COHERENT AND TRAJECTORY-COHERENT STATES OF A DAMPED HARMONIC OSCILLATOR

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Abstract
In this paper we construct the coherent and trajectory-coherent states of a damped harmonic oscillator. We investigate the properties of this states.

1. Introduction.

At present the coherent states are widely used to describe many fields of theoretical physics. The interest and activity in coherent states were revived by the paper of Glauber, who showed that the coherent states could be successfully used for problem of quantum optics. Recently Nieto and Simmons have constructed coherent states for particles in general potential, Hartley and Ray have obtained coherent states for time-dependent harmonic oscillator on the basis of Lewis-Risenfeld theory; Yeon,Um and George have constructed exact coherent states for the damped harmonic oscillator.

Some time ago Bagrov, Belov and Ternov have constructed approximate (for $\hbar \to 0$) solutions of the Schrödinger equation for particles in general potentials, such that the coordinate and momentum quantum-mechanical averages were exact solutions of the corresponding classical Hamiltonian equations; these states were called trajectory-coherent (TCS). The basis of this construction is the complex WKB method by V.P. Maslov.
The aim of this work is to construct the coherent (CS) and trajectory-coherent (TCS) states of a damped harmonic oscillator by using the Caldirola-Kanai Hamiltonian and the complex WKB method. It is shown that these states satisfy the Schrödinger equation exactly.

2. The construction of CS and TCS.

Consider the Schrödinger equation

\[ i\hbar \partial_t \Psi = \hat{H} \Psi, \]  

(1)

where the symbol of operator \( \hat{H} \) - the function \( H(x, p, t) \) is arbitrary real and analytical function of coordinate and momentum. The method of construction TCS in this case has been described in detail in Ref.6, hence we shall illustrate some moments only. For constructing the TCS of the Schrödinger equation it is necessary to solve the classical Hamiltonian system

\[ \dot{x}(t) = \partial_p H(x, p, t), \quad \dot{p}(t) = -\partial_x H(x, p, t), \]  

(2)

and the system in variations (this is the linearization of the Hamiltonian system in the neighbourhood of the trajectory \( x(t), p(t) \))

\[ \dot{w}(t) = -H_{xp}(t)w(t) - H_{xx}(t)z(t), \quad w(0) = b, \]  

\[ \dot{z}(t) = H_{pp}(t)w(t) + H_{px}(t)z(t), \quad z(0) = 1, \]  

(3)

where \( H(x, p, t) \) is the classical Hamiltonian

\[ H_{xp}(t) = \partial_x \partial_p H(x, p, t) \big|_{x=x(t), p=p(t)}, \quad H_{px}(t) = \partial_p \partial_x H(x, p, t) \big|_{x=x(t), p=p(t)}, \]  

\[ H_{xx}(t) = \partial^2_{xx} H(x, p, t) \big|_{x=x(t), p=p(t)}, \quad H_{pp}(t) = \partial^2_{pp} H(x, p, t) \big|_{x=x(t), p=p(t)}, \]  

(4)

\( b \) is complex number obeying the condition \( \text{Im} b > 0 \), and \( x(t), p(t) \) are the solutions of system (2).

Consider the damped harmonic oscillator\(^5\)\(^,\)\(^10\)\(^,\)\(^11\)

\[ H(x, p, t) = \exp(-\gamma t)(2m)^{-1}p^2 + \frac{1}{2} \exp(\gamma t)m\omega_0^2x^2. \]  

(5)

The Lagrangian and mechanical energy are given by\(^5\)

\[ L = \exp(\gamma t)(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega_0^2x^2), \quad E = \exp(-2\gamma t)(2m)^{-1}p^2 + \frac{1}{2}m\omega_0^2x^2. \]  

(5)
We first define the Cauchy problem for equation (1)

\[ |0\rangle_{t=0} = \Psi_0(x, t, \hbar) \bigg|_{t=0} = N \exp\{i\hbar^{-1}(p_0(x - x_0) + \frac{b}{2}(x - x_0)^2)\}, \tag{6} \]

where \( x_0 = x(t) \bigg|_{t=0}, \ p_0 = p(t) \bigg|_{t=0} \).

The function of WKB - solution type (TCS)\(^6\)

\[ |0\rangle = \Psi_0(x, t, \hbar) = N\Phi(t) \exp\{i\hbar^{-1}S(x, t)\}, \tag{7} \]

where \( N = (\text{Im}(\pi\hbar)^{-1})^{-1/4}, \ \Phi(t) = (z(t))^{-1/2}, \)

\[ S(x, t) = \int_0^t \{\dot{x}(t)p(t) - H(x(t), p(t), t)\}dt + p(t)(x - x(t)) + \frac{1}{2}w(t)z^{-1}(t)(x - x(t))^2, \]

and the phase \( S(x, t) \) is the complex - valued function \( (\text{Im}S > 0) \) is the approximate solution of the Cauchy problem (6) for the Schrödinger equation (1). We should note that in the case of the quadratic systems, for example, for Hamiltonian (4) the function (7) is the exact solution of the equation (1).

Solving the differential equations (2),(3) we obtain

\[ x(t) = \cosh \omega t \exp\left(-\frac{1}{2}\gamma t\right), \ p(t) = m(\omega \sinh \omega t - \frac{1}{2}\gamma \cosh \omega t) \exp\left(\frac{1}{2}\gamma t\right), \tag{8} \]

\[ w(t) = m\exp\left(\frac{1}{2}\gamma t\right)\{\delta\omega - \frac{1}{2}\gamma\} \cosh \omega t + (\omega - \frac{1}{2}\gamma\delta) \sinh \omega t\}, \]

\[ z(t) = \exp\left(-\frac{1}{2}\gamma t\right)\{\cosh \omega t + \delta \sinh \omega t\}, \]

\[ \delta = (m\omega)^{-1}(b + \frac{1}{2}\gamma m), \ \omega^2 = \frac{1}{4}\gamma^2 - \omega_0^2. \]

It is easy to check that the function (7), where \( x(t), p(t), w(t), z(t) \) are defined by (8) satisfy the equation (1) exactly as for \( \omega^2 > 0 \) as \( \omega^2 < 0 \) (\( \omega \to i\omega \).
Further, we define annihilation operator $\hat{a}(t)$ and creation operator $\hat{a}^+(t)$ as:

$$\hat{a}(t) = (2\hbar \text{Im} b)^{-1/2}\{z(t)(\hat{p} - p(t)) - w(t)(x - x(t))\}, \quad (9)$$

$$\hat{a}^+(t) = (2\hbar \text{Im} b)^{-1/2}\{z^*(t)(\hat{p} - p(t)) - w^*(t)(x - x(t))\}.$$

It is easy to check that the creation and annihilation operators satisfy the usual Bose permutation rule

$$[\hat{a}, \hat{a}^+] = 1, \quad [\hat{a}, \hat{a}] = [\hat{a}^+, \hat{a}^+] = 0. \quad (10)$$

The complete orthonormal set of the trajectory-coherent states (TCS) are defined as:

$$|n> = (n!)^{-1/2}(\hat{a}^+)^n|0>.$$ \quad (11)

It is not difficult to check the relations

$$<n|m>=\delta_{n,m}, \quad \hat{a}|0>=0,$$ \quad (12)

$$\hat{a}^+|n>=(n+1)^{1/2}|n+1>, \quad \hat{a}|n>=(n)^{1/2}|n-1>.$$ \quad (13)

Using the usual methods\(^1\) we obtain the expression for coherent states (CS)

$$|\alpha> = \hat{A}(\alpha)|0>,$$ \quad (14)

where $\hat{A}(\alpha)$ is the unitary operator

$$\hat{A}(\alpha) = \exp(\alpha \hat{a}^+ - \alpha^* \hat{a}),$$ \quad (15)

$\alpha$ is the complex number; $\hat{a}^+, \hat{a}$ are defined by (9).

Besides, it is follows from (12)-(14), the functions $|\alpha>$ are eigenstates of the operator $\hat{a}$ with eigenvalue $\alpha$, i.e.

$$\hat{a}|\alpha>=\alpha|\alpha>.$$ \quad (16)

### 3. Quantum - mechanical averages and uncertainty relations.

Further we shall find the expressions for quantum - mechanical averages

$$<\hat{x}_{TCS}, <\hat{x}^2_{TCS}, <\hat{p}_{TCS}, <\hat{p}^2_{TCS}, <\hat{x}_{CS},$$

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\[ <\hat{x}^2>_{cs}, <\hat{p}>_{cs}, <\hat{p}^2>_{cs}, <\hat{E}>_{TCS}, <\hat{E}>_{cs}, \]

where, for example,

\[ <\hat{x}>_{TCS} = \langle n | \hat{x} | n \rangle, <\hat{x}>_{cs} = \langle \alpha | \hat{x} | \alpha \rangle. \]

Further, we present the relations expressing the coordinate and momentum operators in terms of the operators \( \hat{a}^+, \hat{a} \)

\[ \hat{x} = x(t) - i(2\text{Im}b)(\hbar)^{-1/2}\{z(t)\hat{a}^+ - z^*(t)\hat{a}\}, \quad (16) \]

\[ \hat{p} = p(t) - i(2\text{Im}b)(\hbar)^{-1/2}\{w(t)\hat{a}^+ - w^*(t)\hat{a}\}. \]

Using (5), (9)-(16), we obtain for quantum-mechanical averages:

\[ <\hat{x}>_{TCS} = x(t), <\hat{p}>_{TCS} = p(t), \]

\[ <\hat{x}>_{cs} = x(t) - i(2\text{Im}b)(\hbar)^{-1/2}\{\alpha^*z(t) - \alpha z^*(t)\}, \quad (17) \]

\[ <\hat{p}>_{cs} = p(t) - i(2\text{Im}b)(\hbar)^{-1/2}\{\alpha^*w(t) - \alpha w^*(t)\}, \]

\[ <\hat{x}^2>_{TCS} = x^2(t) + \frac{\hbar}{2\text{Im}b}(n + \frac{1}{2})|z(t)|^2, \]

\[ <\hat{p}^2>_{TCS} = p^2(t) + \frac{\hbar}{2\text{Im}b}(n + \frac{1}{2})|w(t)|^2, \]

\[ <\hat{x}^2>_{cs} = <x>^2_{cs} + \frac{\hbar}{2\text{Im}b}|z(t)|^2, \]

\[ <\hat{p}^2>_{cs} = <p>^2_{cs} + \frac{\hbar}{2\text{Im}b}|w(t)|^2. \]

\[ <\hat{E}>_{TCS} = \exp(-2\gamma t)(2m)^{-1}p^2(t) + \frac{1}{2}m\omega_0^2x^2(t) + \frac{\hbar}{2\text{Im}b}(n + \frac{1}{2})\{\exp(-2\gamma t)(2m)^{-1}|w(t)|^2 + \frac{1}{2}m\omega_0^2|z(t)|^2\}, \]

\[ <\hat{E}>_{cs} = \exp(-2\gamma t)(2m)^{-1} <p>^2_{cs} + \frac{1}{2}m\omega_0^2 <x>^2_{cs} + \frac{\hbar}{2\text{Im}b}\{\exp(-2\gamma t)(2m)^{-1}|w(t)|^2 + \frac{1}{2}m\omega_0^2|z(t)|^2\}. \]
Now, by calculating the uncertainty in $x$ and $\hat{p}$ in the TCS and CS one finds:

\[
(\Delta \hat{x})_{TCS}^2 = (\hat{x} - < \hat{x} >_{TCS})^2 = \hbar (n + \frac{1}{2}) \frac{|z(t)|^2}{\text{Im}b}, \\
(\Delta \hat{p})_{TCS}^2 = (\hat{p} - < \hat{p} >_{TCS})^2 = \hbar (n + \frac{1}{2}) \frac{|w(t)|^2}{\text{Im}b},
\]

\[
(\Delta \hat{x})_{CS}^2 = (\hat{x} - < \hat{x} >_{CS})^2 = \frac{\hbar |z(t)|^2}{2 \text{Im}b}, \\
(\Delta \hat{p})_{CS}^2 = (\hat{p} - < \hat{p} >_{CS})^2 = \frac{\hbar |w(t)|^2}{2 \text{Im}b}.
\]

So, the Heisenberg’s uncertainty relations is expressed as

\[
(\Delta \hat{x})_{TCS}^2 (\Delta \hat{p})_{TCS}^2 = \hbar^2 (n + \frac{1}{2})^2 \frac{|w(t)z(t)|^2}{(\text{Im}b)^2}, \\
(\Delta \hat{x})_{CS}^2 (\Delta \hat{p})_{CS}^2 = \frac{\hbar^2 |w(t)z(t)|^2}{4 (\text{Im}b)^2}.
\]

For the damped harmonic oscillator (4) we choose $\text{Re}b = 0$ (it is necessary for minimization of the uncertainty relations in the initial instant of time), $\text{Im}b = \mu m |\omega|$, where $\mu > 0$ shows the initial uncertainty of coordinate; and we denote $\theta = \gamma / 2 \omega$.

By using (8),(16)-(21), in the case $\omega^2 = \omega_0^2 - \frac{1}{4} \gamma^2 > 0$ for $(\Delta \hat{x})_{TCS}^2$, $(\Delta \hat{p})_{TCS}^2$, $(\Delta \hat{x})_{CS}^2$, $(\Delta \hat{p})_{CS}^2$ we obtain

\[
(\Delta \hat{x})_{TCS}^2 = \frac{\hbar}{2} \exp(-\gamma t)(\mu m \omega)^{-1} \{1 + \sin^2 \omega t(\theta^2 + \\
+ \mu^2 - 1) + \theta \sin 2\omega t\},
\]

\[
(\Delta \hat{p})_{TCS}^2 = \frac{\hbar}{2} \exp(\gamma t) \mu m \omega \{1 + \frac{1}{\mu^2} \sin^2 \omega t(2\theta^2 + \\
+ \theta^4 + 1 + \mu^2 \theta^2 - \mu^2) - \theta \sin 2\omega t\},
\]

\[
(\Delta \hat{x})_{CS}^2 = \frac{\hbar}{2} \exp(-\gamma t)(\mu m \omega)^{-1} \{1 + \sin^2 \omega t(\theta^2 + \\
+ \mu^2 - 1) + \theta \sin 2\omega t\},
\]

\[
(\Delta \hat{p})_{CS}^2 = \frac{\hbar}{2} \exp(\gamma t) \mu m \omega \{1 + \frac{1}{\mu^2} \sin^2 \omega t(2\theta^2 + \\
+ \theta^4 + 1 + \mu^2 \theta^2 - \mu^2) - \theta \sin 2\omega t\}.
\]
\[ (\Delta \hat{x})^2_{cs} = \frac{\hbar}{2} \exp(\gamma t) \mu m \omega \{1 + \frac{1}{\mu^2} \sin^2 \omega t (2\theta^2 + \theta^4 + 1 + \mu^2 \theta^2 - \mu^2) - \theta \sin 2\omega t\}, \]

and thus the uncertainty relations becomes

\[ (\Delta \hat{x})^2_{tcs} (\Delta \hat{p})^2_{tcs} = \hbar^2 (n + \frac{1}{2})^2 (1 + g(t)), \tag{22} \]

\[ (\Delta \hat{x})^2_{cs} (\Delta \hat{p})^2_{cs} = \frac{\hbar^2}{4} (1 + g(t)), \]

where

\[ g(t) = \left\{ \frac{\theta}{\mu} (\theta^2 + \mu^2 + 1) \sin^2 \omega t + \frac{1}{2\mu} (\theta^2 - \mu^2 + 1) \sin 2\omega t \right\}^2. \tag{23} \]

We should note, that in the special case \( \mu = 1 \) the formula (23) coincides with formula (14) from Ref. 5.

The function (23) is equal to zero, and therefore, the minimization of the uncertainty relations has place in the instants of time

\[ t_{1k} = \frac{\pi k}{\omega}, \quad t_{2k} = \frac{1}{\omega} \arctan \frac{\mu^2 - \theta^2 - 1}{\theta (\mu^2 + \theta^2 + 1)} + \frac{\pi k}{\omega}; \quad k = 0, 1, 2, \ldots. \]

The preceding equation can be solved for the parameter \( \mu > 0 \) in the case

\[ |\theta \tan \omega t| < 1. \tag{24} \]

and therefore, by choosing parameter \( \mu \) we can obtain the minimization for any instant of time \( t \) obeying the condition (24).

In the case \( \omega^2 = \frac{1}{4} \gamma^2 - \omega^2_0 > 0 \) we obtain

\[ (\Delta \hat{x})^2_{tcs} = \hbar (n + \frac{1}{2}) \exp(-\gamma t) (\mu m \omega)^{-1} \{1 + \sinh^2 \omega t (\theta^2 + \mu^2 + 1) + \theta \sinh 2\omega t\}, \]

\[ (\Delta \hat{p})^2_{tcs} = \hbar (n + \frac{1}{2}) \exp(\gamma t) \mu m \omega \{1 + \frac{1}{\mu^2} \sinh^2 \omega t (1 - 2\theta^2 + \theta^4 + \mu^2 \theta^2 + \mu^2) - \theta \sinh 2\omega t\}, \]
\[
(\Delta \hat{x})^2_{cs} = \frac{\hbar}{2} \exp(-\gamma t)(\mu \omega)^{-1}\{1 + \sinh^2 \omega t(\theta^2 + 
+ \mu^2 + 1) + \theta \sinh 2\omega t\},
\]

and for the uncertainty relations we have

\[
(\Delta \hat{p})^2_{cs} = \frac{\hbar}{2} \exp(\gamma t)\mu \omega\{1 + \frac{1}{\mu^2}\sinh^2 \omega t(1 - 2\theta^2 +
+ \theta^4 + \mu^2\theta^2 + \mu^2) - \theta \sinh 2\omega t\},
\]

where

\[
g(t) = \left\{\frac{\theta}{\mu}(\theta^2 + \mu^2 - 1) \sinh^2 \omega t + \frac{1}{2\mu}(\theta^2 - \mu^2 - 1) \sinh 2\omega t\right\}^2. \tag{25}
\]

Now the function \( g(t) \) \( (25) \) is equal to zero only for

\[
t_{01} = 0 \quad , \quad t_{02} = \frac{1}{\omega} \arctanh \frac{\mu^2 - \theta^2 + 1}{\theta(\mu^2 + \theta^2 - 1)}.
\]

This equation can be solved for parameter \( \mu \) in the case

\[
|\theta \tanh \omega t| < 1, \tag{26}
\]

and we can obtain the minimization for any instant of time \( t \) obeying the condition \( (26) \).

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