A SIMPLE CHARACTERIZATION OF DOUBLY TWISTED SPACETIMES

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Abstract. In this note we characterize 1+n doubly twisted spacetimes in terms of 'doubly torqued' vector fields. They extend Bang-Yen Chen’s characterization of twisted and generalized Robertson-Walker spacetimes with torqued and concircular vector fields. The result is a simple classification of 1+n doubly-twisted, doubly-warped, twisted and generalized Robertson-Walker spacetimes.

1. Introduction

Several interesting Lorentzian metrics have a block-diagonal form, with time labelling a foliation with spacelike hypersurfaces. They include doubly-twisted, doubly-warped, twisted, warped, and Robertson-Walker spacetimes [1]. The Frobenius theorem characterizes the vector fields $u_i$ that are hypersurface orthogonal, for which there exist functions $\lambda$ and $f$ such that, locally, $\lambda u_i = \nabla_i f$ (2, p.19). This establishes a dual description of such spacetimes: the special form of the metric allows explicit evaluations, the one in terms of the vector field is covariant. While physicists conceive the vector field as a congruence of timelike trajectories, geometers prefer other vectors, as is here illustrated.

Doubly twisted spacetimes were introduced (and named ‘conformally separable’) by Kentaro Yano in 1940:

$$ds^2 = -b^2(q, t)dt^2 + a^2(t, q)g^\mu\nu(q)dq^\mu dq^\nu.$$ (1)

He showed that the metric structure is necessary and sufficient for the hypersurfaces to be totally umbilical [3]. The spacetime is doubly warped if $b$ only depends on $q$ and $a$ only depends on time $t$.

Ferrando, Presilla and Morales [4] proved that doubly twisted spacetimes are covariantly characterized by the existence of a timelike unit, shear and vorticity free vector field: $u'u_i = -1$ and

$$\nabla_i u_j = \varphi (u_i u_j + g_{ij}) - u_i  \dot{u}_j$$ (2)

where $\dot{u}_j = u^k \nabla_k u_j$ is the acceleration, and $\dot{u}_j u^j = 0$.

In 1979 Bang-Yen Chen introduced twisted spacetimes, eq.(1) with $b = 1$ [5]. Years later he characterized them through the existence of a timelike torqued vector field [6]:

$$\nabla_i \tau_j = \kappa g_{ij} + \alpha_i \tau_j,$$

$$\alpha_i \tau^i = 0$$ (3)

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where $\kappa$ is a scalar field. We gave the equivalent description in terms of torse-forming time-like unit vectors, eq. (2) with $\dot{u}_i = 0$, and obtained the form of the Ricci tensor [7], and unicity of the vector, up to special cases [8].

Generalized Robertson-Walker (GRW) spacetimes were introduced in 1995 by Alías, Romero and Sánchez [9,10]:

$$d\mathbf{s}^2 = -dt^2 + a^2(t)g_{\mu\nu}(\mathbf{q})d\mathbf{q}^\mu d\mathbf{q}^\nu$$

Bang-Yen Chen characterized them through the existence of a timelike concircular vector field $\nabla_i \tau_j = \kappa g_{ij}$ [11,12] (the statement can be weakened, see Prop [2,3]). We gave the alternative characterization (2) with $\dot{u}_i = 0$ and $\nabla_i \varphi = -u_i \dot{\varphi}$, and proved the useful property for the Weyl tensor [13]: $u_m C_{jkl}^m = 0$ if and only if $\nabla_m C_{jkl}^m = 0$. If the Weyl tensor is zero, the spacetime is Robertson-Walker.

All these cases constitute a rich family of manifolds which are mostly studied by geometers. They also appear in physics, as inhomogeneous extensions of the Robertson-Walker metric. In fact, the Einstein equations with a source of imperfect fluid with shear-free and irrotational velocity, lead to doubly-twisted metrics [14,15,16,4,17,18]. The Stephani universes fall in this class [2,19]. The requirement of geodesic flow specialises the metric to twisted, with interesting applications discussed by Coley and McManus [20]. Doubly warped and GRW (or warped) manifolds have an ample geometric literature [21,22,23,24,12].

In this note we present a simple characterization that includes all such spacetimes, and discuss some properties of doubly torqued vectors.

2. Another Characterization

**Theorem 2.1.** A Lorentzian spacetime is doubly-twisted if and only if it admits a timelike vector field, which we name ‘doubly torqued’:

$$\nabla_i \tau_j = \kappa g_{ij} + \alpha_i \tau_j + \tau_i \beta_j$$

with $\alpha_i \tau^i = 0$, $\beta_i \tau^i = 0$, and $u_k = \nabla_i \tau^i$.

**Proof.** We prove the equivalence of (5) with (2).

Let $N = \sqrt{-\tau^i \tau_i}$, and introduce the vector $u_i = \tau_i/N$. Evaluate: $\nabla_i N^2 = -2 \tau^i \nabla_i \tau_j = -2 \kappa \tau_i + 2 \alpha_i N^2$. Then: $\nabla_i N = -\kappa u_i + \alpha_i N$. Next:

$$N \nabla_i u_j = \nabla_i \tau_j - u_j \nabla_i N = \kappa g_{ij} + N \alpha_i u_j + N \alpha_i \beta_j + \kappa u_i u_j - N \alpha_i \beta_j$$

Therefore: $\nabla_i u_j = (\kappa/N)(u_i u_j + g_{ij}) + u_i \beta_j$. Contraction with $u^i$ shows that $\beta_j = -\dot{u}_j$, and eq. (2) is obtained.

Given (2), the corresponding metric is (1). Define $\beta_j = -\dot{u}_j$ and $\tau_i = S u_i$, where $S$ is to be found. Multiply eq. (2) by $S$:

$$\nabla_i (Su_j) - u_j \nabla_i S = \varphi (u_i u_j + g_{ij}) + Su_i \beta_j$$

This is eq. (5) with the vector $\alpha_i = (\nabla_i S)/S + \varphi u_i$.

We must impose the condition: $0 = \alpha_i \tau^i = u^i \nabla_i S - S \varphi$ i.e. $\varphi = u^i \nabla_i \log S$.

In the frame (1) it is $u_0 = -b$, $u_\mu = 0$, and the condition becomes $\varphi = (\partial_0 S)/(Sb)$.

With the Christoffel symbols in appendix we obtain $u_0 = \dot{u}_0 = 0$ and $\dot{u}_\mu = -u^\nu \Gamma^0_{\nu\mu} u_0 = b_\mu/b$. The time component of (2), $\nabla_0 u_0 = \varphi (u_0^2 + g_{00})$, gives $\varphi = (\partial_0 a)/(ab)$. Therefore $S = a$ up to a constant factor. □
There is some freedom in the choice of the doubly torqued vector: multiplication of eq. (5) by a function \( \lambda \) gives an equation for a vector \( \lambda \tau_i \) that is orthogonal to the hypersurfaces and \( \nabla_i (\lambda \tau_j) = (\lambda \kappa) g_{ij} + (\alpha_i + \partial_i \lambda / \lambda) (\lambda \tau_j) + (\lambda \tau_i) \beta_j \). It is doubly torqued provided that:

\[
(6) \quad \tau^i \partial_i \lambda = 0
\]

We show the relation of the special vectors \( \alpha_i \) and \( \beta_i \) with the scale functions \( a \) and \( b \) of the metric. In the coordinate frame \((t, q)\), the vectors are \( \tau_i = (\tau_0, 0) \), \( \alpha_i = (0, \alpha_\mu) \) and \( \beta_i = (0, \beta_\mu) \). The equations (5) are: \( \partial_0 \tau_0 - \Gamma^q_{00} \tau_0 = -k b^2 \), \( \partial_\mu \tau_0 - \Gamma^q_{\mu 0} \tau_0 = \tau_0^1 \alpha_\mu \), \( -\Gamma^q_{\mu 0} \tau_0 = \beta_\mu \tau_0 \) and \( -\Gamma^q_{\mu \nu} \tau_0 = \kappa a^2 g_{\mu \nu}^* \). They can be rewritten as follows:

\[
(7) \quad \beta_\mu = -\partial_\mu \log b \\
(8) \quad \partial_0 \tau_0 = \tau_0 \partial_\mu \log(ab) \\
(9) \quad \partial_\mu \tau_0 = \tau_0 (\alpha_\mu - \beta_\mu) \\
(10) \quad kb = -\partial_i (\tau_0/b)
\]

The second equation is integrated: \( \tau_0(t, q) = c(q)(ab)(t, q) \), where \( c \) is an arbitrary function. Then, the first and third equation give \( \alpha_\mu = \partial_\mu \log (ca) \).

In coordinates \((t, q)\) the condition (6) is \( \tau^0 \partial_0 \lambda = 0 \) and implies that \( \lambda \) does not depend on time \( t \). The transformation \((\tau_0, \alpha_\mu, \beta_\mu, \kappa) \rightarrow (\lambda \tau_0, \alpha_\mu + \partial_\mu \lambda / \lambda, \beta_\mu, \lambda \kappa) \) leaves the above equations unchanged. This freedom is used to put \( c(q) = \pm 1 \). Then: \( \tau_0(t, q) = -(ab)(t, q) \) (if \( \tau^0 > 0 \)). The other equations give:

\[
(11) \quad \alpha_\mu = \frac{\partial}{\partial q^\mu} \log a, \quad \beta_\mu = -\frac{\partial}{\partial q^\mu} \log b, \quad \kappa = \frac{1}{b} \frac{\partial a}{\partial t}
\]

and establish a simple relation of the vectors with the metric. An interesting invariant is:

\[
(12) \quad \tau^k \tau_k = -\frac{1}{b^2} \tau^2_0 = -a^2
\]

For a doubly warped \( 1+n \) metric, the scale functions \( b \) does not depend on time and \( a \) does not depend on \( q \). In this case, the analysis shows that \( \alpha_i \) is either zero or a gradient orthogonal to \( \tau \), that can be absorbed by a rescaling of \( \tau \).

We obtain the characterization:

**Theorem 2.2.** A \( 1+n \) spacetime is doubly warped if and only if there is a timelike vector such that \( \nabla_i \tau_j = \kappa g_{ij} + \tau_i \beta_j \) with \( \tau^i \beta_i = 0 \), and \( \beta_i \) is closed.

**Proof.** In a doubly warped spacetime, \( b(q) \) and \( a(t) \) are given and specialize eq. (5).

Eq. (11) gives: \( \alpha_\mu = 0 \) i.e. \( \alpha_i = 0 \). Eq. (7) with \( \partial_\mu b = 0 \) implies that \( \beta_i \) is closed.

If \( \alpha_i = 0 \) in (5), then eq. (9) gives: \( \partial_0 \tau_0 / \tau_0 - \partial_\mu \log b \), has solution \( \tau_0(t, q) = F(t)b(t, q) \). The result in (8) gives: \( \partial_\mu \log a = \partial_\mu F/F \), so that \( a = a(t) \). \( \beta_i \) closed becomes \( \beta_\mu = -\partial_\mu r(q) \). Eq. (7) gives \( b(t, q) = \exp[r(q) + s(t)] \). The metric \( ds^2 = -e^{2r(q)} e^{2s(t)} dt^2 + a^2(t) g^*_{\mu \nu} dq^\mu dq^\nu \) is doubly warped with a redefinition of time. \( \square \)

In twisted \( 1+n \) spacetimes, \( a \) depends on \( t \) and \( q \), and \( b = 1 \). Then \( \beta_i = 0 \) and we recover Chen’s result (3). In a GRW spacetime the scale function \( a \) only depends on \( t \), so that also the vector \( \alpha_i \) is zero.
Proposition 2.3. In a doubly twisted spacetime with doubly-torqued vector $\tau_i$, if $\beta_i = \nabla_i \theta$ and $\tau^i \nabla_i \theta = 0$ then the spacetime is conformally equivalent to a twisted spacetime.

Proof. Consider the conformal map $\hat{g}_{ij} = e^{2\theta} g_{ij}$. The new Christoffel symbols are

$$\hat{\Gamma}^k_{ij} = \Gamma^k_{ij} + \delta^k_i \partial_j \theta + \delta^k_j \partial_i \theta - g^{kl} g^{ij} \hat{\kappa}_l \theta$$

The vector $\hat{\tau}_i = e^\theta \tau_i$ solves:

$$\hat{\nabla}_i \hat{\tau}_j = \nabla_i (e^\theta \tau_j) - \hat{\tau}_i \partial_j \theta - \hat{\tau}_j \partial_i \theta + g^{kl} \hat{\kappa}_k \theta$$

$$= \hat{\tau}_i \partial_j \theta + e^\theta (\kappa g_{ij} + \alpha_i \tau_j + \tau_i \partial_j \theta) - \hat{\tau}_j \partial_i \theta - \hat{\tau}_i \partial_j \theta + \hat{g}_{ij} \hat{\kappa}_l \hat{\tau}_l$$

$$= (e^\theta \kappa) \hat{g}_{ij} + \alpha_i \hat{\tau}_j$$

The absence of $\beta_i$ characterizes a twisted spacetime. \hfill \Box

A consequence of the proof is the following statement (obvious if regarded on the side of the metric):

Proposition 2.4. Conformal transformations $\hat{g}_{ij} = e^{2\theta} g_{ij}$ map doubly twisted to doubly twisted spacetimes.

The same conformal transformation, with the condition $\tau^i \nabla_i \theta = 0$, maps doubly warped to doubly warped spacetimes, or twisted to twisted spacetimes.

Proof. Given the vector $\tau_i$, consider the vector $\hat{\tau}_i = e^{2\theta} \tau_i$ in the new metric. It solves the equation of a doubly torqued vector:

$$\hat{\nabla}_i \hat{\tau}_j = (\kappa + \hat{\tau}_k \partial_k \theta) \hat{g}_{ij} + (\alpha_i + \hat{h}_i \partial_k \theta) \hat{\tau}_j + \hat{\tau}_i (\beta_j - \hat{h}_j \partial_k \theta)$$

where $\hat{h}_{ij} = \hat{g}_{ij} - \hat{\tau}_i \hat{\tau}_j / \hat{\tau}^2$ is a projection.

If instead the vector $\hat{\tau}_i = e^\theta \tau_i$ is considered, in the new metric it solves

$$\hat{\nabla}_i \hat{\tau}_j = (e^\theta \kappa + \hat{\tau}_k \partial_k \theta) \hat{g}_{ij} + \alpha_i \hat{\tau}_j + \tau_i (\beta_j - \partial_j \theta)$$

If the space is doubly warped ($\alpha_i = 0$ and $\beta$ closed) then, with the condition $\tau^i \partial_i \theta = 0$, the vector $\hat{\tau}_i$ solves the equation for a doubly warped spacetime. The same is true for twisted spacetimes ($\beta_i = 0$). \hfill \Box

3. Conclusion

The introduction of doubly torqued vectors, defined by the equation \[5\], has the virtue of covariantly describing a class of $1+n$ spacetimes in simple manner: doubly-twisted ($\alpha \neq 0$, $\beta \neq 0$), doubly-warped ($\alpha = 0$, $\beta \neq 0$ closed), twisted ($\alpha \neq 0$, $\beta = 0$), generalized Robertson-Walker ($\alpha = \beta = 0$).

We provide some examples from the physics literature.

Stephani universes \[19\] are conformally flat solutions of the Einstein field equations with a perfect fluid source. As in the Robertson-Walker space-time, the hypersurfaces orthogonal to the matter world-lines have constant curvature, but now its value $k$, and even its sign, changes from one hypersurface to another. The line element is doubly twisted:

$$ds^2 = -D(t, x)dt^2 + \frac{R^2(t)}{V^2(t, x)} (dx^2 + dy^2 + dz^2)$$

with $V = 1 + \frac{1}{4} ||x - x_0(t)||^2$, $D = F(t)[V/V - R/R]$, $k = \frac{C^2(t) - 1/F^2(t)}{R^2(t)}$, arbitrary functions of time $C$, $F$, $R$ and $x_0$. 

Another example is the solution by Banerjee et al.  \[15\] for a matter field with shear and vorticity free velocity and heat transfer, in a conformally flat metric:

\[
\begin{align*}
\frac{ds^2}{-dt^2} + X(t)^2(dx^2 + e^{-2\gamma}dy^2 + e^{2\gamma}dz^2)
\end{align*}
\]

With $UV = A(t)||x||^2 + A(t)\cdot x + A_i(t)$, $U = B(t)||x||^2 + B(t)\cdot x + B_i(t)$, where $A$, $B$, $A_i$ and $B_i$ are arbitrary functions of time.

Coley studied the case with no acceleration, $V = 1$, which makes the above metric twisted (\[20\] eq.3.12). The same paper contains an example of GRW spacetime (eq. 1.2), which is also Bianchi VI

\[
\begin{align*}
0
\end{align*}
\]

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These are the Christoffel symbols for the doubly-twisted metric \[11\]:

\[
\begin{align*}
\Gamma^0_{00} &= \frac{\partial b}{\partial b}, & \Gamma^0_{\mu,0} &= \frac{b\mu}{a^2}, & \Gamma^0_{\mu,\nu} &= \frac{\partial a}{a} \delta_\mu^\nu, & \Gamma^0_{\mu,\nu} &= \frac{a\partial a}{b^2} g^*_{\mu\nu}, \\
\Gamma^\mu_{\nu,\rho} &= \Gamma_{\nu,\rho}^\mu + \frac{a\nu}{a} \delta_\mu^\rho + \frac{a\mu}{a} \delta_\nu^\rho - \frac{a^\rho}{a} g^*_{\mu\nu}
\end{align*}
\]

Where $a_\mu = \partial_\mu a$ and $a^\mu = g^{\mu\nu}a_\nu$, and the same is for $b$.

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