Symmetries of p–Branes

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ABSTRACT

Using canonical methods, we study the invariance properties of a bosonic p–brane propagating in a curved background locally diffeomorphic to $M \times G$, where $M$ is spacetime and $G$ a group manifold. The action is that of a gauged sigma model in $p + 1$ dimensions coupled to a Yang–Mills field and a $(p + 1)$–form in $M$. We construct the generators of Yang-Mills and tensor gauge transformations and exhibit the role of the $(p + 1)$–form in cancelling the potential Schwinger terms. We also discuss the Noether currents associated with the global symmetries of the action and the question of the existence of infinite dimensional symmetry algebras, analogous to the Kac-Moody symmetry of the string.

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1. Introduction

The importance of symmetries in physical systems hardly needs to be emphasized. In string theory, for example, the Virasoro and Kac-Moody symmetries play a crucial role in solving and interpreting the theory. In the case of higher $p$–branes, however, no analogues of these symmetries are known. This motivates a systematic study of all invariances of $p$–brane theory.

We choose to work with the bosonic $p$–brane theory discussed in [1]. It describes a $p$–brane propagating in a curved background locally diffeomorphic to $M \times G$, where $M$ is spacetime and $G$ is a group manifold. The $p$–brane is coupled to a metric, a Yang-Mills field and a rank–$(p + 1)$ antisymmetric tensor field on $M$. Strictly speaking our conclusions apply only to this particular theory. However we believe that our main conclusions apply also to fermionic formulations of the $p$–brane. This is certainly true in the case $p = 1$, in which bosonization works. In the higher case the bosonic and fermionic formulations are not equivalent. Instead, the former can be thought of as a low–energy approximation of the latter. We expect that symmetry aspects (such as anomaly cancellations) are properly reflected in the low–energy theory.

If the Yang–Mills and tensor gauge field on $M$ are held fixed, the model has world-volume diffeomorphisms as gauge invariances, and Noether symmetries consisting of those transformations which leave the background fields invariant. For generic background fields, these will be just the group $G_L$ of left multiplications in $G$. If the fields on $M$ are treated as dynamical, the theory is also invariant under diffeomorphisms of $M$ and local $G_R$ transformations, where $G_R$ is the group of right multiplications in $G$. In this case one also has the Noether charges associated with global $G_R$ transformations.

In this theory gauge invariance is achieved via a kind of Green–Schwarz anomaly cancellation mechanism, with the variation of the Wess–Zumino part of the action being compensated by the variation of the term involving the antisymmetric tensor. We will discuss in detail the counterpart of this phenomenon in the canonical approach. We shall construct explicitly the generators of Yang–Mills and tensor gauge transformations for arbitrary $p$–branes and show that they form a closed algebra. In the absence of the tensor gauge field one would find an anomalous extension of the Yang–Mills algebra of the Mickelsson–Faddeev type [2]. In the presence of the tensor gauge field these anomalous extensions cancel; what remains can be identified as a field-dependent tensor gauge transformation. An alternative derivation of the gauge generators is given in [3].

It appears from all this that the true analogue of the Kac–Moody symmetry of the string is the finite dimensional group $G_R$. While the infinite dimensional target space gauge in-
variances do not lead in the case of higher $p$–branes to true symmetry algebras, they are nonetheless interesting for several reasons. For example, the generators of these transformations can be realized as functional differential operators on the space of $p$–branes. These operators play a role in the study of $p$–brane field theory. Furthermore, they can be used to define functional covariant derivatives which act on functionals of the $p$–brane. These covariant derivatives become especially useful in the case of a $\kappa$–invariant super $p$–brane theory [4], because their algebra together with the principle of integrability along null super-planes holds the key to the target space equations of motion [5,6]. Even in the absence of an action, they can be used to construct the first class constraints of the theory, which may actually define the theory. We leave these applications for a future work.

The paper is organized as follows. In section 2 we describe the model. In section 3 we discuss the Noether symmetries of the theory. In particular we find that the algebra of Noether currents of global right and left multiplications have an abelian extension. In section 4 we treat the fields on $M$ as dynamical variables and derive the currents which generate the gauge transformations. The algebra of these currents is found to close, as one would expect in a gauge invariant theory. Throughout this paper we work with arbitrary odd $p$. Explicit formulae for the cases $p = 1, 3, 5$ are collected in an Appendix.

2. The Model

We will restrict our attention to the case of $p$–branes with $p$ odd. The dynamical variables describing the $p$–brane are scalar fields $x^\mu(\sigma)$, $y^m(\sigma)$ and a world-volume metric $\gamma_{ij}(\sigma)$. Here $\sigma^i$ ($i = 0, ..., p$) are the world-volume coordinates, $x^\mu$, $\mu = 0, ..., d - 1$ are coordinates on $M$ and $y^m$, $m = 1, ..., \dim G$ are coordinates on $G$. The background fields are the metrics $g_{\mu\nu}(x)$ and $g_{mn}(y)$ on $M$ and $G$ respectively, a Yang-Mills field $A^a_\mu(x)$ and antisymmetric tensor fields $B_{\mu_1...\mu_{p+1}}(x)$ and $b_{m_1...m_{p+1}}(y)$.

The action for the $p$–brane propagating in curved background can be written as [1]

$$S = \int d^{p+1}\sigma L$$

$$= \int d^{p+1}\sigma \sqrt{-\gamma} \left[ -\frac{1}{2} \left( \gamma^{ij} \partial_i x^\mu \partial_j x^\nu g_{\mu\nu} + \gamma^{ij} D_i y^m D_j y^n g_{mn} \right) + \frac{p-1}{2} + *B + *C - *b \right],$$

(2.1)

where $D_i y^m = \partial_i y^m - \partial_i x^\mu A^a_\mu L^m_a$. In this paper we denote $L^m_a$ (resp. $R^m_a$) the left-invariant (resp. right-invariant) Killing vectors on $G$, satisfying $[L_a, L_b] = f^{c}_{ab} L_c$, $[R_a, R_b] = -f^{c}_{ab} R_c$, $[L_a, R_b] = 0$. The left– (resp. right–) invariant Maurer-Cartan forms are $L_m = g^{-1} \partial_m g = L^a_m T_a$, $R_m = \partial_m g g^{-1} = R^a_m T_a$. We follow the conventions of [1], namely the generators $T_a$ obey $[T_a, T_b] = f^{c}_{ab} T_c$ and the raising and lowering of algebra indices is done with the
invariant tensor $d_{ab} = \text{tr} T_a T_b$. We will use form notation only for objects defined on the world–volume. For example, we will denote $A = \partial_i x^\mu A^i_\mu d\sigma^i$ and $L = \partial_t y^m L_m d\sigma^i$ the pull-backs of the connection on $M$ and of the Maurer-Cartan form on $G$. Similarly, the forms $B$ and $b$ in (2.1) are given by

$$B = \frac{1}{(p+1)!} B_{\mu_1\ldots\mu_{p+1}} \partial_{i_1} x^{\mu_1} \ldots \partial_{i_{p+1}} x^{\mu_{p+1}} d\sigma^{i_1} \wedge \ldots \wedge d\sigma^{i_{p+1}} ,$$

$$b = \frac{1}{(p+1)!} b_{m_1\ldots m_{p+1}} \partial_{i_1} y^{m_1} \ldots \partial_{i_{p+1}} y^{m_{p+1}} d\sigma^{i_1} \wedge \ldots \wedge d\sigma^{i_{p+1}} .$$

The tensor $B_{\mu_1\ldots\mu_{p+1}}$ is arbitrary, while the tensor $b_{m_1\ldots m_{p+1}}$ is defined by the relation

$$\partial_{[m_1} b_{m_2\ldots m_{p+2}]} = -k_p c_p (p+1) ! \text{tr} L_{[m_1} \ldots L_{m_{p+2}]} = -k_p c_p (p+1) ! \text{tr} R_{[m_1} \ldots R_{m_{p+2}]} ,$$

where $k_p$ and $c_p$ are normalization constants discussed in the Appendix. The pulled–back version of (2.3) can be written

$$db + \omega^0_{p+2}(L) = 0 ,$$

where $\omega^0_{p+2}$ is a Chern-Simons form (see the Appendix). The form $C$ in (2.1) is defined as follows. Let $A_t = t A + (1-t) L$ and $F_t = dA_t + A_t^2 = tF + t(t-1)(A-L)^2$. Defining the operator

$$\ell_t = dt(A^a - L^a) \frac{\partial}{\partial F_t^a} ,$$

we have

$$C(A, L) = \int_0^1 \ell_t \omega^0_{p+2}(A_t, F_t) .$$

Explicit forms for the cases $p = 1, 3, 5$ are given in the Appendix. The Lagrangian $\mathcal{L}$ contains the duals of the forms $B, C$ and $b$. The dual of a $p+1$–form $\omega$ is $^*\omega = \frac{1}{(p+1)!} \epsilon^{i_1\ldots i_{p+1}} \omega_{i_1\ldots i_{p+1}}$.

The action is manifestly invariant under world–volume diffeomorphisms and global $G_L$, which infinitesimally is given by $\delta y^m = \epsilon^a R^m_a(y)$, where $\epsilon$ is a constant. If the fields $A^a_\mu$ and $B_{\mu_1\ldots\mu_{p+1}}$ are treated as independent variables, then the action is also invariant under the tensor gauge transformations

$$\delta_\Lambda B_{\mu_1\ldots\mu_{p+1}} = (p+1) \partial_{[\mu_1} \Lambda_{\mu_2\ldots\mu_{p+1}]}$$

and under the target space local $G_R$ transformations

$$\delta_\epsilon y^m = \epsilon^a(x) L^m_a(y) ,$$

$$\delta_\epsilon A^a_\mu = \partial_\mu \epsilon^a + f^a_{\ bc} A^b_\mu \epsilon^c ,$$

$$\delta_\epsilon B_{\mu_1\ldots\mu_{p+1}} = - (p+1) \partial_{[\mu_1} \epsilon^a \phi^a_{\mu_2\ldots\mu_{p+1}]}(A) ,$$

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where $\phi^a(A)$ is a polynomial in $A_\mu$ and $F_{\mu\nu}$ which is defined by equation (A.5). The invariance is seen easily by noting that the variations of the pulled-back fields are

$$
\begin{align*}
\delta L &= d\epsilon + L\epsilon - \epsilon L, \\
\delta A &= d\epsilon + A\epsilon - \epsilon A, \\
\delta B &= -\omega^1_{p+1}(A, \epsilon) + d\Lambda,
\end{align*}
\tag{2.9}
$$

where $\omega^1_{p+1}$ is defined by (A.4). Under the transformations (2.9), we find (up to surface terms) from equation (2.4) $\delta b = -\omega^1_{p+1}(L, \epsilon)$, while $\delta C = \omega^1_{p+1}(A, \epsilon) - \omega^1_{p+1}(L, \epsilon)$, so the action (2.1) is gauge invariant.

The algebra of the gauge transformations (2.8) is derived by using the Wess–Zumino consistency condition (formula (A.7) in the Appendix) and reads

$$
\begin{align*}
[\delta_{\epsilon_1}, \delta_{\epsilon_2}] &= \delta_{[\epsilon_1, \epsilon_2]} + \delta_{\Lambda=\omega^2_\rho}, \\
[\delta_{\epsilon}, \delta_{\Lambda}] &= 0, \\
[\delta_{\Lambda_1}, \delta_{\Lambda_2}] &= 0,
\end{align*}
\tag{2.10a-c}
$$

where $\omega^2_\rho(A, \epsilon_1, \epsilon_2)$ is the 2-cocycle defined in (A.7).

Notice that $\int (C-b)$ is the Wess–Zumino action, whose infinitesimal gauge variation is the consistent anomaly $\omega^1_{p+1}(A, \epsilon)$. The difference between the usual gauged Wess–Zumino model and the present theory resides in the fact that here the gauge field $A$ is not a fundamental variable but rather a composite field, i.e. a fixed functional of the scalar fields $x$. Therefore the anomaly that would be present in the absence of the $B$ field is a so-called sigma model anomaly.

If we treat also the spacetime metric as an independent variable, then the theory is manifestly invariant under target space diffeomorphisms. Unlike the case of the gauge symmetry discussed above, there is no subtle anomaly cancellation involved here, so we shall not discuss this invariance further in this paper.

3. The Noether Symmetries

In this section we will treat the fields $A^a_\mu$, $B_{\mu_1...\mu_{p+1}}$ and $g_{\mu\nu}$ as fixed backgrounds, as is customary when the $p$–brane is treated as a fundamental theory. Since the background fields are not to be varied, the only invariances of the theory are the world–volume diffeomorphisms and the global transformations which leave the background fields invariant. In the case of the string there is in addition Weyl invariance. In this case the Virasoro and the Kac–Moody groups are infinite dimensional global symmetries with associated Noether charges. No such
infinite dimensional symmetries are known in the case of higher \( p \)-branes with the usual action. We shall illustrate this fact below in the case of Kac–Moody symmetry.

For general \( p \)-branes the world–volume diffeomorphisms will have vanishing Noether charges, as one would expect of a gauge invariance. We will not discuss them any further in this paper.

We will only be interested in variations of the form \( \delta x^\mu = 0 \), \( \delta y^m = v^m(y) \). Then the invariance condition reads

\[
0 = \delta S = \int d^{p+1} \sigma \left[ \sqrt{-\gamma^{ij} D_i y^m L_m^a \partial_j y^n L_n^a} - \frac{1}{(p+1)!} \varepsilon^{i_1 \ldots i_p+1} \partial_{i_1} y^{m_1} \ldots \partial_{i_p} y^{m_{p+1}} L_v b_{m_1 \ldots m_{p+1}} \right.
\]

\[
+ \frac{\delta}{\delta L^a_i(\sigma)} \partial_i y^m L_v L^a_m \right]. 
\]

This can be satisfied by choosing \( v \) so that \( L_v b_{m_1 \ldots m_{p+1}} = 0 \) and

\[
L_v L^a_m = (p+1) \partial_{[m_1} \lambda_{m_2 \ldots m_{p+1}]} \]

for some \( p \)-form \( \lambda \) on \( G \). These conditions are satisfied by the global left group action \( v^m = \epsilon^a R^m_a \), where \( \epsilon \) is constant. The Noether current corresponding to this transformation is

\[
J^a_{Ra} = R^m_a \left( \sqrt{-\gamma^{ij} D_i y^m + L^b_m \partial^* C} \right) + \frac{1}{p!} \varepsilon^{i_1 \ldots i_p} \partial_{i_1} y^{m_1} \ldots \partial_{i_p} y^{m_p} \lambda^a_{m_1 \ldots m_p} ,
\]

where the \( p \)-form \( \lambda^a_p \) is defined by \( \lambda_p = \epsilon a \lambda^a_p \). From (2.3) and (3.2) one gets

\[
\lambda^a_{m_1 \ldots m_p} = R^m_a b_{m_1 \ldots m_p} - k_p c_p \frac{(p+2)!}{p+1} \text{tr} T^a R_{m_1} \ldots R_{m_p} .
\]

The momenta canonically conjugate to \( y^m \) are

\[
p_m = \frac{\partial L}{\partial \partial_0 y^m} = \sqrt{-\gamma^{0j} D_j y^m g_{mn}} - \frac{1}{p!} \varepsilon^{r_1 \ldots r_p} \partial_{r_1} y^{m_1} \ldots \partial_{r_p} y^{m_p} b_{m_1 \ldots m_p} + L^a_m \frac{\partial^* C}{\partial L^a_0} ,
\]

where \( g_{mn} = L^a_m L^b_n \delta_{ab} \) is the invariant metric on \( G \) and \( r = 1 \ldots p \) refer to spacelike directions of the world–volume. Then, for \( i = 0 \) the charge density can be rewritten as

\[
J^i_{Ra} = R^m_a \left( p_m + \frac{1}{p!} \varepsilon^{r_1 \ldots r_p} \partial_{r_1} y^{m_1} \ldots \partial_{r_p} y^{m_p} b_{m_1 \ldots m_p} - k_p c_p \frac{(p+2)!}{p+1} \text{tr} R_{m_1} \ldots R_{m_p} \right).
\]
The algebra of the charge densities is

\[ \{ J_{Ra}(\sigma), J_{Rb}(\sigma') \} = -f_{abc} J_{Rc}(\sigma) \delta^{(p)}(\sigma, \sigma') \]

\[ + 2k_p c_p (p + 2) \varepsilon^{r_1 \cdots r_p} \partial_1 \partial_{r_1} \cdots \partial_{r_{p-1}} L_{r_{p-1}} \partial_{r_p} \delta^{(p)}(\sigma, \sigma') . \]

(3.7)

where \( R_r = \partial_r y^m R_m \). Note that the extension integrates to zero on a spacelike surface, so the algebra of the Noether charges \( Q_{Ra} = \int d^p \sigma J_{Ra}(\sigma) \) is (anti-)isomorphic to the Lie algebra of \( G \).

For strings (\( p=1 \)) the Noether symmetry group is much larger. If we set \( v^m = \epsilon^a(\sigma) R^m_a \), condition (3.1) reduces to

\[ (\sqrt{-\gamma} \gamma^{ij} - \varepsilon^{ij}) D_j \epsilon^a = 0 . \]

(3.8)

For \( A_\mu = 0 \) this equation admits all functions of \( \sigma^0 + \sigma^1 \) as solutions. These form an infinite dimensional Kac-Moody algebra with a central extension. It is a Noether symmetry because the corresponding Noether charges are nonvanishing. One could ask whether a similar algebra exists also for higher \( p \)-branes. Restricting our attention for simplicity to the case \( A_\mu = 0 \), the invariance condition (3.1) reduces to

\[ \int d^{p+1} \sigma \left( \sqrt{-\gamma} \gamma^{ij} \delta^{ab} + k_p c_p \frac{(p + 2)!}{p + 1} \varepsilon^{ijk_1 \cdots k_{p-1}} \text{tr} T^a T^b L_{k_1} \cdots L_{k_{p-1}} \right) \partial_j \epsilon^a = 0 . \]

(3.9)

As opposed to the case of eq. (3.8), the coefficient matrix in brackets is now a functional of the fields \( y^m \). For generic fields this matrix is nondegenerate and therefore the only solution is \( \epsilon^a \) constant. Consequently, the corresponding Noether charges are just the usual global Yang–Mills charges.

Finally we note that in the absence of gauge fields the theory would also have a global right \( G \) invariance, with associated Noether charge density

\[ J_{La} = L^m_a \left( p_m + \frac{1}{p!} \varepsilon^{r_1 \cdots r_p} \partial_{r_1} y^{m_1} \cdots \partial_{r_p} y^{m_p} \left( b_{m_1 \cdots m_p} + k_p c_p \frac{(p + 2)!}{p + 1} \text{tr} L_m L_{m_1} \cdots L_{m_p} \right) \right) . \]

(3.10)

Note that using (A.5) and (A.12b) the last term can be rewritten as \( \phi^a_p(L) \). The algebra of these currents is

\[ \{ J_{La}(\sigma), J_{Lb}(\sigma') \} = f_{abc} J_{La}(\sigma) \delta^{(p)}(\sigma, \sigma') \]

\[ - 2k_p c_p (p + 2) \varepsilon^{r_1 \cdots r_p} \text{tr} \left( \{ T_a, T_b \} L_{r_1} \cdots L_{r_{p-1}} \right) \partial_{r_p} \delta^{(p)}(\sigma, \sigma') . \]

(3.11)

For later use we observe that multiplying this formula by \( \epsilon_1(\sigma) \epsilon_2(\sigma') \) (not to be confused with the constant symmetry transformation parameters), integrating over \( \sigma, \sigma' \) and making use of equation (A.12c), we can write

\[ \{ J_{L\epsilon_1}, J_{L\epsilon_2} \} = J_L[\epsilon_1, \epsilon_2] + \omega^2_p(L, \epsilon_1, \epsilon_2) . \]

(3.12)
In the case $p = 3$ this algebra was derived using canonical methods in [7] and from a different point of view in [8].

4. Target Space Gauge Invariance

As mentioned above, if we allow the background fields to be transformed in a suitable way, the theory is invariant under target-space dependent gauge transformations. This invariance arises through a cancellation of anomalous terms. In fact, if one drops the field $x$ and treats $A$ as a fundamental (rather than composite) gauge field, then the action (2.1) (without the term $\ast B$) describes a gauged sigma model with Wess-Zumino term. This model is well-known not to be gauge invariant and moreover its anomaly cannot be cancelled by the introduction of a fundamental $B$ field (the field equation for $B$ would be inconsistent). However, in our model $A$ is a composite gauge field and the resulting anomaly is known as a sigma model anomaly. In this case a composite $B$ field can be meaningfully employed to cancel the anomaly via the Green-Schwarz mechanism (for strings, this was illustrated in [9]).

In this section, we are going to discuss this anomaly-cancellation mechanism at the Hamiltonian level. As is well known, anomalies appear in the Hamiltonian formulation as Schwinger terms in the algebra of the currents which couple to the gauge fields. A convenient way of dealing with this problem is to treat the fields $A^a_\mu$ and $B_{\mu_1...\mu_{p+1}}$ as dynamical variables. In this case the action which describes the dynamics of the extended objects coupled to these fields can be written as

$$S = \int d^{p+1}\sigma \int d^d x \delta^d(x, x(\sigma)) \mathcal{L},$$

(4.1)

where $\mathcal{L}$ is defined as in (2.1), with $A_\mu$ and $B_{\mu_1...\mu_{p+1}}$ now regarded as functions of $x$ rather than $x(\sigma)$. At this point we could also add to $\mathcal{L}$ independent kinetic terms for $A$ and $B$, but we shall not do so here. Note that we could have introduced also a factor $\delta(y, y(\sigma))$ and an integration over $y$. However, since the fields $A_\mu$ and $B_{\mu_1...\mu_{p+1}}$ are $y$-independent, this procedure is unnecessary. Therefore it is always understood that $y = y(\sigma)$.

The equations of motion for the spacetime fields are

$$0 = j^a_\mu(x) = \frac{\delta S}{\delta A^a_\mu(x)} = -\int d^{p+1}\sigma \delta^d(x, x(\sigma)) \sqrt{-\gamma} \gamma^{ij} \partial_i x^\mu D_i y^m I_a^m + \frac{\delta}{\delta A^a_\mu(x)} \int \ast C,$$

(4.2a)

$$0 = j^{\mu_1...\mu_{p+1}}(x) = \frac{\delta S}{\delta B_{\mu_1...\mu_{p+1}}(x)} = \varepsilon^{\mu_1...\mu_{p+1}} \partial_{\mu_1} x^{\mu_1} \cdots \partial_{\mu_{p+1}} x^{\mu_{p+1}}.$$  

(4.2b)

For simplicity of notation from now on the symbol $\int$ of indefinite integration will stand for $\int d^d x \int d^{p+1}\sigma \delta^d(x, x(\sigma))$. Notice that owing to the absence of kinetic terms, the equations
(4.2) do not contain second time derivatives of the fields and are therefore equations of constraint.

In the Hamiltonian formulation of this theory one has primary constraints

\[ P^\mu_a = \frac{\partial^* C}{\partial \partial_0 A_{\mu}^a} = 0, \tag{4.3a} \]
\[ P^{\mu_1\ldots\mu_{p+1}} = 0, \tag{4.3b} \]

where \( P^\mu_a \) and \( P^{\mu_1\ldots\mu_{p+1}} \) are the momenta canonically conjugate to \( A_{\mu}^a \) and \( B^{\mu_1\ldots\mu_{p+1}} \) respectively. These primary constraints arise from the fact that kinetic terms have not been included for \( A_{\mu}^a \) and \( B^{\mu_1\ldots\mu_{p+1}} \). In fact, demanding that the primary constraints have vanishing Poisson brackets with the Hamiltonian, one finds the secondary constraints which are equivalent to the field equations (4.2). Since the theory is gauge invariant, there must exist linear combinations of these constraints which are first class and generate the gauge transformations.

We choose the gauge for world-volume diffeomorphisms such that \( x^0 = \sigma^0 \). Then from (4.2), using (3.5), we obtain

\[ j^0_a(x) = \int d^{p+1} \sigma \delta^d(x, x(\sigma)) L_a^m \left( p_m + \frac{1}{p!} \varepsilon^{r_1\ldots r_p} \partial_{r_1} y^{m_1} \cdots \partial_{r_p} y^{m_p} b_{m_1\ldots m_p} \right) \]
\[ - \left( \frac{\delta \int^* C}{\delta A_0^a(\sigma)} + \frac{\delta \int^* C}{\delta L_0^a(\sigma)} \right), \tag{4.4} \]

\[ j^{0\mu_1\ldots\mu_p}(x) = \int d^{p+1} \sigma \delta^d(x, x(\sigma)) \varepsilon^{r_1\ldots r_p} \partial_{r_1} x^{\mu_1} \cdots \partial_{r_p} x^{\mu_p}. \tag{4.5} \]

In this formula and in the rest of the paper, we use world-volume pullbacks \( A_0^a = \partial_0 x^\mu A_\mu^a \) and \( L_0^a = \partial_0 y^m L_m^a \). Note that because \( dL = -L^2 \) we can assume that no derivatives of \( L \) appear in \( C \), and therefore \( \frac{\delta C(\sigma')}{\delta L_0^a(\sigma)} = \frac{\partial^* C}{\partial L_0^a(\sigma)} \delta(\sigma, \sigma') \).

We find that the first class constraints generating tensor gauge transformations are

\[ G_\Lambda = \int \left( (p+1) \partial_{\mu_1} \Lambda_{\mu_2\ldots\mu_{p+1}} P^{\mu_1\ldots\mu_{p+1}} - \frac{1}{p!} \Lambda_{\mu_1\ldots\mu_{p-1}} j^{0\mu_1\ldots\mu_p} \right) \tag{4.6} \]

and those generating Yang–Mills gauge transformations are

\[ G_\epsilon = \int \left[ D_\mu \epsilon^a \left( \phi^a_\mu - \frac{\partial^* C}{\partial \partial_0 A^a_\mu} \right) + \epsilon^a j^0_a \right. \]
\[ - (p+1)(\partial_{\mu_1} \epsilon^a_\mu) \phi_{\mu_2\ldots\mu_{p+1}}(A) P^{\mu_1\ldots\mu_{p+1}} + \frac{1}{p!} \epsilon^a \phi_{\mu_1\ldots\mu_{p+1}}(A) j^{0\mu_1\ldots\mu_p} \right]. \tag{4.7} \]
where $\phi_p$ is defined in (A.5). These equations can be understood as follows. The coefficients of the momenta $P^\mu_a, p_m$ and $P^{\mu_1...\mu_{p+1}}$ are fixed by the requirement that the Poisson brackets of $G_\epsilon$ with the fields $A^a_\mu, y^m$ and $B^{\mu_1...\mu_{p+1}}$ yield the gauge transformations (2.9). Since the momenta appear linearly in the constraints, this fixes the coefficients of the first three terms. The coefficient of the last term is fixed by the requirement that the generators form a closed algebra.

To prove that this happens we first simplify the form of the generator $G_\epsilon$. We observe that the second term in round brackets in (4.7) and the third term in (4.4) combine as follows:

$$
\int \left( -D_\mu \epsilon^a A^a_\mu + \epsilon^a \delta^* C \right) = \int d^dx \epsilon^a \frac{\partial^* C}{\partial A^a_0} .
$$

(4.8)

The partial derivative on the r.h.s. of this formula means that $C$ should be written in terms of $A$ and $F$ and varied only with respect to $A$. Thus, the first line in (4.7) can be written as

$$
G^{(1)}_\epsilon = \int \left[ D_\mu \epsilon^a P^\mu_a + \epsilon^a L^m_a \left( p_m + \frac{1}{p!} \epsilon^{r_1...r_p} \partial r_1 y^m_{r_1} \cdots \partial r_p y^{m_p} b^{m_{r_1}...m_p} \right) - \epsilon^a V_a \right] ,
$$

(4.9)

where

$$
V_\epsilon = \epsilon^a V_a = \epsilon^a \left( \frac{\partial^* C}{\partial A^a_0} + \frac{\partial^* C}{\partial L^a_0} \right) .
$$

(4.10)

To compute $V$, let us introduce a (graded) derivation $\ell_\epsilon$, defined by $\ell_\epsilon A = \ell_\epsilon L = \epsilon, \ell_\epsilon F = 0$. Then we can write $V_\epsilon = \ell_\epsilon C$. We now use the formula (2.6) for $C$. We observe that $\ell_\epsilon$ anticommutes with $\ell_t$ and that acting on $\omega^0_{p+2}(A_t, F_t)$ it coincides with the operator $\ell_\lambda$ defined in the Appendix of [10] where it is also shown that $\ell_\lambda \omega^0_{p+2} = \hat{\omega}^1_{p+1}$ (see (A.9)). Using these results we obtain

$$
V_\epsilon = - \int_0^1 \ell_t \ell_\epsilon \omega^0_{p+2}(A_t, F_t) = \int_0^1 \ell_t \text{tr} \epsilon d\phi_p(A_t, F_t) .
$$

We now apply to $\phi_p$ the homotopy formula [10]

$$
d_t = dt \frac{d}{dt} = (\ell_t d - d\ell_t)
$$

(4.11)

with $\ell_t$ given by equation (2.5). We find that

$$
V_a = \phi^a_p(A) - \phi^a_p(L) + d\chi^a_{p-1} ,
$$

(4.12)

where

$$
\chi^a_{p-1}(A, L) = \int_0^1 \ell_t \phi^a_p(A_t, F_t) .
$$

(4.13)
The explicit expressions for $\chi_p$ in the cases $p = 1, 3, 5$ is given in (A.13). Substituting (4.12) in (4.9), the second term in $V_a$, which from (A.12b) is equal to $c_p(p + 2)\text{tr} T_a L^p$, combines with the purely $y$-dependent terms in $G_\epsilon$ to yield the generator of the global right multiplications given in (3.10). We can thus write $G_\epsilon = G^{(1)} + G^{(2)}_\epsilon$, with

\begin{equation}
G^{(1)}_\epsilon = \int \left[ D_\mu \epsilon^a P^\mu_a + \epsilon^a (J_L a - d\chi_{p-1}^a (A, L) - \phi_p^a (A)) \right], \tag{4.15a}
\end{equation}

\begin{equation}
G^{(2)}_\epsilon = \int \left[ -(p + 1) \partial_\mu \epsilon^a \phi_{\mu_2 \ldots \mu_{p+1}} (A) P^{\mu_1 \ldots \mu_{p+1}} + \epsilon^a \phi_p^a (A) \right]. \tag{4.15b}
\end{equation}

For the reader’s convenience we recall that $J_L$, $\chi_a$ and $\phi$ are defined in (3.10), (4.13) and (A.5) respectively. Note that the terms of the form $\text{tr} \phi$ cancel in $G_\epsilon$ and one gets

\begin{equation}
G_\epsilon = \left[ D_\mu \epsilon^a P^\mu_a - (p + 1) \partial_\mu \epsilon^a \phi_{\mu_2 \ldots \mu_{p+1}} (A) P^{\mu_1 \ldots \mu_{p+1}} + \epsilon^a (J_L a - d\chi_{p-1}^a) \right]. \tag{4.16}
\end{equation}

Even though for the purpose of computing the algebra it would be more efficient to make use of this simplification, it is instructive to keep $G^{(1)}_\epsilon$ and $G^{(2)}_\epsilon$ separate. In fact, $G^{(1)}_\epsilon$ is identical to the Gauss law operator of the gauged Wess-Zumino-Witten model, except for the fact that the gauge field $A$ is now composite. On the other hand $G^{(2)}_\epsilon$ is a linear combination of the constraints which follow from the existence of the field $B$, and has no analogue in the gauged Wess-Zumino-Witten model.

We will now compute separately the Poisson brackets of $G^{(1)}_\epsilon$ and $G^{(2)}_\epsilon$. Using (3.12) we have

\begin{equation}
\{G^{(1)}_{\epsilon_1}, G^{(1)}_{\epsilon_2}\} = \int \left[ D_\mu [\epsilon_1, \epsilon_2]^a P^\mu_a + [\epsilon_1, \epsilon_2]^a J_L a + \omega_p^2 (L, \epsilon_1, \epsilon_2)
- \delta_{\epsilon_1} (\epsilon_2^a d\chi_{p-1}^a) + \delta_{\epsilon_2} (\epsilon_1^a d\chi_{p-1}^a) - \delta_{\epsilon_1} (\epsilon_2^a \phi_p^a) + \delta_{\epsilon_2} (\epsilon_1^a \phi_p^a) \right]. \tag{4.17}
\end{equation}

The last four terms in this formula arise from the Poisson brackets of the first two terms with the last two terms in (4.15a). Applying the homotopy formula to $\omega_p^2$ one gets the identity

\begin{equation}
\delta_{\epsilon_1} \int \epsilon_2 d\chi_{p-1} - \delta_{\epsilon_2} \int \epsilon_1 d\chi_{p-1} = \int \left( \epsilon_1 \epsilon_2 d\chi_{p-1} - \omega_p^2 (A, \epsilon_1, \epsilon_2) + \omega_p^2 (L, \epsilon_1, \epsilon_2) \right). \tag{4.18}
\end{equation}

On the other hand subtracting algebraically equation (A.10) from (A.7) we find that

\begin{equation}
\delta_{\epsilon_1} \int \epsilon_2 \phi_p - \delta_{\epsilon_2} \int \epsilon_1 \phi_p = \int \left( \epsilon_1 \epsilon_2 \phi_p + \omega_p^2 (A, \epsilon_1, \epsilon_2) - \omega_p^2 (A, \epsilon_1, \epsilon_2) \right). \tag{4.19}
\end{equation}

Substituting in (4.16) we find that the operators $G^{(1)}_\epsilon$ satisfy the algebra

\begin{equation}
\{G^{(1)}_{\epsilon_1}, G^{(1)}_{\epsilon_2}\} = G^{(1)}_{[\epsilon_1, \epsilon_2]} + \int \tilde{\omega}_p^2 (A, \epsilon_1, \epsilon_2). \tag{4.20}
\end{equation}
This agrees with the explicit calculations in the Wess-Zumino-Witten model in two and four dimensions [11]. (It is interesting to note that if one adds to $G^{(1)}_c$ the term $\int \phi^a_p(A)$, then by using (4.19), one finds that the anomalous extension in (4.20) gets replaced by $\omega^2_p(A, \epsilon_1, \epsilon_2)$).

Next, using the Wess–Zumino consistency condition (A.7) for $\omega^{p+1}$, we get

\[
\{G^{(1)}_{\epsilon_1}, G^{(2)}_{\epsilon_2}\} = \int \left[ (-\omega^{1}_{p+1}(A_{\mu}, [\epsilon_1, \epsilon_2]) - d\omega^{2}_{p}(A_{\mu}, \epsilon_1, \epsilon_2)) \right] \mu_1...\mu_{p+1} P^{\mu_1...\mu_{p+1}}
\]

\[
+ \delta_{\epsilon_1} (\epsilon_2^a \phi^a_p) - \delta_{\epsilon_2} (\epsilon_1^a \phi^a_p)
\]

\[
= G^{(2)}_{[\epsilon_1, \epsilon_2]} - G_\Lambda(A, \epsilon_1, \epsilon_2) - \int \hat{\omega}^2_p(A, \epsilon_1, \epsilon_2)
\]

(4.21)

where $G_\Lambda$ is the generator of a field–dependent tensor gauge transformation with parameter $\Lambda(A, \epsilon_1, \epsilon_2) = \omega^2_p(A, \epsilon_1, \epsilon_2)$. Collecting (4.20), (4.21) and observing that $\{G^{(2)}_{\epsilon_1}, G^{(2)}_{\epsilon_2}\} = 0$, we see that the anomalous extensions cancels and we remain with

\[
\{G_{\epsilon_1}, G_{\epsilon_2}\} = G_{[\epsilon_1, \epsilon_2]} - G_\Lambda(A, \epsilon_1, \epsilon_2)
\]

(4.22)

with $G_\epsilon$ and $G_\Lambda$ given in (4.16) and (4.6), respectively. Since evidently $G_\Lambda$ has vanishing Poisson brackets with all other generators, the algebra of the generators of Yang–Mills and tensor gauge transformations closes with field–dependent structure constants. It is isomorphic to the algebra given in (2.10). It is important to stress the difference between the significance of (4.20) and (4.22). The former is referred to in the literature on anomalies as a Mickelsson–Faddeev algebra; the second term on its right hand side cannot be identified with any generator of the algebra and therefore gives rise to an anomalous extension. In the latter, the second term on the right hand side is an already existing generator and therefore should not be regarded as an extension.

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APPENDIX

We collect here some well–known formulae on anomalies, which have been used in the text. In particular we define the cocycles $\omega^k_{p-k+2}$, for $k = 0, 1, 2$, where $p - k + 2$ is the degree of $\omega$ as a form on $M$ (or on the world–volume, if the connection is pulled back) and $k$ is its degree as a form in the space of connections. The Chern–Simons forms $\omega^0_{p+2}$ are defined by the relation

$$d\omega^0_{p+2} = k_p \text{tr} F^2 + \frac{3}{2},$$  \hspace{1cm} (A.1)

where $k_p$ is a normalization constant, depending on the group $G$, which we will not specify (for the case $p = 1$ see, for example, [12]). For $p = 1, 3, 5$ we have

$$\omega^0_3(A) = k_1 \text{tr} \left( FA - \frac{1}{3} A^3 \right), \hspace{1cm} (A.2a)$$

$$\omega^0_5(A) = k_3 \text{tr} \left( F^2 A - \frac{1}{2} FA^3 + \frac{1}{10} A^5 \right), \hspace{1cm} (A.2b)$$

$$\omega^0_7(A) = k_5 \text{tr} \left( F^3 A - \frac{2}{5} F^2 A^3 - \frac{1}{5} FAFA^2 + \frac{1}{5} F^4 - \frac{1}{35} A^7 \right). \hspace{1cm} (A.2c)$$

The functional $C$ defined in (2.6) can be computed explicitly in the cases $p = 1, 3, 5$ by substituting (A.2) into (2.6). We get

$$C_2 = k_1 \text{tr}(AL), \hspace{1cm} (A.3a)$$

$$C_4 = \frac{1}{4} k_3 \text{tr} \left[ 2(FA + AF - A^3)L + ALAL - 2AL^3 \right], \hspace{1cm} (A.3b)$$

$$C_6 = \frac{1}{30} k_5 \text{tr} \left[ (10F^2 A + 10FAF + 10AF^2 - 8FA^3 - 8A^3 F - 4FAF^2 - 4A^2 FA + 6A^6)L 
+ 2F(A^2 L^2 - L^2 A^2 + 3ALAL - 3LALA) - 6A^3 LAL 
+ 3F(LAL^2 - L^2 AL + 2L^3 A - 2AL^3) + 6A^3 L^3 
- 3L^2 A^2 LA + 3A^2 L^2 AL + 2ALALAL + 6L^3 ALA + 6AL^5 \right]. \hspace{1cm} (A.3c)$$

The gauge variations of the Chern–Simons forms defines the consistent anomaly $\omega^1_{p+1}$:

$$\delta_\epsilon \omega^0_{p+2}(A) = d\omega^1_{p+1}(A, \epsilon). \hspace{1cm} (A.4)$$

A general formula for $\omega^1_{p+1}$ can be found in [10]. It can be written in the form

$$\omega^1_{p+1}(A, \epsilon) = \text{tr} d\epsilon \phi_p(A). \hspace{1cm} (A.5)$$

where the $p$-form $\phi_p = \phi_p^a T_a$ is a polynomial in $A$ and $F$. For $p = 1, 3, 5$ this polynomial is given by

$$\phi_1 = - k_1 A, \hspace{1cm} (A.6a)$$

$$\phi_3 = - \frac{1}{2} k_3 (FA + AF - A^3), \hspace{1cm} (A.6b)$$

$$\phi_5 = - \frac{1}{3} k_5 \left[ (F^2 A + FAF + AF^2) - \frac{4}{5} (A^3 F + FA^3) - \frac{2}{5} (A^2 FA + AFA^2) + \frac{3}{5} A^5 \right]. \hspace{1cm} (A.6c)$$
Note that $\phi_p$ is also the coefficient of the term in $C$ linear in $L$. The coboundary of $\omega_{p+1}^1$ defines $\omega_p^2$:

$$\delta_{\epsilon_1} \omega^1(A, \epsilon_2) - \delta_{\epsilon_2} \omega^1(A, \epsilon_1) - \omega^1(A, [\epsilon_1, \epsilon_2]) = d\omega_p^2(A, \epsilon_1, \epsilon_2).$$

(A.7)

For $p = 1, 3, 5$ it is given by

$$\omega_p^2(A, \epsilon_1, \epsilon_2) = - 2k_1 \text{tr} \epsilon_1 d\epsilon_2,$$

(A.8a)

$$\omega_p^2(A, \epsilon_1, \epsilon_2) = - k_3 \text{tr} \{d\epsilon_1, d\epsilon_2\} A,$$

(A.8b)

$$\omega_p^2(A, \epsilon_1, \epsilon_2) = \frac{1}{15} k_5 \text{tr} (5F - 3A^2) [2A\{d\epsilon_1, d\epsilon_2\} - d\epsilon_1 A d\epsilon_2 + d\epsilon_2 A d\epsilon_1].$$

(A.8c)

It is clear from (A.4) and (A.7) that $\omega_{p+1}^1$ and $\omega_p^2$ are only defined up to a closed form. In particular one could add to $\omega_{p+1}^1$ the closed form $-d(\text{tr} \epsilon \phi(A))$ and get

$$\hat{\omega}_{p+1}^1(A, \epsilon) = - \text{tr} \epsilon d\phi_p,$$

(A.9)

which is another form of the consistent anomaly. Applying the coboundary to $\hat{\omega}_{p+1}^1$ defines a different 2-cocycle $\hat{\omega}_p^2$:

$$\delta_{\epsilon_1} \hat{\omega}^1(A, \epsilon_2) - \delta_{\epsilon_2} \hat{\omega}^1(A, \epsilon_1) - \hat{\omega}^1(A, [\epsilon_1, \epsilon_2]) = d\hat{\omega}_p^2(A, \epsilon_1, \epsilon_2).$$

(A.10)

For $p = 1, 3, 5$

$$\hat{\omega}_p^2(A, \epsilon_1, \epsilon_2) = k_1 \text{tr} [\epsilon_1, \epsilon_2] A,$$

(A.11a)

$$\hat{\omega}_p^2(A, \epsilon_1, \epsilon_2) = \frac{1}{2} k_3 \text{tr} \{[\epsilon_1, \epsilon_2] (3A^2) - \epsilon_1 dA \epsilon_2 A - \epsilon_1 A d\epsilon_2 A - \epsilon_1 A d\epsilon_2 A\},$$

(A.11b)

$$\hat{\omega}_p^2(A, \epsilon_1, \epsilon_2) = \frac{1}{2} k_5 \text{tr} \left\{ [\epsilon_1, \epsilon_2] \left[ (5F + 3A^2) - \frac{4}{5} \{A^3, F\} - \frac{2}{5} \{A, A^2\} + \frac{2}{5} A^5 \right] 
- \frac{1}{5} \{d\epsilon_1, d\epsilon_2\} [F, A^2] - \frac{2}{5} (d\epsilon_1 A d\epsilon_2 + d\epsilon_2 A d\epsilon_1)(F A + A^2 - A^3)
+ \frac{1}{5} \{d\epsilon_1, d\epsilon_2\} [F, A^2] - \frac{2}{5} (d\epsilon_2 A d\epsilon_1 + d\epsilon_1 A d\epsilon_2)(F A + A^2 - A^3) \right\}.$$

(A.11c)

These are the cocycles one gets in the Gauss law algebra of an anomalous fermionic theory using the Bjorken-Johnson-Low procedure [13], or in the gauged Wess-Zumino-Witten model at the canonical level [11]. They differ from the cocycles $\omega_p^2$ by a redefinition of the current.

These cocycles assume a simpler form when their argument is $L$ instead of $A$. We have

$$\omega_{p+2}^0(L) = k_p c_p \text{tr} L^{p+2},$$

(A.12a)

$$\omega_{p+1}^1(L, \epsilon) = k_p c_p (p+2) \text{tr} d\epsilon L^p,$$

(A.12b)

$$\omega_p^2(L, \epsilon_1, \epsilon_2) = 2k_p c_p (p+2) \text{tr} \{d\epsilon_1, d\epsilon_2\} L^{p-2},$$

(A.12c)

where $c_p = (-1)^{p+1} \left( \frac{p+3}{2} \right) \Gamma(p+3/2)/\Gamma(p+3)$. 

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The quantity $\chi_p^a$ defined by (4.13) and (2.5) can be calculated inserting (A.6) in (4.13). For the cases $p = 1, 3, 5$ we find

\[
\begin{align*}
\chi_0^a &= 0, \\
\chi_2^a &= -\frac{1}{2}k_3\text{tr} T^a [A, L], \\
\chi_4^a &= k_5\text{tr} T^a \left[ -\frac{1}{6}(\{F, [A, L]\} + AFL - LFA) - \frac{3}{10}([ALA, A] + 2[A^3, L]) \\
&\quad + \frac{1}{10}([A^2, L^2] + 3[ALA, L]) + \frac{3}{10}([LAL, L] + 2[L^3, A]) \right].
\end{align*}
\] (A.13a, A.13b, A.13c)

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