THE POLES AND RESIDUES OF EISENSTEIN SERIES
INDUCED FROM SPEH REPRESENTATIONS

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ABSTRACT. We consider Eisenstein series, on split special orthogonal groups, symplectic groups, or their double covers, induced from Speh representations. Except the metaplectic groups case, their poles were determined by Jiang, Liu, Zhang [JLZ13]. We give another proof for the existence of these poles, which is straightforward and works for double covers of symplectic groups, as well. In case of symplectic groups, or their double covers, we use the same proof to show that for each pole, there is a unique maximal nilpotent orbit, attached to Fourier coefficients admitted by the corresponding residual representation. We find this orbit in each case.

1. Introduction

Let \( \tau \) be an irreducible, automorphic, cuspidal representation of \( \text{GL}_n(\mathbb{A}) \), where \( \mathbb{A} \) is the adele ring of a number field \( F \). Let \( \Delta(\tau, m) \) (\( m \), a positive integer) denote the Speh representation of \( \text{GL}_{mn}(\mathbb{A}) \), attached to \( \tau \). See [MWS99]. This is the representation spanned by the (multi-) residues of Eisenstein series corresponding to the parabolic induction from

\[
\tau|\det|^{s_1} \times \tau|\det|^{s_2} \times \cdots \times \tau|\det|^{s_m},
\]

at the point

\[
\left(\frac{m-1}{2}, \frac{m-3}{2}, \ldots, \frac{m-1}{2}\right).
\]

In this paper, we assume that \( \tau \) is self-dual, and consider Eisenstein series, induced from \( \Delta(\tau, m) \), on symplectic groups, their double covers, or on split special orthogonal groups \( H_{2mn+1}(\mathbb{A}) \) (\( d_0 = 0, 1 \)). More precisely, for \( d_0 = 0 \), let \( H_{2mn} \) denote one of the (algebraic) groups \( \text{Sp}_{2mn} \), or (split) \( \text{SO}_{2mn} \). For \( d_0 = 1 \), let \( H_{2mn+1} = \text{SO}_{2mn+1} \) (split). Denote \( H = H_{2mn+d_0} \). We will write these groups as matrix groups in a standard form, so that the standard Borel subgroups will consist of upper triangular matrices. Let \( Q_{mn} \) be the standard parabolic subgroup of \( H_{2mn+d_0} \), with Levi part isomorphic to \( \text{GL}_{mn} \). In the linear case, let \( f_{\Delta(\tau,m),s} \) be a smooth, holomorphic section of

\[
\rho_{\Delta(\tau,m),s} = \text{Ind}_{Q_{mn}(\mathbb{A})}^{H(\mathbb{A})} \Delta(\tau,m)|\det|^s.
\]

We denote the corresponding Eisenstein series by \( E(f_{\Delta(\tau,m),s}) \). In [JLZ13], Theorem 5.2, the possible poles of the normalized Eisenstein series \( E^*(f_{\Delta(\tau,m),s}) \) in \( \Re(s) \geq 0 \) are determined. Theorem 6.2 in loc. cit. states that these are indeed poles. For example, when \( H = \text{Sp}_{2mn} \), \( L(\tau,\wedge^2, s) \) has a pole at \( s = 1 \), and

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Let \( L(\tau, \frac{1}{2}) \neq 0 \), denote this set by \( \Lambda^{H}_{\tau, \wedge^2, m} \). Then \( \Lambda^{H}_{\tau, \wedge^2, m} \) consists of the elements

\[ e_{k, m}^{H}(\wedge^2) = k, \; k = 1, 2, \ldots, \frac{m}{2}, \; m \text{ even}, \]

\[ e_{k, m}^{H}(\wedge^2) = k - \frac{1}{2}, \; k = 1, 2, \ldots, \frac{m + 1}{2}, \; m \text{ odd}. \]

Note that

\[ 0 < e_{k, m}^{Sp_{2mn}}(\wedge^2) = \frac{m}{2}, \frac{m - 2}{2}, \frac{m - 4}{2}, \ldots \]

In the case of double covers of symplectic groups, we consider the similar Eisenstein series \( E(f_{\Delta(\tau, m) \gamma_\psi, s}) \) on \( Sp_{2n}(\mathbb{A}) \), corresponding to

\[ (1.2) \]

\[ \rho_{\Delta(\tau, m) \gamma_\psi, s} = \text{Ind}_{\mathbb{Q}^{(2)}_{n}(\mathbb{A})}^{Sp_{2mn}(\mathbb{A})} \Delta(\tau, m) \gamma_\psi | \det |^s. \]

Here, \( \gamma_\psi \) is the Weil factor attached to a nontrivial character \( \psi \) of \( F \setminus \mathbb{A} \). The analogous list is the following set \( \Lambda^{H}_{\tau, \wedge^2, m} \) (now, denoting \( H = Sp_{2mn}^{(2)} \)):

\[ e_{k, m}^{H}(\wedge^2) = k - \frac{1}{2}, \; k = 1, 2, \ldots, \frac{m}{2}, \; m \text{ even}, \]

\[ e_{k, m}^{H}(\wedge^2) = k, \; k = 1, 2, \ldots, \frac{m - 1}{2}, \; m \text{ odd}. \]

Thus,

\[ 0 < e_{k, m}^{Sp_{2mn}^{(2)}}(\wedge^2) = \frac{m - 1}{2}, \frac{m - 3}{2}, \frac{m - 5}{2}, \ldots \]

In these cases, we have

**Theorem A:** Assume that \( L(\tau, \wedge^2, s) \) has a pole at \( s = 1 \), and \( L(\tau, \frac{1}{2}) \neq 0 \). Then each point of the set \( \Lambda^{Sp_{2mn}}_{\tau, \wedge^2, m} \) (resp. \( \Lambda^{Sp_{2mn}^{(2)}}_{\tau, \wedge^2, m} \)) is a simple pole of \( E(f_{\Delta(\tau, m), s}) \) (resp. \( E(f_{\Delta(\tau, m) \gamma_\psi, s}) \)), as the section varies.

This theorem was proved in [JLZ13, Theorem 6.2, for symplectic groups (and split special orthogonal groups)]. The proof there is by considering constant terms, and uses Arthur’s classification of the discrete spectrum and the behavior at \( s = 0 \) of normalized intertwining operators. Our proof is different and uses automorphic descent. Moreover, the same ideas of our proof lead us to the main result of this paper, namely the proof and determination of the unique maximal nilpotent orbits attached to Fourier coefficients admitted by the residual representations at each pole in \( \Lambda^{Sp_{2mn}}_{\tau, \wedge^2, m} \) (resp. \( \Lambda^{Sp_{2mn}^{(2)}}_{\tau, \wedge^2, m} \)). We prove the analog of Theorem A for special orthogonal groups, as well, and, similarly, when the symmetric square \( L \)-function \( L(\tau, \wedge^2, s) \) has a pole at \( s = 1 \). Of course, in each case there is a different list of poles.

Our main tool for proving Theorem A is the second identity proven in [GS18]. We review this identity in the various cases in the next section. In the examples above, it exhibits an Eisenstein series on \( Sp_{2n}^{(2)}(\mathbb{A}) \), \( E(f_{\Delta(\tau, i), \gamma_\psi, s}) \), as a descent from an Eisenstein series, \( E(\varphi_{\Delta(\tau, i+1), s}) \), on \( Sp_{2(i+1)n}(\mathbb{A}) \). Similarly, we can express an Eisenstein series on \( Sp_{2n}(\mathbb{A}) \), \( E(f_{\Delta(\tau, i), s}) \), as a descent from an Eisenstein series, \( E(\varphi_{\Delta(\tau, i+1) \gamma_\psi, s}) \), on \( Sp_{2(i+1)n}(\mathbb{A}) \). Now, the proof follows by induction on \( i \), the initial case being the existence of a pole at \( s = \frac{1}{2} \) for the Eisenstein series on
\(\text{Sp}_{2m}(A), E(f_{\Delta(\tau,m),s}),\) which can be proved directly, by considering constant terms. See Prop. \(2.1\). Thus, Theorem A is proved simultaneously for symplectic groups and metaplectic groups. This is why we need to require that \(L(\tau, \frac{1}{2}) \neq 0\), also when we deal with \(\text{Sp}_{2mn}(A)\), when \(L(\tau, \Lambda^2, s)\) has a pole at \(s = 1\). Note, that, in this case, when we compute the constant term of \(E(f_{\Delta(\tau,m)\gamma_0,s})\), along the Siegel radical, then the contribution of the intertwining operator, corresponding to the long Weyl element, is, up to a finite set of places \(S\), containing those at infinity, and outside which the section is unramified,

\[
\prod_{k=1}^{m-1} \frac{L^S(\tau, \Lambda^2, 2s - 2k + 1) L^S(\tau, \sqrt{2}, 2s - 2k + 1)}{L^S(\tau, \Lambda^2, 2s + 2k - 1) L^S(\tau, \sqrt{2}, 2s + 2k)} = m \text{ even};
\]

\[
\prod_{k=1}^{m-1} \frac{L^S(\tau, \Lambda^2, 2s - 2k + 1) L^S(\tau, \sqrt{2}, 2s - 2k + 1)}{L^S(\tau, \Lambda^2, 2s + 2k) L^S(\tau, \sqrt{2}, 2s + 2k - 1)} = m \text{ odd}.
\]

The maximal possible pole of the last product, in each case (when \(L(\tau, \Lambda^2, s)\) has a pole at \(s = 1\)) is at \(s = \frac{m-1}{2}\). But then, when \(m\) is even and \(k = \frac{m}{2}\), \(L^S(\tau, \sqrt{2}, 2s - 2k + 1) = L^S(\tau, \sqrt{2}, 2s + m)\) might vanish (and even to a high order) at \(s = \frac{m-1}{2}\). Similarly, when \(m\) is odd and \(k = \frac{m+1}{2}\), \(L^S(\tau, \sqrt{2}, 2s - 2k + 1) = L^S(\tau, \sqrt{2}, 2s - m + 1)\) might vanish at \(s = \frac{m-1}{2}\).

The proof of the analog of Theorem A is along the same lines when \(L(\tau, \sqrt{2}, s)\) has a pole \(s = 1\). Similarly, we prove it simultaneously for even orthogonal groups and odd orthogonal groups.

The main result of this paper is on the top nilpotent orbits attached to nontrivial Fourier coefficients admitted by the residual Eisenstein series at the above poles. We prove it for symplectic groups and metaplectic groups. For example, denote by \(\mathcal{E}_{\Delta(\tau,m),\Lambda^2,k}\) the representation of \(\text{Sp}_{2mn}(A)\) generated by the residues \(\text{Res}_{s=\tau,m} f_{\Delta(\tau,m),s} E(f_{\Delta(\tau,m),s})\) of the Eisenstein series above, on \(\text{Sp}_{2mn}(A)\). Similarly, we introduce the notations \(\mathcal{E}_{\Delta(\tau,m),\Lambda^2,k}^{(2)}\). Recall that partitions of \(2mn\), where each odd part appears with an even multiplicity, determine nilpotent orbits of the Lie algebra of \(\text{Sp}_{2mn}\) over the algebraic closure of \(F\), and these determine Fourier coefficients along unipotent subgroups. See [GRS03]. For an automorphic representation on a group \(H(A)\), as above, denote by \(\mathcal{O}(\pi)\) the set of maximal partitions corresponding to (nilpotent orbits, attached to) nontrivial Fourier coefficients admitted by \(\pi\). Then

**Theorem B**: With the same assumptions made in Theorem A, with \(H = \text{Sp}_{2mn}\), or \(\text{Sp}_{2mn}^{(2)}\),

\[
\mathcal{O}(\mathcal{E}_{\Delta(\tau,m),\Lambda^2,k}^{H}) = ((2n)^{m-2\ell_k,m(\Lambda^2)}, n^{4\ell_k,m(\Lambda^2)}).
\]

In detail, for \(m\) even, \(1 \leq k \leq \frac{m}{2}\),

\[
\mathcal{O}(\mathcal{E}_{\Delta(\tau,m),\Lambda^2,k}^{\text{Sp}_{2mn}}) = ((2n)^{m-2k}, n^{4k}).
\]

For \(m\) odd, \(1 \leq k \leq \frac{m+1}{2}\),

\[
\mathcal{O}(\mathcal{E}_{\Delta(\tau,m),\Lambda^2,k}^{\text{Sp}_{2mn}}) = ((2n)^{m-2k+1}, n^{4k-2}).
\]
For \( m \) even, \( 1 \leq k \leq \frac{m}{2} \),
\[
O(e_{\Delta(T(m,m)\gamma_m,\wedge^2,k)}^{Sp_{2m}^{(2)}}) = ((2n)^{m-2k+1}, n^{4k-2}).
\]
For \( m \) odd, \( 1 \leq k \leq \frac{m-1}{2} \),
\[
O(e_{\Delta(T(m,m)\gamma_m,\wedge^2,k)}^{Sp_{2m}^{(2)}}) = ((2n)^{m-2k}, n^{4k}).
\]

We prove a similar theorem when \( L(T,\chi^2,s) \) has a pole at \( s = 1 \). The proof should work similarly for orthogonal groups, but we are missing there the analog of Lemma 6 in [GRS03], which we use repeatedly. Otherwise, the proof uses our second identity in [GS18] in the same way as for the proof of Theorem A. Of course, we need to prove separately the initial cases (Theorems 5.1, 5.2). In [JL16], some of the cases of Theorem B are proved, only for \( k \) maximal.

**Notation**

For a positive integer \( k \), let \( w_k \) denote the \( k \times k \) permutation matrix which has 1 along the main anti-diagonal. For a field \( F' \), we write the symplectic group \( Sp_{2k}(F') \) as
\[
Sp_{2k}(F') = \{ g \in GL_{2k}(F') \mid g \left( \begin{array}{cc} -w_k & w_k \\
\end{array} \right) g = \left( \begin{array}{cc} -w_k & w_k \\
\end{array} \right) \}
\]
and the split special orthogonal group \( SO_k(F') \) as
\[
SO_k(F') = \{ g \in SL_{k}(F') \mid g w_k g = w_k \}.
\]
Similarly, we have the adele groups \( Sp_{2k}(A) \), \( SO_k(A) \), where \( A \) is the adele ring of the number field \( F \). For a place \( v \) of \( F \), where \( F_v \neq \mathbb{C} \), we write the metaplectic group \( Sp_{2k}^{(2)}(F_v) \) according to the Ranga Rao cocycle, corresponding to the standard Siegel parabolic subgroup [RR93]. See [GS18], Sec. 1.1.

It will be convenient to denote in general \( H^{(1)}_{\ell}(F') \) any one of the linear groups \( Sp_{\ell}(F') \) (\( \ell \) even), or \( SO_{\ell}(F') \), and, for \( \ell \) even, \( H_{\ell}^{(2)}(F_v) = Sp_{2}^{(2)}(F_v) \). For \( H = H_{\ell}(\epsilon) \), \( \epsilon = 1, 2 \), denote \( \delta_H = -1 \), when \( H \) is symplectic, and \( \delta_H = -1 \), when \( H \) is orthogonal. For the rest of this introduction, we will drop \( F' \) and will consider these groups as algebraic groups over any given field.

Let \( r \leq \left[ \frac{d}{2} \right] \), and let \((r_1, \ldots, r_\ell)\) be a partition of \( r \). We denote by \( Q_{r_1,\ldots, r_\ell} \) the standard parabolic subgroup of \( H_\ell \), whose Levi part, \( M_{r_1,\ldots,r_\ell} \), is isomorphic to \( GL_{r_1} \times \cdots \times GL_{r_\ell} \times H_{\ell-2r} \). We will denote its unipotent radical by \( U_{r_1,\ldots, r_\ell} \). The group \( H_\ell \) will usually be clear from the context. If not, then we denote \( Q_{r_1,\ldots,r_\ell}^{H_{\ell}}, M_{r_1,\ldots,r_\ell}^{H_{\ell}}, U_{r_1,\ldots,r_\ell}^{H_{\ell}} \). Similarly, in \( GL_N \), for a partition \((r_1, \ldots, r_\ell)\) of \( N \), we denote the corresponding standard parabolic subgroup of \( GL_N \) by \( P_{r_1,\ldots, r_\ell} \). We denote its Levi part and unipotent radical by \( L_{r_1,\ldots, r_\ell}, V_{r_1,\ldots, r_\ell} \). We will denote, \( Z_n = V_1 \). This is the standard maximal unipotent subgroup of \( GL_n \).

For a matrix \( a \) in \( GL_r \), \( r \leq \left[ \frac{d}{2} \right] \), we will denote
\[
(1.4) \quad \tilde{a} = \text{diag}(a, I_{\ell-2r}, a^*) \in H_\ell,
\]
where \( a^* = w_{r}^{\frac{1}{2}} a^{-1} w_r \).

We fix a nontrivial character \( \psi \) of \( F \setminus A \). We will use one notation, \( \rho_{\Delta(T,m)\gamma_m^*,s} \), for the representations (1.1), (1.2), where \( \epsilon = 1, 2 \). In the cases (1.1), \( \epsilon = 1 \) and
Consider the attached Eisenstein series $E$. Thus, either each element of the list by $e, L$ stage, this is just a set of points.

\[ \psi_{Z_n}(z) = \psi(z_{1,2} + z_{2,3} + \cdots + z_{n-1,n}). \] (1.5)

2. Preliminaries and statement of the main theorems

1. The set of possible poles of the Eisenstein series $E(f_{\Delta(\tau,m)_{\gamma_{\psi}^{(*)}},s})$

We fix a self-dual, cuspidal representation $\tau$ of $\text{GL}_n(\mathbb{A})$, and a positive integer $m$. Thus, either $L(\tau, \wedge^2, s)$ has a pole at $s = 1$, or $L(\tau, \vee^2, s)$ has a pole at $s = 1$. Note that in the first case $n$ must be even, and the central character of $\tau$, $\omega_\tau$, must be trivial. In the second case, $n$ can be any positive integer and $\omega_\tau^2 = 1$. We will assume that $\tau$ is not the trivial character of $\text{GL}_1(\mathbb{A})$. Let $f_{\Delta(\tau,m)_{\gamma_{\psi}^{(*)}},s}$ denote a smooth holomorphic section of the representation $(1.3)$, or $(1, \overline{2})$, $\rho_{\Delta(\tau,m)_{\gamma_{\psi}^{(*)}},s}$.

Consider the attached Eisenstein series $E(f_{\Delta(\tau,m)_{\gamma_{\psi}^{(*)}},s})$ on $H(\mathbb{A}) = H_{2mn+d_0}(\mathbb{A})$ ($d_0 = 0, 1$). In [JLZ13], Theorem 6.2, the poles of the normalized Eisenstein series $E^*(f_{\Delta(\tau,m)_{\gamma_{\psi}^{(*)}},s})$ in $Re(s) \geq 0$ are determined. We now recall this list. We denote each element of the list by $e_{k,m}(\wedge^2)$, or $e_{k,m}(\vee^2)$. We include the case of metaplectic groups which does not appear in [JLZ13]. In this case, we simply form the set of poles of each partial L-function which appears in the numerator of $(1.3)$. At this stage, this is just a set of points.

Case $\wedge^2$: Assume that $L(\tau, \wedge^2, s)$ has a pole at $s = 1$, and in case $H = \text{Sp}_{2mn}$, assume further that $L(\tau, \frac{1}{2}) \neq 0$.

\[
eq_{k,m}(\wedge^2) = e_{k,m}(\wedge^2) = \frac{m-2k+2}{2} = \\
\begin{cases} 
  k, & k = 1, 2, \ldots, \frac{m}{2}, \text{ m even}, \\
  k - \frac{1}{2}, & k = 1, 2, \ldots, \frac{m+1}{2}, \text{ m odd}. \quad (1 \leq k \leq \lfloor \frac{m+1}{2} \rfloor)
\end{cases}
\]

Case $\vee^2$: Assume that $L(\tau, \vee^2, s)$ has a pole at $s = 1$.

\[
eq_{k,m}(\vee^2) = e_{k,m}(\vee^2) = \frac{m-2k+1}{2} = \\
\begin{cases} 
  k - \frac{1}{2}, & k = 1, 2, \ldots, \frac{m}{2}, \text{ m even}, \\
  k, & k = 1, 2, \ldots, \frac{m+1}{2}, \text{ m odd}. \quad (1 \leq k \leq \lfloor \frac{m}{2} \rfloor)
\end{cases}
\]

\[
eq_{k,m}(\vee^2) = e_{k,m}(\vee^2) = \frac{m+2k+2}{2} = \\
\begin{cases} 
  k, & k = 1, 2, \ldots, \frac{m}{2}, \text{ m even}, \\
  k - \frac{1}{2}, & k = 1, 2, \ldots, \frac{m+1}{2}, \text{ m odd}. \quad (1 \leq k \leq \lfloor \frac{m+1}{2} \rfloor)
\end{cases}
\]

Let $\eta$ be either $\wedge^2$ or $\vee^2$. Denote by $\Lambda_{\tau,\eta,m}^H$ the set of points $e_{k,m}(\eta)$ listed above, in each case. Denote the normalizing factor of the Eisenstein series above by $D_H^\tau(s)$. It is easy to check that $D_H^\tau(s)$ is holomorphic and nonzero at each $e_{k,m}(\eta)$, and so we may replace $E^*(f_{\Delta(\tau,m)_{\gamma_{\psi}^{(*)}},s})$ by $E(f_{\Delta(\tau,m)_{\gamma_{\psi}^{(*)}},s})$. 

\[
\]
Proposition 2.1. Assume that \( L(\tau, \eta, s) \) has a pole at \( s = 1 \). The Eisenstein \( E(f(M(m,m), s)) \), on \( H(\mathfrak{A}) \), has a simple pole at \( s = \frac{m}{2} \), as the section varies, in the following cases:

1. \( \eta = \wedge^2; \) \( H = \text{Sp}_{2mn} \ (L(\tau, \frac{1}{2}) \neq 0) \), \( SO_{2mn} \);
2. \( \eta = \vee^2; \) \( H = \text{Sp}_{2mn}^{(2)} \), \( SO_{2mn+1} \).

Proof. The proof is straightforward, by examining the constant term along the radical \( U^H_{mn} \), and showing that it has a pole at \( s = \frac{m}{2} \). For simplicity of notation, we assume that \( H \) is linear. The modifications in the metaplectic case are easy to make. We have, for \( Re(s) \) sufficiently large, the standard expression of the constant term, along \( U_{mn} \),

\[
E^{U_{mn}}(f(M(m,m), s))(I) = \sum_{w \in Q_{mn}(F)/Q_{mn}(F)} \sum_{\gamma \in M_{mn}(\mathbb{A}) \setminus M_{mn}(F)} \int_{U_{mn}(\mathbb{A}) \setminus U_{mn}(F)} f_\Delta(\tau, \eta, s)(wu\gamma)du,
\]

where \( M_{mn} = M_{mn} \cap w^{-1}Q_{mn}w \) and \( U_{mn} = U_{mn} \cap w^{-1}Q_{mn}w \). We can choose the following representatives \( w = \epsilon_r, 0 \leq r \leq mn \),

\[
\epsilon_r = \omega_0^{mn-r} \left( \begin{array}{cc} I_r & 0 \\ \delta H I_{mn-r} & I_r \end{array} \right).
\]

Here, \( \omega_0 = I_{2mn} \) when \( H = \text{Sp}_{2mn}^{(2)} \); \( \omega_0 = -I_{2mn+1} \), when \( H = SO_{2mn+1} \); \( \omega_0 = \text{diag}(I_{mn-1}, \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), I_{mn-1}) \), when \( H = SO_{2mn} \). (See [GRS11], p. 70.)

The calculation of \( M_{mn}^{(r)} \) and \( U_{mn}^{(r)} \) is done in [GRS11], Sec. 4.3. Thus, \( M_{mn}^{(r)} = P_{r,mn-r}(F) \setminus \text{GL}_{mn}(F) \). Factoring the \( du \)-integration in (2.1), for each \( r \), through \( U_{mn}^{(r)}(F) \setminus U_{mn}^{(r)}(\mathbb{A}) \), and conjugating the elements of \( U_{mn}^{(r)}(\mathbb{A}) \) by \( \epsilon_r \), the corresponding \( du \)-integral, contains the following inner integral, for given \( r, u \in U_{mn}^{(r)}(\mathbb{A}) \), \( \gamma \in P_{r,mn-r}(F) \setminus \text{GL}_{mn}(F) \),

\[
\int_{U_{mn}^{(r)}(\mathbb{A}) \setminus U_{mn}^{(r)}(F)} f_\Delta(\tau, \eta, s)(\epsilon_r, u \gamma)du.
\]

This is an application of the constant term along \( V_{r,mn-r} \) applied to an element of \( \Delta(\tau, m) \). For this to be nonzero, \( r \) must be a multiple of \( n \). This follows from the cuspidality of \( \tau \). Thus in (2.1), only \( r = jn, 0 \leq j \leq m \) contribute. We get

\[
E^{U_{mn}}(f(M(m,m), s))(I) = \sum_{j=0}^{m} \sum_{\gamma} \int_{S_{(m-j)n}^{(r)}(\mathbb{A}) \setminus S_{(m-j)n}^{(r)}(F)} f_\Delta(\tau, \eta, s)(\epsilon_jn \left( \begin{array}{cc} I_{jn} & u_{(m-j)n}(z) \\ \\ \end{array} \right) \gamma)du.
\]

The inner sum is over \( \gamma \in P_{j(n,m-j)n}(F) \setminus \text{GL}_{mn}(F) \), and \( S_{(m-j)n}^{(r)} \) consists of the \( (m-j)n \times (m-j)n \) matrices \( z \) satisfying \( t(w_{(m-j)n}z) = -\delta_H(w_{(m-j)n}z) \). Note that, by our assumptions, in case \( H \) is even orthogonal, \( n \) must be even, and then \( \omega_0^{(m-j)n} = I_{2mn} \).
Assume that the section \( f_{\Delta (\tau, m), s} \) is decomposable, and fix a finite set of places \( S \), containing the Archimedean places, such that outside \( S \), it is unramified (as well as \( \tau \)). Fix a place \( v_0 \in S \), and assume that the local section at \( v_0 \) is supported inside the open cell \( Q_{mn} (F_v) \epsilon_v U_{mn} (F_v) \). For such sections, only one summand in \( j \) remains in (2.23), namely the one with \( j = 0 \), which is the intertwining operator on \( \rho_{\Delta (\tau, m), s} \), corresponding to \( \epsilon_v \).

(2.4) \[ E_{mn} (f_{\Delta (\tau, m), s}) (I) = M (\epsilon_v, s) (f_{\Delta (\tau, m), s}) = \int_{U_{mn} (k)} f_{\Delta (\tau, m), s} (\epsilon_v u) du. \]

Since we know how to compute this intertwining operator locally on unramified sections \( f_{\Delta (\tau, m), s} \), we can directly verify that \( M (\epsilon_v, s) (f_{\Delta (\tau, m), s}) \) has a pole at \( s = \frac{m}{2} \), and hence the Eisenstein series \( E (f_{\Delta (\tau, m), s}) \) has a pole at \( s = \frac{m}{2} \). In more details, let us realize \( \Delta (\tau, m) \) in its (local Whittaker-Speh-Shalika) model with respect to \( (V_{\tau} (F_v), \psi_{\tau} (F_v)) \) (the character \( \psi_{\tau} (F_v) \) is written right after (2.20)). See [CFK18], Theorem 3. It has a unique, unramified function \( W_{\Delta (\tau, m), s} \), taking the value 1 on \( I_{mn} \). Think of \( f_{\Delta (\tau, m), s} \) as a function \( f_{\Delta (\tau, m), s} (h, g) \) on \( H (F_v) \times GL_{mn} (F_v) \), and assume that \( f_{\Delta (\tau, m), s} (f, g) = W_{\Delta (\tau, m), s} (g) \). Note that since \( \tau \) is self-dual, \( \Delta (\tau, m) \) is self-dual. We have

(2.5) \[ M (\epsilon_v, s) (f_{\Delta (\tau, m), s}) = a (\tau, s) f_{\Delta (\tau, m), -s}. \]

Let us write the example where \( H \) symmetric.

(1) For \( H = Sp_{2mn}, m \) even,

\[ a (\tau, s) = \frac{L (\tau, s + \frac{1 - m}{2})} {L (\tau, s + \frac{1 - m}{2})} \prod_{k = 1}^{\frac{m}{2}} \frac{L (\tau, 2s + 2k - 1) L (\tau, 2s + 2k + 1)} {L (\tau, 2s + 2k) L (\tau, 2s + 2k - 2)}. \]

(2) For \( H = Sp_{2mn}, m \) odd,

\[ a (\tau, s) = \frac{L (\tau, s + \frac{1 - m}{2})} {L (\tau, s + \frac{1 - m}{2})} \prod_{k = 1}^{\frac{m}{2}} \frac{L (\tau, 2s + 2k + 1)} {L (\tau, 2s + 2k - 1)} \prod_{k = 1}^{\frac{m - 1}{2}} \frac{L (\tau, 2s + 2k + 1)} {L (\tau, 2s + 2k - 1)}. \]

We see immediately that \( a^S (\tau, s) = \prod_{v \notin S} a (\tau, s) \) has a pole at \( s = \frac{m}{2} \), when we assume that \( L (\tau, 2s + 1) \) has a pole at \( s = 1 \). Recall that in this case we assume that \( L (\tau, \frac{1}{2}) \neq 0 \). The local intertwining operators inside \( S \) can be made holomorphic and nonzero by choosing appropriate local sections at the places of \( S \). All in all, \( M (\epsilon_v, s) (f_{\Delta (\tau, m), s}) \) has a pole at \( s = \frac{m}{2} \), for a good choice of \( f_{\Delta (\tau, m), s} \). The other cases of the proposition are obtained in the same way.

\[ \square \]

Remark: Our argument fails if we want to show that, if \( L (\tau, 2s + 1) \) has a pole at \( s = 1 \), then \( E (f_{\Delta (\tau, m), s}) \), on \( Sp_{2mn} (k) \), has a pole at \( s = \frac{m - 1}{2} \). Indeed, consider \( a^S (\tau, s) \) in the last proof. For example, when \( m \) is even, the factor \( L (\tau, \frac{1 - m}{2}) \) \( L (\tau, 2s + 2k + 1) \), when \( k = \frac{m - 1}{2} \), has a pole at \( s = \frac{m - 1}{2} \). All other \( L \)-functions appearing in the numerator don’t cancel this pole except, maybe, \( L (\tau, 2s + \frac{1 - m}{2}) L (\tau, 2s + 2k + 2) \), which might vanish, and even to a high order. Similarly, the argument fails for the maximal element of \( \Lambda_{\tau, \sqrt{2}, m}, \Lambda_{\tau, \sqrt{2}, m + 1}, \Lambda_{Sp_{2mn}} \).

In Sections 3, 4, we will prove

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Theorem 2.2. Each element in the set \( \Lambda^H_{τ, \eta, m} \) is a (simple) pole of the Eisenstein series \( E(f_{Δ(τ,m)}γ^ι_{ψ,s}) \), on \( H(𝔸) \), as the section varies. In the cases \( η = \lambda^2 \), \( H = \text{Sp}_{2mn} \), \( H = \text{Sp}_{2mn}^{(2)} \), we need to add the assumption that \( L(τ, \frac{1}{2}) \neq 0 \).

As we mentioned in the introduction, this theorem was proved in [JLZ 13], Theorem 6.2, except for metaplectic groups. See our remarks there.

Our main theorem is on the top nilpotent orbits of the residual Eisenstein series above, for symplectic, or metaplectic groups, at the various poles listed in Theorem 2.2. Let \( E^H_{Δ(τ,m)}γ^ι_{ψ,s} \) be the residual representation of \( H(𝔸) \), generated by the residues \( \text{Res}_{e=±1} E(f_{Δ(τ,m)}γ^ι_{ψ,s}) \), as the section varies, at the pole \( e^H_{Δ(τ,m)}(η) \in \Lambda^H_{τ, \eta, m} \). Let \( O' \) be a nilpotent orbit of the Lie algebra of \( H \) over \( F \). It corresponds to a partition \( P' \) of \( 2mn + d_0 \). (Recall that \( H = \text{Sp}_{2mn+d_0} \), \( d_0 = 0, 1 \).) Assume that \( E^H_{Δ(τ,m)}γ^ι_{ψ,s} \) admits a nontrivial Fourier coefficient corresponding to \( O' \). In [GS20], Prop. 3.1, 3.2, we bounded \( O' \) (or \( P' \)), in many cases. Denote by \( O(Δ(τ,m))γ^ι_{ψ,s} \) the set of maximal nilpotent orbits \( O \), supporting \( E^H_{Δ(τ,m)}γ^ι_{ψ,s} \), in the sense that \( E^H_{Δ(τ,m)}γ^ι_{ψ,s} \) admits a nontrivial Fourier coefficient corresponding to \( O \). Our main theorem, which will be proved in Sections 5, 6, is

**Theorem 2.3.** Let \( H = \text{Sp}_{2mn}^{(2)} \).

I. Assume that \( L(τ, \lambda^2, s) \) has a pole at \( s = 1 \), and \( L(τ, \frac{1}{2}) \neq 0 \). Then

\[
O(Δ(τ,m))γ^ι_{ψ,s} = ((2n)^{m-2e^H_{k,m}(\lambda^2)}, n^{2e^H_{k,m}(\lambda^2)}).
\]

II. Assume that \( L(τ, \sqrt{2}, s) \) has a pole at \( s = 1 \), and \( ω_τ = 1 \). Then

\[
O(Δ(τ,m))γ^ι_{ψ,s} =
\begin{cases}
((2n)^{m-2e^H_{k,m}(\sqrt{2})}, n^{2e^H_{k,m}(\sqrt{2})}), & n \text{ even} \\
((2n)^{m-2e^H_{k,m}(\sqrt{2})}, (n+1)^{2e^H_{k,m}(\sqrt{2})}, (n-1)^{2e^H_{k,m}(\sqrt{2})}), & n \text{ odd}.
\end{cases}
\]

III. Assume that \( L(τ, \sqrt{2}, s) \) has a pole at \( s = 1 \), and \( ω_τ \neq 1 \). Then

\[
O(Δ(τ,m))γ^ι_{ψ,s} =
\begin{cases}
((2n)^{m-2e^H_{k,m}(\sqrt{2})}, (n+2)^{2e^H_{k,m}(\sqrt{2})}, (n-2)^{2e^H_{k,m}(\sqrt{2})}), & n \text{ even} \\
((2n)^{m-2e^H_{k,m}(\sqrt{2})}, (n+1)^{2e^H_{k,m}(\sqrt{2})}, (n-1)^{2e^H_{k,m}(\sqrt{2})}), & n \text{ odd}.
\end{cases}
\]
2. Descent of Eisenstein series induced from Speh representations

Our main tool of proof is the second inentity in [GS18]. This identity roughly says that an appropriate descent of an Eisenstein series induced from $\Delta(\tau, i + 1)$ is an Eisenstein series induced from $\Delta(\tau, i)$. This descent goes from $\text{Sp}_{2n(i+1)}(A)$ to $\text{Sp}_{2n(i+1)}(A)$, from $\text{SO}_{2n(i+1)}(A)$ to $\text{SO}_{2n(i+1)}(A)$, and from $\text{SO}_{2n(i+1)+1}(A)$ to $\text{SO}_{2n(i+1)}(A)$. Let us recall this identity, in each case, in detail. For this identity, $\tau$ need not be self-dual. We assume that it is unitary.

Let $H = H^{(i)}_{2n(i+1)+d_0}$. Consider the unipotent radical $U^H_{1n-1+d_0} = U^{H_{2n(i+1)+d_0}}_{1n-1+d_0}$. Write its elements as

\[(2.6)\quad u = \begin{pmatrix} z & x & y \\ I_{2n(i+1)+d_0} & x' & z' \\ 0 & 0 & z^* \end{pmatrix} \in H, \ z \in Z_{n-1+d_0}.\]

Recall that when $d_0 = 0$, $H_{2n(i+1)}$ is symplectic, or even orthogonal, and when $d_0 = 1$, $H_{2n(i+1)+1}$ is odd orthogonal. Consider the character $\psi_{n-1+d_0} = \psi_{n-1+d_0}$ of $U^{H_{2n(i+1)+d_0}}_{1n-1+d_0}(A)$ given by

\[(2.7)\quad \psi_{n-1+d_0}(u) = \psi_{z_{n-1+d_0}}(z) \psi_{x_{n-1+d_0}}(x) e(u),\]

where $\psi_{z_{n-1+d_0}}$ is the Whittaker character [1.5], and $e = e^{H_{2n(i+1)+d_0}}$ is the following column vector.

When $H_{2n(i+1)}$ is symplectic, $e = \begin{pmatrix} 1 \\ 0 \\ : \\ 0 \end{pmatrix} \in F^{2ni+2}$.

When $H_{2n(i+1)}$ is even orthogonal,

\[e = \begin{pmatrix} 0_n \\ 1 \\ : \\ 0_n \end{pmatrix}.\]

When $H_{2n(i+1)+1}$ is odd orthogonal,

\[e = \begin{pmatrix} 0_n \\ 1 \\ 0_n \end{pmatrix}.\]

We now write each identity in detail.

a. $H = \text{Sp}^{(i)}_{2n(i+1)}$

The descent here is via Fourier-Jacobi coefficients. We consider the character $\psi_{2n}$. This character is stabilized by the semi-direct product of $\text{Sp}_{2n}(A)$ and $t(H_{2n+1}(A))$, where $\text{Sp}_{2n}$ is realized inside $\text{Sp}_{2n+1}(A)$ by

\[t(h) = \text{diag}(I_n, h, I_n), h \in \text{Sp}_{2n},\]
and \( \mathcal{H}_{2ni+1} \) is the Heisenberge group in \( 2ni + 1 \) variables. It is realized inside \( H \) by

\[
(2.8) \quad t((x, e)) = \text{diag}(I_{n-1}, \begin{pmatrix} 1 & x & e \\ I_{2ni} & x' & 1 \end{pmatrix}), I_{n-1}) \in H.
\]

We have the projection \( \beta \) from \( U_{1n} = U_{1n-1} \times t(\mathcal{H}_{2ni+1}) \) onto \( \mathcal{H}_{2ni+1} \). Extend the character (2.7) to \( U_{1n}(\mathbb{A}) \) by making it trivial on \( t(\mathcal{H}_{2ni+1}(\mathbb{A})) \). We continue to denote this extension by \( \psi_{n-1} \).

Let \( \omega_{\psi}^{-1} \) be the Weil representation of \( \mathcal{H}_{2ni+1}(\mathbb{A}) \times \text{Sp}^{(2)}_{2ni}(\mathbb{A}) \), associated to \( \psi^{-1} \), i.e. the elements \((0, z)\) of the center of \( \mathcal{H}_{2ni+1}(\mathbb{A}) \) act by multiplication by \( \psi^{-1}(z) \). We let \( \omega_{\psi} \) act on the space of Schwartz-Bruhat functions \( \mathcal{S}(\mathbb{A}^ni) \). For \( \phi \in \mathcal{S}(\mathbb{A}^ni) \), we have the corresponding theta series \( \theta^\phi_{\psi} \), viewed as a function on \( \mathcal{H}_{2ni+1}(\mathbb{A}) \times \text{Sp}^{(2)}_{2ni}(\mathbb{A}) \).

Let \( f_{\Delta(\tau,i+1)\gamma^{(s)},s} \) be a smooth, holomorphic section of \( \rho^H_{\Delta(\tau,i+1)\gamma^{(s)},s} \). Consider the corresponding Eisenstein series \( E(f_{\Delta(\tau,i+1)\gamma^{(s)},s}) \), and apply to it the following Fourier-Jacobi coefficient,

\[
D^\phi_{\psi,n}(E(f_{\Delta(\tau,i+1)\gamma^{(s)},s}))(h)
\]

\[
(2.9) = \int_{U_{1n}(F)\backslash U_{1n}(\mathbb{A})} E(f_{\Delta(\tau,i+1)\gamma^{(s)},s}, u\tilde{t}(h))\psi_{n-1}^{-1}(u)\theta^\phi_{\psi}^{-1}(\beta(u)\tilde{h})du.
\]

Here, \( h \in \text{Sp}^{(3-\epsilon)}_{2ni}(\mathbb{A}) \). When \( \epsilon = 1 \), and \( h \in \text{Sp}^{(2)}_{2ni}(\mathbb{A}) \) projects to \( h' \in \text{Sp}_{2ni}(\mathbb{A}) \), \( \tilde{t}(h) = t(h') \), and \( \tilde{h} = h \). When \( \epsilon = 2 \), \( h \in \text{Sp}_{2ni}(\mathbb{A}) \), \( \tilde{h} \) is any element of \( \text{Sp}^{(2)}_{2ni}(\mathbb{A}) \), which projects to \( h \), and \( t(h) \) projects to \( t(h) \), so that the projection of \( \tilde{h} \) on the second \pm 1 coordinate is the same as that of \( t(h) \). Recall that in the metaplectic case, unipotent subgroups split in the double cover. Thus, we identify \( U_{1n}^{\text{Sp}(2n+1)}(\mathbb{A}) \) as a subgroup of \( \text{Sp}^{(2)}_{2(i+1)n}(\mathbb{A}) \), when \( H = \text{Sp}^{(2)}_{2(i+1)n} \).

Let

\[
(2.10) \quad \alpha_0 = \begin{pmatrix} 0 & I_{ni} & 0 & 0 \\ -I_n & 0 & 0 & 0 \\ 0 & 0 & I_{ni} & 0 \end{pmatrix}.
\]

Denote

\[
(2.11) \quad U'_n = \{ u'_{x,y} = \begin{pmatrix} I_n & x & 0 & y \\ I_{ni} & 0 & 0 & 0 \\ I_{ni} & x' & I_n \end{pmatrix} \in H \}.
\]

Let, for \( g \in H(\mathbb{A}) \),

\[
(2.12) \quad f_{\Delta(\tau,i+1)\gamma^{(s)},s}(g) = \int_{V_{ni,1n}(F)\backslash V_{ni,1n}(\mathbb{A})} f_{\Delta(\tau,i+1)\gamma^{(s)},s}(\tilde{v}g)\psi_{V_{ni,1n}}(v)dv,
\]

where \( \psi_{V_{ni,1n}} \) is the character of \( V_{ni,1n}(\mathbb{A}) \) given by (see (1.5))

\[
(2.13) \quad \psi_{V_{ni,1n}}\left( \begin{pmatrix} I_{ni} & y \\ z & I_n \end{pmatrix} \right) = \psi_{Z_n}(z), \quad z \in Z_n(\mathbb{A}).
\]
Denote, for a finite set of places $S$, containing the Archimedean places, outside which $\tau$ is unramified,
\[
d_{\tau, S}^{\text{Sp}(2n+1)}(s) = L^S(\tau, s + j + 1) \prod_{k=1}^{j+1} L^S(\tau, \wedge^2, 2s + 2k - 1) \prod_{k=1}^{j} L^S(\tau, \text{sym}^2, 2s + 2k);
\]
\[
d_{\tau, S}^{\text{Sp}(4n+1)}(s) = L^S(\tau, s + j + \frac{1}{2}) \prod_{k=1}^{j} L^S(\tau, \wedge^2, 2s + 2k)L^S(\tau, \text{sym}^2, 2s + 2k - 1);
\]
\[
d_{\tau, S}^{\text{Sp}(2n+2)}(s) = \prod_{k=1}^{j} L^S(\tau, \wedge^2, 2s + 2k) \prod_{k=1}^{j+1} L^S(\tau, \text{sym}^2, 2s + 2k - 1);
\]
\[
d_{\tau, S}^{\text{Sp}(4n+2)}(s) = \prod_{k=1}^{j} L^S(\tau, \wedge^2, 2s + 2k - 1)L^S(\tau, \text{sym}^2, 2s + 2k);
\]

The following theorem is proved in [GS18], Theorems 7.1, 7.4, 8.1, 8.3.

**Theorem 2.4.** For $\text{Re}(s)$ sufficiently large, $h \in \text{Sp}^{(3-\epsilon)}(\mathbb{A})$, 
\[
D^\phi_{\psi, n_i}(E(f_{\Delta(\tau, i+1)}\gamma, s))(h) = \sum_{\gamma \in Q_{n_i}(\mathbb{A}) \backslash \text{Sp}_{2n_i}(\mathbb{A})} \Lambda(f_{\Delta(\tau, i+1)}\gamma, s, \phi)((\gamma, 1)h),
\]
where
\[
\Lambda(f_{\Delta(\tau, i+1)}\gamma, s, \phi)(h) = \int_{U_{\psi}(\mathbb{A})} \omega_{\psi^{-1}}(\beta(u)h)\phi(0)f_{\Delta(\tau, i+1)}\gamma, s, \phi(\alpha_0 u f(h))du.
\]

In the sum, $Q_{n_i} = Q^{\text{Sp}_{2n_i}}$. The function $\Lambda(f_{\Delta(\tau, i+1), s, \phi})$, defined for $\text{Re}(s)$ sufficiently large, by the last integral, admits analytic continuation to a meromorphic function of $s$ in the whole plane. It defines a smooth meromorphic section of
\[
(3-\epsilon) \Delta(\tau, i+1)\gamma, s, \phi)
\]

Thus, $D^\phi_{\psi, n_i}(E(f_{\Delta(\tau, i+1)}\gamma, s))$ is the Eisenstein series on $\text{Sp}^{(3-\epsilon)}(\mathbb{A})$, corresponding to the section $\Lambda(f_{\Delta(\tau, i+1)}\gamma, s, \phi)$ of $\rho_{(3-\epsilon)\Delta(\tau, i+1), s, \phi}$.

Let $S$ be a finite set of places, containing the Archimedean places, outside which $\tau$ is unramified. Denote by $E^S_*(\cdot)$ an Eisenstein series normalized outside $S$. Then
\[
D^\psi_{\omega, n_i}(E^S_{\psi, n_i}(f_{\Delta(\tau, i+1)}\gamma, s)) = E^S_*(\Lambda(d_{\tau, S}^{\text{Sp}(2n+1)}(s)f_{\Delta(\tau, i+1)}\gamma, s, \phi)).
\]

That is, $D^\phi_{\omega, n_i}(E^S_{\psi, n_i}(f_{\Delta(\tau, i+1)}\gamma, s))$, is the normalized (outside $S$) Eisenstein series on $\text{Sp}^{(3-\epsilon)}(\mathbb{A})$ corresponding to the section $\Lambda(d_{\tau, S}^{\text{Sp}(2n+1)}(s)f_{\Delta(\tau, i+1)}\gamma, s, \phi)$.

b. $H = \text{SO}_{2n+1}$

Here, the descent is via Bessel coefficients. We take the unipotent radical $U_{1n-1}$. The character $\psi_{n-1}$ is stabilized by $\text{SO}_{2n+1}(\mathbb{A})$, realized as the subgroup of elements $\text{diag}(I_{n-1}, h, I_{n-1})$, with $h \in \text{SO}_{2n+2}(\mathbb{A})$ satisfying $h \cdot e = e$. Let $j$ denote
the isomorphism from $\text{SO}_{2n+1}$ to this stabilizer. Denote

$$t(h) = \begin{pmatrix} I_{n-1} & j(h) \\ I_{n-1} & I_{n-1} \end{pmatrix}, \quad h \in \text{SO}_{2n+1}.$$ \hspace{1cm} (2.15)

Let $f_{\Delta(\tau,i+1),s}$ be a smooth, holomorphic section of $\rho_{\Delta(\tau,i+1),s}$. Consider the corresponding Eisenstein series $E(f_{\Delta(\tau,i+1),s})$, and take its Fourier coefficient along $U_{n-1}$, with respect to $\psi_{n-1}$,

$$\mathcal{D}_{\psi,n}(E(f_{\Delta(\tau,i+1),s}))(h) = \int_{U_{n-1}(F) \backslash U_{n-1}(\mathbb{A})} E(f_{\Delta(\tau,i+1),s}, u \tau(h)) \psi_{n-1}^{-1}(u) du. \hspace{1cm} (2.16)$$

Here, $h \in \text{SO}_{2n+1}(\mathbb{A})$. Let

$$U'_{n-1} = \{ u : u = \begin{pmatrix} I_{n-1} & x \\ I_{n-1} & I_{n-1} \end{pmatrix}, \quad x \in \text{sym}^2(\mathbb{A}) \}.$$ \hspace{1cm} (2.16)

$$\delta_0 = \begin{pmatrix} 0 & I_{n-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \omega_0^{n-1};$$ \hspace{1cm} (2.16)

For $g \in H(\mathbb{A})$, let (see \text{[2.13]})

$$f_{\Delta(\tau,i+1),s}^g(\mathbb{A}) = \int_{V_{n-1}(F) \backslash V_{n-1}(\mathbb{A})} f_{\Delta(\tau,i+1),s}(\mathbb{A}g) \psi_{n-1}^{-1}(v) dv. \hspace{1cm} (2.16)$$

Denote (S as in the last theorem)

$$d_{\tau_{2n+2}}^{\text{SO}_{2n+2},S}(s) = \prod_{k=1}^{j+1} L^S(\tau, \wedge^2, 2s + 2k - 1) \prod_{k=1}^{j} L^S(\tau, \text{sym}^2, 2s + 2k);$$

$$d_{\tau_{2n+2}}^{\text{SO}_{2n+2},S}(s) = \prod_{k=1}^{j} L^S(\tau, \wedge^2, 2s + 2k) L^S(\tau, \text{sym}^2, 2s + 2k - 1).$$

The following theorem is Theorem 5.1 in \text{[GS18]},

**Theorem 2.5.** Let $f_{\Delta(\tau,i+1),s}$ be a smooth, holomorphic section of $\rho_{\Delta(\tau,i+1),s}$. Then, for $\text{Re}(s)$ sufficiently large and $h \in \text{SO}_{2n+1}(\mathbb{A})$,

$$\mathcal{D}_{\psi,n}(E(f_{\Delta(\tau,i+1),s}))(h) = \sum_{h' \in Q_n(F) \backslash \text{SO}_{n+1}(F)} \Lambda(f_{\Delta(\tau,i+1),s})(h'h),$$

where

$$\Lambda(f_{\Delta(\tau,i+1),s})(h) = \int_{U_{n-1}(\mathbb{A})} f_{\Delta(\tau,i+1),s}^g(\mathbb{A}h') \psi_{n-1}^{-1}(u) du dg. \hspace{1cm} (2.16)$$

The function $\Lambda(f_{\Delta(\tau,i+1),s})$, defined for $\text{Re}(s)$ sufficiently large by the last integral, is smooth and admits an analytic continuation to a meromorphic function of $s$ in the whole plane, and defines a smooth meromorphic section of

$$\rho_{\Delta(\tau,i),s} = \text{Ind}_{Q_n(\mathbb{A})}^{\text{SO}_{2n+1}(\mathbb{A})} \Delta(\tau, i) | \det \cdot |^s.$$
Thus, $D_{\psi,n_i}(E(f_{\Delta(\tau,i+1),s}))(h)$ is the Eisenstein series on $SO_{2n_i+1}(\mathbb{A})$, corresponding to the section $\Lambda(f_{\Delta(\tau,i+1),s})$ of $\rho_{\Delta(\tau,i),s}$. Moreover, when we normalize (outside $S$, as above) $E(f_{\Delta(\tau,i+1),s})$ by

$$E_{S}^{*}(f_{\Delta(\tau,i+1),s}) = d_{r}^{SO_{2(n+1);S}(s)}E(f_{\Delta(\tau,i+1),s}),$$

then $D_{\psi,n_i}(E_{S}^{*}(f_{\Delta(\tau,i+1),s}))$ is an Eisenstein series on $SO_{2n_i+1}(\mathbb{A})$, corresponding to $\rho_{\Delta(\tau,i),s}$, and it is normalized outside $S$.

c. $H = SO_{2n(i+1)+1}$

Here the character $\psi_n$ of $U_{1^n}(\mathbb{A})$ is stabilized by $SO_{2n_i}(\mathbb{A})$, where $SO_{2n_i}$ is realized as the subgroup of elements $diag(I_{n_i},h,I_{n_i})$, with $h \in SO_{2n_i+1}$, satisfying

$$h \begin{pmatrix} 0_{n_i} \\ 1 \\ 0_{n_i} \end{pmatrix} = \begin{pmatrix} 0_{n_i} \\ 1 \\ 0_{n_i} \end{pmatrix}.$$  

The isomorphism of this stabilizer and $SO_{2n_i}$ is given by

$$(2.17) \quad j(a \ b \ c \ d) = \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{pmatrix},$$

where $a, b, c, d$ are $n_i \times n_i$ matrices and $(a \ b \ c \ d) \in SO_{2n_i}$. For $h \in SO_{2n_i}$, denote

$$(2.18) \quad t(h) = \begin{pmatrix} I_{n_i} & j(h) \\ & I_{n_i} \end{pmatrix}.$$  

Let $f_{\Delta(\tau,i+1),s}$ be a smooth, holomorphic section of $\rho_{\Delta(\tau,i+1),s}$. Consider the corresponding Eisenstein series $E(f_{\Delta(\tau,i+1),s})$, and apply to it the Fourier coefficient along $U_{1^n}$ with respect to the character $\psi_n$,

$$(2.19) \quad D_{\psi,n_i}(E(f_{\Delta(\tau,i+1),s}))(h) = \int_{U_{1^n}(\mathbb{F}) \backslash U_{1^n}(\mathbb{A})} E(f_{\Delta(\tau,i+1),s},ut(h))\psi_{n}^{-1}(u)du.$$

It turns out that $2.19$ is a sum of two Eisenstein series on $SO_{2n_i}(\mathbb{A})$, with the second term obtained from the first by an outer conjugation. We need some more notations. Let

$$(2.20) \quad \alpha_0 = \begin{pmatrix} 0 & I_{n_i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{n_i} \\ 0 & 0 & (-1)^n & 0 & 0 \\ I_{n_i} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n_i} & 0 \end{pmatrix};$$

Denote

$$(2.21) \quad U_{n}' = \{u_{x_1,x_2:y} = \begin{pmatrix} I_{n_i} & x_1 & x_2 & 0 & y \\ x_1 & I_{n_i} & 0 & 0 \\ 0 & 1 & x'_1 & 0 \\ y & x'_1 & I_{n_i} & x_1' \\ 0 & 0 & x'_2 & I_{n_i} \end{pmatrix} \in H \};$$
\( \omega_0 = \text{diag}(I_{n+i+1}, -1, I_{n+i+1}) \in \text{O}_{2n(i+1)+1}; \) \( \omega'_0 = \text{diag}(I_{n-i-1}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, I_{n-i-1}) \in \text{O}_{2n(i)} \).

We extend \( t \) in (2.18) to \( \text{O}_{2n(i)} \), so that \( t(\omega'_0) \in \text{O}_{2n(i+1)+1} \). Denote \( \omega''_0 = \omega_0 t(\omega'_0) \).

Let, for \( g \in H(\mathbb{A}) \) (see (2.13)),

\[
(2.23) \quad f^\psi_{\Delta(\tau,i+1),s}(g) = \int_{V_{n+1}^n(\mathbb{F}) \setminus V_{n+1}^n(\mathbb{A})} f_{\Delta(\tau,i+1),s}(\hat{v}g) \psi_{n,1}(v) dv.
\]

Denote (\( S \) as before)

\[
d^S_{\text{O}_{2n(2i+1)+1}}(s) = \prod_{k=1}^{j} L^S(\tau, \Lambda^2, 2s + 2k) \prod_{k=1}^{j+1} L^S(\tau, \text{sym}^2, 2s + 2k - 1);
\]

\[
d^S_{\text{O}_{4n+1}}(s) = \prod_{k=1}^{j} L^S(\tau, \Lambda^2, 2s + 2k - 1) L^S(\tau, \text{sym}^2, 2s + 2k).
\]

The next theorem is proved in [GS18], Theorems 6.1, 6.4.

**Theorem 2.6.** For \( \text{Re}(s) \) sufficiently large, \( h \in \text{SO}_{2n(i)}(\mathbb{A}) \),

\[
D_\psi,1(t)(\text{E}(f_{\Delta(\tau,i+1),s}))(h) = (2.24)
\]

\[
\sum_{\gamma \in \mathfrak{Q}_n(\mathbb{F}) / \text{SO}_{2n}(\mathbb{F})} \Lambda^+(f_{\Delta(\tau,i+1),s})(\gamma h) + \sum_{\gamma \in \mathfrak{Q}_n(\mathbb{F}) / \text{SO}_{2n}(\mathbb{F})} \Lambda^-(f_{\Delta(\tau,i+1),s})(\gamma h \omega_0),
\]

where

\[
\Lambda^+(f_{\Delta(\tau,i+1),s})(h) = \int_{U^+_{n,1}(\mathbb{A})} f^\psi_{\Delta(\tau,i+1),s}(\alpha_0 ut(h)) \psi_n^{-1}(u) du;
\]

\[
\Lambda^-(f_{\Delta(\tau,i+1),s})(h) = \int_{U^-_{n,1}(\mathbb{A})} f^\psi_{\Delta(\tau,i+1),s}(\alpha_0 ut(h) \omega_0) \psi_n(u) du.
\]

Both functions \( \Lambda^\pm(f_{\Delta(\tau,i+1),s}) \), defined for \( \text{Re}(s) \) sufficiently large, admit analytic continuations to meromorphic functions of \( s \) in the whole plane. They define smooth meromorphic sections of

\[
\rho_{\Delta(\tau,i),s} = \text{Ind}_{\mathfrak{Q}_n(\mathbb{A})}^{\text{SO}_{2n}(\mathbb{A})} \Delta(\tau,i) |\det|^{-s}.
\]

Thus, \( D_\psi,1(t)(\text{E}(f_{\Delta(\tau,i+1),s})) \) is the sum of two Eisenstein series on \( \text{SO}_{2n(i)}(\mathbb{A}) \); the first corresponds to the section \( \Lambda^+(f_{\Delta(\tau,i+1),s}) \) of \( \rho_{\Delta(\tau,i),s} \), and the second corresponds to the section \( h \mapsto \Lambda^-(f_{\Delta(\tau,i+1),s})(h \omega_0) \) of

\[
\rho_{\omega_0,\Delta(\tau,i)} = \text{Ind}_{\mathfrak{Q}_n(\mathbb{A})}^{\text{SO}_{2n}(\mathbb{A})} \Delta(\tau,i) |\det|^{-s}.
\]

When we normalize (outside \( S \), as before,) \( E(\text{f}_{\Delta(\tau,i+1),s}) \) by

\[
E^*_S(f_{\Delta(\tau,i+1),s}) = d^S_{\text{O}_{2n(2i+1)+1}}(s) E(\text{f}_{\Delta(\tau,i+1),s}),
\]

then \( D_\psi,1(t)(E^*_S(f_{\Delta(\tau,i+1),s})) \), is the sum of two normalized (outside \( S \)) Eisenstein series on \( \text{SO}_{2n(i)}(\mathbb{A}) \); the first corresponds to the section \( \Lambda^+(d^S_{\text{O}_{2n(2i+1)+1}}(s) f_{\Delta(\tau,i+1),s}) \) and the second corresponds to the section

\[
h \mapsto \Lambda^-(d^S_{\text{O}_{2n(2i+1)+1}}(s) f_{\Delta(\tau,i+1),s})(h \omega_0).
\]
3. The sections $\Lambda$

The sections $\Lambda(f_{\Delta(\tau,i+1)}\gamma^{(\tau,i),s})$ in the symplectic, or metaplectic case, $\Lambda(f_{\Delta(\tau,i+1),s})$, in the even orthogonal case, and $\Lambda(\pm(f_{\Delta(\tau,i+1),s})$, in the odd orthogonal case, are decomposable, for decomposable data. See, for example, Theorem 4.7 in [GS18] (for the case $m = 1$). Let us recall this. We will use the notation of [GS18]. Fix an isomorphism $\Lambda(\tau,p)$ in $\Lambda(\tau,i+1)$. Thus, at a place $v$, we denote the local factor of $\Delta(\tau,i)$ at $v$, by $\Delta(\tau,v,i)$. In the case of the double cover of $GL_n(A_v)$, we have the corresponding isomorphism (see (4.14) in [GS18])

$$(2.25) \quad p_{\tau,i} : \otimes_v \Delta(\tau,v,i) \to \Delta(\tau,i).$$

We realize $\Delta(\tau,v,i)$ in the Whittaker-Speh-Shalika model corresponding to $\psi^{-1}_{V_{\tau,v}}$ (see [FGK19], Sec. 2.2). We will call it, for short, the $\psi^{-1}_{V_{\tau,v}}$-model of $\Delta(\tau,v,i)$, and denote it by $W_{\psi^{-1}_{V_{\tau,v}}} \Delta(\tau,v,i)$. It is obtained as follows. Fix a space $E_v$ where $\Delta(\tau_v,i)$ acts. Then, up to scalars multiples, there is a unique (continuous) linear functional $c_v$ on $E_v$, satisfying, for all $e \in E_v$, and all $u \in V_{\tau,v}(F_v)$,

$$(2.26) \quad c_v(\Delta(\tau,v,i)(u)e) = \psi^{-1}_{V_{\tau,v}}(u)c_v(e),$$

where $\psi_v$ is the character of $V_{\tau,v}(F_v)$ given by

$$\psi_v = \begin{pmatrix} I_i & x_{1,2} & x_{1,3} & \cdots & x_{1,n} \\ I_1 & x_{2,3} & \cdots & x_{2,n} \\ \vdots & \ddots & \ddots & \vdots \\ I_{n-1} & x_{n-1,n} \\ I_i & \\
\end{pmatrix} = \psi(\text{tr}(x_{1,2} + x_{2,3} + \cdots + x_{n-1,n})).$$

The corresponding model of $\Delta(\tau_v,i)$ is the space of functions on $GL_n(F_v)$ given by $g \mapsto c_v(\Delta(\tau_v,i)(g)e)$, for $e \in E_v$. Note that when $i = 1$, we get the $\psi^{-1}_{V_v}$-Whittaker model of $\tau_v$.

For each place $v$, let

$$(2.28) \quad \rho_{\Delta(\tau,v,i);\gamma^{(\tau,v),s}} = \text{Ind}_{Q_{m,n}(F_v)}^{H(F_v)}(\Delta(\tau,v,i)\gamma^{(\tau,v),s}_{\psi_v}) \det |^{*\frac{s}{2}} \times \tau_v \gamma^{(\tau,v),s}_{\psi_v} \det |^{*\frac{s}{2}}.$$
where, for $h \in \text{Sp}_{2n}^{(3-\epsilon)}(F_v)$, in the notation of Theorem 2.4
\begin{equation}
\Lambda_v(f_{\Delta(\tau,v,1)},(\psi_v)^{(e)}(\tau),\phi_v)(h)) = \int_{U_{n-1}^0(F_v)} \omega_v^{-1}(\beta(u)h)\phi_v(0)f_{\Delta(\tau,v,1),(\psi_v)^{(e)}}(\alpha_0u\delta(h))du.
\end{equation}
We recall that when $\epsilon = 1$, we view $\Lambda_v(f_{\Delta(\tau,v,1)},(\psi_v)^{(e)}(\tau),\phi_v)$ as an element of $\rho_{\Delta(\tau,v,1),(\psi_v)^{(e)}(\tau),\phi_v}$ via
\begin{equation}
(a, \mu) \mapsto \Lambda_v(f_{\Delta(\tau,v,1)},(\psi_v)^{(e)}(\tau),\phi_v)((\hat{a}, \mu)h) = \mu\gamma_v(\det(a))\det(a)^{s+\frac{n+1}{2}}\int_{U_{n-1}^0(F_v)} \omega_v^{-1}(\beta(u)h)\phi_v(0)f_{\Delta(\tau,v,1),(\psi_v)^{(e)}(\tau),\phi_v}(\alpha_0u\delta(h), a; I_n)du.
\end{equation}
Similarly in the case $\epsilon = 2$ and in the following cases for orthogonal groups.

When $H = \text{SO}_{2n(i+1)}$, for decomposable sections,
\begin{equation}
\Lambda(f_{\Delta(\tau,v,1)},(\psi_v)^{(e)}(\tau),\phi_v)) = \rho_{\tau,v}(\otimes^\epsilon\Lambda_v(f_{\Delta(\tau,v,1)},(\psi_v)^{(e)}(\tau),\phi_v))
\end{equation}
where, for $h \in \text{SO}_{2n(i+1)}(F_v)$, in the notation of (2.10),
\begin{equation}
\Lambda_v(f_{\Delta(\tau,v,1)},(\psi_v)^{(e)}(\tau),\phi_v))(h) = \int_{U_{n-1}^0(F_v)} f_{\Delta(\tau,v,1),(\psi_v)^{(e)}(\tau),\phi_v}(\delta_0u\delta(h))\psi^{-1}_{n-1,v}(u)du.
\end{equation}
When $H = \text{SO}_{2n(i+1)+1}$, for decomposable sections,
\begin{equation}
\Lambda^\pm(f_{\Delta(\tau,v,1)},(\psi_v)^{(e)}(\tau),\phi_v)) = \rho_{\tau,v}(\otimes^\epsilon\Lambda_v^\pm(f_{\Delta(\tau,v,1)},(\psi_v)^{(e)}(\tau),\phi_v))
\end{equation}
where, for $h \in \text{SO}_{2n}(F_v)$, in the notation of (2.13),
\begin{equation}
\Lambda_v^+(f_{\Delta(\tau,v,1)},(\psi_v)^{(e)}(\tau),\phi_v))(h) = \int_{U_{n-1}^0(F_v)} f_{\Delta(\tau,v,1),(\psi_v)^{(e)}(\tau),\phi_v}(\alpha_0u\delta(h))\psi^{-1}_{n,v}(u)du;
\end{equation}
\begin{equation}
\Lambda_v^-(f_{\Delta(\tau,v,1)},(\psi_v)^{(e)}(\tau),\phi_v))(h) = \int_{U_{n-1}^0(F_v)} f_{\Delta(\tau,v,1),(\psi_v)^{(e)}(\tau),\phi_v}(\alpha_0u\delta(h))\psi^{-1}_{n,v}(u)du.
\end{equation}

**Proposition 2.7.** For each place $v$, the local sections $\Lambda_v(f_{\Delta(\tau,v,1)},(\psi_v)^{(e)}(\tau),\phi_v)$ (for $H_v = \text{Sp}_{2n}^{(e)}(F_v)$), $\Lambda_v(f_{\Delta(\tau,v,1)},(\psi_v)^{(e)}(\tau),\phi_v)$ (for $H_v = \text{SO}_{2n}(F_v)$), $\Lambda_v^\pm(f_{\Delta(\tau,v,1)},(\psi_v)^{(e)}(\tau),\phi_v)$ (for $H_v = \text{SO}_{2n(i+1)}(F_v)$), are holomorphic in $\mathbb{C}$.

**Proof.** Consider the case where $H_v = \text{SO}_{2n(i+1)}(F_v)$. This case is treated in detail in [GS18], Sec. 4, where we take there $m = 1$. By Cor. 4.4 in [GS18], it is enough to show the holomorphy of the following functions. Let $\varphi_{\tau,s+\frac{i}{2}}$ be a smooth, holomorphic section of
\begin{equation}
\rho_{\tau,s+\frac{i}{2}} = \text{Ind}_{Q_{n}(F_v)}^{\text{SO}_{2n}(F_v)} \Delta(\tau,v,s+\frac{i}{2})|\det|^{-s}.
\end{equation}
Then it is enough to show the holomorphy (of the continuation of) the following integral,
\begin{equation}
r(\varphi_{\tau,s+\frac{i}{2}}) = \int_{U_{n-1}^0(F_v)} \varphi_{\tau,s+\frac{i}{2}}(\delta_0u)\psi^{-1}_{n-1,v}(u)du,
\end{equation}
where
\begin{equation}
U_{n-1}^0 = \left\{u_{x,y} = \begin{pmatrix} I_{n-1} & x & 0 & y \\ 1 & 0 & 0 & x' \\ 1 & 0 & 0 & x' \\ I_{n-1} & 0 & 0 & 0 \end{pmatrix} \in \text{SO}_{2n}(F_v) \right\}; \psi_{U_{n-1}^0,v}(u_{x,y}) = \psi_v(x_{n-1});
\end{equation}
The integral (2.35) converges absolutely, for $Re(s)$ sufficiently large, and has an
analytic continuation to the whole plane. Indeed, this is a Jacquet integral ([J67]),
defining a Whittaker functional for $\rho_{\tau,s+\frac{1}{2}}$, with respect to the character

\[
\omega_0 = \begin{pmatrix} I_{n-1} & 2 \\ \frac{1}{2} & I_{n-1} \end{pmatrix}; \quad \delta'_0 = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & I_{n-1} \\ I_{n-1} & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix} \omega_0^{n-1}.
\]

The proof of the proposition for the case $H_v = SO_{2n(i+1)+1}(F_v)$ is similar and follows from [GS18], right after (6.35) and (6.36). As for the case $H_v = Sp_{2n(i+1)}(F_v)$, the proof is similar and follows from Prop. 7.2 in [GS18]. Indeed, Eq. (7.18) in [GS18] tells us that it is enough to consider the following integral

(2.37) \[
\begin{pmatrix} z & x & y & c \\ 1 & 0 & y' & 1 \end{pmatrix} \rightarrow \psi_{Z_{n-1},v}(z)\psi_v(x_{n-1} + \frac{1}{2}y_{n-1}).
\]

The proof of the proposition for the case $H_v = SO_{2n(i+1)+1}(F_v)$ is similar and follows from [GS18], right after (6.35) and (6.36). As for the case $H_v = Sp_{2n(i+1)}(F_v)$, the proof is similar and follows from Prop. 7.2 in [GS18]. Indeed, Eq. (7.18) in [GS18] tells us that it is enough to consider the following integral

(2.38) \[
\int \rho(a_0)(f_{\Delta(\tau,s+1),s})(\begin{pmatrix} I_{n_i} & I_n & I_n & I_{n_i} \\ y & I_n & I_n & I_{n_i} \end{pmatrix})\psi_v^{-1}(y_{n,1})dy.
\]

The integration is over the lower Siegel radical of $Sp_{2n}(F_v)$. As in the beginning of the proof of Theorem 7.3 in [GS18], the function, defined for $h \in Sp_{2n}(F_v)$, by

\[f_{\tau,s+\frac{1}{2}}(h) = \rho(a_0)(f_{\Delta(\tau,s+1),s})(diag(I_{n_i},h,I_{n_i}),)\]

is a smooth, holomorphic section of $\rho_{\tau,s+\frac{1}{2}}$. The integral (2.38) becomes

(2.39) \[
\int f_{\tau,s+\frac{1}{2}}(\begin{pmatrix} I_n & I_n \\ y & I_n \end{pmatrix})\psi_v^{-1}(y_{n,1})dy.
\]

Clearly the integral (2.39), which converges absolutely, for $Re(s)$ sufficiently large, is a Jacquet integral defining a Whittaker functional for $\rho_{\tau,s+\frac{1}{2}}$, and hence admits an analytic continuation to a holomorphic function in $C$. The proof in the metaplectic case is entirely similar with simple modifications.

**Corollary 2.8.** The global sections $\Lambda(f_{\Delta(\tau,i+1),\gamma(s),s},\phi)$ in the symplectic, or metaplectic case are holomorphic at the half plane $Re(s) \geq -\frac{1}{2}$. Similarly for $\Lambda(f_{\Delta(\tau,i+1),s})$, in the even orthogonal case, and $\Lambda^\pm(f_{\Delta(\tau,i+1),s})$, in the odd orthogonal case.

**Proof.** Consider, for example, the case $H = Sp_{2n(i+1)}$. Let $f_{\Delta(\tau,i+1),s}$ and $\phi$ be decomposable. We use the same notation as before. Let $S$ be a finite set of places, containing the infinite places, outside which $f_{\Delta(\tau,i+1),s}$ and $\phi$ are unramified. At these places, assume that $\phi_v = \phi_v^0$ and $f^0_{\Delta(\tau,i+1),s}$ are as in Theorem 7.3 in [GS18]. This theorem implies that

(2.40) \[
\Lambda(f_{\Delta(\tau,i+1),s},\phi) = \frac{1}{L^S(\tau,\lambda^2,2s+1+1)p_{\tau,i}(\otimes_v E(\Delta(\tau,i+1),s),\phi_v) \otimes (\otimes_v E(\Delta(\tau,i+1),\gamma(s),s)).}
\]
The corollary follows from Prop. [2.7] and the fact that \( L^S(\tau, \Lambda^2, 2s + i + 1) \) doesn’t vanish at \( \text{Re}(2s + i + 1) \geq 1 \). The other cases are proved similarly, using Theorems 8.2, Theorem 6.3, and Theorem 4.7, Prop. 4.8 and (4.69) (all) in [GS18]. □

**Proposition 2.9.** Fix a place \( v \) and fix a complex number \( s_0 \) with \( \text{Re}(s_0) \geq 0 \). Consider, in each case, the local sections, as in Prop. [2.7], at the point \( s_0 \): 
\[
\Lambda_v(f_{\Delta(\tau, i; 1)} \gamma^{(v)}_{\psi, s_0}, \phi_v) \quad (\text{for } H_v = \text{Sp}_{2n(i+1)}(F_v)), \quad \Lambda_v(f_{\Delta(\tau, i; 1), s_0})
\]
(\text{for } H_v = \text{SO}_{2n(i+1)}(F_v)). \( \Lambda^\pm_\psi(f_{\Delta(\tau, i; 1), s_0}) \) (for \( H_v = \text{SO}_{2n(i+1)+1}(F_v) \)). View these as maps to \( \rho^H_\psi \Delta(\tau, i) \gamma^{(v)}_{\psi, s_0} \). Then, when \( v \) is finite, these are surjective maps.

When \( v \) is Archimedean, their image, in each case, is dense (in the Frechet topology).

**Proof.** We prove the proposition for the case where \( H_v = \text{Sp}_{2n(i+1)}(F_v) \). The proof in the other cases is very similar.

Denote by \( \Lambda_v, s_0 \) be the following bilinear map on \( \mathcal{V}_{\rho_\Delta(\tau, i; 1), s_0} \times \mathcal{S}(F_v) \). Let \( f_0 \) be a function in \( \mathcal{V}_{\rho_\Delta(\tau, i; 1), s} \), and let \( \phi_v \in \mathcal{S}(F_v) \). Let \( f_{\Delta(\tau, i; 1), s} \) be any smooth, holomorphic section of \( \rho_\Delta(\tau, i; 1), s \), such that \( f_{\Delta(\tau, i; 1), s_0} = f_0 \). Then
\[
\Lambda_v(f_0, \phi_v) = \Lambda(f_{\Delta(\tau, i; 1), s}, \phi_v)|_{z = s_0}.
\]

This is well defined, since \( \Lambda(f_{\Delta(\tau, i; 1), s}, \phi_v)|_{z = s_0} \) depends only on \( f_{\Delta(\tau, i; 1), s_0} \). Let \( s_0 \) be an element in the dual of \( \rho_{\Delta(\tau, i), s_0} \), realized as \( \rho_{\Delta(\tau, i), s^*} \). Assume that it is zero on the image of \( \Lambda_v, s_0 \). Then we need to prove that \( s_0 = 0 \). Our assumption is that for all \( f_0 \) and \( \phi_v \),
\[
(2.41) \quad < \Lambda_v, s_0(f_0, \phi_v), s_0 > = 0.
\]

Let us take in (2.41), \( f_0 = \rho(\hat{w}_{n(i+1)} f_{0}', \text{ supported in the open cell, modulo } Q_{m,n}(F_v) \) from the left, in \( \text{Sp}_{2n(i+1)}(F_v) \), and assume that this support is compact. We claim that the support of the function
\[
(2.42) \quad z \mapsto \Lambda_v, s_0(f_0, \phi_v)(J_{2n(i)} u_{ni}(z))
\]
is compact. Here, \( S_{ni}(F_v) \) is the space of \( ni \times ni \) matrices \( z \) over \( F_v \), such that \( u_{ni} z \) is symmetric. Also, \( J'_{2ni} = \begin{pmatrix} -I_n & I_{ni} \\ -I_{ni} & I_n \end{pmatrix} \). By (2.30), we have
\[
(2.43) \quad \Lambda_v, s_0(f_0, \phi_v)(J_{2n(i)} u_{ni}(z)) = \int \psi^{-1}(y_{n,1}) \omega \psi^{-1}(J_{2n(i)} u_{ni}(z), 1) \phi_v(x_n) f_{0}'(J_{2n(i+1)} u_{n(i+1)}(z)) \begin{pmatrix} x \\ z \end{pmatrix}^{w_{n(i+1)}} dx dy.
\]
The integration is over \( x \in M_{n \times n}(F_v) \) and \( y \in S_n(F_v) \). By our assumption on \( f_{0}'(\text{the r.h.s. of (2.43) is compactly supported in } z) \). For such \( f_0 \), we can write the l.h.s. of (2.41) as an absolutely convergent integral along the open cell, modulo \( Q_{ni}^{(2)}(F_v) \) from the left, in \( \text{Sp}_{2n(i+1)}(F_v) \), and we get
\[
(2.44) \quad \int_{S_{ni}(F_v)} < \Lambda_v, s_0(f_0, \phi_v)(J_{2n(i)} u_{ni}(z)), \xi_0(J_{2n(i)} u_{ni}(z)) > dz = 0.
\]
The inner pairing $<,>$ in (2.44) is the bilinear invariant pairing on $\Delta(\tau_v, i) \times \Delta(\tau_v, i)$. By (2.33), we can rewrite (2.44) as

$$(2.45) \quad \int \psi_v^{-1}(y_n, 1) \mathcal{w}_v^{-1}((J_{2n}u_{ni}(z), 1)) \phi_v(x_n)$$

$$<f_0'(J_{2n(i+1)}u_{ni(i+1)}(x'_{n(i+1)})), \xi_0(J_{2n}u_{ni}(z))> dx dy dz = 0.$$

The integration is over $z \in S_{ni}(F_v)$, $x \in M_{n \times ni}(F_v)$ and $y \in S_{n}(F_v)$. This is valid for all $f_0'$ as above and all $\phi_v$. We conclude that $\xi_0$ must vanish on the open cell, modulo $Q_{ni}^{(2)}(F_v)$ from the left, in $Sp_{2ni}(F_v)$, and hence $\xi_0 = 0$. The proposition follows.

\[ \square \]

3. PROOF OF THEOREM 2.22: SYMPLECTIC GROUPS AND METAPELIC GROUPS

We keep the notation of the previous section.

**Theorem 3.1.** Assume that $L(\tau, 0^2, s)$ has a pole at $s = 1$ and $L(\tau, 1_2) \neq 0$. For all integers $k \geq 1$, the elements of the set

$$\Lambda_{\tau, 0^2, 2k-1}^{Sp_{2n}(2k-1)} = \{ \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots, \frac{2k-1}{2} \} = \Lambda_{\tau, 0^2, 2k}^{Sp_{2n}(2k)}$$

are all poles of the Eisenstein series $E^{Sp_{2n}(2k-1)}(f_{\Delta(\tau, 2k-1), s})$, and of the Eisenstein series $E^{Sp_{2n}(2k)}(f_{\Delta(\tau, 2k), s})$, as the sections vary.

**Proof.** We prove the theorem for $\Lambda_{\tau, 0^2, 2k-1}^{Sp_{2n}(2k-1)}$ by induction on $k$. The proof for $\Lambda_{\tau, 0^2, 2k}^{Sp_{2n}(2k)}$ will follow. We start with $k = 1$ and $E^{Sp_{2n}}(f_{\tau, s})$. It has a pole at $s = 1_2$. This is a special case of Prop. 2.1. By Theorem 2.4 for all $h \in Sp_{2n}(A)$,

$$(3.1) \quad D_{\psi, n}^{\phi}(E^{Sp_{2n}}(f_{\Delta(\tau, 2\gamma_{\psi, s}), s}))(h) = E^{Sp_{2n}}(\Lambda(f_{\Delta(\tau, 2\gamma_{\psi, s}, 0)}))(h).$$

Note that all the normalizing factors involved in Theorem 2.4 are holomorphic and nonzero at $s = 1_2$. From Cor. 2.5 it follows that $\Lambda(f_{\Delta(\tau, 2\gamma_{\psi, s}, 0)})$ is holomorphic at $s = 1_2$. From Prop. 2.9 it follows that the map $\Lambda(f_{\Delta(\tau, 2\gamma_{\psi, s}, 0)})(\phi)$ on the space of $\rho_{\Delta(\tau, 2\gamma_{\psi, s}, 0)}$, has image which corresponds to

$$\bigotimes_{v \in S_{n,v}} W(\rho_{\tau_v, 1/2}) \otimes \bigotimes_{v' < \infty} V(\rho_{\tau_v, 1/2}),$$

where $V(\rho_{\tau_v, 1/2})$ denotes the space of $\rho_{\tau_v, 1/2}$, and, for $v \in S_{\infty}, W(\rho_{\tau_v, 1/2})$ is a dense subspace of $V(\rho_{\tau_v, 1/2})$. Taking residues of Eisenstein series at a given point is a continuous map. We conclude that $E^{Sp_{2n}}(\Lambda(f_{\Delta(\tau, 2\gamma_{\psi, s}, 0)}))$ has a pole at $s = 1_2$. Thus, the r.h.s. of (3.1) has a pole at $s = 1_2$ and hence, so does $E^{Sp_{2n}}(f_{\Delta(\tau, 2\gamma_{\psi, s}, 0)})(h)$ (as the section varies). Assume by induction that the theorem holds for $\Lambda_{\tau, 0^2, 2k-1}^{Sp_{2n}(2k-1)}$ and $1 \leq k \leq i$. By Theorem 2.4 for all $1 \leq j \leq i$ and $h \in Sp_{2n(2j-1)}(A)$,

$$(3.2) \quad D_{\psi, n}^{\phi}(E^{Sp_{2n}}(f_{\Delta(\tau, 2j\gamma_{\psi, s}), s}))(h) = E^{Sp_{2n(2j-1)}}(\Lambda(f_{\Delta(\tau, 2j\gamma_{\psi, s}, 0)}))(h).$$

By induction, the r.h.s., as the section varies, has poles at each point of $\Lambda_{\tau, 0^2, 2k-1}^{Sp_{2n}(2k-1)}$. This follows from Cor. 2.5 and Prop. 2.9 as in the case $k = 1$. We conclude from
that $E^{Sp_{2n}(2)}(f_{\Delta(\tau,2j)}(\gamma_0, s), \Lambda_{\tau, \gamma_0, 2j-1}^{Sp_{2n}(2)})$ has poles at each point of $\Lambda_{\tau, \gamma_0, 2j-1}^{Sp_{2n}(2)}$, as the section varies. By Theorem 2.4 for all $h \in Sp_{2n}^{(2)}(\mathbb{A})$,

\[(3.3) \quad \mathcal{D}_{\psi, 2n}^{(2)}(E^{Sp_{2n}(2)}(f_{\Delta(\tau,2j+1)}(\gamma_0, s), \Lambda_{\tau, \gamma_0, 2j+1}^{Sp_{2n}(2)}))(h) = E^{Sp_{2n}(2)}_{\Lambda_{\tau, \gamma_0, 2j+1}^{Sp_{2n}(2)}}(f_{\Delta(\tau,2j+1)}(\gamma_0, s), \Lambda_{\tau, \gamma_0, 2j+1}^{Sp_{2n}(2)}))(h).
\]

We just proved that the r.h.s. of (3.3) has poles at each point of $\Lambda_{\tau, \gamma_0, 2j+1}^{Sp_{2n}(2)}$. Again, this follows from Cor. 2.8 and Prop. 2.9. We conclude from (3.3) that $E^{Sp_{2n}(2)}(f_{\Delta(\tau,2j+1)}(\gamma_0, s), \Lambda_{\tau, \gamma_0, 2j+1}^{Sp_{2n}(2)})$ has poles at each point of $\Lambda_{\tau, \gamma_0, 2j+1}^{Sp_{2n}(2)}$, as the section varies. By Prop. 2.1 $s = \frac{2i+1}{2}$ is a pole of $E^{Sp_{2n}(2)}(f_{\Delta(\tau,2j+1)}(\gamma_0, s), \Lambda_{\tau, \gamma_0, 2j+1}^{Sp_{2n}(2)})$, and hence all elements of $\Lambda_{\tau, \gamma_0, 2j+1}^{Sp_{2n}(2)} = \Lambda_{\tau, \gamma_0, 2j+1}^{Sp_{2n}(2)} \cup \{\frac{2i+1}{2}\}$ are poles of $E^{Sp_{2n}(2)}(f_{\Delta(\tau,2j+1)}(\gamma_0, s), \Lambda_{\tau, \gamma_0, 2j+1}^{Sp_{2n}(2)})$, as the section varies. This completes the proof of the theorem.

d

The remaining cases are proved in the same way.

**Theorem 3.2.** Assume that $L(\tau, \Lambda^2, s)$ has a pole at $s = 1$ and $L(\tau, \frac{1}{2}) \neq 0$. For all integers $k \geq 1$, the elements of the set

$$\Lambda_{\tau, \Lambda^2, 2k}^{Sp_{4n}} = \{1, 2, 3, ..., k\} = \Lambda_{\tau, \Lambda^2, 2k+1}^{Sp_{4n}}$$

are all poles of the Eisenstein series $E^{Sp_{4n}}(f_{\Delta(\tau,2k)}(\gamma_0, s))$, and the Eisenstein series $E^{Sp_{2n}(2)}(f_{\Delta(\tau,2k+1)}(\gamma_0, s))$, as the sections vary.

**Proof.** Since the proof follows the same lines of the proof of Theorem 3.1 we will be brief. When $k = 1$, $E^{Sp_{4n}}(f_{\tau, s})$ has a pole at $s = 1$. This is a special case of Prop. 2.4. For the induction passage, we use Theorem 2.4. For all $h \in Sp_{4n}(\mathbb{A})$,

\[(3.4) \quad \mathcal{D}_{\psi, 2n}^{(2)}(E^{Sp_{4n}(2)}(f_{\Delta(\tau,2i+1)}(\gamma_0, s), \Lambda_{\tau, \gamma_0, 2i+1}^{Sp_{4n}(2)}))(h) = E^{Sp_{4n}(2)}(f_{\Delta(\tau,2i+1)}(\gamma_0, s), \Lambda_{\tau, \gamma_0, 2i+1}^{Sp_{4n}(2)}))(h).
\]

By induction, the r.h.s., as the section varies, has poles at each point of $\Lambda_{\tau, \gamma_0, 2i+1}^{Sp_{4n}(2)}$, and we conclude from (3.4) that $E^{Sp_{4n}(2)}(f_{\Delta(\tau,2i+1)}(\gamma_0, s), \Lambda_{\tau, \gamma_0, 2i+1}^{Sp_{4n}(2)})$ has poles at each point of $\Lambda_{\tau, \gamma_0, 2i+1}^{Sp_{4n}(2)} = \Lambda_{\tau, \gamma_0, 2i+1}^{Sp_{4n}(2)}$, as the section varies. Of course, we need to use Cor. 2.8 and Prop. 2.9, as we did right after (3.4). Next, by Theorem 2.4 for all $h \in Sp_{2n(2i+1)}(\mathbb{A})$,

\[(3.5) \quad \mathcal{D}_{\psi, 2n}^{(2)}(E^{Sp_{4n+(i+1)}}(f_{\Delta(\tau,2i+2)}(\gamma_0, s), \Lambda_{\tau, \gamma_0, 2i+2}^{Sp_{4n+(i+1)}}))(h) = E^{Sp_{4n+(i+1)}}(f_{\Delta(\tau,2i+2)}(\gamma_0, s), \Lambda_{\tau, \gamma_0, 2i+2}^{Sp_{4n+(i+1)}}))(h).
\]

We just proved that the r.h.s. of (3.5) has poles at each point of $\Lambda_{\tau, \gamma_0, 2i+2}^{Sp_{4n+(i+1)}}$ (using Cor. 2.8 and Prop. 2.9). We conclude from (3.5) that $E^{Sp_{4n+i+1}}(f_{\Delta(\tau,2i+2)}(\gamma_0, s))$ has poles at each point of $\Lambda_{\tau, \gamma_0, 2i+2}^{Sp_{4n+i+1}}$, as the section varies. By Prop. 2.1 $s = i + 1$ is a pole of $E^{Sp_{4n+i+1}}(f_{\Delta(\tau,2i+2)}(\gamma_0, s))$, and hence all elements of $\Lambda_{\tau, \gamma_0, 2i+2}^{Sp_{4n+i+1}} = \Lambda_{\tau, \gamma_0, 2i+2}^{Sp_{4n+i+1}} \cup \{i + 1\}$ are poles of $E^{Sp_{4n+i+1}}(f_{\Delta(\tau,2i+2)}(\gamma_0, s))$, as the section varies. This proves the theorem.

d
Theorem 3.3. Assume that \( L(\tau, \sqrt{2}, s) \) has a pole at \( s = 1 \). For all integers \( k \geq 1 \), the elements of the set

\[
\Lambda^{\text{Sp}_{4n+k}}_{\tau, \sqrt{2}, 2k} = \left\{ \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots, \frac{2k-1}{2} \right\} = \Lambda_{\tau, \sqrt{2}, 2k}^{(2)}
\]

are all poles of the Eisenstein series \( E^{\text{Sp}_{4n+k}}_{\tau}(f_{\Delta(\tau, 2k), s}) \), and the Eisenstein series \( E^{\text{Sp}_{2n(2k-1)}}_{\tau}(f_{\Delta(\tau, 2k-1)\gamma_0, s}) \), as the sections vary.

Proof. The proof is similar to the last two proofs, only that here, we prove the theorem by induction on \( k \); for \( \Lambda^{(2)}_{\text{Sp}_{2n(2k-1)}} \), and the theorem for \( \Lambda^{\text{Sp}_{4n+k}}_{\tau, \sqrt{2}, 2k} \) follows from the proof. We will be brief. We start with \( k = 1 \) and \( E^{\text{Sp}_{2n}}_{\tau}(f_{\gamma_0, s}) \). It has a pole at \( s = \frac{1}{2} \) (by Prop. 2.1). For the induction passage, we use Theorem 2.4. For all \( h \in \text{Sp}_{2n(2i-1)}(\Lambda) \),

\[
(3.6) \quad D^h_{\psi, 2n(2i-1)}(E^{\text{Sp}_{4n}}_{\tau}(f_{\Delta(\tau, 2i), s}))(h) = E^{\text{Sp}_{2n(2i-1)}}_{\tau}(\Lambda(f_{\Delta(\tau, 2i), s}, \psi))(h).
\]

By induction, the r.h.s., as the section varies, has poles at each point of \( \Lambda^{\text{Sp}_{2n(2i-1)}}_{\tau, \sqrt{2}, 2i-1} \), and hence, by (3.3), \( E^{\text{Sp}_{4n}}_{\tau}(f_{\Delta(\tau, 2i), s}) \) has poles at each point of \( \Lambda^{\text{Sp}_{4n}}_{\tau, \sqrt{2}, 2i-1} = \Lambda^{\text{Sp}_{4n}}_{\tau, \sqrt{2}, 2i} \), as the section varies. By Theorem 2.4 for all \( h \in \text{Sp}_{4n}(\Lambda) \),

\[
(3.7) \quad D^h_{\psi, 2n(2i-1)}(E^{\text{Sp}_{2n(2i+1)}}_{\tau}(f_{\Delta(\tau, 2i+1)\gamma_0, s}))(h) = E^{\text{Sp}_{4n}}_{\tau}(\Lambda(f_{\Delta(\tau, 2i+1)\gamma_0, s}, \psi))(h).
\]

We just proved that the r.h.s. has poles at each point of \( \Lambda^{\text{Sp}_{4n}}_{\tau, \sqrt{2}, 2i} \) and hence \( E^{\text{Sp}_{2n(2i+1)}}_{\tau}(f_{\Delta(\tau, 2i+1)\gamma_0, s}) \) has poles at each point of \( \Lambda^{\text{Sp}_{4n}}_{\tau, \sqrt{2}, 2i} \) (as the sections vary).

By Prop. 2.1 \( E^{\text{Sp}_{2n(2i+1)}}_{\tau}(f_{\Delta(\tau, 2i+1)\gamma_0, s}) \) has a pole at \( s = \frac{2i+1}{2} \) (as the section varies), and hence all elements of

\[
\Lambda^{\text{Sp}_{2n(2i+1)}}_{\tau, \sqrt{2}, 2i+1} = \Lambda^{\text{Sp}_{4n}}_{\tau, \sqrt{2}, 2i+1} \cup \left\{ \frac{2i+1}{2} \right\}
\]

appear as poles of \( E^{\text{Sp}_{2n(2i+1)}}_{\tau}(f_{\Delta(\tau, 2i+1)\gamma_0, s}) \).

Finally, we have, with a similar proof,

Theorem 3.4. Assume that \( L(\tau, \sqrt{2}, s) \) has a pole at \( s = 1 \). For all integers \( k \geq 1 \), the elements of the set

\[
\Lambda^{\text{Sp}_{2n(2k+1)}}_{\tau, \sqrt{2}, 2k+1} = \{1, 2, 3, \ldots, k\} = \Lambda^{(2)}_{\text{Sp}_{2n, 2k}}
\]

are all poles of the Eisenstein series \( E^{\text{Sp}_{2n(2k+1)}}_{\tau}(f_{\Delta(\tau, 2k+1), s}) \), and of the Eisenstein series \( E^{\text{Sp}_{4n+k}}_{\tau}(f_{\Delta(\tau, 2k)\gamma_0, s}) \), as the sections vary.

4. Proof of Theorem 2.2 orthogonal groups

The proofs are the same as in the previous section, except that instead of Theorem 2.4 we use Theorems 2.5, 2.6.
Theorem 4.1. Assume that $L(\tau, \Lambda^2, s)$ has a pole at $s = 1$. For all integers $k \geq 1$, the elements of the set
\[
\Lambda_{\tau, \Lambda^2, 2k-1}^{SO_{2n(2k-1)}} = \left\{ \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots, \frac{2k-1}{2} \right\} = \Lambda_{\tau, \Lambda^2, 2k}
\]
are all poles of the Eisenstein series $E^{SO_{2n(2k-1)}}(f_{\Delta(\tau, 2k-1), s})$, and of the Eisenstein series, $E^{SO_{4n+1}}(f_{\Delta(\tau, 2k), s})$, as the sections vary.

Proof. We prove the theorem for $\Lambda_{\tau, \Lambda^2, 2k-1}^{SO_{2n(2k-1)}}$ by induction on $k$. The proof of the theorem for $\Lambda_{\tau, \Lambda^2, 2k}$ will follow. Let $k = 1$. The Eisenstein series $E^{SO_{2n}}(f_{\tau, s})$ has a pole at $s = \frac{1}{2}$. This is a special case of Prop. 2.4 (Note that $n$ must be even). By Theorem 2.1 for all $h \in SO_{2n}(A)$,
\[
(4.1) \quad D_{\psi, n}(E^{SO_{2n+1}}(f_{\Delta(\tau, 2), s}))(h) = E^{SO_{2n}}(\Lambda^+(f_{\Delta(\tau, 2), s}))(h) + E^{SO_{2n}}(\Lambda^-(f_{\Delta(\tau, 2), s}))(h^{\omega'_0}).
\]
Note that all the normalizing factors involved in Theorem 2.1 are holomorphic and nonzero at $s = \frac{1}{2}$. From Cor. 2.8 it follows that $\Lambda^\pm(f_{\Delta(\tau, 2), s})$ are holomorphic at $s = \frac{1}{2}$. From Prop. 2.9 we conclude, as we did in the proof of Theorem 3.1, that $E^{SO_{2n}}(\Lambda^\pm(f_{\Delta(\tau, 2), s}))$ have poles at $s = \frac{1}{2}$. (This is the first case of our induction.) We leave it to the reader to check that the residue of the second term in (4.1) does not cancel the first. We conclude that $E^{SO_{4n+1}}(f_{\Delta(\tau, 2), s})$ has a pole at $s = \frac{1}{2}$. Note that $\Lambda_{\tau, \Lambda^2, 2} = \left\{ \frac{1}{2} \right\}$.

Assume that we proved the theorem for $\Lambda_{\tau, \Lambda^2, 2k-1}^{SO_{2n(2k-1)}}$, for all $1 \leq k \leq i$. By Theorem 2.1 for all $h \in SO_{2n(2i-1)}(A)$,
\[
(4.2) \quad D_{\psi, n(2i-1)}(E^{SO_{4n+1}}(f_{\Delta(\tau, 2i), s}))(h) = E^{SO_{2n(2i-1)}}(\Lambda^+(f_{\Delta(\tau, 2i), s}))(h) + E^{SO_{2n(2i-1)}}(\Lambda^-(f_{\Delta(\tau, 2i), s}))(h^{\omega'_0}).
\]

By the induction assumption, and the same arguments as above (using Cor. 2.8 and Prop. 2.9), since the elements of $\Lambda_{\tau, \Lambda^2, 2i-1}^{SO_{2n(2i-1)}}$ are poles of the r.h.s. of (4.2), then they are poles of $E^{SO_{4n+1}}(f_{\Delta(\tau, 2i), s})$ (as the section varies). Thus, as the section varies, $E^{SO_{4n+1}}(f_{\Delta(\tau, 2i), s})$ has poles at each point of $\Lambda_{\tau, \Lambda^2, 2i}^{SO_{2n(2i-1)}} = \Lambda_{\tau, \Lambda^2, 2i-1}^{SO_{2n(2i-1)}}$. Next, by Theorem 2.5
\[
(4.3) \quad D_{\psi, 2n(2i-1)}(E^{SO_{2n(2i-1)}}(f_{\Delta(\tau, 2i+1), s}))(h) = E^{SO_{4n+1}}(\Lambda(f_{\Delta(\tau, 2i+1), s})/(h)).
\]

We just proved that the r.h.s. of (4.3) has poles at each point of $\Lambda_{\tau, \Lambda^2, 2i}^{SO_{2n(2i-1)}}$, as the section varies (using Cor. 2.8 and Prop. 2.9). We conclude from (4.3) that $E^{SO_{2n(2i-1)}}(f_{\Delta(\tau, 2i+1), s})$ has poles at each point of $\Lambda_{\tau, \Lambda^2, 2i}^{SO_{2n(2i-1)}}$, as the section varies. By Prop. 2.3 $s = \frac{2i+1}{2}$ is a pole of $E^{SO_{2n(2i+1)}}(f_{\Delta(\tau, 2i+1), s})$. Now, note that
\[
\Lambda_{\tau, \Lambda^2, 2i+1}^{SO_{2n(2i+1)}} = \Lambda_{\tau, \Lambda^2, 2i}^{SO_{2n(2i-1)}} \cup \left\{ \frac{2i+1}{2} \right\}.
\]

This proves the theorem. □

The remaining cases are proved similarly, and our proofs will be brief.
Theorem 4.2. Assume that \( L(\tau, A^2, s) \) has a pole at \( s = 1 \). For all integers \( k \geq 1 \), the elements of the set

\[
\Lambda^{SO_{2n}}_{\tau, A^2, 2k} = \{1, 2, 3, ..., k\} = \Lambda^{SO_{2n}(2k+1)}_{\tau, A^2, 2k+1}
\]

are all poles of the Eisenstein series \( E^{SO_{2n}}_{\tau, A^2, 2k} (f_{\Delta(\tau, 2k), s}) \), and of the Eisenstein series \( E^{SO_{2n}(2k+1)}_{\tau, A^2, 2k+1} (f_{\Delta(\tau, 2k+1), s}) \), as the sections vary.

Proof. We prove the theorem for \( \Lambda^{SO_{2n}}_{\tau, A^2, 2k} \) by induction on \( k \). The theorem for \( \Lambda^{SO_{2n}(2k+1)}_{\tau, A^2, 2k+1} \) will follow. Consider the Eisenstein series \( E^{SO_{2n}}_{\tau, A^2, 2k} (f_{\Delta(\tau, 2), s}) \). By (4.1), it has a pole at \( s = 1 \). By Theorem 2.5, for all \( h \in SO_{4n}(A) \),

\[
D_{\psi, n}(f_{\Delta(\tau, 3), s})(h) = E^{SO_{4n}}_{\tau, A^2, 2k+1}(\Lambda^+(f_{\Delta(\tau, 3), s}))(h) + E^{SO_{4n}}_{\tau, A^2, 2k+1}(\Lambda^-(f_{\Delta(\tau, 3), s}))(h^{\omega_0}).
\]

As in the last proof, since the r.h.s. of (4.4) has a pole at \( s = 1 \), we conclude that \( E^{SO_{4n+1}}_{\tau, A^2, 2k+1} (f_{\Delta(\tau, 3), s}) \) has a pole at \( s = 1 \). Note that \( \Lambda^{SO_{4n+1}}_{\tau, A^2, 3} = \{1\} \).

Assume that we proved the theorem for \( \Lambda^{SO_{4n}}_{\tau, A^2, 2k} \), for all \( 1 \leq k \leq i \). By (2.3) for all \( h \in SO_{2n(2i+1)+1}(A) \),

\[
D_{\psi, n}(f_{\Delta(\tau, 2i+2), s})(h) = E^{SO_{2n(2i+1)+1}}_{\tau, A^2, 2k+1}(f_{\Delta(\tau, 2i+2), s})(h).
\]

As in the last proof, since, by induction, the r.h.s. of (4.5) has poles at all points of \( \Lambda^{SO_{4n}}_{\tau, A^2, 2i+1} \), as the section varies, so does \( E^{SO_{2n(2i+1)+1}}_{\tau, A^2, 2k+1} (f_{\Delta(\tau, 2i+2), s}) \). Hence \( E^{SO_{2n(2i+1)+1}}_{\tau, A^2, 2k+1} (f_{\Delta(\tau, 2i+2), s}) \) has poles at each point of \( \Lambda^{SO_{4n}}_{\tau, A^2, 2i+1} \). Next, by (2.4) for all \( h \in SO_{2n(2i+1)+1}(A) \),

\[
D_{\psi, n}(f_{\Delta(\tau, 2i+2), s})(h) = E^{SO_{2n(2i+1)+1}}_{\tau, A^2, 2k+1}(f_{\Delta(\tau, 2i+2), s})(h).
\]

We just proved that the r.h.s. of (4.6) has poles at each point of \( \Lambda^{SO_{2n(2i+1)+1}}_{\tau, A^2, 2k+1} \), as the section varies. Hence, so does \( E^{SO_{2n(2i+1)+1}}_{\tau, A^2, 2k+1} (f_{\Delta(\tau, 2i+2), s}) \). By (2.1), \( E^{SO_{2n(2i+1)+1}}_{\tau, A^2, 2k+1} (f_{\Delta(\tau, 2i+2), s}) \) has a pole at \( s = i+1 \). Note that \( \Lambda^{SO_{2n(2i+1)+1}}_{\tau, A^2, 2k+1} \cup \{i+1\} \). This proves the theorem.

Theorem 4.3. Assume that \( L(\tau, \sqrt{2}, s) \) has a pole at \( s = 1 \). For all \( k \geq 1 \), the elements of the set

\[
\Lambda^{SO_{2n}}_{\tau, \sqrt{2}, 2k} = \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, ..., \frac{2k-1}{2}\} = \Lambda^{SO_{2n(2k-1)+1}}_{\tau, \sqrt{2}, 2k-1}
\]

are all poles of the Eisenstein series \( E^{SO_{2n}}_{\tau, \sqrt{2}, 2k} (f_{\Delta(\tau, 2k), s}) \), and of the Eisenstein series \( E^{SO_{2n(2k-1)+1}}_{\tau, \sqrt{2}, 2k-1} (f_{\Delta(\tau, 2k-1), s}) \), as the sections vary.

Proof. We prove the theorem for \( \Lambda^{SO_{2n}}_{\tau, \sqrt{2}, 2k} \) by induction on \( k \). The proof for \( \Lambda^{SO_{2n+1}}_{\tau, \sqrt{2}, 2k} \) will follow. The Eisenstein series \( E^{SO_{2n}+1}(f_{\tau, s}) \) has a pole at \( s = \frac{1}{2} \). Again, this is a special case of Prop. 2.1. By Theorem 2.5, for all \( h \in SO_{2n+1}(A) \),

\[
D_{\psi, n}(f_{\Delta(\tau, 2), s})(h) = E^{SO_{2n+1}}_{\tau, \sqrt{2}, 2k}(f_{\Delta(\tau, 2), s})(h).
\]

Since the r.h.s. has a pole at \( s = \frac{1}{2} \), for some section, so does the Eisenstein series \( E^{SO_{2n}}_{\tau, \sqrt{2}, 2k} (f_{\Delta(\tau, 2), s}) \). This proves the case \( k = 1 \).
By what we just proved, we conclude that each point of \( \Lambda \) is a pole of the Eisenstein series \( E \), as the section varies. Next, by Theorem 2.6 for all \( h \in SO_{4n} \),

\[
D_{\psi,n}(E^{SO_{4n}}(f_{\Delta(\tau,2),s}))(h) = E^{SO_{2(n+1)}}(\Lambda(f_{\Delta(\tau,2),s}))(h).
\]

We conclude that each point of \( \Lambda \) is a pole of \( E^{SO_{4n}}(f_{\Delta(\tau,2),s}) \), as the section varies. Finally, by induction as we did in the proofs of the theorems in Sec. 3. In this section, we will need to prove the following initial cases in order to prove Theorem 2.3.

Theorem 4.4. Assume that \( L(\tau, \sqrt{2}, s) \) has a pole at \( s = 1 \). For all integers \( k \geq 1 \), the elements of the set

\[
\Lambda^{SO_{2n+1}}_{\tau, \sqrt{2}, 2k} = \{ 1, 2, 3, ..., k \} = \Lambda^{SO_{4n+1}}_{\tau, \sqrt{2}, 2k}
\]

are all poles of the Eisenstein series \( E^{SO_{2n+1}}(f_{\Delta(\tau,2k+1),s}) \), and of the Eisenstein series \( E^{SO_{4n+1}}(f_{\Delta(\tau,2k),s}) \), as the sections vary.

5. Top Orbits for \( E\) and \( E' \)

We will need to prove the following initial cases in order to prove Theorem 2.3 by induction as we did in the proofs of the theorems in Sec. 3. In this section, we consider the following residual representations of \( Sp_{2nm} \):

1. When \( L(\tau, \sqrt{2}, s) \) has a pole at \( s = 1 \), and \( L(\tau, \frac{1}{2}) \neq 0 \), we know from Prop. 2.3 that \( E^{Sp_{2nm}}(f_{\Delta(\tau,m),s}) \) has a pole at \( s = \frac{m}{2} \), as the section varies. We will consider the residual representation \( E^{Sp_{2nm}}(\Delta(\tau,m), \sqrt{2} \cdot \frac{m+1}{2}) \) generated by the residues \( R_{\psi_s} E^{Sp_{2nm}}(f_{\Delta(\tau,m),s}) \).

2. When \( L(\tau, \sqrt{2}, s) \) has a pole at \( s = 1 \), and \( L(\tau, \frac{1}{2}) \neq 0 \), we know from Prop. 2.3 that \( E^{Sp_{2nm}}(f_{\Delta(\tau,m),\sqrt{2} \cdot \frac{m+1}{2}}) \) has a pole at \( s = \frac{m}{2} \), as the section varies. We will consider the residual representation \( E^{Sp_{2nm}}(\Delta(\tau,m), \sqrt{2} \cdot \frac{m+1}{2}) \) generated by the residues \( R_{\psi_s} E^{Sp_{2nm}}(f_{\Delta(\tau,m),s}) \).

Theorem 5.1. Assume that \( L(\tau, \sqrt{2}, s) \) has a pole at \( s = 1 \), and \( L(\tau, \frac{1}{2}) \neq 0 \). Then

\[
O(E^{Sp_{2nm}}(\Delta(\tau,m), \sqrt{2} \cdot \frac{m+1}{2})) = (n^2m).
\]

The proof of this theorem was sketched by Ginzburg in [G03], Prop. 3.2. See [L33], Theorem 1.2, for a detailed proof.

Theorem 5.2. Assume that \( L(\tau, \sqrt{2}, s) \) has a pole at \( s = 1 \).
(1) If $\omega_{r} = 1$, then

$$O(\mathcal{E}_{\Delta(r,m)\gamma_{\omega},\nu^{2},[\frac{m+2}{2}]}^{(2)}) = \begin{cases} (n^{2m}), & n \text{ even} \\ ((n+1)^{m},(n-1)^{m}), & n \text{ odd} \end{cases}$$

(2) If $\omega_{r} \neq 1$ (recall that $\omega_{r}^2 = 1$) then

$$O(\mathcal{E}_{\Delta(r,m)\gamma_{\omega},\nu^{2},[\frac{m+2}{2}]}^{(2)}) = \begin{cases} ((n+2)^{m},(n-2)^{m}), & n \text{ even} \\ ((n+1)^{m},(n-1)^{m}), & n \text{ odd} \end{cases}$$

Proof of Theorem 5.2, Part 1:

Assume that $\omega_{r} = 1$. In this case, the proof of the theorem when $n$ is even is the same as that of Theorem 5.1. We prove the theorem when $n = 2n' - 1$ is odd. We first note that $O(\mathcal{E}_{\Delta(r,m)\gamma_{\omega},\nu^{2},[\frac{m+2}{2}]}^{(2)})$ is bounded by $((n+1)^{m},(n-1)^{m})$. This is in [GS20], Prop. 3.2, where it is stated for $m$ even, but the same proof works for $m$ odd as well. The main work of the proof of Theorem 5.2 in this case is to show that $\mathcal{E}_{\Delta(r,m)\gamma_{\omega},\nu^{2},[\frac{m+2}{2}]}^{(2)}$ admits a nontrivial Fourier coefficient, corresponding to the partition $((n+1)^{m},(n-1)^{m})$. The corresponding Fourier coefficient (see [GRS13], Sec. 2) is with respect to the unipotent radical $U_{m,(2m)^{n'-1}} = U_{m,(2m)^{n'-1}}^{Sp_{2m}}$ and the character $\psi'_{U_{m,(2m)^{n'-1}}}(\xi)$ as follows. Write an element of $U_{m,(2m)^{n'-1}}(\bar{k})$ as

$$\begin{pmatrix} I_{m} & x_{1,2} & \cdots & * & * & * & \cdots & * & * \\ I_{2m} & x_{2,3} & * & * & * & \cdots & * & * & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & I_{2m} & x_{n'-1,n'} & * & * & * & \cdots & * & \cdots \\ & I_{2m} & x_{n',n'+1} & * & * & * & \cdots & * & \cdots \\ & I_{2m} & x_{n'-1,n'} & * & * & * & \cdots & * & \cdots \\ & I_{2m} & x_{n',n'+1} & * & * & * & \cdots & * & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & I_{2m} & x_{2,3} & * & * & * & \cdots & * & \cdots \\ & I_{2m} & x_{1,2} & * & * & * & \cdots & * & \cdots \\ & I_{m} & \end{pmatrix}$$

Then

$$\psi'_{U_{m,(2m)^{n'-1}}}(u) = \psi(tr(x_{1,2} \begin{pmatrix} 0_{m \times m} \\ I_{m} \end{pmatrix}) + tr(x_{2,3}) + tr(x_{3,4}) + \cdots tr(x_{n'-1,n'}) + \frac{1}{2} tr(x_{n',n'+1})).$$

Note that $x_{n',n'+1}$ has the form

$$x_{n',n'+1} = \begin{pmatrix} a & b \\ c & w_{n}' aw_{m} \end{pmatrix},$$

where $a, b, c$ are $m \times m$ matrices, so that $\frac{1}{2} tr(x_{n',n'+1}) = tr(a)$.

Our goal is to show that

$$F_{\psi_{U_{m,(2m)^{n'-1}}}}(\xi) = \int_{U_{m,(2m)^{n'-1}}(\bar{k})\setminus U_{m,(2m)^{n'-1}}(\bar{k})} \xi(u) \psi_{U_{m,(2m)^{n'-1}}}(u) du \neq 0,$$
as $\xi$ varies in $\mathcal{E}_{\Delta(\tau,m)\Gamma(V^2, \mathbb{Z})}$. Recall that we identify $U_{m,(2m)^{n' - 1}}(\mathbb{A})$ as a subgroup of $\text{Sp}_{2n}(\mathbb{A})$. Consider the following Weyl element $w_0 \in \text{Sp}_{2n}(F)$. Write $w_0$ as a $2n \times 2n$ matrix of $m \times m$ blocks. Then, for $1 \leq i \leq n$, the $i$-th block row of $w_0$ has the form

$$
(0_{m \times m} \ 0_{m \times m} \ \cdots \ 0_{m \times m} \ I_m \ 0_{m \times m} \ \cdots \ 0_{m \times m})
$$

where $I_m$ appears at the $2i - 1$ position. This determines $w_0$. For example, for $n = 3$

$$
w_0 = 
\begin{pmatrix}
I_m & 0 & 0 & 0 & 0 & 0 \\
0 & I_m & 0 & 0 & 0 & 0 \\
0 & 0 & I_m & 0 & 0 & 0 \\
0 & -I_m & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I_m & 0 & 0 \\
0 & 0 & 0 & 0 & I_m & 0 \\
\end{pmatrix}.
$$

We may conjugate inside the integral (5.3) by $w_0$,

$$
(5.4) \quad \mathcal{F}_{\psi_{U_{m,(2m)^{n' - 1}}}}(\xi) = \int_{U_{m,(2m)^{n' - 1}}(F) \setminus U_{m,(2m)^{n' - 1}}(\mathbb{A})} \xi(w_0u)\psi_{U_{m,(2m)^{n' - 1}}(\mathbb{A})}^{-1}(u)du = \\
= \int_{V(F) \setminus V(\mathbb{A})} \xi(vw_0)\psi_{V}^{-1}(v)dv,
$$

where $V = w_0U_{m,(2m)^{n' - 1}}w_0^{-1}$, and, for $v \in V(\mathbb{A})$,

$$
\psi_{V}(v) = \psi_{U_{m,(2m)^{n' - 1}}(\mathbb{A})}(w_0^{-1}vw_0).
$$

Let us describe the subgroup $V$. Write $v \in V$ as

$$
(5.5) \quad v = \begin{pmatrix} U & X \\ Y & U' \end{pmatrix},
$$

Then $U$ has the form

$$
(5.6) \quad U = \begin{pmatrix}
I_m & u_{1,2} & * & \cdots & * \\
I_m & u_{2,3} & * & \cdots \\
& \ddots & \ddots & \ddots \\
& & I_m & u_{n-1,n} \\
& & & I_m
\end{pmatrix};
$$

the matrix $U'$ has a similar form to (5.6). Next, write the matrices $X$, $Y$, each, as an $n \times n$ matrix of $m \times m$ blocks. Then $X$ and $Y$ have upper triangular shapes. The matrix $Y$ is also such that its main block diagonal and the one above it consist
of zero matrices. Thus, $X$, $Y$ have the forms

$$X = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1,n-1} & a_{n-1,n} & \cdots & a_{n,n} \end{pmatrix}.$$  \hfill (5.7)

$$Y = \begin{pmatrix} 0_{m \times m} & 0_{m \times m} & b_{1,3} & \cdots & b_{1,n} \\ 0_{m \times m} & 0_{m \times m} & b_{2,4} & \cdots & b_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_{m \times m} & 0_{m \times m} & 0_{m \times m} & \cdots & b_{n-2,n} \\ 0_{m \times m} & 0_{m \times m} & 0_{m \times m} & \cdots & 0_{m \times m} \end{pmatrix}.$$  \hfill (5.8)

For $v \in V(\mathcal{A})$ of the form (5.3)-(5.7),

$$\psi_V(v) = \psi(\text{tr}(u_{1,2} + u_{2,3} + \cdots + u_{n',n'+1}) - \text{tr}(u_{n'+1,n'+2} + u_{n'+2,n'+3} + \cdots + u_{n-1,n})).$$  \hfill (5.9)

Note that we have here a special case of (5.8) in [GS20], where in (5.9) in loc. cit. we should replace $r$ by $m$, $n$ by $n'$, and we should also ignore in (5.8) there the second block row and block column. Now we apply a chain of roots exchange, exactly as we did in the second part of the proof of Prop. 6.1 in [GS20], (where we exchanged there $Y_{3,1}^{i-1,j}$ with $X_{1,3}^{j-1,i}$, for $1 < i < j < 2$, and $Y_{1,3}^{j-1,j+1}$ with $X_{1,3}^{j-1,i}$). Thus, we exchange the $(1,3)$ block of $Y$ with the $(2,1)$ block of $X$, then the $(2,4)$, $(1,4)$ blocks of $Y$ with the $(3,2)$, $(3,1)$ blocks of $X$, in this order, and we proceed and exchange, for $1 \leq j \leq n'$, the $(j-2,j)$, $(j-3,j)$,..., $(1,j)$ blocks of $Y$ with the $(j-1,j-2)$, $(j-1,j-3)$,..., $(j-1,1)$ blocks of $X$, in this order. We get, as in Prop. 6.2 in [GS20], that the Fourier coefficient (5.3) (and hence (5.4)), $\mathcal{F}_{\psi_{\psi_U}}(\xi)$ is nontrivial on $\mathcal{E}^{(2)}_{\text{Sp}(2m)} \Delta(\tau,\gamma \nu^2,\beta^{\frac{n-1}{2}})$, if and only if the following Fourier coefficient is nontrivial on $\mathcal{E}^{(2)}_{\text{Sp}(2m)} \Delta(\tau,\gamma \nu^2,\beta^{\frac{n-1}{2}})$.

$$\mathcal{F}_{\psi_{\psi_U}}(\xi) = \int_{V_{n',n'}(\mathcal{F}) \backslash V_{n',n'}(\mathcal{A})} \xi(u) \psi_{\psi_U}^{-1}(u) du,$$

where $V_{n',n'}$ is the subgroup of symplectic matrices of the form (5.6), with $U$ (and $U'$) of the form (5.6), but now $X$, $Y$ are of the following forms,

$$X = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n'} & a_{1,n'+1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ a_{n'-1,1} & \cdots & a_{n'-1,n'} & a_{n'-1,n'+1} & \cdots & * \\ 0 & \cdots & 0 & a_{n',n'} & * & * \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & * & * & & & * \end{pmatrix}.$$  \hfill (5.10)
Recall that $X$ is written as a $n \times n$ matrix of $m \times m$ blocks.

$$
Y = \begin{pmatrix}
0 & 0 & \cdots & 0 & b_{1,n'+1} & b_{1,n'+2} & \cdots & b_{1,n-1} & b_{1,n} \\
0 & 0 & \cdots & 0 & b_{2,n'+1} & b_{2,n'+2} & \cdots & b_{2,n-1} & * \\
& \ddots & & & \vdots & \vdots & & \vdots & \vdots & \\
& & 0 & b_{n'-1,n'+1} & * & * & * \\
& & 0 & 0 & \cdots & 0 & 0 & \\
& & \ddots & & \vdots & \vdots & & \vdots & \\
& & & & 0 & 0 & & & \\
\end{pmatrix}.
$$

(5.11)

The character $\psi_{\gamma_{n',n'}}$ is given by the same formula as in (5.8).

Next, we continue exactly as in the second step, right after Prop. 6.2 in [GS20], meaning that we exchange the $(n'-1, n'+1)$ block of $Y$ "into" the $(n', n'-1)$ block of $X$. Note that if we let $U = I_{mn}$, $X = 0$ and $Y$ as in (5.11), with all blocks being zero, except for $b_{n'-1,n'+1}$, then $w_{mn} b_{n'-1,n'+1}$ must be a symmetric matrix. If we similarly define the subgroup $X^{n'-1,n'}$, then it isomorphic to the additive group of $m \times m$ matrices (as algebraic groups over $F$). Thus, we exchange $Y^{n'-1,n'+1}$ with the subgroup of $X^{n'-1,n'}$, corresponding to the $m \times m$ matrices $x$, such that $w_{mn}x$ is symmetric. It follows that the Fourier coefficient (5.9) is nontrivial on $\mathcal{E}_{\Delta(r,m)\gamma_{n',n'}, \gamma_{n',n'}}^{(2)}$, if and only if the following Fourier coefficient is nontrivial on $\mathcal{E}_{\Delta(r,m)\gamma_{n',n'}, \gamma_{n',n'}}^{(2)}$.

$$
\mathcal{F}_{\psi_{\gamma_{n',n'-1}}}(\xi) = \int_{V^{n',n'-1,0}(F)/V^{n',n'-1,0}(k)} \xi(u) \psi_{\gamma_{n',n'-1}}^{-1}(u) du,
$$

(5.12)

where $V^{n',n'-1,0}$ is the subgroup of symplectic matrices of the form (5.5), with $U$ (and $U''$) of the form (6.6), $Y$ has the form (5.11) with $b_{n'-1,n'+1} = 0$ and $X$ has the form,

$$
X = \begin{pmatrix}
a_{1,1} & a_{1,2} & \cdots & a_{1,n'-1} & a_{1,n'} & a_{1,n'+1} & \cdots & a_{1,n-1} & a_{1,n} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\
a_{n'-1,1} & a_{n'-1,2} & \cdots & a_{n'-1,n'-1} & a_{n'-1,n'} & a_{n'-1,n'+1} & \cdots & * & * \\
0 & 0 & \cdots & a_{n',n'-1} & a_{n',n'} & * & \cdots & * & * \\
& & \vdots & & \vdots & \vdots & & \vdots & \vdots \\
& & & & \vdots & \vdots & & \vdots & \vdots \\
& & & & 0 & * & \cdots & * & * \\
\end{pmatrix},
$$

(5.13)

such that $w_{mn}a_{n',n'-1}$ is symmetric. The character $\psi_{\gamma_{n',n'-1}}$ is given by the same formula as in (5.8). Now, we apply the argument of Prop. 7.1 in [GS20] and get that in (5.12), we may replace $V^{n',n'-1,0}$ by $V^{n',n'-1}$ which is defined as $V^{n',n'-1,0}$, except that in (5.13), we do not restrict $w_{mn}a_{n',n'-1}$ to be symmetric, and let it be arbitrary. The character $\psi_{\gamma_{n',n'-1}}$ is given by the same formula as in (5.8). The point is that when we carry out the Fourier expansion as in Prop. 7.1 in [GS20], we get Fourier coefficients corresponding to partitions of the form $(n + 2, ...)$.

These must be zero on $\mathcal{E}_{\Delta(r,m)\gamma_{n',n'}, \gamma_{n',n'}}^{(2)}$, since $\mathcal{O}(\mathcal{E}_{\Delta(r,m)\gamma_{n',n'}, \gamma_{n',n'}}^{(2)})$ is bounded by $(n + 1)^m (n - 1)^m$. Thus, the Fourier coefficient (5.12) is nontrivial.
on \(E^{(2)}_{\Delta(\tau,m)\gamma_v,\nu^2,|\tau,\nu\rangle^*} \), if and only if the following Fourier coefficient is nontrivial on \\
\(E^{(2)}_{\Delta(\tau,m)\gamma_v,\nu^2,|\tau,\nu\rangle^*} \),

\[
(5.14) \quad F_{\psi_{V^{n',1}}} (\xi) = \int_{V^{n',1}(F) \setminus V^{n',1}(A)} \xi(u) \psi_{V^{n',1}}^{-1}(u) du,
\]

where \(V^{n',1} \) is as \(V^{n',1} \) except that the block \(Y \), written as in (5.11), is such that all blocks in column \(n' + 1 \) (and row \(n' - 1 \)) are zero, and in \(X \), written as in (5.10), we fill in the zero blocks in row \(n' \) (and column \(n' \)). The character \(\psi_{V^{n',1}} \) is given by the same formula as in (5.8). We continue like this, step by step, repeating the arguments of Prop. 7.4, 7.1 in [GS20], and then we exchange roots. We write Fourier expansions, showing at each step that only the trivial character contributes to the Fourier expansion. At each such step, we use the fact that \(O(E^{(2)}_{\Delta(\tau,m)\gamma_v,\nu^2,|\mu,\nu\rangle^*}) \) is bounded by \(((n + 1)^m, (n - 1)^m)\). In the end we get that the Fourier coefficient (5.4) is nontrivial on \(E^{(2)}_{\Delta(\tau,m)\gamma_v,\nu^2,|\mu,\nu\rangle^*} \), if and only if the following Fourier coefficient is nontrivial on \\
\(E^{(2)}_{\Delta(\tau,m)\gamma_v,\nu^2,|\mu,\nu\rangle^*} \),

\[
(5.15) \quad F_{\psi_{U_m^n}} (\xi) = \int_{U_m^n(F) \setminus U_m^n(A)} \xi(u) \psi_{U_m^n}^{-1}(u) du,
\]

where, for \(v = \begin{pmatrix} U & X \\ U^* \end{pmatrix} \in U_m^n(A) \),

with \(U \) as in (5.10), \(\psi_{U_m^n}(v) \) is given by (5.8). Thus, \(F_{\psi_{U_m^n}} \) is the composition of the constant term of \(\xi, \xi_{U_m^n}, \) along \(U_m^n \), and the Fourier coefficient along the unipotent radical \(V_m \) (inside \(GL_m \)), and the character given by (5.8), which we denote now by \(\psi_{U_m^n} \).

As in the proof of Prop. 2.4, and with similar notation, we get that for a decomposable section \(f_{\Delta(\tau,m)\gamma_v,s} \), supported at one place \(v_0 \in S \), inside the open cell \(Q^{(2)}_{mn}(F_{v_0})(\epsilon_0, 1)Q^{(2)}_{mn}(F_{v_0}) \), the constant term along \(U_m^n \) of \(E_{\Delta(\tau,m)\gamma_v,s} \) is as in (2.4),

\[
(5.17) \quad E_{U_m^n}(f_{\Delta(\tau,m)\gamma_v,s})(I) = M(\epsilon_0, s)(f_{\Delta(\tau,m)\gamma_v,s}) = \int_{U_m^n(A)} f_{\Delta(\tau,m)\gamma_v,s}(\epsilon_0 u) du.
\]

Now, with notation similar to the one we used right after (2.4), we have, for \(v \notin S \),

\[
(5.18) \quad M(\epsilon_0, s)(f_{\Delta(\tau,m)\gamma_v,s}) = a(\tau_v, s)f_{\Delta(\tau_v,m)\gamma_v,s}^0.
\]
where, for \( m \) even,
\[
a(\tau_v, s) = \prod_{k=1}^{m/2} \frac{L(\tau_v, \wedge^2, 2s - 2k + 2)L(\tau_v, \vee^2, 2s - 2k + 1)}{L(\tau_v, \wedge^2, 2s + 2k - 1)L(\tau_v, \vee^2, 2s + 2k)}
\]
and for \( m \) odd,
\[
a(\tau_v, s) = \prod_{k=1}^{m-1} \frac{L(\tau_v, \wedge^2, 2s - 2k + 2)L(\tau_v, \vee^2, 2s - 2k + 2)}{L(\tau_v, \wedge^2, 2s + 2k)L(\tau_v, \vee^2, 2s + 2k)}
\]
Note that \( a^S(\tau, s) = \prod_{v \in S} a(\tau_v, s) \) has a pole at \( s = \frac{m}{2} \). Let us take a decomposable section as above, so that, for all \( v \in S \), \( f_{\Delta(\tau_v, m)\gamma_{\psi_v}} \) is supported inside the open cell \( Q_{mn}^{(2)}(F_v)(\epsilon_0, 1)Q_{mn}^{(2)}(F_v) \). Then for \( b \in GL_{mn}(\mathbb{A}) \),
\[
(5.19) \quad [Res_{s=\frac{m}{2}}E^{Sp_{2mn}^{(2)}}(f_{\Delta(\tau_v, m)\gamma_{\psi_v}})]_{U_{mn}}((b, 1)) = \left( Res_{s=\frac{m}{2}}a^S(\tau, s) \right) \prod_{v \in S} M_v(\epsilon_0, s)(f_{\Delta(\tau_v, m)\gamma_{\psi_v}}) |_{s=\frac{m}{2}} f^{0,S}_{\Delta(\tau, m)\gamma_{\psi_v}, -\frac{m}{2}}((b, 1)),
\]
where \( f^{0,S}_{\Delta(\tau, m)\gamma_{\psi_v}, -\frac{m}{2}} = \prod_{v \in S} f_{\Delta(\tau_v, m)\gamma_{\psi_v}, -\frac{m}{2}}^0 \). Since the Fourier coefficient along \( V_{mn} \), with respect to \( \psi_{V_{mn}} \), is nontrivial on \( \Delta(\tau, m) \), then it is nontrivial on the automorphic forms \( (5.19) \), viewed as automorphic forms on the double cover of \( GL_{mn}(\mathbb{A}) \). This proves that the Fourier coefficient \( (5.16) \) is nontrivial on \( E^{Sp_{2mn}^{(2)}}_{\Delta(\tau,m)\gamma_{\psi_v}, \vee^2, |\frac{m+1}{2}|} \). This proves the first part of the theorem.

**Proof of Theorem 5.2, Part 2:**

Assume now, that the quadratic character \( \omega_v \) is nontrivial. Let \( v \) be a finite place, where \( \tau_v \) is unramified, and its central character is the unique, unramified, nontrivial character \( \lambda_v \). Assume that \( n = 2n' \) is even. Since \( \tau_v \) is self dual, we can write \( \tau_v \) as a parabolic induction from an unramified character of the standard Borel subgroup, as follows,
\[
(5.20) \quad \tau_v = \chi_1 \times \cdots \times \chi_{n'-1} \times 1 \times \lambda_v \times \chi_1^{-1} \times \cdots \times \chi_1^{-1},
\]
where \( \chi_i \) are unramified characters of \( F_v^* \). Then the unramified constituent of the factor at \( v \) of \( E^{Sp_{2mn}^{(2)}}_{\Delta(\tau,m)\gamma_{\psi_v}, \vee^2, |\frac{m+1}{2}|} \) is the unramified constituent of the following parabolic induction,
\[
(5.21) \quad Ind_{Q_{(2m)n'-1,2}^{(2)}(F_v)}^{Sp_{2mn}^{(2)}(F_v)} [\otimes_{i=1}^{n'-1} (\chi_i \circ det_{GL_{2mn}}) \otimes (\lambda_v \circ det_{GL_m}) |^{\frac{m}{2}} \otimes |det_{GL_m}|^{\frac{m}{2}} \gamma_{\psi_v}}.
\]
Recall that the pole in question here is at \( s = \frac{m}{2} \). As in the proof of Prop. 3.2 in [GS20], we conclude, from \( (5.21) \), that all symplectic partitions of \( 2mn \) corresponding to nontrivial Fourier coefficients on \( E^{Sp_{2mn}^{(2)}}_{\Delta(\tau,m)\gamma_{\psi_v}, \vee^2, |\frac{m+1}{2}|} \) are bounded by the induced nilpotent orbit corresponding to \( (6.21) \), and this corresponds to the partition \( ((n + 2)^{n}, (n - 2)^{m}) \) of \( 2mn \). See [CM93], Chapter 7. When \( n = 2n' + 1 \) is odd, the analog of \( (5.20) \) is, with similar notation,
\[
(5.22) \quad \tau_v = \chi_1 \times \cdots \times \chi_{n'} \times \lambda_v \times \chi_1^{-1} \times \cdots \times \chi_1^{-1},
\]
In this case, as in the case where \( n \) is odd and the central character is trivial, the associated partition for (5.23) is \(((n + 1)m, (n - 1)m)\), as in the first part of the theorem. What we proved in the first part works exactly the same in this case. This proves the second part of the theorem when \( n \) is odd. Thus, assume that \( n = 2n' \) is even (and \( \omega_r \neq 1 \)). As in the first part, we need to show that \( \mathcal{E}_{\Delta(t,m)\gamma_0 \cdot \nu^2, [m \times m]}^{Sp(2m, \mathbb{A})} \) admits a (nontrivial) Fourier coefficient, corresponding to the partition \(((n + 2)m, (n - 2)m)\). The corresponding Fourier coefficient (see [GRS03], Sec. 2) is with respect to the unipotent radical \( U_{m^2, (2m)^{n-1}} = U_{Sp_{2m}, m^2, (2m)^{n-1}} \) and the character \( \psi_{U_{m^2, (2m)^{n-1}}} \) as follows. Write an element of \( U_{m^2, (2m)^{n-1}}(\mathbb{A}) \) as

\[
(5.24) \quad e = \begin{pmatrix} I_m & x & z \\ u & x' & I_m \end{pmatrix} \in Sp_{2mn}(\mathbb{A}),
\]

where \( u \) has the form (5.1). We keep using the notation in (5.1), only that now, with \( n' = \frac{n}{2} \), \( u \) lies in \( Sp_{2m(n-1)}(\mathbb{A}) \). Write in (5.24),

\[
x = (x_{0,1}, x_{0,2}, \ldots, x_{0,n}),
\]

where \( x_{0,1}, x_{0,n} \in M_{m \times m}(\mathbb{A}), x_{0,2}, \ldots, x_{0,n-1} \in M_{m \times 2m}(\mathbb{A}) \). The character \( \psi_{U_{m^2, (2m)^{n'-1}}} \) is given by

\[
(5.25) \quad \psi_{U_{m^2, (2m)^{n'-1}}}(u) = \psi(tr(x_{0,1}) + tr(x_{1,2})I_m + tr(x_{2,3}) + \cdots + tr(x_{n'-1,n'} + \frac{1}{2} tr(x_{n',n'+1})).
\]

As in (5.3), our goal is to show that

\[
(5.26) \quad \mathcal{F}_{\psi_{U_{m^2, (2m)^{n'-1}}}}(\xi) = \int_{U_{m^2, (2m)^{n'-1}}(\mathbb{A})} \xi(u) \psi_{U_{m^2, (2m)^{n'-1}}}(u) du \neq 0,
\]

as \( \xi \) varies in \( \mathcal{E}_{\Delta(t,m)\gamma_0 \cdot \nu^2, [m \times m]}^{Sp_{2m, \mathbb{A}}}(F) \). The proof is very similar to the first part, and so we will just sketch it. Consider the following Weyl element \( \tilde{w}_0 \in Sp_{2mn}(F) \). Write \( \tilde{w}_0 \) as a \( 2n \times 2n \) matrix of \( m \times m \) blocks. Then the first three block rows form the matrix

\[
(I_{3m} \ 0 \ \cdots \ 0).
\]

For \( 4 \leq i \leq n \), the \( i \)-th block row of \( \tilde{w}_0 \) has the form

\[
\begin{pmatrix}
0_{m \times m} & 0_{m \times m} & \cdots & 0_{m \times m} & I_m & 0_{m \times m} & \cdots & 0_{m \times m}
\end{pmatrix},
\]

and the analog of (5.21) is

\[
(5.23) \quad \text{Ind}_{Q_{(2m)^{n'}}}^{Sp_{2m, (2m)^{n'}}}(\chi_1 \circ \text{det}_{GL_{2m}} \otimes (\lambda_r \circ \text{det}_{GL_{mn}})|\text{det}_{GL_{mn}}|^{\frac{E}{2}}) \gamma_v.
\]
where \( I_m \) appears at the \( 2i - 3 \) position. This determines \( \tilde{w}_0 \). For example, for \( n = 6 \)
\[
\tilde{w}_0 = \begin{pmatrix}
I_{3m} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I_m & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I_m & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I_m & 0 \\
0 & -I_m & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -I_m & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I_m & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I_{3m}
\end{pmatrix}.
\]

We conjugate inside the integral \((5.20)\) by \( \tilde{w}_0 \),
\[
(5.27) \quad \mathcal{F}_{\psi_U^{-1}}(\xi) = \int_{U_{m^2,\langle 2m \rangle^{n' - 1}}(F) \setminus U_{m^2,\langle 2m \rangle^{n' - 1}}(A)} \xi(\tilde{w}_0 U)\psi_U^{-1}(u)du = \int_{V(F) \setminus V(A)} \xi(vw_0)\psi_V^{-1}(v)dv,
\]
where \( V = \tilde{w}_0 U_{m^2,\langle 2m \rangle^{n' - 1}}\tilde{w}_0^{-1} \), and, for \( v \in V(A) \),
\[
\psi_V(v) = \psi_U^{-1}(v) = \psi_U^{-1}(\tilde{w}_0^{-1}v\tilde{w}_0).
\]
The subgroup \( V \) has a similar description to \((5.3) - (5.7)\). In the notation of \((5.3)\), \( U \) has the form \((5.6)\). Write the matrices \( X, Y \), each, as \( n \times n \) matrices of \( m \times m \) blocks. They have the forms
\[
(5.28) \quad X = \begin{pmatrix}
a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n-1} & a_{1,n} \\
a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n-1} & a_{2,n} \\
a_{3,2} & a_{3,3} & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} & \cdots & a_{n-1,n} & a_{n,n}
\end{pmatrix};
\]
\[
Y = \begin{pmatrix}
0_{m \times m} & 0_{m \times m} & 0_{m \times m} & b_{1,4} & b_{1,5} & \cdots & b_{1,n} \\
0_{m \times m} & 0_{m \times m} & 0_{m \times m} & b_{2,4} & b_{2,5} & \cdots & b_{2,n} \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0_{m \times m} & 0_{m \times m} & 0_{m \times m} & 0_{m \times m} & 0_{m \times m} & b_{n-3,n} & \cdots \\
0_{m \times m} & 0_{m \times m} & 0_{m \times m} & 0_{m \times m} & 0_{m \times m} & 0_{m \times m} & 0_{m \times m} \\
\end{pmatrix}.
\]

The matrix \( Y \) has an upper triangular shape, with its main block diagonal and two diagonals above it being zero. The matrix \( X \) is such that its lower \( n - 2 \) block diagonals are zero. Finally, for \( v \in V(A) \) of the form \((5.3), (5.6), (5.28)\),
\[
(5.29) \quad \psi_V(v) = \psi(tr(u_{1,2} + u_{2,3} + \cdots + u_{n',+1,n'+2}) - tr(u_{n'+2,n'+2} + u_{n'+3,n'+4} + \cdots + u_{n-1,n})).
\]
Now we exchange roots ‘from \( Y \) into \( X \)’ as we did in the first part. We exchange the block \( b_{1,4} \) in \( Y \) (notation of \((5.28)\)) to ‘fill’ the block in \( X \) in position \((3,1)\), then the blocks \( b_{2,5}, b_{1,5} \) in \( Y \) to fill in the blocks in \( X \) in positions \((4,1), (4,2)\) (in
this order), and so on, exactly as in the first part. The analogs of (5.10), (5.11) are as follows.

\[
X = \begin{pmatrix}
  a_{1,1} & \cdots & a_{1,n'} & \cdots & a_{1,n+1} & \cdots & a_{1,n}\\
  \vdots & & \vdots & & \vdots & & \vdots \\
  a_{n',1} & \cdots & a_{n',n'} & \cdots & a_{n',n+1} & \cdots & * \\
  0 & \cdots & 0 & a_{n'+1,n'} & * & * \\
  0 & * & * & & & & \\
  \vdots & & \vdots & & \vdots & & \\
  \end{pmatrix}
\]

(5.30)

\[
Y = \begin{pmatrix}
  0 & 0 & \cdots & 0 & b_{1,n'+2} & b_{1,n'+3} & \cdots & b_{1,n-1} & b_{1,n} \\
  0 & 0 & b_{2,n'+2} & b_{2,n'+3} & b_{2,n-1} & * \\
  \vdots & & \vdots & & \vdots & & \\
  0 & b_{n'-1,n'+2} & * & * & * \\
  0 & 0 & \cdots & 0 & 0 & 0 \\
  \vdots & & \vdots & & \vdots & & \\
  \end{pmatrix}
\]

(5.31)

We continue as in the first part, using repeatedly the arguments in the proofs of Prop. 6.2, Prop. 7.1, Prop. 7.4 in [GS20], and the fact that \(\mathcal{O}(\mathcal{E}^{(2)}_{\Delta(\tau,m)\gamma,\sqrt{2}n,n+\frac{1}{2}})\) is bounded by \(((n+2)^m, (n-2)^m)\). In the end we get that the Fourier coefficient (5.27) is nontrivial on \(\mathcal{E}^{(2)}_{\Delta(\tau,m)\gamma,\sqrt{2}n,n+\frac{1}{2}}\) if and only if the Fourier coefficient (5.16) is nontrivial on \(\mathcal{E}^{(2)}_{\Delta(\tau,m)\gamma,\sqrt{2}n,n+\frac{1}{2}}\), and this we proved in the end of the first part. This completes the proof of the second part of the theorem.

6. Proof of Theorem 2.3

I. \(L(\tau, \wedge^2, s)\) has a pole at \(s = 1\) and \(L(\tau, \frac{1}{2}) \neq 0\).

Let us write the statement of the theorem in detail for this case:

1. For \(m\) even, \(1 \leq k \leq \frac{m}{2}\) \((\mathcal{E}_{k,m}^{(2)}(\wedge^2) = k)\)

\[
\mathcal{O}(\mathcal{E}^{(2)}_{\Delta(\tau,m)\wedge^2,k}) = (2n)^{m-2k}, n^{4k}.
\]

2. For \(m\) odd, \(1 \leq k \leq \frac{m+1}{2}\), \((\mathcal{E}_{k,m}^{(2)}(\wedge^2) = k - \frac{1}{2})\)

\[
\mathcal{O}(\mathcal{E}^{(2)}_{\Delta(\tau,m)\wedge^2,k}) = (2n)^{m-2k+1}, n^{4k-2}.
\]

3. For \(m\) even, \(1 \leq k \leq \frac{m}{2}\), \((\mathcal{E}_{k,m}^{(2)}(\wedge^2) = k - \frac{1}{2})\)

\[
\mathcal{O}(\mathcal{E}^{(2)}_{\Delta(\tau,m)\wedge^2,k}) = (2n)^{m-2k+1}, n^{4k-2}.
\]

4. For \(m\) odd, \(1 \leq k \leq \frac{m}{2}\), \((\mathcal{E}_{k,m}^{(2)}(\wedge^2) = k)\)

\[
\mathcal{O}(\mathcal{E}^{(2)}_{\Delta(\tau,m)\wedge^2,k}) = (2n)^{m-2k}, n^{4k}.
\]
Assume that $L(\tau, \Lambda^2, s)$ has a pole at $s = 1$, and $L(\tau, \frac{1}{2}) \neq 0$. We note, first, that the indicated partition, in each case, bounds $O(\mathcal{E}_{\Delta(\tau,m)}^{S_{\nu,2}}(\tau,\Lambda^2,k))$. This is proved in [GS20], Prop. 3.1, when $m$ is even. The proof when $m$ is odd is entirely the same. Thus, all we need is to show that the residual Eisenstein series admits in each case a nontrivial Fourier coefficient corresponding to the indicated top partition. When an automorphic representation $\pi$ admits a nontrivial Fourier coefficient corresponding to a partition $\nu$, we will also say that $\nu$ supports $\pi$.

We prove, first, part I.2 of the theorem, by induction on $m$. The proof then implies part I.3. The proof follows the exact same lines of the proof of Theorem 3.1. We start with $E^{S_{p,2n}}(f_{r,\pi})$. It has a pole at $s = \frac{1}{2}$, and by Theorem 5.1

\[(6.1)\quad O(\mathcal{E}_{\Delta(m,1)}^{S_{p,2n}}) = (n^2)\]

This proves part I.2 of the theorem for $m = 1$.

Assume by induction that, for $m = 2i - 1$ odd, and $1 \leq k \leq \frac{m+1}{2} = i$,

\[(6.2)\quad O(\mathcal{E}_{\Delta(2i-1),\Lambda^2,k}^{S_{p,2m}}) = ((2n)^{m-2k+1}, n^{4k-2})\]

By Theorem 2.4, we have the relation (3.2). By the same argument as in the proof of Theorem 3.1 (right after (3.1)), using Cor. 2.8 and Prop. 2.9, we conclude from (6.2) that $(2n, 2^{2m}) \circ ((2n)^{m-2k+1}, n^{4k-2})$ supports $\mathcal{E}_{\Delta(m,1)}^{S_{p,2n}}$. We used the notion of composition of unipotent classes, defined in the end of Sec. 1 in [GRS03]. Note that the Fourier coefficient used to define $D_{\psi,m(2i-1)}$ in (3.1) corresponds to the partition $(2n, 2^{2i-1})$ of $4n$. By Lemma 6, $(2n)^{m-2k+2}, n^{4k-2}$ supports $\mathcal{E}_{\Delta(m,1)}^{S_{p,2n}}$. This proves that

\[(6.3)\quad O(\mathcal{E}_{\Delta(2i),\Lambda^2,k}^{S_{p,2m}}) = ((2n)^{2i-2k+1}, n^{4k-2})\]

By Theorem 2.4, we have the relation (3.3). Again, as above (using Cor. 2.8 and Prop. 2.9), we conclude from (6.3) that $(2n, 2^{4n}) \circ ((2n)^{2i-2k+1}, n^{4k-2})$ supports $\mathcal{E}_{\Delta(m,1)}^{S_{p,2n+(2i-1)}}$. By Lemma 6, $(2n)^{2i-2k+2}, n^{4k-2}$ supports $\mathcal{E}_{\Delta(2i+1),\Lambda^2,k}$, and hence, for all $1 \leq k \leq i = \frac{m+1}{2}$,

\[(6.4)\quad O(\mathcal{E}_{\Delta(2i+1),\Lambda^2,k}^{S_{p,2n(m+2)}}) = ((2n)^{(2i+1)-2k+1}, n^{4k-2})\]

For $k = i + 1 = \frac{m+3}{2}$, we know from Theorem 5.1 that

\[O(\mathcal{E}_{\Delta(m+2),\Lambda^2,k}^{S_{p,2n(m+2)}}) = (n^{2(m+2)})\]

Thus, (6.4) is valid for $k = \frac{m+3}{2}$, as well. This proves parts I.2, I.3 of the theorem.

The proof of Parts I.1, I.4 is similar and follows the same lines of the proof of Theorem 3.2. One proves part I.1 by induction on $m$ (even) and part I.3 follows from the proof. Note that when $m = 2$, part I.1 is a special case of Theorem 5.1. We omit the details.

II. $L(\tau, \nu^2, s)$ has a pole at $s = 1$, and $\omega_\tau = 1$.

Assume that $L(\tau, \nu^2, s)$ has a pole at $s = 1$, and $\omega_\tau = 1$. The statement of the theorem in detail is the following:
(1) For $m$ even and $1 \leq k \leq \frac{m}{2}$, $(e_{k,m}^{\text{Sp}_{2m}}(\sqrt{2}) = k - \frac{1}{2})$

$$\mathcal{O}(\mathcal{E}_{\Delta(\tau,m),\sqrt{2},k}) = \begin{cases} ((2n)^{m-2k+1}, n^{4k-2}), & n \text{ even} \\ ((2n)^{m-2k+1}, (n+1)^{2k-1}, (n-1)^{2k-1}), & n \text{ odd} \end{cases}$$

(2) For $m$ odd and $1 \leq k \leq \frac{m+1}{2}$, $(e_{k,m}^{\text{Sp}_{2m}}(\sqrt{2}) = k)$

$$\mathcal{O}(\mathcal{E}_{\Delta(\tau,m),\sqrt{2},k}) = \begin{cases} ((2n)^{m-2k}, n^{4k}), & n \text{ even} \\ ((2n)^{m-2k}, (n+1)^{2k}, (n-1)^{2k}), & n \text{ odd} \end{cases}$$

(3) For $m$ even and $1 \leq k \leq \frac{m}{2}$, $(e_{k,m}^{\text{Sp}_{2m}}(\sqrt{2}) = k)$

$$\mathcal{O}(\mathcal{E}_{\Delta(\tau,m)\gamma_0,\sqrt{2},k}) = \begin{cases} ((2n)^{m-2k}, n^{4k}), & n \text{ even} \\ ((2n)^{m-2k}, (n+1)^{2k}, (n-1)^{2k}), & n \text{ odd} \end{cases}$$

(4) For $m$ odd and $1 \leq k \leq \frac{m+1}{2}$, $(e_{k,m}^{\text{Sp}_{2m}}(\sqrt{2}) = k)$

$$\mathcal{O}(\mathcal{E}_{\Delta(\tau,m)\gamma_0,\sqrt{2},k}) = \begin{cases} ((2n)^{m-2k+1}, n^{4k-2}), & n \text{ even} \\ ((2n)^{m-2k+1}, (n+1)^{2k-1}, (n-1)^{2k-1}), & n \text{ odd} \end{cases}$$

We note again that the indicated partition, in each case, bounds $\mathcal{O}(\mathcal{E}_{\Delta(\tau,m)\gamma_0,\sqrt{2},k})$.

This is proved in [GS20], Prop. 3.2, when $m$ is even. The proof when $m$ is odd is entirely the same. Thus, all we need is to show that the residual Eisenstein series admits in each case a nontrivial Fourier coefficient corresponding to the indicated top partition.

We prove first part II.4 of the theorem, by induction on $m$. The proof then implies part II.1. The proof follows the same lines of the proof of Theorem 3.3. We start with $\mathcal{E}_{\text{Sp}_{2m}}^{(2)}(f_{r,s})$. It has a pole at $s = \frac{1}{2}$, and by Theorem 5.2(1),

$$\mathcal{O}(\mathcal{E}_{\tau,\gamma_0,\sqrt{2},1}) = \begin{cases} (n^2), & n \text{ even} \\ (n+1, n-1), & n \text{ odd} \end{cases}$$

This proves part II.4 of the theorem for $m = 1$. Assume by induction that, for $m = 2i - 1$ odd, and $1 \leq k \leq \frac{m}{2} = i$,

$$\mathcal{O}(\mathcal{E}_{\Delta(\tau,2i-1)\gamma_0,\sqrt{2},k}) = \begin{cases} ((2n)^{m-2k+1}, n^{4k-2}), & n \text{ even} \\ ((2n)^{m-2k+1}, (n+1)^{2k-1}, (n-1)^{2k-1}), & n \text{ odd} \end{cases}$$

By Theorem 2.4 we have the relation (3.6). We conclude from (3.3) and the induction assumption, using Cor. 2.8 and Prop. 2.9 that $(2n, 1^{2mn}) \circ ((2n)^{m-2k+1}, n^{4k-2})$, when $n$ is even, and $(2n, 1^{2mn}) \circ ((2n)^{m-2k+1}, (n+1)^{2k-1}, (n-1)^{2k-1})$, when $n$ is odd, supports $\mathcal{E}_{\Delta(\tau,2i),\sqrt{2},k}$. By GRS03, Lemma 6, $(2n)^{2i-2k+1}, n^{4k-2})$, when $n$ is even, and $(2n)^{2i-2k+1}, (n+1)^{2k-1}, (n-1)^{2k-1})$, when $n$ is odd, supports $\mathcal{E}_{\Delta(\tau,2i),\sqrt{2},k}$. Hence, for all $1 \leq k \leq \frac{2i}{2} = i$,

$$\mathcal{O}(\mathcal{E}_{\Delta(\tau,2i)\gamma_0,\sqrt{2},k}) = \begin{cases} ((2n)^{2i-2k+1}, n^{4k-2}), & n \text{ even} \\ ((2n)^{2i-2k+1}, (n+1)^{2k-1}, (n-1)^{2k-1}), & n \text{ odd} \end{cases}$$

By Theorem 2.4 we have the relation (3.7). Again, we conclude from the last equality and (3.7), using Cor. 2.8 and Prop. 2.9 that $(2n, 1^{4i}) \circ ((2n)^{2i-2k+1}, n^{4k-2})$, .
when \( n \) is even, and \((2n, 1^{4n}) \circ ((2n)^{2i} - 2k + 1, (n + 1)^{2k-1}, (n - 1)^{2k-1})\), when \( n \) is odd, supports \( \mathcal{E}^{(2)}_{\Delta(\tau, 2i+1)_{\gamma_0, \nu^2, k}} \). By [GRS03], Lemma 6, \((2n)^{2i} - 2k + 2, (n)^{4k-2}\), when \( n \) is even, and \((2n)^{2i} - 2k + 2, (n + 1)^{2k-1}, (n - 1)^{2k-1}\), when \( n \) is odd, supports \( \mathcal{E}^{(2)}_{\Delta(\tau, 2i+1)_{\gamma_0, \nu^2, k}} \). Hence, for all \( 1 \leq k < \frac{2i+1+1}{2} = i + 1 \),

\[
\mathcal{O}(\mathcal{E}^{(2)}_{\Delta(\tau, 2i+1)_{\gamma_0, \nu^2, k}}) = \begin{cases} 
((2n)^{2i+1} - 2k + 1, (n + 1)^{4k-2}), & n \text{ even} \\
((2n)^{2i+1} - 2k + 1, (n + 1)^{2k-1}, (n - 1)^{2k-1}), & n \text{ odd}
\end{cases}
\]

The last equality is true for \( k = \frac{(2i+1)+1}{2} = i + 1 \), as well, by Theorem [3.21]. This proves parts II.4, II.1 of the theorem.

The proof of parts II.3, II.2 is entirely similar. Its proof and the proof of Theorem 3.4 are parallel. One proves part II.3 by induction on \( m \) (even) and part II.2 follows from the proof. Note that part II.3 for \( m = 2 \) is a special case of Theorem 5.2. We omit the details.

III. \( L(\tau, \nu^2, s) \) has a pole at \( s = 1 \), and \( \omega_{\tau} \neq 1 \).

Assume that \( L(\tau, \nu^2, s) \) has a pole at \( s = 1 \), and \( \omega_{\tau} \neq 1 \). The statement of the theorem in detail is the following:

(1) For \( m \) even and \( 1 \leq k \leq \frac{m}{2} \) \((\mathcal{E}^{(2)}_{\Delta(\tau, m)_{\nu^2, k}}(\nu^2) = k - \frac{1}{2})\)

\[
\mathcal{O}(\mathcal{E}^{(2)}_{\Delta(\tau, m)_{\nu^2, k}}) = \begin{cases} 
((2n)^{m-2k+1}, (n + 2)^{2k-1}, (n - 2)^{2k-1}), & n \text{ even} \\
((2n)^{m-2k+1}, (n + 1)^{2k-1}, (n - 1)^{2k-1}), & n \text{ odd}
\end{cases}
\]

(2) For \( m \) odd and \( 1 \leq k \leq \frac{m}{2} \) \((\mathcal{E}^{(2)}_{\Delta(\tau, m)_{\nu^2, k}}(\nu^2) = k)\)

\[
\mathcal{O}(\mathcal{E}^{(2)}_{\Delta(\tau, m)_{\nu^2, k}}) = \begin{cases} 
((2n)^{m-2k}, (n + 2)^{2k}, (n - 2)^{2k}), & n \text{ even} \\
((2n)^{m-2k}, (n + 1)^{2k}, (n - 1)^{2k}), & n \text{ odd}
\end{cases}
\]

(3) For \( m \) even and \( 1 \leq k \leq \frac{m+1}{2} \) \((\mathcal{E}^{(2)}_{\Delta(\tau, m)_{\gamma_0, \nu^2, k}}(\nu^2) = k)\)

\[
\mathcal{O}(\mathcal{E}^{(2)}_{\Delta(\tau, m)_{\gamma_0, \nu^2, k}}) = \begin{cases} 
((2n)^{m-2k}, (n + 2)^{2k}, (n - 2)^{2k}), & n \text{ even} \\
((2n)^{m-2k}, (n + 1)^{2k}, (n - 1)^{2k}), & n \text{ odd}
\end{cases}
\]

(4) For \( m \) odd and \( 1 \leq k \leq \frac{m+1}{2} \) \((\mathcal{E}^{(2)}_{\Delta(\tau, m)_{\gamma_0, \nu^2, k}}(\nu^2) = k - \frac{1}{2})\)

\[
\mathcal{O}(\mathcal{E}^{(2)}_{\Delta(\tau, m)_{\gamma_0, \nu^2, k}}) = \begin{cases} 
((2n)^{m-2k+1}, (n + 2)^{2k-1}, (n - 2)^{2k-1}), & n \text{ even} \\
((2n)^{m-2k+1}, (n + 1)^{2k-1}, (n - 1)^{2k-1}), & n \text{ odd}
\end{cases}
\]

When \( n \) is odd, this part is the same as the last part. Thus, assume that \( n = 2n' \) is even. In each case, the indicated partition bounds \( \mathcal{O}(\mathcal{E}^{(2)}_{\Delta(\tau, m)_{\gamma_0, \nu^2, k}}) \). The proof is similar to that of Prop. 3.2 in [GS20]. Let us sketch it for \( H = \text{Sp}^{(2)}_{2mn} \). The case of the residue at \( s = \frac{m}{2} \) was proved in the beginning of the proof of Theorem 5.2. The proof for \( 1 \leq k < \frac{m+1}{2} \) is similar. Let \( v \) be a finite place, where \( \tau_v \) is unramified, and its central character is the unique, unramified, nontrivial character \( \lambda_v \). Write \( \tau_v \) as in [5.20]. Then the unramified constituent of the factor at \( v \) of \( \mathcal{E}^{(2)}_{\Delta(\tau, m)_{\gamma_0, \nu^2, k}} \) is the unramified constituent of the following parabolic inductions,
according to whether \( m \) is even, or odd. When \( m \) is even, this is the parabolic induction from the following character of \( Q_{(m+2k)n'-1,(m-2k)n'-1,mz}^{(2)}(F_v) \),

\[
\otimes_{i=1}^{n'-1} (\chi_i \circ \text{det}_{GL_{m+2k}}) \otimes (\chi_i \circ \text{det}_{GL_{m-2k}}) \otimes (\lambda_n \circ \text{det}_{GL_m}) \mid \text{det}_{GL_m} \mid^{k} \mid \text{det}_{GL_m} \mid^{k} \gamma_v;
\]

When \( m \) is odd, this is the parabolic induction from the following character of \( Q_{(m+2k-1)n'-1,(m-2k+1)n'-1,mz}^{(2)}(F_v) \),

\[
\otimes_{i=1}^{n'-1} (\chi_i \circ \text{det}_{GL_{m+2k-1}}) \otimes (\chi_i \circ \text{det}_{GL_{m-2k+1}}) \otimes (\lambda_n \circ \text{det}_{GL_m}) \mid \text{det}_{GL_m} \mid^{k-\frac{2}{i}} \otimes \mid \text{det}_{GL_m} \mid^{k-\frac{2}{i}} \gamma_v;
\]

As in the proof of Prop. 3.2 in [GS20], we conclude, from (6.5), (6.6), that all symplectic partitions of \( 2mn \) corresponding to nontrivial Fourier coefficients on \( E^{Sp_{2n}^{(2)}}(\tau,\gamma_v,\psi,1) \) are bounded by the induced nilpotent orbit corresponding to (6.6), and this corresponds to the partition \((2n)^{m-2k}, (n+2)^{2k}, (n-2)^{2k}) \), when \( m \) is even, and \((2n)^{m-2k+1}, (n+2)^{2k-1}, (n-2)^{2k-1}) \), when \( m \) is odd. See [CM93], Chapter 7. It remains to show that the residual Eisenstein series admits in each case a nontrivial Fourier coefficient corresponding to the indicated top partition.

We now prove part III.4 by induction on \( m \) (even) and part III.1 follows from the proof. The proof is the same as in the last part, except that we need to replace at each place in the proof (when \( n \) is even) the partition \((2n)^{m-2k+1}, n^{4k-2}) \) by the partition \((2n)^{m-2k+1}, (n+2)^{2k-1}, (n-2)^{2k-1}) \). We start with \( E^{Sp_{2n}^{(2)}}(f,\gamma_v) \). It has a pole at \( s = \frac{1}{2} \), and by Theorem (5.2),

\[
\mathcal{O}(E^{Sp_{2n}^{(2)}}_{(\tau,\gamma_v,\psi,1)}) = (n+2, n-2).
\]

This proves part III.4 of the theorem for \( m = 1 \). Assume by induction that, for \( m = 2i - 1 \) odd, and \( 1 \leq k \leq \frac{m+2}{2} = i \) (recall that \( n \) is even),

\[
\mathcal{O}(E^{Sp_{2n}^{(2)}}_{\Delta(\tau,2i-1)\gamma_v,\psi}) = ((2n)^{m-2k+1}, (n+2)^{2k-1}, (n-2)^{2k-1}).
\]

As in the last part, we conclude from (3.40) and the induction assumption, that \((2n, 1^{2m}) \circ ((2n)^{m-2k+1}, (n+2)^{2k-1}, (n-2)^{2k-1}) \) supports \( E^{Sp_{2n}^{(2)}}_{\Delta(\tau,2i),\psi} \). By [GRS03], Lemma 6, \((2n)^{2i-2k+1}, (n+2)^{2k-1}, (n-2)^{2k-1}) \) supports \( E^{Sp_{2n}^{(2)}}_{\Delta(\tau,2i),\psi} \). Hence, for all \( 1 \leq k \leq \frac{2i}{2} = i \),

\[
\mathcal{O}(E^{Sp_{2n}^{(2)}}_{\Delta(\tau,2i),\psi}) = ((2n)^{2i-2k+1}, (n+2)^{2k-1}, (n-2)^{2k-1}).
\]

By Theorem (5.2), we have the relation (3.41). Again, we conclude from the last equality and (3.41) that \((2n, 1^{4n}) \circ ((2n)^{2i-2k+1}, (n+2)^{2k-1}, (n-2)^{2k-1}) \) supports \( E^{Sp_{2n}^{(2)}}_{\Delta(\tau,2i+1)\gamma_v,\psi} \). By [GRS03], Lemma 6, \((2n)^{2i-2k+2}, (n+2)^{2k-1}, (n-2)^{2k-1}) \) supports \( E^{Sp_{2n}^{(2)}}_{\Delta(\tau,2i+1)\gamma_v,\psi} \). Hence, for all \( 1 \leq k < \frac{(2i+1)+1}{2} = i + 1 \),

\[
\mathcal{O}(E^{Sp_{2n}^{(2)}}_{\Delta(\tau,2i+1)\gamma_v,\psi}) = ((2n)^{(2i+1)-2k+1}, (n+2)^{2k-1}, (n-2)^{2k-1}).
\]
The last equality is true for $k = \frac{(2i+1)+1}{2} = i + 1$, as well, by Theorem 5.2. This proves parts III.4, III.1 of the theorem. We leave the proof of parts III.3, III.2 to the reader.

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