Virial coefficients from 2+1 dimensional QED effective actions at finite temperature and density

P. F. Borges\textsuperscript{1*}, H. Boschi-Filho\textsuperscript{2†} and Marcelo Hott\textsuperscript{3‡}

\textsuperscript{(1)} Centro Federal de Educação Tecnológica Celso Suckow da Fonseca
Coordenação de Física, Av. Maracanã, 229, Maracanã
20271-110 Rio de Janeiro, BRAZIL

\textsuperscript{(2)} Instituto de Física, Universidade Federal do Rio de Janeiro
Cidade Universitária, Ilha do Fundão, Caixa Postal 68528
21941-972 Rio de Janeiro, BRAZIL

\textsuperscript{(3)} Departamento de Física e Química, Universidade Estadual Paulista
Campus de Guaratinguetá, Caixa Postal 205
12500-000 Guaratinguetá, São Paulo, BRAZIL

Abstract

From spinor and scalar 2+1 dimensional QED effective actions at finite temperature and density in a constant magnetic field background, we calculate the corresponding virial coefficients for particles in the lowest Landau level. These coefficients depend on a parameter $\theta$ related to the time-component of the gauge field, which plays an essential role for large gauge invariance. The variation of the parameter $\theta$ might lead to an interpolation between fermionic and bosonic virial coefficients, although these coefficients are singular for $\theta = \pi/2$.

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\textsuperscript{*}e-mail: pborges@cefet-rj.br
\textsuperscript{†}e-mail: boschi@if.ufrj.br
\textsuperscript{‡}e-mail: hott@feg.unesp.br
1 Introduction

Gauge invariance in 2+1 spacetime dimensions allows the existence of a mass term for
gauge fields, known as the Chern-Simons (CS) term \[1\]. This term can be generated
dynamically by radiative corrections \[2\] but can not preserve parity and gauge invariance
simultaneously. At finite temperature, gauge invariance takes on a particular form being
large (not infinitesimal) with respect to the compactified Euclidean time, leaving a free
parameter in the theory, usually a constant value of the time-component of the gauge
field. Recently, it has been shown that a resummation of the one-loop graphs is essential
to preserve the large gauge invariance of the finite temperature 2+1 dimensional quantum
electrodynamics (QED\(_3\)) effective action \[3, 4\].

In this letter we compute explicitly the virial coefficients from spinor and scalar QED\(_3\)
effective actions in a constant magnetic field background for particles in the lowest Landau
level, maintaining large gauge invariance.

The virial coefficients obtained here are similar to those found previously \[5\] impos-
ing generalized boundary conditions to scalar and spinor fields in 2+1 dimensions in the
absence of magnetic fields. The possibility of relating these boundary conditions to frac-
tional statistics was envisaged in a quantum mechanical one-dimensional model \[6\]. It
has also been shown in different contexts at finite temperature that if one considers an
imaginary part of the chemical potential, fermions can transmute into bosons \[7\].

In fact, the coefficients found here depend on the time-component of the gauge field
which can be seen as an imaginary part of the chemical potential and whose variation
might lead to an interpolation between fermionic and bosonic coefficients. However, such
coefficients differ from those coming from non-relativistic anyons (quasi-particles with
fractional statistics) \[8, 9, 10, 11, 12\] and are singular for a certain combination of the
time-component of the gauge field and the temperature.

The physical picture that would be responsible for generating the interpolation of the
bosonic and fermionic virial coefficients for both scalar and spinor QED\(_3\) at finite tem-
perature and density is the large gauge invariance together with the cylindrical structure
of the space-time, \(\mathcal{R}^2 \times S^1\). This non-trivial topology inhibits the removal of the time
component of the gauge field leaving a free parameter in the theory.
2 Effective actions at finite temperature and density

In QED the one-loop effective action for the gauge field $A_\nu = A_\nu(x)$ is obtained by integrating out the fermion field. In 2+1 dimensions one has

$$\exp\left\{i\mathcal{S}_{\text{eff}}[A_\nu]\right\} = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\left\{i \int d^3x \bar{\psi}[\gamma^\nu(i\partial_\nu + eA_\nu) - m]\psi - \frac{1}{4}F^\rho_\nu F^\rho_\mu\right\}.$$  

The effective action can be calculated exactly for some configurations of the gauge field in which cases the field is understood as a classical background.

The thermodynamics of a system in thermal equilibrium at a finite temperature $T = \beta^{-1}$ is described by its partition function $Z$ which can be obtained by a Wick rotation of the effective action from Minkowski to Euclidean space with the imaginary time $\tau = -ix^0$ compactified into the interval $[0, \beta]$. Further, to describe the system at finite density we introduce a chemical potential $\mu$ which is identified with the imaginary part of the time component of gauge field $A_\nu$. The free energy

$$\Omega(\beta, \mu) = -\frac{1}{\beta} \ln Z$$

within this prescription is given in terms of the real part of the effective action as

$$\Omega(\beta, \mu) = -\frac{1}{\beta} \Re\{\mathcal{S}_{\text{eff}}(\beta, \mu)\}.$$  

It can be expanded in terms of the fugacity $z = \exp(e\beta\mu)$ as

$$\Omega(\beta, \mu) = -\frac{S}{\beta} \sum_n b_n z^n,$$  

where $S$ is the area and $b_n \equiv b_n(\beta)$ are the cluster coefficients. The pressure can be written as a power expansion of the density as

$$P_\beta = \sum_{n=1}^\infty a_n \rho^n,$$

where $a_n$ are the virial coefficients which can be obtained through the standard relations:

$$b_1 = a_1 b_1;$$

$$b_2 = 2a_1 b_2 + a_2 b_1^2;$$

$$b_3 = 3a_1 b_3 + 4a_2 b_1 b_2 + a_3 b_1^3;$$

$$b_4 = 4a_1 b_4 + 4a_2 (b_2^2 + 6b_1 b_3) + 6a_3 b_1^2 b_2 + a_4 b_1^4;$$

$$\vdots$$

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3 The virial coefficients from fermions

Here we consider the QED\(_3\) effective action for a configuration where the time-component of the gauge field depends only on the Euclidean time \(A_3 = A_3(\tau)\), and choose \(A_j = \frac{1}{2}F_{jk}x^k\) with \(j = 1, 2\) corresponding to a constant magnetic field background \((F_{12} = B)\). In this case it is possible to obtain an exact result for the fermion propagator, the effective action [2], and generate the CS term dynamically preserving large gauge invariance [4]. Further, we work with a \((2 \times 2)\) irreducible representation of Dirac matrices in 2+1 dimensions implying a CS term that explicitly breaks parity invariance. Besides, there is a parity invariant contribution that comes from the Maxwell term which in this case corresponds to a constant magnetic field \(B\).

The effective action at finite temperature and density that comes from the parity violating Lagrangian density can be found to be [15, 16]

\[
S_{PV\text{ eff}}(\beta, \mu) = eBS\frac{m}{4\pi|\mu|}\left\{ie\beta \Xi + G_+(|m| + ie \Xi) - G_+(|m| - ie \Xi)\right\},
\]

(3)

where we defined

\[
G_+(x) = \ln(1 + e^{-\beta x})
\]

and included the chemical potential contribution through \(\Xi = \tilde{A}_3 + i\mu\) where \(\tilde{A}_3\) is a constant related to \(A_3(\tau)\) by the large gauge transformations [4]

\[
\tilde{A}_3 = \frac{2\pi k}{e\beta} + \frac{1}{\beta}\int_0^\beta d\tau A_3(\tau),
\]

(4)

with \(k = 0, \pm 1, \pm 2, \ldots\) being the winding number. Since the finite temperature and density QED effective actions that we discuss here depend on \(A_3(\tau)\) through \(\tilde{A}_3\) they are gauge invariant under large gauge transformations.

The effective action at finite temperature and density that comes from parity invariant Lagrangian can be written as [15, 16]

\[
S_{PI\text{ eff}}(\beta, \mu) = e|B|S\frac{m}{4\pi}\sum_{\ell=0}^{\infty} \sum_{s=1}^{2} \left\{\beta E_{\ell,s} + G_+(E_{\ell,s} + ie \Xi) + G_+(E_{\ell,s} - ie \Xi)\right\},
\]

(5)

where \(E_{\ell,s} = \sqrt{m^2 + 2e|B|(\ell + s - 1)}\) is the energy of the Landau levels.

Now, we can calculate the cluster and virial coefficients from the above spinor effective actions. Expanding \(G_+(x)\) as a power series of the fugacity \(z\), we have that the corresponding contribution of the parity violating (PV) part Lagrangian to the free energy is
given by:
\[
\Omega^{PV}(\beta, \mu) = \frac{eBS}{2\pi\beta |m|} \left\{ e\beta\mu + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-n\beta|m|} \cos(en\beta \tilde{A}_3) \sinh(en\beta\mu) \right\}.
\] (6)

The corresponding cluster coefficients can be obtained by comparing this expression with the expansion (9). Note that in a non-relativistic system one usually finds only positive (or negative) powers of the fugacity \(z\). However, in a relativistic system due to the presence of particles and antiparticles one finds both positive and negative powers of \(z\), as in the above equation. Also the chemical potentials for particles and antiparticles differ only by its sign, as a consequence of charge conservation [17]. The cluster coefficients for the PV Lagrangian are then given by:
\[
b^{PV}_{\pm n} = \mp \frac{(-1)^n}{n} \frac{eB m}{4\pi|m|} e^{-n\beta|m|} \cos(en\beta \tilde{A}_3),
\] (7)

where \(n = 1, 2, ..., \) the signs \(\pm\) refers to particles and antiparticles respectively, with opposite cluster coefficients while \(b_0 = -e^2B \beta m / 4\pi|m|\) does not contribute to the thermodynamics of the system once it depends linearly on \(\beta\).

The action that comes from the parity invariant (PI) Lagrangian includes the summation over all Landau levels. It can be split into the lowest Landau level (LLL) contribution \((E_{0,1} = |m|)\) and the excited states ones as
\[
\Omega^{PI}(\beta, \mu) = -\frac{e|B|S}{4\pi\beta} \Re \left\{ \beta|m| + G_+ (|m| + i\epsilon \Xi) + G_+ (|m| - i\epsilon \Xi) \right. \\
+ 2 \sum_{k=1}^{\infty} \beta E_k + G_+ (E_k + i\epsilon \Xi) + G_+ (E_k - i\epsilon \Xi) \right\},
\] (8)

where \(E_k = \sqrt{m^2 + 2e|B|k}.\) Then, the cluster coefficients in this case can be written as
\[
b^{PI}_n = b^{PI}_{n,LLL} + b^{PI}_{n,excited}
\]
and the LLL case follows similarly to the PV case so that
\[
b^{PI}_{\pm n,LLL} = -\frac{(-1)^n e|B|}{4n\pi} e^{-n\beta|m|} \cos(en\beta \tilde{A}_3),
\] (9)

where \(n = 1, 2, ..., \) and \(\pm\) refers to particles and antiparticles. In this case they have the same cluster coefficients and \(b^{PI}_0|_{LLL} = -e\beta|mB| / 4\pi\) again does not affect the thermodynamics since it also depends linearly on \(\beta\).
Now we can find the virial coefficients corresponding to the sum of PI and PV actions. Assuming that the magnetic field is strong and the temperature low enough to keep the particles in the LLL state we can take the contributions from the cluster coefficients eqs. (1), (2), so that (choosing \( B > 0 \)) we have

\[
b_{\pm n}^{LLL} = -\frac{(-1)^n e B}{4n\pi} \left( 1 \pm \frac{m}{|m|} \right) e^{-n|\beta|m} \cos(en\beta\tilde{A}_3) .
\]

Up to this point the distinction of particles and antiparticles (\( \pm \)) is completely arbitrary. However, due to the factor \( 1 \pm m/|m| \) one can see that only particles or antiparticles will contribute depending on the sign of \( m \) that we choose. This result is closely related to the choice of the non-equivalent irreducible representation of Dirac matrices that we pick up for the PV action. Have we chosen the other inequivalent irreducible representation we would have found the PV action with reversed signs so that the role of particles and antiparticles would be reversed. From now on we take \( m > 0 \) and \( n > 0 \), so that

\[
b_n^{LLL} = -(-1)^n \rho_L e^{-n\beta m} \cos(n\theta) ,
\]

where \( \rho_L = 2\beta\omega_c/\lambda^2 \) with \( \omega_c = |eB|/2m \) being half of the cyclotron frequency and \( \lambda = \sqrt{2\pi\beta/m} \) the thermal wavelength. Furthermore, we have defined \( \theta \equiv e\beta\tilde{A}_3 \), in analogy with the situation without the magnetic field background [3]. One can also define a non-relativistic chemical potential in terms of the relativistic one up to the rest mass \( m \) (in natural units \( \hbar = c = 1 \)). However, this redefinition of the chemical potential does not affect the virial coefficients.

If we substitute the above cluster coefficients into relations (3), we are able to find all virial coefficients. Then, we obtain the virial coefficients

\[
a_2^{LLL} = \frac{1}{2\rho_L}[1 - \tan^2 \theta] ; \quad (11)
\]
\[
a_3^{LLL} = \frac{1}{3(\rho_L)^2}[1 + 3 \tan^4 \theta] ; \quad (12)
\]
\[
a_4^{LLL} = \frac{1}{4(\rho_L)^3}[1 - 3 \tan^4 \theta + 10 \tan^6 \theta] ; \quad (13)
\]
\[
a_5^{LLL} = \frac{1}{5(\rho_L)^4}[1 + 20 \tan^6 \theta + 35 \tan^8 \theta] ; \quad (14)
\]
\[
a_6^{LLL} = \frac{1}{6(\rho_L)^5}[1 - 10 \tan^6 \theta - 105 \tan^8 \theta + 126 \tan^{10} \theta] . \quad (15)
\]
Note that the above expressions for the virial coefficients diverge for $\theta = \pi/2$ and we have not found any apparent reason why it should be so.

4 The virial coefficients from bosons

Now we turn to scalar QED. The effective action for the gauge field $A_\nu$ in this case can be calculated using the same choice we did (a constant magnetic field) for the fermionic case. Here, there is no parity violating Lagrangian once it does not involve Dirac matrices. Then the bosonic effective action is similar to the PI part of the fermionic case and we find

$$S^B_{\text{eff}}(\beta, \mu) = \frac{eSB}{2\pi} \sum_{l=0}^{\infty} \left\{ \beta E_l + G_-(E_l - ie\Xi) + G_-(E_l + ie\Xi) \right\} ,$$

(16)

where the energy of Landau levels is now given by $E_l = \sqrt{m^2 + 2eB(l + 1/2)}$ and we defined

$$G_-(x) = \ln(1 - e^{-\beta x}) .$$

As in the PI part of the fermionic case we can split the contribution of LLL state which is given by $E_0 = \sqrt{m^2 + eB}$ from the excited ones, so that

$$\Omega^B(\beta, \mu) = -\frac{eBS}{2\pi\beta} \Re \left\{ \beta E_0 + G_-(E_0 - ie\Xi) + G_-(E_0 + ie\Xi) \right\} + \sum_{l=1}^{\infty} \beta E_l + G_-(E_l - ie\Xi) + G_-(E_l + ie\Xi) \right\} .$$

(17)

Thus, the cluster coefficients can be written as

$$b^B_n = b^B_n|_{\text{LLL}} + b^B_n|_{\text{excited}} .$$

Note that here, in opposition to the fermionic case, the LLL cluster coefficients depend on the magnetic field $B$ so we use the approximation

$$E_0 = \sqrt{m^2 + eB} \approx m + eB/2m ,$$

valid when $eB << m^2$. Then, we find that the cluster coefficients corresponding to the LLL in this case are given by

$$b^B_{\pm n}|_{\text{LLL}} = \frac{\rho_4}{n} e^{-n\beta(m + \omega_4)} \cos(n\beta 3) ,$$

(18)
where \( n = 1, 2, \ldots \) while \( b_0 = -e\beta B(m + \omega_c)/2\pi \) does not contribute to the thermodynamics. In analogy with the fermionic case we can write the cluster coefficients here as

\[
b_n^B|_{LLL} = \frac{\rho_L}{n} e^{-n\beta(m + \omega_c)} \cos(n\theta) .
\]

Then, we find that the virial coefficients in this case are given by

\[
a_2^B|_{LLL} = -\frac{1}{2\rho_L}[1 - \tan^2 \theta]; \quad (20)
\]
\[
a_3^B|_{LLL} = +\frac{1}{3(\rho_L)^2}[1 + 3\tan^4 \theta]; \quad (21)
\]
\[
a_4^B|_{LLL} = -\frac{1}{4(\rho_L)^3}[1 - 3\tan^4 \theta + 10\tan^6 \theta]; \quad (22)
\]
\[
a_5^B|_{LLL} = -\frac{1}{5(\rho_L)^4}[1 + 20\tan^6 \theta + 35\tan^8 \theta]; \quad (23)
\]
\[
a_6^B|_{LLL} = +\frac{1}{6(\rho_L)^5}[1 - 10\tan^6 \theta - 105\tan^8 \theta + 126\tan^{10} \theta]. \quad (24)
\]

These coefficients are closely related with those obtained in the fermionic case, namely:

\[
a_n^B|_{LLL} = (-1)^{n-1}a_n^{LLL}.
\]

They may interpolate between the bosonic and fermionic coefficients as happens for particles without magnetic field background [3]. An analogous situation happens in the fermionic case discussed previously. These coefficients differ in general from those of anyons [8, 9, 10, 11, 12]. Note that the bosonic virial coefficients are also singular for \( \theta = \pi/2 \), which seems to prevent spontaneous transmutation of bosons into fermions and vice-versa.

5 Conclusions

Virial coefficients of a gas of relativistic particles can be obtained from the fundamental quantum theory of electromagnetic interactions, i.e., QED, which guarantees large gauge invariance in 2+1 dimensions at finite temperature and density. These depend on an arbitrary parameter \( \theta \) that carries a dependence on the temperature and the time-component of the gauge field. We have shown that such parameter could play the role of an interpolation parameter between the bosonic and fermionic virial coefficients, although these virial coefficients are singular for \( \theta = \pi/2 \).
It is important to stress that the approach used here in the calculation of the virial coefficients is based on finite temperature and density effects of the fundamental bosonic and fermionic particles, similar to that discussed in the absence of magnetic fields \[5\] and suggested in the literature \[7\]. In other words, the interpolation discussed here appears only at finite temperature and density, since at zero temperature large gauge transformations become trivial and the (constant) parameter associated with the time component of the gauge field can be removed by an ordinary (infinitesimal) gauge transformation.

Finally, in this work we have not discussed interactions of the fundamental fermions (or bosons) with other fields than the constant magnetic background. This implied that the constant parameter \( \theta \) (or \( \tilde{A}_3 \)) was not fixed here. We expect that the basic characteristics discussed here related to finite temperature and density interpolation between bosonic and fermionic virial coefficients would survive to the presence of other interactions.

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