left( \frac{\dot{S}}{S} \right)^{2} - \frac{1}{2} \frac{\dot{S}}{S} \right) \right\} \phi_{k}, \]
which is unfortunately much too difficult to be analytically solved. We therefore turn to the approach developed in [4, 6]. The background evolution is assumed to be classical \( (D \approx 1) \) with the scale factor given by the usual slow-roll approximation \( a(\eta) = l_{0} |\eta|^{-1-\epsilon} \). In this case, the effective Schrödinger equation \( \left[ \frac{d^{2}}{d\eta^{2}} + E_{k}(\eta) - V(\eta) \right] \phi_{k}(\eta) = 0 \), reads, to first order in \( \lambda_{s} \), as
\[ E_{k}(\eta) = S^{2}k^{2} = \left[ 1 + 2 \lambda_{s} \left( \frac{l_{PL}}{l_{0}} \right)^{s} |\eta|^{s(1+\epsilon)} \right] k^{2}, \tag{47} \]
\[ V(\eta) = \frac{2}{\eta^{2}} + \frac{6}{\kappa \rho_{c}} \frac{1}{l_{0}^{2}} |\eta|^{-2(1-\epsilon)} + \lambda_{s} \left( \frac{l_{PL}}{l_{0}} \right)^{s} \left[ -12 \frac{1}{\kappa \rho_{c}} \frac{1}{l_{0}^{2}} |\eta|^{s-2+2(1+\epsilon)+2} + s(1+2\epsilon)|\eta|^{s(1+\epsilon)-2} - \frac{1}{2} s(s-1+\epsilon(2s-1)) |\eta|^{s(1+\epsilon)-2} \right]. \tag{48} \]
To implement initial conditions, we consider the limit $\eta \to -\infty$ where the adiabatic vacuum holds. Of course, if higher order terms in $\lambda_s$ were to be included, the vacuum would not be the same anymore. However, we have checked that the adiabaticity condition would still be fulfilled in the relevant wavenumber range.

It is possible to solve analytically this equation, at least for one set of parameters: $s = 2$ and $\epsilon = 0$. It becomes

$$\frac{d^2 \phi_k}{d\eta^2} + \left[ \left( 1 + 2\lambda_s \left( \frac{l_{PL}}{l_0} \right)^2 \right) \frac{\gamma^2}{\eta^2} k^2 - \frac{2}{\eta^2} \left( 1 - \frac{3}{\kappa^2} \frac{\rho_c}{l_0^2} \right) \right] \phi_k = 0.$$  (49)

By some appropriate changes of variables, this equation can be turned into a Whittaker equation. The solution can be expressed with Kummer functions and the Wronskian condition $\phi_k \partial_\eta \phi_k^+ - \phi_k^+ \partial_\eta \phi_k = 16i\pi/M_{PL}$ allows one to normalize the modes. The field is then given at the end of inflation by

$$\phi_k(c) = \frac{2\sqrt{2}\pi}{M_{PL}(k\sqrt{2Z})^2} e^{\frac{i\pi}{4}} c \left( 1 + \mu - v, 1 + 2\mu, ic \right),$$  (50)

and the resulting primordial tensor power spectrum is

$$P_T(k) = \frac{16}{M_{PL}^2} k^{3-2\mu} H_0^2 (\sqrt{2Z})^{-2\mu} \left| \frac{\Gamma(b-1)}{\Gamma(a)} e^{-\frac{i\pi}{4}} \right|^2,$$  (51)

with

$$a = \frac{1}{2} + \mu - v = \frac{1}{2} + \frac{3}{4} \sqrt{1 + \frac{8\gamma^2 l_{PL}^2}{9 l_0^2}} + \frac{i}{\sqrt{32Zk^2}} \left( k^2 - Z \left( 1 - 4 \frac{\gamma^2 l_{PL}^2}{l_0^2} \right) \right),$$  (52)

$$b = 1 + 2\mu = 1 + \frac{3}{2} \sqrt{1 + \frac{8\gamma^2 l_{PL}^2}{9 l_0^2}},$$  (53)

$$v = -\frac{i}{\sqrt{32Zk^2}} \left( k^2 - Z \left( 1 - 4 \frac{\gamma^2 l_{PL}^2}{l_0^2} \right) \right),$$  (54)

where $Z = (l_{PL}/l_0)^2 \lambda_s$ and $\gamma^2 = 3/(\kappa \rho_c l_{PL}^2)$. The ultraviolet limit of this spectrum can be easily derived and leads to

$$P_T^{UV}(k) = 16\pi^3 \left( \frac{l_{PL}}{l_0} \right)^2 \left( 1 + \frac{3}{2} \frac{Z}{k^2} (1 - 4\epsilon) \right) k^{-\frac{i\pi}{4}},$$  (55)

with $\omega = \gamma^2 l_{PL}^2/l_0^2$. On the other hand, the infrared limit is given by

$$P_T^{IR}(k) = 16\pi^3 \left( \frac{l_{PL}}{l_0} \right)^2 (Z(1 - 4\omega))^{-\frac{3}{2}} k^3 e^{\frac{i\pi}{4}} \left( 1 - 4\omega \right).$$  (56)

Those results show that the $k \to +\infty$ limit of the power spectrum is in agreement with the general relativistic behavior with the addition of a slight tilt. The ultraviolet spectrum is nearly asymptotically scale invariant. This is not surprising as both the holonomy correction (encoded in the $k^{-\frac{i\pi}{4}}$ term) and the inverse-volume correction (encoded in the $(1 + \frac{1}{2} \frac{Z}{k^2} (1 - 4\epsilon))$ term), taken individually, lead to this behavior. The infrared limit is more interesting as, in this case, the holonomy and inverse-volume corrections lead to very different spectra. The result obtained here shows that the power spectrum is exponentially divergent, in exact agreement with the limit obtained with the inverse-volume correction alone. This proves that, under the standard inflationary background evolution hypothesis, the inverse-volume term strongly dominates over the holonomy one. This is to be contrasted with the background evolution in the very remote past where the holonomy term alone leads to the replacement of the singularity by a bounce.

V. CONCLUSION

This work derives the fully LQC-corrected equation of motion for gravitational waves. This equation is expressed in terms of cosmological variables and is explicitly solved for a given set of parameters in a standard inflationary background. It is shown that the spectrum remains exponentially infrared divergent, as for a pure inverse-volume correction. This reinforces the use of primordial gravitational waves as a strong probe of loop quantum gravity effects. The next step is naturally to build a fully consistent model which includes all the corrections for both the perturbations and the background.
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