Clark measures and de Branges–Rovnyak spaces in several variables

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ABSTRACT
Let \( D \) denote a finite product of \( B_{n_j} \), \( j \geq 1 \), and \( \partial D \) denote the distinguished boundary \( \partial B_{n_1} \times \partial B_{n_2} \times \cdots \times \partial B_{n_k} \). For a non-constant holomorphic function \( b : D \to B_1 \), we study the corresponding family of Clark measures on \( \partial D \). We construct a natural unitary operator from the de Branges–Rovnyak space \( \mathcal{H}(b) \) onto the Hardy space \( \mathcal{H}(\sigma_\partial) \). As an application, for \( D = B_n \) and an inner function \( I : B_n \to B_1 \), we show that the property \( \sigma_1[I] \ll \sigma_1[b] \) is directly related to the membership of an appropriate explicit function in \( \mathcal{H}(b) \).

1. Introduction

Let \( B_n \) denote the open unit ball of \( \mathbb{C}^n \), \( n \geq 1 \), and \( \partial B_n \) denote the unit sphere. We also use symbols \( \mathbb{D} \) and \( \mathbb{T} \) for the unit disc \( B_1 \) and the unit circle \( \partial B_1 \), respectively.

Given \( k \in \mathbb{N} \) and \( n_j \in \mathbb{N}, j = 1, 2, \ldots, k \), let
\[
D = D[n_1, n_2, \ldots, n_k] = B_{n_1} \times B_{n_2} \times \cdots \times B_{n_k} \subset \mathbb{C}^{n_1+n_2+\cdots+n_k}.
\]

Model examples of \( D \) are \( B_n \) and the polydisc \( \mathbb{D}^n \).
Let \( C(z, \zeta) = C_D(z, \zeta) \) denote the Cauchy kernel for \( D \). Recall that
\[
C_{B_n}(z, \zeta) = \frac{1}{(1 - \langle z, \zeta \rangle)^n}, \quad z \in B_n, \quad \zeta \in S_n.
\]

Let \( \partial D \) denote the distinguished boundary \( \partial B_{n_1} \times \partial B_{n_2} \times \cdots \times \partial B_{n_k} \) of \( D \). Then
\[
C_D(z, \zeta) = \prod_{j=1}^k \frac{1}{(1 - \langle z_j, \zeta_j \rangle)^{n_j}}, \quad z = (z_1, z_2, \ldots, z_k) \in D, \quad \zeta = (\zeta_1, \zeta_2, \ldots, \zeta_k) \in \partial D,
\]
where \( z_j = (z_{j,1}, z_{j,2}, \ldots, z_{j,n_j}) \in B_{n_j} \) and \( \zeta_j = (\zeta_{j,1}, \zeta_{j,2}, \ldots, \zeta_{j,n_j}) \in \partial B_{n_j} \).
The corresponding Poisson type kernel is given by the formula

$$P(z, \zeta) = \frac{C(z, \zeta)C(\zeta, z)}{C(z, z)}, \quad z \in \mathcal{D}, \quad \zeta \in \partial \mathcal{D}.$$  

For $\mathcal{D} = B_n$, $P(\cdot, \cdot)$ is often called the Möbius invariant Poisson kernel; see [1] for further details.

Let $M(\partial \mathcal{D})$ denote the space of complex Borel measures on $\partial \mathcal{D}$. For $\mu \in M(\partial \mathcal{D})$, the Cauchy transform $\mu_+$ is defined as

$$\mu_+(z) = \int_{\partial \mathcal{D}} C(z, \zeta) \, d\mu(\zeta), \quad z \in \mathcal{D}.$$  

1.1. Clark measures

Given an $\alpha \in \mathbb{T}$ and a holomorphic function $b : \mathcal{D} \to \mathbb{D}$, the quotient

$$\frac{1 - |b(z)|^2}{|\alpha - b(z)|^2} = \text{Re} \left( \frac{\alpha + b(z)}{\alpha - b(z)} \right), \quad z \in \mathcal{D},$$

is positive and pluriharmonic. Therefore, there exists a unique positive measure $\sigma_\alpha = \sigma_\alpha[b] \in M(\partial \mathcal{D})$ such that

$$P[\sigma_\alpha](z) = \text{Re} \left( \frac{\alpha + b(z)}{\alpha - b(z)} \right), \quad z \in \mathcal{D}.$$  

After the seminal paper of Clark [2], various properties and applications of the measures $\sigma_\alpha$ on the unit circle $\mathbb{T}$ have been obtained; see, for example, reviews [3–6] for further references. Several results related to the Clark measures on the unit sphere $\partial B_n$, $n \geq 2$, are given in [7].

1.2. Model spaces and de Branges–Rovnyak spaces

Let $\Sigma$ denote the normalized Lebesgue measure on $\partial \mathcal{D}$.

**Definition 1.1:** A holomorphic function $I : \mathcal{D} \to \mathbb{D}$ is called inner if $|I(\zeta)| = 1$ for $\Sigma$-a.e. $\zeta \in \partial \mathcal{D}$.

In the above definition, $I(\zeta)$ stands, as usual, for $\lim_{r \to 1^-} I(r\zeta)$. Recall that the corresponding limit exists $\Sigma$-a.e. Also, by the above definition, every inner function is non-constant.

Given an inner function $I$ in $\mathcal{D}$, we have

$$P[\sigma_\alpha](\zeta) = \frac{1 - |I(\zeta)|^2}{|\alpha - I(\zeta)|^2} = 0 \quad \Sigma\text{-a.e.},$$

therefore, $\sigma_\alpha = \sigma_\alpha[I]$ is a singular measure. Here and in what follows, this means that $\sigma_\alpha$ and $\Sigma$ are mutually singular; in brief, $\sigma_\alpha \perp \Sigma$.  

Let $\mathcal{H} \text{ol}(\mathcal{D})$ denote the space of holomorphic functions in $\mathcal{D}$. For $0 < p < \infty$, the classical Hardy space $H^p = H^p(\mathcal{D})$ consists of those $f \in \mathcal{H} \text{ol}(\mathcal{D})$ for which

$$
\|f\|_{H^p}^p = \sup_{0 < r < 1} \int_{\partial \mathcal{D}} |f(r\zeta)|^p d\Sigma(\zeta) < \infty.
$$

As usual, we identify the Hardy space $H^p(\mathcal{D})$, $p > 0$, and the space $H^p(\partial \mathcal{D})$ of the corresponding boundary values.

For an inner function $\theta$ on $\mathcal{D}$, the classical model space $K_\theta$ is defined as

$$
K_\theta = H^2(T) \ominus \theta H^2(T).
$$

Clark [2] introduced and studied a family of useful unitary operators $U_\alpha : K_\theta \to L^2(\sigma_\alpha)$, $\alpha \in \mathbb{T}$.

For an inner function $I$ in $\mathcal{D}$, there are several reasonable generalizations of $K_\theta$. Consider the following direct analog of $K_\theta$:

$$
I^*(H^2) = H^2 \ominus IH^2.
$$

For $\mathcal{D} = B_n$, it is shown in [7, Theorem 5.1] that Clark’s construction appropriately extends to $I^*(H^2)$ and also provides natural unitary operators $T_\alpha : I^*(H^2) \to L^2(\sigma_\alpha)$, $\alpha \in \mathbb{T}$.

Observe that

$$
C(\zeta, z) = (1 - I(z)\overline{I(\zeta)})C(\zeta, z) + \overline{I(z)}I(\zeta)C(\zeta, z) \in I^*(H^2) \oplus IH^2
$$

as functions of $\zeta$. Therefore,

$$
K(z, \zeta) = (1 - I(z)\overline{I(\zeta)})C(z, \zeta)
$$

is the reproducing kernel for the Hilbert space $I^*(H^2)$ at $z \in \mathcal{D}$, that is,

$$
g(z) = \int_{\partial \mathcal{D}} g(w)K(z, w) d\Sigma(w), \quad z \in \mathcal{D},
$$

for all $g \in I^*(H^2)$.

Now, let $b : \mathcal{D} \to \mathbb{D}$ be an arbitrary non-constant holomorphic function. Direct inspection shows that the function

$$
k_b(z, w) = (1 - b(z)b(w))C(z, w)
$$

has the reproducing kernel properties. The corresponding Hilbert space $\mathcal{H}(b) \subset H^2$ is called a de Branges–Rovnyak space. In particular, $I^*(H^2) = \mathcal{H}(I)$ for an inner function $I$. Further details are given in [8, Chapter II] for $\mathbb{D}$ in the place of $\mathcal{D}$.

For $\alpha \in \mathbb{T}$, Sarason [8, Section III-7] introduced unitary operators

$$
U_{b,\alpha} : \mathcal{H}(b) \to H^2(\sigma_\alpha[b])
$$

and closely related partial isometries

$$
V_{b,\alpha} : L^2(\sigma_\alpha[b]) \to \mathcal{H}(b),
$$

where $\mathcal{H}(b) \subset H^2(\mathbb{D})$ is the de Branges–Rovnyak space generated by $b$, $H^2(\sigma_\alpha[b])$ is a Hardy type space. In the present paper, we construct analogous natural operators $U_{b,\alpha}$ and $V_{b,\alpha}$, $\alpha \in \mathbb{T}$, for a given non-constant holomorphic function $b : \mathcal{D} \to \mathbb{D}$; see Theorems 3.1 and 3.2.
1.3. Comparison of Clark measures

Sarason [8, Section III-11] argued with the help of $V_{b, \alpha}$ to compare Clark measures on the unit circle $\mathbb{T}$. To show that the operators $V_{b, \alpha}$ are useful in combination with appropriate results in several complex variables, we obtain the following comparison theorem for the Clark measures on the unit sphere $\partial B_n$.

**Theorem 1.2:** Let $I$ be an inner function in $B_n$ and let $b : B_n \to \mathbb{D}$, $n \geq 2$, be a non-constant holomorphic function. Let $\sigma = \sigma_{\alpha}[I]$ and $\mu = \sigma_{\alpha}[b]$, $\alpha \in \mathbb{T}$, be the corresponding Clark measures and let $K_w(\cdot) = K(\cdot, w)$, where $K(z, w)$ denotes the reproducing kernel for $I^*(H^2)$. Then the following properties are equivalent:

(i) $\sigma \ll \mu$ and $\frac{d\sigma}{d\mu} \in L^2(\mu)$;
(ii) the function

$$\frac{\alpha - b}{\alpha - I} K_w$$

is in the de Branges–Rovnyak space $\mathcal{H}(b)$ for all $w \in B_n$;
(iii) the function

$$\frac{\alpha - b}{\alpha - I} K_w$$

is in $\mathcal{H}(b)$ for some $w \in B_n$.

1.4. Organization of the paper

Auxiliary properties of Clark measures are obtained in Section 2. Operators $U_{b, \alpha}$ and $V_{b, \alpha}$ are constructed in Section 3. The final Section 4 is devoted to the proof of Theorem 1.2.

Theorem 1.2 was announced in extended abstract [9].

2. Cauchy integrals and Clark measures

The following lemma is a particular case of Exercise 1 from [10, Chapter 8].

**Lemma 2.1:** Let $F$ be a holomorphic function on $\mathcal{D} \times \mathcal{D}$. If $F(z, \bar{z}) = 0$ for all $z \in \mathcal{D}$, then $F(z, w) = 0$ for all $(z, w) \in \mathcal{D} \times \mathcal{D}$.

The following key technical proposition is obtained in [7, Proposition 3.5] for $\mathcal{D} = B_n$.

**Proposition 2.2:** Let $b : \mathcal{D} \to \mathbb{D}$ be a holomorphic function and let $\sigma_{\alpha} = \sigma_{\alpha}[b]$, $\alpha \in \mathbb{T}$, be the corresponding Clark measure. Then

$$\int_{\partial \mathcal{D}} C(z, \zeta) C(\zeta, w) \, d\sigma_{\alpha}(\zeta) = \frac{1 - b(z)\overline{b(w)}}{(1 - \overline{\alpha}b(z))(1 - \alpha b(w))} C(z, w)$$

for all $\alpha \in \mathbb{T}$, $z, w \in \mathcal{D}$. 
Proof: The equality
\[ \int_{\partial D} P(z, \zeta) \, d\sigma_\alpha(\zeta) = \frac{1 - |\varphi(z)|^2}{|\alpha - \varphi(z)|^2}, \quad z \in D, \]
and the definition of \( P(z, \zeta) \) provide
\[ \int_{\partial D} C(z, \zeta) C(\zeta, z) \, d\sigma_\alpha(\zeta) = \frac{1 - |\varphi(z)|^2}{|\alpha - \varphi(z)|^2} C(z, z), \quad z \in D. \]
It remains to apply Lemma 2.1. ■

For \( \mu \in M(\partial D) \), recall that \( \mu_+ \) denotes the Cauchy transform of \( \mu \).

**Corollary 2.3:** Let \( \varphi : D \to \mathbb{D}, d \geq 2 \), be a holomorphic function and let \( \sigma_\alpha = \sigma_\alpha[\varphi], \alpha \in \mathbb{T} \). Then
\[ (\sigma_\alpha)_+(z) = \frac{1}{1 - \bar{\alpha} \varphi(z)} + \frac{\alpha \varphi(0)}{1 - \alpha \varphi(0)} \]
for all \( \alpha \in \mathbb{T}, z \in D \).

Proof: Apply Proposition 2.2 with \( w = 0 \). ■

### 3. Clark measures and de Branges–Rovnyak spaces

In this section, \( b : D \to \mathbb{D} \) is an arbitrary non-constant holomorphic function.

#### 3.1. Unitary operators \( U_{b,\alpha} : \mathcal{H}(b) \to H^2(\sigma_\alpha[b]) \)

Fix an \( \alpha \in \mathbb{T} \). Let \( \sigma_\alpha = \sigma_\alpha[b] \) and let \( k_w(\cdot) = k_b(\cdot, w) \), where \( k_b(z, w) \) denotes the reproducing kernel for \( \mathcal{H}(b) \). Define
\[ (U_{b,\alpha} k_w)(\cdot) = (1 - \bar{\alpha} b(w)) C(\cdot, w), \quad w \in D. \]

Let \( H^2(\sigma_\alpha) \) denote the closed linear span of the holomorphic polynomials or, equivalently, of \( C(\cdot, w), w \in D, \) in \( L^2(\sigma_\alpha) \). In other words, \( H^2(\sigma_\alpha) \) is the Hardy space generated by \( \sigma_\alpha \).

**Theorem 3.1:** For each \( \alpha \in \mathbb{T}, U_{b,\alpha} \) has a unique extension to a unitary operator from \( \mathcal{H}(b) \) onto \( H^2(\sigma_\alpha) \).

Proof: Fix an \( \alpha \in \mathbb{T} \). Applying Proposition 2.2, we obtain
\[
(U_{b,\alpha} k_w, U_{b,\alpha} k_z)_{L^2(\sigma_\alpha)} = \int_{\partial D} (1 - \bar{\alpha} b(w)) C(\zeta, w)(1 - \bar{\alpha} b(z)) C(z, \zeta) \, d\sigma_\alpha(\zeta) \\
= (1 - \bar{\alpha} b(w))(1 - \bar{\alpha} b(z)) \int_{\partial D} C(\zeta, w) C(z, \zeta) \, d\sigma_\alpha(\zeta)
\]
\[ = (1 - b(z)b(w))C(z, w) \]
\[ = k_b(z, w) = (k_w, k_z)_{\mathcal{H}(b)}. \]

So, \( U_{b,\alpha} \) extends to an isometric embedding of \( \mathcal{H}(b) \) into \( L^2(\sigma_\alpha) \). Since the linear span of the family \( \{k_w\}_{w \in \mathcal{D}} \) is dense in \( \mathcal{H}(b) \), the extension is unique. Finally, \( U_{b,\alpha} \) maps \( \mathcal{H}(b) \) onto \( H^2(\sigma_\alpha) \) by the definition of \( H^2(\sigma_\alpha) \).

### 3.2. Partial isometries \( V_{b,\alpha} : L^2(\sigma_\alpha [b]) \rightarrow \mathcal{H}(b) \)

Define
\[ (V_{b,\alpha}g)(z) = (1 - \overline{\alpha}b(z))(g\sigma_\alpha)_+(z), \quad g \in L^2(\sigma_\alpha), \quad z \in \mathcal{D}. \] (1)

**Theorem 3.2:** For each \( \alpha \in \mathbb{T} \), formula (1) defines a partial isometry from \( L^2(\sigma_\alpha) \) into \( \mathcal{H}(b) \). The restriction of \( V_{b,\alpha} \) to \( H^2(\sigma_\alpha) \) coincides with \( U_{b,\alpha}^* \); in particular,
\[ V_{b,\alpha}C(\cdot, w)(z) = (1 - \alpha b(w))^{-1}k_b(z, w), \] (2)

where \( k_b(z, w) \) denotes the reproducing kernel for \( \mathcal{H}(b) \).

**Proof:** For \( g(\zeta) = (1 - \alpha \overline{b(w)})C(\zeta, w) \) with \( w \in \mathcal{D} \), the definition of \( U_{b,\alpha} \) and Proposition 2.2 guarantee that
\[ (U_{\alpha}^*g)(z) = (1 - \overline{\alpha}b(z))(g\sigma_\alpha)_+(z), \quad z \in \mathcal{D}. \] (3)

Therefore, (3) holds for all \( g \in H^2(\sigma_\alpha) \); hence, the restriction of \( V_{b,\alpha} \) on \( H^2(\sigma_\alpha) \) coincides with \( U_{b,\alpha}^* \). If \( h \in L^2(\sigma_\alpha) \) and \( h \perp H^2(\sigma_\alpha) \), then \( (h\sigma_\alpha)_+ = 0 \). Therefore, \( V_{b,\alpha} \) maps \( L^2(\sigma_\alpha) \) into \( \mathcal{H}(b) \), as required.

### 3.3. Cauchy transforms of \( f\sigma_\alpha \) with \( f \in H^2(\sigma_\alpha) \)

The following proposition is probably of independent interest.

**Proposition 3.3:** Let \( \alpha \in \mathbb{T} \) and \( b : \mathcal{D} \rightarrow \mathbb{D} \) be a non-constant holomorphic function. Then \( (f\sigma_\alpha)_+ \neq 0 \) for any \( f \in H^2(\sigma_\alpha) \setminus \{0\} \).

**Proof:** Let \( \alpha \in \mathbb{T} \) and \( f \in H^2(\sigma_\alpha) \setminus \{0\} \). Since \( k_b(z, \cdot) \in \overline{\mathcal{H}(b)} \) and \( U_\alpha = U_{b,\alpha} : \mathcal{H}(b) \rightarrow H^2(\sigma_\alpha) \) is unitary, we obtain
\[ U_{\alpha}^*f(z) = \int_{\partial \mathcal{D}} U_{\alpha}^*f(\xi)k_b(z, \xi) \, d\Sigma(\xi) \]
\[ = \int_{\partial \mathcal{D}} (1 - \overline{\alpha}I(z))f(\xi)C(z, \xi) \, d\sigma_\alpha(\xi) \]
\[ = (1 - \overline{\alpha}I(z))(f\sigma_\alpha)_+(z) \]
for \( z \in \mathcal{D} \). Now, assume that \( (f\sigma_\alpha)_+ \equiv 0 \). Then \( U_{\alpha}^*f \equiv 0 \), hence, \( f = 0 \). This contradiction ends the proof.
For an inner function $b$, it is natural to ask whether $U_{b,\alpha} : \mathcal{H}(b) \to L^2(\sigma_\alpha)$ is surjective or, equivalently, $H^2(\sigma_\alpha[b]) = L^2(\sigma_\alpha[b])$.

**Corollary 3.4:** Let $\alpha \in \mathbb{T}$ and $I : \mathcal{D} \to \mathbb{D}$ be an inner function. Then the following properties are equivalent:

(i) $U_{I,\alpha}$ maps $\mathcal{H}(b)$ onto $L^2(\sigma_\alpha)$;

(ii) if $f \in L^2(\sigma_\alpha)$ and $(f \sigma_\alpha)_+ \equiv 0$, then $f = 0$.

**Proof:** (ii) $\Rightarrow$ (i) Assume that (i) does not hold, that is, $H^2(\sigma_\alpha) \neq L^2(\sigma_\alpha)$. Then the definition of $H^2(\sigma_\alpha)$ guarantees that there exists $f \in L^2(\sigma_\alpha) \setminus \{0\}$ such that

$$\int_{\partial \mathcal{D}} f(\zeta) \overline{C(\zeta, z)} \, d\sigma_\alpha(\zeta) = 0$$

for all $z \in \mathcal{D}$. In other words, $(f \sigma_\alpha)_+ \equiv 0$ and we arrive to a contradiction.

By Proposition 3.3, (i) implies (ii), hence, the proof is finished. ■

**Remark 3.1:** Corollary 3.4 reduces to Proposition 3.3 for $\mathcal{D} = B_n$. Indeed, we have $H^2(\sigma_\alpha[I]) = L^2(\sigma_\alpha[I])$ for any $\alpha \in \mathbb{T}$ and any inner function $I$ in the unit ball $B_n$, $n \geq 1$; see [7]. However, this is not the case for many inner functions in the polydisc $\mathbb{D}^n$, $n \geq 2$.

To give a simple example, consider the following inner function: $I(z) = z_1$, $z \in \mathbb{D}^n$, $n \geq 2$. We have

$$\sigma_\alpha = \delta_\alpha(\xi_1) \otimes m(\xi_2) \otimes \cdots \otimes m(\xi_n),$$

where $m$ denotes the normalized Lebesgue measure on $\mathbb{T}$. For all $\alpha \in \mathbb{T}$, the space $H^2(\sigma_\alpha)$ is strictly smaller than $L^2(\sigma_\alpha)$.

### 4. Proof of Theorem 1.2

#### 4.1. Auxiliary results and definitions

##### 4.1.1. Pluriharmonic measures

A measure $\mu \in M(\partial \mathcal{D})$ is called pluriharmonic if the Poisson integral

$$P[\mu](z) = \int_{\partial \mathcal{D}} P(z, \zeta) \, d\mu(\zeta), \quad z \in \mathcal{D},$$

is a pluriharmonic function. Let $PM(\mathcal{D})$ denote the set of all pluriharmonic measures. Clearly, every Clark measure is an element of $PM(\mathcal{D})$.

##### 4.1.2. Totally singular measures

By definition, the ball algebra $A(B_n)$ consists of those $f \in C(B_n)$ which are holomorphic in $B_n$. Let $M_0(S_n)$ denote the set of those probability measures $\rho \in M(S_n)$ which represent the origin for $A(B_n)$, that is,

$$\int_{S_n} f \, d\rho = f(0) \quad \text{for all } f \in A(B_n).$$

Elements of $M_0(S_n)$ are called representing measures.
**Definition 4.1:** A measure $\mu \in M(S_n)$ is said to be *totally singular* if $\mu \perp \rho$ for all $\rho \in M_0(S_n)$.

**Proposition 4.2 ([11, Theorem 10]):** Let $\mu \in PM(S_n)$. Then $\mu^s$ is totally singular.

**Remark 4.1:** For positive pluriharmonic measures, the above theorem was obtained in [12, Chapter 5, Section 3.3.3]. In fact, we will apply Proposition 4.2 to Clark measures, that is, to positive $\mu \in PM(S_n)$.

### 4.1.3. Henkin measures

**Definition 4.3 (see [1, Section 9.1.5]):** We say that $\mu \in M(S_n)$ is a *Henkin measure* if

$$\lim_{j \to \infty} \int_{S_n} f_j \, d\mu = 0$$

for any bounded sequence $\{f_j\}_{j=1}^\infty \subset A(B_n)$ with the following property:

$$\lim_{j \to \infty} f_j(z) = 0 \quad \text{for any } z \in B_n.$$

### 4.2. Proof of Theorem 1.2

We are given an inner function $I$ in $B_n$ and a non-constant holomorphic function $b : B_n \to \mathbb{D}$. Without loss of generality, assume that $\alpha = 1$. So, let $\sigma = \sigma_1[I]$, $\mu = \sigma_1[b]$ and $K_w = K(\cdot, w)$, where $K(z, w)$ denotes the reproducing kernel for $I^*(H^2) = \mathcal{H}(I)$. Applying formula (1) and property (2) with $b = I$, we obtain

$$\left(1 - b\right)(C(\cdot, w)\sigma)_+ = \frac{1 - b}{1 - I} V_I C(\cdot, w) = (1 - \overline{I(w)})^{-1} \frac{1 - b}{1 - I} K_w. \quad (4)$$

Now, we are in position to prove the theorem.

(i)⇒(ii) By the definition of $V_b$, we have

$$V_b \left( \frac{d\sigma}{d\mu} C(\cdot, w) \right) = (1 - b) \left( \frac{d\sigma}{d\mu} C(\cdot, w)\mu \right)_+ = (1 - b)(C(\cdot, w)\sigma)_+.$$

Since $V_b$ maps into $\mathcal{H}(b)$, combination of the above property and (4) provides (ii).

(ii)⇒(iii) This implication is trivial.

(iii)⇒(i) By assumption, we are given a point $w \in B_n$ and a function $q = q_w \in L^2(\mu)$ such that

$$\left(1 - \overline{I(w)}\right)^{-1} \frac{1 - b}{1 - I} K_w = V_b q = (1 - b)(q\mu)_+.$$

The above property and (4) guarantee that

$$(C(\cdot, w)\sigma - q\mu)_+ = 0.$$

Hence,

$$C(w, \cdot)\sigma - \tilde{q}\mu$$
is a Henkin measure. Thus, by the Cole–Range Theorem (see [13] or [1, Theorem 9.6.1]),
there exists a representing measure $\rho$ such that $C(\cdot, w)\sigma - q\mu \ll \rho$; in particular,

$$C(\cdot, w)\sigma - q\mu^s \ll \rho \quad \text{for some } \rho \in M_0(S_n). \quad (5)$$

Recall that $\sigma$ is a singular measure. Therefore, $\sigma$ and $\mu^s$ are totally singular by Proposition 4.2. Hence,

$$C(\cdot, w)\sigma - q\mu^s \text{ is totally singular.}$$

Combining this observation and (5), we conclude that

$$\sigma = \frac{q}{C(\cdot, w)} \mu^s.$$ 

In particular, $\sigma \ll \mu$ and $\frac{d\sigma}{d\mu} \in L^2(\mu)$, as required. The proof of Theorem 1.2 is finished.

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