AN ALTERNATIVE TO VAUGHAN’S IDENTITY

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Abstract. We exhibit an identity that plays the same role as Vaughan’s identity but is arguably simpler.

1. Introduction

Let \( 1_y \) denote the characteristic function of the integers free of prime factors \( \leq y \), The idea is to work with the identity

\[
\Lambda(n) = \log n - \sum_{\ell, m \mid n, \ell, m > 1} \Lambda(\ell) \tag{1.1}
\]

summing it up over integers \( n \leq x \) for which \( 1_y(n) = 1 \), where we might select \( y := \exp(\sqrt{\log x}) \) or larger. In this case the second sum can be written a sum of terms \( 1_y(\ell) \Lambda(\ell) \cdot 1_y(m) \) which have the bilinear structure that is used in “Type II sums”. We will see the identity in action in two key results in analytic number theory:

2. The Bombieri-Vinogradov Theorem

Theorem 1 (The Bombieri-Vinogradov Theorem). If \( x^{1/2}/(\log x)^B \leq Q \leq x^{1/2} \) then

\[
\sum_{q \leq Q} \max_{(a,q) = 1} \left| \pi(x; q, a) - \frac{\pi(x)}{\phi(q)} \right| \ll Q x^{1/2} (\log \log x)^{1/2}. \tag{2.1}
\]

This is a little stronger than the results in the literature (for example Davenport \[1\] has the \((\log \log x)^{1/2}\) replaced by \((\log x)^B\)). The reason for this improvement is the simplicity of our identity, and some slight strengthening of the auxiliary results used in the proof.

Proof. Let \( y = x^{1/\log \log x} \). We will instead prove the following result, in which the \( \psi \) function replaces \( \pi \), and deduce (2.1) by partial summation:

\[
\sum_{q \leq Q} \max_{(a,q) = 1} \left| \psi(x; q, a) - \frac{\psi(x)}{\phi(q)} \right| \ll Q x^{1/2} (\log x)(\log \log x)^{1/2}. \tag{2.2}
\]

Using (1.1) for integers \( n \) with \( 1_y(n) = 1 \), the quantity on the left-hand side of (2.2) is \( \ll S_I + S_{II} + E \) where

\[
S_I = \sum_{q \leq Q} \max_{(a,q) = 1} \left| \sum_{n \equiv a (\mod q)} 1_y(n) \log n - \frac{1}{\phi(q)} \sum_{n \leq x, (n,q) = 1} 1_y(n) \log n \right|
\]

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which is $\ll xu^{-u+o(u)} \ll_A \frac{x}{(\log x)^{1+\epsilon}}$ by the small sieve, where $x/Q = y^u$; and $E$ is the contribution of the powers of primes $\leq y$, which contribute $\ll \pi(y) \log x$ to each sum and therefore $\ll Q \pi(y) \log x \ll_A \frac{x}{(\log x)^{1+\epsilon}}$ in total. Most interesting is

$$S_{II} = \sum_{q \leq Q} \max_{(a,q)=1} \left| \sum_{n \equiv a \pmod{q}} f(n) - \frac{1}{\phi(q)} \sum_{(n,q)=1} f(n) \right|$$

where $f(n) = \sum_{\ell m=n, \ell, m \geq y} \Lambda(\ell)1_y(\ell) \cdot 1_y(m)$. Its bilinearity means that this is a Type II sum, and we can employ the following general result.

**Theorem 2.** For each integer $n \leq x$ we define

$$f(n) := \sum_{\ell m=n} \alpha_\ell \beta_m$$

where $\{\alpha_\ell\}$ and $\{\beta_m\}$ are sequences of complex numbers, for which
- The $\{\alpha_\ell\}$ satisfy the Siegel-Walfisz criterion;
- The $\{\alpha_\ell\}$ are only supported in the range $L_0 \leq \ell \leq x/y$;
- $\sum_{\ell \leq L} |\alpha_\ell|^2 \leq aL$ and $\sum_{m \leq M} |\beta_m|^2 \leq bM$ for all $L, M \leq x$.

For any $B > 0$ we have

$$\sum_{q \leq Q} \max_{(a,q)=1} \left| \sum_{n \equiv a \pmod{q}} f(n) - \frac{1}{\phi(q)} \sum_{(n,q)=1} f(n) \right| \ll (ab)^{1/2} Q^{1/2} \log x, \quad (2.3)$$

where $Q = x^{1/2}/(\log x)^B$, with $x/y \leq \frac{Q^2}{\log x^3}$ and $L_0 \geq y, \exp((\log x)^\epsilon)$.

We deduce that $S_{II} \ll Q^{1/2} (\log x)^{5/4}$ by Theorem 2 since $a \ll \log x$ and $b \ll \frac{1}{\log y} = \frac{\log \log x}{\log x}$.

3. A General Bound for a Sum over Primes

**Proposition 1.** For any given function $F(.)$ and $y \leq x$ we have

$$\left| \sum_{n \leq x \atop p(n) > y} \Lambda(n) F(n) \right| \ll S_I \log x + (S_{II} x(\log x)^{5/4})^{1/2}$$

where $S_I$ is the Type I sum given by

$$S_I := \max_{t \leq x} \left| \sum_{n \leq t \atop p(n) > y} F(n) \right| \leq \sum_{d \geq 1 \atop P(d) \leq y} \left| \sum_{m \leq t/d} F(dm) \right|,$$

and $S_{II}$ is the Type II sum given by

$$S_{II} := \max_{y < L \leq x/y \atop y < m \leq 2x/L} \left| \sum_{2nL < L \leq 2L \atop t < L \leq y} F(tn) \right|.$$

This simplifies, and slightly improves chapter 24 of [1], which is what is used there to bound exponential sums over primes.

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1. This can be obtained by taking the ideas in proving (5) of chapter 28 of [1], along with the method of proof of Theorem 9.16 of [2]; in any case it is only a minor improvement on either of these results. For full details see chapter 51 of [3].
Proof. We again use (1.1) so that
\[ \sum_{\substack{n \leq x \\ p(n) > y}} \Lambda(n) F(n) = \sum_{n \leq x} F(n) \log n - \sum_{\substack{\ell \leq x \\ p(\ell) > y}} \Lambda(\ell) \sum_{m \leq \ell} F(\ell m). \]
where \( p(n) \) denotes the smallest prime factor of \( n \). Now
\[ \sum_{n \leq x} F(n) \log n = \sum_{n \leq x} F(n) \int_1^x \frac{dt}{t} = \int_1^x \sum_{t \leq n \leq x} F(n) \frac{dt}{t} \leq 2 \log x \max_{t \leq x} \left| \sum_{n \leq t} F(n) \right|. \]
Moreover for \( P = \prod_{p \leq y} p \),
\[ \sum_{n \leq t} F(n) = \sum_{n \leq t} F(n) \sum_{d|P, d|n} \mu(d) = \sum_{d|P} \mu(d) \sum_{m \leq t/d} F(dm). \]
For the second sum we first split the sums into dyadic intervals (\( L < \ell \leq 2L, M < m \leq 2M \)) and then Cauchy, so that the square of each subsum is
\[ \leq \sum_{\ell, p(\ell) > y} \Lambda(\ell)^2 \sum_{\substack{m \leq \ell \\ p(m) > y}} F(\ell m)^2 \leq L \log L \sum_{\substack{M < m, n \leq 2M \\ \ell \leq \max(m, n)}} F(\ell m) F(\ell n) \]
\[ \leq x \log x \max_{M < m \leq 2M} \sum_{\substack{m/2 < n \leq 2m \\ p(n) > y}} \sum_{\ell \leq \ell \leq 2L} F(\ell m) F(\ell n) \]
since \( m, n \in (M, 2M) \), and the result follows. \( \square \)

4. Genesis

The idea for using (1.1) germinated from reading the proof of the Bombieri-Vinogradov Theorem (Theorem 9.18) in [2], in which they used Ramaré’s identity, that if \( \sqrt{x} < n \leq x \) and \( n \) is squarefree then
\[ 1_p(n) = 1 - \sum_{\substack{m = n \\ p \ prime \leq \sqrt{x}}} 1 \]
where \( 1_p \) is the characteristic function for the primes, and \( \omega_x(m) = 1 + \sum_{p | m, \ p \leq x} 1. \) They also had to sum this over all integers free of prime factors \( > y \).

References

[1] H.M. Davenport, *Multiplicative number theory* (3rd ed), Graduate Texts in Mathematics, vol. 74, Springer-Verlag, New York, vol. 74, 2000.
[2] J.B. Friedlander and H. Iwaniec, *Opera de Cribro*, AMS Colloquium Publications vol. 57, 2010.
[3] Andrew Granville, *Analytic number theory revealed: The distribution of prime numbers*, American Mathematical Society (to appear)

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