Regularity and Decay of Global Solutions for the Generalized Benney-Lin Equation Posed on Bounded Intervals and on a Half-Line

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Abstract: Initial-boundary value problems for the generalized Benney-Lin equation posed on bounded intervals and on the right half-line were considered. The existence and uniqueness of global regular solutions on arbitrary intervals as well as their exponential decay for small solutions and for a special choice of a bounded interval have been established.

Keywords: Benney-Lin equation; Korteweg-de Vries equation; Kuramoto-Sivashinsky equation; Zakharov-Kuznetsov equation; global solutions

MSC: 35B35; 35K91; 35Q53

1. Introduction

This work concerns the existence and uniqueness of global solutions, regularity and exponential decay rates of solutions to some initial-boundary value problems for the generalized Benney-Lin equation

\[ u_t + \eta D^5_x u + \beta D^4_x u + \alpha D^3_x u + \gamma D^2_x u + u^k u_x = 0, \tag{1} \]

where \( k \geq 1 \) is a natural number.

This equation was deduced in [1] in connection with applications in fluid mechanics and later was exploited in [2] during research in the theory of liquid films. For various values of the coefficients \( \alpha, \eta, \beta, \gamma \) and \( k = 1 \), it presents well-known equations of mathematical physics such as the Korteweg-de Vries equation when \( \eta = \beta = \gamma = 0 \) and \( \alpha = 1 \); the Kawahara equation when \( \gamma = \beta = \alpha = 0 \) and \( \eta = -1 \). In the case \( \eta = \alpha = 0 \) and \( \beta, \gamma \) are positive constants, (1) is the Kuramoto-Sivashinsky equation. In [3], Kuramoto studied the turbulent phase waves and Sivashinsky in [4] obtained an asymptotic equation which simulated the evolution of a disturbed plane flame front. See also [5]. Mathematical results on initial and initial-boundary value problems for various variants of (1) are presented in [6–15], see references there for more information. In [6,9,12,15], Kuramoto-Sivashinsky type equations have been considered which included \( u_{xxx} \) (KdV) term. Initial-boundary value problems for equations of higher orders than (1) have been considered in [16–18]. What concerns (1) with \( k > 1 \), \( \eta = \beta = \gamma = 0 \); \( \alpha = 1 \), called generalized Korteweg-de Vries equations, the Cauchy problem has been studied in [19,20], where it has been proved that for \( k = 4 \), called the critical case, the initial problem is well-posed for small initial data, whereas for arbitrary initial data, solutions may blow-up in a finite time. The generalized KdV equation was intensively studied in order to understand the interaction between the dispersive term and nonlinearity in the context of the theory of nonlinear dispersive evolution equations, see [21]. In [22], an initial-boundary value problem for the generalized KdV equation with an internal damping posed on a bounded interval was studied in the critical case. In [23], an initial-boundary value problem for the generalized KdV equation posed on a half-line was considered, where exponential decay of regular solutions for small
initial data has been established. In [24], the Zakharov-Kuznetsov equation with super critical nonlinear term has been studied.

Our goal here are some initial-boundary value problems for the generalized Benney-Lin (BL) equation

\[ u_t - D_x^5 u + D_x^4 u + D_x^2 u + u^k u_x = 0, \tag{2} \]

where \( k \geq 1 \) is a natural number.

First essential problem that arises while one studies stability of solutions for (2), is a destabilizing effect of \( D_x^2 u \) that may be damped by a dissipative term \( D_x^4 u \) provided an interval \((0, L)\), where (2) is posed, has some specific properties.

The existence and uniqueness of global regular solutions to (2) posed on any bounded interval \((0, L)\) and on the right half-line for \( k \leq 4 \) and any positive time interval \((0, T)\) without smallness conditions for the initial data was established in Theorems 1 and 2. An initial-boundary value problem for (2) posed on a special interval \((0, L)\) was formulated. The existence of a global regular solution, uniqueness and exponential decay rate of small solutions with \( k \leq 8 \) have been established in Theorem 3.

2. Notations and Auxiliary Facts

Let \( L \) be a positive number and \( x \in (0, L) \). We use the standard notations of Sobolev spaces \( W^{k, p} \), \( L^p \) and \( H^k \) for functions and the following notations for the norms [25,26]:

\[ \mathbb{R}^+ = \{ t \in \mathbb{R}; t > 0 \}, \quad \| f \|_2 = (f, f) = \int_0^L |f|^2 \, dx, \quad \| f \|_{L^p} = \int_0^L |f|^p \, dx, \]

\[ \| f \|_{W^{k,p}} = \sum_{0 \leq \alpha \leq k} \| D_x^\alpha f \|_{L^p}, \quad D_x^k = \frac{d^k}{dx^k}, \quad D_x = D_x^1, \quad D_x^0 u = u. \]

We will use also the standard notations:

\[ D_x f = f_x, \quad D_x^2 f = f_{xx}, \quad \frac{\partial}{\partial t} f = f_t. \]

When \( p = 2 \), \( W^{k, p} = H^k \) is a Hilbert space with the scalar product

\[ ((u, v))_{H^k} = \sum_{|j| \leq k} (D^j u, D^j v), \quad \| u \|_\infty = \| u \|_{L^\infty(0, L)} = \text{ess sup}_{(0, L)} |u(x)|. \]

We use the notation \( H^k_0(0, L) \) to represent the closure of \( C_0^\infty(0, L) \), the set of all \( C^\infty \) functions with compact support in \((0, L)\), with respect to the norm of \( H^k(0, L) \).

**Lemma 1.** (Steklov’s Inequality [27].) Let \( v \in H^1_x(0, L) \). Then

\[ \frac{\pi^2}{L^2} \| v \|^2 \leq \| v_x \|^2. \]

**Lemma 2.** (See [18], Lemma 2.2.) Let \( u \) be either \( u \in H^2_x(0, L) \) or \( u \in H^2(\mathbb{R}^+) \), \( u(0) = D_x u(0) = 0 \). Then the following inequality holds:

\[ \| u \|_\infty \leq \sqrt{2} \| D^2 u \|^{\frac{1}{4}} \| u \|^{\frac{3}{4}}. \]

**Lemma 3.** (See [28], p. 125.) Suppose \( u \) and \( D^m u, m \in \mathbb{N} \), belong to \( L^2(0, L) \). Then for the derivatives \( D^i u, 0 \leq i < m \), the following inequality holds:

\[ \| D^i u \| \leq A_1 \| D^m u \|^{\frac{1}{4}} \| u \|^{i - \frac{1}{2}} + A_2 \| u \|, \]

where \( A_1, A_2 \) are constants depending only on \( L, m, i \).
Lemma 4. (Differential form of the Gronwall Inequality.) Let \( I = [t_0, t_1] \). Suppose that functions \( a, b: I \to \mathbb{R} \) are integrable and a function \( a(t) \) may be of any sign. Let \( u: I \to \mathbb{R} \) be a differentiable function satisfying
\[
    u_t(t) \leq a(t)u(t) + b(t), \quad t \in I \quad \text{and} \quad u(t_0) = u_0,
\]
then
\[
    u(t) \leq u_0e^{\int_{t_0}^{t}a(\tau)d\tau} + \int_{t_0}^{t}e^{\int_{s}^{t}a(\tau)d\tau}b(s)ds.
\]

Lemma 5. Let \( f(t) \) be a continuous positive function such that \( f'(t) \) is a measurable, integrable function and
\[
    f'(t) + (\alpha - kf^n(t))f(t) \leq 0, \quad t > 0, \quad n \in \mathbb{N}, \quad (3)
\]
\[
    \alpha - kf^n(0) > 0, \quad k > 0. \quad (4)
\]
Then
\[
    f(t) < f(0)
\]
for all \( t > 0 \).

**Proof.** Obviously, \( f'(0) + (\alpha - kf^n(0))f^n(0) \leq 0 \). Since \( f \) is continuous, there exists \( T > 0 \) such that \( f(t) < f(0) \) for every \( t \in [0, T) \). Suppose that \( f(0) = f(T) \). Integrating (3), we find
\[
    f(T) + \int_0^T (\alpha - kf^n(t))f(t) dt \leq f(0).
\]
Since
\[
    \int_0^T (\alpha - kf^n(t))f(t) dt > 0,
\]
then \( f(T) < f(0) \). This contradicts that \( f(T) = f(0) \). Therefore, \( f(t) < f(0) \) for all \( t > 0 \).

\[\square\]

3. Generalized Benney-Lin Equation Posed on Bounded Intervals

**Problem 1.** Define an interval
\[
    D = x \in (0, L), \quad L > 0; \quad Q_t = (0, t) \times D.
\]

**Lemma 6.** Let \( f \in H^2(D) \cap H^1_0(D) \). Then
\[
    a\|f\|^2 \leq \|f_x\|^2, \quad a^2\|f\|^2 \leq \|f_{xx}\|^2, \quad \|f_x\|^2 \leq \|f_{xx}\|^2, \quad (5)
\]
where \( a = \frac{\pi^2}{12} \).

**Proof.** Making use of Steklov’s inequalities, we get
\[
    \|f_x\|^2 \geq \frac{\pi^2}{12} \|f\|^2 = a\|f\|^2.
\]
On the other hand,
\[
    a\|f\|^2 \leq \|f_x\|^2 = -\int_0^L f f_{xx} dx \leq \|f_{xx}\|\|f\|.
\]
This implies
\[
    a\|f\| \leq \|f_{xx}\| \quad \text{and} \quad a^2\|f\|^2 \leq \|f_{xx}\|^2.
\]
Consequently, \( a\|f_3\|^2 \leq \|f_{xx}\|^2 \).

Proof of Lemma 6 is complete. \(\square\)

In \( Q_t \) consider the following initial-boundary value problem:

\[
\begin{align*}
 u_t - D_5^2 u + D_4^2 u + D_3^2 u + u^k u_x &= 0, \quad (6) \\
 D_4^2 u(0) &= D_4^2 u(L) = D_3^2 u(L) = 0, \quad i = 0, 1; \quad (7) \\
 u(x, 0) &= u_0(x), \quad x \in (0, L). \quad (8)
\end{align*}
\]

**Theorem 1.** Let \( T, L \) be arbitrary positive numbers and a natural \( k \leq 4 \). Given \( u_0 \in H^5(D) \cap H^2_0(D), \ D_3^2 u_0(L) = 0 \). Then the problem (6)–(8) has a unique regular solution

\[
\begin{align*}
 u &\in L^\infty((0, T); H^5_0(D)) \cap L^2((0, T); H^7(D)); \\
u_t &\in L^\infty((0, T); L^2(D)) \cap L^2((0, T); H^6_0(D)).
\end{align*}
\]

**Proof.** Define the space \( W = \{ f \in H^5(D) \cap H^2_0(D), \ D_3^2 f(L) = 0 \} \) and let \( \{ w_i(x), i \in \mathbb{N} \} \) be a countable dense set in \( W \). We can construct approximate solutions to (6)–(8) in the form

\[ u^N(x, t) = \sum_{i=1}^{N} g_i(t)w_i(x). \]

Unknown functions \( g_i(t) \) satisfy the following initial problems:

\[
\begin{align*}
 \frac{d}{dt}(u^N, w_i)(t) + (D_4^2 u^N, w_i)(t) + (D_3^2 u^N, w_i)(t) \\
-(D_5^2 u^N, w_i)(t) + (D_4^2 u^N, w_i)(t) + ((u^N)^k D_5 u^N, w_i)(t) &= 0, \quad i = 1, 2, \ldots \quad (9) \\
g_i(0) &= g_{0i}, \quad i = 1, 2, \ldots \quad (10)
\end{align*}
\]

By Carathéodory’s existence theorem, see [29], there exist solutions of (9)–(10), at least locally in \( t \), hence for all finite \( N \), we can construct an approximate solution \( u^N(x, t) \) of (6)–(8). In [17,18], the existence of local regular solutions to problems similar to (6)–(8) has been proved. Taking this into account, all the estimates will be proved on smooth solutions of (6)–(8). Naturally, the same estimates are true also for approximate solutions \( u^N \).

**Estimate I.** Multiply (6) by \( 2u \) to obtain

\[
\frac{d}{dt}\|u\|^2(t) + 2\|D_5^2 u\|^2(t) + 2(D_4^2 u, u)(t) + |D_3^2(0, t)|^2 = 0. \quad (11)
\]

Estimating third term in (11) by the Cauchy inequality, we get

\[
\frac{d}{dt}\|u\|^2(t) + \|D_5^2 u\|^2(t) + |D_4^2 u(0, t)|^2 \leq \|u\|^2(t). \quad (12)
\]

Dropping second and third terms in (12) and applying Lemma 4, we find

\[\|u\|^2(t) \leq e^t\|u_0\|^2, \quad t \in (0, T). \quad (13)\]

Returning to (12), we obtain

\[
\int_0^T \left[ \|D_5^2 u\|^2(t) + |D_4^2 u(0, t)|^2 \right] dt \leq (1 + T e^T)\|u_0\|^2. \quad (14)
\]

Here and henceforth, \( T \) is an arbitrary positive number.

**Estimate II.**
Differentiate (6) with respect to \( t \), then multiply the result by \( 2u_t \) to get
\[
\frac{d}{dt} \| u_t \|^2(t) + \| D_x^2 u_t \|^2(t) + | D_x^2 u_t(0,t) |^2 \\
\leq \| u_t \|^2(t) + 2(u^k u_{1t}, u_{s1})(t).
\] (15)

**The case \( k < 4 \).**
Making use of Lemmas 2 and 6 for an arbitrary \( \epsilon > 0 \), we estimate
\[
I = 2(u^k u_{1t}, u_{s1})(t) \leq 2 \sup_D \| u(x,t) \|^k \| u_t \| \| u_{s1} \| (t)
\leq 2 \left( 2^{1/2} \| D_x^2 u \|^{1/4} \| u \|^{3/4}(t) \right) \| u_t \| \left( \frac{1}{\epsilon^{1/2}} \| u_{s1} \| (t) \right)
\leq \epsilon \| u_{s1} \|^{2}(t) + \frac{2k}{a\epsilon} \| D_x^2 u \|^{k/2} \| u \|^{3k/2}(t) \| u_t \|^2(t)
\leq \epsilon \| u_{s1} \|^{2}(t) + \frac{2k}{a\epsilon} \left[ k \| D_x^2 u \|^{2}(t) \right] \| u_t \|^2(t) + \frac{4 - k}{4} \| u \|^{6k/(4-k)}(t) \| u_t \|^2(t).
\] (16)

Taking \( \epsilon = \frac{1}{2} \), and substituting \( I \) into (15), we obtain
\[
\frac{d}{dt} \| u_t \|^2(t) + \frac{1}{2} \| D_x^2 u_t \|^2(t) + | D_x^2 u_t(0,t) |^2 \\
\leq \left( \frac{3}{2} + \frac{2k-1}{a} \left[ k \| D_x^2 u \|^{2}(t) \right] + \frac{4 - k}{4} \left( e^T \| u_0 \|^2 \right)^{3k/(4-k)} \right) \| u_t \|^2(t).
\]

By (13), (14), \( \| D_x^2 u \|^2(t) \in L^1(0,T) \) and \( \| u \| \in L^\infty(0,T) \), whence, dropping second and third terms in (16) and making use of Lemma 4, we find that
\[
\| u_t \|^2(t) \leq C_1 \| u_t \|^2(0), \quad t \in (0,T),
\]
where
\[
C_1 = \exp \left\{ \int_0^T \left[ \frac{3}{2} + \frac{2k-1}{a} \left[ k \| D_x^2 u \|^{2}(t) \right] + \frac{4 - k}{4} \left( e^T \| u_0 \|^2 \right)^{3k/(4-k)} \right] dt \right\}.
\]

Returning to (16), we obtain
\[
\int_0^T \left[ \| D_x^2 u_t \|^2(t) + | D_x^2 u_t(0,t) |^2 \right] dt \leq C(T, \| u_0 \|) \| u_t \|^2(0),
\] (17)
where \( \| u_t \|(0) \leq C(\| u_0 \|_W) \) can be estimated directly from (6) on \( t = 0 \).

**The case \( k = 4 \).**

Consider (15) for \( k = 4 \):
\[
\frac{d}{dt} \| u_t \|^2(t) + \| D_x^2 u_t \|^2(t) + | D_x^2 u_t(0,t) |^2 \\
\leq \| u_t \|^2(t) + 2(u^4 u_{1t}, u_{s1})(t).
\] (18)

Acting as in the case \( k < 4 \), we estimate
\[
I_2 = 2(u^4 u_{1t}, u_{s1})(t) \leq 2 \| u \|_\infty^4(t) \| u_t \| \| u_{s1} \| (t)
\leq \frac{2^3}{a^{1/2}} \| D_x^2 u \| (t) \| u \|^{3}(t) \| u_t \| (t) \| D_x^2 u_t \| (t)
\]
Taking \( \epsilon = \frac{1}{4} \) and substituting \( I_2 \) into (18), we get

\[
\frac{d}{dt} \|u_t\|^2(t) + \frac{1}{2} \|D_x^2 u_t\|^2(t) + \|D_x^2 u_t(0,t)\|^2 \leq \left( \frac{3}{2} + \frac{25e^{3T}}{a} \right) \|D_x^2 u\|^2(t) \|u_0\|^6 \|u_t\|^2(t).
\]

By (14), \( \|D_x^2 u\|^2(t) \in L^1(O,T) \), whence, dropping second and third terms in (19) and making use of Lemma 4, we find that

\[
\|u_t\|^2(t) \leq C_2 \|u_t\|^2(0), \quad t \in (0,T),
\]

where

\[
C_2 = \int_0^T \left( \frac{3}{2} + \frac{25e^{3T}}{a} \right) \|D_x^2 u\|^2(t) \|u_0\|^6 \) dt.
\]

Returning to (19), we obtain

\[
\int_0^T \left[ \|D_x^2 u_t\|^2(t) + \|D_x^2 u_t(0,t)\|^2 \right] dt \leq C(T, \|u_0\|) \|u_t\|^2(0).
\]

Here \( \|u_t\|(0) \leq C(\|u_0\|) \) can be estimated directly from (6) on \( t = 0 \).

**Proposition 1.** \( \text{ess sup}_{Q_t} |u(x,t)| \leq M < +\infty, \ t \leq T. \)

**Proof.**

\[
\|D_x^2 u\|^2(t) - \|D_x^2 u_0\|^2 = \int_0^t \frac{d}{ds} \|D_x^2 u\|^2(s) \) ds
\]

\[
\leq \int_0^t 2\|D_x^2 u\|(s) \|D_x^2 u_s\|(s) \) ds \leq \int_0^T \left[ \|D_x^2 u\|^2(s) + \|D_x^2 u_s\|^2(s) \right] ds.
\]

Estimates (13), (14) and (21) prove that \( \|D_x^2 u\|(t) \in L^\infty(0,T) \) and Lemma 2 completes the proof of Proposition 1.

Estimates (13), (14), (20) and (21) and Proposition 1 imply that

\( u \in L^\infty((0,T);H_0^2(D)), \ u_t \in L^\infty((0,T);L^2(D)) \) \( \cap L^2((0,T);H_0^2(D)). \)

It is easy to see that these inclusions do not depend on \( N \), hence, by standard arguments, we can pass to the limit as \( N \to \infty \) in (9), (10) and to prove the existence of weak solution to (6)–(8) \( \{u(x,t)\} \) satisfying the following identity:

\[
(u_t, \phi)(t) + (D_x^2 u, D_x^2 \phi)(t) + (D_x^4 u, \phi)(t)
\]

\[
\quad + (D^2 u_s, D_x^2 \phi)(t) - (D_x^2 u, \phi_s)(t) + (u^k u_x, \phi)(t) = 0, \ t > 0,
\]

where \( \phi(x,y) \) is an arbitrary function from \( H_0^2(D) \).

We can rewrite (22) in the form of a distribution on \( (0,L) \)

\[
D_x^5 u_x - D_x^4 u - D_x^2 u = u_t + D_x^2 u + u^k u_x \equiv f(x,t).
\]

Due to properties of a weak solution \( u(x,t) \), \( f \in L^\infty((0,T);L^2(D)) \). This implies that

\[
I = D_x^5 u - D_x^4 u - D_x^2 u \in L^\infty((0,T);L^2(D)).
\]
Making use of Lemma 3, we find that for an arbitrary \( \epsilon > 0 \)
\[
(1 - 2\epsilon)\|D^5_u\|(t) \leq \frac{C}{\epsilon}(\|u\|(t) + \|f\|(t)).
\]
Choosing \( 4\epsilon = 2 \) and taking into account that \( f, u \in L^\infty((0, T); L^2(D)) \), we get
\[
u \in L^\infty((0, T); H^5(D)).
\]
Taking this into account and that \( u_t \in L^2((0, T); H_0^1(D)) \), we find from (23)
\[
D^5_u = u_t + D^3_x u + D^3_x u + D^2_x u + u^k u_x \in L^2((0, T); H^1(D)),
\]
whence
\[
u \in L^\infty((0, T); H^5(D)) \cap L^2((0, T); H^6(D)).
\]
Returning to (24), one can observe that \( D^5_u \in L^2((0, T); H^2(D)) \). hence
\[
u \in L^\infty((0, T); H^5(D)) \cap L^2((0, T); H^7(D)).
\]
This proves the existence part of Theorem 1.

Remark 1. Assertions of Theorem 1 are true for arbitrary positive numbers \( T, L \), but estimates of solutions depend on \( T \). It means that one can not pass to the limit as \( L \to +\infty \). Hence, we do not have stability results. On the other hand, we have not any smallness restrictions for \( u_0(x) \).

Lemma 7. The regular solution of (6)–(8) is unique.

Proof. Let \( u \) and \( v \) be two distinct solutions to (6)–(8). Denoting \( w = u - v \), we come to the following problem:
\[
w_t - D^3_x w + D^3_x w + D^2_x w + \frac{1}{k + 1} D_x [u^{k+1} - v^{k+1}] = 0,
\]
\[
D^i_x w(0) = D^i_x w(L) = D^2_x w(L) = 0, \quad i = 0, 1;
\]
\[
w(x, 0) = 0.
\]
Multiplying (26) by \( 2w \) and taking into account that
\[
\|D_x w\|^2 = -\|D^2_x w, w\| \leq \frac{1}{2} \left[ \|D^2_x w\|^2 + \|w\|^2 \right],
\]
we find
\[
\frac{d}{dt}\|w\|^2(t) + |D^2_x w(0, t)|^2 + 2\|D^2_x w\|^2(t) + 2(D^2_x w, w)(t)
\]
\[
= \frac{2}{k + 1} ([u^{k+1} - v^{k+1}], D_x w)(t) \leq \|D_x w\|^2(t)
\]
\[
+ \frac{4}{(k + 1)^2}\|u^{k+1} - v^{k+1}\|^2(t)
\]
\[
\leq \frac{1}{2} \|D^2_x w\|^2(t) + \frac{1}{2} \|w\|^2(t) + \frac{4}{(k + 1)^2}\|u^{k+1} - v^{k+1}\|^2(t).
\]
Making use of the functional mean value theorem and Proposition 1, we get
\[ I = \|u^{k+1} - v^{k+1}\|^2 \leq (k + 1)^2 \left( \sup_{Q_r} |u| + \sup_{Q_r} |v| \right)^{2k} \|w\|^2 \]

\[ \leq (k + 1)^2 2^{2k} M^{2k} \|w\|^2. \]

Substituting \( I \) into (29), we obtain
\[ \frac{d}{dt} \|w\|^2(t) + \frac{1}{2} \|D_x^2 w\|^2(t) \leq C\|w\|^2(t), \]
where the constant \( C \) depends on \( M \). Applying Lemma 4, we find that
\[ \|w\|(t) = 0. \]

This proves Lemma 7 and consequently Theorem 1. \( \square \)

**Benney-Lin equation posed on \( \mathbb{R}^+ \).**

In \( Q_t = (0, t) \times \mathbb{R}^+, t \in (0, T) \), consider the following problem:
\[ u_t - D_x^5 u + D_x^3 u + D_x^2 u + u^k u_x = 0, \]
\[ D_x^i u(0, t) = 0, \quad i = 0, 1; \lim_{x \to +\infty} \|u(x)\| = 0; t \in (0, T), \]
\[ u(x, 0) = u_0(x), \quad x \in \mathbb{R}^+. \]

**Theorem 2.** Let \( T \) be an arbitrary positive number and a natural \( k \leq 4 \). Given \( u_0 \in H^5(\mathbb{R}^+), \; u(0) = D_x(0) = 0 \). Then the problem (31) and (33) has a unique regular solution
\[ u \in L^\infty((0, T); H^2(\mathbb{R}^+)) \cap L^2((0, T); H^7(\mathbb{R}^+)); \]
\[ u_t \in L^\infty((0, T); L^2(\mathbb{R}^+)) \cap L^2((0, T); H^7(\mathbb{R}^+)). \]

**Proof.** Define the space \( W = \{ f \in H^5(\mathbb{R}^+), \; f(0) = D_x f(0) = 0 \} \) and let \( \{w_i(x), \; i \in \mathbb{N}\} \) be a countable dense set in \( W \). We can construct approximate solutions to (31)–(33) in the form
\[ u^N(x, t) = \sum_{i=1}^{N} g_i(t) w_i(x). \]

Unknown functions \( g_i(t) \) satisfy the following initial problems:
\[ \frac{d}{dt}(u^N, w_i)(t) + (D_x^4 u^N, w_i)(t) + (D_x^2 u^N, w_i)(t) \]
\[ - (D_x^5 u^N, w_i)(t) + (D_x^3 u^N, w_i)(t) + \left( (u^N)^k D_x u^N, w_i \right)(t) = 0, \]
\[ g_i(0) = g_{0i}, \quad i = 1, 2, \ldots. \]

By Carathéodory’s existence theorem, see [29], there exist solutions of (34) and (35), at least locally in \( t \), hence for all finite \( N \), we can construct an approximate solution \( u^N(x, t) \) of (31)–(33). Taking this into account, all the estimates we will prove will be done on smooth solutions of (31)–(33). Naturally, the same estimates are true also for approximate solutions \( u^N \).

**Estimate I.** Multiplying (31) by \( 2u \) and acting in the same manner as by proving (13) and (14), we obtain
\[ \|u\|^2(t) \leq e^T \|u_0\|^2, \; t \in (0, T); \]
\[ \|w\|^2(t) \leq \|w_0\|^2, \; t \in (0, T); \]
\[ \|D_x^2 w\|^2(t) \leq C \|w\|^2(t), \]
\[ \frac{d}{dt} \|w\|^2(t) + \frac{1}{2} \|D_x^2 w\|^2(t) \leq C\|w\|^2(t), \]
where the constant \( C \) depends on \( M \). Applying Lemma 4, we find that
\[ \|w\|(t) = 0. \]
\[
\int_0^T \left[ \|D_x^2 u\|^2(t) + |D_x^2 u(0,t)|^2 \right] dt \leq (1 + Te^T)\|u_0\|^2.
\]

Here and henceforth, \( T \) is an arbitrary positive number,

\[
\|u\|^2(t) = \int_{\mathbb{R}^+} u^2(x,t) dx.
\]

**Estimate II.** Differentiate (31) with respect to \( t \), then multiply the result by \( 2u_t \) to get:

**The case \( k < 4 \).**

\[
\|u_t\|^2(t) \leq C_1\|u_t\|^2(0), \quad t \in (0, T),
\]

where

\[
C_1 = \exp \left\{ \int_0^T \left( \frac{3}{2} + \frac{2k-2}{a} \left[ k\|D_x^2 u\|^2(t) + \frac{4-k}{4} (e^T\|u_0\|)^{3k/(4-k)} \right] \right) dt \},
\]

\[
\int_0^T \left[ \|D_x^2 u_t\|^2(t) + |D_x^2 u_t(0,t)|^2 \right] dt \leq C(T, \|u_0\|)\|u_t\|^2(0)
\]

and \( \|u_t\|(0) \leq C(\|u_0\|_W) \) can be estimated directly from (31) on \( t = 0 \). **The case \( k = 4 \).**

Acting similarly to the case \( k < 4 \), we find

\[
\|u\|^2(t) \leq e^T\|u_0\|^2, \quad t \in (0, T),
\]

\[
\int_0^T \left[ \|D_x^2 u\|^2(t) + |D_x^2 u(0,t)|^2 \right] dt \leq (1 + Te^T)\|u_0\|^2,
\]

\[
\|u_t\|^2(t) \leq C_3\|u_t\|^2(0), \quad t \in (0, T),
\]

where

\[
C_2 = \int_0^T \left( \frac{3}{2} + \frac{2^5 e^T}{a} \left[ \|D_x^2 u\|^2(t)\|u_0\|^6 \right] \right) dt,
\]

\[
\int_0^T \left[ \|D_x^2 u_t\|^2(t) + |D_x^2 u_t(0,t)|^2 \right] dt \leq C(T, \|u_0\|)\|u_t\|^2(0).
\]

Here \( \|u_t\|(0) \leq C(\|u_0\|_W) \) can be estimated directly from (31) on \( t = 0 \). Independent of \( N \) estimates (39)–(42) and standard arguments allow us to prove the existence of weak solutions to (31)–(33). Regularity and uniqueness of a regular solution can be proved in the same manner as in the proof of Theorem 1. The proof of Theorem 2 is complete.

\[ \square \]

4. **Stability Intervals. Small Solutions**

Our goal in this section is to determine intervals \((0, L)\) which guarantee decay of small solutions as \( t \to +\infty \).

In \( Q_t = (0,t) \times (D), \quad D = (0,L) \), consider the following initial-boundary value problem:

\[
\begin{align*}
D_t^4 u - D_x^5 u + D_t^4 u + D_x^3 u + D_x^2 u + u^k u_x = 0, \\
D_t^4 u(0) = D_t^4 u(L) = D_x^2 u(L) = 0, \quad i = 0,1; \\
u(x,0) = u_0(x), \quad x \in (0,L).
\end{align*}
\]
Theorem 3. Let
\[ a = \frac{\pi^2}{L^2} > 1, \quad \theta = 1 - \frac{1}{a} > 0, \text{ natural } k \leq 8. \]

Given \( u_0 \in H^5(D) \cap H^2_0(D) \), \( D_x^2u_0(L) = 0 \) such that
\[ \theta - \frac{2^{k-2}}{a \theta} \left[ k \| u_0 \|^2 \| u_t \|^2(0) \frac{1}{\theta^2} \right] + (8 - k) \| u_0 \|^2 \frac{1}{8 - k} > 0 \text{ for } k < 8, \]
\[ \theta - \frac{2^8}{a \theta} \| u_0 \|^4 \| u_t \|^2(0) > 0 \text{ for } k = 8. \]

Then (43)–(45) has a unique regular solution
\[ u \in L^\infty(\mathbb{R}^+; H^5(D) \cap H^2_0(D)) \cap L^2(\mathbb{R}^+; H^7(D)); \]
\[ u_t \in L^\infty(\mathbb{R}^+; L^2(D)) \cap L^2(\mathbb{R}^+; H^2_0(D)). \]

Moreover, \( u \) satisfies the following inequalities:
\[ \| u \|^2(t) \leq \| u_0 \|^2 \exp \{-2a^2 \theta t\}, \]
\[ \| u \|^2(t) + 2\theta \int_0^t \| D_x^2u \|^2(\tau)d\tau \leq \| u_0 \|^2, \]
\[ \| u_t \|^2(t) + \frac{\theta}{2} \int_0^t \| D_x^2u_t \|^2(\tau)d\tau \leq \| u_t \|^2(0), t > 0; \]
\[ \| u_t \|^2(0) \leq \| u_t(0) \|^2 = C(\| u_0 \|_{H^5(D)}). \]

**Proof.** To establish the results of Theorem 3, we use as before the same approach based on Faedo-Galerkin’s method and we start with estimates of smooth solution of (43)–(45).

**Estimate I.** Multiply (43) by 2\( u \) to obtain
\[ \frac{d}{dt} \| u \|^2(t) + 2 \| D_x^2u \|^2(t) - 2 \| D_xu \|^2(t) + |D_x^2(0, t)|^2 = 0. \]

Making use of conditions of Theorem 3 and Lemma 6, we get
\[ \frac{d}{dt} \| u \|^2(t) + 2\theta \| D_x^2u \|^2(t) \leq 0. \]

Again, by Lemma 6,
\[ \frac{d}{dt} \| u \|^2(t) + 2a^2\theta \| u \|^2(t) \leq 0. \]

This implies
\[ \| u \|^2(t) \leq \| u_0 \|^2 \exp \{-2a^2 \theta t\}. \]

Returning to (53), we obtain
\[ \| u \|^2(t) + 2\theta \int_0^t \| D_x^2u \|^2(\tau)d\tau \leq \| u_0 \|^2. \]

**Estimate II.** Differentiate (43) by \( t \) and multiply the result by 2\( u_t \) to obtain
\[
\frac{d}{dt} \|u_t\|^2(t) + 2\theta \|D^2_{xt} u_t\|^2(t) + |D^2_{xt} u_t(0, t)|^2 \\
\leq 2(u^k u_t, u_{xt})(t).
\] (56)

The case \(k < 8\).

Making use of Lemma 2, for an arbitrary \(\epsilon > 0\), we estimate

\[
I = 2(u^k u_t, u_{xt})(t) \leq 2 \sup_D |u(x, t)|^k \|u_t\|(t) \|u_{xt}\|(t)
\]

\[
\leq 2 \left(2^{1/2} \|D^2_{xt} u\|^{1/4}(t) \|u\|^{3/4}(t)\right)^k \|u_t\|(t) \frac{1}{\sqrt{u}} \|u_{xt}\|(t)
\]

\[
\leq \epsilon \|u_{xt}\|^2(t) + \frac{2^k}{\epsilon} \|D^2_{xt} u\|^{k/2}(t) \|u\|^{3k/2}(t) \|u_t\|^2(t)
\]

\[
\leq \epsilon \|u_{xt}\|^2(t) + \frac{2^k}{\epsilon} \left[\frac{k \|u_0\|^2 \|u_t\|^2(t)}{\epsilon^2} + \frac{8 - k}{8} \|u\|^{12k/(8-k)}(t)\right] \|u_t\|^2(t).
\]

Rewrite (53) as

\[
\|D^2_{xt} u\|^2(t) \leq \frac{1}{\theta} \|u_t\|(t) \|u\|(t).
\] (57)

Substitute this into \(I\), to get

\[
I = 2(u^k u_t, u_{xt})(t) \leq 2 \sup_D |u(x, t)|^k \|u_t\|(t) \|u_{xt}\|(t)
\]

\[
\leq \epsilon \|u_{xt}\|^2(t) + \frac{2^k - 3}{\epsilon} \frac{k \|u_0\|^2 \|u_t\|^2(t)}{\theta^2}
\]

\[
+ (8 - k) \|u_0\|^{12k/(8-k)} \|u_t\|^2(t).
\]

Taking \(2\epsilon = \theta\) and substituting \(I\) into (56), we obtain

\[
\frac{d}{dt} \|u_t\|^2(t) + \frac{\theta}{2} \|D^2_{xt} u_t\|^2(t) + (\theta - \frac{2^k - 3}{\theta} \frac{k \|u_0\|^2 \|u_t\|^2(t)}{\theta^2})
\]

\[
+ (8 - k) \|u_0\|^{12k/(8-k)} \|u_t\|^2(t) \leq 0.
\] (58)

Making use of (46), (57), positivity of second term in (58) and Lemma 5, we get

\[
\left(\theta - \frac{2^k - 3}{\theta} \frac{k \|u_0\|^2 \|u_t\|^2(t)}{\theta^2} + (8 - k) \|u_0\|^{12k/(8-k)}\right) > 0, \quad t > 0.
\]

Hence (58) becomes

\[
\frac{d}{dt} \|u_t\|^2(t) + \frac{\theta}{2} \|D^2_{xt} u_t\|^2(t) \leq 0.
\] (59)

Integration gives

\[
\|u_t\|^2(t) + \frac{\theta}{2} \int_0^t \|D^2_{xt} u_\tau\|^2(\tau) d\tau \leq \|u_t\|^2(0), \quad t > 0.
\] (60)

On the other hand, by Lemma 6, (59) can be rewritten as

\[
\frac{d}{dt} \|u_t\|^2(t) + \frac{\theta^2}{2} \|u_t\|^2(t) \leq 0.
\] (61)

Hence

\[
\|u_t\|^2(t) \leq \|u_t\|^2(0) \exp\left(-\frac{\theta^2}{2} t\right).
\] (62)
The case $k = 8$.

Since (54), (55) are true for all natural $k$, consider (56) for $k = 8$:

$$\frac{d}{dt} \|u_t\|^2 (t) + 2\theta \|D_x^2 u_t\|^2 (t) + |D_x^2 u_t(0, t)|^2 \leq +2(u^8 u_t, u_{xt})(t).$$

(63)

Making use of Lemma 2, we estimate

$$I = 2(u^8 u_t, u_{xt}) (t) \leq 2 \sup_D |u(x, t)|^8 \|u_t\| (t) \|u_{xt}\| (t)$$

$$\leq 2 \left(2^{1/2} \|D_x^3 u\|^{1/4} (t) \|u\|^{3/4} (t)\right)^8 \|u_t\| (t) \frac{1}{a^{7/2}} \|u_{xt}\| (t)$$

$$\leq 2e \|u_{xt}\|^2 (t) + \frac{28}{ae} \|D_x^4 u\|^4 \|u\|^4 (t) \|u\|^2 (t).$$

Taking $2e = \theta$, making use of (60) and substituting $I$ into (63), we obtain

$$\frac{d}{dt} \|u_t\|^2 (t) + \frac{\theta}{2} \|D_x^2 u_t\|^2 (t)$$

$$+ \left(\theta - \frac{210}{a^{10}} \|u_0\|^{14} \|u_t\|^2 (t)\right) \|u_t\|^2 (t) \leq 0.$$  

(64)

Making use of (47), positivity of second term in (64) and Lemma 5, we get

$$\frac{d}{dt} \|u_t\|^2 (t) + \frac{\theta}{2} \|D_x^2 u_t\|^2 (t) \leq 0.$$  

(65)

By Lemma 6, this implies

$$\|u_t\|^2 (t) \leq \|u_t\|^2 (0) \exp\{-\frac{d^2 \theta}{2} t\};$$

(66)

$$\|u_t\|^2 (t) + \frac{\theta}{2} \int_0^t \|D_x^2 u_{\tau\tau}\|^2 (\tau) d\tau \leq \|u_t\|^2 (0), \ t > 0.$$  

(67)

Acting in the same manner as in the case $k \leq 4$, we find

$$u \in L^\infty((0, T); H^2 (D)) \cap L^2((0, T); H^4 (D)).$$

(68)

This proves the existence part of Theorem 3. Uniqueness of this regular solution has been proved in Lemma 7. It means that the proof of Theorem 3 is complete.

\[ \square \]

5. Conclusions

In this work, we studied initial-boundary value problems for the generalized Benney-Lin Equation (2) posed on bounded intervals and on a half-line. In the case of an interval $(0, L)$, where $L$ is an arbitrary positive number, we proved the existence and uniqueness of a regular solution without smallness conditions on the initial data for $k \leq 4$ and all $t \in (0, T)$, where $T$ is an arbitrary positive number. For a special choice of $(0, L)$, we proved for $k \leq 8$ the existence and uniqueness of small regular solutions as well as their exponential decay as $t \to +\infty$. In the case of the right half-line, we proved for $k \leq 4$ and arbitrary positive number $T$ the existence and uniqueness of a regular solution without smallness conditions for the initial data.

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References
1. Benney, D.J. Long waves on liquid films. J. Math. Phys. 1966, 45, 150–155. [CrossRef]
2. Lin, S.P. Finite amplitude side band stability of a viscous film. J. Fluid Mech. 1974, 63, 417–429. [CrossRef]
3. Kuramoto, Y.; Tsuzuki, T. On the formation of dissipative structures in reaction-diffusion systems. Progr. Theor. Phys. 1975, 54, 687–699. [CrossRef]
4. Sivashinsky, G.I. Nonlinear analysis of hydrodinamic instability in laminar flames-1. Derivation of basic equations. Acta Astronautica 1977, 4, 1177–1206 [CrossRef]
5. Cross, M.C. Pattern formation outside of equilibrium. Rev. Mod. Phys. 1993, 65, 851–1086. [CrossRef]
6. Biagioni, H.A.; Bona, J.L.; Iorio, R.J., Jr.; Scialom, M. On the Korteweg de Vries-Kuramoto-Sivashinsky Equation. Adv. Differ. Equations 1996, 1, 1–20.
7. Larkin, N.A. Correct initial boundary value problems for dispersive equations. J. Math. Anal. Appl. 2008, 344, 1079–1092. [CrossRef]
8. Larkin, N.A.; Simões, M.H. The Kawahara equation on bounded intervals and on a half-line. Nonlinear Anal. 2015, 127, 397–412. [CrossRef]
9. Li, J.; Zhang, B.-Y.; Zhang, Z. A nonhomogeneous boundary value problem for the Kuramoto-Sivashinsky equation in a quarter intervals. J. Math. Anal. Appl. 2004, 297, 169–185. [CrossRef]
10. Larkin, N.A. The Kawarhara equation in a bounded domain. Discret. Contin. Dyn. Syst. Ser. B 2008, 10, 783–799. [CrossRef]
11. Larkin, N.A. Initial-boundary value problems for nonhomogeneous boundary value problems for the Korteweg-de Vries-Kuramoto-Sivashinsky Equation. Stud. Appl. Math. Adv. Math. Suppl. Stud. 1983, 8, 93–128.
12. Larkin, N.A. Correct initial boundary value problems for dispersive equations. J. Math. Anal. Appl. 2008, 344, 1079–1092. [CrossRef]
13. Larkin, N.A.; Luchesi, J. Initial-boundary value problems for generalized dispersive equations of higher orders posed on bounded intervals. J. Appl. Math. Optim. 2019, 83, 1081–1102. [CrossRef]
14. Foneca, G.; Linares, F.; Ponce, G. Global existence for the critical generalized KDV equation. Proc. AMS 2002, 131, 1847–1855. [CrossRef]
15. Martel, Y.; Merle, F. Instability of solutions for the critical generalized Korteweg-de Vries equation. Geom. Funct. Anal. 2001, 11, 74–123. [CrossRef]
16. Jeffrey, A.; Kakinaka, A. Weak nonlinear dispersive waves: A discussion centered around the Korteweg-de Vries equation. SIAM Rev. 1972, 14, 582–643. [CrossRef]
17. Linares, F.; Pazo, A. On the exponential decay of the critical generalized Korteweg-de Vries equation with localized damping. Proc. Am. Math. Soc. 2007, 135, 1515–1522. [CrossRef]
18. Larkin, N.A. Global regular solutions for a generalized KdV equation posed on a half-line. arXiv 2020, arXiv:2006.06645v1.
19. Linares, F.; Pazoto, A. On the exponential decay of the critical generalized Korteweg-de Vries equation with localized damping. Proc. Am. Math. Soc. 2007, 135, 1515–1522. [CrossRef]
20. Coyle, A.V. The problem of cooling of an heterogeneous rigid rod. Commun. Kharkov Math. Soc. Ser. 1986, 2, 136–181. (In Russian)
21. Nirenberg, L. On elliptic partial differential equations. In Annali Della Scuola Normale Superiore di Pisa; Classe di Scienze 3ª Série; Springer: Berlin/Heidelberg, Germany, 1959; pp. 115–162.
22. Coddington, E.; Levinson, N. Theory of Ordinary Differential Equations; MacGraw-Hill: New York, NY, USA, 1955.