A Note on the Radiative and Collisional Branching Ratios in Polarized Radiation Transport with Coherent Scattering

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Abstract

We discuss the implementation of physically meaningful branching ratios between the CRD and partial redistribution contributions to the emissivity of a polarized multi-term atom in the presence of both inelastic and elastic collisions. Our derivation is based on a recent theoretical formulation of partially coherent scattering, and it relies on a heuristic diagrammatic analysis of the various radiative and collisional processes to determine the proper form of the branching ratios. The expression we obtain for the emissivity is \( \epsilon = [\epsilon^{(1)} - \epsilon^{(2)}_{\text{f.s.}}] + \epsilon^{(2)} \), where \( \epsilon^{(1)} \) and \( \epsilon^{(2)} \) are the emissivity terms for the redistributed and partially coherent radiation, respectively, and where “f.s.” implies that the corresponding term must be evaluated assuming a flat-spectrum average of the incident radiation. This result is shown to be in agreement with prior literature on the subject in the limit of the unpolarized multi-level atom.

Key words: line: formation – line: profiles – polarization – radiative transfer

1. Introduction

The formal theory of spectral line formation in a two-term atom (and more generally in a multi-term atom of the \( \Lambda \)-type; see Figure 1)—extended perturbatively to fully include second-order atom–photon processes, to be able to describe partial redistribution (PRD) effects—predicts that the vector radiative transfer (RT) equation acquires a new source term, \( \epsilon^{(2)} \), which describes the coherent (in the broader sense of “memory preserving”) scattering of radiation from the lower term (Casini et al. 2014). This second-order emissivity appears in addition to the usual source term \( \epsilon^{(1)} \) corresponding to the emission of completely redistributed radiation via spontaneous de-excitation of the upper state.

In other words, to the second order of perturbation, the interaction of an atomic system with an incoming beam of photons acquires an additional scattering channel, besides the one corresponding to the pure absorption of a photon with the consequent excitation of the target atom (i.e., \( k + a \rightarrow b \)), where \( k \) is the incoming photon, and \( a \) and \( b \) two atomic states satisfying the energy conservation relation \( \omega_k + \omega_a = \omega_b \). In the new scattering channel available to perturbative second order, the incident photon is instead immediately re-emitted after a virtual excitation of the atom, possibly leaving the atom in a different energy (and polarization) state from the original one (i.e., \( k + a \rightarrow k' + b \), with \( \omega_k + \omega_a = \omega_{k'} + \omega_b \); the condition \( \omega_{k'} = \omega_b \) evidently corresponds to Rayleigh scattering, whereas \( \omega_{k'} \neq \omega_b \) to Raman scattering).

It is verified (see Casini et al. 2014; Casini & Manso Sainz 2016) that the two emissivity terms \( \epsilon^{(1)} \) and \( \epsilon^{(2)} \) evaluate to exactly the same quantity, in the case of spectrally flat incident radiation, if one adopts the solution of the statistical equilibrium (SE) problem for the interacting system restricted to one-photon processes. This implies that including both emissivity terms in the second-order RT while adopting the atomic density matrix solution of the first-order SE problem would generally lead to double counting the energy radiated by the scatterer.

On the other hand, the parallel extension to second-order atom–photon processes of the SE problem in the formalism of Casini et al. (2014) suggests that a partial cancellation of the radiative rate for one-photon absorption occurs in the presence of coherent scattering, with a consequent reduction of the population of the excited state with respect to the case of one-photon processes. In the limit of infinitely sharp lower levels, this population cancellation is found to be total, in agreement with the simple physical argument that no upper state population can be produced when the lower state has an infinite lifetime. One of the expected results of a self-consistent solution of the combined SE+RT problem to perturbative second order is that the radiative energy is conserved even when both emissivity terms \( \epsilon^{(1)} \) and \( \epsilon^{(2)} \) are taken into account.

Recently, the expression for the RT equation including coherent scattering has been applied to modeling various examples of the PRD of polarized radiation in spectral lines from \( \Lambda \)-type transitions formed in a collisionless plasma (Casini & Manso Sainz 2016). Since the formal derivation of the corresponding set of SE equations has not yet been completed, the solution of the first-order SE problem was adopted in that work. This creates no formal issues of energy conservation, in the limit of infinitely sharp lower levels assumed by the model, since no population of the upper state from true absorption of the incoming photons is expected in that case. Hence, one can assume that the scattering of radiation is completely described by \( \epsilon^{(2)} \), and the contribution of \( \epsilon^{(1)} \) to the RT equation can be omitted altogether. Of course, the condition of infinite lifetime of the lower state in deriving the first-order SE solution can only be approximated numerically by using an extremely diluted radiation field to illuminate the scatterer (Casini & Manso Sainz 2016).

On the other hand, in the application of numerical RT including PRD effects to realistic models of optically thick atmospheres, the fraction of the upper state population \( \rho(u) \) that enters the first-order emissivity can be significant (typically of the order of 1% at the bottom of the atmosphere; see Figure 2). This is even more critical in the presence of collisions, as it is

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safe to assume that collisional excitation always leads to the complete redistribution of the energy of the scattered radiation. Hence, in the absence of a formal self-consistent solution of the combined SE+RT problem to perturbative second order, the question naturally arises of how to handle phenomenologically the contribution of spontaneous de-excitation via $\epsilon^{(1)}$ along with the coherent-scattering emissivity $\epsilon^{(2)}$, without impacting energy conservation.

A possible way of dealing with this type of issue is to introduce appropriate weights between the contributions of $\epsilon^{(1)}$ and $\epsilon^{(2)}$ in the RT equation for the Stokes vector $S = (S_0, S_1, S_2, S_3)^T$ of the propagating radiation field, i.e., (see Casini et al. 2014),

\[
\frac{d}{ds} S(\omega_k, \hat{k}') = -K(\omega_k, \hat{k}') S(\omega_k, \hat{k}') + \alpha \epsilon^{(1)}(\omega_k, \hat{k}') + \beta \epsilon^{(2)}(\omega_k, \hat{k}'),
\]

where $K(\omega_k, \hat{k}')$ is the $4 \times 4$ absorption matrix for the outgoing photon of frequency $\omega_k$ and propagation direction $\hat{k}'$, $s$ is the linear coordinate along the propagation path, and finally $\alpha$ and $\beta$ are the (real) weights for the fully incoherent (i.e., completely redistributed) and partially coherent scattering contributions, respectively. Because these weights take the form of probability ratios for the various processes that determine the excitation state of the atom, they are commonly called branching ratios. The choice of the $\alpha$ and $\beta$ weights will generally depend on the properties of the physical system at hand. In particular, they will take different forms depending on whether the system includes, or not, collisions.

In this work we do not address the problem of establishing physically consistent branching ratios for the purely radiative case, and simply adopt $\alpha = 0$ and $\beta = 1$ in such a case (see Casini & Manso Sainz 2016). This choice is supported by the underlying assumption of weak incident radiation, which in turn is consistent with the hypothesis of infinite radiative lifetime for the lower state, and with the approximation of neglecting stimulated effects. We must note that this choice does not create any issues when the ratio $\rho(u)/\rho(l)$ is important, since in practical cases this always happens when the system is close to local thermodynamic equilibrium (LTE), e.g., at the bottom of the atmosphere (see Figure 2). In such a case, the incident radiation field can be assumed to be spectrally flat over a very large interval of the atomic transition’s frequency, and so $\epsilon^{(2)} = \epsilon^{(1)}$, when the first-order density matrix solution is employed. Thus our choice of branching ratios for the purely radiative case correctly reproduces the expected CRD regime of the scattered radiation at LTE.

We develop our approach to the derivation of branching ratios between the CRD and PRD contributions to the RT Equation (1) for the polarized multi-term atom starting first from the case of an unpolarized two-level atom. This is a natural choice that allows us to directly interpret the branching ratios as event probabilities for the various radiative and collisional processes that can be realized within the interacting system. We use a simple diagrammatic representation of the interacting system in order to describe the interplay of radiative and collisional (both inelastic and elastic) processes. After such detailed, albeit heuristic, analysis, at the end we are able to justify very naturally a straightforward extension of our results to the case of the polarized multi-term atom.

2. The Case of the Unpolarized Two-level Atom

We consider the case of a two-level atom with lower level $l$ and upper level $u$, in the presence of both radiative and collisional processes. We have (see, e.g., Landi Degl’Innocenti & Landolfi 2004)

\[
\rho(u) \approx \rho(l) \frac{B_{lu} J(\omega_{ul}) + C_{lu}}{A_{ul} + B_{ul} J(\omega_{ul}) + C_{ul}} = \rho(u)_{\text{rad}} + \rho(u)_{\text{coll}},
\]

where $\rho(u)$ and $\rho(l)$ are the relative populations of the upper and lower states, respectively, which are subject to the normalization of the trace of the atomic density matrix, so that $\rho(u) + \rho(l) = 1$. $J(\omega_{ul})$ is some appropriate average of the ambient radiation intensity at the frequency of the atomic transition (see, e.g., Equation (16)), whereas $A_{ab}$ and $B_{ab}$ are the usual Einstein radiative coefficients for the transition $a \rightarrow b$, and $C_{ab}$ the corresponding collisional rate. We note that the approximate equality in Equation (2) becomes exact in the case of an unpolarized lower state (see Landi Degl’Innocenti & Landolfi 2004).

Equation (2) suggests that a fraction

\[
w_c = \frac{\rho(u)_{\text{coll}}}{\rho(u)} \approx \frac{C_{lu}}{B_{lu} J(\omega_{ul}) + C_{ul}}
\]

Figure 2. Height dependence of the $\rho(u)/\rho(l)$ ratio for the Mg II two-term model atom in the collisional FAL-C atmosphere (Fontenla et al. 1993).

![Figure 1](image1.png)

Figure 1. Schematic diagram of the $\Lambda$-type multi-term atom. In order to correctly describe the polarization properties of the outgoing light in the $u \rightarrow f$ transition, all lower terms $l_i$ that are radiatively or collisionally connected to the upper term $u$, including the final term $f$, must be taken into account.
of the upper state population is contributed by collisional excitation. In that case, there is also a purely collisional contribution to the emitted radiation given by $w_{j} e^{(1)}$. Such a process is represented schematically by the first diagram of Figure 3. In the same figure, diagram (ii) represents the coherent scattering of radiation, which is accounted for by the second-order emissivity $e^{(2)}$, and does not contribute to the population of the upper state, which is only virtually excited. However, in the presence of collisions, the coherence of the scattering process to perturbative second order may be destroyed by the intervention of a colliding particle (diagrams (iii) and (iv)). This mechanism may either trigger the immediate collisional de-excitation of the upper state in the case of an inelastic encounter with rate $C_{n}$ (final state $l$; diagram (iii)), or simply redistribute the internal energy of the upper state (with radiatively induced population $\rho (a)_{\text{rad}} = (1 - w_{j}) \rho (a)$; see Equations (2) and (3)) in the case of an elastic encounter with rate $\Gamma_{u}$ (final state $u$; diagram (iv)). In the first case, there is no radiative contribution to Equation (1), whereas in the second case the upper state eventually decays with the emission of completely redistributed radiation (i.e., associated again with $e^{(1)}$).

The above analysis should be generalized to properly take into account the effects of collisions on atomic polarization. As this requires the introduction of individual multipolar collisional rates $C_{ab}(K)$ in the SE problem, one could think of introducing accordingly specific weights $w_{j}^{(k)}$ for the individual multipolar density matrix components $\rho_{Q}^{(k)}(u)$ in $e^{(1)}$. On the other hand, this is quite inconvenient (as well as ambiguous), and an alternative approach will be presented at the very end, after we discuss the results of our heuristic diagrammatic analysis in the light of other treatments of this problem found in the literature. For the current discussion, we are assuming that the partition coefficient $w_{j}$ applies identically to all components of the atomic density matrix, and therefore it is fully appropriate only in the case of the unpolarized atom.

It is important to observe that when the inelastic collision lifetime of the upper state is included in the computation of the damping parameter of the line profile, then $e^{(2)}$ already contains the proper branching ratio to account for the possibility of collisional de-excitation following the radiative excitation of the upper state (Figure 3(iii)). In order to see this, one must recall that the formal derivation of the second-order emissivity (Casini et al. 2014) leads to the following expression for the atomic-frame redistribution function $R(\omega_{k}, \omega_{k'})$ in terms of the generalized line profiles $\langle \omega_{ab,cde}^{k,k',k''} \rangle$ that appear in $e^{(2)}$ (see Casini et al. 2014, Equation (6)),

$$R(\Omega_{u}, \Omega_{u';} \Omega_{l}, \Omega_{l'}; \Omega_{f}, \Omega_{f'}; \omega_{k}, \omega_{k'}) = i(\Omega_{u} - \Omega_{u'})\left(\Psi_{u,l;f,l'}^{+k+k'-k} + \Psi_{u,l;f,l'}^{k-k'+k'},\right),$$  

where we indicated with $\Omega_{u} = \omega_{u} - i \epsilon_{u}$ the (complex) frequency of the atomic level $a$, which is comprised of the true energy $\omega_{u}$ of the level and its statistical width $\epsilon_{u}$. Generally, $\epsilon_{a}$ represents the inverse of the total lifetime of the quantum state $a$, from which the damping parameter for the corresponding energy level is calculated. Therefore, in the presence of a non-zero probability for collisional de-excitation, the quantity $i(\Omega_{u} - \Omega_{u'})$, which appears at the denominator in the expression of $e^{(2)}$, is augmented by the corresponding collisional rate. Hence, the probability for a coherent scattering event to occur is accordingly reduced, and it is immediately verified that the reduction factor is $(1 - w_{j}) = \rho (a)_{\text{rad}} / \rho (a)$.

We now recall that $e^{(2)} \equiv e^{(1)}$ in the absence of collisions and in the limit of spectrally flat illumination (see Casini et al. 2014, Equations (48) and (49)), showing that $e^{(2)}$ in that case contains exactly $\rho (a)_{\text{rad}} \equiv \rho (a)$. Because in the presence of collisions the denominator in $e^{(2)}$ is augmented by $C_{ab}$ from Equations (2) and (3) and the previous discussion we conclude that, also in this case, the limit of spectrally flat illumination, $e^{(2)}$ contains exactly $\rho (a)_{\text{rad}} \equiv (1 - w_{j}) \rho (a)$. This demonstrates that the possibility that the virtually excited upper state might decay via collisional de-excitation, reducing the contribution of coherent scattering to the RT Equation (1), is already accounted for in the expression of $e^{(2)}$, and that the reduction factor is exactly given by $\rho (a)_{\text{rad}} / \rho (a) = (1 - w_{j})$.

Hence, in the presence of radiative and inelastic collisional processes, $e^{(2)}$ already provides the correct contribution of coherent scattering to the emitted radiation without the need for an additional multiplicative weighting factor. The RT Equation (1) thus becomes $(\alpha = w_{j}, \beta = 1)$,

$$\frac{d}{ds} S(\omega_{k'}, \hat{k'}) = -K(\omega_{k'}, \hat{k'})S(\omega_{k'}, \hat{k'}) + w_{j} e^{(1)}(\omega_{k}, \hat{k'}) + e^{(2)}(\omega_{k'}, \hat{k'}),$$  

where we also recalled Equation (3). We note that, in the limit of spectrally flat illumination, the above equation properly converges to the equation for the polarized RT in the CRD regime, since $e^{(2)} = (1 - w_{j}) e^{(1)}$ in that case.

In the additional presence of elastic collisions, the same analysis as before applies, but now a fully redistributed radiation component is contributed also by $\rho (u)_{\text{rad}}$ according to our diagrammatic analysis presented above (Figure 3(iv)). Since the main effect of elastic collisions on the line shape comes from the perturbation of the energy of the atomic levels, which statistically can be approximated with a Lorentzian distribution, it is sensible to add the elastic collision rate to the damping parameters in the redistribution function.

Similarly, the effect of an elastic collision on an atomic level can be interpreted as a process where the atomic state before the collision is destroyed and immediately recreated into a new (iso-energetic) perturbed state, with a characteristic inverse lifetime $1/\Gamma_{u}$ for the process. Accordingly, the energy of the
atomic level acquires an additional imaginary contribution due to elastic collisions, i.e., \( \Omega_a = \omega_a - i(\epsilon_a + \frac{1}{2}\Gamma_a) \), and the probability for the coherent scattering of the incident radiation is once again reduced by the corresponding modification of the denominator \( i(\Omega_a - \Omega_b^2) \) in the expression of \( e^{(2)} \). Because the inclusion of elastic collisions to the first-order SE problem does not affect the population balance between the upper and lower states, the proposed modification of \( e^{(2)} \), in the limit of spectrally flat illumination, is formally equivalent to the substitution \( (\Omega_a = \Omega_a^2, \text{for the population}) \)

\[
\rho(u)_{\text{rad}} \rightarrow (1 - \gamma)\rho(u)_{\text{rad}}
\]

where

\[
\gamma = \frac{\Gamma_a}{A_{ul} + B_{ul}J(\omega_{ul}) + C_{ul} + \Gamma_a}.
\]

is the probability that the excited atom undergoes an elastic encounter with a colliding particle. This is exactly what we expect from our diagrammatic analysis (see Figure 3(iv)). Therefore, even in the additional presence of elastic collisions, when the corrresponding rate is added to the lifetime \( \epsilon_a \) of the upper level, \( e^{(2)} \) describes the proper contribution of coherent scattering to the emitted radiation without the need of an ad hoc multiplicative branching ratio.

In turn, according to our diagrammatic analysis, the probability (7) that an elastic collision occurs after the atom has been radiatively excited must bring a new contribution of fully redistributed radiation to the RT Equation (5), which is proportional to

\[
\gamma \rho(u)_{\text{rad}} = \gamma (1 - w_c) \rho(u).
\]

Therefore, in the general presence of radiative and collisional processes, both inelastic and elastic, the RT equation for partially coherent scattering by an unpolarized two-level atom becomes,

\[
\frac{d}{ds}S(\omega_f, \hat{\mathbf{k}}') = -K(\omega_f, \hat{\mathbf{k}})S(\omega_i, \hat{\mathbf{k}}') + [w_c + \gamma (1 - w_c)]e^{(1)}(\omega_i, \hat{\mathbf{k}}') + e^{(2)}(\omega_i, \hat{\mathbf{k}}').
\]

3. The Case of the Unpolarized Multi-level Atom of the \( \Lambda \)-type

So far we have concerned ourselves with establishing the choice of branching ratios that allows us to apply the formalism of Casini et al. (2014) to the polarized line formation in two-level atoms. Casini & Manso Sainz (2016) have shown that such a formalism naturally extends to the treatment of the general multi-level atom of the \( \Lambda \)-type (see Figure 1), and so we now want to extend the development of the previous section to such a general model atom. In the spirit of the former development, we do this by considering first the extension to the unpolarized multi-level atom of the \( \Lambda \)-type.

We therefore consider an atom with an upper level \( u \) connected to a set of lower levels \( l_1, l_2, \ldots, l_n \), which are assumed to be isolated from each other, both radiatively and collisionally (see Figure 1). Then, the first-order solution for the population of the upper state is given by (see Equation (2))

\[
\rho(u) \approx \rho(l_i) \frac{B_{lu}J(\omega_{ul}) + C_{lu}}{A_{ul} + B_{ul}J(\omega_{ul}) + C_{ul}}, \quad \forall i = 1, \ldots, n. \tag{9}
\]

(Once again, we note that the above relation is exact in the case of an atom with unpolarized lower states.) We can therefore write

\[
\rho(u) = \sum_{i=1}^{n} \alpha_i \rho(l_i) \frac{B_{lu}J(\omega_{ul}) + C_{lu}}{A_{ul} + B_{ul}J(\omega_{ul}) + C_{ul}}
\]

\[
+ \sum_{i=1}^{n} \alpha_i \rho(l_i) \frac{B_{ul}J(\omega_{ul}) + C_{ul}}{A_{ul} + B_{ul}J(\omega_{ul}) + C_{ul}}
\]

\[
= \rho(u)_{\text{rad}} + \rho(u)_{\text{coll}},
\]

for any choice of the weights \( \alpha_i \) such that \( \sum_i \alpha_i = 1 \).

The above equation demonstrates that it is still possible to operate a separation between the radiative and collisional contributions to \( \rho(u) \) even for the more general model of the \( \Lambda \)-type multi-level atom. This allows us to define \( w_c = \rho(u)_{\text{coll}} / \rho(u) \) like before (see Equation (3)), and to extend the use of Equation (8) to such a model, after performing the formal substitutions

\[
A_{ul} \rightarrow \sum_i A_{ul}, \quad B_{ul}J(\omega_{ul}) \rightarrow \sum_i B_{ul}J(\omega_{ul}),
\]

\[
C_{ul} \rightarrow \sum_i C_{ul},
\]

in Equation (7). Using Equations (9) and (10), we thus arrive at the following generalization of Equation (3),

\[
w_c = \sum_{i=1}^{n} \alpha_i \frac{C_{lu}}{B_{lu}J(\omega_{ul}) + C_{lu}}. \tag{12}
\]

If we limit ourselves to considering the first-order SE+RT problem, the weights \( \alpha_i \) can be arbitrarily chosen, in so far that they satisfy the normalization condition \( \sum_i \alpha_i = 1 \). In contrast, in the application of the branching-ratio formalism to the treatment of partially coherent scattering, the choice of these weights is subject to additional physical constraints, because the second-order emissivity already accounts for the proper branching among all possible transitions \( (u, l_i) \).

Thus, in order to determine the proper expression for the weights \( \alpha_i \), we simply observe that the choice

\[
\alpha_i = \frac{A_{ul} + B_{ul}J(\omega_{ul}) + C_{ul}}{\sum_l [A_{ul} + B_{ul}J(\omega_{ul}) + C_{ul}]} \tag{13}
\]

ensures that the expression for \( \rho(u)_{\text{rad}} \) in Equation (10) becomes formally equivalent to the one that is explicitly contained in \( e^{(2)} \), when the multi-level atom is illuminated by a spectrally flat radiation (see the discussion leading to Equation (5)). We therefore propose Equation (13) as the proper definition of the weights \( \alpha_i \) to be used in the generalized expression (12) for \( w_c \)
4. Comparison with Previous Results

The above results were completely derived within the framework of a recent theory of partially coherent scattering from multi-term atoms in a collisionless plasma developed by Casini et al. (2014; see also Casini & Manso Sainz 2016). It is therefore important to show how those results compare with existing work on the modeling of the PRD of unpolarized radiation in multi-level atoms and in the presence of collisions.

For our comparison, we rely on the work of Uitenbroek (2001) (see also Avrett & Loeser 2008), which has become a standard reference for the numerical modeling of multi-level systems including PRD effects. We report here only the essential formulas from that work, adapted to our notation.

We consider a multi-level atom of the Λ-type, consisting of one upper state \( u \) and a set \( l_1, l_2, \ldots, l_n \) of lower states (see Section 3 and Figure 1). The intensity emissivity in the \( (u, f \equiv l_i) \) branch of the multi-level system of transitions can be written as (Uitenbroek 2001, Equation (6))

\[
\epsilon_i(\omega, \mathbf{k}) = \frac{\hbar \omega/4\pi}{N\rho(u)N_{ul}} \psi_i(\omega, \mathbf{k}) + \frac{1}{\rho(u)} \sum_{n} \rho(l_j) B_{l_ju} J(\omega_{ul}) + \frac{1}{\rho(u)} \sum_{n} [A_{ul} + B_{ul} J(\omega_{ul}) + C_{ul}] + \Gamma_u
\]

where \( N \) is the atomic density, and the generalized line profile \( \psi_i \) is given by (Uitenbroek 2001, Equation (6))

\[
\psi_i(\omega, \mathbf{k}) = \phi_i(\omega, \mathbf{k}) + \frac{1}{\rho(u)} \sum_{n} \rho(l_j) B_{l_ju} J(\omega_{ul}) + \frac{1}{\rho(u)} \sum_{n} [A_{ul} + B_{ul} J(\omega_{ul}) + C_{ul}] + \Gamma_u
\]

where \( \phi_i \) is the atomic density, and the generalized line profile \( \psi_i \) is given by (Uitenbroek 2001, Equation (6))

\[
\psi_i(\omega, \mathbf{k}) = \phi_i(\omega, \mathbf{k}) + \frac{1}{\rho(u)} \sum_{n} \rho(l_j) B_{l_ju} J(\omega_{ul}) + \frac{1}{\rho(u)} \sum_{n} [A_{ul} + B_{ul} J(\omega_{ul}) + C_{ul}] + \Gamma_u
\]

After substitution of Equation (15) into Equation (14), and making use of the definition

\[
J(\omega_{ul}) = \frac{d\omega}{4\pi} \int d\omega_k \phi_j(\omega_k, \mathbf{k}) I(\omega, \mathbf{k}),
\]

for the mean intensity of the incident radiation, we find

\[
\epsilon_i(\omega, \mathbf{k}) = \frac{\hbar \omega/4\pi}{N\rho(u)N_{ul}} \psi_i(\omega, \mathbf{k}) + \frac{1}{\rho(u)} \sum_{n} \rho(l_j) B_{l_ju} J(\omega_{ul}) + \frac{1}{\rho(u)} \sum_{n} [A_{ul} + B_{ul} J(\omega_{ul}) + C_{ul}] + \Gamma_u
\]

We thus have demonstrated that Equation (17), corresponding to the emissivity term of Uitenbroek (2001), is equivalent to the intensity emissivity of our RT Equation (8), when we adopt our definitions (7), (12), and (13).

5. The General Case of the Polarized Multi-term Atom of the Λ-type

The analysis in the previous section suggests a straightforward method for generalizing Equation (8) to include the
effects of atomic polarization. In such a case, one typically cannot write the separation between \( \rho(u)_{\text{rad}} \) and \( \rho(u)_{\text{coll}} \) as simply as in Equation (2), preventing a clear identification of the partition coefficient \( w_\epsilon \) (see the discussion immediately following the description of the diagrams in Figure 3). On the other hand, it is known that in the limit of spectrally flat illumination, regardless of the presence of collisions (whether inelastic and/or elastic), the total emissivity in the RT Equation (8) must be identical to \( \epsilon^{(1)} \).

Relying on this fact, we introduce the quantity

\[
\dot{\epsilon}^{(1)} \equiv \epsilon^{(1)} - \epsilon^{(2)}_{\text{PRD}},
\]

where “f.s.” stands for “flat spectrum.” This is to indicate that the term \( \epsilon^{(2)}_{\text{PRD}} \) is obtained from the general expression for \( \epsilon^{(2)} \) by approximating the incoming Stokes vector \( S(\omega_i, \hat{k}) \) with its spectral and angular average, which can be expressed through the spherical tensors \( J_0^S(\omega_{\text{id}}) \) of the incident radiation field (Landi Degl’Innocenti & Landolfi 2004). Then the redistribution integral over the incident frequency in \( \epsilon^{(2)}_{\text{PRD}} \) can be performed using Equation (15) of Casini et al. (2014).

It is important to observe that \( \dot{\epsilon}^{(1)} = 0 \) in the absence of collisions, owing to the fact that \( \rho(u) = \rho(u)_{\text{rad}} \) in that case. Then, the equality of \( \epsilon^{(1)} \) and of \( \epsilon^{(2)}_{\text{PRD}} \) follows from the analysis of Casini et al. (2014). In the presence of collisions, instead, \( \rho(u) \neq \rho(u)_{\text{rad}} \) (and consequently, when elastic collisions are also present, \( \rho(u) = (1 - \gamma)\rho(u)_{\text{rad}} \); see Equation (6)), and so \( \dot{\epsilon}^{(1)} \neq 0 \) in general. Then, according to our diagrammatic analysis, \( \dot{\epsilon}^{(1)} \) accounts for the completely redistributed radiation produced via all possible excitation processes, with the exception of the radiative process leading to the coherent scattering of the incident radiation (Figure 3(ii)).\(^4\) It follows that \( \dot{\epsilon}^{(1)} \) automatically accounts also for the contribution of the upper-state atomic polarization to the completely redistributed radiation.

Therefore, rather than attempting to define ad hoc weights \( w_\epsilon^{(K)} \) for the individual multipolar components \( \rho_\rho^{K}(u) \) of the atomic density matrix, in order to generalize Equation (8) to the case of a polarized atom, we can simply operate the substitution

\[
[w_\epsilon + \gamma (1 - w_\epsilon)] \epsilon^{(1)} \rightarrow \dot{\epsilon}^{(1)}.
\]

With this generalization, the RT Equation (1) for the polarized multi-term atom of the \( \Lambda \)-type can finally be given,

\[
\frac{d}{ds} S(\omega_i', \hat{k}') = -K(\omega_i', \hat{k}')S(\omega_i', \hat{k}')
+ [\epsilon^{(1)}(\omega_i', \hat{k}') - \epsilon^{(2)}(\omega_i', \hat{k}')_{\text{PRD}}] + \epsilon^{(2)}(\omega_i', \hat{k}'),
\]

where the first-order solution of the atomic density matrix should be used to evaluate the emissivity and absorption coefficients. In particular, in the case of an unpolarized multi-level system, the total contribution to the intensity emissivity in the above equation converges exactly to the emissivity \( \epsilon_i(\omega_i', \hat{k}') \) of Uitenbroek (2001) (see Equation (17)), which we proved to be in agreement with our heuristic diagrammatic analysis of the branching between the CRD and PRD components of the scattered radiation (see Section 2).

6. Summary

We discussed the problem of determining physically meaningful branching ratios between the contributions of completely and partially redistributed radiation (respectively, CRD and PRD) to the formation of polarized spectral lines in collisional plasmas.

Our analysis was based on the results of a recent diagrammatic theory of partially coherent scattering by polarized multi-term atoms of the \( \Lambda \)-type (Casini et al. 2014; Casini & Manso Sainz 2016), which considers all radiation processes to the second order of perturbation in the collisionless limit. The present work provides a heuristic extension of that theory to the collisional case, relying on an intuitive diagrammatic description of the interplay of radiation and collisional processes in spectral line formation to the same order of perturbation.

This work was motivated by the need to devise physically consistent numerical schemes for the modeling of scattering polarization of spectral lines in realistic models of optically thick stellar atmospheres. These schemes are traditionally based on the iteration between the (local) solution of the statistical equilibrium (SE) of the plasma atoms with the (non-local) feedback of the emitted radiation on the plasma excitation. The work of Casini et al. (2014) provides the RT equation for partially coherent scattering, but the corresponding set of SE equations has not yet been derived. Hence, the solution of the SE problem to the perturbative first order (Landi Degl’Innocenti & Landolfi 2004) must be used instead in practical cases (e.g., Casini & Manso Sainz 2016). For this very reason, one must introduce branching ratios between the CRD and PRD contributions to the RT equation of Casini et al. (2014), when redistribution effects in the modeled problem are important, such as in the case of collisional plasmas.

In this paper we showed how this can be done self-consistently, without impacting energy conservation, which is critical for the stability and convergence of numerical schemes for polarized RT in optically thick atmospheres. The guiding principle throughout is the fact that the second-order emissivity term \( \epsilon^{(2)} \), which describes the coherent (in the broader sense of “memory preserving”) scattering of radiation, must converge to the usual term of spontaneous emission in the limit of spectrally structureless illumination and in the absence of collisions.

For the sake of simplicity, we formulated and solved first the problem of the unpolarized two-level atom, to finally arrive at the generalization of the results to the case of the polarized multi-term atom of the \( \Lambda \)-type. For the intermediate case of the unpolarized multi-level atom, our results are found to agree with previous works on the subject (e.g., Uitenbroek 2001).

One notable result of this study is that the expression of the \( \epsilon^{(2)} \) emissivity term already accounts for the proper branching among the various radiative and collisional processes that determine the PRD contribution to the scattered radiation. Hence, this emissivity term always enters the RT equation without the need for a multiplicative weighting factor (see Equation (20)). Only the CRD contribution to the RT equation must be modified in order not to violate energy conservation (see Equation (19)).

\(^4\) This is readily seen in the particular case of an unpolarized atom, since

\[
\rho(u) = \rho(u)_{\text{coll}} + \rho(u)_{\text{rad}} = \rho(u)_{\text{coll}} + \gamma \rho(u)_{\text{rad}} + (1 - \gamma)\rho(u)_{\text{rad}},
\]

and the last addendum corresponds exactly to the contribution of \( \epsilon^{(2)}_{\text{PRD}} \) (see discussion around Equation (6)). Therefore the contribution of \( \dot{\epsilon}^{(1)} \) corresponds to the sum \( \rho(u)_{\text{coll}} + \gamma \rho(u)_{\text{rad}} \) (i.e., diagrams (i) and (iv) of Figure 3).
The final expression \((20)\) of the RT equation for polarized scattering in a multi-term atom of the \(\Lambda\)-type was recently used to model the formation of the Mg II h–k doublet in a magnetized atmosphere, taking into account both inelastic and elastic collisions, but neglecting stimulated emission (del Pino Alemán et al. 2016).

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