Infinitely Many Periodic Solutions for Some N-Body Type Problems with Fixed Energies

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Abstract

In this paper, we apply the Lusternik-Schnirelman theory with local Palais-Smale condition to study a class of N-body problems with strong force potentials and fixed energies. Under suitable conditions on the potential $V$, we prove the existence of infinitely many non-constant and non-collision symmetrical periodic solutions.

Key Words: Periodic solutions, N-body problems, Lusternik-Schnirelman theory, Local Palais-Smale condition.

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1 Introduction and Main Results

The N-body problem is an old and very difficult problem. Many mathematicians studied this problem using many different methods, here we only concern variational methods. In 1975 and 1977, Gordon ([15], [16]) firstly used variational methods to study the periodic solutions of 2-body problems. In [16], he put forward to the strong force condition (SF for short) and got the following Lemma:

Lemma 1.1. (Gordon[16]) Suppose that $V(x)$ satisfies the so called Gordon’s Strong Force condition: There exists a neighborhood $N$ of 0 and a function $U \in C^1(N \setminus \{0\}, \mathbb{R})$ such that:

(i) $\lim_{|x| \to 0} U(x) = -\infty$;

(ii) $-V(x) \geq |
\nabla U(x)|^2$ for every $x \in N \setminus \{0\}$.

Let

$\Lambda \triangleq \{x \mid x(t) \in H^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n), \ x(t) \neq 0, \forall t \in [0, 1]\}$

$\partial \Lambda \triangleq \{x \mid x(t) \in H^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n), \ \exists t_0 \in [0, 1] \text{ s.t. } x(t_0) = 0\}$

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Then we have
\[ \int_0^1 V(x_n) \, dt \to -\infty, \quad \forall x_n \to x \in \partial \Lambda. \]

After Gordon, many researchers used variational methods to study \( N (N \geq 3) \) body problems ([1] – [8], [10] – [19], [22] – [27], etc.).

In [18], P. Majer used Ljusternik-Schnirelman theory (LS for short) with local Palais-Smale condition to seek \( T \)-periodic solutions of the following second order Hamiltonian system:
\[ \ddot{x} + ax + \nabla_x W(t, x) = 0 \quad (1.1) \]
where \( W \) is singular at \( x = 0 \), \( W(t + T, x) = W(t, x) \).

He got the following theorem:

**Theorem 1.2.** Suppose
\begin{enumerate}
  \item[(P1).] \( a < \left( \frac{\pi}{T} \right)^2 \);
  \item[(P2).] \( W \in C^1(S^1_T \times (\mathbb{R}^n \setminus \{0\}); \mathbb{R}) \) satisfies (SF);
  \item[(P3).] \( \exists c, \theta < 2, r > 0, \) such that \( \forall |x| \geq r, \forall t \in S^1_T \)
    \[ W(t, x) \leq c|x|^\theta, \quad (\nabla W_x(t, x) , x) - 2W(t, x) \leq c|x|^\theta \]
  \item[(P4).] \( W(t, x) \leq b. \)
\end{enumerate}

Then the system (1.1) has infinitely many \( T \)-periodic non-collision solutions.

After Majer, Zhang-Zhou ([27]) studied a class of N-Body problems. They considered the following system:
\[ m_i \ddot{x}_i(t) + \nabla_{x_i} V(t, x_1(t), \cdots, x_N(t)) = 0, \quad x_i \in \mathbb{R}^n, \quad i = 1, \cdots, N \quad (1.2) \]
where \( m_i > 0 \) for all \( i \), and \( V \) satisfies the following conditions:
\begin{enumerate}
  \item[(V1).] \( V(t, x_1, \cdots, x_N) = \frac{1}{2} \sum_{1 \leq i \neq j \leq N} V_{ij}(t, x_i - x_j) \);
  \item[(V2).] \( V_{ij} \in C^2(\mathbb{R} \times (\mathbb{R}^n \setminus \{0\}); \mathbb{R}) \), for all \( 1 \leq i \neq j \leq N \);
  \item[(V3).] \( V_{ij}(t, \xi) \to -\infty \) uniformly on \( t \) as \( |\xi| \to 0 \), for all \( 1 \leq i \neq j \leq N \);
  \item[(V4).] \( V_{ij}(t, \xi) \leq 0, \) for all \( t \in \mathbb{R}, \xi \in \mathbb{R}^n \setminus \{0\} \);
  \item[(V5).] \( V_{ij} \) satisfies Gordon’s strong force condition;
  \item[(V6).] There exists an element \( g \) of finite order \( s \) in \( SO(k) \) which has no fixed point other than origin (i.e. 1 is not an eigenvalue of \( g \)), such that
    \[ V(t, x_1, \cdots, x_N) = V(t + T/s, gx_1, \cdots, gx_N), \quad \forall t \in [0, 1], \quad x_i \in \mathbb{R}^n. \]
\end{enumerate}
If the potential \( V \) satisfies (V1) – (V6), and \( x(t) = (x_1(t), \cdots, x_N(t)) \) is a \( T \)-periodic non-collision solution of system (1.2) and satisfies \( x(t + T/s) = (x_1(t + T/s), \cdots, x_N(T + t/s)) = (gx_1(t), \cdots, gx_N(t)) = gx(t) \), we say that \( x(t) \) is a \( g \)-symmetric \( T \)-periodic non-collision solutions (we also call \( (g, T) \) is a non-collision solution for short).

Zhang-Zhou obtained the following theorem:
Theorem 1.3. Suppose the potential $V$ satisfies (V1) – (V6) and $T$ is any positive real number. Then system (1.2) has infinitely many $(g, T)$ non-collision solutions.

Motivated by P. Majer, Zhang S.Q.- Zhou Q. ’s work, we consider the following system:

\[(Ph)\begin{cases} m_i\ddot{u}_i + \nabla u_i V(u_1, \ldots, u_N) = 0, & (1 \leq i \leq N), \\
\frac{1}{2} \sum m_i|\dot{u}_i(t)|^2 + V(u_1(t), \ldots, u_N(t)) = h. & \end{cases}\] (Ph.1)

We have the following theorem:

**Theorem 1.4.** Suppose $V(u_1, \ldots, u_N) = \frac{1}{2} \sum_{1 \leq i \neq j \leq N} V_{ij}(u_i - u_j)$, $V_{ij}(\xi) \in C^1(\mathbb{R}^n \setminus \{0\}, \mathbb{R})$, and satisfies

(V5'). $V_{ij}(\xi) \leq 0, \forall \xi \in \mathbb{R}^n \setminus \{0\}$;

(V6'). $\exists \alpha > 2$ and $r_1 > 0$ such that $(\nabla V_{ij}(\xi), \xi) \geq -\alpha V_{ij}(\xi), \forall 0 < |\xi| \leq r_1$;

(V7'). $\exists c \geq 0, -2 < \theta < 0, r_2 > r_1$, such that when $|\xi| \geq r_2$, there holds

\[|\nabla \xi V_{ij}(\xi), \xi| \leq c|\xi|^\theta.\]

(V8'). $V_{ij}(\xi) = V_{ij}(\xi), \forall \xi \in \mathbb{R}^n \setminus \{0\}$.

Then for any $h > 0$, the system (Ph) has infinitely many non-constant and non-collision periodic solutions.

**Example 1.5.** We take

\[V_{ij}(\xi) = \begin{cases} -\frac{1}{|\xi|^\alpha}, & \text{if } 0 < |\xi| \leq r_1, \\
\text{smooth connecting,} & \text{if } r_1 < |\xi| < r_2, \\
|\xi|^\theta, & \text{if } |\xi| \geq r_2 > r_1. \end{cases}\]

2 Some Lemmas

In this section, we collect some known lemmas which are necessary for the proof of Theorem 1.4.

Let us introduce the following notations:

\[m^* = \min \{m_1, \ldots, m_N\}; \quad H^1 = W^{1,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n).\]

\[E = \{u = (u_1, \ldots, u_N) \mid u_i \in H^1, u_i(t + \frac{1}{2}) = -u_i(t)\}.\]

\[\Lambda_0 = \{u \in E \mid u_i(t) \neq u_j(t), \forall t, \forall i \neq j\}.\]

\[\partial \Lambda_0 = \{u \in E \mid \exists t_0, 1 \leq i_0 \neq j_0 \leq N \text{ s.t. } u_{i_0}(t_0) = u_{j_0}(t_0)\}.\]

\[p(u) = \min_{1 \leq i \neq j \leq N, t \in [0, 1]} |u_i(t) - u_j(t)|.\]

\[\{f \leq c\} = \{u \in \Lambda_0, f(u) \leq c\}.\]
Lemma 2.1. ([1] − [4]) Let $f(u) = \frac{1}{2} \int_0^1 \sum_{i=1}^m m_i |u_i|^2 \, dt \int_0^1 (h - V(u)) \, dt$ and $\tilde{u} \in \Lambda_0$ satisfy $f'(\tilde{u}) = 0$ and $f(\tilde{u}) > 0$. Set
\[
\frac{1}{T^2} = \frac{1}{2} \int_0^1 (h - V(\tilde{u})) \, dt.
\] (2.1)

Then $\tilde{q}(t) = \tilde{u}(t/T)$ is a non-constant $T$-periodic solution for (Ph).

Lemma 2.2. (Palais[20]) Let $\sigma$ be an orthogonal representation of a finite or compact group $G$ in the real Hilbert space $H$ such that for $\forall \sigma \in G$,
\[
f(\sigma \cdot x) = f(x),
\]
where $f \in C^1(H, \mathbb{R})$.

Let $S = \{x \in H \mid \sigma x = x, \forall \sigma \in G\}$, then the critical point of $f$ in $S$ is also a critical point of $f$ in $H$.

By Lemma 2.1–2.2, we have

Lemma 2.3. ([1] − [4]) Assume $V_{ij} \in C^1(\mathbb{R}^n \setminus \{0\}, \mathbb{R})$ satisfies $(V8^*)$. If $\bar{u} \in \Lambda_0$ is a critical point of $f(u)$ and $f(\bar{u}) > 0$, then $\bar{q}(t) = \bar{u}(t/T)$ is a non-constant $T$-periodic solution of (Ph).

Lemma 2.4. ([28]) Let $q \in W^{1,2}(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^n)$ and $\int_0^T q(t) \, dt = 0$, then we have
(i). Poincare-Wirtinger’s inequality:
\[
\int_0^T |\dot{q}(t)|^2 \, dt \geq \left(\frac{2\pi}{T}\right)^2 \int_0^T |q(t)|^2 \, dt.
\] (2.2)

(ii). Sobolev’s inequality:
\[
\text{max}_{0 \leq t \leq T} |q(t)| = |q|_{\infty} \leq \sqrt{\frac{T}{12}} \left(\int_0^T |\dot{q}(t)|^2 \, dt\right)^{1/2}.
\] (2.3)

By the definition of $\Lambda_0$ and Lemma 2.3, for $\forall u \in \Lambda_0$, $(\int_0^1 \sum_{i=1}^N m_i |\dot{u}_i|^2 \, dt)^{1/2}$ is equivalent to the $(H^1)^N = H^1 \times \cdots H^1$ norm:
\[
\| u \|_{(H^1)^N} = \left(\int_0^1 |\dot{u}|^2 \, dt\right)^{1/2} + \left(\int_0^1 |u|^2 \, dt\right)^{1/2}.
\]

So we take norm $\| u \| = (\int_0^1 \sum_{i=1}^N m_i |\dot{u}_i|^2 \, dt)^{1/2}$.
Lemma 2.5. (Coti Zelati[11]) Let \( X = (x_1, \ldots, x_N) \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n, x_i \neq x_j, \forall 1 \leq i \neq j \leq N. \) Then
\[
\sum_{1 \leq i < j \leq N} \frac{m_i m_j}{|x_i - x_j|^\alpha} \geq C_\alpha(m_1, \ldots, m_N)(\sum_{i=1}^N m_i|x_i|^2)^{-\alpha/2},
\]
where \( C_\alpha(m_1, \ldots, m_N) \triangleq C_\alpha = (\sum_{i=1}^N m_i)\frac{-\alpha}{2}(\sum_{1 \leq i < j \leq N} m_i m_j)^{2+\alpha}. \)

Lemma 2.6. ([18]) Let \( X \) be a Banach space with norm \( \| \cdot \| \), \( \Lambda \) be an open subset of \( X \), and suppose a functional \( f : \Lambda \to \mathbb{R} \) is given such that the following conditions hold:

(i). \( \text{Cat}_\Lambda(\Lambda) = +\infty; \)

(ii). \( f \in C^1(\Lambda) \) and \( \forall u_n \to \partial \Lambda, f(u_n) \to +\infty; \)

(iii). \( \forall \lambda \in \mathbb{R}, \text{Cat}_\Lambda(\{ f \leq \lambda \}) < +\infty; \)

Suppose in addition that there exist \( g \in C^1(\Lambda), \beta \in (0, 1) \) and \( \lambda_0 \in \mathbb{R} \) such that

(iv). \( \text{Cat}_\Lambda(\{ f \leq g \}) < +\infty; \)

(v). the PS condition holds in the set \( \{ f \geq g \}; \)

(vi). \( \beta \| f'(u) \| \geq \| g'(u) \|, \forall u \in \{ f = g \geq \lambda_0 \}. \)

Then \( f \) has a sequence \( \{ u_n \} \subset \Lambda \) of critical points such that \( f(u_n) \to +\infty \) and \( f(u_n) \geq g(u_n) - 1. \)

3 The Proof of Theorem 1.4

Lemma 3.1. Let \( \{ u_n \} \subset \Lambda_0 \) and \( u_n \to u \in \partial \Lambda_0. \) Then \( f(u_n) \to +\infty. \)

Proof. First of all, we recall that
\[
f(u_n) = \frac{1}{2} \int_0^1 \sum_{i=1}^N m_i|\dot{u}_n|^2 dt \int_0^1 (h - V(u_n))dt
\]

(1). If \( u = \text{constant}, \) we can deduce \( u = 0 \) by \( u(t + \frac{1}{2}) = -u_i(t). \) By Sobolev’s embedding theorem, we obtain
\[
|u_n|_\infty \to 0, \quad n \to \infty \tag{3.1}
\]
So when \( n \) is large enough, \( 0 \leq |u_n' - u_j'| \leq r_1. \) By (V6'), there exists an \( A > 0 \) such that
\[
V_{ij}(\xi) \leq -\frac{A}{|\xi|^\alpha}, \quad \forall 0 < |\xi| \leq r_1 \tag{3.2}
\]
By \( h > 0 \), Sobolev’s inequality (2.3), (3.2) and Lemma 2.5, we have

\[
f(u_n) = \frac{1}{2} \| u_n \|^2 \int_0^1 (h - \sum_{1 \leq i < j \leq N} V_{ij}(u^i_n - u^j_n)) dt \\
\geq \frac{1}{2} \| u_n \|^2 \int_0^1 \{ - \sum_{1 \leq i < j \leq N} V_{ij}(u^i_n - u^j_n) \} dt \\
\geq \frac{1}{2} \| u_n \|^2 \int_0^1 \sum_{1 \leq i < j \leq N} \frac{A}{|u^i_n - u^j_n|^\alpha} dt \\
\geq \frac{1}{2} \| u_n \|^2 \int_0^1 A \left[ \frac{N(N-1)}{2} \right]^{\frac{2+\alpha}{2}} N^{-\alpha} |u_n|^{-\alpha} dt \\
\geq 3m^* A 2^{-\frac{\alpha}{2}} N(N-1)^{1+\frac{\alpha}{2}} |u_n|^{2-\alpha}
\]

Then by (3.1) and \( \alpha > 2 \), we can deduce

\[
f(u_n) \to +\infty, \quad n \to \infty
\]

(2). If \( u \neq \text{constant} \), by the weakly lower-semi-continuity property for norm, we have

\[
\liminf_{n \to \infty} \int_0^1 \sum_{i=1}^N m_i |\dot{u}^i_n|^2 dt \geq \int_0^1 \sum_{i=1}^N m_i |\dot{u}^i|^2 dt > 0.
\]

Since \( u \in \partial \Lambda_0 \), there exist \( t_0, 1 \leq i_0 \neq j_0 \leq N \) s.t. \( u_{i_0}(t_0) = u_{j_0}(t_0) \). Set

\[
\xi_n(t) = u^i_n(t) - u^j_n(t) \\
\xi(t) = u_{i_0}(t) - u_{j_0}(t)
\]

By \( u_n \to u \), we have \( \xi_n(t) \to \xi(t) \). Then by (V6’) and Lemma 1.1, we have

\[
\int_0^1 V_{i_0j_0}(u^i_n - u^j_n) dt \to -\infty.
\]

Recalling that

\[
V(u_n) = \sum_{i < j} V_{ij}(u^i_n - u^j_n).
\]

So we have

\[
f(u_n) \to +\infty, \quad n \to \infty.
\]

Lemma 3.2. For every \( \lambda \in \mathbb{R} \) there exists a constant \( k = k(\lambda) \) such that

\[
\| u \| \leq k(\lambda)p(u), \quad \forall u \in \{ f \leq \lambda \}
\]

(3.4)
Proof. Recall that
\[ f(u) = \frac{1}{2} \| u \|^2 \int_0^1 (h - V(u))dt \]
Since \( V_{ij} \leq 0 \), we have \( f(u) \geq \frac{1}{2} h \| u \|^2 \). Since \( u \in \{ f \leq \lambda \} \), we can deduce that
\[ \| u \| \leq C. \quad (3.5) \]
If (3.4) is not true, then there exists a sequence \( \{ u_n \} \) such that \( \| u_n \| \geq np(u_n) \). By (3.5), we have
\[ 0 < p(u_n) \leq \frac{C}{n}. \quad (3.6) \]
Let \( n \to \infty \) in (3.6), we have \( p(u_n) \to 0 \). Then there exists a subsequence \( \{ u_{n_k} \} \to u \in \partial \Lambda_\lambda \), by Lemma 3.1, \( f(u_{n_k}) \to +\infty \), which is a contradiction since \( \{ u_{n_k} \} \subset \{ f \leq \lambda \} \).

\[ \square \]

Lemma 3.3. For every \( \lambda \in \mathbb{R} \), the set \( \Lambda_\lambda = \{ u \in \Lambda_0 : \frac{\| u \|}{p(u)} \leq \lambda \} \) is of finite category in \( \Lambda_0 \).

Proof. The proof is almost the same as the proof of Lemma 4.3 of [27]. For the convenience of the readers, we write the complete proof.

It is sufficient that we give a homotopy \( h : [0, 1] \times \Lambda_c \subset \subset \Lambda_0 \).

Take \( \delta \in (0, 1) \) s.t. \( \frac{1}{m^*} \sqrt{\delta c} < 1 \), and define \( \phi(t) = \frac{p(u)}{\delta} \), if \( t \in [0, \delta] \), otherwise \( \phi(t) = 0 \).

Define
\[ h(u, \lambda) = (1 - \lambda)u(t) + \lambda \frac{(u * \phi)(t)}{p(u)}, \quad \forall u \in \Lambda_c, \ 0 \leq \lambda \leq 1 \]
where the convolution \( (u * \phi)(t) = (\int_0^1 u_1 \phi(s)ds, \ldots, \int_0^1 u_N \phi(s)ds) \), then \( h(u, 0) \) is an inclusion and \( h_1(\Lambda_c) \) is paracompact since \( h(u, 1) \) is a convolution operator, so it’s a compact operator. We need to prove \( h(\Lambda_c \times I) \subset \Lambda_0 \).

Suppose this is not true, then \( \exists \lambda_0 \in (0, 1], \ u \in \Lambda_c, \ 1 \leq i_0 \neq j_0 \leq N, \ t_0 \in [0, 1] \), such that
\[ (1 - \lambda_0)(u_{i_0} - u_{j_0}) + \lambda_0 \frac{1}{p(u)}((u_{i_0} - u_{j_0})*\phi)(t_0) \]
Then
\[ (u_{i_0} - u_{j_0})*\phi(t_0) = p(u)(1 - \frac{1}{\lambda_0})(u_{i_0}(t_0) - u_{j_0}(t_0)) \]
So we have
\[ |p(u)(u_{i_0}(t_0) - u_{j_0}(t_0) - ((u_{i_0} - u_{j_0})*\phi)(t_0))| \]
\[ = \frac{p(u)}{\lambda_0} \geq p(u)p(u) = p^2(u). \]
On the other hand, \( \forall u = (u_1, \ldots, u_N) \in \Lambda_c, \ i \neq j, \ t \in [0, 1] \), we have
\[
|p(u)(u_i(t) - u_j(t)) - ((u_i - u_j) * \phi)(t)|
\]
\[
= \left| \frac{p(u)}{\delta} \int_0^\delta (u_i(t) - u_j(t))ds - \frac{p(u)}{\delta}(u_i(t-s) - u_j(t-s))ds \right|
\]
\[
\leq \frac{p(u)}{\delta} \int_0^\delta |u_i(t) - u_j(t) - (u_i(t-s) - u_j(t-s))||ds
\]
\[
\leq p(u) \cdot \sup_{|s| \leq \delta} |u_i(t) - u_j(t) - (u_i(t-s) - u_j(t-s))|
\]
\[
\leq p(u) \sqrt{\delta} \parallel \dot{u}_i - \dot{u}_j \parallel_{L^2} \leq p(u) \sqrt{\delta} \parallel \dot{u}_i \parallel_{L^2} + \parallel \dot{u}_j \parallel_{L^2}
\]
\[
\leq p(u) 2 \sqrt{\delta} \parallel \dot{u} \parallel_{L^2} \leq p(u) \cdot \left( \frac{1}{m^*} \sqrt{\delta} \right) \cdot p(u) < p^2(u)
\]
Which is a contradiction. \( \square \)

**Lemma 3.4.** The functional \( f \) verifies the Palais-Smale condition on \( \Lambda_0 \).

**Proof.** Let \( \{u_n\} \subset \Lambda_0 \) be a P.S. sequence, then up to a subsequence, it converges weakly in \( (H^1)^N \) and uniformly in \( |u|_\infty \) to an element \( u \in \Lambda_0 \) by Lemma 3.1. Hence \( \langle \nabla_x V_i(u^n_i - u^n_j), (u_i - u^n_i - u_j - u^n_j) \rangle \) converges uniformly to zero. Since \( f'(u_n) \to 0 \), and \( u - u_n \) is \( (H^1)^N \)-bounded, and
\[
\langle f'(u), v \rangle = \int_0^1 \sum_{1 \leq i \leq N} m_i \langle \dot{u}_i, \dot{v}_i \rangle dt \int_0^1 (h - V(u))dt - \frac{1}{2} \parallel u \parallel^2 \int_0^1 \langle \nabla_u V(u), v \rangle dt
\]
So we have
\[
\parallel u \parallel^2 - \lim_{n \to \infty} \parallel u_n \parallel^2 = \lim_{n \to \infty} \int_0^1 m_i \langle \dot{u}_i, (\dot{u}_i - \dot{u}_n) \rangle dt
\]
\[
= \lim_{n \to \infty} \int_0^1 (h - V(u_n))dt + \lim_{n \to \infty} \frac{1}{2} \parallel u_n \parallel^2 \int_0^1 \langle \nabla_u V(u_n), (u - u_n) \rangle dt \]
\[
= 0
\]
\( \square \)

**Lemma 3.5.** \( \text{Cat}_{\Lambda_0}(\Lambda_0) = +\infty \)

**Proof.** See the Corollary 3.4 of [27]. \( \square \)

**Lemma 3.6.** There exist a functional \( g \in C^1(\Lambda_0), \beta \in (0, 1) \) and \( \lambda_0 \in \mathbb{R} \) such that:

(i) \( \text{Cat}_{\Lambda_0}\{f \leq g\} < +\infty \);

(ii) the P.S. condition holds in the set \( \{f \leq g\} \);

(iii) \( \beta \parallel f'(u) \parallel \geq \parallel g'(u) \parallel, \quad \forall u \in \{f = g \geq \lambda_0\} \).
Proof. By Sobolev’s embedding theorem, we know there is a \( k_\infty > 0 \) s.t.
\[
\| u \|_\infty \leq k_\infty \| u \|, \forall u \in \Lambda_0.
\]

Take \( \beta \) such that \( \beta > \frac{\theta + 2}{2} \), take \( \gamma > 0 \) such that
\[
\beta \cdot [2\gamma - N(N - 1)c2^{\theta - 2}k_\infty] > \gamma(\theta + 2).
\]

Let \( g(u) = \gamma \| u \|^\theta + 2 \), we shall show that \( \{ f \leq g \} \) is a set of finite category. We take \( 0 < \varepsilon < \frac{h}{2} \) and \( M > 0 \) such that \( \forall s \in \mathbb{R}, \gamma |s|^\theta \leq \varepsilon s^2 + M \).

Define
\[
f_\varepsilon(u) = f(u) - \varepsilon \| u \|^2
\]

Then
\[
\{ f \leq g \} \subset \{ f_\varepsilon(u) \leq M \}
\]

We can use the similar proof of Lemma 3.2 to show that there exists \( k_1 \in \mathbb{R} \) such that
\[
\| u \| \leq k_1 p(u), \forall u \in \{ f_\varepsilon \leq M \}
\]
then by Lemma 3.3,
\[
Cat_{\Lambda_0} \{ f \leq g \} \leq Cat_{\Lambda_0} \{ f_\varepsilon \leq M \} < +\infty
\]

From Lemma 3.4, we know that \((v)\) is satisfied.

Since \( \| u \| \leq k_1 p(u) \), we take \( \lambda_0 =: \gamma(k_1 r_2)^{\theta + 2} \). Then if \( u \in \{ f = g \geq \lambda_0 \} \), so
\[
\| u \| \geq \left( \frac{\lambda_0}{\gamma} \right)^{\frac{1}{\theta + 2}} = k_1 r_2
\]

So we can obtain :
\[
r_2 \leq p(u) \leq |u_i(t) - u_j(t)| \leq 2|u|_\infty \leq 2k_\infty \| u \|
\]

\[
\int_0^1 \langle \nabla u V(u), u \rangle dt
\]
\[
= \int_0^1 \sum_{1 \leq i < j \leq N} \langle \nabla V_{ij}(u_i - u_j), (u_i - u_j) \rangle dt
\]
\[
\leq \int_0^1 \sum_{1 \leq i < j \leq N} c|u_i - u_j|^\theta dt
\]
\[
\leq c \frac{N(N - 1)}{2} (2k_\infty \| u \|)^\theta
\]
(3.7)
Since
\[ \| f'(u) \| \geq \langle f'(u), u \rangle \]
\[ = 2 f(u) - \frac{1}{2} \| u \|^2 \int_0^1 \langle \nabla_u(u), u \rangle dt \tag{3.8} \]

Then by (3.7) and (3.8), we have
\[ \| f'(u) \| \geq [2\gamma - N(N - 1)c2^{\theta-2}k_\infty^\theta] \| u \|^{\theta+1} \tag{3.9} \]

Since \( \| g'(u) \| = \gamma(\theta + 2) \| u \|^{\theta+1} \), from our choice of \( \gamma \), we have
\[ \beta \| f'(u) \| - \| g'(u) \| \geq 0, \forall u \in \{ f = g \geq \lambda_0 \} \tag{3.10} \]

That is, (vi) holds. \( \square \)

References

[1] A.Ambrosetti and V.Coti Zelati, Closed orbits of fixed energy for a class of N-body problems, Ann. Inst. H.Poincare, Analyse Non Lineaire 9(1992), 187-200, Addendum, Ann. Inst. H.Poincare, Analyse Non Lineaire 9(1992), 337-338.

[2] A.Ambrosetti and V.Coti Zelati, Closed orbits of fixed energy for singular Hamiltonian systems, Arch. Rat. Mech. Anal. 112(1990), 339-362.

[3] A.Ambrosetti and V.Coti Zelati, Non-collision periodic solutions for a class of symmetric 3-body type problems, Topol. Methods Nonlinear Anal. 3(1994), 197-207.

[4] A.Ambrosetti and V.Coti Zelati, Periodic solutions for singular Lagrangian systems, Springer, 1993.

[5] A.Ambrosetti and P.H.Rabinowitz, Dual variational methods in critical point theory and applications, J. Funct. Anal. 14(1973), 349-381.

[6] A.Ambrosetti and K.Tanaka and E.Vitillaro, Periodic solutions with prescribed energy for some Keplerian N-body problems, Ann.Inst.H.Poincar Anal. Nonlinaire 11(1994), 613-632.

[7] A.Bahri and P.H.Rabinowitz, Periodic solutions of Hamiltonian systems of three body type. Ann. Inst. H. Poincare Anal. Non Lineaire 8(1991), 561-649.

[8] U.Bessi and V.Coti Zelati, Symmetries and noncollision closed orbits for planar N-body-type problems, Nonlinear Anal. 16(1991), 587-598.

[9] G.Buttazzo and M.Giaquinta and S.Hildebrandt, One-dimensional variational problems, Oxford University Press, 1998.
[10] A.Chenciner and R.Montgomery, A remarkable periodic solutions of the three-body problem in the case of equal masses, Annals of Math, 152(2000), 881-901

[11] V.Coti Zelati, The periodic solutions of N-body type problems, Ann. Inst. H.Poincare Anal. Nonlineaire 7(1990), 477-492.

[12] M.Degiovanni and F.Giannoni, Dynamical systems with Newtonian type potentials, Ann. Sc. Norm. Sup. Pisa 15(1989), 467-494.

[13] G.F.Dell’s Antonio, Classical solutions of a perturbed N-body system, In Top. Nonlinear Anal., M.Matzeu etc. ed., Birkhauser, 1997, 1-86.

[14] E.Fadell and S.Husseini, A note on the category of free loop space, Pro.Am.Math.Soc., 107(1989),527-536.

[15] W.B.Gordon, A minimizing property of Keplerian orbits, Amer. J. Math. 99(1977), 961-971.

[16] W.B.Gordon, Conservative dynamical systems involving strong forces, Trans. Amer. Math. Soc. 204(1975), 113-135.

[17] Y.M.Long and S.Q.Zhang, Geometric characterizations for variational minimization solutions of the 3-body problems, Act Math. Sinica 16(2000), 579-592.

[18] P.Majer, Ljusterni-Schnirelman theory with local Palais-Smale condition and singular dynamical systems, Ann. Inst. Henri Poincare Anal. Non Lineaire 8(1991), 459-476.

[19] P.Majer and S.Terracini, Periodic solutions to some N-body type problems:the fixed energy case. Duke Math.J.69(1993), 683-697.

[20] R.Palais, The principle of symmetric criticality, Comm. Math. Phys. 69(1979),19-30.

[21] J.P.Serre, Homologic singular des espaces fibers, Ann.Math. 54(1951),425-505.

[22] E.Serra and S.Terracini , Collisionless periodic solutions to some 3-body problems, Arch. Rational. Mech. Anal. 120(1992), 305-325.

[23] S.Terracini, Multiplicity of periodic solution with prescribed energy to singular dynamical systems.Ann.Inst.H.Poincar Anal.Nonlineaire 9(1992),597-641.

[24] A.Venturelli, Une caracterisation variationnelle des solutions de Lagrange du problem plan des trois corps, C.R. Acad. Sci. Paris 332(2001), 641-644.

[25] S.Q.Zhang and Q.Zhou, A minimizing property of Lagrangian solutions, Acta Math. Sinica 17(2001), 497-500.
[26] S.Q.Zhang and Q.Zhou, Variational methods for the choreography solution to the three-body problem, Sci.China 45(2002), 594-597.

[27] S.Q.Zhang and Q.Zhou, Symmetric periodic noncollision solutions for N-Body-type problems, Acta Math.Sinica 11(1995),37-43.

[28] W.P.Ziemer, Weakly differentiable functions, Springer, 1989.