Fundamental branes and shock waves

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Abstract

We construct supersymmetric brane solutions in string and M-theory with moduli parameters that depend arbitrarily on the light-cone time. Our investigation aims in understanding time dependent phenomena in gauge theories at strong coupling within the gauge/gravity correspondence. For that reason we use, as a basic ingredient, multicenter supergravity solutions which model the Coulomb branch of the corresponding strongly coupled gauge theories. We introduce the notion of shape invariant motions and show that in a particular limit involving pulse-type motions of finite energy, the solutions represent gravitational shock waves moving on the brane background geometry. We apply the general formalism for D3-branes distributed on a disc and on a sphere as well as for NS5-branes distributed on a ring, all with time varying radii. We examine the problem of open strings attached on moving branes and suggest a mechanism which may be responsible for giving rise at a macroscopic level to gravitational shock waves.

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1 Introduction

The gauge/gravity correspondence [1] has been used over the last years mainly in order to study gauge theories at strong coupling. Going beyond the original proposal on the holographic relation between $N = 4$ SYM and string theory on $AdS_5 \times S^5$ required the introduction of moduli parameters that break conformality, as well as part, if not all, of the supersymmetry and global symmetries. A natural question that will be the main focus of this paper is whether or not we can promote these moduli parameters to functions of time since understanding time dependent phenomena in physics is of immense importance. Given that the vast majority of tests and predictions within the gauge/gravity correspondence concerned supersymmetric field theories and their supergravity duals, we would like, at least in a first step, to introduce time dependence in a supersymmetric manner in addition to satisfying the field equations. This is a strong condition and as a result the time dependence comes in the form of the light-cone time, but it is otherwise arbitrary. In this paper we will be concerned with the construction of supergravity solutions with light-cone dependent moduli parameters. Among the various supergravity solutions with field theory duals we could have chosen to start our investigations, perhaps the most attractive for our purposes are those that deviate the least from the conformal supersymmetric case. To be specific we will use multicenter supergravity solutions representing the gravitational field of a large collection of fundamental branes in string and M-theory with unwrapped all the worldvolume directions. The constant moduli parameters that will be promoted to functions of the light-cone time are nothing but the centers of these branes which from the gauge theoretical point of view represent vacuum expectation values of scalar fields [2, 3, 4].

This paper is organized as follows: In section 2 we present the main idea and results that apply generically to all fundamental branes with centers depending on the light-cone time. In sections 3 and 4 we derive and present in detail the light-cone time dependent supergravity solutions for all fundamental branes of M- and string theory. The emphasis is given on the preservation of a fraction of supersymmetry compared with the generic static case. In section 5 we consider an important subclass of center motions in which the angles between the vectors, defining them in the transverse to the brane space, remain constant. Therefore, in these cases the shape of the distribution of the centers remains invariant. We also consider the limit of sudden changes in which the supergravity solution reduces to that of a shock wave propagating on the background multicenter brane solution and compute its profile in general. In section 6 we explicitly construct the solutions for the cases of D3-branes uniformly and continuously distributed on a disc and on a 3-sphere of varying radii. In addition we construct the shock waves on the maximally supersymmetric spaces $AdS_5 \times S^5$, $AdS_{4,7} \times S^{7,4}$ of the ten-dimensional type-IIB and
eleven-dimensional supergravities, respectively and consider scattering amplitudes in the propagation of scalar fields. In section 7 we consider the case of NS5-branes on a circle of varying radii. We explicitly solve the eigenvalue problem for the massless field spectrum and use the result for the computation of the exact scattering shock wave amplitudes. In section 8 we consider the problem of the first quantized open string with one end on a moving brane and compute exactly its unitary evolution in time for arbitrary brane motion. Finally, we present our conclusions and directions for future work in section 9. The paper is supplemented with an Appendix where we have collected the expressions for the spin connection and the Ricci tensor for the general form of the supergravity backgrounds we consider.

2 The general construction

Before we turn into the description of our brane solutions in detail, we present in this section some general aspects and results of the construction. The supersymmetric branes of M-theory and string theory with unwrapped all the world-volume directions, correspond to supergravity solutions whose metric can be cast in the following general form (for reviews, see, for instance, [5])

$$ds^2 = H^\alpha (-dt^2 + dy_1^2 + \cdots + dy_p^2) + H^{1+\alpha} dx^i dx^i ,$$

$$i = 1, 2, \ldots, d , \quad H = H(x) . \quad (2.1)$$

The dimensionality of the space-time is $D = p + 1 + d$ and $H$ is a harmonic function in the transverse space $\mathbb{R}^d$, i.e. $\partial^2 H = 0$. The numerical parameter $\alpha$ depends on the particular brane and the metric is supported by a non-trivial flux and possibly a dilaton so that the classical supergravity equations of motion are satisfied and a fraction of the maximal supersymmetry is preserved. We may introduce a wave by writing $-dt^2 + dy_1^2 = 2 dudv$, denoting the rest of the directions along the brane by $y^\alpha, \alpha = 2, \ldots, p$ and generalizing the ansatz (2.1) to

$$ds^2 = H^\alpha \left[ dy^\alpha dy^\alpha + 2dudv + F(x, u)du^2 + 2V_i(x, u)dx^i du \right] + H^{1+\alpha} dx^i dx^i ,$$

$$i = 1, 2, \ldots, d , \quad \alpha = 2, \ldots, p , \quad H = H(x, u) , \quad V_i = V_i(x, u) . \quad (2.2)$$

Hence, the various functions are allowed to depend on the transverse space coordinates $x^i$ as well as on the light-cone time $u$. The presence of the wave breaks the Lorentz symmetry along the brane to its $SO(p-1)$ rotational subgroup times an $\mathbb{R}$ factor representing shifts

\footnote{There should be no confusion between the numerical parameter $\alpha$ and the space-time index denoted by the same symbol.}
of $v$. We aim at constructing supersymmetric solutions by supporting the metric (2.2) with appropriate tensor fields. The gravitino and dilatino Killing spinor equations are of the general form

$$
\partial_\mu \epsilon + \frac{1}{4} \omega^{ab}_\mu \Gamma_{ab} \epsilon + (\cdots) \epsilon = 0 ,
$$

$$
\Gamma^\mu \partial_\mu \Phi \epsilon + (\cdots) \epsilon = 0 ,
$$

(2.3)

where the extra terms depend on products of the tensor fields and $\Gamma$-matrices. In these computations we will use the general frame

$$
e^+ = H^{\alpha/2} du , \quad e^- = H^{\alpha/2} \left( dv + V_i dx^i + \frac{F}{2} du \right) ,
$$

$$
e^i = H^{(1+\alpha)/2} dx^i , \quad e^\alpha = H^{\alpha/2} dy^\alpha .
$$

(2.4)

The spin connection necessary for solving the Killing spinor equations as well as the Ricci tensor are computed in the Appendix. We will find that it is indeed possible to construct such solutions and the amount of supersymmetry to be preserved depends on the dimensionality of the brane world-volume. Typically, the solution in the absence of the wave preserves half of the maximal supersymmetry. If the world-volume is a least $(2 + 1)$-dimensional, the presence of the wave requires an extra projection which is provided by

$$
\Gamma^+ \epsilon = 0 .
$$

(2.5)

This conclusion applies to M2- and M5-branes, the NS5-branes, as well as all Dp-branes with $p \geq 2$. Hence in these cases our configurations will preserve $\frac{1}{4}$ of the maximum supersymmetry. For the fundamental string NS1 and the D1-brane the amount of supersymmetry is the same as in the case of no wave, i.e. $\frac{1}{2}$ of the maximum, and (2.5) provides the only projection since $\Gamma^{01} \epsilon = \epsilon$ is equivalent to (2.5). The general form of the Killing spinor that satisfies (2.3) is essentially dictated by the supersymmetry algebra (as noted in a similar context in [6]) and reads

$$
\epsilon = H^{\alpha/4} \epsilon_0 ,
$$

(2.6)

where $\epsilon_0$ is a constant spinor subject to the same projections as $\epsilon$. The Killing spinor eq. determines the form of the tensor fields corresponding to the branes as well as in the case of string branes the dilaton field. However, it is not sufficient to completely determine the unknown functions $V_i$ and $F$ in the background metric (2.2). The reason is essentially that for Lorenzian backgrounds the integrability condition for the Killing spinor equation does not automatically imply the second order supergravity equations of motion. Hence, we have to also employ the Einstein field equations $R_{\mu \nu} = \cdots$. It turns out that, for all fundamental branes of M- and string theory there is a general result

$$
\partial^2 H = \partial^2 F = \partial^2 V_i = 0 ,
$$

(2.7)
that is, all functions entering into the expression for the metric (2.2) are harmonic in the $d$-dimensional transverse space to the branes. There is an additional condition linking together $H$ and $V_i$, namely

$$\dot{H} = \partial_i V_i,$$  \hspace{1cm} (2.8)

where the dot represents the derivative with respect to $u$.\(^2\) Therefore the general asymptotically flat solution representing the gravitational field of $N$ branes has

$$H(x, u) = 1 + \mathcal{A}_{sc} \sum_{a=1}^{N} \frac{1}{|x - x_a(u)|^{d-2}},$$

$$V_i(x, u) = -\mathcal{A}_{sc} \sum_{a=1}^{N} \frac{\dot{x}_a^i(u)}{|x - x_a(u)|^{d-2}},$$  \hspace{1cm} (2.9)

where the $d$-dimensional vectors $x_a, a = 1, 2, \ldots, N$, correspond to the location of the branes in the transverse space. Hence we see that compared with the static case the main difference is that the moduli vectors representing the constant positions of the branes have been promoted to arbitrary functions $x_a(u)$ of the light-cone time.\(^3\) The constant $\mathcal{A}_{sc}$, with units of $(\text{length})^{d-2}$, depends on the Planck scale for M-branes, the string scale and the string coupling constant for the branes of string theory, as well as on numerical factors. This constant does not depend on the fact that we have introduced lightcone dependence to the moduli parameters and has the same value as in the static case (see, for instance, [8]). From now on we scale for convenience the constant $\mathcal{A}_{sc}$ to unity. Also note that, we have not included a constant part in the $V_i$'s since, unlike in $H$, this can be absorbed by a shift of the coordinate $v$. Finally, we note that the functions $F$ and $V_i$ in the general metric (2.2) can all be set to zero by a suitable coordinate transformation (as in the case of backgrounds with a covariantly constant null Killing vector [9] and also in [10, 11]. Indeed, let

$$x^i = f^i(u, x'), \hspace{1cm} v = v' + h(u, x').$$  \hspace{1cm} (2.10)

Then for the new coordinates $(u, v', x'^i)$ we obtain the same metric as in (2.2) but with

$$F'(u, x') = F + 2\dot{h} + \dot{f}^i \dot{f}^i + 2V_i \dot{f}^i,$$

$$V_i'(u, x') = \partial_i h + V_j \partial_i f^j + \partial_i f^j \dot{f}^j.$$  \hspace{1cm} (2.11)

\(^2\)It turns out that this is the most convenient gauge choice to present our results in a clear way. The conditions (2.7) have been written after using (2.8). See also the transformation (2.10)-(2.12) below and the related comment at the end of this section.

\(^3\)Essentially the end result reminds of the explicit construction of manifolds with Lorentzian holonomy in various dimensions, where also the promotion of constant moduli parameters in Euclidean holonomy manifolds into arbitrary functions of the light-cone time, played an essential rôle [7].
Setting $F' = V_i' = 0$ in (2.11), gives a first order system for the unknown functions $h$ and $f^i$ which in principle can be solved. However, there is a price to pay. Namely, the flat transverse space metric $dx^i dx^i$ is replaced by the curved one

$$g'_{ij}(u, x') = \partial_i f^k \partial_j f^k .$$

(2.12)

This last, rather undesirable feature, makes the form of the metric (2.2) preferable for the general discussion. Finally, note that for $f_i = x_i$ in (2.10)-(2.12) we have a gauge-like transformation with parameter $h$ and the transverse space metric remains flat. After choosing (2.8), only $h$’s satisfying $\partial^2 h = 0$ preserve this choice and the conditions (2.7). This remaining freedom still allows to set the function $F = 0$, but we will not do so since it will be necessary for presenting specific classes of examples below. Another reason, not of direct concern to us, is that in the special case that $\partial/\partial u$ is also a Killing vector, the metric (2.2) cannot be written in a way that is manifestly independent of $u$ and simultaneously general, unless $F \neq 0$ (for an analogous discussion see also [11]).

### 3 M-theory branes

In the context of eleven-dimensional supergravity [12] the geometry is supplemented by a 4-form field strength $F_{\mu \nu \rho \sigma}$. The Killing spinor equation arising by setting to zero the gravitino supersymmetry variation is

$$\partial_{\mu} \epsilon + \frac{1}{4} \omega_{\mu}^{ab} \Gamma_{ab} \epsilon - \frac{1}{288} (F_{\nu \rho \lambda \sigma} \Gamma^{\nu \rho \lambda \sigma}_{\mu} - 8 F_{\mu \nu \rho \lambda} \Gamma^{\nu \rho \lambda}) \epsilon = 0 .$$

(3.1)

In addition, the Einstein equations of motion

$$R_{\mu \nu} = \frac{1}{12} \left( (F^2)_{\mu \nu} - \frac{1}{12} g_{\mu \nu} F^2 \right) ,$$

(3.2)

together with that for the 4-form field strength should be satisfied.

#### 3.1 M2-branes

In this case the metric is given by (2.1) with $\alpha = -2/3$ and $d = 8, p = 2$, so that $D = 11$. In the static limit it describes the multicenter generalization of the M2-brane solution of [13]. Working out the details of the Killing spinor eq. (3.1) we find that the usual projection

$$\Gamma^{+\cdot-2} \epsilon = \epsilon ,$$

(3.3)

where all indices are along the brane, should be supplemented with (2.5). This means that $\Gamma^2 \epsilon = \epsilon$. Also we get

$$F_{+2i} = -H^{-7/6} \partial_i H , \quad F_{+2ij} = H^{-2/3} \partial_i V_j ,$$

(3.4)
where all the indices refer to tangent space.

We note that all other M- and string theory brane solutions considered below follow from the M2-brane solution after performing a series of operations involving dimensional reductions, a smearing along a direction transverse to the brane and U-dualities. This step by step procedure is well known and rather straightforward for brane solutions with a single center, but it can also be generalized in the multicenter case. Nevertheless, we preferred to be analytic and pedagogical in our approach so that all different cases will be obtained in an independent way.

### 3.2 M5-branes

In this case the metric is given by (2.1) with \( \alpha = -1/3 \) and \( d = 5 \), \( p = 5 \), so that \( D = 11 \). In the static limit it describes the multicenter generalization of the M2-brane solution of [14]. Working out the details of the Killing spinor eq. (3.1) we find that the usual projection (all indices are transverse to the brane)

\[
\Gamma^{12345} \epsilon = -\epsilon ,
\]

when supplemented with (2.5) gives rise to

\[
F_{ijkl} = H^{-4/3} \epsilon_{ijklm} \partial_m H , \quad F_{+ijk} = -\frac{1}{2} H^{-5/6} \epsilon_{ijklm} \partial_l V_m ,
\]

where all the indices refer to tangent space.

### 4 String theory branes

Our discussion concerning string theory branes will be in the context of type-IIA [15, 16, 17] or IIB [18, 19] supergravities (for a pedagogical treatment in relation also to T-duality in the presence of RR fields, see also [20]).

#### 4.1 Dp-branes (\( p \neq 3 \))

In this case the metric is given by (2.1) with \( \alpha = -1/2 \) and \( d = 9 - p \), so that \( D = 10 \). In the static limit it describes the multicenter generalization of the \( Dp \)-brane solutions of [21, 22]. The geometry is supplemented by a \( (p + 2) \)-form field strength \( F_{p+2} \) in the RR-sector. The Killing spinor equations arising by setting to zero the gravitino and dilatino supersymmetry variations in type-IIA or IIB supergravity according to weather \( p \) is even
or odd, respectively, are (we perform the computation in the string frame) [23]

\[ \partial_\mu \epsilon + \frac{1}{4} \omega^{ab}_\mu \Gamma_{ab} \epsilon - \frac{(-i)^{p+1}}{8(p+2)!} e^\Phi F_{p+2} \cdot \Gamma \Gamma^p \epsilon(\mu(p)) = 0 , \]

\[ \Gamma^\mu \partial_\mu \Phi \epsilon + \frac{3}{4(p+2)!} e^\Phi F_{p+2} \cdot \Gamma \Gamma^p \epsilon(\mu(p)) = 0 , \]

(4.1)

where

\[ F_{p+2} \cdot \Gamma = F_{\mu_1 \ldots \mu_{p+2}} \Gamma^{\mu_1 \ldots \mu_{p+2}} , \]

(4.2)

and

\[ \epsilon_{(1,5)} = i\epsilon^* , \quad \epsilon_{(2,6)} = -\Gamma_{11} \epsilon , \quad \epsilon_{(-1,3,7)} = i\epsilon , \quad \epsilon_{(0,4,8)} = \epsilon . \]

(4.3)

In addition, the Einstein equations of motion

\[ R_{\mu\nu} + 2D_\mu D_\nu \Phi = \frac{e^{2\Phi}}{2(p+1)!} \left( (F^2_{p+2})_{\mu\nu} - \frac{1}{2(p+2)} g_{\mu\nu} F^2_{p+2} \right) , \]

(4.4)

as well as that for the dilaton

\[ R + 4 \left[ D^2 \Phi - (\partial \Phi)^2 \right] = 0 , \]

(4.5)

and the fluxes should be satisfied. Here we are interested in the cases with \( p \neq 3 \). For D3-branes there are some slight changes due to the self-dual 5-form field strength, which will be taken into account below. We find the usual projection

\[ \Gamma^{+-+-p} \epsilon(\mu(p)) = i^{p+1} \epsilon \]

(4.6)

and in addition (2.5), whereas for the \((p+2)\)-form we obtain

\[ F_{+-+-p} = H^{p/4} \partial_5 H^{-1} , \quad F_{+2-\ldots pij} = (-1)^p H^{(p-6)/4} \partial_5 V_j , \]

(4.7)

where all indices refer to the tangent space. Consistency requires also a nontrivial dilaton given by

\[ e^{-2\Phi} = H^{p+3} . \]

(4.8)

### 4.2 D3-branes

In this case we have to satisfy that

\[ F_5 = \pm A + \ast A , \]

(4.9)

for some 5-form field strength \( A \), so that \( F_5 \) is selfdual (anti-selfdual). The supersymmetry variation for the gravitino is

\[ \partial_\mu \epsilon + \frac{1}{4} \omega^{ab}_\mu \Gamma_{ab} \epsilon + \frac{i}{480} F_{\mu_1 \ldots \mu_5} \Gamma^{\mu_1 \ldots \mu_5} \Gamma_\mu \epsilon = 0 . \]

(4.10)
In addition, we should satisfy the Einstein equations of motion
\[
R_{\mu\nu} = \frac{1}{6} (F^2)_{\mu\nu} - \frac{1}{3} \left( (A^2)_{\mu\nu} - \frac{1}{10} g_{\mu\nu} A^2 \right). 
\] (4.11)

Using that
\[
F_{\mu_1...\mu_5} \Gamma^{\mu_1...\mu_5} = \pm 2 A_{\mu_1...\mu_5} \Gamma^{\mu_1...\mu_5}. 
\] (4.12)

and taking into account (4.3), the Killing spinor eq. (4.1) gives the projection
\[
i \Gamma^{+12} \epsilon = \epsilon, 
\] (4.13)
in addition to (2.5), so that \( i \Gamma^{12} \epsilon = \epsilon \). Also
\[
A_{+12i} = \pm \frac{1}{4} H^{-5/4} \partial_i H, \quad A_{+12ij} = \pm \frac{1}{4} H^{-3/4} \partial_i V^j. 
\] (4.14)

Note that the form of the Killing spinor eq. (4.10) when written for \( A \) using (4.12) as well as the field equation (4.11) correspond to (4.1) and (4.4) (for \( p = 3 \) and \( \Phi = \text{const.} \)) after the identification \( A = \mp F_{p+2}/4 \). Also, from (4.11) the scalar curvature \( R = 0 \) which from (4.5) shows that it is consistent to set \( \Phi = \text{const.} \).

### 4.3 String theory NS-branes

In this case the geometry is supplemented by a 3-form field strength \( H_{\mu\nu\rho} \) and a dilaton field. The Killing spinor equations arising by setting to zero the gravitino and dilatino supersymmetry variations are
\[
\partial_{\mu} \epsilon + \frac{1}{4} (\omega_{\mu}^{ab} - \frac{1}{2} H_{\mu}^{ab}) \Gamma_{ab} \epsilon = 0, 
\]
\[
\Gamma^{\mu} \partial_{\mu} \Phi \epsilon - \frac{1}{12} H_{\mu\nu\rho} \Gamma^{\mu\nu\rho} \epsilon = 0. 
\] (4.15)

In addition, the equations of motion
\[
R_{\mu\nu} - \frac{1}{4} (H^2)_{\mu\nu} + 2 D_{\mu} D_{\nu} \Phi = 0, 
\]
\[
D_{\mu} \left( e^{-2\Phi} H_{\nu\rho} \right) = 0, 
\] (4.16)
should be satisfied.

#### 4.3.1 NS1-branes

In this case \( \alpha = -1 \) and \( d = 8 \), \( p = 1 \) so that \( D = 10 \). Using the Killing spinor eq. (4.15) we find that the only projection to be imposed is (2.5). In addition we determine the dilaton field and the antisymmetric tensor field strength as
\[
e^{-2\Phi} = H 
\] (4.17)
and
\[ H_{+i} = -H^{-1} \partial_i H, \quad H_{+ij} = -H^{-1/2} \partial_i V_j, \]  \hspace{1cm} (4.18)
where all the indices refer to tangent space. This can be derived by an antisymmetric tensor field with non-vanishing components
\[ B_{uv} = H^{-1}, \quad B_{ui} = H^{-1} V_i. \] \hspace{1cm} (4.19)
This result was found before in [24].

### 4.3.2 NS5-branes

In this case \( \alpha = 0 \) and \( d = 4, \ p = 5 \), so that \( D = 10 \). From the Killing spinor eq. (4.15) we find that the usual projection (all indices are transverse to the brane)
\[ \Gamma^{1234} \epsilon = -\epsilon, \] \hspace{1cm} (4.20)

together with (2.5). Also we find the dilaton
\[ e^{2\Phi} = H \] \hspace{1cm} (4.21)
and the antisymmetric tensor field strength components
\[ H_{ijk} = H^{-3/2} \epsilon_{ijkl} \partial_l H, \quad H_{+ij} = -\frac{1}{2} H^{-1} \epsilon_{ijkl} \partial_l [k V_l]. \] \hspace{1cm} (4.22)
Again all the indices refer to tangent space and (2.8) provides the necessary for its integrability condition.

## 5 Shape invariant motions and shock waves

In order to proceed we have to specify the vectors \( x_a \) as functions of the light-cone time. Recall that in the static case an arbitrary distribution of these centers breaks completely the rotational \( SO(d-2) \) symmetry of \( \mathbb{R}^{d-2} \). We are mostly interested in cases where a continuous or a discrete subgroup of the full rotational group can be preserved by the center distribution. In the non-static case, even if such a subgroup is at some moment preserved, an arbitrary motion of the centers will cause a further destruction of the symmetry. More interesting are cases where this can be kept under control. For instance, we may consider motions in which in the far past and future at \( u \to \mp \infty \), a given symmetry subgroup \( H_{\mp \infty} \in SO(d-2) \) is preserved. Then, it is interesting to investigate the type of phenomena that arise in the transition between the two. In this
section (and paper) we will be concerned with an even simpler type of center motion in which all of them change in time as a result of a single overall function, that is

\[ x^i_a(u) = r_0(u)x^i_a, \quad (5.1) \]

where the \( x^i_a \)’s on the right hand side are constant moduli. Such a motion leaves invariant the angles between the defining vectors of the centers \( x_a \) and therefore, although their distribution changes in time, it does so in a \textit{shape invariant} way. To better study the solution we perform the following change of variables

\[ x_i \rightarrow r_0(u)x_i, \quad (u,v) \rightarrow (u,v). \quad (5.2) \]

As a result in the metric (2.2) the coefficient of the \( du^2 \) term becomes

\[ F(u,x) = \frac{\dot{r}^2_0}{r_0^{d-2}} \sum_a \frac{x^2_a - 2x \cdot x_a}{|x - x_a|^{d-2}}, \quad (5.3) \]

where we have assumed that initially \( F = 0 \) (but \( V_i \neq 0 \)). Similarly, the coefficient of the \( 2dudx^i \) term changes to

\[ V_i \rightarrow r_0 \dot{r}_0 x^i + \frac{\dot{r}_0}{r_0^{d-3}} \sum_a \frac{x^i - x^i_a}{|x - x_a|^{d-2}}. \quad (5.4) \]

However, this is a total derivative

\[ V_i = -\partial_i \Lambda, \quad \Lambda = \frac{1}{2} r_0 \dot{r}_0 x^2 + \frac{1}{d - 4} \frac{\dot{r}_0}{r_0^{d-3}} \sum_a \frac{1}{|x - x_a|^{d-4}}, \quad (5.5) \]

which can be eliminated by the coordinate shift \( v \rightarrow v + \Lambda \). This, of course affects the coefficient of the \( du^2 \) term. After taken all these into account we find the end result for the metric

\[ ds^2 = H^\alpha \left[ dy^\alpha dy^\alpha + 2dudv + F(x,u)du^2 \right] + r_0^2 H^{1+\alpha} dx^i dx^i, \quad (5.6) \]

where

\[ H(u,x) = 1 + \frac{1}{r_0^{d-2}} \sum_{a=1}^{N} \frac{1}{|x - x_a|^{d-2}} \quad (5.7) \]

and

\[ F(u,x) = -r_0 \dot{r}_0 x^2 - \frac{\dot{r}_0^2}{r_0^{d-2}} \sum_a \frac{x^2_a}{|x - x_a|^{d-2}} + \frac{2}{d - 4} \frac{r_0^{2-d/2}}{r_0^{d/2-1}} \partial_a \left( \frac{\dot{r}_0}{r_0^{d/2-1}} \right) \sum_a \frac{1}{|x - x_a|^{d-4}}. \quad (5.8) \]

This geometry is supported by fluxes with the same non-vanishing tangent space components as in the static case (since \( V_i = 0 \)). In other words for shape invariant motions of...
the centers it turns out that the matter flux fields supporting the metric for a valid gravitational solution to exist, remain unchanged. In the cases of NS1, NS5 and $Dp$-branes (with $p \neq 3$) there is also a dilaton field which however, due to the fact that it depends solely on the harmonic function $H$ in (5.7), it remains $u$-dependent.

Let’s consider the cases of $M2$, $M5$, $D3$ and $NS5$ branes, in the field theory limit in which effectively the unity in (5.7) is ignored and as a consequence also the first term in (5.8). In addition, let’s perform the coordinate transformation

$$u \to \int_{u_0}^{u} du'[r_0(u')]^{4-d}, \quad y^\alpha \to y^\alpha r_0^{2-d/2}, \quad v \to v + (d/4 - 1) \frac{r_0}{r_0} y^\alpha y^\alpha.$$  \hspace{1cm} (5.9)

This transformation leaves the $SO(p - 1)$ rotational subgroup of the Lorentz group along the brane invariant, as we will also see in the final expressions below. Using the relation

$$(\alpha + 1)(d - 2) = 2, \quad \text{for} \quad M2, M5, D3, NS5,$$  \hspace{1cm} (5.10)

the metric takes the form

$$ds^2 = H_0^{\alpha} \left[ dy^\alpha dy^\alpha + 2 du dv + F(y, x, u) du^2 \right] + H_0^{1+\alpha} dx^i dx^i,$$  \hspace{1cm} (5.11)

with

$$H_0 = \sum_a \frac{1}{|x - x_a|^{d-2}}$$  \hspace{1cm} (5.12)

and the profile function $F$ encoding the entire $u$-dependence is given by

$$F(y, x, u) = \left( \frac{\dot{r}_0}{r_0} + \frac{1}{2} (d - 6) \frac{\dot{r}_0^2}{r_0^2} \right) \left[ \frac{1}{2} (d - 4) y^\alpha y^\alpha + \frac{2}{d - 4} \sum_a \frac{1}{|x - x_a|^{d-4}} \right]$$

$$- \frac{\dot{r}_0^2}{r_0^2} \sum_a \frac{x_a^2}{|x - x_a|^{d-2}}.$$  \hspace{1cm} (5.13)

For $d = 4$ the expression above is not valid and, as it turns out, we should perform the replacement

$$\frac{2}{d - 4} \sum_a \frac{1}{|x - x_a|^{d-4}} \to -2 \sum_a \ln |x - x_a|.$$  \hspace{1cm} (5.14)

As a check of the expression in (5.13) recall that the equations of motion are satisfied provided that $D^2 F = 0$, where the Laplacian is defined with respect to the $D$-dimensional metric. This translates to the condition

$$H_0^{-1-\alpha} \partial_\perp^2 F + H_0^{-\alpha} \partial_\parallel^2 F = 0,$$  \hspace{1cm} (5.15)

where the subscripts in the two Laplacians indicate that they are taken with respect to the $d$-dimensional and $(p - 1)$-dimensional spaces perpendicular and along the brane, respectively. Since this equation should be valid for all $u$’s, the two terms in (5.13) having
different \( u \)-dependences should satisfy it separately. This is obvious for the term in the second line in (5.13). The first line also satisfies (5.15) due to the relation
\[
\partial^2 \sum_a \frac{1}{|x - x_a|^{d-4}} = -2(d - 4)H_0 = -2(d - 4) \sum_a \frac{1}{|x - x_a|^{d-2}} .
\]
(5.16)

In the above manipulations we have used that
\[
\frac{1}{2}(d + \alpha D) = 1 + \alpha , \quad (p - 1)(d - 4) = 4 , \quad \text{for } M2, M5, D3 .
\]
(5.17)

In the case of NS5-branes when \( d = 4 \) the necessary identity replacing (5.16) is
\[
\partial^2 \sum_a \ln |x - x_a| = 2 \sum_a \frac{1}{|x - x_a|^2} .
\]
(5.18)

At this point we recall and make precise contact with the solution generating technique of [25]. Accordingly, the construction of a new solution from some seed solution with given matter content, requires the existence of a null Killing vector \( \xi^\mu \) satisfying the additional condition
\[
D_{[\mu} \xi_{\nu]} = \xi_{[\mu} D_{\nu]} S ,
\]
(5.19)

for some scalar function \( S \). If the right hand side vanishes then \( \xi^\mu \) is a covariantly constant null Killing vector. A new solution with the same matter fields can be constructed with metric
\[
G_{\mu\nu} + e^S F \xi_\mu \xi_\nu ,
\]
(5.20)

where we emphasize that the construction uses the Einstein frame metric. The function \( F \) satisfies
\[
\xi^\mu D_\mu F = 0 , \quad D^2 F = 0 ,
\]
(5.21)

that is it has vanishing Lie-derivative along the Killing vector and is required to be harmonic. Other aspects and details of this method have been analyzed in [26]. In our case the null vector is simply \( \xi^\mu \partial_\mu = \partial/\partial v \) and it turns out that the function \( e^{-S} = H_0^\alpha \). Then the deformed metric (5.20) gives precisely the line element in (5.11). Note that even for the case of NS5-branes where there is a nontrivial dilaton it turns that performing the computation in the Einstein frame and then translating back into the string frame gives the correct result in (5.11) with \( \alpha = 0 \). The method of [25] was recently used in [27] to construct new solutions from the single center branes of M- and string theory. These solutions differ from ours (even in the single center cases) due to the fact that they have an essential part angular dependence related to a spherical harmonic.
5.1 The shock wave limit

If the function $r_0(u)$ changes suddenly, say at $u = 0$, we expect to obtain a shock wave geometry\(^4\) since such a change should affect the geometry only in the null hypersurface at $u = 0$. However, it is not immediately clear how the $\delta$-function in the metric arises. If we simply demand that $\dot{r}_0 \sim \delta(u)$ then the square of the $\delta$-function appears and we fail to make sense of it. However, the gauge theory side of the correspondence gives an idea on how to proceed. Recall again that the centers of the harmonic function correspond to vev’s of scalar fields in the gauge theory. From a physical viewpoint a sudden vev change should be associated with a finite energy pulse. Such an energy is proportional to the integral of the $T_{uu}$ component of the energy momentum tensor along the $u$ direction, that is

$$\int du \dot{x}_a \cdot \dot{x}_a \sim \int du \dot{r}_0^2.$$

(5.22)

Therefore we should demand that $\dot{r}_0^2$ behaves like a $\delta$-function so that the integral and consequently the energy of the pulse is finite. Then, $\dot{r}_0/r_0$ has a term behaving as $\delta'(u)$ which as a distribution can be integrated by parts. With this procedure we effectively replace $\ddot{r}_0/r_0$ by $\dot{r}_0^2/r_0^2$. Taking all these into account in (5.13), we end up with the shock wave profile

$$F_{\text{shock}}(y, x, u) = V_{\text{shock}}(y, x) \delta(u),$$

(5.23)

with the non-trivial transverse space function

$$V_{\text{shock}}(y, x) = f \left( \frac{1}{4} (d-4)^2 y^\alpha y^\alpha + \sum_a \frac{1}{|x - x_a|^{d-4}} - \sum_a \frac{x_a^2}{|x - x_a|^{d-2}} \right),$$

(5.24)

where the overall positive constant $f$ arises from $\dot{r}_0^2/r_0^2 = f \delta(u)$. In the case of $d = 4$, corresponding to NS5-branes, only the last term survives (the second term that becomes a constant can be absorbed by a harmless shift of the variable $v$).

A way to construct a shock wave solution in a background geometry is by cutting and pasting a spacetime along the $u = 0$ hypersurface, that is by omitting the $\delta$-function term, replacing $v \rightarrow \hat{v}$ and $dv \rightarrow d\hat{v} - \frac{1}{2} \Theta(u) (\partial_i V_{\text{shock}} dx^i + \partial_\alpha V_{\text{shock}} dy^\alpha)$, where $\Theta(u)$ is the step function. Changing variables as $\hat{v} = v + \frac{1}{2} \Theta(u) V_{\text{shock}}$ we obtain back the standard form we’ve been using. This method was employed in order to construct shock waves on purely gravitational backgrounds in \cite{29} and has been generalized in the presence of matter fields, of a non-vanishing cosmological constant as well as for shock waves in string theory \cite{30, 31}.\(^5\) Also shock wave solutions have been recently constructed in relation to brane-world scenarios and bran-induced gravity \cite{37, 38}. Next, observe that the function

\(^4\)The prototype example of a gravitational shock wave geometry is that of a massless particle moving in the four-dimensional Minkowski spacetime \cite{28}.

\(^5\)An alternative method to obtain a shock wave is to start with a geometry in which the deviation
$F_{\text{shock}}$ depends (except for $d = 4$) not only on the transverse space coordinates $x^i$, but also on the brane variables $y^\alpha$, via the rotational invariant combination $y^\alpha y^\alpha$. This feature has the consequence, as we will see, that momentum along the spatial brane directions transverse to the $(u, v)$-lightcone will not be conserved in scattering processes.

An explicit realization for the function $r_0(u)$ with the desired properties is the following

$$r_0(u) = a + \frac{\epsilon}{2} \tanh \left( \frac{lu}{\epsilon^2} \right), \quad (5.25)$$

where $a$, $l$ and $\epsilon$ three parameter scales. We see that for finite $u$ we have that $r_0(\pm \infty) = a \pm \frac{\epsilon}{2}$ which in the limit of vanishing $\epsilon$ implies that there is no change in $r_0$. However, we easily check that this type of behaviour corresponds to

$$\lim_{\epsilon \to 0} \frac{\dot{r}_0^2}{r_0} = f \delta(u), \quad f = \frac{|l|}{3a^2}. \quad (5.26)$$

With the representation (5.25) one easily verifies that, unlike its square, $\dot{r}_0/r_0$ is zero as a distribution.

Notice that, in the special case with

$$r_0 \sim (\cosh u)^{\frac{d-4}{2}}, \quad (5.27)$$

giving rise to an exponential grow of $r_0$ in the far past and remote future, all $u$-dependence in the metric disappears. Explicitly we have that, up to a numerical constant

$$F_{\text{exp}}(y, x) = \frac{(d-4)^2}{4} y^\alpha y^\alpha + \sum_a \frac{1}{|x - x_a|^{d-4}} - \sum_a \frac{x_a^2}{|x - x_a|^{d-2}}. \quad (5.28)$$

Finally we mention that, as it was shown for the case of shock waves in [31], the modifications of backgrounds we are considering, can be given a string theoretical interpretation as marginal perturbations by a massless vertex operator along the lines discussed in the work of [39].

### 6 Examples of varying brane distributions

Although we have kept the discussion so far as general as possible, it is easier in practice to present some explicit examples in the limit of continuous brane distributions since these from Lorentz invariance is represented by a term containing a small mass parameter. By performing an infinite boost transformation and simultaneously taking the mass parameter to zero in a correlated manner a shock profile of the type (5.23) arises. This method is very interesting from the physical point of view, but at the same time the cases where it can be applied are limited by the very requirement that the unperturbed solution should be Lorentz invariant. For notable applications and more details on this method see [28, 29] and [32]-[36].
have the advantage to be describable by a finite number of moduli parameters. In this section we consider the gravitational solutions that arise from the shape invariant motion of a uniform continuous distribution of D3-branes on a disc and on a three-dimensional spherical shell, both with a \( u \)-dependent radii. For the static case these solutions were first constructed as the extremal limits of rotating D3-brane solutions in [2, 3]. They were also used in several investigations in the literature within the AdS/CFT correspondence starting with the works of [40, 41] and belong to the rich class of examples representing continuous distributions of M- and string theory branes on higher dimensional ellipsoids [42]. It should be possible to construct the solutions with moving moduli parameters in the more general cases as well.

### 6.1 D3-branes on a disc of varying radius

Consider the following parametrization of the transverse to the D3-branes space \( \mathbb{R}^6 \)

\[
\begin{align*}
(x_1, x_2) &= r \cos \theta \sin \psi \left( \frac{\cos \phi_2}{\sin \phi_2} \right), \\
(x_3, x_4) &= r \cos \theta \cos \psi \left( \frac{\cos \phi_3}{\sin \phi_3} \right), \\
(x_5, x_6) &= \sqrt{r^2 + 1} \sin \theta \left( \frac{\cos \phi_1}{\sin \phi_1} \right).
\end{align*}
\]

(6.1)

The ranges of the variables are

\[
0 \leq \theta, \psi < \frac{\pi}{2}, \quad 0 \leq \phi_{1,2,3} < 2\pi, \quad r \geq 0.
\]

(6.2)

In this parametrization the uniform D3-brane distribution occurs at the \( x_5 - x_6 \) plane, or for \( \theta = \pi/2 \) and \( r = 0 \), in a disc of unit radius. This brane distribution breaks the \( SO(6) \) \( \mathcal{R} \)-symmetry to its \( SO(4) \times SO(2) \) subgroup, where the last factor is actually an approximation to a discrete \( Z_N \) group due to the continuum approximation. The metric turns out to be

\[
ds^2 = H^{-1/2}(2du dv + dy_2^2 + dy_3^2 + Fdu^2) + H^{1/2} \frac{r^2 + \cos^2 \theta}{r^2 + 1} dr^2 \\
+ H^{1/2} \left( (r^2 + \cos^2 \theta) d\theta^2 + (r^2 + 1) \sin^2 \theta d\phi_1^2 + r^2 \cos^2 \theta d\Omega_3^2 \right),
\]

(6.3)

where the harmonic function is (all integrals needed in this paper are calculated with the aid of [43])

\[
H = \int_0^1 2dl \int_0^{2\pi} \frac{d\phi}{2\pi \left( \frac{r_6^2 + l^2}{r_6^2 + l^2 - 2r_2l \cos \phi} \right)^2} = \int_0^1 dl \frac{2l(r_6^2 + l^2)}{[(r_6^2 + l^2)^2 - 4r_2^2l^2]^{3/2}} \\
= \frac{1}{2(r_6^2 - r_2^2)} \left[ 1 - \frac{r_6^2 - 1}{\sqrt{(r_6^2 + 1)^2 - 4r_2^2}} \right] = \frac{1}{r^2(r^2 + \cos^2 \theta)}
\]

(6.4)
and the line element for the $S^3$ is explicitly given by

$$dΩ_3^2 = dψ^2 + \sin^2 ψdφ^2 + \cos^2 ψdφ^2 + 2.$$  \hspace{1cm} (6.5)

We have also used the notation

$$r_2^2 = x_5^2 + x_6^2 = (r^2 + 1) \sin^2 θ, \quad r_6^2 = x_1^2 + \cdots + x_6^2 = r^2 + \sin^2 θ.$$  \hspace{1cm} (6.6)

The other sum that is needed can also be computed in the continuous approximation

$$\sum_a \frac{x_a^2}{|x - x_a|^4} = \int_0^1 2dl \int_0^{2π} dφ \frac{l^2}{2π (r_6^2 + l^2 - 2r_2l \cos(φ - ψ))^2}$$

$$= 2 \int_0^1 dl \frac{r_6^2 + l^2}{(r_6^2 + l^2)^2 - 4r_2^2l^2} \frac{1}{2(r_6^2 - r_2^2)} \left[ r_6^2 - \frac{r_6^4 + 3r_6^2 - 4r_2^2}{\sqrt{(r_6^2 + 1)^2 - 4r_2^2}} \right]$$

$$+ \ln \left[ \frac{r_6^2 + 1 - 2r_2^2 + \sqrt{(r_6^2 + 1)^2 - 4r_2^2}}{2(r_6^2 - r_2^2)} \right]$$

$$= \ln \left( 1 + \frac{1}{r^2} \right) - \frac{r^2 - \sin^2 θ}{r^2 (r^2 + \cos^2 θ)}. \hspace{1cm} (6.7)$$

This can be checked to be a harmonic function in $\mathbb{R}^6$. Finally, we also have the sum (computed also in the continuous approximation)

$$\sum_a \frac{1}{|x - x_a|^2} = \int_0^1 2dl \int_0^{2π} dφ \frac{l^2}{2π r_6^2 + l^2 - 2r_2l \cos(φ - ψ)}$$

$$= 2 \int_0^1 dl \frac{2l}{(r_6^2 + l^2)^2 - 4r_2^2l^2}^{1/2}$$

$$= \ln \left[ \frac{r_6^2 + 1 - 2r_2^2 + \sqrt{(r_6^2 + 1)^2 - 4r_2^2}}{2(r_6^2 - r_2^2)} \right]$$

$$= \ln \left( 1 + \frac{1}{r^2} \right). \hspace{1cm} (6.8)$$

Therefore the function $F$ in (6.3) is

$$F = \frac{r_0^2}{r_0} \left[ y_1^2 + y_2^2 + \ln \left( 1 + \frac{1}{r^2} \right) \right] - \frac{r_0^2}{r_0} \left[ \ln \left( 1 + \frac{1}{r^2} \right) - \frac{r^2 - \sin^2 θ}{r^2 (r^2 + \cos^2 θ)} \right]. \hspace{1cm} (6.9)$$

and for the case of a shock wave this is replaced by

$$F_{\text{shock}} = f \left[ y_1^2 + y_2^2 + \frac{r^2 - \sin^2 θ}{r^2 (r^2 + \cos^2 θ)} \right] \delta(u). \hspace{1cm} (6.10)$$
6.2 D3-branes on a sphere of varying radius

Consider the following parametrization of the transverse to the D3-branes space $\mathbb{R}^6$

\[
\begin{align*}
(x_1) &= r \cos \theta \sin \psi \left( \cos \phi_2 \right), \\
(x_2) &= \sin \phi_2, \\
(x_3) &= r \cos \theta \cos \psi \left( \cos \phi_3 \right), \\
(x_4) &= \sin \phi_3, \\
(x_5) &= \sqrt{r^2 - 1} \sin \theta \left( \cos \phi_1 \right), \\
(x_6) &= \sin \phi_1,
\end{align*}
\]

(6.11)

with the same ranges for the variables as in (6.2), except that now $r \geq 1$. In this parametrization the D3-brane distribution occurs at $\theta = 0$ and $r = 1$, in a spherical shell of of unit radius. This brane distribution breaks the $SO(6) \mathcal{R}$-symmetry to its $SO(2) \times SO(4)$ subgroup, where again the last factor is actually an approximation to a discrete $Z_N$ group. The metric turns out to be

\[
ds^2 = H^{-1/2}(dy_1^2 + dy_2^2 + 2dudv + Fdu^2) + H^{1/2} \frac{r^2 - \cos^2 \theta}{r^2 - 1} dr^2
\]

\[+ H^{1/2} \left( (r^2 - \cos^2 \theta)d\theta^2 + (r^2 - 1) \sin^2 \theta d\phi_1^2 + r^2 \cos^2 \theta d\Omega_3^2 \right),\]

(6.12)

where the harmonic function is

\[H = \frac{1}{r^2(r^2 - \cos^2 \theta)},\]

(6.13)

and the line element for the $S^3$ is explicitly given by (6.5). Also in the continuous approximation

\[\sum_a \frac{x_a^2}{|x - x_a|^4} = -\ln \left( 1 - \frac{1}{r^2} \right) - \frac{r^2 + \sin^2 \theta}{r^2(r^2 - \cos^2 \theta)} .\]

(6.14)

This can be checked to be a harmonic function in $\mathbb{R}^6$. Finally, we also have the sum (computed also in the continuous approximation)

\[\sum_a \frac{1}{|x - x_a|^2} = -\ln \left( 1 - \frac{1}{r^2} \right).\]

(6.15)

Therefore the function $F$ in (6.3) is

\[F = \frac{\dot{r}_0}{r_0} \left[ y_1^2 + y_2^2 - \ln \left( 1 - \frac{1}{r^2} \right) \right] + \frac{\dot{r}_0^2}{r_0^2} \left[ \ln \left( 1 - \frac{1}{r^2} \right) + \frac{r^2 + \sin^2 \theta}{r^2(r^2 - \cos^2 \theta)} \right].\]

(6.16)

and for the case of a shock wave this is replaced by

\[F_{\text{shock}} = f \left[ y_1^2 + y_2^2 + \frac{r^2 + \sin^2 \theta}{r^2(r^2 - \cos^2 \theta)} \right] \delta(u).\]

(6.17)
6.3 Shock waves on $AdS_p \times S^q$

The most elementary examples one may construct are those for which we place all brane centers at a single point, i.e. $x_a = 0$. Then from (5.13) we may compute the profile function for all cases of interest and in particular for M2, M5 and D3 branes in the near horizon for which the background geometry is given by the products $AdS_4 \times S^7$, $AdS_7 \times S^4$ and $AdS_5 \times S^5$, respectively [44]. We find the result

\begin{align*}
AdS_4 \times S^7 : & \quad F = \frac{1}{2} \left( \frac{\dot{r}_0}{r_0} + \frac{r_0^2}{\dot{r}_0^2} \right) \left[ 4y_2^2 + \frac{1}{r^4} \right], \\
AdS_7 \times S^4 : & \quad F = \left( \frac{2\dot{r}_0}{r_0} - \frac{\dot{r}_0^2}{r_0^2} \right) \left[ \frac{1}{4} \left( y_2^2 + y_3^2 + y_4^2 + y_5^2 \right) + \frac{1}{r} \right], \quad (6.18) \\
AdS_5 \times S^5 : & \quad F = \frac{\dot{r}_0}{r_0} \left[ y_2^2 + y_3^2 + \frac{1}{r^2} \right].
\end{align*}

However, for any smooth function $r_0(u)$ the fact that the profile function $F$ is non-vanishing is an artifact of the coordinate system we are using. Indeed, since all centers are at a single point at the origin, there is no notion of a shape invariant motion and we might as well use a coordinate system in which $F = 0$ without affecting the background geometry which remains simply of the direct product type $AdS_p \times S^q$. However, let’s consider the shock wave limit. Then, the corresponding profile functions $F_{\text{shock}}$ are given by (6.18) by just replacing all $u$-dependent prefactors by $f\delta(u)$. In that case the coordinate transformation becomes extremely unwieldy both for mathematical computations and for preserving the physical intuition since it involves the use of the step function and its second power. This was already noted for pure gravity shock waves in [29]. We also remark that the shock waves on $AdS_5 \times S^5$ can be easily obtained from the expressions corresponding to D3-branes on a disc, namely from (6.3), (6.9) and (6.10) by rescaling $r \rightarrow \lambda r$, $y_{1,2} \rightarrow y_{1,2}/\lambda$, $v \rightarrow v/\lambda^2$ and then let $\lambda \rightarrow \infty$. This procedure effectively sets the radius of the disc to zero, thus restoring conformality. A similar limiting procedure can also be performed from the expressions corresponding to D3-branes on a sphere, namely from (6.12), (6.16) and (6.17), leading to the same result.

6.4 Field propagation and transitions

It is possible to compute the amplitudes for transitions between different eigenstates due to the presence of the shock wave, in field theory by following a procedure similar to that for the amplitude for scattering by a shock wave in four-dimensional Minkowski spacetime computed in [45] (for further details and developments in relation also to string theory see [46]). Essentially one takes advantage of the fact that the spacetime on either
side of the shock wave is the same except for the fact that the coordinate $v$ appears, as explained, shifted by the function $V_{\text{shock}}(y, x)$ defined in (5.23). For simplicity consider a scalar field $\Psi$. For $u \to 0^-$ the solution has the form

$$u \to 0^- : \quad \Psi_{k, n}(v, x, y) = \Psi_n(x, y)e^{ik_v v},$$

(6.19)

where $\Psi_n$ denotes the complete set of eigenfunctions solving the Laplace equation in the transverse to the lightcone metric and $n$ denotes collectively the corresponding set of quantum numbers. For $u \to 0^+$ due the above shift of the coordinate $v$, the solution should be a linear combination of wavefunctions of the form

$$u \to 0^+ : \quad \Psi_{k, n}(v, x, y) = \Psi_n(x, y)e^{ik_v(v + \frac{1}{2}V_{\text{shock}})},$$

(6.20)

Hence, the scattering amplitude is

$$A_{k, n; k', n'} = \delta(k_v - k'_v)A_{n; n'},$$

(6.21)

where the $\delta$-function expresses energy conservation and the non-trivial part of the amplitude is

$$A_{n; n'} = \int [dxdy]\Psi_n'(x, y)\Psi_n(x, y)e^{\frac{i}{2}k_v V_{\text{shock}}},$$

(6.22)

where the measure factor $[dxdy]$ includes the dilaton factor $e^{-2\Phi}$ if there is one.

As an application, consider a massless scalar propagating on $AdS_5 \times S^5$. Since from (6.18) the profile function of the shock wave does not depend on the Euler angles parametrizing $S^5$ the quantum numbers associated with $S^5$ will be conserved in the scattering process. Hence, we focus on the $S$-wave solution which, when properly normalized, reads

$$\Psi(u, v, y, r) = \frac{1}{(2\pi)^2}e^{i(k_v u + k_v v + k \cdot y)}\Phi_M(z), \quad \Phi_M(z) = M^{1/2}z^2J_2(Mz),$$

(6.23)

with $z = 1/r$, $y = (y_1, y_2)$, $k = (k_1, k_2)$ and $M^2 = -k \cdot k = -2k_u k_v - k^2$. Also $J_2$ is the Bessel function of index 2, which is regular at $z = 0$. The arbitrary overall constant is chosen so that the Dirac-type normalization condition is satisfied

$$\int_0^\infty \frac{dz}{z^3}\Phi_M\Phi_{M'} = \delta(M - M').$$

(6.24)

Then the amplitude from a state with $(k_u, k_v, k)$ to $(k'_u, k'_v, k')$ is given by (6.21) with

$$A_{k, k'; k', k'} = \frac{\sqrt{MM'}}{2\pi} \int_{-\infty}^{+\infty} \! \! d^2y e^{i(k-k') \cdot y} e^{\frac{i}{2}k_v y^2} \int_0^\infty \! \! dz J_2(Mz)J_2(M'z)e^{\frac{i}{2}k_v f z^2}. $$

(6.25)
Note the absence of momentum conservation along the two brane coordinates transverse to the lightcone. This is due to the dependence of the shock wave profile on these coordinates. The fact that one can compute the scattering amplitude off a shock wave on $AdS_5 \times S^5$ explicitly, suggests the interesting possibility to further explore the rôle of shock waves within the AdS/CFT correspondence. Work on that relation has appeared in the literature [47]. We believe that much more remains to be understood.

7 NS5-branes on a circle

In this section we consider NS5-branes with centers on a $N$-polygon situated on the $x_3$-$x_4$ plane in the $R^4$ space transverse to the branes. Following [3] we have

$$x_a = (0, 0, \cos \phi_a, \sin \phi_a), \quad \phi_a = \frac{2\pi a}{N}, \quad a = 0, 1, \ldots, N - 1.$$ (7.1)

This distribution of branes preserves an $SO(2) \times \mathbb{Z}_N$ subgroup of the $SO(4)$ original $\mathcal{R}$-symmetry group when all branes are located at a single point of the transverse space. We will be interested in motions of the centers of the branes preserving this symmetry. Hence, we allow the parameter $r_0$ to depend on $u$ which is a radial motion that preserves the angular distance between the branes. In the continuum limit the branes are distributed on a ring of varying radius $r_0(u)$ situated at the $(34)$-plane and the subgroup of the $\mathcal{R}$-symmetry preserved by our configuration becomes continuous, i.e. $SO(2) \times SO(2)$.

After changing variables as

$$
\begin{pmatrix}
    x_1 \\
    x_2
\end{pmatrix} = \sinh \rho \cos \theta \begin{pmatrix}
    \cos \tau \\
    \sin \tau
\end{pmatrix},
\begin{pmatrix}
    x_3 \\
    x_4
\end{pmatrix} = \cosh \rho \sin \theta \begin{pmatrix}
    \cos \psi \\
    \sin \psi
\end{pmatrix},
$$ (7.2)

with ranges

$$0 \leq \rho < \infty, \quad 0 \leq \theta < \frac{\pi}{2}, \quad 0 \leq \psi, \tau < 2\pi,$$ (7.3)

we find that\(^6\)

$$H = \frac{1}{\sqrt{(x_1^2 + x_2^2 + x_3^2 + x_4^2 - r_0^2(x_3^2 + x_4^2))} = \frac{1}{\sinh^2 \rho + \cos^2 \theta}.$$ (7.4)

In this parametrization the ring is at $\rho = 0$ and $\theta = \pi/2$. After a straightforward computation we find that the 6-dim non-trivial part of the background is

$$d\text{ }s^2 = 2dudv + d\rho^2 + d\theta^2 + \tan^2 \theta d\psi^2 + \tanh^2 \rho d\tau^2 + \frac{1}{1 + \tan^2 \theta \tanh^2 \rho} + Fdu^2,$$

$$B_{\tau\psi} = \frac{1}{1 + \tan^2 \theta \tanh^2 \rho},$$

$$e^{-2\Phi} = r_0^2(u)(\sinh^2 \rho + \cos^2 \theta),$$ (7.5)

\(^6\)Even in the discrete case it is possible to explicitly compute $H$ [3].
where
\[ F = - \left( \frac{\dot{r}_0}{r_0} \right)^2 \frac{1}{\sinh^2 \rho + \cos^2 \theta} - 2 \frac{d}{du} \left( \frac{\dot{r}_0}{r_0} \right) \ln \cosh \rho . \] (7.6)

In the case of the shock wave this should be replaced by
\[ F_{\text{shock}} = -f \frac{1}{\sinh^2 \rho + \cos^2 \theta} \delta(u) . \] (7.7)

and the \( r_0^2(u) \) factor in the dilaton is set to a constant. In the static case, when the function \( F = 0 \), it was shown in [3] that a T-duality transformation with respect to \( \tau \), relates the background corresponding to NS5-branes on a ring to the background for the \( SL(2, \mathbb{R})/U(1) \times SU(2)/U(1) \) exact CFT (actually an orbifold of it, see [48] and for further details [49]). In the non-static case, performing a T-duality transformation with respect to \( \tau \) we find
\[ ds^2 = 2dudv + d\rho^2 + \coth^2 \rho d\omega^2 + d\theta^2 + \tan^2 \theta d\tau^2 + F du^2 \]
\[ e^{-2\Phi} = r_0^2(u) \cos^2 \theta \sinh^2 \rho , \] (7.8)

where \( \omega = \tau + \psi \) and zero antisymmetric tensor. For the case of a shock wave we simply replace \( F \) by \( F_{\text{shock}} \) and set \( r_0(u) \) to a constant.

\section*{7.1 Generalities on transitions}

To see the effect of the time-changing moduli we consider a scalar field propagating in the geometry (7.5). Since we would like our formalism to be applicable to more general cases, let’s consider a general transverse metric. In addition, we will develop the formalism for a general profile function which will be suitable for various approximations schemes. When we specialize to the case of a shock wave it is possible to obtain the exact amplitude we have already seen. In particular, we let our string frame metric be of the form
\[ ds^2 = dy_1^2 + \cdots + dy_4^2 + 2dudv + F(u, x) du^2 + g_{ij}(x) dx^i dx^j , \quad i = 1, 2, 3, 4 . \] (7.9)

Then the standard massless wave equation reads
\[ \frac{1}{e^{-2\Phi} \sqrt{-G}} \partial_{\mu} e^{-2\Phi} \sqrt{-G} G^{\mu\nu} \partial_{\nu} \Psi = 0 , \] (7.10)

where we note that if we write it using the Einstein frame metric the dilaton factor does not appear. Making the ansatz
\[ \Psi(u, v, y, x) = \frac{1}{(2\pi)^{5/2}} e^{i\vec{k} \cdot \vec{y}} e^{i k_v v} \Psi(u, x) , \] (7.11)
we derive an equation for the amplitude \( \Psi(u, x) \) written in the suggestive form

\[
(H^{(0)} + H^{(1)}) \Psi = -2i k_v \dot{\Psi},
\]

(7.12)

where

\[
H^{(0)} = \frac{1}{e^{-2\Phi} \sqrt{g}} \partial_i e^{-2\Phi} \sqrt{g} g^{ij} \partial_j - \hat{k}^2,
\]

(7.13)

denotes the unperturbed Hamiltonian and

\[
H^{(1)} = k^2_v F(u, x) - 2i k_v \dot{\Phi},
\]

(7.14)

is the, not necessarily small, perturbation. Let \( \{ \Psi_n^{(0)} \} \) be a complete set of states that solve the unperturbed problem, with \( n \) denoting collectively all quantum numbers. They have the form

\[
\Psi_n^{(0)}(u, x) = \frac{1}{(2\pi)^{1/2}} \Psi_n^{(0)}(x) e^{-i E_n k^2_v u},
\]

(7.15)

and the amplitude solves the equation

\[
\frac{1}{e^{-2\Phi} \sqrt{g}} \partial_i e^{-2\Phi} \sqrt{g} g^{ij} \partial_j \Psi_n^{(0)} + E_n \Psi_n^{(0)} = 0,
\]

(7.16)

defined completely in the four-dimensional transverse space. Proceeding further we expand the solutions of (7.12) as

\[
\Psi(u, x) = \sum_n a_n(u) \Psi_n^{(0)}(u, x).
\]

(7.17)

Substituting into (7.12) we derive a system of coupled first order equations given by

\[
\dot{a}_m(u) = \frac{i}{2k_v} \sum_n e^{i \omega_{mn} u'} H^{(1)}_{mn} a_n(u), \quad \omega_{mn} = \frac{E_m - E_n}{2k_v},
\]

(7.18)

where the matrix elements are\(^7\)

\[
H^{(1)}_{mn}(u) = \int d^4 x e^{-2\Phi} \sqrt{g} \Psi_m^{(0)*} (x) H^{(1)} \Psi_n^{(0)}(x).
\]

(7.20)

In the special case of shock waves we have using (5.23) that

\[
H^{(1)} = k^2_v \delta(u) V_{\text{shock}}(x)
\]

(7.21)

\(^7\)If the system is initially at the \( i \)-th state and \( H^{(1)} \) can be treated as a perturbation, then the coefficients \( a_m \), with \( m \neq i \) are small compared to \( a_i \approx 1 \). Hence, to first order in perturbation theory

\[
a_m^{(1)}(u) = \frac{i}{2k_v} \int_{-\infty}^u du' e^{i \omega_{mi} u'} H^{(1)}_{mi}(u'), \quad m \neq i.
\]

(7.19)

If we are interested in the final state we just let \( u \to \infty \) in the above integral.
and the amplitude can be exactly computed in this more general approach leading to the same results we have mentioned before. Indeed, when \( H^{(1)} \) is of the form (7.21), the system (7.18) becomes

\[
\dot{a}_m(u) = \frac{i k_v f}{2} \sum_n (V_{\text{shock}})_{mn} a_n(u) \delta(u),
\]

(7.22)

where \((V_{\text{shock}})_{mn}\) is given by an expression similar to (7.20). This system can be readily solved giving

\[
a_m = \int d^4xe^{-2\Phi} \sqrt{g} \Psi^{(0)*}(x) \Psi^{(0)}(x)e^{\frac{i}{\hbar}k_v V_{\text{shock}}(x)}, \quad \text{for } u > 0.
\]

(7.23)

This is indeed of the form (6.22) with the measure \([dxdy]\) being \(d^4xe^{-2\Phi}\), since there is no longitudinal contribution to the shock wave profile and the dilaton field has been properly taken into account.

### 7.2 The spectrum

In order to proceed we need to compute the eigenfunctions and spectrum in (7.16). For our static background (7.5) (with \(F = 0\)) this is possible to do by the method of separation of variables. Let

\[
\Psi^{(0)}(\rho, \theta, \psi, \tau) = \frac{1}{2\pi}e^{im\psi}e^{in\tau}e^{i\Phi}R(\rho).
\]

(7.24)

In this way, after standard manipulations, we obtain two ordinary linear second order differential equations

\[
\frac{1}{\sin 2\theta} \frac{d}{d\theta} \left( \sin 2\theta \frac{d\Theta}{d\theta} \right) + \left( E_1 - m^2 \cot^2 \theta - n^2 \tan^2 \theta \right) \Theta = 0
\]

(7.25)

and

\[
\frac{1}{\sinh 2\rho} \frac{d}{d\rho} \left( \sinh 2\rho \frac{dR}{d\rho} \right) + \left( E_2 - m^2 \tanh^2 \rho - n^2 \coth^2 \rho \right) R = 0,
\]

(7.26)

where \(E_1\) and \(E_2\) arise as separation of variables constants and obey \(E_1 + E_2 = E\), where \(E\) is the energy eigenvalue.

The solution to (7.25) is given in terms of Jacobi Polynomials as

\[
\Theta_{l,m,n}(\theta) = A_{l,m,n}\sin^{|m|} \theta \cos^{|n|} \theta P^{(|m|,|n|)}_{l-|m|,|n|} (\cos 2\theta), \quad l - \frac{|m|}{2} - \frac{|n|}{2} = 0, 1, \ldots
\]

(7.27)

where the normalization constant is

\[
A_{l,m,n}^2 = (2l + 1) \frac{\Gamma(l + \frac{1}{2}|m| + \frac{1}{2}|n| + 1)\Gamma(l - \frac{1}{2}|m| - \frac{1}{2}|n| + 1)}{\Gamma(l + \frac{1}{2}|m| - \frac{1}{2}|n| + 1)\Gamma(l - \frac{1}{2}|m| + \frac{1}{2}|n| + 1)},
\]

(7.28)
so that the state is normalized to one w.r.t. the measure \( d\theta \sin 2\theta \). The spectrum is quantized accordingly as

\[
(E_1)_{l,m,n} = 4l(l + 1) - m^2 - n^2 .
\]

(7.29)

We note here that \( l \) is generally half an integer and for the particularly interesting case with \( m = n = 0 \) we have

\[
\Theta_l(\theta) = \sqrt{2l + 1} P_l(\cos 2\theta) , \quad l = 0, 1, \ldots ,
\]

(7.30)

where \( P_l \) denote the Legendre polynomials.

The differential equation (7.26) can be cast by an appropriate transformation into a hypergeometric differential equation. The particular form of the solution depends very much on the values of \( M_2^2 \). We find convenient to parametrize

\[
(E_2)_{j,m,n} = m^2 + n^2 - 4j(j + 1) ,
\]

(7.31)

so that the total energy is

\[
E_{l,j} = 4l(l + 1) - 4j(j + 1) .
\]

(7.32)

The solution for \( R(\rho) \) is given in terms of hypergeometric functions. There are different ways of representing it. If we let

\[
R = x^{\frac{|m|}{2}}(x - 1)^{|n|/2}F(x) ,
\]

(7.33)

where \( x = \cosh^2 \rho \), we find that the function \( F(x) \) obeys the hypergeometric equation with

\[
a = \frac{|m|}{2} + \frac{|n|}{2} + j + 1 , \quad b = \frac{|m|}{2} + \frac{|n|}{2} - j , \quad c = 1 + |m| ,
\]

(7.34)

in the standard notation.

If we let \( z = 1/\cosh^2 \rho \) and make the transformation

\[
R = z^{j+1}(1 - z)^{|n|/2}F(z) ,
\]

(7.35)

we obtain for \( F \) a hypergeometric differential equation with

\[
a = \frac{|m| + |n|}{2} + j + 1 , \quad b = \frac{|n| - |m|}{2} + j + 1 , \quad c = 2(j + 1) ,
\]

(7.36)

in the standard notation. If we make instead the transformation

\[
\psi_2 = z^{-j}(1 - z)^{|n|/2}F(z) ,
\]

(7.37)
we obtain for $F$ a hypergeometric differential equation with
\begin{equation}
    a = \frac{|m| + |n|}{2} - j, \quad b = \frac{|n| - |m|}{2} - j, \quad c = -2j.
\end{equation}

The latter cases are related by the replacement $j \to -j - 1$.

The different solutions are characterized by the values of $j, m$ and $n$. We will be particularly interested in the case where $j = i\sigma - \frac{1}{2}$. In that case we will obtain an unbound solution that behaves as a plane wave asymptotically. Using (7.35) we find the solution (up to a normalization constant)
\begin{equation}
    R_{\sigma,m,n}(\rho) = z^{1/2}(1 - z)^{|n|/2} \left( z^{i\sigma} e^{i\varphi} {}_2F_1(a, b, c, z) + c.c. \right), \quad z = \frac{1}{\cosh^2 \rho},
\end{equation}

where
\begin{equation}
    a = \frac{1}{2}(|m| + |n| + 1 + i\sigma), \quad b = \frac{1}{2}(|n| - |m| + 1 + i\sigma), \quad c = 1 + 2i\sigma,
\end{equation}

\begin{equation}
    e^{2i\varphi} = \frac{\Gamma(-2i\sigma)\Gamma(a)\Gamma(b)}{\Gamma(2i\sigma)\Gamma(a^*)\Gamma(b^*)}.
\end{equation}

The relative coefficient between the two terms has been fixed so that the solution is regular at $z = 1$ (corresponding to $\rho = 0$). The energy eigenvalue is given as
\begin{equation}
    E_{l,\sigma} = 4l(l + 1) + 4\sigma^2 + 1,
\end{equation}
corresponding to a continuous spectrum that has a mass gap $E_{\text{gap}} = 1$ and a discrete part superimposed on it.

An equivalent way of representing the solution can be found using properties of the hypergeometric functions
\begin{equation}
    R_{\sigma,m,n}(\rho) = \sqrt{\frac{2\sigma}{\pi}} \sinh 2\pi\sigma |\Gamma(a)\Gamma(b)| \cosh |m| \rho \sinh |n| \rho \ {}_2F_1(\alpha, \alpha^*, 1 + |n|, -\sinh^2 \rho),
\end{equation}

\begin{equation}
    \alpha = \frac{|m| + |n| + 1}{2} + i\sigma,
\end{equation}

where $\sigma$ is real number. We have the limiting behaviours
\begin{equation}
    R_{\sigma,m,n}(\rho) \simeq \sqrt{\frac{2\sigma}{\pi}} \sinh 2\pi\sigma |\Gamma(a)\Gamma(b)| |m| \rho^{m|n|}, \quad \text{as} \quad \rho \to 0.
\end{equation}

and
\begin{equation}
    R_{\sigma,m,n}(\rho) \simeq 2 \left( 4^{i\sigma} e^{-i(2\sigma\rho - \varphi)} + c.c. \right) e^{-\rho}, \quad \text{as} \quad \rho \to \infty.
\end{equation}

Therefore, due to the behaviour at infinity, these are unbound states well suited for scattering problems.
A particularly interesting case in which the various integrals appearing below in the evaluation of scattering amplitudes will be possible to explicitly compute, is when the quantum number $\sigma \gg 1$. Then since the corresponding energy $E_2$ is very high the solutions simplifies to that for a plane wave (for notational convenience we let $\sigma \rightarrow \sigma/2$)

$$R_{\sigma}(\rho) = \sqrt{\frac{2}{\pi \sinh 2\rho}} \cos \sigma \rho, \quad \sigma > 0, \quad \rho > 0.$$  \hspace{1cm} (7.45)$$

so that they form a complete orthonormal set in $\mathbb{R}^+$ for $\rho$ and $\sigma$, with measure $\sinh 2\rho$.

### 7.3 Computing the transition amplitude

Returning to our case and concentrating to scattering by the shock wave, we have from (7.7), (7.14) and (7.21) that

$$V_{\text{shock}}(\rho, \theta) = -\frac{1}{\sinh^2 \rho + \cos^2 \theta}. \hspace{1cm} (7.46)$$

It is clear that in computing the amplitude we necessarily have that the quantum numbers $m$ and $n$ do not change. Below we are interested in cases where $m = n = 0$, $l = \text{integer}$ and $\sigma \gg 1$ since then our computations can be performed most easily. We will denote the initial state in the remote past by the quantum numbers $l'$ and $\sigma'$ and the corresponding unprimed quantities $l$ and $\sigma$ will denote the quantum numbers in the remote future. Then the relevant wavefunctions for the computation of the amplitude are given by (7.30) and (7.45). We have in general that the exact amplitude is given by

$$a_{l,\sigma}; l'; \sigma' = \frac{2c_l c_{l'}}{\pi} \int_0^\infty d\rho \cos(\sigma \rho) \cos(\sigma' \rho) \int_0^{\pi/2} d\theta \sin 2\theta e^{\pm k_v f V_{\text{shock}}(\rho, \theta)} P_l(\cos 2\theta) P_{l'}(\cos 2\theta),$$

where $c_l = (2l + 1)^{1/2}$.

#### 7.3.1 Low frequency perturbative expansion

For $k_v f \ll 1$ we may expand the phase factor and to first order in perturbation theory we compute\(^8\)

$$a_{l,\sigma}; l'; \sigma' = -i k_v f c_l c_{l'} \int_0^\infty d\rho \cos \sigma \rho \cos \sigma' \rho \int_0^{\pi/2} d\theta \frac{\sin 2\theta}{\sinh^2 \rho + \cos^2 \theta} P_l(\cos 2\theta) P_{l'}(\cos 2\theta)$$

$$= -i \frac{2k_v f}{\pi} (-1)^{l+l'} c_l c_{l'} \int_0^\infty d\rho \cos(\sigma \rho) \cos(\sigma' \rho) Q_l(\cosh 2\rho) P_{l'}(\cosh 2\rho),$$

\(^8\)To perform the various integrations in (7.47) and (7.49) below we have used [43] and in particular eqs. 7.224(5), 8.825 and 3.983(1).
where in order to for the second line to be valid we have assumed with no loss of generality that \( l' \leq l \). We may further simplify the expression for \( a^{(1)}_{l,\sigma, l',\sigma'} \) when \( l' = 0 \) corresponding to the \( S \)-wave as far as the angular part is concerned. Then

\[
a^{(1)}_{l,\sigma,0,\sigma'} = -\frac{2k_v f}{\pi} (-1)^l c_l \int_0^\infty d\rho \cos(\sigma \rho) \cos(\sigma' \rho) Q_l(\cosh 2\rho)
\]

\[
= -\frac{i k_v f}{\pi} c_l \int_{-1}^1 dt P_n(t) \int_0^\infty d\rho \frac{\cos(\sigma \rho) \cos(\sigma' \rho)}{\cosh 2\rho + t}
\]

\[
= -i \frac{k_v f}{4} c_l \left( \int_0^\pi d\phi P_l(\cos \phi) \frac{\sinh(\sigma_- \phi)}{\sinh(\pi \sigma_-)} + (\sigma_- \rightarrow \sigma_+) \right),
\]

where in the last line we have changed the integration variable as \( t = \cos \phi \) and used the definition \( \sigma_\pm = \frac{1}{2}(\sigma \pm \sigma') \). Recall that both \( \sigma, \sigma' \gg 1 \) and therefore \( \sigma_- \) could be finite, but necessarily \( \sigma_+ \gg 1 \). Hence, although we have kept the corresponding term this in fact should be neglected as we do from now on. The amplitude drops as \( 1/\sigma_- \), for \( \sigma_- \gg 1 \) and remains finite when \( \sigma_- \rightarrow 0 \). For low values of \( l \) we have explicitly that

\[
a^{(1)}_{0,\sigma,0,\sigma'} = -i \frac{k_v f}{4} \frac{1}{\sigma_-} \tanh \left( \frac{\pi}{2} \sigma_- \right),
\]

\[
a^{(1)}_{1,\sigma,0,\sigma'} = i \frac{\sqrt{3} k_v f}{4} \frac{\sigma_-}{1 + \sigma_-^2} \coth \left( \frac{\pi}{2} \sigma_- \right),
\]

\[
a^{(1)}_{2,\sigma,0,\sigma'} = -i \frac{\sqrt{5} k_v f}{4} \frac{1 + \sigma_-^2}{\sigma_- (4 + \sigma_-^2)} \tanh \left( \frac{\pi}{2} \sigma_- \right).
\]

7.3.2 High frequency asymptotic expansion

In this case we have the opposite limit with \( k_v f \gg 1 \) and the phase factor in (7.47) oscillates very rapidly. We may evaluated the leading contribution in the saddle point approximation. To do that note that the integral over the \( \rho \) variable, after changing variables as \( x = e^{-2\rho} \), becomes

\[
\frac{1}{8} \int_0^1 dx (x^{-\frac{1}{2}\sigma'} + x^{+\frac{1}{2}\sigma'})(x^{-\frac{1}{2}\sigma'} + x^{+\frac{1}{2}\sigma'}) e^{ik_v f \psi(x)} , \quad \psi(x) = \frac{-2}{x + x^{-1} + 2 \cos 2\theta}.
\]

The phase factor \( \psi(x) \) has a stationary point at \( x = 1 \), where \( \psi'(1) = 0 \). Using \( \psi''(1) = 1/(4 \cos^2 \theta) \) we obtain that asymptotically the integral behaves as

\[
\sqrt{\frac{\pi}{2|k_v f|}} \frac{1}{\cos^2 \theta} e^{\frac{i \pi}{4}} e^{-\frac{ik_v f}{2 \cos^2 \theta}},
\]

where the sign in the exponent is that same as the sign of \( k_v f \). Next we find the leading behaviour of the remaining integral over \( \theta \). After changing integration variable as \( x =
cos 2θ we have to consider the integral

\[
\int_{-1}^{1} \frac{dx}{x+1} P_l(x) P_{l'}(x) e^{-i k v f x} = \frac{1}{i k v f} \int_{-1}^{1} dx (x+1) P_l(x) P_{l'}(x) \frac{d}{dx} e^{-i k v f x} \\
= \frac{2}{i k v f} e^{-\frac{i}{2} k v f} + O \left( \frac{1}{k v f} \right)^2 ,
\]

where we simply performed an integration by parts and have used that \( P_l(1) = 1 \) for the Legendre polynomials. Putting everything together we find that the amplitude behaves asymptotically as

\[
a_{l,\sigma,l',\sigma'} \approx \sqrt{\frac{8}{\pi}} \sqrt{2l + 1/2 l'} e^{-\frac{i}{2} k v f} .
\]

The amplitude in this limit does not depend on the quantum numbers \( \sigma \) and \( \sigma' \).

8 Open strings ending on moving branes

In this section we will consider an open string with one end attached on a fixed \( Dp \)-brane and the other on a moving \( Dp \)-brane. Our aim is to find a possibly generic mechanism which might be responsible for producing at the macroscopic level a shock wave on the gravitational background of a large number of branes. Our treatment is parallel to that of an open string interacting with its ends with a wave of arbitrary profile [50]. We also note that the philosophy and some of the mathematical techniques we will use have appeared in the present context some time ago in order to demonstrate the solvability of the first quantized string in shock wave backgrounds of arbitrary profile on Minkowski space-time [51]. We will consider a \( d \)-dimensional flat space-time with Minkowski signature so that the directions along the branes satisfy the usual NN boundary conditions. For the coordinates normal to the branes we have to choose DD boundary conditions. We will take the moving brane to have a time-dependent position via the light-cone coordinate \( u = X^0 + X^1 \). With this choice it is possible to solve the string equations of motion in the light-cone gauge in which \( u = \tau \). In this case the other light-cone variable \( v \) is determined as usual in terms of the remaining \( d - 2 \) transverse coordinates denoted by \( X^i \), using the Virasoro constraints. We split the index \( i = (a, I) \) where \( a \) and \( I \) refer to the directions along and normal to the brane, respectively. We will take the spatial world-sheet coordinate \( \sigma \in [0, l] \). Hence, we may easily take the length of the string to zero, i.e. \( l \to 0 \), corresponding to the point particle limit. Then, we have the following boundary conditions

\[
NN : \quad \partial_\sigma X^a(\tau, 0) = \partial_\sigma X^\mu(\tau, l) = 0 , \quad a = 2, \ldots, p .
\]
and
\[
DD : \quad X^I(\tau,0) = 0, \quad X^I(\tau,l) = A^I + f^I(\tau), \quad I = p + 1, \ldots, d, \quad (8.2)
\]
where \(A^I\) are constant vectors and where \(f^I(\tau)\) is a given set of function assuming to behave in the far past and future as
\[
f^I(-\infty) = 0, \quad f^I(+\infty) = B^I. \quad (8.3)
\]
Therefore the ends of the string are stretched in the \(I\)th direction a distance \(A^I\) in the far past, and a distance \(A^I + B^I\) in the far future. The two-dimensional action for the transverse coordinates is
\[
S = \frac{1}{2l} \int_0^l d\tau d\sigma \left( \partial_\tau X^i \partial_\sigma X^i + \partial_\sigma X^i \partial_\tau X^i \right). \quad (8.4)
\]
The solution of the equations of motion
\[
\partial_\tau^2 X^i - \partial_\sigma^2 X^i = 0, \quad (8.5)
\]
for the longitudinal coordinates is simply given by
\[
X^a(\tau,\sigma) = x^a_0 + a^0_n \tau + i \sqrt{l} \sum_{n \neq 0} \frac{a^a_n}{n} \cos(n\pi\sigma/l) e^{-in\pi\tau/l}. \quad (8.6)
\]
Using the general equal time computation relations
\[
[X^i(\tau,\sigma), X^j(\tau,\sigma')] = [P^i(\tau,\sigma), P^j(\tau,\sigma')] = 0, \quad [X^i(\tau,\sigma), P^j(\tau,\sigma')] = i\delta^{ij}\delta(\sigma - \sigma'), \quad (8.7)
\]
where in the momentum is defined as \(P^i = \frac{\partial^i}{\partial\tau}\), we find the usual commutation algebra for the longitudinal mode coefficients
\[
[a^a_n, a^b_m] = n\delta_{n+m} \delta^{ab}, \quad [x^a_0, p^b_0] = i\delta^{ab}, \quad (8.8)
\]
with \(p^a_0 = a^a_0/l\) the zero mode momentum. For the transverse to the brane coordinates we write
\[
X^I(\tau,\sigma) = x^I_0(\tau,\sigma) + \bar{X}^I_0(\tau,\sigma), \quad (8.9)
\]
where \(x^I_0\) is a classical piece that satisfies (8.5) and the DD boundary condition (8.2) with \(A^I = 0\). In addition, we demand that it obeys the initial condition
\[
\lim_{\tau \to -\infty} x^I_0(\tau,\sigma) = 0. \quad (8.10)
\]
\(^9\)Before the light-cone gauge choice is made we may use \(f^I(u(\tau,l))\), instead of \(f^I(\tau)\).
The $\bar{X}^I$'s represent the fluctuations around $x^I_0$ and satisfy the usual DD boundary condition $\bar{X}^I(\tau, 0) = 0$ and $\bar{X}^I(\tau, l) = A^I$. The most general solution for the fluctuations is

$$X^I(\tau, \sigma) = \frac{\sigma}{l} A^I + \sqrt{\frac{l}{\pi}} \sum_{n \neq 0} \frac{a^I_n}{n} \sin(n \pi \sigma / l) e^{-i n \pi \tau / l}.$$  \hfill (8.11)

For the classical part we find that the solution is given by

$$x^I_0(\tau, \sigma) = \sum_{n=1,3}^{\infty} [f^I(\tau + \sigma - nl) - f^I(\tau - \sigma - nl)],$$ \hfill (8.12)

where the sum extends over the positive odd integers. Clearly (8.12) satisfies the equation of motion (8.5) and the appropriate boundary condition (8.2) (with $A^I = 0$). In addition, due to the behaviour (8.3) it vanishes in the far past as demanded by (8.10). In the far future manipulations with the terms in the infinite sum are problematic since each term separately diverges. However, an appropriate regularization procedure yields the expected result $x^I_0(+\infty, \sigma) = B^I \sigma / l$. Therefore in the far future the solution is given by (8.11) with $A^I$ replaced by $A^I + B^I$. The classical expression (8.12) contains no free moduli parameters to be quantized and the equal time computation relations give rise to the algebra for the mode coefficients of the fluctuations

$$[a^I_n, a^J_m] = n \delta_{n+m} \delta^{IJ}.$$ \hfill (8.13)

This define an “in” vacuum and the question is whether a unitary operator $U(\tau)$ exists, that evolves the “in” solution to the full solution (8.9). Namely that

$$X^I(\tau, \sigma) = x^I_0(\tau, \sigma) + \bar{X}^I(\tau, \sigma) = U^{-1}(\tau) \bar{X}^I(\tau, \sigma) U(\tau).$$ \hfill (8.14)

Given the initial condition (8.10) we demand that $U(-\infty) = 1$. Then, the $S$-matrix describing the unitary evolution to the final state is by definition given by

$$S = U(\infty).$$ \hfill (8.15)

For the unitary operator $U(\tau)$ we make the ansatz

$$U(\tau) = e^{iA(\tau)} , \quad A(\tau) = \sum_n \lambda^I_n(\tau) a^I_n , \quad (\lambda^I_{-n})^* = \lambda^I_n,$$ \hfill (8.16)

where the last condition ensures the hermiticity of $A$. In order to compute the coefficients $\lambda^I_n$ we use (8.14) and the fact that the commutator $[A, \bar{X}^I]$ is a $c$-number. We immediately find that

$$x^I_0(\tau, \sigma) = -i[A(\tau), \bar{X}^I(\tau, \sigma)] = -i \sum_n \lambda^I_n(\tau) [a^I_n, \bar{X}^I(\tau, \sigma)]$$

$$= -i \sqrt{\frac{l}{\pi}} \sum_n \lambda^I_n(\tau) \sin \frac{n \pi \sigma}{l} e^{-i n \pi \tau / l}.$$ \hfill (8.17)
Using for the left hand side (8.12) we may express

$$\lambda_n^I(\tau) = -\sqrt{\frac{\pi}{l^3}} \int_0^{2l} d\sigma e^{-in\pi(\tau+\sigma)/l} \sum_{m=1,3} f^I(\tau+\sigma-ml).$$  \hspace{1cm} (8.18)$$

Using the identity

$$\sum_{m=1,3} f^I(\tau+\sigma-ml) = \int_{-\infty}^{\tau} ds \sum_{n} (-1)^n e^{-in\pi\tau} f^I(s) a_n^I,$$

we finally obtain the unitary operator

$$U(\tau) = e^{iA(\tau)}, \quad A(\tau) = -\sqrt{\frac{\pi}{l^3}} \int_{-\infty}^{\tau+l} ds \sum_{n} (-1)^n e^{-in\pi\tau} f^I(s) a_n^I.$$

For the \textit{S}-matrix we have

$$S = e^{iA(\infty)}, \quad A(\infty) = -2 \left( \frac{\pi}{l} \right)^{3/2} \sum_{n} (-1)^n \tilde{f}^I(n\pi/l) a_n^I,$$

where

$$\tilde{f}^I(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau e^{-ik\tau} f^I(\tau),$$

are the Fourier components of the function \textit{f}(\tau). Putting the expression for the \textit{S}-matrix in a normal ordered form, results into

$$S = e^{-\delta} : e^{iA(\infty)} :, \quad \delta = \frac{2\pi^2}{l^3} \sum_{n=1}^{\infty} n|\tilde{f}^I(n\pi/l)|^2.$$  \hspace{1cm} (8.23)$$

The factor \textit{e}^{-\delta} has the interpretation of being the probability amplitude for the string to stay in its ground state. Moreover we may compute the expectation value of the mass square in the remote future assuming that the string was in each ground state in the remote past. The Hamiltonian is simply

$$H_\infty = \frac{(A^I + B^I)^2}{2l^2} + \frac{a_0^I a_0^I}{2} + \frac{\pi}{l} \sum_{n=1} a_n^I a_n^I.$$  \hspace{1cm} (8.24)$$

Therefore the desired expectation value is

$$M^2 = \langle 0 | S^{-1} H_\infty S | 0 \rangle = \frac{(A^I + B^I)^2}{2l^2} + \frac{4\pi^4}{l^4} \sum_{n=1}^{\infty} n^2 |\tilde{f}^I(n\pi/l)|^2.$$  \hspace{1cm} (8.25)$$

Note that in the point particle limit \textit{l} \rightarrow 0 there should be no string excitations at all and \textit{\delta} should tend to zero. Indeed, this is true for all functions \textit{f}^I provided that they
have Fourier coefficients that go to zero fast enough. However, there is a correlated limit which besides sending the length of the string to zero, it also keeps finite the mass of the string associated with the pulse. As a explicit illustrative example consider the profile

\[ f^I(\tau) = f_0^I \frac{\tau_0}{\tau_0^2 + \tau^2}, \quad (8.26) \]

with Fourier components

\[ \tilde{f}^I(k) = \frac{f_0^I}{2\pi} e^{-\tau_0|k|}. \quad (8.27) \]

Also we take \( B^I = 0 \) so that in the far past and future the string is stretched in the same length. Then we compute that

\[ \delta = \frac{\pi}{8l^3} \frac{(f_0^I)^2}{\sinh^2(\pi\tau_0/l)}. \quad (8.28) \]

In addition, we find that the expectation value of the mass square is

\[ M^2 = \frac{A^I A^I}{2l^2} + \frac{\pi^4 (f_0^I)^2}{4l^4} \frac{\cosh(\pi\tau_0/l)}{\sinh^3(\pi\tau_0/l)}. \quad (8.29) \]

The first term is the usual contribution of the stretched string whereas the second is due to the pulse. Clearly, sending \( l \to 0 \) and at the same time keeping \( \tau_0/l = \text{const.} \) and \( f_0^I/l^2 = \text{const.} \) results from (8.28) into \( \delta = 0 \) (also the \( S \)-matrix becomes the identity). Then from (8.26) we see that the pulse becomes a \( \delta \)-function, but of vanishing strength.

On the basis of these we might have expected to obtain just the mass corresponding to the stretched string. However, note that the second, due to the pulse, term in (8.29) remains finite in this correlated limit. Therefore there is a non-trivial effect even though since \( \delta = 0 \) and \( S = 1 \) the probability of the string staying in its ground state is a certainty.\(^{10}\)

We note that the limit we considered should be taken with the order we have explained since if we simply take \( S = 1 \) from the very beginning, the second term in (8.29) will not arise. This term comes from the contribution of the whole massive tower of string states, hence the correlated limit that we took is not really a point particle limit. We think that this mechanism that gives a non-trivial contribution to the mass in a seemingly trivial set up, could be at work in understanding the emergence of shock waves in supergravity solutions from a microscopic point of view as an integrated macroscopic backreaction effect. Obviously more work in this direction is required.

\(^{10}\)A similar conclusion can be reached with a profile similar to that in (5.25). However, we chose not to present the details since the computation involves PolyGamma functions and zeta-function regularization of infinite sums.
9 Concluding remarks

In this paper we emphasized the possibility to promote constant moduli parameters appearing in supergravity duals of supersymmetric gauge theories into arbitrary functions of the light-cone time. We explicitly showed that, for all multicenter fundamental brane solutions of M- and string theory, this can be done in a way that respects not only the field equations but also supersymmetry. In our solutions the branes are located at centers that are functions of the light-cone time. Moreover, we showed that the global symmetries of the solutions can be respected in what we called shape invariant motions. In our construction shock wave propagation on the brane gravitational background arise by sudden changes in the location of the centers of the branes. We gave explicit expressions for the supergravity backgrounds as well as for scattering amplitudes of scalar fields propagating in these geometries. The most natural question that arises is on the precise meaning of these solutions, within the gauge/gravity correspondence, on the gauge theory side. Since the brane centers correspond to vev’s of scalar fields we have a gauge theory with vev’s that depend on the light-cone time. Expanding around such a vacuum gives as usual masses to the scalars and gauge fields in the theory which now are functions of the light-cone time. In that respect, a useful toy model to study is that of a free scalar with a mass that depends on the light-cone time. It turns out that the classical equations of motion and Green’s functions of the theory can be explicitly computed for arbitrary mass profiles [52]. However, in making precise contact with the computations based on supergravity solutions for continuous brane distributions, we need to take into account the entire mass matrix of the matrix valued scalar fields. Moreover, we have to perform the continuous limit of the vev distribution on the gauge theory side. In our investigations it is important, though quite special, to give an interpretation on the gauge theory side of the shock wave on the maximally supersymmetric spaces of string and M-theory of the type \( AdS_p \times S^q \). Also, note the contribution to the mass spectrum of the string modes as a reaction to an external pulse that survived the point particle-like correlated limit (see section 8). Perhaps this can be in the root of a mechanism generating a shock wave classical geometry as an integrated backreaction string effect. We hope to report work along these directions in the future [52].

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A APPENDIX: Spin connection and Ricci tensor

In this appendix we compute the spin connection and the Ricci tensor for the general metric (2.1), since these are necessary for working out the Killing spinor equation (2.3) as well as the Einstein equations of motion.

A.1 The spin connection

Using the structure equations $de + \omega \wedge e = 0$ and the frame (2.4) we find for the spin connection

\[ \omega^{ij} = -\frac{\alpha + 1}{2H} \partial_i H dx_j - \frac{1}{2H} \partial_i V_j du , \]
\[ \omega^{-i} = \frac{\alpha + 1}{2} H^{-1/2} \dot{H} dx^i - \frac{1}{2} H^{-1/2} \left[ \partial_i V_j dx^j + (\partial_i F - 2 \dot{V}_i) du \right] \]
\[ - \frac{\alpha}{2} H^{-3/2} \partial_i H \left[ dv + V \cdot dx^i + \frac{F}{2} du \right], \]
\[ \omega^{i+} = -\frac{\alpha}{2} H^{-3/2} \partial_i H du , \quad \omega^{+i} = \frac{\alpha \dot{H}}{2H} du , \]
\[ \omega^{-\alpha} = \frac{\alpha}{2} \dot{H} dx^\alpha , \quad \omega^{\alpha i} = \frac{\alpha}{2} H^{-3/2} \partial_i H dx^\alpha . \]

where we have denoted tangent and target space space indices by $i, j$ and $\alpha$ (not to be confused with the numerical parameter $\alpha$ in (2.1)).

A.2 The Ricci tensor

The components of the Ricci tensor are

\[ R_{ij} = a_1 \frac{\partial_i H \partial_j H}{H^2} + a_2 \frac{\partial_i \partial_j H}{H} + \delta_{ij} \left[ a_3 \frac{\partial^2 H}{H} + a_4 \left( \frac{\partial H}{H^2} \right)^2 \right] , \]
\[ a_1 = \frac{1}{4} [3(d - 2) + \alpha(\alpha + 4)(D - 2)] , \quad a_2 = 1 - \frac{1}{2} [d + \alpha(D - 2)] , \]
\[ a_3 = -\frac{1}{2} (\alpha + 1) , \quad a_4 = -\frac{1}{4} (\alpha + 1)[d - 4 + \alpha(D - 2)] , \]

\[ R_{iu} = b_1 \frac{\dot{H} \partial_i H}{H^2} + b_2 \frac{\partial_i \dot{H}}{H} + b_3 \frac{1}{H} (\partial_i \partial \cdot V - \partial^2 V_i) + b_4 \frac{1}{H^2} \partial_i V_j \partial_j H \]
\[ + b_5 \frac{(\partial H)^2}{H^3} V_i + b_6 \frac{\partial^2 H}{H^2} V_i , \]
\[ b_1 = \frac{1}{4} [2(d - 1) + \alpha(\alpha + 3)(D - 2)] , \]
\[ b_2 = \frac{-1}{2}[(d - 1) + \alpha(D - 2)], \quad (A.3) \]
\[ b_3 = \frac{1}{2}, \quad b_4 = \frac{1}{4}[d - 4 + \alpha(D - 2)], \]
\[ b_5 = \frac{-\alpha}{4}[d - 4 + \alpha(D - 2)], \quad b_6 = \frac{-\alpha}{2}, \]
\[ R_{uu} = c_1 \frac{\dot{H}^2}{H^2} + c_2 \frac{\dot{H}}{H} + c_3 \frac{1}{H^2} (\partial_i V_j)^2 + c_4 \frac{1}{H^2} \partial_i H (\partial_i F - 2V_i) \]
\[ + c_5 \frac{1}{H} (\partial^2 F - 2\partial \cdot \dot{V}) + c_6 F \frac{\partial^2 H}{H^2} + c_7 F \frac{(\partial H)^2}{H^3}, \]
\[ c_1 = \frac{1}{4}[d + \alpha(\alpha + 2)(D - 2)], \quad c_2 = -\frac{1}{2}[d - 4 + \alpha(D - 2)], \]
\[ c_3 = \frac{1}{4}, \quad c_4 = -\frac{1}{4}[d - 2 + \alpha(D - 2)], \quad (A.4) \]
\[ c_5 = \frac{1}{2}, \quad c_6 = \frac{-\alpha}{2}, \quad c_7 = \frac{-\alpha}{4}[d - 4 + \alpha(D - 2)], \]
\[ R_{uv} = d_1 \frac{(\partial H)^2}{H^3} + d_2 \frac{\partial^2 H}{H^2}, \]
\[ R_{\alpha\beta} = \delta_{\alpha\beta} \left[ d_1 \frac{(\partial H)^2}{H^3} + d_2 \frac{\partial^2 H}{H^2} \right], \quad (A.5) \]
\[ d_1 = -\frac{\alpha}{4}[d - 4 + \alpha(D - 2)], \quad d_2 = \frac{\alpha}{2}. \]

**A.3 The branes**

The various coefficients above for the branes that appear in string and M-theory are

**M2-brane:** \( D = 11, \quad d = 8, \quad \alpha = -\frac{2}{3}, \)

\[ a_1 = -\frac{1}{2}, \quad a_2 = 0, \quad a_3 = -\frac{1}{6}, \quad a_4 = \frac{1}{6}, \]
\[ b_1 = 0, \quad b_2 = -\frac{1}{2}, \quad b_3 = \frac{1}{2}, \quad b_4 = -\frac{1}{2}, \quad b_5 = -\frac{1}{3}, \quad b_6 = \frac{1}{3}, \quad (A.6) \]
\[ c_1 = 0, \quad c_2 = -1, \quad c_3 = \frac{1}{4}, \quad c_4 = 0, \quad c_5 = -\frac{1}{2}, \quad c_6 = \frac{1}{3}, \quad c_7 = -\frac{1}{3}, \]
\[ d_1 = -\frac{1}{3}, \quad d_2 = \frac{1}{3}. \]

**M5-brane:** \( D = 11, \quad d = 5, \quad \alpha = -\frac{1}{3}, \)

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\[ a_1 = \frac{1}{2}, \ a_2 = 0, \ a_3 = -\frac{1}{3}, \ a_4 = \frac{1}{3}, \]
\[ b_1 = 0, \ b_2 = -\frac{1}{2}, \ b_3 = \frac{1}{2}, \ b_4 = -\frac{1}{2}, \ b_5 = -\frac{1}{6}, \ b_6 = \frac{1}{6} \quad \text{(A.7)} \]
\[ c_1 = 0, \ c_2 = -1, \ c_3 = \frac{1}{4}, \ c_4 = 0, \ c_5 = -\frac{1}{2}, \ c_6 = \frac{1}{6}, \ c_7 = -\frac{1}{6}, \]
\[ d_1 = -\frac{1}{6}, \ d_2 = \frac{1}{6}. \]

**Dp-branes (string frame):** \quad D = 10, \quad d = 9 - p, \quad \alpha = -\frac{1}{2},
\[ a_1 = \frac{1}{4}(7 - 3p), \ a_2 = \frac{1}{2}(p - 3), \ a_3 = -\frac{1}{4}, \ a_4 = \frac{1}{8}(p - 1), \]
\[ b_1 = \frac{1}{2}(3 - p), \ b_2 = \frac{1}{2}(p - 4), \ b_3 = \frac{1}{2}, \ b_4 = \frac{1}{4}(1 - p), \]
\[ b_5 = \frac{1}{8}(1 - p), \ b_6 = \frac{1}{4}, \quad \text{(A.8)} \]
\[ c_1 = \frac{1}{4}(3 - p), \ c_2 = \frac{1}{2}(p - 5), \ c_3 = \frac{1}{4}, \ c_4 = \frac{1}{4}(p - 3), \ c_5 = -\frac{1}{2}, \]
\[ c_6 = \frac{1}{4}, \ c_7 = \frac{1}{8}(1 - p), \]
\[ d_1 = \frac{1}{8}(1 - p), \ d_2 = \frac{1}{4}. \]

**NS1-string (string frame):** \quad D = 10, \quad d = 8, \quad \alpha = -1,
\[ a_1 = -\frac{3}{2}, \ a_2 = 1, \ a_3 = 0, \ a_4 = 0, \]
\[ b_1 = -\frac{1}{2}, \ b_2 = \frac{1}{2}, \ b_3 = \frac{1}{2}, \ b_4 = -1, \ b_5 = -1, \ b_6 = \frac{1}{2} \quad \text{(A.9)} \]
\[ c_1 = 0, \ c_2 = 0, \ c_3 = \frac{1}{4}, \ c_4 = \frac{1}{2}, \ c_5 = -\frac{1}{2}, \ c_6 = \frac{1}{2}, \ c_7 = -1, \]
\[ d_1 = -1, \ d_2 = \frac{1}{2}. \]

**NS5-brane (string frame):** \quad D = 10, \quad d = 4, \quad \alpha = 0,
\[ a_1 = \frac{3}{2}, \ a_2 = -1, \ a_3 = -\frac{1}{2}, \ a_4 = 0, \]
\[ b_1 = \frac{3}{2}, \ b_2 = -\frac{3}{2}, \ b_3 = \frac{1}{2}, \ b_4 = 0, \ b_5 = 0, \ b_6 = 0 \quad \text{(A.10)} \]
\[ c_1 = 1, \ c_2 = -2, \ c_3 = \frac{1}{4}, \ c_4 = -\frac{1}{2}, \ c_5 = -\frac{1}{2}, \ c_6 = 0, \ c_7 = 0, \]
\[ d_1 = 0, \ d_2 = 0. \]
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