CHERN CLASSES OF GAUSS-MANIN BUNDLES OF WEIGHT 1 VANISH

HÉLÈNE ESNAULT AND ECKART VIEHWEG

Abstract. We show that the Chern character of a variation of polarized Hodge structures of weight one with nilpotent residues at $\infty$ dies up to torsion in the Chow ring, except in codimension 0.

1. Introduction

Let $f : X \to S$ be a proper smooth family over a smooth base, defined over a field $k$ of characteristic 0. Then the Gauss-Manin bundles $\mathcal{H}^i := R^i f_* \Omega_{X/S}^\bullet$ are endowed with the Gauss-Manin connection, and thereby, e.g. by Chern-Weil theory, their Chern classes die in de Rham cohomology. By Griffiths’ fundamental theorem [9], the Gauss-Manin connection is regular singular, and thinking of $k = \mathbb{C}$, the underlying local monodromies around the components of $T$ are quasi-unipotent. If they are unipotent, then Deligne’s extension $\mathcal{H}^i$ ([4]) has nilpotent residues, and an Atiyah class computation ([6], appendix B) shows that the Chern classes of $\mathcal{H}^i$ also die in de Rham cohomology.

On the other hand, Mumford ([14]) remarked that the Grothendieck-Riemann-Roch theorem applied to the structure sheaf and the dualizing sheaf of a family of curves $f$, yields vanishing, up to torsion, of the Chern classes of $\mathcal{H}^1$, a much stronger information than the vanishing in de Rham cohomology. If one compactifies $f$ as a semistable family of curves $\bar{f}$, Deligne’s extension $\mathcal{H}^1$ is simply $R^1 \bar{f}_* \Omega_{X/S}^\bullet(\log \infty)$ and again, Grothendieck-Riemann-Roch allows to conclude that the Chern classes of $\mathcal{H}^1$ die, up to torsion, in the Chow ring as well ([4]). This led the first author to wonder whether Gauss-Manin bundles in general can yield non-trivial algebraic cycles in Chow groups (see [7]). For example, it is proven in [2] that the algebraic Chern-Simons invariants of Gauss-Manin bundles in characteristic $p > 0$ always die (up to torsion).

The main theorem of this article is

Date: Dec. 31, 2001.

1991 Mathematics Subject Classification. 14K10, 14C15, 14C40.
Theorem 1.1. Let $B$ be a smooth complex variety, with a compactification $\overline{B} \subset B$ such that $\overline{B}$ is smooth and $T := \overline{B} \setminus B$ is a divisor with normal crossings. Let $H^1$ be a variation of polarized pure Hodge structures of weight 1, with unipotent local monodromies along the components of $T$, and let $\overline{H}^1$ be its Deligne’s extension with nilpotent residues. Then one has

$$\text{ch}(\overline{H}^1) \in CH^0(\overline{B}) \otimes \mathbb{Q}.$$ 

Applying Grothendieck-Riemann-Roch to powers of a principal polarization in a family of abelian varieties, van der Geer proved vanishing for $H^1$ ([13]). A reduction to Mumford’s curve case, via the Abel-Jacobi map for genus $\leq 3$ ([16]) and via the Abel-Prym map for $g = 4, 5$ ([12]), yields vanishing for $H^1$. On the other hand, Mumford’s argument for a family of abelian varieties shows immediately that the alternate sum of the Gauss-Manin bundles $H^i$ has vanishing Chern classes, up to torsion. The problem is therefore to find a way to separate the different weights. Van der Geer’s argument seems to be a detour. De Rham cohomology is not coherent cohomology, yet his proof relies of the remarkable identity $\text{Td}(E^\vee) = \text{ch}(f_*L)$, where $E^\vee$ is the Hodge bundle and $L$ is a principal polarization. This identity does not extend across the boundary, as one easily checks for families of elliptic curves.

In this note, we present a way to separate the different weights in the spirit of de Rham cohomology. We mod out the family of abelian varieties by the $(-1)$ action, which has the effect to separate the even from the odd weights. Then we observe that vanishing for the sum of the Gauss-Manin bundles in even weights is equivalent to vanishing for $H^1$ (Lemma 2.1).

The next step is to extend the quotient across the boundary, keeping track both of the Gauss-Manin bundles and of the Riemann-Roch theorem. To this aim, one has to consider a compactification of the universal abelian variety with level $n \geq 3$ structure over the moduli space $A_{g,n}$, and to extend the $(-1)$ action, controlling the fixed points. This is performed by a careful study of $\mathfrak{S}$ (see Theorem 3.1). One has then to understand the effect of the boundary on the Riemann-Roch-Grothendieck theorem. This is Theorem 4.1, which is perhaps of independent interest. The philosophy of this log version is that most of the extra terms one has in the Riemann-Roch formula are killed by the existence of the residues on the relative log 1-forms. The other ones die when one assumes that the relative 1-log forms essentially come from the base of the family.
A version of Theorem 1.1 holds over any field of characteristic \( p \neq 2 \) (see Remark 5.1 and Theorem 5.2), replacing the Chern character of the alternate sum of the Gauß-Manin bundles by the Chern character of the alternate sum of the cohomology of the relative differential forms with log poles.

Acknowledgement: It is a pleasure to thank Luc Illusie for a discussion on his results in log geometry and Valery Alexeev for encouraging us to dig out from [8] some useful geometric information.

2. A numerical computation

Let \( \mathcal{H} \) be a bundle of rank \( 2g \) over a smooth variety \( S \), which is an extension of a bundle \( E \) by its dual \( E^\vee \).

Lemma 2.1. If

\[
\text{ch}\left( \sum_{i=1}^{g} \wedge^{2i} \mathcal{H} \right) \in CH^0(S) \otimes \mathbb{Q}, \quad \text{and} \\
\text{ch}\left( \sum_{i=0}^{g-1} \wedge^{2i+1} \mathcal{H} \right) = 0 \in CH^\bullet(S) \otimes \mathbb{Q},
\]

then one has

\[
\text{ch}(\mathcal{H}) \in CH^0(S) \otimes \mathbb{Q}.
\]

Proof. Let \( K(S) \) denote the \( K \)-group of vector bundles on \( S \). Setting as usual (see [1], [3], for example)

\[
\lambda_t(\mathcal{H}) = \sum_{i=0}^{2g} \lambda^i(\mathcal{H})t^i \in K(S)[[t]] \quad \text{with} \\
\lambda^i(\mathcal{H}) = [\wedge^i \mathcal{H}] \in K(S),
\]

and denoting by \( e_i, i = 1, \ldots, g \), the Chern roots of \( E \), one has

\[
\lambda_t(\mathcal{H}) = \lambda_t(E) \cdot \lambda_t(E^\vee) = \prod_{i=1}^{g} (1 + e_it)(1 + e_i^\vee t).
\]

Thus

\[
\text{ch}(\lambda_t(\mathcal{H})) = \prod_{i=1}^{g} (1 + e_i^a t)(1 + e_i^{-a} t) \in CH^\bullet(S) \otimes \mathbb{Q}
\]

with \( a_i = c_1(e_i) \).
We set
\[
\text{ch}^{\text{even}}(\wedge H) := \frac{1}{2}(\text{ch} \lambda_1(H) + \text{ch} \lambda_{-1}(H))
\]
\[
= \frac{1}{2} \prod_{i=1}^{g}(1 + e^{a_i})(1 + e^{-a_i}) + \frac{1}{2} \prod_{i=1}^{g}(1 - e^{a_i})(1 - e^{-a_i}),
\]
\[
\text{ch}^{\text{odd}}(\wedge H) := \frac{1}{2}(\text{ch} \lambda_1(H) - \text{ch} \lambda_{-1}(H))
\]
\[
= \frac{1}{2} \prod_{i=1}^{g}(1 + e^{a_i})(1 + e^{-a_i}) - \frac{1}{2} \prod_{i=1}^{g}(1 - e^{a_i})(1 - e^{-a_i}).
\]
Thus the assumption is equivalent to
\[
\prod_{i=1}^{g}(1 + e^{a_i})(1 + e^{-a_i}) = 0 \in CH^{\geq 1}(S) \otimes \mathbb{Q}
\]
\[
\prod_{i=1}^{g}(1 - e^{a_i})(1 - e^{-a_i}) = 0 \in CH^\bullet(S) \otimes \mathbb{Q}.
\]
The first relation reads
\[
\prod_{i=1}^{g}(1 + e^{a_i})^2 e^{-a_i}(= 2^{2g}) \in CH^0(S) \otimes \mathbb{Q},
\]
or equivalently
\[
- \sum_{i=1}^{g} a_i + 2 \sum_{i=1}^{g} \log(1 + e^{a_i}) \in CH^0(S) \otimes \mathbb{Q}.
\]
Setting
\[
\psi(t) = \log(1 + e^t)
\]
one has
\[
\psi'(t) = \frac{e^t}{1 + e^t} = 1 - \varphi(t), \text{ with}
\]
\[
\varphi(t) = \frac{1}{1 + e^t} = \frac{1}{2} \sum_{n=0}^{\infty} E_n(0) \frac{t^n}{n!},
\]
where the $E_n(0)$ are the Euler numbers at 0 (see [1], 1.14, (2)).
Vanishing of $\text{ch}(H)$ in $CH^{\geq 1}(S) \otimes \mathbb{Q}$ is equivalent to vanishing of
\[
\text{ch}_{2g}(E) \in CH^{2\bullet}(S) \otimes \mathbb{Q}, \quad \bullet \geq 1.
\]
Thus it is equivalent to the assertion that none of the odd coefficients of the expansion of $\varphi(t)$ is vanishing, that is that $E_{2n-1}(0) \neq 0$ for all
$n \geq 1$. By \cite[1.14, (7)]{ref}, one has
\[
E_{2n-1}(0) = \frac{2(1 - 2^{2n})}{2n} B_{2n}(0),
\]
where $B_n(0)$ are the Bernoulli numbers at 0, and by \cite[1.13, (16)]{ref}, $B_{2n}(0) \neq 0$ for all $n \geq 1$. This concludes the proof. \hfill \qed

3. The geometry of the compactified family of abelian varieties

In this section, we extract from \cite{ref} the necessary geometric information in order to find a model for the compactification of a family of abelian varieties, which will allow us to apply in section 5 a log version of the Grothendieck-Riemann-Roch theorem.

We use the following notations. Fixing the level $n$, we denote by $S = \mathcal{A}_{g,n}$ the moduli stack of abelian varieties with level $n$ structure \footnote{\cite{ref}}. For $n \geq 3$, not divisible by the characteristic of $k$, $S$ is a scheme and it carries a universal family $f : X \to S$ of abelian varieties. We consider one of the compactifications $\tilde{f} : \overline{X} \to \overline{S}$ described on p. 195 of \cite{ref}.

More precisely, the compactification $S \subset \overline{S}$ is determined by a certain polyhedral cone decomposition $\{\sigma_\alpha\}$ of $C(N)$, where $N$ is a free abelian group of rank $g$, $B(N)$ is the space of integer valued symmetric bilinear forms and $C(N) \subset B(N_\mathbb{R})$ is the convex cone of all positive semi-definite symmetric bilinear forms whose radicals
\[
\text{Ker}(b : N_\mathbb{R} \to N_\mathbb{R}^\vee),
\]
are defined over $\mathbb{Q}$ (p. 96, 2.1). The interior $C^o(N)$ consists of the positive definite forms.

Then, $\overline{S}$ being chosen, one considers
\[
\tilde{B}(N) = B(N) \times N_\mathbb{R}^\vee.
\]
A compactification $\tilde{f} : \overline{X} \to \overline{S}$ is determined by the choice of a polyhedral cone decomposition $\{\tau_\beta\}$ of the cone
\[
\tilde{C}(N) = \{(b, \ell), \ell = 0 \text{ on Ker } b \} \subset C(N) \times N_\mathbb{R}^\vee
\]
\[
\subset \tilde{B}(N_\mathbb{R}) = B(N_\mathbb{R}) \times N_\mathbb{R}^\vee
\]
(p. 195, last section). For the existence of $\tilde{f}$ one needs (p. 196, 1.3 (v)) that any $\tau_\beta$ maps into a $\sigma_\alpha$. Recall moreover that one of the conditions on the polyhedral cone decompositions requires $\{\sigma_\alpha\}$ to be $GL(N)$-invariant (p. 96, 2.2), $\{\tau_\beta\}$ to be $GL(N) \rtimes N$-invariant (p. 196, 1.3), and that there are finitely many orbits.
As is underlined on p. 195, l.1, the family $f : X \to S$ does not necessarily extend to a semi-abelian group scheme $G \to S$ embedded in $\overline{f} : \overline{X} \to \overline{S}$. Yet, Remark 1.4 p. 197 asserts that it is possible to further refine the cone decompositions in such a way that the natural section of $\tilde{B}(N) \to B(N)$ respects the cone decomposition, which means that any $\sigma_\alpha \times \{0\}$ is precisely one of the $\tau_\beta$. This guarantees that $f : X \to S$ extends to a semi-abelian group scheme $G \to S$ embedded in $\overline{f} : \overline{X} \to \overline{S}$.

We set $T := \overline{S} \setminus S$, $Y := \overline{X} \setminus X$.

Refining, we may assume $\{\tau_\beta\}$ and $\{\sigma_\alpha\}$ to be smooth (p. 96, 2.3 and p.98, (iii)) and $\{\sigma_\alpha\}$ to satisfy the condition (ii) on p. 97. In particular, this says that both $\overline{S}$, $\overline{X}$ are smooth, that $T$, $Y$ are normal crossings divisors and that the components of $T$ are non-singular (p. 118, 5.8, a).

For $n \geq 3$, it is explained on p.172 and p. 173 how to refine a given polyhedral smooth cone decomposition to force $\overline{S}$ to be a projective and smooth scheme. We remark, that via p. 173, c), whatever projective polyhedral decomposition is chosen to define the compactification, it is always possible to refine it to a finer smooth projective one. Of course, this changes the compactifications of $S$ and $X$. For the family $X$, one first quotes Theorem 1.1 p. 195 which yields $\overline{f} : \overline{X} \to \overline{S}$ as a morphism of algebraic stacks. However, by p. 207, l.4 to 8, we know that $\overline{X}$ is a smooth projective variety and that $\overline{f}$ is then consequently a projective morphism.

We have essentially reached the first part of the following theorem.

**Theorem 3.1** (Faltings-Chai). *Let $k$ be a field containing the $n$-th roots of unity. For $n$ even $\geq 4$ and not divisible by the characteristic of $k$, there is a compactification $\overline{f} : \overline{X} \to \overline{S}$ of the universal family of principally polarized abelian varieties of genus $g$ with level $n$ structure, with the following properties:

1. $\overline{X}$ and $\overline{S}$ are smooth projective varieties.
2. $T$, $Y$ are normal crossings divisors with smooth irreducible components.
3. The sheaf of relative 1-forms $\Omega^1_{\overline{X}/\overline{S}}(\log Y)$ with logarithmic poles along $Y$ is locally free.
4. The Hodge bundle $E = \overline{f}_*\Omega^1_{\overline{X}/\overline{S}}(\log Y)$ is locally free.
5. One has $f^*E = \Omega^1_{X/S}(\log Y)$.
6. One has $R^q\overline{f}_*(\Omega^p_{\overline{X}/\overline{S}}(\log Y)) = \wedge^q E^\vee \otimes \wedge^p E$.\*
7. When the ground field $k$ has characteristic 0, the Gauß-Manin sheaf $R^qf_*\Omega^\bullet_{X/S}(\log Y) =: \mathcal{H}^q$ is Deligne’s extension \[4\] of its restriction to $S$. In particular, it is locally free. The residues of the Gauß-Manin connection are nilpotent.

8. $f : X \to S$ extends to a semi-abelian group scheme $G \to \overline{S}$ embedded into $\overline{f} : \overline{X} \to \overline{S}$.

9. The level $n$-structure sections $S_i$ of $f : X \to S$ extend to disjoint sections $\overline{S}_i$ of $\overline{f} : \overline{X} \to \overline{S}$.

10. The $\iota := (-1) : X \to X$ involution over $S$ extends to an involution, still denoted by $\iota : \overline{X} \to \overline{X}$ over $\overline{S}$.

11. The fixed points of $\iota$ lie in $\bigcup \overline{S}_i$.

Proof. 1., 8. have already been discussed, as well as part of 2. For 3., 4., 5., 6., we refer to Theorem 1.1, p. 195. For 7., we know (p. 218, (4)) that the Hodge to de Rham spectral sequence degenerates, which implies via 6. that $\overline{\mathcal{H}}$ is locally free. On the other hand, restricting to a generic curve $C$ in $\overline{S}$ intersecting $T$ in general points $C \setminus C$, one obtains a family $h : \overline{W} \to \overline{C}$, the fibres of which are all normal crossings divisors. As well known (see \[10\], p. 130, for example) $\overline{\mathcal{H}}$ is the Deligne extension of $\overline{\mathcal{H}}|_C$. On the other hand, 8. implies that the Gauß-Manin connection on $\overline{\mathcal{H}}|_C$ has nilpotent residues.

For 9., 10., 11. and to see that the components of $Y$ can be assumed to be smooth, we need a more precise discussion of the polyhedral cone decomposition and its relation with the compactification. Obviously the three properties hold true over $S$, hence extending them to the boundary is a local question. As on p. 207, 2., we can even replace $\overline{S}$ by the formal completion along a certain stratum $\overline{3}$ and $\overline{f} : \overline{X} \to \overline{S}$ by the pullback family. Doing so, we are allowed to use the description of $\overline{f}$ given p. 201 and p. 203.

Recall that the toroidal embeddings $\overline{F} \to \overline{E}$ given by the polyhedral cone decompositions are stratified by locally closed subschemes $\overline{3}_{\tau_3}$ and $\overline{3}_{\sigma_\alpha}$ (p. 100, 2.5, (iv)). The $\overline{3}_{\sigma_\alpha}$ and $\overline{3}_{\tau_3}$ are orbits under the torus action, and their codimension is equal to the dimension of the $\mathbb{R}$-vectorspace spanned by $\sigma_\alpha$ or $\tau_3$, respectively.

If the fibre $G_0$ of $G$ over the general point of the stratum $\overline{3}$ is a torus, then the formal completions along the stratum, together with the pullback of $\overline{f} : \overline{X} \to \overline{S}$, are obtained by restricting $\overline{F}$ to the formal completion of $\overline{E}$ along a stratum $\overline{3}_{\sigma_\alpha}$, with $\sigma_\alpha \subset C^\circ(N)$, and by taking the quotient by $N$ (p. 201, last section). In general $G_0$ will be an extension of an abelian variety $A$ of dimension $g - r$ by a torus. As sketched on p. 202 - 203, one has to replace $N$ by an $r$-dimensional
quotient lattice $N_\xi$ in this case. In particular one may again assume that $\sigma_\alpha$ lies in the interior of the cone $C(N_\xi)$. Since the combinatorial description remains the same, we drop the $\xi$.

By p. 100, 2.5 (i) and (ix), the category of rational partial polyhedral cone decompositions is equivalent to the category of torus embeddings. Thus composing a given torus embedding with $\iota = (-1)$ corresponds to the action of $\iota$ on the data giving the polyhedral cone decomposition. So we just have to verify that $\iota$ respect those data. $\iota$ acts on $N$ by multiplication with $(-1)$, hence the action is trivial on $B(N)$, and is $(-1)$ on $N^\vee$. An element $\mu \in N$ acts on $(b, \ell) \in \tilde{B}(N) = B(N) \times N$ by

$$(b, \ell) \mapsto (b, \ell + b(\mu, \ell)),$$

while $\gamma \in GL(N)$ acts via

$$(b, \ell) \mapsto (b \circ (\gamma^{-1}, \gamma^{-1}), \ell \circ \gamma^{-1})$$

(p. 196, first section). In particular,

$$(-\text{Id}, 0) \in GL(N) \ltimes N$$

maps $$(b, \ell) \in B(N) \times N^\vee$$

to $(b, -\ell)$. Since the polyhedral cone decompositions are invariant under $GL(N) \ltimes N$, it is invariant under $\iota$.

As for 9., we just remark that a morphism from $\overline{S}$ to the corresponding compactification $\overline{S}_1$ of the moduli stack $\mathcal{A}_{g,1}$ is defined by multiplication with $n$ on the torus (p. 130, 6.7. (6)). In different terms, one keeps the cone decompositions in $B(N_\mathbb{R})$ and $\tilde{B}(N_\mathbb{R}) \times N_\mathbb{R}$, but changes the integral structure by multiplication with $n$ on $N$. To see that the closure of the sections of $f : X \to S$ of order $n$ in $\overline{X}$ are disjoint sections of $\overline{f}$, it is again sufficient to consider the pullback of $\overline{f}$ to formal completions of the strata in $\overline{S}$. By p. 202, first section, the $n$-torsion points are the pull-back of the zero-section of the semi-abelian group scheme over the formal completion.

As already seen, $\iota$ acts trivially on the cone $\sigma_\alpha \subset B(N_\mathbb{R})$, and the fixed points under the $\iota$ involution lie in strata $\mathfrak{F}_{\tau_\beta}$ of $\overline{F}$, the $N$-orbit of which are invariant under $\iota$. We assumed that $\{\tau_\beta\}$ is smooth, that is each cone $\tau_\beta$ is generated by a partial $\mathbb{Z}$-basis

$$((b_1, \ell_1), \ldots, (b_r, \ell_r)) \text{ of } B(N) \times N^\vee.$$ 

Thus one has a $\mu \in N$ with

$$(b_i, -\ell_i) = (b_{j(i)}, \ell_{j(i)} + b_{j(i)}(\mu, \ell_{j(i)}))$$
for $i = 1, \ldots, r$. We have taken an even level $n$. Thus $N = 2 \cdot N'$ for another integral structure, and one has $\mu = 2 \cdot \mu'$ for a $\mu' \in N'$. We obtain

$$b_i = b_{j(i)} \quad \text{and} \quad \ell_i + \ell_{j(i)} = 2b_{j(i)}(\mu', \mu')$$

Twisting the free $\mathbb{Z}$-module $B(N) \times N^\vee$ by $\mathbb{Z}/2$ over $\mathbb{Z}$, the basis elements $(b_i, \ell_i)$ and $(b_{j(i)}, \ell_{j(i)})$ become equal, which implies that $j(i) = i$. This in turn implies that $2\ell_i = 2b_i(\mu', \mu')$, and shows that the dimension of the subspace of $B(N) \times N^\vee$ generated by $\tau_\beta$ is the same as the dimension of its image in $B(N)$. This finally implies that $\text{codim}(3_{\tau_\beta})$ is equal to the codimension of its image in $\overline{E}$. So the fixed points of $\iota$ all lie in the smooth locus of $\overline{f}$.

We verified all the conditions stated in [1], except that $Y$ can still have singular components. However, since 1., 3. - 11. are compatible with the blowing up of non-singular strata of the singular locus of $Y_{\text{red}}$, this last point can be achieved.

4. A Log Version of the Grothendieck-Riemann-Roch Theorem

In this section, we show that the Grothendieck-Riemann-Roch theorem extends to a log version.

Theorem 4.1. Let $\overline{f} : \overline{X} \to \overline{S}$ be a projective morphism of relative dimension $g$ over a field, with $X$, $S$ smooth, compactifying the smooth projective morphism $f : X \to S$, with the following properties:

1. Both $T = \overline{S}\setminus S$ and $Y := (\overline{f}^*(T))_{\text{red}}$ are normal crossings divisors with smooth irreducible components.
2. The sheaves $\Omega^i_{X/S}(\log Y)$ are locally free.
3. There are cycles $W \in CH^g(X) \otimes \mathbb{Q}$, $\xi \in CH^g(S) \otimes \mathbb{Q}$, such that

$$c_g(\Omega^i_{X/S}(\log Y)) = \overline{f}^*(\xi) + W \in CH^g(X) \otimes \mathbb{Q},$$

with the property $W \cdot Y_i = 0 \in CH^{g+1}(X) \otimes \mathbb{Q}$ for all irreducible components of $Z := \overline{f}^*(T) - Y$.

Then one has

$$\sum_i (-1)^i \text{ch}(\sum_j R^j f_* \Omega^{i-j}_{X/S}(\log Y)) \in CH^0(S) \otimes \mathbb{Q}.$$
Proof. We apply the Grothendieck-Riemann-Roch theorem \[3\] to the alternate sum of the sheaves $\Omega^i_{X/S}(\log Y)$. It yields
\[
\sum_i (-1)^i \text{ch}(\sum_j R^j f_* \Omega^{i-j}_{X/S}(\log Y)) = f_* \text{Todd}(T_{X/S}) \cdot (\sum_i (-1)^i \text{ch}(\Omega^i_{X/S}(\log Y))).
\]

We consider the residue sequences
\[
0 \to \mathcal{T}^i \Omega^i_S \to \mathcal{T}^i \Omega^i_S(\log T) \to \mathcal{O}_{\mathcal{T}} \to 0 \quad \text{and} \quad 0 \to \mathcal{O}^1_S \to \mathcal{O}^1_S(\log Y) \to \mathcal{O}_Y \to 0,
\]
where, in order to simplify notations, we have set $\mathcal{T}_j = f^* T_j$ and $\mathcal{O}_T = \bigoplus_j \mathcal{O}_{T_j}$,

for the irreducible components $T_j$ of $T$, and similarly $\mathcal{O}_Y = \bigoplus_i \mathcal{O}_{Y_i}$. Multiplicativity of the Todd class implies
\[
\text{Todd}(T_X) \cdot \text{Todd}(f^* T_S)^{-1} = \text{Todd}(T_{X/S}(\log Y)) \cdot \prod_{q \geq 1} \text{Todd}(\text{Ext}^q(\mathcal{O}_Y, \mathcal{O}_X))^{(-1)^{q+1}} \cdot \prod_{q \geq 1} \text{Todd}(\text{Ext}^q(\mathcal{O}_T, \mathcal{O}_X))^{(-1)^q}.
\]

We define $B$ and $C$ via the exact sequences
\[
0 \to B \to \mathcal{O}_Y \to \mathcal{O}_Y / \mathcal{O}_{\mathcal{T}} \to 0 \quad \text{and} \quad 0 \to C \to \mathcal{O}_T \to B \to 0.
\]

Using again multiplicativity, one obtains
\[
\text{Todd}(T_X) \cdot \text{Todd}(f^* T_S)^{-1} = \text{Todd}(T_{X/S}(\log Y)) \cdot \prod_{q \geq 1} \text{Todd}(\text{Ext}^q(\mathcal{O}_Y / \mathcal{O}_T, \mathcal{O}_X))^{(-1)^{q+1}} \cdot \prod_{q \geq 1} \text{Todd}(\text{Ext}^q(C, \mathcal{O}_X))^{(-1)^q}.
\]

On the other hand, one has the well known relation \[11\]
\[
\text{Todd}(T_{X/S}(\log Y)) \cdot (\sum_i (-1)^i \text{ch}(\Omega^i_{X/S}(\log Y))) = (-1)^g c_g(\Omega^1_{X/S}(\log Y)).
\]

Let us write
\[
\prod_{q \geq 1} \text{Todd}(\text{Ext}^q(\mathcal{O}_Y / \mathcal{O}_T, \mathcal{O}_X))^{(-1)^{q+1}} = 1 + \text{err}
\]

with $\text{err} \in CH^*(X)$. 

The sheaf $\mathcal{O}_Y/\mathcal{O}_T$ is a direct sum of structure sheaves $\mathcal{O}_{Y_1\cap\ldots\cap Y_\ell}$. The surjection $\Omega^1_{X/S}(\log Y) \to \mathcal{O}_{Y_1\cap\ldots\cap Y_\ell}$ implies that
\[ c_g(\Omega^1_{X/S}(\log Y)) \cdot (Y_1 \cap \ldots \cap Y_\ell) = 0 \in CH^{g+\ell}(X). \]
Since $err$ is a sum of terms supported in those strata $Y_1 \cap \ldots \cap Y_\ell$, we conclude
\[ c_g(\Omega^1_{X/S}(\log Y)) \cdot \prod_{q \geq 1} \text{Todd}(\mathcal{E}xt^q(\mathcal{O}_Y/\mathcal{O}_T, \mathcal{O}_X))^{(-1)^q+1} = c_g(\Omega^1_{X/S}(\log Y)). \]
Thus (4.1)
\[ \overline{f}_* \text{Todd}(T_{X/S}) \cdot (\sum_i (-1)^i \text{ch}(\Omega^1_{X/S}(\log Y))) = (-1)^g \overline{f}_*(c_g(\Omega^1_{X/S}(\log Y)) \cdot \prod_{q \geq 1} \text{Todd}(\mathcal{E}xt^q(\mathcal{C}, \mathcal{O}_X))^{(-1)^q}). \]
If $Z = \emptyset$, the sheaf $\mathcal{C}$ is zero, hence $\text{Todd}(\mathcal{E}xt^q(\mathcal{C}, \mathcal{O}_X)) = 1$. This finishes the proof of Remark 4.2.

For the general case we have to study the residue map and the sheaf $\mathcal{O}_Y/\mathcal{O}_T$ more precisely. Let us first fix notations. We write
\[ Y = \sum_{i \in I} Y_i, \quad T = \sum_{j \in J} T_j, \]
where the $Y_i, T_j$ are prime divisors. For $I \subset \mathcal{I}, J \subset \mathcal{J}$, we define
\[ Y_I = \bigcap_{i \in I} Y_i, \quad T_J = \bigcap_{j \in J} T_j. \]
Since $\Omega^1_{X/S}(\log Y)$ is locally free, $\overline{f}$ sends strata to strata. Indeed, choose $J$ maximal with the property that $T_J \supset \overline{f}(Y_I)$. If $Y_I \to T_J$ is not surjective, we find a local parameter $t$ on $\overline{S}$, in a general point of $\overline{f}(Y_I)$, such that $\overline{f}(Y_I)$ lies in the zero locus of $t$ but not in $T_J$. As $\mathcal{O}_S \cdot dt$ is locally split in $\Omega^1_{S}(\log T)$, and the injection
\[ \overline{f}_* \Omega^1_{S}(\log T) \subset \Omega^1_{X}(\log Y) \]
is locally split, $\mathcal{O}_X \cdot dt$ is locally split in $\Omega^1_{X}(\log Y)$ as well. But since by assumption,
\[ t = \sum_{i \in I} y_i \alpha_i, \quad \alpha_i \in \mathcal{O}_X, \]
one finds
\[ dt = \sum_{i \in I} y_i \cdot (d\alpha_i + \frac{dy_i}{y_i} \alpha_i), \]
a contradiction.

This allows to define, for each \( I \subset \mathcal{I} \), the index set \( J(I) \) with the property \( T_J = T(I) \). It yields a \( \mathbb{Q} \)-linear map

\[
\varphi_I : \bigoplus_{j \in J} \mathbb{Q} \cdot T_j \to \bigoplus_{i \in I} \mathbb{Q} \cdot Y_i
\]

\[
\varphi_I(T_j) = \sum_{i \in I} \nu_i^j Y_i \quad \text{where} \quad \overline{f}^* T_j = \sum_{i \in I} \nu_i^j Y_i.
\]

One defines

\[
\delta_Y = \text{codim}(\text{Im}(\varphi_I)) = |I| - \dim(\text{Im}(\varphi_I)).
\]

**Claim 4.3.** One has \( \delta_I > 0 \) if and only if the residue map

\[
\Omega_X^1(\log Y) \to \mathcal{O}_{Y_I}
\]

factors through \( \Omega_{X/S}^1(\log Y) \to \mathcal{O}_{Y_I} \).

**Proof.** We fix \( I, J = J(I) \) as before. The question being local at generic points of \( T_J \) and \( Y_I \), we choose local parameters \( \{t_j\} \) and \( \{y_i\} \) defining \( T_J \) and \( Y_I \) locally. Then the equation of the morphism \( f \) is simply

\[
t_j = \prod_{i \in I} y_i^i \nu_i^j.
\]

As in the beginning of the proof of Theorem 4.1, functoriality maps

\[
\overline{f}^* \Omega_{S}^1(\log T) \xrightarrow{\text{res}} \overline{f}^* (\bigoplus_{j \in J} \mathcal{O}_{T_j}) \to \Omega_{X}^1(\log Y) \xrightarrow{\text{res}} \bigoplus_{i \in I} \mathcal{O}_{Y_i}.
\]

It induces

\[
\bigoplus_{j \in J} \overline{f}^*(\mathcal{O}_{T_j})_{|Y_j} = \bigoplus_{j \in J} \mathcal{O}_{Y_j} \to \bigoplus_{i \in I} \mathcal{O}_{Y_i}
\]

\[
\text{res}(\frac{dt_j}{t_j}) \mapsto \sum_{i \in I} \nu_i^j \text{res}(\frac{dy_i}{y_i}).
\]

\( \square \)

**Claim 4.4.** Let \( i_1, \ldots, i_\mu \in \mathcal{I} \), not necessarily pairwise distinct. Assume that \( Y_I \neq \emptyset \), and \( \delta_I > 0 \), where \( I \subset \mathcal{I} \) is the smallest subset of \( \mathcal{I} \) containing \( \{i_1, \ldots, i_\mu\} \). Then the cycle \( Y_{i_1} \cdots Y_{i_\mu} \in CH^\mu(\mathcal{X}) \) fulfills

\[
c_\alpha(\Omega_{X/S}^1(\log Y)) \cdot Y_{i_1} \cdots Y_{i_\mu} = 0 \in CH^{\mu+\alpha}(\mathcal{X}).
\]

**Proof.** Without loss of generality, we may assume that \( I = \{i_1, \ldots, i_\eta\} \) are pairwise distinct. By Lemma 4.3, one has \( c_\alpha(\Omega_{X/S}^1(\log Y)) \cdot Y_I = 0 \), thus a fortiori \( c_\alpha(\Omega_{X/S}^1(\log Y)) \cdot Y_I \cdot Y_{i_{\eta+1}} \cdots Y_{i_\mu} = 0 \).

\( \square \)
Recall that for the proof of Theorem 4.1, it remains to show that the term
\[ (-1)^g \overline{T}_* (c_g(\Omega_{X/S}^1(\log Y))) \cdot \prod_{q \geq 1} \text{Todd}(\mathcal{E}xt^q(C, \mathcal{O}_X))^{(-1)^q} \]
in formula (4.1) is equal to \((-1)^g \overline{T}_* (c_g(\Omega_{X/S}^1(\log Y)))\). Writing
\[ \tilde{T}_j = \overline{f}^* (T_j) \quad \text{and} \quad Z_j = \tilde{T}_j - (\tilde{T}_j)_{\text{red}} \]
one has a commutative diagram
\[
\begin{array}{cccccc}
0 & \longrightarrow & \bigoplus_j \mathcal{O}_{Z_j}(-\tilde{T}_j)_{\text{red}} & \longrightarrow & \bigoplus_j \mathcal{O}_{\tilde{T}_j} & \longrightarrow & \bigoplus_j \mathcal{O}(\tilde{T}_j)_{\text{red}} & \longrightarrow & 0 \\
\rho_1 & & \downarrow & & \downarrow & & \downarrow \rho_2 & & \\
0 & \longrightarrow & \mathcal{C} & \longrightarrow & \bigoplus_j \mathcal{O}_{\tilde{T}_j} & \longrightarrow & \bigoplus_i \mathcal{O}_{Y_i} & & \\
\end{array}
\]
with exact rows. Since \(\rho_1\) is injective \(\text{Ker} \rho_2 = \text{Coker} \rho_1 =: \mathcal{K}\). Thus
\[
\prod_{q \geq 1} \text{Todd}(\mathcal{E}xt^q(C, \mathcal{O}_X))^{(-1)^q} = 
\prod_{q \geq 1} \text{Todd}(\mathcal{E}xt^q((\bigoplus_j \mathcal{O}_{Z_j}(-\tilde{T}_j)_{\text{red}}, \mathcal{O}_X))^{(-1)^q} \cdot \prod_{q \geq 1} \text{Todd}(\mathcal{E}xt^q(\mathcal{K}, \mathcal{O}_X))^{(-1)^q}.
\]
One has
\[
\mathcal{E}xt^1(\mathcal{O}_{Z_j}(-\tilde{T}_j)_{\text{red}}, \mathcal{O}_X) = \mathcal{O}_X(\tilde{T}_j)/\mathcal{O}_X((\tilde{T}_j)_{\text{red}}),
\]
while \(\mathcal{E}xt^q(\mathcal{O}_{Z_j}(-\tilde{T}_j)_{\text{red}}, \mathcal{O}_X) = 0, \ q \geq 2\).
Thus
\[
\prod_{q \geq 1} \text{Todd}(\mathcal{E}xt^q(\mathcal{O}_{Z_j}(-\tilde{T}_j)_{\text{red}}, \mathcal{O}_X))^{(-1)^q} = 1 + v_j,
\]
where \(v_j \in CH^{2,1}(X)\) can be written as a sum of terms
\[
m_{i_1, \ldots, i_\mu} Y_{i_1} \cdot \ldots \cdot Y_{i_\mu},
\]
where at least one of the \(Y_{i_p}\) lies in \(Z_j\), and \(m_{i_1, \ldots, i_\mu} \in \mathbb{Z}\). On the other hand, one has the exact sequence
\[
0 \rightarrow \mathcal{K} \rightarrow \bigoplus_j \mathcal{O}(\tilde{T}_j)_{\text{red}} \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_Y/\mathcal{O}_{\tilde{T}} \rightarrow 0.
\]
Let us write
\[
Y = \Phi + V \quad \text{and} \quad (\tilde{T}_j)_{\text{red}} = \Phi_j + V_j,
\]
where each component of \(\Phi_j\) maps surjectively onto \(T_j\), where each component of \(V\) and \(V_j\) maps to a lower dimensional strata in \(T\), and
with $\Phi = (\sum_j \Phi_j)_{\text{red}}$. Of course $V_j \subset V$ and $V \subset Z$. Denoting as usual by $\tilde{\cdot}$ the normalization, one has
\[
\prod_{q \geq 1} \text{Todd}(\text{Ext}^q(\mathcal{C}, \mathcal{O}_\mathcal{X}))^{(-1)^q} = \\
\prod_{q \geq 1} [\text{Todd}(\text{Ext}^q(\oplus_j (\mathcal{O}_{\tilde{\Phi}_j} \oplus \mathcal{O}_{V_j}), \mathcal{O}_\mathcal{X}))^{(-1)^q} \\
\cdot \text{Todd}(\text{Ext}^q((\mathcal{O}_{\tilde{\Phi}_j} \oplus \mathcal{O}_{V_j})/\mathcal{O}_\mathcal{X}))^{(-1)^{q+1}} \\
\cdot \text{Todd}(\text{Ext}^q(\mathcal{O}_\mathcal{Y}, \mathcal{O}_\mathcal{X}))^{(-1)^{q+1}} \cdot \text{Todd}(\text{Ext}^q(\mathcal{O}_\mathcal{V}/\mathcal{O}_\mathcal{X}))^{(-1)^{q+1}}].
\]

Next, for $i \in \mathcal{I}$ let us define
\[
N(i) = |\{ j \in \mathcal{J}; Y_i \subset (\tilde{T}_j)_{\text{red}}\}| - 1.
\]
Remark that $N(i)$ is zero except when $Y_i \subset V$. One has
\[
\text{Todd}(\text{Ext}^q(\oplus_j (\mathcal{O}_{\tilde{\Phi}_j} \oplus \mathcal{O}_{V_j}), \mathcal{O}_\mathcal{X}))^{(-1)^q} = \\
\text{Todd}(\text{Ext}^q(\oplus_{i \in \mathcal{I}} \mathcal{O}_{Y_i}^{\oplus N(i)}, \mathcal{O}_\mathcal{X}))^{(-1)^q}.
\]
Moreover, one has
\[
\text{Todd}(\text{Ext}^q((\mathcal{O}_{\tilde{\Phi}_j} \oplus \mathcal{O}_{V_j})/\mathcal{O}_\mathcal{X}))^{(-1)^{q+1}} = 1 + w_j,
\]
where $w_j \in CH^{\geq 1}(\mathcal{X})$ is the sum of terms $m_{i_1, \ldots, i_\mu} Y_{i_1} \cdot \ldots \cdot Y_{i_\mu}$, where at least two of the $i_\mu$ are different.

Assume that $Y_{i_1}, \ldots, Y_{i_\mu}$ are all components of $\Phi_j$. The image of $Y_{i_1,\ldots,i_\mu}$ is one of the strata of $T$. Since $Y$ is a normal crossing divisor, $\overline{f}(Y_{i_1,\ldots,i_\mu}) = T_j$. So $\delta_{i_1,\ldots,i_\mu} > 0$ and by Claim 4.4 the intersection of $c_g(\Omega^1_{\mathcal{X}/\mathcal{S}}(\log Y))$ with the cycle $Y_{i_1} \cdot \ldots \cdot Y_{i_\mu}$ is zero.

Altogether, one has
\[
c_g(\Omega^1_{\mathcal{X}/\mathcal{S}}(\log Y)) \cdot \prod_{q \geq 1} \text{Todd}(\text{Ext}^q(\mathcal{C}, \mathcal{O}_\mathcal{X}))^{(-1)^q} = \\
c_g(\Omega^1_{\mathcal{X}/\mathcal{S}}(\log Y)) \cdot (1 + w),
\]
where the cycle $w \in CH^{\geq 1}(\mathcal{X})$ lies in the subspace generated by products $Y_{i_1} \cdot \ldots \cdot Y_{i_\mu}$ with at least one $Y_{i_\mu}$ contained in $Z$.

We conclude the proof of Theorem 4.1 applying Proposition 4.3. \qed

**Proposition 4.5.** Let $\overline{f} : \overline{X} \to \overline{S}$ be as in Theorem 4.1, $\ell \geq 1$ irreducible components $Y_{i_1}, \ldots, Y_{i_\ell}$ of $Y$, with $Y_{i_\ell} \subset |Z|$, fulfill
\[
\overline{f}_*(c_g(\Omega^1_{\mathcal{X}/\mathcal{S}}(\log Y)) \cdot Y_{i_1} \cdot \ldots \cdot Y_{i_\ell}) = 0 \in CH^{\ell}(\mathcal{S}) \otimes \mathbb{Q}.
\]
By Claim 4.4 we only have to study strata for which $\delta = 0$.
Lemma 4.6. If \( \delta_I = 0 \) and \( i \in I \) is given, then there is a divisor \( \Gamma \) supported in \( T \subset S \), there are indices \( \ell_1, \ldots, \ell_m \in I \setminus I \) and multiplicities \( \beta_1, \ldots, \beta_m \in \mathbb{Q} \) fulfilling

\[
Y_i \cdot Y_I = \overline{f}(\Gamma) \cdot Y_I + \sum_{\nu=1}^{m} \beta_{\nu} Y_{I \cup \{\ell_{\nu}\}} \in CH^{1+|I|}(X) \otimes \mathbb{Q}.
\]

Proof. By formula (4.2), we know that \( Y_i \) lies in the image of \( \varphi_I \). Thus there is a \( \mathbb{Q} \)-divisor \( \Gamma \) supported on \( T \) such that \( Y_i - f^*(\Gamma) \) is supported away of \( Y_a, a \in I \).

Lemma 4.7. For \( \ell \geq 1 \) let \( \mathfrak{P}_\ell \) be the set of all sets of indices \( I \) with \( |I| = \mu \leq \ell \), with \( Y_1 \neq \emptyset \), \( \delta_I = 0 \), and such that there is one \( i \in I \) with \( Y_i \subset |Z| \). For irreducible components \( Y_{i_1}, \ldots, Y_{i_\ell} \) of \( Y \), with \( Y_{i_\ell} \subset |Z| \), there exist multiplicities \( a_I \in \mathbb{Q} \) for \( I \in \mathfrak{P}_\ell \), cycles

\[
\Gamma_I \in CH^{\ell-|I|}(S) \otimes \mathbb{Q}, \quad \text{with} \quad \Gamma_I|_{\overline{S} \setminus T} = 0 \in CH^{\ell-|I|}(S \setminus T) \otimes \mathbb{Q},
\]

such that

\[
(4.3) \quad c_g(\Omega^1_{X/S}(\log Y)) \cdot Y_{i_1} \cdot \cdots \cdot Y_{i_\ell} = c_g(\Omega^1_{X/S}(\log Y)) \cdot \sum_{I \in \mathfrak{P}_\ell} a_I \overline{f}(\Gamma_I) \cdot Y_I.
\]

Proof. Obviously \( \delta_{\{i\}} = 0 \), thus \( \mathfrak{P}_1 = \{ i \in I, Y_i \subset |Z| \} \). So for \( \ell = 1 \) there is nothing to show.

We assume now by induction on \( \ell \) that formula (4.3) holds true for \( \ell \). We want to prove it for \( \ell + 1 \). It remains to show that for \( i \in I \) and \( I \in \mathfrak{P}_\ell \),

\[
(4.4) \quad c_g(\Omega^1_{X/S}(\log Y)) \cdot Y_i \cdot Y_I
\]

has the shape required on the right hand side of formula (4.3).

If \( i \notin I \), then either \( \delta_{\{i\} \cup I} = 0 \), which implies \( \{i\} \cup I \in \mathfrak{P}_{\ell+1} \), or else the expression (4.4) is zero by Claim 4.4.

Thus we assume that \( i \in I \). By Lemma 4.6, one has

\[
Y_i \cdot Y_I = \overline{f}(\Gamma) \cdot Y_I + \sum_{\nu=1}^{m} \beta_{\nu} Y_{I \cup \{\ell_{\nu}\}},
\]

with \( \ell_{\nu} \notin I \). Thus again, either \( \{\ell_{\nu}\} \cup I \in \mathfrak{P}_{\ell+1} \), in which case we are done, or else

\[
c_g(\Omega^1_{X/S}(\log Y)) \cdot Y_{i_{\ell_{\nu}}} = 0.
\]

\qed
Proof of Proposition 4.5. The definition of $\mathfrak{P}_\ell$ implies that the restriction to $X \setminus |Z|$ of the cycles $Y_I \in CH^{|I|}(X) \otimes \mathbb{Q}$ die. In particular one has

$$W \cdot Y_I = 0 \in CH^{g+|I|}(X) \otimes \mathbb{Q}.$$ 

Thus formula (4.3) implies

$$c_g(\Omega^1_{X/S}(\log Y)) \cdot Y^\ell = \tilde{f}^*(\xi) \cdot \sum_{I \in \mathfrak{P}_\ell} a_I \tilde{f}^*(\Gamma_I) \cdot Y_I \in CH^{g+\ell}(X) \otimes \mathbb{Q}.$$ 

On the other hand, $\delta_I = 0$ implies that $|I| \leq |J(I)|$. This in turn implies that the fiber dimension of $Y_I \to T_{J(I)}$ is at least $g$, thus by projection formula, $\tilde{f}_*^*$ of the right hand side of formula (4.3) vanishes as well. This concludes the proof of the proposition. 

Theorem 4.1 implies

**Corollary 4.8.** The assumptions are as in Theorem 4.1, and moreover the following conditions are fulfilled:

1. The characteristic of the ground field $k$ is 0.
2. The Hodge to de Rham spectral sequence

$$E_2^{pq} = R^qf_*\Omega^p_{X/S}(\log Y) \Rightarrow \overline{H}^i := R^i\tilde{f}_*\Omega^p_{X/S}(\log Y)$$

degenerates in $E_2$ and the Gauß-Manin sheaves $\overline{H}^i$ are locally free. Then

$$\sum_i (-1)^i ch(\overline{H}^i) \in CH^0(S) \otimes \mathbb{Q}.$$ 

5. **Vanishing of the Chern character of the Gauß-Manin bundles of weight one.**

In this section, we prove Theorem 1.1, which is the main result of this article.

**Remark 5.1.** Any variation of polarized pure Hodge structures of weight 1 is the Gauß-Manin bundle of a smooth polarized family $f : X \to S$ of abelian varieties ([5]), with $B = S$. Thus Theorem 1.1 may be reformulated over any field of characteristic 0 by saying that

$$ch(\overline{H}^1) \in CH^0(B) \otimes \mathbb{Q},$$

where $\mathcal{H}^1 = R^1f_*\Omega^1_{X/S}$ is the first Gauß-Manin bundle, and $\overline{H}^1$ its Deligne extension with nilpotent residues.
Proof of Theorem 1.1. Let $B \subset \overline{B}$ be as in the theorem. There exists a generically finite morphism $\pi : B_1 \to \overline{B}$ together with a morphism $\psi : B_1 \to \overline{S}$, where $\overline{S}$ is as in Theorem 3.1, with

$$\pi^*H^1 = \psi^*R^1f_*\mathcal{O}_{\overline{X}/\overline{S}}^*(\log Y).$$

Indeed, the polarization of $H^1$ is not necessarily coming from a principal polarization on the underlying family of abelian varieties, and also, this family might not have a level $n$-structure, but both can be achieved after replacing $B$ by a generically finite covering.

Since $\overline{S}$ and $\overline{B}$ are smooth, projection formula for $\psi$ and $\pi$ implies that

$$CH^0(\overline{S})^\bullet \otimes \mathbb{Q} \; \text{and} \; CH^0(\overline{B})^\bullet \otimes \mathbb{Q}$$

are subrings of $CH^\bullet(B_1) \otimes \mathbb{Q}$, thus we may assume that $B = \overline{S}$ and

$$\overline{H}^1 = R^1f_*\mathcal{O}_{\overline{X}/\overline{S}}^*(\log Y).$$

By Corollary 4.8, we know

$$\sum_i (-1)^i \operatorname{ch}(\overline{H}^i) \in CH^0(\overline{S}) \otimes \mathbb{Q}.$$ 

On the other hand, consider as in Theorem 3.1, 11., the involution $\iota$ on $\overline{X}$. By loc. cit. 9., 11., the fixed points of $\iota$ lie in disjoint sections $S_\alpha \subset \overline{X}$ of $\overline{f}$. In particular, setting again $Y = (f(T))_{\text{red}}$ and $Y + Z = \overline{f}(T)$, the sections do not hit the divisor $Z$. We consider the blow up $a : \overline{X} \to \overline{X}$ of the sections $S_\alpha$. Hence $\overline{X}$ is non-singular, the divisor $Y' = a^{-1}Y$ is again a normal crossings divisor, and $(\overline{f} \circ a)^*T$ is reduced in a neighborhood of the exceptional divisors

$$E_\alpha = a^{-1}S_\alpha.$$ 

$\iota$ acts on the relative Zariski tangent space of a section $S_\alpha$ by multiplication with $-1$, hence it induces an action on $\overline{X}$, again denoted by $\iota$. The restriction of $\iota$ to $E_\alpha$ is trivial, and $\iota$ acts fixed point free on $\overline{X} \setminus \cup E_\alpha$. One has

$$R^i(\overline{f} \circ a)_*(\mathcal{O}_{\overline{X}/\overline{S}}^*(\log a^{-1}Y)) = H^i \oplus T_i,$$

where $T_i$ is an algebraically trivial bundle on which $\iota$ acts trivially, and $T_i = 0$ for $i$ odd.

The quotient, $\mathcal{H} = \overline{X}/(\iota)$ is non-singular and $h : \mathcal{H} \to \overline{S}$ is a proper family, smooth over $S \subset \overline{S}$. Here $\mathcal{H}$ stands for Kummer.

The ramification locus $\cup E_\alpha$ of $\overline{X} \to \mathcal{H}$ is contained in the smooth locus of $\overline{X} \to \overline{S}$, hence $h^{-1}(T)$ is a normal crossings divisor, reduced
in a neighborhood of the image of $\cup E_\alpha$. So for $\kappa = (h^{-1}(T))_{\text{red}}$ the sheaf $\Omega^1_{\mathfrak{R}/S}(\log \kappa)$ remains locally free and

$$c_g(\Omega^1_{\mathfrak{R}/S}(\log \kappa)) = c_g(h^*f_*\Omega^1_{\mathfrak{X}/S}(\log Y))$$

in a neighborhood of the multiple locus of $h^{-1}(T)$. Altogether, $h : \mathfrak{R} \to \mathfrak{S}$ satisfies the assumption 1. - 3. in Theorem 4.1, except possibly that the components of $\kappa$ might be singular. As at the very end of the proof of Theorem 3.1, we blow up non-singular strata of $\kappa_{\text{red}}$ to have 1. as well.

The Gauß-Manin bundles $\mathcal{H}^i_{\mathfrak{R}} := R^i h_*\Omega^\bullet_{\mathfrak{R}\mathfrak{X}/\mathfrak{S}}(\log \kappa)$ vanish for $i = 2p + 1$, $p > 0$, and fulfill

$$\mathcal{H}^{2p}_{\mathfrak{R}} = (R^{2p}(f \circ a)_*\Omega^\bullet_{\mathfrak{X}/\mathfrak{S}}(\log a^{-1}Y))^\iota,$$

where $^\iota$ means the invariants under $\iota$. Since $\iota$ acts trivially on $\mathcal{H}^{2p}$, one obtains

$$\mathcal{H}^{2p}_{\mathfrak{R}} = \mathcal{H}^{2p} \oplus \mathcal{T}_{2p}.$$  

Corollary 4.8 implies

$$\text{ch}(\sum_{p \geq 0} \mathcal{H}^{2p}_{\mathfrak{R}}) = (\text{ch}(\sum_{p \geq 0} \mathcal{H}^{2p})) \in CH^0(\mathfrak{S}) \otimes \mathbb{Q}.$$  

Lemma 2.1 implies then

$$\text{ch}(\mathcal{H}^1) \in CH^0(\mathfrak{S}) \otimes \mathbb{Q}.$$  

This concludes the proof.

**Theorem 5.2.** Let $k$ be a field of characteristic $p \neq 2$. Let $\overline{f} : \overline{X} \to \overline{S}$ be a compactified family of abelian varieties as in Theorem 3.1. Then one has

$$\text{ch}(E^\vee \oplus E) = \text{ch}(R^1\overline{f}_*\mathcal{O}_{\overline{X}} \oplus \overline{f}_*\Omega^1_{\overline{X}/\overline{S}}(\log Y)) \in CH^0(\mathfrak{S}) \otimes \mathbb{Q},$$

or equivalently

$$\text{ch}_{2\ell}(E) = \text{ch}_{2\ell}(E^\vee) = 0 \in CH^{2\ell}(\mathfrak{S}) \otimes \mathbb{Q}, \text{ for } \ell \geq 1.$$  

**Proof.** We replace in the proof of Theorem 4.1 the Gauß-Manin bundle $\mathcal{H}^i$ by the sum $\sum_j R^j\overline{f}_*\Omega^{i-j}_{\overline{X}/\overline{S}}(\log Y)$ of the Hodge bundles (see Theorem 3.1, 6.).
REFERENCES

[1] Bateman, H. (compiled by the Bateman Manuscript Project): Higher transcendental functions, vol. I, McGraw-Hill Book Company (1953).

[2] Bloch, S., Esnault, H.: Algebraic Chern-Simons theory. Amer. J. Math. 119 (1997) 903–952.

[3] Borel, A., Serre, J.-P.: Le théorème de Riemann-Roch (d’après Grothendieck), Bull. Soc. Math. France 86 (1958), 97–136.

[4] Deligne, P.: Équations différentielles à points singuliers réguliers. Lecture Notes in Mathematics, 163 Springer-Verlag, Berlin-New York, 1970. iii+133 pp.

[5] Deligne, P.: Théorie de Hodge II, Publ. Math. Inst. Hautes Ét. Sci. 40 (1972) 5–57.

[6] Esnault, H., Viehweg, E.: Logarithmic de Rham complexes and vanishing theorems. Invent. Math. 86 (1986) 161–194.

[7] Esnault, H.: Recent developments on characteristic classes of flat bundles on complex algebraic manifolds. Jahresber. Deutsch. Math.-Verein. 98 (1996) 182–191.

[8] Faltings, G., Chai, C.-L.: Degeneration of Abelian Varieties, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3-te Folge, 22 (1990), Springer Verlag.

[9] Griffiths, P.: Periods of integrals on algebraic manifolds. III. Some global differential-geometric properties of the period mapping. Inst. Hautes Études Sci. Publ. Math. 38 (1970) 125–180.

[10] Griffiths, P. (Editor): Topics in transcendental algebraic geometry. Ann of Math. Stud. 106, Princeton Univ. Press, Princeton, NJ. (1984)

[11] Hirzebruch, F.: Topological Methods in Algebraic Geometry, Grundlehren 131, Springer Verlag, 3-rd edition (1966).

[12] Iyer, J.: preprint Essen (2001).

[13] Mumford, D.: Geometric Invariant Theory, Ergebnisse der Mathematik und ihrer Grenzgebiete, 2-te Folge, 34 (1982), Springer Verlag.

[14] Mumford, D.: Towards an enumerative geometry of the moduli space of curves. Arithmetic and geometry, Vol. II, 271–328, Progr. Math., 36, Birkhäuser Boston, Boston, MA, 1983.

[15] van der Geer, G.: Cycles on the moduli space of abelian varieties. Moduli of curves and abelian varieties, 65–89, Aspects Math., E33, Vieweg, Braunschweig, 1999.

[16] van der Geer, G.: The Chow ring of the moduli space of abelian threefolds. J. Algebraic Geom. 7 (1998) 753–770.

Universität Essen, FB6, Mathematik, 45117 Essen, Germany
E-mail address: esnault@uni-essen.de

Universität Essen, FB6, Mathematik, 45117 Essen, Germany
E-mail address: viehweg@uni-essen.de