A SEMINORMAL FORM FOR PARTITION ALGEBRAS

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Abstract. Using a new presentation for partition algebras [arXiv:1009.1939], we give explicit combinatorial formulae for the seminormal representations of the partition algebras. Our results generalise to the partition algebras the classical formulae given by Young for the symmetric group.

1. Introduction

The partition algebras $A_k(n)$, for $k, n \in \mathbb{Z}_{\geq 0}$, are a family of algebras defined in the work of Martin and Jones in [Mar], [Mar1], [Jo] in connection with the Potts model and higher dimensional statistical mechanics. By [Jo], the partition algebra $A_k(n)$ is in Schur–Weyl duality with the symmetric group $\mathfrak{S}_n$ acting diagonally on the $k$–fold tensor product $V^\otimes k$ of its $n$–dimensional permutation representation $V$. In [Mar2], Martin defined the partition algebras $A_{k+\frac{1}{2}}(n)$ as the centralisers of the subgroup $\mathfrak{S}_{n-1} \subseteq \mathfrak{S}_n$ acting on $V^\otimes k$. Including the algebras $A_{k+\frac{1}{2}}(n)$ in the tower

$$A_0(n) \subseteq A_{\frac{1}{2}}(n) \subseteq A_1(n) \subseteq A_{1+\frac{1}{2}}(n) \subseteq \cdots$$

(1.1)

allowed for the simultaneous analysis of the whole tower of algebras (1.1) using the Jones Basic construction by Martin [Mar2] and Halverson and Ram [HR]. The partition algebras (1.1) have connections to Deligne’s category $\operatorname{Rep}(S_t)$ [CO], and are important examples of cellular algebras [Xi],[DW],[GG].

Halverson and Ram [HR] used Schur–Weyl duality to show that certain diagrammatically defined elements in the partition algebras play an analogous role to the classical Jucys–Murphy elements in the symmetric group. By definition, a seminormal form for $A_k(n)$ or $A_{k+\frac{1}{2}}(n)$ is an irreducible matrix representation relative to a basis of eigenvectors for Jucys–Murphy elements.

Classically, the seminormal form appeared in Young’s construction [Yo] of irreducible representations for the symmetric group $\mathfrak{S}_k$ (see [KI] or [VO] for a modern treatment of the subject). Kosuda [Ko1] has used the presentation in [HR] to compute the seminormal representations for the partition algebra $A_3(n)$. The seminormal representations of the subalgebra of the partition
algebra $A_k(n)$ that acts as centraliser of $G(r, 1, n)$ on the tensor space $V^\otimes k$, for $n \geq k$ and $r > k$, have been constructed by Kosuda [Ko].

In this paper we provide explicit combinatorial formulae for the seminormal representations of the partition algebras $A_k(n)$ and $A_{k+\frac{1}{2}}(n)$. The new approach here is to use the presentation for partition algebras in [En] to compute seminormal representations. For the representations of $A_k(n)$ and $A_{k+\frac{1}{2}}(n)$ which factor through $\mathfrak{S}_k$, our formulae coincide with those given by Young [Yo]. Our construction of seminormal representations provides a partition algebra analogue of the work of Nazarov [Na] for the Brauer algebras and the work of Leduc and Ram [LR] for Brauer and BMW algebras.

Nazarov introduced a remarkable recursion for special central elements in the Brauer algebras and established the relation between these central elements and seminormal representations (see Corollary 3.10 and Proposition 4.2 of [Na]). A similar relation was established by Beliakova and Blanchet for the BMW algebras (see Lemma 7.2 and Lemma 7.4 of [BB]). In this paper, we obtain analogous recursions for central elements in the partition algebras and explain the relation between these central elements and the seminormal representations of the partition algebras.

In §2 we recall the presentations of the partition algebras from [HR] and [En], and state the definition of the Jucys–Murphy elements from [En]. In §3 we show that the Jucys–Murphy elements act triangularly on the partition algebras relative to the Murphy–type bases given in [En1]. In §4 we define seminormal bases for the partition algebras and in §5 we state and prove explicit combinatorial formulae for the images of the generators in the seminormal matrix representations of the partition algebras. In §6 we define central elements (6.3) by contracting powers of the Jucys–Murphy elements in the partition algebras and derive partition algebra analogues of the Nazarov recursions (see Proposition 6.1). In Proposition 6.2 we establish the relation between seminormal matrix entries and the Nazarov–type recursions for central elements in the partition algebras. In §7 we give tables of representing matrices of small rank.

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2. Preliminaries

2.1. Combinatorics. We recall the notation established in [En1]. Let $k$ denote a non–negative integer and $\mathfrak{S}_k$ be the symmetric group acting on $\{1, \ldots, k\}$ on the right. For $i$ an integer, $1 \leq i < k$, let $s_i$ denote the transposition $(i, i + 1)$. Then $\mathfrak{S}_k$ is presented as a Coxeter group by generators $s_1, s_2, \ldots, s_{k-1}$, with the relations

\begin{align*}
    s_i^2 &= 1, & \text{for } i = 1, \ldots, k - 1, \\
    s_is_j &= s_js_i, & \text{for } j \neq i + 1. \\
    s_is_{i+1} &= s_{i+1}s_is_{i+1}, & \text{for } i = 1, \ldots, k - 2.
\end{align*}

An product $w = s_{i_1}s_{i_2}\cdots s_{i_j}$ in which $j$ is minimal is called a reduced expression for $w$ and $j = \ell(w)$ is the length of $w$. If $i, j = 1, \ldots, k$, define

\[ w_{i,j} = \begin{cases} 
    s_is_{i+1}\cdots s_{j-1}, & \text{if } j \geq i, \\
    s_{i-1}s_{i-2}\cdots s_j, & \text{if } i > j.
\end{cases} \]
If \( k > 0 \), a partition \( \lambda = (\lambda_1, \lambda_2, \ldots) \) of \( k \) is a non-increasing sequence \( \lambda_1 \geq \lambda_2 \geq \cdots \geq 0 \), such that \( \sum_{i=1}^\infty \lambda_i = k \); otherwise, if \( k = 0 \), we write \( \lambda = \emptyset \) for the empty partition. The fact that \( \lambda \) is a partition of \( k \) will be denoted by \( \lambda \vdash k \). If \( \lambda \) is a partition, we will also write \( |\lambda| = \sum_{i \geq 1} \lambda_i \).

The integers \( \{\lambda_i | i \geq 1\} \) are the parts of \( \lambda \). If \( \lambda \vdash k \), the Young diagram of \( \lambda \) is the set \[ [\lambda] = \{(i,j) | \lambda_i \geq j \geq 1 \text{ and } i \geq 1 \} \subseteq \mathbb{N} \times \mathbb{N} \].

The elements of \([\lambda]\) are the nodes of \( \lambda \) and more generally a node is a pair \((i,j) \in \mathbb{Z} \times \mathbb{Z}\). The diagram \([\lambda]\) is traditionally represented as an array of boxes with \( \lambda_i \) boxes on the \( i \)-th row.

For example, if \( \lambda = (3, 2) \), then \([\lambda] = \begin{array}{c} \bullet \bullet \\ \bullet \bullet \end{array} \). We will usually identify the partition \( \lambda \) with its diagram and write \( \lambda \) in place of \([\lambda]\). Let \( \lambda \) be a partition. A node \((i,j)\) is an addable node of \( \lambda \) if \((i,j) \not\in \lambda\) and \( \mu = \lambda \cup \{(i,j)\} \) is a partition; in this case \((i,j)\) is also referred to as a removable node of \( \mu \). Let \( A(\lambda) \) and \( R(\lambda) \) respectively denote the set of addable nodes and removable nodes of \( \lambda \).

The dominance \( \trianglerighteq \) on partitions of \( k \) is defined as follows: if \( \lambda \vdash k \) and \( \mu \vdash k \), then \( \lambda \trianglerighteq \mu \) if \( \sum_{i=1}^\infty \lambda_i \geq \sum_{i=1}^\infty \mu_i \) for all \( j \geq 1 \).

We write \( \lambda \triangleright \mu \) to mean that \( \lambda \trianglerighteq \mu \) and \( \lambda \neq \mu \).

Let \( \lambda \vdash k \). A \( \lambda \)-tableau \( t \) from the nodes of the diagram \([\lambda]\) to the integers \( \{1, 2, \ldots, k\} \). A given \( \lambda \)-tableau \( t : [\lambda] \to \{1, 2, \ldots, k\} \) can be represented by labelling the nodes of the diagram \([\lambda]\) with the integers \( 1, 2, \ldots, k \). For example, if \( k = 6 \) and \( \lambda = (3, 2, 1) \),

\[
\begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & 6 \\
\end{array}
\]

represents a \( \lambda \)-tableau. If \( \lambda \vdash k \), let \( t^\lambda \) denote the \( \lambda \)-tableau in which \( 1, 2, \ldots, k \) are entered in increasing order from left to right along the rows of \([\lambda]\). Thus in the previous example where \( k = 6 \) and \( \lambda = (3, 2, 1) \),

\[
\begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & 6 \\
\end{array}
\]

\( \lambda \)-tableau. The symmetric group \( S_k \) acts on the set of \( \lambda \)-tableaux on the right by permuting the integer labels of the nodes of \([\lambda]\). For example,

\[
\begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & 6 \\
\end{array} \quad (2,4)(3,6,5) = \begin{array}{ccc}
1 & 2 & 3 \\
5 & 4 & 6 \\
\end{array}
\]

If \( \lambda \vdash k \), the Young subgroup \( \mathfrak{S}_\lambda \) is defined to be the row stabiliser of \( t^\lambda \) in \( \mathfrak{S}_k \). For instance, when \( k = 6 \) and \( \lambda = (3, 2, 1) \), as in (2.2), then \( \mathfrak{S}_\lambda = \langle s_1, s_2, s_4 \rangle \).

2.2. Partition algebras. In this section we follow the exposition given by Halverson and Ram in [HR]. For \( k = 1, 2, \ldots \), let

\[
A_k = \{ \text{set partitions of } \{1, 2, \ldots, k, 1', 2', \ldots, k'\} \}, \quad \text{and,} \quad A_{k-\frac{1}{2}} = \{ d \in A_k \mid k \text{ and } k' \text{ are in the same block of } d \}.
\]

Any element \( \rho \in A_k \) may be represented as a graph with \( k \) vertices in the top row, labelled from left to right, by \( 1, 2, \ldots, k \) and \( k \) vertices in the bottom row, labelled, from left to right by \( 1', 2', \ldots, k' \), with vertex \( i \) joined to vertex \( j \) if \( i \) and \( j \) belong to the same block of \( \rho \). The representation of a partition by a diagram is not unique; for example the partition

\[
\rho = \{\{1, 1', 3, 4', 5', 6\}, \{2, 2', 3', 4, 5, 6'\}\}
\]
can be represented by the diagrams:
If $\rho_1, \rho_2 \in A_k$, then the composition $\rho_1 \circ \rho_2$ is the partition obtained by placing $\rho_1$ above $\rho_2$ and identifying each vertex in the bottom row of $\rho_1$ with the corresponding vertex in the top row of $\rho_2$ and deleting any components of the resulting diagram which contains only elements from the middle row. The composition product makes $A_k$ into an associative monoid with identity

$$1 = \begin{array}{ccc}
\vdots \\
\end{array}$$

Let $z$ be an indeterminate and $R = \mathbb{Z}[z]$. The partition algebra $A_k(z)$ is the $R$-module freely generated by $A_k$, equipped with the product

$$\rho_1 \rho_2 = z^\ell \rho_1 \circ \rho_2, \quad \text{for } \rho_1, \rho_2 \in A_k,$

where $\ell$ is the number of blocks removed from the middle row in constructing the composition $\rho_1 \circ \rho_2$. Let $A_{k-\frac{1}{2}}(z)$ denote the subalgebra of $A_k(z)$ generated by $A_{k-\frac{1}{2}}$. A presentation for $A_k(z)$ has been given by Halverson and Ram [HR] and East [Ea].

**Theorem 2.1** (Theorem 1.11 of [HR]). If $k = 1, 2, \ldots$, then the partition algebra $A_k(z)$ is the unital associative $R$-algebra presented by the generators

$$p_1, p_{1+\frac{1}{2}}, p_2, \ldots, p_k, s_1, s_2, \ldots, s_{k-1},$$

and the relations

1. **(Coxeter relations)**
   
   (i) $s_i^2 = 1$, for $i = 1, \ldots, k - 1$.
   
   (ii) $s_i s_j = s_j s_i$, if $j \neq i + 1$.
   
   (iii) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, for $i = 1, \ldots, k - 2$.

2. **(Idempotent relations)**
   
   (i) $p_i^2 = z p_i$, for $i = 1, \ldots, k$.
   
   (ii) $p_{i+\frac{1}{2}}^2 = p_{i+\frac{1}{2}}$, for $i = 1, \ldots, k - 1$.
   
   (iii) $s_i p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}} s_i = p_{i+\frac{1}{2}}$, for $i = 1, \ldots, k - 1$.
   
   (iv) $s_i p_i p_{i+1} = p_i p_{i+1} s_i = p_i p_{i+1}$, for $i = 1, \ldots, k - 1$.

3. **(Commutation relations)**
   
   (i) $p_i p_j = p_j p_i$, for $i = 1, \ldots, k$ and $j = 1, \ldots, k$.
   
   (ii) $p_i p_{j+\frac{1}{2}} = p_{j+\frac{1}{2}} p_i$, for $i = 1, \ldots, k - 1$ and $j = 1, \ldots, k - 1$.
   
   (iii) $p_{i+\frac{1}{2}} p_{j+\frac{1}{2}} = p_{j+\frac{1}{2}} p_{i+\frac{1}{2}}$, for $j \neq i, i + 1$.
   
   (iv) $s_i p_j = p_j s_i$, for $j \neq i, i + 1$.
   
   (v) $s_i p_{j+\frac{1}{2}} = p_{j+\frac{1}{2}} s_i$, for $j \neq i - 1, i + 1$.
   
   (vi) $s_i p_{i+\frac{1}{2}} s_i = p_{i+1}$, for $i = 1, \ldots, k - 1$.
   
   (vii) $s_i p_{i+\frac{1}{2}} s_i = s_{i-1} p_{i+\frac{1}{2}} s_{i-1}$, for $i = 2, \ldots, k - 1$.

4. **(Contraction relations)**
   
   (i) $p_{i+\frac{1}{2}} p_j p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}$, for $j = i, i + 1$.
   
   (ii) $p_j p_{j+\frac{1}{2}} p_i = p_i$, for $j = i, i + 1$.

The following identifications have been made in Theorem 2.1:

$$s_i = \begin{array}{ccc}
\vdots \\
\end{array} \begin{array}{ccc}
\vdots \\
\end{array} \begin{array}{ccc}
\vdots \\
\end{array} i \begin{array}{ccc}
\vdots \\
\end{array} i + 1 \begin{array}{ccc}
\vdots \\
\end{array} \begin{array}{ccc}
\vdots \\
\end{array} j \begin{array}{ccc}
\vdots \\
\end{array}$$

and

$$p_j = \begin{array}{ccc}
\vdots \\
\end{array} \begin{array}{ccc}
\vdots \\
\end{array} \begin{array}{ccc}
\vdots \\
\end{array} i \begin{array}{ccc}
\vdots \\
\end{array} i + 1 \begin{array}{ccc}
\vdots \\
\end{array} \begin{array}{ccc}
\vdots \\
\end{array} \begin{array}{ccc}
\vdots \\
\end{array} \begin{array}{ccc}
\vdots \\
\end{array} \begin{array}{ccc}
\vdots \\
\end{array} \begin{array}{ccc}
\vdots \\
\end{array}$$

and

$$p_{i+\frac{1}{2}} = \begin{array}{ccc}
\vdots \\
\end{array} \begin{array}{ccc}
\vdots \\
\end{array} i \begin{array}{ccc}
\vdots \\
\end{array} i + 1 \begin{array}{ccc}
\vdots \\
\end{array} \begin{array}{ccc}
\vdots \\
\end{array} \begin{array}{ccc}
\vdots \\
\end{array}$$
We also recall the presentation for $A_k(z)$ given in [En].

**Theorem 2.2** (Theorem 4.1 of [En]). If $k = 1, 2, \ldots$, then $A_k(z)$ is the unital associative algebra presented by the generators

$$p_1, p_{1+\frac{1}{2}}, p_2, \ldots, p_k, \sigma_2, \sigma_{2+\frac{1}{2}}, \sigma_3, \ldots, \sigma_k,$$

and the relations:

1. **(Involutions)**
   - (a) $\sigma_{1+\frac{1}{2}} = 1$, for $i = 2, \ldots, k - 1$.
   - (b) $\sigma_{1+\frac{1}{2}} = 1$, for $i = 1, \ldots, k - 1$.

2. **(Braid–like relations)**
   - (a) $\sigma_{i+1}\sigma_{j+\frac{1}{2}} = \sigma_{j+1}\sigma_{i+1}$, if $j \neq i + 1$.
   - (b) $\sigma_{i}\sigma_{i+1} = \sigma_{i+1}\sigma_{i}$, if $j \neq i + 1$.
   - (c) $\sigma_{i+\frac{1}{2}}\sigma_{j+\frac{1}{2}} = \sigma_{j+\frac{1}{2}}\sigma_{i+\frac{1}{2}}$, if $j \neq i + 1$.
   - (d) $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$, for $i = 1, \ldots, k - 2$, where

\[
s_\ell = \begin{cases} 
\sigma_{\ell+1}, & \text{if } \ell = 1, \\
\sigma_{\ell+\frac{1}{2}}\sigma_{\ell+1}, & \text{if } \ell = 2, \ldots, k - 1,
\end{cases}
\]

are the Coxeter generators for the symmetric group.

3. **(Idempotent relations)**
   - (a) $p_i^2 = zp_i$, for $i = 1, \ldots, k$.
   - (b) $p_{i+\frac{1}{2}}^2 = p_{i+\frac{1}{2}}$, for $i = 1, \ldots, k - 1$.
   - (c) $\sigma_{i+1}p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}\sigma_{i+1} = p_{i+\frac{1}{2}}\sigma_{i+1}$, for $i = 1, \ldots, k - 1$.
   - (d) $\sigma_{i+\frac{1}{2}}p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}\sigma_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}\sigma_{i+\frac{1}{2}}$, for $i = 1, \ldots, k - 1$.
   - (e) $\sigma_{i+1}p_{i+1} = \sigma_{i+1}p_{i+1}$, for $i = 1, \ldots, k - 1$.
   - (f) $p_ip_{i+1}\sigma_{i+1} = p_{i+1}\sigma_{i+1}p_i$, for $i = 1, \ldots, k - 1$.

4. **(Commutation relations)**
   - (a) $p_ip_j = p_jp_i$, for $i, j = 1, \ldots, k$.
   - (b) $p_{i+\frac{1}{2}}p_{j+\frac{1}{2}} = p_{j+\frac{1}{2}}p_{i+\frac{1}{2}}$, for $i, j = 1, \ldots, k - 1$.
   - (c) $p_{i+\frac{1}{2}}p_j = p_jp_{i+\frac{1}{2}}$, for $j \neq i + 1, i$.
   - (d) $\sigma_ip_j = \sigma_ip_j$, if $j \neq i - 1, i$.
   - (e) $\sigma_ip_{i+\frac{1}{2}} = \sigma_ip_{i+\frac{1}{2}}$, if $j \neq i$.
   - (f) $\sigma_{i+\frac{1}{2}}p_j = \sigma_{i+\frac{1}{2}}p_j$, if $j \neq i, i + 1$.
   - (g) $\sigma_{i+\frac{1}{2}}p_j = p_{j+\frac{1}{2}}\sigma_{i+\frac{1}{2}}$, if $j \neq i - 1$.
   - (h) $\sigma_{i+\frac{1}{2}}p_{i+1}\sigma_{i+1} = \sigma_{i+1}p_{i+1}\sigma_{i+1}$, for $i = 1, \ldots, k - 1$.
   - (i) $\sigma_{i+\frac{1}{2}}p_{i+1}\sigma_{i+1} = \sigma_{i+1}p_{i+1}\sigma_{i+1}$, for $i = 2, \ldots, k - 1$.

5. **(Contraction relations)**
   - (a) $p_{i+\frac{1}{2}}p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}$, for $j = i, i + 1$.
   - (b) $p_{i+\frac{1}{2}}p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}$, for $j = i, i + 1$.

It will be useful to note that $R$-linear map $*: A_k(z) \to A_k(z)$ defined on generators by

$$u^* = u^{-1} \quad \text{(for } u \in \langle \sigma_2, \sigma_{2+\frac{1}{2}}, \ldots, \sigma_k \rangle)$$

and

$$p_i^* = p_i \quad \text{(for } i = 1, \ldots, k) \quad \text{and} \quad p_j^* = p_{j+\frac{1}{2}} \quad \text{(for } j = 1, \ldots, k - 1),$$

is an algebra anti-involution of $A_k(z)$. Restricting the map $*$ from $A_k(z)$ to $A_{k-\frac{1}{2}}(z)$, gives an algebra anti-involution of $A_{k-\frac{1}{2}}(z)$ which we also denote by $*$. 


2.3. Jucys–Murphy elements. Jucys–Murphy elements for the partition algebras were defined in the diagram basis by Halverson and Ram [HR]. The following recursions for Jucys–Murphy elements in $A_k(z)$ and $A_{k+1}(z)$, which have been obtained in [En], are equivalent to the definition of Jucys–Murphy elements given in [HR].

Let $(\sigma_i : i = 1, 2, \ldots)$ and $(L_i : i = 0, 1, \ldots)$ be given by

\[
L_0 = 0, \quad L_1 = p_1, \quad \sigma_1 = 1, \quad \text{and,} \quad \sigma_2 = s_1,
\]

and, for $i = 1, 2, \ldots$,

\[
L_{i+1} = -s_i L_i p_{i+\frac{1}{2}} - p_{i+\frac{1}{2}} L_i s_i + p_{i+\frac{1}{2}} L_i p_{i+1} p_{i+\frac{1}{2}} + s_i L_i s_i + \sigma_{i+1},
\]  

(2.3)

where, for $i = 2, 3, \ldots$,

\[
\sigma_{i+1} = s_{i-1}s_i\sigma_i s_i s_i - s_i p_i - \frac{1}{2} L_i - s_i p_i p_i - \frac{1}{2} s_i - s_i L_i p_{i+\frac{1}{2}} - p_{i+\frac{1}{2}} p_i p_i p_i - \frac{1}{2} s_i - s_i L_i p_{i+\frac{1}{2}} s_i.
\]

(2.4)

Define $(\sigma_{i+\frac{1}{2}} : i = 1, 2, \ldots)$ and $(L_{i+\frac{1}{2}} : i = 0, 1, \ldots)$ by

\[
L_{\frac{1}{2}} = 0, \quad \sigma_{\frac{1}{2}} = 1, \quad \text{and,} \quad \sigma_{1+\frac{1}{2}} = 1,
\]

and, for $i = 1, 2, \ldots$,

\[
L_{i+\frac{1}{2}} = -L_i p_{i+\frac{1}{2}} - p_i p_i - \frac{1}{2} p_i p_i p_i + s_i L_i s_i + \sigma_{i+\frac{1}{2}},
\]

(2.5)

where, for $i = 2, 3, \ldots$,

\[
\sigma_{i+\frac{1}{2}} = s_{i-1}s_i\sigma_{i-\frac{1}{2}} s_i - s_i p_i - \frac{1}{2} L_i - s_i p_i p_i - \frac{1}{2} \sigma_{i+1} - s_i L_i p_{i+\frac{1}{2}} - p_{i+\frac{1}{2}} p_i p_i p_i - \frac{1}{2} \sigma_{i+1} - s_i L_i p_{i+\frac{1}{2}} s_i.
\]

(2.6)

Remark 2.3. The transition from the algebra presentation given in Theorem 2.1 to the presentation in Theorem 2.2 is defined by the equations (2.3)–(2.6).

We collate the following facts from §3 of [En] for later reference.

**Proposition 2.4.** For $i = 1, 2, \ldots$, the following statements hold:

1. $\sigma_{i+\frac{1}{2}} \in A_{i+\frac{1}{2}}(z)$ and $\sigma_{i+1} \in A_{i+1}(z)$.
2. $(\sigma_{i+\frac{1}{2}})^* = \sigma_{i+\frac{1}{2}}$ and $(\sigma_{i+1})^* = \sigma_{i+1}$.
3. $L_{i+\frac{1}{2}} \in A_{i+\frac{1}{2}}(z)$ and $L_{i+1} \in A_{i+1}(z)$.
4. $(L_{i+\frac{1}{2}})^* = L_{i+\frac{1}{2}}$ and $(L_{i+1})^* = \sigma_{i+1}$.
5. $s_i \sigma_i p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}} \sigma_i s_i$.
6. $\sigma_i p_i p_{i+\frac{1}{2}} = s_i L_i p_{i+\frac{1}{2}}$.
7. $\sigma_i p_i p_{i+\frac{1}{2}} = L_i p_{i+\frac{1}{2}}$.
8. $\sigma_i s_i = \sigma_i s_i = \sigma_{i+1} = 1$.
9. $L_{i+1}$ commutes with $A_{i+\frac{1}{2}}(z)$, and $\sigma_{i+1}$ commutes with $A_{i+\frac{1}{2}}(z)$.
10. $L_{i+\frac{1}{2}}$ commutes with $A_i(z)$, and $\sigma_{i+\frac{1}{2}}$ commutes with $A_{i-\frac{1}{2}}(z)$.
11. $(L_{i+\frac{1}{2}} + L_{i+1}) p_i = p_i (L_{i+\frac{1}{2}} + L_{i+1}) = z p_i$.
12. $(L_i + L_{i+1}) p_i = p_i (L_i + L_{i+1}) = z p_i$.
13. The element $z_i = L_i + L_{i+1} + \cdots$ is central in $A_i(z)$.
14. The element $z_{i+\frac{1}{2}} = L_{i+\frac{1}{2}} + L_{i+1} + \cdots$ is central in $A_{i+\frac{1}{2}}(z)$.
15. $p_i \sigma_i p_i = L_i p_i$.
16. $p_i \sigma_i p_i = (z - L_i) p_i$.
17. $p_i \sigma_i p_i = p_i z p_i$.
The next statement gives recursions for the Jucys–Murphy elements in terms of the presentation Theorem 2.2.

**Proposition 2.5.** For \( i = 1, 2, \ldots \), the following statements hold:

1. \( \sigma_{i+1} L_{i+1} - L_i \sigma_i + 1 = -L_i p_{i+\frac{1}{2}} - p_{i+\frac{1}{2}} p_{i+1} + p_{i+\frac{1}{2}} L_i p_{i+1} + 1. \)
2. \( \sigma_{i+\frac{1}{2}} L_{i+\frac{1}{2}} - L_i \sigma_{i+\frac{1}{2}} + 1 = -p_{i+\frac{1}{2}} L_i - (z - L_i - \frac{1}{2}) p_{i+\frac{1}{2}} + 1. \)
3. \( L_{i+1} = -p_{i+1} p_{i+\frac{1}{2}} - p_{i+\frac{1}{2}} p_{i+1} + p_{i+\frac{1}{2}} L_i p_{i+1} + \sigma_{i+1} L_i \sigma_{i+1} + \sigma_{i+1}. \)
4. \( L_{i+\frac{1}{2}} = -p_{i+\frac{1}{2}} p_{i+\frac{1}{2}} - p_{i+\frac{1}{2}} p_{i+1} + (z - L_i - \frac{1}{2}) p_{i+\frac{1}{2}} + \sigma_{i+\frac{1}{2}} L_i \sigma_{i+\frac{1}{2}} + \sigma_{i+\frac{1}{2}}. \)

**Proof.** (1) The definition (2.3), and the relations

\[ \sigma_{i+1}s_i = s_i\sigma_{i+1} = \sigma_{i+\frac{1}{2}} \quad \text{and} \quad \sigma_{i+1}^2 = 1, \]

imply that

\[ s_i L_{i+1} s_i = -L_i p_{i+\frac{1}{2}} - p_{i+\frac{1}{2}} L_i + p_{i+\frac{1}{2}} L_i p_{i+\frac{1}{2}} + L_i + \sigma_{i+1}, \]

and

\[ s_i L_{i+1}\sigma_{i+\frac{1}{2}} = -L_i p_{i+\frac{1}{2}} - p_{i+\frac{1}{2}} L_i \sigma_{i+1} + p_{i+\frac{1}{2}} L_i p_{i+\frac{1}{2}} + L_i \sigma_{i+1} + 1. \]

Since \( L_{i+1} \) commutes with \( \sigma_{i+\frac{1}{2}} \in A_{i+\frac{1}{2}}(x) \),

\[ \sigma_{i+1} L_{i+1} = -L_i p_{i+\frac{1}{2}} - p_{i+\frac{1}{2}} L_i \sigma_{i+1} + p_{i+\frac{1}{2}} L_i p_{i+\frac{1}{2}} + L_i \sigma_{i+1} + 1, \]

and

\[ \sigma_{i+1} L_{i+1} - L_i \sigma_{i+1} = -L_i p_{i+\frac{1}{2}} - p_{i+\frac{1}{2}} L_i \sigma_{i+1} + p_{i+\frac{1}{2}} L_i p_{i+\frac{1}{2}} + L_i \sigma_{i+1} + 1. \]

Making the substitution \( p_{i+\frac{1}{2}} p_{i+1} \sigma_{i+1} = p_{i+\frac{1}{2}} L_i \sigma_{i+1} \) in the last expression gives the required statement.

(2) Since

\[ L_{i+\frac{1}{2}} = -L_i p_{i+\frac{1}{2}} - p_{i+\frac{1}{2}} L_i + (z - L_i - \frac{1}{2}) p_{i+\frac{1}{2}} + s_i L_{i-\frac{1}{2}} s_i + \sigma_{i+\frac{1}{2}}, \]

the relations \( \sigma_{i+1}s_i = s_i\sigma_{i+1} = \sigma_{i+\frac{1}{2}} \) and \( \sigma_{i+1}^2 = 1 \) imply that

\[ \sigma_{i+\frac{1}{2}} L_{i+\frac{1}{2}} = -\sigma_{i+\frac{1}{2}} L_i p_{i+\frac{1}{2}} - p_{i+\frac{1}{2}} L_i + (z - L_i - \frac{1}{2}) p_{i+\frac{1}{2}} + \sigma_{i+1} L_i - \frac{1}{2} s_i + 1. \]

Since \( \sigma_{i+1} \) commutes with \( L_{i-\frac{1}{2}} \in A_{i-\frac{1}{2}}(x) \),

\[ \sigma_{i+\frac{1}{2}} L_{i+\frac{1}{2}} - L_i \sigma_{i+\frac{1}{2}} = -\sigma_{i+\frac{1}{2}} L_i p_{i+\frac{1}{2}} - p_{i+\frac{1}{2}} L_i + (z - L_i - \frac{1}{2}) p_{i+\frac{1}{2}} + 1. \]

Applying the relation \( p_{i+\frac{1}{2}} L_i \sigma_{i+\frac{1}{2}} = p_{i+\frac{1}{2}} p_i \) from Proposition 2.4 to the right hand side of the last equality, gives the required statement.

(3) Using item (7) of Proposition 2.4 and the relation \( \sigma_{i+1}^2 = 1 \), we obtain \( \sigma_{i+1} L_i p_{i+\frac{1}{2}} = p_{i+1} p_{i+\frac{1}{2}} \). Since \( \sigma_{i+1} p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}} \), the statement follows from item (1).

(4) We proceed as in (3), using the relation \( p_{i+\frac{1}{2}} L_i \sigma_{i+\frac{1}{2}} = p_{i+\frac{1}{2}} p_i \). \( \square \)

3. A Murphy Basis

Murphy–type bases for the partition algebras have been given in [En1] (see Theorems 3.2 and 3.3). In this section, we re–write the bases of [En1] and use the new definition of Murphy–type bases to show that the Jucys–Murphy elements (2.3) and (2.5) act triangularly on the partition algebras.

For \( k = 0, 1, \ldots \), let

\[ \hat{A}_k = \hat{A}_{k+\frac{1}{2}} = \{ (\lambda, \ell) \mid \lambda \vdash (k - \ell), \text{ for } \ell = 0, 1, \ldots, k \}. \]
If \((\lambda, \ell), (\mu, m) \in \hat{A}_k\), write \((\lambda, \ell) \trianglerighteq (\mu, m)\) if either (i) \(\ell = m\) and \(\lambda \trianglerighteq \mu\), or (ii) \(\ell > m\).

Following §2 of [HR], build a graph \(\hat{A}\) with

1. vertices on level \(k\): \(\hat{A}_k\),
2. vertices on level \(k + \frac{1}{2}\): \(\hat{A}_{k+\frac{1}{2}} = \hat{A}_k\),
3. an edge \((\lambda, \ell) \to (\mu, m)\) in \(\hat{A}\), for \((\lambda, \ell) \in \hat{A}_k\), \((\mu, m) \in \hat{A}_{k+\frac{1}{2}}\), if \(\lambda = \mu\), or if \(\lambda\) is obtained from \(\mu\) by removing a node,
4. an edge \((\lambda, \ell) \to (\mu, m)\) in \(\hat{A}\), for \((\lambda, \ell) \in \hat{A}_{k+\frac{1}{2}}\), \((\mu, m) \in \hat{A}_{k+1}\), if \(\lambda = \mu\), or if \(\lambda\) is obtained from \(\mu\) by adding a node.

**Definition 3.1.** Let \(j \in \frac{1}{2}\mathbb{Z}_{\geq 0}\). A path to level \(j\) in the graph \(\hat{A}\) is a sequence

\[
\ell = ((\lambda(0), \ell_0), (\lambda(\frac{1}{2}), \ell_{\frac{1}{2}}), (\lambda(1), \ell_1), \ldots, (\lambda(j), \ell_j)),
\]

where (3.1)

\[
(\lambda(0), \ell_0) = (\emptyset, 0) \quad \text{and} \quad (\lambda(r), \ell_r) \to (\lambda(r+\frac{1}{2}), \ell_{r+\frac{1}{2}}), \quad \text{for } r = 0, \frac{1}{2}, 1, \ldots, j - \frac{1}{2}.
\]

We say that \(\ell\) is a path of shape \((\lambda, \ell)\), and write \((\lambda, \ell) = \text{Shape}(\ell)\).

If \((\lambda, \ell) \in \hat{A}_k\), let

\[
\hat{A}_k^{(\lambda, \ell)} = \{ \ell \mid \ell \text{ is a path to level } k, \text{ and } \text{Shape}(\ell) = (\lambda, \ell) \}, \quad \text{and}
\]

\[
\hat{A}_{k+\frac{1}{2}}^{(\lambda, \ell)} = \{ \ell \mid \ell \text{ is a path to level } k + \frac{1}{2}, \text{ and } \text{Shape}(\ell) = (\lambda, \ell) \}.
\]

The definitions of the sets \(\hat{A}_k^{(\lambda, \ell)}\) and \(\hat{A}_{k+\frac{1}{2}}^{(\lambda, \ell)}\), for \((\lambda, \ell) \in \hat{A}_k\), are illustrated in the diagram (3.2) where the first few levels of the graph \(\hat{A}\) are given.

\[
\begin{align*}
&\text{(3.2)} \\
&0 \quad \emptyset \\
&\downarrow \quad \downarrow \\
&1 \quad \emptyset \\
&\downarrow \quad \downarrow \\
&2 \quad \emptyset \\
&\downarrow \quad \downarrow \\
&3 \quad \emptyset
\end{align*}
\]

If \(\ell\) is a path of the form (3.1), we will generally write

\[
\ell = ((\lambda(0), \lambda(\frac{1}{2}), \lambda(1), \ldots, \lambda(j)),
\]

and, if \(r = 0, \frac{1}{2}, 1, \ldots, j - \frac{1}{2}\), define the truncation of \(\ell\) to level \(r\) to be the path

\[
\ell|_r = ((\lambda(0), \lambda(\frac{1}{2}), \lambda(1), \ldots, \lambda(r)).
\]
We can now define the Murphy bases for partition algebras. For $i = 1, 2, \ldots$, let
\[
\beta_i^{(t)} = \frac{p_{i-t+1}p_{i-t+2} \cdots p_i}{\ell \text{ factors}}
\]
and
\[
\delta_{i+\frac{1}{2}}^{(t)} = \frac{p_{i-t+\frac{1}{2}}p_{i-t+\frac{3}{2}} \cdots p_{i+\frac{1}{2}}}{\ell \text{ factors}}
\]
if $\ell \leq i$
and let
\[
\beta_i^{(0)} = 0 \quad \text{and} \quad \delta_{i+\frac{1}{2}}^{(0)} = 0
\]
if $\ell > i$.

For $i = 0, 1, \ldots$, and $(\lambda, \ell) \in \hat{A}_i$, let
\[
c_{\lambda} = \sum_{v \in S_{\lambda}} v
\]
and
\[
x_{\lambda, \ell}^{(i)} = c_{\lambda}\beta_i^{(i)} \in A_i
\]
and
\[
x_{\lambda, \ell}^{(i+\frac{1}{2})} = c_{\lambda}\delta_{i+\frac{1}{2}}^{(i)} \in A_{i+\frac{1}{2}}
\]
Define the two sided ideals
\[
A_{\lambda, \ell}^{(i)} = \sum_{(\mu, m) \sqsupseteq (\lambda, \ell)} A_i x_{\mu, m}^{(i)} A_i
\]
and
\[
A_{\lambda, \ell}^{(i+\frac{1}{2})} = \sum_{(\mu, m) \sqsupseteq (\lambda, \ell)} A_{i+\frac{1}{2}} x_{\mu, m}^{(i+\frac{1}{2})} A_{i+\frac{1}{2}}
\]
where each sum is taken over $(\mu, m) \in \hat{A}_i$ such that $(\mu, m) \sqsupset (\lambda, \ell)$. For $(\lambda, \ell) \in \hat{A}_i$, define the right $A_i$-module
\[
A_{\lambda, \ell} = \left\{ x_{\lambda, \ell}^{(i)} p + A_{\lambda, \ell}^{(i)} \middle| p \in A_i \right\} \subseteq A_i/A_{\lambda, \ell}^{(i+\frac{1}{2})}
\]
and the right $A_{\lambda, \ell}$-module
\[
A_{\lambda, \ell} = \left\{ x_{\lambda, \ell}^{(i+\frac{1}{2})} p + A_{\lambda, \ell}^{(i+\frac{1}{2})} \middle| p \in A_{i+\frac{1}{2}} \right\} \subseteq A_{i+\frac{1}{2}}/A_{\lambda, \ell}^{(i+\frac{1}{2})}
\]
If $\lambda \vdash (i - 1)$ and $\mu \vdash i$, such that $\mu = \lambda \cup \{(j, \mu_j)\}$, let $a_j = \sum_{j=1}^{i} \mu_j$ and define
\[
\bar{a}_{\lambda \rightarrow \mu} = w_{i, a_j} \sum_{r=0}^{\lambda_i} w_{a_i, a_j-r}
\]
and
\[
\tilde{a}_{\lambda \rightarrow \mu} = w_{i, a_j}
\]
so that
\[
c_{\lambda}^{(i-1)} \bar{a}_{\lambda \rightarrow \mu}^{(i)} = (a_{\lambda \rightarrow \mu}^{(i)})^* c_{\mu}^{(i)}.
\]
If $(\lambda, \ell) \in \hat{A}_{i-1}$ and $(\mu, m) \in \hat{A}_i$ and $(\lambda, \ell) \rightarrow (\mu, m)$, let
\[
a_{(\lambda, \ell) \rightarrow (\mu, m)}^{(i)} = \begin{cases} p_{i-1}^{(m)} & \text{if } \ell = m, \\ p_{i-1}^{(m-1)} & \text{if } \ell = m - 1. \end{cases}
\]
Similarly, if $(\lambda, \ell) \in \hat{A}_i$ and $(\mu, m) \in \hat{A}_{i+\frac{1}{2}}$ and $(\lambda, \ell) \rightarrow (\mu, m)$, let
\[
a_{(\lambda, \ell) \rightarrow (\mu, m)}^{(i+\frac{1}{2})} = \begin{cases} p_{i-1}^{(m)} & \text{if } m = n, \\ p_{i-1}^{(m-1)} & \text{if } \ell = m - 1. \end{cases}
\]
Generally, we write $a_{\lambda \rightarrow \mu}^{(i)}$ for $a_{(\lambda, \ell) \rightarrow (\mu, m)}^{(i)}$ and $a_{\lambda \rightarrow \mu}^{(i+\frac{1}{2})}$ for $a_{(\lambda, \ell) \rightarrow (\mu, m)}^{(i+\frac{1}{2})}$.

Let $k \in \mathbb{Z}_{\geq 0}$ and $(\lambda, \ell) \in \hat{A}_k$. For $t = (\lambda^{(0)}, \lambda^{(\frac{1}{2})}, \ldots, \lambda^{(k)}) \in \hat{A}_k^{(\lambda, \ell)}$, let
\[
a_{(k)}^{(i)} = a_{\lambda^{(k)} \rightarrow \lambda^{(i)}}^{(k)} a_{\lambda^{(k-1)} \rightarrow \lambda^{(i-\frac{1}{2})}}^{(k-\frac{1}{2})} \cdots a_{\lambda^{(0)} \rightarrow \lambda^{(i-\frac{1}{2})}}^{(i-\frac{1}{2})},
\]
and, for \( s = (\lambda(0), \lambda(\frac{1}{2}), \ldots, \lambda(k+\frac{1}{2})) \in \hat{A}_{k+\frac{1}{2}}^{(\lambda, \ell)} \), let
\[
a_s^{(k+\frac{1}{2})} = a^{(k+\frac{1}{2})}_{\lambda(k)\rightarrow \lambda(k+\frac{1}{2})} a^{(k)}_{\lambda(k-\frac{1}{2})\rightarrow \lambda(k)} \cdots a^{(\frac{1}{2})}_{\lambda(0)\rightarrow \lambda(\frac{1}{2})}.
\]

The two statements below are Theorem 5.10 of [En1] applied to the algebras \( A_k \) and \( A_{k+\frac{1}{2}} \) respectively. For brevity we have written \( a_t = a_t^{(k)} \) in Theorem 3.2, and \( a_t = a_t^{(k+\frac{1}{2})} \) in Theorem 3.3.

**Theorem 3.2.** If \( k = 1, 2, \ldots, \), then the set
\[
\mathcal{A}_k = \left\{ a_s^x(\lambda, \ell) a_t | s, t \in \hat{A}_k^{(\lambda, \ell)}, (\lambda, \ell) \in \hat{A}_k, and \ell = 0, 1, \ldots, k \right\}
\]
is an \( R \)-basis for \( A_k \). Moreover, the following statements hold:

1. If \( (\lambda, \ell) \in \hat{A}_k \), \( t \in \hat{A}_k^{(\lambda, \ell)} \), and \( p \in A_k \), there exist \( r_u \in R \), for \( u \in \hat{A}_k^{(\lambda, \ell)} \), such that
\[
a_s^x(\lambda, \ell) a_t p = \sum_{u \in \hat{A}_k^{(\lambda, \ell)}} r_u a_s^x(\lambda, \ell) a_t \mod A_k^{(\lambda, \ell)} \text{ for all } s \in \hat{A}_k^{(\lambda, \ell)},
\]

where \( A_k^{(\lambda, \ell)} \) is the \( R \)-module freely generated by
\[
\left\{ a_s^* x^{(\lambda, \ell)} a_t | s, t \in \hat{A}_k^{(\lambda, \ell)}, (\mu, m) \in \hat{A}_k, and (\mu, m) \trianglerighteq (\lambda, \ell) \right\}.
\]

2. If \( (\lambda, \ell) \in \hat{A}_k \), \( s, t \in \hat{A}_k^{(\lambda, \ell)} \), then \( * : a_s^x(\lambda, \ell) a_t \mapsto a_t^* x(\lambda, \ell) a_s \).

**Theorem 3.3.** If \( k = 1, 2, \ldots, \), then the set
\[
\mathcal{A}_{k+\frac{1}{2}} = \left\{ a_s^x(\lambda, \ell) a_t | s, t \in \hat{A}_{k+\frac{1}{2}}^{(\lambda, \ell)}, (\lambda, \ell) \in \hat{A}_{k+\frac{1}{2}}, and \ell = 0, 1, \ldots, k \right\}
\]
is an \( R \)-basis for \( A_{k+\frac{1}{2}} \). Moreover, the following statements hold:

1. If \( (\lambda, \ell) \in \hat{A}_{k+\frac{1}{2}} \), \( t \in \hat{A}_{k+\frac{1}{2}}^{(\lambda, \ell)} \), and \( p \in A_{k+\frac{1}{2}} \), there exist \( r_u \in R \), for \( u \in \hat{A}_{k+\frac{1}{2}}^{(\lambda, \ell)} \), such that
\[
a_s^x(\lambda, \ell) a_t p = \sum_{u \in \hat{A}_{k+\frac{1}{2}}^{(\lambda, \ell)}} r_u a_s^x(\lambda, \ell) a_t \mod A_{k+\frac{1}{2}}^{(\lambda, \ell)} \text{ for all } s \in \hat{A}_{k+\frac{1}{2}}^{(\lambda, \ell)},
\]

where \( A_{k+\frac{1}{2}}^{(\lambda, \ell)} \) is the \( R \)-module freely generated by
\[
\left\{ a_s^* x^{(\lambda, \ell)} a_t | s, t \in \hat{A}_{k+\frac{1}{2}}^{(\lambda, \ell)}, (\mu, m) \in \hat{A}_{k+\frac{1}{2}}, and (\mu, m) \trianglerighteq (\lambda, \ell) \right\}.
\]

2. If \( (\lambda, \ell) \in \hat{A}_{k+\frac{1}{2}} \), \( s, t \in \hat{A}_{k+\frac{1}{2}}^{(\lambda, \ell)} \), then \( * : a_s^x(\lambda, \ell) a_t \mapsto a_t^* x(\lambda, \ell) a_s \).

The next statement gives a filtration of the cell modules for \( A_i \) by \( A_{i-\frac{1}{2}} \)-modules and a filtration of the cell modules for \( A_{i+\frac{1}{2}} \) by \( A_{i} \)-modules.

**Proposition 3.4** (Proposition 4.2 of [En1]). (1) Let \( (\mu, m) \in \hat{A}_i \) and \( \{(\lambda(1), \ell_1), \ldots, (\lambda(t), \ell_t)\} \) be an indexing of the set
\[
\{(\lambda, \ell) \in \hat{A}_{i-\frac{1}{2}} | (\lambda, \ell) \trianglerighteq (\mu, m)\}
\]
such that \( (\lambda(r), \ell_r) \trianglerighteq (\lambda(s), \ell_s) \) whenever \( s > r \). For \( j = 1, \ldots, t \), let
\[
N_j = \sum_{s \in A_{i-\frac{1}{2}}^{(\mu, m)}} (x^{(1)}_{(\mu, m)} + A_{i-\frac{1}{2}}^{(\mu, m)}) a_s^j A_{i-\frac{1}{2}}.
\]
Then
\[ \{0\} = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_p = A^{(\mu,m)}_{i+\frac{1}{2}} \]
is a filtration by right \( A_{i-\frac{1}{2}} \)-modules and, for \( j = 1, \ldots, t \), the \( R \)-linear map
\[ A^{(\lambda,j),j}_{i-\frac{1}{2}} \rightarrow N_j/N_{j-1} \]
\[ (x^{(i-\frac{1}{2})}_{(\lambda,j),j}) + A^{\triangleright(\mu,m)}_{i-\frac{1}{2}} a_u^{(i-\frac{1}{2})} \]lmaps to \( (x^{(i)}_{(\mu,m)} + A^{\triangleright(\mu,m)}_{i+\frac{1}{2}}) a_u^{(i)} + N_{j-1}, \] (3.5)
for \( u \in A^{(\lambda,j),j}_{i-\frac{1}{2}} \) and \( t \in A^{(\mu,m)}_{i+\frac{1}{2}} \) such that \( t|_{i-\frac{1}{2}} = u \), is an isomorphism of right \( A_{i-\frac{1}{2}} \)-modules.

(2) Let \( (\mu,m) \in A_{i+\frac{1}{2}} \) and \( \{(\lambda^{(1)},\ell_1), \ldots, (\lambda^{(t)},\ell_t)\} \) be an indexing of the set
\[ \{(\lambda,\ell) \in A_i \mid (\lambda,\ell) \rightarrow (\mu,m)\} \]
such that \( (\lambda_r,\ell_r) \triangleright (\lambda_s,\ell_s) \) whenever \( s > r \). For \( j = 1, \ldots, t \), let
\[ N_j = \sum_{s \in A^{(\mu,m)}_{i+\frac{1}{2}}} \text{Shape}(s|_i \triangleright (\lambda^{(s)},\ell^{(s)})) \]
Then
\[ \{0\} = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_p = A^{(\mu,m)}_{i+\frac{1}{2}} \]
is a filtration by right \( A_{i} \)-modules and, for \( j = 1, \ldots, p \), the \( R \)-linear map
\[ A^{(\lambda,j),j}_{i} \rightarrow N_j/N_{j-1} \]
\[ (x^{(i)}_{(\mu,m)} + A^{\triangleright(\mu,m)}_{i+\frac{1}{2}}) a_u^{(i)} + N_{j-1}, \] (3.6)
for \( u \in A^{(\lambda,j),j}_{i} \) and \( t \in A^{(\mu,m)}_{i+\frac{1}{2}} \) such that \( t|_i = u \), is an isomorphism of right \( A_{i} \)-modules.

Corollary 3.5. (1) Let \( (\mu,m) \in A_{i} \) and \( (\lambda,\ell) \in A_{i-\frac{1}{2}} \) such that \( (\lambda,\ell) \rightarrow (\mu,m) \). If \( p \in A_{i-\frac{1}{2}} \) and \( x^{(i-\frac{1}{2})}_{(\lambda,\ell)} \in \hat{A}_{i-\frac{1}{2}}^{\triangleright(\lambda,\ell)} \), then there exist \( r_s \in R \), for \( s \in A^{(\mu,m)}_{i} \), such that
\[ x^{(i)}_{(\mu,m)} a^{(i)}_{\lambda \rightarrow \mu} p = \sum_{s \in A^{(\mu,m)}_{i}} r_s x^{(i)}_{(\mu,m)} a^{(i)}_{s} \mod \hat{A}^{\triangleright(\mu,m)}_{i}, \]
where \( r_s = 0 \) unless \( \text{Shape}(s|_{i-\frac{1}{2}}) \triangleright (\lambda,\ell) \).

(2) Let \( (\mu,m) \in A_{i+\frac{1}{2}} \) and \( (\lambda,\ell) \in A_{i} \) such that \( (\lambda,\ell) \rightarrow (\mu,m) \). If \( p \in A_{i} \) and \( x^{(i)}_{(\lambda,\ell)} p \in A^{\triangleright(\lambda,\ell)}_{i} \), then there exist \( r_s \in R \), for \( s \in A^{(\mu,m)}_{i+\frac{1}{2}} \), such that
\[ x^{(i+\frac{1}{2})}_{(\mu,m)} a^{(i+\frac{1}{2})}_{\lambda \rightarrow \mu} p = \sum_{s \in A^{(\mu,m)}_{i+\frac{1}{2}}} r_s x^{(i+\frac{1}{2})}_{(\mu,m)} a^{(i+\frac{1}{2})}_{s} \mod \hat{A}^{\triangleright(\mu,m)}_{i+\frac{1}{2}}, \]
where \( r_s = 0 \) unless \( \text{Shape}(s|_{i}) \triangleright (\lambda,\ell) \).

Proof. (1) Observe that there exists \( b^{(i)}_{\lambda \rightarrow \mu} \in A_{i} \) such that
\[ x^{(i)}_{(\mu,m)} a^{(i)}_{\lambda \rightarrow \mu} = b^{(i)}_{\lambda \rightarrow \mu} x^{(i-\frac{1}{2})}_{(\lambda,\ell)} \]
and then use (1) of Proposition 3.4 to see that if \( p \in \mathcal{A}_{i-\frac{1}{2}} \) and \( x_{(\lambda,\ell)}^{(i-\frac{1}{2})} p \in \mathcal{A}_{i-\frac{1}{2}}^{(\lambda,\ell)} \), then

\[
b_{\lambda 
abla m}^{(i)} x_{(\lambda,\ell)}^{(i-\frac{1}{2})} p \equiv \sum_{s \in \mathcal{A}_{i}^{(\mu,m)}} r_s x_{(\mu,m)}^{(i)} a_s^{(i)} \mod \mathcal{A}_{i}^{(\mu,m)},
\]

where \( r_s = 0 \) unless \( \text{Shape}(s \mid_{i-\frac{1}{2}}) \geq (\lambda, \ell) \). The proof of (2) is similar. \( \square \)

The sets \( \hat{A}_i^{(\lambda,\ell)} \) and \( \hat{A}_{1+\frac{1}{2}}^{(\lambda,\ell)} \) are partially ordered by the following reverse lexicographic order given in 2.7 of [GG1].

**Definition 3.6.** Let \( i \in \frac{1}{2} \mathbb{Z} \) and \( s = (s_0, s_\frac{1}{2}, \ldots, s_i) \) and \( t = (t_0, t_\frac{1}{2}, \ldots, t_i) \) be two paths in \( \hat{A} \).

We write \( s \succ t \) if \( s = t \), or if for the last index \( j \in \frac{1}{2} \mathbb{Z} \) such that \( s^{(j)} \neq t^{(j)} \), we have \( s_j \geq t_j \) in \( \hat{A}_j \). Let \( s \succ t \) denote the fact that \( s \succ t \) and \( s \neq t \); write \( s \succdot t \) if \( s \succ t \) and \( j \in \frac{1}{2} \mathbb{Z} \) is the largest index for which \( s_j = t_j \).

If \( (\lambda, \ell) \in \hat{A}_i \), let \( t^\lambda \) denote the maximal element in \( \hat{A}_i^{(\lambda,\ell)} \) under the partial order \( \succ \). Similarly, if \( (\mu,m) \in \hat{A}_{1+\frac{1}{2}} \), let \( t^\mu \) denote the maximal element in \( \hat{A}_{1+\frac{1}{2}}^{(\mu,m)} \) under the partial order \( \succ \). We obtain the next statement by induction on \( k \).

**Proposition 3.7.** If \( (\lambda, \ell) \in A_i \), then

\[
a_i^{(i)} = w_{i-\ell,1} w_{i-\ell+2,2} \cdots w_{i,\ell} \quad \text{and} \quad a_i^{(i+\frac{1}{2})} = w_{i-\ell+1,1} w_{i-\ell+2,2} \cdots w_{i,\ell} p_{\ell}^{(i)}.
\]

If \( i = 1, 2, \ldots \), and \( (\lambda, \ell) \in \hat{A}_i \), then in light of Proposition 3.7, we define

\[
f_{\lambda}^{(i)} = (w_{i-\ell+1,1} w_{i-\ell+2,2} \cdots w_{i,\ell})^{-1} x_{(\lambda,\ell)}^{(i)} a_i^{(i+\frac{1}{2})} w_{i-\ell+1,1} w_{i-\ell+2,2} \cdots w_{i,\ell} = (a_i^{(i)})^{-1} a_i^{(i)} a_i^{(i)},
\]

and write

\[
f_{\lambda}^{(i)} = (f_{\lambda}^{(i)} + A_i^{(\lambda,\ell)}) \in A_i / A_i^{(\lambda,\ell)}
\]

and

\[
f_{\lambda}^{(i+\frac{1}{2})} = (f_{\lambda}^{(i)} + A_{i+\frac{1}{2}}^{(\lambda,\ell)}) \in A_{i+\frac{1}{2}} / A_{i+\frac{1}{2}}^{(\lambda,\ell)}.
\]

**Example 3.8.** Let \( \lambda = (2,1) \) and \( \ell = 2 \). Then \( (\lambda, \ell) \in \hat{A}_{3+\frac{1}{2}} \) and

\( t^\lambda = (\emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset) \).

If

\( t = (\emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset) \),

then \( a_i^{(5+\frac{1}{2})} = 1 \) and \( a_i^{(5+\frac{1}{2})} = p_4 w_{4,1} p_5 w_{5,2} \). In the diagram presentation of \( A_{3+\frac{1}{2}} \), we have

\[
x_{(\lambda,\ell)}^{(5+\frac{1}{2})} a_i^{(5+\frac{1}{2})} = 1 \quad \text{and} \quad f_{\lambda}^{(5)} = \cdots.
\]
If \((\lambda, \ell) \in \hat{A}_{i-\frac{1}{2}}\) and \((\mu, m) \in \hat{A}_i\), and \((\lambda, \ell) \rightarrow (\mu, m)\), define
\[
p_{\lambda \rightarrow \mu}^{(i)} = \begin{cases} 
    w_{m,i}, & \text{if } \mu = \lambda, \\
    w_{a_j,i}, & \text{if } \mu = \lambda \cup \{(j, \ell_j)\}, 
\end{cases}
\tag{3.7}
\]
and
\[
p_{\lambda \rightarrow \mu}^{(i)} = \begin{cases} 
    p_m w_{m,i}, & \text{if } \mu = \lambda, \\
    \sum_{r=0}^{\lambda_j} w_{a_j-r,a_j} w_{a_j,i}, & \text{if } \mu = \lambda \cup \{(j, \ell_j)\}, 
\end{cases}
\tag{3.8}
\]
where in (3.7) and (3.8) we have written
\[a_j = \ell + \sum_{r=1}^{j} \mu_r.\]

Similarly, if \((\lambda, \ell) \in \hat{A}_i\) and \((\mu, m) \in \hat{A}_{i+\frac{1}{2}}\), and \((\lambda, \ell) \rightarrow (\mu, m)\), let
\[
p_{\lambda \rightarrow \mu}^{(i+\frac{1}{2})} = \begin{cases} 
    1, & \text{if } \lambda = \mu, \\
    p_{m,i} p_{\lambda \rightarrow \mu}^{(i)} w_{i,a_j} \sum_{r=0}^{\mu_j} w_{a_j-a_j-r}, & \text{if } \lambda = \mu \cup \{(j, \ell_j)\}, 
\end{cases}
\tag{3.9}
\]
and
\[
p_{\lambda \rightarrow \mu}^{(i+\frac{1}{2})} = \begin{cases} 
    1, & \text{if } \lambda = \mu, \\
    p_m w_{m,i} p_{\lambda \rightarrow \mu}^{(i)} w_{i,a_j}, & \text{if } \lambda = \mu \cup \{(j, \ell_j)\}, 
\end{cases}
\tag{3.10}
\]
where in (3.9) and (3.10) we have written
\[a_j = \ell + \sum_{r=1}^{j} \lambda_r.\]

The next statement is verified using the braid relations.

**Proposition 3.9.** (1) If \((\lambda, \ell) \in \hat{A}_{i-\frac{1}{2}}, (\mu, m) \in \hat{A}_i\) and \((\lambda, \ell) \rightarrow (\mu, m)\), then
\[f_{\mu}^{(i)} p_{\lambda \rightarrow \mu}^{(i)} = p_{\lambda \rightarrow \mu}^{(i)} f_{\lambda}^{(i-1)}.\]

(2) If \((\lambda, \ell) \in \hat{A}_i, (\mu, m) \in \hat{A}_{i+\frac{1}{2}}\) and \((\lambda, \ell) \rightarrow (\mu, m)\), then
\[f_{\mu}^{(i)} p_{\lambda \rightarrow \mu}^{(i+\frac{1}{2})} = p_{\lambda \rightarrow \mu}^{(i+\frac{1}{2})} f_{\lambda}^{(i)}.\]

For \(t = (\lambda(0), \lambda(\frac{1}{2}), \ldots, \lambda(k)) \in \hat{A}_{k+\frac{1}{2}}(\lambda, \ell)\), let
\[p_t = p_{\lambda(k-\frac{1}{2}) \rightarrow \lambda(k)}^{(k-\frac{1}{2})} p_{\lambda(k-1-\frac{1}{2}) \rightarrow \lambda(k-\frac{1}{2})} \cdots p_{\lambda(0) \rightarrow \lambda(\frac{1}{2})}^{(0)}.\]
Similarly, for \(s = (\lambda(0), \lambda(\frac{1}{2}), \ldots, \lambda(k+\frac{1}{2})) \in \hat{A}_{k+\frac{1}{2}}(\lambda, \ell)\), let
\[p_s = p_{\lambda(k) \rightarrow \lambda(k+\frac{1}{2})}^{(k)} p_{\lambda(k-\frac{1}{2}) \rightarrow \lambda(k)} \cdots p_{\lambda(0) \rightarrow \lambda(\frac{1}{2})}^{(1)}.\]

Let \(i \in \mathbb{Z}_{\geq 0}\) and \((\lambda, \ell) \in \hat{A}_i\). By Theorems 3.2 and 3.3, we may write
\[A_{i}(\lambda, \ell) = \{ f_{\lambda}^{(i)} p \mid p \in A_i \} \]
and
\[A_{i+\frac{1}{2}}(\lambda, \ell) = \{ f_{\lambda}^{(i)} p \mid p \in A_{i+\frac{1}{2}} \}.\]

**Proposition 3.10.** Let \((\mu, m) \in \hat{A}_i\).

(1) The set \(\{ f_{\mu}^{(i)} p_t \mid t \in \hat{A}_{i+\frac{1}{2}}(\mu, m) \}\) is an \(R\)-basis for \(A_{i+\frac{1}{2}}(\mu, m)\).
(2) The set \( \{ f_{i+\frac{1}{2}}^{(i)} p_i \mid t \in \hat{A}_{i+\frac{1}{2}}^{(\mu, m)} \} \) is an \( R \)-basis for \( \hat{A}_{i+\frac{1}{2}}^{(\mu, m)} \).

**Proof.** (1) If \((\lambda, \ell) \in \hat{A}_{i-\frac{1}{2}} \) and \((\lambda, \ell) \rightarrow (\mu, m)\), then
\[
 f_{i+\frac{1}{2}}^{(i)} p_{\lambda \rightarrow \mu} = p_{\lambda \rightarrow \mu} f_{i}^{(i-1)} + \hat{A}_{i}^{(\mu, m)}. \tag{3.11}
\]

If \( p \in \hat{A}_{i-\frac{1}{2}} \) and \( f_{i}^{(i-\frac{1}{2})} p \in \hat{A}_{i+\frac{1}{2}}^{(\lambda, \ell)} \), then by (1) of Corollary 3.5, there exist \( r_s \in R \), for \( s \in \hat{A}_{i}^{(\mu, m)} \), such that
\[
 p_{\lambda \rightarrow \mu} f_{i}^{(i-1)} p = \sum_{s \in \hat{A}_{i}^{(\mu, m)}} r_s f_{i}^{(i)} p \mod \hat{A}_{i}^{(\mu, m)}, \tag{3.12}
\]
where \( r_s = 0 \) unless \( \text{Shape}(s) \triangleright (\lambda, \ell) \). Let \( \{ (\lambda^{(j)}, \ell_j) \mid j = 1, \ldots, t \} \) be an indexing of the set \( \{ (\lambda, \ell) \in \hat{A}_{i-\frac{1}{2}} \mid (\lambda, \ell) \rightarrow (\mu, m) \} \) such that \( (\lambda^{(r)}, \ell_r) \triangleright (\lambda^{(s)}, \ell_s) \) whenever \( s > r \), and define
\[
 N_j = \sum_{r=1}^{j} f_{i+\frac{1}{2}}^{(i)} p_{\lambda(\rightarrow) \rightarrow \mu} A_{i-\frac{1}{2}} \quad \text{ (for } j = 1, \ldots, t \).
\]

By (3.11) and (3.12),
\[
 \{0\} = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_t = \hat{A}_{i}^{(\mu, m)}
\]
is a filtration by \( A_{i-\frac{1}{2}} \)-modules, and, for \( j = 1, \ldots, t \), the map
\[
 A_{i-\frac{1}{2}}^{(\lambda^{(j)}, \ell_j)} \rightarrow N_j / N_{j-1}
\]
\[
f_{i+\frac{1}{2}}^{(i)} p_{\lambda \rightarrow \mu} a + N_{j-1} \quad \text{ (for } a \in A_{i-\frac{1}{2}}) \tag{3.13}
\]
is an isomorphism of \( A_{i-\frac{1}{2}} \)-modules. By induction we may take, for each \( (\lambda, \ell) \in \hat{A}_{i-\frac{1}{2}} \), an \( R \)-basis for \( A_{i-\frac{1}{2}}^{(\lambda, \ell)} \) of the form \( \{ f_{i}^{(i-\frac{1}{2})} p_s \mid s \in \hat{A}_{i}^{(\mu, m)} \} \). Then, using the map (3.13) to push bases for \( A_{i-\frac{1}{2}}^{(\lambda^{(j)}, \ell_j)} \) onto bases for \( N_j / N_{j-1} \), completes the proof of (1).

(2) Regard \((\mu, m)\) as an element of \( \hat{A}_{i+\frac{1}{2}} \). If \((\lambda, \ell) \in \hat{A}_i \) and \((\lambda, \ell) \rightarrow (\mu, m)\), then
\[
 f_{i+\frac{1}{2}}^{(i+\frac{1}{2})} p_{\lambda \rightarrow \mu} = f_{i}^{(i)} p_{\lambda \rightarrow \mu} + A_{i+\frac{1}{2}}^{(\mu, m)}. \tag{3.14}
\]

If \( p \in A_i \) and \( f_{i}^{(i)} p \in A_{i+\frac{1}{2}}^{(\lambda, \ell)} \), then by item (2) of Corollary 3.5, there exist \( r_s \in R \), for \( s \in \hat{A}_{i+\frac{1}{2}}^{(\mu, m)} \), such that
\[
 p_{\lambda \rightarrow \mu} f_{i}^{(i)} p = \sum_{s \in \hat{A}_{i+\frac{1}{2}}^{(\mu, m)}} r_s f_{i}^{(i)} p_s \mod \hat{A}_{i+\frac{1}{2}}^{(\mu, m)} \tag{3.15}
\]
where \( r_s = 0 \) unless \( \text{Shape}(s) \triangleright (\lambda, \ell) \). Let \( \{ (\lambda^{(j)}, \ell_j) \mid j = 1, \ldots, t \} \) be an indexing of the set \( \{ (\lambda, \ell) \in \hat{A}_i \mid (\lambda, \ell) \rightarrow (\mu, m) \} \) such that \( (\lambda^{(r)}, \ell_r) \triangleright (\lambda^{(s)}, \ell_s) \) whenever \( s > r \), and define
\[
 N_j = \sum_{r=1}^{j} f_{i+\frac{1}{2}}^{(i+\frac{1}{2})} p_{\lambda(\rightarrow) \rightarrow \mu} A_i \quad \text{ (for } j = 1, \ldots, t \).
\]

By (3.14) and (3.15),
\[
 \{0\} = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_t = A_{i+\frac{1}{2}}^{(\mu, m)}
\]
is a filtration by $\mathcal{A}_i$–modules, and, for $j = 1, \ldots, t$, the map
\[
\begin{align*}
A^{(\lambda, \ell)}_i & \longrightarrow N_j / N_{j-1} \\
f^{(i)}_{\lambda(\ell)} : a & \mapsto f^{(i)}_{\mu} P^{(i+\frac{1}{2})}_{\lambda(\ell)} \mu a + N_{j-1} \\
\text{(for } a \in \mathcal{A}_i) \end{align*}
\] (3.16)
is an isomorphism of $\mathcal{A}_i$–modules. Using induction as in (1) above completes the proof of (2). □

In Theorems 3.11 and 3.12 the bases (3.3) and (3.4) are expressed relative to the elements $f^{(i)}_{\lambda}$, for $(\lambda, \ell) \in \hat{\mathcal{A}}_i$.

**Theorem 3.11.** If $i = 1, 2, \ldots$, then the set
\[
\mathcal{A}_i = \{ p^*_s f^{(i)}_{\lambda} p_t | s, t \in \hat{A}^{(\lambda, \ell)}_i \text{ and } (\lambda, \ell) \in \hat{\mathcal{A}}_i \}
\]
is an $R$–basis for $\mathcal{A}_i(z)$. Moreover the following statements hold:

(1) If $(\lambda, \ell) \in \hat{\mathcal{A}}_i$, $t \in \hat{A}^{(\lambda, \ell)}_k$ and $p \in \mathcal{A}_k(z)$, then there exist $r_0 \in R$, for $v \in \hat{A}^{(\lambda, \ell)}_k$, such that
\[
p^*_s f^{(i)}_{\lambda} p_t = \sum_{v \in \hat{A}^{(\lambda, \ell)}_k} r_v p^*_s f^{(i)}_{\lambda} p_v \quad \text{mod } \mathcal{A}^{(\lambda, \ell)}_i
\] (3.17)

where $\mathcal{A}^{(\lambda, \ell)}_i$, for $(\lambda, \ell) \in \hat{\mathcal{A}}_i$, is the $R$–module freely generated by
\[
\{ p^*_s f^{(i)}_{\mu} p_t | s, t \in \hat{A}^{(\mu, m)}_i \text{ for } (\mu, m) \in \hat{\mathcal{A}}_i \text{ and } (\mu, m) \triangleright (\lambda, \ell) \}.
\]

(2) If $(\lambda, \ell) \in \hat{\mathcal{A}}_i$ and $s, t \in \hat{A}^{(\lambda, \ell)}_i$, then $*: p^*_s f^{(i)}_{\lambda} p_t \mapsto p^*_t f^{(i)}_{\lambda} p_s$.

**Theorem 3.12.** If $i = 1, 2, \ldots$, then the set
\[
\mathcal{A}_{i+\frac{1}{2}} = \{ p^*_s f^{(i)}_{\lambda} p_t | s, t \in \hat{A}^{(\lambda, \ell)}_{i+\frac{1}{2}} \text{ and } (\lambda, \ell) \in \hat{\mathcal{A}}_{i+\frac{1}{2}} \}
\]
is an $R$–basis for $\mathcal{A}_{i+\frac{1}{2}}(z)$. Moreover the following statements hold:

(1) If $(\lambda, \ell) \in \hat{\mathcal{A}}_{i+\frac{1}{2}}$, $t \in \hat{A}^{(\lambda, \ell)}_{i+\frac{1}{2}}$ and $p \in \mathcal{A}_k(\frac{1}{2})$, then there exist $r_0 \in R$, for $v \in \hat{A}^{(\lambda, \ell)}_{i+\frac{1}{2}}$, such that
\[
p^*_s f^{(i)}_{\lambda} p_t = \sum_{v \in \hat{A}^{(\lambda, \ell)}_{i+\frac{1}{2}}} r_v p^*_s f^{(i)}_{\lambda} p_v \quad \text{mod } \mathcal{A}^{(\lambda, \ell)}_{i+\frac{1}{2}}
\] (3.18)

where $\mathcal{A}^{(\lambda, \ell)}_{i+\frac{1}{2}}$, for $(\lambda, \ell) \in \hat{\mathcal{A}}_{i+\frac{1}{2}}$, is the $R$–module freely generated by
\[
\{ p^*_s f^{(i)}_{\mu} p_t | s, t \in \hat{A}^{(\mu, m)}_{i+\frac{1}{2}} \text{ for } (\mu, m) \in \hat{\mathcal{A}}_{i+\frac{1}{2}} \text{ and } (\mu, m) \triangleright (\lambda, \ell) \}.
\]

(2) If $(\lambda, \ell) \in \hat{\mathcal{A}}_{i+\frac{1}{2}}$ and $s, t \in \hat{A}^{(\lambda, \ell)}_{k+\frac{1}{2}}$, then $*: p^*_s f^{(i)}_{\lambda} p_t \mapsto p^*_t f^{(i)}_{\lambda} p_s$.

If $a = (i, j)$ is a node, let $c(a) = j - i$ be the content of $a$. Let $(\mu, m) \in \hat{A}_k$. For $t = (\mu^{(0)}, \mu^{(1)}, \ldots, \mu^{(k)}) \in \hat{A}_k^{(\mu, m)}$, and $i = 1, \ldots, k$, define
\[
c(i - \frac{1}{2}) = \begin{cases} |\mu^{(i-1)}|, & \text{if } \mu^{(i-1)} = \mu^{(i-\frac{1}{2})}; \\ z - c(a), & \text{if } \mu^{(i-\frac{1}{2})} = \mu^{(i-1)} \setminus \{a\}. \end{cases}
\]
and
\[
c(i) = \begin{cases} \frac{z - |\mu^{(i)}|}{2}, & \text{if } \mu^{(i)} = \mu^{(i-\frac{1}{2})}; \\ c(a), & \text{if } \mu^{(i)} = \mu^{(i-\frac{1}{2})} \cup \{a\}. \end{cases}
\]
If $s = (\mu(0), \mu(\frac{1}{2}), \ldots, \mu(k+\frac{1}{2})) \in \hat{A}_{k+\frac{1}{2}}^{(\mu,m)}$ and $s_i = t$, let
\[
c_t(k + \frac{1}{2}) = \begin{cases} 
|\mu(k)|, & \text{if } \mu(k) = \mu(k+\frac{1}{2}); \\
|z - c(a)|, & \text{if } \mu(k+\frac{1}{2}) = \mu(k) \setminus \{a\},
\end{cases}
\]
and let $c_t(i - \frac{1}{2}) = c_t(i + \frac{1}{2})$ and $c_t(i) = c_t(i)$ for $i = 1, \ldots, k$.

The next statement shows that the elements $\{L_i, L_{i+\frac{1}{2}} \mid i = 0, 1, \ldots\}$ are additive Jucys–Murphy family in the sense of Goodman and Graber (Definition 3.4 of [GG1]).

**Proposition 3.13.** For $(\lambda, \ell) \in \hat{A}_k$, the following statements hold.

1. If $i = 1, 2, \ldots, k$, then
   \[
   f_\lambda^{(k)} L_{i-\frac{1}{2}} \equiv c_\lambda(i - \frac{1}{2}) f_\lambda^{(k)} \mod A_k^{(\lambda,\ell)} \quad \text{and} \quad f_\lambda^{(k)} L_i \equiv c_\lambda(i) f_\lambda^{(k)} \mod A_k^{(\lambda,\ell)} \quad (3.19)
   \]
   and
   \[
   f_\lambda^{(k)} L_i \equiv c_\lambda(i) f_\lambda^{(k)} \mod A_{k+\frac{1}{2}}^{(\lambda,\ell)} \quad \text{and} \quad f_\lambda^{(k)} L_{i+\frac{1}{2}} \equiv c_\lambda(i + \frac{1}{2}) f_\lambda^{(k)} \mod A_{k+\frac{1}{2}}^{(\lambda,\ell)} \quad (3.20)
   \]
2. The central element $z_k = L_1 + L_{1+\frac{1}{2}} + L_2 + \cdots + L_k \in A_k$ acts on $A_k^{(\lambda,\ell)}$ as a scalar multiple of the identity by
   \[
   \ell z + \sum_{b \in \lambda} c(b) + \frac{(k - \ell)(k - \ell - 1)}{2}.
   \]
3. The central element $z_{k+\frac{1}{2}} = L_1 + L_{1+\frac{1}{2}} + L_2 + \cdots + L_{k+\frac{1}{2}} \in A_{k+\frac{1}{2}}$ acts on $A_{k+\frac{1}{2}}^{(\lambda,\ell)}$ as a scalar multiple of the identity by
   \[
   \ell z + \sum_{b \in \lambda} c(b) + \frac{(k - \ell + 1)(k - \ell)}{2}.
   \]

**Proof.** We prove (1). For $j = 1, 2, \ldots$, define $\{L_i^{(j)} \mid j = 0, 1, \ldots\}$ and $\{L_i^{(j)} \mid i = 0, 1, \ldots\}$ by
\[
L_i^{(1)} = L_{i-\frac{1}{2}}, \quad \text{and} \quad L_i^{(j+1)} = w_{j,i} L_{i-\frac{1}{2}} w_{j,i}, \quad \text{for } j = 1, 2, \ldots,
\]
and
\[
L_i^{(1)} = L_i, \quad \text{and} \quad L_i^{(j+1)} = w_{j,i} L_i^{(j)} w_{j,i}, \quad \text{for } j = 1, 2, \ldots.
\]
Since $p_1 L_1 = z p_1$ and $p_1 L_{1+\frac{1}{2}} = 0$, using induction, we obtain
\[
\begin{align*}
p_j L_i^{(j)} &= 0, \quad \text{if } i = 0, \\
p_j L_i^{(j+1)} &= z p_j, \quad \text{if } i = 1, \quad \text{and} \quad p_j L_i^{(j)} &= 0, \quad \text{if } i = 0, \\
p_j L_i^{(j+1)} &= 0, \quad \text{if } i = 1, \quad \text{and} \quad p_j L_i^{(j+1)} &= p_j L_i^{(j+1)}, \quad \text{if } i \geq 2,
\end{align*}
\]
so that
\[
\begin{align*}
p_i^{(\ell)} L_i^{(j)} &= 0, \quad \text{if } i = 0, \\
p_i^{(\ell)} L_{i-\frac{1}{2}}^{(j)} &= z p_i^{(\ell)}, \quad \text{if } 1 \leq i \leq \ell, \quad \text{and} \quad p_i^{(\ell)} L_i^{(j)} &= 0, \quad \text{if } i = 0, \\
p_i^{(\ell)} L_{i-\frac{1}{2}}^{(j+1)} &= p_i^{(\ell)} L_{i-\frac{1}{2}}^{(j+1)}, \quad \text{if } 1 \leq i \leq \ell, \quad \text{and} \quad p_i^{(\ell)} L_{i-\ell+1}^{(j+1)} &= 0, \quad \text{if } \ell < i.
\end{align*}
\]
Thus
\[
f_\lambda^{(k)} L_i = c_\lambda(i) f_\lambda^{(k)} \quad \text{and} \quad f_\lambda^{(k)} L_{i+\frac{1}{2}} = c_\lambda(i + \frac{1}{2}) f_\lambda^{(k)} \quad \text{for } i = 1, \ldots, \ell,
\]
as required. Now suppose that $\ell < i \leq k$. Since the image of $L_{i-\ell}$ under the quotient map $A_{k-\ell} \rightarrow R\hat{S}_{k-\ell}$ is the $(i-\ell)$–th Jucys–Murphy element in $\hat{S}_{k-\ell}$, we have

$$c_\lambda L_{i-\ell} \equiv c_\lambda(i) c_\lambda \mod A_{k-\ell}^{(\lambda,0)}.$$

As $p_k^{(t)} A_{k-\ell}^{(\lambda,0)} \subseteq A_{k-\ell}^{(\lambda,\ell)}$, it now follows that

$$f_\lambda^{(k)} L_i = f_\lambda^{(k)} f_{i-\ell}^{(\ell+1)} \equiv c_\lambda(i) f_\lambda^{(k)} \mod A_{k-\ell}^{(\lambda,\ell)},$$

as required. If $\ell < i \leq k$, then the image of $L_{i-\ell+\frac{1}{2}}$ under the quotient map $A_{k-\ell+\frac{1}{2}} \rightarrow R\hat{S}_{k-\ell}$ is the identity in $\hat{S}_{k-\ell}$, and the above argument also shows that $f_\lambda^{(k)} L_{i+\frac{1}{2}} \equiv f_\lambda^{(k)} \mod A_{k-\ell}^{(\lambda,\ell)}$. This verifies (3.19). The proof of (3.20) is identical.

Writing

$$f_\lambda^{(k)} z_k = \sum_{i=1}^{\ell} f_\lambda^{(k)} (L_{i-\ell} + L_i) + \sum_{i=\ell+1}^{k} f_\lambda^{(k)} (L_{i-\ell+\frac{1}{2}} + L_i),$$

and using (1), proves (2) and (3).

The next statement which shows that the family $\{L_i, L_{i+\frac{1}{2}} \mid i = 1, \ldots, k\}$ acts triangularly on $A_{k+\frac{1}{2}}$, now follows by a direct calculation or by Theorem 3.7 of [GG1].

**Corollary 3.14.** If $(\lambda, \ell) \in \hat{A}_k$ and $t \in \hat{A}_{k+\frac{1}{2}}^{(\lambda,\ell)}$ then the following statements hold.

1. If $i = 1, \ldots, k$, then there exist $r_u \in R$, for $u \in \hat{A}_k^{(\lambda,\ell)}$ with $u > t$, such that

$$f_\lambda^{(i+\frac{1}{2})} p_i L_i = c_i(i) f_\lambda^{(i)} p_i + \sum_{u > t} r_u f_\lambda^{(i+\frac{1}{2})} p_u \quad \text{and} \quad f_\lambda^{(i+\frac{1}{2})} p_i L_{i+\frac{1}{2}} = c_i(k+\frac{1}{2}) f_\lambda^{(i+\frac{1}{2})} p_i + \sum_{u > t} r_u f_\lambda^{(i+\frac{1}{2})} p_u.$$

2. If $i = 1, \ldots, k$, then there exist $r_u \in R$, for $u \in \hat{A}_{k+\frac{1}{2}}^{(\lambda,\ell)}$ with $u > t$, such that

Using the $A_k$–module embedding $A_k^{(\lambda,\ell)} \subseteq A_{k+\frac{1}{2}}^{(\lambda,\ell)}$ for $(\lambda, \ell) \in \hat{A}_k$, Corollary 3.14 shows that the family $\{L_{i-\frac{1}{2}}, L_i \mid i = 1, \ldots, k\}$ acts triangularly on $A_k$.

4. A Seminormal Form

In this section, we use the triangular action of the Jucys–Murphy elements $\{L_i, L_{i+\frac{1}{2}} \mid i = 1, \ldots, k\}$ to define a seminormal form for the partition algebras. Let $\kappa$ denote the field of fractions of $R = \mathbb{Z}[z]$, and

$$A_k(z) = A_k(z) \otimes_R \kappa \quad \text{and} \quad A_{k+\frac{1}{2}}(z) = A_{k+\frac{1}{2}}(z) \otimes_R \kappa$$

denote partition algebras over $\kappa$. We write $A_k = A_k(z)$ and $A_{k+\frac{1}{2}} = A_{k+\frac{1}{2}}(z)$. If $(\lambda, \ell) \in \hat{A}_k$, let

$$A_k^{(\lambda,\ell)} = A_k^{(\lambda,\ell)} \otimes_R \kappa \quad \text{and} \quad A_{k+\frac{1}{2}}^{(\lambda,\ell)} = A_{k+\frac{1}{2}}^{(\lambda,\ell)} \otimes_R \kappa.$$
If \((\lambda, \ell) \in \hat{A}_{k+\frac{1}{2}},\) and \(s, t \in \hat{A}_{k+\frac{1}{2}}^{(\lambda, \ell)},\) define

\[
F_i = \prod_{u \in \hat{A}_{k+\frac{1}{2}}^{(\lambda, \ell)}} \prod_{c_1(i) < c_u(i)} \frac{L_i - c_u(i)}{c_i(i) - c_u(i)},
\]

and

\[
f^{(k+\frac{1}{2})}_t = f^{(k)}_t p_t F_i \quad \text{and} \quad f^{(k+\frac{1}{2})}_s = f^{(k)}_s f^{(k)}_t p_t F_i.
\]

Following §3 of [Mat], or by direct calculation, we obtain the next statement.

**Proposition 4.1.** Let \(k = 1, 2, \ldots,\) and \((\mu, m) \in \hat{A}_{k+\frac{1}{2}}^{(\lambda, \ell)}.

1. If \(t \in \hat{A}_{k+\frac{1}{2}}^{(\mu, m)},\) then there exist \(r_s \in \kappa,\) for \(s \in \hat{A}_{k+\frac{1}{2}}^{(\mu, m)},\) such that

\[
f^{(k+\frac{1}{2})}_t f^{(k+\frac{1}{2})}_s = f^{(k+\frac{1}{2})}_t f^{(k+\frac{1}{2})}_s + \sum_{s \in \hat{A}_{k+\frac{1}{2}}^{(\mu, m)}} r_s f^{(k+\frac{1}{2})}_s f^{(k+\frac{1}{2})}_t \quad \text{where} \quad r_s = 0 \text{ unless } s > t.
\]

2. The sets

\[
\{f^{(k+\frac{1}{2})}_t \mid t \in \hat{A}_{k+\frac{1}{2}}^{(\mu, m)}\} \quad \text{and} \quad \{F_{st} \mid s, t \in \hat{A}_{k+\frac{1}{2}}^{(\mu, m)} \text{ and } (\mu, m) \in \hat{A}_{k+\frac{1}{2}}^{(\lambda, \ell)}\}
\]

are bases over \(\kappa\) for \(\hat{A}_{k+\frac{1}{2}}^{(\mu, m)}\) and \(\hat{A}_{k+\frac{1}{2}}^{(\lambda, \ell)}\) respectively.

3. If \(t \in \hat{A}_{k+\frac{1}{2}}^{(\mu, m)},\) then

\[
f^{(k+\frac{1}{2})}_t L_i = c_i(i) f^{(k+\frac{1}{2})}_t \quad \text{and} \quad f^{(k+\frac{1}{2})}_t L_{i+\frac{1}{2}} = c_i(i + \frac{1}{2}) f^{(k+\frac{1}{2})}_t \quad \text{for } i = 0, 1, \ldots, k.
\]

4. If \((\lambda, \ell) \in \hat{A}_{k+\frac{1}{2}},\) then the symmetric contravariant bilinear form \(\langle \cdot, \cdot \rangle : \hat{A}_{k+\frac{1}{2}}^{(\mu, m)} \times \hat{A}_{k+\frac{1}{2}}^{(\lambda, \ell)} \to \kappa,\) defined by

\[
\langle f^{(k+\frac{1}{2})}_s f^{(k+\frac{1}{2})}_t \rangle_p \equiv f^{(k)}_s p_t f^{(k)}_t \quad \text{mod } \hat{A}_{k+\frac{1}{2}}^{(\mu, m)} \quad \text{for } s, t \in \hat{A}_{k+\frac{1}{2}}^{(\mu, m)}.
\]

satisfies

\[
\langle f^{(k+\frac{1}{2})}_s, f^{(k+\frac{1}{2})}_t \rangle = \delta_{st} f^{(k+\frac{1}{2})}_s, \quad \text{for } s, t \in \hat{A}_{k+\frac{1}{2}}^{(\mu, m)}.
\]

5. If \(s, t, u \in \hat{A}_{k+\frac{1}{2}}^{(\mu, m)},\) then

\[
f^{(k+\frac{1}{2})}_s F_t = \delta_{st} f^{(k+\frac{1}{2})}_s \quad \text{and} \quad f_u F_{st} = \langle f^{(k+\frac{1}{2})}_s, f^{(k+\frac{1}{2})}_t \rangle f^{(k+\frac{1}{2})}_t.
\]

6. If \(t, u \in \hat{A}_{k+\frac{1}{2}}^{(\mu, m)},\) then

\[
F_t F_u = \delta_{tu} F_t \quad \text{and} \quad F_{st} F_{tu} = \langle f^{(k+\frac{1}{2})}_s, f^{(k+\frac{1}{2})}_t \rangle F_{st}.
\]

7. If \((\mu, m) \in \hat{A}_{k+\frac{1}{2}},\) then there is an \(A_k\)-module isomorphism

\[
\hat{A}_{k+\frac{1}{2}}^{(\mu, m)} \cong \bigoplus_{(\lambda, \ell) \in \hat{A}_k} A_{k+\frac{1}{2}}^{(\lambda, \ell)},
\]

determined by the maps (3.16).
If \((\mu, m) \in \hat{A}_k\), the homomorphism (3.16) determines an inclusion \(A^{(\mu, m)}_k \hookrightarrow A^{(\mu, m)}_{k+\frac{1}{2}}\) of \(A_k\)-modules. Therefore, in what follows, we identify the seminormal form of the \(A_{k+\frac{1}{2}}\)-module \(A^{(\mu, m)}_{k+\frac{1}{2}}\), for \((\mu, m) \in \hat{A}_k\), with the \(\kappa\)-vector space generated by

\[
\left\{ f^{(k)}_s = f^{(k+\frac{1}{2})}_t \mid s \in A^{(\mu, m)}_k, t \in A^{(\mu, m)}_{k+\frac{1}{2}} \text{ and } s = t|_k \right\},
\]

with right \(A_k\)-action given by the inclusion \(A_k \subseteq A_{k+\frac{1}{2}}\). The \(A_{k+\frac{1}{2}}\)-module homomorphisms (3.13) determine an \(A_{k+\frac{1}{2}}\)-module isomorphism

\[
A^{(\mu, m)}_k \cong \bigoplus_{(\lambda, \ell) \in \hat{A}_{k+\frac{1}{2}}} A^{(\lambda, \ell)}_{k+\frac{1}{2}}.
\]

If \((\lambda, \ell) \in \hat{A}_{k+\frac{1}{2}}\) and \(t \in \hat{A}^{(\lambda, \ell)}_{k+\frac{1}{2}}\), define the seminormal matrix entries

\[
f^{(k+\frac{1}{2})}_t p_i = \sum_{s \in \hat{A}^{(\lambda, \ell)}_{k+\frac{1}{2}}} p_i(s, t) f^{(k+\frac{1}{2})}_s \quad \text{and} \quad f_t p_{i+\frac{1}{2}} = \sum_{s \in \hat{A}^{(\lambda, \ell)}_{k+\frac{1}{2}}} p_{i+\frac{1}{2}}(s, t) f^{(k+\frac{1}{2})}_s \quad \text{for } i = 1, \ldots, k,
\]

and

\[
f^{(k+\frac{1}{2})}_t \sigma_{i+\frac{1}{2}} = \sum_{s \in \hat{A}^{(\lambda, \ell)}_{k+\frac{1}{2}}} \sigma_{i+\frac{1}{2}}(s, t) f^{(k+\frac{1}{2})}_s, \quad \text{for } i = 1, \ldots, k - 1.
\]

The matrix entries of the transposition \(s_i\) in the seminormal representation of \(A_{k+\frac{1}{2}}(z)\) are obtained by the relation

\[
s_i(s, t) = \sum_{u \in \hat{A}^{(\lambda, \ell)}_{k+\frac{1}{2}}} \sigma_{i+\frac{1}{2}}(s, u) \sigma_{i+\frac{1}{2}}(u, t), \quad \text{for } s, t \in \hat{A}^{(\lambda, \ell)}_{k+\frac{1}{2}}, \text{ and } i = 1, \ldots, k - 1.
\]

In [En2] we have explicitly computed the bilinear form (4.1) with respect to the seminormal basis for \(A_{k+\frac{1}{2}}\). We state the results of these calculations below for reference in Theorem 5.1.

Recall that if \(\mu\) is a partition, then \(A(\mu)\) is the set of addable nodes of \(\mu\) and \(R(\mu)\) is the set of removable nodes of \(\mu\). If \(a = (i, j)\) and \(b = (k, \ell)\) are nodes, write \(a < b\) if \(k < i\), and write \(a > b\) if \(k > i\). If \(\mu\) is a partition and \(a\) is a node, write

\[
A(\mu)^{<a} = \{ b \in A(\mu) \mid b < a \} \quad \text{and} \quad A(\mu)^{>a} = \{ b \in A(\mu) \mid b > a \},
\]

and

\[
R(\mu)^{<a} = \{ b \in R(\mu) \mid b < a \} \quad \text{and} \quad R(\mu)^{>a} = \{ b \in R(\mu) \mid b > a \}.
\]

If \((\lambda, \ell) \in \hat{A}_{i-\frac{1}{2}}, (\mu, m) \in \hat{A}_i, \text{ and } (\lambda, \ell) \rightarrow (\mu, m)\), define

\[
\gamma^{(i)}_{\lambda \rightarrow \mu} = \begin{cases} 
\frac{\prod_{b \in A(\mu)} (z - c(b) - |\mu|)}{\prod_{b \in R(\mu)} (z - c(b) - |\mu|)}, & \text{if } \mu = \lambda, \\
\frac{\prod_{b \in A(\mu)^{<a}} (c(a) - c(b))}{\prod_{b \in R(\mu)^{<a}} (c(a) - c(b))}, & \text{if } \mu = \lambda \cup \{a\}. 
\end{cases}
\]
Similarly, if $(\lambda, \ell) \in \hat{A}_i$, $(\mu, m) \in \hat{A}_{i+\frac{k}{2}}$, and $(\lambda, \ell) \to (\mu, m)$, let

\[
\gamma_{\lambda \to \mu}^{(i + \frac{k}{2})} = \begin{cases} 
1, & \text{if } \mu = \lambda, \\
\frac{(z - c(a) - |\lambda|)(z - c(a) - |\mu| + r)}{(z - c(a) - |\mu|)(z - c(a) - |\lambda| + r)} \times \prod_{b \in R(\mu) \setminus \{a\}} (c(a) - c(b)) \quad & \text{if } \lambda = \mu \cup \{a\}, \\
\end{cases}
\]

where in the last case we have written $a = (j, \lambda_j)$, and

\[
r = \sum_{s > j} \mu_s.
\]

**Theorem 4.2.** Let $(\lambda, \ell) \in \hat{A}_{k + \frac{1}{2}}$. If $t = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(k + \frac{1}{2})}) \in \hat{A}_{k + \frac{1}{2}}^{(\lambda, \ell)}$, then

\[
(f_{t}^{(k + \frac{1}{2})}, f_{t}^{(k + \frac{1}{2})}) = \prod_{i=1}^{k} \gamma_{\lambda^{(i)}}^{(i - \frac{1}{2})} \gamma_{\lambda^{(i)}}^{(i + \frac{1}{2})}. 
\]

5. **Main Results**

Theorems 5.1, 5.2 and 5.3 below give explicit combinatorial formulae for images of generators of the partition algebras in Theorem 2.2 in the seminormal representations. The statement of these results is split into three parts merely to avoid clashes between indices. In Theorem 5.1 we have used the abbreviation $f_t = f_{t}^{(k + \frac{1}{2})}$, for $t \in \hat{A}_{k + \frac{1}{2}}^{(\lambda, \ell)}$.

**Theorem 5.1.** Let $(\lambda, \ell) \in \hat{A}_{k + \frac{1}{2}}$. If $s, t \in \hat{A}_{k + \frac{1}{2}}^{(\lambda, \ell)}$ and $s \neq t$, and $i = 1, 2, \ldots, k$, then

\[
p_i(s, t) = \sqrt{\frac{p_i(s, s)p_i(t, t)(f_{s}, f_{t})}{(f_s, f_s)}} \quad \text{and} \quad p_{i + \frac{1}{2}}(s, t) = \sqrt{\frac{p_{i + \frac{1}{2}}(s, s)p_{i + \frac{1}{2}}(t, t)(f_{s}, f_{t})}{(f_s, f_s)}},
\]

where

\[
p_i(t, t) = \begin{cases} 
\prod_{a \in A(\mu)} (z - c(a) - |\mu|) \prod_{b \in R(\mu)} (z - c(b) - |\mu|), & \text{if } t^{(i-1)} = t^{(i - \frac{1}{2})} = t^{(i)} = \mu, \\
(z - c(\beta) - |\mu| + 1) \prod_{a \in A(\mu)} (c(\beta) - c(a)) \prod_{b \notin \beta} (c(\beta) - c(b)), & \text{if } t^{(i-1)} = t^{(i)} = \mu \text{ and } t^{(i - \frac{1}{2})} = \mu \setminus \{\beta\}, \\
0, & \text{otherwise},
\end{cases}
\]

and,

\[
p_{i + \frac{1}{2}}(t, t) = \begin{cases} 
\prod_{b \in R(\mu)} (z - c(b) - |\mu|) \prod_{a \in A(\mu)} (z - c(a) - |\mu|), & \text{if } t^{(i - \frac{1}{2})} = t^{(i)} = t^{(i + \frac{1}{2})} = \mu, \\
(z + c(\beta) + |\mu| + 1) \prod_{a \in A(\mu)} (c(\beta) - c(b)) \prod_{b \neq \beta} (c(\beta) - c(a)), & \text{if } t^{(i - \frac{1}{2})} = t^{(i + \frac{1}{2})} = \mu \text{ and } t^{(i)} = \mu \cup \{\beta\}, \\
0, & \text{otherwise}.
\end{cases}
\]

**Theorem 5.2.** Let $(\lambda, \ell) \in \hat{A}_{k + \frac{1}{2}}$ and $s, t \in \hat{A}_{k + \frac{1}{2}}^{(\lambda, \ell)}$. For $i = 1, \ldots, k$, the following statements hold:
(1) If $t^{(i-\frac{1}{2})} = t^{(i+\frac{1}{2})}$, and $v \in \mathbb{A}^{(\lambda, \ell)}_{k+\frac{1}{2}}$ satisfies $v^{i+\frac{1}{2}} \sim t$ and $v^{(i-1)} = v^{(i)}$, then
\[
\sigma_{i+\frac{1}{2}}(s, t) = \frac{\delta_{st} + (z - c_i(i - c_i(i - \frac{1}{2})))p_{i+\frac{1}{2}}(s, t) - p_{i+\frac{1}{2}}(s, v)p_i(v, t)}{c_i(i + \frac{1}{2}) - c_i(i - \frac{1}{2})}, \quad \text{if } s \neq v,
\]
and
\[
\sigma_{i+\frac{1}{2}}(v, t) = \frac{p_{i+\frac{1}{2}}(v, t)}{p_{i+\frac{1}{2}}(v, v)p_i(v, v)c_i(i) - \sum_{u \neq v \neq u \neq v} p_i(v, u)\sigma_{i+\frac{1}{2}}(u, t)}.
\]

(2) If $t^{(i-1)} = t^{(i)}$ and $t^{(i-\frac{1}{2})} \neq t^{(i+\frac{1}{2})}$, and $v \in \mathbb{A}^{(\lambda, \ell)}_{k+\frac{1}{2}}$ satisfies $v \sim t$ and $v^{(i-\frac{1}{2})} = v^{(i+\frac{1}{2})}$, then
\[
\sigma_{i+\frac{1}{2}}(s, t) = \frac{\delta_{st} - p_{i+\frac{1}{2}}(s, v)p_i(v, t)}{c_i(i + \frac{1}{2}) - c_i(i - \frac{1}{2})}, \quad \text{if } s \neq v,
\]
and
\[
\sigma_{i+\frac{1}{2}}(v, t) = -\sum_{u \neq v \neq u \neq v} \frac{p_i(v, u)\sigma_{i+\frac{1}{2}}(u, t)}{p_i(v, v)}.
\]

(3) If $t^{(i-1)} \neq t^{(i)}$ and $t^{(i-\frac{1}{2})} \neq t^{(i+\frac{1}{2})}$, and $\sigma_{i+\frac{1}{2}}$ does not exist, then
\[
\sigma_{i+\frac{1}{2}}(s, t) = \frac{\delta_{st}}{c_i(i + \frac{1}{2}) - c_i(i - \frac{1}{2})}.
\]

(4) If $t^{(i-1)} \neq t^{(i)}$ and $t^{(i-\frac{1}{2})} \neq t^{(i+\frac{1}{2})}$, and $\sigma_{i+\frac{1}{2}}$ exists, then
\[
\sigma_{i+\frac{1}{2}}(s, t) = \begin{cases} 
\frac{1}{c_i(i + \frac{1}{2}) - c_i(i - \frac{1}{2})}, & \text{if } s = t, \\
1 - \frac{1}{1 - (c_i(i + \frac{1}{2}) - c_i(i - \frac{1}{2}))^2}, & \text{if } s = t \sigma_{i+\frac{1}{2}} \text{ and } s \succ t, \\
1, & \text{if } s = t \sigma_{i+\frac{1}{2}} \text{ and } t \succ s, \\
0, & \text{otherwise.}
\end{cases}
\] (5.2)

Theorem 5.3. Let $(\lambda, \ell) \in \mathbb{A}_{k+1}$ and $s, t \in \mathbb{A}^{(\lambda, \ell)}_{k+1}$. If $i = 1, \ldots, k$, then the following statements hold.

(1) If $t^{(i-\frac{1}{2})} = t^{(i+\frac{1}{2})}$, and $v \in \mathbb{A}^{(\lambda, \ell)}_{k+\frac{1}{2}}$ satisfies $v^{i+\frac{1}{2}} \sim t$ and $v^{(i)} = v^{(i+1)}$, then
\[
\sigma_{i+1}(s, t) = \frac{\delta_{st} + (p_{i+\frac{1}{2}}(v, v)c_i(i)p_{i+1}(v, v) - c_i(i))p_{i+\frac{1}{2}}(s, v)p_{i+1}(s, v)p_{i+\frac{1}{2}}(v, t)}{c_i(i + 1) - c_i(i)}, \quad \text{if } s \neq v,
\]
and
\[
\sigma_{i+1}(v, t) = \frac{p_{i+\frac{1}{2}}(v, t)}{p_{i+1}(v, v)p_{i+\frac{1}{2}}(v, v)c_i(i) - \sum_{u \neq v \neq u \neq v} p_{i+1}(v, u)\sigma_{i+1}(u, t)}.
\]

(2) If $t^{(i)} = t^{(i+1)}$ and $t^{(i-\frac{1}{2})} \neq t^{(i+\frac{1}{2})}$, and $v \in \mathbb{A}^{(\lambda, \ell)}_{k+\frac{1}{2}}$ satisfies $v^{i+1} \sim t$ and $v^{(i-\frac{1}{2})} = v^{(i+\frac{1}{2})}$, then
\[
\sigma_{i+1}(s, t) = \frac{\delta_{st}}{c_i(i + 1) - c_i(i)}, \quad \text{if } s \neq v,
\]
and
\[
\sigma_{i+1}(v, t) = \frac{p_{i+1}(v, t)}{(c_i(i) - c_i(i + 1))p_{i+1}(v, v)}
\]

(3) If \(t^{i-\frac{1}{2}} \neq t^{(i+\frac{1}{2})}\) and \(t^{(i)} \neq t^{(i+1)}\), and \(t\sigma_{i+1}\) does not exist, then
\[
\sigma_{i+1}(s, t) = \frac{\delta_{st}}{c_i(i + 1) - c_i(i)}.
\]

(4) If \(t^{i-\frac{1}{2}} \neq t^{(i+\frac{1}{2})}\) and \(t^{(i)} \neq t^{(i+1)}\), and \(t\sigma_{i+1}\) exists, then
\[
\sigma_{i+1}(s, t) = \begin{cases}
\frac{1}{c_i(i + 1) - c_i(i)}, & \text{if } s = t, \\
1 - \frac{1}{(c_i(i + 1) - c_i(i))^2}, & \text{if } s = t\sigma_{i+1} \text{ and } s \succ t, \\
1, & \text{if } s = t\sigma_{i+1} \text{ and } t \succ s, \\
0, & \text{otherwise. }
\end{cases}
\]

Remark 5.4. A consistent choice of signs for the square roots in (5.1) can be made as follows. Let \(s \in A_{k+\frac{1}{2}}^{(\lambda, \ell)}\). Suppose that for each \(t, r \in A_{k+\frac{1}{2}}^{\lambda}\), such that \(s \sim t\) and \(s \sim t^\ell r\), a choice of scalars \(\hat{p}_i(t, s)\) and \(\hat{p}_{i+\frac{1}{2}}(r, s)\) has been made, subject to the constraints
\[
\hat{p}_i(t, t) = p_i(t, t) \quad \text{and} \quad \hat{p}_{i+\frac{1}{2}}(r, r) = p_{i+\frac{1}{2}}(r, r),
\]

and
\[
\hat{p}_i(t, s)^2 = \frac{p_i(t, t)p_i(s, s)(f_s, f_s)}{(f_t, f_t)} \quad \text{and} \quad \hat{p}_{i+\frac{1}{2}}(r, s)^2 = \frac{p_{i+\frac{1}{2}}(r, r)p_{i+\frac{1}{2}}(s, s)(f_s, f_s)}{(f_t, f_t)}.
\]

If \(u, v \in \hat{A}_{k+\frac{1}{2}}^{(\lambda, \ell)}\) and \(u \sim s\) and \(v \sim s\), determine uniquely the scalars \(\hat{p}_i(u, t)\) and \(\hat{p}_{i+\frac{1}{2}}(v, r)\) respectively by
\[
\hat{p}_i(u, t)\hat{p}_i(t, s) = \hat{p}_i(u, s)p_i(t, t) \quad \text{and} \quad \hat{p}_{i+\frac{1}{2}}(v, r)\hat{p}_{i+\frac{1}{2}}(r, s) = \hat{p}_{i+\frac{1}{2}}(v, s)p_{i+\frac{1}{2}}(r, r).
\]

Let \(\hat{p}_i \in \text{End}_e(A_{k+\frac{1}{2}}^{(\lambda, \ell)})\) and \(\hat{p}_{i+\frac{1}{2}} \in \text{End}_e(A_{k+\frac{1}{2}}^{(\lambda, \ell)})\) denote the matrices with entries as determined using (5.4), (5.5) and (5.6) above. Propositions 5.7 and 5.8 imply that, with the above choices, the maps
\[
p_i \mapsto \hat{p}_i \quad \text{and} \quad p_{i+\frac{1}{2}} \mapsto \hat{p}_{i+\frac{1}{2}},
\]
respect the relations (2)(i), (2)(ii), (3)(i)–(iii) and (4)(i), (4)(ii) in the presentation of Theorem 2.1, and hence yield an algebra homomorphism \(\langle p_i, p_{i+\frac{1}{2}} \mid i = 1, \ldots, k \rangle \to \text{End}_e(A_{k+\frac{1}{2}}^{(\lambda, \ell)})\).

5.1. Proof of Main Results. Before proving Theorems 5.1, 5.2 and 5.3, we establish some basic properties of the generators \(p_i, p_{i+\frac{1}{2}}, \sigma_{i+\frac{1}{2}}, \sigma_{i+1}\).

Let \(s, t \in A_{k+\frac{1}{2}}^{(\lambda, \ell)}\). Write \(s \sim t\) if
\[
s^{(i)} = t^{(i)}, \quad \text{for } i = 1, \ldots, k, \quad \text{and} \quad s^{(i+\frac{1}{2})} = t^{(i+\frac{1}{2})} \Rightarrow \ell = i,
\]
and write \(s \sim t^\ell\) if
\[
s^{(\ell-\frac{1}{2})} = t^{(\ell-\frac{1}{2})}, \quad \text{for } i = 1, \ldots, k, \quad \text{and} \quad s^{(\ell)} = t^{(\ell)} \Rightarrow \ell = i.
\]

The next statement is obtained using Theorem 4.1 together with the commutativity relations of Proposition 2.4.
Proposition 5.5. Let \((\lambda, \ell) \in \hat{A}_{k+\frac{1}{2}}\) and \(s,t \in \hat{A}_{k+\frac{1}{2}}^{(\lambda,\ell)}\). For \(i = 1, \ldots, k\), the following statements hold:

1. If \(p_i(s,t) \neq 0\), then \(s \sim t\).
2. If \(p_{i+\frac{1}{2}}(s,t) \neq 0\), then \(s \sim t\) and \(s \sim t\).

Using Theorem 4.28 of [HR], or Schur–Weyl duality (cf. Theorem 3.6 of [HR] and Lemma 3.5 of [Na]), we obtain the next statement.

Proposition 5.6. Let \((\lambda, \ell) \in \hat{A}_{k+\frac{1}{2}}\) and \(t \in \hat{A}_{k+\frac{1}{2}}^{(\lambda,\ell)}\). Then, for \(i = 1, \ldots, k\), the following statements hold:

1. If \(t^{i-\frac{1}{2}} = t^{i+\frac{1}{2}}\) then \(p_{i+\frac{1}{2}}(t,t) \neq 0\).
2. If \(t^i = t^{i+1}\) then \(p_{i+1}(t,t) \neq 0\).

The next statement provides a counterpart to Corollary 3.7 of [Na].

Proposition 5.7. Let \((\lambda, \ell) \in \hat{A}_{k+\frac{1}{2}}\). Then, for \(i = 1, \ldots, k\), the following equalities hold:

\[ p_{i+\frac{1}{2}}(s,t)p_{i+\frac{1}{2}}(t,u) = p_{i+\frac{1}{2}}(s,t)p_{i+\frac{1}{2}}(s,u), \quad \text{for } s,t,u \in \hat{A}_{k+\frac{1}{2}}^{(\lambda,\ell)}, \quad (5.7) \]

and

\[ p_i(s,t)p_i(t,u) = p_i(t,u)p_i(s,u), \quad \text{for } s,t,u \in \hat{A}_{k+\frac{1}{2}}^{(\lambda,\ell)}. \quad (5.8) \]

Proof. It suffices to verify (5.7) for \(i = k\). Let \(t \in \hat{A}_{k+\frac{1}{2}}^{(\lambda,\ell)}\). If \(t^{k+\frac{1}{2}} \neq t^{k-\frac{1}{2}}\), then

\[ p_{k+\frac{1}{2}}(s,t) = p_{k+\frac{1}{2}}(t,s) = 0, \quad \text{for all } s \in \hat{A}_{k+\frac{1}{2}}^{(\lambda,\ell)}, \]

so the relation (5.7) clearly holds. Suppose therefore, that \(t^{k+\frac{1}{2}} = t^{k-\frac{1}{2}}\) so that

\[ F_i p_{k+\frac{1}{2}} F_i = p_{k+\frac{1}{2}}(t,t) F_i \neq 0. \quad (5.9) \]

Given \(g(L_k, L_{k+\frac{1}{2}})\), a polynomial in \(L_k\) and \(L_{k+\frac{1}{2}}\) over \(\kappa\), then the Jones basic construction shows that

\[ p_{k+\frac{1}{2}} g(L_k, L_{k+\frac{1}{2}}) p_{k+\frac{1}{2}} = \xi p_{k+\frac{1}{2}}, \quad \text{where } \xi \text{ is a central element in } A_{k-\frac{1}{2}}(z). \]

In particular, if \(v = t^{k-\frac{1}{2}}\), then there exists a polynomial \(g(L_k, L_{k+\frac{1}{2}})\) in \(L_k, L_{k-\frac{1}{2}}\) over \(\kappa\), such that

\[ p_{k+\frac{1}{2}} F_i p_{k+\frac{1}{2}} = g(L_k, L_{k+\frac{1}{2}}) p_{k+\frac{1}{2}} F_i = p_{k+\frac{1}{2}}(t,t) F_i, \quad (5.10) \]

where \(\xi_t\) is a central element in \(A_{k-\frac{1}{2}}(z)\) and \(F_i \in A_{k-\frac{1}{2}}(z)\). Using (5.9) and (5.10),

\[ F_i p_{k+\frac{1}{2}} F_i p_{k+\frac{1}{2}} F_i = \xi_t F_i p_{k+\frac{1}{2}} F_i = p_{k+\frac{1}{2}}(t,t) \xi_t F_i F_0 = p_{k+\frac{1}{2}}(t,t) \xi_t F_i, \]

and

\[ F_i p_{k+\frac{1}{2}} F_i p_{k+\frac{1}{2}} F_i = p_{k+\frac{1}{2}}(t,t) F_i p_{k+\frac{1}{2}} F_i = p_{k+\frac{1}{2}}(t,t)^2 F_i. \]

Since \(F_i\) acts as a matrix unit on \(A_{k-\frac{1}{2}}^{(\lambda,\ell)}\) and \(p_{k+\frac{1}{2}}(t,t) \neq 0\), it follows that

\[ \xi_t \text{ acts on } f_{v}^{(k+\frac{1}{2})} \text{ by scalar multiplication by } p_{k+\frac{1}{2}}(t,t). \]

Since \(\xi_t\) is central in \(A_{k-\frac{1}{2}}(z)\), it follows that

\[ \xi_t \text{ acts on the } A_{k-\frac{1}{2}}^{(\lambda,\ell-1)} \text{ by scalar multiplication by } p_{k+\frac{1}{2}}(t,t). \]
If $s, u \in \hat{A}_{k+\frac{1}{2}}^{(\lambda, \ell)}$, let $e_{su}$ denote the vector space endomorphism of $A_{k+\frac{1}{2}}^{(\lambda, \ell)}$ defined by
\[
  f_r^{(k+\frac{1}{2})} e_{su} = \delta_{rs} f_u^{(k+\frac{1}{2})},
\]
for $r \in \hat{A}_{k+\frac{1}{2}}^{(\lambda, \ell)}$.

Then, as operators on $A_{k+\frac{1}{2}}^{(\lambda, \ell)}$,
\[
  p_{i+\frac{1}{2}} F_0 p_{i+\frac{1}{2}} = F_0 \xi t p_{i+\frac{1}{2}} = \xi t F_0 p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}(t, t) \sum_{s \sim t} p_{i+\frac{1}{2}}(s, u) e_{su},
\]
and
\[
  p_{i+\frac{1}{2}} F_t p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}} F_t p_{i+\frac{1}{2}} = \left( \sum_{s \sim t} p_{i+\frac{1}{2}}(s, t) e_{st} \right) \left( \sum_{u \sim t} p_{i+\frac{1}{2}}(t, u) e_{u} \right)
  = \sum_{s \sim t} p_{i+\frac{1}{2}}(s, t) p_{i+\frac{1}{2}}(t, u) e_{su},
\]
which completes the proof of (5.9). The statement (5.10) follows by an appropriate modification of the proof of (5.9).

Proposition 5.8 is a counterpart to Lemma 3.6 of [Na].

**Proposition 5.8.** Let $(\lambda, \ell) \in \hat{A}_{k+\frac{1}{2}}$ and $t \in \hat{A}_{k+\frac{1}{2}}$.

1. If $ti^{-\frac{1}{2}} = ti^{\frac{1}{2}}$, then
\[
  f_0^{(k+\frac{1}{2})} p_{i+\frac{1}{2}} = \frac{p_{i+\frac{1}{2}}(t, v)}{p_{i+\frac{1}{2}}(t, t)} f_t^{(k+\frac{1}{2})} p_{i+\frac{1}{2}},
\]
for all $v \in \hat{A}_{k+\frac{1}{2}}^{(\lambda, \ell)}$ such that $v i^{\frac{1}{2}} \sim t$,
and the eigenspace for the action of $p_{i+\frac{1}{2}}$ on
\[
  \text{Span}_k \{ f_0^{(k+\frac{1}{2})} | s \in \hat{A}_{k+\frac{1}{2}}^{(\lambda, \ell)} \text{ and } s i^{\frac{1}{2}} \sim t \},
\]
is one-dimensional.

2. If $ti^{-1} = ti$, then
\[
  f_0^{(k+\frac{1}{2})} p_i = \frac{p_i(t, v)}{p_i(t, t)} f_i^{(k+\frac{1}{2})} p_i,
\]
for all $v \in \hat{A}_{k+\frac{1}{2}}^{(\lambda, \ell)}$ such that $v i \sim t$,
and the eigenspace for the action of $p_i$ on
\[
  \text{Span}_k \{ f_0^{(k+\frac{1}{2})} | s \in \hat{A}_{k+\frac{1}{2}}^{(\lambda, \ell)} \text{ and } s \sim t \},
\]
is one-dimensional.

**Proof.** (1) If $s \in \hat{A}_{k+\frac{1}{2}}^{(\lambda, \ell)}$, then
\[
  p_{i+\frac{1}{2}}(s, v) = p_{i+\frac{1}{2}}(s, t) p_{i+\frac{1}{2}}(t, v) p_{i+\frac{1}{2}}(t, t)^{-1},
\]
and
\[
  f_0^{(k+\frac{1}{2})} p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}(t, v) p_{i+\frac{1}{2}}(t, t)^{-1} \sum_{s \sim t} p_{i+\frac{1}{2}}(s, t) f_0^{(k+\frac{1}{2})} = p_{i+\frac{1}{2}}(t, v) p_{i+\frac{1}{2}}(t, t)^{-1} f_0 p_{i+\frac{1}{2}},
\]
as required. The proof of (2) is similar.

**Lemma 5.9.** Let $(\lambda, \ell) \in \hat{A}_{k+\frac{1}{2}}$ and $t \in \hat{A}_{k+\frac{1}{2}}^{(\lambda, \ell)}$. The following statements hold:

1. If $ti^{-1} \neq ti$, then $c_i(i - \frac{1}{2}) + c_i(i) \neq z$. 

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(2) If \( t^{(i-\frac{1}{2})} \neq t^{(i+\frac{1}{2})} \), then \( c_t(i) + c_t(i + \frac{1}{2}) \neq z \).

(3) If \( t^{(i-\frac{1}{2})} \neq \sigma t^{(i+\frac{1}{2})} \), then \( f_t p_i = 0 \).

(4) If \( t^{(i-\frac{1}{2})} \neq t^{(i+\frac{1}{2})} \), then \( f_t p_{i+\frac{1}{2}} = 0 \).

Proof. (1) Let \( (t^{(i-1)}, t^{(i+\frac{1}{2})}, t^{(i-\frac{1}{2})}) = (\mu, \nu, v) \). By definition, \( c_t(i - \frac{1}{2}) + c_t(i) = z \) if and only if either \( \nu = v \) and \( \mu = \nu \), or there exists a node \( b \) such that \( v = \nu \cup \{b\} \) and \( \mu = \nu \cup \{b\} \). The proof of (2) is similar.

(3) Since the Jucys–Murphy elements act diagonally on the seminormal basis, we obtain

\[
f_t^{(k+\frac{1}{2})}(L_{i-\frac{1}{2}} + L_i)p_i = (c_t(i - \frac{1}{2}) + c_t(i))f_t^{(k+\frac{1}{2})}p_i,
\]

while, item (11) of Proposition 2.4 gives

\[
f_t^{(k+\frac{1}{2})}(L_{i-\frac{1}{2}} + L_i)p_i = zf_t^{(k+\frac{1}{2})}p_i.
\]

The last statement is in contradiction to item (1) if \( f_t^{(k+\frac{1}{2})}p_i \neq 0 \). The proof of (4) is similar. \( \square \)

Let \( (\lambda, \ell) \in \hat{A}_{k+\frac{1}{2}} \) and \( s, t \in \hat{A}_{\lambda, \ell}^{(k, \frac{1}{2})} \). Write \( s \preceq t \) if

\[
s^{(\ell+\frac{1}{2})} \neq t^{(\ell+\frac{1}{2})} \Rightarrow \ell = i \quad \text{and} \quad s^{(\ell)} \neq t^{(\ell)} \Rightarrow \ell = i,
\]

and write \( s \succeq t \) if

\[
s^{(\ell+\frac{1}{2})} \neq t^{(\ell-\frac{1}{2})} \Rightarrow \ell = i \quad \text{and} \quad s^{(\ell)} \neq t^{(\ell)} \Rightarrow \ell = i.
\]

The next statement is obtained using Theorem 4.1 together with the commutativity relations of Proposition 2.4.

Proposition 5.10. Let \( (\lambda, \ell) \in \hat{A}_{k+\frac{1}{2}} \) and \( s, t \in \hat{A}_{\lambda, \ell}^{(k, \frac{1}{2})} \). For \( i = 1, \ldots, k \), the following statements hold:

1. If \( \sigma_{i+\frac{1}{2}}(s, t) \neq 0 \), then \( s \preceq t \).
2. If \( \sigma_{i+1}(s, t) \neq 0 \), then \( s \succeq t \).

If \( \lambda, \mu \) are partitions, let \( \lambda \ominus \mu = \lambda \setminus \mu \cup \mu \setminus \lambda \).

Lemma 5.11. If \( (\lambda, \ell) \in \hat{A}_{k+\frac{1}{2}} \) and \( t \in \hat{A}_{\lambda, \ell}^{(k, \frac{1}{2})} \), then the following statements hold:

1. If \( t^{(i-1)} \neq t^{(i+\frac{1}{2})} \) and the nodes \( t^{(i-1)} \ominus t^{(i+\frac{1}{2})} \) are neither in the same row nor the same column, then there exists \( s = t\sigma + \frac{1}{2} \in \hat{A}_{\lambda, \ell}^{(k, \frac{1}{2})} \), uniquely determined by \( s \preceq t \) and

\[
c_{\lambda}(i + \frac{1}{2}) = c_{\lambda}(i - \frac{1}{2}), \quad c_{\lambda}(i - \frac{1}{2}) = c_{\lambda}(i + \frac{1}{2}), \quad \text{and}, \quad c_{\lambda}(i) = c_{\lambda}(i).
\]

2. If \( t^{(i-\frac{1}{2})} \neq t^{(i+1)} \) and the nodes \( t^{(i-\frac{1}{2})} \ominus t^{(i+1)} \) are neither in the same row nor the same column, then there exists \( s = t\sigma_{i+1} \in \hat{A}_{\lambda, \ell}^{(k, \frac{1}{2})} \), uniquely determined by \( s \succeq t \) and

\[
c_{\lambda}(i) = c_{\lambda}(i + 1), \quad c_{\lambda}(i + 1) = c_{\lambda}(i), \quad \text{and}, \quad c_{\lambda}(i + \frac{1}{2}) = c_{\lambda}(i + \frac{1}{2}).
\]

Proof. (1) There is no loss of generality in proving the statement for \( i = k \). We have two cases to consider.

(a) If \( (t^{(k-1)}, t^{(k-\frac{1}{2})}, t^{(k-\frac{1}{2})}) = (\mu, \nu, \lambda) \), where \( \nu = \mu \setminus \{\alpha\} = \lambda \setminus \{\beta\} \) and \( \alpha, \beta \) are neither
in the same row nor in the same column, then $v = \mu \cup \{\beta\} = \lambda \cup \{\alpha\}$ is a partition. Thus $s = t^k_{k+\frac{1}{2}} \in \hat{A}^{(\lambda, \ell)}_{k+\frac{1}{2}}$, given by

$$s^{k+\frac{1}{2}} \simeq t^{k+\frac{1}{2}} \quad \text{and} \quad (g^{(k-1)}, g^{(k)}, g^{(k+\frac{1}{2})}) = (\mu, \mu, \nu, \lambda),$$

satisfies the required properties. By symmetry, we may define $t = \sigma_{k+\frac{1}{2}}$.

(b) If $(t^{(k-1)}, t^{(k)}, t^{(k+1)}) = (\mu, \nu, \lambda, \alpha, \beta)$ are neither in the same row nor in the same column, then $v = \mu \setminus \{\beta\} = \lambda \setminus \{\alpha\}$ is a partition. Now define $s \in \hat{A}^{(\lambda, \ell)}_{k+\frac{1}{2}}$ by

$$s^{k+\frac{1}{2}} \simeq t^{k+\frac{1}{2}} \quad \text{and} \quad (g^{(k-1)}, g^{(k-\frac{1}{2})}, g^{(k)}, g^{(k+\frac{1}{2})}) = (\mu, \nu, \lambda),$$

to obtain the path in $\hat{A}$ with the required properties.

(2) There is no loss of generality supposing that $t \in \hat{A}^{\lambda}_{k+1}$ and proving the statement for $i = k$. Again we have two cases to consider.

(a) If $(t^{(k-1)}, t^{(k)}, t^{(k+1)}) = (\mu, \nu, \lambda, \alpha, \beta)$ are neither in the same row nor in the same column, then $v = \mu \setminus \{\beta\} = \lambda \setminus \{\alpha\}$ is a partition. Thus $s = t^k_{k+\frac{1}{2}} \in \hat{A}^{(\lambda, \ell)}_{k+\frac{1}{2}}$, given by

$$s^{k+\frac{1}{2}} \simeq t^{k+\frac{1}{2}} \quad \text{and} \quad (g^{(k-1)}, g^{(k)}, g^{(k+\frac{1}{2})}) = (\mu, \nu, \lambda),$$

satisfies the required properties. By symmetry, we may define $t = \sigma_{k+1}$.

(b) If $(t^{(k-\frac{1}{2})}, t^{(k)}, t^{(k+\frac{1}{2})}) = (\mu, \nu, \lambda, \alpha, \beta)$ are neither in the same row nor in the same column, then $v = \mu \setminus \{\beta\} = \lambda \setminus \{\alpha\}$ is a partition. Thus $s \in \hat{A}^{(\lambda, \ell)}_{k+\frac{1}{2}}$ defined by

$$s^{k+1} \simeq t^{k+1} \quad \text{and} \quad (g^{(k-\frac{1}{2})}, g^{(k)}, g^{(k+1)}) = (\mu, \nu, \lambda),$$

satisfies the required properties.

\[\square\]

**Proposition 5.12.** Let $(\lambda, \ell) \in \hat{A}_{k+\frac{1}{2}}$. If $t \in \hat{A}^{(\lambda, \ell)}_{k+\frac{1}{2}}$ satisfies $t^{(k-\frac{1}{2})} = t^{(k-1)} \subseteq t^{(k)}$, then

$$f^{(k+\frac{1}{2})}_{\lambda} p_{\sigma_{k+\frac{1}{2}}} = f^{(k+\frac{1}{2})}_{\lambda} p_t.$$  

**Proof.** Let $t^{(k-\frac{1}{2})} = \mu$, where $\lambda = \mu \cup \{(j, \ell)\}$. Then

$$p_{t^{(k-\frac{1}{2})}} = p_{t^{(k-1)}} = 1, \quad \text{and} \quad p_{t^{(k+\frac{1}{2})}} = w_{a_j, k}, \quad \text{where} \quad a_j = \ell + \sum_{r=1}^{j} \lambda_r,$$

If $j = 1, 2, \ldots$, define

$$\sigma^{(1)}_{j-\frac{1}{2}} = \sigma_{j-\frac{1}{2}} \quad \text{and} \quad \sigma^{(r+1)}_{j-\frac{1}{2}} = w_{r, r+j} \sigma^{(r)}_{j-\frac{1}{2}} \quad \text{for} \quad r = 1, 2, \ldots.$$  

Since $\sigma_1 = \sigma_{1+\frac{1}{2}} = 1$, by induction we obtain

$$p_{r} \sigma_{j+\frac{1}{2}}^{(r)} = \begin{cases} p_r, & \text{if} \quad j = 0 \quad \text{or} \quad j = 1, \\ p_r \sigma_{j+\frac{1}{2}}^{(r)} & \text{if} \quad j \geq 2. \end{cases}$$

Thus, as $\sigma_{k-\ell+\frac{1}{2}} \rightarrow 1$ under the map $\hat{A}_{k-\ell+\frac{1}{2}} \rightarrow \kappa G_{k-\ell}$ and $w_{a_j, k}$ commutes with $p_{t}^{(\ell)}$, we have

$$p_{t}^{(\ell)} w_{a_j, k} \sigma_{k-\ell+\frac{1}{2}}^{(r+1)} = p_{t}^{(\ell)} w_{a_j, k} \mod \hat{A}^{\lambda, \ell}_{k+\frac{1}{2}}.$$  

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and 
\[
\alpha_{t,\hat{\ell}}^{(k+\frac{1}{2})} p_t \sigma_k \equiv \alpha_{t,\hat{\ell}}^{(k+\frac{1}{2})} p_t \mod A_{k+\frac{1}{2}}^{(\lambda, \ell)}.
\]
Since \( f_{\lambda}^{(k)} = (\alpha_{t,\hat{\ell}}^{(k+\frac{1}{2})}) \ast_{(k+\frac{1}{2})} (\alpha_{t,\hat{\ell}}^{(k+\frac{1}{2})}) \), the statement follows. \( \Box \)

**Proof of Theorem 5.1.** The given formulae for the diagonal entries of the matrix representations \( p_i \) and \( p_{i+\frac{1}{2}} \) follow from Proposition 6.2 and the expressions (6.19) and (6.18) respectively.

Let \( s, t \in \hat{A}_{k+\frac{1}{2}}^{(\lambda, \ell)} \). We prove our formula for the off-diagonal entry \( p_i(s, t) \). If \( p_i(t, t) = 0 \), then \( p_i(u, t) = 0 \) for all \( u \in \hat{A}_{k+\frac{1}{2}}^{(\lambda, \ell)} \), and the first equality in (5.1) clearly holds. Otherwise, by the contravariance of the bilinear form (4.1), we have
\[
f_t^{(k+\frac{1}{2})} p_t F_{sv} = p_t(s, t) \langle f_s^{(k+\frac{1}{2})}, f_v^{(k+\frac{1}{2})} \rangle f_v^{(k+\frac{1}{2})} \quad \text{for all } v \in \hat{A}_{k+\frac{1}{2}}^{(\lambda, \ell)},
\]
and
\[
f_t^{(k+\frac{1}{2})} p_t F_{sv} = p_t(t, s) \langle f_t^{(k+\frac{1}{2})}, f_v^{(k+\frac{1}{2})} \rangle f_v^{(k+\frac{1}{2})} \quad \text{for all } v \in \hat{A}_{k+\frac{1}{2}}^{(\lambda, \ell)}.
\]
By Proposition 5.7, we have \( p_t(t, s) = p_t(s, t) p_t(t, t) p_t(s, t)^{-1} \), and
\[
p_t(s, t)^2 = \frac{p_t(s, s) p_t(t, t) \langle f_t^{(k+\frac{1}{2})}, f_v^{(k+\frac{1}{2})} \rangle}{\langle f_s^{(k+\frac{1}{2})}, f_v^{(k+\frac{1}{2})} \rangle},
\]
as required. The proof of the second equality in (5.1) is similar. \( \Box \)

**Proof of Theorem 5.2.** Let \( (\lambda, \ell) \in \hat{A}_{k+\frac{1}{2}} \) and \( s, t \in \hat{A}_{k+\frac{1}{2}}^{(\lambda, \ell)} \).

(1) Suppose that \( t^{(i-\frac{1}{2})} = t^{(i+\frac{1}{2})} \) and let \( v \in \hat{A}_{k+\frac{1}{2}}^{(\lambda, \ell)} \) such that \( v \sim t \) and \( v^{(i-1)} = v^{(i)} \). If \( s \neq v \), then, using item (2) of Proposition 2.5 and Lemma 5.9,
\[
\sum_{u \sim t} (c_u(i + \frac{1}{2}) - c_u(i - \frac{1}{2}) \sigma_{i+\frac{1}{2}}(u, t) f_u^{(k+\frac{1}{2})}) = -f_t^{(k+\frac{1}{2})} (p_t p_{i+\frac{1}{2}} + p_{i+\frac{1}{2}} L_i - (z - L_{i-\frac{1}{2}}) p_{i+\frac{1}{2}} - 1)
\]
\[
= f_t^{(k+\frac{1}{2})} - f_t^{(k+\frac{1}{2})} p_{i+\frac{1}{2}} L_i - p_t(v, t) f_v^{(k+\frac{1}{2})} p_{i+\frac{1}{2}} + (z - c_t(i + \frac{1}{2})) f_t^{(k+\frac{1}{2})} p_{i+\frac{1}{2}},
\]
which gives the formula for \( \sigma_{i+\frac{1}{2}}(s, t) \). Next,
\[
f_t^{(k+\frac{1}{2})} \sigma_{i+\frac{1}{2}} p_t p_{i+\frac{1}{2}} = \sum_{u \sim v} \sigma_{i+\frac{1}{2}}(u, t) f_u^{(k+\frac{1}{2})} p_t p_{i+\frac{1}{2}} = \sum_{u \sim v} \sigma_{i+\frac{1}{2}}(u, t) p_t(v, u) f_v^{(k+\frac{1}{2})} p_{i+\frac{1}{2}},
\]
where
\[
f_t^{(k+\frac{1}{2})} \sigma_{i+\frac{1}{2}} p_t p_{i+\frac{1}{2}} = c_t(i) f_t^{(k+\frac{1}{2})} p_{i+\frac{1}{2}} = c_t(i) p_t(v, t) f_v^{(k+\frac{1}{2})} p_{i+\frac{1}{2}} (v, v)^{-1} f_v^{(k+\frac{1}{2})} p_{i+\frac{1}{2}}
\]
and \( f_v^{(k+\frac{1}{2})} p_{i+\frac{1}{2}} \neq 0 \). The formula for \( \sigma_{i+\frac{1}{2}}(v, t) \) now follows.

(2) Suppose that \( t^{(i-1)} = t^{(i)} \) and \( t^{(i+\frac{1}{2})} \neq t^{(i+\frac{1}{2})} \), and let \( v \in \hat{A}_{k+\frac{1}{2}}^{(\lambda, \ell)} \) such that \( v \sim t \) and \( v^{(i-\frac{1}{2})} = v^{(i+\frac{1}{2})} \). Using item (2) of Proposition 2.5 and Lemma 5.9,
\[
\sum_{u \sim t} (c_u(i + \frac{1}{2}) - c_u(i - \frac{1}{2}) \sigma_{i+\frac{1}{2}}(u, t) f_u^{(k+\frac{1}{2})}) = f_t - p_t(v, t) f_v^{(k+\frac{1}{2})} p_{i+\frac{1}{2}}.
\]
which gives the formula for $\sigma_{i+\frac{1}{2}}(s,t)$ when $s \neq u$. Next,

$$f_t^{(k+\frac{1}{2})} \sigma_{i+\frac{1}{2}} p_i p_{i+\frac{1}{2}} = \sum_{u \preceq v} \sigma_{i+\frac{1}{2}}(u,t) f_t^{(k+\frac{1}{2})} p_i p_{i+\frac{1}{2}} = \sum_{u \preceq v} \sigma_{i+\frac{1}{2}}(u,t) p_i (v,u) f_v^{(k+\frac{1}{2})} p_{i+\frac{1}{2}},$$

where

$$f_t^{(k+\frac{1}{2})} \sigma_{i+\frac{1}{2}} p_i p_{i+\frac{1}{2}} = c_i(i) f_t^{(k+\frac{1}{2})} p_{i+\frac{1}{2}} = 0$$

and $f_v^{(k+\frac{1}{2})} p_{i+\frac{1}{2}} \neq 0$.

The formula for $\sigma_{i+\frac{1}{2}}(v,t)$ now follows.

3) Suppose that $t^{i-1} \neq t^i$ and $t^{i-\frac{1}{2}} \neq t^{i+\frac{1}{2}}$ and that $t \sigma_{i+\frac{1}{2}}$ does not exist. By item (2) of Proposition 2.5 and Lemma 5.9,

$$f_t^{(k+\frac{1}{2})} \sigma_{i+\frac{1}{2}} L_{i+\frac{1}{2}} = f_t^{(k+\frac{1}{2})} L_{i-\frac{1}{2}} \sigma_{i+\frac{1}{2}} = f_t^{(k+\frac{1}{2})}.$$

Since

$$u \in \hat{A}_{k+\frac{1}{2}}^{(\lambda,\ell)}$$

and $u \sim t$, then $u = t$, it follows that $\sigma_{i+\frac{1}{2}}(u,t) = 0$ if $u \neq t$, while

$$(c_i(i + \frac{1}{2}) - c_i(i - \frac{1}{2})) \sigma_{i+\frac{1}{2}}(t,t) = 1,$$

as required.

4) Suppose that $t^{i-1} \neq t^i$ and $t^{i-\frac{1}{2}} \neq t^{i+\frac{1}{2}}$, and that $t \sigma_{i+\frac{1}{2}}$ exists. First observe that the hypotheses on $t$ imply that if $u \in \hat{A}_{k+\frac{1}{2}}^{(\lambda,\ell)}$ and $u \sim t$, then $u \in \{t, t \sigma_{i+\frac{1}{2}}\}$. Thus, by Lemma 5.9,

$$f_t^{(k+\frac{1}{2})} \sigma_{i+\frac{1}{2}} = \sigma_{i+\frac{1}{2}}(t,t) f_t^{(k+\frac{1}{2})} + \sigma_{i+\frac{1}{2}}(s,t) f_s^{(k+\frac{1}{2})},$$

where $s = t \sigma_{i+\frac{1}{2}}$.

We now determine $\sigma_{i+\frac{1}{2}}(t,t)$, Lemma 5.9 and the assumptions on $t$ imply that

$$f_t^{(k+\frac{1}{2})} p_i = f_t^{(k+\frac{1}{2})} p_{i+\frac{1}{2}} = 0.$$

Thus item (2) of Proposition 2.5 yields

$$(c_i(i + \frac{1}{2}) - c_i(i - \frac{1}{2})) \sigma_{i+\frac{1}{2}}(t,t) = 1,$$

which gives the required expression for $\sigma_{i+\frac{1}{2}}(t,t)$.

Next, we show that if $s = t \sigma_{i+\frac{1}{2}}$ and $t \sim s$, then $\sigma_{i+\frac{1}{2}}(s,t) = 1$. Since

$$f_t^{(k+\frac{1}{2})} \sigma_{i+\frac{1}{2}} = f_t^{(k+\frac{1}{2})} p_i \sigma_{i+\frac{1}{2}} + \sum_{u \sim t} r_u f_t^{(k+\frac{1}{2})} \sigma_{i+\frac{1}{2}},$$

where $r_u \in \kappa$, for $u \in \hat{A}_{k+\frac{1}{2}}^{(\lambda,\ell)}$,

it suffices to show that

$$\sigma_{i+\frac{1}{2}}(u,s) = 0,$$

if $u \in \hat{A}_{k+\frac{1}{2}}^{(\lambda,\ell)}$ and $u \sim t$, (5.11)

and that there exist $r_u \in \kappa$, for $u \in \hat{A}_{k+\frac{1}{2}}^{(\lambda,\ell)}$, such that

$$f_t^{(k+\frac{1}{2})} p_i \sigma_{i+\frac{1}{2}} = f_t^{(k+\frac{1}{2})} p_i + \sum_{u \sim t} r_u f_t^{(k+\frac{1}{2})} p_u.$$ (5.12)

To prove (5.11), suppose that $u \sim t$. Then

$$f_u^{(k+\frac{1}{2})} \sigma_{i+\frac{1}{2}} = \sum_{i \frac{1}{2}} r_i \frac{1}{2} \sigma(u,v) f_r^{(k+\frac{1}{2})},$$

for $r \in \hat{A}_{k+\frac{1}{2}}^{(\lambda,\ell)}$. (5.13)
If $\sigma_{i+\frac{1}{2}}(s,u) \neq 0$ in the expression (5.13), then $u \simeq s$, so $u \in \{t, s\}$, which leads to a contradiction if $u \approx t > s$. Thus (5.11) holds.

Now we verify (5.12) assuming, without any loss of generality, that $i = k$. There are two cases to consider here.

(a) Suppose that $t^{(i-\frac{1}{2})} \not\subseteq u^{(i+\frac{1}{2})}$. Let

$$(t^{(k-1)}, t^{(k-\frac{1}{2})}, t^{(k)}, t^{(k+\frac{1}{2})}) = (\mu, \mu, \lambda, \lambda) \quad \text{and} \quad (s^{(k-1)}, s^{(k-\frac{1}{2})}, s^{(k)}, s^{(k+\frac{1}{2})}) = (\nu, \nu, \lambda, \lambda),$$

where

$$\lambda = \mu \cup \{(j, \lambda_j)\} \quad \text{and} \quad \nu = \mu \cup \{(j', \nu_{j'})\} \quad \text{and} \quad \nu = \nu \cup \{(j, \lambda_j)\} = \lambda \cup \{(j', \nu_{j'})\}.$$

Then

$$p_{t^{(k-1)}\rightarrow t} = 1 \quad \text{and} \quad p_{u^{(k-\frac{1}{2})} \rightarrow u} = w_{\alpha, k}, \quad \text{where} \quad \alpha = \ell + \sum_{r=1}^{j} \lambda_j,$$

and

$$p_{t^{(k-\frac{1}{2})} \rightarrow t^{(k-1)}} = w_{t, k-1}p_{k-\frac{1}{2}}w_{k-1, a, j} \sum_{r=0}^{\mu_{j'}} w_{\alpha_{j', a, j'-r}}, \quad \text{where} \quad \alpha_{j'} = \ell - 1 + \sum_{r=1}^{j'} \nu_r,$$

while

$$p_{s^{(k-1)} \rightarrow s} = w_{t, k+1}p_{k+1}w_{k, a, j} \sum_{r=0}^{\lambda_{j'}} w_{\alpha_{j', a, j'-r}}, \quad \text{where} \quad \alpha_{j'} = \ell - 1 + \sum_{r=1}^{j'} \nu_r,$$

and

$$p_{s^{(k-\frac{1}{2})} \rightarrow s^{(k)}} = w_{\alpha, k}, \quad \text{and} \quad p_{s^{(k-\frac{1}{2})} \rightarrow s^{(k)}} = 1, \quad \text{where} \quad \alpha = \ell - 1 + \sum_{r=1}^{j} \nu_r.$$

Therefore,

$$f_{\lambda}^{(k)} p_{t^{(k-1)} \rightarrow t^{(k-\frac{1}{2})}} p_{t^{(k-\frac{1}{2})} \rightarrow t} p_{t^{(k-\frac{1}{2})} \rightarrow t^{(k-\frac{1}{2})}} p_{t^{(k-\frac{1}{2})} \rightarrow t^{(k-1)}} \sigma_{k+\frac{1}{2}}$$

$$= f_{\lambda}^{(k)} w_{\alpha, k} w_{t, k-1} p_{k-\frac{1}{2}} w_{k-1, a, j} \left( \sum_{r=0}^{\mu_{j'}} w_{\alpha_{j', a, j'-r}} \right) \sigma_{k+\frac{1}{2}}$$

$$= f_{\lambda}^{(k)} w_{\alpha, k} w_{t, k-1} \sigma_{k+\frac{1}{2}} p_{k+1} w_{k, a, j} \sum_{r=0}^{\mu_{j'}} w_{\alpha_{j', a, j'-r}}.$$

Now, $f_{\lambda}^{(k)} w_{\alpha, k} w_{t, k-1} = f_{\lambda}^{(k)} p_{u}$, where $u \in \mathcal{A}_{k+\frac{1}{2}}$ is given by

$$(u^{(k-\frac{1}{2})}, u^{(k-1)}, u^{(k-\frac{1}{2})}, u^{(k)}, u^{(k+\frac{1}{2})}) = (\mu, \mu, \mu, \lambda, \lambda),$$

and

$$p_{u^{(1)} \rightarrow u} = p_{u^{(1)} \rightarrow u^{(1+\frac{1}{2})}} = \cdots = p_{u^{(k-2)} \rightarrow u^{(k-\frac{3}{2})}} = 1.$$ 

Thus, by Proposition 5.12,

$$f_{\lambda}^{(k+\frac{1}{2})} p_{u} \sigma_{k+\frac{1}{2}} = f_{\lambda}^{(k+\frac{1}{2})} p_{u},$$

and

$$f_{\lambda}^{(k+\frac{1}{2})} w_{\alpha, k} w_{t, k-1} \sigma_{k+\frac{1}{2}} w_{k, a, j} \sum_{r=0}^{\mu_{j'}} w_{a_{j', a, j'-r}} = f_{\lambda}^{(k+\frac{1}{2})} w_{\alpha, k} w_{t, k-1} \sigma_{k+\frac{1}{2}} w_{k, a, j} \sum_{r=0}^{\mu_{j'}} w_{a_{j', a, j'-r}}.$$
There are two further cases to consider here.

(i) If \( j < j' \), then, by the braid relation, the right hand side of the last expression is

\[
\sum_{r=0}^{\mu_j} w_{a_j, a_j - r} = \sum_{r=0}^{\mu_j'} \left( w_{a_j, a_j - r} - 1 \right) w_{a_j, a_j - r}
\]

since, in this case, \( a_j = a_j - 1 \) and \( w_{a_j, a_j - r} = w_{a_j, a_j - r-1} \).

(ii) If \( j' < j \), then using the fact that in this case \( a_j = a_j \) and \( a_j = a_j' \),

\[
\sum_{r=0}^{\mu_j} w_{a_j, a_j - r} = \sum_{r=0}^{\mu_j'} \left( w_{a_j, a_j - r} - 1 \right) w_{a_j, a_j - r}
\]

since, in this case, \( a_j = a_j - 1 \) and \( w_{a_j, a_j - r} = w_{a_j, a_j - r-1} \).

(b) Suppose that \( t^{(k-\frac{1}{2})} \geq t^{(k+\frac{1}{2})} \). We embed \( A^{(\ell, \ell)}_{k+\frac{1}{2}} \) in \( A^{(\ell, \ell+1)}_{k+\frac{1}{2}} \) and let

\[(t^{(k-1)}, t^{(k-\frac{1}{2})}, t^{(k)}, t^{(k+\frac{1}{2})}, t^{(k+1)}) = (\mu, \nu, \nu, \lambda, \lambda),\]

where \( \mu \supseteq \lambda \) and \( \mu \setminus \lambda \) are neither in the same row, nor in the same column. The observe that, as in (3) of Theorem 5.3, \( f_{(k+1)}^{(k+1)} \sigma_{k+1} = f_{(k+1)}^{(k+1)} \). Since \( \sigma_{k+1} \sigma_{k+\frac{1}{2}} = s_k \), we have

\[
\sigma_{k+\frac{1}{2}}(s, t) = s_k(s, t) \quad \text{for all } s \in \mathcal{A}_{k+1}^{(\ell, \ell)}.
\]

Let us suppose that \( s = t \sigma_{k+\frac{1}{2}} \), where \( t \succ s \). Since

\[
f_{(k)}^{(k+1)} s_k = f_{(k)}^{(k+1)} p_t s_k + \sum_{u \in \mathcal{A}_{k+1}^{(\ell, \ell)}} r_u f_{(k)}^{(k+1)} p_u s_k, \quad \text{where } a_u = 0 \text{ unless } u \succ t,
\]

it suffices to show that

\[
s_k(u, s) = 0 \quad \text{if } u \in \mathcal{A}_{k+1}^{(\ell, \ell)} \text{ and } u \succ t, \quad \text{and} \quad f_{(k)}^{(k+1)} p_t s_k = f_{(k+1)}^{(k+1)} p_u.
\]

For the first equality in (5.14), observe that if \( s_k(u, s) \neq 0 \), then \( u^{(k-1)} = s^{(k-1)} \) which, together with \( u^{(k+1)} = s^{(k+1)} \), implies that \( u^{(k+\frac{1}{2})} = s^{(k+\frac{1}{2})} \). Thus \( u \in \{ s \} \), which is impossible under the assumption that \( u \succ t \succ s \).

For the second equality in (5.14), we define \( s \in \mathcal{A}_{k+1}^{(\ell, \ell)} \) by

\[
P_{s^{(k-\frac{1}{2})}} p_{s^{(k-1)}} = p_{(k-\frac{1}{2})}, \quad \text{and} \quad f_{(k+1)}^{(k+1)} p_t s_k = f_{(k+1)}^{(k+1)} p_u.
\]

and

\[
(g^{(k-1)}, g^{(k-\frac{1}{2)}}, g^{(k)}, g^{(k+\frac{1}{2)}}, g^{(k+1)}) = (\mu, \nu, \nu, \lambda, \lambda),
\]

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where
\[ \nu = \lambda \cup \{(j, \nu_j)\} \quad \text{and} \quad \mu = \nu \cup \{(j', \nu_{j'})\} =: \nu \cup \{(j, \nu_j)\}, \]
and \( j < j' \), so that \( s = \sigma_{k+1} \) and \( t > s \). If \( \lambda \vdash k + 1 - \ell \), then
\[
P_{g(k+1) \rightarrow g(k+1)} = P_{g(k+1) \rightarrow g(k+1)} = w_{\ell,k+1}, \quad \text{and} \quad P_{g(k+1) \rightarrow g(k)} = P_{g(k+1) \rightarrow g(k)} = w_{\ell-2,k},
\]
and
\[
P_{g(k) \rightarrow g(k+1)} = w_{\ell-1,k} P_{k+\frac{1}{2}} w_{k,a_j} \sum_{r=0}^{\lambda_j} w_{a_j,a_j-r}, \quad \text{where} \quad a_j = \ell - 2 + \sum_{r=1}^{j} \nu_r,
\]
and
\[
P_{g(k-1) \rightarrow g(k-\frac{1}{2})} = w_{\ell-3,k-1} P_{k-\frac{1}{2}} w_{k-1,a_j'} \sum_{r=0}^{\nu_j'} w_{a_j',a_j'-r}, \quad \text{where} \quad a_j' = \ell - 4 + \sum_{r=1}^{j'} \mu_r,
\]
while
\[
P_{g(k) \rightarrow g(k-\frac{1}{2})} = w_{\ell-1,k} P_{k+\frac{1}{2}} w_{k,a_j} \sum_{r=0}^{\lambda_j} w_{a_j,a_j-r}, \quad \text{where} \quad a_j = \ell - 2 + \sum_{r=1}^{j} \nu_r = a_j' + 1,
\]
and
\[
P_{g(k-1) \rightarrow g(k-\frac{1}{2})} = w_{\ell-3,k-1} P_{k-\frac{1}{2}} w_{k-1,a_j} \sum_{r=0}^{\nu_j} w_{a_j,a_j-r}, \quad \text{where} \quad a_j = \ell - 4 + \sum_{r=1}^{j} \mu_r = a_j - 2.
\]

By the braid relation,
\[
f_{t \lambda}^{(k+1)} P_{g(k+1) \rightarrow g(k+1)} P_{g(k) \rightarrow g(k+1)} P_{g(k-1) \rightarrow g(k)} s_k = f_{t \lambda}^{(k+1)} w_{\ell,k+1} w_{\ell-1,k} P_{k+\frac{1}{2}} w_{k,a_j} \sum_{r=0}^{\nu_j} w_{a_j,a_j-r} \left( \sum_{b=0}^{\nu_j} w_{a_j',a_j'-b} \right),
\]
and
\[
f_{t \lambda}^{(k+1)} P_{g(k) \rightarrow g(k+1)} P_{g(k) \rightarrow g(k)} P_{g(k-1) \rightarrow g(k-\frac{1}{2})} = f_{t \lambda}^{(k+1)} w_{\ell,k+1} w_{\ell-1,k} P_{k+\frac{1}{2}} w_{k,a_j} \sum_{r=0}^{\nu_j} w_{a_j,a_j-r} \left( \sum_{b=0}^{\nu_j} w_{a_j',a_j'-b} \right),
\]
so the fact that \( f_{t \lambda}^{(k+1)} P_{g(k)} s_k = f_{t \lambda}^{(k+1)} P_{g(k)} \) will follow once we have shown that
\[
P_{k+\frac{1}{2}} w_{k,a_j} w_{\ell-2,k} w_{\ell-3,k-1} P_{k-\frac{1}{2}} w_{k-1,a_j} s_k = P_{k+\frac{1}{2}} w_{k,a_j} w_{\ell-2,k} w_{\ell-3,k-1} P_{k-\frac{1}{2}} w_{k-1,a_j}. \tag{5.15}
\]

Considering the left hand side of (5.15), the braid relation gives
\[
P_{k+\frac{1}{2}} w_{k,a_j} w_{\ell-2,k} w_{\ell-3,k-1} P_{k-\frac{1}{2}} w_{k-1,a_j} s_k = P_{k+\frac{1}{2}} w_{\ell-2,k} w_{\ell-3,k-1} P_{k-\frac{1}{2}} w_{k-2} w_{k-1,a_j} s_k
\]
\[= w_{\ell-2,k-1} w_{\ell-3,k-2} P_{k-\frac{1}{2}} w_{k-1} s_k - 2 P_{k-\frac{1}{2}} w_{k-2} w_{k-1,a_j} s_k
\]
\[= w_{\ell-2,k-1} w_{\ell-3,k-2} P_{k-\frac{1}{2}} w_{k-1} s_k - 2 P_{k-\frac{1}{2}} w_{k-2} w_{k-1,a_j} s_k
\]
\[= w_{\ell-2,k-1} w_{\ell-3,k-2} P_{k-\frac{1}{2}} w_{k-1} s_k - 2 P_{k-\frac{1}{2}} w_{k-2} w_{k-1,a_j},
\]

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\[
p_{k+\frac{1}{2}}^{\alpha, j, \ell} = k - 2k^{\alpha, j, \ell} - 3k^{\alpha, j, \ell} - 1 p_{k+\frac{1}{2}}^{\alpha, j, \ell} - 2 w_{k+\frac{1}{2}, \alpha, j} - 2 w_{k+\frac{1}{2}, \alpha, j} - 1 = w_{k+\frac{1}{2}, \alpha, j} - 1 \]
which completes the proof of (5.15). This verifies (5.12) and establishes that \( \sigma_{k+\frac{1}{2}}(s, t) = 1 \) if \( s = \sigma_{k+\frac{1}{2}}^2 \) and \( t \succ s \).

Finally, using the above calculations together with the fact that \( \sigma_{i+\frac{1}{2}}^2 = 1 \), we can verify the given formula for \( \sigma_{i+\frac{1}{2}}(s, t) \) when \( s = \sigma_{i+\frac{1}{2}}^2 \) and \( s \succ t \) by observing that \( c_s(i - \frac{1}{2}) = c_t(i + \frac{1}{2}) \) and \( c_s(i + \frac{1}{2}) = c(i - \frac{1}{2}) \).

**Proof of Theorem 5.3.** Let \( (\lambda, \ell) \in \hat{A}_{k+1} \) and \( s, t \in \hat{A}_{k+1}(\lambda, \ell) \).

(1) Suppose that \( t(i - \frac{1}{2}) = t(i + \frac{1}{2}) \) and let \( v \in \hat{A}_{k+1}(\lambda, \ell) \) such that \( v \twoheadrightarrow t \) and \( v(i) = v(i+1) \). If \( s \neq v \), then by item (1) of Proposition 2.5, and Lemma 5.9,

\[
\sum_{i \leq t} (c_u(i + 1) - c_u(i)) \sigma_{i+1}(u, t) p_{i+j}^{\alpha} f_0^{k+1} = f_1^{k+1}((1) p_{i+\frac{1}{2}} + p_{i+\frac{1}{2}} - p_{i+\frac{1}{2}}) p_{i+\frac{1}{2}} + p_{i+\frac{1}{2}}(v, t) c_i(v) p_{i+1}(v, u) f_0^{k+1} p_{i+\frac{1}{2}},
\]

where

\[
f_0^{k+1} p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}(t, v) f_{i}^{k+1} p_{i+\frac{1}{2}} \text{ and } p_{i+\frac{1}{2}}(v, t) p_{i+\frac{1}{2}}(t, v) = p_{i+\frac{1}{2}}(t, v) p_{i+\frac{1}{2}}(v, v).
\]

This gives the formula for \( \sigma_{i+1}(s, t) \). Next,

\[
f_1^{k+1} \sigma_{i+1} + p_{i+\frac{1}{2}} p_{i+\frac{1}{2}} = \sum_{i \leq t} \sigma_{i+1}(u, t) f_{u}^{k+1} p_{i+1} p_{i+\frac{1}{2}} = \sum_{i \leq t} \sigma_{i+1}(u, t) p_{i+1}(v, u) f_0^{k+1} p_{i+\frac{1}{2}}.
\]

Since

\[
f_0^{k+1} \sigma_{i+1} + p_{i+\frac{1}{2}} p_{i+\frac{1}{2}} = c_t(i) f_1^{k+1} p_{i+\frac{1}{2}} = c_t(i) p_{i+\frac{1}{2}}(v, v) f_0^{k+1} p_{i+\frac{1}{2}},
\]

the formula for \( \sigma_{i+1}(s, t) \) follows.

(2) Suppose that \( t(i) = t(i+1) \) and \( t(i-\frac{1}{2}) \neq t(i+\frac{1}{2}) \), and let \( v \in \hat{A}_{k+1}(\lambda, \ell) \) such that \( v \twoheadrightarrow t \) and \( v(i-\frac{1}{2}) = v(i+\frac{1}{2}) \). By item (1) of Proposition 2.5, and Lemma 5.9,

\[
\sum_{i \leq t} (c_u(i + 1) - c_u(i)) \sigma_{i+1}(u, t) f_{u}^{k+1} = f_1^{k+1},
\]

which gives the formula for \( \sigma_{i+1}(s, t) \) when \( s \neq v \). Next,

\[
f_1^{k+1} \sigma_{i+1} + p_{i+\frac{1}{2}} p_{i+\frac{1}{2}} = \sum_{i \leq v} \sigma_{i+1}(u, t) f_{u}^{k+1} p_{i+1} p_{i+\frac{1}{2}} = \sum_{i \leq v} \sigma_{i+1}(u, t) p_{i+1}(v, u) f_0^{k+1} p_{i+\frac{1}{2}}.
\]

Since

\[
f_1^{k+1} \sigma_{i+1} + p_{i+\frac{1}{2}} p_{i+\frac{1}{2}} = c_t(i) f_1^{k+1} p_{i+\frac{1}{2}} = 0 \text{ and } f_0^{k+1} p_{i+\frac{1}{2}} \neq 0,
\]

the formula for \( \sigma_{i+1}(v, t) \) follows.

(3) Suppose that \( t(i - \frac{1}{2}) \neq t(i + \frac{1}{2}) \) and \( t(i) \neq t(i+1) \) and that \( t \sigma_{i+1} \) does not exist. By item (1) of Proposition 2.5 and Lemma 5.9,

\[
f_1^{k+1} \sigma_{i+1} L_{i+1} - f_1^{k+1} L_i \sigma_{i+\frac{1}{2}} = f_1^{k+1}.
\]
Since
\[ u \in A_{k+1}^{(\lambda, \ell)} \text{ and } u \overset{i+1}{\sim} t \]
implies that \( u = t \),
it follows that \( \sigma_{i+1}(u, t) = 0 \) if \( u \neq t \), while
\[ (c_i(i + 1) - c_i(i)) \sigma_{i+1}(t, t) = 1, \]
as required.

(4) Suppose that \( t(i-\frac{1}{2}) \neq t(i+\frac{1}{2}) \) and \( t(i) \neq t(i+1) \) and that \( t\sigma_{i+1} \) exists. First observe that the hypotheses on \( t \) imply that if \( u \in A_{k+1}^{(\lambda, \ell)} \) and \( u \overset{i+1}{\sim} t \), then \( u \in \{t, t\sigma_{i+1}\} \). Thus by Lemma (5.10),
\[ f_t^{(k+1)} \sigma_{i+1} = \sigma_{i+1}(t, t)f_t^{(k+1)} + \sigma_{i+1}(s, t)f_s^{(k+1)}, \]
where \( s = t\sigma_{i+1} \).
We now determine \( \sigma_{i+1}(t, t) \). Lemma 5.9 and the assumptions on \( t \) imply that \( f_t p_{i+1} = f_t p_{i+\frac{1}{2}} = 0 \). Thus item (1) of Proposition 2.5 gives
\[ (c_i(i + 1) - c_i(i))\sigma_{i+1}(t, t) = 1, \]
as required.

Next, we show that if \( s = t\sigma_{i+1} \) and \( t \succ s \), then \( \sigma(s, t) = 1 \). Now,
\[ f_t^{(k+1)} \sigma_{i+1} = f_t^{(k+1)} p_t \sigma_{i+1} + \sum_{\nu > 1} r_\nu f_t^{(k+1)} p_\nu \sigma_{i+1}, \]
where \( r_\nu \in \kappa \), for \( v \in A_{k+1}^{(\lambda, \ell)} \).

It therefore suffices to show that
\[ \sigma_{i+1}(v, s) = 0, \]
if \( v \in A_{k+1}^{(\lambda, \ell)} \) and \( v \succ t \), \hspace{1cm} (5.16)
and that there exist \( r_\nu \in \kappa \), for \( u \in A_{k+1}^{(\lambda, \ell)} \), such that
\[ f_t^{(k+1)} p_t \sigma_{i+1} = f_t^{(k+1)} p_\nu + \sum_{u \succ s} r_u f_t^{(k+1)} p_u. \hspace{1cm} (5.17)\]

To prove (5.16), suppose that \( v \succ t \). Then
\[ f_v^{(k+1)} \sigma_{i+1} = \sum_{i+1} \sigma_{i+1}(u, v) f_u^{(k+1)}, \hspace{1cm} (5.18)\]
If \( \sigma_{i+1}(s, v) \neq 0 \) in the expression (5.18), then \( i+1 \overset{v}{\sim} s \), so \( v \in \{t, s\} \), which violates the assumption that \( v \succ t \succ s \). Thus (5.16) holds.

Now we verify (5.17). There is no loss of generality in assuming that that \( i = k \). There are two cases to consider here.
(a) If \( t(k-\frac{1}{2}) = t(k) \), let
\[ (t(k-\frac{1}{2}), t(k), t(k+\frac{1}{2}), t(k+1)) = (\mu, \mu, \nu, \lambda) \quad \text{and} \quad (s(k-\frac{1}{2}), s(k), s(k+\frac{1}{2}), s(k+1)) = (\mu, \nu, \lambda, \lambda), \]
where
\[ \lambda = \nu \cup \{(j, \lambda_j)\}, \quad \mu = \nu \cup \{(j', \mu_j')\}, \quad \text{and} \quad \nu = \mu \cup \{(j, v_j)\} = \lambda \cup \{(j', v_j')\}. \]
If \( \lambda \vdash k - \ell \), then
\[ p_t^{(k-\frac{1}{2})} = w_{a_j, k+1} \quad \text{and} \quad p_t^{(k-\frac{1}{2})} = w_{\ell-1, k}, \quad \text{where} \quad a_j = \ell + \sum_{r=1}^{j} \lambda_r, \]
and
\[ p_t^{(k-\frac{1}{2})} = w_{\ell+1} p_{k+1} w_{k+1, a_j} \sum_{r=0}^{\nu_j} w_{a_j', a_j-r}, \quad \text{where} \quad a_j' = \ell - 1 + \sum_{r=1}^{j'} \mu_r, \]

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while
\[ p_{g(k + \frac{1}{2}) \to g(k + 1)} = w_{\ell, k} \quad \text{and} \quad p_{g(k - \frac{1}{2}) \to g(k)} = w_{\alpha_j, k - 1}, \quad \text{where} \quad \alpha_j = \ell - 2 + \sum_{r=1}^{j} v_r, \]
and
\[ p_{g(k) \to g(k + 1)} = w_{\ell - 1, k - 1} p_{k + \frac{1}{2}} w_{k - 1, \alpha_j} \sum_{r=0}^{\lambda'_j} w_{\alpha_j, \alpha_j - r}, \quad \text{where} \quad \alpha'_j = \ell - 2 + \sum_{r=1}^{j'} v_r. \]

The relation \( \sigma_{k+1} = s_k \sigma_{k + \frac{1}{2}} \) together with the braid relation, gives
\[
f_{k+1}^{(k+1)} p_{\ell_{k+1}}^{(k+1)} = f_{k+1}^{(k+1)} w_{\alpha, k+1} w_{\ell, k+1} p_{k+\frac{1}{2}} w_{k-1, \alpha_j} \left( \sum_{r=0}^{\lambda'_j} w_{\alpha_j, \alpha_j - r} \right) w_{\ell-1, k+1} \sigma_{k+\frac{1}{2}}
\]
\[
= f_{k+1}^{(k+1)} w_{\ell-1, k+1} w_{\alpha, k} w_{\ell-1, k-2} p_{k-\frac{1}{2}} w_{k-2, \alpha_j} \left( \sum_{r=0}^{\lambda'_j} w_{\alpha_j, \alpha_j - r} \right) \sigma_{k+\frac{1}{2}}
\]
\[
= f_{k+1}^{(k+1)} w_{\ell-1, k+1} w_{\alpha, k} w_{\ell-1, k-2} p_{k-\frac{1}{2}} w_{k-2, \alpha_j} \left( \sum_{r=0}^{\lambda'_j} w_{\alpha_j, \alpha_j - r} \right) \sigma_{k+\frac{1}{2}}
\]
\[
= f_{k+1}^{(k+1)} p_{u} \sigma_{k+\frac{1}{2}},
\]
where \( u \in A_{k+1}^{(\lambda, \ell)} \) is given by
\[
(u^{(k-1)}, u^{(k-\frac{1}{2})}, u^{(k)}, u^{(k+\frac{1}{2})}, u^{(k+1)}) = (\mu, \nu, \lambda, \lambda).
\]
and
\[
p_{u}^{(k-\frac{1}{2}) \to u^{(1)}} = p_{u}^{(1) \to u^{(1+\frac{1}{2})}} = \cdots = p_{u}^{(k-\frac{1}{2}) \to u^{(k-1)}} = 1.
\]
Since \( v = u \sigma_{k+\frac{1}{2}} \) exists, and
\[
(v^{(k-1)}, v^{(k-\frac{1}{2})}, v^{(k)}, v^{(k+\frac{1}{2})}, v^{(k+1)}) = (\mu, \mu, \nu, \lambda, \lambda),
\]
it follows that \( u \succ v \). By (5.12), there exist \( r_a \in \kappa \), for \( a \in A_{k+1}^{(\lambda, \ell)} \), such that
\[
f_{k+1}^{(k+1)} p_{\ell_{k+1}}^{(k+1)} = f_{k+1}^{(k+1)} p_{\ell}^{(k+1)} = f_{k+1}^{(k+1)} = f_{k+1}^{(k+1)} p_a + \sum_{a \succ v} r_a f_{k+1}^{(k+1)} p_a.
\]
Multiplying both sides of the last expression by
\[
p_{g(k-1)}^{(k-\frac{1}{2})} p_{g(k-\frac{1}{2}) \to g(k-1)} \cdots p_{g(\frac{1}{2}) \to g(1)} = p_{g(k-1)}^{(k-\frac{1}{2})} p_{g(k-\frac{1}{2}) \to g(k-1)} \cdots p_{g(\frac{1}{2}) \to g(1)}
\]
on the right verifies (5.17) in this instance.

(b) If \( t^{(k)} = t^{(k+\frac{1}{2})} \), let \( s = t \sigma_{k+1} \),
\[
(g^{(k-\frac{1}{2})}, g^{(k)}, g^{(k+\frac{1}{2})}, g^{(k+1)}) = (\mu, \nu, \nu, \lambda) \quad \text{and} \quad (t^{(k-\frac{1}{2})}, t^{(k)}, t^{(k+\frac{1}{2})}, t^{(k+1)}) = (\mu, v, v, \lambda)
\]
where
\[
\lambda = \nu \cup \{ (j, \lambda_j) \}, \quad \nu = \mu \cup \{ (j, \lambda_j) \}, \quad \text{and} \quad v = \nu \cup \{ (j, \lambda_j) \}, \quad v = \mu \cup \{ (j, \lambda_j) \}.
\]
and \( j' < j \). If
\[
a_j = \ell + \sum_{r=1}^{j} \lambda_r, \quad \text{and} \quad a_j' = \ell + \sum_{r=1}^{j'} \lambda_r,
\]
34
then
\[ P_{g(k + \frac{1}{2}) \rightarrow g(k + 1)} P_{g(k) \rightarrow g(k + \frac{1}{2})} P_{g(k) \rightarrow g(k)} = w_{a_j^t} w_{a_j^t} w_{a_j^{k+1}} \]
and
\[ P_{g(k + \frac{1}{2}) \rightarrow g(k + 1)} P_{g(k) \rightarrow g(k + \frac{1}{2})} P_{g(k) \rightarrow g(k)} = w_{a_j^t} w_{a_j^{k+1}}. \]
The braid relation and \( s_{k-1} \sigma_{k+1} = \sigma_{k+1} \) give
\[ P_{g(k + \frac{1}{2}) \rightarrow g(k + 1)} P_{g(k) \rightarrow g(k + \frac{1}{2})} \sigma_{k+1} = w_{a_j^t} w_{a_j^{k+1}} \]
\[ = P_{g(k + \frac{1}{2}) \rightarrow g(k + 1)} P_{g(k) \rightarrow g(k + \frac{1}{2})} P_{g(k) \rightarrow g(k)} \sigma_{k+1}. \]
By (5.12), there exist \( r_\alpha \in \kappa \), for \( \alpha \in A^{(i, l)}_{k+1} \), such that
\[ f^{(k+1)}_{t_{i_1}} P_{g(k + \frac{1}{2}) \rightarrow g(k + 1)} P_{g(k) \rightarrow g(k + \frac{1}{2})} P_{g(k) \rightarrow g(k)} \sigma_{k+1} = f^{(k+1)}_{t_{i_1}} p_\alpha + \sum_{\alpha \succ \beta} r_\alpha f^{(k+1)}_{t_{i_1}} p_\beta, \tag{5.19} \]
where \( \nu \in A^{(i, l)}_{k+1} \) is given by
\[ p_\nu = P_{g(k + \frac{1}{2}) \rightarrow g(k + 1)} P_{g(k) \rightarrow g(k + \frac{1}{2})} P_{g(k) \rightarrow g(k)}. \]
Multiplying both sides of the expression (5.19) by
\[ P_{g(k-1) \rightarrow g(k + \frac{1}{2})} P_{g(k - \frac{1}{2}) \rightarrow g(k - 1)} \cdots P_{g(1) \rightarrow g(1)} = P_{g(k-1) \rightarrow g(k - \frac{1}{2})} P_{g(k - \frac{1}{2}) \rightarrow g(k - 1)} \cdots P_{g(1) \rightarrow g(1)} \]
on the right completes the proof (5.17) and establishes that \( \sigma_{i+1}(s, t) = 1 \) if \( s = \omega t_i + \frac{1}{2} \) and \( s \succ t \). Finally, using the above calculations together with the fact that \( \sigma_{i+1}^2 = 1 \), we can verify the formula for \( \sigma_{i+1}(s, t) \) when \( s = \omega t_i + \frac{1}{2} \) and \( s \succ t \) by observing that \( c_i(i) = c_i(i+1) \) and \( c_i(i+1) = c_i(i) \). \( \square \)

6. Central Element Recursions

In this section we obtain partition algebra analogues of the central element recursions obtained by Nazarov [Na] for the Brauer algebras and Beliakova and Blanchet [BB] for the BMW algebras, and explain the relationship between these central element recursions and the seminormal representations of the partition algebras.

We renormalise the Jucys–Murphy elements (2.3) and (2.5) by defining elements
\[ x_{i+\frac{1}{2}} = -\frac{i}{2} + L_{i+\frac{1}{2}} \quad \text{ and } \quad x_{i+1} = -\frac{i+1}{2} + L_{i+1}, \quad \text{ for } i = 0, 1, \ldots. \]
Then, for \( i = 1, 2, \ldots \), we have the relations
\[ x_{i+\frac{1}{2}} = -x_i p_{i+\frac{1}{2}} - p_{i+\frac{1}{2}} x_i - \left( \frac{i}{2} + x_{i-\frac{1}{2}} \right) p_{i+\frac{1}{2}} + s_i x_{i-\frac{1}{2}} s_i + \sigma_{i+\frac{1}{2}}, \tag{6.1} \]
\[ x_{i+1} = -\frac{i}{2} p_{i+\frac{1}{2}} - s_i x_{i+\frac{1}{2}} p_{i+\frac{1}{2}} - p_{i+\frac{1}{2}} x_i s_i + p_{i+\frac{1}{2}} x_{i+\frac{1}{2}} p_{i+\frac{1}{2}} + s_i x_{i-\frac{1}{2}} s_i + \sigma_{i+1}. \tag{6.2} \]
Following §2 of [Na], for \( i = 0, 1, \ldots \), and \( j = 0, 1, \ldots \), define central elements
\[ x_{i+\frac{1}{2}}^{(j)} \in A_i(z) \quad \text{ and } \quad x_{i+1}^{(j)} \in A_{i+\frac{1}{2}}(z). \]
by
\[ x_{i+\frac{1}{2}}^{(j)} p_{i+1} = p_{i+1} (x_{i+\frac{1}{2}})^j p_{i+1} \quad \text{ and } \quad x_{i+1}^{(j)} p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}} (x_{i+1})^j p_{i+\frac{1}{2}}. \tag{6.3} \]
Let \( u \) denote a formal variable, and define the central elements \( W_{i+\frac{1}{2}}(u) \in A_i(z)[[u^{-1}]] \), and \( W_{i+1}(u) \in A_{i+\frac{1}{2}}(z)[[u^{-1}]] \), for \( i = 0, 1, \ldots \), by
\[ W_{i+\frac{1}{2}}(u) = u^{-1} \sum_{j \geq 0} x_{i+\frac{1}{2}}^{(j)} u^{-j} \quad \text{ and } \quad W_{i+1}(u) = \sum_{j \geq 0} x_{i+1}^{(j)} u^{-j}. \]
For $i = 0, 1, \ldots$, we will work with the formal expressions
\[
W_{i + \frac{1}{2}}(u)p_{i+1} = p_{i+1} \frac{1}{u - x_{i+\frac{1}{2}}} p_{i+1} \quad \text{and} \quad W_{i+1}(u)p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}} \frac{u}{u - x_{i+1}} p_{i+\frac{1}{2}}.
\]

The next statement is the partition algebra analogue of Proposition 4.2 of [Na] and (12) of [BB].

**Proposition 6.1.** If $i = 1, 2, \ldots$, then
\[
\frac{W_{i+\frac{1}{2}}(u) + (\frac{1}{2} - u - 1)}{W_{i-\frac{1}{2}}(u)(\frac{1}{2} - u - 1)} = \frac{(u + x_i)^2 - 1}{(u - x_{i-\frac{1}{2}})^2 - 1} (u + x_i)^2,
\]
and
\[
\frac{W_{i+1}(u)}{W_i(u)} = \frac{(u + x_{i+\frac{1}{2}})^2 - 1}{(u - x_{i+\frac{1}{2}})^2 - 1} (u + x_{i+\frac{1}{2}})^2.
\]

**Proof.** We first prove (6.4). From the relation (6.1),
\[
s_i(u - x_{i+\frac{1}{2}}) = \frac{1}{2} p_{i+\frac{1}{2}} + s_i x_{i+\frac{1}{2}} p_{i+\frac{1}{2}} + p_{i+\frac{1}{2}} x_i + x_{i-\frac{1}{2}} p_{i+\frac{1}{2}} + (u - x_{i-\frac{1}{2}}) s_i - \sigma_i + 1
\]
\[
= \sigma_i + 1 p_{i+\frac{1}{2}} p_{i+\frac{1}{2}} + p_{i+\frac{1}{2}} x_i + x_{i-\frac{1}{2}} p_{i+\frac{1}{2}} + (u - x_{i-\frac{1}{2}}) s_i - \sigma_{i+1},
\]
and
\[
\frac{1}{u - x_{i+\frac{1}{2}}} s_i = \frac{1}{u - x_{i+\frac{1}{2}}} \sigma_i + 1 p_{i+\frac{1}{2}} p_{i+\frac{1}{2}} + \frac{1}{u - x_{i+\frac{1}{2}}} p_{i+\frac{1}{2}} x_i + \frac{1}{u - x_{i+\frac{1}{2}}} s_i.
\]

Using the fact that $p_{i+\frac{1}{2}}(u - x_{i+\frac{1}{2}}) = p_{i+\frac{1}{2}}(u + x_i)$, the previous expression gives
\[
\frac{1}{u - x_{i+\frac{1}{2}}} s_i = \frac{1}{u - x_{i+\frac{1}{2}}} \sigma_i + 1 p_{i+\frac{1}{2}} p_{i+\frac{1}{2}} + \frac{1}{u - x_{i+\frac{1}{2}}} p_{i+\frac{1}{2}} x_i + \frac{1}{u - x_{i+\frac{1}{2}}} s_i.
\]

Since $\sigma_{i+1}$ commutes with $x_{i-\frac{1}{2}}$, multiplying both sides of the last expression by $\sigma_{i+1}$ on the left, we obtain
\[
\frac{1}{u - x_{i+\frac{1}{2}}} \sigma_{i+\frac{1}{2}} = \frac{1}{u - x_{i-\frac{1}{2}}} p_{i+\frac{1}{2}} p_{i+\frac{1}{2}} + \frac{1}{u - x_{i-\frac{1}{2}}} p_{i+\frac{1}{2}} x_i + \frac{1}{u - x_{i-\frac{1}{2}}} s_i.
\]

Multiplying each side of (6.6) by $p_{i+1}$ on the right and on the left,
\[
p_{i+1} \sigma_{i+\frac{1}{2}} = \frac{1}{u - x_{i+\frac{1}{2}}} p_{i+1} p_{i+\frac{1}{2}} + \frac{1}{u - x_{i-\frac{1}{2}}} p_{i+\frac{1}{2}} x_i + \frac{1}{u - x_{i-\frac{1}{2}}} s_i.
\]
Using (15) of Proposition 2.4, we obtain $p_{i+1}\sigma_{i+\frac{1}{2}}p_{i+1} = (\frac{3}{2} - x_{i-\frac{1}{2}})p_{i+1}$ which, substituted into (6.7), gives

$$p_{i+1}\sigma_{i+\frac{1}{2}}\frac{1}{u - x_{i+\frac{1}{2}}}p_{i+1} = (\frac{3}{2} - u - 1)\frac{1}{u - x_{i-\frac{1}{2}}}p_{i+1} + p_{i+1} - \frac{1}{u - x_{i-\frac{1}{2}}}p_{i} - \frac{1}{u + x_{i}}p_{i+1} \tag{6.8}$$

Multiplying each side of (6.6) by $p_{i}$ on the left, and applying the algebra anti-involution $\ast$ to the result,

$$\sigma_{i+\frac{1}{2}}\frac{1}{u - x_{i-\frac{1}{2}}}p_{i} = \frac{1}{u + x_{i}}p_{i+\frac{1}{2}}p_{i}W_{i-\frac{1}{2}}(u) + \frac{1}{u - x_{i-\frac{1}{2}}}p_{i+\frac{1}{2}}p_{i} - \frac{1}{u + x_{i}}p_{i+\frac{1}{2}}p_{i}$$

Making the substitution $\sigma_{i+\frac{1}{2}}p_{i} = (\frac{3}{2} + x_{i})p_{i+\frac{1}{2}}p_{i}$ in the last expression gives

$$\sigma_{i+\frac{1}{2}}\frac{1}{u - x_{i-\frac{1}{2}}}p_{i} = \frac{1}{u + x_{i}}p_{i+\frac{1}{2}}p_{i}W_{i-\frac{1}{2}}(u) + \frac{1}{u - x_{i-\frac{1}{2}}}p_{i+\frac{1}{2}}p_{i} + p_{i+\frac{1}{2}}p_{i}$$

Therefore,

$$p_{i+1}\sigma_{i+\frac{1}{2}}\frac{1}{u - x_{i-\frac{1}{2}}}p_{i+1} = \frac{1}{(u + x_{i})^{2}}W_{i-\frac{1}{2}}(u)p_{i+1} + \frac{1}{u + x_{i}}p_{i+1} \tag{6.9}$$

Now,

$$p_{i+1}\sigma_{i+\frac{1}{2}}\frac{1}{u - x_{i-\frac{1}{2}}}\sigma_{i+\frac{1}{2}}p_{i+1} = p_{i+1}\sigma_{i+\frac{1}{2}}\frac{1}{u - x_{i-\frac{1}{2}}}p_{i+1}p_{i+\frac{1}{2}}u + \frac{1}{u - x_{i-\frac{1}{2}}}p_{i+1}$$

Using the relation

$$p_{i+1}\sigma_{i+\frac{1}{2}}\frac{1}{u - x_{i-\frac{1}{2}}} \sigma_{i+\frac{1}{2}}p_{i+1} = s_{i}p_{i}\sigma_{i+1}\frac{1}{u - x_{i-\frac{1}{2}}}p_{i+1}s_{i} = W_{i-\frac{1}{2}}(u)p_{i+1},$$

and substituting (6.8) and (6.9) into (6.10), we obtain

$$W_{i-\frac{1}{2}}(u)p_{i+1} = \frac{1}{(u + x_{i})^{2}}W_{i-\frac{1}{2}}(u)p_{i+1} + (\frac{3}{2} - u - 1)\frac{1}{(u + x_{i})^{2}}p_{i+1} + W_{i+\frac{1}{2}}(u)p_{i+1}$$

$$- \frac{1}{(u - x_{i-\frac{1}{2}})^{2}}p_{i+1} - (\frac{3}{2} - u - 1)\frac{1}{(u - x_{i-\frac{1}{2}})^{2}}p_{i+1}$$
or

\[
\frac{(u + x_i)^2 - 1}{(u + x_i)^2} p_{i+1} W_{i + \frac{1}{2}}(u) - \frac{1}{(u + x_i)^2} p_{i+1} = \frac{(u - x_{i+\frac{1}{2}})^2 - 1}{(u - x_{i+\frac{1}{2}})^2} p_{i+1} W_{i + \frac{1}{2}}(u) - \frac{1}{(u - x_{i+\frac{1}{2}})^2} p_{i+1},
\]

from which the relation (6.4) now follows.

For the proof of (6.5), use the the relation (6.2) to write

\[
(u - x_{i+1}) s_i = \frac{\zeta}{2} p_{i+\frac{1}{2}} + s_i x_i p_{i+\frac{1}{2}} + p_{i+\frac{1}{2}} x_i - p_{i+\frac{1}{2}} x_i p_{i+1} + p_{i+\frac{1}{2}} + s_i(u - x_i) - \sigma_{i+\frac{1}{2}}. \tag{6.11}
\]

Since \((x_{i+1})^j p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}} (x_i)^j p_{i+1} p_{i+\frac{1}{2}}\) for \(j = 0, 1, \ldots\), we obtain

\[
\frac{1}{u - x_{i+1}} p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}} \frac{1}{u - x_i} p_{i+1} p_{i+\frac{1}{2}},
\]

which together with (6.11), gives

\[
s_i \frac{1}{u - x_i} = \frac{\zeta}{2} \frac{1}{u - x_{i+1}} p_{i+\frac{1}{2}} \frac{1}{u - x_i} + \frac{1}{u - x_{i+1}} s_i x_i p_{i+\frac{1}{2}} \frac{1}{u - x_i} - p_{i+\frac{1}{2}} \frac{1}{u - x_i} p_{i+1} p_{i+\frac{1}{2}} \frac{1}{u - x_i} p_{i+1} p_{i+\frac{1}{2}} + p_{i+\frac{1}{2}} u - x_i
\]

\[
+ \frac{1}{u - x_{i+1}} s_i - \sigma_{i+\frac{1}{2}}(u - x_i)(u - x_{i+1}) \frac{1}{u - x_i} p_{i+\frac{1}{2}} u - x_i
\]

Multiplying both sides of the above expression by \(\sigma_{i+\frac{1}{2}}\) on the left,

\[
\sigma_{i+\frac{1}{2}} \frac{1}{u - x_i} = \frac{1}{u - x_{i+1}} p_{i+1} p_{i+\frac{1}{2}} \frac{1}{u - x_i} - p_{i+\frac{1}{2}} \frac{1}{u - x_i} p_{i+1} p_{i+\frac{1}{2}} + p_{i+\frac{1}{2}} \frac{1}{u - x_i} u - x_i \tag{6.12}
\]

and multiplying both sides of the expression (6.12) by \(p_{i+\frac{1}{2}}\) on the right, and then applying the anti-involution * to the result,

\[
p_{i+\frac{1}{2}} \frac{1}{u - x_i} \sigma_{i+\frac{1}{2}} = \frac{W_i(u)}{u} p_{i+\frac{1}{2}} p_{i+\frac{1}{2}} \frac{1}{u - x_i} + \frac{W_i(u)}{u} p_{i+\frac{1}{2}} \frac{1}{u - x_i} \tag{6.13}
\]

Multiplying both sides of the expression (6.12) by \(p_{i+\frac{1}{2}}\) on the right and on the left, and then using (17) of Proposition 2.4 together with the fact that \(p_{i+1}(u + x_{i+\frac{1}{2}}) = p_{i+1}(u - x_{i+1})\), we obtain

\[
p_{i+\frac{1}{2}} \frac{1}{u - x_{i+1}} \sigma_{i+1} p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}} p_{i+\frac{1}{2}} \frac{W_i(u)}{u} - \frac{1}{u + x_{i+\frac{1}{2}}} p_{i+\frac{1}{2}} p_{i+\frac{1}{2}} \frac{1}{u - x_i}
\]

\[
+ \frac{1}{u - x_i} p_{i+\frac{1}{2}} \frac{W_i(u)}{u}. \tag{6.14}
\]
Multiplying (6.12) on the right by \( \sigma_{i+1} \) and substituting for the term appearing on the left hand side of (6.13),

\[
\sigma_{i+1} \frac{1}{u - x_i} \sigma_{i+1} = \frac{W_i(u)}{u} \frac{1}{u + x_i + \frac{1}{2}} p_{i+1} \frac{1}{u + x_i + \frac{1}{2}} + \frac{W_i(u)}{u} \frac{1}{u + x_i + \frac{1}{2}} p_{i+1} p_{i+\frac{1}{2}} \\
- \frac{1}{(u - x_i)(u + x_i + \frac{1}{2})} p_{i+1} p_{i+\frac{1}{2}} \frac{1}{u - x_i} - \frac{1}{(u + x_i + \frac{1}{2})} p_{i+\frac{1}{2}} p_{i+1} \\
+ \frac{W_i(u)}{u} p_{i+\frac{1}{2}} p_{i+1} \frac{1}{u + x_i + \frac{1}{2}} + \frac{W_i(u)}{u} p_{i+\frac{1}{2}} \\
- \frac{1}{u - x_i} p_{i+\frac{1}{2}} p_{i+1} \frac{1}{u - x_i} + \frac{1}{u - x_i} - \frac{1}{(u - x_i)(u - x_{i+1})} \sigma_{i+1}. \tag{6.15}
\]

Since

\[
p_{i+\frac{1}{2}} \sigma_{i+1} \frac{1}{u - x_i} p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}} \left( \frac{W_i(u)}{u} \frac{1}{(u + x_i + \frac{1}{2})^2} + \frac{W_i(u)}{u} \frac{1}{u + x_i + \frac{1}{2}} p_{i+\frac{1}{2}} \\
- \frac{1}{(u - x_i)(u + x_i + \frac{1}{2})} p_{i+\frac{1}{2}} \frac{1}{u - x_i} - \frac{W_i(u)}{u} p_{i+\frac{1}{2}} + \frac{W_i(u)}{u} p_{i+\frac{1}{2}} \frac{1}{u + x_i + \frac{1}{2}} \\
+ \frac{W_i(u)}{u} p_{i+\frac{1}{2}} - \frac{W_i(u)}{u} p_{i+\frac{1}{2}} \frac{1}{u - x_i} + \frac{W_{i+1}(u)}{u} - \frac{1}{u - x_i} p_{i+\frac{1}{2}} \\
+ \frac{1}{(u - x_i)(u + x_i + \frac{1}{2})} p_{i+\frac{1}{2}} \frac{1}{u - x_i} - \frac{1}{(u - x_i)^2} \right).
\]

Since \((u - x_i)p_{i+\frac{1}{2}} = (u + x_i + \frac{1}{2})p_{i+\frac{1}{2}}\), we obtain

\[
\frac{W_i(u)}{u} - \frac{W_i(u)}{u} \frac{1}{(u + x_i + \frac{1}{2})^2} = \frac{W_{i+1}(u)}{u} - \frac{W_{i+1}(u)}{u} \frac{1}{(u - x_i)^2}.
\]

and the statement (6.5) follows. \(\square\)

As an application of the recursions (6.4) and (6.5), the seminormal matrix entries of the contractions \(p_i\) and \(p_{i+\frac{1}{2}}\) can be computed independently of any formula for the dimensions of the irreducible representations of the symmetric group.

Determine a series \(Q_{i+\frac{3}{2}}(u) \in A_i(z)[[u^{-1}]]\) by the recursion (6.4) and

\[
Q_{i+\frac{3}{2}}(u) = Z_{i+\frac{3}{2}}(u) + (\frac{u}{2} - u - 1), \quad \text{and} \quad Q_{\frac{3}{2}}(u) = -\frac{(u + 1 + \frac{u}{2})(u - \frac{u}{2})}{(u + \frac{u}{2})}, \tag{6.16}
\]

and a series \(Q_{i+1}(u) \in A_{i+\frac{1}{2}}(z)[[u^{-1}]]\) by the recursion (6.5), and

\[
Q_{i+1}(u) = \frac{W_{i+1}(u)}{u}, \quad \text{and} \quad Q_{\frac{1}{2}}(u) = \frac{(u + 1 - \frac{u}{2})}{(u + \frac{u}{2})(u - \frac{u}{2})}. \tag{6.17}
\]
If \( i = 0, 1, \ldots, \) and \( \mu \in \hat{A}_i \), denote by
\[
Q_{i+\frac{1}{2}}(u, \mu) \quad \text{the scalar by which } Q_{i+\frac{1}{2}}(u) \text{ acts on the } A_i(z)[[u^{-1}]]-\text{module } A_i^{(\mu, m)}.
\]
Similarly, if \( \mu \in \hat{A}_{i+\frac{1}{2}} \), denote by
\[
Q_{i+1}(u, \mu) \quad \text{the scalar by which } Q_{i+1}(u) \text{ acts on the } A_{i+\frac{1}{2}}(z)[[u^{-1}]]-\text{module } A_{i+\frac{1}{2}}^{(\mu, m)}.
\]

Given Proposition 3.4, the diagonal entries of the contractions \( p_i \) and \( p_{i+\frac{1}{2}} \) in the seminormal matrix representation of \( \hat{A}_{k+\frac{1}{2}}(z) \) are determined by the recursions (6.16) and (6.17) as follows.

If \( (\lambda, \ell) \in \hat{A}_{k+\frac{1}{2}} \) and \( t = (\mu^{(0)}, \mu^{(\frac{1}{2})}, \ldots, \mu^{(k+\frac{1}{2})}) \in \hat{A}_{k+\frac{1}{2}}(\lambda, \ell) \), define
\[
x_i(i) = \begin{cases} 
\frac{1}{2} - |\mu^{(i)}|, & \text{if } \mu^{(i)} = \mu^{(i-\frac{1}{2})} \\
\frac{1}{2} - c(b), & \text{if } \mu^{(i)} = \mu^{(i-\frac{1}{2})} \cup \{b\},
\end{cases}
\]
and
\[
x_i(i - \frac{1}{2}) = \begin{cases}
\frac{1}{2} + |\mu^{(i-1)}|, & \text{if } \mu^{(i-1)} = \mu^{(i-\frac{1}{2})}, \\
\frac{1}{2} - c(b), & \text{if } \mu^{(i+\frac{1}{2})} = \mu^{(i)} \setminus \{b\},
\end{cases}
\]
for \( i = 0, 1, \ldots, k \), which are the eigenvalues of the Jucys–Murphy elements under the normalisation (6.1) and (6.2).

The next statement is a counterpart to (3.6) of [Na] and Lemma 7.4 of [BB].

**Proposition 6.2.** Let \( (\lambda, \ell) \in \hat{A}_{k+\frac{1}{2}} \). The following statements hold:

1. If \( t \in \hat{A}_{k+\frac{1}{2}}^{(\lambda, \ell)} \) satisfies \( t^{(i-\frac{1}{2})} = t^{(i+\frac{1}{2})} = \mu \), then
\[
p_{i+\frac{1}{2}}(t, t) = \text{Res}_{u=x_i(i)} Q_{i}(u, \mu),
\]
for \( i = 1, \ldots, k \).

2. If \( t \in \hat{A}_{k+\frac{1}{2}}^{(\lambda, \ell)} \) satisfies \( t^{(i-1)} = t^{(i)} = \mu \), then
\[
p_{i}(t, t) = \text{Res}_{u=x_i(i-\frac{1}{2})} Q_{i-\frac{1}{2}}(u, \mu),
\]
for \( i = 1, \ldots, k \).

**Proof.** (1) If \( (\lambda, \ell) \in \hat{A}_{k+\frac{1}{2}} \) and \( t \in \hat{A}_{k+\frac{1}{2}}^{(\lambda, \ell)} \) are as in the statement of the proposition, then
\[
\frac{1}{u - x_i(i)} f_{\frac{1}{2}} W_i(u) p_{i+\frac{1}{2}} \frac{1}{u - x_i(i)} = \sum_{s, \sim \sim t} p_{i+\frac{1}{2}}(s, t) f_s p_{i+\frac{1}{2}} = \sum_{s, \sim \sim t} p_{i+\frac{1}{2}}(s, t) u - x_s(i) f_s p_{i+\frac{1}{2}} = \sum_{s, \sim \sim t} p_{i+\frac{1}{2}}(s, t) u - x_s(i) f_s p_{i+\frac{1}{2}} = \sum_{s, \sim \sim t} p_{i+\frac{1}{2}}(s, t) u - x_s(i) f_s p_{i+\frac{1}{2}},
\]
together with the fact that \( p_{i+\frac{1}{2}}(t, t) \neq 0 \), gives the required result. The statement (2) follows similarly.

**Proposition 6.3.** If \( (\mu, m) \in \hat{A}_k \), then
\[
Q_{k+\frac{1}{2}}(u, \mu) = -\frac{(u - |\mu| + 1 + \frac{\mu}{2}) \prod_{a \in A(\mu)} (u + c(a) - \frac{\mu}{2})}{(u - |\mu| + \frac{\mu}{2}) \prod_{b \in R(\mu)} (u + c(b) - \frac{\mu}{2})},
\]
(6.18)
and,
\[
Q_{k+1}(u, \mu) = \frac{(u + |\mu| + 1 + \frac{\mu}{2}) \prod_{b \in R(\mu)} (u - c(b) + \frac{\mu}{2})}{(u + |\mu| - \frac{\mu}{2}) \prod_{a \in A(\mu)} (u - c(a) + \frac{\mu}{2})},
\]
(6.19)
Proof. We prove (6.18). Let \((\mu, m) \in \hat{A}_k\). Since \(f^{(k)}_{\nu}\) is a common eigenvector for the action of \(\{Q_{i+\frac{1}{2}}(u) \mid i = 1, \ldots, k\}\), the recursion (6.4) implies that

\[
f^{(k)}_{\nu}Q_{i+\frac{1}{2}}(u) = Q_{i+\frac{1}{2}}(u)f^{(k)}_{\nu}, \quad \text{for } i = 1, \ldots, m.
\]

Thus it is sufficient to consider the case where \(\mu \vdash k\). Let \(u^{\mu} = (\mu^{(0)}, \mu^{(1)}, \ldots, \mu^{(k)})\). We now verify that

\[
f^{(k)}_{\nu}Q_{i+\frac{1}{2}}(u) = Q_{i+\frac{1}{2}}(u, \mu^{(i)})f^{(k)}_{\nu} \quad \text{for } i = 0, 1, \ldots, k.
\] (6.20)

Let \(i = 1, 2, \ldots, \) and \(\mu^{(i-1)} = \nu\) and \(\mu^{(i)} = \lambda = \nu \cup \{\alpha\}\). Then \(Q_{i+\frac{1}{2}}(u)\) acts on \(f^{(k)}_{\nu}\) by the scalar

\[
Q_{i+\frac{1}{2}}(u, \nu) = \frac{(u + c_1(i))^2 - 1}{(u + c_1(i + 1))^2 - 1} \cdot \frac{(u - c_1(i - \frac{1}{2}))^2}{(u - c_1(i + \frac{1}{2}))^2} = \frac{(u - |\nu| + 1 + \frac{1}{2})}{(u - |\nu| + \frac{1}{2})} \cdot \frac{(u - c_1(i))^2}{(u - c_1(i - \frac{1}{2}))^2} \cdot \frac{(u - c_1(i + 1)^2 - 1}{(u - c_1(i + 1 + \frac{1}{2}))^2}.
\]

Let \(a_1, a_4\) respectively be the nodes above and below \(\alpha\) and \(a_2, a_3\) respectively be the nodes to the right and left of \(\alpha\). Then \(a_1 \in R(\nu) \iff a_2 \not\in A(\lambda)\), and \(a_3 \in R(\nu) \iff a_4 \not\in A(\lambda)\). Since \(c(a_1) = c(\alpha) + 1 = c(a_2)\) and \(c(a_3) = c(\alpha) - 1 = c(a_4)\) the proof of (6.18) is complete. The statement (6.19) is verified similarly. 

7. Tables of Representing Matrices

The tables below give the representing matrices for the generators \(p_i, p_{i+\frac{1}{2}}, \sigma_{i+\frac{1}{2}}, \sigma_{i+1}\) in selected representations of \(A_k\) and \(A_{k+\frac{1}{2}}\) of small rank.
\[
A_1(z) \quad \lambda = \emptyset \\
P_1 \mapsto [z]
\]

\[
A_{1+\frac{1}{2}}(z) \quad \lambda = (1) \\
P_1 \mapsto [0] \\
P_{1+\frac{1}{2}} \mapsto [0] \quad \sigma_{1+\frac{1}{2}} \mapsto [1]
\]

\[
A_2(z) \quad \lambda = (1) \\
P_1 \mapsto \begin{bmatrix} z & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
P_{1+\frac{1}{2}} \mapsto \begin{bmatrix} \frac{1}{z} & \frac{z-1}{z^2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
\sigma_{1+\frac{1}{2}} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\
P_2 \mapsto \begin{bmatrix} 0 & 0 & 0 \\ \frac{z}{z-1} & \frac{z^2(z-2)}{(z-1)^2} & \frac{z-2}{z-1} \\ 0 & \frac{1}{z} & \frac{1}{z-1} \end{bmatrix} \\
\sigma_2 \mapsto \begin{bmatrix} 0 & \frac{1}{z} & \frac{z-2}{z-1} \\ \frac{z}{z-1} & \frac{z-2}{z-1} & \frac{z(2-z)}{(z-1)^2} \\ 1 & -\frac{1}{z} & \frac{1}{z-1} \end{bmatrix}
\]

\[
A_2(z) \quad \lambda = \emptyset \\
P_1 \mapsto \begin{bmatrix} z & 0 \\ 0 & 0 \end{bmatrix} \\
P_{1+\frac{1}{2}} \mapsto \begin{bmatrix} \frac{1}{z} & \frac{z-1}{z^2} \\ 1 & \frac{z-1}{z} \end{bmatrix} \\
\sigma_{1+\frac{1}{2}} \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
P_2 \mapsto \begin{bmatrix} z & 0 \\ 0 & 0 \end{bmatrix} \\
\sigma_2 \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

\[
A_2(z) \quad \lambda = (2) \\
P_1 \mapsto [0] \\
P_{1+\frac{1}{2}} \mapsto [0] \quad \sigma_{1+\frac{1}{2}} \mapsto [1] \\
P_2 \mapsto [0] \quad \sigma_2 \mapsto [1]
\]
\[ A_2(z) \quad \lambda = (1, 1) \]

\[ p_1 \mapsto [0] \]

\[ p_{1+\frac{1}{2}} \mapsto [0] \quad \sigma_{1+\frac{1}{2}} \mapsto [1] \]

\[ p_2 \mapsto [0] \quad \sigma_2 \mapsto [-1] \]

\[ A_{2+\frac{1}{2}}(z) \quad \lambda = \emptyset \]

| \[ p_1 \mapsto \] | \[ \begin{bmatrix} z & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \] | \[ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \] |
| \[ p_{1+\frac{1}{2}} \mapsto \] | \[ \begin{bmatrix} \frac{1}{z} & 0 & \frac{z-1}{z} & 0 & 0 \\ 0 & 1 & \frac{z-1}{z} & 0 & 0 \\ 0 & 0 & 1 & \frac{z(z-2)}{(z-1)^2} & 0 \\ 0 & 0 & 0 & 1 & \frac{z}{z-1} \end{bmatrix} \] | \[ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \] |
| \[ p_2 \mapsto \] | \[ \begin{bmatrix} z & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{z^2(z-2)}{(z-1)^2} & \frac{z(z-2)}{z-1} & 0 \\ 0 & 0 & 1 & \frac{z}{z-1} & 0 \end{bmatrix} \] | \[ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \] |
| \[ p_{2+\frac{1}{2}} \mapsto \] | \[ \begin{bmatrix} \frac{1}{z} & 0 & \frac{z-1}{z} & 0 & 0 \\ 0 & \frac{1}{z} & 0 & \frac{z-1}{z} & 0 \\ 0 & 0 & \frac{1}{z} & 0 & 0 \\ 1 & 0 & \frac{z-1}{z} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \] | \[ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \] |

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\[
A_{2,\frac{1}{2}}(z)
\]

\[
\lambda = (1)
\]

\[
p_1 \mapsto \begin{bmatrix}
    z & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
p_1 + \frac{1}{2} \mapsto \begin{bmatrix}
    1 \\
    z \\
    \frac{z-1}{z^2} \\
    1 \\
\end{bmatrix}
\]

\[
s_{1,\frac{1}{2}} \mapsto \begin{bmatrix}
    1 & 0 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 & 0 \\
    0 & 0 & 1 & 0 & 0 \\
    0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

\[
p_2 \mapsto \begin{bmatrix}
    0 & 0 & 0 & 0 & 0 \\
    0 & \frac{z}{z-1} & \frac{z^2(z-2)}{(z-1)^2} & 0 & 0 \\
    0 & 1 & \frac{z(z-2)}{z-1} & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
s_2 \mapsto \begin{bmatrix}
    0 & 1 & 0 & 0 & 0 \\
    \frac{z}{z-1} & \frac{z-2}{z-1} & \frac{z^2-z}{(z-1)^2} & 0 & 0 \\
    \frac{z}{z-1} & 0 & \frac{z(z-2)}{(z-1)^2} & 0 & 0 \\
    1 & -\frac{1}{z} & \frac{1}{z-1} & 0 & 0 \\
    0 & 0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 0 & -1 \\
\end{bmatrix}
\]

\[
p_{2,\frac{1}{2}} \mapsto \begin{bmatrix}
    0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 \\
    \frac{z-1}{z(z-2)} & \frac{z(z-3)(z-1)}{z(z-2)^2} & \frac{(z-1)^2}{2z^2(z-2)} & 0 & 0 \\
    0 & \frac{1}{z} & \frac{z-3}{z^2} & \frac{z}{z-2} & 0 \\
    0 & 0 & 1 & \frac{z-3}{z-2} & \frac{z}{2z} \\
\end{bmatrix}
\]

\[
s_{2,\frac{1}{2}} \mapsto \begin{bmatrix}
    1 & 0 & 0 & 0 & 0 \\
    0 & -\frac{1}{z-1} & \frac{z}{z-1} & \frac{z(z-3)}{(z-1)(z-2)} & -\frac{1}{2} \\
    0 & \frac{z}{z(z-2)} & \frac{z^3-3z^2+2z-1}{z(z-2)(z-1)} & -\frac{z-3}{z(z-2)^2} & \frac{z-1}{z-2} \\
    0 & \frac{1}{z} & -\frac{1}{z(z-1)} & \frac{z-1}{z(z-2)} & \frac{z}{2z} \\
    0 & -1 & \frac{1}{z-1} & \frac{z-3}{z-2} & \frac{z+1}{2z} \\
\end{bmatrix}
\]
| $A_{z+\frac{1}{2}}(z)$ | $\lambda = (2)$ | $A_{z+\frac{1}{2}}(z)$ | $\lambda = (1, 1)$ |
|----------------------|----------------|----------------------|----------------|
| $p_1 \mapsto [0]$    |                | $p_1 \mapsto [0]$    |                |
| $p_{1+\frac{1}{2}} \mapsto [0]$ | $\sigma_{1+\frac{1}{2}} \mapsto [1]$ | $p_{1+\frac{1}{2}} \mapsto [0]$ | $\sigma_{1+\frac{1}{2}} \mapsto [1]$ |
| $p_2 \mapsto [0]$    | $\sigma_2 \mapsto [1]$ | $p_2 \mapsto [0]$    | $\sigma_2 \mapsto [-1]$ |
| $p_{2+\frac{1}{2}} \mapsto [0]$ | $\sigma_{2+\frac{1}{2}} \mapsto [1]$ | $p_{2+\frac{1}{2}} \mapsto [0]$ | $\sigma_{2+\frac{1}{2}} \mapsto [1]$ |

### $A_3(z)$ for $\lambda = \emptyset$

| $A_3(z)$ | $\lambda = \emptyset$ |
|----------|-------------------------|

| $p_1 \mapsto$ | $\begin{bmatrix} z & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ |
| $p_{1+\frac{1}{2}} \mapsto$ | $\begin{bmatrix} \frac{1}{z} & \frac{z}{z-1} & 0 & 0 & 0 \\ 1 & \frac{z}{z-1} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{z} & \frac{z}{z-1} & 0 \\ 0 & 0 & 1 & \frac{z}{z-1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ |
| $p_2 \mapsto$ | $\begin{bmatrix} z & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{z(z-2)}{(z-1)^2} & \frac{z}{z-1} & 0 \\ 0 & 0 & 1 & \frac{z}{z-1} & 0 \end{bmatrix}$ |
| $p_{2+\frac{1}{2}} \mapsto$ | $\begin{bmatrix} \frac{1}{z} & 0 & \frac{z}{z-1} & 0 & 0 \\ 0 & \frac{1}{z} & 0 & \frac{z}{z-1} & 0 \\ 1 & 0 & \frac{z}{z-1} & 0 & 0 \\ 0 & 1 & \frac{z}{z-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ |
| $p_3 \mapsto$ | $\begin{bmatrix} z & 0 & 0 & 0 & 0 \\ 0 & z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ |
| $\sigma_3 \mapsto$ | $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ |
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