Conditions for local transformations between sets of quantum states

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Abstract

We study the problem of transforming a set of pure bipartite states into another using deterministic LOCC (local operations and classical communication). Necessary conditions for the existence of such a transformation are obtained using LOCC constraints on state transformation, entanglement, and distinguishability. These conditions are shown to be independent but not sufficient. We discuss their satisfiability and classify all possible input-output pairs of sets accordingly. We also prove that strict inclusions hold between LOCC, separable, and positive partial transpose operations for set transformation problems.

1 Introduction

Many of the distinctive features of quantum theory appear in scenarios that impose constraints on quantum operations. One such scenario is the so-called distant lab paradigm in which quantum operations on a composite system are restricted to LOCC (local operations and classical communication), a subset of all quantum operations that one may perform on the system under consideration. The LOCC setup considers a quantum system shared between two or more physically separated observers who can perform any quantum operation on their local subsystems and communicate via classical channels but cannot exchange quantum information. Quantum operations that can be realized in this fashion belong to the LOCC class [1]. Many of the fundamental questions in quantum information theory, especially those related to quantum nonlocality [2, 3, 5, 6, 7], resource theories [8], state discrimination [3, 4, 5, 6, 7, 9, 10, 11, 12, 13],
14, 15, 16, 20, 18, 17, 19], and entanglement distillation [21], are studied within the framework of LOCC.

One of the goals of quantum information theory is to understand the strengths and limitations of LOCC. The motivation mainly comes from the resource theory of entanglement [8, 21], which considers quantum operations belonging to the LOCC class as free but those requiring entangled states, as a resource, expensive. Thus it is necessary to identify which quantum operations can, in fact, be realized by LOCC and which cannot be. The standard approach to do so, in general, is a two-step process: first, we specify a quantum task—a measurement, a unitary operation, or something more general, such as entanglement transformation or state discrimination, and then try to work out, often with the help of known LOCC constraints, whether LOCC can perform this task just as well as global operations, and if LOCC is sub-optimal, only then do we need to call upon entanglement.

Some of the well-known limitations of LOCC manifest themselves in quantum state transformations. For example, LOCC cannot convert a separable state into an entangled state, increase entanglement on average, or increase the Schmidt rank of a bipartite state. These properties may be applied to identify quantum operations not realizable by LOCC. For example, consider the result stating that LOCC implementation of Bell measurement is impossible [11]—here, the proof simply follows from the demonstration that LOCC implementation of Bell measurement implies LOCC conversion of a product state into a Bell state. Likewise, applying the second property, one can show that LOCC cannot convert a nonmaximally entangled state into a maximally entangled state belonging to the same state space with unit probability. Other notable results include quantum nonlocality without entanglement and the proof that LOCC is a strict subset of separable quantum operations [3], the existence of incomparable pairs [22], and the grouping of entangled states into SLOCC (stochastic LOCC) inequivalence classes [21].

One of the fundamental problems in quantum information theory is quantum set transformation, which, simply put, deals with the task of transforming a set of input states into a set of output states. The significance of this problem or variants thereof lies in the fact that many quantum information processing tasks can be naturally understood as set transformations, especially in situations where a quantum channel acts on an ensemble of input or signal states. The general set transformation problem can be described as follows. Suppose that $S_\rho = \{\rho_1, \ldots, \rho_n\}$ is a given set of input states and $S_\sigma = \{\sigma_1, \ldots, \sigma_n\}$ is a set of output states, and the goal is to achieve the transformation $\rho_i \rightarrow \sigma_i$ for every $i$. This, however, is possible only when there exists a quantum operation $\Lambda$ such that $\Lambda (\rho_i) = \sigma_i$ for all $i$. So the question is: what necessary and sufficient conditions must be satisfied for the existence of a $\Lambda$ such that the desired transformation is possible? Note that the scenario we just described does not have restrictions on quantum operations and has been studied before but with limited success [23, 24, 25, 26, 27].

In this paper, we investigate the set transformation problem within the paradigm of LOCC.
The motivation is three-fold. First, LOCC provides the framework to study set transformations in a distributed setting which, as discussed earlier, is the natural setup in many quantum information-theoretic tasks. But this scenario, as far as we know, has not been systematically studied before (a somewhat closely related problem was discussed recently in Ref. [28]). Second is the observation that the LOCC problem has certain distinguishing features that do not occur in the general formulation simply because they arise from orthogonality of states, quantum entanglement, and local distinguishability, none of which is significant when there are no restrictions on quantum operations. Our third motivation stems from the earlier discussion on the need to understand the strengths and limitations of LOCC protocols, which largely depends on investigating how well LOCC protocols perform well-defined quantum tasks, which, in this case, is quantum set transformation.

Specifically, we investigate conditions under which a given set of pure bipartite states can be deterministically converted into another using LOCC. We obtain necessary conditions by applying LOCC constraints on state transformations, entanglement, and distinguishability. We show that they are independent but not sufficient. To demonstrate independence, we consider a proper subset of the necessary conditions and then show that for every such subset, there exists a pair of input and output sets that satisfy the members of the subset under consideration but violate others that do not belong to it. This leads to a natural classification of all possible input-output set pairs with distinct and sometimes overlapping properties. To prove the insufficiency, we present an input-output set pair that satisfies all the necessary conditions but for which the desired set transformation is still impossible.

We also discuss the set transformation problem vis-à-vis other notable classes of quantum operations, namely, the separable operations (SEP) and positive partial-transpose operations (PPT). It is well-known that the following relations hold (see, for example, [1]):

\[
\text{LOCC} \subset \text{SEP} \subset \text{PPT} \subset \text{ALL},
\]

where ALL denotes the set of all possible quantum operations. The above inclusions are strict. However, for specific problems, they may not be; for example, the classes are all equally good for distinguishing two mutually orthogonal pure states because any two orthogonal pure states can be perfectly distinguished by LOCC [10]. So the question here is, are the inclusions strict for set transformations? We will answer this question in the affirmative.

2 Transformations between sets of quantum states

Let \( S_\rho = \{\rho_1, \ldots, \rho_n\} \) and \( S_\sigma = \{\sigma_1, \ldots, \sigma_n\} \) be two sets of quantum states. Suppose that a quantum system is now prepared in one of the states chosen from \( S_\rho \) but we do not know which
state it is. The goal is to find out whether there exists a deterministic quantum transformation\(^1\), a linear completely positive trace-preserving map \(\Lambda\) satisfying \(\Lambda(\rho_i) = \sigma_i\) for all \(i\). Henceforth, we will call \(S_\rho\) the input set, \(S_\sigma\) the output set, and use the notation \(S_\rho \rightarrow S_\sigma\) to indicate \(\Lambda(\rho_i) = \sigma_i\) for all \(i\).

Note that if the input states are mutually orthogonal, the problem is trivial, for one could first learn the identity of the input state with appropriate measurement and then prepare the output state accordingly. So the input states, at the very least, cannot be all mutually orthogonal. The problem, however, mostly remains open, and the answers are known only for specific cases [25, 27].

The fidelity \(f(\rho, \sigma)\) between two quantum states \(\rho\) and \(\sigma\) is defined as
\[
f(\rho, \sigma) = \text{Tr} \sqrt{\sqrt{\sigma} \rho \sigma}.\tag{1}
\]
If \(\rho = |\psi\rangle \langle \psi|\) and \(\sigma = |\phi\rangle \langle \phi|\) correspond to pure states, then
\[
f(\psi, \phi) = |\langle \psi | \phi \rangle|\tag{2}.
\]

The following results have been proved.

**Lemma 1.** [25] If \(S_\rho \rightarrow S_\sigma\) then \(f(\sigma_i, \sigma_j) \geq f(\rho_i, \rho_j)\) for all \(1 \leq i, j \leq n\).

Lemma 1 is a necessary condition that tells us that the pairwise distinguishability can never increase under a deterministic quantum operation. This is, of course, intuitively expected. The condition is also sufficient in the following case [24].

**Lemma 2.** Let \(S_\rho = \{\rho_1, \rho_2\}\) and \(S_\sigma = \{\sigma_1, \sigma_2\}\), where \(\rho_1\) and \(\rho_2\) are pure states. Then \(S_\rho \rightarrow S_\sigma\) if and only if \(f(\sigma_1, \sigma_2) \geq f(\rho_1, \rho_2)\).

Lemma 2 can be improved upon for qubit states. Let \(\|O\|_1 = \text{Tr} \sqrt{O^\dagger O}\) be the trace-norm of an operator \(O\). If \(O\) is Hermitian, then \(\|O\|_1 = \sum |\lambda_i|\), where \(\lambda_i\) are the eigenvalues of \(O\).

**Lemma 3.** [24] Let \(S_\rho = \{\rho_1, \rho_2\}\) and \(S_\sigma = \{\sigma_1, \sigma_2\}\), where \(\rho_1, \rho_2, \sigma_1, \sigma_2\) are qubit states. Then \(S_\rho \rightarrow S_\sigma\) if and only if
\[
\|p_1 \sigma_1 - p_2 \sigma_2\|_1 \leq \|p_1 \rho_1 - p_2 \rho_2\|_1\tag{3}
\]
for all probability distributions \(\{p_1, p_2\}\).

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\(^1\)One may consider a probabilistic formulation in which \(\rho_i \rightarrow \sigma_i\) with some nonzero probability \(p_i\) for every \(i\) [27].
Necessary conditions have also been obtained for sets of pure states, which are also sufficient provided the input states are linearly independent [26]. The good news, however, pretty much ends here. Lemma 2, for example, fails to be sufficient if $\rho_1$ and $\rho_2$ are not pure states [27]. One can also construct explicit counterexamples to show that Lemma 1 is not sufficient either [29]. The necessary and sufficient conditions for transforming a set of pure states to a set of pure or mixed states have also been obtained [27], but they may not easy to deal with in general.

3 LOCC transformations between sets of quantum states

The LOCC set transformation problem can be formulated by making the expected changes to the general formulation: the quantum system is now composite, that is, composed of two or more subsystems, and the quantum operations now belong to the LOCC class.

In this paper, we will only consider bipartite quantum systems $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ of finite dimensions. Let $\mathcal{D}(\mathcal{H})$ denote the set of density operators on $\mathcal{H}$, and let $S_B = \{\rho_1, \ldots, \rho_n\} \subset \mathcal{D}(\mathcal{H})$ be the set of input states and $S_C = \{\varsigma_1, \ldots, \varsigma_n\} \subset \mathcal{D}(\mathcal{H})$ be the set of output states, where $n \geq 2$.

So the question here is the following: Does there exist a deterministic LOCC transformation $L$ such that $L(\rho_i) = \varsigma_i$ for all $i$?\(^2\)

The operational meaning here is also quite clear. We assume that two physically separated observers, Alice and Bob, share a quantum state chosen from $S_B$ but do not know which state they actually share, but their goal is to produce the correct output state with certainty using a LOCC protocol. Then for which input-output pairs of sets they could perform this task?

Before we get to the problem in detail, let us briefly discuss the essential differences between the LOCC and the general problem. We will see that by restricting the class of quantum operations to LOCC, we encounter situations that do not arise otherwise.

Orthogonality of states

Recall that, in the general case, the input states cannot be all pairwise orthogonal because the problem becomes trivial due to the “identify and prepare (IP)” strategy. In the LOCC scenario, as we will now explain, the input states can be mutually orthogonal and yet give rise to nontrivial situations as the IP strategy does not always work.

Let us recall some basic concepts and results from LOCC distinguishability of quantum states which we will also need later in this paper. One of the fundamental results is that a set of bipartite or multipartite orthogonal states cannot always be perfectly distinguished (i.e., without error) by LOCC [3, 10], although they can be, as we know, by a joint measurement on the whole system. Specifically, if a given set of states is perfectly distinguishable by LOCC, then it means that the

\(^2\)Probabilistic formulation is also possible similar to entanglement transformations [30].
unknown state can be identified without error using some LOCC protocol. Now, this can happen either with certainty or with some nonzero probability. We will call a set of orthogonal states locally distinguishable if the members can be perfectly distinguished by LOCC with certainty, else locally indistinguishable. Note that a locally indistinguishable set comes in two varieties. The first is where a LOCC protocol cannot distinguish the states without error. The second one is where a LOCC protocol can, in fact, distinguish the states without error but not with unit probability. This can sometimes be understood as unambiguous state discrimination where one of the outcomes is inconclusive, but all others are conclusive [17]. There are, however, examples that do not quite fit into this picture; for instance, in LOCC discrimination of three Bell states, none of the measurement outcomes is conclusive except one.

We now explain why the IP strategy does not always work in the LOCC scenario. Consider a LOCC set transformation problem where the input states are mutually orthogonal. Therefore, they are either locally indistinguishable or distinguishable. If they are locally indistinguishable, the IP strategy clearly fails, and if they are not, the IP strategy still may not succeed. To see this, suppose the input states are locally distinguishable, and the output states are entangled. Since the input states can be locally distinguished, we know there exists a LOCC protocol that correctly identifies the input state. So in the first step, we will be able to correctly identify the state, but in the process, the initial state, in general, will transform into an intermediate state, the entanglement of which will be crucial to carrying out the next step. That is because the desired output state ought to be prepared by deterministic LOCC from this intermediate state and a necessary condition for this to happen is that the entanglement of the intermediate state must be greater than or equal to that of the output state. So if this condition is not satisfied, which may well be the case, we can never achieve our goal. Therefore, while orthogonal input states make the general problem trivial, the same cannot be said in the LOCC scenario.

**Entanglement**

The entanglement of the states under consideration is another crucial element that differentiates the LOCC scenario from the general problem. Since the input and output states could be entangled, the LOCC restrictions on entanglement transformations come into play. For example, we know that a product state cannot be converted into an entangled state by LOCC with nonzero probability. So if an input state is a product and the corresponding output state is entangled, the desired LOCC conversion at the set level obviously cannot happen.

One of the results we will extensively use in our analysis is Nielsen’s theorem [22] which provides a necessary and sufficient condition for converting a bipartite pure state to another using deterministic LOCC.

*Schmidt decomposition.* Every bipartite pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ can be written as a linear
superposition of biorthogonal product states known as the Schmidt decomposition [21]:

\[ |\psi\rangle = \sum_{i=1}^{r} \lambda_i |a_i\rangle |b_i\rangle, \quad \lambda_i \geq \lambda_{i+1} \geq 0, \quad \sum_{i=1}^{r} \lambda_i^2 = 1 \] (4)

where \( r \leq \min \{ \dim \mathcal{H}_A, \dim \mathcal{H}_B \} \) is the Schmidt number, \( \{ \lambda_i \} \) are the Schmidt coefficients, and \( \{ |a_i\rangle \} \) and \( \{ |b_i\rangle \} \) are orthonormal bases of \( \mathcal{H}_A \) and \( \mathcal{H}_B \) respectively.

**Theorem 1.** (Nielsen) Let \( |\chi\rangle \) and \( |\eta\rangle \) be two bipartite states with Schmidt coefficients given by \( \{ \alpha_i \}_{i=1}^{n} \) and \( \{ \beta_i \}_{i=1}^{m} \), where \( \alpha_i \geq \alpha_{i+1} \) for \( i = 1, \ldots, n - 1 \), and \( \beta_j \geq \beta_{j+1} \) for \( j = 1, \ldots, m - 1 \). Then there exists a deterministic LOCC transformation \( |\chi\rangle \rightarrow |\eta\rangle \) if and only if

\[ \sum_{i=1}^{l} \alpha_i^2 \leq \sum_{i=1}^{l} \beta_i^2 \quad \forall l = 1, \ldots, \min \{ n, m \} . \] (5)

This condition can be easily checked for any pair of pure bipartite states. So if the problem involves such states, Nielsen’s theorem needs to be satisfied for all input-output pairs of states. However, as we will see, even if that is the case, it does not guarantee that the desired LOCC protocol exists at the set level.

**Distinguishability**

How well a given set of states can be distinguished can be quantified by distinguishability measures, such as fidelity or the average probability of success. So let us first explain the concept of a distinguishability measure through a specific one, namely, fidelity [18, 31, 32]. This brief discussion will capture the essential elements of any well-defined measure.

We can associate a given set of states \( S = \{ \rho_1, \ldots, \rho_n \} \) with a probability distribution \( p = \{ p_1, \ldots, p_n \} \), where we assume that \( \rho_i \) appears with probability \( p_i \). Then, for a measurement (POVM) \( \mathcal{M} = \{ M_a \} \) and a guessing strategy \( G: a \rightarrow \tau_a \) (maps the measurement outcomes to new quantum states), the average fidelity is given by

\[ F (S; \mathcal{M}, G) = \sum_{a, i} p_i \text{Tr} (\rho_i M_a) \text{Tr} (\rho_i \tau_a) , \] (6)

where \( 0 \leq F (S; \mathcal{M}, G) \leq 1 \). The fidelity (global optimum) is now obtained by optimizing the
average fidelity over all measurements and guessing strategies:

\[ F(S) = \sup_{M, G} F(S; M, G). \]  

(7)

Conceptually, fidelity quantifies how much can we learn about the system that has been prepared in a state \( \rho_i \) with probability \( p_i \). Note that if the states are orthogonal we can always correctly identify the input state and then \( F(S) = 1 \) for any distribution of prior probabilities. On the other hand, if the states are nonorthogonal, we have \( F(S) < 1 \). The prior probabilities, however, are crucial in computing \( F(S) \), for if we change the prior probabilities, \( F(S) \) will also change, in general.

The above definition of fidelity can be suitably modified to accommodate composite systems and LOCC, SEP, or PPT measurements. Suppose that \( S \) now corresponds to a set of bipartite or multipartite states. Clearly, the definition of (global) fidelity as given by Eq. (7) will not change. The LOCC fidelity or local fidelity (LOCC or local optimum), on the other hand, can be obtained simply by restricting the class of measurements to LOCC. In particular,

\[ F_l(S; M, G) = \sum_{a, i} p_i \text{Tr} (\rho_i M_a) \text{Tr} (\rho_i \tau_a), \quad M \in \text{LOCC}; \]  

(8)

\[ F_l(S) = \sup_{M, G} F_l(S; M, G), \quad M \in \text{LOCC}. \]  

(9)

So LOCC fidelity is obtained by optimizing the average LOCC fidelity over all LOCC measurements and guessing strategies. Note that for a set of locally indistinguishable orthogonal states, \( F(S) = 1 \) but \( F_l(S) < 1 \).

In this paper, however, we will not choose any particular measure for our analysis as any well-defined one will serve our purpose. Let \( D \) be a measure of distinguishability, assumed to be well-defined for any given class of measurements. Then for a given set \( S \) of bipartite or multipartite states with associated probabilities given by \( p = \{p_1, \ldots, p_n\} \), let \( D_g(S) \) and \( D_l(S) \) denote the global and LOCC optimum, respectively. They naturally satisfy \( D_l(S) \leq D_g(S) \) for any choice of probability distribution \( p \).

Suppose under some deterministic quantum operation \( \Lambda \) we have \( S_1 \rightarrow S_2 \) where \( S_1 \) and \( S_2 \) are two sets of bipartite or multipartite states. Since distinguishability cannot increase under deterministic quantum operations we have the following observations:

- If \( \Lambda \) belongs to the set of all quantum operations, then for any \( p \) it holds that
  \[ D_g(S_1) \geq D_g(S_2). \]  
  (10)
• If \( \Lambda \in \text{LOCC} \) then we must also have

\[
D_l(S_1) \geq D_l(S_2)
\]

(11)

for any \( p \).

The second inequality becomes significant when both the input and output sets consist of mutually orthogonal states, for the first inequality cannot help in this case. In such cases, one may apply the second equality to rule out some kinds of transformations. For example, if \( S_1 \) consists of locally indistinguishable states, but the states in \( S_2 \) are locally distinguishable, then it is not possible to achieve \( S_1 \rightarrow S_2 \). More generally, if both are locally indistinguishable and \( D_l(S_1) < D_l(S_2) \), then \( S_1 \rightarrow S_2 \) under LOCC. However, the second inequality will not give us any useful information if both sets are locally distinguishable.

### 4 LOCC transformations between sets of pure states

The discussions in the previous section show that the LOCC problem is indeed special in some ways, for the states could be entanglement, orthogonal, and locally distinguishable or indistinguishable. We will now examine the question of the existence of a deterministic LOCC transformation between two sets of bipartite pure states.

**Necessary conditions**

Necessary conditions are obtained from LOCC constraints on state transformation, entanglement, and distinguishability.

**Proposition 1.** Consider a bipartite quantum system \( \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \) of finite dimensions. Let \( S_\psi = \{ |\psi_1 \rangle, |\psi_2 \rangle, \ldots, |\psi_n \rangle \} \) and \( S_\phi = \{ |\phi_1 \rangle, |\phi_2 \rangle, \ldots, |\phi_n \rangle \} \) be the sets of input and output states, respectively. Suppose there exists a deterministic LOCC under which \( S_\psi \rightarrow S_\phi \). Then:

(a) Theorem 1 holds for every pair \( (|\psi_i \rangle, |\phi_i \rangle) \), \( i = 1, \ldots, n \);

(b) For any probability distribution \( p = \{ p_1, p_2, \ldots, p_n \} \), \( 0 < p_i < 1 \), \( i = 1, \ldots, n \), and the density operators

\[
\rho_\psi,p = \sum_{i=1}^{n} p_i |\psi_i \rangle \langle \psi_i | ,
\]

(12)

\[
\rho_\phi,p = \sum_{i=1}^{n} p_i |\phi_i \rangle \langle \phi_i | ,
\]

(13)
it holds that

\[ E(\rho_{\psi,p}) \geq E(\rho_{\phi,p}), \quad (14) \]

where \( E \) is any well-defined entanglement measure;

(c) For any probability distribution \( p = \{p_1, p_2, \ldots, p_n\}, 0 < p_i < 1, i = 1, \ldots, n \), and the ensembles \( S_{\psi,p} = \{p_i, |\psi_i\rangle\} \) and \( S_{\phi,p} = \{p_i, |\phi_i\rangle\} \), it holds that

\[ D(S_{\psi,p}) \geq D(S_{\phi,p}), \quad (15) \]

where \( D \) is a well-defined measure of distinguishability.

Proof. There is nothing in particular to discuss about (a) which is obvious. Note however that (a) is stronger than a plausible alternative: \( E(\psi_i) \geq E(\phi_i), i = 1, \ldots, n \). That is because if (a) is satisfied then \( E(\psi_i) \geq E(\phi_i) \) for every \( i \), but the converse does not hold. In particular, there exists states \( |\psi\rangle \) and \( |\phi\rangle \) satisfying \( E(\psi) \geq E(\phi) \) but \( |\psi\rangle \not\rightarrow |\phi\rangle \) under deterministic LOCC [22].

We now come to (b). Since \( S_{\psi} \rightarrow S_{\phi} \) is possible by a deterministic LOCC, then for any probability distribution \( p = \{p_1, p_2, \ldots, p_n\}, 0 < p_i < 1, i = 1, \ldots, n \), it holds that

\[ \{(p_1, |\psi_1\rangle), (p_2, |\psi_2\rangle), \ldots, (p_n, |\psi_n\rangle)\} \rightarrow \{(p_1, |\phi_1\rangle), (p_2, |\phi_2\rangle), \ldots, (p_n, |\phi_n\rangle)\}, \]

or equivalently,

\[ \rho_{\psi,p} \rightarrow \rho_{\phi,p}, \]

where \( \rho_{\psi,p} \) and \( \rho_{\phi,p} \) are given by (12) and (13) respectively. Because the entanglement of a state cannot increase under deterministic LOCC, (14) must be satisfied for all \( p \) and any well-defined measure of entanglement \( E \).

To prove (c), once again first note that \( S_{\psi} \rightarrow S_{\phi} \) by a deterministic LOCC. The proof then follows from the fact that the distinguishability of given set of states cannot increase under a deterministic quantum operation, LOCC or otherwise. In other words, the ensemble \( S_{\psi,p} \) cannot be less distinguishable than \( S_{\phi,p} \) for any given probability distribution \( p \). Hence, (15) must be satisfied for any well-defined measure of distinguishability \( D \) and all \( p \).

Let us now look at the implications of (a), (b), and (c). Specifically, we would like to know whether they are independent of each other and what possible relations exist between them. These questions are better answered by considering sets of input-output pairs that satisfy each of them.
Proposition 2. Let $S_a$, $S_b$, and $S_c$ denote the sets of all input-output pairs satisfying conditions (a), (b), and (c), respectively. The relations

$$S_x \setminus (S_y \cup S_z) \neq \emptyset$$

hold for $x \neq y \neq z \in \{a, b, c\}$.

Proof. Equation (16) tells us that for any choice of $x \neq y \neq z \in \{a, b, c\}$, there exist pairs of input and output sets $S_\psi$ and $S_\phi$ that satisfy $(x)$ but do not satisfy $(y)$ or $(z)$. We will now prove (16) for each case.

| $S_\psi$ | $S_\phi$ | Satisfied | Not satisfied |
|---------|---------|-----------|---------------|
| $|\psi_1\rangle = \frac{1}{\sqrt{2}} \left( |00\rangle + |11\rangle \right)$ | $|\phi_1\rangle = \frac{1}{\sqrt{2}} \left( |01\rangle + |10\rangle \right)$ | (a) | (b) and (c) |
| $|\psi_2\rangle = \frac{1}{\sqrt{2}} \left( |00\rangle - |11\rangle \right)$ | $|\phi_2\rangle = |00\rangle$ | $|\phi_3\rangle = |11\rangle$ |
| $|\psi_3\rangle = |01\rangle$ | $|\phi_3\rangle = |11\rangle$ |

(a) is satisfied: Straightforward to check.

(b) is not satisfied: Consider the probability distribution of the form $p = \{p, p, 1 - 2p\}$, $0 < p < 1/2$. Then one finds that $\rho_\psi, p$ is separable for all $0 < p < 1/2$, but $\rho_\phi, p$ is entangled for $4/9 < p < 1/2$. The latter can be obtained by applying the partial transposition criterion [33] or by computing the concurrence [34]. Thus the inequality (14) does not hold whenever $4/9 < p < 1/2$. Hence, (b) is not satisfied.

(c) is not satisfied: To prove this we will use the following result proved in [12]:

Lemma 4. Three orthogonal two-qubit pure states can be distinguished by LOCC if and only if at least two of those states are product states.

By applying the above lemma it immediately follows that $S_\psi$ is locally indistinguishable but $S_\phi$ is locally distinguishable. So inequality (15) is violated for any LOCC measure of distinguishability (discussions on such measures can be found in [18]). Hence, (c) is not satisfied.
\[ S_b \setminus (S_a \cup S_c) \neq \emptyset \]

| \( S_\psi \) | \( S_\phi \) | Satisfied | Not satisfied |
|---|---|---|---|
| \(|\psi_1\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)\) | \(|\phi_1\rangle = \sqrt{\frac{3}{5}} |00\rangle + \sqrt{\frac{2}{5}} |11\rangle + \sqrt{\frac{1}{10}} |22\rangle\) | \(|\phi_2\rangle = |13\rangle\) | \((b)\) | \((a)\) and \((c)\) |
| \(|\psi_2\rangle = \frac{1}{\sqrt{2}} (|22\rangle + |33\rangle)\) | \(|\phi_3\rangle = |23\rangle\) | \(|\phi_4\rangle = |33\rangle\) | | |
| \(|\psi_3\rangle = \frac{1}{\sqrt{2}} (|22\rangle - |33\rangle)\) | | | |
| \(|\psi_4\rangle = \frac{1}{\sqrt{2}} (|23\rangle + |32\rangle)\) | | | |

\((b)\) is satisfied: Consider the density operators \( \rho_{\psi,p} \) and \( \rho_{\phi,p} \) with \( p = \{p_1, p_2, p_3, p_4\}, 0 < p_i < 1, \ i = 1, \ldots , 4 \). Let \( E \) denote the entanglement of formation \([21]\) measured in ebits. We will show that \( E(\rho_{\psi,p}) \geq p_1 \) but \( E(\rho_{\phi,p}) < p_1 \).

First, note that we can distill at least \( p_1 \) ebit from \( \rho_{\psi,p} \). To see this, consider the following LOCC protocol. Alice performs a binary measurement that distinguishes between the two orthogonal subspaces spanned by \( \{|0\rangle, |1\rangle\} \) and \( \{|2\rangle, |3\rangle\} \). If she obtains the first outcome, she and Bob end up sharing the state \(|\psi_1\rangle\) which has one ebit of entanglement. The probability of getting this outcome is \( p_1 \), so the distillable entanglement is at least \( p_1 \) ebit. Since distillable entanglement is a lower bound on the entanglement of formation \([21]\), we have \( E(\rho_{\psi,p}) \geq p_1 \). Now,

\[
E(\rho_{\phi,p}) \leq \sum_{i=1}^{4} p_i E(\phi_i),
\]

\[
= p_1 E(\phi_1),
\]

\[
< p_1
\]

since \( E(\phi_1) < 1 \) (one could easily check) and \( E(\phi_i) = 0 \) for \( i = 2, 3, 4 \). So the inequality \((14)\) is satisfied for all \( p \).

\((a)\) is not satisfied: This follows from the observation that Nielsen criterion is violated for the pair \( (|\psi_1\rangle, |\phi_1\rangle) \).

\((c)\) is not satisfied: \( S_\psi \) is locally indistinguishable, for it contains a locally indistinguishable subset \( \{|\psi_2\rangle, |\psi_3\rangle, |\psi_4\rangle\} \) \([20]\), whereas \( S_\phi \) is locally distinguishable, for a local measurement in the computational basis perfectly distinguishes the states.
\[ S_c \setminus (S_a \cup S_b) \neq \emptyset \]

| \( S_\psi \) | \( S_\phi \) | Satisfied | Not satisfied |
|---|---|---|---|
| \( |\psi_1\rangle = |01\rangle \) | \( |\phi_1\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \) | \( (c) \) | \( (a) \) and \( (b) \) |
| \( |\psi_2\rangle = |10\rangle \) | \( |\phi_2\rangle = |00\rangle \) | | |

The proof that \( (c) \) is satisfied is simple: \( S_\psi \) is a distinguishable set, whereas \( S_\phi \) contains nonorthogonal states, which cannot be perfectly distinguished.

It is also easy to see that \( (a) \) and \( (b) \) are not satisfied. The first follows from the observation that \( |\psi_1\rangle \) is a product state but \( |\phi_1\rangle \) is entangled. And the second follows from the following easily checkable properties: \( \rho_{\psi,p} \) is separable but \( \rho_{\phi,p} \) is entangled (apply the partial-transposition criterion [21]).

We will now show that for any choice of \( x \neq y \neq z \in \{a, b, c\} \), one can find pairs of input and output sets that satisfy both \( (x) \) and \( (y) \) but not \( (z) \).

**Proposition 3.** Let \( S_a, S_b, \) and \( S_c \) denote the sets of all input-output pairs satisfying conditions \( (a) \), \( (b) \), and \( (c) \), respectively. The relations

\[ (S_x \cap S_y) \setminus S_z \neq \emptyset \quad (17) \]

hold for \( x \neq y \neq z \in \{a, b, c\} \).

**Proof.** As before, we will prove (17) for each case.

\[ (S_a \cap S_b) \setminus S_c \neq \emptyset \]

| \( S_\psi \) | \( S_\phi \) | Satisfied | Not satisfied |
|---|---|---|---|
| \( |\psi_1\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \) | \( |\phi_1\rangle = |01\rangle \) | \( (a) \) and \( (b) \) | \( (c) \) |
| \( |\psi_2\rangle = |00\rangle \) | \( |\phi_2\rangle = |10\rangle \) | | |

Clearly, \( (a) \) is satisfied; \( (b) \) is also satisfied because \( \rho_{\psi,p} \) is entangled but \( \rho_{\phi,p} \) is separable for all \( p = \{p, 1 - p\}, p \in (0, 1) \).

Note, however, that \( (c) \) cannot be satisfied. That is because \( S_\psi \) contains nonorthogonal states, which cannot be perfectly distinguished, whereas the states in \( S_\phi \) are mutually orthogonal and
can be perfectly distinguished.

\[(S_a \cap S_c) \setminus S_b \neq \emptyset\]

\[
\begin{array}{|c|c|c|c|}
\hline
S_\psi & S_\phi & \text{Satisfied} & \text{Not satisfied} \\
\hline
|\psi_1\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) & |\phi_1\rangle = \alpha |00\rangle + \beta |11\rangle & (a) \text{ and } (c) & (b) \\
|\psi_2\rangle = \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle) & |\phi_2\rangle = \beta |00\rangle + \alpha |11\rangle & & \\
\hline
\end{array}
\]

where \(\alpha > \beta > 0\) and \(\alpha^2 + \beta^2 = 1\).

Nielsen’s criterion is satisfied for each \(i = 1, 2\), so \((a)\) is satisfied. Condition \((c)\) is also satisfied because the input states are distinguishable but the output states, being nonorthogonal, are not.

On the other hand, for \(p = \frac{1}{2}\), one finds that \(\rho_{\psi,p} = p |\psi_1\rangle \langle \psi_1| + (1 - p) |\psi_2\rangle \langle \psi_2|\) is separable but \(\rho_{\phi,p} = p |\phi_1\rangle \langle \phi_1| + (1 - p) |\phi_2\rangle \langle \phi_2|\) is entangled. So the inequality (14) cannot be satisfied for all probability distributions. Hence, \((b)\) is not satisfied.

\[(S_b \cap S_c) \setminus S_a \neq \emptyset\]

\[
\begin{array}{|c|c|c|c|}
\hline
S_\psi & S_\phi & \text{Satisfied} & \text{Not satisfied} \\
\hline
|\psi_1\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) & |\phi_1\rangle = \sqrt{\frac{4}{5}} |00\rangle + \sqrt{\frac{1}{10}} |11\rangle + \sqrt{\frac{1}{10}} |22\rangle & (b) \text{ and } (c) & (a) \\
|\psi_2\rangle = \frac{1}{\sqrt{2}} (|22\rangle + |33\rangle) & |\phi_2\rangle = |33\rangle & & \\
\hline
\end{array}
\]

Let \(E\) denote the entanglement of formation measured in ebits. Then: \(E (\psi_1) = E (\psi_2) = 1\), \(E (\phi_1) < 1\), and \(E (\phi_2) = 0\).

\((b)\) is satisfied: Consider the density matrices \(\rho_{\psi,p}\) and \(\rho_{\phi,p}\), where \(p = \{p, 1 - p\}, p \in (0, 1)\). First, note that that one ebit can be distilled from \(\rho_{\psi,p}\) for any \(p\) using deterministic LOCC. The protocol is similar to the one discussed earlier, so we omit the details. Because distillable entanglement is a lower bound on the entanglement of formation, we have \(E (\rho_{\psi,p}) \geq 1\). On the other hand, \(E (\rho_{\phi,p}) \leq pE (\phi_1) + (1 - p) E (\phi_2) < p < 1\). Therefore, \(E (\rho_{\psi,p}) > E (\rho_{\phi,p})\) for all \(p\). So \((b)\) is satisfied.

\((c)\) is satisfied: Both sets are distinguishable, locally or globally.

\((a)\) is not satisfied: That is because the transformation \(|\psi_1\rangle \rightarrow |\phi_1\rangle\) is not possible by deterministic LOCC as Nielsen’s criterion is violated.
Remark. Propositions 2 and 3 establish independence of the conditions \((a), (b), \) and \((c)\).

**Proposition 4.** Let \(S_a, S_b, \) and \(S_c\) denote the sets of all input-output pairs satisfying conditions \((a), (b), \) and \((c)\), respectively. Then the following relation holds:

\[
S_a \cap S_b \cap S_c \neq \emptyset.
\]  

(18)

**Proof.** There are many examples of input-output pairs \((S_\psi, S_\phi)\) for which \(S_\psi \rightarrow S_\phi\) by deterministic LOCC. Any such pair satisfies all three conditions \((a), (b), \) and \((c)\) and therefore belongs to the set \(S_a \cap S_b \cap S_c\). A simple example is where the output states \(\{|\phi_i\rangle\}\) are related to the corresponding input states \(\{|\psi_i\rangle\}\) by some fixed local unitary operator:

\[
|\phi_i\rangle = U \otimes V |\psi_i\rangle, \quad i = 1, \ldots, n.
\]

A somewhat more nontrivial and instructive example is the following pair of \(S_\psi\) and \(S_\phi\):

\[
|\psi_1\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \quad |\phi_1\rangle = \sqrt{\frac{4}{5}} |00\rangle + \sqrt{\frac{1}{5}} |11\rangle
\]

\[
|\psi_2\rangle = \frac{1}{\sqrt{2}} (|22\rangle + |33\rangle) \quad |\phi_2\rangle = |22\rangle
\]

It is enough to show there exists a deterministic LOCC transformation under which \(S_\psi \rightarrow S_\phi\). The protocol is an IP strategy. First, Alice (or Bob) performs an orthogonal measurement that distinguishes the subspaces spanned by \(\{|0\rangle, |1\rangle\}\) and \(\{|2\rangle, |3\rangle\}\). The outcome corresponding to \(\{|0\rangle, |1\rangle\}\) implies that they hold \(|\psi_1\rangle\), otherwise, \(|\psi_2\rangle\). Note that the measurement does not change the input state in any way. Once they correctly identify the input, they transform it to the desired output state using a deterministic LOCC protocol. That such a protocol exists for each input-output pair follows from Theorem 1. \(\Box\)

Remark. Note that, although the output set contains an entangled state, the IP strategy works in this case because the local protocol identifying the input state is nondestructive; that is, the input state remains intact during the discrimination process.

5 **Are the necessary conditions sufficient?**

It is reasonable to ask whether the conditions in Proposition 1 are also sufficient. The answer, unfortunately, turns out to be no.
Proposition 5. The set of necessary conditions in Proposition 1 is not sufficient.

Proof. Consider the input and output sets $S_\psi = \{ |\psi_1 \rangle, |\psi_2 \rangle \}$ and $S_\phi = \{ |\phi_1 \rangle, |\phi_2 \rangle \}$, where

$$
|\psi_1 \rangle = \frac{1}{\sqrt{2}} (|00 \rangle + |11 \rangle) \quad |\phi_1 \rangle = \alpha |00 \rangle + \beta |11 \rangle
$$

$$
|\psi_2 \rangle = \frac{1}{\sqrt{2}} (|00 \rangle - |11 \rangle) \quad |\phi_2 \rangle = \beta |00 \rangle - \alpha |11 \rangle
$$

where $\alpha > \beta > 0$ and $\alpha^2 + \beta^2 = 1$.

First, we will show all three conditions in Proposition 1 are satisfied. Then we will prove that the desired set transformation is not possible by deterministic LOCC.

Condition (a) holds as Nielsen’s theorem is satisfied in each case. To show that (b) holds we proceed as follows. Consider the density operators $\rho_{\psi, p}$ and $\rho_{\phi, p}$, where $p = \{p, 1 - p\}$, $p \in (0, 1)$. The respective concurrences are given by

$$
C (\rho_{\psi, p}) = \begin{cases} 
|1 - 2p|, & p \neq \frac{1}{2} \\
0, & p = \frac{1}{2} 
\end{cases} 
$$

(19)

$$
C (\rho_{\phi, p}) = \begin{cases} 
2\alpha\beta|1 - 2p|, & p \neq \frac{1}{2} \\
0, & p = \frac{1}{2} 
\end{cases} 
$$

(20)

It follows from (19) and (20) that $C (\rho_{\psi, p}) \geq C (\rho_{\phi, p})$ for all $p$. Therefore, (b) is satisfied. Since both sets are distinguishable, locally [10] or globally, condition (c) is satisfied as well.

We now prove that there does not exist a separable operation $S$ satisfying $S (\psi_i) = \phi_i$, where $\psi_i = |\psi_i \rangle \langle \psi_i|$ and $\phi_i = |\phi_i \rangle \langle \phi_i|$. Suppose, on the contrary, there exists a separable operation $S = \{A_k \otimes B_k\}$, where $A_k \otimes B_k$ are Kraus operators satisfying $\sum_k A_k^\dagger A_k \otimes B_k^\dagger B_k = I$, $I$ being the identity operator, such that $S (\psi_i) = \phi_i$ for $i = 1, 2$; that is:

$$
S (\psi_i) = \sum_k (A_k \otimes B_k) \psi_i (A_k^\dagger \otimes B_k^\dagger) = \phi_i, \quad i = 1, 2. 
$$

(21)

From (21), it follows that

$$
A_k \otimes B_k |\psi_1 \rangle = \mu_{k1} |\phi_1 \rangle, 
$$

(22)

$$
A_k \otimes B_k |\psi_2 \rangle = \mu_{k2} |\phi_2 \rangle, 
$$

(23)
where $\mu_{k1}, \mu_{k2}$ are complex numbers. For a given $k$, we need to consider two possibilities: $\mu_{k1}, \mu_{k2}$ are both nonzero or one of them is zero.

First suppose that both $\mu_{k1}$ and $\mu_{k2}$ are nonzero. Adding (22) and (23) we get

$$\mathcal{A}_k \otimes \mathcal{B}_k (|\psi_1 \rangle + |\psi_2 \rangle) = \mu_{k1} |\phi_1 \rangle + \mu_{k2} |\phi_2 \rangle.$$  \hspace{1cm} (24)

Simplifying the above equation we arrive at

$$\mathcal{A}_k \otimes \mathcal{B}_k |00 \rangle = \frac{1}{\sqrt{2}} \left[ (\mu_{k1} \alpha + \mu_{k2} \beta) |00 \rangle + (\mu_{k1} \beta - \mu_{k2} \alpha) |11 \rangle \right].$$  \hspace{1cm} (25)

Now the LHS of (25) is a product state. That means the state on the RHS must also be a product state. Since this state already enjoys biorthogonal decomposition, the eigenvalues of the reduced density matrices are easy to obtain. Requiring one of eigenvalues to be zero as the Schmidt rank of a product state is one, we find that either of the conditions

$$\frac{\mu_{k1}}{\mu_{k2}} = -\frac{\beta}{\alpha} \text{ or } \frac{\alpha}{\beta}.$$  \hspace{1cm} (26)

need to hold.

Now subtracting (23) from (22) and simplifying we find that

$$\mathcal{A}_k \otimes \mathcal{B}_k |11 \rangle = \frac{1}{\sqrt{2}} \left[ (\mu_{k1} \alpha - \mu_{k2} \beta) |00 \rangle + (\mu_{k1} \beta + \mu_{k2} \alpha) |11 \rangle \right].$$  \hspace{1cm} (27)

Since the LHS of (27) is a product state, the RHS of (27) must also be a product state, which requires us to satisfy

$$\frac{\mu_{k1}}{\mu_{k2}} = \frac{\beta}{\alpha} \text{ or } -\frac{\alpha}{\beta}.$$  \hspace{1cm} (28)

Now observe that the conditions (26) and (28) both cannot be simultaneously satisfied unless $\alpha = \beta$. Since $\alpha > \beta$ it follows that (22) and (23) both cannot hold for nonzero $\mu_{k1}$ and $\mu_{k2}$. Thus the desired set transformation is not possible by a separable operation satisfying both (22) and (23) for nonzero $\mu_{k1}$ and $\mu_{k2}$.
Next, suppose that $\mu_{k_1} \neq 0$ but $\mu_{k_2} = 0$. Then
\[
A_k \otimes B_k |\psi_1\rangle = \frac{1}{\sqrt{2}} \left( (A_k \otimes B_k |00\rangle + A_k \otimes B_k |11\rangle) = \mu_{k_1} |\phi_1\rangle \right), \tag{29}
\]
\[
A_k \otimes B_k |\psi_2\rangle = \frac{1}{\sqrt{2}} \left( (A_k \otimes B_k |00\rangle - A_k \otimes B_k |11\rangle) \right) = 0. \tag{30}
\]

Adding (29) and (30) and simplifying we find that the LHS is a product state, whereas the RHS is an entangled state as $\mu_{k_1} \neq 0$. Hence, (29) and (30) cannot be both satisfied. A similar argument holds for the case where $\mu_{k_1} = 0$ but $\mu_{k_2} \neq 0$. Therefore, there does not exist a separable operation which could achieve the desired transformation.

So we have proved that there does not exist a separable operation satisfying (22) and (23), which implies that the desired set transformation using a separable operation is impossible. Noting that LOCC is a strict subset of separable operations, the proof is therefore complete. \[\square\]

Propositions 2–5 suggest that one can now classify possible input-output pairs of sets based on the satisfiability of the necessary conditions. This is shown in Fig. 1.
6 Set transformations and classes of quantum operations

Consider a bipartite system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. Let $Q = \{Q_1, Q_2, \ldots\}$ be a quantum operation acting on the set of density operators $\mathcal{D}(\mathcal{H})$. Then for any density operator $\rho \in \mathcal{D}(\mathcal{H})$ we have

$$\rho'_i = Q_i(\rho).$$

Recall the definitions:

- $Q \in \text{SEP}$ if each $Q_i$ is a separable map, which implies that $\rho'_i$ is separable whenever $\rho$ is separable.
- $Q \in \text{PPT}$ if each $Q_i$ is a PPT map, which implies that $\rho'_i$ is PPT whenever $\rho$ is PPT.

The following relations hold:

$$\text{LOCC} \subset \text{SEP} \subset \text{PPT} \subset \text{ALL},$$

where ALL is simply the set of all quantum operations $\{Q\}$. The inclusions are strict. Operationally this means the following: Let $X = \{X\}$ and $Y = \{Y\}$ denote two classes of quantum operations. Then $X \subset Y$ means $Y$ is more powerful than $X$ in the sense that every $X \in X$ can be realized by some $Y \in Y$ but the converse does not hold, i.e., not every $Y \in Y$ can be realized by some $X \in X$.

The purpose of this section is to show that the relations (31) also hold in the context of set transformations. We begin by noting that (31) hold in the case of distinguishing quantum states.

**Lemma 5.** Let $X, Y \in \{\text{LOCC, SEP, PPT, ALL}\}$ satisfy $X \subset Y$. Then there exists a set $S(X, Y)$ of bipartite orthogonal states that can be perfectly distinguished with certainty by some $Y \in Y$ but not by any $X \in X$.

The proof of Lemma 5 follows from examples in the existing literature. Note that it suffices to cover the cases LOCC $\subset$ SEP, SEP $\subset$ PPT, and PPT $\subset$ ALL. We will give one example for each of them.

- $S(\text{LOCC, SEP})$ is an orthogonal product basis exhibiting quantum nonlocality without entanglement [3]. The basis is locally indistinguishable [3, 12] but perfectly distinguishable using a separable measurement (obviously). Such bases exist in all $\mathcal{H}_A \otimes \mathcal{H}_B$ with local dimensions $d_A, d_B \geq 3$. 

19
• $S$(SEP, PPT) is the following set of three states from $\mathcal{H}_A \otimes \mathcal{H}_B$ with local dimensions $d_A = d_B = 4$ [35]:

\[
|\psi_1\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \otimes \left( \sqrt{\frac{2}{3}} |00\rangle + \sqrt{\frac{1}{3}} |11\rangle \right) \\
|\psi_2\rangle = \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle) \otimes \left( \sqrt{\frac{2}{3}} |00\rangle + \sqrt{\frac{1}{3}} |11\rangle \right) \\
|\psi_3\rangle = \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle) \otimes \left( \sqrt{\frac{2}{3}} |00\rangle + \sqrt{\frac{1}{3}} |11\rangle \right)
\]

• $S$(PPT, ALL) is the two-qubit Bell basis. The proof that the Bell basis is not PPT distinguishable follows by appropriately modifying the argument in [11].

We will make use of Lemma 5 to prove the following proposition.

**Proposition 6.** Let $X, Y \in \{\text{LOCC}, \text{SEP}, \text{PPT}, \text{ALL}\}$ satisfy the relation $X \subset Y$. Then there exist input-output pairs of sets $S_\psi$ and $S_\phi$ for which the transformation $S_\psi \rightarrow S_\phi$ is possible with unit probability with a $Y \in Y$ but not with any $X \in X$.

**Proof.** Let $X, Y \in \{\text{LOCC}, \text{SEP}, \text{PPT}, \text{ALL}\}$ satisfy the relation $X \subset Y$. We choose the input set $S_\psi$ as $S(X,Y)$ and assume that its cardinality is $n$. Let the output set $S_\phi$ contain any $n$ orthogonal product states from the computational basis. Then:

• The members of $S(X,Y)$ can be perfectly distinguished with certainty by some $Y \in Y$ but not by any $X \in X$. And, we know from Lemma 5 that such $S(X,Y)$ exists for any choice of $X, Y \in \{\text{LOCC}, \text{SEP}, \text{PPT}, \text{ALL}\}$ satisfying $X \subset Y$.

• The members of $S_\phi$ are locally distinguishable, and therefore both SEP-distinguishable and PPT-distinguishable.

Let us first prove that the transformation $S(X,Y) \rightarrow S_\phi$ is achievable with certainty by some $Y \in Y$. The protocol is simple. We know that the members of $S(X,Y)$ are distinguishable by some $Y \in Y$. So in the first step we simply identify the input state. Since the corresponding output state is product, it can be prepared by LOCC. Therefore, the desired transformation is possible.

We now show that the transformation $S(X,Y) \rightarrow S_\phi$ is not achievable with certainty by any $X \in X$. Suppose, on the contrary, the transformation is, in fact, possible. Then for every $i, i = 1, \ldots, n$, we have $|\psi_i\rangle \rightarrow |\phi_i\rangle$, where $|\psi_i\rangle \in S(X,Y)$ and $|\phi_i\rangle \in S_\phi$. Since $S_\phi$ is locally distinguishable, we can perform a LOCC measurement to identify the output state, which will reveal the identity of the input state given in the beginning. Therefore, we are able to perfectly
distinguish the members of $S(X,Y)$ with certainty using an $X \in X$. This contradicts the fact that $S(X,Y)$ is not perfectly distinguishable with certainty by any $X \in X$. Hence, the proof is complete.

7 Conclusions

The set transformation problem in quantum information theory deals with the question of the existence of a physical transformation that transforms a given set of input states into a set of output states. In this paper, we considered this problem within the LOCC framework and discussed the unique features which arise from orthogonality, entanglement, and local distinguishability of the states under consideration. Specifically, we investigated the problem of transforming a set of pure bipartite states into another using deterministic LOCC. We obtained the necessary conditions for the existence of such a transformation using LOCC constraints on state transformation, entanglement, and distinguishability. We proved the conditions are independent but not sufficient. We discussed their satisfiability and classified all possible input-output pairs of sets accordingly. We also showed the strict inclusion relations that hold for LOCC, separable, and PPT operations apply in the case of set transformations.

We, however, did not attempt to address the most general LOCC set transformation problem. In particular, we left out mixed states and multipartite systems. Not much is known about local mixed-state transformations and local distinguishability, and entanglement of mixed states is also extremely hard to compute. In fact, the main tools we used in this paper will not work that well for mixed states. These are the reasons why we did not consider mixed states.

Multipartite systems also pose considerable challenges, as neither state transformation, entanglement, nor distinguishability properties are well understood, at least not as well as bipartite systems. However, we believe that many of the results presented in this paper could be extended to problems involving multipartite pure states without much difficulty. This scenario could be an exciting avenue for further research.

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