AMENABILITY, COMPLETELY BOUNDED PROJECTIONS, DYNAMICAL SYSTEMS AND SMOOTH ORBITS

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ABSTRACT. We describe a general method to construct completely bounded idempotent mappings on operator spaces, starting from amenable semigroups of completely bounded mappings. We then explore several applications of that method to injective operator spaces, fixed points of completely contractive mappings, Toeplitz operators, dynamical systems and similarity orbits of group representations.

1. Introduction

If an injective von Neumann algebra is acted on by an amenable group then the corresponding fixed point algebra is in turn injective (Theorem 3.16 in Chapter XV in [Ta03]). This fact turns out to play a key role in several proofs in the theory of operator algebras. To give only one example in this connection, see the proof that an injective von Neumann algebra of type III is semidiscrete (§3 in Chapter XV in [Ta03]).

In the present paper we investigate what versions the aforementioned fact might have in the more general framework of operator spaces (see Corollary 3.2 below). Our initial motivation was that it might be useful to have a very general setting where completely bounded projections are associated with actions of semigroups. With the general result at hand (see Theorem 3.1 below) we soon realized that a lot of seemingly unrelated structures in operator theory can now be understood in a unifying manner. Thus such different things as dynamical systems, generalized Toeplitz operators or homogeneous spaces of Lie groups can be looked at from a unique point of view.

We should point out that the technique of averaging over amenable groups has a long history in functional analysis and related areas. Its applications range from representation theory of finite and compact groups (see the so-called Weyl’s unitary trick) to ergodic theory (see [Lu92]) and cohomology of von Neumann algebras (see the papers [SS98] and [SS04]). From this point of view, what we are doing in the present paper is to investigate the relationship between that technique and the idea of completely bounded map.

The structure of the paper is as follows: In Section 2 we introduce the notion of operator $S$-space, which is roughly speaking an operator space $X$ equipped with a semigroup $S$ of completely bounded maps. To each $S$-invariant subspace $Y \subseteq X^*$ and any left invariant mean on the corresponding space of coefficients $C_{X,Y}(S)$ we associate a completely bounded mapping $X \to Y^*$ and we study some basic

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properties of that construction. In Section 3 we prove our main result on existence of completely bounded projections on fixed point subspaces (Theorem 3.1) and then we explore some of the consequences of that theorem (see Theorems 3.4–3.5 and Corollaries 3.6 through 3.10).

Preliminaries

Our basic references for the theory of operator spaces and completely bounded maps are the monographs [ER00], [Pa02] and [BL04]. We shall now recall several basic facts that will be needed in the sequel. If $X$ is a vector space and $p, q \geq 1$ then $M_{p,q}(X)$ is the space of all $p$ by $q$ matrices with entries in $X$ and $M_p(X) = M_{p,p}(X)$. If $X$ and $Y$ are vector spaces, $\varphi : X \to Y$ is a linear mapping and $n \geq 1$ then $\varphi_n : M_n(X) \to M_n(Y)$ is defined by $\varphi_n([x_{ij}]) = [\varphi(x_{ij})]$ for every $[x_{ij}] \in M_n(X)$.

Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{B}(\mathcal{H})$ the $C^*$-algebra of all bounded linear operators on $\mathcal{H}$. Then $M_n(\mathcal{B}(\mathcal{H}))$ has a unique $C^*$-algebra norm $\| \cdot_n$ induced by its identification with $\mathcal{B}(\mathcal{H}^{(n)})$ where $\mathcal{H}^{(n)}$ is the orthogonal sum of $n$ copies of $\mathcal{H}$. An operator space is a complex vector space $X$ endowed with a complete norm $\| \cdot_n$ on every space $M_n(X)$ and with the property that there exists a linear mapping $\varphi : X \to \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ such that $\varphi_n : (M_n(X), \| \cdot_n) \to (M_n(\mathcal{B}(\mathcal{H})), \| \cdot_n)$ is isometric for all $n \geq 1$.

Any closed subspace of $\mathcal{B}(\mathcal{H})$ inherits a canonical structure of operator space. In this case this holds true for $C^*$-algebras. More precisely, if $A$ is a $C^*$-algebra and $n \geq 1$ then $M_n(A)$ has a unique $C^*$-algebra norm that is induced by an arbitrary faithful representation of $A$ on a Hilbert space. If $A$ and $B$ are $C^*$-algebras and $\varphi : A \to B$ is linear then $\varphi$ is said to be completely positive if $\varphi_n$ is a positive map for all $n \geq 1$.

If $X$ and $Y$ are operator spaces and $\varphi : X \to Y$ is linear then $\varphi$ is said to be completely bounded if $\| \varphi \|_{cb} = \sup\{\| \varphi_n \| : n \geq 1\} < \infty$, and completely contractive if $\| \varphi \|_{cb} \leq 1$. Moreover $\varphi$ is completely isometric if $\varphi_n$ is isometric for all $n \geq 1$. For $X$ and $Y$ operator spaces the space $\text{CB}(X, Y)$ of all completely bounded maps between $X$ and $Y$ is a Banach space when endowed with the norm $\| \cdot \|_{cb}$. Moreover it has an operator space structure given by the isomorphisms $M_p(\text{CB}(X, Y)) \cong \text{CB}(X, M_p(Y))$. We shall always denote $\text{CB}(X, X) = \text{CB}(X)$. When $Y = \mathbb{C}$ the space $X^* = \text{CB}(X, \mathbb{C})$ is called the operator space dual of $X$. Now let us consider the operator space $X^{**} = (X^*)^*$. Then it can be shown that the canonical injection $J : X \to X^{**}$ is a complete isometry.

If $X, Y$ and $Z$ are operator spaces and $\varphi : X \times Y \to Z$ is a bilinear map then for all $p, q \geq 1$ one denotes

$$\varphi_{p,q} : M_p(X) \times M_q(Y) \to M_{pq}(Z), \quad \varphi_{p,q}([u_{ij}], [v_{kl}]) = [\varphi(u_{ij}, v_{kl})]_{(i,k)(j,l)}.$$ 

Then $\varphi$ is said to be completely bounded if $\| \varphi \|_{cb} = \sup\{\| \varphi_{p,q} \| : p, q \geq 1\} < \infty$. As in the case of completely bounded linear maps, the space $\text{CB}(X \times Y, Z)$ of all completely bounded bilinear maps $\varphi : X \times Y \to Z$ has an operator space structure; see [ER00] for details.

We shall also use the operator space projective tensor product $X \hat{\otimes} Y$ of two operator spaces $X$ and $Y$ (see [ER00] for the precise definition). All we need to know is that it is an operator space structure on a certain completion of the algebraic tensor product $X \otimes Y$ so that for any operator space $Z$ there is a canonical complete
is a smooth mapping. Now assume that $G$ of degree $\leq 1$ (see e.g., Theorem 8.19 and Corollary 8.3 in [Up85]). In this case we say that $G/H$ and $\text{Ran} Z$ of $G/H$ is a homogeneous space of $A$ and that $G/H$ is a Banach-Lie group. Now let $a \in A$ the group operations (i.e., multiplication and inversion) are smooth. For instance, $\bar U$ equipped with a maximal smooth atlas. If $A$ is a unital associative Banach algebra then its group of invertible elements, denoted by $A^\times$, is a Banach-Lie group. Now let $G$ be a Banach-Lie group and $H$ a subgroup of $G$. We say that $H$ is a Banach-Lie subgroup if there exists a local chart $\varphi: U \to V$ of $G$ such that $\varphi(U \cap H) = V \cap W$, where $U$ is an open neighborhood of $1 \in G$, $V$ is an open subset of the Banach space $Z$, and $W$ is a split subspace of $Z$ (that is, there exists a bounded linear operator $E: Z \to Z$ such that $E^2 = E$ and $\text{Ran} E = W$). If this is the case, then $G/H$ with the quotient topology has a structure of Banach manifold such that the natural projection $\pi: G \to G/H$ is smooth and has smooth local cross-sections on a neighborhood of each point of $G/H$ (see e.g., Theorem 8.19 and Corollary 8.3 in [Up85]). In this case we say that $G/H$ is a homogeneous space of $G$, and the natural transitive action

$$G \times G/H \to G/H, \quad (g_1, g_2 H) \mapsto g_1 g_2 H$$

is a smooth mapping. Now assume that $G = A^\times$ for some unital associative Banach algebra $A$ and that $H$ is an algebraic subgroup of $G$ (of degree $\leq d$) in the sense that there exist an integer $d \geq 1$ and a family $\mathcal{F}$ of polynomial functions on $A \times A$ of degree $\leq d$ such that

$$H = \{ g \in A^\times \mid (\forall f \in \mathcal{F}) \quad f(g, g^{-1}) = 0 \}.$$ 

Denote

$$L(H) = \{ a \in A \mid (\forall t \in \mathbb{R}) \quad \exp(ta) \in H \}.$$
Then $H$ with the topology inherited from $A$ is a Banach-Lie group and $L(H)$ (the Lie algebra of $H$) is a closed subspace of $A$ such that $[a, b] := ab - ba \in L(H)$ whenever $a, b \in L(H)$. If it happens that $L(H)$ is a split subspace of $A$, then $H$ is a Banach-Lie subgroup of $A^\times$. (See the main theorem in [HK77] or Theorem 7.14 in [Up85].)

2. Operator $S$-spaces

We begin this section by introducing some terminology on semitopological semigroups; we refer to [BH67] and [BJM78] for more details.

**Definition 2.1.** For any semigroup $S$ we denote by $\mathcal{F}_b(S)$ the commutative unital $C^*$-algebra of all complex bounded functions on $S$ with the sup norm $\| \cdot \|_\infty$. For each $t \in S$ we define

$$L_t : \mathcal{F}_b(S) \rightarrow \mathcal{F}_b(S) \quad \text{and} \quad R_t : \mathcal{F}_b(S) \rightarrow \mathcal{F}_b(S)$$

by $(L_t f)(s) = f(ts)$ and $(R_t f)(s) = f(st)$ whenever $s \in S$ and $f \in \mathcal{F}_b(S)$.

Now assume that the semigroup $S$ is equipped with a topology. We say that $S$ is a right (respectively, left) topological semigroup if for each $s \in S$ the mapping $S \rightarrow S$, $t \mapsto ts$ (respectively, $t \mapsto st$) is continuous. Moreover $S$ is a semitopological semigroup if it is both left and right topological.

If the semigroup $S$ is equipped with a topology then we denote by $\mathcal{C}_0(S)$ the set of all continuous functions in $\mathcal{F}_b(S)$. When $S$ is a right topological semigroup we denote $LUC_0(S)$ the set of all left uniformly continuous bounded complex functions on $S$. That is, $f \in LUC_0(S)$ if and only if $f \in \mathcal{C}_0(S)$ and the mapping $S \rightarrow \mathcal{C}_0(S)$, $s \mapsto R_s f$, is continuous. Similarly, when $S$ is a left topological semigroup we define the set $RUC_0(S)$ of all right uniformly continuous bounded complex functions on $S$ by the above condition with $R_s$ replaced by $L_s$. Moreover, when $S$ is a semitopological semigroup we shall need the set $UC_0(S) = LUC_0(S) \cap RUC_0(S)$ consisting of all uniformly continuous bounded complex functions on $S$. It is clear that all of the sets $LUC_0(S)$, $RUC_0(S)$ and $UC_0(S)$ are unital $C^*$-subalgebras of $\mathcal{C}_0(S)$.

Next assume again that $S$ is an arbitrary semigroup and let $T$ be any linear subspace of $\mathcal{F}_b(S)$. We say that $T$ is unital if it contains the unit element $1$ of $\mathcal{F}_b(S)$ (i.e., if each constant function belongs to $T$). In this case, a state of $T$ is a linear functional $\mu : T \rightarrow \mathbb{C}$ such that $\| \mu \| = \mu(1) = 1$. Now assume that $T$ is a linear subspace of $\mathcal{F}_b(S)$ that is invariant under the operators $L_t$ for each $t \in S$. We say that a linear functional $\mu : T \rightarrow \mathbb{C}$ is $S$-invariant if $\mu \circ L_t = \mu$ for all $t \in S$. The unital subspace $T$ of $\mathcal{F}_b(S)$ is said to be amenable if it admits an $S$-invariant state. If the space $\mathcal{F}_b(S)$ is amenable, then the semigroup $S$ is said to be amenable.

A topological group $S$ is said to be amenable if the space $RUC_0(S)$ is amenable. For instance the unitary groups of all injective von Neumann algebras with the strong operator topology, and also the unitary groups of all nuclear unital $C^*$-algebras with the weak topology are amenable groups (see [dH79] and [Pat92]). It is known that if $S$ is an amenable locally compact group then the larger space $\mathcal{C}_0(S)$ is amenable (see Theorem 2.2.1 in [Gr69]). □

**Definition 2.2.** Let $S$ be a semigroup and $X$ an operator space. We say that $X$ is an operator $S$-space if it is equipped with a mapping

$$\alpha : S \times X \rightarrow X, \quad (s, x) \mapsto \alpha(s, x) = \alpha_s(x)$$
satisfying the following conditions:
(i) for all $s, t \in S$ we have $\alpha_{st} = \alpha_s \circ \alpha_t$;
(ii) for all $s \in S$ the mapping $\alpha_s : X \to X$ is completely bounded linear, and
moreover $\sup_{s \in S} \|\alpha_s\|_{cb} < \infty$.

We say that $X$ is a dual operator $S$-space if moreover there exists an operator space $X_*$ such that $X = (X_*)^*$ and $(\alpha_s)^* X_* \subseteq X_*$ for all $s \in S$. An equivalent condition is that $\alpha_s : X \to X$ is weak*-continuous for all $s \in S$. In this case, $X_*$ is said to be a predual of the operator $S$-space $X$.

Let $X$ be an operator $S$-space and let $Y \subseteq X^*$ be a closed linear subspace such that $(\alpha_s)^* Y \subseteq Y$ for all $s \in S$. We denote by $\mathcal{C}_{X,Y}(S)$ the smallest unital closed subspace of $\mathcal{F}_b(S)$ that contains all the functions $f_{x,\psi} := \psi(\alpha(., x))$ for $x \in X$ and $\psi \in Y$. We always think of $\mathcal{C}_{X,Y}(S)$ as an operator space with the unique operator space structure that makes the inclusion map $\mathcal{C}_{X,Y}(S) \hookrightarrow \mathcal{F}_b(S)$ into a complete isometry.

Note that for all $s \in S$, $x \in X$ and $\psi \in Y$ we have $L_s(f_{x,\psi}) = f_{x,(\alpha_s)^*\psi}$. Thus $\mathcal{C}_{X,Y}(S)$ is invariant under $L_s$ for all $s \in S$. □

Lemma 2.3. Let $S$ be a semigroup, $X$ an operator $S$-space, $Y \subseteq X^*$ a closed linear subspace such that $(\alpha_s)^* Y \subseteq Y$ for all $s \in S$, and $\mathcal{C}_{X,Y}(S)$ as above. Then the mapping
\[
E^0 : Y \times X \to \mathcal{C}_{X,Y}(S), \quad (\psi, x) \mapsto (\psi \circ \alpha)(., x)
\]
is a completely bounded bilinear mapping and $\|E^0\|_{cb} \leq \sup_{s \in S} \|\alpha_s\|_{cb}$.

Proof. Let $p, q$ be arbitrary integers, denote $p = \{1, 2, \ldots, p\}$ and $q = \{1, 2, \ldots, q\}$, and consider the bilinear mapping
\[
(E^0)_{pq} : M_p(Y) \times M_q(X) \to M_{pq}(\mathcal{F}_b(S))
\]
defined by
\[
(E^0)_{pq}(\psi, x) = [\psi_{ij}(\alpha(., x_{kl}))]_{(i,k),(j,l) \in p \times q}
\]
for $\psi = (\psi_{ij})_{i,j \in p} \in M_p(Y) \subseteq M_p(X^*) \simeq \mathcal{CB}(X, M_p)$ and $x = (x_{kl})_{k,l \in q} \in M_q(X)$. What we have to prove is that the norm of the bilinear mapping $(E^0)_{pq}$ is at most $\sup_{s \in S} \|\alpha_s\|_{cb}$. In fact,
\[
\|(E^0)_{pq}(\psi, x)\|
= \sup_{s \in S} \|\psi_{ij}(\alpha(., x_{kl}))\|_{(i,k),(j,l) \in p \times q}
\]
\[
= \sup_{s \in S} \|\psi_q((\alpha_s)_q(x))\| (\text{compare (1.1.30) in [ER00]})
\]
\[
\leq \sup_{s \in S} \|\psi\|_{cb} \cdot \|\alpha_s\| \cdot \|x\| (\text{see (3.2.5) in [ER00]})
\]
and we are done. □

Definition 2.4. Let $S$ be a semigroup, $X$ an operator $S$-space with the semigroup action $\alpha : S \times X \to X$, and $Y \subseteq X^*$ a closed linear subspace such that $(\alpha_s)^* Y \subseteq Y$ for all $s \in S$. Consider the bilinear map
\[
E^0 : Y \times X \to \mathcal{C}_{X,Y}(S), \quad (\psi, x) \mapsto (\psi \circ \alpha)(., x)
\]
from Lemma 2.3. Then for each bounded linear functional \( \mu : C_{X,Y}(S) \to \mathbb{C} \) we define the mapping

\[
E_{\mu} : X \to Y^*, \quad (E_{\mu}(x))(\psi) = \mu(E(\psi, x)) = \mu((\psi \circ \alpha)(\cdot, x))
\]

for all \( x \in X \) and \( \psi \in Y^* \) \( \Box \)

**Lemma 2.5.** With the notation of Lemma 2.3 and Definition 2.4 the bilinear mapping \( E^0 \) gives rise to a completely bounded linear mapping \( E : Y \hat{\otimes} X \to C_{X,Y}(S) \) such that \( E(\psi \otimes x) = f_{x,\psi} \). Its dual is a completely bounded linear mapping

\[
E^* : C_{X,Y}(S)^* \to CB(X,Y^*)
\]

with \( \|E^*\|_{cb} = \sup_{s \in S} \|\alpha_s\|_{cb} \) and \( E^*(\mu) = E_{\mu} \) for all \( \mu \in C_{X,Y}(S)^* \). In particular, for each \( \mu \in C_{X,Y}(S)^* \) we have \( \|E_{\mu}\|_{cb} \leq \sup_{s \in S} \|\alpha_s\|_{cb} \cdot \|\mu\| \).

**Proof.** Denote \( M := \sup_{s \in S} \|\alpha_s\|_{cb} \). It follows by the above Lemma 2.3 along with Proposition 7.1.2 in [ER00] that the bilinear mapping \( E : Y \times X \to C_{X,Y}(S) \) naturally corresponds to a completely bounded mapping \( E : Y \hat{\otimes} X \to C_{X,Y}(S) \) with \( \|E\|_{cb} \leq M \). Consequently, for the mapping dual to \( E : Y \hat{\otimes} X \to C_{X,Y}(S) \) we have

\[
E^* : C_{X,Y}(S)^* \to (Y \hat{\otimes} X)^* = CB(X,Y^*)
\]

(the last equality follows by Corollary 7.1.5 in [ER00]) and \( \|E^*\|_{cb} = \|E\|_{cb} \leq M \).

Moreover, we have by Corollary 2.2.3 in [ER00] that any continuous linear functional \( \mu : C_{X,Y}(S) \to \mathbb{C} \) is completely bounded and \( \|\mu\|_{cb} = \|\mu\| \). Consequently \( \|E_{\mu}\|_{cb} = \|E^*(\mu)\|_{cb} \leq \|E^*\|_{cb} \cdot \|\mu\| \leq M \cdot \|\mu\| \), and the proof is finished. \( \Box \)

**Lemma 2.6.** Let \( S \) be a semigroup, \( X \) an operator \( S \)-space with the semigroup action \( \alpha : S \times X \to X \), and \( Y \subseteq X^* \) a closed linear subspace such that \( (\alpha_s)^* Y \subseteq Y \) for all \( s \in S \). Let \( E^* : C_{X,Y}(S)^* \to CB(X,Y^*) \) as in Lemma 2.5 and endow the operator space \( CB(X,Y^*) \) with the semigroup action

\[
\gamma : S \times CB(X,Y^*) \to CB(X,Y^*), \quad (s, \theta) \mapsto \gamma(t, \theta) = \gamma_t(\theta) := ((\alpha_t)^* Y)^* \circ \theta.
\]

Then for all \( t \in S \) the diagram

\[
\begin{array}{ccc}
CB(X,Y^*) & \xrightarrow{\gamma_t} & CB(X,Y^*) \\
E^* \uparrow & & \uparrow E^* \\
C_{X,Y}(S)^* & \xrightarrow{L_t^*} & C_{X,Y}(S)^*
\end{array}
\]

is commutative. In particular, if we have a bounded linear functional \( \mu : C_{X,Y}(S) \to \mathbb{C} \) and an element \( t \in S \) satisfying \( \mu \circ L_t = \mu \), then \( ((\alpha_t)^* Y)^* \circ E_{\mu} = E_{\mu} \).

**Proof.** Let \( x \in X \) and \( \psi \in Y \) arbitrary. We have

\[
((\alpha_t)^* Y)^*(E_{\mu}(x), \psi) = (E_{\mu}(x), (\alpha_t)^*(\psi)) = (E_{\mu}(x), \psi \circ \alpha_t) = \mu((\psi \circ \alpha_t)(\alpha(\cdot, x))) = \mu((\psi \circ \alpha_t)(\cdot, x)) (\text{since } \alpha_t \alpha_s = \alpha_{ts})
\]

\[
= \mu(\alpha_t(\psi \circ \alpha)(\cdot, x)) = L_t^* \mu((\psi \circ \alpha)(\cdot, x)) = (E_{L_t^* \mu}(x), \psi),
\]
and the proof is complete. □

**Notation 2.7.** Let $S$ be a semigroup and $X$ an operator space such that there is a semigroup action $\alpha: S \times X \to X$, $(s,x) \mapsto \alpha(s,x) = \alpha_x(s)$, with $\alpha_s \in CB(X)$ for all $s \in S$. Assume that $Y \subseteq X^*$ is a closed linear subspace such that $(\alpha_s)^* Y \subseteq Y$ for all $s \in S$. Then there exists a natural semigroup action

$$S \times Y^* \to Y^*, \quad (s,z) \mapsto ((\alpha_s)^*|_Y)^*(z).$$

We denote

$$X^S := \{ x \in X \mid (\forall s \in S) \quad \alpha_s(x) = x \},$$

and similarly $(Y^*)^S = \{ z \in Y^* \mid (\forall s \in S) \quad ((\alpha_s)^*|_Y)^*(z) = z \}$.

Also we denote by $\iota_Y: X \to Y^*$ the mapping defined by $\iota_Y(x)(y) = \langle y,x \rangle$ for all $x \in X$ and $y \in Y \subseteq X^*$. Thus $\iota_Y$ is the composition between the natural embedding $X \hookrightarrow X^{**}$ and the quotient mapping $X^{**} \to Y^*$, $\psi \mapsto \psi|_Y$. □

**Proposition 2.8.** Let $S$ be a semigroup, $X$ an operator $S$-space with the semigroup action $\alpha: S \times X \to X$, and $Y \subseteq X^*$ a closed linear subspace such that $(\alpha_s)^* Y \subseteq Y$ for all $s \in S$. Assume that the space $C_{X,Y}(S)$ is amenable and pick an $S$-invariant state $\mu \in C_{X,Y}(S)^*$. Then

(i) for all $x \in X^S$ we have $E_\mu(x) = \iota_Y(x)$, and

(ii) $\iota_Y(X^S) \subseteq \text{Ran} \ E_\mu \subseteq (Y^*)^S$.

In particular, if $X$ is a dual operator $S$-space and $Y^* = X$, then $\iota_Y = \text{id}_X$, therefore $E_\mu(x) = x$ for all $x \in X^S$ and $\text{Ran} \ E_\mu = X^S$. On the other hand, if $Y = X^*$ then $\iota_Y$ coincides with the canonical embedding $X \hookrightarrow X^{**}$ and, by this identification, it follows again that $E_\mu(x) = x$ for all $x \in X^S$ and $X^S \subseteq \text{Ran} \ E_\mu \subseteq (X^{**})^S$.

**Proof.** Assertion (i) follows at once in view of the way $E_\mu$ was defined (see Definition 2.4) along with the fact that $\mu(1) = 1$, where $1 \in \mathcal{F}_\mu(S)$ is the function that is constant 1 on $S$.

The first inclusion in assertion (ii) follows by (i). The second inclusion follows by $S$-invariance of $\mu$ along with Lemma 2.6. □

In the following example we show that in the setting of Proposition 2.8 it could happen that $X^S \neq \text{Ran} \ E_\mu$.

**Example 2.9.** Let $G$ be an amenable discrete infinite group, so that $C(G) = \ell^\infty(G)$ is the $C^*$-algebra of all bounded complex functions on $G$, and there exists a $G$-invariant state $\mu: \ell^\infty(G) \to \mathbb{C}$. Consider the Banach space of all absolutely summable complex functions on $G$,

$$\ell^1(G) = \left\{ f: G \to \mathbb{C} \mid ||f|| := \sup_{F \subseteq G \text{ finite}} \sum_{g \in F} |f(g)| < \infty \right\},$$

and for all $g \in G$ define $\alpha_g: \ell^1(G) \to \ell^1(G)$ by $(\alpha_g f)(h) = f(g^{-1}h)$ for $h \in G$ and $f \in \ell^1(G)$. Also for each $f \in \ell^1(G)$ consider the convolution operator

$$C_f: \ell^\infty(G) \to \ell^\infty(G), (C_f b)(h) = \sum_{g \in G} b(g) f(g^{-1}h) \quad \text{for} \ b \in \ell^\infty(G) \text{ and } h \in G.$$
Now define the operator $G$-space $X = \max \ell^1(G)$ (see Section 3.3 in [ER00] for the definition of the functors min and max from Banach spaces to operator spaces) with
\[ \alpha: G \times X \to X, \quad (g, f) \mapsto \alpha_g f. \]
Then $X^* = \min \ell^\infty(G)$ and $X^{**} = \max(\ell^\infty(G))^*$ as operator spaces (see (3.3.13) and (3.3.15) in [ER00]).

Since $G$ is infinite, it follows that the only absolutely summable constant function on $G$ is 0, hence
\[ X^G = \{0\} \neq (X^{**})^G, \]
where $(X^{**})^G$ is just the set of all $G$-invariant continuous linear functionals on the commutative $C^*$-algebra $\ell^\infty(G)$, and this set is different from $\{0\}$ since $G$ is amenable. Moreover, with the notation of Proposition 2.8 we claim that actually
\[ X^G = \{0\} \neq \text{Ran} E_\mu \subseteq (X^{**})^G. \]
In fact it is easy to see that the mapping $E_\mu: \ell^1(G) \to (\ell^\infty(G))^*$ can be equivalently defined in terms of the convolution operators by
\[ (E_\mu f)(b) = \mu(C_f b) \quad \text{for} \quad b \in \ell^\infty(G) \text{ and } f \in \ell^1(G). \]
Hence for $f = \delta_1$ (the characteristic function of $\{1\} \subseteq G$) we have $E_\mu \delta_1 = \mu$, whence $0 \neq \mu \in \text{Ran} E_\mu$, and the above claim is proved.

3. The main results

**Theorem 3.1.** Let $S$ be a semigroup and $X$ an operator $S$-space with the semigroup action $\alpha: S \times X \to X$. Assume that one of the following hypotheses holds:
(a) $X$ is a dual operator $S$-space and $S$ is amenable as a discrete semigroup, or
(b) $X$ is a dual operator $S$-space, $S$ is an amenable locally compact topological group and for each $x \in X$ the mapping $\alpha(\cdot, x)$ is continuous with respect to the weak$^*$-topology of $X$, or
(c) $X$ is a dual operator $S$-space, $S$ is an amenable topological group and the natural action of $S$ on the predual of $X$ is strongly continuous, or
(d) $X$ is an operator $S$-space, $S$ is a compact left topological semigroup which is amenable as a discrete semigroup, and for each $x \in X$ the mapping $\alpha(\cdot, x)$ is continuous, or
(e) $X$ is an operator $S$-space which is separable as a Banach space, $S$ is a compact left topological semigroup which is amenable as a discrete semigroup, and for each $x \in X$ the mapping $\alpha(\cdot, x)$ is weakly continuous, or
(f) $S$ is a compact topological group and for each $x \in X$ the mapping $\alpha(\cdot, x)$ is weakly continuous.

Then there exists a linear map $P: X \to X$ with the following properties:
(i) $P \in CB(X)$ and $\|P\|_{cb} \leq \sup_{s \in S} \|\alpha_s\|_{cb}$,
(ii) $\text{Ran} P = X^S$, and
(iii) $P \circ P = P$.

**Proof.** We first consider the conditions (a)–(c). Let $X_*$ be an operator space predual of $X$ as in Definition 2.2. We are going to make use of Proposition 2.8 for $Y = X_*$. To this end we first make sure that, if either of the conditions (a)–(c) is satisfied,
then the function space $C_{X,Y}(S)$ is amenable. In the case (a), this is obvious since the space $F_b(S)$ of all bounded complex functions on $S$ is amenable. In the case (b), we have $C_{X,Y}(S) \subseteq C_b(S)$ and the space $C_b(S)$ is amenable since the group $S$ is locally compact. Finally, in the case (c) recall that for all $s \in S$, $x \in X$ and $\psi \in Y$ we have $L_s(f,\psi) = f_x(\alpha_s) \circ \psi$. The hypothesis (c) means that the mapping $S \to Y$, $s \mapsto (\alpha_s)^* \psi$, is continuous for all $\psi \in Y$, hence we get $C_{X,Y}(S) \subseteq RUC_b(S)$, while the latter space is amenable since $G$ is an amenable group. Consequently, the space $C_{X,Y}(S)$ is amenable in either of the cases (a)–(c). Now pick an $S$-invariant state $\mu: C_{X,Y}(S) \to \mathbb{C}$ and denote $P = E_\mu: X \to Y^* = X$. Then we have by Proposition 2.8 that $\text{Ran} \, P = X^S$ and $P$ is the identity map on $X^S$, whence the desired properties (ii)–(iii) follow. As for property (i), it is a consequence of Lemma 2.5.

We now address the conditions (d)–(f). We are going to apply Proposition 2.8 with $Y = X^*$. Again we first need to check that the space $C_{X,Y}(S)$ is amenable. In both cases (d) and (e) this is obvious since the whole space $F_b(S)$ is amenable. In the case (f) note that $C_{X,Y} \subseteq C_b(S)$ and the latter space is amenable. Thus the space $C_{X,Y}(S)$ is amenable under either of the conditions (d)–(f), and then we can pick an $S$-invariant state $\mu: C_{X,Y}(S) \to \mathbb{C}$ and denote $P = E_\mu: X \to Y^* = X^*$. We are going to prove that actually $\text{Ran} \, E_\mu \subseteq X$, and then the desired properties (i)–(iii) will follow just as above, by Proposition 2.8 along with Lemma 2.5.

Firstly assume that the condition (d) is satisfied and let $x \in X$ arbitrary. In order to show that $E_\mu(x) \in X$ we have to check that $E_\mu(x): X^* \to \mathbb{C}$ is weak*-continuous. To this end, it is enough to check that $E_\mu(x)$ is weak*-continuous on the unit ball of $X^*$. (See e.g., Corollary 2 to Theorem 6.2 in Chapter IV of [Sch66].) Thus let $\{\psi_j\}_{j \in J}$ be a net in $X^*$ such that $\|\psi_j\| \leq 1$ for all $j \in J$ and $\psi_j \xrightarrow{\text{weak}^*} 0$. Then $\psi_j \xrightarrow{j \in J} 0$ uniformly on the compact subsets of $X$. On the other hand, since $S$ is compact, it follows that $\{\alpha(s,x) \mid s \in S\}$ is a compact subset of $X$, hence $(\psi_j \circ \alpha)(\cdot, x) \xrightarrow{j \in J} 0$ uniformly on $S$. Consequently $(E_\mu(x))(\psi_j) = \mu((\psi_j \circ \alpha)(\cdot, x)) \xrightarrow{j \in J} 0$, and thus $E_\mu(x)$ is weak*-continuous on the unit ball of $X^*$.

In the case (e), first recall that $\mu$ actually extends to an $S$-invariant state of $F_b(S)$, and in particular to an $S$-invariant state $\mu: C_b(S) \to \mathbb{C}$. Thus $\mu$ actually defines a Radon measure on $S$. Next, as in the case (d), we let $x \in X$ arbitrary and check that $E_\mu(x): X \to \mathbb{C}$ is weak*-continuous on the unit ball of $X^*$. Since $X$ is separable, the weak*-topology of the unit ball of $X^*$ is metrizable, hence it is enough to check that, if $\{\psi_j\}_{j \geq 0}$ is a sequence in the unit ball of $X^*$ with $\psi_j \xrightarrow{j \to \infty} 0$ then $\lim_{j \to \infty} (E_\mu(x))(\psi_j) = 0$. But this fact follows by Lebesgue’s dominated convergence theorem, since $(E_\mu(x))(\psi_j) = \mu((\psi_j \circ \alpha)(\cdot, x))$ and $\|((\psi_j \circ \alpha)(\cdot, x))\|_\infty \leq \sup \|\alpha_s\|_{cb} \cdot \|x\|$ for all $j \geq 1$.

In the case (f), since $S$ is a compact group, it follows by Proposition 4.2.2.1 in [Wa72] that for each $x \in X$ the mapping $\alpha(\cdot, x)$ is actually continuous, hence the conclusion follows by (d). Alternatively, note that the invariant state $\mu: C_b(S) \to \mathbb{C}$ is defined by a probability Haar measure on $S$, and use of Proposition 2 and Remark 1 in Chapter III, §4, no. 1 in [Bo65] to show that $\text{Ran} \, E_\mu(x) \in X$ for all $x \in X$. \(\square\)

We note that, under the hypothesis (b) of Theorem 3.1, the mapping $E_\mu: X \to X$
used in the proof shows up in several places in the existing literature. See e.g., Weyl’s unitary trick (that is, the fact that every representation of a compact group is similar to a unitary representation) or, more recently, [DLRZ02] and [OR03].

For the first corollary of Theorem 3.1 we recall that an operator space \( Y \) is said to be injective if for any complete isometry \( \varphi: X_0 \to X \) and every \( \psi_0 \in CB(X_0,Y) \) there exists \( \psi \in CB(X,Y) \) such that \( \psi \circ \varphi = \psi_0 \) and \( \| \psi \|_{cb} = \| \psi_0 \|_{cb} \). (See [ER00] for details.)

Corollary 3.2. Let \( X \) be a dual operator space and \( S \) an amenable semigroup of completely contractive, weak*-continuous linear mappings on \( X \). If \( X \) is an injective operator space, then \( X^S \) is in turn injective.

Proof. It follows by Theorem 3.1 along with condition (a) in Definition 2.2 that there exists a completely contractive projection \( P: X \to X \) with \( \text{Ran} \ P = X^S \). Now the desired conclusion follows by Proposition 4.1.6 in [ER00]. □

It is safe to say that most of the assertions contained in the following two theorems are parts of the folklore of operator algebras. However we would like to show how they follow directly from Theorem 3.1 and to emphasize that the idempotent mappings we construct here are completely bounded. Before going further, we recall that any *-homomorphism of \( C^* \)-algebras is completely contractive and any *-automorphism of a von Neumann algebra is weak*-continuous.

Theorem 3.3. Let \( A \) be a \( C^* \)-algebra, \( G \) a topological group, and \( \alpha: G \to \text{Aut}(A) \) a group homomorphism such that for each \( x \in A \) the map \( g \mapsto \alpha_g(x) \) is continuous with respect to the norm topology of \( A \). (When \( G \) is locally compact, the triple \( (A,G,\alpha) \) with these properties is called in literature a \( C^* \)-dynamical system.) Then the following hold true:

(a) If \( G \) is amenable then there exists a completely bounded idempotent mapping \( Q: A^* \to A^* \) whose range consists of all linear forms \( \phi \in A^* \) which are \( \alpha \)-invariant, i.e., \( \phi(\alpha_g(x)) = \phi(x) \) for all \( g \in G \) and all \( x \in A \). Moreover, if \( A \) is unital, then \( Q(S(A)) \subset S(A) \) hence \( Q \) maps the set of all states of \( A \) onto the set of all \( \alpha \)-invariant states.

(b) If \( G \) is a compact group, then there exists a completely positive and completely contractive idempotent \( P: A \to A \) with \( \text{Ran} \ P = \{ x \in A \mid \alpha_g(x) = x \text{ for all } g \in G \} \).

Proof. (a) Let us consider the action \( \beta \) of \( G \) on the dual space \( A^* \) defined by

\[
\beta(g,\phi)(x) = \phi(\alpha_{g^{-1}}(x))
\]

for all \( x \in A \) and all \( \phi \in A^* \). It easy to see that \( A^* \) becomes, via this action, a dual operator \( G \)-space satisfying Theorem 3.1 item (b) for the case when \( G \) is locally compact or item (c) for the case when \( G \) is a topological group. Now the conclusion follows from that theorem.

b) Follows immediately from Theorem 3.1(f) with \( X = A \). □

Remark 3.4. We refer to [Pe79] for general information of \( C^* \)-dynamical systems. In the case when \( G \) is an amenable locally compact group, the item (a) in Theorem 3.3 holds true under the weaker hypothesis that all the functions \( g \mapsto \phi(\alpha_g(x)) \) are continuous on \( G \) for all \( x \in A \) and all \( \phi \in A^* \). Indeed, in this case we can apply Theorem 3.1 item (b) to the operator \( G \)-space \( A \). □
Theorem 3.5. Let $M$ be a von Neumann algebra, $G$ a topological group and $\alpha : G \to \operatorname{Aut}(M)$ a group homomorphism such that for each $x \in M$ and each $\phi \in M^*$ (the predual of $M$) the functions $g \mapsto \phi(\alpha_g(x))$ are continuous on $G$. (When $G$ is locally compact, the triple $(M, G, \alpha)$ as above is called a $W^*$-dynamical system.) Then the following hold true:

(a) Suppose either $G$ is an amenable locally compact group or $G$ is an amenable (not necessarily locally compact) topological group with the additional hypothesis (in this general case) that for all $\phi \in M^*$ the mapping $g \mapsto \alpha^*_g(\phi)$ is continuous with respect to the norm topology of the predual $M_*$. Then there exists a completely positive unital (hence completely contractive too) idempotent mapping $P : M \to M$ whose range is the fixed point algebra of $\alpha$. In particular, it follows that if $M$ is injective, then the fixed point algebra is also injective.

(b) If $G$ is compact, then there exists a completely contractive idempotent mapping $Q : M_* \to M_*$ whose range is precisely the set of all $\alpha$-invariant normal forms on $M$. Moreover, $Q$ maps the set of all normal states of $M$ onto the set of all normal and $\alpha$-invariant states of $M$. The dual map $Q^* : M \to M$ is a faithful completely positive and normal idempotent mapping whose range is the fixed point algebra of $\alpha$ (faithful means that $\ker P \cap M^+ = \{0\}$).

Proof. a) This follows from Theorem 3.1 item (b) for the case when $G$ is locally compact or from item (c) when $G$ is a topological group.

b) Let us consider, in a similar way as in the proof of the preceding theorem, the action

$$\beta : G \times M_* \to M_*$$

defined by

$$\beta(g, \phi)(x) = \phi(\alpha_{g^{-1}}(x)).$$

Then $M_*$ becomes an operator $G$-space, and moreover, by Proposition 4.2.2.1 in [Wa72], the action $\beta$ is also continuous with respect to the norm topology on $M_*$. Now the existence and other properties (except faithfulness) of $Q^*$ follows from item (f) in Theorem 3.1. The expression of $Q$ as an integral with respect to the Haar measure on a compact group shows that $Q^* = P$, where $P$ is the one from (a). Thence the asserted properties of $Q^*$ follow. □

As another consequence of Theorem 3.1 we now get the following version of Theorem 16(b) in [Ke02]. See [Ke04] for more information on generalized Toeplitz operators.

Corollary 3.6. Let $(S, \cdot)$ be an amenable semigroup, $\mathcal{H}$ be a complex Hilbert space and $\rho : S \to \mathcal{B}(\mathcal{H})$ a norm-continuous mapping such that $\rho(st) = \rho(s)\rho(t)$, $\rho(1) = \operatorname{id}_\mathcal{H}$ and $\|\rho(s)\| \leq 1$ for all $s, t \in S$. Now consider the space of $\rho$-Toeplitz operators

$$\mathcal{T}(\rho) = \{C \in \mathcal{B}(\mathcal{H}) \mid (\forall s \in S) \quad \rho(s)C\rho(s)^* = C\}.$$

Then there exists a completely positive, completely contractive mapping

$$P : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$$

with $\operatorname{Ran} P = \mathcal{T}(\rho)$, $P \circ P = P$ and $P(ADB^*) = AP(D)B^*$ whenever $D \in \mathcal{B}(\mathcal{H})$ and $A$ and $B$ belong to the commutant of $\rho(S)$. 

Proof. First note that condition (a) in Definition 2.2 is satisfied for \( X = B(H) \) with the structure of dual operator \( S \)-space defined by

\[
\alpha: S \times B(H) \to B(H), \quad \alpha(s, A) = \rho(s) A \rho(s)^*.
\]

Clearly \( B(H)S = T(\rho) \), hence Theorem 3.1 shows that there exists an idempotent completely contractive linear mapping \( P: B(H) \to B(H) \) with \( \text{Ran} \, P = T(\rho) \). Now it follows by the very construction of \( P \) that \( P \) is completely positive and \( P(ADB^*) = AP(D)B^* \) whenever \( D \in B(H) \) and \( A \) and \( B \) belong to the commutant of \( \rho(S) \). □

**Corollary 3.7.** Let \( T(\mathbb{T}) \) be the space of all Toeplitz operators on the Hardy space \( H^2(\mathbb{T}) \) associated with the unit disk. Then there exists a completely positive, completely contractive linear mapping \( P: B(H^2(\mathbb{T})) \to B(H^2(\mathbb{T})) \) such that \( P \circ P = P \) and \( \text{Ran} \, P = T(\mathbb{T}) \). In particular \( T(\mathbb{T}) \) is an injective operator space.

Proof. Let \( M_z: H^2(\mathbb{T}) \to H^2(\mathbb{T}), \quad (M_z f)(e^{i\theta}) = e^{i\theta} f(e^{i\theta}), \) the unilateral shift operator. It is well known that \( T(\mathbb{T}) = \{ C \in B(H^2(\mathbb{T})) \mid M_z^* CM_z = C \} \), hence the desired conclusion follows by Corollary 3.4 applied for the Abelian semigroup \( (S, \cdot) = (\mathbb{N}, +) \) and \( \rho: \mathbb{N} \to B(H^2(\mathbb{T})) \), \( \rho(n)C = (M_z^*)^n C \) for all \( n \in \mathbb{N} \). We note that \( (\mathbb{N}, +) \) is amenable since it is Abelian (see Theorem 1.2.1 in [Gr69]). □

As another consequence of Theorem 3.1 we now provide an alternative proof of Theorem 2.4(a) in [AGG02]. In the special case when \( M = B(H) \), the next corollary shows that the set \( C_{\alpha}(\varphi) = \{ X \in B(H) \mid \varphi(X) = X \} \) studied in [Po03] is an injective operator space provided \( \varphi: B(H) \to B(H) \) is a weak*-continuous, completely positive, completely contractive map.

**Corollary 3.8.** Let \( M \) be a \( W^* \)-algebra, \( \alpha: M \to M \) a weak*-continuous completely positive, completely contractive linear mapping, and denote

\[
\mathcal{M}^\alpha = \{ x \in M \mid \alpha(x) = x \}.
\]

Then there exists an idempotent, completely positive, completely contractive, linear mapping \( P: M \to M \) with \( \text{Ran} \, P = \mathcal{M}^\alpha \).

Proof. The existence of a completely contractive projection \( P \) from \( M \) onto \( \mathcal{M}^\alpha \) follows by Theorem 3.1 for \( X = M \), \( S = (\mathbb{N}, +) \) and

\[
\alpha(n, x) = \alpha^n(x)
\]

whenever \( n \in \mathbb{N} \) and \( x \in M \). To conclude the proof, we only have to remark that the idempotent mapping \( P \) given by Theorem 3.1 is completely positive according to its construction, since \( \alpha: M \to M \) is completely positive (see also the construction of \( E_\mu \) in Definition 2.4). □

We now arrive at a corollary that has interesting consequences in providing certain homogeneous spaces with structures of Banach manifolds. See [BR04] and also Corollary 3.10 below.
Corollary 3.9. Let $\mathcal{X}$ be a complex Banach space, $S$ a topological group and
\[
\alpha: S \to \mathcal{B}(\mathcal{X}), \quad s \mapsto \alpha_s,
\]
a norm continuous representation of $S$ by bounded linear operators on $\mathcal{X}$ such that
\[
\alpha_1 = \text{id}_\mathcal{X} \quad \text{and} \quad \sup_{s \in S} \|\alpha_s\| < \infty.
\]
Assume that one of the following hypotheses holds:

(a) $S$ is an amenable topological group and $\mathcal{X}$ is a dual Banach space such that
\[
\alpha_s: \mathcal{X} \to \mathcal{X} \quad \text{is weak}^*\text{-continuous for all } s \in S,
\]
or
(b) $S$ is a compact topological group.

Next denote
\[
\mathcal{X}^S = \{ x \in \mathcal{X} \mid (\forall s \in S) \quad \alpha_s(x) = x \}.
\]
Then there exists a bounded linear operator $P \in \mathcal{B}(\mathcal{X})$ such that
\[
\|P\| \leq \sup_{s \in S} \|\alpha_s\|,
\]
\[
\text{Ran } P = \mathcal{X}^S \quad \text{and} \quad P^2 = P.
\]

Proof. We are going to apply Theorem 3.1 for the operator space $\mathcal{X} = \text{max} \mathcal{X}$. According to (3.3.9) in [ER00] we have an isometric identification $\mathcal{B}(\mathcal{X}) \simeq \mathcal{CB}(\text{max} \mathcal{X})$, hence it follows at once that $\text{max} \mathcal{X}$ is an operator $S$-space. On the other hand, the above identification shows that the desired conclusion will follow as soon as we show that each of the present hypotheses implies one of the conditions of Theorem 3.1 for the operator space $\mathcal{X} = \text{max} \mathcal{X}$.

Actually, it is obvious that the present hypothesis (b) implies that condition (f) in Theorem 3.1 is fulfilled. As for the present hypothesis (a), note that it implies that the condition (c) in Theorem 3.1 is satisfied. In fact, it follows by (3.3.15) in [ER00] that if $\mathcal{Y}$ is a Banach space such that $\mathcal{X} = \mathcal{Y}^*$ then $(\text{min} \mathcal{Y})^* = \text{max} \mathcal{Y}^* = \text{max} \mathcal{X}$, hence $\text{max} \mathcal{X}$ is the dual operator space of $\text{min} \mathcal{Y}$, and we are done. □

The next result is a partial extension of Theorem 4.8 in [CG99] and is also related to Theorems 3.12 and 4.4 in [ACS95]. We note that under hypothesis (a) of this corollary we do not require that the group $G$ should be locally compact, and thus the result holds for infinite-dimensional Lie groups.

Corollary 3.10. Let $A$ be a unital operator algebra and denote by $A^\times$ its group of invertible elements. Consider an amenable topological group $G$ and denote
\[
\mathcal{R} := \{ \rho: G \to A^\times \mid \rho \text{ continuous group homomorphism and } \sup_{g \in G} \|\rho(g)\| < \infty \}.
\]
Assume that one of the following conditions is satisfied:

(i) $A$ is a dual algebra, or
(ii) $G$ is compact.

Then the orbits of the action
\[
A^\times \times \mathcal{R} \to \mathcal{R}, \quad (a, \rho) \mapsto a \cdot \rho(-) \cdot a^{-1},
\]
have natural structures of Banach manifolds that are smoothly acted on by the Banach-Lie group $A^\times$.

Proof. In the proof we need techniques and ideas from Lie theory that were recalled in the Introduction. Fix $\rho \in \mathcal{R}$ and consider its isotropy group
\[
(A^\times)_\rho = \{ a \in A^\times \mid (\forall g \in G) \quad a \cdot \rho(g) \cdot a^{-1} = \rho(g) \}. 
\]
We shall prove that $(A^\times)_\rho$ is a Banach-Lie subgroup of $A^\times$, and then the desired conclusion follows by Theorem 8.19 in [Up85] in view of the natural bijection that exists from $A^\times/(A^\times)_\rho$ onto the orbit of $\rho$.

To show that $(A^\times)_\rho$ is a Banach-Lie subgroup of $A^\times$, we first note that it is an algebraic subgroup of $A^\times$ of degree $\leq 1$ in the sense explained in the Introduction to the present paper. It then follows that $(A^\times)_\rho$ has a structure of Banach-Lie group with the topology inherited from $A^\times$, as a consequence of the main result of [HK77]. The Lie algebra of $A^\times$ is $L(A^\times) = A$, while the Lie algebra of $(A^\times)_\rho$ is

$$L((A^\times)_\rho) = \{a \in A \mid (\forall g \in G) \ a \cdot \rho(g) = \rho(g) \cdot a \} = \rho(G)' ,$$

hence it remains to prove that $\rho(G)'$ has a complement in $A$.

To this end, consider the action of $G$ on $A$ defined by

$$\alpha: G \times A \rightarrow A, \quad \alpha(g, a) = \rho(g)a\rho(g)^{-1}.$$

This action makes $A$ into an operator $G$-space, since $\alpha(g, \cdot)$ is completely bounded on $A$ for all $g \in G$ as an easy consequence of Theorem 17.1.2 in [ER00]. Moreover note that condition (b) in Definition 2.2 is satisfied. In case (i), it follows by Theorem 2.1 in [Bl01] that the multiplication in $A$ is separately weak$^*$-continuous, hence $A$ is actually a dual operator $G$-space. Now we see that in either of the cases (i) and (ii) it follows by Theorem 3.1 that there exists a completely bounded idempotent mapping $P: A \rightarrow A$ with $\text{Ran} \ P = A^G = \rho(G)'$, and we are done. □

We point out that some further smoothness properties of similarity orbits of group representations (in particular existence of complex structures on the unitary orbits) are discussed in [Ma90] and [MS95].

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