FORMATION OF CUSPY DENSITY PROFILES: A GENERIC FEATURE OF COLLISIONLESS GRAVITATIONAL COLLAPSE

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ABSTRACT

Using the formalism of the spherical infall model, the structure of collapsed and virialized dark halos is calculated for a variety of scale-free initial conditions. In spite of the scale-free cosmological nature of the problem, the collapse of individual objects is not self-similar. Unlike most previous calculations, the dynamics used here relies only on adiabatic invariants and not on self-similarity. This Letter focuses on the structure of the innermost part of the collapsed halos and addresses the problem of central density cusps. The slopes of density profiles at 1% of virial radius are calculated for a variety of cosmological models and are found to vary with the mass of the halos, power spectrum of density fluctuations, and Ω₀. The inner slopes range between and with the limiting case of reached for the largest masses. The steep cusps found here correspond to the limiting case where all particles move on radial orbits. The introduction of angular momentum will make the density profile shallower. We expect this to resolve the discrepancy found between the calculated profiles and the ones found in high-resolution N-body simulations, where the exponent ranges from −0.5 to −1.5. The robust prediction here is that collisionless gravitational collapse in an expanding universe is expected to form density cusps and not halos with a core structure.

Subject headings: cosmology: theory — galaxies: formation — large-scale structure of universe — methods: analytical

1. INTRODUCTION

The structure of collapsed and virialized objects such as galaxies and clusters poses a real challenge to our understanding of structure formation in the universe. The basic problem in this field is that of the formation of “dark halos,” namely, the outcome of the collisionless collapse in an expanding universe. This was addressed first by two seminal papers of Gunn & Gott (1972) and Gunn (1977), where the cosmological expansion and the role of adiabatic invariance were first introduced in the context of the formation of individual objects. The next step was made by Fillmore & Goldreich (1984) and Bertschinger (1985), who found analytical predictions for the density of collapsed objects seeded by scale-free primordial initial perturbation in a flat universe. Hoffman & Shaham (1985, hereafter HS) applied and modified these solutions to realistic initial conditions in flat as well as open Friedmann models. Their basic result was that for an Einstein–de Sitter universe and a primordial power spectrum \( P(k) \propto k^n \), the asymptotic solution for the density profile is \( \rho \propto r^{-\alpha} \), where \( \alpha = -3(3 + n)/(4 + n) \), i.e., higher \( n \) yield steeper profiles (similar trend was found for lower \( \Omega_0 \)). This model was improved in a series of papers focusing on the incorporation of the peak formalism of Bardeen et al. (1986, hereafter BBKS) or weakly nonlinear corrections and the refinement of the adiabatic invariance calculations (Ryden & Gunn 1987; Hoffman 1988; Ryden 1988; Zaroubi & Hoffman 1993; Łokas 1998, 2000).

The analytical studies were followed by numerical studies of the collapse problem by the means of N-body simulations, e.g., by Quinn, Salmon, & Zurek (1986) and Crone, Evrard, & Richstone (1994), who confirmed predictions of HS. Recently Navarro, Frenk, & White (1997, hereafter NFW) established that the density profile of dark matter halos forming in different cosmologies follows a universal form that steepens from \( r^{-1} \) near the center of the halo to \( r^{-3} \) at large distances. This result was confirmed by Cole & Lacey (1996), Huss, Jain, & Steinmetz (1999a) and others, although Kravtsov et al. (1998) obtain much shallower inner profiles. Some recent very high resolution cosmological simulations produce steeper density profiles, with inner slopes \( r^{-1.5} \) (Fukushige & Makino 1997, 2000; Moore et al. 1998; Jing & Suto 2000). It seems that the outcome of a collisionless gravitational collapse is a cuspy density profile rather than a core structure, although the exact slope of the cusp is still under debate.

There have been a number of attempts to identify a mechanism responsible for the formation of the cusp, mainly referring to the merging formalism of Lacey & Cole (1993). Syer & White (1998) and Nusser & Sheth (1999) claimed that the universal profile is a result of hierarchical clustering by mergers of smaller halos into bigger ones. However, Moore et al. (1999) performed N-body simulations with a cutoff in the power spectrum at small scales and also obtained halos with cuspy density profiles. This proves that merging and substructure does not play a critical role in the formation of density cusps.

The aim of the present Letter is to demonstrate that the density cusp is indeed a generic feature of collisionless collapse in an expanding universe that emerges also from the highly simplistic spherical infall model. The model is presented in § 2. The resulting inner density profiles are described in § 3, and general discussion follows in § 4.

2. THE MODEL

The spherical infall model is based on two ingredients, namely, the initial conditions and the dynamical model. Given the initial density structure, a dynamical model is needed to describe the spherical collapse in an expanding universe. The
The dynamics of a given mass shell has two phases, before and after shell crossing occurs. The first one is a trivial dynamics of an isolated shell in which energy is conserved. The other is a much more complicated process, which, as shown by Fillmore & Goldreich (1984), can be described by considering the adiabatic invariance of the motion. In the case of scale-free mass distribution, for a given spherical shell, characterized by a maximum radius $r$ and the mass that is confined within it, $m(r)$, and $r_m(r)$ is an adiabatic invariant. The mass enclosed by a shell is composed of the primordial mass that was confined within it, $m_0(r)$, and a secondary component of the mass contributed by external shells spending some fraction of their time within it, $m_{add}(r)$. It follows that the maximum radius shrinks in time by a factor $f = m_0(r)/(m_0(r) + m_{add}(r))$.

However, self-similar solutions have very limited validity in trying to calculate the structure of actual dark halos, although most of earlier work on the subject relied on self-similarity (with the notable exception of Ryden 1988). We use here the spherical infall model of Lokas (2000), which provides a generalization of that proposed by HS. This version of the model used the generalized (not scale-free) statistically expected initial density distribution around an overdense region and introduced a boundary of the evolving protosystem by cutting off the distribution at half interpeak separation.

Here we further modify the model by assuming that the dark halos are seeded by local density maxima (not just overdense regions) and their initial structure is given by the peak statistics (BBKS) of the linear density field smoothed with a Gaussian filter of scale $R$. We assume scale-free initial density fluctuation spectra $P(k) \propto k^n$ and normalize them to $a_0 = 1$. We apply here the peak density profile averaged over curvatures and orientations as given by equation (7.10) of BBKS and use the half-interpeak cutoff scales as calculated in Lokas (2000).

The initial density distribution has to be specified further by adopting appropriate initial conditions. We assume that the starting point in the evolution of every halo is given by the condition for the overdensity within $R$, $\delta = \rho_0 / \rho_c = 0.1$ (a value small enough for the linear theory to be valid), where $\rho$ is the height of the peak and $\sigma \propto R^{-0.5} a_{\text{rms}}^2$ is the rms fluctuation at scale $R$. Once $R$ is chosen, this condition yields the initial redshift $w$.

Our purpose is to study the present properties of dark halos; therefore, we assume that the collapse time (as defined in the spherical top-hat model) for all objects is the present epoch. This determines the mean density inside the presently collapsing shell. Comparing this value with the density distribution around the peak, we determine the radius of this shell (typically a few $R$) and define it to be the virial radius. The present turnaround radius is obtained in a similar way. The mass of the halo is a sum of the mass inside the virial radius and the mass contributed by the shells with maximum radii between the virial radius and the turnaround radius.

Once the initial density structure up to the present turnaround radius is determined, it is straightforward to predict the fiducial mass distribution with all shells at their turnaround radii. We then apply the adiabatic invariance approach to calculate the motion of shells in such distribution and the $m_{\text{add}}$ contributed for each shell by shells of larger maximum radii (see Lokas 2000). In the case of the absence of self-similarity, we find that for our initial mass distribution the adiabatic invariant $J \propto \int_0^{r_f} v(r, r') dr = \text{const} \ (r' \text{ being the shell’s apapsis})$ leads to a more complicated relation between the final radius $r_f$ and mass: $g(r_f)g(r) = f^{1/3}$, where $g(r)$ can be well approximated by $r^{1/2}$ multiplied by a polynomial correction. The collapse factor $F = r_f/r$ has to be obtained by solving numerically this equation and typically yields values of a few percent higher than $f$, with the difference being larger for larger masses. An example of the difference between $F$ and $f$ is shown in Figure 1. It turns out that the proper calculation of the collapse factor in the case of departure from self-similarity is crucial to the formation of central density cusps in dark halos. Previous calculations, where the approach based on self-similarity was used, did not reproduce the density cusps (del Popolo et al. 2000).

We also generalized the improved spherical infall formalism to apply to the open universe models. This is straightforward and involves only some changes in the relation between the maximum radius and the initial radius of the shell (HS), the rate of growth of linear density fluctuations, and the formula for the present age of the universe, which are all well known (e.g., Padmanabhan 1993).

3. RESULTS

In this section we present the results of application of the presented model to the calculation of cusps, i.e., the slopes of the final density profiles at 1% of the virial radius. The most uncertain part of the model is determining the boundaries of the collapsing halo. As mentioned, it is assumed to correspond to the half-interpeak separation. We adopt the same smooth radial cutoff of exponential shape as in Lokas (2000) with a parameter $w$ (a width of the cutoff filter in units of the smoothing scale $R$) as a measure of its sharpness: the smaller the $w$ the sharper the cutoff. One may argue that $w$ should be of the order of unity, since this is the resolution associated with smoothing of the initial density field. However, the concentrations of final density profiles depend strongly on $w$, which significantly reduces the predictive power of the spherical infall model.

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Fig. 1.—Collapse factors $F$ and $f$ as functions of initial radius (in units of the virial radius) for one particular case of the density profile of a halo of mass $M/M_c = 0.78$ obtained in the case of $\Omega_0 = 1$, $n = -1$, $z_0 = 100$, and $w = 0.5$. 

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Given the uncertainty in determining the cutoff, we check how the slopes of cusps depend on $w$. Figure 2 shows the effective slopes $\alpha$ of the final density profiles, $\rho \propto r^{-\alpha}$, at 1% of the virial radius calculated with three different values of $w = 0.5, 1, 2$ in the case of $\Omega_0 = 1, \nu = 3$, and two values of the spectral index $n$. The mass dependence comes from different smoothing scales corresponding to different initial redshifts. The plotted range of masses was obtained with initial redshifts in the range $z_i$!100. The masses were expressed in terms of the so-called present nonlinear mass $M_*$, which depends on the cosmological model and reflects how advanced is the formation of nonlinear objects at present (see NFW and Łokas 2000).

Fortunately, we find that the range of cusp profiles obtained for different masses depends very weakly on $w$. The slopes of the inner profiles vary between $\alpha = -2.3$ and $\alpha = -2$ when going from the smallest to the largest masses, being somewhat steeper for $n = 0$ than for $n = -1$, in agreement with the original prediction of HS.

Next, we explore the dependence of the inner profiles on the initial conditions, taking into account different heights of the peak $\nu$ in the initial density field $\nu$ that gives rise to a bound object. Our condition $\nu a = 0.1$ leads to degeneracy in this picture of structure formation: when $\nu$ is specified, this condition enables us to relate uniquely the mass of the halo to the smoothing scale $R$. If we let $\nu$ vary, halos of the same mass can result from assumptions of different $(\nu, R)$ pairs.

We test the dependence of the inner profiles on the choice of $\nu$ by calculating the cusps for $\nu = 2$ and 4 in addition to the results for $\nu = 3$ presented above. We accordingly adjust the scale of the cutoff in the initial density distribution (it is smaller for lower peaks; see Łokas 2000), but choose only the width $w = 0.5$, since, as we proved with Figure 2, the results depend weakly on its value. The predictions of the inner profiles for different $\nu$ are shown in Figure 3. The trend of steeper cusps for lower masses seen in Figure 2 is preserved for all values of $\nu$, but smaller $\nu$ (with the rest of initial conditions unchanged) typically produce smaller masses, so the curves in Figure 3 are shifted accordingly. Figure 4 shows how the results depend on $\Omega_0$. Again, our predictions agree with the trend of steeper profiles for lower $\Omega_0$ found by HS.

4. DISCUSSION

In the limit of large masses, we reproduce the result of Fillmore & Goldreich (1984) and Zaroubi & Hoffman (1993), who

![Fig. 2. Slopes of the inner profiles of halos as a function of their mass obtained with different parameters $w$ for $\Omega_0 = 1, \nu = 3$, and two spectral indices: $n = -1$ (thick lines) and $n = 0$ (thin lines).](image1)

![Fig. 3. Slopes of the inner profiles of halos as a function of their mass for different heights of the peak $\nu$ for $\Omega_0 = 1$ and two spectral indices: $n = -1$ (thick lines) and $n = 0$ (thin lines).](image2)

![Fig. 4. Slopes of the inner profiles of halos as a function of their mass for different values of $\Omega_0$ (with height of the peak $\nu = 3$) and for two spectral indices: $n = -1$ (thick lines) and $n = 0$ (thin lines).](image3)
found that in the case of scale-free initial density distribution if this distribution is flat enough, the final density profile will be $\rho \propto r^{-2}$. This result can be interpreted in terms of the shape of the initial density distribution; the profiles of halos of larger mass originate from shells that initially were quite close to the peak, so their cumulative density distributions were rather flat. When comparing our results to those of $N$-body simulations, one must be careful about the meaning of the slope of the cusp. In the case of the NFW profile, the limiting inner slope of their fitting formula when $r \to 0$ is $-1$, but this is not equal to the effective slope at 0.01$r_c$, which instead is $\alpha = -(1 + 0.03c)/(1 + 0.01c)$, where $c$ is the concentration, i.e., the ratio of the virial radius to the scale radius $r_c$ (the radius at which the slope is $-2$). Clearly, $r_c = 0.01c$, or $c = 100$ is enough to have $\alpha = -2$. For example, in the range of masses shown in Figure 2 for $n = -1$, the effective inner slopes calculated with the above formula (using the description NFW provide for the dependence of the concentration on mass) change between $\alpha = -2.2$ for the smallest mass and $\alpha = -1.3$ for the largest, that is, they are quite far from the limiting slope of $-1$. Comparison of these values with those predicted by our model shows that reasonable agreement is obtained for smaller masses, while for larger masses NFW predict much shallower cusps. Unfortunately, this conclusion is based on an extrapolation of the $N$-body results to the range of small masses, because in the simulations the halos are typically more massive than those we obtain. A better agreement is expected with those authors of $N$-body simulations, who claim the inner slope to be closer to $-1.5$. In general, however, the spherical infall model predicts inner slopes of the profiles that are steeper than those observed in $N$-body simulations. This result is expected, given the simplistic nature of the model where radial orbits are assumed. This assumption maximizes the secondary mass added to the original mass enclosed within a given shell and therefore reinforces contraction, which steepens the density profile. In reality, angular momentum prevents a given shell from pulling inside some of the inner shells, which results in weaker adiabatic contraction of these shells. This effect was noticed before by Ryden (1988), Sikivie, Tkachev, & Wang (1997), Avila-Reese, Firmani, & Hernandez (1998), and Huss, Jain, & Steinmetz (1999b), who found that addition of angular momentum or tangential velocity dispersion can flatten the density profile even to $\rho \propto r^{-1}$ instead of the $\rho \propto r^{-2}$ expected for radial infall. Comparisons of the theoretically predicted density profiles with observations have recently put the cold dark matter (CDM)-based scenario of structure formation in trouble (Moore et al. 1999). A detailed comparison is beyond the scope of the present Letter, but we conclude with a few comments. The strongest case against the claim that galaxies form via collisionless gravitational collapse is that of the core structure (or shallow cusp) of low surface brightness (LSB) galaxies (Kravtsov et al. 1998; Moore et al. 1999). However, van den Bosch et al. (2000) claim that in most cases observations of LSB galaxies have low resolution and can place little constraints on the inner shapes of density profiles. In the cases with high enough resolution, they find agreement with NFW. Anyway, LSB galaxies are more angular momentum dominated compared to normal galaxies with the same luminosity (McGaugh & de Blok 1999), and therefore we argue that these objects should typically have shallower density cusps. The strongest claims against CDM based on the rotation curves of dwarf galaxies have also been recently challenged (van den Bosch & Swaters 2000) because of observational uncertainties. It is interesting to note here that steep inner profiles similar to those of NFW are roughly consistent with the observations of elliptical galaxies (Lokas & Mamon 2000). Thus, we conclude that the paradigm of collisionless gravitational collapse should not yet be dismissed.

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