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Soliton Solutions for a Nonisospectral Semi-Discrete Ablowitz–Kaup–Newell–Segur Equation

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Received: 4 September 2020; Accepted: 29 October 2020; Published: 31 October 2020

Abstract: In this paper, we study a nonisospectral semi-discrete Ablowitz–Kaup–Newell–Segur equation. Multisoliton solutions for this equation are given by Hirota’s method. Dynamics of some soliton solutions are analyzed and illustrated by asymptotic analysis. Multisoliton solutions and dynamics to a nonisospectral semi-discrete modified Korteweg–de Vries equation are also discussed.

Keywords: nonisospectral semi-discrete Ablowitz–Kaup–Newell–Segur equation; nonisospectral semi-discrete modified Korteweg–de Vries equation; Hirota’s method; soliton solutions; dynamics

1. Introduction

Nonisospectral integrable system [1–3] has attracted a great deal of interest and has been studied for many years from various view points, such as integrability and exact solutions. Compared with the isospectral integrable system, a nonisospectral integrable system is usually related to time-dependent spectral parameters and has time-varying solitary wave solutions. In general, depending on the linear problems, a nonisospectral integrable hierarchy can be derived from the Lax equation, zero curvature equation, or in the frame of Kac–Moody algebra, etc. [4–6]. The nonisospectral flows play the role of master symmetries and can be applied to generate time-dependent symmetries [7,8]. With regard to solutions for the nonisospectral integrable system, various of approaches have been developed, such as Inverse Scattering Transform, Bäcklund transformation, Darboux transformation, Hirota’s method, and the Wronskian technique [9–13].

In recent decades, the semi-discrete integrable system, i.e., the system given by an integrable partial differential-difference equation, has become an increasingly popular topic. Some prototypical examples include Toda lattice, Volterra lattice, semi-discrete nonlinear Schrödinger equation, semi-discrete modified Korteweg–de Vries (mKdV) equation, and so on. Such type of equations relate to many branches of mathematical theories and and play a central role in many fields. For instance, the semi-discrete nonlinear Schrödinger equation (also known as the self-trapping equation) has a number of applications in molecular physics, nonlinear optics, and in other fields [14,15]. The semi-discrete mKdV equation arises in a wide variety of fields, such as plasma physics, electromagnetic waves in ferromagnetic, antiferromagnetic, or dielectric systems.

Generally speaking, a continuous equation could have several discrete versions upon the discretisation procedure. Until now, many results for semi-discrete Ablowitz–Kaup–Newell–Segur (AKNS) type equations have been derived [16–21]. In [22] (see also [23]), Ablowitz and Ladik propose a semi-discrete spectral problem, called the Ablowitz–Ladik spectral problem,

$$\left( \begin{array}{c} \phi_{1,n+1} \\ \phi_{2,n+1} \end{array} \right) = M_n \left( \begin{array}{c} \phi_{1,n} \\ \phi_{2,n} \end{array} \right), \quad M_n = \left( \begin{array}{cc} \lambda & u_n \\ v_n & \frac{1}{\lambda} \end{array} \right)$$ (1)
with spectral parameter $\lambda$ and potentials $u_n = u(n, t)$ and $v_n = v(n, t)$. The spectral problem (1) can be viewed as a semi-discrete counterpart of the famous AKNS spectral problem [24]. By imposing time evolution,

$$
\begin{align*}
\phi_{1,n} &= N_n \phi_{1,n}, \\
\phi_{2,n} &= N_n \phi_{2,n}, \\
M_n &= \left( \begin{array}{cc} A_n & B_n \\ C_n & D_n \end{array} \right),
\end{align*}
$$

then, by zero curve equation $M_{n,t} = N_{n+1}M_n - M_nN_n$, the Ablowitz–Ladik lattice hierarchy can be derived. When $\lambda_t \neq 0$, the corresponding hierarchy is called nonisospectral Ablowitz–Ladik lattice hierarchy [25]. The spectral problem (1), coupled with different time evolution parts, has provided Lax integrabilities for many nonisospectral semi-discrete integrable systems, such as a nonisospectral semi-discrete nonlinear Schrödinger equation, nonisospectral semi-discrete mKdV equation, and so forth. Recently, starting from nonisospectral flows of the Ablowitz–Ladik lattice hierarchy, by suitable linear combinations, Zhang et al. get nonisospectral AKNS flows, which go to the continuous nonisospectral AKNS flows under a continuous limit [26].

Up to now, several works have been done to search for exact solutions for nonisospectral semi-discrete integrable systems. In [27,28], Inverse Scattering Transform was used to construct multisoliton solutions of the nonisospectral Ablowitz–Ladik hierarchy. Recently, $N$-soliton solutions to a nonisospectral Ablowitz–Ladik type equation have been investigated by using the Hirota’s method and Casorati technique [29]. In addition, based on Hankel type determinants, Chen et al. present solutions for a nonisospectral Toda lattice [30] as well as the first and second members in the nonisospectral extended Volterra lattice hierarchy [31].

In this paper, we are interested in the multisoliton solutions for the following nonisospectral semi-discrete AKNS equation [26]:

$$
\begin{align*}
\frac{u_{n,t}}{u_n} &= \frac{1}{2}(1 - u_n v_n) \left( (2n + 3)u_{n+1} - (2n - 1)u_{n-1} \right) - u_n(E - 1)^{-1}(u_{n+1}v_n - u_nv_{n+1}), \\
\frac{v_{n,t}}{v_n} &= \frac{1}{2}(1 - u_n v_n) \left( (2n + 3)v_{n+1} - (2n - 1)v_{n-1} \right) + v_n(E - 1)^{-1}(u_{n+1}v_n - u_nv_{n+1}),
\end{align*}
$$

where the dependent variables $u$ and $v$ are defined on the discrete-continuous coordinates $(n, t) \in \mathbb{Z} \times \mathbb{R}$ and shift operator is defined by $Ef_n = f_{n+1}$. The inverse of the difference operator $E - 1$ is defined by $(E - 1)^{-1}f_n = -\sum_{m=-\infty}^n f_m$ or $(E - 1)^{-1}f_n = \sum_{m=-\infty}^n f_{m+1}$. By performing $v_n = \epsilon u_n$ with $\epsilon = \pm 1$, Equation (3) reduces to the nonisospectral semi-discrete mKdV equation

$$
2u_{n,t} = (1 - \epsilon u_n^2) \left( (2n + 3)u_{n+1} - (2n - 1)u_{n-1} \right).
$$

When $\epsilon = -1$, (4) is viewed as a plus type equation, while, when $\epsilon = 1$, (4) is viewed as a minus type equation. These two equations are related with each other by transformation $u_n \rightarrow iu_n$. In what follows, we call Equation (3) by an nsd-AKNS equation, respectively and Equation (4) with the nsd-mKdV equation for short. We plan to solve the nsd-AKNS Equation (3) by applying Hirota’s method [32], which is a direct and effective approach to construct multisoliton solutions for the integrable systems. The idea is to make a transformation into new variables, so that, in these new variables, multisoliton solutions appear in a particularly simple form.

The outline of this paper is as follows: in Section 2, Hirota’s method is used to derive multisoliton solutions for the nsd-AKNS Equation (3). In Section 3, dynamics of the some soliton solutions are analyzed and illustrated by asymptotic analysis. Section 4 is devoted to the multisoliton solutions and dynamics of the nsd-mKdV Equation (4). Section 5 presents conclusions.
2. Bilinear Form and Multisoliton Solutions for nsd-AKNS Equation (3)

In this section, we transform the nsd-AKNS Equation (3) into bilinear form and present the multisoliton solutions in \( e \)-exponential series form.

Through the dependent variable transformation

\[
 u_n = \frac{g_n}{f_n}, \quad v_n = \frac{h_n}{f_n},
\]

Equation (3) is bilinearized as

\[
 D_t g_n \cdot f_n = \frac{1}{2} \left( (2n + 3)g_{n+1}f_{n-1} - (2n - 1)g_{n-1}f_{n+1} \right) - g_n z_n,
\]

\[
 D_t h_n \cdot f_n = \frac{1}{2} \left( (2n + 3)h_{n+1}f_{n-1} - (2n - 1)h_{n-1}f_{n+1} \right) + h_n z_n,
\]

\[
 f_n^2 - f_{n+1}f_{n-1} = g_n h_n, \tag{6c}
\]

\[
 g_{n+1}h_n - g_n h_{n+1} = z_{n+1}f_n - z_n f_{n+1}, \tag{6d}
\]

where \( z_n \) is an auxiliary variable and \( D \) is the well-known Hirota’s bilinear operator \([32]\) defined by

\[
 D_t^m g \cdot f = (\partial_t - \partial_{t'})^m g(t)f(t')|_{t'=1}.
\]

To obtain multisoliton solutions, we expand \( f_n, g_n, h_n \) and \( z_n \) as

\[
 f_n = 1 + \sum_{j=1}^{\infty} f_n^{(2j)} \epsilon^{2j}, \quad g_n = \sum_{j=1}^{\infty} g_n^{(2j-1)} \epsilon^{2j-1}, \quad h_n = \sum_{j=1}^{\infty} h_n^{(2j-1)} \epsilon^{2j-1}, \quad z_n = 1 + \sum_{j=1}^{\infty} z_n^{(2j)} \epsilon^{2j}.
\]

Substituting the above expansions into bilinear Equations (6), we can list the coefficients of each \( \epsilon^j \),

\[
 g_{n,t}^{(1)} = \frac{1}{2} \left( (2n + 3)g_{n+1}^{(1)} - (2n - 1)g_{n-1}^{(1)} \right) - g_n^{(1)}, \tag{7a}
\]

\[
 g_{n,t}^{(3)} = \frac{1}{2} \left( (2n + 3)g_{n+1}^{(3)} - (2n - 1)g_{n-1}^{(3)} \right) + g_n^{(3)}
 = -D_t g_n^{(1)} \cdot f_n^{(2)} + \frac{1}{2} \left( (2n + 3)g_{n+1}^{(1)} f_n^{(2)} - (2n - 1)g_{n-1}^{(1)} f_{n+1}^{(2)} \right) - g_n^{(1)} z_n^{(2)}, \tag{7b}
\]

\[
 h_{n,t}^{(1)} = \frac{1}{2} \left( (2n + 3)h_{n+1}^{(1)} - (2n - 1)h_{n-1}^{(1)} \right) + h_n^{(1)}, \tag{8a}
\]

\[
 h_{n,t}^{(3)} = \frac{1}{2} \left( (2n + 3)h_{n+1}^{(3)} - (2n - 1)h_{n-1}^{(3)} \right) - h_n^{(3)}
 = -D_t h_n^{(1)} \cdot f_n^{(2)} + \frac{1}{2} \left( (2n + 3)h_{n+1}^{(1)} f_n^{(2)} - (2n - 1)h_{n-1}^{(1)} f_{n+1}^{(2)} \right) + h_n^{(1)} z_n^{(2)}, \tag{8b}
\]

\[
 2f_n^{(2)} - (f_{n+1}^{(2)} + f_{n-1}^{(2)}) = g_n^{(1)} h_n^{(1)}, \tag{9a}
\]

\[
 2f_n^{(4)} - (f_{n+1}^{(4)} + f_{n-1}^{(4)}) = f_{n+1}^{(2)} f_n^{(2)} - f_n^{(2)} f_{n+1}^{(2)} + g_n^{(1)} h_n^{(3)} + g_n^{(3)} h_n^{(1)}, \tag{9b}
\]

\[
 \ldots
\]
Multisoliton solutions can be derived by taking

\[ g_n^{(1)} = \sum_{j=1}^{N} \omega_j e^{\xi_j}, \quad h_n^{(1)} = \sum_{j=1}^{N} \sigma_j e^{\eta_j}, \quad \xi_j = k_j n, \quad \eta_j = l_j n, \]

where \( k_j, l_j, \omega_j, \) and \( \sigma_j \) are undetermined functions with respect to \( t \).

When \( N = 1 \), it is obvious that

\[ g_n^{(1)} = \omega_1 e^{\xi_1}, \quad h_n^{(1)} = \sigma_1 e^{\eta_1}. \]

Taking (12) into (7a) and (8a) implies the time evolution relations

\[
\begin{align*}
  k_{1,t} &= 2 \sinh k_1, & l_{1,t} &= 2 \sinh l_1, \\
  \omega_{1,t} &= (2 \cosh k_1 + \sinh k_1 - 1) \omega_1, & \sigma_{1,t} &= (2 \cosh l_1 + \sinh l_1 + 1) \sigma_1.
\end{align*}
\]

Moreover, from Equations (9a) and (10a), we can successively solve out

\[ f_n^{(2)} = -w_1 \sigma_1 e^{\xi_1 + \eta_1 + \theta_{11}}, \quad z_n^{(2)} = -w_1 \sigma_1 (1 - 2 \cosh k_1 + 2 \cosh l_1) e^{\xi_1 + \eta_1 + \theta_{11}}, \]

with

\[ e^{\theta_{11}} = \frac{1}{4} \cosh^2 \frac{k_1 + l_1}{2} \quad \text{or} \quad \theta_{11} = -2 \ln \left( \frac{2 \sinh \frac{k_1 + l_1}{2}}{2} \right). \]

In addition, Equations (7b)–(10b) admit \( g_n^{(2j-1)} = h_n^{(2j-1)} = f_n^{(2j)} = z_n^{(2j)} = 0 \) \((j \geq 2)\). Thus, from transformation (5), we obtain 1-soliton solution

\[ u_n = \frac{\omega_1 e^{\xi_1}}{1 - w_1 \sigma_1 e^{\xi_1 + \eta_1 + \theta_{11}}}, \quad v_n = \frac{\sigma_1 e^{\eta_1}}{1 - w_1 \sigma_1 e^{\xi_1 + \eta_1 + \theta_{11}}}, \]

where we have taken \( \epsilon = 1 \).

When \( N = 2 \), Equation (11) yields

\[ g_n^{(1)} = \omega_1 e^{\xi_1} + \omega_2 e^{\xi_2}, \quad h_n^{(1)} = \sigma_1 e^{\eta_1} + \sigma_2 e^{\eta_2}. \]

Moreover, one can get

\[
\begin{align*}
  f_n^{(2)} &= -\sum_{j=1}^{2} \sum_{j=1}^{2} w_j \sigma_j e^{\xi_j + \eta_j + \theta_{ij}}, \\
  z_n^{(2)} &= -\sum_{j=1}^{2} \sum_{j=1}^{2} w_j \sigma_j (1 - 2 \cosh k_j + 2 \cosh l_j) e^{\xi_j + \eta_j + \theta_{ij}}, \\
  g_n^{(3)} &= -w_1 w_2 e^{\xi_1 + \xi_2 + \eta_1 + \eta_2} \sum_{j=1}^{2} \sigma_j e^{\theta_{ij} + \theta_{ij}}, \quad h_n^{(3)} = -w_1 w_2 \sigma_1 e^{\eta_1 + \eta_2} \sum_{j=1}^{2} w_j \sigma_j e^{\theta_{ij} + \theta_{ij}}, \\
  f_n^{(4)} &= w_1 w_2 \sigma_1 \sigma_2 e^{\xi_1 + \xi_2 + \eta_1 + \eta_2 + \theta_{12} + \theta_{12} + \theta_{12} + \theta_{12} + \theta_{12} + \theta_{12} + \theta_{12} + \theta_{12}, \\
  z_n^{(4)} &= w_1 w_2 \sigma_1 \sigma_2 (1 - 2 \cosh k_1 + 2 \cosh l_1 + 2 \cosh l_2)
  \times e^{\xi_1 + \xi_2 + \eta_1 + \eta_2 + \theta_{12} + \theta_{12} + \theta_{12} + \theta_{12} + \theta_{12} + \theta_{12}}.
\end{align*}
\]
and \( g_n^{(2j-1)} = h_n^{(2j-1)} = f_n^{(2j)} = z_n^{(2j)} = 0 \) \((j \geq 3)\), where

\[
\begin{align*}
  k_{j,\ell} &= 2 \sinh k_j, \quad \omega_{j,\ell} = (2 \cosh k_j + \sinh k_j - 1) \omega_j, \quad j = 1, 2, \quad (19a) \\
  l_{j,\ell} &= 2 \sinh l_j, \quad \sigma_{j,\ell} = (2 \cosh l_j + \sinh l_j + 1) \sigma_j, \quad j = 1, 2, \quad (19b) \\
  e^{\theta_j} &= \frac{1}{4} \cosh^2 \frac{k_1 + l_1}{2}, \quad e^{\theta_{12}} = 4 \sinh^2 \frac{k_1 - k_2}{2}, \quad e^{\theta_{12}} = 4 \sinh^2 \frac{l_1 - l_2}{2}, \quad i, j = 1, 2. \quad (19c)
\end{align*}
\]

In this case, the truncated series

\[ f_n = 1 + f_n^{(2)} + f_n^{(4)}, \quad g_n = g_n^{(1)} + g_n^{(3)}, \quad h_n = h_n^{(1)} + h_n^{(3)} \quad (20) \]

provide a 2-soliton solutions for Equation (3) through transformation (5), where we have taken \( \varepsilon = 1 \).

In general, the \( N \)-soliton solutions can be given by (5) with

\[
\begin{align*}
  f_{n,N} &= \sum_{k=0}^{N} A_1(k) \exp \left( \frac{\sum_{j=1}^{2N} k_j (\xi_j + \ln \omega_j)}{2} + \frac{\sum_{1 \leq j < s} k_j k_s (\theta_j + \theta_s)}{2} \right), \quad (21a) \\
  g_{n,N} &= \sum_{k=0}^{N} A_2(k) \exp \left( \frac{\sum_{j=1}^{2N} k_j (\xi_j + \ln \omega_j)}{2} + \frac{\sum_{1 \leq j < s} k_j k_s (\sigma_j + \sigma_s)}{2} \right), \quad (21b) \\
  h_{n,N} &= \sum_{k=0}^{N} A_3(k) \exp \left( \frac{\sum_{j=1}^{2N} k_j (\xi_j + \ln \omega_j)}{2} + \frac{\sum_{1 \leq j < s} k_j k_s (\theta_j + \sigma_j)}{2} \right), \quad (21c) \\
  z_{n,N} &= \sum_{k=0}^{N} A_4(k) \left( 1 - 2 \sum_{j=1}^{N} k_j \cosh k_j + 2 \sum_{j=1}^{N} k_j \cosh k_{N+j} \right) \times \exp \left( \frac{\sum_{j=1}^{2N} k_j (\xi_j + \ln \omega_j)}{2} + \frac{\sum_{1 \leq j < s} k_j k_s (\theta_j + \sigma_j)}{2} \right), \quad (21d)
\end{align*}
\]

where \( \xi_{N+j} = \eta_j, \quad \omega_{N+j} = \sigma_j, \quad k_{N+j} = l_j \) and

\[
\begin{align*}
  k_{j,\ell} &= 2 \sinh k_j, \quad \omega_{j,\ell} = (2 \cosh k_j + \sinh k_j - 1) \omega_j, \quad (j = 1, 2, \ldots, N), \quad (22a) \\
  l_{j,\ell} &= 2 \sinh l_j, \quad \sigma_{j,\ell} = (2 \cosh l_j + \sinh l_j + 1) \sigma_j, \quad (j = 1, 2, \ldots, N), \quad (22b) \\
  e^{\theta_j(N+s)} &= -\frac{1}{4} \cosh^2 \frac{k_j + l_s}{2}, \quad (j, s = 1, 2, \ldots, N), \quad (22c) \\
  e^{\theta_{js}} &= -4 \sinh^2 \frac{k_j - k_s}{2}, \quad e^{\theta_{(N+j)(N+s)}} = -4 \sinh^2 \frac{l_j - l_s}{2}, \quad (1 \leq j < s \leq N), \quad (22d)
\end{align*}
\]

and \( A_1(k), A_2(k) \) and \( A_3(k) \) take over all possible combinations of \( k_j = 0, 1 \), \((j = 1, 2, \ldots, 2N)\) and satisfy the following constraints, respectively,

\[
\sum_{j=1}^{N} k_j = \sum_{j=1}^{N} k_{N+j}, \quad \sum_{j=1}^{N} k_j = 1 + \sum_{j=1}^{N} k_{N+j}, \quad 1 + \sum_{j=1}^{N} k_j = \sum_{j=1}^{N} k_{N+j}. \quad (23)
\]

3. Dynamics of 1-Soliton Solution and 2-Soliton Solutions

To proceed, we consider the expressions of functions \( k_j, l_j, \omega_j \) and \( \sigma_j \). For the first equation in (22a), one can find that \(-k_j \) is a solution if \( k_j \) is a solution. By solving the differential Equations (22a) and (22b), we immediately have

\[
\begin{align*}
  k_j &= \ln \left( \frac{1 + e^{2l_j + \eta_j}}{1 - e^{2l_j + \eta_j}} \right), \quad \omega_j = \frac{\alpha_j e^{d + 2c_j}}{(1 - e^{2l_j + \eta_j}) \sqrt{1 - e^{4l_j + \eta_j}}}, \quad (24a) \\
  l_j &= \ln \left( \frac{1 + e^{2l_j + d_j}}{1 - e^{2l_j + d_j}} \right), \quad \sigma_j = \frac{\beta_j e^{3 + 2d_j}}{(1 - e^{2l_j + d_j}) \sqrt{1 - e^{4l_j + d_j}}}, \quad (24b)
\end{align*}
\]
where $c_j, d_j \in \mathbb{R}$ and $\alpha_j, \beta_j \in \mathbb{R} \setminus \{0\}$ ($j = 1, 2, \ldots, N$) are real constants. Here, $\alpha_j$ and $\beta_j$ play the role of phase shifts. One has to assume that $t < \min\{-\frac{c_j}{2}, -\frac{d_j}{2}\}$ in order to guarantee the real properties of $k_j l_j, \omega_j$ and $\sigma_j$. Under the condition $t < \min\{-\frac{c_j}{2}, -\frac{d_j}{2}\}$, one knows that $k_j > 0$ and $l_j > 0$. In addition, one can recognize that $k_j, \omega_j \to +\infty$ as $t \to -\frac{c_j}{2}$ and $k_j, \omega_j \to 0$ as $t \to -\infty$. Similar results also hold for functions $l_j$ and $\sigma_j$.

We now consider the dynamical properties of 1-soliton solution (16) and 2-soliton solutions (20). Without loss of generality, we only focus on the dynamics of $u_n$, since $\psi_n$ behaves similarly. We first pay attention to $u_n$ in (16). For the sake of brevity, we denote $\omega_1 = \text{sgn}(\alpha_1)e^{\theta_1}$ and $\sigma_1 = \text{sgn}(\beta_1)e^{\theta_1}$, i.e.,

\begin{align}
\rho_1 &= t + 2c_1 - \ln \left( (1 - e^{2t + c_1}) \sqrt{1 - e^{4t + 2c_1}} \right) + \ln |\alpha_1|, \quad \text{(25a)} \\
\phi_1 &= 3t + 2d_1 - \ln \left( (1 - e^{2t + d_1}) \sqrt{1 - e^{4t + 2d_1}} \right) + \ln |\beta_1|. \quad \text{(25b)}
\end{align}

Then, we can rewrite $u_n$ (16) as

\begin{equation}
\boxed{u_n = \begin{cases} 
\frac{\text{sgn}(\alpha_1)}{2} e^{\frac{k_1 + \rho_1 + \phi_1 + \theta_{11}}{2}} \text{sech} \frac{k_1 + \rho_1 + \phi_1 + \theta_{11}}{2}, & \alpha_1\beta_1 < 0, \\
\frac{\text{sgn}(\alpha_1)}{2} e^{\frac{k_1 + \rho_1 + \phi_1 + \theta_{11}}{2}} \text{csch} \frac{k_1 + \rho_1 + \phi_1 + \theta_{11}}{2}, & \alpha_1\beta_1 > 0.
\end{cases}} \quad (26)
\end{equation}

For $\alpha_1\beta_1 < 0$, $u_n$ is nonsingular and provides a solitary wave because of the soliton part \( \text{sech} \frac{k_1 + \rho_1 + \phi_1 + \theta_{11}}{2} \). Moreover, the part \( \frac{\text{sgn}(\alpha_1)}{2} e^{\frac{k_1 + \rho_1 + \phi_1 + \theta_{11}}{2}} \) defines a time varying amplitude due to the nonisospectral effect. In terms of the sign of parameter $\alpha_1$, the amplitude of the wave can be positive or negative, which corresponds to soliton or anti-soliton. The top trace is given by the point trace

\begin{equation}
n(t) = \frac{-\rho_1 + \phi_1 + \theta_{11}}{k_1 + l_1}. \quad \text{(27)}
\end{equation}

Furthermore,

\begin{equation}
\frac{dn(t)}{dt} = (\text{csch}(2t + c_1) + \text{csch}(2t + d_1)) \frac{k_1 + l_1 - 2(\rho_1 + \phi_1 + \theta_{11})}{(k_1 + l_1)^2} \quad \text{(28)}
\end{equation}

implies the time-varying velocity of the wave. We illustrate this soliton in Figure 1.

![Figure 1. Shape and motion of 1-soliton given by (26) for $c_1 = 1$ and $d_1 = 1.35$. (a) soliton solution for $\alpha_1 = 1$ and $\beta_1 = -1$; (b) anti-soliton solution for $\alpha_1 = -1$ and $\beta_1 = 1$; (c) waves in blue (solid line) and yellow (dotted line) stand for plot (a) at $t = -0.68$ and $t = -0.75$, respectively.](image)
It is worth noting that this soliton is not in a symmetric shape when $c_1 \neq d_1$. This is because, for any fixed $t$,

$$u_n \sim \begin{cases} O(\text{sgn}(-\beta_1) e^{-1n}), & n \to +\infty, \\ O(\text{sgn}(\alpha_1) e^{1n}), & n \to -\infty. \end{cases} \quad (29)$$

This phenomenon is very similar to the solitons’ behavior of negative order AKNS equation [33].

For $\alpha_1 \beta_1 > 0$, $u_n$ has singularity along the point trace (27) and the velocity is still expressed by (28). We depict this solution in Figure 2.

![Figure 2](image.png)

Figure 2. $u_n$ given by (26) for $c_1 = 1.5, d_1 = 0.01, \alpha_1 = 1$ and $\beta_1 = 1$. (a) shape and motion; (b) a contour plot of (a) with range $n \in [4, 4]$ and $t \in [-1.2, -0.8]$; (c) waves in blue (solid line) and yellow (dotted line) stand for plot (a) at $t = -1.2$ and $t = -1$, respectively.

We next focus on $u_n$ given by (20). Since $k_i, l_j, \omega_j$ and $\sigma_j$ ($j = 1, 2$) are functions of $t$, it is intractable to make asymptotic analysis as usual [34]. Here, we only depict $u_n$ in Figure 3.

![Figure 3](image.png)

Figure 3. 2-soliton solutions for $u_n$ with $c_1 = 0.05, c_2 = 0.06, d_1 = 1, d_2 = 1.2, \alpha_1 = \alpha_2 = 0.1$ and $\beta_1 = \beta_2 = -0.1$. (a) shape and motion; (b) waves in blue (solid line) and yellow (dotted line) stand for plot (a) at $t = -0.82$ and $t = -0.83$, respectively.

4. Multisoliton Solutions for nsd-mKdV Equation (4) and Dynamics

The nsd-mKdV Equation (4) can be derived from the nsd-AKNS Equation (3) by imposing constraint $\text{v}_n = \epsilon u_n$, while it is difficult to derive its soliton solutions by imposing constraint $h_n = \epsilon g_n$ since there is no linear relation between $\omega_j$ and $\sigma_j$. This can be demonstrated by the 1-soliton solution.
To derive the multisoliton solutions for the nsd-mKdV Equation (4), we take the dependent
variable transformation
\[ u_n = \frac{g_n}{f_n}, \tag{30} \]
which yields the bilinear form for Equation (4)
\[
\begin{align*}
D_1 g_n \cdot f_n &= \frac{1}{2} \left( (2n + 3)g_{n+1}f_{n-1} - (2n - 1)g_{n-1}f_{n+1} \right), \\
\left( f_n^2 - f_{n+1}f_{n-1} \right) &= \epsilon g_n^2.
\end{align*}
\tag{31a, 31b}
\]

Analogous to the previous analysis, one can successively derive 1-soliton, 2-soliton, and \( N \)-soliton solutions. The \( N \)-soliton solutions are given by
\[
\begin{align*}
    f_{n,N} &= \sum_{k=0}^{n} A_1(\kappa) \exp \left( \sum_{j=1}^{2N} \kappa_j (\xi_j + \ln \omega_j) + \sum_{1 \leq j < s \leq N} \kappa_j \kappa_s \theta_{js} \right), \\
    g_{n,N} &= \sum_{k=0}^{n} A_2(\kappa) \exp \left( \sum_{j=1}^{2N} \kappa_j (\xi_j + \ln \omega_j) + \sum_{1 \leq j < s \leq N} \kappa_j \kappa_s \theta_{js} \right),
\end{align*}
\tag{32a, 32b}
\]
where \( \xi_{n+j} = \xi_j \), \( \omega_{n+j} = \omega_j \) and
\[
\begin{align*}
    k_{js} &= 2 \sinh k_j, \quad \omega_{j,s} = (2 \cosh k_j + \sinh k_j) \omega_j, \quad (j = 1, 2, \ldots, N), \\
    e^{\theta_{j}(N+s)} &= -\frac{e}{4} \csc \frac{k_j + k_s}{2}, \quad (j, s = 1, 2, \ldots, N), \\
    e^{\theta_{j}(N+j)(N+s)} &= e^{\theta_{js}} = -4 \sinh^2 \frac{k_j - k_s}{2}, \quad (1 \leq j < s \leq N),
\end{align*}
\tag{33a, 33b, 33c}
\]
and \( A_1(\kappa) \) and \( A_2(\kappa) \) are defined by (23).

Equation (33a) gives rise to \( k_j \) (24a) and
\[
\omega_j = \frac{\kappa_j e^{2(t+c_j)}}{(1 - e^{2(t+c_j)}) \sqrt{1 - e^{4t+2c_j}}}, \quad c_j \in \mathbb{R}, \quad \alpha_j \in \mathbb{R} \setminus \{0\}. \tag{34}
\]
Similarly, it is necessary to take \( t < -\frac{c_j}{2} \) (\( j = 1, 2, \ldots, N \)). In terms of (25a), the 1-soliton solution can be expressed as
\[
\begin{align*}
    u_n = \begin{cases}
        \sinh k_1 \sech(\xi_1 + \rho_1 + t - \ln(2 \sinh k_1)), & \epsilon = -1, \\
        \sinh k_1 \csch(\xi_1 + \rho_1 + t - \ln(2 \sinh k_1)), & \epsilon = 1.
    \end{cases}
\end{align*}
\tag{35}
\]
Obviously, solution (35) is nonsingular when \( \epsilon = -1 \) and has singularity when \( \epsilon = 1 \). For \( \epsilon = -1 \), (35) provides a solitary wave with time varying amplitude \( \sinh k_1 \). The top trace is given by the point trace
\[
n(t) = \frac{1}{k_1} (\ln(2 \sinh k_1) - \rho_1 - t), \tag{36}
\]
and the velocity is
\[
\frac{dn(t)}{dt} = \csch(2t + c_1) \frac{k_1 - 2(\rho_1 + t - \ln(2 \sinh k_1))}{k_1^2}. \tag{37}
\]
For \( \epsilon = 1 \), (35) has singularity along with point trace (36) and the velocity is still given by (37).
Figures 4a, c depict the solution (35) with \( \epsilon = -1 \) and \( \epsilon = 1 \), respectively. Figure 4b shows the symmetric shape and the Gaussian-distributed structure with respect to \( n \) for a given \( t \). Figure 4d exhibits the movement of the singularity point.

The 2-soliton solutions for nsd-mKdV Equation (4) read

\[
 u_n = \frac{g_n}{f_n},
\]

in which

\[
 g_n = \omega_1 e^{\xi_1} + \omega_2 e^{\xi_2} - \epsilon \alpha_1 \omega_2 e^{2\xi_1} + \xi_1 + \xi_2 + \xi_{12} + \mu_{12},
\]

\[
 f_n = 1 - \epsilon \alpha_1^2 e^{2\xi_1} + \xi_1 - 2 \epsilon \alpha_1 \omega_2 e^{\xi_1} + \xi_2 - \epsilon \alpha_2^2 e^{2\xi_2} + \xi_2 + \alpha_1^2 \omega_2 e^{2\xi_1} + 2\xi_2 + \xi_1 + 2\xi_{12} + \xi_2 + 2\mu_{12},
\]

where \( k_j, \omega_j (j = 1, 2) \) and \( \epsilon \mu_{12} \) are given by (24a), (34) and (19c), respectively. In addition,

\[
 \xi_1 = -2 \ln(2 \sinh k_1), \quad \xi_2 = -2 \ln(2 \sinh k_2), \quad \xi_{12} = -2 \ln \left( \frac{2 \sinh \left( \frac{k_1 + k_2}{2} \right)}{2} \right).
\]

When \( \epsilon = -1 \), solution (38) is nonsingular. Figures 5 and 6 depict soliton–soliton interaction, respectively, and soliton–anti-soliton interaction of the nsd-mKdV Equation (4) with \( \epsilon = -1 \).
Figure 6. $u_n$ given by (38) with $\epsilon = -1$ for $c_1 = 1$, $c_2 = 0.6$, $\alpha_1 = 1$ and $\alpha_2 = -1$. (a) shape and motion; (b) a contour plot of (a) with range $n \in [-10, 20]$ and $t \in [-1.8, -0.8]$; (c) waves in blue (solid line) and yellow (dotted line) stand for plot (a) at $t = -1$ and $t = -0.8$, respectively.

When $\epsilon = 1$, solution (38) has singularity, which appears at the point trace

\[
\{(n, t)|1 + \omega_1^2 \omega_2^2 e^{2(n+1)} + \omega_1^2 e^{2(n+1)} + 2 \omega_1 \omega_2 e^{n+1} + \omega_2^2 e^{n+1} + \omega_1^2 \omega_2^2 e^{n+1} + \omega_2^2 e^{n+1}\}. 
\]

Figures 7 and 8 depict soliton–soliton interaction, respectively, soliton–anti-soliton interaction of the nsd-mKdV Equation (4) with $\epsilon = 1$.

Figure 7. $u_n$ given by (38) with $\epsilon = 1$ for $c_1 = 1$, $c_2 = 0.8$ and $\alpha_1 = \alpha_2 = 1$. (a) shape and motion; (b) a contour plot of (a) with range $n \in [-5, 15]$ and $t \in [-1.2, -0.6]$; (c) waves in blue (solid line) and yellow (dotted line) stand for plot (a) at $t = -1$ and $t = -0.8$, respectively.

Figure 8. $u_n$ given by (38) with $\epsilon = 1$ for $c_1 = 1$, $c_2 = 0.8$, $\alpha_1 = 1$ and $\alpha_2 = -1$. (a) shape and motion; (b) a contour plot of (a) with range $n \in [-10, 20]$ and $t \in [-1.8, -0.8]$; (c) waves in blue (solid line) and yellow (dotted line) stand for plot (a) at $t = -1$ and $t = -1.2$, respectively.
5. Conclusions

In this paper, we have discussed the multisoliton solutions for nsd-AKNS Equation (3) by applying the Hirota’s method. Dynamical behaviors for some obtained soliton solutions are identified through asymptotic analysis. In terms of the sign of product $\alpha_1 \beta_1$, we present two kinds of 1-soliton solution for nsd-AKNS Equation (3). One is nonsingular and the other is singular. Nonisospectral effects on amplitude, top trace/singularity point trace, and velocity of these two 1-soliton solutions are analyzed emphatically. The nonsingular 1-soliton exhibits a nonsymmetric structure when $c_1 \neq d_1$. Since functions $k_j$, $l_j$, $\omega_j$ and $\sigma_j$ ($j = 1, 2$) depend on $t$, the asymptotic analysis on 2-soliton solutions to nsd-AKNS Equation (3) is not given in the present paper. With the help of Hirota’s method, we also present the multisoliton solutions to the nsd-mKdV Equation (4). For the plus type nsd-mKdV Equation (4) ($\epsilon = -1$), the resulting multisoliton solutions are nonsingular, while, for the minus type nsd-mKdV Equation (4) ($\epsilon = 1$), the resulting multisoliton solutions are singular. Although the multisoliton solutions to the nsd-mKdV Equation (4) have appeared in Refs. [27,28], there is little description on the dynamics of 1-soliton solutions. To understand the 2-soliton solutions better, we give some figures. The detailed discussion on the applications of the graphs will be given in the near future. We hope that the results given in this article might attract some attention in the areas of semi-discrete integrable system and nonisospectral integrable system. Particularly, this work will be beneficial to developing other approaches to study the multi-component nonisospectral semi-discrete integrable system.

Funding: This project is supported by the Natural Science Foundation of Zhejiang Province (Nos. LY18A010033, LY17A010024) and the National Natural Science Foundation of China (No. 11401529).

Acknowledgments: We are very grateful to the reviewers for the invaluable and expert comments.

Conflicts of Interest: The author declares no conflict of interest.

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