TOPOLOGICAL CASIMIR EFFECT IN POWER-LAW FRW COSMOLOGIES

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We investigate the vacuum expectation values of the field squared and the energy-momentum tensor for a massless scalar field with general curvature coupling parameter in spatially flat Friedmann-Robertson-Walker universes with an arbitrary number of toroidally compactified dimensions. When the comoving lengths of the compact dimensions are short compared to the Hubble length, the topological parts coincide with those for a conformal coupling. This limit corresponds to the adiabatic approximation. In the opposite limit of large comoving lengths of the compact dimensions, in dependence of the curvature coupling parameter, two regimes are realized with monotonic or oscillatory behavior of the vacuum expectation values.

Keywords: Topological Casimir effect; Friedmann-Robertson-Walker cosmology

1. Introduction

In the present talk, based on Ref.,¹ we consider an exactly soluble problem for the topological Casimir effect on background of Friedmann-Robertson-Walker (FRW) universes with a power-law scale factor. The vacuum polarization and the particle creation in the FRW cosmological models with trivial topology have been considered in a large number of papers (see Refs.²,³). In particular, the vacuum expectation values of the field squared and the energy-momentum tensor in models with power law scale factors have been discussed in Refs.⁴

In most work on the topological Casimir effect in cosmological backgrounds, the results for the corresponding static counterparts were used replacing the static length scales by comoving lengths in the cosmological bulk. This procedure is valid in conformally invariant situations or under the assumption of a quasi-adiabatic approximation. For non-conformal fields
the calculations should be done directly within the framework of quantum field theory on time-dependent backgrounds (for the topological Casimir effect in toroidally compactified de Sitter spacetime see Refs.\textsuperscript{5}).

The paper is organized as follows. In the next section we evaluate the vacuum expectation value (VEV) of the field squared in spatially flat FRW model with topology $R^p \times (S^1)^q$. In Sec. 3 we consider the VEV of the energy-momentum tensor. The main results are summarized in Sec. 4.

2. VEV of the field squared

We consider a scalar field with curvature coupling parameter $\xi$ evolving on background of the $(D + 1)$-dimensional spatially flat FRW spacetime with power law scale factor $a(t) = \alpha t^\nu$. In addition to the synchronous time coordinate $t$ it is convenient to introduce the conformal time $\tau$ in accordance with $t = |\alpha(1 - c)\tau|^{1/(1 - c)}$. Here we assume that $c \neq 1$. Note that one has $0 \leq \tau < \infty$ for $0 < c < 1$ and $-\infty < \tau \leq 0$ for $c > 1$.

We will assume that the spatial coordinates $z^l$, $l = p + 1, \ldots, D$, are compactified to $S^1$: $0 \leq z^l \leq L_q$, and for the other coordinates we have $-\infty < z^l < +\infty$, $l = 1, \ldots, p$. Hence, we consider the spatial topology $R^p \times (S^1)^q$ with $p + q = D$. Along the compact dimensions we will consider the boundary conditions $\varphi(\tau, z_p, z_q + \mathbf{e}_l L) = e^{2\pi i \alpha_l} \varphi(\tau, z_p, z_q)$ with constant phases $\alpha_l$, where $z_p = (z^1, \ldots, z^p)$, $z_q = (z^{p+1}, \ldots, z^D)$, and $\mathbf{e}_l$, $l = p + 1, \ldots, D$, is the unit vector along the direction $z^l$.

For the VEV of the field squared we have the following decomposition:

$$
\langle \varphi^2 \rangle_{p, q} = \langle \varphi^2 \rangle_{FRW} + \langle \varphi^2 \rangle_{p, q}^{(t)} + \langle \varphi^2 \rangle_{p, q}^{(t)} = \sum_{j=p}^{D-1} \Delta_{j+1} \langle \varphi^2 \rangle_{j, D-j},
$$

where $\langle \varphi^2 \rangle_{FRW} = \langle \varphi^2 \rangle_{D, 0}$ is the VEV in the spatial topology $R^D$ and the part $\langle \varphi^2 \rangle_{p, q}^{(t)}$ is induced by the nontrivial topology. For the topological part induced due to the compactness of the $z^{p+1}$ direction we have

$$
\Delta_{p+1} \langle \varphi^2 \rangle_{p, q} = \frac{4A\eta^{2b}}{(2\pi)^{p+3}V_{q-1}} \sum_{n_{p-1} \in \mathbb{Z}_{q-1}} \int_0^\infty dy y \left[ I_{-\nu}(y\eta) + I_{\nu}(y\eta) \right] 
\times K_{\nu}(y\eta) \sum_{n=1}^\infty \frac{\cos(2\pi n \alpha_{p+1})}{(n L_{p+1})^{p-1}} f_{(p-1)/2}(n L_{p+1}) \sqrt{y^2 + k_{n_{p-1}}^2},
$$

where $I_{\nu}(z)$ and $K_{\nu}(z)$ are the modified Bessel functions, $f_{b}(x) = x^b K_{b}(x)$, and $A = \alpha^{1-D}[\alpha|1 - \alpha|]^{(D-1)c/(c-1)}$. Here the notations $\eta = |\tau|$, $b = (cD - 1)/(2(c - 1))$, $V_{q-1} = L_{p+2}, \ldots, L_D$ and $n_{p-1} = (n_{p+2}, \ldots, n_D)$, $k_{n_{p-1}}^2 = \ldots$.
\[ \sum_{l=p+2}^{D}(2\pi n_l/L_l)^2, \]
are introduced. The parameter \( \nu \) is defined as
\[ \nu = \frac{1}{2|1-c|}\sqrt{(cD - 1)^2 - 4\xi D c[(D + 1)c - 2]}. \tag{3} \]

In figure Fig. 1, we have plotted the ratio \( \langle \varphi^2 \rangle_{D-1,1}/\langle \varphi^2 \rangle_{D-1,1} \) for the special case of topology \( R^{D-1} \times S^1 \) as a function of \( L/\eta \), with \( L = L_D \) being the length of the compact dimension, for untwisted \( D = 3 \) scalar field \( (\alpha_D = 0) \) and for various values of the parameter \( \nu \). Note that the ratio \( L/\eta \) is related to the comoving length of the compact dimension, measured in units of the Hubble length, by \( L/\eta = (|1 - c|/c) L_{(c)}/r_H \). Figure 1 clearly shows that the adiabatic approximation for the topological part is valid only for small values of the ratio \( L(c)/r_H \).

In the limit \( L_{(c)}^D \ll r_H \) the topological part coincides with that for a conformal coupling and behaves like \( \langle \varphi^2 \rangle_{p,q} \propto t^c(1-D) \). This limit corresponds to the adiabatic approximation. In the opposite limit the behavior of the VEV is qualitatively different for real and imaginary values of the parameter \( \nu \). For real values, the topological part behaves as \( \langle \varphi^2 \rangle_{p,q} \propto t^{2(c-1)\nu - cD+1} \). In the limit \( L_{(c)}^D \gg r_H \) and for imaginary values \( \nu \) the asymptotic behavior is oscillatory: \( \langle \varphi^2 \rangle_{p,q} \propto t^{1-cD} \cos[2|\nu|(c-1)\ln(t/t_0) + \psi] \).
3. VEV of the energy-momentum tensor

For the VEV of the energy-momentum tensor we have the formula

\[
(T^{k}_{i})_{p,q} = \langle T^{k}_{i}\rangle_{FRW} + \langle T^{k}_{i}\rangle_{TR}, \quad \langle T^{k}_{i}\rangle_{p,q} = \sum_{j=p}^{D-1} \Delta_{j+1}(T^{k}_{i})_{j,D-j},
\]

where \(\langle T^{k}_{i}\rangle_{FRW}\) is the part corresponding to the uncompactified FRW spacetime and \(\langle T^{k}_{i}\rangle_{p,q}\) is induced by the nontrivial topology. The first term is well investigated in the literature. Here for the topological part we have (no summation over \(i\))

\[
\Delta_{p+1}(T^{k}_{i})_{p,q} = \frac{4A\Omega^{-2}}{(2\pi)^{(p+3)/2}V(1)} \sum_{n_{p-q} \in \mathbb{Z}} \sum_{n=1}^{\infty} \frac{\cos(2\pi n \alpha_{p+1})}{(n L_{p+1})^{p+1}} \int_{0}^{\infty} dy y^{3-b} \times \left[ f_{(p-1)/2}(z) F^{(i)}(\eta y) - f_{(p-1)/2}(z) \frac{\tilde{I}_{\nu}(\eta y) \tilde{K}_{\nu}(\eta y)}{(n L_{p+1+y})^{2}} \right]_{z=n L_{p+1} \sqrt{y^{2}+w^{2}}},
\]

with \(\tilde{K}_{\nu}(z) = z^{b} K_{\nu}(z), \quad \tilde{I}_{\nu}(z) = z^{b} [I_{\nu}(z) + I_{-\nu}(z)],\) and

\[
F^{(0)}(z) = \frac{1}{2} \tilde{T}_{\nu} \tilde{K}_{\nu} + \frac{Dz^{c}}{z(1-c)} (I_{\nu} \tilde{K}_{\nu})' - \frac{1}{2} \left[ 1 - \frac{\xi D(D-1)c^{2}}{z^{2}(1-c)^{2}} \right] \tilde{I}_{\nu} \tilde{K}_{\nu},
\]

\[
F^{(l)}(z) = 2 \left( \xi - \frac{1}{4} \right) \tilde{T}_{\nu} \tilde{K}_{\nu} - \frac{1}{z} \frac{c^{\xi}}{1-c} (I_{\nu} \tilde{K}_{\nu})' + 2 \left[ \xi - \frac{1}{4} - Dc^{\xi} \frac{(D-2)(\xi - Dc^{\xi} + \xi c)}{z^{2}(1-c)^{2}} \right] \tilde{I}_{\nu} \tilde{K}_{\nu},
\]

where \(l = 1, \ldots, D.\) In Eq. (5) we have used the notations

\[
f_{p}^{(0)}(z) = 0, \quad f_{p}^{(i)}(z) = f_{(p+1)/2}(z), \quad f_{p}^{(l)}(z) = (n L_{p+1} k)^{2} f_{(p+1)/2}(z), \quad f_{p}^{(l)}(z) = -p f_{(p+1)/2}(z) - z^{2} f_{(p+1)/2}(z),
\]

where \(i = 1, 2, \ldots, p\) and \(l = p + 2, \ldots, D.\)

For \(L^{(c)} \ll r_{H}\) the topological part behaves like \(\langle T^{k}_{i}\rangle_{p,q} \propto t^{-c(D+1)}\) and to the leading order the stresses along the uncompactified dimensions are equal to the vacuum energy density. In the limit \(L^{(c)} \gg r_{H}\) and for real values \(\nu\) the asymptotic has the form \(\langle T^{k}_{i}\rangle_{p,q} \propto t^{(c+1)\nu-cD-1}\). The corresponding vacuum stresses are isotropic and the equation of state for the topological parts in the vacuum energy density and pressures is of the barotropic type. For \(L^{(c)} \gg r_{H}\) and for imaginary values \(\nu\) we have the asymptotic behavior \(\langle T^{k}_{i}\rangle_{p,q} \propto t^{-cD-1} \cos[2|\nu|(c-1) \ln(t/t_{0}) + \psi].\)
4. Conclusion

We have investigated one-loop quantum effects for a scalar field with general curvature coupling, induced by toroidal compactification of spatial dimensions in spatially flat FRW cosmological models with power law scale factor. General boundary conditions with arbitrary phases are considered along compact dimensions. The boundary conditions imposed on possible field configurations change the spectrum of vacuum fluctuations. Among the most important characteristics of the vacuum state are the expectation values of the field squared and the energy-momentum tensor. Though the corresponding operators are local, due to the global nature of the vacuum these VEVs carry an important information on the global structure of the background spacetime. We present the VEVs as the sum of the function for topologically trivial FRW model and the topological part. The latter is finite in the coincidence limit and in this way the renormalization of the VEVs is reduced to that for the FRW universe with trivial topology. The topological parts are given by formulae (1) and (2) for the field squared and by formulae (4), (5) for the energy density and the stresses. A.L.M. gratefully acknowledges the organizers of the conference QFEXT09 for the opportunity to present this paper.

References

1. A. A. Saharian and A. L. Mkhitaryan, arXiv:0908.3291.
2. N. D. Birrel and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1982); A. A. Grib, S. G. Mamayev, and V. M. Mostepanenko, *Vacuum Quantum Effects in Strong Fields* (Friedmann Laboratory Publishing, St. Petersburg, 1994).
3. V. M. Mostepanenko and N.N. Trunov, *The Casimir Effect and Its Applications* (Clarendon, Oxford, 1997); K. A. Milton, *The Casimir Effect: Physical Manifestation of Zero-Point Energy* (World Scientific, Singapore, 2002); M. Bordag, G. L. Klimchitskaya, U. Mohideen and V. M. Mostepanenko, *Advances in the Casimir Effect* (Oxford University Press, Oxford, 2009).
4. M. Bordag, J. Lindig, V. M. Mostepanenko, and Yu. V. Pavlov, *Int. J. Mod. Phys. D* **6**, 449 (1997); M. Bordag, J. Lindig, and V. M. Mostepanenko, *Class. Quantum Grav.* **15**, 581 (1998).
5. A. A. Saharian and M. R. Setare, Phys. Lett. B **659**, 367 (2008); S. Bellucci and A. A. Saharian, Phys. Rev. D **77**, 124010 (2008); A. A. Saharian, *Class. Quantum Grav.* **25**, 165012 (2008); E. R. Bezerra de Mello and A. A. Saharian, *JHEP* **12**, 081 (2008).