MINIMAL MODELS OF CFT ON $\mathbb{Z}_N$-SURFACES

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Abstract

The conformal field theory on a $\mathbb{Z}_N$-surface is studied by mapping it on the branched sphere. Using a coulomb gas formalism we construct the minimal models of the theory.

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1  Introduction

In recent years there have been many physical applications of the theory of algebraic surfaces in a wide spectrum of subjects, such as string theory [1], conformal field theory [2], solutions of the Einstein equations [3], integrable models [4], the theory of defects [5]. The best known and simplest algebraic curves are the hyper-elliptic surfaces (HESs) [6]; therefore it is not a surprise that the majority of the references cited above deals with them. Recently, HESs have also appeared in the Seiberg-Witten theory of 4-dimensional gauge theories [7].

However, interesting physical applications are not limited to HES but include other algebraic surfaces as well [8]. In fact, one would like to understand quantum field theories on curved space-time with non-trivial topology.

HESs have a $\mathbb{Z}_2$ automorphism. An immediate generalization is therefore the study of $\mathbb{Z}_N$-surfaces (ZNSs), i.e. surfaces that will have a $\mathbb{Z}_N$ automorphism. Although many mathematical aspects of ZNSs are not so tractable as in the case of HES, they are still characterized by a high degree of symmetry and there are certain physical constructions on them, such as the Conformal Field Theory (CFT), that can be studied relatively easy.

The subject of the present paper is to discuss minimal models (MMs) of CFT on ZNSs. Surprisingly, these have not been discussed in the existing literature. We thus construct the basic features of the CFT on the ZNSs, such as the corresponding graded Virasoro algebra, the Coulomb Gas Formalism (CGF), the minimal models and their Kac spectrum. Our construction generalizes the construction of Crnkovic et all. [2] on HESs. In a forthcoming publication [9], we will examine integrable perturbations of these models.

Incidentally, we notice that Krichever and Novikov [10] have studied an important generalization of the Virasoro algebra (KN algebra) on general Riemann surfaces. Their construction has been furthermore studied and extended by various other people in several directions. However, the graded algebra (3.21) of ZNSs enjoys a particular simple structure, a property that it is not shared by the general KN algebra.

2  A primer to $\mathbb{Z}_N$-Symmetric Riemann Surfaces

A $\mathbb{Z}_N$-symmetric Riemann surface $X_g^{(N)}$ is defined as the set of all points $(z, y)$ such that $y(z)$ satisfies the following equation

$$ y^N(z) = \prod_{i=1}^{h} (z - w_i)^{n_i}, \quad n_i > 0. \quad (2.1) $$

Notice that, without loss of generality, we can assume that

$$ 0 < n_i < N, \quad \forall i, \quad (2.2) $$

since if $n_{i_0} = q N + r$ we can define a new variable $\zeta = y/(z - w_{i_0})^q$, in terms of which the power $r$ of the factor $z - w_{i_0}$ satisfies (2.2). A surface for which all $n_i$ are 1 is called non-singular; if at least one of them differs from unity, then the surface is called singular.

1Of course, one can find Riemann surfaces with an even larger symmetry, e.g. $\mathbb{Z}_N \times \mathbb{Z}_M$. Increasing the symmetry we restrict ourselves to smaller portions of the moduli space of Riemann surfaces; nevertheless, surfaces of higher symmetry may be important as well.
Obviously, a singular surface can be considered a special case of a non-singular one, in the case some of the branch points $w_i$ coincide.

We assume that the point at infinity is not a branch point. This assumption is not essential but it is introduced to avoid extra (purely technical) complications. In terms of equations, it implies

$$\sum_i n_i = 0 \mod N.$$  \hfill (2.3)

Another technical assumption is the requirement that the G.C.D. of $N$ with each of $n_i$ must be 1. In particular, this is true for the non-singular surfaces or for surfaces with prime $N$.

The genus $g$ of a $\mathbb{Z}_N$-surface can be calculated very easily using the Riemann-Hurwitz theorem. According to this theorem, if the function $f : M \to N$ maps the Riemann surface $M$ of genus $g$ to the Riemann surface $N$ of genus $\gamma$ and the degree of the mapping is $n$, then

$$g = n(\gamma - 1) + 1 + \frac{B}{2},$$  \hfill (2.4)

where $B$ is the ramification index. In our case, the function $z : X^{(N)}_g \to S^2$, maps the the $\mathbb{Z}_N$-surface $X^{(N)}_g$ with genus $g$ onto 2-sphere with $\gamma = 0$ such that $n = N$. Now, since every point of the $S^2$ is covered exactly $N$ times and therefore the branch number is $N - 1$ and since there are $h$ branch points, we conclude that $B = h(N - 1)$. Substituting in equation (2.4), we find

$$g = \frac{(N - 1)(h - 2)}{2}.$$  \hfill (2.5)

The ZNS (2.1) has $h$ (complex) parameters. Three of them can be mapped by $SL(2, \mathbb{R})$ invariance to 0, 1, $\infty$. Therefore, the moduli space $M_{\text{ZMS}}$ of ZNSs has dimension

$$\dim M_{\text{ZMS}} = h - 3 = \frac{2g}{N - 1} - 1.$$  

Comparing to the dimension of the moduli space $\mathcal{M}$ of generic Riemann surfaces

$$\dim \mathcal{M} = \begin{cases} 1, & g = 1, \\ 3g - 3, & g > 1, \end{cases}$$

we conclude that ZNSs do not exhaust all Riemann surfaces. However, notice that all genus $g = 1, 2$ are HESs ($N = 2$).

The number of independent holomorphic differentials on $X^{(N)}_g$ equals the genus $g$. To count them, we write down all such independent differentials:

$$\Omega_{ij} = \frac{z^{j-1} dz}{y_l},$$  \hfill (2.6)

where $l = 1, 2, \ldots, N - 1$ and where we introduced the quantity\footnote{The symbol $\{x\}$ denotes the fractional part of $x$. If $\lfloor x \rfloor$ is the integer part of $x$ then $x = \lfloor x \rfloor + \{x\}$. Obviously, $0 \leq \{x\} < 1.$} \begin{equation} y_l = \prod_{i=1}^{h} (z - w_i)^{\frac{\gamma_i}{N}}. \end{equation}
The index $j$ counts the number of independent holomorphic differentials for a fixed $l$; in particular $j = 1, 2, \ldots, \leq j_{\text{max}}(l)$ with

$$j_{\text{max}}(l) = \begin{cases} \sum_i \left\{ \frac{in_i}{N} \right\} - 1, & \text{if } \sum_i \left\{ \frac{in_i}{N} \right\} > 1, \\ 0, & \text{otherwise} \end{cases} \quad (2.8)$$

Notice that the genus $g$ of $X_g^{(N)}$ equals the total number of independent holomorphic differentials

$$g = \sum_{l=1}^{N-1} \lfloor j_{\text{max}}(l) \rfloor = \frac{(N - 1)(h - 2)}{2}, \quad (2.9)$$

as it should be.

### 3 CFT on $\mathbb{Z}_N$-Surfaces

#### 3.1 The Algebra

We label the $N$ sheets of the Riemann $\mathbb{Z}_N$-surface $X_g^{(N)}$ by the numbers $l = 0, 1, \ldots, N - 1$:

$$y^{(l)}(z) = \omega^l \prod_{j=1}^{h} (z - w_j)^{\frac{n_j}{N}}, \quad (3.10)$$

where $\omega$ is the $N$-th root of unity

$$\omega = e^{\frac{2\pi i}{N}}. \quad (3.11)$$

Let $\{A_a, B_a\}$ be the basic cycles for the monodromy group of the surface. As we encircle the point $w_i$ along the contours $A_a, B_a$, in the case of an $A_a$ cycle we stay on the same sheet, while in the case of a $B_a$ cycle we pass from the $l$-th sheet to the $(l + n_i)$-th one. We shall denote the process of encircling the points $w_i$ on the cycles $A_a, B_a$ by the symbols $\hat{\pi}_{A_a}, \hat{\pi}_{B_a}$ respectively. Then

$$\hat{\pi}_{A_a} y^{(l)} = y^{(l)} \quad \text{and} \quad \hat{\pi}_{B_a} y^{(l)} = y^{(l + n_i)}. \quad (3.12)$$

Here these generators form a group of monodromy that in our case of $N$-sheeted covering of the sphere coincides with the $\mathbb{Z}_N$ group:

$$\hat{\pi}_{A_a}^N = \hat{\pi}_{B_a}^N = 1. \quad (3.13)$$

We consider the energy-momentum tensor $T(z, y)$ which is a single valued field on the ZNS. Projecting on the plane, it becomes a multi-valued field

$$T^{(l)}(z) \equiv T(z, y^{(l)}(z)).$$

We now view the representation of $T$ on each the $N$ sheets, $T^{(l)}(z)$, as a field on the sphere.

The monodromy properties (3.12) along the cycles $A_a, B_a$ imply that the following boundary conditions should be satisfied by the energy-momentum tensor:

$$\hat{\pi}_{A_a} T^{(l)} = T^{(l)}, \quad \hat{\pi}_{B_a} T^{(l)} = T^{(l + 1)}. \quad (3.14)$$

\footnote{When algebraic operations are indicated on subscripts or superscripts that appear inside parentheses, then the result is always meant mod $N$.}
It is convenient to pass to a basis, in which the operators \( \hat{\pi}_{A_n}, \hat{\pi}_{B_n} \) are diagonal\footnote{Notice that in this equation, as well as (3.14), we assumed that \( n_i = 1, \forall i \). In the general case, one has only to substitute equation (3.15) with \( T_{(k)} = \sum_{l=0}^{N-1} c_{kl} T^{(l)} \), where \( c_{ik} \) are some constants.}

\[
T_{(k)} = T_{(N-k)}^\dagger(z) = \sum_{l=0}^{N-1} \omega^{-kl} T^{(l)} .
\] (3.15)

Then

\[
\hat{\pi}_{A_n} T_{(k)} = T_{(k)} , \quad \hat{\pi}_{B_n} T_{(k)} = \omega^k T_{(k)} .
\] (3.16)

The corresponding operator product expansions (OPEs) of the \( T_{(k)} \) fields can be determined by taking into account the OPEs of \( T^{(l)} \). On the same sheet, the OPEs of the two fields \( T^{(l)}(z), T^{(l')}(z) \), are the same as that on the sphere, while on different sheets they do not correlate; this is a statement of the fact that two points \( z, w \) can be close enough only if they are on the same sheet:

\[
T^{(l)}(z) T^{(l')}(w) = \left[ c/2 \left( \frac{\hat{c}}{2} \right)^2 + \frac{2}{(z-w)^2} + \frac{\partial_w T^{(l)}(w)}{z-w} \right] \delta^{ll'} + \text{reg} .
\] (3.17)

Thus, in the diagonal basis the OPEs can be found to be

\[
T_{(k)}(z) T_{(k')}(w) = \frac{c/2 \delta_{(k+k'),0}}{(z-w)^2} + \frac{2 T_{(k+k')}(w)}{(z-w)^2} + \frac{\partial_w T_{(k+k')}(w)}{z-w} + \text{reg} ,
\] (3.18)

where

\[
c = N \hat{c} ,
\]

and \( \hat{c} \) is the central charge in the OPE \( T^{(l)}(z) T^{(l)}(w) \). It is seen from (3.18) that \( T_{(k')}, k' \neq 0 \) is a primary field with respect to \( T_{(0)} \).

To write the algebra (3.18) in the graded form we first notice that the CFT on the sphere should have \( N \) sectors, labeled by an integer \( s = 0, 1, \ldots, N - 1 \). Consequently, the mode expansion of \( T_{(k)} \) is given by

\[
T_{(k)}(z) = \sum_{n \in \mathbb{Z}} z^{n-2-(\frac{k}{N})} L_{n-\{\frac{k}{N}\}}^{(k)} .
\] (3.19)

Inverting the above relation we have

\[
L_{n-\{\frac{k}{N}\}}^{(k)} = \int \frac{dz}{2\pi i} z^{n+1-(\frac{k}{N})} T_{(k)}(z) .
\] (3.20)

Standard calculations lead to the following algebra for the operators \( L_{n-\{\frac{k}{N}\}}^{(k)} \):

\[
\left[ L_{n-\{\frac{k}{N}\}}^{(k)}, L_{n'-\{\frac{k'}{N}\}}^{(k')} \right] = \left( n - n' - \{ \frac{ks}{N} \} + \{ \frac{k's'}{N} \} \right) L_{n+n'-\{\frac{k}{N}\}-\{\frac{k'}{N}\}}^{(k+k')} + \frac{c}{12} \left[ \left( n - \{ \frac{ks}{N} \} \right)^3 - \left( n - \{ \frac{ks}{N} \} \right) \right] \delta_{n+n'-\{\frac{k}{N}\}-\{\frac{k'}{N}\},0} \delta^{(k+k'),0} .
\] (3.21)

The operators \( \overline{T}_{n-\{\frac{k}{N}\}}^{(k)} \) satisfy the same relations and \( \overline{T}_{n-\{\frac{k}{N}\}}^{(k)} \) commute with \( L_{n-\{\frac{k}{N}\}}^{(k)} \).
3.2 The Representations

To describe the representations of the algebra (3.21), it is necessary to consider separately the non-twisted sector with \( k = 0 \) and the twisted sectors with \( k \neq 0 \). In order to write the \([V_k]\) representation of the algebra (3.21) in a more explicit form, it is convenient to consider the highest weight states.

(i) In the \( k = 0 \) sector, the highest weight state \( | \bar{\Delta}_{[0]} \rangle \equiv | \Delta^{(0)}_{[0]}, \Delta^{(1)}_{[0]}, \ldots \Delta^{(N-1)}_{[0]} \rangle \) is determined with the help of a primary field \( V_{[0]} \) by means of the formula

\[
| \bar{\Delta}_{[0]} \rangle = V_{[0]} | \emptyset \rangle .
\]  

(3.22)

Using the definition of vacuum, it is easy to see that

\[
L_0^{(k)} | \bar{\Delta}_{[0]} \rangle = \Delta^{(k)}_{[0]} | \bar{\Delta}_{[0]} \rangle ;
\]  

(3.23a)

\[
L_n^{(k)} | \bar{\Delta}_{[0]} \rangle = 0 \quad n \geq 1 , \quad k = 0, 1, \ldots, N .
\]  

(3.23b)

Thus, the Verma module over the algebra (3.21) in the \( k = 0 \) sector is obtained by the action of any number of \( L_{-n}^{(p)} \), \( p = 0, 1, 2, \ldots, N - 1 \), operators with \( n > 0 \) on the state (3.22).

The first null state is at level 1:

\[
| \chi \rangle = \sum_{k=0}^{N-1} c_k L_{-1}^{(k)} | \bar{\Delta}_{[0]} \rangle , \quad c_0 = 1 .
\]  

(3.24)

Acting with the operators \( L_{1}^{(N-l)} \), \( l = 1, 2, \ldots, N - 1 \) on \( | \chi \rangle \) we find

\[
\sum_{k=0}^{N-1} c_k \Delta^{(N-l+k)}_{[0]} = 0 .
\]  

(3.25)

This system of equations has a non-trivial solution for the constants \( c_k \), only iff

\[
\det[\Delta^{(N-l+k)}_{[0]}] = 0 .
\]  

(3.26)

The last condition has \( N \) solutions, with the \( l \)-th solution \( (l = 0, 1, \ldots, N - 1) \) being

\[
\Delta^{(k)}_{[0]} = \omega^{-kl} \Delta^{(0)}_{[0]} .
\]  

(3.27)

(ii) In the \( k \neq 0 \) sectors, we define the vector of highest weight \( | \Delta_{[k]} \rangle \) of the algebra to be

\[
| \Delta_{[k]} \rangle = V_{[k]} | \emptyset \rangle ,
\]  

(3.28)

where \( V_{[k]} \) is a primary field with respect to \( T \). In analogy with the non-twisted sector we obtain

\[
L_0^{(0)} | \Delta_{[k]} \rangle = \Delta_{[k]} | \Delta_{[k]} \rangle ,
\]  

(3.29a)

\[
L_n^{(p)} | \Delta_{[k]} \rangle = 0 , \quad n \geq 1 , \quad p = 0, 1, \ldots, N - 1 .
\]  

(3.29b)

\[5\]This assumes that \( N \) is a prime number since otherwise the product \( kp \) may be divisible by \( N \); in this case, we have to take into account additional zero modes. For example, if \( N = 4 \) and \( k = p = 2 \) then \( \{ \frac{k}{N} \} = 0 \) and the primary state in the sector \( k = 2 \) should carry an additional weight. In the following, for simplicity, we shall always assume that \( N \) is a prime number. The generalization to composite numbers is straightforward.
Thus, the Verma module over the algebra $[3.21]$ is obtained by the action of any number of $L^{(k)}_{-m-L^{(k)}_{\frac{k}{N}}}$, $k, p = 0, 1, \ldots, N - 1$, operators with $m > -\{\frac{k}{N}\}$ on the state $[3.28]$.

We notice that for a twisted primary state $|\Delta_{[k]}\rangle$, the descendent state $|\Delta'_{[k]}\rangle = L^{(k')}_{-n-L^{(k')}_{\frac{k'}{N}}}|\Delta_{[k]}\rangle$, satisfies

$$L^{(0)}_{0}|\Delta'_{[k]}\rangle = (\Delta + n + \frac{kk'}{N})|\Delta_{[k]}\rangle .$$

The tower of states in the twisted sectors is seen in the following table (we have omitted the primary state for simplicity):

| level         | $k = 1$ sector                                      | $k = 2$ sector                                      | $k = 3$ sector                                      | ... |
|---------------|----------------------------------------------------|----------------------------------------------------|----------------------------------------------------|-----|
| $\Delta + \frac{1}{N}$ | $L^{(1)}_{-1/N}$                                  | $L^{(\frac{2}{N})}_{-1/N}$                        | ...                                                | ... |
| $\Delta + \frac{2}{N}$ | $L^{(1)}_{-1/N}L^{(1)}_{-1/2N}$                   | $L^{(\frac{1}{N})}_{-1/2N}L^{(\frac{1}{N})}_{-2/3N}$ | ...                                                | ... |
| $\Delta + \frac{3}{N}$ | $L^{(1)}_{-1/N}L^{(1)}_{-1/2N}L^{(1)}_{-2/3N}$ | $L^{(\frac{1}{N})}_{-1/2N}L^{(\frac{1}{N})}_{-2/3N}$ | ...                                                | ... |
| $\Delta + \frac{4}{N}$ | $L^{(1)}_{-1/N}L^{(1)}_{-1/2N}L^{(1)}_{-2/3N}$ | $L^{(\frac{1}{N})}_{-1/2N}L^{(\frac{1}{N})}_{-2/3N}$ | ...                                                | ... |

For each sector $s$ there is a permutation $P_s$ of $(1,2,\ldots,N-1)$, $k \mapsto P_s(k)$, such that

$$\left\{\frac{P_s(k)s}{N}\right\} = \left\{\frac{k}{N}\right\} .$$

In this way, the previous table can be rewritten more compactly:

| level         | $s$ sector                                      |
|---------------|------------------------------------------------|
| $\Delta + \frac{1}{N}$ | $L^{(P_s(1))}_{-1/N}$                        |
| $\Delta + \frac{2}{N}$ | $L^{(P_s(1))}_{-1/N}L^{(P_s(1))}_{-1/2N}$ | $L^{(P_s(2))}_{-1/2N}$                          |
| $\Delta + \frac{3}{N}$ | $L^{(P_s(1))}_{-1/N}L^{(P_s(1))}_{-1/2N}L^{(P_s(1))}_{-2/3N}$ | $L^{(P_s(2))}_{-2/3N}L^{(P_s(3))}_{-2/3N}$ |
| $\Delta + \frac{4}{N}$ | $L^{(P_s(1))}_{-1/N}L^{(P_s(1))}_{-1/2N}L^{(P_s(1))}_{-2/3N}L^{(P_s(1))}_{-2/3N}$ | $L^{(P_s(2))}_{-2/3N}L^{(P_s(3))}_{-2/3N}$ |

The simplest null state in the $s$-th sector is

$$|\chi\rangle = L^{(P_s(1))}_{-1/N}|\Delta_{[s]}\rangle .$$

From the fact that $\langle \chi | \chi \rangle = 0$, we derive that

$$\Delta_{[s]} = \frac{c}{24} \left(1 - \frac{1}{N^2}\right) .$$

Notice that the weight of the lowest null state is independent of the sector. In fact, this result is generic to all null states and it is seen to be an immediate consequence of $[3.31]$ and the algebra $[3.21]$. 


This can be checked explicitly, by finding the weights the next simplest null state:

\[
|\psi\rangle = \left( L^{(P_1,1)}_{-1/N} L^{(P_1,1)}_{-1/N} + b L^{(P_2,1)}_{-2/N} \right) |\Delta_{[s]}\rangle .
\] (3.34)

After some calculations, one finds that

\[
\Delta_{[s]} = \frac{c}{24} \left( 1 - \frac{1}{N^2} \right) ,
\] (3.35a)

\[
\Delta_{[s]} = \frac{1}{16N} \left[ 5 + \frac{c}{3} (2N^2 - 5) \pm \sqrt{(1 - \frac{c}{N})(25 - \frac{c}{N})} \right] .
\] (3.35b)

Of course, one of the weights corresponds to the null state at the previous level.

### 3.3 Coulomb Gas Formalism

Using reference [12], Dotsenko and Fateev [13] gave the complete solution for the minimal model correlation functions on the sphere. They were able to write down the integral representation for the conformal blocks of the chiral vertices in terms of the correlation functions of the vertex operators of a free bosonic scalar field \( \Phi(z, \bar{z}) = \phi(z) \bar{\phi}(\bar{z}) \) coupled to a background charge \( \alpha_0 \). This construction has become known as the Coulomb Gas Formalism (CGF). In the present case, this approach is also applicable by considering a Coulomb gas for each sheet separately but coupled to the same background charge:

\[
T^{(l)}(z) = -\frac{1}{2} (\partial_z \phi^{(l)}(z))^2 + i \alpha_0 \partial_z^2 \phi^{(l)}(z) .
\] (3.36)

where

\[
\langle \phi^{(l)}(z) \phi^{(l')}(w) \rangle = -\delta^{ll'} \ln(z - w) ,
\] (3.37a)

\[
\hat{\pi}_{A_a} \partial_z \phi^{(l)} = \partial_z \phi^{(l)} , \quad \hat{\pi}_{B_a} \partial_z \phi^{(l)} = \omega^k \partial_z \phi^{(l+1)} .
\] (3.37b)

In this case \( \hat{c} = 1 - 12\alpha_0^2 \) and thus

\[
c = N \left( 1 - 12\alpha_0^2 \right) .
\] (3.38)

Passing to the basis which diagonalizes the operators \( \hat{\pi}_{A_a} , \hat{\pi}_{B_a} \), i.e.

\[
\phi_{(k)} = \sum_{l=0}^{N-1} \omega^{-kl} \phi^{(l)} ,
\] (3.39a)

\[
\langle \phi_{(k)}(z) \phi_{(k')}(w) \rangle = -N \delta_{(k+k'),0} \ln(z - w) ,
\] (3.39b)

\[
\hat{\pi}_{A_a} \partial_z \phi_{(k)} = \partial_z \phi_{(k)} , \quad \hat{\pi}_{B_a} \partial_z \phi_{(k)} = \omega^k \partial_z \phi_{(k)} ,
\] (3.39c)

we finally obtain the bosonization rule for the operators \( T_{(k)} \) in the diagonal basis

\[
T_{(k)} = -\frac{1}{2N} \sum_{s=0}^{N-1} \partial_z \phi_{(s)} \partial_z \phi_{(k-s)} + i \alpha_0 \partial_z^2 \phi_{(k)} .
\] (3.40)

In analogy with the Virasoro algebra, we can expand the fields \( \phi_{(s)} \) in modes

\[
\partial_z \phi_{(k)}(z) = \sum_{n \in \mathbb{Z}} z^{n-1+\{\frac{k}{N}\}} a_{-n-\{\frac{k}{N}\}}^{(k)} .
\] (3.41)
From the formulas written above, one can easily see that

\[ a^{(k)}_{n - \{ \frac{k}{N} \}}, a^{(k')}_{n' - \{ \frac{k'}{N} \}} = N(n - \{ \frac{k}{N} \}) \delta_{n + n' - \{ \frac{k}{N} \} - \{ \frac{k'}{N} \}, 0} \delta^{(k + k'), 0}. \]  

(3.42)

Equipped with this result, we can calculate the correlation function of two bosons in the \( l \)-th sector:

\[ \langle \partial_z \phi(l)(z) \partial_w \phi(l')(w) \rangle(l) = \delta_{(k + k'), 0} N \frac{(1 - \{ \frac{k}{N} \}) (\frac{w}{z})^{-\{ \frac{k}{N} \}} + \{ \frac{k}{N} \} (\frac{w}{z})^{1 - \{ \frac{k}{N} \}}}{(z - w)^2}. \]  

(3.43)

Using this result, we obtain the expectation value of the energy-momentum tensor on the \( l \)-th sector to be

\[ \langle T_{(0)}(z) \rangle(l) = \frac{\sum_{k=0}^{N-1} \left( \{ \frac{k}{N} \} - \{ \frac{k}{N} \} \right)^2}{4} \frac{1}{z^2} = \frac{N^2 - 1}{24N} \frac{1}{z^2}. \]  

(3.44)

To simulate the properties of the different sectors, one defines the twist fields \( \Sigma_i(z, \bar{z}|k) = \sigma_i(z|k) \sigma_i(\bar{z}|k), l, k = 1, 2, \ldots, N - 1 \). These fields are primary fields

\[ T_{(0)}(z) \sigma_i(w|k) = \frac{\Delta_{ik}}{(z - w)^2} \sigma_i(w|k) + \frac{\partial_w \sigma_i(w|k)}{z - w} + \text{reg}, \]  

(3.45)

and create branch cuts. More precisely, the twist field \( \sigma_i(z; k) \) react (i.e. has non-trivial OPE) with \( \partial_z \phi(k)(z) \) and \( \partial_z \phi(N-k)(z) \) only:

\[ \partial_z \phi(k)(z) \sigma_i(w|k) = (z - w)^{\left\{ \frac{k}{N} \right\} - \left\{ \frac{1}{N} \right\}} \sigma_i(w|k) + \text{reg}. \]  

(3.46)

From the formulas written above, one can easily see that

\[ \Delta_{ik} = \frac{1}{4} \left[ \{ \frac{k}{N} \} - \{ \frac{k}{N} \} \right]^2. \]  

(3.47)

In the \( k = 0 \) (non-twisted) sector, we now represent the primary fields with charges \( \alpha_{(s)}^{(k)} \) \( s = 0, 1, \ldots, N - 1 \), by

\[ V_{(0)}(z) = e^{i \sum_{s=0}^{N-1} \alpha_{(0)}^{(s)} \phi_{(s)}(z)}. \]  

(3.48)

A simple calculation gives

\[ T_{(k)}(z) V_{(0)}(w) = \frac{\Delta_{(0)}}{(z - w)^2} V_{(0)}(w) + \frac{i \sum_{s=0}^{N-1} \alpha_{(0)}^{(s-k)} \partial_w \phi_{(s)}(w)}{z - w} V_{(0)}(w) + \text{reg}, \]  

(3.49)

where

\[ \Delta_{(0)}^{(k)} \left( \alpha_{(0)}^{(s)} \right) \equiv \Delta_{(0)}^{(k)} = \frac{N}{2} \sum_{s=0}^{N-1} \alpha_{(0)}^{(s-k)} \alpha_{(0)}^{(-s)} - N \alpha_0 \alpha_{(0)}^{(-k)}. \]  

(3.50)

---

\(^6\)If \( N \) is not a prime, let \( q(N, l) \) be the G.C.D. of \( N \) and \( l \) and \( N = q N', l = q l' \). Then we observe that

\[ \frac{1}{4} \sum_{k=0}^{N-1} \left[ \frac{kl}{N} \right] \left( \frac{kl}{N} \right)^2 = \frac{q}{4} \sum_{k=0}^{N'-1} \left[ \frac{k}{N'} \right] \left( \frac{k}{N'} \right)^2 = \frac{N'^2 - q^2}{24N'^2}. \]  

9
So

$$|\Delta_{[0]}⟩ ≡ \lim_{z \to 0} V_{[0]}(z)|\emptyset⟩ .$$

(3.51)

It is easy to see that for the states $|\Delta_{[0]}⟩$ thus constructed, equations (3.23a) and (3.23b) are satisfied. We notice the following symmetry

$$\Delta_{[0]}^{(k)} \left( 2\alpha_0^0 - \alpha_{[0]}^{(s)} \right) = \Delta_{[0]}^{(s)} \left( \alpha_{[0]}^{(s)} \right) .$$

(3.52)

In the $k \neq 0$ sectors, we propose

$$V_{[k]}(z) = e^{i\alpha_{[k]}(z)} \sigma_k(z|1)\sigma_k(z|2)\ldots\sigma_k(z|N - 1) .$$

(3.53)

The product of the twist fields is implied by the form of the energy-momentum tensor

$$T_{(0)} = -\frac{1}{2N} \partial_z \phi_{(0)} \partial_z \phi_{(0)} + i\alpha_0 \partial^2_z \phi_{(0)} - \frac{1}{2N} \sum_{s \neq 0} \partial_z \phi_{(s)} \partial_z \phi_{(N-s)} .$$

(3.54)

In representation theory, the first two terms (untwisted boson) requires the vertex operator, while each additional term introduces its own twist field.

Defining

$$|\Delta_{[k]}⟩ ≡ \lim_{z \to 0} V_{[k]}(z)|\emptyset⟩ ,$$

(3.55)

equations (3.29a) and (3.29b) are satisfied with

$$\Delta_{[k]} \left( \alpha_{[k]} \right) ≡ \Delta_{[k]} = \frac{N^2}{2} \alpha_{[k]}^2 - N \alpha_0 \alpha_{[k]} + \frac{N^2 - 1}{24N} .$$

(3.56)

This weight has also the symmetry

$$\Delta_{[k]} \left( 2\alpha_0 - \alpha_{[k]} \right) = \Delta_{[k]} \left( \alpha_{[k]} \right) .$$

(3.57)

3.4 The Screening Charges

We can easily construct screening charges. In the $k = 0$ sector, a screening charge must have singular terms with all $T_{(k)}$ which can be exressed as a total derivative, i.e.

$$Q = \int \frac{dw}{2\pi i} S(w)$$

(3.58)

is a screening charge iff

$$T_{(k)}(z)S(w) = \text{const} \partial_w \left( \frac{S(w)}{z - w} \right) + \text{reg} , \quad \forall k ,$$

(3.59)

since in this case

$$[T_{(k)}(z), Q] = \int \frac{dw}{2\pi i} T_{(k)}(z)S(w) = 0 .$$

(3.60)

This will be true if

$$\sum_{s=0}^{N-1} \alpha_{[0]}^{(s-k)} \partial_w \phi_{(s)}(w) = \Delta_{[0]}^{(k)} \sum_{s=0}^{N-1} \alpha_{[0]}^{(s)} \partial_w \phi_{(s)}(w) ,$$

(3.61)
or

\[ \sum_{s=0}^{N-1} \left\{ \alpha_{(s-k)}^{(s)} - \Delta_{(0)}^{(s)} \right\} \partial_w \phi_s(w) = 0 . \]  

(3.62)

The solutions to equation (3.62) can be found easily:

\[ \Delta_{(0)}^{(s)} = \omega^{-kl} , \quad \alpha_{(s)}^{(s)} = \alpha \omega^s l , \]  

(3.63)

where \( l = 0, 1, \ldots, N - 1 \) and \( \alpha \) is a constant to be determined by (3.50). In particular, equation (3.50) reduces to the quadratic equation

\[ \frac{N^2}{2} \alpha^2 - N \alpha_0 \alpha - 1 = 0 , \]  

(3.64)

which has the two solutions

\[ \alpha_+(N) \equiv \alpha_+ = \frac{\alpha_0}{N} \pm \frac{1}{N} \sqrt{\alpha_0^2 + 2} . \]  

(3.65)

For later convenience, let us note the identities

\[ \alpha_+ \alpha_- = -\frac{2}{N^2} , \quad \alpha_+ + \alpha_- = \frac{2\alpha_0}{N} . \]  

(3.66)

Therefore we conclude that we have the following \( 2N \) screening vertices

\[ S^\pm_l(z) = e^{i\alpha_\pm \sum_{s=0}^{N-1} \omega^s l \phi_s(z)} , \]  

(3.67)

defining the charges \( Q^\pm_l \) by (3.58).

Of course, in the \( k \neq 0 \) sectors we must only screen the charge for the zeroth field; the corresponding screening charges \( Q^+_l \), \( Q^-_l \) arise essentially as products of \( Q^+_l \), \( Q^-_l \).

### 3.5 Null States

Having the screening charges we can construct many null states \([14]\). Let

\[ |\chi^\pm_l \rangle = Q^\pm_l \Delta_{(0)} (2\alpha_0 \delta^0 - \alpha^{(s)} - \alpha_\pm \omega^sl) \]

\[ = \oint_{\mathcal{C}} \frac{dz}{2\pi i} S^\pm_l(z) e^{i \sum_s [2\alpha_0 \delta^0 - \alpha^{(s)} - n\alpha_\pm \omega^sl] \phi_s(0)} |\emptyset\rangle \]

\[ = \oint_{\mathcal{C}} \frac{dz}{2\pi i} : e^{i \alpha_\pm \sum_s \omega^s l \phi_s(z)} : e^{i \sum_s [2\alpha_0 \delta^0 - \alpha^{(s)} - n\alpha_\pm \omega^sl] \phi_s(0)} : |\emptyset\rangle \]

for some closed contour \( \mathcal{C} \). We can rewrite the last integral in the form

\[ |\chi^\pm_l \rangle = \oint_{\mathcal{C}} \frac{dz}{2\pi i} e^{i \sum_s \omega^s l \phi_s(z)} : e^{2i \sum_s [2\alpha_0 \delta^0 - \alpha^{(s)} - n\alpha_\pm \omega^sl] \phi_s(0)} : |\emptyset\rangle \]

The above integral will be well defined and non-vanishing iff

\[ N\alpha_\pm [2\alpha_0 - \sum_s \alpha^{(s)} \omega^{-sl} - N\alpha_\pm] = -n_l - 1 , \]  

(3.68)

where \( n_l = 0, 1, 2, \ldots \). In particular, this condition implies

\[ |\chi^\pm_l \rangle \propto \partial_z^{n_l} : e^{i \sum_s [2\alpha_0 \delta^0 - \alpha^{(s)}] \phi_s(0)} : |\emptyset\rangle . \]  

(3.69)
This clearly shows that $|\chi^\pm\rangle$ is a descendant of the state $|\Delta_{[0]}(2\alpha_0\delta^{sl}-\alpha^{(s)})\rangle = |\Delta_{[0]}(\alpha^{(s)})\rangle$. Since it is also a highest weight state, it must be a null vector. Notice that for $n_l = 0$ we obtain the state $|\Delta_{[0]}(\alpha^{(s)})\rangle$ and therefore we shall ignore this value.

One can consider the more general case

$$
|\chi^\pm\rangle = (Q^\pm)^n|\Delta_{[0]}(2\alpha_0\delta^{sl}-\alpha^{(s)}-n\alpha_\pm\omega^{sl})\rangle
$$

$$
= \oint_{C_1} \frac{dz_1}{2\pi i} S^+_1(z_1) \oint_{C_2} \frac{dz_2}{2\pi i} S^+_2(z_2) \ldots \oint_{C_n} \frac{dz_n}{2\pi i} S^+_n(z_n) e^{i\sum_s(2\alpha_0\delta^{sl}-\alpha^{(s)}-n\alpha_\pm\omega^{sl})}[\phi^{(s)}(0)|\emptyset\rangle,
$$

where $C_1, C_2, \ldots, C_n$ are some closed contours. This integral will be now well defined and non-vanishing if

$$(n-1)\alpha_\pm^2 N^2 + N\alpha_\pm [2\alpha_0 - \sum_s\alpha^{(s)}\omega^{-sl} - nN\alpha_\pm] = -n_l - 1 ,
$$

where $n_l = 0, 1, 2, \ldots$. The integer $n$ cancels in the above equation, the solution being

$$\alpha^{(s)} = \alpha_\pm \sum_{l=0}^{N-1} \frac{1-n_l}{2}\omega^{sl}.
$$

Therefore, the most general result is

$$\alpha_{[0]}^{(s)} = \begin{bmatrix}
n_0 & m_0 \\
n_1 & m_1 \\
\ldots & \ldots \\
n_{N-1} & m_{N-1}
\end{bmatrix}
= \left(\sum_{l=0}^{N-1} \frac{1-n_l}{2}\omega^{sl}\right)\alpha_+ + \left(\sum_{l=0}^{N-1} \frac{1-m_l}{2}\omega^{sl}\right)\alpha_-,
$$

where $n_l, m_l$ take the values 1, 2, $\ldots$. The corresponding weights are

$$\Delta_{[0]}^{(s)} = \begin{bmatrix}
n_0 & m_0 \\
n_1 & m_1 \\
\ldots & \ldots \\
n_{N-1} & m_{N-1}
\end{bmatrix}
= \frac{N^2}{2} \sum_{l=0}^{N-1} \left(\frac{1-n_l}{2}\alpha_+ + \frac{1-m_l}{2}\alpha_-\right)^2 \omega^{-sl}
- N\alpha_0 \sum_{l=0}^{N-1} \left(\frac{1-n_l}{2}\alpha_+ + \frac{1-m_l}{2}\alpha_-\right) \omega^{-sl}.
$$

Formulae (3.72) and (3.73) consist Kac’s spectrum in the $k = 0$ sector. Notice that the weight $\Delta_{[0]}^{(s)}$ corresponding to the actual energy-momentum tensor is real, while the rest weights are in general complex. However, they satisfy the expected condition

$$\Delta_{[0]}^{(s)*} = \Delta_{[0]}^{(N-s)}
$$

Similarly, one can check that the most general result for the null states in the $k \neq 0$ sectors is given by

$$\alpha_{[k]}[n,m] = \frac{N-n}{2}\alpha_+ + \frac{N-m}{2}\alpha_-,
$$

where $n, m$ take the values 1, 2, $\ldots$. The corresponding weights are

$$\Delta_{[k]}[n,m] = \frac{N}{8} \left[(n\alpha_+ + m\alpha_-)^2 - N^2(\alpha_+ + \alpha_-)^2\right] + \frac{N^2-1}{24N}.
$$

Formulae (3.75) and (3.76) consist Kac’s spectrum in the $k \neq 0$ sector.
3.6 Correlation Functions and Fusion Rules

The $L$-leg correlation function

$$\langle \Phi_1 [n_1 m_1] \Phi_2 [n_2 m_2] \ldots \Phi_L [n_L m_L] \rangle$$  \hspace{1cm} (3.77)

of the $L$-fields in the $k = 0$ sector will be non-vanishing if there are natural numbers $M_l, N_l$ such that the correlation function

$$\langle V_{[0]}(\bar{\alpha}_1)V_{[0]}(\bar{\alpha}_2)\ldots V_{[0]}(\bar{\alpha}_L)\prod_l (Q_i^+)^{M_l} \prod_l (Q_i^-)^{N_l} \rangle$$ \hspace{1cm} (3.78)

is non-vanishing.

The non-vanishing 3-leg correlation functions will provide the fusion rules. In particular, the field $\Phi [n m]$ will appear in the OPE of $\Phi [n' m']$ and $\Phi [n'' m'']$ under the conditions the functions

$$(3.79a) \quad \langle V_{[0]}(\bar{\alpha}^*)V_{[0]}(\bar{\alpha}')V_{[0]}(\bar{\alpha}'')\prod_l (Q_i^+)^{M_l} \prod_l (Q_i^-)^{N_l} \rangle$$

$$(3.79b) \quad \langle V_{[0]}(\bar{\alpha})V_{[0]}(\bar{\alpha}'')V_{[0]}(\bar{\alpha}''')\prod_l (Q_i^+)^{M_l} \prod_l (Q_i^-)^{N_l} \rangle$$

$$(3.79c) \quad \langle V_{[0]}(\bar{\alpha})V_{[0]}(\bar{\alpha})V_{[0]}(\bar{\alpha}''')\prod_l (Q_i^+)^{M_l} \prod_l (Q_i^-)^{N_l} \rangle$$

are non-vanishing. For $(3.79a)$, charge conservation implies that

$$0 = \sum_l \omega^{st} \left[ \alpha_+ \left( N_l + \frac{1 - n_l' - n_l'' + n_l}{2} \right) + \left( M_l + \frac{1 - m_l' - m_l'' + m_l}{2} \right) \right] .$$ \hspace{1cm} (3.80)

From these equations, $s = 0, 1, \ldots, N - 1$, we find

$$N_l + \frac{1 - n_l' - n_l'' + n_l}{2} = 0 ,$$ \hspace{1cm} (3.81a)

$$M_l + \frac{1 - m_l' - m_l'' + m_l}{2} = 0 .$$ \hspace{1cm} (3.81b)

Equivalently

$$n_l \leq n_l' + n_l'' - 1 , \quad -n_l' - n_l'' + n_l = \text{odd} ,$$ \hspace{1cm} (3.82a)

$$m_l \leq m_l' + m_l'' - 1 , \quad -m_l' - m_l'' + m_l = \text{odd} .$$ \hspace{1cm} (3.82b)

In the same way, equation $(3.79b)$ gives

$$n_l' \leq n_l + n_l'' - 1 , \quad +n_l' - n_l'' - n_l = \text{odd} ,$$ \hspace{1cm} (3.83a)

$$m_l' \leq m_l + m_l'' - 1 , \quad +m_l' - m_l'' - m_l = \text{odd} .$$ \hspace{1cm} (3.83b)

and equation $(3.79c)$

$$n_l'' \leq n_l' + n_l - 1 , \quad -n_l' + n_l'' - n_l = \text{odd} ,$$ \hspace{1cm} (3.84a)

$$m_l'' \leq m_l' + m_l - 1 , \quad -m_l' + m_l'' - m_l = \text{odd} .$$ \hspace{1cm} (3.84b)
Taking into account all previous results, the OPEs are

$$\Phi [n' m'] \Phi [n'' m''] = \sum_{n_0 = |n_0' - n_0''| + 1}^{n_0' + n_0'' - 1} \cdots \sum_{m_0' + m_0'' - 1}^{n_N' - n_{N-1}' + 1} \sum_{n_0 + n_0' + n_0'' = \text{odd}} \cdots \sum_{m_0 + m_0' + m_0'' = \text{odd}} \Phi [n m] . \quad (3.85)$$

Now, let us introduce the symbol

$$\sigma_{(k)}(z) = \sigma_k(z|1)\sigma_k(z|2) \cdots \sigma_k(z|N - 1) . \quad (3.86)$$

These fields satisfy the following algebra:

$$\sigma_{(k)}(z)\sigma_{(l)}(w) = \frac{\sigma_{(k+l)}(z)}{(z - w)\Delta} + \cdots , \quad k + l \neq N , \quad (3.87a)$$

$$\sigma_{(k)}(z)\sigma_{(N-k)}(w) = \sum C_{\vec{p}}(z - w)^{\nu} e^{i\vec{p}\vec{\phi}(w)} . \quad (3.87b)$$

Using the screening charges, one can now write down the fusion rules among any fields in the theory.

### 3.7 Minimal Models

In the case

$$\alpha_+ = \sqrt{2} N \sqrt{\frac{p}{q}} , \quad \alpha_- = -\sqrt{2} N \sqrt{\frac{q}{p}} , \quad (3.88)$$

where $q, p$ are two relatively prime natural numbers, we notice the following symmetry

$$\Delta^{(k)}_{[0]} \begin{bmatrix} n_0 & m_0 \\ n_1 & m_1 \\ \vdots & \vdots \\ n_{N-1} & m_{N-1} \end{bmatrix} = \Delta^{(k)}_{[0]} \begin{bmatrix} q - n_0 & p - m_0 \\ n_1 & m_1 \\ \vdots & \vdots \\ n_{N-1} & m_{N-1} \end{bmatrix} = \cdots \Delta^{(k)}_{[0]} \begin{bmatrix} n_0 & m_0 \\ n_1 & m_1 \\ \vdots & \vdots \\ q - n_{N-1} & p - m_{N-1} \end{bmatrix} ,$$

in the untwisted sector and

$$\Delta^{(k)}_{[s]} [n m] = \Delta^{(k)}_{[s]} [q - n p - m] ,$$

in the twisted sectors. Therefore the corresponding fields have to be identified. In this case the OPEs close with a finite number of fields

$$1 \leq n_l \leq q , \quad 1 \leq m_l \leq p , \quad s = 0 , \quad (3.89a)$$

$$1 \leq n \leq q , \quad 1 \leq m \leq p , \quad s \neq 0 . \quad (3.89b)$$

We call the corresponding models the minimal models. Among them, there is a series of models $q = p + 1$ which is also unitary. In this case, the central charge is

$$c = N \left( 1 - \frac{6}{p(p + 1)} \right) . \quad (3.90)$$
Unitary representations can be constructed using the coset construction \[15\] (also see the recent review see \[16\]) based on the group
\[G = SU(2) \times SU(2) \times \ldots \times SU(2),\]
\[
\hat{G}_{p-2} \times \hat{G}_1 \quad (3.91)
\]
\[\hat{G}_{p-1}\]

at the indicated levels (for \(SU(2)\)), one recovers the central charge given by equation (3.90).

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