Controlled Geometry via Smoothing

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Abstract

We prove that Riemannian metrics with a uniform weak norm can be smoothed to having arbitrarily high regularity. This generalizes all previous smoothing results. As a consequence we obtain a generalization of Gromov’s almost flat manifold theorem. A uniform Betti number estimate is also obtained.

1 Introduction

An ultimate goal in geometry is to achieve a classification scheme, using natural geometric quantities to characterize the topological type or diffeomorphism type of Riemannian manifolds. While this grand scheme seems to be an impossible dream, its basic philosophy has been a driving force in many important developments in Riemannian geometry. The sphere theorems and various topological finiteness theorems are typical examples. These results are concerned with control of global topology of manifolds, and a crucial point therein is to control, uniformly, the local topology.

Control of local topology often follows from control of local geometry. Here, by local geometry, we mean the local behavior of the metric tensor. On the other hand, control of local geometry is frequently also the essential ingredient for control of global geometry, such as in Cheeger-Gromov’s compactness theorem and its various extensions, which can be named geometric finiteness theorems. Notice that some rudimental topological finiteness results are direct corollaries of geometric finiteness theorems. But the significance of the latter goes beyond this. In any case, control of local geometry is obviously a key topic. An interesting and important aspect of this topic is various degrees of control of local geometry needed or available in different situations. In [P], the first author introduced a sequence of norms which provide a certain quantitative measure for local geometric control. These norms can be defined

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either in terms of the $C^{k,\alpha}$-norms or the $L^{k,p}$-norms for functions, and they are defined on a given scale. For example, the $C^{k,\alpha}$-norm of a Riemannian manifold on scale $r$ is bounded, if it is covered by coordinate charts of size comparable to $r$ such that the metric tensor expressed in the coordinates is uniformly bounded in $C^{k,\alpha}$-norm, and that the coordinate transition functions are uniformly bounded in $C^{k+1,\alpha}$-norm. Note that the local topology is uniformly trivial if one of these norms on some scale is bounded. To admit richer topological and geometric structures under norm bounds, we shall introduce a weak version of these norms. The essential new feature is that we allow coordinate maps to have double points. In spirit, this is similar to replacing an injectivity radius bound by a conjugate radius bound. (Of course, e.g. a weak (harmonic) $C^{0,\alpha}$ bound is so weak that it is far from implying a conjugate radius bound.)

Indeed, our basic theme is to try to find the minimal degree of control of local geometry under which interesting geometric and topological consequences can be drawn. Traditional geometric conditions such as curvature bounds imply various degrees of local geometric control, so we can think from their perspectives. Historically, sectional curvature bounds were the first to be systematically studied. They can roughly be compared with weak $C^2$-norm bounds, at least the latter imply the former. Since understanding of sectional curvature bounds has been reached on a good level, it is natural to try to find the minimal degree of local geometric control under which a metric can be approximated by metrics with sectional curvature bounds or weak $C^2$-norm bounds (or better bounds). In this paper, we present a result towards this goal, along with some applications. The local geometric control we need is as weak as a bound on the weak harmonic $C^{0,\alpha}$-norm, or a bound on the weak $L^{1,p}$-norm. These do appear to be the sought-after minimal degree of local geometric control in our set-up.

To formulate the result precisely, we introduce the following classes of Riemannian manifolds. (The definition of the weak norms are given in §2.)

**Definition.** Given $n \geq 2$, $0 < \alpha < 1$, $p > n$ and function $Q : (0, \infty) \to [0, \infty)$ which is nondecreasing in $r$ and satisfies $\lim_{r \to 0} Q(r) = 0$, we define

$$\mathcal{M}(n, \alpha, Q) = \left\{ (M, g) \mid (M, g) \text{ is a complete Riemannian manifold, } \dim M = n, \text{ the weak harmonic } C^{0,\alpha} \text{ norm } \|(M, g)\|_{C^{0,\alpha}, r}^{W,h} \leq Q(r) \right\},$$

and

$$\mathcal{M}(n, p, Q) = \left\{ (M, g) \mid (M, g) \text{ is a complete Riemannian manifold, } \dim M = n, \text{ the weak } L^{1,p} \text{ norm } \|(M, g)\|_{L^{1,p}, r}^{W} \leq Q(r) \right\}.$$

**Remark 1.** By Anderson-Cheeger’s work [AC] manifolds with a lower bound for Ricci curvature and a positive lower bound for conjugate radius belong to these two classes.
Remark 2. It is unknown whether (weak) $L^{1,p}$ bounds imply (weak) harmonic $C^{0,\alpha}$ bounds. (By the Sobolev embedding, they do imply $C^{0,\alpha}$ bounds, but not yet harmonic $C^{0,\alpha}$ bounds.) In other words, it is unknown whether controlled $C^{0,\alpha}$ harmonic coordinates exist under the assumption of a (weak) $L^{1,p}$ bound. This question seems rather subtle. We plan to return to it in the future. At the moment, we consider the said two bounds as independent conditions. Note that one can further ask whether (weak) $C^{0,\alpha}$ bounds imply (weak) harmonic $C^{0,\alpha}$ bounds. There does not seem to be any evidence to support an answer in the affirmative.

Theorem 1.1 For every manifold $(M, g)$ in $\mathcal{M}(n, \alpha, Q)$ and positive numbers $\epsilon$, there are metrics $g_\epsilon$ on $M$ such that

\[
e^{-\epsilon} g \leq g_\epsilon \leq e^\epsilon g,\]
\[
\| (M, g_\epsilon) \|_{C^{0,\alpha}, r}^W \leq 2Q(r),\]
\[
\| (M, g_\epsilon) \|_{C^{k,\alpha}, r}^W \leq \tilde{Q},\]

where $k$ is an arbitrary positive integer, $0 < r \leq 1$ and $\tilde{Q} = \tilde{Q}(n, k, \epsilon, \alpha, Q(r))$ denotes a positive number depending only on $n, k, \epsilon, \alpha$ and $Q(r)$.

Theorem 1.2 For every closed manifold $(M, g)$ in $\mathcal{M}(n, p, Q)$ and positive numbers $\epsilon$, there are metrics $g_\epsilon$ on $M$ such that

\[
e^{-\epsilon} g \leq g_\epsilon \leq e^\epsilon g,\]
\[
\| (M, g_\epsilon) \|_{L^{1,p}, r}^W \leq 2Q(r),\]
\[
\| (M, g_\epsilon) \|_{L^{k,p}, r}^W \leq \tilde{Q},\]

where $k$ is an arbitrary positive integer, $0 < r \leq 1$ and $\tilde{Q} = \tilde{Q}(n, k, \epsilon, p, Q(r))$ denotes a positive number depending only on $n, k, \epsilon, p$ and $Q(r)$.

Remark Theorem 1.2 actually also holds for complete, non-compact manifolds. This will be shown in a paper by the third author.

Thus a metric with some regularity (given by the weak norm) can be deformed or smoothed to a nearby one with arbitrarily high regularity. In particular, manifolds with a lower bound on Ricci curvature and a positive lower bound on conjugate radius can be smoothed. Previous smoothing results have been concerned with metrics with various curvature bounds, and involved two independent techniques: the embedding method and the Ricci flow. The embedding technique in smoothing as used by Cheeger-Gromov [CG] consists of embedding (or immersing) a given manifold into a Euclidean space and then perturbing it suitably by a smoothing operation, which is
based on the classical convolution process. The smoothing result in [CG] is that metrics on closed manifolds with lower and upper bounds on sectional curvatures and a positive lower bound on injectivity radius can be smoothed to metrics with bounds on all derivatives of the Riemann curvature tensor. Later, by embedding into a Hilbert space instead of a finite dimensional space, Abresch [A] was able to remove the condition on injectivity radius and extend to complete manifolds. More recently Shen [Sh] showed that manifolds with a lower bound on sectional curvatures and a positive lower bound on injectivity radius can be smoothed to having two-sided sectional curvature bounds. The technique of Ricci flow is based on the fundamental work of Hamilton [H]. Using this technique, Bemelmans-Min Oo-Ruh [BMR] obtained the same result as in [CG] without injectivity radius lower bound, and Shi [S] obtained the same result as in [A]. Later work considers metrics with other kinds of curvature bound. For example, in [Y1, Y2] Yang dealt with integral bounds on sectional curvatures. In [DWY], Ricci curvature bounds were treated.

By virtue of the available constructions of controlled harmonic coordinates under various curvature bounds, all these smoothing results are consequences of Theorem 1.1 or Theorem 1.2.

As typical applications we present the following two results.

**Theorem 1.3 (Betti number estimate)** For the class of manifolds \( M^n \) in \( \mathcal{M}(n, \alpha, Q) \) or in \( \mathcal{M}(n, p, Q) \), and satisfying \( \text{diam}_M \leq D \), we have the estimate for the Betti numbers

\[
\sum_i b^i(M^n) \leq C(n, D, \alpha, Q) \quad \text{or} \quad C(n, D, p, Q),
\]

(1.7) and the estimate for the number of isomorphism classes of rational homotopy groups

\[
\pi_q(M) \otimes Q \leq C(n, q, D, \alpha, Q) \quad \text{or} \quad C(n, q, D, p, Q) \quad \text{for} \quad q \geq 2.
\]

(1.8) follows from Theorem 1.1, 1.2 and Gromov’s uniform betti number estimate regarding sectional curvature [G2]. This estimate can also be proved directly using Toponogov type comparison estimate introduced in [W], see [PW] for details. In [W] the same estimate (1.7) is given for the class of manifolds satisfying \( \text{Ric}_M \geq -(n-1)H \), \( \text{conj} \geq r_0 \) and \( \text{diam}_M \leq D \). (1.8) follows from Theorem 1.1, 1.2 and the results in [R].

**Theorem 1.4** There exists an \( \epsilon = \epsilon(n, \alpha, Q) \) or \( \epsilon(n, p, Q) > 0 \) such that if a manifold \( M^n \) belongs to \( \mathcal{M}(n, \alpha, Q) \) or \( \mathcal{M}(n, p, Q) \) and \( \text{diam} \leq \epsilon \), then \( M \) is diffeomorphic to an infranilmanifold.

This generalizes Gromov’s almost flat manifold theorem [G1] as well as its generalization in [DWY]. (The proof is simple: combine Theorem 1.1 with [G1].)
The proof of Theorem 1.1 and Theorem 1.2 uses the embedding method in [A]. Roughly speaking, we embed a given Riemannian manifold into the Hilbert space of $L^2$-functions on it, and then use the embedding to pull back the $L^2$-metric of the Hilbert space. The crucial point is of course to find a suitable embedding, such that the pull-back metric will enjoy nice properties. In [A], the embedding is defined in terms of distance functions. In our situation, these functions are not appropriate, and we employ instead solutions of a canonical geometric partial differential equation. Now if e.g. the harmonic $C^{0,\alpha}$-norm of the manifold is bounded, then a uniform pointwise bound on sectional curvatures will hold for the pull-back metric, and hence we can apply the smoothing results for metrics with sectional curvature bounds as given e.g. in [A] or [S].

If we only assume that the weak harmonic $C^{0,\alpha}$-norm of the manifold is bounded, i.e. it is in the class $\mathcal{M}(n, \alpha, Q)$, the global embedding is generally not under control. To remedy the situation, we follow the idea in [A] of employing instead local embeddings. In [A], Abresch uses the exponential map to lift local patches of the manifold and his local embeddings are exactly embeddings of these lifted patches. In our situation, the exponential map is not suitable. Our substitute for it is the coordinate maps. Thus we use them to lift local patches, and construct embeddings of the lifted patches via the same geometric partial differential equation as mentioned before. To make sure that the pull-back metrics induced by these local embeddings descend to the local patches and that the resulting metrics patch together to define a metric globally, it is crucial to require the embeddings to be equivariant under isometries. Since our embeddings are defined in terms of solutions of a canonical geometric PDE, they naturally share this equivariance property.

Basically, the above scheme also works for manifolds in the class $\mathcal{M}(n, p, Q)$, but some modifications are necessary. As before, the said pull-back metrics descend to yield a new metric on the underlying manifold. But these metrics satisfy here an integral bound on sectional curvatures rather than a pointwise bound. This is a new situation. To handle it, we apply the Ricci flow and follow the arguments in [DWY]. A pointwise bound on Ricci curvature is used in several places in [DWY]. Since no such bound is available in our current situation, the arguments in [DWY] need to be improved and modified. The result we thus arrive at not only completes the smoothing scheme for the class $\mathcal{M}(n, p, Q)$, but also provides some new understanding of short time existence of the Ricci flow.

# 2 Norm, Weak Norm and Smoothing

Fix an integer $k \geq 0$ and a number $0 \leq \alpha \leq 1$. The $C^{k,\alpha}$-norm of an $n$-dimensional Riemannian manifold $(M, g)$ on scale $r$, $\|(M, g)\|_{C^{k,\alpha}, r}$, is defined to be the infimum of positive numbers $Q$ such that there exist embeddings:

$$\varphi_s : B(0, r) \subset \mathbb{R}^n \to U_s \subset M$$
\((B(0, r))\) denotes the closed ball of radius \(r\) centered at the origin with images \(U_s, s \in \mathcal{S}\) (an index set), with the following properties:

1) \(e^{-Q} \delta_{ij} \leq g_{s,ij} \leq e^Q \delta_{ij}\),
2) Every metric ball \(B(p, \frac{r}{10} e^{-Q})\), \(p \in M\) lies in some set \(U_s\),
3) \(r|j| + \alpha \|\partial^j g_{s,ij}\|_{C^\alpha} \leq Q\) for all multi-indices \(j\) with \(0 \leq |j| \leq k\).

Here \(g_{s,ij}\) denote the coefficients of \(g_s = \varphi_s^* g\) on \(B(0, r)\), and \(\delta_{ij}\) are the Kronecker symbols.

Note that this definition is slightly different from the corresponding one in [P], where in addition the (rescaled) \(C^{k+1,\alpha}_+\)-norm of the transition functions are required to be under control. For convenience, we can call the \(C^{k,\alpha}\)-norm (of Riemannian manifolds) as defined in [P] the \textit{strong} \(C^{k,\alpha}\)-norm. (Note however that the ”strong” harmonic \(C^{k,\alpha}\)-norm is equivalent to the harmonic \(C^{k,\alpha}\)-norm.)

We define the harmonic \(C^{k,\alpha}\)-norm on scale \(r\), \(\|(M, g)\|_{h}^{C^{k,\alpha}, r}\), by requiring additionally the following

4) \(\varphi_s^{-1} : U_s \to \mathbb{R}^n\) is harmonic, which is equivalent to saying that

\[4')\] \(\text{id} : B(0, r) \to B(0, r)\) is harmonic with respect to \(g_s\) on the domain and the Euclidean metric on the target, which is in turn equivalent to saying that

\[\sum_i \partial_i (g_s^{ij} \sqrt{|\det g_{s,ij}|}) = 0\]

for all \(j\).

If \(k \geq 1\) and \(p > n\) (when \(k = 1\)) or \(p > \frac{n}{2}\) (when \(k \geq 2\)), then we define the \(L^{k,p}\)-norm on the scale of \(r\), \(\|(M, g)\|_{L^{k,p}, r}\), by retaining 1) and 2), and replacing 3) by

\[3')\] \(r|j| + \frac{\alpha}{2} \|\partial^j g_{s,ij}\|_{L^p} \leq Q\) for all \(1 \leq |j| \leq k\).

The harmonic \(L^{k,p}\)-norm is defined similarly. For any choice of these norms, it is clear that the local topology is trivial on some uniform scale for any class of manifolds with uniformly bounded norm. (Note that the injectivity radius may not be uniformly positive though.) To allow nontrivial local topology, we introduce the weak norms \(\|\|_{W}^{C^{k,\alpha}, r}\) and \(\|\|_{W}^{L^{k,p}, r}\), which are defined in identical ways except that each \(\varphi_s : B(0, r) \to U_s\) is assumed to be a \textit{local} diffeomorphism instead of diffeomorphism. The corresponding weak harmonic norms \(\|\|_{W,h}^{C^{k,\alpha}, r}\) and \(\|\|_{W,h}^{L^{k,p}, r}\) are defined in a similar way, with 4) being replaced by 4').

Note that (weak) harmonic norms dominate (weak) norms on the same scale. We also have \(\|\|_{W,r} \leq \|\|_{r}\) and \(\|\|_{W,h,r} \leq \|\|_{h}\). All norms are continuous and non-decreasing in \(r\). If \((M, g)\) is sufficiently smooth, these norms converge to zero as \(r \to 0\). Furthermore, (weak) \(C^{k,\alpha}\) \((L^{k,p})\) norms vary continuously in the \(C^{k,\alpha}\) \((L^{k,p})\) topology of Riemannian manifolds. See [P] for the relevant details.

We point out that \(\mathbb{R}^n\) is the only space with norm = 0 on all scales. And flat manifolds are the only spaces with weak norm = 0 on all scales.

Conventional geometric conditions such as curvature bounds imply norm bounds. Such implications are mostly contained in constructions of controlled harmonic co-
ordinates and are a crucial ingredient for various compactness theorems. To have a clear perspective, we collect these results in the following proposition.

**Proposition 2.1** There is a $Q(H, i_0, r, p)$ with $\lim_{r \to 0} Q(H, i_0, r, p) = 0$ such that for manifolds with

a) $|K| \leq H$, $\text{inj} \geq i_0$, then $\|(M, g)\|_{L^2, p, r}^b \leq Q(H, i_0, r, p)$;

b) $|K| \leq H$, then $\|(M, g)\|_{L^2, p, r}^{W,h} \leq Q(H, r, p)$;

c) $|\text{Ric}| \leq (n - 1)H$, $\text{inj} \geq i_0$, then $\|(M, g)\|_{L^2, p, r}^b \leq Q(H, i_0, r, p)$;

d) $\text{Ric} \geq -(n - 1)H$, $\text{inj} \geq i_0$, then $\|(M, g)\|_{L^2, p, r}^{W,h} \leq Q(H, i_0, r, p)$;

e) $\text{Ric} \geq -(n - 1)H$, $\text{conj} \geq i_0$, then $\|(M, g)\|_{L^2, p, r}^{W,h} \leq Q(H, i_0, r, p)$.

These results follow from works of Jost-Karcher [JK], Anderson [An] and Anderson-Cheeger [AC].

We now turn to the smoothing question. As explained in the introduction, our strategy is to first achieve sectional curvature bounds by embedding into the Hilbert space of $L^2$-functions. This is done in the next two sections. The higher regularity smoothing then easily follows from known smoothing results. Consider $(M, g) \in \mathcal{M}(n, \alpha, Q)$. We have a collection of local diffeomorphisms

$$\varphi_s : B(0, r) \to U_s \subset M$$

satisfying 1), 2), 3) and 4$'$).

In the next section we will construct a canonical embedding

$$F_s : (B(0, r), g_s) \to L^2(B(0, r), g_s),$$

where $g_s = \varphi_s^* g$. We use $F_s$ to pullback the $L^2$ metric of $L^2(B(0, r), g_s)$ to produce a new metric $\tilde{g}_s$ on $B(0, r)$. This construction works for general metrics on $B(0, r)$, and has the following equivariance property, which will be proved in §4. Namely, if $g_1$, $g_2$ are two metrics on $B(0, r)$ such that there is an isometric embedding

$$\psi : (B(0, r), g_1) \to (B(0, r), g_2)$$

and if $\tilde{g}_1$, $\tilde{g}_2$ are obtained via the above construction, then

$$\psi : (B(0, r), \tilde{g}_1) \to (B(0, r), \tilde{g}_2)$$

is also an isometric embedding. Granted this (see Proposition 4.4) we have

**Proposition 2.2** There exists a smooth metric $\bar{g}$ on $M$ such that the pullback of $\bar{g}$ by $\varphi_s$ is exactly $\tilde{g}_s$.

**Proof.** Let $r_1 = \frac{r_0}{10} e^{-Q}$. Then for every $p \in M$, $B(p, r_1) \subset U_s$ for some $s$. It follows that there exists a $\tilde{p} \in B(0, r)$ such that $B_{g_s}(\tilde{p}, r_1) \subset B(0, r)$ and $\varphi_s(\tilde{p}) = p$. 

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We now define the metric $\bar{g}$ as follows. If $X, Y \in T_p M$, then

$$\bar{g}(X, Y) = \tilde{g}_s \left( ((\varphi_s)_s|\tilde{p})^{-1}(X), ((\varphi_s)_s|\tilde{p})^{-1}(Y) \right).$$

To show that this metric is well-defined, let $\tilde{p}'$ be another such point, i.e. for some $s'$, $B_{g_{s'}}(\tilde{p}', r_1) \subset B(0, r)$ and $\varphi_{s'}(\tilde{p}') = p$. Let $r_4 = \frac{r}{20}e^{-4Q}$ and $r_3 = \frac{r}{20}e^{-3Q}$. Denote $g_0$ the Euclidean metric. Then we can show that

**Lemma 2.3** There is an isometric embedding

$$\psi : (B_{g_0}(\tilde{p}, r_4), g_s) \to (B_{g_{s'}}(\tilde{p}', r_3), g_{s'}).$$

**Proof.** First $\psi$ can be defined as follows. Since $g_s$ is $e^Q$-quasi-isometric to $g_0$,

$$B_{g_0}(\tilde{p}, r_4) \subset B_{g_s}(\tilde{p}, r_3). \quad (2.1)$$

For any point $q \in B(\tilde{p}, r_4)$, connect $q$ to the center point $\tilde{p}$ with a curve $\tilde{\gamma}$ in $B_{g_0}(\tilde{p}, r_4)$ such that the length of $\tilde{\gamma}_{l_{g_s}(\tilde{\gamma})} < r_3$. Since

$$\varphi_s : (B_{g_s}(\tilde{p}, r_3), g_s) \to B(p, r_3)$$

is a local isometry and $\varphi_s(\tilde{p}) = p$. From \cite{[2,4,7]} $\varphi_s$ maps the curve $\tilde{\gamma}$ to a curve $\gamma$ in $B(p, r_3)$ starting with $p$ and $l(\gamma) < r_3$. Again since $\varphi_{s'}$ is a local isometry and $\varphi_{s'}(\tilde{p}') = p$. The curve $\gamma$ then can be lifted via $\varphi_{s'}$ to a curve in $B_{g_{s'}}(\tilde{p}', r_3)$ starting with $\tilde{p}'$. The other end point of this curve is defined to be the image of $q$. (Note that, in general, lifting can not be done for incompletely space. Here the map is a local isometry and the curve starts from the center, and we have control on the length of the curve and the size of the metric ball, so it will not hit the boundary during lifting.) Now we will show that $\psi$ is well-defined, i.e. the image is independent of the choices of the curve $\tilde{\gamma}$. If $\tilde{\gamma}$ is another curve in $B_{g_0}(\tilde{p}, r_4)$ connecting $q$ to the center point $\tilde{p}$ with $l_{g_s}(\tilde{\gamma}) < r_3$, we ca $g_s$ is $e^Q$-quasi-isometric to $g_0$. Then $\varphi_s$ maps $H(s, t)$ to a homotopy $H(s, t)$ in $B(p, 2r_3)$ with $l(H(s, \cdot)) < 2r_3$ for each $s$. Therefore $H(s, t)$ can be lifted via $\varphi_{s'}$ to a homotopy in $B_{g_{s'}}(\tilde{p}', 2r_3)$ starting with $\tilde{p}'$. By the (localized) homotopy lifting lemma the other end points are all the same. Therefore $\psi$ is well-defined.

Next we show that $\psi$ is one-to-one. Let $r_2 = \frac{r}{20}e^{-2Q}$. Then

$$B_{g_{s'}}(\tilde{p}', r_3) \subset B_{g_0}(\tilde{p}', r_2) \subset B_{g_{s'}}(\tilde{p}', \frac{1}{2}r_1).$$

Since $B_{g_0}(\tilde{p}', r_2)$ is an Euclidean ball one can construct “inverse” $\phi$ similarly as above:

$$\phi : (B_{g_{s'}}(\tilde{p}', r_3), g_{s'}) \to (B_{g_s}(\tilde{p}, \frac{1}{2}r_1), g_s).$$
Thus $\psi$ is one-to-one. That $\psi$ is an isometric embedding follows from the construction.

Now using the equivariance, we have

$$\psi^*\tilde{g}_s = \tilde{g}_s.$$  

Therefore

$$\tilde{g}_{s'} \left( (\varphi_{s'})^*|\tilde{p})^{-1}(X), ((\varphi_{s'})^*|\tilde{p})^{-1}(Y) \right) = \tilde{g}_s \left( (\varphi_s)^*|\tilde{p})^{-1}(X), ((\varphi_s)^*|\tilde{p})^{-1}(Y) \right).$$

To show that $\varphi_s^*\tilde{g} = \tilde{g}_s$, consider

$$\varphi_s : B(0, \frac{9}{10}r) \to U_s.$$ 

In particular, for any $\tilde{p} \in B(0, \frac{9}{10}r)$, $B_{\tilde{g}_s}(\tilde{p}, r_1) \subset B(0, r)$, and therefore $\tilde{p}$ can be used to define the metric $\tilde{g}$ at $\varphi_s(\tilde{p})$. It follows from the definition that

$$\varphi_s^*\tilde{g} = \tilde{g}_s.$$  

Finally, note that the smoothness of the metric $\tilde{g}$ is an immediate consequence of (2.2).

3 Embedding I

We continue with the above manifold $(M, g)$. Let $\Omega = B(0, r) \subset \mathbb{R}^n$ with a pull back metric $\varphi^*g$. For convenience, this metric will be denoted by $g$. It is easy to see that $\|\langle \Omega, g \rangle\|_{H^{\alpha, \beta}} \leq Q(r)$. We are going to construct an equivariant embedding of $(\Omega, g)$ into $L^2(\Omega, g)$ by associating to every point $p \in \Omega$ a geometric function $f_p \in L^2(\Omega) \equiv L^2(\Omega, g)$, which depends nicely on $p$. A natural choice seems to be the distance function measured from $p$. Indeed, it is used by Abresch in [A]. However, under our rather weak assumptions on the metric it is impossible to have uniform control of the second order derivative of the distance function, which is needed to ensure that the pull-back metric induced by the embedding satisfies a sectional curvature bound. In fact, one can not even expect differentiability of the distance function in balls of uniform size. Our substitute for the distance function is solutions of a canonical geometric partial differential equation. Those solution functions have the crucial equivariance (like the distance functions) and enjoy better regularity. Many choices of “canonical” PDE solutions are possible, e.g. in [A] Green’s function.
is suggested. But Green’s function is inconvenient because of its singularity. We shall employ a very simple and nicely-behaved PDE.

Denote
\[ \Omega_1 = \Omega \setminus \bigcup_{q \in \partial \Omega} B_g(q, r_0), \]
where \( r_0 = \frac{\pi}{10} \). \((B^g(q, \cdot)) \) denotes the closed geodesic ball of center \( q \) and radius \( \cdot \) measured in \( g \).) Then for \( s \in \Omega_1 \), let \( h_s \in L^2_0(\Omega) \) be the unique weak solution of the following Dirichlet boundary value problem:

\[
\begin{cases}
\Delta h_s = -1 & \text{in } B^g(s, i_0) \\
h_s \equiv 0 & \text{on } \partial B^g(s, i_0).
\end{cases}
\] (3.1)

Here the Laplace operator is defined with respect to the metric \( g \). The function \( h_s \) will be extended to be zero outside the geodesic ball.

Since the harmonic \( C^{0, \alpha} \)-norm of \((\Omega, g)\) is uniformly bounded, it is easy to see that a uniform Poincare inequality holds on the balls \( B^g(s, i_0) \) with dependence on \( i_0 \). A simple integration argument then yields a uniform estimate of the Sobolev norm of \( h_s \). Uniform interior \( C^{2, \alpha} \) estimates then follow readily, because in harmonic coordinates the Laplace operator takes the form \( \Delta = g^{ij} \partial_i \partial_j \). We also have a uniform \( L^\infty \) estimate up to boundary, but it seems impossible to obtain better estimate up to boundary because the control of the geometry of the boundary is very weak. At a first glance this appears to threaten to destroy the embedding scheme. Fortunately we have a way to get around it. On the other hand, we can not obtain control of the dependence of \( h_s \) on the center \( s \). To remedy this, we shall take a suitable average of \( h_s \) over \( s \). The resulting new family of functions will depend nicely on the center.

Now let us state a few basic properties of the functions \( h_s \) in the following proposition, which will be proved at the end of this section. Here, as before, we work under the assumption \( \|{(\Omega, g)}\|_{C_0, \alpha, r} \leq Q(r) \).

**Proposition 3.1** Let \( \bar{h}_s(p) \) be the solution of equation (3.1) with respect to the canonical Euclidean metric \( g_0 \) on the Euclidean ball \( B^{g_0}(s, i_0) \). Then for any \( \epsilon > 0 \) and fixed \( 0 < R < 1 \), there is an \( r_0 = r_0(\epsilon, R, Q) > 0 \) such that if \( i_0 \leq r_0 \),

\[
|h_s(p) - \bar{h}_s(p)| < \epsilon i_0^2,
\] (3.2)

\[
\left| \frac{\partial}{\partial p} h_s(p) - \frac{\partial}{\partial p} \bar{h}_s(p) \right| < \epsilon i_0
\] (3.3)

for all \( s \) and all \( p \) with \( d_{g_0}(s, p) \leq R i_0 \). It will follow from the proof that \( B_{g_0}(s, Ri_0) \subset B_g(s, i_0) \) so that these estimate make sense. Also

\[
\left| \frac{\partial^2}{\partial p^2} h_s(p) \right| \leq C(n, Q, R), \quad \left| \frac{1}{i_0} \frac{\partial}{\partial p} h_s(p) \right| \leq C(n, Q, R).
\] (3.4)
Note that
\[ h_s(p) = \frac{1}{2n}(i_0^2 - d_{g_0}^2(s, p)). \]  
(3.5)

Therefore \( \frac{2n}{i_0^2} h_s(p) \leq \frac{1}{6} \) when \( d_{g_0}(s, p) \geq \sqrt{\frac{3}{2}} i_0 \). Choosing \( R = \frac{10}{11} \) in Proposition 3.4, we have \( \frac{2n}{i_0^2} h_s(p) < \frac{1}{3} \) when \( \sqrt{\frac{3}{2}} i_0 \leq d_{g_0}(s, p) \leq \frac{10}{11} i_0 \) and \( i_0 \) is sufficiently small.

Let \( \beta = \beta_n \in C_0^\infty([0, \infty)) \) be the cut off function.

\[ \beta_n(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq \frac{1}{4} \\ B_n & \text{if } t \geq \frac{1}{2} \end{cases}, \]

where \( B_n \) is a constant which will be determined later.

Then \( \beta \left( \frac{2n}{i_0} h_s(p) \right) = 0 \) near the sphere \( d_g(s, p) = \frac{9}{10} i_0 \) for all \( i_0 \) small. (Note that \( d_g \) converges to \( d_{g_0} \) when \( i_0 \to 0 \).) We define a new function which is \( \beta \left( \frac{2n}{i_0} h_s(p) \right) \) restricted to the ball \( B(s, \frac{9}{10} i_0) \) and identically zero outside. For simplicity we still denote this new function by \( \beta \left( \frac{2n}{i_0} h_s(p) \right) \). As mentioned before, we have no control of the dependence of \( h_s \) on the center \( s \). The said average function is given as follows

\[ f_p(q) = \int_\Omega \beta \left( \frac{2n}{i_0} h_s(p) \right) \beta \left( \frac{2n}{i_0} h_s(q) \right) ds. \]

(3.6)

Note that \( f_p(q) \) is symmetric in \( p \) and \( q \) and is \( C^{2,\alpha} \) uniformly bounded in both variables.

Now we define the embedding

\[ F : \Omega_1 \to L^2(\Omega, g) \]

\[ p \to i_0^{-\frac{3}{2}n+1} f_p(q) \]

Note that

\[ d_{v_p} F : q \mapsto 2n_i_0^{-\frac{3}{2}n} \int_\Omega \beta' \left( \frac{2n}{i_0} h_s(p) \right) \langle \frac{1}{i_0} \nabla_{v_p} h_s(p), v_p \rangle \beta \left( \frac{2n}{i_0} h_s(q) \right) ds, \]

(3.7)

\[ \nabla_{v_{p,wp}}^2 F : q \mapsto 4n_i_0^{-\frac{3}{2}n-1} \int_\Omega \beta'' \left( \frac{2n}{i_0} h_s(p) \right) \langle \frac{1}{i_0} \nabla h_s(p), v_p \rangle \langle \frac{1}{i_0} \nabla h_s(p), w_p \rangle \beta \left( \frac{2n}{i_0} h_s(q) \right) ds \]

\[ + 2n_i_0^{-\frac{3}{2}n-1} \int_\Omega \beta' \left( \frac{n}{i_0} \right) \nabla_{v_{p,wp}}^2 h_s(p)(v_p, w_p) \beta \left( \frac{n}{i_0} h_s(q) \right) ds. \]

(3.8)

We first show that when \( \Omega \) is an Euclidean domain, we can normalize \( \beta \) so that \( F \) is an isometric imbedding. In this case the imbedding function

\[ \bar{F}_p(q) = \int_\Omega \beta \left( 1 - \frac{d_{g_0}^2(p, s)}{i_0^2} \right) \beta \left( 1 - \frac{d_{g_0}^2(q, s)}{i_0^2} \right) ds. \]

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By the symmetry of the integration domain, $B(p, \frac{\sqrt{2}i_0}{2}) \cap B(q, \frac{\sqrt{2}i_0}{2})$, and the integrand, $\tilde{f}_p(q)$ depends only on $d_{g_0}(p, q)$ (and $\beta$). We can write $\tilde{f}_p(q) = i_0^{-n/2} \bar{f}(n, \frac{1}{i_0}d_{g_0}(p, q))$, $d_{\bar{v}} \bar{F}(q) = i_0^{-n/2} \bar{f}(n, \frac{1}{i_0}d_{g_0}(p, q))\langle \nabla d_{g_0}(p, q), v_p \rangle$. Then

$$\|d_{\bar{v}} \bar{F}\|^2_{L^2(\Omega)} = i_0^{-n} \int_{B(p, 2i_0)} \bar{f}^2(n, \frac{1}{i_0}d_{g_0}(p, q))\langle \nabla d_{g_0}(p, q), v_p \rangle^2 dq$$

$$= i_0^{-n} \int_0^{2i_0} r^{n-1} \int_{S^{n-1}} \bar{f}^2(n, \frac{1}{i_0}r)\langle \xi, v \rangle^2 d\xi dr$$

$$= \frac{i_0^{-n} \vol(S^{n-1})|v|^2}{n} \int_0^{2i_0} r^{n-1} \bar{f}^2(n, \frac{1}{i_0}) dr$$

$$= \frac{\vol(S^{n-1})|v|^2}{n} \int_0^2 r^{n-1} \bar{f}^2(n, r) dr.$$  

Choose $B_n$ in the definition of $\beta$ so that $\frac{\vol(S^{n-1})}{n} \int_2^r r^{-1} f^2(n, r) dr = 1$. Then we will have achieved the following.

**Lemma 3.2** $F$ is an isometric embedding.

With the above choice of $\beta$ we will show that $F$ is an almost isometric embedding when $\Omega$ is not necessarily an Euclidean domain and the second derivative of $F$ is also uniformly bounded. More precisely we have

**Proposition 3.3** For any given $\epsilon_0 > 0$, there exists an $r_0 > 0$ such that

$$(1 + \epsilon_0)^{-2}|v|^2 \leq \|d_{\bar{v}} F\|^2_{L^2(\Omega)} \leq (1 + \epsilon_0)^2 |v|^2,$$

for all $v \in T_p \Omega$ and $0 < i_0 \leq r_0$. And

$$\|\nabla_{\bar{v}, w} F\|^2_{L^2(\Omega)} \leq C(n, \alpha, Q) i_0^{-2} |v|^2 \cdot |w|^2.$$  

**Proof.** By definition

$$\|d_{\bar{v}} F\|^2_{L^2(\Omega)} = \int_{B(p, 2i_0)} |d_{\bar{v}} F(q)|^2 dq.$$  

Now the volume element of the metric $g$ is comparable with Euclidean one. Namely

$$e^{-Q(i_0)} \vol_{R^n} \leq \vol_g \leq e^{Q(i_0)} \vol_{R^n}.  \quad (3.11)$$

Therefore it suffices to prove that $|d_{\bar{v}} F(q)|$ is close to $|d_{\bar{v}} \bar{F}(q)|$ when $i_0$ is small, which follows from $(3.10)$ and Proposition 3.1.

$(3.10)$ also follows from $(3.8)$, $(3.11)$ and Proposition 3.1. 

**Proof of Proposition 3.1.** First we introduce some new functions. Let $\tilde{h}_{s,i_0}(p)$ be the solutions of $(3.1)$ on $B_{i_0^{-2}}(s, 1)$ with respect to the scaled metrics $i_0^{-2} g$, and...
Therefore \( C \) uniformly in \( s \) all \( \bar{s} \).

Consequently, for a fixed \( R \) with the smooth dependence of \( \bar{\tilde{s}} \) with respect to \( s \) and for any fixed \( 0 < R < 1 \), deduce the following: for each sequence of centers \( s \) converging weakly to \( \bar{\tilde{s}} \), we have the following estimates

\[
\| \tilde{h}_{s,i_0} \|_{L^2(B_{g(s,1)})} \leq C(n, Q(1)) \tag{3.14}
\]

and for any fixed \( 0 < R < 1 \),

\[
\| \tilde{h}_{s,i_0} \|_{C^2(B_g(s,R))} \leq C(n, Q(1), R). \tag{3.15}
\]

On the other hand, the hypothesis \( \|(B_{i_0 - 2g}(s, 1), i_0^{-2g}g)\|_{C^{0, \alpha}} \leq Q(i_0) \) implies that

\[
e^{-Q(i_0)} d_{i_0 - 2g_0}(p, s) \leq d_{i_0 - 2g}(p, s) \leq e^{Q(i_0)} d_{i_0 - 2g_0}(p, s).
\]

Therefore

\[
B_{i_0 - 2g_0}(s, e^{-Q(i_0)} R) \subset B_{i_0 - 2g}(s, R) \subset B_{i_0 - 2g_0}(s, e^{Q(i_0)} R).
\]

From these estimates and the uniqueness of the weak solution \( \tilde{h}_s \) it is easy to deduce the following: for each sequence of centers \( s_k \) converging to some center \( s_0 \) and each sequence \( i_0(k) \) converging to zero, the corresponding rescaled solutions \( \tilde{h}_{s_k,i_0(k)} \) converge weakly to \( \tilde{h}_{s_0} \). Moreover, by the Arzela-Ascoli theorem, they also converge uniformly in \( C^1 \) on proper compact subsets of \( B_{g_0}(s_0, 1) \). This convergence fact along with the smooth dependence of \( \tilde{h}_s \) on \( s \) then imply that the \( \tilde{h}_{s,i_0} \) converge uniformly with respect to \( s \) in \( C^1 \) on proper compact subsets of \( B_{g_0}(s, 1) \) as \( i_0 \) goes to zero. Consequently, for a fixed \( R \in (0, 1) \), given any \( \epsilon > 0 \), there is an \( r_0 > 0 \) such that for all \( s \) and \( p \) with \( d_{g_0}(p, s) < R \), if \( i_0 \leq r_0 \), then

\[
| \tilde{h}_s(p) - \tilde{h}_s(p) | < \epsilon. \tag{3.16}
\]

Similarly,

\[
| \frac{\partial}{\partial p} \tilde{h}_s(p) - \frac{\partial}{\partial p} \tilde{h}_s(p) | < \epsilon. \tag{3.17}
\]

Hence for all \( s \) and all \( p \) with \( d_{g_0}(s, p) \leq R r_0 \),

\[
| h_s(p) - \tilde{h}_s(p) | < \epsilon i_0^2,
\]

\[
| \frac{\partial}{\partial p} h_s(p) - \frac{\partial}{\partial p} \tilde{h}_s(p) | < \epsilon i_0.
\]

(3.14) just follows from (3.15) and (3.12).
4 Embedding II

In this section we study the geometry of $F(\Omega_1)$ as a submanifold in $L^2(\Omega)$. We will prove, among other things, two important properties of $F(\Omega_1)$. That is, the induced metric of $F(\Omega_1)$ has uniformly bounded sectional curvature and the embedding $F$ is equivariant.

The geometry of $F(\Omega_1)$ is completely determined by the second fundamental form of its embedding into $L^2(\Omega)$, which in turn can be described by the family of orthogonal projections. $P(y): L^2(\Omega) \to T_y F(\Omega_1) \subset L^2(\Omega), y \in F(\Omega^1)$. We have

**Lemma 4.1** The sectional curvature of $F(\Omega_1)$ is given by the following formula:

$$R(z_1, z_2)z_3 = [d_{z_1}P, d_{z_2}P]z_3, \quad z_1, z_2, z_3 \in T_z F(\Omega_1).$$  \hspace{1cm} (4.1)

**Proof.** Since $P^2 = P$, one has

$$(d_{z_1}P)P + P(d_{z_1}P) = d_{z_1}P.$$  \hspace{1cm} (4.2)

Let $\nabla$ be the connection on $F(\Omega_1)$ and $d_{z_1}$ the directional derivative on the $L^2$ space. Then

$$\nabla_{z_1}z_2 = P(d_{z_1}z_2) = d_{z_1}z_2 - (1 - P)(d_{z_1}(Pz_2)) = d_{z_1}z_2 - (1 - P)[(d_{z_1}P)z_2 + P(d_{z_1}z_2)] = d_{z_1}z_2 - (1 - P)(d_{z_1}P)(Pz_2) = d_{z_1}z_2 - (d_{z_1}P)z_2.$$  

Here we have used (4.2) in the last equation. Therefore

$$\nabla_{z_1} = d_{z_1} - (d_{z_1}P).$$  \hspace{1cm} (4.3)

Now formula (4.1) follows from (4.3) and the definition of the curvature tensor. \hfill \blacksquare

**Proposition 4.2** Let $\alpha_0 = (1 + \epsilon_0)^2 C(n, \alpha, Q)$. Here $\epsilon_0, r_0, C$ are the same constants as in Proposition 3.3. Then for all $0 < i_0 < r_0$,

$$\|d_y P\|_{op} \leq \alpha_0 i_0^{-1} \|\hat{y}\|,$$  \hspace{1cm} (4.4)

**Proof.** Since $(1 - P(F(p))) d_{w_p} F = 0$,

$$d_{w_p} F \cdot d_{w_p} F = (1 - P(F(p))) \nabla_{w_p, w_p}^2 F.$$  

By (3.9) and (3.10), $\|d_y P\|_{op} \leq \alpha_0 i_0^{-1} \|\hat{y}\|.$ \hfill \blacksquare

Therefore the metric $\tilde{g} = F^* g_{L^2}$, the metric on $\Omega_1$ obtained by pulling back the $L^2$ metric, has bounded sectional curvatures.

To prove the equivariance, we first note:
Lemma 4.3 Let $h_s(p)$ be the function defined in (3.1), and let $\psi : \Omega \to \Omega'$ be an isometric embedding. Then

$$h_{\psi(s)}(\psi(p)) = h_s(p).$$ (4.5)

Proof. Since equation (3.1) is invariant under isometry, this follows from the uniqueness of solutions to (3.1). \qed

Let $(\Omega, g)$ be as before and $F : \Omega \to L^2(\Omega)$ the embedding defined in §3. With the above lemma, we can now prove

**Proposition 4.4** If $\psi : (\Omega, g) \to (\Omega', g')$ is an isometric embedding, then

$\psi : (\Omega, \tilde{g}) \to (\Omega', \tilde{g}')$

is also an isometric embedding.

Proof. First, we assume $\psi$ is actually an isometry. Then

$$F \circ \psi(p) = f_{\psi(p)},$$

where the function

$$f_{\psi(p)}(q) = \int_{\Omega} \beta \left( \frac{2n}{t_0^2} h_s(\psi(p)) \right) \beta \left( \frac{2n}{t_0^2} h_s(q) \right) ds.$$

$$= \int_{\Omega} \beta \left( \frac{2n}{t_0^2} h_{\psi^{-1}(s)}(p) \right) \beta \left( \frac{2n}{t_0^2} h_{\psi^{-1}(s)}(\psi^{-1}(q)) \right) ds.$$

Here we have used Lemma 4.3. Since $\psi$ is an isometry, a change of coordinates yields

$$f_{\psi(p)}(q) = f_p(\psi^{-1}(q)).$$

It follows then that

$$F \circ \psi = (\psi^{-1})^* \circ F,$$

where we have denoted by $(\psi^{-1})^*$ the map on $L^2(\Omega)$ induced by $\psi^{-1}$. Therefore

$$\psi^* F^* g_{L^2} = (F \circ \psi)^* g_{L^2} = F^* ((\psi^{-1})^*)^* g_{L^2} = F^* g_{L^2}.$$

This proves the equivariance when $\psi$ is an isometry. Since $\Omega, \Omega'$ are both domains of $R^n$, the general statement follows by applying the above to $\psi : \Omega \to \psi(\Omega)$. \qed

**Proof of Theorem 1.1.** This theorem is a consequence of Proposition 2.2, Lemma 4.1, Proposition 4.2 and Proposition 3.3. \qed
5 Proof of Theorem 1.2

We consider \((M, g) \in \mathcal{M}(n, p, Q)\) and \(\Omega = B(0, r) \subset \mathbb{R}^n\) with the pull-back metric \(\varphi_s^* g\), where \(\varphi_s\) is a coordinate map. For convenience, we shall again denote the pull-back metric by \(g\). We have the inequality \(\| (\Omega, g) \|_{L^{1,p}, r} \leq Q(r)\).

We employ the same embedding of \((\Omega, g)\) as before. Thus we use the same functions \(h_s \in L^{1,2}_0(\Omega)\), as given by (3.1). But the estimates for \(h_s\) are different now. By the \(L^p\) elliptic theory we have uniform \(L^{2,p}\) estimates for \(h_s\) in the interior. A result similar to Proposition 3.1 then holds, namely we have the estimates (3.2), (3.3) and the second one in (3.4), while the first one in (3.4) is replaced by an estimate on the \(L^p\)-norm of the second order derivative. Now it is clear that our smoothing process produces a metric \(\bar{g}\) on \(M\) such that the lifted metrics \(\varphi_s^* \bar{g}\) on \(B(0, r/2)\) have uniformly bounded Sobolev constant and uniformly \(L^p\)-bounded sectional curvatures.

Next we apply the Ricci flow to deform \(\bar{g}\). For this purpose, we assume that \(M\) is closed. We appeal to the arguments in [DWY]. There, manifolds with a pointwise bound on Ricci curvature and a conjugate radius bound are treated. These conditions are used to show that controlled harmonic coordinates exist on lifted local patches, where the lifting is given by the exponential map. In these coordinates, the Ricci curvature bound then implies an \(L^p\)-bound on sectional curvatures. In our situation, we do have an \(L^p\)-bound on sectional curvatures. But the Ricci curvature bound is also used in several other places in [DWY]. Since this bound is not available here, we need to modify the arguments in [DWY].

In the key Proposition 3.1 (uniform short time existence of the Ricci flow with a priori control) in [DWY], we drop the estimate (3.4) on Ricci curvature. (Note that the conclusion of the proposition without (3.4) still suffices for our purpose.) We claim that the proposition then holds in our new situation. For convenience, we shall call this proposition the "Key Proposition". In [DWY], the proof of Key Proposition is based on four lemmata: Lemmata 3.2, 3.3, 3.4 and 3.5. Now let’s take a look at these lemmata in our new situation. Lemma 3.2 (pointwise estimate for Riemann curvature tensor) holds without change. The proof of Lemma 3.3 (\(L^p\)-estimate for Riemann curvature tensor) depends on a covering estimate, i.e. an estimate for certain covering number and multiplicity regarding geodesic balls in the lifted patches. In [DWY], the estimate comes from the Bishop-Gromov covering argument, which depends on a pointwise lower bound for Ricci curvature. Now we do not have such a bound. But we still have a covering estimate, which follows from the properties of our coordinates and the basic control over the pull-back metrics as given by (the modified versions of) Propositions 3.1 and 3.2 (in the present paper).

Another ingredient in the proof of Lemma 3.3 is an isometry correspondence between geodesic balls on different lifted patches. Now it is given by Lemma 2.3. (The proof of this correspondence given in [DWY] does not work here.) Thus Lemma 3.3 also holds. Since we have dropped the estimate (3.4) about Ricci curvature in Key Proposition, Lemma 3.4 is no longer needed. Finally, note that the proof for
Lemma 3.5 (estimate of the Sobolev constant) in [DWY] goes by computing the change rate of the Sobolev constant along the flow. In [DWY], this rate is controlled by a uniform bound on Ricci curvature, which is not valid here. However, Lemma 3.2 contains an estimate for Ricci curvature at positive time $t$, namely it is dominated by a constant times $t^{-1/2}$. Since the function $t^{-1/2}$ is integrable at 0, it is clear that the change rate of the Sobolev constant is still under control without a uniform Ricci curvature bound, and hence Lemma 3.5 carries over. (An alternative way of handling the Sobolev constant is to apply Yang’s estimate for it in [Y2], which uses only an $L^p$-bound on Ricci curvature and a positive lower bound on (local) volume. But that is more involved.)

We leave to the reader to formulate precisely the independent result implied by the above proof about short time existence of the Ricci flow.

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