A random matrix approach to decoherence

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Abstract

In order to analyze the effect of chaos or order on the rate of decoherence in a subsystem, we aim to distinguish effects of the two types of dynamics by choosing initial states as random product states from two factor spaces representing two subsystems. We introduce a random matrix model that permits to vary the coupling strength between the subsystems. The case of strong coupling is analyzed in detail, and we find no significant differences except for very low-dimensional spaces.

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1 Introduction

The discussion of decoherence phenomena has been centered around problems of quantum optics and atomic physics, which usually implied the use of smooth, often Gaussian, wave packets. Yet recent ideas on how to process quantum information changed that situation to some extent. Indeed in this context it is desirable to store a maximal amount of information in product states of qubits, such that it can be processed by unitary time evolution. During the process, the relevant states are changed in a complicated way. Hence, they will most certainly resemble random states much more than Gaussian wave packets. As the conservation of coherence is essential for any quantum information application, we have a clear interest in understanding decoherence of random states in such situations. The standard tool in this context is the study of “fidelity” which involves a change in the Hamiltonian, and a number of interesting results are available [1, 2, 3].

The fundamental question, how the integrability or chaoticity of the corresponding classical system i.e. “quantum chaos” affects the process of decoherence (see e.g. [4, 5, 6]), appears in a different light, because the behaviour of random states becomes relevant. We shall focus on this aspect, concentrating our interest on the effects of the chosen dynamics. At first glance, the semi-classical context implicit in this view may seem odd after the reference to quantum computing, but note that in many-body systems some effects of the chaotic dynamics tend to survive even for very low lying states [7], where semi-classical arguments certainly do not apply.

In a recent letter [8] we proposed to discuss decoherence in a random matrix model, in order to separate effects of the Hamiltonian from those of the choice of the initial pure state. Such a model is ideally suited to discuss the behaviour of random states. In this context we do not embrace the scheme of fidelity analysis [2], but simply follow decoherence as a function of time for a fixed Hamiltonian. We introduce decoherence by performing partial traces over a subsystem instead of considering non-unitary time evolution. What we show, could therefore more strictly be considered

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as a study of the progress of entanglement of a product wave function under an integrable or chaotic time evolution.

In section 2, we present our random matrix model, which allows for a variety of situations both with respect to the nature of the subsystems and the coupling strength between them. Note that models based on random matrix theory (RMT) have been used to discuss dissipation (see [1] and references therein), but these aspects will not be discussed here. In section 3 we shall analyze a special situation of strong coupling between the two subsystems. Though not particularly relevant for quantum computation, it has the advantage of a transparent mathematical structure which allows to obtain non-perturbative analytic results. These are compared to numerical calculations in section 4. As it turns out, the results are typical for other situations as well. In particular for weak coupling, analyzed numerically in the second part of section 4, we are lead to essentially the same conclusions as in the strong coupling case. Our approach to the problem with (largely) random wave functions actually separates rather nicely the effects of the dynamics from those of the functions considered, and in section 5 we shall conclude with some considerations on this point, as well as with a comparison of our findings with those of Refs. [2, 3].

2 The random matrix model

Properties of chaos and integrability of a classical system manifest themselves both in the spectrum and in the wave functions of the corresponding quantum system. While the former is invariant the latter are basis dependent. This does not mean that the latter are irrelevant in a semi-classical context; they certainly reflect the special features of the dynamics such as KAM tori or short periodic orbits. However, while localized wave packets may feel a strong influence on their dynamics, random wave functions would be rather insensitive to such localized features.

To construct our RMT model we start from the standard quantum chaos conjecture, that the classical ensembles [10,11] (e.g. the Gaussian orthogonal ensemble (GOE) for time reversal invariant systems) describe the universal features of a quantum system, whose classical counterpart is chaotic [12]. For classically integrable systems we expect a random quantum spectrum if we exclude harmonic oscillators [13]. The random spectrum can be combined with the orthogonal invariance to give the so called Poisson orthogonal ensemble (POE) [14], extending in this way the classical ensembles to the integrable case. This concept provides the ideal tool to describe the evolution of random wave functions, which is one of the main purposes of the present work.

Consider a Hamiltonian consisting of three terms $H = H^{(0)} + \lambda V^{(1,2)}$ with $H^{(0)} = h^{(1)} + h^{(2)}$, where the two terms of $H^{(0)}$ act on different degrees of freedom of the system; we may refer to them as the central system and the environment. Note that the $h^{(i)}$ may act on one or several degrees of freedom each, and the interaction $V^{(1,2)}$ may or may not induce chaos. Indeed the total system may be integrable and separable in a different set of coordinates.

Taking this Hamiltonian as a quantum operator we shall denote by $\mathcal{H}_1$ and $\mathcal{H}_2$ the Hilbert spaces of the central system and the environment with dimensions $n$ and $m$, on which $h^{(1)}$ and $h^{(2)}$ respectively act. The total Hamiltonian $H$ acts on the product space $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2$ with dimension $N = nm$. We write the basis states of $\mathcal{H}_1$ as kets with Latin letters such as $|i\rangle$ and those of $\mathcal{H}_2$ as kets with Greek letters such as $|\mu\rangle$. The states $|i,\mu\rangle = |i\rangle|\mu\rangle$ with indices conveniently written as pairs, form an eigenbasis of $H^{(0)}$. $H$ is diagonal in a different basis, which we enumerate by a single index $\alpha$. Thus $H_{i\mu,i',\mu'} = \sum_\alpha O_{i\mu,\alpha} E_\alpha O_{i',\mu',\alpha}$ where $E_\alpha$ denotes elements of the diagonal energy matrix $E$ and $O$ the orthogonal transformation between the two bases.

We distinguish between strong and weak interaction. The interaction strength is usually discussed in terms of the spreading width, which indicates the width of the distribution of the expansion coefficients of the eigenstates of $H$ in terms of those of $H^{(0)}$. This spreading width is a semi-classical quantity in the sense that it can be calculated from a phase space integral [15]. It is closely related to the width of the LDOS (the density of the eigenstates of $H_0$ expressed in terms of those of $H$) if the spreading width is small, i.e. if the eigenstates of $H_0$ contain a dominant amplitude in the eigenbasis of $H$ we clearly have a weak coupling case, where perturbative calculations should yield
the correct answer. A detailed study will be published elsewhere, but we shall give some numerical results in section 3.

We now determine appropriate matrix ensembles for the three terms in the Hamiltonian. As mentioned before, $h^{(1)}$ could describe the central system and $h^{(2)}$ the environment, e.g. the heat bath. In any case both $h^{(1)}$ and $h^{(2)}$ could pertain to either of the above ensembles, the GOE or the POE, and $V^{(1,2)}$ could typically be symmetric with independent Gaussian distributed matrix elements.

Finally we must specify the initial states. As mentioned before, we are interested in states of complicated structure, but require that they were initially pure. Hence we choose two random states in $H_1$ and $H_2$ and define the initial state as the product state of both. The probability measure for the random states is the orthogonally invariant one.

The situation first studied by Zurek and coworkers [4] could be simulated as a weak coupling case using a GOE with high level density for $h^{(2)}$ and a GOE or a POE respectively for $h^{(1)}$. However, we would have to use initially smooth wave packets. In this respect the states, whose evolution is discussed here, are entirely different.

3 Purity decay in the strong coupling regime

In the case of strong coupling $h^{(1)}$ and $h^{(2)}$ determine the factor spaces $H_1$ and $H_2$ only. Their spectral properties are irrelevant except for their relative spectral density in the energy region where the wave packet lives. The total Hamiltonian which is essentially equal to the interaction $V^{(1,2)}$ will be given by the GOE for chaotic systems and by the POE for integrable ones. Both ensembles are given by matrices of the form $OEO^T$, where $E$ is a diagonal energy matrix, and $O$ is a orthogonal matrix distributed according to the Haar measure of the orthogonal group. For the POE the distribution of the energies has complicated correlations and a semi circle density, while they are independently Gaussian distributed for the GOE.

As the level density has no relation to the chaoticity or integrability of the system, we shall unfold both spectra to have uniform density with the variance $\langle E^2 \rangle$ normalized to one in the ensemble average. As a consequence, the length of the spectrum is $2\sqrt{3}$ and the average level spacing is $d = 2\sqrt{3}/N$. One reason to fix the energy scale in this way is, that the variance is easy to control even in the weak coupling regime. Note that decoherence depends on the behaviour of the level density as a function of energy. Our random matrix model allows also to choose this behaviour differently, e.g. as dictated by the classical or quantum Hamiltonian.

The case of strong coupling both for integrable and chaotic systems was modeled in [15] with two-dimensional anharmonic oscillators, and the systems considered in [6] might be close to this domain. We shall also use equidistant "picket fence" spectra to complete the range of possible spectral correlations. The latter is important, because the spectra of low dimensional systems are much stiffer [16, 17] than the universal random matrix ensembles would predict.

The dimensions $n, m$ and $N$ of our Hilbert spaces are chosen finite. If we think of Hamiltonians with infinite spectra this truncation is the only reference we make to a choice of the initial wave function.

The entanglement of the two subsystems will be measured in terms of the purity defined as [3]

$$I(t) = \text{Tr}_1[\text{Tr}_2(\rho(t))]^2 = \text{Tr}_2[\text{Tr}_1(\rho(t))]^2.$$  \hspace{1cm} (1)

Here $\text{Tr}_1$ indicates the trace with respect to the first (Latin) index and $\text{Tr}_2$ the one over the second (Greek) index. The definition of the purity $I$ is related to the idempotency defect or linear entropy, defined as $1 - I$ [3].

We are interested in the time-evolution of an initially pure, i.e. non-entangled state. Therefore we construct the initial density matrix $\rho_{\mu,\nu,\mu',\nu'}(0)$ from a "product state" which is pure with respect to both pairs of indices, and we have $I(0) = \text{Tr}_1[\text{Tr}_2(\rho(0))]^2 = 1$. Denoting by $\Delta$ the diagonal matrix with entries $\Delta_\alpha = \exp[i\mu ]a_\alpha$ ($\hbar$ is set equal to one), we find in the basis of double indices
$$\rho(t) = O\Delta O^* \rho(0) O \Delta^* O^*$$ and by consequence

$$I(t) = \text{Tr}_1[\text{Tr}_2(O\Delta O^* \rho(0) O \Delta^* O^*) \text{Tr}_2(O\Delta O^* \rho(0) O \Delta^* O^*)] .$$

(2)

We take the averages involving energies and the averages involving states separately for different terms of the sum. In principle we do not need to specify \( \rho(0) \) in more detail. After performing the averages, the result must be independent of the two factor states used to construct \( \rho(0) \). This is due to the orthogonal invariance of the total Hamiltonian \( H \) used, and the invariance of the purity with respect to independent orthogonal transformations in the factor spaces. Hence, without loss of generality we may set \( \rho(0)_{11,11} = 1 \) with all other matrix elements being zero. With this initial condition we obtain the ensemble averaged purity

$$I(t) = \sum_{\alpha, \beta, \gamma, \delta} A_{\alpha, \beta; \gamma, \delta} B_{\alpha, \beta; \gamma, \delta}$$

(3)

in terms of the two averages

$$A_{\alpha, \beta; \gamma, \delta} = \langle \Delta_{\alpha} \Delta_{\beta}^* \Delta_{\gamma}^* \Delta_{\delta} \rangle = \langle \exp[i t(E_{\alpha} + E_{\gamma} - E_{\beta} - E_{\delta})] \rangle$$

(4)

$$B_{\alpha, \beta; \gamma, \delta} = \sum_{\mu, \nu, i, j} \langle O_{\mu, \alpha} O_{11, \alpha} O_{11, \beta} O_{j, \mu} O_{j, \nu} O_{11, \beta} O_{11, \gamma} O_{11, \delta} O_{11, \nu} \rangle .$$

(5)

The averages are connected only because one may force indices to be equal and thus reduce the other to a special case; as we shall see below five different terms exist. Note that we omitted the average symbols on the purity itself for convenience.

The obviously relevant time scales are the Heisenberg time \( d^{-1} = N/(2\sqrt{3}) \), and the inverse length of the spectrum, which is \( 1/(2\sqrt{3}) \). Using these we obtain four different regimes for the time evolution, because of the modulus operation inherent in the exponential \( \exp[i H t] \):

1) Short times, \( t \ll 1 \): Here, perturbation theory can be applied, and we will find the expected \( t^2 \) dependence with a factor given to leading order by the variance of the energy eigenvalues \( \langle E_2^2 \rangle = 1 \).

2) First filling of the unit circle at \( t = \pi/\sqrt{3} \): We will find a quadratic minimum for the purity with value \( I_{\text{min}} = 1/n + 1/m + 0(N^{-1}) \).

3) Long times, \( 2\pi/d \gg t \gg \pi/\sqrt{3} \): In this region, the spectrum has wended many times around the unit circle. This acts as a random number generator eliminating correlations. Hence we will obtain a result similar to the one for 2) though sub-leading terms may be different.

4) Poincaré recurrence at \( t = 2\pi/d \): At this point, a picket fence spectrum will cause exact recurrence, while even for a GOE spectrum the recurrence is essentially wiped out. Yet for low-dimensional systems with their long-range stiffness [16 17] and for models involving harmonic oscillators this part may well be important.

First we shall calculate the energy average \( A_{\alpha, \beta; \gamma, \delta} \). The result does not depend on the values of the indices, but only on whether certain indices are equal or not. If two indices of the energies coincide, we get either 0 if they have opposite signs or twice the energy if their signs are equal. It may readily be seen that five terms are possible (the rightmost equality is valid for a random spectrum only):

$$S_1(t) = \langle \exp[-it (E_1 - E_2 + E_3 - E_4)] \rangle = f^4(t)$$

$$S_2(t) = \langle \exp[-it (E_1 - E_2)] \rangle = f^2(t)$$

$$S_3(t) = \langle \exp[-it (2E_1 - E_2 - E_3)] \rangle = f(2t) f^2(t)$$

$$S_4(t) = \langle \exp[-it 0] \rangle = 1$$

$$S_5(t) = \langle \exp[-2it (E_1 - E_2)] \rangle = f^2(2t) .$$

(6)
The first equality in each line determines the special case at hand and the second one gives the result for a random spectrum. There \( f(t) = \sin(\sqrt{3}t)/\sqrt{3}t \) is the Fourier transform of the level density, which we assumed to be uniform. For GOE spectra the evaluation is more difficult, but some general considerations hold for any kind of spectrum. For long times all terms except \( S_1 \) go to zero. For short times, on the other hand, \( S_1 \) dominates because it has the largest weight. We now consider the four time regimes:

In the short time limit we expand the exponential. Due to the symmetry of the energy distribution, the linear terms in \( t \) vanish while quadratic ones survive. These are of two types. Each exponential associated with a given index has a quadratic term, and indices in the linear terms of two exponentials may coincide. This implies that we only need the well-known averages over monomials of fourth order in the group elements \([18]\) to obtain

\[
I(t) \sim 1 - 2(E_0^2) t^2 \left[ 1 - (n + m + 1)/(N + 2) \right].
\] (7)

In the last factor we seem to have a \( 1/N \) correction. Yet if \( n \) and \( m \) grow as \( \sqrt{N} \) the correction is of order \( 1/\sqrt{N} \). If one of the two dimensions is kept constant, the other becomes proportional to \( N \), and the second term is of order 1. Terms resulting from correlations of the energies are truly of order 1/\( N \) and were omitted.

The next time scale is that of the first filling of the unit circle, for which the first minimum of the function \( f^2(t) \) is reached. We have a complicated interplay of different terms and it seems that we would need the average over the orthogonal group \( B_{\alpha,\beta,\gamma,\delta} \) completely. However, this can be avoided using the following trick: For uniform density of the spectrum the energy eigenvalues are essentially the eigenphases of a circular ensemble. For the case of GOE fluctuations the corresponding ensemble is known as the circular orthogonal ensemble (COE) \([11]\), which is the ensemble of unitary symmetric \( N \times N \) matrices \( S \). This ensemble has a unique invariant measure. By identifying the energies with \( \alpha,\beta,\gamma,\delta \) and \( S \) with \( O_{\mu,\nu} \), we obtain

\[
I_{\infty} = \langle \text{Tr}_1 [\text{Tr}_2 (S \rho(0) S^*) \text{Tr}_2 (S \rho(0) S^*)] \rangle.
\] (8)

where \( S_{\mu,\nu} = \sum_\alpha O_{\mu,\alpha} \exp[iE_\alpha(2\pi/\Gamma)]O_{\nu,\alpha} \). The ensemble average originally given as one over states and spectra is thus given in terms of averages over four symmetric unitary COE matrices, two of which are complex conjugate. Such averages are calculated in \([10]\) and we obtain

\[
I_{\infty} \approx \frac{(n + m)N^2 + [3(n + m) + 2]N - 2(n + m - 1)}{N(N + 1)(N + 3)}.
\] (9)

As we shall see below this is slightly lower than the long time limit, while for the integrable (POE) case the long time limit and the value \( I_{\infty} \) coincide. We shall calculate the long time limit next.

For \( t \gg 1 \), the process of stretching and taking modulo \( 2\pi \) is a reasonably efficient randomizer for a fluctuating set of numbers with correlations such as a GOE spectrum. Therefore the eigenphases on this time scale are random both for the GOE and the POE. Thus only the fourth term survives, where the indices of energies in conjugate terms coincide. The energy dependence, and therefore the time dependence, drops out and we are left with averages over the orthogonal group. Only two-vector terms \( i.e. \) averages over elements from two rows of the matrix occur. These have been calculated \([20]\), and we find

\[
I_{\infty} = \frac{(n + m)N^3 + 3[4(n + m) + 3]N^2 + [35(n + m) + 57]N + 48}{(N + 1)(N + 2)(N + 4)(N + 6)}.
\] (10)

For the POE this result holds equally at time \( t = \pi/\sqrt{3} \), which we have discussed above for the GOE, though it will oscillate for larger times. The value for the GOE lies slightly below the one for POE. What we see is a weak signature of the correlation hole, characteristic of chaotic systems.

When evaluating the large \( N \) limit of these expressions, we have to take into account the \( n \) and \( m \) always occur in the form \((n + m)\). As \( N = nm \) goes to infinity, \( n + m \) can behave as any power
The average purity as a function of time for $m = n = 4$. The ensembles considered are: POE (long dashed line), GOE (solid line), and picket fence (short dashed line). The value $I_\infty$ is denoted by a thin dotted horizontal line; the value $I_{\text{min}}$ at the first minimum by a thin upright cross. We clip a small interval of the ordinate, to see the difference between GOE and POE in more detail. To observe the behaviour of the purity at short times, see Fig. 2.

$N^q$, with $1/2 \leq q \leq 1$. The two extremes are realized if $n$ and $m$ increase simultaneously, such that the ratio $n/m$ remains constant, or if one is fixed such that the other becomes proportional to $N$. Keeping this in mind we can still expand both expressions and find that the first minimum for the GOE case will always be slightly lower, but the difference will diminish as $1/N^2$, while purity itself will either diminish as $1/\sqrt{N}$ or reach a constant value $1/n$ or $1/m$. We can therefore conclude that the effects of spectral correlations are quite insignificant except for the smallest dimensions in both subsystems.

For times of the order of the Heisenberg time we expect a very different behaviour. More precisely, if $t = 2\pi/d = N\pi/\sqrt{3}$, we have exact revival in the case of the a picket fence spectrum, while there will be no particular signature for a random spectrum. In the case of GOE fluctuations one should consider the width of the $k$th neighbour spacing distribution. It is known to increase logarithmically. For $k = 1$ it has a width of $\approx 1.25d$ and for $k = 8$ its width already is $\approx 1.85d$ [21]. Due to the increasing broadness, we will see no Poincaré revival in the case of a GOE spectrum either. We should though note two facts: First we will find an additional partial revival at half the time mentioned for a picket fence spectrum, because of the terms $S_3(t)$ and $S_5(t)$, which contain eigen-energies with a factor two. Second, and more important, the long-range stiffness of spectra in low-dimensional systems [22] implies a saturation of the width of the $k$th neighbour spacing distribution and could therefore lead to recurrence effects.

4 Numerical results

First we consider a strong coupling situation with small dimensions, where all the effects we predict are most notable. For this purpose we choose the case $n = m = 4$ and show the corresponding
Figure 2: The average purity as in figure 1, but for the GOE case only (thick solid line). The average purity plus and minus the standard deviation (dotted lines), and the purity for the first member from the ensemble (dashed line).

time evolution of purity in Fig. 1. The results are displayed for spectra with random and GOE like fluctuations as well as picket fence spectra. The ensemble size is $2 \times 10^6$. As expected, at the beginning all curves are equal, and therefore we choose a scale where the quadratic dependence at the origin is not visible. The theoretical predictions are well fulfilled. The first minimum occurs at $t = \frac{\pi}{\sqrt{3}}$ for all spectral ensembles. Its depth for GOE correlations coincides with the COE result indicated by a cross, while both the GOE and the POE case take the asymptotic value given by a dotted line. In the POE case, the oscillations are essentially due to the form of $f(t) = \sin(\sqrt{3}t)/(\sqrt{3}t)$ (cf. eq. (6)). However, the GOE and picket fence cases follow this behaviour in a similar fashion. It is essentially a “diffraction” effect due to the sharp cutoff of the level density. E. g. for a semi-circle density the effect is attenuated, and for a Gaussian density it disappears. Therefore the experimental significance of these oscillations is limited to special situations, but the rapid decay on the time scale indicated does not depend on this fact. The first minimum is lower for GOE like fluctuations and for picket fence spectra than for random spectra, though the effect is only a few percent. The results coincide with our theoretical predictions both at the minimum and in the asymptotic region. This is also true for the recurrences, which are only seen for picket fence spectra both at $t = N\pi/\sqrt{3}$ and more weakly at $t = N\pi/(2\sqrt{3})$.

If we neglect oscillations, the rise of purity after the first minimum follows roughly that of the Fourier transform of the two-point function with appropriate scaling. This is not surprising, because we may expect that a cluster expansion of the correlations relevant for the difference from the random case is dominated by the two-point function. Yet it is important to note that the differences between the POE and GOE case lie within a few percents for such small dimensions. If $n$ or $m$ is increased any further, the differences become practically invisible. In [8] we show a figure for $n = m = 10$, which confirms this fact. We conclude that decoherence of random states is insensitive to chaos or integrability, except for very small systems.

Concerning noticeable differences between the integrable and chaotic cases, the news get even worse, if we study the fluctuations of the purity as a function of time. Fig. 2 shows the GOE
case for \( n = m = 4 \) as above but including the standard deviation of the purity plotted as a band around its average. For illustration we have also included the first member of the ensemble into the figure. The standard deviation is big, and it is fair to say, that any medium and long time features shown by a particular initial state in a chaotic system are most likely fluctuations. The picture for a random spectrum is essentially the same and thus the same conclusion holds for a random wave function in an integrable system.

Calculations with larger dimensions show that the fluctuations of the purity diminish, but not as fast as the difference resulting from spectral correlations. This is shown to diminish with \( N^{-2} \).

Finally let us discuss some numerical results for weak coupling: We construct our ensembles in the eigenbasis of \( H^{(0)} \). Then \( h^{(1)} \) and \( h^{(2)} \) are diagonal with elements taken from the spectrum of the GOE or the POE. In both cases, the spectra are unfolded to uniform density. The matrix elements of \( V^{(1,2)} \) are Gaussian distributed random variables. As the diagonal elements are set to zero, we have: \( \langle (V^{(1,2)}_{\alpha\beta})^2 \rangle = 1 - \delta_{\alpha\beta} \). It is assumed, that the diagonal elements are effectively absorbed into the Hamiltonians \( h^{(1)} \) and \( h^{(2)} \).

The resulting level density of the total system becomes quite complicated. However, it is still a simple task to obtain its variance:

\[
\langle E^2 \rangle = \frac{1}{N} \text{Tr} H^2 = \langle e^2 \rangle + \langle \mu^2 \rangle + \lambda^2 (N - 1)
\]  

(11)

where \( \langle e^2 \rangle \) and \( \langle \mu^2 \rangle \) denote the variances of the spectra of the subsystems, which are equal to one half. In what follows, the energy scale is again chosen such that \( \langle E^2 \rangle = 1 \).

While the randomness of the initial states was irrelevant in the previous examples of strong coupling (due to the orthogonal invariance of the total Hamiltonian), here it makes an important difference. In distinction to eigenstates of \( H^{(0)} \) or localized wave packets, we may expect that random initial states decohere faster. The reason is most easily understood looking at the LDOS. There a random initial state overlaps typically with many more eigenstates of the total system, as localized wave packets or eigenstates of \( H^{(0)} \). In the course of time this leads to faster decoherence.

Again small dimensions provide us with some surprises: Fig. 3 shows results for the dimensions \( n = m = 4 \). The average purity is displayed for three different combinations of spectra used in \( h^{(1)} \) and \( h^{(2)} \), namely for GOE–GOE, GOE–POE, and POE–POE. The coupling parameter is \( \lambda = 0.03 \). We find that for short and intermediate times the three curves have the same behaviour, but the asymptotic values differ: In contrast to the intuitive picture, that decoherence should be strongest in the chaotic case, the asymptotic value of the purity is lowest for the POE–POE case, which corresponds to two integrable subsystems. We get an intermediate value for the GOE–POE case, and the highest value is obtained for the GOE–GOE case, corresponding to two chaotic subsystems.

Yet again the significance of this difference is limited. This becomes clear by means of Fig. 4. There we show again the standard deviation of the purity in a similar fashion as in Fig. 2. This is done for the GOE–GOE case from Fig. 3. In addition the purity curve for a single Hamiltonian from the ensemble is plotted. As can be seen in the figure, the size of the fluctuations of this curve is very well described by the standard deviation.

We see that the band is extremely wide and much bigger than the difference in the purity between the different combinations of spectral statistics. Note that the corresponding figure for the other two cases from Fig. 3 (not shown) yield the same conclusion.

In Fig. 5 we show again the average purity for the three different combinations of spectral statistics in the subsystems, but here the dimensions are \( n = m = 10 \), and the coupling parameter is \( \lambda = 0.01 \). The smaller coupling parameter (as compared to the previous case, where \( n = m = 4 \)) is meant to compensate approximately the increase of the total dimension \( N \). Note that the variance of the spectrum of \( \lambda V^{(1,2)} \), which may serve as a measure for the strength of the perturbation, increases linearly with \( N \).

The results shown in Fig. 5, are very similar to the previous ones. However, the differences between the three curves at large times has diminished. Note that an increase of \( \lambda \) would further
Figure 3: The average purity as a function of time for $n = m = 4$, in the weak coupling case ($\lambda = 0.03$). It is plotted for three different combinations of ensembles for $h^{(1)}$ and $h^{(2)}$: GOE–GOE (solid line), GOE–POE (dashed line), and POE–POE (dotted line).

Figure 4: The average purity as in Fig. 3, but for the GOE–GOE case only (thick solid line). The average purity plus and minus the standard deviation (dotted lines), and the purity for the first member from the ensemble (dashed line).
Figure 5: The average purity as a function of time for $n = m = 10$ in the weak coupling case ($\lambda = 0.01$). It is plotted for three different combinations of ensembles for $h^{(1)}$ and $h^{(2)}$: GOE–GOE (solid line), GOE–POE (dashed line), and POE–POE (dotted line).

reduce these differences. The standard deviations of the purity curves also diminish (not shown), but still exceed the differences in the asymptotic average values. Hence, there is probably no chance to see an effect of the different spectral statistics of the subsystems without doing averages over rather large samples. Nevertheless a detailed analysis of the different decays in weak coupling is an interesting open question.

5 Conclusions

The influence of integrability versus chaos of the dynamics of a system on the entanglement of a random product wave function was considered. The type of dynamics was expressed in terms of spectral statistics, as this is the only invariant property of quantum dynamics, which is sensitive to the above distinction. We conclude that the effect of the type of dynamics on random states is small, and vanishes rapidly with the dimension of the Hilbert spaces involved. For quantum information processing, where random states are typical, this means that the distinction between chaos and integrability is not relevant to the rate of entanglement of two subsystems and thus for the decoherence in the central system.

Some effects have been seen for small systems, and these are interesting in themselves. In particular we find very large fluctuations for individual purity decays. One may argue that the fluctuations seen for initial packets in the chaotic area of phase space in $\lambda$, could be interpreted as fluctuations, but clearly a large statistical calculation would be necessary.

A comparison with the work of Prosen and Žnidarič $\lambda, \lambda$ is more difficult. As the Hamiltonian is disturbed we are looking at non-linear effects even in quantum mechanics. After all this is what the fidelity concept was designed for. In this context our result is important because, the reduction in the loss of fidelity, that is obtained in $\lambda, \lambda$ for chaotic dynamics, does not imply increased decoherence in the forward time evolution scheme with a fixed Hamiltonian, as one might have
inferred from the common concept, that chaos enhances decoherence.

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