Perfect Folding of Graphs

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Perfect Folding of Graphs

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I. Introduction

Let $G = (V, E)$ be a graph, where $V$ is the set of its vertices and $E$ is the set of its edges. Two distinct vertices $u, v \in V$ are called independent if $\{u, v\}$ is not an edge in $G$. Two vertices $u, v$ are called neighbors (adjacent) if $\{u, v\}$ is an edge in $G$. The degree (valency) of a vertex is the number of edges with the vertex as an end point. A graph with no loops or multiple edges is called a simple graph. A graph is said to be connected if every pair of vertices has a path connecting them otherwise the graph is disconnected. A graph $H = (V', E')$ is called induced subgraph of $G = (V, E)$ if $V' \subseteq V$ and $\{u, v\}$ is an edge in $H$ wherever $u$ and $v$ are distinct vertices in $V$ and $\{u, v\}$ is an edge in $G$, $H$ is called proper if $H \neq G$. A cycle graph is a graph that consists of a single cycle, or in other words, some number of distinct vertices connected in a closed chain. The cycle graph with $n$ vertices is denoted by $C_n$. The number of vertices in $C_n$ equals the number of edges, and every vertex has degree 2. The wheel graph $W_n$ or $n$-wheel is a graph that contains a cycle of order $n-1$, and for which every graph vertex in the cycle is connected to one other graph vertex which is called the hub. A bipartite graph is a graph whose vertex set can be split into two sets $A$ and $B$ in such a way that each edge of the graph joins a vertex in $A$ to a vertex in $B$. A vertex coloring of a

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graph $G=(V,E)$ is a way of coloring the vertices of the graph such that no two adjacent vertices share the same color. A clique of a graph $G$ is a maximal complete subgraph. In this case each pair of vertices of the clique are adjacent. The clique number $W(G)$ of a graph is the number of graph vertices in the largest clique of $G$, [8]. The clique number of a cycle graph $C_n$, $n$ odd is 3 and 2 otherwise. For a wheel graph $W_n$, $n$ is even the clique number is 4 and is 3 otherwise. The chromatic number of a graph $G$ is the smallest number of colors needed to color the vertices of a graph $G$ so that no two adjacent vertices share the same color, and is often denoted by $\chi(G)$. A graph $G$ is called perfect if for every induced subgraph $H$ of $G$, $\chi(H) = W(H)$. Note that if $G$ is a perfect graph, then every induced subgraph of $G$ is also perfect,[2].

II. Perfect Folding

**Definition (2-1)**

Let $G_1$ and $G_2$ be two simple graphs and $f : G_1 \rightarrow G_2$ be continuous map. Then $f$ is called a graph map, if

(i) For each vertex $v \in V(G_1)$, $f(v)$ is a vertex in $V(G_2)$.

(ii) For each edge $e \in E(G_1)$, $dim(f(e)) \leq dim(e)$, [3].

**Definition (2-2)**

A graph map $f : G_1 \rightarrow G_2$ is called a graph folding if and only if $f$ maps vertices to vertices and edges to edges, i.e., if

(i) For each vertex $v \in V(G_1)$, $f(v)$ is a vertex in $V(G_2)$.

(ii) For each edge $e \in E(G_1)$, $f(e)$ is an edge in $E(G_2)$,[4].

Note that if the vertices of an edge $e=(u,v) \in E(G_1)$ are mapped to the same vertex, then the edge $e$ will collapse to this vertex and hence we cannot get a graph folding. In other words, any graph folding cannot maps edges to loops but it may maps loops, if there is any, to loops.

**Definition (2-3)**

Let $G$ and $H$ be simple connected graphs. We call a graph folding $f: G \rightarrow H$ perfect folding if its image $f(G)$ is a perfect subgraph of $H$.

In general the image of a graph folding $f: G \rightarrow H$ is not a perfect graph e.g., if $G_1$ is the imperfect graph shown in Fig.(1-a), where $V(G_1)={v_1, v_2, v_5, v_4, v_5, v_6, v_7}$ and $E(G_1)={e_1, e_2, e_3, e_4, e_5, e_6, e_7}$. Then the graph folding $f: G_1 \rightarrow G_1$ defined by $f(v_6, v_7)={v_5, v_4}$ and $f(e_6, e_7)={e_2, e_4}$ is not a perfect folding. While if we consider the imperfect graph $G_2$ shown in Fig.(1-b), where $V(G_2)={u_1,..., u_7}$ and $E(G_2)={e_1,..., e_7}$.
Then the graph folding \( g: G_2 \rightarrow G_2 \) defined by \( g\{u_1, u_4\} = \{u_6, u_6\} \) and \( g\{e_4, e_7\} = \{e_5, e_6\} \) is a perfect folding. The omitted vertices and edges in this example and through the paper will be mapped to themselves.

Let \( G \) be a simple connected graph such that the number of \( E(G) \geq 2 \). If the chromatic number \( \chi(G) \) is equal to two, then \( G \) can be perfectly folded.

**Proof**

From [5], any simple connected graph \( G \) such that \( E(G) \geq 2 \) and \( \chi(G)=2 \) can be folded to an edge. In this case \( \chi(f(G)) = W(f(G)) = 2 \), and thus the graph \( G \) can be perfectly folded to an edge.

**Example (2-5)**

The cubic graph \( G \) with \( \chi(G) = W(G) = 2 \), shown in Fig. (2) can be folded to an edge by the graph folding \( f(v_1, ..., v_8) = (v_1, v_2, v_1, v_2, v_1, v_2, v_1, v_2) \). This folding can be done by the composition of a sequence of foldings \( f_1, f_2, f_3 \) and \( f_4 \), see Fig.(2). And hence the graph folding is a perfect.
Any folding of a bipartite graph (complete) is a perfect folding.

Proof
This follows from the fact that the chromatic number of a bipartite graph is equal to two, and thus it can be perfectly folded.

Example (2-7)
Consider the bipartite graph $G$ shown in Fig.(3). A graph folding $f: G \rightarrow G$ defined by $f \{v_1, v_3\} = \{v_2\}$ and $f \{e_1, e_4\} = \{e_2, e_3\}$ is a perfect folding.

The chromatic number of a cycle graph $C_n$, $n > 2$ where $n$ is odd is 3 while that for $n$ even is 2, \[1\].
**Theorem (3-1)**

Any folding of a cycle graph $C_n$ of an even number of edges is a perfect folding.

**Proof**

This follows from the fact that $\chi(C_n)$, $n$ is an even number is equal to two. Thus $C_n$ can be perfectly folded.

**Example (3-2)**

Consider the cycle graph $C_4$ where $\chi(C_4) = W(C_4) = 2$, the graph folding $f: C_4 \rightarrow C_4$ defined by $f\{v_1,v_4\} = \{v_3,v_2\}$ and $f\{e_i\} = \{e_3\}$, $i=1,2,4$ is a perfect folding, see Fig.(4).

![Figure 4](image)

It should be noted that the cycle graph $C_3$ cannot be folded, [4].

**Theorem (3-3)**

Let $G = C_n$, $n > 3$ be a cycle graph of an odd number of edges (vertices). Then $G$ can be perfectly folded to $C_3$.

**Proof**

Since $G = C_n$ has an odd number of edges (vertices). Thus the graph $C_n$ has three color classes, say $V_1$, $V_2$ and $V_3$. We can color the vertices of $C_n$ alternatively with the two colors of $V_1$ and $V_2$ except the last two edges one will join a vertex colored by the color of $V_2$ and a vertex colored by the color of $V_3$ and the other edge will join a vertex colored by the color of $V_3$ and a vertex colored by the color of $V_1$. Thus the number of vertices of color class $V_j = \text{the number of vertices color class}$.
\( V_2 = (n-1)/2 \), but \( V_3 \) has only one vertex \( w \). We can define a graph folding \( f : C_n \rightarrow C_n, n \) is odd, by mapping vertices of \( V_1 \) to a vertex of \( V_1 \), say \( u \), and mapping the vertices of \( V_2 \) to a vertex of \( V_2 \), say \( v \), finally mapping \( w \) into itself. Thus we have three vertices \( u, v, w \) and hence three edges in the image i.e., we have \( C_3 \). But \( \chi(C_3) = W(C_3) = 3 \), i.e., the graph folding \( f \) is perfect.

**Example (3-4)**

Let \( G = C_5 \) and \( h : G \rightarrow G \) be the graph folding defined by \( h\{v_5,v_4\} = \{v_3, v_1\} \) and \( h\{e_3\} = \{e_2\}, i = 3, 4 \) is a perfect folding, see Fig.(5). This can be done by the composition of the two graph foldings \( h_1 : C_5 \rightarrow C_5 \) defined by \( h_1\{v_5\} = \{v_3\}, h_1\{e_3\} = \{e_4\} \) and \( h_2 : h_1(C_5) \rightarrow h_1(C_5) \) defined by \( h_2\{v_4\} = \{v_1\}, h_2\{e_4\} = \{e_2\} \).

**Figure 5**

**IV. Perfect Folding of Wheel Graphs**

The chromatic number of a wheel graph \( W_n \) if \( n \) is odd is 3 and 4 if \( n \) is even, [1].

**Theorem (4-1)**

Any wheel graph \( W_n \) of an odd number of vertices can be perfectly folded to \( C_3 \).

**Proof**

A wheel graph \( W_n \) of order \( n, n \) is an odd number, is a graph that contains a cycle of even order \( n - 1 \), and each vertex in the cycle is
connected to the hub. In this case the chromatic number $\chi(W_n) = 3$, thus the graph $W_n$ can be colored by using three colors $A$, $B$ and $C$. One color for the hub, say $A$, and the vertices of the even cycle $C_{n-1}$ can be colored alternatively with two colors $B$ and $C$, i.e., if the set of vertices of the cycle $C_{n-1}$ is $V(C_{n-1}) = \{v_1, v_2, ..., v_{n-1}\}$, then the colors $B$ and $C$ have the following vertices, $B=\{v_1, v_3, ..., v_{n-2}\}$ and $C=\{v_2, v_4, ..., v_{n-1}\}$. Now we can define a graph folding by mapping the vertices of $B$ to a vertex of $B$, the vertices of $C$ to a vertex of $C$ and the hub onto itself. The image of this map will contain three vertices, three edges and thus we have $C_3$, i.e., the graph folding is perfect.

**Example (4-2)**

Consider the wheel graph $W_7$ and the graph folding $f: W_7 \rightarrow W_7$ defined by $f\{v_i\} = \{v_1\}$, $i=3,5$, $f\{v_j\} = \{v_2\}$, $j=4,6$ and $f\{e_k\} = \{e_1, e_1, e_1, e_1, e_1, e_1, e_7, e_7, e_8, e_7, e_8, e_7, e_8\}$, $k=1, ..., 12$. This graph folding is perfect, see Fig.(6).

![Figure 6](image)

It should be noted that the wheel graph of an even number of vertices cannot be folded, [4], and hence cannot be perfectly folded.

**V. The Clique Number and Perfect Folding**

The chromatic number of any graph is equal to or greater than its clique number, i.e., $\chi(G) \geq W(G)$. For connected graphs $2 \leq W(G) \leq \chi(G) \leq n$, where $n$ is the number of vertices of the graph $G$, [7].

**Theorem (5-1)**

Let $G$ be a simple connected graph, if the clique number $W(G)$ equal to the chromatic number $\chi(G)$ equal to 2 and $E(G) \geq 2$, then the graph $G$ can be perfectly folded.
Proof
It is immediately follows from Theorem (2-4) and since $\chi(G)=2$, then $G$ can be perfectly folded.

Example (5-2)
Consider the cycle graph $C_6$ shown in Fig.(7). A graph folding $f: C_6 \to C_6$ defined by $f\{v_2,v_3,v_4,v_5\} = \{v_6,v_1,v_6,v_1\}$ and $f\{e_i\} = \{e_6\}$, $i=1,...,5$ is a perfect folding.

![Figure 7](image)

Theorem (5-3)
Let $G$ be a simple connected graph such that no. $V(G) = n$. If $2 < W(G) = \chi(G) = k < n$, then the graph can be perfectly folded to a clique of order $k$.

Proof
Let $W(G) = \chi(G) = k$, then we have a maximal complete subgraph of $k$ vertices. This complete graph cannot be folded, [3]. These vertices must be colored by different colors $A_1, A_2, ..., A_k$. Now the other $(n-k)$ vertices of $G$, will be colored by the colors $A_1, ..., A_m$, $m \leq k$ in such a way that any edge will joins two vertices of different colors. So we can define a sequence of graph folding $f_i: G \to G_i$, where $G_i = f_i(G_{i-1})$, $i = 1,...,m$, $G_0 = G$, by mapping the $(n-k)$ vertices to other vertices but of the same color, until we get the $k$-clique which cannot be folded any more. And hence $W(f_i(G_i)) = \chi(f_i(G_i)) = k$, i.e., the graph folding is a perfect.

Example (5-4)
Consider the house graph $G$ with 5 vertices and 6 edges shown in Fig.(8), where $2 < W(G) = \chi(G) = 3 < n = 5$. This graph can be folded to a triangle by the graph folding $f: G \to G$ defined by $f\{v_4,v_5\} = \{v_2,v_3\}$ which is a perfect folding.
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