Covariant Linear Perturbations in a Concordance Model

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Abstract

We present the complete solution of the first order metric and density perturbation equations in a spatially flat ($K = 0$), Friedmann-Robertson-Walker (FRW) universe filled with pressureless ideal fluid, in the presence of cosmological constant. We use covariant linear perturbation formalism and the comoving gauge condition to obtain the field and conservation equations. The solution contains all modes of the perturbations, i.e. scalar, vector and tensor modes, and we show that our results are in agreement with the Sachs & Wolfe metric perturbation formalism.

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1 Introduction

Recent cosmological measurements by the Wilkinson Microwave Anisotropy Probe (WMAP) [1], Ia type supernovae observations [2] and other sky surveys eg. Sloan Digital Sky Survey (SDSS) [3] seem to prove today that the expansion of the universe is accelerating. Although there exist many different approaches explaining this phenomenon, the most accepted description in the literature is the concordance model. In this model the universe is flat
In this work our aim is to obtain the complete first order solution of the perturbations of a concordance model filled with pressureless matter in Bardeen’s covariant formalism \[4\]. Hu & Sugiyama \[5\] claim that in the presence of cosmological constant one must use a numerical approach to get the result. Recent calculations of Perjés et al. \[6\] yield all the $C^\infty$ solution of the problem following the original work of Sachs & Wolfe \[7\]. Here we show that the complete analytic solution of the perturbed field equations can also be obtained using the covariant linear perturbation formalism, in agreement with the results of Perjés et al. \[6\]. Providing the solution of the problem in this formalism has the advantage that recent model calculations of the fluctuations of the Cosmic Microwave Background Radiation (CMBR) prefer using Bardeen’s covariant linear formalism rather than the perturbation approach of Sachs & Wolfe.

In Sec. 2, we present the well known solution of the background quantities in a flat, homogeneous and isotropic universe and give a brief description of the covariant linear perturbation formalism. The perturbed field equations and their solution for scalar, vector and tensor modes are presented in Sec. 3. In Sec. 4 we build up the complete first order solution of the metric and show the agreement with the results of \[6\].

2 Background metric and linear perturbations in covariant formalism

As background solution we use the homogeneous and isotropic FRW metric

$$ds^2 = a^2(\eta)(-d\eta^2 + \gamma_{\mu\nu}dx^\mu dx^\nu) ,$$

where $\gamma_{\mu\nu}$ is the three metric of a space with constant spatial curvature $K$, and $\eta$ is the conformal time variable. Introducing the comoving time coordinate $t$ by the relation $a d\eta = dt$, the scale factor $a(t)$ can be expressed in the case of a flat ($K = 0$) space and pressureless ($p = 0$) matter source as \[8\]

$$a = a_0 \sinh^{2/3}(Ct + C_0) , \quad \rho a^3 = \mathcal{C}_M , \quad a_0 = \left(\frac{\mathcal{C}_M}{\Lambda}\right)^{1/3} , \quad C = \frac{\sqrt{3\Lambda}}{2} . \quad (2)$$
Throughout this paper we use units in which the gravitational constant $G = 1/8\pi$, the speed of light $c = 1$ and we set $C_0 = 0$. Roman indices run from 0 to 3 and Greek indices run from 1 to 3. We use $\gamma_{\mu\nu}$ and its inverse to raise and lower spatial indices of first order quantities.

To obtain the perturbed field equations we use Bardeen’s covariant linear formalism [4] as it is presented by Hu [9]. Hereafter we discuss only the main steps of this description to derive the basic equations.

For a spatially flat FRW universe the general perturbations of the metric tensor have the following form

\begin{align*}
    g_{00} &= -a^2(1 + 2A), \\
    g_{0\alpha} &= -a^2B_{\alpha}, \\
    g_{\alpha\beta} &= a^2[(1 + 2H_L)\gamma_{\alpha\beta} + 2H_{T\alpha\beta}].
\end{align*}

The functions $A$, $H_L$, $B_{\alpha}$ and $H_{T\alpha\beta}$ give a complete representation of the metric, where $H_{T\alpha\beta}$ is a $3 \times 3$ trace-free tensor.

We define an effective energy momentum tensor in the Einstein equations with cosmological constant as

\begin{equation}
    R_{ik} - \frac{1}{2}Rg_{ik} = \tilde{T}_{ik} = T_{ik} + \Lambda g_{ik}. \tag{4}
\end{equation}

The energy momentum tensor $\tilde{T}_{ik}$ in the case of a homogeneous and isotropic ideal fluid source can be generally perturbed as

\begin{align*}
    \tilde{T}_0^0 &= -\tilde{\rho} - \delta\rho, \\
    \tilde{T}_0^\alpha &= (\tilde{\rho} + \tilde{p})(v_\alpha - B_{\alpha}), \\
    \tilde{T}_\alpha^0 &= -(\tilde{\rho} + \tilde{p})v^\alpha, \\
    \tilde{T}_\alpha^\beta &= (\tilde{\rho} + \delta\rho)v^\beta + \tilde{\rho}\Pi_{\alpha\beta}^\beta,
\end{align*}

where $\delta\rho$, $\delta p$ and $v_\alpha$ are the density, pressure and velocity perturbations, respectively, and we introduce the effective energy density $\tilde{\rho}$ and pressure $\tilde{p}$ as new variables with the following definition

\begin{equation}
    \tilde{\rho} = \rho + \Lambda, \quad \tilde{p} = p - \Lambda. \tag{6}
\end{equation}

The $\Pi_{\alpha\beta}$ tensor represents the anisotropic stress perturbations.

In covariant linear formalism the Einstein equations can be decoupled into a set of ordinary differential equations by employing scalar, vector and
tensor eigenmodes of the Laplacian operator which form a complete set. In a spatially flat ($K = 0$) universe, these eigenmodes are plane waves

\[ Q^{(0)} = \exp(i k \cdot x), \]
\[ Q^{(\pm 1)} = \frac{-i}{\sqrt{2}} (\hat{e}_1 \pm i \hat{e}_2)_a \exp(i k \cdot x), \]
\[ Q^{(\pm 2)} = -\sqrt{\frac{3}{8}} (\hat{e}_1 \pm i \hat{e}_2)_a (\hat{e}_1 \pm i \hat{e}_2)_\beta \exp(i k \cdot x), \]

with the unit vectors $e_1$ and $e_2$ spanning the plane transverse to the wave vector $k$. For an arbitrary scalar, vector and tensor function the components of the $k$th eigenmode become

\[ F(x) = F(k) Q^{(0)}, \quad F_a(x) = \sum_{m=-1}^{1} F^{(m)}(k) Q_a^{(m)}, \quad F_{a\beta}(x) = \sum_{m=-2}^{2} F^{(m)}(k) Q_{a\beta}^{(m)}. \]

(8)

In the plane wave expansion there are relations between the curl free vectors and longitudinal components of tensors, and the scalar and vector modes of the eigenfunctions as follows

\[ Q^{(0)} = -k^{-1} \nabla_\alpha Q^{(0)}, \]
\[ Q^{(0)}_{a\beta} = (k^{-2} \nabla_\alpha \nabla_\beta + \frac{1}{3} \gamma_{a\beta}) Q^{(0)}, \]
\[ Q^{(\pm 1)}_{a\beta} = -\frac{1}{2k} [\nabla_\alpha Q^{(\pm 1)}_\beta + \nabla_\beta Q^{(\pm 1)}_\alpha], \]

where $\nabla$ is the covariant derivative operator with respect to $\gamma_{a\beta}$.

In the perturbed space-time we choose comoving coordinates. Using this gauge the following conditions hold

\[ A = 0, \quad v^\alpha = 0. \]

(10)

In further calculations we are interested in a spatially flat ($K = 0$) universe filled with pressureless ($p = 0$) ideal fluid and vanishing anisotropic stress perturbations ($\Pi^{\alpha}_{\beta} = 0$).

### 3 The field equations and their solution

In this section we solve the Einstein equations for scalar, vector and tensor modes. First we present the field equations using the conditions discussed at
the end of the previous section, then we give the complete solution for each mode of the perturbations.

Scalar modes

The field equations for scalar modes have the following form

\[ k^2 \left[ H_L + \frac{1}{3} H_T + \dot{a} \left( \frac{B}{k} - \frac{a \dot{H}_T}{k^2} \right) \right] = \frac{a^2}{2} \left[ \delta \rho - 3 \dot{a} \rho \frac{B}{k} \right] , \]

\[ k^2 \left( H_L + \frac{1}{3} H_T \right) + \left( \frac{a}{dt} + 2 \dot{a} \right) (kB - a \dot{H}_T) = 0 , \quad (11) \]

\[ a \dot{H}_L + \frac{1}{3} a \dot{H}_T = \frac{a^2}{2} \rho \frac{B}{k} , \quad \left[ \frac{a}{dt} + \dot{a} \right] \left( a \dot{H}_L + \frac{kB}{3} \right) = -\frac{a^2}{6} \delta \rho , \]

and the scalar modes of the conservation equations become

\[ \left( \frac{a}{dt} + 3 \dot{a} \right) \delta \rho = -3 \rho a \dot{H}_L , \quad \left( \frac{a}{dt} + 4 \dot{a} \right) \rho \frac{B}{k} = 0 . \quad (12) \]

For the scalar quantities the solution of the field equations gives

\[ \delta \rho = \frac{\cosh(Ct)}{\sinh^3(Ct)} \left[ K_1(k) - K_2(k) I(t) \right] , \quad B = \frac{k}{C_M} \frac{B_0(k)}{a} , \]

\[ H_L = -\frac{a^3}{3C_M} \coth(Ct) \left[ K_1(k) - K_2(k) I(t) \right] + V(k) , \quad (13) \]

\[ H_T = -3H_L - \frac{3B_0(k)}{2Ca_0^3} \coth(Ct) + H_0(k) , \]

where \( K_1(k) \), \( K_2(k) \), \( B_0(k) \), \( V(k) \) and \( H_0(k) \) are arbitrary scalar functions depending on spatial coordinates. The function \( I(t) \) is the following integral of the time variable

\[ I(t) = 2^{-2/3} \sqrt{3\Lambda} \int_0^t \frac{\sinh^{2/3}(C\tau)}{\cosh^{2}(C\tau)} \, d\tau . \quad (14) \]

The Legendre normal form of the elliptic integral \( I(t) \) is given in the appendix of \[ \text{[6].} \]
Vector modes

The Einstein equations for vector modes are

\[
\left( k B^{(\pm 1)} - a \dot{H}_T^{(\pm 1)} \right) = -2a^2 \rho \frac{B^{(\pm 1)}}{k}, \tag{15}
\]

\[
\left[ a \frac{d}{dt} + 2 \dot{a} \right] \left( k B^{(\pm 1)} - a \dot{H}_T^{(\pm 1)} \right) = 0 \tag{16}
\]

and the conservation equations become

\[
\left[ a \frac{d}{dt} + 4 \dot{a} \right] \rho \frac{B^{(\pm 1)}}{k} = 0. \tag{17}
\]

Equations (15) - (17) are not independent. Inserting Eq. (15) into Eq. (16) we get Eq. (17). The general solution of this system of equations is

\[
B^{(\pm 1)} = \frac{k B_0^{(\pm 1)}}{C_M \ a}, \tag{18}
\]

\[
H_T^{(\pm 1)} = \frac{B_0^{(\pm 1)}}{C a_0^2} \left[ \frac{k^2}{C_M} J(t) - \frac{2}{a_0} \ coth(C t) \right] + H_0^{(\pm 1)}, \tag{19}
\]

where \(B_0^{(\pm 1)}(k)\), and \(H_0^{(\pm 1)}(k)\) are arbitrary functions depending on only spatial coordinates. The function \(J(t)\) is also an elliptic integral of the time variable \(t\), and can be expressed in terms of \(I(t)\) as follows

\[
J(t) = -\frac{3}{21/3} I(t) - 3 \sinh^{-1/3}(Ct) \cosh^{-1}(Ct). \tag{20}
\]

Tensor modes

For tensor modes we get a source-free gravitational wave propagation equation

\[
a^2 \ddot{H}_T^{(\pm 2)} + 3a \dot{a} \dot{H}_T^{(\pm 2)} + k^2 H_T^{(\pm 2)} = 0 \tag{20}
\]

as is the case in the absence of cosmological constant. The general solution of the wave equation in the presence of \(\Lambda\) is given in Eq.(50) of [6].
4 The complete form of the metric

Having in hand the solution for all modes of the perturbations, Eqs. (13) and (18), we can build up the general form of the metric (3). The first order part of the metric takes the following form

\[
H_L \gamma_{\alpha\beta} + H_T \alpha_{\beta} = H_T^{(+2)} Q_{\alpha\beta}^{(+2)} + H_T^{(-2)} Q_{\alpha\beta}^{(-2)}
\]

\[
- \frac{i}{2k} \left[ B_0^{(+1)} \left( \frac{k^2 J(t)}{C M} - \frac{2}{a_0} \coth(Ct) \right) + H_0^{(+1)} \right] \left( k_{\alpha} Q_{\beta}^{(+1)} + k_{\beta} Q_{\alpha}^{(+1)} \right)
\]

\[
- \frac{i}{2k} \left[ B_0^{(-1)} \left( \frac{k^2 J(t)}{C M} - \frac{2}{a_0} \coth(Ct) \right) + H_0^{(-1)} \right] \left( k_{\alpha} Q_{\beta}^{(-1)} + k_{\beta} Q_{\alpha}^{(-1)} \right)
\]

\[
- \left\{ \left[ \frac{a_0^3}{C M} \coth(Ct) [K_1 - K_2 I(t)] - 3V - \frac{3B_0}{2C a_0^2} \coth(Ct) + H_0 \right] \frac{k_{\alpha} k_{\beta}}{k^2} \right\} Q^{(0)},
\]

and

\[
g_{0\alpha} = -\frac{k_{\alpha}}{C M} \left[ B_0 Q_{\alpha}^{(0)} + B_0^{(+1)} Q_{\alpha}^{(+1)} + B_0^{(-1)} Q_{\alpha}^{(-1)} \right].
\]

To show the equivalence of the metric (21) and (22) with the \( C^\infty \) solution in [6], we choose the space dependent integrational functions as follows,

\[
K_1 = -\frac{k^2 C M}{2a_0^3} A_k, \quad K_2 = -\frac{3k^2 C M}{2^{7/3}a_0^3 C^2} B_k,
\]

\[
B_0 = -k C M C_k^{(0)} = 0, \quad B_0^{(\pm1)} = -k C M C_k^{(\pm1)},
\]

\[
H_0 = 3V = \frac{3}{2} B_k, \quad H_0^{(\pm1)} = 0.
\]

Here \( A_k, B_k, C_k^{(0)} \) and \( C_k^{(\pm1)} \) are the components of the \( k \)th eigenmode of the spatial functions in Eqs. (52-54) of [6]. Inserting the values in Eq. (23) into Eqs. (21) and (22) we find the agreement of the two solutions up to the remaining gauge freedom, which is not completely fixed in [6].

5 Concluding remarks

The result of our work is relevant in the context of studying the CMBR fluctuations in the picture of first order approximation of a concordance model.
We presented the complete form of the metric perturbations with cosmological constant in Bardeen’s covariant linear formalism. We concluded that all the analytic solution can be obtained in agreement with the Sachs & Wolfe approach. This result can be useful in predicting the power spectrum of the fluctuations in the microwave background radiation.

We dedicate this paper to the memory of our supervisor Prof. Zoltán Perjés who took part actively in the first period of the calculations. Regrettably we had to finish writing the manuscript without his help and suggestions.

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