About the value of the two dimensional Levy’s constant

Yitwah Cheung\textsuperscript{1} \quad Nicolas Chevallier\textsuperscript{2}

\textsuperscript{1}Yau Mathematical Sciences Center, Tsinghua University
\textsuperscript{2}Département de Mathématiques, Université de Haute Alsace

July 6, 2021

Abstract

We give a numerical approximation of the Lévy constant on the growth of the denominators of the best Diophantine approximations in dimension 2 with respect to the euclidean norm. This constant is expressed as an integral on a surface of dimension 7. We reduce the computation of this integral to a triple integral, whose numerical evaluation was carried out in [3].

1 Introduction

In 1936, Aleksandr Khintchin showed that there exists a constant $K$ such that the denominators $(q_n)_{n \geq 0}$ of the convergents of the continued fraction expansions of almost all real numbers $\theta$ satisfy

$$\lim_{n \to \infty} \frac{1}{n} \ln q_n = K$$

Soon afterward, Paul Lévy gave the explicit value of the constant,

$$K = \frac{\pi^2}{12 \ln 2}.$$ 

In [2], this result is extended to the denominators of best Diophantine approximations to vectors in $\mathbb{R}^d$ and even to matrices in $M_{d,c}(\mathbb{R})$. The value of the limit is given by an integral $\int_S d\mu_S$ over a codimension one submanifold $S$ in the space of lattices $\text{SL}(d+1, \mathbb{R})/\text{SL}(d+1, \mathbb{Z})$ (see Section 2.2 below). However, apart from the case of $d = 1$, this integral is very difficult to calculate. The aim of this document is to give a numerical approximation of the integral associated with best Diophantine approximations to vectors in $\mathbb{R}^2$.

This document is organized as follows. We first give the definition of the submanifold $S$ together with a parametrization of $S$. Then we give an explicit formula for the measure $\mu_S$ induced by the flow. The two difficult parts of the work are the explicit description of the domain of integration, i.e., the subset of parameters corresponding to $S$, and the calculation of the integral $\int_S d\mu_S$. This is done in the two last sections. For more details on best Diophantine approximation we refer the reader to [2].
2 Definitions of $S$ and its parametrization

2.1 Definition

The surface $S$ is the set of unimodular lattices $\Lambda$ in $\mathbb{R}^3$ such that there exist two independent vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ in $\Lambda$ such that:

- $|u_3|$ and $\sqrt{v_1^2 + v_2^2}$ are $< r = |v_3| = \sqrt{u_1^2 + u_2^2}$,
- the only nonzero points of $\Lambda$ in the cylinder $C(r) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \max(\sqrt{x_1^2 + x_2^2}, |x_3|) \leq r\}$ are $\pm u$ and $\pm v$.

Let $\mu_G$ be the Haar measure in $G = \text{SL}(3, \mathbb{R})$. Let $\mu$ be the measure in the space of unimodular lattices $\text{SL}(3, \mathbb{R})/\text{SL}(3, \mathbb{Z})$ invariant by the left action of $G$, induced by $\mu_G$. In turn, let $\mu_S$ be the measure induced by $\mu$ and the diagonal flow $g_t = \text{diag}(e^t, e^t, e^{-2t}), t \in \mathbb{R}$ on $S$.

2.2 The main formula

In [2], it is proved that for Lebesgue-almost all $\theta \in \mathbb{R}^2$,

$$\lim_{n \to \infty} \frac{1}{n} \ln q_n(\theta) = \frac{2\mu(\text{SL}(3, \mathbb{R})/\text{SL}(3, \mathbb{Z}))}{\mu_S(S)}$$

where $(q_n(\theta))_n$ is the sequence of best approximation denominators of $\theta$ associated with the standard Euclidean norm in $\mathbb{R}^2$.

2.3 Parametrization of $S$

Let $\Lambda$ be a lattice in $S$ and let $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ be the two vectors associated with $\Lambda$ by the definition of $S$. By Lemma 10 of [2], there exists a vector $w \in \Lambda$ such that $u, v, w$ is a basis of $\Lambda$. We can suppose $u_3$ and $v_3 \geq 0$ w.l.g.. There is a rotation $k_\theta$ in $\text{SO}(3, \mathbb{R})$ that fixes $e_3$ and such that $rk_\theta e_1 = (u_1, u_2, 0)$ where $r = \sqrt{u_1^2 + u_2^2}$. If $M$ is the $3 \times 3$ matrix whose columns are $u, v$ and $w$, we have

$$\Lambda = M \mathbb{Z}^3,$$

$$M = r k_\theta \begin{pmatrix} 1 & a_1 & c_1 \\ 0 & a_2 & c_2 \\ b & 1 & c_3 \end{pmatrix}$$

where $b \in [0, 1], a_1^2 + a_2^2 < 1$ and

$$\det M = r^3((1 - a_1 b)(-c_2) - a_2 (bc_1 - c_3)) = 1.$$
Therefore, we can parametrize $S$ with the seven parameters
$$\theta, b, a_1, a_2, c_1, c_2, c_3.$$ 

The problem is now to find a subset of parameters $\Omega_7$ such that
- $\pm u$ and $\pm v$ are the only nonzero vector of $\Lambda$ in the cylinder $C(r)$,
- for every $\Lambda \in S$ there exists exactly one 7-tuple $(\theta, b, a_1, a_2, c_1, c_2, c_3)$ such that $\Lambda = M\mathbb{Z}^3$, see the section about the Domain of integration.

3  Induced measure on $S$

We use Siegel normalization of the Haar measure on $G = \text{SL}(3, \mathbb{R})$, see [1] Lecture XV. For a Borel set $B \subset \text{SL}(3, \mathbb{R})$,
$$\mu_G(B) = \text{Lebesgue}_{\mathbb{R}^9}(B')$$
where $B' = \{ tM : M \in B, t \in [0, 1] \}$. Consider the parametrization of $\text{GL}_+(3, \mathbb{R})$ given by
$$M = g_t r k_\theta \begin{pmatrix} 1 & a_1 & c_1 \\ 0 & a_2 & c_2 \\ b & 1 & c_3 \end{pmatrix}.$$ 

In these coordinates the standard volume form in $\mathbb{R}^9$ is
$$3r^8 d(r, t, \theta, a_1, a_2, b, c_1, c_2, c_3).$$ 

We wish to replace $r$ with the homogeneous coordinate $\rho := \Delta^{1/3}$ where
$$\Delta = \det M = r^3 \eta,$$ 
and $\eta := (1 - a_1 b)(-c_2) - a_2 (bc_1 - c_3) > 0$.

Rewrite Lebesgue volume form in the new coordinates as
$$\frac{\Delta^2 d(\Delta, t, \theta, a_1, a_2, b, c_1, c_2, c_3)}{\eta^3} = \frac{3\rho^8 d(\rho, t, \theta, a_1, a_2, b, c_1, c_2, c_3)}{\eta^3}.$$ 

The local form of Haar measure with Siegel’s normalization is
$$d\mu_G = \int_0^1 \frac{3\rho^8 d(t, \theta, a_1, a_2, b, c_1, c_2, c_3)}{\eta^3} d\rho = \frac{d(t, \theta, a_1, a_2, b, c_1, c_2, c_3)}{3((1 - a_1 b)(-c_2) - a_2 (bc_1 - c_3))^3}.$$ 

Using this normalization, we can quote Siegel’s formula and bypass the computation of the first return time:
$$\text{vol}(\text{SL}(3, \mathbb{R})/\text{SL}(3, \mathbb{Z})) = \frac{\zeta(2)\zeta(3)}{3}.$$ 

\footnote{This was obtained by computing the determinant of a 9-by-9 matrix. The factor $r^8$ is expected as $r$ is homogeneous of degree one. For $g_t = \exp(\text{diag}(\lambda_1 t, \lambda_2 t, \lambda_3 t))$ the leading coefficient generalizes to $\lambda_1 - \lambda_3$.}
The local form for the induced measure on the transversal $S$ is now

$$d\mu_S = \frac{d(\theta, a_1, a_2, b, c_1, c_2, c_3)}{3((1 - a_1 b)(-c_2) - a_2 (bc_1 - c_3))^3}.$$ 

Levy’s constant is $d = 2$ times the average return time (see the main formula), or

$$K = L_{2,1} = \frac{2\zeta(2)\zeta(3)}{3\mu_S(\Omega_7)}$$

where $\Omega_7$ is any domain of integration parametrizing the transversal $S$.

4 Domain of integration

The suspension of the transversal gives a fundamental domain for the action of $\text{SL}(3, \mathbb{Z})$ that is invariant under is $\text{SO}(2)$ rather than $\text{SO}(3)$. The quotient of the transversal by the circle action is a 6 dimensional space that we shall realize as a fiber bundle over the 3 dimensional base $\Omega_2 \times [0, 1[$ where

$$\Omega_2 = \{a = a_1 + ia_2 \in \mathbb{C} : |a| < 1, |a - 1| \geq 1\}$$

parametrizes the possible configurations of $v = (a_1, a_2, 1)$ and $u = (1, 0, b)$ on the boundary of the standard unit cylinder $C_0 = C(1)$. Given a lattice in our transversal $S$ we rescale and rotate so that the systole cylinder is given by $C_0$ with the vectors $u$ and $v$ on its $\partial_+-$ and $\partial_- -$faces. Ambiguity in the normal form can be ignored since it occurs on a set of positive codimension. The lattice is determined by specifying the third vector $w = (c_1, c_2, c_3)$ that forms an integral basis for the lattice, positively oriented by requiring $c_2 < 0$. We have some freedom for the choice of $w$, which we will describe next. Once this choice is made, the projection from the 6 dimensional total space to the base will have been specified.

The space of possibilities for $w$ is a connected component of the complement of the union of $C_0$ with all its translates under the action of the discrete subgroup $\mathbb{Z}u + \mathbb{Z}v$. The topological boundary is tessellated by the domain $G(a, b)$ where this surface meets $\partial C_0$. The orthogonal projection of $G(a, b)$ onto the $(c_1, c_3)$-plane is a rectilinear domain that we will denote by

$$F(a, b).$$

The choice of $w$ is such that its orthogonal projection lies in $F(a, b)$; that is,

$$(c_1, c_3) \in F(a, b) \quad \text{and} \quad -c_2 \geq \sqrt{1 - c_1^2}.$$ 

$F(a, b)$ is rectilinear since curved arcs on $\partial G(a, b)$ map to horizontal segments.
4.1 Intersection patterns

For the precise description of $F(a,b)$ we need to know which cylinders appear on $\partial G(a,b)$. By symmetry, it suffices to consider the case $a_2 > 0$.

**Lemma 1.** $C_{\pm u}, C_{\pm v}, C_{\pm(u+v)}$ and $C_{\pm(v-u)}$ are the only translates whose intersection with $C_0$ has nonempty interior, unless $|2a-1| < 2$, when there is an additional pair $C_{\pm(2v-u)}$.

**Proof.** The hypothesis on $C_{mu+nv}$ is satisfied by $(m,n)$ in the intersection of the infinite strip $|bm+n| < 2$ and an ellipse that is contained in the vertical strip $|m| < 2|a_1|/|a|$. The pair $(2, -1)$ lies in the ellipse if and only if $|2a-1| < 2$. □

**Lemma 2.** $C_{\pm(2v-u)}$ is disjoint from $\partial C_0 \setminus U$ where $U$ is the union of the cylinders in Lemma 7.

**Proof.** By symmetry, it suffices to show $C_{2v-u} \cap \partial C_0 \subset \text{Int } C_{v-u} \cup C_v$, which in terms of euclidean disks is the condition $D_0 \cap D_{2a-1} \subset \text{Int } D_a \cup D_{a-1}$. For this, it is enough to verify that $D_0 \cap D_{2a-1}$ is contained in the open disk of radius $1/2$ centered at $a-1/2$, i.e.

\[
\left| \chi_- (2a - 1) - a + \frac{1}{2} \right| < \frac{1}{2}
\]

which is equivalent to $|2a-1| < 2$. Here, the notation

\[
\chi_{\pm}(a) := \frac{a}{2} \pm \frac{ia}{|a|} \sqrt{1 - \frac{|a|^2}{4}}
\]

denotes the two points where the circles $|z| = 1$ and $|z-a| = 1$ meet. □

To show that a cylinder meets $\partial G(a,b)$ in an arc involves checking that the arc is disjoint from the other $7$ cylinders. There does not seem to be an easier way to carry out this tedious task, e.g., to verify that the cylinders $C_{\pm u}$ and $C_{\pm v}$ appear on $\partial G(a,b)$ in every possible scenario. We shall skip the verification of these intuitive claims and simply identify the criteria for the appearance of the other $4$ cylinders. It is perhaps surprising that the answer is always determined by the sign of $|a-\xi| - 1$ where $\xi$ denotes the sixth root of unity in the first quadrant.

**Lemma 3.** $C_{\pm(u-v)}$ appear iff $|a-\xi| < 1$ while $C_{\pm(u+v)}$ appear iff $|a-\xi| > 1$.

**Proof.** Considerations of elementary nature lead to the following characterizations: $C_{u-v}$ appears iff $d(1-a, \xi) < 1$ while $C_{v-u}$ appears iff $d(a - 1, -\xi) < 1$; $C_{u+v}$ appears iff $d(a + \xi, 1) > 1$ while $C_{u-v}$ appears iff $d(-a - \xi, -1) > 1$. In each case the sign of $|a-\xi| - 1$ is nonzero. □

The shape of $F(a,b)$ also depends on how the cylinders intersect with $C_0$.

**Lemma 4.** $G(a,b)$ has overlap with the bottom of $C_0$ iff $|a+\xi| < 1$.

**Proof.** Note that the point where the curved faces of $C_{-v}$ and $C_{-u-v}$ meet the plane $z = -1$ is described by the complex number $-a - \xi$. □
Figure 1: Three subregions of $\Omega_2^+$.  

Let us divide $\Omega_2^+ := \{a \in \Omega_2 : a_2 \geq 0\}$ into the following subregions as depicted in Figure 1:

- $\Omega_1^+ = \{a \in \Omega_2 : |a - \xi| < 1\}$
- $\Omega_{II}^+ = \{a \in \Omega_2 : |a - \xi| \geq 1, |a + \xi| \geq 1\}$
- $\Omega_{III}^+ = \{a \in \Omega_2 : |a + \xi| < 1, \text{Im } a \geq 0\}$

Next, we describe $G(a, b)$ in each of the 3 main cases.

CASE $a \in \Omega_1^+$: $G(a, b)$ is bounded by the cylinders $C_{\pm u}, C_{\pm v}$ and $C_{\pm (v-u)}$ and consists of two subregions $G_+ \sqcup G_0$ joined along an arc, labeled $\oplus$, along the top rim of $C_0$. The arcs along the boundary of each subregion in counter-clockwise order are labeled

- $G_+ : \oplus, u, v, v - u$
- $G_0 : \oplus, v - u, -u, -v, u - v, u - v, u$

CASE $a \in \text{Int } \Omega_{II}^+$: $G(a, b)$ is bounded by the cylinders $C_{\pm u}, C_{\pm v}$ and $C_{\pm (v+u)}$ and consists of two subregions as in the previous case, with arcs along the boundary of the subregions labeled

- $G_+ : \oplus, u, v + u, v$
- $G_0 : \oplus, v, -u, -u, -v - u, -v, -v, u$

CASE $a \in \Omega_{III}^+$: $G(a, b)$ is bounded by the same cylinders in the previous case but now consists of three subregions $G_+ \sqcup G_0 \sqcup G_-$ bounded by the following arcs

- $G_+ : \oplus, v + u, v$
- $G_0 : \oplus, v, -u, -v - u, -v, -v, u, v + u, v + u$
- $G_- : \ominus, -v - u, -v$

\[\text{Later, we shall show that the projection } F(a, b) \text{ forms a fundamental domain for the action of } \mathbb{Z}(1, b) + \mathbb{Z}(-a_1, 1), \text{ which indirectly shows that none of the arcs on } \partial F(a, b) \text{ can be blocked by the other cylinders.}\]
where $\otimes$ is an arc along the bottom rim of $C_0$ joining $G_0$ and $G_-$. In the next section, we use the above description of $G(a, b)$ to arrive at an explicitly described region $\tilde{F}(a, b)$ that is \textit{a priori} only known to contain $F(a, b)$. We will verify that $\tilde{F}(a, b)$ is a fundamental domain for the action of $\mathbb{Z}(1, b) + \mathbb{Z}(-a, 1)$, from which it follows that $\tilde{F}(a, b)$ provides an equivalent definition of $F(a, b)$. In anticipation of this conclusion and since there is no further need to distinguish between the two sets, we shall drop the overscript when referring to $\tilde{F}(a, b)$ in the following section.

### 4.2 Explicit description of $F(a, b)$

Each vertical arc on $\partial F(a, b)$ is the projection (in the $y$-direction) of a linear segment on $\partial G(a, b)$ along which one of the cylinders in Lemma 1 is transverse to $C_0$. Using complex notation for the projection (in the $z$-direction) of the linear segment, we arrive at the following table

| vertical arc | $z_0$ | $\kappa(z_0)$ |
|--------------|-------|---------------|
| $C_u$        | $\xi$ | $1/2$         |
| $C_{-u}$     | $-\xi$| $-1/2$        |
| $C_v$        | $\chi_+(a)$ | $\kappa(a)$ |
| $C_{-v}$     | $\chi_-(a)$ | $\kappa(-a)$ |
| $C_{u-v}$    | $\chi_+(1-a)$ | $\kappa(1-a)$ |
| $C_{v-u}$    | $\chi_+(a-1)$ | $\kappa(a-1)$ |
| $C_{a+v}$    | $\chi_-(a+1)$ | $\kappa(a+1)$ |
| $C_{-u-v}$   | $\chi_+(-a-1)$ | $\kappa(-a-1)$ |

where vertical arcs are labelled by the corresponding transverse cylinder and $\kappa$ is given by

$$\kappa(a) := \text{Re} \chi_+(a) = \frac{a_1}{2} - \frac{a_2}{|a|} \sqrt{1 - \frac{|a|^2}{4}}.$$ 

In each case, $F(a, b)$ can be described as some larger rectangle $[a, b] \times [c, d]$ with two or more “corners” removed. These “corners” will be described by the notation:

$$\begin{align*}
NW(x, y) := [a, x[ \times ]y, d] & \quad NE(x, y) := ]x, b] \times ]y, d] \\
SW(x, y) := [a, x[ \times ]c, y[ & \quad SE(x, y) := ]x, b] \times ]c, y[ 
\end{align*}$$

For $a \in \Omega_1^+$, $F(a, b)$ is given by

$$\left[ \kappa(a-1), \frac{1}{2} \right] \times [0, 1] \quad \text{minus} \quad SW(-1/2, 1-b) \cup SE(\kappa(1-a), b)$$

while for $a \in \text{Int } \Omega_1^+$, it is given by

$$\left[ \kappa(a), \frac{1}{2} \right] \times [-b, 1] \quad \text{minus} \quad SW(-1/2, 1-b) \cup SE(\kappa(-a), 0)$$

7
Figure 2: $F(a, b)$ in the case $a = \frac{3i}{10} \in \Omega_I^+$ and $b = .3$

Figure 3: $F(a, b)$ in the case $a = \frac{-9+3i}{10} \in \Omega_{II}^+$ and $b = .3$
Figure 4: $F(a, b)$ in the case $a = \frac{-10+i}{20} \in \Omega^+_{III}$ and $b = .4$
and for \( a \in \Omega_{III}^+ \) by
\[
\left[ -\frac{1}{2}, \frac{1}{2} \right] \times [-1, 1] \quad \text{minus} \quad NW(\kappa(a), 0) \cup NE(\kappa(1 + \bar{a}), b) \cup SW(\kappa(-a - 1), -b) \cup SE(\kappa(\bar{a}), 0).
\]

It is easy to verify directly from the explicit description that \( F(a, b) \) is a fundamental domain for the action of \( \mathbb{Z}(1, b) + \mathbb{Z}(a_1, 1) \). From this, it follows that area \( F(a, b) = 1 - a_1b \). This last claim can also be checked directly: Indeed, in the case \( a \in \Omega_{III}^+ \) this boils down to the identity \( \kappa(\bar{a}) - \kappa(a) = -a_1 \), while for \( a \in \Omega_{II}^+ \) it is the same identity with \( 1 - \bar{a} \) instead of \( a \). The case \( a \in \Omega_{III}^+ \) follows by observing that the total width of NE and NW corners is
\[
\left( \kappa(a) + \frac{1}{2} \right) + \left( \frac{1}{2} - \kappa(\bar{a} + 1) \right) = \kappa(\bar{a}) - \kappa(-a - 1).
\]

5 **Computation of \( \mu_S(\Omega_7) \)**

We have now established the closed form expression
\[
\mu_S(\Omega_7) = \frac{4\pi}{3} \int_{\Omega_7^+} da_1 da_2 \int_0^1 db \int_{F(a,b)} dc_1 dc_3 \int_0^{\infty} \frac{dc_2}{\sqrt{1- c_1^2} ((1 - a_1b)c_2 - a_2(bc_1 - c_3))^3}
\]
\[
= \frac{2\pi}{3} \int_{\Omega_7^+} da_1 da_2 \int_0^1 db \left( \frac{1}{1 - a_1b} \int_{F(a,b)} \frac{dc_1 dc_3}{\Xi^2} \right)
\]
where
\[
\Xi := (1 - a_1b)\sqrt{1 - c_1^2} - a_2(bc_1 - c_3).
\]

Green’s theorem applied to \( \frac{1}{\Xi^2} = \frac{\partial Q}{\partial c_1} - \frac{\partial P}{\partial c_3} \) with \( Q = 0 \) and \( P = \frac{1}{a_2 \Xi} \) implies
\[
\mu_S(\Omega_7) = \int_{\Omega_7^+} da_1 da_2 \int_0^1 db \left( \frac{2\pi/3}{a_2(1 - a_1b)} \int_{\partial F(a,b)} \frac{dc_1}{\Xi} \right).
\]

The substitution
\[
c_1 = \frac{2\tau}{1 + \tau^2} \quad \text{and} \quad dc_1 = \frac{2(1 - \tau^2)}{(1 + \tau^2)^2} d\tau
\]
and the observation
\[
(1 + \tau^2)\Xi = (1 - a_1b)(1 - \tau^2) - a_2(bc_1 - c_3)(1 + \tau^2)
\]
readily leads to
\[
\frac{dc_1}{\Xi} = \frac{2(1 - \tau^2)}{(1 + \tau^2)^2} d\tau
\]
\[
= \frac{(1 + \tau^2)(1 - a_1b + a_2c_3 - 2a_2b\tau - (1 - a_1b - a_2c_3)\tau^2)}{(A + B\tau + C(a_2b + (1 - a_1b - a_2c_3)\tau) + D)} d\tau
\]

\( ^3 \text{Remark: it can be shown that } \Xi > 1/24. \)
where
\[ B = C = \frac{a_2 b A}{1 - a_1 b}, \quad D = -a_2 c_3 A, \quad \text{and} \quad A = \frac{2(1 - a_1 b)}{(1 - a_1 b)^2 + a_2^2 b^2}. \]

Since \( A \tan^{-1} \tau + \frac{\pi}{2} \ln(1 + \tau^2) \) depends on \( c_1 \) but not on \( c_3 \), the sum over the corners of \( F(a, b) \) with alternating sign vanishes. The same reasoning applies to the first term of the decomposition
\[
\frac{C}{2} \ln \left\{ (1 - a_1 b + a_2 c_3) - 2a_2 b \tau - (1 - a_1 b - a_2 c_3) \tau^2 \right\} = \frac{C}{2} \ln(1 + \tau^2) + \frac{C}{2} \ln \Xi.
\]

The second term also vanishes when summed over the corners of \( F(a, b) \) with alternating sign. To see this, we consider the pairing of the corners of \( F(a, b) \) induced by the identification of vertical edges, noting that paired vertices are summed with opposite signs, and that \( \Xi \) is constant on each pair. It follows that
\[
\mu_S(\Omega_7) = \int_{\Omega_2^+} \frac{da}{b^2} \int_0^1 db \frac{4\pi/3}{(1 - a_1 b)^2 + a_2^2 b^2} \int_{\partial F(a,b)} \frac{(a_2 d\tau)}{1 - a_1 b + a_2 c_3 - 2a_2 b \tau - (1 - a_1 b - a_2 c_3) \tau^2}.
\]

Since we can factor
\[
(\phi_+ - 2a_2 b \tau - \phi_- \tau^2) \phi_- = (\sqrt{D} + a_2 b + \phi_- \tau)(\sqrt{D} - a_2 b - \phi_- \tau)
\]
where \( \phi_\pm = 1 - a_1 b \pm a_2 c_3 \) and
\[
D := (1 - a_1 b)^2 + a_2^2 (b^2 - c_3^2) \geq |a|^2 b^2 - 2a_1 b + 1 - a_2^2 \geq a_2^2 \left( \frac{1}{|a|^2} - 1 \right) > 0
\]
and noting that
\[
\frac{d}{d\tau} \left( \ln \left\{ \frac{\sqrt{(1 - a_1 b)^2 + a_2^2 (b^2 - c_3^2)} + a_2 b + (1 - a_1 b - a_2 c_3) \tau^2}{1 - a_1 b + a_2 c_3 - 2a_2 b \tau - (1 - a_1 b - a_2 c_3) \tau^2} \right\} \right) = \frac{2 \sqrt{(1 - a_1 b)^2 + a_2^2 (b^2 - c_3^2)}}{1 - a_1 b + a_2 c_3 - 2a_2 b \tau - (1 - a_1 b - a_2 c_3) \tau^2}
\]
the rational function of \( \tau \) can readily be integrated to yield
\[
\mu_S(\Omega_7) = \int_{\Omega_2^+} da \int_0^1 db \sum_{\nu} \frac{(-1)^\nu 2\pi c_3/3}{(1 - a_1 b)^2 + a_2^2 b^2} \ln \left( \frac{\phi_+ - 2a_2 b \tau - \phi_- \tau^2}{(\sqrt{D} + a_2 b + \phi_- \tau^2)^2} \right)
\]
where the sum is over the corners of \( F(a, b) \) labelled in such a way that the sign of, say \((1/2, b)\) in the case \( a \in \Omega_7 \) and \((1/2, 0)\) in the other two cases, is given a plus sign.

**Remark.** \( \sqrt{D} + a_2 b + \phi_- \tau > 0 \). (Proof. First note that \( \phi_+ - 2a_2 b \tau - \phi_- \tau^2 > 0 \) because \( \Xi > 0 \). In the case \( \phi_- > 0 \) the factors \( \sqrt{D} \pm (a_2 b + \phi_- \tau) \) have the same sign and cannot both be negative since their average is \( \sqrt{D} > 0 \). In the case \( \phi_- \leq 0 \), it suffices to show that \( |\phi_-| \leq a_2 b \) (since \( D > 0 \) and \( |\tau| \leq 1 \)) and this follows from \( D = a_2^2 b^2 - |\phi_-| \phi_+ > 0 \) and \( |\phi_-| \leq \phi_+ \).)
When evaluating the integrand, it is convenient to pair up terms with the same $c_3$ that are joined by a horizontal segment on $\partial F(a, b)$. If $\tau_- < \tau_+$ distinguishes the endpoints, then the integrand of the triple integral is the expression

$$
\frac{2\pi c_3/\sqrt{D}}{(1 - a_1 b)^2 + a_2^2 b^2} \ln \left\{ \left( \frac{\phi_+ - 2a_2 b \tau_+ + \phi_- \tau_+^2}{\phi_+ - 2a_2 b \tau_- + \phi_- \tau_-^2} \right) \left( \frac{\sqrt{D} + a_2 b + \phi_- \tau_-}{\sqrt{D} + a_2 b + \phi_- \tau_+} \right)^2 \right\}
$$

summed over $c_3 \in \{b, 1 - b, 1\}$ with signs $\{+, +, -\}$ for $a \in \Omega_1^+$; $c_3 \in \{-b, 1 - b, 1\}$ with signs $\{+, +, -\}$ for $a \in \Omega_1^+$; and $c_3 \in \{-1, -b, b\}$ with signs $\{+, +, -, -\}$ for $a \in \Omega_{III}$.

| $c_3$ | sign | $c_1(\tau_-)$ | $c_1(\tau_+)$ |
|-------|------|----------------|----------------|
| I     | 1    | $\kappa(a - 1)$ | .5             |
|       | $1 - b$ | $\kappa(a - 1)$ | .5             |
|       | $b$    | $\kappa(1 - a)$ | .5             |
| II    | 1    | $\kappa(a)$   | .5             |
|       | $1 - b$ | $\kappa(a)$   | .5             |
|       | $-b$  | -.5           | $\kappa(-a)$  |
| III   | 1    | $\kappa(a)$   | $\kappa(a + 1)$ |
|       | $b$    | $\kappa(a + 1)$ | .5             |
|       | $-b$  | -.5           | $\kappa(-a - 1)$ |
|       | $1$    | $\kappa(-a - 1)$ | $\kappa(-a)$ |

Note that $\tau$ as a function of $c_1$ is given by $\tau = \frac{1 - \sqrt{1 - c_1^2}}{c_1}$ apart from the removable singularity at $c_1 = 0$, so that

$$
\tau(a) = \frac{2a_1 - \text{sgn}(a_1)|a|\sqrt{4 - |a|^2}}{|a|^2 + \text{sgn}(a_1)2a_2}.
$$

The expression that was fed into Octave is

$$
3\mu_S(\Omega_7) = \int_{\Omega_1^+} da \int_0^1 db \sum_{c_3} \frac{(-1)^\nu 2\pi c_3}{(1 - a_1 b)^2 + a_2^2 b^2 \sqrt{D}} \ln \frac{1 - x}{1 + x}
$$

$$
= \int_{\Omega_1^+} da \int_0^1 db \sum_{c_3} \frac{(-1)^\nu 2\pi c_3(\tau_+ - \tau_-)}{(1 - a_1 b)^2 + a_2^2 b^2 (\phi_+ - \tau_+ \phi_- - a_2 b (\tau_+ + \tau_-))} \frac{1}{x} \ln \frac{1 - x}{1 + x}
$$

where

$$
x = \frac{(\tau_+ - \tau_-)\sqrt{D}}{\phi_+ \tau_+ \phi_- - a_2 b (\tau_+ + \tau_-)}
$$

and the value obtained by numerical integration is

$$
3\mu_S(\Omega_7) = 3.49277983865703...
$$

which, using $\zeta(2) = 1.2020569031...$ and $\zeta(3) = 1.649340668...$, leads to

$$
L_{2,1} = 1.13525697416719...
$$

which is the value reported in [3].

---

[3] In [3], the factor of 3 is missing.
References

[1] Siegel, Carl Ludwig. Lectures on the geometry of numbers. Springer-Verlag, Berlin, 1989.

[2] Yitwah Cheung, Nicolas Chevallier, Lévy-Khintchin Theorem for best simultaneous Diophantine approximations, arXiv (2019)

[3] Seraphine Xieu, SFSU Applied Math Project, http://math.sfsu.edu/cheung/xieu-amp.pdf