Markovian loop clusters on the complete graph and coagulation equations

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Abstract

Poissonian ensembles of Markov loops on a finite graph define a random graph process in which the addition of a loop can merge more than two connected components. We study Markov loops on the complete graph derived from a simple random walk killed at each step with a constant probability. Using a component exploration procedure, we describe the asymptotic distribution of the connected component size of a vertex at a time proportional to the number of vertices, show that the largest component size undergoes a phase transition and establish the coagulation equations associated to this random graph process.

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Key words. Poisson point process of loops, random graph process, branching processes, coalescent process, coagulation equations.

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Introduction

We consider the Poissonian ensembles of Markov loops on the complete graph $K_n$ derived from a simple random walk killed at a constant rate $\kappa_n = n\varepsilon$ (loops can be viewed as excursions of the simple random walk with a random starting point, up to re-rooting). Poissonian ensembles are seen as a Poisson point process of loops indexed by ‘time’. The edges crossed by loops before time $t$ define a subgraph $G_t^{(n)}$ of $K_n$. The evolution of the connected components of $G_t^{(n)}$ also named loop clusters, defines a coalescent process with multiple mergers: the addition of a loop of length $k$ can merge up to $k$ clusters at the same time.

The notion of Poissonian ensembles of Markov loops (loop soups) was introduced by Lawler and Werner in [14] in the context of two dimensional Brownian motion (it already appeared informally in [27]). Loop clusters induced by a Brownian loop soup were used to give a construction of conformal loop ensembles (CLE) in [30] and [24]. Some general properties of loop clusters on finite and countable graphs were presented in [16] (see [15], [13] and [28] for studies of other aspects of Markov loops on graphs).

The random graph process $(G_t^{(n)})_{t \geq 0}$ depends on the ‘killing parameter’ $\varepsilon$. For large values of $n$ and $\varepsilon$, the proportion of loops of length 2 is near 1, hence we can expect that $(G_t^{(n)})_{t \geq 0}$ behaves like the random graph process introduced by Erdös and Rényi in [7]. In [16], the asymptotic distribution of the cover time (the first time when $G_t^{(n)}$ has no isolated vertex) and the coalescence time (the first time when $G_t^{(n)}$ is connected) were studied showing in particular that $\varepsilon(\varepsilon + 1)n \log(n)$ is a sharp threshold function for connectivity of the random graph process $(G_t^{(n)})_{t \geq 0}$. In comparison, the threshold function for the connectivity of the Erdös-Rényi random graph with $n$ vertices is $\frac{1}{2}n \log(n)$ (see [7]). In this paper, we study the (connected) component sizes of $G_t^{(n)}$ and the hydrodynamic behavior of the associated coalescent process.

Let us review some properties related to our study in the case of the Erdös-Rényi random graph. We follow the presentation given by J. Bertoin in [3], chapter 5. To make the link with the multiplicative coalescent simpler, we consider a variant of the Erdös-Rényi random graph constructed as follows: let $\{X_{\{x,y\}}, x \neq y\}$ be a family of independent random variables indexed by the edges of $K_n$ with Exponential(1)-distribution. We define an increasing family of random graphs denoted $(\mathcal{H}(n, t))_{t \geq 0}$ by setting: $\varepsilon$ is an edge of $\mathcal{H}(n, t)$ if and only if $X_\varepsilon \leq t$.

More precisely, if $G(n, N)$ denotes a random graph obtained by forming $N$ links between $n$ labelled vertices, each of the $\left(\begin{array}{c} N \\ 2 \end{array}\right)$ graphs being equally likely, then the probability that $G(n, \lfloor \frac{1}{2}(\log(n) + c) \rfloor)$ is a connected graph converges to $\exp(-e^{-c})$ as $n$ tends to $+\infty$ for every $c \in \mathbb{R}$.

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Let \( c_t^{(n)}(x) \) denote the component size of a vertex \( x \) of \( \mathcal{H}(n, \frac{t}{n}) \) and let \( c_{1,t}^{(n)} \geq c_{2,t}^{(n)} \) denote the two largest component sizes.

Phase transition

1. Assume that \( t < 1 \).
   - For every vertex \( x \), \( c_t^{(n)}(x) \) converges in distribution to the total population size of a Galton-Watson process with one progenitor and Poisson(\( t \)) offspring distribution.
   - Let \( I_t \) be the value at 1 of the Cramér function of the Poisson(\( t \))-distribution: \( I_t = t - 1 - \log(t) \).
     For every \( a > I_t^{-1} \), \( \mathbb{P}(c_{1,t}^{(n)} \geq a \log(n)) \) converges to 0 as \( n \) tends to \(+\infty\).

2. Assume that \( t > 1 \) and denote by \( q_t \) the extinction probability of a Galton-Watson process with one progenitor and Poisson(\( t \)) offspring distribution.
   For every \( a \in [1/2, 1[ \), there exist \( b > 0 \) and \( c > 0 \) such that
   \[
   \mathbb{P}(|c_{1,t}^{(n)} - (1 - q_t)n| \geq na^{\alpha}) + \mathbb{P}(c_{2,t}^{(n)} \geq c \log(n)) = O(n^{-b}).
   \]

This phase transition was first proved by Erdős and Rényi in [8]. The statements we present are taken from [29] where proofs are based on the use of branching processes.

Coalescent process. The component sizes of \( \mathcal{H}(n, \frac{t}{n}) \) evolve as a multiplicative coalescent process with binary aggregations: aggregations of more than two components at one time do not occur and the aggregation rate of two components is proportional to the product of their sizes.

- For any \( t > 0 \) and \( x \in \mathbb{N}^* \), the average number of components of size \( x \) in \( \mathcal{H}(n, \frac{t}{n}) \) converges in \( L^2 \) to
  \[
  n(t, x) = \frac{(tx)^{x-1}e^{-tx}}{x.x!} \quad \forall x \in \mathbb{N}^* \text{ and } t \in [0, 1].
  \]
  The value \( xn(x,t) \) is equal to the probability that \( x \) is the total population size of a Galton-Watson process with one progenitor and Poisson(\( t \)) offspring distribution\(^2\).

\(^2\)For \( t \leq 1 \), \( \{xn(x,t), x \in \mathbb{N}^* \} \) is a probability distribution called Borel-Tanner distribution with parameter \( t \).
• \( \{n(x, \cdot), \ x \in \mathbb{N}^+\} \) is solution on \( \mathbb{R}_+ \) of the Flory’s coagulation equations with multiplicative kernel:

\[
\frac{d}{dt} n(x, t) = \frac{1}{2} \sum_{y=1}^{x-1} g(x - y)n(y, t)n(x - y, t) - \sum_{y=1}^{+\infty} xyn(t, x)n(t, y) - xn(t, x) \sum_{y=1}^{+\infty} y(n(0, y) - n(t, y)) \tag{1}
\]

Up to time 1, this solution coincides with the solution of the Smoluchowski’s coagulation equations with multiplicative kernel starting from the monodisperse state:

\[
\frac{d}{dt} n(x, t) = \frac{1}{2} \sum_{y=1}^{x-1} g(x - y)n(y, t)n(x - y, t) - xn(t, x) \sum_{y=1}^{+\infty} y(n(t, y)). \tag{2}
\]

Equations (2) introduced by Smoluchowski in [25] are used for example to describe aggregations of polymers in an homogeneous medium where diffusion effects are ignored. The first term in the right-hand side describes the formation of a particle of mass \( x \) by aggregation of two particles, the second sum describes the ways a particle of mass \( x \) can be aggregated with another particle. If the total mass of particles decreases after a finite time, the system is said to exhibit a ‘phase transition’ called ‘gelation’: the loss of mass is interpreted as the formation of infinite mass particles called gel. Smoluchowski’s equations do not take into account interactions between gel and finite mass particles. Equations (1) introduced by Flory in [9] are a modified version of the Smoluchowski’s equations with an extra term describing the loss of a particle of mass \( x \) by ‘absorption’ in the gel. Let \( T_{gel} \) denote the largest time such that the Smoluchowski’s coagulation equations with monodisperse initial condition has a solution which has the mass-conserving property\(^3\). Then \( T_{gel} = 1 \) and \( T_{gel} \) coincides with the smallest time when the second moment \( \sum_{x=1}^{+\infty} x^2 n(x, t) \) diverges (see [20]). Let us note that the random graph process \( (\mathcal{H}(n; \frac{1}{t}))_{t \geq 0} \) is equivalent to the microscopic model introduced by Marcus [18] and further studied by Lushnikov [17] (see [5] for a first study of the relationship between these two models and [1] for a review, [22], [21] and [10] for convergence results of Marcus-Lushnikov’s model to (1)).

The aim of this paper is to show that similar statements hold for the loop model if the Poisson\((t)\)-distribution is replaced by the compound Poisson distribution with probability-generating function \( s \mapsto \exp\left( \frac{(1-s)}{e(x+1)} \right) \) (i.e. the distribution of a Poisson\((\frac{t}{e(x+1)})\)-distributed number

\(^{3}\text{Different definitions of the ‘gelation time’ } T_{gel} \text{ are used in the literature: the gelation time is sometimes defined as the smallest time when the second moment diverges (see [1])}\)
of independent random variables with geometric distribution on \( \mathbb{N}^* \) of parameter \( \frac{\varepsilon}{\varepsilon+1} \). Phase transition in the loop model occurs at \( t = \varepsilon^2 \). Hydrodynamic behavior of the coalescent process associated to the loop model is described by new coagulation equations in which more than two particles can collide at the same time.

Section 1 is devoted to a presentation of the Markov loop model on the complete graph and the statement of the main results (Theorem 1.1, Theorem 1.2 and Proposition 1.2). In section 2, we describe the component exploration process used to compute the component size of a vertex and to construct the associated Galton-Watson process. The asymptotic distribution of the component size of a vertex is studied in Section 3 and the proof of Theorem 1.1 is presented.

In Section 4, we prove Theorem 1.2 which presents some properties of the largest component size in the two phases, \( t < \varepsilon^2 \) and \( t > \varepsilon \). Section 5 is devoted to the proof of Proposition 1.2 that describes the hydrodynamic behaviour of the coalescent process.

1 Description of the model and main results

1.1 Setting

We consider the complete graph \( K_n \) with \( n \) vertices. The set of vertices is identified with \([n] = \{1, \ldots, n\}\). We add to each vertex \( x \) a self-loop \( \{x, x\} \). This defines an undirected graph denoted \( \bar{K}_n \). We consider the loop sets induced by a simple random walk on \( \bar{K}_n \) killed at each step with probability \( \frac{\varepsilon}{\varepsilon+1} \) with \( \varepsilon > 0 \). In other words, the graph \( \bar{K}_n \) is endowed with unit conductances and a uniform killing measure with intensity \( \kappa_n = n\varepsilon \). The transition matrix \( P \) of the random walk is defined by \( P_x,y = \frac{1}{n(\varepsilon+1)} \) for every \( x, y \in [n] \).

A discrete based loop \( \ell \) of length \( k \in \mathbb{N}^* \) on \( \bar{K}_n \) is defined as an element of \([n]^k\). A discrete loop is an equivalent class of based loops for the following equivalent relation: the based loop of length \( k \), \((x_1, \ldots, x_k)\) is equivalent to the based loop of length \( k \) \((x_i, x_k, x_1, \ldots, x_{i-1})\) for every \( i \in \{2, \ldots, k\} \). We associate to each based loop \( \ell = (x_1, \ldots, x_k) \) of length \( k \geq 2 \) the weight \( \tilde{\mu}(\ell) = \frac{1}{k} P_{x_1,x_2} P_{x_2,x_3} \cdots P_{x_k,x_1} = \frac{1}{k(n(\varepsilon+1))^k} \). This defines a measure \( \tilde{\mu} \) on the set of discrete based loops of length at least 2, which is invariant by the shift and therefore induces a measure \( \mu \) on the set of discrete loops of length at least 2 denoted by \( \mathcal{DL}([n]) \). The Poisson loop sets on \( \bar{K}_n \) is defined as a Poisson point process \( \mathcal{DP} \) with intensity \( \text{Leb} \times \mu \) on \( \mathbb{R}_+ \otimes \mathcal{DL}([n]) \). For \( t > 0 \), we denote by \( \mathcal{DL}_t^{(n)} \) the projection of the set \( \mathcal{DP} \cap ([0, t] \times \mathcal{DL}([n])) \). The loop set \( \mathcal{DL}_t^{(n)} \) on \( \bar{K}_n \) defines a subgraph denoted by \( \mathcal{G}_t^{(n)} \).

Let \( C_t^{(n)}(x) \) denote the connected component of the vertex \( x \) in the random graph \( \mathcal{G}_t^{(n)} \). The
aim of the paper is to study the size of $C(n)^{(ni)}(x)$ as $n$ tends to $+\infty$. The loop sets defined by this model are slightly different from the loop sets on the complete graph studied in [16] since we add a self-loop $\{x, x\}$ at each vertex $x$. But the partitions induced by the loop clusters on $\bar{K}_n$ or $K_n$ have the same distribution recalled in the following proposition:

**Proposition 1.1.** For every $t > 0$, let $C_t$ denote the partition induced by the connected components of $\mathcal{G}_t^{(n)}$ and let $\hat{C}_t$ denote the partition induced by the loop sets on $K_n$ associated to the transition matrix $\hat{P}$ defined by: $\hat{P}_{xy} = \frac{1}{n-1+\varepsilon} \mathbf{1}_{\{x \neq y\}}$ for every $x, y \in [n]$. The Markov processes $(C_t)_t$ and $(\hat{C}_t)_t$ have the same distribution:

- If $\pi$ is a partition of the set of vertices $[n]$ with $k$ blocks $B_1, \ldots, B_k$ then the probability that $C_t$ is finer than $\pi$ is
  \[
  \mathbb{P}_{\pi_0}(C_t \preceq \pi) = \left(\frac{\varepsilon}{\varepsilon + 1}\right)^t \prod_{i \in I} (1 - \frac{|B_i|}{n(\varepsilon + 1)})^{-t} \mathbf{1}_{\{\pi_0 \preceq \pi\}}.
  \]

- From state $\pi = \{B_i, i \in I\} \in \mathfrak{P}([1:n])$, the only possible transitions of $(C_t)_{t \geq 0}$ are to partitions $\pi \oplus J$ obtained by merging blocks indexed by a subset $J$ of $I$ with $L \geq 2$ elements. The transition rate from $\pi$ to $\pi \oplus J$ is equal to:

  \[
  \tau_{\pi, \pi \oplus J}^{(n)} = \sum_{k \geq L} \frac{1}{kn^k(\varepsilon + 1)^k} \left(\begin{array}{c} k \\ k_1, \ldots, k_L \end{array}\right) \prod_{u=1}^L |B_{j_u}|^{k_u}
  \]

where $W_k(J)$ is the set of $k$-tuples of $J$ in which each element of $J$ appears.

Proposition 1.1 can be proved as in [16] where the distribution of $(\hat{C}_t)_{t > 0}$ is computed.

### 1.2 Main results

In the Erdős-Rényi random graph process, the connected component of a vertex $x$ can be compared with a Galton-Watson process with ancestor $x$ in which any vertices $y$ connected by an edge to a vertex $z$ are seen as offspring of $z$. Unlike edges in Erdős-Rényi random graph, loops can cross a vertex several times and intersect other loops at several vertices. But these events are rare enough so that the component $C^{(n)}(x)$ for large $n$ can still be compared with a Galton-Watson process with ancestor $x$ in which offspring of a vertex $z$ corresponds to vertices $y \neq z$
crossed by loops in $\mathcal{D}L_{nt}^{(n)}$ that pass through $z$ and the number of offspring can be approximated by

$$\sum_{\ell \in \mathcal{D}L_{nt}^{(n)} \text{ s.t. } x \in \ell} (|\ell| - 1).$$

Before stating the main results, let us introduce some notations.

- For a positive real $\lambda$ and a probability distribution $\nu$ on $\mathbb{R}$, let $\text{CPois}(\lambda, \nu)$ denote the compound Poisson distribution with parameters $\lambda$ and $\nu$: $\text{CPois}(\lambda, \nu)$ is the probability distribution of $\sum_{i=1}^{N} X_i$, where $N$ is a Poisson distributed random variable with expected value $\lambda$ and $(X_i)_i$ is a sequence of independent random variables with law $\nu$ that are independent of $N$.

- For $p \in [0, 1]$, let $G_{N^*}(p)$ denote the geometric distribution on $\mathbb{N}^*$ with parameter $p$ (its probability mass function is $(1 - p)^{k-1}p \forall k \in \mathbb{N}^*$).

- For $u \in \mathbb{N}^*$, $\varepsilon > 0$ and $t > 0$, let $T_{\varepsilon,t}^{(u)}$ denote the total number of descendants of a Galton-Watson process with family size distribution $\text{CPois}(\frac{t}{\varepsilon(\varepsilon+1)}, G_{N^*}(\frac{\varepsilon}{\varepsilon+1}))$ and $u$ ancestors.

### 1.2.1 Component sizes

The following theorem shows in particular, that for $t \in [0; \varepsilon^2]$ the component size of a vertex at time $nt$ converges in distribution to $T_{\varepsilon,t}^{(1)}$.

**Theorem 1.1.** Let $\varepsilon$ and $t$ be two positive reals. Let $(k_n)_n$ be a sequence of positive numbers.

$$\mathbb{P}(|C_{nt}^{(n)}(x)| \leq k_n) - \mathbb{P}(T_{\varepsilon,t}^{(1)} \leq k_n) = O\left(\frac{k^2}{n}\right).$$

**Remark 1.1.** Let $t$ and $\varepsilon$ be two positive reals.

1. A Galton-Watson process with family size distribution $\text{CPois}(\frac{t}{\varepsilon(\varepsilon+1)}, G_{N^*}(\frac{\varepsilon}{\varepsilon+1}))$ is subcritical if and only if $t < \varepsilon^2$. Let $q_{\varepsilon,t}$ denote the extinction probability of such a Galton-Watson process starting with one ancestor. It is a decreasing function of $t$ and an increasing function of $\varepsilon$.

Moreover,

$$\begin{align*}
\mathbb{P}(T_{\varepsilon,t}^{(u)} = u) &= e^{-\frac{ut}{\varepsilon(\varepsilon+1)}} \\
\mathbb{P}(T_{\varepsilon,t}^{(u)} = k) &= \frac{u}{k} e^{-\frac{kt}{\varepsilon(\varepsilon+1)}} \sum_{j=1}^{k-u} \binom{k-u-1}{j-1} \left(\frac{kt}{\varepsilon+1}\right)^j \forall k \geq u+1.
\end{align*}$$

For $t \leq \varepsilon^2$, $T_{\varepsilon,t}^{(u)}$ is almost surely finite and for $t > \varepsilon^2$, $\mathbb{P}(T_{\varepsilon,t}^{(u)} < \infty) = q_{\varepsilon,t}^u < 1$. (see appendix A for a detailed description of the properties of such a Galton-Watson process.)
2. The convergence result in Theorem 1.1 still holds if \( t \) and \( \varepsilon \) are replaced in the statement by two positive sequences \((t_n)_n\) and \((\varepsilon_n)_n\) that converge to \( t \) and \( \varepsilon \) respectively.

Theorem 1.1 is used to show that the component sizes of \((G_{nt}^{(n)})_{t \geq 0}\) undergo a phase transition at \( t = \varepsilon^2 \) similar to the phase transition of the Erdős-Rényi random graph process.

**Theorem 1.2.** Let \( C_{nt,1}^{(n)} \) and \( C_{nt,2}^{(n)} \) denote the first and second largest components of the random graph \( G_{nt}^{(n)} \).

1. **Subcritical regime.** Assume that \( 0 < t < \varepsilon^2 \). Set \( h(t) = \sup_{\theta \in [0, \log(\varepsilon + 1)]} \left( \theta - \log(L_{t,\varepsilon}(\theta)) \right) \) where \( L_{t,\varepsilon} \) is the moment-generating function of the compound Poisson distribution \( \text{CPois}(\frac{t}{\varepsilon^2 + 1}, G_{N^*}(\frac{\varepsilon}{\varepsilon^2 + 1})) \).
   For every \( a > 1/h(t) \), \( \mathbb{P}(|C_{nt,1}^{(n)}| > a \log(n)) \) converges to 0 as \( n \) tends to +\( \infty \).

2. **Supercritical regime.** Assume that \( t > \varepsilon^2 \) and let \( q_{\varepsilon,t} \) denote the extinction probability of a Galton-Watson process with one progenitor and \( \text{CPois}(\frac{t}{\varepsilon^2 + 1}, G_{N^*}(\frac{\varepsilon}{\varepsilon^2 + 1})) \) offspring distribution.
   For every \( a \in ]1/2, 1[ \), there exist \( b > 0 \) and \( c > 0 \) such that
   \[
   \mathbb{P}[|C_{nt,1}^{(n)}| - (1 - q_{\varepsilon,t})n| \geq n^a] + \mathbb{P}[|C_{nt,2}^{(n)}| \geq c \log(n)] = O(n^{-b}).
   \]

### 1.2.2 Coagulation equations

We turn to the hydrodynamic behavior of the coalescent process generated by the loop sets. Components of size \( k \) can be seen as a cluster of \( k \) particles of unit mass; at the same time, several clusters of masses \( k_1, \ldots, k_j \) can merge into a unique cluster of mass \( k_1 + \ldots + k_j \) at a rate proportional to the product \( k_1 \cdot \ldots \cdot k_j \). The initial state corresponds to the monodisperse configuration \((n \text{ particles of unit mass})\). The following proposition describes the asymptotic limit of the average number of components of size \( k \) at time \( nt \) as the number of particles \( n \) tends to +\( \infty \):

**Proposition 1.2.** For \( k \in \mathbb{N}^* \), \( n \in \mathbb{N} \) and \( t > 0 \), let \( \rho_{\varepsilon,t}^{(n)}(k) = \frac{1}{nk} |\{ x \in [n], \ C_{nt}^{(n)}(x) = k \}| \) be the average number of components of size \( k \) and let \( \rho_{\varepsilon,t}(k) = \frac{1}{k} \mathbb{P}(T_{\varepsilon,t}^{(1)} = k) \).

1. \( (\rho_{\varepsilon,t}^{(n)}(k))_n \) converges to \( \rho_{\varepsilon,t}(k) \) in \( L^2 \) for every \( t > 0 \).
2. \((\rho_{\varepsilon,t}(k), \ k \in \mathbb{N}^* \text{ and } t \geq 0)\) is a solution to the following coagulation equations:

\[
\frac{d}{dt}\rho_t(k) = \sum_{j=2}^{+\infty} \frac{1}{(\varepsilon + 1)^j} G_j(\rho_t, k)
\]

where

\[
G_j(\rho_t, k) = \frac{1}{j} \left( \sum_{(i_1, \ldots, i_j) \in (\mathbb{N}^*)^j \atop i_1 + \cdots + i_j = k} \prod_{u=1}^{j} i_u \rho_t(i_u) \right) I_{\{j \leq k\}} - k \rho_t(k) \left( \sum_{i=1}^{+\infty} i \rho_t(i) \right)^{j-1} - k \rho_t(k) \sum_{h=1}^{j-1} \left( \frac{j-1}{h} \right) \left( \sum_{i=1}^{+\infty} i (\rho_0(i) - \rho_t(i)) \right)^h \left( \sum_{u=1}^{+\infty} u \rho_t(u) \right)^{j-1-h}
\]

**Remark 1.2.**

1. Consider a medium with integer mass particles and let \(\rho_t(k)\) denote the density of mass \(k\) particles at time \(t\). Equation (6) describes the evolution of \(\rho_t(k)\) if for every \(j \geq 2\) the number of aggregations of \(j\) particles of mass \(i_1, \ldots, i_j\) in time interval \([t, t+dt]\) is assumed to be

\[
\frac{1}{j(\varepsilon + 1)^j} \rho_t(i_1) \cdots \rho_t(i_j) K_j(i_1, \ldots, i_j) dt
\]

where \(K_j(i_1, \ldots, i_j) = i_1 \cdots i_j\) is the multiplicative kernel.

The first sum of \(G_j\) describes the formation of a particle of mass \(k\) by aggregation of \(j\) particles, the second sum describes the ways a particle of mass \(k\) can be aggregated with \(j-1\) other particles. The third term in the definition of \(G_j\) is null if the total mass is preserved. Otherwise, the decrease of the total mass can be interpreted as the appearance of a ‘gel’ and the third term describes the different ways a particle of mass \(k\) can be aggregated with the gel and other particles.

2. \(\rho_{\varepsilon,t}\) defined by \(\rho_{\varepsilon,t}(k) = \frac{1}{k} P(T_{\varepsilon,t}^{(1)} = k)\) for every \(k \in \mathbb{N}^*\) gives an explicit solution of (6) with mass-conserving property on the interval \([0, \varepsilon^2]\). Its second moment \(\sum_{k=1}^{+\infty} k^2 \rho_{\varepsilon,t}(k) = (1 - \frac{t}{\varepsilon^2})^{-1}\) diverges as \(t\) tends to \(\varepsilon^2\).

Let us note that the set of equations

\[
\frac{d}{dt}\rho_t(k) = G_2(\rho_t, k), \quad \forall k \in \mathbb{N}^*
\]

corresponds to the Flory’s coagulation equations with the multiplicative kernel (see equation (1)).
The following proposition shows that for every \( j \geq 2 \), an approximation of the solution of the set of equations
\[
\frac{d}{dt}\rho_t(k) = G_j(\rho_t, k), \quad \forall k \in \mathbb{N}^*
\]
can be constructed by considering only loops of length \( j \):

**Proposition 1.3.** Let \( j \) be an integer greater than or equal to 2. The set of loops of length \( j \) before time \( t \) defines a subgraph of \( G_t(n) \) denoted by \( G_t(n,j) \). For \( k \in \mathbb{N}^* \), \( n \in \mathbb{N} \) and \( t > 0 \), let \( \rho^{(n,j)}_{t,x}(k) \) be the average number of components of size \( k \) in the random graph \( G_{nt(x+1)^j} \).

1. \( (\rho^{(n,j)}_{t,x}(k))_n \) converges to \( \rho^j_t(k) = e^{-tk} \frac{(tk)^{k-1}}{k^2(k-1)!} \mathbb{1}_{\{k-1 \in (j-1)\mathbb{N}\}} \) in \( L^2 \) for every \( t > 0 \).

2. \( (\rho^j_t(k), k \in \mathbb{N}^* \text{ and } t \geq 0) \) is a solution to the following coagulation equations:
\[
\frac{d}{dt}\rho_t(k) = G_j(\rho_t, k)
\]
where \( G_j \) is defined by equation (7).

The study of the random graph process defined by the Poisson ensemble of loops of a fixed length is postponed to Appendix B.

## 2 Component exploration procedure and associated Galton-Watson process

In this section, we describe a component exploration procedure modeled on the Karp [12] and Martin-Löf [19] exploration algorithm. The aim of this procedure is to find \( C_t(n)(x) \) and to construct a Galton-Watson process, the total population size of which bounds the size of the component \( |C_t(n)(x)| \).

### 2.1 Component exploration procedure

For every subset of vertices \( V \) let \( \mathcal{D}L_t(V) \) denote the subset of loops before time \( t \) included in \( V \) and let \( \mathcal{D}L_{t,x}(V) \) denote the subset of those that also pass through \( x \). Let define the set of ‘neighbours’ of \( x \) in \( V \) as
\[
N_{t,x}(V) = \{y \in V \setminus \{x\}, \exists f \in \mathcal{D}L_{t,x}(V) \text{ that passes through } y\}.
\]

In each step of the algorithm, a vertex is either active, explored or neutral. Let \( A_k \) and \( H_k \) be the sets of active vertices and explored vertices in step \( k \) respectively. In step 0, vertex \( x_1 = x \)
is said to be active \((A_0 = \{x_1\})\) and other vertices are neutral. In step 1, every neighbour is declared active and the vertex \(x\) is said to be an explored vertex: \(A_1 = N_{t,x}([n])\) and \(H_1 = \{x_1\}\). In step \(k\), let us assume that \(A_{k-1}\) is non-empty. Let \(x_k\) denote the smallest active vertex in \(A_{k-1}\). We add the neutral vertices \(z \in N_{t,x_k}([n] \setminus H_{k-1})\) to \(A_{k-1}\) and change the status of \(x_k\): \(A_k = A_{k-1} \cup N_{t,x_k}([n] \setminus H_{k-1}) \setminus \{x_k\}\) and \(H_k = H_{k-1} \cup \{x_k\}\). In particular, \(|A_k| = |A_{k-1}| + \xi_{t,k}^{(n)}\) with \(\xi_{t,k}^{(n)} = |N_{t,x_k}([n] \setminus H_{k-1}) \setminus A_{k-1}|\). The process stops in step \(T_t^{(n)} = \min(k, A_k = \emptyset)\). By construction, \(T_t^{(n)} = \min(k, \sum_{i=1}^k \xi_{t,i}^{(n)} \leq k - 1)\), the component of \(x\) is \(C_t^{(n)}(x) = H_{T_t^{(n)}}\) and its size is \(T_t^{(n)}\) (an example is presented in Figure 1).

Figure 1: a component of \(\mathcal{DL}_t^{(n)}\) formed by four based loops \(\ell_1 = (1, 2, 3, 4), \ell_2 = (2, 5, 2, 3), \ell_3 = (3, 6, 4)\) and \(\ell_4 = (6, 7)\).

The steps of exploration procedure for this component are

- **Step 1**: \(x_1 = 1\) and \(A_1 = \{2, 3, 4\}\) so that \(\xi_{t,1}^{(n)} = 3\).
- **Step 2**: \(x_2 = 2\) and \(A_2 = \{3, 4, 5\}\) so that \(\xi_{t,2}^{(n)} = 1\).
- **Step 3**: \(x_3 = 3\) and \(A_3 = \{4, 5, 6\}\) so that \(\xi_{t,3}^{(n)} = 1\).
- **Step 4**: \(x_4 = 4\) and \(A_4 = \{5, 6\}\) so that \(\xi_{t,4}^{(n)} = 0\).
- **Step 5**: \(x_5 = 5\) and \(A_5 = \{6\}\) so that \(\xi_{t,5}^{(n)} = 0\).
- **Step 6**: \(x_6 = 6\) and \(A_6 = \{7\}\) so that \(\xi_{t,6}^{(n)} = 1\).
- **Step 7**: \(x_7 = 7\) and \(A_7 = \emptyset\) so that \(\xi_{t,7}^{(n)} = 0\).

### 2.2 The Galton-Watson process associated to a component

The random variable \(\xi_{t,k}^{(n)}\) is bounded above by

\[
\xi_{t,k}^{(n,1)} = \sum_{\ell \in \mathcal{DL}_{t,x_k}([n] \setminus H_{k-1})} (|\ell| - 1)
\]

in which a same vertex is counted as many times as it appears in loops \(\ell \in \mathcal{DL}_{t,x_k}([n] \setminus H_{k-1})\).

To obtain identically distributed random variables in each step, we have to consider also in step \(k\), loops that pass through \(x_k\) and \(H_{k-1}\) before time \(t\). Let denote this set of loops \(\mathcal{DL}_{t,x_k,H_{k-1}}\).

We define \(\xi_{t,k}^{(n,2)} = \sum_{\ell \in \mathcal{DL}_{t,x_k,H_{k-1}}} (|\ell| - 1)\) and set

\[
\xi_{t,k}^{(n)} = \xi_{t,k}^{(n,1)} + \xi_{t,k}^{(n,2)} = \sum_{\ell \in \mathcal{DL}_{t,x_k}([n])} (|\ell| - 1).
\]
The distribution of \( \zeta_{t,k} \) is the compound Poisson distribution \( \text{CPois}(t, \beta_{n,\varepsilon, \nu_{n,\varepsilon}}) \) with
\[
\beta_{n,\varepsilon} = \mu(\{ \ell \in \mathcal{DL}(\mathbb{N}), x \in \ell \}) = \mu(\mathcal{DL}(\mathbb{N})) = \mu(\mathcal{DL}(\mathbb{N}) \setminus \{ x \}) = \log(1 + \frac{1}{n}) - \frac{1}{n(\varepsilon + 1)}.
\]
\[
\nu_{n,\varepsilon}(j) = \frac{1}{\beta_{n,\varepsilon}} \mu(\{ \ell \in \mathcal{DL}(\mathbb{N}), x \in \ell \text{ and } |\ell| = j + 1 \}) = \frac{1 - (1 - \frac{1}{n})^{j+1}}{\beta_{n,\varepsilon}(j+1)(\varepsilon + 1)^{j+1}} \quad \forall j \in \mathbb{N}^*.
\]

**Example 1.** For the component drawn in Figure 1, the random variables associated with the first three steps of the exploration procedure are \( \zeta_{t,1}^{(n,1)} = 3, \zeta_{t,1}^{(n,2)} = 0, \zeta_{t,2}^{(n,1)} = 3, \zeta_{t,2}^{(n,2)} = 3, \zeta_{t,3}^{(n,1)} = 2 \) and \( \zeta_{t,3}^{(n,2)} = 6 \).

Let \( \mathcal{F}_k = \sigma(H_j, A_j, j \leq k) \). Let us note that the random variables \( \zeta_{t,j}^{(n)} \) and \( \zeta_{t,k}^{(n)} \) for \( j < k \) are not independent since a same loop can belong to \( \mathcal{DL}_{t,x_k, H_{k-1}} \) and \( \mathcal{DL}_{t,x_j, H_{j-1}} \). Nevertheless, as disjoint subsets of loops in \( \mathcal{DL}_t^{(n)} \), are independent, the random variables \( \zeta_{t,j}^{(n,1)} \) for \( j \leq k \) are independent conditionally on \( \mathcal{F}_k \), and the random variable \( \zeta_{t,k}^{(n,1)} \) is independent of \( \zeta_{t,k}^{(n,2)} \) conditionally on \( \mathcal{F}_k \). Therefore, by using independent copies of the Poisson point processes \( \mathcal{DL} \), we can construct a sequence of nonnegative random variables \( \zeta_{t,k}^{(n,2)} \) such that for every \( k \):

- \( \zeta_{t,k}^{(n,2)} \) has the same distribution as \( \zeta_{t,k}^{(n,2)} \) and is independent of \( \zeta_{t,k}^{(n,1)} \) conditionally on \( \mathcal{F}_k \).
- \( \zeta_{t,k}^{(n)} = \zeta_{t,k}^{(n,1)} + \zeta_{t,k}^{(n,2)} \) are independent with distribution \( \text{CPois}(\beta_{n,\varepsilon, \nu_{n,\varepsilon}}) \).

Set \( \bar{T}_t^{(n)} = \min(k, \zeta_{t,1}^{(n)} + \ldots + \zeta_{t,k}^{(n)} = k - 1) \). By construction, \( \bar{T}_t^{(n)} \geq |C_t^{(n)}(x)| \). If \( \zeta_{t,1}^{(n)} \) is seen as the number of offspring of an individual \( I \) and \( \zeta_{k}^{(n)} \) for \( k \geq 2 \) as the number of offspring of the \( k \)-th individual explored by a breadth-first algorithm of the family tree of \( I \), then \( \bar{T}_t^{(n)} \) is the total number of individuals in the family tree of \( I \). We call \( \zeta_{t,k}^{(n)} \) the associated Galton-Watson process (a bijection between Galton-Watson trees and lattice walks was described by T. E. Harris [11] in Section 6, see also Section 6.2 in [23] for a review).

### 3 Approximation of component sizes

The number of neighbours of a vertex is used to approximate the number of vertices added in each step of the exploration process of a component. We begin this section by studying its asymptotic distribution. Next, we prove Theorem 1.1. Its proof is divided into two steps: we give an upper bound of the deviation between the cumulative distribution function of \( |C_{n,t}^{(n)}(x)| \) and of the total population size of the associated Galton-Watson process and then we study the asymptotic distribution of the Galton-Watson process associated to \( |C_{n,t}^{(n)}(x)| \). We end this section by a proof of Corollary 3.1.
3.1 Neighbours of a vertex

Let $V_n$ be a subset of vertices in $\bar{K}_n$ and let $x$ be another vertex. The aim of this section is to show that the number of neighbours of $x$ in $[n] \setminus V_n$ at time $nt$ (denoted by $|N_{nt,x}(\lfloor n \rfloor \setminus V_n)|$) converges in distribution to the compound Poisson distribution $CPois\left(\frac{t}{\varepsilon(\varepsilon + 1)}, G_{N^\ast}(\frac{t}{\varepsilon + 1})\right)$ if $\frac{V_n}{n}$ tends to 0.

The number of neighbours of $x$ in $\lfloor n \rfloor \setminus V_n$ at time $t$ is equal to $\sum_{\ell \in D_{L_t,x}(\lfloor n \rfloor \setminus V_n)} |\ell| - 1$ except if there are loops in $D_{L_x}(\lfloor n \rfloor \setminus V_n)$ that cross a same vertex several times or that cross another loop of $D_{L_t,x}(\lfloor n \rfloor \setminus V_n)$ in a vertex $y \neq x$. The following lemma yields an upper bound for the probability that such an event occurs:

**Lemma 3.1.** Let $x$ be a vertex. Set $G_{n,t}$ be the event ‘there exists a loop in $D_{L_t,x}(\lfloor n \rfloor)$ that crosses a same vertex several times or that intersects another loop in $D_{L_t,x}(\lfloor n \rfloor)$ at a vertex $y \neq x’.’

$$P(G_{n,t}) \leq \frac{t}{n^2 \varepsilon^3 (\varepsilon + 1)} + \frac{t^2}{n^3 \varepsilon^4}.$$  

**Proof.** We study separately the following two events:

- $G_{n,t}^{(1)}$: ‘there exists a vertex $y \neq x$ which is crossed several times by loops in $D_{L_t,x}(\lfloor n \rfloor)$’
- $G_{n,t}^{(2)}$: ‘there exists a loop in $D_{L_t,x}(\lfloor n \rfloor)$ that crosses $x$ several times’.

To compute $P(G_{n,t}^{(1)})$, we introduce the random variable $S_{t,x}$ as the total length of loops in $D_{L_t,x}(\lfloor n \rfloor)$ minus the number of times these loops pass through $x$: $S_{t,x} = \sum_{\ell \in D_{L_t,x}(\lfloor n \rfloor)} M_x(\ell)$ where $M_x(\ell)$ denotes the number of vertices different from $x$ in a loop $\ell$. Since the vertices that form a loop are chosen independently with the uniform distribution on $\lfloor n \rfloor$,

$$P(G_{n,t}^{(1)}) = 1 - \mathbb{E}(\prod_{i=0}^{S_{t,x}-1} (1 - \frac{i}{n-1})) \leq \frac{1}{2(n-1)} \mathbb{E}(S_{t,x}(S_{t,x} - 1)).$$

By Campbell’s formula, the probability-generating function of $S_{t,x}$ is

$$\mathbb{E}(u^{S_{t,x}}) = \exp\left(\sum_{\ell \in D_{L_t,x}(\lfloor n \rfloor)} (u^{M_x(\ell)} - 1)t\mu(\ell)\right).$$

The $\mu$-measure of loops in $D_{L_t,x}(\lfloor n \rfloor)$ of length $j$ that cross $i$ times the vertex $x$ is $\binom{j}{i} \frac{(n - 1)^j - i}{j(n(\varepsilon + 1))^j}$. 

Using the binomial formula, we obtain that \( \sum_{\ell \in \mathcal{L}_{x,t}} (u^{M_{\ell}} - 1) \mu(\ell) \) is equal to:

\[
\sum_{j=2}^{+\infty} \sum_{i=1}^{j} (u^{j-i} - 1) \left( \frac{j}{i} \right) \frac{(n-1)^{j-i}}{j(n+1)(n+2)}
\]

\[
= \sum_{j=2}^{+\infty} \frac{1}{j(n+1)(n+2)} \left( (u(n-1)+1)^j - u^j(n-1) - n^j + (n-1)^j \right)
\]

\[
= -\log(1 - \frac{1}{n+1}) - \log(1 - \frac{u(n-1)+1}{n+1}) + \log(1 - \frac{u(n-1)}{n+1}).
\]

Therefore, \( \mathbb{E}(u^{S_{t,x}}(S_{t,x} - 1)) = (1 + \frac{1}{n+1})^{-t} (1 - \frac{1}{n+1})^{-t}. \) In particular,

\[
\mathbb{E}(S_{t,x}(S_{t,x} - 1)) = \frac{t(n-1)^2(2n+1)}{(n+1)^2(n+2)^2}.
\]

Thus \( \mathbb{P}(G_{n,t,k}^{(1)}) \leq \frac{t(n+1)}{n+2}. \)

To study \( G_{n,t}^{(2)} \), we set \( N_x(\ell) \) the number of times a loop \( \ell \) passes through the vertex \( x \). We have \( \mathbb{P}(G_{n,t}^{(2)}) = 1 - \exp(-t\mu(\ell \in \mathcal{L}(n)), N_x(\ell) \geq 2) \). We have already seen in Lemma 3.3 that

\[
\mu(\ell \in \mathcal{L}(n)), N_x(\ell) \geq 1 = \log(1 + \frac{1}{n+1}) - \frac{1}{n+1}.
\]

Finally

\[
\mu(\ell \in \mathcal{L}(n)), N_x(\ell) = 1 = \sum_{j=2}^{+\infty} \frac{1}{j(n+1)(n+2)} = \frac{n-1}{n+1} = \frac{1}{n+1}.
\]

Therefore, \( \mathbb{P}(G_{n,t}^{(2)}) \leq \frac{t\mu(\ell \in \mathcal{L}(n)), N_x(\ell) \geq 2}{} = \frac{t}{(n+1)} \leq \frac{t}{n+1}. \)

The distribution of \( \sum_{\ell \in \mathcal{L}_{x,t}} (|\ell| - 1) \) is described in the following proposition:

**Proposition 3.1.** Let \( V_n \) be a subset of vertices and let \( x \) be another vertex.

(i) The random variable \( \sum_{\ell \in \mathcal{L}_{x,n}(n \setminus V_n)} (|\ell| - 1) \) is \( CPois(n^2b_n, \nu_n) \)-distributed where:

\[
b_n = \frac{1}{n} + \log(1 + \frac{1}{n+1}) \text{ and } \nu_n(j) = \frac{1}{b_n(j+1)} (1 - \frac{|V_n|}{n})^{j+1} + \frac{1}{b_n(j+1)} (1 - \frac{|V_n|}{n})^{j+1} \forall j \in \mathbb{N}^*.
\]

(ii) \( \mathbb{d}_{TV}(CPOIS(n^2b_n, \nu_n), CPOIS(\frac{t}{n+1}, \mathcal{G}_{n,t}^{(2)})) \leq \frac{t}{n+1} \left( \frac{|V_n|}{n} + \frac{1}{2n} \right). \)

**Proof.** (i) By definition of the Poisson loop set, \( \sum_{\ell \in \mathcal{L}_{x,n}(n \setminus V_n)} (|\ell| - 1) \) has a compound Poisson distribution, \( b_n = \mu(\ell \in \mathcal{L}(n \setminus V_n), x \in \ell) \) and for every \( j \in \mathbb{N}^* \),

\[
\nu_n(j) = \frac{1}{b_n} \mu(\ell \in \mathcal{L}(n \setminus V_n), x \in \ell, |\ell| = j+1).
\]
(ii) The total variation distance between two compound Poisson distributions can be bounded as follows using coupling arguments:

**Lemma 3.2.** Let $p_1$ and $p_2$ be two probability measures on $\mathbb{N}$ and let $\lambda_1$ and $\lambda_2$ be two positive reals such that $\lambda_1 < \lambda_2$. Then

\[
\text{d}_{TV}(\text{CPois}(\lambda_1, p_1), \text{CPois}(\lambda_2, p_2)) \leq 1 - e^{-(\lambda_2 - \lambda_1)} + \lambda_1 \text{ d}_{TV}(p_1, p_2).
\]

**Proof of Lemma 3.2.** By Strassen’s theorem, there exist two independent sequences $(X_i)_{i \in \mathbb{N}^*}$ and $(Y_i)_{i \in \mathbb{N}^*}$ of i.i.d. random variables with distributions $p_1$ and $p_2$ respectively such that $\text{d}_{TV}(p_1, p_2) = P(X_i \neq Y_i)$ for every $i \in \mathbb{N}$. Let $Z_1$ and $Z_2$ be two independent Poisson-distributed random variables with parameters $\lambda_1$ and $\lambda_2 - \lambda_1$ respectively which are independent of the two sequences $(X_i)_{i}$ and $(Y_i)_{i}$. Set $Z = Z_1 + Z_2$. Then

\[
\mathbb{P}(\sum_{i=1}^{Z_1} X_i \neq \sum_{i=1}^{Z_2} Y_i) \leq \mathbb{P}(Z_2 > 0) + \mathbb{P}(\sum_{i=1}^{Z_1} X_i \neq \sum_{i=1}^{Z_2} Y_i)
\]

and

\[
\mathbb{P}(\sum_{i=1}^{Z_1} X_i \neq \sum_{i=1}^{Z_2} Y_i) \leq \sum_{k=0}^{\infty} \mathbb{P}(Z_1 = k) \sum_{i=1}^{k} \mathbb{P}(X_i \neq Y_i) = \mathbb{E}(Z_1) \text{ d}_{TV}(p_1, p_2).
\]

We apply Lemma 3.2 with $\lambda_1 = nb_n t$, $\lambda_2 = \frac{t}{\varepsilon(x+1)}$, $p_1 = \nu_n$ and $p_2 = G_{\mathbb{N}^*}(\frac{\varepsilon}{\varepsilon+1})$; We have:

\[
\text{d}_{TV}(p_1, p_2) = \frac{\varepsilon}{2} \sum_{k=1}^{\infty} (\varepsilon + 1)^{-k} |a_{k,n}|
\]

with

\[
a_{k,n} = 1 - \frac{1}{(k+1)\varepsilon(\varepsilon + 1)b_n} \left( (1 - \frac{|V_n|}{n})^{k+1} - (1 - \frac{|V_n| + 1}{n})^{k+1} \right).
\]

As, for $x \in [0, 1]$ and $k \in \mathbb{N}^*$, \((1-x)^{k+1} - (1-x - \frac{1}{n})^{k+1} \leq \frac{k+1}{n} \text{ and } 1 - \frac{n}{k+1} \left( (1-x)^{k+1} - (1-x - \frac{1}{n})^{k+1} \right) \leq k(x + \frac{1}{2n}),\)

we obtain:

\[
\forall k \in \mathbb{N}^*, \quad -\frac{1}{\varepsilon^2} \left( \frac{|V_n|}{n} + \frac{1}{2n} \right) \leq nb_n a_{k,n} \leq \frac{k}{\varepsilon(\varepsilon + 1)} \left( \frac{|V_n|}{n} + \frac{1}{2n} \right).
\]
Therefore,
\[
d_{TV}(\text{CPois}(\lambda_1, p_1), \text{CPois}(\lambda_2, p_2)) \leq 1 - \exp \left( -\frac{t}{\varepsilon^2} \left( \frac{|V_n|}{n} + \frac{1}{2n} \right) \right) + \frac{t}{2\varepsilon^2} \left( \frac{|V_n|}{n} + \frac{1}{2n} \right) \leq \frac{3t}{2\varepsilon^2} \left( \frac{|V_n|}{n} + \frac{1}{2n} \right).
\]

In summary, it follows from Lemma 3.1 and Proposition 3.1 the following result for the number of neighbours of a vertex:

**Proposition 3.2.** Let \( x \in [n] \) be a vertex and let \( V_n \subset [n] \setminus \{x\} \) be a subset of vertices. The total variation distance between the distribution of \( |N_{nt,x}([n] \setminus V_n)| \) and the \( \text{CPois}\left( \frac{t}{(\varepsilon+1)} , \mathcal{G}_{n,t}^{*}(\frac{t}{\varepsilon+1}) \right) \) distribution is smaller than
\[
\frac{3t}{2\varepsilon^2} \left( \frac{|V_n|}{n} + \frac{1}{2n} \right) + \frac{t(\varepsilon+1)}{n\varepsilon^3} + \frac{t^2}{n\varepsilon^4}.
\]

### 3.2 Comparison between a component size and the associated Galton-Watson process

The aim of this section is to prove that small component sizes at time \( nt \) are well approximated by \( \bar{T}_{nt}^{(n)} \) which has the same distribution as the total population size of a Galton-Watson process with offspring distribution \( \text{CPois}(nt\beta_{n,\varepsilon}, \nu_{n,\varepsilon}) \) and a single ancestor (first step of the proof of Theorem 1.1):

**Proposition 3.3.** Let \( x \) be a vertex. There exist two polynomial functions \( B_1 \) and \( B_2 \) such that for every \( t > 0, \varepsilon > 0 \) and \( k, n \in \mathbb{N}^* \),
\[
|\mathbb{P}(|C_{nt}^{(n)}(x)| \leq k) - \mathbb{P}(\bar{T}_{nt}^{(n)} \leq k)| \leq B_1(t, \varepsilon) \frac{k^2}{n} + B_2(t, \varepsilon) \frac{k^4}{n^2}.
\]

Let us recall that the number of new vertices added in the \( j \)-th step of the exploration procedure is \( \xi_{t,j}^{(n)} = |N_{t,x_{j}}([n] \setminus H_{j-1}) \setminus A_{j-1}| \) where \( A_{j-1} \) and \( H_j = \{x_1, \ldots, x_{j-1}\} \) are respectively the set of active vertices and explored vertices in step \( j - 1 \). We have already seen one source of difference between \( \xi_{t,j}^{(n)} \) and \( \xi_{t,j}^{(n)} = \sum_{\ell \in DL_{t,x_{j}}([n])} (|\ell| - 1) \). It is described by the event

\( G_{n,t,j} \): ‘there exists a loop in \( DL_{t,x_{j}}([n] \setminus H_{j-1}) \) that crosses a same vertex several times or that crosses another loop in \( DL_{t,x_{j}}([n] \setminus H_{j-1}) \) at a vertex \( y \neq x_{j} \).’

By Lemma 3.1, the probability of this event is bounded by: \( \frac{t}{n\varepsilon^2}(\varepsilon + 1) + \frac{t^2}{n\varepsilon^3} \).

There are two other sources of difference described by the following events:
• \{ζ^{(n,2)}_{t,j} > 0\}: ‘there exists a loop passing through \(x_j\) and through already explored vertices \(H_{j-1}\),

• \(F_{n,t,j}\): ‘there exists a loop in \(\mathcal{DL}_{t,x_j}([-n] \setminus H_{j-1})\) (i.e. passing through \(x_j\) but not through \(H_{j-1}\)) which intersects active vertices \(A_{j-1}\),

The probability of these two events can be bounded by using the following lemma:

**Lemma 3.3.** Let \(A\) be a subset of vertices and let \(x\) be another vertex. For every \(t > 0\),

\[
\mathbb{P}(\exists \ell \in \mathcal{DL}_{t,x}([-n]), \ell \text{ intersects } A) = 1 - \left(1 + \frac{|A|}{nε|A| + 1}\right)^{-t}.
\]

**Proof.** Let \(F_{A,x}\) be the subset of loops \(\ell\) which intersect \(A\) and pass through \(x\).

\[
\mathbb{P}(\exists \ell \in \mathcal{DL}_{t,x}([-n]), \ell \text{ intersects } A) = 1 - \exp(-t\mu(F_{A,x}))
\]

and

\[
\mu(F_{A,x}) = \mu(\mathcal{DL}_{x}([-n])) - \mu(\mathcal{DL}_{x}([-n] \setminus A)).
\]

For a subset \(V\) of \(v\) vertices, set \(β_{n,ε,v} = \mu(\mathcal{DL}_{x}([-n]) \setminus V)):

\[
β_{n,ε,v} = \mu(\mathcal{DL}([-n]) \setminus V)) - \mu(\mathcal{DL}([-n] \setminus (V \cup \{x\})))
\]

\[
= -\log(1 - \frac{n - v}{n(ε + 1)}) - \frac{n - v}{n(ε + 1)} + \log(1 - \frac{n - v - 1}{n(ε + 1)}) + \frac{n - v - 1}{n(ε + 1)}
\]

\[
= -\log(1 + \frac{v}{nε}) + \log(1 + \frac{v + 1}{nε}) - \frac{1}{n(ε + 1)}.
\]

Then,

\[
\mu(F_{A,x}) = \log(1 + \frac{|A|}{nε|A| + 1}).
\]

With the help of these estimates, we prove Proposition 3.3.

**Proof of Proposition 3.3.** As \(|C_{nt}^{(n)}(x)| \leq ˜T_{nt}^{(n)}\),

\[
|\mathbb{P}(\{C_{nt}^{(n)}(x) \leq k\}) - \mathbb{P}(\tilde{T}_{nt}^{(n)} \leq k)| = \mathbb{P}(\{C_{nt}^{(n)}(x) \leq k\} \text{ and } \tilde{T}_{nt}^{(n)} > k).
\]

It is bounded above by

\[
\mathbb{P}(\{C_{nt}^{(n)}(x) \leq k\} \text{ and } \exists j \leq |C_{nt}^{(n)}(x)|, \xi_{nt,j}^{(n)} < \tilde{σ}_{nt,j}^{(n)} \leq \sum_{j=1}^{k} E_1(\{C_{nt}^{(n)}(x) \geq j\}) \mathbb{P}(\xi_{nt,j}^{(n)} < \tilde{σ}_{nt,j}^{(n)} | \mathcal{F}_{j-1})).
\]
We have seen that
\[ P(\xi(n)_{nt,j} < \bar{\zeta}(n)_{nt,j} | F_{j-1}) \leq P(\xi(n)_{nt,j} > 0 | F_{j-1}) + P(F_{nt,j} | F_{j-1}) + P(G_{nt,j} | F_{j-1}) \]
with the notations introduced page 17. By Lemma 3.3
\[ P(F_{nt,j} | F_{j-1}) \leq \frac{t|A_{j-1}|}{n\varepsilon^2} \]
and by Lemma 3.1
\[ P(\bar{\zeta}(n)_{nt,j} > 0 | F_{j-1}) \leq \frac{t(j-1)}{n\varepsilon^2} \]
Therefore,
\[ P(|C_{nt}(x)| \leq k \text{ and } \bar{T}_{nt} > k) \leq \frac{t}{n\varepsilon^2} \sum_{j=1}^{k} \left( E(|A_{j-1}| \mid \{C_{nt}(x) \geq j\}) + (j + 1 + \frac{t}{\varepsilon^2}) P(\bar{\zeta}(n)_{nt,j} \geq j | F_{j-1}) \right) \]
By construction \(|A_{j-1} - 1| = \sum_{i=1}^{j-1} (s_{nt,i}^{(n)} - 1)\). Let us recall that \(s_{nt,i}^{(n)}\) has nonnegative integer values, it is bounded above by \(\bar{s}_{nt,i}^{(n)}\) and the conditional law of \(\bar{s}_{nt,i}^{(n)}\) given \(F_{i-1}\) is equal to the law of \(\zeta_{nt,1}^{(n)}\). Thus,
\[ E(1_{\{C_{nt}(x) \geq j\}} | A_{j-1} - 1) \leq \sum_{i=1}^{j-1} E(1_{\{C_{nt}(x) \geq i\}} E(\bar{\zeta}_{nt,i}^{(n)} | F_{i-1})) \leq (j - 1) E(\zeta_{nt,1}^{(n)}) . \quad (9) \]
To conclude we note that \((E(\zeta_{nt,1}^{(n)}))_n\) converges to \(\frac{t}{\varepsilon} \), the expectation of \(\text{CPois}(\frac{t}{\varepsilon(\varepsilon + 1)}, \mathbb{N}^*)\) as \(n\) tends to \(+\infty\). Therefore, there exist positive reals \(A, B, C\) such that for every \(n \in \mathbb{N}, t \geq 0\) and \(\varepsilon > 0\),
\[ |P(|C_{nt}(x)| \leq k) - P(\bar{T}_{nt} \leq k)| \leq \frac{k^2 t}{n\varepsilon^2} (A + \frac{B}{\varepsilon} + \frac{Ct}{\varepsilon^2}) . \]

### 3.3 The total progeny of the Galton-Watson process associated to a component

Recall that the offspring distribution of the Galton-Watson process associated to a component at time \(nt\) is the compound Poisson distribution \(\text{CPois}(tn\beta, \nu)\) with:
\[ \beta = \log(1 + \frac{1}{n\varepsilon}) - \frac{1}{n(\varepsilon + 1)} . \]
and
\[ \nu(j) = \frac{1 - (1 - \frac{1}{n})^j + 1}{\beta(j + 1)(\varepsilon + 1)^j + 1} \quad \forall j \in \mathbb{N}^* . \]
We have shown (Proposition 3.1) that the compound Poisson distribution $\text{CPois}(tn^\beta, \nu_{n}, \epsilon_{n}, \nu_{n}, \epsilon_{n})$ is close to the $\text{CPois}(\frac{t}{\epsilon (\epsilon + 1)}, G_{N^*}(\frac{\epsilon}{\epsilon + 1}))$-distribution for large $n$. We now consider the distribution of the total number of individuals in a Galton-Watson process with one ancestor and offspring distribution $\text{CPois}(tn^\beta, \nu_{n}, \epsilon_{n}, \nu_{n}, \epsilon_{n})$. Let us state a general result for the comparison of the total number of individuals in two Galton-Watson processes:

**Lemma 3.4.** Let $\nu_1$ and $\nu_2$ be two probability distributions on $\mathbb{N}$. Let $d_{TV}$ denote the total variation distance between probability measures. Let $T_1$ and $T_2$ be the total population sizes of the Galton-Watson processes with one ancestor and offspring distribution $\nu_1$ and $\nu_2$ respectively. For every $k \in \mathbb{N}^*$, $|P(T_1 \geq k) - P(T_2 \geq k)| \leq 2d_{TV}(\nu_1, \nu_2) \sum_{i=1}^{k-1} P(T_2 \geq i)$.

**Proof.** We follow the proof of Theorem 3.20 in [29] which states an analogous result between binomial and Poisson branching processes. The proof is based on the description of the total population size by means of the hitting time of a random walk and coupling arguments. By Strassen’s theorem, there exist two independent sequences $(X_i)_{i \in \mathbb{N}^*}$ and $(Y_i)_{i \in \mathbb{N}^*}$ of i.i.d. random variables with distributions $\nu_1$ and $\nu_2$ respectively such that $d_{TV}(\nu_1, \nu_2) = P(X_i \neq Y_i)$ for every $i \in \mathbb{N}$. Let $\tau_1 = \min(n, X_1 + \ldots + X_n = n - 1)$ and $\tau_2 = \min(n, Y_1 + \ldots + Y_n = n - 1)$. $\tau_1$ and $\tau_2$ have the same law as $T_1$ and $T_2$ respectively. Let $k \in \mathbb{N}^*$.

$$P(\tau_1 \geq k \text{ and } \tau_2 < k) = P(\exists i \leq k-1, Y_i \neq X_i \text{ and } \tau_1 \geq k) \leq \sum_{i=1}^{k-1} P(X_j = Y_j \forall j \leq i-1, X_i \neq Y_i \text{ and } \tau_1 \geq k).$$

Since $\{X_j = Y_j \forall j \leq i-1 \text{ and } \tau_1 \geq k\} \subset \{\tau_2 \geq i\}$ and since $\{\tau_2 \geq i\}$ depends only on $Y_1, \ldots, Y_{i-1}$, we obtain:

$$P(\tau_1 \geq k \text{ and } \tau_2 < k) \leq \sum_{i=1}^{k-1} P(\tau_2 \geq i) P(X_i \neq Y_i) = d_{TV}(\nu_1, \nu_2) \sum_{i=1}^{k-1} P(\tau_2 \geq i).$$

Similarly,

$$P(\tau_1 < k \text{ and } \tau_2 \geq k) \leq \sum_{i=1}^{k-1} P(X_i \neq Y_i \text{ and } \tau_2 \geq i) \leq d_{TV}(\nu_1, \nu_2) \sum_{i=1}^{k-1} P(\tau_2 \geq i).$$

From Lemma 3.4 and Proposition 3.1, we obtain:
Proposition 3.4. Let \( t > 0 \) and \( n \in \mathbb{N}^* \). Let \( T_i^{(n)} \) and \( T_i \) denote the total number of individuals in a Galton-Watson process with one ancestor and offspring distribution \( CPois(t \beta_{n, \varepsilon}, \nu_{n, \varepsilon}) \) and \( CPois(t, \mathcal{G}_n(\varepsilon, n, \varepsilon + 1)) \) respectively.

\[
|\mathbb{P}(T_{nt}^{(n)} \geq k) - \mathbb{P}(T_t \geq k)| \leq \frac{3t(k - 1)}{2n \varepsilon^2} \quad \text{for every } k \in \mathbb{N}^*
\]

Theorem 1.1 follows from Propositions 3.3 and 3.4.

3.4 Asymptotic distribution of two component sizes

Theorem 1.1 can be extended to a joint limit theorem for the sizes of two components following the proof used in [3] for the Erdős-Rényi random graph process:

Corollary 3.1. Let \( x \) and \( y \) be two distinct vertices of \( \bar{K}_n \). For every \( t > 0, j, k \in \mathbb{N}^* \), \( \mathbb{P}(|C_{nt}^{(n)}(x)| = j \text{ and } |C_{nt}^{(n)}(y)| = k) \) converges to \( \mathbb{P}(T_{\varepsilon, t}^{(1)} = i) \mathbb{P}(T_{\varepsilon, t}^{(1)} = j) \) as \( n \) tends to \(+\infty\).

Proof of Corollary 3.1. The proof is similar to the proof presented in [3] for the Erdős-Rényi random graph. It is based on the following properties:

(i) the vertices of \( \bar{K}_n \) play the same role,

(ii) for every subset \( A \) of \([n]\), the loop set inside \( A \) at time \( t \), \( \mathcal{D}\mathcal{L}_t^{(n)}(A) \) is associated with the restriction of \( \mu \) to \( A \) (denoted \( \mu(A) \)) and is independent of \( \mathcal{D}\mathcal{L}_t^{(n)} \setminus \mathcal{D}\mathcal{L}_t^{(n)}(A) \). The measure \( \mu(A) \) can be seen as the loop measure on \( \bar{K}_n \) with vertex set \( A \), unit conductances and killing measure \( \kappa_{n, A} = |A| \varepsilon, n|A| = n - |A| + n \varepsilon \). Let us note that \( \varepsilon, n|A| = \varepsilon + \frac{n - |A|}{|A|}(\varepsilon + 1) \) converges to \( \varepsilon \) if \( |A|/n \) tends to \( 1 \).

Let \( x \) and \( y \) be two distinct vertices and let \( j, k \) be two nonnegative integers. We have to study the convergence of \( \mathbb{P}(|C_{\varepsilon, nt}^{(n)}(x)| = j \text{ and } |C_{\varepsilon, nt}^{(n)}(y)| = k) \). First, let us note that by (i), for every \( n \geq j \), \( \mathbb{P}(y \in C_{\varepsilon, nt}^{(n)}(x) \mid |C_{\varepsilon, nt}^{(n)}(x)| = j) = \frac{j - 1}{n - 1} \). Therefore, \( \mathbb{P}(y \in C_{\varepsilon, nt}^{(n)}(x) \text{ and } |C_{\varepsilon, nt}^{(n)}(x)| = j) \) converges to 0 as \( n \) tends to \(+\infty\).

By (ii), \( \mathbb{P}(|C_{\varepsilon, nt}^{(n)}(y)| = k \mid y \notin C_{\varepsilon, nt}^{(n)}(x) \text{ and } |C_{\varepsilon, nt}^{(n)}(x)| = j) = \mathbb{P}(|C_{\varepsilon, n-t, nt}^{(n-j)}(y)| = k) \). The convergence result stated in Theorem 1.1 still holds if \( t \) is replaced by the sequence \( (t_n)_n \) defined by \( t_n = (1 - \frac{1}{n}) t \forall n \in \mathbb{N}^* \), and if \( \varepsilon \) is replaced by the sequence \( (\varepsilon, n_j)_n \) which converges to \( \varepsilon \). Therefore, \( \mathbb{P}(y \notin C_{\varepsilon, nt}^{(n)}(x) \text{ and } |C_{\varepsilon, nt}^{(n)}(x)| = j \text{ and } |C_{\varepsilon, nt}^{(n)}(y)| = k) \) which is equal to

\[
\mathbb{P}(|C_{\varepsilon, nt}^{(n)}(y)| = k \mid y \notin C_{\varepsilon, nt}^{(n)}(x) \text{ and } |C_{\varepsilon, nt}^{(n)}(x)| = j) \times (1 - \mathbb{P}(y \in C_{\varepsilon, nt}^{(n)}(x) \mid |C_{\varepsilon, nt}^{(n)}(x)| = j)) \mathbb{P}(|C_{\varepsilon, nt}^{(n)}(x)| = j).
\]
converges to $\mathbb{P}(T_{\varepsilon,t}^{(1)} = k)\mathbb{P}(T_{\varepsilon,t}^{(1)} = j)$.

\[4\] \textbf{Phase transition}

This section is devoted to the proof of Theorem 1.2. The expectation of the compound Poisson distribution $\text{CPOis}(\frac{t}{\varepsilon(t+1)}, \mathcal{G}_{N^{*}}(\frac{\varepsilon}{\varepsilon+1}))$ is $\frac{t}{\varepsilon}$. Thus the limiting Galton-Watson process associated to a component is subcritical or supercritical depending on whether $t$ is smaller or larger than $\varepsilon^2$.

\[4.1\] \textbf{The subcritical regime}

An application of the component exploration procedure and a Chernov bound allow to prove that when $t < \varepsilon^2$, the largest component size at time $nt$ is at most of order $\log(n)$ with probability that converges to 1:

\[\text{Theorem (1.2.(i)).} \text{ Let } 0 < t < \varepsilon^2. \text{ Set } h(t) = \sup_{\theta \in [0, \log(\varepsilon+1)]} (\theta - \log(L_{t,\varepsilon}(\theta))) \text{ where } L_{t,\varepsilon} \text{ is moment-generating function of the compound Poisson distribution } \text{CPOis}(\frac{t}{\varepsilon(t+1)}, \mathcal{G}_{N^{*}}(\frac{\varepsilon}{\varepsilon+1})). \text{ For every } a > (h(t))^{-1}, \mathbb{P}(\max_{x \in [n]} |C_{nt}^{(n)}(x)| > a(\log(n)) \text{ converges to 0 as } n \text{ tends to } +\infty.\]

\[\text{Proof.} \text{ Let } k \in \mathbb{N}^{*}. \text{ By construction of the random variables } \xi_{t,j}^{(n)} \text{ and } \bar{\zeta}_{t,j}^{(n)}, \]

\[\mathbb{P}(|C_{lt}^{(n)}(v)| > k) \leq \mathbb{P}(\sum_{i=1}^{k} \xi_{t,i}^{(n)} \geq k) \leq \mathbb{P}(\sum_{i=1}^{k} \bar{\zeta}_{t,i}^{(n)} \geq k).\]

The moment-generating function of $\bar{\zeta}_{nt,x}^{(n)}$ is finite on $[0, \log(\varepsilon+1)]$ and is equal to

\[\mathbb{E}(e^{\theta \bar{\zeta}_{nt,x}^{(n)}}) = \exp \left( -nt \left( \log(1 + \frac{1}{n\varepsilon}) - e^{-\theta} \log(1 + \frac{e^\theta}{n(\varepsilon + 1 - e^\theta)}) \right) \right).\]

The moment-generating function of the compound Poisson distribution $\text{CPOis}(\frac{t}{\varepsilon(t+1)}, \mathcal{G}_{N^{*}}(\frac{\varepsilon}{\varepsilon+1}))$ is $L_{t,\varepsilon}(\theta) = \exp(-\frac{t}{\varepsilon} + \frac{t}{\varepsilon+1}e^{-\theta})$ for every $\theta \in [0, \log(1+\varepsilon)]$. Therefore, $\mathbb{E}(e^{\theta \bar{\zeta}_{nt,x}^{(n)}}) = L_{t,\varepsilon}(\theta) \exp(tg_{n}(\theta))$ where

\[g_{n}(\theta) = \sum_{j=1}^{+\infty} \frac{e^{\theta j} - 1}{(j + 1)(\varepsilon + 1)^{j+1}} \left( 1 - (1 - \frac{1}{n})^{j+1} - \frac{j + 1}{n} \right) \leq 0 \text{ for } \theta \geq 0.\]

By Markov’s inequality:

\[\mathbb{P}(|C_{nt}^{(n)}(x)| > k) \leq \mathbb{E}(e^{\theta \bar{\zeta}_{nt,x}^{(n)}}) e^{-k\theta} \leq \exp \left( -k(\theta - \log(L_{t,\varepsilon}(\theta))) \right) \quad \forall 0 < \theta < \log(\varepsilon + 1).\]

\[\text{h}(t) \text{ is the value of the Cramér function at 1 of } \text{CPOis}(\frac{t}{\varepsilon(t+1)}, \mathcal{G}_{N^{*}}(\frac{\varepsilon}{\varepsilon+1})). \text{ As the expectation of } \text{CPOis}(\frac{t}{\varepsilon(t+1)}, \mathcal{G}_{N^{*}}(\frac{\varepsilon}{\varepsilon+1})) \text{ is } \frac{t}{\varepsilon}, \text{ h}(t) \text{ is positive for } t < \varepsilon^2 \text{ and vanishes at } t = \varepsilon^2.\]
We deduce that for every \( k \in \mathbb{N}^* \), \( \mathbb{P}(|C^{(n)}_{nt}(x)| > k) \leq \exp(-kh(t)) \). In particular, for every \( a > 0 \), \( \mathbb{P}(\max_{v \in [n]} |C^{(n)}_{nt}(v)| > a \log(n)) \leq n^{-ah(t)} \exp(h(t)) \) which completes the proof.

4.2 The supercritical regime

When \( t > \varepsilon^2 \), the Galton-Watson process with family size distribution \( \text{CPois} \left( \frac{t}{\varepsilon(\varepsilon+1)^t}, \mathcal{G}_{\varepsilon+1} \right) \) is supercritical. Let \( q_{t,\varepsilon} \) be the extinction probability of this Galton-Watson process starting with one ancestor.

We show that there is a constant \( c > 0 \) such that with high probability there is only one component with more than \( c \log(n) \) vertices, and the size of this component is equivalent to \( n(1-q_{t,\varepsilon}) \):

**Theorem (1.2.(ii)).** Let \( C^{(n)}_{nt,m_1} \) and \( C^{(n)}_{nt,m_2} \) denote the first and second largest components of the random graph \( G^{(n)}_{nt} \). Assume that \( t > \varepsilon^2 \).

For every \( a \in ]1/2; 1[ \), there exist \( \delta > 0 \) and \( c > 0 \) such that

\[
\mathbb{P}( |C^{(n)}_{nt,m_1}| - n(1-q_{t,\varepsilon}) | \geq a \log(n)) + \mathbb{P}( |C^{(n)}_{nt,m_2}| \geq c \log(n)) = O(n^{-\delta}).
\]

The proof consists of four steps:

1. In the first step, we show that a vertex has a component of size greater than \( c \log(n) \) with a probability equivalent to the Galton-Watson process survival probability \( 1-q_{t,\varepsilon} \).

**Proposition 4.1.** Let \( X \) denote a \( \text{CPois} \left( \frac{t}{\varepsilon(\varepsilon+1)^t}, \mathcal{G}_{\varepsilon+1} \right) \)-distributed random variable with \( t > \varepsilon^2 \). Set \( I_t = \sup_{\theta \geq 0} (-\log \mathbb{E}(e^{-\theta X}) - \theta) \).

For every \( a > I_t^{-1} \), \( \mathbb{P}( |C^{(n)}_{nt,m_1}| \geq a \log(n)) = 1-q_{t,\varepsilon} + O \left( \frac{\log^2(n)}{n} \right) \).

2. For \( k \in \mathbb{N} \), let \( Z_{nt}(k) \) denote the number of vertices that belong to a component of size greater than or equal to \( k \) at time \( nt \). In the second step, we study the first two moments of \( Z_{nt}(k) \) in order to prove:

**Proposition 4.2.** For every \( b \in ]1/2; 1[ \), there exists \( \delta > 0 \) such that if \( a > I_t^{-1} \) then

\[
\mathbb{P}( |Z_{nt}(a \log(n)) - n(1-q_{t,\varepsilon}) | > n^b) = O(n^{-\delta}).
\]

3. The aim of the third step is to prove that with high probability, there is no component of size between \( c_1 \log(n) \) and \( c_2 n^\beta \) for any constant \( \beta \in ]1/2, 1[ \). More precisely, we show the following result on \( A_k(v) \), the set of active vertices in step \( k \) of the exploration of the component of a vertex \( v \):
Proposition 4.3. Let $\beta \in ]1/2, 1[$. For every $0 < c_2 < \min(1, \frac{1}{2\beta} - 1)$, there exists $\delta(c_2) > 0$ such that for $c_1 > \delta^{-1}(c_2)$,

$$\mathbb{P} \left( \exists v \in [n], A_{c_1 \log(n)}(v) \neq \emptyset \text{ and } \exists k \in [c_1 \log(n), n^\beta], |A_k(v)| \leq c_2 k \right) = O(n^{1-c_1 \delta(c_2)}).$$

4. In the fourth step, we deduce from Proposition 4.3 that with high probability there exists at most one component of size greater than $a \log(n)$:

Proposition 4.4. For every $0 < c_2 < \min(1, \frac{1}{2\beta} - 1)$, there exists $\delta(c_2) > 0$ such that for $c_1 > \delta^{-1}(c_2)$,

$$\mathbb{P} \left( \text{there exist two distinct components of size greater than } c_1 \log(n) \right) = O(n^{1-c_1 \delta(c_2)}).$$

Assertion (ii) of Theorem 1.2 is then a direct consequence of Proposition 4.2 and Proposition 4.4 since $Z_{nt}(c_1 \log(n))$ is equal to the size of the largest component on the event:

$$\{|Z_{nt}(c_1 \log(n)) - n(1-q_{t,\epsilon})| \leq n^b\} \cap \{\text{there is at most one component of size greater than } c_1 \log(n)\}.$$ 

The first two steps of the proof of assertion (ii) of Theorem 1.2 are similar to the first two steps detailed in [29] for the Erdős-Rényi random graph. The last two steps follow the proof described in [4] for the Erdős-Rényi random graph.

Proof of Proposition 4.1. Let $v$ be a vertex. By Theorem 1.1, for every $a > 0$,

$$\mathbb{P}(|C_{nt}^{(n)}(v)| \geq c \log(n)) = \mathbb{P}(T_{\epsilon,t}^{(1)} \geq c \log(n)) + O\left(\frac{\log^2(n)}{n}\right).$$

Moreover, $\mathbb{P}(T_{\epsilon,t}^{(1)} = +\infty) = 1 - q_{t,\epsilon}$. To complete the proof, we use the following result on the total progeny of a supercritical Galton-Watson process stated in [29]:

Theorem (3.8 in [29]). Let $T$ denote the total progeny of a Galton-Watson process with family size distribution $\nu$. Assume that $\sum_{k \in \mathbb{N}} k\nu(k) > 1$.

Then $I = \sup_{\theta \geq 0} \left( -\theta - \log(\sum_{k=0}^{+\infty} e^{-\theta x} \nu(k)) \right)$ is positive and $\mathbb{P}(k \leq T < +\infty) \leq \frac{e^{(1-e^{-1})I}}{1-e^{-1}}$.

Therefore, for every $c > I^{-1}$, $\mathbb{P}(c \log(n) \leq T_{\epsilon,t}^{(1)} < +\infty) = O(n^{-1})$ and

$$\mathbb{P}(|C_{nt}^{(n)}(v)| \geq c \log(n)) = 1 - q_{t,\epsilon} + O\left(\frac{\log^2(n)}{n}\right).$$

$\square$
**Proof of Proposition 4.2.** We shall use Bienaymé-Chebyshev inequality to bound

\[ \mathbb{P}(|Z_{nt}(a \log(n)) - n(1 - q_t(x))| > n^b). \]

As \( Z_{nt}(k) = \sum_{x \in [n]} \mathbb{I}_{\{|C_{nt}^{(n)}(x)| \geq k\}}, \) we deduce from Proposition 4.1 that if \( a > I_t^{-1} \) then

\[ \mathbb{E}(Z_{nt}(a \log(n))) = -n(1 - q_t(x)) + O(\log^2(n)). \]

We proceed as in [29] to bound the variance of \( Z_{nt}(k). \) The computations are based on the properties (i) and (ii) of \( G_t^{(n)} \) stated in the proof of Corollary 3.1 which implies that

\[ \mathbb{P}(|C_{nt}^{(n)}(y)| < k \mid y \notin C_{nt}^{(n)}(x) \text{ and } |C_{nt}^{(n)}(x)| = h) - \mathbb{P}(|C_{nt}^{(n)}(y)| < k) \]

is bounded above by the probability that there exist loops \( \ell \in \mathcal{D}C_{nt}^{(n)} \) passing through the two subsets of vertices \( \{1, \ldots, k\} \) and \( \{k + 1, \ldots, k + h\}. \)

We now explain in detail the computations. The variance of \( Z_{nt}(k) \) is equal to the variance of

\[ \sum_{x,y \in [n]} \mathbb{I}_{\{|C_{nt}^{(n)}(x)| < k\}}. \]

Therefore,

\[ \text{Var}(Z_{nt}(k)) = \sum_{x,y \in [n]} \left( \mathbb{P}(|C_{nt}^{(n)}(x)| < k \text{ and } |C_{nt}^{(n)}(y)| < k) - \mathbb{P}(|C_{nt}^{(n)}(x)| < k) \mathbb{P}(|C_{nt}^{(n)}(y)| < k) \right). \]

We split \( \mathbb{P}(|C_{nt}^{(n)}(x)| < k \text{ and } |C_{nt}^{(n)}(y)| < k) \) into two terms depending on whether the vertices \( x \) and \( y \) belong to a same component or not: \( \text{Var}(Z_{nt}(k)) = S_n^{(1)}(k) + S_n^{(2)}(k) \) where

\[ S_n^{(1)}(k) = \sum_{x,y \in [n]} \mathbb{P} \left[ |C_{nt}^{(n)}(x)| < k \text{ and } y \in C_{nt}^{(n)}(x) \right], \]

\[ S_n^{(2)}(k) = \sum_{x,y \in [n]} \left( \mathbb{P} \left[ C_{nt}^{(n)}(x) < k, |C_{nt}^{(n)}(y)| < k \text{ and } y \notin C_{nt}^{(n)}(x) \right] - \mathbb{P} \left[ C_{nt}^{(n)}(x) < k \right] \mathbb{P} \left[ |C_{nt}^{(n)}(y)| < k \right] \right). \]

First, \( S_n^{(1)}(k) = n \mathbb{E}(|C_{nt}^{(n)}(1)| \mathbb{I}_{\{|C_{nt}^{(n)}(1)| < k\}) \leq nk. \)

We consider now the following term in \( S_n^{(2)}(k): \)

\[ \mathbb{P} \left[ |C_{nt}^{(n)}(x)| < k, |C_{nt}^{(n)}(y)| < k \text{ and } y \notin C_{nt}^{(n)}(x) \right] = \sum_{h=1}^{k-1} \mathbb{P} \left[ |C_{nt}^{(n)}(x)| = h, |C_{nt}^{(n)}(y)| < k \text{ and } y \notin C_{nt}^{(n)}(x) \right]. \]

For an integer \( h < k, \)

\[ \mathbb{P} \left[ |C_{nt}^{(n)}(x)| = h, |C_{nt}^{(n)}(y)| < k \text{ and } y \notin C_{nt}^{(n)}(x) \right] \]

\[ \leq \mathbb{P} \left[ |C_{nt}^{(n)}(x)| = h \right] \mathbb{P} \left[ |C_{nt}^{(n)}(y)| < k \mid y \notin C_{nt}^{(n)}(x) \text{ and } |C_{nt}^{(n)}(x)| = h \right]. \]
Let $G_{nt}^{(n,h)}$ denote the random graph generated by the loops included in the subset of vertices $[n-h]$ and let $C_{nt}^{(n,h)}(1)$ denote the component of the vertex 1 in $G_{nt}^{(n,h)}$. By the properties of the Poisson loop ensemble,

$$\mathbb{P}[|C_{nt}^{(n)}(y)| < k \mid y \not\in C_{nt}^{(n)}(x) \text{ and } |C_{nt}^{(n)}(x)| = h] = \mathbb{P}[|C_{nt}^{(n,h)}(1) < k] .$$

We can couple $G_{nt}^{(n,h)}$ and $G_{nt}^{(n)}$ by adding to $G_{nt}^{(n,h)}$, $h$ vertices and the loops of an independent Poisson point process on $\mathbb{R}^+ \otimes DL([n])$ at time $nt$ that are not included in $[n-h]$. Therefore, $\mathbb{P}(|C_{nt}^{(n,h)}(1)| < k) - \mathbb{P}(|C_{nt}^{(n)}(1)| < k)$ is equal to the probability that the component of the vertex 1 in $G_{nt}^{(n,h)}$ is smaller than $k$ and that the component of 1 in $G_{nt}^{(n)}$ is greater or equal to $k$. This probability is bounded above by the probability that there exist loops $\ell \in DL^{(n)}$ passing through the two subsets of vertices $\{1, \ldots, k\} \text{ and } \{n-h+1, \ldots, n\}$. Therefore,

$$\mathbb{P}(|C_{nt}^{(n,h)}(1)| < k) - \mathbb{P}(|C_{nt}^{(n)}(1)| < k) \leq 1 - \exp\left(-nt(\mu(\ell \in DL([n]), \ell \text{ intersects } [k] \text{ and } \{k+1, \ldots, k+h\})\right)$$

$$= 1 - \left(1 + \frac{kh}{n\varepsilon(k+h+n\varepsilon)}\right)^{-nt} \leq 1 - \left(1 + \frac{k^2}{n^2\varepsilon^2}\right)^{-nt} .$$

We deduce that $S_{n}^{(2)}(k) \leq n^2 \mathbb{P}(|C_{nt}^{(n)}(1)| < k)\left(1 - \left(1 + \frac{k^2}{n^2\varepsilon^2}\right)^{-nt}\right)$ and

$$\text{Var}(Z_{nt}(k)) \leq nk + n^2\left(1 - \left(1 + \frac{k^2}{n^2\varepsilon^2}\right)^{-nt}\right).$$

Let us note that for every $\delta > 0$, $\frac{\text{Var}(Z_{nt}(a \log(n)))}{n^{1+\delta}}$ converges to 0 as $n$ tends to $+\infty$. Therefore, Bienaymé-Chebyshev inequality is sufficient to complete the proof. \(\square\)

**Proof of Proposition 4.3.** Let $\beta \in [1/2, 1[$. The idea of the proof is to lower bound the number of new active vertices at the first steps of the component exploration procedure by considering only loops inside a subset of $m_n = n - \lfloor 2n^\beta \rfloor$ vertices. For large $n$, the Galton-Watson associated to this component exploration procedure is still supercritical.

Let $\tau = T_t^{(n)} \wedge \min\{k \in \mathbb{N}^*, \sum_{i=1}^k \xi^{(n)}_{t,i} \geq 2n^\beta\}$. On the event $\{k \leq \tau\}$, the number of neutral sites at step $k$ is greater than $m_n$. Let $U_k$ denote the set of the $m_n$ first neutral vertices at step $k$ and let $Y_{t,k+1}^{(n)}$ denote the number of vertices $y \in U_k$ which are crossed by a loop $\ell \in DL_{t,x_k}(U_k \cup \{x_k\})$. On the event $\{k \leq \tau\}$, $Y_{t,k+1}^{(n)} \leq \xi_{t,k+1}^{(n)}$. Therefore, $\sum_{i=1}^{k \wedge \tau} Y_{t,i}^{(n)} \leq \sum_{i=1}^{k \wedge \tau} \xi_{t,i}^{(n)}$.

For a vertex $v$, set

$$\Omega_{c_1, c_2}^{(n)}(v) = \{A_{c_1 \log(n)}(v) \neq \emptyset \text{ and } \exists k \in [c_1 \log(n), n^\beta], |A_k(v)| \leq c_2k\}.$$
On the event \( \{ k \leq \tau \text{ and } |A_k(v)| \leq c_2 k \} \), \( \sum_{i=1}^{k} Y_{t,i}^{(n)} \) is bounded above by \((c_2 + 1)k - 1\). Thus,

\[
P(\Omega_{c_1,c_2}^{(n)}(v)) \leq \sum_{k=c_1 \log(n)}^{n^3} \mathbb{E} \left( \mathbb{P}(A_{c_1 \log(n)}(v) \neq \emptyset \text{ and } |A_k(v)| \leq c_2 k \mid F_{k-1}) \mathbb{I}_{(k \leq \tau)} \right)
\]

\[
\leq \sum_{k=c_1 \log(n)}^{n^3} \mathbb{P} \left( \sum_{i=1}^{k} Y_{t,i}^{(n)} \leq (c_2 + 1)k - 1 \right).
\]

where \((Y_{t,i}^{(n)})_i\) denotes a sequence of independent random variables distributed as \(|N_{\ell,1}(\lceil m_n + 1 \rceil)|\). The last step consists in establishing an exponential bound for

\[
p_{n,k} := \mathbb{P} \left( \sum_{i=1}^{k} Y_{t,i}^{(n)} \leq (c_2 + 1)k - 1 \right)
\]

uniformly on \(n\). A such exponential bound is an easy consequence of the following two facts:

(i) \((c_2 + 1)\) is smaller than the expectation of the CPois\((\frac{t}{e(c+1)}, G_{N^*}(\frac{e}{c+1}))\) distribution.

(ii) \((Y_{t,i}^{(n)})_n\) converges in law to the CPois\((\frac{t}{e(c+1)}, G_{N^*}(\frac{e}{c+1}))\) distribution (Proposition 3.2).

For every \(\theta > 0\), \(p_{n,k} \leq \exp(kA_n(-\theta))\) where \(A_n(\theta) = \log \left( \mathbb{E}(e^{\theta(Y_{t,1}^{(n)}-(c_2+1))}) \right)\). Let \(Y\) be CPois\((\frac{t}{e(c+1)}, G_{N^*}(\frac{e}{c+1}))\)-distributed random variable. Set \(\Lambda(\theta) = \log \left( \mathbb{E}(e^{\theta Y-(c_2+1)}) \right)\) for \(\theta < \log(1 + \varepsilon)\). As \(c_2 + 1 < \mathbb{E}(Y), \Lambda'(0) > 0\) and thus there exists \(u^* < \Lambda(u^*) < 0\). Set \(\delta = \frac{1}{2}\Lambda(u^*)\). By assertion (ii), \(\Lambda_n(u^*)\) converges to \(\Lambda(u^*)\), hence there exists \(n^*\) such that for every \(n \geq n^*\) and \(k \in \mathbb{N}^*, p_{n,k} \leq \exp(-k\delta)\). We deduce that for \(n \geq n^*\),

\[
P \left( \bigcup_{v \in \lceil n \rceil} \Omega_{c_1,c_2}^{(n)}(v) \right) \leq n \mathbb{P}(\Omega_{c_1,c_2}^{(n)}(1)) \leq n^{1-c_1\delta}(1 - e^{-\delta})^{-1}
\]

which converges to 0 if \(c_1 > \delta^{-1}\).

\[\square\]

**Proof of Proposition 4.4.** Let \(\Omega_{c_1,c_2}^{(n)}\) denote the event

\[
\{ \exists x \in \lceil n \rceil \text{ such that } A_{c_1 \log(n)}(x) \neq \emptyset \text{ and } 3k \in [c_1 \log(n), n^3] \text{ such that } |A_k(x)| \leq c_2 k \}.
\]

It occurs with probability \(O(n^{1-c_1\delta(c_2)})\) by Proposition 4.3.

Assume that \(\Omega_{c_1,c_2}^{(n)}\) does not hold and that there exist two vertices \(x_1\) and \(x_2\) the components of which are different and are both of size greater than \(c_1 \log(n)\). The subsets of active vertices in step \(n^3\), \(A_{n^3}(x_1)\) and \(A_{n^3}(x_2)\), are disjoint and both of size greater than \(c_2 n^3\). It means that
no loop $\ell \in \mathcal{DL}^{(n)}_{nt}$ passes through $A_{n,\beta}(x_1)$ and $A_{n,\beta}(x_2)$. Note that if $F_1$ and $F_2$ are two disjoint subsets of vertices then

$$
P(\exists \ell \in \mathcal{DL}^{(n)}_{nt}, \ell \text{ intersects } F_1 \text{ and } F_2) = \exp \left(- nt \mu(\ell \in \mathcal{DL}([n])), \ell \text{ intersects } F_1 \text{ and } F_2 \right)$$

$$
= \left(1 - \frac{|F_1||F_2|}{(1 + \frac{|F_1|}{n})(1 + \frac{|F_2|}{n})}\right)^{nt} \leq \exp \left(- \frac{t|F_1||F_2|}{n(\varepsilon + 1)^2}\right).
$$

Therefore there exists two different components of size greater than $c_1 \log(n)$ with a probability smaller than the sum of $P(\Omega_{\epsilon_1,\epsilon_2}^{(n)})$ and

$$
\mathbb{E} \left( \sum_{x_1, x_2 \in [n], A_{n,\beta}(x_1) \cap A_{n,\beta}(x_2) = 0, |A_{n,\beta}(x_1)| > c_1 \log(n), |A_{n,\beta}(x_2)| > c_1 \log(n)} P(\exists \ell \in \mathcal{DL}^{(n)}_{nt}, \ell \text{ intersects } A_{n,\beta}(x_2) \text{ and } A_{n,\beta}(x_2) | \mathcal{F}_{n,\beta}) \right)
$$

$$
\leq n^2 \exp \left(- \frac{t c_1^2 n^{2\beta-1}}{(\varepsilon + 1)^2}\right).
$$

\begin{align*}
\end{align*}

5 Hydrodynamic behavior of the coalescent process

This section is devoted to the proof of Proposition 1.2.

1. Let $t > 0$. First, we prove that $\rho_{\epsilon,t}^{(n)}(k) = \frac{1}{nk}\{|x \in [n], |C_{nt,\epsilon}^{(n)}(x)| = k\}$ converges in $L^2$ to $\rho_{\epsilon,t}(k) = \frac{1}{k} \mathbb{P}(T_{\epsilon,t}^{(1)} = k)$. Theorem 1.1 and Corollary 3.1 imply the convergence of the first two moments of $\rho_{\epsilon,t}^{(n)}(k)$ to $\rho_{\epsilon,t}(k)$ and $(\rho_{\epsilon,t}(k))^2$ respectively and thus the $L^2$ convergence of $(\rho_{\epsilon,t}^{(n)}(k))_n$. Indeed, $\mathbb{E}(\rho_{\epsilon,t}^{(n)}(k)) = \frac{1}{k} \mathbb{P}(|C_{nt,\epsilon}^{(n)}(1)| = k)$ converges to $\rho_{\epsilon,t}(k)$. The second moment is

$$
\mathbb{E}((\rho_{\epsilon,t}^{(n)}(k))^2) = \frac{1}{nk^2} \mathbb{P}(|C_{nt,\epsilon}^{(n)}(1)| = k) + (1 - \frac{1}{n}) \frac{1}{k^2} \mathbb{P}(|C_{nt,\epsilon}^{(n)}(1)| = k \text{ and } |C_{nt,\epsilon}^{(n)}(2)| = k).
$$

The first term converges to 0 and the second term converges to $(\rho_{\epsilon,t}(k))^2$.

2. It remains to show that $\{\rho_{\epsilon,t}, t \in \mathbb{R}_+\}$ is solution of the coagulation equations:

$$
\frac{d}{dt} \rho_t(k) = \sum_{j=2}^{+\infty} \frac{1}{(\varepsilon + 1)^j} G_j(\rho_t, k)
$$
where
\[ G_j(\rho_t, k) = \frac{1}{j} \left( \sum_{(i_1, \ldots, i_j) \in \mathbb{N}^j} \prod_{u=1}^j \rho_t(i_u) \mathbb{I}_{\{j \leq k\}} - k \rho_t(k) \left( \sum_{i=1}^{+\infty} i \rho_t(i) \right)^{j-1} \right) \]

\[ - k \rho_t(k) \sum_{h=1}^{+\infty} \binom{k-j-1}{h} \left( k \rho_t(i) \right)^{j-h} \left( \sum_{u=1}^{+\infty} u \rho_t(u) \right)^{j-1-h}. \]

By definition of \( \rho_{\varepsilon,t} \), \( G_j(\rho_{\varepsilon,t}, k) = \frac{1}{j} \mathbb{P}(T_{\varepsilon,t}^{(j)} = k) - k \rho_{\varepsilon,t}(k) \) where \( T_{\varepsilon,t}^{(j)} \) is the total progeny of a Galton-Watson process with family size distribution \( \text{C\!P\!o\!i\!s}\left(\frac{t}{\varepsilon(\varepsilon+1)}, G_{\mathbb{N}^*}(\frac{\varepsilon}{\varepsilon+1})\right) \) and \( j \) ancestors.

The probability distribution of \( T_{\varepsilon,t}^{(j)} \) is computed in the appendix (Lemma A.1):
\[
\begin{cases}
P(T_{\varepsilon,t}^{(j)} = j) = e^{-\frac{jt}{\varepsilon(\varepsilon+1)}} \\
P(T_{\varepsilon,t}^{(j)} = k) = j \sum_{h=1}^{k-j} \frac{\binom{k-j-1}{h}}{h!} \frac{1}{\varepsilon(\varepsilon+1)}^h \quad \forall k \geq j+1.
\end{cases}
\]

We deduce that
\[
\sum_{j=2}^{+\infty} \frac{1}{(\varepsilon+1)^j} G_j(\rho_{\varepsilon,t}, k) = e^{-\frac{kt}{\varepsilon(\varepsilon+1)}} \left( 1 + \frac{1}{h!} \sum_{h=1}^{k-2} \frac{\binom{k-1}{h}}{\varepsilon(\varepsilon+1)^h} \sum_{j=2}^{k-h} \binom{k-j-1}{h-1} \right) - \frac{k}{\varepsilon(\varepsilon+1)} \rho_{\varepsilon,t}(k).
\]

By using that \( \binom{m}{k} = \sum_{j=k-1}^{m-1} \binom{j}{k-1} \) for every \( 1 \leq k \leq m-1 \), we obtain that
\[
\sum_{j=2}^{+\infty} \frac{1}{(\varepsilon+1)^j} G_j(\rho_{\varepsilon,t}, k) \text{ is equal to } \frac{d}{dt} \rho_{\varepsilon,t}(k).
\]

### A Some properties of the Galton-Watson process with offspring distribution \( \text{C\!P\!o\!i\!s}(\lambda, G_{\mathbb{N}^*}(\rho)) \)

Let \( \lambda \) be a positive number and let \( p \in ]0,1[. \) In this section we describe some properties of the Galton-Watson process with offspring distribution \( \text{C\!P\!o\!i\!s}(\lambda, G_{\mathbb{N}^*}(\rho)) \) which are useful in the study of the component sizes of a random graph. In our model, the parameters are \( \lambda = \frac{t}{\varepsilon(\varepsilon+1)} \) and \( p = \frac{\varepsilon}{\varepsilon+1} \).

**Average number of offspring.** The expectation of the compound Poisson distribution \( \text{C\!P\!o\!i\!s}(\lambda, G_{\mathbb{N}^*}(\rho)) \) is \( \frac{\lambda}{p} \), hence the Galton-Watson process with offspring distribution \( \text{C\!P\!o\!i\!s}(\lambda, G_{\mathbb{N}^*}(\rho)) \) is subcritical if \( \frac{\lambda}{p} < 1. \)
Extinction probability. Let $\rho$ denote the extinction probability of this Galton-Watson process with one ancestor. As the probability-generating function of $\text{CPois}(\lambda, \mathcal{G}_{N^*}(p))$ is
\[
\phi(s) = \exp(-\lambda \frac{1-s}{1-s+sp}) \quad \forall s < \frac{1}{1-p},
\]
$\rho$ is the smallest positive solution to the equation $\exp(-\lambda \frac{1-s}{1-s+sp}) = s$.

Total progeny distribution. Let us first recall the general result on the total population size of a Galton-Watson proved by Dwass in [6].

Theorem. Consider a branching process with offspring distribution $\nu$ and $u \geq 1$ ancestors. Let $T$ denote its total progeny and let $(X_n)_n$ be a sequence of independent random variables with distribution $\nu$.
\[
\forall k \geq u, \quad P(T = k) = \frac{u}{k} P(X_1 + \ldots + X_k = k - u).
\]

Recall that in the supercritical case (i.e. $\sum k \nu(k) > 1$), $P(T < +\infty) = \rho_u < 1$ if $\rho$ denote the extinction probability of the branching process starting from one ancestor.

Using this theorem, we obtain:

Lemma A.1. Let $T(u)$ denote the total progeny of a Galton-Watson process with $u$ ancestors and with offspring distribution $\text{CPois}(\lambda, \mathcal{G}_{N^*}(p))$. Then,
\[
\begin{align*}
P(T(u) = u) &= e^{-u\lambda} \\
P(T(u) = k) &= \frac{u}{k} e^{-k\lambda}(1-p)^{k-u} \sum_{j=1}^{k-u} \binom{k-u-1}{j-1} \frac{1}{j!} \left(\frac{k\lambda p}{1-p}\right)^j \quad \forall k \geq u + 1.
\end{align*}
\]

Proof. In our setting, the sum $X_1 + \ldots + X_k$ appearing in the Dwass’s theorem has the same distribution as $\sum_{i=0}^{k} Y_i$ where $Z_k$ is a Poisson$(k\lambda)$-distribution random variable and $(Y_i)_i$ is a sequence of independent random variables with $\mathcal{G}_{N^*}(p)$-distribution.

Therefore, $P(T(u) = u) = P(Z = 0) = e^{-u\lambda}$ and for every $k \geq u + 1$,
\[
P(T(u) = k) = \frac{u}{k} \sum_{j=1}^{k-u} P(Z = j) P(Y_1 + \ldots + Y_j = k - u)
\]

with $P(Y_1 + \ldots + Y_j = k - u) = \binom{k-u-1}{j-1}(1-p)^{k-u-j}p^j$. \hfill \Box
Dual Galton-Watson process. A supercritical Galton-Watson process conditioned to become extinct is a subcritical Galton-Watson process:

Theorem ([2], Theorem 3, p. 52). Let \((Z_n)_n\) be a supercritical Galton-Watson process with one ancestor. Let \(\phi\) denote the generating function of its offspring distribution and let \(q\) denote its extinction probability. Assume that \(\phi(0) > 0\). Then, \((Z_n)_n\) conditioned to become extinct has the same law as a subcritical Galton-Watson process with one ancestor and offspring generating function \(s \mapsto \frac{1}{q} \phi(qs)\).

If the offspring distribution is \(\text{CPois}(\lambda, G_{n^*}(p))\), we obtain:

Lemma A.2. Let \(Z\) be a Galton-Watson process with offspring distribution \(\text{CPois}(\lambda, G_{n^*}(p))\). Assume that \(\lambda p > 1\) and let \(q\) denote the extinction probability of \(Z\). Then \(Z\) conditioned to become extinct has the same law as the subcritical Galton-Watson process with family size distribution \(\text{CPois}(\tilde{\lambda}, G_{n^*}(\tilde{p}))\) where \(\tilde{\lambda} = \lambda q \frac{p}{\tilde{p}}\) and \(\tilde{p} = q(1 - p)\).

In particular, if \(Z\) is the Galton-Watson process with family size distribution \(\text{CPois}(\tilde{\lambda}, G_{n^*}(p))\) and one ancestor for \(t > \varepsilon^2\) then \(Z\) conditioned to become extinct is a subcritical Galton-Watson process with family size distribution \(\text{CPois}(\tilde{\lambda}, G_{n^*}(\tilde{p}))\) where \(\tilde{\lambda} = \lambda q \frac{p}{\tilde{p}}\).

Let us note that in the Erdős-Rényi random graph \(\mathcal{H}(n, t\frac{1}{n})\) with \(t > 1\), the ‘dual’ of the Galton-Watson process with Poisson(\(t\)) offspring distribution corresponds to the limit of the Galton-Watson process associated to the component of a vertex outside the ‘maximal component’ of \(\mathcal{H}(n, t\frac{1}{n})\) (see, for example, [26]). It is also the case for the random graph \(\mathcal{G}^{(n)}_{n^*}(A)\) has the same law as a Poisson loop set defined on \(\overline{K_m}\) endowed with unit conductances and a uniform killing measure with intensity \(\tilde{\kappa}_n = \frac{m}{n} (\tilde{\epsilon} + 1)\) at time \(m_n t_n\) where \(\tilde{\epsilon} = \varepsilon + (\frac{m}{m_n} - 1)(\varepsilon + 1)\) and \(t_n = \frac{m}{m_n} \rightarrow \frac{1}{q(1-p)}\).

B The random graph process defined by loops of fixed length

Let \(j\) be an integer greater than or equal to 2. In this section, we consider the random graph \(\mathcal{G}^{(n,j)}_t\) defined by the set of loops of length \(j\) at time \(t\). The study of \(\mathcal{G}^{(n)}_t\) detailed in the paper can be conducted in the same manner on \(\mathcal{G}^{(n,j)}_t\). We present in this section the main results and justify Proposition 1.3.

For a vertex \(x\), let \(C^{(n,j)}_t(x)\) denote the connected component of a vertex \(x\) in \(\mathcal{G}^{(n,j)}_t\) ans set \(\mathcal{D}_t^j([n]) = \{\ell \in \mathcal{D}_t^j([n]), |\ell| = j\}\). A Galton-Watson process can be constructed by
using the component exploration procedure described in Section 2 to explore $C^{(n,j)}_t(x)$: the
offspring distribution of this Galton-Watson process is the distribution of $(j - 1)|\mathcal{DL}^{(j)}_{t,x}(\lfloor n \rfloor)|$.
Set $\beta^{(j)}_{n,\varepsilon} = \mu(\mathcal{DL}^{(j)}_{t,x}(\lfloor n \rfloor))$:
\[
\beta^{(j)}_{n,\varepsilon} = \frac{1}{j(\varepsilon + 1)^j}(1 - (1 - \frac{1}{n})^j).
\]
The random variable $|\mathcal{DL}^{(j)}_{t,x}(\lfloor n \rfloor)|$ is a Poisson$(t\beta^{(j)}_{n,\varepsilon})$-distributed random variable. Therefore,
$|\mathcal{DL}^{(j)}_{tn(\varepsilon + 1),x}(\lfloor n \rfloor)|$ converges in distribution to Poisson$(t)$ as $n$ tends to $+\infty$. Let $\nu_{t,j}$ be the
distribution of $(j - 1)Y$ where $Y$ denotes a Poisson$(t)$-distributed random variable. Let $T^{(u,j)}_t$ denote the total progeny of a Galton-Watson process with $u$ ancestors and offspring distribution $\nu_{t,j}$. The size of the connected component of $x$ in $G^{(n,j)}_{tn(\varepsilon + 1)^j}$ can be compared to $T^{(1,j)}_t$.

**Theorem B.1.** Let $\varepsilon$ and $t$ be two positive reals. 
If $(k_n)_n$ is a sequence of positive numbers such that $\frac{k^2_n}{n}$ converges to 0, then
\[
\mathbb{P}(\lfloor C^{(n,j)}_{nt(\varepsilon + 1)^j}(x) \rfloor \leq k_n) - \mathbb{P}(T^{(1,j)}_t \leq k_n)
\]
converges to 0.

We deduce the following joint limit theorem:

**Corollary B.1.** Let $x$ and $y$ be two distinct vertices of $\tilde{K}_n$. For every $t > 0$, $k, h \in \mathbb{N}^*$,
$\mathbb{P}(\lfloor C^{(n,j)}_{nt(\varepsilon + 1)^j}(x) \rfloor = k$ and $\lfloor C^{(n,j)}_{nt(\varepsilon + 1)^j}(y) \rfloor = h$) converges to $\mathbb{P}(T^{(1,j)}_t = k) \mathbb{P}(T^{(1,j)}_t = h)$ as $n$ tends to $+\infty$.

**Proof of Proposition 1.3.**

1. Let $k$ be a positive integer. The average number of components of size $k$ in the random
graph $G^{(n,j)}_{tn(\varepsilon + 1)^j}$ is $\rho^{(n,j)}_{t,x}(k) = \frac{1}{nk}\lfloor \{x \in \lfloor n \rfloor, \lfloor C^{(n,j)}_{nt(\varepsilon + 1)^j}(x) \rfloor = k \rfloor \rfloor$.
Set $\rho^{(j)}_{t}(k) = \frac{1}{k}\mathbb{P}(T^{(1,j)}_t = k)$. By Theorem B.1 and Corollary B.1, the first two moments of
$\rho^{(n,j)}_{t,x}(k)$ converge to $\rho^{(j)}_{t}(k)$ and $(\rho^{(j)}_{t}(k))^2$ respectively. Therefore, $(\rho^{(n,j)}_{t,x}(k))_n$ converges
to $\rho^{(j)}_{t}(k)$ in $L^2$ as $n$ tends to $+\infty$.

2. To complete the proof of Proposition 1.3, we compute the distribution of $T^{(u,j)}_t$ for $u \in \mathbb{N}^*$,
using Dwass’s Theorem:
\[
\begin{cases}
\mathbb{P}(T^{(u,j)}_t = u + (j - 1)k) = \frac{u}{k}(u + (j - 1)k)^k k^{-1} e^{-(u + (j - 1)k)t} & \forall k \in \mathbb{N} \\
\mathbb{P}(T^{(u,j)}_t = h) = 0 & \text{if } h - u \notin (j - 1)\mathbb{N}.
\end{cases}
\]
As $1 + (j - 1)k = j + (j - 1)(k - 1)$,

$$\frac{d}{dt} \rho_t^{(j)}(1 + (j - 1)k) = \frac{1}{j} P(T_t^{(j,j)} = 1 + (j - 1)k) - P(T_t^{(1,j)} = 1 + (j - 1)k)$$

Therefore, $(\rho_t^{(j)}(k))_{t \geq 0}$ is solution of equation (8) for every $k \in \mathbb{N}^*$.

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