Isotropic–Nematic Phase Transitions in Gravitational Systems. II. Higher Order Multipoles

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Abstract
The gravitational interaction among bodies orbiting in a spherical potential leads to the rapid relaxation of the orbital planes’ distribution, a process called vector resonant relaxation. We examine the statistical equilibrium of this process for a system of bodies with similar semimajor axes and eccentricities. We extend the previous model of Roupas et al. by accounting for the multipole moments beyond the quadrupole, which dominate the interaction for radially overlapping orbits. Nevertheless, we find no qualitative differences between the behavior of the system with respect to the model restricted to the quadrupole interaction. The equilibrium distribution resembles a counterrotating disk at low temperature and a spherical structure at high temperature. The system exhibits a first-order phase transition between the disk and the spherical phase in the canonical ensemble if the total angular momentum is below a critical value. We find that the phase transition erases the high-order multipoles, i.e., small-scale structure in angular momentum space, most efficiently. The system admits a maximum entropy and a maximum energy, which lead to the existence of negative temperature equilibria.

Key words: galaxies: evolution – galaxies: nuclei – galaxies: structure – Galaxy: center – Galaxy: nucleus – stars: kinematics and dynamics

1. Introduction
Most galaxies contain a supermassive black hole (SMBH) at their centers with mass \( M \approx 10^6–10^{10} M_\odot \). In many cases, the SMBH is surrounded by a very dense stellar system of a few parsecs called the nuclear star cluster (NSC; Kormendy & Ho 2013). Understanding the processes within the NSC may be key to understanding the growth of SMBHs along with the large-scale structure of the host galaxy, which is regulated by the SMBH.

The dynamical evolution in such systems is governed by several processes operating on different timescales (see Rauch & Tremaine 1996; Hopman & Alexander 2006; Gurkan & Hopman 2007; Eilon et al. 2009; Kocsis & Tremaine 2011, 2015). At leading order, bodies follow eccentric, near-Keplerian orbits with period \( t_{\text{orb}} \). On longer timescales, \( t_{\text{aps}} \), the Newtonian gravity of the spherical mass distribution and general relativistic effects lead to apsidal precession. On even longer timescales, \( t_{\text{vrr}} \), the nonspherical components of the gravitational field of the stellar system lead to diffusion in the orientation of the orbits without changing the magnitude of angular momenta and the mechanical (or binding) energy of the orbits around the SMBH, resulting in the conservation of the orbital eccentricities and semimajor axes. This process is called vector resonant relaxation (VRR). On still longer timescales, \( t_{\text{srr}} \), the nonaxisymmetric torques between the stellar orbits lead to a diffusion in the eccentricities (or, equivalently the magnitudes of the angular momenta), a process called scalar resonant relaxation (SRR). Due to the close encounters of the bodies, the two-body relaxation changes the semimajor axes or Keplerian energies of the orbits on the longest timescale. Therefore, the dynamical evolution is strongly collisional at the \( t_{\text{vrr}} \) timescale and above. These timescales are estimated as follows (Kocsis & Tremaine 2011):

1. orbital period: \( t_{\text{orb}} \sim (GM/r^2)^{-1/2} \sim 1–10^4 \) years;
2. apsidal precession: \( t_{\text{aps}} \sim t_{\text{orb}} M/(Nm) \sim 10^3–10^4 \) years;
3. orbital plane reorientation: \( t_{\text{vrr}} \sim t_{\text{orb}} M/\sqrt{N m} \sim 10^6–10^7 \) years;
4. angular momentum diffusion: \( t_{\text{srr}} \sim t_{\text{orb}} M/m \sim 10^8–10^9 \) years; and
5. Keplerian energy diffusion: \( t_{2\text{-body}} \sim t_{\text{orb}} M/(Nm^2) \sim 10^9–10^{10} \) years.

More generally, this hierarchy applies for collisional gravitational systems dominated by a point mass or spherical potential (e.g., planetary systems, moon systems, etc.). Globular clusters may also exhibit VRR (Y. Meiron & B. Kocsis 2018, in preparation).

The statistical physics of gravitational systems have been examined by several previous works (see e.g., Nakamura 2000; Chavanis 2002; Arad & Lynden-Bell 2005; Levin et al. 2014; Touma & Tremaine 2014; Fouvré et al. 2015, 2017; Tremaine 2015; Bar-Or & Alexander 2016; Rocha Filho 2016; Sridhar & Touma 2016a, 2016b, 2017 and references therein). Recently, in a companion paper, Roupas et al. (2017) examined the statistical equilibrium of orbital planes after the VRR process had been completed for one-component system in which the semimajor axis and eccentricities were the same for all objects. They truncated the interaction at the quadrupole order in a multipole expansion. These simplifying assumptions have led to a tractable model in which the statistical equilibrium structure of phase space could be completely mapped out. The results showed that the stable distribution of bodies close to zero temperature represents an axisymmetric thin disk in which the bodies may orbit in both senses and where the disk thickness increases with temperature. Anisotropic (i.e., biaxial) long-lived metastable equilibria were also found, which consist of two disks with a relative inclination larger than 90°. If the total angular momentum is smaller than a critical value, the system exhibits a first-order phase transition in the canonical ensemble from the disk phase to a nearly spherical distribution.
Furthermore, the results showed that negative temperature equilibria are possible and are stable, a curious phenomenon in condensed matter physics (see in Braun et al. 2013; Dunkel & Hilbert 2013; Campisi 2015; Cerino et al. 2015; Frenkel & Warren 2015; Poulter 2016).

This is the second paper in this series, in which we extend Roupas et al. (2017) to examine the statistical equilibrium of orbital planes due to VRR by relaxing the quadrupole-interaction approximation. Kocsis & Tremaine (2015) have shown that the quadrupole approximation dominates the dynamics for radially widely separated orbits, but for systems with radially overlapping orbits the contribution of higher \( \ell \) multipoles dominates for mutual inclinations less than \( O(1/\ell) \).

In this case, the Hamilton-equations of motion of VRR are mathematically similar to that of a point vortex system on the sphere (see Equation B84 in Appendix B in Kocsis & Tremaine 2015). In this study, we restrict our attention to an axisymmetric distribution of angular momentum vectors and keep the simplifying assumption of a single-component system with the same semimajor axis, eccentricity, and mass for all objects (Roupas et al. 2017).\(^2\) We may expect that the distribution is strongly modified by higher order multipoles in the case where the distribution forms a thin disk. But since the first-order phase transition takes place between a thick disk and a spheroidal structure, one may expect a similar phenomenon to also take place in models with higher multipoles. Similarly, the negative temperature states are broadly distributed in inclination, which may indicate that they are not greatly affected by high-order moments. We verify these expectations quantitatively in this paper.

Important astronomical applications of VRR include the origin and evolution of a thin disk of massive stars in the Galactic center (see in Bartko et al. 2009; Kocsis & Tremaine 2011; Haas & Šubr 2014a, 2014b; Yelda et al. 2014; Sridhar & Touma 2016b, 2017). Furthermore, VRR affects the distribution of putative stellar mass black holes in NSCs, which may represent important sources of gravitational waves (GWs) for existing and upcoming instruments: LIGO,\(^3\) VIRGO,\(^4\) KARGA,\(^5\) and LISA\(^6\) (O’Leary et al. 2009; Antonini & Perets 2012; McKernan et al. 2013, 2014b, 2014a; Bartos et al. 2017; Hoang et al. 2017; McKernan et al. 2017). The distribution of event rates as a function of mass and eccentricity may depend on the structure of the dynamical environments in which these GW sources form (Gondan et al. 2017).

### 2. Dynamics of VRR

#### 2.1. Hamiltonian

To examine VRR, the \( N \)-body gravitational Hamiltonian is averaged over the apsidal precession timescale. During this process, the Keplerian Hamiltonian of orbits around the SMBH are conserved for each orbit, respectively; hence these conserved terms in the Hamiltonian may be dropped. This yields (Kocsis & Tremaine 2015)

\[
H_{\text{VRR}} = -\frac{1}{2} \sum_{i \neq j} \frac{Gm_i m_j}{|r_i(t) - r_j(t')|} \left|_{t, t'} \right.
\]

\[
= -\frac{1}{2} \sum_{i \neq j} \sum_{\ell=0}^{\infty} J_{ij\ell} P_{\ell}(\hat{L}_i \cdot \hat{L}_j),
\]

where \( i, j = 1, \ldots, N \) by which we express the results in, \( G \) is the gravitational constant, and the distance \( r \) is measured from the SMBH, and \( P_{\ell}(x) \) is Legendre polynomials. The dynamical variables are the directions of angular momentum vectors \( \hat{L}_i \). Coupling coefficients are generally given by Kocsis & Tremaine (2015), which are nonzero for even \( \ell \) and depend on the mass, semimajor axis, and eccentricity of the bodies.

In this paper, we examine the case of radially overlapping orbits. In this case, the coupling coefficients follow asymptotically

\[
J_{ij\ell} = J^0_{ij\ell} / \ell^2
\]

for large and even \( \ell \), where \( J^0_{ij\ell} \) does not depend on \( \ell \) (see Appendices B5 and B6 in Kocsis & Tremaine 2015). For all odd \( \ell \), \( J_{ij\ell} = 0 \). Note that this approximation to utilize this expression for all even \( \ell \), which is complementary to the quadrupolar approximation, is expected to be sufficient to predict the qualitative behavior of radially overlapping systems, as the net contribution of small \( \ell \) terms is subdominant in this case (see Figure B1 in Kocsis & Tremaine 2015). The numerical value of \( J^0_{ij\ell} \) is not important, as we express the results in those units. The sum over \( \ell \) asymptotically simplifies to

\[
H_{\text{VRR}} \approx -\sum_{i \neq j} \frac{J^0_{ij\ell}}{2} \left[ 1 - x_1 - x_2 + x_1^2 \ln \left( 1 + \frac{1}{x_1} \right) \right.
\]

\[
+ x_2^2 \ln \left( 1 + \frac{1}{x_2} \right) \right],
\]

where we used the generator function of Legendre polynomials as discussed in Appendix B6 in Kocsis & Tremaine (2015), and introduced \( x_i = 1 + \hat{L}_i \cdot \hat{L}_j / 2 \)^1/2.

#### 2.2. Mean-field Theory

We formulate the mean-field model for \( N \) bodies with identical mass \( m \), semimajor axis \( a \), eccentricity \( e \), and angular momentum magnitudes \( l = m_i \sqrt{G M a (1 - e^2)} \). In this case, \( J_{ij\ell} \) in Equations (1)–(2) does not vary with \( i \) and \( j \). To separate the Hamiltonian, we expand \( P_{\ell}(\hat{L}_i \cdot \hat{L}_j) \) using spherical harmonics (Jackson 1975),

\[
H_{\text{VRR}} = -\frac{1}{2} \sum_{i \neq j} \sum_{\ell=0}^{\infty} J_{ij\ell} \frac{4\pi}{2\ell + 1} \times \sum_{m=-\ell}^{\ell} Y^m_{\ell}(\hat{L}_i) Y^m_{\ell}(\hat{L}_j).
\]

The system is characterized by the total number of bodies \( N \), the total angular momentum \( L \), and the total orbit- and apsidal-precession-averaged VRR energy \( E \). We define the number of bodies with the direction of angular momentum unit vector \( n \)
oriented within an infinitesimal solid angle element $d\Omega$ to be $f(n)d\Omega$. The particle number, total angular momentum, and energy are, respectively,

$$N = \int d\Omega f(n),$$  \hspace{1cm} (5)

$$L = \int d\Omega f(n)n,$$  \hspace{1cm} (6)

$$E = \frac{1}{2}N\langle \varepsilon \rangle,$$  \hspace{1cm} (7)

where $\langle \varepsilon \rangle$ denotes the mean energy

$$\varepsilon(n) = -\frac{1}{2}JN \sum_{\ell=-\infty}^{\infty} \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} (Y_{\ell m}^m) Y_{\ell m}^m(n),$$  \hspace{1cm} (8)

where the configuration average over all particles for any function $X(n)$ is defined by

$$\langle X \rangle = \frac{1}{N} \int d\Omega f(n)X.$$  \hspace{1cm} (9)

### 2.3. Statistical Equilibrium

We determine the equilibrium distribution of $f(n)$ by extremizing the Boltzmann entropy

$$S = -k_B \int d\Omega f(n) \ln f(n),$$  \hspace{1cm} (10)

with respect to the equilibrium distribution for fixed total energy $E$, total angular momentum $L$, and particle number $N$. This gives

$$\delta S / k_B + \alpha \delta N - \beta \delta E + \gamma \cdot \delta L = 0,$$  \hspace{1cm} (11)

where $\alpha$, $\beta$, and $\gamma$ are the Lagrange multipliers of the constraints (Equations (5)-(7)). The resulting equilibrium distribution is (Roupas et al. 2017)

$$f(n, \beta, \gamma, N) = N e^{-\beta E - \gamma \cdot L} / \int d\Omega e^{-\beta E - \gamma \cdot L}.$$  \hspace{1cm} (12)

The order parameters satisfy the mean-field self-consistency equation

$$\langle Y_{\ell m}^m \rangle = \frac{1}{N} \int d\Omega Y_{\ell m}^m(n)f(n),$$  \hspace{1cm} (13)

where $f(n)$ is the equilibrium distribution function given by the order parameters in Equations (8) and (12). To avoid confusion, note that here $\langle Y_{\ell m}^m \rangle$ in the left-hand side of Equation (13) labels the unknown parameters that we determine by solving this equation, and the right-hand side depends on this parameter through the equilibrium distribution $f(n)$ via $\varepsilon(n)$. Since the right-hand side depends only on $\langle Y_{\ell m}^m \rangle$ with even $\ell$, the $\langle Y_{\ell m}^m \rangle$ unknowns form a closed set of equations. Once determined, the distribution function follows from Equations (8) and (12). Note that, given $\langle Y_{\ell m}^m \rangle$, the odd moments $\langle Y_{\ell m}^m \rangle$ may be evaluated directly from Equation (13). They are also generally nonzero if $L \neq 0$ since this implies that $\gamma \neq 0$.

In equilibrium, the entropy and free energy are

$$S(E, L) / k_B = 2\beta E - \gamma \cdot L + N \ln Z_0 - N \ln N,$$  \hspace{1cm} (14)

$$F(T, L) = -S / k_B = -E + \gamma \cdot L - N \ln Z_0,$$  \hspace{1cm} (15)

where $T = \beta^{-1}$ is the temperature and

$$Z_0 = \int d\Omega e^{-\beta E - \gamma \cdot L}.$$  \hspace{1cm} (16)

### 2.4. Axisymmetric System

In the rest of the paper, we restrict our investigation for simplicity to an axisymmetric ($m = 0$) radially overlapping system. Roupas et al. (2017) found that, for $\ell = 2$, the maximum entropy configuration at any nonzero temperature is always axisymmetric, but at low temperatures nonaxisymmetric metastable equilibria also exist that may be long-lived. Metastable equilibria may be studied by solving the nonlinear self-consistency system of Equations (13) with $m = 0$, but this lies beyond the scope of this paper. We examine the canonical equilibrium distribution for fixed ($\beta, N, L$). In this case, the self-consistency equations for the order parameters are

$$\langle Y_{\ell m}^m \rangle = \frac{\int d\Omega e^{-\beta \varepsilon(\theta) - \gamma \cdot L} Y_{\ell m}^m(n)}{\int d\Omega e^{-\beta \varepsilon(\theta) - \gamma \cdot L}}.$$  \hspace{1cm} (17)

$$\frac{L}{Nl} = \frac{\int d\Omega e^{-\beta \varepsilon(\theta) - \gamma \cdot L} \cos \theta}{\int d\Omega e^{-\beta \varepsilon(\theta) - \gamma \cdot L} \cos \theta},$$  \hspace{1cm} (18)

$$\varepsilon(\theta) = -\sum_{\ell = 0}^{\infty} \frac{J/2}{\ell^2/2+1} \langle Y_{\ell m}^m \rangle \langle Y_{\ell m}^m \rangle.$$  \hspace{1cm} (19)

Entropy extrema have the property that $\gamma$ is parallel with $L$, which we choose along $\theta = 0$ (see Appendix B in Roupas et al. 2017).

By definition, the total angular momentum is bounded between $0 \leq L/(NL) \leq 1$; therefore, the order parameters are bounded\footnote{We use the definition $Y_{\ell m}^m(\theta) = \sqrt{2(2\ell+1)/4\pi} P_{\ell m}(\cos \theta)$, where $-1/2 \leq P_{\ell m}(\cos \theta) \leq 1$, and $\langle \cos \theta \rangle^2 = \langle \cos^2 \theta \rangle$ for $n > 1$ integer due to the Cauchy–Schwarz inequality.} between

$$\sqrt{\frac{2\ell+1}{4\pi}} \leq P_{\ell m}(L/(NL)) \leq \sqrt{\frac{2\ell+1}{4\pi}}.$$  \hspace{1cm} (20)

The total energy is also bounded

$$-\frac{\pi^2}{48} \leq \frac{E}{J^2 N^2} \leq 0,$$  \hspace{1cm} (21)

$$-\frac{\pi^2}{48} \leq \frac{E}{J^2 N^2} \leq -\sum_{\ell, \text{even}} P_{\ell m}(L/(NL))^2 \frac{2\ell^2}{2\ell^2} \frac{N}{N},$$  \hspace{1cm} (22)

$$\frac{L}{NL} \leq \frac{L_0}{NL},$$  \hspace{1cm} (23)

where we use Equations (7) and (20) and defined $L_0/(NL) = 0.5488$. Note that the upper bound on the VRR energy, which leads to the existence of negative temperature equilibria.

### 2.5. Numerical Method

We determine the axisymmetric mean-field equilibrium distribution by numerically satisfying Equation (17) for fixed $\beta$ and $L/NL$. To solve for $\langle Y_{\ell m}^m \rangle$, we look for the zeros of the...
following function:

\[ F_\ell(Y) = \langle Y_\ell^0 \rangle - \frac{\int d\Omega \ Y_L^0(n)e^{-\beta l(\theta)+\beta Y_\ell}}{\int d\Omega \ e^{-\beta l(\theta)+\beta Y_\ell}}, \]  

where \( Y \) is the set of all order parameters. Using the first-order Newton’s method, we Taylor’s expand the equilibrium around some nearby order parameters to leading order

\[ F_\ell(Y^{\text{eq}}) \approx F_\ell(Y) + \sum_{\ell'} \frac{\partial F_\ell(Y)}{\partial (Y_{\ell'}^0)} \left( Y_{\ell'}^{0,\text{eq}} - \langle Y_{\ell'}^0 \rangle \right). \]  

The left-hand side vanishes by definition at the equilibrium. The order parameter can be determined iteratively. In the \( M \)th iteration,

\[ Y_{\ell [M+1]}^0 = Y_{\ell [M]}^0 - \sum_{\ell'} \left( \frac{\partial F_\ell(Y_{\ell'})}{\partial (Y_{\ell'}^0)} \right)^{-1} F_\ell(Y_{\ell [M]}), \]  

where \( \cdot^{-1} \) denotes the inverse matrix. The order parameters may be calculated with this method for any number of harmonics and tolerance. We carry out calculations by truncating the harmonics at \( \ell_{\text{max}} = 10, 38, \) and 58, respectively. Deviations for \( \ell_{\text{max}} = 10 \) and 38 from 58 are of orders of \( \mathcal{O}(10^{-3}) \) and \( \mathcal{O}(10^{-6}) \), respectively, for \( \beta J^2 N = 50 \) and \( L/N = 0.1 \).

### 3. Results

We determine the numerical solution for the one-component axisymmetric system at fixed \( (\beta, N, L) \). The results are qualitatively very similar to that for the quadrupole interaction (Roupas et al. 2017). We direct the reader to that paper for extensive discussions. To compare the two results, the following conversions are needed: \( J_{\text{Roupas}} = 2J^2/3 \) and \( \langle q \rangle_{\text{Roupas}} = \sqrt{16\pi/45} \langle Y_2^0 \rangle \).

#### 3.1. Order Parameters and Phase Transition

Figures 1 and 2 show the order parameters for axisymmetric one-component systems, \( \langle Y_\ell^0 \rangle \) for even \( \ell \), which solve Equations (17)–(19) as a function of temperature, assuming a truncation of the harmonics at \( \ell_{\text{max}} = 10 \). Different colors show different fixed total angular momentum per particle \( s = L/(N\ell) \), where we introduced \( s = \cos \theta \). The \( \ell \) harmonics are labeled with small numbers. Solid and dashed lines belong to stable and metastable/unstable equilibria in the canonical ensemble.8 The right panel is a zoom-in of the left panel, which highlights the phase transition shown with dotted lines. At low temperatures, the system forms an ordered phase, which is analogous to the nematic phase of liquid crystals.9 The high-temperature phase represents a nearly spherical disordered phase. We find that if the total angular momentum is less than a critical total angular momentum \( L_C/(N\ell) = 0.188 \), there is a discontinuous change in the order parameters, i.e., the multipole moments \( \langle Y_\ell^0 \rangle \), at the phase transition temperature. The phase transition in this case is first order. The phase transition becomes second order at \( L = L_C \) and a critical temperature \( k_B T_C/(J^2 N) = 0.05883 \), and there is a continuous series of equilibria at higher \( L \). Figure 2 and the right panel of Figure 1 show that the absolute change of the order parameters at the phase transition is most prominent for the quadrupole \( \ell = 2 \). The phase transition erases the high-order multipoles (i.e., small-scale anisotropies) much more than the low-order multipoles approximately exponentially with \( |Y_\ell| \propto \exp(-\alpha \ell) \), where \( 1 \lesssim \alpha \lesssim 3 \), higher \( \alpha \) values correspond to higher temperatures and lower total angular momentum.

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8 Due to the nonadditivity of the system, the stability of equilibria are different in the canonical and the microcanonical ensembles, a phenomenon known as ensemble inequivalence (Campa et al. 2014; Roupas et al. 2017). For VRR, there is no phase transition in the microcanonical ensemble, where the system changes continuously from the flattened state to a nearly spherical state along the S-shaped curves shown in the right panel of Figure 1 for all \( \ell \).

9 The quadrupole approximation of the VRR Hamiltonian is similar to the Maier–Saque model of liquid crystals, where the ordered phase is called nematic (see Roupas et al. 2017 for a discussion).
The highest entropy configuration satisfies $\beta = 0, T \to \pm \infty$. In this case, the angular momentum vector distribution function is identical to that of Roupas et al. (2017), $f(n) \propto e^{\sqrt{\gamma} \cos \theta}$, and so the order parameters are

$$\langle Y_0^\ell \rangle = \sqrt{\frac{2\ell + 1}{4\pi}} \int_{-1}^{1} ds P_\ell(s) e^{s \sqrt{\gamma}} \quad (T \to \pm \infty),$$

where $P_\ell(s)$ are Legendre polynomials and $\gamma$ is given by

$$L = \frac{\int_{-1}^{1} ds s e^{s \sqrt{\gamma}}}{\int_{-1}^{1} ds e^{s \sqrt{\gamma}}} = \coth(\sqrt{\gamma}) - \frac{1}{\sqrt{\gamma}} \quad (T \to \pm \infty).$$

The moments are exponentially suppressed as a function of $\ell$ for $\ell \gtrsim (\sqrt{\gamma})^{1/2}$. Note that, while the even moments fully parameterize the distribution function (Equations (8) and (12)), the odd $\ell$ moments are also generally nonzero.

Figure 3 shows the angular momentum vector distribution for axisymmetric equilibria at various temperatures. It is sharply peaked near the $\pm z$ axis in the low-temperature ordered phase ($k_B T/J^z N = 0.05$, left panel). This configuration corresponds to a counterrotating thin disk in physical space, in which the ratio of bodies orbiting in the two senses is set by the value of the total angular momentum. The angular momentum distribution in the high-temperature disordered phase ($k_B T/J^z N = 0.07$) is shown in the middle panel. The distinction between the two cases is clear for low total angular momentum, which exhibits a phase transition. The right panel shows negative temperature equilibria, which is similar to the positive temperature disordered phase.

3.2. Thermodynamical Properties and Instability

To explore the phase transition and the stability of states, we calculate the thermodynamical properties of canonical equilibria.

The left panel of Figure 4 shows the caloric curve, i.e., the inverse temperature as a function of total energy from Equation (7). There is a discontinuous change in energy between the marked points that characterize a first-order phase transition, which is the latent heat. The temperature of the phase transition is shown with dotted lines. Between the filled and empty circles (local minima and maxima), the system is in metastable equilibria, i.e., superheated or supercooled. The right panel shows the free energy (Equation (15)) as a function of temperature. The free energy for a fixed temperature is
minimized for stable equilibria (solid lines). As the order parameters are changed continuously along the series of equilibria with fixed $\langle s \rangle$ from the low-energy ordered states (see Figure 1), the free energy increases with temperature, then goes around a triangular shape. The phase transition takes place at the lower right vertex, where the solid curve has a discontinuous derivative. The temperature and free energy are constant along the phase transition. We find that the series of equilibria shown in Figure 4 is qualitatively very similar to Figure 11 in Roupas et al. (2017), see further discussion about the first-order phase transition therein.

Figure 4 shows the phase diagram of the system.

3.3. Negative Temperature Equilibria

To understand the origin of negative temperature equilibria in Figures 1 and 3, Figure 6 shows the entropy as a function of total energy. By definition, entropy is a decreasing function of energy at negative temperatures, marked with solid gray lines. The origin of the negative temperature states stems from the fact that the VRR energy and the entropy are bounded from above, which leads to a decreasing entropy at energies higher than that of the maximum entropy state. A first-order phase transition appears where the $S(E)$ function is convex. A careful analysis shows that this is indeed the case between the large dots connected by a dashed curve on the blue curve. Note that, for this series of equilibria, the negative temperature states subtend a rather small range of energies above the phase transition. Negative temperature states appear more prominently at higher total angular momenta or high $\langle s \rangle$, which do not admit a phase transition. The negative temperature region is

10 Generally because the phase space is compact (see Equation (21)).
always the highest entropy for fixed energy and is therefore stable in the microcanonical ensemble. Where \( S(E) \) is concave, showing that there is no phase transition at negative temperatures. The qualitative shape of \( S(E) \) is very similar in the quadrupole-interaction approximation, discussed in detail in Roupas et al. (2017).

The angular momentum vector distribution of negative temperature states is approximately isotropic for small total angular momentum \( L/(N!) \). The \( \langle Y_{\ell}^{m} \rangle \)-order parameters have alternating signs as a function of even \( \ell \) for negative temperatures (Figure 1), they are continuous functions of \( \beta = 1/T \) over \( \beta = 0 \) and \( \langle Y_{\ell}^{m} \rangle \neq 0 \) for even \( \ell \) for \( T = 0 \). Figures 1 and 2 show that \( \langle Y_{\ell}^{m} \rangle \) are highly suppressed for increasing even \( |\ell| \) at negative temperatures.

4. Discussion

In this paper, we have examined the statistical equilibrium distribution of orbital planes of gravitating bodies orbiting around a central massive object. We have considered a one-component system, in which all bodies have approximately equal semimajor axes and eccentricities, but relaxed the quadrupole-interaction approximation applied recently in Roupas et al. (2017). The higher multipole moments modify the interaction significantly for radially overlapping orbits, particularly for a low mutual inclination (Kocsis & Tremaine 2015). While the interaction Hamiltonian in the quadrupole approximation is that of liquid crystals (see Roupas et al. 2017 for detailed discussions), for the case of overlapping orbits, the VRR interaction resembles a vortex system (see Equation B84 in Appendix B in Kocsis & Tremaine 2015). Despite these differences, we found no qualitative difference in their statistical physics behavior between the case limited to the quadrupole approximation and that including high-order harmonics. All of the physical features seen in that model (Roupas et al. 2017) is also present in the complementary model for overlapping orbits in which we extrapolated the dominant asymptotic contribution of higher harmonics \( \ell \to \infty \) to all even \( \ell \):

1. Low-temperature axisymmetric equilibria resemble thin counterrotating disks in physical space in which the bodies may orbit in either sense, which is set by the total angular momentum.
2. Below a critical total angular momentum, \( L_C \approx 0.18 Nl \), the system exhibits a first-order phase transition in the canonical ensemble between the ordered phase, which resembles a thick disk, and the disordered phase, which resembles a spheroidal distribution.
3. The disordered phase for \( L < L_C \) is not completely isotropic if the total angular momentum is nonzero, but the multipole moments are exponentially suppressed as a function of the harmonic number \( \ell \). Thus, the phase transition erases the small-scale features most efficiently.
4. The system admits a maximum energy and maximum entropy, which leads to the existence of negative temperature states.

Possible differences may occur at low temperatures for anisotropic states. For example, various arrangements of razor thin disks are expected to be stable during VRR discussed in Roupas et al. (2017). However, since the high-temperature states generally do not have sharp density peaks in angular momentum direction space, we expect their behavior to be described well by the quadrupole approximation, at least for one-component systems. This expectation may not hold for multicomponent systems, where heavier objects may be expected to form much thinner disks (Roupas et al. 2017). A study of anisotropic or multicomponent equilibria lies beyond the scope of this paper.

This result provides a point of comparison for interpreting the results of more detailed investigations on the equilibrium configurations of complex astrophysical systems with a distribution of overlapping orbits with different mass, eccentricity, and semimajor axis. It may have applications in explaining the origin of a thin disk of massive stars in the Galactic center (Bartko et al. 2009; Kocsis & Tremaine 2011; Haas & Šubr 2014a, 2014b; Yelda et al. 2014; Panamarev et al. 2018; Perets et al. 2018) and the distribution of possible putative stellar mass black holes lurking in NSCs. These elusive objects may be possibly observed by future X-ray observations including Chandra, XMM-Newton, and NuSTAR (Bartos et al. 2013), and may represent important sources of mergers for GW detectors including LIGO, VIRGO, KARGA, and LISA (O’Leary et al. 2009; Antonini & Perets 2012; McKernan et al. 2013, 2014b, 2014a; Bartos et al. 2017; Hoang et al. 2017; McKernan et al. 2017). The distribution of GW event rates may carry information on the structure of the dynamical environments in which these GW sources form (Gondan et al. 2017).

We conclude that for one-component axisymmetric systems, the thermodynamics of VRR is qualitatively well described by the quadrupole approximation.

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