The John Theorem for Simplex

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Abstract. In this paper, we give a description of the John contact points of a regular simplex. We prove that the John ellipsoid of any simplex is ball if and only if this simplex is regular and that the John ellipsoid of a regular simplex is its inscribed ball.

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1. Introduction

In 1948, F. John proved that every convex body (i.e., a compact, convex subset with nonempty interior) in $\mathbb{R}^n$ contains only one maximal (in volume) ellipsoid, which is known as the John ellipsoid. When the John ellipsoid is the unit ball $B_n^2$, F. John has proved the following theorem.

**Theorem 1.** [John] Let $C$ be a convex body in $\mathbb{R}^n$. The ellipsoid of maximal volume in $C$ is $B_n^2$, if and only if $C$ contains $B_n^2$ and there are some points $(u_i)_{i=1}^m$ on the boundary of $C$ and positive numbers $(c_i)_{i=1}^m$ so that

\[
a) \quad \sum_{i=1}^m c_i u_i \otimes u_i = I_n, \quad \text{and} \\
b) \quad \sum_{i=1}^m c_i u_i = 0.
\]

Here, $I_n$ is the identity map on $\mathbb{R}^n$ and, for any unit vector $u$, $u \otimes u$ is the rank-one orthogonal projection onto the span of $u$, i.e., the map $x \rightarrow \langle x, u \rangle u$. The $u_i$'s of the theorem are the intersection points of the unit sphere $S^{n-1}$ with the boundary of $C$.

Condition a) shows that the $(u_i)_{i=1}^m$ behave rather like an orthonormal basis in that we can resolve the Euclidean norm as a (weighted) sum of square of inner products. This condition is equivalent to the statement that, for all $x$ in $\mathbb{R}^n$,

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\[ |x|^2 = \sum_{i=1}^{m} c_i \langle x, u_i \rangle^2, \]  
\[ (1) \]

where \( \langle \cdot, \cdot \rangle \) is the usual Euclidean inner product and \( |\cdot| \) is the induced norm by this inner product.

By a simple computation, we know that equality (1) is equivalent to
\[ x = \sum_{i=1}^{m} c_i \langle x, u_i \rangle u_i. \]

(2)

For detail, see \([Ba1]\) and in \([Ba3]\), one can find a modern proof of Theorem 1.

**Definition 1** Suppose that \( A_1, \ldots, A_{n+1} \in \mathbb{R}^n \) be affinely independent then the convex hull, denoted by \( A \), of these points is called a simplex, i.e.,
\[ A = \{ x \mid x = \sum_{i=1}^{n+1} \lambda_i A_i, \ \sum_{i=1}^{n+1} \lambda_i = 1, \ \lambda_i \geq 0 \}, \]
and if all \( |A_i A_j|, \ i \neq j \) are equal, then we call \( A \) a regular simplex.

In Theorem 1., \( (u_i)_{1}^{m} \) is usually called the contact points. For the unit cube \([-1,1]^{n} \) in \( \mathbb{R}^n \), the maximal ellipsoid is \( B_2^n \) as one would expect, so the contact points are the standard basis vectors \((e_1, \ldots, e_n)\) of \( \mathbb{R}^n \) and their negatives. However, even for the simplest nonsymmetric convex body-simplex, there is no nature description of the contact points.

In this paper, we give a description of the contact points for a regular simplex and the main results are the following theorems.

**Theorem 2.** The John ellipsoid of a regular simplex is its inscribed ball.

**Theorem 3.** For any simplex in \( \mathbb{R}^n \), the John ellipsoid of this simplex is ball if and only if the simplex is regular.

2. The Proof of Main Results

First we introduce the following definition[C].

**Definition 2[C]** Suppose that \( A \) is an \( n \)-dimensional simplex with vertexes \( \{A_1, \ldots, A_{n+1}\} \), \( M \) is a point in \( \mathbb{R}^n \). Denote by \( V_i, i = 1, \ldots, n+1 \), the volume of the simplex with vertexes \( \{A_1, \ldots, A_{i-1}, M, A_{i+1}, \ldots, A_{n+1}\} \), and if the dimensions of
\[ \text{con}\{A_1, \ldots, A_{i-1}, M, A_{i+1}, \ldots, A_{n+1}\} \quad \text{and} \quad \text{con}\{A_1, \ldots, A_{i-1}, M, A_{i+1}, \ldots, A_{n+1}\} \cap A \]
are both \( n \), then the following ratio is called the barycentric coordinates of \( M \),
\[ V_1 : V_2 : \ldots : V_{n+1}. \]

Suppose that \( \{A_1, \ldots, A_{n+1}\} \) are the vertexes of a regular simplex \( A \) and \( B_2^n \) is its inscribed ball. Denote by \( \{B_i, i = 1, \ldots, n+1\} \), the tangent points of \( B_2^n \) with the face generated by the
convex hull of \( \{A_1, ..., A_{i-1}, A_i, ..., A_n \} \) and denote by \( \{u_i, i = 1, ..., n+1\} \) the outer normal unit vectors of these facet respectively. According to the Definition 2, we have the barycentric coordinates of \( B_i \) as follows

\[
\left( \frac{1}{n}, ..., \frac{1}{n}, 0, \frac{1}{n}, ..., \frac{1}{n} \right),
\]

where 0 is in the \( i \)-th \((i = 1, ..., n+1) \) position.

**Proof of Theorem 2.**

According to the Theorem 1., it suffices to prove that the tangent points of a regular simplex with its inscribed ball satisfied the condition a) and b).

Now suppose that \( A \) is a regular simplex with vertexes \( \{A_1, A_2, ..., A_{n+1}\} \) and \( B_2^n \) is its inscribed ball. Denote by \( \{B_i, i = 1, ..., n+1\} \) the tangent points which is opposite to \( \{A_i, i = 1, ..., n+1\} \) respectively. From the above discussion, the barycentric coordinates of \( B_i \) is

\[
\left( \frac{1}{n}, ..., \frac{1}{n}, 0, \frac{1}{n}, ..., \frac{1}{n} \right),
\]

where 0 is in the \( i \)-th \((i = 1, ..., n+1) \) position.

Obviously, the barycentric coordinates of the origin is \( (1, ..., 1) \).

Let \( c_i = \frac{n}{n+1}, i = 1, ..., n+1, \) then

\[
\sum_{i=1}^{n+1} c_i B_i = (1, ..., 1).
\]

Thus the condition b) is satisfied.

Next, we will prove that \( u_i \) satisfied the condition a) of Theorem 1., that is, for any \( x \in \mathbb{R}^n \), the following equality holds,

\[
x = \sum_{i=1}^{n+1} c_i \langle x, u_i \rangle u_i,
\]

where \( c_i = \frac{n}{n+1}, i = 1, ..., n+1. \)

Because \( A \) is a \( n \)-dimensional simplex, the space spaned by the \( n+1 \) vectors \( \{u_i, i = 1, ..., n+1\} \) must be \( \mathbb{R}^n \), i.e.,

\[
\text{Span}\{u_1, u_2, ..., u_{n+1}\} = \mathbb{R}^n.
\]

So for any \( x \in \mathbb{R}^n \), there must exist \( n+1 \) real numbers \( \alpha_1, ..., \alpha_{n+1} \) such that

\[
x = \alpha_1 u_1 + ... + \alpha_{n+1} u_{n+1}.
\]
Thus, we can get
\[
\begin{align*}
\langle u_1, x \rangle &= \alpha_1 \langle u_1, u_1 \rangle + \alpha_2 \langle u_1, u_2 \rangle + \ldots + \alpha_{n+1} \langle u_1, u_{n+1} \rangle, \\
\langle u_2, x \rangle &= \alpha_1 \langle u_2, u_1 \rangle + \alpha_2 \langle u_2, u_2 \rangle + \ldots + \alpha_{n+1} \langle u_2, u_{n+1} \rangle, \\
& \vdots \\
\langle u_{n+1}, x \rangle &= \alpha_1 \langle u_{n+1}, u_1 \rangle + \alpha_2 \langle u_{n+1}, u_2 \rangle + \ldots + \alpha_{n+1} \langle u_{n+1}, u_{n+1} \rangle.
\end{align*}
\]

Denote \( \alpha = (\alpha_1, \ldots, \alpha_{n+1}) \), \( \beta = (\langle u_1, x \rangle, \ldots, \langle u_{n+1}, x \rangle) \), and

\[
D = \begin{pmatrix}
\langle u_1, u_1 \rangle & \langle u_1, u_2 \rangle & \cdots & \langle u_1, u_{n+1} \rangle \\
\langle u_2, u_1 \rangle & \langle u_2, u_2 \rangle & \cdots & \langle u_2, u_{n+1} \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle u_{n+1}, u_1 \rangle & \langle u_{n+1}, u_2 \rangle & \cdots & \langle u_{n+1}, u_{n+1} \rangle
\end{pmatrix},
\]

then the above equation system can be written as

\[
D \alpha^T = \beta^T, \tag{4}
\]

where \( \alpha^T, \beta^T \) represent respectively the transform of \( \alpha \) and \( \beta \).

Observe that every element of \( D \), \( \langle u_i, u_j \rangle \), is the cosine of angle of two outer normal unit vectors. Denote by \( F_i, F_j \) the faces whose outer normal unit vectors are \( u_i, u_j \) respectively. Obviously, the angle of \( u_i, u_j \) is mutually complementary with the angle of \( F_i, F_j \), i.e.,

\[
\langle u_i, u_j \rangle = - \cos \angle(F_i, F_j),
\]

where \( \angle(F_i, F_j) \) represents the dihedral angle of \( F_i, F_j \).

For \( \cos \angle(F_i, F_j) \), we have the following equality,

\[
\cos \angle(F_i, F_j) = \frac{S_{ji}}{S_j},
\]

where \( S_j \) is the \((n-1)\)-dimensional volume of face \( F_j \), and \( S_{ji} \) is the volume of the projection \( F_j \) to \( F_i \) along \( u_i \).

For \( A \) is regular simplex, the \( n \) \((n-1)\)-dimensional volumes of all the projections \( F_j, j \neq i \) to \( F_i \) along \( u_i \) are equal. So \( \frac{S_{ji}}{S_j} = \frac{1}{n} \). Thus we get the \( D \), i.e.

\[
D = \begin{pmatrix}
1 & -\frac{1}{n} & \cdots & -\frac{1}{n} \\
-\frac{1}{n} & 1 & \cdots & -\frac{1}{n} \\
\vdots & \vdots & \ddots & \vdots \\
-\frac{1}{n} & -\frac{1}{n} & \cdots & 1
\end{pmatrix} \quad \text{\((n+1)\times(n+1)\)}
\]

It follows from condition b) and (4) that

\[
\begin{align*}
D \alpha^T &= \beta^T \\
\sum_{i=1}^{n+1} \langle u_i, x \rangle &= 0.
\end{align*}
\]
Let \( \alpha = \left( \frac{n}{n+1} \langle u_1, x \rangle, ..., \frac{n}{n+1} \langle u_{n+1}, x \rangle \right) \) in the above equation system, we know that \( \alpha \) is a solution of this equation system. So every point \( x \in \mathbb{R}^n \) can be represented as the form of (2). The proof of Theorem 2. is completed.

To prove Theorem 3., we need the following Brascamp-Lieb inequality, which is the generalization of convolution inequality.

**Theorem 4.**[BL] Suppose that \( (u_i)_1^m \) is a sequence of unit vector in \( \mathbb{R}^n \), \( (c_i)_1^m \) is a sequence of positive real numbers and they satisfied the following equality
\[
\sum_{i=1}^{m} c_i u_i \otimes u_i = I_n.
\]

If \( f_i : R \rightarrow [0, \infty) \), \( i = 1, ..., m \) is a sequence of integrable functions, then
\[
\int_{\mathbb{R}^n} \prod_{i=1}^{m} f_i(\langle u_i, x \rangle)^{c_i} dx \leq \prod_{i=1}^{m} \left( \int_{R} f_i \right)^{c_i}. \tag{5}
\]

F.Barthe get a necessary condition for the equality holds in Theorem 4.

**Theorem 5.**[Bar] Suppose that \( (u_i)_1^m \) is a sequence of unit vector in \( \mathbb{R}^n \), \( (c_i)_1^m \) a sequence of positive real numbers, and they satisfied the following equality
\[
\sum_{i=1}^{m} c_i u_i \otimes u_i = I_n.
\]

If \( (f_i)_1^m \) is a sequence functions, not all zero in \( L_1(R) \), and all \( (f_i)_1^m \) are not the density function of Gauss distribution, then the necessary condition for the equality hold in (5) is
\[
m = n,
\]
and
\[
(u_i)_1^m
\]
is a orthonormal basis of \( \mathbb{R}^n \).

**Proof of Theorem 3.**

The "if" part of Theorem 3. can be obtained from Theorem 2. directly. So it is sufficient to prove that if the John ball of the simplex \( C \) is \( B^n_2 \) then \( C \) is regular.

Firstly, we observe that if the John ball of the simplex \( C \) is \( B^n_2 \), then \( B^n_2 \) is the inscribed ball of \( C \). If not, without lost of generalization, suppose that \( B^n_2 \) is not tangent with face \( F_i \). Let \( u_i \) be the outer normal unit vector of \( F_i \), then there must exist a positive number \( \varepsilon \), such that \( B^n_2 \) is not tangent with any faces of \( C \) when \( B^n_2 \) move \( \varepsilon \) along \( u_i \). At this time, there must exist another positive number \( r > 1 \) such that \( rB^n_2 \) be the John ball of \( C \). This contradicts with the fact that \( B^n_2 \) is the John ball of the simplex \( C \).
Because the inscribed ball of $C$ is its John ball, by Theorem 1., there exist a sequence positive real numbers $(c_i)_{1}^{n+1}$ and a sequence of unite vectors $(u_i)_{1}^{n+1}$ on the boundary of $C$ such that

$$\sum_{i=1}^{n+1} c_i u_i \otimes u_i = I_n,$$  \hspace{1cm} (6)$$

and

$$\sum_{i=1}^{n+1} c_i u_i = 0.$$  \hspace{1cm} (7)

Denote by $K = \{x \in R^n : \langle x, u_i \rangle \leq 1, 1 \leq i \leq n+1\}$, then $K$ is also the simplex in $R^n$. Because $(u_i)_{1}^{n+1}$ are the contact points of $C$ and $B^n_2$, 

$$C \subset \{x \in R^n : \langle x, u_i \rangle \leq 1, 1 \leq i \leq m\} = K.$$  

Observe that $B^n_2$ is also the inscribed ball of $C$ and that $K, C$ have the same tangent points $(u_i)_{1}^{n+1}$ with $B^n_2$, so 

$$C = K.$$  

Next, we will show that $K$ is regular simplex. 

In the following discussion, $R^{n+1}$ will be regarded as $R^n \times R$. For each $i$ let 

$$v_i = \sqrt{\frac{n}{n+1}} (-u_i, \frac{1}{\sqrt{n}}) \in R^{n+1}, \hspace{0.5cm} i = 1, ..., n+1,$$

$$d_i = \frac{n+1}{n} c_i, \hspace{0.5cm} i = 1, ..., n+1,$$

then $v_i$ is a unit vector and the identities (6) and (7), together yield that

$$\sum_{i=1}^{n+1} d_i v_i \otimes v_i = I_{n+1}.$$  

Define a sequence functions $(f_i)_{1}^{n+1}$ as follows,

$$f_i(t) = \begin{cases} e^{-t}, & \text{if } t \geq 0, \\ 0, & \text{if } t < 0. \end{cases}$$

For any $x \in R^{n+1}$, let 

$$F(x) = \prod_{i=1}^{n+1} f_i(\langle v_i, x \rangle)^{d_i},$$

by Theorem 4., we have

$$\int_{R^n} F(x) dx \leq \prod_{i=1}^{n+1} (\int_{R} f_i)^{d_i} = 1.$$  \hspace{1cm} (8)
Some of the above technique are from Ball. Using the similar discussion in [Ba2], we get the integration of $F$ in the hyperplane $\{x : x_{n+1} = r \geq 0\}$

$$e^{-\frac{r}{\sqrt{n}}}Vol(\frac{r}{\sqrt{n}}K) = e^{-\frac{r}{\sqrt{n}}}r^nVol(K).$$

So by (8)

$$1 \geq Vol(K) \int_0^\infty e^{-\frac{r}{\sqrt{n}}}r^n dr = \frac{Vol(K)n!}{\sqrt{n}(n+1)^{n+1}},$$

i.e.,

$$Vol(K) \leq \frac{n^n(n+1)^{n+1}}{n!}. \tag{9}$$

Observe that the right hand of (9) is exactly the volume of the regular simplex whose inscribed ball is $B_2^n$.

Observe the construction of $(fi)_{1}^{n+1}$, and the Theorem 5. for (8), thus we can get the condition for the equality holds in (9) and that is $(vi)_{1}^{n+1}$ is a sequence of orthonormal basis of $R^{n+1}$. For any two vectors of this basis

$$v_i = \sqrt{\frac{n}{n+1}}(-u_i, \frac{1}{\sqrt{n}}),$$

and

$$v_j = \sqrt{\frac{n}{n+1}}(-u_j, \frac{1}{\sqrt{n}}),$$

we have

$$0 = \langle v_i, v_j \rangle = \frac{n}{n+1}(u_i, u_j) + \frac{1}{n}. \tag{10}$$

So

$$\langle u_i, u_j \rangle = -\frac{n+1}{n^2}, i \neq j,$$

is a constant. Because that $(u_i)_{1}^{n+1}$ are the normal vectors of the $n + 1$ faces of the simplex $K$, $K$ is a regular simplex.

This completes the proof of the theorems.

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