REMARKS ON CONNECTED COMPONENTS OF MODULI OF REAL POLARIZED K3 SURFACES

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Abstract. We have finalized the old (1979) results from [11] about enumeration of connected components of moduli or real polarized K3 surfaces.

As an application, using recent results of [13] (see also math.AG/0312396), we completely classify real polarized K3 surfaces which are deformations of real hyper-elliptically polarized K3 surfaces. This could be important in some questions, because real hyper-elliptically polarized K3 surfaces can be constructed explicitly.

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1. Introduction

In [11], enumeration of connected components of moduli of real polarized K3 surfaces $(X, P')$ had been considered. We review these results in Section 2.1.

Using Global Torelli Theorem for K3 surfaces due to Piatetski-Shapiro and Shafarevich [14] and epimorphicity of Torelli map for K3 surfaces due to Vic. Kulikov [7], it had been shown in [11] that the connected component of moduli is determined by the isomorphism class of the action of the anti-holomorphic involution $\varphi$ in $H_2(X(\mathbb{C}), \mathbb{Z})$ with considering the corresponding polarization $P' \in H_2(X(\mathbb{C}), \mathbb{Z})$ which satisfies $\varphi(P') = -P'$. This reduces enumeration of connected components to a purely arithmetic problem.

To solve this problem, in [14] the invariants

\[(r, a, \delta_\varphi; k, n, \delta_P, \delta_\varphi P)\]

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of the genus of this action had been introduced and classified. They have very clear geometric
meaning. For example, \( k \in \mathbb{N} \) is defined by the condition that \( P = P'/k \) is primitive in the
Picard lattice, and \( n = P^2 \) is the primitive degree, \( n > 0 \) and \( n \equiv 0 \mod 2 \). The invariants
\((r, a, \delta, \varphi)\) determine the topological type of \( X(\mathbb{R}) \) which is an orientable compact surface.
The invariant \( \delta_{\varphi} P \in \{0, 1\} \), and \( \delta_{\varphi} P = 0 \) if and only if \( X(\mathbb{R}) \sim P \mod 2 \) in \( H_2(X(\mathbb{C}), \mathbb{Z}) \).
All possible invariants \((r, a, \delta, \varphi)\) are given in Figure 1 and describe connected components of
moduli of real Kählerian (i.e. without polarization) K3 surfaces. See Sect. 2.2 for other
details.

The invariants \((r, a, \delta, \varphi)\) give main invariants of connected components of moduli of real po-
larized K3 surfaces. It had been shown in [11], that the invariants \((r, a, \delta, \varphi)\) “almost always”
determine the connected component of moduli uniquely, i.e. the moduli space of real po-
larized K3 surfaces with these invariants is connected. More exactly, by [11], this is true if
\( r \leq 18 - \delta P \). Here the invariant \( r \in \mathbb{N} \) takes values in the range \( 1 \leq r \leq 20 \) and \( \delta P \in \{0, 1\} \).
Thus, only for very few cases when \( r = 18 \) or \( r = 19, 20 \) (look at Figure 1) the connectedness
was not known after the results of [11].

In Sect. 2.2 we finalize this uniqueness result. We show that the same uniqueness is valid
if \( r \leq 18 \). To prove this, we use results of Miranda and Morrison [8], [9] and of D. James [4]
where the general analogue of Witt’s theorem had been proved for indefinite even lattices of
the rank at least three. In [11] its particular case (Theorem 1.14.2 in [11]) had been used.
Real K3 surfaces which satisfy the condition \( r \leq 18 \) can be described purely topologically:
\[ X(\mathbb{R}) \neq T_1 \Pi (T_0)^8, (T_0)^9, (T_0)^{10} \]
Here \( T_g \) denotes a compact orientable surface of the genus \( g \). Thus, if the topological type is
different from these three types, then the connected component of moduli of real polarized
K3 surfaces is determined uniquely by the genus invariants \((r, a, \delta, \varphi)\).

If \( X(\mathbb{R}) = T_1 \Pi (T_0)^8, (T_0)^9 \) or \( (T_0)^{10} \), then the uniqueness result above surely is not valid
in general, for any primitive degree \( n \), and one has to introduce some additional to \((r, a, \delta, \varphi)\)
invariants of connected components of moduli. We do it in Theorems 5—9 introducing the
additional invariants which determine the connected component of moduli uniquely. Here
we use the discriminant forms technique which had been developed in [11].

For real polarized K3 surfaces having the topological type \( X(\mathbb{R}) = T_1 \Pi (T_0)^8 \) or \( (T_0)^9 \) some
connected components of moduli (and their K3) are especially remarkable and are called
standard or different from standard only over 2. They are determined by the especially
simple these additional invariants, and are important for our further considerations. See
their definition after formulations of the theorems 5—9.

In Sect. 3 using recent results of [13], we apply these improvements of the old results of
[11] to answer the following interesting question: Which real polarized K3 surfaces are
deformations of real hyper-elliptically polarized K3 surfaces \((X, \tilde{P})\)? Here \((X, \tilde{P})\) is hyper-
elliptically polarized when the general curve of the corresponding complete linear system \( |\tilde{P}| \)
is hyper-elliptic, and then the linear system gives a double covering. Roughly speaking, when
a connected component of moduli of real polarized K3 surfaces contains a hyper-elliptically
polarized K3 surface?
This could be important in some questions because real hyper-elliptically polarized K3 surfaces can be constructed explicitly as double coverings of the relatively minimal rational surfaces \( Y = \mathbb{P}^2 \) or \( Y = \mathbb{F}_n \) for \( n = 0, 1, 2, 4 \) ramified in a non-singular curve \( A \in (-2K_Y) \) where \( K_Y \) denotes the canonical class of \( Y \). The corresponding complete linear system \( |P| \) is then the preimage of some standard complete linear system of \( Y \).

In [13] all possible genus invariants (1.1) of these deformations had been described (in [13] they were called as deformations of general K3 double rational scrolls which is equivalent). Using these results and the results above about enumeration of connected component of moduli of real polarized K3 surfaces, we can classify these K3 surfaces completely.

We get the following result where the first statement (i) had been obtained in [13].

**Theorem 1.** A real polarized K3 surface \((X, P')\) is a deformation of a general real K3 double rational scroll (equivalently, of a real hyper-elliptically polarized K3 surface) if and only if one of conditions (i)—(iv) below satisfies:

(i) The primitive degree \( n = 2 \) or \( 4 \) (see [13]),

(ii) The primitive degree \( n \geq 6 \), and \( X(\mathbb{R}) \neq T_1 (T_0)^8, (T_0)^9, (T_0)^{10} \), and \( X(\mathbb{R}) \not\sim P \mod 2 \) in \( H_2(X(\mathbb{C}), \mathbb{Z}) \) if \( X(\mathbb{R}) = (T_0)^k \).

(iii) The primitive degree \( n \geq 6 \), and \( X(\mathbb{R}) = T_1 (T_0)^8 \), and \((X, P')\) is standard if \( n \equiv 0, 2 \mod 8 \), and \((X, P')\) is different from a standard only over 2 if \( n \equiv 4, 6 \mod 8 \).

(iv) The primitive degree \( n \geq 6 \), and \( X(\mathbb{R}) = (T_0)^9 \), and \( X(\mathbb{R}) \not\sim P \mod 2 \) in \( H_2(X(\mathbb{C}), \mathbb{Z}) \), and \((X, P')\) is standard.

2. Enumeration of connected components of moduli of real polarized K3 surfaces

Here we finalize results of [14] about description of connected components of moduli of real polarized K3 surfaces.

2.1. Reminding of known results about connected components of moduli of real polarized K3 surfaces. Here we review results of [14].

Let \((X, P')\) be a real polarized K3 surface. Here \( X \) is an algebraic K3 surface, and \( P' \) a very ample divisor class on \( X \), defined over the field \( \mathbb{R} \) of real numbers. I. e. \((X, P')\) is a complex polarized K3 surface together with an anti-holomorphic involution \( \varphi \) of \( X \) such that \( \varphi^*(P') = -P' \). We want to describe connected components of moduli of the pairs \((X, P')\).

Let \( L = H_2(X(\mathbb{C}), \mathbb{Z}) \) be the homology lattice of \( X \) with the intersection pairing. It is an even unimodular lattice of signature \((3, 19)\). This characterizes the lattice up to isomorphisms. The polarization \( P' \in L \) is an element of \( L \) with the \((P')^2 > 0 \). The anti-holomorphic involution \( \varphi \) acts in \( L \), and \( \varphi(P') = -P' \). The triplet \((L, \varphi, P')\) considered up to natural isomorphisms is called the polarized integral K3 involution corresponding to the real polarized K3 surface \((X, P')\). Here another triplet \((\tilde{L}, \tilde{\varphi}, \tilde{P}')\) is isomorphic to \((L, \varphi, P')\) if there exists an isomorphism \( f : L \to \tilde{L} \) of lattices (i. e. preserving the intersection pairing) such that \( f(P') = \tilde{P}' \) and \( \tilde{\varphi} f = f \varphi \). Further we denote by \( L^\varphi \) and \( L_{\varphi} \) the eigenvalue 1 and \(-1\) parts.
respectively of the action of \( \varphi \) in \( L \). The integral polarized K3 involution \((L, \varphi, P')\) satisfies the conditions: \( L^\varphi \) is hyperbolic (i.e. it has exactly one positive square), and \( P' \in L_{\varphi} \) (e.g. see \([9, 11]\)).

By Theorem 3.10.1 in \([11]\), we have the following main result which is based on Global Torelli Theorem \([14]\) and epimorphicity of Torelli map \([7]\) for K3 surfaces.

**Theorem 2.** (see Theorem 3.10.1 in \([11]\)) Connected components of moduli of real polarized K3 surfaces are in one to one correspondence with isomorphism classes of integral polarized involutions \((L, \varphi, P')\) such that \( L \) is an even unimodular lattice of signature \((3, 19)\), \( L^\varphi \) is hyperbolic, \( P' \in L_{\varphi} \) (i.e. \( \varphi(P') = -P' \)) and \((P')^2 > 0\).

Thus, description of connected components of moduli of real polarized K3 surfaces is equivalent to a purely arithmetic problem of classification of the integral polarized involutions above. Further we call them as integral polarized K3 involutions. To solve this problem, in \([11]\), their invariants

\[
(r, a, \delta_{\varphi}; k; n, \delta_P, \delta_{\varphi}P)
\]

were introduced. Here \( r = \text{rk} \ L^\varphi \in \mathbb{N}; \ ((L^\varphi)^*/L^\varphi) = (\mathbb{Z}/2\mathbb{Z})^a \) where \( a \geq 0 \) is an integer; \( \delta_{\varphi} \in \{0, 1\} \) is equal to 0 if and only if \( x \cdot \varphi(x) \equiv 0 \mod 2 \) for any \( x \in L \). They are all invariants of the corresponding pair \((L, \varphi)\). Here \( k \in \mathbb{N} \) is defined by the condition that \( P = P'/k \) is a primitive element of \( L \); here the primitive degree \( n = P^2 = (P'/k)^2 \) is an even natural number; here \( \delta_P \in \{0, 1\} \) is equal to 0 if and only if \( P \cdot L_{\varphi} \equiv 0 \mod 2 \); here \( \delta_{\varphi}P \in \{0, 1\} \) is equal to 0 if and only if \( x \cdot \varphi(x) \equiv x \cdot P \) for any \( x \in L \). The invariants \((2.1)\) give all invariants of the genus of the corresponding integral polarized K3 involutions: for equal invariants \((2.1)\), the corresponding integral polarized K3 involutions are isomorphic over \( \mathbb{R} \) and the rings \( \mathbb{Z}_p \) of \( p \)-adic integers for any prime \( p \).

See \([9, 11]\) and \([11]\) about geometric meaning of the invariants \((2.1)\). We only mention the following where we denote by \( T_g \) an orientable compact surface of the genus \( g \). We have

\[
X(\mathbb{R}) = \begin{cases} 
\emptyset & \text{if } (r, a, \delta_{\varphi}) = (10, 10, 0) \\
T_1 \amalg T_1 & \text{if } (r, a, \delta_{\varphi}) = (10, 8, 0) \\
T_g \amalg (T_0)^k & \text{otherwise, where } \\
& g = (22 - r - a)/2, \ k = (r - a)/2 
\end{cases}
\]

\[
(2.3) \quad X(\mathbb{R}) \sim 0 \mod 2 \text{ in } H_2(X(\mathbb{C}); \mathbb{Z})
\]

if and only if \( \delta_{\varphi} = 0 \), and

\[
(2.4) \quad X(\mathbb{R}) \sim P \mod 2 \text{ in } H_2(X(\mathbb{C}); \mathbb{Z})
\]

if and only if \( \delta_{\varphi}P = 0 \). Here \( X(\mathbb{R}) = X(\mathbb{C})^\varphi \) is the fixed points set for the corresponding anti-holomorphic involution \( \varphi \) on the complex K3 surface \( X(\mathbb{C}) \).

In \((11, \text{ Theorem 3.4.3})\), the genus invariants \((2.1)\) of the integral polarized K3 involutions were classified: One should set \( l_{(+)} = 3, l_{(-)} = 19, t_{(+)} = 1 \) and \( t_{(-)} = r - 1 \) in this theorem. We have
Theorem 3. (see Theorem 3.4.3 in [11]) The invariants (2.1) give complete genus invariants of integral polarized K3 involutions.

There exists a real polarized K3 surface with the genus invariants (2.1) if and only if the invariants satisfy the conditions 0.(1)—(7) and I. (1)—(19) listed below.

0. Conditions on \((r, a, \delta_\varphi)\):
(1) \(1 \leq r \leq 20, 0 \leq a \leq \min\{r, 22 - r\}\);
(2) \(r + a \equiv 0 \mod 2\); if \(\delta_\varphi = 0\), then \(r \equiv 2 \mod 4\);
(3) if \(a = 0\), then \(\delta_\varphi = 0\) and \(r \equiv 2 \mod 8\);
(4) if \(a = 1\), then \(r \equiv 1, 3 \mod 8\);
(5) if \((a = 2, r \equiv 6 \mod 8)\), then \(\delta_\varphi = 0\);
(6) if \((a = r, \delta_\varphi = 0)\), then \(r \equiv 2 \mod 8\);
(7) if \((a = 22 - r, \delta_\varphi = 0)\), then \(r \equiv 2 \mod 8\).

I. Conditions on \(n, \delta_P, \delta_\varphi P\):
General conditions:
(1) \(n > 0\) and \(n \equiv 0 \mod 2\);
(2) if \((n \equiv 2 \mod 4, \delta_P = 0)\), then \(\delta_\varphi = 1\);
(3) if \(\delta_\varphi P = 0\), then \((\delta_P = 0, \delta_\varphi = 1, r \equiv 2 + n/2 \mod 4)\).

Relations near the boundary \(a = 22 - r\):
(4) if \(a = 22 - r\), then \(\delta_P = 0\);
(5) if \((a = 22 - r, \delta_\varphi P = 0)\), then \(r \equiv 2 + n/2 \mod 8\);
(6) if \((a = 20 - r, n \equiv 0 \mod 4, \delta_P = 1, \delta_\varphi = 0)\), then \(r \equiv 2 \mod 8\).

Relations near the boundary \(a = 0\):
(7) if \(a = 0\), then \(\delta_P = 1\);
(8) if \((a = 1, n \equiv 0 \mod 4)\), then \(\delta_P = 1\);
(9) if \((a = 1, \delta_P = 0, n \equiv \pm 2 \mod 8)\), then \(r \equiv 2 \pm 1 \mod 8\);
(10) if \((a = 2, \delta_P = 0, n \equiv \pm 2 \mod 8)\), then \(r \equiv 2, 2 \pm 2 \mod 8\);
(11) if \((a = 2, \delta_P = 0, n \equiv 0 \mod 8)\), then \(r \equiv 2 \mod 8\);
(12) if \((a = 3, \delta_P = 0, n \equiv 0 \mod 8)\), then \(r \equiv 1, 3 \mod 8\);
(13) if \((a = 2, \delta_P = 0, n \equiv 4 \mod 8, r \equiv 2 \mod 8)\), then \(\delta_\varphi = 0\);
(14) if \((a = 1, \delta_P = 0)\), then \(\delta_\varphi P = 0\);
(15) if \((a = 2, \delta_P = 0, n \equiv 4 \mod 8, r \equiv 0 \mod 4)\), then \(\delta_\varphi P = 0\);
(16) if \((a = 3, \delta_P = 0, n \equiv \pm 2 \mod 8, r \equiv 2 \pm 5 \mod 8)\), then \(\delta_\varphi P = 0\);
(17) if \((a = 2, \delta_P = 0, n \equiv 0 \mod 8, r \equiv 2 \mod 8, \delta_\varphi = 1)\), then \(\delta_\varphi P = 0\);
(18) if \((a = 4, \delta_P = 0, n \equiv 0 \mod 8, r \equiv 6 \mod 8, \delta_\varphi = 1)\), then \(\delta_\varphi P = 0\).

Relations near the boundaries \(a = 0\) and \(a = 22 - r\):
(19) if \(r = 20\), then \(n = 2^\epsilon p_1^{a_1} \cdots p_m^{a_m}\), where \(\epsilon \leq 2\), \(p_i\) a prime, \(p_i \equiv 1 \mod 4\).

We remark that the invariants \((r, a, \delta_\varphi)\) satisfying the conditions 0.(1)—(7) classify, up to isomorphism, all integral involutions \((L, \varphi)\) satisfying the condition: \(L^\varphi\) is hyperbolic. All
these invariants are listed in Figure 1. They classify connected components of moduli of real Kählerian (and then without a polarization) K3 surfaces.

All polarization conditions I.(1)—(19) depend on \( n \mod 8 \) except two conditions: the condition I.(5) depends on \( n \mod 16 \) and may happen only on the boundary \((r + a = 22, \delta_\varphi = 1)\) (see Figure 1); the condition I.(19) depends on prime decomposition of \( n \) and may happen only in one point \((r, a, \delta_\varphi) = (20, 2, 1)\) (see Figure 1). Thus, depending on \( n \mod 8 \) or \( n \mod 16 \), all genus invariants (2.1) can be easily enumerated similarly to Figure 1. See Figures 33—41 in [13].

At last, it was shown in [11] that “almost in all cases” the genus invariants (2.1) determine the isomorphism class of an integral polarized K3 involution, and then (by Theorem 2) the moduli space of real polarized K3 surfaces with this invariants is connected. Exactly we have (see Theorem 3.3.1 in [11]):

The moduli space of real polarized K3 surfaces with the fixed genus invariants (2.1) is connected if \( r \leq 18 - \delta_\varphi \). In particular, it is connected if \( r \leq 17 \).

Looking at Figure 1 one can see that there are very few points \((r, a, \delta_\varphi)\) which don’t satisfy this condition. But, there are still many.

In the next Section 2.2 we shall improve and finalize this connectedness result. When it is not valid, we shall enumerate connected components of moduli of real polarized K3 surfaces by additional invariants.

2.2. The enumeration of connected components of moduli of real polarized K3 surfaces. We have the following significant improvement of the old connectedness result above where the last statement of the theorem follows from (2.2) and Figure 1.

**Theorem 4.** The genus invariants (2.1) determine the isomorphism class of an integral polarized K3 involution if \( r \leq 18 \).

In particular (by Theorem 2) the moduli space of real polarized K3 surfaces with the fixed genus invariants (2.1) is connected if \( r \leq 18 \).
In particular, the moduli space of real polarized K3 surfaces \((X, P')\) with the fixed genus invariants (2.1) is connected if \(X(\mathbb{R})\) is different from \(T_1 \sqcup (T_0)^9\) (i.e. \((r, a) = (19, 1)\)), and \((T_0)^9\) (i.e. \((r, a) = (19, 3)\)), and \((T_0)^{10}\) (i.e. \((r, a) = (20, 2)\)).

**Proof.** Fix the genus invariants (2.1) satisfying \(r \leq 18\). The invariants determine the genus of the lattices \(L^r\) and \(L_{\varphi, P} = (L^r \oplus \mathbb{Z}P)^\perp\). We denote by \(\mathcal{q}_S\) the discriminant quadratic form of an even lattice \(S\) (see [11]). The statement follows if the lattices \(L^r\) and \(L_{\varphi, P}\) are unique in their genus, and the canonical homomorphisms

\[
O(L^r) \rightarrow O(q_{L^r}), \quad O(L_{\varphi, P}) \rightarrow O(q_{L_{\varphi, P}})
\]

are epimorphic. See the proof of Theorem 3.3.1 in [11] (or look at Remark 1.6.2 of [12]). Here \(O\) denotes the full orthogonal group.

The lattice \(M = L_{\varphi, P}\) is an indefinite lattice of the rank at least 3 since \(\operatorname{rk} M = 22 - r - 1\) and \(r \leq 18\). Its discriminant group \(A_M = M^* / M\) and the discriminant form on that group are calculated in [11] (see also [12]). Its \(p\)-component is a cyclic group \((\mathbb{Z}/p^k\mathbb{Z})\) for any odd prime \(p\). Its 2-component is isomorphic to \((\mathbb{Z}/2\mathbb{Z})^{k_1} \oplus (\mathbb{Z}/2^k\mathbb{Z})\) where \(k \geq 0\).

Assume that \(\operatorname{rk} M \leq 17\). Then \(\operatorname{rk} M = 21 - r \geq 4\), and the required statement for \(M = L_{\varphi, P}\) follows from Theorem 1.14.2 in [11] (or Theorem 1.2' in [10]).

Assume that \(r = 18\). Then \(\operatorname{rk} M = 3\). If the 2-component of \(A_M\) is a cyclic group, then the statement again follows from Theorem 1.14.2 in [11] (or Theorem 1.2' in [10]). For instance, it is valid if the invariant \(\delta_P = 0\).

Assume that \(r = 18\), but the 2-component of \(A_M\) is not cyclic. Since \(M\) is even, it follows that the 2-component of \(A_M\) is isomorphic to \((\mathbb{Z}/2\mathbb{Z})^2 \oplus (\mathbb{Z}/2^k\mathbb{Z})\) where \(k \geq 1\). In this case, the statement follows from the general results announced by Miranda and Morrison in [8], [9]. The proofs are contained in James [4]. Remark that we need a very particular case of the general results of Miranda—Morrison and James which is a little bit stronger than Theorem 1.14.2 in [11].

It follows the theorem.  \(\square\)

Now let us assume that \(r \geq 19\). Then the lattice \(M = L_{\varphi, P}\) has the rank 2 or 1, and the statement of Theorem 4 surely is not valid (see below). We must introduce some additional invariants.

By Theorem 3 (see Figure 1), if \(r \geq 19\), then \((r, a, \delta_{\varphi}) = (19, 1, 1), (19, 3, 1)\) or \((20, 2, 1)\). Assuming one of this cases, we apply the following general construction which uses the technique of discriminant forms from [11].

We fix the invariants (2.1) with the \((r, a, \delta_{\varphi})\) where \(r \geq 19\). The invariants \((r, a, \delta_{\varphi})\) define the isomorphism class of the lattice \(L^r\). The additional invariants \((n, \delta_P, \delta_{\varphi P})\) define the lattice

\[
L_{\varphi, P} = [L^r, P]_{pr}
\]
which is a primitive sublattice in \( L \) generated by \( L^\varphi \) and \( P \). We denote by

\[
q = q_{L^\varphi, P}
\]

the discriminant quadratic form of the lattice \( L^\varphi, P \). We introduce the group

\[
O(L^\varphi)_P = \{ f \in O(L^\varphi, P) \mid f(P) = P \} \subset O(L^\varphi),
\]

and calculate the image of the natural homomorphism

\[
(2.7) \quad O(L^\varphi)_P \to \overline{O(L^\varphi)}_P \subset O(q),
\]

which we denote as \( \overline{O(L^\varphi)}_P \). The lattice \( L_{\varphi, P} \) is orthogonal to \( L^\varphi, P \) in the even unimodular lattice \( L \). Then it has the signature \((1, 20 - r)\) and the discriminant quadratic form which is isomorphic to \((-q)\). This fixes the genus of the lattice \( L^\varphi, P \). Any lattice \( L^\varphi, P \) with these invariants can be taken, and its class of isomorphism is the main additional invariant of the integral polarized K3 involution.

Since \( L^\varphi, P \) and \( L_{\varphi, P} \) are orthogonal complements to each other in the even unimodular lattice \( L \), it defines the canonical isomorphism

\[
(2.8) \quad \tau : q_{L_{\varphi, P}} \cong -q
\]

of the finite quadratic forms. Any of them can be taken, and it defines another invariant of an integral polarized K3 involutions. Any such a pair

\[
(2.9) \quad (L_{\varphi, P}, \tau)
\]

can be taken, and it defines the isomorphism class of an integral polarized K3 involution with the given genus invariants. More exactly,

\[
(2.10) \quad L = [L^\varphi, P \oplus L_{\varphi, P}, \{ x_+^* \oplus x_-^* \mid x_+^* \in (L^\varphi, P)^*, \ x_-^* \in (L_{\varphi, P})^* \text{ and } \tau(x_+^* + L_{\varphi, P}) = x_+^* + L^\varphi, P}]]
\]

with the action of the involution \( \varphi \) which is +1 on \( L^\varphi \), and −1 on \( P \) and \( L_{\varphi, P} \). Other speaking

\[
L/ (L^\varphi, P \oplus L_{\varphi, P}) \subset (L^\varphi, P)^*/L^\varphi, P \oplus (L_{\varphi, P})^*/L_{\varphi, P}
\]

is the graph of the isomorphism \( \tau \).

Two such pairs \((L_{\varphi, P}, \tau)\) and \((\widetilde{L}_{\varphi, P}, \widetilde{\tau})\) define isomorphic integral polarized K3 involutions if and only if there exists an isomorphism \( f : L_{\varphi, P} \to \widetilde{L}_{\varphi, P} \) of lattices such that for the induced isomorphism \( \overline{f} : q_{L_{\varphi, P}} \to q_{\widetilde{L}_{\varphi, P}} \) of their discriminant forms, one has

\[
(2.11) \quad \overline{\tau} \overline{f} = g \tau
\]

where \( g \in O(L^\varphi)_P \).

Thus, for the fixed isomorphism class \( L_{\varphi, P} \), the number of classes of pairs is equal to the number of double cosets

\[
(2.12) \quad O(L_{\varphi, P}) \backslash O(q) / O(L^\varphi)_P
\]

where \( O(L_{\varphi, P}) \) is the image of \( O(L_{\varphi, P}) \) in \( O(q) \) by using \( \tau \).
Thus, the actual number of classes of integral polarized K3 involutions and connected components of moduli of real polarized K3 surfaces is equal to

\begin{equation}
\sum_{\text{classes of genus } (1,20-r,-q)} \sharp (\mathcal{O}(L_{\varphi,P}) \backslash \mathcal{O}(q)/\mathcal{O}(L_{\varphi})_P).
\end{equation}

We remark that

\begin{equation}
O(q) = \prod_p O(q_p),
\end{equation}

where \(q_p\) is the non-trivial \(p\)-component of \(q\) for a prime \(p\). The \(p\)-component \(q_p\) is a quadratic form on a cyclic group if \(p|n\) is odd, and then \(O(q_p) = \{\pm 1_p\}\) where \(-1_p\) means minus identity on the \(p\)-component \(q_p\). The same is valid for \(q_2\) if the 2-component is also cyclic. We also remark that \(O(L_{\varphi,P})\) acts only on 2-component of \(q\), and it is trivial almost in all cases. Thus, we have

\begin{equation}
O(q) = O(q_2) \prod_{\text{odd } p|n} \{\pm 1_p\}, \quad O(L_{\varphi,P}) \subset O(q_2).
\end{equation}

Below we calculate these invariants for each of three cases \((r,a,\delta_{\varphi})\) with \(r \geq 19\).

We use notations: \(\langle A \rangle\) denote a lattice defined by a symmetric integral matrix \(A\). We denote \(U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\). It is a unique class of an even unimodular lattice of the signature \((1,1)\). The lattice \(E_8\) is a negative definite even unimodular lattice of the rank 8. It is defined by the corresponding diagram \(E_8\). For a lattice \(K\), we denote by \(K(m)\) a lattice obtained from the lattice \(K\) by multiplication of the form of \(K\) by \(m \in \mathbb{Q}\). By \(\oplus\) we denote the orthogonal sum of lattices.

2.2.1. The case \((r,a,\delta_{\varphi}) = (19,1,1)\). In this case \(L_{\varphi} = U \oplus 2E_8 \oplus \langle -2 \rangle\).

The case \(\delta_{\varphi} = 1\). Then \(L_{\varphi,P} = L_{\varphi} \oplus \mathbb{Z}P = U \oplus 2E_8 \oplus \langle -2 \rangle \oplus \langle n \rangle\). Since \(U\) and \(E_8\) are unimodular, the discriminant form

\begin{equation}
q = q_{L_{\varphi,P}} = q_{\langle -2 \rangle \oplus \langle n \rangle}.
\end{equation}

The group \(O(L_{\varphi,P})_P\) is trivial since it acts only on the orthogonal part \(q_{\langle -2 \rangle}\) which has the group of order 2. The hyperbolic lattice \(\langle 2 \rangle \oplus \langle -n \rangle\) has the discriminant quadratic form which is isomorphic to \(-q\). Thus, \(L_{\varphi,P}\) is any lattice having the same genus. Here \(n > 0\) can be any even number, the invariant \(\delta_{\varphi,P} = 1\) (see Theorem 3). We naturally identify \(-q = -q_{\langle -2 \rangle \oplus \langle n \rangle} = q_{\langle 2 \rangle \oplus \langle -n \rangle}\). Then an isomorphism \(\tau : q_{L_{\varphi,P}} \rightarrow -q\) is identified with the isomorphism \(\tau : q_{L_{\varphi,P}} \rightarrow q_{\langle 2 \rangle \oplus q_{\langle -n \rangle}}\).

Thus, we finally obtain the result.

**Theorem 5.** If \((r,a,\delta_{\varphi},\delta_{P}) = (19,1,1,1)\) (equivalently, \(X(\mathbb{R}) = T_1 \amalg (T_0)^8\) and \(\delta_{P} = 1\)), then \(\delta_{\varphi,P} = 1\), \(n\) any positive even integer, \(k\) any positive integer, and the isomorphism
classes of integral polarized K3 involutions (equivalently connected components of moduli of real polarized K3 surfaces) with these invariants are in one to one correspondence with pairs

$$(L_{\varphi,P}, \tau)$$

where $L_{\varphi,P}$ is any lattice of the genus $\langle 2 \rangle \oplus \langle -n \rangle$ and $\tau : q_{L_{\varphi,P}} \to q_{\langle 2 \rangle \oplus q_{\langle -n \rangle}}$ is any isomorphism of quadratic forms considered up to the natural action of $O(L_{\varphi,P})$.

Thus, the number of connected components of moduli is equal to

$$\sum_{L_{\varphi,P} \text{ of genus } \langle 2 \rangle \oplus \langle -n \rangle} \#(O(q_{L_{\varphi,P}})/O(L_{\varphi,P})).$$

For our further study, the following definition is very important. A connected component of moduli from Theorem 5 is called standard if $L_{\varphi,P} = \langle 2 \rangle \oplus \langle -n \rangle$ (it defines the identity of the corresponding Gauss class group of binary lattices, e.g. see [3]) and $\tau$ is induced by the identity isomorphism of the lattice $\langle 2 \rangle \oplus \langle -n \rangle$. Any other connected component of moduli with the lattice $L_{\varphi,P} \sim \langle 2 \rangle \oplus \langle -n \rangle$ will be different from the standard one by an automorphism of $O(q_{\langle 2 \rangle \oplus q_{\langle -n \rangle}})$ (up to $O(L_{\varphi,P})$), and it can be labelled by the automorphism. If the automorphism belongs to the 2-component $O(q_{\langle 2 \rangle \oplus q_{\langle -n \rangle}})$ of the automorphism group, we say that this component is different from the standard only over 2. Of course, we use the same names for real polarized K3 surfaces from these connected components of moduli. Similar definitions we use in all cases below when we introduce the standard connected component of moduli.

We remark that the 2-component $O(q_{\langle 2 \rangle \oplus q_{\langle -n \rangle}})$ is trivial if $n \equiv 2 \mod 8$, it is $(\mathbb{Z}/2)^2$ if $n \equiv 0 \mod 16$, and it is $\mathbb{Z}/2$ in the remaining cases.

Thus, the number of connected components of moduli which are different from the standard only over 2 is one if $n \equiv 2 \mod 8$, it is at most four if $n \equiv 0 \mod 16$, and it is at most two in the remaining cases.

For example, Theorem 5 gives only one connected component of moduli which is the standard one if $n = 2, 4, 6$ or 8.

The case $\delta_P = 0$. Since $(r, a, \delta_\varphi) = (19, 1, 1)$ and $\delta_P = 0$, by Theorem 3 (see relations I.(2),(9),(19)) we have $\delta_{\varphi P} = 0$, $P^2 = n \equiv 2 \mod 8$, and the lattice $\mathbb{Z}P = \langle n \rangle$.

We again write $L_\varphi = U \oplus 2E_8 \oplus \langle -2 \rangle$. We denote by $e$ the generator of the summand $\langle -2 \rangle$ with $e^2 = -2$. Since $\delta_P = 0$ and $a = 1$,

$$L^{\varphi,P} = U \oplus 2E_8 \oplus (\langle -2 \rangle \oplus \langle n \rangle)(1/2, 1/2)$$

where $\langle -2 \rangle \oplus \langle n \rangle(1/2, 1/2) \supset \langle -2 \rangle \oplus \langle n \rangle$ is the overlattice of the index 2 defined by

$$(2.17) \quad \langle -2 \rangle \oplus \langle n \rangle(1/2, 1/2) = [e, e/2 + P/2] = \left\langle \begin{array}{cc} -2 & -1 \\ -1 & \frac{n-2}{4} \end{array} \right\rangle.$$

Since $U$ and $E_8$ are unimodular and the lattice $(2.17)$ is unimodular over 2, it follows

$$(2.18) \quad q_{L^{\varphi,P}} = q \left( \left\langle \begin{array}{cc} -2 & -1 \\ -1 & \frac{n-2}{4} \end{array} \right\rangle \right) = \oplus_{\text{odd } p \mid n} q(n)_p.$$
Here we denote by $q(S)$ (or $b(S)$) the discriminant quadratic (or bilinear) form of a lattice $S$.

Then the group $O(L^\varphi)_P$ is again trivial, and we obtain similarly to the previous case

**Theorem 6.** If $(r, a, \delta_\varphi, \delta_P) = (19, 1, 1, 0)$ (equivalently, $X(\mathbb{R}) = T_1 \Pi (T_0)^8$ and $\delta_P = 0$), then $\delta_\varphi P = 0$, $n$ any positive integer such that $n \equiv 2 \mod 8$, $k$ any positive integer, and the isomorphism classes of integral polarized K3 involutions (equivalently connected components of moduli of real polarized K3 surfaces) with these invariants are in one to one correspondence with pairs

$$(L_{\varphi,P}, \tau)$$

where $L_{\varphi,P}$ is any lattice of the genus $\begin{pmatrix} 2 & 1 \\ 1 & 2 -n/4 \end{pmatrix}$, and

$$\tau : q_{L_{\varphi,P}} \to q \left( \begin{pmatrix} 2 & 1 \\ 1 & 2 -n/4 \end{pmatrix} \right) = \bigoplus_{\text{odd } p \mid n} q(-n)_p$$

is any isomorphism of quadratic forms considered up to the natural action of $O(L_{\varphi,P})$.

Thus, the number of connected components of moduli is equal to

$$\sum_{L_{\varphi,P} \text{ of genus } \begin{pmatrix} 2 & 1 \\ 1 & 2 -n/4 \end{pmatrix}} \sharp(O(q_{L_{\varphi,P}}) / O(L_{\varphi,P})).$$

We define the standard connected component of moduli when $L_{\varphi,P} = \begin{pmatrix} 2 & 1 \\ 1 & 2 -n/4 \end{pmatrix}$ and $\tau$ is induced by the identity isomorphism of the lattice.

In this case, any connected component of moduli which is different from the standard only over 2 coincides with the standard one.

For example, Theorem 6 gives only one connected component of moduli which is the standard one if $n = 2$.

2.2.2. The case $(r, a, \delta_\varphi) = (19, 3, 1)$. By Theorem 3 relation I,(5), we have: if $\delta_\varphi P = 0$, then $n \equiv 2 \mod 16$. It is the only relation between genus invariants $(2.1)$ which we have in this case.

We apply a modification of the general construction above.

Since $(r, a, \delta_\varphi) = (19, 3, 1)$, the lattice $L_\varphi$ where $\varphi = -1$ is an even 2-elementary lattice of signature $(2,1)$ and with $a = 3$. It follows that $L_\varphi(1/2)$ is an even unimodular lattice of signature $(2,1)$. This lattice is odd, and we have

$$L_\varphi(1/2) \cong (1) \oplus (1) \oplus (-1).$$

by classification of unimodular indefinite lattices.

The element $P \in L_\varphi(1/2)$ is then a primitive element with $P^2 = n/2$. Since both lattices $L^\varphi$ and $L_\varphi$ are unique in their genus and for both of them the canonical homomorphisms

$$O(L^\varphi) \to O(q_{L^\varphi}), \quad O(L_\varphi) \to O(q_{L_\varphi})$$

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are epimorphic, any automorphism of $L\varphi$ can be extended to an automorphism of $L$ which commutes with the involution $\varphi$.

Thus, two integral polarized involutions are isomorphic if and only if the corresponding to them elements $P \in L\varphi(1/2)$ are conjugate by $O(L\varphi)$. The invariant $\delta_{\varphi P}$ has then the following meaning. We have $P_{L\varphi(1/2)}^+ = L\varphi, P(1/2)$ is odd if $\delta_{\varphi P} = 1$, and it is even if $\delta_{\varphi P} = 0$. The last case is possible only if $n \equiv 2 \mod 16$.

The sublattice $\mathbb{Z}P \subset L\varphi(1/2)$ has the discriminant bilinear form $b_{(n/2)}$. Then the lattice $L\varphi, P(1/2)$ has the discriminant bilinear form $-b_{(n/2)}$ and signature $(1, 1)$.

If $\delta_{\varphi P} = 1$, this lattice is odd, and has the genus of the lattice
\[\langle 1 \rangle \oplus \langle -n/2 \rangle.\]
Thus $L\varphi, P$ has the genus of the lattice
\[\langle 2 \rangle \oplus \langle -n \rangle.\]

If $\delta_{\varphi P} = 0$, we should change the lattice $\langle 1 \rangle \oplus \langle -n/2 \rangle$ to make it even, but having the same bilinear discriminant form. Let us denote by $e_1$ and $e_2$ the generators of $\langle 1 \rangle$ and $\langle -n/2 \rangle$ respectively. The new lattice will be $[2e_1, 2e_2, (e_1 + e_2)/2]$. This changes the lattice only over $2$. The new lattice has the basis $\{2e_1, (e_1 + e_2)/2\}$, and has the matrix
\[\begin{pmatrix} 4 & 1 & -n/8 \\ 1 & 2 -n/8 & -n/8 \end{pmatrix}.\]

It is even and unimodular over 2 if $n \equiv 2 \mod 16$. Thus, the lattice $L\varphi, P$ has the genus of the lattice
\[\langle 8 \rangle \oplus \langle 2, 2-n/4 \rangle.\]

Finally, we obtain

**Theorem 7.** If $(r, a, \delta_{\varphi}, \delta_{\varphi P}) = (19, 3, 1, 1)$ (equivalently, $X(\mathbb{R}) = (T_0)^9$ and $X(\mathbb{R}) \not\sim P$ mod 2 in $H_2(X(\mathbb{C}), \mathbb{Z})$), then $\delta_P = 0$, $n$ any positive even integer, $k$ any positive integer, and the isomorphism classes of integral polarized K3 involutions (equivalently connected components of moduli of real polarized K3 surfaces) with these invariants are in one to one correspondence with pairs
\[(L\varphi, P, \tau)\]
where $L\varphi, P$ is any lattice of the genus $\langle 2 \rangle \oplus \langle -n \rangle$, and
\[\tau : b_{L\varphi, P(1/2)} \to b_{(1) \oplus \langle -n/2 \rangle} = b_{\langle -n/2 \rangle}\]
is any isomorphism of bilinear forms considered up to the natural action of $O(L\varphi, P)$.

Thus, the number of connected components of moduli is equal to
\[\sum_{L\varphi, P \text{ of genus } \langle 2 \rangle \oplus \langle -n \rangle} \sharp(O(b_{L\varphi, P(1/2)})/O(L\varphi, P)).\]
We define the standard connected component of moduli when \( L_{\varphi,P} = \langle 2 \rangle \oplus \langle -n \rangle \) and \( \tau \) is induced by the identity isomorphism of the lattice.

We remark that the 2-component \( O(b_{(-n/2),2}) \) is trivial if \( n \equiv 2 \mod 4 \) or \( n \equiv 4 \mod 8 \), it is \( \mathbb{Z}/2 \) if \( n \equiv 8 \mod 16 \), and it is \( \mathbb{Z}/2 \times \mathbb{Z}/2 \) if \( n \equiv 0 \mod 16 \).

Thus, the number of connected components of moduli which are different from the standard only over 2 is one if \( n \equiv 2 \mod 4 \) or \( n \equiv 4 \mod 8 \); it is at most two if \( n \equiv 8 \mod 16 \); it is at most four if \( n \equiv 0 \mod 16 \).

For example, Theorem 7 gives only one connected component of moduli which is the standard one if \( n = 2 \), 4, 6 or 8.

**Theorem 8.** If \((r, a, \delta, \delta_P) = (19, 3, 1, 0)\) (equivalently, \( X(\mathbb{R}) = (T_0)^9 \) and \( X(\mathbb{R}) \sim P \mod 2 \) in \( H_2(X(\mathbb{C}), \mathbb{Z}) \)), then \( \delta_P = 0 \), \( n \) any positive integer such that \( n \equiv 2 \mod 16 \), \( k \) any positive integer, and the isomorphism classes of integral polarized K3 involutions (equivalently connected components of moduli of real polarized K3 surfaces) with these invariants are in one to one correspondence with pairs

\[(L_{\varphi,P}, \tau)\]

where \( L_{\varphi,P} \) is any lattice of the genus

\[
\begin{pmatrix}
8 & 2 \\
2 & 2-n/4
\end{pmatrix},
\]

and

\[
\tau : b_{L_{\varphi,P}(1/2)} \to b \left( \begin{pmatrix}
4 & 1 \\
1 & 2-n/8
\end{pmatrix} \right) = b_{(-n/2)}
\]

is any isomorphism of bilinear forms considered up to the natural action of \( O(L_{\varphi,P}) \).

Thus, the number of connected components of moduli is equal to

\[
\sum_{L_{\varphi,P} \text{ of genus} \langle 8, 2, 2-n/4 \rangle} \#(O(b_{L_{\varphi,P}(1/2)})/O(L_{\varphi,P})).
\]

We define the standard connected component of moduli when \( L_{\varphi,P} = \langle 8, 2, 2-n/4 \rangle \) and \( \tau \) is induced by the identity isomorphism of the lattice.

Since \( n \equiv 2 \mod 4 \) in this case, any connected component of moduli which is different from the standard only over 2 is the standard one.

For example, Theorem 8 gives only one connected component of moduli which is the standard one if \( n = 2 \).
2.2.3. The case \( (r, a, \delta_\varphi) = (20, 2, 1) \). By Theorem \( \ref{thm:n} \) relations I.(4),(11),(15),(19), we have: 
\[
\delta_P = 0; \quad n = 2^\epsilon p_1^{\alpha_1} \cdots p_m^{\alpha_m}, \quad \text{where } 1 \leq \epsilon \leq 2, \ p_i \text{ a prime, } p_i \equiv 1 \text{ mod } 4; \quad \delta_\varphi P = 1 \text{ if } n \equiv 2 \text{ mod } 4 \ (\text{i. e. } \epsilon = 1), \text{ and } \delta_\varphi P = 0 \text{ if } n \equiv 0 \text{ mod } 4 \ (\text{i. e. } \epsilon = 2).
\]

This case is very similar to the previous one. Since \( (r, a, \delta_\varphi) = (20, 2, 1) \), the lattice \( L_\varphi \) is a positive definite even 2-elementary lattice of the rank 2 and with \( a = 2 \). It follows that \( L_\varphi(1/2) \) is an even unimodular positive definite lattice of the rank 2. This lattice is odd and \( L_\varphi(1/2) \cong (1) \oplus (1) \) by classification of unimodular lattices of a small rank.

We have the same results as for previous case.

Two integral polarized involutions are isomorphic if and only if the corresponding to them elements \( P \in L_\varphi(1/2) \) are conjugate by \( O(L_\varphi) \). We have \( P_{L_\varphi(1/2)}^\perp = L_\varphi P(1/2) \) is odd if \( \delta_\varphi P = 1 \), and it is even if \( \delta_\varphi P = 0 \).

The sublattice \( \mathbb{Z}P \subset L_\varphi(1/2) \) has the discriminant bilinear form \( b_{(n/2)} \). Then the lattice \( L_\varphi P(1/2) \) has the discriminant bilinear form \(-b_{(n/2)}\), and it is positive definite of the rank one. It follows that \( L_\varphi P \cong \langle n \rangle \), and the bilinear form \( b_{(n/2)} \) is isomorphic to \(-b_{(n/2)}\). This is equivalent to the condition on \( n \) above.

Thus, we obtain

**Theorem 9.** If \( (r, a, \delta_\varphi) = (20, 2, 1) \) (equivalently, \( X(\mathbb{R}) = (T_0)^{10} \)), then \( \delta_P = 0; \ \delta_\varphi P = 1 \) (equivalently, \( X(\mathbb{R}) \not\sim P \) mod 2 in \( H_2(X(\mathbb{C}), \mathbb{Z}) \)) if \( n \equiv 2 \) mod 4, and \( \delta_\varphi P = 0 \) (equivalently, \( X(\mathbb{R}) \sim P \) mod 2 in \( H_2(X(\mathbb{C}), \mathbb{Z}) \)) if \( n \equiv 0 \) mod 4; \( n = 2^\epsilon p_1^{\alpha_1} \cdots p_m^{\alpha_m}, \) where \( \epsilon = 1 \) or 2, \( p_i \) a prime, \( p_i \equiv 1 \) mod 4; \( k \) any positive integer. Isomorphism classes of integral polarized K3 involutions (equivalently connected components of moduli of real polarized K3 surfaces) with these invariants are in one to one correspondence with isomorphisms \( \tau : b_{(n/2)} \to -b_{(n/2)} \) of finite bilinear forms considered up to \( \pm 1 \). (The lattice \( L_\varphi P = \langle n \rangle \). )

Thus, the number of connected components of moduli is equal to \( 2^{\max\{0,m-1\}} \) where \( m \) is the number of different odd prime divisors of \( n \).

In this case, we don’t have a notion of a standard connected component of moduli.

## 3. Deformations of real hyper-elliptically polarized K3 surfaces.

Here we apply results above to classify real polarized K3 surfaces which are deformations of real hyper-elliptically polarized K3 surfaces. This question had been studying in \[13\]. Using results of Section 2 we will be able to finalize these results.

First let us formulate the problem exactly. Let \((X, \widetilde{P})\) be a K3 surface with an ample divisor class \( \tilde{P} \) (not necessarily very ample) such that the linear system \(|\tilde{P}|\) does not have fixed components. Let \( P \) be the corresponding primitive element of the Picard lattice of \( X \) such that \( \tilde{P} = mP \) where \( m \in \mathbb{N} \). As above, \( n = P^2 \) is the primitive degree. A pair \((X, \widetilde{P})\) is called a hyper-elliptically polarized K3 surface if the complete linear system \(|\tilde{P}|\) does not give an embedding of \( X \) into a projective space.

Let \((X, \widetilde{P})\) be a hyper-elliptically polarized K3 surface. By the results of Saint-Donat [15], a general curve of the linear system \(|\tilde{P}|\) is hyper-elliptic in this case, and the linear system \(|\tilde{P}|\) gives a double covering \(|\tilde{P}| : X \to Y \subset \mathbb{P}^N \) onto a rational surface \( Y \) where
\( N = \dim |\tilde{P}| = \tilde{P}^2/2 + 1 = m^2n/2 + 1 \). Taking \( P' = kP \) where \( k > m \) a sufficiently large (by \cite{15}, it is enough to take \( k \geq 3 \)), we obtain a polarized K3 surface \((X, kP)\) (i.e. \( kP \) is very ample). It is naturally also called hyper-elliptically polarized. Then any polarized K3 surface in the same connected component of moduli as \((X, kP)\) can be considered as a deformation of the hyper-elliptically polarized K3 surface \((X, kP)\) (or just of the \((X, \tilde{P})\)).

It is easy to see that any complex polarized K3 surface is such a deformation because the moduli space of complex polarized K3 surfaces \((X, kP)\) of the degree \( n = P^2 > 0 \) is connected by Global Torelli Theorem for K3 surfaces \cite{14}. It is empty if \( n = 2 \) and \( k = 1 \) or 2.

We ask similar question for real polarized K3 surfaces: \textit{When a real polarized K3 surface is a deformation a hyper-elliptically polarized K3 surface?} It could be important because reduces some questions to hyper-elliptically polarized K3 surfaces which are much simpler and can be constructed explicitly.

We can reformulate this question as follows. Let us denote by \( M_{n,k} \) the moduli space of real polarized K3 surfaces \((X, kP)\) where \( P \) is primitive, \( P^2 = n \) and \( k \in \mathbb{N} \) (i.e. \( kP \) is very ample). When \( n = 2 \), we assume that \( k \geq 3 \), since \( M_{2,1} \) and \( M_{2,2} \) are empty. We have the obvious embedding of moduli spaces

\[(3.1) \quad M_{n,k_1} \subset M_{n,k_2} \text{ if } k_1 \leq k_2 \]

and the induced map for the sets of their connected components of moduli. By Theorem 2 \cite{14} gives an isomorphism on the sets of connected components. The connected components of \( M_{n,k} \) are the same for any \( k \). Moreover, it is known that difference in \( M_{n,k} \) for different \( k \) is only in codimension \( \geq 1 \) (the corresponding K3 surfaces have Picard number \( \geq 2 \)). Thus, even when \( M_{n,k} \) does not have hyper-elliptically polarized K3 surfaces, we still can consider these K3 surfaces as deformations of hyper-elliptically polarized K3 surfaces if the same connected component of moduli of \( M_{n,k_2} \) for \( k_2 > k \) has hyper-elliptically polarized K3 surfaces. Thus, our question does not depend on \( k \) and can be formulated as follows:

\textit{Which connected components of } \( M_{n,k} \) \textit{for } \( k \gg 0 \) \textit{contain hyper-elliptically polarized K3 surfaces }\((X, kP)\), \textit{i.e. } \( P \) \textit{is primitive and the linear system }\{|\tilde{P}|\} \textit{is hyper-elliptic for } \tilde{P} = mP \textit{where } m = 1 \textit{ or } 2?\]

This question had been studying in \((\cite{13}, \text{Sect. } 8)\). Using results of Section 2 here we finalize these results.

The exposition in \((\cite{13}, \text{Sect. } 8)\) was very short, and first we clarify the general considerations in \cite{13}.

By Theorem 5.2 in \cite{15}, for an ample \( \tilde{P} = mP \), the linear system \(|\tilde{P}|\) is hyper-elliptic in the following and only the following cases:

(i) \( n = 2 \), \( m = 1 \) or 2. Then \( Y = \mathbb{P}^2 \), \(|\tilde{P}| : X \to Y = \mathbb{P}^2\) is a double covering ramified in a degree 6 curve, \( P \) is the preimage of a line in \( \mathbb{P}^2 \).

(ii) \( n \geq 4 \), there exists an elliptic curve \( C \) on \( X \) such that \( \tilde{P} \cdot C = 2 \).
It follows that for \( n = 2 \) all polarized K3 surfaces are hyper-elliptic, and then they are deformations of hyper-elliptic ones.

Let us assume that \( n \geq 4 \). Since \( C \) is an elliptic curve, \( C^2 = 0 \). Thus \( \tilde{P} \) and \( C \) have the Gram matrix \( 2 \begin{pmatrix} n/2 & 1 \\ 1 & 0 \end{pmatrix} \) where the matrix \( \begin{pmatrix} n/2 & 1 \\ 1 & 0 \end{pmatrix} \) has the determinant \(-1\), and it is unimodular. It follows that the 2-dimensional primitive sublattice \( S \subset S_X \) generated by \( \tilde{P} \) and \( C \) in the Picard lattice \( S_X \) of \( X \) is a 2-elementary lattice, i.e. \( S^* / S \cong (\mathbb{Z}/2\mathbb{Z})^a \) where \( a \leq 2 \).

Since \( P \) is ample, \( P_{S_X}^1 \) has no elements with square \(-2\). It follows that \( S_{S_X}^1 \) also has no elements with square \(-2\). By Global Torelli Theorem for K3 surfaces [14], there exists an involution \( \tau \) of \( X \) which is identity on \( S \) and which is \(-1\) on the orthogonal complement \( S^1 \) in \( H_2(X(\mathbb{C}), \mathbb{Z}) \). This involution is non-symplectic, \( Y = X/\{1, \tau\} \) is a rational surface and the quotient map \( \pi : X \to Y \) is a double covering of \( Y \) ramified in a non-singular curve \( A \in |--2KY|\). See [1] for details. Depending on the isomorphism class of \( S \), we obtain 5 cases (see the general classification in [1], or see [13]):

**Case** \( \mathbb{F}_1 \): \( S \cong \langle 2 \rangle \oplus \langle -2 \rangle \). Then \( Y = X/\{1, \tau\} = \mathbb{F}_1 \) is a blow-up of \( \mathbb{P}^2 \) in one point. Denoting by \( c \) the class of \( C \), we obtain that the elliptic pencil \(|c|\) is the preimage of the rational pencil on \( \mathbb{F}_1 \). We denote by \( h \) the class of the preimage of a line \( l \) in \( \mathbb{P}^2 \) and by \( e \) the class of the preimage of the exceptional section \( s \) of \( \mathbb{F}_1 \). Thus, we have \( h^2 = 2, e^2 = -2 \) and \( h \cdot e = 0 \). Then \( c = h - e \). Since \( \tilde{P} \cdot c = 2 \) and \( \tilde{P} \cdot e > 0 \), it follows that \( \tilde{P} = P = n_1c + e = n_1h + (1 - n_1)e \) where \( n = P^2 = 4n_1 - 2 \equiv 2 \mod 4 \) and \( n \geq 6 \). By Riemann-Roch Theorem on \( \mathbb{F}_1 \), it is easy to see that the linear system \(|P|\) is the preimage of the linear system from \( \mathbb{F}_1 \). It follows that the map \(|\tilde{P}| : X \to Y \subset \mathbb{P}^N\) is the quotient map.

**Case** \( \mathbb{F}_4 \): \( S \cong U \). Then \( Y = X/\{1, \tau\} = \mathbb{F}_4 \) where \( \mathbb{F}_4 \) is a relatively minimal rational surface with the exceptional section \( s \) such that \( s^2 = -4 \). We denote \( E = \pi^{-1}(s) \) and by \( e \) its class. The elliptic pencil \(|C|\) is the preimage by \( \pi \) of the the rational pencil of \( \mathbb{F}_4 \). We denote its class by \( c \). Then \( c^2 = 0, e^2 = -2 \) and \( c \cdot e = 1 \). Since \( mP \cdot c = 2 \) and \( P \cdot e > 0 \), we get two cases:

**Case** \((\mathbb{F}_4)^{(1)}\): \( m = 2 \) and \( \tilde{P} = 2P \) where \( P = n_1c + e \) where \( n_1 \geq 3 \). Then \( n = P^2 = 2n_1 - 2 \). We have \( n \equiv 0 \mod 2 \) and \( n \geq 6 \).

**Case** \((\mathbb{F}_4)^{(2)}\): \( m = 1 \) and \( \tilde{P} = P = n_1C + 2E \) where \( n_1 \geq 5 \) and \( n_1 \equiv 1 \mod 2 \). Then \( n = P^2 = 4n_1 - 8 \). We have \( n \equiv 4 \mod 8 \) and \( n \geq 12 \).

Using Riemann-Roch Theorem on \( \mathbb{F}_4 \), it is easy to see that in both these cases the linear system \(|\tilde{P}|\) is the preimage by \( \pi \) of the corresponding linear system from \( \mathbb{F}_4 \). It follows that \(|\tilde{P}| : X \to Y \subset \mathbb{P}^N\) is the quotient map.

**Case** \( \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1 \): \( S \cong U(2) \) and \( Y = X/\{1, \tau\} \cong \mathbb{P}^1 \times \mathbb{P}^1 \). We denote by \( e_1 \) and \( e_2 \) classes of preimages of \( pt \times \mathbb{P}^1 \) and \( \mathbb{P}^1 \times pt \). We have \( e_1^2 = e_2^2 = 0 \) and \( e_1 \cdot e_2 = 2 \). Then \( C \) has the class of \( e_1 \) or \( e_2 \), and we obtain two cases. Either \( \tilde{P} = P = n_1e_1 + e_2, n_1 \geq 1 \) (case \((\mathbb{F}_0)^{(1)}\)) or \( \tilde{P} = P = e_1 + n_1e_2, n_1 \geq 1 \) (case \((\mathbb{F}_0)^{(2)}\)). Then \( n = P^2 = 4n_1 \) where \( n \equiv 0 \mod 4 \) and \( n \geq 4 \). In both cases the linear system \(|\tilde{P}|\) is the preimage of the corresponding
linear system from $\mathbb{F}_0$. It follows that $|\tilde{P}| : X \to Y \subset \mathbb{P}^N$ is the quotient map. (Over $\mathbb{R}$, it is important to distinguish the cases $(\mathbb{F}_0)_{(1)}$ and $(\mathbb{F}_0)_{(2)}$ which are symmetric over $\mathbb{C}$.)

**Case $\mathbb{F}_2$:** $S \cong U(2)$ and $Y = X/\{1, \tau\} \cong \mathbb{F}_2$. This case can be considered as the degeneration of the previous one. We denote by $c$ the class of the preimage of the exceptional section of the relatively minimal rational surface $\mathbb{F}_2$ (it is union of two disjoint non-singular rational curves on $X$ which are conjugate by $\tau$; then the Picard lattice of $X$ has the rank at least three), and by $e$ the class of the preimage of the rational pencil of $\mathbb{F}_2$. We have $c^2 = -4$, $c^2 = 0$ and $c \cdot e = 2$. Then $C$ has the class $c$, and $\tilde{P} = P = n_1 c + e$ where $n_1 \geq 3$. Then $n = 4n_1 - 4$ where $n \equiv 0 \mod 4$ and $n \geq 8$. Again $|\tilde{P}|$ is the preimage of the corresponding linear system from $\mathbb{F}_2$ and $|\tilde{P}| : X \to Y \subset \mathbb{P}^N$ is the quotient map.

These considerations show that classification of hyper-elliptically polarized K3 surfaces $(X, \tilde{P})$ is equivalent to classification of K3 surfaces with non-symplectic involution $(X, \tau)$ when $Y = X/\{1, \tau\} \cong \mathbb{F}_2$ or $\mathbb{F}_r$ for $r = 0, 1, 2, 4$, by picking up $|\tilde{P}| = \pi^*|Q|$ where $|Q|$ is the appropriate linear system of $Y$. Considering their deformations gives the corresponding polarized K3 surfaces we are looking for. The following statement shows that we can drop $\mathbb{F}_2$ from the consideration.

**Lemma 10.** Any real polarized K3 surface which is a deformation of a real hyper-elliptically polarized K3 surface coming from $\mathbb{F}_2$ is also a deformation of a real hyper-elliptically polarized K3 surface coming from $\mathbb{F}_0$.

**Proof.** Let $(X, \tilde{P})$ be a real hyper-elliptically polarized K3 surface where $\tilde{P} \in S \cong U(2)$ comes from $Y = X/\{1, \tau\}$ where $Y$ is $\mathbb{F}_2$. Let $\varphi$ be the corresponding anti-holomorphic involution on $X$ which commutes with $\tau$. Thus, the triplet $(X, \tau, \varphi)$ is the real K3 surface with the non-symplectic involution $\tau$. If $Y \cong \mathbb{F}_2$, the Picard lattice $S_X$ contains $S$, but it has the rank at least 3. Changing complex structure of $X$ a little and applying Global Torelli Theorem for K3 surfaces [14], we can construct another real K3 surface with a non-symplectic involution $(\tilde{X}, \tilde{\tau}, \tilde{\varphi})$ which has the same action of involutions on $H_2(\tilde{X}, \mathbb{Z})$, but has the Picard lattice $S_{\tilde{X}} = S$ of the rank two. Then $\tilde{X}/\{1, \tilde{\tau}\} \cong \mathbb{F}_0$, and the same class $\tilde{P}$ comes from $\mathbb{F}_0$. Both these real polarized K3 surfaces give the same integral polarized K3 involutions $(L, \varphi = \tilde{\varphi}, \tilde{P})$ and, by Theorem 2 the same connected component of moduli of real polarized K3 surfaces. This finishes the proof. ☐

Hyper-elliptically polarized K3 surfaces $(X, \tilde{P})$ where $\tilde{P}$ comes from $Y = X/\{1, \tau\}$ where $Y \cong \mathbb{F}_r$, $r = 0, 1, 4$, were called in [13] as *general K3 double rational scrolls* because $|\tilde{P}|$ defines an embedding $\mathbb{F}_r = Y \subset \mathbb{P}^N$ as a rational scroll (the degree of $Y$ is equal to $N - 1$, see [15]), and $X$ is its double covering having in general the smallest possible Picard number 2. By above considerations and Lemma 10 deformations of real hyper-elliptically polarized K3 surfaces (which we want to classify) are equivalent to deformations of general real K3 double rational scrolls.
In [13], classification of connected components of moduli of general real K3 double rational scrolls had been obtained. Their K3 polarized deformations had been studying. Using results of Sect. 2, here want to finalize the results about deformations.

The main result of [13] about deformations is

**Theorem 11.** (see Theorem 30 in [13]) A genus invariant (2.1) of a real polarized K3 surface can be obtained as a deformation of some general real K3 double rational scroll (equivalently, of a real hyper-elliptically polarized K3 surface) if and only if the following condition is valid:

\[ n \leq 4 \text{ if either } (r, a) = (20, 2) \text{ or } r + a = 22 \text{ and } \delta_{\varphi P} = 0. \]

Equivalently,

\[ n \leq 4 \text{ if either } X(\mathbb{R}) = (T_0)^{10} \text{ or } X(\mathbb{R}) = (T_0)^k \text{ and } X(\mathbb{R}) \sim P \text{ mod } 2 \text{ in } H_2(X(\mathbb{C}), \mathbb{Z}). \]

Since for \( n = 2 \) or \( n = 4 \), the genus invariants give only one connected component (see Sect. 2), we then obtain from Theorem 13 the following result which is trivial for \( n = 2 \) and had been proved for \( n = 4 \) in [13].

**Theorem 12.** (see [13]) Any real polarized K3 surface \((X, P')\) of the primitive degree \( n = 2 \) or \( n = 4 \) is a deformation of a general real K3 double rational scroll (equivalently, of a real hyper-elliptically polarized K3 surface).

Thus, further we can assume that \( n \geq 6 \). Applying the connectedness Theorem 4, from Theorem 11 we get the result.

**Theorem 13.** Let \((X, P')\) be a real polarized K3 surface of the primitive degree \( n \geq 6 \), and \( X(\mathbb{R}) \neq T_1 \coprod (T_0)^8, (T_0)^9 \).

Then \( X \) is a deformation of a general real K3 double rational scroll (equivalently of a real hyper-elliptically polarized K3 surface) if and only if \( X(\mathbb{R}) \neq (T_0)^{10} \) and \( X(\mathbb{R}) \not\sim P \text{ mod } 2 \text{ in } H_2(X(\mathbb{C}), \mathbb{Z}) \) when \( X(\mathbb{R}) = (T_0)^k \).

Now let us assume that \( n \geq 6 \) and \( X(\mathbb{R}) = T_1 \coprod (T_0)^8 \). Equivalently, \((r, a, \delta_{\varphi}) = (19, 1, 1)\). Then we should apply invariants of Theorems 3 and 6.

By results of [13] (especially see Sect. 8.1 in [13]), there are three types of the deformations, and they can be described symbolically as shown (see also considerations above).

**The Case of** \((\mathbb{F}_4)^{(1)}\): \( n \in (\mathbb{F}_4)^{(1)} \) if and only if \( n \equiv 0 \mod 2 \) and \( n \geq 4 \); \( P = (n/2+1)C + E \); \( D_n : ((\mathbb{F}_4)^{(1)}; r = 19, a = 1, H = 0, \delta_{\varphi S} = 1) \Rightarrow (n; r = 19, a = 1, \delta_P = 1, \delta_{\varphi} = 1, \delta_{\varphi P} = 1). \)

In this case

\[ L = [C, E] \oplus [g_1, g_2; (g_1 + g_2)/2] \oplus U \oplus 2E_8 \]

where \( C^2 = 0, E^2 = -2 \) and \( C \cdot E = 1, g_1^2 = 2, g_2^2 = -2 \) and \( g_1 \perp g_2 \). The involutions \( \tau \) and \( \varphi \) are characterized by their eigenvalue 1 parts \( L^\tau = [C, E] \) and \( L^\varphi = [g_2] \oplus U \oplus 2E_8 \). Then the triplet \((L, \tau, \varphi)\) has the required invariants.
The lattice \( L^\varphi \cdot P = \mathbb{Z}P \oplus \mathbb{Z}g_2 \oplus U \oplus 2E_8 \) is \( \langle n \rangle \oplus \langle -2 \rangle \) modulo the unimodular lattice \( U \oplus 2E_8 \). Here \( P \) and \( g_2 \) are the standard generators of \( \langle n \rangle \) and \( \langle -2 \rangle \) respectively. Its orthogonal complement in \( L \) is \( L_{\varphi, P} = \mathbb{Z}Q \oplus \mathbb{Z}g_1 \) where \( Q = (n/2 - 1)C - E \) and \( Q^2 = -n \). It follows that \( L_{\varphi, P} = \langle -n \rangle \oplus \langle 2 \rangle \) where \( Q \) and \( g_2 \) are the standard generators of \( \langle -n \rangle \) and \( \langle 2 \rangle \) respectively.

We have \( Q/n + P/n = C \in L \) and \( g_2/2 + g_1/2 = (g_1 + g_2)/2 \in L \). It follows that the connected component of moduli is standard.

The Case of \( (\mathbb{F}_4)^{(2)} \): \( n \in (\mathbb{F}_4)^{(2)} \) if and only if \( n \equiv 4 \mod 8 \) and \( n \geq 12 \); \( P = (n/4 + 2)C + 2E \);
\[ D_n : ((\mathbb{F}_4)^{(2)}; r = 19, a = 1, H = 0, \delta_{\varphi S} = 1) \implies (n; r = 19, a = 1, \delta_P = 1, \delta_{\varphi} = 1, \delta_{\varphi P} = 1) \]

In this case, everything is the same as in the previous one, only \( P = (n/4 + 2)C + 2E \) and \( Q = (n/4 - 2)C - 2E \). Thus,
\[ \frac{P}{n/2} + \frac{Q}{n/2} = C \in L, \quad \frac{P}{4} + \left( -\frac{Q}{4} \right) = C + E \in L \]
where \( n \equiv 4 \mod 8 \). It follows that the connected component of moduli is different from the standard one by \( (1) \) in the 2-component of the discriminant form.

The Case of \( \mathbb{F}_1 \): \( n \in \mathbb{F}_1 \) if and only if \( n \equiv 2 \mod 4 \) and \( n \geq 6 \); \( P = ((n + 2)/4)h + \langle (2 - n)/4 \rangle e \);
\[ D_n : (\mathbb{F}_1; r = 19, a = 1, H = [h], \delta_{\varphi S} = 0, v = h) \implies (n; r = 19, a = 1, \delta_P, \delta_{\varphi} = 1, \delta_{\varphi P}) \] where \( \delta_{P} = \delta_{\varphi P} = \begin{cases} 
0 & \text{if } n \equiv 2 \mod 8 \\
1 & \text{if } n \equiv -2 \mod 8 
\end{cases} \)

In this case, the lattice \( L = [h, e, g_1, g_2; (h + g_2)/2, (e + g_1)/2] \oplus U \oplus 2E_8 \)
where \( h, e, g_1, g_2 \) are orthogonal to each other and \( h^2 = 2, \ e^2 = -2, \ g_1^2 = 2, \ g_2^2 = -2 \). The involutions \( \tau \) and \( \varphi \) are characterized by their eigenvalue 1 parts \( L^\tau = [h, e] \) and \( L^\varphi = \mathbb{Z}g_2 \oplus U \oplus 2E_8 \). Then the triplet \( (L, \tau, \varphi) \) has the required invariants.

Assume that \( n \equiv -2 \mod 8 \). Then \( L^\varphi \cdot P = \mathbb{Z}P \oplus \mathbb{Z}g_2 \oplus U \oplus 2E_8 \) which is \( \langle n \rangle \oplus \langle -2 \rangle \) modulo the unimodular lattice \( U \oplus 2E_8 \). Here \( P \) and \( g_2 \) are the standard generators of \( \langle n \rangle \) and \( \langle -2 \rangle \) respectively. Its orthogonal complement in \( L \) is \( L_{\varphi, P} = \mathbb{Z}Q \oplus \mathbb{Z}g_1 \) where \( Q = ((n - 2)/4)h - ((n + 2)/4)e \) and \( Q^2 = -n \). Thus, \( L_{\varphi, P} = \langle -n \rangle \oplus \langle 2 \rangle \) where \( P \) and \( g_1 \) are the standard generators of \( \langle -n \rangle \) and \( \langle 2 \rangle \) respectively.

We have
\[ e = \frac{n - 2}{2n} P - \frac{n + 2}{2n} Q, \quad h = \frac{n + 2}{2n} P + \frac{2 - n}{2n} Q. \]
It follows
\[ \frac{P}{n/2} + \frac{Q}{n/2} = h - e \in L, \]
and
\[-\frac{n}{2} \left( \frac{g_2}{2} \right) + \frac{n - 2}{4} \left( \frac{Q}{2} \right) = -\frac{n}{2} \left( \frac{h + g_2}{2} \right) + \frac{n + 2}{8} P \in L.\]

This shows that the connected component of moduli is different from the standard one by the non-trivial automorphism of the 2-component of the discriminant form which has the automorphism group \( \mathbb{Z}/2 \) in this case.

If \( n \equiv 2 \mod 8 \), then the 2-component of the discriminant form is trivial. In this case, our (or the same) calculations show that the connected component is standard.

Thus, we finally get the result.

**Theorem 14.** Let \((X, P')\) by a real polarized \(K3\) surface of the primitive degree \(n \geq 6\), and \(X(\mathbb{R}) = T_1 \Pi (T_0)^8\).

Then \(X\) is a deformation of a general real \(K3\) double rational scroll (equivalently, of a real hyper-elliptically polarized \(K3\) surface) if and only if the following conditions satisfy
(a) if \(n \equiv 0, 2 \mod 8\), then \((X, P')\) is standard;
(b) if \(n \equiv 4, 6 \mod 8\), then \((X, P')\) is different from a standard only over 2.

Now let us assume that \(n \geq 6\) and \(X(\mathbb{R}) = (T_0)^9\). Equivalently, \((r, a, \delta) = (19, 3, 1)\). Then we should apply invariants of Theorems 7 since \(\delta_{\varphi P} = 1\).

By results of [13] (especially see Sect. 8.1 in [13]), there are two types of the deformations, and they can be described symbolically as shown (see also considerations above).

The Case of hyperboloid \(\mathbb{H}(1)\): \(n \in \mathbb{H}(1)\) if and only if \(n \equiv 0 \mod 4\) and \(n \geq 4\); \(P = (n/4)e_1 + e_2\);
\[D_n : (\mathbb{H}(1); r = 19, a = 3, H = [e_1, e_2], \delta_{\varphi S} = 1) \implies (n; r = 19, a = 3, \delta_P = 0, \delta = 1, \delta_{\varphi P} = 1).\]

In this case
\[L = [e_1, e_2, e_1', e_2'; (e_1 + e_1')/2, (e_2 + e_2')/2] \oplus [g, g'; (g + g')/2] \oplus 2E_8\]
where \(e_1^2 = e_2^2 = 0\), \(e_1 \cdot e_2 = 2\); \((e_1')^2 = (e_2')^2 = 0\), \(e_1' \cdot e_2' = -2\); \([e_1, e_2] \perp [e_1', e_2']\); \(g^2 = 2\), \((g')^2 = -2\), \(g \perp g'\). The involutions \(\tau\) and \(\varphi\) are characterized by \(L^\tau = [e_1, e_2]\) and \(L^\varphi = [e_1', e_2'] \oplus [g'] \oplus 2E_8\). Then the triplet \((L, \tau, \varphi)\) has the required invariants.

The lattice \(L_{\varphi} = [e_1, e_2, g]\). We have \(\mathbb{Z}P = \langle n \rangle\) where \(P = (n/4)e_1 + e_2\) is the standard generator of \(\langle n \rangle\). Its orthogonal complement in \(L_{\varphi}\) is \(L_{\varphi, P} = \mathbb{Z}Q \oplus \mathbb{Z}g\) where \(Q = (n/4)e_1 - e_2\) and \(Q^2 = -n\). Thus \(L_{\varphi, P} = \langle -n \rangle \oplus \langle 2 \rangle\) where \(Q\) and \(g\) are the standard generators of \(\langle -n \rangle\) and \(\langle 2 \rangle\) respectively. We have
\[Q \cdot n/2 + P \cdot n/2 = e_1 \in L.\]

It follows (see Theorem 7) that the connected component of moduli is standard.

The Case of \(\mathbb{F}_1\): \(n \in \mathbb{F}_1\) if and only if \(n \equiv 2 \mod 4\) and \(n \geq 6\); \(P = ((n + 2)/4)h + ((2 - n)/4) e;\)
\[D_n : (\mathbb{F}_1; r = 19, a = 3, H = [h, e], \delta_{\varphi S} = 1) \implies (n; r = 19, a = 3, \delta_P = 0, \delta = 1, \delta_{\varphi P} = 1).\]
In this case
\[
L = [h, e, h', e'; (h + h')/2, (e + e')/2] \oplus [g, g'; (g + g')/2] \oplus 2E_8
\]
where \(h^2 = 2, e^2 = -2, h \perp e; (h')^2 = -2, (e')^2 = 2, h' \perp e'; [h, e] \perp [h', e']; g^2 = 2, (g')^2 = -2, g \perp g'.\) The involutions \(\tau\) and \(\varphi\) are characterized by \(L^\tau = [h, e]\) and \(L^\varphi = [h', e'] \oplus [g'] \oplus 2E_8.\) Then the triplet \((L, \tau, \varphi)\) has the required invariants.

The lattice \(L_\varphi = [h, e, g]\). We have \(\mathbb{Z}P = \langle n \rangle\) where \(P = ((n + 2)/4)h + ((2 - n)/4)e\) is the standard generator of \(\langle n \rangle\). Its orthogonal complement in \(L_\varphi\) is \(L_{\varphi, P} = \mathbb{Z}Q \oplus \mathbb{Z}g\) where \(Q = ((n - 2)/4)h - ((n + 2)/4)e\) and \(Q^2 = -n\). Thus \(L_{\varphi, P} = \langle -n \rangle \oplus \langle 2 \rangle\) where \(Q\) and \(g\) are the standard generators of \(\langle -n \rangle\) and \(\langle 2 \rangle\) respectively. We have
\[
\frac{Q}{n/2} + \frac{P}{n/2} = h - e \in L.
\]

It follows (see Theorem [7]) that the connected component of moduli is standard.

Thus, we obtain the result.

**Theorem 15.** Let \((X, P')\) by a real polarized K3 surface of a primitive degree \(n \geq 6,\) and \(X(\mathbb{R}) = (T_0)^9.\)

Then \(X\) is a deformation of a general real K3 double rational scroll (i.e. of a real hyper-elliptically polarized K3 surface) if and only if \(X(\mathbb{R}) \not\sim P\) in \(H_2(X(\mathbb{C}), \mathbb{Z})\) and \((X, P')\) is standard.

Let us unify all these results in one final statement.

**Theorem 16.** A real polarized K3 surface \((X, P')\) is a deformation of a general real K3 double rational scroll (equivalently, of a real hyper-elliptically polarized K3 surface) if and only if one of conditions (i)—(iv) below satisfies:

(i) The primitive degree \(n = 2\) or \(4.\)

(ii) The primitive degree \(n \geq 6,\) and \(X(\mathbb{R}) \neq T_1 \Pi (T_0)^8, (T_0)^9, (T_0)^{10},\) and \(X(\mathbb{R}) \not\sim P\) mod 2 in \(H_2(X(\mathbb{C}), \mathbb{Z})\) if \(X(\mathbb{R}) = (T_0)^k.\)

(iii) The primitive degree \(n \geq 6,\) and \(X(\mathbb{R}) = T_1 \Pi (T_0)^8,\) and \((X, P')\) is standard if \(n \equiv 0, 2\) mod 8, and \((X, P')\) is different from standard only over 2 if \(n \equiv 0, 4, 6\) mod 8.

(iv) The primitive degree \(n \geq 6,\) and \(X(\mathbb{R}) = (T_0)^9,\) and \(X(\mathbb{R}) \not\sim P\) mod 2 in \(H_2(X(\mathbb{C}), \mathbb{Z})\), and \((X, P')\) is standard.

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