Paley type inequality on Hardy spaces in the Dunkl setting

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Abstract
We investigate $\lambda$-Hilbert transform, $\lambda$-Possion integral and conjugate $\lambda$-Poisson integral on the atomic Hardy space in the Dunkl setting and establish a new version of Paley type inequality which extends the results in [11] and [14].

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1 Introduction and preliminaries

For $0 < p < \infty$, $L_p^\lambda(\mathbb{R})$ is the set of measurable functions satisfying $\|f\|_{L_p^\lambda} = \left( c_\lambda \int_{\mathbb{R}} |f(x)|^p |x|^{2\lambda} dx \right)^{1/p} < \infty$, $c_\lambda^{-1} = 2^{\lambda+1/2}\Gamma(\lambda+1/2)$, and $p = \infty$ is the usual $L^\infty(\mathbb{R})$ space. For $\lambda \geq 0$, The Dunkl operator on the line is:

$$D_x f(x) = f'(x) + \lambda [f(x) - f(-x)]$$

involving a reflection part. The associated Fourier transform for the Dunkl setting for $f \in L_1^\lambda(\mathbb{R})$ is given by:

$$(\mathcal{F}_\lambda f)(\xi) = c_\lambda \int_{\mathbb{R}} f(x) E_\lambda(-ix\xi) |x|^{2\lambda} dx, \quad \xi \in \mathbb{R}, \quad f \in L_1^\lambda(\mathbb{R}).$$

$E_\lambda(-ix\xi)$ is the Dunkl kernel

$$E_\lambda(iz) = j_{\lambda-1/2}(z) + \frac{iz}{2\lambda+1} j_{\lambda+1/2}(z), \quad z \in \mathbb{C}$$

where $j_\alpha(z)$ is the normalized Bessel function

$$j_\alpha(z) = 2^\alpha \Gamma(\alpha + 1) \frac{J_\alpha(z)}{z^\alpha} = \Gamma(\alpha + 1) \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n}}{n! \Gamma(n + \alpha + 1)}.$$

Since $j_{\lambda-1/2}(z) = \cos z$, $j_{\lambda+1/2}(z) = z^{-1} \sin z$, it follows that $E_0(iz) = e^{iz}$, and $\mathcal{F}_0$ agrees with the usual Fourier transform. We assume $\lambda > 0$ in what follows. And the associated $\lambda$-translation in Dunkl setting is

$$\tau_y f(x) = c_\lambda \int_{\mathbb{R}} (\mathcal{F}_\lambda f)(\xi) E(ix\xi) E(iy\xi) |\xi|^{2\lambda} d\xi, \quad x, y \in \mathbb{R}.$$
The \(\lambda\)-convolution \(f \ast \lambda g(x)\) of two appropriate functions \(f\) and \(g\) on \(\mathbb{R}\) associated to the \(\lambda\)-translation \(\tau_{i}\) is defined by

\[
(f \ast \lambda g)(x) = c_{\lambda} \int_{\mathbb{R}} f(t) \tau_{i}g(-t)(t^{2\lambda})dt.
\]

The "Laplace Equation" associated with the Dunkl setting is given by:

\[
(\Delta_{\lambda} u)(x, y) = (D_{x}^{2} + D_{y}^{2}) u(x, y) = \left(\frac{\partial^{2}}{\partial x^{2}} + \frac{\partial^{2}}{\partial y^{2}}\right) u + \frac{\lambda}{x} \frac{\partial u}{\partial x} - \frac{\lambda}{x^{2}} (u(x, y) - u(-x, y)).
\]

A \(C^2\) function \(u(x, y)\) satisfying \(\Delta_{\lambda} u = 0\) is \(\lambda\)-harmonic. When \(u\) and \(v\) are \(\lambda\)-harmonic functions satisfying \(\lambda\)-Cauchy-Riemann equations:

\[
\left\{\begin{array}{l}
D_{x}u - \partial_{y}v = 0, \\
\partial_{y}u + D_{x}v = 0
\end{array}\right.
\]

the function \(F(z) = F(x+iy) = u(x,y) + iv(x,y)\) is a \(\lambda\)-analytic function. We define the Complex-Hardy spaces \(H^{p}_{\lambda}(\mathbb{R}^{2}_{+})\) to be the set of \(\lambda\)-analytic functions \(F = u + iv\) on \(\mathbb{R}^{2}_{+}\) satisfying

\[
\|F\|_{H^{p}_{\lambda}(\mathbb{R}^{2}_{+})} = \sup_{y>0} \left\{ c_{\lambda} \int_{\mathbb{R}} |F(x + iy)|^{p}x^{2\lambda}dx \right\}^{1/p} < \infty.
\]

Throughout this paper, for any \(x_{0} \in \mathbb{R}\), \(\delta_{0} > 0\), \(I(x_{0}, \delta_{0})\) is an Euclid interval with center \(x_{0}\) and radius \(\delta_{0}\): \(I(x_{0}, \delta_{0}) = \{ x : |x - x_{0}| < \delta_{0} \} \) with \(\delta_{0} < |x_{0}/2|\), and \(I\) denotes as the set \(I = (I(x_{0}, 4\delta_{0}) \cup I(-x_{0}, 4\delta_{0}))^{c}\).

In this paper we define the atomic Hardy space in the Dunkl setting. A class of fundamental functions that we will call atoms will be introduced as following. For \(\frac{2\lambda}{2\lambda + 1} < p \leq 1\), a Lebesgue measure function \(a(x)\) is a \(\lambda\)-atom, if it satisfies the following conditions

(i) \(\|a(x)\|_{L^{\infty}_{\mu}} \lesssim \frac{1}{|I(x_{0}, \delta_{0})|^{1/p}}\),
(ii) \(\sup \mu(a(x)) \subseteq I(x_{0}, \delta_{0})\), with \(\delta_{0} < |x_{0}/2|\),
(iii) \(\int_{\mathbb{R}} t^{k}a(t)|t|^{2\lambda}dt = 0 \quad (k = 0, 1, 2, 3 \ldots \kappa), \quad \kappa \geq 2 \left((2\lambda + 1)\frac{1-p}{p}\right)\).

Let \(\frac{2\lambda}{2\lambda + 1} < p \leq 1\). Our Hardy type space \(H^{p}_{\lambda}(\mathbb{R})\) is constituted by all those \(f \in S'(\mathbb{R})\) that can be represented by

\[
f = \sum_{k} \lambda_{k}a_{k}(x),
\]

being \(\lambda_{j} \in \mathbb{C}\) and \(a_{j}\) is a \(\lambda\)-atom, for all \(j \in \mathbb{N}\), where \(\sum_{j=0}^{\infty} |\lambda_{j}|^{p} < \infty\) and the series in (4) converges in \(S'(\mathbb{R})\). We define on \(H^{p}_{\lambda}(\mathbb{R})\) the norm \(\| \cdot \|_{H^{p}_{\lambda}(\mathbb{R})}\) by

\[
\|f\|_{H^{p}_{\lambda}(\mathbb{R})} = \inf \left(\sum_{k} |\lambda_{k}|^{p}\right)^{1/p},
\]

where the infimum is taken over all those sequences \(\{\lambda_{j}\}_{j \in \mathbb{N}} \subseteq \mathbb{C}\) such that \(f\) is given by (4) for certain \(\lambda\)-atom \(a_{j}, j \in \mathbb{N}\).

We will establish the following version of Paley type inequality. Let \(\frac{2\lambda}{2\lambda + 1} < p \leq 1, \ k \geq p\) then there exists \(C > 0\) such that for every \(f \in H^{k}_{\lambda}(\mathbb{R})\),

\[
\int_{0}^{\infty} |(\mathcal{F}_{\lambda}f)(\xi)^{k}|\xi^{(2\lambda + 1)(k-1-k/p)+2\lambda}d\xi \leq C\|f\|^{p}_{H^{k}_{\lambda}(\mathbb{R})}.
\]

When \(k = p\), we could obtain

\[
\int_{0}^{\infty} |(\mathcal{F}_{\lambda}f)(\xi)^{p}|\xi^{(2\lambda + 1)(p-2)+2\lambda}d\xi \leq C\|f\|^{p}_{H^{p}_{\lambda}(\mathbb{R})}.
\]

This property is studied in [1, 2] for the classical case, in [11] for the Dunkl setting on high dimension, in [14] for the Dunkl setting on the upper half plane.
Our result is different to [11, 14] that the center of atoms in [11] are fixed at the origin, and the functions in [14] are λ-analytic functions.

Throughout the paper we use the classic notation. $S'(\mathbb{R})$ and $S(\mathbb{R})$ are the space of tempered distributions on $\mathbb{R}$ and the Schwartz space on $\mathbb{R}$ respectively. $C$ will denote a positive constant not necessary the same in each occurrence. We use $A \lesssim B$ to denote the estimate $|A| \leq CB$ for some absolute universal constant $C > 0$, which may vary from line to line, $A \gtrsim B$ to denote the estimate $|A| \geq CB$ for some absolute universal constant $C > 0$, $A \sim B$ to denote the estimate $|A| \leq C_1B$, $|A| \geq C_2B$ for some absolute universal constant $C_1, C_2$.

2 Kernels and transforms associated with the Dunkl setting

The following properties are studied in [14]:

**Proposition 2.1.** [14] For $f \in L^1_\lambda(\mathbb{R}) \cap L^\infty_\lambda(\mathbb{R}), \ x \in \mathbb{R}$, $y \in (0, \infty)$, we can define $\lambda$-Hilbert transform, $\lambda$-Poisson integral and conjugate $\lambda$-Poisson integral by

\[
\mathcal{H}_\lambda f(x) = c_\lambda \lim_{\epsilon \to 0^+} \int_{|t-x| > \epsilon} f(t)h(x, t)|t|^{2\lambda}dt,
\]

\[
(P_f)(x, y) = (f * \lambda P_y)(x) = c_\lambda \int_{\mathbb{R}} f(t)(\tau_x P_y)(-t)|t|^{2\lambda}dt,
\]

\[
(Q_f)(x, y) = (f * \lambda Q_y)(x) = c_\lambda \int_{\mathbb{R}} f(t)(\tau_x Q_y)(-t)|t|^{2\lambda}dt,
\]

where the $\lambda$-Hilbert kernel is defined as:

\[
h(x, t) = \frac{\lambda \Gamma(\lambda + 1/2)}{2^{\lambda-1/2}\pi} (x-t)^{1-\lambda}|x|^\lambda|t|^{2\lambda+1}ds,
\]

$\lambda$-Poisson kernel $(\tau_x P_y)(-t)$ has the representation

\[
(\tau_x P_y)(-t) = \frac{\lambda \Gamma(\lambda + 1/2)}{2^{\lambda-1/2}\pi} \int_0^\pi \frac{y(1 + \text{sgn}(xt) \cos \theta)}{(y^2 + x^2 + t^2 - 2|xt| \cos \theta)^{\lambda+1}} \sin^{2\lambda} \theta d\theta,
\]

and $(\tau_x Q_y)(-t)$ is the conjugate $\lambda$-Poisson kernel, with the following representation:

\[
(\tau_x Q_y)(-t) = \frac{\lambda \Gamma(\lambda + 1/2)}{2^{\lambda-1/2}\pi} \int_0^\pi \frac{(x-t)(1 + \text{sgn}(xt) \cos \theta)}{(y^2 + x^2 + t^2 - 2|xt| \cos \theta)^{\lambda+1}} \sin^{2\lambda} \theta d\theta.
\]

$Pf(x, y)$ and $Qf(x, y)$ satisfy the generalized Cauchy-Riemann system (3) on $\mathbb{R}_+^2$, and $Pf(x, y) + iQf(x, y)$ is a $\lambda$-analytic function on the upper half plane.

The Hardy-Littlewood type inequalities for $F \in H^p_\lambda(\mathbb{R}_+^2)$ are studied in [14]

**Theorem 2.2.** [14] Let $\frac{2\lambda}{2\lambda+1} < p \leq 1$, $k \geq p$ then there exists $C > 0$ such that for every $f \in H^k_\lambda(\mathbb{R})$,

\[
\int_0^\infty |(\mathcal{F}_\lambda F)(\xi)|^k|\xi|^{(2\lambda+1)(k-1-k/p)+2\lambda}d\xi \leq c\|F\|_{H^k_\lambda(\mathbb{R}_+^2)}^p.
\]

When $k = p$, we could obtain

\[
\int_0^\infty |(\mathcal{F}_\lambda F)(\xi)|^k|\xi|^{(2\lambda+1)(p-2)+2\lambda}d\xi \leq c\|F\|_{H^k_\lambda(\mathbb{R}_+^2)}^p.
\]

**Proposition 2.3.** [14] If $\frac{2\lambda}{2\lambda+1} < p < l \leq +\infty$, $\delta = \frac{1}{p} - \frac{1}{l}$, and $F(x, y) \in H^p_\lambda(\mathbb{R}_+^2)$, $p \leq k < \infty$, then

\[
(\int_0^\infty y^{k\delta(1+2\lambda)-1} \left( \int_{\mathbb{R}} |F(x, y)|^{l}|x|^{2\lambda}dx \right)^{\frac{1}{l}} dy)^{\frac{1}{l}} \leq c\|F\|_{H^k_\lambda(\mathbb{R}_+^2)}.
\]
Obviously the following inequality holds:

\[
\left( \int_{\mathbb{R}} |F(x, y)|^p |x|^{2\lambda} \, dx \right)^{1/p} \leq cy^{-\frac{1}{p-1}}(1+2\lambda) \|F\|_{H^\lambda_{x,y}(\mathbb{R}^2_+)}.
\]  

(8)

When \( p = \infty \), we could deduce the following from (8)

\[
\sup_{x \in \mathbb{R}} |F(x, y)| \leq cy^{-\frac{1}{p}}(1+2\lambda) \|F\|_{H^\lambda_{x,y}(\mathbb{R}^2_+)}.
\]

(iii) If \( 1 \leq p < \infty \) and \( F(x, y) = u(x, y) + iv(x, y) \in H^\lambda_{x,y}(\mathbb{R}^2_+) \), then \( F(x, y) \) is the \( \lambda \)-Poisson integrals of its boundary values \( F(x) \), and \( F(x) \in L^p_{x,y}(\mathbb{R}) \).

**Proposition 2.4.**

\[
\int_I \frac{1}{||x| - |x_0||^k} \, dx \lesssim \delta_0^{1-k} \quad (k > 1).
\]

Proof.

\[
\int_I \frac{1}{||x| - |x_0||^k} \, dx \leq \int_{I(0,4\delta_0)^c} \frac{1}{||x| - |x_0||^k} \, dx = \int_{I(0,4\delta_0)^c} \frac{1}{|x|^k} \, dx \lesssim \delta_0^{1-k}.
\]

\( \square \)

**Proposition 2.5.**

\[
\int_{-1}^{1} (1 - bs)^{-\lambda-1} (1 + s)(1 - s^2)^{\lambda-1} \, ds \leq C \frac{1}{1 - |b|}, \quad \forall \, -1 < b < 1, \, \lambda > 0
\]

\( C \) is depend on \( \lambda \), and independent on \( b \). (\( C \sim 1/\lambda \))

Proof. CASE 1: when \( 0 \leq b < 1 \).

It is obvious to see that when \( 0 \leq b < 1, \, \lambda > 0 \),

\[
\int_{-1}^{0} (1 - bs)^{-\lambda-1} (1 + s)(1 - s^2)^{\lambda-1} \, ds \lesssim 1.
\]

By the formula of integration by parts and \( 1 - s \leq 1 - bs \) when \( 1 \geq s \geq 0 \) ( \( 0 \leq b < 1, \, \lambda > 0 \) ), we obtain:

\[
\left| \int_{0}^{1} (1 - bs)^{-\lambda-1} (1 + s)(1 - s^2)^{\lambda-1} \, ds \right| \lesssim \left| \int_{0}^{1} (1 - bs)^{-\lambda-1} (1 - s)^{\lambda-1} \, ds \right| \lesssim \frac{1}{\lambda + 1} \int_{0}^{1} (1 - bs)^{-2} \, ds \lesssim \frac{1}{1 - b}.
\]

Next we need to prove when \( -1 < b \leq 0 \)

\[
\int_{-1}^{1} (1 - bs)^{-\lambda-1} (1 + s)(1 - s^2)^{\lambda-1} \, ds \lesssim \frac{1}{1 - b}
\]

CASE 2: when \( -1 < b \leq 0 \).

Obviously the following inequality holds:

\[
\int_{0}^{1} (1 - bs)^{-\lambda-1} (1 + s)(1 - s^2)^{\lambda-1} \, ds \lesssim 1.
\]

By the formula of integration by parts and \( 1 + s \leq 1 - bs \) when \( -1 \leq s \leq 0 \) ( \( -1 < b \leq 0, \, \lambda > 0 \) ), we obtain:

\[
\left| \int_{-1}^{0} (1 - bs)^{-\lambda-1} (1 + s)(1 - s^2)^{\lambda-1} \, ds \right| \lesssim \left| \int_{-1}^{0} (1 - bs)^{-\lambda-1} (1 + s)^{\lambda} \, ds \right| \lesssim \frac{1}{\lambda + 1} - b \int_{-1}^{0} (1 - bs)^{-1} \, ds \lesssim -\ln(1 + b) \lesssim \frac{1}{1 + b}.
\]
By CASE 1 and CASE 2, the inequality:

\[
\int_{-1}^{1} (1 - bs)^{-\lambda - 1}(1 + s)(1 - s^2)^{\lambda - 1} ds \leq C \frac{1}{1 - |b|}, \quad \forall -1 < b < 1, \; \lambda > 0
\]

holds. Hence the proposition holds.

Thus we could obtain the following Proposition 2.6 and Proposition 2.5

**Proposition 2.6.**

\[
\int_{-1}^{1} (1 - bs)^{-\lambda - 1}(1 - s)(1 - s^2)^{\lambda - 1} ds \leq C \frac{1}{1 - |b|}, \quad \forall -1 < b < 1, \; \lambda > 0
\]

\(C\) is depend on \(\lambda\), and independent on \(b\).

The following Proposition 2.7 could be obtained in a way similar to Proposition 2.5

**Proposition 2.7.**

\[
\int_{-1}^{1} (1 - bs)^{-\lambda - 1}(1 - s^2)^{\lambda - 1/2} ds \leq C \frac{1}{1 - |b|}, \quad \forall -1 < b < 1, \; \lambda > 0
\]

\(C\) is depend on \(\lambda\), and independent on \(b\). (In fact \(C \sim \frac{2}{2\lambda + 1}\))

**Theorem 2.8** (\(\lambda\)-Hilbert transform). For \(\frac{2\lambda}{2\lambda + 1} < p \leq 1\), if \(a(t)\) is a \(p_\lambda\)-atom, with vanishing order \(\kappa \geq 2 \left(2\lambda + 1\right)\left(\frac{1 - p}{p}\right)\) then the following holds:

\[
\int_{\mathbb{R}} |\mathcal{H}_\lambda a(x)|^p |x|^{2\lambda} dx \leq C;
\]

\(C\) is depend on \(\lambda\) and \(p\).

**Proof.** Assume first that \(x_0 > 0\). Let \(\kappa = 2 \left(2\lambda + 1\right)\left(\frac{1 - p}{p}\right)\). Thus \(\kappa\) is an even integer. Let \(n = \kappa/2\). We could write the above integral as:

\[
\int_{\mathbb{R}} |\mathcal{H}_\lambda a(x)|^p |x|^{2\lambda} dx = \int_{I(x_0, 4\delta_0) \cup I(-x_0, 4\delta_0)} |\mathcal{H}_\lambda a(x)|^p |x|^{2\lambda} dx + \int_{(I(x_0, 4\delta_0) \cup I(-x_0, 4\delta_0))^c} |\mathcal{H}_\lambda a(x)|^p |x|^{2\lambda} dx
\]

\[
= I + II.
\]

We could see the following inequality holds:

\[
4^{2\lambda + 1} \int_{x_0 - \delta_0}^{x_0 + \delta_0} |x|^{2\lambda} dx = \int_{4x_0 - 4\delta_0}^{4x_0 + 4\delta_0} |x|^{2\lambda} dx \geq \int_{4x_0 - 4\delta_0}^{4x_0 + 4\delta_0} |x|^{2\lambda} dx.
\]

By [14] [Theorem 5.7], together with (9) we obtain:

\[
I = \int_{I(x_0, 4\delta_0) \cup I(-x_0, 4\delta_0)} |\mathcal{H}_\lambda a(x)|^p |x|^{2\lambda} dx
\]

\[
\leq \left(\int_{I(x_0, 4\delta_0) \cup I(-x_0, 4\delta_0)} |\mathcal{H}_\lambda a(x)|^p |x|^{2\lambda} dx\right)^{p/2} \left(\int_{I(x_0, 4\delta_0) \cup I(-x_0, 4\delta_0)} |x|^{2\lambda} dx\right)^{1-p/2}
\]

\[
\leq C.
\]

Next we need to prove:

\[
II = \int_{(I(x_0, 4\delta_0) \cup I(-x_0, 4\delta_0))^c} |\mathcal{H}_\lambda a(x)|^p |x|^{2\lambda} dx \leq C.
\]
By Proposition 2.1, when $x \in (I(x_0, 4\delta_0) \cup I(-x_0, 4\delta_0))^c$ we could write $\mathcal{H}_\lambda a(x)$ as:

$$\mathcal{H}_\lambda a(x) = c_\lambda \int a(t) h(x, t) |t|^{2\lambda} dt.$$ 

Next we need to estimate $\mathcal{H}_\lambda a(x)$ when $x \in I = (I(x_0, 4\delta_0) \cup I(-x_0, 4\delta_0))^c$ ($\delta_0 < |x_0/2|$). Notice that $t \in \text{supp} a(t) \subseteq I(x_0, \delta_0)$. When $x \geq 0$, or $x < -2x_0$, the following inequality

$$|x - x_0| \lesssim \langle (x, x_0)_s \rangle^{1/2}$$

holds. It is also obvious to see that the following inequalities hold:

$$|x - x_0 s| \lesssim \langle (x, x_0)_s \rangle^{1/2},$$

$$|xs - x_0| \lesssim \langle (x, x_0)_s \rangle^{1/2},$$

$$|x_0 + t - 2xs| \leq |x_0 - xs| + |t - xs| \leq \langle (x, x_0)_s \rangle^{1/2} + \langle (x, x_0)_s \rangle^{1/2} + |t - x_0| \leq 3 \langle (x, x_0)_s \rangle^{1/2}.$$ 

From (15), we could obtain the following inequality:

$$|\delta_1| \leq 3|t - x_0| \langle (x, x_0)_s \rangle^{1/2}.\quad (16)$$

For $\delta_0 < |x_0/2|$, and $x \in I = (I(x_0, 4\delta_0) \cup I(-x_0, 4\delta_0))^c$, from (16), we could have:

$$\frac{|\delta_1|}{\langle x, x_0 \rangle} \leq 3\frac{|t - x_0|}{\langle (x, x_0)_s \rangle^{1/2}} \leq 3\frac{|t - x_0|}{||x| - |x_0||} \leq \frac{3\delta_0}{4\delta_0} = 3/4.\quad (17)$$

We could see that:

$$\frac{x - t}{\langle x, t \rangle_{s+1}^{\lambda_{s+1}}} = \frac{x - x_0}{\langle x, t \rangle_{s}^{\lambda_{s}}} + \frac{x_0 - t}{\langle x, t \rangle_{s+1}^{\lambda_{s+1}}},$$

$$= A + B.\quad (18)$$

By Taylor expansion of $\left(1 + \frac{\delta_1}{\langle x, x_0 \rangle}\right)^{-\lambda-1}$, when $x \in [-2x_0, 0] \cap (I(x_0, 4\delta_0) \cup I(-x_0, 4\delta_0))^c$, we could obtain:

$$A = \frac{x - x_0}{\langle x, x_0 \rangle_{s}^{\lambda_{s}+1}} \left(1 + \frac{\delta_1}{\langle x, x_0 \rangle_s}\right)^{\lambda_{s}+1}$$

$$= \frac{x - x_0}{\langle x, x_0 \rangle_{s}^{\lambda_{s}+1}} \left[1 + \frac{\lambda + 1}{1} \left(-\frac{\delta_1}{\langle x, x_0 \rangle_s}\right)^{1} + \frac{(\lambda + 1)(\lambda + 2)}{2!} \left(-\frac{\delta_1}{\langle x, x_0 \rangle_s}\right)^{2} + \cdots + \frac{(\lambda + 1)_n}{(n)!} \left(-\frac{\delta_1}{\langle x, x_0 \rangle_s}\right)^{n}\right],$$

and

$$B = \frac{x_0 - t}{\langle x, x_0 \rangle_{s}^{\lambda_{s}+1}} \left(1 + \frac{\delta_1}{\langle x, x_0 \rangle_s}\right)^{\lambda_{s}+1}$$

$$+ \frac{x_0 - t}{\langle x, x_0 \rangle_{s+1}^{\lambda_{s+1}}} \left[1 + \frac{\lambda + 1}{1} \left(-\frac{\delta_1}{\langle x, x_0 \rangle_s}\right)^{1} + \frac{(\lambda + 1)(\lambda + 2)}{2!} \left(-\frac{\delta_1}{\langle x, x_0 \rangle_s}\right)^{2} + \cdots + \frac{(\lambda + 1)_{n-1}}{(n-1)!} \left(-\frac{\delta_1}{\langle x, x_0 \rangle_s}\right)^{n-1} + \frac{(\lambda + 1)_n}{(n)!} \left(1 + \frac{1}{1 + \xi_2}\right)^{\lambda_{n+1}} \left(-\frac{\delta_1}{\langle x, x_0 \rangle_s}\right)^{n}\right].$$
Thus from Proposition 2.1, Formulas (17, 18, 19, 20, 12, 15, 16), together with the vanishing property of \( a(t) \), we obtain: for \( x \in [-2x_0, 0] \cap (I(x_0, 4\delta_0) \cup I(-x_0, 4\delta_0))^c \),

\[
|\mathcal{H}_{\delta_0} a(x) | \lesssim \int \int_{-1}^{1} |a(t)| \left| \frac{x-x_0}{(x,x_0)^{\lambda+1}} \right| \left( \frac{1}{(x,x_0)^{\lambda+1}} \right)^{n+1} \left| (1+s)(1-s^2)^{\lambda-1} ds \right| \! |t|^{2\lambda} dt \\
+ \int \int_{-1}^{1} |a(t)| \left| \frac{x_0-t}{(x,x_0)^{\lambda+1}} \right| \left( \frac{1}{(x,x_0)^{\lambda+1}} \right)^{n} \left| (1+s)(1-s^2)^{\lambda-1} ds \right| \! |t|^{2\lambda} dt \\
\lesssim |I(x_0, \delta_0)|^{(1/p)} \int_{-1}^{1} \frac{(\delta_0)^{n+1}}{(x,x_0)^{n+2+\lambda+1}} (1+s)(1-s^2)^{\lambda-1} ds.
\]

(22)

It is easy to see the following inequality holds:

\[
\int_{-1}^{1} \frac{(\delta_0)^{n+1}(1+s)(1-s^2)^{\lambda-1}}{(x,x_0)\!^{n+2+\lambda+1}} ds \leq \int_{-1}^{1} \frac{(\delta_0)^{n+1}(1+s)(1-s^2)^{\lambda-1}}{||x|-|x_0||^n (x,x_0)^{\lambda+1}} ds.
\]

(23)

By Proposition (2.5), we obtain the following

\[
\int_{-1}^{1} \frac{(1+s)(1-s^2)^{\lambda-1}}{(x^2+x_0^2-2x_0x)^{\lambda+1}} ds = \left( \frac{1}{x^2+x_0^2} \right)^{\lambda+1} \int_{-1}^{1} (1+s)(1-s^2)^{\lambda-1} \left( 1 - \frac{2xx_0s}{x^2+x_0^2} \right)^{-\lambda-1} ds \leq C \frac{1}{(x^2+x_0^2)^{\lambda}||x|-|x_0||^2}.
\]

(24)

Thus from Formulas (23, 24) we have

\[
|I(x_0, \delta_0)|^{(1/p)} \int_{-1}^{1} \frac{(\delta_0)^{n+1}}{(x,x_0)^{n+2+\lambda+1}} (1+s)(1-s^2)^{\lambda-1} ds \\
\lesssim |I(x_0, \delta_0)|^{(1/p)} \frac{(\delta_0)^{n+1}}{||x|-|x_0||^{n+2}||x|+|x_0||^{2\lambda}}.
\]

(25)

When \( x \in [-2x_0, 0] \cap (I(x_0, 4\delta_0) \cup I(-x_0, 4\delta_0))^c \), we obtain:

\[
|\mathcal{H}_{\delta_0} a(x) | \leq C |I(x_0, \delta_0)|^{(1/p)} \frac{(\delta_0)^{n+1}}{||x|-|x_0||^{n+2}||x|+|x_0||^{2\lambda}}.
\]

(26)

Then we need to consider the case when \( x \in [-2x_0, 0] \cap (I(x_0, 4\delta_0) \cup I(-x_0, 4\delta_0))^c \). We could see that

\[
\frac{x-t}{(x,t)^{\lambda+1}} = \frac{x-x_0s}{(x,x_0)^{\lambda+1}} + \frac{x_0-t}{(x,t)^{\lambda+1}} + \frac{x_0(s-1)}{(x,x_0)^{\lambda+1}} = C + D + E.
\]

(27)

By the Taylor expansion, we could obtain:

\[
C = \frac{x-x_0s}{(x,x_0)^{\lambda+1}} \left[ \frac{1}{1+\frac{\delta_1}{(x,x_0)^{\lambda+1}}} \right]^{\lambda+1} \\
= \frac{x-x_0s}{(x,x_0)^{\lambda+1}} \left[ \frac{\lambda+1}{1} \left( \frac{-\delta_1}{(x,x_0)^{\lambda+1}} \right) + \frac{(\lambda+1)(\lambda+2)}{2!} \left( \frac{-\delta_1}{(x,x_0)^{\lambda+1}} \right)^2 + \cdots \right] \\
+ \frac{\lambda+1}{(n)!} \left( \frac{-\delta_1}{(x,x_0)^{\lambda+1}} \right)^n + \frac{\lambda+1}{(n+1)!} \left( \frac{1}{1+\xi_3} \right)^{\lambda+n+2} \left( \frac{-\delta_1}{(x,x_0)^{\lambda+1}} \right)^{n+1},
\]

(28)
By (17), \( \xi_3, \xi_4, \xi_5 \in [-3/4, 3/4] \). Thus:

\[
\left( \frac{1}{1+\xi_i} \right) \leq \left( \frac{1}{1-3/4} \right) \leq 4 \text{ for } i = 3, 4, 5. \tag{31}
\]

Thus Formulas (17, 31, 27, 28, 29, 30, 12, 15, 16), together with the vanishing property of \( a(t) \), we obtain: for \( x \in [-2x_0, 0] \cap (I(x_0, 4\delta_0) \cup (-x_0, 4\delta_0)) \)

\[
\begin{align*}
|\mathcal{H}_\lambda a(x)| &\lesssim \int_{-1}^{1} |a(t)| \bigg| \frac{x-x_0s}{\langle x, x_0 \rangle_{s}^{\lambda+1}} \left( \frac{-\delta_1}{\langle x, x_0 \rangle_s} \right)^{n+1} \bigg| (1+s)(1-s^2)^{\lambda-1}ds |t|^{2\lambda}dt \\
&\quad + \int_{-1}^{1} |a(t)| \bigg| \frac{x_0-t}{\langle x, x_0 \rangle_{s}^{\lambda+1}} \left( \frac{-\delta_1}{\langle x, x_0 \rangle_s} \right)^{n+1} \bigg| (1+s)(1-s^2)^{\lambda-1}ds |t|^{2\lambda}dt \\
&\quad + \int_{-1}^{1} |a(t)| \bigg| \frac{x_0(s-1)}{\langle x, x_0 \rangle_{s}^{\lambda+1}} \left( \frac{-\delta_1}{\langle x, x_0 \rangle_s} \right)^{n+1} \bigg| (1+s)(1-s^2)^{\lambda-1}ds |t|^{2\lambda}dt \\
&\lesssim |I(x_0, \delta_0)|_{\lambda}^{1-(1/p)} \int_{-1}^{1} \frac{(\delta_0)^{n+1}}{\langle x, x_0 \rangle_{s}^{\lambda+1}} |1+s|(1-s^2)^{\lambda-1}ds \\
&\quad + \int_{-1}^{1} |a(t)| \left| \frac{x_0(s-1)}{\langle x, x_0 \rangle_{s}^{\lambda+1}} \left( \frac{-\delta_1}{\langle x, x_0 \rangle_s} \right)^{n+1} \right| (1+s)(1-s^2)^{\lambda-1}ds |t|^{2\lambda}dt.
\end{align*}
\tag{32}
\]

Notice that \( \langle x, x_0 \rangle_s = x^2 + x_0^2 - 2x_0s \geq (1-s^2)x_0^2 \) holds for \( \forall x \in \mathbb{R} \). Thus we could obtain the following inequality:

\[
\begin{align*}
&\int_{-1}^{1} |a(t)| \left| \frac{x_0(s-1)}{\langle x, x_0 \rangle_{s}^{\lambda+1}} \left( \frac{-\delta_1}{\langle x, x_0 \rangle_s} \right)^{n+1} \right| (1+s)(1-s^2)^{\lambda-1}ds |t|^{2\lambda}dt \\
&\lesssim \int_{-1}^{1} |a(t)| \left| \frac{x_0(s-1)}{x_0(1-s^2)^{1/2} \langle x, x_0 \rangle_{s}^{\lambda+1}} \right| (1+s)(1-s^2)^{\lambda-1}ds |t|^{2\lambda}dt \\
&\lesssim |I(x_0, \delta_0)|_{\lambda}^{1-(1/p)} \int_{-1}^{1} \frac{(\delta_0)^{n+1}}{\langle x, x_0 \rangle_{s}^{\lambda+1}} (1-s^2)^{-1/2}ds \\
&\lesssim |I(x_0, \delta_0)|_{\lambda}^{1-(1/p)} \int_{-1}^{1} \frac{(\delta_0)^{n+1}}{||x|-x_0||^{\lambda+1} \langle x, x_0 \rangle_{s}^{\lambda+1}} (1-s^2)^{-1/2}ds.
\end{align*}
\tag{33}
\]
By Proposition \ref{prop:2}, Formula (33) implies that:
\[
|I(x, \delta_0)|^{1-(1/p)} \int_{-1}^{1} \frac{(\delta_0)^{n+1}}{||x| - |x_0||^{n+2}} (1 - \kappa^2)^{1/2} ds 
\leq |I(x, \delta_0)|^{1-(1/p)} \frac{(\delta_0)^{n+1}}{||x| - |x_0||^{n+2}} ||x| + |x_0||^{2\lambda}. \tag{34}
\]
Formulas (25, 34) imply the following inequality holds when \(x \in [-2x_0, 0] \cap (I(x, 4\delta_0) \cup I(-x, 4\delta_0))^c\):
\[
|\mathcal{H}_\lambda a(x)| \leq C |I(x, \delta_0)|^{1-(1/p)} \frac{(\delta_0)^{n+1}}{||x| - |x_0||^{n+2}} ||x| + |x_0||^{2\lambda}. \tag{35}
\]
From Formulas (35, 26), we obtain that:
\[
|\mathcal{H}_\lambda a(x)| \leq C |I(x, \delta_0)|^{1-(1/p)} \frac{(\delta_0)^{n+1}}{||x| - |x_0||^{n+2}} ||x| + |x_0||^{2\lambda} \tag{36}
\]
holds for \(x \in (I(x, 4\delta_0) \cup I(-x, 4\delta_0))^c\). Because \(\frac{2\lambda}{2\lambda + 1} < p \leq 1\), we could get \(0 \leq 2\lambda(1-p) \leq \frac{2\lambda}{2\lambda + 1} < 1\). Thus the following holds:
\[
|x|^{2\lambda(1-p)} \leq ||x| - |x_0||^{2\lambda(1-p)} + |x_0|^{2\lambda(1-p)}.
\]
Therefore:
\[
II \leq C |I(x, \delta_0)|^{p-1} \int_{\mathbb{R}} \left( \frac{(\delta_0)^{n+1}}{||x| - |x_0||^{n+2}} \right)^p (||x| - |x_0||^{2\lambda(1-p)} + |x_0|^{2\lambda(1-p)}) dx 
\approx |I(x, \delta_0)|^{p-1} \int_{\mathbb{R}} \left( \frac{(\delta_0)^{n+1}}{||x| - |x_0||^{n+2}} \right)^p (||x| - |x_0||^{2\lambda(1-p)}) dx 
+ |I(x, \delta_0)|^{p-1} \int_{\mathbb{R}} \left( \frac{(\delta_0)^{n+1}}{||x| - |x_0||^{n+2}} \right)^p (|x_0|^{2\lambda(1-p)}) dx 
= II_1 + II_2. \tag{37}
\]
\(\kappa\) and \(p\) satisfy the relation: \(\kappa = 2 \left( 2\lambda + 1 \right) \frac{1-p}{p} \), \(n = \kappa/2\). Therefore we could have:
\[
(n+2)p + 2\lambda(p-1) > 1 and (n+2)p > 1.
\]
Then the integral of \(II_1\) and \(II_2\) converge:
\[
II_1 = C |I(x_0, \delta_0)|^{p-1} \int_{\mathbb{R}} \left( \frac{(\delta_0)^{n+1}}{||x| - |x_0||^{n+2}} \right)^p (||x| - |x_0||^{2\lambda(1-p)}) dx 
\leq C |I(x_0, \delta_0)|^{p-1} (\delta_0)^{(n+1)} \int_{\mathbb{R}} \frac{1}{||x| - |x_0||^{(n+2)p+2\lambda(p-1)}} dx 
\leq C |I(x_0, \delta_0)|^{p-1} (\delta_0)^{(n+1)} \int_{4\delta_0}^{\infty} \frac{1}{r^{(n+2)p+2\lambda(p-1)}} dr 
\leq C, \tag{38}
\]
\[
II_2 = C |I(x_0, \delta_0)|^{p-1} \int_{\mathbb{R}} \left( \frac{(\delta_0)^{n+1}}{||x| - |x_0||^{n+2}} \right)^p (|x_0|^{2\lambda(1-p)}) dx 
\leq C x_0^{2\lambda(p-1)} (\delta_0)^{p-1} |x_0|^{2\lambda(1-p)} (\delta_0)^{(n+1)} (\delta_0)^{-(n+2)p+1} 
\leq C. \tag{39}
\]
Thus from Formulas (10, 37, 38, 39), the theorem is proved.
Theorem 2.9. If $a(t)$ is a $p_\lambda$-atom, with vanishing order $\kappa \geq 2 \left( 2 \lambda + 1 \right) \frac{1 - p}{p}$ then the following holds:

$$\frac{2\lambda}{2\lambda + 1} < p \leq 1 \quad \int_{\mathbb{R}} |(a * \lambda P_y)^\rho(x)|x|^{2\lambda} \, dx \leq C,$$

$C$ is depend on $\lambda$ and $p$.

Proof. Assume first that $x_0 > 0$. Let $\kappa = 2 \left( 2 \lambda + 1 \right) \frac{1 - p}{p}$. Thus $\kappa$ is an even integer. Let $n = \kappa/2$.

We could write the above integral as:

$$\int_{\mathbb{R}} |(a * \lambda P_y)^\rho(x)|x|^{2\lambda} \, dx = \int_{I(x_0, 4\delta_0) \cup (x_0, 4\delta_0)} |(a * \lambda P_y)^\rho(x)|x|^{2\lambda} \, dx$$

$$+ \int_{I(x_0, 4\delta_0) \cup (x_0, 4\delta_0)} |(a * \lambda P_y)^\rho(x)|x|^{2\lambda} \, dx = III + IV.$$

By [14] (Theorem 3.8) and Formula (9), we could get

$$III = \int_{I(x_0, 4\delta_0) \cup (x_0, 4\delta_0)} |(a * \lambda P_y)^\rho(x)|x|^{2\lambda} \, dx$$

$$\leq \left( \int_{I(x_0, 4\delta_0) \cup (x_0, 4\delta_0)} |(a * \lambda P_y)^\rho(x)|x|^{2\lambda} \, dx \right)^{p/2} \left( \int_{I(x_0, 4\delta_0) \cup (x_0, 4\delta_0)} |x|^{2\lambda} \, dx \right)^{1-p/2}$$

Next we estimate the following integer:

$$IV = \int_{I(x_0, 4\delta_0) \cup (x_0, 4\delta_0)} |(a * \lambda P_y)^\rho(x)|x|^{2\lambda} \, dx.$$

By Proposition 2.1, when $x \in I(x_0, 4\delta_0) \cup (x_0, 4\delta_0)$ we could write $(a * \lambda P_y)(x)$ as:

$$(a * \lambda P_y)(x) = c_{\lambda} \int a(t)(\tau_{x}P_y)(-t)|t|^{2\lambda} \, dt.$$

Notice that $t \in \text{supp } a(t) \subseteq I(x_0, \delta_0)$. By the Taylor expansion of formula $\left( 1 + \frac{\delta}{x,x_0} \right)^{-\lambda - 1}$, we could get the following

$$\frac{y}{(x_0,x)^{\lambda+1}} = \frac{y}{(x_0,x)^{\lambda+1}} \left( 1 + \frac{\delta}{x,x_0} \right)^{\lambda+1}$$

$$= \frac{y}{(x_0,x)^{\lambda+1}} \left[ 1 + \frac{\lambda}{x,x_0} \left( -\frac{\delta}{x,x_0} \right) + \frac{\lambda + 1}{2!} \left( -\frac{\delta}{x,x_0} \right)^2 + \cdots \right]$$

$$+ \frac{(\lambda + 1)n}{(n)!} \left( -\frac{\delta}{x,x_0} \right)^n + \frac{(\lambda + 1)n+1}{(n+1)!} \left( 1 + \frac{\delta}{x,x_0} \right)^{\lambda+2}$$

We could see that:

$$\left| \frac{\delta}{x,x_0} \right| \leq \left| \frac{3|x|}{(x_0,x)^{1/2}} \right| \leq \left| \frac{3|x|}{x_0^{1/2}} \right| \leq \left| \frac{3\delta_0}{4\delta_0} \right| = 3/4.$$ (41)

From (41), we could have: $\xi \in [-3/4, 3/4]$. Thus:

$$\left( \frac{1}{1 + \xi} \right) \leq \left( \frac{1}{1 - 3/4} \right) \leq 4,$$

$$\frac{y}{(x_0,x)^{1/2}} \leq \frac{1}{2(x_0,x)^{1/2}} \text{and } \frac{1}{(x_0,x)^{1/2}} \leq \frac{1}{1 + \xi}$$ (42)

From (42), we could have:

$$\left| \frac{1}{1 + \xi} \right|^{\lambda+2} \frac{y}{(x_0,x)^{\lambda+1}} \frac{\delta_1^{n+1}}{(x,x_0)^{\lambda+1}} \leq C \frac{y}{(x_0,x)^{\lambda+1}} \frac{|t-x_0|^{n+1}}{(x_0,x)^{n+1}}$$

$$\leq C \frac{|t-x_0|^{n+1}}{(x_0,x)^{n+2+\lambda+1}}.$$
Since we have
\[ |x_0 + t - 2s| \leq |x_0 - x| + |t - x| \]  
(43)
\[ \leq ((x, x_0)_s)^{1/2} + ((x, x_0)_s)^{1/2} + |t - x_0| \]
\[ \leq 3((x, x_0)_s)^{1/2} , \]
then the following inequality holds:
\[ |\delta_1| \leq 3|t - x_0|((x, x_0)_s)^{1/2} . \]  
(44)
Thus for \( x \in (I(x_0, 4\delta_0) \cup I(-x_0, 4\delta_0))^c \), by Proposition 2.1 and Formula (40), we could deduce the following
\[
(a * \lambda P_y)(x) = c_2 2^{\lambda + 1/2} \pi^{-1} \lambda \Gamma(\lambda + 1/2) \int a(t) \int_I^1 \frac{y}{\langle x, x_0 \rangle_{y,s}}^{\lambda + 1} \left[ 1 + \frac{\lambda + 1}{1} \left( -\delta_1 \right) \left( \frac{1}{(x, x_0)_{y,s}} \right)^{1/2} \right] + \frac{(\lambda + 1)(\lambda + 2)}{2!} \left( -\delta_1 \right)^{2} \frac{1}{\langle x, x_0 \rangle_{y,s}} + \cdots + \frac{(\lambda + 1)n}{(n)!} \left( -\delta_1 \right)^{n} \frac{1}{\langle x, x_0 \rangle_{y,s}} + \frac{(\lambda + 1)n+1}{(n+1)!} \left( 1 + \frac{1}{\langle x, x_0 \rangle_{y,s}} \right) \lambda^{n+1} \left( -\delta_1 \right)^{n+1} \left( (1 + s)(1 - s^2)^{\lambda-1} ds \right) t^{2\lambda} dt.
\]
Then the above formula together with the vanishing property of \( a(t) \) we obtain:
\[ |(a * \lambda P_y)(x)| \leq C|I(x_0, \delta_0)|^{1/(1/p)} \int_{-1}^{1} \frac{(\delta_0)^{n+1}}{(\langle x, x_0 \rangle_{s})^{n/2+\lambda+1}} (1 + s)(1 - s^2)^{\lambda-1} ds.
\]  
(45)
By (25),
\[ |(a * \lambda P_y)(x)| \leq C|I(x_0, \delta_0)|^{1/(1/p)} \frac{(\delta_0)^{n+1}}{|x| - |x_0|^{n/2+\lambda+1} |x| + |x_0|^{2\lambda}} \]  
(46)
holds for \( x \in (I(x_0, 4\delta_0) \cup I(-x_0, 4\delta_0))^c \). Thus the Theorem 2.9 is proved in the same way as Theorem 2.8. This proves the theorem. □

**Theorem 2.10.** If \( a(t) \) is a \( p_\lambda \)-atom, with vanishing order \( \kappa \geq 2 \left( 2\lambda + 1 \frac{1-p}{p} \right) \) then
\[ \left( \frac{2\lambda}{2\lambda + 1} < p \leq 1 \right) \int_R |(a * \lambda Q_y)|^p |x|^{2\lambda} dx \leq C , \]
\( C \) is depend on \( \lambda \) and \( p \).

**Proof.** Assume first that \( x_0 > 0 \). Let \( \kappa = 2 \left( 2\lambda + 1 \frac{1-p}{p} \right) \). Thus \( \kappa \) is an even integer. Let \( n = \kappa/2 \). We could write the above integral as:
\[
\int_R |(a * \lambda Q_y)|^p |x|^{2\lambda} dx = \int_{I(x_0, 4\delta_0) \cup I(-x_0, 4\delta_0)} |(a * \lambda Q_y)|^p |x|^{2\lambda} dx + \int_{I(x_0, 4\delta_0) \cup I(-x_0, 4\delta_0)^c} |(a * \lambda Q_y)|^p |x|^{2\lambda} dx = V + VI.
\]
We could have the estimation:
\[
V = \int_{I(x_0, 4\delta_0) \cup I(-x_0, 4\delta_0)} |(a * \lambda Q_y)|^p |x|^{2\lambda} dx \leq \left( \int_{I(x_0, 4\delta_0) \cup I(-x_0, 4\delta_0)} |(a * \lambda Q_y)|^2 |x|^{2\lambda} dx \right)^{p/2} \left( \int_{I(x_0, 4\delta_0) \cup I(-x_0, 4\delta_0)} |x|^{2\lambda} dx \right)^{1-p/2} \leq C \left( \int_R |a(x)|^2 |x|^{2\lambda} dx \right)^{p/2} (|I(x_0, 4\delta_0)|_\lambda)^{1-(p/2)} 2^{1-(p/2)} \leq C .
\]
Next we estimate the following integer:

\[
V_I = \int_{(I(x_0, 4\delta_0) \cup I(-x_0, 4\delta_0))^c} \left| (a \ast \chi_{Q_y}) \right| |x|^{2\lambda} \, dx.
\]

By Proposition 2.1, when \( x \in (I(x_0, 4\delta_0) \cup I(-x_0, 4\delta_0))^c \) we could write \((a \ast \chi_{Q_y})(x)\) as:

\[ (a \ast \chi_{Q_y})(x) = c_\lambda \int a(t)(\tau_x Q_y)(-t)|t|^{2\lambda} \, dt, \]

where \( supp(t) \subseteq I(x_0, \delta_0) \). Notice the following holds

\[
x - t \left( \frac{1}{\langle x, x \rangle_{y,s}^{\lambda+1}} \left( 1 + \frac{\delta_1}{\langle x, x \rangle_{y,s}^{\lambda+1}} \right) \right)_{y,s}^\lambda = \frac{x - t}{\langle x, x \rangle_{y,s}^{\lambda+1}} + \frac{x_0 - t}{\langle x, x \rangle_{y,s}^{\lambda+1}} + \cdots \tag{47}
\]

By the Taylor expansion of \((1 + \delta_1)\)\(^{\lambda-1}\), when \( x \in [-2x_0, 0]^c \cap (I(x_0, 4\delta_0) \cup I(-x_0, 4\delta_0))^c \), we could obtain:\(\exists \xi\)

\[
F = \left[ \frac{x - x_0}{\langle x, x \rangle_{y,s}^{\lambda+1}} + \frac{x_0 - t}{\langle x, x \rangle_{y,s}^{\lambda+1}} \left( 1 + \frac{\delta_1}{\langle x, x \rangle_{y,s}^{\lambda+1}} \right) \right]_{y,s}^\lambda \tag{48}
\]

and

\[
G = \left[ \frac{x_0 - t}{\langle x, x \rangle_{y,s}^{\lambda+1}} \left( 1 + \frac{\delta_1}{\langle x, x \rangle_{y,s}^{\lambda+1}} \right) \right]_{y,s}^\lambda \tag{49}
\]

We have to estimate the size of formula \(\delta_1\)\((x, x_0)_{y,s}\). Since \(|\delta_1| \leq 3|t-x_0|\left(\langle x, x_0 \rangle_{y,s}\right)^{1/2}\), and \( |x-x_0| \approx \left(\langle x, x_0 \rangle_{y,s}\right)^{1/2} \), when \( x \in [-2x_0, 0]^c \cap (I(x_0, 4\delta_0) \cup I(-x_0, 4\delta_0))^c \). Then:

\[
\left| \frac{\delta_1}{\langle x, x \rangle_{y,s}^{\lambda+1}} \right| \leq \frac{3|t-x_0|}{\left(\langle x, x_0 \rangle_{y,s}\right)^{1/2}} \leq \frac{3|t-x_0|}{\left|\langle x, x_0 \rangle_{y,s}\right|^{1/2}} \leq \frac{3\delta_0}{4\delta_0} = 3/4,
\]

\[
\left( \frac{1}{1+\xi_1} \right) \leq \left( \frac{1}{1+\xi_2} \right) \leq 4 \quad \text{and} \quad \left( \frac{1}{1+\xi_2} \right) \leq \left( \frac{1}{1+\xi_2} \right) \leq 4.
\]

Let \(dv(s)\) denote \((1 + s)(1 - s)^{\lambda-1}ds\). Then by Proposition 2.1 and Formulas (47, 48, 49) for \( x \in [-2x_0, 0]^c \cap (I(x_0, 4\delta_0) \cup I(-x_0, 4\delta_0))^c \), we could get

\[
(a \ast \chi_{Q_y})(x) \approx \int_{-1}^{1} a(t) \left( \frac{x - x_0}{\langle x, x \rangle_{y,s}^{\lambda+1}} + \frac{x_0 - t}{\langle x, x \rangle_{y,s}^{\lambda+1}} \left( 1 + \frac{\delta_1}{\langle x, x \rangle_{y,s}^{\lambda+1}} \right) \right)_{y,s}^\lambda \, dt + \cdots \tag{48}
\]

\[
(a \ast \chi_{Q_y})(x) \approx \int_{-1}^{1} a(t) \left( \frac{x_0 - t}{\langle x, x \rangle_{y,s}^{\lambda+1}} \left( 1 + \frac{\delta_1}{\langle x, x \rangle_{y,s}^{\lambda+1}} \right) \right)_{y,s}^\lambda \, dt + \cdots \tag{49}
\]
Taylor expansion, we could obtain:

\[ |(a \ast \lambda Q_y)(x)| \lesssim \int \int_{-1}^{1} |a(t)| |x - x_0| x \left( \frac{-\delta_1}{(x, x_0)_s} \right)^{n+1} \, dv(s)|t|^{2\lambda} \, dt \]

\[ + \int \int_{-1}^{1} |a(t)| \frac{x_0 - t}{(x, x_0)_s} \left( \frac{-\delta_1}{(x, x_0)_s} \right)^{n} \, dv(s)|t|^{2\lambda} \, dt \]

\[ \leq C |I(x_0, \delta_0)|^{1-(1/p)} \int_{-1}^{1} \left( \frac{\delta_0^{n+1}}{(x, x_0)_s} \right)^{\lambda+1} (1 + s)(1 - s^2)^{\lambda-1} \, ds. \tag{50} \]

We could see that:

\[ \frac{x - t}{(x, t)_y} = \frac{x - x_0}{(x, t)_y} + \frac{x_0 - t}{(x, t)_y} + \frac{x_0(s - 1)}{(x, t)_y} \]

\[ = C_1 + D_1 + E_1. \tag{51} \]

Then we need to estimate \( (a \ast \lambda Q_y)(x) \) when \( x \in [-2x_0, 0] \cap (I(x_0, 4\delta_0) \cup I(-x_0, 4\delta_0))^c \). By the Taylor expansion, we could obtain:

\[ C_1 = \frac{x - x_0}{(x, x_0)_y} \left( 1 + \frac{\delta_1}{(x, x_0)_y} \right)^{\lambda+1} \]

\[ = \frac{x - x_0}{(x, x_0)_y} \left[ 1 + \frac{\lambda + 1}{(x, x_0)_y} \left( \frac{-\delta_1}{(x, x_0)_y} \right)^{1} + \frac{(\lambda + 1)(\lambda + 2)}{2!} \left( \frac{-\delta_1}{(x, x_0)_y} \right)^{2} + \cdots \right. \]

\[ + \left. \frac{\lambda + 1}{(n)!} \left( \frac{-\delta_1}{(x, x_0)_y} \right)^{n} \right], \tag{52} \]

\[ D_1 = \frac{x_0 - t}{(x, t)_y} \left( 1 + \frac{\delta_1}{(x, x_0)_y} \right)^{\lambda+1} \]

\[ = \frac{x_0 - t}{(x, x_0)_y} \left[ 1 + \frac{\lambda + 1}{(x, x_0)_y} \left( \frac{-\delta_1}{(x, x_0)_y} \right)^{1} + \frac{(\lambda + 1)(\lambda + 2)}{2!} \left( \frac{-\delta_1}{(x, x_0)_y} \right)^{2} + \cdots \right. \]

\[ + \left. \frac{\lambda + 1}{(n - 1)!} \left( \frac{-\delta_1}{(x, x_0)_y} \right)^{n-1} \right], \tag{53} \]

and

\[ E_1 = \frac{x_0(s - 1)}{(x, x_0)_y} \left( 1 + \frac{\delta_1}{(x, x_0)_y} \right)^{\lambda+1} \]

\[ = \frac{x_0(s - 1)}{(x, x_0)_y} \left[ 1 + \frac{\lambda + 1}{(x, x_0)_y} \left( \frac{-\delta_1}{(x, x_0)_y} \right)^{1} + \frac{(\lambda + 1)(\lambda + 2)}{2!} \left( \frac{-\delta_1}{(x, x_0)_y} \right)^{2} + \cdots \right. \]

\[ + \left. \frac{\lambda + 1}{(n)!} \left( \frac{-\delta_1}{(x, x_0)_y} \right)^{n} \right]. \tag{54} \]

Notice that \( 0 < \langle x, x_0 \rangle_s \leq \langle x, x_0 \rangle_y \). Then by (17), we could have: \( \xi_3, \xi_4, \xi_5 \in [-3/4, 3/4] \). Thus:

\[ \left( \frac{1}{1 + \xi_i} \right) \leq \left( \frac{1}{1 - 3/4} \right) \leq 4 \text{ for } i = 3, 4, 5. \tag{55} \]

Notice that \( 0 < \langle x, x_0 \rangle_s \leq \langle x, x_0 \rangle_y \). Thus Formulas (51, 52, 53, 17, 55, 12, 15, 16), together
with the vanishing property of $a(t)$, we obtain the following inequality:

\[
\begin{align*}
|\langle a * \lambda Q_y \rangle(x)| & \lesssim \int \int_{-1}^{1} |a(t)| \frac{|x - x_0 s|}{\langle x, x_0 \rangle^\Lambda} \left( \frac{-\delta_1}{\langle x, x_0 \rangle} \right)^{n+1} \frac{(1 + s)(1 - s^2)^{\Lambda-1} ds \, |t|^2 \, dt}{\langle x, x_0 \rangle^\Lambda} \\
& + \int \int_{-1}^{1} |a(t)| \frac{|x_0 - t|}{\langle x, x_0 \rangle^\Lambda} \left( \frac{-\delta_1}{\langle x, x_0 \rangle} \right)^{n} \frac{(1 + s)(1 - s^2)^{\Lambda-1} ds \, |t|^2 \, dt}{\langle x, x_0 \rangle^\Lambda} \\
& + \int \int_{-1}^{1} |a(t)| \frac{|x_0(s - 1)|}{\langle x, x_0 \rangle^\Lambda} \left( \frac{-\delta_1}{\langle x, x_0 \rangle} \right)^{n+1} \frac{(1 + s)(1 - s^2)^{\Lambda-1} ds \, |t|^2 \, dt}{\langle x, x_0 \rangle^\Lambda} \\
& \lesssim |I(x_0, \delta_0)| \frac{1}{\lambda} (1/p) \int_{-1}^{1} \frac{(\delta_0)^{n+1}}{\langle \langle x, x_0 \rangle \rangle^{n/2 + \Lambda}} \frac{(1 + s)(1 - s^2)^{\Lambda-1} ds}{\langle x, x_0 \rangle^\Lambda} \\
& + \int \int_{-1}^{1} |a(t)| \frac{|x_0(s - 1)|}{\langle x, x_0 \rangle^\Lambda} \left( \frac{-\delta_1}{\langle x, x_0 \rangle} \right)^{n+1} \frac{(1 + s)(1 - s^2)^{\Lambda-1} ds \, |t|^2 \, dt}{\langle x, x_0 \rangle^\Lambda}.
\end{align*}
\]

for $x \in [-2x_0, 0] \cap (I(x_0, 4\delta_0) \cup I(-x_0, 4\delta_0))^c$. Similar to Formulas (32, 56) could imply that: for $x \in [-2x_0, 0] \cap (I(x_0, 4\delta_0) \cup I(-x_0, 4\delta_0))^c$

\[
|\langle a * \lambda Q_y \rangle(x)| \lesssim |I(x_0, \delta_0)| \frac{1}{\lambda} (1/p) \frac{(\delta_0)^{n+1}}{\langle x, x_0 \rangle^\Lambda} \frac{(1 + s)(1 - s^2)^{\Lambda-1} ds}{\langle x, x_0 \rangle^\Lambda}.
\]

Formulas (25, 50) imply that: for $x \in [-2x_0, 0] \cap (I(x_0, 4\delta_0) \cup I(-x_0, 4\delta_0))^c$

\[
|\langle a * \lambda Q_y \rangle(x)| \lesssim |I(x_0, \delta_0)| \frac{1}{\lambda} (1/p) \frac{(\delta_0)^{n+1}}{\langle x, x_0 \rangle^\Lambda} \frac{(1 + s)(1 - s^2)^{\Lambda-1} ds}{\langle x, x_0 \rangle^\Lambda}.
\]

Thus similar to (36), we could obtain that

\[
VI = \int_{I(x_0, 4\delta_0) \cup I(-x_0, 4\delta_0)^c} |\langle a * \lambda Q_y \rangle|^p(x)|x|^{2\Lambda} \, dx \leq C.
\]

This proves the theorem. \qed

In the way similar to Theorem 2.9 and Theorem 2.10, we could obtain the following Proposition

**Proposition 2.11.** If $a(t)$ is a $p_\lambda$-atom, then

\[
\left( \int \left| \partial_y \left( P(a_k)(x, y) \right) \right|^p |x|^{2\Lambda} \, dx \right)^{1/p} \lesssim \frac{1}{y}, \quad \left( \int \left| \partial_y \left( Q(a_k)(x, y) \right) \right|^p |x|^{2\Lambda} \, dx \right)^{1/p} \lesssim \frac{1}{y},
\]

where $C$ is a constant depend on $\lambda$ and $p$.

### 3 Main result

**Proposition 3.1.** Let $\{f_k\}_k (k \in \mathbb{N})$ be a sequence of $\lambda$-analytic functions on the set $S \cap \{y > 0\}$. If $\sum_k |\lambda_k| |f_k|, \sum_k |\lambda_k||\partial_x f_k|$ and $\sum_k |\lambda_k||\partial_y f_k|$ converge uniformly on the set $S \cap \{y > 0\}$, then $\sum_k \lambda_k f_k (k \in \mathbb{N})$ is a $\lambda$-analytic function on the set $S \cap \{y > 0\}$.

**Proof.** We denote $D_z$ as

\[
D_z = \frac{1}{2} (D_x + iD_y).
\]

A function $F(z) = F(x, y) = u(x, y) + iv(x, y)$ is a $\lambda$-analytic function if and only if $F(z)$ satisfies the following $\lambda$-Cauchy-Riemann equations:

\[
\begin{aligned}
D_x u - \partial_y v &= 0, \\
\partial_y u + D_x v &= 0.
\end{aligned}
\]

Then the $\lambda$-Cauchy-Riemann equations could be replaced by $D_z F(z) = 0$. Thus a function $F(z) = F(x, y) = u(x, y) + iv(x, y)$ is a $\lambda$-analytic function if and only if

\[
D_z F(z) = 0.
\]
Notice that \( \sum_k |\lambda_k||f_k|, \sum_k |\lambda_k||\partial_x f_k| \) and \( \sum_k |\lambda_k||\partial_y f_k| \) (\( k \in \mathbb{N} \)) converge uniformly on the set \( S \cap \{ y > 0 \} \). Thus we could have

\[
\sum_k \lambda_k \partial_y f_k = \partial_y \left( \sum_k \lambda_k f_k \right), \quad \sum_k \lambda_k \partial_x f_k = \partial_x \left( \sum_k \lambda_k f_k \right)
\]
on the set \( S \cap \{ y > 0 \} \). Thus \( \sum_k \lambda_k f_k \) (\( k \in \mathbb{N} \)) is a \( \lambda \)-analytic function on the set \( S \cap \{ y > 0 \} \).

**Lemma 3.2.** For \( 0 < p \leq 1, i \in \mathbb{N} \), if \( \sum_i |a_i|^p < \infty \), then

\[
\left( \sum_i |a_i|^p \right)^p \leq \sum_i |a_i|^p.
\]

**Proof.** Without loss of generality, we may assume that

\[
\sum_i |a_i|^p = 1.
\]

In this case, we have \( |a_i| \leq 1 \) for every \( i \), so

\[
\left( \sum_i |a_i|^p \right)^p \leq \sum_i |a_i|^p \leq 1.
\]

Then we could obtain:

\[
\left( \sum_i |a_i|^p \right)^p \leq \sum_i |a_i|^p.
\]

This proves the Lemma.

Let

\[
f = \sum_k \lambda_k a_k(x) \in H^p_\lambda(\mathbb{R}).
\]

By Proposition 2.1, \( Pa(x,y) + iQa(x,y) \) is a \( \lambda \)-analytic function. Together with Theorem 2.9, \( P(a_k)(x,y) + iQ(a_k)(x,y) \in H^p_\lambda(\mathbb{R}^2) \). From Proposition 2.3, we could deduce that

\[
\sup_{x \in \mathbb{R}} |P(a_k)(x,y) + iQ(a_k)(x,y)| \leq cy^{-(1/p)(1+2\lambda)}
\]

Then by Lemma 3.2, we could deduce the following for \( y > 0 \)

\[
\sup_{x \in \mathbb{R}} \left| \sum_k \lambda_k \left( P(a_k)(x,y) + iQ(a_k)(x,y) \right) \right| \leq cy^{-(1/p)(1+2\lambda)} \left( \sum_k |\lambda_k|^p \right)^{1/p}.
\]

Notice that the following holds

\[
\left| \partial_y \left( P(a_k)(x,y) \right) \right| = \left| Dx \left( Q(a_k)(x,y) \right) \right|, \quad \left| \partial_y \left( Q(a_k)(x,y) \right) \right| = \left| Dx \left( P(a_k)(x,y) \right) \right|
\]

Then by Proposition 3.1 and Proposition 2.11, we could deduce that \( \sum_k \lambda_k \left( P(a_k)(x,y) + iQ(a_k)(x,y) \right) \in H^p_\lambda(\mathbb{R}^2) \). Thus for \( f(x) \in H^p_\lambda(\mathbb{R}) \), we could write \( Pf(x,y), Qf(x,y) \) as

\[
Pf(x,y) = \sum_k \lambda_k P(a_k)(x,y), \quad Qf(x,y) = \sum_k \lambda_k Q(a_k)(x,y).
\]

Then by Theorem 2.9 and Theorem 2.10, we could obtain

\[
\sup_{y > 0} \left( \int_{\mathbb{R}} \left| \sum_k \lambda_k a_k * \lambda P_y(x) + i \sum_k \lambda_k a_k * \lambda Q_y(x) \right|^p |x|^{2\lambda} dx \right)^{1/p} \lesssim \sum_k |\lambda_k|^p.
\]

Thus we could obtain the following Proposition:
Proposition 3.3. For any \( f \in H^p_\lambda(\mathbb{R}^2) \), \( \frac{2\lambda}{2\lambda+1} < p \leq 1 \), we could deduce that \( Pf(x,y) + iQf(x,y) \in H^p_\lambda(\mathbb{R}^2) \) and the following holds:

\[
\sup_{y > 0} \left( \int_{\mathbb{R}} |f \ast_\lambda P_y(x) + i f \ast_\lambda Q_y(x)|^p |x|^{2\lambda} \, dx \right) \leq C \|f\|_{H^p_\lambda(\mathbb{R})}.
\]

By Proposition 3.3 and Theorem 2.2, we could obtain the main result in this paper.

Theorem 3.4. For any \( f \in H^p_\lambda(\mathbb{R}^2) \), \( \frac{2\lambda}{2\lambda+1} < p \leq 1 \), we could deduce the following:

\[
\int_{\mathbb{R}} |(\mathcal{F}_\lambda f)(\xi)|^{p} |\xi|^{(2\lambda+1)(k-1-k/p)+2\lambda} \, d\xi \leq c \|f\|_{H^p_\lambda(\mathbb{R})}^p.
\]

When \( k = p \), we could obtain

\[
\int_{0}^{\infty} |(\mathcal{F}_\lambda f)(\xi)|^{p} |\xi|^{(2\lambda+1)(p-2)+2\lambda} \, d\xi \leq c \|f\|_{H^p_\lambda(\mathbb{R})}^p.
\]

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