Sobolev and isoperimetric inequalities for submanifolds in weighted ambient spaces

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Abstract In this paper, we prove Sobolev and isoperimetric inequalities for submanifold in weighted manifold. Our results generalize the Hoffman–Spruck’s inequalities (Hoffman and Spruck in Commun Pure Appl Math 27:715–727, 1974).

Keywords Weighted manifold · Sobolev inequality · Weighted mean curvature

Mathematics Subject Classification 53C42 · 53B25

1 Introduction

A lot of topics in geometric analysis, such as, Ricci flow, mean curvature flow, anisotropic mean curvature and optimal transportation theory, are related to submanifolds in weighted manifolds, see for instance [5, 10, 18, 20, 21, 24] and references therein. We recall a weighted manifold $(\tilde{M}, g, d\tilde{\mu})$ is a Riemannian manifold $(\tilde{M}, g)$ endowed with a weighted volume form $d\tilde{\mu} = e^{-f} d\tilde{M}$, where $d\tilde{M}$ is the volume element induced by the metric $g$ and $f$ is a real-valued smooth function on $\tilde{M}$, sometimes called the density of $\tilde{M}$. In this paper, following the papers of Hoffman and Spruck [15] and Michael and Simon [19], we will study Sobolev and isoperimetric inequalities for immersed submanifolds in weighted ambient spaces.

Let $x : M^m \to M$ be an isometric immersion of a complete manifold with (possibly nonempty) boundary $\partial M$ in the weighted manifold $(\tilde{M}, g, d\tilde{\mu})$. Gromov [13] introduced the

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extrinsic object associate with the immersion $x$, called weighted mean curvature vector field $H_f$, given by

$$H_f = H + \tilde{\nabla} f^\perp,$$

where $H$ is the mean curvature vector of the submanifold and $\perp$ denotes the orthogonal projection onto the normal bundle $TM^\perp$. In this context, it is natural to consider the first and second variations for the weighted area functional,

$$\text{vol}_f(\Omega) = \int_{\Omega} d\mu,$$

where $d\mu = e^{-f(x)}dM$ and $\Omega$ is a bounded domain. In Bayle [2], obtained the first variational formulae

$$\frac{d}{dt} \bigg|_{t=0} \text{vol}_f(\Omega_t) = \int_{\Omega} \langle H_f, V \rangle d\mu,$$

where $V$ is the variational field. Thus, the $f$-mean curvature vector appears naturally from a variational context.

**Example 1** Consider the weighted Euclidean space $(\mathbb{R}^n, d\tilde{\mu} = e^{-\frac{|x|^2}{4}}dx)$, where $|\cdot|$ denotes the Euclidean norm and $dx$ the Euclidean volume element. We recall that an isometric immersion $F: M \to \mathbb{R}^n$ is a self-shrinker if its mean curvature vector satisfies $2H = -F^\perp$. It is simple to show this definition is equivalent to say that $F$ is $(\frac{|x|^2}{4})$-minimal.

To state our main theorem, we need some definitions and notations. Let $K : \mathbb{R} \to [0, \infty)$ be a nonnegative, even and continuous function and let $h$ be the solution of the following Cauchy Problem:

$$\begin{cases} 
   h'' + K h = 0; \\
   h(0) = 0, h'(0) = 1.
\end{cases} \quad (1)$$

Let $r_0 = r_0(K) > 0$ and $s_0 = s_0(K) > 0$ be defined as follows: $(0, r_0)$ is an interval where $h$ is increasing and $(0, s_0) = h(0, r_0)$. Assume that the radial curvatures of $\bar{M}$ with base point $\bar{\xi}$ satisfy

$$(K_{\text{rad}})(\bar{\xi}) \leq K(r_\xi), \quad (2)$$

for all $\bar{\xi} \in \bar{M}$, where $r_\xi = d_{\bar{M}}(\cdot, \bar{\xi})$ is the distance in $\bar{M}$ from $\bar{\xi}$. For more details about radial curvatures and their uses, see for instance [14].

Our main theorem says the following.

**Theorem 1** Assume that $\bar{M}$ satisfies (2) and that $f^* = \sup_M f < +\infty$. Let $\varphi$ be a compactly supported nonnegative $C^1$ function on $M^m$ that vanishes along the boundary $\partial M$. Then, there exists a positive constant $S$, depending only on $m$ and $K$, such that the following inequality holds:

$$\left( \int_M \varphi^{\frac{m-p}{m}} d\mu \right)^{\frac{m-p}{m}} \leq S \frac{e^{f_\varphi}}{m} \int_M \left( |\nabla \varphi| + \varphi |H_f - \tilde{\nabla} f| \right)^p d\mu,$$

for all $1 \leq p < m$, provided that there exists $\kappa \in (0, 1)$ satisfying:

$$\begin{cases} 
   \bar{J} := \left( \frac{\omega_m^{-1} e^{f^*}}{1 - \kappa} \text{vol}_f(\text{supp}(\varphi)) \right)^{\frac{1}{\kappa}} \leq s_0; \\
   h^{-1}(\bar{J}) \leq 2 R_0,
\end{cases} \quad (3)$$
where \( R_0 = \min(\text{Inj}_\varphi, r_0) \), \( \omega_m \) is the volume of the unit ball in \( \mathbb{R}^m \) and \( \text{Inj}_\varphi \) is the minimum of the injectivity radius of \( \bar{M} \) restricted to the points of \( \text{supp} \varphi \). Furthermore, the constant \( S \) is given by

\[
S = \frac{2^m m}{\kappa (m-1) s_0} \left( \frac{\omega_m^{-1}}{1 - \kappa} \right)^{\frac{1}{m}} .
\]  

(4)

**Remark 1**  It is simple to see that if \( \bar{M} \) is a Hadamard manifold, then the injectivity radius of \( \bar{M} \) is infinite, so \( \text{Inj}_\varphi = +\infty \), and we can take \( \kappa = 0 \), hence the solution \( h \) of (1) is given by

\[
h(t) = t \text{ defined on any positive interval } (0, r_0). \]

Thus, condition (3) is always satisfied and \( r_0/s_0 = 1 \). In this case, we can choose \( S = S_0 \) by

\[
S_0 = \min_{\kappa \in (0, 1)} S = \frac{2^m (m + 1) \omega_m^{-1}}{m - 1} .
\]  

(5)

If \( \bar{M} \) is the sphere \( S^n(1/b) \subset \mathbb{R}^{n+1} \) of radius \( 1/b > 0 \), then we can take \( \kappa = b^2 \). In this case, \( h(t) = b^{-1} \sin(t/b) \text{ defined on the interval } (0, \pi/(2b)) \). Hence, \( r_0/s_0 = \pi/2 \). Thus, we see that Theorem 1 improve Hoffman–Spruck’s inequality [15] even when \( f \equiv 0 \). The question about the optimal constant \( S \) in Theorem 1 remains open, even for \( f \equiv 0 \) and \( M \) being a minimal surfaces in \( \mathbb{R}^3 \). For more details about this problem, see [3] and [4].

**Remark 2**  We believe curvature assumption (2), that is, curvature bounded from above by the nonnegative function \( \kappa \) instead of a nonnegative constant as in Hoffman and Spruck [15], adds considerable flexibility to the theorem. For instance, if the curvature is bounded above by a positive constant \( b \), the functions for which the Hoffman–Spruck’s Sobolev inequality holds are required to have supports whose size is severely bounded in terms of \( b \). So, for instance, a small perturbation of a Hadamard manifold in the sense that somewhere some bumps with positive curvature are allowed would dramatically reduce the size of the supports of the functions in the Sobolev inequality, whereas, if the bumps are not too bad, so that the radial curvature can be controlled by a nice \( \kappa \), the constraints on the supports can be much milder and it can even happen that no constraint is necessary (if \( r_0 = s_0 = +\infty \)).

Curvature condition (2) is far from being new. Indeed, Chapter 2 of Pigola–Rigoli–Setti’s book [22] is almost all dedicated to show how condition (2) is good enough to obtain useful Hessian, Laplacian and volume comparison theorems. Furthermore, Abresch [1], Itokawa et al. [16], Kondo and Ohta [23], among others, assumed the curvature condition \( (K_{rad})_{\xi} = \kappa(r_{\xi}) \) on a complete manifold obtaining strong topological obstructions.

A consequence of Theorem 1 is the following isoperimetric inequality.

**Theorem 2**  Assume that \( \bar{M} \) satisfies (2) and that \( M \) is compact with (possibly nonempty) boundary. Then, it holds

\[
\vol_f(M)^{\frac{m-1}{m}} \leq S e^{rac{\mu}{m}} \left( \vol_f(\partial M) + \int_M |H_f - \bar{\nabla} f| d\mu \right) ,
\]  

(6)

provided that there exists \( \kappa \in (0, 1) \) satisfying:

\[
\begin{align*}
\bar{J} &= \left( \frac{\omega_m^{-1} e^f}{1 - \kappa} \right)^{\frac{1}{m}} \leq s_0; \\
h^{-1}(\bar{J}) &\leq 2 R_0,
\end{align*}
\]  

(7)

where \( R_0 = \min(\text{Inj}_M, r_0) \). \( f^* = \sup_M f \). \text{Inj}_M \text{ is the minimum of the injectivity radius of } \bar{M} \text{ restricted to the points of } M, \text{ and } S \text{ is the constant as given in (4).}
By Theorem 2, it is simple to show that if $M^m$ is a closed self-shrinker contained in a Euclidean ball $B \subset \mathbb{R}^n$ of radius $R$, then it holds that $e^{R^2/4} R \geq 2/S_0$ and \( \text{vol}_{(|x|^2/4)}(M)^{1/m} \geq 2e^{R^2/4}/(S_0 R) \), where $S_0$ is the positive constant as in (5). Since the round spheres $S^m(\sqrt{2m}) \subset \mathbb{R}^{m+1}$ of radius $\sqrt{2m}$ are examples of $(|x|^2/4)$-minimal hypersurfaces, the term “$|H_f - \bar{\nabla} f|$” that appears in Theorems 1 and 2 cannot be replaced by “$|H_f|$”. Further examples of immersed self-shrinkers spheres can be found in [9].

We can also see that the hypothesis “$f^* < \infty$” is essential in Theorems 1 and 2. Consider the weighted Euclidean space $(\mathbb{R}^3, e^{-f} \, dx)$. If we take the function $f(x) = |x|^2/2$, then the plane $P = \mathbb{R}^2 \subset \mathbb{R}^3$ has finite $f$-volume, $H_f = 0$ and $\bar{\nabla} f = x$; hence, $|H_f - \bar{\nabla} f|$ has finite $L^2_\mu$-norm. However, if $f \in C^1(\mathbb{R}^3)$ satisfies $f^* < \infty$ and $\sup_P |\nabla f| < \infty$, then, by Theorem 2 and coarea formula, we can show that $P$ has infinite $f$-volume. More generally, we have the following

**Theorem 3** Let $\tilde{M}$ be a complete weighted manifold $(\tilde{M}, d\mu = e^{-f} d\tilde{M})$ with injectivity radius bounded from below by a positive constant and radial sectional curvatures satisfying (2), for some even function $0 \leq K \in C^0(\mathbb{R})$. Let $M^m$ be a complete noncompact manifold isometrically immersed in $\tilde{M}$. Assume that $f^* < \infty$ and $|H_f - \bar{\nabla} f| \in L^p_\mu(M)$, for some $m \leq p \leq \infty$. Then, each end of $M$ has infinite $f$-volume.

Theorem 3, for the case that $\tilde{M}$ has bounded geometry, i.e. injectivity radius bounded below by a positive constant and sectional curvatures bounded above, was proved by: (1) Frensel [11] and by do Carmo et al. [8] for the case that the mean curvature vector field is bounded in norm (the case $p = \infty$); (2) Fu and Xu [12] for the case that the total mean curvature is finite (the case $p = m$); and Cheung and (3) Leung [7] for the case that the mean curvature vector has finite $L^p$-norm for some $p > m$. A unified proof of (1), (2) and (3) is found in [6].

We were informed of an independent manuscript of Impera and Rimoldi [17] which proves a similar version of Theorem 1 for the case that $M$ is a hypersurface in a weighted manifold $\tilde{M}$ with nonpositive sectional curvature. The authors thank them for useful comments.

## 2 Preliminaries

We assume the notations in the introduction. Consider the following

**Definition 1** Let $X : M^m \rightarrow T\tilde{M}$ be a $C^1$ vector field. The $f$-divergence of $X$ is defined by:

$$D_f X = e^f \text{div}_M(e^{-f} X^T)$$

By direct computations, the following holds.

**Proposition 1** Let $X : M^m \rightarrow T\tilde{M}$ be a $C^1$-vector field and $g \in C^1(M)$. Then, it holds:

(A) $D_f X = \text{div}_M X + (H - \nabla f, X) = \text{div}_M X + \{H_f - \bar{\nabla} f, X\}$, where $\nabla f = (\bar{\nabla} f)^T$ is the gradient vector field of the restriction $f|_M$;

(B) $D_f (g X) = g D_f X + (X, \nabla g)$.

Fix a point $\xi \in M$ and consider $r_\xi = d_{\tilde{M}}(\cdot, \xi)$ the distance function in $\tilde{M}$ from $\xi$. Assume that the radial curvature of $\tilde{M}$ with basis point $\xi$ satisfies

$$\langle \hat{K}_{\text{rad}}(\xi) \rangle \leq K(r_\xi),$$

where $K : \mathbb{R} \rightarrow [0, \infty)$ is a nonnegative even continuous function. Let $h : (0, r_0) \rightarrow (0, s_0)$ be the increasing function as defined in (1).
Let $B = B_{r_0}(\xi)$ be the geodesic ball of $\bar{M}$ with center $\xi$ and radius $r_0$. Consider the radial vector field

$$X_\xi = h(r_\xi)\bar{\nabla}r_\xi,$$

(9)
defined on $B \cap V$, where $V$ is a normal neighborhood of $\xi$ in $\bar{M}$ and $\bar{\nabla}r_\xi$ is the gradient vector field of $r_\xi$ in $\bar{M}$. By the Hessian comparison theorem (see Theorem 2.3 page 29 of [22]), we have that in $B$ the following holds

$$\text{Hess}_{r_\xi}(v, v) \geq \frac{h'(r_\xi)}{h(r_\xi)} \left(1 - \langle \bar{\nabla}r_\xi, v \rangle^2\right),$$

(10)
for all vector field $v \in T\bar{M}$ with $|v| = 1$.

**Proposition 2** Under the notations above, it holds that

$$\mathcal{D}_f X_\xi \geq mh'(r_\xi) + h(r_\xi)\{H_f - \bar{\nabla}f, \bar{\nabla}r_\xi\}.$$  

(11)

**Proof** Using Proposition 1, we have

$$\mathcal{D}_f X_\xi = h(r_\xi)\mathcal{D}_f \bar{\nabla}r_\xi + h'(r_\xi)|\nabla r_\xi|^2.$$  

(12)
Furthermore, using (10), we obtain

$$\mathcal{D}_f \bar{\nabla}r_\xi = \text{div}_M \bar{\nabla}r_\xi + \{H_f - \bar{\nabla}f, \bar{\nabla}r_\xi\} \geq \frac{h'(r_\xi)}{h(r_\xi)}(m - |\nabla r_\xi|^2) + \{H_f - \bar{\nabla}f, \bar{\nabla}r_\xi\}.$$  

(13)
Combining (12) and (13), the result follows.

Let $M$ be a complete manifold with (possibly nonempty) boundary $\partial M$ and let $\varphi : M \to [0, \infty)$ be a compactly supported nonnegative $C^1$ function such that $\varphi|_{\partial M} = 0$. Let $\lambda \in C^1(\mathbb{R})$ be a nonnegative and nondecreasing function satisfying $\lambda(t) = 0$, for $t \leq 0$. We define the following real-variable functions:

$$\phi_\xi (R) = \phi_{\xi, \varphi, \lambda}(R) = \int_M \lambda(R - r_\xi(x))\varphi \, d\mu;$$

$$\psi_\xi (R) = \psi_{\xi, \varphi, \lambda}(R) = \int_M \lambda(R - r_\xi(x))|\nabla \varphi + \varphi(H_f - \bar{\nabla}f)| \, d\mu;$$

$$\bar{\phi}_\xi (R) = \bar{\phi}_{\xi, \varphi}(R) = \int_{M \cap B_R(\xi)} \varphi \, d\mu;$$

$$\bar{\psi}_\xi (R) = \bar{\psi}_{\xi, \varphi}(R) = \int_{M \cap B_R(\xi)} |\nabla \varphi + \varphi(H_f - \bar{\nabla}f)| \, d\mu.$$

Our first lemma says the following.

**Lemma 1** It holds that

$$-\frac{d}{dR}(h(R)^{-m}\phi_\xi(R)) \leq h(R)^{-m}\psi_\xi(R),$$

for all $0 < R < R_0 = \min(\text{Inj}_\varphi, r_0)$.

**Proof** We denote by $r = r_\xi$ and let $X = X_\xi$ be defined in $B_{R_0}(\xi)$. Using part (B) of Proposition 1, we obtain that

$$\mathcal{D}_f(\lambda(R - r)\varphi X) = \lambda(R - r)\varphi\mathcal{D}_f X + \langle \nabla(\lambda(R - r)\varphi), X \rangle$$

$$= \lambda(R - r)\varphi\mathcal{D}_f X + \lambda(R - r)\langle \nabla \varphi, X \rangle - \lambda'(R - r)\varphi \langle \nabla r, X \rangle.$$  

(14)
Since $\text{supp} \varphi$ is compact and $\varphi|_{\partial M} = 0$, using part (A) of Proposition 1 and the divergence theorem, we obtain
\[
\int_{\partial M} \mathcal{D}_f (\lambda (R - r) \varphi X) d\mu = 0.
\] (15)
Thus, by (14) and (15), we obtain
\[
\int_{\partial M} \lambda (R - r) \varphi \mathcal{D}_f X d\mu = \int_{\partial M} \lambda' (R - r) \varphi h(r) \{ \nabla r, \tilde{\nabla} r \} d\mu
- \int_{\partial M} \lambda (R - r) h(r) \{ \nabla \varphi, \tilde{\nabla} r \} d\mu.
\] (16)
Using that:
(a) the functions $\lambda$ and $\lambda'$ are nonnegative;
(b) the function $h$ is positive and increasing in $(0, r_0)$;
(c) $\lambda (R - r(x)) = \lambda' (R - r(x)) = 0$ in the subset $\{ x \in M \mid r(x) \geq R \}$.
Since $h'' = -K h \leq 0$ in $(0, r_0)$, we have that $h'$ is nonincreasing in $(0, r_0)$.
By using (a), (c) and Proposition 2, we obtain that
\[
\int_{\partial M} \lambda (R - r) \varphi \mathcal{D}_f X d\mu \geq m h'(R) \varphi (R) + \int_{\partial M} \lambda (R - r) \varphi h(r) \{ H_f - \tilde{\nabla} f, \tilde{\nabla} r \}.
\]
Thus, since $|\nabla r| \leq 1$, using (16), (a) and (c) we obtain
\[
 mh'(R) \varphi_{\xi} (R) \leq h(R) \int_{\partial M} \lambda' (R - r) \varphi d\mu
- \int_{\partial M} \lambda (R - r) h(r) \{ \nabla \varphi + \varphi (H_f - \tilde{\nabla} f), \tilde{\nabla} r \}
\leq h(R) \left( \frac{d}{dR} \varphi_{\xi} (R) + \psi_{\xi} (R) \right)
\]
This implies that
\[
\frac{d}{dR} \left( h(R)^{-m} \varphi_{\xi} (R) \right) = h(R)^{-m} \left( \frac{d\varphi_{\xi}}{dR} (R) - m \frac{h'(R)}{h(R)} \varphi_{\xi} (R) \right)
\geq h(R)^{-m} \left( \frac{d\varphi_{\xi}}{dR} (R) - \left( \frac{d\varphi_{\xi}}{dR} (R) + \psi_{\xi} (R) \right) \right)
= - h(R)^{-m} \psi_{\xi} (R).
\]
Lemma 1 is proved.

Take $\kappa \in (0, 1)$ and let $J = J(\kappa, \varphi, f) \geq 0$ be the constant defined by
\[
J = \left( \frac{\omega_m^{-1} e^f}{1 - \kappa} \int_M \varphi d\mu \right)^{\frac{1}{m}}.
\] (17)
Note that $J$ differs from $\bar{J}$ in that $\text{vol}_f (\text{supp} \varphi)$ is replaced by $\int_M \varphi d\mu$.

Our next lemma is the following result.

**Lemma 2** Fix $\xi \in M$ satisfying $\varphi(\xi) \geq 1$. Assume that $0 < J < s_0$ and set $\alpha = \alpha(\kappa, \varphi) \in (0, r_0)$ given by $h(\alpha) = J$. Assume further that $t \alpha \leq R_0 = \min (\text{Inj}_{\varphi}, r_0)$, for some $t > 1$.
Then, there exists $R \in (0, \alpha)$ such that
\[
\bar{\varphi}_{\xi} (tR) \leq \frac{2\alpha}{\kappa} t^{m-1} \bar{\psi}_{\xi} (R).
\] (18)
Proof By Lemma 1,
\[- \frac{d}{dR} (h(R) - m \phi_{\xi}(R)) \leq h(R) - m \psi_{\xi}(R), \tag{19}\]
for all $0 < R < R_0$.

Note that $0 < \alpha \leq R_0$. Given $\sigma \in (0, \alpha)$, integrating the both sides of (19) on the interval $(\sigma, \alpha)$, we obtain
\[ h(\sigma) - m \phi_{\xi}(\sigma) \leq h(\alpha) - m \phi_{\xi}(\alpha) + \int_{\sigma}^{\alpha} h(\tau) - m \psi_{\xi}(\tau) d\tau. \tag{20}\]
Take $0 < \epsilon < \sigma$ and let $\lambda : \mathbb{R} \to [0, 1]$ be a nondecreasing $C^1$ function satisfying:
\[
\begin{align*}
\lambda(t) &= 1, & \text{for all } t \geq \epsilon; \\
\lambda(t) &= 0, & \text{for all } t \leq 0; \\
0 &\leq \lambda(t) \leq 1, & \text{for all } t.
\end{align*}
\tag{21}\]
Consider this function $\lambda$ in the definitions of $\phi_{\xi} = \phi_{\xi, \psi, \lambda}$ and $\psi_{\xi} = \psi_{\xi, \psi, \lambda}$. By (20) and (21), we obtain
\[ \phi_{\xi}(\sigma) = \int_M \lambda(\sigma - r_{\xi}) \varphi d\mu = \int_{M \cap B_\sigma(\xi)} \lambda(\sigma - r_{\xi}) \varphi d\mu \]
\[ \geq \int_{M \cap B_{\sigma-\epsilon}(\xi)} \varphi d\mu = \int_{M \cap B_{\sigma-\epsilon}(\xi)} \phi d\mu \]
\[ = \bar{\phi}_{\xi}(\sigma - \epsilon). \tag{22}\]
Since $0 \leq \lambda(t) \leq 1$, for all $t$, and $\lambda(R - r_{\xi}(x)) = 0$ in $\{ x \in M \mid r_{\xi}(x) \geq R \}$, we have that $\phi_{\xi}(\sigma) \leq \phi_{\xi}(\sigma)$ and $\psi_{\xi}(\sigma) \leq \psi_{\xi}(\sigma)$. Thus, by (20) and (22), we obtain the following.
\[ h(\sigma) - m \bar{\phi}_{\xi}(\sigma - \epsilon) \leq h(\alpha) - m \bar{\phi}_{\xi}(\alpha) + \int_{0}^{\alpha} h(\tau) - m \bar{\psi}_{\xi}(\tau) d\tau. \tag{23}\]
Since the inequality (23) does not depend on $\lambda$, we can take $\epsilon \to 0$. Thus, we obtain
\[ \sup_{\sigma \in (0, \alpha)} (h(\sigma) - m \bar{\phi}_{\xi}(\sigma)) \leq h(\alpha) - m \bar{\phi}_{\xi}(\alpha) + \int_{0}^{\alpha} h(\tau) - m \bar{\psi}_{\xi}(\tau) d\tau. \tag{24}\]
Now suppose that Lemma 2 is false. Then, it holds that
\[ \bar{\psi}_{\xi}(R) < \frac{C}{2^\alpha} t^{1-m} \bar{\phi}_{\xi}(tR), \]
for all $R \in (0, \alpha)$. Multiplying both sides of this inequality by $h(R)^{-m}$, integrating on $(0, \alpha)$ and using the change of variable $\sigma = tR$, we obtain
\[ \int_{0}^{\alpha} h(R)^{-m} \bar{\psi}_{\xi}(R) dR < \frac{C}{2^\alpha} t^{-m} \int_{0}^{\alpha t} \left( h\left( \frac{\sigma}{t} \right) \right)^{-m} \bar{\phi}_{\xi}(\sigma) d\sigma. \tag{25}\]
Given $0 < \sigma < t\alpha \leq R_0$, using that $h'' = -Kh \leq 0$, we have that $h$ is concave and increasing on $(0, \alpha)$. Thus, we obtain the following.
\[
\begin{align*}
\text{If } \sigma &\in (0, \alpha) \text{ then } h(t^{-1} \sigma) \geq t^{-1} h(\sigma), \text{ for all } t \geq 1; \\
\text{If } \sigma &\in (\alpha, t\alpha) \text{ then } 0 < \frac{\sigma}{\alpha} < 1 \text{ and } \frac{\sigma}{\tau} = \frac{\sigma}{t\alpha} \alpha, \text{ which implies that } h\left( \frac{\sigma}{\tau} \right) \geq \frac{\sigma}{\alpha} h(\alpha). \tag{26}\end{align*}
\]
Using (26), we obtain
\[
\int_0^{1a} \left( h \left( \frac{\sigma}{t} \right) \right)^{-m} \bar{\phi}_\xi (\sigma) d\sigma \leq t^m \int_0^a h(\sigma)^{-m} \bar{\phi}_\xi (\sigma) d\sigma \\
+ \left( \frac{h(\sigma)}{t^a} \right)^{-m} \int_0^{1a} \sigma^{-m} \bar{\phi}_\xi (\sigma) d\sigma.
\]

Since \( \bar{\phi}_\xi (\sigma) \leq \int_M \varphi d\mu \) and \( \int_0^a \sigma^{-m} d\sigma \leq \frac{\alpha^{1-m}}{m-1} \), we obtain
\[
\int_0^{1a} \left( h \left( \frac{\sigma}{t} \right) \right)^{-m} \bar{\phi}_\xi (\sigma) d\sigma \leq t^m \int_0^a h(\sigma)^{-m} \bar{\phi}_\xi (\sigma) d\sigma \\
+ t^m \frac{h(\sigma)^{m-1}}{m-1} \int_M \varphi d\mu.
\]  

It follows from (25) and (27) the following inequality.
\[
\frac{2}{\kappa} \int_0^a h(R)^{-m} \bar{\psi}_\xi (R) dR < \frac{h(\sigma)^{-m}}{m-1} \int_M \varphi d\mu + \frac{1}{\alpha} \int_0^a h(\sigma)^{-m} \bar{\phi}_\xi (\sigma) d\sigma \\
\leq \frac{h(\sigma)^{-m}}{m-1} \int_M \varphi d\mu + \sup_{\sigma \in (0,\alpha)} (h(\sigma)^{-m} \bar{\phi}_\xi (\sigma)).
\]  

Using (24) and (28), we obtain
\[
\frac{2}{\kappa} \sup_{\sigma \in (0,\alpha)} (h(\sigma)^{-m} \bar{\phi}_\xi (\sigma)) < \frac{2}{\kappa} (h(\alpha)^{-m} \bar{\phi}_\xi (\alpha)) + \frac{h(\sigma)^{-m}}{m-1} \int_M \varphi d\mu \\
+ \sup_{\sigma \in (0,\alpha)} (h(\sigma)^{-m} \bar{\phi}_\xi (\sigma));
\]

hence, we obtain
\[
\left( \frac{2}{\kappa} - 1 \right) \sup_{\sigma \in (0,\alpha)} (h(\sigma)^{-m} \bar{\phi}_\xi (\sigma)) < \frac{2}{\kappa} (h(\alpha)^{-m} \bar{\phi}_\xi (\alpha)) + \frac{h(\sigma)^{-m}}{m-1} \int_M \varphi d\mu.
\]  

We recall that \( h(0) = 0, h'(0) = 1 \) and \( h(\alpha) = J = \left( \frac{\omega^m e^{f^*}}{1-\kappa} \int_M \varphi e^{-f} dM \right)^{\frac{1}{m}} \). Thus, we obtain
\[
\begin{aligned}
&h(\alpha)^{-m} \bar{\phi}_\xi (\alpha) \leq h(\alpha)^{-m} \int_M \varphi d\mu = (1-\kappa)\omega_me^{-f^*}; \\
&\sup_{\sigma \in (0,\alpha)} (h(\sigma)^{-m} \bar{\phi}_\xi (\sigma)) \geq \limsup_{\sigma \to 0} (h(\sigma)^{-m} \bar{\phi}_\xi (\sigma)) = \omega_m \left( \varphi(\xi)e^{-f(\xi)} \right) \geq \omega_m e^{-f^*}.
\end{aligned}
\]

Thus, using (29), we obtain
\[
\left( \frac{2}{\kappa} - 1 \right) < \frac{2(1-\kappa)}{\kappa} + \frac{1-\kappa}{m-1},
\]
that is, \( 1 < \frac{1-\kappa}{m-1} \), which is a contradiction. Lemma 2 is proved.
3 Proof of Theorem 1

Consider the set $A = \{ \xi \in M \mid \varphi(\xi) \geq 1 \}$. Take $t > 2$ so that $t\alpha \leq R_0 = \min(\text{Inj}_\varphi, r_0)$ and set $\beta \in [\frac{2}{t}, 1)$. Consider the sequence $R_j = \beta^j \alpha$, with $j = 0, 1, \ldots$, and define the collection of subsets

$$A_j = \left\{ \xi \in A \mid \tilde{\phi}_\xi(t R) \leq \frac{2\alpha}{\kappa} t^{m-1} \tilde{\psi}_\xi(R), \text{ for some } R \in [\beta R_j, R_j) \right\}.$$

By Lemma 2, $A = \bigcup_{j=0}^{\infty} A_j$. Consider the sequence of subsets $F_k \subset A$, with $k = 0, 1, \ldots$, defined inductively as follows: (I): $F_0 = \emptyset$; (II): Assume that $F_0, \ldots, F_{k-1}$ are defined, with $k \geq 1$. For each $\ell \geq 0$, let $S_\ell(\xi) = M \cap B_\ell(\xi)$. Consider

$$D_k = \tilde{A}_k - \bigcup_{j=1}^{k-1} \cup_{\xi \in F_j} S_\beta R_j(\xi).$$

Claim There exists a finite subset $F_k \subset D_k$ satisfying:

(i) $D_k \subset \cup_{\xi \in F_k} S_{\beta R_k}(\xi)$;
(ii) the collection $B_{R_k}(\xi)$, with $\xi \in F_k$ and $k \geq 1$, are pairwise disjoint;
(iii) $A \subset \bigcup_{j=1}^{\infty} \cup_{\xi \in F_j} S_\beta R_k(\xi)$.

Proof Note that $D_k$ is compact, since $A$ is compact and $D_k$ is closed. We define

$$B = \left\{ F = [\xi_0] \mid \xi_0 \in D_k, S_{R_k}(\xi) \cap S_{R_k}(\xi') = \emptyset, \xi \neq \xi' \in F \right\}.$$ 

It is easy to see that, by Zorn’s Lemma, $B$ has a maximal element. Denote by $F_0$ a maximal element of $B$. It holds that $D_k \subset \cup_{\xi \in F_0} S_{\beta R_k}(\xi)$, otherwise there were $\xi \in D_k - \cup_{\xi' \in F} S_{\beta R_k}(\xi')$ and $F_0 \cup \{ \xi \}$ belongs $B$, contradicting a maximality of $F$. Now, using the compactness of $D_k$, we obtain $F_k = \{ \xi, \ldots, \xi_N \} \subset F$ completing the construction.

For each $\xi \in F_k$, it holds that $\tilde{\phi}_\xi(t R) \leq \frac{2\alpha}{\kappa} t^{m-1} \tilde{\psi}_\xi(R)$, for some $R \in (\beta R_k, R_k]$. This implies that

$$\tilde{\phi}_\xi(t \beta R_k) \leq \tilde{\phi}_\xi(t R) \leq \frac{2\alpha}{\kappa} t^{m-1} \tilde{\psi}_\xi(R) \leq \frac{2\alpha}{\kappa} t^{m-1} \tilde{\psi}_\xi(R_k).$$

Thus, since $\varphi(\xi) \geq 1$, for all $\xi \in A$, by the Claim, we have

$$\text{vol}_f(A) \leq \int_A \varphi \mu \leq \sum_{k=1}^{\infty} \sum_{\xi \in F_k} \tilde{\phi}_\xi(t \beta R_k) \leq \sum_{k=1}^{\infty} \sum_{\xi \in F_k} \frac{2\alpha}{\kappa} t^{m-1} \tilde{\psi}_\xi(R_k)$$

$$= \frac{2\alpha}{\kappa} t^{m-1} \int_{\cup_{\xi \in F_k} S_{R_k}(\xi)} |\nabla \varphi + \varphi(H_f - \tilde{\nabla} f)| d\mu \leq \frac{2\alpha}{\kappa} t^{m-1} \int_M |\nabla \varphi + \varphi(H_f - \tilde{\nabla} f)| d\mu.$$

Now, for each $s > 0$, we define the set $A^s = \{ \xi \in M \mid \varphi(\xi) \geq s \}$, and let $\tilde{J} = \tilde{J}(\kappa, \varphi)$ be given by

$$\tilde{J} = \left( \frac{\omega_m^{-1} e^{f^*}}{1 - \kappa} \text{vol}_f(\text{supp}(\varphi)) \right)^{\frac{1}{m}}.$$

Assume that $0 < \tilde{J} < s_0$, for some $\kappa \in (0, 1)$ and let $\tilde{\alpha} \in (0, r_0)$ be given by $h(\tilde{\alpha}) = \tilde{J}$. Assume further that $t\tilde{\alpha} < R_0$, for some $t > 2$. 

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Fix $\epsilon > 0$ and let $\delta = \delta(\cdot, \epsilon) : \mathbb{R} \to [0, 1]$ be a nondecreasing $C^1$ function satisfying:
\[
\begin{align*}
0 < \delta(t) &< 1, \quad \text{for all } t \in (-\epsilon, 0); \\
\delta(t) & = 0, \quad \text{for all } t \in (-\infty, -\epsilon]; \\
\delta(t) & = 1, \quad \text{for all } t \in [0, \infty).
\end{align*}
\]

For all $s > \epsilon$, we consider the function $\eta = \eta(\cdot, \epsilon, s) : M \to \mathbb{R}$ given by
\[
\eta(\xi) = \delta(\varphi(\xi) - s).
\]

It is easy to see that the following statements hold:

(i) $\eta \in C^1(M)$;
(ii) $0 \leq \eta(\xi) \leq 1$, for all $\xi \in M$;
(iii) $\text{supp} \eta \subset \text{supp} \varphi$;
(iv) $\eta(\xi) = 1$ if, and only if, $\varphi(\xi) \geq s$.

In particular, if $\text{supp} \eta \neq \emptyset$ then $0 < J(\kappa, \eta) \leq \bar{J}(\kappa, \varphi) < r_0$; hence, Lemmas 1 and 2 apply (with $J = J(\kappa, \eta)$ and $\alpha = \alpha(\kappa, \eta)$). By the previous argument with $\phi$ replaced by $\eta$, we obtain
\[
\text{vol}_f(A^s) = \text{vol}(\{\xi \mid \eta(\xi) = 1\}) \leq \frac{2\alpha}{\kappa} r^{m-1} \int_M |\nabla \varphi + \varphi(H_f - \bar{\nabla} f)|d\mu.
\]

We recall that the function $h$ satisfies $h(0) = 0$, $h'(0) = 1$ and $h : [0, r_0) \to [0, s_0)$ is increasing and concave. Thus, the inverse function $h^{-1} : [0, s_0) \to [0, r_0)$ is increasing, convex and satisfies $h^{-1}(0) = 0$ and $(h^{-1})'(0) = 1$ hence, $h^{-1}(\tau) \leq \frac{r_0}{s_0} \tau$, for all $\tau \in (0, s_0)$, which implies
\[
\alpha = h^{-1}(J) \leq \frac{r_0}{s_0} J = C_1 \left( \int_M \eta d\mu \right)^{\frac{1}{m}},
\]

where $C_1 = \frac{r_0}{s_0} \left( \frac{\omega_n^{-1}}{1 - \kappa} \right)^{\frac{1}{m}} e^{\frac{r_0}{s_0}}.$

Note that $s^{\frac{m}{m-1}} \delta(\varphi - s) \leq (\varphi + \epsilon)^{\frac{m}{m-1}}$, for all $s > \epsilon.$ Thus, by (30) and (31), we obtain
\[
\begin{align*}
s^{\frac{1}{m-1}} \text{vol}_f(A^s) &\leq \frac{2C_1}{\kappa} r^{m-1} \left( s^{\frac{m}{m-1}} \int_M \eta d\mu \right)^{\frac{1}{m}} \int_M |\nabla \eta + \eta(H_f - \bar{\nabla} f)|d\mu \\
&= C_2 \left( \int_M (\varphi + \epsilon)^{\frac{m}{m-1}} d\mu \right)^{\frac{1}{m}} \int_M |\nabla \eta + \eta(H_f - \bar{\nabla} f)|d\mu \\
&\leq C_2 \left( \int_M (\varphi + \epsilon)^{\frac{m}{m-1}} d\mu \right)^{\frac{1}{m}} \int_M |\nabla \eta + \eta(H_f - \bar{\nabla} f)|d\mu,
\end{align*}
\]

for all $s > \epsilon$, where $C_2 = \frac{2C_1}{\kappa} r^{m-1}$. Furthermore,
\[
\int_0^\infty s^{\frac{1}{m-1}} \text{vol}_f(A_s) ds = \int_0^\infty \int_{\xi \in M \mid \varphi(\xi) \geq s} s^{\frac{1}{m-1}} d\mu ds
\]
\[
= \int_{\{\xi, s\} \in M \times \mathbb{R} \mid 0 < s \leq \varphi(\xi)} s^{\frac{1}{m-1}} d\mu ds
\]
\[
= \int_M \int_0^{\varphi(\xi)} s^{\frac{1}{m-1}} ds d\mu
\]
\[
= \frac{m - 1}{m} \int_M \varphi^{\frac{m}{m-1}} d\mu.
\]
Furthermore, since $\|\nabla S\| = \max_{0 \leq \delta(t) \leq 1, \delta(t-s) = 0} \|\nabla S\|$ we obtain the constant $S$.

Therefore, we obtain

\[
\int_{e}^{\infty} \int_{M} |\nabla \eta| \, d\mu \, ds = \int_{M} \int_{e}^{\infty} |\nabla \eta| \, d\mu \, ds
\]

Since $0 \leq \delta(t) \leq 1$, for all $t$, and $\delta(t-s) = 0$, for all $s \geq t + \epsilon$, we obtain

\[
\int_{e}^{\infty} \int_{M} |\nabla \eta| \, d\mu \, ds \leq \int_{M} \int_{e}^{\infty} \delta (\varphi - s) \, |\nabla \eta| \, d\mu \, ds
\]

Furthermore, since $|\nabla \eta| = \delta' (\varphi - s) \, |\nabla \varphi| = - \frac{d}{ds} \delta (\varphi - s) \, |\nabla \varphi|$, we obtain from the fundamental theorem of calculus the following.

\[
\int_{e}^{\infty} \int_{M} |\nabla \eta| \, d\mu \, ds \leq \int_{M} \int_{e}^{\infty} \delta' (\varphi (\xi) - s) \, |\nabla \varphi| \, d\mu
\]

Therefore, we obtain

\[
\left( \int_{M} \varphi^{m-p} \, d\mu \right)^{\frac{m-1}{m}} \leq \frac{mC_2}{m-1} \int_{M} (|\nabla \varphi| + \varphi (|H_f - \tilde{V} f|)) \, d\mu.
\]

To finish the proof of Theorem 1, we apply (34) to the function $\varphi^\gamma$, where $\gamma > 1$ is a constant to be defined. By Hölder inequality, we obtain

\[
\left( \int_{M} \varphi^{\frac{m}{m-1}} \, d\mu \right)^{\frac{m-1}{m}} \leq C_3 \int_{M} \varphi^{\gamma - 1} (|\nabla \varphi| + \varphi (|H_f - \tilde{V} f|)) \, d\mu
\]

where $C_3 = \frac{mC_2}{m-1}$ and $q = \frac{p}{m-1}$. Take $1 < p < m$ and let $\gamma = \frac{p(n-1)}{mp}$. We have that $\frac{m}{mp} = q (\gamma - 1) = \frac{mp}{m-1}$ and $\frac{m-1}{q} = \frac{m-p}{mp}$. Therefore,

\[
\left( \int_{M} \varphi^{\frac{mp}{m-p}} \, d\mu \right)^{\frac{m-p}{mp}} \leq C_3 \int_{M} (|\nabla \varphi| + \varphi (|H_f - \tilde{V} f|))^{p} \, d\mu.
\]

We obtain the constant $S$ as in (4) by taking $t \to 2$ in

\[
\lim_{t \to 2} C_3 = \frac{2mr_0}{k_0} \left( \frac{\omega_{m-1}^{1}}{1 - \kappa} \right)^{\frac{m}{m}} e^{i^*} = Se^{i^*}.
\]
Theorem 1 is proved.

4 Proof of Theorem 2

Proof Consider the neighborhood $V = \{ x \mid \rho(\xi) = d_M(\xi, \partial M) < \epsilon \}$. Take $A > 1$ and let $\varphi = \varphi(\cdot, \epsilon) : M \to \mathbb{R}$ be a nonnegative $C^1$ function satisfying:

(i) $\varphi(x) = 1$, if $d_M(x, \partial M) \geq \epsilon$;
(ii) $0 < \varphi(x) < 1$ and $|\nabla \varphi| \leq A\epsilon^{-1}$, if $0 < d_M(x, \partial M) < \epsilon$;
(iii) $\varphi|_{\partial M} = 0$.

By Theorem 1, we obtain

$$\frac{1}{S} \left( \int_{\{ \rho(\xi) \geq \epsilon \}} d\mu \right)^{\frac{m-1}{m}} \leq \int_M |\nabla \varphi|d\mu + \int_M |H_f - \tilde{\nabla} f|d\mu,$$

provided that condition (7) holds. Using that $|\nabla \rho| = 1$, everywhere in $V$, it follows from the coarea formula that

$$\int_M |\nabla \varphi|d\mu = \int_M |\nabla \varphi|e^{-f}dM \leq \frac{A}{\epsilon} \int_0^\epsilon \int_{\{ \rho(\xi) = \tau \}} e^{-f}d\mathcal{H}^{m-1}$$

Since $\partial M = \{ \xi \mid d_M(\xi, \partial M) = 0 \}$, by taking $\epsilon \to 0$, we obtain that

$$\int_M |\nabla \varphi|d\mu \leq A \int_{\partial M} e^{-f}d\mathcal{H}^{m-1} = A \text{vol}_f(\partial M).$$

Therefore, since $A > 1$ is arbitrary, it holds

$$\frac{1}{S} \text{vol}_f(M)^{\frac{m-1}{m}} \leq \text{vol}_f(\partial M) + \int_M |H_f - \tilde{\nabla} f|d\mu$$

Theorem 2 is proved.

5 Proof of Theorem 3

Let $K \in C^0(\mathbb{R})$ be a nonnegative even function such that the radial curvatures of $\tilde{M}$ satisfy $\tilde{K}_{\text{rad}}(\xi) \leq K(r(\xi))$, for all $\xi \in M$. Let $h : (0, r_0) \to (0, s_0)$ be an increasing solution of (1) with $(0, s_0) = h(0, r_0)$. Assume by contradiction that an end $E$ of $M$ has finite $f$-volume. Let $B = B_{\lambda_0}(\xi)$ be a geodesic ball of $M$ of radius $\lambda_0$ and center $\xi$. Take $\lambda_0$ sufficiently large so that $\partial E \subset B$ and $\text{vol}_f(E - B) < A$, where $0 < A < 1$ is a small constant satisfying

$$\tilde{J}_A = \left( \frac{\omega_{m-1}^e f^*}{1 - \kappa} A \right)^{\frac{1}{m}} \leq s_0;$$

$$h^{-1}(\tilde{J}_A) \leq 2 \min(\text{inj}_M, r_0),$$

for some $\kappa \in (0, 1)$. Moreover, we take $\lambda_0$ sufficiently large satisfying further

$$\|H_f - \tilde{\nabla} f\|_{L^p(E-B)} < C, \quad \text{if } m \leq p < \infty;$$

$$\|H_f - \tilde{\nabla} f\|_{L^\infty(E-B)\text{vol}_f(E-B)\frac{1}{p}} < C, \quad \text{if } p = \infty,$$

where $2C = (Se_{\frac{p}{m}})^{-1}$. 

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Now take \( \lambda_1 > \lambda_0 \) sufficiently large so that \( d_M(\partial E, x) > 2\lambda_0 \), for all \( x \in E - B_{2\lambda_1} \). For all \( q \in E - B_{2\lambda_1} \), we obtain that the ball \( B_{\lambda_1}(q) \subset E - B \). In particular, by (35), Theorem 2 applies to \( B_r(q) \), for all \( 0 < r < \lambda_1 \). By Hölder inequality, we obtain that

\[
\int_{B_r(q)} |H_f - \bar{\nabla} f| \, d\mu \leq \|H_f - \bar{\nabla} f\|_{L^{p-1}_f(E - B)} \text{vol}_f(B_r(q))^{\frac{p-1}{p}}, \quad \text{if } m \leq p < \infty; \\
\int_{B_r(q)} |H_f - \bar{\nabla} f| \, d\mu \leq \|H_f - \bar{\nabla} f\|_{L^{\infty}_f(E)} \text{vol}_f(E - B)^\frac{1}{m} \text{vol}_f(B_r(q))^{\frac{m-1}{m}}.
\]

(37)

Furthermore, if \( m \leq p < \infty \) then, since \( \text{vol}_f(B_r(q)) \leq \text{vol}_f(E - B) < 1 \) and \( \frac{(p-1)}{p} \geq \frac{(m-1)}{m} \), it holds that \( \text{vol}_f(B_r(q))^{\frac{p-1}{p}} \leq \text{vol}_f(B_r(q))^{\frac{m-1}{m}} \). Thus, by Theorem 2, and using (36) and (37), we obtain \( C \text{vol}_f(B_r(q))^{\frac{m-1}{m}} \leq \text{vol}_f(\partial B_r(q)) \). Thus, by using the coarea formula,

\[
\frac{d}{dr} \text{vol}_f(B_r(q))^{\frac{1}{m}} = m^{-1} \text{vol}_f(B_r(q))^{-(1 - \frac{1}{m})} \text{vol}_f(\partial B_r(q)) \geq c,
\]

(38)

for all \( 0 < r < \lambda_1 \), where \( c = \frac{C}{m} > 0 \). Hence, \( \text{vol}_f(B_r(q)) \geq (c r)^m \), for all \( 0 < r < \lambda_1 \) and \( q \in E - B_{2\lambda_1} \).

Since \( M \) is complete and \( E \) is an unbounded connected component of \( M \), we can take \( k \in E \cap (B_{2k\lambda_1} - B_{2k\lambda_1 - 1}) \), for all \( k = 2, 3, \ldots \). Note also that \( B_{r_i}(q_k) \subset E - B_{2k\lambda_1} \) and \( B_{r_i}(q_k) \cap B_{r_j}(q_j) = \emptyset \), if \( k \neq l \), where \( 0 < r_i < \frac{1}{2}\lambda_1 \). Thus, we obtain \( \text{vol}_f(E) \geq \sum_{k=2}^{\infty} \text{vol}_f(B_{r_i}(q_k)) \geq \sum_{k=2}^{\infty} (cr_1)^m = \infty \), which is a contradiction. Theorem 3 is proved.

**Remark 3** In order to obtain (38), it was used that the derivative of the weighted volume of \( B_r(q) \) is the weighted area of \( \partial B_r(q) \). Although it is a well-known fact, it is worth to mention the main steps to prove it. First, the function \( \gamma(x) = d(x, q) \), with \( x \in M \), is Lipschitzian; hence, it has derivative almost everywhere in \( M \). Furthermore, since \( M \) is complete, one has \( \partial B_r(q) = \{ x \in M; d(x, q) = r \} \) and, by Gauss Lemma, the gradient vector of \( \gamma \) must satisfies \( |\nabla \gamma| = 1 \) at the differentiable points of \( \gamma \). Thus, using coarea formula,

\[
\text{vol}_f(B_r) = \int_{B_r} |\nabla \gamma| e^{-f} \, dM \\
= \int_{E} \int_{\{ x \in B_r; \gamma(x) = r \}} e^{-f} \, ds^2 d\tau = \int_{0}^{r} \text{vol}_f(\partial B_r(q)) d\tau.
\]

Hence, we have that \( \frac{d}{dr} \text{vol}_f(B_r(q)) = \text{vol}_f(\partial B_r(q)) \).

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**References**

1. Abresch, U.: Lower curvature bounds, Toponogov’s theorem, and bounded topology. Ann. Sci. École Norm. Sup. 18, 651–670 (1985)
2. Bayle, V.: Propriétés de concavité du profil isopérimétrique et applications, Thèse de Doctorat (2003)
3. Castillon, P.: Submanifolds, isoperimetric inequalities and optimal transportation. J. Funct. Anal. 259(1), 79–103 (2010)
4. Choe, J.: Isoperimetric Inequalities of Minimal Submanifolds, Global Theory of Minimal Surfaces, Clay Math. Proc., vol. 2, pp. 325–369. Amer. Math. Soc., Providence, RI (2005)
5. Cheng, X., Mejia, T., Zhou, D.: Stability and Compactness for Complete f-minimal Surfaces, arXiv:12108076v1 [math.DG] (2012)
6. Cavalcante, M.P., Mirandola, H., Vitorio, F.: The Non-parabolicity of Infinite Volume Ends. Preprint, to appear in Proc. Amer. Math. Soc. (2012)
7. Cheung, L.-F., Leung, P.-F.: The mean curvature and volume growth of complete noncompact submanifolds. Differ. Geom. Appl. 8(3), 251–256 (1998)
8. do Carmo, M.P., Wang, Q., Xia, C.: Complete submanifolds with bounded mean curvature in a Hadamard manifold. J. Geom. Phys. 60(1), 142–154 (2010)
9. Drugan, G.: An immersed $S^2$ self-shrinker, arXiv:1304.0032. To appear in Trans. of Amer. Math. Soc. (2013)
10. Espinar, J.M.: Manifolds with Density, Applications and Gradient Schrödinger Operators, arXiv:1209.6162v6 [math.DG] (2012)
11. Frensel, K.: Stable complete surfaces with constant mean curvature. Bull. Braz. Math. Soc. 27, 129–144 (1996)
12. Fu, H.-P., Xu, H.-W.: Total curvature and $L^2$ harmonic 1-forms on complete submanifolds in space forms. Geom. Dedicata 144, 129–140 (2010)
13. Gromov, M.: Isoperimetric of waists and concentration of maps. Geom. Funct. Anal. 13(1), 178–215 (2003)
14. Greene, R.E., Wu, H.: Function Theory on Manifolds Which Possess a Pole. Lecture Notes in Mathematics, vol. 699. Springer, Berlin (1979)
15. Hoffman, D., Spruck, J.: Sobolev and isoperimetric inequalities for Riemannian submanifolds. Commun. Pure. Appl. Math. 27, 715–727 (1974)
16. Itokawa, Y., Machigashira, Y., Shiohama, K.: Generalized Toponogov’s Theorem for Manifolds with Radial Curvature Bounded Below. Explorations in complex and Riemannian geometry, 121130, Contemp. Math., 332, Amer. Math. Soc., Providence, RI (2003)
17. Impera, D., Rimoldi, M.: Stability Properties and Topology at Infinity of $f$-Minimal Hypersurfaces, arXiv:1302.6160v1 [math.DG] (2013)
18. Morgan, F.: Manifolds with density. Not. Am. Math. Soc. 52(8), 853–858 (2005)
19. Michael, J.H., Simon, L.M.: Sobolev and mean-value inequalities on generalized submanifolds of $\mathbb{R}^n$. Commun. Pure Appl. Math. 26, 361-379 (1973)
20. Monteau, O., Wang, J.: Analysis of weighted Laplacian and Applications to Ricci Solitons, arXiv:1112.3027v1. To appear in Comm. Anal. Geom. (2011)
21. Monteau, O., Wang, J.: Geometry of Manifolds with Densities, arXiv:1211.3996v1 [math.DG] (2012)
22. Pigola, S., Rigoli, M., Setti, A.G.: Vanishing and Finiteness Results in Geometric Analysis, Progress in Mathematics, vol. 266. Birkhäuser Verlag, Basel, A generalization of the Bochner technique (2008)
23. Kondo, K., Ohta, S.-I.: Topology of complete manifolds with radial curvature bounded from below. Geom. Funct. Anal. 17, 1237–1247 (2007)
24. Wei, G., Wylie, W.: Comparison geometry for the Bakry–Emery Ricci tensor. J. Differ. Geom. 83(2), 377–405 (2009)