DARBOUX TRANSFORMATION FOR
CLASSICAL ACOUSTIC SPECTRAL PROBLEM

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Abstract. We study discrete isospectral symmetries for the classical acoustic spectral problem in spatial dimensions one and two, by developing a Darboux (Moutard) transformation formalism for this problem. The procedure follows the steps, similar to those for the Schrödinger operator. However, there is no one-to-one correspondence between the two problems. The technique developed enables one to construct new families of integrable potentials for the acoustic problem, in addition to those already known.

The acoustic problem produces a non-linear Harry Dym PDE. Using the technique, we reproduce a pair of simple soliton solutions of this equation. These solutions are further used to construct a new positon solution for this PDE. Furthermore, using the dressing chain approach, we build a modified Harry Dym equation together with its LA-pair.

As an application, we construct some singular and non-singular integrable potentials (dielectric permitivity) for the Maxwell equations in a 2D inhomogeneous medium.

INTRODUCTION

This note develops the Darboux transformation and dressing chain formalism for the classical acoustic spectral problem (below just the “acoustic problem”) and the related Harry Dym (HD) equation. It treats the problem in the same vein as it is done for the Schrödinger operator and the related KdV (mKdV) hierarchies. The acoustic problem and the Schrödinger operator are closely connected. This connection constituted a base for the approach to the acoustic problem and the HD equation in [1-3]. However, as is discussed below, the relation between the problems is not utterly straightforward.

The acoustic problem describes wave propagation in inhomogeneous acoustic or electromagnetic media and just like the Schrödinger equation is non-integrable for an arbitrary potential. For applications, it is important to be able to construct integrable potentials, which result in solutions with given properties or asymptotic behavior. For instance, for the purposes of transmission of information, reflexionless potentials are important. These potentials are such that the problem admits solutions, which asymptote to $e^{-ipx}$ as $x \to -\infty$ and $T(p)e^{ipx}$ as $x \to \infty$, with the passage coefficient $T(p) \in \mathbb{C}$ being one in absolute value. This was recently studied in a work by Novikov [4], which has drawn our attention to the problem.

The latter work constructs a family of so-called B-potentials for the acoustic problem via a semi-classical solution ansatz. We show that these potentials naturally come up as

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a result of a one-step Darboux transform on the “vacuum background”. In continuous
media electrodynamics, potentials of the acoustic problem can be interpreted as the
medium’s dielectric permittivity. The B-potentials in question possess point singularities,
and hence their physical meaning is not entirely clear. On the other hand, the dressing
technique enables one to construct a variety of potentials, which are neither B-potentials,
nor singular. We illustrate it by a single act of dressing chain closing (in dimension one
and two) which yields regular integrable potentials (dielectric permittivity).

The acoustic problem on the real line is described by the following ODE:

$$\psi_{xx} = \frac{\lambda}{u^2(x)} \psi.$$  \hfill (1)

This equation models wave propagation in non-homogeneous (acoustic or electromagnetic)
media. Consider for instance the Maxwell equations in a medium without external
sources with the standard notations \((E, H)\) for the electromagnetic field, as well as
\(D = \varepsilon E, \quad B = \kappa H\). Suppose, the medium is isotropic but inhomogeneous with \(\kappa \equiv 1\) and
\(\varepsilon = \varepsilon(x, y, z)\). Then

$$\text{rot } B = \frac{1}{c} \frac{\partial D}{\partial t}, \quad \text{rot } E = -\frac{1}{c} \frac{\partial B}{\partial t}, \quad \text{div } D = \text{div } B = 0.$$  \hfill (2)

One can easily exclude the quantity \(B\) from (2) to obtain an equation connecting the
quantities \(E\) and \(D\):

$$\text{rot } \text{rot } E = -\frac{1}{c^2} \frac{\partial^2 D}{\partial t^2}. \quad \hfill (3)$$

For the electric field \(E\) one writes \(E = e^{i\omega t} \psi\), with \(\psi = \psi(x, y, z)\) and taking into account
the last equation of (2) obtains:

$$\nabla \left( \frac{(\psi, \nabla) \varepsilon}{\varepsilon} \right) + \Delta \psi = -\frac{\omega^2}{c^2} \varepsilon \psi,$$  \hfill (4)

where \(\Delta\) is a three dimensional Laplassian.

The equation (1) follows if one lets \(\varepsilon = \varepsilon(x)\), \(\psi = (0, 0, \psi(x))\), \(\lambda = -\omega^2/c^2, u^{-2}(x) = \varepsilon(x)\). The dielectric permittivity \(\varepsilon(x)\) will be henceforth referred to as a potential.

Alternatively, one can choose \(\varepsilon = \varepsilon(x, y)\) as well as \(\psi = (0, 0, \psi(x, y))\). If this is the
case (4) is reduced to a linear PDE

$$\Delta \psi = \lambda \varepsilon \psi,$$  \hfill (5)

where \(\Delta\) is a two dimensional Laplassian.

Hence, studying the equations (1,5) is of interest for continuous media electrodynamics. A similar case can be made in acoustics, whence comes the original name of the
equation (1). In both cases the equations describe transmission of signals, and are quite
relevant for applications.

For the Schrödinger equation one of the most efficient ways of building potentials
allowing exact solutions, is the method of factorization, or the Darboux transformation.
Developing a similar formalism for the equation (1) appears to be a natural thing to do.
Its basics are presented in the next two sections of this article. In particular, we derive
the related generalized Crum formulae, build chains of discrete symmetries and study
their simple closing. We argue that there is no one-to-one correspondence between this
formalism and the well-known technique for the Schrödinger equation [5,6,10].
The acoustic problem (1) is also interesting from the integrable systems viewpoint. It is known that this equation represents the L-equation of the Lax pair (or the LA-pair) for the nonlinear Harry Dym (HD) PDE: see [1-3] and references therein. We illustrate the Darboux transformation technique for the ODE (1) by constructing a new positon solution of the HD equation from a pair of its simple soliton solutions.

The Darboux transformation is known not only as a way of finding exact solutions of nonlinear equations, but also as a resource for proliferation of these equations, that is building new integrable PDEs together with their Lax pairs by means of the dressing chain technique. As an illustration, we construct a modified Harry Dym (mHD) equation (and its Lax pair) which has a remarkably simple form.

In the last section we turn to the equation (5). The Moutard transformations for this equation provide a simple method for construction of exact solutions of the Maxwell equations (2) with a dielectric permitivity \( \epsilon = \epsilon(x, y) \), which in general has a complicated singularity structure. However, a simple periodic closing of a dressing chain, generated by the Moutard transforms results in a regular integrable 2D dielectric permitivity.

**Discrete symmetries of the one-dimensional acoustic problem**

The Darboux transformation technique makes use of the existence of specific discrete isospectral symmetries of the equation under consideration and is standard in the theory of integrable PDEs: e.g. [5,6] and references therein.

We start out with a standard non-linear substitution [1,2,4] due to which typically the solutions of (1) will be represented parametrically:

\[
  u(x) = v_y(y), \quad x = v(y). \tag{6}
\]

This reduces the equation (1) to

\[
  \psi_{yy} = U \psi_y + \lambda \psi, \tag{7}
\]

with \( U = v_{yy}/v_y \). The above quantity \( U(y) \) will be referred to as a potential as well as the dielectric permitivity function \( \epsilon(x) = \frac{1}{u^2(x)} \) mentioned earlier.

In spite of the fact that (7) is easily reducible to the Shrödinger operator, from the point of view of finding integrable potentials this connection is not trivial. Let us address this issue in more detail.

A substitution

\[
  \psi \rightarrow \sqrt{v_y} \psi,
\]

transforms (7) into the stationary Schrödinger equation

\[
  \psi_{yy} = (\lambda + V(y)) \psi, \tag{7.1}
\]

where the potential \( V(y) \) is related to the potential \( U(y) \) of the acoustic problem (7) via

\[
  V = \frac{U^2 - 2U_y}{4}. \tag{7.2}
\]

Linearizing (7.2) with a substitution \( U = -2p_y/p \), one sees that \( p(y) \) in turn satisfies (7.1) with \( \lambda = 0 \).

The Darboux transformation for (7.1) is well known and at the first sight it may appear that developing an independent technique for the acoustic problem (7) is superfluous.
However, the following argument shows that this is not the case. Namely, there is no one-to-one correspondence between the problems (7) and (7.1).

Indeed, let \( U(y) \) be a specific potential for the acoustic problem (7), not depending on any free parameters. From (7.2) one can get (uniquely) the Shrödinger potential \( V(y) \) and further substitute it into (7.1). In order to reconstruct the initial potential \( U(y) \), equation (7.1) should be solved with \( \lambda = 0 \). Let the solution be \( p = p(y, C_1, C_2) \), depending on a pair of constants \( C_1, C_2 \). One of these constants, say \( C_1 \), plays the normalizing role and can be omitted. Then the restored potential \( U = U(y, C_2) \) will depend not only on \( y \), but the free parameter \( C_2 \) as well. Hence, a single potential in the Shrödinger operator generates the whole family of potentials for the acoustic problem, and in order to single out a specific potential for the latter one would have to subsequently develop some selection mechanism by studying the sequence of maps \( U(y) \to V(y) \to U(y, C_2) \to U(y) \). This necessity gets bypassed if one develops the Darboux transform formalism directly apropos of the operator (7) without using (7.1), and this is done in this and the following section.

Following Shabat [5] we seek elementary discrete symmetries of the equation (7) effecting the change

\[
\psi \to \psi^{(1)} = f \psi_y + g \psi, \tag{8}
\]

for some \( \lambda \)-independent functions \( f \) and \( g \) of \( y \).

One easily verifies that there are three distinct discrete symmetries of the type (8) for the equation (7). They are

\[
\psi \to \psi^{(1)} = \frac{\psi}{v}, \quad v \to v^{(1)} = \frac{1}{v}; \quad \psi \to \psi^{(1)} = \frac{\psi_y}{v_y}, \quad v \to v^{(1)} = \int \frac{dy}{v_y} \tag{9}
\]

and

\[
\psi \to \psi^{(1)} = \frac{\psi_1 \psi_y}{\psi_{1,y}} - \psi, \quad v_y \to v_y^{(1)} = v_y \left( \frac{\psi_1}{\psi_{1,y}} \right)^2, \quad U \to U^{(1)} = U + 2D \ln \frac{\psi_1}{\psi_{1,y}}. \tag{10}
\]

In the latter equation \( \psi_1 = \psi_1(y, \lambda_1) \) is a particular solution of (7) with the spectral parameter value \( \lambda_1 \), further referred to as a "prop solution", \( D = \partial_y \) and \( \psi_{1,y} = D \psi_1 \).

Note that the former two symmetries (9) define the new quantity \( U^{(1)} = \frac{v^{(1)}_y}{v^{(1)}_y} \) in a way independent of any solution \( \psi(y, \lambda) \) of (7). These symmetries arise as a particular case of (8) as the result of gauging corresponding to the choice of \( f \) or \( g \) alternatively zero. These symmetries have a trivial kernel in the solution space of (7). According to the terminology of [5] we call the symmetries (9) T-symmetries, sometimes referred to as Schlesinger transforms\(^1\).

On the other hand, the transformation (10) alias the Darboux transformation, which [5] calls an S-symmetry, does have a non-trivial kernel on the solution space of (7) (one can let \( \psi = \psi_1 \) in the first equation of (10) and get zero). This property will be essential in the sequel.

Along the way, we shall use the popular term "dressing" for the application procedure of the transformation (10) to a triple \( (\psi, v, U) \), the resulting pair \( (\psi^{(1)}, v^{(1)}, U^{(1)}) \) being referred to as the "dressed" one.

\(^1\)In the context of soliton solutions, the T-symmetries play the part of explicitly invertible Bäcklund transforms [7].
Despite a nearly trivial countenance, the Darboux transform (10) has a remarkable capacity to enable one to engineer potentials with arbitrary discrete spectra ad hoc. Indeed, suppose it is possible to solve the equation (7) formally (namely, obtaining among others some “non-physical” solutions which are not in $L^2$) for some potential $U$ and all $\lambda \in \mathbb{R}$. Suppose, $\psi_1(y, \lambda_1)$ is such a solution. Let us denote its linearly independent counterpart as $\hat{\psi}_1(y, \lambda_1)$, i.e.

$$\hat{\psi}_1 = \psi_1 \int dy \frac{v_y}{\psi_1^2}.$$ 

Dressing $\hat{\psi}_1$ according to (10), we find

$$\hat{\psi}_1^{(1)} = \frac{v_y}{\psi_{1,y}}.$$

Therefore, if one comes up with a non-physical prop solution $\psi_1$ by requiring that its derivative $\psi_{1,y}$ be strictly positive and rapidly growing as $|y| \to \infty$, then in the spectrum of the dressed potential $U^{(1)}$ there will appear a level $\lambda_1$, not present in the original spectrum for $U$. Since the principle for the choice of the value of $\lambda_1$ is such that this value is not to be present in the physical spectrum for $U$, repeating the dressing procedure $n$ times will result in a potential $U^{(n)}$ possessing $n$ new pre-chosen levels $\lambda_j$, $j = 1, \ldots, n$. See (13,14) in the sequel.

Conversely, the function $\hat{\psi}_1^{(1)}$ generates an inverse transformation (undressing) to (10). Thus, one can as well remove some pre-chosen levels from the spectrum of a potential.

**Crum formulae and dressing chains for the classical acoustic spectral problem**

Below we present the formulae describing an $n$-step dressing procedure for any $n \geq 1$, whose analogues are known for the Schrödinger equation as the Crum formulae [8]. We derive them for equation (7) following the procedure exposed in [9].

A single act of dressing (10) can be iterated $n$ times to yield a triple $(\psi^{(n)}, v^{(n)}, U^{(n)})$. One starts out by dressing a triple $(\psi, v, U) \equiv (\psi^{(0)}, v^{(0)}, U^{(0)})$ corresponding to a spectral parameter $\lambda$ with a prop function $\psi_1 \equiv \psi^{(0)}_1$, which is a formal solution of (7) with a spectral parameter value $\lambda_1$ and the potential $U^{(0)}$. The resulting solution $\psi^{(1)}$ solves (7) with the dressed potential $U^{(1)}$ (and the same spectral parameter $\lambda$). On the $j$th step, $j = 1, \ldots, n$ one uses some prop solution $\psi^{(j-1)}_j$ which solves (7) with a pre-dressed potential $U^{(j-1)}$ and a spectral parameter value $\lambda_j$ to produce the $j$ times dressed solution $\psi^{(j)}$ and potential $U^{(j)}$ (as well as the function $v^{(j)}$ with $U^{(j)} = v^{(j)}_{yy}/v^{(j)}_y$). Note that the spectral parameter $\lambda$ in the dressed equations for $\psi^{(j)}$ is the same for all $j = 1, \ldots, n$.

It’s easy to see that the $n$ times dressed solution $\psi^{(n)}$ shall have the form

$$\psi^{(n)} = \sum_{j=1}^n a_j D^j \psi + (-1)^n \psi,$$ 

with the functions-coefficients $a_j$ to be found, which of course will depend on the choice of the prop solutions $\psi^{(j-1)}_j$. It follows from (10) that

$$U^{(n)} = U + 2D \ln a_n,$$
for
\[ \psi^{(n)} = \prod_{j=1}^{n} \frac{\psi_j^{(j-1)}}{D\psi_j^{(j-1)}} D^n \psi^{(0)} + \ldots + (-1)^n \psi^{(0)}, \quad U^{(n)} = U^{(0)} + 2D \ln \prod_{j=1}^{n} \frac{\psi_j^{(j-1)}}{D\psi_j^{(j-1)}}, \]

where the ellipses in the first formula stand for the terms containing the derivatives of \( \psi^{(0)} \) of orders from 1 through \( n - 1 \).

So far the choice of the prop solutions \( \psi_i^{(i-1)} \) has been quite arbitrary. But suppose now that the original equation (7) possesses \( n \) distinct formal solutions \( \psi_j \), corresponding to spectral parameter values \( \lambda_j, j = 1, \ldots, n \). Let \( \psi_j \equiv \psi_j^{(0)} \) and consider the following dressing procedure (which will be further used for the dressing chain construction):

\[
\begin{array}{cccccccc}
\psi^{(0)} & \psi_1^{(0)} & \psi_2^{(0)} & \ldots & \psi_{n-1}^{(0)} & \psi_n^{(0)} & U^{(0)} \\
\psi^{(1)} & 0 & \psi_2^{(1)} & \ldots & \psi_{n-1}^{(1)} & \psi_n^{(1)} & U^{(1)} \\
\vdots & & & & & & \vdots \\
\psi^{(n-1)} & 0 & 0 & \ldots & 0 & \psi_n^{(n-1)} & U^{(n-1)} \\
\psi^{(n)} & 0 & 0 & \ldots & 0 & 0 & U^{(n)} \\
\end{array}
\]

Namely, for \( j = 1, \ldots, n \) on the above diagram (*) every new line \( j + 1 \) is obtained by dressing the functions from the preceding line \( j \) by (10) with a prop solution \( \psi_j^{(j-1)} \) marked in \textbf{bold}.

Zeroes, proliferating as one moves down the diagram stem from the non-trivial kernel property of the S-symmetry, and it is this property that now enables one to find the unknown functions \( a_j \). Indeed, substitution of any \( \psi_j = \psi_j^{(0)} \) for \( \psi \) in the right hand side of (11) shall yield zero. Hence, the coefficients \( a_j \) satisfy a system of \( n \) independent linear algebraic equations, namely

\[ \sum_{k=1}^{n} a_k D^k \psi_j + (-1)^n \psi_j = 0, \quad j = 1, \ldots, n. \]

Solving it by the Kramer rule and substituting the result into (11) and (12), we end up having:

\[ U^{(n)} = U + 2D \ln \frac{\Delta_n}{\tilde{\Delta}_n}, \quad \text{i.e.} \quad v^{(n)}_y = v_y \left( \frac{\Delta_n}{\tilde{\Delta}_n} \right)^2, \quad \psi^{(n)} = \frac{\tilde{\Delta}_{n+1}}{\Delta_n}, \quad (13) \]

where \( \Delta_n, \tilde{\Delta}_n \) are determinants of square \( n \times n \) matrices, whereas \( \tilde{\Delta}_{n+1} \) – of an \( n+1 \times n+1 \) matrix as follows:

\[ \Delta_n = \begin{vmatrix} D\psi_1 & \ldots & D^n \psi_1 \\ \vdots & \ddots & \vdots \\ D\psi_n & \ldots & D^n \psi_n \end{vmatrix}, \quad \tilde{\Delta}_n = \begin{vmatrix} \psi_1 & \ldots & D^{n-1} \psi_1 \\ \vdots & \ddots & \vdots \\ \psi_n & \ldots & D^{n-1} \psi_n \end{vmatrix}, \]

\[ \tilde{\Delta}_{n+1} = \begin{vmatrix} \psi & \ldots & D^n \psi \\ \psi_1 & \ldots & D^n \psi_1 \\ \vdots & \ddots & \vdots \\ \psi_n & \ldots & D^n \psi_n \end{vmatrix}. \quad (14) \]
Note that the linearity of (7) makes the choice of the sign before $\psi^{(n)}$ irrelevant.

The obtained formulae (13,14) make it possible to find rich families of exact solutions of (1). The easiest case is dressing from the birthday suit, or on the vacuum background, assuming $U = 0$ (hence $x = c_1 y + c_2$, $u(x) = c_1$, where $c_1$ and $c_2$ are arbitrary constants).

Such a natural rigging yields gratis all the B-potentials reported in [4], moreover the formulae for their computation derived therein turn out to be particular cases of (13,14) with merely $\psi \equiv 0$. For instance $\psi_1 = \sinh \xi_1$, $\psi_2 = \cosh \xi_2$ (here $n = 2$, $y_j$ are constants, $\xi_i = k_j(y - y_j)$, $j = 1,2$) yield a reflexionless B-potential with a power $2/3$ singularity: $\epsilon(2) = \frac{1}{|u^{(2)}|^2}$, where

$$u^{(2)} = \left( \frac{k_2 \sinh \xi_1 \sinh \xi_2 - k_1 \cosh \xi_1 \cosh \xi_2}{k_1 \sinh \xi_1 \sinh \xi_2 - k_2 \cosh \xi_1 \cosh \xi_2} \right)^2.$$ 

The same $\psi_1$ and $\psi_2 = \sinh \xi_2$ yield another potential with a power $4/5$ singularity:

$$u^{(2)} = \left( \frac{k_2 \sinh \xi_1 \cosh \xi_2 - k_1 \sinh \xi_2 \cosh \xi_1}{k_1 \sinh \xi_1 \cosh \xi_2 - k_2 \sinh \xi_2 \cosh \xi_1} \right)^2.$$ 

By construction, these potentials have only two levels $\lambda_{1,2}$.

In addition, all the regular reflexionless potentials can be also built by the formulae (13,14) once again by dressing $U = 0$. Here comes the proof. The passage coefficient for a regular $n$-level reflexionless potential can be expressed by a well-known formula

$$T_n(p) = \prod_{j=1}^{n} \frac{k_j - ip}{k_j + ip},$$

with $\lambda_j = k_j^2$. As has been pointed out earlier, the levels $\lambda_j$ can be successivly removed from the spectrum by means of the inverse of the Darboux transformation (still having the form (10)), each application of which will kill a term in the product. Successively applying this procedure $n$ times, for the passage coefficients we have

$$T_n(p) \to T_{n-1}(p) \to \ldots \to T_0(p) = 1.$$ 

This proves our assertion, because the case $R(p) = 0$ and $T(p) = 1$ for the reflection and the passage coefficient for all $p$ is feasible with $U = 0$ only. It’s worth reiterating the point that the argument above owes itself to the fact that the S-symmetry possesses a non-trivial kernel in the space of solutions of (7).

Matveev and Salle [6] find super-reflexionless potentials for the KdV equation, alias positons. In the same vein one can operate on the equations (1,7). In order to do so, one should use the formulae (13,14) with $n = 2$ choosing the prop solutions $\psi_{1,2}$ respectively as $\psi_1(y, \lambda_1)$ and $\psi_1(y, \lambda_1 + \delta)$, and then letting $\delta \to 0$. If withal $U = 0$ and $\psi_1$ generates a single soliton potential, then (13) defines a single positon potential. The next section describing the positon solutions of the HD equation contains the aforementioned computation.

$^{2}$Note that the expressions for $u^{(2)}$ are parametric. In order to interpret the formulae correctly, the reader is referred back to (1,6,7). The orders of the singularities pertain to the potential $\epsilon(x)$, which is a zero of the function $u(x)$ and a singularity of the potential $U(y) = v_{yy}/v_y$, where the function $v(y)$ solves the equation $u[v(y)] = v_y(y)$.
In addition to B-potentials, various other interesting ones can be produced. For instance, one can construct soluble potentials with a finite equidistant spectrum. One can also investigate potentials which change in a specific simple way under the Darboux transform, e.g.

\[ U \rightarrow U + \text{const} \quad \text{or} \quad U \rightarrow \text{const} U. \]

For the Schrödinger equation, the former transformation is shape-invariant and for \( n = 1 \) results in the harmonic oscillator potential. Let's develop an analogue for the model (1,7) under investigation. Suppose,

\[ U^{(1)} = U + \frac{2}{\omega^2}, \]

for a constant \( \omega \). Then we can obtain parametrically the function \( u(x) \) from (1) as follows:

\[ u(x) = \alpha z \exp \left( -\omega^2 z - \kappa^2 z \right), \quad x = x_0 - \alpha \omega^2 \int dz \exp \left( -\omega^2 z - \kappa^2 z \right), \]

where \( \kappa, x_0, \alpha \) are real constants and \( z = \exp(-y/\omega^2) \). The prop function \( \psi_1 \) rendering the potential \( U^{(1)} \) from \( U \) has the countenance \( \psi_1 = \exp(-\omega^2 z) \) and solves (7) with an eigenvalue \( \lambda_1 = b^2/\omega^2 \). It's easy to verify that the dielectric permitivity \( \epsilon(x) = 1/u^2(x) \) has a second order pole at \( x = x_0 \).

The theory of the Darboux transformation for the Schrödinger equation utilizes the concept of dressing chains of discrete symmetries and their closing. The work of Veselov and Shabat [10] elucidates how the dressing chain closing method can be used in order to obtain various potentials with meaningful mathematical physics. Namely, a simple closing procedure leads one to the harmonic oscillator potential (resulting also in a shape-invariant change of potential). A more complicated closing results in finite-gap potentials as well as the fourth and the fifth Painleve equations, see [10].

Dressing chains can be written out for the equation (7) as well. Let us introduce a sequence \( \{f_n\}_{n \geq 1} \) of functions as follows:

\[ f_n = D \ln \psi_n^{(n-1)}, \]

with the quantity \( \psi_n^{(n-1)} \) as it has been introduced in the diagram (*) above (where it appeared in bold). In particular, it corresponds to the pre-chosen value \( \lambda_n \) of the spectral parameter.

One can verify by hand starting from \( n = 1 \) that

\[ U^{(n)} = U - 2D \ln \prod_{j=1}^{n} f_j. \]

Besides, direct substitution shows that \( f_n \) satisfies the equation

\[ f'_n + f_n^2 - U^{(n)} f_n = \lambda_n, \]

where \( f' = Df \). The two latter relations imply the recursion connecting \( f_n \) and \( f_{n+1} \) as follows:

\[ (f_n f_{n+1})' = f_n f_{n+1} (f_n - f_{n+1}) + \lambda_{n+1} f_n - \lambda_n f_{n+1}. \quad \text{(15)} \]

This equation (15) represents a dressing chain for the acoustic problem.
In a way analogous to the theory of dressing chains for the Schrödinger equation [10] we are interested in T-periodic chain closing, namely imposing the condition $f_{n+T} = f_n$ for an integer $T \geq 1$. We shall consider here the easiest case $T = 1$.

Given the spectral parameter values $\lambda_{1,2}$, one obtains a one-parameter family of potentials, indexed by a constant $c$:

$$ U = \frac{-(\lambda_1 - \lambda_2)^2 y^2 + 2c(\lambda_2 - \lambda_1)y + 6\lambda_1 - 2\lambda_2 - c^2}{2[(\lambda_1 - \lambda_2)y + c]}. $$

If $\lambda_2 = 3\lambda_1 > 0$ and $c = 0$, we can express the function $u(x)$ parametrically:

$$ x(y) = \frac{\sqrt{\pi}}{2\alpha} \text{Erf}(\alpha y), \quad u(y) = \exp\left(-\alpha^2 y\right), $$

where $\alpha^2 = -\lambda/2 > 0$.

It is known that for the Schrödinger equation, a nontrivial chain closing operation with $T > 1$ results in finite gap potentials [10]. Such potentials for the HD equation are due to Dmitrieva [3]. A close analogy between the Schrödinger equation and the acoustic problem [1,2,11] suggests that one can expect results similar to those of [10] apropos of the analysis of higher order chain closing for $T > 1$. We expect that potentials built in such a way can have interesting physical applications, such as for instance a model of wave propagation in media whose dielectric permittivity is a periodic function of a single spatial variable.

**HD AND mHD EQUATIONS**

The 1+1 HD equation

$$ u_t = u^3u_{xxx} + \beta u_x, \tag{16} $$

with some real constant $\beta$, has been studied quite extensively since late 70’s: see [1-3] and references therein. It arises in the study of evolution equations solvable via the spectral transforms method based on the string rather than the Schrödinger equation. The principal approach to it has been based on its relation to the KdV, mKdV and other more classical hierarchies of integrable PDEs [2,3]. However, as was shown above, this relation is not entirely straightforward, and the direct approach developed herein enables one to produce new solutions of the HD equation in addition to those already known. As an example, below we construct a simple positon solution.

The acoustic problem (1) is the first equation in the LA-pair for the HD equation (16), the full pair being

$$ \begin{cases} 
    \psi_{xx} = \frac{\lambda}{u^2} \psi, \\
    \psi_t = (4\lambda u + \beta) \psi_x - 2\lambda u_x \psi. 
\end{cases} \tag{17} $$

The coordinate change (6) in the presence of time dependence becomes

$$ t \to t, \quad x \to v(y,t), $$

thus

$$ \partial_x \to \frac{1}{v_y} \partial_y, \quad \partial_t \to \partial_t - \frac{v_t}{v_y} \partial_y. $$
After this change (17) becomes

\[
\begin{align*}
\psi_{yy} &= \frac{v_{yy}}{v_y} \psi_y + \lambda \psi, \\
\psi_t &= \left(\frac{v_t + \beta}{v_y} + 4\lambda\right) \psi_y - \frac{2\lambda v_{yy}}{v_y} \psi,
\end{align*}
\]

and the HD equation (16) transforms to

\[
v_y(v_t v_{yy} - v_y v_t) + 3v_{yy}^2 + v_y(v_{4y} v_y - 4v_{3y} v_{yy} + \beta v_{yy}) = 0.
\]

with the notations \(v_{3y}, v_{4y}\) for the partial derivatives in \(y\) of order 3 and 4 respectively.

The goal now is to extend the Darboux transformation (10) for the equation (7) alias the first equation in (18), so that it agrees with the second equation in the Lax pair. One just includes the \(t\)-dependencies in (10). At this point it only provides the value of the partial derivative

\[v^{(1)}_y = \left(\frac{\psi}{\psi_{1,y}}\right)^2 \equiv A(y, t),\]

rather than the dressed quantity \(v^{(1)}(y, t)\) of interest. Hence, let \(v^{(1)}_t = B(y, t)\) be unknown and let’s assume that \(v^{(1)}\) satisfies the second equation of the pair (18) with the dressed according to (10) function \(\psi^{(1)}\), the quantities \((\lambda, \beta)\) remaining the same. One can express the unknown quantity \(B\) as follows:

\[B = \left(\frac{\psi}{\psi_{1,y}}\right)^2 \left(\beta + 4\mu v_y + v_t - 2v_y U_y\right) + \frac{4\psi v_{yy}}{\psi_{1,y}} - 4v_y - \beta,
\]

and verify that \(B_y = A_t\). It follows that

\[v^{(1)}(y, t) = \int A dy + B dt,
\]

with a closed 1-form under the integral.

Hence, the Darboux transformation (10,20) is an S-symmetry for the LA-pair (18), and therefore for the HD equation (19).

This enables one to construct exact solutions for this equation. We exemplify it with a single soliton solution and a single positon solution.

Let \(v = y, \lambda_1 = k^2, \psi_1(y, t, k) = \sinh[\phi(y, t, k)]\) with \(\phi = k \left(y + (4k^2 + \beta)t\right)\), then by (20) one has

\[v^{(1)} = \frac{1}{k^3} (\phi - \tanh \phi) - (4 + \beta)t.
\]

The function \(v^{(1)}(y, t, k)\) determines a single soliton B-potential \(U^{(1)} = v^{(1)}_{yy}/v^{(1)}_y\), mentioned in the previous section.

The single positon potential is obtained from two distinct soliton solutions \(\psi_1(y, t, k)\) and \(\psi_1(y, t, k + \delta)\), using them as the prop functions \(\psi_{1,2}\) in the formulae (13,14) with \(n = 2\) and taking the limit as \(\delta \to 0\). Namely,

\[v^{(2)}_y = v_y \left(\frac{\psi_{1,yy} \psi_1 - \psi_{1,yy} \psi_1}{\psi_1 - \psi_1}\right)^2,
\]
where the subscript $k$ means differentiation by $k$. Taking $\psi_1$ explicitly as the hyperbolic sine in the previous example results in

$$v_y^{(2)} = k^4 \left( \frac{\sinh(2\phi) + 2\tilde{\phi}}{\sinh(2\phi) - 2\tilde{\phi}} \right)^2,$$

with $\tilde{\phi} = k \left( y + (12k^2 + 2\beta) t \right)$.

It is well known that the dressing formalism enables one to produce hierarchies of integrable PDEs. Borisov and Zykov [12] proposed a technique for proliferation of integrable equations, which they applied to the KdV and the Sine-Gordon (SG) equations. The technique is based on the discrete symmetries dressing chain closing. The main idea of the approach is as follows. The equation (for illustration purposes let us take the KdV equation) is written as a compatibility condition of a pair of equations, further denoted as $L_1$ and $A_1$. Each of these equations is quadratic in the auxiliary field. Using invariance of the pair with respect to the Darboux transformation (which is viewed as a discrete symmetry), a second pair $L_2$, $A_2$ of equations is built. Excluding the potentials from $L_1$, $L_2$, and $A_1$, $A_2$, it is possible to obtain two equations, which [12] calls an $x$ and a $t$ chain, respectively. (We further use the notations $C_x$ and $C_t$ instead.) If a potential is excluded from $L_1$ and $A_1$, one ends up with a modified equation mKdV. The equations $C_x$ and $C_t$ can be converted into the Lax pair for the mKdV equation in two ways, the Darboux transformation being already known.

This procedure can be repeated, producing new equations with their LA-pairs. In this vein, the equations $m^2$KdV and $m^3$KdV were obtained. The former becomes the exponential Calogero-Degasperis equation [11] after an exponential change, the latter contains an elliptic equation of the same authors.

In spite of its simplicity, the technique described is very powerful. This can be illustrated by the following examples. First, see [13], the $m^N$KdV equations with $N = 0, ..., 3$ together with the Krichever-Novikov equation exhaust (modulo a contact transformation) all the integrable equations of the form $u_t + u_{xxx} + f(u_{xx}, u_x, u) = 0$. Second, applying their approach to the SG equation, the authors of [12] have succeeded to come up with a new (!) nonlinear equation already on the second step. This equation has a non-trivial Bäcklund transform, admitting an interesting $2\pi$-kink-shelf solution.

The same technique was shown to be applicable to the study of considerably more difficult (1+2)-dimensional nonlinear PDEs. For instance, in [14] the proliferation procedure was successfully adapted to the Kadomtsev-Petviashvili and Boiti-Leon-Pempinelli equations.

Let us apply this formalism to the HD equation. First note that the LA-pair for (19) can be written as a system of two Ricatti equations:

$$\begin{cases}
g_y = -\lambda g^2 - U g + 1, \\
g_t = \lambda \left( 2U_y - \frac{v_t + \beta}{v_y} - 4\lambda \right) g^2 - \left( \frac{v_t}{v_y} + 4\lambda U \right) g + \frac{v_t + \beta}{v_y} + 4\lambda.
\end{cases}$$

The second summand in the right hand side of the second equation has a term, denoted as $v_{ty}$, representing a fairly long expression which can be derived from (19). The function $g = g(y, t)$ is connected with the solution $\psi$ of (18) as $g = \psi_y/\psi$. Excluding the function $v$ from (1.1) and returning to the old variables via $x = g$, $u = g_y$, we obtain a modified Harry Dym (mHD) equation:

$$u_t = u^3 u_{3x} + 3u^2 u_x u_{xx} - 3\lambda^2 x u_x^2 - \frac{3u^2(3u u_{xx} u_x^2)}{x^2} + \frac{6u^3 u_x}{x^2} + \frac{3u^2(1 - u_x^2)}{x^3}.\quad (22)$$
(By analogy with the equation mKdV in [12], we call (1.2) the mHD equation.) As one can see, this equation has a different countenance than the HD equation. However, omission of all the summands but the first one in the right hand side of (22), yields the HD equation (16) with $\beta = 0$. Note that (22) can be rewritten quite nicely in new variables $x = 1/z, u(x, t) = \sqrt{\theta(z, t)}$:

$$
\left(\theta^{-1/2}\right)_t = \frac{3\lambda^2}{z} + z^3 \left(\frac{1}{2} z^3 \theta_{3z} + \frac{9}{2} z^2 \theta_{zz} + 9z \theta_z + 3\theta - 3\right).
$$

The formula (23) can be simplified even further by changing

$$
\theta = e^{-2\xi} \eta(\xi, t) + 1, \quad y = \log z.
$$

As the result, it becomes

$$
\left[(e^{-2\xi} \eta + 1)^{-1/2}\right]_t = 3\lambda^2 e^{-\xi} + \frac{1}{2} e^\xi \left(\eta_{3\xi} - \eta_\xi\right).
$$

However, we will be considering the mHD equation in the form (23). Note than in the stationary $\theta_t = 0$ case, it reduces to a linear ODE!

The dressing chain method produces not only the equation (23), but also its LA-pair. It is constructed as follows. Return to the chain (15) and let $f_n = 1/g, f_{n+1} = \Psi, \lambda_n = \lambda, \lambda_{n+1} = \mu$. Considering $\mu$ as a spectral parameter, one can see that (15) can be viewed as an L-equation of the LA-pair for equation (23). One should also define the second, non-stationary chain $C_t$ for the functions $g_n$ (in terms of the dynamical equation in (18)) and build the A-equation. Omitting the lengthy but straightforward computation, we present the LA-pair for equation (23), written in the variables $t, z$:

$$
\begin{cases}
\Psi_z = \frac{\mu}{z^2 \sqrt{\theta}} \Psi^2 + \left(\frac{1}{z} + \frac{1}{z \sqrt{\theta}} - \frac{\mu}{z^3 \sqrt{\theta}}\right) \Psi - \frac{1}{z^2 \sqrt{\theta}}, \\
\Psi_t = \mu a \Psi^2 + b \Psi + c,
\end{cases}
$$

where

$$
a = -4\mu + 2\lambda - \frac{\lambda^2}{z} - 2\lambda \sqrt{\theta} + (\theta - 1) z^2 + 2z^3 \theta_z + \frac{1}{2} z^4 \theta_{zz},
$$

$$
b = 4 \left(\frac{\lambda}{z} - z - z \sqrt{\theta}\right) \mu + \frac{1}{2} z^5 \theta_{zz} + 2z^4 \theta_z + (\theta - \frac{3}{2} \theta_{zz} - 1) z^3 - 3\lambda z^2 \theta_z + 3\lambda (1 - \theta) z - \frac{3\lambda^2}{z} + \left(\frac{\lambda}{z}\right)^3,
$$

$$
c = 4\mu - \frac{1}{2} z^4 \theta_z z - 3z^2 \theta_z + \left(1 - 3\theta - 2 \sqrt{\theta}\right) z^2 - 2\lambda + \left(\frac{\lambda}{z}\right)^2.
$$

Note that the spectral parameter in (24) is $\mu$, whereas $\lambda$ enters the non-linear equation (23). As one can see, the LA-pair for (23) also has the form of a pair of Riccati equations. These equations can be simultaneously linearized in order to represent (23) as a compatibility condition for two linear equations, as it is done in the theory of solitons.

We will not proceed further with the mHD equation. To conclude this section, we would like to repeat the statement that the exact solutions of (23) are easily found via the dressing technique, and this procedure can be extended in order to produce the mHD equation and its LA-pair. It is worth emphasizing that the equations HD and (23) are members of different hierarchies. Thus, discrete symmetries enable one to establish connections between different integrable equation hierarchies, promoting the unification of knowledge about them.
Moutard transformations

We devote this last section to equation (5) which has been obtained from the Maxwell equations (2) in the case of an isotropic but inhomogeneous in two directions \((x, y)\) medium.

Clearly, a PDE (5) is harder to investigate than an ODE (1). Nevertheless, its analysis in terms of the Darboux transform (10) known also as the Moutard transformation [6] is quite similar to its ODE cousin. Below we shall present the relevant formulae without the derivation details.

Let \(\psi = \psi(x, y)\) and \(\phi = \phi(x, y)\) be two particular solutions of (5), namely

\[
\Delta \psi - \lambda \epsilon \psi = \Delta \phi - \lambda \epsilon \phi = 0. \tag{25}
\]

We choose the function \(\phi\) as a prop solution. Then the following transformations represent an analogue of (10):

\[
\psi \rightarrow \psi^{(1)} = \frac{\theta[\psi, \phi]}{\phi}, \quad \epsilon \rightarrow \epsilon^{(1)} = \epsilon - 2\lambda \Delta \ln \phi, \tag{26}
\]

where

\[
\theta[\psi, \phi] = \int_{\Gamma} dx_\mu \epsilon_{\mu\nu} (\phi \partial_\nu \psi - \psi \partial_\nu \phi). \tag{27}
\]

Above, the following (standard) tensor notations have been used: \(\mu \in \{1, 2\}\), \(x_\mu \in \{x, y\}\), \(\partial_\mu = \partial / \partial x_\mu\), \(\epsilon_{\mu\nu}\) is a fully antisymmetric tensor with \(\epsilon_{12} = 1\), summation is implied over repeated indices. Note that a one-form, which is being integrated in the formula (27) is closed in the case when \(\psi\) and \(\phi\) are solutions of (26). Hence, the shape of the contour of integration \(\Gamma\) in (27) is irrelevant.

One can verify by direct substitution of (26,27) into the formulæ (28) below, that the dressed function \(\psi^{(1)}\) satisfies the dressed equation (25) (with the potential \(\epsilon^{(1)}(x, y)\) and the same spectral parameter value \(\lambda\)).

The Moutard transformation (26) can be iterated several times, and the result can be expressed via Pfaffian forms [6]. Instead, we direct our interest to the Maxwell equations (2). A straightforward computation (recall, \(\lambda = -\frac{\epsilon^2}{\omega^2}\)) yields the expressions for the dressed electric and magnetic fields \(E^{(1)}, B^{(1)}\):

\[
E^{(1)} = e^{i\omega t} \left(0, 0, \psi^{(1)}\right), \quad B^{(1)} = \frac{c}{\omega} e^{i\omega t} \left(-\psi^{(1)}_y, \psi^{(1)}_x, 0\right), \quad D^{(1)} = \epsilon^{(1)} E^{(1)}. \tag{28}
\]

On the basis of (28), one can build a variety of exact solutions of the Maxwell equations. As a simple example let’s dress \(\epsilon = 0\). This isn’t quite a medium, but one can easily proceed with formal calculations (26-28) which result in a new “medium” whose dielectric permitivity \(\epsilon^{(1)}(x, y)\) and the stationary component of the field \(\psi^{(1)}\) are as follows:

\[
\epsilon^{(1)} = -\frac{8\epsilon^2}{\omega^2} \frac{a'(z)b'(\bar{z})}{(a(z) + b(\bar{z}))^2}, \quad \psi^{(1)} = \frac{a(z)\beta(\bar{z}) - a(z)b(\bar{z}) + \xi(z, \bar{z})}{a(z) + b(\bar{z})}, \tag{29}
\]

where

\[
\xi(z, \bar{z}) = \int dz \left(\alpha(z)a'(z) - a(z)\alpha'(z)\right) + \int d\bar{z} \left(\beta'(\bar{z})b(\bar{z}) - b(\bar{z})\beta'(\bar{z})\right),
\]

\[
\alpha(z) = a(z) + b(\bar{z}), \quad \beta(z) = a(\bar{z}) + b(z),
\]

\[
\alpha'(z) = \frac{a'(z)}{a(z) + b(\bar{z})}, \quad \beta'(z) = \frac{b'(z)}{a(z) + b(\bar{z})}.
\]
$a(z), \alpha(z), b(\bar{z}), \beta(\bar{z})$ are arbitrary functions of $z = x + iy, \bar{z} = x - iy$. Note that the function $\psi^{(1)}$ from (26,28) provides in fact a general solution of the dressed equation, for it is described in terms of two arbitrary functions $\alpha(z)$ and $\beta(\bar{z})$. To ensure that the quantities found correspond to a physical non-absorbing medium, one should require that the dressed dielectric permittivity function $\epsilon^{(1)}$ be real. This imposes an extra restriction to the quantities $a(z)$ and $b(\bar{z})$, namely $b(\bar{z}) = \overline{a(z)}$. Generally speaking, the functions $\epsilon^{(1)}$ and $\psi^{(1)}$ from (29) will have singularities along certain curves in the $(x, y)$-plane.

The reflectionless B-potentials for the one-dimensional problem (1) above, possess point singularities on the real line (corresponding to zeroes of the function $u(x)$). Clearly, their 2D-analogues, such as (29) for the equation (5) allow a much more diverse structure of singularities on the real plane. On the other hand, not requiring that the quantity $\epsilon^{(1)}$ be real, one obtains an absorbing medium which may not be devoid of interest for physical applications.

In conclusion, let us study a dressing chain generated by the Moutard transformations (26). A simple periodic closing of the dressing chain results in a regular dielectric permittivity, similar to the 1D case studied above.

Denote $f_n = \ln \phi, f_{n+1} = \ln \psi^{(1)}$. Then after a straightforward computation

$$\Delta (f_n + f_{n+1}) = \|\nabla f_n\|^2 - \|\nabla f_{n+1}\|^2,$$

where $\| \cdot \|$ is the Euclidean norm.

The chain (30) is closely related to that of Veselov and Shabat [10] for the Schrödinger equation. Choosing $f_n$ specifically as

$$f_n = \sqrt{\lambda_n} y + \int dx g_n(x),$$

and substituting it into (30) ($\lambda_n$ being constant), we obtain for the quantities $g_n(x)$ the following expression

$$(g_n + g_{n+1})' = g_n^2 - g_{n+1}^2 + \lambda_n - \lambda_{n+1},$$

matching the corresponding formula of [10].

The simplest periodic closing of the dressing chain (30) is $f_{n+1} = F(x, y) = f_n$, which implies that the latter function $F$ is harmonic, and that the regular dielectric permittivity function in the corresponding medium is given by the formula

$$\epsilon(x, y) = \frac{c^2}{\omega^2} (F_x^2 + F_y^2).$$
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