Lyapunov type inequalities and their applications for quasilinear impulsive systems

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ABSTRACT
A novel Lyapunov-type inequality for Dirichlet problem associated with the quasilinear impulsive system involving the \((p_j, q_j)\)-Laplacian operator for \(j = 1, 2\) is obtained. Then utility of this new inequality is exemplified in finding disconjugacy criterion, obtaining lower bounds for associated eigenvalue problems and investigating boundedness and asymptotic behaviour of oscillatory solutions. The effectiveness of the obtained disconjugacy criterion is illustrated via an example. Our results not only improve the recent related results but also generalize them to the impulsive case.

1. Introduction
In this paper, we obtain a Lyapunov type inequality for the following Dirichlet problem associated with the quasilinear impulsive system involving the \((p_j, q_j)\)-Laplacian operator for \(j = 1, 2\):

\[
\begin{aligned}
&\left( h_1(t) \left| u(t) \right|^{p_1-2} u(t) \right)'' - \left( m_1(t) \left| u(t) \right|^{q_1-2} u(t) \right)'' = f(t) \left| u(t) \right|^{\beta-2} \left| v(t) \right|^\gamma \\
&\left( h_2(t) \left| v(t) \right|^{p_2-2} v(t) \right)'' - \left( m_2(t) \left| v(t) \right|^{q_2-2} v(t) \right)'' = g(t) \left| v(t) \right|^{\beta-2} \left| u(t) \right|^\gamma \\
&-\Delta \left( h_1(t) \left| u(t) \right|^{p_1-2} u(t) + m_1(t) \left| u(t) \right|^{q_1-2} u(t) \right)_{t=\tau_i}^{=} = a_i \left| u(t) \right|^{\delta-2} \left| u(t) \right|^\delta, \quad i \in \mathbb{N}, \\
&-\Delta \left( h_2(t) \left| v(t) \right|^{p_2-2} v(t) + m_2(t) \left| v(t) \right|^{q_2-2} v(t) \right)_{t=\tau_i}^{=} = b_i \left| v(t) \right|^{\delta-2} \left| v(t) \right|^\delta, \quad i \in \mathbb{N}.
\end{aligned}
\]

Throughout this section, we assume that

(i) \(h_j, m_j, f, g \in PLC[t_0, \infty) = \{ \omega : [t_0, \infty) \to \mathbb{R} \text{ is continuous on each interval } (t_i, t_{i+1}), \text{ the limits } \omega(t_j) \text{ exist and } \omega(t_i^-) = \omega(t_i) \text{ for } i \in \mathbb{N} \}, h_j, m_j > 0, j = 1, 2;\)

(ii) \(p_j, q_j > 1, j = 1, 2 \) and \(\alpha, \beta, \gamma, \theta > 0 \) are real numbers,

(iii) \(\{\tau_i\} \) is a strictly increasing sequence of real numbers for \(i \in \mathbb{N},\)

(iv) \(a_i, b_i \) are sequence of real numbers for \(i \in \mathbb{N}.\)

Definition 1.1: By a solution \(w(t) = (u(t), v(t))\) of system (1) on the interval \([t_0, \infty)\), we mean a nontrivial pair

of continuous functions \((u(t), v(t))\) defined on \([t_0, \infty)\) such that \((h_1(t)|u(t)|^{p_1-2}u(t), (m_1(t)|u(t)|^{q_1-2}u(t)), (h_2(t)|v(t)|^{p_2-2}v(t), (m_2(t)|v(t)|^{q_2-2}v(t)) \in PLC[t_0, \infty)\) satisfying (1) for \(t \geq t_0.\)

Definition 1.2 ([24]): The solution \(w(t) = (u(t), v(t))\) of system (1) has a zero at the point \(c\) if both components of the solution \(w\) have a zero at this point.

We also need the following definitions.

Definition 1.3 ([24]): System (1) is called disconjugate on an interval \([a, b]\) if and only if there is no real nontrivial solution \(w(t) = (u(t), v(t))\) of system (1) having two or more zeros on \([a, b]\).

Definition 1.4 ([24]): A nontrivial solution \(w(t) = (u(t), v(t))\) of system (1) is bounded on \([t_0, \infty)\) if both components of \(w\) are bounded on \([t_0, \infty)\). If at least one component of \(w\) is not bounded on \([t_0, \infty)\), then this solution is called unbounded.

Definition 1.5 ([24]): A nontrivial solution \(w(t) = (u(t), v(t))\) of system (1) is said to be oscillatory if both components of \(w\) are oscillatory on \([T_0, \infty)\), i.e., if for each \(T > T_0\) there is a point \(T_1 \in (T, \infty)\) such that \(u(T_1) = v(T_1) = 0.\) If either at least one component of \(w\) is not oscillatory or they are oscillatory but they become zero at different points, this solution is called nonoscillatory.

Definition 1.6 ([24]): A nontrivial solution \(w(t) = (u(t), v(t))\) of system (1) tends to zero as \(t \to \infty\) if both components of \(w\) tend to zero as \(t \to \infty\). If at least one component of \(w\) does not approach zero as \(t \to \infty\), then this solution does not approach zero as \(t \to \infty\).
Lyapunov type inequality is one of the main tools to investigate asymptotic behaviours, such as oscillation, disconjugacy, stability, of solutions of differential equations and to analyse boundary and eigenvalue problems. It was established by Lyapunov [1] and generalized to linear impulsive case in [2]. For a comprehensive exhibition of the results, we refer two surveys [3,4] and references therein. The half linear version of Lyapunov inequality was obtained in [5–9]. To the best of our knowledge, although many results have been obtained for quasilinear systems [10–23], there is little known for the impulsive quasilinear systems [24]. Although there is a large body of literature on quasilinear systems that we can not cover completely, the results in [10,11,24] and in [22] are worth mentioning due to their contribution to these subject.

Recall that the numbers \( \mu, \mu' \geq 1 \) are said to be conjugate if \( \frac{1}{\mu} + \frac{1}{\mu'} = 1 \). In the sequel, we denote \( r^+(t) = \max(r(t), 0) \) and \( r^+_1 = \max(r, 0) \).

**Theorem 1.7 ([10]):** In system (1), let \( h_1(t) = h_2(t) = 1, m_1(t) = m_2(t) = 0, f(t), g(t) > 0, \alpha = \beta = \gamma, \alpha + \beta = 1 \), \( \alpha = \beta = 0, i \in \mathbb{N} \), and \( p_1' \) and \( p_2' \) be conjugate numbers for \( p_1 \) and \( p_2 \), respectively. If system (1) has a real nontrivial solution \( (u(t), v(t)) \) such that \( u(a) = u(b) = v(a) = v(b) = 0, a, b \in \mathbb{R} \) with \( a < b \) are consecutive zeros, and \( u, v \) are not identically zero on \( [a, b] \), then we have the following Lyapunov type inequality

\[
2^{\alpha + \beta} \leq (b - a)^{\frac{\alpha}{p_1} + \frac{\beta}{p_2}} \left( \int_a^b f(t) \, dt \right)^{\frac{\alpha}{p_1}} \left( \int_a^b g(t) \, dt \right)^{\frac{\beta}{p_2}}.
\]

**Theorem 1.8 ([11]):** In system (1), let \( m_1(t) = m_2(t) = 0, \alpha = \beta = \gamma, \alpha + \beta = 1 \), \( \alpha = \beta = 0, i \in \mathbb{N} \), and \( p_1' \) and \( p_2' \) be conjugate numbers for \( p_1 \) and \( p_2 \), respectively. If system (1) has a real solution \( (u(t), v(t)) \) such that \( u(a) = u(b) = v(a) = v(b) = 0, a, b \in \mathbb{R} \) with \( a < b \) are consecutive zeros, and \( u, v \) are not identically zero on \( [a, b] \), then we have the following Lyapunov type inequality

\[
2^{\alpha + \beta} \leq \left( \int_a^b h_1^{1-p_1'}(t) \, dt \right)^{\frac{\alpha}{p_1'}} \left( \int_a^b h_2^{1-p_2'}(t) \, dt \right)^{\frac{\beta}{p_2'}} \times \left( \int_a^b f^+(t) \, dt \right)^{\frac{\alpha}{p_1'}} \left( \int_a^b g^+(t) \, dt \right)^{\frac{\beta}{p_2'}}.
\]

**Theorem 1.9 ([24]):** In system (1), let \( m_1(t) = m_2(t) = 0 \) and \( p_1' \) and \( p_2' \) be conjugate numbers for \( p_1 \) and \( p_2 \), respectively and \( (e_1, e_2) \) be a nontrivial solution of the homogenous system

\[
e_1(\alpha - p_1) + e_2\beta = 0,
\]

\[
e_1\beta + e_2(\gamma - p_2) = 0,
\]

where \( e_k > 0 \) for \( k = 1, 2 \) and \( e_1^2 + e_2^2 > 0 \). If the system (1) has a real nontrivial solution \( (u(t), v(t)) \) such that \( u(a) = u(b) = v(a) = v(b) = 0, a, b \in \mathbb{R} \) with \( a < b \) are consecutive zeros, and \( u, v \) are not identically zero on \( [a, b] \), then we have the following Lyapunov type inequality

\[
2^{\alpha + \beta} \leq \left( \int_a^b h_1^{1-p_1'}(t) \, dt \right)^{\frac{\alpha}{p_1'}} \left( \int_a^b h_2^{1-p_2'}(t) \, dt \right)^{\frac{\beta}{p_2'}} \times \left( \int_a^b f^+(t) \, dt \right)^{\frac{\alpha}{p_1'}} \left( \int_a^b g^+(t) \, dt \right)^{\frac{\beta}{p_2'}}.
\]

**Theorem 1.10 ([22]):** In system (1), let \( h_1(t) = h_2(t) = 1, f(t), g(t) > 0, \alpha = \beta = \gamma, \alpha + \beta = 1 \), \( i \in \mathbb{N} \) and \( \frac{\alpha}{p_1} + \frac{\beta}{p_2} = 1 \). If the system (1) has a real nontrivial solution \( (u(t), v(t)) \) such that \( u(a) = u(b) = v(a) = v(b) = 0, a, b \in \mathbb{R} \) with \( a < b \) are consecutive zeros, and \( u, v \) are not identically zero on \( [a, b] \), then we have the following Lyapunov type inequality

\[
\left( \frac{1}{2} \int_a^b f(t) \, dt \right)^{\frac{2\alpha}{p_1+q_1}} \left( \frac{1}{2} \int_a^b g(t) \, dt \right)^{\frac{2\beta}{p_2+q_2}} \geq \left[ \min \left\{ \frac{2^{p_1}}{(b-a)^{p_1-1}}, \frac{2^{p_2}}{(b-a)^{p_2-1}} \right\} \right]^{\frac{2\alpha}{p_1+q_1}} \times \left[ \min \left\{ \frac{2^{p_1}}{(b-a)^{p_1-1}}, \frac{2^{p_2}}{(b-a)^{p_2-1}} \right\} \right]^{\frac{2\beta}{p_2+q_2}}.
\]

Since our main interest is Lyapunov type inequality for system (1), we assume the existence of nontrivial solution of this system. Our main purpose is to establish Lyapunov type inequality for the impulsive system of differential equations (1) satisfying Dirichlet boundary conditions. Although our motivation comes from the papers of [11,22,24], our results not only extend the results of such papers to the impulsive case but also improve them.

### 1.1. Lyapunov type inequality

In the sequel, we assume that

\[
\frac{2\alpha}{p_1+q_1} + \frac{2\beta}{p_2+q_2} = 1
\]

and

\[
\frac{2\alpha}{p_1+q_1} + \frac{2\gamma}{p_2+q_2} = 1.
\]

...
For the sake of convenience, we define the following integral operator

\[
M(s, k, \mu) = \left( \int_{a}^{s} (k(t))^{-\frac{\mu}{\mu'}} \, dt \right)^{1-\mu} + \left( \int_{s}^{b} (k(t))^{-\frac{\mu}{\mu'}} \, dt \right)^{1-\mu},
\]

where \( s \in (a, b) \), \( k \) is a real-valued continuous function such that \( k(t) > 0 \) for all \( t \in \mathbb{R} \), and \( \mu, \mu' > 1 \) are conjugate numbers.

**Remark 1.1:** For a given number \( \mu \) and function \( k \), set \( F(s) = M(s, k, \mu) \) for \( s \in (a, b) \). \( F(s) \) obtains its minimum at the point \( s \in (a, b) \) such that

\[
\int_{a}^{s} (k(t))^{-\frac{\mu}{\mu'}} \, dt = \int_{s}^{b} (k(t))^{-\frac{\mu}{\mu'}} \, dt.
\]

holds. Thus, we have

\[
F(s) \geq F_{\text{min}}(s) = 2 \left( \int_{a}^{b} (k(t))^{-\frac{\mu}{\mu'}} \, dt \right)^{1-\mu} = 2N(s, k, \mu).
\]

**Remark 1.2:** Since the function \( h(t) = t^{1-\mu} \) is convex for \( t > 0 \), Jensen’s inequality \( h(\frac{x+y}{2}) \leq \frac{h(x) + h(y)}{2} \) with \( y = \int_{a}^{c} (h(t))^{-\frac{\mu}{\mu'}} \, dt \) and \( z = \int_{c}^{b} (h(t))^{-\frac{\mu}{\mu'}} \, dt \) implies

\[
M(c, h_{1}, p_{1}) = \left( \int_{a}^{c} (h_{1}(t))^{-\frac{\mu}{\mu'}} \, dt \right)^{1-p_{1}} + \left( \int_{c}^{b} (h_{1}(t))^{-\frac{\mu}{\mu'}} \, dt \right)^{1-p_{1}} \geq 2^{p_{1}} \left( \int_{a}^{b} (h_{1}(t))^{-\frac{\mu}{\mu'}} \, dt \right)^{1-p_{1}} = 2^{p_{1}} N(b, h_{1}, p_{1}).
\]

Similarly, we obtain

\[
M(c, m_{j}, q_{j}) \geq 2^{q_{j}} \left( \int_{a}^{b} (m_{j}(t))^{-\frac{\mu}{\mu'}} \, dt \right)^{1-q_{j}} = 2^{q_{j}} N(b, m_{j}, q_{j}), \ j = 1, 2
\]

and

\[
M(d, h_{2}, p_{2}) \geq 2^{p_{2}} \left( \int_{a}^{b} (h_{2}(t))^{-\frac{\mu}{\mu'}} \, dt \right)^{1-p_{2}} = 2^{p_{2}} N(b, h_{2}, p_{2}).
\]

Now we are ready to give the main result of this paper as follows.

**Theorem 1.11:** Assume that the condition (3) holds. Let \( p_{j}' \) and \( q_{j}' \) be conjugate numbers for \( p_{j} \) and \( q_{j} \), \( j = 1, 2 \), respectively and \( (e_{1}, e_{2}) \) be a nontrivial solution of the homogenous system

\[
e_{1} \left( \alpha - \frac{p_{1} + q_{1}}{2} \right) + e_{2} \beta = 0,
\]

\[
e_{1} \beta + e_{2} \left( \gamma - \frac{p_{2} + q_{2}}{2} \right) = 0,
\]

where \( e_{k} \geq 0 \) for \( k = 1, 2 \) and \( e_{1}^{2} + e_{2}^{2} > 0 \). If the system (1) has a real nontrivial solution \( w(t) = (u(t), v(t)) \) such that \( w(a) = w(b) = 0 \), \( a, b \in \mathbb{R} \) with \( a < b \) are consecutive zeros, then we have the following Lyapunov type inequality

\[
\left( \int_{a}^{b} f^{+}(t) \, dt + \sum_{a \leq \tau < b} a_{\tau}^{+} \right) e_{1} + \left( \int_{a}^{b} g^{+}(t) \, dt + \sum_{a \leq \tau < b} b_{\tau}^{+} \right) e_{2} \geq 2^{e_{1}+e_{2}} \left[ \min \left\{ 2^{p_{1}} N(b, h_{1}, p_{1}), 2^{q_{1}} N(b, m_{1}, q_{1}) \right\} \right] e_{1} \times \left[ \min \left\{ 2^{p_{2}} N(b, h_{2}, p_{2}), 2^{q_{2}} N(b, m_{2}, q_{2}) \right\} \right] e_{2}.
\]

**Proof:** Multiplying the first equation of system (1) by \( u \) and integrating from \( a \) to \( b \) and using \( f^{+}(t) = \max(f(t), 0) \) and \( a_{\tau}^{+} = \max(a_{\tau}, 0) \), we have

\[
\int_{a}^{b} h_{1}(t) \left| u'(t) \right|^{p_{1}} \, dt + \int_{a}^{b} m_{1}(t) \left| u'(t) \right|^{q_{1}} \, dt 
\]

\[
\leq \int_{a}^{b} f^{+}(t) \left| u(t) \right|^{a} \left| v(t) \right|^{b} \, dt + \sum_{a \leq \tau < b} a_{\tau}^{+} \left| u(t_{\tau}) \right|^{a} \left| v(t_{\tau}) \right|^{b}.
\]

Let

\[
\left| u(c) \right| = \max_{a \leq t \leq b} \left| u(t) \right|
\]

and

\[
\left| v(d) \right| = \max_{a \leq t \leq b} \left| v(t) \right|
\]

then from (6), we have

\[
\int_{a}^{b} h_{1}(t) \left| u'(t) \right|^{p_{1}} \, dt + \int_{a}^{b} m_{1}(t) \left| u'(t) \right|^{q_{1}} \, dt 
\]

\[
\leq \left| u(c) \right|^{a} \left| v(d) \right|^{b} \left[ \int_{a}^{b} f^{+}(t) \, dt + \sum_{a \leq \tau < b} a_{\tau}^{+} \right].
\]

Similarly from the second equation of system (1) and by using \( g^{+}(t) = \max(g(t), 0) \) and \( b_{\tau}^{+} = \max(b_{\tau}, 0) \), we get

\[
\int_{a}^{b} h_{2}(t) \left| u'(t) \right|^{p_{2}} \, dt + \int_{a}^{b} m_{2}(t) \left| u'(t) \right|^{q_{2}} \, dt 
\]

\[
\leq \left| u(c) \right|^{a} \left| v(d) \right|^{b} \left[ \int_{a}^{b} g^{+}(t) \, dt + \sum_{a \leq \tau < b} b_{\tau}^{+} \right].
\]
On the other hand by employing Hölder inequality with indices \( p'_1 \) and \( p_1 \), one can obtain

\[
|u(c)| = \left| \int_a^c u'(t) \, dt \right| \leq \int_a^c |u'(t)| \, dt
\]

\[
= \int_a^c |\frac{1}{h_1^c(t)}| h_1^c(t) \frac{1}{p_1} |u'(t)| \, dt
\]

\[
\leq \left( \int_a^c \frac{1}{h_1^c(t)} \, dt \right)^{\frac{1}{p_1}} \left( \int_a^c h_1(t) |u'(t)|^{p_1} \, dt \right)^{\frac{1}{p'_1}}
\]

or

\[
|u(c)|^{p_1} \left( \int_a^c \frac{1}{h_1^c(t)} \, dt \right)^{-\frac{p_1}{p'_1}} \leq \int_a^c h_1(t) |u'(t)|^{p_1} \, dt.
\]

Similarly, by using Hölder inequality with indices \( p'_1 \) and \( p_1 \), one can obtain

\[
|u(c)|^{p_1} \left( \int_a^b \frac{1}{h_1^b(t)} \, dt \right)^{-\frac{p_1}{p'_1}} \leq \int_a^b h_1(t) |u'(t)|^{p_1} \, dt.
\]

Adding (8) and (9) together yields

\[
|u(c)|^{p_1} \left[ \left( \int_a^c \frac{1}{h_1^c(t)} \, dt \right)^{1-p_1} + \left( \int_a^b \frac{1}{h_1^b(t)} \, dt \right)^{1-p_1} \right] \leq \int_a^b h_1(t) |u'(t)|^{p_1} \, dt.
\]

Repeating the above procedure with

\[
|v(d)| = \left| \int_a^d v'(t) \, dt \right| \leq \int_a^d |v'(t)| \, dt
\]

\[
= \int_a^d \frac{1}{h_2^d(t)} h_2^d(t) \frac{1}{p_1} |v'(t)| \, dt
\]

\[
\leq \left( \int_a^d \frac{1}{h_2^d(t)} \, dt \right)^{\frac{1}{p_1}} \left( \int_a^d h_2(t) |v'(t)|^{p_1} \, dt \right)^{\frac{1}{p'_1}}
\]

and

\[
|v(d)|^{p_1} \left( \int_a^d \frac{1}{h_2^d(t)} \, dt \right)^{-\frac{p_1}{p'_1}} \leq \int_a^d h_2(t) |v'(t)|^{p_1} \, dt.
\]

one can obtain the following inequality

\[
|u(v)|^{p_1} \left[ \left( \int_a^d \frac{1}{h_2^d(t)} \, dt \right)^{1-p_1} + \left( \int_a^b \frac{1}{h_2^b(t)} \, dt \right)^{1-p_1} \right] \leq \int_a^b h_2(t) |v'(t)|^{p_1} \, dt.
\]

Moreover, the similar process implies the following inequalities

\[
|u(c)|^{q_1} \left[ \left( \int_a^c \frac{1}{m_1^c(t)} \, dt \right)^{1-q_1} + \left( \int_a^b \frac{1}{m_1^b(t)} \, dt \right)^{1-q_1} \right] \leq \int_a^b m_1(t) |u'(t)|^{q_1} \, dt
\]

and

\[
|v(d)|^{q_2} \left[ \left( \int_a^d \frac{1}{m_2^d(t)} \, dt \right)^{1-q_2} + \left( \int_a^b \frac{1}{m_2^b(t)} \, dt \right)^{1-q_2} \right] \leq \int_a^b m_2(t) |v'(t)|^{q_2} \, dt.
\]

Adding (10) and (12), we have

\[
|u(c)|^{p_1} M(c, h_1, p_1) + |u(c)|^{q_1} M(c, m_1, q_1)
\]

\[
\leq \int_a^b h_1(t) |u'(t)|^{p_1} \, dt + \int_a^b m_1(t) |u'(t)|^{q_1} \, dt.
\]

By using (7), we get

\[
|u(c)|^a |v(d)|^b \left[ \int_a^b f^+(t) \, dt + \sum_{a \leq t < b} a_i^+ \right]
\]

\[
\geq |u(c)|^{p_1} M(c, h_1, p_1) + |u(c)|^{q_1} M(c, m_1, q_1)
\]

\[
\geq \left[ |u(c)|^{p_1} + |u(c)|^{q_1} \right] \min \{ M(c, h_1, p_1), M(c, m_1, q_1) \}
\]

\[
\geq 2 |u(c)|^{p_1} |u(c)|^{q_1} \min \{ M(c, h_1, p_1), M(c, m_1, q_1) \},
\]

where we have used \( A + B \geq 2 \sqrt{AB} \) with \( A = |u(c)|^{p_1} \) and \( B = |u(c)|^{q_1} \). Hence we can conclude that

\[
|u(c)|^{a-p_1+q_1} |v(d)|^b \left[ \int_a^b f^+(t) \, dt + \sum_{a \leq t < b} a_i^+ \right]
\]

\[
\geq 2 \min \{ M(c, h_1, p_1), M(c, m_1, q_1) \}.
\]

The above process is applied to inequality (11) and inequality (13) to obtain the following inequality

\[
|u(c)|^a |v(d)|^{b-p_1+q_2} \left[ \int_a^b g^+(t) \, dt + \sum_{a \leq t < b} b_i^+ \right]
\]

\[
\geq 2 \min \{ M(d, h_2, p_2), M(d, m_2, q_2) \}.
\]
Raising inequalities (14) and (15) by \( e_1 \) and \( e_2 \), respectively, then multiplying the resulting inequalities yield
\[
|u(c)|^{\alpha_1} |v(d)|^{\alpha_2} + |u(d)|^{\beta_1} |v(c)|^{\beta_2} \
\geq 2^{\alpha_1 + \alpha_2} \left[ \min \{M(c, h_1, p_1), M(c, m_1, q_1)\} \right]^{\alpha_1} \cdot \left[ \min \{M(d, h_2, p_2), M(d, m_2, q_2)\} \right]^{\alpha_2}.
\]

(16)

Now, we choose \( e_1 \) and \( e_2 \) such that \( |u(c)| \) and \( |v(d)| \) cancels out, i.e., they solve the homogeneous linear system (4). Based on the results obtained in Remark 1.1 and Remark 1.2 and inequality (16), the desired result can be obtained.

The following corollaries provide new Lyapunov type inequalities for the particular cases of system (1). Since system (4) has infinitely many solutions \((e_1, e_2)\), assuming different conditions on the relations between \( \alpha, \beta, \theta, \gamma, p_1 \) and \( q_j, j = 1, 2 \) yields more inequalities than we will show.

**Corollary 1.12:** Assume that the condition (3) holds. Let \( p_j^* \) and \( q_j^* \) be conjugate numbers for \( p_j \) and \( q_j, j = 1, 2 \), respectively. Suppose that
\[
\alpha + \theta = \frac{p_1 + q_1}{2}, \quad \beta + \gamma = \frac{p_2 + q_2}{2}.
\]

(17)

If system (1) has a real solution \( w(t) = (u(t), v(t)) \) such that \( w(a) = w(b) = 0, a, b \in \mathbb{R} \) with \( a < b \) are consecutive zeros, then we have the following Lyapunov type inequality
\[
\left( \int_a^b f^+(t) \, dt + \sum_{a \leq t < b} a_j^+ \right) \left( \int_a^b g^+(t) \, dt + \sum_{a \leq t < b} b_j^+ \right) \
\geq 4 \left[ \min \{2^{p_1} N(b, h_1, p_1), 2^{q_1} N(b, m_1, q_1)\} \right] \cdot \left[ \min \{2^{p_2} N(b, h_2, p_2), 2^{q_2} N(b, m_2, q_2)\} \right].
\]

**Proof:** From the proof of Theorem 1.11, we see that condition (17) implies that \( e_1 = \theta, e_2 = \frac{p_1 + q_1}{2} - \alpha \) is a non zero solution of (4). Now, Corollary 1.13 is a direct consequence of Theorem 1.11.

**Corollary 1.13:** Assume that the condition (3) holds. Let \( p_j^* \) and \( q_j^* \) be conjugate numbers for \( p_j \) and \( q_j, j = 1, 2 \), respectively. Suppose that
\[
\frac{2\alpha}{p_1 + q_1} + \frac{2\gamma}{p_2 + q_2} = 1, \quad \beta \theta = \alpha \gamma.
\]

(18)

If system (1) has a real solution \( w(t) = (u(t), v(t)) \) such that \( w(a) = w(b) = 0, a, b \in \mathbb{R} \) with \( a < b \) are consecutive zeros, then we have the following Lyapunov type inequality
\[
\left( \int_a^b f^+(t) \, dt + \sum_{a \leq t < b} a_j^+ \right)^{\frac{p_1 + q_1}{2} - \gamma} \
\times \left( \int_a^b g^+(t) \, dt + \sum_{a \leq t < b} b_j^+ \right)^{\beta} \
\geq 2^{\frac{p_1 + q_1}{2} - \gamma + \beta} \cdot \left[ \min \{2^{p_1} N(b, h_1, p_1), 2^{q_1} N(b, m_1, q_1)\} \right]^{\frac{p_1 + q_1}{2} - \gamma} \
\times \left[ \min \{2^{p_2} N(b, h_2, p_2), 2^{q_2} N(b, m_2, q_2)\} \right]^{\beta}.
\]

**Proof:** From the proof of Theorem 1.11, we see that condition (18) implies that \( e_1 = \theta, e_2 = \frac{p_1 + q_1}{2} - \alpha \) is a non zero solution of (4). Now, Corollary 1.14 is a direct consequence of Theorem 1.11.

**Corollary 1.14:** Assume that the condition (3) holds. Let \( p_j^* \) and \( q_j^* \) be conjugate numbers for \( p_j \) and \( q_j, j = 1, 2 \), respectively. Suppose that condition (18) holds. If system (1) has a real solution \( w(t) = (u(t), v(t)) \) such that \( w(a) = w(b) = 0, a, b \in \mathbb{R} \) with \( a < b \) are consecutive zeros, then we have the following Lyapunov type inequality
\[
\left( \int_a^b f^+(t) \, dt + \sum_{a \leq t < b} a_j^+ \right)^{\frac{p_1 + q_1}{2} - \gamma} \
\times \left( \int_a^b g^+(t) \, dt + \sum_{a \leq t < b} b_j^+ \right)^{\beta} \
\geq 2^{\frac{p_1 + q_1}{2} - \gamma + \beta} \cdot \left[ \min \{2^{p_1} N(b, h_1, p_1), 2^{q_1} N(b, m_1, q_1)\} \right]^{\frac{p_1 + q_1}{2} - \gamma} \
\times \left[ \min \{2^{p_2} N(b, h_2, p_2), 2^{q_2} N(b, m_2, q_2)\} \right]^{\beta}.
\]

**Proof:** From the proof of Theorem 1.11, we see that condition (18) implies that \( e_1 = \theta, e_2 = \frac{p_1 + q_1}{2} - \alpha \) is a non zero solution of (4). Now, Corollary 1.14 is a direct consequence of Theorem 1.11.

**Corollary 1.15:** Assume that the condition (3) holds. Let \( p_j^* \) and \( q_j^* \) be conjugate numbers for \( p_j \) and \( q_j, j = 1, 2 \), respectively. Suppose that
\[
e_1 \left( \frac{\alpha - p_1 + q_1}{2} \right) + e_2 \alpha = 0,
\]
\[
e_1 \beta + e_2 \left( \frac{\beta - p_2 + q_2}{2} \right) = 0,
\]
(19)

where \( e_k \geq 0 \) for \( k = 1, 2 \) and \( e_1^2 + e_2^2 > 0 \). If the system (1) has a real nontrivial solution \( w(t) = (u(t), v(t)) \) such
that \( w(a) = w(b) = 0, a, b \in \mathbb{R} \) with \( a < b \) are consecutive zeros, then we have the following Lyapunov type inequality

\[
\left( \int_a^b f^+(t) \, dt + \sum_{a \leq \sigma_i < b} a_i^+ \right)^{\epsilon_1} \times \left( \int_a^b g^+(t) \, dt + \sum_{a \leq \sigma_i < b} b_i^+ \right)^{\epsilon_2} \\
\geq 2^{\epsilon_1 + \epsilon_2} \left[ \min \left\{ 2^{\delta_1} N(b, h_1, p_1), 2^{\delta_1} N(b, m_1, q_1) \right\} \right]^{\epsilon_1} \times \left[ \min \left\{ 2^{\delta_2} N(b, h_2, p_2), 2^{\delta_2} N(b, m_2, q_2) \right\} \right]^{\epsilon_2}.
\]

**Remark 1.3:** Since no sign condition is assumed for \( f \) and \( g \), Theorem 1.11 is an impulsive generalization and improvement of [22, Theorem 2.1]. Since system (1) is more general than system (15) of [11] and system (6.1) of [24], Theorem 1.11 extends [11, Corollary 2] and [24, Theorem 6.1.1].

**Remark 1.4:** In the absence of impulse effect, Theorem 1.11 still improves [22, Theorem 2.1], which implies that Theorem 1.11 is new even for the nonimpulsive case.

**Remark 1.5:** In view of \( r^+(t) \leq |r(t)| \) and \( r_i^+ \leq |r_i| \), we may replace the Lyapunov type inequality (5) by

\[
\left( \int_a^b |f(t)| \, dt + \sum_{a \leq \sigma_i < b} |a_i| \right)^{\epsilon_1} \times \left( \int_a^b |g(t)| \, dt + \sum_{a \leq \sigma_i < b} |b_i| \right)^{\epsilon_2} \\
\geq 2^{\epsilon_1 + \epsilon_2} \left[ \min \left\{ 2^{\delta_1} N(b, h_1, p_1), 2^{\delta_1} N(b, m_1, q_1) \right\} \right]^{\epsilon_1} \times \left[ \min \left\{ 2^{\delta_2} N(b, h_2, p_2), 2^{\delta_2} N(b, m_2, q_2) \right\} \right]^{\epsilon_2}.
\]

### 2. Applications

In this section, we give some applications of Lyapunov type inequalities which are used as a handy tool in studying of the qualitative nature of solutions.

#### 2.1. Disconjugacy

In this part by using Lyapunov type inequality (5) obtained in Section 1.1, we establish a disconjugacy criterion for system (1).

**Theorem 2.1:** Assume that the condition (3) holds. Let \( p_j^i \) and \( q_j^i \) be conjugate numbers for \( p_j \) and \( q_j \), \( j = 1, 2 \), respectively and \( (\epsilon_1, \epsilon_2) \) be a nontrivial solution of the homogenous system (4). If

\[
\left( \int_a^b f^+(t) \, dt + \sum_{a \leq \sigma_i < b} a_i^+ \right)^{\epsilon_1} \times \left( \int_a^b g^+(t) \, dt + \sum_{a \leq \sigma_i < b} b_i^+ \right)^{\epsilon_2} \\
< 2^{\epsilon_1 + \epsilon_2} \left[ \min \left\{ 2^{\delta_1} N(b, h_1, p_1), 2^{\delta_1} N(b, m_1, q_1) \right\} \right]^{\epsilon_1} \times \left[ \min \left\{ 2^{\delta_2} N(b, h_2, p_2), 2^{\delta_2} N(b, m_2, q_2) \right\} \right]^{\epsilon_2}
\]

holds, then system (1) is disconjugate on \( [a, b] \).

**Proof:** Suppose on the contrary that there is a real solution \( w(t) = (u(t), v(t)) \) with nontrivial \( (u(t), v(t)) \) having two zeros \( s_1, s_2 \in [a, b] \) \( (s_1 < s_2) \) such that \( (u(t), v(t)) \neq 0 \) for all \( t \in (s_1, s_2) \). Applying Theorem 1.11 we see that

\[
\left( \int_{s_1}^{s_2} f^+(t) \, dt + \sum_{s_1 \leq \sigma_i < s_2} a_i^+ \right)^{\epsilon_1} \times \left( \int_{s_1}^{s_2} g^+(t) \, dt + \sum_{s_1 \leq \sigma_i < s_2} b_i^+ \right)^{\epsilon_2} \\
> 2^{\epsilon_1 + \epsilon_2} \left[ \min \left\{ 2^{\delta_1} N(s_2, h_1, p_1), 2^{\delta_1} N(s_2, m_1, q_1) \right\} \right]^{\epsilon_1} \times \left[ \min \left\{ 2^{\delta_2} N(s_2, h_2, p_2), 2^{\delta_2} N(s_2, m_2, q_2) \right\} \right]^{\epsilon_2}.
\]

Since \( N(s, k, \mu) \geq N(b, k, \mu) \) for \( s \leq b \), we obtain

\[
\left( \int_a^b f^+(t) \, dt + \sum_{a \leq \sigma_i < b} a_i^+ \right)^{\epsilon_1} \times \left( \int_a^b g^+(t) \, dt + \sum_{a \leq \sigma_i < b} b_i^+ \right)^{\epsilon_2} \\
\geq 2^{\epsilon_1 + \epsilon_2} \left[ \min \left\{ 2^{\delta_1} N(b, h_1, p_1), 2^{\delta_1} N(b, m_1, q_1) \right\} \right]^{\epsilon_1} \times \left[ \min \left\{ 2^{\delta_2} N(b, h_2, p_2), 2^{\delta_2} N(b, m_2, q_2) \right\} \right]^{\epsilon_2}.
\]

Clearly, the last inequality contradicts (20). The proof is complete.

**Remark 2.1:** If we consider a particular case, where \( m_1(t) = m_2(t) = 0 \), we obtain system (6.1) in [24]. In this case, inequality (5) reduces to the following form

\[
\left( \int_a^b f^+(t) \, dt + \sum_{a \leq \sigma_i < b} a_i^+ \right)^{\epsilon_1} \times \left( \int_a^b g^+(t) \, dt + \sum_{a \leq \sigma_i < b} b_i^+ \right)^{\epsilon_2} \\
\geq 2^{\delta_1 \epsilon_1 + \delta_2 \epsilon_2} \left[ N(b, h_1, p_1) \right]^{\epsilon_1} \left[ N(b, h_2, p_2) \right]^{\epsilon_2}
\]

and disconjugacy criterion becomes exactly the same as in Theorem 6.3.1 of [24]. This implies that Theorem 2.1
generalizes the previous disconjugacy criterion given in [24].

**Example 2.2:** Let us consider system (1) with \( h_1(t) = e^t, \) \( h_2(t) = t + 1, m_1(t) = t^2, m_2(t) = t^4, f(t) = \sin t, g(t) = \cos t \) and \( p_1 = p_2 = q_1 = q_2 = 2, \) \( \alpha = \beta = \gamma = \theta = 1. \) Then condition (17) is valid and \( e_1 = e_2 = 1. \) Moreover if we choose \( a = \frac{T}{4}, \) \( b = 4\pi \) and \( a_i = \left( -\frac{1}{i} \right), b_i = \left( -\frac{1}{i} \right), t_i = \frac{T}{4}, i \in \mathbb{N}, \) then the assumptions (i)–(iv) are satisfied. In this case system (1) is reduced to the following second-order linear system of impulsive differential equations

\[
- \left( e^t u' \right)' - \left( t^2 u' \right)' = \sin t \text{sgn}(u) |v|, \quad t \neq \frac{i\pi}{2},
\]

\[
- \left( (t + 1) v' \right)' - \left( t^4 v' \right)' = \cos t \text{sgn}(u) |v|, \quad t \neq \frac{i\pi}{2},
\]

\[
- \Delta \left( e^t + t^2 \right) u' \bigg|_{t = i\pi} = \left( -\frac{1}{i} \right) \text{sgn}(u) |v|, \quad i \in \mathbb{N},
\]

\[
- \Delta \left( t^4 + t + 1 \right) v' \bigg|_{t = i\pi} = \left( -\frac{1}{i} \right) \text{sgn}(v) |u|, \quad i \in \mathbb{N}. \tag{21}
\]

If we compute all the terms of inequality (20), then we will show that all the conditions of Theorem 2.1 are satisfied. Observe that

\[
\left( \int_a^b f^+(t) \, dt + \sum_{a \leq t < b} a_i^+ \right) \left( \int_{\frac{T}{4}}^{4\pi} \sin t^+ \, dt \right) + \sum_{\frac{T}{4} \leq t < 4\pi} \left( -\frac{1}{i} \right) \leq 4.623773448
\]

and

\[
\left( \int_a^b g^+(t) \, dt + \sum_{a \leq t < b} b_i^+ \right) \left( \int_{\frac{T}{4}}^{4\pi} \cos t^+ \, dt \right) + \sum_{\frac{T}{4} \leq t < 4\pi} \left( -\frac{1}{i} \right) \leq 4.969083695.
\]

Therefore the left hand side of inequality (20) is \( \text{LHS} \geq 22.97591725. \)

On the other hand

\[
2^{e_1+e_2} \left[ \min \left\{ 2^{r_1} N(b, h_1, p_1), 2^{r_1} N(b, m_1, q_1) \right\} \right]^{e_1} \times \left[ \min \left\{ 2^{r_1} N(b, h_2, p_2), 2^{r_1} N(b, m_2, q_2) \right\} \right]^{e_2} \geq 64 \min \left\{ \frac{1}{\int_{\frac{T}{4}}^{4\pi} \, dt}, \frac{1}{\int_{\frac{T}{4}}^{4\pi} \, dt} \right\} \times \min \left\{ \frac{1}{\int_{\frac{T}{4}}^{4\pi} \, dt}, \frac{1}{\int_{\frac{T}{4}}^{4\pi} \, dt} \right\} \geq 26.43874243.
\]

Therefore the right hand side of inequality (20) is \( \text{RHS} \geq 26.43874243. \) Since all the conditions of Theorem 2.1 are satisfied, we can conclude that system (1) is disconjugate on \( \left[ \frac{T}{4}, 4\pi \right]. \) This result can be visualized in Figure 1. If we impose the initial conditions \( u \left( \frac{T}{4} \right) = v \left( \frac{T}{4} \right) = 0, \) \( u' \left( \frac{T}{4} \right) = v' \left( \frac{T}{4} \right) = 1 \) to the system (21), the numerical solution of system (21) can be shown as in Figure 1. In this figure, the red and blue curves represent the solutions \( u \) and \( v, \) respectively. These solutions have zeros only at \( t = \frac{T}{4} \) and they are different than zero when \( t > \frac{T}{4}. \) Therefore, the solution \( w(t) = (u(t), v(t)) \) of system (21) can not be zero when \( t > \frac{T}{4}. \) This implies that solution \( w(t) = (u(t), v(t)) \) of system (21) is disconjugate on \( \left[ \frac{T}{4}, 4\pi \right]. \) Since the solution \( w(t) = (u(t), v(t)) \) is continuous at all points in \( \left[ \frac{T}{4}, 4\pi \right] \) but its derivative has jumps at the jump points \( t_i = \frac{T}{4}, i \in \mathbb{N}, \) the edges on the graph of solution \( w(t) = (u(t), v(t)) \) occur at the impulse points \( t_i = \frac{T}{4}, i \in \mathbb{N}. \)

**2.2. Eigenvalue problems**

Now, we present an application of the obtained Lyapunov-type inequality for system (1). By using techniques similar to the technique in Napoli and Pinacon [10], we establish the following result which gives lower bounds for eigenvalues of the associated eigenvalue problem of system (1). The proof of the following theorem is based on the Lyapunov type inequality derived in Theorem 1.11.

Let \( f(t) = \lambda \alpha_{r_1}(t), \) \( g(t) = \mu \beta_{r_2}(t), \) \( a_i = \lambda \alpha_{c_1} \) and \( b_i = \mu \beta_{c_2}, \) where \( r_1, r_2 > 0 \) and \( c_{i_k} > 0, k = 1, 2. \) Then system (1) reduces to the following impulsive eigenvalue problem

\[
- \left( h_1(t) \left| u' \right|^{q_1-2} u' \right)' - \left( m_1(t) \left| u \right|^{q_1-2} u \right)' = \lambda \alpha_{r_1}(t) \left| u \right|^{eta-2} u, \quad t \neq t_i
\]

\[
- \left( h_2(t) \left| v' \right|^{q_2-2} v' \right)' - \left( m_2(t) \left| v \right|^{q_2-2} v \right)' = \mu \beta_{r_2}(t) \left| v \right|^{eta-2} v, \quad t \neq t_i
\]

\[
- \Delta \left( h_1(t) \left| u' \right|^{q_1-2} u' - m_1(t) \left| u \right|^{q_1-2} u \right)_{t=t_i} = -\lambda \alpha_{c_1} \left| u \right|^{eta-2} u, \quad i = 1, 2, \ldots, m
\]

\[
- \Delta \left( h_2(t) \left| v' \right|^{q_2-2} v' - m_2(t) \left| v \right|^{q_2-2} v \right)_{t=t_i} = -\mu \beta_{c_2} \left| v \right|^{eta-2} v, \quad i = 1, 2, \ldots, m
\]

\[
u(a) = u(b) = v(a) = v(b) = 0. \tag{22}
\]

**Definition 2.3:** A pair \((\lambda, \mu)\) is called an eigenvalue of (22) if there is a corresponding solution \((u, v)\) such that \( u, v \neq 0 \) on \((a, b).\)

**Theorem 2.4:** Assume that the condition (3) holds. Let \( p_j \) and \( q_j \) be conjugate numbers for \( p_j \) and \( q_j, j = 1, 2, \) respectively and \((e_1, e_2)\) be a nontrivial solution of the homogeneous system (4) and

\[
\int_a^b r_k(t) \, dt + \sum_{a \leq t < b} c_k > 0, \quad k = 1, 2. \tag{23}
\]

Then there exists a function \( h(\lambda) = \frac{1}{\lambda} \left( \frac{e_1}{e_2} \right) \) such that \( |\mu| \geq h(\lambda) \) for every eigenvalue pair \((\lambda, \mu)\) of the system.
(22) where the constants $C$ and $D$ are given as

$$
C = 2^e_1 + e_2 \left[ \min \left\{ 2^p N(b, h_1, p_1), 2^q N(b, m_1, q_1) \right\} \right]^{e_1} \\
\times \left[ \min \left\{ 2^p N(b, h_2, p_2), 2^q N(b, m_2, q_2) \right\} \right]^{e_2},
$$

$$
D = \left( \int_a^b r_1(t) \, dt + \sum_{a \leq t < b} c_{11} \right)^{-e_1} \\
\times \left( \int_a^b r_2(t) \, dt + \sum_{a \leq t < b} c_{12} \right)^{-e_2}.
$$

Proof: Let $(\lambda, \mu)$ be an eigenvalue pair and $(u, v)$ be the corresponding eigenfunctions of the system (22). If we apply Lyapunov inequality obtained in Theorem 1.11 for system (22), we get

$$
\begin{align*}
C & \leq \left( \int_a^b |\lambda| \alpha r_1(t) \, dt + \sum_{a \leq t < b} |\lambda| \alpha c_{11} \right)^{e_1} \\
\times \left( \int_a^b |\mu| \beta r_2(t) \, dt + \sum_{a \leq t < b} |\mu| \beta c_{12} \right)^{e_2} \\
= & \left( \int_a^b r_1(t) \, dt + \sum_{a \leq t < b} c_{11} \right)^{e_1} \\
\times \left( \int_a^b r_2(t) \, dt + \sum_{a \leq t < b} c_{12} \right)^{e_2} \quad (|\lambda| \alpha)^{e_1} (|\mu| \beta)^{e_2}. \\
(24)
\end{align*}
$$

For the eigenvalue $\mu$, we can find the following lower bound as

$$
|\mu| \beta \geq C_1^{\frac{1}{e_2}} (|\lambda| \alpha)^{-\frac{e_1}{e_2}} \left( \int_a^b r_1(t) \, dt + \sum_{a \leq t < b} c_{11} \right)^{-\frac{e_1}{e_2}} \\
\times \left( \int_a^b r_2(t) \, dt + \sum_{a \leq t < b} c_{12} \right)^{-1}.
$$

Also by rearranging terms in (24), we obtain

$$
|\lambda| \alpha |\mu| \beta \geq \frac{CD}{\alpha^{e_1} \beta^{e_2}}.
$$

Since the proofs of following corollaries are the same as that of Theorem 2.4, they are omitted.

**Corollary 2.5:** Assume that the condition (3) holds. Let $p_j$ and $q_j$ be conjugate numbers for $p_j$ and $q_j, j = 1, 2$, respectively. Suppose that (17) and (23) hold. Then there exists a function $h_1(\lambda) = \frac{1}{\beta} \left( \frac{C_1 D_1}{|\lambda| \alpha} \right)^{\frac{1}{e_2}}$ such that $|\mu| \geq h_1(\lambda)$ for every eigenvalue pair $(\lambda, \mu)$ of the system (22) where the constants $C_1$ and $D_1$ are given as

$$
C_1 = 2^e_1 \left[ \min \left\{ 2^p N(b, h_1, p_1), 2^q N(b, m_1, q_1) \right\} \right]^{e_1} \\
\times \left[ \min \left\{ 2^p N(b, h_2, p_2), 2^q N(b, m_2, q_2) \right\} \right]^{e_2},
$$

$$
D_1 = \left( \int_a^b r_1(t) \, dt + \sum_{a \leq t < b} c_{11} \right)^{-1} \\
\times \left( \int_a^b r_2(t) \, dt + \sum_{a \leq t < b} c_{12} \right)^{-1}.
$$

**Corollary 2.6:** Assume that the condition (3) holds. Let $p_j$ and $q_j$ be conjugate numbers for $p_j$ and $q_j, j = 1, 2$, respectively. Suppose that (18) and (23) hold. Then there exists a function $h_2(\lambda) = \frac{1}{\beta} \left( \frac{C_2 D_2}{|\lambda| \alpha} \right)^{\frac{1}{e_2}}$ such that $|\mu| \geq h_2(\lambda)$ for every eigenvalue pair $(\lambda, \mu)$ of the system (22) where the constants $C_2$ and $D_2$ are given as

$$
C_2 = 2^{p_1+q_1 - \alpha + \theta} \\
\times \left[ \min \left\{ 2^p N(b, h_1, p_1), 2^q N(b, m_1, q_1) \right\} \right]^{p_1+q_1 - \alpha} \\
\times \left[ \min \left\{ 2^p N(b, h_2, p_2), 2^q N(b, m_2, q_2) \right\} \right]^{\frac{p_1+q_1 - \alpha}{2}}.
$$
Corollary 2.7: Assume that the condition (3) holds. Let \( p_j' \) and \( q_j' \) be conjugate numbers for \( p_j \) and \( q_j \), respectively, \( \alpha = 0 \) and \( \beta = \gamma \) and \( (e_1, e_2) \) be a nontrivial solution of the homogenous system (19). Suppose that (23) holds. Then there exists a function \( h_3(\lambda) = \frac{1}{\lambda^2} \left( \frac{C_3(t, s)}{f \left( t, s \right)} \right)^{1/2} \) such that \( |\mu| \geq h_3(\lambda) \) for every eigenvalue pair \( (\lambda, \mu) \) of the system (22) where the constants \( C_3 \) and \( D_3 \) are given as

\[
C_3 = 2^{e_1+e_2} \left[ \min \left\{ 2^{p_1} N(b, h_1, p_1), 2^{q_1} N(b, m_1, q_1) \right\} \right]^{e_1} \\
D_3 = \left( \int_a^b r_1(t) \ dt + \sum_{a < t < b} c_{11} \right)^{-e_1} \\
\times \left( \int_a^b r_2(t) \ dt + \sum_{a < t < b} c_{11} \right)^{-e_2}.
\]

2.3. Asymptotic behaviour of oscillatory solutions

In this section as an application of Lyapunov type inequality given in Section 1.1, we establish the following results to study the asymptotic behaviour of the oscillatory solutions of system (1).

Theorem 2.8: Assume that the condition (3) holds. Let \( p_j' \) and \( q_j' \) be conjugate numbers for \( p_j \) and \( q_j \), respectively and \( (e_1, e_2) \) be a nontrivial solution of the homogenous system (4). Let

\[
D_2 = \left( \int_a^b r_1(t) \ dt + \sum_{a < t < b} c_{11} \right)^{-e_1} \\
\times \left( \int_a^b r_2(t) \ dt + \sum_{a < t < b} c_{11} \right)^{-e_2}.
\]

Then for every \( M_1 \), we can find \( T = T(M_1) \) such that \( |w(t)| > M_1 \) for all \( t > T \). Since \( w \) is oscillatory, there exists an interval \((t_1, t_2)\) with \( t_1 \geq T \) such that \( w(t_1) = w(t_2) = 0 \). By using Lyapunov inequality for \( t_1 \geq T \), we get

\[
\left( \int_{t_1}^{t_2} f^+(t) \ dt + \sum_{a \leq t \leq t_2} a_i^+ \right)^{e_1} \\
\times \left( \int_{t_1}^{t_2} g^+(t) \ dt + \sum_{a \leq t \leq t_2} b_i^+ \right)^{e_2} \\
\geq 2^{e_1+e_2} \left[ \min \left\{ 2^{p_1} N(t_2, h_1, p_1), 2^{q_1} N(t_2, m_1, q_1) \right\} \right]^{e_1} \\
\times \left[ \min \left\{ 2^{p_2} N(t_2, h_2, p_2), 2^{q_2} N(t_2, m_2, q_2) \right\} \right]^{e_2}.
\]

Since \( N(s, k, \mu) \leq N(t_2, k, \mu) \) for \( s \geq t_2 \), we obtain

\[
\left( \int_a^s f^+(t) \ dt + \sum_{a \leq t \leq s} a_i^+ \right)^{e_1} \\
\times \left( \int_a^s g^+(t) \ dt + \sum_{a \leq t \leq s} b_i^+ \right)^{e_2} \\
\geq 2^{e_1+e_2} \left[ \min \left\{ 2^{p_1} N(s, h_1, p_1), 2^{q_1} N(s, m_1, q_1) \right\} \right]^{e_1} \\
\times \left[ \min \left\{ 2^{p_2} N(s, h_2, p_2), 2^{q_2} N(s, m_2, q_2) \right\} \right]^{e_2}.
\]

or

\[
2^{e_1+e_2} \leq \left( \int_a^\infty f^+(t) \ dt + \sum_{a \leq t < \infty} a_i^+ \right)^{e_1} \\
\times \left( \int_a^\infty g^+(t) \ dt + \sum_{a \leq t < \infty} b_i^+ \right)^{e_2} \\
\times \left[ \min \left\{ 2^{p_1} N(\infty, h_1, p_1), 2^{q_1} N(\infty, m_1, q_1) \right\} \right]^{-e_1} \\
\times \left[ \min \left\{ 2^{p_2} N(\infty, h_2, p_2), 2^{q_2} N(\infty, m_2, q_2) \right\} \right]^{-e_2} \\
< 1,
\]

where \( N(\infty, k, \mu) = \left( \int_a^\infty (k(t))^{\frac{1}{\mu}} \ dt \right)^{1/\mu} \). Then every oscillatory solution \( w(t) = (u(t), v(t)) \) of system (1) is bounded and approaches zero as \( t \to \infty \).

Proof: First we prove the boundedness of oscillatory solution \( w(t) = (u(t), v(t)) \). Let us suppose that \( w(t) \) is oscillatory but not bounded. Then

\[
\limsup_{t \to \infty} |w(t)| = \infty.
\]

Since \( w \) has arbitrarily large zeros, there exists an interval \((t_1, t_2)\) with \( t_1 \geq T \), where there is sufficiently large, such that \( w(t_1) = w(t_2) = 0 \). The remainder of the proof is similar to above, hence it is omitted.

\[\blacksquare\]
The following corollaries and their proofs follow easily from Theorem 2.8 and its proof, respectively.

**Corollary 2.9:** Assume that the condition (3) holds. Let \( p_j \) and \( q_j \) be conjugate numbers for \( p_j \) and \( q_j, j = 1, 2 \), respectively. Suppose that (17) holds. Let

\[
\left( \int_{t_1}^{\infty} f^+(t) \, dt + \sum_{t_1 < \infty} a_j^+ \right)^{\alpha} \times \left( \int_{t_1}^{\infty} g^+(t) \, dt + \sum_{t_1 < \infty} b_j^+ \right)^{\beta - \alpha} \times \left[ \min \left\{ 2^{p_j} N(\infty, h_1, p_1), 2^{q_j} N(\infty, m_1, q_1) \right\} \right]^{-1} \\
\times \left[ \min \left\{ 2^{p_j} N(\infty, h_2, p_2), 2^{q_j} N(\infty, m_2, q_2) \right\} \right]^{-1} < \infty,
\]

where \( N(\infty, k, \mu) = \left( \int_{t_1}^{\infty} (k(t))^\mu \, dt \right)^{\frac{1 - \mu}{\mu}} \). Then every oscillatory solution \( w(t) = (u(t), v(t)) \) of system (1) is bounded and approaches zero as \( t \to \infty \).

**Corollary 2.10:** Assume that the condition (3) holds. Let \( p_j \) and \( q_j \) be conjugate numbers for \( p_j \) and \( q_j, j = 1, 2 \), respectively. Suppose that (18) holds. Let

\[
\left( \int_{t_1}^{\infty} f^+(t) \, dt + \sum_{t_1 < \infty} a_j^+ \right)^{\theta} \times \left( \int_{t_1}^{\infty} g^+(t) \, dt + \sum_{t_1 < \infty} b_j^+ \right)^{\frac{p_j + q_j}{\theta} - \alpha} \times \left[ \min \left\{ 2^{p_j} N(\infty, h_1, p_1), 2^{q_j} N(\infty, m_1, q_1) \right\} \right]^{-\theta} \\
\times \left[ \min \left\{ 2^{p_j} N(\infty, h_2, p_2), 2^{q_j} N(\infty, m_2, q_2) \right\} \right]^{-\frac{p_j + q_j}{\theta}} < \infty,
\]

where \( N(\infty, k, \mu) = \left( \int_{t_1}^{\infty} (k(t))^\mu \, dt \right)^{\frac{1 - \mu}{\mu}} \). Then every oscillatory solution \( w(t) = (u(t), v(t)) \) of system (1) is bounded and approaches zero as \( t \to \infty \).

**Corollary 2.11:** Assume that the condition (3) holds. Let \( p_j \) and \( q_j \) be conjugate numbers for \( p_j \) and \( q_j, j = 1, 2 \), respectively, \( \alpha = \theta \) and \( \beta = \gamma \) and \((e_1, e_2)\) be a nontrivial solution of the homogenous system. Suppose that (19) holds. Let

\[
\left( \int_{t_1}^{\infty} f^+(t) \, dt + \sum_{t_1 < \infty} a_j^+ \right)^{e_1} \times \left( \int_{t_1}^{\infty} g^+(t) \, dt + \sum_{t_1 < \infty} b_j^+ \right)^{e_2} \times \left[ \min \left\{ 2^{p_j} N(\infty, h_1, p_1), 2^{q_j} N(\infty, m_1, q_1) \right\} \right]^{-e_1} \\
\times \left[ \min \left\{ 2^{p_j} N(\infty, h_2, p_2), 2^{q_j} N(\infty, m_2, q_2) \right\} \right]^{-e_2} < \infty,
\]

where \( N(\infty, k, \mu) = \left( \int_{t_1}^{\infty} (k(t))^\mu \, dt \right)^{\frac{1 - \mu}{\mu}} \). Then every oscillatory solution \( w(t) = (u(t), v(t)) \) of system (1) is bounded and approaches zero as \( t \to \infty \).

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No potential conflict of interest was reported by the author.

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