Entanglement of finite cyclic chains at factorizing fields

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We examine the entanglement of cyclic spin 1/2 chains with anisotropic XYZ Heisenberg couplings of arbitrary range at transverse factorizing magnetic fields. At these fields the system exhibits a degenerate symmetry-breaking separable ground state (GS). It is shown, however, that the side limits of the GS pairwise entanglement at these fields are actually non-zero in finite chains, corresponding such fields to a GS spin-parity transition. These limits exhibit universal properties like being independent of the pair separation and interaction range, and are directly related to the magnetization jump. Illustrative exact results are shown for chains with I) full range and II) nearest neighbor couplings. Global entanglement properties at such points are also discussed.

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Quantum entanglement is well recognized as a fundamental resource in quantum information science \[1\]. It also provides a new perspective for the analysis of quantum many-body systems, allowing to identify the genuine quantum correlations \[2–4\]. An important result in this context is in contrast with the correlation length, the genuine quantum correlations \[2–4\]. An important result in this context is in contrast with the correlation length, the genuine quantum correlations \[2–4\].

We consider a cyclic chain of \(n\) qubits or spins interacting through an XYZ Heisenberg coupling with arbitrary common range in a transverse magnetic field \(b\). Denoting with \(s^i\) the spin at site \(i\), the Hamiltonian reads

\[
H = bS_z - \sum_{i<j} r_{j-i} (v_x s^i_x s^j_x + v_y s^i_y s^j_y + v_z s^i_z s^j_z),
\]

\[
= bS_z - \sum_{i<j} r_{j-i} \left(\frac{1}{2} (v_+ s^i_+ s^j_+ + v_- s^i_- s^j_- + h.c.) + v_z s^i_z s^j_z\right),
\]

where \(S = \sum_{i=1}^{n} s^i, v_{\pm} = (v_x \pm v_y)/2\) and \(r_l = r_{n-l}\) for \(l = j - i = 1, \ldots, n - 1\). Without loss of generality we can here assume \(b \geq 0\) and \(v_x \geq |v_y|\) (i.e., \(v_{\pm} \geq 0\), with \(r_l\) arbitrary. We will be interested in the attractive (ferromagnetic) case \(r_l \geq 0\ \forall\ l\), although the following considerations are general. Since \(H\) conserves the \(S_z\)-parity,

\[
[H, P_z] = 0, \quad P_z = \exp[i\pi(S_z + n/2)],
\]

its nondegenerate eigenstates will have definite parity \(P_z = \pm 1\).

Let us now examine the conditions for which a completely symmetric separable state of the form

\[
\theta = \prod_{i=1}^{n} \left(\cos \frac{1}{2}\theta |\uparrow_i\rangle + \sin \frac{1}{2}\theta |\downarrow_i\rangle\right) = \exp[i\theta S_y](|0\rangle),
\]

where \(S^i_y = 1/2 (|\uparrow_i\rangle \langle \downarrow_i| + |\downarrow_i\rangle \langle \uparrow_i|)\) and \(|0\rangle = \prod_{i} |\downarrow_i\rangle\), can be an exact eigenstate of \(H\). This state is fully aligned along an axes \(z'\) forming an angle \(\theta\) with the \(z\) axes, such that \(S_{z'}(\theta) = -\frac{1}{2}n(\theta),\) breaking parity symmetry for \(\theta \in (0, \pi)\). Replacing \(s^i_{z',z} = s^i_{z',z'} \cos \theta \pm s^i_{z',z''} \sin \theta\) in \(H\), it is easily seen that these conditions are

\[
\cos \theta = \pm \sqrt{\lambda}, \quad \chi \equiv \frac{v_y}{v_x - v_z},
\]

\[
b = r(v_x - v_z) \cos \theta, \quad r \equiv \frac{1}{n} \sum_{l=1}^{n-1} r_l,
\]

where \(v = (v_x, v_y, v_z)\) is the vector of interaction strengths and \(\lambda = \chi^2 + 2\chi + 1\). The state is characterized by the parameter \(\lambda\) and the parameter \(0 \leq \chi \leq 1\). It is a separable state for \(\chi = 0\), and a pure state for \(\chi = 1\).
where Eq. (5) is required for $\theta \in (0, \pi)$, i.e., $\chi \in [0, 1)$ (in the $XXZ$ case $\chi = 1$ ($v_x = v_y$) both $|0\rangle$ and $|\pi\rangle$ are trivial eigenstates of $H$ for all fields $b$). Such parity breaking separable eigenstates is then feasible for $\chi \in [0, 1)$ (i.e., $v_z \leq v_y < v_x$ or $v_x > |v_y|$) and $b = \pm b_s$, with

$$b_s \equiv r(v_x - v_z)$$

(6)

the factorizing field. The state $|\theta\rangle$ will depend on the anisotropy $\chi$ but not on the factors $r_i$, being then independent of the interaction range.

It is also apparent that both $|\theta\rangle$ and $|\pi - \theta\rangle = P_c|\pi\rangle$ are degenerate eigenstates of $H$ at $b = b_s$, with energy

$$\langle \theta | H | \theta \rangle = -\frac{1}{4} n[b \cos \theta + \frac{1}{2} r(v_x \sin^2 \theta + v_z \cos^2 \theta)]$$

(7)

Hence, at $b = \pm b_s$ two levels of opposite parity necessarily cross, enabling the formation of these eigenstates.

Let us remark that in the attractive case $r_i \geq 0 \forall i$, the state that minimizes $\langle H \rangle$ among separable states (i.e., the mean field approximate GS) is precisely of the form $|\theta\rangle \forall b$, with $|\theta\rangle$ determined by Eq. (5) if $b < b_s = r(v_x - v_z)$ (parity-breaking solution) and $\theta = 0$ otherwise. Hence, in this case the factorizing field can be seen as that where the mean field GS becomes an exact eigenstate (i.e., the exact GS, as shown below).

The states $|\pm \theta\rangle$ will then form a basis of the corresponding eigenspace at $b = b_s$ (assumed of dimension 2), which is non-orthogonal for $\theta \neq \pi/2$: $\langle -\theta | \theta \rangle = \cos \theta$. A proper orthonormal basis conserving parity symmetry is provided by the entangled states

$$|\theta_{\pm}\rangle = \frac{|\theta\rangle \pm |\pi - \theta\rangle}{\sqrt{2(1 \pm \cos \theta)}}$$

(8a)

$$= \sum_{k \text{ even}} \sqrt{2} \sin \frac{k \theta}{2} \frac{\cos^{n-k} \theta}{\cos \theta} \frac{1}{2} S_+^k |0\rangle,$$

(8b)

which satisfy $P_c |\theta_{\pm}\rangle = \pm |\theta_{\pm}\rangle$ and are the actual eigenstates of $H$ in each parity subspace at $b = b_s$. These states (and not the states $|\pm \theta\rangle$) are the actual limits of the corresponding exact eigenstates $|\Psi_{\pm}(b)\rangle$ (which have definite parity) for $b \to b_s$.

In the attractive case $r_i \geq 0 \forall i$ (with $|v_y| \leq v_x$), the states $|\theta_{\pm}\rangle$ (and hence $|\pm \theta\rangle$) are ground states of $H$ at $b = b_s$: The exact GS $|\Psi_0^b(b)\rangle$ in each parity subspace must have expansion coefficients all of the same sign in the standard computational basis (i.e., that of separable states with definite values of $\{s^x_i\}$) in order to minimize the average energy, since the average of the off-diagonal $XY$ term in $H$ can only increase (or eventually stay constant) for different signs (as $r_{j-1} \geq 0$, $v_x \geq 0$) while those of the diagonal terms $b_s$ and $v_z s^x_i s^y_i$ are sign independent. Hence, $|\Psi_0^b(b)\rangle$ cannot be orthogonal to $|\theta_{\pm}\rangle$, whose expansion coefficients in this basis are all non-zero and of the same sign (Eq. (5b)), and must then coincide with $|\theta_{\pm}\rangle$ at $b = b_s$.

Thus, in the attractive case $|\theta_{\pm}\rangle$ represent the side limits $\lim_{b \to b_s} |\Psi_0^b(b)\rangle$ of the exact GS $|\Psi_0^b(b)\rangle$ in the whole space at $b = b_s$, which undergoes there a $|\theta_-\rangle \to |\theta_+\rangle$ parity transition (actually the last parity transition as $b$ increases, as will be shown in the examples).

The pairwise entanglement in the states $|\theta_{\pm}\rangle$ depends essentially on the overlap $\langle -\theta | \theta \rangle$. When orthogonal ($\theta = \pi/2$), they are generalized GHZ states $|\Psi_0^b(b)\rangle$, which, although globally entangled, exhibit no pairwise entanglement ($n > 2$). Moreover, in this case the normalized projector onto the space spanned by the states $|\theta_{\pm}\rangle$,

$$\rho_0 = \frac{1}{4} (|\theta_+\rangle \langle \theta_+ | + |\theta_-\rangle \langle \theta_- |),$$

(9)

which represents in the attractive case the $T \to 0$ limit of the thermal mixed state $\rho(T) \propto \exp[-H/kT]$ at $b = b_s$, is fully separable (i.e., a convex combination of projectors onto separable states) as $\rho_0 = \frac{1}{4} (|\theta\rangle \langle \theta | + |\pi - \theta\rangle \langle \pi - \theta |)$ for $\pi/2$. In contrast, for $\theta \in (0, \pi/2)$ both states $|\theta_{\pm}\rangle$ as well as the mixed state $|\Psi_0^b(b)\rangle$ will be shown to exhibit a uniform non-zero entanglement between any two spins (note that the projector onto this subspace is no longer the sum of the individual projectors $|\pm \theta\rangle \langle \pm \theta |$ when $\langle -\theta | \theta \rangle \neq 0$).

Let us first evaluate the pairwise concurrence $\tilde{C}$ (a measure of pairwise entanglement) in the states $|\theta_{\pm}\rangle$. As a consequence of (2) and the cyclic nature of $H$, the reduced two spin density matrix $\rho_{\text{GS}}$ in any non-degenerate eigenstate or in $|\rho(T), \text{p} \rangle$, will commute with the reduced parity $P_\text{p}^{ij} = e^{i \pi (s^x_i + s^y_i + 1)}$ and depend just on $l = |i - j|$. The ensuing concurrence $C_l \equiv C_{\rho_p}$ takes the form

$$C_l = 2 \max |\alpha^+ | - p_l, |\alpha^- | - q_l, 0 |$$

(10)

where $\alpha^+ = \langle s^+_l s^+_j \rangle, p_l = \frac{1}{2} - \langle s^+_l \rangle, q_l = [(\frac{1}{2} - p_l)^2 - (\langle s^+_l \rangle)^2]^{1/2}$. If $|\alpha^+ | > p_l$ (i.e., $|\alpha^- | > q_l$) $C_l$ is of even (odd) parity type, i.e., parallel (antiparallel) $|\Psi_0^b(b)\rangle$, as in Bell state $\propto |\uparrow \uparrow \rangle + |\downarrow \downarrow \rangle$ ($|\uparrow \downarrow \rangle + |\downarrow \uparrow \rangle$). Just one of these inequalities can be satisfied in a given state.

In the states $|\Psi_0^b(b)\rangle$, $\alpha^+ = \frac{1}{2} \sin^2 \theta \gamma^+ \delta^+, p_l = \alpha^+_l, \langle s^+_l \rangle = -i \cos \theta \gamma^+ \delta^+$, with $\gamma^+ \delta^+ = \frac{1}{1 + \cos^2 \theta}$ and $\nu = \pm$. We then obtain $C_l (|\theta_{\pm}\rangle) = C_{\pm} \forall l$, with (assuming $\theta \in (0, \pi/2)$)

$$C_{\pm} = \sin^2 \theta \cos^n \theta$$

(11a)

$$= (1 - x) \chi^{n/2}$$

(11b)

Thus, $C_- > C_+ > 0$, with $C_+ (C_-)$ parallel (antiparallel). Note that for $\theta \to 0$ ($\chi \to 1$),

$$C_+ \to 0, \quad C_- \to 2/n,$$

as in this limit $|\theta_+ \rangle \to |0\rangle$ but $|\theta_- \rangle \to |1\rangle \equiv \frac{1}{\sqrt{2}} S_+ |0\rangle$, which is an $W$-state $|W\rangle (2/n$ is in fact the maximum value that can be attained by the concurrence in fully symmetric states $|10\rangle$). As $\theta$ increases, $C_-$ decreases while $C_+$ becomes maximum at $\theta \approx 1.6/\sqrt{n}$ (see Eq. (13)), vanishing both for $\theta \to \pi/2$ ($\chi \to 0$) if $n > 2$.

In the attractive case the values $\{11\}$ represent the universal side limits $C_{\pm} = \lim_{b \to b_s} C_l (\theta_{\pm})$ of the GS concurrences $C_l (\theta_{\pm})$, valid for any separation $l$ or
interaction range. For \( \chi \rightarrow 1 \) they correctly approach those for the \( |1\rangle \rightarrow |0\rangle \) transition taking place at \( b = b_c \) in the XXZ limit, \( \text{II} \) (where \( b \rightarrow b_c \)).

The concurrence jump \( C_- - C_+ \) determines, notably, the concurrence \( C_0 \equiv C(\rho_0) \) in the GS mixture, \( \text{III} \),

\[
C_0 = \frac{1}{2}(C_- - C_+) = (1 - \chi) \frac{\chi^{n-1}}{1 - \chi^n}, \tag{12}
\]
(see also Eq. \( \text{17} \)), which is of antiparallel type. It is a decreasing function of \( \theta \), starting at \( 1/n \) for \( \theta \rightarrow 0 \). In the attractive case, Eq. \( \text{12} \) represents the common \( T \rightarrow 0 \) limit of the thermal concurrences \( C_1(T) \) at \( b = b_s \) for any separation \( l \) and coupling range.

Although for fixed \( \chi < 1 \), \( C \) becomes exponentially small as \( n \) increases, the rescaled concurrences \( nC_{\pm} \) remain finite for small anisotropy \( \chi = 1 - \delta/n \) for large \( n \) and fixed \( \delta \), we obtain from \( \text{III} \) the \( n \)-independent limits

\[
\begin{align*}
C_+ &\equiv nC_+ \equiv \delta e^{-\delta/2}/(1 \pm e^{-\delta/2}), \\
C_0 &\equiv nC_0 \equiv \delta e^{-\delta}/(1 - e^{-\delta}),
\end{align*}
\]
depicted in Fig. \( \text{I} \). While \( c_- \) and \( c_0 = c_- (2\delta)/2 \) are decreasing functions of \( \delta \), \( c_+ \) is maximum at \( \delta = 2(1 + w(e^{-1})) \approx 2.56 \), where \( c_+ = 2w(e^{-1}) \approx 0.56 \) \((w(x) = \text{the productlog function, such that } x = we^w)\). We note also that \( c_0 > c_+ \) for \( \delta < 2 \ln 2 \).

The mean rescaled concurrence \( (c_+ + c_-)/2 \) determines, remarkably, the total magnetization jump at \( b = b_s \):

\[
\Delta M \equiv \langle \theta_- | S_z | \theta_- \rangle - \langle \theta_+ | S_z | \theta_+ \rangle
= n \sin^2 \theta \frac{\cos^{-n} \theta}{1 - \cos 2n \theta} = \frac{1}{2}(c_+ + c_-) \sqrt{n}, \tag{15a}
\]
which represents as well the slope of the energy gap \( \Delta E \approx (b - b_s)\Delta M \) between the odd and even GS at \( b = b_s \). For large \( n \) and fixed \( \delta \),

\[
\Delta M \approx (c_+ + c_-)/2 = \delta e^{-\delta/2}/(1 - e^{-\delta}), \tag{15b}
\]
remaining finite and providing a direct way to determine the average rescaled concurrence at \( b_s \).

As illustration, we now show exact results for the concurrence in I) a fully connected chain with constant \( r_1 \) \( \text{II} \) and II) a chain with nearest neighbor coupling \( (r_1 = \delta_{1,1} + \delta_{1,n-1}) \). In I) we set \( r_1 = 2/\{n(1 - n) \} \) such that \( r = 1 \) in I and II (Eq. \( \text{5} \)). The factorizing field \( \text{II} \) and the energy \( \text{II} \) are then the same in I and II for fixed \( v_{x,y,z} \). We will consider \( v_x > 0 \) and \( v_z = 0 \) (XY case).

In I, the GS can be obtained numerically by diagonalizing \( H \) in the subspace of maximum total spin states (to which it belongs) as \( [H, S^2] = 0 \):

\[
H_I = bS_z - \sum_{\mu=x,y} v_{\mu}(S^2_{\mu} - \frac{1}{4}n)/(n - 1).
\]
The fixed parity GS is then of the form \( \sum_{k \text{ even (odd)}} w_k k^l | 0 \rangle \), leading to \( l \) independent elements \( \alpha_l^+ = \langle S^+_{l,1} \rangle / c_n, \alpha_l^- = (n^2/4 - \langle S^2_{l,1} \rangle / c_n, \langle S^+_{l,1} S^+_{l+1} \rangle = (\langle S^2_{l,1} \rangle - n/4)/c_n, \) with \( c_n = n(n - 1) \).

In II, the Hamiltonian can be solved analytically for any finite \( n \) by means of the Jordan-Wigner transformation \( \text{13} \), which allows to rewrite \( H, \) for each value \( \{\pm\} \) of the parity \( P_{z} \), as a quadratic form in fermion operators \( c_i^\dagger, c_i \) defined by \( c_i^\dagger = s_i \exp[-i\pi \sum_j s_j^+ s_j^-] \):

\[
H_{II}^\dagger = \sum_{k \in K_\pm} \frac{1}{2} (b c_i^\dagger c_i - \frac{1}{2} v_i^2) + v_i S_i^z + h.c.,
\]
where \( n + 1 \equiv 1, \eta_i^+ = 1, \eta_i^- = 1 - 2\delta_{in} \) and

\[
\lambda_{k}^2 = (b + v_k \cos \omega_k)^2 + v_k^2 \sin^2 \omega_k, \quad \omega_k = 2\pi/n,
\]
with \( K_+ = \{\frac{1}{2}, \ldots, n - \frac{1}{2}\}, \quad K_- = \{0, \ldots, n - 1\} \) (i.e., \( k \) half-integer (integer) for positive (negative) parity). The diagonal form \( \text{10} \) is obtained through a discrete parity-dependent Fourier transform \( \text{11} \)

\[
c^\dagger_i = \frac{e^{i\pi/4}}{\sqrt{n}} \sum_{k \in K_\pm} e^{-i\omega_k j} t_k^\dagger, \quad \text{followed by a BCS transformation}
\]

\[
\rho_{1,k} = u_k a_k^\dagger + u_k a_{n-k}^- = u_k a_{n-k}^- - v_k a_k^\dagger, \quad \text{to quasiparticle fermion operators}
\]

\[
a_k, a_k^\dagger, \quad \text{with } a_k^2, v_k^2 = \frac{1}{4}[1 \pm (b - v_k \cos \omega_k)/\lambda_{k}].
\]

For \( b \geq 0 \) we set \( \lambda_{k} \geq 0 \) for \( k \neq 0 \) and \( \lambda_0 = v_k - b \), such that the quasiparticle vacuum in \( H_{II}^\dagger \) is odd and the lowest energies for each parity are \( E_{II}^\dagger = -\frac{1}{2} \sum_{k \in K_{\pm}} \lambda_{k} \). At \( b = b_s = \sqrt{v_x v_y} \) (Eq. \( \text{10} \)), \( \lambda_{k} = v_k + b \cos \omega_k \) and \( E_{II}^\dagger = E_{II}^\dagger = -vn_+ /2, \) in full agreement with Eq. \( \text{7} \).

The concurrences in the fixed parity GS can be obtained from the contractions \( f_i \equiv \langle c_i^\dagger c_i^\dagger \rangle \pm \frac{1}{2} \delta_{ij}, \) \( g_i \equiv \langle c_i^\dagger c_i \rangle \pm \) and the use of Wick’s theorem \( \text{13} \), leading to \( \langle s_i^+ s_j^- \rangle = f_i^2 - f_i^2 + g_i^2 \) and \( \alpha_l^+ = \frac{1}{2} \text{det} \langle A_l^+ \rangle = \frac{1}{2} \text{det} \langle A_l^- \rangle \), with \( \langle A_l^\pm \rangle_{ij} = 2f_{i-j-\pm 1} + g_{i-\pm 1} \).

In both I and II, as \( b \) increases from \( 0^+ \), \( [n/2] \) GS parity transitions \( \pm \to \mp \) take place if \( \chi \in (0, 1] \) (as in the XXZ case \( \text{11} \)), the last one \( (\rightarrow +) \) at \( b = b_c. \)
They are clearly visible for low \( n \) if \( \chi \) is not small, i.e., if \( \delta = n(1 - \chi) \) is not too large, as seen in Fig. 2 for \( n = 10 \) qubits. For \( b \to b_5^\pm \), all GS concurrences \( C_l^\pm \) are seen to approach the same side limits (11) in both I and II, which are non-negligible. For \( \delta = 2.5 \), \( C_l^\pm \) reaches in fact its maximum for \( b \to b_s \approx 0.87v_x \) in I, and also in II if \( l > 2 \). For \( \delta = 5 \) the side limits are still noticeable but nearly coincident, implying a negligible \( c_0 \) (Eq. 12).

The behavior of \( C_l^\pm \) for \( n = 50 \) qubits at the same values of \( \delta \) (now \( \chi = 0.95 \) and 0.9), depicted in Fig. 3 is seen to be the same as in Fig. 2 in the vicinity of \( b_s \). All pairs become entangled as \( b \to b_s \), with \( C_l \) approaching the common values (11) in I and II, now well approximated by Eqs. (13).

Within each parity subspace, the factorizing field is distinguished as that where all GS concurrences \( C_l^\pm \) cross at the values (14) (rather than vanish), as seen in the top panel of Fig. 3. Moreover, in II the ordering of concurrences \( C_l^\pm \) becomes inverted at \( b = b_s \): \( C_l^+ \) (\( C_l^- \)) increases with increasing separation \( l \) for \( b \) just above (below) \( b_s \), as \( C_l^\pm (b) \) is linear close to \( b_s \). Note that \( C_l^\pm \) vanishes at a lower field \( b_l^+ < b_s \), becoming antiparallel for \( b < b_l^+ \).

On the other hand, at sufficiently low temperatures, \( b_s \) can be identified as the field where all thermal concurrences \( C_l(T) \) cross at the value (12), as seen in the bottom panel of Fig. 3. \( C_l(T) \) vanishes at a slightly larger \( l \)-dependent field \( b_l(T) > b_s \), remaining antiparallel until \( b_l(T) \). To understand this effect, we note that in a general mixture \( q|\theta_+\rangle\langle\theta_+| + (1 - q)|\theta_-\rangle\langle\theta_-|, \ q \in [0, 1] \)

the concurrence \( C(q) \equiv C_l(\rho_q) \) is

\[
C(q) = |1 - q/q_c|C_\pm, \qquad q_c = \frac{1}{2}(1 + \cos\theta),
\]

which generalizes Eq. (12) (recovered for \( q = 1/2 \), \( C(q) \) is antiparallel (parallel) for \( q < q_c \) (\( > q_c \)) and zero at \( q = q_c = 1/2 \), where \( \rho_q \) becomes completely separable \( (\rho_q = \frac{1}{2}|\theta\rangle\langle\theta| + |\theta\rangle\langle\theta|) \). Separability requires then a slightly greater weight in \( |\theta_+\rangle \) due to its lower concurrence. Hence, at low \( T > 0 \) \( C_l(T) \) vanishes and changes from antiparallel to parallel at a slightly larger field \( b_l(T) > b_s \), where the positive parity GS has a higher weight in the thermal mixture. In case II this entails the surprising result that in the narrow interval \( b_s < b < b_l(T) \), the thermal concurrence \( C_l(T) \) will increase with increasing \( l \), since it is still driven by the lowest odd state \( \forall \ l \).

Let us finally mention that for any chain partition \( (L, n - L) \), the Schmidt number of the states (8) is 2. Their Schmidt decomposition (1) is

\[
|\theta_\pm\rangle = \sqrt{p_{L\pm}^\theta}|\theta_\pm^L\rangle|\theta_\mp^L\rangle + \sqrt{p_{L\pm}^{-\theta}}|\theta_\pm^{-L}\rangle|\theta_\mp^{-L}\rangle, \quad \theta_{L\pm}^\theta = \frac{1 + \nu \cos\theta}{2(1 + \cos\theta)}, \quad \nu = \pm (19)
\]

where \( |\theta_\pm^L\rangle = (|\theta^L\rangle \pm |\theta_{-L}\rangle)/\sqrt{2(1 + \cos\theta)} \) denotes the analogous fixed parity states for \( L \) spins and \( p_{L\pm}^\theta \) the eigenvalues of the ensuing reduced density. The entanglement between \( L \) and \( n - L \) spins can be measured.
just an increasing function of \( (\text{square root of the tangle for a pure state}) \), which is \( \theta \) where we have replaced \( \cos \theta = \sqrt{\chi} \). Hence, \( C_L^\pm \geq C_L^\pm \), with \( C_L^\pm \) increasing functions of \( \theta \), i.e., decreasing functions of \( \chi \), in contrast with the pairwise concurrences \( C_L \). For \( \theta \to \pi/2 \) (\( \chi \to 0 \)), \( C_L^\pm \to 1 \) (GHZ limit), whereas for \( \theta \to 0 \) (\( \chi \to 1 \)), \( C_L^\pm \to 0 \) but

\[
C_L^\pm \to 2\sqrt{L(n-L)/n},
\]

(W-state limit), in which case \( S_L^x \approx (L/n)(1-\log_2(L/n)) \) for \( L \ll n \). Thus, within the bounds imposed by a Schmidt number 2, the behavior of \( S_L^x \) and \( C_L^\pm \) with \( L \) is “non-critical” (i.e. saturated) for low \( \chi \) (large \( \delta \)) and “critical” (non-saturated) for \( \chi \to 1 \) (low \( \delta \)) and negative parity. It is also verified that \( C_L^\pm \geq \sqrt{n-1}C_L^\pm \) (in agreement with the general inequality \( C_L^2 \geq \sum_{i=1}^{n-1} C_i^2 \) [13]), saturation reached for \( \theta \to 0 \), where \( C_L^\pm/C_L^\pm \approx \sqrt{n-1}[1+1/\theta^2(n-2)] \).

In summary, we have shown that due to the \( S_2 \) parity conservation, the GS of finite cyclic chains with attractive couplings of the form (11) remains entangled as the factoring field \( b_s \) is approached, undergoing at \( b_s \) the last parity transition and exhibiting for \( b \to b_s^\pm \) universal entanglement properties, “intermediate” between those of GHZ and W-states. This field plays thus the role of a “quantum critical field” for small chains, with the pairwise concurrence reaching infinite range and approaching distinct side limits which are independent of the pair separation and interaction range. Their average is directly measurable through the GS magnetization jump [10], which provides then a signature of the present effects, while their difference determines the concurrence of the GS mixture [12]. These effects remain appreciable for increasing \( n \) if the anisotropy becomes sufficiently small (finite \( \delta \)), i.e., for chains close to the XXZ limit. Moreover, within a fixed parity subspace (and also at sufficiently low \( T > 0 \)), \( b_s \) is singled out as the field where all pairwise concurrences cross, the ordering with separation becoming inverted as \( b \) crosses \( b_s \). Type, range and even ordering of the pairwise entanglement can thus be controlled by tuning the field around \( b_s \).

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