MARCINKIEWICZ REGULARITY FOR SINGULAR PARABOLIC
p-LAPLACE TYPE EQUATIONS WITH MEASURE DATA

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Abstract. We consider quasilinear parabolic equations with measurable coefficients when the right-hand side is a signed Radon measure with finite total mass, having $p$-Laplace type:

$$u_t - \text{div} \ a(Du, x, t) = \mu \quad \text{in } \Omega \times (0, T) \subset \mathbb{R}^n \times \mathbb{R}.$$ 

In the singular range $\frac{2n}{n+1} < p \leq 2 - \frac{1}{n+1}$, we establish regularity estimates for the spatial gradient of solutions in the Marcinkiewicz spaces, under a suitable density condition of the right-hand side measure.

1. Introduction

We study some integrability results of the spatial gradient of solutions to nonlinear parabolic problems with measure data, having the $p$-Laplace type:

$$u_t - \text{div} \ a(Du, x, t) = \mu \quad \text{in } \Omega_T,$$

where $\mu$ is a signed Radon measure with finite total mass, that is, $|\mu|(\Omega_T) < \infty$.

As usual the unknown is $u : \Omega_T \to \mathbb{R}$, $u = u(x, t)$, where $\Omega_T = \Omega \times (0, T)$ is a cylindrical domain with a bounded, open subset $\Omega \subset \mathbb{R}^n$, $n \geq 2$, and $T > 0$. We write $Du := D_x u$. The vector field $a = a(\xi, x, t) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is assumed to be measurable in $(x, t)$, continuous in $\xi$, and subject to the structure conditions

$$\begin{align*}
|a(\xi, x, t)| &\leq \Lambda_1 |\xi|^{p-1}, \\
\langle a(\xi_1, x, t) - a(\xi_2, x, t), \xi_1 - \xi_2 \rangle &\geq \Lambda_0 \left( |\xi_1|^2 + |\xi_2|^2 \right)^{\frac{p-2}{2}} |\xi_1 - \xi_2|^2
\end{align*}$$

for almost every $(x, t) \in \mathbb{R}^n \times \mathbb{R}$, for any $\xi_1, \xi_2, \xi \in \mathbb{R}^n$ and for some constants $\Lambda_1 \geq \Lambda_0 > 0$. A typical prototype of (1.1) is the parabolic $p$-Laplace equation

$$u_t - \text{div} \ (|Du|^{p-2} Du) = \mu.$$

According to the range of $p$, the equation (1.3) can be divided into three types. If $p = 2$, then (1.3) just becomes the classical heat equation. At points where $|Du| = 0$, the diffusivity coefficient $|Du|^{p-2}$ vanishes if $p > 2$ and it blows up if $1 < p < 2$. Thus (1.3) for $p > 2$ is called degenerate parabolic $p$-Laplace equation. When $1 < p < 2$, (1.3) is referred to as singular parabolic $p$-Laplace equation.

In this paper, we shall deal with the singular range

$$\frac{2n}{n+1} < p \leq 2 - \frac{1}{n+1}.$$
while the other singular range \( 2 - \frac{1}{n+1} < p < 2 \) has been considered in [6], and the degenerate range \( p \geq 2 \) has been treated in [4,11]. The lower bound on \( p \) in (1.4) is natural and sharp, since it is related to the existence of the (Barenblatt) fundamental solution (see for instance [47, Chapter 11.4]). Indeed, the fundamental solution \( \Gamma \) is explicitly given for \( \frac{2n}{n+1} < p < 2 \) by
\[
\Gamma(x,t) = t^{-n\alpha} \left[ c(n,p) + \frac{2-p}{p} \alpha^{\frac{1}{p-1}} \left( \frac{|x|}{t^\alpha} \right)^{\frac{p-1}{p}} \right],
\]
where \( \alpha := \frac{1}{p(n+1) - 2n} \). The solution \( \Gamma \) is well defined when \( \alpha > 0 \); that is, \( p > \frac{2n}{n+1} \).

Our aim of this paper is to establish Marcinkiewicz estimates for the spatial gradient of solutions to the singular parabolic measure data problems (1.1) satisfying (1.2) and (1.4), under some decay conditions on the right-hand side measure (see Section 2.2). To formulate our main results, we define certain so-called Morrey-type condition for a measure as follow. For a signed Radon measure \( \mu \) with finite total mass, we say that \( \mu \) satisfies a Morrey-type condition, written \( \mu \in \mathcal{L}^{1,\kappa}(\Omega_T) \), provided that
\[
|\mu|(Q_r(z_0)) \leq C_0 r^{N-\kappa} \quad (0 \leq \kappa \leq N, \ C_0 \geq 1)
\]
holds for any standard parabolic cylinder \( Q_r(z_0) := B_r(x_0) \times (t_0 - r^2, t_0 + r^2) \subset \Omega_T \) of parabolic dimension \( N := n + 2 \). Then we shall prove
\[
\mu \in \mathcal{L}^{1,\kappa}(\Omega_T) \quad \text{for} \quad \kappa_c < \kappa \leq N \quad \implies \quad Du \in \mathcal{M}^\gamma_{loc}(\Omega_T, \mathbb{R}^n),
\]
where two constants \( \kappa_c = \kappa_c(n, \Lambda_0, \Lambda_1, p) \geq 1 \) and \( \gamma = \gamma(n,p,\kappa) \geq 1 \) are determined explicitly later. Here \( \mathcal{M}^\gamma(\Omega_T, \mathbb{R}^n) \) is the Marcinkiewicz space (see (2.2) for definition).

To prove our main results, we use some covering arguments via a so-called maximal function free technique introduced in [1] (see Section 4). This approach is suitable to the situation in which it occurs the lack of homogeneity (roughly speaking it scales differently in time and space) of nonlinear parabolic problems, such as \( p \)-Laplacian with \( p \neq 2 \) or porous medium equation. In this paper, this covering arguments are considered under the following intrinsic parabolic cylinders:
\[
(1.5) \quad \frac{s}{r^2} = \lambda^{2-p} \quad \text{with} \quad \int_{Q_{r,s}(z_0)} |Du|^{\theta} \, dx \, dt \approx \lambda^{\theta} \quad \text{for some} \ \theta \in (0,1),
\]
where \( Q_{r,s}(z_0) := B_r(x_0) \times (t_0 - s, t_0 + s) \). We point out that the spatial gradient of a solution \( u \) of (1.1) may not belong to the \( L^1 \) space under (1.4). For this reason, we need a notion of solution in a renormalized sense (see Definition 2.1). Also, in this circumstance, we show a decay estimate of the upper-level sets of \( |Du| \) (see Section 4.2), alongside with (1.5) and difference estimates (see Section 3).

There are several results concerning gradient regularity for parabolic \( p \)-Laplace type equations with measure data for \( p > 2 - \frac{1}{n+1} \), as follows.

- Potential estimates: e.g. [21] for \( p = 2 \), [30,31] for \( p \geq 2 \), and [29] for \( 2 - \frac{1}{n+1} < p \leq 2 \).
- Marcinkiewicz estimates: e.g. [4,11] for \( p \geq 2 \), and [6] for \( 2 - \frac{1}{n+1} < p < 2 \).
- Fractional differentiability: e.g. [5,7,12] for \( p = 2 \).
- Calderón-Zygmund type estimates: e.g. [14,35,36] for \( p = 2 \), and [15] for \( p > 2 - \frac{1}{n+1} \).
Compared to the results mentioned above, there are few regularity results for the case $\frac{2n}{n+1} < p < 2 - \frac{1}{n}$. We refer to [40] for global Calderón-Zygmund type estimates on nonsmooth domains. It is worthwhile to note that there are regularity estimates (see [19, 37–39]) for elliptic measure data problems with $1 < p \leq 2 - \frac{1}{n}$.

The paper is organized as follows. In Section 2, we introduce notation, terminologies and renormalized solutions, and we present our main results. In Section 3, we collect comparison estimates between our $p$-Laplace type problem and its reference problems. Finally, in Section 4, we prove our main results, by deriving decay estimates via covering arguments under intrinsic parabolic cylinders.

2. Preliminaries and main results

2.1. Notation and definitions. We start with notations. Let us denote by $c$ a universal positive constant, which may change from line to line. Let $B_r(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < r\}$ and $I_r(t_0) := (t_0 - r^2, t_0 + r^2)$. For $\lambda > 0$, we denote by

$$Q^\lambda_r(x_0) := \left\{ x \in \mathbb{R}^n : |x - x_0| < \frac{\lambda r}{2} \right\} \times I_r(t_0)$$

the intrinsic parabolic cylinder in $\mathbb{R}^n \times \mathbb{R} =: \mathbb{R}^{n+1}$ with center $z_0 := (x_0, t_0) \in \mathbb{R}^{n+1}$. When $p = 2$ or $\lambda = 1$, $Q^\lambda_r(x_0) \equiv Q_r(x_0)$. Also we have $Q^\lambda_r(x_0) \subset Q_r(x_0)$ if $\lambda \geq 1$ and $p \leq 2$. The concept of intrinsic means, roughly speaking, that the size of parabolic cylinders depends on the solution of a given PDE in some integral average sense; in particular, the formulation (1.5) can be rewritten as

$$\int_{Q^\lambda_r(x_0)} |Du|^\theta \, dxdt \approx \lambda^\theta \quad \text{for some } \theta \in (0, 1).$$

Under this setting, we shall consider various a priori estimates for the spatial gradient of a solution later. We refer to [16, 31, 46] for further discussion about intrinsic scalings.

Let us define the truncation operator

$$T_k(s) := \max \{-k, \min\{k, s\}\} \quad \text{for any } k > 0 \text{ and } s \in \mathbb{R}. \quad (2.1)$$

For each set $Q \subset \mathbb{R}^{n+1}$, $|Q|$ is the $(n + 1)$-dimensional Lebesgue measure of $Q$ and $\chi_Q$ is the usual characteristic function of $Q$. For $f \in L^1_{loc}(\mathbb{R}^{n+1})$, $\tilde{f}_Q$ stands for the integral average of $f$ over a parabolic cylinder $Q \subset \mathbb{R}^{n+1}$; that is,

$$\tilde{f}_Q := \int_Q f(z) \, dxdt := \frac{1}{|Q|} \int_Q f(z) \, dxdt, \quad \text{where } z := (x, t).$$

Let us introduce a nonlinear parabolic capacity (see [3, 20, 26, 44] for details), which is necessary to define our solution later. For every open subset $Q \subset \Omega_T$, the $p$-parabolic capacity of $Q$ is defined by

$$\text{cap}_p(Q) := \inf \left\{ \|u\|_W : u \in W, u \geq \chi_Q \text{ a.e. in } \Omega_T \right\},$$

where $W := \left\{ u \in L^p(0, T; W_0^{1,p}(\Omega)) : u_t \in L^{p'}(0, T; W^{-1,p'}(\Omega)) \right\}$ endowed with the norm $\|u\|_W := \|u\|_{L^p(0, T; W_0^{1,p}(\Omega))} + \|u_t\|_{L^{p'}(0, T; W^{-1,p'}(\Omega))}$. Here $p' := \frac{p}{p-1}$.

Let $\mathcal{M}_b(\Omega_T)$ (or $\mathcal{M}_b(\Omega), \mathcal{M}_b(0, T)$) be the space of all signed Radon measures on $\Omega_T$ (or $\Omega, (0, T)$, respectively) with finite total mass. Let $\mathcal{M}_s(\Omega_T)$ be the subspace of $\mathcal{M}_b(\Omega_T)$ of the measures that are absolutely continuous with respect to the $p$-parabolic capacity, let $\mathcal{M}_s(\Omega_T)$ be the space of finite signed Radon measures in $\Omega_T$.
with support on a set of zero $p$-parabolic capacity, and let $C_b(\Omega_T)$ be the space of all bounded and continuous functions on $\Omega_T$. A measure $\mu \in \mathcal{M}_b(\Omega_T)$ can be written as a sum of two measures as follows: $\mu = \mu_a + \mu_s$, where $\mu_a \in \mathcal{M}_a(\Omega_T)$ and $\mu_s \in \mathcal{M}_s(\Omega_T)$, see [23, Lemma 2.1]. Also, $\mu_a \in \mathcal{M}_a(\Omega_T)$ if and only if $\mu_a = f + \mu_o + \text{div} G$, where $f \in L^1(\Omega_T)$, $g \in L^p(0,T;W^{1,p}_0(\Omega))$ and $G \in L^p(\Omega_T)$, see [20, 28]. We denote by $\mu^+$ and $\mu^-$ the positive and negative parts of a measure $\mu \in \mathcal{M}_s(\Omega_T)$, respectively. We write $|\mu| := \mu^+ + \mu^-$. We say that a sequence $\{\mu_k\} \subset \mathcal{M}_b(\Omega_T)$ converges tightly (or in the narrow topology of measures) to $\mu \in \mathcal{M}_b(\Omega_T)$ if

$$
\lim_{k \to \infty} \int_{\Omega_T} \varphi \ d\mu_k = \int_{\Omega_T} \varphi \ d\mu \quad \text{for every} \ \varphi \in C_b(\Omega_T).
$$

Finally, we define a certain function space. For $0 < \gamma < \infty$, the space $\mathcal{M}^\gamma(\Omega_T, \mathbb{R}^l)$ is the so-called Marcinkiewicz space (or the weak-$L^\gamma$ space), defined as the set of all measurable maps $f : \Omega_T \to \mathbb{R}^l$ such that

$$
\|f\|_{\mathcal{M}^\gamma(\Omega_T, \mathbb{R}^l)} := \sup_{\lambda > 0} \lambda \cdot \left( \int_{\Omega_T} \text{1}_{\{f(z) > \lambda\}} \ dz \right)^{\frac{1}{\gamma}} < \infty.
$$

We observe the following connection between the Marcinkiewicz and Lebesgue spaces: $L^\gamma(\Omega_T, \mathbb{R}^l) \subset \mathcal{M}^\gamma(\Omega_T, \mathbb{R}^l) \subset L^{\gamma-\varepsilon}(\Omega_T, \mathbb{R}^l)$ for any $\varepsilon \in (0, \gamma)$. We refer to [25, Chapter 1] for various properties for the Marcinkiewicz space.

### 2.2. Main results

We start by introducing a suitable notion of a solution. Our solution $u$ will be treated in a very weak sense because our solution does not generally belong to the usual energy space. Moreover, under (1.4), the spatial gradient of a solution may not be in $L^1(\Omega_T)$ (see [29, Section 1.3]). To overcome this situation, we introduce the following: if $u$ is a measurable function defined in $\Omega_T$ such that $u$ is finite almost everywhere and $T_k(u) \in L^p(0,T;W^{1,p}_0(\Omega))$ for any $k > 0$, then there exists a unique measurable function $U$ such that $DT_k(u) = U\chi_{\{|u| < k\}}$ a.e. in $\Omega_T$ for all $k > 0$. We define the spatial gradient of $u$ as the function $U$ and denote $Du := U$. Now we define the following notion of a solution.

**Definition 2.1** (See [43]). Let $\mu = \mu_a + \mu_s \in \mathcal{M}_b(\Omega_T)$, where $\mu_a \in \mathcal{M}_a(\Omega_T)$ and $\mu_s \in \mathcal{M}_s(\Omega_T)$. A function $u \in L^1(\Omega_T)$ is a renormalized solution of the Cauchy-Dirichlet problem

$$
\begin{cases}
\text{div} a(Du, x, t) = \mu & \text{in} \ \Omega_T, \\
u = 0 & \text{on} \ \partial_p \Omega_T,
\end{cases}
$$

satisfying (1.2) and (1.4) if $T_k(u) \in L^p(0,T;W^{1,p}_0(\Omega))$ for every $k > 0$ and the following property holds: for any $k > 0$ there exist two sequences $\{\nu_k^+\}, \{\nu_k^-\}$ of nonnegative measures in $\mathcal{M}_a(\Omega_T)$ such that

$$
\nu_k^+ \to \mu_a^+, \quad \nu_k^- \to \mu_a^- \quad \text{tightly as} \ k \to \infty
$$

and

$$
\int_{\Omega_T} T_k(u) \varphi_t \ dx dt + \int_{\Omega_T} \langle a(DT_k(u), x, t), D\varphi \rangle \ dx dt = \int_{\Omega_T} \varphi \ d\mu_k
$$

for every $\varphi \in W \cap L^\infty(\Omega_T)$ with $\varphi(\cdot, T) = 0$, where $\mu_k := \mu_a + \nu_k^+ - \nu_k^-$. Here the parabolic boundary of $\Omega_T$ is $\partial_p \Omega_T := (\partial \Omega \times [0,T]) \cup (\Omega \times \{0\})$. We refer to [40, Section 1.1] and the references given there for further discussion of renormalized solutions.
Next, we define a density condition of a measure.

**Definition 2.2.** (i) For \( \mu \in \mathcal{M}_b(\Omega_T) \), we say that \( \mu \) satisfies a Morrey-type condition, written \( \mu \in \mathcal{L}^{1,\kappa}(\Omega_T) \), provided that

\[
|\mu|(Q_r(z_0)) \leq C_0 r^{N-\kappa} \quad (0 \leq \kappa \leq N := n + 2, \ C_0 \geq 1)
\]

holds for any standard parabolic cylinder \( Q_r(z_0) \subset \Omega_T \).

(ii) Similarly, for \( \mu_1 \in \mathcal{M}_b(\Omega) \), we define

\[
\mu_1 \in \mathcal{L}^{1,\kappa_1}(\Omega) \iff |\mu_1|(B_r(x_0)) \leq C_1 r^{n-\kappa_1} \quad (0 \leq \kappa_1 \leq n, \ C_1 \geq 1)
\]

holds for any ball \( B_r(x_0) \subset \Omega \).

(iii) Also, for \( \mu_2 \in \mathcal{M}_b(0,T) \), we define

\[
\mu_2 \in \mathcal{L}^{1,\kappa_2}(0,T) \iff |\mu_2|(I_r(t_0)) \leq C_2 r^{2-\kappa_2} \quad (0 \leq \kappa_2 \leq 2, \ C_2 \geq 1)
\]

holds for any interval \( I_r(t_0) \subset (0,T) \).

Note that \( \mathcal{L}^{1,N}(\Omega_T) \equiv \mathcal{M}_b(\Omega_T) \). For example, the Dirac measure charging a point in \( \Omega_T \) belongs to \( \mathcal{L}^{1,N}(\Omega_T) \).

We are ready to state the first main result of this paper.

**Theorem 2.3.** Let \( \frac{2n}{n+1} < p \leq 2 - \frac{1}{n+1} \). There is a constant \( \kappa_c = \kappa_c(n,\Lambda_0,\Lambda_1,p) \geq 1 \) such that if \( u \) is a renormalized solution of the problem (2.3) under (1.2) with \( \mu \in \mathcal{L}^{1,\kappa}(\Omega_T) \) for \( \kappa_c \leq \kappa \leq N \), then

\[
Du \in \mathcal{M}_{loc}^\gamma(\Omega_T,\mathbb{R}^n), \quad \text{where} \quad \gamma := \frac{\kappa}{\kappa - 1}\max\left\{ p - 1, \frac{1}{2}\left( p - \frac{n(2-p)}{\kappa} \right) \right\}.
\]

Moreover, for any given \( \theta \) satisfying

\[
\max\left\{ \frac{n+2}{2(n+1)}, \frac{n(2-p)}{2} \right\} < \theta < p - \frac{n}{n+1} \leq 1,
\]

there is a constant \( c = c(n,\Lambda_0,\Lambda_1,p,\kappa,C_0,\theta) \geq 1 \) such that

\[
||Du||_{M^\gamma(Q_{2r},\mathbb{R}^n)} \leq c R^\gamma \left\{ \left[ \frac{|\mu|(Q_{2r})}{|Q_{2r}|} \right]^d + \left( \int_{Q_{2r}} |Du|^\theta \ dx\ dt \right)^{\frac{d}{\theta}} + 1 \right\}
\]

for any standard parabolic cylinder \( Q_{2r} \equiv Q_{2r}(z_0) \subset \Omega_T \), where the scaling deficit \( d \) is defined by

\[
d := \frac{2\theta}{2\theta - n(2-p)}.
\]

Theorem 2.3 provides a precise quantitative estimate of the spatial gradient of a renormalized solution in terms of the Marcinkiewicz space, under the assumption that the measure on the right-hand side satisfies the Morrey-type condition. Roughly speaking, the less concentrated the measure \( \mu \) is (i.e. the smaller \( \kappa \) is), the better the integrability of \( Du \) is (i.e. the bigger \( \gamma \) is).

**Remark 2.4.** (i) Note that the value of \( \gamma \) in (2.5) is

\[
\gamma = \begin{cases} \frac{\kappa(p-1)}{\kappa - 1} & \text{if } 1 < \kappa \leq n, \\ \frac{\kappa}{2(\kappa - 1)} \left( p - \frac{n(2-p)}{\kappa} \right) & \text{if } n \leq \kappa \leq N. \end{cases}
\]
(ii) When $\kappa = N$, the value $\gamma$ has the minimum $p - \frac{n}{n+1}$. Also, $\gamma > p(1 + \sigma)$ when $\kappa < N$, see Section 4.3 for details. Here $\sigma$ is the constant coming from a higher integrability for homogeneous problems (Lemma 3.2). Thus, we have

\[ p - \frac{n}{n+1} \leq \gamma < p(1 + \sigma) \text{ for } \kappa < \kappa_2 \leq N. \]

(iii) As $p \leq \frac{2n}{n+1}$, the constant $c$ in (2.7) blows up.

(iv) The ranges of both exponents $p$ and $\theta$ in Theorem 2.3 come from combining Lemmas 3.1 and 3.2 below. Also, the exponent $\theta$ is not empty since $p > \frac{2n}{n+1}$; see Remark 3.6 for details.

(v) The scaling defect like (2.8) occurs when we study regularity theories for PDE having anisotropic structures such as parabolic $p$-Laplace ($p \neq 2$) equations, see for example [1, 4, 6, 10, 11, 13, 15, 27, 29–31, 40].

Theorem 2.3 also gives the following direct consequence when $\mu$ is merely a finite signed Radon measure (i.e. $\mu \in L^{1,N}(\Omega_T) \equiv \mathcal{M}_b(\Omega_T)$). This result was found in [2] without a local estimate.

**Corollary 2.5.** Let $\frac{2n}{n+1} < p \leq 2 - \frac{1}{n+1}$. If $u$ is a renormalized solution of (2.3) under (1.2) satisfying $\mu \in \mathcal{M}_b(\Omega_T)$, then $Du \in \mathcal{M}_{\text{loc}}^{\gamma,1}(\Omega_T, \mathbb{R}^n)$. Moreover, the estimate (2.7) holds for $\kappa = N$ and $\gamma = p - \frac{n}{n+1}$.

**Remark 2.6.** In the case $\mu \in \mathcal{M}_b(\Omega_T)$, there are many results regarding the existence of a solution, such that

\[ Du \in L^q_{\text{loc}}(\Omega_T, \mathbb{R}^n) \quad \text{for all } q < p - \frac{n}{n+1}, \]

see for instance [8, 9, 41, 42, 45]. We also refer to [34, 40] for further discussions in the literature. Thus Corollary 2.5 provides a sharp integrability of $Du$.

Next, if the measure $\mu$ can be decomposed into space and time components (see (2.9) below), then we obtain the following Marcinkiewicz bound.

**Theorem 2.7.** Let $\frac{2n}{n+1} < p \leq 2 - \frac{1}{n+1}$ and let $u$ be a renormalized solution of the problem (2.3) under (1.2). Suppose that the following decomposition holds:

\[ \mu = \mu_1 \otimes \mu_2, \]

where $\mu_1 \in L^\infty(\Omega)$ and $\mu_2 \in L^{1,\kappa_2}(0, T)$ for some $\kappa_2 \in (1, 2]$. Then there exists a constant $\kappa_{2,c} = \kappa_{2,c}(n, \Lambda_0, \Lambda_1, p) \geq 1$ such that for $\kappa_{2,c} < \kappa_2 \leq 2$, we have

\[ Du \in \mathcal{M}_{\text{loc}}^{\gamma_2}(\Omega_T, \mathbb{R}^n), \quad \text{where } \gamma_2 \equiv \frac{p\kappa_2}{2(\kappa_2 + 1)}. \]

Moreover, for any $Q_{2R}(z_0) \Subset \Omega_T$, a local estimate similar to (2.7) holds with $\gamma_2$ replacing $\gamma$.

**Remark 2.8.** (i) By analogy with Remark 2.4 (ii), we see that $p \leq \gamma_2 < p(1 + \sigma)$ for $\kappa_{2,c} < \kappa_2 \leq 2$.

(ii) Note that $\gamma_2 \geq \gamma$ for $1 < \kappa = \kappa_2 \leq 2$. Under (2.9), (2.10) gives a higher integrability compared to (2.5).

(iii) In the case $\mu_1 \in L^{1,\kappa_1}(\Omega)$ for $\kappa_1 \in (1, n]$ and $\mu_2 \in L^\infty(0, T)$, we can obtain an integrability of $Du$ like (2.10). However, in our approach, this integrability is not improved compared to Theorem 2.3; see [6, Section 1] for detailed explanations.
Theorem 2.3 can be improved under more regular vector field than (1.2). We consider the vector field $a = a(\xi, x, t)$ measurable in $(x, t)$ and $C^1$-regular in $\xi$, satisfying

$$(2.11) \begin{cases} |a(\xi, x, t)| + |\xi||D_\xi a(\xi, x, t)| \leq \Lambda_1|\xi|^{p-1}, \\ \Lambda_0|\xi|^{p-2}|\eta|^2 \leq \langle D_\xi a(\xi, x, t)\eta, \eta \rangle \end{cases}$$

for a.e. $(x, t) \in \mathbb{R}^n \times \mathbb{R}$, for every $\eta \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n \setminus \{0\}$ and for some $\Lambda_1 \geq \Lambda_0 > 0$. Note that (2.11) implies the monotonicity condition (1.2)$_2$. For an improvement of Theorem 2.3, we also need the following condition. Let $\delta, R_0 > 0$. We say that the vector field $a(\xi, x, t)$ is $(\delta, R_0)$-BMO if

$$(2.12) \sup_{t_1, t_2 \in \mathbb{R}, 0 < r \leq R_0, y \in \mathbb{R}^n} \sup_{t_1 \leq t \leq t_2} \int_{B_r(y)} \Theta (a, B_r(y))(x, t) \, dx \, dt \leq \delta,$$

where

$$\Theta (a, B_r(y))(x, t) := \sup_{\xi \in \mathbb{R}^n \setminus \{0\}} \frac{|a(\xi, x, t)|}{|\xi|^{p-1}} - \int_{B_r(y)} \frac{|a(\tilde{\xi}, \tilde{x}, t)|}{|\xi|^{p-1}} \, d\tilde{x}.$$ 

We remark that the map $x \mapsto \frac{a(\xi, x, t)}{|\xi|^{p-1}}$ is of BMO (Bounded Mean Oscillation) such that its BMO seminorm is less than $\delta$, uniformly in $\xi$ and $t$. This condition allows merely measurability in $t$-variable and discontinuity in $x$-variable. It also includes VMO (Vanishing Mean Oscillation) condition.

Finally, we obtain the following regularity result.

**Theorem 2.9.** Let $\frac{2n}{n+1} < p \leq 2 - \frac{1}{n+1}$. Assume that the vector field $a$ satisfies (2.11) and a $(\delta, R_0)$-BMO condition for some $\delta, R_0 > 0$. If $u$ be a renormalized solution of the problem (2.3) with $\mu \in L^{1,\alpha}(\Omega_T)$ for $1 < \alpha \leq N$, then we have $Du \in M^\gamma_{loc}(\Omega_T, \mathbb{R}^n)$, where $\gamma$ is given by (2.5) with the range $p - \frac{n}{n+1} \leq \gamma < \infty$. Moreover the estimate (2.7) holds.

We emphasize that all the main results in this section are consistent with the results in [6] for the singular case $2 - \frac{1}{N+1} < p < 2$. Specifically, the exponents $\gamma$ and $\gamma_2$ in (2.5) and (2.10) are precisely the same as those in [6], and so is the Marcinkiewicz estimate (2.7) with $\theta = 1$. Thus, our results naturally extend the results in [6] to another singular case $\frac{2n}{n+1} < p \leq 2 - \frac{1}{n+1}$. We refer to [4,11] for the degenerate case $p \geq 2$. See also [32,33] for counterparts of elliptic problems.

3. **Comparison estimates**

In this section we derive comparison estimates between the problem (2.3) and its references problems, under the assumptions on the vector field $a(\xi, x, t)$ and the measure $\mu$. (see Propositions 3.5 and 3.8). From Definition 2.1, we may regard $T_k(u) \in L^p(0, T; W^{1,p}_0(\Omega))$ as a weak solution of (2.4) with $\mu_k \in L^{p'}(0, T; W^{-1,p'}(\Omega))$. Throughout this section, we replace $T_k(u)$ by $u$ and $\mu_k$ by $\mu$.

Let $w$ be the unique weak solution to the Cauchy-Dirichlet problem

$$\begin{cases} w_t - \text{div} a(Dw, x, t) = 0 \quad \text{in } Q^\lambda_{4r}(z_0) \cap \Omega_T, \\ w = u \quad \text{on } \partial_p Q^\lambda_{4r}(z_0), \end{cases}$$

where the vector field $a$ satisfies (1.2). In this section, we for simplicity omit the center $z_0$ in $Q^\lambda_{4r}(z_0)$.

We first give a comparison estimate for $Du - Dw$ as follows:
Lemma 3.1 (See [40, Lemma 3.1]). Let $\frac{3n+2}{2n+2} < p \leq 2 - \frac{1}{n+1}$, let $u$ be a weak solution of (2.4) and let $w$ as in (3.1) under (1.2). Then there exists a constant $c = c(n, \Lambda_0, p, \theta) \geq 1$ such that

$$
\left( \int_{Q_{4r}^2} |Du - Dw|^\theta \, dx \, dt \right)^{\frac{1}{\theta}} \leq c \left[ \frac{\mu(Q_{4r}^2)}{Q_{4r}^{\frac{n+2}{n+1}}} \right]^{\frac{n+2}{p(n+1)-n}} 
+ c \left[ \frac{\mu(Q_{4r}^2)}{Q_{4r}^{\frac{n+2}{n+1}}} \right] (\int_{Q_{4r}^2} |Du|^\theta \, dx \, dt)^{\frac{1}{\theta} - \frac{1}{2 - p(n+1)}}
$$

for any constant $\theta$ such that $\frac{n+2}{2(n+1)} < \theta < p - \frac{n}{n+1} \leq 1$.

We point out that Lemma 3.1 also holds for $p > 2 - \frac{1}{n+1}$; see [29, Lemma 4.3] for $2 - \frac{1}{n+1} < p \leq 2$, and [31, Lemma 4.1] for $p \geq 2$.

We next introduce a higher integrability result for $Du$.

Lemma 3.2 (See [27, 40]). Let $\frac{2n}{n+2} < p \leq 2$ and let $\frac{n(2-p)}{2} < \theta < p$. If $w$ is the weak solution of (3.1) under (1.2) satisfying

$$
\int_{Q_{4r}^2} |Dw|^\theta \, dx \, dt \leq c_w \lambda^{\theta}
$$

for some constant $c_w \geq 1$, then there exist two constants $\sigma = \sigma(n, \Lambda_0, \Lambda_1, p, \theta) > 0$ and $c = c(n, \Lambda_0, \Lambda_1, p, \theta, c_w) \geq 1$ such that

$$
\int_{Q_{4r}^2} |Dw|^p(1+\sigma) \, dx \, dt \leq c \lambda^{p(1+\sigma)}.
$$

We remark that Lemma 3.2 also holds for $p \geq 2$, see [15, 27].

If the measure $\mu$ satisfies a Morrey-type condition (see Definition 2.2), then we have the following relation:

Lemma 3.3. Let $\lambda \geq 1$ and let $\mu \in \mathcal{L}^{1,\kappa}(\Omega_T)$ for some $1 < \kappa \leq N := n + 2$. Assume that $\gamma$ is given by (2.5). If the relation

$$
(3.2) \quad \left[ \frac{\mu(Q_{4r}^2)}{Q_{4r}^{\frac{n+2}{n+1}}} \right]^{\frac{1}{\gamma}} \leq \delta \lambda
$$

holds for some constant $\delta \in (0, 1)$, then we have

$$
\left[ \frac{\mu(Q_{4r}^2)}{Q_{4r}^{\frac{n+2}{n+1}}} \right]^{\frac{n+2}{p(n+1)-n}} \leq c_0 \gamma^{(n-1)(n+2)} \delta^{-\frac{n(n-1)(n+2)}{2}} \lambda
$$

for some constant $c = c(n, p, \kappa, C_0) \geq 1$, where $C_0$ is given by Definition 2.2(i).

Proof. For simplicity, we write $\beta := p(n+1) - n$. We compute

$$
(3.3) \quad \left[ \frac{\mu(Q_{4r}^2)}{Q_{4r}^{\frac{n+2}{n+1}}} \right]^{\frac{n+2}{\beta}} = \left[ \frac{\mu(Q_{4r}^2)}{Q_{4r}^{\frac{n+2}{n+1}}} \right]^{\frac{n+2}{\beta}} |Q_{4r}^2|^{\frac{1}{\beta}}
$$

$$
= \left[ \frac{\mu(Q_{4r}^2)}{Q_{4r}^{\frac{n+2}{n+1}}} \right]^{\alpha \frac{n+2}{\beta}} \left[ \frac{\mu(Q_{4r}^2)}{Q_{4r}^{\frac{n+2}{n+1}}} \right]^{(1-\alpha) \frac{n+2}{\beta}} |Q_{4r}^2|^{\frac{1}{\beta}},
$$

where $\alpha \in (0, 1)$ is to be determined later.
First, we assume that $1 < \kappa \leq n$. The intrinsic parabolic cylinder $Q^\lambda_{4r}$ can be covered by finitely many (at most $2 \lfloor N/2 \rfloor$) standard parabolic cylinders with radius $\lambda \sqrt{2} 4r$. Combining this property and Definition 2.2 (i), we deduce

$$\frac{|\mu|(Q^\lambda_{4r})}{|Q^\lambda_{4r}|} \leq c \lambda^{2-p} (\lambda \sqrt{2} 4r)^{N-\kappa} \leq c \lambda^{\frac{(2-p)}{2} - \kappa}.$$

Inserting (3.2) and (3.4) into the right-hand side of (3.3) yields

$$\frac{|\mu|(Q^\lambda_{4r})}{|Q^\lambda_{4r}|} \leq c(\delta) \left( \gamma_{n(1-n)} \left( \frac{\lambda^{\frac{n}{p}}}{N} \right)^{2} \right) \left( \lambda^{\frac{(2-p)}{2} - \kappa} \right)^{\gamma_{n(1-n)}}.$$

for some constant $c = c(n, p, \kappa, C_0) \geq 1$, where $\tilde{\alpha} := \frac{(n+2)(1-\kappa)}{p}$. On the other hand, we assume that $n \leq \kappa \leq N$. Definition 2.2 (i) provides

$$\frac{|\mu|(Q^\lambda_{4r})}{|Q^\lambda_{4r}|} \leq c(\delta) \left( \gamma_{n(1-n)} \left( \frac{\lambda^{\frac{n}{p}}}{N} \right)^{2} \right) \left( \lambda^{\frac{(2-p)}{2} - \kappa} \right)^{\gamma_{n(1-n)}}.$$

Similar to (3.5), we insert (3.2) and (3.6) into the right-hand side of (3.3), to discover

$$\frac{|\mu|(Q^\lambda_{4r})}{|Q^\lambda_{4r}|} \leq c(\delta) \left( \gamma_{n(1-n)} \left( \frac{\lambda^{\frac{n}{p}}}{N} \right)^{2} \right) \left( \lambda^{\frac{(2-p)}{2} - \kappa} \right)^{\gamma_{n(1-n)}}.$$

Now we fix $\alpha \in (0, 1)$ such that $\tilde{\alpha} = 0$; that is, $\alpha = \frac{n-1}{n}$. From such a choice of $\alpha$ and the definition of $\gamma$, both (3.5) and (3.7) imply

$$\frac{|\mu|(Q^\lambda_{4r})}{|Q^\lambda_{4r}|} \leq c(\delta) \left( \gamma_{n(1-n)} \left( \frac{\lambda^{\frac{n}{p}}}{N} \right)^{2} \right) \left( \lambda^{\frac{(2-p)}{2} - \kappa} \right)^{\gamma_{n(1-n)}}.$$

which completes the proof.

If the measure $\mu$ admits a favorable decomposition, we obtain

**Lemma 3.4.** Let $\lambda \geq 1$. Assume that the measure $\mu$ has the following decomposition

$$\mu = \mu_1 \otimes \mu_2,$$

where $\mu_1 \in L^\infty(\Omega)$ and $\mu_2 \in L^{1, \kappa_2}(0, T)$ for some $\kappa_2 \in (1, 2]$. Also, assume that $\gamma_2$ is given by (2.10). If the relation

$$\left( \frac{|\mu|(Q^\lambda_{4r})}{|Q^\lambda_{4r}|} \right)^{\frac{1}{\gamma_2}} \leq \delta \lambda,$$

holds for some constant $\delta \in (0, 1)$, then we have

$$\frac{|\mu|(Q^\lambda_{4r})}{|Q^\lambda_{4r}|} \leq c(\delta) \left( \gamma_{n(1-n)} \left( \frac{\lambda^{\frac{n}{p}}}{N} \right)^{2} \right) \left( \lambda^{\frac{(2-p)}{2} - \kappa} \right)^{\gamma_{n(1-n)}}.$$

for some constant $c = c(n, p, \kappa, C_2, ||\mu_1||_{L^\infty(\Omega)}) \geq 1$, where $C_2$ is given by Definition 2.2 (iii).
Proof. For simplicity, we write $\beta := p(n+1) - n$. Our assumption implies
\begin{equation}
\frac{|\mu_1|(|Q_{4r}^\lambda|)}{|Q_{4r}|} \leq \|\mu_1\|_{L^\infty(\Omega)} \quad \text{and} \quad \frac{|\mu_2|(I_{4r})}{|I_{4r}|} \leq cr^{-\kappa_2}.
\end{equation}
Inserting (3.8) and (3.10) into the right-hand side of (3.3) yields
\begin{align*}
\left[ \frac{|\mu_1|(|Q_{4r}^\lambda|)}{|Q_{4r}|} \right]^{\frac{\alpha+2}{\alpha}} & \leq c(\delta) \frac{\gamma_2 \alpha(n+2)}{\lambda} r^{-\kappa_2(1-\alpha)(n+2)} \left( \frac{\gamma_2 \alpha(2\kappa_2(n+2)+n)}{\lambda r^{n+2}} \right)^{\frac{\alpha}{\beta}} \\
& \leq c(\delta) \frac{\gamma_2 \alpha(n+2)}{\lambda} r^{-\kappa_2(1-\alpha)(n+2)} \left( \frac{\gamma_2 \alpha(2\kappa_2(n+2)+n)}{\lambda r^{n+2}} \right)^{\frac{\alpha}{\beta}}
\end{align*}
for some constant $c = c(n, p, \kappa, C_2, \|\mu_1\|_{L^\infty(\Omega)}) \geq 1$. Then we choose $\alpha \in (0, 1)$ such that $\frac{(n+2)(1-\kappa_2(1-\alpha))}{\kappa_2} = 0$; that is, $\alpha = \frac{\kappa_2 - 1}{\kappa_2}$. This and the definition of $\gamma_2$ yield the desired estimate (3.9).

Combining all the previous results, we derive

**Proposition 3.5.** Let $\frac{2n}{n+1} < p \leq 2 - \frac{1}{n+1}$, $\lambda \geq 1$, $0 < \delta < 1$, and let $\theta$ be a constant such that $\max\left\{\frac{2n+2}{2(n+1)}, \frac{n(2-p)}{2}\right\} < \theta < p - \frac{n}{n+1} \leq 1$. Assume that $\gamma$ is given by (2.5). If $u$ and $w$ are weak solutions of (2.4) and (3.1), respectively, satisfying (1.2), $\mu \in L^{1,\kappa}(\Omega_T)$ for some $1 < \kappa \leq N$,
\begin{equation}
\int_{Q_{4r}^\lambda} |Du|^\theta \, dx \, dt \leq \lambda^\theta \quad \text{and} \quad \left[ \frac{|\mu|(Q_{4r}^\lambda)}{|Q_{4r}|} \right]^{\frac{\alpha}{\beta}} \leq \delta \lambda,
\end{equation}
then there are constants $\sigma = \sigma(n, \Lambda_0, \Lambda_1, p, \theta) > 0$ and $c = c(n, \Lambda_0, \Lambda_1, p, \theta, \kappa, C_0) \geq 1$ such that
\begin{equation}
\int_{Q_{4r}^\lambda} |Du - Dw|^\theta \, dx \, dt \leq c \delta \sigma^\alpha \lambda^\theta \quad \text{and} \quad \int_{Q_{2r}^\lambda} |Dw|^{p(1+\sigma)} \, dx \, dt \leq c \lambda^{p(1+\sigma)},
\end{equation}
where $\sigma_0 = \sigma_0(n, p, \theta, \kappa) > 0$.

**Remark 3.6.** When we combine Lemmas 3.1 and 3.2, the range of $p$ in Proposition 3.5 is valid when $\frac{2n}{n+1} < p \leq 2 - \frac{1}{n+1}$, not max\{\frac{3n+2}{2n+2}, \frac{2n}{n+2}\} < p \leq 2 - \frac{1}{n+1}$, since the exponent $\theta$ in Proposition 3.5 exists only when $p - \frac{n}{n+1} > \max\{\frac{2n+2}{2(n+1)}, \frac{2n}{2(n+1)}\}$. Note that $\frac{2n}{n+1} \geq \max\{\frac{3n+2}{2n+2}, \frac{2n}{2n+2}\}$, where the equality holds if and only if $n = 2$.

To prove Theorem 2.9, we need a more comparison estimate as follows. Assume that the vector field $a$ satisfies (2.11) and a $(\delta, R_0)$-BMO condition for some $R_0 > 4r$ and $\delta \in (0, 1)$. We consider the unique weak solution $v$ to the coefficient frozen problem
\begin{equation}
\begin{cases}
v_t - \text{div} \, \bar{a}_{B^r_{2r}}(Dv, t) = 0 & \text{in } Q^\lambda_{2r}, \\
v = w & \text{on } \partial_p Q^\lambda_{2r},
\end{cases}
\end{equation}
where a freezing operator $\bar{a}_{B^r_{2r}} = \bar{a}_{B^r_{2r}}(\xi, t) : \mathbb{R}^n \times (-4r^2, 4r^2) \to \mathbb{R}^n$ is given by
\begin{equation}
\bar{a}_{B^r_{2r}}(\xi, t) := \int_{B^r_{2r}} a(\xi, x, t) \, dx.
\end{equation}
Then the operator $\bar{a}_{B^r_{2r}}$ satisfies (2.11).

Now we derive the following comparison result between (3.1) and (3.11):
Lemma 3.7. Let $p > \frac{2n}{n+2}$. Assume that the vector field $a$ satisfies (2.11) and a $(\delta, R_0)$-BMO condition for some $R_0 > 4r$ and $\delta \in (0, 1)$. If $w$ and $v$ are weak solutions of (3.1) and (3.11), respectively, then there is a constant $c = c(n, \Lambda_0, \Lambda_1, p) \geq 1$ such that

$$
(3.12) \quad \int_{Q^2_r} |Dw - Dv|^p \, dxdt \leq c_1 \sigma^1 \lambda^p \quad \text{and} \quad \|Dv\|_{L^\infty(Q^2_r)} \leq c \lambda,
$$

where $\sigma_1 = \sigma_1(n, \Lambda_0, \Lambda_1, p) > 0$.

Proof. The first estimate of (3.12) follows from Lemma 3.2, (2.12), and [13, Lemma 3.10]. For interior regularity results (see [16–18]), the second estimate of (3.12) holds.

Finally, we combine Lemmas 3.1, 3.2, 3.3 and 3.7 to obtain the following regularity estimate:

Proposition 3.8. Let $\frac{2n}{n+1} < p \leq 2 - \frac{1}{n+1}$, $\lambda \geq 1$, and let $\theta$ be a constant such that $\max\left\{\frac{n+2}{2(n+1)}, \frac{n+1}{2(n+1)}\right\} < \theta < p - \frac{n}{n+1} < 1$. Assume that the vector field $a$ satisfies (2.11) and a $(\delta, R_0)$-BMO condition for some $R_0 > 4r$ and $\delta \in (0, 1)$. If $u$, $w$ and $v$ are weak solutions of (2.4), (3.1) and (3.11), respectively, satisfying $\mu \in L^{1, \kappa}(\Omega_T)$ for some $1 < \kappa \leq N$,

$$
\int_{Q^2_r} |Du|^{\theta} \, dxdt \leq \lambda^{\theta} \quad \text{and} \quad \left[\frac{\mu((Q^2_r)\setminus N)^{\frac{1}{\theta}}}{|Q^2_r|} \right]^{\frac{1}{\theta}} \leq \delta \lambda,
$$

where $\gamma$ is given by (2.5), then there is a constant $c_0 = c_0(n, \Lambda_0, \Lambda_1, p, \theta, \kappa, C_0) \geq 1$ such that

$$
\int_{Q^2_r} |Du - Dv|^{\theta} \, dxdt \leq c_0 \sigma_2 \lambda^{\theta} \quad \text{and} \quad \|Dv\|_{L^\infty(Q^2_r)} \leq c_0 \lambda,
$$

where $\sigma_2 = \sigma_2(n, \Lambda_0, \Lambda_1, p, \theta, \kappa) > 0$.

Remark 3.9. In the case $\mu = \mu_1 \otimes \mu_2$, where $\mu_1 \in L^\infty(\Omega)$ and $\mu_2 \in L^{1, \kappa_2}(0, T)$ for some $\kappa_2 \in (1, 2)$, Propositions 3.5 and 3.8 also hold with $\gamma_2$ replacing $\gamma$, by using Lemma 3.4 instead of Lemma 3.3.

4. Proofs of main results

In this section, we derive Marcinkiewicz estimates (Theorems 2.3, 2.7 and 2.9) for spatial gradient of a renormalized solution $u$ of the problem (2.3). For this, we employ a so-called stopping-time argument introduced in [1], to obtain decay estimates on the upper-level set of $|Du|$.

We consider a renormalized solution $u$ of (2.3). We denote by $u_k := T_k(u)$ ($k \in N$) the truncation of $u$ and $\mu_k \in L^{p'}(0, T; W^{-1,p'}(\Omega))$ the corresponding measure given in (2.4). We also denote by $w_k$ and $v_k$ the corresponding weak solutions of (3.1) and (3.11), respectively. We know that $\mu_k = \mu_k^+ - \mu_k^- + \nu_k^+ - \nu_k^-$ for $k \in N$. Since $\mu_k^+ + \nu_k^+ \to \mu_\ast^+ + \mu_\ast^+$ tightly as $k \to \infty$, we have

$$
\limsup_{k \to \infty} |\mu_k|(K) \leq |\mu|(K) \quad \text{for every compact subset } K \subset \Omega_T.
$$

Let $\frac{2n}{n+1} < p \leq 2 - \frac{1}{n+1}$, let $\theta$ be a constant such that (2.6) holds, and take $Q_{2R} \equiv Q_{2R}(y_0, \tau_0) \subset \Omega_T$. Assume that $\mu \in L^{1, \kappa}(\Omega_T)$ for some $\kappa \in (1, N)$, where
\( N := n + 2 \). We consider a parameter \( \lambda_0 \) to be defined, such that

\[
\lambda_0 := \left( \int_{Q_{2R}} |Du|^\theta \, dx \, dt \right)^{\frac{1}{\theta}} + \frac{1}{\delta} \left[ \frac{\mu((Q_{2R})]}{|Q_{2R}|} \right]^{\frac{1}{\gamma}} + 1,
\]

where the constant \( d \) is given by (2.8) and \( \gamma \) is given by (2.5). The number \( \delta \in (0, 1) \) will be determined later as a universal constant depending only on \( n, \Lambda_0, \Lambda_1, p, \theta, \kappa \) and \( C_0 \).

### 4.1 Stopping-time arguments

For \( \Lambda > \lambda_0 \) and \( r \in (0, 2R] \), we define

\[
E(r, \Lambda) := \{ z \in Q_r : |Du(z)| > \Lambda \}.
\]

For fixed radii \( R \leq R_1 < R_2 \leq 2R \), the relation \( Q^\lambda_{R_1}(z_0) \subset Q_{R_2} \subset Q_{2R} \) holds whenever \( z_0 \in Q_{R_1}, r \in (0, R_2 - R_1] \) and \( \lambda \in [\lambda_0, \infty) \). Fix \( z_0 \in E(R_1, 4\Lambda) \). For almost every such point, Lebesgue’s differentiation theorem implies

\[
\lim_{s \to 0^+} \left( \int_{Q^\lambda_{R_1}(z_0)} |Du|^\theta \, dx \, dt \right)^{\frac{1}{\theta}} + \frac{1}{\delta} \left[ \frac{\mu((Q^\lambda_{R_2})]}{|Q^\lambda_{R_2}|} \right]^{\frac{1}{\gamma}} \geq |Du(z_0)| > 4\Lambda,
\]

where the symbol \( |Q| \) denotes the parabolic closure of \( Q \) defined as \( |Q| := Q \cup \partial P \). We consider

\[
\lambda > B\lambda_0, \text{ where } B := \left( \frac{320R}{R_2 - R_1} \right)^{\frac{4N}{\theta}} > 1.
\]

For any radius \( s \) with

\[
\frac{R_2 - R_1}{160} \leq s \leq \frac{R_2 - R_1}{2},
\]

we see from (4.2), (4.4), (4.5) and (2.8) that

\[
\left( \int_{Q^\lambda_{R_1}(z_0)} |Du|^\theta \, dx \, dt \right)^{\frac{1}{\theta}} + \frac{1}{\delta} \left[ \frac{\mu((Q^\lambda_{R_2})]}{|Q^\lambda_{R_2}|} \right]^{\frac{1}{\gamma}} \leq \left( \frac{|Q_{2R}|}{|Q^\lambda_{R_1}|} \right)^{\frac{1}{\theta}} \left( \int_{Q_{2R}} |Du|^\theta \, dx \, dt \right)^{\frac{1}{\theta}} + \frac{1}{\delta} \left( \frac{|Q_{2R}|}{|Q^\lambda_{R_1}|} \right)^{\frac{1}{\gamma}} \left[ \frac{\mu((Q_{2R})]}{|Q_{2R}|} \right]^{\frac{1}{\gamma}} \leq \left( \frac{2R}{s} \right)^{\frac{N}{\theta}} \lambda^\frac{n(2-p)}{2-p} \lambda_0 \leq \left( \frac{320R}{R_2 - R_1} \right)^{\frac{N}{\theta}} \lambda^\frac{n(2-p)}{2-p} \lambda_0 \leq \lambda < 4\Lambda,
\]

where we used the inequality \( \theta < \gamma \), see (2.6) and Remark 2.4 (ii). According to (4.3), (4.6) and the (absolute) continuity of the integral and the measure, there exists a maximal radius \( r_{z_0} \in (0, \frac{R_2 - R_1}{4\Lambda}) \) such that

\[
\left( \int_{Q^\lambda_{r_{z_0}}(z_0)} |Du|^\theta \, dx \, dt \right)^{\frac{1}{\theta}} + \frac{1}{\delta} \left[ \frac{\mu((Q^\lambda_{r_{z_0}}(z_0))]}{|Q^\lambda_{r_{z_0}}(z_0)}| \right]^{\frac{1}{\gamma}} = 4\Lambda
\]

and

\[
\left( \int_{Q^\lambda_{R_1}(z_0)} |Du|^\theta \, dx \, dt \right)^{\frac{1}{\theta}} + \frac{1}{\delta} \left[ \frac{\mu((Q^\lambda_{R_2}(z_0))]}{|Q^\lambda_{R_2}(z_0)}| \right]^{\frac{1}{\gamma}} < 4\Lambda \text{ for any } s \in \left( r_{z_0}, \frac{R_2 - R_1}{2} \right).
\]
4.2. Decay estimates. The goal of this subsection is to derive a decay estimate on an upper-level set of $|Du|$, see (4.22) below. Let $z_0 \in E(R_1, 4\lambda)$, let $r_{z_0} \in (0, \frac{R_1 - R_2}{160})$ be a maximal radius as in (4.7). For $\lambda > B\lambda_0$, the upper-level set $E(R_1, 4\lambda)$ can be covered by a family $\mathcal{F} \equiv \{Q^\lambda_{4r_{z_0}}(z_0)\}_{z_0 \in E(R_1, 4\lambda)}$. By the standard Vitali covering lemma (see e.g. [10, Theorem C.1] or [22, Theorem 1.24]), there exists a countable subfamily $\{Q^\lambda_{4r_{z_i}}(z_i)\}_{i \in \mathbb{N}} \subset \mathcal{F}$ consisting of pairwise disjoint cylinders such that

$$E(R_1, 4\lambda) \setminus \mathcal{N} \subset \bigcup_{i=1}^{\infty} Q^\lambda_{20r_{z_i}}(z_i) \subset Q_{R_2},$$

where $\mathcal{N}$ is a Lebesgue measure zero set; that is, $|\mathcal{N}| = 0$. For simplicity, we denote

$$Q^0_i := Q^\lambda_{4r_{z_i}}(z_i), \quad Q^1_i := Q^\lambda_{r_{z_i}}(z_i), \quad Q^2_i := Q^\lambda_{20r_{z_i}}(z_i),$$
$$Q^3_i := Q^\lambda_{40r_{z_i}}(z_i), \quad \text{and } Q^4_i := Q^\lambda_{80r_{z_i}}(z_i).$$

Note that since $160r_{z_i} < R_2 - R_1 \leq R$, we have $Q^4_i \subset Q_{R_2} \subset Q_{2R}$.

We now fix $H \geq 4$ to be chosen later and we estimate

$$|E(R_1, H\lambda)| \leq \sum_{i=1}^{\infty} |Q^2_i \cap E(R_2, H\lambda)|. \tag{4.9}$$

We first split into

$$|Q^2_i \cap E(R_2, H\lambda)| = \left| \left\{ z \in Q^2_i : |Du| > H\lambda \right\} \right|$$
$$\leq \left| \left\{ z \in Q^2_i : |Du - Du_k| > \frac{H\lambda}{3} \right\} \right|$$
$$+ \left| \left\{ z \in Q^2_i : |Du_k - Dw_{k,i}| > \frac{H\lambda}{3} \right\} \right|$$
$$+ \left| \left\{ z \in Q^2_i : |Dw_{k,i}| > \frac{H\lambda}{3} \right\} \right|$$
$$=: I_1 + I_2 + I_3, \tag{4.10}$$

where $w_{k,i}$ is the weak solution of the Cauchy-Dirichlet problem

$$\begin{cases}
\partial_t w_{k,i} - \text{div} a(Dw_{k,i}, x, t) = 0 & \text{in } Q^4_i, \\
w_{k,i} = u_k & \text{on } \partial_\nu Q^4_i.
\end{cases} \tag{4.11}$$

From the absolute continuity of the Lebesgue integral and (4.8), for each $\varepsilon \in (0, 1)$ we have

$$I_1 \leq \frac{3^\theta |Q^2_i|}{(H\lambda)^\theta} \int_{Q^2_i} |Du - Du_k|^\theta \, dx \, dt = \frac{3^\theta |Q^2_i|}{(H\lambda)^\theta} \int_{Q^2_i} \chi_{\{|Du| > k\}} |Du|^\theta \, dx \, dt$$
$$\leq \frac{c\varepsilon}{H^\theta} |Q^2_i|, \tag{4.12}$$

for $k$ large enough. Moreover, applying Proposition 3.5 with $r = 20r_{z_i}$ and using (4.1) and (4.8), we deduce

$$I_2 \leq \frac{3^\theta |Q^2_i|}{(H\lambda)^\theta} \int_{Q^2_i} |Du_k - Dw_{k,i}|^\theta \, dx \, dt \leq \frac{c\delta^{\rho, \sigma_0}}{H^\theta} |Q^2_i|, \tag{4.13}$$

where $\delta^{\rho, \sigma_0}$ is a constant depending on $\rho, \sigma_0, \varepsilon$.
where \( \sigma_0 = \sigma_0(n, p, \theta, \kappa) > 0 \), and
\[
I_3 \leq \left( \frac{3}{H\lambda} \right)^{(p(1+\sigma))} \int_{Q_1^2} |Dw_{k,i}|^{p(1+\sigma)} \, dx \, dt \leq \frac{c}{H^p(1+\sigma)} |Q_1^2|
\]
for some two constants \( c = c(n, \Lambda_0, \Lambda_1, p, \theta, \kappa, C_0) \geq 1 \).

Plugging \((4.12)-(4.14)\) into \((4.10)\), we obtain
\[
|Q^2_t \cap E(R_2, H\lambda)| \leq \left( \frac{c\varepsilon}{H^\theta} + \frac{c\varepsilon\theta_{\sigma_0}}{H^\theta} + \frac{c}{H^p(1+\sigma)} \right) |Q_1^2|.
\]
Now we will estimate \( |Q_1^2| \). Recalling \((4.7)\), we have then either
\[
2\lambda \leq \left( \int_{Q_0^1} |Du|^{\bar{\theta}} \, dx \, dt \right)^{\frac{1}{\bar{\theta}}} \quad \text{or} \quad 2\lambda \leq \frac{1}{\delta} \left[ \frac{\mu(|Q^0_1|)}{|Q^0_1|} \right]^{\frac{1}{\bar{\theta}}}.
\]
We assume that the first case of \((4.16)\) holds. Then it follows
\[
(2^\theta - 1)\lambda^\theta \leq \frac{1}{|Q_1^2|} \int_{Q_0^1 \cap \{|Du| > \lambda\}} |Du|^\theta \, dx \, dt
\]
\[
\leq \left( \int_{Q_0^1} |Du|^{\bar{\theta}} \, dx \, dt \right)^{\frac{1}{\bar{\theta}}} \left( \frac{|Q_0^1 \cap \{|Du| > \lambda\}|}{|Q_0^1|} \right)^{1-\frac{1}{\bar{\theta}}}
\]
for any \( \bar{\theta} \in \left( \theta, \frac{p}{n+1} \right) \). Applying Proposition 3.5 with \( \bar{\theta} \) instead of \( \theta \) and utilizing \((4.1)\) and \((4.8)\), we deduce
\[
\int_{Q_0^1} |Du|^{\bar{\theta}} \, dx \, dt \leq \int_{Q_0^1} |Du - Dw_k|^{\bar{\theta}} \, dx \, dt + \int_{Q_0^1} |Dw_k - Dw_{k,i}|^{\bar{\theta}} \, dx \, dt
\]
\[
+ \int_{Q_0^1} |Dw_{k,i}|^{\theta} \, dx \, dt \leq c\lambda^{\bar{\theta}}
\]
for \( k \) large enough. Inserting this estimate into \((4.17)\), we obtain
\[
|Q_0^1| \leq c |Q_0^1 \cap \{|Du| > \lambda\}|.
\]
If the second case of \((4.16)\) holds, then we see
\[
|Q_0^1| \leq \frac{|\mu(|Q^0_1|)|}{(2\delta\lambda)^\bar{\theta}}.
\]
Assertions \((4.18)\) and \((4.19)\) yield
\[
|Q_1^2| = 20^N |Q_0^1| \leq c |Q_1^1 \cap E(R_2, \lambda)| + \frac{c|\mu(|Q^0_1|)|}{(2\delta\lambda)^\bar{\theta}}.
\]
We combine \((4.15)\) and \((4.20)\) to obtain
\[
|Q_1^2 \cap E(R_2, H\lambda)| \leq \left( \frac{c\varepsilon}{H^\theta} + \frac{c\varepsilon\theta_{\sigma_0}}{H^\theta} + \frac{c}{H^p(1+\sigma)} \right) |Q_1^1 \cap E(R_2, \lambda)| + \frac{c|\mu(|Q^1_1|)|}{(2\delta\lambda)^\bar{\theta}}.
\]
Since the cylinders \( \{Q^1_1\} \) are pairwise disjoint, we have from \((4.9)\) and \((4.21)\) that
\[
|E(R_1, H\lambda)| \leq \left( \frac{c\varepsilon}{H^\theta} + \frac{c\varepsilon\theta_{\sigma_0}}{H^\theta} + \frac{c}{H^p(1+\sigma)} \right) |E(R_2, \lambda)| + \frac{c|\mu(|Q_{2R}|)}{(2\delta\lambda)^\bar{\theta}}.
\]
for some constant \( c = c(n, \Lambda_0, \Lambda_1, p, \theta, \kappa, C_0) \geq 1 \).

4.3. **Marcinkiewicz estimates.** Before Theorems 2.3, 2.7 and 2.9, we introduce the following technical assertion:

**Lemma 4.1** (See [24, Lemma 6.1]). Let \( \phi : [r, \rho] \to \mathbb{R}_{\geq 0} \) be a nonnegative bounded function. Assume that for \( r \leq t < s \leq \rho \) we have
\[
\phi(t) \leq \vartheta \phi(s) + A(s - t)^{-\beta} + C
\]
with \( 0 \leq \vartheta < 1, A, C \geq 0, \) and \( \beta > 0 \). Then there holds
\[
\phi(r) \leq c(\vartheta, \beta) \left[ A(\rho - r)^{-\beta} + C \right].
\]

Now we prove Theorems 2.3, 2.7 and 2.9.

**Proof of Theorem 2.3.** For \( r \in (0, 2R] \), we define the upper-level set
\[
E_l(r, \lambda) := \{ z \in Q_r : t_1(|Du|) > \lambda \},
\]
where \( t_1 \) is the truncation operator (2.1). Note that \( E_l(r, \lambda) = E(r, \lambda) \) for \( l > \lambda \). Then it follows from (4.22) that
\[
|E_l(R_1, H\lambda)| \leq \left( \frac{c\varepsilon}{H^{\theta - \gamma}} + \frac{c \delta \sigma_0}{H^{\theta - \gamma}} + \frac{c}{H^{p(1 + \sigma)}} \right) |E_l(R_2, \lambda)| + \frac{c|\mu|(Q_{2R})}{(\delta \lambda)^{\gamma}},
\]
whenever \( l > H\lambda \), where \( \sigma_0 = \sigma_0(n, p, \theta, \kappa) > 0 \). Multiplying (4.23) by \( (H\lambda)^\gamma \), we have
\[
(H\lambda)^\gamma |E_l(R_1, H\lambda)| \leq \left( \frac{c\varepsilon}{H^{\theta - \gamma}} + \frac{c \delta \sigma_0}{H^{\theta - \gamma}} + \frac{c}{H^{p(1 + \sigma)}} \right) \lambda^\gamma |E_l(R_2, \lambda)|
\]
\[
+ \frac{cH^\gamma}{(\delta \lambda)^{\gamma}} |\mu|(Q_{2R})
\]
for some constant \( c = c(n, \Lambda_0, \Lambda_1, p, \theta, \kappa, C_0) \geq 1 \). First choose \( H \) sufficiently large such that
\[
\frac{c\varepsilon}{H^{\theta - \gamma}} \leq \frac{1}{4} \quad \text{and} \quad H \geq 4,
\]
and then choose \( \varepsilon \) and \( \delta \) sufficiently small such that
\[
\frac{c\varepsilon}{H^{\theta - \gamma}} \leq \frac{1}{4} \quad \text{and} \quad \frac{c \delta \sigma_0}{H^{\theta - \gamma}} \leq \frac{1}{4}.
\]
From (4.24), \( \gamma \) have to be chosen such that \( \gamma < p(1 + \sigma) \). Thus, we choose the critical value \( \kappa_c \) such that \( \gamma = p(1 + \sigma) \); that is,
\[
\frac{\kappa_c}{\kappa_c - 1} \max \left\{ p - 1, \frac{1}{2} \left( p - \frac{n(2 - p)}{\kappa_c} \right) \right\} = p(1 + \sigma).
\]
Taking the supremum with respect to \( \lambda > B\lambda_0 \), we obtain
\[
\sup_{\lambda > B\lambda_0} \lambda^\gamma |E_l(R_1, \lambda)| \leq \frac{3}{4} \sup_{\lambda > B\lambda_0} \lambda^\gamma |E_l(R_2, \lambda)| + c|\mu|(Q_{2R}),
\]
whenever \( l > H\lambda \). Here \( \lambda_0 \) is given by (4.2). Recalling (2.2), we see from (4.4) that
\[
|T_l(|Du|)|_{M_\gamma(Q_{R_1}, \mathbb{R}^n)} \leq \frac{3}{4} |T_l(|Du|)|_{M_\gamma(Q_{R_2}, \mathbb{R}^n)} + c|\mu|(Q_{2R}) + (HB\lambda_0)^\gamma |Q_{R_1}|
\]
\[
\leq \frac{3}{4} |T_l(|Du|)|_{M_\gamma(Q_{R_2}, \mathbb{R}^n)} + c|\mu|(Q_{2R}) + c\lambda_0^{\gamma} \left( \frac{R}{R_2 - R_1} \right)^{\frac{\kappa_c}{\kappa_c - 1}} R^N.
\]
for all \( R \leq R_1 < R_2 \leq 2R \). Applying Lemma 4.1 and letting \( l \to \infty \), we discover
\[
\|Du\|_{\mathcal{M}^\gamma(Q_{2R},\mathbb{R}^n)} \leq c|\mu|(Q_{2R}) + c\lambda_0^\gamma R^N
\]
\[
\leq cR^N \frac{|\mu|(Q_{2R})}{|Q_{2R}|} + cR^N \left( \int_{Q_{2R}} |Du|^\theta \, dx \, dt \right)^{\frac{d}{\theta}}
\]
\[
+ cR^N \left[ \frac{|\mu|(Q_{2R})}{|Q_{2R}|} \right]^d,
\]
which completes the proof. \( \square \)

**Proof of Theorem 2.7.** In view of Lemma 3.4 and Remark 3.9, we replace \( \gamma \) by \( \gamma_2 \) in Sections 4.1–4.2. Proceeding as in Sections 4.1–4.2 and Proof of Theorem 2.3 above, we can obtain Theorem 2.7. \( \square \)

**Proof of Theorem 2.9.** We proceed as in Section 4. We only mention the parts that change in Section 4. We choose \( H \geq \max\{4,3c_0\} \), where the constant \( c_0 \) is given by Proposition 3.8. Instead of (4.10), we split into
\[
|Q_i^2 \cap E(R_2, H\lambda)| = \left| \left\{ z \in Q_i^2 : |Du| > H\lambda \right\} \right|
\]
\[
\leq \left| \left\{ z \in Q_i^2 : |Du - Du_k| > \frac{H\lambda}{3} \right\} \right|
\]
\[
+ \left| \left\{ z \in Q_i^2 : |Du_k - Dv_{k,i}| > \frac{H\lambda}{3} \right\} \right|
\]
\[
+ \left| \left\{ z \in Q_i^2 : |Dv_{k,i}| > \frac{H\lambda}{3} \right\} \right|
\]
\[
=: J_1 + J_2 + J_3,
\]
where \( v_{k,i} \) is the weak solution of the Cauchy-Dirichlet problem
\[
\begin{cases}
\partial_t v_{k,i} - \text{div} \ a_{B_{w_{k,i}}}(Dv_{k,i}, t) = 0 & \text{in } Q_i^3, \\
v_{k,i} = w_{k,i} & \text{on } \partial_p Q_i^3.
\end{cases}
\]
Here \( w_{k,i} \) is the weak solution of (4.11). Then we see from Proposition 3.8 and the choice of \( H \) that \( J_3 = 0 \). Also, we can estimate \( J_1 \) and \( J_2 \) similar to the estimates of \( I_1 \) and \( I_2 \) in Section 4.2. Performing the rest of Section 4.2, we obtain, instead of (4.22), the following decay estimate
\[
|E(R_1, H\lambda)| \leq \left( \frac{c\varepsilon}{H^\rho} + \frac{c\delta^\sigma}{H^\delta} \right) |E(R_2, \lambda)| + \frac{c|\mu|(Q_{2R})}{(\delta\lambda)^\mu},
\]
where \( \sigma_2 = \sigma_2(n, \Lambda_0, \Lambda_1, p, \theta, \kappa) > 0 \). Proceeding as in Proof of Theorem 2.3 above with (4.25) replacing (4.23), we deduce Theorem 2.9. We remark that we obtain Theorem 2.9 without (4.24); that is, \( \gamma < \infty \). \( \square \)

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