Placement of M-Sequences over the Field \( F_p \) in the Space \( R^n \)

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Abstract: Spread spectrum communication systems are widely used today in a variety of applications for different purposes such as access of same radio spectrum by multiple users (multiple access), anti-jamming capability (so that signal transmission cannot be interrupted or blocked by spurious transmission from enemy), interference rejection, secure communications, multi-path protection, etc. Several spreading codes are popular for use in practical spread spectrum systems. One of these important codes is Maximal Sequence (M-sequence) length codes, These are the longest codes that can be generated by a shift register of a specific length, The number of 1-s in the complete sequence and the number of 0-s will differ by one, Further, the auto-correlation of an m-sequence is -1, another interesting property of an M-sequence is that the sequence, when added (modulo-2) with a cyclically shifted version of itself, results in another shifted version of the original sequence. Hence, the M-sequences are also known as, pseudo-noise or PN sequences. Current article study placement M-Sequences over the finite field \( F_p \) or Mp-Sequences (where \( p \) is a prime number) in the space \( R^n \), these sequences can be generated as a closed set under the addition. These sequences form additive groups with the corresponding null sequence that was generated by the feedback shift registers. Such Mp-Sequences see a great application in the forward links of communication channels. Furthermore, they form coders and decoders that combine the information during the connection process with the backward links of these channels. These sequences scrutinize the transmitted information to enable it to reach the receivers in an accurate form. This study has defined eminent surfaces in the vector space ‘R’ having dimensions of ‘n’ as quadratic forms, spheres, and planes that contain these sequences.

Keywords: M-Sequences, Coefficient of Correlation, Orthogonal Sequences, Additive Group, Code, Field \( F_p \), Space \( R^n \)

1. Introduction

The sequence of order \( k \) that is homogeneously linear has been supported by recurring:

\[
a_{n+k} + \sum_{i=0}^{k-1} \lambda_i a_{n+i} = 0, \lambda_i \in F_p = \{0,1,2,...,p-1\}, p \text{ is prime}
\]

(1)

With the characteristic polynomial;

\[f(x) = x^k + \lambda_{k-1} x^{k-1} + \lambda_{k-2} x^{k-2} + ... + \lambda_1 x + \lambda_0\]  

(2)

If \( f(x) \) is prime and \( a_1, a_2, a_3, ... a_k \) are not equal to zero, then the sequence \( a_1, a_2, a_3, ... \) is periodic with the maximum length or period \( p^k - 1 \). The calculations preformed using mod \( p \) is called Mp-Sequence over \( F_p \), and the number of all these non-zero sequences corresponding to all cases of \( a_1, a_2, ... a_k \) is \( p^k - 1 \). These Mp-Sequences can be widely used in communication channels, and coders, and decoders by mod \( p \) as. The Mp-Sequences have zero sequence form an additive group, when the addition is pre-formed at mod \( p \).

Example 1. The sequence over \( F_3 \) of Linear Recurring;

\[a_{n+2} + a_{n+1} + 2a_n = 0 \text{ or } a_{n+2} = 2a_{n+1} + a_n\]  

(3)

The prime factor with the characteristic equation \( x^2 + x + 2 = 0 \) as well as the characteristic polynomial \( f(x) = x^2 + x + 2 \) produce \( F_2 \). If \( \alpha \) is a root of \( f(x) \) and generates \( F_{3^2} \), then this characteristic equation can be solved as \( (\alpha^n, \alpha^{3n}) \). While equation (3) has a general solution that is given by;

\[a_n = 2\alpha a^n + (1 + \alpha)a^{3n}\]  

(4)

while the sequence is periodic with the period \( 3^2 - 1 = 8 \).
Initially it has position \( a_1 = 1, a_2 = 0 \), then \( S_1 = (1 \ 0 \ 1 \ 2 \ 2 \ 0 \ 2 \ 1) \); whereas, by the cyclic permutations on
\[
S_2 = (1 \ 1 \ 0 \ 1 \ 2 \ 2 \ 0 \ 2); \quad S_3 = (2 \ 1 \ 1 \ 0 \ 1 \ 2 \ 2 \ 0); \quad S_4 = (0 \ 2 \ 1 \ 1 \ 0 \ 1 \ 2 \ 2); \quad S_5 = (2 \ 0 \ 2 \ 1 \ 1 \ 0 \ 1); \quad S_6 = (0 \ 1 \ 2 \ 2 \ 0 \ 2 \ 1 \ 1)
\]
(5)

The first two digits represent the initial position for each of the sequences for the feedback register. In the given example the resulting sequences is: 1012021110122).

**Example 2:** The Sequence over \( F_5 \) as the Linear Recurring is;
\[
an_{n+2} + an_{n+1} + 2an = 0 \quad \text{or} \quad an_{n+2} = 4an_{n+1} + 3an \quad (6)
\]
The characteristic equation of this sequence is \( x^2 + x + 2 = 0 \) and its characteristic polynomial is \( g(x) = x^2 + x + 2 \) which is a prime polynomial generates \( F_2 \), for the initial position \( 0(1) \) we have the following sequence;
\[
014434023313041121032242014\ldots
\]
(7)

**Example 3:** The Sequence over \( F_2 \) of Linear Recurring is given by;
\[
an_{n+3} = an_{n+1} + an \quad \text{or} \quad an_{n+3} + 2an_{n+1} + an = 0 \quad (8)
\]
The prime factor with the characteristic equation \( x^3 + x + 1 = 0 \) and the polynomial characteristic \( f(x) = x^3 + x + 1 \) produces the field \( F_2 \) by considering sequence that is periodic within the period \( 2^3 - 1 = 7 \). Initially, \( a_1 = 1, a_2 = 0, a_3 = 0 \), thereby, \( S_1 = (1001101) \), while with the cyclic permutations of \( S_1 \) has \( S = \{S_1, S_2, S_3, S_4, S_5, S_6, S_7\} \) where;
\[
S_2 = (1100101), S_3 = (1110010), S_4 = (0111001) \quad (9)
\]
\[
S_5 = (1011100), S_6 = (0101110), S_7 = (0010111) \quad (10)
\]

There are first three digits that represent the initial position for each of these sequences for the feedback register [1-10]. Some of the famous surfaces are in the vector space \( \mathbb{R}^n \) having prime \( n \) dimension. The quadratic forms are between the family of famous surfaces and their equation is as follows;
\[
f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} a_i x_i^2 + 2 \sum_{1 \leq i < j \leq n} a_{ij} x_i x_j + \sum_{i=1}^{n} b_i x_i + d = 0 \quad (11)
\]
The original point can shift and rotate the axes; therefore, we can write equation (11) as symmetric equations for some symmetric axes and symmetric planes;
\[
\sum_{i,j=1}^{n} a_{ij} x_i x_j = k, \quad a_{ij} = a_{ji}; i,j = 1,2,\ldots,n \quad (12)
\]

The above equation (11 or 12) represents some of the eminent surfaces in the space \( R^n \) that are Ellipsoids, Elliptobalsoids, Spheres, or Lines [11-14].

The equation of the Ellipsoid in the space \( R^n \) with the center \( o \) is;
\[
x_1^2 + x_2^2 + \cdots + x_n^2 = 1 \quad (13)
\]

The equation of the sphere in the space \( R^n \) with the center \( o \) and the radius \( r \) is;
\[
x_1^2 + x_2^2 + \cdots + x_n^2 = r^2 \quad (14)
\]
The equation of the plane in the space \( R^n \) is;
\[
\sum_{i=1}^{n} a_i x_i = b \quad (15)
\]

2. Research Method and Materials

The Ultimately Periodic Sequence \( a_0, a_1, \ldots \) in \( F_p = \{0, 1, 2, \ldots, p-1\} \) along with \( r \) which, represents periodic or the smallest period given as; \( a_{n+r} = a_n; n = 0, 1, \ldots \) \{1-4\}. Euler function \( \varphi(n) \) represents relative prime number of the natural numbers through \( n \). \{5-8\}. The code \( C \) of the form \( [n, k, d] \) is used, if each element (Codeword) has the length \( n \). The Rank \( k \) is the number of information components and minimum distance is \( d \) \{9\}. Two vectors are; \( x = (x_0, x_1, \ldots, x_{n-1}) \) and \( y = (y_0, y_1, \ldots, y_{n-1}) \) on \( F_p \) having \( n \) length. The coefficient of correlations functions of \( x \) and \( y \) is indicated by \( R_{xy} ; \{4,5\} \)
\[
R_{xy} = \sum_{i=0}^{n-1} (-1)^{x_i+y_i} \quad (16)
\]
where, \( x_i + y_i \) is computed mod \( p \). The periodic sequence \( (a_i)_{i=0}^{n} \) over \( F_p \) with \( r \) period comprises of the properties within “Ideal Auto Correlation” but only in the case when there is a “periodic auto Correlation”.

\[ R_a(r) \] as given below for \( p > 2 \):
\[
R_a(r) = \left\{ \begin{array}{ll} \frac{r}{p} & \text{for } r \equiv 0 \mod p \\ 0 & \text{otherwise and } r \neq \frac{r}{2} \end{array} \right. \quad (17)
\]
When \( [6, 7] \);
\[
R_a(r) = \sum_{\tau=r}^{n-r} (-1)^{(a(r+\tau)+a(\tau)} \quad (18)
\]
Any Periodic Sequence \( a_0, a_1, \ldots \) over \( F_p \) having prime polynomial, represents an ideal autocorrelation and orthogonal cyclic code \{8, 9\}. 

**Theorem 1.**
\[ \text{i. If } a_0, a_1, \ldots \text{ is a sequence of order } k \text{ in } F_p, \text{ which is homogeneously linear recurring and satisfying equation (1) and } \lambda_0 \neq 0, \text{ then it is a periodic sequence.} \]
\[ \text{ii. If the current sequence is homogeneous and linear recurring then it will be periodic, and if its characteristic polynomial } f(x) \text{ is prime and the period of the sequence is } r \text{ then } r | \text{ord } f(x). \]
\[ \text{iii. If } f(x) \text{ is prime characteristic polynomial of the sequence then } r \text{ the period of the sequence is given by } r = p^k - 1, \text{ known as } M_p - \text{ Sequence over } F_p \text{ or briefly } M_p - \text{Sequence. } [10-11] \]

**Theorem 2** (Fermat’s theorem).

If there is the finite field \( F \), having \( q \) elements, then all the elements of \( F \) will satisfy this equation [12];
\[ x^q = x \quad (19) \]

Theorem 3.

g(x) representing the prime characteristic polynomial over \( \mathbb{F}_p \) of the (H. L. R. S.) \( a_0, a_1, \ldots \) for \( k \) degree, while the root of \( g(x) = 0 \) in \( \mathbb{F}_q \) considering any splitting field \( \mathbb{F}_p \), then the general term of the sequence is:

\[ a_n = \sum_{i=1}^{k} C_i (a^{p^{i-1}})^n \quad (20) \]

And the coefficients \( C_i \) determined in \( \mathbb{F}_q \) uniquely. [13]

Theorem 4.

For any prime \( p \) and positive integers \( m \) and \( n \),

\[ (q^m - 1) | (q^n - 1) \Leftrightarrow m \mid n \quad (21) \]

If \( F_q \) represents a field within the \( q = p^n \) is the number of elements in it then the order of any subfields of it is of the form \( p^m \), where \( m \) is positive and \( m \mid n \), or \( m \) is positive divisor of \( n \). Also, through inverse, if \( m \mid n \) and \( m \) is positive, then there is a subfield \( F_q \) of Order \( p^m \). In any field, if \( \beta^2 = \beta \) then \( \beta = 0 \) or 1. [14]

Theorem 5.

The number of unitary irreducible polynomials of degree \( m \) and order \( e \) in \( F_q[x] \) equals to \( \varphi(e)/m \) if \( e \geq 2 \), where \( m \) is the order of \( q \) mod \( e \), it equals 2 in case \( m = e = 1 \), and equals to zero elsewhere. [15, 16]

Theorem 6.

i. The necessary and sufficient condition for the square matrix \( A = [a_{ij}] \) to be similar to the diagonal matrix \( D = [d_{ij}] \) (T. e: There is a regular matrix \( B = [b_{ij}] \) such, \( A = B D B^{-1} \) or \( A = D B^{-1} D^{-1} \) ) is the matrix \( A \) is symmetric and has \( n \) independent characteristic vectors.

ii. If the independent characteristic vectors of the matrix \( A \) are \( V_1 = (v_{11}, v_{12}, \ldots, v_{1n})^T \), \ldots, \( V_n = (v_{n1}, v_{n2}, \ldots, v_{nn})^T \) corresponding to the characteristic values \( \lambda_1, \ldots, \lambda_n \) then:

\[
B = [V_1, V_2, \ldots, V_n] \quad \text{and} \quad D = \begin{bmatrix}
\lambda_1 & 0 & \ldots & 0 \\
0 & \lambda_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \lambda_n
\end{bmatrix}
\]

And if the axis \( Y_1, Y_2, \ldots, Y_n \) corresponding to the independent characteristic vectors \( V_1, V_2, \ldots, V_n \), then the quadratic form \( \sum_{i,j=1}^{n} a_{ij} X_i X_j = X^T A X \) can be shown as:

\[
\sum_{i,j=1}^{n} a_{ij} X_i X_j = \sum_{i=1}^{n} \lambda_i Y_i^2 = Y^T D Y
\]

Where, \( Y = [Y_1, Y_2, \ldots, Y_n] \). [17-19] Corollary 7.

All non-zero elements in \( GF(p^n) \) satisfy the equation \( x^p = x \), or the equation \( x^{p^n-1} = 1 \mid p - 1 \mod p \). The corresponding matrix of a linear recurring sequence generated by prime polynomial or any power of the matrix satisfies the following condition:

\[
R_j + R_{j+\frac{n}{2}} = 0 \mod p : j = 1, 2, 3, \ldots, \frac{n}{2} \quad (23)
\]

When \( R_j \) is the \( j^\text{th} \) row.

Sum of all the entries in any row (or any one period) of the \( Mp \)-Sequence is equal to zero \( \mod p \). Practical examples show that the repetition of any non-zero element in one period is \( p^{\alpha-1} \), but the repetition of zero in one period is \( p^{\alpha-1} - 1 \). [20]

\[
\sum_{i=1}^{n-1} a_i^2 = 0 \mod p \quad (24)
\]

3. Results and Discussions (Findings)

3.1. Quadratic Forms

The general form of second degree equations is;

\[
f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} a_{ii} x_i^2 + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} a_{ij} x_i x_j + \sum_{i=1}^{n} b_i x_i + d = 0 \quad (25)
\]

If \( a_{ij} = a_{ji} \) it can write (25) by the form:

\[
f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} a_{ii} x_i^2 + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} a_{ij} x_i x_j + \sum_{i=1}^{n} b_i x_i + d = 0 \quad (26)
\]

By translation of the axes, the first degree terms can be removed; or

\[
f(x'_1, x'_2, \ldots, x'_n) = \sum_{i=1}^{n} a_{ii} x'_i^2 + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} a_{ij} x'_i x'_j + k' = 0 \quad (27)
\]

Or,

\[
\sum_{i,j=1}^{n} a_{ij} X_i X_j + k' = 0 \quad (28)
\]

Or,

\[
\sum_{i,j=1}^{n} a_{ij} X_i X_j = k \quad (29)
\]

The form,

\[
\sum_{i,j=1}^{n} a_{ij} X_i X_j = k \quad (30)
\]

When, \( a_{ij} = a_{ji} \) is called as “Quadratic Form” and can be written as;

\[
= [X_1 X_2 \ldots X_n] \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix} [X_1] \quad (31)
\]

Where;

\[
A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}, X = [X_1] \quad (32)
\]

And the equation (29) becomes:

\[
\sum_{i,j=1}^{n} a_{ij} X_i X_j = X^T A X = k \quad (33)
\]
The equation (33) is symmetric for the origin of coordinate and if there are symmetric axes it pass from the origin coordinates and the equation (29) becomes:

$$X^TAX = k$$ (34)

The equation of the sphere in the space $\mathbb{R}^n$ having a center as the origin point and $r$ as the radius:

$$x_1^2 + x_2^2 + \cdots + x_n^2 = r^2 \text{ or } \sum_{i=1}^{n} x_i^2 = r^2$$ (35)

The equation of the plane in the space $\mathbb{R}^n$ is:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n + d = 0 \text{ or } \sum a_ix_i + d = 0$$ (36)

**3.2. Return a Quadratic Form to Its Regular Form**

In theorem 6, if the matrix $A$ of the quadratic form contains $n$ independent characteristic vectors $V_1, V_2, \ldots, V_n$ consequently to the characteristic values $\lambda_1, \lambda_2, \ldots, \lambda_n$ respectively, and if the axis $Y_1, Y_2, \ldots, Y_n$ corresponding to these independent characteristic vectors, then the quadratic form:

$$\sum_{i,j=1}^{n} a_{ij}X_iX_j = X^TAX$$

Can be shown as;

$$\sum_{i,j=1}^{n} a_{ij}X_iX_j = \sum_{i=1}^{n} \lambda_i Y_i^2 = Y^TDY \text{ when } D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$ (38)

Moreover, the equation:

$$\sum_{i,j=1}^{n} a_{ij}X_iX_j = k$$ (39)

becomes:

$$\sum_{i=1}^{n} \lambda_i Y_i^2 = k$$ (40)

**3.3. Mp-Sequences as a Points on the Quadratic Surface**

Each of the Mp-Sequences $(a_i)_n$ over $\mathbb{F}_p$ generated by the prime polynomial $f(x)$ of degree $k$ has the period $r = p^k - 1$; where as, each of these non-zero elements has repetition $p^{k-1}$ and its zero element have the repetition $p^{k-1} - 1$.

One family of the functions of the quadratic forms;

$$F(x_1, x_2, \ldots, x_n) \equiv \sum_{i=1}^{n} a_{ii}x_i^2 + 2 \sum_{i=1,j>i}^{n} a_{ij}x_ix_j = C$$ (41)

As a surface of second degree in the space $\mathbb{R}^n$, the coefficients were determined as $a_{ii}, i = 1, 2, \ldots, n, a_{ij}, i = 1, 2, \ldots, n - 1 & j > i$ and $C'$ for the Mp-Sequences, when $n = p^k - 1$ and $x_i = a_i, i = 1, 2, \ldots, n$, will be on the surface. The function;

$$F(x_1, x_2, \ldots, x_n) \equiv \sum_{i=1}^{n} x_i^2 + \sum_{i=1,j>i}^{n-1} x_ix_j = C$$ (42)

The constant $C$ was determined for the sequences which will lie on the surface (42).

It is known that;

$$\sum_{i=1}^{n-1} x_i^2 + 2 \sum_{i=1,j>i}^{n-1} x_ix_j = (x_1 + x_2 + \cdots + x_n)^2$$ (43)

Or;

$$\sum_{i=1,j>i}^{n-1} x_ix_j = (x_1 + x_2 + \cdots + x_n)^2 - \sum_{i=1}^{n-1} x_i^2$$ (44)

Or;

$$\sum_{i=1}^{n} x_i^2 + \sum_{i=1,j>i}^{n-1} x_ix_j = (x_1 + x_2 + \cdots + x_n)^2 - 2\sum_{i=1}^{n-1} x_i^2$$ (45)

Or;

$$\sum_{i=1}^{n} x_i^2 + \sum_{i=1,j>i}^{n} x_ix_j = (x_1 + x_2 + \cdots + x_n)^2 + \sum_{i=1}^{n} x_i^2$$ (46)

It is known that;

$$\begin{cases} \binom{1+2+\ldots+n}{n} = (n(n+1))/2 \\ \binom{1^2+2^2+\ldots+n^2}{n} = (n(n+1)(2n+1))/6 \\ \binom{1^3+2^3+\ldots+n^3}{n} = [(n(n+1))/2]^2 \end{cases}$$ (47)

Thus:

$$x_1 + x_2 + \cdots + x_n = p^{k-1}(1 + 2 + \cdots + (p-1)) = p^{k-1} \frac{(p-1)p}{2}$$

$$x_1 + x_2 + \cdots + x_n = p^{k-1} \frac{(p-1)}{2}$$ (48)

$$(x_1 + x_2 + \cdots + x_n)^2 = \frac{p^{2k}(p-1)^2}{4}$$

$$\sum_{i=1}^{n} x_i^2 = p^{k-1}(1^2 + 2^2 + \cdots + (p-1)^2) = p^{k-1} \frac{(p-1)(2p-1)}{6}$$
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\[
(x_1 + x_2 + \ldots + x_n)^2 + \sum_{i=1}^{n} x_i^2 = p^k(p-1)(2p-1) \tag{49}
\]

\[
(x_1 + x_2 + \ldots + x_n)^2 + \sum_{i=1}^{n} x_i^2 = \frac{p^{2k}(p-1)^2}{4} \cdot \frac{p^k(p-1)(2p-1)}{6} \tag{50}
\]

\[
(x_1 + x_2 + \ldots + x_n)^2 + \sum_{i=1}^{n} x_i^2 = p^k(p-1)[3p^k(p-1)+2(2p-1)] \tag{51}
\]

The Mp-Sequences lie on the surface if;

\[
\frac{p^k(p-1)[3p^k(p-1)+2(2p-1)]}{24} \tag{52}
\]

Or the equation (42) will be;

\[
F(x) \equiv \sum_{i=1}^{n} x_i^2 + \sum_{i=j,i>1}^{n-1} x_i x_j = \frac{p^k(p-1)[3p^k(p-1)+2(2p-1)]}{24} \tag{53}
\]

Or;

\[
\sum_{i=1}^{n} x_i^2 + 2 \sum_{i=j,i>1}^{n-1} x_i x_j = \frac{p^k(p-1)[3p^k(p-1)+2(2p-1)]}{24} \tag{54}
\]

The equation can be written as;

\[
\begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{bmatrix}

= \frac{p^k(p-1)[3p^k(p-1)+2(2p-1)]}{24} \tag{55}
\]

The Matrix can be determined;

\[
A = \begin{bmatrix}
1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots \\
\frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2}
\end{bmatrix} \tag{56}
\]

For matrix \( A \) characteristic polynomial is represented as \( I_n \) and given by;

\[
I_n = \begin{vmatrix}
\lambda - 1 & -\frac{1}{2} & -\frac{1}{2} & \cdots & -\frac{1}{2} \\
-\frac{1}{2} & \lambda - 1 & -\frac{1}{2} & \cdots & -\frac{1}{2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \cdots & \lambda - 1
\end{vmatrix} = 0 \tag{57}
\]

Subtracting the last column from the all columns results in

\[
I_n = \begin{vmatrix}
\lambda - \frac{1}{2} & 0 & 0 & \cdots & -\frac{1}{2} \\
0 & \lambda - \frac{1}{2} & 0 & \cdots & -\frac{1}{2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{1}{2} - \lambda & \frac{1}{2} - \lambda & \frac{1}{2} - \lambda & \cdots & \lambda - 1
\end{vmatrix} = 0 \tag{58}
\]

The determinant is computed according to the first row;

\[
I_n = \left( \lambda - \frac{1}{2} \right) I_{n-1} + (-1)^{n+1} \left( \frac{1}{2} \right) \begin{vmatrix}
0 & \left( \lambda - \frac{1}{2} \right) & 0 & \cdots & 0 \\
0 & 0 & \left( \lambda - \frac{1}{2} \right) & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\left( \frac{1}{2} - \lambda \right) & \left( \frac{1}{2} - \lambda \right) & \left( \frac{1}{2} - \lambda \right) & \cdots & \left( \frac{1}{2} - \lambda \right)
\end{vmatrix}_{(n-1)} \tag{59}
\]

Computation of the determinant according to the first column;
\[ I_n = (\lambda - \frac{1}{2}) I_{n-1} + (-1)^n \left( \frac{1}{2} \right) (\lambda - n - \frac{1}{2}) \left( \lambda - \frac{1}{2} \right)^{n-2} \] 

Thus;

\[ I_n = (\lambda - \frac{1}{2}) I_{n-1} + (-1)^{2n+3} \left( \frac{1}{2} \right) (\lambda - \frac{1}{2})^{n-1} \] 

\[ I_n = (\lambda - \frac{1}{2})^{n-1} \left( (\lambda - \frac{1}{2}) - n \frac{1}{2} \right) \] 

First the rule is to be proved true for \( n = 3 \) is;

\[ I_3 = \begin{vmatrix} (\lambda - 1) & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & (\lambda - 1) & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & (\lambda - 1) \end{vmatrix} = \begin{vmatrix} (\lambda - \frac{1}{2}) & 0 & -\frac{1}{2} \\ 0 & (\lambda - \frac{1}{2}) & -\frac{1}{2} \\ \left( \frac{1}{2} - \lambda \right) & \left( \frac{1}{2} - \lambda \right) & (\lambda - 1) \end{vmatrix} \]

\[ I_3 = (\lambda - \frac{1}{2}) \left[ (\lambda - \frac{1}{2})(\lambda - 1) - \frac{1}{4}(\lambda - \frac{1}{2}) \right] - (-1)^4 \left( \frac{1}{2} \right) (\lambda - \frac{1}{2})^2 = (\lambda - \frac{1}{2})^2 (\lambda - 1 - \frac{1}{2} - \frac{1}{2}) = (\lambda - \frac{1}{2}) \left( (\lambda - \frac{1}{2}) - 3 \frac{1}{2} \right) \] 

Therefore, the rule is proven true for \( n = 3 \). Secondly, the rule is true on \( n-1 \) and also true for \( n \), which means:

\[ I_{n-1} = (\lambda - \frac{1}{2})^{n-2} \left( (\lambda - \frac{1}{2}) - (n - 1) \frac{1}{2} \right) \] 

It is true and will prove that;

\[ I_n = (\lambda - \frac{1}{2})^{n-1} \left( (\lambda - \frac{1}{2}) - n \frac{1}{2} \right) \]

\[ I_n = (\lambda - \frac{1}{2}) I_{n-1} + (-1)^{2n+3} \left( \frac{1}{2} \right) (\lambda - \frac{1}{2})^{n-1} \]

\[ I_n = (\lambda - \frac{1}{2}) \left( \lambda - \frac{1}{2} \right)^{n-2} \left( (\lambda - \frac{1}{2}) - (n - 1) \frac{1}{2} \right) + (-1)^{2n+3} \left( \frac{1}{2} \right) (\lambda - \frac{1}{2})^{n-1} \]

\[ I_n = (\lambda - \frac{1}{2})^{n-1} \left[ (\lambda - \frac{1}{2}) - (n - 1) \frac{1}{2} \right] + (-1)^{2n+3} \left( \frac{1}{2} \right) (\lambda - \frac{1}{2})^{n-1} \]

\[ I_n = (\lambda - \frac{1}{2})^{n-1} \left[ (\lambda - \frac{1}{2}) - (n - 1) \frac{1}{2} - \frac{1}{2} \right] \]

Thus;

\[ I_n = (\lambda - \frac{1}{2})^{n-1} \left[ (\lambda - \frac{1}{2}) - n \frac{1}{2} \right] \] 

The corresponding characteristic vectors were determined as;

\[ \lambda_i = \frac{1}{2}, i = 1, 2, ..., (n - 1) \text{ and } \lambda_n = (n + 1) \frac{1}{2} \] 

For \( \lambda_i = \frac{1}{2} \)

The characteristic equation becomes;

\[ (\lambda - \frac{1}{2})^{n-1} \left[ \lambda - (n + 1) \frac{1}{2} \right] = 0 \] 

The characteristic values are;

\[ \begin{cases} -\frac{1}{2} x_1 - \frac{1}{2} x_2 - \frac{1}{2} x_3 + \cdots + (-\frac{1}{2} x_n) = 0 \\ -\frac{1}{2} x_1 - \frac{1}{2} x_2 - \frac{1}{2} x_3 + \cdots + (-\frac{1}{2} x_n) = 0 \\ \vdots \end{cases} \]
\[ x_1 + x_2 + x_3 + \cdots + x_n = 0 \]  
(70)

And the characteristic vectors include:

\[
\begin{align*}
\vec{V}_1 &= (1, -1, 0, \ldots, 0)_n \\
\vec{V}_2 &= (1, 0, -1, \ldots, 0)_n \\
\vdots \\
\vec{V}_{n-1} &= (1, 0, 0, \ldots, -1)_n
\end{align*}
\]  
(71)

And the vectors are not orthogonal.

For \( x_1 = x_2 = \cdots = x_n \), the characteristic vector is: \( \vec{v}_1 = (1, 1, \ldots, 1)_n \). The vectors: \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \) are linearly independent because their determinant not equal to zero:

\[
\begin{bmatrix}
1 & -1 & 0 & \cdots & 0 \\
1 & 0 & -1 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
1 & 0 & 0 & \cdots & -1 \\
1 & 1 & 1 & 1 & 1
\end{bmatrix}
\]  
(73)

By adding all columns to the first column:

\[
\begin{bmatrix}
0 & -1 & 0 & \cdots & 0 \\
0 & 0 & -1 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & -1 \\
1 & 1 & 1 & 1 & 1
\end{bmatrix}
= (-1)^{n+1} \mathbf{n} \cdot (-1)^{-n} \neq 0
\]  
(74)

The vectors: \( \{\vec{V}_1, \vec{V}_2, \ldots, \vec{V}_n\} \) are linearly independent.

According to the new coordinates \( m_1, m_2, \ldots, m_n \) corresponding of the potential vectors \( \{\vec{V}_1, \vec{V}_2, \ldots, \vec{V}_n\} \) with the vectors unit \( \{\vec{1}_1, \vec{1}_2, \ldots, \vec{1}_n\} \):

\[
\vec{1}_1 = \left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, \ldots, 0 \right)_n \\
\vec{1}_2 = \left( \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, \ldots, 0 \right)_n \\
\vec{1}_{n-1} = \left( \frac{1}{\sqrt{n}}, 0, 0, \ldots, -\frac{1}{\sqrt{n}} \right)_n \\
\vec{1}_n = \left( \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}} \right)_n
\]  
(75)

While,

\[
\overrightarrow{oM} = x_1 \vec{1}_1 + x_2 \vec{1}_2 + \cdots + x_n \vec{1}_n = X_1 \vec{1}_1 + X_2 \vec{1}_2 + \cdots + X_n \vec{1}_n
\]  
(76)

Thus:

\[
\begin{align*}
x_1 &= \frac{1}{\sqrt{2}} X_1 + \frac{1}{\sqrt{2}} X_2 + \frac{1}{\sqrt{2}} X_3 + \cdots + \frac{1}{\sqrt{2}} X_{n-1} + \frac{1}{\sqrt{n}} X_n \\
x_2 &= -\frac{1}{\sqrt{2}} X_1 + 0 + 0 + \cdots + 0 + \frac{1}{\sqrt{n}} X_n \\
x_3 &= 0 - \frac{1}{\sqrt{2}} X_2 + 0 + \cdots + 0 + \frac{1}{\sqrt{n}} X_n \\
&\vdots \\
x_n &= 0 + 0 + \cdots - \frac{1}{\sqrt{n}} X_{n-1} + \frac{1}{\sqrt{n}} X_n
\end{align*}
\]

After addition

\[
(x_1 + \cdots + x_n) = \sqrt{n} X_n \text{ or } X_n = \frac{\beta^{k-1}}{2\sqrt{n}}
\]  
(77)

And:

\[
\left( \sum_{i=1}^{n-1} \frac{1}{\sqrt{2}} X_i + \frac{1}{\sqrt{n}} X_n \right)^3 + \left( -\frac{1}{\sqrt{2}} X_1 + \frac{1}{\sqrt{n}} X_n \right)^3 + \cdots + \left( -\frac{1}{\sqrt{2}} X_n + \frac{1}{\sqrt{n}} X_n \right)^3 =
\]

\[
x_1^3 + \cdots + x_n^3
\]  
(78)

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_n
\end{bmatrix} =
\begin{bmatrix}
\frac{1}{\sqrt{2}} & 0 & \cdots & \frac{1}{\sqrt{n}} \\
-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \cdots & \frac{1}{\sqrt{n}} \\
0 & -\frac{1}{\sqrt{2}} & \cdots & \frac{1}{\sqrt{n}} \\
\vdots & \vdots & \cdots & \vdots \\
0 & 0 & \cdots & -\frac{1}{\sqrt{2}}
\end{bmatrix}
\begin{bmatrix}
X_1 \\
X_2 \\
\vdots \\
X_n
\end{bmatrix}
\]  
(79)
And the equation (54 or 55) using theorem 6, becomes;

\[ \frac{1}{2} X_1^2 + \frac{1}{2} X_2^2 + \cdots + \frac{1}{2} X_{n-1}^2 + \frac{(n+1)}{2} X_n^2 = \frac{p^{k(p-1)(2p-1)}}{2^4} \]

Or;

\[ X_1^2 + X_2^2 + \cdots + X_{n-1}^2 + (n+1)X_n^2 = \frac{p^{k(p-1)(2p-1)}}{12} \]

The sphere with the center is considered at \(0,0,\ldots,0\) and the radius \(r\);

\[ x_1^2 + x_2^2 + \cdots + x_n^2 = r^2 \text{ or } \sum_{i=1}^{n} x_i^2 = r^2 \]  

The \(M_p\)-Sequences of the period \(p^k - 1\) lie on the surface of the sphere (The sphere has the same center for both the coordinates) but only in the following case;

\[ r^2 = p^{k-1} \left[ \frac{(p-1)p(2p-1)}{6} \right] = \frac{p^{k(p-1)(2p-1)}}{6} \]  

Or;

\[ x_1^2 + x_2^2 + \cdots + x_n^2 = \frac{p^{k(p-1)(2p-1)}}{6} \]

Or;

\[ X_1^2 + X_2^2 + \cdots + X_n^2 = \frac{p^{k(p-1)(2p-1)}}{6} \]

We can get the same result from equation (81) after subtraction \(nX_n^2 = \frac{p^{2k(p-1)^2}}{4n}\).

The intersection of (81) and (85) gives;

\[ nX_n^2 = \frac{p^{k(p-1)(2p-1)}}{12} - \frac{p^{k(p-1)(2p-1)}}{6} \]

Or;

\[ \left( \sum_{i=1}^{n-1} \frac{1}{\sqrt{2}} X_i + \frac{1}{\sqrt{n}} X_n \right)^3 + \left( -\frac{1}{\sqrt{2}} X_1 + \frac{1}{\sqrt{n}} X_n \right)^3 + \cdots + \left( -\frac{1}{\sqrt{2}} X_1 + \frac{1}{\sqrt{n}} X_n \right)^3 = \frac{p^{k+1}(p-1)^2}{4} \]

3.4. The Regular Surfaces for \(M_p\)-Sequences

In the original coordinate \(a x_1, x_2, \ldots, x_n\), the system of equations of the surfaces is;

\[ x_1 + x_2 + \cdots + x_n = \frac{p^k(p-1)}{2} \]

The equations of the system is;

\[ \left\{ \begin{array}{l}
\sum_{i=1}^{n} X_i^2 + \sum_{i=1, j> i}^{n-1} X_i X_j = \frac{p^{k(p-1)(2p-1)}}{24} \\
X_1^2 + X_2^2 + \cdots + X_n^2 = \frac{p^{k(p-1)(2p-1)}}{6} \\
X_1^3 + X_2^3 + \cdots + X_n^3 = \frac{p^{k+1}(p-1)^2}{4}
\end{array} \right\} \]

These are pending with the equation (93) and cannot give any new information about other variables \(x_k, x_{k+1}, \ldots, x_n\). The first equation (93) can define the variables by using the equations of recurring linear sequence. The equations (93) and (94) become for the new coordinate \(aX_1, X_2, \ldots, X_n\) as;
\[
\begin{align*}
X_n &= \frac{p^k(p-1)}{2\sqrt{n}} \\
X_1^2 + X_2^2 + \cdots + (n+1)X_n^2 &= \frac{p^k(p-1)[2p^k(p-1)+2(2p-1)]}{12} \\
X_1^3 + X_2^3 + \cdots + X_n^3 &= \frac{p^k(p-1)(2p-1)}{6}
\end{align*}
\]

(95)

The system (94) and system (95) are pending in the equation (93). For the sequence \(x_1, x_2, x_3, \ldots, x_n\) over \(F_p\) with the period \(n = p^k - 1\) the first \(x_1, x_2, \ldots, x_k\) (the initial position) are given and if it can find the other \(x_{k+1}\) to \(x_n\), then finding the other \(x_{n+1}\) to \(x_n\) is very easy from the relations:

\[
\begin{align*}
\text{if } x_i &= 0 \text{ then } x_{\frac{n+1}{2}} = 0, i = 1,2, \ldots, \frac{n}{2} \\
\text{if } x_i &= \neq 0 \text{ then } x_{\frac{n+1}{2}} = p - x_i, i = 1,2, \ldots, \frac{n}{2}
\end{align*}
\]

(96)

Thus it needs other \(\frac{n}{2} - 1 - k\) independent equations (surfaces) for all one period of the sequence from the Linear Recurring Sequence;

\[a_{n+k} + \sum_{i=1}^{k-1} \lambda_i a_{n+i} = 0, \lambda_i \in F_p = \{0,1,2,\ldots,(p-1)\}, p \text{ is prime} \]

(97)

The solution of this system calculated by mod \(p\).

Example 4: In the space \(R^3\) (for \(n = 8\)) with the coordinate \(ax_1x_2 \ldots x_n\), the \(M_2\)-Sequences (over \(F_3\)) are generated by the recurring equations;

\[a_{n+2} + a_{n+1} + 2a_n = 0 \text{ or } a_{n+2} = 2a_{n+1} + a_n \]

(98)

\[
\begin{align*}
S_1 &= \{1 0 1 2 0 2 0 2\}; S_2 &= \{2 1 0 1 2 0 2 0\}; S_3 &= \{2 1 1 0 1 2 2 0\}; S_4 &= \{0 2 1 1 0 1 2 2\}; S_5 &= \{2 0 2 1 1 0 2\}; \\
S_6 &= \{2 2 0 2 1 1 0\}; S_7 &= \{1 2 2 0 2 1 1\}; S_8 &= \{0 1 2 2 0 2 1\}
\end{align*}
\]

(99)

It can be seen that the sequence satisfies the equations of systems (94) and (95) For \(S_1 = \{1 0 1 2 0 2 0 2\}\), and further for initial position: \(a_1 = 1 = x_1, a_2 = 0 = x_2\) and system (94) it can find all other \(x_i, i = 3, \ldots, 8\) as following:

From Corollary 6. (2); If \(x_i = 0\) then \(x_{i+4} = 0\) and if \(x_i \neq 0\) then.

\[x_{i+4} = 3 - x_i, \text{ thus: } x_5 = 3 - x_1 = 2 \text{ and } x_6 = x_2 = 0, x_7 = 3 - x_3, \text{ and } x_8 = 3 - x_4, \text{ and the first equation becomes:}
\]

\[x_3 = 2x_2 + x_1 = 1 \text{ then } x_7 = 3 - 1 = 2 \& x_4 = 2a_3 + a_2 = 2 \]

(102)

From \(x_4 = 2\) give as \(x_8 = 3 - 2 = 1\). The equation (93) and system (94) became;

\[
\begin{align*}
\sum_{i=1}^{n} x_i &= 9 \\
\sum_{i=1}^{n} x_i^2 + \sum_{i=1,j\neq i}^{n} x_ix_j &= 48 \\
x_1^3 + x_2^3 + \cdots + x_8^3 &= 15 \\
x_1^3 + x_2^3 + \cdots + x_8^3 &= 27
\end{align*}
\]

(103)

Example 5: In the space \(R^{24}\) (for \(n = 24\)) with the coordination of \(ax_1x_2 \ldots x_n\), the \(M_3\)-Sequences (over \(F_3\)) are generated by the recurring equations;

\[a_{n+2} + a_{n+1} + 2a_n = 0 \text{ or } a_{n+2} = 4a_{n+1} + 3a_n \]

(104)

With the characteristic equation \(x^2 + x + 2 = 0\) and the prime characteristic polynomial \(g(x) = x^2 + x + 2\), it generates \(F_{2^2}\) with the initial position \(0 1\) which results in sequence; 0144340233104121032242 014... With the period \(5^2 - 1 = 24\), the sequence is satisfied by;
\[
\begin{aligned}
\sum_{i=1}^{24} x_i &= 50 \\
\sum_{i=1}^{24} x_i^2 + \sum_{i=1,j\neq i}^{24} x_ix_j &= 1325 \\
x_1^2 + x_2^2 + \ldots + x_{24}^2 &= 150 \\
x_1^3 + x_2^3 + \ldots + x_{24}^3 &= 500
\end{aligned}
\] (105)

The regular quadratic Surfaces for \(M_2\)-Sequences in \(F_2\) the equation (93) and system (94) becomes;

\[
\begin{aligned}
x_1 + x_2 + \ldots + x_n &= 2k^{-1} \\
\sum_{i=1}^{n} x_i^2 + \sum_{i=1,j \neq i}^{n} x_ix_j &= \frac{2k^{2k+2}}{n} \\
x_1^2 + x_2^2 + \ldots + x_n^2 &= 2k^{-1} \\
x_1^3 + x_2^3 + \ldots + x_n^3 &= 2k^{-1}
\end{aligned}
\] (106) (107)

The equations of systems (105) and (107) are pending in the first equation and for the sequence \(x_1, x_2, x_3, \ldots, x_9\) over \(F_2\) with the period \(2^k - 1\). While, the first \(x_8, x_9\) (the initial position) are given and it can find the other \(x_{k+1}\) to \(x_n\). Therefore, other \(n-1-k\) independent equation (surface) is needed to find all one period of the sequence and it can get from the Linear Recurring Sequence;

\[
a_{n+k} + \sum_{i=1}^{k-1} \lambda_i a_{n+i} = 0, \lambda_i \in F_2 = \{0,1\}
\] (108)

Example 6: Suppose the Sequence of Linear Recurring is;

\[
a_{n+3} = a_{n+1} + a_n \text{ or } a_{n+3} + a_{n+1} + a_n = 0
\] (109)

With the characteristic equation \(x^3 + x + 1 = 0\) and the polynomial characteristic \(f(x) = x^3 + x + 1\) represent a prime and generate the field \(F_2^3\). This sequence represents a periodic within the period \(2^3 - 1 = 7\). Initial position: \(a_1 = 1, a_2 = 0, a_3 = 0\), then \(S_1 = (1 \, 0 \, 0 \, 1 \, 0 \, 1 \, 1)\) and when it consider the cyclic per mutations on \(S_1\), it has \(S = \{S_1, S_2, S_3, S_4, S_5, S_6, S_7\}\). Where;

\[
\begin{aligned}
S_2 &= (1 \, 1 \, 0 \, 0 \, 1 \, 0 \, 1) \, S_3 &= (1 \, 1 \, 1 \, 0 \, 0 \, 1) \, S_4 &= (0 \, 1 \, 1 \, 1 \, 0) \\
S_5 &= (1 \, 0 \, 1 \, 1 \, 1 \, 0) \, S_6 &= (0 \, 1 \, 0 \, 1 \, 1 \, 1) \, S_7 &= (0 \, 0 \, 1 \, 0 \, 1 \, 1)
\end{aligned}
\] (110)

There are first three digits that represent the initial position for each of these sequences and generates these sequences for the feedback register. The sequences satisfies the equations (94) and (95) for \(p = 2\) and \(k = 3\), and further for the first sequence \(S_1 = (1 \, 0 \, 0 \, 1 \, 0 \, 1)\);

\[
\begin{aligned}
\sum_{i=1}^{n} x_i &= 4 \\
\sum_{i=1,j \neq i}^{n} x_ix_j &= 10 \\
x_1^2 + x_2^2 + \ldots + x_n^2 &= 4 \\
x_1^3 + x_2^3 + \ldots + x_n^3 &= 4
\end{aligned}
\] (111)

New system can be developed from the first equation in latest system and from the linear recurring sequence;

\[
\begin{aligned}
x_4 + x_5 + x_6 + x_7 &= 3 \\
x_4 + x_2 + x_1 &= 0 \\
x_5 + x_3 + x_2 &= 0 \\
x_6 + x_4 + x_3 &= 0
\end{aligned}
\] (112)

\[
\begin{aligned}
\sum_{i=1}^{n} x_i^2 &= \frac{p^k(p-1)}{2} \\
\sum_{i=1,j \neq i}^{n} x_ix_j &= \frac{p^k(p-1)[3p^k(p-1)+2(2p-1)]}{24} \\
x_1^2 + x_2^2 + \ldots + x_n^2 &= \frac{p^k(p-1)(2p-1)}{6} \\
x_1^3 + x_2^3 + \ldots + x_n^3 &= \frac{p^{k+1}(p-1)^2}{4}
\end{aligned}
\] (115)

4. Conclusion

a) The above theorems have concluded that the finding of the surfaces (planes) that contains \(M_p\)-Sequences is;

\[
\sum_{i=1}^{n} x_i = \frac{p^k(p-1)}{2}
\] (114)

b) The formula of finding the (Ellipsoids 2., Spheres 3., Others 4.) the surfaces contains the \(M_p\)-Sequences that are shown below;

\[
\begin{aligned}
X_n &= \frac{p^k(p-1)}{2\sqrt{n}} \\
X_1^2 + X_2^2 + \ldots + (n+1)X_n &= \frac{p^k(p-1)(3p^k(p-1)+2(2p-1))}{12} \\
X_1^2 + X_2^2 + \ldots + X_n^2 &= \frac{p^k(p-1)(2p-1)}{6}
\end{aligned}
\] (116) (117)

(\(\sum_{i=1}^{n} \frac{1}{\sqrt{V_i}} + \frac{1}{\sqrt{V_n}}X_n\))^2 + (\(-\frac{1}{\sqrt{V}}X_1 + \frac{1}{\sqrt{V_n}}X_n\))^3 + \ldots + (\(-\frac{1}{\sqrt{V}}X_n + \frac{1}{\sqrt{V_n}}X_n\))^3 = \frac{p^{k+1}(p-1)^2}{4}
c) The equations are satisfied by the given examples of sequences on $F_2$, $F_3$ and $F_5$.

Limitation: This method of compose sequences is useful for only binary sequences and the addition on the sequences computed by “mod 2” also used Microsoft Word 2010 and the Microsoft equation 3.0 for written the math equations.

The method for reading a page which has a block will be according to the following direction as in figure 1.

Figure 1. Method reading page with block.

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