Abstract

It has been shown by Strahov and Fyodorov that averages of products and ratios of characteristic polynomials corresponding to Hermitian matrices of a unitary ensemble, involve kernels related to orthogonal polynomials and their Cauchy transforms. We will show that, for the unitary ensemble \( \hat{Z}_n | \det M |^{2\alpha} e^{-n V(M)} dM \) of \( n \times n \) Hermitian matrices, these kernels have universal behavior at the origin of the spectrum, as \( n \to \infty \), in terms of Bessel functions. Our approach is based on the characterization of orthogonal polynomials together with their Cauchy transforms via a matrix Riemann-Hilbert problem, due to Fokas, Its and Kitaev, and on an application of the Deift/Zhou steepest descent method for matrix Riemann-Hilbert problems to obtain the asymptotic behavior of the Riemann-Hilbert problem.

1 Introduction

Characteristic polynomials of random matrices are useful to make predictions about moments of the Riemann-Zeta function, see \[8, 18, 19, 21\]. Another domain where they are of great value is quantum chromodynamics, see for example \[2, 3, 9, 33\]. In this paper we consider characteristic polynomials \( \det(x - M) \) of random matrices taken from the following unitary ensemble of \( n \times n \) Hermitian matrices \( M \), cf. \[4, 5, 25\]

\[
\frac{1}{\hat{Z}_n} | \det M |^{2\alpha} e^{-n V(M)} dM, \quad \alpha > -1/2.
\]

(1.1)

Here \( dM \) is the associated flat Lebesgue measure on the space of \( n \times n \) Hermitian matrices, and \( \hat{Z}_n \) is a normalization constant. The confining potential \( V \) in (1.1) is a real valued function with enough increase at infinity, for example a polynomial of even degree with positive leading coefficient. This unitary ensemble induces a probability density function on the \( n \) eigenvalues \( x_1, \ldots, x_n \) of \( M \), see \[26\]

\[
P^{(n)}(x_1, \ldots, x_n) = \frac{1}{Z_n} \prod_{j=1}^{n} w_n(x_j) \Delta^2(x_1, \ldots, x_n),
\]

where \( \Delta(x_1, \ldots, x_n) = \prod_{i<j}(x_j - x_i) \) stands for the Vandermonde determinant, where \( Z_n \) is a normalization constant (the partition function), and where \( w_n \) is the following varying weight on the real line

\[
w_n(x) = |x|^{2\alpha} e^{-n V(x)}.
\]

(1.2)
The unitary ensemble \( (1.1) \) is relevant in three-dimensional quantum chromodynamics [33], and has been investigated before in [4, 5, 20, 25, 28], where universal behavior for local eigenvalue correlations is established in various regimes of the spectrum, as \( n \to \infty \).

It is known that averages of products and ratios of characteristic polynomials are intimately related to orthogonal polynomials and their Cauchy transforms, see \([7, 8, 17, 27, 30]\). Let \( \pi_{j,n}(x) = x^j + \cdots \) be the \( j \)-th degree monic orthogonal polynomial with respect to \( w_n \). There is an integral representation for the monic orthogonal polynomials, which appears already in the work of Heine in 1878, see for example [31],

\[
\pi_{n,n}(x) = \int \cdots \int \prod_{j=1}^{n} (x - x_j) P^{(n)}(x_1, \ldots, x_n) dx_1 \cdots dx_n.
\]

So, the monic orthogonal polynomial \( \pi_{n,n}(x) \) can be understood as the average of the characteristic polynomial \( \det(x - M) \) over the unitary ensemble \( (1.1) \)

\[
\langle \det(x - M) \rangle_M = \pi_{n,n}(x).
\]

Here, the brackets are used to denote the average over the ensemble \( (1.1) \) of random matrices \( M \).

A first generalization of this formula was obtained by Brézin and Hikami [8], and also by Mehta and Normand [27]. They have derived a determinantal formula for the average of both products and ratios of characteristic polynomials in terms of orthogonal polynomials. A further generalization was obtained by Fyodorov and Strahov [17], who derived a determinantal formula for the average of both products and ratios of characteristic polynomials in terms of both orthogonal polynomials and their Cauchy transforms. Here, the ratios gave rise to the Cauchy transforms. For explicit formulas and streamlined proofs of these results we refer to [7].

Recently, Strahov and Fyodorov [30] showed, see also [7] for an alternative proof, that the averages of characteristic polynomials of \( n \times n \) Hermitian matrices, are governed by kernels related to orthogonal polynomials and their Cauchy transforms

\[
h_{j,n}(z) = \frac{1}{2\pi i} \int \frac{\pi_{j,n}(x)}{x - z} w_n(x) dx, \quad \text{for } \text{Im } z \neq 0. \tag{1.3}
\]

Namely, kernels \( W_{I,n+m} \) made of orthogonal polynomials, kernels \( W_{II,n+m} \) made of both orthogonal polynomials and their Cauchy transforms, and kernels \( W_{III,n+m} \) made of Cauchy transforms of orthogonal polynomials. See Table II for the explicit expressions of these kernels. This connection between the averages of characteristic polynomials and the three kernels is given by, see [7, 30]

\[
\left\langle \prod_{i=1}^{k} \frac{\det(x_i - M) \det(y_i - M)}{\det(x_i - M)} \right\rangle_M = \left( \frac{c_{n+k-1,n}}{\prod_{j=n}^{n+k-1} c_{j,n}} \right)^k \frac{1}{\Delta(\hat{x}) \Delta(\hat{y})} \det(W_{I,n+k}(x_i, y_j))_{1 \leq i, j \leq k},
\]

\[
\left\langle \prod_{i=1}^{k} \frac{\det(y_i - M)}{\det(x_i - M)} \right\rangle_M = (-1)^{\frac{k(k-1)}{2}} \left( \frac{c_{n-1,n}}{\Delta^2(\hat{x}) \Delta^2(\hat{y})} \right)^k \det(W_{II,n}(x_i, y_j))_{1 \leq i, j \leq k},
\]

and

\[
\left\langle \prod_{i=1}^{2k} \frac{1}{\det(x_i - M)} \right\rangle_M = (-1)^{k} \left( \frac{c_{n-k-1,n}}{(2k)!} \right)^k \prod_{l=n-k}^{n-1} c_l \sum_{\sigma \in \mathcal{S}_{2k}} \frac{\det(W_{III,n-k}(x_{\sigma(i)}, x_{\sigma(k+j)}))_{1 \leq i, j \leq k}}{\Delta(x_{\sigma(1)}, \ldots, x_{\sigma(k)}) \Delta(x_{\sigma(k+1)}, \ldots, x_{\sigma(2k)})},
\]

where \( c_l \) are coefficients and \( \mathcal{S}_{2k} \) is the symmetric group of degree \( 2k \).
Finite kernels

| Kernel | Expression |
|--------|------------|
| $W_{I,n+m}(\zeta, \eta)$ | $\frac{\pi_{n+m,n}(\zeta)\pi_{n+m-1,n}(\eta) - \pi_{n+m-1,n}(\zeta)\pi_{n+m,n}(\eta)}{\zeta - \eta}$ |
| $W_{II,n+m}(\zeta, \eta)$ | $\frac{h_{n+m,n}(\zeta)\pi_{n+m-1,n}(\eta) - h_{n+m-1,n}(\zeta)\pi_{n+m,n}(\eta)}{\zeta - \eta}$ |
| $W_{III,n+m}(\zeta, \eta)$ | $\frac{h_{n+m,n}(\zeta)h_{n+m-1,n}(\eta) - h_{n+m-1,n}(\zeta)h_{n+m,n}(\eta)}{\zeta - \eta}$ |

Table 1: Expressions for the finite kernels $W_{I,n+m}, W_{II,n+m}$ and $W_{III,n+m}$, cf. [30].

where $\hat{x} = (x_1, \ldots, x_k)$, $\hat{y} = (y_1, \ldots, y_k)$, where $c_{j,n} = -2\pi i \gamma_{j,n}^2$ with $\gamma_{j,n}$ the leading coefficient of the $j$-th degree orthonormal polynomial with respect to $w_n$, and where $S_{2k}$ is the permutation group of the index set $\{1, \ldots, 2k\}$. There are also explicit formulas for averages containing non-equal number of characteristic polynomials in the numerator and the denominator, in terms of these kernels, see [30] for details. Strahov and Fyodorov [30] used this connection, together with the Riemann-Hilbert (RH) approach, to establish universal behavior, as $n \to \infty$, for the averages of characteristic polynomials of random matrices taken from the unitary ensemble

$$\frac{1}{Z_n} e^{-n \text{tr} V(M)} dM,$$

in the bulk of the spectrum.

It is the goal of this paper to establish universal behavior as $n \to \infty$, for the kernels $W_{I,n+m}, W_{II,n+m}$ and $W_{III,n+m}$ (and thus also for the averages of characteristic polynomials) associated to the unitary ensemble [1.1], appropriate scaled at the origin such that the asymptotic eigenvalue density at the origin is 1. This scaling limit is called the origin of the spectrum by various authors, see for example [4, 6, 20, 25]. It will turn out that this universal behavior is described in terms of the Bessel kernels given in Table 2. For the case $\alpha = 0$, our results agree with those of Strahov and Fyodorov [30].

The issue of universality at the origin of the spectrum for the averages of characteristic polynomials, corresponding to Hermitian matrices of the unitary ensemble [1.1], was also considered by Akemann and Fyodorov [6]. They showed, on a physical level of rigor using Shohat’s method, that the asymptotic behavior near the origin, as $n \to \infty$, of the orthogonal polynomials and their Cauchy transforms are expressed in terms of Bessel and Hankel functions, see [6] for details. However, explicit expressions for the universal behavior of the three kernels $W_{I,n+m}, W_{II,n+m}$ and $W_{III,n+m}$ at the origin of the spectrum have not been given yet, which we will determine on a mathematical level of rigor using the RH approach, as in [30].

In [6] was assumed that the potential $V$ is an even polynomial with positive leading coefficient, and that the spectrum support is only one interval. In this paper, we can allow $V$ to be quite arbitrary, and assume the following

$$V : \mathbb{R} \to \mathbb{R}$$

is real analytic,

$$\lim_{|x| \to \infty} \frac{V(x)}{\log(x^2 + 1)} = +\infty,$$

$$\psi(0) > 0,$$

where $\psi = V'$. There is also a transversality condition $\psi(0) > 0$.

$$\lim_{|x| \to \infty} \frac{V(x)}{\log(x^2 + 1)} = +\infty,$$

$$\psi(0) > 0,$$
where \( \psi \) is the density of the equilibrium measure \( \mu_V \) in the presence of the external field \( V \), see \[12, 13, 29\]. The equilibrium measure \( \mu_V \) has compact support, it is supported on a finite union of intervals (since \( V \) is real analytic), and it is absolutely continuous with respect to the Lebesgue measure, i.e. \( d\mu_V(x) = \psi(x)dx \). The importance of \( \psi \) lies in the fact that its density \( \psi \) is the limiting mean eigenvalue density of the unitary ensemble \( \{n\} \). Assumption \[14\] then states that the mean eigenvalue density does not vanish at the origin.

Our results are given by the following three theorems. We use \( \mathbb{C}_+ \) and \( \mathbb{C}_- \) to denote the upper and lower half-plane, respectively.

**Theorem 1.1** Fix \( m \in \mathbb{Z} \), let \( W_{I,n+m} \) be the kernel given in Table 1 and let \( \gamma_{j,n} > 0 \) be the leading coefficient of the \( j \)-th degree orthonormal polynomial with respect to \( w_n \). For \( \zeta, \eta \in \mathbb{C} \)

\[
\gamma_{n+m-1,n}^2 \frac{1}{n\psi(0)} W_{I,n+m} \left( \frac{\zeta}{n\psi(0)}, \frac{\eta}{n\psi(0)} \right) = \left( n\psi(0) \right)^{2\alpha} e^{nV(0)} \left( e^{\frac{V'(0)(\zeta+\eta)}{2}} J_{\alpha,\alpha}^{(1)}(\zeta, \eta) + O(1/n) \right), \tag{1.8}
\]

as \( n \to \infty \), where the Bessel kernel \( \mathbb{J}_{\alpha,\alpha}(\zeta, \eta) \) is given in Table 2. The error term holds uniformly for \( \zeta \) and \( \eta \) in compact subsets of \( \mathbb{C} \).

**Theorem 1.2** Fix \( m \in \mathbb{Z} \), let \( W_{II,n+m} \) be the kernel given in Table 1 and let \( \gamma_{j,n} > 0 \) be the leading coefficient of the \( j \)-th degree orthonormal polynomial with respect to \( w_n \). Then the following holds.

| Limiting Bessel kernels | Case \( \alpha = 0 \) |
|-------------------------|---------------------|
| \( \mathbb{J}_{\alpha,\alpha}^{(1)}(\zeta, \eta) \) | \( \pi^{\alpha-\frac{1}{2}} J_{\alpha-\frac{1}{2}}(\pi \zeta) J_{\alpha-\frac{1}{2}}(\pi \eta) - \frac{1}{2(\zeta-\eta)} \) |
| \( \mathbb{J}_{\alpha,\alpha}^{(1)}(\zeta, \eta) \) | \( \sin(\pi(\zeta-\eta)) \) |
| \( \mathbb{J}_{\alpha,\alpha}^{(2)}(\zeta, \eta) \) | \( -\frac{\text{ie}^{-i\pi(\zeta-\eta)}}{2\pi(\zeta-\eta)} \) |
| \( \mathbb{J}_{\alpha,\alpha}^{(2)}(\zeta, \eta) \) | 0 |
| \( \mathbb{J}_{\alpha,\alpha}^{(2)}(\zeta, \eta) \) | 0 |

Table 2: Expressions for the limiting Bessel kernels. Here, \( J_\nu \) is the usual \( J \)-Bessel function of order \( \nu \), and \( H^{(1)}_\nu \) and \( H^{(2)}_\nu \) are the Hankel functions of order \( \nu \) of the first and second kind, respectively. The right column denotes the expressions in case \( \alpha = 0 \).
(a) For \( \zeta \in \mathbb{C}_+ \) and \( \eta \in \mathbb{C} \),
\[
\gamma_{n+m-1,n}^2 \frac{\zeta - \eta}{n\psi(0)} W_{III,n+m} \left( \frac{\zeta}{n\psi(0)}, \frac{\eta}{n\psi(0)} \right) = (\zeta - \eta)e^{-\frac{\psi'(0)}{2n\psi(0)}(\zeta-\eta)} \mathbb{J}_{\alpha,III}^+(\zeta, \eta) + O(1/n), \tag{1.9}
\]
as \( n \to \infty \), where the Bessel kernel \( \mathbb{J}_{\alpha,III}^+(\zeta, \eta) \) is given in Table 2. The error term holds uniformly for \( \zeta \) and \( \eta \) in compact subsets of \( \mathbb{C}_+ \) and \( \mathbb{C} \), respectively.

(b) For \( \zeta \in \mathbb{C}_- \) and \( \eta \in \mathbb{C} \),
\[
\gamma_{n+m-1,n}^2 \frac{\zeta - \eta}{n\psi(0)} W_{III,n+m} \left( \frac{\zeta}{n\psi(0)}, \frac{\eta}{n\psi(0)} \right) = (\zeta - \eta)e^{-\frac{\psi'(0)}{2n\psi(0)}(\zeta-\eta)} \mathbb{J}_{\alpha,III}^-(\zeta, \eta) + O(1/n), \tag{1.10}
\]
as \( n \to \infty \), where the Bessel kernel \( \mathbb{J}_{\alpha,III}^-(\zeta, \eta) \) is given in Table 2. The error term holds uniformly for \( \zeta \) and \( \eta \) in compact subsets of \( \mathbb{C}_- \) and \( \mathbb{C} \), respectively.

**Theorem 1.3** Fix \( m \in \mathbb{Z} \), let \( W_{III,n+m} \) be the kernel given in Table 4 and let \( \gamma_{j,n} > 0 \) be the leading coefficient of the \( j \)-th degree orthonormal polynomial with respect to \( w_n \). Then the following holds.

(a) For \( \zeta, \eta \in \mathbb{C}_+ \),
\[
\gamma_{n+m-1,n}^2 \frac{1}{n\psi(0)} W_{III,n+m} \left( \frac{\zeta}{n\psi(0)}, \frac{\eta}{n\psi(0)} \right) = \left( \frac{1}{n\psi(0)} \right)^{2\alpha} e^{-n\psi(0)} \left( e^{-\frac{\psi'(0)}{2n\psi(0)}(\zeta+\eta)} \mathbb{J}_{\alpha,III}^+(\zeta, \eta) + O(1/n) \right), \tag{1.11}
\]
as \( n \to \infty \), where the Bessel kernel \( \mathbb{J}_{\alpha,III}^+(\zeta, \eta) \) is given in Table 2. The error term holds uniformly for \( \zeta \) and \( \eta \) in compact subsets of \( \mathbb{C}_+ \).

(b) For \( \zeta \in \mathbb{C}_+ \) and \( \eta \in \mathbb{C}_- \),
\[
\gamma_{n+m-1,n}^2 \frac{1}{n\psi(0)} W_{III,n+m} \left( \frac{\zeta}{n\psi(0)}, \frac{\eta}{n\psi(0)} \right) = \left( \frac{1}{n\psi(0)} \right)^{2\alpha} e^{-n\psi(0)} \left( e^{-\frac{\psi'(0)}{2n\psi(0)}(\zeta+\eta)} \mathbb{J}_{\alpha,III}^-(\zeta, \eta) + O(1/n) \right), \tag{1.12}
\]
as \( n \to \infty \), where the Bessel kernel \( \mathbb{J}_{\alpha,III}^-(\zeta, \eta) \) is given in Table 2. The error term holds uniformly for \( \zeta \) and \( \eta \) in compact subsets of \( \mathbb{C}_+ \) and \( \mathbb{C}_- \), respectively.

(c) For \( \zeta, \eta \in \mathbb{C}_- \),
\[
\gamma_{n+m-1,n}^2 \frac{1}{n\psi(0)} W_{III,n+m} \left( \frac{\zeta}{n\psi(0)}, \frac{\eta}{n\psi(0)} \right) = \left( \frac{1}{n\psi(0)} \right)^{2\alpha} e^{-n\psi(0)} \left( e^{-\frac{\psi'(0)}{2n\psi(0)}(\zeta+\eta)} \mathbb{J}_{\alpha,III}^-(\zeta, \eta) + O(1/n) \right), \tag{1.13}
\]
as \( n \to \infty \), where the Bessel kernel \( \mathbb{J}_{\alpha,III}^-(\zeta, \eta) \) is given in Table 2. The error term holds uniformly for \( \zeta \) and \( \eta \) in compact subsets of \( \mathbb{C}_- \).
Remark 1.4 In case $\alpha = 0$ we can simplify the expressions for the limiting Bessel kernels, using the facts that, see [1]

$$J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \sin z, \quad J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \cos z, \quad H^{(1)}_{\frac{1}{2}}(z) = -i\sqrt{\frac{2}{\pi z}} e^{iz},$$

$$H^{(1)}_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} e^{iz}, \quad H^{(2)}_{\frac{1}{2}}(z) = i\sqrt{\frac{2}{\pi z}} e^{-iz}, \quad H^{(2)}_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} e^{-iz}.$$}

We then obtain the kernels given in the right column of Table 2. This is in agreement with the results of Strahov and Fyodorov [30]. Note however that in [30] the second and the third kernel are multiplied with an extra factor $-2\pi i$.

Remark 1.5 As noted before, it has been shown by Strahov and Fyodorov [30], see also [7], that

$$\left\langle \frac{\det(\nu \eta) - M}{\det(\nu \eta - M)} \right\rangle_M = 2\pi i\gamma^2_n \frac{\zeta - \eta}{n\nu(0)} W_{II,n} \left( \frac{\zeta}{n\nu(0)}, \frac{\eta}{n\nu(0)} \right).$$

Then it follows from [15], Table 2 and [1] formula 9.1.3, that for $\zeta \in \mathbb{C}_+$,

$$\left\langle \frac{\det(\nu \eta) - M}{\det(\nu \eta - M)} \right\rangle_M = \frac{\pi^2 \gamma^2}{2} \left( J_{\alpha + \frac{1}{2}}(\pi\zeta)Y_{\alpha - \frac{1}{2}}(\pi\zeta) - J_{\alpha - \frac{1}{2}}(\pi\zeta)Y_{\alpha + \frac{1}{2}}(\pi\zeta) \right) + O(1/n),$$

as $n \to \infty$, where $Y_\nu$ is the Bessel function of the second kind of order $\nu$. By [1] formula 9.1.16], the right hand side of this equation is $1 + O(1/n)$, as it should be. Similarly we find the same result for $\zeta \in \mathbb{C}_-$.  

The proofs of these theorems are based on the characterization of orthogonal polynomials with respect to the weight (1.2), together with their Cauchy transforms via a $2 \times 2$ matrix RH problem for $Y$, due to Fokas, Its and Kitaev [16], and on an application of the Deift/Zhou steepest descent method [15] for matrix RH problems. See [10, 22] for an excellent exposition. This technique was used before by Deift et al. [13] to establish universality for the local eigenvalue correlations in unitary random matrix ensembles [14] in the bulk of the spectrum. Strahov and Fyodorov [30] used this method also to establish universality for the three kernels $W_{I,n+m}, W_{II,n+m}$ and $W_{III,n+m}$ in the bulk of the spectrum.

In a previous paper [25] together with A.B.J. Kuijlaars, the asymptotic analysis of the RH problem for $Y$, corresponding to the weight (1.2), has already been done. An essential step in the analysis is the construction of the parametrix near the origin, which gives us the behavior of $Y$ near the origin. Here, the Bessel functions come in. In [25], the behavior of the first column of $Y$ (with the orthogonal polynomials as entries) was determined near the origin for positive (real) values, and used to establish universality for the local eigenvalue correlations at the origin of the spectrum, in terms of a Bessel kernel. Here, we determine the behavior of the first column of $Y$, as well as the second column of $Y$ (with the Cauchy transforms of orthogonal polynomials as entries) in a full neighborhood of the origin, and use this in a similar fashion to prove our results.

The rest of the paper is organized as follows. In Section 2 we give a short overview of the asymptotic analysis of the corresponding RH problem for $Y$. In Section 3 we determine the behavior of $Y$ near the origin, in terms of Bessel functions. This will be used in the last section to prove our results.
2 The corresponding RH problem

In this section we recall the matrix RH problem for $Y$, due to Fokas, Its and Kitaev \cite{16}, which characterizes the orthogonal polynomials with respect to the weight \cite{12}, together with their Cauchy transforms. We also give a short overview of the Deift/Zhou steepest descent method \cite{10} \cite{15} to obtain the asymptotic behavior of $Y$. For details we refer to \cite{13} \cite{25}, see also \cite{10} \cite{14}.

Our point of interest lies in the asymptotic behavior, as $n \to \infty$, of the orthogonal polynomials $\pi_{n+m,n}$ of degree $n+m$ with respect to the weight $w_n$, for any fixed $m \in \mathbb{Z}$. So, in contrast to the RH problem in \cite{13} \cite{25}, we have to modify the asymptotic condition at infinity of the RH problem, and leave the jump condition unchanged. However, this will not create any problems. We seek a $2 \times 2$ matrix valued function $Y = Y^{(n+m,n)}$ that satisfies the following RH problem, cf. \cite{10} \cite{13} \cite{14} \cite{16} \cite{25}.

**RH problem for $Y$:**

(a) $Y : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2 \times 2}$ is analytic.

(b) $Y$ possesses continuous boundary values for $x \in \mathbb{R} \setminus \{0\}$ denoted by $Y_+(x)$ and $Y_-(x)$, where $Y_+(x)$ and $Y_-(x)$ denote the limiting values of $Y(z')$ as $z'$ approaches $x$ from above and below, respectively, and

$$
Y_+(x) = Y_-(x) \begin{pmatrix} 1 & |x|^{2\alpha}e^{-nV(x)} \\ 0 & 1 \end{pmatrix}, \quad \text{for } x \in \mathbb{R} \setminus \{0\}. \quad (2.1)
$$

(c) $Y$ has the following asymptotic behavior at infinity:

$$
Y(z) = (I + O(1/z)) \begin{pmatrix} z^{n+m} & 0 \\ 0 & z^{-(n+m)} \end{pmatrix}, \quad \text{as } z \to \infty. \quad (2.2)
$$

(d) $Y$ has the following behavior near the origin:

$$
Y(z) = \begin{cases} 
O \left( \frac{1}{|z|^{2\alpha}} \right), & \text{if } \alpha < 0, \\
O \left( \frac{1}{1} \right), & \text{if } \alpha > 0,
\end{cases} \quad (2.3)
$$

as $z \to 0$, $z \in \mathbb{C} \setminus \mathbb{R}$.

**Remark 2.1** The $O$-terms in condition (d) of the RH problem are to be taken entrywise. So for example $Y(z) = O \left( \frac{1}{|z|^{2\alpha}} \right)$ means that $Y_{11}(z) = O(1)$, $Y_{12}(z) = O(|z|^{2\alpha})$, etc. This condition is used to control the behavior of $Y$ near the origin. In the following we will not go into detail about this condition, and refer to \cite{23} \cite{32} for details.

The unique solution of the RH problem for $Y$, see \cite{16} (for condition (d) we refer to \cite{23}), is then given by

$$
Y(z) = Y^{(n+m,n)}(z) = \begin{pmatrix} \pi_{n+m,n}(z) & h_{n+m,n}(z) \\ -2\pi i \gamma_{n+m-1,n} \pi_{n+m-1,n}(z) & -2\pi i \gamma_{n+m-1,n}^2 h_{n+m-1,n}(z) \end{pmatrix}, \quad (2.4)
$$

for $z \in \mathbb{C} \setminus \mathbb{R}$, where $\pi_{j,n}$ is the $j$-th degree monic orthogonal polynomial with respect to $w_n$, where $\gamma_{j,n}$ is the leading coefficient of the $j$-th degree orthonormal polynomial with respect to $w_n$, and where $h_{j,n}$ is the Cauchy transform of $\pi_{j,n}$, see \cite{13}. 

7
Remark 2.2 The superscript \( n + m \) in the notation \( Y^{(n+m,n)} \) corresponds to the asymptotic condition (c) at infinity of the RH problem, which yields that the orthogonal polynomials in the solution \( 2.3 \) of the RH problem have degree \( n + m \) and \( n + m - 1 \). The superscript \( n \) corresponds to the jump condition (b), which yields that the orthogonality is with respect to \( w_n \).

Remark 2.3 We note that the first column of \( Y \) contains the orthogonal polynomials, and the second column their Cauchy transforms. So, from Table 1 and (2.4), the kernel \( W_{I,n+m} \) depends only on the first column of \( Y \), the kernel \( W_{II,n+m} \) on both the first and the second column, and the kernel \( W_{III,n+m} \) only on the second column, as follows:

\[
W_{I,n+m}(\zeta, \eta) = \frac{1}{\gamma^2_{n+m-1,n}} \frac{1}{-2\pi i(\zeta - \eta)} \begin{vmatrix}
Y_{11}(\zeta) & Y_{11}(\eta) \\
Y_{21}(\zeta) & Y_{21}(\eta)
\end{vmatrix},
\]

(2.5)

\[
W_{II,n+m}(\zeta, \eta) = \frac{1}{\gamma^2_{n+m-1,n}} \frac{1}{-2\pi i(\zeta - \eta)} \begin{vmatrix}
Y_{12}(\zeta) & Y_{11}(\eta) \\
Y_{22}(\zeta) & Y_{21}(\eta)
\end{vmatrix},
\]

(2.6)

and

\[
W_{III,n+m}(\zeta, \eta) = \frac{1}{\gamma^2_{n+m-1,n}} \frac{1}{-2\pi i(\zeta - \eta)} \begin{vmatrix}
Y_{12}(\zeta) & Y_{12}(\eta) \\
Y_{22}(\zeta) & Y_{22}(\eta)
\end{vmatrix}.
\]

(2.7)

The asymptotic analysis of the RH problem for \( Y \) includes a series of transformations \( Y \mapsto T \mapsto S \mapsto R \) to obtain a RH problem for \( R \) normalized at infinity (i.e. \( R(z) \rightarrow I \) as \( n \rightarrow \infty \)), and with jumps uniformly close to the identity matrix, as \( n \rightarrow \infty \). Then \( [10, 13, 14] \), \( R \) is also uniformly close to the identity matrix, and by unfolding the series of transformations we obtain the asymptotic behavior of \( Y \).

Before we can give an overview of the series of transformations, we need some properties of the equilibrium measure \( \mu_V \) for \( V \). Here, we closely follow \( [25] \), see also \( [12, 13] \). The support of \( \mu_V \) consists of a finite union of intervals, say \( \bigcup_{j=1}^{N+1}[b_{j-1}, a_j] \), and we define its interior as \( J = \bigcup_{j=1}^{N+1} (b_{j-1}, a_j) \). The \( N + 1 \) intervals of \( J \) are referred to as the bands. The density \( \psi \) of the equilibrium measure is given by

\[
\psi(x) = \frac{1}{2\pi i} R^{1/2}(x) h(x), \quad \text{for } x \in J,
\]

(2.8)

with \( h \) real analytic on \( \mathbb{R} \), and where \( R \) is the \( 2(N+1) \)-th degree monic polynomial with the endpoints \( a_j, b_j \) of \( J \) as zeros,

\[
R(z) = \prod_{j=1}^{N+1} (z - b_{j-1})(z - a_j).
\]

(2.9)

We use \( R^{1/2} \) to denote the branch of \( \sqrt{R} \) which behaves like \( z^{N+1} \) as \( z \rightarrow \infty \), and which is defined and analytic on \( \mathbb{C} \setminus \bar{J} \). In (2.8), \( R^{1/2} \) is used to denote the boundary value of \( R^{1/2} \) on \( J \) from above. The equilibrium measure satisfies the Euler-Lagrange variational conditions, which state that there exists a constant \( \ell \in \mathbb{R} \) such that

\[
2 \int \log |x - s| \psi(s) ds - V(x) = \ell, \quad \text{for } x \in \bar{J},
\]

(2.10)

\[
2 \int \log |x - s| \psi(s) ds - V(x) \leq \ell, \quad \text{for } x \in \mathbb{R} \setminus \bar{J}.
\]

(2.11)

If the inequality in (2.11) is strict for every \( x \in \mathbb{R} \setminus \bar{J} \), and if \( h(x) \neq 0 \) for every \( x \in \bar{J} \), then \( V \) is called regular. Otherwise, there are a finite number of points, called singular points of \( V \), such
that \( h \) vanishes there, i.e. a singular point in \( \tilde{J} \), or such that we obtain equality in \((2.11)\), i.e. a singular point in \( \mathbb{R} \setminus \tilde{J} \).

Let \( \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) be the Pauli matrix. Following [13], see also [25], we define the \( 2 \times 2 \) matrix valued function

\[
T(z) = e^{-(n+m)\frac{\tau}{2} \sigma_3} Y(z) e^{(n+m)\frac{\tau}{2} \sigma_3} e^{-(n+m)g(z)\sigma_3}, \quad \text{for } z \in \mathbb{C} \setminus \mathbb{R},
\]

\[(2.12)\]

where \( \ell \) is the constant that appears in the Euler-Lagrange variational conditions \((2.10)\) and \((2.11)\), and where the scalar function \( g \) is defined by

\[
g(z) = \int \log(z-s) \psi(s) ds, \quad \text{for } z \in \mathbb{C} \setminus (-\infty, a_{N+1}].
\]

\[(2.13)\]

Note the small difference in the definition \((2.12)\) of \( T \) with its definition in [13, 25], which comes from the modified asymptotic condition \((c)\) of the RH problem for \( Y \). For the case \( m = 0 \), both definitions agree. It is known [13, 25] that \( T \) is normalized at infinity and satisfies the jump relation \( T_+(x) = T_-(x)v^{(1)}(x) \) for \( x \in \mathbb{R} \setminus \{0\} \), where

\[
v^{(1)}(x) = \begin{cases} 
\begin{pmatrix}
|x|^{2\alpha} e^{mV(x)} & 0 \\
0 & e^{(n+m)(g_+(x)-g_-(x))}
\end{pmatrix}, & x \in \tilde{J} \setminus \{0\} \\
\begin{pmatrix}
e^{-2\pi i(n+m)\Omega_j} |x|^{2\alpha} e^{mV(x)} e^{(n+m)(g_+(x)+g_-(x)-V(x)-\ell)} & 0 \\
0 & e^{2\pi i(n+m)\Omega_j}
\end{pmatrix}, & x \in (a_j, b_j), \\
\begin{pmatrix}
1 & |x|^{2\alpha} e^{mV(x)} e^{(n+m)(g_+(x)+g_-(x)-V(x)-\ell)} \\
0 & 1
\end{pmatrix}, & x < b_0 \text{ or } x > a_{N+1}.
\end{cases}
\]

\[(2.14)\]

The constant \( \Omega_j \) is the total \( \mu_V \)-mass of the \( N + 1 - j \) largest bands.

The second transformation is referred to as the opening of the lens. Define [25] for every \( z \in \mathbb{C} \setminus \mathbb{R} \) lying in the region of analyticity of \( h \) the scalar function

\[
\phi(z) = \frac{1}{2} \int_{z}^{a_{N+1}} R^{1/2}(s) h(s) ds,
\]

\[(2.15)\]

where the path of integration does not cross the real axis. Then [25], on the bands, \( \phi \) is purely imaginary and satisfies

\[
2\phi_+(x) = -2\phi_-(x) = g_+(x) - g_-(x), \quad \text{for } x \in J,
\]

\[(2.16)\]

so that \( 2\phi \) and \(-2\phi \) provide analytic extensions of \( g_+ - g_- \) into the upper half-plane and lower half-plane, respectively. The opening of the lens is based on the factorization of the jump matrix \( v^{(1)} \) on the bands, see \((2.14)\), into the following product of three matrices, cf. [25]

\[
\begin{pmatrix}
e^{-(n+m)(g_+(x)-g_-(x))} & |x|^{2\alpha} e^{mV(x)} \\
0 & e^{(n+m)(g_+(x)-g_-(x))}
\end{pmatrix} \times \begin{pmatrix}
1 & 0 \\
|x|^{2\alpha} e^{mV(x)} e^{-2(n+m)\phi_-(x)} & 1
\end{pmatrix} \times \begin{pmatrix}
1 & 0 \\
|x|^{2\alpha} e^{mV(x)} e^{-2(n+m)\phi_+(x)} & 1
\end{pmatrix},
\]

\[
\left. \begin{pmatrix}
e^{-(n+m)(g_+(x)-g_-(x))} & |x|^{2\alpha} e^{mV(x)} \\
0 & e^{(n+m)(g_+(x)-g_-(x))}
\end{pmatrix} \times \begin{pmatrix}
1 & 0 \\
|x|^{2\alpha} e^{mV(x)} e^{-2(n+m)\phi_-(x)} & 1
\end{pmatrix} \times \begin{pmatrix}
1 & 0 \\
|x|^{2\alpha} e^{mV(x)} e^{-2(n+m)\phi_+(x)} & 1
\end{pmatrix}. \right\}
\]
We take an analytic continuation of the factor $|x|^{2\alpha}e^{mV(x)}$ by defining for $z$ in the region of analyticity of $V$,

$$
\omega(z) = \begin{cases} 
(-z)^{2\alpha}e^{mV(z)}, & \text{if } \Re z < 0, \\
     z^{2\alpha}e^{mV(z)}, & \text{if } \Re z > 0,
\end{cases}
$$

(2.17)

with principal branches of powers. We now open the lens. Let $\Sigma$ be the lens shaped contour, as shown in Figure 1 going through the endpoints $a_i, b_j$ of $J$, going trough the origin, and also going through the singular points of $V$ in $J$. Of course we take the lens shaped regions to lie within the region of analyticity of $\phi$ and $V$.

Define, cf. [25]

$$
S(z) = \begin{cases} 
T(z), & \text{for } z \text{ outside the lens}, \\
T(z) \left( -\omega(z)^{-1}e^{-2(n+m)\phi(z)} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \right), & \text{for } z \text{ in the upper parts of the lens}, \\
T(z) \left( \omega(z)^{-1}e^{-2(n+m)\phi(z)} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right), & \text{for } z \text{ in the lower parts of the lens}.
\end{cases}
$$

(2.18)

As for the first transformation $Y \mapsto T$, there is small difference in the definition [2.18] for $S$ with its definition in [25], which comes from the modified asymptotic condition (c) of the RH problem for $Y$. For the case $m = 0$, again both definitions agree. Then [26], the matrix valued function $S$ is normalized at infinity and satisfies the jump relation $S_+(z) = S_-(z)\nu^{(2)}(z)$ for $z \in \Sigma$, where

$$
\nu^{(2)}(z) = \begin{cases} 
\begin{pmatrix} 1 & 0 \\ -|z|^{-2\alpha}e^{-mV(z)} & 1 \end{pmatrix}, & z \in \Sigma \cap \mathbb{C}_+, \\
\begin{pmatrix} 0 & |z|^{2\alpha}e^{mV(z)} \\ -|z|^{-2\alpha}e^{-mV(z)} & 0 \end{pmatrix}, & z \in J \setminus \{0\}, \\
\begin{pmatrix} e^{-2\pi i(n+m)\Omega_j} & |z|^{2\alpha}e^{mV(z)}e^{(n+m)(g_+(z)+g_-(z)-V(z)-l)} \\ 0 & e^{2\pi i(n+m)\Omega_j} \end{pmatrix}, & z \in (a_j, b_j) \\
\begin{pmatrix} 1 & |z|^{2\alpha}e^{mV(z)}e^{(n+m)(g_+(z)+g_-(z)-V(z)-l)} \\ 0 & 1 \end{pmatrix}, & z < b_0 \text{ or } z > a_{N+1}.
\end{cases}
$$

(2.19)

For $z$ in a neighborhood of a regular point $x \in J$ we have, cf. [26],

$$
\Re \phi(z) > 0, \quad \text{if } \Im z \neq 0,
$$

and for every regular point in $\mathbb{R} \setminus \bar{J}$ we have from the Euler-Lagrange variational condition [2.11], cf. [13]

$$
g_+(x) + g_-(x) - V(x) - l < 0, \quad \text{for } x \in \mathbb{R} \setminus \bar{J}.
$$
So, we expect that the leading order asymptotics are determined by a RH problem for $P^{(\infty)}$, normalized at infinity, that satisfies the jump relation $P^+_{(\infty)}(x) = P^-_{(\infty)}(x)v_{(\infty)}(x)$ for $x \in (b_0, a_{N+1})$, where

$$v_{(\infty)}(x) = \begin{cases} \begin{pmatrix} 0 & |x|^{2\alpha}e^{mV(x)} \\ -|x|^{-2\alpha}e^{-mV(x)} \\ e^{-2\pi i(n+m)\Omega_j} & 0 \\ 0 & e^{2\pi i(n+m)\Omega_j} \end{pmatrix}, & \text{for } x \in J \setminus \{0\}, \\
\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, & \text{for } x \in (a_j, b_j), j = 1 \ldots N. \end{cases}$$

(2.20)

The solution of this RH problem is referred to as the parametrix for the outside region, and is constructed using a Szegő function on multiple intervals associated to $|x|^{2\alpha}e^{mV(x)}$, cf. [25], and using Riemann-Theta functions, cf. [13], see also [11]. For our purpose here, we do not need the explicit formulas for $P^{(\infty)}$, and refer to [13, 25] for details.

Before we can do the third transformation, we have to be careful since the jump matrices for $S$ and $P^{(\infty)}$ are not uniformly close to each other near 0, near the endpoints $a_i, b_j$ of $J$, and near the singular points of $V$. To solve this problem, we surround these points by small non-overlapping disks, say of radius $\delta > 0$, and within each disk we construct a parametrix $P$ satisfying the following local RH problem.

**RH Problem for $P$ near $x_0$ where $x_0$ is 0, an endpoint of $J$, or a singular point of $V$:**

1. $P(z)$ is defined and analytic for $z \in \{|z-x_0| < \delta_0 \} \setminus \Sigma$ for some $\delta_0 > \delta$.
2. $P$ satisfies the same jump relations as $S$ does on $\Sigma \cap \{|z-x_0| < \delta\}$.
3. There is $\kappa > 0$ such that, as $n \to \infty$,
   $$P(z) \left(P^{(\infty)}\right)^{-1}(z) = I + O(1/n^\kappa),$$
   uniformly for $|z-x_0| = \delta$.
4. $SP^{-1}$ has a removable singularity at $x_0$.

For regular endpoints and the origin we can take $\kappa = 1$ in (2.21). It is known that this local RH problem is solvable for every $x_0$. For the endpoints of $J$ and the singular points of $V$ we refer to [13], for the origin we refer to [25]. For our purpose here, it suffices to know the explicit formula for the parametrix near the origin.

We will now give the explicit formula for the parametrix $P$ near the origin, see [25, Section 5] for details, see also [32, Section 4]. This is an essential step in the asymptotic analysis of the RH problem since it allows us to determine the behavior of $Y$ near the origin, which will be the main tool to prove our results. Introduce the scalar function

$$f(z) = \begin{cases} i\phi(z) - i\phi_+(0), & \text{if } \text{Im } z > 0, \\
-i\phi(z) - i\phi_+(0), & \text{if } \text{Im } z < 0, \end{cases}$$

(2.22)

which is defined and analytic in a neighborhood of the origin. The behavior of $f$ near the origin [25, Section 5] is given by

$$f(z) = \pi\psi(0)z + O(z^2), \quad \text{as } z \to 0.$$ 

(2.23)

Let $U_\delta$ be the disk with radius $\delta$ around the origin, with $\delta > 0$ sufficiently small such that $U_\delta$ lies in the region of analyticity of $\phi$ and $V$. Since $f'(0) = \pi\psi(0) > 0$ we can choose $\delta$ also sufficiently small such that $f$ is a conformal mapping on $U_\delta$ onto a convex neighborhood of 0. We have that $f(x)$ is real and positive (negative) for $x \in U_\delta$ positive (negative).
Decompose $f(U_\delta)$ into eight regions I–VIII, as shown in the right of Figure 2 divided by eight straight rays

$$\Gamma_j = \{ \zeta \in \mathbb{C} \mid \arg \zeta = (j - 1)\frac{\pi}{4} \}, \quad j=1,\ldots,8.$$  

This in turn divides the disk $U_\delta$ into eight regions $\Gamma'$–VIII' as the pre-images under $f$ of I–VIII, as shown in the left of Figure 2. Sector I' and IV' correspond to the right and left upper part of the lens inside $U_\delta$, respectively, sector V' and VIII' to the left and right lower part of the lens inside $U_\delta$, respectively.

Let $\Psi_\alpha$ be the piecewise analytic matrix valued function [32, Section 4], see also [25, Section 5], that satisfies the jump relation $\Psi_\alpha,+,\zeta) = \Psi_\alpha,−,\zeta)v_\alpha(\zeta)$ for $\zeta \in \bigcup \Gamma_j$, where

$$v_\alpha(\zeta) = \begin{cases} 
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{for } \zeta \in \Gamma_1 \cup \Gamma_5, \\
\begin{pmatrix} 1 & 0 \\ e^{-2\pi i\alpha} & 0 \end{pmatrix}, & \text{for } \zeta \in \Gamma_2 \cup \Gamma_6, \\
e^{\pi i\alpha\sigma_3}, & \text{for } \zeta \in \Gamma_3 \cup \Gamma_7, \\
\begin{pmatrix} 1 & 0 \\ e^{2\pi i\alpha} & 0 \end{pmatrix}, & \text{for } \zeta \in \Gamma_4 \cup \Gamma_8,
\end{cases}$$

and that has the following behavior near the origin,

$$\Psi_\alpha(\zeta) = O\left(\frac{|\zeta|^\alpha}{|\zeta|^\alpha} \frac{|\zeta|^\alpha}{|\zeta|^\alpha}\right), \quad \text{as } \zeta \to 0,$$

if $\alpha < 0$, and

$$\Psi_\alpha(\zeta) = \begin{cases} 
O\left(\frac{|\zeta|^\alpha}{|\zeta|^\alpha} \frac{|\zeta|^{-\alpha}}{|\zeta|^{-\alpha}}\right), & \text{as } \zeta \to 0 \text{ for } \frac{\pi}{4} < |\arg \zeta| < \frac{3\pi}{4}, \\
O\left(\frac{|\zeta|^{-\alpha}}{|\zeta|^{-\alpha}} \frac{|\zeta|^{-\alpha}}{|\zeta|^{-\alpha}}\right), & \text{as } \zeta \to 0 \text{ for } 0 < |\arg \zeta| < \frac{\pi}{4} \text{ and } \frac{3\pi}{4} < |\arg \zeta| < \pi,
\end{cases}$$

if $\alpha > 0$. The behavior of $\Psi_\alpha$ near the origin will ensure that part (d) of the RH problem for $P$ is satisfied, see [25, 22] for details. The matrix valued function $\Psi_\alpha$ is constructed out of Bessel functions of order $\alpha \pm \frac{1}{2}$, and its explicit formula for $0 < \arg \zeta < \frac{\pi}{4}$ is given by

$$\Psi_\alpha(\zeta) = \frac{1}{2} \sqrt{-\pi} \zeta^{1/2} \begin{pmatrix} H^{(2)}_{\alpha+\frac{1}{2}}(\zeta) & -iH^{(1)}_{\alpha+\frac{1}{2}}(\zeta) \\
H^{(2)}_{\alpha-\frac{1}{2}}(\zeta) & -iH^{(1)}_{\alpha-\frac{1}{2}}(\zeta) \end{pmatrix} e^{-(\alpha+\frac{1}{2})\pi i\sigma_3}. \quad (2.24)$$
For $\frac{\pi}{4} < \arg \zeta < \frac{\pi}{2}$ it is given by

$$\Psi_\alpha(\zeta) = \begin{pmatrix} \sqrt{\pi} \zeta^{1/2} I_{\alpha+\frac{1}{2}}(\zeta e^{-\frac{\pi i}{4}}) & -\frac{1}{\sqrt{\pi}} \zeta^{1/2} K_{\alpha+\frac{1}{2}}(\zeta e^{-\frac{\pi i}{4}}) \\ -i \sqrt{\pi} \zeta^{1/2} I_{\alpha-\frac{1}{2}}(\zeta e^{-\frac{\pi i}{4}}) & -\frac{i}{\sqrt{\pi}} \zeta^{1/2} K_{\alpha-\frac{1}{2}}(\zeta e^{-\frac{\pi i}{4}}) \end{pmatrix} e^{-\frac{1}{2} \pi i \sigma_3},$$

(2.25)

where $I_\nu$ and $K_\nu$ are the modified Bessel functions of order $\nu$. See [32, Section 4] for the explicit expressions of $\Psi_\alpha$ in the other sectors of the complex plane. Also define the piecewise analytic function $W$ by

$$W(z) = \begin{cases} z^\alpha e^{m V(z)} & \text{if } z \in \text{III',IV',V',VI'}, \\ (-z)^\alpha e^{m V(z)} & \text{if } z \in \text{I',II',VII',VIII'}. \end{cases}$$

(2.26)

And finally, define the following matrix valued function, analytic in a neighborhood of the disk $U_\delta$, $E_{n+m,n}(z) = E(z)e^{(n+m)\phi_+(0)\sigma_3} e^{-\frac{n}{2} \sigma_3} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$,

(2.27)

where the matrix valued function $E$ is given by [25 (5.27)-(5.30)].

Then, cf. [25, Section 5], the parametrix near the origin is defined by $P(z) = E_{n+m,n}(z)\Psi_\alpha((n+m)\phi(z))W(z)^{-\sigma_3} e^{-(n+m)\phi(z)\sigma_3}$.

(2.28)

**Remark 2.4** In contrast to [25, Section 5], we evaluate the matrix valued function $\Psi_\alpha$ in $(n+m)f(z)$ instead of in $nf(z)$. This comes from the fact that, in order that the matching condition (c) of the RH problem for $P$ is satisfied, we need to cancel out the factor $e^{-(n+m)\phi(z)\sigma_3}$ instead of $e^{-n\phi(z)}$. This follows in essence from the modified asymptotic condition (c) of the RH problem for $Y$. For the case $m = 0$, the definition (2.28) of the parametrix $P$ near the origin agrees with its definition in [25, Section 5].

Now, we have all the ingredients to give the third transformation. Define [13, 25] the $2 \times 2$ matrix valued function $R$ as

$$R(z) = \begin{cases} S(z) \left( P(\infty) \right)^{-1}(z) & \text{for } z \text{ outside the disks}, \\ S(z)P^{-1}(z) & \text{for } z \text{ inside the disks}. \end{cases}$$

(2.29)

Then [13, 25], $R$ is normalized at infinity, and analytic on the entire plane except for jumps on the reduced system of contours $\Sigma_R$, as shown in Figure 3 and except for possible isolated singularities at the endpoints $a_i$, $b_j$ of $J$, at the singularities of $V$ and at 0. However, from condition (d) of the RH problem for $P$, these singularities are removable, so that $R$ is analytic on $\mathbb{C} \setminus \Sigma_R$. It is known [13, 25] that the jumps of $R$ on $\Sigma_R$ are uniformly close to the identity matrix as $n \to \infty$. This implies [13], see also [10, 14]

$$R(z) = I + O(1/n^\kappa), \quad \text{as } n \to \infty,$$

(2.30)

uniformly for $z \in \mathbb{C} \setminus \Sigma_R$, where $\kappa$ is the constant that appears in the matching condition (c) of the RH problem for $P$. By tracing back the steps $Y \mapsto T \mapsto S \mapsto R$ we obtain the asymptotic behavior of $Y$ in all regions of the complex plane, as $n \to \infty$. 

Figure 3: Part of the contour $\Sigma_R$. The points $z_1$ and $z_2$ are singular points of $V$. 

...
3 Behavior of $Y$ near the origin

In this section we unravel, as in [25, Lemma 7.1], the series of transformations $Y \mapsto T \mapsto S \mapsto R$, see Section 2, to determine the behavior of the first and the second column of $Y$ inside the disk $U_{\delta}$. This behavior will be the main tool to prove our results. Note that the second column of $Y$ has jumps on the real axis, see [2.11]. So, for the behavior of the second column of $Y$ inside the disk $U_{\delta}$ we have to distinguish between the upper and lower parts of $U_{\delta}$.

For notational convenience we introduce the $2 \times 2$ matrix valued function, cf. [25, Lemma 7.1]

$$M(z) = M_{n+m,n}(z) = R(z)E_{n+m,n}(z), \quad \text{for } z \in U_{\delta},$$

where $E_{n+m,n}$ is given by [22.27]. For the case $m = 0$, the $M$-matrix defined by (3.1) corresponds to the $M$-matrix in [25, Lemma 7.1]. It is known that $M$ is analytic on $U_{\delta}$, that each entry of $M$ is uniformly bounded in $U_{\delta}$ as $n \to \infty$, and that $\det M \equiv 1$, cf. [25, Lemma 7.1].

We also need the following lemma.

**Lemma 3.1** For $z \in U_{\delta}$,

$$2g(z) - 2\phi(z) - \ell = V(z).$$

**Proof.** Let $H(z) = 2g(z) - 2\phi(z) - \ell - V(z)$, which is defined and analytic for $z \in U_{\delta} \setminus \mathbb{R}$. For $x \in (-\delta, \delta) \subset J$ we have by (2.13)

$$H_+(x) = H_-(x) = g_+(x) + g_-(x) - \ell - V(x),$$

so that $H$ is analytic in the entire disk $U_{\delta}$. For $x \in (-\delta, \delta)$ we have by (2.13)

$$g_+(x) + g_-(x) = 2 \int \log |x - s| \psi(s) ds.$$

Inserting this into (3.3) and using the Euler-Lagrange variational condition (2.10), we have that $H(x) = 0$ for $x \in (-\delta, \delta)$. This implies from the uniqueness principle that $H \equiv 0$ on $U_{\delta}$, which proves the lemma. \qed

First, the behavior of the first column of $Y$ inside the disk $U_{\delta}$ is given by the following theorem.

**Theorem 3.2** Fix $m \in \mathbb{Z}$. For $z \in U_{\delta}$ and $n$ sufficiently large, the first column of $Y = Y^{(n+m,n)}$ is given by

$$
\begin{pmatrix}
Y_{11}(z) \\
Y_{21}(z)
\end{pmatrix} = z^{-\alpha} e^{\frac{V(z)}{2}} \sqrt{\pi} e^{-\frac{z^2}{4}} e^{(n+m)\frac{3}{2}\sigma_3} M(z) \\
\times \begin{pmatrix}
((n+m)f(z))^{1/2} J_{\alpha+\frac{1}{2}}((n+m)f(z)) \\
((n+m)f(z))^{1/2} J_{\alpha-\frac{1}{2}}((n+m)f(z))
\end{pmatrix},
\tag{3.4}
$$

Here, $J_\nu$ is the $J$-Bessel function of order $\nu$, $f$ is given by (2.22), and $M$ is given by (3.1).

**Proof.** Let $z$ be in sector I’ of the disk $U_{\delta}$, see Figure 2. Unfolding the series of transformations $Y \mapsto T \mapsto S \mapsto R$ we obtain by (2.12), (2.18), (2.28) and (2.29)

$$
Y(z) = e^{(n+m)\frac{3}{2}\sigma_3} R(z) E_{n+m,n}(z) \Psi_\alpha((n+m)f(z)) W(z)^{-\sigma_3} \\
\times e^{-(n+m)\phi(z)} \sigma_3 \begin{pmatrix}
1 & 0 \\
\omega(z)^{-1} e^{-2(n+m)\phi(z)} & 1
\end{pmatrix} e^{-(n+m)\frac{3}{2}\sigma_3} e^{(n+m)\Phi(z)} \sigma_3.
\tag{3.5}
$$
Note that \( \omega(z) = z^{2\alpha} e^{mV(z)} \), see (2.17), and that \( W(z) = (-z)^{\alpha}e^m\frac{V(z)}{2} = z^{\alpha}e^{-\pi i\alpha}\frac{e^mV(z)}{2} \), see (2.20). Inserting this into (3.6) and using (3.1) and (3.2), the first column of \( Y \) is then given by

\[
\begin{pmatrix} Y_{11}(z) \\ Y_{21}(z) \end{pmatrix} = z^{\alpha} e^m\frac{V(z)}{2} e^{(n+m)\frac{\pi i}{2}} M(z) \Psi_\alpha((n+m)f(z)) e^{\pi i\alpha 3} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\] (3.6)

Since \( f(z) \) is in sector I of \( f(U_\delta) \), see Figure 2, we have for \( n \) sufficiently large (namely \( n+m > 0 \)) that \( 0 < \arg(n+m)f(z) < \pi/4 \). So, we have to use (2.24) to evaluate \( \Psi_\alpha((n+m)f(z)) \). From (3.6) and [1] formulas 9.1.3 and 9.1.4, which connect the Hankel functions of the first and second kind with the ordinary \( J \)-Bessel functions, we then establish (3.4) in sector II' of \( U_\delta \).

Now, let \( z \) be in sector II' of \( U_\delta \). Similarly as in sector I', we obtain by (2.12), (2.18), (2.28) and (2.29)

\[
Y(z) = e^{(n+m)\frac{\pi i}{2}} R(z) E_{n+m,n}(z) \Psi_\alpha((n+m)f(z)) \\
\times W(z)^{-\sigma_3} e^{(n+m)\phi(z)\sigma_3} e^{-(n+m)\frac{\pi i}{2}} e^{(n+m)g(z)\sigma_3}.
\]

Since \( W(z) = z^{\alpha} e^{-\pi i\alpha}\frac{e^mV(z)}{2} \), see (2.20), and using (3.1) and (3.2), the first column of \( Y \) is then given by

\[
\begin{pmatrix} Y_{11}(z) \\ Y_{21}(z) \end{pmatrix} = z^{\alpha} e^m\frac{V(z)}{2} e^{(n+m)\frac{\pi i}{2}} M(z) \Psi_\alpha((n+m)f(z)) e^{\pi i\alpha 3} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\] (3.7)

Since \( \pi/4 < \arg(n+m)f(z) < \pi/2 \) for \( n \) sufficiently large, we have to use (2.24) to evaluate \( \Psi_\alpha((n+m)f(z)) \). This implies, using [1] formula 9.6.3, which connects the modified Bessel function \( J_{\alpha\pm\frac{1}{2}} \) with the Bessel function \( J_{\alpha\pm\frac{1}{2}} \), that

\[
\Psi_\alpha((n+m)f(z)) e^{\pi i\alpha 3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \sqrt{\pi} e^{-\frac{\pi i}{4}} \begin{pmatrix} ((n+m)f(z))^{1/2} J_{\alpha+\frac{1}{2}}((n+m)f(z))e^{-\frac{\pi i}{4}} \\
\times \frac{1}{i}((n+m)f(z))^{1/2} J_{\alpha-\frac{1}{2}}((n+m)f(z))e^{-\frac{\pi i}{4}} \end{pmatrix}.
\] (3.8)

Inserting this into (3.7), we establish (3.4) also in sector II' of \( U_\delta \).

In the other sectors of the disk \( U_\delta \) the calculations are similar, and are left as an easy exercise for the careful reader.

\[\square\]

**Remark 3.3** For the case \( m = 0 \), this theorem agrees with [25] Lemma 7.1.

**Remark 3.4** It is not quite clear from (3.4) that the first column of \( Y \) is analytic in the entire disk \( U_\delta \), which must be the case since it has polynomials as entries, see (2.24). Obviously, it is analytic on \( U_\delta\setminus(-\delta,0) \). From [1] formula 9.1.35 we have for \( x \in (-\delta,0) \)

\[
\begin{pmatrix} Y_{11,+}(x) \\ Y_{21,+}(x) \end{pmatrix} = \begin{pmatrix} Y_{11,-}(x) \\ Y_{21,-}(x) \end{pmatrix} = |x|^{-\alpha} e^m\frac{V(x)}{2} \sqrt{\pi} e^{-\frac{\pi i}{4}} e^{(n+m)\frac{\pi i}{2}} M(x) \\
\times \begin{pmatrix} -(n+m)f(x))^{1/2} J_{\alpha+\frac{1}{2}}(-(n+m)f(x)) \\
\times \frac{1}{i}(-(n+m)f(x))^{1/2} J_{\alpha-\frac{1}{2}}(-(n+m)f(x)) \end{pmatrix}.
\] (3.9)

So, the first column of \( Y \) is analytic in the entire disk \( U_\delta \) except for a possible isolated singularity at the origin. Since \( J_{\alpha\pm\frac{1}{2}}(z) = O(z^{\alpha\pm\frac{1}{2}}) \) as \( z \to 0 \), see [1] formula 9.1.10 this singularity is removable, which implies that the first column of \( Y \) is analytic in the entire disk.
Next, the behavior of the second column of $Y$ in the upper part of the disk $U_\delta$ is given by the following theorem.

**Theorem 3.5** Fix $m \in \mathbb{Z}$. For $z \in U_\delta \cap \mathbb{C}_+$ and $n$ sufficiently large, the second column of $Y = Y^{(n+m,n)}$ is given by

\[
\begin{pmatrix}
Y_{12}(z) \\
Y_{22}(z)
\end{pmatrix}
= z^\alpha e^{-n\frac{\sqrt{\pi}}{2}} e^{(n+m)\frac{\sigma_3}{2}} M(z) e^{\alpha \frac{\pi i \sigma_3}{2}}
\begin{pmatrix}
((n+m)f(z))^{1/2} H^{(1)}_{\alpha + \frac{1}{2}}((n+m)f(z)) \\
((n+m)f(z))^{1/2} H^{(1)}_{\alpha - \frac{1}{2}}((n+m)f(z))
\end{pmatrix}.
\]  

(3.10)

Here, $H^{(1)}_{\nu}$ is the Hankel function of the first kind of order $\nu$, $f$ is given by \((2.22)\), and $M$ is given by \((3.11)\).

**Proof.** Let $z$ be in sector $\Gamma$ of $U_\delta$. Instead of \((3.6)\) we get for the second column of $Y$

\[
\begin{pmatrix}
Y_{12}(z) \\
Y_{22}(z)
\end{pmatrix}
= z^\alpha e^{-n\frac{\sqrt{\pi}}{2}} e^{(n+m)\frac{\sigma_3}{2}} M(z) \Psi_\alpha((n+m)f(z)) e^{\pi i \sigma_3} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]  

(3.11)

Since $0 < \arg(n+m)f(z) < \pi/4$ for $n$ sufficiently large, we have to use \((2.24)\) to evaluate $\Psi_\alpha((n+m)f(z))$. Inserting this into \((3.11)\), we obtain \((3.10)\) for this choice of $z$.

Now, let $z$ be in sector $\Pi'$ of $U_\delta$. Instead of \((3.7)\) the second column of $Y$ is given by

\[
\begin{pmatrix}
Y_{12}(z) \\
Y_{22}(z)
\end{pmatrix}
= z^\alpha e^{-n\frac{\sqrt{\pi}}{2}} e^{(n+m)\frac{\sigma_3}{2}} M(z) \Psi_\alpha((n+m)f(z)) e^{\pi i \sigma_3} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]  

(3.12)

Since $\pi/4 < \arg(n+m)f(z) < \pi/2$ for $n$ sufficiently large, we have to use \((2.25)\) to evaluate $\Psi_\alpha((n+m)f(z))$. From [1 formula 9.6.4], which connects the modified Bessel function $K_{\alpha \pm \frac{1}{2}}$ with the Hankel function $H^{(1)}_{\alpha \pm \frac{1}{2}}$ of the first kind, we then have

\[
\Psi_\alpha((n+m)f(z)) e^{\pi i \sigma_3} \begin{pmatrix} 0 \\ 1 \end{pmatrix}
= \frac{1}{\sqrt{\pi}} e^{-\frac{\pi i}{2}} \begin{pmatrix}
((n+m)f(z))^{1/2} K_{\alpha + \frac{1}{2}}((n+m)f(z)) e^{\frac{-\pi i}{2}} \\
((n+m)f(z))^{1/2} K_{\alpha - \frac{1}{2}}((n+m)f(z)) e^{\frac{-\pi i}{2}}
\end{pmatrix}
\]

\[
= \frac{1}{2} \sqrt{\pi} e^{\frac{-\pi i}{2}} \begin{pmatrix}
((n+m)f(z))^{1/2} H^{(1)}_{\alpha + \frac{1}{2}}((n+m)f(z)) \\
((n+m)f(z))^{1/2} H^{(1)}_{\alpha - \frac{1}{2}}((n+m)f(z))
\end{pmatrix}.
\]

Inserting this into \((3.12)\), equation \((3.10)\) is proven in this sector as well.

Similarly, we can prove \((3.10)\) in the other sectors of the upper part of $U_\delta$. \qed

And finally, the behavior of the second column of $Y$ in the lower part of the disk $U_\delta$ is given by the following theorem.
Theorem 3.6 Fix $m \in \mathbb{Z}$. For $z \in U_\delta \cap \mathbb{C}_-$ and $n$ sufficiently large, the second column of $Y = Y^{(n+m,n)}$ is given by

$$
\begin{bmatrix}
Y_{12}(z) \\
Y_{22}(z)
\end{bmatrix}
= -z^\alpha e^{-nV(z)} \frac{1}{2} \sqrt{\pi} e^{-\frac{\pi}{4} e^{(n+m)\frac{3}{2}} M(z)}
\times \begin{pmatrix}
((n+m)f(z))^{1/2} H_{\alpha+\frac{1}{2}}^{(2)}((n+m)f(z)) \\
((n+m)f(z))^{1/2} H_{\alpha-\frac{1}{2}}^{(2)}((n+m)f(z))
\end{pmatrix}.
$$

(3.13)

Here, $H_\nu^{(2)}$ is the Hankel function of the second kind of order $\nu$, $f$ is given by (2.22), and $M$ is given by (3.1).

Proof. The proof is similar to the proofs of Theorem 3.2 and Theorem 3.5.

Remark 3.7 By (2.1), the jump relation for the second column of $Y$ is

$$
\begin{pmatrix}
Y_{12,+}(x) \\
Y_{22,+}(x)
\end{pmatrix}
- \begin{pmatrix}
Y_{12,-}(x) \\
Y_{22,-}(x)
\end{pmatrix}
= \begin{pmatrix}
Y_{11}(x) \\
Y_{21}(x)
\end{pmatrix} |x|^{2\alpha} e^{-nV(x)}, \quad \text{for } x \in \mathbb{R} \setminus \{0\}.
$$

(3.14)

For $x \in (0,\delta)$ one can check easily, using (3.2), (3.10), (3.13) and [1, formulas 9.1.3 and 9.1.4] that (3.14) is satisfied. For $x \in (-\delta,0)$ it follows from (3.9), (3.10), (3.13) and [1, formulas 9.1.3, 9.1.4 and 9.1.39] that (3.14) is satisfied.

Remark 3.8 Theorems 3.2, 3.5 and 3.6 give because of (2.4), after straightforward calculations, the behavior near the origin of the orthogonal polynomials and their Cauchy transforms. It has been shown before by Akemann and Fyodorov [6] that the behavior near the origin of the orthogonal polynomials is given in terms of the $J$-Bessel functions $J_{\alpha+\frac{1}{2}}$, and that the behavior near the origin of their Cauchy transforms is given in terms of the Hankel functions $H_{\alpha+\frac{1}{2}}^{(1)}$ of the first kind in the upper half-plane, and in terms of the Hankel functions $H_{\alpha-\frac{1}{2}}^{(2)}$ of the second kind in the lower half-plane. However, in [6] this was done on a physical level of rigor, and under the assumption that the eigenvalue density was supported on only one interval.

4 Proof of Theorem 1.1–1.3

In this section we prove the universal behavior at the origin of the spectrum for the three kernels $W_{I,n+m}$, $W_{II,n+m}$ and $W_{III,n+m}$, in terms of the Bessel kernels given by Table 2. Similar as in [24, 25], where we have investigated local eigenvalue correlations, we do this by using the connection of these kernels with the solution of the RH problem for $Y$, see (2.5)–(2.7), and by using the behavior of $Y$ near the origin, derived in the previous section.

We first need the following lemma’s.

Lemma 4.1 Let $M$ be the matrix valued function given by (3.1), and let $\zeta, \eta \in \mathbb{C}$. Then, each entry $M_{ij}$ of $M$ satisfies

$$
M_{ij} \left( \frac{\zeta}{n^\psi(0)} \right) - M_{ij} \left( \frac{\eta}{n^\psi(0)} \right) = O \left( \frac{\zeta - \eta}{n} \right), \quad \text{as } n \to \infty,
$$

(4.1)

uniformly for $\zeta$ and $\eta$ in compact subsets of $\mathbb{C}$.
Proof. Let $\zeta, \eta \in \mathbb{C}$, denote $\zeta_n = \frac{\zeta}{n \psi(0)}$ and $\eta_n = \frac{\eta}{n \psi(0)}$, and let $\gamma$ be a positively oriented simple closed contour in $U_\delta$ going around the origin. Then, since $M$ is analytic on $U_\delta$, an application of Cauchy’s formula shows that

$$M_{ij}(\zeta_n) - M_{ij}(\eta_n) = (\zeta_n - \eta_n) \frac{1}{2\pi i} \oint_{\gamma} \frac{M_{ij}(z)}{(z - \zeta_n)(z - \eta_n)} dz,$$

for $\zeta$ and $\eta$ in compact subsets of $\mathbb{C}$ and $n$ sufficiently large. Since $M_{ij}$ is uniformly bounded in $U_\delta$ as $n \to \infty$, see Section 3, the integral is uniformly bounded for $\zeta$ and $\eta$ in compact subsets of $\mathbb{C}$ as $n \to \infty$. This proves the lemma.

Lemma 4.2 Fix $m \in \mathbb{Z}$. Let $\zeta, \eta \in \mathbb{C}$, and denote

$$\tilde{\zeta}_n = (n + m)f \left( \frac{\zeta}{n \psi(0)} \right), \quad \text{and} \quad \tilde{\eta}_n = (n + m)f \left( \frac{\eta}{n \psi(0)} \right).$$

Then,

$$\zeta^{-\alpha} \tilde{\zeta}_n^{1/2} J_{\alpha + \frac{1}{2}}(\tilde{\zeta}_n) = \zeta^{-\alpha} (\pi \zeta)^{1/2} J_{\alpha + \frac{1}{2}}(\pi \zeta) + O(1/n), \quad \text{(4.2)}$$

as $n \to \infty$, uniformly for $\zeta$ in compact subsets of $\mathbb{C}$. The left hand side of (4.2) is uniformly bounded for $\zeta$ in compact subsets of $\mathbb{C}$ as $n \to \infty$. Also

$$\zeta^{-\alpha} \tilde{\zeta}_n^{1/2} J_{\alpha + \frac{1}{2}}(\tilde{\zeta}_n) - \eta^{-\alpha} \tilde{\eta}_n^{1/2} J_{\alpha + \frac{1}{2}}(\tilde{\eta}_n)$$

$$= \zeta^{-\alpha} (\pi \zeta)^{1/2} J_{\alpha + \frac{1}{2}}(\pi \zeta) - \eta^{-\alpha} (\pi \eta)^{1/2} J_{\alpha + \frac{1}{2}}(\pi \eta) + O \left( \frac{\zeta - \eta}{n} \right), \quad \text{(4.3)}$$

as $n \to \infty$, uniformly for $\zeta$ and $\eta$ in compact subsets of $\mathbb{C}$.

Proof. By (4.2) it follows that $\tilde{\zeta}_n = \pi \zeta (1 + O(1/n))$ as $n \to \infty$, uniformly for $\zeta$ in compact subsets of $\mathbb{C}$. Inserting this into the left hand side of (4.2) we easily obtain estimates (4.2), cf. [25, Lemma 7.2]. Since $J_\nu(\zeta) = \zeta^\nu H_\nu(\zeta)$ with $H_\nu$ entire, see [11, formula 9.1.10], we have that $\zeta^{-\alpha + \frac{1}{2}} J_{\alpha + \frac{1}{2}}(\zeta)$ is entire. This implies by (4.2) that the left hand side of (4.2) is uniformly bounded for $\zeta$ in compact subsets of $\mathbb{C}$ as $n \to \infty$.

Let $K_1, K_2$ be compact subsets of $\mathbb{C}$, and let $\gamma$ be a positively oriented simple closed contour with $K_1$ and $K_2$ in its interior. Define

$$q_n(z) = z^{-\alpha} \tilde{z}_n^{1/2} J_{\alpha + \frac{1}{2}}(\tilde{z}_n) - z^{-\alpha} (\pi z)\tilde{z}_n^{1/2} J_{\alpha + \frac{1}{2}}(\pi z), \quad \text{(4.4)}$$

with $\tilde{z}_n = (n + m)f(\frac{z}{n \psi(0)})$. Note that $q_n$ is analytic in an open neighborhood of the interior of $\gamma$ for $n$ sufficiently large. An application of Cauchy’s theorem then shows that

$$q_n(\zeta) - q_n(\eta) = (\zeta - \eta) \frac{1}{2\pi i} \oint_{\gamma} \frac{q_n(z)}{(z - \zeta)(z - \eta)} dz,$$

for $\zeta \in K_1$ and $\eta \in K_2$, and $n$ sufficiently large. Since $q_n(z) = O(1/n)$ as $n \to \infty$ uniformly for $z$ in compact subsets of $\mathbb{C}$, see (4.2) and (4.3), and since $\zeta$ and $\eta$ are not close to the contour $\gamma$, the lemma is then proven.

Now, we are ready to prove the universal behavior at the origin of the spectrum for the kernel $W_{I,n+m}$. 

18
Proof of Theorem 1.1. Let \( \zeta, \eta \in \mathbb{C} \), denote

\[
\zeta_n = \frac{\zeta}{n \psi(0)}, \quad \eta_n = \frac{\eta}{n \psi(0)}, \quad \tilde{\zeta}_n = (n + m) f(\zeta_n), \quad \text{and} \quad \tilde{\eta}_n = (n + m) f(\eta_n),
\]

and let \( Y = Y^{(n+m,n)} \). Similar considerations as in \eqref{eq:24} \cite{24, 25}, using \eqref{eq:25} and the behavior \eqref{eq:26} of the first column of \( Y \) inside the disk \( U_\delta \), show that,

\[
\widehat{W}_{I,n+m}(\zeta, \eta) = \gamma_{n+m-1,n}^2 \frac{1}{n \psi(0)} W_{I,n+m}(\zeta_n, \eta_n)
\]

\[
= \frac{1}{-2\pi i(\zeta - \eta)} \det \begin{pmatrix} Y_{11}(\zeta_n) & Y_{11}(\eta_n) \\ Y_{21}(\zeta_n) & Y_{21}(\eta_n) \end{pmatrix}
\]

\[
= (n \psi(0))^{2\alpha} e^{\frac{2}{7}(V(\zeta_n)+V(\eta_n))} \frac{1}{2(\zeta - \eta)} \times \det \begin{pmatrix} M(\zeta_n) \left( \frac{\zeta^{1/2}J_{\alpha+1/2}(\tilde{\zeta}_n)}{\zeta^{1/2}J_{\alpha+1/2}(\tilde{\eta}_n)} \right) & 0 \\ \frac{\zeta^{1/2}J_{\alpha-1/2}(\tilde{\zeta}_n)}{\zeta^{1/2}J_{\alpha-1/2}(\tilde{\eta}_n)} & 0 \end{pmatrix} + M(\eta_n) \begin{pmatrix} 0 & \eta^{1/2}J_{\alpha+1/2}(\tilde{\eta}_n) \\ 0 & \eta^{1/2}J_{\alpha-1/2}(\tilde{\eta}_n) \end{pmatrix}. \tag{4.5}
\]

The matrix in the determinant can be written as, cf. \eqref{eq:24} \cite{24, 25},

\[
M(\zeta_n) \begin{pmatrix} \frac{\zeta^{1/2}J_{\alpha+1/2}(\tilde{\zeta}_n)}{\zeta^{1/2}J_{\alpha+1/2}(\tilde{\eta}_n)} & \eta^{1/2}J_{\alpha+1/2}(\tilde{\eta}_n) \\ \frac{\zeta^{1/2}J_{\alpha-1/2}(\tilde{\zeta}_n)}{\zeta^{1/2}J_{\alpha-1/2}(\tilde{\eta}_n)} & \eta^{1/2}J_{\alpha-1/2}(\tilde{\eta}_n) \end{pmatrix}
\]

\[
+ M(\zeta_n)^{-1}(M(\eta_n) - M(\zeta_n)) \begin{pmatrix} 0 & \eta^{1/2}J_{\alpha+1/2}(\tilde{\eta}_n) \\ 0 & \eta^{1/2}J_{\alpha-1/2}(\tilde{\eta}_n) \end{pmatrix}.
\]

Since \( \det M \equiv 1 \) and each entry of \( M \) is uniformly bounded for \( \zeta \) in compact subsets of \( \mathbb{C} \) as \( n \to \infty \), see Section \ref{sec:3} each entry of \( M(\zeta_n)^{-1} \) is uniformly bounded for \( \zeta \) in compact subsets of \( \mathbb{C} \) as \( n \to \infty \). Using Lemma \ref{lem:4.1} and the fact that \( \eta^{1/2}J_{\alpha+1/2}(\tilde{\eta}_n) \) is uniformly bounded for \( \eta \) in compact subsets of \( \mathbb{C} \) as \( n \to \infty \), see Lemma \ref{lem:4.2} we then obtain

\[
M(\zeta_n)^{-1}(M(\eta_n) - M(\zeta_n)) \begin{pmatrix} 0 & \eta^{1/2}J_{\alpha+1/2}(\tilde{\eta}_n) \\ 0 & \eta^{1/2}J_{\alpha-1/2}(\tilde{\eta}_n) \end{pmatrix} = \begin{pmatrix} 0 & O \left( \frac{\zeta^{1/2}}{n} \right) \\ 0 & O \left( \frac{\zeta^{1/2}}{n} \right) \end{pmatrix}.
\]

Using the facts that \( \det M \equiv 1 \) and that \( \zeta^{1/2}J_{\alpha+1/2}(\tilde{\zeta}_n) \) is uniformly bounded for \( \zeta \) in compact subsets of \( \mathbb{C} \) as \( n \to \infty \), see Lemma \ref{lem:4.2} we then find

\[
\widehat{W}_{I,n+m}(\zeta, \eta) = (n \psi(0))^{2\alpha} e^{\frac{2}{7}(V(\zeta_n)+V(\eta_n))}
\]

\[
	imes \left[ \begin{pmatrix} \zeta^{1/2}J_{\alpha+1/2}(\tilde{\zeta}_n) & \eta^{1/2}J_{\alpha+1/2}(\tilde{\eta}_n) \\ \zeta^{1/2}J_{\alpha-1/2}(\tilde{\zeta}_n) & \eta^{1/2}J_{\alpha-1/2}(\tilde{\eta}_n) \end{pmatrix} + O(1/n) \right]. \tag{4.6}
\]

We can now replace \( z^{1/2}J_{\alpha+1/2}(\tilde{\zeta}_n) \) by \( z^{1/2}(\pi z)^{1/2}J_{\alpha+1/2}(\pi z) \) for \( z = \zeta, \eta \), and obtain the limiting Bessel kernel \( \mathbb{J}_{\alpha,f}(\zeta, \eta) \) given in Table \ref{tab:2}. However, then we make an error which does not hold uniformly for \( \zeta \) and \( \eta \) close to each other. To solve this problem we will work as in \eqref{eq:24} \cite{24, 25}. We subtract the second column in the determinant from the first one. From \eqref{eq:4.3} and
the fact that \( \eta^{-\alpha} \eta^{-1/2} J_{\alpha+\frac{1}{2}}(\eta_n) \) is uniformly bounded for \( \eta \) in compact subsets of \( \mathbb{C} \) as \( n \to \infty \), the term inside the brackets in (4.6) is then given by

\[
\frac{1}{2(\zeta - \eta)} \det \left( \begin{array}{cc}
\zeta^{-\alpha} (\pi \zeta)^{1/2} J_{\alpha+\frac{1}{2}}(\pi \zeta) & \eta^{-\alpha} \eta^{-1/2} J_{\alpha+\frac{1}{2}}(\eta_n) \\
\zeta^{-\alpha} (\pi \zeta)^{1/2} J_{\alpha-\frac{1}{2}}(\pi \zeta) & \eta^{-\alpha} \eta^{-1/2} J_{\alpha-\frac{1}{2}}(\eta_n) \end{array} \right) + O(1/n).
\]

Using the fact that

\[
\eta^{-\alpha} \eta^{-1/2} J_{\alpha+\frac{1}{2}}(\eta_n) = \eta^{-\alpha} (\pi \eta)^{1/2} J_{\alpha+\frac{1}{2}}(\pi \eta) + O(1/n),
\]

and the fact that

\[
\frac{\zeta^{-\alpha} (\pi \zeta)^{1/2} J_{\alpha+\frac{1}{2}}(\pi \zeta) - \eta^{-\alpha} (\pi \eta)^{1/2} J_{\alpha+\frac{1}{2}}(\pi \eta)}{\zeta - \eta}
\]

remains bounded for \( \zeta \) and \( \eta \) in compact subsets of \( \mathbb{C} \), which follows since \( z^{-\alpha}(\pi z)^{1/2} J_{\alpha+\frac{1}{2}}(z) \) is entire, we then easily obtain that the term inside the brackets in (4.6) is given by

\[
\frac{1}{2(\zeta - \eta)} \det \left( \begin{array}{cc}
\zeta^{-\alpha} (\pi \zeta)^{1/2} J_{\alpha+\frac{1}{2}}(\pi \zeta) & \eta^{-\alpha} (\pi \eta)^{1/2} J_{\alpha+\frac{1}{2}}(\pi \eta) \\
\zeta^{-\alpha} (\pi \zeta)^{1/2} J_{\alpha-\frac{1}{2}}(\pi \zeta) & \eta^{-\alpha} (\pi \eta)^{1/2} J_{\alpha-\frac{1}{2}}(\pi \eta) \end{array} \right) + O(1/n).
\]

The first term in (4.7) is exactly the limiting Bessel kernel \( J_{\alpha,I}(\zeta, \eta) \), see Table 2 From (4.6) and (4.7) we then obtain

\[
\tilde{W}_{I,n+m}(\zeta, \eta) = (n \psi(0))^{2\alpha} e^{\frac{n}{2}(V(\zeta_n) + V(\eta_n))} \left( J_{\alpha,I}(\zeta, \eta) + O(1/n) \right),
\]

as \( n \to \infty \), uniformly for \( \zeta \) and \( \eta \) in bounded subsets of \( \mathbb{C} \). Note that

\[
e^{\frac{n}{2}(V(\zeta_n) + V(\eta_n))} = e^{nV(0) + \frac{V'(0)}{2}(\zeta+n)+O(1/n)}
\]

\[
= e^{nV(0)} e^{V'(0)(\zeta+n)}(1+O(1/n)), \quad \text{as} \quad n \to \infty,
\]

uniformly for \( \zeta \) and \( \eta \) in compact subsets of \( \mathbb{C} \). Inserting this into (4.8), and using the fact that \( J_{\alpha,I}(\zeta, \eta) \) is bounded for \( \zeta \) and \( \eta \) in compact subsets of \( \mathbb{C} \) the theorem is then proven. \( \square \)

In order to prove Theorem \( \ref{thm:main} \) we also need the following lemma, which is analogous to Lemma \( \ref{lem:MainLemma} \).

**Lemma 4.3** Fix \( m \in \mathbb{Z} \). Let \( \zeta \in \mathbb{C}_+ \), and denote \( \tilde{\zeta}_n = (n+m)\zeta(\frac{m}{\psi(0)}) \). Then,

\[
\zeta^{\alpha} \tilde{\zeta}_n^{1/2} H^{(1)}_{\alpha+\frac{1}{2}}(\tilde{\zeta}_n) = \zeta^{\alpha} (\pi \zeta)^{1/2} H^{(1)}_{\alpha+\frac{1}{2}}(\pi \zeta) + O(1/n),
\]

as \( n \to \infty \), uniformly for \( \zeta \) in compact subsets of \( \mathbb{C}_+ \). The left hand side of (4.9) is uniformly bounded for \( \zeta \) in compact subsets of \( \mathbb{C}_+ \) as \( n \to \infty \).

**Proof.** Recall, cf. the proof of Lemma \( \ref{lem:MainLemma} \) that \( \tilde{\zeta}_n = (\pi \zeta)(1 + O(1/n)) \). Inserting this into the left hand side of (4.9) and using the fact that \( H^{(1)}_{\alpha+\frac{1}{2}} \) is analytic on \( \mathbb{C} \setminus (-\infty, 0] \) we easily obtain estimate (4.9). Since \( H^{(1)}_{\alpha+\frac{1}{2}} \) is analytic on \( \mathbb{C} \setminus (-\infty, 0] \), we have that \( \zeta^{\alpha} (\pi \zeta)^{1/2} H^{(1)}_{\alpha+\frac{1}{2}}(\pi \zeta) \) is bounded for \( \zeta \) in compact subsets of \( \mathbb{C} \setminus (-\infty, 0] \), and thus in particular in compact subsets of \( \mathbb{C}_+ \). Together with estimate (4.9) this implies that the left hand side of (4.9) remains uniformly bounded for \( \zeta \) in compact subsets of \( \mathbb{C}_+ \) as \( n \to \infty \). \( \square \)
Proof of Theorem 1.2 Let \( \zeta \in \mathbb{C}_+, \eta \in \mathbb{C} \), denote
\[
\zeta_n = \frac{\zeta}{n^{\psi(0)}}, \quad \eta_n = \frac{\eta}{n^{\psi(0)}}, \quad \tilde{\zeta}_n = (n + m)f(\zeta_n), \quad \tilde{\eta}_n = (n + m)f(\eta_n),
\]
and let \( Y = Y^{(n+m,n)} \). Instead of equation (4.5), we obtain from (2.6), from the behavior (3.1) of the first column of \( Y \) inside \( U_\delta \), and from the behavior (3.10) of the second column of \( Y \) in the upper part of \( U_\delta \),
\[
\tilde{W}_{II,n+m}(\zeta, \eta) = \gamma_{n+m-1,n}^2 \zeta - \eta n^{\psi(0)} \tilde{W}_{II,n+m}(\zeta, \eta_n)
= \frac{1}{-2\pi i} \det \begin{pmatrix} Y_{12}(\zeta_n) & Y_{11}(\eta_n) \\ Y_{22}(\zeta_n) & Y_{21}(\eta_n) \end{pmatrix}
= \frac{1}{4} e^{-\frac{i}{2}(V(\zeta_n) - V(\eta_n))}
\]
\times \det \begin{pmatrix} M(\zeta_n) \left( \zeta \zeta_n^{1/2} H^{(1)}_{\alpha+1/2}(\tilde{\zeta}_n) \eta \eta_n^{1/2} J_{\alpha-1/2}(\tilde{\eta}_n) \right) + M(\eta_n) \left( 0 \eta^{-\alpha} \eta_n^{1/2} J_{\alpha-1/2}(\tilde{\eta}_n) \right) \end{pmatrix},
\]
We now rewrite the matrix in the determinant as was done in the proof of Theorem 1.1. Using also the fact that \( \zeta \zeta_n^{1/2} H^{(1)}_{\alpha+1/2}(\tilde{\zeta}_n) \) is uniformly bounded for \( \zeta \) in compact subsets of \( \mathbb{C}_+ \) as \( n \to \infty \), see Lemma 4.3, we obtain instead of equation (4.10), in a similar fashion, the following
\[
\tilde{W}_{II,n+m}(\zeta, \eta) = e^{-\frac{i}{2}(V(\zeta_n) - V(\eta_n))}
\]
\times \begin{pmatrix} \frac{1}{4} \det \begin{pmatrix} \zeta \zeta_n^{1/2} H^{(1)}_{\alpha+1/2}(\tilde{\zeta}_n) \eta \eta_n^{1/2} J_{\alpha-1/2}(\tilde{\eta}_n) \end{pmatrix} + O(1/n) \end{pmatrix}, \quad (4.10)
\]
as \( n \to \infty \), uniformly for \( \zeta \) and \( \eta \) in compact subsets of \( \mathbb{C}_+ \) and \( \mathbb{C} \), respectively. We now insert the fact that, see Lemma 4.3,
\[
\zeta \zeta_n^{1/2} H^{(1)}_{\alpha+1/2}(\tilde{\zeta}_n) = \zeta \pi^{1/2} H^{(1)}_{\alpha+1/2}(\pi \zeta) + O(1/n),
\]
into (4.10), and use the fact that \( \eta^{-\alpha} \eta_n^{1/2} J_{\alpha-1/2}(\tilde{\eta}_n) \) is uniformly bounded for \( \eta \) in compact subsets of \( \mathbb{C} \) as \( n \to \infty \), see Lemma 1.2, to obtain
\[
\tilde{W}_{II,n+m}(\zeta, \eta) = e^{-\frac{i}{2}(V(\zeta_n) - V(\eta_n))}
\times \begin{pmatrix} \frac{1}{4} \det \begin{pmatrix} \zeta \pi^{1/2} H^{(1)}_{\alpha+1/2}(\pi \zeta) \eta^{-\alpha} \eta_n^{1/2} J_{\alpha-1/2}(\tilde{\eta}_n) \end{pmatrix} + O(1/n) \end{pmatrix}. \quad (4.11)
\]
Inserting, see Lemma 1.2,
\[
\eta^{-\alpha} \eta_n^{1/2} J_{\alpha-1/2}(\tilde{\eta}_n) = \eta^{-\alpha} (\pi \eta)^{1/2} J_{\alpha+1/2}(\pi \eta) + O(1/n),
\]

into (4.11), and using the fact that \( \zeta^\alpha (\pi \zeta)^{1/2} H_{\alpha+1/2}^{(1)} (\pi \zeta) \) is uniformly bounded for \( \zeta \) in compact subsets of \( \mathbb{C}_+ \) as \( n \to \infty \), we then obtain

\[
\hat{W}_{II,n+m}(\zeta, \eta) = e^{-\frac{n}{2}(V(\zeta_n) - V(\eta_n))} \left[ \frac{1}{4} \det \begin{pmatrix} \zeta^\alpha (\pi \zeta)^{1/2} H_{\alpha+1/2}^{(1)} (\pi \zeta) & \eta^{-\alpha} (\pi \eta)^{1/2} J_{\alpha+1/2}^{(1)} (\pi \eta) \\ \zeta^\alpha (\pi \zeta)^{1/2} H_{\alpha-1/2}^{(1)} (\pi \zeta) & \eta^{-\alpha} (\pi \eta)^{1/2} J_{\alpha-1/2}^{(1)} (\pi \eta) \end{pmatrix} + O(1/n) \right]
\]

\[
= e^{-\frac{n}{2}(V(\zeta_n) - V(\eta_n))} \left( (\zeta - \eta) J_{\alpha,II}^+(\zeta, \eta) + O(1/n) \right),
\]

as \( n \to \infty \), uniformly for \( \zeta \) and \( \eta \) in compact subsets of \( \mathbb{C}_+ \) and \( \mathbb{C} \), respectively. In (4.12), \( J_{\alpha,II}^+(\zeta, \eta) \) is the Bessel kernel given in Table 2. Note that

\[
e^{-\frac{n}{2}(V(\zeta_n) - V(\eta_n))} = e^{-\frac{V'(0)}{2\psi(0)}(\zeta - \eta) + O(1/n)},
\]

\[
e^{-\frac{V'(0)}{2\psi(0)}(\zeta - \eta)} (1 + O(1/n)), \quad \text{as } n \to \infty,
\]

uniformly for \( \zeta \) and \( \eta \) in compact subsets of \( \mathbb{C}_+ \) and \( \mathbb{C} \), respectively. Inserting this into (4.12) and using the fact that \( (\zeta - \eta) J_{\alpha,II}^+(\zeta, \eta) \) is bounded for \( \zeta \) and \( \eta \) in compact subsets of \( \mathbb{C}_+ \) and \( \mathbb{C} \), respectively, the first part of the theorem is then proven.

The second part of the theorem can be treated in the same way, using the behavior of the second column of \( Y \) in the lower part of the disk \( U_\delta \), instead of in the upper part of the disk.

We leave it as an exercise for the careful reader to prove the universal behavior at the origin of the spectrum for the kernel \( W_{III,n+m} \).

**Proof of Theorem 1.3** The proof is similar to the proofs of Theorem 1.1 and Theorem 1.2.

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