On Geometric Paradox in Quantization of Natural Dynamical Systems

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Abstract

The initial Schrödinger variational approach (1926) to quantization of the natural Hamilton mechanics in $2n$-dimensional phase space is revised in the modern paradigm of quantum mechanics in application to the system the Hamilton function of which is a positive-definite quadratic form of the $n$ momenta with the coefficients depending on the canonically conjugate coordinates in the generic Riemannian configuration space $V_n$. The quantum Hamiltonian thus obtained has a paradoxical potential term depending on choice of coordinates in $V_n$, which was discovered first by B. DeWitt in 1952 in the framework of canonical quantization of the system by a particular ordering of the basic operators of observables of momenta and coordinates. It is shown that the Schrödinger approach in the standard paradigm of quantum mechanics determines uniquely the ordering selected by DeWitt among a continuum of other possibilities to determine the quantum Hamiltonian.

Two conceptually important particular classes of observables of localization in $V_n$ are considered in detail. It is shown that, in general, the quantum-mechanical potential does not vanish even in the Euclidean configuration space $V_n$ except the case when Cartesian coordinates are taken as the observables of localization of the system. It is noted also that, in the quasi-classical approach to the quantization considered by DeWitt in 1957, the quantum Green function (propagator) is also non-unique and depends on the choice of a line in $V_n$ connecting these points. All three formalisms have the same local asymptotics of the quantum Hamiltonian if the normal Riemannian coordinates are used, at least implicitly, to localize the system in $V_n$.

Keywords: Hamilton function; Quantization; Quantum-Mechanical Potential; Observables of Localization; Quantum Anomaly of (non-relativistic) General Covariance.
1 Preface.

A little-known, but at least, theoretically existing paradoxical phenomenon of the quantum-mechanical potential (it will further be denoted as QMP) arises generically in quantization of a particular class of conservative Hamilton systems which are called the natural systems. They are the systems whose the (classical) Hamiltonians functions \( H^{(\text{nat})}(q,p) \) are the (positive-definite) inhomogeneous quadratic forms of momenta \( p_a, \ a, b, \ldots = 1, 2, \ldots, n \) with coefficients \( \omega_{ab}(q) \equiv \omega_{ba}(q) \) and an external potential \( V^{(\text{ext})}(q) \) of which are sufficiently smooth functions of coordinates \( q^a \) in an \( n \)-dimensional configuration space \( V_n \), that is

\[
H^{(\text{nat})}(q,p) = \frac{1}{2m} \omega_{ab}(q)p_ap_b + V^{(\text{ext})}(q),
\]

where \( m \) is a constant of dimension of mass, \( q^a, p_b \) are canonically conjugate coordinates on the phase space \( P_{2n} \equiv T^*V_n \) which has thus a symplectic structure. The functions \( \omega_{ab}(q) \) and \( V^{(\text{ext})}(q) \) are supposed to be tensor fields on \( V_n \) of types \( \left( \begin{array}{c} 2 \\ 0 \end{array} \right) \) and \( \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \) (i.e. a scalar) correspondingly w.r.t. the arbitrary sufficiently smooth transformations of \( q^a \in V_n \). Then, \( V_n \) is endowed by a Riemannian structure with the metric form

\[
ds^2(\omega) = \omega_{ab}(q) dq^a dq^b, \tag{2}
\]

where \( \omega_{ab} \) of \( V_n \) is determined as usual: \( \omega_{ac} \omega^{cb} = \delta^b_a \).

In the present paper, the so called heuristical level of mathematics traditional in the theoretical physics is adopted.

2 Canonical quantization of natural systems

Let \( s_{2n} \) be an appropriate subalgebra of the Poisson algebra of real functions \( f \in C^\infty(P_{2n}) \) which are often called Hamiltonians in the mathematically rigorous texts. In the present paper oriented to physics, they will be called classical observables and the term (classical) Hamiltonian will be kept for \( H^{(\text{nat})}(q,p) \) in view of its straightforward relation to the natural Hamilton mechanics'. According to [1], pp 425 - 434, existence on \( V_n \) of the natural
measure $\omega^{1/2}dq^1...dq^n$, $\omega$ being the determinant of the matrix $||\omega_{ab}||$, allows to construct a quantum counterpart to the generic natural system by the linear map

$$Q : s_{2n} \ni f \xrightarrow{Q} \hat{f}$$

(operators in a pre-Hilbert space $\mathcal{H}$),

satisfying the following conditions:

1. $Q^{-1} \xrightarrow{Q} \hat{1}$ (the identity operator in $\mathcal{H}$);
2. $\{f, g\}_h \xrightarrow{Q} i\hbar^{-1}[\hat{f}, \hat{g}] \xrightarrow{\text{def}} \left[ i\hbar^{-1}(\hat{f}\hat{g} - \hat{g}\hat{f}) \right]$ where $\{f, g\}_h \equiv \{f, g\}_0 + O(\hbar)$ is an antisymmetric bilinear functional of $f$ and $g$ and $\{f, g\}_0 \equiv \{f, g\}$ is the Poisson bracket in $\mathcal{P}_{2n}$;
3. $\hat{f} \xrightarrow{Q} (\hat{f})^\dagger$ (the Hermitean conjugation of $\hat{f}$ with respect to the scalar product in $\mathcal{H}$);
4. a complete set of classical observables ($\text{maximal Abelian subalgebra}$) $f^{(1)}, ..., f^{(n)} : f^{(a)} \in s_{2n}$, is mapped to a complete set (in the sense by Dirac [8], Chapter.III) of commuting operators $\hat{f}^{(1)}, ..., \hat{f}^{(n)}$.

If a particular system of canonically conjugate coordinates $q^a, p_b$ is fixed in $s_{2n} \equiv T^*V_n$ and the coordinates $q^a$ in $V_n$ are taken as the complete set of the classical observables of localisation of the system and operators $\hat{q}^{(a)} \equiv q^a \cdot \hat{1}$ in $\mathcal{H} \sim L^2(\Sigma; \mathbb{C}; \omega^{1/2}dq^a)$ are their quantum counterparts under condition (Q3), then the condition (Q2) will be fulfilled for a map $Q$ of the classical observables of canonically conjugate coordinates (the Darboux coordinates) $q^a, p_b$ on $T^*V_n$ to the symmetric operators (a configuration space representation of $n$-dimensional Heisenberg algebra).

$$q^a \xrightarrow{Q} \hat{q}^a \xrightarrow{\text{def}} q^a \cdot \hat{1}, \quad p_b \xrightarrow{Q} \hat{p}_b \xrightarrow{\text{def}} -i\hbar \left( \frac{\partial}{\partial q^b} + \frac{1}{4} \frac{\partial}{\partial q^b} \ln \omega \right).$$

(4)

The quantization map $Q$ of the metric tensor $\omega(q)$ can be be defined following the Von Neumann rule ([9], p. 313) for functions of commuting operators $\hat{A}_1, ..., \hat{A}_N$:

$$f(\hat{A}_1, ..., \hat{A}_N) \xrightarrow{Q} \hat{f} \xrightarrow{\text{def}} f(\hat{A}_1, ..., \hat{A}_N).$$

(5)

Thus, the components of the metric tensor on $V_n$ is also considered and quantized as an observable:

$$\omega_{ab}(q^1, ..., q^n) \xrightarrow{Q} \hat{\omega}_{ab}(q^1, ..., q^n) \xrightarrow{\text{def}} \omega_{ab}(q^1, ..., q^n) \cdot \hat{1}. \quad (6)$$

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1 As in the classical as well as in the quantum-mechanical context, the term "observable" means that, basically, a classical physical procedure exists, at least, speculative. In particular, the set of observables $\{q^a\}$ fixes a localization of the system in $V_n$. 

3
In 1952, B. C. DeWitt [2] had taken the following Hermitean product of these operators as the quantization map  of the generic natural Hamilton function \( H^{(\text{nat})}(q,p) \):

\[
H^{(\text{nat})}(q,p) \xrightarrow{Q} \hat{H}^{(\text{DW})}(\hat{q},\hat{p}) \overset{\text{def}}{=} \frac{1}{2m}\hat{p}_a\omega^{ab}(\hat{q})\hat{p}_b + V^{(\text{ext})}(\hat{q}).
\]

Then, in terms of the representations (1), (3) (the Schrödinger representation) as differential operators on \( L^2(\Sigma;\mathbb{C};\omega^{\frac{1}{2}}d^nq) \):

\[
\hat{H}^{(\text{DW})} \equiv -\frac{\hbar^2}{2m}\Delta(\omega)(q) + V^{(\text{qm:DW})}(q) \cdot \mathbf{i} + V^{(\text{ext})}(q) \cdot \mathbf{i},
\]

where

\[
\Delta(\omega)(q) \overset{\text{def}}{=} \omega^{-\frac{1}{2}}\frac{\partial}{\partial q^a}\left(\omega^{\frac{1}{2}}\omega^{ab}\frac{\partial}{\partial q^a}\right) \equiv \omega^{ab}\nabla^{(\omega)}_a\nabla^{(\omega)}_b
\]

is the Laplace-Beltrami operator for \( V_n \), \( \nabla^{(\omega)}_a \) is the metric connection (covariant derivative) in \( V_n \), and

\[
V^{(\text{qm:DW})}(q) \overset{\text{def}}{=} -\frac{\hbar^2}{2m}\omega^{-\frac{1}{2}}\frac{\partial}{\partial q^a}(\omega^{ab}\frac{\partial}{\partial q^b}\omega^{\frac{1}{2}})
\]

is just the DeWitt version of QMP. DeWitt perceived this potential term as an extraordinary and unsatisfactory result because \( V^{(\text{qm:DW})}(q) \) is generically not invariant w.r.t. to change of coordinates \( q^a \), contrary to \( H^{(\text{nat})}(q,p) \) and \( \Delta(\omega)(q) \). (Of course, there are particular choices of the coordinates, e.g., such that the coordinate condition \( \frac{\partial}{\partial q^a}\omega = 0 \) is satisfied and thus \( V^{(\text{qm:DW})}(q) = 0 \) for that choice.) The presence of this potential in the quantum Hamiltonian obviously makes its spectrum dependent on the choice of the coordinates \( q^a \) that is on a method of observation of localization of the system in \( V_n \) in the classical mechanics. An amazing result not only for the 1950s but also for today’s quantum physics, isn’t it?.

On the other hand, in the quantum field theory, the phenomenon of quantum anomalies is well known when some conservation law related to a symmetry that is valid in the classical theory ceases to be satisfied if the quantum effects are correctly taken into account. So the phenomenon of QMP may apparently be called the quantum anomaly of covariance of the natural Hamilton dynamics w.r.t. the coordinate transformations in the configuration space \( V_n \) (non-relativistic general covariance).

As soon as \( \hat{H}^{(\text{DW})} \) is determined, one may transform it, at least locally, to another curvilinear coordinates \( \tilde{q}^a \) by substitution \( q^a = q^a(\tilde{q}) \) as if all terms
in (8) were invariant w.r.t. to the substitution (i.e. are scalars). Thus, it is very important to distinguish the phenomenon of QMP generated by fixation particular observables $q^a$ from the "technical" change of an expression for the Hamilton operator changed by transition to some new coordinate variables as a appropriate tool to solve or to study the Schrödinger equation with QMP.

DeWitt took $\hat{H}^{(DW)}$ as "the simplest method of symmetrising" of the product of operators $\hat{q}, \hat{p}, \hat{\omega}$. It is a matter of taste, however. There is an one-parametric family of the hermitean orderings of the product, which satisfy the Correspondence Principle:

$$\hat{H}^{(\nu)} = \frac{2 - \nu}{8m} \omega^{ab}(\hat{q})\hat{p}_a\hat{p}_b + \frac{\nu}{4m} \hat{p}_a\omega^{ab}(\hat{q})\hat{p}_b + \frac{2 - \nu}{8m} \hat{p}_a\hat{p}_b\omega^{ab}(\hat{q}) \quad (11)$$

$$\hat{H}^{(Sch)} + V^{(qm;\nu)}(q) \cdot \mathbf{1} \quad (12)$$

$$V^{(qm;\nu)}(q) \equiv V^{(qm)}(q) + \hbar^2 \frac{(\nu - 2)}{8m} \partial_a\partial_b \omega^{ab}. \quad (13)$$

This is a typical ambiguity that arises not only in the canonical formalism of quantization and constitutes one of the main problems for the transition from classical to quantum theory in the Hilbert-space-based formalisms.

Again, the potential $V^{(qm;\nu)}(q)$ depends generically on the choice of coordinates $q^a$ for any value of parameter $\nu$ except the case when $V_n$ is Euclidean and Cartesian coordinates $x^i$ are taken as $q^a$ in it.

In 1957, DeWitt [10] had returned to the problem of quantization in the quasi-classical formalism, which he considered as leading to the result invariant w.r.t. the arbitrary transformations of coordinates in $V_n$. However, in the quasi-classical approach, these coordinates correspond to the "technical" changes of coordinates mentioned above. Dependence on the choice of "observation" is included into a propagator (a two-point Green function). The quasi-classical approach and its local coincidence with the result of the above canonical approach will be discussed in Section 5 below. But first I will state a strong argument in favor of $\hat{H}^{(DW)}$ arising in the Schrödinger variational approach to quantization [3] after a recent review of it in the modern paradigm of quantum mechanics [5].
The Schrödinger variational quantization of natural systems and its modernization

It seems to be a little-known fact that quantization of the generic natural system was considered first by E. Schrodinger in the third of his five papers submitted to "Annalen der Physik" in the first half of 1926, by which he had founded the wave mechanics\[4\]. The main purpose of his study was to construct the quantum dynamics which plays the same role w.r.t. the classical dynamics that the wave theory of light does w.r.t. the geometric optics. To this end, he had taken the following positive-definite functional of the \(\psi(q)\) from the normalized space \(\mathcal{H}(r)\) of real (!) functions

\[
J^{(Sch)}[\psi] = \int_{V_n} \frac{1}{2} \omega^2 d^n q \left\{ \frac{\hbar^2}{2m} \left( \partial_a \psi \omega^{ab} \partial_b \psi \right) + \psi^2 V^{(ext)}(q) \right\}, \tag{14}
\]

\(\psi(q) \in \mathcal{H}(r), \quad \partial_a \overset{\text{def}}{=} \frac{\partial}{\partial q^a}\)

subordinated to the additional condition

\[
\int_{V_n} \omega^2 d^n q \psi^2 = 1, \tag{15}
\]

Schrödinger interpreted \(J\) as the energy mean value of the real field \(\psi(q)\) corresponding to the quantum state of the natural system under consideration. Then the Euler-Lagrange equation with the right-hand term of (14) and accounting condition (15) with a Lagrange multiplier \(E\) comes to

\[
\hat{H}^{(Sch)} \psi = E \psi, \quad \hat{H}^{(Sch)} \overset{\text{def}}{=} -\frac{\hbar^2}{2m} \Delta^{(\omega)} + V^{(ext)}, \tag{16}
\]

The Schrödinger Hamiltonian \(\hat{H}^{(Sch)}\) looks a very satisfactory quantum counterpart to \(H^{(mat)}(q,p)\) : it depends on the observables of localization \(q^a\) in the configuration space \(V_n\) only through the Laplace-Beltrami operator \(\Delta^{(\omega)}\) and the external potential \(V^{(ext)}(q)\), and is invariant w.r.t. to choice of coordinates \(q^a\). In few weeks after publication of Schrödinger’s work, Eq. (16) had been used by F. Reiche \[11\] to calculate the spectrum of a spherical top considering its configuration space as the three-dimensional sphere with the identified antipodal points. Later, this result was developed for calculation of spectra of more complex molecules.
The point, however, is that Schrödinger considered the wave functions \( \psi(q) \) as real physical fields forming

\[
\mathcal{H} \sim \mathcal{H}^{(r)} \sim L^2(V_n; \mathbb{R}; \sqrt{\omega} d^n q).
\]  

(17)

His last paper among the mentioned basic ones \cite{4} was devoted just to the search for physical meaning of the wave function \( \psi(q) \in \mathcal{H}^{(r)} \).

In the modern QM, however, the space of wave functions is the pre-Hilbert space of complex-valued functions

\[
\mathcal{H} \sim \mathcal{H}^{(c)} \sim L^2(V_n; \mathbb{C}; \sqrt{\omega} d^n q).
\]  

(18)

The variational Schrödinger quantization has been reconsidered in the modern paradigm of non-relativistic quantum mechanics based on \( \mathcal{H}^{(c)} \) in \cite{5} . It unambiguously lead to the Hamiltonian \( \hat{H}^{(DW)}(\hat{q}, \hat{p}) \) introduced by DeWitt.\footnote{Apparently, Schrödinger’s approach to quantization of the generic natural systems was not known to DeWitt since he never referred to it.} Thus, QMP \( V^{(qm)}(q) \) , or, equivalently, \( \nu = 2 \) in the general rule of ordering and DeWitt’s choice \cite{7} of the quantum Hamiltonian are distinguished unambiguously and the problem of ordering of the elements of the Heisenberg algebra and the operator \( \hat{\omega}^{ab} \) in the Hamilton operator is solved.

Since the QMP depends on choice of observables of localization \( \hat{q}^a \) in the configuration space \( V_n \), consider two important particular cases: quasi-Cartesian (or normal Riemannian) coordinates and small deformations of the Cartesian coordinates \( x^i \) in \( E_n \).

4 Quasi-Cartesian, or normal Riemannian, coordinates in \( V_n \)

These coordinates are denoted further as \( y^{(a)} \). Let \( q_0 \in V_n \) is an origin of the coordinates \( y^{(a)}(q) \) determined by the components of the tangent vector in \( q_0 \) to the geodesic line from \( q_0 \) to \( q \) in an orthonormal frame \( \lambda_b^{(a)} \) at \( q_0 \) as follows

\[
y^{(a)}(q) \overset{\text{def}}{=} s(q, q_0) \left( \frac{dq^b}{ds} \right)_{q=q_0} \lambda_b^{(a)} ,
\]  

(19)

where \( s(q, q_0) \) is the geodesic distance between \( q_0 \) and \( q \), see, e.g., \cite{6}, Ch.II, Sec.8 .
Geodesic lines have no intrinsic curvatures. Thus, the coordinates \( y^{(a)} \) completely determined by the Riemannian structure of \( V_n \). In a vicinity of the origin \( q_0 \)

\[
 ds^2(\omega) = \left( \delta_{ij} - \frac{1}{3} R^{(\omega)}_{(ijkj)}(q_0) y^{(k)} y^{(l)} + O(s^3) \right) dy^{(i)} dy^{(j)}, \tag{20}
\]

where \( R^{(\omega)}_{abcd} \) is the Riemann–Christoffel curvature tensor of \( V_n \) determined so that \( \nabla \omega (\nabla \omega) a \nabla \omega b = R^{(\omega)}_{abcd} f_{d} \) for any twice differentiable 1-form \( f_{c} \). From here, the asymptotic expression of zero-order in \( s \) for QMP, Eq. \( (10) \), is

\[
 V^{\text{qm;DW}}(y) = \frac{\hbar^2}{2m} \left( \frac{1}{6} R^{(\omega)}(q_0) + O(s) \right) = \frac{\hbar^2}{2m} \left( \frac{1}{6} R^{(\omega)}(q) + O(s) \right) \tag{21}
\]

\( R^{(\omega)}(q) \) is the Ricci scalar curvature of \( V_n \). (In the expression for \( V^{(\text{qm})}(y) \), it is used that \( R^{(\omega)}(q_0) = R^{(\omega)}(y) + O(s) \).) However, it should not be missed that Eq.\((16)\) is a result of the special choice of observation of localization in \( V_n \). Then, the zero-order asymptotic quantum Hamiltonian in the time-independent Schrödinger equation is

\[
 \hat{H}^{\text{DW}}(y) = -\frac{\hbar^2}{2m} \left( \left( \frac{\partial}{\partial y^{(1)}} \right)^2 + \left( \frac{\partial}{\partial y^{(2)}} \right)^2 + \ldots + \left( \frac{\partial}{\partial y^{(n)}} \right)^2 \right) + V^{\text{qm;DW}}(y) + V^{\text{(ext)}}(y) + O(s), \tag{22}
\]

\( Nb! \) The expression for QMP is valid only for the canonical quantization with the particular choice \( \hat{q}^{a} \equiv y^{a} \) of the phase space coordinates:

\[
 y^{(a)} \rightarrow \hat{y}^{(a)} \equiv y^{(a)} \cdot \hat{1}, \quad p^{(b)} \rightarrow \hat{p}^{(b)} \equiv -i\hbar \left( \frac{\partial}{\partial y^{(b)}} + O(s) \right). \tag{23}
\]

After fixing thus the asymptotic method of observation of localization of the quantum system in \( V_n \), one may transform quasi-Cartesian coordinates \( y^{(a)} \) to arbitrary curvilinear coordinates \( \hat{q}^{a} \) in a small domain at the origin \( q_0 \) and thus represent there the asymptotic Hamiltonian as

\[
 \hat{H}^{\text{DW}} \approx -\frac{\hbar^2}{2m} \left( \Delta^{(\hat{\omega})} + \frac{1}{6} R^{(\hat{\omega})}(\hat{q}) \right) + \hat{V}^{(\text{ext})}(\hat{q}) \tag{24}
\]

where \( \Delta^{(\hat{\omega})} \) is the Laplace-Beltrami operator in ”arbitrary” coordinates \( \hat{q}^{a} \). However, recall again that Eq.\((21)\) is a result of ”technical” transformation to coordinates \( \hat{q}^{a} \) from the coordinates \( y^{a} \) initially taken for quantization as observables of localization in \( V_n \), and that \( y^{a} \) are in fact two-point functions determined by the geodesic line between \( q_0 \) and \( q \).
5 QMP in the Euclidean configuration space

QMP is generally non-zero even in the Euclidean $V_n$ if the observables $q^a$ are not Cartesian coordinates. Denoting the latters as $x^a$, let us calculate the first order approximation of QMP for the case in which $q^a = x^a + \epsilon f^a(x)$ where $\epsilon$ is a small parameter and $f^a(x)$ are arbitrary sufficiently differentiable functions. This means a small deformation of the system of Cartesian coordinates. A little algebra gives for this case

$$\hat{H}(DW;\epsilon) = -\frac{\hbar^2}{2m} \left( \Delta (\omega) + \frac{\epsilon \hbar}{4m} \Delta \text{Tr} \left\| \partial_a f^b \right\| + O(\epsilon^2) \right),$$

where the second summand is the asymptotic QMP and $\Delta$ is the Laplace operator in the Cartesian coordinates $x^a$ (as well in $q^a$ in the approximation under consideration).

This particular example shows more clearly that QMP is tightly related to the conceptual problem of measurement in quantum mechanics. Contrary to the case of Subsection 3.1, where the measurement of the localization in $V_n$ is determined in terms of the interior Riemannian structure of the natural system, now a non-trivial QMP is generated by introduction of an external vector field slightly deforming the Cartesian coordinates. It may have also a practical meaning, since a real physical observation of localization can never be precise even the Cartesian coordinates are taken for that.

6 Quasi-classical quantization

In 1957, DeWitt [10] had returned to a thorough study of the relation between (non-relativistic) classical and quantum dynamics’ in the curved configuration space. Considering the evolution in time of the wave function $\psi(q,t) \in L^2(V_n; \mathbb{C}; \omega^{1/2} d^n q)$ as determined, in general, by a propagator $G(q,t|q',t')$:

$$\psi(q,t) = \int_{V_n} \omega^{1/4}(q') d^n q' G(q,t|q',t') \psi(q',t'), \quad q,q' \in V_n,$$

DeWitt had generalized the Pauli construction [12] of $G(q,t|q',t')$ for a charge in an e.m. field in the Cartesian coordinates in the flat $V_n$ to the case of the generic natural system:

$$G(q,t|q',t') =$$

$$= \omega^{-1/4}(q) D^{1/2}(q,t|q',t') \omega^{-1/4}(q') \exp \left( -\frac{i}{\hbar} S(q,t|q',t') \right), \quad (27)$$
\[ D(q, t|q', t') \overset{\text{def}}{=} \det \left( -\frac{\partial^2 S}{\partial q^i \partial q'^j} \right) \] (the Van Vleck determinant)

and \( S(q, t|q', t') \) is a solution of the Hamilton-Jacobi equation for \( H^{(\text{nat})}(q, p) \).
(In fact, initially he had included interaction with an external vector field into the Hamiltonian, but, for the sake of simplicity, it will be omitted here as well as the term with the external scalar potential \( V^{(\text{ext})} \) as DeWitt had done in the final part of his paper.)

Construction (27) is in fact the postulate of quasi-classical (or, WKB) quantisation. Using the Hamilton-Jacobi equation for the natural system DeWitt had obtained that the quasi-classical propagator \( G(q, t|q', t') \) "nearly satisfies the Schrödinger equation":

\[ i\hbar \frac{\partial}{\partial t} G(q, t|q', t') = \left( -\frac{\hbar^2}{2m} \Delta^{(\omega)} + \tilde{V}^{(\text{qm})}(q, t|q', t') \right) G(q, t|q', t') \quad (28) \]

where

\[ \tilde{V}^{(\text{qm})}(q, t|q', t') \overset{\text{def}}{=} \frac{\hbar^2}{2m} \frac{\partial_i \left( \omega^\frac{1}{2}(q) \partial^i \left( \omega^{-\frac{1}{2}}(q) D^\frac{1}{2}(q, t|q', t') \right) \right)}{\omega^\frac{1}{2}(q) D^\frac{1}{2}(q, t|q', t')} \quad (29) \]

In fact, Eq. (28) is just the Schrödinger equation with QMP \( \tilde{V}^{(\text{qm})}(q, t|q', t') \) for the propagator \( G(q, t|q', t') \), and this QMP is evidently a two-point function of \( q \) and \( q' \) and, in general, is a functional of the line connecting them. So one sees that \( \tilde{V}^{(\text{qm})} \) only looks as a scalar with respect to transformations of the coordinates in neighborhoods of the points \( q^a \) and \( q'^a \). However, one should keep in mind that the two-point functions \( G(q, t|q', t') \) and \( D(q, t|q', t') \) are correspondingly a bi-scalar and a bi-density of the weight \( -1 \) at both points.\(^3\) In a more general context, they depend on a choice of the bi-scalar \( S(q, t|q', t') \) of the classical action. In the simplest case under consideration, it determines the geodesic dynamics in \( V_n \) and therefore the situation is equivalent to introduction of the Riemannian coordinates \( y^a \) as basic observables in Section 4. The system of arbitrary coordinates \( \{ q^a \} \) in Eqs. (28), (29) plays

\(^3\) An excellent introduction to the technique of bi-scalars, bi-tensors and bi-densities is given in the paper *Radiation Damping in a Gravitational Field* \([13]\) by B. DeWitt and R. Brehme. However, their main conclusion is not correct that the general–relativistic Lorentz–Dirac equation has no term including values of Riemann–Christoffel curvature tensor at the point of localization and thus the principle of equivalence is fulfilled. This was corrected in \([14]\) for the case of the electric charge and in \([15]\) for the case of point-like charge of the scalar field which is more relevant to the context of the present paper.
the technical role mentioned there. Therefore, it is not surprising that, from Eq. (29), DeWitt comes to the following zero order asymptotics:

\[ \tilde{V}(q_m)(q,t|q',t') = \frac{\hbar^2}{2m} \frac{1}{6} R^{(\omega)}(q) + o(q - q') + o(t - t'), \]

that is just to \( V^{(qm)}(y) \) in Eq. (21), Section 4. Thus, in this approximation, DeWitt’s quasiclassical result coinide with canonical QMP in the version of ordering of the basic operators determined by the modern Schrödinger QMP \( V^{(qm)}(q) \), Eq. (10), i.e. with \( \nu = 2 \) chosen in Eq. (13).

The quasi-classical local asymptotic by DeWitt had been focused on the use in the path integration approach to quantum mechanics. In this aspect, it has attracted quite a lot of attention and has been applied to more general non-relativistic mechanical systems, which demand to take into account the asymptotic terms of higher orders of \( q - q' \) in the quasi-classical propagators. A large analytical review of these systems and problems related to them in the path integration is given by L. V. Prokhorov [16] in 1982. He repeatedly noted ambiguities appearing in the formalism. In paper [5], a uniquely determined causal quantum propagator for the generic natural system have been constructed in the original Feynman formulation of the path integral, which supports again the result of the present paper. But it should be the matter of a special publication.

7 Conclusions

The quantum-mechanical potential (QMP), arising at quantization of the generic classical natural system, violates the symmetry (general covariance) of the original classical Hamiltonian under generic change of the observable of localization of the system in \( V_n \). (So, the Darboux coordinates in the phase space are determined, too). This phenomenon may be interpreted as a quantum anomaly of general covariance. It may to take place in a much wider range of dynamical systems where geometry of the phase space comes into play. From this point of view, the subject of the present paper has a great potential for development and improvement.

Returning to the natural systems and the formalisms of quantization considered in the present paper, namely DeWitt’s canonical and quasi-classical approaches, and the modernized Schrödinger variational approach, the following conclusions need to be emphasized.

1. When \( R \neq 0 \), the modernized Schrödinger variational quantization leads
not only to the same QMP which the canonical approach by DeWitt gives but distinguishes it unambiguously (requires $\nu = 2$ in Eq. (12)) among infinitely many other possibilities of ordering of the basic operators $\hat{q}, \hat{p}, \hat{\omega}$ in the quantum Hamilton operator $\hat{H}^{(\omega)}$.

2. The variational and canonical quantizations give rise to a closed expression for QMP contrary to the quasi-classical approach which is asymptotic and, in fact, is implicitly related to the choice of normal Riemannian (quasi-Cartesian) coordinates.

3. The both former approaches lead to the generically non-vanishing QMP even in the Euclidean configuration space $E_n$ except the case when the Cartesian coordinates are taken for localization of the quantum system. This effect can be understood from the physical point of view: any coordinate line, except the Cartesian ones, has intrinsic curvatures which can be considered as manifestation(s) of external forces on trajectories of the thought instruments for observation of localization of the system.

4. The most intriguing result is that, when the normal Riemannian coordinates $y^{(a)}$ are taken in $V_n$, it follows

$$V^{(\text{qm})}(y) = -\hbar^2 \frac{1}{2m} R(y) + O(s),$$

from all three considered formalisms of quantization as the first non-vanishing term of the local asymptotics of QMP in a vicinity of the origin of the coordinates. (Comparison the following terms of the asymptotic deserves calculation.) In its geometric meaning, this is just the term that R.Penrose [17] introduced into the massless general-relativistic Klein-Gordon equation (the general-relativistic generalization of the Schrödinger equation) for it to become conformally covariant. (Penrose has considered, of course, only the four-dimensional space-time, i.e. $n = 3$.) However, as it was shown in [18], a much more comprehensive study of the role of conformal covariance in the quantum field theory of a scalar field with $m^2 \geq 0$ and arbitrary space dimension $n$, the conformal covariance of the equation in the particular case of $m^2 = 0$ requires that the additional term must be

$$-\frac{\hbar^2}{2m} \frac{n-1}{4n} R,$$

that is it coincides with (31) on the appropriate space sections of the globally-static space-time only if $n = 3$. Since the early 1970s the general-relativistic scalar field equation with this additional term (exact, not asymptotic) has been called later as the equation with conformal coupling (the term Penrose-Chernikov-Tagirov equation can also be met in the literature) and widely
applied in the inflationary cosmology. However, a mystery is that the asymptotic QMP arises in the non-relativistic quantum mechanics without any visible connection with general relativity and conformal symmetry and with the same coefficient $1/6$ in front of the Ricci scalar curvature $R$ of the Riemannian configuration space $V_n$ i.e. for any space dimension $n$. Thus, non-relativistic quantum mechanics distinguishes the dimension of real space $n = 3$ among all thought Lorentzian possibilities.

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