Characterisations for split graphs and unbalanced split graphs

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Abstract

We introduce a characterisation for split graphs by using edge contraction. First, we use it to prove that any \((2K_2, \text{claw})\)-free graph with \(\alpha(G) \geq 3\) is a split graph. Next, we apply it to characterise any pseudo-split graph. Finally, by using edge contraction again, we characterise unbalanced split graphs, which we use to characterise Nordhaus–Gaddum graphs.

1. Introduction

Given graph \(G\), we denote its vertex set by \(V(G)\) (\(V\) in short) if the underlying graph is clear. Similarly, the edge set of \(G\) is \(E(G)\) (\(E\) in short). For vertex \(v \in V\), the set of vertices adjacent to \(v\) in \(G\) is denoted by \(N(v)\). Further, for a set of vertices, \(S \subseteq V\), the set of vertices adjacent to \(v\) in \(S\) is denoted by \(N_S(v)\) and defined as \(N_S(v) = N(v) \cap S\). The neighbour of \(S\), denoted by \(N(S)\), is \(\bigcup_{v \in S} N(v) \setminus S\). A vertex set, \(S\), is called dominating if \(N(S) \cup S = V\).

We denote a cycle graph, a complete graph, and an edgeless graph by \(C_n\), \(K_n\), and \(E_n\), respectively, where \(n\) is the order of the graph. Graph \(G[S]\) has \(S\) as the vertex set, and two vertices are adjacent if and only if they are adjacent in \(G\). \(G[S]\) is called the vertex-induced subgraph of \(G\). A clique is a vertex set that induces a complete subgraph in \(G\). A set of mutually nonadjacent vertices is independent. Furthermore, if two edges have no common vertex, they are called independent. We denote a graph with four vertices and two independent edges by \(2K_2\).

Graph \(G\) is called \(H\)-free if every vertex-induced (induced in short) subgraph of \(G\) is not isomorphic to graph \(H\). Furthermore, \(G\) is \((H_1, H_2, \ldots, H_k)\)-free if every vertex-induced subgraph of \(G\) is not isomorphic to any graph \(H_i\), where \(1 \leq i \leq k\). For two disjoint graphs, \(G\) and \(H\), the graph constructed from \(G \cup H\) by adding edges from any vertex in \(G\) to any vertex in \(H\) can be denoted by \(G \lor H\).

*This document is the results of the research project funded by the European Social Fund (ESF).
By identifying two adjacent vertices, \( u \) and \( v \), or contracting the edge between them, we obtain a graph constructed from \( G \) by adding an edge between \( u \) and every neighbour of \( v \). Then, \( v \), along with all loops, is deleted, followed by the deletion of all but one edge that forms multiple edges between any two vertices. We denote the graph obtained by identifying \( u \) and \( v \) as \( G/uv \). If \( e \) is the edge between \( u \) and \( v \), then we denote graph \( G/uv \) by \( G/e \). In this study, we assume that any graph is a connected simple graph, unless otherwise stated. For any other notion, we follow [2]. Long proofs are divided into small claims in which only the ambiguous aspects are proven, for example, Lemmas 7, Lemma 8 and Theorem 12.

The KS-partition of a graph is a partition of the vertex set, where \( K \) is a clique and \( S \) is an independent set. Graph \( G \) is called split if it admits a KS-partition. Split graphs were introduced in [8] and characterised as follows:

**Theorem 1.** [8] Graph \( G \) is split if and only if \( G \) is \((2K_2, C_5, C_4)\)-free.

In addition, split graphs were characterised in [8] as chordal graphs whose complements are also chordal. Furthermore, these graphs were characterised by their degree sequences in [10].

In Section 2, we present the characterisation of the split graphs. Then, in Section 3, we use it to prove that any \((2K_2, \text{claw})\)-free graph with \( \alpha(G) \geq 3 \) is a split graph. Finally, in Section 4, we characterise unbalanced split graphs by using edge contraction, and we use this to characterise Nordhaus–Gaddum graphs.

### 2. Split graphs characterisation

**Proposition 2.** Let \( G \) be a graph, \( C \subseteq V(G) \) with \( u \in C \) and \( v \notin C \), where \( u \) and \( v \) are adjacent. If \( N_C(v) \setminus \{u\} \subseteq N_C(u) \), then \( G[C] \) is isomorphic to \( G/uv[C] \).

**Proposition 3.** Let \( G \) be a graph and \( C \subseteq V(G) \). If \( u \) and \( v \notin C \), where \( u \) and \( v \) are adjacent, then \( G[C] \) is isomorphic to \( G/uv[C] \).

**Proposition 4.** Let \( G \) be a graph and \( C \subseteq V(G) \). If \( C \) is not dominant, then there is an edge, \( e \in E(G) \), such that \( G[C] \) is isomorphic to \( G/e[C] \).

**Proof.** If \( C \) is not dominant, then there is a vertex, \( v \notin C \), that is not adjacent to any vertex in \( C \). Because \( G \) is connected, there is a vertex, \( u \notin C \), such that \( u \) is adjacent to \( v \). Hence, according to Proposition 3, there exists an edge, \( e \in E(G) \), such that \( G[C] \) is isomorphic to \( G/e[C] \). \( \square \)

**Proposition 5.** If \( G \) is a graph that is isomorphic to \( C_n \) with \( n \geq 4 \), then \( G/e \) is isomorphic to \( C_{n-1} \) for any edge, \( e \in E \).

**Proposition 6.** If \( G \) is a graph that is isomorphic to \( K_n \) with \( n \geq 2 \), then \( G/e \) is isomorphic to \( K_{n-1} \) for any edge, \( e \in E \).
Before we characterise the split graphs, we present a list of special graphs in Figure 1. These graphs have an interesting property; although they are not split graphs, we can construct one if we contract an edge in any of them. We prove that these are the only graphs that possess such a property.

$$H_1^l (K_{2,l}, l \geq 2) \quad H_2 (W_4) \quad H_3 (E_2 \vee C_4)$$

$$H_4 (2K_2) \quad H_5 (P_5) \quad H_6 (\text{hammer}) \quad H_7 (\text{butterfly})$$

Figure 1

**Lemma 7.** Given graph $G$, if $G$ has an induced $C_4$, then either $G$ is isomorphic to one of $\{H_1^l : l \geq 2\}$, $H_2$, and $H_3$ or there is an edge, $e \in E(G)$, such that $G/e$ has an induced $C_4$.

Proof. Let $C = \{p, q, r, s\} \subseteq V(G)$ that induces $C_4$ with edges $pq, qr, rs$, and $sp$.

Based on Proposition 3, we observe the following claim.

**Claim 7.1.** If there are two adjacent vertices not in $C$, then there is an edge, $e \in E(G)$, such that $G/e$ has an induced $C_4$.

By applying Proposition 4, the following claim is achieved.

**Claim 7.2.** If $C$ is not dominant, then there is an edge, $e \in E(G)$, such that $G/e$ has an induced $C_4$.

By using Proposition 2, we obtain the following claim.

**Claim 7.3.** If $v \notin C$ such that $|N_C(v)| = 1$ or $v$ is adjacent to exactly two adjacent vertices in $C$ or $|N_C(v)| = 3$, then there is an edge, $e \in E(G)$, such that $G/e$ has an induced $C_4$. 

3
Claim 7.4. If for every \( v \notin C \), \( v \) is adjacent to exactly the same two nonadjacent vertices in \( C \), then \( G \) is isomorphic to \( H_4^1 \), where \( l = |V(G) \setminus C| + 2 \).

Claim 7.5. If \( |V(G) \setminus C| = 1 \), and the vertex not in \( C \) is adjacent to all vertices in \( C \), then \( G \) is isomorphic to \( H_2 \).

Claim 7.6. If \( |V(G) \setminus C| = 2 \), and each of the two vertices not in \( C \) is adjacent to all vertices in \( C \), then \( G \) is isomorphic to \( H_3 \).

Claim 7.7. If \( |V(G) \setminus C| = 3 \) and each of the three vertices in \( V(G) \setminus C \) is adjacent to all vertices in \( C \), then there is an edge, \( e \in E(G) \), such that \( G/e \) has an induced \( C_4 \).

Proof. If \( \{u, v, w\} \subseteq V(G) \setminus C \) such that \( \{|N_C(u)| = |N_C(v)| = |N_C(w)| = 4\)\), then \( \{p, r, v\} \) induces \( C_4 \) in \( G/uq \). (□)

Claim 7.8. If there are two vertices, \( u \) and \( v \notin C \), where \( |N_C(u)| = |N_C(v)| = 2 \) and \( N_C(u) \cap N_C(v) = \{\} \), then there is an edge, \( e \in E(G) \), such that \( G/e \) has an induced \( C_4 \).

Proof. If \( v \) is adjacent to exactly two adjacent vertices in \( C \), then according to Claim 7.3, there is an \( e \in E(G) \) such that \( G/e \) has an induced \( C_4 \). Therefore, we assume that \( G \) has no vertex that is adjacent to exactly two adjacent vertices in \( C \). Let \( v \notin C \) be adjacent to exactly two nonadjacent vertices in \( C \), say \( p, r \). If there is a vertex, \( u \notin C \), that is adjacent to two different nonadjacent vertices in \( C \), say \( q, s \), then the vertex set, \( \{p, r, s\} \), induces \( C_4 \) in \( G/uq \). (□)

The following claim is proved in a similar way to the proof of Claim 7.8.

Claim 7.9. If there are two vertices, \( u \) and \( v \notin C \), where \( N_C(v) = 2 \) and \( N_C(u) = 4 \), then there is an edge, \( e \in E(G) \), such that \( G/e \) has an induced \( C_4 \).

Based on Claims 7.1, . . . , 7.9, the proof is complete. □

Lemma 8. Given graph \( G \), if \( G \) has an induced \( 2K_2 \), then either \( G \) is isomorphic to one of \( 2K_2 \), \( P_5 \), Hammer, Butterfly, and \( C_6 \) or there is an edge, \( e \in E(G) \), such that \( G/e \) has an induced \( 2K_2 \) or \( C_4 \).

Proof. Let \( C = \{p, q, r, s\} \subseteq V(G) \), where \( pq, rs \in E(G) \), and \( C \) induces \( 2K_2 \) in \( G \).

Based on Proposition 3, we observe the following claim.

Claim 8.1. If there are two adjacent vertices not in \( C \), then there is an edge, \( e \in E(G) \), such that \( G/e \) has an induced \( 2K_2 \).

By applying Proposition 4, the following claim is achieved.

Claim 8.2. If \( C \) is not dominant, then there is an edge, \( e \in E(G) \), such that \( G/e \) has an induced \( 2K_2 \).

By using Proposition 2, we obtain the following claim.
Claim 8.3. If $v \notin C$, such that $N_C(v) = 1$ or $v$ is adjacent to exactly two adjacent vertices in $C$, then there is an edge, $e \in E(G)$, such that $G/e$ has an induced $2K_2$.

Claim 8.4. If $V = C$, then $G$ is isomorphic to $2K_2$.

Claim 8.5. If $|V \setminus C| = 1$ and the vertex in $V \setminus C$ is adjacent to exactly two nonadjacent vertices in $C$, then $G$ is isomorphic to $P_5$.

Claim 8.6. If $|V \setminus C| = 1$ and the vertex not in $C$ is adjacent to exactly three vertices in $C$, then $G$ is isomorphic to Hammer.

Claim 8.7. If $|V \setminus C| = 1$ and the vertex not in $C$ is adjacent to all vertices in $C$, then $G$ is isomorphic to Butterfly.

Claim 8.8. If there are two vertices, $u$ and $v \notin C$, such that $|N_C(u)| = |N_C(v)| = 2$, then $G$ is isomorphic to $C_6$ or there is an edge, $e \in E(G)$, such that $G/e$ has an induced $2K_2$ or $C_4$.

Proof. If $u$ or $v$ is adjacent to two adjacent vertices in $C$, then according to Claim 8.3, there is an edge, $e \in E(G)$, such that $G/e$ has an induced $2K_2$. Therefore, we assume that neither $u$ nor $v$ is adjacent to two adjacent vertices in $C$. If $N_C(u) = N_C(v)$, say $N_C(u) = \{p, r\}$, then vertex set $\{p, r, u, v\}$ induces $C_4$ in $G$. According to Lemma 7, there is an edge, $e \in E(G)$, such that $G/e$ has an induced $C_4$. If $|N_C(u) \cap N_C(v)| = 1$, then there is a vertex set in $G$ that induces $C_5$. Thus, according to Proposition 5, there is an edge, $e \in E(G)$, such that $G/e$ has an induced $C_4$. If $N_C(u) \cap N_C(v) = \{\} = \{p, r\}$ and $N_C(v) = \{q, s\}$, then vertex set $\{p, q, r, s, u, v\}$ induces $C_6$ in $G$. If $w \notin C$, where $|N_C(w)|$ equals either 3 or 4, then $G$ has an induced $C_4$. Thus, $G$ is isomorphic to $C_6$. 

Claim 8.9. If there are two vertices, $u$ and $v \notin C$, such that $|N_C(u)| = |N_C(v)| = 3$, then there is an edge, $e \in E(G)$, such that $G/e$ has an induced $C_4$.

Proof. For any two vertices, $u$ and $v \notin C$, such that $|N_C(u)| = |N_C(v)| = 3$, $G$ has an induced $C_4$ or $C_5$. Thus, according to Lemma 7 and Proposition 5, there is an edge, $e \in E(G)$, such that $G/e$ has an induced $C_4$. 

Claim 8.10. If there are two vertices, $u$ and $v \notin C$, such that $|N_C(u)| = |N_C(v)| = 4$, then there is an edge, $e \in E(G)$, such that $G/e$ has an induced $C_4$.

Proof. For any two vertices, $u$ and $v \notin C$, such that $|N_C(u)| = |N_C(v)| = 4$, $G$ has an induced $C_4$. Thus, according to Lemma 7, there is an edge, $e \in E(G)$, such that $G/e$ has an induced $C_4$. 

In a similar way to the proof of Claim 8.9, we can prove the following claim.

Claim 8.11. If there are two vertices, $u$ and $v \notin C$, such that $|N_C(u)| = 2$ and $|N_C(v)| = 3$, then there is an edge, $e \in E(G)$, such that $G/e$ has an induced $C_4$. 

5
In a similar way to the proof of Claim 8.10, we can prove the following two claims.

**Claim 8.12.** If there are two vertices, $u$ and $v \notin C$, such that $|N_C(u)| = 2$ and $|N_C(v)| = 4$, then there is an edge, $e \in E(G)$, such that $G/e$ has an induced $C_4$.

**Claim 8.13.** If there are two vertices, $u$ and $v \notin C$, such that $|N_C(u)| = 3$ and $|N_C(v)| = 4$, then there is an edge, $e \in E(G)$, such that $G/e$ has an induced $C_4$.

Based on Claims 8.1, . . . , 8.13, the proof is complete.

**Theorem 9.** Given a connected graph, $G$, that is not isomorphic to any graph in Figure 1, $G$ is split if and only if $G/e$ is split for any $e \in E(G)$.

**Proof.** Let $G$ be a split graph with a $KS$-partition for $V$. An edge in $E(G)$ is either connecting two vertices in $K$ or a vertex each in $K$ and $S$. Let $u, v \in K$. According to Proposition 6, the order of the complete subgraph induced by $K$ in $G$ decreases by 1 in $G/uv$. Hence, we can partition $V(G/uv)$ into $\{K \setminus \{v\}, S\}$. Consequently, $G/uv$ is a split graph. Let $e$ be an edge in $E(G)$ between a vertex in $K$ and a vertex in $S$, say $v$. We can partition $V(G/e)$ into $\{K, S \setminus \{v\}\}$. Thus, $G/e$ is a split graph.

Conversely, we prove that if $G$ is not split, then $G/e$ is not split for at least an edge, $e \in E$. Consequently, and based on Theorem 1, we prove that if $G$ is not split, then $G/e$ has an induced $2K_2$, $C_5$, or $C_4$.

According to Proposition 5, if $G$ has an induced $C_5$, then there is an edge, $e \in E(G)$, such that $G/e$ is not split.

According to Lemma 7, if $G$ has an induced $C_4$, then either $G$ is isomorphic to one of $\{H_l: l \geq 2\}$, $H_2$, and $H_3$ or there is an edge, $e \in E(G)$, such that $G/e$ is not split.

According to Lemma 8, if $G$ has an induced $2K_2$, then either $G$ is isomorphic to one of $2K_2$, $P_5$, Hammer, Butterfly or there is an edge, $e \in E(G)$, such that $G/e$ is not split.

Theorem 9 shows that we can partition the set of all graphs into the following three parts:

- The set of split graphs: the contraction of an edge in any graph in this set constructs a split graph.
- The set of graphs presented in Figure 1: these are non-split graphs in which the contraction of an edge in any graph constructs a split graph.
- The set of all non-split graph not presented in Figure 1: the contraction of at least one edge in any graph in this set constructs a non-split graph.

This partition is presented in Figure 2.
3. \((2K_2, \text{claw})\)-free graphs

A graph is called a claw if it has four vertices and three edges with one vertex adjacent to the other three. The clique number of a graph is the maximum cardinality of a clique in the graph. The independent number of graph \(G\) is the maximum cardinality of an independent set in \(G\) and is denoted by \(\alpha(G)\).

Moreover, graph \(G\) is called perfect if the clique number of \(H\) equals its chromatic number for any vertex-induced subgraph \(H\) of \(G\). A hole is a vertex-induced cycle in \(G\) with a length of at least four. An antihole is a vertex-induced complement of a hole. In [5], it was shown that

**Theorem 10.** A graph is perfect if and only if it contains neither an odd hole nor an odd antihole.

In [3], it was proved that

**Theorem 11.** If \(G\) is connected \((2K_2, \text{claw})\)-free with \(\alpha(G) \geq 3\), then \(G\) is perfect.

We use the characterisation in Theorem 9 to prove the following:

**Theorem 12.** If \(G\) is connected \((2K_2, \text{claw})\)-free with \(\alpha(G) \geq 3\), then \(G\) is split.

**Proof.** For the sake of contradiction, we assume that \(G\) is a connected \((2K_2, \text{claw})\)-free graph with \(\alpha(G) \geq 3\), but it is not split. According to Theorem 9, \(G\) is either isomorphic to a graph in Figure 1: \(H_i\), where \(1 \leq i \leq 7\), or there is an edge, \(e \in E\), where \(G/e\) is not split.

**Claim 12.1.** Graphs \(H_4, H_5, H_6, \text{ and } H_7\) are not \(2K_2\)-free.

**Claim 12.2.** Graph \(H_1^l\), where \(l \geq 2\), is not claw-free if \(\alpha(H_1^l) \geq 3\).
Proof. The vertex set of $H_1$ is a union of set $C = \{p, q, r, s\}$ that induces $C_4$ and $l - 2$ vertices that are adjacent to exactly two nonadjacent vertices in $C$, say $q, s$. If $\alpha(H_1) \geq 3$, then $l \geq 3$. Let $v \notin C$. Vertex set $\{p, q, r, v\}$ induces a claw.

Claim 12.3. Each graph $H_2$ and $H_3$ has no independent set with a cardinality larger than 2.

Based on Claims 12.1, 12.2, and 12.3, the following claim follows.

Claim 12.4. If $G$ is a connected $(2K_2, \text{claw})$-free graph with $\alpha(G) \geq 3$, then $G$ is not isomorphic to any graph in Figure 1.

Claim 12.5. If $G$ is $2K_2$-free, then $G/e$ is $2K_2$-free for any $e \in E(G)$.

Claim 12.6. If $G$ has a vertex set, $C$, that induces $C_4$, then $G$ has an induced $C_4$ or $C_5$. Further, if $G$ is $C_5$-free, then $G$ has either vertex $v$ that is adjacent to exactly one vertex, two adjacent vertices, or three vertices in $C$ or two adjacent vertices, $u$ and $v \in \mathcal{V} \setminus C$.

Claim 12.7. If $G$ is $(2K_2, \text{claw})$-free with $\alpha(G) \geq 3$, then $G/e$ is $C_5$-free for any $e \in E(G)$.

Proof. For any edge, $e \in E(G)$, if $G/e$ has an induced $C_5$, then $G$ has an induced $C_5$ or $C_6$. According to Theorem 11, $G$ does not have a vertex set that induces $C_5$. Moreover, because $G$ is $2K_2$-free, $G$ has no vertex set that induces $C_6$. (□)

Let $C = \{p, q, r, s\} \subseteq \mathcal{V}$ that induces $C_4$ in $G$ with edges $pq, qr, rs$, and $sp$.

Claim 12.8. $C$ is dominating.

Proof. For the sake of contradiction, we assume that $C$ does not dominate. Therefore, there is a vertex, $v \notin C$, that is not adjacent to any vertex in $C$. Because $G$ is connected, $v$ is adjacent to vertex $u \notin C$. If $u$ is adjacent to no more than one vertex in $C$, say $p$, or exactly two adjacent vertices in $C$, say $p, q$, then $\{r, s, u, v\}$ induces $2K_2$ in $G$, and this is a contradiction. Otherwise, $u$ is adjacent to exactly either two nonadjacent vertices in $C$, say $p, r$, or at least three vertices in $C$, say $p, q, r$, and $\{p, r, u, v\}$ induces a claw in $G$, which is a contradiction. (□)

Claim 12.9. If $v \notin C$, such that $N_C(v) = 1$ or $v$ is adjacent to exactly two nonadjacent vertices in $C$, then $G$ is not claw-free.

Proof. We assume that $v$ is adjacent to exactly either a vertex in $C$, say $p$, or two nonadjacent vertices in $C$, say $p$ and $r$. Hence, $\{p, q, s, v\}$ induces a claw in $G$. (□)

Claim 12.10. If $u$ and $v \notin C$, where $|N_C(u)| = |N_C(v)| = 2$, then $u$ and $v$ are adjacent.
Proof. For the sake of contradiction, we assume that there are two vertices, \( u \) and \( v \), where \( |N_C(u)| = |N_C(v)| = 2 \) but \( u \) and \( v \) are not adjacent. According to Claim 12.9, neither \( u \) nor \( v \) is adjacent to two nonadjacent vertices in \( C \). Thus, let \( u \) be adjacent to \( p \) and \( q \). If \( v \) is adjacent to \( p \) and \( q \), then \( \{p, s, u, v\} \) induces a claw in \( G \), which is a contradiction. Otherwise, \( v \) is adjacent to \( r \) or \( s \); then, \( \{p, r, u, v\} \) or \( \{q, s, u, v\} \) induces \( 2K_2 \) in \( G \), which is a contradiction. \( \square \)

Claim 12.11. If \( S \) is an independent set in \( G \) with \( |S| \geq 3 \), then \( S \cap C \) is empty.

Proof. For the sake of contradiction, we assume that there is an independent set, \( S \), in \( G \) with \( |S| \geq 3 \) and \( S \cap C \) is not empty. If \( |S \cap C| = 2 \), then there is a vertex in \( S \) that is adjacent to exactly one or two nonadjacent vertices in \( C \), which contradicts Claims 12.8 and 12.9. Otherwise, \( |S \cap C| = 1 \). Let \( |S \cap C| = \{p\} \) and \( u, v \in S \setminus \{p\} \). According to Claims 12.8 and 12.10, at most one of \( u \) and \( v \) is adjacent to exactly two adjacent vertices in \( C \). If \( u \) and \( v \) are adjacent to \( q \) or \( s \), then \( \{p, q, u, v\} \) (or \( \{p, s, u, v\} \)) induces a claw in \( G \), which is a contradiction. \( \square \)

Claim 12.12. If \( v \notin C \) and \( |N_C(v)| = 2 \), then \( v \notin S \), where \( S \) is an independent set in \( G \) and \( |S| \geq 3 \).

Proof. According to Claim 12.9, \( v \) is not adjacent to two nonadjacent vertices in \( C \). Thus, for the sake of contradiction, we assume that there is a vertex, \( v \notin C \), that is adjacent to exactly two adjacent vertices in \( C \), say \( p \) and \( q \), and \( v \in S \), where \( S \) is an independent set in \( G \) with \( |S| \geq 3 \). According to Claim 12.11, \( S \) does not have any vertex from \( C \). Let \( u, w \in S \setminus \{v\} \). According to Claims 12.8 and 12.10 and because \( |N_C(v)| = 2 \), \( N_C(u) \) and \( N_C(w) \) are at least equal to \( 3 \). If \( u \) and \( w \) are adjacent to \( p \) (or \( q \)), then \( \{p, u, v, w\} \) (or \( \{q, u, v, w\} \)) induces a claw in \( G \), which is a contradiction. Otherwise, w.l.o.g. \( u \) is adjacent to \( p, r \), and \( s \), and \( w \) is adjacent to \( q, r \), and \( s \). Therefore, \( \{p, r, v, w\} \) induces \( 2K_2 \) in \( G \), which is a contradiction. \( \square \)

Claim 12.13. If \( v \notin C \) and \( |N_C(v)| = 3 \), then \( v \notin S \), where \( S \) is an independent set in \( G \) and \( |S| \geq 3 \).

Proof. For the sake of contradiction, we assume that there is a vertex, \( v \notin C \), with \( |N_C(v)| = 3 \) and \( v \in S \), where \( S \) is an independent set in \( G \) and \( |S| \geq 3 \). According to Claims 12.8, 12.11, and 12.12, if \( u \in S \), then \( |N_C(u)| \geq 3 \). Let \( u, w \in S \setminus v \). According to the pigeonhole principle, \( |N_C(u) \cap N_C(v) \cap N_C(w)| \geq 1 \). Let \( p \in N_C(u) \cap N_C(v) \cap N_C(w) \); then, \( \{p, u, v, w\} \) induces a claw in \( G \), which is a contradiction. \( \square \)

Claim 12.14. If \( v \notin C \) and \( |N_C(v)| = 4 \), then \( v \notin S \), where \( S \) is an independent set in \( G \) and \( |S| \geq 3 \).

Proof. For the sake of contradiction, we assume that there is a vertex, \( v \notin C \), with \( |N_C(v)| = 4 \) and \( v \in S \) where \( S \) is an independent set in \( G \) and \( |S| \geq 3 \). According to Claims 12.8, 12.11, 12.12, and 12.13, if \( u \in S \), then \( N_C(u) = 4 \).
Then, any three vertices in $S$ and any vertex in $C$ form a vertex set of cardinality 4 that induces a claw in $G$, which is a contradiction.  

Based on Claims 12.8, . . . , 12.14, we obtain the following claim.

**Claim 12.15.** If $G$ is $(2K_2,\text{claw})$-free with $\alpha(G) \geq 3$, then $G/e$ is $C_4$-free for any $e \in E(G)$.

Based on Claims 12.4, 12.5, 12.6, 12.7, and 12.15, the proof is complete.  □

4. Unbalanced split graphs

A *star* denoted by $S_n$ is a graph constructed by $E_1 \lor E_n$, where $n$ is a nonpositive integer. A split graph, $G$, is called balanced split if $G$ has a KS-partition, where $|K| = \omega(G)$ and $|S| = \alpha(G)$, and unbalanced split, otherwise. Based on the work in [10], the following theorem appears in [9] and its proof is presented in [4]:

**Theorem 13** ([10],[9],[4]). For any KS-partition of split graph $G$, exactly one of the following holds: (i) $|K| = \omega(G)$ and $|S| = \alpha(G)$. (ii) $|K| = \omega(G) - 1$ and $|S| = \alpha(G)$. (iii) $|K| = \omega(G)$ and $|S| = \alpha(G) - 1$.

Moreover, in (i), the KS-partition is unique.

**Theorem 14.** Let $G$ be a split graph that is not isomorphic to any $S_n$ for $n \geq 2$. Then, $G$ is an unbalanced split if and only if there is an edge, $e \in E(G)$, such that $\omega(G/e) = \omega(G) - 1$ and $G/e$ is an unbalanced split.

Proof. Let $G$ be an unbalanced split graph, then $V(G)$ can be partitioned into a KS-partition, where $|K| = \omega(G) - 1$ and $|S| = \alpha(G)$. For any two adjacent vertices, $u$ and $v \in K$, $V(G/uv)$ can be partitioned into $KS$-partitions, where $|K'| = \omega(G) - 2$ and $|S| = \alpha(G)$. Thus, $G/uv$ is an unbalanced split with $\omega(G/uv) = \omega(G) - 1$.

Conversely, we prove that if $G$ is a balanced split, then for any edge, $e$, $G/e$ is a balanced split or $\omega(G/e) = \omega(G)$. Because $G$ is a balanced split, then based on Theorem 13, $V(G)$ can be partitioned into the unique KS-partition, where $|K| = \omega(G)$ and $|S| = \alpha(G)$. For any two adjacent vertices, $u$ and $v \in K$, $V(G/uv)$ can be partitioned into $K'S$-partitions, where $|K'| = \omega(G) - 1$ and $|S| = \alpha(G)$. Thus, either $G/uv$ is a balanced split with $|K'| = \omega(G) - 1$ or $G/uv$ is an unbalanced split with $|K'| = \omega(G)$. For any two adjacent vertices, $u \in K$ and $v \in S$, $V(G/uv)$ can be partitioned into $K'S$-partitions, where $|K| = \omega(G)$ and $|S| = \alpha(G) - 1$. Thus, $\omega(G/uv) = \omega(G)$. □

4.1. Pseudo-split graphs

A graph is called $G$ pseudo-split if $G$ is $(2K_2, C_4)$-free. In [1], the family of $(2K_2, C_4)$-free graphs was investigated and later referred to as pseudo-split graphs in [11]. Different authors have characterised pseudo-split graphs as follows.
Theorem 15 ([1], [11]). A graph is a pseudo-split if and only if its vertex set can be partitioned into three sets, $A, B,$ and $C$, such that $A$ induces a clique, $B$ induces an independent set, and $C$ induces $C_5$ or is empty, such that there are all possible edges between $A$ and $C$, and there are no edges between $B$ and $C$.

The following result is obtained from Theorems 9 and 15:

Theorem 16. Given a connected graph, $G$, that is not isomorphic to any graph in Figure 1, $G$ is pseudo-split if and only if

- $G/e$ is split for any $e \in E(G)$ or
- $V(G)$ can be partitioned into three sets, $A, B,$ and $C$, such that $A$ induces a clique, $B$ induces an independent set, and $C$ induces $C_5$, where there are all possible edges between $A$ and $C$, and there are no edges between $B$ and $C$.

4.2. Nordhaus–Gaddum graphs

A graph $G$ is called Nordhaus–Gaddum (NG in short) if $\chi(G) + \chi(\overline{G}) = |V(G)| + |V(\overline{G})| + 1$. NG graphs were investigated and characterised in [7], [12], [6], and [4]. Based on the work in [6], the authors in [4] proved the following characterisation for NG graphs:

Theorem 17 ([4]). Graph $G$ is an NG-graph if and only if $G$ is a pseudo-split but not a balanced split.

In other words, the aforementioned theorem can be formulated as follows:

Theorem 18 ([4]). Graph $G$ is an NG-graph if and only if

- $G$ is unbalanced split or
- $V(G)$ can be partitioned into three sets, $A, B,$ and $C$, such that $A$ induces a clique, $B$ induces an independent set, and $C$ induces $C_5$, where there are all possible edges between $A$ and $C$, and there are no edges between $B$ and $C$.

Based on Theorems 14 and 18, the following result is obtained:

Theorem 19. Graph $G$ that is not isomorphic to any $S_n$ for $n \geq 2$ is an NG-graph if and only if

- there is an edge, $e \in E(G)$, such that $\omega(G/e) = \omega(G) - 1$ and $G/e$ is unbalanced split, or
- $V(G)$ can be partitioned into three sets, $A, B,$ and $C$, such that $A$ induces a clique, $B$ induces an independent set, and $C$ induces $C_5$, where there are all possible edges between $A$ and $C$, and there are no edges between $B$ and $C$. 
5. Acknowledgments

The author would like to sincerely thank Peter Tittmann for his constructive suggestions and discussions.

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