A no-ghost theorem for the bosonic Nappi-Witten string

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Abstract

We prove a no-ghost theorem for a bosonic string propagating in Nappi-Witten spacetime. This is achieved in two steps. We first demonstrate unitarity for a class of \(N\!W\!/U(1)\) modules: the norm of any state which is primary with respect to a chosen timelike \(U(1)\) is non-negative. We then show that physical states—states satisfying the Virasoro constraints—in a class of modules of an affinisation of the Nappi-Witten algebra are contained in the \(N\!W\!/U(1)\) modules. Similar to the case of strings on \(AdS_3\), in order to saturate the spectrum obtained in light-cone quantization we are led to include modules with energy not bounded from below, which are related to modules with energy bounded from below by spectral flow automorphisms.
I. INTRODUCTION

For a string propagating in a target space-time $M$ with metric of Lorentzian signature, unitarity of the space of physical states has been shown only for relatively few choices of $M$ beyond the flat case and $\text{dim}(M) > 2$. The first non-trivial case to be investigated was $M = SL(2, \mathbb{R})$, where the string in conformal gauge is described by the WZW model on the group $SL(2, \mathbb{R})$. The analysis of the $SL(2, \mathbb{R})$ WZW model has so far spanned two decades. And although the structure of the conformal field theory is still not completely understood, the question of unitarity of the corresponding string is by now settled [1, 2, 3, 4, 5]. Beyond the $SL(2, \mathbb{R})$-string there are some scattered results in the literature, such as a class of non-compact coset models [6, 7, 8].

We add another example to this list by considering a string on the Nappi-Witten group, $NW$, a non-semisimple group with dimension 4. It was first shown in [9] that although the group is not semisimple, there exist a non-degenerate invariant bilinear form on its Lie algebra $\mathfrak{nw}$ (in fact there is, up to an overall normalisation, a one-parameter family of such forms), and there is a WZW model defined on it. An underlying assumption is that the space of states of the WZW model can be fully decomposed into certain types of representations of a centrally extended loop algebra $\hat{\mathfrak{nw}}$ of $\mathfrak{nw}$, namely prolongation modules of unitary simple modules, and modules related to these by spectral flow automorphisms of $\hat{\mathfrak{nw}}$. We show that the space of physical string states is unitary for a certain subset of these modules. The allowed representations include subsets of the unitary representations of the highest and lowest weight, all representations of the continuous series, the trivial representation, and all representations obtained from these by spectral flow. The proof of the no-ghost theorem follows closely the proof for the $SL(2, \mathbb{R})$ string, and there are many similarities between the two theories.

In a broader context of interest the Nappi-Witten model appears as an example of an exactly solvable string theory on the maximally symmetric backgrounds of plane-polarized gravitational waves [11, 12, 13, 14]. Classically the solvability comes with the realization that such models become free in the light-cone gauge. Nappi-Witten
model, on the other hand, has the added merit of being exactly solvable at the quantum level as a WZW model \([9]\). In fact interactions of closed strings in such a background has been analyzed with in a covariant manner \([10]\). In particular correlation functions of an arbitrary number of scattering closed strings are given, utilizing a set of free-field realization of the algebra introduced also in the same paper. Nappi-Witten space being the closest analogue of four-dimensional Minkowski spacetime yet encompassing the three-form gauge field is sure to provide more interesting playgrounds for connecting string theory with observations.

The paper is organized as follows. In section \(\text{III}\) we show that the states in the relevant \(\nabla_\infty\)-representations which are primary with respect to a chosen timelike \(\hat{u}(1)\) (i.e., states in a certain \(\text{NW}/U(1)\) state space) have non-negative norm. This section follows the main idea of the corresponding procedure for \(SL(2,\mathbb{R})\) presented in \([15]\), but requires more work due to the non-semisimplicity of \(\nabla_\infty\). Section \(\text{III}\) contains the proof that physical states, i.e. states satisfying the Virasoro constraints, are contained in the \(\text{NW}/U(1)\) state space, assuming that we complete our theory with an arbitrary unitary CFT to get total central charge 26. This section follows closely \([2]\) for the un-flowed representations, and the proof goes through with minor modifications, while the extension of the proof to the spectral flowed representations becomes relatively straightforward and is very similar to the analogous proof for \(SL(2,\mathbb{R})\) \([4]\).

\section{Unitarity of \(\text{NW}/U(1)\) Modules}

When describing a critical bosonic string in terms of a CFT, the physical string states are the states in the CFT Hilbert space satisfying the Virasoro constraints:

\begin{align}
L_n|\psi\rangle &= 0 \quad \text{for } n > 0 \\
(L_0 - 1)|\psi\rangle &= 0 \quad \text{(on-shell condition)}.
\end{align}

For target manifolds with metric of Lorentzian signature the CFT will generically be non-unitary. And therefore one must show that the space of physical states is unitary,
i.e. contains no states with negative norm. The no-ghost theorem for flat (Minkowski) space was originally proved in the “old covariant quantization” scheme by Goddard and Thorn [16], see also extended discussion in [17]. For a group manifold $G$, more precisely for a string based on a WZW model on a non-compact group $G$ with either exactly one compact or exactly one non-compact generator, the proof has two parts. Part I involves showing that states in the coset $G/U(1)$ (i.e. states in the gauged $G/U(1)$ WZW model) have non-negative norm if the $U(1) \subset G$ subgroup is chosen to span a timelike curve in $G$.

Recall that the space of states in a gauged $G/H$ WZW model is the subspace of the space of states of the $G$ WZW model spanned by highest weight states of $H$. In a $G/U(1)$ model we are thus considering vectors in representations of $\widehat{\mathfrak{g}}_k$, $\mathfrak{g} = \text{Lie}(G)$, that are annihilated by all positive modes $J_n$, $n > 0$ of the $U(1)$ current $J$. One can only expect unitarity of the $G/U(1)$ state space if $G$ has at most one timelike direction. For the case of $G = SO(2,1)$ this was first established in [15], using in particular a certain completeness relation. More precisely one uses the fact that a projection down to $U(1)$ primaries and descendents is the identity. In the Nappi-Witten group we first show this equality in detail, finding an explicit expression for an arbitrary state as a $U(1)$ descendent. Part II of the proof is more formal, and one can partly use results from [16]. This subsection is devoted to Part I and we leave Part II to the next section.

A. Current algebra in Nappi-Witten space

The Nappi-Witten group $NW$ is four dimensional group obtained by centrally extending the two-dimensional Poincaré Group. Its Lie algebra $\mathfrak{nw}$ can be expressed in terms of the anti-hermitian generators, $J^+, J^-, J, T$, with commutation relations:

$$\begin{align*}
[J^+, J^-] &= 2i \, T, \\
[J, J^+] &= i \, J^+, \\
[J, J^-] &= -i \, J^-, \\
[T, Q^\alpha] &= 0,
\end{align*}$$

(3)

where we use $Q^\alpha$ to denote an arbitrary generator. One important aspect of $\mathfrak{nw}$ is that it is not semisimple [26] and consequently admits (up to normalization) two independent
invariant bilinear forms which we call $\Omega$ and $\kappa$:

$$
\Omega = \begin{pmatrix}
0 & 2 & 0 & 0 \\
2 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
\end{pmatrix}, \quad \kappa = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -2 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}.
$$

We then obtain a family of non-degenerate invariant bilinear forms, $\Omega_b = \Omega - \frac{b}{2} \kappa$. It is easy to see that $J + \alpha T$ is a timelike vector w.r.t. $\Omega_b$ when $b + 2\alpha < 0$. As a result, we should prove that the highest weight states of some choice of $U(1)$ current algebra generated by $J + \alpha T$ such that $b + 2\alpha < 0$, i.e. states $|\Psi\rangle$ such that $(J_n + \alpha T_n)|\Psi\rangle = 0$ when $n > 0$, have non-negative norm.

There are four types of unitary representations, $V_{p^+, p^-}^+, V_{p^-, p^-}^-, V_{0^+, p^-}^0, V_{0^-, p^-}^0$, classified by the eigenvalues of the central generator $T = ip^+$ and the quadratic Casimir $C^{(2)}$ (These are also nicely summarized in Appendix B1 of [10]):

- The representations $V_{p^+, p^-}^+$ are highest weight representations (by which we here mean representations generated from an eigenvector of $T$ and $J$, annihilated by $J^+$) with highest weight vector $|p^+, p^-\rangle$ such that $T|p^+, p^-\rangle = ip^+|p^+, p^-\rangle$ and $J|p^+, p^-\rangle = ip^-|p^+, p^-\rangle$. They are unitary for $p^+ > 0$, $p^- \in \mathbb{R}$, and the quadratic Casimir has the eigenvalue $C^{(2)} = -2p^+(p^- + \frac{1}{2})$. Since this class of representations will play the main role in the rest of the paper we provide a summary of their structure.

We obtain bases $\{|r\rangle\}_{r \in \mathbb{N}_0}$ of the vector spaces $V_{p^+, p^-}^+$ by acting with $J^-$ on the vacua $|0\rangle := |p^+, p^-\rangle$ for fixed $p^+, p^-$, which are annihilated by $J^+$.

$$
|r\rangle = (-iJ^-)^r|0\rangle
$$

(5)
The generators act on the representation as

\[ T|r⟩ = i|p^+⟩ - r⟩ \]
\[ J^+|r⟩ = 2i|p^+⟩ - r⟩ - 1⟩ \]
\[ J^-|r⟩ = i|r⟩ + 1⟩ \]
\[ J|r⟩ = i(p^- - r)|r⟩. \]

There is an inner product on \( V^{p^+, p^-} \) that takes the following form on the basis elements

\[ \langle s|r⟩ = (2p^+)^r!\delta_{r,s}. \]

* There are also lowest weight representations, \( V_{p^+, p^-} \) generated from a lowest weight vector (annihilated by \( J^- \)) \( |p^+, p^-⟩ \). These are unitary for \( p^+ < 0, p^- \in \mathbb{R} \). It can be shown [10] that there exists a non-degenerate invariant pairing between \( V^{p^+, p^-}_+ \) and \( V^{p^+, p^-}_- \), so the latter is equivalent to the contragredient of the former. \( C^{(2)} \) takes the value \(-2p^+(p^- - \frac{1}{2})\) on \( V^{p^+, p^-}_- \).

* The representations \( V^{α, p^-}_α \) have neither highest nor lowest weight vectors, and are characterized by \( p^- \in [0, 1) \) and \( α \in \mathbb{R} \) such that not both are zero simultaneously, where the quadratic casimir takes the value \( C^{(2)} = -α^2 \). In analogy with \( SL(2, \mathbb{R}) \) we will refer to this class of representations as the continuous series.

* Finally, the representation \( V^{0, 0} \) is the trivial representation.

The Nappi-Witten currents obey the operator product relations

\[ J^+(z)J^-(w) \sim \frac{2iT(w)}{z - w} + \frac{k}{(z - w)^2}, \]
\[ J(z)J+(w) \sim \frac{iJ^+(w)}{z - w}, \]
\[ J(z)J^- (w) \sim \frac{-iJ^-(w)}{z - w}, \]
\[ J(z)T(w) \sim \frac{\frac{1}{2}k}{(z - w)^2}, \]
\[ J(z)J(w) \sim \frac{\frac{1}{2}kb}{(z - w)^2}. \]
where we assume the central element $k \in \mathbb{R}$. By standard manipulations, we can convert them into commutation relations for the Laurent modes

\begin{align}
[J^+_m, J^-_n] &= 2iT_{m+n} + km\delta_{m+n}, \quad (16) \\
[J_m, J^+_n] &= iJ^+_{m+n}, \quad (17) \\
[J_m, J^-_n] &= -iJ^-_{m+n}, \quad (18) \\
[J_m, T_n] &= \frac{k}{2}m\delta_{m+n}, \quad (19) \\
[J_m, J_n] &= \frac{kb}{2}m\delta_{m+n}. \quad (20)
\end{align}

The OPE of the energy-momentum tensor $\mathcal{E}(z)$ with a primary operator $Q(w)$ of conformal dimension one is

$$Q(z)\mathcal{E}(w) \sim \frac{Q(w)}{(z-w)^2}, \quad (21)$$

so demanding that the currents are Virasoro primaries of conformal dimension one restricts the “Sugawara form” of the energy-momentum tensor to

$$\mathcal{E}(z) = \frac{1}{k} \left( JT(z) + TJ(z) + \frac{J^+J^-(z) + J^-J^+(z)}{2} + \left( \frac{2}{k} - b \right) TT(z) \right). \quad (22)$$

The commutation relations of Virasoro generators with current modes are

$$[L_m, Q^\alpha_n] = -nQ^\alpha_{m+n}. \quad (23)$$

A standard calculation confirms that the central charge takes the value $c = 4$, independently of the values of $k$ and $b$. It is straightforward to check that one can set the parameter $b$ in $\Omega_b$ to any real value by an automorphism of $\mathfrak{nw}$, and we thus expect that different values of $b$ correspond to equivalent CFT’s. See Remark 1 at the end of section II C. For simplicity we will work with the case $b = 2/k$, such that the energy-momentum tensor has no term $TT(z)$. For the rest of the paper the name $\widehat{\mathfrak{nw}}$ denotes solely the Lie algebra given in (16)–(20) with $b = 2/k$. 

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B. Constraints from unitarity at low excitations

The representations that we consider as candidates for the space of states of the Nappi-Witten WZW model are the prolongation modules \( \hat{\mathcal{V}}_{p^+,p^-} \), \( \hat{\mathcal{V}}_{p^+,p^-} \) and \( \hat{\mathcal{V}}_{0,p^-} \) of \( V_{p^+,p^-} \), \( V_{p^+,p^-} \) and \( V_{0,p^-} \), respectively (plus the spectral flow of these representations, see section III). These are the \( \hat{\mathfrak{w}} \)-modules obtained by the requirement that any vector in the (horizontal) \( \mathfrak{w} \)-module is annihilated by the positive modes of the currents, \( J^+_n, J^-_n, T_n, n > 0 \). Note that, similar to the \( SL(2,\mathbb{R}) \)-case, the prolongation modules constructed from \( V_{p^+,p^-} \) and \( V_{p^+,p^-} \) for \( p^+ \neq 0 \) (i.e. the highest resp. lowest weight modules) are in fact equivalent to the Verma modules of \( \hat{\mathfrak{w}} \) with highest resp. lowest weights \( (p^+, p^-) \).

Some interesting observations follow by analyzing the Virasoro condition \( (L_0 - 1)|\psi\rangle = 0 \). Acting with \( L_0 \) on a state \( |\psi\rangle \) at level (mode number) \( N \) in \( \hat{\mathcal{V}}_{p^+,p^-} \) gives

\[
L_0|\psi\rangle = \left( \frac{-2p^+(p^- + \frac{1}{2})}{k} + N \right)|\psi\rangle
\]  

(24)

Applying the Virasoro condition \( (L_0 - 1)|\psi\rangle = 0 \), we see that, since the only requirement on \( p^- \) is that it is real, this imposes no obvious constraint on the allowed representations. An analogous conclusion follows from looking at a lowest weight representation. If instead \( |\psi\rangle \in \hat{\mathcal{V}}_{0,p^-} \), the same Virasoro condition reads

\[
\left( \frac{-\alpha^2}{k} + N - 1 \right)|\psi\rangle = 0.
\]

(25)

If \( k > 0 \) we may potentially find for any \( N \in \mathbb{N} \) a representation with a physical state at that level. If \( k < 0 \), however, the only representations where excited states may be physical are those where \( \alpha = 0 \). Furthermore, these representations only allow \( N = 1 \). It is easily seen that as long as \( p^- \neq 0 \) the only states in such a representation that satisfy the constraints \( L_n|\psi\rangle = 0, n > 0 \), are the states \( T_{-1}|\chi\rangle \) where \( |\chi\rangle \in V_{0,p^-} \). It is equally easy to see that all of these states have zero norm. If \( \alpha \neq 0 \) then only vectors in the horizontal representation may satisfy the Virasoro conditions, and these have positive norm. Note that these conclusions still hold if we include a contribution to \( L_0 \) from another unitary
CFT since such contributions are necessarily positive. To conclude, we have shown that if the level $k$ is negative, all physical states in the modules $\hat{V}_{\alpha}^{p-}$ have non-negative norm.

Next, consider the norms of some of the first excited string states in $\hat{V}_{\perp}^{p+,p-}$.

$$
\| -iJ_{-1}^{-1}|r\rangle\|^2 = -\langle r| [J_1^{+}, J_{-1}^{-}] + J_{-1}^{-1} J_1^{+} |r\rangle = (2p^{+} - k)\langle r|r\rangle \quad (26)
$$

$$
\| -iJ_{-1}^{+}|r\rangle\|^2 = -\langle r| [J_1^{-}, J_{-1}^{+}] + J_{-1}^{+} J_1^{-} |r\rangle = (-2p^{+} - k)\langle r|r\rangle \quad (27)
$$

$$
\| -i(J_{-1} - T_{-1})|r\rangle\|^2 = -\langle r| J_1 J_{-1} - T_1 J_{-1} + J_1 T_{-1} |r\rangle = (-k - 1)\langle r|r\rangle \quad (28)
$$

$$
\| -i(J_{-1} + T_{-1})|r\rangle\|^2 = -\langle r| J_1 J_{-1} - T_1 J_{-1} - J_1 T_{-1} |r\rangle = (k - 1)\langle r|r\rangle \quad (29)
$$

The first observation is that there is no possible value of $k$ such that none of these states have negative norm. Since $p^{+} > 0$, the first two states can have non-negative norm only if $k < 0$ and $p^{+} \leq -\frac{k}{2}$. In fact these conditions are necessary since the state $J_{-1}^{+}(p^+, p^-)$ is physical for suitable values of $p^-$. It then follows that the last two states both have negative norm if $k \in (0, -1)$, and if $k \leq -1$ only the last state has negative norm. It is then natural to guess that we should also require $k \leq -1$. Let us, however, investigate this a bit further, and temporarily let the parameter $b$ be arbitrary. Consider the combination $R_{-1} = J_{-1} + \alpha T_{-1}$ for some $\alpha \in \mathbb{R}$, we then have

$$
\| R_{-1}|r\rangle\|^2 = \frac{k}{2}(b + 2\alpha)\langle r|r\rangle.
$$

It follows that for any $b \in \mathbb{R}$ there are combinations of the currents $J$ and $T$ producing positive norm states, and other producing negative norm states. The picture in the $J-T$ plane with a metric given by $\Omega_b$ is as follows. For $b = 0$ the lightcone is aligned with the $J$ and $T$ axes. For $b > 0$ the lightcone is compressed, and when $b > 2$ both combinations $J \pm T$ are spacelike. Similarly when $b < 0$ the lightcone is widened, and for $b < -2$ both combinations $J \pm T$ are timelike. For $b = \pm 2$ the lightcone is aligned with the $T$ and $J \mp T$ axes. With our choice $b = 2/k$ we thus get that both combinations $J \pm T$ are timelike if $k \in (-1, 0)$, but there are still other combinations that are spacelike. Demanding
unitarity at the first excited level thus only gives the necessary conditions
\[ p^+ \leq -\frac{k}{2} \]  
\[ k < 0. \]  

C. Unitarity of NW/U(1) states

We will now show that the conditions (30) and (31) are sufficient to ensure the absence of negative norm states in the coset module built from \( \hat{V}_{+}^{p^+p^-} \) or \( \hat{V}_{-}^{p^+p^-} \), i.e. the subspace spanned by states which are primary with respect to the timelike \( \hat{u}(1) \) current algebra generated by the modes \( \{J_n - T_n\}_{n \in \mathbb{Z}} \). For brevity we only consider the case of \( \hat{V}_{+}^{p^+p^-} \), the same result follows with obvious modifications for \( \hat{V}_{-}^{p^+p^-} \).

Let \( |N; r, p^+, p^-\rangle \) be a general state, where \( N \) is the excited string level and \( r \) is defined by \( J_0 |N; r, p^+, p^-\rangle = i(p^ -- r) |N; r, p^+, p^-\rangle \). Such a state obeys
\[ \langle N; r, p^+, p^-|L_0|N; r, p^+, p^-\rangle = (-\frac{2p^+(p^- + \frac{1}{2})}{k} + N)\langle N; r, p^+, p^-|N; r, p^+, p^-\rangle. \]  

Since
\[ L_0 = \frac{1}{k}\left[\frac{1}{2}(J^+_0 J^-_0 + J^-_0 J^+_0) + (J_0 T_0 + T_0 J_0)
+ \sum_{n \geq 1}\left(J^+_n J^-_n + J^-_n J^+_n + 2J^-_n T_n + 2T^-_n J_n\right)\right], \]  
we find that
\[ \langle N; r|L_0|N; r\rangle = \frac{1}{k}\langle N; r|J^-_0 J^+_0 \rangle + \sum_{n \geq 1}\langle J^+_n J^-_n + J^-_n J^+_n + 2J^-_n T_n + 2T^-_n J_n\rangle |N; r\rangle
+ \frac{1}{k}(-p^+ + 2p^+ r - 2p^+ p^-)\langle N; r|N; r\rangle. \]  

From (32) and (34) we get
\[ \langle N; r|N; r\rangle = \frac{\langle N; r|J^-_0 J^+_0 \rangle + \sum_{n \geq 1}\langle J^+_n J^-_n + J^-_n J^+_n + 2J^-_n T_n + 2T^-_n J_n\rangle |N; r\rangle}{Nk - 2p^+ r} \]  

As follows from (30), (31), and \( r \geq -N \), the denominator is manifestly negative for \( k < 0 \). We prove by induction that the numerator is also negative.
The inductive step amounts to showing that the numerator is negative for given values of \( N \) and \( r \) if it is negative for all excitation levels less than \( N \) and \( J_0 \) eigenvalues smaller than \( r \) (note that \( r \) is bounded from below by \(-N\)). For \( N = 0 \) all states are \( U(1) \)-primaries, and those states form a unitary \( \mathfrak{m} \mathfrak{w} \)-module. The only state with \( r = -N \) is \((J^-)^N |0; p^+, p^-\rangle\), and it is easily checked that this state is a \( U(1) \) primary, and that it has positive norm. The proof of the inductive step follows the procedure described in [15], where it was applied to \( SL(2, \mathbb{R}) \). The idea is to express the norm of a state of the form \( J_n^0 |N; r\rangle \), \( n > 0 \), in terms of norms of states with lower \( N \) and larger \( r \). In the case of \( SL(2, \mathbb{R}) \) it follows easily since any state is a linear combination of descendents of primaries of the timelike \( \hat{u}(1) \). In our case, since \( \mathfrak{m} \mathfrak{w} \) is not semisimple, this does not follow trivially. As a first step we therefore show explicitly that any state in \( \hat{V}_p^+, p^- \) can itself be written as a combination of \( \hat{u}(1) \)-descendent states. The same statement holds, with a similar proof, for \( \hat{V}_p^+, p^- \).

**Lemma 1** Any state \( |\Psi\rangle \) in \( \hat{V}_p^+, p^- \) can be written

\[
|\Psi\rangle = |h^0\rangle + \sum_{n > 0} (J_n - T_n)|h^1_n\rangle \\
+ \sum_{n_1, n_2 > 0} (J_{n_1} - T_{n_1})(J_{n_2} - T_{n_2})|h^2_{n_1, n_2}\rangle + \ldots 
\]

(36)

where the states \( |h^i_{n_1, \ldots, n_i}\rangle \) are annihilated by any \( J_n - T_n \) for \( n > 0 \), and only finitely many terms on the right hand side are non-zero.

Proof: We first show that the Lemma follows if we can show that any state obtained by acting with one of the operators \( Q_m \in \{ J^+_m, J^-_m, J_m, T_m \} \), where \( m > 0 \), on a \( \hat{u}(1) \)-primary state can be written on the required form. Any state at level \( N = 0 \) is already a \( \hat{u}(1) \) primary state. Any state at level \( N = 1 \) is obtained as a linear combination of states of the form \( Q_{-1}|h\rangle \) where \( |h\rangle \) is a state at level \( N = 0 \). The statement follows by induction since any state at level \( N_0 \) is a linear combination of states of the form \( Q_{-m}|\Phi_N\rangle \), and if we assume that we can write any state at level \( N < N_0 \) on the required
form, then we have

\[ Q_{-m}|\Phi_N\rangle = Q_{-m}|h^0\rangle + \sum_{n>0} Q_{-m}(J_{-n} - T_{-n})|h_n^1\rangle + \ldots \]

Using the commutation relations of \( \hat{w} \) we can re-write this expression as a sum of terms where every term consists of some factors of negative \( \hat{u}(1) \)-modes acting on a state \( Q_{-s}|h\rangle \) for some \( s > 0 \), where \( |h\rangle \) is a \( \hat{u}(1) \)-primary state, and the induction is complete.

We must now show that any state of the form \( Q_{-m}|h\rangle \), where \( |h\rangle \) is a primary state, can be written on the required form. For \( Q = J \) or \( T \) this follows trivially. Note that the combination \( J_{-m} + (1 - \frac{2}{k})T_{-m} \) commutes with any timelike generator \( J_n - T_n \). The decompositions

\[
J_{-m} = \frac{k}{2(k-1)}(J_{-m} + (1 - \frac{2}{k})T_{-m}) + \frac{k-2}{2(k-1)}(J_{-m} - T_{-m}) , \tag{37}
\]

\[
T_{-m} = \frac{k}{2k-2}(J_{-m} + (1 - \frac{2}{k})T_{-m}) - \frac{k}{2k-2}(J_{-m} - T_{-m}) \tag{38}
\]

thus immediately put \( Q_{-m}|h\rangle \) on the required form. It remains to investigate when \( Q = J^\pm \). For simplicity we restrict to \( J^- \), the other case follows analogously. Our strategy is to recursively construct the first term on the right hand side of (36), and the result will be of a form where the Lemma becomes obvious. Consider therefore a state \( J_{-m}|h_N\rangle \) with \( |h_N\rangle \) a \( \hat{u}(1) \) primary state at level \( N \). Note that \( J_{-N}|h_N\rangle \) is either zero or a \( \hat{u}(1) \) primary state, and that the state \( |h_{(1)}\rangle \) defined as

\[
|h_{(1)}\rangle := \left( J_{-N-1} + \frac{2i}{k}T_{-1}J_{-N} \right) |h_N\rangle
\]

is a primary state. Thus, using (38) we see that \( J_{-N-1}|h_N\rangle = |h_{(1)}\rangle - \frac{2i}{k}T_{-1}J_{-N}|h_N\rangle \) has the required form. Similarly, the state \( |h_{(2)}\rangle \) defined by

\[
|h_{(2)}\rangle := \left( J_{-N-2} + \frac{2i}{k}T_{-1}J_{-N-1} + \frac{1}{2}\left( \frac{2i}{k} \right)^2 (T_{-1})^2 J_{-N} \right) + \frac{2i}{2k}T_{-2}J_{-N}|h_N\rangle
\]

is primary, so using (38) and the decomposition of \( J_{-N-1}|h_N\rangle \) we see that \( J_{-N-2}|h_N\rangle \) has the required form. Recursively one can then show that any state \( J_{-m}|h_N\rangle \) can be written
on the required form by constructing a primary state \( |h_{(m+N)}\rangle \) given by the general expression:

\[
|h_{(m+N)}\rangle := \sum_{n_1, n_2, \ldots, n_{m+N} \geq 0} \frac{1}{n_1! \cdots n_{m+N}!} \left( \frac{2i}{k} \right)^{n_1} \left( \frac{2i}{k} \right)^{n_2} \cdots \left( \frac{2i}{(m + N)k} \right)^{n_{m+N}} \times (T_1)^{n_1} \cdots (T_{m-N})^{n_{m+N}} J_{-m+n_1+2n_2+\ldots+(m+N)n_{m+N}} |h_N\rangle.
\] (39)

There are only a finite number of non-zero terms on the right hand side since \( N < \infty \).

The first term in this expression is \( J_{-m} |h_N\rangle \), and using recursively the same expression together with (38) on every other term one obtains the required form for \( J_{-m} |h_N\rangle \). By changing every factor \((2i)^{n_j} \) to \((-2i)^{n_j}\), and \( J_- \) to \( J_+ \) in (39) one obtains the analogous formula for \( J_+ |h_N\rangle \). This completes the proof of the Lemma.

Let \( \mathcal{H}^{p+,p-} \subset \hat{V}_+^{p+,p_-} \) denote the subspace spanned by \( \hat{u}(1) \)-primary states. Choose a linear projector \( P \) on \( \hat{V}_+^{p+,p_-} \) down to \( \mathcal{H}^{p+,p-} \), implying that \( (J_n - T_n)P = 0 \) for all \( n > 0 \). It is clear from Lemma [1] that we can choose \( P \) such that also \( P(J_{-n} - T_{-n}) = 0 \) for all \( n > 0 \), and one can show that this implies \( P^\dagger = P \). Using Lemma [1] it is then not difficult to show the completeness relation

\[
\text{id}_{\hat{V}_+^{p+,p_-}} = P + \sum_{n>0} \frac{1}{(-k-1)n}X_{-n}PX_n
\]

\[
+ \frac{1}{2!} \sum_{n_1, n_2 \geq 0} \frac{1}{(-k-1)n_1} \frac{1}{(-k-1)n_2} X_{-n_1}X_{-n_2}PX_{n_1}X_{n_2}
\]

\[+ \cdots \] (40)

where \( X_n := J_n - T_n \). The idea is now to insert (40) into each term on the right hand side of (35), affording a sum over intermediate states. Consider first the effect of inserting (40) in the middle of the term \( \langle N, r | J_{-p}^+ J_p^- | N, r \rangle \). After commuting the factors of \( (J - T) \) to the left and right, a typical term in the resulting sum is

\[
\langle N, r | \left[ \frac{1}{n!} (k-1)^{-m} \frac{1}{n_1 \cdots n_m} J_{-(p+n_1+\cdots+n_m)}^+ P J_{(p+n_1+\cdots+n_m)}^- \right] | N, r \rangle
\] (41)

We define \( P = p + n_1 + \cdots + n_m \). The properties of \( P \) implies

\[
\langle N, r | J_{-p}^+ P J_p^- | N, r \rangle = -\|PJ_{-p}^- | N, r \rangle \|^2.
\]

13
where $J^+_p|N, r\rangle$ has level $N - P < N$ if $P > 0$. If every primary state of level strictly lower than $N$ has non-negative norm it follows that $\langle N, r|J^+_p P J^-_p|N, r\rangle \leq 0$. The same result follows by inserting (40) in the expression $\langle N, r|J^-_p P J^+_p|N, r\rangle$ for $p > 0$. Inserting the completeness relation in the term $\langle N, r|J^-_0 P J^+_0|N, r\rangle$ gives a sum of terms with the same property as above, plus the term $-\|P J^+_0|N, r\|^2$. The state $J^+_0|N, r\rangle$ also has level $N$, but it has lower $J_0$-eigenvalue, and we can assume that all primaries at level $N$ but with $J_0$-eigenvalue lower than $r$ have non-negative norm it follows that $\langle N, r|J^-_0 P J^+_0|N, r\rangle \leq 0$.

Finally, using (37) and (38) and the notation $Y_n = J_n + (1 - \frac{2}{k})T_n$, it follows that

$$\langle N, r|2T_n J_n + 2J_n T_n|N, r\rangle = \left(1 - \frac{1}{k}\right)^{-1} \langle N, r|Y_n Y_n|N, r\rangle,$$

where $Y_n|N, r\rangle$ is again a $\hat{u}(1)$ primary state. Summing up the terms in (35), we find that the operators between in-state and out-state are of the form

$$\mathcal{O} = \sum_{P \geq 1} \sum_{m \geq 0} \sum_{n_1 \geq 0, \ldots, n_m \geq 0} \frac{1}{m!} (k - 1)^{-m} \frac{1}{n_1 \cdots n_m} J^+_p P J^-_p$$

$$+ \sum_{P \geq 1} \sum_{m \geq 0} \sum_{n_1 \geq 0, \ldots, n_m \geq 0} \frac{1}{m!} (k - 1)^{-m} \frac{1}{n_1 \cdots n_m} J^-_p P J^+_p$$

$$+ \left(1 - \frac{1}{k}\right)^{-1} \sum_{n \geq 1} Y_n Y_n$$

$$= \sum_{P \geq 1} F_P(k) J^+_P P J^-_P + \sum_{P \geq 0} F_{P-1}(k) J^-_P P J^+_P + \left(1 - \frac{1}{k}\right)^{-1} \sum_{n \geq 1} Y_n Y_n,$$

(42)

where $F_P(k) = (1 + \frac{1}{(k-1)}) (1 + \frac{1}{2(k-1)}) \cdots (1 + \frac{1}{P(k-1)})$ for $P > 0$ and $F_0(k) = 1$. In particular $F_P(k) > 0$ for $k < 0$, as well as $(1 - 1/k) > 0$. Since all states at $N = 0$ as well as the $\hat{u}(1)$-primary states with $r = -N$ have non-negative norm, it follows by induction that the numerator of (35) is negative for all $N \in \mathbb{N}_0$ and $r \geq -N$. Hence, we conclude that there are no negative norm states in $\mathcal{H}^{p^+, p^-} \subset \hat{V}^{p^+, p^-}$ when $p^+ \leq -\frac{k}{2}$ and $k < 0$.

The same result holds in the case of $\hat{V}^{p^+, p^-}$ for $\frac{k}{2} \leq p^+$. This completes the first part of the proof of our no-ghost theorem for bosonic strings on the Nappi-Witten group.

**Remark 1** Denote by $\hat{\mathfrak{u}}_b$ the algebra (16)–(20) with an arbitrary $b \in \mathbb{R}$. There is an automorphism $J_0 \mapsto J_0 + \mu T_0$ of the horizontal subalgebra that shifts the value of $b$
according to $b \mapsto b + 2\mu$, which we took as a sign that different values of $b$ correspond to equivalent CFT’s. The argument can be made stronger by considering a change of basis of $\hat{\mathfrak{nw}}_b$ where $J_n$ is replaced with $\tilde{J}_n = J_n + \mu T_n$. This is not an automorphism, but changes only the commutator (20) to

$$[\tilde{J}_m, \tilde{J}_n] = \frac{1}{2} k(b + 2\mu) m \delta_{m+n}.$$ 

In other words, we have shown that $\hat{\mathfrak{nw}}_b \cong \hat{\mathfrak{nw}}_{b+2\mu}$ for any $\mu \in \mathbb{R}$. When discussing highest weight modules we have used the triangular decomposition $\hat{\mathfrak{nw}}_b = \hat{\mathfrak{nw}}_b^- \oplus \hat{\mathfrak{nw}}^0_b \oplus \hat{\mathfrak{nw}}_b^+$ where all positive (negative) modes are in the $+$ ($-$) parts, and in addition $J_0^- \in \hat{\mathfrak{nw}}_b^-$, $J_0^+ \in \hat{\mathfrak{nw}}_b^+$. Note that the change of basis above preserves the triangular decomposition. This implies in particular that the $\hat{\mathfrak{nw}}_{b+2\mu}$-module $\hat{\mathcal{V}}_{p^+,p^-}$ is the same thing as the $\hat{\mathfrak{nw}}_{b+2\mu}$-module $\hat{\mathcal{V}}_{p^+,p^-+\mu p^+}$. Using this fact it is not difficult to check explicitly that for any $b = \frac{2}{k} + 2\mu$ such that the vector $J - T$ is still timelike, the proof of unitarity above still holds. For other values of $b$ one must choose a different $\hat{\mathfrak{u}}(1)$ subalgebra, but this is straightforward. With the same constraint on $b$ one can show that also the results of the next section holds, and we expect similarly that this extends straightforwardly to arbitrary values of $b$.

III. NO-GEST THEOREM FOR NAPPI-WITTEN STRING THEORIES

The idea of the proof of the no-ghost theorem follows [2], see also [3], using also classic results by Goddard and Thorn [16].

Denote by $\hat{\mathcal{V}}_{p^+,p^-}$ the subspace of $\hat{\mathcal{V}}_{p^+,p^-}$ spanned by states of excitation level $N$ or less. Furthermore, let $\mathcal{T} \subset \mathcal{H}_{p^+,p^-}$ be the subspace of $\hat{\mathfrak{u}}(1)$ primary states that are also Virasoro primary, i.e. annihilated by $L_n$ with $n > 0$. Use the notation $X_n := J_n - T_n$ for the generators of the timelike $\hat{\mathfrak{u}}(1)$. For the rest of this section it is important that we complete the Nappi-Witten CFT with a unitary CFT with $c = 22$ such that the total central charge takes the value 26.
Lemma 2  The set of states of the form

$$|\{\lambda, \mu, \psi\} := L^{\lambda_1}_{-1} \cdots L^{\lambda_p}_{-p}X^{\mu_1}_{-1} \cdots X^{\mu_p}_{-p}|\psi\rangle$$  \hspace{1cm} (43)$$

where $|\psi\rangle \in T$ has level $N_\psi \leq N$ and $\sum_{a=1}^{p} a(\lambda_a + \mu_a) + N_\psi = N$ for a fixed $N$, form a basis for $\hat{V}^{p^+,p^-}_{+(N)}$ if $p^+ \in (0, -\frac{k}{2})$.

Proof: We first prove that states of the form (43) are linearly independent, and then proceed by showing that they span $\hat{V}^{p^+,p^-}_{+(N)}$. The field $E^X(z) = \frac{1}{2(1-k)} XX(z)$ defines an energy-momentum tensor for a $U(1)$ ($c = 1$) theory with current $X(z)$. The Laurent modes $L^X_n$ are given by $L^X_n = \frac{1}{2(1-k)} \sum_{m \in \mathbb{Z}} :X_m X_{n-m}:$. The conventional coset construction gives a representation of the Virasoro algebra with central charge $c = 25$ generated by $L^c_n = L_n - L^X_n$ since $L^c_m$ commutes with $X_n$. Since $L^X_n$ are bilinears in modes $X_m$ we might as well switch the generators $L_{-n}$ on the right hand side of (43) to the generators $L^c_{-n}$. Since $L^c_n$ and $X_m$ commute, the states of this form span a highest weight representation of $\text{Vir} \oplus \hat{u}(1)$ with $c = 25$. The $L^c_0$-eigenvalues on $\hat{V}^{p^+,p^-}_{+(N)}$ are of the form $h^c = -\frac{2p^+(p^+-\frac{1}{2})}{k} + \frac{(p^+ + r-p^-)^2}{2(1-k)} + N$ with $N \in \mathbb{N}_0$ and $r \in \mathbb{Z}$ such that $r \geq -N$. We immediately see that if $p^- + \frac{1}{2} \geq 0$ then $h^c \geq 0$ (for $k < 0$). One checks that with the constraints $k < 0$, $p^+ \leq -k/2$, $N \geq 0$, $r \geq -N$, $h^c$ as a function of $p^-$ can only change sign if $p^+ = -k/2$ (and then only if $r = -N$). In other words, for $0 < p^+ < -k/2$ the values of $h^c$ are positive. The Kac determinant of the representation of $\text{Vir} \oplus \hat{u}(1)$ is the product of Kac determinants of the $\hat{u}(1)$ part and the Virasoro part. Since the Kac determinant of the $\hat{u}(1)$ part is always non-zero, and the Virasoro contribution for $c = 25$ vanishes only for $h^c \leq 0$, it follows that states of the form (43) are indeed linearly independent.

Let $N$ be the total level of a state of the form (43), i.e. there are contributions $\sum_{a} a(\lambda_a + \mu_a)$ from the oscillators, and $N_\psi$ from $|\psi\rangle$ adding up to $N$. The states of the form (43) with $N = 0$ obviously span $\hat{V}^{p^+,p^-}_{+(0)}$. Assume that the states at level $N - 1$ or less span $\hat{V}^{p^+,p^-}_{+(N-1)}$. Let $B_{(N)}$ denote the linear span of states of the form (43) of level $N$ or less and where $N_\psi < N$. Since the Kac determinant is non-zero there are no null states in $B_{(N)}$, so $\hat{V}^{p^+,p^-}_{+(N)} \cong B_{(N)} \oplus B_{(N)}^{\perp}$ where $B_{(N)}^{\perp}$ is the orthogonal complement of $B_{(N)}$ in
\( V_{p^+, p^-}^{+(N_s)} \). Pick an arbitrary \(|u\rangle \in B_{(N_s)}^+\). Then for any \(|v_s\rangle \in V_{p^+, p^-}^{+(N-s)}, s > 0\), we have

\[
0 = (|u\rangle, L_{-s}|v_s\rangle) = (L_s|u\rangle, |v_s\rangle).
\]

Since by assumption the inner product is non-degenerate on \( V_{p^+, p^-}^{+(N-s)} \) for \( s > 0 \), we conclude that \( L_s|u\rangle = 0 \) for any \( s \in [1, N] \). We also have trivially that \( L_n|u\rangle = 0 \) for \( n > N \) so we conclude that \( L_n|u\rangle = 0 \) for any \( n > 0 \). One shows similarly that \( X_n|u\rangle = 0 \) for any \( n > 0 \). We have thus shown that \( B_{(N)}^+ = T \cap V_{p^+, p^-}^{+(N)} \), and the lemma follows. □

A spurious state is a linear combination of states of the form (43) with \( \lambda \neq 0 \). Recall the result of Goddard and Thorn [16] that when \( c = 26 \), the space of on-shell spurious states (i.e. spurious states \(|s\rangle\) satisfying \((L_0 - 1)|s\rangle = 0\) is closed under the action of \( L_n \) with \( n > 0 \). It follows that if \(|\psi\rangle = |\phi\rangle + |s\rangle\) is on shell with \(|s\rangle\) spurious and on-shell, the condition \( L_n|\psi\rangle = 0 \) for \( n > 0 \) is equivalent to \( L_n|\phi\rangle = 0 \) and \( L_n|s\rangle = 0 \). Thus a general physical state \(|\psi\rangle\) is a combination of a physical state \(|\chi\rangle\) of the form (43) with \( \lambda = 0 \) and a physical spurious state \(|s\rangle\). It remains to show:

**Lemma 3** Let \(|\chi\rangle\) be a physical state of the form (43) with \( \lambda = 0 \). Then it follows that \(|\chi\rangle \in \mathcal{T}\).

Proof: Fix a state \(|\varphi\rangle \in \mathcal{T}\) and denote the \( \hat{u}(1) \) Verma module with highest weight state \(|\varphi\rangle\) by \( V_{\varphi}^X \), such that \(|\chi\rangle \in V_{\varphi}^X\). Call the corresponding \( c = 1 \) Virasoro Verma module \( V_{\varphi}^{Vir} \), and let \( L_0|\varphi\rangle = h_{\varphi}|\varphi\rangle\). If \( h_{\varphi} \neq 0 \), the module \( V_{\varphi}^{Vir} \) is irreducible, and it is well known that it is isomorphic as a graded vector space to \( V_{\varphi}^X \) (with \( L_0\)-grading in both cases). Since \( \hat{V}_{p^+, p^-} \) is a representation of \( Vir^c \oplus Vir^X \) any highest weight state must simultaneously be a \( Vir^c \) and a \( Vir^X \) highest weight state, so any physical state must in particular be annihilated by all operators \( L_n^X \) with \( n > 0 \). It follows that if \( h_{\varphi} \neq 0 \), the only state in \( V_{\varphi}^X \) that may be physical is \(|\varphi\rangle\), so \(|\chi\rangle \in \mathcal{T}\). If \( h_{\varphi} = 0 \) we must do a bit more work. Recall that \( h_{\varphi} = \frac{(p^+ - p^- + r)^2}{2(1-k)} \) where \( r \in \mathbb{Z} \), and if \(|\varphi\rangle\) is at level \( N \) we have the constraint \( r \geq -N \). In fact there is only one state in \( \hat{V}_{p^+, p^-} \) with \( r = -N \) namely \((J^+_1)^N|p^+, p^-\rangle\), the state on the right border in the \((N + 1)\)'st row in the following figure.
A simple calculation shows that all right-border states lie in $T$, and thus have positive norm. Now, assume $|\chi\rangle$ is a physical state of the form (43) at level $N$ with $h_{\varphi} = 0$. This implies $p^+ - p^- = P$, i.e. $p^- = p^+ - P$, for some integer $P \leq N$. The on-shell condition reads

$$-\frac{2p^+(p^+ - P + \frac{1}{2})}{k} \leq 1 - N$$

implying

$$P \geq p^+ + \frac{1}{2} + \frac{l}{2p^+}(N - 1).$$

The conditions $p^+ > 0$, $2p^+/|k| < 1$ then gives $P > N - 1$, so $P = N$. Since the right-border states lie in $T$ this finishes the proof of the lemma. \qed

There is a one-parameter family of automorphisms $\{\theta_s\}_{s \in \mathbb{Z}}$ of $\hat{\mathfrak{w}}$ called spectral flow, acting on our basis of choice as

$$\begin{align*}
\theta_s : J_n^+ &\mapsto J_n^{\pm,s} \\
\theta_s : J_n &\mapsto J_n \\
\theta_s : T_n &\mapsto T_n - is\frac{k}{2}\delta_{n,0}.
\end{align*}$$

Via the Sugawara construction $\theta_s$ induces automorphisms of the Virasoro algebra acting on the Sugawara generators as

$$\begin{align*}
\theta_s : L_n &\mapsto L_n - isJ_n + is\frac{2}{k}T_n + s^2\frac{1}{2}\delta_{n,0}.
\end{align*}$$
If $\hat{V}$ is a $\hat{\mathfrak{nw}}$ module where $\rho_V$ is the representation morphism (i.e. we write the action of $X \in \hat{\mathfrak{nw}}$ on $v \in \hat{V}$ as $\rho_V(X)v$), denote by $(s)\hat{V}$ the $\hat{\mathfrak{nw}}$ module with representation morphism $\rho_V^{(s)} := \rho_V \circ \theta_s$. We will be interested in extending the results obtained so far for $(s)\hat{V}_{\pm}^{p^+, p^-}$, $(s)\hat{V}_{\alpha}^{0, p^-}$, and $(s)\hat{V}_{0, 0}^{0, 0}$, which for brevity will be referred to as “flowed representations”.

We first make two observations regarding the flowed representations.

**Remark 2**

(i) If $T_0$ has eigenvalue $ip^+$ on the module $V$, the $T_0$ eigenvalue on $(s)\hat{V}$ will be $i(p^+ - s \frac{k}{2})$. Thus even with the constraint $p^+ \in (0, -\frac{k}{2})$ for $\hat{V}_{\pm}^{p^+, p^-}$ and the analogous constraint on $\hat{V}_\alpha^{0, p^-}$, including all flowed representations indicated above the spectrum of $T_0$ becomes $i\mathbb{R}$ (note that to get the eigenvalues $s \frac{k}{2}$ we must include the flowed representations $(s)\hat{V}_{\alpha}^{0, p^-}$ or $(s)\hat{V}_{0, 0}^{0, 0}$, however only the former lead to a unitary space of physical states). This is the spectrum obtained by light cone quantization [10].

(ii) We have $(-1)^s \hat{V}_+^{p^+, p^-} \cong \hat{V}_-^{p^+, p^- + \frac{k}{2}}$. This follows from the fact that $\hat{V}_{\pm}^{p^+, p^-}$ are Verma modules, and therefore determined by their highest resp. lowest weight vectors. The property $\theta_s \circ \theta_t = \theta_{s+t}$ implies that concentrating on $\hat{V}_+^{p^+, p^-}$ ($\hat{V}_-^{p^+, p^-}$) it is enough to consider flowed representations with $s > 0$ ($s < 0$).

**Lemma 4** Fix $s \in \mathbb{N}$. States of the form in $(s)\hat{V}_+^{p^+, p^-}$ for all levels $N$ span $(s)\hat{V}_+^{p^+, p^-}$. 

Proof: Since $\theta_s(X_n) = X_n$ for $n \neq 0$, a state in $\hat{V}_+^{p^+, p^-}$ is $\hat{u}(1)$ primary iff the same vector is also $\hat{u}(1)$ primary in the flowed representation. By rewriting $\theta_s(L_n)$ in terms of $X_n$ and $Y_n$ and using $[X_m, Y_n] = 0$, so $Y_{-n}|\psi\rangle$ is $\hat{u}(1)$ primary if $|\psi\rangle$ is, the Lemma follows immediately since the underlying vector space of $\hat{V}_+^{p^+, p^-}$ coincides with that of $(s)\hat{V}_+^{p^+, p^-}$. 

Next, observe that $\theta_s$ preserves the Hermiticity properties of our generators. Since $(s)\hat{V}_+^{p^+, p^-}$ has a cyclic vector, namely the highest weight vector of $\hat{V}_+^{p^+, p^-}$, we conclude
that the inner products on $\hat{V}_+^{p^+,p^-}$ and $(s)\hat{V}_+^{p^+,p^-}$ coincide. If we can show that a state of the form (43) in $(s)\hat{V}_+^{p^+,p^-}$ with $\lambda = 0$ lies in $T$ we have managed to show a no-ghost theorem for the flowed representation.

If we can show an analogue of Lemma 3 for $(s)\hat{V}_+^{p^+,p^-}$ then we have managed to show a no-ghost theorem for the flowed representation.

Lemma 5 Let $(s)T \subset (s)\hat{V}_+^{p^+,p^-}$, $s \in \mathbb{N}$, be the subspace spanned by vectors annihilated by all $X_n$ and $L_n$ for $n > 0$, i.e. all vectors in $\hat{V}_+^{p^+,p^-}$ annihilated by $\theta_s(X_n)$ and $\theta_s(L_n)$ for $n > 0$, and let $|\chi\rangle \in (s)\hat{V}_+^{p^+,p^-}$ be a physical state of the form (43) with $\lambda = 0$. Then $|\chi\rangle \in (s)T$.

Proof: The strategy of the proof is equivalent to the proof of Lemma 3 and we need to investigate the situation where the physical state $|\chi\rangle$ is a descendent of $|\varphi\rangle$ with $\text{Vir}^X$ weight $h_\varphi = 0$. In $(s)\hat{V}_+^{p^+,p^-}$ we have

$$h_\varphi = \frac{(p^+ - p^- - \frac{sk}{2} + r)^2}{2(1 - k)},$$

where again $r \geq -N$ if $|\varphi\rangle$ is of level $N$. If $h_\varphi = 0$ we have $p^- = p^+ - \frac{sk}{2} + r$. The $L_0$-eigenvalues of states in $(s)\hat{V}_+^{p^+,p^-}$ are of the form

$$h = -\frac{2p^+ (p^- + \frac{1}{2}) + s(p^- - r) - \frac{2p^+}{k} + \frac{s^2}{2} + N},$$

and the mass-shell condition states that $h \leq 1$. Inserting the expression for $p^-$ in the mass-shell condition we get

$$-\frac{2p^+}{k} \left( p^+ - \frac{k}{2} + \frac{1}{2} + r \right) + s \left( p^+ - \frac{k}{2} - \frac{2p^+}{k} + \frac{s}{2} \right) \leq 1 - N. \quad (48)$$

Discarding strictly positive terms on the left hand side we get the condition $-\frac{2p^+}{k}r < 1 - N$, and since $-\frac{2p^+}{k} \in (0,1)$ we conclude that $r = -N$, so it is again a “right-border” state. Such a state is also in $(s)\hat{V}_+^{p^+,p^-}$ annihilated by all $X_n$ and $L_n$ such that $n > 0$, and it follows that $|\chi\rangle \in (s)T$. \hfill \Box
In section II B we saw that unitarity for $\hat{V}_0^{0,0}$ follows immediately from the on-shell condition. We now turn to the corresponding flowed representations $(s)\hat{V}_0^{0,0}$. Note first that Lemma 2 also holds in the representations $\hat{V}_0^{0,0}$ (the relevant $L_0^c$-eigenvalues $h^c$ are trivially positive), and hence also in the flowed representations $(s)\hat{V}_0^{0,0}$ respectively $(s)\hat{V}_0^{0,0}$ by the same argument as in Lemma 4. We are thus left to show the analogue of Lemma 5 for these representations, which again reduces to analyzing Vir$_X$ descendents of weight 0 primaries $|\varphi\rangle$. In the representation $(s)\hat{V}_0^{0,0}$ we get $h\varphi = \frac{-(p^- - n + \frac{1}{2}s)k}{2(1-k)^2}$, and $h\varphi = 0$ implies $p^- - n = -\frac{s}{2}k$. A corresponding $L_0$-eigenvalue then takes the form

$$h = -\frac{\alpha^2}{k} + \frac{s^2}{2}(1 - k) + N,$$

and since $1 - k > 0$ the on-shell condition implies $N = 0$, so all physical states lie in the horizontal submodule $V_0^{0,0}$ which is unitary.

We summarize the results above in the following

**Theorem 1** *The space of physical states for the bosonic string on the Nappi-Witten group with level $k$ is unitary if $k < 0$ in the following cases*

- the space $(s)\hat{V}_+^{0,0}$ when $s \in \mathbb{N}_0$, $p^+ \in (0, -\frac{1}{2})$, $p^- \in \mathbb{R}$
- the space $(s)\hat{V}_-^{0,0}$ when $s \in -\mathbb{N}_0$, $p^+ \in (0, \frac{1}{2})$, $p^- \in \mathbb{R}$
- the space $(s)\hat{V}_0^{0,0}$ when $s \in \mathbb{Z}$, $\alpha, p^- \in \mathbb{R}$

**IV. DISCUSSION**

In this paper, we presented the proof of a no-ghost theorem for the bosonic string in Nappi-Witten spacetime. The proof involves two parts. The first part consists of showing that the states in a class of representations of the coset $NW/U(1)$ have non-negative norm, where the $U(1)$ subgroup corresponds to a timelike direction in the Nappi-Witten spacetime. The proof of this part follows closely the discussion of the unitarity of the
$SO(2, 1)/U(1)$ modules by Dixon et al \cite{15}. In the second part, physical states satisfying the Virasoro constrains are shown to lie within the coset modules if we assume the total Hilbert space of the NW WZW model decomposes completely in a certain set of representations of the NW current algebra. The proof in this part follows a more direct and clear proof of unitarity of the $SU(1, 1)$ bosonic string theory given in \cite{2} than the one given in \cite{1}.

A crucial ingredient for the first part is that every state in the current algebra representation can be expanded as a sum of terms, each of which is obtained by acting a product of negative mode operators of the timelike generator $J_n - T_n$ $(n > 0)$ on a state annihilated by any $J_n - T_n$ $(n > 0)$. One nice property of the Nappi-Witten group is that we can construct this expansion explicitly.

In addition to establishing a unitary spectrum one would also like to identify physical configurations corresponding to these states. In particular we expect to find analogues of the long string states in $AdS_3$ \cite{4} for $p^+ \in \frac{k}{2}Z$, which with our proposal for the representation content would correspond to states in $(s)V_{\alpha}^{0, p^-}$. Another obvious continuation is the construction of modular invariant partition functions. We refer to a future publication for both topics \cite{24}.

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[1] S. Hwang, No-ghost theorem for SU(1,1) string theories, Nucl. Phys. B 354 (1991) 100.

[2] S. Hwang, Cosets as gauge slices in SU(1,1) strings, Phys. Lett. B 276 (1992) 451, [arXiv:hep-th/9110039]

[3] J. M. Evans, M. R. Gaberdiel and M. J. Perry, The no-ghost theorem for AdS(3) and the stringy exclusion principle, Nucl. Phys. B 535, 152 (1998) [arXiv:hep-th/9806024].

[4] J. M. Maldacena and H. Ooguri, Strings in AdS(3) and SL(2,R) WZW model. I, J. Math. Phys. 42, 2929 (2001) [arXiv:hep-th/0001053].

[5] J. M. Maldacena and H. Ooguri, Strings in AdS(3) and the SL(2,R) WZW model. III: Correlation functions, Phys. Rev. D 65, 106006 (2002) [arXiv:hep-th/0111180].

[6] J. M. Maldacena, G. W. Moore and N. Seiberg, Geometrical interpretation of D-branes in gauged WZW models, JHEP 0107, 046 (2001) [arXiv:hep-th/0105038].

[7] J. Björnsson and S. Hwang, On the unitarity of gauged non-compact WZNW strings, Nucl. Phys. B 797, 464 (2008) [arXiv:0710.1050 [hep-th]].

[8] J. Björnsson and S. Hwang, On the unitarity of gauged non-compact world-sheet supersymmetric WZNW models, Nucl. Phys. B 812, 525 (2009) [arXiv:0802.3578 [hep-th]].

[9] C. R. Nappi and E. Witten, A WZW model based on a nonsemisimple group, Phys. Rev. Lett. 71, 3751 (1993) [arXiv:hep-th/9310112].

[10] Y. K. Cheung, L. Freidel and K. Savvidy, Strings in gravimagnetic fields, JHEP 0402, 054 (2004) [arXiv:hep-th/0309005].

[11] R. R. Metsaev, Type IIB Green-Schwarz superstring in plane wave Ramond-Ramond background, Nucl. Phys. B 625, 70 (2002) [arXiv:hep-th/0112044].

[12] M. Blau, J. Figueroa-O’Farrill, C. Hull and G. Papadopoulos, A new maximally supersymmetric background of IIB superstring theory, JHEP 0201, 047 (2002) [arXiv:hep-th/0110242].

[13] D. Berenstein, J. M. Maldacena and H. Nastase, Strings in flat space and pp waves from
\[ N = 4 \text{ super Yang Mills}, \text{JHEP} \ 0204, \ 013 (2002) \ [\text{arXiv:hep-th/0202021}]. \]

[14] J. G. Russo and A. A. Tseytlin, \textit{On solvable models of type IIB superstring in NS-NS and R-R plane wave backgrounds}, JHEP \textbf{0204}, 021 (2002) [arXiv:hep-th/0202179].

[15] L. J. Dixon, M. E. Peskin and J. Lykken, \textit{N=2 Superconformal Symmetry and SO(2,1) Current Algebra}, Nucl. Phys. B \textbf{325} (1989) 329.

[16] P. Goddard and C. Thorn, \textit{Compatibility of the Dual Pomeron with Unitarity and the Absence of Ghosts in the Dual Resonance Model} Phys. Lett. \textbf{40B} (1972) 235.

[17] C. Thorn, \textit{A Detailed Study Of The Physical State Conditions In Covariantly Quantized String Theories}, Nucl. Phys. B \textbf{286} (1987) 61.

[18] E. Kiritsis and C. Kounnas, \textit{String propagation in gravitational wave backgrounds}, Phys. Lett. B \textbf{320}, 264 (1994) [Addendum-ibid. B \textbf{325}, 536 (1994)] [arXiv:hep-th/9310202].

[19] K. Sfetsos, \textit{Gauging a nonsemisimple WZW model}, Phys. Lett. B \textbf{324}, 335 (1994) [arXiv:hep-th/9311010].

[20] K. Sfetsos, \textit{Exact string backgrounds from WZW models based on nonsemisimple groups}, Int. J. Mod. Phys. A \textbf{9}, 4759 (1994) [arXiv:hep-th/9311093].

[21] D. I. Olive, E. Rabinovici and A. Schwimmer, \textit{A Class of string backgrounds as a semi-classical limit of WZW models}, Phys. Lett. B \textbf{321}, 361 (1994) [arXiv:hep-th/9311081].

[22] N. Mohammedi, \textit{On bosonic and supersymmetric current algebras for nonsemisimple groups}, Phys. Lett. B \textbf{325}, 371 (1994) [arXiv:hep-th/9312182].

[23] J. M. Figueroa-O’Farrill and S. Stanciu, \textit{Nonsemisimple Sugawara constructions}, Phys. Lett. B \textbf{327}, 40 (1994) [arXiv:hep-th/9402035].

[24] G. Chen, Y–K E. Cheung, Z. Fan, J. Fjelstad and S. Hwang, \textit{in preparation}

[25] There seems to be no universally accepted notion of the Nappi-Witten group, in this paper we will take it to be the unique connected and simply connected group with the given Lie algebra.

[26] See \([19, 20, 21, 22, 23]\) for a general construction of WZW models with non-semisimple algebras.