Nonequilibrium Kosterlitz-Thouless Transition in a Three-Dimensional Driven Disordered System

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In this Letter, we have demonstrated a three-dimensional Kosterlitz-Thouless (KT) transition in the random field XY model driven at a uniform velocity. By employing the spin-wave approximation and non-perturbative renormalization group approach, in the weak disorder regime, the three-dimensional driven random field XY model is found to exhibit a quasi-long-range order phase, wherein the correlation function shows power-law decay with a non-universal exponent that depends on the disorder strength and the driving velocity. We further develop a phenomenological theory of the KT transition by taking into account the effect of vortices.

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Introduction.— Two-dimensional (2D) systems with a global $U(1)$ symmetry such as liquid $^4$He films [1], superconducting arrays of Josephson junctions [2], and trapped atomic gases [3] exhibit a topologically ordered phase, which is characterized by power-law decay of the correlation function with a continuously varying exponent. The transition from such a quasi-long-range order (QLRO) phase to a disordered phase is called the Kosterlitz-Thouless (KT) transition [4, 5]. The peculiarity of this transition comes from the fact that it is caused by the structural changes in topological defects or vortices. It is intriguing to understand the role of spatial dimensionality in the realization of the KT transition because the geometries and interactions of the topological defects crucially depend on the spatial dimensions. In the first step toward clarifying this problem, we ask whether there exists a topologically ordered phase and a KT transition in higher dimensions.

It is well known that a spatially inhomogeneous disorder can significantly change the large-scale physics of phase ordering systems. A remarkable example for disorder-induced phase structures is a QLRO known as the Bragg glass phase in the $(4-\epsilon)$-dimensional random field XY model [6-11]. Although this is reminiscent of the spin-wave model exhibits a QLRO phase, wherein the correlation function shows power-law decay with an exponent that depends on the disorder strength and the driving velocity. This QLRO phase resembles the topologically ordered phase in the 2D pure XY model. In the second part, we develop a phenomenological theory of the KT transition by taking into account the effect of the vortices. The change in the vortex structure at the transition point is also discussed. The essential ingredients of the theory are the introduction of a renormalized elastic constant and a dimensional reduction property, which correlates the large-scale behavior of $D$-dimensional driven disordered system with that of the $(D-1)$-dimensional pure system.

Model.— Let $\phi(r) = (\phi^1(r), \phi^2(r))$ be a two component real vector field. The Hamiltonian of the XY model with a quenched random field $h(r) = (h^1(r), h^2(r))$ is given by

$$H[\phi; h] = \int d^D r \left[ \frac{1}{2} K \nabla \phi^\alpha \nabla \phi^\alpha + U(\rho) - \mathbf{h} \cdot \phi \right],$$

where $U(\rho) = \frac{1}{2} \rho \rho_m^2 - 1$ is a local interaction potential and $\rho = |\phi|^2/2$ is the field amplitude. The random field $h^\alpha(r)$ obeys a mean-zero Gaussian distribution with $\langle h^\alpha(r) h^\beta(r') \rangle = \delta_{\alpha\beta} \delta(r-r')$. The dynamics of the field $\phi(r, t)$ is described by

$$\partial_t \phi^\alpha + v \partial_x \phi^\alpha = - \frac{\delta H}{\delta \phi^\alpha} + \xi^\alpha(r, t),$$

where $v$ is a uniform and time-independent driving velocity, and $\xi^\alpha(r, t)$ represents the thermal noise that satisfies $\langle \xi^\alpha(r, t) \xi^\beta(r', t') \rangle_T = 2 T \delta^{\alpha\beta} \delta(r-r') \delta(t-t')$. We call this model the driven random field XY model (DR-FXYM). This model can be considered to describe the dynamics of liquid crystals or superfluid Bose gases flowing in a random environment [12, 13].
It is convenient to introduce the spin-wave model of the DRFXYM to investigate the phase structure in the weak disorder regime. We define the single-valued phase parameter $u \in (-\infty, \infty)$ by $(\phi^1, \phi^2) = (\cos u, \sin u)$. The dynamics of $u(r, t)$ is described by
\[
\partial_t u + v \partial_x u = K \nabla^2 u + F(r; u) + \xi(r, t),
\]
where $F(r; u) = -h^1(r) \sin u + h^2(r) \cos u$ is a random force, which satisfies $(F(r; u) F(r'; u'))_R = \hat{h}_0^2 \cos(u - u') \delta(r - r')$. This model was also introduced in the context of the moving Bragg glass in Refs. [15–17] to describe the dynamics of the displacement field of an elastic lattice driven in a random pinning potential. The spin-wave model is valid only when the order parameter varies slowly in space. Thus, it is not reliable in the strong disorder regime.

It can be easily shown that the lower critical dimension of the DRFXYM is three because the disconnected Greens function for the phase parameter can be expected to behave as $G_u(q) \equiv \langle (u(r) - u(0)) \rangle \sim (K^2 q^2 + v^3 q^3)^{-1}$, whose $q$-integral exhibits an infrared-divergence below three dimensions. (...) represents the average of the distribution functions of the nonequilibrium steady state and the random field. From the analogy to the 2D pure XY model, the 3D-DRFXYM is expected to exhibit a QLRO in the weak disorder regime. In fact, a naive calculation analogous to the case of the 2D pure XY model [18] shows that the correlation function $C(r) \equiv \langle \exp[i(u(r) - u(0)) \rangle$, which is anisotropic due to the driving, decays with power-law form $C(r) \sim r^{-\eta}$, where the exponent here are given by $\eta_1 = \hat{h}_0^2 / (8 \pi K v)$ for the direction parallel to the driving velocity and $\eta_2 = \hat{h}_0^2 / (4 \pi K v)$ for the perpendicular direction [19]. The result reasonably agrees with that of numerical simulations in which Eq. (3) is numerically solved and the correlation function $C(r)$ is calculated [18]. However, it is known that in disordered systems, such naive approaches fail in the large length scale because of the non-perturbative effects associated with the presence of multiple meta-stable states [20].

**RG analysis of the spin-wave model.** — We have applied the NPRG formalism in this work [21]. This approach enables us to deal with the critical behaviors of nonequilibrium systems [22, 23] and disordered systems [24, 25] in a systematic way. We first recall how the equation of motion (3) can be cast into a field theory. By introducing the replicated fields $\{U_a\}_{a=1} = \{u_a, \hat{u}_a\}$, the disorder averaged action is given by
\[
S[\{U_a\}] = \sum_a \int_{rt} \hat{u}_a \left[ \partial_t u_a - T \hat{u}_a + v \partial_x u_a - K \nabla^2 u_a \right] - \frac{1}{2} \sum_{a,b} \int_{rtt'} \hat{u}_{a,t} \hat{u}_{b,t'} \Delta_B(u_{a,t} - u_{b,t'}),
\]
where $\Delta_B(u) = \hat{h}_0^2 \cos u$ is the second cumulant of the bare random force. The NPRG formalism is based on an exact flow equation for the scale-dependent effective action $\Gamma_k[\{U_a\}]$, which includes only high-energy modes with momenta larger than the running scale $k$. As $k$ goes from the cutoff scale $\Lambda$ to zero, $\Gamma_k$ interpolates between the bare action Eq. (3) and the usual effective action, which is defined by the Legendre transform of the generating functional associated with Eq. (3). The exact flow equation for $\Gamma_k$ is given by
\[
\partial_k \Gamma_k = \frac{1}{2} \text{Tr} \left[ \partial_k R_k(q) \left( \Gamma^{(2)}_k + R_k(q) \right)^{-1} \right],
\]
where a $2n \times 2n$ matrix $R_k(q)$ is an infrared cutoff function, which has a constant value proportional to $k^2$ for $q \ll k$ and rapidly deceases for $q > k$. $\Gamma^{(2)}_k$ is the second functional derivative of $\Gamma_k$ and $\text{Tr}$ represents an integration over momentum and frequency as well as a sum over replica indices and the two conjugate fields $\{u, \hat{u}\}$ [26].

In order to solve Eq. (5), we have to introduce an approximation for the functional form of $\Gamma_k$. According to Ref. [26], $\Gamma_k$ is expanded by increasing the number of free replica sums as
\[
\Gamma_k[\{U_a\}] = \sum_{p=1}^\infty \sum_{a_1, \ldots, a_p} \left( \frac{-1}{p!} \right)^{p-1} \Gamma_{p,k}[U_{a_1}, \ldots, U_{a_p}].
\]
We employ the following functional form for the one-replica part,
\[
\Gamma_{1,k} = \int_{rt} \hat{u} \left[ X_k(\partial_t u - T_k \hat{u}) + v \partial_x u - K \nabla^2 u \right],
\]
where $X_k$ and $T_k$ are the scale-dependent relaxation coefficient and temperature. For the multi-replica part,
\[
\Gamma_{p,k} = \int_{rt_1 \ldots r_{tp}} \hat{u}_{a_1,t_1} \ldots \hat{u}_{a_p,t_p} \Delta_{p,k}(u_{a_1,t_1}, \ldots, u_{a_p,t_p}),
\]
where $\Delta_{p,k}(u_{1}, \ldots, u_{p})$ is the $p$-th cumulant of the renormalized random force, which is a fully symmetric, periodic function satisfying $\Delta_{p,k}(u_1 + \lambda, \ldots, u_p + \lambda) = \Delta_{p,k}(u_1, \ldots, u_p)$ for an arbitrary $\lambda$. Although the bare random force is chosen as Gaussian, the higher order cumulants can be generated along the renormalization group (RG) flow. Note that the elastic constant $K$ and the driving velocity $v$ in Eq. (3) are not renormalized [16]. Insertion of Eqs. (7) and (8) into Eq. (5) leads to the RG equations for $X_k$, $T_k$, and $\Delta_{p,k}$. In the following, we consider the zero-temperature case $T = T_k = 0$, because the temperature is irrelevant at three dimensions [18].

We express the RG equations in a scaled form by introducing renormalized dimensionless quantities. In the following, the cutoff scale $\Lambda$ is set to unity. The transverse momentum $q_L$ and longitudinal momentum $q_0$ are measured in units of $k$ and $k^2$, respectively, considering the anisotropy of the system due to the driving. The renormalized dimensionless cumulants are defined.
by $\Delta_{2,k}(u_1, u_2) = C_{D}^{-1}K^{-1}v^{-1}k^{-D+3}A_{2,k}(u_1, u_2)$ and
$\Delta_{l,k}(u_1, u_2, u_3) = C_{D}^{-1}K^{-1}v^{-2}k^{2D-4}A_{l,k}(u_1, u_2, u_3)$, where $C_{D}^{-1} \equiv (D/4) 2^{D+1/2} \Gamma(D/2)$. We employ an
infrared cutoff function independent of $I$ where

\[ R_k(q) = K k^2 r \left( \frac{q_{\perp}}{k} \right)^2 \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \otimes I_n, \] (9)

where $I_n$ is the $n \times n$ unit matrix, which acts on the space of the replica index. For simplicity, we take
$r(y) = (1 - y) \Theta(1 - y)$, where $\Theta(x)$ is the step function. We also use the simplified notations, $\Delta_i(u_a - u_b) \equiv \Delta_{i}(u_a, u_b)$ and
$\Delta'_i(u_a - u_b) \equiv 1/2\{\partial_{u_i} \Delta_i(u_a, u_a, u_b) + \partial_{u_i} \Delta_i(u_b, u_a, u_b)\}$. The RG equations for $\Delta$ and $\Delta'_i$ are given by

\[ \partial_t \Delta(u) = -(D - 3)\Delta(u) - 2I_{1}^{(-)}(z_{\parallel})\Delta'_i(u) + I_2^{(+)}(z_{\parallel})\Delta''(u)(\Delta(0) - \Delta(u)) - I_2^{(-)}(z_{\parallel})\Delta'(u)^2, \] (10)

and

\[ \partial_t \Delta'_i(u) = -(2D - 4)\Delta'_i(u) + O(\Delta_1) + O(\Delta_2 \Delta_3) - 2I_3^{(+)\parallel}(z_{\parallel})[\Delta''(u)\Delta'(u)(\Delta(u) - \Delta(0))]' - I_3^{(-)\parallel}(z_{\parallel})[\Delta'(u)^3 - \Delta'(0)^2\Delta'(u)]', \] (11)

where $l = -\ln k$ and $z_{\parallel} \equiv v^{-2}K k^2 = v^{-2}K e^{-2l}$, which is related to the ratio between the longitudinal elastic term $K \tilde{\mathbf{z}} \mathbf{u}$ and the advection term $v \partial_{x} u$ [20]. The functions $I_{n}^{(\pm)}(z)$ are defined by $I_{n}^{(-)\parallel}(z) = n \pi^{-1} \int dx (1 + z x^2 + \text{i}(z x^2 - \text{i}x)^{-n+1})$ and $I_{n}^{(+)}(z) = \pi^{-1} \int dx \sum_{j=1}^{n}(1 + z x^2 + \text{i}(z x^2 - \text{i}x)^{-n+1})^{-j}$. One can easily check that $I_{n}^{(+)}(0) = 1$ while $I_{n}^{(-)\parallel}(z) \sim z^n$ for a small $z$. The scale-dependent anomalous dimension $\eta_{\perp,i}$, which corresponds to the exponent of the transverse correlation function $C(\tau_{\perp})$, is given by $\eta_{\perp,i} = \Delta_{i}(0)$.

The RG equation for $\Delta_{p}$ contains $\Delta_{p+1}$. Thus, we have an infinite hierarchy of the coupled RG equations. However, in Eq. (11), the contribution of $\Delta_{p}$ vanishes in the large length scale because $I_{1}^{(-)\parallel}(z_{\parallel}) \sim e^{-2l}$. In general, it can be shown that $\Delta_{p+1}$ is irrelevant in the RG equation for $\Delta_{p}$ always appears in the form of $I_{1}^{(-)\parallel}(z_{\parallel})\Delta'_{p+1}$. This implies that the infinite hierarchy of the RG equations is decoupled in the large scale limit, which is rather surprising. Note that such a decoupling does not occur in the equilibrium case.

We proceed to a detailed analysis of Eqs. (10) and (11) at $D = 3$. In the large length scale, $z_{\parallel}$ can be set to zero. Thus, we have a simple RG equation given by

\[ \partial_t \Delta(u) = \Delta''(u)(\Delta(0) - \Delta(u)), \] (12)

which is also derived in Refs. [16, 17] from the 1-loop perturbative calculation. In the equilibrium case, it is known that the renormalized cumulant $\Delta(u)$ exhibits a non-analytic behavior, which is the consequence of the presence of multiple meta-stable states [6, 27]. Such a singularity also arises in the nonequilibrium case. The left panel in Fig. 1 shows the schematic picture of the RG evolution of $\Delta(u)$ corresponding to Eq. (12). In a finite renormalization scale, it develops linear cusps at $u = 2\pi m$ and evolves into a parabolic profile. In the large scale limit, $\Delta(u)$ eventually becomes a constant. Therefore, invoking the decoupling property mentioned above, we obtain a family of stable fixed points $\Delta(u) = \Delta_{c}$.

\[ \Delta_{c} = \Delta_{c}(0) + \Delta_{c}(1)u + \Delta_{c}(2)u^2 + \cdots, \]

where $\Delta_{c}(i)$ is the $i$-th order correction to the bare disorder strength $h_{0}^{2}/(4\pi K v)$ for a small $z$. The scale-dependent anomalous dimension $\eta_{\perp,i}$, which corresponds to the exponent of the transverse correlation function $C(\tau_{\perp})$, is given by $\eta_{\perp,i} = \Delta_{i}(0)$.

The right panel in Fig. 1 shows the RG trajectories calculated from Eqs. (10) and (11). We have ignored the higher order terms proportional to $\Delta_{2}\Delta_{3}$ and $\Delta_{4}$ in Eq. (11). In order to follow the RG evolution of the non-analytic solution, $\Delta(u)$ and $\Delta'(u)$ are approximated by parabolic forms [28]. For a weak disorder $\Delta_{B}(0) < \Delta_{c}$, $\Delta_{i}(0) = \eta_{\perp,i}$ flows into a finite value $\Delta(\infty) = \eta_{\perp,i}$. Especially, in the weak disorder limit, $\eta_{\perp,i}$ is equal to $\Delta(0)$, which is a finite value. This regime corresponds to a disordered phase. However, it may be recalled that the spin-wave model is invalid in the strong disorder regime because it cannot describe the vortices. The QLRO phase with a large $\eta_{\perp,i}$ is expected to be broken by the proliferation of the vortices.

**Effect of the vortices.**— We construct a phenomenological theory valid near the transition point by taking
into account the effect of the vortices. We employ two assumptions. The first assumption is that the dominant effect of the vortices is to renormalize the elastic constant $K$. This means that the phenomenological theory is obtained by the replacement of $K$ by $K_{\text{eff}}$ in the spin-wave model. The effective elastic constant $K_{\text{eff}}$ is always smaller than $K$. The second assumption is that $K_{\text{eff}}$ obeys the RG equations similar to the 2D pure XY model. This assumption is the consequence of the following argument. From Eq. (2), a steady state $\phi^{(s)}(x, r_\perp)$ satisfies the following equation:

$$v\partial_x \phi = K\nabla^2_x \phi - U'(\rho)\phi + h(x, r_\perp),$$

where we have omitted the longitudinal elastic term $K\partial_x^2 u$ because it is negligible compared to the advection term $v\partial_x u$ in the large length scale. By considering the coordinate $x$ as a fictitious time, one can find that $\phi^{(s)}(x, r_\perp)$ is identical to a dynamical solution of the 2D pure XY model with temperature $T_{\text{eff}} = h_\rho^2/(2v)$. If we assume that all steady states contribute with equal weight, we can conclude that the large-scale behavior of the 3D-DRFXYM is identical to that of the 2D pure XY model. This is an analogue of the dimensional reduction property in equilibrium cases, which states that the critical behavior of $D$-dimensional disordered system is identical to that of $(D-2)$-dimensional pure system [24, 27]. However, the physics of the 3D-DRFXYM can be different from that of the 2D pure XY model in general because each steady state satisfying Eq. (13) contributes with non-trivial weight to the averaged quantities. In fact, it is known that the non-analytic behavior of the renormalized cumulant $\Delta(u)$ leads to the breakdown of the dimensional reduction [24, 27]. In our case, the dimensional reduction recovers in the large length scale because the cusps of $\Delta(u)$ disappear in the $l \to \infty$ limit (See the left panel in Fig. 1).

From the second assumption, the RG equations for $K_{\text{eff}}$ can be obtained by replacing $T$ in the RG equations for the 2D pure XY model [3] with $T_{\text{eff}} = \Delta(0)/(2v) = 2\pi K_{\text{eff}} \Delta(0)$. Thus, we have

$$\frac{\Delta(0)}{d\ln K_{\text{eff}}^{-1}} = \pi^2 y^2,$$

$$\frac{dy}{d\ln K_{\text{eff}}^{-1}} = \left(2 - \frac{1}{2\Delta(0)}\right)y,$$

where $y$ is the fugacity of the vortices. Note that, in the first equation in Eq. (13), $K_{\text{eff}}^{-1}$ alone is modified by the vortices. Thus, the derivative operator does not act on the dimensionful disorder strength $\Delta(0)$. The RG equations for $\Delta(u)$ and $\Delta'_u(u)$ are obtained from Eqs. (10) and (11) by replacing $K$ with $K_{\text{eff}}$ and by adding terms $\Delta(u)\partial_u \ln K_{\text{eff}}^{-1}$ and $\Delta'_u(u)\partial_u \ln K_{\text{eff}}^{-1}$, respectively. The bare value of the vortex fugacity $y_0$ is also obtained from the same replacement $T \to 2\pi K \Delta(0)$ in the 2D pure XY model [5]. Thus, we have $y_0 = \exp[-(\pi/4)\Delta_B(0)^{-1}]$.

The RG trajectories for the spin-wave model with the correction of the vortices are shown in Fig. 2. The trajectories that started from the weak disorder regime $\Delta_B(0) < \Delta_{\text{KT}}$ flow into a line of fixed points. This regime corresponds to the topologically ordered phase. The trajectories that started from the strong disorder regime $\Delta_B(0) > \Delta_{\text{KT}}$ diverge $\Delta(0), y \to \infty$. This regime corresponds to a disordered phase. At $\Delta_B(0) = \Delta_{\text{KT}}$, the anomalous dimension $\eta_\perp$ is $1/4$ as in the conventional KT transition. This value agrees with the value obtained from the previous numerical study in Ref. [19].

Let us consider the change in the vortex structure at the transition point. It may be recalled that the KT transition of the 2D pure XY model is understood as the dissociation of tightly bounded vortex-antivortex pairs. This vortex dissociation picture can be also applied to the 3D-DRFXYM. According to the dimensional reduction property, in the large length scale, a snapshot of the 3D-
DRFXYM is identical to a space-time trajectory of the 2D pure XY model by considering the coordinate $x$ as a fictitious time. Thus, in the topologically ordered phase, the system is considered as a dilute gas of small vortex rings, whose transverse sections correspond to bounded vortex-antivortex pairs. At the transition point, such vortex rings break up and tangled vortex lines percolate along the driving direction. The schematic picture is given in Fig. 3.

Conclusion.— By employing the RG approach and a phenomenological argument, we have shown that the 3D random field XY model exhibits a novel type of topological phase transition when it is driven at a uniform velocity. We hope that, in the future, such a transition will be observed in experimental studies for liquid crystals or quantum gases flowing in a random environment.

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**SUPPLEMENTAL MATERIAL**

**The RG equations for \( \Delta_2 \) and \( \Delta_3 \)**

In this section, we sketch the derivation of the RG equations for the renormalized dimensionless cumulants \( \Delta_2 \) and \( \Delta_3 \). Our approach is based on the non-perturbative functional renormalization group (NP-FRG) formalism developed for disordered systems in Ref. [24].

The bare action for the spin-wave model is given by

\[
S[[U_a]] = \sum_a \int_{rt} \dot{u}_a \left[ \partial_t u_a - T \dot{u}_a + v \partial_x u_a - K \nabla^2 u_a \right] - \frac{1}{2} \sum_{a,b} \int_{rt} \dot{u}_a \dot{u}_{b,t} \Delta_B(u_a,t - u_{b,t'}). \tag{S1}
\]

By introducing source fields \( J_a = \dot{t}(J_a, \dot{J}_a), a = 1, ..., n \), the generating functional reads

\[
Z[[J_a]] = \int \prod_a D U_a \exp \left[ -S[[U_a]] + \sum_a \int_{rt} t J_a \cdot U_a \right]. \tag{S2}
\]

The effective action is defined as a Legendre transform:

\[
\Gamma[[U_a]] = -\ln Z[[J_a]] + \sum_a \int_{rt} t J_a \cdot U_a, \tag{S3}
\]

where \( U_a \) and \( J_a \) are related by \( U_a = \delta \ln Z[[J_a]]/\delta J_a \). Since \( \Gamma[[U_a]] \) gives the renormalized vertices, its zero momentum limit defines the renormalized disorder.

In the NP-FRG formalism, we introduce a one-parameter family of the effective action \( \Gamma_k \) in which only high-energy modes with \( q > k \) are included. In order to define \( \Gamma_k \), we add, to the original action, a momentum-dependent mass term

\[
\Delta S_k[[U_a]] = \frac{1}{2} \sum_{a,b} \int_{q,\omega} t U_a(q, \omega) R_k(q)_{ab} U_b(-q, -\omega), \tag{S4}
\]

where \( R_k(q) \) is a \( 2n \times 2n \) matrix,

\[
R_k(q) = \begin{pmatrix} 0 & R_k(q) \\ R_k(q) & 0 \end{pmatrix} \otimes I_n, \tag{S5}
\]

where \( R_k(q) \) is an infrared cutoff function, which has a constant value proportional to \( k^2 \) for \( q \ll k \) and rapidly decreases for \( q > k \), and \( I_n \) is the \( n \times n \) unit matrix, which acts on the space of the replica index. We employ an infrared cutoff function independent of \( q_x \),

\[
R_k(q) = K k^2 r \left( \frac{|q_x|^2}{k^2} \right), \tag{S6}
\]

with \( r(y) = (1 - y)\Theta(1 - y) \), where \( \Theta(x) \) is the step function. The scale-dependent generating functional reads

\[
Z_k[[J_a]] = \int \prod_a D U_a \exp \left[ -S[[U_a]] - \Delta S_k[[U_a]] + \sum_a \int_{rt} t J_a \cdot U_a \right]. \tag{S7}
\]

Then, the scale-dependent effective action is defined as

\[
\Gamma_k[[U_a]] = -\ln Z_k[[J_a]] + \sum_a \int_{rt} t J_a \cdot U_a - \Delta S_k[[U_a]], \tag{S8}
\]

where \( U_a \) and \( J_a \) are related by \( U_a = \delta \ln Z_k[[J_a]]/\delta J_a \). It can be shown that \( \Gamma_{k=0} = \Gamma \) and \( \lim_{k \to \infty} \Gamma_k = S \). The exact flow equation for \( \Gamma_k \) is given by

\[
\partial_k \Gamma_k = \frac{1}{2} \text{Tr} \left[ \partial_k R_k(q) \left( \Gamma_k^2 + R_k(q) \right)^{-1} \right], \tag{S9}
\]
where \( \Gamma^{(2)} \) is the second functional derivative of \( \Gamma_k \) and \( \text{Tr} \) represents an integration over momentum and frequency as well as a sum over replica indices and the two conjugate fields \( \{ u, \tilde{u} \} \).

By inserting the replica sum expansion Eq. (6) in the main text into Eq. (S9), we obtain the hierarchy of the exact flow equations for \( \Gamma_{p,k} \), which are given in Appendix of Ref. [25]. From the functional form Eq. (8) in the main text, the RG equation for \( \tilde{\Delta}_p \) is obtained from

\[
\partial_k \tilde{\Delta}_{p,k}(u_1, \ldots, u_p) = \frac{1}{f_p} \Xi_{\delta u_1 \ldots \delta u_p} \partial_k \Gamma_{p,k}[U_1, \ldots, U_p],
\]

where the functional derivative is evaluated for a uniform field configuration: \( u_{1,rt} \equiv u_1, \ldots, u_{p,rt} \equiv u_p \) and \( \dot{u}_{1,rt} \equiv 0, \ldots, \dot{u}_{p,rt} \equiv 0 \). \( \mathcal{V} \) and \( \mathcal{T} \) are space and time volumes, respectively. In order to express the RG equations in compact forms, we introduce notations for the Green’s function,

\[
\begin{align*}
g_{11}(q) &= 2X_k T_k \{ M(q)^2 + q_x^2 v^2 \}^{-1}, \\
g_{12}(q) &= (M(q) - i q v)^{-1}, \\
g_{21}(q) &= (M(q) + i q v)^{-1}, \\
g_{22}(q) &= 0,
\end{align*}
\]

where \( M(q) = K q^2 + R_k(q) \).

For the zero temperature case \( T = T_k = 0 \), the RG equation for \( \tilde{\Delta}_2 \) is given as follows:

\[
\partial_l \tilde{\Delta}_2(u_1, u_2) = \frac{1}{2} \int_q \partial_l R_k(q) \left[ (1) + (2) + (3) + (4) + \text{perm}(u_1, u_2) \right],
\]

where \( l = -t = -\ln k \) and we have used simplified notations such as \( \tilde{\Delta}^{(1)}_{2}(u_a, u_b) \equiv \partial_{u_1} \partial_{u_2} \tilde{\Delta}_2(u_a, u_b) \). The RG equation for \( \tilde{\Delta}_3 \) is given as follows:

\[
\begin{align*}
\partial_l \tilde{\Delta}_3(u_1, u_2, u_3) &= \frac{1}{2} \int_q \partial_l R_k(q) \left[ (A - 1) + (B - 1) + \ldots + (B - 6) \\
&\quad + (C - 1) + \ldots + (C - 5) + \text{perm}(u_1, u_2, u_3) \right],
\end{align*}
\]

where \( A = 1 - 1 \tilde{\Delta}_4^{(100)}(u_1, u_2, u_3) \{ g_{21}(q)^2 + g_{12}(q)^2 \} \),

\( B - 1 = 2 \tilde{\Delta}_2^{(1)}(u_1, u_2) \Delta_3^{(100)}(u_1, u_2, u_3) \{ g_{21}(q)^2 g_{12}(q) + g_{21}(q) g_{12}(q)^2 \} \),

\( B - 2 = 2 \Delta_2^{(10)}(u_1, u_2) \Delta_3^{(100)}(u_1, u_2, u_3) \{ g_{21}(q)^3 + g_{12}(q)^3 \} \),

\( B - 3 = 2 \Delta_2^{(11)}(u_1, u_2) \Delta_3^{(100)}(u_2, u_1, u_3) \{ g_{21}(q)^2 g_{12}(q) + g_{21}(q) g_{12}(q)^2 \} \),

\( B - 4 = 2 \Delta_2^{(10)}(u_1, u_2) \Delta_3^{(100)}(u_2, u_1, u_3) \{ g_{21}(q)^3 + g_{12}(q)^3 \} \),

\( B - 5 = 2 \Delta_2^{(1)}(u_1, u_2) \Delta_3^{(100)}(u_2, u_1, u_3) \{ g_{21}(q)^2 g_{12}(q) + g_{21}(q) g_{12}(q)^2 \} \),

\( B - 6 = 2 \Delta_2^{(1)}(u_1, u_1) \Delta_3^{(100)}(u_1, u_2, u_3) \{ g_{21}(q)^2 g_{12}(q) + g_{21}(q) g_{12}(q)^2 \} \).

\( C - 1 = -2 \tilde{\Delta}_2^{(100)}(u_1, u_2, u_3) \{ g_{21}(q)^2 + g_{12}(q)^2 \} \),

\( C - 2 = -2 \tilde{\Delta}_2^{(10)}(u_1, u_2) \Delta_3^{(100)}(u_1, u_2, u_3) \{ g_{21}(q)^2 g_{12}(q) + g_{21}(q) g_{12}(q)^2 \} \),

\( C - 3 = -2 \tilde{\Delta}_2^{(11)}(u_1, u_2) \Delta_3^{(100)}(u_2, u_1, u_3) \{ g_{21}(q)^3 + g_{12}(q)^3 \} \),

\( C - 4 = -2 \tilde{\Delta}_2^{(10)}(u_1, u_2) \Delta_3^{(100)}(u_2, u_1, u_3) \{ g_{21}(q)^2 g_{12}(q) + g_{21}(q) g_{12}(q)^2 \} \),

\( C - 5 = -2 \tilde{\Delta}_2^{(11)}(u_1, u_2) \Delta_3^{(100)}(u_1, u_2, u_3) \{ g_{21}(q)^3 + g_{12}(q)^3 \} \).
The integrals in the RG equations are calculated as follows:

\[
\frac{1}{2} \int_{q} \partial_{t} R_{k}(q) \{ ng_{21}(q)^{n+1} + ng_{12}(q)^{n+1} \} = C_{D-1} K^{-n+1} k^{D-2n+1} v^{-1} I_{n}^{(-)}(z_{||}, k),
\]

\[
\frac{1}{2} \int_{q} \partial_{t} R_{k}(q) \sum_{j=1}^{n} 2g_{21}(q)^{n+1-j} g_{12}(q)^{j} = C_{D-1} K^{-n+1} k^{D-2n+1} v^{-1} I_{n}^{(+)}(z_{||}, k),
\]  

(S14)

where \( C_{D}^{-1} = \left( D/4 \right) 2^{D+1} \pi^{D/2} \Gamma(D/2), \) \( z_{||, k} = v^{-2} K^{2} k^{2} = v^{-2} K^{2} e^{-2l}, \) and

\[
I_{n}^{(-)}(z) = \frac{n}{\pi} \int_{-\infty}^{\infty} dx \left( 1 + z x^{2} + ix \right)^{-(n+1)},
\]

\[
I_{n}^{(+)}(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} dx \sum_{j=1}^{n} (1 + z x^{2} + ix)^{-(n+1-j)}(1 + z x^{2} - ix)^{-j}.
\]  

(S15)

In terms of the dimensionless cumulants \( \Delta_{2,k}(u_1, u_2) = C_{D-1} K^{-1} v^{-1} k^{D-3} \tilde{\Delta}_{2,k}(u_1, u_2) \) and \( \Delta_{3,k}(u_1, u_2, u_3) = C_{D-1}^{2} K^{-1} v^{-2} k^{2D-4} \tilde{\Delta}_{3,k}(u_1, u_2, u_3) \), we obtain Eqs. (10) and (11) in the main text.

**Numerical integration of the RG equations**

In this section, we explain the numerical method to solve Eqs. (10) and (11) in the main text. Standard numerical schemes cannot be applicable to follow the RG evolution of the renormalized cumulants due to the generation of cusps. Noting the fact that \( \Delta(u) \) evolves into a parabolic profile, we approximate the non-analytic solutions by parabolic functions,

\[
\Delta_{1}(u) = a_{1}(l)(u - \pi)^{2} + b_{1}(l),
\]

\[
\Delta_{3,1}(u) = a_{2}(l)(u - \pi)^{2} + b_{2}(l),
\]  

(S16)

for \( u \in [0, 2\pi] \). By substituting Eq. (S16) into Eqs. (10) and (11), we have

\[
\frac{da_{1}}{dl} = \left\{ 2I_{2}^{(+)}(z_{||}) + 4I_{2}^{(-)}(z_{||}) \right\} a_{1}^{2} - 2I_{1}^{(-)}(z_{||}) a_{2},
\]

\[
\frac{da_{2}}{dl} = -2a_{2} - 24\left\{ I_{3}^{(+)}(z_{||}) + I_{3}^{(-)}(z_{||}) \right\} a_{1}^{3},
\]

\[
\frac{d}{dl} \Delta(0) = -4\pi^{2}I_{2}^{(-)}(z_{||}) a_{1}^{2} - I_{1}^{(-)}(z_{||}) \Delta_{1}'(0),
\]

\[
\frac{d}{dl} \Delta_{3}(0) = -2\Delta_{3}'(0) - 16\pi^{2} \left\{ I_{3}^{(+)}(z_{||}) + I_{3}^{(-)}(z_{||}) \right\} a_{1}^{3}.
\]  

(S17)

where we have ignored the higher order terms. We numerically solve this set of differential equations with the following initial condition: \( \Delta(0) = \Delta_{B}(0) = \delta_{0}^{2}/(4\pi K v), \) \( \Delta_{3}(0) = 0, \) \( a_{1} = (1/2)\Delta_{B}'(u = \pi) = \Delta_{B}(0)/2, \) and \( a_{2} = 0. \)