The Limits of Information

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Abstract

Black holes have their own thermodynamics including notions of entropy and temperature and versions of the three laws. After a light introduction to black hole physics, I recollect how black hole thermodynamics evolved in the 1970’s, while at the same time stressing conceptual points which were given little thought at that time, such as why the entropy should be linear in the black hole’s surface area. I also review a variety of attempts made over the years to provide a statistical mechanics for black hole thermodynamics. Finally, I discuss the origin of the information bounds for ordinary systems that have arisen as applications of black hole thermodynamics.

Keywords: Black holes, entropy, second law, information, information bound.

A theory is the more impressive the greater the simplicity of its premises, the more different kinds of things it relates, and the more extended its area of applicability. Therefore, the deep impression which classical thermodynamics made upon me. It is the only physical theory of a universal content concerning which I am convinced that within the framework of applicability of its basic concepts, it will never be overthrown . . .

A. Einstein, Autobiographical Notes

Introduction

Although the citation from Einstein captures the professed attitude of many physicists, many others would today regard it as a basically sentimental statement. For we have become accustomed to regard thermodynamics as a straight consequence of statistical mechanics and the atomic hypothesis. The usual paradigm, inferred from myriad examples - ideal gas, black body radiation, a superfluid, etc. - is that a system made up of a multitude of similar parts - molecules, electrons, phonons - with weak interactions and with no initial correlations between constituents (Boltzmann’s Stosszahl-Ansatz), will automatically exhibit thermodynamic behavior. Thus - so the claim - it is statistical mechanics, not thermodynamics, which is the theory of “universal content”. The advent of black hole thermodynamics thirty years ago seems, however, to have turned the tables on this “modern” assessment of thermodynamics’ secondary status.

In effect, black holes provide a second paradigm of thermodynamics. Black hole thermodynamics has meaning already at the classical level. It possesses a first law (conservation of energy, of momentum, of angular momentum and of electric charge), as well as a second law (in a generalized version) and a third law which delimits the kingdom of black holes. Black hole thermodynamics is no ordinary thermodynamics.

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Gravitation is all important in it, while in traditional thermodynamics gravitation is a nuisance which is customarily ignored. Information about a black hole’s interior is not just practically unavailable, as in the garden variety thermodynamic system; rather, there are physical barriers forbidding acquisition of information sequestered in a black hole. And, again unlike everyday thermodynamic systems, a black hole is a monolithic systems with no parts. Granted, as the usual description has it, a black hole is usually the product of the collapse of ordinary, thermodynamic, matter, but that infallen matter becomes invisible and thermodynamically irrelevant. To top it off, we are not sure today, despite vociferous claims to the contrary, whether there exists a statistical mechanics which reduces to black hole thermodynamics in some limit. Einstein would have been pleased: thermodynamics seems to stand by itself in the black hole domain.

And the issue is not just academic. Black hole thermodynamics seems to tell us things about mundane physical systems. For instance, a very pragmatic question in modern technology is how much information can be stored by *whatever* means in a cube of *whatever* composition one centimeter on the side. As we shall see, black hole thermodynamics has suggested upper bounds on this quantity; these are the limits of information.

**What is a Black Hole?**

Ask what is a black hole and you will get many answers. The purist will say: a solution of Einstein’s theory of gravity - general relativity - representing a spacetime which over most of its extension is like the one we are familiar with from special relativity, but which includes a region of finite spatial extent whose interior and boundary are totally invisible from the rest of the spacetime. Colloquially that invisible region is a black hole. Others will define the black hole differently: a tear in the fabric of spacetime or a gravitational soliton. The first of these emphasizes the familiar notion that whatever goes into a black hole cannot return. The second hints at the property that that invisible region behaves in many ways like an ordinary object: it is localized, can move, can be scattered, attracts other objects, etc.

It is fairly certain that a black hole can form from the collapse of an old star; the X-ray sources observed in our galaxy since the 1970’s include a class of some dozens of rapidly flickering ones, each of which is thought to harbor a black hole of stellar mass. It is also fairly certain that quasars, those extremely luminous beacons in distant parts of the universe, of which some 5000 have been catalogued, represent active regions in galaxies, each of which is energized by a very massive black hole at its center. Our own Milky Way, although no quasar, is known to have a modestly massive black hole at its core. Finally, it is suspected that microscopic scale black holes formed in the universe when it was extremely dense from the strong density fluctuations which would naturally arise then. None of these primordial black holes have yet been spotted. However, what the above list emphasizes is that black holes may be found with a wide range of masses, from $10^{15}$ g for the primordial ones to $10^{41}$ g for those in quasars, and perhaps higher.

Much of what we know about the physics of black holes is based on exact solutions to Einstein’s theory, the Schwarzschild, Reissner-Nordström, Kerr, and Kerr-Newman solutions (Misner, Thorne and Wheeler 1973). The most general of these, the Kerr-Newman solution, represents a black hole possessing mass $m$, electric charge $q$ and angular momentum $j$. The black hole can move, and so it can also have linear momentum, but I shall gloss over this possibility here. One feature of the black hole
phenomenon which is clear from the Kerr-Newman solution is the (event) horizon, the boundary of the invisible region. The warping of spacetime, archetypical of Einstein’s description of gravity, causes the local light cones familiar from special relativity to tilt so that their exterior boundary lies tangent to the horizon. Thus, in Wheeler’s words (Ruffini and Wheeler 1971), the horizon acts like a “one-way membrane”: no object or light ray that crosses it inward-going can ever recross it outwardly. For in special relativity language that would be tantamount to moving faster than light. As a result of this purely geometric obstacle, information that enters the black hole is permanently trapped inside the horizon, and is unrecoverable to any observer outside the black hole precisely because no signals, whatever their nature, can cross the horizon outward. A horizon constitutes a barrier to information flow.

How big is the horizon? From Schwarzschild’s solution representing a spherical black hole with exactly one parameter, mass $m$, we learn that its radius is $R_h = \frac{2Gmc^{-2}}{\sqrt{M^2 - Q^2 - a^2}}$, where $G$ is Newton’s constant and $c$ the speed of light. For a solar mass black hole $R_h \approx 3$ km. From the three parameters, $m$, $q$ and $j$, of the more general (an aspherical) Kerr-Newman black hole (KNBH) we can form three lengths: $M = Gmc^{-2}$, $Q = \sqrt{Gq}c^{-2}$ and $a = jm^{-1}c^{-1}$. In terms of these the area of the horizon $A$ is given by the important formula

$$A = 4\pi(R_+^2 + a^2); \quad R_+ \equiv M + \sqrt{M^2 - Q^2 - a^2}$$

(1)

From this one can get an idea of the generic size of the horizon; for $m$ a solar mass, $(A/4\pi)^{1/2}$ is always of order 1 km.

When charge $\delta q$ is added to a KNBH, its parameter $q$ grows by $\delta q$; likewise when angular momentum $\delta J$ is added, $j$ grows by $\delta q$. And $m$ grows by whatever energy (and work) was added. The laws of energy, charge and angular momentum retain their usual meaning. Not so other respected laws of the physicist. For instance, baryon number conservation is ‘transcended’ by black holes (Ruffini and Wheeler 1971; Bekenstein 1972a). Addition of baryon number to a KNBH is followed only by changes of its $m$, $q$ and $j$ as appropriate, but there is no unique way to reconstruct from these changes how much baryon number was lost into the black hole. The black hole forgets how many baryons it has swallowed. In fact, it forgets everything but the energy, charge and angular momentum it ever acquired. Wheeler refers to this poverty of characteristics as baldness: ‘black holes have no hair’ (Ruffini and Wheeler 1971).

After much evidence was garnered in its support in the 1970’s and 80’s, black hole baldness was put in question in the 1990’s by the discovery of what is termed ‘hairy black holes’: solutions representing quiescent black holes more complicated than the KNBH ones, and sometimes with extra parameters (Heusler 1996). I have argued (Bekenstein 1997) that since most of these hairy black holes are unstable, Wheeler’s basic idea remains: a black hole has just a few parameters, whose number is independent of the black hole size.

**Black Hole Thermodynamics**

A black hole can form from collapse of an extremely complex mess of atoms, ions, radiation. Yet it transpires that once this object has settled down to the only stable available black hole configuration - the KNBH one - it must be specifiable by just three numbers: $m$, $q$ and $j$. This is a paradox. But we are familiar with a similar situation. A cup of hot tea is an agglomeration of trillions of molecules of a number
of species, all dancing around violently. We know, though, that from a macroscopic point of view we may describe all there is to describe about the equilibrium situation by giving energy $E$, volume $V$ and the mole fractions of sucrose, and the various compounds found in tea essence and in lemon. A few variables describe the big mess. Why not try to describe black holes by thermodynamics given the essential similarity of the situations? This was one of the motivations for black hole thermodynamics (Bekenstein 1973).

A second paradox played an important role. In 1971 Wheeler stressed to me - I was then one of his novice doctoral students - that black holes seem to provide a mechanism for violating the second law of thermodynamics. Mix the cup of hot tea with one of cold water and so create entropy. Then dump the lukewarm mix into a black hole. The newly made entropy disappears permanently from our sight - for we have no interest to follow it into the black hole and be lost. The black hole does change, of course. But how can we figure out the amount of lost entropy from the changes in $m$, $q$ and $j$? Not in a unique way! So, Wheeler concluded triumphantly, the perfect crime - erasing an increase of entropy - has been perpetrated. My incredulity as to whether such a useful and universal law as the second law could be so easily brushed aside sent me into a feverish struggle to straighten out the situation.

My basic idea was to ascribe entropy to a black hole, not the entropy of matter that has gone down the black hole, but some function of the “observables” $m$, $q$ and $j$ with the requisite properties. And the central desirable property is, of course, that black hole entropy $S_{BH}$ should tend to grow. Here a nice candidate presented itself. Christodoulou (1970) and Floyd and Penrose (1971) had given strong evidence that, at least in classical physics, the area $A$ of a black hole horizon cannot decrease. Hawking (1971) made this into a theorem. I thus concluded that $S_{BH} = f(A)$ with $f$ real, positive and monotonically increasing in its argument. The choice $f(A) = \text{const} \times \sqrt{A}$ beckoned because it implies for $q = 0$ and $j = 0$ that $S_{BH} \propto m$ in harmony with the extensive character of entropy in ordinary thermodynamics. I rejected this choice, though, because conservation of energy implies that when two Schwarzschild black holes $a$ and $b$ merge, the mass of the resulting black hole is below $m_a + m_b$ given that some losses to gravitational radiation should occur. But with $S_{BH} \propto m$ this would say that the total black hole entropy would decrease! The next obvious choice, $S_{BH} = \text{const} \times A$ exhibits no such problem, and so I adopted it. Further, on Wheeler’s suggestion I took the proportionality constant of the order $\ell^{-2}$ with $\ell$ the Planck length $(G\hbar/c^3)^{1/2} \approx 2 \cdot 10^{-33}$ cm (theoretical physicists, unlike chemists and engineers, measure temperature in energy units, in which case Boltzmann’s constant $k$ is unity and entropy dimensionless).

With black hole entropy so defined, I could claim to have resolved Wheeler’s paradox. We knew from Hawking’s theorem that the infall of the tea would cause an increase in the area of the horizon, i.e. and an increment $\delta S_{BH}$ of the black hole entropy,. I claimed that in these and similar situations $\delta S_{BH}$ would exceed the ordinary entropy lost to the black hole. More precisely, I formulated the generalized second law (GSL): the sum of black hole entropy and ordinary entropy outside black holes never decreases. A number of examples showed the law worked (Bekenstein 1972b, 1973, 1974).

The above historical argument for the form of black hole entropy evidently has logical gaps in it; for example, it very much guesses the form of $f(A)$. If we had to define an entropy for a black hole ab initio, what could we say about it on purely
classical grounds? To answer this, take the differential of $A$ in Eq. (1) and rearrange terms to get

$$d(mc^2) = \Theta dA + \Phi dq + \Omega dj$$

(2)

$$\Theta \equiv c^4(2GA)^{-1}(R_+ - M)$$

(3)

$$\Phi \equiv (R_+^2 + a^2)^{-1}R_+ q$$

(4)

$$\Omega \equiv (R_+^2 + a^2)^{-1}M^{-1}j$$

(5)

What is the significance of $\Phi$ and $\Omega$? If the black hole is charged, it is surrounded by an electrical potential. It is found that as one approaches the horizon, this potential approaches $\Phi$ which is thus aptly termed the electric potential of the black hole. Further, if the black hole rotates, it is found that it tends to drag infalling objects around it in the sense of its rotation. As each such objects falls to the horizon, its angular frequency approaches $\Omega$, which is thus appropriately identified as the rotational angular frequency of the black hole.

Eq. (2) reminds us of the first law of thermodynamics for an object of energy $E$, charged with potential $\tilde{\Phi}$ and rotating with frequency $\tilde{\Omega}$:

$$dE = TdS + \tilde{\Phi}dq + \tilde{\Omega}dj$$

(6)

Comparing Eqs. (2) and (6) and taking into account the mass-energy equivalence tells us that if we want to regard a KNBH as a thermodynamic system and Eq. (2) as the first law of thermodynamics for it, we must identify the differential of its entropy $dS_{BH}$ multiplied by its temperature $T_{BH}$ (whatever that may mean) with $\Theta dA$. But then, mathematically, we have no choice but to take $S_{BH} = f(A)$; no more complicated dependence will do (I owe this point G. Gour). Further, obviously $T_{BH} = \Theta / f'(A)$. Since we want $T_{BH}$ non-negative, we must take $f'(A) > 0$. Thus we have recovered part of my original claim by a different, stricter route. Note that it will not do, as have a number of people, to redefine black hole entropy as depending on $m$, $q$ and $j$ in a more complicated way than just through $A$. This clashes with the first law!

What can we say about $f$ on classical grounds? I introduced it as a property of a KNBH, but it actually is more widely applicable. Consider slowly lowering towards a black hole with $j = 0$ but $q \neq 0$ from opposite sides two equal massive and possibly charged objects, or instead lowering a massive symmetric ring lying in the hole’s equatorial plane. By symmetry the hole stays immobile, but it should become distorted. Because the process is slow (adiabatic) it is subject to a rule that the horizon area will not change (Bekenstein 1998; Mayo 1998). And the process looks reversible, so the black hole entropy should not change. All this indicates that also for a distorted KNBH, $S_{BH} = f(A)$ with the old $f$. In other words, stationary distortion does not introduce extra variables in the black hole entropy of a nonrotating KNBH. A similar conclusion can be established for a $j \neq 0$ distorted KNBH.

Now imagine a number of such black holes in vacuum held at rest at some distance one from the other. There is no evident source of entropy besides the horizons, and we understand entropy of independent systems to be additive, so we can write $S = \sum_i f(A_i)$. We now make a the assumption that the entropy contributed by a black hole is not changed by its motion and associated dynamical changes, so that the formula applies also when the black holes fall toward each other (by Hawking’s result, though, the individual $A_i$ are then on the increase). Consider two of the black holes, $a$ and $b$, which fall together and merge into a single one. Horizon area does
not jump up suddenly, but grows smoothly (Hawking 1971). Thus at the “moment” of merging, the area of the new black holes is \( A_{\text{new}} = A_a + A_b \). Our assumption allows us to use \( S_{\text{BH}} = f(A) \) for the new black hole. In the process of merging, as during the infall, gravitational radiation (and perhaps electromagnetic one if the holes are charged) is emitted. But this is coherent radiation, and should have negligible entropy. Thus we are led to assume that just at merger \( f(A_a + A_b) = f(A_a) + f(A_b) \).

We can verify by differentiating with respect to \( A_a \), and taking account that \( A_b \) could be anything, that this necessarily requires \( f(A) \propto A \) with no possibility of adding a constant (as one can do, classically, for entropy). Thus the area law of black hole entropy is required by classical physics.

But what about the coefficient of the area in this law? This seems to require quantum physics for its determination. I mentioned Wheeler’s suggestion that the right order of magnitude of \( S_{\text{BH}} \) should be gotten if we just divide \( A \) by the Planck length squared \( \ell^2 \). Support for this came from the observation (Bekenstein 1973) that when an elementary particle (a thing whose dimension is of order its Compton length) is very softly deposited at the horizon of any KNBH, the minimal increase of \( S_{\text{BH}} \) as just calibrated, is of order unity. Since an elementary particle should carry no more than a unit or so of entropy, the GSL would work, but it would not if we took the \( \ell \) to be much larger than Planck’s length. Also it made no sense to take \( \ell \) smaller than Planck’s length, which is regarded as the smallest scale on which smooth spacetime (a must for the black hole concept) is a reasonable paradigm. Nothing could be said about the pure numerical coefficient \( \eta \) in the proposed formula \( S_{\text{BH}} = \eta \ell^{-2} A \), which, incidentally, established the black hole temperature I mentioned before as \( T_{\text{BH}} = \eta^{-1} \ell^2 \Theta \). What this temperature meant operationally was not clear, though I did study the matter in detail (Bekenstein 1973, 1974).

Notice the restriction \( Q^2 + a^2 \leq M^2 \) required for the expression for horizon area to make sense. Black holes that just saturate this limit are termed ‘extremal’. Since \( \Theta \) in (3) vanishes for the extremal black holes, all these have \( T_{\text{BH}} = 0 \). However, \( S_{\text{BH}} \) does not vanish: the Nernst-Simon statement of the third law of thermodynamics is thus violated by black holes. However, all evidence is consistent with the conclusion that the unattainability statement (\( T = 0 \) cannot be reached by a finite chain of operations) is obeyed. For black holes the two statements are not equivalent!

Hawking Radiation

Hawking had been a leader of the vociferous opposition to black hole thermodynamics. In a joint paper, “The Four Laws of Black Hole Mechanics”, Bardeen, Carter and Hawking (1973) argued against a thermodynamic interpretation of formulae like (2) and in favor of a purely mechanical one. Ironically, many uninformed authors still cite that paper as one of the sources of black hole thermodynamics! By his own account Hawking (1988) was trying to discredit black hole thermodynamics when he set out to investigate the behavior of quantum fields in the gravitational field of a spherical body which is collapsing to a Schwarzschild black hole. To his surprise he found (Hawking 1974) that the incipient black hole is in a radiating state, emitting spontaneously and steadily all types of radiation in nature with a thermal spectrum (basically Planck or Fermi according to whether bosons or fermions are emitted). A black hole is hot!

How can a black hole possibly radiate? Is it not defined as a region out of which nothing can come out? Hawking also provided an intuitive explanation of why no such problem exists. In the strong gravitation vicinity of the incipient horizon, pairs
of quanta are created from vacuum, just as an electron-positron pairs are created from vacuum in a sufficiently strong electric field. And because of the strong gravitation, one member of a pair may have negative energy overall (counting its big negative gravitational binding energy). Since negative energy particles cannot exist in the familiar spacetime far from a black hole, those pair members have an ephemeral lifetime and soon enough get swallowed by the hole. Their partners, though, necessarily have positive energies (since each pair came out of nothing with zero total energy); they may thus escape to large distance, and those that do constitute the Hawking radiation. It is pretty clear that Hawking emission is a quantum process.

Hawking’s result, rederived over the years by a number of workers using a score of different approaches, left him no choice but to accept the verity of black hole thermodynamics. The temperature of Hawking’s radiance, the same for all species of quanta, came out to be $4\ell^2\Theta$. This jibed with my proposal for black hole temperature provided $\eta = \frac{1}{4}$. A large number of calculations since have verified this value for the black hole temperature. Hawking’s work thus served to calibrate the black hole entropy formula.

Hawking’s justly acclaimed discovery also proffered a striking confirmation of the GSL. As used up to that point, this law was checked only for situations when ordinary entropy is lost into a black hole, being overcompensated by a growth in black hole entropy. But Hawking’s radiance disclosed an unexpected possibility. The Hawking radiation slowly drain the black hole’s mass energy $m$. For a Schwarzschild hole, $S_{\text{BH}} \propto m^2$, so the black hole entropy decreases. The only way the GSL can hold, then, is for the emergent radiation to carry enough entropy to overcompensate the loss. And it does, as checked by Hawking (1976) and by me (Bekenstein 1975). In 1971 when the GSL was formulated, the Hawking radiance was undreamed of. The just described success thus amounts to confirmation of a bona fide prediction. Successes such as this have made the GSL a pillar of black hole physics in the eyes of gravity and string theorists.

**Black Hole Statistical Mechanics?**

In statistical mechanics, the entropy of an ordinary object is a measure of the number of states available to it, for example, the logarithm of the number of quantum states that it may access given its energy. This is the statistical meaning of entropy. What, in this sense, does black hole entropy represent? Is there a black hole statistical mechanics?

Black hole entropy is large; for instance, a solar mass black hole has $S_{\text{BH}} \approx 10^{79}$ whereas the sun has $S \approx 10^{57}$. Early on I expressed the view (Bekenstein 1973, 1975) that black hole entropy is the logarithm of the number of quantum configurations of any matter that could have served as its origin. The ‘any matter’ qualification is in entire harmony with the ‘no hair’ principle. A Schwarzschild black hole of mass $m$ can have come from a mass $m$ of atomic hydrogen, or a mass $m$ of electron-positron plasma, or a mass $m$ of photons, or for that matter any combination of these and other compositions adding up to mass $m$. One cannot distinguish the various possibilities by measuring anything about the hole. There is obviously much more entropy here (many more possible states) than in a mass $m$ of a pure ‘substance’. This interpretation would go a long way towards explaining why black hole entropy is large. But, you counter, $S_{\text{BH}}$ varies like mass squared, not like mass, which is the reason for its bigness. So let us compare a black hole of very small mass with the same mass of ordinary stuff. Indeed, a black hole of mass $10^{15}$ g has an entropy
similar to that of $10^{15}$ g of ordinary matter. However, such a black hole is only about $10^{-13}$ cm across. Matter cannot collapse to make it because the hole is no bigger than the Compton wavelengths of elementary particles. It transpires that nature would not allow a black hole to form whose entropy is not large by “matter standards”. The suggested interpretation of black hole entropy thus has an air of self-consistency about it.

But it cannot be the whole truth. Suppose we start with a Schwarzschild black hole. It is, of course, emitting Hawking radiation at some rate, and would thus lose mass (and entropy) in the course of time. But suppose (Fiola, et al 1994) we arrange for a stream of matter to pour into it at just such a rate as to balance the mass loss, but without adding charge or angular momentum to the black hole. So the black hole does not change in time, and neither does its entropy. But surely the inflowing matter is bringing into the black hole fresh quantum states; yet this is not reflected in a growth of $S_{BH}$! True, radiation is streaming out; could it be carrying away those additional quantum states, or their equivalent? If we continue thinking of the Hawking radiation as originating outside the horizon, this does not sound possible. We are left with the realization that the proposed black hole entropy interpretation is not handling this example well.

An alternative interpretation is that black hole entropy is the entropy of quantum fluctuations of material fields in the vicinity of the horizon. Thorne and Zurek (1985) first proposed this “quantum atmosphere” picture, and the idea was further developed by ’t Hooft (1985) and many others. It has the advantage that the linear dependence on $S_{BH}$ on horizon area is automatic. However, the coefficient $\eta$ in the formula comes out formally infinite unless one admits that the fluctuations are suppressed in a layer next to the horizon. The right order of magnitude of $\eta$ is obtained if that layer’s thickness is of order of the Planck length. Although this scale might have been expected, one can hardly derive the value of $\eta$ this way. An added problem is that the entropy would come out proportional to the number of material fields in nature; different conceptions of nature would lead to different coefficients, yet Hawking’s original inference of the coefficient $\frac{1}{4}$ leaves no room for such freedom.

An improved approach along this line is that of Carlip (1999) and Solodukhin (1999). Here the focus is on fluctuations of the gravitational field, not matter. The propinquity of the horizon makes these obey the laws of a conformal field theory in two spatial dimensions (that being the dimensionality of the horizon). Now this sort of theory has been thoroughly investigated, and using that formalism Carlip and Solodukhin show that the number of states associated with the fluctuations, when translated into an entropy, is exactly the accepted formula, coefficient and all.

How can we understand intuitively a part of this important result? Suppose, as has been suggested (Bekenstein and Mukhanov 1995), and verified by many workers in a variety of ways, that the area of the horizon is quantized with uniformly spaced levels of order of the squared Planck length: $A = \alpha \ell^2 n$ with $\alpha$ a positive pure number and $n = 1, 2, \cdots$. This suggests that the horizon is to be thought of as a patchwork of patches with area $\alpha \ell^2$. If every patch can have, say, 2 distinct states, then a black hole with area $A = \alpha \ell^2 n$ can be in any of $2^n$ ‘surface’ states. As usual, degeneracy makes a contribution to the entropy. If there is no other contribution, then $S_{BH} = \ln 2^n = \ln 2 \cdot (\alpha \ell^2)^{-1} A$ and we have recovered the area law (Bekenstein 1999, Sorkin 1998). Further, from Hawking we know that $\alpha^{-1} \ln 2 = \frac{1}{4}$ so that the horizon quantization law is $A = 4 \ln 2 \cdot \ell^2 n$. 
This uniform spacing of the horizon area spectrum is not universally accepted. The Ashtekar (loop gravity) school of quantum gravity derives a rather more complicated spectrum. Surprisingly enough, a statistical mechanics of the horizon based on this spectrum does recover the law $S_{\text{BH}} \propto A$, although without giving the coefficient (Ashtekar, et al. 1998). Of late a claim that loop quantum gravity actually does lead to a uniformly spaced area spectrum has appeared (Alekseev, et al. 2000). In the face of such basic disagreement, it is unclear what to make of the just mentioned black hole statistical mechanics.

String theorists have in the last years claimed to have completely clarified the statistical mechanical origin of the formula $S_{\text{BH}} = \frac{1}{4} A/\ell^2$. String theory regards elementary particles like the electron or the photon as vibrations of more fundamental entities, the strings. These are one-dimensional objects which move in higher (10 or 26) dimensional spacetime. String theory admits other solutions to its equations, the Dirichlet branes - or branes for short. Strings are to branes as cords are to membranes in our humble three dimensional space. String theorists have found that arrays of branes can have black hole properties. Actually these brane black holes carry parameters different from those of the Kerr-Newman black hole. Nevertheless, string theorists are able to identify what might be called the horizon area $A$ and to establish its dependence on these new parameters. And they are able to count the number of different brane configurations that correspond to a particular $A$. The logarithm of this “degeneracy” is taken to be the entropy associated with that set of configurations. Apart from well understood higher corrections, this turns out to coincide with $\frac{1}{4} A/\ell^2$ (Strominger and Vafa 1996).

Despite this triumph, this brand of black hole statistical mechanics has drawbacks. First, the program is cleanly executable only for extremal black holes (some claims to have escaped this restriction have not attained general recognition), but we already know that extremal black holes are pathological. For example, one cannot reach such a black hole state starting from the other black holes. Second, it is unclear in what sense the conglomerations of branes being considered are the same as the black holes one would find, say, in the aftermath of gravitational collapse in nature. When impressed by the reproduction of the black hole entropy formula from branes, I am thus reminded of the story told about George Gamow, the colorful Russian-American physicist who with Ralph Alpher predicted in 1948 that the universe should be full of thermal radiation with temperature $5-10^9$ K. In 1965 Penzias and Wilson discovered the celebrated $3^0$ K microwave background radiation (for which discovery they later shared the Nobel prize with Ryle). Gamow was asked how he felt about this confirmation of his prediction. “Well”, he retorted, “if you have lost a nickel, and somebody has found a nickel, it does not prove it is the same nickel”. Are brane black holes and traditional black holes the same nickel?

The Bounds on Information

How much information can be stored by whatever means in a cube of whatever composition one centimeter on the side? Foreseeable technology making use of atomic manipulation would suggest an upper bound of around $10^{20}$ bits. But as technology takes advantage of unforeseen paradigms, this number could - and will - go up. For example, we might one day harness the atomic nucleus as an information cache. Can the bound go up without limit? Thirty years ago we would not have known what to answer. But with black hole thermodynamics some definite answers are forthcoming.
First, by information theory, the maximal information $I_{\text{max}}$ a system can hold, reachable if we know in detail its state, is numerically (up to a factor $\ln 2$) just the maximal entropy $S_{\text{max}}$ it could hold under the complementary circumstance that we know nothing about its internal state. Now suppose our information cache is coaxed into collapsing into a black hole. Obviously its surface area $A$ will shrink in the process. The resulting black hole has an entropy $\frac{1}{4}A_{\text{BH}}/\ell^2$ with $A_{\text{BH}} < A$. But by the GSL this entropy must exceed $S_{\text{max}}$. Thus $I_{\text{max}} \ln 2 = S_{\text{max}} < S_{\text{BH}} < \frac{1}{4}A/\ell^2$, or

$$I_{\text{max}} < \frac{A}{4\ell^2 \ln 2}. \quad (7)$$

This bound was inferred by ’t Hooft (1993) and Susskind (1995) by following very much the previous reasoning; ’t Hooft termed it ‘holographic’. By now Eq.(7) has come to be part and parcel of a whole philosophy - the holographic principle espoused by string and gravity theorists - of what constitutes an acceptable physical theory. But as an information bound, the holographic one seems exaggerated. Our standard one centimeter cube is only required by it to hold no more than some $10^{65}$ bits. It is hard to conceive how any technology can ever span the gulf of 45 orders of magnitude between foreseeable information capacity and this figure.

A more efficient information bound came up already in 1980 from my turning around the logic that supplied support for the GSL (Bekenstein 1981). Suppose we have faith in the validity of the GSL. Drop an information cache of overall radius $R$ and total mass-energy $E$ gently into a black hole so that it causes a minimum of horizon area increase. This minimum is determined by purely mechanical considerations; it is $8\pi G R E c^{-4}$. In accordance with the GSL, demand now that the corresponding increase in black hole entropy be no smaller than the maximum entropy the cache can hold, namely $I_{\text{max}} \ln 2$. The result is the bound (Bekenstein 1981)

$$I_{\text{max}} < \frac{2\pi R E}{c h \ln 2} \quad (8)$$

Unlike (7), this bound does not contain $G$; it looks very ‘everyday’ indeed. In fact, for simple closed systems (8) can also be derived from quantum statistical considerations without even mentioning black holes (Bekenstein and Schiffer 1990). And (8) is generally a tighter bound than (7). For instance, it requires our one centimeter cube, if made of ordinary materials, to hold no more than about $10^{38}$ bits. This is 27 orders of magnitude tighter than the holographic requirement, and “only” 18 orders above the foreseeable information capacity. I believe better bounds can be found.

The logic leading to both of the above bounds obviously assumes that the ordinary entropy mentioned in the GSL is the total entropy of the system at all levels. As we know, were we to ignore the atomic, nuclear, quark, and possibly deeper degrees of freedom of matter, and compute the entropy of its molecules by statistical mechanics, we would miss out contributions to the total entropy. How do we know that the GSL “sees” all these? Because it is a gravitational law (for example, $G$ appears in the black hole entropy contribution), and gravitation, unlike other interactions, is aware of all degrees of freedom because they all gravitate. In the end this is what allows black hole thermodynamics to inform us about subtle aspects of ordinary physical systems, like the limits of information.

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