AN EVALUATION HOMOMORPHISM FOR QUANTUM TOROIDAL $\mathfrak{gl}_n$ ALGEBRAS

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Abstract. We present an affine analog of the evaluation map for quantum groups. Namely we introduce a surjective homomorphism from the quantum toroidal $\mathfrak{gl}_n$ algebra to the quantum affine $\mathfrak{gl}_n$ algebra completed with respect to the homogenous grading. We give a brief discussion of evaluation modules.

1. Introduction

For an arbitrary Lie algebra $\mathfrak{g}$ and a non-zero constant $u$, we have the evaluation map

$$\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \rightarrow \mathfrak{g}, \quad g \otimes t^k \mapsto u^k g.$$  

The evaluation map is a surjective homomorphism of Lie algebras. It plays a prominent role in representation theory of current algebras and various constructions in mathematics and physics.

When $\mathfrak{g} = \mathfrak{gl}_n$, a quantum version of the evaluation map $U_q \widehat{\mathfrak{g}} \rightarrow U_q \mathfrak{g}$ was introduced in [J]. This map is used to construct simplest possible representations of the quantum affine algebra $U_q \widehat{\mathfrak{gl}}_n$ called evaluation modules. The evaluation modules are central for many studies. For example, the $R$ matrix of the celebrated six vertex model is an intertwiner for tensor products of two evaluation modules. It is well known that the quantum evaluation map does not exist for simple Lie algebras $\mathfrak{g}$ other than in type $A$.

In this note we give explicit formulas for an affine analog of the quantum evaluation map.

Let $\mathcal{E}'_n = \mathcal{E}'_n(q_1, q_2, q_3)$ be the quantum toroidal algebra associated to $\mathfrak{gl}_n$. It depends on complex parameters $q_1, q_2, q_3$ such that $q_1 q_2 q_3 = 1$ and a central elements $C$ (we set the second central element to 1), see Section 2. Let $U_q \widehat{\mathfrak{gl}}_n$ be the quantum affine algebra associated to $\mathfrak{gl}_n$, see Section 3. It depends on a complex parameter $q$ and it has a central element $K$. We always assume $q^2 = q_3$. An easy well known fact is that there is an inclusion of algebras $v : U'_q \widehat{\mathfrak{gl}}_n \rightarrow \mathcal{E}'_n(q_1, q_2, q_3)$ such that $v(K) = C$.

We consider $U'_q \widehat{\mathfrak{gl}}_n$ in the Drinfeld new realization, and we denote by $\tilde{U}'_q \widehat{\mathfrak{gl}}_n$ its completion with respect to the homogeneous grading, see (4.1). We also impose the following key relation to the central elements of $\mathcal{E}'_n$ and $\tilde{U}'_q \widehat{\mathfrak{gl}}_n$:

$$(1.1) \quad K^2 = C^2 = q_3^n.$$  

We then construct a surjective algebra homomorphism depending on a non-zero complex number $u$:  

$$\text{ev}_u : \mathcal{E}'_n \rightarrow \tilde{U}'_q \widehat{\mathfrak{gl}}_n$$  

such that  

$$(1.2) \quad \text{ev}_u \circ v = id.$$
We call the homomorphism $\text{ev}_u$ the quantum affine evaluation map.

The usual quantum evaluation map is written in the Chevalley generators. In contrast, the affine evaluation map is described purely in terms of currents. The quantum toroidal algebra $\mathcal{E}'_n$ is generated by series $E_i(z), F_i(z), K_i^\pm (z)$, $i = 0, 1, \ldots, n - 1$, and by $C$. The subalgebra generated by currents with non-zero indices together with $K_0^\pm (z)$ is isomorphic to $U'_q \widehat{\mathfrak{sl}}_n$. Then by (1.2) the images of $E_i(z), F_i(z), K_i^\pm (z)$ with $i \neq 0$ and $K_0^\pm (z)$ are already trivially known. The only issue is to determine the image of $E_0(z)$ and of $F_0(z)$. It turns out that the crucial ingredient is the “fused currents” studied in [FJMM1] in relation to the construction of a subalgebra $\mathcal{E}'_2$ inside a completion of $\mathcal{E}'_n$. These fused currents commute with $E_i(z), F_i(z), i = 2, 3, \ldots, n - 2$, and have simple rational commutation relations with $E_j(z), F_j(z)$ for $j = 1$ and $j = n - 1$, see Section 4. Then it is possible to modify the fused currents with suitable exponentials of Cartan generators to achieve the desired commutation relations.

As mentioned above, the fused currents and their modification require the completion of the quantum toroidal algebra. However, the completed algebra still acts on highest weight $U'_q \widehat{\mathfrak{gl}}_n$ modules and therefore every such representation becomes an evaluation representation of the quantum toroidal algebra. Moreover, this representation of $\mathcal{E}'_n$ is also a highest weight module. We discuss the highest weight in Section 7. It turns out that some evaluation representations appeared already in [FJMM2].

We note that due to various automorphisms, the evaluation homomorphism can be written in different forms. In addition, one can similarly write a different evaluation map using a different completion (e.g. the one acting in lowest weight modules). But it seems that the only possibility to change the condition (1.1) is to replace $q^4$ with $q^n$.

The paper is constructed as follows. We give definitions of quantum toroidal and quantum affine algebras in Sections 2 and 3. In Section 4 we recall the fused currents of [FJMM1] and study their commutation relations with other currents. We give the evaluation map in Secton 5 and prove it is well-defined in Section 6. Finally we discuss the evaluation modules in Section 7.

After we finished the paper, we were directed to reference [Mi2] which contains our main result. Therefore, the main findings of this paper are not new.

2. Quantum Toroidal $\mathfrak{gl}_n$

In this section we introduce the quantum toroidal algebra. First we prepare some notation.

We fix $q^{1/2}, d^{1/2} \in \mathbb{C}^\times$ and set $q_1^{1/2} = q^{-1/2}d^{1/2}, q_2^{1/2} = q, q_3^{1/2} = q^{-1/2}d^{-1/2}$, so that $q_1^{1/2}q_2^{1/2}q_3^{1/2} = 1$.

Let $n$ be a positive integer. For $n \geq 2$, let $(a_{ij})_{i,j \in \mathbb{Z}/n\mathbb{Z}}$ be the Cartan matrix of type $A^{(1)}_{n-1}, \Lambda_i$ and $\bar{\alpha}_i$ the simple weights and the simple roots of $\mathfrak{sl}_n$, $\bar{P} = \oplus_{i=1}^{n-1} \mathbb{Z}\bar{\lambda}_i$ the weight lattice, and $(, ) : \bar{P} \times \bar{P} \to \mathbb{Q}$ the standard symmetric bilinear form such that $(\bar{\alpha}_i, \bar{\alpha}_i) = 2$. When $n = 1$, we set $a_{0,0} = 0, \bar{P} = 0$.

For $r \neq 0$ we set

$$a_{i,j}(r) = \left[\frac{r}{r}\right] \times \left( (q^r + q^{-r})\delta_{i,j}^{(n)} - d^r\delta_{i,j-1}^{(n)} - d^{-r}\delta_{i,j+1}^{(n)} \right)$$

where $[x] = (q^x - q^{-x})/(q - q^{-1})$ and $\delta_{i,j}^{(n)}$ is Kronecker’s delta modulo $n$. 

Define further functions \( g_{i,j}(z, w) \) by
\[
\begin{align*}
\text{if } n \geq 3 : & \quad g_{i,j}(z, w) = \begin{cases} 
  z - q_1 w & (i \equiv j - 1), \\
  z - q_2 w & (i \equiv j), \\
  z - q_3 w & (i \equiv j + 1), \\
  z - w & (i \neq j, j \pm 1),
\end{cases} \\
\text{if } n = 2 : & \quad g_{i,j}(z, w) = \begin{cases} 
  z - q_2 w & (i \equiv j), \\
  (z - q_1 w)(z - q_3 w) & (i \neq j),
\end{cases} \\
\text{if } n = 1 : & \quad g_{0,0}(z, w) = (z - q_1 w)(z - q_2 w)(z - q_3 w),
\end{align*}
\]
and set \( d_{i,j} = d^{\pm 1} (i \equiv j \mp 1, n \geq 3) \), \( d_{i,j} = -1 (i \neq j, n = 2) \), and \( d_{i,j} = 1 \) (otherwise).

The quantum toroidal algebra of type \( \mathfrak{gl}_n \), which we denote by \( \mathcal{E}'_n \), is a unital associative algebra generated by \( E_{i,k}, F_{i,k}, H_{i,r} \) and invertible elements \( q^h, C \), where \( i \in \mathbb{Z}/n\mathbb{Z}, k \in \mathbb{Z}, r \in \mathbb{Z}\setminus\{0\} \), \( h \in \mathbb{P} \).

We set \( K_i = q^{\alpha_i} \) for \( 1 \leq i \leq n - 1 \) and
\[
K_0 = \prod_{i=1}^{n-1} K_i^{-1}.
\]

We present below the defining relations in terms of generating series
\[
E_i(z) = \sum_{k \in \mathbb{Z}} E_{i,k}z^{-k}, \quad F_i(z) = \sum_{k \in \mathbb{Z}} F_{i,k}z^{-k},
\]
\[
K_i^\pm(z) = K_i^{\pm 1} \tilde{K}_i^\pm(z), \quad \tilde{K}_i^\pm(z) = \exp \left( \pm (q - q^{-1}) \sum_{r \geq 0} H_{i,\pm r} z^r \right).
\]

The relations are as follows.

\( C, q^h \) relations
\[
\begin{align*}
(2.2) & \quad C \text{ is central, } q^h q^{h'} = q^{h+h'} \ (h, h' \in \mathbb{P}), \quad q^0 = 1, \\
(2.3) & \quad q^h E_i(z)q^{-h} = q^{(h,\alpha_i)} E_i(z), \quad q^h F_i(z)q^{-h} = q^{-(h,\alpha_i)} F_i(z) \ (h \in \mathbb{P}).
\end{align*}
\]

\( K-K, K-E \) and \( K-F \) relations
\[
\begin{align*}
(2.4) & \quad K_i^\pm(z) K_j^\pm(w) = K_j^\pm(w) K_i^\pm(z), \\
(2.5) & \quad g_{i,j}(C^{-1}z, w)K_i^-(z)K_j^+(w) = g_{i,j}(w, Cz^{-1})K_j^+(w)K_i^-(z), \\
(2.6) & \quad d_{i,j}g_{i,j}(z, w)K_i^+(C^{-1/2}z)E_j(w) + g_{j,i}(w, z)E_j(w)K_i^+(C^{-1/2}z) = 0, \\
(2.7) & \quad d_{j,i}g_{j,i}(w, z)K_i^+(C^{-1/2}z)F_j(w) + g_{i,j}(z, w)F_j(w)K_i^+(C^{-1/2}z) = 0.
\end{align*}
\]

\( E-F \) relations
\[
(2.8) \quad [E_i(z), F_j(w)] = \frac{\delta_{i,j}}{q - q^{-1}} (\delta(C^{z \frac{w}{2}}) K^+_i(w) - \delta(C^{\frac{z}{w}}) K^-_i(z)).
\]
$E-E$ and $F-F$ relations

$$[E_i(z), E_j(w)] = 0, \quad [F_i(z), F_j(w)] = 0 \quad (i \neq j, j \pm 1),$$

$$d_{i,j} g_{i,j}(z, w) E_i(z) E_j(w) + g_{j,i}(w, z) E_j(w) E_i(z) = 0,$$

$$d_{j,i} g_{j,i}(w, z) F_i(z) F_j(w) + g_{i,j}(z, w) F_j(w) F_i(z) = 0.$$

**Serre relations** For $n \geq 3$,

$$\text{Sym}_{z_1, z_2} [E_i(z_1), [E_i(z_2), E_{i \pm 1}(w)]_q]_q^{-1} = 0,$$

$$\text{Sym}_{z_1, z_2} [F_i(z_1), [F_i(z_2), F_{i \pm 1}(w)]_q]_q^{-1} = 0.$$

For $n = 2$, $i \neq j$,

$$\text{Sym}_{z_1, z_2, z_3} [E_i(z_1), [E_i(z_2), [E_i(z_3), E_j(w)]_q^2]]_{q-2} = 0,$$

$$\text{Sym}_{z_1, z_2, z_3} [F_i(z_1), [F_i(z_2), [F_i(z_3), F_j(w)]_q^2]]_{q-2} = 0.$$

For $n = 1$,

$$\text{Sym}_{z_1, z_2, z_3} z_2 z_3^{-1} [E_0(z_1), [E_0(z_2), E_0(z_3)]] = 0,$$

$$\text{Sym}_{z_1, z_2, z_3} z_2 z_3^{-1} [F_0(z_1), [F_0(z_2), F_0(z_3)]] = 0.$$

In the above we set $[A, B]_p = AB - pBA$ and

$$\text{Sym} f(x_1, \ldots, x_N) = \frac{1}{N!} \sum_{\pi \in S_N} f(x_{\pi(1)}, \ldots, x_{\pi(N)}).$$

Under the $C, q$ relations (2.3), the $K-K$, $K-E$ and $K-F$ relations (2.4)–(2.7) are equivalently written as

$H-E$, $H-F$, and $H-H$ relations  For $r \neq 0$,

$$[H_{i,r}, E_j(z)] = a_{i,j}(r) C^{-(r+|r|)/2} z^r E_j(z),$$

$$[H_{i,r}, F_j(z)] = -a_{i,j}(r) C^{-(r-|r|)/2} z^r F_j(z),$$

$$[H_{i,r}, H_{j,s}] = \delta_{r+s,0} \cdot a_{i,j}(r) \eta_r, \quad \eta_r = \frac{C^r - C^{-r}}{q - q^{-1}}.$$
3. Quantum affine algebra $U_q\widehat{gl}_n$

In this section we recall the quantum affine algebra $U_q\widehat{gl}_n$ in Drinfeld new realization and the embedding into $E_n'$.

The quantum affine algebra $U_q\widehat{gl}_n$ is defined by generators $x_{i,k}^\pm$, $1 \leq i \leq n - 1$, $k \in \mathbb{Z}$, $h_{i,r}$, $0 \leq i \leq n - 1$, $r \in \mathbb{Z} \setminus \{0\}$, $K$ and $q^h$, $h \in \overline{P}$, with the defining relations

\[ C \text{ is central, } \quad q^h q^{h'} = q^{h+h'} \quad (h, h' \in \overline{P}) , \quad q^0 = 1 , \]
\[ q^h x_i^\pm(z) q^{-h} = q^{\pm(h,\alpha_i)} x_i^\pm(z) \quad (h \in \overline{P}) , \]
\[ [h_{i,r}, h_{j,s}] = \delta_{r+s,0} \frac{[ra_{ij}]}{r} K^r - K^{-r} , \]
\[ [h_{i,r}, x_j^\pm(z)] = \pm \frac{[ra_{ij}]}{r} K^{-(r\pm|r|)/2} z^r x_j^\pm(z) , \]
\[ [x_i^+(z), x_j^-(w)] = \frac{\delta_{ij}}{q - q^{-1}} \left( \delta(K\frac{w^r}{z}) \phi_i^+(w) - \delta(K\frac{z}{w}) \phi_i^-(z) \right) , \]
\[ (z - q^{\pm a_{ij}} w) x_i^+(z) x_j^+(w) + (w - q^{\pm a_{ij}} z) x_j^+(w) x_i^+(z) = 0 , \]
\[ [x_i^+(z), x_j^+(w)] = 0 \quad \text{if } a_{ij} = 0 , \quad \text{Sym}_{z_1,z_2} [x_i^+(z_1), x_j^+(z_2), x_j^+(w)] q^{-1} = 0 \quad \text{if } a_{ij} = -1 . \]

Here we set $x_i^+(z) = \sum_{k \in \mathbb{Z}} x_{i,k}^+ z^{-k}$, $\phi_i^+(z) = K_i^1 \exp(\pm(q - q^{-1}) \sum_{r>0} h_{i,\pm r} z^r)$.

We have an embedding $v : U_q\widehat{gl}_n \rightarrow E_n'$ given by

\[ x_i^+(z) \mapsto E_i(d^{-i} z) , \quad x_i^-(z) \mapsto F_i(d^{-i} z) , \quad \phi_i^+(z) \mapsto K_i^+(d^{-i} z) \]

and $K \mapsto C$, $q^h \mapsto q^h$. Hereafter we identify $U_q\widehat{gl}_n$ with the subalgebra of $E_n'$ by $v$, and regard $E_i(z), F_i(z), 1 \leq i \leq n - 1$, $K_i^\pm(z), 0 \leq i \leq n - 1$, $C$ and $q^h$, $h \in \overline{P}$, as generators of $U_q\widehat{gl}_n$.

In the standard definition, the quantum affine $gl_n$ algebra contains a split central element corresponding to $I \otimes t^0$ of $gl_n[t, t^{-1}]$ and the degree operator. The algebra $U_q\widehat{gl}_n$ defined above is obtained by setting this central element to one, cf. (2.1), and dropping the degree operator.

4. Fused currents

We recall the fused currents of [FJMM1] and compute the commutation relations with generators of $U_q\widehat{gl}_n$.

Both $E_n'$ and $U_q\widehat{gl}_n$ are $\mathbb{Z}$-graded algebras by the degree assignment

\[ \deg E_{i,k} = \deg F_{i,k} = k , \quad \deg H_{i,r} = r , \quad \deg C = \deg q^h = 0 . \]

Denote by $\overline{U}_q\widehat{gl}_n$ the completion of $U_q\widehat{gl}_n$ with respect to this grading.
Following [FJMM1] let us introduce the following elements of $\tilde{U}_q^r\hat{g}_n$:

\begin{align}
(4.2) & \quad E(z) = \prod_{i=1}^{n-2} \left(1 - \frac{z_i}{z_{i+1}}\right) \cdot E_{n-1}(q_3^{n/2-1}z_{n-1}) \cdots E_2(q_3^{n/2-2}z_2) E_1(q_3^{n/2-1}z_1) \Bigg|_{z_1 = \cdots = z_{n-1} = z}, \\
(4.3) & \quad F(z) = \prod_{i=1}^{n-2} \left(1 - \frac{z_{i+1}}{z_i}\right) \cdot F_1(q_3^{-n/2+1}z_1) F_2(q_3^{-n/2+2}z_2) \cdots F_{n-1}(q_3^{-n/2-1}z_{n-1}) \Bigg|_{z_1 = \cdots = z_{n-1} = z}, \\
(4.4) & \quad K^\pm(z) = \prod_{i=1}^{n-1} K^\pm_i(q_3^{-n/2+i}z).
\end{align}

When $n = 2$ we have $E(z) = E_1(z)$, $F(z) = F_1(z)$, $K^\pm(z) = K^\pm_1(z)$.

The following result is a special case of the construction in [FJMM1].

**Proposition 4.1.** [FJMM1] For $n \geq 2$, the currents (4.2)—(4.4) satisfy

\begin{align}
(4.5) & \quad [E(z), F(w)] = \frac{1}{q - q^{-1}} \left( \delta\left(\frac{w}{z}\right)K^+(z) - \delta\left(\frac{z}{w}\right)K^-(z) \right), \\
(4.6) & \quad (z - q_3w)E(z)E(w) + (w - q_3z)E(w)E(z) = 0, \\
(4.7) & \quad (w - q_3z)F(z)F(w) + (z - q_3w)F(w)F(z) = 0, \\
(4.8) & \quad [E(z), E_i(w)] = [E(z), F_i(w)] = 0 \quad (2 \leq i \leq n - 2), \\
(4.9) & \quad [E_i(z), F(w)] = [F_i(z), F(w)] = 0 \quad (2 \leq i \leq n - 2).
\end{align}

In addition we calculate the other commutation relations.

**Proposition 4.2.** For $n \geq 3$, we have

\begin{align}
(4.10) & \quad (z - q_3^{-n/2}q_1^{-1}w)E_1(z)E(w) = q(z - q_3^{-n/2+1}w)E(w)E_1(z), \\
(4.11) & \quad (z - q_3^{n/2-1}w)E_{n-1}(z)E(w) = q(z - q_3^{n/2}q_1^{-1}w)E(w)E_{n-1}(z), \\
(4.12) & \quad (z - q_3^{-n/2+1}w)F_1(z)F(w) = q^{-1}(z - q_3^{-n/2}q_1^{-1}w)F(w)F_1(z), \\
(4.13) & \quad (z - q_3^{n/2}w)F_{n-1}(z)F(w) = q^{-1}(z - q_3^{n/2-1}w)F(w)F_{n-1}(z), \\
(4.14) & \quad (z - C^{-1}q_3^{-n/2}w)E(z), F_1(w)] = (z - Cq_3^{-n/2+1}w)[E(z), F_{n-1}(w)] = 0, \\
(4.15) & \quad (z - Cq_3^{-n/2+1}w)[E_1(z), F(w)] = (z - C^{-1}q_3^{-n/2-1}w)[E_{n-1}(z), F(w)] = 0.
\end{align}

**Proof.** As an example we consider (4.14). From the defining relations (2.8) we have

\begin{align*}
(q - q^{-1})[E_{n-1}(q_3^{n/2-1}z_{n-1}) \cdots E_1(q_3^{-n/2+1}z_1), F_1(w)] \\
= E_{n-1}(q_3^{n/2-1}z_{n-1}) \cdots E_2(q_3^{-n/2+2}z_2) \left( \delta\left(\frac{q_3^{n/2-2}w}{z_1}\right)K^+_1(w) - \delta\left(q_3^{-n/2+1}z_1\right)K^-_1(q_3^{-n/2+1}z_1) \right).
\end{align*}

Upon multiplying by $z_1 = C^{-1}q_3^{n/2-1}w$, the second term vanishes. The first term does not have a pole at $z_1 = z_2$ because $K^+(w)$ is a power series in $w^{-1}$ placed at the rightmost. Multiplying further by $\prod_{i=1}^{n-2}(1 - z_i/z_{i+1})$ and setting $z_1 = \cdots = z_{n-1} = z$ we find

\begin{align*}
(z - C^{-1}q_3^{-n/2-1}w)[E(z), F_1(w)] = 0,
\end{align*}

as desired.
The rest of the relations can be shown in a similar manner.

5. Evaluation map

In this section we present our main result: the quantum affine evaluation map.
From now on, we consider the quotient algebra of $\mathcal{E}_n'$ by the relation

$$C = q^{n/2}_3.$$  

Denote this quotient by $\mathcal{E}'_n$. We consider also the quotient of $\tilde{U}'_{q_3}\tilde{gl}_n$ by the relation $K = \kappa$, where $\kappa \in \mathbb{C}^\times$. Denote this quotient by $\tilde{U}'_{q_3,\kappa}\tilde{gl}_n$.

Introduce currents $A_{\pm}(z) = \sum_{r>0} A_{\pm}^r z^{\mp r}$, $B_{\pm}(z) = \sum_{r>0} B_{\pm}^r z^{\mp r}$ in $U'_{q_3}\tilde{gl}_n$ by setting

$$A_{-r} = \eta_r^{-1} C^{-r}(H_{0,-r} + \sum_{i=1}^{n-1} q_3^{ir} H_{i,-r}), \quad A_r = -\eta_r^{-1} (H_{0,r} + \sum_{i=1}^{n-1} q_3^{(n-i)r} H_{i,r}),$$

$$B_{-r} = -\eta_r^{-1} (H_{0,-r} + \sum_{i=1}^{n-1} q_3^{-(n-i)r} H_{i,-r}), \quad B_r = \eta_r^{-1} C^r (H_{0,r} + \sum_{i=1}^{n-1} q_3^{-ir} H_{i,r}).$$

Set further

$$\mathcal{K} = q^{-\tilde{\Lambda}_1+\tilde{\Lambda}_{n-1}} = \prod_{i=1}^{n-1} K_i^{(2i-n)/n}.$$  

Note that $\mathcal{K}$ commutes with $E(z), F(z)$.

We are now in a position to state the main result of the present note.

**Theorem 5.1.** Let $u \in \mathbb{C}^\times$, and set $\kappa = q_3^{n/2}$. The following assignment gives a homomorphism of algebras $ev_u : \mathcal{E}'_n \to \tilde{U}'_{q_3,\kappa}\tilde{gl}_n$ such that $ev_u \circ v = id$:

\begin{align*}
(5.1) & \quad E_0(z) \mapsto u^{-1} e^{A_0(z)} F(z) e^{A_0(z)\mathcal{K}}, \quad F_0(z) \mapsto u e^{B_0(z)} E(z) e^{B_0(z)\mathcal{K}}^{-1}, \\
(5.2) & \quad E_i(z) \mapsto E_i(z), \quad F_i(z) \mapsto F_i(z) \quad (i = 1, \ldots, n-1), \\
(5.3) & \quad K_i^\pm(z) \mapsto K_i^\pm(z) \quad (i = 0, 1, \ldots, n-1), \quad q^h \mapsto q^h \quad (h \in \mathbb{P}).
\end{align*}

**Remark.** When $n = 1$ we set formally $E(z) = F(z) = \mathcal{K} = 1$. Then (5.1) is nothing but the known vertex operator realization of $\mathcal{E}'_1$ for $C = q_3^{1/2}$.

Theorem 5.1 is proved in Section 6.

In the above theorem we have chosen the currents $E_0(z), F_0(z)$ to play a special role. In view of the cyclic symmetry of $\mathcal{E}'_n$ which sends $E_i(z) \mapsto E_{i+1}(z)$, $F_i(z) \mapsto F_{i+1}(z)$, $K_i^\pm(z) \mapsto K_{i+1}^\pm(z)$, we could have started with $E_i(z), F_i(z)$ for any $i$.

The algebra $\mathcal{E}'_n$ has also a symmetry of interchanging $q_1$ with $q_3$, see (2.9) in [FJMM1]. This means that it is easy to write another evaluation homomorphism when

$$C = q_1^{n/2}.$$  

This is parallel to the fact that there are two evaluation homomorphisms $U'_{q}\tilde{gl}_n \to U_q gl_n$. 

We also remark that the evaluation map is clearly graded with respect to the homogeneous degree, see (4.1) and commutes with the automorphism of changing of spectral parameter, see (2.7) in [FJMM1].

6. Proof

In this section, we prove Theorem 5.1. To simplify the notation we consider \( ev = ev_1 \).

We shall need commutation relations between \( A_\pm(z), B_\pm(z) \) and \( E_i(w), F_i(w) \).

First we have:

\[
[A_\pm(z), E_i(w)] = [A_\pm(z), F_i(w)] = [B_\pm(z), E_i(w)] = [B_\pm(z), F_i(w)] = 0, \quad 2 \leq i \leq n - 2.
\]

Other relations are given as follows.

\[
\begin{align*}
(6.1) \quad e^{A_+(z)} E_1(w) e^{-A_+(z)} &= \frac{z - q_3^{-1}w}{z - q_1w} E_1(w), \quad e^{A_+(z)} F_1(w) e^{-A_+(z)} = \frac{z - Cq_1w}{z - Cq_3^{-1}w} F_1(w), \\
(6.2) \quad e^{B_+(z)} E_{n-1}(w) e^{-B_+(z)} &= \frac{z - C^{-1}q_1^{-1}w}{z - C^{-1}q_3w} E_{n-1}(w), \quad e^{B_+(z)} F_{n-1}(w) e^{-B_+(z)} = \frac{z - q_3w}{z - q_1^{-1}w} F_{n-1}(w), \\
(6.3) \quad e^{-A_-(z)} E_{n-1}(w) e^{A_-(z)} &= \frac{w - q_3^{-1}z}{w - q_1z} E_{n-1}(w), \quad e^{-A_-(z)} F_{n-1}(w) e^{A_-(z)} = \frac{w - Cq_1z}{w - Cq_3^{-1}z} F_{n-1}(w), \\
(6.4) \quad e^{-B_-(z)} E_1(w) e^{B_-(z)} &= \frac{w - C^{-1}q_1^{-1}z}{w - C^{-1}q_3z} E_1(w), \quad e^{-B_-(z)} F_1(w) e^{B_-(z)} = \frac{w - q_3z}{w - q_1^{-1}z} F_1(w).
\end{align*}
\]

We also have

\[
\begin{align*}
(6.5) \quad e^{A_+(z)} e^{A_-(w)} &= \frac{(z - w)^2}{(z - q_2w)(z - q_2^{-1}w)} e^{A_-(w)} e^{A_+(z)}, \\
(6.6) \quad e^{A_+(z)} e^{B_-(w)} &= \frac{z - Cq_2w}{(z - Cw)(z - C^{-1}w)} e^{B_-(w)} e^{A_+(z)}, \\
(6.7) \quad e^{B_+(z)} e^{A_-(w)} &= \frac{(z - Cq_2w)(z - C^{-1}q_2^{-1}w)}{(z - Cw)(z - C^{-1}w)} e^{A_-(w)} e^{B_+(z)}, \\
(6.8) \quad e^{B_+(z)} e^{B_-(w)} &= \frac{(z - w)^2}{(z - q_2w)(z - q_2^{-1}w)} e^{B_-(w)} e^{B_+(z)}.
\end{align*}
\]

Let us verify the relations involving \( E_0(z), F_0(z) \) case-by-case.

\( C, q^h \) relations. These are easy to check.

**H-E and H-F relations.** The relation

\[
[ev(H_{i,r}), ev(E_0(z))] = a_{i,0}(r) C^{-r} z^r ev(E_0(z)) \quad (r > 0)
\]
follows from $C = q_3^{n/2}$ and

$$[H_{i,r}, e^{A_-(z)}] = z^r e^{A_-(z)} \times C^{-r} (a_{i,0}(r) + \sum_{j=1}^{n-1} a_{i,j}(r) q_3^{j/r}),$$

$$[H_{i,r}, F(z)] = -z^r F(z) \times \sum_{j=1}^{n-1} a_{i,j}(r) (q_3^{-n/2+j})^r.$$ 

The relations for $r < 0$ and for $[\text{ev}(H_{i,r}), \text{ev}(F_0(z))]$ can be verified in a similar manner.

### $E$-$F$ relations.

First consider the relation

$$[\text{ev}(E_0(z)), \text{ev}(F_i(w))] = 0 \quad (i \neq 0).$$

We have

$$\text{ev}(E_0(z))\text{ev}(F_i(w)) = e^{A_-(z)}F(z)F_i(w)e^{A_+(z)} \mathcal{K} \times \left\{ \begin{array}{ll} q(z - Cq_1w)/(z - Cq_3^{-1}w) & (i = 1) \\ q^{-\delta_{i,n-1}} & (2 \leq i \leq n - 1) \end{array} \right.,$$

$$\text{ev}(F_i(w))\text{ev}(E_0(z)) = e^{A_-(z)}F_i(w)F(z)e^{A_+(z)} \mathcal{K} \times \left\{ \begin{array}{ll} 1 & (1 \leq i \leq n - 2) \\ (w - Cq_3z)/(w - Cq_3^{-1}z) & (i = n - 1) \end{array} \right.,$$

so that the relations reduce to (4.12), (4.13) and (4.9). The case $[\text{ev}(E_i(z)), \text{ev}(F_0(w))] = 0 (i \neq 0)$ is similar.

Using (6.1), (6.4) and (6.6) we obtain

$$\text{ev}(E_0(z))\text{ev}(F_0(w)) = e^{A_-(z)}F(z)e^{A_+(z)} \mathcal{K} \cdot e^{B_-(z)}E(z)e^{B_+(z)} \mathcal{K}^{-1} = e^{A_-(z) + B_-(w)}F(z)E(w)e^{A_+(z) + B_+(w)}.$$

Computing $\text{ev}(F_0(w))\text{ev}(E_0(z))$ similarly and using further (4.5), we find

$$[\text{ev}(E_0(z)), \text{ev}(F_0(w))] = \frac{1}{q - q^{-1}}$$

$$\times e^{A_-(z) + B_-(w)} (-\delta(Cz/w)K^+(z) + \delta(Cw/z)K^-(w))e^{A_+(z) + B_+(w)}.$$

Noting that

$$e^{A_-(z) + B_-(Cz)} = K_0^-(z), \quad e^{A_+(Cw) + B_+(w)} = K_0^+(w),$$

$$e^{A_-(Cw) + B_-(w)} = K_0K^-(w)^{-1}, \quad e^{A_+(z) + B_+(Cz)} = K_0^{-1}K^+(z)^{-1},$$

we obtain the desired result.

### $E$-$E$ and $F$-$F$ relations.

To check the quadratic relations

$$d_{0,j}g_{0,j}(z,w)\text{ev}(E_0(z))\text{ev}(E_j(w)) + g_{j,0}(w,z)\text{ev}(E_j(w))\text{ev}(E_0(z)) = 0,$$

we proceed in the same way as above; using (6.1), (6.3) we bring $A_+(z)$ to the right, $A_-(z)$ to the left, and apply (4.10), (4.11). Verification of the $F$-$F$ relations is entirely similar.
Serre relations. Let us check the Serre relations assuming \( n \geq 3 \). We have
\[
e^{-A_- (w)} \text{ev}(E_1(z_1)), \left[ \text{ev}(E_1(z_2)), \text{ev}(E_0(w)) \right]_q \ e^{-A_+ (w)}
\]
\[
= E_1(z_1) E_1(z_2) F(w) - (q + q^{-1}) q^{-1} \frac{w - q_3^{-1} z_2}{w - q_1 z_2} E_1(z_1) F(w) E_1(z_2)
\]
\[
+ q^{-2} \frac{w - q_3^{-1} z_1}{w - q_1 z_1} \frac{w - q_3^{-1} z_2}{w - q_1 z_2} F(w) E_1(z_1) E_1(z_2).
\]

In view of (4.15), we can move \( F(w) \) to the right without producing delta functions. After simplification, the right hand side becomes
\[
- \frac{(1 - q_2^{-1}) w}{(w - q_1 z_1)(w - q_1 z_2)} (z_1 - q_2 z_2) E_1(z_1) E_1(z_2) F(w).
\]

Symmetrizing in \( z_1, z_2 \) we obtain 0 due to the quadratic relations for \( E_1(z) \).

Likewise we compute
\[
e^{-A_- (z_1) - A_- (z_2)} [\text{ev}(E_0(z_1)), \left[ \text{ev}(E_0(z_2)), \text{ev}(E_1(w)) \right]_q] \ e^{-A_+ (z_1) - A_+ (z_2)}
\]
\[
= \frac{z_1 - q_2^{-1} z_2}{z_1 - q_2 z_2} \left( q^{-1} \frac{z_1 - q_3^{-1} w z_2}{z_1 - q_1 w} - q_3^{-1} \frac{w}{z_1 - q_1 w} \right) F(z_1) F(z_2) E_1(w)
\]
\[
- (q + q^{-1}) q^{-1} \frac{z_1 - q_3^{-1} w}{z_1 - q_1 w} F(z_1) E_1(w) F(z_2) - E_1(w) F(z_1) F(z_2)
\]
\[
= \frac{q_2^{-1} (q_1 - q_3^{-1}) w}{(z_1 - q_1 w)(z_2 - q_1 w)} (z_1 - q_2^{-1} z_2) E_1(w) F(z_1) F(z_2).
\]

Due to (4.7), the last line vanishes after symmetrization.

Serre relations in the remaining cases (including the case \( n = 2 \)) can be verified by the same argument. We omit further details.

The proof is over.

7. Evaluation modules

In this section we define and discuss the evaluation modules.

Recall the grading (4.1). We say that a \( U_q \hat{\mathfrak{gl}}_n \) module \( V \) is admissible if for any \( v \in V \) there exists an \( N \) such that \( xv = 0 \) holds for any \( x \in U_q \hat{\mathfrak{gl}}_n \) with \( \deg x > N \). Algebra \( U_q \hat{\mathfrak{gl}}_n \) has a well-defined action on admissible modules.

The quantum affine evaluation map \( \text{ev}_u \) goes from the quantum toroidal \( \mathfrak{gl}_n \) algebra to the (completed) quantum affine \( \mathfrak{gl}_n \) algebra, provided the central element of the former has the special value \( C = q_3^3 \). Note that for the latter this value for the central element is completely arbitrary as \( q \) and \( q_3 \) are independent variables.

It follows that any admissible representation \( V \) of \( U_q \hat{\mathfrak{gl}}_n \) on which \( K \) acts as an arbitrary scalar \( \kappa \) can be pulled back by \( \text{ev}_u \) to a representation of \( \mathcal{E}'_n \), by choosing \( q_3 \) so that \( \kappa = q_3^{3/2} \). We call the resulting \( \mathcal{E}'_n \) module evaluation module and denote it by \( V(u) \).

An example of admissible modules is given by highest weight modules.
A $U'_q \hat{gl}_n$ module $V$ is a highest weight module of highest weight $(\kappa_0, \ldots, \kappa_{n-1}) \in (\mathbb{C}^\times)^n$ if it is generated by a cyclic vector $v$ satisfying
\[ xv = 0 \quad \text{if} \; \deg x > 0, \; \quad x_{i,0}^+ v = 0 \quad (i = 1, \ldots, n-1), \]
\[ K_i v = \kappa_i v \quad (i = 1, \ldots, n-1), \quad K v = \prod_{i=0}^{n-1} \kappa_i \cdot v. \]

Highest weight modules of $\mathcal{E}'_n$ are defined in terms of generators $\theta^{-1}(E_i(z)), \theta^{-1}(F_i(z)), \theta^{-1}(K_i^\pm(z))$ obtained by applying Miki’s automorphism $\theta$ which interchanges the vertical and horizontal subalgebras [Mi1], see also [FJMM1]. Let $\mathbf{P} = (P_0(z), \ldots, P_{n-1}(z)) \in \mathbb{C}(z)^n$ be an $n$-tuple of rational functions which are regular at $z^{\pm 1} = \infty$ and satisfy $P_i(0) P_i(\infty) = 1$. An $\mathcal{E}'_n$ module $W$ is a highest weight module of highest weight $\mathbf{P}$ if it is generated by a cyclic vector $w$ satisfying
\[ \theta^{-1}(E_i(z)) w = 0 \quad (i = 0, 1, \ldots, n-1), \]
\[ \theta^{-1}(K_i^\pm(z)) w = P_i(z) w \quad (i = 0, 1, \ldots, n-1). \]

In the last line, $P_i(z)$ stands for its expansion at $z^{\pm 1} = \infty$. Highest weight modules of $\mathcal{E}'_n$ were studied in detail in [FJMM2].

It turns out that evaluation highest weight $U'_q \hat{gl}_n$ module is a highest weight $\mathcal{E}'_n$ module. The following proposition describes the corresponding highest weight.

**Proposition 7.1.** Let $V$ be a highest weight $U'_q \hat{gl}_n$ module with highest weight $(\kappa_0, \ldots, \kappa_{n-1})$. Let $V(u)$ be the evaluation $\mathcal{E}'_n$ module. Then $q_3^n = \prod_{i=0}^{n-1} \kappa_i^2$ and $V(u)$ is a highest weight module with highest weight
\[\left( \frac{1 - v/z}{1 - \kappa_0^2 v/z}, \frac{1 - q_3^{-1} \kappa_0^2 v/z}{1 - \kappa_1^2 v/z}, \ldots, \frac{1 - q_3^{-n+1} \left( \prod_{i=1}^{n-2} \kappa_i^2 v/z \right)}{1 - q_3^{-n+1} \left( \prod_{i=0}^{n-1} \kappa_i^2 v/z \right)} \right)\]
for an appropriate choice of $v \in \mathbb{C}^\times$.

**Proof.** The proposition is proved similarly to Theorem 5.7 in [MY]. \qed

In the case of evaluation modules defined for $C = q_1^{n/2}$, the highest weight should read
\[\left( \frac{1 - \kappa_0^{-2} v/z}{1 - v/z}, \frac{1 - q_1 (\kappa_0 \kappa_1)^{-2} v/z}{1 - q_1 \kappa_0^{-2} v/z}, \ldots, \frac{1 - q_1^{n-1} \left( \prod_{i=0}^{n-2} \kappa_i^{-2} \right) v/z}{1 - q_1^{n-1} \left( \prod_{i=0}^{n-1} \kappa_i^{-2} \right) v/z} \right)\]
where $q_1^n = \prod_{i=0}^{n-1} \kappa_i^{-2}$.

It follows from Proposition 7.1 that modules $\mathcal{V}(u_1, \ldots, u_n)$ and $\mathcal{G}_{\mu, \nu}(k)$ in [FJMM2] are evaluation Verma and Weyl type modules respectively. Note that [FJMM2] deals with lowest weight modules and level of these modules is $q_1^{n/2}$. Therefore, the evaluation map should be modified accordingly.

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