Quasinormal mode and spectroscopy of a Schwarzschild black hole surrounded by a cloud of strings in Rastall gravity

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Abstract

We obtain the solution of a static spherically symmetric black hole surrounded by a cloud of strings in Rastall gravity and study the influence of the string parameter $a$ on the event horizon and the Hawking temperature. Through the analysis of the black hole metric, we find that the static spherically symmetric black hole solution surrounded by a cloud of strings in Rastall gravity can be transformed into the static spherically symmetric black hole solution surrounded by quintessence in Einstein gravity when the parameter $\beta$ in Rastall gravity is positive, which provides a possibility for the string fluid to be a candidate of dark energy. We use the high order WKB-Padé approximation to calculate the quasinormal mode frequencies of the odd parity gravitational perturbation for this kind of black holes. The results show that the increase of the string parameter $a$ makes the gravitational wave decay more slowly in Rastall gravity. In addition, we use two methods, which are based on the adiabatic invariant integral and the periodic property of outgoing waves, respectively, to derive the area spectrum and entropy spectrum of the black hole model. The results are consistent with the spectroscopy of the Schwarzschild black hole given by Beckenstein, which indicates that the parameter $\beta$ and the string parameter $a$ lead to no effects on the area spectrum and entropy spectrum of the black hole model.

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1 Introduction

The gravitational waves from binary black hole mergers and neutron stars detected by LIGO and Virgo collaborations [1,2], as well as the first black hole photograph released recently, have fueled our interest in black holes. A black hole is a magic celestial body predicted by Einstein’s general relativity and is a spacetime region where the light cannot escape. An important means to obtain the basic properties of black holes is to study the quasinormal modes in the spacetime of black holes. In the past few decades, the quasinormal modes of various classical or semiclassical black holes have been studied extensively and deeply under the framework of Einstein’s theory of gravity. Up to now, the conservation law that the covariant derivative of the energy-momentum tensor in Einstein gravity is zero has only been tested [3] in the Minkowski spacetime or specifically in a weak gravitational field limit, so the covariant derivative may not be zero in a strong gravitational field. Therefore, it is especially necessary for the analysis of gravitational waves to study the quasinormal modes of black holes in the framework of other gravitational theories including the case that the covariant derivative of the energy-momentum tensor is not zero.

Rastall gravity is a gravitational theory obtained by modifying the vanishing of the covariant derivative of the energy-momentum tensor to be nonvanishing. In this gravitational theory, the covariant derivative of the energy-momentum tensor is directly proportional [4] to the derivative of the Ricci scalar, i.e. $T^\mu{}_{\nu,\mu} \propto R^{,\nu}$. In this respect, Rastall gravity can be seen [5] as a phenomenological implementation of some quantum effects in the curved spacetime background. In recent years, Rastall gravity has been applied to cosmology. It was found [6,7] that Rastall gravity theory is consistent with various observational data in the cosmological context and it gives some new and interesting results at the cosmological level. For instance, the evolution of small dark matter fluctuations is the same as that in the $\Lambda$CDM model. But in Rastall’s theory, the dark energy is clustered. This characteristic leads [8] to inhomogeneities in the evolution of dark matter in a nonlinear region, which is different from the standard CDM model.

According to string theory, the basic elements of the nature are not point particles but extended one-dimensional objects. Therefore, it is essential to understand what are the gravitational effects induced by a collection of strings. The first study of a cloud of strings as the source of the gravitational field gave [9] the exact integral expression of the general solution of the Schwarzschild black hole surrounded by a spherically symmetric string clouds under Einstein gravity, and the following work focused on [10,11] the construction of black hole solutions. So it is natural to explore the gravitational effects of strings as basic objects in a gravitational theory that goes beyond Einstein gravity. For example, the 5-dimensional and n-dimensional black holes surrounded by a cloud of strings in Lovelock gravity have been analyzed [12,13].

The investigation of black hole entropy/area quantum has an important physical meaning because it can provide [14] a window to find an effective way to quantize gravity. Beckenstein was the first to suggest [15] that the area of black holes should be quantized. That is, if the black hole horizon area was dealt with as an adiabatic invariant, the area spectrum of black holes was proved to be equidistant and quantized in units $\Delta A = 8\pi \hbar$. There are a lot of papers about the area spectrum and entropy spectrum of black holes, see, for instance, refs. [14, 16, 17, 18, 19, 20, 21, 22].

In this paper, our aim is to analyze the quasinormal modes of black holes surrounded by a cloud of
strings in Rastall gravity. Therefore, we first work out the exact solution of a static spherically symmetric black hole surrounded by a cloud of strings in Rastall gravity. Then we make a simple analysis of the characteristics of the black hole model. According to the method of Regge and Wheeler [23], we derive the quasinormal modes of the odd parity gravitational perturbation for this black hole model and calculate the corresponding quasinormal mode frequencies by using the high order WKB-Padé approximation [24, 25]. Then we focus on the influence of a cloud of strings on the real and imaginary parts of the quasinormal mode frequencies. In addition, we use the method of the adiabatic invariant integral [14] and that of the periodic property of outgoing waves [22] to derive the area spectrum and entropy spectrum of the black hole model, respectively, so as to give the influence of Rastall gravity and of a cloud of strings on the area spectrum and entropy spectrum of the black hole.

The paper is organized as follows. In Sect. 2, we first find out the exact solution of a static spherically symmetric black hole surrounded by a cloud of strings in Rastall gravity. Then, we analyze in Sect. 3 the characteristics of the black hole model for different values of the two parameters related to the Rastall gravity and string, where the case of the Einstein theory is added as a contrast. In Sect. 4, we calculate the quasinormal mode frequencies of the odd parity gravitational perturbation for the black hole model. In Sect. 5, we use the method of the adiabatic invariant integral to compute the area spectrum and entropy spectrum of the black hole model, and then utilize the method of the periodic property of outgoing waves to calculate the same quantities and obtain the same results in Sec. 6. Finally, we make a simple summary in Sect. 7. Throughout this paper, we use the units \( c = G = k_B = 1 \) and the sign convention \((+,−,−,−)\).

## 2 Schwarzschild black hole surrounded by a cloud of strings in Rastall gravity

According to Rastall gravity, the field equations take [4] the following forms,

\[
G_{\mu \nu} + \beta g_{\mu \nu}R = \kappa T_{\mu \nu},
\]

\[
T^{\mu \nu} ; \mu = \lambda R^{\nu},
\]

where \( \beta \equiv \kappa \lambda \), \( \lambda \) is the Rastall parameter and \( \kappa \) is the Rastall gravitational coupling constant. From Eq. (1) and Eq. (2) one can get

\[
R = \frac{\kappa}{4\beta - 1} T,
\]

\[
T^{\mu \nu} ; \mu = \frac{\beta}{4\beta - 1} T^{\nu},
\]

where \( R \) is the Ricci scalar and \( T = T^\mu_\mu \) is the trace of the energy-momentum tensor. When \( \lambda = 0 \) and \( \kappa = 8\pi G \), the Einstein general relativity and the conservation of the energy-momentum tensor can be recovered. It should be noted that \( \beta = \frac{1}{6} \) and \( \beta = \frac{1}{4} \) are not allowed [26].

In order to derive a Schwarzschild black hole solution surrounded by a cloud of strings in Rastall theory, we consider the following static spherically symmetric metric,

\[
ds^2 = f(r)dt^2 - f^{-1}(r)dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2),
\]

3
and obtain the non-vanishing components of the Rastall tensor defined by $H_{\mu \nu} \equiv G_{\mu \nu} + \beta g_{\mu \nu}R$ as follows:

$$H^0_0 = G^0_0 + \beta R = -\frac{rf'(r) + f(r) - 1}{r^2} + \beta R,$$

(6)

$$H^1_1 = G^1_1 + \beta R = -\frac{rf'(r) + f(r) - \frac{1}{r}}{r^2} + \beta R,$$

(7)

$$H^2_2 = G^2_2 + \beta R = -\frac{rf''(r) + 2f'(r)}{2r} + \beta R,$$

(8)

$$H^3_3 = G^3_3 + \beta R = -\frac{rf''(r) + 2f'(r)}{2r} + \beta R,$$

(9)

where $R = \frac{[(r^2 f''(r) + 4rf'(r) + 2f(r) - 2)]}{r^2}$, and $f'(r)$ and $f''(r)$ stand for the first and second derivatives of $f(r)$ with respect to $r$, respectively.

The first solution of black holes with a cloud of strings in Einstein gravity was given by Letelier [9] in which the action of a string in the evolution of spacetime reads

$$S = \int m \sqrt{-\gamma} d\xi^0 d\xi^1,$$

(10)

where $m$ is a dimensionless constant that is related to the tension of the string. And $\gamma$ is the determinant of the induced metric,

$$\gamma_{ab} = g_{\mu \nu} \frac{\partial x^\mu}{\partial \xi^a} \frac{\partial x^\nu}{\partial \xi^b},$$

(11)

where $x^\mu = x^\mu(\xi^a)$ describes a two-dimensional string world sheet $\Sigma$, and $\xi^0$ and $\xi^1$ are timelike and spacelike parameters, respectively. The string bivector which is associated to the string world sheet is defined by

$$\Sigma^{\mu \nu} = \epsilon^{ab} \frac{\partial x^\mu}{\partial \xi^a} \frac{\partial x^\nu}{\partial \xi^b},$$

(12)

where $\epsilon^{ab}$ is the two-dimensional Levi-Civita symbol with $\epsilon^{01} = -\epsilon^{10} = 1$.

Now we consider a cloud of strings with world sheets. The energy-momentum tensor of a cloud of strings characterized by a proper density $\rho$ reads

$$T^{\mu \nu} = \frac{\rho \Sigma^{\mu \beta} \Sigma_{\beta}^\nu}{\sqrt{-\gamma}},$$

(13)

where $\gamma = \frac{1}{2} \Sigma^{\mu \nu} \Sigma_{\mu \nu}$. Because the string cloud is spherically symmetric, this restricts the density $\rho$ and the string bivector $\Sigma^{\mu \nu}$ to be only a function of $r$. In this situation, the only non-vanishing components of the string bivector are $\Sigma^{01}$ and $\Sigma^{10}$, which are linked by $\Sigma^{01} = -\Sigma^{10}$. Thus the only non-vanishing components of the energy-momentum tensor of a cloud of strings take the forms,

$$T^0_0 = T^1_1 = -\rho \Sigma^{01}.$$    (14)
That is, the energy-momentum tensor of a cloud of strings with the spherical symmetry can be written as

\[ T^\mu_\nu = \begin{pmatrix} \rho_s(r) & \rho_s(r) \\ 0 & 0 \end{pmatrix}. \]  

(15)

After substituting Eq. (15) into Eq. (4), we can get the following equation,

\[ \frac{d\rho_s}{dr} + \frac{2\rho_s}{r} = \frac{2\beta}{4\beta - 1} \frac{d\rho_s}{dr}, \]  

(16)

from which we deduce the solution,

\[ \rho_s(r) = \frac{b}{r^{2(4\beta - 1)}}, \]  

(17)

where \( b \) is an integration constant which is linked to the energy density of the cloud of strings.

Considering the line element given by Eq. (5), we rewrite the Rastall field equations as follows:

\[ -rf'(r) + f(r) - \frac{1}{r^2} + \beta R = \kappa \rho_s, \]  

(18)

\[ -rf''(r) + 2f'(r) + \beta R = 0, \]  

(19)

and obtain the general solution of the metric function,

\[ f(r) = 1 - \frac{C_1}{r} + \frac{1-2\beta}{r^\beta} C_2 + \frac{4\kappa b(\beta - \frac{1}{2})^2}{(8\beta^2 + 2\beta - 1)r^{2\beta - 1}}, \]  

(20)

where \( C_1 \) and \( C_2 \) are two constants to be determined.

When \( \beta = b = 0 \), Eq. (20) should go back to the Schwarzschild vacuum solution in Einstein gravity. Moreover, Eq. (20) should not be divergent at \( \beta = 0 \). The two constraints require \( C_1 = 2M \) and \( C_2 = 0 \). Therefore, we reach the final form of the function,

\[ f(r) = 1 - \frac{2M}{r} + \frac{4a(\beta - \frac{1}{2})^2}{(8\beta^2 + 2\beta - 1)r^{2\beta - 1}}, \]  

(21)

where the string parameter \( a \) is defined as \( a \equiv \kappa b \). It is easy to check that this result turns [9] back to the case of Einstein gravity when \( \beta = 0 \).

### 3 Characteristics of the black hole model

According to Eq. (21), we can compute the radius of the event horizon from the equation \( f(r_H) = 0 \), which gives the relationship between the black hole mass and the radius,

\[ M = \frac{1}{2} r_H \left( 1 + \frac{4a(\beta - \frac{1}{2})^2 r_H^{-2\beta - 1}}{8\beta^2 + 2\beta - 1} \right). \]  

(22)
Using \( \kappa_{r} = \left. \frac{1}{2} \frac{d f(r)}{d r} \right|_{r=r_H} \), we find that the surface gravity of the Schwarzschild black hole surrounded by a cloud of strings in Rastall gravity is

\[
\kappa_{r_H} = \frac{1}{2r_H} + \frac{a(1-2\beta)r_H^{4\beta}}{2(4\beta-1)r_H^4}.
\]

(23)

Then we can calculate the corresponding Hawking temperature through the surface gravity as follows,

\[
T_{BH} = \frac{\hbar \kappa_{r_H}}{2\pi} = \frac{\hbar}{4\pi r_H} \left( 1 + \frac{a(1-2\beta)r_H^{4\beta}}{4\beta-1} \right).
\]

(24)

In the following, we analyze the characteristics of the black hole model for different values of the two parameters \( \beta \) and \( a \).

In Fig. 1.1 and Fig. 1.2, we draw the graphs of function \( f(r) \) with respect to \( r \) for different values of \( a \) when \( \beta > 0 \) and \( \beta < 0 \), respectively. It should be noted that the curves with \( \beta = 0 \) in Fig. 1.1 and Fig. 1.2 represent the case in Einstein gravity, which is given as a contrast. From Fig. 1.1, we can see that when \( \beta > 0 \), \( f(r) \) in Rastall gravity is significantly lower than that in Einstein gravity at the same value of \( a \). For example, when the string parameter \( a = 0.1 \), the red curve representing Rastall gravity is significantly lower than the black curve representing Einstein gravity in Fig. 1.1. Moreover, as the value of \( a \) increases, \( f(r) \) in Rastall gravity keeps decreasing. When \( a \) is small, there are generally two horizons as shown by the red curve in Fig. 1.1, where the first is the event horizon of the black hole, and the second is what we call the string horizon which is similar to the quintessence horizon of the spherically symmetric black hole surrounded by quintessence in ref. \cite{27}. However, two horizons will shrink to one or even no horizons appear when \( a \) is large, see, for instance, the blue curve for the former case and the green curve for the latter in Fig. 1.1. From Fig. 1.2, we can see that when \( \beta < 0 \), \( f(r) \) in Rastall gravity is significantly higher than that in Einstein gravity at the same value of \( a \). For example, when we compare the two curves in red and black \( (a = 0.1) \) and the two curves in green and purple \( (a = 0.9) \), we notice that the red curve representing Rastall gravity is significantly higher than the black curve representing Einstein gravity in Fig. 1.2, and the same situation exists for the green and purple curves. Moreover, as the value of \( a \) increases, \( f(r) \) in Rastall gravity also keeps decreasing, which is same as the case of \( \beta > 0 \), see the red, blue and green curves of Fig. 1.2. Nonetheless, the black hole still maintains one event horizon in Rastall gravity when \( a \) is large, see, for instance, the green curve in Fig. 1.2, which is quite different from the case of \( \beta > 0 \).

In Fig. 2.1 and Fig. 2.3, we draw the graphs of the Hawking temperature \( T_{BH} \) with respect to the event horizon radius \( r_H \) for different values of \( a \) when \( \beta > 0 \) and \( \beta < 0 \), respectively. In Fig. 2.2 and Fig. 2.4, we draw the graphs of the Hawking temperature \( T_{BH} \) with respect to the event horizon radius \( r_H \) for different values of \( a \) and \( \beta \) in Rastall gravity and in Einstein gravity when \( \beta > 0 \) and \( \beta < 0 \), respectively. From Fig. 2.1 and Fig. 2.2, we can see that \( T_{BH} \) in Rastall gravity keeps decreasing as the value of \( a \) increases, see the curves in red, blue and green. Moreover, when \( \beta > 0 \), at the same value of \( a \), \( T_{BH} \) in Rastall gravity is higher than that in Einstein gravity at a small \( r_H \) and then lower than that in Einstein gravity at a large \( r_H \), see the three pairs of curves in red and black, in blue and orange, and in green.
and purple, respectively, in Fig. 2.2. From Fig. 2.3 and Fig. 2.4, we can see that $T_{BH}$ in Rastall gravity also keeps decreasing as the value of $a$ increases. Moreover, when $\beta < 0$, at the same value of $a$, $T_{BH}$ in Rastall gravity is lower than that in Einstein gravity at a small $r_H$ and then higher than that in Einstein gravity at a large $r_H$, see the three pairs of curves in red and black, in blue and orange, and in green and purple, respectively, in Fig. 2.4.

In addition, we find that when $\beta > 0$, $T_{BH}$ decreases monotonically with respect to the increase of $r_H$, see Fig. 2.1 and Fig. 2.2, which means that the black hole temperature will increase continuously during the whole stage of evaporation, especially the Hawking temperature $T_{BH}$ will diverge when $r_H$ tends to zero. However, when $\beta < 0$ and the parameter $a$ takes a suitable value, say, $a = 0.5$ which corresponds to the blue curves in Fig. 2.3 and Fig. 2.4, $T_{BH}$ increases at first and then decreases, and finally tends to zero with respect to the increase of $r_H$, which means that at the initial state of evaporation, the black hole temperature increases with respect to the decrease of $r_H$, and after the black hole temperature increases to a finite maximum, it quickly drops to zero at $r_H = r_0$ (the event horizon radius of the extreme black hole), leaving a frozen black hole. The former case ($\beta > 0$) is similar to the situation of a classical Schwarzschild black hole in Einstein gravity, while the latter ($\beta < 0$) is similar to the situation of a noncommutative Schwarzschild black hole $[28]$ in Einstein gravity.

At the end of this section, we try to establish the relationship between the static spherically symmetric black hole solution surrounded by a cloud of strings in Rastall gravity and the static spherically symmetric black hole solution surrounded by quintessence in Einstein gravity. Our result is that the two
Fig. 1.2 Graph of function $f(r)$ with respect to $r$ for different values of $a$. Here we choose $M = 1$ and $\beta = -0.1$, and add $\beta = 0$ as a contrast.

Fig. 2.1 Graph of the Hawking temperature $T_{BH}$ with respect to the event horizon radius $r_H$ for different values of $a$. Here we choose $M = \bar{h} = 1$ and $\beta = 0.1$. 
Fig. 2.2 Graph of the Hawking temperature $T_{BH}$ with respect to the event horizon radius $r_H$ for different values of $a$ and $\beta$. Here we choose $M = \hbar = 1$.

Fig. 2.3 Graph of the Hawking temperature $T_{BH}$ with respect to the event horizon radius $r_H$ for different values of $a$. Here we choose $M = \hbar = 1$ and $\beta = -0.1$. 

$$T_{BH}$$

$$T_{BH}$$

$$T_{BH}$$

$$T_{BH}$$

$T_{BH}$
Fig. 2.4 Graph of the Hawking temperature $T_{BH}$ with respect to the event horizon radius $r_H$ for different values of $a$ and $\beta$. Here we choose $M = \hbar = 1$.

solutions exchange to each other when their parameters are connected to each other in a specific formula. Therefore, this result would provide a possibility for the string fluid to be a candidate of dark energy if the parameter $\beta$ in Rastall gravity is positive.

For a static spherically symmetric exact solution of Einstein’s equations describing the black holes surrounded by the quintessential matter under the condition of additivity and linearity in energy momentum tensor, the metric takes [29] the following form,

$$ds^2 = \left(1 - \frac{2M}{r} - \frac{\tilde{c}}{r^3\omega_q + 1}\right) dt^2 - \left(1 - \frac{2M}{r} - \frac{\tilde{c}}{r^3\omega_q + 1}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (25)$$

where $M$ is the black hole mass, $\omega_q$ is the quintessential state parameter, and $\tilde{c}$ is an integration constant which is linked to the energy density of the quintessence. We compare Eq. (21) with Eq. (25) and find that the static spherically symmetric black hole solution surrounded by a cloud of strings in Rastall gravity can be transformed into the static spherically symmetric black hole solution surrounded by quintessence in Einstein gravity. The specific transformation relationship is as follows:

$$\beta = \frac{3\omega_q + 1}{6\omega_q - 2}, \quad (26)$$

$$a = -\frac{9\tilde{c}}{2} \left(\omega_q + \omega_q^2\right). \quad (27)$$
Now we draw the graphs of $\beta$ and $\frac{a}{\epsilon}$ as a function of $\omega_q$. Note that the range of parameter $\omega_q$ related to the quintessence field is constrained [29] from $-1$ to $-\frac{1}{3}$.

In Fig. 3.1 and Fig. 3.2, we draw the graphs of the parameter $\beta$ and the parameter $\frac{a}{\epsilon}$ with respect to the quintessential state parameter $\omega_q$, respectively. From Fig. 3.1, we can see that $\beta$ decreases monotonically with the increase of $\omega_q$. From Fig. 3.2, we can see that $\frac{a}{\epsilon}$ increases at first and then decreases with the increase of $\omega_q$, where it reaches the maximum $\frac{a}{\epsilon} = 1.125$ when $\omega_q = -0.5$.

4 Quasinormal mode frequency of gravitational perturbation

The gravitational perturbation of black holes was first studied by Regge and Wheeler [23] for the odd parity type of the spherical harmonics, and then extended to the even parity type by Zerilli [30]. In the gravitational perturbation, $g_{\mu\nu}$ is usually used to represent the background metric and $h_{\mu\nu}$ the perturbation. As $h_{\mu\nu}$ is very small when it compares to $g_{\mu\nu}$, the canonical form for the odd parity perturbation is

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & h_0(r) \\ 0 & 0 & 0 & h_1(r) \\ 0 & 0 & 0 & 0 \\ h_0(r) & h_1(r) & 0 & 0 \end{pmatrix} \exp(-i\omega t)\sin^\theta \frac{\partial}{\partial \theta} P_l(\cos\theta),$$

where $l$ is the angular quantum number, $\omega$ is the complex quasinormal mode frequency, and $P_l(\cos\theta)$ is the Legendre function. $h_0(r)$ and $h_1(r)$ are two independent components of $h_{\mu\nu}$, and we now derive the desired Schrödinger-like equation, i.e. the master equation.

One can compute $R_{\mu\nu}$ from $g_{\mu\nu}$ and similarly calculate $R_{\mu\nu} + \delta R_{\mu\nu}$ from $g_{\mu\nu} + h_{\mu\nu}$, and thus deduce [31] $\delta R_{\mu\nu}$ as follows:

$$\delta R_{\mu\nu} = -\delta^\beta_{\mu\nu} + \delta^\beta_{\mu\nu},$$
where

\[ \delta \Gamma^{\alpha} \beta_{\gamma} = \frac{1}{2} g^{\alpha \nu} (h_{\beta \nu; \gamma} + h_{\gamma \nu; \beta} - h_{\beta \gamma; \nu}). \]  

(30)

By substituting Eq. (28) into Eq. (29), we obtain two independent perturbation field equations for \( h_0(r) \) and \( h_1(r) \),

\[ \frac{i \omega h_0(r)}{f(r)} + \frac{d(f(r)h_1(r))}{dr} = 0, \]  

(31)

\[ \frac{i \omega}{f(r)} \left( \frac{dh_0(r)}{dr} + i \omega h_1(r) - \frac{2h_0(r)}{r} \right) + (l - 1)(l + 2) \frac{h_1(r)}{r^2} = 0, \]  

(32)

where the former corresponds to \( \delta R_{23} = 0 \) and the latter to \( \delta R_{13} = 0 \), respectively, and \( f(r) \) has been given in Eq. (21). Eliminating \( h_0(r) \) by \( h_1(r) \) in the above two equations, and then defining \( \Psi(r) = \frac{f(r)h_1(r)}{r} \) and \( \frac{dr}{dr} = \frac{1}{f(r)} \), we finally reach the master equation,

\[ \frac{d^2\Psi(r)}{dr^2} + [\omega^2 - V(r)]\Psi(r) = 0, \]  

(33)

where the effective potential \( V(r) \) reads

\[ V(r) = \left( 1 - \frac{2M}{r} + \frac{4a(\beta - \frac{1}{2})^2}{(8\beta^2 + 2\beta - 1)r^{2\beta - 1}} \right) \left( \frac{l(l + 1)}{r^2} - \frac{6M}{r^3} - \frac{2a r^{4\beta - 2}}{2\beta + 1} + \frac{4a \beta r^{4\beta - 2}}{2\beta + 1} \right). \]  

(34)

In order to let the effective potential satisfy \( V \to 0 \) at \( r \to \infty \), the condition \( \beta < \frac{1}{6} \) is needed, see also ref. [26]. In addition, we can easily see that both \( f(r) \) and \( V(r) \) are divergent when \( \beta = -0.5 \). Therefore, we set the range of \( \beta \) in this paper be \( -0.5 < \beta < \frac{1}{6} \), which conforms to the entropy positivity.
As to the range of $a$, it depends on the positivity or negativity of $\beta$. When $\beta > 0$, at first, a large string parameter $a$ will result in the black hole having no event horizons; secondly, the larger $a$ is, the less reliable the quasinormal mode frequencies calculated by the high order WKB-Padé approximation are; thirdly, the static spherically symmetric black hole solution surrounded by a cloud of strings in Rastall gravity can be transformed into the static spherically symmetric black hole solution surrounded by quintessence in Einstein gravity. Considering the three aspects of restriction, we take four typical values, $a = 0, 0.0001125, 0.001125, 0.01125$. By using the relation Eq. (27) at the extreme point in Fig. 3.2, i.e. $a = 1.125 \tilde{c}$, we have $\tilde{c} = 0, 0.0001, 0.001, 0.01$ which are just the values set in refs. [33, 34, 35, 36]. When $\beta < 0$, the range of $a$ is $0 \leq a < 1$. In the following, we use the high order WKB-Padé approximation to calculate the quasinormal mode frequencies of the odd parity gravitational perturbation for this black hole model when $\beta > 0$ and $\beta < 0$, respectively.

As for the high order WKB-Padé approximation, it was first proposed [24] by Matyjasek and Opala and then developed by Konoplya et al. [25] through a special averaging treatment. In this treatment, a quantity $\Delta_k = \frac{1}{2} \left| \frac{\omega_{k+1} - \omega_{k-1}}{\omega_{k}} \right|$ is defined for the error estimation and it is positively correlated with the relative error $E_k = \left| \frac{\omega_k - \omega}{\omega} \right| \times 100\%$ of the quasinormal mode frequencies, where $k$ stands for the order number, $\omega_k$ the $k$-th order quasinormal mode frequency calculated by the WKB approximation or the WKB-Padé approximation, and $\omega$ the accurate value of the quasinormal mode frequency. In our calculations, we at first give the quasinormal mode frequencies from the 1st to 13th orders, and then work out the error estimations of the 13 order’s frequencies, and finally pick out such an order’s frequency that has the smallest relative error. The results are shown in Table 1, Table 2 and Table 3. Here we have two notes: (1) For $\beta < 0$, the barrier height of the effective potential will disappear if the string parameter $a$ goes to one, so that we make the restriction, $a \leq 0.92$; (2) The order number with the smallest relative error is not less than 3.

In Fig. 4.1 and Fig. 4.2, we draw the graphs of real parts and imaginary parts of quasinormal mode frequencies with respect to the parameter $\beta$ for different values of $a$ when $\beta > 0$. From the two figures, we can see that both the real part and the absolute value of the imaginary part decrease with the increase of $\beta$. And at the same value of $\beta$, both the real part and the absolute value of the imaginary part decrease with the increase of $a$ when $a > 0$. In addition, the intersections on the vertical axis in Fig. 4.1 and Fig. 4.2 represent the real parts and imaginary parts of quasinormal mode frequencies in Einstein gravity, respectively. By comparison, we can find that for the same string parameter $a$, the real part and the absolute value of the imaginary part of quasinormal mode frequencies in Rastall gravity is greater or smaller than that in Einstein gravity, which depends on the value of the parameter $\beta$. For example, in Fig. 4.2, for the green curve with the string parameter $a = 0.01125$, we can easily see that when $\beta = 0.05$, the absolute value of the imaginary part of quasinormal mode frequencies in Rastall gravity is greater than that of Einstein gravity ($\beta = 0$), while when $\beta = 0.14$, the absolute value of the imaginary part of quasinormal mode frequencies in Rastall gravity is smaller than that of Einstein gravity. It should be noted that from $\beta = 0$ to $\beta \neq 0$, both the real part and the absolute value of the imaginary part have a jump, which is caused by the change of the black hole spacetime from the Einstein’s to Rastall’s gravity.

In Fig. 4.3 and Fig. 4.4, we draw the graphs of real parts and imaginary parts of quasinormal mode frequencies with respect to the parameter $a$ for different values of $\beta$ when $\beta < 0$, where the black curves
Table 1: The quasinormal mode frequencies of gravitational perturbation in a Schwarzschild black hole surrounded by a cloud of strings in Rastall gravity for $\beta > 0$, where $\beta = 0$ corresponds to the case in Einstein gravity. Here we choose $l = 2$ and $n = 0$.

| $a$ | $\beta$ | $\omega$ | $a$ | $\beta$ | $\omega$ |
|-----|---------|----------|-----|---------|----------|
| 0   | 0       | 0.373609 - 0.0889758i | 0.001125 |
|     | 0.01    | 0.373609 - 0.0889758i | 0.01 |
|     | 0.02    | 0.373609 - 0.0889758i | 0.02 |
|     | 0.04    | 0.373609 - 0.0889758i | 0.04 |
|     | 0.06    | 0.373609 - 0.0889758i | 0.06 |
|     | 0.08    | 0.373609 - 0.0889758i | 0.08 |
|     | 0.10    | 0.373609 - 0.0889758i | 0.10 |
|     | 0.12    | 0.373609 - 0.0889758i | 0.12 |
|     | 0.14    | 0.373609 - 0.0889758i | 0.14 |
|     | 0.163   | 0.373609 - 0.0889758i | 0.163 |

Fig. 4.1 Graph of real parts of quasinormal mode frequencies with respect to the parameter $\beta$ for different values of $a$. Here we choose $l = 2$ and $n = 0$. 

\[ \text{Re}(\omega) \]
Table 2: The quasinormal mode frequencies of gravitational perturbation in a Schwarzschild black hole surrounded by a cloud of strings in Rastall gravity for $\beta = -0.05$, $-0.1$, and $-0.2$, where $\beta = 0$ corresponds to the case in Einstein gravity. Here we choose $l = 2$ and $n = 0$.

| $a$ | $\beta$ | 0       | -0.05   | -0.1    | -0.2    |
|-----|---------|---------|---------|---------|---------|
| 0   | 0.00    | 0.373609 - 0.0889758i | 0.373609 - 0.0889758i | 0.373609 - 0.0889758i | 0.373609 - 0.0889758i |
| 0.01| 0.368085 - 0.087252i | 0.369137 - 0.0918532i | 0.369285 - 0.0904483i | 0.368865 - 0.0919081i |
| 0.02| 0.362589 - 0.0855446i | 0.364108 - 0.0902656i | 0.364416 - 0.0904483i | 0.363614 - 0.0903886i |
| 0.04| 0.351676 - 0.0821788i | 0.354169 - 0.0871444i | 0.354835 - 0.0875207i | 0.353381 - 0.0874398i |
| 0.06| 0.340873 - 0.0788784i | 0.34439 - 0.0840947i | 0.345461 - 0.0846746i | 0.343495 - 0.0846075i |
| 0.08| 0.330197 - 0.0756167i | 0.334771 - 0.0811161i | 0.33629 - 0.0819209i | 0.334713 - 0.0792728i |
| 0.10| 0.319614 - 0.0724516i | 0.325311 - 0.0782084i | 0.32732 - 0.0792209i | 0.324713 - 0.0792728i |
| 0.12| 0.309144 - 0.0693517i | 0.316011 - 0.0753711i | 0.318549 - 0.0766105i | 0.315794 - 0.0767613i |
| 0.16| 0.288541 - 0.0633437i | 0.297889 - 0.0699056i | 0.301593 - 0.0716148i | 0.298845 - 0.0720283i |
| 0.2  | 0.26841 - 0.0576046i | 0.280404 - 0.0647158i | 0.2854 - 0.0669097i | 0.28301 - 0.0676553i |
| 0.24 | 0.248753 - 0.0521208i | 0.263552 - 0.0597971i | 0.26995 - 0.062483i | 0.268211 - 0.0636125i |
| 0.28 | 0.229587 - 0.0469078i | 0.247332 - 0.0551445i | 0.255221 - 0.0583228i | 0.254374 - 0.0598725i |
| 0.32 | 0.210927 - 0.041954i | 0.231742 - 0.0507523i | 0.241191 - 0.0544172i | 0.241431 - 0.0564102i |
| 0.36 | 0.192783 - 0.0372704i | 0.216778 - 0.0466146i | 0.227839 - 0.0507544i | 0.229319 - 0.0532026i |
| 0.40 | 0.175176 - 0.0328518i | 0.202437 - 0.042725i | 0.215143 - 0.0473226i | 0.217978 - 0.0502287i |
| 0.44 | 0.158125 - 0.0287039i | 0.188715 - 0.0390768i | 0.203081 - 0.0441103i | 0.207353 - 0.0474692i |
| 0.48 | 0.141648 - 0.0248258i | 0.175607 - 0.0356629i | 0.191632 - 0.0411064i | 0.197395 - 0.0449066i |
| 0.52 | 0.125769 - 0.0212199i | 0.163108 - 0.0324757i | 0.180772 - 0.0382997i | 0.188055 - 0.0425247i |
| 0.56 | 0.110513 - 0.0178881i | 0.151212 - 0.0295076i | 0.17048 - 0.0356796i | 0.17929 - 0.0403089i |
| 0.6  | 0.0959109 - 0.0148325i | 0.139016 - 0.0264807i | 0.160735 - 0.0332356i | 0.17106 - 0.0382493i |
| 0.64 | 0.0819965 - 0.0102551 | 0.128469 - 0.0240121i | 0.150258 - 0.0305205i | 0.166313 - 0.0377104i |
| 0.68 | 0.0688098 - 0.0055797i | 0.119097 - 0.0222195i | 0.141694 - 0.0284855i | 0.158857 - 0.0358193i |
| 0.72 | 0.0563983 - 0.0034378i | 0.110604 - 0.0202067i | 0.136355 - 0.0276866i | 0.151847 - 0.0340561i |
| 0.76 | 0.0448199 - 0.00541499i | 0.101448 - 0.0181322i | 0.128415 - 0.0257666i | 0.145252 - 0.0324105i |
| 0.80 | 0.0341463 - 0.0037434i | 0.0928593 - 0.0162353i | 0.12093 - 0.0239845i | 0.139042 - 0.0308733i |
| 0.84 | 0.0244707 - 0.00424274i | 0.0848235 - 0.0145061i | 0.11388 - 0.022331i | 0.13319 - 0.0294357i |
| 0.88 | 0.0159194 - 0.00136915i | 0.0773248 - 0.0129344i | 0.107242 - 0.0207974i | 0.127673 - 0.0280901i |
| 0.92 | 0.00867958 - 0.000610903i | 0.0703466 - 0.0115101i | 0.100996 - 0.0193755i | 0.122468 - 0.0268294i |
Table 3: The quasinormal mode frequencies of gravitational perturbation in a Schwarzschild black hole surrounded by a cloud of strings in Rastall gravity for $\beta = -0.3$, $-0.4$, $-0.45$, and $-0.48$. Here we choose $l = 2$ and $n = 0$.

| a  | $\beta$ | -0.3               | -0.4               | -0.45              | -0.48              |
|----|---------|--------------------|--------------------|--------------------|--------------------|
|    |         | 0.373609 - 0.0889758i | 0.373609 - 0.0889758i | 0.373609 - 0.0889758i | 0.373609 - 0.0889758i |
| 0  | -0.3    | 0.367189 - 0.0915334i | 0.361474 - 0.0901381i | 0.350238 - 0.08735i  | 0.320214 - 0.0798757i |
| 0.01| -0.4    | 0.360369 - 0.0896678i | 0.349494 - 0.0870222i | 0.329026 - 0.0819572i | 0.279643 - 0.0696907i |
| 0.02| -0.45   | 0.347298 - 0.0861063i | 0.327555 - 0.0813368i | 0.293198 - 0.0728776i | 0.222814 - 0.0554557i |
| 0.04| -0.48   | 0.334939 - 0.0827568i | 0.307968 - 0.0762845i | 0.264138 - 0.0655402i | 0.189981 - 0.048794i  |
| 0.06|         | 0.323242 - 0.0796033i | 0.290391 - 0.0717698i | 0.240123 - 0.0594953i | 0.162264 - 0.0416428i |
| 0.08|         | 0.312164 - 0.0766315i | 0.274544 - 0.0677148i | 0.219966 - 0.0544342i | 0.141527 - 0.0362999i |
| 0.1 |         | 0.301663 - 0.0738279i | 0.260193 - 0.0640558i | 0.208183 - 0.0508471i | 0.125439 - 0.0321589i |
| 0.12|         | 0.282239 - 0.0686782i | 0.235238 - 0.0577227i | 0.179845 - 0.0457637i | 0.102118 - 0.026163i  |
| 0.16|         | 0.264701 - 0.0640687i | 0.214316 - 0.052427i  | 0.158094 - 0.0401645i | 0.084043 - 0.0220346i |
| 0.2 |         | 0.248815 - 0.0599279i | 0.201504 - 0.0506146i | 0.140897 - 0.03575i  | 0.065327 - 0.0167261i |
| 0.24|         | 0.234383 - 0.0561951i | 0.18584 - 0.0465641i  | 0.126975 - 0.0321843i | 0.061541 - 0.0167261i |
| 0.28|         | 0.221236 - 0.0528188i | 0.172272 - 0.0430709i | 0.115484 - 0.0292467i | 0.058309 - 0.0149206i |
| 0.32|         | 0.209226 - 0.0497554i | 0.160418 - 0.0400309i | 0.105845 - 0.0267863i | 0.052624 - 0.0134635i |
| 0.36|         | 0.198227 - 0.0469347i | 0.149984 - 0.0373638i | 0.097646 - 0.0246968i | 0.047938 - 0.0122632i |
| 0.4 |         | 0.192323 - 0.0465152i | 0.140735 - 0.035007i  | 0.090596 - 0.022901i  | 0.044012 - 0.0112576i |
| 0.44|         | 0.182774 - 0.0440444i | 0.132487 - 0.0329108i | 0.084467 - 0.0213419i | 0.040674 - 0.0104028i |
| 0.48|         | 0.173969 - 0.0417802i | 0.12509 - 0.0310356i  | 0.079092 - 0.0199758i | 0.037775 - 0.0096352i |
| 0.52|         | 0.165834 - 0.0396999i | 0.118422 - 0.029349i  | 0.074343 - 0.0187695i | 0.036151 - 0.0082523i |
| 0.56|         | 0.1583 - 0.0377839i  | 0.112384 - 0.0278248i | 0.070116 - 0.0176968i | 0.033910 - 0.0077398i |
| 0.6 |         | 0.151311 - 0.0360139i | 0.106893 - 0.0264412i | 0.066318 - 0.0167638i | 0.031927 - 0.0072868i |
| 0.64|         | 0.144813 - 0.0343785i | 0.10188 - 0.0251801i  | 0.062924 - 0.0158728i | 0.030163 - 0.0068331i |
| 0.68|         | 0.138761 - 0.0326611i | 0.097286 - 0.0240264i | 0.059839 - 0.0150913i | 0.028577 - 0.00652176i |
| 0.72|         | 0.133115 - 0.0314513i | 0.093064 - 0.0229672i | 0.057036 - 0.0143811i | 0.0271506 - 0.0061954i |
| 0.76|         | 0.127838 - 0.0301391i | 0.089169 - 0.0219918i | 0.054476 - 0.013733i  | 0.0258583 - 0.00590082i |
| 0.8 |       | 0.122899 - 0.0289151i | 0.085563 - 0.0210908i | 0.05213 - 0.0131393i  | 0.0246822 - 0.00563226i |
| 0.84|         | 0.118267 - 0.0277717i | 0.082228 - 0.0202562i | 0.0499723 - 0.0125934i | 0.0236074 - 0.00538686i |
| 0.88|         | 0.113919 - 0.0267018i | 0.0791237 - 0.0194812i | 0.0479814 - 0.0120898i | 0.0226214 - 0.00516174i |
Fig. 4.2 Graph of negative imaginary parts of quasinormal mode frequencies with respect to the parameter $\beta$ for different values of $a$. Here we choose $l = 2$ and $n = 0$.

represent the real part and the absolute value of the imaginary part in Einstein gravity. From the two figures, we can see that, on the one hand, both the real part and the absolute value of the imaginary part decrease with the increase of $a$; and on the other hand, when $\beta$ is small, e.g. $\beta = -0.48$ (the grey curve), both the real part and the absolute value of the imaginary part decrease fast at a small $a$, e.g. $a < 0.4$, and then decrease very slowly at a large $a$, e.g. $a > 0.85$, which makes the real part and the absolute value of the imaginary part in Rastall gravity larger than that in Einstein gravity. We also find that for the same value of $a$, both the real part and the absolute value of the imaginary part increase at first and then decrease with the decrease of $\beta$. In addition, we notice that for the same string parameter $a$, the real part and the absolute value of the imaginary part of quasinormal mode frequencies in Rastall gravity are greater or smaller than that in Einstein gravity, depending on the value of the parameter $\beta$. For example, in Fig. 4.4, when the string parameter $a = 0.4$, we can easily see that the absolute value of the imaginary part of quasinormal mode frequencies in Rastall gravity (the red curve corresponding to $\beta = -0.05$) is larger than that of Einstein gravity (the black curve corresponding to $\beta = 0$), while the absolute value of the imaginary part of quasinormal mode frequencies in Rastall gravity (the orange curve corresponding to $\beta = -0.45$) is smaller than that of Einstein gravity (the black curve corresponding to $\beta = 0$).

Therefore, we have the following conclusions:

1. The increase of the string parameter $a$ in Rastall gravity makes the gravitational wave decay more slowly.

2. For the same string parameter $a$, the gravitational waves in Rastall gravity decay more slowly or faster than that in Einstein gravity, which depends on the value of the parameter $\beta$.

3. For $\beta < 0$, the gravitational waves in Rastall gravity decay faster than that in Einstein gravity as $a$ approaches 1.
Fig. 4.3 Graph of real parts of quasinormal mode frequencies with respect to the parameter $a$ for different values of $\beta$. Here we choose $l = 2$ and $n = 0$.

Fig. 4.4 Graph of negative imaginary parts of quasinormal mode frequencies with respect to the parameter $a$ for different values of $\beta$. Here we choose $l = 2$ and $n = 0$. 
5 Area and entropy spectra via adiabatic invariance

In this section, based on the method of adiabatic invariance of black holes \cite{37} for deriving the quantized entropy spectrum and its later improvement \cite{14}, we study the area spectrum and entropy spectrum of a Schwarzschild black hole surrounded by a cloud of strings in Rastall gravity. By applying the Wick rotation in Lorentzian time and transforming \(t \rightarrow -i\tau\), we write the Euclideanized form of the metric,

\[
ds^2 = -f(r)\,d\tau^2 - f^{-1}(r)\,dr^2 - r^2(d\theta^2 + \sin^2\theta\,d\phi^2),
\]

(35)

where \(f(r)\) is shown in Eq. (21) and \(\tau\) means the Euclidean time. For a neutral and static spherically symmetric black hole spacetime, the only dynamic freedom of adiabatic invariants is the radial coordinate \(r\), so the adiabatic invariant can be expressed \cite{38} as

\[
\mathcal{L} = \oint p_r\,dq_r = \oint \int_0^{p_r} dp_r'\,dr, \tag{36}
\]

where \(p_r\) is the conjugate momentum of the coordinate \(q_r\).

It is known that the equation of motion of a massless particle is a radial null geodesic \(\dot{r} = \frac{dr}{d\tau}\) \cite{39} and the equation of motion of a massive particle is the phase velocity \(\dot{r} = v_p\) \cite{40}. However, if one only considers the outgoing path, one does not care \cite{41} whether the particle has mass or not. According to Hamilton’s canonical equation, we have

\[
\dot{r} = \frac{dr}{d\tau} = \frac{dH'}{dp_r'}, \tag{37}
\]

where \(H' = M'\) and \(M' = M - \omega'\) is the mass of the black hole from which a particle with energy \(E' = \omega'\) tunnels through its horizon. By substituting Eq. (37) into Eq. (36), we obtain the following formula,

\[
\mathcal{L} = \oint p_r\,dq_r = \oint \int_0^{H} \frac{dH'}{r}\,dr = \oint \int_0^{M} dM'\,d\tau. \tag{38}
\]

Since any background spacetime with the horizon in Kruskal coordinates has periodicity with respect to the Euclidean time, we assume that the particles moving in such a background spacetime have the same periodicity as the background spacetime, and this period satisfies \cite{42} the following relationship,

\[
T = \oint d\tau = \frac{2\pi}{\kappa_{rH}} = \frac{\hbar}{T_{BH}}. \tag{39}
\]

Using Eq. (39), we rewrite Eq. (38) as follows:

\[
\mathcal{L} = \hbar \int_0^{M} \frac{dM'}{T_{BH}} = \hbar \int_0^{r_H} \frac{1}{r_H} \,dr_H \,dr_H'. \tag{40}
\]

Next, we deduce the following formula from Eq. (22),

\[
\frac{dM}{dr_H} = \frac{1}{2} \left(1 + \frac{a(1 - 2\beta)r_H^{4\beta}}{4\beta - 1}\right). \tag{41}
\]
After substituting Eq. (24) and Eq. (41) into Eq. (40), we finally recast this adiabatic invariant as follows:

$$L = \pi r_H^2 = \frac{A}{4},$$  \hspace{1cm} (42)$$

where $A$ is the area of the event horizon of the black hole. In addition, we know from the Bohr-Sommerfeld quantization rule its quantized form, $L = 2\pi n\hbar$, $n = 0, 1, 2, 3\ldots$, so we have the formulations of area and entropy of the black hole,

$$A_n = 8\pi n\hbar, \quad S_n = \frac{A_n}{4\hbar} = 2\pi n.$$  \hspace{1cm} (43)$$

As a result, we reach the area spectrum and the entropy spectrum of the Schwarzschild black hole surrounded by a cloud of strings in Rastall gravity,

$$\Delta A = A_n - A_{n-1} = 8\pi\hbar, \quad \Delta S = S_n - S_{n-1} = 2\pi.$$  \hspace{1cm} (44)$$

### 6 Area and entropy spectra via gravitational wave periodicity

In this section, we investigate the area spectrum and entropy spectrum of a Schwarzschild black hole surrounded by a cloud of strings in Rastall gravity by utilizing the method of the periodic property of the outgoing gravitational wave \[22\]. We have two ways to calculate wave functions. The first is to substitute the assumed form \[43\] of wave functions, $\Phi = \frac{1}{4\pi \omega} R_{\omega}(r,t)Y_{l,m}(\theta,\phi)$, and Eq. (5) into the Klein-Gordon equation,

$$g^{\mu\nu} \partial_\mu \partial_\nu \Phi - \frac{m^2}{\hbar^2} \Phi = 0,$$  \hspace{1cm} (45)$$

and the other is to resort the Hamilton-Jacobi equation,

$$g^{\mu\nu} \partial_\mu S \partial_\nu S + m^2 = 0,$$  \hspace{1cm} (46)$$

where the wave function $\Phi$ and the action $S$ satisfy the following relationship:

$$\Phi = \exp\left[\frac{i}{\hbar} S(t,r,\theta,\phi)\right].$$  \hspace{1cm} (47)$$

Because the spacetime of the black hole we consider is spherically symmetric, the action $S$ can be decomposed \[44,45\] as

$$S(t,r,\theta,\phi) = -Et + W(r) + J(\theta,\phi),$$  \hspace{1cm} (48)$$

where $E = -\frac{\partial S}{\partial t}$ represents the energy of the emitted particles observed at infinity. Considering $J(\theta,\phi) = 0$, $W(r) = i\pi E/f'(r_H)$ and only the outgoing wave near the outside horizon, one can express the wave function $\Phi$ there as follows,

$$\Phi = \exp\left(-\frac{i}{\hbar} Et\right) \Psi(r_H).$$  \hspace{1cm} (49)$$
where $\Psi(r_H) = \exp \left[ -\frac{\pi E}{h f'(r_H)} \right]$ and $f'(r_H) = \frac{df(r)}{dr} \big|_{r=r_H}$. From the above function, we find that $\Phi$ is a periodic wave function with the period,

$$T = \frac{2\pi \hbar}{E}. \quad (50)$$

Considering the relation $E = \hbar \omega$, we obtain

$$T = \frac{2\pi}{\omega}. \quad (51)$$

In addition, we know that $T = \frac{\hbar}{T_{BH}}$ from Eq. (50), so we derive the following formula,

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{\frac{h}{T_{BH}}} = \frac{2\pi T_{BH}}{\hbar}. \quad (52)$$

According to Hod’s idea [16], the change in the area of the event horizon of the Schwarzschild black hole surrounded by a cloud of strings in Rastall gravity is

$$\Delta A = 8\pi r_H \, dr_H = 8\pi r_H \, dM \frac{dr_H}{dM} = 8\pi r_H \hbar \omega \frac{dr_H}{dM}. \quad (53)$$

Substituting Eq. (41) and Eq. (52) into Eq. (53), we work out

$$\Delta A = 8\pi r_H \frac{2\pi T_{BH}}{\hbar} \frac{dr_H}{dM} = 16\pi^2 r_H T_{BH} \frac{dr_H}{dM} = 16\pi^2 r_H \frac{\hbar}{2\pi r_H} = 8\pi \hbar. \quad (54)$$

Then using the Bekenstein-Hawking area law, we deduce the corresponding black hole entropy spectrum,

$$\Delta S = \frac{\Delta A}{4\hbar} = 2\pi. \quad (55)$$

7 Conclusion

We first give the static spherically symmetric black hole solution surrounded by a cloud of strings in Rastall gravity, and then analyze this black hole model. We plot the graphs of the function $f(r)$ with respect to $r$ and the graphs of the Hawking temperature $T_{BH}$ with respect to the event horizon radius $r_H$ for different values of $a$ when $\beta > 0$ and $\beta < 0$, respectively. We find that the increase of the string parameter $a$ causes the function $f(r)$ and the Hawking temperature $T_{BH}$ to decrease in Rastall gravity. Through the analysis of the black hole metric, we deduce that the static spherically symmetric black hole solution surrounded by a cloud of strings in Rastall gravity can be transformed into the static spherically symmetric black hole solution surrounded by quintessence in Einstein gravity when $\beta > 0$, which provides a possibility for a string fluid as a candidate of the dark energy. However, when $\beta < 0$, the spacetime of this black hole model is a kind of asymptotically flat spacetime. We use the high order WKB-Padé approximation to calculate the quasinormal mode frequencies of the odd parity gravitational perturbation for this model. Then we plot the graphs of real parts and negative imaginary parts of quasinormal mode frequencies with respect to the parameter $\beta$ for different values of $a$ when $\beta > 0$, and we also plot the graphs of real parts and negative imaginary parts of quasinormal mode frequencies
with respect to the parameter $a$ for different values of $\beta$ when $\beta < 0$. We conclude that the increase of the string parameter $a$ in Rastall gravity causes the decay of the gravitational wave to slow down. In addition, we use the method of the adiabatic invariant integral and the method of the periodic property of outgoing waves to derive the area spectrum and entropy spectrum of the black hole model, respectively. The results are consistent with the spectroscopy of the Schwarzschild black hole given by Beckenstein, which shows that the parameter $\beta$ and the string parameter $a$ do not affect the area spectrum and entropy spectrum of the black hole. This result is similar to that of ref. [38], in which the area spectrum and entropy spectrum of the quantum-corrected Schwarzschild black hole surrounded by quintessence in Einstein gravity are not affected by the quantum correction parameter and the quintessence parameter.

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