CLIQUES AND CONSTRUCTORS IN “HATS” GAME. I

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The following general variant of deterministic “Hats” game is analyzed. Several sages wearing colored hats occupy the vertices of a graph, the kth sage can have hats of one of h(k) colors. Each sage tries to guess the color of his own hat merely on the basis of observing the hats of his neighbors without exchanging any information. A predetermined guessing strategy is winning if it guarantees at least one correct individual guess for every assignment of colors.

For complete graphs and cycles, the problem of describing the function h(k) for which the sages win is solved in the present paper. A “theory of constructors,” i.e., a collection of theorems demonstrating how one can construct new graphs for which the sages win is developed. A new game “Rook check” equivalent to the Hats game on a 4-cycle is introduced and completely analyzed.

Bibliography: 11 titles.

1. Introduction

The “Hats” game is an interesting mathematical puzzle attracting attention of many mathematicians for many years. In the classical version of the problem, there is a set of n ≥ 2 players (sages) and adversary who puts a hat of one of n colors on the head of each sage. Each sage sees the hats of the other sages, but does not see his own hat. Taking into account this information only, he tries to guess the color of the hat he is wearing. The goal of the sages is to guarantee that at least one of them guesses the color of his hat correctly whatever the hat arrangement is. The players are allowed to discuss and fix a strategy before the hat assignment. After that any communication is prohibited. When the sages simultaneously say their guesses, the winning condition is checked (is it true that at least one of the sages guesses is correct). The problem is “whether the sages can guarantee a win?”

The answer to the above problem is “Yes!”. It can be justified gracefully. Let us enumerate the sages and identify the colors of hats with integers modulo n. Every sage sees every hat except his own. Let the ith sage check the hypothesis that the sum of all colors, including his own, equals i modulo n and say the corresponding remainder. It is clear that the hypothesis of exactly one of the sage is true, regardless of the hat arrangement. Thus this sage guesses correctly the color of his hat.

A natural generalization of this problem is a game in which every sage can see only some part of the others. Formally, let the sages be located in the vertices of some graph ("visibility graph"), the sage i can see a color of the hat of the sage j if and only if there is edge (i, j) in the graph. This generalization was introduced in [1] and further was studied in a number of papers [2, 3, 4]. For example, the connection of this “Hats” game with dynamical systems and coding theory was analyzed in [3]. M. Farnik defines HG(G) in his PhD thesis [5] as the maximal number of hat colors, for which the sages can guarantee a win. He estimated HG(G) in terms of the maximum degree and chromatic number of the graph G. In [6], N. Alon et al. studied HG(G) for some classes of graphs, using mainly probabilistic methods. The connection between HG(G) and other graph parameters was considered by Bosek et al. in [7].

W. Szczechla obtained in [4] a complicated result that in case of three colors, the sages can win on cycles with n vertices if and only if n is divisible by 3 or n = 4. A complete list of graphs on which sages win in case of three colors can be found in [8].

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Translated from Zapiski Nauchnykh Seminarov POMI, Vol. 488, 2019, pp. 66–96. Original article submitted November 21, 2019.

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In addition to the above, a lot of other variants of the “Hats” game were considered. For example, M. Krzywkowski described in [9] 36 variants of the game rules and most of them are probabilistic. Description of important results and applications of this game can be found in the same paper.

In the present paper we consider a modification of the classical deterministic game on a graph, in which the sages have different number of possible hat colors. This modification is not only of its own interest but allows one to find more simple strategies in the classical game where the number of colors is constant.

We introduce the following notation.

\( G = \langle V, E \rangle \) is a visibility graph, i.e., a graph at the vertices of which the sages are located. We often identify the sages and vertices of \( G \).

\( h: V \to \mathbb{N} \) is a hat function, or “hatness” for short; \( h(v) \) is the number of possible colors for the hat of sage \( v \). For the sage \( A \in V \), the value \( h(A) \) is called the hatness of \( A \). We assume that the list of colors using in the game is known in advance, and the color of the hat of \( A \) is one of the first \( h(A) \) colors in this list. We will often identify the set of possible hat colors of \( A \) with the set of integers modulo \( h(A) \).

**Definition.** The “Hats” game is a pair \( \mathcal{H}G = \langle G, h \rangle \), where \( G \) is a visibility graph and \( h \) is a hat function. The sages are located at the vertices of \( G \) and participate in a test. During the test, every sage \( v \) gets a hat of one of \( h(v) \) colors. The sages try to guess the colors of their own hats. If for each hats arrangement at least one of guesses is correct, we say that the sages win or the game is winning, we call the graph in this case also winning; note that this property depends also on the hat function. A game in which the sages have no winning strategy is said to be losing.

A game \( \mathcal{H}G_1 = \langle G_1, h_1 \rangle \) is a subgame of the game \( \mathcal{H}G = \langle G, h \rangle \) if \( G_1 \) is a subgraph of the graph \( G \) and \( h_1 = h \big|_{V(G_1)} \).

When the adversary puts hats on the heads of all sages, i.e., assigns a possible hat color to every sage, we obtain hats arrangement. Formally, every hats arrangement is a function \( \varphi: V(G) \to \mathbb{Z} \), where \( 0 \leq \varphi(v) \leq h(v) - 1 \) for all \( v \in V(G) \).

We use standard notations of graph theory: \( C_n \) is an \( n \)-vertex cycle graph, \( P_n \) is an \( n \)-vertex path, \( P_n(AB) \) is an \( n \)-vertex path with ends \( A \) and \( B \), \( K_n \) is a complete graph with \( n \) vertices, \( N(v) \) or \( N_G(v) \) is a set of neighbors of the vertex \( v \) in the graph \( G \).

Denote by \( G^A \) the graph, in which one of the vertices is \( A \). This notation, is used to emphasize that graphs under consideration share common vertex \( A \).

We denote by \( \langle G, k \rangle \) the game on the graph \( G \) with constant hat function which is equal to \( k \). For example, the classical game described in the first paragraph is in these terms, just \( \langle K_n, n \rangle \).

In the second section, we consider the “Hats” game on complete and “almost complete” graphs. The main result here is Theorem 2.1.

In the third section, we develop “theory of constructors” which is a set of theorems that allow one to construct new winning graphs from existing ones.

In the fourth section, we develop a new elegant approach to the “Hats” game. We describe a new game “Rook check” which is, in fact, equivalent to the “Hats” game on a 4-cycle. It expands the arsenal of combinatorial tools for constructing strategies. We present a complete research of the game “Rook check” and discuss some of its variations.

In the fifth section, we analyze the “Hats” game on cycles with arbitrary hat functions.
2. “Hats” game on complete graphs

In this section, we describe the game on a complete graph with vertices \( A_1, A_2, \ldots, A_n \) and arbitrary hat function \( h \). Let \( a_i = h(A_i) \). The following theorem completely solves the problem of which hat functions on a complete graph the sages win.

**Theorem 2.1.** Let the hatnesses of \( n \) sages located in vertices of complete graph be \( a_1, a_2, \ldots, a_n \). Then the sages win if and only if

\[
\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} \geq 1. \tag{1}
\]

**Proof.** The necessity of condition (1) is obvious: for each strategy of the sages, the \( i \)th sage guesses correctly his own color in this hats arrangement. For each sage \( i \), we split the set of all hats arrangements into subsets of \( a_i \) elements in the following way. Delete the color \( c_i \) of the \( i \)th sage from every hats arrangement. For the remaining set \( c = (c_1, \ldots, c_i-1, \bar{c}_i, c_{i+1}, \ldots, c_n) \) (the symbol “bar” means that this color is omitted), put

\[ A^i_c = \{ (c_1, \ldots, c_{i-1}, \ell, c_{i+1}, \ldots, c_n) \mid 0 \leq \ell \leq a_i - 1 \}. \]

The set \( A^i_c \) consists of “potentially possible” hats arrangements from the point of view of the \( i \)th sage: he sees that the colors of the other sages form a set \( c \) and mentally appends to it all possible colors \( \ell \) of his own hat. Bearing in mind the application of Hall’s theorem, we associate the sets \( A^i_c \) with “girls” and hats arrangements with “boys”. The boy \( s \) and girl \( A^i_c \) know each other if the hats arrangement \( s \) is an element of \( A^i_c \). Every boy knows \( n \) girls, and for each \( i \) every boy knows exactly one girl of type \( A^i_c \). Every girl \( A^i_c \) knows exactly \( a_i \) boys.

Let us prove that there exists a matching sending each boy to a girl. It suffices to check the theorem condition, i.e., every \( m \) boys know together at least \( m \) girls. Consider an arbitrary set of \( m \) boys. Since for each \( i \), the girl \( A^i_c \) knows exactly \( a_i \) boys, any \( m \) boys know in total at least \( \frac{m}{a_i} \) girls of type \( A^i_c \) for each \( i \). Summing over \( i \), we find that the total number of girls familiar with these \( m \) boys is at least

\[ \frac{m}{a_1} + \frac{m}{a_2} + \ldots + \frac{m}{a_n} \geq m. \]

This shows that the condition of the Hall theorem is satisfied.

Thus, there exists a matching that assigns to each hats arrangement a set of type \( A^i_c \). Note that if \( \frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} = 1 \), then this matching selects, in fact, one element in each \( A^i_c \). Otherwise, if \( \frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} > 1 \), then “there are lonely girls”, i.e., no elements are selected in some sets \( A^i_c \).

The constructed matching allows to define a strategy for the sages. Let the \( j \)th sage act by the rule: looking at hats of the other sages, i.e., at the set of colors

\[ c = (c_1, \ldots, c_{j-1}, c_{j+1}, \ldots, c_n), \]

he reconstructs the set \( A^j_c \) which, in fact, consists of all possible ways to supplement the set \( c \) to the hats arrangement on the whole graph. The sage should say the color marked in set \( A^j_c \) by our matching (if there is no marked element, he says the color arbitrarily). Since each hats arrangement is mapped by our matching to the selected element of one of sets \( A^i_c \), the \( i \)th sage guesses correctly his own color in this hats arrangement. \( \square \)
Corollary 2.1.1. Let $G$ be an almost complete graph obtained from a complete graph $K_n$ with vertices $A_1, A_2, \ldots, A_n$ by removing the edge $A_{n-1}A_n$. Let the $i$th sage get hat of one of $a_i$ colors. If the graph $G$ is winning, then

$$\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} - \frac{1}{a_{n-1}a_n} \geq 1. \quad (2)$$

Proof. The fraction of total number of the arrangements for which $A_{n-1}$ or $A_n$ guesses correctly, is equal to

$$\frac{1}{a_{n-1}} + \frac{1}{a_n} - \frac{1}{a_{n-1}a_n}.$$

Indeed, let us fix hat colors for the sages $A_1, \ldots, A_{n-2}$. Then the answers of $A_{n-1}$ and $A_n$ are determined by the strategy. It is not difficult to see that there are exactly $a_{n-1} + a_n - 1$ hats arrangements among $a_{n-1}a_n$ possible arrangements for $A_{n-1}$ and $A_n$, where either $A_{n-1}$ or $A_n$ (possibly both) guesses correctly. As for the other sages, each sage $A_k$ guesses correctly on $\frac{1}{a_k}$ fraction of all arrangements. So if inequality (2) does not hold, then there exists a hats arrangement, where no one guesses correctly. \qed

In paper [8], the authors explain how to reduce the problem of finding winning strategies for sages on graphs (generally speaking, this problem is very cumbersome) to SAT (the Boolean satisfiability problem). This makes it possible to study winning strategies by computer for small graphs quite efficiently.

Remark. Inequality (2) is not sufficient for sages to win on almost complete graphs. For $n = 4$, an almost complete graph with hat function $a_1 = 3$, $a_2 = 6$, $a_3 = 3$, $a_4 = 4$ (the edge $A_3A_4$ is removed) is losing, though it satisfies inequality (2) (in this case, equality holds).

Definition. The strategy of sages is said to be precise if for each hats arrangement exactly one of sage guesses is correct.

Corollary 2.1.2. Precise strategies exist if and only if the visibility graph is complete and the hat function satisfies the equality

$$\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} = 1. \quad (3)$$

Proof. Let the sages act according to some strategy. If the graph contains two nonadjacent vertices $A$ and $B$, then we put arbitrary hats to all sages except for $A$ and $B$. Now the answers of $A$ and $B$ are determined by the strategy. Let us give them those hats for which their guesses are correct. With this hats arrangement, the sages $A$, $B$ and, possibly, someone else, guess correctly. Therefore, the strategy is not precise. The fact that the existing of precise strategy on complete graph is equivalent to equality (3) follows from the proof of Theorem 2.1. \qed

Finally, we note one more amazing corollary of Theorem 2.1. We ask what is the maximum number of hats given to a sage in a winning graph on $n$ vertices? To make the question meaningful we require that the hat function makes the graph simple, i.e., for each its subgraph the sages do not win on this subgraph. Obviously, it suffices to find the maximum number for complete graphs.

Thus the question is equivalent to the following number-theoretical combinatorial problem: given $n$, find $\max(a_1, a_2, \ldots, a_n)$, where the positive integers $a_i$ satisfy relation (1). The solution of this problem is known, namely, this maximum is determined by Sylvester’s sequence $(s_n)$:

$$s_0 = 2, \quad s_n = 1 + \prod_{i=0}^{n-1} s_i,$$
and \( \max(a_1, a_2, \ldots, a_n) = s_n - 1 \). The proof can be found in [10].

Sylvester’s sequence grows very quickly, for example, \( s_8 \) is a 27-digit number. Thus if 8 sages are going to win in the Hats game on the complete graph, then one of them can be given a 27-digit number of hats! In recreational mathematics the phrases “number 8” and “huge numbers” are associated with the story of the inventor of chess, who asked to be given \( 2^{64} - 1 \) wheat grains as reward. The number \( 2^{64} - 1 \) has “only” 21 digits. In fairness, we note that both sequences grow as \( C^{2^n} \), where \( C \) is a constant.

3. Constructors

In this section we describe several constructors. Each constructor is a theorem providing a tool which allows to construct new winning games by combining several graphs in a new graph.

3.1. Product

**Definition.** Let \( G_1 = \langle V_1, E_1 \rangle \) and \( G_2 = \langle V_2, E_2 \rangle \) be two graphs sharing a common vertex \( A \). A sum of \( G_1, G_2 \) with respect to \( A \) is the graph \( \langle V_1 \cup V_2, E_1 \cup E_2 \rangle \). The sum is denoted by \( G_1 +_\Delta G_2 \).

Let \( \mathcal{H}G_1 = \langle G_1, h_1 \rangle \) and \( \mathcal{H}G_2 = \langle G_2, h_2 \rangle \) be two games such that \( V_1 \cap V_2 = \{A\} \). The game \( \mathcal{H}G = \langle G_1 +_\Delta G_2, h \rangle \), where \( h(v) = h_i(v) \) for \( v \in V(G_i) \setminus \{A\} \) and \( h(A) = h_1(A) \cdot h_2(A) \) (Fig. 1), is called a product of \( \mathcal{H}G_1 \) and \( \mathcal{H}G_2 \) with respect to \( A \). This product is denoted by \( \mathcal{H}G_1 \times_A \mathcal{H}G_2 \).

In such constructions, it is convenient to define the color of the vertex \( A \) as a pair \((c_1, c_2)\), where \( 0 \leq c_1 \leq h_1(A) - 1 \) and \( 0 \leq c_2 \leq h_2(A) - 1 \). In this case, we say that \( A \) has composite color.

![Fig. 1. Product of games.](image)

**Theorem 3.1** (on game products). Let \( \mathcal{H}G_1 = \langle G_1^A, h_1 \rangle \) and \( \mathcal{H}G_2 = \langle G_2^A, h_2 \rangle \) be two games such that \( V(G_1) \cap V(G_2) = \{A\} \). If the sages win in \( \mathcal{H}G_1 \) and \( \mathcal{H}G_2 \), then they also win in \( \mathcal{H}G = \mathcal{H}G_1 \times_A \mathcal{H}G_2 \).

**Proof.** Let the hat of the sage \( A \) have composite color \((c_1, c_2)\), where \( c_i \) is the hat color of \( A \) in \( \mathcal{H}G_i \). We fix winning strategies for \( \mathcal{H}G_1 \) and \( \mathcal{H}G_2 \) and construct strategy for the game \( \mathcal{H}G_1 \times_A \mathcal{H}G_2 \). Let all the sages in the graph \( G_i \setminus \{A\} \) play according to the winning strategy for the game \( \mathcal{H}G_i \) (the neighbors of \( A \) in \( G_i \) look only at the component \( c_i \) of the composite color of \( A \)). The sage \( A \) plays in accordance with both strategies by giving composite answer \((c_1, c_2)\), where \( c_i \) \((i = 1, 2)\) corresponds to his winning strategy for the game \( \mathcal{H}G_i \) (for calculating the answer, the sage \( A \) looks only on his neighbors in the graph \( G_i \)).

The presented strategy is winning, because either someone from \( G_1 \setminus \{A\} \) or \( G_2 \setminus \{A\} \) guesses correctly, or \( A \) guesses correctly both components of his color. \(\square\)

**Corollary 3.1.1.** Let \( G \) be a tree. The sages win in the game \( \langle G, h \rangle \), where \( h(v) = 2^{\deg(v)} \).

**Proof.** The sages win in the classical game \( \langle P_2, 2 \rangle \). Multiplying \( |E(G)| \) copies of this game, we get the required result. \(\square\)
Corollary 3.1.1 was proved in [7, Theorem 11] by induction.

In the sequel, we use the following notation for a hat function taking a constant value on the whole graph except for several vertices. Let $A$, $B$, $C$ be some vertices of the graph. The notation $h_{A^2B^2C^3}$ represents a hat function, for which $h(A) = 2$, $h(B) = 2$, $h(C) = 3$ (superscript), and $h(V) = 4$ for all other $v \in V(G)$ (subscript).

The following corollary is an important special case of the previous one.

**Corollary 3.1.2.** The sages win in the game $\langle P_4(AB), h_{A^2B^2} \rangle$.

We note, by the way, that together with Theorem 3.1 this corollary is a strengthened analog of a lemma on “pushing a hint” [8, Lemma 10]. Namely, if we consider the hatness 2 of the vertex $A$ as a hint which bounds the number of colors for the sage $A$ (there should be 4 colors, but we simplify the game for this sage), then we can “push” this hatness 2 along the path $AB$, where all the other sages have hatness 4. As a result, we see that in the graph $G_1 + P_4(AB)$ this hatness 2 “moves” from the vertex $A$ to the vertex $B$.

3.2. Substitution. The following constructor removes a vertex of a graph $G_1$ and put a graph $G_2$ on its place.

**Definition.** Let $G_1$ and $G_2$ be two graphs without common vertices. A substitution of $G_2$ into $G_1$ in place of a vertex $v$ is defined to be the graph $(G_1 \setminus \{v\}) \cup G_2$ with additional edges connecting each vertex of $G_2$ with each neighbor of $v$, see Fig. 2. We denote the substitution by $G_1[v := G_2]$.

![Fig. 2. A substitution.](image)

**Theorem 3.2.** Let the sages win in the games $\mathcal{H}G_1 = \langle G_1, h_1 \rangle$ and $\mathcal{H}G_2 = \langle G_2, h_2 \rangle$. Let $G$ be the graph of the substitution $G_1[v := G_2]$, where $v \in G_1$ is an arbitrary vertex. Then the game $\mathcal{H}G = \langle G, h \rangle$ is winning, where

$$h(u) = \begin{cases} h_1(u) & \text{if } u \in G_1, \\ h_2(u) \cdot h_1(v) & \text{if } u \in G_2. \end{cases}$$

**Proof.** Let $f_1$ and $f_2$ be winning strategies in the games $\mathcal{H}G_1$ and $\mathcal{H}G_2$, respectively.

Let each sage $u$ of the subgraph $G_2$ of $G$ get a hat of composite color $(c_1, c_2)$, where $0 \leq c_1 \leq h_1(v) - 1$ and $0 \leq c_2 \leq h_2(u) - 1$. These sages can calculate the coordinates of their composite colors independently: the sage $v$ finds the colors $c_1$ and $c_2$ by using strategies $f_2(v)$ and $f_1(v)$, respectively. In particular, this means that all the sages of $G_2$ say composite colors with the same first component.

Those of the other sages of $G$, who are not the neighbors of $v$, play in accordance with the strategy $f_1$. After the substitution, the sages of $G_1$, who are neighbors of $v$, see that instead of one neighbor $v$ they have now $|V_2|$ neighbors (and, generally speaking, with different hat colors). These sages do as follows. They see all the hats of the sages of $G_2$ and know their strategies. Therefore, they understand which of the sages of $G_2$ guesses the second coordinate of his color. Denote this player by $w$ (if there are several winners, then they choose, for example, the first winner in the pre-compiled list). Then each former neighbor of $v$ looks...
only at \( w \), more precisely, at the first component of \( w \)’s color, and plays in accordance with strategy \( f_1 \).

As a result, either someone from the subgraph \( G_1 \setminus \{v\} \) guesses correctly, or \( w \) guesses both components of his color correctly.

\[ \square \]

**Corollary 3.2.1.** The sages win in the games shown in Fig. 3.

**Proof.** We apply Theorem 3.2 to the games \( \mathcal{G}_1 = \langle P_2, 2 \rangle \) and \( \mathcal{G}_2 = \langle P_n(AB), h_4^{A2B2} \rangle \).

Note, that the win of the sages of the first graph (Fig. 3) also follows from Theorem 2.1.

**3.3. Attaching a vertex of hatness 2.** The following theorems-constructors allow to obtain new winning or losing graphs by adding small constructions to the existing graph.

**Theorem 3.3.** Let \( \langle G, h \rangle \) be a winning game and \( B, C \in V(G) \). Then the sages win in the game \( \langle G', h' \rangle \), where \( G' \) is the graph obtained from \( G \) by adding a vertex \( A \) and edges \( AB, AC \) (Fig. 4), and the hat function is given by formula

\[
h'(v) = \begin{cases} 2 & \text{if } v = A, \\ h(v) + 1 & \text{if } v = B \text{ or } C, \\ h(v) & \text{otherwise}. \end{cases}
\]

\[ \Rightarrow \]

**Fig. 4.** Attaching a vertex of hatness 2.

**Proof.** Let us describe a winning strategy. After using the constructor, the sages \( B \) and \( C \) have a new hat. Let \( A \) say “1” if he sees at least one hat of new color on \( B \) and \( C \); otherwise, \( A \) says “0”. If \( B \) and \( C \) see a hat of color 0 on \( A \), then they both say the new color. Thus, if \( A \)’s color is 0, then one of \( A, B, \) and \( C \) wins. If \( A \)’s color is 1, then \( B \) and \( C \) may think that they have not a new color, and therefore, can play on their graph \( G \) according with the strategies of the game \( \langle G, h \rangle \).

\[ \square \]

**Corollary 3.3.1.** Let \( G \) be a cycle \( C_n \) \((n \geq 4)\), and let \( B, A, \) and \( C \) be three consequent vertices of the cycle. Then the sages win in the game \( \langle G, h_4^{B3A2C3} \rangle \).

**Proof.** By Corollary 3.1.2, the sages win on \( P_{n-1}(CB) \) with hatnesses \( 2, 4, \ldots, 4, 2 \). Attaching \( A \) to this graph gives a winning graph by Theorem 3.3.

This corollary strengthens the lemma “on hint \( A - 1 \) for cycle” [8, Lemma 9] without any technical calculations.

The following theorem generalizes Theorem 3.3.
Theorem 3.4. Let \( \langle G_1, h_1 \rangle \), \( \langle G_2, h_2 \rangle \) be two winning games. Let \( A_1, A_2, \ldots, A_k \in V_1 \) and \( B_1, B_2, \ldots, B_m \in V_2 \). Let \( G' = \langle V', E' \rangle \) be a graph obtained by adding all the edges \( A_iB_j \) to the graph \( G_1 \cup G_2 \): \( V' = V_1 \cup V_2 \), \( E' = E_1 \cup E_2 \cup \{ A_iB_j, i = 1, \ldots, k; j = 1, \ldots, m \} \) (Fig. 5). Then the sages win in the game \( \langle G', h' \rangle \), where

\[
h'(u) = \begin{cases} 
  h_1(u) & \text{if } u \in G_1 \setminus \{ A_1, A_2, \ldots, A_k \}, \\
  h_2(u) & \text{if } u \in G_2 \setminus \{ B_1, B_2, \ldots, B_m \}, \\
  h_1(u) + 1 & \text{if } u \in \{ A_1, A_2, \ldots, A_k \}, \\
  h_2(u) + 1 & \text{if } u \in \{ B_1, B_2, \ldots, B_m \}.
\end{cases}
\]

\[\text{Fig. 5. Stitching of two graphs, } k = 2, \ m = 3.\]

Proof. As in Theorem 3.3, one new color has been added for the sages \( A_i \) and one for the sages \( B_j \) with respect to initial games. Let this color be red. For each \( i \), let \( A_i \) say that he has red hat if he sees at least one red hat on the sages \( B_j \); in the opposite case, let \( A_i \) see at his neighbors in \( G_1 \) only and play in accordance with winning strategy on \( G_1 \). For each \( j \), if the sage \( B_j \) sees at least one red hat on \( A_i \), then he sees at his neighbors in \( G_2 \) only and plays in accordance with winning strategy on \( G_2 \). In the opposite case, \( B_j \) says that he has red hat. It is easy to check that this strategy is winning. \(\square\)

The following constructor shows that if the vertices \( B \) and \( C \) in Theorem 3.3 are adjacent, then the numbers of colors for these vertices can greatly be increased.

Theorem 3.5. Let \( \langle G, h \rangle \) be a winning game, and let \( BC \) be an edge of the graph \( G \). Consider a graph \( G' = \langle V', E' \rangle \) obtained by adding a new vertex \( A \) and two new edges to \( G \): \( V' = V \cup \{ A \} \), \( E' = E \cup \{ AB, AC \} \). Then the sages win in the game \( \langle G', h' \rangle \) (see Fig. 6), where

\[
h'(v) = \begin{cases} 
  2 & \text{if } v = A, \\
  2h(v) & \text{if } v = B \text{ or } v = C, \\
  h(v) & \text{otherwise}.
\end{cases}
\]

\[\text{Fig. 6. Attaching a vertex of hatness 2 to edge } BC.\]
Proof. Let the sages $B$ and $C$ have composite colors $(c, \epsilon)$, where $c$ is a possible hat color in the game $\langle G, h \rangle$ and $\epsilon \in \{0, 1\}$. Let the sage $A$ say the color $c(A) = \epsilon_B + \epsilon_C \mod 2$. Let the sages $B$ and $C$ look at the colors of their neighbors in the graph $G$, calculate the colors $c(B), c(C)$ in accordance with their winning strategy in the game $\langle G, h \rangle$, and take these colors as the first coordinates of their composite colors. By seeing $A$’s hat as well as each other hat, the sages $B$ and $C$ can calculate the values $\epsilon_B$ and $\epsilon_C$ for which $A$ does not guess correctly; they take these bits as the second components. □

3.4. Attaching vertices of hatnesses 2 and 3, connected by an edge. Apparently, it is hard to determine whether the graph obtained by attaching a new fragment via two independent “jumpers” is winning. We are able to do this for very small fragment only.

Theorem 3.6. Let $\mathcal{HG} = \langle G, h \rangle$ be a winning game, and let $Z, C \in V$ be two vertices of the graph $G$. Consider a graph $G' = (V', E')$ obtained by adding a new path $ZABC$ to $G$: $V' = V \cup \{A, B\}$, $E' = E \cup \{ZA, AB, BC\}$ (Fig. 7). Then the sages win in the game $\mathcal{HG}' = \langle G', h' \rangle$, where

$$h'(v) = \begin{cases} 2 & \text{if } v = A, \\ 3 & \text{if } v = B, \\ 2h(v) & \text{if } v = Z, \\ h(v) + 1 & \text{if } v = C, \\ h(v) & \text{otherwise.} \end{cases}$$

Proof. Let a sage $X$ get a hat of color $c_X$. Consider the color $c_Z$ as composite: $c_Z = (\epsilon, c)$, where $\epsilon_Z \in \{0, 1\}$ and $c$ is one of the $h(Z)$ colors in the game $\mathcal{HG}$. Let us describe a winning strategy.

- If $c_B \neq 2$, then $A$ says $B$’s hat color; otherwise he says the color $\epsilon_Z$.
- The sage $B$ says “2” if he sees the hat of new color on the sage $C$. Otherwise he says $1 - c_A$.
- If $c_B \neq 2$, then $C$ says new color; otherwise $C$ uses strategy of the game $\mathcal{HG}$.
- Let $Z$ take the first bit $\epsilon_Z \neq c_A$ of his color and find the second component of his color in accordance with his strategy in $\mathcal{HG}$.
- The sages in $V(G) \setminus \{C, Z\}$ use strategy of $\mathcal{HG}$. It is assumed here that the neighbors of $Z$ see at the second component of $Z$’s color only and the neighbors of $C$ do not distinguish the new color and 0th color.

Now we consider all variants for the pairs $(c_A, c_B)$ and check that the strategy is winning.

In the cases $(0, 0)$ and $(1, 1)$, $A$ guesses correctly.

In the cases $(0, 1)$ and $(1, 0)$, $B$ or $C$ guesses correctly.

In the cases $(0, 2)$ and $(1, 2)$, $A$ guesses correctly if $c_A = \epsilon_Z$, and $B$ guesses correctly if $C$ has hat of new color. In the remaining cases, the sages on $G$ use the strategy of $\mathcal{HG}$, and one of them guesses correctly. □

Fig. 7. Adding a new path $ZABC$. 

Proof. Let the sages $B$ and $C$ have composite colors $(c, \epsilon)$, where $c$ is a possible hat color in the game $\langle G, h \rangle$ and $\epsilon \in \{0, 1\}$. Let the sage $A$ say the color $c(A) = \epsilon_B + \epsilon_C \mod 2$. Let the sages $B$ and $C$ look at the colors of their neighbors in the graph $G$, calculate the colors $c(B)$, $c(C)$ in accordance with their winning strategy in the game $\langle G, h \rangle$, and take these colors as the first coordinates of their composite colors. By seeing $A$’s hat as well as each other hat, the sages $B$ and $C$ can calculate the values $\epsilon_B$ and $\epsilon_C$ for which $A$ does not guess correctly; they take these bits as the second components. □
Corollary 3.6.1. Let $G$ be a cycle $C_n$ ($n \geq 4$), and $A$, $B$, $C$ three consequent vertices of the cycle. Then the sages win in the game $(G, h_{AB}B_3C^3)$. 

Proof. Follows from Theorem 3.6 applied to $P_{n-2}(ZC)$ with hatnesses 2, 4, ... , 4, 2 (this game is winning by Corollary 3.1.2).

3.5. Attaching a leaf of large hatness. Our latest constructor is great in that it works both for winning games and losing games. It claims that attaching to a graph a leaf with hatness greater than or equal to 2 does not affect the result of the game.

Definition. Let $HG = (G, h)$ be a game and $A \in V(G)$. The game $HG$ with hint $A^*$ is defined as follows. The sages play on the graph $G$ with hat function $h$, but during the test the adversary comes up to the sage $A$ and says the color defined by the strategy $c(A)$.

Proof. In one direction the statement is obvious: if $HG$ is winning for any two different colors ($c_1$, $c_2$), then the sages win in the game $(G, h_{AB}B_3C^3)$. We present here its proof to make the paper self contained.

Theorem 3.8. Let $HG_1 = (G_1, h_1)$, $B \in V(G_1)$, $G_2 = G_1 + P_2(AB)$, and $HG_2 = (G_2, h_2)$, where $h_2\big|_{V(G_1)} = h_1$ and $h_2(A) \geq 3$. The game $HG_1$ is winning if and only if $HG_2$ is winning.

Proof. In one direction the statement is obvious: if $HG_1$ is winning, then $HG_2$ is also winning. (The sages on the subgraph $G_1$ win.)

Now we prove that if $HG_2$ is winning, then $HG_2$ is also winning. We demonstrate that if the sages win in $HG_2$, then they can also win in $HG_1$ with hint $B^*$.

Let $f_2$ be a winning strategy for $HG_2$. In order to construct a winning strategy for the game $HG_1$ with hint $B^*$, we first define a strategy $f_2$ for the sages on $V(G_1) \setminus \{B\}$. Second, for any two different colors $(b_1, b_2)$ that can occur in the hint we define a strategy of $B$. Since $h_2(A) \geq 3$, for each pair of colors $(b_1, b_2)$, $b_1 \neq b_2$, we can find a color $a$ such that $A$ cannot say “$a$” if he sees that the $B$’s hat is of color $b_1$ or $b_2$. Let $B$, having received the hint $(b_1, b_2)$, say the color defined by the strategy $f_2$ when he sees the hat of color $a$ on $A$ and the colors of the other neighbors in $G_1$ are given by the current arrangement.

"A theory of the Hats game with hint" (where the hatnesses of each sage is equal to 3) is developed by Kokhas and Latyshev in [8]. The following lemma from [8] remains almost unchanged in the case of arbitrary hatnesses. We present here its proof to make the paper self contained.

Lemma 3.7. The hint $A^*$ does not affect the result of the “Hats” game.

Proof. Assume that the sages win with hint $A^*$. Let us fix a strategy of all sages except for $A$ in the game with hint $A^*$; we construct a strategy of $A$ so that the sages win without hints.

Assume that the adversary gives a hat of color $x$ to $A$ so that there exists a hat arrangement in which $A$ gets a hat of color $x$, his neighbors get hats of colors $u$, $v$, $w$ ... , the other sages also get hats of some colors, and no one (except for $A$) guesses correctly. We want that in this case $A$ guesses the color of his hat correctly, i.e., his strategy satisfies the requirement $f_A(u, v, w, ... ) = x$.

These requirements for different hat arrangements do not contradict each other. Indeed, if there exists another hat arrangement where the neighbors still have colors $u$, $v$, $w$ ... and $A$ gets another color $y$, then the sages cannot win with hint $A^*$, because the adversary can inform $A$ that he has a hat of color $x$ or $y$ and then choose one of the two hat arrangements for which $A$ does not guess his color correctly.

Proof. In one direction the statement is obvious: if $HG_1$ is winning, then $HG_2$ is also winning. (The sages on the subgraph $G_1$ win.)

Now we prove that if $HG_2$ is winning, then $HG_2$ is also winning. We demonstrate that if the sages win in $HG_2$, then they can also win in $HG_1$ with hint $B^*$.

Let $f_2$ be a winning strategy for $HG_2$. In order to construct a winning strategy for the game $HG_1$ with hint $B^*$, we first define a strategy $f_2$ for the sages on $V(G_1) \setminus \{B\}$. Second, for any two different colors $(b_1, b_2)$ that can occur in the hint we define a strategy of $B$. Since $h_2(A) \geq 3$, for each pair of colors $(b_1, b_2)$, $b_1 \neq b_2$, we can find a color $a$ such that $A$ cannot say “$a$” if he sees that the $B$’s hat is of color $b_1$ or $b_2$. Let $B$, having received the hint $(b_1, b_2)$, say the color defined by the strategy $f_2$ when he sees the hat of color $a$ on $A$ and the colors of the other neighbors in $G_1$ are given by the current arrangement.

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This strategy is winning in the game $\mathcal{HG}_1$ with hint $B^*$. Indeed, let the hat arrangement on $G_1$ be fixed and $B$ get a hint $(b_1, b_2)$. Consider the corresponding hat arrangement on $G_2$ (we give the hat of color $a$ to $A$). Then all the sages on $G_2$ use the strategy $f_2$ (and $A$ does not guess correctly). Therefore, someone on $G_1$ guesses correctly. Thus for the hat arrangement and the hint under consideration, the sages on $G_1$ win, and hence the sages win with hint $B^*$. Then they win in the game $\mathcal{HG}_1$ by Lemma 3.7. 

□

4. Blind chess

In this section we present a new game which is in fact a special case of the Hats game on 4-cycle. This game gives us a whole class of new games on cooperative guessing. All you need to change in the initial Hats game is the target of guessing. Here we replace the guessing of marked element in the set (i.e., a color of hat) with making a check to invisible king! In general, the sages can try to perform any actions, for which in the absence of information one hundred percent success is not guaranteed.

Rook check.

**Definition.** The game “Rook check.” Two chess players $L$ and $R$ are sitting opposite each other and there is a chessboard on the wall behind each of them. Each chess player does not see his own board (which is behind him) but sees the board of the other chess player. The referee places a black king on each of these boards. So the players see the king on another board but do not see the king in their own board. After that, each chess player, independently of the other, point to one square of his own chessboard and the referee puts a white rook on this square. If at least one of the kings is under attack by the rook (or the rook is placed on the square where the king is), then the chess players both win, otherwise they lose.

Chessboards of the players can be different and have arbitrary sizes, which are known to the players. As in the game “Hats”, the chess players determine public deterministic strategy in advance. The referee knows this strategy and plays against the chess players.

Let us explain how Rook check game relates to the Hats game. Let a graph $G$ be the 4-cycle $ABCDA$ with hat function $h$. In fact, $G$ is a complete bipartite graph $K_{2,2}$ with parts $\{A, C\}$ and $\{B, D\}$. The pair of players $A$ and $C$ is called a chess player $L$; his board has size $h(A) \times h(C)$. The pair $B$ and $D$ is called a chess player $R$; his board has size $h(B) \times h(D)$.

The hat colors of $A$ and $C$ can be interpreted as coordinates of the cell where the king is placed. Since $A$ and $C$ do not see each other, they know nothing about the king placement on their board. The pair of colors that $A$ and $C$ say can be interpreted as a cross on the chessboard, i.e., a configuration consisting of one horizontal and one vertical line, or, which is the same, a position for chess rook. It is clear that one or both chess players guess their colors if and only if the king is under the attack of the rook. Similar interpretations are valid for $B$ and $D$.

Thus the Hats game on the cycle $ABCDA$ with hat function $h$ is equivalent to the game Rook check on the boards $L(h(A) \times h(C))$ and $R(h(B) \times h(D))$. It is clear that the result of the game does not depend on which board is the left and which is the right.

Generally, we can define Rook check in the case where $n$ chess players are in the vertices of an arbitrary graph: each player has his own chessboard but sees only the boards of his neighbors (and does not see his own chessboard). The aim of the players is similar, they want at least one of kings to be under attack. This game is equivalent to the Hats game on a “doubled” graph. We do not discuss this game here.

Let us return to the game of two players on the boards $L(a \times c)$ and $R(b \times d)$. We use the following standard notations.
Let us number the cells of the board $L(a \times c)$ from left to right from top to bottom see Fig. 8a, where we use the boards $L(2 \times 3)$ and $R(3 \times 4)$ as examples. Let the strategy of the chess player $R$ be given by the table as in Fig. 8b. Put $ac$ labels $r_i$ in the cells of $R(b \times d)$ (a cell can contain several labels $r_i$), where the index $i$ runs over all numbers of the cells of $L(a \times c)$. The label $r_i$ means that $R$, seeing that his partner’s king is on the $i$th cell of $L(a \times c)$, puts the rook on the cell of $R(b \times d)$ with the label $r_i$.

The strategy of $L$ is also given with the help of $R(b \times d)$, see Fig. 8c. Here there is a number from 1 to $ac$ in each cell of $R(b \times d)$; the numbers denote the cells of $L(a \times c)$. Each cell of $R(b \times d)$ contains exactly one number, some numbers from 1 to $ac$ can be absent in this table and some numbers can repeat. When $L$ sees that $R$’s king is located on $R(3 \times 4)$ in the cell labeled $k$, he puts the rook on the $k$th cell of the board $L(a \times c)$.

To avoid misunderstandings in notations, we use labels of type “letter $r$ with index” for $R$, and of type “number” for $L$. The lines on the board $L$ are called rows and columns, whereas the lines on the board $R$ are called verticals and horizontals.

| 1 | 2 | 3 |
|---|---|---|
| 4 | 5 | 6 |

Fig. 8. Notation for strategies.

Definition. Let the king be in the $i$th cell of the board $L(a \times c)$. A cell of $L(a \times c)$ is said to be $i$-weak if the rook does not attack the king from this cell. For example, the cells 5 and 6 on the board $L(2 \times 3)$ (Fig. 8a) are 1-weak.

Lemma 4.1. Let $L(a \times c)$ and $R(b \times d)$ be the boards in the game Rook check. A strategy is winning if and only if for each $i$, $1 \leq i \leq ac$, all cells on $R(b \times d)$, labeled by the numbers of $i$-weak cells belong to the cross with center $r_i$.

Proof. Let cell $\ell$ of $L(a \times c)$ be $i$-weak and the referee put the kings on the cell $i$ of $L(a \times c)$ and the cell of $R(c \times d)$ labeled $\ell$. Then the player $L$ according to his strategy puts the rook on the cell $\ell$ of $L(a \times c)$, and it does not attack the king. In the same time, the player $R$ puts his rook on the cell of $R(c \times d)$, labeled by $r_i$. The players win if and only if this rook attacks the king, i.e., the cell labeled by $\ell$ is in the cross with center $r_i$. □

This lemma provides the following property of winning strategies: if a cell $\ell$ on $L(a \times c)$ is simultaneously $i$-weak, $j$-weak, etc., then all the cells on $R(b \times d)$ labeled by $\ell$ (if they exist) are located in the intersection of the crosses with centers $r_i$, $r_j$, etc. For example, for the strategies in Fig. 8 (we prove below that they are winning) both cells labeled by 1 on $R(3 \times 4)$ belong to the intersection of the crosses $r_5$ and $r_6$ (the shaded area in Fig. 8b), because the cells 5 and 6 on $L(2 \times 3)$ are 1-weak.

The following theorem gives a complete analysis of the game Rook check for two players. We assume that the number of verticals of each board does not exceed the number of verticals and that the left board has the shortest vertical size.

Theorem 4.2. The chess players win in the game “rook check” on the following boards:

Win1) if one of the boards has sizes $1 \times k$, where $k$ is an arbitrary positive integer;
Win2) $L(2 \times k)$ and $R(2 \times m)$, where $k$ and $m$ are arbitrary positive integers;
Win3) $L(3 \times 3)$, $R(3 \times 3)$;
Win4) $L(2 \times 3)$, $R(3 \times 4)$;

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Win5) \(L(2 \times 4), R(3 \times 3)\);
Win6) \(L(2 \times 2), R(k \times m), \text{ where } \min(k, m) \leq 4\).

The chess players lose on the following boards:
Lose1) \(L(2 \times 3), R(4 \times 4)\);
Lose2) \(L(2 \times 3), R(3 \times 5)\);
Lose3) \(L(2 \times 4), R(3 \times 4)\);
Lose4) \(L(2 \times 5), R(3 \times 3)\);
Lose5) \(L(3 \times 3), R(3 \times 4)\);
Lose6) \(L(2 \times 2), R(5 \times 5)\).

For boards of other sizes the question if the sages win can be answered by comparing with these cases. For example, the chess players lose on the boards \(L(3 \times 4)\) and \(R(3 \times 4)\), because they lose even in the “smaller” case Lose3). The chess players win on the boards \(L(2 \times 3)\) and \(R(3 \times 3)\), because they win even on larger boards (as in the case Win3)).

Proof of the theorem.

**Win1)** This statement is trivial.

**Win2)** In the “Hats” language, the hat function of two neighbor sages in the corresponding 4-cycle equals 2; these sages provide a win, even not looking at the others.

**Win3)** This statement is a retelling to the language of the game Rook check the known statement that the sages win on 4-cycle if everyone is given hats of three colors ([2, 4]). For example, the strategy of the sages, described in [11], looks in Rook check language as follows. If a chess player sees that the king of his partner is in the central cell of the board, then he puts his rook on the center too. Otherwise he puts his rook on the cell, where the arrow leading from partner’s king shows (in the auxiliary diagram for this chess player), see Fig. 9. The coordinates of cells in the figure correspond to the numbers of the hat colors. Thus, the chess player \(L\), seeing that the king of his partner is located in the cell \((2, 2)\) (this case corresponds to the bold arrow in Fig. 9 on the left).

![Strategy diagram](image)

Fig. 9. Four sages stand around non-transparent baobab...

**Win4), Win5)** The strategies presented in Fig. 8 and 10 satisfy Lemma 4.1 (direct check). So the chess players win.

![Strategy diagram](image)

Fig. 10. Winning strategy for game on \(L(2 \times 4), R(3 \times 3)\).
Win6) In the “Hats” language, this case means that the 4-cycle contains path $P_3$ with hat function 2, $x$, 2, where $x \leq 4$. The sages win on such path by Corollary 3.1.2.

Lose1) We show that the players have no winning strategy in this case.

Fix a strategy of $R$, see, for instance, Fig 11b. Let us try to understand how the strategy of $L$ looks like, namely, where can the cells with labels 1, 2, and 3 be located on $R(4 \times 4)$.

By Lemma 4.1, the cells with label 1 belong to the intersection of the crosses $r_5$ and $r_6$, the cells with label 2 belong to the intersection of the crosses $r_4$ and $r_6$, and the cells with label 3 belong to the intersection of $r_4$ and $r_5$.

![Diagram](image)

a) Cell numbers on board $L$

b) Strategy of player $R$

Fig. 11. It happens that this strategy is losing.

We note that the union of pairwise intersection of any three crosses (possibly, coinciding) on $R(4 \times 4)$ contains at most 8 cells. Indeed, let us examine the cases.

1. If the centers of the crosses belong to different verticals and horizontals, then each pairwise intersection consists of two cells (in the example in Fig. 11b, the intersection of the crosses $r_5$ and $r_6$ is shaded), so we have at most 6 cells totally.

2. If the centers of any two crosses do not coincide and two centers belong to the same horizontal or vertical (as the crosses $r_4$ and $r_5$ in Fig. 11b), then the intersection of these two crosses contains 4 cells and adding of the third cross (say, $r_6$) can give 4 more cells to the union of pairwise intersections only if the center of the third cross and one of the first two centers are on the same line (as $r_4$ and $r_6$ in Fig. 11b). In this case, we have 8 cells, and 7 of them belong to one cross (the cross $r_4$ in our example).

3. If the centers of some two crosses coincide, then the intersection of these crosses contains 7 cells. For any location of the third center the set of pairwise intersections does not increase.

Thus, for the cells with labels 1, 2, 3 on $R(4 \times 4)$, there are at most 8 positions, and similarly, for the cells with labels 4, 5, 6, there are at most 8 positions too. Since $R(4 \times 4)$ contains 16 cells, we have 8 positions for the labels 1, 2, 3, and 8 positions for the labels 4, 5, 6. But as was established by trying all possible cases 1–3, 8 positions can be realized only as a set “whole cross plus one cell”. It remains to observe that it is impossible to cover $R(4 \times 4)$ completely by two crosses and two additional cells.

Lose2) As in Lose1), we check that the union of pairwise intersections of any three crosses (possibly, coinciding) on $R(3 \times 5)$ contains at most 8 cells. The cases, in which this union contains 7 or 8 cells, are drawn in Fig. 12; these are the cases, when the centers of two crosses belong to the same vertical or the same horizontal (including the case, when both centers are in one cell). In all these cases, the union of pairwise intersections of three crosses occupies one whole horizontal of the board, and it occupies less than a half of cells in each of the two other horizontals. This means that the union of two such sets cannot cover the board completely.

Lose3) The argument below was proposed by Oleg Chemokos.

We fix some strategies of the chess players $L$ and $R$ and verify that one can find positions for the kings such that both kings avoid a check. In our standard notation, each cell $i$ on $L(2 \times 4)$ determines three $i$-weak cells (see Fig. 10a). This set of three weak cells can consist of any three cells in one row.

The strategy of $L$ is given by labeling each cell on $R(3 \times 4)$. The cells of $R(3 \times 4)$, containing the labels corresponding to the first row of $L(2 \times 4)$ are painted white, and the other cells are
painted black. Without loss of generality we may assume that the number of white cells on
the board is less than or equal to the number of black cells. The following three cases cover
all the possibilities, for which this inequality can be realized.

1. One of the horizontals of $R(3 \times 4)$ (for definiteness the first one) contains three white cells $u_1, u_2, u_3$ and one more horizontal (the second) contains two white cells $u_4$ and $u_5$. Then the first row of $L(2 \times 4)$ contains a cell $\ell$ such that the label $\ell$ occurs in the first two horizontals of $R(3 \times 4)$ at most once, and, moreover, if so, then the label $\ell$ occurs in the first horizontal, say, in the cell $u_1$. The other cells of the first row on $L(2 \times 4)$ are $(\ell + 4)$-weak, and the cells $u_2, u_3, u_4$ and $u_5$ belong to the same cross by Lemma 4.1, which is not true.

2. Each horizontal of $R(3 \times 4)$ contains two white cells. Then we choose a cell $\ell$ in the first row of $L(2 \times 4)$, such that the label $\ell$ occurs on $R(3 \times 4)$ at most once (for definiteness, in the third horizontal). The other cells in the first row of $L(2 \times 4)$ are $(\ell + 4)$-weak, and the corresponding labels in the first two horizontals of $R(3 \times 4)$ again do not lie in the same cross.

3. One horizontal contains four white cells and two other horizontals contain one white cell each. Then we replace “black” and “white” and consider the first case.

The obtained contradiction proves that the strategy is losing.

Lose4) We number the cells of the board $L$, as in Fig. 13a. The strategy of the chess player $\mathcal{L}$ is given by writing the number from 1 to 10 in each cell of $R(3 \times 3)$ (these are the numbers of cells on $L(2 \times 5)$). Since $L(2 \times 5)$ has two rows only, there exist two horizontals on $R(3 \times 3)$ and two cells in each of them, such that the four labels in these cells correspond to the cells (some of them can coincide) belonging to the same row of $L(2 \times 5)$. Let $j$th cell in the other row be $i$-weak with respect to all these cells.

For example, let the labels 1, 2, 3, 4 be located on $R(3 \times 3)$ as in Fig. 13b. Then the number 10 is 1-, 2-, 3-, and 4-weak simultaneously. This means that the rook in the cell $r_{10}$ of $R(3 \times 3)$ attacks the cells with labels 1, 2, 3, and 4. But this is impossible: to attack labels 1 and 2, it must be located in the upper row of $R(3 \times 3)$, and to attack labels 3 and 4, it must be located in the bottom row.

By the same reason the general case is also impossible the cell $r_j$ must be located in two horizontals of $R(3 \times 3)$ simultaneously.
Lose5) Assume that the chess players have a winning strategy. We number the cells of $L(3 \times 3)$ by numbers from 1 to 9. Then the strategy of $R$ is specified by a placement of nine symbols: $r_1, r_2, \ldots, r_9$ on $R(3 \times 4)$. And the strategy of $L$ is specified by writing a number from 1 to 9 in each cell of $R(3 \times 4)$.

Claim 1. If the cells $u, v, w$ belong to three different rows and three different columns of $L(3 \times 3)$, then the labels $r_u, r_v, r_w$ belong to three different horizontals of $R(3 \times 4)$.

Indeed, each cell of $L(3 \times 3)$ is either $u$-weak, or $v$-weak, or $w$-weak. By Lemma 4.1 this implies that each label on $R(3 \times 4)$ belongs to $r_u, r_v, r_w$-cross. This is possible if the labels $r_u, r_v, r_w$ are in different horizontals only.

Claim 2. There are two possible cases of the placement of symbols $r_1, r_2, \ldots, r_9$ on $R(3 \times 4)$:

1) either the symbols $r_1, r_2, r_3$ are located in one horizontal of $R(3 \times 4)$, the symbols $r_4, r_5, r_6$ are located in another horizontal, and the symbols $r_7, r_8, r_9$ are in the third one;

2) or the symbols $r_1, r_4, r_7$ are located in one horizontal of $R(3 \times 4)$, the symbols $r_2, r_5, r_8$ are located in another horizontal, and the symbols $r_3, r_6, r_9$ are in the third one.

The claim is proved by moderately nasty brute force with the help of Claim 1.

Put rooks in all cells $r_i$ of $R(3 \times 4)$ (we put in a cell as many rooks as there are symbols $r_i$ in it). Each cell $i$ of $L(3 \times 3)$ determines four $i$-weak cells which are located in two rows and two columns.

Claim 3. Each cell of $R(3 \times 4)$ (let it contain a label $i$) is under attack when the rook stands on cells labeled by $r_i$, where $j$ is an $i$-weak number. Two of this “dangerous” cells are located in the same horizontal, and the other two belong to another horizontal.

This claim follows from Claim 2 which means that we put several rooks on some cells.

Now we prove that no winning strategy with these properties exist. By Claim 2, the first horizontal of $R(3 \times 4)$ contains at most three labels $r_i$. Therefore the first horizontal of $R(3 \times 4)$ contains an “empty” cell, i.e., the cell containing no symbols $r_i$; denote it by $a$. For definiteness let it be in the fourth vertical (Fig. 14). By Claim 3, four rook attacks are directed at this cell, and two of these four rooks are in one horizontal, and another two are in another horizontal.

This means that two rooks are certainly located in one of the cells of the fourth vertical. For definiteness, let this cell be located in the second horizontal. By Claim 2, the second horizontal contains three rooks in total, and we have established that two of them are in one cell. Therefore there are two “empty” cells in the second horizontal. Let us choose the one above which there is no more than one rook stands in the first horizontal. Let this cell be in the first column; denote it $b$. There are four rook attacks from two pairs of rooks located in two rows directed to chosen call. One pair of rooks is obviously located in the second horizontal, and another pair is located in the third horizontal (there is at most one rook above the cell $b$ in the first horizontal). Now we see that one of the cells in the third horizontal, in the second or in the third vertical, cannot gather four rook attacks from two different horizontals, a contradiction.

Lose6) Assume that the chess players have a winning strategy. The strategy of $R$ is given in standard notation by placement of the four symbols $r_1, r_2, r_3, r_4$ on $R(5 \times 5)$. There is at least one cell $Q$ of $R(5 \times 5)$, not belonging to any of four crosses determined by these symbols. The strategy of $L$ is specified by writing a number from 1 to 4 in each cell of $R(5 \times 5)$. Without loss of generality, we assume that the cell $Q$ is labeled by 1. Let the referee put the kings in the cell $Q$ of $R(5 \times 5)$ and in the cell 4 of $L(2 \times 2)$. Then the player $L$ puts his rook in the cell 1 of $L(2 \times 2)$, and the player $R$ puts the rook in the cell 4 of $R(5 \times 5)$. None of the rooks attacks the king. The chess players lost.

The theorem is completely proved. □
**Queen check.** Consider a variation of the game, where the players put queens instead of rooks. Call this game *Queen check*.

**Lemma 4.3.** *The players win in Queen check game on the boards $L(4 \times 5)$ and $R(4 \times 5)$.*

*Proof.* Paint the cells of both boards as shown in Fig. 15a. Let both chess players put their queens only on cells marked with queens, and let the first chess player act under the assumption “the kings are on cells of the same color,” and the second under the assumption “the kings are on cells of different colors.” □

However, one can also use usual chess coloring instead of “exotic” coloring as above. Indeed, the queen on the cell $c_2$ holds under attack all the cells of the same color in chessboard coloring! And the same is true for the cell $c_3$, Fig. 15b.

![Fig. 15. “Queen check” on $4 \times 5$ board.](image)

We found by computer that in Queen check game, the chess players win on $L(4 \times 4)$ and $R(5 \times 5)$ and lose on $L(3 \times 4)$ and $R(7 \times 7)$, and $L(4 \times 5)$ and $R(5 \times 5)$.

The following statement was suggested to us by S. Berlov. It generalizes the argument of the previous lemma.

**Lemma 4.4.** *Let us consider a variation of Queen check game in which five chess players are located so that each of them sees the boards of all the others but does not see his own. All the boards have size $11 \times 11$. As in the initial game, the referee puts one king on each board, and the chess players simultaneously point to cells on their own boards, where the queen has to be put. The chess players win in this game.*

*Proof.* On an $11 \times 11$ board, one can place 5 queens that keep all the cells under attack (for example, $b_4$, $d_{10}$, $f_6$, $h_2$, and $j_8$). During the game, the chess players put their queens on the these 5 positions only.

We number these positions from 0 to 4. In each cell of $11 \times 11$ board, we place the number of any of these queens that holds this cell under attack. We assume that this labeling is applied to all boards. When the referee puts the king on some cell of a board, the label of this cell is called the *weight* of the king.

The strategy of the chess players is as follows: let the $k$th player check the hypothesis that the sum of the weights of all kings equals $k$ modulo 5”. Each player sees all the kings except for his own and calculates the weight of his king at which the hypothesis is correct. Then the player puts his queen on the position, which number equals the calculated weight. □

**Check with other chess figures.** In the games “Bishop check” or “Knight check” the check declaration means that the chess player guesses the color of the cell in which the king is standing. Therefore the chess players can win in these games only on small boards, where all cells of each color can be attacked from one point.

Consider the game King check (the referee puts on the board a “good king” and the chess player puts on the board an “evil king” who must put the good one in check.
Theorem 4.5. For the King check game on the boards $L(a \times b)$ and $R(c \times d)$, denote by $\ell$ the number of elements in the maximal set of cells on $L(a \times b)$, such that no two cells can be under attack of the same king. Define a number $r$ for $R(c \times d)$ analogously. Then the chess players win if and only if $\ell = r = 2$ or one of the numbers $\ell$, $r$ equals 1.

Proof. Choose sets $S_L$, $S_R$ of cells on $L(a \times b)$ and $R(c \times d)$ so that no two cells in these sets can be under attack of the same king, $|S_L| = \ell$, $|S_R| = r$. Let the referee make things easier for the chess players by promising that he will place the kings in the cells of the sets $S_L$ and $S_R$ only. Since the “evil king” cannot attack two cells simultaneously, we may assume that the chess players just try to guess where the “kind king” stands, or, which is the same, to guess hat colors in the Hats game on the graph $P_2$ with hatnesses $\ell$ and $r$, which is possible only if $\ell = r = 2$ or one of the numbers $\ell$, $r$ equals 1.

It remains to show that in these cases the chess players win. For $\ell = 1$ or $r = 1$, this is obvious. The maximal possible board for $\ell = r = 2$ is $3 \times 6$, because no two corners of a $4 \times 4$ board as well as no two of the three cells with coordinates 1, 4, 7 of a $1 \times 7$ board are attacked by the same king. On a $3 \times 6$ board, the chess players easily win by splitting the board into two halves of sizes $3 \times 3$ and checking the hypotheses “kind kings are in the same/different halves”.

5. Analysis of Hats game on a cycle

According to results of W. Szczepański [4], the sages have some difficulties in the game on cycle $C_n$ already in the case, when all the hatnesses are equal to 3. In this case, the win of the sages is possible only if $n = 4$ or $n$ is divisible by 3. If one of the sages on any cycle has hatness 4 (and all other have hatness 3), the sages lose [4, Corollary 8].

The following theorem gives the list of games on cycles containing a vertex of hatness 2, where the sages win.

Theorem 5.1. Let $G$ be a cycle $C_n$, and $h$ the hat function such that $2 \leq h(v) \leq 4$ for all vertices $v$. Let $A \in V(G)$ and $h(A) = 2$. Then the game $HG = \langle G, h \rangle$ is winning in the following cases:

1. $n = 3$;
2. there is one more vertex with hatness 2 other than $A$;
3. both neighbors of $A$ have hatness 3;
4. one neighbor of $A$ and the vertex following it are of hatness 3.

Proof. If $h_1(v) \leq h_2(v)$ for all $v \in V(G)$, then the winning in $\langle G, h_2 \rangle$ implies the winning in $\langle G, h_1 \rangle$, or, which is the same, the losing in $\langle G, h_1 \rangle$ implies the losing in $\langle G, h_2 \rangle$. This is obvious, because the winning strategy for $\langle G, h_2 \rangle$ can be used as a winning strategy for $\langle G, h_1 \rangle$, in which instead of “non-existing” colors the sages say any “existing” color. Therefore, to prove the theorem, it suffices to check the winning for the cases when the hat function is “maximal” (in the sense of definition in Sec. 3).

For each statement of the theorem, we give below the maximal hat functions and the proofs that the sages win. We recall that $h_4^{AB2}$ is the hat function which values are equal to 4 in all the vertices other than $A$ and $B$, where $h(A) = 2$ and $h(B) = 2$.

1. $C_3$ with hatnesses 2, 4, 4. The sages win by Corollary 3.2.1.
2. The game $\langle C_n, h_4^{AB2} \rangle$ is winning, because it contains a path with hatnesses 2, 4, ... , 4, 2, where the sages win by Corollary 3.1.2.
3. The game $\langle C_n, h_4^{AB2BC3} \rangle$, where $B$ and $C$ are the neighbors of $A$ is winning by Corollary 3.3.1.
The game $\langle C_n, h^{A2B3C3} \rangle$, where $A$, $B$, $C$ are three consequent vertices, is winning by Corollary 3.6.1.

\[ \square \]

**Conjecture 5.2.** Let $G$ be a cycle $C_n$ and let $h$ be a hat function such that $2 \leq h(v) \leq 4$ for every vertex $v$. Let $A \in V(G)$ be such that $h(A) = 2$. Then the game $HG = \langle G, h \rangle$ is winning only in the cases listed in Theorem 5.1.

To prove the conjecture, it suffices to prove that the following two games are losing.

1. $\langle C_n, h^{A2B4C4} \rangle$ ($n \geq 4$), where $B$ and $C$ are the neighbors of $A$. The loss in this game for $n = 4$ is proved in Theorem 4.2, in the language of Rook check game. For $n \leq 7$, the loss was checked on computer by reduction to SAT [8]. This allows us to assume that for $n \geq 8$ the game is losing too, but we have no proof for this fact.

2. $\langle G_n, h^{A2B4C3D4} \rangle$ ($n \geq 4$), where the sages $B$ and $C$ are the neighbors of $A$, and $D \neq A$ is the second neighbor of $C$. The loss of this game for $n = 4$ is proved in Theorem 4.2. For $n \leq 7$, the loss was checked by computer. This allows us to assume that for $n \geq 8$, the game is losing, but we still have no proof for this fact too.

6. Conclusion

In the present paper, we certainly prove that the variation of the Hats game in question is a real gem of combinatorics. The firework of ideas that arise when considering different approaches to the game are mesmerizing and awakens the imagination. In the same time, the computational complexity of the game prevents the advancement of up nasty hypotheses and reliably protects the game from being full studied.

Translated by the authors.

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