Generalized solutions to a chemotaxis-Navier-Stokes system with arbitrary superlinear degradation

Mengyao Ding\textsuperscript{1}, Johannes Lankeit\textsuperscript{2}

\textsuperscript{1} School of Mathematical Sciences, Peking University, 100871 Beijing, PR China
\textsuperscript{2} Leibniz Universität Hannover, Institut für Angewandte Mathematik, Welfengarten 1, 30167 Hannover, Germany

Abstract. In this work, we study a chemotaxis-Navier-Stokes model in a two-dimensional setting as below,

\begin{align}
  n_t + u \cdot \nabla n &= \Delta n - \nabla \cdot (n \nabla c) + f(n), & x \in \Omega, t > 0, \\
  c_t + u \cdot \nabla c &= \Delta c - c + n, & x \in \Omega, t > 0, \\
  u_t + \kappa(u \cdot \nabla)u &= \Delta u + \nabla P + n \nabla \phi, & x \in \Omega, t > 0, \\
  \nabla \cdot u &= 0, & x \in \Omega, t > 0.
\end{align}

(0.1)

Motivated by a recent work due to Winkler, we aim at investigating generalized solvability for the model without imposing a critical superlinear exponent restriction on the logistic source function $f$. Specifically, it is proven in the present work that there exists a triple of integrable functions $(n, c, u)$ solving the system globally in a generalized sense provided that $f \in C^1([0, \infty))$ satisfies $f(0) \geq 0$ and $f(n) \leq r n - \mu n^\gamma$ ($n \geq 0$) with any $\gamma > 1$. Our result indicates that persistent Dirac-type singularities can be ruled out in our model under the aforementioned mild assumption on $f$. After giving the existence result for the system, we also show that the generalized solution exhibits eventual smoothness as long as $\mu/r$ is sufficiently large.

Keywords: chemotaxis; fluid; logistic source; generalized solution; eventual smoothness

Mathematics Subject Classification 2020: 92C17; 35K55; 35A01; 35D99
1 Introduction

In this article, we consider the following Keller-Segel-Navier-Stokes system

\[
\begin{align*}
    n_t + u \cdot \nabla n &= \Delta n - \nabla \cdot (n \nabla c) + f(n), & x \in \Omega, \; t > 0, \\
    c_t + u \cdot \nabla c &= \Delta c - c + n, & x \in \Omega, \; t > 0, \\
    u_t + \kappa (u \cdot \nabla) u &= \Delta u + \nabla P + n \nabla \phi, & x \in \Omega, \; t > 0, \\
    \nabla \cdot u &= 0, & x \in \Omega, \; t > 0, \\
    \partial_\nu n = \partial_\nu c &= u = 0, & x \in \partial \Omega, \; t > 0, \\
    n(x,0) &= n_0(x), c(x,0) = c_0(x), u(x,0) = u_0(x), & x \in \Omega,
\end{align*}
\]

(1.1)

which models a chemotactically active species in a fluid environment, in a bounded domain \( \Omega \subset \mathbb{R}^2 \). While global classical solvability cannot be expected for arbitrary initial data due to the propensity for blow-up inherent in the chemotaxis subsystem, solvability in a generalized sense has recently been proven in the fluid-free setting, [36]. In the present work we show that a framework of generalized solvability can even cope with the coupling to a Navier–Stokes fluid \((\kappa = 1)\), despite the latter negatively affecting a priori known regularity properties of the system.

The system. The quantities in (1.1) denote a density \( n \) of cells (for example, of bacteria); the concentration \( c \) of a chemical signal towards higher concentrations of which the cells move, as indicated by the cross-diffusive “chemotaxis term”; and the velocity field \( u \) and pressure \( P \) of the fluid, which transports both cells and chemical, and in turn is affected by buoyancy forces driven by differences in density between cells and fluid. The given function \( \phi \in W^{2,\infty}(\Omega) \) models the gravitational potential causing the buoyance, and \( f \in C^1([0, \infty)) \) is used to describe the growth of the population and will be a generalization of the typical choices of logistic-type reproduction. Without fluid motions included, the model (1.1) turns to be the system

\[
\begin{align*}
    n_t &= \Delta n - \nabla \cdot (n \nabla c) + f(n), & x \in \Omega, \; t > 0, \\
    c_t &= \Delta c - c + n, & x \in \Omega, \; t > 0,
\end{align*}
\]

(1.2)

considered in bounded domains \( \Omega \subset \mathbb{R}^d \), which has attracted great attention for decades (cf. e.g. the surveys [18, 2, 24]). When \( f(s) \equiv 0 \), the system – then the classical Keller–Segel system – admits distinct solution behaviour, from global existence to blow-up of solutions, depending on the spatial dimension and the initial data and their mass (cf. the above-mentioned surveys and references therein, in particular [29, 27, 16, 26, 41, 37]).

To what extent exactly the presence of logistic-type terms \( f(s) = rs - \mu s^\gamma \) with \( \gamma > 1, \mu > 0, r \in \mathbb{R} \), in the first equation of (1.1) or (1.2) hinders the occurrence of blow-up is still subject of ongoing research. Several partial results, however, are known:

In the most prototypical case of \( \gamma = 2 \), solutions are global and classical if \( d \leq 2 \), [28], and if \( d \geq 3 \) according to [38] there is \( \mu_0 > 0 \) such that \( \mu > \mu_0 \) ensures global classical solvability. The dependence of an upper bound for \( \mu_0 \) on other parameters in the system has been investigated by Xiang in [47] following the approach of [38].

Without imposing any largeness restriction on \( \mu \), the solvability in the weak sense was determined in [21] for any \( d \geq 3 \) and \( r > 0 \), the eventual smoothness of these weak solutions was also discussed under the condition that \( d = 3 \) and \( r \) is sufficiently small in [21].

In parabolic–elliptic relatives of the system – for which, in principle, boundedness results similar to those reported for (1.2) are available (see [32]) – blow-up has been detected in some cases: First in [39] for \( d \geq 5 \) and \( 0 < \gamma < \frac{3}{2} + \frac{1}{2d-2} \) in a system where the second equation of (1.2) is replaced by
$0 = \Delta c - \frac{1}{\kappa} f_0 n + n$, but also for systems with $0 = \Delta c - c + n$ and in $d \geq 3$ such results have more recently been obtained, \cite{43, 10, 4}.

Recently, considering a suitably designed framework of generalized solvability, Winkler \cite{36} showed that global solution $(n, c)$ with $n \in L^1_{loc}(\mathbb{R}^d \times [0, \infty))$ can be constructed under the mere hypothesis that $f$ satisfies $f(0) \geq 0$ and

$$\frac{f(s)}{s} \to -\infty \quad \text{as} \quad s \to \infty.$$  \hfill (1.3)

This result indicates that the mild condition (1.3) can rule out the occurrence of persistent Dirac-type singularities in the model (1.2). Inspired by \cite{36}, the present work is devoted to proving that there is no critical superlinear exponent on the logistic function $f$ for ensuring the generalized solvability when the model is coupled by the Navier-Stokes equation.

The interest in the coupling of fluid equations to chemotaxis systems, although initially focussed on systems where the signal substance is consumed (see the overview in \cite{2, Sec. 4.1.1} or the derivation in \cite{1}) also extends to settings with signal production, see e.g. \cite{19, 8}, in the context of broadcast spawning of corals, or \cite{3, 46, 48, 25, 35, 49}. If the fluid flow is governed by the full Navier–Stokes equations, the resulting system is (1.1) with $\kappa = 1$. Classical solvability necessarily can only be expected in settings where both the Keller–Segel subsystem and the Navier–Stokes equations have classical solutions. For a small-data existence result in the case of $f \equiv 0$ see \cite{48}. Several further findings concerning classical or weak solutions rest on stronger nonlinear diffusion, \cite{3, 25, 49}, a decaying sensitivity function \cite{35, 25, 49} or, especially for $d \geq 3$, simplification of the fluid flow to the Stokes equation \cite{35, 49}.

When the logistic term $f(s) = rs - \mu s^2$ is involved, Tao and Winkler \cite{31} proved that the model (1.1) with $d = 2$ admits a global and bounded classical solution regardless of the size of $r \geq 0, \mu > 0$. For $d \geq 3$ with a Stokes-governed flow ($\kappa = 0$), classical solutions were found in \cite{30} if $\mu > 23$. These results, however, rely on $\gamma = 2$. For less than quadratic degradation terms in $f$, generalized solutions have recently been found (see \cite{6} for $d = 2$ and \cite{34} for $d = 3$), but only for the case of Stokes-fluid.

Motivated by \cite{36}, the purpose of the present paper is to study the global solvability of the model (1.1) under a mild assumption on the logistic function.

For the initial data, we assume

$$\left\{ \begin{array}{l} n_0 \in L^1(\Omega) \quad \text{with} \quad n_0 \geq 0 \quad \text{in} \quad \Omega \quad \text{and} \quad n_0 \not\equiv 0, \\
_0 \in \mathcal{D}(\Delta + 1) \quad \text{for some} \quad \sigma \in (0, \frac{1}{d}), \\
_0 \in L^2(\Omega; \mathbb{R}^d) \quad \text{and} \quad \nabla \cdot u_0 = 0 \quad \text{in} \quad \mathcal{D}'(\Omega), \end{array} \right.$$  \hfill (1.4)

where $\Delta$ is the Neumann-Laplacian on $L^2(\Omega)$ (see also Sec. 4.2). With these, our main result reads as follows.

**Theorem 1.1.** Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary. Suppose that $f \in C^1([0, \infty))$ is such that

$$f(0) = 0, \quad f(s) \leq rs - \mu s^\gamma \quad \text{for all} \quad s \geq 0$$  \hfill (1.5)

with $r \in \mathbb{R}$, $\mu > 0$ and $\gamma > 1$. Then for any given initial data $(n_0, c_0, u_0)$ satisfying (1.4), there exist functions

$$\left\{ \begin{array}{l} n \in L^1_{loc}(\mathbb{R}^d \times [0, \infty)), \\
c \in L^2_{loc}(\mathbb{R}^d \times W^{1,2}(\Omega)), \\
u \in L^2_{loc}(\mathbb{R}^d \times W^{1,2}(\Omega; \mathbb{R}^d)) \end{array} \right.$$  \hfill (1.6)
with the property that \((n, c, u)\) forms a global generalized solution of (1.1) in the sense of Definition 2.3 below.

Whereas Theorem 1.1 answers the question of existence rather completely, the regularity of solutions as guaranteed by (1.6) is far from that of classical solutions. On the other hand, for \(\gamma = 2\) the system (at least without fluid) is known to regularize its solutions: It has been shown ([23]) that even initial data in \(L^1 \times W^{1,2}\) result in solutions that are smooth in \(\Omega \times (0, \infty)\) in two-dimensional domains \(\Omega\), and in a three-dimensional setting solutions become smooth after some waiting time if \(r\) is not too large, [21]. Also in related chemotaxis-fluid systems with logistic sources it has been demonstrated that for large times regularity or even convergence can be achieved, see e.g. [22, 33] for a related system with signal consumption and [44] for a stabilization result concerning solutions of (1.1) with \(\gamma = 2\) and small \(r\). Here we show that the same eventual smoothness already occurs in systems with much weaker degradation (\(\gamma > 1\)), again under a smallness condition on \(r\) (cf. [44, 22]):

**Theorem 1.2.** Let \(\Omega \subset \mathbb{R}^2\) be a bounded domain with smooth boundary. Let \(\gamma > 1\). Then there is \(\mu_0 = \mu_0(\gamma, \Omega) > 0\) such that for every function \(f\) fulfilling (1.5) with \(r \in \mathbb{R}\) and \(\mu > \mu_0 r_+\) and for all initial data \((n_0, c_0, u_0)\) as in (1.4), the global generalized solution \((n, c, u)\) of (1.1) constructed in Theorem 1.1 satisfies

\[
n \in C^{2,1}(\overline{\Omega} \times [T, \infty)), \quad c \in C^{2,1}(\overline{\Omega} \times [T, \infty)), \quad u \in C^{2,1}(\overline{\Omega} \times [T, \infty))
\]

with some \(T > 0\).

**Remark 1.3.** For the fluid-free system it has recently been shown that for \(\gamma > 1\) (more generally, for \(\gamma \geq 2 - \frac{2}{n}\)) and sufficiently large \(\frac{\mu}{n}\), solutions \((n, c)\) essentially converge to the spatially homogeneous equilibrium in \(L^1(\Omega) \times L^2(\Omega)\) as \(t \to \infty\), [45]. The study relies on a Lyapunov functional in which, when evaluated for solutions of (1.1), all fluid-terms vanish immediately. It is therefore most likely to be expected that solutions of (1.1) have the same property.

## 2 The solution concept

While the notion of solution for the first equation in (1.1) is somewhat more involved, the second and third equation can be understood in a rather usual weak sense. In order to indicate the space to which \(u\) belongs, we introduce the solenoidal subspace of \(L^2(\Omega; \mathbb{R}^2)\) as

\[
L_\sigma^2(\Omega) := \{ \varphi \in L^2(\Omega; \mathbb{R}^2) \mid \nabla \cdot \varphi = 0 \text{ in } D'(\Omega) \}
\]

and abbreviate \(W^{1,2}_{0,\sigma}(\Omega; \mathbb{R}^2) = W^{1,2}_{0}(\Omega; \mathbb{R}^2) \cap L_\sigma^2(\Omega)\).

**Definition 2.1.** A pair \((u, n)\) of functions

\[
\begin{cases}
  u \in L^2_{\text{loc}}([0, \infty); W^{1,2}_{0,\sigma}(\Omega; \mathbb{R}^2)) \\
  n \in L^1_{\text{loc}}(\overline{\Omega} \times [0, \infty))
\end{cases}
\]

satisfying \(n \geq 0\) is said to globally solve the equation

\[
\begin{cases}
  u_t + \kappa(u \cdot \nabla) u = \Delta u + \nabla P + n \nabla \phi, & \text{in } \Omega, t > 0, \\
  \nabla \cdot u = 0, & \text{in } \Omega, t > 0, \\
  u = 0, & \text{on } \partial \Omega, t > 0, \\
  u(0) = u_0, & \text{in } \Omega
\end{cases}
\]

where

\[
\kappa = \begin{cases}
  \frac{n}{n + 1}, & \text{in } \Omega \\
  \frac{n}{n + 1} + \frac{1}{n + 1}, & \text{on } \partial \Omega
\end{cases}
\]
in the weak sense if
\[-\int_\Omega u_0 \varphi(0) - \int_0^\infty \int_\Omega u_\varphi - \kappa \int_0^\infty \int_\Omega u \otimes u \cdot \nabla \varphi = \int_0^\infty \int_\Omega n \Delta \varphi - \int_0^\infty \int_\Omega \nabla u \cdot \nabla \varphi \] (2.2)
holds for every \( \varphi \in C_0^\infty (\Omega; \mathbb{R}^2) \) with \( \nabla \cdot \varphi \equiv 0 \) in \( \Omega \times (0, \infty) \).

**Definition 2.2.** A triple \((n, c, u)\) of functions
\[
\begin{cases}
  n \in L^1_{\text{loc}}(\Omega \times [0, \infty)), \\
  c \in L^2_{\text{loc}}([0, \infty); W^{1,2}(\Omega)), \\
  u \in L^2_{\text{loc}}([0, \infty); W^{1,2}_0(\Omega; \mathbb{R}^2))
\end{cases}
\] (2.3)
satisfying \( n \geq 0, c \geq 0 \) is said to globally solve the problem
\[
\begin{cases}
  c_t + u \cdot \nabla c = \Delta c - c + n, & x \in \Omega, t > 0, \\
  \partial_n c = 0, & x \in \partial \Omega, t > 0, \\
  c(0) = c_0, & x \in \Omega
\end{cases}
\] in the weak sense if
\[-\int_\Omega c_0 \varphi(0) - \int_0^\infty \int_\Omega c \varphi_t = -\int_0^\infty \int_\Omega c \varphi \cdot \nabla \varphi + \int_0^\infty \int_\Omega c u \cdot \nabla \varphi + \int_0^\infty \int_\Omega n \varphi - \int_0^\infty \int_\Omega c \varphi \] (2.4)
holds for every \( \varphi \in C_0^\infty (\widetilde{\Omega} \times [0, \infty)) \).

With the two latter components weakly solving the corresponding equations, the generalized solution \((n, c, u)\) of (1.1) is exhibited as \( n \) satisfies the first equation of (1.1) in a certain form which was first established in [42]. It combines a logarithmic supersolution property of \( n \) (cf. also [20]) with an upper estimate for its mass.

**Definition 2.3.** Let a triple \((n, c, u)\) of functions
\[
\begin{cases}
  n \in L^1_{\text{loc}}(\Omega \times [0, \infty)), \\
  c \in L^1_{\text{loc}}([0, \infty); W^{1,2}(\Omega)), \\
  u \in L^1_{\text{loc}}([0, \infty); W^{1,2}_0(\Omega; \mathbb{R}^2))
\end{cases}
\] (2.5)
satisfy \( n \geq 0, c \geq 0 \) and be such that
\[f(n) \in L^1_{\text{loc}}(\Omega \times [0, \infty)).\] (2.6)
Then \((n, c, u)\) will be called a global generalized solution of (1.1) if (2.3)-(2.2) are satisfied, and if
\[
\int_\Omega n(\cdot, t) \leq \int_\Omega n_0 + \int_0^t \int_\Omega f(n) \quad \text{for a.e. } t > 0,
\] (2.7)
and if
\[ (n + 1)^{-1} |\nabla n|^2 \in L^1_{\text{loc}}(\Omega \times [0, \infty)) \] (2.8)
and if
\[
\int_0^\infty \int_\Omega \ln(n + 1) \varphi_t + \int_\Omega \ln(n_0 + 1) \varphi(0)
\]
holds for each nonnegative function $\varphi \in C_0^\infty(\Omega \times [0, \infty))$.

Like its famous precursor, the concept of renormalized solutions (see [7], transferred to the context of chemotaxis systems in [42]), this notion of solvability rests on the idea that it may be easier to derive a priori estimates for or pass to the limit in integrals involving not the solution itself, but a nonlinear transformation thereof, here $\ln(n+1)$ instead of $n$.

For the solutions in [36, 6, 34] (dealing with (1.1) in a fluid-free variant or for a Stokes fluid), a combined quantity of the form $\phi(n)\psi(c)$, with $\phi$ being a bounded, decreasing and convex function, was employed. This additional step further away from classical solvability was not necessary for Definition 2.3 or Theorem 1.1. This is because in the two-dimensional case, we can investigate the regularity of solutions more precisely by applying fractional powers of the operator $\mathcal{L} = -\Delta + 1$, which allows us to construct the generalized solution closer to a classical one. Let us focus on the component $c$ and specifically discuss how the above idea is performed: When studying the energy development of $c$, the time-space estimates are established by testing the equation by $c$ or $\mathcal{L} c$ in the existing literature. But the resulting $L^\infty L^2$-boundedness of $c$ can not give all desired estimates and the $L^\infty L^2$-boundedness of $\mathcal{L}^{1/2} c$ seems to require the assumption $\gamma \geq 2$. To handle this difficulty, we turn to the test function $\mathcal{L}^\beta c$ with a proper exponent $\beta$ ensuring that the estimates of $\|\mathcal{L}^{(\beta+1)/2} c\|_{L^\infty L^2}$ can be built on the condition $\gamma > 1$ and suffice to proceed to the compactness arguments. Here, we remark that the aforementioned reasoning needs to be performed under the assumption of two-dimensionality due to the appearance of the convection term $u \cdot \nabla c$.

3 Solutions of an approximate system

In the following, we fix $\kappa = 1$, $\lambda > 1$, $\sigma \in (0, \frac{1}{2})$ (cf. (1.4)), $r \in \mathbb{R}$, $\mu > 0$, $\gamma > 1$, $f$ fulfilling (1.5) and let $n_0$, $c_0$, $u_0$ be as in (1.4). We furthermore introduce a family of functions

\[
\begin{cases}
  n_{0\varepsilon} &\in C^1(\overline{\Omega}) \text{ with } n_{0\varepsilon} \geq 0 \text{ in } \Omega \text{ and } n_{0\varepsilon} \neq 0, \\
  c_{0\varepsilon} &\in C^1(\overline{\Omega}) \cap D((-\Delta + 1)^{2\sigma}) \text{ with } c_{0\varepsilon} \geq 0 \text{ in } \Omega, \\
  u_{0\varepsilon} &\in C^1(\overline{\Omega}; \mathbb{R}^2) \text{ with } \nabla \cdot u_{0\varepsilon} = 0 \text{ in } \Omega \text{ and } u_{0\varepsilon} = 0 \text{ on } \partial\Omega
\end{cases}
\]

(3.1)
satisfying

\[
\int_{\Omega} n_{0\varepsilon} \leq 2 \int_{\Omega} n_0 \text{ for each } \varepsilon \in (0, 1).
\]

(3.2)

and

\[
n_{0\varepsilon} \to n_0 \text{ in } L^1(\Omega), \quad c_{0\varepsilon} \to c_0 \text{ in } D((-\Delta + 1)^{2\sigma}) \text{ as well as } u_{0\varepsilon} \to u_0 \text{ in } L^2(\Omega)
\]

(3.3)
as $\varepsilon \to 0$, where $-\Delta$ stands for the Neumann-Laplacian in $L^2(\Omega)$, see (4.24).
We then intend to construct the generalized solution whose existence Theorem 1.1 claims as limit of a sequence of solutions of the approximate system

\[
\begin{align*}
  n_{xt} + u_x \cdot \nabla n_x &= \Delta n_x - \nabla \cdot (n_x \nabla c_x) + f(n_x) - \varepsilon n_x^2, & x \in \Omega, \ t > 0, \\
  c_{xt} + u_x \cdot \nabla c_x &= \Delta c_x - c_x + \frac{n_x}{1 + \varepsilon n_x}, & x \in \Omega, \ t > 0, \\
  u_{xt} + \kappa(u_x \cdot \nabla) u_x &= \Delta u_x + \nabla P_x + n_x \nabla \phi, & x \in \Omega, \ t > 0, \\
  \nabla \cdot u_x &= 0, & x \in \Omega, \ t > 0, \\
  \partial_n n_x &= \partial_n c_x = 0, u_x = 0, & x \in \partial \Omega, \ t > 0, \\
  n_x(x, 0) = n_{0e}(x), c_x(x, 0) = c_{0e}(x), u_x(x, 0) = u_{0e}(x), & x \in \Omega,
\end{align*}
\]

(3.4)

The modifications in its first two equations (if compared to (1.1)) ensure global classical solvability:

**Lemma 3.1.** Let \( \varepsilon > 0 \). Then there exist functions \( (n_x, c_x, u_x) \) satisfying \( n_x, c_x \geq 0 \) in \( \overline{\Omega} \times [0, \infty) \) and

\[
\begin{align*}
  n_x &\in C^0([0, \infty) \times \overline{\Omega}) \cap C^{2,1}([0, \infty) \times (0, \infty)), \\
  c_x &\in C^0([0, \infty) \times \overline{\Omega}) \cap C([0, \infty); D((-\Delta + 1)^{2\sigma}) \cap C^{2,1}([0, \infty) \times (0, \infty)), \\
  u_x &\in C^0([0, \infty) \times \overline{\Omega} \times \mathbb{R}^2) \cap C^{2,1}([0, \infty) \times (0, \infty)),
\end{align*}
\]

(3.5)

together with some \( P_x \in C^{1,0}([0, \infty) \times \overline{\Omega}) \) such that \( (n_x, c_x, u_x, P_x) \) is a classical and global solution of (3.4). This solution is unique within the indicated class, up to addition of spatially constant functions to \( P_x \).

**Proof.** This is covered by the setting of [31], with local existence and regularity proven along the lines of [40, Lemma 2.1]. \( \square \)

Given any \( \varepsilon > 0 \), by \( (n_x, c_x, u_x, P_x) \) we refer to this solution. In a first step we ensure that it satisfies an integral identity resembling that of Definition 2.3:

**Lemma 3.2.** Let \( \varepsilon > 0 \). Then for any \( \varphi \in C^\infty(\overline{\Omega} \times (0, \infty)) \), we have

\[
\int_\Omega \partial_t \ln(n_x + 1) \varphi = \int_\Omega \nabla \ln(n_x + 1)^2 \varphi - \int_\Omega (n_x + 1)^{-1} \nabla n_x \cdot \nabla \varphi + \int_\Omega \ln(n_x + 1) u_x \cdot \nabla \varphi - \int_\Omega (n_x + 1)^{-2} n_x \nabla n_x \cdot \nabla c_x \varphi + \int_\Omega (n_x + 1)^{-1} n_x \nabla c_x \cdot \nabla \varphi + \int_\Omega (n_x + 1)^{-1} f(n_x) \varphi - \varepsilon \int_\Omega (n_x + 1)^{-1} n_x^2 \varphi \quad \text{in (0, \infty)}.
\]

(3.6)

In lieu of proof we give the following more general lemma, from which Lemma 3.2 results upon inserting \( F(s) = \ln s, F'(s) = \frac{1}{s} \) and \( -F''(s) = \frac{1}{s^2} \).

**Lemma 3.3.** Let \( F \in C^2((0, \infty)) \). Then for every \( \varepsilon > 0 \) and \( \varphi \in C^\infty(\overline{\Omega} \times (0, \infty)) \),

\[
\int_\Omega \partial_t F(n_x + 1) \varphi = -\int_\Omega F''(n_x + 1)|\nabla n_x|^2 \varphi - \int_\Omega F'(n_x + 1) \nabla n_x \cdot \nabla \varphi + \int_\Omega F(n_x + 1) u_x \cdot \nabla \varphi + \int_\Omega F'(n_x + 1) n_x \nabla n_x \cdot \nabla c_x \varphi + \int_\Omega F''(n_x + 1) n_x \nabla c_x \cdot \nabla \varphi + \int_\Omega F'(n_x + 1) f(n_x) \varphi - \varepsilon \int_\Omega n_x^2 F'(n_x + 1) \varphi \quad \text{in (0, \infty)}.
\]
Proof. Applying the chain rule and inserting (3.4) gives that

\[ \varphi \partial_t F(n_\varepsilon + 1) = \varphi F'(n_\varepsilon + 1) \left( \Delta n_\varepsilon - \mathbf{u}_\varepsilon \cdot \nabla n_\varepsilon - \nabla \cdot (n_\varepsilon \nabla c_\varepsilon) + f(n_\varepsilon) - \varepsilon n_\varepsilon^2 \right) \]  

in \( \Omega \times (0, \infty) \). Integration by parts shows that

\[ \int_{\Omega} F'(n_\varepsilon + 1) \Delta n_\varepsilon \varphi = -\int_{\Omega} F''(n_\varepsilon + 1) |\nabla n_\varepsilon|^2 \varphi - \int_{\Omega} F'(n_\varepsilon + 1) \nabla n_\varepsilon \cdot \nabla \varphi \]  

and, since \( \nabla \cdot \mathbf{u}_\varepsilon = 0 \) in \( \Omega \times (0, \infty) \),

\[ -\int_{\Omega} (n_\varepsilon + 1)^{-1} \mathbf{u}_\varepsilon \cdot \nabla n_\varepsilon \varphi = -\int_{\Omega} \mathbf{u}_\varepsilon \cdot \nabla F(n_\varepsilon + 1) \varphi = \int_{\Omega} F(n_\varepsilon + 1) \mathbf{u}_\varepsilon \cdot \nabla \varphi, \]  

(3.8)

hold true in \( (0, \infty) \), as does

\[ -\int_{\Omega} F'(n_\varepsilon + 1) \nabla \cdot (n_\varepsilon \nabla c_\varepsilon) \varphi = -\int_{\Omega} F''(n_\varepsilon + 1) n_\varepsilon \nabla n_\varepsilon \cdot \nabla c_\varepsilon \varphi + \int_{\Omega} F'(n_\varepsilon + 1) n_\varepsilon \nabla c_\varepsilon \cdot \nabla \varphi. \]  

(3.9)

(3.10)

Integrating (3.7) over \( \Omega \) and inserting (3.9), (3.8) and (3.10), we conclude the proof. \( \square \)

4 Uniform estimates

In this first step of the proof we derive the following integrability properties of \( n_\varepsilon, f(n_\varepsilon) \). This result will serve as fundamental ingredient for the forthcoming estimates.

Lemma 4.1. Let \( T > 0 \). Then there exists \( C = C(T) > 0 \) ensuring

\[ \int_0^T \int_\Omega |f(n_\varepsilon)| \leq C \quad \text{for all } \varepsilon \in (0, 1). \]  

(4.1)

Moreover, we have

\[ \sup_{\varepsilon \in (0, 1)} \sup_{t \geq 0} \int_{\Omega} n_\varepsilon(\cdot, t) < \infty \]  

(4.2)

and

\[ \int_{\Omega} n_\varepsilon(\cdot, t) \leq \int_{\Omega} n_{0, \varepsilon} + \int_0^t \int_{\Omega} f(n_\varepsilon) \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1). \]  

(4.3)

Proof. This results from an integration of (3.4) in exactly the same way as in [36, Lemma 4.1], since \( \nabla \cdot \mathbf{u}_\varepsilon = 0 \) in \( \Omega \times (0, T) \) and \( \mathbf{u}_\varepsilon = 0 \) on \( \partial \Omega \times (0, T) \) make the only integral containing the fluid term vanish. \( \square \)

Based on the structural assumption on \( f \), we use an interpolation inequality to state the following result as a direct application of Lemma 4.1.

Lemma 4.2. Let \( T > 0 \). There exists \( C = C(T) > 0 \) such that

\[ \|n_\varepsilon\|_{L^\gamma((0,T);L^\gamma(\Omega))} \leq C \quad \text{for all } \varepsilon \in (0, 1). \]  

(4.4)

Moreover, for any \( p, q > 1 \) satisfying \( p \leq \gamma \) and \( \frac{1}{p} + \frac{2}{q} = 1 \), we can find a positive constant \( C = C(p,q,T) \) such that

\[ \|n_\varepsilon\|_{L^p((0,T);L^q(\Omega))} \leq C \quad \text{for all } \varepsilon \in (0, 1). \]  

(4.5)
Proof. The claim \((4.4)\) can be directly achieved by combining the assumption \((1.5)\) on the form of \(f\) with \((4.1)\). Let \(p, q > 1\) be such that \(p \leq \gamma\) and \(\frac{1}{p} + \frac{2 - 1}{q} \geq 1\). By using the Hölder inequality, we have

\[
\int_0^T \left( \int_\Omega n^\varepsilon(x,t) dx \right)^\frac{q}{q-1} \leq \int_0^T \left( \int_\Omega \left( \int_\Omega \left( \int_0^T n^\varepsilon(x,t) dx \right)^q \right)^\frac{q}{p(q-1)} \right)^\frac{q}{q-1} \leq \left( \sup_{\varepsilon \in (0,T)} \|n^\varepsilon(\cdot, t)\|_{L^1(\Omega)} \right)^\frac{q}{p(q-1)} \int_0^T \left( \int_\Omega n^\varepsilon(x,t) dx \right)^\frac{q}{p(q-1)} (4.6)
\]

for all \(\varepsilon \in (0,1)\). Since the condition \(\frac{1}{p} + \frac{2 - 1}{q} \geq 1\) ensures \(\frac{q}{p(q-1)} \leq 1\), it holds that

\[
\left( \int_\Omega n^\varepsilon(x,t) dx \right)^\frac{q}{p(q-1)} \leq \int_\Omega n^\varepsilon(x,t) dx + 1 \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1).
\]

Thus, we can derive \((4.5)\) from \((4.2)\), \((4.4)\), \((4.6)\) and \((4.7)\).

Taking \(q = 2\) and \(p = \frac{2}{\gamma - 1}\) in the above lemma, we immediately have the following corollary.

**Corollary 4.3.** Let \(T > 0\). If \(\gamma \in (1,2)\), then there exists \(C = C(T) > 0\) such that

\[
\|n^\varepsilon\|_{L^2(0,T); L^{\frac{2}{\gamma - 1}}(\Omega)} \leq C \quad \text{for all } \varepsilon \in (0,1).
\]

We note that the derivation of Lemma 4.2 relies on \(f\) growing at least as fast as a superlinear power, as opposed to growing superlinearly only. The result in the form of \((4.8)\) will be a crucial ingredient in the following proofs. In order to apply this corollary, in several of the upcoming lemmata we will assume \(\gamma < 2\). They will still allow to conclude Theorem 1.1 for \(\gamma \geq 2\), too (see proof of Theorem 1.1 at the end of Section 5).

### 4.1 Estimates for \(\{u^\varepsilon\}_{\varepsilon \in (0,1)}\)

The main difference between \((1.1)\) and \((1.2)\) is the additional presence of \(u\). Fortunately, \((4.2)\), \((4.4)\), \((4.8)\) imply some \(\varepsilon\)-independent boundedness of \(u^\varepsilon\) and \(\nabla u^\varepsilon\).

**Lemma 4.4.** Let \(T > 0\). If \(\gamma \in (1,2)\), we can find a positive constant \(C = C(T)\) such that

\[
\int_\Omega |u^\varepsilon(\cdot, t)|^2 \leq C \quad \text{for all } t \in (0,T) \text{ and } \varepsilon \in (0,1)
\]

and

\[
\int_0^T \int_\Omega |\nabla u^\varepsilon(x,t)|^2 dxdt \leq C \quad \text{for all } \varepsilon \in (0,1).
\]

**Proof.** By using the imbedding \(W^{1,2}_0(\Omega) \hookrightarrow L^{\frac{2}{\gamma - 2}}(\Omega)\) and the Poincaré inequality, we can find \(c_1 = c_1(\gamma, \Omega) > 0\) such that

\[
\|u^\varepsilon\|_{L^{\frac{2}{\gamma - 2}}(\Omega)} \leq c_1 \|\nabla u^\varepsilon\|_{L^2(\Omega)} \quad \text{for all } \varepsilon \in (0,1).
\]
Testing (3.4) by \( u_\varepsilon \), noticing \( \nabla \cdot u_\varepsilon = 0 \) in \( \Omega \times (0, T) \) and defining \( c_2 := \| \phi \|_{L^\infty(\Omega)} \), we use the Hölder inequality and Young’s inequality to find
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |u_\varepsilon|^2 + \int_\Omega |\nabla u_\varepsilon|^2 = \int_\Omega n_\varepsilon u_\varepsilon \cdot \nabla \phi \leq c_2 \| u_\varepsilon \|_{L^\infty(\Omega)} \| n_\varepsilon \|_{L^\infty(\Omega)} \leq c_1 c_2 \| \nabla u_\varepsilon \|_{L^2(\Omega)} \| n_\varepsilon \|_{L^\infty(\Omega)} \leq \frac{1}{2} \int_\Omega |\nabla u_\varepsilon|^2 + \frac{c_1^2 c_2^2}{2} \| n_\varepsilon \|_{L^\infty(\Omega)}^2 \quad \text{in} \ (0, T) \ \text{for all} \ \varepsilon \in (0, 1),
\]
which directly shows that
\[
\frac{d}{dt} \int_\Omega |u_\varepsilon|^2 + \int_\Omega |\nabla u_\varepsilon|^2 \leq c_1^2 c_2^2 \| n_\varepsilon \|_{L^\infty(\Omega)}^2 \quad \text{in} \ (0, T) \ \text{for all} \ \varepsilon \in (0, 1). \quad (4.11)
\]
An integration of (4.11) over \((0, \tau)\) with any \( \tau \in (0, T) \) gives that
\[
\int_\Omega |u_\varepsilon(\cdot, \tau)|^2 + \int_0^\tau \int_\Omega |\nabla u_\varepsilon|^2 \leq c_1^2 c_2^2 \int_0^\tau \| n_\varepsilon(\cdot, t) \|_{L^\infty(\Omega)}^2 \, dt + \int_\Omega |u_0\varepsilon|^2 \leq c_1^2 c_2^2 \int_0^\tau \| n_\varepsilon(\cdot, t) \|_{L^\infty(\Omega)}^2 \, dt + \int_\Omega |u_0\varepsilon|^2 \quad \text{for all} \ \varepsilon \in (0, 1). \quad (4.12)
\]
Thus, by utilizing (4.8) and (3.3), we can deduce the existence of \( c_3 = c_3(\gamma, T, \Omega) > 0 \) such that
\[
\sup_{\tau \in (0, T)} \int_\Omega |u_\varepsilon(\cdot, \tau)|^2 + \int_0^\tau \int_\Omega |\nabla u_\varepsilon|^2 \leq c_3 \quad \text{for all} \ \varepsilon \in (0, 1), \quad (4.13)
\]
as desired.

**Corollary 4.5.** If \( \gamma \in (1, 2) \), letting \( T > 0 \), we can find \( C = C(T) > 0 \) satisfying
\[
\int_0^T \int_\Omega |u_\varepsilon|^4 \leq C \quad \text{for all} \ \varepsilon \in (0, 1). \quad (4.14)
\]

**Proof.** An application of the Gagliardo-Nirenberg inequality allows us to find \( c_1 > 0 \) satisfying
\[
\| u_\varepsilon \|_{L^4(\Omega)} \leq c_1 \| \nabla u_\varepsilon(\cdot, t) \|_{L^2(\Omega)}^{\frac{1}{2}} \| u_\varepsilon(\cdot, t) \|_{L^1(\Omega)}^{\frac{1}{2}} \quad \text{for all} \ \varepsilon \in (0, 1), \quad (4.15)
\]
which together with Lemma 4.4 implies that for some \( c_2 = c_2(T) > 0 \)
\[
\int_0^T \int_\Omega |u_\varepsilon(x, t)|^4 \, dx \, dt \leq c_1^4 \int_0^T \| \nabla u_\varepsilon(\cdot, t) \|_{L^2(\Omega)}^2 \| u_\varepsilon(\cdot, t) \|_{L^2(\Omega)}^2 \, dt \leq c_1^4 \left( \sup_{\varepsilon \in (0, T)} \| u_\varepsilon(\cdot, t) \|_{L^2(\Omega)} \right)^2 \int_0^T \| \nabla u_\varepsilon(\cdot, t) \|_{L^2(\Omega)}^2 \, dt \leq c_2 \quad (4.16)
\]
for all \( \varepsilon \in (0, 1) \).

**Lemma 4.6.** Let \( T > 0 \). If \( \gamma \in (1, 2) \), then \( \{ u_\varepsilon \}_{\varepsilon \in (0, 1)} \) is relatively compact with respect to the strong topology in \( L^2((0, T); L^2(\Omega; \mathbb{R}^2)) \).
Proof. We multiply (3.4) by $\xi \in C_0^\infty(\Omega; \mathbb{R}^2)$ with $\nabla \cdot \xi = 0$ and integrate by parts to get
\[
\int_{\Omega} u_{\epsilon t}(\cdot, t)\xi = \int_{\Omega} -\nabla u_{\epsilon} \cdot \nabla \xi - \kappa \int_{\Omega} (u_{\epsilon} \cdot \nabla) u_{\epsilon} \xi + \int_{\Omega} n_{\epsilon} \nabla \phi \cdot \xi \\
\leq \|\nabla u_{\epsilon}(\cdot, t)\|_{L^2(\Omega)} \|\nabla \xi\|_{L^2(\Omega)} + \kappa \|u_{\epsilon}(\cdot, t)\|_{L^2(\Omega)} \|\nabla u_{\epsilon}(\cdot, t)\|_{L^2(\Omega)} \|\xi\|_{L^{\infty}(\Omega)} + \|n_{\epsilon}(\cdot, t)\|_{L^1(\Omega)} \|\nabla \phi\|_{L^{\infty}(\Omega)} \|\xi\|_{L^{\infty}(\Omega)}
\]
for all $t \in (0, T)$ and $\epsilon \in (0, 1)$.

Thus, there exists $c_1 > 0$ such that
\[
\|u_{\epsilon t}(\cdot, t)\|_{(W^{1, \infty}_\epsilon(\Omega))'} \leq c_1 \left( \|\nabla u_{\epsilon}(\cdot, t)\|_{L^2(\Omega)} + \|u_{\epsilon}(\cdot, t)\|_{L^2(\Omega)} \|\nabla u_{\epsilon}(\cdot, t)\|_{L^2(\Omega)} + \|n_{\epsilon}(\cdot, t)\|_{L^1(\Omega)} \right)
\]
for all $t \in (0, T)$ and $\epsilon \in (0, 1)$.

This along with (4.3), (4.9) and (4.10) indicates that
\[
\{u_{\epsilon t}\}_{\epsilon \in (0, 1)} \text{ is bounded in } L^2(0, T; (W^{1, \infty}_\epsilon(\Omega) \cap L^2_\gamma(\Omega))^').
\]

Since $W^{1, 2}_\epsilon(\Omega) \cap L^2_\gamma(\Omega)$ is compactly embedded into $L^2(\Omega)$, and $L^2(\Omega)$ is continuously embedded in $(W^{1, \infty}_\epsilon(\Omega) \cap L^2_\gamma(\Omega))^*$, we obtain the desired result by (4.10) and (4.18) due to the Aubin-Lions lemma. \qed

4.2 Estimates for $\{c_{\epsilon}\}_{\epsilon \in (0, 1)}$

Lemma 4.7. Let $T > 0$. If $\gamma \in (1, 2)$, we can find a positive constant $C = C(T)$ such that
\[
\int_{\Omega} c_{\epsilon}^2(\cdot, t) \leq C \text{ for all } t \in (0, T) \text{ and } \epsilon \in (0, 1)
\]
and
\[
\int_{0}^{T} \int_{\Omega} |\nabla c_{\epsilon}(x, t)|^2 \, dx \, dt \leq C \text{ for all } \epsilon \in (0, 1).
\]

Proof. Testing (3.4) by $c_{\epsilon}$ and using the H"older inequality, we have that
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} c_{\epsilon}^2 + \int_{\Omega} |\nabla c_{\epsilon}|^2 + \int_{\Omega} c_{\epsilon}^2 \leq \int_{\Omega} \frac{n_{\epsilon} c_{\epsilon}}{1 + \varepsilon n_{\epsilon}} \leq \|c_{\epsilon}\|_{L^{\frac{1}{\gamma}}(\Omega)} \|n_{\epsilon}\|_{L^{\frac{2}{(1-\gamma)}}(\Omega)}
\]
in $(0, T)$ and for all $\epsilon \in (0, 1)$,

(4.21)

as long as $\gamma \in (1, 3)$. The Sobolev imbedding $W^{1, 2}(\Omega) \hookrightarrow L^{\frac{2}{(1-\gamma)}}(\Omega)$ allows us to find $c_1 = c_1(\Omega) > 0$ such that
\[
\|c_{\epsilon}(\cdot, t)\|_{L^{\frac{2}{(1-\gamma)}}(\Omega)} \leq c_1 \left( \|\nabla c_{\epsilon}\|_{L^2(\Omega)} + \|c_{\epsilon}\|_{L^2(\Omega)} \right)
\]
for all $t \in (0, T)$ and $\epsilon \in (0, 1)$.

Substituting this in (4.21) and using Young’s inequality yields
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} c_{\epsilon}^2 + \int_{\Omega} c_{\epsilon}^2 + \int_{\Omega} |\nabla c_{\epsilon}|^2 \leq c_1 \left( \|\nabla c_{\epsilon}\|_{L^2(\Omega)} + \|c_{\epsilon}\|_{L^2(\Omega)} \right) \|n_{\epsilon}\|_{L^{\frac{2}{(1-\gamma)}}(\Omega)} \leq \frac{1}{2} \int_{\Omega} |\nabla c_{\epsilon}|^2 + \frac{1}{2} \int_{\Omega} c_{\epsilon}^2 + c_1^2 \|n_{\epsilon}\|^2_{L^{\frac{2}{(1-\gamma)}}(\Omega)}
\]
in $(0, T)$ for all $\epsilon \in (0, 1)$,
which directly tells us that

\[
\frac{d}{dt} \int_\Omega c_\epsilon^2 + \int_\Omega c_\epsilon^2 + \int_\Omega |\nabla c_\epsilon|^2 \leq 2c_\epsilon^2 \|n_\epsilon\|^2_{L^{\infty}(\Omega)} \quad \text{in} \ (0, T) \ \text{for all} \ \epsilon \in (0, 1). \quad (4.22)
\]

It can be deduced from the variation-of-constants formula that

\[
\frac{d}{dt} \left( e^t \int_\Omega c_\epsilon^2(x, t)dx \right) + e^t \int_\Omega |\nabla c_\epsilon|^2 \leq 2c_\epsilon^2 e^t \|n_\epsilon\|^2_{L^{\infty}(\Omega)} \quad \text{in} \ (0, T) \ \text{for all} \ \epsilon \in (0, 1). \quad (4.23)
\]

For any \( \tau \in (0, T] \), integrating (4.23) with respect to the time-variable over the interval \((0, \tau)\) shows that

\[
\int_0^\tau \int_\Omega e^t |\nabla c_\epsilon(x, t)|^2 dx \leq 2c_\epsilon^2 \int_0^\tau \int_\Omega \|n_\epsilon(\cdot, t)\|^2_{L^{\infty}(\Omega)} dt + \|c_\epsilon\|^2_{L^2(\Omega)} \quad \text{for all} \ \epsilon \in (0, 1).
\]

As \( \gamma < 2 \), this combined with (4.8) and (3.3) guarantees our claims.

For the treatment of \( c_\epsilon \), we need slightly more regularity than could be achieved from testing (3.4) by \( c_\epsilon \) as in the previous lemma. For the next testing procedure, we let \( \mathcal{L} \) denote the operator \( -\Delta + 1 \) with Neumann boundary conditions, that is the sectorial operator defined by

\[
\mathcal{L} u := -\Delta u + u \quad \text{for} \ u \in D(\mathcal{L}) := \{ u \in W^{2, 2}(\Omega) : \partial_\nu u = 0 \ \text{on} \ \partial \Omega \}
\]

and briefly recall some of its properties:

**Lemma 4.8.** (i) \( \mathcal{L} \) has closed fractional powers \( \mathcal{L}^\alpha \ (\alpha > 0) \). They are self-adjoint.

(ii) For any \( \alpha \in (0, 1) \), the domain \( D(\mathcal{L}^\alpha) \) is continuously embedded in \( W^{2, 2}(\Omega) \).

(iii) For all \( \alpha, \beta, \delta \in \mathbb{R} \) satisfying \( \beta < \alpha < \delta \), there is \( C > 0 \) such that

\[
\|\mathcal{L}^\alpha \varphi\|_{L^2(\Omega)} \leq C\|\mathcal{L}^\beta \varphi\|_{L^2(\Omega)}^{\frac{\alpha - \beta}{\delta - \beta}}\|\mathcal{L}^\delta \varphi\|_{L^2(\Omega)}^{\frac{\alpha - \beta}{\delta - \alpha}} \quad \text{for all} \ \varphi \in D(\mathcal{L}^\beta).
\]

**Proof.** This lemma is a summary of well-known properties of \( \mathcal{L} \), which can be found in, e.g., [15, 9], [11, Theorem 6.7] and [9, Theorem 14.1].

Before giving Lemma 4.10, we state the following elementary result allowing to derive boundedness by means of an argument of ordinary differential inequities which we will employ in several places while studying the development of certain energy-like functionals.

**Lemma 4.9.** Let \( M_1, M_2 > 0 \). Then there is \( C = C(M_1, M_2) > 0 \) with the property that whenever for some \( T_* \in (0, \infty) \) and \( \tau \in [0, T_*) \) the function \( y \in C^0([\tau, T_*]) \cap C^1((\tau, T_*)) \) is nonnegative and satisfies

\[
y'(t) + h(t) \leq a(t)y(t) + b(t), \quad t \in (\tau, T_*), \quad (4.25)
\]

where \( h, a, b \) are nonnegative integrable functions with \( a \) and \( b \) satisfying

\[
\int_\tau^{T_*} a(t) dt \leq M_1 \quad \text{and} \quad \int_\tau^{T_*} b(t) dt \leq M_2, \quad (4.26)
\]

then it holds that

\[
\sup_{t \in (\tau, T_*)} y(t) \leq C y(\tau) + C \quad \text{and} \quad \int_\tau^{T_*} y(t) dt \leq C y(\tau) + C. \quad (4.27)
\]
Proof. Nonnegativity of \( h \) on \( (\tau, T_\ast) \) combined with (4.25) ensures
\[
y'(t) \leq a(t)y(t) + b(t), \quad t \in (\tau, T_\ast).
\]

An integration of this differential inequality gives that for any \( t \in (\tau, T_\ast) \),
\[
y(t) \leq y(\tau) e^{\int_\tau^t a(s)ds} + \int_\tau^t e^{\int_s^t a(\sigma)ds} b(s)ds \leq y(\tau) e^{M_1} + \int_\tau^t e^{M_1} b(s)ds \leq y(\tau) e^{M_1} + M_2 e^{M_1}. \quad (4.28)
\]

Integrating (4.25) over \( (\tau, T_\ast) \) and applying (4.28) we infer that
\[
\int_\tau^{T_\ast} h(t)dt \leq \int_\tau^{T_\ast} a(t)y(t)dt + \int_\tau^{T_\ast} b(t)dt + y(\tau) \leq (y(\tau)e^{M_1} + M_2 e^{M_1}) \int_\tau^{T_\ast} a(t)dt + M_2 + y(\tau) \leq y(\tau)(M_1 e^{M_1} + 1) + M_1 M_2 e^{M_1} + M_2. \quad (4.29)
\]

With taking \( C = (M_1 + 1)\max\{e^{M_1}, M_2 e^{M_1}\} \) in the claim of this lemma, we can end the proof by virtue of (4.28) and (4.29).

Lemma 4.9 will find its first application during the derivation of the following estimates for the second solution component.

**Lemma 4.10.** Let \( T > 0 \). If \( \gamma \in (1, 2) \), there exist positive constants \( \beta > 0 \) and \( C = C(T) > 0 \) such that
\[
\int_0^T \int_{\Omega} |\mathcal{L}^{\beta} c_\varepsilon(x, t)|^2 dx dt \leq C \quad \text{for all } \varepsilon \in (0, 1). \quad (4.30)
\]

**Proof.** With \( \gamma > 1 \) and \( \sigma > 0 \) (fixed at the beginning of Section 3), we take \( \beta = \min\{2\sigma, \gamma - 1\} \) and obtain from (3.3) that
\[
\int_{\Omega} |\mathcal{L}^{\beta} c_\varepsilon|^2 \leq c_1 \quad \text{for all } \varepsilon \in (0, 1) \quad (4.31)
\]

with some \( c_1 = c_1(\sigma, \gamma, \Omega) > 0 \). By testing (3.4)\_2 by \( \mathcal{L}^{\beta} c_\varepsilon \) and utilizing the self-adjointness of \( \mathcal{L} \) and its powers, it follows that for all \( \varepsilon \in (0, 1) \)
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\mathcal{L}^{\beta} c_\varepsilon|^2 + \int_{\Omega} |\mathcal{L}^{\beta+\frac{1}{2}} c_\varepsilon|^2 = -\int_{\Omega} (\mathcal{L}^{\beta} c_\varepsilon) \nabla c_\varepsilon \cdot u_\varepsilon + \int_{\Omega} \frac{n_\varepsilon \mathcal{L}^{\beta} c_\varepsilon}{1 + \varepsilon n_\varepsilon} \quad \text{in } (0, T). \quad (4.32)
\]

An application of Hölder’s inequality implies that
\[
-\int_{\Omega} \mathcal{L}^{\beta} c_\varepsilon \nabla c_\varepsilon \cdot u_\varepsilon \leq \|\mathcal{L}^{\beta} c_\varepsilon\|_{L^4(\Omega)} \|\nabla c_\varepsilon\|_{L^2(\Omega)} \|u_\varepsilon\|_{L^4(\Omega)} \quad \text{in } (0, T) \quad \text{for all } \varepsilon \in (0, 1). \quad (4.33)
\]

Utilizing the Sobolev imbedding inequality (of [5, Theorem 6.7]) and Lemma 4.8, we can see
\[
\|\mathcal{L}^{\beta} c_\varepsilon\|_{L^4(\Omega)} \leq c_2 \|\mathcal{L}^{\beta} c_\varepsilon\|_{W^{1, 2} \Omega} \leq c_3 \|\mathcal{L}^{\beta+\frac{1}{2}} c_\varepsilon\|_{L^2(\Omega)} \leq c_4 \|\mathcal{L}^{\beta+\frac{1}{2}} c_\varepsilon\|_{L^2(\Omega)}^{1-\beta+\frac{1}{2}} \quad \text{in } (0, T) \quad \text{for all } \varepsilon \in (0, 1) \quad (4.34)
\]

13
with positive constants $c_2, c_3, c_4$ only depending on $\sigma, \gamma$ and $\Omega$. Similarly, we also have $c_5, c_6 > 0$ such that

$$\|\nabla c_\varepsilon\|_{L^2(\Omega)} \leq c_5 \|\mathcal{L}_t^\varepsilon c_\varepsilon\|_{L^2(\Omega)}$$

$$\leq c_6 \|\mathcal{L}_\varepsilon c_\varepsilon\|_{L^2(\Omega)}^{1-\beta} \|\mathcal{L}_t^\varepsilon c_\varepsilon\|_{L^2(\Omega)}^\beta \quad \text{in} \ (0, T) \quad \text{for} \ \varepsilon \in (0, 1). \quad (4.35)$$

Inserting (4.34) and (4.35) into (4.33) and applying Young’s inequality give that

$$- \int_{\Omega} \mathcal{L}_t^\varepsilon c_\varepsilon \nabla c_\varepsilon \cdot u_\varepsilon \leq c_4 c_6 \|\mathcal{L}_\varepsilon c_\varepsilon\|_{L^2(\Omega)} \|\mathcal{L}_t^\varepsilon c_\varepsilon\|_{L^2(\Omega)} \|u_\varepsilon\|_{L^4(\Omega)}$$

$$\leq \frac{1}{4} \|\mathcal{L}_\varepsilon c_\varepsilon\|_{L^2(\Omega)}^2 + \frac{27}{4} c_4^4 c_6^4 \|\mathcal{L}_t^\varepsilon c_\varepsilon\|_{L^2(\Omega)}^2 \|u_\varepsilon\|_{L^4(\Omega)}^4 \quad (4.36)$$

in $(0, T)$ for all $\varepsilon \in (0, 1)$. Since $\beta \leq \gamma - 1$, Lemma 4.8 (ii) ensures the existence of $c_7, c_8 > 0$ satisfying

$$\|\mathcal{L}_t^\varepsilon c_\varepsilon\|_{L^\frac{2}{1-\gamma}(\Omega)} \leq c_7 \|\mathcal{L}_t^\varepsilon c_\varepsilon\|_{W^{2-\gamma, 2}(\Omega)} \leq c_8 \|\mathcal{L}_t^\varepsilon c_\varepsilon\|_{L^2(\Omega)} \quad \text{in} \ (0, T) \quad \text{for} \ \varepsilon \in (0, 1). \quad (4.37)$$

Thus, it can be obtained by the H"{o}lder inequality (due to $\gamma \in (1, 3)$ ensuring $\frac{2}{1-\gamma} > 1$ and $\frac{2}{\gamma-1} > 1$) and Young’s inequality that

$$\int_{\Omega} n_\varepsilon \mathcal{L}_t^\varepsilon c_\varepsilon \leq \|n_\varepsilon\|_{L^\frac{2}{2-\gamma}(\Omega)} \|\mathcal{L}_t^\varepsilon c_\varepsilon\|_{L^\frac{2}{2-\gamma}(\Omega)}$$

$$\leq c_8^2 \|n_\varepsilon\|_{L^\frac{2}{2-\gamma}(\Omega)}^2 + \frac{1}{4 c_8^4} \|\mathcal{L}_t^\varepsilon c_\varepsilon\|_{L^\frac{2}{2-\gamma}(\Omega)}^2$$

$$\leq c_8^2 \|n_\varepsilon\|_{L^\frac{2}{2-\gamma}(\Omega)}^2 + \frac{1}{4} \|\mathcal{L}_t^\varepsilon c_\varepsilon\|_{L^2(\Omega)}^2 \quad \text{in} \ (0, T) \quad \text{for} \ \varepsilon \in (0, 1).$$

Together with (4.32), (4.35) and (4.37), this shows that

$$\frac{d}{dt} \int_{\Omega} \|\mathcal{L}_t^\varepsilon c_\varepsilon\|^2 + \int_{\Omega} \|\mathcal{L}_\varepsilon c_\varepsilon\|^2$$

$$\leq c_9 \|n_\varepsilon\|_{L^\frac{2}{2-\gamma}(\Omega)}^2 + c_9 \|\mathcal{L}_t^\varepsilon c_\varepsilon\|_{L^2(\Omega)}^2 \|u_\varepsilon\|_{L^4(\Omega)}^4 \quad \text{in} \ (0, T) \quad \text{for} \ \varepsilon \in (0, 1) \quad (4.38)$$

with $c_9 = \max\{2c_8^2, \frac{27}{4} c_4^4 c_6^4\} > 0$. In order to apply Lemma 4.9 with

$$y(t) = \|\mathcal{L}_t^\varepsilon c_\varepsilon(\cdot, t)\|_{L^2(\Omega)}$$

$$a(t) = c_0 \|u_\varepsilon(\cdot, t)\|_{L^4(\Omega)}$$

$$b(t) = c_0 \|n_\varepsilon(\cdot, t)\|_{L^\frac{2}{2-\gamma}(\Omega)}, \quad t \in (0, T),$$

and $\tau = 0$, we note that Corollary 4.5 and Corollary 4.3 ensure (4.26), as long as $\gamma \in (1, 2)$, and by (4.31), we may conclude (4.30) from (4.27).

The final outcome of the previous bounds on $c_\varepsilon$ is summarized in the following compactness statement, which will be directly applicable in the convergence proofs in Section 5.

**Lemma 4.11.** Assume $\gamma \in (1, 2)$. Let $T > 0$. Then $\{c_\varepsilon\}_{\varepsilon \in (0, 1)}$ is relatively compact with respect to the strong topology in $L^2((0, T); W^{1, 2}(\Omega))$.

**Proof.** Multiplying the second equation in (3.4) by $\xi \in C_0^\infty(\Omega)$ and integrating by parts show that

$$\left| \int_{\Omega} c_{4t}(\cdot, t) \xi \right| = - \int_{\Omega} \nabla c_{\varepsilon} \cdot u_\varepsilon \xi - \int_{\Omega} \nabla c_{\varepsilon} \cdot \nabla \xi - \int_{\Omega} c_{\varepsilon} \xi + \int_{\Omega} \frac{n_\varepsilon}{1 + \varepsilon n_\varepsilon} \xi$$

14
\[
\leq \|\nabla c_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \|u_\varepsilon\|_{L^2(\Omega)} \|\xi\|_{L^\infty(\Omega)} + \|c_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \|\xi\|_{L^2(\Omega)} + \|n_\varepsilon(\cdot,t)\|_{L^1(\Omega)} \|\xi\|_{L^\infty(\Omega)}
\]
(4.39)

We deduce from (4.39) and the embedding \(W^{1,3}_0(\Omega) \to L^\infty(\Omega) \cap W^{1,2}_0(\Omega)\) that there is \(c_1 > 0\) satisfying
\[
\|c_{t\varepsilon}(\cdot,t)\|_{(W^{1,2}_0(\Omega))^*} \leq c_1 \left( \|c_\varepsilon(\cdot,t)\|_{W^{1,3}_0(\Omega)} + \|n_\varepsilon(\cdot,t)\|_{L^1(\Omega)} \right) \quad \text{for all } t \in (0,T) \text{ and } \varepsilon \in (0,1),
\]
which along with Lemma 4.10 and (4.2) leads to the observation that
\[
\{c_{t\varepsilon}\}_{\varepsilon \in (0,1)} \text{ is uniformly bounded in } L^2\left((0,T); (W^{1,3}_0(\Omega))^*\right).
\]
(4.40)

Since \(D(L^{2+1})\) is compactly embedded into \(W^{1,2}(\Omega)\) with \(\beta > 0\) determined in Lemma 4.10, the claim results from (4.30), (4.40) and the Aubin-Lions lemma.

4.3 Estimates for \(\{n_\varepsilon\}_{\varepsilon \in (0,1)}\)

We begin this subsection with the uniform integrability involving \(n_\varepsilon\).

**Lemma 4.12.** Let \(T > 0\). Then
\[
\{n_\varepsilon\}_{\varepsilon \in (0,1)} \text{ is uniformly integrable over } \Omega \times (0,T).
\]
(4.41)

Moreover, we have that both
\[
\left\{(n_\varepsilon + 1)^{-1} n_\varepsilon^2\right\}_{\varepsilon \in (0,1)} \quad \text{and} \quad \left\{(n_\varepsilon + 1)^{-1} f(n_\varepsilon)\right\}_{\varepsilon \in (0,1)} \text{ are uniformly integrable over } \Omega \times (0,T).
\]
(4.42)

**Proof.** We let \(g(z) = z\), \(g(z) = \frac{z^2}{1+z}\) or \(g(z) = \frac{|f(z)|}{1+z}\) for the proofs of (4.41) or the first or second part of (4.42), respectively. In each of these cases, there is \(c_1 > 0\) such that
\[
|g(z)| = g(z) \leq c_1 + c_1 \varepsilon^{\max\{1,\gamma-1\}} \quad \text{for every } z \geq 0
\]
and according to Lemma 4.2, we therefore can find \(c_2 > 0\) satisfying
\[
\int_0^T \int_\Omega |g(n_\varepsilon)|^{\max\{1,\gamma-1\}} \leq c_2 \quad \text{for every } \varepsilon \in (0,1),
\]
by the de la Vallée-Poussin theorem proving uniform integrability of \(\{g(n_\varepsilon)\}_{\varepsilon \in (0,1)}\). \(\square\)

In order to conclude \(L^1\)-convergence from uniform integrability by means of Vitali’s convergence theorem, we will additionally require some convergence in measure. Aiming to obtain this along a subsequence obtained from application of an Aubin-Lions lemma to \(\ln(n_\varepsilon + 1)\), we thus prepare the following estimates of derivatives of the latter.

**Lemma 4.13.** Let \(T > 0\) and \(\gamma \in (1,2)\). Then we can find \(C = C(T) > 0\) such that
\[
\int_0^T \int_\Omega |\nabla \ln (n_\varepsilon + 1)|^2 \leq C \quad \text{for all } \varepsilon \in (0,1)
\]
(4.43)

and
\[
\int_0^T \|\partial_t \ln (n_\varepsilon(\cdot,t) + 1)\|_{(W^{1,3}_0(\Omega))^*} \leq C \quad \text{for all } \varepsilon \in (0,1).
\]
(4.44)
Proof. We choose $\varphi \equiv 1$ in Lemma 3.2 and integrate (3.6) with respect to $t \in (0, T)$ to get
\[
\int_0^T \int_\Omega |\nabla \ln (n_\varepsilon + 1)|^2 = \int_\Omega \ln (n_\varepsilon (\cdot, T) + 1) - \int_\Omega \ln (n_0 \varepsilon + 1) + \int_0^T \int_\Omega n_\varepsilon (n_\varepsilon + 1)^{-2} \nabla n_\varepsilon \cdot \nabla c_\varepsilon \\
- \int_0^T \int_\Omega (n_\varepsilon + 1)^{-1} f(n_\varepsilon) + \varepsilon \int_0^T \int_\Omega (n_\varepsilon + 1)^{-1} n_\varepsilon^2 \quad \text{for all } \varepsilon \in (0, 1),
\]
(4.45)
where the term involving $\nabla c_\varepsilon$ can be estimated due to $\frac{n_\varepsilon}{1+n_\varepsilon} \leq 1$ and by Young’s inequality according to
\[
\int_0^T \int_\Omega (n_\varepsilon + 1)^{-2} \nabla n_\varepsilon \cdot \nabla c_\varepsilon \leq \frac{1}{2} \int_0^T \int_\Omega |\nabla \ln (n_\varepsilon + 1)|^2 + \frac{1}{2} \int_0^T \int_\Omega |\nabla c_\varepsilon|^2 \quad \text{for all } \varepsilon \in (0, 1).
\]
(4.46)
Lemma 4.1 yields a constant $c_1 > 0$ such that
\[
- \int_0^T \int_\Omega (n_\varepsilon + 1)^{-1} f(n_\varepsilon) \leq \int_0^T \int_\Omega |f(n_\varepsilon)| \leq c_1 \quad \text{for all } \varepsilon \in (0, 1)
\]
(4.47)
and
\[
\varepsilon \int_0^T \int_\Omega (n_\varepsilon + 1)^{-1} n_\varepsilon^2 \leq \int_0^T \int_\Omega n_\varepsilon \leq c_1 \quad \text{for all } \varepsilon \in (0, 1).
\]
(4.48)
Inserting (4.46)-(4.48) into (4.45) and invoking Lemma 4.7, we arrive at the claim (4.43).

We now turn to the assertion (4.44). Taking $\xi \in C^\infty (\overline{\Omega})$, we may invoke Lemma 3.2 for the function $\varphi$ defined by $\varphi (\cdot, t) = \xi$ for all $t > 0$, obtaining
\[
\left| \int_\Omega \partial_t \ln (n_\varepsilon (\cdot, t) + 1) \xi \right| \\
= \left| \int_\Omega (n_\varepsilon + 1)^{-2} |\nabla n_\varepsilon|^2 \xi - \int_\Omega (n_\varepsilon + 1)^{-1} \nabla n_\varepsilon \cdot \nabla \xi \\
+ \int_\Omega \ln (n_\varepsilon + 1) \cdot \nabla c_\varepsilon - \int_\Omega (n_\varepsilon + 1)^{-2} \nabla n_\varepsilon \cdot \nabla c_\varepsilon \xi \\
+ \int_\Omega (n_\varepsilon + 1)^{-1} n_\varepsilon \nabla \nabla c_\varepsilon \cdot \nabla \xi + \int_\Omega (n_\varepsilon + 1)^{-1} f(n_\varepsilon) \xi - \varepsilon \int_\Omega (n_\varepsilon + 1)^{-1} n_\varepsilon^2 \xi \right| \\
\leq \| \nabla \ln (n_\varepsilon + 1) (\cdot, t) \|_{L^2(\Omega)} \| \xi \|_{L^\infty (\Omega)} + \| \nabla \ln (n_\varepsilon + 1) (\cdot, t) \|_{L^2(\Omega)} \| \nabla \xi \|_{L^2(\Omega)} \\
+ \| \ln (n_\varepsilon + 1) (\cdot, t) \|_{\mathcal{L}^1(\Omega)} \| \nabla c_\varepsilon (\cdot, t) \|_{\mathcal{L}^1(\Omega)} \| \xi \|_{L^\infty (\Omega)} \\
+ \| \nabla c_\varepsilon (\cdot, t) \|_{\mathcal{L}^1(\Omega)} \| \nabla \xi \|_{L^2(\Omega)} + \| f(n_\varepsilon (\cdot, t)) \|_{\mathcal{L}^1(\Omega)} \| \xi \|_{L^\infty (\Omega)} + \| n_\varepsilon (\cdot, t) \|_{\mathcal{L}^1(\Omega)} \| \xi \|_{L^\infty (\Omega)}
\]
(4.49)
for all $t \in (0, T)$ and $\varepsilon \in (0, 1)$, where the second inequality relies on $\frac{n_\varepsilon}{1+n_\varepsilon} \leq 1$. This in conjunction with Young’s inequality, Poincaré’s inequality and the Sobolev embedding inequality ensures the existence of $c_3 = c_3 (\Omega) > 0$ such that
\[
\left| \int_\Omega \partial_t \ln (n_\varepsilon (\cdot, t) + 1) \xi \right| \leq c_3 \left( 1 + \| u_\varepsilon (\cdot, t) \|_{L^2(\Omega)}^2 + \| \nabla \ln (n_\varepsilon + 1) (\cdot, t) \|_{L^2(\Omega)}^2 + \| \nabla c_\varepsilon (\cdot, t) \|_{L^2(\Omega)}^2 \\
+ \| f(n_\varepsilon (\cdot, t)) \|_{\mathcal{L}^1(\Omega)} + \| n_\varepsilon (\cdot, t) \|_{\mathcal{L}^1(\Omega)}^2 \right) \| \xi \|_{W^{1,2}(\Omega)}
\]
for all $t \in (0, T)$ and $\varepsilon \in (0, 1)$. This in conjunction with Young’s inequality, Poincaré’s inequality and the Sobolev embedding inequality ensures the existence of $c_3 = c_3 (\Omega) > 0$ such that
\[
\left| \int_\Omega \partial_t \ln (n_\varepsilon (\cdot, t) + 1) \xi \right| \leq c_3 \left( 1 + \| u_\varepsilon (\cdot, t) \|_{L^2(\Omega)}^2 + \| \nabla \ln (n_\varepsilon + 1) (\cdot, t) \|_{L^2(\Omega)}^2 + \| \nabla c_\varepsilon (\cdot, t) \|_{L^2(\Omega)}^2 \\
+ \| f(n_\varepsilon (\cdot, t)) \|_{\mathcal{L}^1(\Omega)} + \| n_\varepsilon (\cdot, t) \|_{\mathcal{L}^1(\Omega)}^2 \right) \| \xi \|_{W^{1,2}(\Omega)}
\]
for all \( t \in (0, T) \) and \( \varepsilon \in (0, 1) \), which directly gives that
\[
\| \partial_t (n_\varepsilon(-, t) + 1) \|_{L^2(\Omega)},
\]
\[
\leq c_3 \left( 1 + \| u_\varepsilon(-, t) \|^2_{L^2(\Omega)} + \| \nabla \ln (n_\varepsilon + 1)(-, t) \|^2_{L^2(\Omega)} + \| \nabla c_\varepsilon(-, t) \|^2_{L^2(\Omega)} + \| f(n_\varepsilon(-, t)) \|_{L^1(\Omega)} + \| n_\varepsilon(-, t) \|_{L^1(\Omega)} \right)
\]
for all \( t \in (0, T) \) and \( \varepsilon \in (0, 1) \).
(4.50)

By integrating (4.50) in time, we can derive the desired estimate from (4.43) and Lemma 4.1 together with Lemmata 4.4 and 4.10.

**Lemma 4.14.** Let \( T > 0 \) and assume \( \gamma \in (1, 2) \). Then there is \( C = C(T) > 0 \) such that
\[
\int_0^T \int_\Omega \| \nabla (n_\varepsilon + 1)^{-1} \|^2 \leq C \quad \text{for all } \varepsilon \in (0, 1).
\]
(4.51)

**Proof.** According to Lemma 3.3 applied to \( F(s) = \frac{1}{s} \),
\[
\int_\Omega \partial_t (n_\varepsilon(-, t) + 1)^{-1} \varphi = -2 \int_\Omega (n_\varepsilon + 1)^{-2} \| \nabla n_\varepsilon \|^2 \varphi + \int_\Omega (n_\varepsilon + 1)^{-2} \nabla n_\varepsilon \cdot \nabla \varphi + \int_\Omega (n_\varepsilon + 1)^{-1} u_\varepsilon \cdot \nabla \varphi + 2 \int_\Omega (n_\varepsilon + 1)^{-3} n_\varepsilon \nabla n_\varepsilon \cdot \nabla c_\varepsilon \varphi + \int_\Omega (n_\varepsilon + 1)^{-2} n_\varepsilon \nabla c_\varepsilon \cdot \nabla \varphi - \int_\Omega (n_\varepsilon + 1)^{-2} f(n_\varepsilon) \varphi + \varepsilon \int_\Omega (n_\varepsilon + 1)^{-2} n_\varepsilon^2 \varphi \quad \text{in } (0, T) \text{ and } \varepsilon \in (0, 1).
\]
(4.52)

We first take \( \varphi \equiv 1 \), integrate (4.52) with respect to the time-variable and use Young’s inequality to obtain that
\[
2 \int_0^T \int_\Omega (n_\varepsilon + 1)^{-2} \| \nabla n_\varepsilon \|^2 \leq \int_\Omega (n_0\varepsilon(-) + 1)^{-1} + 2 \int_0^T \int_\Omega (n_\varepsilon + 1)^{-3} n_\varepsilon \nabla n_\varepsilon \cdot \nabla c_\varepsilon - \int_0^T \int_\Omega (n_\varepsilon + 1)^{-2} f(n_\varepsilon) + \varepsilon \int_0^T \int_\Omega (n_\varepsilon + 1)^{-2} n_\varepsilon^2
\]
\[
\leq \int_\Omega (n_0\varepsilon(-) + 1)^{-1} + \int_0^T \int_\Omega (n_\varepsilon + 1)^{-2} \| \nabla n_\varepsilon \|^2 + \int_0^T \int_\Omega \| \nabla c_\varepsilon \|^2 + \varepsilon \int_0^T \int_\Omega (n_\varepsilon + 1)^{-2} n_\varepsilon^2 \quad \text{for all } \varepsilon \in (0, 1).
\]

This in conjunction with (4.20) and (4.1) shows that
\[
\int_0^T \int_\Omega (n_\varepsilon + 1)^{-2} \| \nabla n_\varepsilon \|^2 \leq \int_\Omega (n_0\varepsilon(-) + 1)^{-1} + \int_0^T \int_\Omega \| \nabla c_\varepsilon \|^2 + \int_0^T \int_\Omega |f(n_\varepsilon)| + \varepsilon \int_0^T \int_\Omega (n_\varepsilon + 1)^{-2} n_\varepsilon^2
\]
\[
\leq c_1 \quad \text{for all } \varepsilon \in (0, 1)
\]
(4.53)

with some \( c_1 = c_1(T) > 0 \). 
\qed
5 Global existence of generalized solution. Proof of Theorem 1.1

This section is devoted to proving the global existence of a generalized solution to (1.1). To achieve this goal, we shall provide necessary convergence properties for all components in the two succeeding lemmata. First, the next result concerned with \( \{c_\varepsilon\}_{\varepsilon \in (0,1)} \) and \( \{u_\varepsilon\}_{\varepsilon \in (0,1)} \) directly follows from lemmata of the previous section:

**Lemma 5.1.** If \( \gamma \in (1,2) \), there exist \( \{\varepsilon_j\}_{j \in \mathbb{N}} \subset (0,1) \) as well as functions

\[
\begin{align*}
\{ u_\varepsilon \} &\in L^2_{loc}([0,\infty); W^{1,2}(\Omega; \mathbb{R}^2)) \\
\{ c_\varepsilon \} &\in L^2_{loc}([0,\infty); W^{1,2}(\Omega))
\end{align*}
\]

with \( c \geq 0 \) a.e. in \( \Omega \times (0,\infty) \) such that \( \varepsilon_j \searrow 0 \) as \( j \to \infty \), and

\[
\begin{align*}
&c_\varepsilon \to c & \text{in } L^2_{loc}(\overline{\Omega} \times [0,\infty)) \text{ and a.e. in } \Omega \times (0,\infty), \\
&\nabla c_\varepsilon \to \nabla c & \text{in } L^2_{loc}(\overline{\Omega} \times [0,\infty)) \text{ and a.e. in } \Omega \times (0,\infty), \\
&u_\varepsilon \to u & \text{in } L^2_{loc}(\overline{\Omega} \times [0,\infty)) \text{ and a.e. in } \Omega \times (0,\infty), \\
&\nabla u_\varepsilon \to \nabla u & \text{in } L^2_{loc}(\overline{\Omega} \times [0,\infty)) \text{ and a.e. in } \Omega \times (0,\infty)
\end{align*}
\]

as \( \varepsilon = \varepsilon_j \searrow 0 \).

**Proof.** The relative compactness of \( \{c_\varepsilon\}_{\varepsilon \in (0,1)} \) in \( L^2((0,T); W^{1,2}(\Omega)) \) assured in Lemma 4.11 for every \( T > 0 \) guarantees the existence of a sequence along which \( c_\varepsilon \) and \( \nabla c_\varepsilon \) converge in \( L^2_{loc}(\overline{\Omega} \times [0,\infty)) \) (implying a.e. convergence along a further subsequence), and the bounds from Lemma 4.4 can be used to conclude (5.4) and (5.5) along a suitable sequence. \( \square \)

Regarding convergence of the first component, we note the following:

**Lemma 5.2.** There exist \( \{\varepsilon_j\}_{j \in \mathbb{N}} \subset (0,1) \) and a function

\[
n \in L^1_{loc}(\overline{\Omega} \times [0,\infty))
\]

such that \( \{\varepsilon_j\}_{j \in \mathbb{N}} \) is a subsequence of the sequence found in Lemma 5.1, and

\[
n_\varepsilon \to n \text{ in } L^1_{loc}(\overline{\Omega} \times [0,\infty)) \text{ and a.e. in } \Omega \times (0,\infty).
\]

Moreover, we have

\[
\begin{align*}
\ln(n_\varepsilon + 1) &\to \ln(n + 1) & \text{in } L^2_{loc}(\overline{\Omega} \times [0,\infty)), \\
\nabla \ln(n_\varepsilon + 1) &\to \nabla \ln(n + 1) & \text{in } L^2_{loc}(\overline{\Omega} \times [0,\infty)), \\
\nabla (n_\varepsilon + 1)^{-1} &\to \nabla (n + 1)^{-1} & \text{in } L^2_{loc}(\overline{\Omega} \times [0,\infty)), \\
(n_\varepsilon + 1)^{-1}f(n_\varepsilon) &\to (n + 1)^{-1}f(n) & \text{in } L^1_{loc}(\overline{\Omega} \times [0,\infty)), \\
(n_\varepsilon + 1)^{-1}n^2_\varepsilon &\to (n + 1)^{-1}n^2 & \text{in } L^1_{loc}(\overline{\Omega} \times [0,\infty))
\end{align*}
\]

as \( \varepsilon = \varepsilon_j \searrow 0 \).

**Proof.** It follows by Lemma 4.13 that with some \( c_1 = c_1(T) > 0 \),

\[
\int_0^T \int_{\Omega} |\nabla \ln(n_\varepsilon + 1)|^2 + ||\partial_t \ln(n_\varepsilon + 1)||_{W^{1,2}(\Omega)}^2 \leq c_1
\]

\( 18 \)
holds for every $\varepsilon \in (0, 1)$. The Aubin-Lions lemma enables us to find a function $g \in L^2_{\text{loc}}([0, \infty); W^{1,2}(\Omega))$ and a subsequence of $\{\varepsilon_j\}_{j \in \mathbb{N}}$ from Lemma 5.1 (which we do not relabel) satisfying

\[ \ln (n_{\varepsilon} + 1) \to g, \quad \nabla \ln (n_{\varepsilon} + 1) \to \nabla g \quad \text{in } L^2_{\text{loc}}(\Omega \times [0, \infty)), \]  
(5.14)

as $\varepsilon = \varepsilon_j \to 0$ and, along a subsequence, $\ln(n_{\varepsilon} + 1) \to g$ a.e. in $\Omega \times (0, \infty)$. Letting $n = e^g - 1$, we clearly have that $n_{\varepsilon} \to n$ a.e. in $\Omega \times (0, \infty)$. According to the Vitali convergence theorem, we can apply Lemma 4.12 to derive that furthermore

\[ n_{\varepsilon} \to n \quad \text{in } L^1_{\text{loc}}(\Omega \times [0, \infty)). \]  
(5.15)

With the estimate (4.51) at our disposal, we conclude that (5.10) holds along a subsequence. Moreover, the almost everywhere convergence along a further subsequence, as entailed by (5.15) combined with the continuity of $f$ ensures that

\[ (n_{\varepsilon} + 1)^{-1} f(n_{\varepsilon}) \to (n + 1)^{-1} f(n) \quad \text{a.e. in } \Omega \times (0, \infty). \]  
(5.16)

In view of (4.42) and (5.16), the Vitali convergence theorem guarantees the assertion (5.11). Based on the uniform integrability of $(n_{\varepsilon} + 1)^{-1} n_{\varepsilon}^2$, we can obtain (5.12) by the same reasoning. \hfill \Box

Consequences for certain “mixed terms” that appear in the definition of solutions are as follows:

**Lemma 5.3.** Assume that $\gamma \in (1, 2)$. Let $n, c, u$ be given in Lemma 5.1 and Lemma 5.2. There exist $\{\varepsilon_j\}_{j \in \mathbb{N}} \subset (0, 1)$ such that

\[ (n_{\varepsilon} + 1)^{-2} n_{\varepsilon} \nabla n_{\varepsilon} \cdot \nabla c_{\varepsilon} \to (n + 1)^{-2} n \nabla n \cdot \nabla c \quad \text{in } L^1_{\text{loc}}(\Omega \times [0, \infty)), \]  
(5.17)

\[ (n_{\varepsilon} + 1)^{-1} n_{\varepsilon} \nabla c_{\varepsilon} \to (n + 1)^{-1} n \nabla c \quad \text{in } L^1_{\text{loc}}(\Omega \times [0, \infty)), \]  
(5.18)

\[ \ln (n_{\varepsilon} + 1) u_{\varepsilon} \to \ln (n + 1) u \quad \text{in } L^1_{\text{loc}}(\Omega \times [0, \infty)), \]  
(5.19)

\[ \varepsilon_{\varepsilon} u_{\varepsilon} \to c u \quad \text{in } L^1_{\text{loc}}(\Omega \times [0, \infty)), \]  
(5.20)

\[ u_{\varepsilon} \otimes u_{\varepsilon} \to u \otimes u \quad \text{in } L^1_{\text{loc}}(\Omega \times [0, \infty)). \]  
(5.21)

as $\varepsilon = \varepsilon_j \searrow 0$.

**Proof.** It can be verified that

\[ (n_{\varepsilon} + 1)^{-2} n_{\varepsilon} \nabla n_{\varepsilon} = \nabla \ln (n_{\varepsilon} + 1) + \nabla (n_{\varepsilon} + 1)^{-1}, \]  
(5.22)

which in conjunction with (5.9) and (5.10) gives that

\[ (n_{\varepsilon} + 1)^{-2} n_{\varepsilon} \nabla n_{\varepsilon} \to \nabla \ln (n + 1) + \nabla (n + 1)^{-1} \quad \text{in } L^2_{\text{loc}}(\Omega \times [0, \infty)) \]  
(5.23)

as $\varepsilon = \varepsilon_j \searrow 0$. This with (5.3) directly guarantees (5.17). Since $(n_{\varepsilon} + 1)^{-1} n_{\varepsilon} \leq 1$ in $\Omega \times (0, \infty)$ for all $\varepsilon \in (0, 1)$ and $n_{\varepsilon} \to n$ a.e. in $\Omega \times (0, \infty)$, the dominated convergence theorem tells us that as $\varepsilon = \varepsilon_j \searrow 0$,

\[ (n_{\varepsilon} + 1)^{-1} n_{\varepsilon} \to (n + 1)^{-1} n \quad \text{in } L^2_{\text{loc}}(\Omega \times [0, \infty)). \]  
(5.24)

Hence, (5.18) follows by a combination of (5.24) and (5.3). The assertion (5.19) can be obtained by using (5.8) and (5.4). The convergence (5.20) and (5.21) are immediate results of (5.2) and (5.4). \hfill \Box

Finally, all the convergence properties shown in the above two lemmata allow us to give the proof of our main result.
Proof of Theorem 1.1. If $\gamma \geq 2$, then $f$ satisfies (1.5) also for any $\gamma \in (1, 2)$ (possibly with a larger value of $r$); therefore the previous lemmata remain applicable. Let $\varphi \in C_0^\infty(\Omega \times [0, \infty))$ be an arbitrarily fixed nonnegative test function. According to the lower semicontinuity of $L^2$ norms with respect to weak convergence and the weak convergence of $\sqrt{\varphi} \ln(n_\varepsilon + 1)$ (by (5.9)), we have

$$
\int_0^\infty \int_\Omega |\nabla \ln (n_\varepsilon + 1)|^2 \varphi \leq \liminf_{\varepsilon \to 0} \int_0^\infty \int_\Omega |\nabla \ln (n_\varepsilon + 1)|^2 \varphi.
$$

(5.25)

Due to Lemma 4.12 (and boundedness of $\varphi$), $\left\{ (n_\varepsilon + 1)^{-1} n_\varepsilon^2 \varphi \right\}_{\varepsilon \in (0,1)}$ is uniformly integrable over $\Omega \times (0, T)$. This combined with the Vitali convergence theorem entails that $(n_\varepsilon + 1)^{-1} n_\varepsilon^2 \varphi \to (n + 1)^{-1} n^2 \varphi$ in $L^1_{loc}(\Omega \times [0, \infty))$, and thus

$$
\lim_{\varepsilon \to 0} \varepsilon \int_0^\infty \int_\Omega (n_\varepsilon + 1)^{-1} n_\varepsilon^2 \varphi = 0.
$$

(5.26)

Hence, by integrating (3.6) with respect to the time-variable, we utilize (5.25) and (5.26) to get

$$
\int_0^\infty \int_\Omega |\nabla \ln (n_\varepsilon + 1)|^2 \varphi \\
\leq \liminf_{\varepsilon \to 0} \left\{ \int_0^\infty \int_\Omega |\nabla \ln (n_\varepsilon + 1)|^2 \varphi - \varepsilon \int_0^\infty \int_\Omega (n_\varepsilon + 1)^{-1} n_\varepsilon^2 \varphi \right\} \\
= \liminf_{\varepsilon \to 0} \left\{ - \int_0^\infty \int_\Omega \ln (n_\varepsilon + 1) \varphi_t - \int_\Omega \ln (n_0 + 1) \varphi(\cdot, 0) \\
- \int_0^\infty \int_\Omega \ln (n_\varepsilon + 1) u_\varepsilon \cdot \nabla \varphi + \int_0^\infty \int_\Omega (n_\varepsilon + 1)^{-1} n_\varepsilon \nabla n_\varepsilon \cdot \nabla \epsilon \varphi \\
+ \int_0^\infty \int_\Omega (n_\varepsilon + 1)^{-1} \nabla n_\varepsilon \cdot \nabla \varphi - \int_\Omega (n_\varepsilon + 1)^{-1} n_\varepsilon \nabla \epsilon_\varepsilon \cdot \nabla \varphi \\
- \int_0^\infty \int_\Omega (n_\varepsilon + 1)^{-1} f(n_\varepsilon) \right\}.
$$

(5.27)

On the right-hand side of (5.27), we can use the convergence properties previously derived and applying (5.8), (3.3), (5.19), (5.17), (5.9), (5.18) and (5.11), we obtain that

$$
\int_0^\infty \int_\Omega \ln (n_\varepsilon + 1) \varphi_t + \int_\Omega \ln (n_0 + 1) \varphi(0) \\
\leq - \int_0^\infty \int_\Omega \ln (n_\varepsilon + 1) u_\varepsilon \cdot \nabla \varphi - \int_0^\infty \int_\Omega |\nabla \ln (n_\varepsilon + 1)|^2 \varphi \\
+ \int_0^\infty \int_\Omega (n_\varepsilon + 1)^{-1} \nabla n_\varepsilon \cdot \nabla \varphi + \int_0^\infty \int_\Omega (n_\varepsilon + 1)^{-1} n_\varepsilon \nabla \epsilon_\varepsilon \cdot \nabla \varphi \\
- \int_0^\infty \int_\Omega (n_\varepsilon + 1)^{-1} n_\varepsilon \nabla \epsilon_\varepsilon \cdot \nabla \varphi - \int_\Omega (n_\varepsilon + 1)^{-1} f(n_\varepsilon) \varphi.
$$

(5.28)

The estimate

$$
\int_0^\infty \int n(\cdot, t) \leq \int_0^{n_0} + \int_0^t \int f(n) \quad \text{for a.e. } t > 0
$$

(5.29)

results from (5.7) by Fatou’s lemma in the same way as detailed in [36, p.20]. In view of (5.28) and (5.29), we can assert the function $n$ satisfies the conditions required in Definition 2.3. Based on Lemma 5.1, (5.7), (5.20) and (5.21), it is easy to verify the functions $c$ and $u$ satisfy the corresponding equations in the weak sense, as exhibited in Definition 2.2 and Definition 2.1. □
6 Eventual smoothness

In this section, we focus on investigating the eventual regularity properties of the generalized solution \((n, c, u)\). The proof will be based on the eventual quasi-energy functional

\[
\int_\Omega n \ln n + \frac{1}{2} \int_\Omega |\nabla c|^2.
\]

So as to ensure that it is a quasi-energy functional, we will need smallness of the mass \(\int_\Omega n\). This is where the smallness of \(r\) (or largeness of \(\mu\)) matters. As soon as then, finally, (eventual) boundedness of \(n\) in \(L^p(\Omega)\) is achieved for large \(p\), we can rely on standard procedures to prove regularity of the solution components in the corresponding space-time domain.

In order to state a value for \(\mu_0\) in Theorem 1.2, we introduce \(C^*\) (only depending on the domain \(\Omega\)) as the best constant in the following Gagliardo-Nirenberg inequality:

\[
\|\varphi\|_{L^4(\Omega)} \leq C^*(\|\nabla \varphi\|^\frac{2}{3}_{L^2(\Omega)}\|\varphi\|^\frac{1}{3}_{L^2(\Omega)} + \|\varphi\|_{L^2(\Omega)}) \quad \forall \varphi \in W^{1,2}(\Omega). \tag{6.1}
\]

Since most lemmata in Section 4 included the condition \(\gamma \in (1, 2)\), which we want to avoid in the following, let us firstly collect some bounds that later proofs will rely on without this assumption.

**Lemma 6.1.** Let \(T > 0\). If \(\gamma \in (1, \infty)\), there is \(C > 0\) such that for every \(\varepsilon \in (0, 1)\),

\[
\int_0^T \int_\Omega |\nabla u\varepsilon|^2 \leq C, \quad \tag{6.2}
\]

\[
\int_0^T \int_\Omega |\nabla c\varepsilon|^2 \leq C. \quad \tag{6.3}
\]

**Proof.** In the same manner as during the proof of Theorem 1.1, we remark that if \(f\) satisfies (1.5) for some \(\gamma \geq 2\), \(f\) satisfies (1.5) also for smaller values (in \((1, 2)\)) of \(\gamma\), if \(r\) and \(\mu\) are adjusted as necessary.

Therefore, the lemmata of Section 4 are applicable and (6.2) follows from (4.10), (6.3) from (4.20).

The next lemma is used to demonstrate that at large times, the \(L^1(\Omega)\)-norm of \(n\) can be controlled by the system parameters.

**Lemma 6.2.** Then

\[
\limsup_{t \to \infty} \sup_{\varepsilon \in (0, 1)} \int n\varepsilon(x, t)dx \leq |\Omega|\left(\frac{r\varepsilon}{\mu}\right)^{\frac{1}{\gamma}}. \tag{6.4}
\]

**Proof.** Integrating (3.4)_1 over \(\Omega\), we have

\[
\frac{d}{dt} \int_\Omega n\varepsilon \leq r \int_\Omega n\varepsilon - \mu \int_\Omega n\varepsilon^\gamma \leq r \int_\Omega n\varepsilon - \frac{\mu}{|\Omega|^\frac{1}{1-\gamma}} \left(\int_\Omega n\varepsilon\right)^\gamma \quad \text{in} \ (0, \infty) \ \text{for all} \ \varepsilon \in (0, 1). \tag{6.5}
\]

We let \(y \in C^0([0, \infty)) \cap C^1((0, \infty))\) denote the solution of the initial value problem \(y' = ry - \frac{\mu}{|\Omega|^\frac{1}{1-\gamma}} y^\gamma\), \(y(0) = 2 \int_\Omega n_0\) and note that \(\int_\Omega n\varepsilon \leq y\) on \((0, \infty)\) for every \(\varepsilon \in (0, 1)\) due to (6.5) and (3.2). Since \(y(t) \to |\Omega| \left(\frac{r\varepsilon}{\mu}\right)^{\frac{1}{\gamma}}\) as \(t \to \infty\), this shows (6.4).
In the treatment of the derivatives of $|\nabla c_\varepsilon|^2$, the integral involving $u_\varepsilon$ does not vanish like it did so often before. Therefore, further estimates for $u_\varepsilon$ are required. The course of action is similar to that applied for $c_\varepsilon$ in Lemma 4.10. We again firstly introduce a suitable operator and collect a few basic results, analogous to Lemma 4.8:

With $D(A) := W^{2,2}(\Omega; \mathbb{R}^2) \cap W^{1,2}_0(\Omega; \mathbb{R}^2) \cap L^2_\sigma(\Omega)$, we let $A := -\Delta$ denote the realization of the Stokes operator on $D(A)$. Therein, $\mathcal{P}$ stands for the Helmholtz projection from $L^2(\Omega; \mathbb{R}^2)$ to $L^2_\sigma(\Omega)$.

**Lemma 6.3.** (i) $A$ is a sectorial and positive self-adjoint operator, it possesses closed fractional powers $A^\alpha$ defined on $D(A^\alpha)$ with $\alpha \in \mathbb{R}$, where the norm is given by $D(A^\alpha) := \|A^\alpha(\cdot)\|_{L^2(\Omega)}$.

(ii) For any $\alpha \in (0, 1)$, the domain $D(A^\alpha)$ is continuously embedded in $L^2_\sigma(\Omega) \cap (W^{2\alpha,2}(\Omega; \mathbb{R}^2))$.

(iii) For all $\alpha, \beta, \delta \in \mathbb{R}$ satisfying $\beta < \alpha < \delta$, there is $C > 0$ such that

$$\|A^\alpha \varphi\|_{L^2(\Omega)} \leq C \|A^\beta \varphi\|_{L^2(\Omega)} \|A^\delta \varphi\|_{L^2(\Omega)},$$

for all $\varphi \in D(A^\beta)$.

(iv) Let $p > 1$. If $\mathcal{P}$ denotes the Helmholtz projection from $L^p(\Omega; \mathbb{R}^2)$ to $L^p_\sigma(\Omega)$, then $\mathcal{P}$ is a bounded linear operator.

**Proof.** [13, Proposition 4.1] and [9, Theorem 14.1]; and, for (iv), [12, Thm. 1 and Thm. 2].

Finally relying on Lemma 4.9, we then obtain the following regularity information on $u_\varepsilon$:

**Lemma 6.4.** For any $T > 1$, there exists $C > 0$ such that

$$\int_0^T \int_\Omega |A^{\frac{1}{2}} u_\varepsilon(x, t)|^2 \, dx \, dt \leq C \quad \text{for all } \varepsilon \in (0, 1)$$

(6.6)

with $\delta = \min\{1/2, \gamma - 1\}$.

**Proof.** By (6.2), there exists $c_1 > 0$ such that

$$\int_0^1 \int_\Omega |\nabla u_\varepsilon(x, t)|^2 \, dx \, dt \leq c_1 \quad \text{for all } \varepsilon \in (0, 1).$$

(6.7)

For any fixed $\varepsilon \in (0, 1)$, we can thus find $t_\varepsilon \in (0, 1)$ satisfying

$$\|A^{\frac{1}{2}} u_\varepsilon(\cdot, t_\varepsilon)\|_{L^2(\Omega)} \leq c_1.$$

(6.8)

First, we rewrite (3.4) for and (3.4) in the form that

$$u_\varepsilon - A u_\varepsilon = -\kappa \mathcal{P}(u_\varepsilon \cdot \nabla) u_\varepsilon + \mathcal{P} n_\varepsilon \nabla \phi, \quad \varepsilon \in (0, 1), \ t > 0.$$  

(6.9)

Multiplying $A^4 u_\varepsilon$ to (6.9) and using the self-adjointness of the operator $A$, we have that

$$\frac{1}{2} \frac{d}{dt} \int_\Omega |A^{\frac{1}{2}} u_\varepsilon|^2 + \int_\Omega |A^{\frac{3}{2}} u_\varepsilon|^2$$

$$= -\kappa \int_\Omega \mathcal{P}(u_\varepsilon \cdot \nabla) u_\varepsilon \cdot A^4 u_\varepsilon$$

$$+ \int_\Omega \mathcal{P}(n_\varepsilon \nabla \phi) \cdot A^4 u_\varepsilon$$

in $(0, \infty)$ for all $\varepsilon \in (0, 1)$.

(6.10)

With $p > 2$ taken to satisfy $\frac{2}{p} + \delta < 1$, we have

$$-\int_\Omega \mathcal{P}(u_\varepsilon \cdot \nabla) u_\varepsilon \cdot A^4 u_\varepsilon \leq \|A^4 u_\varepsilon\|_{L^{p,2}(\Omega)} \|\mathcal{P}(u_\varepsilon \cdot \nabla) u_\varepsilon\|_{L^{p,2}(\Omega)}$$

22
\[ \leq c_2\|A^\delta u_c\|_{L^{\infty}(\Omega)} \|u_c \cdot \nabla u_c\|_{L^{\frac{2p}{p+2}}(\Omega)} \]
\[ \leq c_2\|A^\delta u_c\|_{L^{\infty}(\Omega)} \|\nabla u_c\|_{L^2(\Omega)} \|u_c\|_{L^p(\Omega)} \quad \text{in } (0, \infty) \text{ for all } \varepsilon \in (0, 1) \quad (6.11) \]

with \( c_2 = c_2(\frac{2}{p+2}) > 0 \) originating from Lemma 6.3(iv). By using the Sobolev inequality [5, Theorem 6.7] and Lemma 6.3(ii) and (iii), we obtain
\[
\|A^\delta u_c\|_{L^{\frac{2p}{p+2}}(\Omega)} \leq c_3\|A^\delta u_c\|_{W^{\frac{1}{2}, \frac{p}{p+2}}(\Omega)} 
\leq c_4\|A^{\frac{\delta}{2} + \frac{\beta}{2}} u_c\|_{L^2(\Omega)} 
\leq c_5\|A^{\frac{\delta}{2} + \frac{\beta}{2}} u_c\|_{L^2(\Omega)} \|A^{\frac{\delta}{2}} u_c\|_{L^2(\Omega)} \quad \text{in } (0, \infty) \text{ for all } \varepsilon \in (0, 1) \quad (6.12) \]

with positive constants \( c_3, c_4, c_5 \) depending on \( \sigma, \gamma, p \) and \( \Omega \). Similarly, still by the Sobolev imbedding inequality and Lemma 6.3(ii), (iii), we can find \( c_6, c_7, c_8 > 0 \) such that
\[
\|u_c\|_{L^p(\Omega)} \leq c_6\|u_c\|_{W^{1, \frac{p}{2}}(\Omega)} 
\leq c_6\|A^{\frac{\delta}{2} - \frac{\beta}{2}} u_c\|_{L^2(\Omega)} 
\leq c_8\|A^{\frac{\delta}{2} - \frac{\beta}{2}} u_c\|_{L^2(\Omega)} \|A^{\frac{\delta}{2}} u_c\|_{L^2(\Omega)} \quad \text{in } (0, \infty) \text{ for all } \varepsilon \in (0, 1) \quad (6.13) \]

A substitution of (6.12) and (6.13) into (6.11) and Young’s inequality give that with \( c_9 = \kappa^2 c_3^2 c_4^2 c_5^2 \)
\[
-\kappa \int_{\Omega} P(u_c \cdot \nabla) u_c \cdot A^\delta u_c 
\leq \kappa c_6 c_8\|A^{\frac{\delta}{2} - \frac{\beta}{2}} u_c\|_{L^2(\Omega)} \|\nabla u_c\|_{L^2(\Omega)} \|A^{\frac{\delta}{2}} u_c\|_{L^2(\Omega)} 
\leq \frac{1}{4}\|A^{\frac{\delta}{2} + \frac{\beta}{2}} u_c\|_{L^2(\Omega)}^2 + c_9\|\nabla u_c\|_{L^2(\Omega)}^2 \|A^{\frac{\delta}{2}} u_c\|_{L^2(\Omega)}^2 \quad (6.14) \]

in \( (0, \infty) \) for all \( \varepsilon \in (0, 1) \).

We deal with the remaining term \( \int_{\Omega} P(n_c \nabla \phi) \cdot A^\delta u_c \) for the cases \( \gamma < 1, 2 \) and \( \gamma > 2 \) separately, beginning with \( \gamma \in (1, 2) \):

Since \( \delta \leq \gamma - 1 \), the Sobolev imbedding inequality and Lemma 6.3(ii) ensure the existence of \( c_{10}, c_{11} \) satisfying
\[
\|A^\delta u_c\|_{L^{\infty}(\Omega)} \leq c_{10}\|A^\delta u_c\|_{W^{1, \frac{\gamma}{2}}(\Omega)} 
\leq c_{11}\|A^{\frac{1}{2}} u_c\|_{L^2(\Omega)} \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1) \quad (6.15) \]

It follows by the Hölder inequality and Young’s inequality that in \( (0, \infty) \) with some \( c_{12} > 0 \) taken from an application of Lemma 6.3(iv) to \( L^{\frac{2p}{p+2}}(\Omega) \),
\[
\int_{\Omega} P(n_c \nabla \phi) \cdot A^\delta u_c 
\leq \|P(n_c \nabla \phi)\|_{L^{\infty}(\Omega)} \|A^\delta u_c\|_{L^{\frac{2p}{p+2}}(\Omega)} 
\leq c_{12}\|\nabla \phi\|_{L^{\infty}(\Omega)} \|n_c\|_{L^{\frac{2p}{p+2}}(\Omega)} \|A^\delta u_c\|_{L^{\frac{2p}{p+2}}(\Omega)} 
\leq c_{12}^2 c_{12}^2\|\nabla \phi\|_{L^{\infty}(\Omega)}^2 \|n_c\|_{L^{\frac{2p}{p+2}}(\Omega)}^2 + \frac{1}{4}\|A^\delta u_c\|_{L^{\frac{2p}{p+2}}(\Omega)}^2 \quad \text{for all } \varepsilon \in (0, 1) \]

which in conjunction with (6.15) tells us that
\[
\int_{\Omega} P(n_c \nabla \phi) \cdot A^\delta u_c 
\leq c_{13}\|n_c\|_{L^{\frac{2p}{p+2}}(\Omega)}^2 + \frac{1}{4}\|A^{\frac{1}{2}} u_c\|_{L^2(\Omega)}^2 \quad \text{in } (0, \infty) \text{ for all } \varepsilon \in (0, 1) \quad (6.16) \]
with $c_{13} = c_1^2 c_2^2 \|\nabla \phi\|_{L^\infty(\Omega)} > 0$.
In the case $\gamma \geq 2$, we instead introduce $c_{14} > 0$ such that

$$\|A^4 \mathbf{u}_\epsilon\|_{L^2(\Omega)} \leq c_{14} \|A^{\frac{4+\delta}{2}} \mathbf{u}_\epsilon\|_{L^2(\Omega)} \quad \text{in } (0, \infty) \text{ for all } \epsilon \in (0, 1)$$

and, with $c_{15} = c_{14}^2 \|\nabla \phi\|_{L^\infty(\Omega)}^2$, estimate

$$\int_\Omega \mathcal{P}(n_\epsilon \nabla \phi) \cdot A^4 \mathbf{u}_\epsilon \leq \|\mathcal{P}(n_\epsilon \nabla \phi)\|_{L^2(\Omega)} \|A^4 \mathbf{u}_\epsilon\|_{L^2(\Omega)}$$

$$\leq \|\nabla \phi\|_{L^\infty(\Omega)} \|n_\epsilon\|_{L^2(\Omega)} \|A^4 \mathbf{u}_\epsilon\|_{L^2(\Omega)}$$

$$\leq c_{14}^2 \|\nabla \phi\|_{L^\infty(\Omega)}^2 \|n_\epsilon\|_{L^2(\Omega)}^2 + \frac{1}{4c_{14}} \|A^4 \mathbf{u}_\epsilon\|_{L^2(\Omega)}^2$$

$$\leq c_{15} \|n_\epsilon\|_{L^2(\Omega)}^2 + \frac{1}{4} \|A^{\frac{4+\delta}{2}} \mathbf{u}_\epsilon\|_{L^2(\Omega)}^2. \quad (6.17)$$

In both cases $\gamma \in (1, 2)$ and $\gamma \geq 2$, we therefore can conclude from (6.10), (6.14) and either (6.16) or (6.17) that

$$\frac{d}{dt} \int_\Omega |A^\frac{4+\delta}{2} \mathbf{u}_\epsilon(x, t)|^2 dx + \int_\Omega |A^{\frac{4+\delta}{2}} \mathbf{u}_\epsilon(x, t)|^2 dx \leq a(t) \|A^\frac{4+\delta}{2} \mathbf{u}_\epsilon(\cdot, t)|_{L^2(\Omega)}^2 + b(t) \quad (6.18)$$

for all $t > 0$ and $\epsilon \in (0, 1)$, where

$$a_\epsilon(t) = 2c_9 \|\nabla \mathbf{u}_\epsilon(\cdot, t)\|_{L^2(\Omega)}^2 \quad \text{and} \quad b_\epsilon(t) = \begin{cases} 2c_{13} \|n_\epsilon(\cdot, t)\|_{L^2(\Omega)}^2, & \gamma \in (1, 2), \\ 2c_{15} \|n_\epsilon(\cdot, t)\|_{L^2(\Omega)}^2, & \gamma \geq 2. \end{cases}$$

We observe that the finiteness of $\sup_{\epsilon \in (0, 1)} \int_0^T a_\epsilon(t) dt$ is covered by (6.2) and that of $\sup_{\epsilon \in (0, 1)} \int_0^T b_\epsilon(t) dt$ by either Corollary 4.3 or (4.4), so that Lemma 4.9 is applicable and, if combined with (6.8), ensures (6.6).

With this better regularity of $\mathbf{u}_\epsilon$ ensured, we are able to deal with the integrals involving convective terms arising during an investigation of the temporal evolution of

$$\int_\Omega n_\epsilon \ln n_\epsilon + \frac{1}{2} \int_\Omega |\nabla c_\epsilon|^2.$$

**Lemma 6.5.** Let $C_\epsilon > 0$ be given in (6.1). If $\mu > r_+ (16C_\epsilon^4 |\Omega|/3)^{-1}$, then there is $t_1 > 1$ such that for any $T > t_1$, we can find $C > 0$ ensuring

$$\int_\Omega |\nabla c_\epsilon(\cdot, t)|^2 \leq C \quad \text{for all } t \in (t_1, T) \text{ and } \epsilon \in (0, 1) \quad (6.19)$$

and

$$\int_{t_1}^T \int_\Omega n^2_\epsilon(x, t) dx dt + \int_{t_1}^T \int_\Omega |\Delta c_\epsilon(x, t)|^2 dx dt \leq C \quad \text{for all } \epsilon \in (0, 1). \quad (6.20)$$

**Proof.** By utilizing (3.4)_1 and integrating by parts, we have

$$\frac{d}{dt} \int_\Omega n_\epsilon \ln n_\epsilon = \int_\Omega n_\epsilon \ln n_\epsilon + \int_\Omega n_\epsilon t$$

24
Thus, we have

\[ \int n_\varepsilon \Delta c_\varepsilon \leq \int n_\varepsilon^2 + \frac{1}{4} \int |\Delta c_\varepsilon|^2 \quad \text{in} \ (0, \infty) \text{ for all } \varepsilon \in (0, 1). \] (6.22)

Thus, it can be obtained by (6.21) that

\[ \frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla c_\varepsilon|^2 + \int_\Omega |\Delta c_\varepsilon|^2 + \frac{1}{2} \int_\Omega |\nabla c_\varepsilon|^2 \\
= \int_\Omega u_\varepsilon \cdot \nabla c_\varepsilon \Delta c_\varepsilon + \frac{n_\varepsilon}{1 + \varepsilon n_\varepsilon} \Delta c_\varepsilon \\
\leq \frac{1}{2} \int_\Omega |\Delta c_\varepsilon|^2 + \int_\Omega |u_\varepsilon \cdot \nabla c_\varepsilon|^2 + \int_\Omega n_\varepsilon^2 \quad \text{in} \ (0, \infty) \text{ for all } \varepsilon \in (0, 1), \] (6.24)

where the Sobolev imbedding inequality tells us that for \( \delta = \min\{\frac{1}{2}, \gamma - 1\} \) as in Lemma 6.4 and with some \( c_3 = c_3(\gamma, \Omega) > 0 \),

\[ \int_\Omega |u_\varepsilon \cdot \nabla c_\varepsilon|^2 \leq \|u_\varepsilon\|_{L^\infty(\Omega)}^2 \|\nabla c_\varepsilon\|_{L^2(\Omega)}^2 \]
\[ \leq c_3 \|A^{\frac{1+\delta}{2}} u_\varepsilon\|_{L^2(\Omega)}^2 \|\nabla c_\varepsilon\|_{L^2(\Omega)}^2 \quad \text{in} \ (0, \infty) \text{ for all } \varepsilon \in (0, 1). \]

Thus, we have

\[ \frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla c_\varepsilon|^2 + \frac{1}{2} \int_\Omega |\Delta c_\varepsilon|^2 + \int_\Omega |\nabla c_\varepsilon|^2 \\
\leq c_3 \|A^{\frac{1+\delta}{2}} u_\varepsilon\|_{L^2(\Omega)}^2 \|\nabla c_\varepsilon\|_{L^2(\Omega)}^2 + \int_\Omega n_\varepsilon^2 \quad \text{in} \ (0, \infty) \text{ for all } \varepsilon \in (0, 1). \] (6.25)

A combination of (6.23) and (6.25) yields that

\[ \frac{d}{dt} \left( \int n_\varepsilon \ln n_\varepsilon + \frac{1}{2} \int |\nabla c_\varepsilon|^2 \right) + 4 \int |\nabla \sqrt{n_\varepsilon}|^2 + \frac{1}{4} \int |\Delta c_\varepsilon|^2 \\
\leq c_3 \|A^{\frac{1+\delta}{2}} u_\varepsilon\|_{L^2(\Omega)}^2 \|\nabla c_\varepsilon\|_{L^2(\Omega)}^2 + 2 \int_\Omega n_\varepsilon^2 + c_2 \quad \text{in} \ (0, \infty) \text{ for all } \varepsilon \in (0, 1). \] (6.26)

An application of the Gagliardo-Nirenberg inequality (6.1) shows that

\[ \|n_\varepsilon\|_{L^2(\Omega)}^2 = \sqrt{n_\varepsilon} \|\nabla c_\varepsilon\|_{L^2(\Omega)}^2 \]
\[ \leq 8C_4^4 \|\nabla \sqrt{\n_\varepsilon}\|_{L^2(\Omega)}^2 \|\sqrt{n_\varepsilon}\|_{L^2(\Omega)}^2 + 8C_4^4 \sqrt{n_\varepsilon} \|\sqrt{n_\varepsilon}\|_{L^2(\Omega)}^2 \quad \text{in} \ (0, \infty) \text{ for all } \varepsilon \in (0, 1). \] (6.27)
Therefore,
\[
\frac{d}{dt}\left( \int_{\Omega} n_{\varepsilon} \ln n_{\varepsilon} + \frac{1}{2} \int_{\Omega} |\nabla c_{\varepsilon}|^2 \right) + \left( 4 - 16C_4^4 \int_{\Omega} n_{\varepsilon} \right) \int_{\Omega} |\nabla \sqrt{n_{\varepsilon}}|^2 + \frac{1}{4} \int_{\Omega} |\Delta c_{\varepsilon}|^2 \\
\leq c_3 \|4^{4-1} u_{\varepsilon}\|_{L^2(\Omega)}^2 \|\nabla c_{\varepsilon}\|_{L^2(\Omega)}^2 + c_2 + 16C_4^4 \int_{\Omega} n_{\varepsilon} \quad \text{in } (0, \infty).
\]  
(6.28)

We now employ Lemma 6.2 to find \( t_1 > 2 \) such that \( \int_{\Omega} n_{\varepsilon}(\cdot, t) < \frac{1}{16C_4^4} \) for every \( t > t_1 - 1 \) and every \( \varepsilon \in (0, 1) \), which is possible since the assumption \( \mu > r_++(16C_4^4|\Omega|/3)^{3^{-1}} \) ensures that \( |\Omega|(\frac{r_+}{\mu})^{3^{-1}} < \frac{3}{16C_4^4} \). We thus deduce from (6.28) that for \( c_4 = c_2 + 3 \)
\[
\frac{d}{dt}\left( \int_{\Omega} n_{\varepsilon} \ln n_{\varepsilon} + \frac{1}{2} \int_{\Omega} |\nabla c_{\varepsilon}|^2 \right) + \left( 4 - 16C_4^4 \int_{\Omega} n_{\varepsilon} \right) \int_{\Omega} |\nabla \sqrt{n_{\varepsilon}}|^2 + \frac{1}{4} \int_{\Omega} |\Delta c_{\varepsilon}|^2 \\
\leq c_3 \|4^{4-1} u_{\varepsilon}\|_{L^2(\Omega)}^2 \|\nabla c_{\varepsilon}\|_{L^2(\Omega)}^2 + c_4 \quad \text{in } (t_1 - 1, T) \text{ for all } \varepsilon \in (0, 1).
\]  
(6.29)

Based on (4.4) and (6.3), there is \( c_5 > 0 \) such that
\[
\int_{t_1-1}^{t_1} \int_{\Omega} n_{\varepsilon} \ln n_{\varepsilon} + \int_{t_1-1}^{t_1} \int_{\Omega} |\nabla c_{\varepsilon}|^2 \leq c_5 \quad \text{for every } \varepsilon \in (0, 1).
\]  
(6.30)

Thus, for any fixed \( \varepsilon \in (0, 1) \), there is \( t_{\varepsilon} \in (t_1 - 1, t_1) \) such that
\[
\int_{\Omega} (n_{\varepsilon} \ln n_{\varepsilon})(x, t_{\varepsilon})dx + \int_{\Omega} |\nabla c_{\varepsilon}(x, t_{\varepsilon})|^2 dx \leq c_5.
\]  
(6.31)

We observe that \( z + z \ln z \geq 0 \) for every \( z > 0 \), and recalling (6.31) and (6.6) (for the latter additionally relying on \( t_{\varepsilon} > t_1 - 1 > 1 \)), we utilize the boundedness result given in Lemma 4.9 to find a constant \( c_6 > 0 \) ensuring
\[
\int_{\Omega} |\nabla c_{\varepsilon}(\cdot, t)|^2 \leq c_6 \quad \text{for all } t \in (t_{\varepsilon}, T) \text{ and } \varepsilon \in (0, 1)
\]  
(6.32)

and
\[
\int_{t_{\varepsilon}}^{T} \int_{\Omega} |\Delta c_{\varepsilon}|^2 + \int_{t_{\varepsilon}}^{T} \int_{\Omega} |\nabla \sqrt{n_{\varepsilon}}|^2 \leq c_6 \quad \text{for all } \varepsilon \in (0, 1).
\]  
(6.33)

This together with (6.27) implies the existence of \( c_7 > 0 \) satisfying
\[
\int_{t_{\varepsilon}}^{T} \int_{\Omega} n_{\varepsilon}^2 \leq \int_{t_{\varepsilon}}^{T} \int_{\Omega} n_{\varepsilon}^2 \leq c_7 \quad \text{for all } \varepsilon \in (0, 1),
\]
as desired.

Motivated by [31, Lemma 3.8], the two following results are used to establish higher integrability of \( n_{\varepsilon} \).

**Lemma 6.6.** Let \( C_6 > 0 \) be as specified in (6.1). Assume that \( \mu > r_+(16C_4^4|\Omega|/3)^{3^{-1}} \). Let \( t_1 > 1 \) be as specified in Lemma 6.5. Let \( \tilde{t} > t_1 \text{ and } T > \tilde{t} \). If there exist \( p \geq 2 \) and \( L > 0 \) such that
\[
\int_{\tilde{t}-1}^{\tilde{t}} \int_{\Omega} n_{\varepsilon}^p(x, t)dxdt \leq L \quad \text{for all } \varepsilon \in (0, 1),
\]  
(6.34)
then we can find $C > 0$ satisfying
\[
\int_\Omega n^p_\varepsilon(x,t) \leq C \quad \text{for all } t \in (\tilde{t}, T) \text{ and } \varepsilon \in (0, 1),
\] (6.35)

and
\[
\int_\tilde{t}^T \int_\Omega n^{p+\gamma-1}_\varepsilon(x,t) dx dt \leq C \quad \text{for all } \varepsilon \in (0, 1).
\] (6.36)

**Proof.** Due to (6.34), for any fixed $\varepsilon \in (0, 1)$, we can find $\tilde{t}_\varepsilon \in (\tilde{t} - 1, \tilde{t})$ satisfying
\[
\int_\Omega n^p_\varepsilon(x,\tilde{t}_\varepsilon) dx \leq L \quad \text{for all } \varepsilon \in (0, 1).
\] (6.37)

We multiply (3.4) by $n^{p-1}_\varepsilon$ and integrate by parts to see that
\[
\frac{1}{p} \frac{d}{dt} \int_\Omega n^p_\varepsilon = -(p - 1) \int_\Omega n^{p-2}_\varepsilon |\nabla n_\varepsilon|^2 - \frac{1}{p} \int_\Omega u_\varepsilon \cdot \nabla n^p_\varepsilon - \frac{p - 1}{p} \int_\Omega \nabla n^p_\varepsilon \cdot \nabla c_\varepsilon
\]
\[
+ r \int_\Omega n^{p-1}_\varepsilon - \mu \int_\Omega n^{p+\gamma-1}_\varepsilon - \varepsilon \int_\Omega n^{p+1}_\varepsilon
\]
\[
\leq \frac{-4(p - 1)}{p^2} \int_\Omega |\nabla \tilde{u}_\varepsilon|^2 + \frac{p - 1}{p} \int_\Omega n^p_\varepsilon \cdot \Delta c_\varepsilon + r \int_\Omega n^p_\varepsilon - \mu \int_\Omega n^{p+\gamma-1}_\varepsilon
\] (6.38)
in $(0, \infty)$ for any $\varepsilon \in (0, 1)$; here we used the fact $\nabla \cdot u_\varepsilon = 0$ in $\Omega \times (0, \infty)$. It follows by Hölder’s inequality that
\[
\int_\Omega n^p_\varepsilon \cdot \Delta c_\varepsilon \leq \left( \int_\Omega n^{2p}_\varepsilon \right)^{\frac{1}{p}} \left( \int_\Omega |\Delta c_\varepsilon|^2 \right)^{\frac{1}{2}} \text{ in } (0, \infty) \text{ for all } \varepsilon \in (0, 1).
\] (6.39)

The Gagliardo-Nirenberg inequality and (4.2) allow us to find $c_1, c_2 > 0$ such that
\[
\left( \int_\Omega n^{2p}_\varepsilon \right)^{\frac{1}{p}} \leq \|n_\varepsilon\|_{L^4(\Omega)}^2
\]
\[
\leq c_1 \|\nabla n_\varepsilon\|_{L^2(\Omega)} \|n_\varepsilon\|_{L^2(\Omega)} + c_1 \|n_\varepsilon\|_{L^2(\Omega)}^2
\]
\[
\leq c_1 \|\nabla n_\varepsilon\|_{L^2(\Omega)} \|n_\varepsilon\|_{L^2(\Omega)} + c_2 \text{ in } (0, \infty) \text{ for all } \varepsilon \in (0, 1).
\] (6.40)

In view of (6.39) and (6.40), we infer from Young’s inequality that
\[
\int_\Omega n^p_\varepsilon \cdot \Delta c_\varepsilon \leq \frac{2(p - 1)}{p^2} \int_\Omega |\nabla \tilde{u}_\varepsilon|^2 + c_3 \|\Delta c_\varepsilon\|_{L^2(\Omega)}^2 \int_\Omega n^p_\varepsilon
\]
\[
+ c_3 \int_\Omega |\Delta c_\varepsilon|^2 + c_3 \text{ in } (0, \infty) \text{ for all } \varepsilon \in (0, 1)
\] (6.41)

with $c_3 = \max\{\frac{\mu^2}{8(p-1)}, \frac{\mu^2}{4}\}$. It is easy to verify that
\[
\int_\Omega n^p_\varepsilon \leq \frac{\mu}{2} \int_\Omega n^{p+\gamma-1}_\varepsilon + c_4 \text{ in } (0, \infty) \text{ for all } \varepsilon \in (0, 1)
\] (6.42)

with $c_4 = |\Omega| \sup_{r>0}(rs^p - \frac{\mu}{2} s^{p+\gamma-1})$. We derive from (6.38), (6.41) and (6.42) that
\[
\frac{1}{p} \frac{d}{dt} \int_\Omega n^p_\varepsilon + \frac{\mu}{2} \int_\Omega n^{p+\gamma-1}_\varepsilon
\]
\[ \leq c_3 \| \Delta c_3 \|_{L^2(\Omega)} \int_{\Omega} n_{\epsilon}^p + c_3 \int_{\Omega} |\Delta c_3|^2 + c_3 + c_4 \quad \text{in } (0, \infty) \text{ for all } \epsilon \in (0, 1). \]  

(6.43)

Based on (6.37) and (6.20), we apply Lemma 4.9 to (6.43) to find \( c_5 > 0 \) such that

\[ \sup_{t \in [\tilde{t}_\epsilon, T]} \int_{\Omega} n_{\epsilon}^p(x, t) dx \leq c_5 \]  

(6.44)

and

\[ \int_{[\tilde{t}_\epsilon, T]} \int_{\Omega} n_{\epsilon}^{p + \gamma - 1}(x, t) dx dt \leq c_5 \]  

(6.45)

hold for all \( \epsilon \in (0, 1) \). Hence, the proof is complete, because \((\tilde{t}, T) \subset (\tilde{t}_\epsilon, T)\) for any \( \epsilon \in (0, 1) \).

**Corollary 6.7.** Let \( C_* > 0 \) be as specified in (6.1) and assume that \( \mu > r_+ (16C_*^3 |\Omega|/3)^{\gamma - 1} \). Let \( t_1 > 1 \) be as provided by Lemma 6.5. Then for each \( p \geq 2 \), there exists \( T_p \geq t_1 + 3 \) with the property that for any \( T > T_p \), we can find \( C > 0 \) satisfying

\[ \int_{\Omega} n_{\epsilon}(\cdot, t) \leq C \quad \text{for all } t \in (T_p, T) \text{ and } \epsilon \in (0, 1). \]

**Proof.** Taking (6.20) as a starting estimate, we can utilize Lemma 6.6 to perform an iteration procedure which results in the desired outcome.

With these preparations, we are in the position to prove Theorem 1.2.

**Proof of Theorem 1.2.** Let \( \mu_0 := (16C_*^3 |\Omega|/3)^{\gamma - 1} \) with the pure constant \( C_* > 0 \) defined in (6.1). Let the assumptions in Theorem 1.2 hold and \( t_1 > 0 \) be the time point specified in Lemma 6.5. Since \( \mu > \mu_0 r_+ \), Corollary 6.7 guarantees the existence of \( \tilde{T} \geq t_1 + 3 \) such that for any \( T > \tilde{T} + 2 \), there exists \( c_1 > 0 \) satisfying

\[ \| n_{\epsilon}(\cdot, t) \|_{L^{\gamma}(\Omega)} \leq c_1 \quad \text{for all } t \in (\tilde{T}, T) \text{ and } \epsilon \in (0, 1). \]  

(6.46)

By utilizing smoothing properties of the semigroups \( \{e^{tA}\}_{t \geq 0} \) and \( \{e^{t\Delta}\}_{t \geq 0} \), we can see that

\[ \| u_{\epsilon}(\cdot, t) \|_{L^\infty(\Omega)} \leq c_1 \quad \text{for all } t \in (\tilde{T} + 1, T) \text{ and } \epsilon \in (0, 1) \]  

(6.47)

and

\[ \| \nabla c_{\epsilon}(\cdot, t) \|_{L^\infty(\Omega)} \leq c_2 \quad \text{for all } t \in (\tilde{T} + 1, T) \text{ and } \epsilon \in (0, 1), \]  

(6.48)

where the detailed proofs for (6.47) and (6.48) can be found in [31, Lemma 3.11 and 3.12]. With the aid of (6.46)-(6.48), we can adopt the arguments used in [31, Lemma 3.13] to deduce that

\[ \| n_{\epsilon}(\cdot, t) \|_{C^\theta(\bar{\Omega})} \leq c_3 \quad \text{for all } t \in (\tilde{T} + 2, T) \text{ and } \epsilon \in (0, 1). \]  

(6.49)

with \( \theta \in (0, 1) \). By involving the regularity theory for parabolic equations and the Stokes semigroup [14, 17], it can be obtained that \( n_{\epsilon}, c_{\epsilon} \) and \( u_{\epsilon} \) enjoy the uniform Schauder estimates as below,

\[ \| n_{\epsilon} \|_{C^{2+\lambda,1+\frac{\lambda}{2}}(\bar{\Omega} \times [\tilde{T} + 2, T])} + \| c_{\epsilon} \|_{C^{2+\lambda,1+\frac{\lambda}{2}}(\bar{\Omega} \times [\tilde{T} + 2, T])} + \| u_{\epsilon} \|_{C^{2+\lambda,1+\frac{\lambda}{2}}(\bar{\Omega} \times [\tilde{T} + 2, T])} \leq c_3 \quad \text{for all } \epsilon \in (0, 1) \]

with some \( \lambda \in (0, 1) \) and \( c_3 = c_3(T) > 0 \), here we refer the reader to [22, Section 3] for more details. According to the Arzelà-Ascoli theorem, we can claim that \( (n, c, u) \) together with some \( P \) classically solves (1.1) on the time interval \((\tilde{T} + 2, T)\) and satisfies that

\[ \| n \|_{C^{2+\lambda,1}(\bar{\Omega} \times [\tilde{T} + 2, T])} + \| c \|_{C^{2+\lambda,1}(\bar{\Omega} \times [\tilde{T} + 2, T])} + \| u \|_{C^{2+\lambda,1}(\bar{\Omega} \times [\tilde{T} + 2, T])} \leq c_3. \]

Recalling that \( T > \tilde{T} + 2 \) is arbitrarily chosen, we conclude the proof.
Acknowledgements

Mengyao Ding is supported by the National Natural Science Foundation of China (12071009).

References

[1] N. Bellomo, A. Bellouquid, and N. Chouhad. From a multiscale derivation of nonlinear cross-diffusion models to Keller-Segel models in a Navier-Stokes fluid. Math. Models Methods Appl. Sci., 26(11):2041–2069, 2016.

[2] N. Bellomo, A. Bellouquid, Y. Tao, and M. Winkler. Toward a mathematical theory of Keller–Segel models of pattern formation in biological tissues. Mathematical Models and Methods in Applied Sciences, 25(09):1663–1763, 2015.

[3] T. Black. Global very weak solutions to a chemotaxis-fluid system with nonlinear diffusion. SIAM J. Math. Anal., 50(4):4087–4116, 2018.

[4] T. Black, M. Fuest, and J. Lankeit. Relaxed parameter conditions for chemotactic collapse in logistic-type parabolic-elliptic Keller-Segel systems. Z. Angew. Math. Phys., 2021. arXiv:2005.12089.

[5] E. Di Nezza, G. Palatucci, and E. Valdinoci. Hitchhiker’s guide to the fractional Sobolev spaces. Bull. Sci. Math., 136(5):521–573, 2012.

[6] M. Ding and W. Lv. Generalized solutions to a chemotaxis-fluid system with arbitrary superlinear degradation. preprint.

[7] R. J. DiPerna and P. L. Lions. On the Cauchy problem for Boltzmann equations: Global existence and weak stability. Annals of Mathematics, 130(2):321–366, 1989.

[8] E. Espejo and T. Suzuki. Reaction enhancement by chemotaxis. Nonlinear Anal. Real World Appl., 35:102–131, 2017.

[9] A. Friedman. Partial differential equations. Holt, Rinehart and Winston, Inc., New York-Montreal, Que.-London, 1969.

[10] M. Fuest. Approaching optimality in blow-up results for Keller-Segel systems with logistic-type damping. arXiv:2007.01184.

[11] D. Fujiwara. On the asymptotic behaviour of the Green operators for elliptic boundary problems and the pure imaginary powers of some second order operators. Journal of the Mathematical Society of Japan, 21(4):481–522, 1969.

[12] D. Fujiwara and H. Morimoto. An $L_r$-theorem of the Helmholtz decomposition of vector fields. J. Fac. Sci. Univ. Tokyo Sect. IA Math., 24(3):685–700, 1977.

[13] Y. Giga and T. Miyakawa. Solutions in $L_r$ of the Navier-Stokes initial value problem. Archive for Rational Mechanics and Analysis, 89(3):267–281, Sept. 1985.

[14] Y. Giga and H. Sohr. Abstract $L_p$ estimates for the Cauchy problem with applications to the Navier-Stokes equations in exterior domains. J. Functional Analysis, 102(1):72 – 94, 1991.

[15] D. Henry. Geometric theory of semilinear parabolic equations, volume 840 of Lecture Notes in Mathematics. Springer-Verlag, Berlin-New York, 1981.

[16] M. Herrero and J. Velázquez. A blow-up mechanism for a chemotaxis model. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 24(4):633–683 (1998), 1997.

[17] M. Hieber and J. Prüss. Heat kernels and maximal $L^p$-$L^q$ estimates for parabolic evolution equations. Comm. Partial Differential Equations, 22(9-10):1647–1669, 1997.

[18] D. Horstmann. From 1970 until present: the Keller-Segel model in chemotaxis and its consequences. I. Jahresber. Deutsch. Math.-Verein., 105(3):103–165, 2003.

29
A. Kiselev and L. Ryzhik. Biomixing by chemotaxis and enhancement of biological reactions. *Comm. Partial Differential Equations*, 37(2):298–318, 2012.

E. Lankeit and J. Lankeit. On the global generalized solvability of a chemotaxis model with signal absorption and logistic growth terms. *Nonlinearity*, 32(5):1569–1596, 2019.

J. Lankeit. Eventual smoothness and asymptotics in a three-dimensional chemotaxis system with logistic source. *J. Differential Equations*, 258(4):1158–1191, 2015.

J. Lankeit. Long-term behaviour in a chemotaxis-fluid system with logistic source. *Math. Models Methods Appl. Sci.*, 26(11):2071–2109, 2016.

J. Lankeit. Immediate smoothing and global solutions for initial data in $L^1 \times W^{1,2}$ in a Keller–Segel system with logistic terms in 2d. *Proc. Roy. Soc. Edinburgh Sect. A*, 2020.

J. Lankeit and M. Winkler. Facing low regularity in chemotaxis systems. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 122:35–64, 2020.

J. Liu and Y. Wang. Boundedness and decay property in a three-dimensional Keller-Segel-Stokes system involving tensor-valued sensitivity with saturation. *J. Differential Equations*, 261(2):967–999, 2016.

N. Mizoguchi and M. Winkler. Finite-time blow-up in the two-dimensional Keller-Segel system. *preprint*.

T. Nagai, T. Senba, and K. Yoshida. Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis. *Funkcial. Ekvac.*, 40(3):411–433, 1997.

K. Osaki, T. Tsujikawa, A. Yagi, and M. Mimura. Exponential attractor for a chemotaxis-growth system of equations. *Nonlinear Anal. Theory Methods Appl.*, 51(1):119–144, Oct. 2002.

K. Osaki and A. Yagi. Finite dimensional attractor for one-dimensional Keller-Segel equations. *Funkcial. Ekvac.*, 44(3):441–469, 2001.

Y. Tao and M. Winkler. Boundedness and decay enforced by quadratic degradation in a three-dimensional chemotaxis-fluid system. *Z. Angew. Math. Phys.*, 66(5):2555–2573, 2015.

Y. Tao and M. Winkler. Blow-up prevention by quadratic degradation in a two-dimensional Keller-Segel-Navier-Stokes system. *Z. Angew. Math. Phys.*, 67(6):Art. 138, 23, 2016.

J. I. Tello and M. Winkler. A chemotaxis system with logistic source. *Commun. Partial Differ. Equ.*, 32(6):849–877, June 2007.

Y. Wang. Global solvability and eventual smoothness in a chemotaxis-fluid system with weak logistic-type degradation. *Math. Models Methods Appl. Sci.*, 30(6):1217–1252, 2020.

Y. Wang, M. Winkler, and Z. Xiang. Global solvability in a three-dimensional Keller-Segel-Stokes system involving arbitrary superlinear logistic degradation. *Adv. Nonlinear Anal.*, 10(1):707–731, 2021.

Y. Wang and Z. Xiang. Global existence and boundedness in a Keller-Segel-Stokes system involving a tensor-valued sensitivity with saturation. *J. Differential Equations*, 259(12):7578–7609, 2015.

M. Winkler. *L^1* solutions to parabolic-Keller-Segel systems involving arbitrary superlinear degradation. *preprint*.

M. Winkler. Aggregation vs. global diffusive behavior in the higher-dimensional Keller-Segel model. *J. Differential Equations*, 248(12):2889–2905, 2010.

M. Winkler. Boundedness in the higher-dimensional parabolic-parabolic chemotaxis system with logistic source. *Comm. Partial Differential Equations*, 35(8):1516–1537, 2010.

M. Winkler. Blow-up in a higher-dimensional chemotaxis system despite logistic growth restriction. *J. Math. Anal. Appl.*, 384(2):261–272, 2011.

M. Winkler. Global large-data solutions in a chemotaxis-(Navier-)Stokes system modeling cellular swimming in fluid drops. *Comm. Partial Differential Equations*, 37(2):319–351, 2012.
[41] M. Winkler. Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller-Segel system. *J. Math. Pures Appl. (9)*, 100(5):748–767, 2013.

[42] M. Winkler. Large-data global generalized solutions in a chemotaxis system with tensor-valued sensitivities. *SIAM J. Math. Anal.*, 47(4):3092–3115, 2015.

[43] M. Winkler. Finite-time blow-up in low-dimensional Keller-Segel systems with logistic-type superlinear degradation. *Z. Angew. Math. Phys.*, 69(2):Art. 69, 40, 2018.

[44] M. Winkler. A three-dimensional Keller-Segel-Navier-Stokes system with logistic source: global weak solutions and asymptotic stabilization. *J. Funct. Anal.*, 276(5):1339–1401, 2019.

[45] M. Winkler. Attractiveness of constant states in logistic-type Keller-Segel systems involving subquadratic growth restrictions. *Adv. Nonlinear Stud.*, 20(4):795–817, 2020.

[46] C. Wu and Z. Xiang. The small-convection limit in a two-dimensional Keller-Segel-Navier-Stokes system. *J. Differential Equations*, 267(2):938–978, 2019.

[47] T. Xiang. How strong a logistic damping can prevent blow-up for the minimal Keller-Segel chemotaxis system? *J. Math. Anal. Appl.*, 459(2):1172–1200, 2018.

[48] H. Yu, W. Wang, and S. Zheng. Global classical solutions to the Keller-Segel-Navier-Stokes system with matrix-valued sensitivity. *J. Math. Anal. Appl.*, 461(2):1748–1770, 2018.

[49] J. Zheng. Boundedness in a three-dimensional chemotaxis-fluid system involving tensor-valued sensitivity with saturation. *J. Math. Anal. Appl.*, 442(1):353–375, 2016.