ALL PAROVICHENKO SPACES MAY BE
SOFT-PAROVICHENKO

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To the memory to Phil Zenor, one of founders of this journal

Abstract. It is shown that, assuming the Continuum Hypothesis,
every compact Hausdorff space of weight at most \( c \) is a remainder
in a soft compactification of \( \mathbb{N} \).

We also exhibit an example of a compact space of weight \( \aleph_1 \) —
hence a remainder in some compactification of \( \mathbb{N} \) — for which it is
consistent that is not the remainder in a soft compactification of \( \mathbb{N} \).

Introduction

A compactification, \( \gamma \mathbb{N} \), of the discrete space \( \mathbb{N} \) of natural numbers is
said to be soft if for all pairs \( \langle A, B \rangle \) of disjoint subsets of \( \mathbb{N} \) the following
holds: if \( \text{cl } A \cap \text{cl } B \neq \emptyset \) then there is an autohomeomorphism \( h \) of \( \gamma \mathbb{N} \) such
that \( h[A] \cap B \) is infinite and \( h \) is the identity on the remainder \( \gamma \mathbb{N} \setminus \mathbb{N} \).

Banakh asked in [1] whether every Parovichenko space is soft-Parovi-
chenko, where a Parovichenko space is defined to be a remainder in some
compactification of \( \mathbb{N} \) and, naturally, a soft-Parovichenko space is a re-
mainder in some soft compactification of \( \mathbb{N} \). Parovichenko’s classic the-
orem, from [7], characterizes, assuming \( \text{CH} \), the Parovichenko spaces as
the compact Hausdorff spaces of weight at most \( c \).

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Example 1. The Čech-Stone compactification, $\beta N$, of $\mathbb{N}$ is soft, vacuously; hence $\beta N \setminus \mathbb{N}$ is soft-Parovichenko.

At the other end of the spectrum the one-point compactification $\alpha N$ is soft too as every permutation of $\mathbb{N}$ determines an autohomeomorphism of $\alpha N$.

Remark 1. As remarked by Banakh in [1]: if $\delta N$ is a compactification of $\mathbb{N}$ with the property that whenever $x \in \text{cl} A$, where $x \in \delta N \setminus \mathbb{N}$ and $A \subseteq \mathbb{N}$, there is a sequence in $A$ that converges to $x$, then $\delta N$ is a soft compactification.

Indeed, if $x \in \text{cl} A \cap \text{cl} B$ and $S$ and $T$ are subsets of $A$ and $B$ respectively that converge to $x$ then one takes a permutation $h$ of $\mathbb{N}$ that interchanges $S$ and $T$ and is the identity outside $S \cup T$. The extension of $h$ by the identity on $X$ is an autohomeomorphism.

In [2] the reader can find more information on the background of the problem of remainders in soft compactifications, including, in Theorem 9.5, some classes of compact spaces that are soft-Parovichenko:

- Parovichenko and of character less than $p$
- perfectly normal
- of weight less than $p$

In each case one obtains the stronger statement that every compactification with the space as its remainder is soft, because in each case the compactification satisfies the condition in Remark 1. The cardinal number $\mathfrak{p}$ is equal to the cardinal number $\mathfrak{t}$ discussed in Section 2 below.

1. Applying the Continuum Hypothesis

In this section we prove the statement in the abstract. The Continuum Hypothesis (CH) implies that every Parovichenko space is soft-Parovichenko.

Let $X$ be compact Hausdorff and of weight $\aleph_1$. We may assume $X$ is embedded in the Tychonoff cube $[0,1]^\omega_1$ and, for technical convenience, that $X \subseteq \{0\} \times [0,1][1,\omega_1]$.

Our aim will be to construct a sequence $\langle f_\alpha : \alpha < \omega_1 \rangle$ of functions from $\mathbb{N}$ to $[0,1]$ such that the Čech-Stone extension $\beta f$ of its diagonal map $f : \mathbb{N} \to [0,1]^{\omega_1}$ satisfies $\beta f[\beta N \setminus \mathbb{N}] = X$. To make sure that $f[N]$ is discrete we demand that $f_0(n) = 2^{-n}$ for all $n$. In this way $f[N] \cup X$ will be a compactification of $\mathbb{N}$ with $X$ as its remainder.

Ensuring softness. To ensure softness of the compactification we take our inspiration from Remark 1.

Along with the functions $f_\alpha$ we construct an almost disjoint family $S$ of subsets of $\mathbb{N}$ such that in the end every $S \in S$ converges to a point $x_S$ of $X$. 

In addition we ensure that whenever \( \langle A, B \rangle \) is a pair of disjoint subsets of \( N \) whose closures intersect then there will be two sets \( S \) and \( T \) in \( S \) such that \( S \cap A \) and \( T \cap B \) are infinite, and \( x_S = x_T \). As in Remark 1 a permutation of \( N \) that interchanges \( S \cap A \) and \( T \cap B \) and is the identity outside these sets gives an autohomeomorphism of the compactification as required.

**The construction.** We let \( \langle (A_\alpha, B_\alpha) : \alpha < \omega_1 \rangle \) enumerate the set of ordered pairs of disjoint infinite subsets of \( N \). We shall construct

- a sequence \( \langle f_\alpha : \alpha < \omega_1 \rangle \) of functions from \( N \) to \([0, 1]\)
- a sequence \( \langle S_\alpha : \alpha < \omega_1 \rangle \) of subsets of \( N \)
- a sequence \( \langle x_\alpha : \alpha < \omega_1 \rangle \) of points in \( X \)
- a sequence \( \langle K_\alpha : \alpha < \omega_1 \rangle \) of subsets of \( N \)

For each \( \delta \) we let \( g_\delta : N \to [0, 1]^\delta \) be the diagonal map of \( \langle f_\alpha : \alpha < \delta \rangle \), and, for bookkeeping purposes, \( I \) will be the set of \( \delta \) for which the closures of \( g_\delta[A_\delta] \) and \( g_\delta[B_\delta] \) intersect.

The sequences should satisfy the following conditions.

1. If \( \alpha < \delta \) then the set \( g_\delta[S_\alpha] \) converges to the point \( x_\alpha \mid \delta \)
2. If \( \delta \in I \) then there are \( \alpha, \beta \leq \delta \) such that \( x_\alpha \mid \delta = x_\beta \mid \delta \), and both intersections \( S_\alpha \cap A_\delta \) and \( S_\beta \cap B_\delta \) are infinite
3. For all \( \delta \) the family \( \{S_\alpha : \alpha < \delta\} \cup \{K_\delta\} \) is almost disjoint
4. If \( \alpha < \beta \) then \( K_\beta \subseteq^* K_\alpha \)
5. If \( \alpha \in \omega_1 \) then \( \beta g_\alpha[N_\alpha^*] = \beta g_\alpha[K_\alpha^*] = X \mid \alpha \)

In condition 2 we do not exclude the possibility that \( \alpha = \beta \).

At each stage \( \delta \) we choose the set \( S_\delta \), construct the function \( f_\delta \), and determine the set \( K_{\delta+1} \) as a subset of \( K_\delta \). This means that in case \( \delta \) is a limit we must construct \( K_\delta \) first.

**Making \( K_\delta \) if \( \delta \) is a limit.** Let \( \langle U_\alpha : n < \omega \rangle \) be a sequence of finite families of basic open sets in \([0, 1]^\delta \) such that for all \( n \) we have \( X \mid \delta \subseteq \bigcup U_\alpha \) and \((X \setminus \delta) \cap U \neq \emptyset \) for all \( U \in U_\alpha \), and such that for every open set \( O \) around \( X \mid \delta \) there is an \( n \) such that \( \bigcup U_\alpha \subseteq O \). For every \( n \) there is a finite set \( F_\delta \) such that every member of \( U_\alpha \) has its support in \( F_\alpha \). Let \( \langle \delta_\alpha : n < \omega \rangle \) be a strictly increasing sequence of ordinals that converges to \( \delta \) and such that \( F_\delta \subseteq \delta_\alpha \) for all \( n \).

The family \( U_\alpha \) can also be considered to be a family of basic open sets in the product \([0, 1]^{\delta_\alpha} \). The condition that \( \beta g_\alpha[K_{\delta_\alpha}^*] = X \mid \delta_\alpha \) for all \( n \) translates into two things:

- for every \( n \) there is a natural number \( N_\alpha \) such that \( g_\delta(k) \in \bigcup U_\alpha \) for \( k \in K_{\delta_\alpha} \setminus N_\alpha \)
- for every \( U \in U_\alpha \) the set \( \{k \in K_{\delta_\alpha} : g_\delta(k) \in U\} \) is infinite
But then the same holds with \( g_5 \) replacing \( g_6 \).

Using this we determine a strictly increasing sequence \( \langle M_n : n < \omega \rangle \) of natural numbers such that \( M_n \supseteq N_n \) for all \( n \), such that \( K_{\delta_{n+1}} \setminus M_{n+1} \subseteq K_{\delta_n} \) for all \( n \), and such that for every \( U \in \mathcal{U}_n \) there is a \( k \in K_{\delta_n} \cap [M_n, M_{n+1}) \) such that \( g_5(k) \in U \).

We let \( K_\delta = \bigcup_{n < \omega} (K_{\delta_n} \cap [M_n, M_{n+1}) \). By construction \( \beta g_5[K_\delta^\ast] = X \), and because \( K_\delta \subseteq K_{\delta_n} \) for all \( n \) the set is also almost disjoint from all \( S_\alpha \) for \( \alpha < \delta \).

The actual construction. Now let \( \delta \in \omega_1 \) and assume that everything has been constructed up to and/or including \( \delta \).

If the closures of \( g_5[A_\delta] \) and \( g_5[B_\delta] \) intersect then we add \( \delta \) to the set \( I \) and determine \( S_\delta \) by considering a few cases.

First shrink \( A_\delta \) and \( B_\delta \) to infinite sets \( C \) and \( D \) such that the closures of \( g_5[C] \) and \( g_5[D] \) intersect in exactly one point of \( X \setminus \delta \), this point is going to grow into \( x_\delta \), so we denote it \( x_\delta \setminus \delta \). Note that the union \( g_5[C] \cup g_5[D] \) converges to \( x_\delta \setminus \delta \).

The cases that can occur are

- both \( C \) and \( D \) are almost disjoint from the \( S_\alpha \) with \( \alpha < \delta \); in this case we let \( S_\delta \) be an infinite subset of \( C \cup D \) that meets both \( C \) and \( D \) in an infinite set and is such that \( K_\delta \setminus S_\delta \) contains an infinite set that converges to \( x_\delta \setminus \delta \).
- \( C \) is almost disjoint from the \( S_\alpha \) with \( \alpha < \delta \), but \( D \) is not; in this case we have a \( \beta < \delta \) such that \( S_\beta \cap D \) is infinite, and so \( x_\beta \setminus \delta = x_\delta \setminus \delta \). Now let \( S_\delta \) be an infinite subset of \( C \) as in the previous case.
- \( D \) is almost disjoint from the \( S_\alpha \) with \( \alpha < \delta \), but \( C \) is not; in this case we have an \( \alpha < \delta \) such that \( S_\alpha \cap D \) is infinite, and so \( x_\alpha \setminus \delta = x_\beta \setminus \delta \). Now let \( S_\delta \) be an infinite subset of \( D \) as in the previous cases.
- neither \( C \) nor \( D \) is almost disjoint from the \( S_\alpha \) with \( \alpha < \delta \); this means that condition 2 is already met. We let \( S_\delta \) be an infinite subset of \( K_\delta \) that converges to some \( x_\delta \setminus \delta \), again subject to the condition from the first three cases.

In all four cases let \( K_{\delta+1} = K_\delta \setminus S_\delta \). Then \( \beta g_5[K_{\delta+1}] = X \), because of the condition on \( S_\delta \). That condition is met automatically if \( x_\delta \setminus \delta \) is not an isolated point of \( X \setminus \delta \).

In case the closures do not intersect we choose an arbitrary point \( x_\delta \setminus \delta \) of \( X \setminus \delta \) and choose an infinite subset \( S_\delta \) of \( K_\delta \) that converges to this point and such that \( K_\beta \setminus S_\delta \), which will be \( K_{\delta+1} \), contains an infinite set that converges to \( x_\delta \setminus \delta \) as well.

In all cases we still have \( \beta g_5[K_{\delta+1}^\ast] = X \).
Before we proceed to the definition of \( f_\delta \) we first choose the \( \delta \)-coordinates of the points \( x_\alpha \) for \( \alpha \leq \delta \). For each \( \alpha \) we check whether there is a \( \beta < \alpha \) such that \( x_\alpha | \delta = x_\beta | \delta \). If that is the case then we must let \( x_\alpha(\delta) = x_\beta(\delta) \). In the other case we ensure that \( (x_\alpha | \delta, x_\alpha(\delta)) \in X | (\delta+1) \). This then introduces the demand that \( f_\delta[S_\alpha] \) converge to \( x_\alpha(\delta) \).

To specify the function \( f_\delta \) we proceed much as in the construction of \( K_\delta \) for limit \( \delta \).

We take a sequence \( \{U_n : n < \omega \} \) of finite families of basic open sets in \([0,1]^{\delta+1}\) as follows. First take an increasing sequence \( \{F_n : n < \omega \} \) of finite subsets of \( \delta + 1 \) such that \( \delta \in F_0 \) and \( \bigcup_n F_n = \delta + 1 \).

Next we let \( B_n \) be the family of all products \( \prod_{\alpha \leq \delta} I_\alpha \), where \( I_\alpha \) is an interval of the form \([0,2^{-n}), (i \cdot 2^{-n}, (i+1) \cdot 2^{-n}) \) or \((1 - 2^{-n}, 1)\) in \([0,1] \) if \( \alpha \in F_n \) and \( I_\alpha = [0,1] \) if \( \alpha \notin F_n \).

We let \( U_n = \{B \in B_n : B \cap (X | (\delta+1)) \neq \emptyset \} \) and we write every \( U \in U_n \) as \( V_U \times I_U \), where \( V_u \) is in \([0,1]^\delta\) and \( I_U \) is an interval in \([0,1] \).

For every \( n \) we let \( S_n = \{S_\alpha : \alpha \in F_n \} \cup \{K_{\delta+1} \} \) and we take an \( N_n \) such that

- for all distinct \( X \) and \( Y \) in \( S_n \) the intersection \( X \cap Y \) is contained in \( N_n \)
- for all \( k \geq N_n \) we have \( g_\delta(k) \in \bigcup \{V_U : U \in U_n \} \)
- for all \( U \in U_n \) and all \( \alpha \leq \delta : \) if \( x_\alpha | \delta \in V_U \) then \( g_\delta(k) \in V_U \) for all \( k \in S_\alpha \setminus N_n \)

Because \( \partial g_\delta[K_{\delta+1}] = X \) we know that for every \( U \in U_n \) the set \( \{k \in K_{\delta+1} : g_\delta(k) \in V_U \} \) is infinite. Using this we take a strictly increasing sequence \( \{M_n : n < \omega \} \) of natural numbers such that \( M_n \geq N_n \) for all \( n \) and we can define \( f_\delta \) on \( K_{\delta+1} \) such that for all \( n \) and \( U \in U_n \) there a \( k \in K_{\delta+1} \cap [M_n, M_{n+1}) \) such that \( (g_\delta(k), f_\delta(k)) \in U \).

We define \( f_\delta \) on \( S_n \cap [M_n, M_{n+1}) \) whenever \( \alpha \in F_n \); because \( M_n \geq N_n \) there will be no interference with the values that we specified on \( K_{\delta+1} \) and between the different \( S_\alpha \)s.

For each \( \alpha \in F_n \) we can simply define \( f_\delta(k) = x_\alpha(\delta) \) for \( k \in S_\alpha \cap [M_n, M_{n+1}) \).

For all \( k \in [M_n, M_{n+1}) \) that are not in \( K_{\delta+1} \), nor in any of the \( S_\alpha \) for some \( \alpha \leq \delta \), we simply choose \( f_\delta(k) \) in such a way that \( (g_\delta(k), f_\delta(k)) \in \bigcup U_n \).

To see that \( f_\delta[S_n] \) converges to \( x_\alpha(\delta) \) it suffices to observe that \( f_\delta \) has the constant value \( x_\alpha(\delta) \) on the intersection \( S_\alpha \cap [M_n, \omega) \), where \( n \) is such that \( \alpha \in F_n \).
2. Some examples of weight $\aleph_1$

One half of Parovichenko’s characterization is a ZFC result: every compact space of weight $\aleph_1$ is a remainder of $\mathbb{N}$. Our proof is in the spirit of Blaszczyk and Szymanski’s proof of this statement in [3]. The main difference is in the number of tasks that need to be done to construct the map $f : \mathbb{N} \to [0, 1]^{\omega_1}$. To make $X$ a remainder involves just $\aleph_1$ many tasks, whereas to make it a soft remainder seems to involve $\aleph$ many tasks as there are that many pairs of subsets of $\mathbb{N}$, hence the need to assume CH.

In this section we present some examples to show that without CH not every compact space of weight $\aleph_1$ is automatically a soft remainder of $\mathbb{N}$.

The ordinal space $\omega_1 + 1$. This relatively simple space already offers some difficulties regarding softness. To be sure: it is a soft remainder, but the proof requires us to consider two cases, depending on the value of the small cardinal $t$.

- if $t > \aleph_1$ then every compactification of $\mathbb{N}$ with $\omega_1 + 1$ as a remainder is soft,
- if $t = \aleph_1$ then some but not all compactifications of $\mathbb{N}$ with $\omega_1 + 1$ as a remainder are soft.

All compactifications of $\mathbb{N}$ with $\omega_1 + 1$ as a remainder have roughly the same structure, as described by Franklin and Rajagopalan in [6].

Let $\gamma\mathbb{N}$ be such a compactification, where we assume that $\mathbb{N}$ and $\omega_1 + 1$ are disjoint. We apply normality to find, for every $\alpha < \omega_1$ open sets $U_\alpha$ and $V_\alpha$ with disjoint closures such that $[0, \alpha] \subseteq U_\alpha$ and $[\alpha + 1, \omega_1] \subseteq V_\alpha$. Then $\gamma\mathbb{N} \setminus (U_\alpha \cup V_\alpha)$ is a compact subset of $\mathbb{N}$, and hence finite. By adding this finite set to $U_\alpha$ we can in fact assume that $U_\alpha \cup V_\alpha = \gamma\mathbb{N}$ for all $\alpha$.

We let $T_\alpha = U_\alpha \cap \mathbb{N}$ for all $\alpha$. The sequence $\langle T_\alpha : \alpha < \omega_1 \rangle$ has the following property:

$(\ast)$ if $\beta < \alpha$ then $T_\beta \setminus T_\alpha$ is finite and $T_\alpha \setminus T_\beta$ is infinite.

Conversely, every such sequence determines a compactification of $\mathbb{N}$. To this end write $T_{\omega_1} = \mathbb{N}$ and then define a topology on $\mathbb{N} \cup (\omega_1 + 1)$ as follows: the points of $\mathbb{N}$ are isolated and if $\alpha \leq \omega_1$ then the sets

$$(\beta, \alpha] \cup (T_\alpha \setminus (T_\beta \cup F))$$

where $\beta < \alpha$ and $F \subseteq \mathbb{N}$ is finite form a local base at $\alpha$. To ensure that the ordinal $0$ is not an isolated point we tacitly assume that $T_0$ is infinite. The sets $U_\alpha = T_\alpha \cup [0, \alpha]$ and $V_\alpha = (\mathbb{N} \setminus T_\alpha) \cup [\alpha + 1, \omega_1]$ are as in the previous paragraph.

Before we investigate how the softness of $\gamma\mathbb{N}$ depends on the sequence $\langle T_\alpha : \alpha < \omega_1 \rangle$ we make a short digression on the cardinal number $t$ alluded to above.
A sequence $\langle T_\alpha : \alpha < \delta \rangle$ that satisfies property (⋆) above is also called a tower. The cardinal number $t$ is defined to be the minimum ordinal $\delta$ for which there is a complete tower: a tower $\langle T_\alpha : \alpha < \delta \rangle$ with the additional property that if $S$ is such that $T_\alpha \setminus S$ is finite for all $\alpha$ then $\mathbb{N} \setminus S$ is finite.

Thus, $t > \aleph_1$ means that every tower $\langle T_\alpha : \alpha < \omega_1 \rangle$ is incomplete: there is an infinite set $R$ such that $T_\alpha \cap R$ is finite, for all $\alpha$, whereas $t = \aleph_1$ means that there is some tower $\langle T_\alpha : \alpha < \omega_1 \rangle$ that is complete, that is, for which no such infinite set exists.

Now let $A$ and $B$ be disjoint subsets of $\mathbb{N}$ such that $\text{cl} A \cap \text{cl} B \neq \emptyset$ in $\gamma \mathbb{N}$. We consider a few cases.

**Case 1:** there is an $\alpha \in \omega_1$ that is in the closure of $A$ and $B$. Then $A$ and $B$ have infinite intersections with $U_\alpha$ and we can take infinite subsets $C$ of $A \cap U_\alpha$ and $D$ of $B \cap U_\alpha$ such that $\text{cl} C = C \cup \{\alpha\}$ and $\text{cl} D = D \cup \{\alpha\}$. Take any permutation $h$ of $\mathbb{N}$ that interchanges $C$ and $D$ and is the identity outside $C \cup D$. Then $h$ extends to an autohomeomorphism of $\gamma \mathbb{N}$ that is the identity on $\mathbb{N} \setminus \mathbb{N}$. By construction $h[A] \cap B$ contains $D$.

**Case 2:** no such $\alpha$ can be found; hence $\text{cl} A \cap \text{cl} B = \{\omega_1\}$. In this case we know that for every $\alpha$ the intersections $V_\alpha \cap A$ and $V_\alpha \cap B$ are both infinite. Now we split into two subcases.

**Subcase 2A:** there are infinite sets subsets $C$ of $A$ and $D$ of $B$ such that $C \cup D \subseteq^* V_\alpha$ for all $\alpha$. Then $C$ and $D$ converge to $\omega_1$ and, as above, interchanging $C$ and $D$ will witness softness.

By the definition of $t$ this subcase occurs, no matter how $\mathbb{N}$ is constructed, if $t > \aleph_1$. This then proves the first statement at the beginning of this subsection.

**Subcase 2B:** one of $A$ and $B$ does not contain an infinite set as in Subcase 2A, say $A$ to be definite. We claim that $\omega_1 \cap \text{cl} A$ is closed and unbounded in $\omega_1$. That it is closed is clear. To show unboundedness let $s \in \omega_1$. The set $A \cap V_s$ is infinite, but by assumption there is a $\beta$ such that $A \cap V_{\alpha} \setminus V_{\beta}$ is infinite. It closure intersects $\omega_1$ and is inside $V_{\alpha} \setminus V_{\beta}$, hence $\text{cl} A$ intersects the interval $[\alpha + 1, \beta]$.

It now follows that $\omega_1 \cap \text{cl} B$ is closed and bounded in $\omega_1$; say $\omega_1 \cap \text{cl} B \subseteq [0, \alpha]$. This means that $D = B \cap V_\alpha$ has $\omega_1$ as its only accumulation point and hence that it converges to $\omega_1$.

In this case the pair $\langle D, A \rangle$ witnesses non-softness: if $h : \gamma \mathbb{N} \to \gamma \mathbb{N}$ is a homeomorphism that is the identity on $\omega_1$ then $h[D]$ converges to $\omega_1$ and so $h[D] \subseteq^* V_\beta$ for all $\beta$ and by our assumption on $A$ this means that $h[D] \cap A$ is finite.

Note that there is no **Subcase 2C:** in that subcase $\omega_1 \cap \text{cl} A$ and $\omega_1 \cap \text{cl} B$ would both be closed and unbounded and this would bring us back to Case 1.
We shall show that under the assumption $t = \aleph_1$ one can construct two compactifications, $\gamma_1 N$ and $\gamma_2 N$, of $N$ and with remainder $\omega_1 + 1$ such that in $\gamma_1 N$ Case 1 always occurs, so $\gamma_1 N$ is soft, and in $\gamma_2 N$ there is a pair $\langle A, B \rangle$ where Subcase 2b occurs, hence $\gamma_2 N$ is not soft.

By the assumption $t = \aleph_1$ there is a complete tower $\langle T_\alpha : \alpha < \omega_1 \rangle$.

We let $\gamma_1 N$ be the compactification of $N$ determined by this tower. Let $A$ and $B$ be disjoint subsets of $N$ whose closures intersect and assume $\omega_1 \in \text{cl} A \cap \text{cl} B$. Because neither $A$ nor $B$ contains an infinite subset as in Subcase 2a the argument in Subcase 2b applies to $A$ and $B$ to show that we are in Case 1 and that $\gamma_1 N$ is a soft compactification of $N$.

To construct the promised non-soft compactification $\gamma_2 N$ we take the sum $\gamma_1 N \oplus \alpha N$ of $\gamma_1 N$ and the one-point compactification $\alpha N$, and identify the points $\omega_1$ and $\infty$.

We let $A$ be the copy of $N$ from $\gamma_1 N$ and $B$ the copy of $N$ from $\alpha N$. This pair witnesses Subcase 2b above and thus shows that $\gamma_2 N$ is not a soft compactification of $N$.

**Remark 2.** As mentioned above the cardinal number $t$ is equal to the number $p$ mentioned in the introduction.

A consequence of this is that we could have concluded right away that $t > \aleph_1$ implies that every compactification of $N$ with remainder $\omega_1 + 1$ is soft. We believe the argument that we gave above is more instructive.

Nevertheless we do record here for future use that under the assumption $t > \aleph_1$ every compactification of $N$ whose remainder has weight $\aleph_1$ or less is soft. This means that if we want to prove, in ZFC, that some specific compact space of weight $\aleph_1$ is soft-Parovichenko we can (or rather must) assume that $t = \aleph_1$.

**The compact ordered space $\omega_1 + 1 + \omega_*$.** The compact ordered space $K = \omega_1 + 1 + \omega_*$ is of weight $\aleph_1$ and hence is a Parovichenko space. It is only slightly more complicated than $\omega_1 + 1$ but, as we shall see, its softness is already undecidable.

We think of $K$ as the quotient of $(\omega_1 + 1) \times 2$ obtained by identifying $\langle \omega_1, 0 \rangle$ and $\langle \omega_1, 1 \rangle$ to one point, which we call $\Omega$.

Note that $(\omega_1 + 1) \times 2$ is a soft remainder, as a sum of two soft compactifications of $N$ is again a soft compactification. This example will therefore show two things: a compact space of weight $\aleph_1$ need not be a soft remainder, and the continuous image of a soft remainder need not be a soft remainder itself.

It is easy to exhibit some compactification of $N$ with remainder $K$ that is not soft, assuming $t = \aleph_1$ of course: take the compactification $\gamma_1 N$ from the previous subsection. We take the quotient of $\gamma_1 N \times 2$ obtained
by identifying $\omega_1 \times 2$ to one point. The sets $A = \mathbb{N} \times \{0\}$ and $B = \mathbb{N} \times \{1\}$ witness non-softness of the resulting compactification: if $C \subseteq A$ is infinite then its closure contains at least one point of $\omega_1 \times \{0\}$; any homeomorphism that maps $C$ into $B$ will move that point to $\omega_1 \times \{1\}$.

We shall show that the principle (NT) from [4] implies that every compactification of $\mathbb{N}$ with remainder $K$ contains two sets that behave like $\mathbb{N} \times \{0\}$ and $\mathbb{N} \times \{1\}$ in the above example.

To formulate (NT) we need to define the notion of a weakly $\sigma$-bounded family of infinite subsets of $\mathbb{N}$: given a family $A$ of infinite subsets of $\mathbb{N}$ we let $A \downarrow$ denote the family of infinite sets $X$ for which there is a member of $A$ that contains it. We call $A$ weakly $\sigma$-bounded if for every countable subfamily $X$ of $A \downarrow$ there is an $A \in A$ such that $A \cap X$ is infinite for all $X \in X$.

The principle (NT) states the following:

for each weakly $\sigma$-bounded subfamily $A$ of $\mathcal{P}(\mathbb{N})$ and each subfamily $B$ of $A$ of cardinality at most $\aleph_1$ there is a subset $C$ of $\mathbb{N}$ such that $C \cap B$ is infinite for all $B \in B$, and for every infinite subset $D$ of $C$ there is an $A \in A$ such that $A \cap D$ is infinite.

In [4] this principle was shown to be consistent with $\mathfrak{c} = \mathfrak{b} = \aleph_2$.

Now let $\delta \mathbb{N} = \mathbb{N} \cup K$ be a compactification, where $K$ is the remainder. For every $\alpha$ we choose pairwise disjoint subsets sets $L_\alpha$, $M_\alpha$ and $R_\alpha$ of $\mathbb{N}$ such that, with closures taken in $\gamma \mathbb{N}$

- $[0, \alpha] \times \{0\} \subseteq \text{cl}L_\alpha$,
- $[\alpha + 1, \omega_1] \times 2 \subseteq \text{cl}M_\alpha$,
- $[0, \alpha] \times \{1\} \subseteq \text{cl}R_\alpha$, and
- the three closures are pairwise disjoint.

We apply the principle (NT) to the families $L = \{L_\alpha : \alpha < \omega_1\}$ and $R = \{R_\alpha : \alpha < \omega_1\}$, and the subfamilies $B_L = \{L_{\alpha+1} \setminus L_\alpha : \alpha < \omega_1\}$ of $L^\downarrow$ and $B_R = \{R_{\alpha+1} \setminus R_\alpha : \alpha < \omega_1\}$ of $R^\downarrow$ respectively.

The families $L$ and $R$ are clearly weakly $\sigma$-bounded: if $B$ is a countable family of infinite sets such that for all $B \in B$ there is an $\alpha_B$ with $B \subseteq L_{\alpha_B}$ then take $\alpha = \sup_B \alpha_B$; the set $L_\alpha$ is as required because $B \subseteq^* L_\alpha$ for all $B \in B$. The same argument works for $R$.

The families $B_L$ and $B_R$ are of cardinality $\aleph_1$ and refine $L$ and $R$, respectively. The principle (NT) then guarantees there are subsets $C_L$ and $C_R$ of $\mathbb{N}$ such that

- $B \cap C_L$ is infinite, for all $B \in B_L$, and, likewise $B \cap C_R$ is infinite, for all $B \in B_R$, and
for every infinite subset $D$ of $C_L$ (or $C_R$) there is an $L \in \mathcal{L}$ (or an $R \in \mathcal{R}$) such that $D \cap L$ (or $D \cap R$) is infinite

We derive some consequences from this.

For every $\alpha$ the set $L_{\alpha+1} \setminus L_\alpha$ converges to the point $\langle \alpha + 1, 0 \rangle$, hence $\langle \alpha + 1, 0 \rangle \in \text{cl} C_L$. It follows that the point in the middle, $\Omega$, is in the closure of $C_L$. And by a symmetric argument $\Omega \in \text{cl} C_R$ also.

The intersection $C_L \cap C_R$ is finite. For if it were infinite then by the second condition in (NT) there is an $\alpha$ such that $L_\alpha \cap C_L \cap C_R$ is infinite, and by a second application of that condition there is a $\beta$ such that $L_\alpha \cap C_L \cap C_R \cap R_\beta$ is infinite. But $L_\alpha \cap R_\beta$ is finite, contradiction. So we may as well assume that $C_L$ and $C_R$ are disjoint.

Now let $h$ be an automorphism of $\gamma \mathbb{N}$ with the property that $h[C_L] \cap C_R$ is infinite. Take an $\alpha$ such that $h[C_L] \cap C_R \cap R_\alpha$ is infinite. Then take a $\beta$ such that $L_\beta \cap C_L \cap h^+[C_R \cap R_\alpha]$ is infinite. Take $\gamma \leq \beta$ such that $\langle \gamma, 0 \rangle$ is in the closure of the latter set; then $h(\gamma, 0)$ is in the closure of $R_\alpha$. Hence certainly $h(\gamma, 0) \neq \langle \gamma, 0 \rangle$.

The Cantor cube $2^{\omega_1}$ and the Tychonoff cube $[0,1]^{\omega_1}$. Two natural compact spaces of weight $\aleph_1$ to consider are the Cantor and Tychonoff cubes $2^{\omega_1}$ and $[0,1]^{\omega_1}$. We shall show that both are soft-Parovichenko.

Since the ordered space $\omega_1 + 1 + \omega_1^*$ is a subspace of both cubes this shows that subspaces of soft-Parovichenko spaces need not be soft-Parovichenko themselves.

As mentioned in Remark 2 we can assume $t = \aleph_1$.

We apply Theorem 3 from [5] and take a set $\mathcal{F}$ of functions from $\mathbb{N}$ to $\mathbb{N}$ that is of cardinality $\omega$ and independent, which means that given $f_1$, ..., $f_k$ in $\mathcal{F}$ and $n_1$, ..., $n_k$ in $\mathbb{N}$ there is an $m \in \mathbb{N}$ such that $f_i(m) = n_i$ for all $i$. One readily checks that this is equivalent to the following: the image of the diagonal map $e : \mathbb{N} \to \mathbb{N}^{\mathcal{F}}$ of the family $\mathcal{F}$ is dense where we consider the product topology on $\mathbb{N}^{\mathcal{F}}$ and the discrete topology on $\mathbb{N}$.

We take an injective sequence $\langle f_\alpha : \alpha < \omega_1 \rangle$ of elements of $\mathcal{F}$ and a complete tower $\langle T_\alpha : \alpha < \omega_1 \rangle$ in $\mathbb{N}$.

We make a technical adjustment to these two sequences, as follows.

Let $N = \{(m,n) \in \mathbb{N}^2 : n \leq m\}$. We define a new tower $\langle L_\alpha : \alpha < \omega_1 \rangle$ and a new independent sequence $\langle F_\alpha : \alpha < \omega_1 \rangle$ of functions on $N$. For every $\alpha$ let

- $L_\alpha = N \cap (T_\alpha \times \mathbb{N})$, and
- define $F_\alpha$ by

$$F_\alpha(m,n) = \begin{cases} f_\alpha(n) & \text{if } m \not\in T_\alpha \text{ and } f_\alpha(n) \neq 0, \\ 0 & \text{otherwise} \end{cases}$$
It is elementary to verify that \( \langle I_\alpha : \alpha < \omega_1 \rangle \) is also a complete tower and that the sequence \( \langle F_\alpha : \alpha < \omega_1 \rangle \) is again independent. We have additionally created some interplay between the two: if \( \beta \geq \alpha \) then \( I_\alpha \setminus F_\beta^{-1}(0) \) is finite.

We adjust the two sequences once more, this time without renaming them. We identify \( N \) with \( N \) via some bijection; and we let the codomains of the functions \( F_\alpha \) be the set \( Q \) of rational numbers in \([0, 1]\), via some bijection that sends 0 to 0.

We let \( e : N \to [0, 1]^{\omega_1} \) be the diagonal map of the sequence \( \langle F_\alpha : \alpha < \omega_1 \rangle \), defined by \( e(n)(\alpha) = F_\alpha(n) \). By the remark above the image set \( e[N] \) is dense in \( Q^{\omega_1} \) where \( Q \) carries the discrete topology, so \( e[N] \) is certainly dense in \([0, 1]^{\omega_1} \).

It follows that \( \beta e : \beta N \to [0, 1]^{\omega_1} \) induces a surjection from \( \beta N \setminus N \) onto \([0, 1]^{\omega_1} \); this surjection determines a compactification of \( N \) with \([0, 1]^{\omega_1} \) as its remainder.

The compactification can be visualised as a subspace of the Alexandroff double \( A([0, 1]^{\omega_1}) \) of \([0, 1]^{\omega_1} \): the underlying set of \( A([0, 1]^{\omega_1}) \) is \([0, 1]^{\omega_1} \times 2 \) with \([0, 1]^{\omega_1} \times \{0\} \) being the set of isolated points; our compactification \( \gamma N \) of \( N \) then is

\[
(\{0\}^{\omega_1} \times \{1\}) \cup (e[N] \times \{0\}).
\]

We identify \( N \) with \( e[N] \times \{0\} \) and \([0, 1]^{\omega_1} \) with \([0, 1]^{\omega_1} \times \{1\} \). We can save on notation by observing for \( x \in [0, 1]^{\omega_1} \) and \( A \subseteq N \) we have \( x \in eA \) iff \( x \) is an accumulation point of \( e[A] \); and \( A \) converges to \( x \) iff \( e[A] \) converges to \( x \).

Let \( A \) and \( B \) be disjoint in \( N \) and let \( x \in [0, 1]^{\omega_1} \) be in the intersection of their closures in \( \gamma N \).

Let \( M \) be a countable elementary substructure of \( H(\mathbb{R})^{+} \) that contains our tower, our independent sequence of functions, and \( A, B \) and \( x \). Let \( \delta = M \cap \omega_1 \).

If \( H \) is a finite subset of \( \delta \) and \( \varepsilon > 0 \) then the basic open set

\[
O(x, H, \varepsilon) = \{ y : (\forall \alpha \in H) (|y_\alpha - x_\alpha| < \varepsilon) \}
\]

meets \( e[A] \) and \( e[B] \) in infinite sets. By elementarity there is an \( \alpha \in \delta \) such that \( I_\alpha \), and hence also \( I_\delta \), has infinite intersections with these infinite sets.

Write \( \delta \) as the union of an increasing sequence \( \langle H_n : n < \omega \rangle \) of finite sets. By the remark above we can find sequences \( \langle a_n : n < \omega \rangle \) and \( \langle b_n : n < \omega \rangle \) in \( A \) and \( B \) such that \( e(a_n) \in O(x, H_n, 2^{-n}) \cap A \cap I_\delta \) and \( e(b_n) \in O(x, H_n, 2^{-n}) \cap B \cap I_\delta \) for all \( n \).

Let \( y \) be the point defined by \( y \upharpoonright \delta = x \upharpoonright \delta \) and \( y(\alpha) = 0 \) for \( \alpha \geq \delta \). We claim that \( \langle a_n : n < \omega \rangle \) and \( \langle b_n : n < \omega \rangle \) converge to \( y \). Indeed, let \( H \) be a finite subset of \( \omega_1 \) and \( \varepsilon > 0 \). Fix \( K \) such that \( H \cap \delta \subseteq H_K \). Also,
using the fact that $I_{\delta} \setminus F_{\alpha}(0)$ is finite for $\alpha \in H \setminus \delta$ find $L \geq K$ such that $F_{\alpha}(a_n) = F_{\alpha}(b_n) = 0$ when $n \geq L$ and $\alpha \in H \setminus \delta$. Then $n \geq L$ implies $a_n, b_n \in O(y, H, \varepsilon)$.

Now take the permutation $h$ of $D$ that interchanges $a_n$ and $b_n$ for all $n$ and leaves the other elements in their places.

This shows that $[0, 1]^{\omega_1}$ is soft-Parovichenko.

To show that $2^{\omega_1}$ is soft Parovichenko one can use basically the same argument as above. The only change that needs to be made is to the diagonal map $e$: let

$$e(n)(\alpha) = \begin{cases} 1 & \text{if } F_\alpha(n) \neq 0, \text{ and} \\ 0 & \text{if } F_\alpha(n) = 0 \end{cases}$$

3. Remarks and questions

The title of this paper uses the words ‘may be’ rather than the word ‘are’ and the previous two sections show why that is. Under the Continuum Hypothesis the ‘are’ is justified but not in general.

In ZFC all compact spaces of weight $\aleph_1$ are Parovichenko, but as we have seen the space $\omega_1 + 1 + \omega_1^*$ is a Parovichenko space that is consistently not soft-Parovichenko.

This state of affairs suggests various further questions about the nature of soft remainders of $\mathbb{N}$.

In exploring the possible parallels between the classes of Parovichenko spaces and soft-Parovichenko spaces we saw that the Continuum Hypothesis simply implies that there is no difference.

The class of Parovichenko spaces is closed under continuous images; the class of soft-Parovichenko spaces is not, consistently.

The class of Parovichenko spaces is, consistently, not closed under subspaces: in the Cohen model where $\aleph = \aleph_2$ the ordinal space $\omega_2 + 1$ is not remainder of $\aleph$ even though it is a subspace of the Parovichenko space $2^{\omega_2}$.

We have seen that the same holds for soft-Parovichenko spaces.

A fair number of known Parovichenko spaces is also soft-Parovichenko; see the list in the introduction. Notably absent in that list are the separable compact spaces, so that will be our first question:

**Question 1.** Is every separable compact space soft-Parovichenko?

There are a few spaces in this class that are worth singling out. Given that every space of weight less than $t$ is a soft remainder we ask in particular

**Question 2.** Are the cubes $2^t$ and $[0, 1]^t$ soft-Parovichenko?
We can ask this for every cardinal in the interval \([t, c]\) but instead we ask whether there is some relationship between these cardinals.

**Question 3.** If \(\kappa < \lambda\) and \(2^\lambda\) is a soft remainder is then \(2^\kappa\) soft as well? Likewise for the Tychonoff cubes.

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