Self-avoiding walks and connective constants in small-world networks

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Long-distance characteristics of small-world networks have been studied by means of self-avoiding walks (SAW’s). We consider networks generated by rewiring links in one- and two-dimensional regular lattices. The number of SAW’s \(u_n\) was obtained from numerical simulations as a function of the number of steps \(n\) on the considered networks. The so-called connective constant, \(\mu = \lim_{n \to \infty} u_n/u_{n-1}\), which characterizes the long-distance behavior of the walks, increases continuously with disorder strength (or rewiring probability, \(p\)). For small \(p\), one has a linear relation \(\mu = \mu_0 + ap\), \(\mu_0\) and \(a\) being constants dependent on the underlying lattice. Close to \(p = 1\) one finds the behavior expected for random graphs. An analytical approach is given to account for the results derived from numerical simulations. Both methods yield results agreeing with each other for small \(p\), and differ for \(p\) close to 1, because of the different connectivity distributions resulting in both cases.

I. INTRODUCTION

Our world is formed by networks of different types (social, biological, technological, economic), whose characterization has launched in last years the emergence of models incorporating the basic ingredients of real-life networks [1–3]. In particular, social networks form the substratum where processes such as information spreading or disease propagation take place. One expects that the structure of these complex networks will play an important role in such dynamical processes, which are usually studied by means of stochastic dynamics and random walks. Some processes, such as navigation and exploratory behavior are neither purely random nor totally deterministic, and can be also described by walks on graphs [4]. In this context, the generic properties of deterministic navigation [5] and directed self-avoiding walks [6] in random networks have been analyzed recently.

In last years, networks displaying the ”small-world” effect have been intensively studied [7–11]. Watts and Strogatz [7,12] proposed for this kind of networks a model based on a locally connected regular lattice, in which a fraction \(p\) of the links between nearest-neighbor sites are replaced by new random connections, thus creating long-range “shortcuts”. Hence, one has in the same network a local neighborhood (as for regular lattices) and some global properties of random graphs [13]. The small-world effect is usually measured by the scaling behavior of the characteristic path length \(\ell\), defined as the average of the distance between any two sites. In small-world networks, one has a logarithmic increase of \(\ell\) with the network size, as happens for random graphs [2,13,14].

This short global length scale changes strongly the behavior of statistical physical problems on small-world networks, as compared with regular lattices (where one has \(\ell \sim N^{1/d}\), \(N\) being the system size and \(d\) the lattice dimension). Among these problems, one finds signal propagation [7], spread of infections [15,16], and random spreading of information [17–19]. Site and bond percolation [16,20], as well as the Ising [21–23] and XY models [24], have been also studied in these networks. Most of the published work on small worlds has focused on networks obtained from one-dimensional lattices (rings). Small-world networks built by rewiring lattices of higher dimensions have been employed to study percolation, as a model of disease propagation [20,25]. Several characteristics of random walks on this kind of networks have been analyzed in connection with diffusion processes [26,27]. In particular, some properties of these walks, such as the probability of returning to the origin, were found to be intermediate between those corresponding to fractals and Cayley trees [28].

In this paper we study self-avoiding walks in small-world networks built up from one- and two-dimensional regular lattices. A self-avoiding walk (SAW) is defined as a walk along the bonds of a given network which can never intersect itself. The walk is restricted to moving to a nearest-neighbor site during each step, and the self-avoiding condition constrains the walk to occupy only sites which have not been previously visited in the same walk [29]. SAW’s have been used for modeling the large-scale properties of long-flexible macromolecules in solution [30], as well as for the study of polymers trapped in porous media, gel electrophoresis, and size exclusion chromatography, which deal with the transport of polymers through membranes with small pores [31]. They have been also employed to characterize complex crystal structures [32] and to analyze critical phenomena in lattice models [29,33]. Universal constants for SAW’s have been discussed by Privman et al. [34].

The paper is organized as follows. In Sec.II we give some basic definitions and concepts related to SAW’s.
Sec. III we present results for SAW's on simulated small-world networks, and in Sec. IV we give an approximate analytical procedure to calculate the number of SAW's on this kind of networks. The paper closes with some conclusions in Sec. V.

II. BASIC DEFINITIONS

For regular lattices, the number $u_n$ of different SAW's starting from a generic site has an asymptotic dependence for large $n$: \[ u_n \sim n^{\gamma-1} \mu^n , \]
where $\gamma$ is a critical exponent which depends on the lattice dimension, and $\mu$ is the so-called “connective constant” or effective coordination number of the considered lattice [35,36]. In general, for a lattice with coordination number (or connectivity) $z$, one has $\mu \leq z - 1$. This parameter $\mu$ can be obtained from Eq. (1) by the limit:

\[ \mu = \lim_{n \to \infty} \frac{u_n}{u_{n-1}} . \]

The connective constant depends upon the particular topology of each lattice, and has been determined very accurately for two-dimensional (2D) and three-dimensional (3D) lattices [37]. In the following, we will consider Eq. (2) as a definition of the connective constant $\mu$ for any network. [Note that the limit in Eq. (2) is well defined provided that the mean connectivity is finite.]

For random and small-world networks the number of SAW’s of length $n$ depends on the considered starting node of the network. In the sequel we will call $u_n$ the average number of SAW’s of length $n$, i.e. the mean value obtained (for each $n$) by averaging over the network sites. For small-world networks one expects $\mu$ values larger than that corresponding to the starting regular lattice. In particular, $\mu$ is expected to increase with $p$ and approach the value corresponding to random lattices with mean connectivity $z$ as $p \to 1$.

For regular lattices all nodes have the same connectivity, i.e., the same number of nearest neighbors. However, for $p > 0$ different connectivities $m$ are possible, giving rise to a probability distribution for which analytic expressions have been found [19,21]. For a (large) random network with mean connectivity $z$, the connectivity distribution $P_{rd}(m)$ follows a Poisson law [2,13]:

\[ P_{rd}(m) = \frac{z^m e^{-z}}{m!} . \]

For random networks, the connective constant can be obtained in a straightforward manner, due to the absence of correlations between links. For $n = 1$ one has obviously $u_1^d = z$. Now, given a generic node and a link going out from it, the connectivity distribution $Q_{rd}(m)$ for the other end of the link in a random graph is given by

\[ Q_{rd}(m) = \frac{m}{z} P_{rd}(m) . \]

Then, the average number of two-step SAW’s is given by

\[ u_2^d = z \sum_{m>1} (m-1)Q_{rd}(m) , \]

and we find $u_2^d = z^2$. Using the same procedure for $n > 2$, one has:

\[ u_n^d = z^n , \]

and thus for (large) random networks one finds $\mu = z$. In connection with this, we note that for a Bethe lattice (or Cayley tree) with connectivity $z$, the number of SAW’s is given by $u_{BL}^d = z(z-1)^{n-1}$, and one has $\mu = z - 1$ (see, e.g. Ref. [38]).

III. NUMERICAL SIMULATIONS

The networks studied here have been generated from three different regular lattices: one-dimensional (1D) ring with coordination number $z = 4$ and 6, and 2D square lattice ($z = 4$). To construct our small-world networks, we consider in turn each of the bonds in the starting lattice and substitute it with a given probability $p$ by a new bond. This means that one end of the bond (chosen at random) is changed to a new node taken randomly from the entire network. We impose three conditions: 1) no two nodes can have more than one bond connecting them, 2) no node can be connected by a link to itself, and 3) isolated sites (with zero links) are not allowed. This method keeps constant the total number of links in the rewired networks (and consequently the average connectivity $z$). The total number of rewired links is $1/2 p N$ on average.

For networks generated in the present way there is a crossover size $N^* \sim p^{-1}$ that separates the large- and small-world regimes [39,40], and the small-world behavior appears for any finite value of $p$ (0 < $p$ < 1) as soon as the network is large enough. Our networks included $1 \times 10^5$ sites in 1D and $300 \times 300$ sites for the 2D system, so that we were in the small-world regime (system size $N > N^*$). In the sequel we will call $L$ the side length of the considered lattices, i.e. $L = N^{1/d}$. Periodic boundary conditions were assumed. We note that our networks differ from those discussed by Watts and Strogatz [7] in that these authors left untouched $z/2$ links per site. For our simulated small-world networks we have obtained the average number $u_n$ of SAW’s up to $n = 21$. Since $u_n$ increases with $p$, this maximum $n$ was reduced to $n = 14$ close to $p = 1$, in order to carry out averages over nodes of the generated networks. These numbers of steps in the SAW’s are sufficient to obtain the connective constant $\mu$ with enough accuracy for our present purposes. In fact, the larger is $p$, the faster the ratio $u_n/u_{n-1}$ converges with $n$. 


First of all, we calculate the mean-squared end-to-end distance of the walks on our small-world networks:

$$R_n^2 = \langle (r_n - r_0)^2 \rangle ,$$  \hspace{1cm} (7)

where $\langle \rangle$ indicates an average over $n$-steps SAW’s with different starting sites and for different network realizations with given $p$. Here, $r_n$ refers to the position of site $n$ in the $d$-dimensional Euclidean space of the underlying regular lattice, and $r_0$ is the position of the origin for the considered walk. For SAW’s on the 1D lattices considered here, one has $R_n^2 \sim bn^2$, whereas for the 2D square lattice $R_n^2 \sim bn^{3/2}$ [29,35], with a lattice-dependent constant $b$ of order unity in all cases.

In Fig. 1 we present the ratio $R_n^2/L^2$ as a function of the number of steps $n$ for SAW’s on 1D small-world networks with $z = 4$, for several values of the rewiring probability $p$. $R_n^2$ increases with $n$ much faster than for the corresponding regular lattice due to the random connections introduced by the rewiring of links. (Note that for a regular lattice with $L = 10^5$, we have for $n = 20$ a ratio $R_n^2/L^2 \sim 10^{-7}$.) In fact, if we call $f_n$ the fraction of $n$-steps paths that include at least one rewired link, we have:

$$R_n^2 \approx b(1-f_n)n^2 + \frac{1}{12}f_n dL^2 .$$  \hspace{1cm} (8)

The first and second terms on the r.h.s. come from SAW’s without and with rewired links, respectively. The second term amounts, apart from the fraction $f_n$, to the average Euclidean distance between any pair of sites in a $d$-dimensional box with side length $L$. Thus, in the limit of large networks (as those considered here, with $L \gg n$) one has $R_n^2 \approx f_n dL^2/12$. In the course of our numerical simulations we have checked Eq. (8) by calculating independently $f_n$ and $R_n$ as a function of $n$. We found that both sides of this equation coincide within error bars (which are smaller than the symbol size in Fig. 1). Then, $R_n^2$ can be considered as a measure of the fraction of SAW’s containing rewired links, and converges to $dL^2/12$ for large $n$ in the rewired networks ($f_n \to 1$).

We now turn to the number $u_n$ of SAW’s on these networks. In Fig. 2 we show $u_n$ as a function of the path length $n$ for several rewiring probabilities $p$, as derived from numerical simulations. From top to bottom: $p = 1, 0.3, 0.1, 0.03$, and 0. Dotted lines are guides to the eye.
connectivity. Small-world networks generated from 1D regular lattices with probability $p$ whole region between function of the rewiring probability $p$. Different symbols represent results obtained for small-world networks generated from 1D regular lattices with connectivity $z = 4$ (squares) and $z = 6$ (circles), as well as from a 2D square lattice (diamonds). Dotted lines are guides to the eye. Error bars are less than the symbol size. Results for $\mu$ obtained by means of the analytical method described in Sec. IV are plotted as dashed lines.

For $p = 1$ our numerical procedure gives in all cases connective constants $\mu$ clearly lower than the mean connectivity $z$. In fact, we found $\mu = 3.69, 5.73,$ and $3.70 \pm (0.01)$ for networks rewired from 1D lattices with $z = 4$ and 6, and from a 2D lattice $(z = 4)$, respectively. The obtained values for networks with $z = 4$ coincide within error bars, irrespective of the starting 1D or 2D lattice, indicating that for $p = 1$ the resulting rewired networks lost memory of the starting regular lattice. However, the $\mu$ values obtained for simulated networks with $p = 1$ contrast with those expected for random networks, that coincide in each case with the average connectivity $z$, as explained above. This occurs because our networks with $p = 1$ are not true (poissonian) random networks, since they still keep memory of the starting regular lattices [19]. This memory effect is due mainly to the fact that one rewrites onely one end of each link, maintaining the other end on its original site. Hence, the connectivity distribution found for our rewired networks with $p = 1$

does not coincide with that given above in Eq. (3).

For $p = 0$ and 1 is found for the networks with $z = 6$. In fact, we find in this case an increase in $\mu$ of 2.13 to be compared with 1.48 and 1.07 in 1D and 2D rewired networks with $z = 4$.


to the average number of nodes in “shell” $n$, which scales as $p^{1/d}$ [19]. Such a dependence is related to the scaling with $d$ of the characteristic length scale of small-world networks, namely the average distance between ends of shortcuts $\xi$, given by $\xi = (pz)^{-1/d}$ [20].

**IV. ANALYTICAL APPROXIMATION**

We now derive an approximate analytical expression that allows us to calculate $u_n$ in a small-world network,

FIG. 3. Connective constant $\mu$ as a function of the rewiring probability $p$. Different symbols represent results obtained for small-world networks generated from 1D regular lattices with connectivity $z = 4$ (squares) and $z = 6$ (circles), as well as from a 2D square lattice (diamonds). Dotted lines are guides to the eye. Error bars are less than the symbol size. Results for $\mu$ obtained by means of the analytical method described in Sec. IV are plotted as dashed lines.

FIG. 4. Dependence upon the rewiring probability $p$ of the change in connective constant, $\Delta \mu = \mu - \mu_0$, with respect to the corresponding regular lattices. Symbols represent the same networks as in Fig. 3. Dashed lines were obtained from analytical calculations by using Eq. (13).
assuming the sequence \( \{u_i^0\} \) for the underlying regular lattice to be known. For a given path length \( n \) we obtain the mean number of SAW’s \( u_n \) by considering all possible sequences of unrewired and rewired links in small-world networks. With this purpose, we calculate the probability of reaching a rewired link as a function of the walk length. In particular, we will obtain the conditional probability that the \( i \)'th link in a SAW is a rewired one, assuming that link \( i \) is an unrewired one \((i > 1)\).

We first note that in a regular lattice the ratio \( c_i = u_i^0 / u_i^{0,1} \) obviously depends on \( i \). This ratio \( c_i \) measures the average number of available links starting from the \((i-1)\)-th site in a generic SAW on the underlying regular lattice, and allows us to calculate the number of possible unrewired links in a SAW on a rewired network. Taking into account that the fraction of unrewired connections in the whole rewired network is \( 1-p \), then the mean number \( q_i \) of available unrewired links going out from site \( i-1 \) in a SAW is \( q_i = c_i(1-p) \). On the other side, the mean number of rewired links going out from an arbitrary node reached by an unrewired link is given by (see Appendix A)

\[
w_i = p(z - \frac{1}{2}).
\]

(Contrary to \( q_i \), the mean number \( w_i \) is independent of \( i \).) Therefore, the conditional probability that link \( i \) is a rewired one, assuming that link \( i-1 \) is an unrewired one, is given by

\[
\bar{p}_i = \frac{w_i}{w_i + q_i} = \frac{z'p}{z'p + c_i(1-p)},
\]

with \( z' = z - 1/2 \). For \( i = 1 \), we take \( \bar{p}_1 = p \). This probability \( \bar{p}_i \) is shown in Fig. 5 for the three types of networks considered here, for a rewiring probability \( p = 0.1 \). Thus, the average number of \( i \)-steps SAW’s that do not include rewired connections is

\[
A_i = (1 - \bar{p}_1) \ldots (1 - \bar{p}_i) u_i^0,
\]

and the number of those consisting of \( i-1 \) unrewired links and a rewired one in step \( i \) is

\[
B_i = (1 - \bar{p}_1) \ldots (1 - \bar{p}_{i-1}) \bar{p}_i u_i^0.
\]

Now we note that a rewired connection in step \( i \) in a (large) small-world network ends on a random site of the network (most probably far away from the sites already visited in the same walk). This means that in step \( i+1 \) one begins most probably with a situation similar to that found in step \( i = 1 \). Finally, the average number of \( n \)-steps SAW’s is given by

\[
u_n = \sum_{i_1 + \ldots + i_j = n} B_{i_1} B_{i_2} \ldots B_{i_{j-1}} A_{i_j},
\]

where the sum is extended to all possible combinations of indexes \( i_1, \ldots, i_j \) with sum equal to \( n \), including null indexes, for which we have \( B_0 = A_0 = 1 \). Each term in Eq. (13) represents a sequence of unrewired and rewired links, and thus the sum includes \( 2^n \) terms. The SAW’s corresponding to the general term in the sum include \( j-1 \) rewired links (in steps number \( i_1, i_1 + i_2, i_1 + \ldots + i_{j-1} \)). As an example, Eq. (13) gives for \( n = 2 \): \( u_2 = B_2 + B_2^2 + B_1 A_1 + A_2 \).

Note that the above equations, although rather accurate, are not exact. There are two reasons for this: First, each ratio \( c_i \) is an average number of allowed links starting from a site reached in \( i-1 \) steps, but the actual number of such allowed links depends on the particular site under consideration. Second, finite-size effects should appear, unless the considered networks are large enough.

Values of connective constants \( \mu \) derived from \( u_n \) obtained with this procedure are presented in Figs. 3 and 4 as a function of the rewiring probability \( p \) (dashed lines). There appears to be good agreement with \( \mu \) derived from numerical simulations (symbols) for \( p < 0.2 \). For larger \( p \) the connective constants deduced from the analytical method are larger than those yielded by the simulations. (For \( z = 6 \) there is a region around \( p = 0.3 \) where the analytical results are slightly lower than those found for the simulated networks.)

For \( p = 1 \) our analytical procedure gives \( \mu = z \), as for random networks. On the other side, our numerical simulations for small-world networks gave in all cases in the limit \( p = 1 \) connective constants \( \mu \) clearly lower than the mean connectivity \( z \), as indicated above and shown in Fig. 3. This difference between both procedures is due to the above-mentioned fact that simulated networks with
$p = 1$ do not have a poissonian connectivity distribution, which is implicitly assumed in the analytical method for this value of $p$. As a matter of fact, in this case we have $A_i = 0$ for $i > 0$, $B_1 = 0$ for $i > 1$, and $B_1 = z$, giving in Eq. (13) $u_n = z^n$, as for random networks [see Eq. (6)]. In this sense, our analytical procedure to calculate the number of SAW’s gives a better interpolation between regular lattices and random graphs.

In the context of this analytical approach, it is natural to expect close to $p = 0$ a linear dependence of the connective constant $\mu$ on $p$, as found from our numerical simulations (see Sec. III), irrespective of the underlying lattice. By expanding the probability $\tilde{p}_i$ [Eq. (10)] to first order in $p$ one finds for $i > 1$: $\tilde{p}_i = z'p/c_i + O(p^2)$. Introducing this expression for $\tilde{p}_i$ into Eqs. (11), (12), and (13), and keeping terms up to first order in $p$, we find:

$$u_n = \sum_{i=0}^{n} B_i A_{n-i} \quad (p \ll 1), \quad (14)$$

with the corresponding linearized expressions for $B_i$ and $A_{n-i}$. This expression includes contributions of SAW’s containing zero (term $A_i$, for $i = 0$) and one rewired links (all other terms, $i = 1, ..., n$). In this way, close to $p = 0$ we find for the derivative $d\mu/dp$ the values 7.6 ($d = 1, z = 4$), 14.3 ($d = 1, z = 6$), and 6.3 ($d = 2$). Thus, it is clear that the functional dependence of $\mu$ upon $p$ for $p \ll 1$ does not change with the dimension of the underlying lattice.

V. CONCLUSIONS

Self-avoiding walks provide us with an adequate tool to study the long-range characteristics of small-world networks. For large networks, the number of SAW’s increases asymptotically as $u_n \sim \mu^n$, provided that one considers system sizes $L \gg n$. For small-world networks generated from a given lattice, the effective connectivity $\mu$ ranges from the value of the regular lattice to $\mu = z$ for random graphs. For small $p$ this effective connectivity follows a linear relation $\mu = \mu_0 + ap$, $a$ being a constant dependent on the underlying lattice.

We have developed an analytical procedure to obtain the number of SAW’s in small-world networks. This method is based on calculating probabilities of finding rewired or unrewired links in the walks, and gives results in good agreement with numerical simulations for $p \lesssim 0.2$. The results of both methods differ for larger $p$, since they assume in practice different connectivity distributions. Our analytical method gives in this respect a correct interpolation between regular lattices and random graphs. On the contrary, the rewiring (simulated) process gives rise to non-poissionan connectivity distributions, even for a rewiring probability $p = 1$, yielding in this case networks with connective constants $\mu$ lower than the average connectivity $z$.

Both analytical calculations and simulations similar to those presented here can be useful to characterize other kinds of networks of current interest, such as scale-free networks, whose properties are known to depend on the asymptotic form of the connectivity distribution for large connectivities.

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APPENDIX A: DERIVATION OF CONDITIONAL PROBABILITIES

Here we calculate a conditional probability related to the number of possible connections starting from a generic node in a SAW, and necessary to derive Eq. (9) in our analytical approach in Sec. IV. In particular, we will obtain the average number of possible rewired links going out from a generic node $X$ (so called for definiteness), assuming that this node was reached in a SAW through an unrewired link in step $n$. Here we mean by possible links all those connections that are available for step $n + 1$ in a SAW (leading to nodes not previously visited in the same walk).

In a rewired network, the links with one end on node $X$ can be classified for our present purpose into three types:

1) Links which were not rewired and remain as in the original lattice. The probability distribution for the number $r$ of these connections is given by:

$$P_1(r) = \binom{z}{r} (1-p)^r p^{z-r}, \quad r = 0, ..., z. \quad (A1)$$

2) Rewired links for which the reference node $X$ was not changed. Following the above notation, there are $z - r$ rewired links, from which $s$ keep one end on site $X$. The probability distribution for $s$ is

$$P_2(s) = \binom{z-r}{s} \left(\frac{1}{2}\right)^{z-r}, \quad s = 0, ..., z-r. \quad (A2)$$

3) New (rewired) links arriving at site $X$. The distribution of the number $v$ of these connections is (for large system size $N$)

$$P_3(v) = \frac{1}{v!} (tz)^v e^{-zt}, \quad v \geq 0, \quad (A3)$$

with $t = p/2$.

Thus, given a site with $r$ unrewired connections, the number of rewired links is $x = s + v$, where $s$ depends on $r$ and $v$ is independent of $r$. The probability distribution for $x$ is:

$$Q_2^{(r)}(x) = \sum_{s=0}^{s_{max}} P_2(s) P_3(x-s), \quad (A4)$$
with \( s_{\text{max}} = \min(z - r, x) \). Then, we have

\[
Q_2(r)(x) = \left( \frac{1}{2} \right)^{z-r} e^{-zt} \sum_{s=0}^{s_{\text{max}}} (z - r)^{s} \frac{1}{(x-s)!} (tz)^{x-s}.
\]  

(A5)

Hence, the probability distribution for the number of outgoing rewired links (assuming that the incoming link was an unrewired one) is

\[
Q_{\text{out}}(x) = \sum_{r=1}^{z} Q_{\text{in}}(r) Q_2(r)(x).
\]  

(A6)

where \( Q_{\text{in}}(r) \) is the probability of reaching a node having \( r \) unrewired links:

\[
Q_{\text{in}}(r) = \frac{r}{z} \left( \frac{z}{r} \right) (1 - p)^{r-1} p^{z-r}.
\]  

(A7)

[\( Q_{\text{in}}(r) \) is given, apart from a normalization constant, by the product \( rP_1(r) \).]

Finally, the mean number of outgoing rewired links [calculated with the probability distribution \( Q_{\text{out}}(x) \)] is

\[
\langle x \rangle = \sum_{x=1}^{\infty} x Q_{\text{out}}(x).
\]  

(A8)

Introducing expression (A6) into Eq. (A8), and after a straightforward but somewhat lengthy algebra, one finds the mean value:

\[
\langle x \rangle = p(z - 1/2).
\]  

(A9)

This average value \( \langle x \rangle \) is called \( w_i \) in the main text, and the last result is Eq. (9) there.

[1] S. H. Strogatz, Nature 410, 268 (2001).
[2] R. Albert and A. L. Barabási, Rev. Mod. Phys. 74, 47 (2002).
[3] S. N. Dorogovtsev and J. F. F. Mendes, Adv. Phys. 51, 1079 (2002); S. N. Dorogovtsev and J. F. F. Mendes, Evolution of Networks: From Biological Nets to the Internet and WWW (Oxford University, Oxford, 2003).
[4] G. M. Viswanathan, S. V. Buldyrev, S. Havlin, M. G. E. da Luz, E. P. Raposo, and H. E. Stanley, Nature 401, 911 (1999).
[5] G. F. Lima, A. S. Martinez, and O. Kinouchi, Phys. Rev. Lett. 87, 010603 (2001).
[6] S. B. Santra, W. A. Seitz, and D. J. Klein, Phys. Rev. E 63, 067101 (2001).
[7] D. J. Watts and S. H. Strogatz, Nature 393, 440 (1998).
[8] L. F. Lago-Fernández, R. Huerta, F. Corbacho, and J. A. Sigüenza, Phys. Rev. Lett. 84, 2758 (2000).
[9] V. Latora and M. Marchiori, Phys. Rev. Lett. 87, 198701 (2001).
[10] M. E. J. Newman, J. Stat. Phys. 101, 819 (2000).
[11] R. Albert, H. Jeong, and A. L. Barabási, Nature 401, 130 (1999); A. L. Barabási and R. Albert, Science 286, 509 (1999).
[12] D. J. Watts, Small Worlds (Princeton University Press, Princeton, 1999).
[13] B. Bollobás, Random Graphs (Academic Press, New York, 1985).
[14] D. S. Callaway, M. E. J. Newman, S. H. Strogatz, and D. J. Watts, Phys. Rev. Lett. 85, 5468 (2000).
[15] M. Kuperman and G. Abramson, Phys. Rev. Lett. 86, 2909 (2001).
[16] C. Moore and M. E. J. Newman, Phys. Rev. E 61, 5678 (2000).
[17] S. A. Pandit and R. E. Amritkar, Phys. Rev. E 63, 041104 (2001).
[18] J. Lahtinen, J. Kertész, and K. Kaski, Phys. Rev. E 64, 057105 (2001).
[19] C. P. Herrero, Phys. Rev. E 66, 046126 (2002).
[20] M. E. J. Newman and D. J. Watts, Phys. Rev. E 60, 7332 (1999).
[21] A. Barrat and M. Weigt, Eur. Phys. J. B 13, 547 (2000).
[22] M. Gitterman, J. Phys.: Condens. Matter 10, 4405 (2000).
[23] C. P. Herrero, Phys. Rev. E 65, 066110 (2002).
[24] B. J. Kim, H. Hong, P. Holme, G. S. Jeon, P. Minnhagen, and M. Y. Choi, Phys. Rev. E 64, 056135 (2001).
[25] M. E. J. Newman, I. Jensen, and R. M. Ziff, Phys. Rev. E 65, 021904 (2002).
[26] S. Jespersen and A. Blumen, Phys. Rev. E 62, 6270 (2000).
[27] F. Jasch and A. Blumen, Phys. Rev. E 64, 066104 (2001).
[28] S. Jespersen, I. M. Sokolov, and A. Blumen, Phys. Rev. E 62, 4405 (2000).
[29] C. Domb, Adv. Chem. Phys. 15, 229 (1969).
[30] P. G. de Gennes, Scaling Concepts in Polymer Physics (Cornell University, Ithaca, 1979).
[31] S. B. Lee, H. Nakanishi, and Y. Kim, Phys. Rev. B 39, 9561 (1989); H. Nakanishi and J. Moon, Physica A 191, 309 (1992).
[32] C. P. Herrero, J. Phys.: Condens. Matter 7, 8897 (1995).
[33] K. Kremer, A. Baumgärtner, and K. Binder, J. Phys. A 15, 2879 (1981).
[34] V. Privman, P. C. Hohenberg, and A. Aharony, in Phase Transitions and Critical Phenomena, vol. 14, edited by C. Domb and J. L. Lebowitz (Academic, London, 1991).
[35] D. S. McKenzie, Phys. Rep. 27, 35 (1976).
[36] D. C. Rapaport, J. Phys. A 18, 113 (1985).
[37] A. D. Sokal, in Monte Carlo and Molecular Dynamics Simulations in Polymer Science, edited by K. Binder (Oxford University, Oxford, 1995).
[38] D. Stauffer and A. Aharony, Introduction to Percolation Theory, 2nd ed. (Taylor and Francis, London, 1992).
[39] M. Barthélémy and L. A. N. Amaral, Phys. Rev. Lett. 82, 3180 (1999); ibid, 5180 (1999).
[40] M. A. de Menezes, C. F. Moukarzel, and T. J. P. Penna, Europhys. Lett. 50, 574 (2000).