OPEN MAPS: SMALL AND LARGE HOLES
WITH UNUSUAL PROPERTIES

KEVIN G. HARE∗
Department of Pure Mathematics, University of Waterloo
Waterloo, Ontario, Canada N2L 3G1

NIKITA SIDOROV
School of Mathematics, The University of Manchester
Oxford Road, Manchester M13 9PL, United Kingdom

(Communicated by Bryna Kra)

Abstract. Let $X$ be a two-sided subshift on a finite alphabet endowed with a mixing probability measure which is positive on all cylinders in $X$. We show that there exists an arbitrarily small finite overlapping union of shifted cylinders which intersects every orbit under the shift map.

We also show that for any proper subshift $Y$ of $X$ there exists a finite overlapping union of shifted cylinders such that its survivor set contains $Y$ (in particular, it can have entropy arbitrarily close to the entropy of $X$). Both results may be seen as somewhat counter-intuitive.

Finally, we apply these results to a certain class of hyperbolic algebraic automorphisms of a torus.

1. Introduction. This paper is concerned with an area of dynamics which is usually referred to as “maps with holes”, “open maps” or “open dynamical systems”. Let $X$ be a compact (or pre-compact) metric space and $f : X \to X$ be a map with positive topological entropy. Let $H \subset X$ be an open set which we regard as a hole.

In this area of dynamics, we consider the set of points that do not fall into a “hole” under iterations of a map. This is a well studied area with numerous papers. The first paper in this area was [18], where they considered a game of billiards with a hole somewhere in the table. The rate at which a random billiard ball “escaped” was considered, and depended upon both the size and the location of the hole. It is clear that if a hole is enlarged, then the rate of escape will not decrease, and often increases. In [1, 5], it was shown that the location could sometimes play a more significant role in the escape rate than the size of the hole. An important factor in the escape rate is the set of periodic points of the map that fall into the hole [3]. For a good history of these problems, see [9, 10, 11].

The problem that we study, while related, is not exactly the same. We are concerned with the set of points that will always avoid a hole, that is, that will never escape. The precise connection between the theory of escape rates and our results is explained in detail in Remark 7 at the end of the paper.

2010 Mathematics Subject Classification. Primary: 28D05; Secondary: 37B10.
Key words and phrases. Open dynamical system, subshift, unavoidable set.
Research of K. G. Hare was supported by NSERC Grant RGPIN-2014-03154.
∗ Corresponding author: Kevin G. Hare.
Formally, we denote by \( J(H) \) the set of all points in \( X \) whose \( f \)-orbit does not intersect \( H \) and call it the \textit{survivor set}. Clearly, a survivor set is \( f \)-invariant, and in a number of recent papers certain dynamical properties of the map \( \mathcal{J}(H) \) have been studied – see, e.g., [2] and references therein.

It seems that a more immediate issue here is the “size” of a survivor set. At first sight, it would seem plausible that if \( H \) is “large”, then \( J(H) \) is countable – or even empty. On the other hand, if it is “small”, then one might expect the Hausdorff dimension of \( J(H) \) to be positive.

The starting point for this line of research has been the case when \( X = [0, 1] \) and \( T(x) = 2x \mod 1 \), the doubling map with \( T(1) = 1 \). Assume our hole to be connected, so we have \( H = (a, b) \subseteq (0, 1) \). We denote \( J(H) \) by \( J(a, b) \).

\textbf{Theorem 1.1} ([12]).

1. We always have \( \dim_H J(a, b) > 0 \) if

\[
 b - a < \frac{1}{4} \cdot \prod_{n=1}^{\infty} \left( 1 - 2^{-2^n} \right) \approx 0.175092,
\]

and this bound is sharp.

2. If \( b - a \geq \frac{1}{2} \), then \( J(a, b) = \{0, 1\} \).

A similar claim holds for the \( \beta \)-transformation \( x \mapsto \beta x \mod 1 \) for \( \beta \in (1, 2) \) – see [7]. Thus, in the one-dimensional setting one’s naive expectations prove to be spot on.

The situation is however very different for the baker’s map. Namely, put \( X = [0, 1]^2 \); the baker’s map \( B : X \rightarrow X \) is the natural extension of the doubling map, conjugate to the shift map on the set of bi-infinite sequences. We have

\[
 B(x, y) = \begin{cases} 
 (2x, \frac{y}{2}) & \text{if } 0 \leq x < \frac{1}{2}, \\
 (2x - 1, \frac{y + 1}{2}) & \text{if } \frac{1}{2} \leq x < 1.
\end{cases}
\]

We will say that an open set \( H \) is a \textit{complete trap} for the baker’s map if \( J(H) \) does not contain any points except, possibly, an orbit on the boundary of \( X \). (In which case it is symbolically \( \ldots 0000 \ldots 1111 \ldots 11110000 \ldots \) or \( \ldots 00001111 \ldots \)).

\textbf{Theorem 1.2}. [8, Theorem 2.2] For any \( \varepsilon > 0 \) there exists a connected complete trap \( H \) such that its Lebesgue measure \( \text{Leb} \) is less than \( \varepsilon \).

By comparison, we also know

\textbf{Theorem 1.3}. [8, Theorem 2.4] If \( H \) is a hole whose closure is disjoint from the boundary of the square, then \( \dim_H \mathcal{J}(H) > 0 \). In particular, for any \( \varepsilon > 0 \) there exists \( H \) such that \( \text{Leb}(\mathcal{J}(H)) > 1 - \varepsilon \).

The purpose of this note is to extend these two results from the full shift on two symbols to more general subshifts (Theorems 2.2 and 3.1) and apply these to the generalized Pisot toral automorphisms (Section 4).

2. Symbolic model: Small holes. Let \( \mathcal{A} \) be a finite alphabet. Let \( \sigma : \mathcal{A}^\mathbb{N} \rightarrow \mathcal{A}^\mathbb{N} \) denote the left shift, i.e.,

\[
 \sigma(\ldots x_{-2}x_{-1}x_0, x_1x_2x_3 \ldots) = \ldots x_{-2}x_{-1}x_0x_1, x_2x_3 \ldots
\]

\[1\]If we regard the doubling map as the map \( z \mapsto z^2 \) on the unit circle \( S^1 \), then \( \frac{1}{2} \) needs to be replaced with \( \frac{2}{3} \), in which case \( \mathcal{J}(a, b) \subset \{1\} \).
We define distance on \( A^\mathbb{N} \) with \( A = \{0, 1, \ldots, k - 1\} \) in the usual way as
\[
d(x, y) = \sum_{i \in \mathbb{Z}} \frac{|x_i - y_i|}{k^{|i|}}.
\]
This topology is equivalent to the product topology. We say that \( X \subset A^\mathbb{N} \) is a **subshift** if \( X \) is closed under this topology, and \( \sigma(X) = X \). Suppose \( \mu \) is a probability \( \sigma \)-invariant measure on \( X \) which we assume to be positive on all cylinders. We denote the set of admissible words of length \( n \) for \( X \) by \( \mathcal{L}_n(X) \) and put \( \mathcal{L}(X) = \bigcup_{n \geq 0} \mathcal{L}_n(X) \), the *language* of \( X \).

The topological entropy of a subshift \( X \) is defined by the formula
\[
h(X) = \lim_{n \to \infty} \frac{\log \# \mathcal{L}_n(X)}{n}.
\]
Recall that a subshift \((X, \sigma)\) is called **irreducible** if, for every ordered pair of words \( u \) and \( v \in \mathcal{L}(X) \), there is a \( w \in \mathcal{L}(X) \) with \( uwv \in \mathcal{L}(X) \). We say a subshift is **sofic** if there exists a regular language (i.e., a language accepted by a finite automaton) such that \( X \) is the set of all infinite sequences that do not contain a subword from this regular language. For more detailed see [15, Chapter 3].

An irreducible sofic subshift is known to have a unique measure of maximal entropy \( \mu \) (see [26]), which has the following property (see [13, Lemma 4.8]): there exists \( \theta \in (0, 1) \) such that
\[
\mu([w_{-n+1} \ldots w_0, w_1 \ldots w_n]) \geq \theta^n, \quad \forall w_{-n+1} \ldots w_n \in \mathcal{L}_{2n}(X).
\]
(Clearly, \( \theta = \frac{1}{2} \exp(-h(X)) \), in view of the Shannon–McMillan–Breiman theorem.)

Let \( w = w_1w_2 \ldots w_n \in \mathcal{L}(X) \); we denote by \([w]\) the cylinder given by all \( \{x_j\}_{j \in \mathbb{Z}} \in X \) with \( x_1 = w_1, x_2 = w_2, \ldots, x_n = w_n \).

Recall that \( \mu \) is called **non-atomic** if \( \mu(\{x\}) = 0 \) for any \( x \in X \). We say that a measure \( \mu \) is **mixing** if for any pair of cylinders \([w], [w']\) we have \( \mu(\sigma^{-n}[w] \cap [w']) \to \mu([w])\mu([w']) \) as \( n \to +\infty \). Similarly to the case of the baker’s map, we say that a Borel \( A \subset X \) is a **complete trap** if it intersects all orbits in \( X \), i.e., for any \( x \in X \) there exists an \( n \in \mathbb{Z} \) such that \( \sigma^n x \in A \).

**Remark 1.** Notice that a mixing measure positive on all cylinders is always non-atomic. Indeed, assume that \( \mu(A) > 0 \), where \( A = \{x\} \). Then by mixing, \( \mu(\sigma^{-n}A \cap A) \to \mu(A)^2 \), which is impossible unless \( x \) is a fixed point for \( \sigma \). If \( x \) is such, then it must have \( x_i \equiv j \) for all \( i \in \mathbb{Z} \) and we have \( \mu(A) = 1 \), which is impossible, since \( \mu \) cannot be supported by a single orbit, being positive on all cylinders.

**Remark 2.** A similar notion exists in Combinatorics on Words, namely, unavoidable sets. A subset \( S \) of \( A^* \) is called **unavoidable** if any word in \( A^* \) contains an element of \( S \) as a factor. These have been introduced by M.-P. Schützenberger in [21], and various results, mostly regarding their size, have been proved since – see, e.g., [6] and references therein.

In the case of \([0, 1] \) and \([0, 1]^2 \) in Section 1, we found connected components with unusual properties. If \( X \) is a subshift, then \( X \) is totally disconnected, so \( X \) has no non-trivial connected components. Instead we consider what we believe is a natural analog of connected, in the case of subshifts. That is, being a finite union of overlapping shifted cylinders. More formally

**Definition 2.1.** Let \( C = \{\sigma^{-i_1}[w_{i_1}], \ldots, \sigma^{-i_n}[w_{i_n}]\} \) be a finite collection of shifted cylinders. We say that \( C \) is **overlapping** if for all \( i \) and \( j \) there exists \( i = i_1, i_2, \ldots, i_n = j \) such that \( \mu(\sigma^{-i}[w_{i_k}] \cap \sigma^{-j}[w_{j_{k+1}}]) > 0 \).
The main result of this section is that for each two-sided subshift with mild assumptions there exist arbitrarily small overlapping complete traps. More precisely:

**Theorem 2.2.** Let $X$ be a two-sided subshift on a finite alphabet $A$ endowed with a shift-invariant probability measure $\mu$ on $X$ which we assume to be positive on all cylinders and mixing (hence non-atomic) in $X$.

Then for any $\epsilon > 0$ there exists a finite union of shifted cylinders $C := \bigcup_{i=1}^{n} \sigma^{-r_i} [w_i]$ such that

1. $\mu(C) < \epsilon$.
2. $C$ is a complete trap.
3. for all $i, j$ we have $\mu(\sigma^{-r_i}[w_i] \cap \sigma^{-r_j}[w_j]) > 0$.

This will be proven by a series of lemmas.

**Lemma 2.3.** Assume that $\mu$ is a shift-invariant non-atomic measure on $X$ and $w = w_1w_2\ldots$ is such that $w_1\ldots w_n \in \mathcal{L}_n(X)$ for all $n \geq 1$. Then $\mu[w_1\ldots w_n] \to 0$ as $n \to \infty$.

**Proof.** Since $\mu$ is shift-invariant, we have

$$\mu[w_1\ldots w_n] = \mu[w_1\ldots w_{n/2}] \cdot w_{n/2+1}\ldots w_n,$$

so the claim follows from $\mu$ being non-atomic.

**Lemma 2.4.** Let $\mu$ is a shift-invariant non-atomic measure on a subshift $X$. Let $C = \bigcup \sigma^{-r_i}[w_i]$ be a finite union of shifted cylinders with all $r_i \geq 0$. For all $\epsilon > 0$ we can write $C = \bigcup [w'_i]$ as a disjoint union of cylinders where $w'_i$ have equal length, and $\mu([w'_i]) < \epsilon$ for all $i$.

**Proof.** It is worth observing that the $\sigma^{-r_i}[w_i]$ may overlap, and the $w_i$ may have different lengths. For $N \geq |w_i| + r_i$, we see that we can find a prefix $u$ of length $r_i$ and suffix $v$ of length $N - r_i - |w_i|$ such that $uw_i v \in \mathcal{L}(X)$ is a word of length $N$. We let $S$ be the set of all such $uw_i v$ for all pairs $w_i$ and $r_i$. This gives us a set $C = \bigcup_{w_i \in S} [w'_i]$ of disjoint union of cylinders of equal length. The fact that we can choose $N$ sufficiently large so that $\mu([w]) < \epsilon$ follows from Lemma 2.3.

**Lemma 2.5.** Let $X$ be a two-sided subshift on a finite alphabet $A$ endowed with a shift-invariant probability measure $\mu$ on $X$ which we assume to be positive on all cylinders and mixing (hence non-atomic) in $X$. Let $C = \bigcup [w_i]$ be a finite union of disjoint cylinders and a complete trap. Then there exists $r_i \geq 0$ such that such that

1. $C' = \bigcup \sigma^{-r_i}[w_i]$ is a complete trap,
2. $\mu(C') \leq \mu(C)$ and
3. for all $i, j$ we have $\mu(\sigma^{-r_i}[w_i] \cap \sigma^{-r_j}[w_j]) > 0$.

**Proof.** As $\mu$ is positive on all cylinders, we see that $\mu([w_i]) > 0$ for all choices $i$. For all $w_i$ and $w_j$ there will exist a $M_{i,j}$ such that for all $n \geq M_{i,j}$ we have $\mu([w_i]) \cap \sigma^{-n}([w_j])) > 0$. Let $M := \max_{i,j} M_{i,j}$. Taking $r_i = M \cdot i$ satisfies the desired properties.

**Lemma 2.6.** Let $n \geq 2$, $0 \leq x_1 \leq \cdots \leq x_n \leq \frac{1}{2}$ with $\sum_{i=1}^{n} x_i = 1$. Then there exists $k \in \{1, \ldots, n\}$ such that $\sum_{i=1}^{k} x_i \geq \frac{1}{4}$ and $\sum_{i=k+1}^{n} x_i \geq \frac{1}{4}$.

**Proof.** If $x_1 \geq \frac{1}{4}$, then we take $k = 1$. Otherwise let $k$ be such that $\sum_{i=1}^{k-1} x_i < \frac{1}{4}$ and $\sum_{i=1}^{k} x_i \geq \frac{1}{4}$. Then $\sum_{i=1}^{k} x_i = 1 - \sum_{i=1}^{k-1} x_i - x_k \geq \frac{1}{2} - \sum_{i=1}^{k-1} x_i \geq \frac{1}{4}$. 

\[\boxdot\]
Lemma 2.7. Let $X$ be a two-sided subshift on a finite alphabet $A$ endowed with a shift-invariant probability measure $\mu$ on $X$ which we assume to be positive on all cylinders and mixing (hence non-atomic) in $X$.

Then for any $\varepsilon > 0$ there exists a $C := \bigcup[w_i]$, a finite union of disjoint cylinders such that

1. $\mu(C) < \varepsilon$.
2. $C$ is a complete trap.

Proof. Put $C_0 = \bigcup_{a \in A} [a]$. Clearly $C_0 = X$, and hence is a complete trap.

We proceed by induction. Assume we have a finite collection of cylinders $C_n = \bigcup \sigma^{-r_i}[w_i]$ which is a complete trap. Using Lemma 2.4 we write $C_n = \bigcup[w'_i]$, a disjoint union of cylinders where $\mu([w'_i]) < \mu(C_n)/2$ and all $w'_i$ are the same length, say $N$.

By Lemma 2.6 and the fact that all cylinders have $\mu([w']) < \mu(C_n)/2$, we can partition the set of cylinders into two sets $C'_n$ and $C''_n$ where

$$\mu(C_n)/4 < \min \left\{ \mu \left( \bigcup_{w' \in C'_n} [w'] \right), \mu \left( \bigcup_{w'' \in C''_n} [w''] \right) \right\}.$$ 

Since $\mu$ is mixing, there exists $\ell$ such that $\mu(\sigma^{-\ell}[w'] \cap [w'']) \geq \frac{1}{2} \mu([w']) \mu([w''])$ for all $[w'] \in C'_n$ and all $[w''] \in C''_n$. Put

$$C_{n+1} = \left( \bigcup_{w' \in C'_n} [w'] \right) \bigcup \left( \bigcup_{w'' \in C''_n} \sigma^{-\ell}[w''] \right)$$

We have

$$\mu(C_{n+1}) = \sum_{w' \in C'_n} \mu([w']) + \sum_{w'' \in C''_n} \mu([w'']) - \sum_{w' \in C'_n, w'' \in C''_n} \mu([w'] \cap \sigma^{-\ell}[w'']) \leq \mu(C'_n) + \mu(C''_n) - \sum_{w' \in C'_n, w'' \in C''_n} \frac{1}{2} \mu([w']) \mu([w''])$$

$$= \mu(C_n) - \frac{1}{2} \mu(C'_n) \mu(C''_n) \leq \mu(C_n) - \frac{1}{32} \mu(C_n)^2$$

Put $t_n = \mu(C_n)$. Then we have

$$t_{n+1} \leq t_n - \frac{t_n^2}{32}. \tag{2}$$

Clearly, $t_n$ is decreasing and positive; let $L = \lim_{n \to \infty} t_n$. Then $L \leq L - L^2/32$, whence $L = 0$.

Choose $n$ such that $\mu(C_n) < \varepsilon$ and use Lemma 2.4 to write $C := C_n$. This union in the desired form.

Remark 3. It follows from the proof of Lemma 2.7 that $\mu$ being mixing can be replaced with the following, weaker, condition: there exists a $\delta > 0$ such that for all cylinders $[w], [w'] \subset X$ we have $\liminf_{n \to \infty} \mu(\sigma^{-n}[w] \cap [w']) \geq \delta \mu([w]) \mu([w'])$ – provided we assume that $\mu(\{x\}) = 0$ for any fixed point $x \in X$.

Remark 4. The sequence $t_n$ in (2) tends to 0 as $\approx 1/n$. This is consistent with the theory of unavoidable sets (see Remark 2), where it is shown for that the minimal size of an unavoidable set (for the full shift on $A$) is $\gg |A|^n/n$ ([21, 16]).
3. **Symbolic model: Large holes.** The goal of this section is to extend Theorem 1.3 to more general subshifts.

**Theorem 3.1.** Let $X$ be a subshift endowed with a mixing probability measure of maximal entropy $\mu$. Let $Y \subset X$ be a subshift such that $0 < h(Y) < h(X)$ (i.e., a proper subshift of $X$).

Then for any $\varepsilon > 0$ there exists a finite overlapping union of cylinders $G$ such that $\mu(G) > 1 - \varepsilon$ and $Y \subset J(G)$. (So, in particular, $h(J(G)) > 0$.)

**Proof.** Let $L_n(Y)$ be as above. Since $h(Y) < h(X)$, we have $\frac{\#L_n(Y)}{\#L_n(X)} \to 0$ as $n \to \infty$.

Let $\Sigma_n = \{[w] : w \in L_n(X)\}$ and $\Sigma'_n = \Sigma_n \setminus \{[w] : w \in L_n(Y)\}$. We have

$$
\mu(\Sigma'_n) = \mu([w] : w \in L_n(X)) - \mu([w] : w \in L_n(Y)) = 1 - \mu([w] : w \in L_n(Y)) \to 1,
$$

as $n \to \infty$.

We see that $Y \subset J(\Sigma'_n)$ and thus, $h(J(\Sigma'_n)) > 0$. To get overlapping, let $[w^*] \in \Sigma'_n$ be such that $\mu([w^*])$ is minimized. We note here that $\mu([w^*]) > 0$ as $\mu$ is positive on all cylinders. Consider

$$
G_n = (\Sigma'_n \setminus \{[w^*]\}) \cup \{\sigma^{-K}([w^*])\}
$$

for $K$ sufficiently large so that $\mu([w] \cap \sigma^{-K}([w^*])) < 0$ for all $[w] \in \Sigma'_n$ (by mixing). That is, so that $G_n$ is overlapping.

We have $h(J(G_n)) > 0$ as $Y \subset J(G_n)$. Let $S_n$ be the number of cylinders in $\Sigma'_n$. Clearly $S_n \to \infty$ as $n \to \infty$. We see that $\mu(G_n) > \frac{S_n - 1}{S_n} \mu(\Sigma'_n) \to 1$ as $n \to \infty$. Furthermore, $G_n$ is overlapping. Taking $n$ sufficiently large, we get that $G := G_n$ has the desired property.

**Remark 5.** The condition $0 < h(Y) < h(X)$ may be seen as a symbolic analogue of $\mathcal{H}$ being disjoint from the boundary of the square for the baker’s map.

4. **Application: Hyperbolic toral automorphisms.**

4.1. **Background.** Let $\beta > 1$ be non-integer and $\tau_\beta$ be the $\beta$-transformation, i.e., the map from $[0,1)$ onto itself acting by the formula

$$
\tau_\beta(x) = \beta x \mod 1.
$$

As is well known, to “encode” it, one needs to apply the greedy algorithm in order to obtain the digits in the $\beta$-expansion, namely,

$$
x = \pi_\beta((a_n)_{n=1}^\infty) := \sum_{n=1}^\infty a_n \beta^{-n}, \quad (3)
$$

where $a_n = \lfloor \beta^{n-1} \rfloor$, $n \geq 1$. Then the one-sided left shift $\sigma^+_{\beta}$ on the space $X^+_{\beta}$ of all possible sequences $(a_n)_{n=1}^\infty$ which can be obtained this way, is isomorphic to $\tau_\beta$, with the conjugating map given by (3). We will call $\sigma^+_{\beta}$ the $\beta$-shift. It is obviously a proper subshift of the full shift on $\prod_{n=1}^\infty \{0,1,\ldots,\lfloor \beta \rfloor\}$.

W. Parry in his seminal paper [17] proved the following. Let the sequence $(d_n)_{n=1}^\infty$ be defined as follows: let $1 = \sum_{k} d'_k \beta^{-k}$ be the greedy expansion of 1, i.e., $d'_n = \lfloor \beta^{n-1} \rfloor$, $n \geq 1$; if the tail of the sequence $(d'_n)$ differs from 0$^\infty$, then we put $d_n \equiv d'_n$. Otherwise let $k = \max \{j : d'_j > 0\}$, and $(d_1,d_2,\ldots) := (d'_1,\ldots,d'_{k-1},d'_k-1)^\infty$.
Then 
\[ X_\beta^+ = \{(a_n)_{n=1}^\infty \in \{0,1,\ldots,|\beta|\}^\infty : a_na_{n+1}\cdots < d_1d_2, \ n \geq 1\}, \]
(where \(\prec\) stands for the lexicographic order) and the following diagram commutes:

\[
\begin{array}{ccc}
X_\beta^+ & \xrightarrow{\sigma_\beta} & X_\beta^+ \\
\pi_\beta & & \pi_\beta \\
\downarrow & & \downarrow \\
[0,1) & \xrightarrow{\tau_\beta} & [0,1)
\end{array}
\]

We restrict our attention to those \(\beta\) where assume that \((d_n')_1^\infty\) does not have unbounded strings of 0s. It follows that there exists \(\ell = \ell(\beta) \geq 1\) such that

\[ u,v \in \mathcal{L}(X_\beta^+) \implies u\ell^d v \in \mathcal{L}(X_\beta^+). \tag{4} \]

Similar to before, let \(w = w_1w_2\ldots w_n \in \mathcal{L}(X_\beta^+)\); we denote by \([w]^+\) the cylinder given by all \(\{x_j\}_{j \in \mathbb{N}} \in X_\beta^+\) with \(x_1 = w_1, x_2 = w_2, \ldots, x_n = w_n\). As also proved in [17, 19], there exists a unique probability measure \(\mu_\beta\) invariant under the \(\tau_\beta\) which is equivalent to the Lebesgue measure, with a density bounded from 0 and \(\infty\). Furthermore, \(\mu_\beta(\pi_\beta([w]^+)) \asymp \beta^{-n}\) for any cylinder \([w]^+ = [w_1 \ldots w_n]^+ \subset X_\beta^+\), provided \((d_n')_1^\infty\) does not have unbounded strings of 0s.

Let \(m \geq 2\) and \(M\) be an \(m \times m\) matrix with integer entries and determinant \(\pm 1\). Then \(M\) determines the algebraic automorphism of the \(m\)-torus \(\mathbb{T}^m := \mathbb{R}^m/\mathbb{Z}^m\), which we will denote by \(T_M\). That is, \(T_Mx = Mx \mod \mathbb{Z}^m\). We assume \(M\) to have a characteristic polynomial \(p\) irreducible over \(\mathbb{Q}\).

Assume that \(T_M\) is hyperbolic, i.e., that \(p\) has no roots of modulus 1. Let \(t\) be a homoclinic point for \(T_M\), i.e., \(T_M^n t \to 0\) as \(n \to \pm \infty\). Let \(X\) be a two-sided subshift on a finite alphabet, and define the map \(\phi_t : X \to \mathbb{T}^m\) as follows:

\[ \phi_t(a) = \sum_{n \in \mathbb{Z}} a_n T_M^{-n} t, \tag{5} \]

where \(a = (a_n)_{n \in \mathbb{Z}}\). These maps have been studied in [20, 23, 24, 25]. Note first that \(\phi_t\) is well defined, since, as is well known, \(T_M^n T_M \to 0\) at an exponential rate, so this bi-infinite series converges. Also, \(\phi_t\) is Hölder continuous, for the same reason. Most importantly, we have \(\phi_t \sigma = T_M \phi_t\), i.e., \(\phi_t\) semiconjugates the shift and \(T_M\). It is known (see [20]) that one can choose \(L \geq 1\) large enough so that if \(X = \{-L, \ldots, L\}^\mathbb{Z}\) is the full shift, then \(\phi\) is surjective.

Assume now that \(p\) has one real root of modulus greater than 1 (\(\beta\), say) and the rest are less than 1 in modulus. Then \(\beta\) is called a Pisot number (a Pisot unit, to be more precise, in view of \(\det M = \pm 1\)) and \(T_M\) a Pisot automorphism. In this case we have a natural choice for \(X\), namely, \(X = X_\beta\), i.e., the natural extension of \(X_\beta^+\) endowed with the measure \(\mu_\beta\), the natural extension of \(\pi_\beta^{-1}(\mu_\beta^+)\) to \(X_\beta\). (This is the measure of maximal entropy for the subshift.) Let \(\sigma_\beta : X_\beta \to X_\beta\) denote the corresponding left shift. Note that since \(\beta\) is Pisot, \((d_n')_1^\infty\) is eventually periodic and therefore, cannot contain unbounded strings of 0s (see, e.g., [4]).

As is well known, any homoclinic point \(t\) can be obtained by projecting a point in \(\mathbb{Z}^m\) onto the leaf of the unstable manifold for \(T_M\) passing through 0 along the stable manifold. In the Pisot case this implies that \(T_M t = \beta t\), whence (5) can be written as

\[ \phi_t(a) = \sum_{n \in \mathbb{Z}} a_n \beta^{-n} t. \tag{6} \]
It has been shown independently in [20] and [23] that $\phi_t$ is surjective and finite-to-one, i.e., there exists $M \geq 1$ such that $\phi_t^{-1}(x)$ is at most $M$ points for any $x \in \mathbb{T}^m$. Furthermore, $(X, \sigma_\beta, \mu_\beta)$ is known to be irreducible sofic in this setting ([4]).

Recall, we define distance on a subshift $(X, \mu, \sigma)$ with the alphabet $A = \{0, 1, \ldots, k - 1\}$ in the usual way as

$$d(x, y) = \sum_{i \in \mathbb{Z}} \frac{|x_i - y_i|}{k^{|i|}}.$$ 

We denote distance on $\mathbb{T}^m$ by $|x - y|$. We say that a map $\phi : X \to \mathbb{T}^m$ is $\alpha$-Hölder continuous if there exists a $C > 0$ such that for all $x, y \in X$ we have

$$|\phi(x) - \phi(y)| \leq Cd(x, y)^\alpha.$$ 

4.2. Auxiliary results.

**Lemma 4.1.** Let $(X, \mu, \sigma)$ be an irreducible sofic subshift endowed with the measure of maximal entropy, on the alphabet $A$ of cardinality $k$. Assume we have an $\alpha$-Hölder continuous map $\phi : X \to \mathbb{T}^m$ such that $\phi \sigma = T_\alpha \phi$.

Then there exist $C, \kappa > 0$ such that for any Borel set $A \subset X$ we have

$$\mathcal{H}_m(\phi(A)) \leq C \mu(A)^\kappa.$$ 

Here $\mathcal{H}_m$ is the Haar measure on $\mathbb{T}^m$ (= the $m$-dimensional Lebesgue measure restricted to $\mathbb{T}^m$).

**Proof.** Without loss of generality assume $A = \{0, 1, \ldots, k - 1\}$. Since $\phi$ semi-conjugates $\sigma$ and $T_\alpha$, it suffices to prove our claim for an arbitrary cylinder $A = [w_{-n} \ldots w_0, w_1 \ldots w_n] \subset X$. By (1), we have $\mu(A) \asymp \theta^n$; we also have $\text{diam} A \asymp k^{-n}$.

Hence

$$\text{diam} \phi(A) = O(k^{-n\alpha}) = O(\mu(A)^\gamma),$$

with $\gamma = -\alpha \log k$. Furthermore,

$$\mathcal{H}_m(\phi(A)) = O(\text{diam} \phi(A)^m) = O((\mu(A)^\kappa),$$

with $\kappa = m\gamma$.

**Corollary 1.** The claim of Lemma 4.1 holds for $(X_\beta, \mu_\beta, \sigma_\beta)$.

**Lemma 4.2.** For any cylinder $\sigma^k[w] \subset X_\beta$ its image under $\phi_t$ is path connected.

**Proof.** Since $T_\alpha$ is continuous, we have that if $\phi_t([w])$ is connected, then so is $\phi_t(\sigma^k([w]))$ for all $k \in \mathbb{Z}$, in view of $\phi_t\sigma_\beta^k = T_\alpha^k\sigma_\beta$. So, without loss of generality, we may prove the result for $[w]$ only.

Recall that $\pi_\beta(X_\beta^+) = [0, 1)$, a path connected set. This implies that for any cylinders $[w]^+ = [w_1 w_2 \ldots w_N]^+$ and $[w']^+ = [w'_1 w'_2 \ldots w'_N]^+$ in $X_\beta^+$, we have that there exists a chain of cylinders in $X_\beta^+$, all of length $N$, namely, $[w^{(j)}]^+ = [w]^+, [w^{(j+1)}]^+ \subset [w^{(j)}]^+$, such that $\pi([w^{(j)}]^+) \cap [w^{(j+1)}]^+ \neq \emptyset$ for $j = 1, \ldots, k - 1$. The key observation to see this is, for $w_n \neq 0$ that $w_{1} w_{2} \ldots w_{n-1} w_{n} 0 0 0 \ldots = w_{1} w_{2} \ldots w_{n-1} (w_{n} - 1) d_{1} d_{2} d_{3} \ldots$ and hence $[w_{1} w_{2} \ldots w_{n-1} w_{n}]^+ \cap [w_{1} w_{2} \ldots w_{n-1} (w_{n} - 1)]^+ \neq \emptyset$. Similarly $[w_{1} w_{2} \ldots w_{n-1} 0] + \cap [w_{1} w_{2} \ldots (w_{n-1} - 1) d_{1}]^+ \neq \emptyset$, $[w_{1} w_{2} \ldots w_{n-2}]^+ \cap [w_{1} w_{2} \ldots (w_{n-2} - 1) d_{1} d_{2}]^+ \neq \emptyset$, etc.

Now the claim follows from (6); indeed, for any two cylinders $[w], [w']$ in $X_\beta$, we can use the same chain as above (to be more precise, their two-sided analogues), and the images $[w^{(j)}]$ and $[w^{(j+1)}]$ under $\phi_t$ will intersect as well.

\[ \square \]
Lemma 4.3. For any cylinder $C \subset X_\beta$ the image $\phi_t(C)$ has non-empty interior.

Proof. Let $\Xi_n$ denote the set of all cylinders $[w_{-n+1} \ldots w_0, w_1 \ldots w_n]$ in $X_\beta$. Since $\phi_t$ is surjective, there exists $C = [w_{-n+1} \ldots w_n] \in \Xi_n$ such that $\phi_t(C)$ has non-empty interior. Notice that

$$\phi_t(C) \subset \phi_t([0 \ldots 0, 0 \ldots 0]) + \sum_{-n+1}^{n} w_j \beta^{-j} t,$$

as the word with all 0s has no constraints in $X$ which proves the claim, since any cylinder contains a longer one, hence $2n$ for now. This implies that $[0 \ldots 0, 0 \ldots 0]$ has the property in question. Hence so does $[0 \ldots 0, 0 \ldots 0] \subset \Xi_{n+\ell}$.

Now, by (4), for any word $w'_{-n+1} \ldots w_n' \in \mathcal{L}_2(X_\beta)$ we have

$$\phi_t([w'_{-n+1} \ldots w_n']) \supset \phi_t([0 \ldots 0, 0 \ldots 0]) + \sum_{-n+1}^{n} w'_j \beta^{-j} t,$$

which proves the claim, since any cylinder contains a longer one, hence $2n \geq \ell$ is not a real constraint.

Let $\beta_1 = \beta, \beta_2, \ldots, \beta_m$ be the Galois conjugates of $\beta$ and put

$$A_\beta = \{x \in X_\beta : x_i = 0 \text{ for all } i \geq 1\}.$$

Then $\phi_t(A_\beta)$ lies in $W$, the $(m-1)$-dimensional span of the eigenvectors for $\beta_2, \ldots, \beta_m$. Recall that $|\beta_j| < 1$ for $j \in \{2, \ldots, m\}$. Notice that since $\prod_{j}^m |\beta_j| = 1$, we have

$$\min\{|\beta_2|, \ldots, |\beta_m|, |\beta^{-1}|\} = |\beta^{-1}|.$$

(7)

Corollary 2. The set $\phi_t(A_\beta)$ has non-empty interior in $W$.

Lemma 4.4. There exists $c = c(M) > 0$ such that for any cylinder $C \subset \Xi_n$ we have that $\phi_t(C)$ contains a cube whose sides are aligned with the axes, with side $c \beta^{-n}$.

Proof. Let $C = [w_{-n+1} \ldots w_0, w_1 \ldots w_n]$. Consider

$$C' := \sigma_{-\beta}^{-n-\ell+1}(C) = [x_\ell = w_{-n+1}, \ldots, x_{2n+\ell-1} = w_n].$$

By (4),

$$\phi_t(C') \supset \phi_t(A) + \phi_t([0^\ell, w_{-n+1} \ldots w_n]).$$

By Corollary 2, the first summand contains an $(m-1)$-dimensional cube with side $c_1$, say. The second summand contains an interval which is transversal to $\phi_t(A)$, of length $\geq c_2 \beta^{-2n}$.

This implies that $\phi_t(C')$ contains a box with dimensions $c_1 \times \ldots \times c_n \times c_2 \beta^{-2n}$. Now, $C = \sigma_{\beta}^{n+\ell-1}(C')$. The map $T_{h_M}$ contracts on $W$, with the contraction ratios $\geq \beta^{-1}$—see (7). On the one-dimensional eigenspace corresponding to $\beta$, it expands by $\beta$. Hence follows the claim. □

4.3. Toral automorphisms with small holes.

Definition 4.5. Let $X$ be a compact metric space, $T : X \to X$ be a continuous invertible map and $\mu$ be an ergodic $T$-invariant probability measure. We say that the dynamical system $(X, \mu, T)$ possesses Property S if for any $\varepsilon > 0$ there exists an open connected subset $A$ of $X$ such that

1. $\mu(A) < \varepsilon$;
2. for all, except, possibly, a countable set of $x \in X$, there exists $n = n(x) \in \mathbb{Z}$, such that $T^n x \in A$. 

Theorem 4.8. Let \( \varphi \) be a H"older continuous, finite-to-one map. None of these appears to produce an explicit symbolic coding space.

Suggested by various authors - see [22, Section 4] for more detail. Unfortunately, models (similar to the one described above for the Pisot automorphisms) have been considered in the literature.

Theorem 4.7. Any generalized Pisot toral automorphism has Property S.

Proof. Suppose \( T = T_M \) is a generalized Pisot. Since the orbit of \( T \) is the same as that of \( T^{-1} \) for any \( x \in \mathbb{T}^n \), it is clear from the definition that a complete trap for \( T \) is a complete trap for \( T^{-1} \).

Let \( S = -T \) and put \( \mathcal{E} = \mathcal{D}' \cup (-\mathcal{D}') \), where \( \mathcal{D}' \) is constructed as above. Clearly, \( \mathcal{E} \) is open and has a small measure if \( \mathcal{D}' \) does. Since \( S^n = T^n \) if \( n \) is even and \( -T^n \) otherwise, we have that \( \mathcal{E} \) is a complete trap for \( S \) if \( \mathcal{D}' \) is such for \( T \). To make \( \mathcal{E} \) connected, we connect \( \mathcal{D}' \) and \(-\mathcal{D}'\) by an open “tunnel”, which can be made arbitrarily small in measure. Let \( \mathcal{E}' \) denote the resulting set, which is clearly small in measure and a complete trap for \( S \).

Now, the same \( \mathcal{E}' \) works for the case \( U = -T^{-1} \), which completes the proof. \( \square \)

Corollary 3. Any hyperbolic automorphism of \( T^2 \) or \( T^3 \) has Property S.

Proof. Clearly, any hyperbolic \( 2 \times 2 \) matrix has one eigenvalue of modulus greater than one and one less than one. This makes it generalized Pisot.

For a \( 3 \times 3 \) hyperbolic matrix \( M \) we have that one of its eigenvalues is less than one in modulus and two greater than one or the other way round. In either case, \( T_M \) is generalized Pisot so we can apply Theorem 4.7. \( \square \)

Consider now a general hyperbolic toral automorphism \( T_M \). Arithmetic symbolic models (similar to the one described above for the Pisot automorphisms) have been suggested by various authors - see [22, Section 4] for more detail. Unfortunately, none of these appears to produce an explicit symbolic coding space.

The following result has been proved by S. Le Borgne:

Theorem 4.8. [14] There exists an irreducible sofic shift \((X, \nu, \sigma)\) and a surjective, H"older continuous, finite-to-one map \( \psi : X \to \mathbb{T}^n \) which semiconjugates \( \sigma \) and \( T_M \).

This result combined with Theorem 2.2 yields that \( T_M \) possesses Property S, except that a hole may not be connected. This is because it is unclear whether \( \psi(C) \) is connected for each cylinder \( C \subset X \) (S. Le Borgne, private communication).

Remark 6. It is shown in [8] that if we restrict our class of holes to convex ones for the baker’s map, then there exists \( \delta > 0 \) such that for any \( H \) of Lebesgue measure less than \( \delta \), we have \( \dim_H f(H) > 0 \). It would be interesting to establish...
an analogous result for hyperbolic toral automorphisms. In this case the class of holes in question would be probably geodesically convex ones.\footnote{\textsuperscript{2}}

4.4. Toral automorphisms with large holes.

\textbf{Lemma 4.9.} Let $Y$ be a subshift of $X_\beta$ and let, as above, $h(Y)$ stand for its topological entropy ($\log$ base $\beta$). Then

$$\dim_H(\phi_t(Y)) \geq 2h(Y).$$

\textit{Proof.} Let $m$ stand for a measure of maximal entropy for $Y$. Then by the Shannon–McMillan–Breiman theorem, for any $\varepsilon > 0$, we have a set $Y' \subset Y$ of full $m$-measure and all $x \in Y'$,

$$\beta^{-2k(h(Y)+\varepsilon)} \leq \mu[x_{-k+1} \ldots x_0, x_1 \ldots x_k] \leq \beta^{-2k(h(Y)-\varepsilon)}, \quad k \geq n(x).$$

Let $N_0$ be large enough that $\mu(S) > 1/2$, where $S = \{x \in Y' : n(x) \leq N_0\}$.

By Lemma 4.4, the set $\phi_t([x_{-k+1} \ldots x_0, x_1 \ldots x_k])$ contains a cube with sides $c\beta^{-k}$, whence its (normalized) $s$-Hausdorff measure $H^s$ is $c^s\beta^{-sk}$. Therefore,

$$H^s(\phi_t(S)) \geq c^s\beta^{-2k(h(Y)+\varepsilon)} \cdot \beta^{-ks} = c^s\beta^{-k(2h(Y)-s+\varepsilon)}.$$ 

Now the claim follows from the definition of the Hausdorff dimension. \hfill $\square$

\textbf{Theorem 4.10.} Let $M \in GL(m, \mathbb{Z})$ and $T = T_M : \mathbb{T}^m \to \mathbb{T}^m$ be the corresponding automorphism which we assume to be generalized Pisot.

Then for any $\varepsilon > 0$ there exists an open connected hole $H \subset \mathbb{T}^m$ such that $H_m(H) > 1 - \varepsilon$ and $\dim_H(F(H)) > 0$.

\textit{Proof.} Assume first $T$ to be Pisot. Then it follows from Theorem 3.1 and Lemma 4.9 that $H = \phi_t(G)$ has positive Hausdorff dimension. Lemma 4.1 ensures that the complement of $H$ has measure $H_m$ as small as we please.

Now let $S$ be generalized Pisot. For $S = T^{-1}$ we have the same survivor set for any hole so the same $H$ does the job.

Assume $S = -T$ now. We will modify our proof of Theorem 3.1 in such a way that we will get $H \subset \mathbb{T}^m$ with $H = -H$. Namely, put for any $x \in X_\beta$,

$$M(x) = \{y \in X_\beta : \phi_t(y) = \pm \phi_t(x)\}.$$ 

Since $\phi_t$ is finite-to-one, there exists $M \geq 1$ such that $\#M(x) \leq M$ for all $x \in X_\beta$. Now let $Y$ be a subshift of $X_\beta$ from Theorem 3.1; for instance, we can take $Y = X_{\beta'}$ where $1 < \beta' < \beta$. Put

$$\widetilde{Y} = \{M(x) : x \in Y\}.$$ 

Roughly speaking, $\widetilde{Y}$ consists of all elements of $Y$ together with their – possibly multiple – negatives. We claim that $\widetilde{Y}$ is shift-invariant (hence a subshift). Indeed, let $y \in \widetilde{Y}$; then we have $\phi_t(y) = \pm \phi_t(x)$ for some $x \in Y$. Thus,

$$\phi_t(\sigma(y)) = T\phi_t(y) = \pm T\phi_t(x) = \pm \phi_t(\sigma(x)),$$

whence $\sigma(y) \in \widetilde{Y}$, since $\sigma(x) \in Y$.

Furthermore, $h(\widetilde{Y}) = h(Y)$, since we only add at most $M$ sequences for each $x \in Y$. By our construction, $-\phi_t(\widetilde{Y}) = \phi_t(\widetilde{Y})$. Following the proof of Theorem 3.1, we set $\Sigma_n := \Sigma_n \setminus \{w : w \in \mathcal{L}(Y)\}$.

\footnote{\textsuperscript{2} A subset $E$ of a Riemmanian manifold is said to be \textit{geodesically convex} if, given any two points in $E$, there is a minimizing geodesic contained within $E$ that joins those two points.}
As before, we have that \( \mu(\tilde{\Sigma}_n) \to 1 \) as \( n \to \infty \). Moreover we see that \( \tilde{Y} \subset J(\tilde{\Sigma}_n) \) and \( h(J(\tilde{\Sigma}_n)) > 0 \).

Note that if \( K \) is even then \( S^K = T^K \). To get connectedness, we again let \( [w^*] \in \tilde{\Sigma}_n \) be such that \( \mu([w^*]) \) is minimized and let

\[
G_n = (\tilde{\Sigma}_n \setminus \{[w^*]\}) \cup \{\sigma^{-K}([w^*])\}
\]

with \( K \) sufficiently large and even. The rest of the proof follows as before.

The case where \( S = -T^{-1} \) follows in exactly the same way because the orbit of \(-T\) and \(-T^{-1}\) are the same.

**Remark 7.** Recall the definition of the escape rate for an invertible map \( T : X \to X \) with an invariant measure \( m \) and a hole \( H \). Put

\[
e(x) = e_H(x) = \min\{n \geq 0 : T^n x \in H\}
\]

and \( E_n = \{x \in X : e(x) > n\} \). Now, put \( \delta = \delta(T, H, m) = \lim_{n \to \infty} m(E_n)^{1/n} \). The quantity

\[
e(T, H, m) = -\log \delta(T, H, m)
\]

is usually referred to as the *escape rate*. Theorem 4.7 says that there exist arbitrarily small connected holes \( H \) such that \( e(T, H, m) = +\infty \) for a wide class of toral automorphisms \( T \).

To our best knowledge, this effect has not been studied in the literature (see Section 1).

**REFERENCES**

[1] V. S. Afraimovich and L. A. Bunimovich, *Which hole is leaking the most: A topological approach to study open systems*, *Nonlinearity*, 23 (2010), 643–656.

[2] R. Alcaraz Barrera, *Topological and ergodic properties of symmetric sub-shifts*, *Discrete Contin. Dyn. Syst. Ser. A*, 34 (2014), 4459–4486.

[3] O. F. Bandtlow, O. Jenkinson and M. Pollicott, *Periodic points, escape rates and escape measures*, in *Ergodic Theory, Open Dynamics, and Coherent Structures*, Springer Proc. Math. Stat. 70 (2014), 41–58.

[4] A. Bertrand-Mathis, Développement en base \( \theta \), répartition modulo un de la suite \( (x_\theta^n)_n \geq 0 \); langages codés et \( \theta \)-shift, *Bull. Soc. Math. Fr.*, 114 (1986), 271–323.

[5] L. A. Bunimovich and A. Yurchenko, *Where to place a hole to achieve a maximal escape rate*, *Israel J. Math.*, 182 (2011), 229–252.

[6] J.-M. Champarnaud, G. Hansel and D. Perrin, *Unavoidable sets of constant length*, *Int. J. Algebra Comput.*, 14 (2004), 241–251.

[7] L. Clark, \( \beta \)-transformation with a hole, *Discreet Cont. Dyn. Sys. A*, 36 (2016), 1249–1269.

[8] L. Clark, K. G. Hare and N. Sidorov, *The baker’s map with a convex hole*, *Nonlinearity* 31 (2018), 3174–3202.

[9] M. F. Demers and L.-S. Young, *Escape rates and conditionally invariant measures*, *Nonlinearity*, 19 (2006), 377–397.

[10] M. F. Demers, *Dispersing billiards with small holes*, in *Ergodic Theory, Open Dynamics, and Coherent Structures*, Springer Proc. Math. Stat. 70 (2014), 137–170.

[11] A. Ferguson and M. Pollicott, *Escape rates for Gibbs measures*, *Ergodic Theory Dynam. Systems*, 32 (2012), 961–988.

[12] P. Glendinning and M. Pollicott, *Escape rates for asymptotically hyperbolic systems*, *Ergodic Theory Dynam. Systems*, 35 (2015), 1208–1228.

[13] W.-G. Hu and S.-S. Lin, *The doubling map with asymmetrical holes*, *Ergodic Theory Dynam. Systems*, 35 (2015), 1208–1228.

[14] S. Le Borgne, *Un codage sofique des automorphismes hyperboliques du tore*, *Bol. Soc. Bras. Mat.*, 30 (1999), 61–93.

[15] D. Lind and B. Marcus, *An Introduction to Symbolic Dynamics and Coding*, Cambridge University Press, 1995.
OPEN MAPS: SMALL AND LARGE Holes

[16] J. Mykkeltveit, A proof of Golomb’s conjecture for the de Bruijn graph, J. Combin. Theory B, 13 (1972), 40–45.

[17] W. Parry, On the $\beta$-expansions of real numbers, Acta Math. Acad. Sci. Hung., 11 (1960), 401–416.

[18] G. Pianigiani and J. A. Yorke, Expanding maps on sets which are almost invariant. Decay and chaos, Trans. Amer. Math. Soc., 252 (1979), 351–366.

[19] A. Rényi, Representations for real numbers and their ergodic properties, Acta Math. Acad. Sci. Hung., 8 (1957), 477–493.

[20] K. Schmidt, Algebraic codings of expansive group automorphisms and two-sided beta-shifts, Monatsh. Math., 129 (2000), 37–61.

[21] M.-P. Schützenberger, On the synchronizing properties of certain prefix codes, Inf. Contr., 7 (1964), 23–36.

[22] N. Sidorov, Arithmetic Dynamics, in Topics in Dynamics and Ergodic Theory, LMS Lecture Notes Ser. 310 (2003), 145–189.

[23] N. Sidorov, Bijective and general arithmetic codings for Pisot toral automorphisms, J. Dynam. Control Systems, 7 (2001), 447–472.

[24] N. Sidorov and A. Vershik, Ergodic properties of Erdős measure, the entropy of the golden-shift, and related problems, Monatsh. Math., 126 (1998), 215–261.

[25] N. Sidorov and A. Vershik, Bijective arithmetic codings of hyperbolic automorphisms of the 2-torus, and binary quadratic forms, J. Dynam. Control Systems, 4 (1998), 365–399.

[26] B. Weiss, Intrinsically ergodic systems, Bull. Amer. Math. Soc., 76 (1970), 1266–1269.

Received February 2018; revised June 2018.

E-mail address: kghare@uwaterloo.ca
E-mail address: sidorov@manchester.ac.uk