Robbins and Ardila meet Berstel

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Abstract

In 1996, Neville Robbins proved the amazing fact that the coefficient of $X^n$ in the Fibonacci infinite product

$$\prod_{n \geq 2} (1 - X^{F_n}) = (1 - X)(1 - X^2)(1 - X^3)(1 - X^5)(1 - X^8) \cdots = 1 - X - X^2 + X^4 + \cdots$$

is always either $-1$, $0$, or $1$. The same result was proved later by Federico Ardila using a different method.

Meanwhile, in 2001, Jean Berstel gave a simple 4-state transducer that converts an “illegal” Fibonacci representation into a “legal” one. We show how to obtain the Robbins-Ardila result from Berstel’s with almost no work at all, using purely computational techniques that can be performed by existing software.

1 Introduction

The goal of this paper is to show how to prove an amazing 1996 result of Robbins [9] on the coefficients of a Fibonacci infinite product, namely that the coefficient of $X^n$ in

$$\prod_{n \geq 2} (1 - X^{F_n}) = (1 - X)(1 - X^2)(1 - X^3)(1 - X^5)(1 - X^8) \cdots = 1 - X - X^2 + X^4 + \cdots$$

is always either $-1$, $0$, or $1$ [9]. A different proof was given later by Ardila [1]. The novelty of our approach is that it is purely “computational”, using algebraic techniques on automata that can be carried out by existing software, starting from a construction of Jean Berstel. With this approach one can also prove new results (see Section 6).
2 Fibonacci representation

Let us start with the basics of Fibonacci representation (also known as Zeckendorf representation) [6, 12]. Every natural number has an essentially unique representation as a sum of Fibonacci numbers \( n = \sum_{0 \leq i < t} e_i F_{i+2} \), provided that no two consecutive Fibonacci numbers are used. (Here, as usual, we write \( F_0 = 0 \), \( F_1 = 1 \), and \( F_n = F_{n-1} + F_{n-2} \) for \( n \geq 2 \).) If \( n \) is written this way, we define \((n)_F\) to be \( e_{t-1}e_{t-2}\cdots e_0\), a binary string called the canonical Fibonacci representation of \( n \). The map \( n \to (n)_F \) gives a bijection between \( \mathbb{N} \) and the strings specified by the regular expression \( CF := \epsilon + 1(0+01)^* \)—that is, the set of all binary having no two consecutive 1’s that do not start with 0. We also define, for a binary string \( x = b_1 \cdots b_t \), the map \([n]_F \) to be \( \sum_{1 \leq i \leq t} b_i F_{t-i+2} \). Thus, for example, \((43)_F = 10010001 \) and \([0010001101]_F = 43 \).

3 From Berstel’s transducer to a linear representation

We start with Berstel’s transducer [2]. When rewritten as a DFA \( M \), it becomes the following:

![Figure 1: Berstel’s DFA M for Fibonacci normalization.](image)

Inputs to \( M \) consist of strings of pairs of letters. The automaton \( M \) accepts if the string spelled out by the first components of the input—an arbitrary string of 0’s and 1’s—evaluates to the same number as the the canonical Fibonacci representation spelled out by the second components. (Here, as usual, accepting states are denoted by double circles.) More formally, if \( x = x_1 \cdots x_i \) and \( y = y_1 \cdots y_i \), define \( x \times y \) to be the string of pairs \([x_1, y_1] \cdots [x_i, y_i] \). On input \( x \times y \), the automaton accepts iff

\[ x \in \{0,1\}^* \text{ and } y \in 0^*C_F \text{ and } [x]_F = [y]_F. \] (1)

Now suppose \( y \) is a canonical Fibonacci representation for \( n \). Let us count the number \( r(n) \) of strings \( x \) such that \( M \) accepts \( x \times y \). As Berstel observed, this is the number of binary strings \( x \) such that \(|x| = |y|\) and \([x]_F = [y]_F \). In other words, this is the number of Fibonacci partitions of \( n \): the number of ways to write \( n \) as a sum of Fibonacci numbers,
where order does not matter. For example, \( r(8) = 3 \), corresponding to the three accepted strings

\[
[1, 1][0, 0][0, 0][0, 0], [0, 1][1, 0][1, 0][0, 0][0, 0], [0, 1][1, 0][0, 0][1, 0][1, 0]
\]

and the three Fibonacci partitions \( 8 = 5 + 3 = 5 + 2 + 1 \).

If we now define \((\mu(a))_{i,j}\) as the number of paths labeled \([\ast, a]\) (where the star \(\ast\) means any symbol) from state \(i\) to state \(j\) of \(M\), we get a so-called linear representation \((v, \mu, w)\) for \(r(n)\):

\[
v = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}; \quad \mu(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}; \quad \mu(1) = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad w = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.
\]

Here \(\mu\) is a morphism, that is, a map satisfying \(\mu(x)\mu(y) = \mu(xy)\) for all strings \(x\) and \(y\). If \((n)_F = x\), then \(r(n) = v\mu(x)w\). This gives a very efficient way to compute \(r(n)\): write \(n\) as its canonical Fibonacci representation \(x\), multiply the matrices \(\mu(0)\) and \(\mu(1)\) according to the bits of \(x\), and then pre- and post-multiply by the vectors \(v\) and \(w\). The rank of a linear representation \((v, \mu, w)\) is the dimension of the vector \(v\); in this case it is 4.

Notice that \(r(n)\) is just the coefficient of \(X^n\) in the following Fibonacci power series:

\[
\prod_{i \geq 2} (1 + X^{F_i}) = (1 + X)(1 + X^2)(1 + X^3)(1 + X^5)(1 + X^8) \cdots.
\]

Of course, \(r(n)\) is unbounded.

Robbins took this power series and modified it to

\[
\prod_{i \geq 2} (1 - X^{F_i}) = (1 - X)(1 - X^2)(1 - X^3)(1 - X^5)(1 - X^8) \cdots
\]

\[
= \sum_{n \geq 0} a(n)X^n, \quad (2)
\]

so that \(a(n)\) is the coefficient of \(X^n\) in this new series. He observed that if \(r_e(n)\) is the number of Fibonacci partitions using an even number of terms, and \(r_o(n)\) is the number of Fibonacci partitions using an odd number of terms, then clearly \(r(n) = r_e(n) + r_o(n)\). Furthermore, Robbins noted that Eq. (2) gives \(a(n) = r_e(n) - r_o(n)\). By adding these two equations we get \(a(n) = 2r_e(n) - r(n)\). Since we already know how to compute \(r(n)\), to compute \(a(n)\) we only need to know \(r_e(n)\).

We can find a linear representation for \(r_e(n)\) using a trivial modification of Berstel’s automaton. It suffices to create a new automaton \(M'\) accepting those pairs \(x \times y\) exactly as before, but constrained by the number of 1’s in \(x\) being even. This amounts to performing a cross product construction of \(M\) with the following simple automaton, where again \(\ast\) matches any symbol:
This cross product can be computed automatically with software that manipulates automata, such as Grail [7].

This gives the new automaton $M'$ below, which accepts those inputs over the alphabet $\Sigma_2^*$ that satisfy the condition (1) and also have an even number of 1’s in the first components. The names of the states match those in Fig. 1, together with the parity (0 or 1) of the number of 1’s in the first coordinate.

The next step is to find the linear representation corresponding to the automaton $M'$. It is $(v', \mu', w')$, as given below. Again, this can be computed “automatically” just by counting paths in the transition diagram of $M'$.

$$v = \begin{bmatrix} 10000000 \end{bmatrix}; \quad \mu'(0) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} ; \quad \mu'(1) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} ; \quad w' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

This representation has rank 8.

One can also use the program Walnut, written by Hamoon Mousavi [8], to produce the linear representation directly. We give the details now. First, we write a Walnut regular expression specifying that $x$ has an even number of 1’s:
Next, after having stored the automaton in Figure 1 in \textit{Walnut} format in the directory \texttt{Automata Library}, under the name $\$berst$, we issue the following command \texttt{eval fibeven y "?msd\_fib $even1(x) & $berst(x,y)"}. The linear representation can then be found in the file \texttt{fibeven.mpl} in the \texttt{Result} directory.

Next, we can construct a linear representation for the function $a(n) = 2r_e(n) - r(n)$, by just combining the ones for $r_e$ and $r$:

$$v'' = \begin{bmatrix} 2v & -v' \end{bmatrix}; \quad \mu''(0) = \begin{bmatrix} \mu(0) & 0 \\ 0 & \mu'(0) \end{bmatrix}; \quad \mu''(1) = \begin{bmatrix} \mu(1) & 0 \\ 1 & \mu'(1) \end{bmatrix}; \quad w'' = \begin{bmatrix} w \\ w' \end{bmatrix}. $$

This gives us a linear representation for $a(n)$ of rank 12.

Next, we minimize this linear representation, using the algorithm given by Berstel and Reutenauer \cite{3}. We get the equivalent rank-4 linear representation $(y, \gamma, z)$ for $a(n)$ below:

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}; \quad \gamma(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{bmatrix}; \quad \gamma(1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad z = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 0 \end{bmatrix}.$$ 

4 \hspace{1em} \textbf{Finishing up the proof}

Finally, we can use breadth-first search (aka the “semigroup trick” of \cite{5}) to verify that the set of all products of the form $y \gamma(x)$, $x \in \{0,1\}^*$ is finite. We find that the resulting semigroup $S$ is of cardinality 15. (We remark that the semigroup generated by the two matrices $\gamma(0)$ and $\gamma(1)$ has cardinality 207.) Each member of $S$ is a vector, and we can easily check that the dot product of each vector with $z$ gives only 0, 1, $-1$. The result of Robbins is now proved. Furthermore, the linear representation $(y, \gamma, z)$ provides a simple algorithm to compute $a(n)$.

5 \hspace{1em} \textbf{Going further}

We can prove even more. The semigroup $S = \{y \gamma(x) : x \in \{0,1\}^*\}$ allows us to construct a finite automaton $A$ that computes $a(n)$ in the following way: on input the canonical Fibonacci representation $(n)_F$, the automaton arrives at a state with output $a(n)$. Here the states are named $y \gamma(x)$ for some $x$, the initial state is $y$, transitions are given by $\delta(u, a) = u \cdot \gamma(a)$, and the output of the state named $y \gamma(x)$ is $y \gamma(x)z$. The automaton $A$ is hence algorithmically constructible, and the result is displayed below:
The automaton $A$ gives us a lot of information about how $a(n)$ behaves. For example, Ardila proved that “almost all” $n$ have $a(n) = 0$. We can easily prove this using the DFA $A$ as follows: clearly, almost all natural numbers $n$ have a Fibonacci representation containing the block $t = 01001001$. Now all we have to check is that $t$ is a synchronizing word (see [11]) for $A$: the action of $t$ on every state $q$ maps $q$ to state 3, which has output 0.

Furthermore, the representations we have obtained for $r(n)$, $r_e(n)$, and $a(n)$ give us most of the other results of Robbins, without the need for inductions or case analysis.

**Theorem 1.**

(a) $r(F_n) = \lfloor n/2 \rfloor$ for $n \geq 2$.

(b) $r_e(F_n) = \lfloor n/4 \rfloor$ for $n \geq 1$.

(c) $a(F_n - 1) = \begin{cases} 1, & \text{if } n \equiv 1, 2 \pmod{4}; \\
-1, & \text{if } n \equiv 0, 3 \pmod{4}; \end{cases}$ and $n \geq 1$.

**Proof.**

(a) Since $(F_n)_F = 10^{n-2}$, it follows that $r(F_n) = v\mu(1)\mu(0)^{n-2}w$. Hence $r(F_n)$ is a linear combination of the entries of $\mu(0)^{n-2}$. But each entry of the matrices $\mu(0)^t$, considered as a sequence indexed by $t$, satisfies a linear recurrence whose annihilating polynomial is the minimal polynomial of $\mu(0)$, and hence so do the values $r(F_n)$. We can use computer algebra software, such as Maple, to compute this minimal polynomial; it is $X(X + 1)(X - 1)^2$. By the fundamental theorem of linear recurrences we know that $r(F_n) = c_1n + c_2 + c_3(-1)^n$ for $n \geq 2$. Solving for the constants gives us $c_1 = 1/2$, $c_2 = -1/4$, and $c_3 = 1/4$. Hence $r(F_n) = \lfloor n/2 \rfloor$ for $n \geq 2$, as desired.
(b) Here we play the same game, but for the linear representation \((v', \mu', w')\). We get a minimal polynomial of \(X(X + 1)(X^2 + 1)(X - 1)^2\) for \(\mu'(0)\). Hence \(r_{e}(F_n) = c_1 n + c_2 + c_3(-1)^n + c_4 i^n + c_5(-i)^n\), where \(i = \sqrt{-1}\). Solving for the constants gives \(c_1 = 1/4, c_2 = -3/8, c_3 = 1/8, c_4 = 1/8 - i/8, c_5 = 1/8 + i/8\), which gives the desired result.

(c) The Fibonacci representation for \(F_{n-1}\) is given by \((10)^{n/2-1}\) if \(n \geq 2\) is even, and \((10)^{(n-3)/2}\) if \(n \geq 3\) is odd. Examining the path in the automaton \(A\) labeled by 101010\(\cdots\), the result follows immediately.

The same ideas can be used to easily prove the equality \(r(F_{n-1}^2 - 1) = F_n\) for \(n \geq 2\) from [10] and to prove the theorems in [4].

6 Some new results

Another advantage to this method is that with exactly the same techniques, we can go on to study additional variations on the Fibonacci infinite product. For example, we could study Fibonacci partitions with the number of parts congruent to 0, 1, or 2 \((\text{mod } 3)\).

Exactly the same computational techniques we have described here then easily prove the following new result. Let \(r_{m,i}(n)\) denote the number of Fibonacci partitions of \(n\) having the number of parts congruent to \(i\) \((\text{mod } m)\).

**Theorem 2.** We have \(r_{3,i}(n) - r_{3,i+1}(n) \in \{-1, 0, 1\}\) for \(i \in \{0, 1, 2\}\).

**Proof.** We use the following Walnut commands to construct the automata and matrices:

```
reg three1 {0,1} "0*(10*10*10*)":
def fib3 "$msd_fib $three1(x) & $berst(x,y)":
eval fib3m y "$msd_fib $fib3(x,y)" :
```

Once we have a linear representation \((v, \mu, w)\) for \(r_{3,0}(n)\), we can easily construct ones for \(r_{3,1}(n)\) and \(r_{3,2}(n)\), by modifying the final states specified in the vector \(w\), and then one can easily construct linear representations for the difference \(r_{3,i}(n) - r_{3,i+1}(n)\) for \(i = 0, 1, 2\). Then we proceed as before, minimizing the linear representations, and using the “semigroup trick” to prove finiteness. In this case the size of the semigroup is 61.

However, for number of parts modulo 4, the boundedness property no longer holds. The following result is also easily proved by our method:

**Theorem 3.** Let \(d(n) = r_{4,0}(n) - r_{4,2}(n)\). Then \(d(n) = -16^k\) for \(n = [((100)^{8k+1}1]_F\) and \(d(n) = 4 \cdot 16^k\) for \(n = [((100)^{8k+5}1]_F\).
Proof. We use the same ideas. This time the minimized linear representation for \(d(n)\) has rank 8:

\[
v = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
-1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\
\end{bmatrix} ; \quad \mu(0) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\
\end{bmatrix} ; \quad w = \begin{bmatrix} 1 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
-1 \\
-1 \\
\end{bmatrix} .
\]

It is now easy to prove by induction that

\[
\mu((100)^{8n+5}1) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 \cdot 16^k \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} ,
\]

from which the claim \(d(n) = 4 \cdot 16^k\) for \(n = [(100)^{8k+5}1]_F\) follows immediately. The other is handled similarly. □

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