An area bound for surfaces in Riemannian manifolds

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Abstract: Let $M$ be a compact Riemannian manifold not containing any totally geodesic surface. Our main result shows that then the area of any complete surface immersed into $M$ is bounded by a multiple of its extrinsic curvature energy, i.e. by a multiple of the integral of the squared norm of its second fundamental form.

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1 Introduction

Let $M$ be a compact differentiable manifold of dimension $n \geq 3$, with a smooth Riemannian metric $\bar{g}$. For a smooth immersion $f : F \to M$ of a surface $F$ into $M$ we consider its extrinsic curvature energy

$$E(f) = \frac{1}{2} \int_F |A|^2 \, d\text{vol}_2.$$  

Here $\text{vol}_2 = \text{vol}_2^f$ is the measure associated to the Riemannian metric $g = f^* \bar{g}$ induced on $F$, $A$ is the normal-valued second fundamental form of $f$, and $|A|$ denotes the euclidean norm of the tensor field $A$. Clearly, $E(f) \geq 0$ with equality if and only if $f$ is a totally geodesic immersion. By the Gauß equation we have

$$\frac{1}{2} |A|^2 = |A^\circ|^2 + K_g - K_g^f = \frac{1}{2} |\bar{H}|^2 + K_g^f - K_g,$$

where $K_g$ is the Gaussian curvature of $g$, and $K_g^f$ denotes the sectional curvature of $\bar{g}$ on the tangent planes of $f$. The second fundamental form is decomposed as $A = A^\circ + \frac{1}{2} g \otimes \bar{H}$, where $A^\circ$ is trace-free and $\bar{H} = \text{trace}(A)$ is the mean curvature vector of $f$. If $f$ is an immersion of a closed surface $F$ into euclidean $\mathbb{E}^n$, the functional $E(f)$ reduces essentially to twice the Willmore energy, more precisely by the Gauß-Bonnet theorem

$$E(f) = \int_F |A^\circ|^2 \, d\text{vol}_2 + 2\pi \chi(F) = \frac{1}{2} \int_F |\bar{H}|^2 \, d\text{vol}_2 - 2\pi \chi(F).$$

In general, the area of an immersed surface is not bounded in terms of its energy. For example, consider the sequence $f_i$ of immersions into the flat torus $M = \mathbb{E}^n/\mathbb{Z}^n$ given by projecting $\lambda_i f$, where $f$ is a fixed immersion into $\mathbb{E}^n$ and $\lambda_i > 0$ goes to infinity. The scale invariance of $E$ implies that $E(f_i) = E(f) < \infty$ for all $i \in \mathbb{N}$, while the surface areas go to infinity. However, we prove as a main result that this behavior is rather special. In the following, an immersion $f : F \to M$ is called complete if the Riemannian metric $f^* \bar{g}$ is complete.

**Theorem 1.1 (Area bound)** Let $(M, \bar{g})$ be a compact Riemannian manifold of dimension $n \geq 3$. Assume that $(M, \bar{g})$ does not admit any complete, totally geodesic surface immersions. Then there is a constant $C = C(M, \bar{g}) < \infty$ such that $\text{vol}_2^f(F) \leq C E(f)$ for every complete, immersed surface $f : F \to M$.

For a generic metric $\bar{g}$ on $M$ there are no totally geodesic submanifolds of dimension $1 < k < \dim M$ at all in $(M, \bar{g})$. This was proved recently by Murphy & Wilhelm [23] for $n \geq 4$, for $n = 3$ there is a sketch by Bryant [4]. See also [19] where Lytchak and Petrunin give a short proof valid for all $n$ in the appendix.
In the more general context of $m$-varifolds in an $n$-dimensional Riemannian manifold, a related mass bound in terms of curvature energies was proved by A. Mondino [20]. Assuming that the bound fails, he applies a compactness argument to construct a nonzero $m$-varifold with vanishing generalized second fundamental form, see Theorem 4.1 in [20]. However, it is not clear how that (a priori non-rectifiable) varifold relates to totally geodesic immersions, and the above generic nonexistence is not immediate for such a varifold.

Our paper deals with the scale-invariant $L^2$ integral of the second fundamental form. The analysis would be simpler in the case of the $L^p$ integral with $p > 2$, employing Langer’s local graph representations [17], see also Breuning [3]. In the critical case $p = 2$ there is an almost graphical description due to Simon [28], and there are also local bilipschitz parametrizations by Toro [30], Hélein [13], and Müller & Šverák [22]. However, it remains unclear how to apply these results to a sequence with mass going to infinity.

Here we prove the following local version of Theorem 1.1 that is an important step in the proofs of Theorem 1.1 and Theorem 1.3.

**Theorem 1.2** Let $(M, \bar{g})$ be a compact Riemannian manifold of dimension $n \geq 3$. Assume that $(M, \bar{g})$ does not contain any totally geodesic surface. Then for every $r > 0$ there exists a constant $c(r) = c(r, M, \bar{g}) > 0$ such that $E(f|B(p, r)) \geq c(r)$ holds for every complete surface immersion $f : F \to M$ and every intrinsic metric ball $B(p, r)$ of radius $r$ on $F$. In particular, if $E(f) < \infty$, then $F$ is compact.

If we relax the condition on $(M, \bar{g})$ in Theorem 1.1 by allowing totally geodesic immersions of $S^2$, we obtain the following slightly weaker consequence.

**Theorem 1.3** Let $(M, \bar{g})$ be a compact Riemannian manifold of dimension $n \geq 3$. Assume that $(M, \bar{g})$ does not admit any complete, totally geodesic immersions of connected surfaces other than $S^2$ or $\mathbb{RP}^2$. Then, for every constant $D$, there exists a constant $C = C(M, \bar{g}, D)$ such that $E(f) < D$ implies $\text{vol}_2^2(F) < C$ for every complete, connected, immersed surface $f : F \to M$.

Our work is partly motivated by the joint paper [16] of Mondino, Schygulla and the second author. For a compact, three-dimensional Riemannian manifold $(M, \bar{g})$, they consider the problem of minimizing $E(f)$ in the class $[S^2, M]$ of immersions $f : S^2 \to M$. They prove existence under the two assumptions:

1. $E(f) < 4\pi$ for some $f \in [S^2, M]$,
2. For some minimizing sequence $f_i \in [S^2, M]$ the surface areas $\text{vol}_2^2(S^2)$ remain bounded.

The approach in [16] follows L. Simon [28]. In recent work by Guodong Wei [31] the result is reproved employing results from [22, 13, 15, 21] on conformal parametrizations. Combining the result from [16] with Theorem 1.3 we obtain

**Corollary 1.4** Let $(M, \bar{g})$ be a three-dimensional, compact Riemannian manifold that admits no complete, totally geodesic immersions of connected surfaces other than $S^2$ or $\mathbb{RP}^2$. If (1.4) is satisfied, there exists $f \in [S^2, M]$ minimizing $E$ on $[S^2, M]$. 

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Condition (1.4) holds if $\text{Scal}_g(x) > 0$ for some $x \in M$. Condition (1.5) is an easy consequence of the Gauß equations when $(M, \bar{g})$ has positive sectional curvature. In particular, (1.6) proves the existence of a minimizer in $[S^2, M]$, if $M$ is compact and has positive sectional curvature. We recover this result, since, by the Bonnet-Myers and the Gauß-Bonnet theorems, a complete, connected, totally geodesic immersed surface in a compact manifold of positive sectional curvature is of type $S^2$ or $\mathbb{R}P^2$.

Next we shortly outline the proofs of Theorems 1.1–1.3. These proofs are by contradiction. To prove Theorem 1.2 we assume that there exists a sequence $f_i : F_i \to M$ of complete surface immersions and a sequence of intrinsic metric balls $B(p_i, r) \subseteq F_i$ of fixed radius $r > 0$ such that $\lim_{i \to \infty} E(f_i|B(p_i, r)) = 0$. Then we prove that the set of limit points of sequences $f_i(x_i)$ with $x_i \in B(p_i, r)$ contains a totally geodesic surface, see Theorem 9.4 and Theorem 13.3 for a statement concerning Hausdorff convergence. This proof relies on the results of Sections 2–8 that will be reviewed below. We will also consider the following stronger assumption:

$$\text{(1.6)} \quad \text{There exist sequences } p_i \in F_i, R_i \to \infty, \text{such that } \lim_{i \to \infty} E(f_i|B(p_i, R_i)) = 0.$$  

Under assumption (1.6) we can prove that $M$ admits a complete, totally geodesic surface immersion, see Corollary 10.5. To derive Theorem 1.1 from this last result we use a Voronoi type covering argument to verify assumption (1.6). Here we need to generalize the well-known Bol-Fiala type area estimates from parallel sets to weakly starshaped sets, cf. Section 11. In Section 13 we prove Theorem 1.3 by contradiction. Here assumption (1.6) is easily seen to be satisfied, so that we obtain a complete, totally geodesic surface immersion $f : N \to M$. Then we show that $N$ is not diffeomorphic to $S^2$ or $\mathbb{R}P^2$. This is inspired by G. Reeb’s stability theorem 25 from the theory of foliations.

Now we review the contents of Sections 2–8 on which the proofs of Theorems 1.1–1.3 are based. In Section 2 we use Gronwall’s inequality to conclude that arclength-parametrized curves $\gamma : [a, b] \to M$ with small total absolute geodesic curvature are $C^1$-close to geodesics. In Section 3 we consider a complete immersion $f : F \to M$ such that $E(f)/\text{vol}_k^4(F)$ is small where $k = \dim F$. We use the invariance of the Liouville measure under the geodesic flow to find a subset of the unit tangent bundle $SF$ of large Liouville measure such that the $f$-images of geodesics in $F$ with initial vectors in this subset are $C^1$-close to geodesics in $M$. More generally, we prove a similar statement for configurations $(c_1, c_2, t)$ where $c_1, c_2$ are geodesics in $F$ with $c_2(0) = c_1(t)$. To relate the estimates on the Liouville measure to the geometry of $F$, we need lower estimates on the volume of intrinsic metric balls in $F$, see Sections 6 and 8. While the results in Sections 2 and 3 are true for manifolds $F$ of arbitrary dimension we can prove these volume estimates and much of the following only if $\dim F = 2$. This involves some new results on the geometry of surfaces in euclidean space $\mathbb{E}^n$ that depend on bounds on their energy, see Sections 4 and 5. In Section 7 we consider a sequence of complete surface immersions $f_i : F_i \to M$, and assume the existence of a sequence of balls $B(p_i, R_i) \subseteq F_i$ such that $R_i \to \infty$ and $\liminf_{i \to \infty} E(f_i|B(p_i, R_i)) < \frac{\pi}{4}$. Relying on ideas by T. Shioya 27 we prove that a subsequence of the sequence $(F_i, p_i)$ converges to a proper, pointed length space $(Y, y_0)$ with respect to pointed Gromov-Hausdorff convergence and that $Y$ has locally finite 2-dimensional Hausdorff measure. Moreover, we can assume that the immersions $f_i$ converge to a 1-Lipschitz map $f : Y \to M$. On the other hand, the results of Section 3 can be used to see that the Hausdorff dimension of $f(Y)$ is at least three, unless
f(Y) ⊆ M contains a totally geodesic surface, see Section 9.

Notation. Here we collect some of the notation used throughout the paper. The unit sphere of a euclidean vector space E is denoted by SE. If dimE = k then αk−1 is the (k−1)-volume of SE. If M is a manifold and 2 ≤ k ≤ dimM then πG : GkM → M denotes the Grassmann bundle of k-dimensional linear subspaces in TM. For Riemannian manifolds (M, g), we let π : SM → M denote the unit sphere bundle. The intrinsic metric ball in M with center p ∈ M and radius r > 0 is denoted by B(p, r). If dimM = m then vol_m is the Riemannian volume on M, and Hk, 1 ≤ k ≤ dimM, denotes k-dimensional Hausdorff measure on M. Geodesics will always be parametrized by arclength, and cv will denote the geodesic with initial vector v ∈ SM. If γ is a curve in M then P_t^γ : T_γ(0)M → T_γ(t)M denotes parallel translation along γ from γ(0) to γ(t). The injectivity radius of M will be denoted by injrad (M).

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2 $L^1$-almost geodesics

Let (M, g) denote a compact Riemannian manifold with Levi-Civita connection ∇. In this section we consider arclength-parametrized curves in M for which the $L^1$-norm of the covariant derivative of $\dot{γ}$ is small. Using Gronwall’s inequality we will easily see that such curves are $C^1$-close to geodesics. This will be applied to geodesics of submanifolds of M along which the norm of the second fundamental form has small integral.

On the unit tangent bundle SM we consider the Sasaki metric $\bar{g}$ induced by g, see [20], and the distance function $d^{SM}$ induced by $\bar{g}$.

Lemma 2.1 There exist constants $B > 0$, $C > 0$, such that the following holds for all arclength-parametrized $C^2$-curves $γ : [a, b] → M$. If $c : [a, b] → M$ is the geodesic with initial vector $\dot{c}(a) = \dot{γ}(a)$ and $t ∈ [a, b]$, then

$$d^{SM}(\dot{γ}(t), \dot{c}(t)) \leq Be^{C(t−a)} \int_a^t |\nabla_a \dot{γ}|(s) \, ds.$$ 

Proof: We recall the following qualitative version of Gronwall’s inequality. Let $(N, h)$ be a compact Riemannian manifold with induced distance $d^N$, and let X be a $C^1$-vector field on N with flow $Φ : N × R → N$. Then there exist constants $B > 0$, $C > 0$ such that the following holds for all $C^1$-curves $β : [a, b] → N$ and all $t ∈ [a, b]$:

$$d^N(β(t), Φ_t(β(a))) \leq Be^{C(t−a)} \int_a^t |\dot{β}(τ) − X(β(τ))| \, dτ.$$ 

We apply this to the case $N = SM$, $h = \bar{g}$, and to the vector field X on SM whose flow $Φ$ is the geodesic flow on SM. Given an arclength-parametrized $C^2$-curve $γ : [a, b] → M$, we consider $β : [a, b] → SM$, $β(t) = \dot{γ}(t)$. By the definition of the Killing metric $\bar{g}$, we have

$$|\dot{β}(t) − X(β(t))|^g = |\nabla_a \dot{γ}(t)|^g.$$ 

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Hence \[2.1\] implies our claim.

The following statement is a simple consequence of the dependence of solutions of linear ODEs on the equation.

**Lemma 2.2** Suppose \(\gamma_i : [-R, R] \to M\) is a sequence of \(C^1\)-curves that converge to \(\gamma : [-R, R] \to M\) uniformly in the \(C^1\)-topology. Let \(W_i\) be parallel vector fields along \(\gamma_i\) such that \(\lim_{i \to \infty} W_i(0) = w \in T_0 \gamma(0) M\). Then the \(W_i\) converge uniformly to the parallel vector field \(W\) along \(\gamma\) with \(W(0) = w\).

These lemmas will be applied in the following situation. We consider a sequence of complete immersions \(f_i : F_i \to M\) of \(k\)-dimensional manifolds \(F_i\) into \(M\). The second fundamental forms of the \(f_i\) will be denoted by \(A_i\).

**Proposition 2.3** Let \(v_i \in SF_i\) be a sequence such that \(\lim_{i \to \infty} df_i(v_i) = v \in SM\) exists. Assume that \(R > 0\) and \(\lim_{i \to \infty} \int_{-R}^R |A_i| \circ c_{v_i}(t) \, dt = 0\). Then the following statements (a)–(c) are true.

(a) The curves \(f_i \circ c_{v_i}[-R, R]\) converge to the geodesic \(c_v[-R, R]\) uniformly in the \(C^1\)-topology.

(b) If \(W_i : [-R, R] \to TM\) are parallel vector fields along \(c_{v_i}[-R, R]\) and \(\lim_{i \to \infty} df_i(W_i(0)) = w \in TM\) exists, then the vector fields \(df_i \circ W_i\) along \(f_i \circ c_{v_i}[-R, R]\) converge uniformly to the parallel vector field \(W\) along \(c_{v}[-R, R]\) with \(W(0) = w\).

(c) If \(\lim_{i \to \infty} df_i(T_{c_{v_i}(0)} F_i) = \tilde{L} \in G_k M\) exists, then the curves \(L_i : [-R, R] \to G_k M\), \(L_i(t) = df_i(T_{c_{v_i}(t)} F_i),\) converge uniformly to the curve \(L : [-R, R] \to G_k M\), where \(L(t) = \mathcal{P}^e_t(\tilde{L})\) is the parallel translate of \(\tilde{L}\) along \(c_v\).

**Proof:**

(a) By the definition of the second fundamental form \(A_i\), we have

\[
\nabla_{\partial_t} (f_i \circ c_{v_i}) = A_i(c_{v_i}, \dot{c}_{v_i}).
\]

Hence (a) is a consequence of Lemma \[2.1\] and the convergence of the \((M, g)\)-geodesics \(c_{df_i(v_i)}[-R, R]\) to \(c_v[-R, R]\).

(b) Let \(Z_i : [-R, R] \to TM\) denote the parallel vector fields along \(f_i \circ c_{v_i}[-R, R]\) with \(Z_i(0) = df_i(W_i(0))\). Then we have

\[
\langle df_i \circ W_i, Z_i \rangle(t) = \langle df_i \circ W_i, Z_i \rangle(0) + \int_0^t \langle A_i(c_{v_i}, W_i), Z_i \rangle(\tau) \, d\tau.
\]

Hence our assumption \(\lim_{i \to \infty} \int_{-R}^R |A_i| \circ c_{v_i}(t) \, dt = 0\) implies that \(\langle df_i \circ W_i, Z_i \rangle\) converges uniformly on \([-R, R]\) to the constant \(\langle Z_i(0) \rangle = |W_i(0)|^2 = |Z_i(0)|^2\). Since \(|(df_i \circ W_i)(t)| = |W_i(0)| = |Z_i(t)|\), this implies \(\lim_{i \to \infty} |df_i \circ W_i - Z_i| = 0\) uniformly on \([-R, R]\). Using (a), our assumption \(\lim_{i \to \infty} df_i(W_i(0)) = w = W(0)\), and Lemma \[2.2\] we see that the \(Z_i\), and hence \(df_i \circ W_i\), converge uniformly to \(W\).

(c) This is a direct consequence of (b).
Remark 2.4 In the applications of Proposition 2.3 in Sections 7, 9, 10, and 13, the hypothesis \( \lim_{i \to \infty} \int_{-R}^{R} |A_i| |c_{v_i}(t)| dt = 0 \) will hold. By the Cauchy-Schwarz inequality this implies \( \lim_{i \to \infty} \int_{-R}^{R} |A_i| \circ c_{v_i}(t) dt = 0 \).

3 Estimates arising from integral geometry

We consider a \( k \)-dimensional, complete Riemannian manifold \((F,g)\). The unit tangent bundle of \( F \) will be denoted by \( \pi : SF \to F \) with fibers \( S_pF = \pi^{-1}(p) \) for \( p \in F \). Our estimates will follow from the invariance of the Liouville measure \( L \) on \( SF \) under the geodesic flow \( \Phi : SF \times \mathbb{R} \to SF \). We recall that locally the Liouville measure \( L \) is the product of the Riemannian volume \( \text{vol}_k \) on \( F \) and the standard \((k-1)\)-volume \( \text{vol}_{S_pF} \) on the euclidean spheres \( S_pF \). For \( v \in SF \), we let \( c_v : \mathbb{R} \to F \) denote the geodesic with initial vector \( \dot{c}_v(0) = v \), i.e. \( c_v(t) = \pi \circ \Phi(v,t) \). “Measurability” will be understood with respect to the Borel \( \sigma \)-algebra. Lebesgue measure on \( \mathbb{R} \) will be denoted by \( \lambda \).

Lemma 3.1 Let \( h : F \to [0, \infty] \) be measurable. For arbitrary \( \varepsilon > 0 \), \( R > 0 \) consider the set

\[
V_{\varepsilon,R}(h) = \left\{ v \in SF \mid \int_{-R}^{R} h(c_v(t)) \, dt \geq \varepsilon \right\}.
\]

Then we have

\[
L(V_{\varepsilon,R}(h)) \leq \frac{\alpha_{k-1} 2R}{\varepsilon} \int_{F} h \, d\text{vol}_k,
\]

where \( \alpha_{k-1} \) denotes the \((k-1)\)-volume of the unit sphere in euclidean \( k \)-space.

PROOF: The invariance of \( L \) under \( \Phi \) implies that the integral

\[
\int_{SF} h(c_v(t)) \, dL(v)
\]

is independent of \( t \in \mathbb{R} \). From the definition of \( L \) we obtain

\[
\int_{SF} h(c_v(t)) \, dL(v) = \int_{SF} h(c_v(0)) \, dL(v) = \alpha_{k-1} \int_{F} h \, d\text{vol}_k.
\]

This implies

\[
L(V_{\varepsilon,R}(h)) \cdot \varepsilon \leq \int_{SF} \left( \int_{-R}^{R} h(c_v(t)) \, dt \right) dL(v) = \alpha_{k-1} 2R \int_{F} h \, d\text{vol}_k.
\]

Lemma 3.2 If \( B \subseteq F \) is a Borel set and \( R > 0 \), then

\[
(L \times \lambda)(\{(v,t) \in SF \times [-R,R] \mid c_v(t) \in B\}) = \alpha_{k-1} 2R \text{vol}_k(B).
\]
Proof: As in the preceding proof we see that $\mathcal{L}(\{v \in SF|c_v(t) \in B\})$ is independent of $t \in \mathbb{R}$. \hfill \Box

We will apply Lemma 3.1 in the following situation. We will consider an isometric immersion $f : F \to M$ of $F$ into a Riemannian manifold $M$, and let $h$ be the squared norm $|A|^2$ of the second fundamental form of $f$. Under appropriate conditions on $E(f) = \frac{1}{2} \int_F |A|^2 d\text{vol}_k$, we can use Lemma 3.1 to find a large set of vectors $v \in SF\setminus V_{\epsilon,R}(|A|^2)$, i.e. vectors $v \in SF$ for which $\int_{-R}^R |A|^2 \circ c_v(t) \, dt < \epsilon$. If additionally $\sqrt{2R\epsilon}$ is small, then Proposition 2.3 shows that for these vectors $v \in SF$ the curve $f \circ c_v|[-R,R]$ is $C^1$-close to a geodesic in $M$.

In the proof of the existence of totally geodesic surfaces in Section 9 we will need pairs of geodesics $c, \tilde{c}$ in $F$ such that $\tilde{c}(0) = c(t)$ for some $t \in \mathbb{R}$ and such that both $f \circ c$ and $f \circ \tilde{c}$ are $C^1$-close to geodesics in $M$ on appropriate intervals $[-R,R]$ resp. $[-r,r]$. We will encode such pairs of geodesics as follows. We consider $SF^2 = \bigsqcup_{p \in F}(S_pF \times S_pF)$ and the bundle $\hat{\pi} : SF^2 \times \mathbb{R} \to F, \hat{\pi}(v,w,t) = \pi(v)\pi(w)$. $\mathcal{L}^2$ will denote the measure on $SF^2$ that is the product of $\text{vol}_k$ with two factors $vol_{S_pF}$. A tuple $(v,w,t) \in SF^2 \times \mathbb{R}$ encodes the pair of geodesics $c = c_v$ and $\tilde{c}$ with $\tilde{c}(0) = \mathcal{P}^{cv}_t(w)$, where $\mathcal{P}^{cv}_t : S_{c_v}(0)F \to S_{c_v(t)}F$ denotes parallel transport along $c_v$.

Given $\epsilon > 0$, $R > 0$, $r > 0$, we set

$$V^2_{\epsilon,R,r}(h) = \{(v,w,t) \in SF^2 \times [-R,R]|v \in V_{\epsilon,R}(h) \text{ or } \mathcal{P}^{cv}_t(w) \in V_{\epsilon,r}(h)\}.$$  \hfill (3.1)

So $(v,w,t) \in (SF^2 \times [-R,R])\setminus V^2_{\epsilon,R,r}(h)$ implies $\int_{-R}^R |A|^2 \circ c_v(t) \, dt < \epsilon$ and $\int_{-r}^r |A|^2 \circ \tilde{c}_\tilde{w}(t) \, dt < \epsilon$, where $\tilde{w} = \mathcal{P}^{cv}_t(w)$.

Lemma 3.3 \quad $(\mathcal{L}^2 \times \lambda)(V^2_{\epsilon,R,r}(h)) \leq \alpha_k^{-1} \frac{4R}{\epsilon}(R + r) \int_{-R}^R h \, d\text{vol}_k$.

Proof: We will prove that

$$\mathcal{L}^2 \times \lambda\{(v,w,t) \in SF^2 \times [-R,R]|v \in V_{\epsilon,R}(h)\} = \alpha_k^{-1}2R \lambda(V_{\epsilon,R}(h))$$  \hfill (3.2)

and that

$$\mathcal{L}^2 \times \lambda\{(v,w,t) \in SF^2 \times [-R,R]|\mathcal{P}^{cv}_t(w) \in V_{\epsilon,r}(h)\} = \alpha_k^{-1}2R \lambda(V_{\epsilon,r}(h)).$$  \hfill (3.3)

Then our claim will follow by combining (3.2), (3.3) with Lemma 3.1. Equation (3.2) is a direct consequence of Fubini’s theorem. To prove equation (3.3) note that the orthogonality of $\mathcal{P}^{cv}_t : T_{c_v}(p)F \to T_{c_v(t)}F$ and the invariance of $\mathcal{L}$ under the geodesic flow imply that, for every $t \in \mathbb{R}$, we have

$$\mathcal{L}^2\{(v,w) \in SF^2|\mathcal{P}^{cv}_t(w) \in V_{\epsilon,r}(h)\} = \int_{SF} \text{vol}_{S_{c_v(t)}F}(V_{\epsilon,r}(h) \cap S_{c_v(t)}) \, d\mathcal{L}(v) = \int_{SF} \text{vol}_{S_{c_v(t)}F}(V_{\epsilon,r}(h) \cap S_{c_v(t)}F) \, d\mathcal{L}(v) = \alpha_k^{-1} \mathcal{L}(V_{\epsilon,r}(h)).$$

Integrating this equation over $t \in [-R,R]$ we obtain (3.3). \hfill \Box

We will also need the following localized version of Lemma 3.3.

Lemma 3.4 \quad $(\mathcal{L}^2 \times \lambda)((V^2_{\epsilon,R,r}(h) \cap \hat{n}^{-1}(B(p,r))) \leq \alpha_k^{-1} \frac{4R}{\epsilon}(R + r) \int_{|B(p,R + 2r)|} h \, d\text{vol}_k$.
Proof: We consider \( \tilde{h} = h \cdot \chi_{B(p,R+2r)} \), and show that \( \psi^2 \subseteq \psi^1(B(p,r)) \subseteq \psi^2 \). Indeed, if \((v,w,t) \in \psi^2 \cap \psi^1(B(p,r)) \) and \(v \in \psi^1 \), then \(c_v(t) \in B(p,R+r) \) for all \(|t| \leq R \), hence \(h \circ c_v(t) = \tilde{h} \circ c_v(t) \) for \(|t| \leq R \), so that \(v \in \psi^1 \). If \((v,w,t) \in \psi^2 \cap \psi^1(B(p,r)) \) and \(\psi^1(w) \in \psi^1 \), then \(c_v(t) \in B(p,R+r) \) and, as above, we see that \(\psi^1(w) \in \psi^1 \).

We define the measurable function \(H = H_{\psi^1} : F \to [0,\infty)\) as the fibrewise volume of \(\psi^1 \), i.e.

\[
H(p) = (\text{vol}_{S^pF} \times \text{vol}_{S^pF} \times \lambda)(\psi^2 \cap (S^pF \times S^pF \times \mathbb{R})).
\]

Then Lemma 3.3 resp. Lemma 3.4 imply

\[
\int_F H \, d\text{vol}_k \leq \alpha_{k-1}^\frac{4R}{\epsilon} (R + r) \int_F h \, d\text{vol}_k,
\]

and

\[
\int_{B(p,r)} H \, d\text{vol}_k \leq \alpha_{k-1}^\frac{4R}{\epsilon} (R + r) \int_{B(p,R+2r)} h \, d\text{vol}_k.
\]

Definition 3.5 The set \(G_{\psi^1} \) of \((\psi^1)\)-good points” consists of all \(p \in F\) such that \(H(p) < \epsilon\), i.e. \(G_{\psi^1} \subseteq H^{-1}([0,\epsilon))\).

Note that \(p \in G_{\psi^1} \) iff – up to a set of measure smaller than \(\epsilon\) – the tuples \((v,w,t) \in S^pF \times S^pF \times [-R,R] \) encode geodesics \(c = c_v \) and \(\tilde{c} = \psi^1(w) \), such that \(\int_{-R}^{R} h \circ c(t) \, dt < \epsilon\) and \(\int_{-r}^{r} h \circ \tilde{c}(t) \, dt < \epsilon\). In particular, if \(r \leq \tilde{r}\) and \(R \leq \tilde{R}\), then \(G_{\psi^1} \subseteq G_{\psi^1} \). This obvious inclusion will be used at various instances without further notice.

The following proposition is a consequence of inequalities (3.5) and (3.6). The following estimates hold for all \(p \in F\), all \(\epsilon > 0\), \(R > 0\), \(r > 0\), and all measurable functions \(h : F \to [0,\infty]\).

Proposition 3.6 (a) \(\text{vol}_k(F \setminus G_{\psi^1}(h)) \leq \alpha_{k-1}^\frac{4R}{\epsilon^2} (R + r) \int_F h \, d\text{vol}_k\)

(b) \(\text{vol}_k(B(p,r) \setminus G_{\psi^1}(h)) \leq \alpha_{k-1}^\frac{4R}{\epsilon^2} (R + r) \int_{B(p,R+2r)} h \, d\text{vol}_k\).

Proof: Since \(F \setminus G_{\psi^1}(h) \subseteq H^{-1}([\epsilon,\infty))\) we have \(\epsilon \text{vol}_k(F \setminus G_{\psi^1}(h)) \leq \int_F H \, d\text{vol}_k\). Hence (a) is a consequence of (3.5). Similarly (b) follows from (3.6).

Assuming that \(\int_F h \, d\text{vol}_k\) is small we want to use Proposition 3.6 to conclude that \(G_{\psi^1}(h)\) is almost dense in \(F\). For this we need a lower bound on the volume of metric balls. For \(k = 2\) and under appropriate assumptions on \((F,g)\), such a lower bound will be proved in Section 6, see Proposition 6.3.

While Proposition 3.6 suffices for the proof of the existence of (pieces of) totally geodesic surfaces, a slightly different estimate will be useful in the proof of the existence of complete totally geodesic surfaces in Section 10. Here we need not only “good” points \(p\), but points.
such that additionally $c_v(t)$ is “good” for most $(v, t) \in S_p F \times [-R, R]$. In this application it will not be necessary to discriminate between the roles of $R$ and $r$. So we will set $R = r$, and abbreviate

\[(3.7) \quad G_{\varepsilon, R}(h)) = G_{\varepsilon, R}(h).\]

For fixed $\varepsilon > 0$, $R > 0$, and $h$, we define $l : SF \to [0, 2R]$ by

\[l(v) = \lambda\{t \in [-R, R]|c_v(t) \notin G_{\varepsilon, R}(h)\} = 2R - \lambda\{t \in [-R, R]|c_v(t) \in G_{\varepsilon, R}(h)\}.\]

Finally we set

\[(3.8) \quad \tilde{G}_{\varepsilon, R}(h) = \{q \in G_{\varepsilon, R}(h)|\text{vol}_{S_q F}(l^{-1}(\varepsilon, 2R) \cap S_q F) < \varepsilon\}.\]

Note that $q \in G_{\varepsilon, R}(h)$ is in $\tilde{G}_{\varepsilon, R}(h)$ iff – up to a set of vol$_{S_q F}$-measure smaller than $\varepsilon$ – the vectors $v \in S_q F$ satisfy

\[(3.9) \quad \lambda\{t \in [-R, R]|c_v(t) \in G_{\varepsilon, R}(h)\} \geq 2R - \varepsilon.\]

As a consequence of Proposition 3.6 and Lemma 3.2 we obtain:

**Corollary 3.7** vol$_k(B(p, R) \setminus \tilde{G}_{\varepsilon, R}(h)) \leq c_k(R, \varepsilon) \int_{B(p, 5R)} h \text{dvol}_k,$

where $c_k(R, \varepsilon) = \alpha_{k-1}^2 \frac{8R^2}{\varepsilon^3} (1 + \alpha_{k-1} \frac{6R}{\varepsilon^2}).$

**Proof:** We are going to prove that

\[(3.10) \quad \text{vol}_k(B(p, R) \cap (G_{\varepsilon, R} \setminus \tilde{G}_{\varepsilon, R})) \leq \alpha_{k-1}^3 \frac{48R^3}{\varepsilon^4} \int_{B(p, 5R)} h \text{dvol}_k.\]

Combined with Proposition 3.6(b) this estimate implies our claim.

To prove (3.10) we first note that

\[\{(v, t) \in \pi^{-1}(B(p, R)) \times [-R, R]|c_v(t) \notin G_{\varepsilon, R}(h)\} \subseteq \{(v, t) \in SF \times [-R, R]|c_v(t) \in B(p, 2R) \setminus G_{\varepsilon, R}(h)\}.\]

Applying Lemma 3.2 to the right hand side of this inclusion and using the definitions of $l$ and $\tilde{G}_{\varepsilon, R}(h))$, we obtain

\[\alpha_{k-1}^2 2R \text{vol}_k(B(p, 2R) \setminus G_{\varepsilon, R}(h)) \geq (L \times \lambda)(\{(v, t) \in \pi^{-1}(B(p, R)) \times [-R, R]|c_v(t) \notin G_{\varepsilon, R}(h)\}) = \int_{\pi^{-1}(B(p, R))} l(v) \text{dL}(v) \geq \varepsilon^2 \text{vol}_k(B(p, R) \cap (G_{\varepsilon, R}(h) \setminus \tilde{G}_{\varepsilon, R}(h))).\]

Now we apply Proposition 3.6(b) to the first term in this chain of inequalities and obtain (3.10).
4 A lower bound for the total absolute geodesic curvature of simple closed curves on surfaces in euclidean spaces

From now on \( F \) will denote a 2-dimensional manifold, in contrast to the preceding section where \( \dim F = k \) was arbitrary. In this section we consider a complete, connected surface \( F \) immersed into euclidean space \( \mathbb{E}^n \) and a smooth simple closed curve \( \Gamma \) on \( F \). The total absolute geodesic curvature \(|K|(\Gamma)\) of \( \Gamma \) is defined by \(|K|(\Gamma) = \int_{\partial t} |\kappa_g(q)| \, ds(q)\), where \(|\kappa_g(q)|\) denotes the absolute value of the geodesic curvature of \( \Gamma \) at \( q \in \Gamma \). The second fundamental form of the immersion of \( F \) into \( \mathbb{E}^n \) will be denoted by \( A \). The tubular neighborhood \( \Gamma^t \) of \( \Gamma \) of radius \( t > 0 \) is the set of points \( x \in F \) that can be joined to \( \Gamma \) by a curve on \( F \) of length at most \( t \), i.e. \( \Gamma^t = \{ x \in F | d^F(x, \Gamma) \leq t \} \).

**Proposition 4.1** Given \( c \in (0, \frac{1}{3}\pi) \) there exists \( \delta = \delta(c) > 0 \) and \( \beta = \beta(c) \geq 1 \) such that the following holds for every \( n \in \mathbb{N} \), every complete surface immersion \( f : F \to \mathbb{E}^n \), and every smooth, simple closed curve \( \Gamma \) on \( F \) of length \( l \). If \( \int_{\Gamma^t} |A|^2 \, dvol_2 \leq c \), then \(|K|(\Gamma) \geq \delta\).

**Remark 4.2** More precisely, we will prove the following inequality for \( \Gamma \subseteq F \) as above and for all \( t > 0 \):

\[
|K|(\Gamma) \geq \frac{3}{5} \left( \frac{4}{3} \pi - \int_{\Gamma^t} |A|^2 \, dvol_2 \right) - \frac{8l}{5t}.
\]

From \((4.1)\) one easily concludes that Proposition \((4.1)\) holds, e.g. for \( \delta(c) = \frac{1}{5} \left( \frac{4}{3} \pi - c \right) \) and \( \beta(c) = \frac{1}{\frac{4}{3} \pi - c} \). From the proof of \((4.1)\) it is clear that \((4.1)\) is not sharp.

The proof of Proposition \((4.1)\) is based on Fenchel’s inequality for the total curvature of closed curves in \( \mathbb{E}^n \), the Gauß-Bonnet formula, and the Bol-Fiala technique that provides area bounds for \( \Gamma^t \) depending on integral bounds on the Gaussian curvature \( K \). In the smooth case the ideas by G. Bol [2] and Fiala [10] were developed and extended by P. Hartman [12], see also chapter 4 of the book [29] and chapter 2 of the book [6]. We use Hartman’s results to treat the difficulties arising from the non-differentiability of the distance function \( d^F(\cdot, \Gamma) \) from \( \Gamma \).

Since the proof of inequality \((4.1)\) is a combination of several estimates we first give a rough outline. The starting point is Fenchel’s inequality that implies

\[
\int_0^t |K|(\partial \Gamma^\tau) \, d\tau + \int_{\Gamma^t} |A| \, dvol_2 \geq 2\pi t
\]

provided \( \partial \Gamma^t \neq \emptyset \). Now one would like to compare \(|K|(\partial \Gamma^t)\) to \(|K|(\Gamma)\), using the Gauss-Bonnet formula and the fact that \( \int_{\Gamma^t} |K| \, dvol_2 \leq \frac{1}{2} \int_{\Gamma^t} |A|^2 \, dvol_2 \) can be assumed to be small. Here one encounters two problems. The first problem is that the Euler characteristic of \( \Gamma^t \) might be negative. The second problem is that the boundary terms in the Gauß-Bonnet formula involve the geodesic curvature, and not its absolute value. To overcome these problems we treat separately the signed total curvature \( K(\partial \Gamma^t) \) and its positive part \( K^+(\partial \Gamma^t) \). Since, for most \( \tau \in (0, t), \frac{d}{d\tau}(\text{length}(\partial \Gamma^\tau)) = K(\partial \Gamma^\tau) \), Hartman’s results imply

\[
\int_0^t K(\partial \Gamma^\tau) \, d\tau > -2l
\]
see Lemma 4.4. On the other hand, \( \mathcal{K}^+(\partial \Gamma^t) \) can be bounded above by \( |\mathcal{K}|(\Gamma) \) and \( \frac{1}{2} \int_{\Gamma^t} |A|^2 \, d\text{vol}_2 \), see Lemma 4.3. This is a consequence of the Gauß-Bonnet formula and the fact that \( \partial \Gamma^t \) has constant distance from \( \Gamma \). Since \( |\mathcal{K}|(\partial \Gamma^t) = 2\mathcal{K}^+(\partial \Gamma^t) - \mathcal{K}(\partial \Gamma^t) \), the preceding inequalities combine to the inequality

\[
2|\mathcal{K}|(\Gamma)t \geq (2\pi - \int_{\Gamma^t} |A|^2 \, d\text{vol}_2)t - \int_{\Gamma^t} |A| \, d\text{vol}_2 - 2l.
\]

Finally, we use \( \int_{\Gamma^t} |A| \, d\text{vol}_2 \leq (\int_{\Gamma^t} |A|^2 \, d\text{vol}_2)^{\frac{1}{2}} \vol_2(\Gamma^t)^{\frac{1}{2}} \), and the Bol-Fiala type estimate

\[
\vol_2(\Gamma^t) \leq 2lt + \left( \frac{1}{2} |\mathcal{K}|(\Gamma) + \frac{1}{4} \int_{\Gamma^t} |A|^2 \, d\text{vol}_2 \right) t^2,
\]

see (4.9), to obtain (4.1).

Now we give the details of the proof of (4.1). Let \( \bar{R} = \sup_{x \in F} d^F(x, \Gamma) \in (0, \infty) \). Then \( \partial \Gamma^t \neq \emptyset \) for \( t \in (0, \bar{R}) \). P. Hartman [12] introduced the set \( \mathcal{N}\mathcal{E} = \mathcal{N}\mathcal{E}(\Gamma) \subseteq (0, \bar{R}) \) of non-exceptional values of the distance function from \( \Gamma \), and proved that \( \mathcal{N}\mathcal{E} \) is open and of full measure in \( (0, \bar{R}) \). Note that if \( t \in \mathcal{N}\mathcal{E} \) then \( \partial \Gamma^t \) is free of focal points of \( \Gamma \), and for every \( q \in \partial \Gamma^t \) there exist at most two geodesics of length \( t \) joining \( q \) to \( \Gamma \). Moreover, if there are two such geodesics they intersect at \( q \) at an angle smaller than \( \pi \). This implies that \( \partial \Gamma^t \) is a piecewise smooth submanifold of \( F \), and that the set \( Q^t \subseteq \partial \Gamma^t \) of points in the neighborhood of which \( \partial \Gamma^t \) is not smooth, is finite. Moreover, \( \Gamma^t \) has a concave angle at each \( q \in Q^t \). For \( t \in \mathcal{N}\mathcal{E} \) we let \( \kappa^t_q : \partial \Gamma^t \setminus Q^t \to \mathbb{R} \) denote the geodesic curvature of \( \partial \Gamma^t \setminus Q^t \) with respect to the normal pointing out of \( \Gamma^t \), and \( (\kappa^t_q)^{+} \) its positive part.

**Lemma 4.3** If \( t \in \mathcal{N}\mathcal{E} \) then \( \int_{\partial \Gamma^t \setminus Q^t} (\kappa^t_q)^{+} \, ds \leq |\mathcal{K}|(\Gamma) + \int_{\Gamma^t} K^- \, d\text{vol}_2 \).

**Proof:** Let \( J \subseteq \partial \Gamma^t \setminus Q^t \) be a compact interval on which \( \kappa^t_q \) is positive. The shortest connections to \( \Gamma \) of the end-points of \( J \), together with \( J \) and the nearest point projection \( \text{pr}(J) \subseteq \Gamma \) of \( J \) to \( \Gamma \) bound a rectangle \( R_J \) in \( \Gamma^t \). We choose an arclength-parametrization \( \gamma \) of \( \Gamma \), and we let \( n_J \) denote the unit normal along \( \text{pr}(J) \) pointing into \( R_J \). Then the Gauß-Bonnet formula yields

\[
\int_J (\kappa^t_q)^+(s) \, ds = \int_{\text{pr}(J)} \langle \nabla_\gamma \gamma, n_J \rangle \, ds - \int_{R_J} K \, d\text{vol}_2.
\]

If \( J, J' \) are two such intervals and \( J \cap J' = \emptyset \), then \( \vol_2(R_J \cap R_{J'}) = 0 \) and \( n_J |\text{pr}(J) \cap \text{pr}(J') = -n_{J'} |\text{pr}(J) \cap \text{pr}(J') \). Hence we have

\[
\int_{\text{pr}(J) \cap \text{pr}(J')} \langle \nabla_\gamma \gamma, n_J \rangle \, ds = -\int_{\text{pr}(J) \cap \text{pr}(J')} \langle \nabla_\gamma \gamma, n_{J'} \rangle \, ds.
\]

So, if we let \( B \subseteq \Gamma \) denote the set of points \( x \in \Gamma \) such that there exists precisely one \( q \in \partial \Gamma^t \setminus Q^t \) with \( \kappa^t_q(x) > 0 \) and \( \text{pr}(q) = x \), then (4.2) and (4.3) imply

\[
\int_{\partial \Gamma^t \setminus Q^t} (\kappa^t_q)^+ \, ds \leq \int_B |\nabla_\gamma \gamma| \, ds + \int_{\Gamma^t} K^- \, d\text{vol}_2 \leq |\mathcal{K}|(\Gamma) + \int_{\Gamma^t} K^- \, d\text{vol}_2.
\]
If \( t \in \mathcal{NE}(\Gamma) = \mathcal{NE} \) and \( q \in Q^t \subseteq \partial \Gamma^t \), we let \( \theta_q \in (0, \pi) \) denote the angle at \( q \) between the two shortest connections from \( q \) to \( \Gamma \). Then \( \pi + \theta_q \in (\pi, 2\pi) \) is the inner angle of \( \Gamma^t \) at \( q \). The total curvature \( K(t) \) of \( \partial \Gamma^t \) as the boundary of \( \Gamma^t \) is defined by

\[
K(t) = \int_{\partial \Gamma^t \setminus Q^t} \kappa_g^t \, ds - \sum_{q \in Q^t} \theta_q,
\]

Lemma 4.4 If \( t \in \mathcal{NE} \subseteq (0, \bar{R}) \), then \( \int_0^t K(\tau) \, d\tau > -2l \), where \( l = \text{length}(\Gamma) \).

Note. Since \( \mathcal{NE} \) has full measure in \((0, \bar{R})\), \( K(\tau) \) is defined for almost all \( \tau \in (0, \bar{R}) \).

Proof: For \( \tau \in \mathcal{NE} \) we let \( l(\tau) \) denote the length of \( \partial \Gamma^\tau \). Note that \( \lim_{\tau \downarrow 0} l(\tau) = 2l \). A standard calculation shows that \( l|\mathcal{NE} \) is smooth and that

\[
l'(\tau) = \int_{\partial \Gamma^\tau \setminus Q^\tau} \kappa_g^\tau \, ds - \sum_{q \in Q^\tau} 2 \tan \frac{\theta_q}{2},
\]

see e.g. [29], Theorem 4.1. Applying Hartman’s Corollary 6.1 in [12] we obtain

\[
l(t) \leq 2l + \int_0^t l'(\tau) \, d\tau \quad \text{for all } t \in \mathcal{NE},
\]

Since \( 2 \tan \frac{x}{2} \geq x \) for \( x \geq 0 \), we conclude from (4.4) that \( \tau' \leq K(\tau) \) for \( \tau \in \mathcal{NE} \). Hence (4.5) implies that

\[
l(t) \leq 2l + \int_0^t K(\tau) \, d\tau \quad \text{for } t \in \mathcal{NE}.
\]

This proves our claim since \( l(t) > 0 \) for \( t \in \mathcal{NE} \).

Lemma 4.5 If \( t \in (0, \infty) \) and \( \int_{\Gamma^t} |A|^2 \, d\text{vol}_2 < 8\pi \), then \( t < \bar{R} \) and \( \partial \Gamma^t \neq \emptyset \).

Proof: If \( t \geq \bar{R} \), then \( \Gamma^t = \Gamma^{\bar{R}} = F \), since \( \bar{R} = \sup_{x \in F} d^F(x, \Gamma) \). In particular, \( F \) is compact. Then the well-known results by Chern-Lashoff [7] imply that

\[
\int_F |A|^2 \, d\text{vol}_2 \geq 8\pi.
\]

Proof of inequality (4.1): Since (4.1) is trivially true if \( \int_{\Gamma^t} |A|^2 \, d\text{vol}_2 \geq 8\pi \), we may assume that \( \int_{\Gamma^t} |A|^2 \, d\text{vol}_2 < 8\pi \), so that \( t < \bar{R} \) by Lemma 4.5. For \( \tau \in \mathcal{NE} \cap (0, t) \) we use Lemma 4.3 to estimate

\[
\int_{\partial \Gamma^\tau \setminus Q^\tau} |\kappa_g^\tau| \, ds = \int_{\partial \Gamma^\tau \setminus Q^\tau} (2(\kappa_g^\tau)^+ - \kappa_g^\tau) \, ds \leq 2|K|_t(\Gamma) + 2 \int_{\Gamma^t} K^- \, d\text{vol}_2 - \int_{\partial \Gamma^\tau \setminus Q^\tau} \kappa_g^\tau \, ds.
\]

For \( q \in \partial \Gamma^\tau \setminus Q^\tau \) we let \( \kappa^\tau(q) \geq 0 \) denote the curvature at \( q \) of the curve \( f|\partial \Gamma^\tau \setminus Q^\tau \) in \( \mathbb{E}^n \). Note that \( \kappa^\tau(q) \leq |\kappa_g^\tau(q)| + |A(q)| \). Hence Fenchel’s inequality yields

\[
2\pi n(\tau) \leq \int_{\partial \Gamma^\tau \setminus Q^\tau} (|\kappa_g^\tau| + |A|) \, ds + \sum_{q \in Q^\tau} \theta_q,
\]
where \( n(\tau) \geq 1 \) denotes the number of connected components of \( \partial \Gamma^\tau \). Combining inequalities \((4.6), (4.7)\) and \( n(\tau) \geq 1 \) we obtain

\[
2\pi \leq -K(\tau) + 2|\mathcal{K}|(\Gamma) + 2 \int_{\Gamma^\tau} K^- d\text{vol}_2 + \int_{\partial \Gamma^\tau} |A| ds.
\]

Integrating this inequality over \( N \mathcal{E} \cap (0, t) \), and using Lemma 4.4 and \( K^- \leq \frac{1}{2} |A|^2 \), and the Cauchy-Schwarz inequality on \( \int_{\Gamma^\tau} |A| d\text{vol}_2 \), we obtain

\[
(4.8) \quad 2\pi t \leq 2l + (2|\mathcal{K}|(\Gamma) + \int_{\Gamma^\tau} |A|^2 d\text{vol}_2) t + \left(\int_{\Gamma^\tau} |A|^2 d\text{vol}_2\right)^{\frac{1}{2}} \cdot \text{vol}_2(\Gamma^\tau)^{\frac{1}{2}}
\]

To estimate \( \text{vol}_2(\Gamma^t)^{\frac{1}{2}} \) we recall that, by \((4.4)\) and Lemma 4.3,

\[
l'(\tau) \leq |\mathcal{K}|(\Gamma) + \int_{\Gamma^\tau} K^- d\text{vol}_2 \leq |\mathcal{K}|(\Gamma) + \frac{1}{2} \int_{\Gamma^\tau} |A|^2 d\text{vol}_2.
\]

Combined with \((4.5)\) this implies

\[
(4.9) \quad \text{vol}_2(\Gamma^t) = \int_0^t l(\tau) d\tau \leq 2lt + \frac{1}{2} \left( |\mathcal{K}|(\Gamma) + \frac{1}{2} \int_{\Gamma^\tau} |A|^2 d\text{vol}_2 \right) t^2
\]

and, hence,

\[
\text{vol}_2(\Gamma^t)^{\frac{1}{2}} \leq \left( \frac{1}{2} |\mathcal{K}|(\Gamma) + \frac{1}{4} \int_{\Gamma^\tau} |A|^2 d\text{vol}_2 \right)^{\frac{1}{2}} + t \left( \frac{1}{2} |\mathcal{K}|(\Gamma) + \frac{1}{4} \int_{\Gamma^\tau} |A|^2 d\text{vol}_2 \right)^{-\frac{1}{2}}.
\]

Inserting this into \((4.8)\) and abbreviating \( \frac{1}{2} |\mathcal{K}|(\Gamma) = k, \frac{1}{4} \int_{\Gamma^\tau} |A|^2 d\text{vol}_2 = a \), we obtain

\[
2\pi t \leq 2\left( 1 + \left( \frac{a}{k + a} \right)^{\frac{1}{2}} \right) l + (4(k + a) + 2(a^2 + ka)^{\frac{1}{2}}) t \leq 4l + (5k + 6a)t.
\]

This last inequality is equivalent to \((4.1)\). \(\square\)

## 5 Contractibility of intrinsic metric balls on surfaces in euclidean spaces

Using Proposition 4.1 we will prove the following result that is the key to obtain a lower bound on the area of balls on surfaces in a compact Riemannian manifold, see Section 6.

**Proposition 5.1** Given \( c \in (0, \frac{4}{3}\pi) \) there exists \( \alpha = \alpha(c) \in (0, 1) \) such that the following holds for every \( n \in \mathbb{N} \), every complete surface immersion \( f : F \to \mathbb{E}^n \), every \( p \in F \) and every \( R > 0 \). If \( \int_{B(p, R)} |A|^2 d\text{vol}_2 \leq c \), then the intrinsic metric ball \( B(p, \alpha R) \subseteq F \) is simply connected.

**Remark 5.2** We do not know the supremum \( c_0 \) of all constants \( c \) for which the statement in Proposition 5.1 holds. By Proposition 5.1 we have \( c_0 \geq \frac{4}{3}\pi \), while the example of the catenoid shows that \( c_0 \leq 8\pi \).
Remark 5.3 Our proof provides an explicit value for $\alpha(c)$, namely

$$\alpha(c) = \left( \frac{2}{1 - \cos \delta(c)} + 2\beta(c) + 1 \right)^{-1},$$

where $\delta(c) = \frac{1}{5} \left( \frac{3}{2} \pi - c \right)$ and $\beta(c) = \frac{4}{5\pi - c}$.

The proof of Proposition 5.1 depends crucially on Proposition 4.1. We will assume that $B(p, r)$ is not simply connected. Then we will prove in a series of lemmas that in some larger ball $B(p, R)$ we can find a simple geodesic loop $\gamma$ that has an angle smaller than $\delta(c)$ at its base point, i.e. $|K|(/) < \delta(c)$. This will contradict Proposition 4.1. Here, the main idea is that the angle of a geodesic loop determines the speed at which it can be shortened by letting its base point vary. This is a purely intrinsic argument that does not depend on the assumption that $\dim F = 2$. In the final step, however, we use the Gauß-Bonnet formula to see that $B(p, r)$ is not contractible in $B(p, R)$. Here and in the sequel a geodesic loop in a Riemannian manifold $M$ means a geodesic $\gamma : [0, L] \to M$ such that $\gamma(0) = \gamma(L)$, and $\gamma(0)$ is called the base point of the loop. The loop will be called simple if $\gamma|[0,L]$ is injective.

Lemma 5.4 Let $\gamma : [0, l] \to M$ be a geodesic loop in a Riemannian manifold $M$ that is shortest among all non-contractible loops based at $\gamma(0)$. Then $\gamma$ is simple.

Proof: Otherwise there exist $0 \leq s_1 < s_2 < l$ such that $\gamma(s_1) = \gamma(s_2)$. Reversing the orientation of $\gamma$, if necessary, we may assume that $s_1 < l - s_2$. Note that the loop $\gamma|[s_1, s_2]$ is non-contractible since otherwise the loop $(\gamma|[0, s_1]) \ast (\gamma|[s_2, l])$ were non-contractible, based at $\gamma(0)$, and shorter than $\gamma$. Hence the loop $\tilde{\gamma} = (\gamma|[0, s_2]) \ast (\gamma|[0, s_1])^{-1}$ is non-contractible, based at $\gamma(0)$, and has length $s_2 + s_1 \leq l$. In particular, $\tilde{\gamma}$ is geodesic, in contradiction to the fact that $\tilde{\gamma}(s_2) \neq -\tilde{\gamma}(s_1)$.

Lemma 5.5 Let $B(p, r)$ be a metric ball in a complete Riemannian manifold, and suppose $B(p, r)$ is not simply connected. Then, among the arclength-parametrized loops in $B(p, r)$ with base point $p$ that are non-contractible in $B(p, r)$, there exists a shortest one, and every such shortest loop is a simple geodesic loop of length smaller than $2r$.

Proof: The existence of a geodesic loop $\gamma : [0, l] \to B(p, r)$ that is shortest among all non-contractible loops in $B(p, r)$ based at $p$, and of length $l < 2r$, is proved in [27], Lemma 3.1. Lemma 5.3 applied to $M = B(p, r)$, shows that every such loop is simple.

Lemma 5.6 Suppose $\delta \in (0, \pi)$, $r > 0$, and $R \geq \left( \frac{2}{1 - \cos \delta} + 1 \right) r$. Let $B(p, R)$ be a metric ball of radius $R$ in a complete Riemannian manifold, and assume that there exists a loop $\gamma_0$ based at $p$ of length smaller than $2r$ that is not contractible in $B(p, R)$. Then there exists a simple geodesic loop $\gamma : [0, l] \to B(p, R)$ of length $l < 2r$ such that the angle $\angle(\hat{\gamma}(0), \hat{\gamma}(l))$ that $\gamma$ makes at $\gamma(0) = \gamma(l)$ is smaller than $\delta$.

Proof: For $\rho \in [0, R - r)$ we let $l(\rho)$ denote the infimum of the lengths of loops in $B(p, R)$ with base-points in $B(p, \rho)$ that are not contractible in $B(p, R)$. Note that $l(\rho)$ is non-increasing and 2-Lipschitz. Since $l(0) < 2r$ by assumption, the infimum $l(\rho)$ is achieved by a geodesic loop $\gamma_\rho : [0, l(\rho)] \to B(p, R + \rho) \subseteq B(p, R)$. Lemma 5.3 implies that $\gamma_\rho$ is simple. Whenever $l'(\rho)$ exists, the first variation formula implies that $l'(\rho) \leq -1 + \cos \delta_\rho$. 
where $\delta_\rho = \angle(\gamma_\rho(0), \gamma_\rho(l(\rho)))$. If, contrary to our claim, we had $\delta_\rho \geq \delta$ for all $\rho \in [0, R - r]$, we would have $0 \leq l(R - r) < 2r - (1 - \cos \delta)(R - r)$, in contradiction to our assumption $R \geq \left(\frac{2}{1 - \cos \delta} + 1\right) r$.

**Proof of Proposition 5.1** Contrary to the claims of Proposition 5.1 and Remark 5.3, we assume that there exist $c \in (0, \frac{\pi}{2})$, a complete surface immersion $f : F \to \mathbb{E}^n$, $p \in F$ and $R > 0$, such that $\int_{B(p,R)} |K|^2 \, d\text{vol}_2 \leq c$, while $B(p, R)$ is not simply connected where $\alpha = \left(\frac{2}{1 - \cos(c)} + 2\beta(c) + 1\right)^{-1}$. We set $r = \alpha R$ and apply Lemma 5.5 to obtain a simple geodesic loop $\gamma_0 : [0, l_0] \to B(p, r)$ of length $l_0 < 2r$ such that $\angle(\gamma(0), \gamma(l_0)) > \pi$, in contradiction to $\int_D K \, d\text{vol}_2 \leq \frac{1}{\alpha} \int_{B(p,R)} |K|^2 \, d\text{vol}_2 \leq \frac{1}{\alpha} c < \pi$. Now we can apply Lemma 5.6 to $\delta(c)$, $\rho$ and $\tilde{R} = \left(\frac{2}{1 - \cos(c)} + 1\right) r < R$ to obtain a simple geodesic loop $\gamma : [0, \tilde{l}] \to B(p, \tilde{R})$ of length $\tilde{l} < 2r$ such that $\angle(\gamma(0), \gamma(\tilde{l})) < \delta(c)$. Smoothing the possible corner of $\gamma$ at $\gamma(0) = \gamma(\tilde{l})$, we obtain a smooth, simple closed curve $\Gamma \subseteq B(p, \tilde{R})$ of total absolute geodesic curvature $|\mathcal{K}|(\Gamma) < \delta(c)$ and of length $\tilde{l} < 2r$. Note that, by the choice of $\rho = \alpha R$ and $\tilde{R}$, the tubular neighborhood $\Gamma^{\beta(c)}$ of $\Gamma$ is contained in $B(p, R)$ so that the assumption $\int_{\Gamma^{\beta(c)}} |K|^2 \, d\text{vol}_2 \leq c$ in Proposition 4.1 is satisfied. Hence the inequality $|\mathcal{K}|(\Gamma) < \delta(c)$ contradicts Proposition 4.1.

\section{Area of intrinsic metric balls on surfaces in Riemannian manifolds}

We consider a complete surface immersion $f : F \to M$ into a Riemannian manifold $M$. We assume that the sectional curvature of $M$ is bounded below by some negative number $k < 0$. We will derive upper and lower estimates for the area of metric balls on $F$, that depend on assumptions on the energy. The upper estimate is standard, and has its origin in an idea that goes back to work by G. Bol \cite{Bol} and by F. Fiala \cite{Fiala}.

**Proposition 6.1** Let $F$ be a complete Riemannian surface, and let $K$ denote its Gaussian curvature. Then the following holds for every number $k < 0$, for every $p \in F$ and every $r > 0$:

$$
\text{vol}_2(B(p, r)) \leq \frac{1}{-k} \left(2\pi + \int_{B(p,r)} (K - k) \, d\text{vol}_2\right) \left(\cosh(\sqrt{-k} r) - 1\right).
$$

**Proof:** This follows easily from case 1) of Theorem 2.4.2 in \cite{Cheeger}. Note also that Corollary 11.5 will generalize Proposition 6.1.

**Corollary 6.2** Let $f : F \to M$ be a complete surface immersion into a Riemannian manifold $M$ with sectional curvature bounded below by $k < 0$. Then the following holds for every $p \in F$ and every $r > 0$:

$$
\text{vol}_2(B(p, r)) \leq \frac{1}{-k} \left(2\pi + E(f|B(p,r))\right) \left(\cosh(\sqrt{-k} r) - 1\right).
$$
Proof: By the Gauß equations we have $K \geq k - \frac{1}{2}|A|^2$, cf. (1.2). Hence we have $(K - k) \leq \frac{1}{2}|A|^2$ and $\int_{B(p,r)}(K - k)^-\,d\text{vol}_2 \leq E(f|B(p,r))$. So our claim is reduced to Proposition 6.1.

Our lower area estimate depends on the result of Section 5.

**Proposition 6.3** Let $M$ be a compact Riemannian manifold. Then there exist $r_0 > 0$ and $\beta \geq 1$ such that the following holds for every complete surface immersion $f : F \to M$ and every $p \in F$. If $r \in (0, r_0)$ and $E(f|B(p, \beta r)) \leq \frac{r}{4}$, then $B(p, r)$ is simply connected, and $\text{vol}_2(B(p, r)) \geq \frac{\pi}{2}r^2$.

Proof: Let $i : M \to \mathbb{E}^n$ be an isometric immersion of $M$ into some euclidean space, and let $C > 0$ be an upper bound for the norm of the second fundamental form $A$ of $i$. Then the second fundamental form $A$ of $\tilde{f} = i \circ f$ can be orthogonally decomposed as $A = A + f^*\tilde{A}$, where $\tilde{A}$ denotes the second fundamental form of $f$. This implies $|\tilde{A}|^2 \leq |A|^2 + C^2$. Hence, using Corollary 6.2 we obtain

$$E(\tilde{f}|B(p,r)) \leq E(f|B(p,r)) + \frac{C^2}{2k}(2\pi + E(f|B(p,r)))(\cosh(\sqrt{-k}r) - 1),$$

where $k < 0$ denotes a lower bound for the sectional curvature of $M$. According to the Gauß equations we can take $k = -C^2$, so that the preceding inequality reduces to

$$E(\tilde{f}|B(p,r)) \leq \pi(\cosh(Cr) - 1) + \frac{1}{2}E(f|B(p,r))(\cosh(Cr) + 1).$$

Hence we can find $r_1 > 0$ so that $r \in (0, r_1)$ and $E(f|B(p,r)) \leq \frac{r}{4}$ imply

$$E(f|B(p,r)) \leq \frac{\pi}{2};$$

namely $r_1 = \frac{1}{r}\text{arcosh}(\frac{11}{8})$. Now we can apply Proposition 5.1 to find $\alpha = \alpha(\pi)$ so that $B(p, \alpha r)$ is simply connected whenever $r \in (0, r_1)$ and $E(f|B(p,r)) \leq \frac{r}{4}$. We set $\beta = \alpha^{-1}$ and $r_0 = \alpha r_1$, and conclude that $B(p, r)$ is simply connected whenever $r \in (0, r_0)$ and $E(f|B(p, \beta r)) \leq \frac{r}{4}$. This proves our first claim. For these $r$ we can apply Proposition 3.2(3) in [27], and obtain

$$\text{vol}_2(B(p,r)) \geq (\pi - \int_{B(p,r)} K^+\,d\text{vol}_2)r^2.$$ 

If $r \in (0, r_0)$ and $E(f|B(p, \beta r)) \leq \frac{r}{4}$, we use $K^+ \leq \frac{1}{2}|\tilde{A}|^2$ and (6.1) to see that $\int_{B(p,r)} K^+\,d\text{vol}_2 \leq E(\tilde{f}|B(p,r)) \leq \frac{r}{2}$. Hence (6.2) implies $\text{vol}_2(B(p,r)) \geq \frac{\pi}{2}r^2$. □

7. Gromov-Hausdorff convergence

We consider a sequence of complete surface immersions $f_i : F_i \to M$ into a compact Riemannian manifold $(M, g)$. We assume that the $F_i$ and $M$ are connected, and let $d_i$ resp. $d^M$ denote the distance functions induced by $f_i^*g$ resp. by $g$. Under appropriate assumptions on the energy of the $f_i$ we will study pointed Gromov-Hausdorff convergence, abbreviated as GH-convergence, of the metric spaces $(F_i, d_i)$ to a limit metric space. We
will rely on the concept of GH-convergence as explained by M. Gromov in [11], Section 6. Except for some technical details much of the material in this section is a special case of [27], Section 3. Here, we present complete details since, on the one hand, the results from Section 6 make the proofs in our special case considerably simpler than the ones given in [27], while, on the other hand, the results in [27] do not apply directly in our situation.

We start by recalling some notions from metric geometry. A metric space \( X \) is **proper** if the compact subsets of \( X \) are precisely the closed and bounded subsets of \( X \). \( X \) is a **length space** if any two points \( x, y \in X \) can be joined by a curve in \( X \) of length \( d(x, y) \). A sequence of compact metric spaces \( X_i \) is **uniformly compact** if the diameters of the \( X_i \) are uniformly bounded, and if, for every \( \varepsilon > 0 \), there exists \( N \in \mathbb{N} \) such that each \( X_i \) has an \( \varepsilon \)-dense subset with at most \( N \) elements. We recall Gromov’s compactness criterion [11], p. 64:

Let \((X_i, x_i)_{i \in \mathbb{N}}\) be a sequence of proper, pointed metric spaces. If for each \( r > 0 \) the sequence of balls \((B(x_i, r))_{i \in \mathbb{N}}\) is uniformly compact, then a subsequence of the sequence \((X_i, x_i)_{i \in \mathbb{N}}\) GH-converges to a proper, pointed metric space.

Finally, we note the following well-known fact, see e.g. [5], Theorem 7.5.1:

**Remark 7.1** If the sequence \((X_i, x_i)_{i \in \mathbb{N}}\) GH-converges to a complete, pointed metric space \( Y \), and if all the \( X_i \) are length spaces, then so is \( Y \).

If \( X \) is a metric space and \( \varepsilon > 0 \), we define \( \alpha_X(\varepsilon) \in \mathbb{N} \cup \{\infty\} \) by

\[
\alpha_X(\varepsilon) = \text{minimal number of elements of an } \varepsilon\text{-dense subset of } X.
\]

If \( A \) is a subset of \( X \), we consider \( A \) with the metric induced from \( X \). This defines \( \alpha_A(\varepsilon) \) for subsets \( A \) of \( X \).

The following lemma relies on the area estimates obtained in Section 6. In particular, the statement involves the constants \( r_0 > 0 \) and \( \beta \geq 1 \) from Proposition 6.3. As in Section 6, we let \( k < 0 \) denote a lower bound for the sectional curvature of \( M \).

**Lemma 7.2** Let \( f : F \to M \) be a complete surface immersion, and suppose \( p \in F \) and \( \tilde{R} > 0 \) are such that \( E(f|B(p, \tilde{R})) \leq \frac{\pi}{4} \). If \( 0 < R < \tilde{R} \) and \( 0 < \varepsilon < 2\min\{r_0, \frac{\tilde{R}-R}{\beta}\} \), then

\[
\alpha_{B(p, \tilde{R})}(\varepsilon) \leq \frac{18}{-k} \left( \cosh(\sqrt{-k} \tilde{R}) - 1 \right) \varepsilon^{-2}.
\]

**Proof:** From Corollary 6.2 we know that

\[
\text{vol}_2(B(p, \tilde{R})) \leq \frac{9\pi}{-4k} \left( \cosh(\sqrt{-k} \tilde{R}) - 1 \right).
\]

If \( D \subseteq B(p, \tilde{R}) \) is \( \varepsilon \)-dense in \( B(p, \tilde{R}) \) and \( \#D = \alpha_{B(p, \tilde{R})}(\varepsilon) \), then \( B(x, \frac{\varepsilon}{2}) \cap B(y, \frac{\varepsilon}{2}) = \emptyset \) for \( x \neq y \) in \( D \). Since \( \frac{\varepsilon}{2} < \tilde{R} - R \) we have \( B(x, \frac{\varepsilon}{2}) \subseteq B(p, \tilde{R}) \) for all \( x \in D \). This implies

\[
\text{vol}_2(B(p, \tilde{R})) \geq \sum_{x \in D} \text{vol}_2(B(x, \frac{\varepsilon}{2})).
\]
Finally, we have \( \frac{\varepsilon}{2} < r_0 \) and \( E(f|B(x, \beta_2^x)) \leq \frac{\varepsilon}{2} \) for \( x \in D \), since \( B(x, \beta_2^x) \subseteq B(p, R + \beta_2^x) \subseteq B(p, R) \). So we can apply Proposition 6.3 to obtain
\[
\text{vol}_2(B(x, \frac{\varepsilon}{2})) \geq \frac{\pi}{8} \varepsilon^2 \quad \text{if} \ x \in D.
\]

Now inequalities (7.2)–(7.4) imply our claim. \( \square \)

Using Gromov’s compactness criterion and Lemma 7.2 we obtain:

**Proposition 7.3**  (a) Suppose \( 0 < R < \tilde{R} \) and \( p_i \in F_i \) is a sequence such that \( \liminf_{i \to \infty} E(f_i|B(p_i, \tilde{R})) < \frac{\varepsilon}{4} \). Then a subsequence of the sequence \( (B(p_i, \tilde{R}), p_i)_{i \in \mathbb{N}} \) GH-converges to a connected, compact, pointed metric space \( (Y_R, y_0) \) of finite 2-dimensional Hausdorff measure.

(b) Suppose \( p_i \in F_i \) is a sequence such that \( \liminf_{i \to \infty} E(f_i|B(p_i, R)) < \frac{\varepsilon}{4} \) holds for all \( R > 0 \). Then a subsequence of the sequence \( (F_i, p_i)_{i \in \mathbb{N}} \) GH-converges to a proper, pointed length space \( (Y, y_0) \) of locally finite 2-dimensional Hausdorff measure.

**Proof:** Lemma 7.2 shows that the sequence \( (B(p_i, \tilde{R}))_{i \in \mathbb{N}} \) is uniformly compact, in case (a) for \( R < \tilde{R} \), and in case (b) for all \( R > 0 \). So the statements concerning GH-convergence follow from Gromov’s compactness criterion. The estimate \( \alpha_{B(p_i, R)}(\varepsilon) \leq c(k, R)\varepsilon^{-2} \) from Lemma 7.2 implies a similar estimate for the limit spaces \( Y_R \) resp. \( Y \). This proves the statements about the 2-dimensional Hausdorff measure, see Section 2.3 and the proof of Theorem 3.1 in [27]. Remark 7.1 shows that \( Y \) is a length space. \( \square \)

**Remark 7.4** In [27] T. Shioya studies the topological structure of GH-limit spaces of sequences of compact Riemannian surfaces with bounds on the total absolute curvature and the diameter. These results do not apply directly in the situation of Proposition 7.3 since the assumptions are not satisfied. In Section 13 we will prove that, under the conditions (13.1)–(13.3), the limit space \( Y \) in Proposition 7.3(b) admits a locally isometric map onto a complete, totally geodesic surface in \( M \), see Proposition 13.3.

To define a notion of convergence for the sequence of maps \( f_i \) we resort to **definite GH-convergence**, as defined in [11], p.66. If \( (X_i, x_i)_{i \in \mathbb{N}} \) GH-converges to \( (Y, y_0) \) we can choose and will fix a sequence of metrics \( \delta_i \) on the disjoint unions \( X_i \cup Y \) with the following properties:

(a) The inclusions \( (X_i, d_i) \to (X_i \cup Y, \delta_i) \) and \( (Y, d_Y) \to (X_i \cup Y, \delta_i) \) are isometric.

(b) For every \( r > 0 \) and every \( \varepsilon > 0 \) the following holds for almost all \( i \in \mathbb{N} \):
\[
\delta_i(x_i, y_0) < \varepsilon, \quad \text{and} \quad B(x_i, r) \text{ is contained in the } \varepsilon\text{-neighborhood of } Y, \quad \text{and} \quad B(y_0, r) \text{ is contained in the } \varepsilon\text{-neighborhood of } X_i \text{ (both neighborhoods with respect to } \delta_i). \]

**Definition 7.5** Suppose \( (X_i, x_i)_{i \in \mathbb{N}} \to (Y, y_0) \) with respect to **definite GH-convergence**. Then we define:

(a) A sequence \( x_i \in X_i \) converges to \( y \in Y \), i.e. \( \lim_{i \to \infty} x_i = y \), if \( \lim_{i \to \infty} \delta_i(x_i, y) = 0 \).

(b) Let \( f_i : X_i \to Z \) be a sequence of maps into a metric space \( Z \). Then the \( f_i \) converge to a map \( f : Y \to Z \) if \( \lim_{i \to \infty} f_i(x_i) = f(y) \), whenever the sequence \( x_i \in X_i \) converges to \( y \in Y \).
Slightly generalizing the usual proof of the Arzelà-Ascoli theorem, we obtain:

**Proposition 7.6** (a) Suppose the sequence \((B(p_i, \bar{R}), p_i)\) converges to \((Y_R, y_0)\) with respect to definite GH-convergence. Then a subsequence of the sequence \((f_i|B(p_i, \bar{R}))_{i \in \mathbb{N}}\) converges to a distance-nonincreasing map \(f : Y_R \to M\).
(b) Suppose the sequence \((F_i, p_i)_{i \in \mathbb{N}}\) converges to \((Y, y_0)\) with respect to definite GH-convergence. Then a subsequence of the sequence \((f_i)_{i \in \mathbb{N}}\) converges to a distance-nonincreasing map \(f : Y \to M\).

Similarly, the Arzelà-Ascoli theorem implies:

**Proposition 7.7** Suppose the sequence \((B(p_i, \bar{R}), p_i)_{i \in \mathbb{N}}\) converges to \((Y_R, y_0)\) and \(\lim_{i \to \infty} (f_i|B(p_i, \bar{R})) = f : Y_R \to M\). Let \(L > 0\), and let \(w_i \in SF_i\) be a sequence such that \(c_{w_i}([-L, L]) \subseteq B(p_i, \bar{R})\) for all \(i \in \mathbb{N}\), and \(\lim_{i \to \infty} f_i^L|A_i|^2 \circ c_{w_i}(t)\) \(dt = 0\), and \(\lim_{i \to \infty} d_f(w_i) = w \in SM\). Then a subsequence of the \((c_{w_i}([-L, L]))_{i \in \mathbb{N}}\) converges to a curve in \(Y_R\), and every limit curve \(c : [-L, L] \to Y_R\) satisfies \(f \circ c = c_w([-L, L])\) and \(d_{Y_R}(c(s), c(t)) = |t - s|\) for all \(s, t \in [-L, L]\) with \(|t - s| \leq \text{injrad}(M)\). In particular, we have \(c_w([-L, L]) \subseteq f(Y_R)\).

**Proof:** Since the GH-convergence of \((B(p_i, \bar{R}), p_i)_{i \in \mathbb{N}}\) to \((Y_R, y_0)\) can be realized by Hausdorff convergence within some fixed compact metric space, cf. [11], p. 65, the usual Arzelà-Ascoli theorem implies the existence of a limit curve \(c : [-L, L] \to Y_R\), say \(\lim_{i \to \infty} c_{w_i}([-L, L]) = c\). If \(t_i \in [-L, L]\) and \(\lim_{i \to \infty} t_i = t\), then \(\lim_{i \to \infty} c_{w_i}(t_i) = c(t)\), while \(\lim_{i \to \infty} f_i \circ c_{w_i}(t_i) = c_w(t)\) by Proposition 7.3. From Definition 7.3(b) we see that this implies \(f_i \circ c(t) = c_w(t)\). Since \(c\) and \(f\) are \(1\)-Lipschitz we conclude that \(d_{Y_R}(c(s), c(t)) = |t - s|\) if \(s, t \in [-L, L]\) and \(|t - s| \leq \text{injrad}(M)\).

**8 Asymptotic density of good points**

We continue to consider a sequence of complete surface immersions \(f_i : F_i \to M\) into a compact Riemannian manifold \(M\). We assume that there exist \(\bar{R} > 0\) and a sequence \(p_i \in F_i\) such that \(\lim_{i \to \infty} E(f_i|B(p_i, \bar{R})) = 0\). Relying on the volume estimate from Proposition 3.6 and the lower estimate for the area of balls from Proposition 6.3, we will prove that there exists a sequence \(\varepsilon_i \downarrow 0\) such that the sets of “\(|A_i|^2, \varepsilon_i, R, r\)-good points” \(G_{\varepsilon_i, R, r}(|A_i|^2)\), cf. Definition 3.5, are asymptotically dense in \(B(p_i, r)\), whenever \(R > 0, r > 0\) and \(R + 2r < \bar{R}\). In combination with the results from Section 7, this will be used to prove the existence of a totally geodesic surface in \(M\). As a prerequisite for the proof of a complete totally geodesic surface in \(M\), we will then assume the existence of a sequence \(\bar{R}_i \to \infty\) such that \(E(f_i|B(p_i, \bar{R}_i)) \to 0\) and prove the existence of sequences \(\varepsilon_i \downarrow 0, R_i \to \infty\) such that the sets \(G_{\varepsilon_i, R_i}(|A_i|^2)\), cf. 3.7, are asymptotically dense in \(B(p_i, R_i)\).

**Definition 8.1** Let \((X_i, d_i)\) be a sequence of metric spaces, and, for each \(i \in \mathbb{N}\), let \(B_i\) and \(G_i\) be subsets of \(X_i\). Then the sequence \(G_i\) is called asymptotically dense in the sequence \(B_i\) if

\[
\lim_{i \to \infty} (\sup_{x \in B_i} \{d_i(x, G_i)|x \in B_i\}) = 0.
\]
Put differently, this condition says that there is a sequence $\delta_i \downarrow 0$ such that each $B_i$ is contained in the $\delta_i$-neighborhood of $G_i$. In the following we will depend on the constants $r_0 > 0$, $\beta \geq 1$ from Proposition 6.3.

Lemma 8.2 Assume there exist $\tilde{R} > 0$ and a sequence $p_i \in F_i$ such that
\[
\lim_{i \to \infty} E(f_i|B(p_i, \tilde{R})) = 0.
\]
Then there exists a sequence $\varepsilon_i \downarrow 0$ such that the sequence $G_{\varepsilon_i, R, r}(|A_i|^2)$ is asymptotically dense in the sequence $B(p_i, r)$, whenever $R > 0$, $r > 0$ and $R + 2r < \tilde{R}$.

Proof: We choose a sequence $\varepsilon_i \downarrow 0$ such that $\lim_{i \to \infty} (\varepsilon_i^{-1} E(f_i|B(p_i, \tilde{R}))) = 0$. Since $R + 2r < \tilde{R}$ we can use Proposition 6.3(b) to find $r > r$ such that
\[
(8.1) \lim_{i \to \infty} (\text{vol}_2(B(p_i, \tilde{r}) \setminus G_{\varepsilon_i, R, r}(|A_i|^2))) = 0.
\]
If our claim were not true we could find $\varepsilon > 0$ and an infinite set $I \subseteq \mathbb{N}$ such that, for each $i \in I$, there exists $q_i \in B(p_i, r)$ with $B(q_i, \varepsilon) \cap G_{\varepsilon_i, R, r}(|A_i|^2) = \emptyset$. We may assume that $\varepsilon < \min\{t_0, \tilde{r} - r\}$ and $r + \beta \varepsilon < \tilde{R}$. Then $B(q_i, \beta \varepsilon) \subseteq B(p_i, \tilde{R})$ and, hence, $E(f_i|B(q_i, \beta \varepsilon)) \leq \frac{\pi}{4}$ for almost all $i \in I$. For these infinitely many $i \in I$ Proposition 6.3 implies $\text{vol}_2(B(q_i, \varepsilon)) \geq \frac{\pi}{4} \varepsilon^2$. On the other hand, since $\varepsilon < \tilde{r} - r$, we have $B(q_i, \varepsilon) \subseteq B(p_i, \tilde{r}) \setminus G_{\varepsilon_i, R, r}(|A_i|^2)$, in contradiction to (8.1).

Similarly, we can prove:

Lemma 8.3 Assume there exist sequences $\tilde{R}_i \to \infty$ and $p_i \in F_i$ such that
\[
\lim_{i \to \infty} E(f_i|B(p_i, \tilde{R}_i)) = 0.
\]
Then there exist sequences $\varepsilon_i \downarrow 0$, $R_i \to \infty$ such that the sequence $\tilde{G}_{\varepsilon_i, R_i}(|A_i|^2)$ is asymptotically dense in the sequence $B(p_i, R_i)$.

Proof: We can find sequences $\varepsilon_i \downarrow 0$ and $R_i \to \infty$ such that $5R_i < \tilde{R}_i$ for all $i \in \mathbb{N}$ and $\lim_{i \to \infty} \left(\frac{R_i^2}{5^2} E(f_i|B(p_i, \tilde{R}_i))\right) = 0$. Then we can use Corollary 3.7 to conclude that
\[
\lim_{i \to \infty} (\text{vol}_2(B(p_i, R_i + 1) \setminus \tilde{G}_{\varepsilon_i, R_i}(|A_i|^2))) = 0.
\]
This implies the asymptotic density of $\tilde{G}_{\varepsilon_i, R_i}(|A_i|^2)$ in $B(p_i, R_i)$ as in the proof of Lemma 8.2.

9 \ Existence of totally geodesic surfaces and proof of Theorem 1.2

In this section we prove Theorem 1.2. We assume that there exists a sequence of complete surface immersions $f_i : F_i \to M$ with points $p_i \in F_i$ and $\tilde{R} > 0$ such that $\lim_{i \to \infty} E(f_i|B(p_i, \tilde{R})) = 0$, and prove that $M$ contains (a piece of) totally geodesic surface.

We start with a rough outline of the proof. According to Lemma 8.2 we may assume that $p_i \in G_{\varepsilon_i, R, r}(|A_i|^2)$, where $\varepsilon_i \downarrow 0$ and $R + 2r < \tilde{R}$. Since $M$ is compact we may assume that the sequence $(df_i(T_{p_i}F_i))_{i \in \mathbb{N}}$ converges in the Grassmann bundle $\pi_G : G_2M \to M$ of 2-dimensional linear subspaces in $TM$, say $\lim_{i \to \infty} df_i(T_{p_i}F_i) = P \in (G_2M)_p$, where

$p = \lim_{i \to \infty} f_i(p_i)$.

We intend to prove that $N_P(R) = N = \{c_n(t) \mid t \in SP, |t| < R\} \subseteq M$ is a totally geodesic, embedded surface if $R$ is smaller than the injectivity radius of $M$ at $p$. 

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According to Propositions 7.3 and 7.6 we may assume that the sequence of compact metric spaces \((B(p_i, R + r), p_i)_{i \in \mathbb{N}}\) converges to a compact metric space \((Y, y_0)\), and that the maps \(f_i|B(p_i, R + r)\) converge to a 1-Lipschitz map \(f: Y \to M\). Since \(Y\) has finite 2-dimensional Hausdorff measure, the same is true for \(f(Y)\). Now, for fixed \(w \in SP\), we define maps \(\bar{w}: SP \times (0, R) \to SM, \bar{w}(v, t) = \mathcal{P}_t^{c_v}(w)\), and \(\psi^w = \psi: SP \times (0, R) \times (-r, r) \to M, \psi(v, t, s) = c_{\bar{w}(v,t)}(s)\). Then \(\psi(SP \times (0, R) \times \{0\}) = N \setminus \{p\}\). Using our assumption \(p_i \in G_{\varepsilon_i, R, r}(I_i(2))\) and Proposition 7.4 we prove that \(\psi(SP \times (0, R) \times (-r, r)) \subseteq f(Y)\). In particular, we see that \(\text{rk}(d\psi) = 2\) in a neighborhood of \(SP \times (0, R) \times \{0\}\). Then standard calculus implies that \(\psi(v, t, s) = c_{\bar{w}(v,t)}(s) \in \psi(SP \times (0, R) \times \{0\}) \subseteq N\) for small enough \(|s|\) and \(\bar{w}(v, t) \in S_{c_v(t)}N\). Since \(\bar{w}(v, t)\) is an arbitrary vector in \(S_{c_v(t)}N\), this proves that \(N\) is totally geodesic.

Here are the precise formulation and the proof.

**Proposition 9.1** Let \(f_i: F_i \to M\) be a sequence of complete surface immersions into a compact Riemannian manifold. Suppose \(\varepsilon_i \downarrow 0, R > 0, r > 0,\) and \(p_i \in G_{\varepsilon_i, R, r}(I_i(2))\) is a sequence such that \(\lim_{i \to \infty} E(f_i|B(p_i, R + 2r)) = 0\) and \(\lim_{i \to \infty} df_i(T_{p_i}F_i) = P \in (G_2M)_p\). If \(R\) is smaller than the injectivity radius of \(M\) at \(p\), then \(N_P(R) = N = \{c_v(t)|v \in SP, |t| < R\} \subseteq M\) is a totally geodesic, embedded surface.

**Remark 9.2** The condition “\(R\) smaller than the injectivity radius of \(M\) at \(p\)” is imposed in order to have a convenient description of the totally geodesic surface \(N\). The methods used in Section 10 indicate that without this condition one should be able to find a complete surface immersion \(f: F \to M\) and \(q \in F\) such that \(f|B(q, R)\) is totally geodesic.

**Proof:** According to Propositions 7.3 and 7.6 we can find a subsequence such that the sequence \((B(p_i, R + r), p_i)_{i \in \mathbb{N}}\) converges with respect to definite GH-convergence and such that the sequence \((f_i|B(p_i, R + r))_{i \in \mathbb{N}}\) converges to a 1-Lipschitz map \(f: Y \to M\). We fix an arbitrary \(w \in SP\) and define maps \(\bar{w}: SP \times (0, R) \to TM, \bar{w}(v, t) = \mathcal{P}_t^{c_v}(w)\), and \(\psi: SP \times (0, R) \times (-r, r) \to M, \psi(v, t, s) = c_{\bar{w}(v,t)}(s)\), as above. Since \(R\) is smaller than the injectivity radius of \(M\) at \(p\), we know that \(N\) is an embedded submanifold and \(\psi|SP \times (0, R) \times \{0\}\) is a diffeomorphism onto \(N \setminus \{p\}\). We want to prove that \(\bar{w}(v, t) \in S_{c_v(t)}N\) and that \(\psi(SP \times (0, R) \times (-r, r)) \subseteq f(Y)\). Since \(p_i \in G_{\varepsilon_i, R, r}(I_i(2))\) we know that the sets \(S_{p_i}F_i \times S_{p_i}F_i \times [-R, R]\) are asymptotically dense in \(S_{p_i}F_i \times S_{p_i}F_i \times [0, R]\), cf. 3.1 and Definition 3.5. Hence, given \((v, w, t) \in SP \times SP \times (0, R)\), we can find a sequence \((v_i, w_i, t_i) \in S_{p_i}F_i \times S_{p_i}F_i \times (0, R)\) such that

\[
\lim_{i \to \infty} (df_i(v_i), df_i(w_i), t_i) = (v, w, t),
\]

and such that \(\int_{-R}^R |A_i|^2 \circ c_{\bar{w}_i}(t) \, dt < \varepsilon_i\) and \(\int_{-r}^r |A_i|^2 \circ c_{\bar{w}_i}(s) \, ds < \varepsilon_i\), where \(\bar{w}_i = \mathcal{P}_t^{c_v}(w_i)\), cf. 3.1. Now Proposition 2.3 implies that

\[
\lim_{i \to \infty} df_i(\bar{w}_i) = \mathcal{P}_t^{c_v}(w) = \bar{w}(v, t),
\]

and Proposition 7.4 implies that

\[
\psi(v, t, s) = c_{\bar{w}(v,t)}(s) \in f(Y)
\]

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if $s \in (-r, r)$. Since $f(Y)$ has finite 2-dimensional Hausdorff measure we conclude that $\text{rk}(d\psi) \leq 2$. Since $\text{rk}(d\psi(v, t, 0)) \geq 2$ for $(v, t) \in SP \times (0, R)$, we can find a neighborhood of $SP \times (0, R) \times \{0\}$ in $SP \times \mathbb{R}^2$ on which $d\psi$ has constant rank 2. Using the local normal form of maps of constant rank we find, for every $(v, t) \in SP \times (0, R)$, some $\delta > 0$ such that $\psi(v, t, s) \in \psi(SP \times (0, R) \times \{0\}) = N \setminus \{p\}$ for $|s| < \delta$. Since $N$ is an embedded submanifold this implies that $\tilde{w}(v, t) = \frac{\partial \psi}{\partial s}(v, t, 0) \in S_{c_w(t)}N$. Hence $\mathcal{P}^w_t(SP) = S_{c_w(t)}N$, and the preceding argument shows that every geodesic with initial vector in $S_{c_w(t)}N$ lies locally in $N$. This proves that $N$ is totally geodesic.

\begin{remark}
The preceding proof shows additionally that under the assumptions of Proposition 9.1 every $q \in N$ is the limit of a sequence $(f_i(q_i))_{i \in \mathbb{N}}$ with $q_i \in B(p_i, R) \subseteq F_i$.
\end{remark}

\begin{theorem}
Let $f_i : F_i \to M$ be a sequence of complete surface immersions into a compact Riemannian manifold. Assume the existence of a sequence $p_i \in F_i$ and of $\tilde{R} > 0$ such that $\lim_{i \to \infty} E(f_i(B(p_i, \tilde{R}))) = 0$. Then there exists a 2-plane $P \in G_2M$ such that $N_P(R) = \{c_w(t)\} \subseteq \{v \in SP, |t| < R\}$ is a totally geodesic, embedded surface in $M$, whenever $R > 0$ is smaller than the minimum of $\tilde{R}$ and the injectivity radius of $M$ at the foot point $\pi_G(P)$ of $P$.
\end{theorem}

\begin{proof}
Since $R < \tilde{R}$ we may choose $r > 0$ such that $R + 2r < \tilde{R}$. Now Lemma 8.2 provides sequences $\varepsilon_i \downarrow 0$ and $q_i \in G_{\varepsilon_i, R+r}(\{|A_i|^2\})$ such that $\lim_{i \to \infty} d_{f_i}(q_i, p_i) = 0$. Hence we have $\lim_{i \to \infty} E(f_i(B(q_i, R + 2r))) = 0$. Since $G_2M$ is compact there exists a subsequence of $d_{f_i}(T_{q_i}F_i) \in G_2M$ converging to some $P \in G_2M$. So our claim follows from Proposition 9.1.
\end{proof}

\begin{remark}
If we consider the case of $k$-dimensional immersions with $k > 2$, our proof will go through under the following additional assumptions:
1) A uniform upper bound on $\text{vol}_k(B(p_i, \tilde{R}))$.
2) An analogue of Proposition 6.3.
\end{remark}

The following proposition will be used to prove the existence of complete, totally geodesic surfaces, see Proposition 10.4.

\begin{proposition}
Suppose there are sequences $\varepsilon_i \downarrow 0$, $R_i \to \infty$, and $p_i \in \tilde{G}_{\varepsilon_i, R_i}(\{|A_i|^2\}) \subseteq F_i$ such that $\lim_{i \to \infty} E(f_i|B(p_i, R_i)) = 0$ and such that $\lim_{i \to \infty} d_{f_i}(T_{p_i}F_i) = P \in G_2M$ exists. Given $(v, t) \in SP \times \mathbb{R}$ set $\tilde{P} = \tilde{P}(v, t) = \mathcal{P}^w_t(P)$. Then $N_{\tilde{P}}(R) = \{c_w(s)\} \subseteq \{v \in SP, |t| < R\}$ is a totally geodesic, embedded surface in $M$, provided $R > 0$ is smaller than the injectivity radius of $M$ at $c_w(t)$.
\end{proposition}

\begin{proof}
Since $p_i \in \tilde{G}_{\varepsilon_i, R_i}(\{|A_i|^2\})$ we can find sequences $v_i \in S_{p_i}F_i$, $t_i \in \mathbb{R}$ such that $\lim_{i \to \infty} (d_{f_i}(v_i), t_i) = (v, t)$ and $\int_{-R_i}^{R_i} |A_i|^2 \circ c_{v_i}(s) \, ds < \varepsilon_i$ and $c_{v_i}(t_i) \in G_{\varepsilon_i, R_i}(\{|A_i|^2\})$, cf. (3.7) and (3.8). Now Proposition 2.3(c) implies that $\lim_{i \to \infty} d_{f_i}(T_{c_{v_i}(t_i)}F_i) = \mathcal{P}^w_t(P) = \tilde{P}$. Since $c_{v_i}(t_i) \in G_{\varepsilon_i, R_i}(\{|A_i|^2\}) = G_{\varepsilon_i, R_i, R_i}(\{|A_i|^2\})$, we see that the assumptions of Proposition 9.1 are satisfied for the sequence $c_{v_i}(t_i)$ and every choice of $R > 0$, $r > 0$. So Proposition 9.1 implies our claim.
\end{proof}
10 Existence of complete, totally geodesic surfaces

We continue to consider a sequence of complete surface immersions \( f_i : F_i \to M \) into a compact Riemannian manifold \((M, g)\). In this section we prove a global version of Proposition 9.1. Under the assumption that there exist sequences \( (M, g_i) \) such that \( \lim_{i \to \infty} E(f_i \| B(p_i, R_i)) = 0 \), we prove the existence of a complete, totally geodesic surface immersion into \( M \). In view of Proposition 9.6 this amounts to piecing together local totally geodesic surfaces. This is reminiscent of the construction of the leaves of a foliation. Indeed, for general Riemannian manifolds \((M, g)\), there exists a distribution \( \mathcal{D}_k = \mathcal{D}_k(M, g) \) in the tangent bundle of the Grassmann bundle \( \pi_G : G_k M \to M \) such that the integral manifolds \( \mathcal{L} \) of \( \mathcal{D}_k \) correspond to totally geodesic immersions \( \pi_G|\mathcal{L} : \mathcal{L} \to M \). Under our assumptions we will prove that there exists a complete leaf \( L \subseteq G_2 M \) of \( \mathcal{D}_2(M, g) \), i.e. a leaf \( L \) such that \( (\pi_G|L)^*g \) is a complete Riemannian metric. Then \( \pi_G|L : L \to M \) is a complete, totally geodesic surface immersion.

We start by collecting some facts concerning integral manifolds of a general distribution \( \mathcal{D} \) of fiber dimension \( k \) in the tangent bundle of a general manifold \( M \). We assume that all considered objects are smooth (\( = C^\infty \)). So, \( \mathcal{D} : M \to G_k M \) is a section of the Grassmann bundle \( \pi_G : G_k M \to M \). An integral manifold of \( \mathcal{D} \) is an immersion \( j : L \to M \) of a connected manifold \( L \) such that \( dj(T_x L) = \mathcal{D}_{j(x)} \) for all \( x \in L \). Although we will not use this in our proofs, we note the following important property of integral manifolds.

Remark 10.1 Let \( j : L \to M \) be an injective integral manifold of \( \mathcal{D} \). Suppose \( P \) is a manifold and \( h \in C^\infty(P, M) \) satisfies \( h(P) \subseteq j(L) \). Then \( j^{-1} \circ h \in C^\infty(P, L) \).

An integral manifold \( j : L \to M \) of \( \mathcal{D} \) is called maximal if \( j \) is injective and if the following is true: If \( \tilde{j} : \tilde{L} \to M \) is an integral manifold of \( \mathcal{D} \) and \( j(L) \cap \tilde{j}(\tilde{L}) \neq \emptyset \), then \( \tilde{j}(\tilde{L}) \subseteq j(L) \). If \( j : L \to M \) is a maximal integral manifold of \( \mathcal{D} \), then \( j(L) \subseteq M \) is called a leaf of \( \mathcal{D} \).

One defines a manifold structure on \( j(L) \) by declaring \( j \) a diffeomorphism. The topology of this manifold structure is finer, and often strictly finer, than the topology of \( j(L) \) as a subspace of \( M \). Remark 10.1 shows that this manifold structure of a leaf is in fact independent of the parametrization \( j \).

Proposition 10.2 Suppose \( p \in M \) lies in the image of some integral manifold of \( \mathcal{D} \). Then there exists a maximal integral manifold \( j : L \to M \) such that \( p \in j(L) \).

If \( \mathcal{D} \) is completely integrable, i.e. if \( \mathcal{D} \) satisfies the Frobenius condition, then this is proved in textbooks treating foliations. The more general version stated here admits a similar proof.

In particular, if \( j_0 : L_0 \to M \) is an integral manifold of \( \mathcal{D} \), then there exists a leaf \( L \) of \( \mathcal{D} \) containing \( j_0(L_0) \). Indeed, \( q \in L \) iff there exists a finite sequence \( j_1 : L_1 \to M, \ldots, j_n : L_n \to M \) of integral manifolds of \( \mathcal{D} \) such that \( j_{i-1}(L_{i-1}) \cap j_i(L_i) \neq \emptyset \) for \( 1 \leq i \leq n \) and \( q \in j_n(L_n) \).

Next we briefly recall how \( k \)-dimensional, totally geodesic immersions into an \( m \)-dimensional Riemannian manifold \((M, g)\) are related to a \( k \)-dimensional distribution \( \mathcal{D}_k = \mathcal{D}_k(M, g) \) in the tangent bundle of the Grassmann bundle \( G_k M \). So, from now on, \( G_k M \) will play the role of the manifold \( M \) in the preceding paragraph. Since the bundle \( \pi_G : G_k M \to M \) is associated to the principal \( \mathcal{O}(m) \)-bundle of orthonormal frames there is a natural horizontal distribution...
\( \mathcal{H} \subseteq T(G_k M) \) induced by the Levi-Civita connection of \( g \), cf. [14], Chapter II, pp. 87–88. Explicitly \( \mathcal{H} \) is given as follows. If \( P \in G_k M \) and \( v \in T_{\pi_G(p)} M \), choose a \( C^1 \)-curve \( \gamma : \mathbb{R} \to M \) such that \( \dot{\gamma}(0) = v \), and let \( P_\gamma(t) = P_{\gamma}(P) \) denote the parallel transport of \( P = P_{\gamma}(0) \) along \( \gamma \). Then \( \tilde{P}_\gamma(0) \in T_P(G_k M) \) is independent of the choice of \( \gamma \) with \( \dot{\gamma}(0) = v \), and defines a linear map \( H_P : T_{\pi_G(p)} M \to T_P(G_k M) \), \( H_P(v) = \tilde{P}_\gamma(0) \), satisfying \( d\pi_G \circ H_P(v) = v \) for all \( v \in T_{\pi_G(p)} M \). Then the \( (m \text{-dimensional}) \) horizontal distribution \( \mathcal{H} \subseteq T(G_k M) \) is given by \( \mathcal{H}_P = H_P(T_{\pi_G(p)} M) \) for all \( P \in G_k M \). Note that a \( C^1 \)-curve \( P : I \to G_k M \) is horizontal, i.e. \( \tilde{P}(t) \in \mathcal{H}_P(t) \) for all \( t \in I \), if and only if \( P(t) \) is parallel along \( (\pi_G \circ P)(t) \). Now we consider the \( k \)-dimensional distribution \( D_k = D_k(M, g) \subseteq \mathcal{H} \) defined by 
\[
D_P = H_P(P)
\]
for all \( P \in G_k M \).

Note: We have \( V \in D_P \) if and only if \( V \in \mathcal{H}_P \) and \( d\pi_G(V) \in P \).

**Lemma 10.3** Let \( j : N \to M \) be a totally geodesic immersion of a connected, \( k \)-dimensional manifold \( N \), and define \( J : N \to G_k M \) by \( J(p) = dj(T_p N) \). Then \( \pi_G \circ J = j \) and \( J \) is an integral manifold of \( D \), i.e. \( dJ(T_p N) = D_{J(p)} \) for all \( p \in N \). Conversely, if \( J : N \to G_k M \) is an integral manifold of \( D \), then \( \pi_G \circ J : N \to M \) is a totally geodesic immersion and \( J(p) = d(\pi_G \circ J)(T_p N) \) for all \( p \in N \).

**Proof:** Suppose first that \( j : N \to M \) is a totally geodesic immersion and define \( J : N \to G_k M \) by \( J(p) = dj(T_p N) \). Let \( \tilde{\gamma} \) be a \( C^1 \)-curve in \( N \) and \( \gamma = j \circ \tilde{\gamma} \). Since \( j \) is a totally geodesic immersion we see that \( J(\tilde{\gamma}(t)) = dj(T_{\tilde{\gamma}(t)} N) \) is parallel along \( \gamma \). By the definition of \( \mathcal{H}_{\gamma(t)} \) this implies that \( (J \circ \tilde{\gamma})(t) \in \mathcal{H}_{J(\gamma(t))} \). This proves that \( dJ(T_p N) \subseteq \mathcal{H}_{J(p)} \) for all \( p \in N \). Additionally we have \( d\pi_G \circ dJ = dj \), so that the note preceding Lemma 10.3 shows that \( dJ(T_p N) = D_{J(p)} \) for all \( p \in N \). Conversely, suppose that \( J : N \to G_k M \) satisfies \( dJ(T_p N) = D_{J(p)} \) for all \( p \in N \), and set \( j = \pi_G \circ J \). Then we have for all \( p \in N \):
\[
dJ(T_p N) = d\pi_G(D_{J(p)}) = d\pi_G(H_{J(p)}(J(p))) = J(p).
\]
In particular, \( j \) is an immersion. To see that \( j \) is totally geodesic note that, for every \( C^1 \)-curve \( \tilde{\gamma} \) in \( N \), the curve 
\[
t \to dj(T_{\tilde{\gamma}(t)} N) = (J \circ \tilde{\gamma})(t) \in G_k M
\]
is horizontal, i.e. parallel along \( \gamma = j \circ \tilde{\gamma} \). This proves that \( j \) is totally geodesic.

\( \square \)

Using Proposition 9.6 and the preceding discussion we will prove:

**Proposition 10.4** Suppose \( \varepsilon_i \downarrow 0 \), \( \rho_i \to \infty \) and \( p_i \in \tilde{G}_{\varepsilon_i, \rho_i}(|A_i|^2) \) are sequences such that \( \lim_{i \to \infty} E(f_i | B(p_i, \rho_i)) = 0 \) and such that \( \lim_{i \to \infty} df_i(T_{p_i} F_i) = P \in G_2 M \) exists. Then there exists a leaf \( L \) of \( D_2(M, g) \) such that \( P \in L \) and such that \( (\pi_G | L)^* g \) is a complete Riemannian metric on \( L \). In particular, \( \pi_G | L : L \to M \) is a complete, totally geodesic surface immersion. Moreover, for every \( P \in L \) there exists a sequence \( q_i \in G_{\varepsilon_i, \rho_i}(|A|^2) \) such that \( \lim_{i \to \infty} df_i(T_{q_i} F_i) = \tilde{P} \) and \( d_i(p_i, q_i) \leq d^L(P, \tilde{P}) \), where \( d^L \) denotes the distance on \( L \) induced by \( (\pi_G | L)^* g \).

**Corollary 10.5** Suppose \( p_i \in F_i \), \( R_i \to \infty \) are sequences such that \( \lim_{i \to \infty} E(f_i | B(p_i, R_i)) = 0 \). Then there exists a complete, totally geodesic surface immersion into \( M \).
Proof of Corollary 10.5 assuming Proposition 10.4. According to Lemma 8.3 we can find sequences $\varepsilon_i \downarrow 0$, $\rho_i \to \infty$ and $p_i < R_i - 1$, and $q_i \in G_{\varepsilon_i, \rho_i}(|A_i|^2)$ such that $\lim_{i \to \infty} d_i(q_i, p_i) = 0$. Then Proposition 10.4 applies to a subsequence of the sequence $q_i$. \qed

Proof of Proposition 10.4. From Proposition 9.6 together with Lemma 10.3 we conclude the following. For every $(v, t) \in SP \times \mathbb{R}$ there exists an integral manifold of $D_2(M, g)$ containing $P_v(t) = P^v_\mathbb{R}(P)$. Hence, by Proposition 10.2 there exists a leaf $L$ of $D_2(M, g)$ containing $P_v(\mathbb{R})$. Moreover, the way $L$ is constructed (resp. Remark 10.1) implies that for all $v \in SP$ the curve $P_v$ is a smooth curve in $L$. Since $\pi_G \circ P_v$ is the $g$-geodesic $c_v$, we see that $P_v : \mathbb{R} \to L$ is a geodesic with respect to $(\pi_G|L)^*g$. Hence every geodesic of $(L, (\pi_G|L)^*g)$ through $P$ is defined on all of $\mathbb{R}$. So, by the Hopf-Rinow theorem, $(L, (\pi_G|L)^*g)$ is a complete Riemannian manifold. Lemma 10.3 implies that $\pi_G|L$ is a totally geodesic immersion. Finally, using the Hopf-Rinow theorem again, we obtain, for every $\tilde{P} \in L$, a geodesic $P_v, v \in SP$, such that $P_v(t) = \tilde{P}$, where $t = d^L(P, \tilde{P})$. Since $p_i \in G_{\varepsilon_i, \rho_i}(|A_i|^2)$, we can find a sequence $(v_i, t_i) \in S_p F_i \times \mathbb{R}$ such that $c_{v_i}(t_i) \in G_{\varepsilon_i, \rho_i}(|A_i|^2)$, $\lim_{i \to \infty} d_f(v_i) = v$, $t_i \uparrow t$, and $\int_{0}^{t_i} |A_i|^2 \circ c_{v_i}(s) \, ds < \varepsilon_i$. Using Proposition 2.3(c) we see that $\lim_{i \to \infty} d_f(T_{c_{v_i}(t_i)} F_i) = P_v(t) = \tilde{P}$. We set $q_i = c_{v_i}(t_i)$. Then $q_i \in G_{\varepsilon_i, \rho_i}(|A|^2)$ and $d_i(p_i, q_i) \leq t_i \leq t = d^L(P, \tilde{P})$, and $\lim_{i \to \infty} d_f(T_{q_i} F_i) = \tilde{P}$. \qed

If $L$ is a non-compact leaf of $D_k(M, g)$ such that $(\pi_G|L)^*g$ is complete, one can prove the existence of additional complete leaves in the closure of $L$. In this context the notions “lamination” and “lamination structure” from [1], Section 2 (D1), seem appropriate.

Proposition 10.6 Let $(M, g)$ be a compact Riemannian manifold, and suppose $L_0 \subseteq G_kM$ is a leaf of $D_k(M, g)$ such that $(\pi_G|L)^*g$ is complete. Let $S$ denote the closure of $L_0$ in $G_k M$. Then there exists a unique $C^\infty$-lamination structure $\mathcal{L}$ on $S$ with tangent distribution $D_k(M, g)|_S$. If $L$ is a leaf of $\mathcal{L}$, then $\pi_G|L$ is a complete, totally geodesic immersion.

Remark 10.7 Since $D_k(M, g)|_S$ is the tangent distribution of $\mathcal{L}$, the leaves of $\mathcal{L}$ are precisely the leaves of $D_k(M, g)$ that are contained in $S$.

Remark 10.8 If $L_0$ is compact, then $L_0 = S$ is the only leaf of $\mathcal{L}$. If all leaves of $\mathcal{L}$ are noncompact, then $\mathcal{L}$ will have uncountably many leaves.

Proof of Proposition 10.6. The lamination structure $\mathcal{L}$ on $S$ will be provided by [1], Proposition 2.7, once we know that conditions (a) and (b) in this proposition hold in our situation. First note that, since $L_0$ is assumed complete, there exists an integral manifold of $D_k(M, g)$ through every $P \in S$. This, together with the fact that $S$ is closed, implies conditions (a) and (b). If $L$ is a leaf of $\mathcal{L}$, then $L$ is a leaf of $D_k(M, g)$, see Remark 10.7 and hence $\pi_G|L$ is a totally geodesic immersion by Lemma 10.3. It is a general fact that the leaves of a lamination (in the sense of [1]) in a complete Riemannian manifold are complete. In our situation we can also argue that for every leaf $L$ of $\mathcal{L}$, every $P \in L$ and every $v \in SP$, we have $P_v(t) = P^v_\mathbb{R}(P) \in L$ for $|t| < \text{injrad}(M, g)$. This implies that actually $P_v(\mathbb{R}) \subseteq L$. Since the $P_v, v \in SP$, are the geodesics of $(L, (\pi_G|L)^*g)$ through $P$, we see that $(L, (\pi_G|L)^*g)$ is complete. \qed
11 An upper area estimate for weakly starshaped domains

Upper estimates for the area of parallel sets of curves on surfaces under assumptions on the integrated Gaussian curvature go back to work by G. Bol [2] and F. Fiala [10]. Rigorous proofs for smooth surfaces were given by P. Hartman [12], see also [6], Chapter 2, and [29], Chapter 4. We will slightly extend these results from parallel sets to sets that will be called weakly starshaped. This generalization to weakly starshaped sets is used in the proof of Theorem 1.1. For a proof of Theorem 1.3 the known estimate for the area of metric balls is sufficient, see Sections 6 and 13.

In this section we will consider a domain $S$ with smooth, compact and connected boundary $\partial S \subseteq S$ on a complete Riemannian surface $(F, g)$. So, the boundary $\partial S$ of $S$ is a simple closed curve that will be denoted by $\Gamma$. We introduce the following notation that will be used throughout this section. The geodesic curvature of $\Gamma$ with respect to the normal pointing into $S$ will be denoted by $\kappa : \Gamma \to \mathbb{R}$. We set $L = \text{length}(\Gamma)$, $K(\Gamma) = \int_{\Gamma} \kappa(q) \, ds(q)$, $|K|(\Gamma) = \int_{\Gamma} |\kappa(q)| \, ds(q)$, and $K^+(\Gamma) = \int_{\Gamma} \kappa^+(q) \, ds(q)$, where $s$ denotes arclength on $\Gamma$. The distance function $d^F : S \to [0, \infty)$ from $\Gamma = \partial S$ is defined by

$$d^F(p) = \min\{d(p, x) | x \in \Gamma\},$$

where $d$ denotes the distance on $F$ induced by $g$. For $t > 0$ we set

$$\Gamma^t = \{p \in S | d^F(p) \leq t\} = (d^F)^{-1}([0, t])$$

and

$$P\Gamma^t = \{p \in S | d^F(p) = t\} = (d^F)^{-1}(\{t\}).$$

Obviously, we have $\partial \Gamma^t \subseteq \Gamma \cup P\Gamma^t$, where strict inclusion may hold for exceptional $t$.

**Definition 11.1** A subset $C \subseteq S$ is weakly $\Gamma$-starshaped, if $C$ is closed, $\Gamma \subseteq C$, and the following holds for every $p \in C$ and every geodesic $c : [0, d^F(p)] \to S$ that is a shortest connection from $p$ to $\Gamma$:

$$c(t) \in \text{Int}(C) \quad \text{for all } t \in (0, d^F(p)).$$

**Remark 11.2** We do not assume that shortest connections from points in $C$ to $\Gamma$ are unique. So, $\Gamma$ may not be a retract of $C$. This is the reason for the attribute “weakly”.

**Remark 11.3** If $C_1$ and $C_2$ are weakly $\Gamma$-starshaped, then so is $C_1 \cap C_2$. The collars $\Gamma^t$ of $\Gamma$ are weakly $\Gamma$-starshaped.

As in [6], Theorem 2.4.2, the area estimate depends on an arbitrary chosen number $k \in \mathbb{R}$. In our application, $k$ will be a negative lower bound for the sectional curvature of the ambient manifold $M$. So we present the estimate only in the case $k < 0$, although the cases $k = 0$ and $k > 0$ can be treated similarly, see [6], Theorem 2.4.2, for the corresponding formulae. If $B \subseteq F$ is measurable we set

$$\omega_k^-(B) = \int_B (K-k)^{-} \, d\text{vol}_2,$$
where $K$ denotes the Gaussian curvature of $F$. For $C \subseteq S$ and $t > 0$ we set

$$C^t = C \cap \Gamma^t \text{ and } PC^t = C \cap P\Gamma^t = \{p \in C|d\Gamma(p) = t\}.$$  

The aim of this section is to prove:

**Proposition 11.4** Suppose $t > 0$, $k < 0$, and $C \subseteq S$ is weakly $\Gamma$-starshaped. Then

$$\text{vol}_2(C^t) \leq \frac{1}{-k}(K^+(\Gamma) + \omega_k^-(C^t))(\cosh(\sqrt{-kt}) - 1) + \frac{L}{\sqrt{-k}} \sinh(\sqrt{-kt})$$

and

$$\mathcal{H}^1(PC^t) \leq \frac{1}{\sqrt{-k}}(K^+(\Gamma) + \omega_k^-(C^t)) \sinh(\sqrt{-kt}) + L \cosh(\sqrt{-kt}).$$

Note: In the case $C = \Gamma^t$ and $\kappa \geq 0$, the first estimate reduces to [6], Theorem 2.4.2, case $k < 0$. If $\kappa \geq 0$ is not assumed, it is weaker, since the estimate in [6], Theorem 2.4.2, contains the term $K(\Gamma)$ instead of $K^+(\Gamma)$.

Specializing Definition 11.1 we say that a closed subset $C \subseteq F$ is **weakly $p$-starshaped**, if $p \in \text{Int}(C)$ and every shortest geodesic $c : [0, d(p,q)] \to F$ from a point $q = c(0) \in C$ to $p$ satisfies $c(t) \in \text{Int}(C)$ for all $t \in (0,d(p,q)]$. Then $C$ is weakly $\Gamma$-starshaped, where $S = F \setminus B(p, \varepsilon)$, $\Gamma = \partial S = \partial B(p, \varepsilon)$, provided $B(p, \varepsilon) \subseteq \text{Int}(C)$ and $\varepsilon$ is smaller than the injectivity radius at $p$. So, in the limit $\varepsilon \downarrow 0$ Proposition 11.4 implies:

**Corollary 11.5** If $C \subseteq F$ is weakly $p$-starshaped, and $C \subseteq B(p, r)$ for some $r > 0$, then the following holds for every $k < 0$:

$$\text{vol}_2(C) \leq \frac{1}{-k}(2\pi + \omega_k^-(C))(\cosh(\sqrt{-kr}) - 1).$$

Next we assume that $f : F \to M$ is a complete surface immersion into a Riemannian manifold with sectional curvature bounded below by $k < 0$. Then the Gauß equation implies $K \geq k - \frac{1}{2}|A|^2$, cf. (11.2), and hence

$$\omega_k^-(B) \leq E(f|B) \tag{11.1}$$

for every measurable subset $B \subseteq F$. So, in this situation, the inequality in Corollary 11.5 says:

$$\text{vol}_2(C) \leq \frac{1}{-k}(2\pi + E(f|C))(\cosh(\sqrt{-kr}) - 1). \tag{11.2}$$

Finally, we note that $C = \overline{B(p,r)}$ is weakly $p$-starshaped and $\partial B(p,r) \subseteq PC^r$. Hence Proposition 11.4 together with (11.1) imply:

$$\mathcal{H}^1(\partial B(p,r)) \leq \frac{1}{\sqrt{-k}}(2\pi + E(f|B(p,r)) \sinh(\sqrt{-kr}). \tag{11.3}$$

At first sight one may have the impression that the proof given in [12] covers the more general case of a weakly $\Gamma$-starshaped set $C$. However, there are some technical problems concerning the analytical properties of the function $t \to \mathcal{H}^1(PC^t)$. To circumvent these
problems we first approximate $C$ by a weakly $\Gamma$-starshaped subset of $C$ that does not intersect the cut locus of $\Gamma$, and that has a smooth and generic boundary. For such sets the original ideas from [2] and [10] apply directly, and Proposition 11.4 will follow from this by approximation.

In order to describe this approximation process we introduce the following tools that are taken from [29], Chapter 4. We let $N$ denote the unit normal field along $\Gamma$ that points into $S$ and define

$$Z : \Gamma \times [0, \infty) \to F, \quad Z(p, t) = \exp(tN(p)).$$

The distance function $\rho$ to the cut locus of $\Gamma$,

$$\rho : \Gamma \to (0, \infty], \quad \rho(p) = \sup\{t|d^F(Z(p, t)) = t\}$$

is continuous. The set

$$CL(\Gamma) = \{Z(p, \rho(p))|p \in \Gamma, \rho(p) < \infty\}$$

is the cut locus of $\Gamma$ (in $S$). If we restrict $Z$ to the set $D = \{(p, t)|p \in \Gamma, 0 \leq t < \rho(p)\}$, then $Z|D$ is a diffeomorphism onto $S\setminus CL(\Gamma)$.

It obviously suffices to prove Proposition 11.4 for compact weakly $\Gamma$-starshaped sets $C$, since we can replace $C$ by $C \cap \Gamma^r$ for $r > t$. So, in the sequel we will assume that $C \subseteq \Gamma^r$ for some $r > 0$. We then define a function $g = g_C : \Gamma \to (0, r]$ by

$$g(p) = \begin{cases} \rho(p) & \text{if } \rho(p) < \infty \text{ and } Z(p, \rho(p)) \in C \\ \inf\{t > 0|Z(p, t) \notin C\} & \text{otherwise.} \end{cases}$$

**Remark 11.6** Since $C$ is weakly $\Gamma$-starshaped, $Z$ maps the set $\{(p, t)|p \in \Gamma, 0 < t < g(p)\} \subseteq D$ diffeomorphically onto $\text{Int}(C) \setminus CL(\Gamma)$, while $Z(\text{graph}(g)) = (\partial C \setminus \Gamma) \cup (C \cap CL(\Gamma))$.

In particular, Remark 11.6 implies

$$\text{vol}_2(\partial C) = 0. \quad \text{(11.4)}$$

**Lemma 11.7** The function $g_C : \Gamma \to (0, r]$ is continuous.

**Proof:** Suppose the sequence $p_i \in \Gamma$ converges to $p \in \Gamma$. Since $C$ is closed and $Z(p_i, g(s_i)) \in C$ for all $i \in \mathbb{N}$, we have $Z(p, t) \in C$ for every limit point $t$ of the sequence $g(p_i) \in (0, r]$. Since $g(p_i) \leq \rho(p_i)$ we conclude that $t \leq \lim_{i \to \infty} \rho(p_i) = \rho(p)$. Hence the definition of $g$ implies $t \leq g(p)$. So, to prove continuity of $g$, it suffices to show that the assumption $t < g(p)$ leads to a contradiction. Indeed, if $t < \tau < g(p)$ then $Z(p, \tau) \in \text{Int}(C)$, and hence $Z(p_i, \tau) \in C$ for almost all $i \in \mathbb{N}$. Since $\tau < g(p) \leq \rho(p) = \lim_{i \to \infty} \rho(p_i)$, this implies $\tau \leq g(p_i)$ for almost all $i \in \mathbb{N}$. Since $t$ is a limit point of the $g(p_i)$, this contradicts our assumption $t < \tau$.

Given $t > 0$ we let $D_1 Z(p, t)$ denote the derivative of the curve $p \in \Gamma \to Z(p, t)$ with respect to arclength on $\Gamma$. Then, if $A \subseteq \Gamma$ is measurable, we have

$$\mathcal{H}^1(Z(A \times \{t\})) \leq \int_A |D_1 Z(p, t)| \, ds(p), \quad \text{(11.5)}$$

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Let \( C \subseteq \Gamma^r \) be weakly \( \Gamma \)-starshaped, and \( \delta > 0 \). Then there exists a weakly \( \Gamma \)-starshaped set \( \tilde{C} \subseteq C \setminus CL(\Gamma) \) with the following properties:

\(\text{(i) } g_C(p) - \delta < g_{\tilde{C}}(p) \text{ for all } p \in \Gamma.\)

\(\text{(ii) } g_{\tilde{C}} : \Gamma \to (0, r) \text{ is a smooth Morse function.}\)

\(\text{(iii) } \text{vol}_2(C \setminus \tilde{C}) \leq l^2 L \delta, \text{ where } l \text{ denotes a common Lipschitz constant for } Z|\Gamma \times [0,r] \text{ and } D_1 Z|\Gamma \times [0,r].\)

\(\text{(iv) } L_C(t) \leq L_{\tilde{C}}(t - \delta) + lL \delta, \text{ whenever } t > \delta.\)

**Proof:** Since, by Lemma 11.7 \( g_C \) is continuous, it is a standard result from Morse theory that there exists a smooth Morse function \( \tilde{g} : \Gamma \to (0, r) \) such that \( g_C(p) - \delta < \tilde{g}(p) < g_C(p) \) for all \( p \in \Gamma \). Then we set

\[\dot{\tilde{C}} = \{Z(p,t) | p \in \Gamma, 0 \leq t \leq \tilde{g}(p)\}.\]

Since \( \tilde{g} < g_C \leq \rho \) we have \( \dot{\tilde{C}} \subseteq C \setminus CL(\Gamma) \). Moreover, \( \dot{\tilde{C}} \) is weakly \( \Gamma \)-starshaped and \( g_{\tilde{C}} = \tilde{g} \). This takes care of properties (i) and (ii). To prove (iii) note that \( C \setminus \dot{\tilde{C}} = \{Z(p,t) | \tilde{g}(p) < t \leq g_C(p)\} \), cf. Remark 11.6. Since \( Z|\Gamma \times [0,r] \) is L-Lipschitz we obtain (iii). Finally note that \( g_C - \delta < \tilde{g} \) implies that \( A_C \subseteq A_{\tilde{C}}^{t-\delta} \) for all \( t > \delta \). Hence, if \( t > \delta \) then

\[L_C(t) = \int_{A_C} |D_1 Z(p,t)| \, ds(p) \leq \int_{A_{\tilde{C}}^{t-\delta}} |D_1 Z(p,t-\delta)| \, ds(p) + \int_{\Gamma} |D_1 Z(p,t) - D_1 Z(p,t-\delta)| \, ds(p) \leq L_{\tilde{C}}(t - \delta) + lL \delta.\]

Note that \( PT^i \setminus CL(\Gamma) \) is a smooth 1-dimensional submanifold of \( F \), if \( PT^i \setminus CL(\Gamma) \neq \emptyset \). We let \( \kappa^i : PT^i \setminus CL(\Gamma) \to \mathbb{R} \) denote the geodesic curvature of \( PT^i \setminus CL(\Gamma) \) with respect to the outward pointing unit normal grad \((d^F)\).

**Lemma 11.9** Let \( C \subseteq \Gamma^r \setminus CL(\Gamma) \) be weakly \( \Gamma \)-starshaped, and assume that \( g = g_C \) is a smooth Morse function. Then the function \( v : (0, \infty) \to (0, \infty), v(t) = \text{vol}_2(C^t) \) is \( C^1 \) with derivative \( v' = L_C \), and \( v \) is smooth on the set of regular values of \( g \). If \( t > 0 \) is a regular value of \( g \), then \( v''(t) \leq \int_{PC^t} \kappa^i(q) \, ds(q) \).
Moreover, (11.8) implies that (11.7) \( \lim_{t \to 0} \frac{1}{h} \int_{I_i(t)} |D_1 Z(p, t) - D_1 Z(p, t)| \, ds(p) = 0 \).

Now suppose \( h > 0 \) and \( [t, t+h] \subseteq (t_-, t_+) \). Then \( A_{C}^{t+h} = g^{-1}([t+h, \infty)) \subseteq g^{-1}([t, \infty)) = A_{C}^{t} \), and hence \( I_i(t+h) \subseteq I_i(t) \) for \( 1 \leq i \leq l \). Using (11.8) we obtain

\[
L_C'(t) = \lim_{h \to 0} \frac{1}{h} \left( \sum_{i=1}^{l} \int_{I_i(t+h)} |D_1 Z(p, t+h) - D_1 Z(p, t)| \, ds(p) - \int_{I_i(t)} |D_1 Z(p, t)| \, ds(p) \right)
\]

Next we show that \( v \) is \( C^1 \), and \( v' = L_C \). Since \( g''(p) \neq 0 \) at every singular point \( p \in \Gamma \) of \( g \), one easily concludes that \( L_C \) is continuous on all of \((0, \infty)\). Moreover, \( PC^t = C \cap PT^t \) are the level sets of the distance function \( d^t \) restricted to \( C \). Hence the coarea formula, see e.g. [9], Theorem 3.2.22, implies that

\[
v(t) = \int_{0}^{t} L_C(\tau) \, d\tau.
\]

Since \( L_C \) is continuous we see that \( v \) is \( C^1 \) and \( v' = L_C \). Since \( L_C \) is smooth on the set of regular values of \( g \), the same holds for \( v \). Moreover, (11.9) implies that \( g''(p) \leq \int_{PC^t} \kappa'(q) \, ds(q) \) for every regular value \( t \) of \( g \).

**Proof of Proposition 11.4** First we treat the case that \( C \) satisfies the assumptions in Lemma 11.9. Then the general case will easily follow from Lemma 11.8. We intend to use the Gauß-Bonnet formula to prove that the following differential inequality for the function \( v(t) = \text{vol}_2(C^t) \) is valid for regular values \( t \) of \( g = g_C \).

\[
v''(t) + kv(t) \leq K^+(\Gamma) + \omega_C^{-1}(C^t).
\]

As in the proof of Lemma 11.9 we consider a maximal interval \( (t_-, t_+) \) of regular values of \( g \), and find \( l \in \mathbb{N} \) and disjoint intervals \( I_i(t), 1 \leq i \leq l \), in \( \Gamma \) such that \( g^{-1}([t, \infty)) = \bigcup_{i=1}^{l} I_i(t) \). Then the sets \( C_i^t = Z(I_i(t) \times [0, t]), 1 \leq i \leq l \), are disjoint subsets of \( C^t \). Each \( C_i^t \) is a simply
connected domain with piecewise smooth boundary consisting of \(PC_t^t = Z(I_t(t) \times \{t\})\), and the two geodesic arcs \(Z(\partial I_t(t) \times [0, t])\). Since \(\partial C_t^t\) has right angles at its four corners, the Gauß-Bonnet formula applied to \(C_t^t\) and summation over \(i\) imply

\[
\int_{\bigcup_{i=1}^t C_t(i)} K d\text{vol}_2 + \int_{PC^t} \kappa^t(q) ds(q) - \int_{\bigcup_{i=1}^t I_t(i)} \kappa(p) ds(p) = 0.
\]

Now the preceding equality, Lemma \[11.9\] and the obvious inequality

\[
k v(t) \leq \int_{\bigcup_{i=1}^t C_t(i)} K d\text{vol}_2 + \omega_k(C^t)
\]

combined prove \[11.10\].

Next we show that the differential inequality \[11.10\] implies the estimates claimed in Proposition \[11.4\]. We fix \(t > 0\). The function \(v(0, t]\) is \(C^1\) and smooth except at the finitely many critical points of the Morse function \(g_C\). For every \(a \geq \kappa^+(\Gamma) + \omega_k(C^t)\) and every regular value \(\tau \in [0, t]\) of \(g_C\), \(v\) satisfies the differential inequality

\[
v''(\tau) + kv(\tau) \leq a,
\]

cf. \[11.10\]. This allows us to compare \(v |[0, t]\) and \(v' |[0, t]\) to the solution \(w(\tau) = w_{k,a,L}(\tau) = \frac{a}{k} \cosh(\sqrt{-k} \tau) - 1 + \frac{a}{k} \sinh(\sqrt{-k} \tau)\) of the initial value problem \(w'' + kw = a\), \(w(0) = 0, w'(0) = L\). So, if \(a \geq \kappa^+(\Gamma) + \omega_k(C^t)\), we obtain

\[
(11.11) \quad v(\tau) \leq w_{k,a,L}(\tau) \quad \text{and} \quad v'(\tau) \leq w'_{k,a,L}(\tau) \quad \text{for all} \quad \tau \in [0, t].
\]

Finally, if \(C\) is only weakly \(\Gamma\)-starshaped and \(t > 0\) and \(\delta \in (0, t)\), we use Lemma \[11.8\] to approximate \(C^t\) by a set \(\tilde{C} = \tilde{C}_\delta \subseteq C^t\) that satisfies the assumptions of Lemma \[11.9\]. We set \(a = \kappa^+(\Gamma) + \omega_k(C)\). Then \(a \geq \kappa^+(\Gamma) + \omega_k(C)\) and \(\tilde{C} = \tilde{C}^t\), so that \[11.11\] implies \(\text{vol}_2(\tilde{C}) \leq w_{k,a,L}(t)\) and \(L_{\tilde{C}}(\tau) \leq w'_{k,a,L}(\tau)\) for \(\tau \in [0, t]\). Letting \(\delta \downarrow 0\) and using Lemma \[11.8\] (iii) and (iv), we see that \(\text{vol}_2(C^t) \leq w_{k,a,L}(t)\) and \(L_C(t) \leq w'_{k,a,L}(t)\). Since \(\mathcal{H}^1(PC^t) \leq L_C(t)\) by \[11.6\], the preceding inequalities prove Proposition \[11.4\].

12 Proof of Theorem \[1.1\]

In this section we prove Theorem \[1.1\] by combining Corollary \[10.5\] with Corollary \[12.3\] below. The main new result is the following Proposition \[12.1\] that is based on the results of the preceding section.

For \(k < 0\) we abbreviate \(\frac{1}{k} \cosh(\sqrt{-k} r) - 1\) by \(g_k(r)\).

**Proposition 12.1** Let \(f : F \to M\) be an isometric immersion of a compact Riemannian surface \((F, g)\) into a Riemannian manifold \((M, \bar{g})\) with sectional curvature bounded below by \(k < 0\). If \(R > 0\) and \(E(f)g_k(R) < \text{vol}_2(F)\), then there exists \(p \in F\) such that

\[
E(f|B(p, R/2)) \leq 2\pi \frac{E(f)g_k(R)}{\text{vol}_2(F) - E(f)g_k(R)}.
\]

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Remark 12.2 If \((F,g)\) is noncompact and complete, and if \(E(f) < \infty\), then \(\inf_{p \in F} E(f|B(p,R)) = 0\) holds for every \(R > 0\).

Corollary 12.3 Let \(f_i : F_i \rightarrow M\) be a sequence of complete surface immersions into a Riemannian manifold \((M,\bar{g})\) with sectional curvature bounded below. If \(E(f_i) < \infty\) for all \(i \in \mathbb{N}\) and \(\lim_{i \to \infty} E(f_i)/\text{vol}_2(F_i) = 0\), then there exist sequences \(p_i \in F_i\) and \(R_i \rightarrow \infty\) such that \(\lim_{i \to \infty} E(f_i|B(p_i,R_i)) = 0\).

Proof of Corollary 12.3 assuming Proposition 12.1:
Since \(\lim_{i \to \infty} E(f_i)/\text{vol}_2(F_i) = 0\) we can find a sequence \(R_i \rightarrow \infty\) such that
\[
\lim_{i \to \infty} g_k(2R_i)E(f_i)/\text{vol}_2(F_i) = 0,
\]
where \(k < 0\) denotes a lower bound for the sectional curvature of \((M,\bar{g})\). Then Proposition 12.1 together with Remark 12.2 provide points \(p_i \in F_i\) such that \(\lim_{i \to \infty} E(f_i|B(p_i,R_i)) = 0\).

To prove Proposition 12.1 we use a decomposition of \(F\) into Voronoi cells stemming from an \(R\)-net on \(F\), and apply Proposition 11.4 to the Voronoi cells. First we recall some facts concerning these concepts. Given \(R > 0\) we consider an \(R\)-net \(N \subseteq F\), i.e. we have \(\bigcup_{p \in N} B(p,R) = F\) and \(d(p,q) \geq R\) for all \(p \neq q\) in \(N\). Then the Voronoi cell of \(p \in N\) is defined by
\[
C_p = C_p(N) = \{x \in F|d(p,x) = \min_{q \in N} d(q,x)\}.
\]

As direct consequences of these definitions we have:

\[
\bigcup_{p \in N} C_p = F \tag{12.1}
\]

\[
\text{If } p \in N \text{ then } \overline{B(p,R/2)} \subseteq C_p \subseteq B(p,R). \tag{12.2}
\]

Lemma 12.4 If \(p \in N\) then \(C_p\) is weakly \(p\)-starshaped, and \(\text{Int}(C_p) = \{x \in F|d(p,x) < d(q,x) \text{ for all } q \in N \setminus \{p\}\}\).

The proof of Lemma 12.4 relies on the following fact that is a direct consequence of the regularity of arclength-parametrized shortest connections.

Fact 12.5 Let \(c : [0, r] \rightarrow F\) be a shortest geodesic from \(c(0) = x\) to \(c(r) = p\), \(r = d(p,x)\). If \(q \in F \setminus \{p\}\) and \(d(q,x) = d(p,x),\) then \(d(p,c(s)) < d(q,c(s))\) for all \(s \in (0,r]\).

Proof: First, we have \(d(p,c(s)) = r - s = d(q,x) - s \leq d(q,c(s))\) by the triangle inequality. Now, contrary to our claim, assume that \(d(p,c(s)) = d(q,c(s))\), and let \(\tilde{c} : [0, r - s] \rightarrow F\) be a shortest geodesic from \(\tilde{c}(0) = c(s)\) to \(\tilde{c}(r - s) = q\). Joining \(\tilde{c}\) to \(c|[0,s]\) we obtain an arclength-parametrized curve of length \(r = d(q,x)\) from \(x\) to \(q\). This implies that this curve is a geodesic, in particular \(\tilde{c}(0) = c(s)\) and \(c(t) = \tilde{c}(t - s)\) for all \(t \in [0,r]\). So, \(p = c(r) = \tilde{c}(r - s) = q\), in contradiction to our assumption \(p \neq q\).

Proof of Lemma 12.4 We start by proving the statement concerning \(\text{Int}(C_p)\). If \(x \in F\) and \(d(p,x) < d(q,x)\) for all \(q \in N \setminus \{p\}\), then there exists \(\varepsilon > 0\) such that \(d(p,x) < d(q,x) - \varepsilon\) for all
for all $s \in N\{p\}$, since $N$ is discrete. This implies $B(x, \varepsilon) \subseteq C_p$, so that $x \in \text{Int}(C_p)$. Conversely, assume that $x \in \text{Int}(C_p)$ and $d(p, x) = d(q, x)$ for some $q \in N\{p\}$. Reversing the role of $p$ and $q$ in Fact 12.5 we let $c$ be a shortest geodesic from $x = c(0)$ to $q$. Then Fact 12.5 implies that $d(q, c(s)) < d(p, c(s))$ for $s \in (0, d(x, q)]$. In particular, we have $c(s) \notin C_p$ for small $s > 0$, in contradiction to $c(0) = x \in \text{Int}(C_p)$. Finally, we prove that $C_p$ is weakly $p$-starshaped. Suppose $x \in C_p$, $r = d(x, p)$, and $c : [0, r] \to F$ is a shortest geodesic from $x = c(0)$ to $p = c(r)$. We intend to show that, for every $q \in N\{p\}$, we have $d(c(s), p) < d(c(s), q)$ for all $s \in (0, r]$. This will prove $c(s) \in \text{Int}(C_p)$ for all $s \in (0, r]$ by the preceding argument. If $q \in N\{p\}$ and $r = d(p, x) < d(q, x)$, then $d(p, c(s)) = r - s < d(q, x) - s \leq d(q, c(s))$ for all $s \in (0, r]$. If $q \in N\{p\}$ and $d(p, x) = d(q, x)$, then Fact 12.5 implies $d(c(s), p) < d(c(s), q)$ for all $s \in (0, r]$. □

As a consequence of the characterization of $\text{Int}(C_p)$ in Lemma 12.4 we obtain

\begin{equation}
(12.3) \text{ If } p \neq q \text{ are in } N \text{ then } C_p \cap C_q = (\partial C_p) \cap (\partial C_q).
\end{equation}

We note that (12.1)-(12.2), Lemma 12.4, and Fact 12.5, are true in more general contexts, e.g. in complete, symmetric Finsler manifolds of arbitrary dimension.

From (11.4) and (12.3) we obtain $\text{vol}_2(C_p \cap C_q) = 0$ for all $p \neq q$ in $N$, and hence (12.1) implies

\begin{equation}
(12.4) \text{ vol}_2(F) = \sum_{p \in N} \text{vol}_2(C_p).
\end{equation}

**Proof of Proposition 12.1** We set $\delta = \min_{p \in N} E(f|C_p)/\text{vol}_2(C_p)$. Then we use (12.1) and (12.4) to see that $E(f) \geq \delta \text{vol}_2(F)$, i.e.

\begin{equation}
(12.5) \delta \leq E(f)/\text{vol}_2(F).
\end{equation}

In particular, our assumption implies $\delta g_k(R) < 1$. Choose $p \in N$ such that $E(f|C_p) = \delta \text{vol}_2(C_p)$. Using Lemma 12.4, (11.2) and (12.2), we obtain

$$E(f|C_p) = \delta \text{vol}_2(C_p) \leq \delta(2\pi + E(f|C_p))g_k(R)$$

and, equivalently,

$$E(f|C_p)(1 - \delta g_k(R)) \leq 2\pi \delta g_k(R).$$

According to (12.2) and (12.5) this implies

$$E(f|B(p, R/2)) \leq 2\pi \frac{E(f)g_k(R)}{\text{vol}_2(F) - E(f)g_k(R)}.$$  □

**Proof of Theorem 1.1** We argue by contradiction, and assume that there exists a sequence $f_i : F_i \to M$ of complete surface immersions into a compact Riemannian manifold $M$ such that $E(f_i) < \infty$ for all $i \in \mathbb{N}$ and $\lim_{i \to \infty} E(f_i)/\text{vol}_2(F_i) = 0$. Then Corollary 12.3 provides sequences $p_i \in F_i$ and $R_i \to \infty$ such that $\lim_{i \to \infty} E(f_i|B(p_i, R_i)) = 0$. Now we can use Corollary 10.5 to obtain a complete, totally geodesic surface immersion into $M$, in contradiction to the assumption made in Theorem 1.1. □
13 Hausdorff convergence and proof of Theorem 1.3

We continue to consider a sequence of complete surface immersions \( f_i : F_i \to M \) into a compact Riemannian manifold \((M, g)\). We will assume that \( p_i \in F_i \) is a sequence satisfying the assumptions of Proposition 10.3 i.e. there exist sequences \( \varepsilon_i \downarrow 0 \), \( \rho_i \to \infty \) such that

\[
\begin{align*}
\text{(13.1)} & \quad \lim_{i \to \infty} E(f_i|B(p_i, \rho_i)) = 0, \text{ and} \\
\text{(13.2)} & \quad p_i \in \bar{G}_{\varepsilon_i, \rho_i}(|A_i|^2), \text{ and} \\
\text{(13.3)} & \quad \lim_{i \to \infty} df_i(T_{p_i}F_i) = P_0 \in G_2M.
\end{align*}
\]

Then Proposition 10.3 provides a leaf \( L \) of \( D_2(M, g) \) such that \( P_0 \in L \) and \( \pi_G|L : L \to M \) is a complete, totally geodesic surface immersion. Under these assumptions we will prove

**Theorem 13.1** For every \( R > 0 \) the sequence of compact sets \( f_i(B(p_i, R)) \subseteq M \) Hausdorff converges to \( \bar{N}_{P_0}(R) = \{c_v(t)|v \in SP_0, |t| \leq R\} \).

**Remark 13.2** If \( \bar{L}_{P_0}(R) \) denotes the closed metric ball in \((L, (\pi_G|L)^*g)\) with center \( P_0 \in L \) and radius \( R > 0 \), then \( \pi_G(\bar{L}_{P_0}(R)) = \bar{N}_{P_0}(R) \).

For the proof of Theorem 1.3 we rely on the results on pointed Gromov-Hausdorff convergence from Section 7 in particular on Propositions 7.3(b) and 7.6(b). So, we will choose a subsequence, denoted by the same symbols, such that the sequence of pointed metric spaces \((F_i, p_i)_{i \in \mathbb{N}}\) converges to a proper length space \((Y, y_0)\) with respect to (definite) Gromov-Hausdorff convergence, and such that the immersions \( f_i \) converge to a 1-Lipschitz map \( f : (Y, y_0) \to (M, p_0) \) where \( p_0 = \pi_G(P_0) \). The distance functions on \( F_i \) will be denoted by \( d_i \), and the distance functions on \( Y \) resp. \( M \) by \( d \) resp. \( dM \). The distance functions on \( F \cup Y \) determining the definite convergence will be denoted by \( \delta_i \).

The following proposition will play a crucial role in the proof of Theorem 1.3. Its proof is given following Corollary 13.12.

**Proposition 13.3** There exists a covering map \( \tilde{f} : (Y, y_0) \to (L, P_0) \) such that \( \pi_G \circ \tilde{f} = f \). In fact, \( \tilde{f} \) is a local isometry from \((Y, d)\) onto \( L \) with the distance induced by \((\pi_G|L)^*g\).

In this section we will call a sequence \( q_i \in F_i \) good resp. very good if there exist sequences \( \varepsilon_i \downarrow 0 \), \( \rho_i \to \infty \) and such that \( q_i \in G_{\varepsilon_i, \rho_i}(|A_i|^2) \) resp. \( q_i \in \bar{G}_{\varepsilon_i, \rho_i}(|A_i|^2) \), see Definition 3.5 (3.7) and (3.8). As a consequence of Lemma 8.3 we have:

\[
\text{(13.4)} \quad \text{Every } y \in Y \text{ is the limit of a very good sequence } q_i \in F_i.
\]

In a series of lemmas we will analyse the properties of the limit map \( f : (Y, y_0) \to (M, p_0) \). This will lead to a proof of Proposition 13.3 and Theorem 1.3.

**Lemma 13.4** Suppose \( q_i \in F_i \) is a good sequence converging to \( y \in Y \), and \( P \in G_2M \) is a limit plane of the sequence \( df_i(T_{q_i}F_i) \). If \( 0 < r < \text{injrad}(M) \), then \( f(\partial B(y, r)) \supseteq SP(r) = \{c_v(r)|v \in SP\} \). Moreover, \( f(B(y, r)) \supseteq N_P(r) = \{c_v(t)|(v, t) \in SP \times [0, r]\} \) for all \( r > 0 \).
Proof: Since $P$ is a limit plane of the sequence $df_i(T_q,F_i)$, we can find a subsequence, denoted by the same symbols, and, for every $v \in SP$, a sequence $w_i \in S_q,F_i$ such that $\lim_{i,\to \infty} df_i(w_i) = v$. Since $q_i$ is a good sequence we can approximate the $w_i$ by $v_i \in S_q,F_i$ such that $\lim_{i,\to \infty} df_i(v_i) = \lim_{i,\to \infty} df_i(w_i) = v$ and $\lim_{i,\to \infty} \int_{R}^{\infty} |A_i|^2 \circ c_{v_i}(t) \, dt = 0$ for every $R > 0$. We choose $R$ such that $r < R < \text{injrad}(M)$. Proposition 2.3 implies that the curves $f_i \circ c_v([-R,R]) \otimes_{c_i} c_i([-R,R])$, while Proposition 7.7 provides a limit curve $c : [-R,R] \to Y$ such that $f \circ c \in \tau([-R,\infty]$ and $\|df_t\| \leq \sigma$ for all $t \in [-R,\infty]$. This implies that $c(r) \in \partial B(y,r)$ and $f \circ c(r) = c_v(r)$, hence $f(\partial B(y,r)) \supseteq S_P(r)$. Similarly, we obtain $f(B(y,r)) \supseteq N_P(r)$ for all $r > 0$. \hfill \Box

The following lemma is a consequence of Proposition 6.3. In this section we let $k > 0$ denote an upper bound for the absolute values of the sectional curvatures of $M$.

Lemma 13.5 Let $y \in Y$ and $r \in (0,r_0)$, where $r_0 > 0$ is given by Proposition 6.3. Then there exists a closed, 1-Lipschitz curve $\gamma : [0,l] \to Y$ such that $\gamma([0,l]) \supseteq \partial B(y,r)$ and $l \leq 2\pi \frac{\sinh(\sqrt{k}r)}{\sqrt{k}}$.

Proof: We choose a sequence $q_i \in F_i$ converging to $y$. According to [12], Lemma 5.2, we can choose a sequence $r_i$ of non-exceptional values of the distance functions $d_i(q_i,\cdot)$ such that $\lim_{i,\to \infty} r_i = r$. Then [12], Proposition 6.1, see also [29], Theorem 4.4.1, together with Proposition 6.3 imply that $\partial B(q_i,r_i)$ is a piecewise smooth, simple, closed curve. We let $\gamma_i : [0,l_i] \to \partial B(q_i,r_i)$ be a parametrization of $\partial B(q_i,r_i)$ by arclength, in particular $l_i = \mathcal{H}^1(\partial B(q_i,r_i))$. Then (13.1) and (11.3) imply

$$\limsup_{i,\to \infty} l_i \leq 2\pi \frac{\sinh(\sqrt{k}r)}{\sqrt{k}}.$$ 

Choosing a subsequence we may assume that the sequence $l_i$ converges to $l \leq 2\pi \frac{\sinh(\sqrt{k}r)}{\sqrt{k}}$ and, by the Arzelà-Ascoli theorem, that the $\gamma_i$ converge uniformly to a closed, 1-Lipschitz curve $\gamma : [0,l] \to Y$. It remains to be shown that $\partial B(y,r) \supseteq \gamma([0,l])$. To prove this we argue by contradiction, and assume:

(13.5) There exists $z \in \partial B(y,r)$ and $\delta > 0$ such that $\delta_i(\partial B(q_i,r_i),z) \geq \delta$

for infinitely many $i \in \mathbb{N}$.

Since $z \in \partial B(y,r)$ there exists $z' \in Y$ such that $d(y,z') > r$ and $d(z,z') < \delta/2$, in particular $d(y,z') < r + \delta/2$. Now we choose a sequence $x'_i \in F_i$ converging to $z'$, and estimate

$$|d_i(q_i,x'_i) - d(y,z')| \leq \delta_i(q_i,y) + \delta_i(z',x'_i),$$

where $r + \delta/2 > d(y,z') > r = \lim_{i,\to \infty} r_i$, and $\lim_{i,\to \infty} \delta_i(q_i,y) = 0 = \lim_{i,\to \infty} \delta_i(z',x'_i)$. Hence, for almost all $i \in \mathbb{N}$, we have

$$r_i < d_i(q_i,x'_i) < r + \delta/2.$$ 

For these $i \in \mathbb{N}$ we choose a shortest geodesic from $q_i$ to $x'_i$ and, on this geodesic, the point $x_i$ with $d_i(q_i,x_i) = r_i$. Then $x_i \in \partial B(q_i,r_i)$, and

$$\delta_i(x_i,z) \leq d_i(x_i,x'_i) + \delta_i(x'_i, z') + d(z',z).$$
Since \( d_i(x_i, x_i') = d_i(q_i, x_i') - r_i < r - r_i + \delta/2 \), \( \lim_{i \to \infty} \delta_i(x_i', z') = 0 \), and \( d(z', z) < \delta/2 \), we obtain \( \delta_i(x_i, z) < \delta \) for almost all \( i \in \mathbb{N} \), in contradiction to (13.5). So, we have \( \lim_{i \to \infty} \delta_i(\partial B(q_i, r_i), z) = 0 \) for all \( z \in \partial B(y, r) \). Hence, for every \( z \in \partial B(y, r) \), we can find a sequence \( s_i \in [0, l_i] \) such that \( \lim_{i \to \infty} \gamma_i(s_i) = z \). Then a subsequence of the \( s_i \) converges to some \( s \in [0, l] \) and, by the uniform convergence of \( \gamma_i \) to \( \gamma \), we have \( \gamma(s) = \lim_{i \to \infty} \gamma_i(s_i) = z \).

Lemma 13.5 has the following important consequences.

**Lemma 13.6** Suppose \( q_i \in F_i \) is a good sequence converging to \( y \in Y \). Then the sequence \( d_f(T_{q_i}, F_i) \) converges in \( G_2M \).

**Proof:** Since \( G_2M \) is compact it suffices to show that the sequence \( d_f(T_{q_i}, F_i) \) cannot have two different limit planes \( P \neq P' \). If this were the case we would have \( f(\partial B(y, r)) \supset S_P(r) \cup S_{P'}(r) \) for \( r \in (0, \text{injrad}(M)) \), see Lemma 13.4. Since \( f \) is 1-Lipschitz and \( P \neq P' \) this would imply

\[
\liminf_{r \to 0} \frac{\mathcal{H}^1(\partial B(y, r))}{2\pi r} \geq 2,
\]

while Lemma 13.5 implies

\[
\limsup_{r \to 0} \frac{\mathcal{H}^1(\partial B(y, r))}{2\pi r} \leq 1.
\]

According to (13.4) and Lemma 13.6 the following definition makes sense.

**Definition 13.7** We define \( \tilde{f} : (Y, y_0) \to (G_2M, P_0) \) by \( \tilde{f}(y) = P \) iff \( \lim_{i \to \infty} d_f(T_{q_i}, F_i) = P \) for every good sequence \( q_i \in F_i \) with \( \lim_{i \to \infty} q_i = y \). In particular, \( \tilde{f} \) is a lift of \( f \), i.e. \( \pi_G \circ \tilde{f} = f \).

To formulate the next lemma we choose \( r_1 > 0 \) such that

\[
(13.6) \quad 2\pi \frac{\sinh(\sqrt{k}r)}{\sqrt{k}} - 2\pi \frac{\sin(\sqrt{k}r)}{\sqrt{k}} < 2r \quad \text{for} \quad r \in (0, r_1),
\]

and set

\[
(13.7) \quad r_2 = \min \left\{ r_0, r_1, \text{injrad}(M), \frac{\pi}{\sqrt{k}} \right\}.
\]

**Lemma 13.8** If \( y, z \in Y \) and \( 0 < d(y, z) < r_2 \), then \( f(y) \neq f(z) \).

**Remark 13.9** Once Proposition 13.3 will be proven, we will know that \( f(y) \neq f(z) \) under the weaker condition \( 0 < d(y, z) < 2\text{injrad}(M) \).

**Proof:** We argue by contradiction and assume that \( y, z \in Y \), \( 0 < d(y, z) < r_2 \), but \( f(y) = f(z) \). We set \( d(y, z) = r \) and, in a first step, we additionally assume that \( z \in \partial B(y, r) \). Lemma 13.5 provides a closed, 1-Lipschitz curve \( \gamma : [0, l] \to Y \) such that \( \gamma([0, l]) \supset \partial B(y, r) \) and \( l \leq 2\pi \frac{\sinh(\sqrt{k}r)}{\sqrt{k}} \). From Lemma 13.4 we conclude that \( (f \circ \gamma)([0, l]) \supset S_P(r) \), where
Our next aim is to prove that, for sufficiently small $r > \tilde{\pi}$, we have

\[ \frac{2\pi \sinh(\sqrt{k}r)}{\sqrt{k}} \geq l \geq \text{length}(f \circ \gamma) \geq \mathcal{H}^1(S_P(r)) + 2r. \]  

Since $r < \min\{\text{injrad}(M), \frac{\pi}{\sqrt{k}}\}$, the Rauch comparison theorem implies

\[ \mathcal{H}^1(S_P(r)) \geq 2\pi \frac{\sin(\sqrt{k}r)}{\sqrt{k}}. \]

Since $0 < r < r_1$, inequalities \((13.8)\) and \((13.9)\) contradict \((13.6)\). Finally we treat the case that $f(y) = f(z)$, $0 < d(y, z) = r < r_2$, but $z \notin \partial B(y, r)$. Since $Y$ is a length space we can find a shortest connection in $Y$ from $y$ to $z$, and on this shortest connection, a sequence of points $z_n \neq z$ converging to $z$. Then $z_n \in \partial B(y, r_n)$, where $r_n = d(y, z_n)$, $r_n \uparrow r$, and $d^M(f(z_n), f(z)) \leq r - r_n$. Now we repeat the argument used above with $\partial B(y, r)$ replaced by $\partial B(y, r_n)$, and note that

\[ d^M(f(z_n), S_P(r_n)) \geq r_n - (r - r_n). \]

So we obtain

\[ 2\pi \frac{\sinh(\sqrt{k}r)}{\sqrt{k}} \geq 2\pi \frac{\sin(\sqrt{k}r)}{\sqrt{k}} + 2(2r_n - r) \]

and, for $n \to \infty$, the same contradiction as before. \(\square\)

Our next aim is to prove that, for sufficiently small $r > 0$ and for all $y \in Y$, $P = \tilde{f}(y)$, we have

\[ f(B(y, r)) = N_P(r) = \{c_v(t) | (v, t) \in SP \times [0, r)\}, \]

see Corollary \((13.12)\). We set $r_c = \min\{\text{convrad}(M), \frac{1}{2}\text{injrad}(M)\}$, where $\text{convrad}(M)$ denotes the convexity radius of $M$. For every $p \in M$ and every $r \in (0, r_c)$ we have a smooth function $b_{p, r}$, defined on the unit tangent bundle over $B(p, r)$ by

\[ b_{p, r}(v) = \sup\{t > 0 | c_v(s) \in B(p, r) \text{ for all } s \in [0, t]\} \in (0, 2r). \]

A totally geodesic disk in $B(p, r)$ is a 2-dimensional, totally geodesic, connected submanifold $N$ of $B(p, r)$ such that $\tilde{N} \cap N \subset \partial B(p, r)$. So, if $N$ is a totally geodesic disk in $B(p, r)$ and $q \in N$ then

\[ N = \{c_v(t) | v \in S_qN, 0 \leq t < b_{p, r}(v)\}. \]

If $z \in f^{-1}(B(p, r))$ and $Q = \tilde{f}(z)$, then

\[ N(z, p, r) = \{c_v(t) | v \in SQ, 0 \leq t < b_{p, r}(v)\} \]

is the totally geodesic disk in $B(p, r)$ determined by $z$. We note the following facts concerning these notions.

\begin{align*}
(13.10) & \quad \text{If } N \neq \tilde{N} \text{ are totally geodesic disks in } B(p, r) \text{ and } q \in N \cap \tilde{N}, \text{ then } T_qN \neq T_q\tilde{N}. \\
& \quad \text{Moreover, } \mathcal{H}^2(N \cap \tilde{N}) = 0. \\
(13.11) & \quad \text{If } 0 < r < \tilde{r} < r_c \text{ and } z \in f^{-1}(B(p, r)), \text{ then } N(z, p, r) = N(z, p, \tilde{r}) \cap B(p, r). 
\end{align*}
As a consequence of the Rauch comparison theorem, see (13.9), we have:

\[
(13.12) \quad \text{If } P \in G_2M \text{ and } r \leq \min\{\text{injrad}(M), \pi/\sqrt{k}\}, \text{ and}
\]
\[N_P(r) = \{c_v(t) | (v, t) \in SP \times [0, r)\}, \text{ then}
\]
\[\mathcal{H}^2(N_P(r)) \geq 2\pi \left(1 - \cos(\sqrt{kr})\right) = a(r) > 0.
\]

From Proposition 7.3 we know that \(\mathcal{H}^2(B(y, r)) < \infty\) for every \(y \in Y, r > 0\). This implies the following preliminary result:

**Lemma 13.10** Suppose \(y \in Y, f(y) = p, \text{ and } 0 < r < r_c\). Then there exist \(n \in \mathbb{N}\) and \(z_1, \ldots, z_n \in B(y, r)\) such that

\[f(B(y, r)) \subseteq \bigcup_{j=1}^{n} N(z_j, p, r).
\]

**Proof:** We will show that \(\{N(z, p, r) | z \in B(y, r)\}\) is a finite set. This will prove our claim since obviously \(f(B(y, r)) \subseteq \bigcup_{z \in B(y, r)} N(z, p, r)\). We choose \(\rho > 0\) such that \(r + \rho < r_c\), and assume that \(w_1, \ldots, w_k \in B(y, r)\) are such that \(N(w_1, p, r), \ldots, N(w_k, p, r)\) are pairwise different. We set \(\tilde{f}(w_1) = P_1, \ldots, \tilde{f}(w_k) = P_k\). From Lemma 13.4 we know that \(N_P(\rho) \subseteq f(B(w_j, \rho)) \subseteq f(B(y, r + \rho)).\) Moreover \(N_P(\rho) \subseteq N(w_j, p, r + \rho)\). Using (13.11) we see that also \(N(w_j, p, r + \rho), 1 \leq j \leq k,\) are pairwise different. Hence (13.10) implies \(\mathcal{H}^2(N_P(\rho) \cap N_P(\rho)) = 0\) if \(1 \leq j < j' \leq k\). Using (13.12) we obtain

\[\mathcal{H}^2(B(y, r + \rho)) \geq \mathcal{H}^2(f(B(y, r + \rho)) \geq \sum_{j=1}^{k} \mathcal{H}^2(N_P(\rho)) \geq \rho a(\rho).
\]

Hence we have \(\#\{N(z, p, r) | z \in B(y, r)\} \leq a(\rho)^{-1} \mathcal{H}^2(B(y, r + \rho)) < \infty.\)

**Lemma 13.11** Suppose \(N\) is a totally geodesic surface in \(M, V \subseteq Y\) is open and \(f(V) \subseteq N\). Then \(\tilde{f}(z) = T_{f(z)}N\) for all \(z \in V\).

**Proof:** First note that Lemma 13.4 implies that \(\tilde{f}(y) = T_{f(y)}N\) for all \(y \in V\). If \(z \in \tilde{V}\) we can find \(r \in (0, r_2/2)\) such that for every \(q \in N\) with \(\delta N(q, f(z)) < r\) there exists \(v \in S_q N\) such that \(c_v(t) = f(z)\) where \(t = \delta N(q, f(z))\). Since \(z \in \tilde{V}\) there exists \(y \in V \cap B(z, r)\). Then \(f(y) \in N\) and \(\delta N(f(y), f(z)) < r\), so that we can find \(v \in S_{f(y)}N\) such that \(c_v(t) = f(z)\) if \(t = \delta N(f(y), f(z))\). Now let \(q_i \in F_i\) be a very good sequence converging to \(y\). Then there exists a sequence \((v_i, t_i) \in S_q F_i \times \mathbb{R}\) such that \(c_{v_i}(t_i)\) is a good sequence, and such that \(\lim_{i \to \infty} (df_i(v_i), t_i) = (v, t)\) and \(\lim_{i \to \infty} \int_{t_i-1}^{t_i+1} |A_s|^2 \circ c_{v_i}(s) \, ds = 0\). Then Proposition 2.3 implies that \(\lim_{i \to \infty} f_i \circ c_{v_i}(t_i) = c_v(t) = f(z)\), and that

\[
(13.13) \quad \lim_{i \to \infty} df_i(T_{c_{v_i}(t_i)}F_i) = P_t^{c_v}(\tilde{f}(y)) = P_t^{c_v}(T_y N) = T_{f(z)}N.
\]

Choosing a subsequence we may assume that the sequence \(c_{v_i}(t_i)\) converges to some \(w \in Y\). Since \(d(y, w) \leq \lim_{i \to \infty} t_i = t\), we have \(d(z, w) < r + t < r_2\). Moreover \(f(w) = \lim_{i \to \infty} f_i(c_{v_i}(t_i)) = f(z)\), so that \(w = z\) by Lemma 13.8. Hence (13.13) implies \(\tilde{f}(z) = \tilde{f}(w) = T_{f(z)}N\). 

\[\square\]
Corollary 13.12 Suppose \( y \in Y \), \( P = \tilde{f}(y) \), and \( 0 < r < r_c \). Then \( f(B(y, r)) = N_P(r) \).

Proof: From Lemma 13.10 we obtain totally geodesic disks \( N_1, \ldots, N_n \) in \( B(f(y), r) \) such that \( f(B(y, r)) \subseteq \bigcup_{j=1}^n N_j \). Obviously, we may assume that \( N_j \neq N_k \) for \( 1 \leq j < k \leq n \). Since the \( N_j \) are closed subsets of \( B(f(y), r) \), the sets \( V_j = \{ z \in B(y, r) | f(z) \in N_j \} \) are open in \( B(y, r) \), and \( f(B(y, r) \setminus \bigcup_{j=1}^n V_j) \subseteq \bigcup_{j<k}(N_j \setminus N_k) \). Since \( H^2(N_j \setminus N_k) = 0 \) by (13.10), and \( H^2(f(W)) > 0 \) for every open set \( \emptyset \neq W \subseteq Y \) by Lemma 13.4, we see that \( B(y, r) \subseteq \bigcup_{j=1}^n V_j \). Since \( Y \) is a length space, the metric ball \( B(y, r) \) is connected. Hence, if \( n > 1 \) there exist \( 1 \leq j < k \leq n \) and \( z \in \tilde{V}_j \cap \tilde{V}_k \cap B(y, r) \). Now Lemma 13.11 implies that \( \tilde{f}(z) = T_{f(z)}N_j = T_{f(z)}N_k \), in contradiction to \( N_j \neq N_k \) and (13.10). Hence we have \( n = 1 \) and \( f(B(y, r)) \subseteq N_1 \). Since \( f(y) \in N_1 \) and \( P = \tilde{f}(y) \), we see that \( N_1 = N(P, f(y), r) = N_P(r) \). Finally, Lemma 13.4 implies that \( N_P(r) \subseteq f(B(y, r)) \), so that indeed \( f(B(y, r)) = N_P(r) \).

Remark 13.13 It is conceivable that the proof of Corollary 13.12 can be shortened by use of results from the theory of spaces of bounded integral curvature. In particular, the estimate in [8], Corollary 3.2(1), would easily imply Corollary 13.12. However, due to the global assumptions in [8], it is not directly applicable in our situation.

Proof of Proposition 13.3 We will prove that the lift \( \tilde{f} : (Y, y_0) \to (G_2M, P_0) \) of \( f \) given by Definition 13.4 is a locally isometric covering map onto the leaf \( L \) of \( D_2(M, g) \) through \( P_0 \). First note that, by Lemma 13.11 and Corollary 13.12 the sets \( \tilde{f}^{-1}(L) \) and \( \tilde{f}^{-1}(G_2M \setminus L) \) are both open in \( Y \). Since \( Y \) is connected and \( y_0 \in \tilde{f}^{-1}(L) \) we see that \( \tilde{f}(Y) \subseteq L \). Let \( d^L \) denote the distance on \( L \) induced by \( (\pi_G|L)^*g \). We will prove that \( d(y, z) = d^M(f(y), f(z)) = d^L(\tilde{f}(y), \tilde{f}(z)) \), if \( y, z \in Y \) and \( d(y, z) < \min\{r_2, r_c\} \). We set \( P = \tilde{f}(y) \) and \( \rho = d^M(f(y), f(z)) \). Then Corollary 13.12 implies that \( f(z) \in S_P(\rho) \). From Lemma 13.4 we obtain \( w \in B(y, \rho) \) such that \( f(w) = f(z) \). Since \( d(z, w) \leq 2d(y, z) < r_2 \), Lemma 13.3 implies \( z = w \), hence \( d(y, z) = d(y, w) = \rho = d^M(f(y), f(z)) \). Using this and Lemma 13.4 we see that, for \( 0 < r < \frac{1}{2}\min\{r_2, r_c\} \), and for every \( y \in Y \), \( f|B(y, r) \) is an isometry from \( B(y, r) \) onto \( N_P(r) \) where \( P = \tilde{f}(y) \). Finally, since \( r < r_c \leq \text{injrad}(M) \) we know that \( \pi_G|L_P(r) \) is an isometry from the metric ball \( L_P(r) \) with center \( P \) and radius \( r \) in \( L \) onto \( N_P(r) \). Hence \( \tilde{f}|B(y, r) = (\pi_G|L_P(r))^{-1} \circ f|B(y, r) \) is an isometry from \( B(y, r) \) onto \( L_P(r) \) with the distance \( d^L \). Since \( L \) is connected this implies that \( \tilde{f} : Y \to L \) is a covering map.

Proof of Theorem 13.1 In view of Lemma 13.4 it suffices to derive a contradiction from the assumption that there exist \( R > 0 \) and a sequence \( q_i \in \overline{B(p_i, R)} \) such that the sequence \( f_i(q_i) \) has a limit point in \( M \setminus N_{P_0}(R) \). If this is the case we can choose a subsequence, denoted by the same symbols, such that all of the following statements hold:

(i) \( \lim_{i \to \infty} f_i(q_i) \) exists and \( \lim_{i \to \infty} f_i(q_i) \notin N_{P_0}(R) \).

(ii) The sequence \( (F_i, p_i)_{i \in \mathbb{N}} \) converges to \( (Y, y_0) \) with respect to (definite) pointed Gromov-Hausdorff convergence, and the sequence \( (f_i)_{i \in \mathbb{N}} \) converges to \( f : Y \to M \).

(iii) The sequence \( (q_i)_{i \in \mathbb{N}} \) converges to some \( y \in \overline{B(y_0, R)} \).

Then we have \( f(y) = \lim_{i \to \infty} f_i(q_i) \notin N_{P_0}(R) \). On the other hand, \( (Y, d) \) is a length space and \( \tilde{f} : Y \to L \) is locally isometric, so that \( \tilde{f}(\overline{B(y_0, R)}) \subseteq L_{P_0}(R) \). Hence \( f(y) = \pi_G(\tilde{f}(y)) \in \pi_G(L_{P_0}(R)) = N_{P_0}(R) \), cf. Remark 13.2.

We mention the following consequence of Proposition 13.3.
Corollary 13.14 Under the assumptions \([13.1)-(13.3)\) we consider the sets \(\Omega_i(R) = \{(x,x')| (x,x') \subseteq B(p_i,R), d_i(x,x') \leq \text{injrad}(M)\}\). Then

\[
\lim_{i \to \infty} \left( \sup_{(x,x') \in \Omega_i(R)} |d_i(x,x') - d_M(f_i(x), f_i(x'))| \right) = 0
\]

for every \(R > 0\).

**Proof:** Otherwise we can find \(R > 0\) and a sequence \((x_i, x'_i) \in \Omega_i(R)\) such that \(\limsup_{i \to \infty} |d_i(x_i, x'_i) - d_M(f_i(x_i), f_i(x'_i))| > 0\). Choosing a subsequence we may assume that actually \(\lim_{i \to \infty} |d_i(x_i, x'_i) - d_M(f_i(x_i), f_i(x'_i))| > 0\) and, additionally, that the \(f_i : F_i \to M\) converge to \(f : Y \to M\), and that \(\lim_{i \to \infty} x_i = y \in Y\), \(\lim_{i \to \infty} x'_i = y' \in Y\). Then we have \(\lim_{i \to \infty} d_i(x_i, x'_i) \leq \text{injrad}(M)\) and \(d_M(f(y), f(y')) = \lim_{i \to \infty} d_M(f_i(x_i), f_i(x'_i))\), and, hence \(|d(y,y') - d_M(f(y), f(y'))| > 0\). In contradiction to this inequality we will now show that \(d(y,y') = d_M(f(y), f(y'))\). Indeed, since \(Y\) is a length-space there exists an arclength-parametrized curve \(c : [0,d(y,y')] \to Y\) from \(c(0) = y\) to \(c(d(y,y')) = y'\). Then Proposition 13.3 implies that \(\tilde{f} \circ c\) is a geodesic in \(L\) and, hence, \(f \circ c = \pi_G \circ \tilde{f} \circ c\) is a geodesic in \(M\). This geodesic \(f \circ c\) has length \(d(y,y') \leq \text{injrad}(M)\) and connects \(f(y)\) to \(f(y')\). This implies

\[
d(y,y') = \text{length}(f \circ c) = d_M(f(y), f(y')).
\]

Finally, we present a proof for Theorem 1.3. It depends on Proposition 13.3 and on an idea that is originally due to G. Reeb [25], namely Reeb’s stability theorem.

**Proof of Theorem 1.3:** We argue by contradiction and assume that there exists a sequence of complete, connected surface immersions \(f_i : F_i \to M\) such that \(\text{vol}_2(F_i) \to \infty\), while the sequence \(E(f_i)\) is bounded. Under this assumption the well-known Vitali covering argument provides sequences \(\tilde{p}_i \in F_i\) and \(R_i \to \infty\) such that \(\lim_{i \to \infty} E(f_i|B(\tilde{p}_i, R_i)) = 0\). Using Lemma 8.3 we obtain sequences \(p_i \in F_i\), \(\varepsilon_i \downarrow 0\) and \(p_i \to \infty\) such that \([13.1]\) and \([13.2]\) hold. Choosing a subsequence we can assume that \(\lim_{i \to \infty} d_{f_i}(T_{p_i} F_i) = P_0\) exists, i.e. also \([13.3]\) is satisfied. Now Proposition 13.3 provides a locally isometric covering map \(\tilde{f} : (Y,y_0) \to (L,P_0)\) from a Gromov-Hausdorff limit space \(Y\) onto the leaf \(L\) of \(D_2(M,g)\) through \(P_0\). We want to show that \(L\) is not homeomorphic to \(S^2\) or \(\mathbb{R}P^2\). Otherwise \(Y\) would be compact. Since the \(F_i\) are connected this would imply that there exists \(D > 0\) such that \(\text{diam}(F_i) \leq D\) and, by Corollary 6.2 \(\text{vol}_2(F_i) \leq \frac{1}{k}(2\pi + E(f_i))(\cosh(\sqrt{2k}D) - 1)\) for infinitely many \(i \in \mathbb{N}\), in contradiction to our assumption \(\text{vol}_2(F_i) \to \infty\). Hence \(\pi_G|L\) is a complete, totally geodesic immersion into \(M\), where \(L\) is a connected surface different from \(S^2\) or \(\mathbb{R}P^2\). This contradicts our assumption on \(M\).

**Remark 13.15** Similarly we see that, under the assumptions made above, the lamination structure \(L\) on \(S = \tilde{L}\) from Proposition 10.6 does not have a leaf homeomorphic to \(S^2\) or \(\mathbb{R}P^2\).

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