Fifth-degree B-spline solution for nonlinear fourth-order problems with separated boundary conditions

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Abstract. In this paper, we discussed a fifth-degree B-spline solution for the numerical solution to nonlinear fourth-order boundary value problems (BVPs) with separated boundary conditions. Two numerical examples are given to illustrate the efficiency and performance of the method. The method gives accurate results for both the linear and nonlinear cases.

1. Introduction
In a recent paper, Loghmani and Alavizadeh [1] have considered the use of the B-spline for solving linear and nonlinear two-point BVPs. This problem belongs to the following general class of BVPs and many problems (such as the plate deflection theory and the problem of bending of a plate on an elastic foundation) in engineering are formulated with this class [2-4]:

\[ y^{(4)} + p(x)y(x) = q(x), \quad A \leq x \leq B, \]  \hspace{2cm} (1)

\[ y(x) = S_i, \quad y'(x) = S_2, \quad y''(x) = S_3, \quad y'''(x) = S_4. \]  \hspace{2cm} (2)

The analytical solution to the above problem for which several authors have considered some numerical methods cannot be found for all \( p(x) \) and \( q(x) \). Several authors have considered some numerical methods for this problem. For example, Simos and Papakaliatikis [3,4] have studied this problem using different techniques such as Runge-Kutta Verner method and the finite difference method. Recently, L.Xu [8] has also applied He’s [9-13] variational iteration method to nonlinear fourth-order boundary value problems.

In our previous work [5], we have examined the heat equation using third degree B-splines. In the present paper, fifth-degree B-spline used to solve fourth-order nonlinear BVPs.

2. The fifth-degree B-splines

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In this section, the fifth-degree B-splines are used to construct numerical solutions to the nonlinear fourth-order problems with separated boundary conditions discussed in sections 3 and 4. A detailed description of B-spline functions generated by subdivision can be found in [6].

Consider equally-spaced knots of a partition \( \pi: a = x_0 < x_1 < \ldots < x_n = b \) on \([a,b]\). Let \( S_j[\pi] \) be the space of continuously-differentiable, piecewise, fifth-degree polynomials on \( \pi \), that is, \( S_j[\pi] \) is the space of fifth-degree splines on \( \pi \). Consider the B-splines basis in \( S_j[\pi] \). The fifth-degree B-splines are defined as

\[
B_5(x) = \frac{1}{120h^5} \left\{ \begin{array}{l}
x^5 - 5x^4 + 30hx^3 - 60h^2x^2 + 60h^3x - 30h^4x + 6h^5 \\
-5x^4 + 30hx^3 - 60h^2x^2 + 60h^3x - 30h^4x + 6h^5 \\
10x^5 - 120hx^4 + 540h^2x^3 - 1140h^3x^2 + 1170h^4x - 474h^5 \\
-10x^4 + 180hx^3 - 1260h^2x^2 + 4260h^3x - 6930h^4x + 4386h^5 \\
5x^5 - 120hx^4 + 1140h^2x^3 - 5340h^3x^2 + 12270h^4x - 10974h^5 \\
-x^4 + 30hx^3 - 360h^2x^2 + 2160h^3x - 6480h^4x + 7776h^5
\end{array} \right. \\
0 \leq x \leq h \\
h \leq x \leq 2h \\
2h \leq x \leq 3h \\
3h \leq x \leq 4h \\
4h \leq x \leq 5h \\
5h \leq x \leq 6h
\]

\( B_{i,1}(x) = B_0(x - (i-1)h) \), \( i = 2,3,... \)

To solve Eq(1), \( B_i, B_i', B_i'', B_i''' \) and \( B_i^{(iv)} \) evaluated at the nodal points are needed. Their coefficients are summarized in Table 1.

| \( x_i \) | \( x_{i+1} \) | \( x_{i+2} \) | \( x_{i+3} \) | \( x_{i+4} \) | \( x_{i+5} \) | \( x_{i+6} \) |
|-------|-------|-------|-------|-------|-------|-------|
| \( B_i \) | 0 | 1 | 26 | 66 | 26 | 1 | 0 |
| \( B_i' \) | 0 | 5/h | 502/h | 0 | -50/h | -5/h | 0 |
| \( B_i'' \) | 0 | 20/h^2 | 40/h^2 | -120/h^2 | 40/h^2 | 20/h^2 | 0 |
| \( B_i''' \) | 0 | 60/h^3 | -120/h^3 | 0 | 120/h^3 | -60/h^3 | 0 |
| \( B_i^{(iv)} \) | 0 | 120/h^4 | 480/h^4 | 720/h^4 | -480/h^4 | 120/h^4 | 0 |

3. Spline solutions for nonlinear boundary-value problems

Consider a fourth-order nonlinear BVPs of the form

\[
y^{(4)}(x) + f(x, y, y', y'', y''') = 0, \tag{4}
\]

with boundary conditions.

\[
a_0y(0) + a_1y(1/2) + a_2y(1) = A_0, \quad y(1) = A_1, \tag{5}
\]

\[
b_0y'(0) + b_1y'(1/2) + b_2y'(1) = B_0, \quad y'(1) = B_1, \tag{6}
\]

where \( f \) is a given nonlinear function of \( y \), \( a_i(i = 0, 2), b_i(i = 0, 2), A_i(i = 0, 1) \) and \( B_i(i = 0, 1) \) are finite real constants. We seek a function \( S(x) \) that approximates the solution to Eq.(4), which may be represented as
\[ S(x) = \sum_{j=5}^{n-1} C_j B_j(x) \]  

(7)

where \( C_j \) are unknown real coefficients and \( B_j(x) \) are fifth-degree B-spline functions. Let \( x_0, x_1, ..., x_n \) be \( n+1 \) grid points in interval \([a,b]\) so that \( x_i = a + ih, \ i = 1, 2, ..., n, \ x_0 = a, x_n = b, h = (b - a)/n. \) The approximate solution (7) is substituted in Eqs.(4-6) and evaluated at the grid points \( x_0, x_1, ..., x_n. \) This leads to a nonlinear system of equations of the form

\[ \sum_{j=5}^{n-1} C_j B_j^{(i)}(x_i) = f\left(\sum_{j=5}^{n-1} C_j B_j(x_i)\right), \quad i = 0, 1, ..., n \]  

(8)

Then it is obtained as in the following nonlinear system

\[ \sum_{j=5}^{n-1} C_j B_j(x_i) = A_i, \quad \text{for } x = 1, \]  

(9)

\[ \sum_{j=5}^{n-1} C_j B_j'(x_i) = B_i, \quad \text{for } x = 1, \]  

(10)

\[ a_0 \sum_{j=5}^{n-1} C_j B_j(0) + a_1 \sum_{j=5}^{n-1} C_j B_j(l/2) + a_2 \sum_{j=5}^{n-1} C_j B_j(l) = A_0, \]  

(11)

\[ b_0 \sum_{j=5}^{n-1} C_j B_j'(0) + b_1 \sum_{j=5}^{n-1} C_j B_j'(l/2) + b_2 \sum_{j=5}^{n-1} C_j B_j'(x_i) = B_0, \]  

(12)

\[ \sum_{j=5}^{n-1} C_j B_j^{(i)}(x_i) + f\left(\sum_{j=5}^{n-1} C_j B_j(x_i)\right) = 0, \quad i = 0, 1, ..., n . \]  

(13)

The values of the spline functions at the knots \( \{x_j\}_{j=0}^{n} \) are determined using Table 1 with substitution in Eqs.(9-13). Thus a system of \( n + 5 \) nonlinear equations in the \( (n + 5) \) unknowns \( C_{-5}, C_{-4}, ..., C_{n-1} \) is obtained. The approximate solution (7) is obtained by solving the nonlinear system using Levenberg-Marquardt optimization method [7] and MATLAB 6.5.

4. Numerical results

In this section, the method discussed in Sections 2 and 3 is tested on the following problems from the literature [1] and the absolute error in the analytical solutions is calculated. All computations were carried out using MATLAB 6.5.

4.1. Example 1

We consider the linear boundary value problem

\[ y^{(e)}(x) + xy(x) = -(8 + 7x + x^3)e^x, \quad 0 \leq x \leq 1 \]  

(14)

with the boundary conditions,

\[ y(0) + y(l/2) + y(l) = \frac{e^{l/2}}{4}, \quad y(l) = 0 , \]  

(15)
\[ y'(0) + 3y'(l/2) - 2y'(l) = 1 + \frac{3e^{l/2}}{4} + 2e, \quad y'(l) = 0, \quad (16) \]

The exact solution to this problem is \( y(x) = x(l + x)e^x \). The observed maximum absolute errors for various values of \( n \) are given in Table 2. The numerical results are illustrated in Figure 1.

| Table 2. The maximum absolute errors. |
|----------------------------------------|
| \( n \) | Example 1 | Example 2 |
|---------|-----------|-----------|
| 11      | 6.00643249x10^{-5} | 1.95314316x10^{-5} |
| 21      | 1.57587578x10^{-5} | 4.82227410x10^{-6} |
| 31      | 6.97503288x10^{-6} | 2.13858644x10^{-6} |

**Figure 1.** Results for \( n = 21 \) for example 1.

### 4.2. Example 2

We consider the nonlinear boundary value problem

\[ y^{(4)}(x) + y'(x)y'''(x) - 4(y'(x))^2 y''(x) = 0, \quad 0 \leq x \leq l, \quad (17) \]

with the boundary conditions,

\[ y(0) + y(l/2) + y(l) = \ln(3), \quad y(l) = 0, \quad (18) \]
\[ y'(0) + 3y'(l/2) - 2y'(l) = 2, \quad y'(l) = l/2, \quad (19) \]

The exact solution to this problem is \( y(x) = \ln(l + x) \). The observed maximum absolute errors for various values of \( n \) are given in Table 2. The numerical results are illustrated in Figure 2.
5. Numerical results

In this paper, we survey B-spline method for the solution of the BVPs associated with the fourth-order nonlinear differential equations. The fifth-degree B-spline has been tested on fourth-order nonlinear BVPs and the maximum absolute errors have been tabulated. As is evident from the numerical results, the present method approximates the exact solution very well. Also the method gives accurate results in both linear and nonlinear boundary conditions.

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