The transformations of non-abelian gauge fields under translations

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I consider infinitesimal translations $x^\alpha = x^\alpha + \delta x^\alpha$ and demand that Noether’s approach gives a symmetric energy-momentum tensor as it is required for gravitational sources. This argument determines the transformations of non-abelian gauge fields under translations to differ from the usually assumed invariance by the gauge transformation,

$$A^{\alpha}_{\gamma}(x') - A^{\alpha}_{\gamma}(x) = \partial_{\gamma} \left[ \delta x^\beta A^{\alpha \beta}(x) \right] + C^\alpha_{\beta \gamma} \delta x^\beta A^{\gamma \beta}(x) A^{\alpha}_{\gamma}(x)$$

where the $C^\alpha_{\gamma \beta}$ are the structure constants of the gauge group.

As in [1] the observation is that the conventionally assumed invariance of the gauge fields under translations

$$x'{}^\alpha = x^\alpha + \delta x^\alpha$$

can be enlarged by gauge transformations and the general form reads

$$A^{\alpha}_{\gamma}(x') = A^{\alpha}_{\gamma}(x) + \partial_{\gamma} \epsilon^\alpha(x) + C^\alpha_{\beta \gamma} \epsilon^\beta(x) A^{\beta}_{\gamma}.$$  

Repeating the arguments of Noether’s theorem in the version of [2] and requesting a symmetric energy-momentum tensor determines the gauge transformation uniquely and leads to the transformation law stated in the abstract

$$A^{\alpha}_{\gamma}(x') = A^{\alpha}_{\gamma}(x) + \partial_{\gamma} \left[ \delta x^\beta A^{\alpha \beta}(x) \right] + C^\alpha_{\beta \gamma} \delta x^\beta A^{\gamma \beta}(x) A^{\alpha}_{\gamma}(x).$$  

The remainder of this letter is devoted to the derivation of this equation and my treatment follows closely [1] where a few additional steps can be found.

First, let us consider general fields $\psi_k$ and recall the derivation of the relativistic Euler Lagrange equations from the action principle. The action is a four dimensional integral over a scalar Lagrangian density

$$A = \int d^4x \mathcal{L}(\psi_k, \partial_\alpha \psi_k).$$  

Variations of the fields are defined as functions

$$\delta \psi_k(x) = \psi'_k(x) - \psi_k(x)$$

which are non-zero for some localized space-time region. The action is required to vanish under such variations

$$0 = \delta A = \sum_k \int d^4x \left[ (\delta \psi_k) \frac{\partial \mathcal{L}}{\partial \psi_k} + (\delta \partial_\alpha \psi_k) \frac{\partial \mathcal{L}}{\partial (\partial_\alpha \psi_k)} \right].$$

Integration by parts allows to factor $\delta \psi_k$ out and, because all the $\delta \psi_k$ are independent, we arrive at the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \psi_k} - \partial_\gamma \frac{\partial \mathcal{L}}{\partial (\partial_\gamma \psi_k)} = 0.$$  

Together with the anti-symmetry of $F^{a \beta}_{\gamma}$ in the Lorentz indices, the Euler-Lagrange equations imply the relation

$$\partial_\gamma \frac{\partial \mathcal{L}}{\partial (\partial_\alpha A^a_{\gamma})} = C^b_{\alpha c} A^c_{\beta} \frac{\partial \mathcal{L}}{\partial (\partial_\beta A^b_{\alpha})}.$$  

Noether’s theorem applies to transformations of the coordinates for which the transformations of the field functions are also known and we introduce, in addition to the $\delta \psi_k$, a second type of variations which combines space-time and their corresponding field variations

$$\overline{\delta} \psi_k(x) = \psi'_k(x) - \psi_k(x).$$  

Using

$$\psi'_k(x) = \psi'_k(x) + \delta x^\alpha \partial_\alpha \psi_k(x)$$

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we find a relation between the variations (12) and (8)
\[ \delta \psi_k(x) = \delta \psi_k(x) + \delta x^\alpha \partial_\alpha \psi_k(x). \]  
(13)
For a scalar field \( \psi \) symmetry under translations means
\[ \delta \psi(x) = \psi'(x') - \psi(x) = 0. \]  
(14)
But for the gauge fields we allow
\[ \delta A^\alpha,\gamma(x) = A^\alpha,\gamma(x') - A^\alpha,\gamma(x) \]
\[ = \partial_\gamma \epsilon^\alpha(x) + C^\alpha_{\beta\gamma} \epsilon^\beta(x) A^\beta,\gamma(x) \]  
(15)
and equation (13) becomes
\[ \delta A^\alpha = \partial_\gamma \epsilon^\alpha(x) + C^\alpha_{\beta\gamma} \epsilon^\beta(x) A^\beta,\gamma(x) - \delta x^\alpha \partial_\alpha A^\alpha,\gamma(x). \]  
(16)
As the Lagrange density is a scalar, we get for its combined variation (12)
\[ 0 = \delta \mathcal{L} = \mathcal{L}'(x') - \mathcal{L}(x) = \delta \mathcal{L} + \delta x^\alpha \partial_\alpha \mathcal{L} \]  
(17)
where besides (14) we used the relation (13). Our aim is to factor an overall variation \( \delta x^\alpha \) out. For \( \delta \mathcal{L} \) we proceed as in equation (3), where the \( \psi_k \) fields are now replaced by the gauge fields \( A^\alpha,\gamma \)
\[ \delta \mathcal{L} = (\delta A^\alpha,\gamma) \frac{\partial \mathcal{L}}{\partial A^\alpha,\gamma} + (\delta \partial_\alpha A^\alpha,\gamma) \frac{\partial \mathcal{L}}{\partial (\partial_\alpha A^\alpha,\gamma)}. \]
Using the Euler-Lagrange equation (10) to eliminate \( \partial \mathcal{L}/\partial A^\alpha,\gamma \), we get
\[ \delta \mathcal{L} = \partial_\alpha \left[ (\delta A^\alpha,\gamma) \frac{\partial \mathcal{L}}{\partial (\partial_\alpha A^\alpha,\gamma)} \right]. \]
Let us collect all terms which contribute to \( \delta \mathcal{L} \) in equation (14). We find (note that \( \partial_\beta \delta x^\alpha = 0 \) holds for all combinations of indices \( \alpha, \beta \))
\[ 0 = \delta \mathcal{L} = \partial_\alpha \left[ (\delta A^\alpha,\gamma) \frac{\partial \mathcal{L}}{\partial (\partial_\alpha A^\alpha,\gamma)} + \delta x^\alpha \mathcal{L} \right] = \]
\[ \partial_\alpha \left[ (\partial_\gamma \epsilon^\alpha(x) + C^\alpha_{\beta\gamma} \epsilon^\beta(x) A^\beta,\gamma) \frac{\partial \mathcal{L}}{\partial (\partial_\alpha A^\alpha,\gamma)} \right] \]
\[ + \delta x_\beta \partial_\alpha \left[ -(\partial^\beta A^\alpha,\gamma) \frac{\partial \mathcal{L}}{\partial (\partial_\alpha A^\alpha,\gamma)} + g^{\alpha\beta} \mathcal{L} \right] \]
where equation (16) was used. To be able to factor \( \delta x_\beta \) also out of the first bracket on the right-hand side, one has to request
\[ \epsilon^\alpha(x) = \delta x_\beta B^{\alpha \beta}(x) \]  
(18)
where \( B^{\alpha \beta}(x) \) is a not yet determined gauge field. With this we get
\[ 0 = \delta x_\beta \partial_\alpha \left[ (\partial^\beta A^\alpha,\gamma) \frac{\partial \mathcal{L}}{\partial (\partial_\alpha A^\alpha,\gamma)} - (\partial_\gamma B^{\alpha \beta} + C^\alpha_{\beta\gamma} B^{\beta \gamma}) \frac{\partial \mathcal{L}}{\partial (\partial_\alpha A^\alpha,\gamma)} - g^{\alpha\beta} \mathcal{L} \right]. \]  
(19)
Equation (11) implies that the contribution from the gauge transformations is a total divergence,
\[ \partial_\gamma \left( \frac{\partial \mathcal{L}}{\partial (\partial_\alpha A^\alpha,\gamma)} B^{\alpha \beta} \right) = \]
\[ \frac{\partial \mathcal{L}}{\partial (\partial_\alpha A^\alpha,\gamma)} \left[ \partial_\gamma B^{\alpha \beta} + C^\alpha_{\beta\gamma} B^{\beta \gamma} A^\gamma,\beta \right]. \]  
(20)
As the variations \( \delta x_\beta \) in (13) are independent, the energy-momentum tensor
\[ \theta^{\alpha\beta} = \frac{\partial \mathcal{L}}{\partial (\partial_\alpha A^\alpha,\gamma)} \left[ \partial_\beta A^\alpha,\gamma - (\partial_\gamma B^{\alpha \beta} + C^\alpha_{\beta\gamma} B^{\beta \gamma} A^\gamma,\beta) \right] \]
\[ - g^{\alpha\beta} \mathcal{L} \]  
(21)
gives the conserved currents
\[ \partial_\alpha \theta^{\alpha\beta} = 0. \]  
(22)
We demand that \( \theta^{\alpha\beta} \) is symmetric. The Lagrangian term \( g^{\alpha\beta} \mathcal{L} \) is manifestly symmetric and we have to deal with the other contributions. We note that
\[ \frac{\partial \mathcal{L}}{\partial (\partial_\alpha A^\alpha,\gamma)} = - F^{\alpha \gamma} \]
with \( F^{\alpha \gamma} \) given by equation (2). Therefore, the choice
\[ B^{\alpha \beta}(x) = A^{\alpha \beta}(x) \]  
(23)
leads to
\[ \theta^{\alpha\beta} = F^{\alpha \gamma} F^{\gamma \beta} - g^{\alpha\beta} \mathcal{L} \]  
(24)
where we used the anti-symmetry of the structure constant under interchange of \( b \) and \( c \). The tensor (24) is symmetric because of
\[ F^{\alpha \gamma} F^{\gamma \beta} = F^{\alpha \beta} F^{\beta \alpha}. \]
In conclusion, I have derived the transformation behavior (6) by demanding that the energy-momentum tensor from Noether’s theorem comes out symmetric. To the many arguments why gauge invariance is needed, this adds another one: It is needed to make the energy-momentum distribution under local translational variations symmetric.

**Note added**

After posting this manuscript Prof. Jackiw kindly informed me that my result is a special case of his work [7], see [8] for details. Prof. Hehl communicated that the use of 1-Forms leads directly to a symmetric energy-momentum tensor, see for instance [8].
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