On sequences of positive integers containing no $p$ terms in arithmetic progression

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Abstract

We use topological ideas to show that, assuming the conjecture of Erdős [4] on subsets of positive integers having no $p$ terms in arithmetic progression (A. P.), there must exist a subset $M_p$ of positive integers with no $p$ terms in A. P. with the property that among all such subsets, $M_p$ maximizes the sum of the reciprocals of its elements.

1 Introduction

A famous conjecture of Erdős asserts that if $A$ is a subset of the positive integers having the property that $\sum_{a \in A} \frac{1}{a} = \infty$, then $A$ must contain arithmetic progressions of arbitrary length. A special case of the conjecture, when $A$ is the set of prime numbers, was recently proved by Green and Tao [3]. This implies that

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if a subset $A$ of the set of positive integers contains no arithmetic progression of length $p$, where $p \geq 3$ is a fixed integer, then the sum $\sum_{a \in A} \frac{1}{a}$ must converge. In this paper we assume the Erdős conjecture and deduce a much stronger consequence of it. We ask whether the sum above can be arbitrarily large as the sets $A$ vary. Our first theorem answers the question in the negative.

Joseph L. Gerver [1] proved that for every $\epsilon > 0$, there exists for all but a finite number of integers $p \geq 3$, sets $S_p$ of positive integers, containing no arithmetic progression of $p$ terms, such that $\sum_{a \in S_p} \frac{1}{a} > (1 - \epsilon)p \log p$. The set $S_p$ is the sequence $\{a_n\}$ where $a_1 = 1$ and for $n \geq 1$, $a_{n+1}$ is the smallest positive integer bigger than $a_n$ such that no $p$ elements of $a_1, a_2, \ldots, a_{n+1}$ in arithmetic progression. He guessed in that paper that for any prime $p$, the set $S_p$ may indeed maximize the sum of the reciprocals of the elements of a set of positive integers having no $p$ terms in arithmetic progression. On the other hand Joseph L. Gerver and L. Thomas Ramsey [2] showed heuristically that the set $S_p$ is not maximizing the above sum for composite $p$. A corollary to our second theorem says that the Erdős conjecture implies the existence of a set of positive integers containing no $p$ terms in arithmetic progression which maximizes the above sum.

In rest of the paper, $p$ is any fixed integer greater than or equal to 3.

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2 Main Results

Theorem 1. Let $\mathcal{A}_p$ be the collection of all subsets of $\mathbb{N}$ having no arithmetic progression of length $p$. Then, under the assumption of the Erdős conjecture,
there is an absolute constant $B_p$ such that

$$\operatorname{Sup} \left\{ \sum_{a \in A} \frac{1}{a} : A \in \mathcal{A}_p \right\} \leq B_p. \tag{1}$$

For further discussion, we need a topological structure on $\mathcal{A}_p$. First we note that there is a natural one-to-one correspondence between the power set $\mathcal{P}(\mathbb{N})$ and the set $\{0, 1 \}^\mathbb{N}$ of all sequences of 0s and 1s; namely, given any subset $A \subset \mathbb{N}$, we send it to the sequence $\{\delta_A(n)\}_{n=1}^\infty$, where

$$\delta_A(n) = \begin{cases} 1 & \text{if } n \in A \\ 0 & \text{otherwise} \end{cases}.$$

Since $\{0, 1 \}^\mathbb{N}$ is compact by Tychonoff’s theorem, the above identification makes $\mathcal{P}(\mathbb{N})$ into a compact topological space. In this topology, a sequence $\{A_n\}$ of subsets converges to $A$ if, for any given $k$, there is some $N_k$ such that, whenever $n \geq N_k$,

$$\delta_{A_n}(j) = \delta_A(j) \text{ for } j = 1, 2, \ldots, k. \tag{2}$$

Proposition 4 below says that $\mathcal{A}_p$ is a compact subspace of $\mathcal{P}(\mathbb{N})$. For any set $A \in \mathcal{A}_p$, let us denote the sum $\sum_{a \in A} \frac{1}{a}$ (which converges if we assume Erdős conjecture) by $\mu(A)$. Then we have the following theorem.

**Theorem 2.** The map $A \mapsto \mu(A)$ between $\mathcal{A}_p$ and $[0, B_p]$ is continuous.

Since $\mathcal{A}_p$ is compact, theorem 2 implies the following corollary.

**Corollary 3.** Under the assumption of the Erdős conjecture, there is a set $M_p \in \mathcal{A}_p$ such that

$$\mu(X) \leq \mu(M_p) \text{ for all } X \in \mathcal{A}_p. \tag{3}$$

That is, the supremum of the set $\{\mu(X) : X \in \mathcal{A}_p\}$ is attained.

## 3 Proofs

In this section, we shall present the proofs of theorem 1 and theorem 2. First we prove a proposition that will be needed later.
Proposition 4. \( \mathcal{A}_p \) is a compact subspace of \( \mathcal{P}(\mathbb{N}) \).

Proof. Since \( \mathcal{P}(\mathbb{N}) \) is compact, it is enough to show that \( \mathcal{A}_p \) is closed. Let \( \{A_n\} \) be sequence in \( \mathcal{A}_p \) converging to some \( A \in \mathcal{P}(\mathbb{N}) \). We need to show that \( A \in \mathcal{A}_p \). Let us denote

\[ A_n = \{a_1^{(n)}, a_2^{(n)}, \cdots\} \quad \text{and} \quad A = \{a_1, a_2, \cdots\}, \]

where the terms in the sequences are written in the increasing order. Suppose, if possible, that \( A \notin \mathcal{A}_p \). So there is an arithmetic progression \( \{a_{k_1}, a_{k_2}, \cdots, a_{k_p}\} \subset A \). We shall obtain a contradiction from this. Since \( A_n \rightarrow A \), by the criterion (2) for convergence, we must have, for any given \( k \), some integer \( N_k \) such that,

\[
a^{(n)}_j = a_j \quad \text{for} \quad j = 1, 2, \cdots, k
\]

(4)

for all \( n \geq N_k \). In particular, if \( k = k_p \), we have, for \( n \geq N_{k_p} \),

\[
a^{(n)}_{k_i} = a_{k_i} \quad \text{for} \quad i = 1, 2, \cdots, p.
\]

(5)

Since \( \{a_{k_i} : i = 1, 2, \cdots, p\} \) is an arithmetic progression, the above implies that \( A_n \notin \mathcal{A}_p \) for \( n \geq N_{k_p} \), which is a contradiction. So \( A \in \mathcal{A}_p \) as was required to be proved.

Proof of theorem 1

Proof. We shall prove this by contradiction. Let \( A_0 = A \in \mathcal{A}_p \) be any finite set with \( \sum_{a \in A} \frac{1}{a} = L > 0 \). For example, we can take \( A_0 = \{1\} \). If we assume that the statement of the theorem is not true, then we shall show that there is a finite set \( B \supset A \), \( B \in \mathcal{A}_p \) with

\[
\sum_{b \in B} \frac{1}{b} \geq L + 1.
\]

(6)
This will result in a contradiction to the conjecture of Erdös in the following manner. Repeating this process that produces \( B \) recursively, we get an increasing sequence of sets \( A_0 \subset A_1 \subset A_2 \subset \cdots \), each of them finite and they all are in \( \mathcal{A}_p \). Moreover,

\[
\sum_{a \in A_j} \frac{1}{a} \geq L + j.
\]

Now the set \( A_\infty = A_0 \cup A_1 \cup A_2 \cup \cdots \) must be in \( \mathcal{A}_p \) since any given collection of \( p \) elements in \( A_\infty \) must also belong to \( A_n \) for some \( n \), so those elements can not be in arithmetic progression. On the other hand, the sum \( \sum_{a \in A} \frac{1}{a} \) must diverge as it is bigger than any fixed number. So all that is now left to prove the theorem is to produce such a set \( B \), given \( A \).

Let \( N \) be the maximum of the elements of \( A \). If the theorem is untrue, then there must exist a set \( E \in \mathcal{A}_p \) such that

\[
\sum_{e \in E} \frac{1}{e} \geq 2N. \quad (7)
\]

In fact, we may take \( E \) to be a finite set; since, if \( E \) is infinite, the tail of the convergent sum will be small. Now define

\[
B = A \sqcup 2NE, \quad (8)
\]

where \( \sqcup \) denotes disjoint union, and \( 2NE = \{2Ne : e \in E\} \). Clearly \( B \) is a finite set containing \( A \), and

\[
\sum_{b \in B} \frac{1}{b} = \sum_{a \in A} \frac{1}{a} + \sum_{e \in E} \frac{1}{2Ne} \geq L + 1 \quad (9)
\]

by (7, 8). Now to show that \( B \in \mathcal{A}_p \), we first note that since \( A \in \mathcal{A}_p \) and \( E \in \mathcal{A}_p \), no \( p \) elements of either \( A \) or \( 2NE \) can be in arithmetic progression. Suppose, if possible, that \( b_1, b_2, \cdots, b_p \in B \) are in A.P., where \( b_1, b_2, \cdots, b_k \in A \), and \( b_{k+1}, b_{k+2}, \cdots, b_p \in 2NE \). If \( k \geq 2 \), then

\[
b_{k+1} - b_k = b_k - b_{k-1}. \quad (10)
\]
Now, $b_k - b_{k-1} \leq N - 1$ since $b_k, b_{k-1} \in A$ and $N$ is the maximum of the elements of $A$. But the right hand side, $b_{k+1} - b_k \geq b_{k+1} - N \geq 2N - N = N$, a contradiction. If $k = 1$, then

$$b_2 - b_1 = b_3 - b_2,$$

or equivalently,

$$b_1 = 2b_2 - b_3. \quad (12)$$

But $b_1 \leq N$, while $2b_2 - b_3$ is a multiple of $2N$ as both $b_2, b_3 \in 2NE$. So we arrive at a contradiction again. Hence we conclude that $B$ cannot have an arithmetic progression of length $p$.

For proving theorem 2, we first prove a lemma.

**Lemma 5.** Given any $\varepsilon > 0$, there exist a natural number $N$ such that for any $A \in \mathcal{A}_p$ with $\min A \geq N$,

$$\sum_{a \in A} \frac{1}{a} < \varepsilon. \quad (13)$$

Note: In the above, $\min A$ denotes the smallest element in $A$.

**Proof.** Suppose, if possible, the lemma is not correct. Then there exists some $\varepsilon > 0$ such that for any given integer $M \geq 1$, there is a set $R \in \mathcal{A}_p$ depending on $M$ with the following properties:

$$\mu(R) = \sum_{r \in R} \frac{1}{r} > \varepsilon, \quad (14)$$

and

$$\min R \geq 2M. \quad (15)$$

For that $\varepsilon$, we choose a set $A \in \mathcal{A}_p$ satisfying

$$\mu(A) > M_p - \frac{\varepsilon}{12}. \quad (16)$$
where $M_p = \text{Sup}\{\mu(A) : A \in \mathcal{A}_p\} < \infty$ by Theorem 1. Let $A = \{a_1, a_2, \cdots\}$ where $a_1 < a_2 < \cdots$. Since $\sum_{a \in A} \frac{1}{a} < \infty$, there is some $n_0$ such that
\[
\sum_{n=n_0+1}^{\infty} \frac{1}{a_n} < \frac{\varepsilon}{12}.
\] (17)

Let $A_1 = \{a_1, a_2, \cdots, a_{n_0}\}$. Then
\[
\mu(A_1) > M_p - \frac{\varepsilon}{6}.
\] (18)

by (16) and (17)

Now we take $M = a_{n_0}$ and write, $R = R_1 \sqcup R_2 \sqcup R_3 \sqcup R_4$ where
\[
R_j = R \cap \bigcup_{i=0}^{\infty} \left( (j \cdot 3^i M, (j + 1) \cdot 3^i M) \right); j = 1, 2, 3, 4.
\] (19)

In other words,
\[
R_1 = R \cap \{ [2M, 3M) \sqcup [6M, 9M) \sqcup [18M, 27M) \sqcup \cdots \},
R_2 = R \cap \{ [3M, 4M) \sqcup [9M, 12M) \sqcup [27M, 36M) \sqcup \cdots \},
R_3 = R \cap \{ [4M, 5M) \sqcup [12M, 15M) \sqcup [36M, 45M) \sqcup \cdots \},
R_4 = R \cap \{ [5M, 6M) \sqcup [15M, 18M) \sqcup [45M, 54M) \sqcup \cdots \}.
\]

We have, \[\text{Max } A_1 = M < 2M \leq \text{Min } R,\] (20)

which implies $R \cap A_1 = \phi$, the empty set. Also, it is easy to check that no $p$ elements of $A_1 \sqcup R_j, 1 \leq j \leq 4$, can be in an arithmetic progression. So $A_1 \sqcup R_j \in \mathcal{A}_p$.

Since $\mu(R) > \varepsilon$, we must have
\[
\mu(R_j) > \frac{\varepsilon}{4}
\] (21)

for some $j, 1 \leq j \leq 4$. 

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For that \( j \),
\[
\mu(A_1 \sqcup R_j) = \mu(A_1) + \mu(R_j) > M_p + \frac{\varepsilon}{12}
\]  \( (22) \)
from \( (18) \). This is a contradiction to the fact that \( M_p \) is the supremum of the set \( \{ \mu(A) : A \in \mathcal{A}_p \} \). This proves the lemma.

\[ \square \]

Now we conclude this paper with the proof of theorem 2.

**Proof of theorem 2**

*Proof.* Suppose \( \{ A_n \} \subset \mathcal{A}_p \) be a sequence and \( A_n \longrightarrow A \). We need to show that \( \mu(A_n) \longrightarrow \mu(A) \).

Let us write the set \( A \) as, \( A = \{ a_1, a_2, a_3, \cdots \} \) where \( a_1 < a_2 < a_3 < \cdots \) and similarly for the sets \( A_n \), we write them as, \( A_n = \{ a_1^{(n)}, a_2^{(n)}, \cdots \} \). Note that if the set \( A \) is finite, then \( A_n = A \) for large enough \( n \) and there is nothing left to prove. Let \( \varepsilon > 0 \) be any given real number. The lemma above allows us to select an \( N \) such that for any set \( X \in \mathcal{A}_p \) with \( \operatorname{Min} X \geq N \), we must have
\[
\sum_{x \in X} \frac{1}{x} < \frac{\varepsilon}{2}.
\]  \( (23) \)
Let \( n_0 \) be an integer such that \( a_{n_0} \geq N \). Since \( A_n \longrightarrow A \), there is some \( N_0 \) such that \( a_k^{(n)} = a_k \) for \( 1 \leq k \leq n_0 \) and all \( n \geq N_0 \). Now, for \( n \geq N_0 \),
\[
\left| \mu(A_n) - \mu(A) \right| = \left| \sum_{k=n_0+1}^{\infty} \frac{1}{a_k^{(n)}} - \sum_{k=n_0+1}^{\infty} \frac{1}{a_k} \right|
\]
\[
\leq \sum_{k=n_0+1}^{\infty} \frac{1}{a_k^{(n)}} + \sum_{k=n_0+1}^{\infty} \frac{1}{a_k} < \varepsilon
\]  \( (24) \)
by \( (23) \). Hence \( \mu(A_n) \longrightarrow \mu(A) \).

\[ \square \]
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