APPROXIMATION BY THE AMPLITUDE AND FREQUENCY OPERATORS

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Abstract. We study approximating properties of the amplitude and frequency operators (suns) of the form
\[ \sum_{k=1}^{n} \mu_k h(\lambda_k z), \quad \mu_k, \lambda_k \in \mathbb{C}, \]
where \( h \) is a function analytic in a neighbourhood of the origin (a basis function). Such sums are used as a tool for 2n-multiple interpolation (at the node \( z = 0 \)) and approximation of individual analytic functions \( f \), and also as operators of differentiation, integration, extrapolation, etc., which act on certain classes of functions. In the former case the approximation is achieved by choosing appropriate amplitudes \( \mu_k \) and frequencies \( \lambda_k \), depending on \( f \) and \( h \). In the latter one the parameters \( \mu_k \) and \( \lambda_k \) are universal, not depending on the individual function and determined only by the nature of a problem under consideration. In both cases the opportunity to construct the necessary amplitude and frequency sum rides on the solvability of the following associated discrete moment problem
\[ \sum_{k=1}^{n} \mu_k \lambda_k^m = \alpha_m, \quad m = 0, \ldots, 2n - 1, \]
with unknown \( \mu_k \) and \( \lambda_k \) and given \( \alpha_m \).

As is well known, the above-mentioned system is fairly often inconsistent. In order to overcome this difficulty, we propose a regularization method for the interpolation problem, which consists in addition of the special binomial \( c_1 z^{n-1} + c_2 z^{2n-1} \) to the amplitude and frequency sum. It turns out that an appropriate choice of the parameters \( c_{1,2} \) leads in some cases to a new associated discrete moment problem, which is already regularly solvable and has a quite simple explicit form of solution. Note that the regularization of the amplitude and frequency sum results in that the corresponding n-point interpolation formula with the nodes \( \lambda_k z \) becomes precise on polynomials of degree \( \leq 2n - 1 \) (i.e. the formula’s remainder is of the order \( O(z^{2n}) \) and this is twice the order of usual n-point interpolation).

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1. Introduction and statement of the problem

In [11–14] the so-called \( h \)-sums of the form

\[
H_n(\{\lambda_k\}, h; z) = \sum_{k=1}^{n} \lambda_k h(\lambda_k z), \quad z, \lambda_k \in \mathbb{C}, \quad n \in \mathbb{N},
\]

were studied. Hereinafter \( h(z) \) denotes a function analytic in the disc \(|z| < \rho (\rho > 0)\)
and \( \lambda_k \) are parameters, being independent on \( z, \lambda \). As well as the \( h \)-sums, the amplitude and frequency operators (sums) were also obtained there (in contrast to the case of the \( h \)-sums, when \(|\lambda_k| \) were already determined recursively as follows. Let \( P_{l,0} = 1, v_{l,1} = -1 \)
for \( k = 1, 2, \ldots \)

\[ P_{l,k} = \lambda P_{l,k-1} + v_{l,k}, \quad v_{l,k} = -1 - \sum_{j=1}^{k-1} \left(1 - \frac{j}{k}\right) v_{l,j}, \quad v_{2,k} = -\frac{1}{k^2} - \sum_{j=1}^{k-1} \frac{v_{2,j}}{k(k-j)}. \]

In the present paper we consider the natural generalization of the \( h \)-sums — amplitude and frequency operators (sums) of the form

\[
H_n(z) = H_n(\{\mu_k\}, \{\lambda_k\}, h; z) := \sum_{k=1}^{n} \mu_k h(\lambda_k z), \quad \mu_k, \lambda_k \in \mathbb{C},
\]

where the amplitudes \( \mu_k \) and frequencies \( \lambda_k \) are parameters, being independent on each other. The number \( n \) is called an order of the amplitude and frequency operator, if there are no zeros between the numbers \( \mu_k \) and, moreover, the numbers \( \lambda_k \) are pairwise distinct (otherwise the order of the operator is less than \( n \)). As well as the \( h \)-sums, the amplitude and frequency sums are going to be used both for approximation of individual functions \( f \) analytic near the origin (then \( \lambda_k = \lambda_k(f; h,n) \), \( \mu_k = \mu_k(f; h,n) \)), and as special operators (of differentiation, extrapolation, etc.), which act on certain classes of functions (and then \( \lambda_k = \lambda_k(n), \mu_k = \mu_k(n) \)).

Adding of the parameters \( \mu_k \) allow us to state a problem of \( 2n \)-multiple (Pade) interpolation at \( z = 0 \) by \( H_n \) (in contrast to the case of the \( h \)-sums, when \( n \)-multiple interpolation is only possible). Indeed, let

\[
f(z) = \sum_{m=0}^{\infty} f_m z^m, \quad h(z) = \sum_{m=0}^{\infty} h_m z^m, \quad \text{where } f_m = 0, \text{ if } h_m = 0.
\]
For $m \in \mathbb{N}_0$ let us introduce numbers $s_m = s_m(h, f)$ as follows
\begin{equation}
(1.4) \quad s_m(h, f) = 0, \text{ if } f_m = 0; \quad s_m(h, f) = f_m/h_m, \text{ if } f_m \neq 0.
\end{equation}

Since in $|z| < \rho \min_{k=1, \ldots, n} |\lambda_k|^{-1}$ the operator (1.3) has the form
\begin{equation}
(1.5) \quad H_n(z) = \sum_{k=1}^n \mu_k \sum_{m=0}^\infty h_m(\lambda_k z)^m = \sum_{m=0}^\infty h_m \left( \sum_{k=1}^n \mu_k \lambda_k^m \right) z^m,
\end{equation}
for the $2n$-multiple interpolation problem
\begin{equation}
(1.6) \quad f(z) = H_n(\{\mu_k\}, \{\lambda_k\}, h; z) + O(z^{2n}), \quad z \to 0,
\end{equation}
the following conditions on the so-called \textit{generalized power sums} (or moments) $S_m$ should be satisfied:
\begin{equation}
(1.7) \quad S_m := \sum_{k=1}^n \mu_k \lambda_k^m = s_m, \quad m = 0, 2n - 1.
\end{equation}

The system (1.7) with unknown $\lambda_k, \mu_k$ and given $s_m$ is well known as the \textit{discrete moment problem}. Classical works of Prony, Silvester, Ramanujan and papers of many contemporary researchers are devoted to a problem of solvability of the system (see [28, 29, 38, 39, 42, 43, 52]). The moment problem is bound up with Hankel forms, orthogonal polynomials, continued fractions, Gaussian quadratures and Padé approximants (a detailed review of the connections is given in [38, 39] and also in the next section of the present paper).

Let us suppose that the system (1.7) is solvable. Then, following [39], we call the system and its solution \textit{regular} if all $\lambda_k$ are pairwise distinct and all $\mu_k$ do not vanish. In the case of regularity of the system (1.7) we call the problem of the $2n$-multiple interpolation (1.6) \textit{regularly solvable}. One of the methods to solve regular systems (1.7) is due to Prony [42] (see also [40, 41]). We now produce it as in [30]. Consider the product of determinants
\begin{equation}
(1.8) \quad \begin{vmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & \mu_1 & \mu_2 & \cdots & \mu_n \\
0 & \mu_1 \lambda_1 & \mu_2 \lambda_2 & \cdots & \mu_n \lambda_n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \mu_1 \lambda_1^{n-1} & \mu_2 \lambda_2^{n-1} & \cdots & \mu_n \lambda_n^{n-1}
\end{vmatrix}
\begin{vmatrix}
1 & \lambda^2 & \cdots & \lambda^n \\
1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^n \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^n
\end{vmatrix}
\end{equation}

Due to the regularity of the system (1.7), the former determinant does not vanish and the latter one does only for $\lambda = \lambda_k$ (by properties of the Vandermonde determinant). On the other hand, a direct multiplication of the determinants and taking into account (1.7) give the determinant, which is a polynomial of $\lambda$ of degree $\leq n$:
\begin{equation}
(1.9) \quad G_n(\lambda) := \sum_{m=0}^n g_m \lambda^m = \begin{vmatrix}
1 & \lambda & \lambda^2 & \cdots & \lambda^n \\
s_0 & s_1 & s_2 & \cdots & s_n \\
s_1 & s_2 & s_3 & \cdots & s_{n-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
s_{n-1} & s_n & s_{n+1} & \cdots & s_{2n-1}
\end{vmatrix}.
\end{equation}

We call $G_n$ a \textit{generating polynomial} (for functional properties of such polynomials, including orthogonality, completeness, etc., see [39]). Therefore, the numbers $\lambda_k$ are simple roots of the generating polynomial $G_n$. After solving the equation $G_n(\lambda) = 0$, we substitute the numbers found into the system (1.7). Finally, deleting any $n$ rows from (1.7) leads to a linear system of equations with the unknowns $\mu_k$. Let us state known formulas for calculation of the numbers $\mu_k$ (see, for example, [15, 24]). By $\sigma_m$ and $\sigma_m^{(k)}$ we denote elementary symmetric polynomials of the form
\begin{equation}
\sigma_m = \sigma_m(\lambda_1, \ldots, \lambda_n) = \sum_{1 \leq j_1 < \cdots < j_m \leq n} \lambda_{j_1} \cdots \lambda_{j_m}, \quad m = 1, n,
\end{equation}
\[
\sigma_0 = 1, \quad \sigma_m^{(k)} = \sigma_m(\lambda_1, \ldots, \lambda_k, 0, \lambda_{k+1}, \ldots, \lambda_n), \quad k = 1, \ldots, n.
\]

**Lemma 1.** The numbers \( \mu_k \) can be calculated as the scalar products \( \mu_k = (L_k \cdot S) \), where \( S = (s_0, \ldots, s_{n-1}) \) and

\[
L_k = \frac{\gamma_n}{G_n(\lambda_k)} \left( (-1)^{n-1} \sigma_{n-1}^{(k)}, \ldots, (-1)^{n-m} \sigma_{n-m}^{(k)}, \ldots, -\sigma_1^{(k)}, 1 \right).
\]

**Proof.** If \( V = V(\lambda_1, \lambda_2, \ldots, \lambda_n) \) is the Vandermonde matrix of the first \( n \) equations of the system (1.7), then (see [15, 24]) the elements of the \( k \)th row \( L_k \) of the matrix \( V^{-1} \) have the form (1.10).

We now formulate a known criterion of regularity in terms of roots of the polynomial \( G_n \). Originally the criterion was obtained in algebraic form by Silvester [52] (see also [28, Ch. 5], [29]); then Lyubich [38, 39] presented it in analytical terms, being used in the present paper.

**Theorem (criterion of regularity).** The system (1.7) is regular if and only if the generating polynomial \( G_n \) has exactly \( n \) pairwise distinct roots. Moreover, the regular system has a unique solution.

This theorem immediately leads to the following proposition about the regular solvability of the \( 2n \)-multiple (Padé) interpolation problem for the function \( f \) analytic in a neighbourhood of \( z = 0 \).

**Theorem 1.** Let the generating polynomial \( G_n \), constructed using the numbers \( s_m = s_m(h, f) \), \( m = 0, 2n-1 \), have exactly \( n \) pairwise distinct roots. Then the amplitude and frequency operator \( H_n \) is determined from the system (1.7) uniquely and realizes \( 2n \)-multiple interpolation (1.6) of the function \( f \) at the node \( z = 0 \).

2. **The amplitude and frequency operators in several classical problems.**

We now consider several classical problems, which are bound up with approximating properties of the amplitude and frequency operators.

**2.1. Hamburger moment problem.** In connection with Theorem 1 there arises a natural question about constructing of interpolating amplitude and frequency operators with real \( \lambda_k \) and \( \mu_k \) (in particular, with \( \mu_k > 0 \)). This question is well-studied [2, 3, 26, 49] and decided by discretization of the classical Hamburger moment problem: for a given sequence of real numbers \( \{s_m\}, m \in \mathbb{N}_0 \), find a non-negative Borel measure \( \mu \) on \( \mathbb{R} \) such that

\[
s_m = \int_{-\infty}^{\infty} \lambda^m d\mu(\lambda), \quad m \in \mathbb{N}_0.
\]

Namely, the following criterion due to Hamburger holds [2, 3, 26, 49]: for \( m = 0, 2n-1 \) a solution to the problem (2.1) with a spectrum, consisting of exactly \( n \) pairwise distinct points \( \lambda_1, \ldots, \lambda_n \), exists and is unique (i.e. is regular) if and only if the leading principal minors \( \Delta_k \) of order \( k \) of the infinite Hankel matrix \((s_{i+j})_{i,j=0}^{\infty}\) satisfy

\[
\Delta_1 > 0, \quad \Delta_2 > 0, \quad \ldots \quad \Delta_n > 0, \quad \Delta_{n+1} = \Delta_{n+2} = \ldots = 0.
\]

From this it follows that the discrete moment problem (1.7) is regularly solvable in real numbers \( \lambda_k \) (this is equivalent to the fact that the polynomial (1.9) has \( n \) pairwise distinct real roots) and \( \mu_k > 0 \) if and only if the sequence (1.4) satisfies the first \( n \) inequalities in (2.2). In this case the sequence \( \{s_m\}_{m=0}^{2n-1} \) is called positive.

**2.2. Chebyshev-Gauss quadratures.** For a function \( f \) analytic in a \( \rho \)-neighbourhood of the origin let us assume

\[
F(x) := \frac{1}{x} \int_{-x}^{x} f(t) dt, \quad 0 \leq x < \rho.
\]
We now construct the interpolating amplitude and frequency operator $H_n(\{\mu_k\}, \{\lambda_k\}, f; x)$ for $F(x)$. From (1.4) we get the (positive) moment sequence $s_m = \frac{1}{m+1}, m = 0, 2n - 1$, and then consider the corresponding discrete moment problem (1.7). It is well-known that it is regular for any $n$ and the generating polynomial $G_n$ of the form (1.9) differs from the Legendre polynomial $P_n$ (we will write it in the Rodrigues’ form below) only in a constant [47, §42, [27, Ch. 7, §2]:

$$G_n(x) = P_n(1)P_n(x), \quad P_n(x) := \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2 - 1)^n.$$ 

Thereby the frequencies $\lambda_k$ are real, pairwise distinct and lie in $(-1, 1)$ (as roots of the Legendre polynomials, forming an orthogonal system on $[-1, 1]$). The amplitudes $\mu_k$ are determined on basis of the numbers $\lambda_k$ by the formulas (see [1], [27, Ch. 10, §3]):

$$\mu_k = \frac{2}{(1 - \lambda_k^2)^{\mu_k}} > 0.$$

Thus we obtain the interpolation formula

$$(2.3) \quad \frac{1}{x} \int_{-x}^{x} f(t) \, dt = \sum_{k=1}^{n} \mu_k f(\lambda_k x) + r_n(x), \quad r_n(x) = O(x^{2n}),$$

which is the Gaussian quadrature for each fixed $x$. The amplitudes and frequencies depend only on $n$ but not on $f$. As is well known [27], the Gaussian quadratures are of highest algebraic degree of accuracy among all formulas of the form (2.3) and are precise on polynomials of degree $\leq 2n - 1$, i.e. $r_n(x) \equiv 0$ on them.

In a similar manner one can obtain interpolation formulas for integrals with the classical weights. For example, for

$$F(x) := \int_{-x}^{x} \frac{f(t)}{\sqrt{x^2 - t^2}} \, dt, \quad 0 \leq x < \rho,$$

the even moments $s_{2m}$ equal $4^{-m} \pi \frac{(2m)!}{(m)!^2}, m = 0, n - 1$, and the odd moments equal zero. In this case $G_n(x) = (-2^{1-n} \pi)^n T_n(x)$, where $T_n(x) = \cos(n \arccos x)$ are the Chebyshev polynomials of the first kind. After calculating of the amplitudes $\mu_k$ on basis of the numbers $\lambda_k$ we get the known Chebyshev-Gauss quadrature [27, Ch. 10] for real $x$ such that $0 \leq x < \rho$:

$$\int_{-x}^{x} \frac{f(t)}{\sqrt{x^2 - t^2}} \, dt = \frac{\pi}{n} \sum_{k=1}^{n} f(\lambda_k x) + O(x^{2n}), \quad \lambda_k = \cos \frac{2k-1}{2n} \pi.$$

The characteristic property of this is the equality of the amplitudes $\mu_k = \pi/n$.

\textbf{2.3. \textit{Padé approximants.}} The \textit{Padé approximants} as well as the Gaussian quadratures are closely related with the classical moment problem (see, for instance, [16]). If for interpolation of a function $f$ analytic in a neighbourhood of the origin we choose $h(z) = (z - 1)^{-1}$ as a basis function in the amplitude and frequency operator $H_n(\{\mu_k\}, \{\lambda_k\}, h; z)$, then we get the sequence of moments $s_m = - f_m, \quad m = 0, 2n - 1$. If the generating polynomial (1.9) for the sequence has exactly $n$ pairwise roots, then Theorem 1 yields the following interpolation identity

$$(2.4) \quad f(z) = \sum_{k=1}^{n} \frac{\mu_k}{\lambda_k z - 1} + O(z^{2n}).$$

This is a classical \textit{Padé approximant} of order $[(n - 1)/n]$. The \textit{Padé approximants} of order $[m/n]$, i.e. interpolating rational functions of the form $P_n(f; z)/Q_n(f; z)$, and their numerous modifications were studied in many papers (see references in the well-known monograph [6]). Note that a method of solving the problem (1.7)
proposed by Ramanudjan [43] is quite equivalent to constructing of the interpolation formula (2.4), i.e. of a Padé approximant of order \([(n-1)/n]\) (see [38,39]).

2.4. Exponential sums. Let \(h(z) = \exp(z)\) be a basis function in the amplitude and frequency operator \(H_\nu(\{\mu_k\}, \{\lambda_k\}, h; z)\) and \(f\) be a function, which we are going to interpolate. The corresponding moments are \(s_m = mlf_m, m = 0, 2n-1\). Let the problem (1.7) for the moments be regular. Then the interpolation formula holds:

\[
f(z) = \sum_{k=1}^n \mu_k e^{\lambda_k z} + O(z^{2n}).
\]

(In particular, in such a form this is already contained in [10] and [39].)

Approximation of functions by the exponential sums was first considered by Prony [42]. At present numerous works are devoted to this method and its various applications (see, for example, [7–9, 25, 41, 50] and references therein). A vast investigation of the exponential series, launched very likely by Carleman (see [35]), was conducted in a scientific school of Leont’ev [31, 32]. It is also appropriate to mention here that members of the school actively studied several generalizations of the exponential series as well (see, for instance, [20, 21, 33, 48] and a comprehensive bibliography in [34]), which were first introduced by Gelfond [19]. Namely, they inquired into a question of the completeness of the infinite systems \(\{h(\lambda_k z)\}\), where \(h\) are entire functions and \(\lambda_k\) are given numbers. Thus, they actually considered representations of analytic functions \(f\) by the amplitude and frequency sums of infinite order \(H_\infty(\{\mu_k\}, \{\lambda_k\}, h; z)\) and properties of these representations (domain of convergence, admissible classes of the numbers \(\lambda_k\) and functions \(f\), computing methods for \(\mu_k\) on basis of given \(\lambda_k\), etc.). In contrast to this approach we consider approximations by the amplitude and frequency sums of finite order and respective errors, and the parameters \(\lambda_k\) and \(\mu_k\) are not given but uniquely determined by the functions \(f\) and \(h\). Moreover, in different applications we think of the amplitude and frequency sums as operators with fixed (universal) numbers \(\lambda_k\) and \(\mu_k\), which are determined by the analytic nature of these operators.

3. Regularization of the problem of interpolation by the amplitude and frequency operators

3.1. The main difficulty of the 2n-multiple interpolation problem emerges when the conditions of regularity (see Theorem 1) are not satisfied. In particular, in this case the problem can be insoluble. In order to avoid this difficulty, we propose a regularization method for the discrete moment problem, which consists in a certain variation of the right hand sides \(s_m\) of the system (1.7), namely, in adding to them of the generalized power sums \(\sigma \sum_{k=1}^\nu \alpha_k (r\beta_k)^m\), where the parameters \(\alpha_k, \beta_k\) are independent of \(s_m\), and \(\sigma\) and \(r\) depend only on \(\max(|s_m|)\). (Another approach is described below in Remark 1.) From the point of view of the interpolation problem it corresponds to that there appears a regularly solvable problem of the from (1.6), (1.7), where \(f\) is exchanged for a new varied function \(\tilde{f}\), differing from \(f\) by the amplitude and frequency sum \(\sigma H_\nu(\{\alpha_k\}, \{r\beta_k\}, h; z)\). We emphasize that in the framework of the interpolation problem the variation of \(s_m = s_m(h, f)\) is universal in the sense that it depends only on \(\max(|s_m|)\) but not on functional properties of \(f\) and \(h\). Moreover, as we will see below, an appropriate choice of \(\alpha_k, \beta_k, \sigma, r\) yields the difference of \(f\) and \(\tilde{f}\) only by a certain binomial. Let us describe the regularization method in more details. Instead of the function \(f\) to be interpolated, for which the problem (1.7) is not regular, we introduce the varied function

\[
\tilde{f}(z) := f(z) + \sigma H_\nu(\{\alpha_k\}, \{r\beta_k\}, h; z), \quad p \in \mathbb{C}, \quad \nu \in \mathbb{N},
\]
where $\alpha_k$, $\beta_k$, $\sigma$, $r$ are constants. Since

$$
\tilde{f}(z) = \sum_{m=0}^{\infty} f_m z^m + \sigma \sum_{k=1}^{\nu} \alpha_k \sum_{m=0}^{\infty} h_m (r \beta_k z)^m = \sum_{m=0}^{\infty} (s_m + \tau_m) h_m z^m,
$$

where $\tau_m := \sigma \sum_{k=1}^{\nu} \alpha_k (r \beta_k)^m$ and $s_m = s_m(h, f)$ as before (see (1.4)), instead of (1.7) we obtain the system

$$(3.2) \quad \sum_{k=1}^{n} \mu_k \lambda_k^m = s_m + \tau_m, \quad m = 0, 2n - 1,$$

which differs from the system (1.7) by adding of the regularizing summands $\tau_m$ (if the initial system is regular, then it is natural to suppose that $\sigma = \tau_m \equiv 0$). Now our question is to find the numbers $\tau_m$ such that the conditions of Theorem 1 are satisfied. Let us assume that we have done this, then by Theorem 1 we obtain the interpolation identity

$$
\tilde{f}(z) = H_n(\{\mu_k\}, \{\lambda_k\}, h; z) + O(z^{2n}).
$$

Returning to the initial interpolation problem and taking into account (3.1) yield

$$(3.3) \quad f(z) = H_n(\{\mu_k\}, \{\lambda_k\}, h; z) - \sigma H_v(\{\alpha_k\}, \{\beta_k\}, h; z) + O(z^{2n}).$$

At the same time it is reasonable to choose $\nu$ as small as possible. However, there is a natural restriction on $\nu$, which is seen from the following assertion (cf. §4 in [39]).

**Lemma 2.** It is necessary for the regularity of the varied system (3.2) that

$$
\nu \geq n - \text{rank}(s_{i+j-2})_{i,j=1}^{n}.
$$

**Proof.** Let us consider the two Hankel matrices

$$
\mathcal{H} := \left( s_{i+j-2} \right)_{i,j=1}^{n}, \quad \mathcal{R} := \sigma \left( \sum_{k=1}^{\nu} \alpha_k (r \beta_k)^{i+j-2} \right)_{i,j=1}^{n}.
$$

For the regularity of the system (3.2) it is necessary that the coefficient of $\lambda^0$ of the corresponding generating polynomial $G_n$ does not vanish, i.e. $\det(\mathcal{H} + \mathcal{R}) \neq 0$, $\text{rank}(\mathcal{H} + \mathcal{R}) = n$. It is well known [22, §0.4.5] that for any $n \times n$-matrices $A$ and $B$

$$(3.4) \quad \text{rank}(A + B) \leq \min\{\text{rank} A + \text{rank} B; n\},$$

hence if the system (3.2) is regular, then $n \leq \text{rank} \mathcal{H} + \text{rank} \mathcal{R}$. It remains to note that $\text{rank} \mathcal{R} \leq \nu$. Indeed, the following representation is valid:

$$
\mathcal{R} = \sigma \sum_{k=1}^{\nu} \alpha_k (r \beta_k)^{i+j-2} \left( s_{i+j-2} \right)_{i,j=1}^{n} = \sum_{k=1}^{\nu} \alpha_k C(k), \quad C(k) := \sigma (r \beta_k)^{i+j-2} \left( s_{i+j-2} \right)_{i,j=1}^{n},
$$

where $\text{rank} C(k) = 1$ (each next row is resulted from the previous one by multiplying by $r \beta_k$). From this by the inequality (3.4) we obtain the bound for $\text{rank} \mathcal{R}$. \hfill \blacksquare

**3.2.** Let us examine the case $\nu = n$. Thus the formula (3.3) has $2n$ summands (if $\sigma \neq 0$, $r \neq 0$), and in this sense the amplitude and frequency sums generally speaking are not at an advantage over the $h$-sums of order $2n$. However, an appropriate choice of $\alpha_k$, $\beta_k$, $\sigma$, $r$ can essentially simplify the latter sum in (3.3). Indeed, let $p$ and $q$ be non-zero complex numbers and let for $k = 1, n$

$$(3.5) \quad \alpha_k = \beta_k = \exp \left( \frac{2\pi (k-1)i}{n} \right), \quad r = \left( \frac{q}{p} \right)^{1/n}, \quad \sigma = \frac{p^2}{nq}.$$ 

The number $r$ is any of $n$ values of the root. Then, as it can be easily seen, we obtain in (3.2):

$$(3.6) \quad \tau_{n-1} = p, \quad \tau_{2n-1} = q, \quad \tau_m = 0 \quad \text{for other} \quad m.$$
In fact
\[ \tau_m = \frac{p^2}{nq} \sum_{k=1}^{n} \exp \left( \frac{2\pi(k-1)i}{m+1} \right), \]
where the sum of exponents equals \( n \) or zero if \( m+1 \) is divisible by \( n \) or not. Consequently,
\begin{equation}
\sigma H_n(\{\alpha_k\}, \{r\beta_k\}, h; z) = p h_{n_1} z_{n_1} + q h_{2n_1} z_{2n_1} + O(z^{2n}).
\end{equation}
Thus, the regularity of the varied problem \( (3.2) \) yields the validity of the interpolation formula
\begin{equation}
f(z) = H_n(\{\mu_k\}, \{\lambda_k\}, h; z) - p h_{n_1} z_{n_1} - q h_{2n_1} z_{2n_1} + O(z^{2n}).
\end{equation}
In order to obtain a main result in this direction, we show that the above-mentioned problem is indeed regular for a certain choice of the parameters \( p \) and \( q \). We now give a possible way of such a choice. For \( \alpha_k \) and \( \beta_k \) from \( (3.5) \) the generating polynomial of the system \( (3.2) \) has the form
\begin{equation}
G_n(\lambda) = G_n(p, q; \lambda) := \begin{vmatrix}
1 & \lambda & \cdots & \lambda^{n-1} & \lambda^n \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & s_1 & \cdots & s_{n-1} + p & s_n \\
1 & s_2 & \cdots & s_n & s_{n+1} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
1 & s_{n-1} + p & s_n & \cdots & s_{2n-2} & s_{2n-1} + q \\
\end{vmatrix}.
\end{equation}
For the parameters \( p \) and \( q \), being sufficiently large in modulus (comparing with the moments \( s_k \) and independently of each other), this polynomial obviously has roots arbitrarily close to those of the polynomial for \( s_k = 0, k = 0, 2n_1 - 1 \). This latter polynomial, as it can be easily checked by expanding the determinant along the first row, is of the form
\[- (-1)^{n(n+1)/2} p^n (\lambda^n - q/p),\]
and all its \( n \) roots are pairwise distinct. From here, the formula \( (3.8) \) and Theorem 1 we obtain the following result.

**Theorem 2.** For \( p \) and \( q \) sufficiently large in modulus the varied problem \( (3.2) \) with the parameters \( (3.6) \) has a regular solution \( \{\mu_k\}, \{\lambda_k\} \). Moreover, for the constants \( c_1 = -ph_{n_1} \) and \( c_2 = -qh_{2n_1} \) the following interpolation formula holds:
\[ f(z) = c_1 z^{n_1} + c_2 z^{2n_1} + \sum_{k=1}^{n} \mu_k h(\lambda_k z) + O(z^{2n}). \]

**Remark 1.** The regularization described with the parameters from \( (3.5) \) and \( (3.6) \) is actually equivalent to adding of the special binomial \( c_1 z^{n_1} + c_2 z^{2n_1} \) with non-vanishing coefficients \( c_{1,2} \) to the function \( f \). In several cases in what follows we will expand the class of regularizable problems by choosing \( c_1 \) and \( c_2 \) from some other considerations and not necessarily non-vanishing. In particular, in the extrapolation problem, which will be discussed below, it will be reasonable to put \( c_2 = 0 \).

**Remark 2.** The conditions on \( p \) and \( q \) mentioned in Theorem 2 are quite qualitative and need additional specification in practice. Several methods to solve this problem will be proposed below in particular applications. In a general case one can be guided by the following considerations. The leading coefficient \( g_n = g_n(p) \) of the polynomial \( G_n \) is obviously a polynomial of \( p \) of degree \( n \), hence \( \deg G_n(p, q; \lambda) = n \) for all \( p \) except ones from the set
\begin{equation}
\Pi := \{ p : g_n(p) = 0 \},
\end{equation}
containing no more than \( n \) points. It is possible to obtain some estimates for boundaries of the set \( \Pi \), using the known property of the non-singularity of a matrix with
strict diagonal dominance (see the Levy–Desplanque theorem in [22, Th. 6.1.10]).
Namely, if we chose \( p \) satisfying the inequality \(|s_{n-1} + p| > \sum_{j=1,j \neq n-1}^{n} |s_{n-1} - j|\), then for any \( i \in \mathbb{N} \), the determinant for \( g_n \) is strict diagonally dominant and hence \( g_n \neq 0 \).

For this it is sufficient, for example, to take

\[
p > n \max_{k=0,2n-1} |s_k|.
\]

We now suppose that the generating polynomial (3.9) is exactly of degree \( n \). Then the question about “separation” of its multiple roots arises. As it is easily seen,

\[
G_n(p, q; \lambda) = S(p; \lambda) + qT(p; \lambda),
\]

where the polynomial \( S \) is of order \( n \) and the polynomial \( T \) is of degree \( \leq n - 1 \).
Both polynomials depend only on \( p \). We now prove the following assertion.

**Lemma 3.** Suppose that in each multiple root (if any) of the polynomial \( G_n \) the polynomial \( T \) either does not vanish or has simple root. Then there exists an arbitrarily small variation \( \delta \neq 0 \) of the parameter \( q \) such that the polynomial \( G_n(p, q + \delta; \lambda) \) has \( n \) simple roots.

**Proof.** Let us assume that \( \lambda_0 \) is an \( s \)-multiple (\( s \geq 2 \)) root of the polynomial \( G_n(p, q; \lambda) \). Then in a sufficiently small neighbourhood of the root the polynomial \( G_n(p, q + \delta; \lambda) = G_n(p, q; \lambda) + \delta T(p; \lambda) \) has the form

\[
G_n(p, q + \delta; \lambda) = (\lambda - \lambda_0)s(\alpha + O(\lambda - \lambda_0) + \delta(t_0 + t_1(\lambda - \lambda_0) + O((\lambda - \lambda_0)^2)),
\]

where \( \alpha \neq 0, |t_0| + |t_1| \neq 0 \) and values \( O(\lambda - \lambda_0), O((\lambda - \lambda_0)^2), \lambda - \lambda_0 \), are independent of \( \delta \). We choose small \( \varepsilon > 0 \) and \( \delta = \delta(\varepsilon) \) in such a way that in the disc \(|\lambda - \lambda_0| < 2\varepsilon\) the polynomial \( G_n \) has no roots, distinct from \( \lambda_0 \) (we take into account the continuous dependence of the roots on the parameter \( \delta \)), and

\[
[G_n(p, q; \lambda)] > |\delta T(p; \lambda)|, \quad |\lambda - \lambda_0| = \varepsilon.
\]

By Rouche’s theorem the polynomial \( G_n(p, q + \delta; \lambda) \) has exactly \( s \) roots in the disc \(|\lambda - \lambda_0| < \varepsilon\); from now on we denote them by \( \hat{\lambda}_k \). If \( t_0 \neq 0 \), then these roots satisfy the equation

\[
(\lambda - \lambda_0)^s = -\frac{\delta t_0}{\alpha}(1 + O(\varepsilon)), \quad \varepsilon \to 0.
\]

If \( t_0 = 0, t_1 \neq 0 \), then \( \hat{\lambda}_1 = \lambda_0 \) and other roots satisfy the equation

\[
(\lambda - \lambda_0)^{s-1} = -\frac{\delta t_1}{\alpha}(1 + O(\varepsilon)), \quad \varepsilon \to 0.
\]

In any case we get \( s \) simple roots. Suppose now that \( \varepsilon \) and \( |\delta| \) are sufficiently small in order that the method works simultaneously for all multiple roots, but all simple roots remain simple (it is possible due to the continuous dependence of the roots on \( \delta \)). Then we get the polynomial \( G_n(p, q + \delta; \lambda) \) with \( n \) simple roots.

It follows from the above that the following conjecture is very likely: a set of the parameters \( (p, q) \) such that the interpolation problem under consideration is regularly solvable is everywhere dense in \( \mathbb{C}^2 \). But now we can only formulate the following statement, which is a supplement to Theorem 2.

**Theorem 3.** Let \( p \notin \Pi \) (see (3.10)) and the conditions of Lemma 3 be satisfied.

Then there exists an arbitrarily small variation \( \delta \neq 0 \) of the parameter \( q \) such that the varied problem (3.2) with \( \tau_{n-1} = p, \tau_{2n-1} = q + \delta \) (all other \( \tau_m = 0 \)) has a regular solution \( \{\mu_k\}, \{\lambda_k\} \), and for constants \( c_1 = -ph_{n-1} \) and \( c_2 = -(q + \delta)h_{2n-1} \) the following interpolation formula holds:

\[
f(z) = c_1z^{n-1} + c_2z^{2n-1} + \sum_{k=1}^{n} \mu_k h(\lambda_k z) + O(z^{2n}).
\]
4. Numerical Differentiation by the Amplitude and Frequency Operators

4.1. As an application of the regularization method we consider the problem of $2n$-multiple interpolation of the function $zf'(z)$ by the amplitude and frequency operators $H_n$ with a basis function $f$. (As before we suppose that $f$ is defined and holomorphic in a neighbourhood of the origin.) Solution to the problem could allow to obtain a high-accuracy formula of numerical differentiation with local accuracy $O(z^{2n})$. However, in this case the discrete moment problem (1.7) with $s_m = m$, $m = 0, 2n - 1$, is non-regular (the generating polynomial (1.9) is of degree less than $n$ for $n = 1$ and $n \geq 3$ since the algebraic adjunct to $\lambda^n$ obviously vanishes, and has a double root $\lambda = 1$ for $n = 2$; both cases do not satisfy Theorem 1). We now apply the above-mentioned regularization method of adding a certain binomial (see Remark 1). More precisely, for some complex parameters $p$ and $q$ we consider the varied function

\[ \tilde{f}(z) := zf'(z) + pf_{n-1}z^{n-1} + qf_{2n-1}z^{2n-1}, \quad zf'(z) = \sum_{m=0}^{\infty} m_f m z^m. \]

From here we get a set of varied moments (see (1.4)) to determine the interpolating sum $H_n([\mu_k], [\lambda_k], f; z)$:

\[ s_m = m, \quad m \neq n - 1, 2n - 1; \quad s_{n-1} = n - 1 + p, \quad s_{2n-1} = 2n - 1 + q, \]

which are independent of $f$. Hence

\[ \hat{G}_n(\lambda) := \sum_{m=0}^{n} \hat{g}_m \lambda^m = \begin{vmatrix} 1 & \lambda & \ldots & \lambda^{n-1} & \lambda^n \\ 0 & 1 & \ldots & 2 & n \\ 1 & 2 & \ldots & n & n + 1 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ n - 1 + p & n & \ldots & 2n - 2 & 2n - 1 + q \end{vmatrix}. \]

If for some $p$ and $q$ the generating polynomial $\hat{G}_n(\lambda)$ has exactly $n$ pairwise distinct roots $\lambda_1, \ldots, \lambda_n$, then by Theorem 1 the varied interpolation problem for a derivative becomes regular and

\[ zf'(z) = H_n([\mu_k], [\lambda_k], f; z) - pf_{n-1}z^{n-1} - qf_{2n-1}z^{2n-1} + O(z^{2n}), \]

where $\mu_k$ are calculated on basis of the moments (4.2) from (1.7) and using formulas from Lemma 1.

4.2. In the case under consideration the coefficients $\hat{g}_m$ can be found explicitly.

Lemma 4. Let $\kappa := (-1)^{n(n+1)/2}p^{n-3}$. Then for $n \geq 1$ coefficients of the polynomial (4.3) have the form

\[ \hat{g}_n = \kappa p \left( p^2 + n(n-1)p + \frac{n^2(n^2-1)}{12} \right), \]

\[ \hat{g}_0 = -\kappa \left( p^2 q + (2n - 1)p^2 + (n - 1)^2p q - \frac{n(n-1)}{6} p + \frac{(n-2)n(n-1)^2}{12} q \right), \]

\[ \hat{g}_m = -\kappa \left( (2n - (m+1))^2 p^2 - (n - (m + 1))^2 p q - \frac{n(n+1)}{6} \left( \frac{2(n+1)}{3} - (m + 1) \right) q + \frac{n(n-1)}{2} \left( \frac{2(n+1)}{3} - (m + 1) \right) q \right), \quad m = 1, n - 1. \]

Proof. The identities for $n = 1, 2$ are checked directly. From now on, $n \geq 3$.

Let us first prove the identity for $\hat{g}_n$ by a direct calculation of the algebraic adjunct $(-1)^n D$ to $\lambda^n$ in the determinant (4.3).
We now show that the characteristic polynomial $P_n(\lambda) = \det(A - \lambda I)$ of the matrix

$$A := \begin{pmatrix}
n - 1 & n - 2 & n - 3 & \ldots & 0 \\
1 & n - 1 & n - 2 & \ldots & 1 \\
0 & 0 & 1 & \ldots & \ldots \\
0 & 0 & 0 & \ldots & 1
\end{pmatrix}$$

has the form

$$P_n(\lambda) = (-1)^n \lambda^{n-2} \left( \lambda^2 - n(n - 1)\lambda + \frac{n^2(n^2 - 1)}{12} \right).$$

It is known that for any matrix $B$

$$\det(B - \lambda I) = (-1)^n (\lambda^n - b_1 \lambda^{n-1} + b_2 \lambda^{n-2} + \ldots + (-1)^n b_n),$$

where $b_j$ is a sum of all $j$-rowed diagonal minors of the matrix $B$ (see, for instance, [23, §3.10]). In particular, in terms of traces of the matrices $B$ and $B^2$ we have

$$b_1 = \text{Tr}B, \quad b_2 = \frac{1}{2} (\text{Tr}B)^2 - \text{Tr}B^2.$$

For our matrix $A$ all minors of size greater than 2 are zero since subtracting a row from any other row gives a constant row, consequently, rank $A = 2$. Therefore, coefficients of terms with powers less than $n-2$ in det $(A - \lambda I)$ are zero. Furthermore, obviously $\text{Tr}A = n(n-1)$. We now consider the coefficient of $\lambda^{n-2}$. It is easily seen (by direct multiplying the $k$th row with the $k$th column of the matrix $A$) that

$$\text{Tr}A^2 = \sum_{k=1}^{n} \sum_{m=0}^{n-1} ((n-1)^2 - (m - k + 1)^2) = \frac{1}{6} n^2 (n-1) (5n - 7).$$

It follows that

$$\frac{1}{2} (\text{Tr}A^2 - \text{Tr}A^2) = \frac{1}{4} \left( n^2(n-1)^2 - \frac{1}{6} n^2 (n-1) (5n - 7) \right) = \frac{n^2(n^2-1)}{12}.$$

This completes the proof of the formula (4.5).

Let us return to the determinant $D$. Its matrix is mirror symmetric with respect to $A$ (i.e. its columns are placed in a reversed order) and can be obtained by a right multiplication of $A$ with the anti-diagonal identity matrix. As is known, the determinant of the $n \times n$ anti-diagonal identity matrix is equal to $(-1)^{n(n-1)/2}$.

Hence

$$D = (-1)^{\frac{n(n-1)}{2}} \det(A + pI) = (-1)^{\frac{n(n-1)}{2}} P_n(-p).$$

This and (4.5) yield the desired formula for $\tilde{g}_n = (-1)^n D$, $n \geq 3$.

Suppose that

$$\hat{\Pi} := \left\{ 0; \frac{1}{2} \left( 1 - n + \sqrt{\frac{2}{3} (n-1)(n-3)} \right), \frac{1}{2} \left( 1 - n - \sqrt{\frac{2}{3} (n-1)(n-3)} \right) \right\}.$$

Note that if $p \notin \hat{\Pi}$, then $\tilde{g}_n \neq 0$. Let $p \notin \hat{\Pi}$. If $\tilde{g}_n$ are supposed to be known, then we determine the desired identities for other $n$ coefficients from the following system of $n$ linear equations

$$\sum_{m=0}^{n} s_{v-m} \tilde{g}_{n-m} = 0, \quad v = \frac{n}{2n - 1}.$$

These equations for every $v$ are resulted from the sum of multiplications algebraic adjuncts to elements of the first row of the determinant (4.3) (generally, of the determinant (1.9)) with corresponding elements of the $(v + 2)$th row (see also [18, 38, 39, 44]). The linear system (4.10) with unknowns $\tilde{g}_0, \ldots, \tilde{g}_{n-1}$ has a non-singular matrix (its determinant equals the coefficient $\tilde{g}_n \neq 0$, $p \notin \hat{\Pi}$), hence has a unique solution.
Therefore in order to complete the proof of Lemma 4 it is sufficient to verify (4.10) by direct substituting of moments (4.2) and coefficients given in Lemma 4. This verifying is quite easy and comes to calculation of the two sums \( \sum_{m=0}^{n} m^{\nu} \), \( \nu = 1, 2 \), so we do not dwell on it.

Finally, let \( p \in \Pi \). The case \( p = 0 \) is out of interest since then we have \( \hat{G}_n(\lambda) \equiv 0 \) (see (4.3) for \( n \geq 3 \)). In the case \( p \in \Pi \setminus \{0\} \) the system (4.10) is homogeneous and has a singular matrix. Therefore it has infinitely many non-zero solutions, one of which is given in Lemma 4 (it follows from the continuous dependence of the coefficients \( \hat{g}_m \) on \( p \)).

4.3. The following lemma about factorization of the coefficients of the polynomial \( \hat{G}_n \) is fundamental since it allows to use Theorem 1 in the problem under consideration.

**Lemma 5.** Let \( n \geq 3 \), \( p \notin \Pi \) (see (4.9)) and

\[
q = q_0(p) := -2 \frac{p(3p + n^2 - 1)}{(n-1)(n-2)}.
\]

Then the ratios \( \hat{g}_m/\hat{g}_n \) for \( m = 1, n \) are independent of \( p \) and \( \hat{g}_0/\hat{g}_n \) depend on \( p \) linearly. More precisely, the generating polynomial \( \hat{G}_n(\lambda) \) has the form

\[
\hat{g}_n \left( \lambda^n - \frac{6\lambda(\lambda^{n-1} - (n-1)\lambda + n - 2)}{(n-1)(n-2)(\lambda-1)^2} + 2 + \frac{6p}{(n-1)(n-2)} \right).
\]

There is an arbitrarily small variation of the parameter \( p \) such that all roots of the polynomial \( \hat{G}_n(\lambda) \) are pairwise distinct.

**Proof.** By Lemma 4 solving of the equation \( \hat{g}_{n-1} = 0 \), being linear with respect to \( q \), yields (4.11). Substituting of \( q \) obtained into other coefficients gives

\[
\hat{g}_0 = \hat{g}_n \left( 2 + \frac{6p}{(n-1)(n-2)} \right),
\hat{g}_m = -6 \hat{g}_n \frac{n-1-m}{(n-1)(n-2)},
\]

where \( \hat{g}_n = (-1)^{n(n+1)/2} \frac{p^{n-2}}{12} \left( p^2 + n(n-1)p + \frac{n(n-1)}{12} \right) \neq 0 \) as \( p \notin \Pi \). Hence

\[
\hat{G}_n(\lambda) = \hat{g}_n \left( \lambda^n - \frac{6}{(n-1)(n-2)} \sum_{m=1}^{n-1} (n-m-1)\lambda^m + 2 + \frac{6p}{(n-1)(n-2)} \right),
\]

which yields (4.12) after calculation of the sum. The conclusion of Lemma about the simplicity of the roots follows immediately from Lemma 3.

4.4. It is known \([18,38,39,44]\) that the identities (4.10) are valid for all \( v \geq n \) (they are called the generalized Newton’s formulas), i.e.

\[
s_v = -\frac{1}{\hat{g}_n} \sum_{m=0}^{n-1} s_{v-n+m}\hat{g}_m, \quad v = n, n+1, \ldots.
\]

After substituting the moments (4.2) and coefficients from Lemma 5 into (4.13), direct calculations give the expression of the 20th moment:

\[
s_{2n} = 2n - C_n(p), \quad C_n(p) := \frac{6np}{(n-1)(n-2)}.
\]

Therefore, for the function \( \tilde{f} \) as in (4.1) the local interpolation error \( r_n(z) \), which was denoted in (4.4) just as \( O(z^{2n}) \), has the following more precise form

\[
\tilde{f}(z) - \sum_{k=1}^{n} \mu_k f(\lambda_k z) = \sum_{m=n}^{\infty} (m-s_m)f_m z^m = C_n(p)f_{2n}z^{2n} + O(z^{2n+1}).
\]

From the foregoing, we get the following assertion.
Theorem 4. For \( n \geq 3, \) any \( p_0 \in \mathbb{C} \) and arbitrarily small \( \varepsilon > 0 \) there exists a value of the parameter \( p, |p - p_0| \leq \varepsilon, \ p \notin \Pi \) (see (4.9)), such that
\[
(4.15) \quad z f'(z) = \sum_{k=1}^{n} \mu_k f(\lambda_k z) - pf_{n-1}z^{n-1} - qf_{2n-1}z^{2n-1} + r_n(z),
\]
where \( r_n(z) = O(z^{2n}) \) and has the form (4.14), \( q = q_0(p) \) (see (4.11)), the frequencies \( \lambda_k \) are pairwise different roots of the polynomial (4.12) and amplitudes \( \mu_k \) are determined uniquely by Lemma 1. Moreover, \( \mu_k = \mu_k(p, n) \) and \( \lambda_k = \lambda_k(p, n) \) do not depend on the form of the analytic function \( f \) and are universal in this sense.

The interpolation formula is precise on polynomials of order \( \leq 2n - 1, \) i.e. \( r_n(z) \equiv 0 \) for the polynomials \( f \) with \( \deg f \leq 2n - 1 \) in (4.15).

Note that the method, which we consider in this section, can be easily extended to interpolation of the functions \( z^\nu f^{(\nu)}(z), \) \( \nu \geq 2, \) hence we can obtain numerical formulas for differentiation of higher order.

4.5. We now make a few observations on a practical use of Theorem 4. The remainder in the interpolation formula for numerical differentiation (4.15) has quite high infinitesimal order, \( O(z^{2n}) \), and it is achieved by knowing only \( n \) values of \( f \) and the two fixed values of its derivatives (at \( z = 0 \)). A traditional interpolation approach with such a number of known values allows to get, generally speaking, only the order \( O(z^{n+2}) \). In other words, the formula (4.15) is precise on polynomials of degree \( \leq 2n - 1, \) whereas usual \((n + 2)\)-point interpolation formulas are precise only on polynomials of degree \( \leq n + 1 \).

Another important feature of the formula (4.15) consists in that the variable interpolation nodes \( \lambda_k z \) depend only on \( z, \) at which we calculate \( z f'(z) \), and are independent of \( f \) (in this sense as it was already mentioned above the frequencies \( \lambda_k = \lambda_k(p, n) \), generating the nodes, are universal for the whole class of functions \( f \) analytic in a neighbourhood of the origin).

It is seen from the formula (4.15) that its precision fundamentally depends on the exactness of the values \( f_\nu = f^{(\nu)}(0)/\nu! \), \( \nu = n - 1, 2n - 1 \) (of course we assume that values of the function \( f \) are known). In several particular cases this difficulty could be overcome. For instance, if it is known a priori that the function \( f \) is even (odd), then there is no necessity in calculation of \( f_{n-1} \) and \( f_{2n-1} \) for even (odd) \( n \) since then \( pf_{n-1}z^{n-1} + qf_{2n-1}z^{2n-1} \equiv 0 \) (\( pf_{n-1}z^{n-1} \equiv 0 \)) and then the error is of local order \( O(z^{2n}) \) (\( O(z^{2n-1}) \)). In the more general case, when \( f \) is even or odd, for even \( n \) the formula (4.15) can be applied to the even auxiliary function \( \omega(z) = f^2(z) \). Then the corresponding coefficients \( \omega_{n-1} = \omega_{2n-1} = 0 \) and
\[
2zf(z)f'(z) = \sum_{k=1}^{n} \mu_k f^2(\lambda_k z) + O(z^{2n}).
\]
In the most general case, for a systematic use of (4.15) for each fixed function \( f \) it is required previously to calculate the regularizing binomial \( pf_{n-1}z^{n-1} + qf_{2n-1}z^{2n-1} \). To do so any known formula for numerical differentiation of an analytic function at \( z = 0 \) can be used. For example, with the help of the finite differences of order \( \nu \) one gets
\[
f_\nu = \frac{(-1)^\nu}{\nu!z^\nu} \sum_{j=0}^{\nu} (-1)^j C_\nu^j f(zj) + O(z).
\]
There are many other formulas of interpolation type for multiple numerical differentiation, which exploit only values of functions (i.e. without calculation of any regularizing binomials). We now cite only several results close in form to (4.15).
In [4, 5, 51], the \( n \)-point formulas for numerical differentiation of real functions are of the form

\[
(4.16) \quad f^\nu = \frac{1}{x^\nu} \sum_{k=1}^n \mu_k f(\lambda_k x) + O(x^{n-\nu}), \quad \nu = 1, 2, \quad n \geq \nu + 1,
\]

where real nodes \( \lambda_k x, |\lambda_k - \lambda_j| > 1 \), minimize the generalized power sums \( s_\nu = \sum_{k=1}^n \mu_k \lambda_k^\nu \) \( (\nu \geq n + 1) \), and this corresponds to the minimization of the remainder. In [45] interpolation formulas of Lagrange type for numerical differentiation were constructed on basis of special non-uniformly distributed nodes, which also minimize the remainder. Formulas for numerical differentiation of analytic functions were obtained in [36, 37] by contour integrals. There were constructed formulas of the type similar to (4.16) and was shown that if \( \lambda_k = \exp(2\pi i k/n), \quad k = 1, n \), then under an appropriate choice of \( \mu_k \) their formula would be precise on polynomials of degree \( \leq n + \nu - 1 \). Apart of (1.2) given above, the following formula is obtained in [13] with the help of the \( f \)-sums:

\[
f^\nu = \frac{1}{x^\nu} \sum_{k=1}^{N} \lambda_k f(\lambda_k z) + O(z^{n-\nu}), \quad N = \left\lfloor \frac{n}{\nu + 1} \right\rfloor, \quad n > \nu + 6,
\]

where the numbers \( \lambda_k \) do not depend on \( f \) and are non-zero roots of the polynomial \( P_n(\lambda) = \sum_{k=0}^N (-1)^k \lambda^{n-k} / (\nu + 1)k! \) (in [13] estimates for the remainder are also given). Other results of this type were obtained in [11, 17].

An important role in calculations by the formula (4.15) is played by absolute values of the amplitudes \( \mu_k \) and frequencies \( \lambda_k \). We now get some estimates for the frequencies.

**Lemma 6.** For the roots \( \lambda_k \) of the polynomial (4.12) we have

\[
|\lambda_k| \leq 1 + \frac{O(1)}{\sqrt{n}}, \quad O(1) > 0, \quad n \to \infty.
\]

More precisely, for \( n \geq 3 \)

\[
|\lambda_k| \leq \Lambda := (2\delta)^{3 \sqrt{n-2}}, \quad \delta := 1 + \frac{3|p|}{(n-1)(n-2)}, \quad p \notin \Pi.
\]

**Proof.** Let us estimate absolute value of the sum of the last three terms in the brackets in (4.12):

\[
V := |\frac{-6\lambda (\lambda^{n-1} - (n-1)\lambda + n - 2)}{(n-1)(n-2)(\lambda - 1)^2} + 2 + \frac{6p}{(n-1)(n-2)}| \leq |\lambda|^{n} \left( \frac{6(1 + (n-1)|\lambda|^{2-n} + (n-2)|\lambda|^{1-n})}{(n-1)(n-2)(|\lambda| - 1)^2} + \frac{2\delta}{|\lambda|^n} \right).
\]

It is easily seen that \( (2\delta)^{\sqrt{n-2}} - 1 \geq \frac{3\ln 2}{\sqrt{n-2}} \) for \( \delta > 1 \). Therefore substituting of \( |\lambda| = \Lambda \) into the latter expression yields

\[
V \leq |\lambda|^{n} \left( \frac{6(1 + (2n - 3)/(2\delta)^3)^{\sqrt{n-2}}}{(3\ln 2)^3(n-1)} + \frac{1}{(2\delta)^{\sqrt{n-2}}-1} \right).
\]

It is also easy to check that for \( n \geq 3 \) and \( \delta > 1 \) the expression in the brackets is less than one, hence \( V < \Lambda^6 \) for above-mentioned \( |\lambda| = \Lambda \). From here by Rouche’s theorem it follows that all roots of the polynomial (4.12) lie in the disc \( |\lambda| < \Lambda \). ■

Lemmas 1 and 6 can be used to obtain estimates for the amplitudes \( \mu_k \) as well, however, this problem needs more trenchant analysis and we do not dwell on it in the present paper.
5.

Numerical Extrapolation by the Amplitude and Frequency Operators

5.1. Let us briefly describe the idea of extrapolation. Let \( a > 0, \ p, q \in \mathbb{R}, \ f(z) \) be a function analytic in the disc \( |z| < \rho, \ \rho > 0. \) Consider the auxiliary problem of multiple interpolation of the function \( f(az) \) in a neighbourhood of \( z = 0 \) by the amplitude and frequency operator \( H_n((\mu_k), (\lambda_k), f; z), \) i.e., choosing \( f \) as a basis function. As in the case of differentiation, we get a non-regular discrete moment problem. To regularize it, we introduce the varied function

\[
(5.1) \quad \tilde{f}(z) := f(az) + pf_{n-1}z^{n-1} + qf_{2n-1}z^{2n-1}, \quad f(z) := \sum_{m=0}^{\infty} f_mz^m,
\]

with some parameters \( p \) and \( q, \) which are non-zero simultaneously. By the same approach that we used at the beginning of Section 3, in order to construct the interpolating sum \( H_n((\mu_k), (\lambda_k), f; z) \) we find the sequence of varied moments:

\[
(5.2) \quad s_k = a^k, \ k \neq n-1, 2n-1, \ k \in \mathbb{N}; \quad s_{n-1} = a^{n-1} + p, \quad s_{2n-1} = a^{2n-1} + q,
\]

and then construct the generating polynomial

\[
(5.3) \quad \mathcal{G}_n(\lambda) := \sum_{m=0}^{n} \hat{g}_m \lambda^m = \begin{vmatrix} 1 & \lambda & \ldots & \lambda^{n-1} & \lambda^n \\ 1 & a & \ldots & a^{n-1} + p & a^n \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ a^{n-1} + p & a^n & \ldots & a^{2n-2} & a^{2n-1} + q \end{vmatrix}.
\]

If for some \( a > 0, \ p, q \) the polynomial \( \mathcal{G}_n(\lambda) \) has pairwise distinct roots \( \lambda_1, \ldots, \lambda_n, \) then by Theorem 1 the varied problem for the function (5.1) turns out to be regularly solvable, so the following interpolation formula holds

\[
(5.4) \quad f(az) = H_n((\mu_k), (\lambda_k), f; z) - pf_{n-1}z^{n-1} - qf_{2n-1}z^{2n-1} + O(z^{2n}).
\]

(Of course, we assume that all arguments of the function \( f \) lie in the disc \( |z| < \rho \) of its analyticity.) Suppose also that the inequalities \( |\lambda_k| < \delta a \) with some \( \delta \in (0, 1) \) are valid for all \( k = 1, n. \) Then it is natural to call the formula (5.4) extrapolational since values of the function \( f \) at the points \( \zeta = az \) are approximated by values of this function at the points \( \lambda_k z, \) which lie in the disc \( \{ \zeta : |\zeta| < \delta |\zeta| \}, \) \( \delta |\zeta| < |\zeta|. \) In the present section we will obtain such an extrapolation formula and quantitative estimate for its remainder.

Let us begin a formal description. As in Section 4, we first analyze the coefficients and roots of the polynomial \( \mathcal{G}_n(\lambda). \)

5.2. The following assertion gives an explicit form of the coefficients \( \hat{g}_m. \)

**Lemma 7.** Let \( \kappa := (-1)^{n(n+1)/2}p^{n-2}, \ p \neq 0, \ -na^{n-1}. \) For \( \mathcal{G}_n \) we have

\[
(5.5) \quad \hat{g}_n = \kappa p (na^{n-1} + p), \quad \hat{g}_0 = -\kappa (a^{2n-1}p + (n-1)a^{n-1}q + pq), \quad \hat{g}_m = -\kappa a^{n-1-m} (a^n p - q), \quad m = 1, n - 1.
\]

**Proof.** A method of proof is the same as in Lemma 4. Let us first prove the identity for \( \hat{g}_n \) by a direct computation of the algebraic adjunct \((−1)^n D\) to the element \( \lambda^n \) in the determinant (5.3). For this we show that for the matrix

\[
A := \begin{pmatrix} a^{n-1} & a^{n-2} & a^{n-3} & \ldots & 1 \\ a^n & a^{n-1} & a^{n-2} & \ldots & a \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ a^{2n-2} & a^{2n-3} & a^{2n-4} & \ldots & \lambda 
\end{pmatrix}
\]

the characteristic polynomial \( P_n(\lambda) = \text{det}(A - \lambda I) \) has the form

\[
(5.6) \quad P_n(\lambda) = (-1)^n \lambda^{n-1} (\lambda - na^{n-1}).
\]
Indeed, in this case rank $A = 1$ since any two rows of $A$ are proportional. Therefore, by (4.6) and (4.7) coefficients of terms with powers less than $n - 1$ are zero in $\det(A - \lambda I)$. To prove (5.6), it remains to note that $\text{Tr}A = na^{n-1}$.

Now we return to the determinant $D$. In the same way as in Lemma 4, (4.8) and (5.6) yield the desired formula for $\hat{g}_n$.

If $\hat{g}_n$ are supposed to be known, then other $n$ coefficients as above in the case of differentiation can be found from the system (4.10) of $n$ linear equations (with exchange of $\hat{\cdot}$ by $\cdot$). This system with respect to unknowns $\hat{g}_0, \ldots, \hat{g}_{n-1}$ has a unique solution for $p \neq 0$, $-na^{n-1}$, since its determinant is equal to $\hat{g}_n \neq 0$. Thus, it is only sufficient to check (4.10) by a direct substitution of values (5.2) and (5.5), which is quite easy.

5.3. Everywhere below we suppose that $q = 0$; then, as we will see below, $\mathcal{G}_n$ has exactly $n$ pairwise distinct roots and its coefficients can be calculated by quite simple formulas.

**Lemma 8.** Let $p > 0$ and $q = 0$, $a > 0$. Then

\begin{equation}
\hat{g}_n = kp (na^{n-1} + p) \neq 0, \quad \hat{g}_m = -kpa^{2n-1-m}, \quad m = 0, n-1,
\end{equation}

and the generating polynomial has the form

\begin{equation}
\mathcal{G}_n(\lambda) = \hat{g}_n \left( \lambda^n - \frac{a^{2n-1}}{na^{n-1} + p} \sum_{m=0}^{n-1} a^m \right) = \hat{g}_n \left( \lambda^n - \frac{a^n}{na^{n-1} + p} \lambda^n - a^n \right).
\end{equation}

Moreover, $\mathcal{G}_n$ has exactly $n$ pairwise distinct roots.

**Proof.** The representation (5.8) is obtained by a direct substitution of $q = 0$ for $p > 0$ into the formulas (5.5) from Lemma 7. It remains to show that the polynomial (5.8) has no multiple roots. Let us write $\mathcal{G}_n(\lambda)$ in the form

\begin{equation}
\mathcal{G}_n(\lambda) = \hat{g}_n \frac{P_{n+1}(\lambda)}{\lambda - a}, \quad P_{n+1}(\lambda) := \lambda^{n+1} - a \left( 1 + \frac{a^{n-1}}{na^{n-1} + p} \right) \lambda^n + \frac{a^{2n}}{na^{n-1} + p}.
\end{equation}

The set of roots of the polynomial $P_{n+1}$ contains all roots of the polynomial $\mathcal{G}_n$ and one more root $\lambda = a$. If the polynomial $P_{n+1}$ has a multiple root, then this root is a root of the derivative $P'_{n+1}$ as well. However,

\begin{equation}
P'_{n+1}(\lambda) = (n+1)\lambda^n \left( \lambda - \lambda^* \right), \quad \lambda^* := \frac{a^n}{a^{n+1} + p},
\end{equation}

and, as it can be easily seen, in both roots $0$ and $\lambda^*$ of the derivative $P'_{n+1}$ values of $P_{n+1}$ (for $p > 0$) do not vanish, more precisely, $P_{n+1}(0) > 0$, $P_{n+1}(\lambda^*) < 0$. Consequently, $P_{n+1}$ and hence $\mathcal{G}_n$ do not have multiple roots.

5.4. We now estimate the roots of the polynomial (5.8) (for $q = 0$ as before).

**Lemma 9.** For given $p > 0$ and $a > 0$ the roots $\lambda_k$ of the polynomial $\mathcal{G}_n$ satisfy the inequality

\begin{equation}
|\lambda_k| < \delta a < a, \quad \delta = \delta(n, a, p) := \frac{1}{\sqrt{1 + \frac{\ma}{\na^{n-1} + p}}}, \quad k = 1, n.
\end{equation}

**Proof.** For $|\lambda| = \delta a$ we have

\begin{equation}
\frac{a^{2n-1}}{na^{n-1} + p} \sum_{m=0}^{n-1} a^m |\lambda|^m = \frac{a^{2n-1}}{na^{n-1} + p} \sum_{m=0}^{n-1} |\delta|^m < \frac{a^{n-1}}{na^{n-1} + p} a^n = (\delta a)^n = |\lambda|^n.
\end{equation}

From this, taking into account the first identity in (5.8), we get by Rouché’s theorem that all $n$ roots of the polynomial $\mathcal{G}_n$ lie in the disc $|\lambda| < \delta a$. ■
Remark 3. Let us mention several properties of $\delta = \delta(n,a,p)$, which plays a leading role in the process of extrapolation. For a fixed $n$ we obviously have

\begin{equation}
\delta \in (0, 1), \quad \delta = \left(\frac{n}{p}\right)^{1/n}a^{1-\frac{1}{n}}\left(1 - O\left(\frac{a^{n-1}}{p}\right)\right), \quad \frac{a^{n-1}}{p} \to 0,
\end{equation}

where $O\left(a^{n-1}/p\right)$ is a positive real value.

From this, in particular, it follows that if the ratio $a^{n-1}/p$ decreases, then all roots $\lambda_k$ become closer to the origin. For example, for a fixed $p$ and $a \to 0$ the largest absolute value of roots $\lambda_k$ is bounded by a value of the order $a^{2-1/n}$.

5.5. Let us estimate the extrapolation remainder (5.4) and then formulate the main result of the section. To do so, we need estimates for the generalized power sums $s_v$, $v \geq 2n$, for given sums (5.2) with indexes $\leq 2n - 1$ (where $q = 0$).

Lemma 10. For $p > 0$

\begin{equation}
0 \leq s_v \leq a^v, \quad v \geq 2n.
\end{equation}

Proof. We prove this result by induction, basing on (5.7) and the identities (4.13) for $v \geq 2n$ (with exchange of ‘$\cdot’$ by ‘$’). For $v = 2n$ from (5.2) and (5.7) we get

\[ s_{2n} = \frac{n a^{3n-1}}{n a^{n-1} + p} \leq a^{2n}. \]

Further, suppose that the inequality $s_v \leq a^v$ is valid for all $v = 2n, N$ (hence for all $v = n, N$). Under the assumption, we obtain

\[ s_{N+1} = \frac{a^{2n-1}}{n a^{n-1} + p} \sum_{m=0}^{n-1} s_{N-n+m+1} a^m \leq \frac{n a^{N+n}}{n a^{n-1} + p} \leq a^{N+1}, \]

which completes the proof. \hfill ■

Now, using (5.11), we estimate the extrapolation remainder (5.4), where $q = 0$:

\[ |r_n(z)| = \left| f(az) + p f_{n-1} z^{n-1} - \sum_{k=1}^{n} \mu_k f(\lambda_k z) \right| = \left| \sum_{m=2n}^{\infty} (a^m - s_m) f_m z^m \right| \leq \sum_{m=2n}^{\infty} |f_m| |az|^m. \]

Note that this estimate is independent of $p$. Summarizing, we can formulate the main result of §5.

Theorem 5. Let $f$ be analytic in the disc $|z| < \rho$, $a > 0$, $p > 0$. Then for $|z| < \rho/a$ the following extrapolation formula holds

\begin{equation}
f(az) = \sum_{k=1}^{n} \mu_k f(\lambda_k z) - p f_{n-1} z^{n-1} + r_n(z), \quad |r_n(z)| \leq \sum_{m=2n}^{\infty} |f_m| |az|^m,
\end{equation}

where the frequencies $\lambda_k$ are pairwise distinct roots of the polynomial (5.8), and (see (5.9))

\[ |\lambda_k z| < \delta a |z| = \frac{a |z|}{\sqrt[2n]{1 + |z|^{2n}}} < a |z|. \]

The amplitudes $\mu_k$ are uniquely defined by Lemma 1. Moreover, $\lambda_k = \lambda_k(a,p,n)$ and $\mu_k = \mu_k(a,p,n)$ do not depend on the form of the analytic function $f$ and are universal in this sense.

The extrapolation formula (5.12) is precise on polynomials of degree $2n - 1$, i.e. $r_n(z) \equiv 0$ for the polynomials $f$ with $\text{deg} f \leq 2n - 1$. 
Remark 4. In the case $f_{n-1} = 0$ the extrapolation formula has a particular simple form and high degree of accuracy. For instance, for even (odd) functions and even (odd) $n$ we have

$$f(az) = \sum_{k=1}^{n} \mu_k f(\lambda_k z) + r_n(z), \quad |\lambda_k| < \delta a, \quad |r_n(z)| \leq \frac{\infty}{m=2n} \sum |f_m| |az|^m.$$ 

Calculation of the coefficient $f_{n-1}$ in a general case was discussed at the end of previous section.

Remark 5. We can choose $q$ in a different manner (i.e. $q \neq 0$). For example, if $q = a^n p - \bar{q}_n / \kappa$, then coefficients of the polynomial $G_n$ can be also calculated easily:

$$\bar{g}_0 = -\bar{g}_n p_0, \quad p_0 := (n a^{n-1} + p) (\kappa p a^n / \bar{g}_n - 1) + a^{n-1};$$

$$\bar{g}_m = -a^{n-1-m} \bar{g}_n, \quad m = 0, n - 1.$$ 

The corresponding generating polynomial has the form

$$G_n(p; \lambda) = \bar{g}_n \left( \lambda^n - a^{n-1} \sum_{m=1}^{n-1} \left( \frac{\lambda}{a} \right)^m - p_0 \right).$$

By an arbitrarily small variation of the parameter $p$ in the function $p_0(p)$, it can be achieved, as in the proof of Lemma 5 above, that the polynomial $G_n(p; \lambda)$ has only simple roots. However, in this case there is no extrapolation since the inequalities $|\lambda_k| < a$ are not valid anymore; we have an interpolation formula of the form (5.12).

Remark 6. Realizing the $n$-point simple or multiple extrapolation (interpolation) on basis of the Lagrange polynomials or other similar approaches, one usually can obtain extrapolation (interpolation) formulas, being precise on polynomials of degree $n - 1$ (see, for instance, [12, 14, 46]). However, our extrapolation formula is precise on polynomials of degree $\leq 2n - 1$. It is interesting that the doubling of precision is gained by adding of just one regularizing power term $p f_{n-1} z^{n-1}$.

We also empathize that due to Remark 3 if $p \to \infty$ and other parameters are fixed, then the extrapolation nodes tend to the point $z = 0$, but at the same time the theoretical error of extrapolation does not increase (see (5.12)) since it is independent of $p$. An analogous phenomenon of convergence of nodes to the origin was noted in similar extrapolation problems in [12, 14].

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