Dynamics of two-dimensional time-periodic Euler fluid flows

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Abstract: This paper investigates the dynamics of time-periodic Euler flows in multi-connected, planar fluid regions which are “stirred” by the moving boundaries. The classical Helmholtz theorem on the transport of vorticity implies that if the initial vorticity of such a flow is generic among real-valued functions in the $C^k$-topology ($k \geq 2$) or is $C^\omega$ and nonconstant, then the flow has zero topological entropy. On the other hand, it is shown that for constant initial vorticity there are stirring protocols which always yield time-periodic Euler flows with positive entropy. These protocols are those that generate flow maps in pseudo-Anosov isotopy classes. These classes are a basic ingredient of the Thurston-Nielsen theory and a further application of that theory shows that pseudo-Anosov stirring protocols with generic initial vorticity always yield solutions to Euler’s equations for which the sup norm of the gradient of the vorticity grows exponentially in time. In particular, such Euler flows are never time-periodic.

§1 Introduction. The motion of fluids provides intuition and basic terminology for Dynamical Systems theory, and the ideas and results of low dimensional dynamics continue to have many important applications in Fluid Mechanics. In this paper we explore a particular case of a general question connecting Fluid Mechanics and Dynamical Systems:

Question 1: Are fluid flows typical dynamical systems, or do they have special, distinguishing dynamical features?

This question is obviously vague and can be made precise in a number of ways depending on the dimension, fluid model and time dependence of the velocity fields. In this paper we study the dynamics of the Poincaré maps of time-periodic Euler flows (Question 2 in §3).

There is a great deal known Question 1 for steady (i.e. time-independent) Euler flows in two and three dimensions. Much of this work is summarized in the book of Arnol’d and Khesin ([AK]). In the interval since its publication significant progress has been made by Etnyre and Ghrist on steady 3D Euler flows (see [GK] for a summary). In particular, they have shown that every nonsingular $C^\omega$-Euler flow on the three sphere has a periodic orbit that is unknotted ([EG]). This is a feature of Euler flows which is not shared by every volume preserving flow and so provides a distinguishing dynamical feature for the class of 3D steady Euler flows.

It is a common heuristic that the dynamics of surface diffeomorphisms share many of the features of 3D flows, and so it is natural from the Dynamical System’s standpoint to study the time $T$-maps of $T$-periodic 2D Euler flows. The lower dimension results in topological simplification, but the time periodicity adds a new feature. In addition,
we shall primarily be concerned with the situation in which the region occupied by the fluid changes during its evolution as the fluid is “stirred” by various protocols. Thus the time $T$-flow maps can be in different isotopy classes, and the insights provided by the Thurston-Nielsen theory of these classes will be central to our analysis.

This paper views Fluid Mechanics from the Dynamical System’s prospective. From this admittedly skewed point of view, Fluid Mechanics is the study of families of diffeomorphisms (the flow maps) which have the special property that their vector fields satisfy equations such as Navier-Stokes, Euler or Stokes. These equations are, of course, derived from physical principles, and the resulting systems are studied because they have been found to agree very well with the behavior of real fluids. Question 1 asks whether these physically based conditions restrict the possible dynamics a fluid can manifest as compared to a general dynamical system.

For example, in many physical situations the amount of expansion or compression of a fluid during its evolution is very small and can be neglected. Thus fluid maps are often assumed to be area or volume preserving. As is well known, the dynamics of area-preserving systems differ from general dynamical systems in many respects, for example, they have no attractors. Thus in formulating the corresponding precise version of Question 1 in two dimensions one should ask whether the dynamics of fluids are distinguished from those of a general area-preserving system.

The version of Question 1 studied here assumes that the fluid motion satisfies the Euler equations. These equations are based on the assumption of a fluid with no viscosity and thus there is no energy dissipation. While we restrict attention here to $C^k$-classical solutions, lesser regularity and various kinds of weak solutions have been an object of much study and are of central importance. Classical results about Euler fluid motions and their consequences for the dynamics are given in §3 and §4. Theorem 2 states a fundamental result of Helmholtz and Kelvin from the mid 19th century. It says that Euler fluid motions are characterized by their preservation of vorticity (or curl) and circulation integrals. These distinguishing features are metric dependent, but nonetheless provide a key to essential topological properties of Euler fluid motions. Proposition 4 shows that as a consequence of the preservation of vorticity, a time-periodic Euler fluid motion whose initial vorticity is a typical $C^r$-function or else is $C^\omega$ and nonconstant must have zero topological entropy (the $C^\omega$ case is equivalent to Remark 4 in [BS]).

Amongst the Euler flows with non-generic vorticity those with zero vorticity are of particular interest. These systems are much studied as a consequence of their mathematical tractability and because it is often argued that a fluid at rest has zero vorticity and so if the vorticity is conserved, it will be zero for all time. Using standard potential theory, Proposition 3 shows that systems with constant vorticity and periodic stirring protocols always give rise to periodic Euler fluid motions. Since conditions that ensure periodic solutions to the Euler equations are very rare, this result is useful in guaranteeing the existence of at least one interesting class of time $T$-Euler flow maps.

Using these results in the real analytic case yields a dichotomy that is somewhat similar to that of Arnol’d for steady 3D Euler flows (see II §1 in [AK]). For 2D time-periodic Euler flows, if the vorticity is nonconstant, then the dynamics are “integrable” and the entropy is zero. One may have chaotic dynamics in the constant vorticity case,
but only if the fluid is stirred

The next section, §5, contains results with the opposite conclusion from that of Proposition 3. As a consequence of the Thurston-Nielsen theory of surface automorphisms, Theorem 7 shows that for generic initial vorticity there are large classes of periodic stirring protocols which never give rise to periodic Euler fluid motions. Further, these stirring protocols result in exponential growth of the sup norm of the gradient of the vorticity. As a consequence of the preservation of vorticity one expects this general type of behavior in any chaotic Euler flow with generic smooth vorticity. The attractive feature of Theorem 7 is that the topology of the stirrer motion allows one to get concrete results on the exponential growth.

It is anticipated that most of this paper’s readers will be familiar with Dynamical Systems and not Fluid Mechanics, so we have made some attempt to include basic Fluid Mechanics from a Dynamical Systems point of view. A second reason for including statements and proofs of classic results is that the case under consideration here, multi-connected fluid regions with moving boundaries, is not usually discussed in the standard texts.

There are many first class books on mathematical Fluid Mechanics. The books [AK], [Ba], [C], [G], [MB], [MP], and [Se] provide a sample with a variety of emphases. For a survey of Fluid Mechanics and mathematical structures with a point of view similar to this paper, see [Bd2]. The motion of planar fluids stirred by topologically complex protocols has been studied in [BAS1], [FCB], and [V].

§2 Fluid motions and Hamiltonian systems. In this paper a fluid motion, \((M_t, \phi_t)\), consists of a smooth family of smooth planar fluid regions, \(M_t\), with the Euclidean metric and a smooth one-parameter family of diffeomorphisms, \(\phi_t : M_0 \to M_t\), with \(\phi_0 = \text{id}\). The diffeomorphisms should be thought of as describing the evolution of fluid particles: the particle at the point \(z \in M_0\) at time 0 is at the point \(\phi_t(z) \in M_t\) at time \(t\).

In our fluid region families the outer boundary of \(M_t\) will always be a fixed, smooth simple closed curve \(C_0\). In addition, \(m\) disks, each of radius \(\epsilon\), (the stirrers) are excluded from the fluid region, with the case \(m = 0\) being allowed. The evolution of the fluid region is determined by the rigid motion of the excluded disks. The stirrers always remain circles of radius \(\epsilon\) and their centers move along paths given by \(\alpha_i : [0, \infty) \to D\), for \(i = 1, \ldots, m\). It is assumed that no two disks collide and no disk collides with the outer boundary. Because of the slip boundary conditions for Euler fluid motions, rotations of the boundary do not affect the fluid motion and so will be ignored. The collection of paths \((\alpha_i) = (\alpha_1, \ldots, \alpha_m)\) is called a stirring protocol. Since the outer boundary remains fixed, specifying a stirring protocol specifies the fluid regions \(M_t\), and vice versa. The inner boundary circles at time \(t\) are denoted \(C_{it}\) and have velocity \(\dot{\alpha}_i\). To simplify notation we let \(\alpha_0\) be a constant function, so the velocity of \(C_0\) is identically zero: \(\dot{\alpha}_0 \equiv 0\).

The velocity or vector field \(X\) of the fluid motion at the point \(\phi_t(z)\) at time \(t\) is defined by

\[
X(\phi_t(z), t) := \frac{\partial \phi_t}{\partial t}(z).
\]

Note that in contrast to what is usual in Dynamical Systems, the vector field \(X\) may, and usually does, depend on time. Thus \(\phi_t\) is a flow in the usual Dynamical System sense if and only \(X\) is time-independent. In that case the fluid motion is called steady. A family
of smooth vector field vector fields $X(\cdot, t)$ on $M_t$ can always be integrated to give a family of diffeomorphisms $\phi_t$. The fact that the fluid at time $t$ is contained solely in $M_t$, i.e. the image of $\phi_t$ is $M_t$, is equivalent to the boundary conditions

$$X \cdot n_i = \dot{\alpha}_i \cdot n_i,$$

(2)

where $n_i$ is the unit normal to the boundary circle $C_{it}$. Without comment in the sequel we will go back and forth between a family of velocity fields satisfying these conditions and the corresponding diffeomorphisms and will use both $(M_t, \phi_t)$ and $(M_t, X)$ to denote the fluid motion.

At this point the notion of a fluid motion is very general and it is not required to satisfy any particular equation. The first restriction on a fluid motion comes from assuming that the fluid neither expands nor contracts during its evolution. A fluid motion is called incompressible if each diffeomorphism $\phi_t$ in the fluid motion is area-preserving, or equivalently, if $\text{div}(X) \equiv 0$.

In two dimensions area preservation is closely related to being a Hamiltonian system. Recall that the family $\phi_t$ is called Hamiltonian if there exist a family of real-valued functions $H_t$, so that the the velocity field $X$ satisfies $X(\cdot, t) = J \nabla H_t$ for each $t$, where $J =\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. In this case each diffeomorphism $\phi_t$ is also called Hamiltonian. In many cases, a system being Hamiltonian forces additional dynamical properties beyond those of area preservation. (There is a vast literature arising from the Arnol’d conjecture and subsequent developments. See, for example, [MS] and [FH].) However, the distinction between area preservation and Hamiltonian on surfaces is associated with the flux across generators of homology that are not associated with boundary curves. Thus the distinction disappears in genus zero. This is the content of first part of the next lemma. We include it here as it introduces a standard construction of the stream function which we shall need later, and because the motion of the boundary curves introduce an element not contained in the usual results. The second part of the lemma concerns a related question: Is every area preserving diffeomorphism $f : M_0 \to M_0$ (which may permute the inner boundaries) Hamiltonian?

**Lemma 1:**

(a) The fluid motion $(M_t, \phi_t)$ is Hamiltonian if and only if it is incompressible.

(b) If $f : M_0 \to M_0$ is an area-preserving diffeomorphism, then there is a Hamiltonian fluid motion $(M_t, \phi_t)$ with $f = \phi_1$.

**Proof:** For part (a), first note that Hamiltonian systems are always area preserving. For the converse, at each fixed time, on the boundary component $C_{it}$ the vector $\dot{\alpha}_i$ is a constant and so by direct calculation,

$$0 = \oint_{C_{it}} \dot{\alpha}_i \cdot d\mathbf{n}_i.$$  

(3)

Thus if for each $t$ we let $Y = -JX$, then $\text{curl}(Y) = \text{div}(X) = 0$ and using (2) and (3), $0 = \oint_{C_{it}} Y \cdot d\mathbf{r}$. So for each $t$, $Y$ is a curl-free field which has zero circulation around each boundary curve. Thus we may fix $z_0$ and unambiguously define

$$\Psi(z, t) = \int_{z_0}^{z} Y \cdot d\mathbf{r}$$

(4)
which will satisfy $Y = \nabla \Psi$. Therefore, $X = J \nabla \Psi$, which is to say that $\Psi$ is the Hamiltonian that generates $\phi_t$.

For part (b), first note that since $M_0$ is a disk with holes removed it is standard that we may find a 1-periodic protocol $(\alpha_i)$ with corresponding regions $M_i$ and a fluid motion $\hat{\psi}_t : M_0 \to M_1$ with $\hat{\psi}_1$ isotopic to $f$ on $M_0 = M_1$ ([Bi]). In addition, since for each $t$ the region $M_t$ has the same area as $M_0$, using a theorem of Moser ([M]) we may find a diffeomorphism $g_t : M_t \to M_t$, isotopic to the identity, with $g_t(\hat{\psi}_t \nu_0) = \nu_t$, where $\nu_t$ is the Euclidean area form on $M_t$. Note that Moser’s theorem is stated for closed manifolds, but the proof also works for manifolds with boundary and yields a family $g_t$ which is smooth in $t$. If we let $\psi_t = g_t \circ \hat{\psi}_t$, then $(M_t, \psi_t)$ is an area-preserving fluid motion.

By construction, $\psi_1^{-1} \circ f$ is area preserving and isotopic to the identity on $M_0$. By a similar argument using Moser’s theorem (cf. remark 1.4.C in [P]) we may find a family of area preserving diffeomorphisms $h_t : M_0 \to M_0$ with $h_0 = id$ and $h_1 = \psi_1^{-1} \circ f$. The incompressible and thus Hamiltonian fluid motion required for (b) is $\phi_t = \psi_t \circ h_t$. $\square$

In Fluid Mechanics the Hamiltonian $\Psi$ is called the stream function. The role of Lemma 1 in this paper is to show that in formalizing Question 1 for our fluid regions it suffices to consider area-preserving systems; we do not need to say that we are looking for Euler models of a general time-periodic Hamiltonian system.

§3 Euler fluid motions and diffeomorphisms. In its simplest interpretation, the Euler equation is Newton’s second law, $F = ma$, applied to each fluid particle with a force resulting from the gradient of the pressure. As is common, we assume a uniform fluid density of one. What we call an Euler fluid motion here should properly be called an incompressible, constant density Euler flow. We omit the adjectives as understood and use the terminology “fluid motion” to avoid confusion with the Dynamical Systems notion of flow. Euler flow is also sometimes called perfect or ideal fluid flow. From this point onward all fluid motions are required to have velocity fields which are $C^k$ for $k \geq 3$.

The Euler equations for the fluid motion: An incompressible fluid motion $(M_t, \phi_t)$ is called Euler if there is a smooth family of smooth functions $p_t : M_t \to \mathbb{R}$ (the pressure) so that $p_t$ and $\phi_t$ satisfy

$$\frac{\partial^2 \phi_t}{\partial t^2}(z) = -\nabla p_t(\phi_t(z)).$$

It is more conventional to express the Euler equations in term of the velocity field. The acceleration term becomes the material derivative of the vector field defined as

$$\frac{DX}{Dt} := \frac{\partial^2 \phi_t}{\partial t^2} = \frac{\partial (X \circ \phi_t)}{\partial t} = \frac{\partial X}{\partial t} + \nabla_X X,$$

where $\nabla_X X$ is the directional or covariant derivative of $X$ in the direction of $X$.

The Euler equations for velocity fields: A fluid motion $(M_t, X)$ is a solution to the incompressible Euler equation if there is a smooth family of smooth functions $p_t : M_t \to \mathbb{R}$
(the pressure) so that $p_t$ and $X$ satisfy

$$\frac{DX}{Dt} = -\nabla p_t$$

$$\text{div}(X) = 0$$

$$X \cdot n_i = \dot{\alpha}_i \cdot n_i$$ on the boundary.

It is easy to check that a fluid motion is incompressible Euler if and only if its velocity field is a solution to the incompressible Euler equations. Since the fluid regions, $M_t$, are determined by the stirring protocol, $(\alpha_i)$, we shall also say that the Euler fluid motion $(M_t, X)$ is an Euler solution which is compatible with the given stirring protocol.

The basic questions of existence, uniqueness, and regularity for solutions to the Euler equations have been much studied (see [C], [MB], [MP], or [Kh] for expositions of various results). For the case of interest here, multiply connected planar regions with moving boundary, existence and uniqueness of a global weak and classical solutions were obtained by Kozono [Ko] and He and Hsiao [HH].

The class of dynamical systems to be studied here are the time $T$-diffeomorphisms of time-periodic Euler fluid motions. A stirring protocol is said to be $T$-periodic if after time $T$, the set of stirrers return to their initial position, and then the stirring process repeats. Note that since the stirrers are indistinguishable, we only require the set of stirrers to come back to itself, not each individual stirrer. In terms of the paths $\alpha_i$, this definition says that $\alpha_i(t + T) = \alpha_{\sigma(i)}(t)$ for some permutation $\sigma$ of $\{1, 2, \ldots, m\}$.

**Definition of Euler Diffeomorphisms:** A diffeomorphism $f : M_0 \to M_0$ is called an Euler diffeomorphism, if $f = \phi_T$ for an incompressible Euler fluid motion $(M_t, \phi_t)$ whose velocity field $X(z, t)$ is $T$-periodic and compatible with a $T$-periodic stirring protocol.

Note that the definition requires not just a periodic protocol, but the compatible Euler velocity field must be periodic as well. The usual definition of a Hamiltonian diffeomorphism requires that $f = \phi_1$ for a Hamiltonian family $\phi_t$. This in, in fact, equivalent to requiring $f = \phi_1$ for a family $\phi_t$ that is generated by a family of Hamiltonians $H_t$ that are 1-periodic, $H_{t+1} = H_t$, and so the corresponding vector fields $X(z, t)$ are also 1-periodic (see, for example, page 37 in [P]). On the other hand, as a consequence of Theorem 7 there are diffeomorphisms, $f$, that are the time 1-map of an Euler fluid motion, but that Euler fluid motion cannot have a periodic velocity field. The general question of which area preserving diffeomorphisms are the time 1-maps of perhaps aperiodic Euler fluid motions is very difficult, but see Brenier [Br] and Shnirelman [Sh] for reports on significant progress. The requirement that an Euler diffeomorphism be generated by a periodic velocity field is included in our definition to insures that the iterates of $f$ describe the dynamics of the fluid motion.

We can now give a precise statement of the version of Question 1 of interest here.

**Question 2:** Given an area preserving diffeomorphism $g : M_0 \to M_0$, is there an Euler diffeomorphism $f$ that is topologically conjugate to $g$?

As is usual in Dynamical Systems, we only require $f$ and $g$ to have the same dynamics up to topological change of coordinates. This is essential because being an Euler fluid
motion is not even preserved under smooth changes of coordinates because the acceleration, or equivalently, the covariant derivative, depends on the metric.

The next theorem collects classical results that are basic to understanding the dynamics of Euler fluid motions. The vorticity of a velocity field is its curl and is denoted \( \omega_t(z) := \text{curl}(X(z,t)) \). In two dimensions the vorticity is a real-valued function and \( \omega_t = -\Delta \Psi \), where \( \Psi \) is the stream function of \( X(z,t) \). Recall that the push forward of a scalar field \((0\text{-form}) \, s \) under a diffeomorphism \( f \) is \( f_*s = s \circ f^{-1} \).

**Theorem 2:** (Helmholtz-Kelvin) An incompressible fluid motion \((M_t, \phi_t)\) with velocity field \( X \) and vorticity \( \omega_t \) is Euler if and only if its vorticity is passively transported,

\[
\phi_t_* \omega_0 = \omega_t, \tag{5}
\]

and circulations around all smooth simple closed curves \( C \) are preserved under the flow,

\[
\frac{d}{dt} \oint_{\phi_t(C)} X \cdot dr = 0. \tag{6}
\]

**Proof.** First observe that standard two-dimensional vector identities yield

\[
\frac{DX}{Dt} = \frac{\partial X}{\partial t} + \nabla \left( \frac{\|X\|^2}{2} \right) - \omega JX
\]

Since \( \text{curl}(JX) = \text{div}(X) = 0 \), taking the curl yields,

\[
\text{curl} \left( \frac{DX}{Dt} \right) = \frac{\partial \omega}{\partial t} + \nabla \omega \cdot X = \frac{\partial \omega(\phi_t(z),t)}{\partial t}. \tag{7}
\]

Now if \( X \) is Euler, then \( \text{curl}(\frac{DX}{Dt}) = -\text{curl}(\nabla p) = 0 \), and so by (7), \( \omega \) is constant on orbits, which is equivalent to (5).

The transport theorem for simple closed curves \( C \) is

\[
\frac{\partial}{\partial t} \oint_{\phi_t(C)} X \cdot dr = \oint_{\phi_t(C)} \frac{DX}{Dt} \cdot dr. \tag{8}
\]

So if \( X \) is Euler,

\[
\frac{\partial}{\partial t} \oint_{\phi_t(C)} X \cdot dr = -\oint_{\phi_t(C)} \nabla p \cdot dr = 0,
\]

and circulations are preserved.

For the converse, first note that if vorticity is transported, then (7) yields \( \text{curl}(\frac{DX}{Dt}) = 0 \). In addition, if circulation integral are conserved then (8) yields

\[
0 = \oint_{\phi_t(C)} \frac{DX}{Dt} \cdot dr.
\]
Thus $\frac{DX}{Dt}$ is a curl-free field whose circulation around each boundary component is zero. Thus as in (4), for each $t$ there exists a smooth function $-p_t$ with $\frac{DX}{Dt} = -\nabla p_t$, and further, since all the data is varying smoothly with $t$, so does $p_t$. □

By Green’s theorem, if $C$ bounds a disk in $M_0$, then (5) implies (6), but for multi-connected regions (6) is stronger. In particular, the requirement that circulation integrals be preserved is needed for multi-connected regions. This condition is often not included in standard texts in the vorticity form of the Euler equations because the regions under consideration are usually simply connected or else solutions with discontinuous pressure are allowed.

§4 Constant and generic initial vorticity. In this section we take the first steps on Question 2 with a pair of results that give dynamical information about Euler fluid motions based on the nature of the initial vorticity distributions. Since vorticity distribution coupled with the circulations determine the velocity field and the vorticity is passively transported by the Helmholtz-Kelvin Theorem, it is not surprising that the initial vorticity distribution very much influence the dynamics.

In view of the definition of Euler diffeomorphism we restrict attention to periodic stirring protocols. As noted above, in general, a periodic stirring protocol does not suffice to insure that a compatible Euler solution is periodic. However, as a consequence of elementary potential theory it does suffice if the initial vorticity is constant. This is content of the first part of Proposition 3. The second part of the proposition concerns the situation with stationary boundaries and is a special case of a more general well-known classical result (see, for example, Proposition 8.2.2 in [AMR], or page 33 in [MP]).

**Proposition 3:** Assume that $(\alpha_i)$ is a $T$-periodic stirring protocol with corresponding fluid regions $M_t$. Given a real number $\Omega$ and a vector $(\Gamma_0, \Gamma_1, \ldots, \Gamma_m) \in \mathbb{R}^{m+1}$ with $\sum \Gamma_i = \Omega \text{ area}(M_0)$, there exists a unique $T$-periodic incompressible Euler fluid motion $(M_t, X)$ with $\omega_0 \equiv \Omega$ and $\oint_{C_{\alpha_i}} X \cdot d\mathbf{r} = \Gamma_i$, for $i = 0, \ldots, m$. In particular, if the inner boundaries are stationary, then the solution $X$ is steady (time-independent) and so $h_{top}(\phi_t) = 0$ for all $t$.

**Proof:** Fix $t$. By standard arguments (eg. a simple modification of the proof of Theorem 2.2 in [MP]), given the planar region with smooth boundaries $M_t$, the circulations $\Gamma_i$ with $\sum \Gamma_i = \Omega \text{ area}(M_t)$, and the vectors $\alpha_i$, there is a unique $C^\infty$-function $\Psi_t$ with $\Delta \Psi_t = -\Omega$, $(J \nabla \Psi) \cdot \mathbf{n}_i = \dot{\alpha}_i \cdot \mathbf{n}_i$ on each boundary component $C_{\alpha}$, and $\oint_{C_{\alpha}} (J \nabla \Psi) \cdot d\mathbf{r} = \Gamma_i$ for all $i$. Note that the condition on the sum of the circulations is necessary by Green’s theorem.

Now define $X = J \nabla \Psi_t$, and let $\phi_t$ be it’s fluid motion. By construction, $\phi_t$ transports vorticity and preserves circulation integrals and so by Theorem 2 is an Euler fluid motion. Since the $M_t$ are a $T$-periodic family, $X$ is $T$ periodic. Uniqueness follow from uniqueness of each $\Psi_t$ and Theorem 2.

If the boundaries of the fluid region do not move, then $M_t = M_0$ for all $t$, and so $\Psi_t = \Psi_0$ for all $t$, and so $X$ is time-independent. Thus $\phi_t$ is a two-dimensional flow in usual Dynamical Systems sense, and so has zero topological entropy ([Y]). □

The next result concerns the dynamics when the initial vorticity is typical amongst all functions in the $C^k$ category, $2 \leq k \leq \infty$. The basic idea, as was observed by Brown
and Samelson [BS], is that as a consequence of the Kelvin-Helmholtz Theorem, an Euler diffeomorphism preserves the level sets of its initial vorticity distribution. Since smooth, one-dimensional manifolds cannot support chaotic dynamics, if the level sets are smooth manifolds, the diffeomorphism has zero topological entropy. The precise version of this idea we give makes use of a frequently invoked theorem of Katok.

**Proposition 4:** If $f$ is an Euler diffeomorphism whose velocity field $X$ has initial vorticity $\omega_0$ which has finitely many critical points, then $h_{\text{top}}(f) = 0$.

**Proof.** By Theorem 1, and the fact that $X$ is $T$-periodic, $\phi_T \omega_0 = \omega_T = \omega_0$. Now assume to the contrary that $h_{\text{top}}(f) > 0$. As a consequence of a theorem of Katok [K2], since $f$ is smooth, it must have a transverse homoclinic point to a hyperbolic periodic point $p$. Now since $\omega_0 = \omega_0 \circ f$, and $\omega_0$ is continuous, it must be constant on the closure of the union of the stable and unstable manifolds of the orbit of $p$. And since these intersect transversally at the homoclinic orbit and $\omega_0$ is smooth, every point on the homoclinic orbit must be a critical point for $\omega_0$, and so there are infinitely many of them, contrary to assumption. ⊓⊔

The class of functions allowed for $\omega_0$ in the proposition contains the Morse functions, and so contains an open dense set in $C^k(M_0, \mathbb{R})$, $2 \leq k \leq \infty$. Thus it represents the typical case in the $C^k$-topology. It important to note what the proposition does not say. It does not show that amongst the initial vorticities that give rise to periodic Euler solutions, the typical case has zero topological entropy. Such a stronger result would require understanding conditions on the initial vorticity that insure or even allow periodicity of the Euler velocity field (cf. Question 3 in §6). Also note that any condition on a smooth invariant function which guarantees zero entropy can be substituted for the assumption of finitely many critical points. Finding such conditions is an interesting general Dynamical Systems question.

A nonconstant real analytic function has finitely many critical points in any compact set and so we have a corollary that via Katok’s theorem is equivalent to Remark 4 in [BS].

**Corollary 5:** (Brown and Samelson) If $f$ is an Euler diffeomorphism whose velocity field is $C^\omega$ and has nonconstant initial vorticity, then $h_{\text{top}}(f) = 0$.

§5 **Pseudo-Anosov stirring protocols.** In this section we use the fact that the topological character of a $T$-periodic stirring protocol determines the isotopy class of the time $T$-map of any compatible fluid motion, and this information can be used with the Thurston-Nielsen theory to obtain classes of periodic stirring protocols for which any compatible Euler solution is never periodic.

We begin by quickly reviewing standard material about braids, isotopy classes and Thurston-Nielsen theory. For more information on braids and isotopy classes see [Bi] or [BL]. For general material on the Thurston Nielsen theory see [T], [FLP], and [CB]. For a survey of dynamical applications of the TN theory see [Bd1], and for fluid mechanical applications see [BAS1] and [BAS2].

For definiteness we fix the geometry of the fluid regions. In this section $M_0$ is always the unit disk with three small round disks of radius $\epsilon$ centered at $p_1 = -1/2$, $p_2 = 0$, and $p_3 = 1/2$ removed:

$$M_0 := \{z \in \mathbb{C} : |z| \leq 1, \text{ and } |z - p_i| \geq \epsilon \text{ for } i = 1, 2, 3\}.$$
We shall also assume that any stirring protocol is 1-periodic. Such a protocol can be associated with braid on three strands. This standard construction proceeds by considering the three-dimensional traces of the paths defined by \((\alpha_i(t), t)\) for \(0 \leq t \leq 1\) and \(i = 1, 2, 3\). This defines a physical braid on three strands. By examining the crossing of the strands projected onto a plane, we obtain a braid word in the braid group on three strings, \(B_3\). Note that changing the projection plane changes the braid word by a conjugacy, and so the construction actually only defines a conjugacy class in \(B_3\), which is known as the \textit{braid type} of the protocol. The distinction between the braid and its conjugacy class is usually not of consequence here, so we will just refer to the “braid” of the protocol.

The braid in turn determines the isotopy class in the mapping class group of \(M_0\). The Thurston-Nielsen (TN) theory contains a classification of surface isotopy classes into three types: finite order, pseudoAnosov (pA), and reducible. A stirring protocol is said to have one of these types if its corresponding isotopy class does. Since we restrict attention here to \(M_0\), we can use a simplified version of the theory which depends on the fact that \(M_0\) is a two-fold cover of the torus with 4 disks removed ([Bi], [BW], [K1], see §1.8 in [Bd1] for an exposition).

The TN-type of an isotopy class represented by an element of \(B_3\) can be computed directly using the homomorphism \(\chi : B_3 \to SL(2, \mathbb{Z})\) with

\[
\chi(\sigma_1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \chi(\sigma_2) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}
\]

Note that this is just the Burau representation with the substitution \(t = -1\), and so can be interpreted as the action on homology in the two-fold cover, the torus. The result we need is that the braid word \(\beta \in B_3\) represents a pA class if and only if the largest eigenvalue \(\lambda'\) of the matrix \(\chi(\beta)\) is real with magnitude larger than one, or equivalently, if \(|\text{trace}(\chi(\beta))| > 2\). The number \(\lambda = |\lambda'|\) is called the \textit{expansion constant} of the pA class.

PseudoAnosov isotopy classes have a number of special properties. Of central importance here is the fact that a pA isotopy class always induces exponential word growth on \(\pi_1\). More precisely, let \(F_3 \cong \pi_1(M_0)\) be the free group on three generators and \(\rho : F^3 \to F^3\) be induced by a pA isotopy class, i.e. \(\rho = f_* : \pi_1(M_0) \to \pi_1(M_0)\) for any (and thus every) homeomorphism \(f\) in the class. If for \(w \in F_3\), \(\ell(w)\) denotes the number of letters in the reduced word for \(w\), then for every nontrivial \(w \in F_3\),

\[
\lim_{n \to \infty} \frac{\ell(\rho^n(w))}{n^{1/2}} = \lambda,
\]

where to be explicit, \(\rho^n(w)\) denotes repeated composition of \(\rho\) applied to \(w\), not multiplication in the group.

One consequence of this induced growth on \(\pi_1\) is that any homeomorphism in the class has topological entropy greater than \(\log(\lambda)\). A second consequence concerns essential curves and arcs. An \textit{essential simple closed curve} is one that is neither contractible nor boundary parallel. An arc with its endpoints on the boundary is \textit{essential} if it cannot be contracted to a point by a homotopy that keeps its endpoints on the boundary. As a consequence of (9), under a homeomorphism in a pA class, no iterate of an essential arc or
curve is homotopic to itself. Here as in the rest of the paper all homotopies are required to keep the boundary fixed set-wise.

A third consequence of (9) is the exponential growth of the length of essential curves and arcs under iteration. If $\gamma$ is a smooth essential arc or closed curve and $f_1, f_2, \ldots$ is a sequence of diffeomorphisms all in the same $\pi$ isotopy class with expansion constant $\lambda$, then there is a positive constant $k$ so that

$$\ell((f_1 \circ f_2 \ldots \circ f_n)(\gamma)) \geq k\lambda^n \tag{10}$$

for all $n > 0$, where $\ell(\cdot)$ now denotes the Euclidean length. The equivalence of (9) and (10) follows from the fact that the word length of an element in $\pi_1$ gives a uniform bound on the displacement caused by its corresponding deck transformation in the universal cover (see [Mi1] and [FLP], exposé 10, §II). This result is usually stated for the repeated application of a single diffeomorphism, but since the composition above induces the same action on $\pi_1$ as a repeated composition, the proof is the same. For what follows it is important to note that the constant $k$ depends only on the homotopy class of the essential arc or closed curve.

In order to get the estimate in Theorem 7 we require (generic) hypothesis on the initial vorticity. Recall that $\mu : M_0 \to \mathbb{R}$ is a Morse function if its Hessian is nonsingular at all critical points. Morse functions are open and dense in $C^k(M_0, \mathbb{R})$, $2 \leq k \leq \infty$. (see [Mi2], [H], or [Ma] for more details on Morse Theory). A connected component of a level set $\mu^{-1}(r)$ is called a critical set if it contains a critical point and a regular set if it does not. Regular sets are either simple closed curves or arcs with both endpoint on the boundary of $M_0$.

Informally, an essential regular strip is a collection of parallel regular arcs. More precisely, an essential regular strip is defined to be a nontrivial interval of $\mu$ values $I$ and a collection of essential regular arcs $K_r$ for each $r \in I$ so that each $K_r$ is a connected component of the level set $\mu^{-1}(r)$ and $S := \cup K_r$ is compact and connected. An essential regular annulus is defined similarly using regular closed curves. If $I = [c, d]$, the boundary of an essential regular annulus consists of the two regular closed curves $K_c$ and $K_d$. The boundary of an essential regular strip is the union of four arcs: the two segments in the boundary of $M_0$ that consists of the endpoints of the regular arcs in the strip and the two regular arcs $K_c$ and $K_d$.

Note that if $K$ is an essential regular arc (resp. closed curve), then it always has a neighborhood (one-sided, if $K$ is tangent to the boundary) that is an essential regular strip (annulus). All regular sets in an essential regular strip or annulus are in the same homotopy class. It is worth noting that not every Morse function on $M_0$ has essential regular sets, see Figure 1a.

**Lemma 6:** Let $M_0$ be the disk with three holes as defined above. For all $2 \leq k \leq \infty$ there is a dense, open set $G \subset C^k(M_0, \mathbb{R})$ such that each $\mu \in G$ has an essential regular set.

**Proof:** Let the set $G$ consists of Morse functions $\mu \in C^k(M_0, \mathbb{R})$ for which

(1) Each critical value comes from exactly one critical point, i.e. if $c_1$ and $c_2$ are distinct critical points, then $\mu(c_1) \neq \mu(c_2)$. 

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Figure 1a: Sample level sets of a Morse function with no essential regular set.

Figure 1b: Components of level sets from the proof of Lemma 6

(2) No critical point is on the boundary and no critical set is tangent to the boundary.

(3) Restricted to each boundary circle \( \mu \) is a Morse function with one-dimensional domain.

Standard arguments show that \( G \) is open dense in the Morse functions, and thus is open dense in \( C^k(M, \mathbb{R}) \). By condition (1) each critical set of a \( \mu \in G \) contains exactly one critical point and thus there are three possibilities for the topology of a critical set: an ‘\( \alpha \)’ with the endpoints of all its arms on the boundary, an ‘\( \alpha \)’ with the endpoint of its two arms on the boundary, and an ‘\( \infty \)’. By condition (2) any critical set that intersects the boundary must be an ‘\( \alpha \)’ or ‘\( \infty \)’ with its endpoints on the boundary, and by condition (3) a given level set can only intersect the boundary in a finite set of points.

Now assume contrary to the conclusion of the lemma that \( \mu \in G \) has no essential regular set. We claim that this implies that for each boundary circle, \( C_i \), there is a regular closed curve, \( B_i \), that is homotopically parallel to \( C_i \). The proof proceeds by drawing a number of conclusions about the possible level sets of \( \mu \) and their configurations all based on the same general argument: if the assertion is not true, then near to the level set under consideration there would be another component of a level set that is an essential arc or closed curve, contrary to the assumption on \( \mu \). For example, the endpoints of the arms of a critical set of type ‘\( \alpha \)’ must be on the same boundary component, for if they were not, we could move a little off the essential path in the \( \alpha \) from endpoint to endpoint and find an essential arc for \( \mu \). Other assertions that follow from the same argument are: A critical set of type ‘\( \alpha \)’ must either be contractible into a boundary circle or else contain its legs inside its closed loop and that closed loop is homotopically parallel to the boundary circle (see Figure 1b). A critical set of type ‘\( x \)’ must have all its endpoints on the same boundary circle and must be contractible into that circle (see Figure 1b). A regular arc must have both of its endpoints on the same boundary curve, be contractible into that boundary circle, and not be tangent to any other boundary circle. A regular closed curve can be tangent to one and only one boundary circle, but then it must be contractible into that boundary circle or else be homotopically parallel to it.
Now for $i = 0, \ldots, 3$, let
\[ F_i = \{ K : K \text{ is a component of a level set and } K \cap C_i \neq \emptyset \}. \]

By construction, each $F_i$ is connected and using the observations of the previous paragraph, for $i \neq j$, $F_i \cap F_j = \emptyset$. In addition, $U = (\cup F_i)^c$ is open because a component of a level set not intersecting a boundary curve implies that all level sets near it also do not intersect the boundary curve. Thus each $F_i$ is a compact, connected set that is disjoint from the other $F_j$. This implies that the topological frontier of each $F_i$ consists of a finite number of components of level sets which are disjoint from the other $F_i$. Looking at the list of possible elements of an $F_i$ we see that this frontier can only be a closed curve tangent and parallel to $C_i$ or else a critical set of $'\alpha'$-type with its arms on $C_i$ and its loop parallel to $C_i$ as on the right of Figure 1b. In either case, just outside the frontier there is a regular closed curve, $B_i$, that is parallel to $C_i$, proving the claim above.

If we let $M'$ be the multi-connected region whose boundary is the union of the $B_i$, then $M'$ is homeomorphic to $M_0$ and has regular closed curves for its boundary. As a consequence, all the components of level sets of $\mu$ in $M'$ are either regular closed curves or else critical sets of type $'\infty'$. For $i = 0, \ldots, 3$, let $E_i$ be the closure of the set of regular closed curves $K$ which are homotopically parallel to the boundary circle $C_i$ and focus on a fixed $i$. The frontier of $F_i$ gives one component of the frontier of $E_i$. The rest of the frontier of $E_i$ must consist of components of critical sets of type $'\infty'$ and one of the loops of the $\infty$ must be parallel to $C_i$. If the other loop of the $\infty$ bounds a disk or is parallel to $C_i$, then nearby the $\infty$ there is a regular closed curve, parallel to $C_i$, contradicting the fact that the $\infty$ was on the boundary of $E_i$. On the other hand, if the other loop of the $\infty$ encloses a boundary circle other than $C_i$, then nearby there would be an essential regular closed curve, contrary to the assumption on $\mu$. Thus in either case we have a contradiction to the assumption that $\mu$ has no regular sets.  

The next result gives conditions under which a periodic stirring protocol gives rise to Euler fluid motions which are never periodic. On one hand, by Proposition 4, if the initial vorticity has finitely many critical points then any Euler diffeomorphism has zero topological entropy. On the other hand, any diffeomorphism in a pA isotopy class must have positive entropy. Thus any Euler fluid motion whose vorticity has finitely many critical points and is compatible with a pA stirring protocol cannot have a periodic velocity field. With the additional generic condition on the initial vorticity given in Lemma 6, one gets more data about this non-periodicity in the form of an estimate of the growth rate of the gradient of the vorticity.

**Theorem 7:** (PseudoAnosov protocols) Assume that $(\alpha_i)$ is a stirring protocol of pseudo-Anosov type with expansion constant $\lambda$, and $X$ is a compatible Euler solution with initial vorticity $\omega_0$. If $\omega_0$ has finitely many critical points, then $X$ is not periodic. If in addition, $\omega_0 \in G$, the open dense set specified in Lemma 6, then there is a positive constant $c$ with
\[ \| \nabla \omega_n \|_{C^0} \geq c \lambda^n, \]
for all $n \in \mathbb{N}$.  

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Proof: The first statement was proved in the paragraph above the theorem so assume that \( \omega_0 \in G \) from Lemma 6 and let \( S_0 \) be an essential regular strip for \( \omega_0 \). The case where \( \omega_0 \) has an essential regular annulus is similar. Since \( S_0 \) is an essential regular strip there is an interval of \( \omega_0 \) values, \( I = [\Omega_L, \Omega_R] \), so that for \( r \in I \), \( \omega_0^{-1}(r) \cap S_0 := K_{0r} \) is an essential regular arc and \( \mu(K_{0r}) = r \).

Now fix \( n > 0 \). By Theorem 2, \( S_n := \phi_n(S_0) \) is an essential regular strip for \( \omega_n \) which is made up of the essential regular arcs \( K_{nr} := \phi_n(K_{0r}) \). If \( \ell_n \) is the minimum Euclidean length of any of the \( K_{nr} \), it follows from (10) that there is a constant \( k \) depending only on the homotopy class of \( S_0 \) so that

\[
\ell_n \geq k\lambda^n. \tag{11}
\]

Since \( S_n \) is a regular strip, it is equipped with a pair of orthogonal foliations by arcs, namely, the regular arcs \( K_{nr} \) and trajectories of the gradient flow of \( \omega_n \) restricted to \( S_n \). This implies that we may find smooth function \( a_1, a_2 : I \to \mathbb{R} \) and a diffeomorphism

\[
F : \{(a, r) : r \in I \text{ and } a_1(r) \leq a \leq a_2(r)\} \to S_n
\]

so that:

1. \( F \) restricted to \([a_1(r), a_2(r)] \times \{r\}\) parameterizes \( K_{nr} \).
2. For \( i = 1, 2 \), \( F(a_i(r), r) \) parameterizes the intersection of \( S_n \) with a boundary component of \( M_n \).
3. \( F(\{a\} \times I) \) is a trajectory of the gradient flow of \( \omega_n \) restricted to \( S_n \).

For such an \( F \), \( \frac{\partial F}{\partial a} \) is orthogonal to \( \frac{\partial F}{\partial r} \), and since \( F^{-1} \) restricted to a \( F(\{a\} \times I) \) is exactly \( \omega_n \),

\[
\left\| \frac{\partial F}{\partial r}(a, r) \right\| = \frac{1}{\|\nabla w_n(F(a, r))\|}.
\]

Using the fact that \( \phi_n \) preserves area

\[
\text{area}(S_0) = \text{area}(S_n)
\]

\[
= \int_{\Omega_L}^{\Omega_R} \int_{a_1(r)}^{a_2(r)} \det(DF) \, da \, dr
\]

\[
= \int_{\Omega_L}^{\Omega_R} \int_{a_1(r)}^{a_2(r)} \frac{\|\frac{\partial F}{\partial a}\|}{\|\nabla w_n(F(a, r))\|} \, da \, dr
\]

\[
\geq \frac{1}{\|\nabla \omega_n(z_n)\|} \int_{\Omega_L}^{\Omega_R} \int_{a_1(r)}^{a_2(r)} \|\frac{\partial F}{\partial a}\| \, da \, dr
\]

\[
\geq \frac{(\Omega_R - \Omega_L)\ell_n}{\|\nabla \omega_n(z_n)\|},
\]

where \( z_n \) is such that \( \|\nabla \omega_n(z_n)\| = \max\{\|\nabla \omega_n(z)\| : z \in S_n\} \). Thus using (11),

\[
\|\nabla \omega_n(z_n)\| \geq \frac{(\Omega_R - \Omega_L)k\lambda^n}{\text{area}(S_0)}
\]

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and note that none of the constants depend on $n$. □

Although we restricted attention here to the case of three stirrers, there is a similar result for pA protocols with more stirrers and for protocols for which the TN representative in the isotopy class has at least one pA component.

In one sense Theorem 7 says that for pA protocols any compatible Euler velocity fields are diverging as $t \to \infty$ because $\|X\|_{C^2} \to \infty$, or alternatively, $\|\Delta X\|_{C^0} \to \infty$. Examining the situation more carefully one sees that this divergence is a result of the “piling up” of levels sets of $\omega$ and thus the graph of the vorticity is acquiring sharp ridges packed closely together. Thus the vorticity is becoming more evenly distributed and regular in the sense that in many places the local mean vorticities are approaching the global mean. Now if the fluid motion were time-periodic and strong mixing with respect to Lebesgue measure, this would be the behavior of $\mu \circ \phi_t$ for any $L^2$-function $\mu$. While the fluid motion is certainly not periodic and strong mixing it is worth noting that the proof of Theorem 7 indicates that this behavior is the result of each time one advance map of the fluid being isotopic to a pA map and so is, in a certain sense, the “memory” of the strong mixing of the pA map.

§6 Discussion and Questions. We begin by summarizing the contributions of this paper to Question 1, and more generally, to the understanding of the dynamics of Euler diffeomorphisms. The results fall into various cases based on the initial vorticity and the isotopy class induced by the stirrer motion. For expositional simplicity we continue to restrict the discussion to fluid regions with the topology of $M_0$ from §5, but similar conclusions can be drawn for systems with more than 3 stirrers.

As a starting point, recall that the set of area preserving diffeomorphisms with positive topological entropy on a genus zero surface is dense and open in the $C^\infty$ topology ([W], see [F] for other generic conditions). Thus in addressing Question 2 we shall primarily focus on the question of the existence of Euler diffeomorphisms with positive entropy.

The simplest case is that of stationary boundaries, i.e. no stirring. From Proposition 3, we know that in this case constant initial vorticity yields only steady Euler fluid motions and thus zero entropy. From Proposition 4 we know that whatever the stirring protocol, for typical initial vorticity a time-periodic Euler fluid motion also has zero entropy. Thus any Euler model for chaotic dynamics in this class comes from vorticity that is atypical and non-constant.

At the other extreme of stirring, for pA protocols Theorem 7 says that typical initial vorticity never gives rise to Euler diffeomorphisms. So once again to find Euler models for chaotic dynamics we must have atypical initial vorticity. Amongst these atypical systems, perhaps the most attractive for additional study are those with constant vorticity. By Proposition 3 we know that these systems do, in fact, always have periodic Euler velocity fields. In addition, since the class is pA, the resulting Euler diffeomorphisms all have positive entropy. Further, Proposition 3 also shows that for a fixed protocol there is only a 4-parameter family of such Euler diffeomorphisms. Amongst these constant vorticity solutions the zero vorticity case is especially attractive because in that case the stream function is harmonic and the methods of complex analysis can be brought to bear. These systems will be the subject of a subsequent paper. Very interesting numerical results on these systems are contained in [FCB].
Between the two extremes of stirring are the finite order protocol in which the stirrers move in paths that are topologically the same as those generated by circulating them in order around a circle. This case also requires further investigation.

In the $C^\omega$-case the situation is somewhat clearer. For nonconstant initial vorticity the entropy is always zero (Corollary 5). As already noted, for constant vorticity one always has periodic compatible Euler velocity fields. For a pA protocol the resulting Euler diffeomorphism always have positive entropy, but stationary boundaries yield zero entropy. For finite order protocols presumably one can have both types of behavior.

This situation is somewhat reminiscent of Arnol’d’s dichotomy for $C^\omega$-steady 3D Euler flows. In that case if the vorticity is generic, then the Bernoulli function gives an integral of motion and forces zero entropy. The non-generic case is when the vorticity in aligned with the velocity field giving what is termed a Beltrami flow. These flows can have positive entropy and much progress has been made by Etnyre and Ghrist based on the observation that Beltrami flows can be identified with the Reeb flows of contact forms. In the analogy to 2D periodic Euler fluid motions the Beltrami case corresponds to zero vorticity, and there also one has nice additional structure, namely, the stream function is harmonic.

While the entire picture is not yet clear, especially in the $C^k$-case, at this point it appears that Euler diffeomorphisms, especially ones with positive entropy, are rather rare. Thus it seems likely that Euler diffeomorphisms cannot manifest all the dynamical behavior of area preserving diffeomorphisms. However, we do not, as of yet, have a specific dynamical obstruction that would give a negative answer to Question 2.

We close with some questions stimulated by Question 2. All the questions except the last assume the situation studied here: planar, multi-connected regions with perhaps moving boundaries.

**Question 3:** How common are time-periodic Euler velocity fields which are not steady? Give explicit examples of initial vorticity distributions and periodic stirring protocols so that the compatible Euler solution is also periodic.

It is worth noting that the results in this paper have only used the fact that the vorticity is passively transported by Euler fluid motions and not the important additional information that this function is actually the curl of the velocity fields generating the motion. In particular, in certain cases the curl will make level sets of the vorticity “curl” up and not return to themselves periodically, and so prevent the periodicity of the Euler solution.

**Question 4:** Study the dynamics of Euler diffeomorphism arising from smooth, atypical initial vorticity such as those with with areas of constant vorticity. Are there any chaotic, time-periodic Euler flows with stationary boundaries or which are compatible with finite order protocols?

It would also be very interesting to lessen the regularity assumptions of this paper and study, for example, Questions 3 and 4 for systems whose vorticity is the indicator function of a region with smooth boundary (this case is often referred to as “vortex patches”). It is known that such initial data gives rise to flow maps that are Hölder homeomorphisms and the boundary curve of a vortex patch remains smooth throughout the evolution. Thus the resulting Euler homeomorphisms have special characteristics which could be valuable
in understanding their dynamics. In this context it is useful to note that the Thurston-Nielsen theory is a theory about homeomorphisms, and so, for example, if the boundary of a vortex patch is an essential curve, then a compatible Euler velocity field cannot be periodic under a pA stirring protocol.

It is also worth remarking that another kind of singular Euler fluid motion, point vortices, do have periodic solutions with positive entropy in the sense that there are relative periodic solutions of the 3-vortex problem on the cylinder for which the motion of the vortices may be treated as stirrers with a protocol of pA type ([BAS2]). Thus the induced velocity field on the cylinder is time-periodic after a space translation, and the time $T$-map has positive entropy. It is likely that such periodic solutions also exist in the disk and plane without the geometric phase. Can these examples be smoothed while maintaining the periodicity? Note that Proposition 4 and the property of pA classes given in (10) force strong restrictions on how the smoothing can be done.

**Question 5:** Study Question 1 for the time $T$-diffeomorphisms of time-periodic quasistationary Stokes flow in multi-connected region with periodic stirring protocols.

The quasistationary Stokes equation is derived from assumptions at the other extreme from those behind the Euler equation. Instead of assuming the viscosity is negligible, one assumes that the vorticity dominates and that the acceleration or inertial term can be neglected. The velocity field is thus required to satisfy:

\[
\Delta X = -\nabla p_t \\
\text{div}(X) = 0 \\
X = \dot{\alpha}_i \text{ on the boundary.}
\]

where $p_t$ is again the pressure. Note that the boundary conditions correspond to the fluid sticking to the boundary without slipping; this is a consequence of the nonzero viscosity. Solutions to the quasistationary Stokes equation are characterized by the stream function $\Psi$ being biharmonic at each time, $\Delta^2 \Psi = 0$. Thus in terms of mathematical structure, quasistationary Stokes flows are similar to time-periodic zero vorticity Euler fluid motions. Both are time-periodic Hamiltonian systems whose Hamiltonian has additional structure. In the zero vorticity Euler case the Hamiltonian is harmonic, while for Stokes flow it is biharmonic. Note, however, the nontrivial difference in the boundary conditions. Stokes flows with a pA stirring protocol are studied in [BAS1], [FCB], and [V].

**Question 6:** In three dimensions the Helmholtz-Kelvin theorem says that the vorticity (now a vector field) is transported. Thus with generic initial vorticity a 3D time-periodic Euler fluid motion preserves a nontrivial vector field. What restrictions does this place on the dynamics of 3D Euler diffeomorphisms?

**Acknowledgments:** The author would like to thank Hassan Aref for pointing out that versions of Proposition 4 and Corollary 5 are contained in [BS] and Igor Mezić for the remark after Theorem 7 about the asymptotics of the vorticity.
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