ESTIMATES FOR INTEGRALS OF DERIVATIVES OF RATIONAL FUNCTIONS IN MULTIPLY CONNECTED DOMAINS IN THE PLANE

A. D. BARANOV AND I. R. KAYUMOV

Abstract. We obtain estimates for integrals of derivatives of rational functions in multiply connected domains in the plane. A sharp order of the growth is found for the integral of the modulus of the derivative of a finite Blaschke product in the unit disk. We also extend the results of E. P. Dolzhenko about the integrals of the derivatives of rational functions to a wider class of domains, namely, to domains bounded by rectifiable curves without zero interior angles, and show the sharpness of the obtained results.

1. Introduction

70 years ago S. N. Mergelyan [1] showed that there exists a bounded analytic function $f$ in the disk $D = \{ |z| < 1 \}$ such that

$$ I(f) := \int_D |f'(z)| \, dA(z) = \infty, $$

where $dA(z) = \frac{1}{\pi} dx dy$, $z = x + iy$.

This problem was further investigated by W. Rudin [2] who constructed an infinite Blaschke product

$$ B(z) = \prod_{k=1}^{\infty} \frac{|z_k|}{z_k - z} \frac{z_k - \overline{z}}{1 - z \overline{z}}, $$

such that $I(B) = \infty$ and, moreover, $\int_{0}^{1} |B'(re^{i\theta})| \, dr = \infty$ for a.e. $\theta \in [0, 2\pi]$. A similar, but more explicit example, was given by G. Piranian [3].

It is then natural to ask what happens if $B$ is a finite Blaschke product of degree $n$? It is obvious, that, for any fixed $n$, the quantity $I(B)$ is bounded, but it cannot be uniformly bounded with respect to $n$, since any bounded function in $D$ is a locally uniform limit of finite Blaschke products. We find the sharp order of growth for such integrals. Namely, we have

Theorem 1. Let $B$ be a finite Blaschke product of degree $n$. Then

$$ I(B) \leq \pi (1 + \sqrt{\log n}). $$

Key words and phrases. Rational function, conformal map, Blaschke product, Hardy space, John domain.

A. D. Baranov was supported by Ministry of Science and Higher Education of the Russian Federation, agreement No 075-15-2021-602. I. R. Kayumov was supported by Ministry of Science and Higher Education of the Russian Federation, agreement No 075-15-2019-1619.
On the other hand, there exists an absolute constant \( c > 0 \) such that for any \( n \in \mathbb{N} \) there exists a finite Blaschke product of degree \( n \) satisfying \( I(B) \geq c(1 + \sqrt{\log n}) \).

The proof of sharpness of this inequality is based on subtle results of N. G. Makarov [4] and R. Bañuelos and C. N. Moore [5] on boundary behaviour of functions from the Bloch space.

It should be noted that there exists a vast literature dealing with the membership of the derivatives of the Blaschke products to various functional spaces, e.g., Bergman-type spaces (see [6, 7, 8] and the references therein). However, most of these results concern infinite products and the conditions are formulated in terms of their zeros.

Since a Blaschke product is a bounded rational function in the unit disk, the problem about the estimates of the derivatives of Blaschke products is related to a more general question about the integrals of bounded rational functions studied for the first time by E. P. Dolzhenko [9] for sufficiently smooth domains. We will say that a curve belongs to the class \( K \) if it is a closed Jordan curve with continuous curvature \( k(s) \) satisfying a Hölder condition as the function of the arc length \( s \). Let \( G \) be a finitely connected domain whose boundary curves belong to the class \( K \). Assume that \( 1 \leq p \leq 2 \) and let \( R \) be a rational function of degree at most \( n \) with the poles outside \( \overline{G} \). Dolzhenko [9, Theorem 2.2] showed that there exists a constant \( C \) depending only on the domain \( G \) and on \( p \) such that

\[
(1.2) \quad \int_G |R'(w)|^p \, dA(w) \leq C n^{p-1} \| R \|^p_{H^\infty(G)}, \quad p \in (1, 2],
\]

\[
(1.3) \quad \int_G |R'(w)| \, dA(w) \leq C \ln(n + 1) \| R \|_{H^\infty(G)}.
\]

Here we denote by \( H^\infty(G) \) the space of all bounded analytic functions in \( G \), and \( \| f \|_{H^\infty(G)} = \sup_{w \in G} |f(w)| \).

Later, inequalities for the derivatives of rational functions (mainly in the disk) were studied in the papers by V. V. Peller [10], S. Semmes [11], A. A. Pekarskii [12, 13], V. I. Danchenko [14, 15] and by many other authors (see, e.g., [16, 17, 18, 19, 20]). A short proof of the Dolzhenko inequalities for the case of the disk when the \( H^\infty \)-norm is replaced by the weaker \( BMOA \)-norm can be found in [19].

In the present article the inequalities (1.2) and (1.3) are proved under substantially weaker restrictions on the domain, namely, under the condition that the domain has no zero interior angles (more precisely, for the John class domains – see the definition in §3).

**Theorem 2.** Let \( G \) be a finitely connected John domain with the rectifiable boundary and let \( 1 \leq p \leq 2 \). Then there exists a constant \( C > 0 \), depending on the domain \( G \) and on \( p \), such that for any rational function \( R \) of degree at most \( n \) the inequalities (1.2) and (1.3) hold.

The sharpness of (1.2) is seen already on the simplest example of the function \( R(z) = z^n \) in the disk (obviously, we can consider polynomials as a special case of rational functions with the pole at infinity). The question about sharpness of the estimate (1.3) in the conditions of Theorem 2 remains open. However, it turns out that under some additional
regularity of the domain \( G \) inequality (1.3) can be improved.

**Theorem 3.** Let \( G \) be a simply connected domain such that \( \varphi' \in H^2 \), where \( \varphi \) is the conformal map of the disk \( \mathbb{D} \) onto \( G \). Then there exists a constant \( C > 0 \) depending on the domain \( G \) such that for any rational function \( R \) of degree at most \( n \) one has

\[
\int_G |R'(w)|^2 dA(w) \leq C \ln(n+1)\|R\|_{H^\infty(G)}.
\]

As follows from Theorem 1, the dependence on \( n \) in this inequality is sharp.

Finally, we give a statement for the case \( p > 2 \). Here \( q \) is the conjugate exponent, i.e., \( 1/p + 1/q = 1 \).

**Theorem 4.** Let \( G \) be a bounded simply connected domain and put \( G_\rho = \{ z \in G : \text{dist} (z, \partial G) > \rho \} \). Then for any rational function \( R \) of degree at most \( n \) and \( p > 2 \) one has

\[
\|R'\|_{A^p(G_\rho)} = \left( \int_{G_\rho} |R'(w)|^p dA(w) \right)^{1/p} \leq n^{1/p} \rho^{1/p-1/q} \|R\|_{H^\infty(G)}.
\]

In [9] inequality (1.5) was established for domains of the class K, but, as our result shows, no restrictions on the regularity of the domain are required.

In [20] another generalization of the Dolzhenko inequality for the case \( p > 2 \) was obtained for the rational functions in the disk. Let \( R \) be a rational function of degree at most \( n \) whose poles lie in the complement of the disk \( \{|z| < 1 + \rho\} \). The following inequality follows directly from [19, Theorem 8.2]:

\[
\|R'\|_{A^p(\mathbb{D})} \leq C(p) n^{1/q} \rho^{1/p-1/q} \|R\|_{BMOA};
\]

here \( BMOA \) denotes the analytic space of functions of bounded mean oscillation in the disk. It is interesting to note that the dependence on \( n \) in Theorem 4 is substantially weaker (since in this case the function \( R \) is assumed to be bounded on a larger set).

In §5 more general inequalities are obtained for the weighted norms of the derivatives of rational functions, where the weight is given as some power of the distance to the boundary of the domain.

A suitable toolbox for the study of such inequalities is provided by the theory of the Hardy spaces. For \( p > 0 \) the Hardy space \( H^p \) is the set of all analytic functions in \( \mathbb{D} \) satisfying \( \|f\|_{H^p} < \infty \), where

\[
\|f\|_{H^p} := \sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p dt.
\]

Note that for \( p \geq 1 \) the last quantity defines a norm with respect to which \( H^p \) is a Banach space.
2. ESTIMATE FOR THE INTEGRAL OF THE MODULUS OF THE DERIVATIVE OF A FINITE BŁASCHKE PRODUCT

In the proof of Theorem 1 we will use the following simple lemma.

**Lemma 1.** Assume that the function \(g(z) = \sum_{k=0}^{\infty} b_k z^k\) is analytic in the disk \(\mathbb{D}\). If \(\|g\|_\infty \leq 1\) and \(p(z) = \sum_{k=0}^{n} b_k z^k, n \geq 2\), then there exists an absolute constant \(C_0\) such that

\[|p(z)| \leq C_0, \quad |z| \leq 1 - 2 \log n/n,\]

and

\[|g'(z) - p'(z)| \leq C_0, \quad |z| \leq 1 - 2 \log n/n.\]

**Proof.** Since \(|b_k| \leq 1\) and \(|z|^k \leq 1/n^2\) for \(|z| \leq 1 - 2 \log n/n\) and \(k \geq n\), the function \(|g(z) - p(z)|\) admits a uniform estimate for \(|z| \leq 1 - 2 \log n/n\), and the first estimate follows. Clearly,

\[\sum_{k=n}^{\infty} (k + 1)|b_{k+1} z^k| \leq \frac{|z|^n}{(1 - |z|)^2} + \frac{n|z|^n}{1 - |z|} \leq \text{const}, \quad |z| \leq 1 - 2 \log n/n,
\]

and the second inequality is proved.

**Proof of Theorem.** We use the following well-known facts:

\[\int_0^{2\pi} |B'(re^{it})|dt \leq 2\pi n, \quad r \in [0, 1],\]

for any finite Blaschke product of degree at most \(n\) and

\[\int_{\{s < |z| < 1\}} |B'(z)|^2 (1 - |z|^2) \, dA(z) = \sum_{n=1}^{\infty} \frac{n}{n+1} |a_n|^2 \leq \|f\|_{H^2}^2
\]

for any function \(f(z) = \sum_{n \geq 0} a_n z^n\) in the Hardy space \(H^2\).

Let \(s \in [0, 1]\). We have

\[\int_{\{s < |z| < 1\}} |B'(z)| \, dA(z) = \int_0^{2\pi} \int_s^1 |B'(re^{it})| r \, dr \, dt \leq 2\pi n \int_s^1 r \, dr = \pi n (1 - s^2).\]

In the remaining part we apply the Cauchy–Schwarz inequality:

\[\int_{\{0 < |z| \leq s\}} |B'(z)| \, dA(z)
\]

\[\leq \left( \int_{\{0 < |z| \leq s\}} |B'(z)|^2 (1 - |z|^2) \, dA(z) \right)^{1/2} \left( \int_{\{0 < |z| \leq s\}} \frac{dA(z)}{1 - |z|^2} \right)^{1/2}
\]

\[\leq \sqrt{\pi} \sqrt{2\pi} \int_0^s \frac{r \, dr}{1 - r^2} = \pi \sqrt{\log \frac{1}{1 - s^2}}.
\]
Thus,
\[ I(B) \leq \pi n(1 - s^2) + \pi \sqrt{\log \frac{1}{1 - s^2}}. \]

Taking \( s^2 = 1 - 1/n \) we obtain \([\text{[]}]\).

The estimate from below can be obtained by the methods based on the Makarov law of the iterated logarithm \([4]\). Recall that the Bloch class \( \mathcal{B} \) consists of functions analytic in \( \mathbb{D} \) with finite seminorm \( \|f\|_\mathcal{B} = \sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| \). In \([5]\) R. Bañuelos and C.N. Moore, answering a question of N.G. Makarov and F. Przytycki, constructed a function \( f(z) = \sum_{k=1}^{\infty} a_k z^k \) in the Bloch class such that its asymptotic entropy admits the lower bound
\[ \liminf_{r \to 1^-} \frac{\sum_{k=1}^{\infty} |a_k|^2 r^{2k}}{\log \frac{1}{1-r}} > 0, \]

whereas for all \( \zeta \) with \( |\zeta| = 1 \) one has
\[ \limsup_{r \to 1^-} \frac{f(r\zeta)}{\sqrt{\log \frac{1}{1-r} \log \log \frac{1}{1-r}}} = 0. \]

Moreover, in \([5]\) p. 852–853 a sequence of polynomials
\[ p_n(z) = \sum_{k=4}^{4n+1} a_k z^k = \sum_{j=1}^{n} b_j(z), \quad \text{where} \quad b_j(z) = \sum_{k=4^j}^{4^{j+1}-1} a_k z^k, \]
is constructed such that \( \|b_j\|_\infty \leq 1 \),
\[ \sum_{k=1}^{4^{n+1}-1} |a_k|^2 \geq c \log m, \quad \|p_n\|_H^\infty \leq C \sqrt{\log m}, \]

where \( m = \deg p_n = 4^{n+1} - 1 \) and \( C, c > 0 \) are some absolute positive constants.

It is not difficult to deduce from \( \|b_j\|_\infty \leq 1 \) that \( \sup_n \|p_n\|_B < \infty \). Indeed, by the Schwarz lemma \( |b_j(rz)| \leq r^{4^j} \), whereas, by the classical Bernstein inequality, \( |b'_j(rz)| \leq 4^j r^{4^j} \). It is well known (and easy to show) that \( \sum_{j=1}^{\infty} 4^j r^{4^j} \leq C_1/(1 - r^2) \) for some constant \( C_1 > 0 \) and, thus, \( \sup_n \|p_n\|_B < \infty \). Without loss of generality we may assume that \( \|p_n\|_B \leq 1 \).

Let \( r = 1 - 1/m \). Then, for some absolute constants \( C', c' > 0 \),
\[ c' \log \frac{1}{1-r} \leq \sum_{k=1}^{m} |a_k|^2 r^{2k} \leq 2 \int_{|z|<r} |p'_n(z)|^2 (1 - |z|^2) \, dA(z) \leq C' \int_{|z|<r} |p'_n(z)| \, dA(z). \]

Now put \( q_n = p_n/(C \sqrt{\log m}) \). Then \( \|q_n\|_\infty \leq 1 \) and
\[ \int_{|z|<1-1/m} |q'_n(z)| \, dA(z) \geq c_1 \sqrt{\log m} \]

for some absolute constant \( c_1 > 0 \). Since
\[ \int_{1-2 \log m/m < |z| < 1-1/m} |q'_n(z)| \, dA(z) \leq \int_{1-2 \log m/m < |z| < 1-1/m} \frac{dA(z)}{1 - |z|^2} = O(\log \log m), \]
we have, for some $c_2 > 0$,

\begin{equation}
\int_{|z| < 1 - 2\log m/m} |q_n'(z)| \, dA(z) \geq c_2 \sqrt{\log m}.
\end{equation}

Let $B$ be a Blaschke product of degree at most $m + 1$ such that its first $m$ Taylor coefficients coincide with respective coefficients of the polynomial $q_n$. By Lemma 1,

$$|q_n'(z) - B'(z)| \leq 2C_0, \quad |z| \leq 1 - 2\log m/m.$$ 

Hence, it follows from (2.2) that

$$\int_D |B'(z)| \, dA(z) \geq c_3 \sqrt{\log m}$$

for some absolute constant $c_3$. Theorem 1 is proved.

3. Estimates for integrals of rational functions

Recall that a finitely connected domain $\Omega$ is said to be a John domain if there exists a constant $C > 0$ such that any points $a, b \in \Omega$ can be connected by a curve $\gamma$ in $\Omega$ with the following property: for any $x \in \gamma$,

$$\min \left( \text{diam } \gamma(a, x), \text{diam } \gamma(x, b) \right) \leq C \text{dist}(x, \partial \Omega).$$

Here $\gamma(a, x)$ and $\gamma(x, b)$ denote the corresponding subarcs of $\gamma$. For equivalent definitions and properties of John domains see, e.g., \cite{21, 22}. Essentially, this definition means that the domain has no zero interior angles. In particular, a domain is a John domain if it satisfies the cone condition: one can touch each boundary point from inside of the domain by some sufficiently small triangle with fixed angles.

In what follows we will essentially use the following property of simply connected John domains: if $\varphi$ is the conformal map of $\mathbb{D}$ onto a simply connected John domain, then

\begin{equation}
|\varphi'(z)| \leq \frac{C}{(1 - |z|)^{\alpha}}
\end{equation}

for some $\alpha \in (0, 1)$ and $C > 0$ (see \cite{22}, pp. 96–100).

In the proof of Theorem 2 we will use the following simple lemma:

**Lemma 2.** Let $g$ be a bounded and at most $n$-valent function in $\mathbb{D}$. Then

$$\frac{1}{2\pi} \int_0^{2\pi} |g(re^{it})|^2 \, dt \leq \frac{n}{1 - r} \|g\|_{H^\infty(\mathbb{D})}^2.$$ 

**Proof.** Let $g(z) = \sum_{k=0}^\infty a_k z^k$. Then

$$\frac{1}{2\pi} \int_0^{2\pi} |g(re^{it})|^2 \, dt = \sum_{k=1}^\infty k^2 |a_k|^2 r^{2k} \leq \frac{1}{1 - r} \sum_{k=1}^\infty k |a_k|^2.$$
Here we used the elementary inequality \( kr^{k-1} (1-r) \leq 1, \ k \geq 1, \ r \in [0, 1) \). Since \( g \) is at most \( n \)-valent in \( \mathbb{D} \), we have, in virtue of the classical area theorem,

\[
\sum_{k=1}^{\infty} k|a_k|^2 \leq n\|g\|_{H^\infty(\mathbb{D})}^2.
\]

**Proof of Theorem 2.** Assume first that the domain \( G \) is bounded. Without loss of generality we may assume that \( G \) is a simply connected domain with a rectifiable boundary. Indeed, making smooth cuts (and controlling the angles) one can easily represent our domain as a finite union of simply connected John domains.

Let \( w = \varphi(z) \) be a conformal map of \( \mathbb{D} \) onto \( G \). Since the boundary of \( G \) is rectifiable, we have \( \varphi' \in H^1 \). Also, \( \varphi' \) satisfies inequality (3.1).

By the change of the variable,

\[
\int_G |R'(w)|^p \, dA(w) = \int_{\mathbb{D}} |R'(\varphi(z))|^{p}|\varphi'(z)|^2 \, dA(z) = \int_{\mathbb{D}} |(R \circ \varphi)'(z)|^p|\varphi'(z)|^{2-p}dA(z).
\]

It is obvious, that for \( p = 2 \) the last integral does not exceed \( n\|R\|_{H^\infty(G)}^2 \), since the function \( R \circ \varphi \) is at most \( n \)-valent in the disk \( \mathbb{D} \).

Let us split the last integral into integrals the integrals over the set \( \{|z| \leq r_n\} \) and over the set \( \{r_n < |z| < 1\} \), where \( r_n = 1 - \frac{1}{(n+1)^p} \) and the number \( K > 0 \) is to be chosen later.

**Estimate of the integral over the set \( \{|z| \leq r_n\} \).** Let \( M = \|R\|_{H^\infty(G)} \). Set

\[
J := \int_{\{|z| \leq r_n\}} |(R \circ \varphi)'(z)|^p|\varphi'(z)|^{2-p}dA(z).
\]

For \( p = 1 \) we use the estimate \( (1 - |z|^2)|R \circ \varphi)'(z)| \leq M \). We have

\[
J \leq \frac{M}{\pi} \int_0^{r_n} \frac{1}{1-r} \int_0^{2\pi} |\varphi'(re^{it})| \, dt \, dr \leq 2\|\varphi'\|_{H^1} M \int_0^{r_n} \frac{dr}{1-r} = 2K\|\varphi'\|_{H^1} \log(n+1) M.
\]

In the case \( 1 < p < 2 \) consider separately the integrals over the sets \( \{|z| \leq 1 - \frac{1}{n+1}\} \) and \( \{1 - \frac{1}{n+1} < |z| \leq r_n\} \). Since \( \varphi' \in H^1 \), we have \( \varphi' \in H^{2-p} \) and \( \|\varphi'\|_{H^{2-p}} \leq \|\varphi'\|_{H^1} \). Hence,

\[
\int_{\{|z| \leq 1 - \frac{1}{n+1}\}} |(R \circ \varphi)'(z)|^p|\varphi'(z)|^{2-p}dA(z) \leq 2\|\varphi'\|_{H^1}^{2-p} \left( \int_0^{1 - \frac{1}{n+1}} \frac{M^p}{(1-r)^{p}} \, dr \right) \leq 2\|\varphi'\|_{H^{2-p}(p-1)^{-1}(n+1)^{p-1}M^p}.
\]

To estimate the integral over the set \( \{1 - \frac{1}{n+1} < |z| \leq r_n\} \) we use the Hölder inequality with exponents \((2-p)^{-1}\) and \((p-1)^{-1}\):

\[
J \leq 2 \int_{1 - \frac{1}{n+1}}^{r_n} \left( \frac{1}{2\pi} \int_0^{2\pi} |(R \circ \varphi)'(re^{it})|^{\frac{p}{(p-1)}} \, dt \right)^{p-1} \left( \frac{1}{2\pi} \int_0^{2\pi} |\varphi'(re^{it})| \, dt \right)^{2-p} \, dr.
\]
Using successively the inequality \((1 - |z|^2)|(R \circ \varphi)'(z)| \leq M\) and Lemma 2 we get
\[ J \leq 2\|\varphi'\|_{H^1}^{2-p} M^{p-2(p-1)} \int_{1-\frac{1}{n+1}}^{r_n} \frac{1}{(1-r)^{2-p+\frac{\alpha}{n+1}}} \left( \frac{1}{2\pi} \int_0^{2\pi} |(R \circ \varphi)'(re^{i\theta})|^2 d\theta \right)^{p-1} dr \]
\[ \leq 2\|\varphi'\|_{H^1}^{2-p} M^p n^{p-1} \int_{1-\frac{1}{n+1}}^{r_n} \frac{dr}{1-r} \]
\[ = 2\|\varphi'\|_{H^1}^{2-p} M^p n^{p-1} \]

**Estimate of the integral over the set** \(\{r_n < |z| < 1\}\). In this case the argument applies to all \(p \in [1, 2]\). Choose \(\delta\) such that \(0 < \delta < 1 - p/2\). Then \(2 - p - \delta \in (0, 1)\). Let \(\beta\) be the exponent conjugate to \((2 - p - \delta)^{-1}\). One has, by (3.1),
\[ I := \int_{r_n < |z| < 1} |(R \circ \varphi)'(z)|^p |\varphi'(z)|^{2-p} dA(z) \]
\[ \leq C^{\delta} \int_{r_n < |z| < 1} \frac{|(R \circ \varphi)'(z)|^p |\varphi'(z)|^{2-p-\delta}}{(1-|z|)^{\alpha \delta}} dA(z) \]
\[ \leq 2C^{\delta} \int_{r_n}^{1} \frac{1}{(1-r)^{\alpha \delta}} \left( \frac{1}{2\pi} \int_0^{2\pi} |(R \circ \varphi)'(re^{i\theta})|^p d\theta \right)^{1/\beta} \left( \frac{1}{2\pi} \int_0^{2\pi} |\varphi'(re^{i\theta})| d\theta \right)^{2-p-\delta} dr. \]
Note that it follows from \(\delta < 1 - p/2\) that \(p\beta > 2\). Applying the estimate \((1 - |z|^2)|(R \circ \varphi)'(z)| \leq M\) and Lemma 2, we get
\[ I \leq 2C^{\delta} \|\varphi'\|_{H^1}^{2-p-\delta} M^{\frac{p\beta - 2}{p\beta}} \int_{r_n}^{1} \frac{1}{(1-r)^{\alpha \delta + \frac{p\beta - 2}{p\beta}}} \left( \frac{1}{2\pi} \int_0^{2\pi} |(R \circ \varphi)'(re^{i\theta})|^2 d\theta \right)^{1/\beta} dr \]
\[ \leq 2C^{\delta} \|\varphi'\|_{H^1}^{2-p-\delta} M^p n^{1/\beta} \int_{r_n}^{1} \frac{dr}{(1-r)^{\alpha \delta + \frac{p\beta - 2}{p\beta}}} \]
\[ = 2C^{\delta} \|\varphi'\|_{H^1}^{2-p-\delta} M^p n^{1/\beta} \int_{r_n}^{1} \frac{dr}{(1-r)^{\alpha \delta + p - \frac{1}{\beta}}} \]
It remains to notice that \(\alpha \delta + p - \frac{1}{\beta} = \alpha \delta + p - (1 - (2 - p - \delta)) = 1 - (1 - \alpha) \delta\), whence
\[ (3.2) \quad I \leq 2(1 - \alpha)^{-1} \delta^{-1} C^{\delta} \|\varphi'\|_{H^1}^{2-p-\delta} M^p n^{1/\beta} (1-r_n)^{(\alpha - 1)\delta}. \]
If we fix \(\delta \in (0, 1 - p/2)\) and choose a sufficiently large \(K\) in \(r_n = 1 - \frac{1}{(n+1)\kappa}\), we conclude that \(I \leq C^{\delta} \|\varphi'\|_{H^1}^{2-p-\delta} M^p\) (and even \(o(1)\) as \(n \to \infty\)).

We now consider the case when \(\infty \in G\). It is clear that such domain can be represented as a union (possibly with an intersection) of the complement of a disk of sufficiently large radius with a simply connected bounded John domain. The statement is already proved for bounded simply connected domains, while for the complement of a disk it follows from the results of Dolzhenko cited above. Theorem 2 is proved.
4. Proofs of Theorems 3 and 4

Proof of Theorem 3. As in the proof of Theorem we set \( r_n = 1 - \frac{1}{n+1} \), where \( K > 0 \).
Since \( \varphi' \in H^2 \subset H^1 \) and the condition (3.1) is satisfied with \( \alpha = 1/2 \), one can use the estimate (3.2) for the integral over the set \( \{ r_n < |z| < 1 \} \) established in the proof of Theorem 2. For a sufficiently large \( K \) this integral is uniformly bounded over \( n \) (and even tends to zero as \( n \to \infty \)).
Thus, it suffices to estimate
\[
J := \int_{0 < |z| \leq r_n} |(R \circ \varphi)'(z)| |\varphi'(z)| dA(z).
\]
By the Cauchy–Schwarz inequality,
\[
J \leq \left( \int_{0 < |z| \leq r_n} (1 - |z|) |(R \circ \varphi)'(z)|^2 dA(z) \right)^{1/2} \left( \int_{0 < |z| \leq r_n} \frac{|\varphi'(z)|^2}{1 - |z|} dA(z) \right)^{1/2}
\leq M \left( \int_{0 < |z| \leq r_n} \frac{|\varphi'(z)|^2}{1 - |z|} dA(z) \right)^{1/2} \leq \sqrt{2} M \| \varphi' \|_{H^2} \left( \int_0^{r_n} \frac{dr}{1 - r} \right)^{1/2}
= \sqrt{2} M \| \varphi' \|_{H^2} \sqrt{K \ln(n + 1)}.
\]
Theorem 3 is proved.

Proof of Theorem 4. Let \( \varphi \) be a conformal map of \( \mathbb{D} \) onto \( G \) and let \( D_\rho = \varphi^{-1}(G_\rho) \).
Since \( \rho \leq (1 - |z|^2)|\varphi'(z)|, z \in D_\rho \), we have
\[
\int_{G_\rho} |R'(\zeta)|^p dA(\zeta) = \int_{D_\rho} |(R \circ \varphi)'(z)|^p |\varphi'(z)|^{2-p} dA(z)
\leq \rho^{2-p} \int_{D_\rho} |(R \circ \varphi)'(z)|^p (1 - |z|^2)^{p-2} dA(z)
\leq \rho^{2-p} M^{p-2} \int_{D_\rho} |(R \circ \varphi)'(z)|^2 dA(z) \leq \rho^{2-p} n M^p.
\]
At the last step we used the fact that \( R \circ \varphi \) covers each point of the disk of radius \( M \) with multiplicity at most \( n \). Theorem 4 is proved.

5. Weighted inequalities of Dolzhenko and Peller type

As a natural generalization of the Dolzhenko inequalities one can consider weighted
integrals of derivatives of rational functions. Similar inequalities were studied extensively in
the setting of the Bergman (or Besov) spaces. E.g., a well-known inequality by V.V. Peller [10] states that for a rational function \( R \) of degree \( n \) with poles outside \( \mathbb{D} \) one has
\[
\| R \|_{B^1_p} \leq C n^{1/p} \| R \|_{BMOA}.
\]
where $B_p^{1/p}$ is the Besov space, $p > 0$, $C = C(p)$. In particular, for $1 < p < \infty$,

$$
\int_{\mathbb{D}} |R'(z)|^p (1 - |z|)^{p-2} dA(z) \leq C n \|R\|_{H^\infty}^p.
$$

Various proofs and generalizations of this inequality can be found in [11, 12, 13, 19].

Using the methods of §3 one can obtain more general weighted estimates where the weight equals to some power of the distance to the boundary. To formulate the corresponding result, we set, for a bounded domain $G \subset \mathbb{C}$ and $z \in G$,

$$
d_G(z) := \text{dist}(z, \partial G).
$$

For $p \geq 1$, $\beta \in \mathbb{R}$ and a function $f$ analytic in $G$ put

$$
I_{p,\beta}(f) := \int_G |f'(\zeta)|^p d_G^\beta(\zeta) dA(\zeta)
$$

(in general, the quantity $I_{p,\beta}(f)$ can be infinite). We are interested in the estimates of the form

$$
I_{p,\beta}(R) \leq C \Psi(n) \|R\|_{H^\infty}^p,
$$

which hold for all rational functions $R$ of degree at most $n$ with poles outside $\overline{G}$ and with a constant $C$, depending on $G$, $p$ and $\beta$, but not on $n$ and $R$. Here $\Psi$ is some function depending on $n$ only. It is easy to see that such estimates are possible only for $\beta \geq p - 2$; it is seen already from the example $G = \mathbb{D}$ and rational fractions $R(\zeta) = \frac{1}{\zeta-\lambda}$ that for $\beta < p - 2$ the integral $I_{p,\beta}(f)$ does not admit the estimate depending only on $n$, one has also to take into account the distance from the poles of $R$ to $\partial G$.

To simplify the notations, in what follows we write $X(R, n) \lesssim Y(R, n)$, if $X(R, n) \leq CY(R, n)$ with a constant $C$, depending only from $G$, $p$ and $\beta$, but not on $n$ and $R$.

**Theorem 5.** Let $G$ be a simply connected bounded domain, $\varphi$ is the conformal map of $\mathbb{D}$ onto $G$, $p \geq 1$, $\beta \geq p - 2$. The following estimates hold true.

1. If $\beta > p - 1$ and $\varphi' \in H^\gamma$ for some $\gamma > 1$, then $I_{p,\beta}(R) \lesssim \|R\|_{H^\infty}^p$, i.e., the dependence on $n$ disappears.

2. If $\beta = p - 1$, $1 \leq p < 2$ and $\varphi' \in H^{\frac{2}{p-1}}$, then

$$
I_{p,\beta}(R) \lesssim (\log n)^{1-\frac{p}{2}} \|R\|_{H^\infty}^p.
$$

If $\beta = p - 1$, $p \geq 2$ and $\varphi' \in H^\infty$, then $I_{p,\beta}(R) \lesssim \|R\|_{H^\infty}^p$.

3. If $p - 2 \leq \beta < p - 1$, $p \geq 2$, $\varphi' \in H^1$ and $G$ is a John domain, then

$$
I_{p,\beta}(R) \lesssim n^{p-1-\beta} \|R\|_{H^\infty}^p.
$$

The dependence on $n$ in the inequalities in Theorem 5 is sharp already for the case of the unit disk. In statement 3 the optimal growth is attained on $R(z) = z^n$, while the sharpness of the inequality in statement 2 can be shown by considering the Bañuelos–Moore construction (as $R$ one can take a polynomial or a Blaschke product).
Note that statement 3 of the theorem does not cover the case $p - 2 \leq \beta < p - 1$ and $1 < p < 2$. In this case it would be sufficient to prove the following analogue of Peller’s inequality:

$$
\int_{\mathbb{D}} |(R \circ \varphi)'(z)|^p (1 - |z|)^{p-2} dA(z) \lesssim n ||R||_{H^{\infty}(G)},
$$

where $G = \varphi(\mathbb{D})$ is a John domain and $R$ is a rational function of degree at most $n$ with the poles outside $\overline{G}$. However, we do not know whether this inequality holds true.

**Proof.** Put $M = ||R||_{H^{\infty}(G)}$. Let us make the change of the variable $\zeta = \varphi(z)$. Taking into account that $d_G(\zeta) \leq |\varphi'(z)| (1 - |z|^2)$, we obtain

$$
I_{p,\beta}(R) \lesssim \int_{\mathbb{D}} |(R \circ \varphi)'(z)|^p |\varphi'(z)| (1 - |z|)^\beta dA(z).
$$

Statement 3 follows from Theorem 2. Indeed, $1 < p - \beta \leq 2$ and, using the inequality $|(R \circ \varphi)'(z)|(1 - |z|) \leq M$, we obtain

$$
I_{p,\beta}(R) \leq M^\beta \int_{\mathbb{D}} |(R \circ \varphi)'(z)|^{p-\beta} |\varphi'(z)|^{2-(p-\beta)} dA(z) \lesssim n^{p-\beta-1} M^p.
$$

Let us prove statement 1. The quantity $d_G(z)$ is bounded, so it suffices to prove the statement for $\beta \in (p - 1, p - 2 + \gamma]$. For such $\beta$ it follows from the inequality $p - \beta < 1$ and the inclusion $\varphi' \in H^\gamma \subset H^{2-p+\beta}$ that

$$
I_{p,\beta}(R) \lesssim M^p \int_0^{2\pi} \left( \int_0^1 |\varphi'(re^{it})|^{2-p+\beta} dr \right)^{\frac{1}{p-\beta}} \lesssim M^p.
$$

At the first step we used the inequality $|(R \circ \varphi)'(z)|(1 - |z|) \leq M$.

Consider the most interesting statement 2: $\beta = p - 1$. Let $1 \leq p < 2$. Put $s = 1 - \frac{1}{n}$. Applying the Hölder inequality with exponents $\frac{2}{p}$ and $\frac{2}{2-p}$ and inequality (2.11), we get

$$
\int_{\{|z| \leq s\}} |(R \circ \varphi)'(z)|^p |\varphi'(z)| (1 - |z|)^{p-1} dA(z)
\lesssim \left( \int_{\{|z| \leq s\}} |(R \circ \varphi)'(z)|^2 (1 - |z|) dA(z) \right)^{\frac{p}{2}} \left( \int_{\{|z| \leq s\}} |\varphi'(z)|^{\frac{2}{p}} dA(z) \right)^{1 - \frac{p}{2}}
\lesssim M^p \left( \int_0^s \left( \int_0^{2\pi} |\varphi'(re^{it})|^2 dt \right) \frac{r dr}{1-r} \right)^{\frac{1}{2}} \lesssim (\log n)^{1 - \frac{p}{2}} M^p.
$$

It remains to estimate the integral over the set $\{s < |z| < 1\}$:

$$
\int_{\{s < |z| < 1\}} |(R \circ \varphi)'(z)|^p |\varphi'(z)| (1 - |z|)^{p-1} dA(z)
\leq M^{p-1} \int_{\{s < |z| < 1\}} |(R \circ \varphi)'(z)| \cdot |\varphi'(z)| dA(z)
\lesssim M^{p-1} \left( \int_{\{s < |z| < 1\}} |(R \circ \varphi)'(z)|^2 dA(z) \right)^{1/2} \left( \int_{\{s < |z| < 1\}} |\varphi'(z)|^2 dA(z) \right)^{1/2} \lesssim M^p.
$$
In the last inequality we used the fact that
\[
\int_{\{s<|z|<1\}} |(R \circ \varphi)'(z)|^2 dA(z) \lesssim nM^2,
\]
since \(R \circ \varphi\) covers the disk of radius \(M\) with multiplicity at most \(n\), as well as the inclusion \(\varphi' \in H^2\).

The case \(p \geq 2\) is trivial:
\[
I_{p,p-1}(R) \lesssim M^{p-2} \int_{\mathbb{D}} |(R \circ \varphi)'(z)|^2 (1 - |z|) dA(z) \lesssim M^p,
\]
i.e., the quantity \(I_{p,p-1}(R)\) is uniformly bounded over \(R\) and \(n\).

Theorem 5 is proved.

References

[1] S. N. Mergelyan, “On an integral connected with analytic functions”, Izvestiya Akad. Nauk SSSR. Ser. Mat., 15 (1951), no. 5, 395–400 (Russian).
[2] W. Rudin, “The radial variation of analytic functions”, Duke Math. J., 22 (1955), no. 2, 235–242.
[3] G. Piranian, “Bounded functions with large circular variation”, Proc. Amer. Math. Soc., 19 (1968), no. 6, 1255–1257.
[4] N. G. Makarov, “Probability methods in the theory of conformal mappings”, Algebra i Analiz, 1 (1989), no. 1, 3–59; English transl.: Leningrad Math. J., 1 (1990), no. 1, 1–56.
[5] R. Bañuelos, C. N. Moore, “Mean growth of Bloch functions and Makarov’s law of the iterated logarithm”, Proc. Amer. Math. Soc., 112 (1991), 851–854.
[6] A. Aleman, D. Vučetić, “On Blaschke products with derivatives in Bergman spaces with normal weights”, J. Math. Anal. Appl., 361 (2010), no. 2, 492–505.
[7] D. Protas, “Blaschke products with derivative in function spaces”, Kodai Math. J., 34 (2011), no. 1, 124–131.
[8] D. Protas, “Derivatives of Blaschke products and model space functions”, Canad. Math. Bull., 63 (2020), no. 4, 716–725.
[9] E. P. Dolzhenko, “Rational approximations and boundary properties of analytic functions”, Mat. Sb. (N.S.), 69(111) (1966), no. 4, 497–524 (Russian).
[10] V. V. Peller, “Hankel operators of class \(S_p\) and their applications (rational approximation, Gaussian processes, the problem of majorizing operators)”, Mat. Sb. (N.S.), 113(155) (1980), no. 4(12), 538–581; English transl.: Math. USSR-Sb., 41 (1982), no. 4, 443–479.
[11] S. Semmes, “Trace ideal criteria for Hankel operators, and applications to Besov spaces”, Integr. Equat. Oper. Theory, 7 (1984), no. 2, 241–281.
[12] A. A. Pekarskii, “Inequalities of Bernstein type for derivatives of rational functions, and inverse theorems of rational approximation”, Mat. Sb., 24(166):4(8) (1984), 571–588; English transl.: Math. USSR-Sb., 52 (1985), no. 2, 557–574.
[13] A. A. Pekarskii, “New proof of the Semmes inequality for the derivative of the rational function”, Mat. Zametki, 72 (2002), no. 2, 258–264; English transl.: Math. Notes, 72 (2002), no. 2, 230–236.
[14] V. I. Danchenko, “An integral estimate for the derivative of a rational function”, Izvestiya Russ. Akad. Nauk SSSR. Ser. Mat., 43 (1979), no. 2, 277–293; English transl.: Math. USSR-Izv., 14 (1980), no. 2, 257–273.
[15] V. I. Danchenko, “Several integral estimates of the derivatives of rational functions on sets of finite density”, Mat. Sb., 187 (1996), no. 10, 33–52; English transl.: Sb. Math., 187 (1996), no. 10, 1443–1463.
[16] E. M. Dyn’kin, “Inequalities for rational functions”, J. Approx. Theory, 91 (1997), 349–367.
[17] E. M. Dyn’kin, “Rational functions in Bergman spaces”. In: V. P. Havin, N. K. Nikolski (eds.) Complex Analysis, Operators, and Related Topics, Operator Theory: Advances and Applications, vol. 113, pp. 77–94. Birkhäuser, Basel, 2000.

[18] A. Baranov, R. Zarouf, “A Bernstein-type inequality for rational functions in weighted Bergman spaces”, Bull. Sci. Math., 137 (2013), no. 4, 541–556.

[19] A. Baranov, R. Zarouf, “The differentiation operator from model spaces to Bergman spaces and Peller type inequalities”, J. Anal. Math., 137 (2019), no. 1, 189–209.

[20] A. Baranov, R. Zarouf, “$H^\infty$ interpolation and embedding theorems for rational functions”, Integr. Equat. Oper. Theory, 91 (2019), article number 18.

[21] O. Martio, J. Sarvas, “Injectivity theorems in plane and space”, Ann. Acad. Sci. Fenn. Ser. A I Math., 4 (1979), no. 2, 383–401.

[22] Ch. Pommerenke, Boundary Behaviour of Conformal Maps, Springer-Verlag New-York, 1992.

Saint Petersburg State University
Email address: anton.d.baranov@gmail.com

Kazan Federal University
Email address: ikayumov@kpfu.ru