ABSTRACT HOMOMORPHISMS FROM LOCALLY COMPACT GROUPS TO DISCRETE GROUPS

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Abstract. We show that every abstract homomorphism \( \varphi \) from a locally compact group \( L \) to a graph product \( G_{\Gamma} \), endowed with the discrete topology, is either continuous or \( \varphi(L) \) lies in a 'small' parabolic subgroup. In particular, every locally compact group topology on a graph product whose graph is not 'small' is discrete. This extends earlier work by Morris-Nickolas.

We also show the following. If \( L \) is a locally compact group and if \( G \) is a discrete group which contains no infinite torsion group and no infinitely generated abelian group, then every abstract homomorphism \( \varphi : L \to G \) is either continuous, or \( \varphi(L) \) is contained in the normalizer of a finite nontrivial subgroup of \( G \). As an application we obtain results concerning the continuity of homomorphisms from locally compact groups to Artin and Coxeter groups.

1. Introduction

We investigate the following type of question.

Let \( L \) be a locally compact group and let \( G \) be a discrete group. Under what conditions on the group \( G \) is an abstract (i.e. not necessarily continuous) homomorphism \( \varphi : L \to G \) automatically continuous?

There are many results in this direction in the literature, see [11], [17], [21] or [30]. In particular, Dudley [21] proved that every abstract homomorphism from a locally compact group to a free group is automatically continuous. This was generalized by Morris and Nickolas [30]. They proved that every abstract homomorphism from a locally compact group to a free product of groups is either continuous, or the image of the homomorphism is conjugate to a subgroup of one of the factors of the free product.

Our first aim is to prove similar results for the case where the codomain \( G \) of an abstract homomorphism \( L \to G \) is a graph product of arbitrary groups. The precise definition of a graph product is given in Section 2 below. For graph products we allow infinite graphs with a uniform bound on the sizes cliques.

Throughout, \( L \) denotes a Hausdorff locally compact group with identity component \( L^0 \), and \( G \) denotes a discrete group. We call \( L \) almost connected if the totally disconnected group \( L/L^0 \) is compact. By an abstract homomorphism we mean a group homomorphism between topological groups which is not assumed to be continuous. We remark that every abstract homomorphism whose codomain is discrete is open.

**Proposition A.** Let \( \varphi : L \to G_{\Gamma} \) be an abstract homomorphism from an almost connected locally compact group \( L \) to a finite dimensional graph product \( G_{\Gamma} \). Then \( \varphi(L) \) lies in a complete parabolic subgroup of \( G_{\Gamma} \).

Using Proposition A, we show the following more general result.

**Theorem B.** Let \( \varphi \) be an abstract homomorphism from a locally compact group \( L \) to a finite dimensional graph product \( G_{\Gamma} \). Then either \( \varphi \) is continuous and open, or \( \varphi(L) \) lies in a conjugate
of a parabolic subgroup $G_{S,\text{lk}(S)}$, where $S \neq \emptyset$ is a clique. If every composite $L \xrightarrow{\varphi} G_{\Gamma} \xrightarrow{r} G_v$ is continuous, then $\varphi$ is continuous and open.

In particular, every locally compact group topology on a finite dimensional graph product $G_{\Gamma}$ is discrete, unless $\Gamma$ is contained in the link of a clique. In the latter case, $G_{\Gamma}$ is a direct product of vertex groups and a smaller graph product, and then a locally compact topology on $G_{\Gamma}$ may indeed be nondiscrete.

Our remaining results deal with a certain class $G$ of discrete groups. Let $G$ denote the class of all groups $G$ with the following two properties:

(i) Every torsion subgroup $T \subseteq G$ is finite, and

(ii) Every abelian subgroup $A \subseteq G$ is a (possibly infinite) direct sum of cyclic groups.

The abelian subgroups $A$ in such a group are thus of the form $A = F \times \mathbb{Z}(J)$, where $F$ is a finite abelian group and $\mathbb{Z}(J)$ is free abelian of (possibly infinite) rank $\text{card}(J)$. We remark that subgroups of free abelian groups are again free abelian [24, A1.9].

We study abstract homomorphisms from locally compact groups to groups in this class. We show in Section 7 that the class $G$ is huge. It is closed under finite products, under coproducts, and under passage to subgroups, see Proposition 7.1. For example, every finitely generated hyperbolic group, every right-angled Artin group, every Artin group of finite type and every Coxeter group is in this class, see Propositions 7.2, 7.3 and 7.4. Furthermore, the groups $\text{GL}_n(\mathbb{Z})$, $\text{Out}(F_n)$ and the mapping class groups $\text{Mod}(S_g)$ of compact orientable surfaces of genus $g$ are in this class, see Proposition 7.5.

We obtain the following results.

**Proposition C.** Let $\varphi$ be an abstract homomorphism from a locally compact group $L$ to a group $G$ in the class $G$. Then $\varphi$ factors through the canonical projection $\pi : L \to L/L^\circ$. If $L$ is almost connected, then $\varphi(L)$ is finite.

**Theorem D.** Let $\varphi$ be an abstract homomorphism from a locally compact group $L$ to a group $G$ in the class $G$. Then either $\varphi$ is continuous, or $\varphi(L)$ lies in the normalizer of a finite non-trivial subgroup of $G$.

The following is an immediate consequence of Theorem D.

**Corollary E.** Every abstract homomorphism from a locally compact group $L$ to a torsion free group $G$ in the class $G$ is continuous. In particular, every abstract homomorphism from a locally compact group to a right-angled Artin group or to an Artin group of finite type is continuous.

Related results on abstract homomorphisms into right-angled Artin groups and into Artin groups of non-exceptional finite type were recently proved in [18].

Our proofs depend heavily on a theorem of Iwasawa on the structure of connected locally compact groups and on a theorem of van Dantzig on the existence of compact open subgroups in totally disconnected groups. For the proof of Proposition A we use the structure of the right-angled building $X_{\Gamma}$ associated to a graph product $G_{\Gamma}$.

2. **Graph products**

In this section we briefly present the main definitions and properties concerning graph products. These groups are defined by presentations of a special form.

A *simplicial graph* $\Gamma = (V, E)$ consists of a set $V$ of *vertices* and a set $E$ of 2-element subsets of $V$ which are called *edges*. We allow infinite graphs. Given a subset $S \subseteq V$, the *graph generated by $S$* is the graph with vertex set $S$ and edge set $E|_S = \{\{v, w\} \mid v, w \in S\}$. We call $S$ a *clique* if $E|_S = \{\{v, w\} \mid v, w \in S \text{ with } v \neq w\} = \binom{S}{2}$. We count the empty set as a
clique. We say that \( \Gamma \) has \textit{finite dimension} if there is a uniform upper bound on the cardinality of cliques in \( \Gamma \). For a subset \( S \subseteq V \) we define its \textit{link} as

\[
\text{lk}(S) = \{ w \in V \mid \{v, w\} \in E \text{ for all } v \in S \}.
\]

**Definition 2.1.** Let \( \Gamma \) be a simplicial graph, as defined above. Suppose that for every vertex \( v \in V \) we are given a nontrivial abstract group \( G_v \). The \textit{graph product} \( G_\Gamma \) is the group obtained from the coproduct of the \( G_v \), for \( v \in V \), by adding the commutator relations \( gh = hg \) for all \( g \in G_v, h \in G_w \) with \( \{v, w\} \in E \), i.e.

\[
G_\Gamma = \coprod_{v \in V} G_v / \langle \langle [G_v, G_w] \text{ for } \{v, w\} \in E \rangle \rangle.
\]

Graph products are special instances of graphs of groups, and in particular colimits in the category of groups [20, §5]. We call the graph product \textit{finite dimensional} if \( \Gamma \) has finite dimension as defined above, i.e. if there is an upper bound on the size of cliques in \( \Gamma \).

The first examples to consider are the extremes. If \( E = \emptyset \), then \( G_\Gamma \) is the coproduct of the groups \( G_v \), for \( v \in V \). On the other hand, if \( E = \binom{V}{2} \) is the set of all 2-element subsets of \( V \), then \( G_\Gamma \) is the direct sum of the \( G_v \), for \( v \in V \). So graph products interpolate between coproducts and direct sums of groups.

### 2.2. Parabolic subgroups

Let \( \Gamma = (V,E) \) be a simplicial graph, let \( G_\Gamma \) denote the graph product of a family of groups \( \{G_v \mid v \in V\} \) and let \( S \) be a subset of \( V \). The subgroup \( G_S \) of \( G_\Gamma \) generated by the \( G_v \), for \( v \in S \), is again a graph product, corresponding to the subgraph \( \Gamma' = (S,E|_S) \). This follows from the Normal Form Theorem [23, Thm. 3.9]. There is also a retraction homomorphism

\[
r_S : G_\Gamma \to G_S
\]

which is obtained by substituting the trivial group for \( G_v \) for all \( v \in V - S \) [3, Section 3].

If \( S \subseteq V \) is a subset (resp. a clique), then \( G_S \) is called a \textit{special parabolic subgroup} (resp. a \textit{special complete parabolic subgroup}). The conjugates in \( G_\Gamma \) of the special (complete) parabolic subgroups are called (complete) parabolic subgroups. We note that parabolic subgroups behave well. For \( R, S \subseteq V \) and \( a, b \in G_\Gamma \) there exists \( T \subseteq R \cap S \) and \( c \in G_\Gamma \) such that

\[
(1) \quad aG_Ra^{-1} \cap bG_Sb^{-1} = cG_Tc^{-1},
\]

see [3, Lemma 3.4]. If \( gG_Sg^{-1} \subseteq G_S \), then by [3, Lemma 3.9]

\[
(2) \quad gG_Sg^{-1} = G_S.
\]

Let \( X \) be a subset of \( G_\Gamma \). If the set of all parabolic subgroups containing \( X \) has a minimal element, then this minimal parabolic subgroup containing \( X \) is unique by the remarks above. In this case, it is called the \textit{parabolic closure} of \( X \) and denoted by \( \text{Pc}(X) \). The parabolic closure always exists if \( \Gamma \) is finite or if \( X \) is finite [3, Proposition 3.10].

Let \( H \subseteq G_\Gamma \) be a subgroup. We denote by \( \text{Nor}_{G_\Gamma}(H) \) the normalizer of \( H \) in \( G_\Gamma \). For a parabolic subgroup of \( G_\Gamma \) there is a good description of the normalizer.

**Lemma 2.3.** [3, Lemma 3.12 and Proposition 3.13]

(i) Let \( H \subseteq G_\Gamma \) be a subgroup. Suppose that the parabolic closure of \( H \) in \( G_\Gamma \) exists. Then \( \text{Nor}_{G_\Gamma}(H) \subseteq \text{Nor}_{G_\Gamma}(\text{Pc}(H)) \).

(ii) Let \( G_S \) be a non-trivial special parabolic subgroup of \( G_\Gamma \). Then \( \text{Nor}_{G_\Gamma}(G_S) = G_{\text{lk}(S)} \).

\(^1\)We need nontrivial vertex groups in order to obtain a building. Alternatively, one may remove all vertices \( v \) from \( \Gamma \) whose vertex group \( G_v \) is trivial, without changing the resulting graph product.
3. Actions on cube complexes

A detailed description of CAT(0) cube complexes can be found in [12] and in [33]. Let $C$ be the class of finite dimensional CAT(0) cube complexes and let $A$ be the subclass of $C$ consisting of simplicial trees. Inspired by Serre’s fixed point property $F_A$, Bass introduced the property $F_A'$ in [5]. A group $G$ has property $F_A'$ if every simplicial action of $G$ on every member $T$ of $A$ is locally elliptic, i.e. if each $g \in G$ fixes some point on the tree $T$. A generalization of property $F_A'$ was defined in [27]. A group $G$ has property $F_C'$ if every simplicial action of $G$ on every member $X$ of $C$ is locally elliptic, i.e. if each $g \in G$ fixes some point on $X$. Bass proved in [5] that every profinite group has property $F_A'$. His result was generalized by Alperin to compact groups in [1] and to almost connected locally compact groups in [2].

The next result was proved by Caprace in [14, Theorem 2.5].

**Proposition 3.1.** Let $X$ be a locally Euclidean CAT(0) cell complex with finitely many isometry types of cells, and $L$ be a compact group acting as an abstract group on $X$ by cellular isometries. Then every element of $L$ is elliptic. In particular, every compact group has property $F_C'$. 

We recall that a group $G$ is called *divisible* if $\{g^n \mid g \in G\} = G$ holds for all integers $n \geq 1$. Another result which we will need in order to prove Proposition A is the following.

**Lemma 3.2.** [14, Theorem 2.5 Claim 7] Every divisible group has property $F_C'$. 

The following result is due to Sageev and follows from the proof of Theorem 5.1 in [33], see also [27, Theorem A].

**Proposition 3.3.** Let $G$ be a finitely generated group acting by simplicial isometries on a finite dimensional CAT(0) cube complex. If the $G$-action is locally elliptic, then $G$ has a global fixed point.

The last fact we need for the proof of Proposition A concerning global fixed points is the following easy consequence of the Bruhat-Tits Fixed Point Theorem [28, Lemma 2.1].

**Lemma 3.4.** Suppose that a group $H$ acts isometrically on a complete CAT(0) space. If $H = H_1 H_2 \cdots H_r$ is a product of finitely many subgroups $H_j$ each fixing some point in $X$, then $H$ has a global fixed point.

3.5. Graph products, cube complexes and the building. Associated to finite dimensional graph products are certain finite dimensional CAT(0) cube complexes. We briefly describe the construction of these spaces. For a graph product $G_\Gamma$ we consider the poset

$$P = \{gG_T \mid g \in G_\Gamma \text{ and } T \text{ is a clique}\},$$

ordered by inclusion (we recall that we allow empty cliques). The group $G_\Gamma$ acts by left multiplication on this poset and hence simplicially on the flag complex $X_\Gamma$ associated to this coset poset. This flag complex has a canonical cubical structure. With respect to this structure $X_\Gamma$ is the Davis realization of a right-angled building, [20, Theorem 5.1]. By [20, Theorem 11.1] the Davis realization of every building is a complete CAT(0) space. Hence $X_\Gamma$ is a finite dimensional CAT(0) cube complex, and $G_\Gamma$ acts isometrically on $X_\Gamma$. The chambers of $X_\Gamma$ correspond to the cosets of the trivial subgroup, i.e. to the elements of $G_\Gamma$. The $G_\Gamma$-stabilizer of a chamber (a maximal cube) is therefore trivial. The vertices of $X_\Gamma$ correspond to the cosets of the $G_S$, where $S \subseteq V$ is an inclusion-maximal clique. The action of $G_\Gamma$ on $X_\Gamma$ preserves the canonical cubical structure.

One nice property of this action is the following: if a subgroup $H \subseteq G_\Gamma$ has a global fixed point in $X_\Gamma$, then there exists a vertex in $X_\Gamma$ which is fixed by $H$. This follows from the fact that the action is type preserving. Furthermore, the stabilizer of a vertex $gG_T$ is equal to $gG_T g^{-1}$. 

Lemma 3.6. Let $G_{\Gamma}$ be a finite dimensional graph product and let $H$ be a subgroup. If the action of $H$ on the building $X_{\Gamma}$ is locally elliptic, then $H$ has a global fixed point.

Proof. For each finite subset $X \subseteq H$, the finitely generated group $\langle X \rangle$ acts locally elliptically on $X_{\Gamma}$. Thus $\langle X \rangle$ has by Proposition 3.3 a fixed vertex $g_{S_X}$, for some $g \in G_{\Gamma}$ and some maximal clique $S$. It follows that the parabolic closure of $X$ is of the form $Pc(X) = g_{S_X}g^{-1}$, where $S_X$ is a clique depending uniquely on $X$. Since there is an upper bound on the size of cliques in $\Gamma$, there exists a finite set $Z \subseteq H$ such that $Z$ is maximal among all cliques $S_X$, for $X \subseteq H$ finite. We claim that $Z \subseteq \text{Pc}(Z)$.

Let $h \in H$ and put $X = Z \cup \{h\}$. If we put $\text{Pc}(X) = aG_{S_X}a^{-1}$ and $\text{Pc}(Z) = bG_{S_Z}b^{-1}$, then

$$aG_{S_X}a^{-1} \supseteq bG_{S_Z}b^{-1}$$

because $X \supseteq Z$. Then $S_X \supseteq S_Z$ holds by (2.2)(1). From the maximality of $S_Z$ we conclude that $S_Z = S_X$. Then $aG_{S_X}a^{-1} = bG_{S_Z}b^{-1}$ by (2.2)(2). It follows that $H \subseteq \text{Pc}(Z) = bG_{S_Z}b^{-1}$, and thus $H$ has a global fixed point. □

4. The proofs of Proposition A and Theorem B

Proposition A. Let $\varphi : L \to G_{\Gamma}$ be an abstract homomorphism from an almost connected locally compact group $L$ to a finite dimensional graph product $G_{\Gamma}$. Then $\varphi(L)$ lies in a complete parabolic subgroup of $G_{\Gamma}$.

Proof. The group $L$ acts via

$$L \to G_{\Gamma} \to \text{Isom}(X_{\Gamma})$$

isometrically and simplicially on the right-angled building $X_{\Gamma}$.

Suppose first that $L$ is compact. Then the $L$-action is by Proposition 3.1 locally elliptic. Hence there is a global fixed point by Lemma 3.6.

Suppose next that $L$ is connected. By Iwasawa’s decomposition [26, Theorem 13] we have

$$L = H_1H_2 \cdots H_rK,$$

where $K$ is a connected compact group and $H_i \cong \mathbb{R}$ for $i = 1, \ldots, r$. Each group $H_i$ has a fixed point by Lemma 3.2 and $K$ has a fixed point by the result in the previous paragraph. Hence $L$ has a fixed point by Lemma 3.4.

Now we consider the general case. If $L$ is almost connected, then the identity component $L^0$ has a global fixed point by the previous paragraph. The fixed point set $Z \subseteq X_{\Gamma}$ of $L^0$ is a convex CAT(0) cube complex, because the $L$-action is simplicial and type-preserving. By Proposition 3.1 the action of $L/L^0$ on $Z$ is locally elliptic. Hence the action of $L$ on $Z$ is locally elliptic and by another application of Lemma 3.6 shows that $L$ has a global fixed point.

Now we may prove Theorem B.

Theorem B. Let $\varphi$ be an abstract homomorphism from a locally compact group $L$ to a finite dimensional graph product $G_{\Gamma}$. Then either $\varphi$ is continuous and open, or $\varphi(L)$ lies in a conjugate of a parabolic subgroup $G_{\text{Sjik}(S)}$, where $S \neq \emptyset$ is a clique. If every composite $L \xrightarrow{\varphi} G_{\Gamma} \xrightarrow{\pi} G_v$ is continuous, then $\varphi$ is continuous and open.

Proof. Let $L^0$ be the connected component of the identity in $L$. We distinguish two cases.

Case 1: $\varphi(L^0)$ is not trivial.

By Proposition A we know that $\varphi(L^0) \subseteq gG_Tg^{-1}$ where $T \subseteq V$ is a clique and $g \in G_{\Gamma}$. Hence $\text{Pc}(\varphi(L^0)) = hG_Sh^{-1}$, where $\emptyset \neq S \subseteq T$ and $h \in G_{\Gamma}$. Since $\varphi(L)$ normalizes $\varphi(L^0)$, we have by Lemma 2.3 that $\varphi(L) \subseteq \text{Nor}_{G_{\Gamma}}(\text{Pc}(\varphi(L)))$. This normalizer is of the form $hG_{\text{Sjik}(S)}h^{-1}$, for some $h \in G_{\Gamma}$. We note that in Case 1, the homomorphism $\varphi$ is not continuous.

Case 2: $\varphi(L^0)$ is trivial.
Then \( \varphi \) factors through an abstract homomorphism \( \psi : L/L^o \to G_\Gamma \), and \( L/L^o \) is a totally disconnected locally compact group. By van Dantzig’s Theorem [10, III\S4, No. 6] there exists a compact open subgroup \( K \) in \( L/L^o \).

Subcase 2a: There is a compact open subgroup \( K \subseteq L/L^o \) such that \( \psi(K) \) is trivial. Then the kernel of \( \psi \) is open in \( L/L^o \) and hence \( \psi \) and \( \varphi \) are continuous.

Subcase 2b: There is no compact open subgroup \( K \)

Let \( \mathcal{K} \) denote the collection of all compact open subgroups of \( L/L^o \). We note that \( L \) acts on \( \mathcal{K} \) by conjugation. For \( K \in \mathcal{K} \) we put \( \text{Pc}(\psi(K)) = gG_Kg^{-1} \). Thus \( S_K \) is a clique in \( \Gamma \) which depends uniquely on \( K \). We choose \( M \in \mathcal{K} \) in such a way that \( S_M \) is minimal and we note that \( S_M \neq \emptyset \). Given \( a \in L \) we have \( M \cap aMa^{-1} \in \mathcal{K} \) and

\[
\text{Pc}(\psi(aMa^{-1})) = (a) \text{Pc}(\psi(M))\psi(a)^{-1}.
\]

From [2.2](1) and

\[
\text{Pc}(\psi(M)) \supseteq \text{Pc}(\psi(M) \cap \psi(aMa^{-1})) \subseteq \text{Pc}(\psi(aMa^{-1}))
\]

we obtain that

\[
S_M \supseteq S_M\cap aMa^{-1} \subseteq S_Ma^{-1}.
\]

Since both \( S_Ma^{-1} \) and \( S_M \) are minimal we conclude that

\[
S_M = S_M\cap aMa^{-1} = S_Ma^{-1}
\]

and that

\[
\text{Pc}(\psi(M)) = \text{Pc}(\psi(aMa^{-1})) = (a) \text{Pc}(\psi(M))\psi(a)^{-1}.
\]

Therefore \( \psi(a) \) normalizes \( \text{Pc}(\psi(M)) \), whence

\[
\varphi(L) = \psi(L/L^o) \subseteq hG_{S_M\cup \text{lk}(S_M)}h^{-1},
\]

for some \( h \in G_\Gamma \) by Lemma [2.3].

Suppose now towards a contradiction that \( \varphi \) is not continuous, but that each composite \( L \xrightarrow{\varphi} G_\Gamma \xrightarrow{r} G_v \) is continuous. Then \( \varphi(L) \subseteq gG_{S_M\cup \text{lk}(S)}g^{-1} \) for some nonempty clique \( S \). There is a direct product decomposition

\[
G_{S_M\cup \text{lk}(S)} = G_S \times G_{\text{lk}(S)} = \prod_{v \in S} G_v \times G_{\text{lk}(S)}
\]

and therefore \( \varphi \) factors as a product of commuting homomorphisms

\[
\varphi(a) = g^{-1} \prod_{v \in S} \varphi_v(a)\varphi_{\text{lk}(S)}(a)g^{-1},
\]

with \( \varphi_v = \varphi \circ r_v \) and \( \varphi_{\text{lk}(S)} = \varphi \circ r_{\text{lk}(S)} \). Here we use the retractions \( r \) introduced in Section [2].

Since the \( \varphi_v \) are all continuous, \( \varphi_{\text{lk}(S)} \) is not continuous. Hence we find a clique \( T \subseteq \text{lk}(S) \) such that \( \varphi_{\text{lk}(S)}(L) \subseteq hG_{T\cup \text{lk}(T)\cup \text{lk}(S)}h^{-1} \). But then \( S \cup T \) is a clique (because \( T \subseteq \text{lk}(S) \)) which is strictly bigger than \( S \). If we continue in this fashion, we end up after finitely many steps with an empty link, because \( \Gamma \) has finite dimension. Thus \( \varphi \) is a finite product of commuting continuous homomorphisms, and therefore itself continuous. This is a contradiction. \( \square \)

5. The proof of Proposition C

We consider abstract homomorphisms from locally compact groups \( L \) into groups \( G \) which are in the class \( \mathcal{G} \). We recall from the introduction that for such a group \( G \), every torsion subgroup \( T \subseteq G \) is finite, and every abelian subgroup \( A \subseteq G \) is a (possibly infinite) direct sum of cyclic groups. In particular, such a group \( G \) has no nontrivial divisible abelian subgroups.
Proposition C. Let $\varphi$ be an abstract homomorphism from a locally compact group $L$ to a group $G$ in the class $\mathcal{G}$. Then $\varphi$ factors through the canonical projection $\pi : L \to L/L^\circ$. If $L$ is almost connected, then $\varphi(L)$ is finite.

Proof. We first show that every homomorphism $\rho : K \to G$ has finite image if $K$ is compact. Suppose that $g \in K$. We claim that $\rho(g)$ has finite order. The subgroup $H = \langle g \rangle$ is compact abelian, whence $\rho(H) = F \times \mathbb{Z}^J$, where $F$ is a finite abelian group. By Dudley’s result [21], a compact group has no nontrivial free abelian quotients. Therefore $\rho(H)$ is finite and in particular, $\rho(g)$ has finite order. Since $G$ contains no infinite torsion groups, $\rho(K)$ is finite.

Now we show that $\varphi(L^\circ)$ is trivial. By Iwasawa’s Theorem [26] Theorem 13] there is a decomposition $L^\circ = H_1 \cdots H_r K$, where $H_j \cong \mathbb{R}$ for $j = 1, \ldots, r$ and where $K$ is a compact connected group. The groups $H_1, \ldots, H_r$ are abelian and divisible. From our assumptions on the class $\mathcal{G}$ we see that the abelian groups $\varphi(H_j)$ are trivial, for $j = 1, \ldots, r$. The compact group $K$ is connected and therefore divisible [24 Theorem 9.35]. A finite divisible group is trivial, and therefore $\varphi(K)$ is trivial as well. This shows that $\varphi(L^\circ)$ is trivial.

The first paragraph of the present proof shows then that $\varphi(L)$ is finite if $L/L^\circ$ is compact. □

6. The proof of Theorem D

We are now ready to prove Theorem D.

Theorem D. Let $\varphi$ be an abstract homomorphism from a locally compact group $L$ to a group $G$ in the class $\mathcal{G}$. Then either $\varphi$ is continuous, or $\varphi(L)$ lies in the normalizer of a finite non-trivial subgroup of $G$.

Proof. Let $L^\circ$ be the connected component of the identity in $L$. By Proposition C the homomorphism $\varphi$ factors through a homomorphism $\psi : L/L^\circ \to G$. The totally disconnected locally compact group $L/L^\circ$ contains by van Dantzig’s Theorem [10 III §4, No. 6] compact open subgroups. We distinguish two cases.

Case 1: $\psi(K)$ is trivial for some compact open subgroup $K \subseteq L/L^\circ$.

Then the kernel of $\psi$ is open and therefore $\psi$ and $\varphi$ are continuous.

Case 2: $\psi(K)$ is nontrivial for every compact open subgroup $K \subseteq L/L^\circ$.

By Proposition C, the image $\psi(K)$ of such a group $K$ is finite. Among the compact open subgroups of $L/L^\circ$ we choose $M$ such that $\psi(M)$ is minimal. Given $g \in L/L^\circ$, we have then that $\psi(gM^g^{-1}) = \psi(M \cap gM^g^{-1}) = \psi(M)$. It follows that $\psi(g)$ normalizes $\psi(M)$. □

7. Some remarks on the class $\mathcal{G}$

In this last section we show that the class $\mathcal{G}$ contains many groups.

Proposition 7.1. The class $\mathcal{G}$ is closed under passage to subgroups, under passage to finite products, and under passage to arbitrary coproducts.

Proof. If $H \subseteq G \in \mathcal{G}$, then clearly $H \in \mathcal{G}$. If $G_1, \ldots, G_r \in \mathcal{G}$ and if $T \subseteq \prod_{j=1}^r G_j$ is a torsion group, then the projection $\pi_j(T) = T_j \subseteq G_j$ is also a torsion group. Hence $T \subseteq \prod_{j=1}^r T_j$ is finite. Similarly, if $A \subseteq \prod_{j=1}^r G_j$ is abelian, then $A$ is contained in the product $\prod_{j=1}^r \pi_j(A)$, which is a direct sum of a finite abelian group and a free abelian group. Hence $A$ itself is a direct sum of a finite abelian group and a free abelian group. Finally suppose that $(G_j)_{j \in J}$ is a family of groups in $\mathcal{G}$. By Kurosh’s Subgroup Theorem [4], every subgroup of the coproduct $\coprod_{j \in J} G_j$ is itself a coproduct $F \ast \coprod_{j \in J} g_j U_j g_j^{-1}$, were $F$ is a free group, $U_j \subseteq G_j$ is a subgroup and the $g_j$ are elements of $\coprod_{j \in J} G_j$. If such a group is abelian, then it is either cyclic or conjugate to a subgroup of one of the free factors. □

Proposition 7.2. Every hyperbolic group $G$ is in the class $\mathcal{G}$. 
Proof. By a theorem of Gromov [22, Chap. 8 Cor. 36], every torsion subgroup of a hyperbolic group is finite. Furthermore, every abelian subgroup of a hyperbolic group is finitely generated.

Proposition 7.3. Let $A$ be an Artin group. If $A$ is a right-angled Artin group or an Artin group of finite type, then $A$ is torsion free and every abelian subgroup of $A$ is finitely generated.

Proof. Every right-angled Artin group is torsion free by [23, Corollary 3.28]. Moreover $A$ is a CAT(0) group, see [16]. Hence every abelian subgroup of $A$ is finitely generated, see [12, II Corollary 7.6]. If $A$ is an Artin group of finite type, then $A$ is torsion free by [13]. By [8, Corollary 4.2], every abelian subgroup of $A$ is finitely generated.

We note that it is an open question if every Artin group is torsion free [15, Conjecture 12].

Proposition 7.4. Let $W$ be a Coxeter group. Then every torsion subgroup of $W$ is finite and every abelian subgroup of $W$ is finitely generated.

Proof. It was proved in [31, Theorem 14.1] that Coxeter groups are CAT(0) groups. Hence every abelian subgroup of $W$ is finitely generated [12, II Corollary 7.6] and the order of finite subgroups of $W$ is bounded [12, II Corollary 2.8(b)]. Let $T \subseteq W$ be a torsion group. Since $W$ is a linear group [19, Corollary 6.12.11] and since every finitely generated linear torsion group is finite [34, I], it follows that every finitely generated subgroup of $T$ is finite. Since the order of finite subgroups of $T$ is bounded, $T$ is finite.

Proposition 7.5. The groups $GL_n(\mathbb{Z})$, the groups $Out(F_n)$ of outer automorphisms of free groups and the mapping class groups $Mod(S_g)$ of orientable surfaces of genus $g$ are in the class $G$.

Proof. Since $GL_n(\mathbb{Z})$ is a linear group, it follows that every finitely generated torsion subgroup is finite [34, I]. Since the order of finite subgroups in $GL_n(\mathbb{Z})$ is bounded [29], we obtain that every torsion subgroup of $GL_n(\mathbb{Z})$ is finite. It was proved in [?] that every abelian subgroup of $GL_n(\mathbb{Z})$ is finitely generated. Hence $GL_n(\mathbb{Z})$ is in the class $G$.

The kernel of the map $Out(F_n) \to GL_n(\mathbb{Z})$ which is induced by the abelianization of $F_n$ is torsion free [7]. Since every torsion subgroup of $GL_n(\mathbb{Z})$ is finite, it follows that every torsion subgroup of $Out(F_n)$ is finite. Every abelian subgroup of $Out(F_n)$ is finitely generated, see [6]. Thus $Out(F_n)$ is in the class $G$.

It was proved in [9, Theorem A] that every abelian subgroup of $Mod(S_g)$ is finitely generated. Further, it was proved in [32, Theorem 1] that $Mod(S_g)$ is a linear group. Hence every finitely generated torsion subgroup is finite [34, I]. We know by [25] that the order of finite subgroups in $Mod(S_g)$ is bounded. Therefore every torsion subgroup of $Mod(S_g)$ is finite. Thus $Mod(S_g)$ is in the class $G$.

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