Study of the Mechanism of Tangent Bifurcation in Voltage Mode Controlled DCM Buck Converter

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Abstract. Tangent bifurcation is a special bifurcation in nonlinear dynamic systems. The investigation of the mechanism of the tangent bifurcation in voltage mode controlled buck converters operating in discontinuous conduction mode (DCM) is performed. The one-dimensional discrete iterative map of the buck converter is derived. Based on the tangent bifurcation theorem, the conditions of producing the tangent bifurcation in DCM buck converters are deduced mathematically. The mechanism of the tangent bifurcation in DCM buck is exposed from the viewpoint of nonlinear dynamic systems. The tangent bifurcation in the DCM buck converter is verified by numerical simulations such as discrete iterative maps, bifurcation map and Lyapunov exponent. The simulation results are in agreement with the theoretical analysis, thus validating the correctness of the theory.

1. Introduction
In recent years, the research in chaos exhibited in the field of power electronics has been the hot spots. DC-DC converters are a kind of piece-wise and nonlinear system. They exhibit various bifurcation and chaos behavior under some operating conditions, such as period-doubling bifurcation [1]-[5], Hopf bifurcation [6]-[8], border collision bifurcation [9]-[11], tangent bifurcation [12]-[13] and chaos behavior [14]-[18]. Bifurcation is a complex structure in nonlinear system. The chaos is characteristic of non-repeat, uncertainty and is extreme sensitive to initial conditions. These nonlinear phenomena make the nonlinear dynamic characteristics of DC-DC converter more complex. Deep investigation of these nonlinear phenomena is of great benefit to understanding the nonlinear behavior and practical design.

The period-doubling bifurcation in DC-DC converters is reported in many published papers. On the other hand, the tangent bifurcation, which is a special bifurcation, has been less investigated. The most studies of tangent bifurcation mainly focus on the numerical simulation modeling. The main approaches used for simulation include bifurcation diagram, Lyapunov exponent. The essential mechanism causing tangent bifurcation was not analyzed in these simulation methods. However, no rigorous attempts have been made to analyze formally the essential mechanism leading to the tangent bifurcation in DC-DC converters.

Buck converters in voltage mode controlled mode are a kind of important converters with wide applications. Although the work in [13] gives no theoretical insights into the underlying cause of tangent
bifurcation in such system, it does prompt the important question of what mechanism may give rise to tangent bifurcation behavior. This paper attempts to answer to this question in the light of the theories of nonlinear dynamic systems. The investigation of the mechanism of the tangent bifurcation in voltage mode controlled buck converters operating in continuous conduction mode (DCM) is deeply studied. In fact, there are strict stability criteria and the conditions leading to the tangent bifurcation in mathematics based on the theories of nonlinear dynamic systems [14]-[15]. Based on the tangent bifurcation theorem, the conditions leading to the tangent bifurcation in the discrete iterative model of the buck converter are demonstrated mathematically. Discrete iterative maps, bifurcation diagram, Lyapunov exponent are done to analyze the mechanism and evolution of leading to the tangent bifurcation. The simulation results are in agreement with the theoretical analysis, thus validating the correctness of the theory. The methods proposed in the paper can also be suitable to analysis of the tangent bifurcation and chaos of other kinds of converter circuits.

2. Discrete Iterative Map of a Buck Converter

The schematic diagram of voltage mode controlled buck converter is shown in Fig.1. The main circuit consists of a switch S, a diode D, a capacitor C, an inductor L and the load resistor R. The controlled circuit consists two comparators, a feedback proportional gain k. X is the expected output voltage, D is the duty cycle in steady state. All the components in the buck converter circuit are ideal, no parasitic effects are considered.

![Fig.1. Circuit configuration of Voltage mode controlled buck converter](image)

The buck converter operates in discontinuous conduction mode. Hence, there are three circuit states depending on whether S is closed or open. Assume that the circuit is at the switch state 1 when the switch S is off and diode D is on, at the switch state 2 when S is on and D is off. The two switch states toggle periodically, and at the switch state 3 when S and D are off. The three switch states toggle periodically.

The buck converter is controlled under the voltage mode. The discrete iterative model of the buck converter can be derived as follows[7]:

\[ x_{n+1} = f(k, x_n) = \alpha x_n + \frac{\beta h(d_n)^2 (E - x_n)}{x_n} \]  

(1)

Where \( x_n \) is the voltage across the capacitor at \( t=nT (n=0,1,2,3,...) \).

\[ \alpha = 1 - \frac{T}{RC} + \frac{T^2}{2C^2R^2} \]  

(2)

\[ \beta = \frac{T^2}{2LC} \]  

(3)
\[ d_n = D - k(x_n - X) \]  
(4)

\[ h(d_n) = \begin{cases} 
0, & d_n < 0 \\
1, & d_n > 1 \\
d_n, & 0 \leq d_n \leq 1
\end{cases} \]  
(5)

Based on the discrete iterative equation, the quadratic and three times discrete iterative model of the buck converter can be derived.

The quadratic discrete iterative can be written in the form of

\[ x_{n+2} = f(k, x_{n+1}) = f^{(2)}(k, x_n) = \alpha x_{n+1} + \frac{\beta h(d_{n})^2 E}{x_{n+1}} \]  
(6)

Where

\[ d_{n} = D - k(x_{n+1} - X) \]  
(7)

\[ h(d_{n}) = \begin{cases} 
0, & d_n < 0 \\
1, & d_n > 1 \\
d_n, & 0 \leq d_n \leq 1
\end{cases} \]  
(8)

The other parameters are the same as (2)-(3).

The three times discrete iterative can be calculated by the following equation

\[ x_{n+3} = f(k, x_{n+2}) = f^{(3)}(k, x_n) = \alpha x_{n+2} + \frac{\beta h(d_{n})^2 E}{x_{n+2}} \]  
(9)

Where,

\[ d_{n} = D - k(x_{n+2} - X) \]  
(10)

\[ h(d_{n}) = \begin{cases} 
0, & d_n < 0 \\
1, & d_n > 1 \\
d_n, & 0 \leq d_n \leq 1
\end{cases} \]  
(11)

The other parameters are the same as (2)-(3).

3. The Conditions Leading to Tangent Bifurcation

The circuit parameters of the buck converter are listed in Table 1.

| Table 1. Circuit parameters | Values |
|-----------------------------|--------|
| Circuit Components          | Values |
| Switching period T          | 333.33μs|
| Input Voltage E             | 33V    |
| Load Resistor R             | 12.5Ω  |
| Inductor L                  | 208μH  |
| Capacitor C                 | 222μF  |
| Output volgae X             | 25V    |
| Duty cycle D                | 0.2874 |

Based on the table 1, the \( \alpha \) and \( \beta \) in (2)-(3) can be calculated, the results are as follows:

\( \alpha=0.8872, \beta=1.2. \)
3.1 A Theorem of Tangent Bifurcation

Theorem 1 [14]-[15] (Tangent Bifurcation). Assume that \( f \) is a \( C^2 \) function from \( R^2 \) to \( R \). We write \( f(x, \mu) = f_\mu(x) \). Assume that there is a bifurcation value \( \mu^* \) that has a fixed point \( x^* \) with derivative equal to one.

(a). \( f(x^*, \mu^*) = x^* \).

(b). \( f_\mu'(x^*) = 1 \).

(c). The second derivative \( f_\mu''(x^*) \neq 0 \), so the graph of \( f_\mu'' \) lies on one side of the diagonal for \( x \) near \( x^* \).

(d). The graph of \( f_\mu \) is moving up or down as the parameter \( \mu \) varies, or more specifically, \( \frac{\partial f}{\partial \mu}(x^*, \mu^*) \neq 0 \).

The tangent bifurcation takes place in the nonlinear system at the fixed point \((x^*, \mu^*)\).

3.2 A fixed point

The fixed point of the three times discrete iterative is calculated by the following equation

\[
f^{(3)}(k, x_n) - x_n = 0
\]

(12)

3.3 Instability boundary

The Instability boundary of the converter is calculated by the following equation

\[
\frac{\partial}{\partial x} f^{(3)}(k, x_n) \bigg|_{x=x^*} = 1
\]

by substituting of circuit parameters into (13)-(14), we have

\[
k^* = 0.2166 \quad x_1^* = 23.76, x_2^* = 26.23, x_3^* = 29.57
\]

The results show that there are 3 fixed point in the three times discrete iterative of DCM voltage mode controlled buck converter. They are \( x_1^* = 23.76, x_2^* = 26.23, x_3^* = 29.57 \). In addition, when \( k = k^* = 0.2166 \), these three fixed points are exactly tangent to the diagonal line, and the slope at the tangent point is exactly + 1.

From (9), the partial derivative can be worked out

\[
\frac{\partial}{\partial k} f^{(3)}(k, x_n) = \alpha \frac{\partial x_{n+2}}{\partial k} + \frac{\partial}{\partial k} \left( \beta h(d_{n3})^2 E(E-x_{n+2}) \right) x_{n+2}
\]

(14)

Substituting of (1)-(11) and circuit parameters into (15), gives

\[
\frac{\partial}{\partial k} f^{(3)}(k, x_n) \bigg|_{k=0.2116, x_1^* = 23.76} = 17.398 \neq 0
\]

\[
\frac{\partial}{\partial k} f^{(3)}(k, x_n) \bigg|_{k=0.2116, x_2^* = 26.23} = -3.3150 \neq 0
\]

\[
\frac{\partial}{\partial k} f^{(3)}(k, x_n) \bigg|_{k=0.2116, x_3^* = 29.57} = 19.694 \neq 0
\]

From (9), the second partial derivative can also be worked out

\[
\frac{\partial^2}{\partial x_n^2} f^{(3)}(k, x_n) = \alpha \frac{\partial^2 x_{n+2}}{\partial x_n^2} + \frac{\partial^2}{\partial x_n^2} \left( \beta h(d_{n3})^2 E(E-x_{n+2}) \right) x_{n+2}
\]

(15)

Substituting of (1)-(11) and circuit parameters into (16), gives

\[
\frac{\partial^2}{\partial x_n^2} f^{(3)}(k, x) \bigg|_{k=0.2116, x_1^* = 23.76} = 7.738 \neq 0
\]
\[
\frac{\partial^2}{\partial x_n^2} f^{(3)}(k, x) \bigg|_{k=0.2116, x_n^*=26.23} = 75.177 \neq 0
\]
\[
\frac{\partial^2}{\partial x_n^2} f^{(3)}(k, x) \bigg|_{k=0.2116, x_n^*=29.57} = -4.612 \neq 0
\]

In summary, the voltage mode controlled buck converter operating in DCM satisfies the hypothesis of theorem 1. (a). There are fixed points, \( x_1^* = 23.76, x_2^* = 26.23, x_3^* = 29.57 \). (b). When \( k=0.2166 \), the system is at the instability boundary, which leading to bifurcation. (c). In the fixed points, the partial derivatives of parameter \( k \) are not equal to 0. (d). In the fixed points, the second derivative is not equal to 0. Therefore, the discrete iterative map of \( f(x_n) \) undergoes the tangent bifurcation at the fixed point, and the tangent bifurcation behavior occurs in this system.

4. Simulations

The evolution process of tangent bifurcation in voltage mode controlled DCM buck converter with voltage feedback gain as parameter is verified by bifurcation diagrams, Lyapunov exponent and discrete iterative maps. The mechanism of tangent bifurcation in the voltage mode controlled buck converter operating in DCM is analyzed.

The horizontal direction is \( k \) which is between 0.1 and 0.26, the vertical direction is the output voltage \( X \) which ranges from 23V and 33V. The bifurcations, subharmonics and chaotic behavior are indicated in the diagram. As shown in Fig.2, the buck converter goes through period 1, period 2 and eventually exhibits chaos. The period-1 solution is stable until \( k=0.12 \) whereupon a period doubling bifurcation takes place. The converter eventually goes to chaos when \( k=0.173 \). It can be interestingly observed that a small periodic window, which also exhibits period doubling cascade, is embedded in the chaos region. In the periodic widow, the converter experiences period-3 to period-6 and so on just above \( k=0.2116 \). The phenomenon that system transits from chaos to period 3 is known as tangent bifurcation.
In Fig. 3, the larger of the Lyapunov exponents is plotted as a function of the parameter \( k \) over the same range as in Fig.3. It is well known that the presence of chaos is signaled by positive Lyapunov exponent. A negative Lyapunov exponent is characteristic of dissipative (non-conservative) systems, which exhibit point stability. A Lyapunov exponent of zero is characteristic of a cycle-stable system. In this case, the orbits maintain their separation. The tangent bifurcation will be happened when the Lyapunov exponent is changed from the started positive value to zero then to negative value. At \( k=0.12 \), where the fixed point changes from attracting to repelling and an attracting periodic orbit is born, the Lyapunov exponent is 0. Just above \( k=0.173 \), the Lyapunov exponent is positive, which means that the system is chaotic. This is the same range in which the bifurcation diagram given in Fig.2 showed a whole interval. For larger values of \( k \), above 0.2116, there is another short parameter interval in which there is an attracting period-3 orbit and the Lyapunov exponent is negative. Therefore, the tangent bifurcation will be happened.

The Bifurcation diagram with \( 0.2116<k<0.226 \) is shown in Fig.4. When \( 0.2116<k<0.2182 \), there is an attractive periodic 3 orbit. The bifurcation takes place at \( k=0.2182 \) where the period 3 bifurcates to period 6. The system enters chaos for \( k>0.226 \).
Fig. 5 is the discrete iterative map of \( f(k, x_n) \) with \( k = 0.2116 \). It can be seen that there is an intersection point between the iterative curve and the diagonal line. The slope at this intersection is less than -1. Therefore, this fixed point is stable.

Fig. 6 Discrete iterative map of \( f^3(k, x_n) \)

The Discrete iterative map of \( f^3(k, x_n) \) is shown in Fig. 6. When \( k = 0.2116 \), there are four intersection points between the three times iterative curve and the diagonal line. One fixed point is the one in iterative map. The other three fixed points are tangent to the diagonal line and the slope at the tangent point is + 1. The buck converter is at the instability boundary, the bifurcation will occur. When \( k > 0.2116 \), the three points of tangent will cross the diagonal. Thus, there will be six fixed points. The slope of three fixed points and diagonals is greater than 1, they are not stable fixed points. The slope of another three fixed points and diagonals is less than 1, they are stable fixed points.

5. Conclusion
The mechanism of tangent bifurcation in the voltage mode controlled buck converter operating in DCM is explored in this paper. Based on the discrete iterative map of the buck converter, the one-dimensional discrete iterative maps of \( f(x_n) \) and \( f^3(x_n) \) have been derived. By the tangent bifurcation theorem, it is demonstrated in mechanism that the tangent bifurcation will happen inevitably in the buck converter. The discrete iterative maps, bifurcation diagram with voltage feedback gain \( k \) as parameter, Lyapunov exponent are used to verify the phenomenon. It has been shown that tangent bifurcation does exist for this system. The method presented in the paper provides the theoretical basics for analyzing the tangent
bifurcation and chaos. It has generality and can be also used to analyze the tangent bifurcation of other kinds of DC-DC converters.

Acknowledgments
The project sponsored by Natural Science Foundation of Guangxi Province (2014GXNSFBA118277), Promotion Project of Guangxi Young Teachers(2017KY0030).

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