Extended Supersymmetric $\sigma$-Model
Based on the $SO(2N+1)$ Lie Algebra
of the Fermion Operators

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Extended supersymmetric $\sigma$-model is proposed, basing on the
$SO(2N+1)$ Lie algebra of fermion operators composed of creation-
annihilation operators and pair operators. The canonical trans-
formation, the extension of the $SO(2N)$ Bogoliubov transforma-
tion to the $SO(2N+1)$ group, is introduced. Embedding the
$SO(2N+1)$ group into an $SO(2N+2)$ group and using $SO(2N+2)$
$U(N+1)$ coset variables, we investigate a new aspect of the supersymmetric $\sigma$-
model on the Kähler manifold of the symmetric space $SO(2N+2)$
$U(N+1)$. We construct a Killing potential which is the extension of the
Killing potential in the $SO(2N)$ $U(N)$ coset space given by van Holten et
al. to that in the $SO(2N+2)$ $U(N+1)$ coset space. The Killing potential plays
an important role to see behaviour of the vacuum expectation
value of the $\sigma$-model fields. Bosonization of the $SO(2N+1)$ Lie
operators is made. The vacuum functions for these bosons are
expressed in terms of the corresponding Kähler potential and a
$U(1)$ phase.

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Plan of the Talk

1. Introduction
2. The SO(2N+1) Lie algebra of fermion operators and the Bogoliubov transformation
3. Embedding into an SO(2N+2) group
4. $\sigma$-model on the SO(2N+2)/U(N+1) coset manifold
5. Expression for SO(2N+2)/U(N+1) Killing potential
6. Discussions and concluding remarks
7. Appendix
   A. Bosonization of SO(2N+2) Lie operators
   B. Vacuum function for bosons
   C. Differential forms for bosons over SO(2N+2)/U(N+1) coset space
Supersymmetric $\sigma$-Model

1. The supersymmetric extension of nonlinear models was first given by Zumino, by introducing scalar fields in a complex Kähler manifold. [1].

2. The higher dimensional nonlinear $\sigma$-models defined on symmetric spaces and on hyper Kähler manifolds have been intensively studied [2, 3, 4, 5].

3. van Holten et al. have discussed a supersymmetric $\sigma$-models on the Kähler coset spaces. They have presented a way of constructing the Killing potentials on the coset spaces $SO(2N)/U(N)$ [2].

4. Higashijima et al. have given Ricci-flat metrics on compact Kähler manifolds, $SU(N)/[SU(N-M)\times U(M)]$, $SO(2N)/U(N)$ and $Sp(N)/U(N)$ [3].

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1 Introduction

The set of the fermion operators composed of annihilation-creation and pair operators forms a larger Lie algebra, Lie algebra of the \( SO(2N+1) \) group.

\[ \xrightarrow{\text{Group extension of the } SO(2N) \text{ Bogoliubov transformation for fermions}} \]

The fermion Lie operators are mapped into the regular representation of the \( SO(2N+1) \) group and are represented by Bose operators.

The Bose images of the fermion Lie operators are expressed by closed first order differential forms.

We give an extended supersymmetric \( \sigma \)-model on Kähler coset space \( \frac{G}{H} = \frac{SO(2N+2)}{U(N+1)} \), basing on the \( SO(2N+1) \) Lie algebra of the fermion operators. Embedding the \( SO(2N+1) \) group into an \( SO(2N+2) \) group and using the \( \frac{SO(2N+2)}{U(N+1)} \) coset variables, we investigate a new aspect of the supersymmetric \( \sigma \)-model on the Kähler manifold of the symmetric space \( \frac{SO(2N+2)}{U(N+1)} \).

We construct Killing potential which is the extension of Killing potential in the \( \frac{SO(2N)}{U(N)} \) coset space given by van Holten et al. to that in the \( \frac{SO(2N+2)}{U(N+1)} \) coset space:

\[ \xrightarrow{\text{Killing potential is equivalent with}} \]

**Generalized density matrix**

Its diagonal-block part : A reduced scalar potential with

**Fayet-Iliopoulos term**

The reduced scalar potential is optimized to see behaviour of the vacuum expectation value of the \( \sigma \)-model fields.
The SO(2N+1) Lie algebra of fermion operators and the Bogoliubov transformation

\( c_\alpha \) and \( c^\dagger_\alpha \), \( \alpha = 1, \ldots, N \): Annihilation and creation operators of the fermion

\[
\{ c_\alpha, c^\dagger_\beta \} = \delta_{\alpha \beta}, \quad \{ c_\alpha, c_\beta \} = \{ c^\dagger_\alpha, c^\dagger_\beta \} = 0. \tag{2.1}
\]

The set of fermion operators consisting of annihilation-creation operators and pair operators:

\[
\begin{align*}
E^{\alpha \beta} &= c^\dagger_\alpha c_\beta - \frac{1}{2}\delta_{\alpha \beta}, & E^{\alpha \dagger} &= c^\dagger_\alpha c^\dagger_\beta, & E_{\alpha \beta} &= c_\alpha c_\beta, \\
E^{\alpha \dagger} &= E^{\beta \alpha}, & E^{\alpha \beta} &= E^{\dagger \beta \alpha}, & E_{\alpha \beta} &= -E_{\beta \alpha}.
\end{align*} \tag{2.2}
\]

The set of fermion operators (2.2) form an \( SO(2N + 1) \) Lie algebra.

As a consequence of the anti-commutatin relation (2.1), the commutation relations for the fermion operators (2.2) in the \( SO(2N + 1) \) Lie algebra:

\[
[E^{\alpha \beta}, E^{\gamma \delta}] = \delta_{\gamma \beta}E^{\alpha \delta} - \delta_{\alpha \delta}E^{\gamma \beta}, \quad (U(N) \text{ algebra}) \tag{2.3}
\]

\[
\begin{align*}
[E^{\alpha \beta}, E_{\gamma \delta}] &= \delta_{\alpha \delta}E^{\beta \gamma} - \delta_{\beta \gamma}E^{\alpha \delta}, \\
[E^{\alpha \beta}, E_{\gamma}^\dagger] &= \delta_{\alpha \delta}E^{\beta \gamma} + \delta_{\beta \gamma}E^{\alpha \delta} - \delta_{\alpha \gamma}E^{\beta \delta} - \delta_{\beta \delta}E^{\alpha \gamma}, \\
[E_{\alpha \beta}, E_{\gamma \delta}] &= 0, \quad \{ \}
\end{align*} \tag{2.4}
\]

5
\[ [c^\dagger_\alpha, c_\beta] = 2E^\alpha_{\beta}, \quad [c_\alpha, c_\beta] = 2E_{\alpha\beta}, \]
\[ [c_\alpha, E^\beta_\gamma] = \delta_{\alpha\beta}c_\gamma, \quad [c_\alpha, E_{\beta\gamma}] = 0, \]
\[ [c_\alpha, E^{\beta\gamma}] = \delta_{\alpha\beta}c^\dagger_\gamma - \delta_{\alpha\gamma}c^\dagger_\beta. \]  

\( n: \) fermion number operator \( n = c^\dagger_\alpha c_\alpha: \)
\[ \{c_\alpha, (-1)^n\} = \{c^\dagger_\alpha, (-1)^n\} = 0. \]  

Operator \( \Theta \) defined by \( \Theta \equiv \theta^\alpha c^\dagger_\alpha - \bar{\theta}^\alpha c_\alpha: \) Due to the relation \( \Theta^2 = -\bar{\theta}^\alpha \theta^\alpha, \)
\[ e^\Theta = Z + X_\alpha c^\dagger_\alpha - \bar{X}_\alpha c_\alpha, \quad \bar{X}_\alpha X_\alpha + Z^2 = 1, \]
\[ Z = \cos \theta, \quad X_\alpha = \frac{\theta^\alpha}{\theta} \sin \theta, \quad \theta^2 = \bar{\theta}^\alpha \theta^\alpha. \]  

From (2.1), (2.6) and (2.7), we have
\[ e^\Theta(c_\alpha, c^\dagger_\alpha, \frac{1}{\sqrt{2}})(-1)^n e^{-\Theta} = (c_\beta, c^\dagger_\beta, \frac{1}{\sqrt{2}})(-1)^n G_X, \]
\[ G_X \equiv \begin{bmatrix} \delta_{\beta\alpha} - \bar{X}_\beta X_\alpha & \bar{X}_\beta \bar{X}_\alpha & -\sqrt{2}Z \bar{X}_\beta \\ X_\beta X_\alpha & \delta_{\beta\alpha} - X_\beta \bar{X}_\alpha & \sqrt{2}Z X_\beta \\ \sqrt{2}Z X_\alpha & -\sqrt{2}Z \bar{X}_\alpha & 2Z^2 - 1 \end{bmatrix}. \]  

From (2.8) and the commutability of \( U(g) \) with \((-1)^n, \)
\[ U(G)(c_\alpha, c^\dagger_\alpha, \frac{1}{\sqrt{2}})(-1)^n U^\dagger(G) = (c_\beta, c^\dagger_\beta, \frac{1}{\sqrt{2}})(-1)^n \begin{bmatrix} A^\alpha_{\beta\alpha} & B^\alpha_{\bar{\beta}\alpha} & -\frac{x_\beta}{\sqrt{2}} \\ B^\alpha_{\beta\alpha} & \bar{A}^\alpha_{\bar{\beta}\alpha} & \frac{x_\beta}{\sqrt{2}} \\ \frac{y_\alpha}{\sqrt{2}} & -\frac{\bar{y}_\alpha}{\sqrt{2}} & z \end{bmatrix}, \]  

(2.9)
\[
A_{a\beta} = a_{a\beta} - \bar{X}_\alpha Y_\beta = a_{a\beta} - \frac{\bar{x}_\alpha y_\beta}{2(1 + z)}, \\
B_{a\beta} = b_{a\beta} + X_\alpha Y_\beta = b_{a\beta} + \frac{x_\alpha y_\beta}{2(1 + z)}, \\
x_\alpha = 2ZX_\alpha, \quad y_\alpha = 2ZY_\alpha, \quad z = 2Z^2 - 1.
\]

\[
U(G)(c, c^\dagger, \frac{1}{\sqrt{2}})U^\dagger(G) = (c, c^\dagger, \frac{1}{\sqrt{2}})(z - \rho)G, \quad (2.11)
\]

\[
G \equiv \begin{bmatrix}
A & \bar{B} & -\bar{x} \\
B & \bar{A} & \frac{x}{\sqrt{2}} \\
y & -\bar{y} & \frac{z}{\sqrt{2}}
\end{bmatrix}, \quad G^\dagger G = GG^\dagger = 1_{2N+1}, \quad \text{det } G = 1, \quad (2.12)
\]

\[
U(G)U(G') = U(GG'), \quad U(G^{-1}) = U^{-1}(G) = U^\dagger(G), \\
U(1_{2N+1}) = \mathbb{I}_G, \quad (2.13)
\]

\[
(c, c^\dagger, \frac{1}{\sqrt{2}}) : (2N+1)-\text{dimensional row vector } ((c_\alpha), (c_\alpha^\dagger), \frac{1}{\sqrt{2}}) \\
A = (A^\alpha_\beta) \text{ and } B = (B_\alpha\beta) : N \times N \text{ matrices.} \\
U(G) : \text{nonlinear transformation with a } q\text{-number gauge } z - \rho: \\
\rho = x_\alpha c_\alpha^\dagger - \bar{x}_\alpha c_\alpha \text{ and } \rho^2 = -\bar{x}_\alpha x_\alpha = z^2 - 1.
\]

The matrix $G$ is a matrix belonging to the $SO(2N+1)$ group.
When $z = 1$, the $G$ becomes an $SO(2N)$ matrix $g$.

$SO(2N + 1)$ WF $|G> = U(G)|0>$:

$$
|G> = <0 |U(G) |0> (1 + r_{\alpha} c_{\alpha}^\dagger) \exp(\frac{1}{2} \cdot q_{\alpha\beta} c_{\alpha}^\dagger c_{\beta}^\dagger) |0>,
$$

$$
r_{\alpha} = \frac{1}{1 + z} (x_{\alpha} + q_{\alpha\beta} \bar{x}_{\beta}), \quad q = ba^{-1},
$$

$$
<0 |U(G) |0> = \Phi_{00}(G) = \sqrt{\frac{1+z}{2}} \left[ \det(1_{N} + q^\dagger q) \right]^{-\frac{1}{4}} e^{i \frac{T}{2}}.
$$

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3 Embedding into an SO(2N+2) group

Projection operator $P_{\pm}$ onto the sub-spaces of even and odd fermion numbers

$$P_{\pm} = \frac{1}{2}(1 \pm (-1)^n), \quad P_{\pm}^2 = P_{\pm}, \quad P_{+}P_{-} = 0,$$  \hspace{1cm} (3.1)

Operators with the supuruous index 0:

$$E^0_0 = -\frac{1}{2}(-1)^n = \frac{1}{2}(P_- - P_+),$$

$$E^\alpha_0 = c^\dagger_\alpha P_+ = P_+ c^\dagger_\alpha, \quad E^{\alpha 0} = -c^\dagger_\alpha P_+ = -P_- c^\dagger_\alpha.$$ \hspace{1cm} (3.2)

Indices $p, q \cdots$ running over $N + 1$ values $0, 1, \cdots N$.

Unified notation : $E^p_q, E_{pq}$ and $E^{pq}$.

The $SO(2N+1)$ group is embedded into an $SO(2N+2)$ group. The embedding leads us to an unified formulation of the $SO(2N+1)$ regular representation in which paired and unpaired modes are treated in an equal way.

$(N+1) \times (N+1)$ matrices $A$ and $B$ :

$$A = \begin{bmatrix} A & -\bar{x} \\ y & 1+z \\ 2 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} B & \frac{x}{2} \\ y & \frac{1-z}{2} \end{bmatrix}, \quad y = x^* a - x^\dagger b.$$ \hspace{1cm} (3.3)

Imposition of the ortho-normalization of the $G$

Matrices $A$ and $B$ satisfy the ortho-normalization condition

$$G = \begin{bmatrix} A & B \\ \bar{B} & \bar{A} \end{bmatrix}, \quad G^\dagger G = GG^\dagger = 1_{2N+2}, \quad \det G = 1,$$  \hspace{1cm} (3.4)

$$A^\dagger A + B^\dagger B = 1_{N+1}, \quad A^\dagger B + B^\dagger A = 0, \quad AA^\dagger + BB^\dagger = 1_{N+1}, \quad AB^\dagger + BA^\dagger = 0.$$ \hspace{1cm} (3.5)
Decomposition of the matrices $\mathcal{A}$ and $\mathcal{B}$:

$$\mathcal{A} = \begin{bmatrix} 1_N - \frac{\bar{x} \bar{r}^T}{2} & -\frac{\bar{x}}{2} \\ \frac{(1 + z) \bar{r}^T}{2} & 1 + \frac{z}{2} \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \quad \mathcal{B} = \begin{bmatrix} 1_N + \frac{\bar{x} \bar{r}^T q^{-1}}{2} & \frac{x}{2} \\ -\frac{(1 + z) \bar{r}^T q^{-1}}{2} & 1 - \frac{z}{2} \end{bmatrix} \begin{bmatrix} b & 0 \\ 0 & 1 \end{bmatrix},$$

(3.6)

$$\mathcal{A}^{-1} = \begin{bmatrix} a^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1_N & \bar{x} \\ -\bar{r}^T & 1 + \bar{z} \end{bmatrix}. \quad (3.7)$$

$SO(2N+2)$ $U(N+1)$ coset variable with the $N+1$-th component:

$$Q = \mathcal{B} \mathcal{A}^{-1} = \begin{bmatrix} q & r \\ -\bar{r}^T & 0 \end{bmatrix} = -Q^T. \quad (3.8)$$

$SO(2N+1)$ variables $q_{\alpha\beta}$ and $r_{\alpha}$: Independent variables of the $SO(2N+2)$ $U(N+1)$ coset space.
Matrix elements of $\mathcal{Q}$ and $\overline{\mathcal{Q}}$: Co-ordinates on the $\text{SO}(2N+2)/\text{U}(N+1)$ coset manifold, on which the real line element can be well defined by a hermitian metric tensor on the coset manifold

$$ds^2 = \mathcal{G}_{pq\,rs} d\mathcal{Q}^{pq} d\overline{\mathcal{Q}}^{rs} \ (\mathcal{Q}^{pq} = \mathcal{Q}_{pq} \text{ and } \overline{\mathcal{Q}}^{rs} = \overline{\mathcal{Q}}_{rs}). \quad (4.1)$$

The hermitian metric tensor $\mathcal{G}_{pq\,rs}$ is locally given through a real scalar function, Kähler potential:

$$\mathcal{K}(\mathcal{Q}^\dagger, \mathcal{Q}) = \ln \det \left(1_{N+1} + \mathcal{Q}^\dagger \mathcal{Q}\right), \quad (4.2)$$

Expression for the components of the metric tensor

$$\mathcal{G}_{pq\,rs} = \frac{\partial^2 \mathcal{K}(\mathcal{Q}^\dagger, \mathcal{Q})}{\partial \mathcal{Q}^{pq} \partial \overline{\mathcal{Q}}^{rs}} = \left\{ (1_{N+1} + \mathcal{Q}^\dagger \mathcal{Q})^{-1}\right\}_{sp} \left\{ (1_{N+1} + \mathcal{Q}^\dagger \mathcal{Q})^{-1}\right\}_{qr} \quad (4.3)$$

$$- (r \leftrightarrow s) - (p \leftrightarrow q) + (p \leftrightarrow q, \ r \leftrightarrow s).$$

In two/four-dimensional space-time, the simplest representation of $\mathcal{N} = 1$ supersymmetry is a scalar multiplet $\phi = \{\mathcal{Q}, \psi_L, H\}$ where $\mathcal{Q}$ and $H$ are complex scalars and $\psi_L \equiv \frac{1}{2}(1 + \gamma_5)\psi$ is a left-handed chiral spinor defined through a Majorana spinor:

$$\phi = \mathcal{Q} + \overline{\theta}_R \psi_L + \overline{\theta}_R \theta_L H. \quad (4.4)$$

General theory of the supersymmetric $\sigma$-model can be constructed from the $[N]$ scalar multiplets $\phi^{[\alpha]} = \{\mathcal{Q}^{[\alpha]}, \psi_L^{[\alpha]}, H^{[\alpha]}\}$ ($[\alpha] = 1, \cdots, [N]$).

Supersymmetry transformations:

$$\delta \mathcal{Q}^{[\alpha]} = \bar{\varepsilon}_R \psi_L^{[\alpha]}, \quad \delta \psi_L^{[\alpha]} = \frac{1}{2}(\delta \mathcal{Q}^{[\alpha]} \varepsilon_R + H^{[\alpha]} \varepsilon_L), \quad \delta H^{[\alpha]} = \bar{\varepsilon}_L \delta \psi_L^{[\alpha]}, \quad (4.5)$$

$\varepsilon$: Majorana spinor parameter.
Following Zumino [1] and van Holten et al. [2], Lagrangian of a supersymmetric $\sigma$-model: Complex scalar fields:

\[ Q^{[\alpha]} = 1, \ldots, \frac{N(N+1)}{2} (= [N]) \]

and Spinors $\psi^{[\alpha]}_L$ and $\bar{\psi}^{[\alpha]}_L$:

\[
\mathcal{L}_{\text{chiral}} = -G^{[\alpha][\beta]} \left( \partial_\mu \bar{Q}^{[\beta]} \partial_\mu Q^{[\alpha]} + \bar{\psi}^{[\beta]}_L \gamma_\mu \psi^{[\alpha]}_L \right) \\
+ W;[\alpha][\beta] \bar{\psi}^{[\beta]}_R \psi^{[\alpha]}_L + \bar{W};[\alpha][\beta] \bar{\psi}^{[\beta]}_L \psi^{[\alpha]}_R \\
- G^{[\alpha][\delta]} \bar{W};[\alpha] \bar{W};[\alpha] + \frac{1}{2} R^{[\alpha][\beta][\gamma][\delta]} \bar{\psi}^{[\beta]}_L \gamma_\mu \psi^{[\alpha]}_L \bar{\psi}^{[\delta]}_L \gamma_\mu \psi^{[\gamma]}_L,
\]

and a Kähler covariant derivative is $D_\mu \psi^{[\alpha]}_L = \partial_\mu \psi^{[\alpha]}_L + \Gamma^{[\alpha]}_\beta_\gamma \psi^{[\beta]}_L \partial_\mu Q^{[\gamma]}$.

**Auxiliary fields** $H^{[\alpha]}$ are eliminated through their field equations

\[
H^{[\alpha]} = \Gamma^{[\alpha]}_\beta_\gamma \bar{\psi}^{[\beta]}_R \psi^{[\gamma]}_L + G^{[\alpha][\delta]} \bar{W};[\alpha].
\]

Right-handed chiral spinor $\psi_R : \psi_R = C\psi^T_L$.
Expression for $\text{SO}(2N+2)/\text{U}(N+1)$ Killing potential

$\text{SO}(2N+2)$ infinitesimal left transformation of an $\text{SO}(2N+2)$ matrix $G$ to $G'$:

$$G' = (1_{2N+2} + \delta G)G = \begin{bmatrix} 1_N + \delta A & \delta \bar{B} \\ \delta B & 1_N + \delta \bar{A} \end{bmatrix}G$$

$$= \begin{bmatrix} A + \delta AA + \delta \bar{B} \delta B & \bar{B} + \delta AB \delta \bar{A} \\ \delta \bar{B} + \delta \bar{A} \delta B & \bar{A} + \delta AA + \delta B \delta \bar{B} \end{bmatrix}.$$  \hfill (5.1)

Define a $\text{SO}(2N+2)/\text{U}(N+1)$ coset variable $Q' (= B'A'^{-1})$ in the $G'$ frame. $Q'$ is calculated infinitesimally as

$$Q' = B'A'^{-1} = (B + \delta \bar{A}B + \delta B\bar{A})(A + \delta AA + \delta B\delta B)^{-1}$$

$$= (Q + \delta \bar{A}Q + \delta B)(1_{N+1} + \delta A + \delta \bar{B}Q)^{-1}$$

$$= Q + \delta B - Q\delta A + \delta \bar{A}Q - Q\delta \bar{B}Q.$$ \hfill (5.2)

The Kähler metrics admit a set of holomorphic isometries, Killing vectors, $\mathcal{R}^{i[\alpha]}(Q)$ and $\bar{\mathcal{R}}^{i[\alpha]}(\bar{Q})$ ($i = 1, \cdots, \dim G$),

$$\mathcal{R}^i_{[\bar{\alpha}]}(Q), \ [\alpha] + \bar{\mathcal{R}}^i_{[\alpha]}(\bar{Q}), \ [\bar{\alpha}] = 0, \ \mathcal{R}^i_{[\bar{\alpha}]}(Q) = G_{[\alpha][\bar{\alpha}]\bar{\alpha}}\mathcal{R}^{i[\alpha]}(Q).$$ \hfill (5.3)

These isometries define infinitesimal symmetry transformations:

$\delta Q = Q' - Q = \mathcal{R}(Q)$ and $\delta \bar{Q} = \bar{\mathcal{R}}(\bar{Q})$ such that $G'(Q, \bar{Q}) = G(Q, \bar{Q})$.

Killing equation (5.3) is the necessary and sufficient condition for an infinitesimal co-ordinate transformation

$$\delta Q^{[\alpha]} = (\delta B - \delta A^r Q - Q\delta A + Q\delta B^r \bar{Q})^{[\alpha]} = \xi_i \mathcal{R}^{i[\alpha]}(Q),$$

$$\delta \bar{Q}^{[\alpha]} = \xi_i \bar{\mathcal{R}}^{i[\alpha]}(\bar{Q}).$$ \hfill (5.4)

$\xi_i$: infinitesimal and global group parameter. Due to the Killing equation, the Killing vectors $\mathcal{R}^{i[\alpha]}(Q)$ and $\bar{\mathcal{R}}^{i[\alpha]}(\bar{Q})$ can be written locally as the gradient of
some real scalar function, the Killing potentials $\mathcal{M}^i(Q, \bar{Q})$ such that

$$R_{[\alpha]}^i(Q) = -iM_{[\alpha]}^i, \quad \bar{R}_{[\alpha]}^i(\bar{Q}) = iM_{[\alpha]}^i.$$  \tag{5.5}

According to van Holten et al. and using the infinitesimal $SO(2N + 2)$ matrix $\delta \mathcal{G}$ (A.3), the Killing potential $M_\sigma$:

$$M_\sigma(\delta A, \delta B, \delta B^\dagger) = \text{Tr} \left( \delta \mathcal{G} \tilde{M}_\sigma \right) = \text{tr} \left( \delta A M_{\sigma \delta A} + \delta B M_{\sigma \delta B} + \delta B^\dagger M_{\sigma \delta B^\dagger} \right),$$

$$\tilde{M}_\sigma \equiv \begin{bmatrix} M_{\sigma \delta A} & M_{\sigma \delta B^\dagger} \\ -M_{\sigma \delta B} & -M_{\sigma \delta A^\dagger} \end{bmatrix}, \quad M_{\sigma \delta A} = \tilde{M}_{\sigma \delta A} + \left( \tilde{M}_{\sigma \delta A^\dagger} \right)^T,$$  \tag{5.6}

Introduce $(N + 1)$-dimensional matrices $\mathcal{R}(Q; \delta \mathcal{G})$, $\mathcal{R}_T(Q; \delta \mathcal{G})$ and $\mathcal{X}$:

$$\mathcal{R}(Q; \delta \mathcal{G}) = \delta B - \delta A^\dagger Q - Q \delta A + Q \delta B^\dagger Q, \quad \mathcal{R}_T(Q; \delta \mathcal{G}) = -\delta A^\dagger + Q \delta B^\dagger,$$  \tag{5.7}

$$\mathcal{X} = (1_{N+1} + QQ^\dagger)^{-1} = \mathcal{X}^\dagger.$$  \tag{5.8}

Killing potential $M_\sigma$:

$$-i M_\sigma(Q, \bar{Q}; \delta \mathcal{G}) = -\text{tr} \Delta \left( Q, \bar{Q}; \delta \mathcal{G} \right),$$

$$\Delta \left( Q, \bar{Q}; \delta \mathcal{G} \right) \equiv \mathcal{R}_T(Q; \delta \mathcal{G}) - \mathcal{R}(Q; \delta \mathcal{G}) Q^\dagger \mathcal{X}$$

$$= \left( Q \delta A Q^\dagger - \delta A^\dagger - \delta B Q^\dagger + Q \delta B^\dagger \right) \mathcal{X}. \tag{5.8}$$

$$-i M_{\sigma \delta B} = -\mathcal{X} Q, \quad -i M_{\sigma \delta B^\dagger} = Q^\dagger \mathcal{X}, \quad -i M_{\sigma \delta A} = 1_{N+1} - 2 Q^\dagger \mathcal{X} Q. \tag{5.9}$$
Their components:

\[
\begin{align*}
-i \tilde{M}_{\sigma B} &= -\mathcal{X} Q, & -i \tilde{M}_{\sigma B^\dagger} &= Q^\dagger \mathcal{X}, \\
-i \tilde{M}_{\sigma A} &= -Q^\dagger \mathcal{X} Q, & -i \tilde{M}_{\sigma A^\dagger} &= Q.
\end{align*}
\]

(5.10)

Introduction of a \((2N + 2) \times (N + 1)\) isometric matrix \(U\) by

\[
U^\dagger = [B^\dagger, A^\dagger], \quad (U^\dagger U = 1_{N+1}).
\]

(5.11)

To make clear meaning of the Killing potential, introduce a \((2N + 2) \times (2N + 2)\) matrix:

\[
W = U U^\dagger = \begin{bmatrix} \mathcal{R} & \mathcal{K} \\ -\bar{\mathcal{K}} & 1_{N+1} - \bar{\mathcal{R}} \end{bmatrix} = W^\dagger, \quad \mathcal{R} = B B^\dagger, \quad \mathcal{K} = B A^\dagger,
\]

(5.12)

which satisfies the idempotency relation:

\[
W^2 = W.
\]

The matrix \(W\) is a natural extension of the generalized density matrix in the \(SO(2N)\) CS rep to the \(SO(2N + 2)\) CS rep.

\[
\mathcal{A} = (1_{N+1} + Q^\dagger Q)^{-\frac{1}{2}} \mathcal{U}, \quad \mathcal{B} = Q(1_{N+1} + Q^\dagger Q)^{-\frac{1}{2}} \mathcal{U}, \quad \mathcal{U} \in U(N+1),
\]

(5.13)

The Killing potential \(-i \tilde{M}_\sigma\) is given by the generalized density matrix as

\[
-i \tilde{M}_\sigma = \begin{bmatrix} -\mathcal{R} & -\bar{\mathcal{K}} \\ \mathcal{K} & -(1_{N+1} - \mathcal{R}) \end{bmatrix} \rightarrow -i \tilde{M}_\sigma = \begin{bmatrix} \mathcal{R} & \mathcal{K} \\ -\bar{\mathcal{K}} & 1_{N+1} - \bar{\mathcal{R}} \end{bmatrix}.
\]

(5.14)

To our great surprise, the expression for the Killing potential just becomes equivalent with the generalized density matrix.
The inverse matrix $\mathcal{X}$ leads to the form
\[
\mathcal{X} = \begin{bmatrix}
Q_{q q}^{\dagger} & Q_{q r} \\
Q_{q r}^{\dagger} & Q_{r r}^{\dagger}
\end{bmatrix}, \quad \chi = (1_N + q q^{\dagger})^{-1} = \chi^{\dagger},
\] (5.15)

\[
Q_{q q}^{\dagger} = \chi - \frac{1 + z}{2} \chi (r r^{\dagger} - q \bar{r} r^{\dagger} q^{\dagger}) \chi, \quad Q_{q r} = \frac{1 + z}{2} \chi q \bar{r}, \quad Q_{r r}^{\dagger} = \frac{1 + z}{2}.
\] (5.16)

Introduction of auxiliary function: \( \lambda = r r^{\dagger} - q \bar{r} r^{\dagger} q^{\dagger} \).

Killing potential $\mathcal{M}_\sigma$ expressed in terms of $q$, $r$ and $1 + z = 2 Z^2$ as,
\[
-i \mathcal{M}_{\sigma \delta B} = \begin{bmatrix}
-\chi q + Z^2 (\chi \lambda q + \chi q \bar{r} r^{\dagger}) & -\chi r + Z^2 \chi \lambda r \\
-Z^2 (r^{\dagger} q^{\dagger} \chi - r^{\dagger} r) & -Z^2 r^{\dagger} q^{\dagger} \chi r
\end{bmatrix},
\] (5.17)

\[
-i \mathcal{M}_{\sigma \delta A} = \begin{bmatrix}
-2 q^{\dagger} \chi q + 2 Z^2 (q^{\dagger} \chi \lambda q + q^{\dagger} \chi q \bar{r} r^{\dagger} + \bar{r} r^{\dagger} q^{\dagger} \chi q - \bar{r} r^{\dagger}) & -2 q^{\dagger} \chi r + 2 Z^2 (q^{\dagger} \chi \lambda r + \bar{r} r^{\dagger} q^{\dagger} \chi r) \\
-2 r^{\dagger} \chi q + 2 Z^2 (r^{\dagger} \chi \lambda q + r^{\dagger} \chi q \bar{r} r^{\dagger}) & 1 - 2 r^{\dagger} \chi r + 2 Z^2 r^{\dagger} \chi \lambda r
\end{bmatrix}.
\] (5.18)

Identities and relations:
\[
r^{\dagger} q^{\dagger} \chi r = 0, \quad r^{\dagger} q \bar{r} = 0, \quad r^{\dagger} \chi r = \frac{1 - Z^2}{Z^2}, \quad r^{\dagger} \chi \lambda r = \left(\frac{1 - Z^2}{Z^2}\right)^2,
\] (5.19)

\[
1 - 2 r^{\dagger} \chi r + 2 Z^2 r^{\dagger} \chi \lambda r = 2 Z^2 - 1,
\] (5.20)

\[
\chi \lambda \chi r = \frac{1 - Z^2}{Z^2} \chi r, \quad r^{\dagger} \chi \lambda \chi = \frac{1 - Z^2}{Z^2} r^{\dagger} r, \quad q^{\dagger} \chi q = 1_N - \bar{\chi}.
\] (5.21)
We get compact forms of the Killing potential $\mathcal{M}_\sigma$ as,

$$-i\mathcal{M}_{\sigma \delta B}^{(\sigma \delta B^i)} = \begin{bmatrix}
-\chi q + Z^2 (\chi r^i \chi q + \chi q r^i \chi) & -Z^2 \chi r \\
(q^i \chi - Z^2 (q^i \chi r^i \chi + \bar{\chi} r^i \chi^i q^i \chi)) & (-Z^2 \bar{\chi} r) \\
Z^2 r^i \bar{\chi} & 0 \\
(Z^2 r^i \chi) & (0)
\end{bmatrix}, \quad (5.22)$$

$$-i\mathcal{M}_{\sigma \delta A} = \begin{bmatrix}
1_N & -2q^i \chi q + 2Z^2 (q^i \chi r^i \chi q - \bar{\chi} r^i \chi^i q^i \chi) & -2Z^2 q^i \chi r \\
-2Z^2 r^i \chi q & 2Z^2 - 1
\end{bmatrix}.$$  \quad (5.23)

Introduction of a gauge covariant derivative:

$$D_\mu Q^{[\alpha]} = \partial_\mu Q^{[\alpha]} - g_i A^i_\mu R^{[\alpha]}(Q),$$

$$D_\mu \psi_L^{[\alpha]} = \partial_\mu \psi_L^{[\alpha]} - g_i A^i_\mu R^{[\alpha]}(Q)\psi_L^{[\beta]} + \Gamma^{[\alpha]}_{[\beta][\gamma]} \psi_L^{[\beta]} \partial_\mu Q^{[\gamma]}.$$  \quad (5.24)

Introduction of gauge fields in Lagrangian, via the gauge covariant derivatives, the $\sigma$-model is no longer invariant under the supersymmetry transformations.

To restore the supersymmetry, it is necessary to add the terms

$$\Delta \mathcal{L}_{\text{chiral}} = 2G_{[\alpha][\alpha]} \left( R^i_{[\alpha]}(Q) \bar{\psi}_L^{[\alpha]} \lambda_R^i + \bar{R}^i_{[\alpha]}(Q) \bar{\lambda}_R^i \psi_L^{[\alpha]} \right)$$

$$-g_i \text{tr} \left\{ D^i (\mathcal{M}^i + \xi^i) \right\},$$

where $\xi_i$ are Fayet-Iliopoulos parameters:

**Full Lagrangian:**

$$\mathcal{L} = -\text{tr} \left\{ \frac{1}{4} F_{\mu \nu}^i F_{\mu \nu}^i + \frac{1}{2} \bar{\lambda}^i \slashed{\partial} \lambda^i - \frac{1}{2} D^i D^i \right\} + \mathcal{L}_{\text{chiral}}(\partial_\mu \rightarrow \mathcal{D}_\mu) + \Delta \mathcal{L}_{\text{chiral}}.$$

(5.26)
Eliminating the auxiliary field $D^i$ by $D^i = -g_i(M^i + \xi^i)$,

**Scalar potential:**

$$V_{SC} = -\frac{1}{2}g_i^2 \text{tr} \left\{ (M^i + \xi^i)^2 \right\}.$$  \hspace{1cm} (5.27)

A reduced scalar potential arising from the gauging of $SU(N + 1) \times U(1)$ including a Fayet-Iliopoulos term with parameter $\xi$ is of the special interest:

$$V_{\text{redSC}} = \frac{g_{U(1)}^2}{2(N + 1)} (\xi - iM_Y)^2 + \frac{g_{SU(N+1)}^2}{2} \text{tr} (-iM_t)^2.$$  \hspace{1cm} (5.28)

New quantities $\text{tr} (-iM_t)^2$ and $-iM_Y$ are defined below.

\[
\begin{align*}
\text{tr}(-iM_t)^2 &= \text{tr}(-iM_{\sigma\delta})^2 - \frac{1}{N+1}(-iM_Y)^2, \\
-iM_Y &= \text{tr}(-iM_{\sigma\delta}), \\
\text{tr}(-iM_{\sigma\delta}) &= -N + 2\text{tr}(\chi) + 2Z^2\text{tr}(\chi^rr^\dagger) - 4Z^2\text{tr}(\chi^rr^\dagger\chi) \\
&\hspace{1cm}+ 2Z^2 - 1, \\
\text{tr}(-iM_{\sigma\delta})^2 &= N - 4\text{tr}(\chi) + 4\text{tr}(\chi\chi) + 12Z^2\text{tr}(\chi^rr^\dagger\chi) \\
&\hspace{1cm} - 16Z^2\text{tr}(\chi\chi^rr^\dagger\chi) \\
&\hspace{1cm} - 4Z^4r^\dagger\chi\chi r \cdot \text{tr}(\chi^rr^\dagger) + 8Z^4r^\dagger\chi\chi r \cdot \text{tr}(\chi^rr^\dagger\chi) \\
&\hspace{1cm} + 1 - 4Z^4r^\dagger\chi\chi r, \\
\end{align*}
\]  \hspace{1cm} (5.29)
Calculate approximately the quantities $r^\dagger\chi^{} r$ and $\text{tr}(rr^\dagger)$ as

\[
 r^\dagger\chi^{} r = \frac{1}{4Z^4}(x^\dagger + x^r q^\dagger)\chi^{}(x + q\bar{x}) = \frac{1}{4Z^4}x^\dagger \chi^{} x
\]

\[
 \approx \frac{1}{4Z^4}\left\{\frac{1}{N} + \text{tr}(q^\dagger q)\right\}^{-1} x^\dagger x = \frac{1 - Z^2}{Z^2}\left\{\frac{1}{N} + \langle q^\dagger q \rangle\right\}^{-1}
\]

\[
 \equiv \frac{1 - Z^2}{Z^2} \langle \chi \rangle,
\]

\[
\text{tr}(rr^\dagger) = r^\dagger r = \frac{1}{4Z^4}(x^\dagger + x^r q^\dagger)(x + q\bar{x}) = \frac{1}{4Z^4}x^\dagger \chi^{-1} x
\]

\[
 \approx \frac{1 - Z^2}{Z^2} \frac{1}{\langle \chi \rangle} \equiv \langle rr^\dagger \rangle.
\]

Approximating $\text{tr}(\chi rr^\dagger)$ as $\langle \chi \rangle \text{tr}(rr^\dagger), \text{tr}(-iM_{\sigma\delta A})$ and $\text{tr}(-iM_{\sigma\delta A})^2$ are computed as

\[
 \begin{align*}
 \text{tr}(-iM_{\sigma\delta A}) &= 1 - N + 2(2Z^2 - 1) \langle \chi \rangle, \\
 \text{tr}(-iM_{\sigma\delta A})^2 &= 1 + N - 4(2Z^2 - 1) \langle \chi \rangle + 4(2Z^4 - 1) \langle \chi \rangle^2.
\end{align*}
\]

Final form of the reduced scalar potential:

\[
 V_{\text{red SC}} = \frac{g_{U(1)}^2}{2(N+1)} \left\{\xi + 1 - N + 2(2Z^2 - 1) \langle \chi \rangle\right\}^2
 \]

\[
 + 2g_{SU(N+1)}^2 \frac{N}{N+1} \left[N - 2(2Z^2 - 1) \langle \chi \rangle + \left\{2(N-1)Z^4 + 4Z^2 - (N+2)\right\} \langle \chi \rangle^2\right].
\]
To see behaviour of the vacuum expectation value of the $\sigma$-fields, it is very important to analyze the form of the reduced scalar potential.

Variation of the reduced scalar potential with respect to $Z$ and $\langle \chi \rangle$:

\[
g_{U(1)}^2 \left\{ \xi + 1 - N + 2(2Z^2 - 1) \langle \chi \rangle \right\} \\
-2g_{SU(N+1)}^2 \left\{ 1 - ((N - 1)Z^2 + 1) \langle \chi \rangle \right\} = 0, \tag{5.34}
\]

\[
g_{U(1)}^2 \left\{ \xi + 1 - N + 2(2Z^2 - 1) \langle \chi \rangle \right\} (2Z^2 - 1) \\
-2g_{SU(N+1)}^2 [2Z^2 - 1 - \{2(N-1)Z^4 + 4Z^2 - (N+2)\} \langle \chi \rangle] = 0. \tag{5.35}
\]

$g^2$-independent relation:

\[
\left\{ 1 - ((N - 1)Z^2 + 1) \langle \chi \rangle \right\} (2Z^2 - 1) \\
- [2Z^2 - 1 - \{2(N - 1)Z^4 + 4Z^2 - (N + 2)\} \langle \chi \rangle] = 0, \tag{5.36}
\]

which reads

\[
(N + 1)(Z^2 - 1) \langle \chi \rangle = 0 \quad \longrightarrow \quad Z^2 = 1. \tag{5.37}
\]
To find proper solutions for the extended supersymmetric $\sigma$-model, rescaling Goldstone fields $Q$ by mass parameter, introduce $(N + 1)$-dimensional matrices $R_f(Q_f; \delta G), R_{fT}(Q_f; \delta G)$ and $\chi_f$:

\[
R_f(Q_f; \delta G) = \frac{1}{f} \delta B - \delta A^T Q_f - Q_f \delta A + f Q_f \delta B^\dagger Q_f,
\]

\[
R_{fT}(Q_f; \delta G) = -\delta A^T + f Q_f \delta B^\dagger,
\]

\[
\chi_f = (1_{N+1} + f^2 Q_f Q_f^\dagger)^{-1} = \chi^\dagger,
\]

\[
Q_f = \begin{bmatrix}
q & \frac{1}{f} r_f \\
-\frac{1}{f} r_f^T & 0
\end{bmatrix},
\]

\[
r_f = \frac{1}{2Z^2} (x + f q \bar{x}),\ f \equiv \frac{1}{m_\sigma}.
\]

Due to the rescaling, the Killing potential $M_\sigma$ is deformed as

\[
-i M_{f\sigma} (Q_f, \bar{Q}_f; \delta G) = -\text{tr} \Delta_f (Q_f, \bar{Q}_f; \delta G),
\]

\[
\Delta_f (Q_f, \bar{Q}_f; \delta G) \equiv R_{fT}(Q_f; \delta G) - R_f(Q_f; \delta G) f^2 Q_f^\dagger \chi_f
\]

\[
= \left( f^2 Q_f \delta A Q_f^\dagger - \delta A^T - f \delta B Q_f^\dagger + f Q_f \delta B^\dagger \right) \chi_f.
\]

A $f$-deformed Killing potential $M_{f\sigma}$:

\[
-i M_{f\sigma \delta B} = -f \chi_f Q_f,
\]

\[
-i M_{f\sigma \delta B^\dagger} = f Q_f^\dagger \chi_f,
\]

\[
-i M_{f\sigma \delta A} = 1_{N+1} - 2f^2 Q_f^\dagger \chi_f Q_f.
\]
The inverse matrix $\chi_f$ leads to

$$\chi_f = \begin{bmatrix} Q_{fqq}^\dagger & Q_{fqr} \\ Q_{fqr}^\dagger & Q_{frf}^\dagger \end{bmatrix}, \quad \chi_f = (1_N + f^2 qq^\dagger)^{-1} = \chi_f^\dagger,$$

(5.41)

$$Q_{fqq}^\dagger = \chi_f - Z^2 \chi_f (r_f r_f^\dagger - f^2 q\bar{r} f r_f^\dagger) \chi_f,$$

(5.42)

$$Q_{qrf} = f Z^2 \chi_f q\bar{r}_f, \quad Q_{rf}^\dagger = Z^2.$$

Introduce $f$-deformed auxiliary function:

$$\lambda_f = r^\dagger r - f^2 q\bar{r} r^\dagger q^\dagger = \lambda_f^\dagger.$$

(5.43)

$$-i\mathcal{M}_{f\delta A} = \begin{bmatrix} 1_N & -2 q^\dagger \chi_f q + 2 Z^2 (q^\dagger \chi_f \lambda_f \chi_f q + q^\dagger \chi_f q\bar{r}_f r_f^\dagger q^\dagger r_f^\dagger) + r_f r_f^\dagger q^\dagger \chi_f q - 1 \sum f^2 \bar{r}_f r_f^\dagger, \\ 1_N & 1 - 2 Z^2 r_f^\dagger \chi_f q + 2 Z^2 r_f^\dagger \chi_f q\bar{r}_f r_f^\dagger, \end{bmatrix}^\dagger \begin{bmatrix} 1_N & -2 q^\dagger \chi_f q + 2 Z^2 (q^\dagger \chi_f \lambda_f \chi_f q + q^\dagger \chi_f q\bar{r}_f r_f^\dagger q^\dagger r_f^\dagger) + r_f r_f^\dagger q^\dagger \chi_f q - 1 \sum f^2 \bar{r}_f r_f^\dagger, \\ 1_N & 1 - 2 Z^2 r_f^\dagger \chi_f q + 2 Z^2 r_f^\dagger \chi_f q\bar{r}_f r_f^\dagger, \end{bmatrix}.$$

(5.44)

Identities and relations:

$$r_f^\dagger q^\dagger \chi_f r_f = 0, \quad r_f^\dagger \chi_f q\bar{r}_f = 0,$$

(5.45)

$$r_f^\dagger \chi_f r_f = \frac{1 - Z^2}{Z^2}, \quad r_f^\dagger \chi_f \lambda_f \chi_f r_f = \left(1 - \frac{Z^2}{Z^2}\right)^2,$$

(5.46)

$$\chi_f \lambda_f \chi_f r_f = \frac{1 - Z^2}{Z^2} \chi_f r_f, \quad r_f^\dagger \chi_f \lambda_f \chi_f = \frac{1 - Z^2}{Z^2} r_f^\dagger \chi_f, \quad q^\dagger \chi_f q = \frac{1}{f^2} (1_N - \bar{\chi}_f).$$

(5.47)
Compact form of the $f$-deformed Killing potential $\mathcal{M}_{f\sigma\delta A}$:

\[-i\mathcal{M}_{f\sigma\delta A} = \left[ 1_{N} - 2q^\dagger \chi_f q + 2Z^2 \left( q^\dagger \chi_f r_f r_f^\dagger \chi_f q - \frac{1}{f^2} \bar{\chi}_f r_f r_f^\dagger \bar{\chi}_f \right) - 2\frac{1}{f} Z^2 q^\dagger \chi_f r_f \right] \]

\[\begin{bmatrix}
1_{N} - 2q^\dagger \chi_f q + 2Z^2 \left( q^\dagger \chi_f r_f r_f^\dagger \chi_f q - \frac{1}{f^2} \bar{\chi}_f r_f r_f^\dagger \bar{\chi}_f \right) - 2\frac{1}{f} Z^2 q^\dagger \chi_f r_f & -2\frac{1}{f} Z^2 r_f^\dagger \chi_f q \\
-2\frac{1}{f} Z^2 r_f^\dagger \chi_f q & \frac{1}{f^2} (2Z^2 - 1) + 1 - \frac{1}{f^2}
\end{bmatrix} \]  

A $f$-deformed reduced scalar potential:

\[V_{f\text{redSC}} = \frac{g^2_U(1)}{2(N+1)} (\xi - i\mathcal{M}_{fY})^2 + \frac{g^2_{SU(N+1)}}{2} \text{tr} (-i\mathcal{M}_{fI})^2 + \frac{1}{N+1} (-i\mathcal{M}_{fY})^2,\]

\[\begin{align*}
\text{tr} (-i\mathcal{M}_{fI})^2 &= \text{tr} (-i\mathcal{M}_{f\sigma\delta A})^2 - \frac{1}{N+1} (-i\mathcal{M}_{fY})^2, \\
-i\mathcal{M}_{fY} &= \text{tr} (-i\mathcal{M}_{f\sigma\delta A}),
\end{align*}\]  

\[\text{tr} (-i\mathcal{M}_{f\sigma\delta A}) = -N + 2\text{tr}(\chi_f) + 2f \frac{1}{f^2} Z^2 \text{tr}(\chi_f r_f r_f^\dagger)\]

\[\begin{align*}
-4f \frac{1}{f^2} Z^2 \text{tr}(\chi_f r_f r_f^\dagger \chi_f) + \frac{1}{f^2} (2Z^2 - 1) + 1 - \frac{1}{f^2},
\end{align*}\]
\[ \text{tr}(-i\mathcal{M}_{f\sigma\delta\Lambda})^2 = N - 4\frac{1}{f^2}(1 - \frac{1}{f^2})N - 4\frac{1}{f^4}\text{tr}(\chi_f) + 4\frac{1}{f^4}\text{tr}(\chi_f\chi_f) \]

\[ + 4\frac{1}{f^2}(1 - \frac{1}{f^2})Z^2\text{tr}(\chi_f r_f r_f^\dagger) + 12\frac{1}{f^4}Z^2\text{tr}(\chi_f r_f r_f^\dagger\chi_f) \]

\[ - 16\frac{1}{f^4}Z^2\text{tr}(\chi_f\chi_f r_f r_f^\dagger\chi_f) \]

\[ - 4\frac{1}{f^4}Z^4 r_f^\dagger\chi_f\chi_f r_f \cdot \text{tr}(\chi_f r_f r_f^\dagger) + 8\frac{1}{f^4}Z^4 r_f^\dagger\chi_f\chi_f r_f \cdot \text{tr}(\chi_f r_f r_f^\dagger\chi_f) \]

\[ + \frac{1}{f^4} + 2(1 - \frac{1}{f^2})(2Z^2 - 1) + (1 - \frac{1}{f^2})^2 - 4\frac{1}{f^4}Z^4 r_f^\dagger\chi_f\chi_f r_f. \]

Identity:

\[ r_f^\dagger\chi_f r_f = \frac{1}{4Z^4}(x^\dagger\chi_f x + x^T q^\dagger\chi_f q\bar{x}) = \frac{1}{4Z^4}x^T\bar{x} = \frac{1 - Z^2}{Z^2}, \quad (5.52) \]

Approximate formulas for the quantities \( r_f^\dagger\chi_f\chi_f r_f \) and \( \text{tr}(r_f r_f^\dagger) \):

\[ r_f^\dagger\chi_f\chi_f r_f = \frac{1}{4Z^4}(x^\dagger + fx^T q^\dagger)\chi_f\chi_f(x + fq\bar{x}) = \frac{1}{4Z^4}x^\dagger\chi_f x \]

\[ \approx \frac{1}{4Z^4} \left\{ \frac{1}{N}(N + f^2\text{tr}(q^\dagger q)) \right\}^{-1} x^\dagger x \equiv \frac{1 - Z^2}{Z^2} <\chi_f>, \quad (5.53) \]

\[ \text{tr}(r_f r_f^\dagger) \approx r_f^\dagger r_f = \frac{1}{4Z^4}(x^\dagger + fx^T q^\dagger)(x + fq\bar{x}) = \frac{1}{4Z^4}x^\dagger\chi_f^{-1} x \]

\[ \approx \frac{1 - Z^2}{Z^2} \frac{1}{<\chi_f>}, \quad (5.54) \]
\[ \text{tr}(\chi f r f r_f^\dagger) \approx <\chi_f> \text{tr}(r f r_f^\dagger), \quad \text{tr}(-iM_{f\sigma\delta A}) \text{ and } \text{tr}(-iM_{f\sigma\delta A})^2 \text{ are given as} \]

\[
\begin{align*}
\text{tr}(-iM_{f\sigma\delta A}) &= 1 - N + 2 \left\{ \frac{1}{f^2}(2Z^2 - 1) + 1 - \frac{1}{f^2} \right\} <\chi_f>, \\
\text{tr}(-iM_{f\sigma\delta A})^2 &= 1 + N - 4 \frac{1}{f^2} \left( 1 - \frac{1}{f^2} \right) N + 2 \left( 1 - \frac{1}{f^2} \right)^2 (2Z^2 - 1) \\
&\quad - 4 \frac{1}{f^2} \left\{ \frac{1}{f^2}(2Z^2 - 1) - \left( 1 - \frac{1}{f^2} \right) \right\} <\chi_f> \\
&\quad + 4 \frac{1}{f^4}(2Z^4 - 1) <\chi_f>^2.
\end{align*}
\]

Final form of \( f \)-deformed reduced scalar potential:

\[
V_{f\text{redSC}} = \frac{g^2}{2(N+1)} \left[ \xi + 1 - N + 2 \left\{ \frac{1}{f^2}(2Z^2 - 1) + 1 - \frac{1}{f^2} \right\} <\chi_f> \right]^2 \\
+ 2 \frac{g^2}{N+1} \left[ N - \frac{1}{f^2} \left\{ 1 + \frac{1}{f^2} - \left( 1 - \frac{1}{f^2} \right) N \right\} (2Z^2 - 1) <\chi_f> \\
+ \frac{1}{f^4} \left\{ 2(N - 1)Z^4 + 4Z^2 - (N + 2) \right\} <\chi_f>^2 \\
- \frac{1}{f^2} \left( 1 - \frac{1}{f^2} \right) N(N + 1) + \frac{1}{2} \left( 1 - \frac{1}{f^2} \right)^2 (N + 1)(2Z^2 - 1) \\
- \left( 1 - \frac{1}{f^2} \right) \left\{ 1 - \frac{1}{f^2} - \left( 1 + \frac{1}{f^2} \right) N \right\} <\chi_f> \\
- \left( 1 - \frac{1}{f^2} \right) \left\{ \frac{2}{f^2}(2Z^2 - 1) + 1 - \frac{1}{f^2} \right\} <\chi_f>^2 \right].
\]

(5.56)
Variation of $f$-deformed reduced scalar potential with respect to $Z$ and $<\chi_f>$,

\[
g_{U(1)}^2 \left[ \xi + 1 - N + 2 \left\{ \frac{1}{f^2}(2Z^2 - 1) + 1 - \frac{1}{f^2} \right\} <\chi_f> \right] \\
-2g_{SU(N+1)}^2 \left\{ \frac{1}{2} \left\{ 1 + \frac{1}{f^2} - \left( 1 - \frac{1}{f^2} \right) N \right\} \right\} \\
- \frac{1}{f^2} \left\{ (N - 1)Z^2 + 1 \right\} <\chi_f> \\
- \left( 1 - \frac{1}{f^2} \right) <\chi_f> + \frac{1}{4} f^2 \left( 1 - \frac{1}{f^2} \right)^2 (N + 1) \frac{1}{<\chi_f>} \right] = 0, \tag{5.57}
\]

\[
g_{U(1)}^2 \left[ \xi + 1 - N + 2 \left\{ \frac{1}{f^2}(2Z^2 - 1) + 1 - \frac{1}{f^2} \right\} <\chi_f> \right] \\
\left\{ \frac{1}{f^2}(2Z^2 - 1) + 1 - \frac{1}{f^2} \right\} \\
-2g_{SU(N+1)}^2 \left\{ \frac{1}{2} f^2 \left\{ 1 + \frac{1}{f^2} - \left( 1 - \frac{1}{f^2} \right) N \right\} (2Z^2 - 1) \\
- \frac{1}{f^4} \left\{ 2(N - 1)Z^4 + 4Z^2 - (N + 2) \right\} <\chi_f> \\
+ \frac{1}{2} \left( 1 - \frac{1}{f^2} \right) \left\{ 1 - \frac{1}{f^2} - \left( 1 + \frac{1}{f^2} \right) N \right\} \\
+ \left( 1 - \frac{1}{f^2} \right) \left\{ 2 \frac{1}{f^2}(2Z^2 - 1) + 1 - \frac{1}{f^2} \right\} <\chi_f> \right] = 0. \tag{5.58}
\]
$g^2$-indepedent relation:

\[
\left[ -\frac{1}{f^4} (N+1)(Z^2 - 1) + 3 \frac{1}{f^2} \left( 1 - \frac{1}{f^2} \right) (2Z^2 - 1) \\
- \frac{1}{f^2} \left( 1 - \frac{1}{f^2} \right) \left\{ (N - 1)Z^2 + 1 \right\} + 2 \left( 1 - \frac{1}{f^2} \right)^2 \right] <\chi_f> \\
= 1 - \frac{1}{f^4} - \left( 1 - \frac{1}{f^2} \right)^2 N + \frac{1}{4}f^2 \left( 1 - \frac{1}{f^2} \right)^2 (N + 1) \\
\left\{ \frac{1}{f^2}(2Z^2 - 1) + 1 - \frac{1}{f^2} \right\} \frac{1}{<\chi_f>},
\]

which reads, a proper solution for the $Z^2$.

\[
\left[ \frac{8}{f^2} \left\{ \left( 1 - \frac{1}{f^2} \right) - \frac{1}{4} \right\} <\chi_f> \frac{1}{2} \left( 1 - \frac{1}{f^2} \right)^2 (N+1) \frac{1}{<\chi_f>} \right] Z^2 \\
= 1 - \frac{1}{f^4} - \left( 1 - \frac{1}{f^2} \right)^2 N - \left\{ \frac{1}{f^4}(N+1) + 2 \left( 1 - \frac{1}{f^2} \right) \right\} <\chi_f> \\
- \frac{1}{4}f^2 \left( 1 - \frac{1}{f^2} \right)^3 (N+1) \frac{1}{<\chi_f>},
\]

Solution of the $so(2N+2)_{U(N+1)}$ supersymmetric $\sigma$-model. $f = 1$, (5.60) \rightarrow a simple solution (5.37).
Another equation for $Z^2$:

$$\left\{ 2g_{U(1)}^2 + (N - 1)g_{SU(N+1)}^2 \right\} \langle \chi_f \rangle = Z^2$$

$$= -\frac{1}{2}g_{U(1)}^2 \left[ f^2(\xi + 1 - N) - 2f^2 \left\{ \frac{1}{f^2} - \left( 1 - \frac{1}{f^2} \right) \right\} \langle \chi_f \rangle \right]$$

$$+ g_{SU(N+1)}^2 \left[ \frac{1}{2}f^2 \left\{ 1 + \frac{1}{f^2} - \left( 1 - \frac{1}{f^2} \right) N \right\} - f^2 \langle \chi_f \rangle \right]$$

$$+ \frac{1}{4}f^4 \left( 1 - \frac{1}{f^2} \right)^2 (N+1) \frac{1}{\langle \chi_f \rangle}.$$  

(5.61)

Ultimate goal of determining $\langle \chi_f \rangle$:

$$\left\{ 2g_{U(1)}^2 + (N - 1)g_{SU(N+1)}^2 \right\} \left\{ \frac{1}{f^2}(N - 1) + 2 \right\} \langle \chi_f \rangle^3$$

$$+ \left[ g_{U(1)}^2 \left[ 2 - f^2 - 2 \left\{ 1 - \frac{1}{f^4} - \left( 1 - \frac{1}{f^2} \right)^2 N \right\} \right] \right] \langle \chi_f \rangle^2$$

$$+ g_{SU(N+1)}^2 \left[ f^2 - (N - 1) \left\{ 1 - \frac{1}{f^4} - \left( 1 - \frac{1}{f^2} \right)^2 N \right\} \right] \langle \chi_f \rangle$$

$$- \frac{1}{2} \left[ g_{U(1)}^2 f^2 \left\{ \xi + 1 - N - \left( 1 - \frac{1}{f^2} \right)^3 (N+1) \right\} \right]$$

$$+ g_{SU(N+1)}^2 f^2 \left\{ 1 + \frac{1}{f^2} - \left( 1 - \frac{1}{f^2} \right) N - \frac{1}{2} \left( 1 - \frac{1}{f^2} \right)^3 (N^2 - 1) \right\} \langle \chi_f \rangle$$

$$- \frac{1}{4}g_{SU(N+1)}^2 f^4 \left( 1 - \frac{1}{f^2} \right)^2 (N + 1) = 0.$$  

(5.62)
Discussions and concluding remarks

To approach an approximate solution for $\langle \chi_f \rangle$, 
put $g_U^2(1) = g_{SU(N+1)}^2$ and neglect the terms $(2 - \frac{1}{f^2})^n$, $(n = 2, 3)$, 
since we consider a small fluctuation of $f$ around 1:

\[
(N + 1) \left\{ \frac{1}{f^2}(N - 1) + 2 \right\} <\chi_f>^2
- \left\{ \left( 1 - \frac{1}{f^4} \right) (N + 1) - 2 \right\} <\chi_f>
- \frac{1}{2} f^2 \left\{ \xi + 2 + \frac{1}{f^2} - \left( 2 - \frac{1}{f^2} \right) N \right\} = 0,
\]

(6.1)

\[
8 \frac{1}{f^2} \left\{ \left( 1 - \frac{1}{f^2} \right) - \frac{1}{4} \right\} <\chi_f> Z^2
= 1 - \frac{1}{f^4} - \left\{ \frac{1}{f^4}(N+1) + 2 \left( 1 - \frac{1}{f^2} \right) \right\} <\chi_f>.
\]

(6.2)

Equation (6.1) is easily solved as

\[
<\chi_f> = \frac{1}{2} \frac{f^2}{N^2 - 1 + 2f^2(N+1)} \left\{ \left( 1 - \frac{1}{f^4} \right) (N+1) - 2 \pm \sqrt{D_{<\chi_f>}} \right\},
\]

\[
D_{<\chi_f>} \equiv 2 \left\{ N^2 - 1 + 2f^2(N+1) \right\} \xi + \left\{ \left( 1 - \frac{1}{f^4} \right) (N+1) - 2 \right\}^2
+ 2 \left\{ N^2 - 1 + 2f^2(N+1) \right\} \left\{ 2 + \frac{1}{f^2} - \left( 2 - \frac{1}{f^2} \right) N \right\}.
\]

(6.3)
\( N = 5 \), The case I: \( f = 1.01 \) and \( f = 0.99 \)

\[
< \chi_f > = \begin{cases} 
(28.0 \times \sqrt{18.612\xi - 38.677} - 25.0) \times 10^{-3}, & (f = 1.01), \\
(27.0 \times \sqrt{17.880\xi - 32.353} - 30.0) \times 10^{-3}, & (f = 0.99), 
\end{cases}
\]

\[
Z^2 = \begin{cases} 
3.220 - \frac{1}{1.295 \times \sqrt{18.612\xi - 38.677} - 1.156}, & (f = 1.01), \\
2.980 + \frac{1}{13.635 \times \sqrt{17.880\xi - 32.353} - 14.140}, & (f = 0.99). 
\end{cases}
\]

Noticing \( 0 < Z^2 < 1 \), after a fine tuning for the parameter \( \xi \),

\[
Z^2 = 0.448, \quad < \chi_f > = 7.844 \times 10^{-3}, \quad (f = 1.01, \ \xi = 2.152),
\]

\[
Z^2 = 0.629, \quad < \chi_f > = -8.481 \times 10^{-3}, \quad (f = 0.99, \ \xi = 1.845),
\]

\( N = 5 \), The case II: \( f = 1.001 \) and \( f = 0.999 \)

\[
< \chi_f > = \begin{cases} 
(28.0 \times \sqrt{18.012\xi - 35.256} - 27.0) \times 10^{-3}, & (f = 1.001), \\
(27.0 \times \sqrt{17.988\xi - 34.736} - 28.0) \times 10^{-3}, & (f = 0.999), 
\end{cases}
\]

\[
Z^2 = \begin{cases} 
3.020 - \frac{1}{13.860 \times \sqrt{18.012\xi - 35.256} - 13.365}, & (f = 1.001), \\
2.980 + \frac{1}{13.635 \times \sqrt{17.988\xi - 34.736} - 14.140}, & (f = 0.999), 
\end{cases}
\]

\[
Z^2 = 0.763, \quad < \chi_f > = 0.916 \times 10^{-3}, \quad (f = 1.001, \ \xi = 2.0125),
\]

\[
Z^2 = 0.264, \quad < \chi_f > = -0.730 \times 10^{-3}, \quad (f = 0.999, \ \xi = 1.9880),
\]
Figure 1: We have used the symbols $<\chi_f>_1$, $<\chi_f>_2$ and $<\chi_f>_3$ to denote the three solutions of equation (5.62). The solutions $<\chi_f>_2$ and $<\chi_f>_3$ are always negative. On the other hand, $<\chi_f>_1$ is always positive. We have used the symbols $Z^2_1$, $Z^2_2$ and $Z^2_3$ to denote the values of the $Z^2$ parameter associated with the solutions $<\chi_f>_1$, $<\chi_f>_2$ and $<\chi_f>_3$, respectively, according to equation (5.60). For $f=1.01$ we can always find an interval for $\xi$ where the conditions $<\chi_f>_1 > 0$ and $0 < Z^2_1 < 1$ are both satisfied. However, for $f=5$ and $f=10$ this seems to be not possible anymore.
We have given an extended supersymmetric $\sigma$-model on Kähler coset space $\frac{G}{H} = \frac{SO(2N+2)}{U(N+1)}$, basing on the $SO(2N+1)$ Lie algebra of the fermion operators. Embedding the $SO(2N+1)$ group into an $SO(2N+2)$ group and using the $\frac{SO(2N+2)}{U(N+1)}$ coset variables, we have investigated a new aspect of the supersymmetric $\sigma$-model on the Kähler manifold of the symmetric space $\frac{SO(2N+2)}{U(N+1)}$.

We have constructed a Killing potential which is just the extension of the Killing potential in the $\frac{SO(2N)}{U(N)}$ coset space to that in the $\frac{SO(2N+2)}{U(N+1)}$ coset space. To our great surprise, it has been shown that the Killing potential is equivalent with the generalized density matrix which is an important clue to fermion many-body problems. Its diagonal-block matrix is related to a reduced scalar potential with the Fayet-Iliopoulos term. The reduced scalar potential has been optimized to see behaviour of the vacuum expectation value of the $\sigma$-model fields. We have got, however, a too simple solution $Z^2 = 1$.

To find proper solutions for the extended supersymmetric $\sigma$-model, after rescaling Goldstone fields by a mass parameter, minimization of the reduced scalar potential has been made. Fayet-Iliopoulos term makes a crucial role to acquire proper solutions for $Z^2$. To get proper solutions for wide range of a rescaling parameter $f$, we have solved the cubic equation for $<\chi_f>$.

We have given bosonization of the $SO(2N+2)$ Lie operators, vacuum functions and differential forms for their bosons expressed in terms of the $\frac{SO(2N+2)}{U(N+1)}$ coset variables, a $U(1)$ phase and the Kähler potential. This provides a powerful tool of describing the Goldstone bosons but accompanying fermionic modes. The effectiveness of $\frac{SO(2N+2)}{U(N+1)}$ Kähler manifold is expected to open a new field for exploration of low-energy elementary particle physics by the supersymmetric $\sigma$-model.
A  Bosonization of SO(2N+2) Lie operators

Fermion state vector $|\Psi> \quad$ corresponding to a function $\Psi(G) \in SO(2N+2)$:

$$|\Psi> = \int U(G)|0><0|U^+(G)|\Psi>dG = \int U(G)|0>\Psi(G)dG.$$  \hspace{1cm} (A.1)

The $G$ is given by (3.3) and (3.4) and the $dG$ is an invariant group integration. When an infinitesimal operator $I_G + \delta\hat{G}$ and a corresponding infinitesimal unitary operator $U(1_{2N+2} + \delta G)$ is operated on $|\Psi>$, using $U^{-1}(1_{2N+2} + \delta G) = U(1_{2N+2} - \delta G)$, it transforms $|\Psi>$ as

$$U(1_{2N+2} - \delta G)|\Psi> = (I_G - \delta\hat{\delta}G)|\Psi>$$

$$= \int U(G)|0><0|U^+((1_{2N+2} + \delta G)G)|\Psi>dG$$

$$= \int U(G)|0>\Psi((1_{2N+2} + \delta G)G)dG = \int U(G)|0>(1_{2N+2} + \delta G)\Psi(G)dG,$$  \hspace{1cm} (A.2)

where

$$1_{2N+2} + \delta G = \begin{bmatrix} 1_{N+1} + \delta A & \delta B \\ \delta B & 1_{N+1} + \delta A^{\dagger} \end{bmatrix},$$

$$\delta A^{\dagger} = -\delta A, \quad \text{tr}\delta A = 0, \quad \delta B = -\delta B^{\dagger},$$

$$\delta\hat{G} = \delta A_{q}^{p}E_{p}^{q} + \frac{1}{2}\left(\delta B_{pq}E_{qp} + \delta B_{pq}E_{qp}\right),$$

$$\delta G = \delta A_{q}^{p}E_{p}^{q} + \frac{1}{2}\left(\delta B_{pq}E_{qp} + \delta B_{pq}E_{qp}\right).$$  \hspace{1cm} (A.3)

The operation of $I_G - \delta\hat{G}$ on the $|\Psi>$ in the fermion space corresponds to the left multiplication by $1_{2N+2} + \delta G$ for the variable of the $G$ of the function $\Psi(G)$.

For a small parameter $\epsilon$, Representation on the $\Psi(G)$:

$$\rho(e^{\epsilon\delta G})\Psi(G) = \Psi(e^{\epsilon\delta G}) = \Psi(G + \epsilon\delta GG) = \Psi(G + dG),$$  \hspace{1cm} (A.4)
which leads us to a relation \( dG = \epsilon \delta G \).

\[
dG = \begin{bmatrix} dA \\ dB \\ d\bar{A} \end{bmatrix} = \epsilon \begin{bmatrix} \delta AA + \delta B\bar{B} & \delta A\bar{B} + \delta B\bar{A} \\ \delta B\bar{A} + \delta \bar{A}B & \delta \bar{A}\bar{A} + \delta B\bar{B} \end{bmatrix},
\]

\[
dA = \epsilon \frac{\partial A}{\partial \epsilon} = \epsilon (\delta AA + \delta \bar{B}B), \quad dB = \epsilon \frac{\partial B}{\partial \epsilon} = \epsilon (\delta B\bar{A} + \delta \bar{A}B).
\]

Differential representation of \( \rho(\delta G), d\rho(\delta G) \):

\[
d\rho(\delta G)\Psi(G) = \begin{bmatrix} \frac{\partial A^p}{\partial \epsilon} \frac{\partial}{\partial A^p} + \frac{\partial B_{pq}}{\partial \epsilon} \frac{\partial}{\partial B_{pq}} + \frac{\partial \bar{A}^q}{\partial \epsilon} \frac{\partial}{\partial \bar{A}^q} + \frac{\partial B_{pq}}{\partial \epsilon} \frac{\partial}{\partial B_{pq}} \end{bmatrix}\Psi(G).
\]

Explicit forms of the differential representation: \( d\rho(\delta G)\Psi(G) = \delta G\Psi(G) \).

Each operator in \( \delta G \) is expressed in a differential form:

\[
E^p_{\ q} = \bar{B}_{qr} \frac{\partial}{\partial B_{qr}} - B_{qr} \frac{\partial}{\partial B_{pr}} - \bar{A}^q_r \frac{\partial}{\partial \bar{A}^p_r} + A^p_r \frac{\partial}{\partial A^q_r},
\]

\[
E_{pq} = -\bar{A}^p_r \frac{\partial}{\partial B_{qr}} - B_{qr} \frac{\partial}{\partial A^p_r} - \bar{A}^q_r \frac{\partial}{\partial \bar{A}^p_r} + B_{pr} \frac{\partial}{\partial A^q_r}.
\]

Definition of the boson operators \( A^p_{\ q} \) and \( \bar{A}^p_{\ q} \), etc.:

\[
A = \frac{1}{\sqrt{2}} \left( A + \frac{\partial}{\partial A} \right), \quad A^{\dagger} = \frac{1}{\sqrt{2}} \left( \bar{A} - \frac{\partial}{\partial A} \right),
\]

\[
\bar{A} = \frac{1}{\sqrt{2}} \left( \bar{A} + \frac{\partial}{\partial A} \right), \quad A^{\dagger} = \frac{1}{\sqrt{2}} \left( A - \frac{\partial}{\partial A} \right),
\]

\[
[A, \ A^{\dagger}] = 1, \quad [\bar{A}, \ A^{\dagger}] = 1,
\]

\[
[A, \ \bar{A}] = [A, \ A^{\dagger}] = 0, \quad [A^{\dagger}, \ \bar{A}] = [A^{\dagger}, \ A^{\dagger}] = 0.
\]
The differential operators (A.7) can be converted into a boson operator representation

\[
\begin{align*}
\mathcal{E}^p_q &= B^T_{pq} - \mathcal{A}_{pr}^T \mathcal{A}_{qr}^q - \mathcal{A}_{qr}^T \mathcal{A}_{pr}^q = B^T_{pr} B_{qr} - \mathcal{A}_{qr}^T \mathcal{A}_{pr}^q, \\
\mathcal{E}_{pq} &= \mathcal{A}_{qr}^r B_{pr} - \mathcal{A}_{qr}^r B_{pr} + \mathcal{B}_{qr}^T \mathcal{B}_{pr}^q - \mathcal{A}_{qr}^T \mathcal{A}_{pr}^q = B_{pr}^T B_{qr} - \mathcal{A}_{qr}^T \mathcal{A}_{pr}^q,
\end{align*}
\]  

(A.9)

by using the notation \( A_{r+1}^{pt} = B_{pr}^T \) and \( B_{pr+1} = A_{pr}^T \) to use a suffix \( \tilde{r} \), \((\tilde{r} = 0, 1, \ldots 2N)\). Then we have the \textbf{boson images} of the fermion \( SO(2N + 1) \) Lie operators as

\[
\begin{align*}
\mathcal{E}_{\alpha\beta}^\alpha &= B_{\alpha r}^T B_{\beta \tilde{r}} - \mathcal{A}_{\tilde{r} r}^\alpha \mathcal{A}_{\alpha \tilde{r}}, \\
\mathcal{E}_{\alpha\beta} &= \mathcal{A}_{\tilde{r} r}^\alpha B_{\beta \tilde{r}} - \mathcal{A}_{\tilde{r} r}^\beta B_{\alpha \tilde{r}}, \\
c_{\alpha} &= \mathcal{A}_{\tilde{r} r}^\alpha (\mathcal{A}_{r 0}^0 - B_{0 \tilde{r}}) + (\mathcal{A}_{0 \tilde{r}}^0 - B_{0 \tilde{r}}^0) B_{\alpha \tilde{r}} = \mathcal{A}_{\tilde{r} r}^\alpha \mathcal{Y}_{\tilde{r}} + \mathcal{Y}_{\tilde{r}}^\alpha B_{\alpha \tilde{r}}.
\end{align*}
\]

(A.10)

This representation involves, in addition to the original \( A_{\alpha \beta} \) and \( B_{\alpha \beta} \) bosons, their complex conjugate bosons and the \( \mathcal{Y}_{\tilde{r}} \) bosons. The complex conjugate bosons arise from the use of matrix \( G \) as the variables of representation and the \( \mathcal{Y}_{\tilde{r}} \) bosons arise from extension of algebra from \( SO(2N) \) to \( SO(2N + 1) \) and embedding of the \( SO(2N + 1) \) into \( SO(2N + 2) \).

\[
\frac{\partial}{\partial A_{pq}^p} \det \mathcal{A} = (\mathcal{A}^{-1})_q^p \det \mathcal{A}, \quad \frac{\partial}{\partial A_{pq}^p} (\mathcal{A}^{-1})_r^s = -(\mathcal{A}^{-1})_s^q (\mathcal{A}^{-1})_p^r, \quad (A.11)
\]

we get the relations which are valid when operated onto functions on the right coset \( SO(2N + 2) / SU(N + 1) \)

\[
\begin{align*}
\frac{\partial}{\partial B_{pq}^r} &= \sum_{p > q} (\mathcal{A}^{-1})^q_r \frac{\partial}{\partial Q_{pr}}, \\
\frac{\partial}{\partial A_{pq}^p} &= -\sum_{r > s} Q_{rp} (\mathcal{A}^{-1})_s^q \frac{\partial}{\partial Q_{sr}} - \frac{i}{2} (\mathcal{A}^{-1})_p^q \frac{\partial}{\partial \tau}.
\end{align*}
\]

(A.12)
B Vacuum function for bosons

The function \( \Phi_{00}(\mathcal{G}) \) in \( \mathcal{G} \in SO(2N + 2) \) corresponds to the free fermion vacuum function in the physical fermion space.

\[
\left( \mathcal{E}_{q}^{p} + \frac{1}{2} \delta_{pq} \right) \Phi_{00}(\mathcal{G}) = \mathcal{E}_{pq} \Phi_{00}(\mathcal{G}) = 0, \quad \Phi_{00}(1_{2N+2}) = 1.
\] (B.1)

The vacuum function \( \Phi_{00}(\mathcal{G}) \) to satisfy (B.1) is given by \( \Phi_{00}(\mathcal{G}) = [\det(\bar{\mathcal{A}})]^{\frac{1}{2}} \).

**The proof:**

\[
\left( \mathcal{E}_{q}^{p} + \frac{1}{2} \delta_{pq} \right) [\det(\bar{\mathcal{A}})]^{\frac{1}{2}}
\]

\[
= \frac{1}{2} \delta_{pq} [\det(\bar{\mathcal{A}})]^{\frac{1}{2}}
\]

\[
+ \left( \bar{B}_{pr} \frac{\partial}{\partial B_{qr}} - B_{qr} \frac{\partial}{\partial B_{pr}} - \bar{A}_{r}^{q} \frac{\partial}{\partial \bar{A}_{r}^{p}} + A_{r}^{p} \frac{\partial}{\partial A_{r}^{q}} \right) [\det(\bar{\mathcal{A}})]^{\frac{1}{2}}
\] (B.2)

\[
= \frac{1}{2} \delta_{pq} [\det(\bar{\mathcal{A}})]^{\frac{1}{2}} - \bar{A}_{r}^{q} \frac{\partial}{\partial A_{r}^{p}} [\det(\bar{\mathcal{A}})]^{\frac{1}{2}}
\]

\[
= \frac{1}{2} \delta_{pq} [\det(\bar{\mathcal{A}})]^{\frac{1}{2}} - \frac{1}{2} [\det(\bar{\mathcal{A}})]^{\frac{1}{2}} \bar{A}_{r}^{q} \frac{\partial}{\partial A_{r}^{p}} \det(\bar{\mathcal{A}})
\]

\[
= \frac{1}{2} \delta_{pq} [\det(\bar{\mathcal{A}})]^{\frac{1}{2}} - \frac{1}{2} [\det(\bar{\mathcal{A}})]^{\frac{1}{2}} (\bar{\mathcal{A}} \bar{\mathcal{A}}^{-1})_{qp} \det(\bar{\mathcal{A}}) = 0,
\]

\[
\mathcal{E}_{pq} [\det(\bar{\mathcal{A}})]^{\frac{1}{2}}
\]

\[
= \left( \bar{A}_{r}^{p} \frac{\partial}{\partial B_{qr}} - B_{qr} \frac{\partial}{\partial \bar{A}_{r}^{p}} - \bar{A}_{r}^{q} \frac{\partial}{\partial B_{pr}} + B_{pr} \frac{\partial}{\partial \bar{A}_{r}^{q}} \right) [\det(\bar{\mathcal{A}})]^{\frac{1}{2}} = 0.
\] (B.3)
The vacuum functions $\Phi_{00}(G)$, $G \in SO(2N+1)$ and $\Phi_{00}(g)$, $g \in SO(2N)$ satisfy

$$c_{\alpha} \Phi_{00}(G) = \left( E^{\alpha}_{\beta} + \frac{1}{2} \delta_{\alpha\beta} \right) \Phi_{00}(G) = E_{\alpha\beta} \Phi_{00}(G) = 0, \; \Phi_{00}(1_{2N+1}) = 1,$$

(B.4)

$$\left( e^{\alpha}_{\beta} + \frac{1}{2} \delta_{\alpha\beta} \right) \Phi_{00}(g) = e_{\alpha\beta} \Phi_{00}(g) = 0, \; \Phi_{00}(1_{2N}) = 1.$$

(B.5)

By using the $SO(2N+2)$ Lie operators $E^{pq}$, the expression for the $SO(2N+1)$ WF $| G >$ is converted to a form quite similar to the $SO(2N)$ WF $| g >$ as

$$| G > = < 0 | U(G) | 0 > \exp \left( \frac{1}{2} \cdot Q_{pq} E^{pq} \right) | 0 >,$$

(B.6)

where we have used the nilpotency relation $(E^{\alpha0})^2 = 0$. Equation (B.6) leads to the property $U(G)| 0 > = U(G)| 0 >$. On the other hand,

$$\det \mathcal{A} = \frac{1+z}{2} \det a, \quad \det \mathcal{B} = 0.$$

(B.7)

**Vacuum function $\Phi_{00}(G)$ expressed in terms of the Kähler potential:**

$$< 0 | U(G) | 0 > = \Phi_{00}(G) = \left[ \det(\mathcal{A}) \right]^{\frac{1}{2}} = e^{-\frac{1}{4}K(Q, Q)} e^{-i\frac{\tau}{2}},$$

(B.8)

$$\Phi_{00}(G) = \Phi_{00}(G) = \sqrt{\frac{1+z}{2}} \left[ \det(\bar{a}) \right]^{\frac{1}{2}} = \sqrt{\frac{1+z}{2}} e^{-\frac{1}{4}K(q, q)} e^{-i\frac{\tau}{2}}.$$

(B.9)
C Differential forms for bosons over $SO(2N+2)/U(N+1)$ coset space

The boson images of the fermion $SO(2N+2)$ Lie operators $\mathcal{E}_q^p$, etc. can be represented by the closed first order differential forms over the $SO(2N+2)/U(N+1)$ coset space in terms of the $Q_{pq}$ and the phase variable $\tau \left( = \frac{i}{2} \ln \left[ \frac{\det(A^*)}{\det(A)} \right] \right)$ of the $U(N+1)$ as

$$\mathcal{E}_q^p = Q_{pr} \frac{\partial}{\partial Q_{qr}} - Q_{qr} \frac{\partial}{\partial Q_{pr}} - i \delta_{pq} \frac{\partial}{\partial \tau},$$

$$\mathcal{E}_{pq} = Q_{pr} Q_{sq} \frac{\partial}{\partial Q_{rs}} - \frac{\partial}{\partial Q_{pq}} - i Q_{pq} \frac{\partial}{\partial \tau}, \quad \mathcal{E}^{pq} = \bar{\mathcal{E}}_{pq}. \tag{C.1}$$

The phase variable $\tau$ is identical with the phase variable $\tau \left( = \frac{i}{2} \ln \left[ \frac{\det(a^*)}{\det(a)} \right] \right)$ of the $U(N)$, due to the first equation of (B.7).

The images of the fermion $SO(2N+1)$ Lie operators:

$$\mathcal{E}_\alpha^\beta = \mathcal{E}_\beta^\alpha, \quad E_{\alpha\beta} = \mathcal{E}_{\alpha\beta}, \quad E^{\alpha\beta} = \mathcal{E}^{\alpha\beta}, \tag{C.2}$$

The representations of the $SO(2N+1)$ Lie operators in terms of the variables $q_{\alpha\beta}$ and $r_\alpha$:

$$\mathcal{E}_\alpha^\beta = e_\alpha^\beta + \bar{r}_\alpha \frac{\partial}{\partial \bar{r}_\beta} - r_\beta \frac{\partial}{\partial r_\alpha}, \quad e_\alpha^\beta = \bar{q}_{\alpha\gamma} \frac{\partial}{\partial \bar{q}_{\beta\gamma}} - q_{\beta\gamma} \frac{\partial}{\partial q_{\alpha\gamma}} - i \delta_{\alpha\beta} \frac{\partial}{\partial \tau}, \tag{C.3}$$

$$\mathcal{E}_{\alpha\beta} = e_{\alpha\beta} + (r_\alpha q_{\beta\xi} - r_\beta q_{\alpha\xi}) \frac{\partial}{\partial r_{\xi}}, \quad e_{\alpha\beta} = q_{\alpha\gamma} q_{\delta \beta} \frac{\partial}{\partial q_{\gamma \delta}} - \frac{\partial}{\partial \bar{q}_{\alpha \beta}} - i q_{\alpha \beta} \frac{\partial}{\partial \tau}, \tag{C.3}$$

$$c_\alpha = \frac{\partial}{\partial \bar{r}_\alpha} + \bar{r}_\xi \frac{\partial}{\partial \bar{q}_{\alpha \xi}} + (r_\alpha r_\xi - q_{\alpha \xi}) \frac{\partial}{\partial r_\xi} - q_{\alpha \xi} r_\eta \frac{\partial}{\partial q_{\xi \eta}} + i r_\alpha \frac{\partial}{\partial \tau}, \quad c_\alpha^\dagger = -\bar{c}_\alpha. \tag{C.4}$$
The vacuum function $\Phi_{00}(G)$ in $G \in SO(2N + 1)$ is given in (B.4).

$$
\begin{align*}
c_\alpha \Phi_{00}(G) & = 0, \quad c_\alpha^\dagger \Phi_{00}(G) = \bar{r}_\alpha \Phi_{00}(G), \quad c_\alpha^\dagger = -\bar{c}_\alpha,
\end{align*}
$$
and the property $U(\mathcal{G})|0> = U(G)|0>$. 

**Exact identities:**

$$
\begin{align*}
c_\alpha U(\mathcal{G})|0> &= \left(-r_\alpha + r_\alpha r_\xi c_\xi^\dagger - q_\alpha q_\xi c_\xi^\dagger\right) \cdot U(G)|0>, \quad (C.6) \\
{c_\alpha^\dagger} U(\mathcal{G})|0> &= -c_\alpha^\dagger \cdot U(G)|0>.
\end{align*}
$$

Succefully using these identities, on the $U(\mathcal{G})|0>$, operators $c_\alpha$ and $c_\alpha^\dagger$ are shown to satisfy exactly the anti-commutation relations of the fermion annihilation-creation operators:

$$
\begin{align*}
(c_\alpha^\dagger c_\beta + c_\beta c_\alpha^\dagger) U(\mathcal{G})|0> &= \delta_{\alpha\beta} \cdot U(\mathcal{G})|0>, \quad (C.7) \\
(c_\alpha c_\beta + c_\beta c_\alpha) U(\mathcal{G})|0> &= (c_\alpha^\dagger c_\beta^\dagger + c_\beta^\dagger c_\alpha^\dagger) U(\mathcal{G})|0> = 0. \quad (C.8)
\end{align*}
$$

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