Thompson’s semigroup and the first Hochschild cohomology

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Abstract. In this paper, we apply the theory of algebraic cohomology to study the amenability of Thompson’s group $F$. We introduce the notion of unique factorization semigroup which contains Thompson’s semigroup $S$ and the free semigroup $F_n$ on $n$ generators ($\geq 2$). Let $\mathcal{B}(S)$ and $\mathcal{B}(F_n)$ be the Banach algebras generated by the left regular representations of $S$ and $F_n$, respectively. It is proved that all derivations on $\mathcal{B}(S)$ and $\mathcal{B}(F_n)$ are automatically continuous, and every derivation on $\mathcal{B}(S)$ is induced by a bounded linear operator in $L^2(S)$, the weak closed Banach algebra consisting of all bounded left convolution operators on $l^2(S)$. Moreover, we show that the first continuous Hochschild cohomology group of $\mathcal{B}(S)$ with coefficients in $L^2(S)$ vanishes. These conclusions provide positive indications for the left amenability of Thompson’s semigroup.

Key words. Amenability, derivation, Banach algebra, Thompson semigroup, cohomology group.

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1. Introduction

The cohomology theory of associative algebras was initiated by Hochschild [9, 10, 11] in 1945 in terms of multilinear maps into a bimodule and coboundary operators. In 1953, after a discussing with Singer at a conference, Kaplansky went on from there to write his paper [20] proposing some problems about derivations on $C^*$-algebras and von Neumann algebras. He showed in [20] that every derivation on a von Neumann algebra $\mathcal{M}$ of type I is inner, which may be restated in cohomological terms that the first continuous cohomology group of $\mathcal{M}$ vanishes.

Recall that a derivation of a Banach algebra $A$ (over the complex field $\mathbb{C}$) with coefficients in a Banach $A$-bimodule $X$ is a linear map $D$ from $A$ into $X$ satisfying $D(AB) = AD(B) + D(A)B$ for all $A, B$ in $A$. We say that $D$ is inner when there is an element $T$ in $X$ such that $D(A) = AT - TA$ for each $A$ in $A$. Let $Z^1(A, X)$ be the space of all (continuous) derivations from $A$ into $X$ and $B^1(A, X)$ be the space of all inner derivations, respectively. It is clear that $B^1(A, X)$ is a linear subspace of $Z^1(A, X)$. The first (continuous) Hochschild cohomology group of $A$ with coefficients in $X$ is then defined to be the following quotient vector space:

$$H^1(A, X) = \frac{Z^1(A, X)}{B^1(A, X)}.$$

The study of cohomology of operator algebras started with the Kadison-Sakai Theorem [17, 23]: Every derivation on a von Neumann algebra $\mathcal{M}$ is inner, i.e., $H^1(\mathcal{M}, \mathcal{M}) = 0$. A similar

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problem is the case when the bimodule is \( B(\mathcal{H}) \), which is equivalent to Kadison’s similarity problem \([16]\). To study the classification of von Neumann algebras, from 1968 to 1972, Johnson, Kadison and Ringrose \([18, 19, 14, 13]\) proved a series of technical results of the cohomology groups of von Neumann algebras. In particular, they showed that \( H^n(\mathcal{M}, \mathcal{M}) = 0 \) for all \( n \geq 1 \) when \( \mathcal{M} \) is a hyperfinite von Neumann algebra. Due to this fact, Kadison and Ringrose conjectured that these cohomology groups are zero for all von Neumann algebras. With the aid of the theory of completely bounded cohomology, this conjecture can be ultimately reduced to the case when \( \mathcal{M} \) is a factor of type II\(_1\) with separable predual. In \([23]\), Sinclair and Smith showed that the conjecture holds for von Neumann algebras with Cartan subalgebras and separable preduals. Later in 2003, Christensen et al. \([4]\) proved that the continuous cohomology groups \( H^2(\mathcal{M} \otimes \mathcal{N}, \mathcal{M} \otimes \mathcal{N}) \) vanishes for arbitrary type II\(_1\) von Neumann algebras \( \mathcal{M} \) and \( \mathcal{N} \). Note that the free group factor \( L_{\mathcal{F}} \) satisfies none of the above cases and the higher order cohomology groups of \( L_{\mathcal{F}} \) are still unknown.

The cohomology of Banach algebras are different from that of von Neumann algebras in two main aspects: the automatically continuity of derivations and the cohomology groups. It was conjectured by Kaplansky in \([20]\) (which was finally proved by Sakai in \([22]\)) that every derivation on a C*-algebra is continuous. While, derivations on a Banach algebra are not necessarily continuous. In \([1]\), Bade and Curtis constructed several examples of Banach algebras on which not all derivations are continuous. On the other hand, Johnson and Sinclair \([15]\) showed that the continuity of derivations still holds for semisimple Banach algebras. Up to now, there are no examples showing that the cohomology groups of a von Neumann algebra are non-trivial. Let \( \mathcal{A}(\mathbb{D}) \) be the set of all complex-valued functions that are continuous on the closed unit disk and analytic in the interior. Then \( \mathcal{A}(\mathbb{D}) \) endowed with the supremum norm is a unital Banach algebra. The second cohomology group of \( \mathcal{A}(\mathbb{D}) \) is non-trivial \([12, \text{Proposition 9.1}]\).

Thompson’s group \( \mathcal{F} = \langle X_0, X_1, X_2, \ldots | X_i^{-1}X_jX_i = X_{j+1}, \ j > i \rangle \) was firstly introduced by Richard Thompson in 1965 \([3]\). It was conjectured by Geoghegan around 1979 that: (i) the group \( \mathcal{F} \) contains no non-abelian free groups; (ii) \( \mathcal{F} \) is not amenable. Statement (i) was obtained by Brin and Squier \([2]\) in 1985 while (ii) still remains unknown. Many research works nowadays are developed to answering this question due to two main reasons: (1) \( \mathcal{F} \) is related to many branches of mathematics such as geometric group theory; (2) the amenability problem is one of the most significant research areas in mathematics. Every nontrivial element in \( \mathcal{F} \) can be represented as a unique normal form:

\[
X_0^{\alpha_0}X_1^{\alpha_1}\cdots X_n^{\alpha_n}X_n^{-\beta_n}\cdots X_0^{-\beta_0},
\]

where \( \alpha_0, \ldots, \alpha_n, \beta_0, \ldots, \beta_n \), and \( n \) are natural numbers such that (i) exactly one of \( \alpha_n \) and \( \beta_n \) is nonzero and (ii) if \( a_k > 0 \) and \( b_k > 0 \) for some integer \( k \) with \( 0 < k < n \), then \( a_{k+1} > 0 \) or \( b_{k+1} > 0 \). For example, \( X_0X_1X_0^{-1} \) and \( X_0X_1X_2^{-1}X_0^{-1} \) are two normal forms while \( X_0X_2X_3^{-1}X_0^{-1} \) is not. The amenability of Thompson’s group \( \mathcal{F} \) has been an open problem for more that 40 years. We refer the readers to \([3]\) for more details about it.
Let $G$ be a locally compact group and $l^\infty(G)$ be the space of all the bounded complex-valued functions on $G$. Then $l^\infty(G)$ is a commutative $C^*$-algebra. The group $G$ is said to be **amenable** if there is a state $\mu$ on $l^\infty(G)$ such that $\mu(gf) = \mu(f)$, where $f \in l^\infty(G)$, $g \in G$ and $(gf)(h) = f(g^{-1}h)$ for each $h$ in $G$. The state $\mu$ is then called a **left invariant mean** on $l^\infty(G)$. Additive group of integers $(\mathbb{Z}, +)$ is amenable while the free group $F_2$ on two generators is not amenable. In [12], Johnson characterized the amenable group $G$ through the first Hochschild cohomology groups of $l^1(G)$, the space of all absolute-summable complex-valued functions on $G$, with coefficients in the dual Banach $l^1(G)$-bimodules: Let $G$ be a locally compact group. Then $G$ is amenable if and only if $H^1(l^1(G), \mathcal{X}^*) = 0$ for each Banach $l^1(G)$-bimodule $\mathcal{X}$.

In this paper, we shall apply the theory of cohomology to study the amenability of groups. In Section 2, we provide some basic definitions and theorems related to semigroup algebras and amenable semigroups. In Section 3, we introduce the notion of unique factorization semigroup and give three classical examples: Thompson’s semigroup, free semigroups and the amenable semigroup $T$ (See Example 3.12). The continuity of derivations (See Proposition 4.1) and the cohomology groups of $\mathcal{B}(S)$ (See Theorem 5.8 and Theorem 5.10) are the main contents of this paper and are discussed in Sections 4 and 5. We end this paper with some further discussions.

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2. Preliminaries

Group algebras and group actions on manifolds are two major sources for the construction of operator algebras. In applications, generalizations of groups (group algebras), such as semigroups (semigroup algebras), are also used. A brief description of semigroup algebras follows.

The Hilbert space $\mathcal{H}$ is $l^2(S)$, the space of all square-summable complex valued functions on the semigroup $S$. The semigroup $S$ (with unit $e$) is assumed to be discrete such that $\mathcal{H}$ is separable. The family of functions $(\delta_s)_{s \in S}$ forms an orthonormal basis of $\mathcal{H}$, where $\delta_s$ is 1 at the semigroup element $s$ and 0 elsewhere. For each $f$ and $g$ in $\mathcal{H}$, let $L_f$ be the left convolution operator defined as $L_fg = f \ast g$, where $f \ast g(s) = \sum_{uv = s} f(u)g(v)$ for each $s$ in $S$. Let $\mathcal{L}(S)$ be the set of all bounded left convolution operators on $\mathcal{H}$. Then $\mathcal{L}(S)$ is a subalgebra of $B(\mathcal{H})$.

In general, $\mathcal{L}(S)$ is a non-self-adjoint algebra. Similarly, let $\mathcal{R}(S)$ be the subalgebra of $B(\mathcal{H})$ consisting of all bounded right convolution operators. Then $\mathcal{L}(S)' = \mathcal{R}(S)$ and $\mathcal{R}(S)' = \mathcal{L}(S)$, which implies that $\mathcal{L}(S)$ and $\mathcal{R}(S)$ are both weak-closed algebras. For each $s$ in $S$, the operator $L_{\delta_s}$ is an isometry on $\mathcal{H}$ and is denoted by $L_s$ in this paper for convenience. Let $\mathfrak{B}(S)$ be the Banach algebra generated by $\{L_s : s \in S\}$ in $B(\mathcal{H})$. Then $\mathfrak{B}(S)$ is a Banach subalgebra of $\mathcal{L}(S)$.

Specific examples for such Banach algebras result from choosing for $S$ any of the free semigroup $\mathcal{F}_n$ on $n$ generators ($\geq 2$), Thompson’s semigroup $\mathcal{S}$, or the multiplicative semigroup of natural numbers $(\mathbb{N}, \cdot)$. The algebraic structures of $\mathfrak{B}(S)$ can reflect the structures of $S$ well. In [4], Dong, Huang and Xue proved that the maximal ideal space of the commutative Banach algebra $\mathfrak{B}(\mathbb{N})$ is homeomorphic to the Cartesian product of unit closed disk indexed by primes (see Theorem 1.1 in [4]). They pointed out that this result implies the fundamental theorem of arithmetic. Analogously, to study the cohomology of the Banach algebras $\mathfrak{B}(S)$, $\mathfrak{B}(\mathcal{F}_2)$ and $\mathfrak{B}(\mathcal{T})$ can help us to understand the properties of the corresponding semigroups well.

In this paper, we shall show that derivations on $\mathfrak{B}(S)$, $\mathfrak{B}(\mathcal{F}_2)$ and $\mathfrak{B}(\mathcal{T})$ are automatically continuous, and every derivation on $\mathfrak{B}(S)$ is spatial and induced by a bounded operator in $\mathcal{L}(S)$. Comparing with a result of Kadison [17, Theorem 4] that every derivation on a $C^*$-algebra is spatial, we give a non-trivial example in the case of Banach algebras. Moreover, the first cohomology group of $\mathfrak{B}(S)$ with coefficients in $\mathcal{L}(S)$ is shown to be zero, which gives a positive indication for the left amenability of Thompson’s semigroup.

In the following, we recall several concepts and results about amenable semigroups. We say that a discrete cancellative semigroup $S$ is left (resp. right) amenable if there exists a left (resp. right) invariant mean on $l^\infty(S)$. For example, the additive semigroup of natural numbers is amenable while the free semigroup on $n$ generators ($\geq 2$) is not. A left (resp. right) Fo\l\emph{ner} net of $S$ is a net of non-empty finite subsets $\{F_\alpha\}$ in $S$ such that for any $s \in S$,

$$\lim_\alpha \frac{|sF_\alpha \Delta F_\alpha|}{|F_\alpha|} = 0 \quad (\text{resp. } \lim_\alpha \frac{|F_\alpha s \Delta F_\alpha|}{|F_\alpha|} = 0).$$

It was firstly proved for groups by Fo\l\emph{ner}, and was then generalized to discrete cancellative semigroups by Frey [7] that $S$ is left (resp. right) amenable if and only if $S$ has a left (resp. right ) Fo\l\emph{ner}’s net. In [3], Fo\l\emph{ner} proved that every subgroup of an amenable group is still
amenable. For semigroups, it is not always true. In [7], Frey gave an example of a left amenable semigroup which contains a semigroup that is not left amenable. Moreover, he proved the following lemma.

**Lemma 2.1.** Let $S$ be a cancellative semigroup such that $S$ contains no free subsemigroup on two generators. If $S$ is left amenable, then every subsemigroup of $S$ is also left amenable.

### 3. Unique factorization semigroup

**Definition 3.1.** Let $S$ be a discrete semigroup. We say that $S$ is a **unique factorization semigroup** if there exists a subset $\{X_1, X_2, X_3, \ldots\}$ of $S$ such that

(i) every non-trivial element in $S$ can be uniquely written as $X_{i_1}^{\alpha_1} \cdots X_{i_n}^{\alpha_n}$, where $\alpha_i (1 \leq i \leq n)$ are positive integers and $i_1 < \cdots < i_n$;

(ii) if $e = X_1^{\beta_1} \cdots X_n^{\beta_n}$ for some non-negative integers $\beta_i (1 \leq i \leq n)$, then $\beta_i$ must be zero. The subset $\{X_1, X_2, X_3, \ldots\}$ is then called a **basis** of $S$.

For example, the multiplicative semigroup of natural numbers is a unique factorization semigroup and the set of all primes is the unique basis up to reorder. It is also clear that Thompson’s semigroup $S = \langle X_0, X_1, \cdots \mid X_jX_i = X_iX_{j+1}, i < j \rangle$ is a unique factorization semigroup with the basis $\{X_n \in S : n \in \mathbb{N}\}$. Next, we introduce some definitions and properties of $S$ that will be frequently used in Sections 4 and 5.

**Definition 3.2.** Let $X = X_0^{\alpha_0}X_1^{\alpha_1} \cdots X_n^{\alpha_n}$ be an element in $S$, where $\alpha_i (1 \leq i \leq n)$ are positive integers and $i_1 < \cdots < i_n$. We define the **index of $X$ at the $i$th position** as $\text{ind}_i(X) := \alpha_i$ and the **index** of $X$ is given by $\text{ind}(X) := \sum_{i=0}^{n} \text{ind}_i(X)$. The **index** of the unit element $e$ is defined to be zero.

It is clear that $\text{ind}_0(uv) = \text{ind}_0(u) + \text{ind}_0(v) = \text{ind}_0(vu)$ and $\text{ind}(uv) = \text{ind}(u) + \text{ind}(v) = \text{ind}(vu)$ for all $u$ and $v$ in $S$. In general, $\text{ind}_i(uv) \neq \text{ind}_i(u) + \text{ind}_i(v)$ for $i \geq 2$. For example, $\text{ind}_2(X_0X_2X_1) = \text{ind}_2(X_0X_1X_2) = 0 \neq 1 = \text{ind}_2(X_0X_2) + \text{ind}_2(X_1)$.

**Definition 3.3.** [26] Let $u, v, w \in S$. We call $u$ a **divisor** of $v$ if $v = uw$ which we denote by $u|v$.

For example, $X_1|X_0X_2$ while $X_2 \nmid X_0X_2$. It is clear that if $u|v$ then $\text{ind}(u) \leq \text{ind}(v)$.

**Lemma 3.4.** [26] The relation “$|$” is a partial order on $S$.

**Proof.** Let $u, v, w \in S$. Next, we verify the following three axioms of the partial order.

(1) (Reflexivity.) $u = ue$ implies $u|u$.

(2) (Antisymmetry.) If $u|v$ and $v|u$, then $u = vw_1$ and $v = uw_2$ for some $w_1, w_2$ in $S$. Then we have $w_2w_1 = e$, which implies $w_1 = w_2 = e$. Thus $u = v$.

(3) (Transitivity.) Suppose that $u|v$ and $v|w$, then $v = uw_1$ and $w = vw_2$ for some $w_1, w_2$ in $S$. We have $w = uw_1w_2$. Thus $u|w$.

As a result, we conclude that “$|$” is a partial order. □
Lemma 3.5. Let $X$ be an element in $\mathcal{S}$ such that $X_0|X$. Then for each $n \in \mathbb{N}$, we have

(i) $X_1^{-n}XX_1^n \in \mathcal{S}$ if and only if $X_1^n|X$;

(ii) $X_1^nXX_1^{-n} \in \mathcal{S}$ if and only if $X = YX_1^n$ for some $Y \in \mathcal{S}$.

Proof. (i) If $X_1^n|X$, then it is clear that $X_1^{-n}XX_1^n \in \mathcal{S}$. Conversely, we have $X_1^n|X$. If $X_1^n$ is not a divisor of $X$, since $X_1^{-n}X_0 = X_0X_2^{-n}$, we have that the normal form of $X_1^{-n}XX_1^n$ is $ZX_j^{-m}$ for some $Z \in \mathcal{S}$ and $j \geq 2$, $m \geq 1$. This leads to a contradiction. Thus $X_1^n|X$. (ii) Assume that $W = X_1^nXX_1^{-n} \in \mathcal{S}$, then we have $X_0|W$ and $X = X_1^{-n}WX_1^n \in \mathcal{S}$. From (i), we have $X_1^n|W$. Let $Y = X_1^{-n}W$, then $X = YX_1^n$. The other direction is obvious. We complete the proof. \( \square \)

Finally, with the aid of the index we introduce a total order on $\mathcal{S}$ which plays an important role in Sections 11 and 12.

Definition 3.6. Let $u, v \in \mathcal{S}$. We write $u \prec v$ if one of the following conditions holds:

(i) $\text{ind}(u) < \text{ind}(v)$;

(ii) $\text{ind}(u) = \text{ind}(v)$ and $\text{ind}_0(u) > \text{ind}_0(v)$;

(iii) $\text{ind}(u) = \text{ind}(v)$, $\text{ind}_0(u) = \text{ind}_0(v)$, and there exists a positive integer $i$ such that $\text{ind}_j(u) = \text{ind}_j(v)$ whenever $j < i$ while $\text{ind}_i(u) > \text{ind}_i(v)$.

We use $u \preceq v$ to denote $u \prec v$ or $u = v$.

For example, $X_0 < X_1 < X_0X_1 < X_0X_2 < X_1X_2$. The relation “$\preceq$” is a total order on $\mathcal{S}$ with the following properties.

Lemma 3.7. We have the following:

(i) There exists a unique minimal element in every non-empty subset of $\mathcal{S}$ under the total order;

(ii) Let $u_i$ and $v_i$ ($1 \leq i \leq n$) be 2n elements in $\mathcal{S}$. If $u_i \preceq v_i$ for each $1 \leq i \leq n$, then $\Pi_{i=1}^n u_i \preceq \Pi_{i=1}^n v_i$. The equality holds if and only if $u_i = v_i$ for each $1 \leq i \leq n$.

Proof. It is clear that (i) holds. We now give the proof of (ii). Firstly, we consider the case when $n = 2$. If $\text{ind}(u_1) < \text{ind}(v_1)$ or $\text{ind}(u_2) < \text{ind}(v_2)$, then $\text{ind}(u_1u_2) = \text{ind}(u_1) + \text{ind}(u_2) < \text{ind}(v_1v_2) + \text{ind}(v_2)$. This implies $u_1u_2 < v_1v_2$. In the case when $\text{ind}(u_1) = \text{ind}(v_1)$ and $\text{ind}(u_2) = \text{ind}(v_2)$, if $\text{ind}_0(u_1) > \text{ind}_0(v_1)$ or $\text{ind}_0(u_2) > \text{ind}_0(v_2)$, then $\text{ind}_0(u_1u_2) = \text{ind}_0(u_1) + \text{ind}_0(u_2) > \text{ind}_0(v_1v_2)$. This also implies $u_1u_2 < v_1v_2$. Hence, we may assume that $\text{ind}_0(u_1) = \text{ind}_0(v_1)$ and $\text{ind}_0(u_2) = \text{ind}_0(v_2)$. If either $\text{ind}(u_1)$ or $\text{ind}(u_2)$ is zero, then it is trivial. Thus we may further assume that $\text{ind}(u_1) = \text{ind}(v_1) \geq 1$ and $\text{ind}(u_2) = \text{ind}(v_2) \geq 1$.

Case I: $u_2 = v_2$. If $u_1 = v_1$, then it is proved. Otherwise, let $u_1 = X_0^{\alpha_0} \cdots X_n^{\alpha_n}$ and $v_1 = X_0^{\beta_0} \cdots X_n^{\beta_n}$. By the definition of the total order, there exists an $i > 0$ such that $\alpha_j = \beta_j$ whenever $j < i$ and $\alpha_i > \beta_i$. Since $\{X_l \in \mathcal{S} : l \in \mathbb{N}\}$ is a basis of $\mathcal{S}$, it only needs to prove $u_1X_l < v_1X_l$ for each $X_l \in \mathcal{S}$. In fact, we have

$$u_1X_l = \begin{cases} X_0^{\alpha_0} \cdots X_i^{\alpha_{i-1}} X_{i+1}^{\alpha_{i+1}} X_{i+2}^{\alpha_{i+2}} \cdots X_n^{\alpha_n} & l < i, \\ X_0^{\alpha_0} \cdots X_i^{\alpha_{i-1}} X_{i+1}^{\alpha_{i+1}} X_{i+2}^{\alpha_{i+2}} \cdots X_n^{\alpha_n} & l = i, \\ X_0^{\alpha_0} \cdots X_i^{\alpha_{i-1}} X_{i+1}^{\alpha_{i+1}} X_{i+2}^{\alpha_{i+2}} \cdots X_n^{\alpha_n} & l > i, \end{cases}$$

(1)
and

\[ v_1 X_l = \begin{cases} X_0^β_0 \cdots X_{i-1}^β_{i-1} X_i^β_i \cdots X_{m+1}^β_m & l < i, \\ X_0^β_0 \cdots X_{i-1}^β_{i-1} X_i^β_i \cdots X_{m+1}^β_m & l = i, \\ X_0^β_0 \cdots X_{i-1}^β_{i-1} X_i^β_i \cdots X_{j_s}^β_{j_s} & l > i, \end{cases} \]

where \( i < i_1 < \cdots < i_t \) and \( i < j_1 < \cdots < j_s \). By comparing equations (1) with (2), we have \( u_1 X_l < u_2 X_l \).

**Case II:** \( u_2 < v_2 \). Let \( u_2 = X_0^γ_0 \cdots X_k^γ_k \) and \( v_2 = X_0^ω_0 \cdots X_l^ω_l \), then there exists an \( h > 0 \) such that \( γ_j = ω_j \) whenever \( j < h \) and \( γ_h > ω_h \). By the first case, it can be reduced to the case when \( u_2 = X_h^γ \cdots X_k^γ \) and \( v_2 = X_h^ω \cdots X_l^ω \). If \( u_1 = v_1 = X_0^α_0 \cdots X_n^α_n \), then

\[ u_1 u_2 = \begin{cases} X_0^α_0 \cdots X_n^α_n X_h^γ \cdots X_l^γ & h > n, \\ X_0^α_0 \cdots X_n^α_n X_{h+1}^γ \cdots X_{l+1}^γ & h \leq n, \end{cases} \]

and

\[ v_1 v_2 = \begin{cases} X_0^α_0 \cdots X_n^α_n X_h^ω \cdots X_l^ω & h > n, \\ X_0^α_0 \cdots X_n^α_n X_{h+1}^ω \cdots X_{l+1}^ω & h \leq n, \end{cases} \]

where \( h < h_1 < \cdots < h_{\text{max}(t,s)} \). By comparing equations (3) with (4), we have \( u_1 u_2 < v_1 v_2 \). The proof of the case when \( u_1 < v_1 \) is similar, we omit it here. Moreover, we can obtain from the above process directly that \( u_1 u_2 = v_1 v_2 \) if and only if \( u_1 = v_1 \) and \( u_2 = v_2 \).

The general case when \( n > 2 \) can be obtained by induction. \( \square \)

We now turn to the free semigroup \( F_n \) on \( n \geq 2 \) generators \( a_i \) (\( 1 \leq i \leq n \)). The following two definitions on \( F_n \) are parallel to Definitions 3.2 and 3.6 and Lemma 3.10 is parallel to Lemma 3.7.

**Definition 3.8.** Let \( g = \prod_{j=1}^m \prod_{k=1}^n a_{jk}^{i_{jk}} \) be an element in \( F_n \), where \( i_{jk} \) (\( 1 \leq j \leq m, 1 \leq k \leq n \)) are non-negative integers. The **index** of \( g \) is defined as \( \text{ind}(g) := \sum_{j=1}^m \sum_{k=1}^n i_{jk} \).

It is clear that \( \text{ind}(gh) = \text{ind}(g) + \text{ind}(h) \) for all \( g \) and \( h \) in \( F_n \).

**Definition 3.9.** Let \( g = \prod_{j=1}^m \prod_{k=1}^n a_{jk}^{i_{jk}} \) and \( h = \prod_{j=1}^m \prod_{k=1}^n a_{jk}^{i_{jk}'} \) be two elements in \( F_n \). We write \( g \prec h \) if one of the following conditions holds:

(i) \( \text{ind}(g) < \text{ind}(h) \);

(ii) \( \text{ind}(g) = \text{ind}(h) \) and there exist some \( j_0 \) (\( 1 \leq j_0 \leq m \)) and \( k_0 \) (\( 1 \leq k_0 \leq n \)) such that \( i_{j_0 k_0} > i_{j_0 k_0}' \) and \( i_{jk} = i_{jk}' \) whenever \( j < j_0 \) or \( j = j_0 \) and \( k < k_0 \).

We use \( g \preceq h \) to denote \( g \prec h \) or \( g = h \).

The relation “\( \preceq \)” is a **total order** on \( F_n \).

**Lemma 3.10.** We have the following:

(i) There exists a unique minimal element in each subset of \( F_n \) under the total order;

(ii) Let \( g_i \) and \( h_i \) (\( 1 \leq i \leq m \)) be \( 2m \) elements in \( F_n \). If \( g_i \preceq h_i \) for each \( i \) (\( 1 \leq i \leq m \)), then \( \prod_{i=1}^m g_i \preceq \prod_{i=1}^m h_i \). Moreover, the equality holds if and only if \( g_i = h_i \) for each \( i \) (\( 1 \leq i \leq m \)).
Proposition 3.11. For \( n \geq 2 \), free semigroup \( F_n \) on \( n \) generators \( a_i \) (\( 1 \leq i \leq n \)) is a unique factorization semigroup\(^2\).

Proof. Let \( X_i = a_i \) (\( 1 \leq i \leq n \)), then \( X_1 < X_2 < \cdots < X_n \). Let \( X_{n+1} \) be the minimal element of \( F \) such that \( X_{n+1} \) cannot be represented as \( X_i^{a_i} \cdots X_n^{a_n} \) for any non-negative integers \( i_j \) (\( 1 \leq j \leq n \)). Then \( X_n < X_{n+1} \). Let \( X_{n+2} \) be the minimal element of \( F \) such that \( X_{n+2} \) cannot be represented as \( X_i^{a_i} \cdots X_{n+1}^{a_{n+1}} \) for any non-negative integers \( i_j \) (\( 1 \leq j \leq n + 1 \)). Then \( X_{n+1} < X_{n+2} \). Continuing this process, we can obtain a subset \( \{X_1, X_2, \ldots \} \) of \( F \) such that \( X_i < X_{i+1} \) for each \( i \geq 1 \). Then every element can be written as the product \( X_1^{a_1} \cdots X_n^{a_n} \) for some non-negative integers \( i_j \) (\( 1 \leq j \leq n \)). We shall show the uniqueness by induction on the index. It is clear that the uniqueness holds for index \( \leq 1 \). Assume that the uniqueness holds for index \( \leq k \) (\( k \geq 1 \)). Let \( g \in F_n \) and \( \text{ind}(g) = k + 1 \). Suppose that \( g \) has two different forms:

\[
g = X_1^{\alpha_1} \cdots X_n^{\alpha_n} = X_1^{\beta_1} \cdots X_n^{\beta_m},
\]

where \( \alpha_i \) (\( 1 \leq i \leq n \)) and \( \beta_j \) (\( 1 \leq j \leq m \)) are positive integers, \( i_1 < \cdots < i_n \) and \( j_1 < \cdots < j_m \). If \( i_1 = j_1 \), then

\[
X_1^{\alpha_1-1} \cdots X_n^{\alpha_n} = X_1^{\beta_1-1} \cdots X_n^{\beta_m},
\]

which implies that \( n = m \), \( i_t = j_t \) and \( \alpha_t = \beta_t \) (\( 1 \leq t \leq n \)). This leads to a contradiction.

We may assume that \( i_1 < j_1 \). Then there exists some \( l \) (\( 1 \leq l \leq n - 1 \)) such that \( (X_i^{\alpha_l} \cdots X_i^{\alpha_l})^{-1} X_{j_l} \) belongs to \( F_n \) and \( 1 \leq \text{ind}((X_i^{\alpha_l} \cdots X_i^{\alpha_l})^{-1} X_{j_l}) < \text{ind}(X_{i_l+1}) \), or \( (X_i^{\alpha_l} \cdots X_{i_l-1}^{\alpha_l})^{-1} X_{j_l} \) (\( 1 \leq l \leq n \)) belongs to \( F_n \) for some \( \alpha_l' \) (\( 1 \leq \alpha_l' < \alpha_l \)) and \( 1 \leq \text{ind}((X_i^{\alpha_l'} \cdots X_{i_l-1}^{\alpha_l'} \cdots X_i^{\alpha_l})^{-1} X_{j_l}) < \text{ind}(X_{i_l}) \). In the first case, we have \( (X_i^{\alpha_l'} \cdots X_{i_l-1}^{\alpha_l})^{-1} X_{j_l} = X_{s_1}^{\gamma_1} \cdots X_{s_t}^{\gamma_t} \) for some positive integers \( \gamma_1, \ldots, \gamma_t \) and \( s_1 < \cdots < s_t < \text{min}\{i_{l+1}, j_l\} \). Then

\[
X_i^{\alpha_{l+1}} \cdots X_i^{\alpha_n} = X_{s_1}^{\gamma_1} \cdots X_{s_t}^{\gamma_t} X_{j_l}^{\beta_1-1} \cdots X_{j_m}^{\beta_m},
\]

which leads to a contradiction. In the second case, we have \( (X_i^{\alpha_l'} \cdots X_{i_l-1}^{\alpha_l})^{-1} X_{j_l} = X_{s_1}^{\gamma_1} \cdots X_{s_t}^{\gamma_t} \) for some positive integers \( \gamma_1, \ldots, \gamma_t \) and \( s_1 < \cdots < s_t < \text{min}\{i_l, j_l\} \). Then

\[
X_i^{\alpha_l} \cdots X_{i_l+1}^{\alpha_l} \cdots X_i^{\alpha_n} = X_{s_1}^{\gamma_1} \cdots X_{s_t}^{\gamma_t} X_{j_l}^{\beta_1-1} \cdots X_{j_m}^{\beta_m},
\]

which also leads to a contradiction. Above all, we complete the proof. \( \square \)

It is known that Thompson’s semigroup is not right amenable while the left amenability is still unknown. The free semigroup \( F_n \) is neither left nor right amenable. For completeness, we construct the following example.

Proposition 3.12. Let \( T \) be the semigroup generated by

\[
A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix}
\]

in \( \text{GL}_2(\mathbb{Z}) \). Then \( T \) is a non-commutative unique factorization semigroup. Moreover, \( T \) is both left and right amenable.

\(^2\)We thank D. Wu for his useful suggestion on the proof of Example 3.11.
Proof. It can be verified directly that $AB = BA, AC = CB, BC = CA$. Hence $\mathcal{T}$ is non-commutative and every element in $\mathcal{T}$ can be written as $A^{\alpha_1}B^{\alpha_2}C^{\alpha_3}$ for some natural numbers $\alpha_1, \alpha_2$ and $\alpha_3$. Let $X$ be a matrix in $\mathcal{T}$ with two different representations: $X = A^{\alpha_1}B^{\alpha_2}C^{\alpha_3} = A^{\beta_1}B^{\beta_2}C^{\beta_3}$, where $\alpha_i$ and $\beta_i$ ($1 \leq i \leq 3$) are natural numbers. Take the determinant at the both side, we have $2^{2^{\alpha_1}} \cdot 2^{2^{\alpha_2}} \cdot (-6)^{\alpha_3} = 2^{2^{\beta_1}} \cdot 2^{2^{\beta_2}} \cdot (-6)^{\beta_3}$. Hence $\alpha_3 = \beta_3$ and $\alpha_1 + \alpha_2 = \beta_1 + \beta_2$. Then we have $A^{\alpha_1-\beta_1}B^{\beta_2-\alpha_2}$, which implies $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$. Thus $\mathcal{T}$ is a unique factorization semigroup. For each positive integer $N \geq 2$, let $F_N = \{A^{\alpha_1}B^{\alpha_2}C^{\alpha_3} \in \mathcal{T} \mid 0 \leq \alpha_i \leq N - 1, i = 1, 2, 3\}$, have

$$AF_N \cap F_N = \{A^{\alpha_1}B^{\alpha_2}C^{\alpha_3} \in \mathcal{T} \mid 1 \leq \alpha_1 \leq N - 1, 0 \leq \alpha_i \leq N - 1, i = 2, 3\},$$

$$BF_N \cap F_N = \{A^{\alpha_1}B^{\alpha_2}C^{\alpha_3} \in \mathcal{T} \mid 1 \leq \alpha_2 \leq N - 1, 0 \leq \alpha_i \leq N - 1, i = 1, 3\},$$

$$CF_N \cap F_N = \{A^{\alpha_1}B^{\alpha_2}C^{\alpha_3} \in \mathcal{T} \mid 1 \leq \alpha_3 \leq N - 1, 0 \leq \alpha_i \leq N - 1, i = 1, 2\}$$

and $|AF_N \cap F_N| = |BF_N \cap F_N| = |CF_N \cap F_N| = N^2(N - 1)$. Then

$$\lim_{N \to \infty} \frac{|AF_N \cap F_N|}{|F_N|} = \lim_{N \to \infty} \frac{|BF_N \cap F_N|}{|F_N|} = \lim_{N \to \infty} \frac{|CF_N \cap F_N|}{|F_N|} = \lim_{N \to \infty} \frac{N^2(N - 1)}{N^3} = 1.$$ 

Since $A, B,$ and $C$ are generators of $\mathcal{T}$, then $(F_N)_{N \in \mathbb{N}}$ is a left Følner sequence of $\mathcal{T}$. Thus $\mathcal{T}$ is left amenable. On the other hand, we have

$$F_NA \cap F_N = \{A^{\alpha_1}B^{\alpha_2}C^{\alpha_3} \in \mathcal{T} \mid 1 \leq \alpha_1 \leq N - 1, 0 \leq \alpha_i \leq N - 1, i = 2, 3, \alpha_3 \text{ even}\}$$

$$\cup \{A^{\alpha_1}B^{\alpha_2}C^{\alpha_3} \in \mathcal{T} \mid 1 \leq \alpha_1 \leq N - 1, 0 \leq \alpha_i \leq N - 1, i = 1, 3, \alpha_3 \text{ odd}\},$$

$$F_NB \cap F_N = \{A^{\alpha_1}B^{\alpha_2}C^{\alpha_3} \in \mathcal{T} \mid 1 \leq \alpha_2 \leq N - 1, 0 \leq \alpha_i \leq N - 1, i = 1, 3, \alpha_3 \text{ even}\}$$

$$\cup \{A^{\alpha_1}B^{\alpha_2}C^{\alpha_3} \in \mathcal{T} \mid 1 \leq \alpha_2 \leq N - 1, 0 \leq \alpha_i \leq N - 1, i = 2, 3, \alpha_3 \text{ odd}\},$$

$$F_NC \cap F_N = \{A^{\alpha_1}B^{\alpha_2}C^{\alpha_3} \in \mathcal{T} \mid 1 \leq \alpha_3 \leq N - 1, 0 \leq \alpha_i \leq N - 1, i = 1, 2\}.$$ 

Similarly, $(F_N)_{N \in \mathbb{N}}$ is also a right Følner sequence of $\mathcal{T}$. Thus $\mathcal{T}$ is both left and right amenable. We complete the proof. \qed

4. The continuity of derivations

In this section, we shall prove the following theorem.

**Theorem 4.1.** Derivations on the Banach algebras $\mathfrak{B}(S)$ and $\mathfrak{B}(F_n)$ are continuous.

For the sake of proof, we firstly introduce the following definition.

**Definition 4.2.** A discrete semigroup $S$ is said to be lower stable if there is a total order “$\preceq$” on $S$ such that

(i) There exists a unique minimal element in each subset of $S$ under the total order;

(ii) Let $u_i$ and $v_i$ ($1 \leq i \leq n$) be $2n$ elements in $S$. If $u_i \preceq v_i$ for each $1 \leq i \leq n$, then $\Pi_{i=1}^n u_i \preceq \Pi_{i=1}^n v_i$. Moreover, the equality holds if and only if $u_i = v_i$ for each $1 \leq i \leq n$.

By Lemmas 3.7 and 3.10, we have that Thompson’s semigroup $S$ and the free semigroup $F_n$ ($n \geq 2$) are both lower stable.
Definition 4.3. A Banach algebra $A$ is said to be **semisimple** if its Jacobson radical $J$ equals zero, where

\[ J = \bigcap \text{maximal ideals of } A \]

\[ I = \bigcap \text{maximal left ideals of } A \]

\[ I_r = \bigcap \text{maximal right ideals of } A \]

The following lemma is a characterization of semisimple Banach algebras.

**Lemma 4.4.** Let $A$ be a Banach algebra with the unit $I$. If $A$ contains no non-zero quasi-nilpotent operators, then $A$ is semisimple.

**Proof.** Let $T$ be a non-zero element in $A$. Let $\lambda$ be a non-zero spectrum of $T$, then at least one of $(\lambda I - T)A$ and $A(\lambda I - T)$ is properly contained in $A$. We may assume that the right ideal $(\lambda I - T)A$ is properly contained in $A$, then there exists a maximal right ideal $I_r$ of $A$ such that $(\lambda I - T)A \subseteq I_r \subseteq A$. Thus $T$ does not belong to $I_r$. This implies $T$ is not in the Jacobson radical $J$ of $A$. Consequently, we have $J = 0$. This completes the proof. \(\square\)

**Lemma 4.5.** Let $S$ be a lower stable discrete semigroup and $B(S)$ be the Banach algebra generated by \(\{L_s : s \in S\}\) in $B(l^2(S))$. Then $B(S)$ is semisimple.

**Proof.** Let $L_f$ be a non-zero element in $B(S)$. We claim that the spectral radius $r(L_f) > 0$. By the definition of lower stable semigroup, there is a unique minimal element $X$ of the subset \(\{X \in S : f(X) \neq 0\}\) under the total order. Then we have

\[ f \ast \cdots \ast f(X^n) = f(X)^n \]

for each $n \geq 1$. Therefore,

\[ r(L_f) = \lim_{n \to \infty} \|L_f^n\|^{1/n} \geq \lim_{n \to \infty} \| f \ast \cdots \ast f \|_2^{1/n} \]

\[ \geq \lim_{n \to \infty} \| f \ast \cdots \ast f(X^n) \|^{1/n} = |f(X)| > 0. \]

By Lemma 4.3, we obtain that $B(S)$ is semisimple. \(\square\)

**Corollary 4.6.** The Banach algebras $B(S)$ and $B(F_n)$ are semisimple.

The semigroup $T$ in Example 3.12 may not be lower stable. However, the Banach algebra $B(T)$ is semisimple. The proof is similar with that of Proposition 4.5, we omit it here. Now, we are ready to prove Proposition 4.1.

**Proof of Proposition 4.1.** Theorem 4.1 of [15] states that derivations on a semisimple Banach algebra are continuous, then by Corollary 4.6 we can obtain the conclusion. \(\square\)
5. The first cohomology group of $\mathfrak{B}(S)$

A derivation $D$ on a Banach algebra $\mathfrak{B} \subseteq B(\mathcal{H})$ is said to be **spatial** if there exists a bounded operator $T$ in $B(\mathcal{H})$ such that $D(A) = TA - AT$ for each $A$ in $\mathfrak{B}$. In [17], Kadison proved that all derivations on a $C^*$-algebra are spatial. In this section, we will prove that all derivations on the Banach algebra $\mathfrak{B}(S)$ are spatial and are induced by bounded operators in $\mathcal{L}(S)$ (Corollary 5.9). Moreover, the first continuous cohomology group of $\mathfrak{B}(S)$ with coefficients in $\mathcal{L}(S)$ is shown to be zero (Theorem 5.8).

The Hilbert space $\mathcal{H}$ is $l^2(\mathcal{S})$. Recall that $\mathfrak{B}(S)$ is the Banach algebra generated by $\{L_s : s \in S\}$ in $B(\mathcal{H})$ and $\mathcal{L}(S) = \{L_f \in B(\mathcal{H}) : f \in \mathcal{H}\}$. We use $\sum_{X \in S} f(X)X$ and $\sum_{X \in S} f(X)X^*$ to denote $L_f$ and its adjoint operator for convenience. It is clear that $XX^*$ is the projection from $\mathcal{H}$ onto the closure of the subspace span$\{\delta_Y : Y \in S, X|Y\}$ and $X^*X = I$ (the identity operator). Recall that “$\sim$” is the partial order of Thompson’s semigroup introduced in Section 3 and “$X|Y$” means that there exists a $Z$ in $S$ such that $Y = XZ$.

Let $(\mathbb{N}, +)$ be the additive semigroup of natural numbers. The notation $\beta \mathbb{N}$ denotes the maximal ideal space of $l^\infty(\mathbb{N})$, and the elements in $\beta \mathbb{N} \setminus \mathbb{N}$ are called **free ultrafilters**. Let $\omega \in \beta \mathbb{N} \setminus \mathbb{N}$ be a given free ultrafilter. For any $n \in \mathbb{N}$ and any $f$ in $l^\infty(\mathbb{N})$, we define $E_n(f) = \frac{1}{n} \sum_{i=0}^{n-1} f(i)$. Then, for each given $f$, the function $n \to E_n(f)$ gives rise to another function in $l^\infty(\mathbb{N})$. By Gelfand-Naimark theorem, we have $l^\infty(\mathbb{N}) \cong C(\beta \mathbb{N})$. Thus $E_n(f)$ is a continuous function on $\beta \mathbb{N}$. The limit of $E_n(f)$ at $\omega$ is denoted by $E_\omega(f)$ or the integral $\int_\mathbb{N} f(n)dE_\omega(n)$. Then $E_\omega$ is an invariant mean defined on $l^\infty(\mathbb{N})$, i.e., $\int_\mathbb{N} f(n)dE_\omega(n) = \int_\mathbb{N} f(n + m)dE_\omega(n)$ for each $m \geq 1$, satisfying $E_\omega(f) = \lim_{n \to \infty} f(n)$ if the limit exists. Let $D$ be a continuous derivation from the Banach algebra $\mathfrak{B}(S)$ into $\mathcal{L}(S)$. For each $\xi, \eta \in \mathcal{H}$, we define

$$
\langle A\xi, \eta \rangle := \int_\mathbb{N} \langle (X_0^*)^n D(X_0^{n+1}) \xi, \eta \rangle dE_\omega(n).
$$

Then $A$ is a bounded linear operator on $\mathcal{H}$. Moreover, we have

$$
\langle A\xi, \eta \rangle = \int \langle (X_0^*)^{n+1} D(X_0^{n+2}) \xi, \eta \rangle dE_\omega(n)
$$

$$
= \langle D(X_0) \xi, \eta \rangle dE_\omega(n) + \int \langle (X_0^*)^{n+1} D(X_0^{n+1}) X_0 \xi, \eta \rangle dE_\omega(n)
$$

$$
= \langle D(X_0) \xi, \eta \rangle + \langle X_0^* AX_0 \xi, \eta \rangle,
$$

and it follows that

$$
D(X_0) = A - X_0^* AX_0.
$$

Similarly, we define

$$
\langle B\xi, \eta \rangle := \int_\mathbb{N} \langle (X_1^*)^n D(X_1^{n+1}) \xi, \eta \rangle dE_\omega(n),
$$

then we have $B \in B(\mathcal{H})$ and

$$
D(X_1) = B - X_1^* B X_1.
$$
5.1. The case of $X_0$. In this subsection, we will prove the following lemma.

**Lemma 5.1.** There exists some $\hat{A}$ in $\mathcal{L}(S)$ such that $D(X_0) = \hat{A}X_0 - X_0\hat{A}$.

**Proof.** Let

$$D(X_0) = \sum_{X_0|X} f(X)X + \sum_{X_0\nmid X} f(X)X.$$ By the fact that $X_1X_0 = X_0X_2$, we have

$$D(X_1)X_0 + X_1D(X_0) = X_0D(X_2) + D(X_0)X_2.$$ It follows that

$$X_1\sum_{X_0|X} f(X)X = \sum_{X_0|X} f(X)XX_2.$$ Since $X_1X \prec XX_2$ whenever $X_0 \nmid X$, hence $f(X) = 0$ in this case. Therefore,

$$D(X_0) = \sum_{X_0|X} f(X)X = X_0L_{f_1}$$ for some $L_{f_1}$ in $\mathcal{L}(S)$. By further computations, we can obtain that $D(X^n_0) = X^n_0L_{f_n}$ for some $L_{f_n}$ in $\mathcal{L}(S)$. We define

$$\langle \hat{A}\xi, \eta \rangle := -\int_N \langle (X^n_0)X_0^{n+1}D(X^n_0)\xi, \eta \rangle dE_\omega(n),$$ then $\hat{A} \in B(H)$. For each $T$ in $\mathcal{R}(S)$, we have

$$\langle T\hat{A}\xi, \eta \rangle = -\int_N \langle (X^n_0)X_0^{n+1}D(X^n_0)\xi, T^n\eta \rangle dE_\omega(n)$$

$$= -\int_N \langle (X^n_0)X_0^{n+1}D(X^n_0)T\xi, \eta \rangle dE_\omega(n)$$

$$= \langle \hat{A}T\xi, \eta \rangle.$$ It follows that $\hat{A} \in \mathcal{R}(S)' = \mathcal{L}(S)$. By equation (5), we have

$$\langle A\xi, \eta \rangle = \int_N \langle (X^n_0)X_0^{n+1}D(X^n_0)\xi, \eta \rangle dE_\omega(n)$$

$$= \int_N \langle (X^n_0)X_0^{n+1}D(X^n_0)\xi, X^n_0\eta \rangle dE_\omega(n)$$

$$= -\langle \hat{A}\xi, X^n_0\eta \rangle,$$ which implies $A = -X_0\hat{A}$. Then by equation (6), we have $D(X_0) = \hat{A}X_0 - X_0\hat{A}$. This completes the proof. □
5.2. **The case of $X_1$.** The following lemma is the main conclusion of this subsection.

**Lemma 5.2.** There exists some $\widehat{B}$ in $\mathcal{L}(S)$ such that $D(X_1) = \widehat{B}X_1 - X_1\widehat{B}$.

We need Lemmas 5.3 and 5.4 to obtain Lemma 5.2.

**Lemma 5.3.** $D(X_1) = B - X_1^*BX_1$ for some $B$ in $\mathcal{L}(S)$.

*Proof.* Let $D(X_1) = L_f$ in $\mathcal{L}(S)$, then by the continuity of $D$ we have $f(X_1^n) = 0$ for each $n \in \mathbb{N}$. For each $m \geq 1$, we have $(X_1^n)^m - D(X_1^n) = \sum_{i=0}^{m-1}(X_1^n)^i L_f X_1^n$ and $\| (X_1^n)^m - D(X_1^n) \| \leq \| D \|$. Let $h_1$ and $h_2$ be two elements in $S$. If $h_2h_1^{-1} = X_1^k$ for some $k \in \mathbb{N}$, then

$$\lim_{n,m \to \infty} \left\langle \left( (X_1^n)^m - (X_1^n)^{-1} D(X_1^n) \right) \delta_{h_1}, \delta_{h_2} \right\rangle = \lim_{n \to \infty} \sum_{i=1}^{m} f(X_1^n h_2 h_1^{-1} X_1^{-i}) = 0.$$ 

If $h_2h_1^{-1} \neq X_1^k$ for any $k \in \mathbb{N}$, then $X_1^n h_2 h_1^{-1} X_1^{-i} \notin S$ whenever $i$ is sufficiently large. Therefore,

$$\lim_{n,m \to \infty} \left\langle \left( (X_1^n)^m - (X_1^n)^{-1} D(X_1^n) \right) \delta_{h_1}, \delta_{h_2} \right\rangle = \lim_{n \to \infty} \sum_{i=1}^{m} f(X_1^n h_2 h_1^{-1} X_1^{-i}) = 0.$$ 

We define

$$\langle T\delta_{h_1}, \delta_{h_2} \rangle := \lim_{m \to \infty} \langle (X_1^n)^m - D(X_1^n) \delta_{h_1}, \delta_{h_2} \rangle.$$ 

Then $T$ is the weak-topology limit of $(X_1^n)^m - D(X_1^n)$ in $B(H)$. We have the following claim.

**Claim:** $T = L_{T\delta_e} \in \mathcal{L}(S)$. To prove this claim, we distinguish two cases:

**Case I:** $h_2h_1^{-1} \in S$. We have

$$\langle T\delta_{h_1}, \delta_{h_2} \rangle = \sum_{n=0}^{\infty} f(X_1^n h_2 h_1^{-1} X_1^{-n}) = \langle T\delta_e, \delta_{h_2h_1^{-1}} \rangle = \langle T\delta_e \ast \delta_{h_1}, \delta_{h_2} \rangle.$$ 

**Case II:** $h_2h_1^{-1} \notin S$. If $X_1^n h_2 h_1^{-1} X_1^{-n} \notin S$ for any $n \in \mathbb{N}$, then

$$\langle T\delta_{h_1}, \delta_{h_2} \rangle = \langle T\delta_e \ast \delta_{h_1}, \delta_{h_2} \rangle = 0.$$ 

On the other hand, there exists a natural number $n$ such that $X_1^{n+1} h_2 h_1^{-1} X_1^{-n-1} \in S$ and $X_1^n h_2 h_1^{-1} X_1^{-i} \notin S$ whenever $i \leq n$. Let $X = X_1^{n+1} h_2 h_1^{-1} X_1^{-n-1}$, then $X_0\hat{X}$. By Lemma 3.3 we have that $X_1^n X_1^n \notin S$ for any $n \geq 1$ and $X_1^n X_1^n \notin S$ whenever $n > \text{ind}(X)$. Let $m$ be an even number such that $m >> \text{ind}(X)$, then we have

$$\sum_{i=1}^{m} \left\langle (D(X_1^m)) \delta_e, \delta_{X_1^{m-i} X_1^{-i}} \right\rangle^2 = \sum_{i=1}^{m} \sum_{j=-(i-1)}^{m-i} f(X_1^i X_1^{-j})^2 = \sum_{i=1}^{m} \sum_{j=0}^{\text{ind}(X)} f(X_1^i X_1^{-j})^2.$$ 

Note that $X_1^{m-i} X_1^{-i} \neq X_1^{-j} X_1^{-j}$ whenever $1 \leq i < j \leq m$, then we have

$$\sum_{i=1}^{m} \sum_{j=0}^{\text{ind}(X)} f(X_1^i X_1^{-j})^2 \leq \left\| D(X_1^m) \delta_e \right\|^2 \leq \left\| D \right\|^2.$$
for any \( m \). This implies that \( \sum_{j=0}^{ \text{ind}(X) } f(X_1^jXX_1^{-j}) = 0 \). Therefore,
\[
\langle T \delta_{h_1}, \delta_{h_2} \rangle = \sum_{j=0}^{ \infty } f(X_1^jh_2h_1^{-1}X_1^{-j}) = \sum_{j=0}^{ \text{ind}(X) } f(X_1^jXX_1^{-j}) = 0.
\]
From the above discussion, we have \( \langle T \delta_{h_1}, \delta_{h_2} \rangle = \langle T \delta_e \ast \delta_{h_1}, \delta_{h_2} \rangle \) for all \( h_1 \) and \( h_2 \) in \( S \). Thus the claim holds.

By equation (7), we have \( B = T \). This completes the proof. \( \Box \)

**Lemma 5.4.** Let \( D(X_1) = L_f \in \mathcal{L}(S) \), then
\[
D(X_1) = \sum_{X_0|X} f(X)X + \sum_{X_0|X, X_1|X} f(X)X.
\]

**Proof.** Let \( D(X_1) = \sum_{X_0|X} f(X)X + \sum_{X_0|X, X_1|X} f(X)X + \sum_{X_0|X, X_1|X} f(X)X \). By the fact that \( X_1X_3 = X_2X_1 \), we have
\[
D(X_1)X_3 + X_1D(X_3) = D(X_2)X_1 + X_2D(X_1).
\]
Then
\[
\sum_{X_0|X, X_1|X} f(X)XX_3 = X_2 \sum_{X_0|X, X_1|X} f(X)X.
\]
Let \( X \) be the minimal element of the set \( \{ X \in S : X_0 \uparrow X, X_1 \uparrow X, f(X) \neq 0 \} \). Then \( X_2X \prec XX_3 \). It follows that \( f(X) = 0 \) when \( X_0 \uparrow X \) and \( X_1 \uparrow X \). This completes the proof. \( \Box \)

Now, we are ready to prove Lemma 5.2.

**Proof of Lemma 5.2.** By Lemma 5.3 we have \( D(X_1) = B - X_1^*BX_1 \) for some \( B = L_g \in \mathcal{L}(S) \). We may assume that \( g(e) = 0 \), where \( e \) is the unit element of \( S \). Let
\[
L_g = \sum_{X_1|X} g(X)X + \sum_{X_1|X, X_0|X} g(X)X + \sum_{X_1|X, X_0|X} g(X)X.
\]
Then
\[
D(X_1) = \sum_{X_1|X} g(X)X - X_1^* \sum_{X_1|X} g(X)XX_1 + \sum_{X_1|X, X_0|X} g(X)X - X_1^* \sum_{X_1|X, X_0|X} g(X)XX_1
\]
\[
+ \sum_{X_1|X, X_0|X} g(X)X - X_1^* \sum_{X_1|X, X_0|X} g(X)XX_1.
\]
Since \( D(X_1) \in \mathcal{L}(S) \) and \( X_1^{-1}XX_1 \notin S \) when \( X_1 \uparrow X \) and \( X_0 \uparrow X \), we have \( \sum_{X_1|X, X_0|X} g(X)X = 0 \). Therefore,
\[
D(X_1) = \sum_{X_1|X} g(X)X - X_1^* \sum_{X_1|X} g(X)XX_1 + \sum_{X_1|X, X_0|X} g(X)X - X_1^* \sum_{X_1|X, X_0|X} g(X)XX_1.
\]
By Lemma 5.4, we have
\[ \sum_{x_1 \mid x \neq x_0 \mid x} g(x)X - X_1^* \sum_{x_1 \mid x \neq x_0 \mid x} g(x)XX_1 = 0. \]

Let \( X \) be the minimal element of the set \( \{ x \in S : x_1 \uparrow x, x_0 \uparrow x, g(x) \neq 0 \} \), then \( X \prec X_1^{-1}XX_1 \) since \( X \neq e \). It follows that \( g(X) = 0 \) when \( x_1 \uparrow x \) and \( x_0 \uparrow x \). Thus \( B = \sum_{x_1 \mid x} g(x)X \). Let \( \widehat{B} = -X_1^*B \), then \( \widehat{B} \in \mathcal{L}(S) \) and \( D(X_1) = B - X_1^*BX_1 = X_1X_1^*B - X_1^*BX_1 = \widehat{B}X_1 - X_1\widehat{B} \). We complete the proof. \( \square \)

5.3. Conditional expectation.

**Definition 5.5.** Let \( \mathfrak{B} \) be a Banach algebra and \( \mathcal{A} \) be a Banach subalgebra of \( \mathfrak{B} \). Let \( E : \mathfrak{B} \rightarrow \mathcal{A} \) be a contraction such that

(i) \( E(I) = I \);

(ii) \( E(A_1BA_2) = A_1E(B)A_2 \) whenever \( A_1, A_2 \in \mathcal{A} \) and \( B \in \mathfrak{B} \).

Then \( E \) is described as a **conditional expectation** from \( \mathfrak{B} \) onto \( \mathcal{A} \).

**Definition 5.6.** Let \( \mathcal{L}_0(S) \) be the subset of \( \mathcal{L}(S) \) such that for each \( L_f \) in \( \mathcal{L}_0(S) \), \( f(X) = 0 \) if \( X \neq X_0^n \).

It is not difficult to check that \( \mathcal{L}_0(S) \) is a Banach subalgebra of \( \mathcal{L}(S) \). We have the following theorem.

**Theorem 5.7.** Let \( E \) be the map: \( \mathcal{L}(S) \rightarrow \mathcal{L}_0(S) \), \( \sum_{x \in S} f(X)X \rightarrow \sum_{n=0}^{\infty} f(X_0^n)X_0^n \). Then \( E \) is a well-defined conditional expectation from \( \mathcal{L}(S) \) onto \( \mathcal{L}_0(S) \).

**Proof.** For each \( g \in L^2(S) \) such that \( g(X) = 0 \) when \( X \neq X_0^n \), we have
\[ \left\| \sum_{n=0}^{\infty} f(X_0^n)X_0^n \sum_{n=0}^{\infty} g(X_0^n)\delta X_0^n \right\|_2^2 \leq \left\| L_f \right\|_2^2 \leq \left\| L_f \right\|_2 \left\| g \right\|_2^2. \]

This implies that \( \sum_{n=0}^{\infty} f(X_0^n)X_0^n \) is bounded on the Hilbert subspace \( L^2(S_1) \), where \( S_1 \) is the subsemigroup of \( S \) generated by \( X_0 \). The norm of \( \sum_{n=0}^{\infty} f(X_0^n)X_0^n \) is bounded by \( \| L_f \| \). Let \( \mathcal{F} \) be the Fourier transform: \( Z \rightarrow S^1 \), \( n \rightarrow e^{2\pi i\theta}, \theta \in [0, 1) \). This induces the following isomorphisms [S]:
\[
\begin{align*}
\mathbb{Z} & \subset \mathbb{C} \mathbb{Z} \subset l^1(\mathbb{Z}) \subset C^*(\mathbb{Z}) \subset \mathcal{L}(\mathbb{Z}) \subset l^2(\mathbb{Z}) \subset \cdots \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
\mathbb{S}^1 & \subset \mathbb{C}[\mathbb{Z}]_{\mathbb{S}^1} \subset R(\mathbb{S}^1) \subset C(\mathbb{S}^1) \subset L^\infty(\mathbb{S}^1) \subset l^2(\mathbb{S}^1) \subset \cdots \\
\end{align*}
\]

Restricting the Fourier transform on \( \mathbb{N} \), we have
\[
\begin{align*}
\mathbb{C}\mathbb{N} & \subset l^1(\mathbb{N}) \subset B(\mathbb{N}) \subset \mathcal{L}(\mathbb{N}) \subset l^2(\mathbb{N}) \subset \cdots \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
\mathbb{C}[\mathbb{Z}]_{\mathbb{D}} & \subset H^1(\mathbb{D}) \subset H_c(\mathbb{D}) \subset H^\infty(\mathbb{D}) \subset H^2(\mathbb{D}) \subset \cdots \\
\end{align*}
\]
Since $S_1$ is isomorphic to $\mathbb{N}$, therefore $\sum_{n=0}^{\infty} f(X_0^n)\delta_n$ belongs to $\mathcal{L}(\mathbb{N})$ and $\mathcal{F}(\sum_{n=0}^{\infty} f(X_0^n)\delta_n)$ is in $H^\infty(\mathbb{D})$. It follows that $\mathcal{F}(\sum_{n=0}^{\infty} f(X_0^n)\delta_n)$ belongs to $L^\infty(S^1)$. Thus $\sum_{n=0}^{\infty} f(X_0^n)\delta_n$ is a bounded operator on $l^2(\mathbb{Z})$ and belongs to $\mathcal{L}(\mathbb{Z})$. Since the subgroup $\langle X_0 \rangle$ of Thompson’s group $\mathcal{F}$ is isomorphic to $\mathbb{Z}$, we have that $\sum_{n=0}^{\infty} f(X_0^n)X_0^n$ is bounded on $l^2(\mathcal{F})$. Thus $\sum_{n=0}^{\infty} f(X_0^n)X_0^n$ is in $L_0(S)$. Then $E$ is well-defined. It is clear that conditions (i) and (ii) in Definition 5.5 hold for $E$. We complete the proof.

5.4. Proof of main results. In this subsection, we shall show the main conclusions (Theorems 5.8 and 5.10) of this paper. When the $\mathfrak{B}(S)$-bimodule is $\mathcal{L}(S)$, the first continuous cohomology group of $\mathfrak{B}(S)$ vanishes.

**Theorem 5.8.** $H^1(\mathfrak{B}(S), \mathcal{L}(S)) = 0$.

The following result is an immediate corollary of Theorems 5.1 and 5.8.

**Corollary 5.9.** Derivations on the Banach algebra $\mathfrak{B}(S)$ are spatial and induced by linear operators in $\mathcal{L}(S)$.

Let $\mathcal{M}(S)$ be the subset of $\mathcal{L}(S)$ such that $[T, A] \in \mathfrak{B}(S)$ for each $T \in \mathcal{M}(S)$ and $A \in \mathfrak{B}(S)$. Then $\mathcal{M}(S)$ is a norm closed linear subspace of $\mathcal{L}(S)$ containing $\mathfrak{B}(S)$. Then the first cohomology group of $\mathfrak{B}(S)$ is characterized as the linear space $\mathcal{M}(S)$ module $\mathfrak{B}(S)$.

**Theorem 5.10.** $H^1(\mathfrak{B}(S), \mathfrak{B}(S)) = \mathcal{M}(S)/\mathfrak{B}(S)$.

The following two lemmas are crucial to obtain the above results.

**Lemma 5.11.** Let $D$ be a continuous derivation from the Banach algebra $\mathfrak{B}(S)$ into $\mathcal{L}(S)$. If $D(X_0) = AX_0 - X_0A$ and $D(X_1) = AX_1 - X_1A$ for some $A$ in $B(l^2(S))$, then $D(T) = AT - TA$ for each $T$ in $\mathfrak{B}(S)$.

**Proof.** For each $n \geq 1$, if $D(X_0^n) = AX_0^n - X_0^nA$, then

$$D(X_0^{n+1}) = X_0D(X_0^n) + D(X_0)X_0^n = AX_0^{n+1} - X_0^{n+1}A.$$ 

Therefore, by induction we have $D(X_0^n) = AX_0^n - X_0^nA$ for any $n \geq 1$. By the definition of Thompson’s semigroup, we have $X_1X_0^m = X_0^mX_{m+1}$ for any $m \geq 1$. Then

$$D(X_1)X_0^m + X_1D(X_0^m) = D(X_0^m)X_{m+1} + X_0^mD(X_{m+1}).$$

By computation, we have $D(X_{m+1}) = AX_{m+1} - X_{m+1}A$. Similarly, it can be proved that $D(X) = AX - XA$ for any $X$ in $S$. By linearity, we have $D(T) = AT - TA$ for each $T$ in the semigroup algebra $C\!\!\!\!S$. Since $C\!\!\!\!S$ is a dense subalgebra of $\mathcal{B}(S)$, we obtain that $D(T) = AT - TA$ for each $T$ in $\mathfrak{B}(S)$ by the continuity of $D$. 

The following lemma is a generalization of the above conclusion.

**Lemma 5.12.** Let $D$ be a continuous derivation from the Banach algebra $\mathfrak{B}(S)$ into $\mathcal{L}(S)$. If $D(X_0) = AX_0 - X_0A$ and $D(X_1) = BX_1 - X_1B$ for some $A$ and $B$ in $\mathcal{L}(S)$, then there exists an operator $C$ in $\mathcal{L}(S)$ such that $D(T) = CT - TC$ for each $T$ in $\mathfrak{B}(S)$. 

Proof. By the definitions of Thompson’s semigroup and derivations, we have

\[ X_1X_0 = X_0X_2, \ X_2X_0 = X_0X_3, \ X_2X_1 = X_1X_3 \]

and

\[
\begin{align*}
D(X_1)X_0 + X_1D(X_0) &= D(X_0)X_2 + X_0D(X_2), \\
D(X_2)X_0 + X_2D(X_0) &= D(X_0)X_3 + X_0D(X_3), \\
D(X_2)X_1 + X_2D(X_1) &= D(X_1)X_3 + X_1D(X_3).
\end{align*}
\]

By the above equations, we have

\[
D(X_2) = X_0^*X_1(A - B)X_0 + X_0^*(B - A)X_1X_0 + (AX_2 - X_2A),
\]

\[
D(X_3) = (X_0^*)^2X_1(A - B)X_0^2 + (B - A)X_1X_0 + (AX_3 - X_3A),
\]

\[
D(X_3) = X_1^*X_0^*X_1(A - B)X_0X_1 + X_1^*X_0^*(B - A)X_1X_0X_1 + X_1^*(A - B)X_2X_1 + X_1^*X_2(B - A)X_1 + (BX_3 - X_3B).
\]

It follows that

\[
(X_0^*)^2X_1(A - B)X_0^2 + (X_0^*)^2(B - A)X_1X_0^2 + (A - B)X_3 + X_3(B - A) =
\]

\[
X_1^*X_0^*X_1(A - B)X_0X_1 + X_1^*X_0^*(B - A)X_1X_0X_1 + X_1^*(A - B)X_2X_1 + X_1^*X_2(B - A)X_1.
\]

Let \( A - B = L_f \in \mathcal{L}(S) \) and \( L_g = L_f - f(e)I - \sum_{n \geq 0} \sum_{m \geq 1} f(X_n^m)X_n^m \). Then we have

\[
(X_0^*)^2X_1L_gX_0^2 - (X_0^*)^2L_gX_1X_0^2 + L_gX_3 - X_3L_g = X_1^*X_0^*X_1L_gX_0X_1 - X_1^*L_gX_1X_0X_1
\]

\[
+ X_1^*L_gX_2X_1 - X_1^*X_2L_gX_1 + \left( \sum_{n \geq 2} \sum_{m \geq 1} f(X_n^m)X_{n+1}^m \right) X_3 - X_3 \left( \sum_{n \geq 2} \sum_{m \geq 1} f(X_n^m)X_{n+1}^m \right) X_3
\]

\[
+ X_3 \left( \sum_{n \geq 2} \sum_{m \geq 1} f(X_n^m)X_n^m \right) - \left( \sum_{n \geq 2} \sum_{m \geq 1} f(X_n^m)X_n^m \right) X_3.
\] (9)

If \( g \neq 0 \), then take the minimal element \( X \) of the set \{ \( X \in S \mid g(X) \neq 0 \} \) under the total order. By the definition of \( L_g \), we have \( X = X_{i_1}^{\alpha_1} \cdots X_{i_t}^{\alpha_t} \), where \( i_1 < i_2 < \cdots < i_t \), \( t \geq 2 \), and \( \alpha_1, \ldots, \alpha_t \geq 1 \).

**Case I:** \( i_1 \geq 3 \). We have \( X_3X \prec XX_3 \). Taking \( \langle \cdot, \delta_e, \delta_{XX_3} \rangle \) at the both sides of equation (9), we obtain that \( g(X) = 0 \).

**Case II:** \( i_1 = 1 \) or \( 2 \). We have \( XX_3 \prec X_3X \). Analogously, taking \( \langle \cdot, \delta_e, \delta_{XX_3} \rangle \) at the both sides of equation (9), we also have \( g(X) = 0 \).

**Case III:** \( i_1 = 0 \). If \( X_1XX_2X_1 \), then taking \( \langle \cdot, \delta_e, \delta_{XX_2X_1} \rangle \), we can obtain that \( g(X) = 0 \). If \( X_1 \nmid XX_2X_1 \), by the normal form of Thompson’s group \( F \) there exist \( Y \) and \( X_k \) in \( S \) such that \( X_{i_1}^{-1}XX_2X_1 = YX_k^{-1} \), where \( X_{i_1}^{-1} \) and \( X_k^{-1} \) are in \( F \). Then taking \( \langle \cdot, \delta_{X_k}, \delta_Y \rangle \), we can also obtain that \( g(X) = 0 \).
It follows from the above discussion that $g = 0$. This leads to
\[
\left(\sum_{n \geq 2} \sum_{m \geq 1} f(X_n^m)X_{n+1}^m\right) X_3 - X_3 \left(\sum_{n \geq 2} \sum_{m \geq 1} f(X_n^m)X_{n+1}^m\right) \\
+ X_3 \left(\sum_{n \geq 2} \sum_{m \geq 1} f(X_n^m)X_n^m\right) - \left(\sum_{n \geq 2} \sum_{m \geq 1} f(X_n^m)X_n^m\right) X_3
= 0.
\]

It is not difficult to verify that $f(X_2^m) = 0$ and $f(X_n^m) = f(X_3^m)$ for all $n \geq 4$ and $m \geq 1$. Since $f \in L^2(S)$, we have $f(X_n^m) = 0$ for any $n \geq 2$ and $m \geq 1$. Therefore, $A - B = \sum_{m=1}^{\infty} f(X_0^m)X_0^m + \sum_{m=1}^{\infty} f(X_1^m)X_1^m + f(e)I$. Let
\[
C = A - \sum_{m=1}^{\infty} f(X_0^m)X_0^m = B + \sum_{m=1}^{\infty} f(X_1^m)X_1^m + f(e)I.
\]

Then by Theorem 5.7 we have $C \in L(S)$ and
\[
D(X_0) = CX_0 - X_0C, \quad D(X_1) = CX_1 - X_1C.
\]

By Lemma 5.11 we have $D(T) = CT - TC$ for each $T$ in $\mathfrak{B}(S)$.

**Proof of Theorem 5.8.** Let $D$ be a continuous derivation from $\mathfrak{B}(S)$ into $L(S)$. By Lemmas 5.1, 5.2 and 5.12, there exists a $C$ in $L(S)$ such that $D(T) = CT - TC$ for each $T$ in $\mathfrak{B}(S)$. Thus $D$ is an inner derivation and $H^1(\mathfrak{B}(S), L(S)) = 0$. We complete the proof.

**Proof of Theorem 5.10.** Let $D$ be a derivation on the Banach algebra $\mathfrak{B}(S)$, then $D$ is continuous. By Lemmas 5.1, 5.2 and 5.12 there exists an operator $C$ in $L(S)$ such that $D(T) = CT - TC$ for each $T$ in $\mathfrak{B}(S)$. This induces the following map:
\[
\pi : \quad H^1(\mathfrak{B}(S), \mathfrak{B}(S)) \quad \mapsto \quad \frac{\mathcal{M}(S)}{\mathfrak{B}(S)} \quad \mapsto \quad C + \mathfrak{B}(S).
\]

We shall show that $\pi$ is well-defined. If there exist two operators $C_1$ and $C_2$ in $L(S)$ such that $D(T) = C_1T - TC_1 = C_2T - TC_2$ for each $T$ in $\mathfrak{B}(S)$, then $C_1 - C_2 \in L(S) \cap \mathfrak{B}(S)' = L(S) \cap R(S) = CI$. Thus $C_1 + \mathfrak{B}(S) = C_2 + \mathfrak{B}(S)$. Now if $\overline{D_1} = \overline{D_2}$, then $D_1 - D_2$ is an inner derivation of $\mathfrak{B}(S)$. There exists an operator $C_3$ in $\mathfrak{B}(S)$ such that $(D_1 - D_2)(T) = C_3T - TC_3$ for each $T$ in $\mathfrak{B}(S)$. Assume that $D_1(T) = C_1'T - TC_1'$ and $D_2(T) = C_2'T - TC_2'$, where $C_1'$ and $C_2'$ are in $L(S)$, then $(C_1' - C_2')T - T(C_1' - C_2') = C_3T - TC_3$. It follows that $C_1' - C_2'$ belongs to $L(S) \cap \mathfrak{B}(S)' = CI$. Thus $C_1' - C_2'$ belongs to $\mathfrak{B}(S)$, that is $C_1' + \mathfrak{B}(S) = C_2' + \mathfrak{B}(S)$. The map $\pi$ is a well-defined and is a group homomorphism. If $\overline{\pi(D)} = 0$, then there exists an operator $C'$ in $\mathfrak{B}(S)$ such that $D(T) = C'T - TC'$, which means that $D$ is an inner derivation. The map $\pi$ is injective. The surjectivity of $\pi$ is obvious. It follows from the above discussion that $\pi$ is a group isomorphism. We complete the proof.
6. Further discussions

We now recall the definition of higher order continuous Hochschild cohomology for Banach algebras. Let \( \mathcal{M} \) be a Banach algebra and \( \mathcal{X} \) be a Banach \( \mathcal{M} \)-bimodule. The space of all \( n \)-linear (continuous) maps from \( n \)-fold Cartesian product \( \mathcal{M}^n = \mathcal{M} \times \cdots \times \mathcal{M} \) into \( \mathcal{X} \) is denoted by \( L^n(\mathcal{M}, \mathcal{X}) \) for \( n \geq 1 \), while \( L^0(\mathcal{M}, \mathcal{X}) \) is defined to be \( \mathcal{X} \).

The coboundary operator \( \partial^n : L^n(\mathcal{M}, \mathcal{X}) \to L^{n+1}(\mathcal{M}, \mathcal{X}) \) is defined, for \( n \geq 1 \), by

\[
\partial^n \phi(a_1, a_2, \ldots, a_{n+1}) = a_1 \phi(a_2, \ldots, a_{n+1}) + \sum_{i=1}^{n} (-1)^i \phi(a_1, \ldots, a_{i-1}, a_i a_{i+1}, a_{i+2}, \ldots, a_{n+1}) + (-1)^{n+1} \phi(a_1, \ldots, a_n)a_{n+1}
\]

where \( \phi \in L^n(\mathcal{M}, \mathcal{X}) \) and \( a_1, a_2, \ldots, a_{n+1} \in \mathcal{M} \). When \( n = 0 \), we define \( \partial^0 \) by

\[
\partial^0 x(m) = mx - xm \quad (x \in \mathcal{X}, \ m \in \mathcal{M}).
\]

It is routine to check that \( \partial^n \partial^{n-1} : L^{n-1}(\mathcal{M}, \mathcal{X}) \to L^{n+1}(\mathcal{M}, \mathcal{X}) \) is zero for all \( n \geq 1 \), and so \( \text{Im}(\partial^{n-1} : L^{n-1}(\mathcal{M}, \mathcal{X}) \to L^n(\mathcal{M}, \mathcal{X})) \) is a linear subspace of \( \text{Ker}(\partial^n : L^n(\mathcal{M}, \mathcal{X}) \to L^{n+1}(\mathcal{M}, \mathcal{X})) \). The \( n \)th Hochschild cohomology group \( H^n(\mathcal{M}, \mathcal{X}) \) is then defined to be the following quotient space

\[
\frac{\text{Ker}(\partial^n : L^n(\mathcal{M}, \mathcal{X}) \to L^{n+1}(\mathcal{M}, \mathcal{X}))}{\text{Im}(\partial^{n-1} : L^{n-1}(\mathcal{M}, \mathcal{X}) \to L^n(\mathcal{M}, \mathcal{X}))}
\]

for \( n \geq 1 \). We end this paper by proposing some problems for future study:

- What are the higher order cohomology groups \( H^n(\mathcal{B}(S), \mathcal{B}(S)) \) for \( n \geq 2 \)?
- When \( n \geq 2 \), does \( H^n(\mathcal{B}(S), \mathcal{L}(S)) = 0 \)? The first step to calculate the high order cohomology groups should be the following. Given a 2-cocyle \( \phi \), we need to modify it by a 1-coboundary such that \( \phi \) is \( X_0 \)-multimodular, i.e., \( \phi(X_0 A, B) = X_0 \phi(A, B) \), \( \phi(A X_0, B) = \phi(A, X_0 B) \), and \( \phi(A, B X_0) = \phi(A, B) X_0 \) for all \( A, B \in \mathcal{B}(S) \).
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