Landau Damping of Electrostatic Waves in Arbitrarily Degenerate Quantum Plasmas

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We carry out a systematic study of the dispersion relation for linear electrostatic waves in an arbitrarily degenerate quantum electron plasma. We solve for the complex frequency spectrum for arbitrary values of wavenumber $k$ and level of degeneracy $\mu$. Our finding is that for large $k$ and high $\mu$ the real part of the frequency $\omega_r$ grows linearly with $k$ and scales with $\mu$ only because of the scaling of the Fermi energy. In this regime the relative Landau damping rate $\gamma/\omega_r$ becomes independent of $k$ and varies inversely with $\mu$. Thus, damping is weak but finite at moderate levels of degeneracy for short wavelengths.

Introduction. Although the field of plasma physics has traditionally been concerned with systems of classical particles, the treatment of collective interactions of charged quantum particles has been of interest for many years [1–4]. More recently, interest in quantum plasma physics has increased due to the possible importance of collective plasma effects in microelectronic systems [5]; the experimental realization of warm, dense matter [6]; and in the description of astrophysical compact objects [7].

Generally, quantum effects are expected to be significant for physics at the smallest scales. For plasma physics, this means systems in which microscopic scales such as the Debye length $\lambda_D \equiv (T/4\pi n e^2)^{1/2}$ and gyroradius $\rho_e \equiv m_e c/\omega_B e B$ are on the order of or smaller than the de Broglie wavelength of the particles, $\lambda_{DB} \equiv h/(2\pi m_e c T)^{1/2}$. The proper treatment of such microscopic scales requires kinetic theory. For this reason, developing a kinetic theory of quantum plasma physics is of special interest.

One classic textbook problem of classical kinetic plasma physics is the propagation, dispersion, and collisionless Landau damping of linear electrostatic electron waves in unmagnetized plasmas [8]. This problem has proven useful in illustrating the effects of quantum physics on plasma phenomena as well [9, 10]. Landau damping in the quantum case is also of interest as it is potentially observable in experiments involving nanoscale electronics, plasmonic devices and warm dense matter [5, 6, 11, 12] and of importance in dense astrophysical plasmas [13]. Plasma dispersion and damping has been observed in solid-state plasmas [14–17]. On the theoretical front, the linear quantum longitudinal dispersion relation has been studied extensively in the literature in a variety of limiting cases, including: the completely degenerate limit with phase velocities above $c$ and near the Fermi velocity; at arbitrary phase velocities in one-dimensional plasmas for several temperatures [18, 19]; and the high phase velocity limit for arbitrarily degenerate quantum plasmas [8, 21] including the ion-acoustic mode [22]. In addition, there have recently been studies of the onset of nonlinearity and particle trapping in quantum Landau damping [23, 24]. However, a systematic examination of the linear dispersion relation for all wavelengths in arbitrarily degenerate plasmas has so far not appeared in the literature. There is, for example, no study of short-wavelength wave propagation or damping in moderately degenerate plasmas. For this reason, a thorough study of the dispersion of linear electrostatic waves of any wavelength in both partially and completely degenerate quantum plasmas is called for.

In this paper, we study linear one-dimensional (1D) longitudinal waves in an arbitrarily degenerate, unmagnetized electron gas, with a neutralizing uniform ion background. We map out numerically the complex dispersion relation over a broad range of wavenumbers $k$ and levels of degeneracy $\mu/T$, $\mu$ being the chemical potential and $T$ the temperature. We then use this map of the complex frequency $\omega = \omega_r - i\gamma$ to analyze the key differences between non-degenerate and degenerate cases and discuss the possibility of Landau damping in the degenerate limit. We find that for long wavelengths $\omega_r$ is approximately constant and the damping rate $\gamma$ is exponentially suppressed for all values of degeneracy. At short wavelengths however, both $\omega_r$ and $\gamma$ grow linearly with $k$ so that the relative Landau damping rate $\gamma/\omega_r$ becomes independent of $k$. The short-wavelength relative damping rate exceeds unity in the classical case, but decreases as $(\mu/T)^{-1}$ as $\mu \to \infty$. This implies that it may be important to account for both the presence of electrostatic waves and their damping in the description of a real partially degenerate plasma, depending on the specific level of degeneracy. This additionally implies that electrostatic waves of arbitrarily short wavelength are able to propagate in a degenerate plasma.

Theory of Linear Electrostatic Waves in a Quantum Plasma. Quantum kinetic plasma physics can be cast in a particularly convenient form by utilizing the phase-space formulation of quantum mechanics [25]. One defines the 1-body Wigner function $F(x, p; t) \equiv W(\hat{\rho}) \equiv \int d^3y e^{-i\phi/\hbar}(x + y/\rho)(x - y)$ [26], where $\rho \equiv \sum_i a_i |\psi_i\rangle \langle \psi_i|$ is the density operator. The operation $W$, called the Wigner transformation, defines the quantum...
phase space and $F$ acts as a quasi-distribution function.

For a system with a scalar potential $V$ and Hamiltonian $H = p^2/2m + V$, the Wigner function evolves according to the Moyal equation \cite{[18, 27]},

$$
\frac{\partial F(x, p, t)}{\partial t} = -\frac{\hbar}{2} \sin \frac{h}{2} \left( \frac{\partial}{\partial x_F} \cdot \frac{\partial F}{\partial p} - \frac{\partial}{\partial x_H} \frac{\partial F}{\partial p} \right) F(x, p, t) H(x, p, t),
$$  

(1)

where the subscripts indicate the function upon which a given derivative operates and the sine of differential operators is defined by its power series. This equation becomes the Vlasov equation in the classical limit $\hbar \to 0$, with corrections of higher order in $\hbar$ acting as scattering/non-phase-space-volume-preserving terms.

Using Eq. (1) linearized about a background equilibrium distribution function $F_0$, one can obtain the longitudinal dielectric function \cite{[11, 11, 23]},

$$
\epsilon = 1 + \frac{\omega_p^2 m_e}{2\hbar^2} \int d^3v \frac{F_0 (\mathbf{v} + \Delta) - F_0 (\mathbf{v} - \Delta)}{\omega - \mathbf{v} \cdot \mathbf{k}},
$$  

(2)

where $\Delta = \hbar \mathbf{k}/m_e$ and $\omega_p = (4\pi n e^2/m_e)^{1/2}$ is the classical electron plasma frequency. We integrate Eq. (2) over the directions perpendicular to the wave propagation $\mathbf{k}$, obtaining a 1D problem with a reduced equilibrium distribution function of the velocity parallel to $\mathbf{k}$,

$$
f_0 (v) = \int d^2v_\perp F_0 (v).$$

The remaining integral is along a contour $C$ following the real $v_\| \perp$ axis and diverting into the negative $\text{Im} (v_\|)$ half-plane to encompass the singularity at $v_\| = \omega/k$. This diversion must be done in such a way that the contour also avoids any singularities in the function $f_0$.

The dielectric function in Eq. (2) differs from the classical case in two ways. First, the velocity derivative of the distribution function is replaced with a finite difference in the plasmon momentum $\hbar \mathbf{k}$. Second, quantum mechanics can influence the background distribution $F_0$, which for arbitrarily degenerate plasmas in thermal equilibrium is a Fermi-Dirac distribution $F_0 (v, \mu) \propto \left[1 + \exp \left(\frac{v^2}{2m_e T} - \mu/T\right)\right]^{-1}$. The corresponding reduced 1D distribution, defined by integrating over the perpendicular directions, is

$$
f_0 (v, \mu) = \frac{N (\mu)}{v_T} \text{Ln} \left(1 + e^{-m_e v^2/2T + \mu/T}\right),
$$  

(3)

where $N (\mu) = \left[\sqrt{\pi} \text{Li}_{3/2} (-e^{\mu/T})\right]^{-1}$ is the degeneracy-dependent normalization, thus normalizing $f_0$ to 1, and $v_T = (T/m_e)^{1/2}$ is the classical electron thermal velocity. Here $\text{Li}_{3/2}$ is the polylogarithm function of order $3/2$. The ratio $\mu/T$ determines the level of degeneracy of the system: a Maxwellian is recovered in the limit $\mu/T \to -\infty$ and a distribution with $f_0 (v > v_F) \equiv 0$ in the degenerate limit $\mu/T \to +\infty$, where $v_F = (2E_F/m_e)^{1/2}$ is the Fermi velocity with $E_F = (\hbar^2/2m_e) (2\pi n)^{2/3}$ the Fermi energy.

By making a transformation to non-dimensional variables $x = v_\| /v_T$, $k \to k \lambda_D$, and $\omega \to \omega/\omega_p$ in Eq. (2) along with Eq. (3), we obtain

$$
\epsilon (\omega, k) = \begin{cases} 
1 - N (\xi) \frac{1}{\pi k^3} \int_{-\infty}^{\infty} \text{Ln} \left[\frac{1 + xe^{-(x+H)k}}{1 + xe^{-(x-H)k}}\right] \frac{dx}{x - \omega/\sqrt{2k}} & \text{Im} (\omega) > 0 \\
1 - N (\xi) \frac{1}{\pi k^3} \left( P \int_{-\infty}^{\infty} \text{Ln} \left[\frac{1 + xe^{-(x+H)k}}{1 + xe^{-(x-H)k}}\right] \frac{dx}{x - \omega/\sqrt{2k}} + i\pi \text{Ln} \left[\frac{1 + xe^{-(x-H)k}}{1 + xe^{-(x+H)k}}\right]\right) & \text{Im} (\omega) = 0 \\
1 - N (\xi) \frac{1}{\pi k^3} \int_{-\infty}^{\infty} \text{Ln} \left[\frac{1 + xe^{-(x+H)k}}{1 + xe^{-(x-H)k}}\right] \frac{dx}{x - \omega/\sqrt{2k}} + 2i\pi \text{Ln} \left[\frac{1 + xe^{-(x-H)k}}{1 + xe^{-(x+H)k}}\right]\right) & \text{Im} (\omega) < 0
\end{cases}
$$  

(4)

Here $P$ is the principal value, $\xi \equiv \exp (\mu/T)$, and $H \equiv \sqrt{\pi} \lambda_D/2\lambda_D = 8\pi \sqrt{2}\omega_p/T$ is the dimensionless quantum recoil parameter.

In Eq. (3), there are three parameters: $k$, $\xi$ (or $\mu$), and $H$. The important physical length and time scales are $\lambda_D$ and $\omega_p^{-1}$. Additionally, there exists a natural scale for wave phenomena given by the screening length $\lambda_s = \langle v \rangle /\omega_p$ where $\langle v \rangle = \int |v| F_0 (v, \mu/T) d^3v$ is the average speed of particles in the plasma. The level of degeneracy determines $\langle v \rangle$ and thus the screening length is dependent upon degeneracy,

$$
\lambda_s = \text{Li}_{5/2} \left[-e^{\mu/T}\right] \lambda_D,
$$  

(5)

which becomes the Debye length and Thomas-Fermi length $\lambda_{TF} = v_F/\omega_p$ in the non-degenerate and degenerate limits, respectively. The dependence of this screening length on the chemical potential is shown in the inset in Fig. 1. All of the above parameters and scales depend...
on $n$ and $T$. 

The physical parameters of the system we describe are the density and temperature $n$ and $T$, but these parameters are manifest in the dielectric function Eq. (4) through the dimensionless parameters $\mu \left( n, T \right) / T$ and $H \left( n, T \right)$. For simplicity we have limited this study to the case of small $H = 10^{-3}$, varying only $\mu$ while keeping $H$ fixed. This means that $n$ and $T$ cannot take arbitrary values. In future work it will be beneficial to map the dispersion relation for arbitrary values of $n$ and $T$ (and hence of $\mu$ and $H$), but for now we focus on the mathematical properties of the dispersion relation without considering all possible physical parameter regimes.

The quantum kinetic theory utilized in this paper is applicable for weakly-coupled non-relativistic plasmas without spin. The first condition requires a large number of particles to be present within a screening radius $n\lambda^3 \gg 1$, and the second condition requires that the thermal and Fermi velocities not approach the speed of light. In the zero temperature (totally degenerate) case this has the effect of setting minimum and maximum values for the density, namely $n\lambda^3_F = \pi^2 m_e e^2$ is the Bohr radius and $v_F/c = 1$. Numerically this means $2.2 \times 10^{23} \text{cm}^{-3} \ll n \ll 7 \times 10^{28} \text{cm}^{-3}$. The extension to relativistic and spin plasmas is an ongoing effort.

Results. We evaluate the dielectric function $\epsilon$ from Eq. (1) by direct numerical integration. Taking the usual logarithm branch cut in the negative horizontal direction, all integrals along the real axis are well-defined. We first solve $\epsilon \left( \omega, k, \mu, H \right) = 0$ for $\omega$ for single values of $k$, $\mu/T$ and $H$ in the classical and $\omega/k \gg v_F$ limits using Newton’s Method with a starting point given by an analytical approximation. We then solve for the entire classical dispersion relation by iterating to larger values of $k$. We then iterate to larger values of $\mu/T$ using the results of the next-smallest degeneracy. In this way we map out $\omega$ in the entire $(k, \mu)$ plane with fixed value of $H = 10^{-3}$.

A complication arises due to the dependence of the screening length $\lambda_*$ on $\mu$, varying between $\lambda_D$ in the classical case and $\lambda_T F \left( \mu \right)$ in the degenerate regime. We thus consider the dependence of $\gamma_F$ and $\gamma$ on $\mu$ and $k\lambda_*$ instead of $k\lambda_D$. This dependence is shown for several values of degeneracy in Fig. 1. In all cases, for small $k$, $\omega_F$ is approximately $\omega_F$ and weakly dependent on $k$, and for $k\lambda_* \gg 1$ grows linearly with $k$. The damping rate $\gamma$ is exponentially suppressed for small $k$ at all degeneracies, and also grows linearly at large $k$. The effect of increasing degeneracy on $\omega \left( k \right)$ is to change the sharpness of the transition between flat and linear growth of $\omega_F \left( k \right)$ near $k\lambda_* = 1$, and to shift $\gamma \left( k \right)$ to larger $k$.

The dependence of $\omega$ on $\mu/T$ for fixed $k\lambda_*$ is shown in Fig. 2 and it is seen from examination of $\omega_F$ that there are small- and large-$\mu$ regimes connected by a transition region. This transition is caused by the change in shape of the distribution function from Maxwellian to Fermi-Dirac. The scaling of the distribution function with $\mu$ is thus accounted for by the change in $\lambda_*$. Since the primary goal of this paper is to understand the dependence of Landau damping on the level of degeneracy, a quantity of particular interest is $\Gamma \left( \mu \right) = \lim_{k \rightarrow \infty} \gamma \left( k, \mu \right) / \omega_F \left( k, \mu \right)$. We find that the quantity $\Gamma$ is approximately constant for small $\mu/T$ and then turns downwards to decrease inversely with degeneracy for larger $\mu/T$ approaching $\Gamma \left( \mu \right) \approx 1.3 \left( \mu/T \right)^{-\frac{1}{2}}$.

We have confirmed that the results obtained with our general numerical method agree well with the previously published analytical results in various limiting cases. For example, our calculations reproduce the analytical approximations obtained by Melrose and Mushtaq [9] for the dependence of both $\omega_F$ and $\gamma$ on $k$ and $\mu$ in the long-wavelength $k\lambda_* \approx 1$ limit for arbitrary degeneracy. Also, in the degenerate limit $(\mu/T \rightarrow \infty)$ our present results agree well with the solutions for $\omega_F$ derived for long and moderate wavelengths (small- and moderate $k$) by [10, 19] using a completely degenerate top-hat equilibrium distribution with $F_0 \left( v > v_F \right) = 0$. For larger wavenumbers, Ref. [14] claimed that there is a critical maximum wavenumber above which no solutions exist in the fully degenerate case, due to the non-analyticity of the distribution function. Our present study (conducted for plasmas with arbitrarily large, but finite degeneracy)
Figure 2. Dependence of $\omega_r$ (a) and the relative damping rate $\omega_r/\gamma$ (b) on level of degeneracy with $k\lambda^*$ fixed. Red (dotted): $k\lambda^* = 0$, orange (dashed): $k\lambda^* = 1$ blue (large dashes): $k\lambda^* = 3$, black (solid): $k\lambda^* = 8$.

cannot confirm this result for the parameter regimes we have explored.

Discussion and Conclusions. In conclusion, in this paper we analyzed the complex frequency spectrum of linear longitudinal electrostatic waves in an arbitrarily degenerate electron plasma with a stationary neutralizing positive background. Using an appropriate analytical form for the dispersion relation from quantum kinetic theory, we developed a numerical procedure to solve for the real part of the frequency $\omega_r$ and the Landau damping rate $\gamma$ as functions of the wavenumber $k$ and the level of degeneracy $\mu/T$. We found that there are two ways in which degeneracy affects the dispersion relation: (1) through the change in the shape of the distribution function from Maxwellian to a Fermi-Dirac profile, and (2) the increased characteristic velocity as a function of $\mu/T$. Above a certain level of degeneracy, the distribution function assumes an approximately constant shape; beyond this, the real part of the frequency only depends on degeneracy through the scaling of the characteristic wavelength $\lambda^*$. By accounting for this scaling we isolated the effect of the shape of the Fermi-Dirac function on the complex frequency $\omega = \omega_r - i\gamma$.

We showed that for wavelengths shorter than $\lambda^*$ both $\omega_r$ and $\gamma$ grow linearly with $k$ for all levels of degeneracy with the relative damping rate $\gamma/\omega_r$ decreasing as $1.3 (\mu/T)^{-1}$, independent of $k$ as $\mu/T \to \infty$. This means that electrostatic waves in such systems propagate but have finite damping rates. Since any real system occurs at finite temperature, the choice of whether to account for the presence of electrostatic waves, and whether to account for their damping, should be informed by the specific temperature and density of a degenerate system. This effect will be of importance not just for pure electrostatic waves, but for other plasma waves in which collisionless kinetic damping occurs.

A thorough understanding of Landau damping in the linear regime is important before moving on to nonlinear effects. The realization that nonlinear particle trapping is suppressed in certain parameter regimes in quantum models gives additional impetus to study the linear regime of Landau damping, in contrast to classical plasmas where the damping more quickly gives way to nonlinearity. Additionally, the method used in this study can be easily extended to further studies of linear waves and instabilities in quantum Fermi-Dirac plasmas. Specifically, the inclusion of additional populations of electrons or of mobile ions can be pursued in order to understand streaming instabilities and ion-acoustic waves in quantum plasmas. A study of these instabilities will be presented in a future publication.

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