BIFURCATION AND MULTIPLICITY RESULTS FOR CRITICAL $p$-LAPLACIAN PROBLEMS

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Abstract. We prove a bifurcation and multiplicity result that is independent of the dimension $N$ for a critical $p$-Laplacian problem that is an analog of the Brezis–Nirenberg problem for the quasilinear case. This extends a result in the literature for the semilinear case $p = 2$ to all $p \in (1, \infty)$. In particular, it gives a new existence result when $N < p^2$. When $p \neq 2$ the nonlinear operator $-\Delta_p$ has no linear eigenspaces, so our extension is non-trivial and requires a new abstract critical point theorem that is not based on linear subspaces. We prove a new abstract result based on a pseudo-index related to the $Z_2$-cohomological index that is applicable here.

1. Introduction and main results

Elliptic problems with critical nonlinearities have been widely studied in the literature. Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with Lipschitz boundary.
In the celebrated paper [4], Brézis and Nirenberg considered the problem

\begin{equation}
\begin{cases}
-\Delta u = \lambda u + |u|^{2^*-2} u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\end{equation}

when \( N \geq 3 \), where \( 2^* = 2N/(N-2) \) is the critical Sobolev exponent. Among other things, they proved that this problem has a positive solution when \( N \geq 4 \) and \( 0 < \lambda < \lambda_1 \), where \( \lambda_1 > 0 \) is the first Dirichlet eigenvalue of \(-\Delta\) in \( \Omega \). Capozzi et al. [6] extended this result by proving the existence of a nontrivial solution for all \( \lambda > 0 \) when \( N \geq 4 \). The existence of infinitely many solutions for all \( \lambda > 0 \) was established by Fortunato and Jannelli [12] when \( N \geq 4 \) and \( \Omega \) is a ball, and by Devillanova and Solimini [9] when \( N \geq 7 \) and \( \Omega \) is an arbitrary bounded domain (see also Schechter and Zou [18]).

García Azorero and Peral Alonso [13], Egnell [10], and Guedda and Véron [14] studied the corresponding problem for the \( p \)-Laplacian

\begin{equation}
\begin{cases}
-\Delta_p u = \lambda |u|^{p-2} u + |u|^{p^*-2} u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\end{equation}

when \( 1 < p < N \), where \( \Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u) \) is the \( p \)-Laplacian of \( u \) and \( p^* = Np/(N-p) \). They proved that this problem has a positive solution when \( N \geq p^2 \) and \( 0 < \lambda < \lambda_1 \), where \( \lambda_1 > 0 \) is the first Dirichlet eigenvalue of \(-\Delta_p\) in \( \Omega \). Degiovanni and Lancelotti [8] extended their result by proving the existence of a nontrivial solution when \( N \geq p^2 \) and \( \lambda > \lambda_1 \) is not an eigenvalue, and when \( N^2/(N+1) > p^2 \) and \( \lambda \geq \lambda_1 \) (see also Arioli and Gazzola [1]). The existence of infinitely many solutions for all \( \lambda > 0 \) was recently established by Cao et al. [5] when \( N > p^2 + p \) (see also Wu and Huang [19]).

On the other hand, Cerami et al. [7] proved the following bifurcation and multiplicity result for problem (1.1) that is independent of \( N \) and \( \Omega \). Let \( 0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \rightarrow +\infty \) be the Dirichlet eigenvalues of \(-\Delta\) in \( \Omega \), repeated according to multiplicity, let

\[
S = \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\|\nabla u\|^2}{\|u\|^2_{L^2}}
\]

be the best constant for the Sobolev imbedding \( H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega) \) when \( N \geq 3 \), and let \( |\cdot| \) denote the Lebesgue measure in \( \mathbb{R}^N \). If \( \lambda_k \leq \lambda < \lambda_{k+1} \) and

\[
\lambda > \lambda_{k+1} - \frac{S}{|\Omega|^{2/N}},
\]

and \( m \) denotes the multiplicity of \( \lambda_{k+1} \), then problem (1.1) has \( m \) distinct pairs of nontrivial solutions \( \pm u_j^\lambda \), \( j = 1, \ldots, m \), such that \( u_j^\lambda \rightarrow 0 \) as \( \lambda \nearrow \lambda_{k+1} \) (see [7, Theorem 1.1]).