ON THE FRAMEWORK OF $L_p$ SUMMATIONS FOR FUNCTIONS

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Abstract. We develop the framework of $L_p$ operations for functions by introducing two primary new types $L_{p,s}$ summations for $p > 0$: the $L_{p,s}$ convolution sum and the $L_{p,s}$ Asplund sum for functions. The first type is defined as the linear summations of functions in terms of the $L_p$ coefficients $(C_{p,\lambda,t}, D_{p,\lambda,t})$, the so-called the $L_{p,s}$ suprema-convolution when $p \geq 1$ and the $L_{p,s}$ inf-sup-convolution when $0 < p < 1$, respectively. The second type $L_{p,s}$ summation is created by the $L_p$ averages of bases for $s$-concave functions. We show that they are equivalent in the case $s = 0$ (log-concave functions) and $p \geq 1$. For the former type $L_{p,s}$ summation, we establish the corresponding $L_p$-Borell-Brascamp-Lieb inequalities for all $s \in [-\infty, \infty]$ and $p \geq 1$. Furthermore, in summarizing the conditions for these new types of $L_p$-Borell-Brascamp-Lieb inequalities, we define a series of the $L_{p,s}$ concavity definitions for functions and measures. On the other hand, for the latter type $L_{p,s}$ Asplund summation, we discover the integral formula for $L_{p,s}$ mixed quermassintegral for functions via tackling the variation formula of quermassintegral of functions for $p \geq 1$.

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2010 Mathematics Subject Classification. Primary: 52A39, 52A40, 46N10; Secondary: 28A75, 26D15.

Key words and phrases. $s$-concave function, $L_{p,s}$ suprema-convolution, $L_{p,s}$ inf-sup-convolution, $L_{p,s}$ Asplund summation, $L_p$-Borell-Brascamp-Lieb inequality, Projection for functions, Quermassintegral for functions, $L_{p,s}$ mixed quermassintegral, $L_{p,s}$ concavity for function.

The first named author was supported in part by the Zuckerman STEM Leadership program.
1. Introduction

Following the seminal books and surveys of Gardner [30, 31], Artstein-Avidan, Giannopoulos, and Milman [5], and conventions of Schneider [57], the Brunn-Minkowski theory of convex bodies and functions will be given firstly as the geometric background.

1.1. Background for convex bodies. We will focus on the $n$-dimensional Euclidean space $\mathbb{R}^n$, together with the origin “$o$” and the usual Euclidean norm $\|x\| = \sqrt{\langle x, x \rangle}$ where $\langle \cdot, \cdot \rangle$ denotes for the standard inner product for vectors in $\mathbb{R}^n$. The unit ball $B^n_2$ whose volume is $\omega_n$ with boundary the unit sphere $S^{n-1} = \partial B^n_2$. A subset $K$ of $\mathbb{R}^n$ is said to be a convex body if it is a compact, convex set with non-empty interior (containing the origin $o$), and the set of all convex bodies in $\mathbb{R}^n$ will be denoted as $\mathcal{K}^n_o$ endowed with the Lebesgue measure (volume) $\text{vol}(\cdot)$, and $\mathcal{K}^n_o$ denotes those containing the origin in their interiors.

To each $K \in \mathcal{K}^n_o$, we associate three correspondingly uniquely determined functions: the convex indicator function $I_K$, characteristic function $\chi_K$ and the support function $h_K$. The convex indicator function $I_K$ and characteristic function $\chi_K$ associated to $K \in \mathcal{K}^n_o$ are defined, respectively, by

$$I_K(x) = \begin{cases} 0, & \text{if } x \in K, \\ +\infty, & \text{if } x \notin K. \end{cases} \quad \text{and} \quad \chi_K(x) = \begin{cases} 1, & \text{if } x \in K, \\ 0, & \text{if } x \notin K. \end{cases}$$

The support function of $K \in \mathcal{K}^n_o$, $h_K: S^{n-1} \to \mathbb{R}$ is defined as $h_K(u) = \sup_{y \in K} \langle u, y \rangle$.

In [29] Firey introduced the following generalization of the Minkowski combination of convex bodies: for $K, L \in \mathcal{K}^n_o$, $p \geq 1$ and $\alpha, \beta \geq 0$, $\alpha \cdot_p K + \beta \cdot_p L$, the $L_p$ Minkowski sum is defined as the convex body whose support function is $h_{\alpha \cdot_p K + \beta \cdot_p L}(u) = (\alpha h_K(u)^p + \beta h_L(u)^p)^{\frac{1}{p}} = (\alpha^{\frac{1}{p}} h_K(u) + \beta^{\frac{1}{p}} h_L(u))^\frac{1}{p}$. Additionally, Firey established the so-called $L_p$-Brunn-Minkowski inequality for convex bodies when $p \geq 1$: given $K, L \in \mathcal{K}^n_o$ and $\alpha, \beta \geq 0$, $\text{vol}_n(\alpha \cdot_p K + \beta \cdot_p L)^\frac{1}{n} \geq \alpha \text{vol}_n(K)^\frac{1}{n} + \beta \text{vol}_n(L)^\frac{1}{n}$. The operations $+_p$ and $\cdot_p$ were generalized in [31] by Lutwak, Yang and Zhang to the setting of non-convex sets (measurable sets) in $\mathbb{R}^n$; i.e., for any $\alpha, \beta \geq 0$ and any measurable subsets $A, B \subset \mathbb{R}^n$,

$$\alpha \cdot_p A + \beta \cdot_p B = \left\{ x^{\frac{1}{p}} (1 - \lambda)^{\frac{1}{p}} x + \beta^{\frac{1}{p}} \lambda^{\frac{1}{p}} y: x \in A, y \in B, 0 \leq \lambda \leq 1 \right\}$$

$$= \bigcup_{0 \leq \lambda \leq 1} \left( \alpha^{\frac{1}{p}} (1 - \lambda)^{\frac{1}{p}} A + \beta^{\frac{1}{p}} \lambda^{\frac{1}{p}} B \right),$$

5.1. Projection for function and $L_{p,s}$ Asplund summation

5.3. Variation formula of general quermassintegrand for functions and $p \geq 1$

6. Acknowledgment

References
where \( \frac{1}{p} + \frac{1}{q} = 1 \). Moreover, they also showed that this definition of the \( L_p \) combination agrees with the original one defined by Firey for \( A, B \in \mathcal{K}_n^{(o)} \). Moreover, Lutwak in [39,40] developed a deep study of the \( L_p \)-Brunn-Minkowski theory which parallels and generalizes the traditional Brunn-Minkowski theory in essence. In particular, for a convex body \( K \in \mathcal{K}_n^{(o)} \), the Kubota’s integral formula expresses the quermassintegral \( W_j(K) \) for \( j \in \{0, 1, \cdots, n-1\} \) as

\[
W_j(K) = c_{n,j} \int_{G_{n,n-j}} \text{vol}_{n-j}(K|H)d\nu_{n-j}(H).
\]

Here \( c_{n,j} = \frac{\omega_n}{\omega_{n-j}} \), \( K|H \) is the projections of \( K \) on the \((n-j)\)-dimensional hyperplane \( H \) belonging to the Grassmannian manifold \( G_{n,n-j} \)—the \((n-j)\)-dimensional subspaces of \( \mathbb{R}^n \) equipped with the Haar probability measure \( \nu_{n-j} \). In [39], the mixed \( p \)-quermassintegrals of two convex bodies \( K, L \in \mathcal{K}_n^{(o)} \) is defined naturally as the variation formula of \( W_j \) with respect to the \( L_p \) Minkowski sum for convex bodies, i.e.,

\[
W_p, j(K, L) = \frac{p}{n-j} \cdot \frac{d}{d \varepsilon} W_j(K + \varepsilon \cdot p L) \bigg|_{\varepsilon = 0} = \frac{1}{n-j} \int_{S^{n-1}} h_L(u)^p h_K(u)^{1-p} dS_j(K, u),
\]

where \( S_j(K, \cdot) \) is the \( j \)-th surface area measure for \( K \) defined on \( S^{n-1} \). If \( j = 0 \), it recovers the classical \( L_p \) mixed volume for convex bodies, and \( S_0(K, \cdot) = S(K, \cdot) \) is the surface area measure on \( S^{n-1} \).

The Brunn-Minkowski theory has parallel “liftings” to the theory of functions through the convex indicator function \( I_K: \mathbb{R}^n \to \mathbb{R}^+ \cup \{+\infty\} \), characteristic function \( \chi_K(x) \) associated to \( K \in \mathcal{K}_n^{(o)} \) and many others, see references [13,14,16,19,21,23,35] and measures [2,12,32,35,37,38,41,43,45,47,49], etc.

One similar parallel definition for “Minkowski summation” of convex bodies for functions—supremal-convolution is defined as follows. For more information please see references for example [5,14,16,21].

### 1.2. Supremal-convolution for functions

To begin with, given \( s \in [-\infty, \infty] \), \( a, b \geq 0 \), the \( s \)-mean of \( a \) and \( b \) with respect to nonnegative coefficients \( \alpha, \beta \geq 0 \) is denoted as

\[
M_\alpha^{(\alpha,\beta)}(a, b) = \begin{cases} 
(a \alpha^s + \beta b^s)^{\frac{1}{s}}, & \text{if } s \neq 0, \pm \infty, \\
\alpha a^\beta, & \text{if } s = 0, \\
\max\{a, b\}, & \text{if } s = +\infty, \\
\min\{a, b\}, & \text{if } s = -\infty,
\end{cases}
\]

whenever \( ab > 0 \), and \( M_\alpha^{(\alpha,\beta)}(a, b) = 0 \) otherwise. A measure \( \mu \) on \( \mathbb{R}^n \) is \( s \)-concave if, for any Borel sets \( A, B \subset \mathbb{R}^n \) and any \( t \in [0, 1] \), one has that \( \mu((1-t)A+tB) \geq M_s^{(1-t, t)}(\mu(A), \mu(B)) \); and a measure \( \mu \) on \( \mathbb{R}^n \) is log-concave (when \( s = 0 \)) if, for any Borel sets \( A, B \subset \mathbb{R}^n \) and any \( t \in [0, 1] \), one has log(\( \mu((1-t)A+tB) \)) \geq (1-t) \log(\mu(A)) + t \log(\mu(B)) \), or equivalently, \( \mu((1-t)A+tB) \geq \mu(A)^{1-t} \mu(B)^t \).

A function \( f: \mathbb{R}^n \to \mathbb{R}_+ \) is \( s \)-concave if, for all \( x, y \in \mathbb{R}^n \) and any \( t \in [0, 1] \), one has that \( f((1-t)x + ty) \geq M_s^{(1-t, t)}(f(x), f(y)) \). The case when \( s = 0 \) and \( s = -\infty \) are referred to
as log-concave and quasi-concave functions, respectively. Note that quasi-concavity of $f$ is equivalent to the condition that the super-level sets $C_f(r) := \{x \in \mathbb{R}^n : f(x) \geq r\}$ are convex for any constant $r > 0$. Moreover, any $s$-concave function with its maximum at the origin is radially decreasing.

An important inequality of the Brunn-Minkowski type for functions which links $s$-concave measures (with $s$-concave density functions) is the Borell-Brascamp-Lieb inequality (see [14, 16, 36, 47, 52]). Let $t \in [0,1]$ and $s \in [-1/n, \infty]$. Given a triple of measurable functions $h, f, g: \mathbb{R}^n \rightarrow \mathbb{R}_+$ satisfying the condition

\[
(3) \quad h((1-t)x + ty) \geq M_s^{(1-t),t}(f(x),g(y))
\]

for any $x, y \in \mathbb{R}^n$, there is

\[
(4) \quad \int_{\mathbb{R}^n} h(x)dx \geq M_s^{(1-t),t} \left( \int_{\mathbb{R}^n} f(x)dx, \int_{\mathbb{R}^n} g(x)dx \right).
\]

The case when $s = 0$ is referred to as the Prékopa-Leindler inequality and was proven firstly by Leindler in [36] and Prékopa in [47]. The minimal function satisfying the condition (3) of Borell-Brascamp-Lieb inequality is the supremal-convolution of the functions $f$ and $g$ (or $s$-supremal-convolution); that is, the function $m_{t,s}: \mathbb{R}^n \rightarrow \mathbb{R}_+$ defined by

\[
(5,3) \quad m_{t,s}(z) = \sup_{z=(1-t)x+ty} M_s^{(1-t),t}(f(x),g(y)).
\]

The introduction of the supremal-convolution of functions leads to the following notions of addition and scalar multiplication of functions: given $s \in [-\infty, \infty]$, $f, g: \mathbb{R}^n \rightarrow \mathbb{R}_+$,

\[
(f \oplus_s g)(z) = \sup_{z=x+y} \begin{cases} (f(x)^s + g(y)^s)^{\frac{1}{s}}, & \text{if } s \neq 0, \pm \infty, \\ f(x)g(y), & \text{if } s = 0, \\ \max\{f(x), g(y)\}, & \text{if } s = +\infty, \\ \min\{f(x), g(y)\}, & \text{if } s = -\infty, \end{cases}
\]

and for $\alpha > 0$,

\[
(\alpha \times_s f)(x) = \begin{cases} \alpha^{\frac{s}{\alpha}} f \left( \frac{x}{\alpha} \right), & \text{if } s \neq 0, \pm \infty, \\ f(x)^\alpha, & \text{if } s = 0, \\ f(x), & \text{if } s = \pm \infty. \end{cases}
\]

The $s$-concavity is closed under the supremal-convolution operation; i.e., $f \oplus_s g$ and $\alpha \times_s f$ defined above are $s$-concave whenever $f$ and $g$ are as well (see [13, Proposition 2.1]). In addition, for any non-empty sets $A, B \subset \mathbb{R}^n$ and $\alpha, \beta > 0$, we have that $(\alpha \times_s \chi_A) \oplus_s (\beta \times_s \chi_B) = \chi_{\alpha A + \beta B}$ whenever $\alpha + \beta = 1$. Denote the total mass of $f$ as $I(f) = \int_{\mathbb{R}^n} f(x)dx$. Then based on this supremal-convolution definition, the Borell-Brascamp-Lieb inequality (4) asserts that for any $s \geq -1/n$,

\[
(5) \quad I(((1-t) \times_s f) \oplus_s (t \times_s g)) \geq M_s^{(1-t),t}(I(f), I(g)).
\]
In [56], the authors established the $L_p$-Borell-Brascamp-Lieb inequality based on the geometric extension of $L_p$ Minkowski sum with respect to measurable sets in $\mathbb{R}^n$ using $L_p$ coefficients of Lutwak, Yang, and Zhang in [41]; that is, let $p, q, \lambda \geq 0$ and $f, g, h : \mathbb{R}^n \to \mathbb{R}_+$ be a triple of bounded integrable functions. For simplicity, we denote the $L_p$ coefficients for $\lambda \in [0, 1]$ and $t \in [0, 1]$ as

$$C_{p,\lambda,t} := (1-t)^{\frac{s}{p}}(1-\lambda)^{\frac{s}{q}}, \quad D_{p,\lambda,t} := t^{\frac{s}{p}}\lambda^{\frac{s}{q}}$$

in later context. Suppose, in addition, that this triple satisfies the $L_p$-Borell-Brascamp-Lieb inequality condition

$$h(C_{p,\lambda,tx} + D_{p,\lambda,ty}) \geq \left[C_{p,\lambda,t}f(x)^s + D_{p,\lambda,t}g(y)^s\right]^{\frac{1}{s}}$$

for every $x \in \text{supp}(f)$, $y \in \text{supp}(g)$ and every $\lambda \in [0, 1]$. Then the following integral inequality holds:

$$I(h) \geq M_{\frac{(1-t),t}{t}}(I(f), I(g)).$$

Naturally, the authors in [56] gave the definition of $L_{p,s}$ supremal-convolution of $f : \mathbb{R}^n \to \mathbb{R}_+$ and $g : \mathbb{R}^n \to \mathbb{R}_+$ for $s \geq 0$ and $p \geq 1$, i.e.,

$$(f \oplus_{p,s} g)(z) := \sup_{0 \leq \lambda \leq 1} \left( \sup_{z = (1-\lambda)^{\frac{s}{q}}x + \lambda^{\frac{s}{q}}y} M_{s}^{\left((1-\lambda)^{\frac{s}{q}},\lambda^{\frac{s}{q}}\right)}(f(x), g(y)) \right)$$

$$= \sup_{0 \leq \lambda \leq 1} \left( [(1-\lambda)^{\frac{s}{q}} \times_s f] \oplus_{s} [\lambda^{\frac{s}{q}} \times_s g](z) \right),$$

where $1/p + 1/q = 1$. And given any scalar $\alpha > 0$, the scalar multiplication $\times_{p,s}$ satisfies

$$(\alpha \times_{p,s} f)(x) = \alpha^{s/p} f\left(\frac{x}{\alpha^{1/p}}\right).$$

Therefore the $L_p$-Borell-Brascamp-Lieb inequality concludes that for any $s \geq 0$,

$$I((1-t) \times_{p,s} f) \oplus_{p,s} (t \times_{p,s} g) \geq M_{\frac{(1-t),t}{t}}(I(f), I(g)),\]$$

which is a $L_p$ generalization of formula (5).

Another type of summations—Asplund summation (or $L_1$ Asplund summation) for functions using infimal convolution (□) for base functions, and its $L_p$ extensions for log-concave functions [28,53,54] with $L_p$ averages for base functions is defined as follows. In the following, we list some basics of Asplund summation for functions first.

1.3. **Asplund summation of s-concave function.** Consider the following class of bounded s-concave functions:

$$\mathcal{F}_s(\mathbb{R}^n) = \{ f : \mathbb{R}^n \to \mathbb{R}_+, f \text{ is s-concave, u.s.c}, f \in L^1(\mathbb{R}^n), f(o) = \|f\|_\infty > 0 \},$$

where u.s.c. stands for upper semi-continuous. The class $\mathcal{F}_0(\mathbb{R}^n)$, is the class of all such log-concave functions, and $\mathcal{F}_{-\infty}(\mathbb{R}^n)$ is the class of all such quasi-concave functions.

To begin with, we will introduce reasonable base classes of convex functions (see [7,8,21,50,51] for example). Denote the set of proper (non-empty domain) convex functions
\( u : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) that are lower semi-continuous by \( \text{Cvx}(\mathbb{R}^n) \). The infimal convolution of \( u, v \in \text{Cvx}(\mathbb{R}^n) \) is the convex function defined by

\[
(u \square v)(x) = \inf_{y \in \mathbb{R}^n} \{ u(x - y) + v(y) \},
\]

which should be viewed as an addition on the class \( \text{Cvx}(\mathbb{R}^n) \). Moreover, the scalar multiplication satisfies

\[
(\alpha \times u)(x) = \alpha u(x/\alpha).
\]

To understand the infimal convolution geometrically, we can see that the function \( u \square v \) whose epigraph is the Minkowski sum of the epigraphs of \( u \) and \( v \) [21][23]:

\[
\text{epi}(u \square v) = \text{epi}(u) + \text{epi}(v),
\]

where \( \text{epi}(u) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : y \geq u(x)\} \) and “+” denotes the Minkowski sum in \( \mathbb{R}^n \).

The classical Legendre transformation \( u^*: \text{Cvx}(\mathbb{R}^n) \to \text{Cvx}(\mathbb{R}^n) \) is given by

\[
u^*(x) = \sup_{y \in \mathbb{R}^n} [\langle x, y \rangle - u(y)].\]

It is easy to check that \((I_K)^* = h_K\) for \( K \in \mathcal{K}_n \). For an extensive list of the properties of the Legendre transformation please see [21, 22, 48, 50, 51] for reference. A crucial connection between the infimal convolution and Legendre transformation on the class \( \text{Cvx}(\mathbb{R}^n) \) is

\[
((\alpha \times u) \square (\beta \times v)) = (\alpha u^* + \beta v^*)^*
\]

for \( \alpha, \beta \geq 0 \).

Alternatively, the class of super-coercive geometric convex functions (originally considered in [8] where a second duality transformation was discovered and classified) is defined as

\[
C_s(\mathbb{R}^n) = \left\{ u \in \text{Cvx}(\mathbb{R}^n) : u(o) = 0, \lim_{x \to \infty} \frac{u(x)}{\|x\|} = +\infty \right\} \subset \text{Cvx}(\mathbb{R}^n).
\]

Denote \( C_s(\mathbb{R}^n)^* = \{ u : \mathbb{R}^n \to \mathbb{R}_+, \text{u is convex, proper, } u(o) = 0 \} \), where the class \( C_s(\mathbb{R}^n)^* \) can be thought of as the dual space of \( C_s(\mathbb{R}^n) \) via the Legendre transform.

In [53] Rotem established a connection between members of \( \mathcal{F}_s(\mathbb{R}^n) \) and \( C_s(\mathbb{R}^n) \) for any \( s \in [-\infty, \infty] \). Given \( f \in \mathcal{F}_s(\mathbb{R}^n) \), the base function for \( f \) is defined as [53, Definition 8],

\[
u_f : \mathcal{F}_s(\mathbb{R}^n) \to C_s(\mathbb{R}^n)\]

such that

\[
f(x) = (1 - s u_f(x))^{\frac{1}{s_+}},
\]

where \( a_+ = \max\{a, 0\} \). When \( s = 0 \), \( f(x) = e^{-u_f(x)} \). In particular, for \( f = \chi_K \) for some \( K \in \mathcal{K}_n^{(0)} \), \( u_f = I_K \). In [53] the following operations—Asplund summation \( \ast_s \) and \( \cdot_s \) for \( s \)-concave functions were considered: given \( f, g \in \mathcal{F}_s(\mathbb{R}^n) \) and \( \alpha > 0 \),

\[
u_{f \ast_s g}(x) = (u_f \square v_g)(x) \quad \text{and} \quad \nu_{\alpha \cdot_s f}(x) = \alpha u_f \left( \frac{x}{\alpha} \right).
\]
In particular, \[53\], Proposition 10 asserts that, for any \(t \in [0,1]\) and \(f, g \in \mathcal{F}_s(\mathbb{R}^n)\), the supremal-convolution coincides with the Asplund summation with coefficients \(((1-t),t)\); that is
\[
((1-t) \cdot_s f) \star_s (t \cdot_s g) = [1 - su_{((1-t) \cdot_s f) \star_s (t \cdot_s g)}]^{1/s} = [((1-t) \times_s f) \oplus_s (t \times_s g)];
\]
or equivalently, using equality (11), the above equality can be more explicitly stated as
\[
[((1-t) \times_s f) \oplus_s (t \times_s g)] = \left[1 - s((1-t)u^*_f + tv^*_g)^{1/s}\right].
\]

For \(u \in C_s(\mathbb{R}^n)\), consider the integral operator \(J_s: C_s(\mathbb{R}^n) \to \mathbb{R}_+\) defined by
\[
J_s(u) = \int_{\mathbb{R}^n} [1 - su(x)]^{1/s} dx.
\]

Then the Borell-Brascamp-Lieb inequality implies that, for any \(s \geq -1/n\) and \(u, v \in C_s(\mathbb{R}^n)\), one has that
\[
J_s([(1-t) \times u] \square (t \times v)) \geq M^{(1-t),t}_{1/n}(J_s(u), J_s(v)).
\]

In [28] and \[54\], the authors proposed the \(L_p\) summations in terms of the base functions for \(s = 0\) (log-concave functions), and the solutions to the corresponding Minkowski type problems for \(p \geq 1\) and \(0 < p < 1\) are also proposed and solved, respectively. Based on these two types of summations for functions above, i.e., the \(L_p\) supremal-convolution for \(p \geq 1\), \(s \geq 0\) and Asplund summation for \(s\)-concave functions including log-concave case, we consider more complicated cases of summations for \(s\)-concave functions for various cases for \(p\) and \(s\).

1.4. Main results. Our paper mainly focus on \(L_p\) functional theory which naturally extending \(L_p\) Brunn-Minkowski theory for convex bodies (measurable sets \((\mathcal{H})\) in the geometric setting. These include two types of \(L_p\) additions for functions, the \(L_p\)-Borell-Brascamp-Lieb type inequalities, and the \(L_{p,s}\) concavity for functions and measures, and the corresponding variation formula in terms of the new defined \(L_p\) Asplund summation (perturbation) for \(s\)-concave functions, etc. Particularly, Section \[2\] focuses on detailed definitions of the \(L_p\) sum for functions, such as for \(s\)-concave functions for \(p \geq 1\) and \(0 < p < 1\), respectively. In summary, we introduce the following new definitions

1. \(p \geq 1\) and \(s \in [-\infty, \infty]\), the \(L_{p,s}\) sup-convolution,
2. \(0 < p < 1\) and \(s \in [-\infty, \infty]\), the \(L_{p,s}\) inf-sup-convolution,
3. \(p \geq 1\) and \(s \in (-\infty, \infty)\), \(L_{p,s}\) Asplund summation for \(s\)-concave functions,
4. \(0 < p < 1\) and \(s \in (-\infty, \infty)\), \(L_{p,s}\) Asplund summation for \(s\)-concave functions.

More in detail, we extend the \(L_{p,s}\) sup-convolution from \(s \in [0, \infty)\) to \(s \in [-\infty, \infty]\) in \[7\] and \[8\], and analyze the corresponding properties for \(p \geq 1\). Based on the definition of \(L_p\) Minkowski sum for convex bodies for \(0 < p < 1\) using Wulff shapes (or Aleksandrov bodies), we give a functional version for the summation—\(L_{p,s}\) inf-sup-convolution accordingly. Let \(0 < p < 1\), \(1/p + 1/q = 1\) and \(s \in [-\infty, \infty]\). Given Borel measurable functions \(f, g: \mathbb{R}^n \to \mathbb{R}_+\),
o ∈ int(supp(f)), o ∈ int(supp(g)) and α, β > 0, we define the $L_{p,s}$ inf-sup-convolution of \( f \) and \( g \) based on (1) (replace “sup” to “inf”) as

$$\left[ α \times_{p,s} f \oplus_{p,s} β \times_{p,s} g \right](z) = \inf_{0 \leq \lambda \leq 1} \left[ \sup_{z = \frac{1}{1-\lambda} f(x) + \frac{\lambda}{1-\lambda} y} M_s^{\left[(1-\lambda)^{\frac{1}{p}}, \lambda^{\frac{1}{p}}\right)} \left( \alpha^{\frac{1}{p}} f(x), \beta^{\frac{1}{p}} g(y) \right) \right].$$

Elementary properties are also provided by a detailed analysis for this new sum in Proposition 2.9.

Moreover, following the method of $L_p$ Asplund summation for log-concave functions when $p \geq 1$ [28] and $0 < p < 1$ [54], we introduce the $L_{p,s}$ Asplund summation for $s$-concave functions using $L_p$ addition for base functions. Let $p > 0$. Given $\alpha, \beta \geq 0$ and $u, v \in C_s(\mathbb{R}^n)$, the $L_p$ additions of $u, v$ (base functions), a generalization of (1), is

$$[(\alpha \boxtimes_p u) \boxplus_p (\beta \boxtimes_p v)](x) := \{(\alpha(u^*(x))^p + \beta(v^*(x))^p)^{1/p}\}^*$$

and the $L_{p,s}$ Asplund summation for functions in $\mathcal{F}_s(\mathbb{R}^n)$ is defined as follows. (i) For $p \geq 1$, $s \in (-\infty, \infty)$, given $f, g \in \mathcal{F}_s(\mathbb{R}^n)$, we define the $L_{p,s}$ Asplund summation with weights $\alpha, \beta \geq 0$ as

$$(\alpha \cdot_{p,s} f) \ast_{p,s} (\beta \cdot_{p,s} g) := \left(1 - s [(\alpha \boxtimes_p u_f) \boxplus_p (\beta \boxtimes_p v_g)] \right)^{\frac{1}{p}}.$$

(ii) For $0 < p < 1$, $s \in (-\infty, \infty)$, given $f, g \in \mathcal{F}_s(\mathbb{R}^n)$ with $h_f, h_g \geq 0$, we define the $L_{p,s}$ Asplund summation $\alpha \cdot_{p,s} f \ast_{p,s} \beta \cdot_{p,s} g$ with weights $\alpha, \beta \geq 0$ as

$$\alpha \cdot_{p,s} f \ast_{p,s} \beta \cdot_{p,s} g := A \left[\left(\alpha h_f^p + \beta h_g^p\right)^{1/p}\right],$$

where $h_f$ is the support function of $f$ and $A[\cdot]_s$ denotes the $s$-Aleksandrov function. See detailed definitions of $h_f$ and $A[\cdot]_s$ for explanation in Section 2.

Inspired by (12) we verify that when $p \geq 1$ and $s = 0$, the $L_{p,s}$ supremal-convolution agrees with the $L_{p,s}$ Asplund summation through base functions. However, for $s \neq 0$, these two summations differs with each other as the relation only works for inequalities.

For these two types of $L_p$ summations for functions, it is much difficult to obtain the variation formula for $L_{p,s}$ sup-convolution for $p \geq 1$ and $L_{p,s}$ inf-sup-convolution for $0 < p < 1$ with delicate $L_p$ coefficients. Instead, we focus on the corresponding $L_p$-Borell-Brascamp-Lieb type inequalities in different circumstances in Section 3. Specially, we study new $L_p$-Borell-Brascamp-Lieb type inequalities for functions extending works in [56] to the generalizing the result in [56] from $s = 0$ to $s \in [-\infty, \infty]$ in different methods. In detail, our main goals are to solve

1. General $L_p$-Borell-Brascamp-Lieb inequality in terms of $\Omega$-total mass.
2. Proof of $L_p$-Borell-Brascamp-Lieb inequality using mass transportation.
3. Proof of $L_p$-Borell-Brascamp-Lieb inequality using classical Borell-Brascamp-Lieb inequality.

Our first main result generalizes a theorem of Bobkov, Colesanti and Fragałá [13 Theorem 4.2] and we obtain the following theorem.
Theorem 1.1. Let $\Omega : \mathcal{B} \to \mathbb{R}_+$ be $\alpha$-concave. Let $p,q \in [1,\infty]$ be such that $1/p + 1/q = 1$. Let $\alpha \in [-1,\infty)$ and $\gamma \in [-\alpha,\infty)$. Suppose that $h,f,g : \mathbb{R}^n \to \mathbb{R}_+$ are a triple of integrable Borel measurable (respectively, quasi-concave functions) that satisfy the condition

$$h(C_{p,\lambda,t}x + D_{p,\lambda,t}y) \geq (C_{p,\lambda,t}f(x)^\alpha + D_{p,\lambda,t}g(y)^\alpha)^{\frac{1}{\alpha}}$$

for every $x \in \text{supp}(f)$, $y \in \text{supp}(g)$, and $\lambda \in (0,1)$ whenever $f(x)g(y) > 0$. Then the following inequality holds:

$$\tilde{\Omega}(h) \geq \left[(1-t)\tilde{\Omega}(f)^\beta + t\tilde{\Omega}(g)^\beta\right]^{\frac{1}{\beta}}, \quad \beta = \frac{p\alpha\gamma}{\alpha + \gamma}.$$

Please see definitions for $\mathcal{B}$ and $\tilde{\Omega}$ in Subsection 3.1 for details. Moreover, our results extend the $L_p$-Borell-Brascamp-Lieb inequality originally appearing in [56] for the case $s \geq 0$ and Borell-Brascamp-Lieb inequality for $s \leq -1/n$ in [26] stated as:

Theorem 1.2. Let $p \geq 1$, $-\infty < s < \infty$, and $t \in (0,1)$. Let $f,g,h : \mathbb{R}^n \to \mathbb{R}_+$ be a triple of functions and corresponding properties related to measure. One typical example with respect to measure we list here is

Definition 1.3. Let $p \geq 1$, $1/p + 1/q = 1$, $t \in [0,1]$, and $s \in [-\infty,\infty]$. We say that a non-negative measure $\mu$ on $\mathbb{R}^n$ is $L_{p,s}$-concave if, for any pair of Borel measurable sets $A,B \subset \mathbb{R}^n$, one has

$$\mu(C_{p,\lambda,t}A + D_{p,\lambda,t}B) \geq M_s(C_{p,\lambda,t},D_{p,\lambda,t})(\mu(A),\mu(B))$$

for every $\lambda \in [0,1]$ and $t \in [0,1]$. Similarly, if $s = -\infty$, the measure $\mu$ is said to be $L_{p,s}$-quasi-concave, and if $s = 0$, the measure $\mu$ is said to be $L_{p,s}$-log-concave.

On the other hand, it is more reasonable to research on the variation formula for $L_{p,s}$ summations using base functions with linear coefficients. Therefore Section 5 concentrates on the definition of $L_{p,s}$ mixed quermassintegral of functions equivalent to the derivative of quermassintegral for functions which is similar to the theory of convex bodies by Lutwak in [39]. The main works we finish are proposing and proving:

1. Projection for functions and corresponding properties related to $L_p$ summations;
2. Integral representation for $L_{p,s}$ mixed quermassintegral for functions.
By the definition of projection of functions and analyzing the properties of the projections for functions with respect to \( L_{p,s} \) in Subsection 5.1 and \( L_{p,s} \) Asplund summation Subsection 5.2 in certain circumstances, we provide the definition of quermassintegral for functions as well as the variation formula—the \( L_{p,s} \) mixed quermassintegral for functions in \( \mathcal{F}_s(\mathbb{R}^n) \) in Subsection 5.3. That is, for \( j \in \{0, \ldots, n-1\} \), the \( j \)-th quermassintegral of function \( f \), is defined as

\[
W_j(f) := c_{n,j} \int H f(x) dx \nu_{n,n-j}(H)
\]

and the \( L_{p,s} \) mixed quermassintegral for \( s \)-concave functions \( f, g \in \mathcal{F}_s(\mathbb{R}^n) \) has the definition of

\[
W_{p,j}^s(f, g) := \lim_{\varepsilon \to 0} \frac{W_j(f *_{p,s} \varepsilon_{p,s} g) - W_j(f)}{\varepsilon}.
\]

Through the process of finding the variation formula for the general \( \Omega \)-\( j \)th-quermassintegral in terms of the base functions \( u \equiv_{p,s} \varepsilon_{p,s} v \), and thus the \( \Omega \)-\( L_{p,s} \) mixed quermassintegral of \( u, v \in C_s(\mathbb{R}^n) \), \( W_{p,j}^s(u, v) \), where \( u, v \) denote the base functions for \( f \) and \( g \) correspondingly, we obtain the integral representation formula with respect to the \( L_{p,s} \) mixed quermassintegral for \( f, g \in \mathcal{F}_s(\mathbb{R}^n) \). That is, \( p \geq 1, j \in \{0, \ldots, n-1\} \), and \( s \in (-\infty, \infty) \), let \( f = (1 - su)_+^{1/s}, g = (1 - sv)_+^{1/s} \) such that \( u, v \in C_s(\mathbb{R}^n) \) and \( u \in C^{2,+}(\mathbb{R}^n) \), and \( \psi \in C_c(\mathbb{R}^n) \) with \( \psi = v^* \). Then the \( L_{p,s} \) mixed quermassintegral for \( f, g \in \mathcal{F}_s(\mathbb{R}^n) \) has the following integral representation:

\[
W_{p,j}^s(f, g) = \frac{1}{n-j} \int_{\mathbb{R}^n} \frac{[1 - su_H(x)]^{1/s} \psi_H(\nabla u_H(x))^p}{\|x\|^j} \phi_H(\nabla u_H(x))^{1-p} dx
\]

For \( s = 0 \), the above becomes

\[
W_{p,j}^0(f, g) = \frac{1}{n-j} \int_{\mathbb{R}^n} \frac{e^{-u_H(x)} \psi_H(\nabla u_H(x))^p \phi_H(\nabla u_H(x))^{1-p}}{\|x\|^j} dx.
\]

(Please see definitions for \( W_{p,j}^s, u_H \) and \( \psi_H \) in Subsection 5.3 for detailed information.) When \( j = 0 \) and \( s = 0 \), it recovers the integral interpretation of variation formulas in [28] and [54], for \( p \geq 1 \) and \( 0 < p < 1 \), respectively.

2. Functional \( L_p \) operations for \( p > 0 \)

In this section, we will first extend the original definitions for \( L_{p,s} \) suprema-convolution for functions in [56] for \( s \geq 0 \) to all \( s \in [-\infty, \infty] \) and \( p \geq 1 \). For \( 0 < p < 1 \), we propose a brand new definition of \( L_{p,s} \) summation for functions—the \( L_{p,s} \) inf-sup-convolution in Subsection 2.1. We verify that these \( L_{p,s} \) convolutions satisfy elegant properties by \( L_p \) coefficients \( (C_{p,\lambda,t}, D_{p,\lambda,t}) \). In Subsection 2.2, we introduce the \( L_{p,s} \) Asplund summation through the base functions for \( s \)-concave functions inspired by the case of log-concave functions [28, 53, 54] for \( p \geq 1 \) and \( 0 < p < 1 \), respectively. Furthermore in Subsection 2.3, we compare definitions proposed in Subsections 2.1 and 2.2 and prove that for log-concave functions \( (s = 0) \), these two summations for \( p \geq 1 \) are equivalent to each other.
2.1. \textbf{General $L_{p,s}$ supremal-convolution for $p > 0$.} The focus on this section is to highlight functional operations of addition and scalar multiplication which generalize the supremal-convolution $\oplus_s$ and $\times_s$ to the $L_{p,s}$ setting and returns to the $L_p$ Minkowski combination for convex bodies in the geometric setting for well selected functions originally discovered in [56].

(i) \textbf{General $L_{p,s}$ supremal-convolution for $p \geq 1$.} Firstly, we extend the range of $s \in [0, \infty]$ in [56] to more general setting $s \in [-\infty, \infty]$ without changing the original formulas for $p \geq 1$; that is:

\textbf{Definition 2.1.} Let $p \geq 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and $s \in [-\infty, \infty]$. Let $f, g \colon \mathbb{R}^n \to \mathbb{R}_+$ be measurable functions. We define the $L_{p,s}$ supremal-convolution of $f$ and $g$ by

\begin{equation}
[f \oplus_{p,s} g](z) := \sup_{0 \leq \lambda \leq 1} \left( \sup_{z \in \mathbb{R}^n} M_{s}^{(\frac{1}{q}, \frac{1}{q})} \left( f(x), g(y) \right) \right) = \sup_{0 \leq \lambda \leq 1} \left( (1 - \lambda) \frac{1}{q} \times_s f \right) \oplus_s \left( \lambda \frac{1}{q} \times_s g \right)(z).
\end{equation}

Moreover, a scalar multiplication is defined by for $\alpha \geq 0$,

\begin{equation}
(\alpha \times_{p,s} f)(x) := \sup_{\tau \in [0,1]} \left( \left( \frac{\alpha \tau^{\frac{1}{q}} \times_s f \right)(x) \right) = \alpha^{\frac{q}{p}} f \left( \frac{x}{\alpha^{1/p}} \right),
\end{equation}

where we set explicitly

\begin{equation}
(0 \times_{p,s} f)(x) = \chi_{\{0\}}(x).
\end{equation}

More generally, for $\alpha, \beta \geq 0$, the $L_{p,s}$ supremal-convolution of the functions $f$ and $g$ with respect to $\alpha$ and $\beta$ is denoted as

\begin{equation}
[\alpha \times_{p,s} f] \oplus_{p,s} [\beta \times_{p,s} g].
\end{equation}

Heuristically, $[\alpha \times_{p,s} f] \oplus_{p,s} [\beta \times_{p,s} g]$ should be understood as evaluating averages of functions over the $L_p$ Minkowski combination of the supports of $f$ and $g$, that is, over the set $\alpha \cdot_p \text{supp}(f) +_p \beta \cdot_p \text{supp}(g)$ in [1]. We illustrate the following example on how the functional operations $\times_{p,s}, \oplus_{p,s}$ naturally extend $\cdot_p, +_p$ in the geometric background for $p \geq 1$.

\textbf{Example 2.2.} Suppose that $p \geq 1$, $s \in [-\infty, \infty]$ and $t \in (0,1)$. Let $1/p + 1/q = 1$, and let $f = \chi_A$ and $g = \chi_B$ be characteristic functions of Borel sets $A, B \subset \mathbb{R}^n$, respectively, and set

\begin{equation}
h_{p,t,s} = ((1 - t) \times_{p,s} f) \oplus_{p,s} (t \times_{p,s} g).
\end{equation}

Then $h_{p,t,s} = \chi_{(1-t)\cdot_p A +_p t\cdot_p B}$.

As the above example shows, there’s a natural embedding on the class of Borel measurable sets equipped with the operations $\cdot_p, +_p$ in [1] into the family of measurable functions equipped with the operations $\times_{p,s}, \oplus_{p,s}$ in [11] and [13], respectively for $p \geq 1$.

It was shown in [56] that $[(1 - t) \times_{p,s} f] \oplus_{p,s} [t \times_{p,s} g] \in \mathcal{F}_s(\mathbb{R}^n)$ whenever $f, g \in \mathcal{F}_s(\mathbb{R}^n)$ for $s \in [-\infty, \infty]$. Except for these properties, the next proposition concerns some key properties of the operations $\times_{p,s}$ and $\oplus_{p,s}$.
Proposition 2.3. Let \( f, g, h : \mathbb{R}^n \to \mathbb{R}_+ \) be arbitrary, not identically zero, functions defined on \( \mathbb{R}^n \), let \( s \in [-\infty, \infty] \), \( p \geq 1 \), and \( \alpha, \beta, \gamma > 0 \). Then the following hold:

(a) Homogeneity:

\[
(\alpha \times_{p,s} f) \oplus_{p,s} (\beta \times_{p,s} g) = (\alpha + \beta) \times_{p,s} \left[ \left( \frac{\alpha}{\alpha + \beta} \times_{p,s} f \right) \oplus_{p,s} \left( \frac{\beta}{\alpha + \beta} \times_{p,s} g \right) \right]
\]

for \( s \neq \pm \infty \).

(b) Measurability: \( (\alpha \times_{p,s} f) \oplus_{p,s} (\beta \times_{p,s} g) \) is measurable whenever both \( f \) and \( g \) are Borel measurable.

(c) Commutativity: \( (\alpha \times_{p,s} f) \oplus_{p,s} (\beta \times_{p,s} g) = (\beta \times_{p,s} g) \oplus_{p,s} (\alpha \times_{p,s} f) \).

Remark 2.4. Here by definitions of \( \oplus_{p,s} \) and \( \times_{p,s} \), we can show that

\[
[(\alpha \times_{p,s} f) \oplus_{p,s} (\beta \times_{p,s} g)] \oplus_{p,s} (\gamma \times_{p,s} h) \neq (\alpha \times_{p,s} f) \oplus_{p,s} [(\beta \times_{p,s} g) \oplus_{p,s} (\gamma \times_{p,s} h)]
\]

by the definition of \( L_{p,s} \) suprema-convolution while when \( p = 1 \) the equality holds in [13]. The core difference is the complex coefficients in \( L_p \) case \( (C_{p,\lambda}, D_{p,\lambda}) \) leading to delicate computation for \( p \geq 1 \).

Proof of Proposition 2.3. We give a detailed proof of (a)-(b) following similar steps of the case \( p = 1 \) in [13] and (c) is omitted for simplicity. For (a), we assume that \( s \in \mathbb{R} \setminus \{0\} \) (the other cases follow by continuity), \( p \geq 1 \) and \( 1/p + 1/q = 1 \). Observe, when \( \bar{z} = \frac{z}{(\alpha + \beta)^{1/p}} \), we have

\[
(\alpha + \beta) \times_{p,s} \left[ \left( \frac{\alpha}{\alpha + \beta} \times_{p,s} f \right) \oplus_{p,s} \left( \frac{\beta}{\alpha + \beta} \times_{p,s} g \right) \right](z) = (\alpha + \beta) \times_{p,s} \left[ \left( \frac{\alpha}{\alpha + \beta} \times_{p,s} f \right) \oplus_{p,s} \left( \frac{\beta}{\alpha + \beta} \times_{p,s} g \right) \right](\bar{z})
\]

\[
= (\alpha + \beta)^{1/p} \sup_{0 \leq \lambda \leq 1} \left\{ \sup_{\bar{z} = (\frac{\alpha}{\alpha + \beta})^{(1-\lambda)} x + (\frac{\beta}{\alpha + \beta})^{\lambda} y} M_{s_{\lambda}} \left( \frac{\alpha}{\alpha + \beta} \times_{p,s} f \right) \oplus_{p,s} \left( \frac{\beta}{\alpha + \beta} \times_{p,s} g \right)(f(x, g(y))) \right\}
\]

\[
= \sup_{0 \leq \lambda \leq 1} \left\{ \sup_{\bar{z} = (\frac{\alpha}{\alpha + \beta})^{(1-\lambda)} x + (\frac{\beta}{\alpha + \beta})^{\lambda} y} M_{s_{\lambda}} \left( \frac{\alpha}{\alpha + \beta} \times_{p,s} f \right) \oplus_{p,s} \left( \frac{\beta}{\alpha + \beta} \times_{p,s} g \right)(f(x, g(y))) \right\}
\]

as desired.

For (b), suppose that \( f, g \) are Borel measurable functions. Let \( a > 0 \) and set

\( h(z) := [(\alpha \times_{p,s} f) \oplus_{p,s} (\beta \times_{p,s} g)](z) \).

We need to show that the level set \( \{ x \in \mathbb{R}^n : h(x) < a \} \) is measurable for any fixed constant \( a > 0 \). Observe that by (14) and (15),

\[
\{ z \in \mathbb{R}^n : h(z) < a \} = \{ z \in \mathbb{R}^n : [(\alpha \times_{p,s} f) \oplus_{p,s} (\beta \times_{p,s} g)](z) < a \}
\]
ON THE FRAMEWORK OF $L_p$ SUMMATIONS FOR FUNCTIONS

$\{z \in \mathbb{R}^n : \sup_{0 \leq \lambda \leq 1} \left[ \alpha \frac{1}{p} (1 - \lambda)^{\frac{1}{q}} \times_s f \oplus_s \beta \frac{1}{q} \lambda \times_s g \right] (z) < a \}$

$\bigcap_{\lambda \in [0,1] \cap \mathbb{Q}} \{z \in \mathbb{R}^n : \left[ \alpha \frac{1}{p} (1 - \lambda)^{\frac{1}{q}} \times_s f \oplus_s \beta \frac{1}{q} \lambda \times_s g \right] (z) < a \}$,

where $\mathbb{Q}$ denotes all rational numbers in $\mathbb{R}$.

It follows from the fact \[13, Page 139\] that the functions of the form (case for $p = 1$)

$\alpha \times_s f \oplus_s \beta \times_s g$

are measurable whenever $f$ and $g$ are Borel measurable, and the countable intersection of measurable sets remains measurable, as desired.

$\square$

We can see from above that the $L_p$ coefficients $(C_{p,\lambda,t}, D_{p,\lambda,t})$ are well defined and have elegant properties. In fact, we can see that it has a close relation with the $p$-mean of parameters in the following lemma. Recall that

$C_{p,\lambda,t} = (1 - t)^{1/p}(1 - \lambda)^{1/q}, \quad D_{p,\lambda,t} = t^{1/p}\lambda^{1/q}$.

Lemma 2.5. Let $a, b \geq 0$.

(1) Let $p \geq 1$. For $t \in [0, 1]$, we have

\[ \sup_{0 \leq \lambda \leq 1} [C_{p,\lambda,t}a + D_{p,\lambda,t}b] = ((1 - t)a^p + tb^p)^{1/p}; \]

(2) Let $p < 0$. For $t \in (0, 1)$, we have

\[ \sup_{0 \leq \lambda \leq 1} [C_{p,\lambda,t}a + D_{p,\lambda,t}b] = ((1 - t)a^p + tb^p)^{1/p}; \]

(3) Let $0 < p < 1$. For $t \in (0, 1)$, we have

\[ \inf_{0 \leq \lambda \leq 1} [C_{p,\lambda,t}a + D_{p,\lambda,t}b] = ((1 - t)a^p + tb^p)^{1/p}. \]

Proof. Consider the function

$F(\lambda) := C_{p,\lambda,t}a + D_{p,\lambda,t}b$.

Observe that $F$ is concave for $p \geq 1$ and $p < 0$ with maximum $((1 - t)a^p + tb^p)^{1/p}$, and that $F$ is convex for $0 < p < 1$ with the same formula for minimum. Therefore, we obtain the $p$-mean values on the right hand side of the equalities.

$\square$

Note that we replace $1 - t$ and $t$ by general coefficients $\alpha > 0$ and $\beta > 0$, then similar results holds naturally.

(ii) $L_{p,s}$ inf-sup-convolution for $0 < p < 1$. In the following, we address an extension on the $L_p$ convolution when $p \in (0, 1)$ under the inspiration of Lemma 2.5. To begin with, we recall that in \[11\] the authors extended the definition of the $L_p$ Minkowski combinations due
to Firey to the case of \( p \in (0, \infty) \). They considered, for convex bodies \( K, L \in \mathcal{K}^n_{(o)} \) and scalars \( \alpha, \beta > 0 \), the Wulff shape \([15]\) given by

\[
\alpha \cdot_p K + \beta \cdot_p L = \bigcap_{u \in \mathbb{S}^{n-1}} \left\{ x \in \mathbb{R}^n : \langle x, u \rangle \leq (\alpha h_K(u)^p + \beta h_L(u)^p)^{\frac{1}{p}} \right\} = \left[ (\alpha h_K(u)^p + \beta h_L(u)^p)^{\frac{1}{p}} \right],
\]

where the Wulff shape of a function \( f \in C^+(S^{n-1}) \) is

\[
[f] = \bigcap_{u \in \mathbb{S}^{n-1}} \left\{ x \in \mathbb{R}^n : \langle x, u \rangle \leq f(u) \right\}.
\]

It is clear that the above definition is equivalent to Firey’s original definition in the case \( p \geq 1 \). Here we present an analogue of the definition \((16)\) to general non-empty Borel sets in \( \mathbb{R}^n \) as follows with \( 0 < p < 1 \). That is,

**Definition 2.6.** For \( p \in (0, 1) \), Borel sets \( A, B \) each having the origin as an interior point, and scalars \( \alpha, \beta > 0 \), we define \( L_p \) summation for \( A \) and \( B \) as

\[
\alpha \cdot_p A + \beta \cdot_p B := \bigcap_{0 \leq \lambda \leq 1} \alpha^{\frac{1}{p}}(1 - \lambda)^{\frac{1}{q}} A + \beta^{\frac{1}{p}}(1 - \lambda)^{\frac{1}{q}} B,
\]

where \( \frac{1}{p} + \frac{1}{q} = 1 \).

It can be checked that formula \((17)\) naturally glues with the definition of the \( L_p \) Minkowski combination \((p \geq 1)\) due to Lutwak, Yang and Zhang in \([11]\) when one takes \( p = 1 \). Similar to \([11]\), we have the following result.

**Proposition 2.7.** The definitions \((16)\) and \((17)\) coincide on the class \( \mathcal{K}^n_{(o)} \) for \( 0 < p < 1 \).

**Proof.** Let \( K, L \in \mathcal{K}^n_{(o)} \) and \( \alpha, \beta > 0 \). Then

\[
\bigcap_{0 \leq \lambda \leq 1} \alpha^{\frac{1}{p}}(1 - \lambda)^{\frac{1}{q}} K + \beta^{\frac{1}{p}}(1 - \lambda)^{\frac{1}{q}} L
= \bigcap_{0 \leq \lambda \leq 1} \bigcap_{u \in \mathbb{S}^{n-1}} \left\{ x \in \mathbb{R}^n : \langle x, u \rangle \leq h_{\alpha^{\frac{1}{p}}(1 - \lambda)^{\frac{1}{q}} K(u) + h_{\beta^{\frac{1}{p}}(1 - \lambda)^{\frac{1}{q}} L(u)} \right\}
= \bigcap_{0 \leq \lambda \leq 1} \bigcap_{u \in \mathbb{S}^{n-1}} \left\{ x \in \mathbb{R}^n : \langle x, u \rangle \leq \alpha^{\frac{1}{p}}(1 - \lambda)^{\frac{1}{q}} h_K(u) + \beta^{\frac{1}{p}}(1 - \lambda)^{\frac{1}{q}} h_L(u) \right\}
= \bigcap_{u \in \mathbb{S}^{n-1}} \left\{ x \in \mathbb{R}^n : \langle x, u \rangle \leq \inf \left\{ \alpha^{\frac{1}{p}}(1 - \lambda)^{\frac{1}{q}} h_K(u) + \beta^{\frac{1}{p}}(1 - \lambda)^{\frac{1}{q}} h_L(u) : 0 \leq \lambda \leq 1 \right\} \right\}.
\]

Using Lemma \([22, 25]\) (3), we see that

\[
\inf \left\{ \alpha^{\frac{1}{p}}(1 - \lambda)^{\frac{1}{q}} h_K(u) + \beta^{\frac{1}{p}}(1 - \lambda)^{\frac{1}{q}} h_L(u) : 0 \leq \lambda \leq 1 \right\} = (\alpha h_K(u)^p + \beta h_L(u)^p)^{\frac{1}{p}}.
\]

This confirms the assertion of this proposition. \( \square \)
With the formula (17) and the above proposition in hand, we are in a position to define a functional counterpart of the $L_p$ Minkowski combination in the setting $p \in (0, 1)$ that coincides with Definition 2.1 in the case $p = 1$, which we refer to as the $L_{p,s}$ inf-sup convolution for functions.

**Definition 2.8.** Let $0 < p < 1$, $1/p + 1/q = 1$ and $s \in [-\infty, \infty]$. Given Borel measurable functions $f, g : \mathbb{R}^n \to \mathbb{R}_+$, each having support containing the origin in their interior, we define the $L_{p,s}$ inf-sup-convolution of $f$ and $g$ with weights $\alpha, \beta > 0$ to be

$$
(18) \ [\alpha \times_{p,s} f \oplus_{p,s} \beta \times_{p,s} g](z) := \inf_{0 \leq \lambda \leq 1} \left[ \sup_{z = \alpha \frac{1}{(1-\lambda)} (\alpha + \beta) \times_{p,s} f + \beta \frac{1}{\lambda} \times_{p,s} g} M_s \left( (1-\lambda) \frac{1}{\frac{1}{\alpha} + \frac{1}{\beta} \times_{p,s} g} \right) \right].
$$

The next result concerns some critical properties of the $L_{p,s}$ inf-sup-convolution (18) and the proofs are similar to those of Proposition 2.3.

**Proposition 2.9.** Let $h, f, g : \mathbb{R}^n \to \mathbb{R}_+$ be Borel measurable, not identically zero with supports containing the origin in their interiors, functions defined on $\mathbb{R}^n$. Let $s \in [-\infty, \infty]$, $0 < p < 1$, and $\alpha, \beta, \gamma > 0$. Then the following hold:

(a) Homogeneity:

$$(\alpha \times_{p,s} f) \oplus_{p,s} (\beta \times_{p,s} g) = (\alpha + \beta) \times_{p,s} \left[ \left( \frac{\alpha}{\alpha + \beta} \times_{p,s} f \right) \oplus_{p,s} \left( \frac{\beta}{\alpha + \beta} \times_{p,s} g \right) \right]$$

for $s \neq \pm \infty$.

(b) Measurability: $(\alpha \times_{p,s} f) \oplus_{p,s} (\beta \times_{p,s} g)$ is measurable whenever both $f$ and $g$ are Borel measurable.

(c) Commutativity: $(\alpha \times_{p,s} f) \oplus_{p,s} (\beta \times_{p,s} g) = (\beta \times_{p,s} g) \oplus_{p,s} (\alpha \times_{p,s} f)$.

(d) When $f = \chi_A$ and $g = \chi_B$ for some pair of non-empty Borel sets $A, B \subset \mathbb{R}^n$ with the origin in their interior, and $t \in (0, 1)$, one has that

$$
[(1 - t) \times_{p,s} f \oplus_{p,s} t \times_{p,s} g] = \chi_{(1 - t) \cdot p_A + t \cdot p_B}.
$$

**Proof.** For (a)-(c), the proofs follow similar lines to Proposition 2.3 (a)-(c) with $p \geq 1$ by changing “$\sup$" to “$\inf$", and “$\cap$” to “$\cup$” correspondingly with $0 < p < 1$. The proof of (d) follows similar lines to Proposition 2.7. □

**Remark 2.10.** Associativity doesn’t hold as $[(\alpha \times_{p,s} f) \oplus_{p,s} (\beta \times_{p,s} g)] \oplus_{p,s} (\gamma \times_{p,s} h) \neq (\alpha \times_{p,s} f) \oplus_{p,s} [(\beta \times_{p,s} g) \oplus_{p,s} (\gamma \times_{p,s} h)]$ with the $L_p$ coefficients.

### 2.2. $L_{p,s}$ Asplund summation for $s$-concave functions for $p > 0$.

Next, we present the definition of $L_{p,s}$ Asplund summation for $s$-concave functions in a similar way to [28] ($s = 0$, log-concave function) with $p \geq 1$ using base functions.

(i) $L_{p,s}$ Asplund summation for $s$-concave functions for $p \geq 1$. Recall the $L_p$ (or $L_{p,0}$) Asplund summations for functions using the $L_p$ operations $\square_p$ for convex functions defined...
in [28] as follows. Let \( p \geq 1 \). Given \( \alpha, \beta > 0 \) and \( u, v \in C_s(\mathbb{R}^n) \),

\[
[(\alpha \boxplus_p u) \boxplus_p (\beta \boxplus_p v)'](x) := \{(\alpha(u^*(x))^p + \beta(v^*(x))^p)^{1/p}\}^*.
\]

In the case \( p = 0 \), it becomes

\[
[(\alpha \boxplus_0 u) \boxplus_0 (\beta \boxplus_0 v)'](x) := [(u^*(x))^\alpha(v^*(x))^\beta]^*.
\]

Therefore, we give the \( L_{p,s} \) Asplund summation for functions in \( F_s(\mathbb{R}^n) \) in the same manner.

**Definition 2.11.** For \( p \geq 1, s \in (-\infty, \infty) \), given \( f, g \in F_s(\mathbb{R}^n) \), we define the \( L_{p,s} \) Asplund summation as

\[
(\alpha \cdot_{p,s} f) \ast_{p,s} (\beta \cdot_{p,s} g) := (1 - s[(\alpha \boxplus_p u) \boxplus_p (\beta \boxplus_p v)])^{1/2},
\]

where \( u \) and \( v \) are base functions for \( f \) and \( g \), respectively.

It was shown in [28, Proposition 3.2] that \( [(\alpha \boxplus_p u) \boxplus_p (\beta \boxplus_p v)] \in C_s(\mathbb{R}^n) \) whenever \( u, v \in C_s(\mathbb{R}^n) \) and \( p \geq 1 \). A similar definition for \( p \in (0,1) \) was introduced in [54]. Here we give a similar \( L_{p,s} \) Asplund summation for \( s \)-concave functions for \( 0 < p < 1 \) using the Legendre transformation for the convex functions.

(ii) \( L_{p,s} \) Asplund summation for \( s \)-concave functions for \( 0 < p < 1 \). We follow the notations in [54] and define the support function for \( f = (1 - su_f)^{1/s} \in F_s(\mathbb{R}^n) \) to be

\[ h_f = (u_f)^\ast, \]

which is the Legendre transformation of the base function for \( f \). Moreover, we propose the definition of \( s \)-Aleksandrov function here.

**Definition 2.12.** Let \( u : \mathbb{R}^n \to [0, \infty] \) be a lower semi-continuous function (which may or may not be convex) with \( u(x) \geq u(o) = 0 \) for all \( x \in \mathbb{R}^n \). For \( s \in (-\infty, \infty) \), the \( s \)-Aleksandrov Function of \( u \) is \( A[u]_s = (1 - su^\ast)^{1/s} \).

Note that \( f \) is the largest \( s \)-concave function with \( h_f \leq (u_f)^\ast \). We then define the \( L_{p,s} \) Asplund summation for \( 0 < p < 1 \) using the Legendre transformation of the base functions, i.e., \( h_f = (u_f)^\ast \) as follows.

**Definition 2.13.** For \( 0 < p < 1, s \in (-\infty, \infty) \), given \( f, g \in F_s(\mathbb{R}^n) \) with \( h_f, h_g \geq 0 \), we define the \( L_{p,s} \) Asplund summation \( \alpha \cdot_{p,s} f \ast_{p,s} \beta \cdot_{p,s} g \) with weights \( \alpha, \beta \geq 0 \) as

\[
\alpha \cdot_{p,s} f \ast_{p,s} \beta \cdot_{p,s} g = A \left[(\alpha h_f^p + \beta h_g^p)^{1/p}\right]^s.
\]

The above definition recovers the results of Asplund summation for log-concave functions in [54] if \( s = 0 \).

One of our main results is that we show the \( L_p \) supremal-convolution and the \( L_{p,s} \) Asplund summation using the base functions coincide with each other in \( F_0(\mathbb{R}^n) \) which generalized the results of [53, Proposition 10] in next subsection. This connection is however only works for the coefficients \( 1 - t \) and \( t \), while for more general coefficients it needs more homogeneity restrictions. See more references in [28,53,54].
2.3. Relation between $L_{p,s}$ convolutions and $L_{p,s}$ Asplund summation. Next inspired by [12] for $p = 1$, we compare the $L_{p,s}$ convolutions and $L_{p,s}$ Asplund summations for functions defined above for different cases of $p$ and $s$. Together with the fact that established by Artstein-Avidan and Milman [7,8] that the Legendre transformation is the only duality on the class $\text{Cvx}(\mathbb{R}^n)$, that is, the only transformation that satisfies the conditions:

$$u^{**} = u \quad \text{and} \quad u^* \geq v^* \quad \text{whenever} \ u, v \in \text{Cvx}(\mathbb{R}^n) \ \text{satisfy} \ u \leq v,$$

we give a detailed proof of the following properties.

**Proposition 2.14.** (1) Let $p \geq 1$, $f, g \in F_0(\mathbb{R}^n)$ $(s = 0)$ be of the form $f = e^{-u}$ and $g = e^{-v}$ for some $u, v \in C_s(\mathbb{R}^n)$. Then the following equality holds:

$$[(1 - t) \times_{p,0} f] \oplus_{p,0} [t \times_{p,0} g](z) = e^{-\left(M_p^{(1-t),z}(u^*(z),v^*(z))\right)}.$$

(2) Let $p \geq 1$, $s \neq 0$, $f, g \in F_s(\mathbb{R}^n)$ be of the form $f = (1 - su)^{1/s}$ and $g = (1 - sv)^{1/s}$, for some $u, v \in C_s(\mathbb{R}^n)$. Then the following inequality holds:

$$[(1 - t) \times_{p,s} f] \oplus_{p,s} [t \times_{p,s} g](z) \leq (1 - s(M_p^{(1-t),z})(u^*(z),v^*(z)))^{1/s}.$$

(3) Let $0 < p < 1$, $f, g \in F_s(\mathbb{R}^n)$ be of the form $f = (1 - su)^{1/s}$ and $g = (1 - sv)^{1/s}$ for some $u, v \in C_s(\mathbb{R}^n)$. Then the following inequality holds:

$$[(1 - t) \times_{p,s} f] \oplus_{p,s} [t \times_{p,s} g](z) \leq (1 - s(M_p^{(1-t),z})(u^*(z),v^*(z)))^{1/s} \quad \text{for} \quad s > 0;$$

$$[(1 - t) \times_{p,s} f] \oplus_{p,s} [t \times_{p,s} g](z) \geq (1 - s(M_p^{(1-t),z})(u^*(z),v^*(z)))^{1/s} \quad \text{for} \quad s < 0.$$

**Proof.** (1) For $p \geq 1$, it follows from the definition of $L_{p,s}$ supremal-convolution for $s = 0$ in Definition 2.7 and (9) that

$$[(1 - t) \times_{p,0} f] \oplus_{p,0} [t \times_{p,0} g](z) = \sup_{0 \leq \lambda \leq 1} \left[ (C_{p,\lambda,t} \times_0 f) \oplus_0 (D_{p,\lambda,t} \times_0 g)(z) \right]$$

$$= \sup_{0 \leq \lambda \leq 1} \left[ \sup_{z = C_{p,\lambda,t} x + D_{p,\lambda,t} v} e^{-C_{p,\lambda,t} u(x) + D_{p,\lambda,t} v(y)} \right]$$

$$= \sup_{0 \leq \lambda \leq 1} e^{-\inf_{z = C_{p,\lambda,t} x + D_{p,\lambda,t} v}[C_{p,\lambda,t} u(x) + D_{p,\lambda,t} v(y)]}$$

$$= \sup_{0 \leq \lambda \leq 1} e^{-\inf_{0 \leq \lambda \leq 1}[C_{p,\lambda,t} u^*(z) + D_{p,\lambda,t} v^*(z)]^*}$$

$$= e^{-\left(\sup_{0 \leq \lambda \leq 1}(C_{p,\lambda,t} u^*(z) + D_{p,\lambda,t} v^*(z))\right)^*}$$

$$= e^{-\left(M_p^{(1-t),z}(u^*(z),v^*(z))\right)^*}.$$
(2) For $p \geq 1$ and $s > 0$, we observe similarly that

$$
\left(\left[(1 - t) \times_{p,s} f\right] \oplus_{p,s} [t \times_{p,s} g]\right)(z)^* \\
= \sup_{0 \leq \lambda \leq 1} \left[\left((C_{p,\lambda,t} \times_s f) \oplus_s (D_{p,\lambda,t} \times_s g)\right)(z)\right] \\
= \sup_{0 \leq \lambda \leq 1} \sup_{z = C_{p,\lambda,t}x + D_{p,\lambda,t}y} \left[C_{p,\lambda,t} - sC_{p,\lambda,t}u(x) + D_{p,\lambda,t} - sD_{p,\lambda,t}v(y)\right] \\
= \sup_{0 \leq \lambda \leq 1} \left[C_{p,\lambda,t} + D_{p,\lambda,t} - s \inf_{z = C_{p,\lambda,t}x + D_{p,\lambda,t}y} \left(C_{p,\lambda,t}u(x) + D_{p,\lambda,t}v(y)\right)\right] \\
\leq \sup_{0 \leq \lambda \leq 1} \left(C_{p,\lambda,t} + D_{p,\lambda,t} - s \inf_{0 \leq \lambda \leq 1} \left(C_{p,\lambda,t}u^*(z) + D_{p,\lambda,t}v^*(z)\right)\right)^* \\
\leq 1 - s \left(M_p^{(1-t)}(u^*(z), v^*(z))\right)^*.
$$

Above we have used the Hölder’s identity $C_{p,\lambda,t} + D_{p,\lambda,t} \leq 1$, [11], [51] Theorem 11.23 (d) together with the Fenchel-Moreau theorem, and Lemma [2,5] (1).

For $s < 0$, it can be proved in a similar way as

$$
\left(\left[(1 - t) \times_{p,s} f\right] \oplus_{p,s} [t \times_{p,s} g]\right)(z)^* \\
= \inf_{0 \leq \lambda \leq 1} \left[\left((C_{p,\lambda,t} \times_s f) \oplus_s (D_{p,\lambda,t} \times_s g)\right)(z)\right] \\
= \inf_{0 \leq \lambda \leq 1} \inf_{z = C_{p,\lambda,t}x + D_{p,\lambda,t}y} \left[C_{p,\lambda,t} - sC_{p,\lambda,t}u(x) + D_{p,\lambda,t} - sD_{p,\lambda,t}v(y)\right] \\
= \inf_{0 \leq \lambda \leq 1} \inf_{z = C_{p,\lambda,t}x + D_{p,\lambda,t}y} \left[C_{p,\lambda,t} + D_{p,\lambda,t} - s \sup_{z = C_{p,\lambda,t}x + D_{p,\lambda,t}y} \left(C_{p,\lambda,t}u(x) + D_{p,\lambda,t}v(y)\right)\right] \\
\geq \inf_{0 \leq \lambda \leq 1} \inf_{z = C_{p,\lambda,t}x + D_{p,\lambda,t}y} \left[C_{p,\lambda,t} + D_{p,\lambda,t} - s \inf_{0 \leq \lambda \leq 1} \left(C_{p,\lambda,t}u^*(z) + D_{p,\lambda,t}v^*(z)\right)\right] \\
\geq 1 - s \left(M_p^{(1-t)}(u^*(z), v^*(z))\right)^*,
$$

as desired.

(3) For $0 < p < 1$ and $s > 0$, we compute

$$
\left(\left[(1 - t) \times_{p,s} f\right] \oplus_{p,s} [t \times_{p,s} g]\right)(z)^* \\
= \inf_{0 \leq \lambda \leq 1} \left[\left((C_{p,\lambda,t} \times_s f) \oplus_s (D_{p,\lambda,t} \times_s g)\right)(z)\right] \\
= \inf_{0 \leq \lambda \leq 1} \inf_{z = C_{p,\lambda,t}x + D_{p,\lambda,t}y} \left[C_{p,\lambda,t} - sC_{p,\lambda,t}u(x) + D_{p,\lambda,t} - sD_{p,\lambda,t}v(y)\right] \\
= \inf_{0 \leq \lambda \leq 1} \inf_{z = C_{p,\lambda,t}x + D_{p,\lambda,t}y} \left[C_{p,\lambda,t} + D_{p,\lambda,t} - s \inf_{z = C_{p,\lambda,t}x + D_{p,\lambda,t}y} \left(C_{p,\lambda,t}u(x) + D_{p,\lambda,t}v(y)\right)\right].
$$
Above we have used the reverse Hölder’s identity $C_{p,\lambda,t} + D_{p,\lambda,t} \geq 1$ for $0 < p < 1$, \cite{11, 51 Theorem 11.23 (d)}, and Lemma \ref{lem:2.5} (3).

For $s < 0$, it can be proved in a similar way as

\[
[(1 - t) \times_{p,s} f] \oplus_{p,s} [t \times_{p,s} g](z) = \inf_{0 \leq \lambda \leq 1} C_{p,\lambda,t} + D_{p,\lambda,t} - s(C_{p,\lambda,t} u^*(z) + D_{p,\lambda,t} v^*(z))^s
\]

\[
\geq \inf_{0 \leq \lambda \leq 1} (C_{p,\lambda,t} + D_{p,\lambda,t}) - s \sup_{0 \leq \lambda \leq 1} (C_{p,\lambda,t} u^*(z) + D_{p,\lambda,t} v^*(z))^s
\]

\[
\geq 1 - s(M_p((1-t),t)(u^*(z), v^*(z)))^s.
\]

as desired. \qed

3. New $L_p$-Borell-Brascamp-Lieb type inequalities

In this section, we will present several $L_p$-Borell-Brascamp-Lieb type inequalities related to the $L_{p,s}$ supremal-convolution. Firstly, in Subsection 3.1 we extend the Borell-Brascamp-Lieb inequality in \cite{13} [Theorem 4.1 and 4.2] to the $L_3$ case as Theorem 3.1 and Theorem 3.3. Secondly, we give different improvement methods of $L_p$-Borell-Brascamp-Lieb inequality in \cite{56} for $s \geq 0$ to $s \in [-\infty, \infty]$ including using mass transportation with matrix inequality and applying the result of classical Borell-Brascamp-Lieb inequality in Subsection 3.2.

3.1. A Novel $L_p$-Borell-Brascamp-Lieb inequality for $p \geq 1$. Recall in \cite{13 Page 22} that functional $\Omega : \mathcal{B} \to \mathbb{R}_+$ (for example, capacity or the measure of a set in $\mathbb{R}^n$), where $\mathcal{B}$ denotes the class of Borel subsets of $\mathbb{R}^n$, is said to be monotone if,

\[
\Omega(A_0) \leq \Omega(A_1), \quad \text{whenever } A_0 \subset A_1,
\]

and $\gamma$-concave, with $\gamma \in [-\infty, \infty]$ and $t \in [0, 1]$, if

\[
\Omega((1 - t)A_0 + tA_1)) \geq M_{\gamma}((1-t),t)(\Omega(A_0), \Omega(A_1))
\]

for all Borel sets $A_0, A_1 \in \mathcal{B}$, with $\Omega(A_0), \Omega(A_1) > 0$. We always take the convention that $\Omega(\emptyset) = 0$ and $\Omega(A) > 0$ implies that $A$ is non-empty. For example, given a compact set $S$ in
the $n$-dimensional Euclidean space $\mathbb{R}^n$ ($n \geq 2$), the variational $p$-capacity of $S$ for $p \in (1, n)$ is defined by

$$\text{Cap}_p(S) = \inf \left\{ \int_{\mathbb{R}^n} |\nabla f|^p \, dx : f \in C_c^\infty(\mathbb{R}^n) \text{ and } f(x) \geq 1 \text{ for all } x \in S \right\},$$

where $C_c^\infty(\mathbb{R}^n)$ denotes the class of all infinitely differentiable functions with compact support in $\mathbb{R}^n$. If $\Omega = \text{Cap}_p$, then $\text{Cap}_p$ is $1 - p$ concave for $p \in (1, n)$ (as shown in \cite[Theorem 1]{13}) on the class $\mathcal{K}_n^o$ of convex bodies in $\mathbb{R}^n$.

Remark 3.2. The version of Theorem 3.1 for $C \in \mathbb{P}$, the class of Borel measurable (respectively, quasi-concave functions) that satisfy the condition of Remark 3.2.

Theorem 3.3. (Lp,γ Borell-Brascamp-Lieb inequality in $\mathbb{R}$) Let $p, q \in [1, \infty]$ be such that $1/p + 1/q = 1$. Let $t \in (0, 1)$, $\alpha \in [-1, +\infty]$ and $\gamma \in [-\alpha, \infty)$. Let $h, f, g : (0, \infty) \to \mathbb{R}_+$ be a triple of integrable functions that satisfy the condition

$$h \left( (C_{p,\lambda,t} x^\gamma + D_{p,\lambda,t} y^\gamma)^\frac{1}{\gamma} \right) \geq \left| [C_{p,\lambda,t} f(x)^\alpha + D_{p,\lambda,t} g(y)^\alpha]^\frac{1}{\alpha} \right|$$

for every $x \in \text{supp}(f), y \in \text{supp}(g)$, and $\lambda \in (0, 1)$ whenever $f(x)g(y) > 0$. Then the following inequality holds:

$$\tilde{\Omega}(h) \geq \left[ (1 - t)\tilde{\Omega}(f)^\beta + t\tilde{\Omega}(g)^\beta \right]^\frac{1}{\beta}, \quad \beta = \frac{p\alpha\gamma}{\alpha + \gamma}.$$
for every \( x \in \text{supp}(f) \), \( y \in \text{supp}(g) \), and \( \lambda \in (0, 1) \) whenever \( f(x)g(y) > 0 \). Then the following integral inequality holds:

\[
\int_0^\infty h(x)dx \geq \left( (1-t) \left( \int_0^\infty f(x)dx \right)^\beta + t \left( \int_0^\infty g(x)dx \right)^\beta \right)^{\frac{1}{\beta}},
\]

where \( \beta = \frac{p\alpha}{\alpha+\gamma} \).

The proof of Theorem 3.3 is postponed until the next section, as it requires division into several steps. For now we prove Theorem 3.1 firstly.

**Proof of Theorem 3.1.** We denote by

\[ C_m(r) = \{ x \in \mathbb{R}^n : m(x) \geq r \} \]

the super-level set for any Borel measurable function \( m \). By the hypothesis (20) placed on the triple of functions \( h, f, g \), one has

\[
C_h(\tau_\lambda^a) \supset C_p,\lambda,tf(r) + D_p,\lambda,tg(s), \quad \tau_\lambda^a := [C_p,\lambda,tf^a + D_p,\lambda,ts^a]^{\frac{1}{a}}, \quad a \in [-\infty, \infty]
\]

holds for all \( \lambda \in (0, 1) \) whenever \( r, s > 0 \) satisfies \( \Omega(C_f(r)) > 0 \) and \( \Omega(C_g(s)) > 0 \). Indeed, if for some fixed \( \lambda_0 \in (0, 1) \), \( z \in C_p,\lambda_0,tf(r) + D_p,\lambda_0,tg(s) \), then there exist some \( x \in C_f(r) \) and \( y \in C_g(s) \) such that

\[ z = C_p,\lambda_0,tx + D_p,\lambda_0,ty. \]

Using the assumption (20), we have that

\[
h(z) = h(C_p,\lambda_0,tx + D_p,\lambda_0,ty)
\geq [C_p,\lambda_0,tf(x)^a + D_p,\lambda_0,tg(y)^a]^{\frac{1}{a}}
\geq \tau_\lambda^a \lambda_0,
\]

which establishes the inclusion (23) for every fixed \( \lambda_0 \in (0, 1) \).

Consider the functions \( \bar{h}, \bar{f}, \bar{g} : (0, \infty) \to \mathbb{R}_+ \), the composition of \( \Omega \) and super level sets, defined, respectively, by

\[
\bar{h}(r) := \Omega(C_h(r)), \quad \bar{f}(r) := \Omega(C_f(r)), \quad \bar{g}(r) := \Omega(C_g(r)).
\]

By the monotonicity and \( \alpha \)-concavity of \( \Omega \) and the inclusion (23), this triple of functions satisfy

\[
\bar{h} \left( [C_p,\lambda,tf^\gamma + D_p,\lambda,ts^\gamma]^{\frac{1}{\gamma}} \right)
= \Omega(C_h(\tau_\lambda^a))
\geq \Omega(C_p,\lambda,tf(r) + D_p,\lambda,tg(s))
\geq \left[ C_p,\lambda,\Omega(C_f(r))^a + (1 - C_p,\lambda,t) \Omega \left( \frac{D_p,\lambda,tf_s}{1 - C_p,\lambda,t} \right)^{\frac{1}{a}} \right]^{\frac{1}{a}}
\geq \left[ C_p,\lambda,tf(r)^a + D_p,\lambda,tg(s)^a \right]^{\frac{1}{a}}
\]
holds for every \( r, s > 0 \) and \( \lambda \in (0,1) \) whenever \( \tilde{f}(r)\tilde{g}(s) > 0 \). Above we have used Hölder’s inequality to conclude that for \( p \geq 1 \), \( C_{p,\lambda,t} + D_{p,\lambda,t} \leq 1 \).

Therefore, the triple of functions \( \{ \tilde{f}, \tilde{g}, \tilde{h} \} \) satisfy the hypothesis of Theorem 3.3 (21), and therefore

\[
\tilde{\Omega}(h) = \int_0^\infty \tilde{h}(r)dr \\
\geq \left[ (1-t) \left( \int_0^\infty \tilde{f}(r)dr \right) + t \left( \int_0^\infty \tilde{g}(r)dr \right) \right]^{\frac{1}{\beta}} \\
= \left[ (1-t)\Omega(f) + t\Omega(g) \right]^{\frac{1}{\beta}},
\]

where \( \beta = \frac{p\alpha}{\alpha + \gamma} \), as desired formula (22). \( \square \)

The proof of Theorem 3.3 is inspired by the work of Ball [9] and Bobkov, Colesanti, and Fragała [13] for different cases for \( \gamma \) with details as follows.

**Proof of Theorem 3.3**

1. **The case \( \gamma = 1 \)**. Assume that \( \gamma = 1 \) and let \( \alpha \in [-1,\infty] \). For the case \( \alpha \geq 0 \), we already handled in our other paper [56]. Therefore, we may assume that \( \alpha \in [-1,0) \).

   Fix \( \lambda \in (0,1) \). As all functions involved are integrable, we may assume, without loss of generality, that \( f \) and \( g \) are bounded with non-zero maximums. Set

\[
M_\lambda = [C_{p,\lambda,t}\|f\|_\infty^\alpha + D_{p,\lambda,t}\|g\|_\infty^\alpha]^{\frac{1}{\alpha}}.
\]

Using the assumptions placed on the triple \( f, g, \tilde{h} \) (21), we see that, for any \( x \in \text{supp}(f) \) and \( y \in \text{supp}(g) \), one has

\[
h(C_{p,\lambda,t}x + D_{p,\lambda,t}y) \\
\geq [C_{p,\lambda,t}f(x)^\alpha + D_{p,\lambda,t}g(y)^\alpha]^{1/\alpha} \\
= M_\lambda \left[ C_{p,\lambda,t} \left( \frac{\|f\|_\infty}{M_\lambda} \right)^\alpha \right]^{\frac{1}{\alpha}} f(x)^\alpha + D_{p,\lambda,t} \left( \frac{\|g\|_\infty}{M_\lambda} \right)^\alpha g(y)^\alpha \right]^{1/\alpha} \\
= M_\lambda \left[ (1-\theta)\tilde{f}(x)^\alpha + \theta\tilde{g}(y)^\alpha \right]^{\frac{1}{\alpha}}, \quad \theta = D_{p,\lambda,t} \left( \frac{\|g\|_\infty}{M_\lambda} \right)^\alpha, \quad \tilde{f} = \frac{f}{\|f\|_\infty}, \quad \tilde{g} = \frac{g}{\|g\|_\infty}, \\
\geq M_\lambda \min\{\tilde{f}(x), \tilde{g}(y)\}.
\]

Therefore, by letting \( h_\lambda := \frac{h}{M_\lambda} \), we see that

\[
\{ h_\lambda \geq \eta \} \supset C_{p,\lambda,t}\{ \tilde{f} \geq \eta \} + D_{p,\lambda,t}\{ \tilde{g} \geq \eta \}
\]

for all \( \eta \in [0,1] \) whenever \( x \in \{ \tilde{f} \geq \eta \} \) and \( y \in \{ \tilde{g} \geq \eta \} \). Hence, using Fubini’s theorem and the Brunn-Minkowski inequality in dimension one \( \text{vol}_1(A + B) \geq \text{vol}_1(A) + \text{vol}_1(B) \) where \( \text{vol}_1 \)
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denotes the volume of set in $\mathbb{R}$, we see that

$$\int_0^\infty h(x)dx = M_\lambda \int_0^\infty h_\lambda(x)dx$$

$$= M_\lambda \int_0^1 \operatorname{vol}_1\{h_\lambda \geq \eta\}d\eta$$

$$= M_\lambda \int_0^1 \operatorname{vol}_1(C_{p,\lambda,t}\{\bar{f} \geq \eta\} + D_{p,\lambda,t}\{\bar{g} \geq \eta\})d\eta$$

$$= M_\lambda \left(C_{p,\lambda,t}\left(\int_0^\infty \bar{f}(x)dx\right) + D_{p,\lambda,t}\left(\int_0^\infty \bar{g}(x)dx\right)\right)$$

$$\quad = \left[(1 - \lambda) \left(\left(\frac{1 - t}{1 - \lambda}\right)^{\frac{1}{p}} \|f\|_\infty\right)^{\alpha} + \lambda \left(\left(\frac{t}{\lambda}\right)^{\frac{1}{p}} \|g\|_\infty\right)^{\alpha}\right]^{\frac{1}{\alpha}} \times$$

$$\quad \left[(1 - \lambda) \left(\frac{1 - t}{1 - \lambda}\right)^{\frac{1}{p}} \left(\int_0^\infty \bar{f}(x)dx\right) + \lambda \left(\left(\frac{t}{\lambda}\right)^{\frac{1}{p}} \left(\int_0^\infty \bar{g}(x)dx\right)\right)\right]$$

$$\geq \left[C_{p,\lambda,t}\left(\int_0^\infty f(x)dx\right)^{\frac{\alpha}{\alpha + 1}} + D_{p,\lambda,t}\left(\int_0^\infty g(x)dx\right)^{\frac{\alpha}{\alpha + 1}}\right]^\frac{\alpha + 1}{\alpha},$$

where in the last line we have used the fact that $\alpha > -1$ together with the generalized Hölder inequality; i.e., for all $u_1, u_2, v_1, v_2 \geq 0$ and $\lambda \in (0, 1)$, $t \in [0, 1]$, it holds

$$(24) \quad M_{\alpha_1}^{C_{p,\lambda,t},D_{p,\lambda,t}}(u_1, v_1)M_{\alpha_2}^{C_{p,\lambda,t},D_{p,\lambda,t}}(u_2, v_2) \geq M_{\alpha_0}^{C_{p,\lambda,t},D_{p,\lambda,t}}(u_1u_2, v_1v_2),$$

whenever

$$\alpha_1 + \alpha_2 > 0, \quad \frac{1}{\alpha_0} = \frac{1}{\alpha_1} + \frac{1}{\alpha_2}.$$

Therefore, as $\lambda$ was arbitrarily fixed, we actually proved that

$$\int_0^\infty h(x)dx \geq \sup_{0<\lambda<1} \left[C_{p,\lambda,t}\left(\int_0^\infty f(x)dx\right)^{\frac{\alpha}{\alpha + 1}} + D_{p,\lambda,t}\left(\int_0^\infty g(x)dx\right)^{\frac{\alpha}{\alpha + 1}}\right]^\frac{\alpha + 1}{\alpha}.$$

By optimizing over $\lambda$, with $\alpha \in (-1, 0)$, together with Lemma 2.5 (2), we see

$$\int_0^\infty h(x)dx \geq \left[(1 - t) \left(\int_0^\infty f(x)dx\right)^{\frac{p\alpha}{\alpha + 1}} + t \left(\int_0^\infty g(x)dx\right)^{\frac{p\alpha}{\alpha + 1}}\right]^\frac{\alpha + 1}{\alpha p},$$

which completes the proof for $\gamma = 1$.

(2) **The case** $\gamma = 0$. Suppose that $\gamma = 0$. Consider the functions $m, d, n : \mathbb{R} \setminus \{0\} \to \mathbb{R}_+$ defined by

$$m(x) := h(e^x)e^x, \quad d(x) := f(e^x)e^x, \quad n(x) := g(e^x)e^x.$$
Then, for any \( e^x \in \text{supp}(f) \), \( e^y \in \text{supp}(g) \), and \( \lambda \in (0, 1) \), applying the assumption (21), one has

\[
m(C_{p,\lambda,t}x + D_{p,\lambda,t}y) = h\left(e^{C_{p,\lambda,t}x + D_{p,\lambda,t}y} \right) \geq \left[ f(e^x) e^y \right]^{C_{p,\lambda,t}} \left[ g(e^y) e^x \right]^{D_{p,\lambda,t}} = d(x)^{C_{p,\lambda,t}} n(y)^{D_{p,\lambda,t}}.
\]

(25)

Recall that the \( L_p \)-Prékopa-Leindler inequality for product measures [56] with quasi-concave densities states that let \( f, g, h : \mathbb{R}^n \to \mathbb{R}_+ \) be a triple of measurable functions, with \( f, g \) weakly unconditional and positively decreasing, that satisfy the condition

\[
h(C_{p,\lambda,t}x + D_{p,\lambda,t}y) \geq f(x)^{C_{p,\lambda,t}} g(y)^{D_{p,\lambda,t}}
\]

for every \( x \in \text{supp}(f) \), \( y \in \text{supp}(g) \), and every \( 0 < \lambda < 1 \). The following integral inequality holds:

\[
\int_{\mathbb{R}^n} h d\mu \geq \sup_{0 < \lambda < 1} \left\{ \left[ \left( \frac{1 - t}{1 - \lambda} \right) \left( \frac{t}{\lambda} \right) \right]^\frac{1}{\lambda} \left( \int_{\mathbb{R}^n} f(\lambda x) \frac{1}{\lambda} dx \right)^{1 - \lambda} \left( \int_{\mathbb{R}^n} g(\lambda x) \frac{1}{\lambda} dx \right)^\lambda \right\}.
\]

(26)

According to inequality (25), the triple of functions \((m, d, n)\) satisfy the condition in dimension 1 (26), and therefore

\[
\int_{\mathbb{R}^n} m dx \geq \sup_{0 < \lambda < 1} \left\{ \left[ \left( \frac{1 - t}{1 - \lambda} \right) \left( \frac{t}{\lambda} \right) \right]^\frac{1}{\lambda} \left( \int_{\mathbb{R}^n} d(\lambda x) \frac{1}{\lambda} dx \right)^{1 - \lambda} \left( \int_{\mathbb{R}^n} n(\lambda x) \frac{1}{\lambda} dx \right)^\lambda \right\}.
\]

(27)

Therefore by choosing \( \lambda = t \), we can see that

\[
\int_{\mathbb{R}^n} m(x) dx \geq \left( \int_{\mathbb{R}^n} d(x) dx \right)^{1 - t} \left( \int_{\mathbb{R}^n} n(x) dx \right)^t.
\]

Finally, note that

\[
\int_{\mathbb{R}^n} m(x) dx = \int_{\mathbb{R}^n} h(x) e^{x} dx = \int_{0}^{\infty} h(x) dx,
\]

and the same with the pairs \((d, f)\) and \((n, g)\). This completes the proof of the theorem in the case \( \gamma = 0 \).

Next we consider \( \gamma \neq 0, 1 \). Suppose that \( \gamma \in (-\infty, 1) \setminus \{0\} \). Let \(-\gamma \leq \alpha \leq \infty\) with \( \gamma > -\infty \). Consider the triple of functions \( w, u, v \) defined by

\[
w(x) = h(x^{1/\gamma}), \quad u(x) = f(x^{1/\gamma}), \quad v(x) = g(x^{1/\gamma}).
\]

Using the assumption (21), we see that

\[
w(C_{p,\lambda,t}x + D_{p,\lambda,t}y) = h\left((C_{p,\lambda,t}x + D_{p,\lambda,t}y)^{\frac{1}{\gamma}}\right) \geq \left[ C_{p,\lambda,t}f(x^{1/\gamma})^{\alpha} + D_{p,\lambda,t}g(y^{1/\gamma})^{\alpha} \right]^\frac{1}{\alpha} = \left[ C_{p,\lambda,t}u(x)^{\alpha} + D_{p,\lambda,t}v(y)^{\alpha} \right]^\frac{1}{\alpha}
\]

(28)
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holds whenever $x^{1/\gamma} \in \text{supp}(f)$, $y^{1/\gamma} \in \text{supp}(g)$, and any $\lambda \in (0, 1)$.

Set $\delta = \frac{\gamma}{1-\gamma}$, and fix $\lambda \in (0, 1)$. Let

$$A = [C_{p, \lambda, t} + D_{p, \lambda, t}]^{\frac{1}{\alpha}}, B = [C_{p, \lambda, t} + D_{p, \lambda, t}]^{\frac{1}{\beta}},$$

and

$$\theta = \frac{D_{p, \lambda, t}}{C_{p, \lambda, t} + D_{p, \lambda, t}} \in [0, 1].$$

Then, for any $z = C_{p, \lambda, t}x + D_{p, \lambda, t}y$, with $x^{1/\gamma} \in \text{supp}(f)$ and $y^{1/\gamma} \in \text{supp}(g)$, the generalized Hölder inequality (24) and inequality (28) yield that

$$w(z)^{\frac{1}{\delta}} \geq [C_{p, \lambda, t}u(x)^{\alpha} + D_{p, \lambda, t}v(y)^{\alpha}]^{\frac{1}{\alpha}} \times [C_{p, \lambda, t}(x^{1/\delta})^{\beta} + D_{p, \lambda, t}(y^{1/\delta})^{\beta}]^{\frac{1}{\beta}} \geq AB[(1 - \theta)u(x)^{\alpha} + \theta v(y)^{\alpha}]^{\frac{1}{\alpha}} (1 - \theta)(x^{1/\delta})^{\beta} + \theta(y^{1/\delta})^{\beta}]^{\frac{1}{\beta}} \geq AB[(1 - \theta)(u(x)x^{1/\delta})^{\alpha_0} + \theta(v(y)y^{1/\delta})^{\alpha_0}]^{\frac{1}{\alpha_0}} = [C_{p, \lambda, t}(u(x)x^{1/\delta})^{\alpha_0} + D_{p, \lambda, t}(v(y)y^{1/\delta})^{\alpha_0}]^{\frac{1}{\alpha_0}}$$

where $\alpha_0$ is defined by

$$\frac{1}{\alpha_0} = \frac{1}{\alpha} + \frac{1}{\delta} = \frac{1}{\alpha} + \frac{1}{\gamma} - 1.$$

Therefore, the triple

$$w(z)^{\frac{1}{\delta}}, u(x)x^{1/\delta}, v(y)y^{1/\delta}$$

satisfy the conditions of the $L_p$-Borell-Brascamp-Lieb inequality (30), provided $\alpha_0 \geq -1$; in which case, we would have

$$\int_0^\infty w(z)^{1/\delta} dz \geq (1 - t) \left( \int_0^\infty u(x)x^{1/\delta} dx \right)^{\beta} + t \left( \int_0^\infty v(y)y^{1/\delta} dy \right)^{\beta} \geq \int_0^\infty u(x)x^{1/\delta} dx = \int_0^\infty f(x^{1/\gamma}) x^{1/\gamma - 1} dx = |\gamma| \int_0^\infty f(x) dx,$$

where $\beta = \frac{\alpha + \gamma}{\alpha + \gamma + \gamma}$. Finally, using the fact that

$$\int_0^\infty u(x)x^{1/\delta} dx = \int_0^\infty f(x^{1/\gamma}) x^{1/\gamma - 1} dx = |\gamma| \int_0^\infty f(x) dx,$$

and the same with the pairs $(u, h)$, and $(v, g)$, we would have inequality (22), as desired. Therefore (29) concludes the inequality (22) of Theorem 3.3, provided that

(a) $\alpha + \delta > 0$;

(b) $\alpha_0 \geq -1$.

For the remain cases to $\gamma$, they have similar proofs for (3) The case $0 < \gamma < 1$, (4) The case $-\infty < \gamma < 0$, and (4) The case $\gamma = -\infty$ in [13, Page 19] by using $L_p$ coefficients.

\[ \square \]

Remark 3.4. If $p = 1$, it recovers the result of Theorem 4.1 in [13].
In the following, we consider several consequences of Theorem 3.1 for certain choices of the functional $\tilde{\Omega}$. The first consequence comes by choosing $\tilde{\Omega}(\cdot) = \mu(\cdot)$ a $\alpha$-concave measure on $\mathbb{R}^n$ with $\alpha \geq -1$. We obtain a $L_p$-Borell-Brascamp-Lieb type inequality for integrals of functions when integrated with respect to $\mu$.

**Corollary 3.5.** Let $p, q \in [1, \infty]$ be such that $1/p + 1/q = 1$. Suppose that $\alpha \geq -1$ and suppose that $\mu$ is an $\alpha$-concave measure on the class of Borel measurable subsets of $\mathbb{R}^n$ (respectively, $K_{(o)}^n$). Let $\gamma \geq -\alpha$. Suppose that $h, f, g: \mathbb{R}^n \to \mathbb{R}_+$ are a triple of integrable Borel measurable (respectively, quasi-concave functions) that satisfy the condition

$$h(C_p, \lambda, t x + D_p, \lambda, t y) \geq \left[ C_p, \lambda, t f(x)^\alpha + D_p, \lambda, t g(y)^\alpha \right]^{\frac{t}{\beta}}$$

for every $x \in \text{supp}(f)$, $y \in \text{supp}(g)$, and $\lambda \in (0, 1)$ whenever $f(x)g(y) > 0$. Then the following inequality holds:

$$\int_{\mathbb{R}^n} h(x) d\mu(x) \geq \left[ (1 - t) \left( \int_{\mathbb{R}^n} f(x) d\mu(x) \right)^\beta + t \left( \int_{\mathbb{R}^n} g(x) d\mu(x) \right)^\beta \right]^{\frac{1}{\beta}} , \beta = \frac{p\alpha\gamma}{\alpha + \gamma}.$$

**3.2. New proofs of $L_p$-Borell-Brascamp-Lieb type inequality.** The main goal of Subsections 3.2 is to extend the $L_p$-Borell-Brascamp-Lieb inequality appearing in [50] for the range $s \geq 0$, to the range $[-\infty, \infty]$ using different methods of proof. Particularly, these proof process are more concise than our previous works in [56] using the level sets and $L_p$ Brunn-Minkowski inequality in geometric setting for $s \geq 0$. Here we also include the case for $s < 0$ to complement the $L_p$-Borell-Brascamp-Lieb inequality for $s$. The result reads as follows.

**Theorem 3.6.** Let $p \geq 1$, $-\infty < s < \infty$, and $t \in (0, 1)$. Let $f, g, h: \mathbb{R}^n \to \mathbb{R}_+$ be a triple of bounded integrable functions. Suppose, in addition, that this triple satisfies the condition

$$h(C_p, \lambda, t x + D_p, \lambda, t y) \geq \left[ C_p, \lambda, t f(x)^s + D_p, \lambda, t g(y)^s \right]^{\frac{t}{\beta}}$$

for every $x \in \text{supp}(f)$, $y \in \text{supp}(g)$ and every $\lambda \in [0, 1]$. Then the following integral inequality holds:

$$I(h) \geq \begin{cases} M_{\gamma_1}^{(1-t), t}(I(f), I(g)), & \text{if } s \geq -\frac{1}{n}, \\ \min \left\{ [C_p, \lambda, t]^{\frac{1}{s}} I(f), [D_p, \lambda, t]^{\frac{1}{s}} I(g) \right\}, & \text{if } s < -\frac{1}{n}, \end{cases}$$

for $0 \leq \lambda \leq 1$, where $\gamma_1 = p\gamma$ and $\gamma = \frac{s}{1 + ns}$.

By the definitions of $L_{p,s}$ supremal-convolution, we conclude that

$$I((1 - t) \times_{p,s} f \oplus_{p,s} t \times_{p,s} g) \geq \begin{cases} M_{\gamma_1}^{(1-t), t}(I(f), I(g)), & \text{if } s \geq -\frac{1}{n}, \\ \min \left\{ [C_p, \lambda, t]^{\frac{1}{s}} I(f), [D_p, \lambda, t]^{\frac{1}{s}} I(g) \right\}, & \text{if } s < -\frac{1}{n}. \end{cases}$$

(i) **Proof of $L_p$-Borell-Brascamp-Lieb type inequality for $s \in [-1/n, \infty)$ using mass transportation.** As is known that the method of mass transportation is widely used in proving functional inequalities, such as the Prékopa-Leindler inequality and Borell-Brascamp-Lieb inequality in [10, 11, 30, 60], etc. Since the $L_p$-Borell-Brascamp-Lieb inequality includes
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the typical case for $s = 0$—the Prékopa-Leindler inequality, and $p = 1$—the Borell-Brascamp-Lieb inequality, we attempt to using the mass transportation method to solve Theorem 3.6 and show that the case for $s \geq -1/n$ works in an analogous approach accordingly.

Before proving the theorem, we require the so-called Minkowski determinant inequality (see [5]) for matrices.

**Lemma 3.7.** Let $A, B$ be $n \times n$ positive symmetric semi-definite matrices, and $a, b \geq 0$. Then one has that

$$\det(aA + bB)^{\frac{1}{n}} \geq a \det(A)^{\frac{1}{n}} + b \det(B)^{\frac{1}{n}}.$$ 

**Proof of Theorem 3.6 for $s \geq -1/n$.** Without loss of generality, we may assume that $I(f), I(g) = 1$, and denote probability measures $\mu$ and $\nu$ defined on $\mathbb{R}^n$ satisfying $d\mu(y) = f(y)dy$ and $d\nu(y) = g(y)dy$. Suppose that $\rho$ is the uniform measure on $[0, 1]^n$. Recall the proof due to F. Barthe in [60, Page 188-189] relies on the concept of mass transportation. Since $\mu, \nu$ are probability measure on $\mathbb{R}^n$ which are absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$ there exist two convex functions $\varphi_1, \varphi_2: \mathbb{R}^n \to \mathbb{R}$, whose gradient maps $\nabla \varphi_1$ and $\nabla \varphi_2$, respectively transport $\rho$ to $\mu$ and $\rho$ to $\nu$, i.e., $(\nabla \varphi_1)\rho = \mu$ and $(\nabla \varphi_2)\rho = \nu$. The change of variable formulas lead to the following results a.e. on $[0, 1]^n$:

$$f(\nabla \varphi_1(x)) \det(\text{Hess } \varphi_1(x)) = 1, \quad g(\nabla \varphi_2(x)) \det(\text{Hess } \varphi_2(x)) = 1,$$

where $\text{Hess } \varphi_i, \ i \in \{1, 2\}$ are the Aleksandrov Hessians defined a.e. and are symmetric non-negative semi-definite.

Fix any $\lambda \in [0, 1]$, and set $\varphi_\lambda = C_{p, \lambda, t} \varphi_1 + D_{p, \lambda, t} \varphi_2$. By the change of variable, together with (24), (30) and Lemma 3.7, we see that

$$\int_{\mathbb{R}^n} h(y)dy \geq \int_{[0, 1]^n} h(y)dy$$

$$= \int_{[0, 1]^n} h(\nabla \varphi_\lambda(x)) \det(\text{Hess } \varphi_\lambda(x))dx$$

$$\geq \int_{[0, 1]^n} h(C_{p, \lambda, t} \nabla \varphi_1(x) + D_{p, \lambda, t} \nabla \varphi_2(x))M_\lambda^{(C_{p, \lambda, t}, D_{p, \lambda, t})} (\det(\text{Hess } \varphi_1(x)), \det(\text{Hess } \varphi_2(x))) dx$$

$$\geq \int_{[0, 1]^n} M_\lambda^{(C_{p, \lambda, t}, D_{p, \lambda, t})} (f((\nabla \varphi_1)(x)), g((\nabla \varphi_2)(x)))$$

$$\times M_\lambda^{(C_{p, \lambda, t}, D_{p, \lambda, t})} (\det(\text{Hess } \varphi_1(x)), \det(\text{Hess } \varphi_2(x))) dx$$

$$\geq \int_{[0, 1]^n} M_\lambda^{(C_{p, \lambda, t}, D_{p, \lambda, t})} (f((\nabla \varphi_1)(x)) \det(\text{Hess } \varphi_1(x)), g((\nabla \varphi_2)(x)) \det(\text{Hess } \varphi_2(x))) dx$$

$$= \int_{[0, 1]^n} [C_{p, \lambda, t} + D_{p, \lambda, t}] \frac{1}{1+s} dx.$$
Therefore, as $\lambda$ is arbitrary in $[0, 1]$, we conclude that

$$\int_{\mathbb{R}^n} h(y)dy \geq \sup_{0 \leq \lambda \leq 1} [C_{p,\lambda,t} + D_{p,\lambda,t}]^{\frac{1}{n+1}} \geq 1,$$

where if $s \geq 0$, we choose $\lambda = t$, and if $-1/n \leq s < 0$, we apply the Hölder inequality $C_{p,\lambda,t} + D_{p,\lambda,t} \leq 1$ for $p \geq 1$, completing the proof.

(ii) Proof of $L_p$-Borell-Brascamp-Lieb inequality using classical Borell-Brascamp-Lieb inequality. In the following, we will give another proof of $L_p$-Borell-Brascamp-Lieb inequality in Theorem 3.6 for $s \in (-\infty, \infty)$ by applying classic Borell-Brascamp-Lieb (BBL) inequality, which is different from but a more concise proof than [56] for $s \geq 0$. Firstly, for $s \leq -1/n$, we require the following lemma of the Borell-Brascamp-Lieb inequality in [26, Lemma 3.3].

Lemma 3.8. Let $f, g : \mathbb{R}^n \to \mathbb{R}_+$ be integrable functions, $-\infty < s < -1/n$, and $0 \leq t \leq 1$. Then

$$\int_{\mathbb{R}^n} \sup_{z=(1-t)x + ty} [(1-t)f(x)^s + tg(y)^s]^{1/s}dz \geq \min \{(1-t)^{n+1/s}I(f), t^{n+1/s}I(g)\}. \tag{33}$$

Furthermore, we conclude from this lemma by the definition of supreimal-convolution as

$$I((1-t) \times_s f \oplus_s t \times_s g) \geq \min \{(1-t)^{n+1/s}I(f), t^{n+1/s}I(g)\},$$

which complement the result for $s$ in (5).

Proof of Theorem 3.6. First we provide the proof for $s \geq -1/n$ using the classical Borell-Brascamp-Lieb inequality. Fix $t \in (0, 1)$. For $\lambda \in [0, 1]$, let

$$\bar{x} := \left(1 - \frac{t}{1-t}\right)^{\frac{1}{n}}x, \quad \bar{y} := \left(\frac{t}{\lambda}\right)^{\frac{1}{n}}y$$

and

$$\bar{f}(\bar{x}) := \left(1 - \frac{t}{1-t}\right)^{\frac{1}{n}}f(x), \quad \bar{g}(\bar{y}) := \left(\frac{t}{\lambda}\right)^{\frac{1}{n}}g(y).$$

Then we have

$$\int_{\mathbb{R}^n} \sup_{0 \leq \lambda \leq 1} \sup_{z=(1-t)\lambda x + t\lambda y} [(1-t)^{1/p}(1-\lambda)^{1/q}f(x)^s + t^{1/p}(1-\lambda)^{1/q}g(y)^s]^{1/s}dz$$

$$\geq \sup_{0 \leq \lambda \leq 1} \int_{\mathbb{R}^n} \sup_{z=(1-t)\lambda x + t\lambda y} [(1-t)^{1/p}(1-\lambda)^{1/q}f(x)^s + t^{1/p}(1-\lambda)^{1/q}g(y)^s]^{1/s}dz$$

$$= \sup_{0 \leq \lambda \leq 1} \int_{\mathbb{R}^n} \sup_{z=(1-t)\lambda x + t\lambda y} \left\{ (1-t)\left[(1-\lambda)^{1/q}_f(x)^s + t[(1-\lambda)^{1/q}g(y)^s]\right]^{1/s}dz \right\}$$

$$= \sup_{0 \leq \lambda \leq 1} \int_{\mathbb{R}^n} \sup_{z=(1-t)\bar{x} + t\bar{y}} (1-t)[\bar{f}(\bar{x})^s + t\bar{g}(\bar{y})^s]^{1/s}dz.$$
Remark 3.9. It can be checked easily that if \( 0 \leq \lambda \leq 1 \), it recovers the result of Lemma 3.8 and the classic Borell-Brascamp-Lieb inequality. Moreover, this method of proof to introduce \( \tilde{f}(\bar{x}) \) and \( \tilde{g}(\bar{y}) \) also works in Theorem 3.3 but only for \( n = 1 \) and \( \gamma = 1 \).

4. APPLICATIONS OF \( L_p \)-BORELL-BRASCAMP-LIEB INEQUALITY

The goal of this section is to provide several functional analytic and measure theoretic consequences of the topics discussed in Section 3. Based on the restriction conditions on \( L_p \)-Borell-Brascamp-Lieb type inequalities, we define the following concavity definitions in \( L_p \) case for functions. It is inspired that if \( h = f = g \) in the Borell-Brascamp-Lieb inequality
condition, it recovers the $s$-concavity definition. Therefore, by letting $h = f = g$ in the $L_p$-Borell-Brascamp-Lieb inequality condition, we provide the $L_{p,s}$ concavity definitions.

**Definition 4.1.** Let $p \geq 1$, $1/p + 1/q = 1$, and $s \in [-\infty, +\infty]$.

1. We say that a function $f : \mathbb{R}^n \to \mathbb{R}_+$ is $L_{p,s}$-concave if, for any pair $x,y \in \mathbb{R}^n$, one has
   
   \[ f(C_{p,\lambda,t}x + D_{p,\lambda,t}y) \geq M_s^{(C_{p,\lambda,t},D_{p,\lambda,t})}(f(x), f(y)) \]

   for every $\lambda \in [0,1]$ and $t \in [0,1]$. In this case,
   
   \[ f(z) \geq \sup_{0 \leq \lambda \leq 1} \sup_{t \in [0,1]} \{ M_s^{(C_{p,\lambda,t},D_{p,\lambda,t})}(f(x), f(y)) : z = C_{p,\lambda,t}x + D_{p,\lambda,t}y \}. \]

2. Similarly, if $s = -\infty$, the function $f$ is said to be $L_p$-quasi-concave if, for any pair $x,y \in \mathbb{R}^n$, one has
   
   \[ f(C_{p,\lambda,t}x + D_{p,\lambda,t}y) \geq \min \{ f(x), f(y) \} \]

   for every $\lambda \in [0,1]$ and $t \in [0,1]$.

3. If $s = 0$, the function $f$ is said to be $L_p$-log-concave, if for any pair $x,y \in \mathbb{R}^n$, one has
   
   \[ f(C_{p,\lambda,t}x + D_{p,\lambda,t}y) \geq f(x)^{C_{p,\lambda,t}} f(y)^{D_{p,\lambda,t}} \]

   for every $\lambda \in [0,1]$ and $t \in [0,1]$.

4. We call the function $f$ is said to be $L_{p,s}$-quasi-concave if, for any pair $x,y \in \mathbb{R}^n$, one has
   
   \[ f(C_{p,\lambda,t}x + D_{p,\lambda,t}y) \geq \min \{ C_{p,\lambda,t} f(x), D_{p,\lambda,t} f(y) \} \]

   for every $\lambda \in [0,1]$ and $t \in [0,1]$.

It is easy to see that (4) recovers the definition of (2) if $s = 0$, and it is inspired by the result of $L_p$ Borell-Brascamp-Lieb inequality for $s < -1/n$ in Theorem 3.6.

**Proposition 4.2.** Let $p \geq 1$ and $s > 0$. If $f : \mathbb{R}^n \to \mathbb{R}_+$ is an $s$-concave function whose support contains the origin in its interior, then $f$ is also $L_{p,s}$-concave.

**Proof.** We only show the proof for $s \neq 0, \pm \infty$ as these cases are essentially identical. Let $t, \lambda \in [0,1]$, $1/p + 1/q = 1$. Then, for any $x,y \in \mathbb{R}^n$ belonging to the support of $f$, we see that

\[
 f(C_{p,\lambda,t}x + D_{p,\lambda,t}y) = f \left( C_{p,\lambda,t}x + \frac{D_{p,\lambda,t}}{1 - C_{p,\lambda,t}} y \right) \\
 \geq \left[ C_{p,\lambda,t} f(x) + (1 - C_{p,\lambda,t}) f \left( \frac{D_{p,\lambda,t}}{1 - C_{p,\lambda,t}} y \right) \right]^{\frac{1}{s}} \\
 \geq M_{s}^{(C_{p,\lambda,t},D_{p,\lambda,t})}(f(x), f(y)),
\]

where in the last step we used the fact that the support of $f$ contains the origin in its interior together with Hölder’s inequality, as required. \qed

We have similar definitions for measures with the $L_p$ coefficients.
Definition 4.3. Let \( p \geq 1, \frac{1}{p} + 1/q = 1, \) and \( s \in [-\infty, +\infty] \). We say that a non-negative measure \( \mu \) on \( \mathbb{R}^n \) is \( L_{p,s} \)-concave if, for any pair of Borel measurable sets \( A, B \subset \mathbb{R}^n \), one has
\[
\mu (C_{p,\lambda,t}A + D_{p,\lambda,t}B) \geq M_s^{(C_{p,\lambda,t}D_{p,\lambda,t})}(\mu(A), \mu(B))
\]
for every \( \lambda \in [0, 1] \) and \( t \in [0, 1] \). Similarly, if \( s = -\infty \), the measure \( \mu \) is said to be \( L_{p,s} \)-quasi-concave if, for any pair of compact \( A, B \subset \mathbb{R}^n \), one has
\[
\mu (C_{p,\lambda,t}A + D_{p,\lambda,t}B) \geq \min \{\mu(A), \mu(B)\}
\]
for every \( \lambda \in [0, 1] \) and \( t \in [0, 1] \). Furthermore, if \( s = 0 \), the measure \( \mu \) is said to be \( L_{p,s} \)-log-concave if, for any pair of compact \( A, B \subset \mathbb{R}^n \), one has
\[
\mu (C_{p,\lambda,t}A + D_{p,\lambda,t}B) \geq \mu(A)^{C_{p,\lambda,t}} \mu(B)^{D_{p,\lambda,t}}
\]
for every \( \lambda \in [0, 1] \) and \( t \in [0, 1] \). Moreover, we call the measure \( \mu \) is said to be \( L_{p,s} \)-quasi-concave if, for any pair of compact \( A, B \subset \mathbb{R}^n \), one has
\[
\mu (C_{p,\lambda,t}A + D_{p,\lambda,t}B) \geq \min \{C_{p,\lambda,t} \mu(A), D_{p,\lambda,t} \mu(B)\}
\]
for every \( \lambda \in [0, 1] \) and \( t \in [0, 1] \).

The next result concerns convolutions concavities related to the \( L_{p,s} \)-concave functions (see also [55] Pages 643-644 for the case \( p = 1 \)).

Theorem 4.4. Let \( p \geq 1, \frac{1}{p} + 1/q = 1, \) t \( \in [0, 1] \), and \( s, \beta \in [-\infty, +\infty] \) be such that \( s + \beta \geq 0 \). Let \( f, g: \mathbb{R}^n \to \mathbb{R}_+ \) be \( L_{p,s} \)-concave and \( L_{p,\beta} \)-concave, respectively. Then the convolution of \( f \) and \( g \),
\[
(f * g)(z) = \int_{\mathbb{R}^n} f(x)g(z-x)dx,
\]
satisfies one of the following:
1. \( L_{p,(s+\beta-1+n)^{-1}} \)-concave whenever \( \frac{s\beta}{s+\beta} \in \left[-\frac{1}{n}, +\infty\right) \);
2. \( L_{p,(s+\beta-1+n)^{-1}} \)-quasi-concave whenever \( \frac{s\beta}{s+\beta} \in \left(-\infty, -\frac{1}{n}\right) \).

Proof. Let \( t \in [0, 1] \). Since \( f, g \) are \( L_{p,s} \)-concave and \( L_{p,\beta} \)-concave, respectively, the condition indicate that for fixed \( v, w \in \mathbb{R}^n \),
\[
f(z) \geq \sup_{0 \leq \lambda \leq 1} \left[ \sup_{z=C_{p,\lambda,t}x+D_{p,\lambda,t}y} M_s^{(C_{p,\lambda,t}D_{p,\lambda,t})}(f(x), f(y)) \right],
\]
\[
g(C_{p,\lambda,t}v + D_{p,\lambda,t}w - z) \geq \sup_{0 \leq \lambda \leq 1} \left[ \sup_{z=C_{p,\lambda,t}x+D_{p,\lambda,t}y} M_\beta^{(C_{p,\lambda,t}D_{p,\lambda,t})}(g(v - x), g(w - y)) \right].
\]
Therefore, by applying the generalized Hölder inequality, the formula (34) for \( \gamma = \frac{s\beta}{s+\beta} \geq -1/n \), and (33) for \( \gamma < -1/n \), we obtain
\[
(f * g)(C_{p,\lambda,t}v + D_{p,\lambda,t}w)
\]
\[
= \int_{\mathbb{R}^n} f(z)g(C_{p,\lambda,t}v + D_{p,\lambda,t}w - z)dz
\]
\[
\geq \int_{\mathbb{R}^n} \sup_{0 \leq \lambda \leq 1} \left[ \sup_{z = C_{p,\lambda,t}x + D_{p,\lambda,t}y} M_{\alpha}^{(C_{p,\lambda,t}x, D_{p,\lambda,t}y)}(f(x), f(y)) M_{\beta}^{(C_{p,\lambda,t}x, D_{p,\lambda,t}y)}(g(v - x), g(w - y)) \right] \, dz
\]

\[
\geq \int_{\mathbb{R}^n} \sup_{0 \leq \lambda \leq 1} \left[ \sup_{z = C_{p,\lambda,t}x + D_{p,\lambda,t}y} M_{\gamma}^{(C_{p,\lambda,t}x, D_{p,\lambda,t}y)}(f(x)g(v - x), f(y)g(w - y)) \right] \, dz
\]

\[
= \begin{cases} 
\sup_{0 \leq \lambda \leq 1} \left[ C_{p,\lambda,t}((f * g)(v))^\gamma + D_{p,\lambda,t}((f * g)(w))^\gamma \right]^{\frac{1}{\gamma}}, & \text{if } \gamma \geq -\frac{1}{n}, \\
\min \left\{ C_{p,\lambda,t}^{\frac{1}{\gamma}} (f * g)(v), D_{p,\lambda,t}^{\frac{1}{\gamma}} (f * g)(w) \right\}, & \text{if } \gamma < -\frac{1}{n},
\end{cases}
\]

for all \(0 \leq \lambda \leq 1\), where \(\gamma_0 = \frac{\gamma}{1+n}\gamma = (s^{-1} + \beta^{-1} + n)^{-1}\). Therefore,

\[
(f * g)(C_{p,\lambda,t}v + D_{p,\lambda,t}w)
\]

\[
(35)
\]

for all \(0 \leq \lambda \leq 1\). Therefore, \(f * g\) is \(L_p(s^{-1} + \beta^{-1} + n)^{-1}\)-concave whenever \(\frac{s}{s+\beta} \in \left[-\frac{1}{n}, +\infty\right)\), and is \(L_p(s^{-1} + \beta^{-1} + n)\)-quasi-concave whenever \(\frac{s}{s+\beta} \in (-\infty, -\frac{1}{n})\).

By the series of \(L_p,s\) concavity definitions, we deduce from Theorem 3.6 and formula (34) that, if a measure has a density that is \(L_p,s\)-concave for \(s \geq -1/n\), then the measure itself is \(L_p,1/n\)-concave, and \(L_p,1/n\)-quasi-concave for \(s < -1/n\). Therefore, we have the following extension of the \(L_p\) version of Brunn’s concavity principle (see [5] and [16] for \(p = 1\)).

**Corollary 4.5.** Let \(K \subset \mathbb{R}^n\) be a convex body containing the origin in its interior, \(H\) is a \((n - j)\)-dimensional subspace of \(\mathbb{R}^n\), and \(j \in \{0, \ldots, n - 1\}\). Let \(\mu\) be a measure on \(\mathbb{R}^n\) whose density is \(L_p,s\)-concave for some \(s \in [-\infty, +\infty]\); i.e., \(d\mu(x)/dx = f(x)\) and \(f(x)\) is \(L_p,s\)-concave. The function \(\Omega : H \rightarrow \mathbb{R}_+\) given by

\[
\Omega(x) = \mu(K \cap (x + H)), \quad x \in H
\]

satisfies

1. \(\Omega\) is a \(L_p,\gamma\)-concave function on its support for \(s \geq -\frac{1}{n-j}\);
2. \(\Omega\) is a \(L_p,\gamma\)-quasi-concave function on its support for \(s < -\frac{1}{n-j}\)

where \(\gamma = \frac{s}{1+(n-j)s}\).

Another \(L_p,s\) concavity definition only works in 1-dimension space \(\mathbb{R}\) by the restriction of parameter \(\gamma\), which is not applicable for measures either. Recall the condition in \(L_p,\gamma\) Borell-Brascamp-Lieb inequality in \(\mathbb{R}\) (21), that is,

\[
h \left( (C_{p,\lambda,t}x + D_{p,\lambda,t}y)^\gamma \right)^{\frac{1}{\gamma}} \geq [C_{p,\lambda,t}f(x)^\gamma + D_{p,\lambda,t}g(y)^\gamma]^{\frac{1}{\gamma}},
\]

we define the following concavity definitions.

**Definition 4.6.** Let \(p \geq 1, 1/p + 1/q = 1\), and \(s \in [-\infty, +\infty]\).
(1) We say that a function $f : \mathbb{R} \to \mathbb{R}_+$ is $L_{p,s}^\gamma$-concave if, for any pair $x, y \in \mathbb{R}$, one has
\[
 f \left((C_{p,\lambda,t}x^\gamma + D_{p,\lambda,t}y^\gamma)^{\frac{1}{\gamma}}\right) \geq M_s^{(C_{p,\lambda,t},D_{p,\lambda,t})}(f(x), f(y))
\]
for every $\lambda \in [0, 1]$ and $t \in [0, 1]$.

(2) Similarly, if $s = -\infty$, the function $f$ is said to be $L_{p}^\gamma$-quasi-concave if, for any pair $x, y \in \mathbb{R}$, one has
\[
 f \left((C_{p,\lambda,t}x^\gamma + D_{p,\lambda,t}y^\gamma)^{\frac{1}{\gamma}}\right) \geq \min(f(x), f(y))
\]
for every $\lambda \in [0, 1]$ and $t \in [0, 1]$.

(3) If $s = 0$, the function $f$ is said to be $L_{p,s}^\gamma$-log-concave if, for any pair $x, y \in \mathbb{R}$, one has
\[
 f \left((C_{p,\lambda,t}x^\gamma + D_{p,\lambda,t}y^\gamma)^{\frac{1}{\gamma}}\right) \geq f(x)^{C_{p,\lambda,t}} f(y)^{D_{p,\lambda,t}}
\]
for every $\lambda \in [0, 1]$ and $t \in [0, 1]$.

(4) We call the function $f$ is said to be $L_{p,s}^\gamma$-quasi-concave if, for any pair $x, y \in \mathbb{R}$, one has
\[
 f \left((C_{p,\lambda,t}x^\gamma + D_{p,\lambda,t}y^\gamma)^{\frac{1}{\gamma}}\right) \geq \min(C_{p,\lambda,t} f(x), D_{p,\lambda,t} f(y))
\]
for every $\lambda \in [0, 1]$ and $t \in [0, 1]$.

It is easy to see that $L_{p,s}^\gamma$ coincides with $L_{p,s}$ concavity when $\gamma = 1$ and $n = 1$.

5. Integral representation of $L_{p,s}$ mixed quermassintegral for functions

In this section, we mainly focus on the extension of $L_p$ Brunn-Minkowski theory including mixed $p$-quermassintegrals and their integral representation formulas for convex bodies in $[10]$ to the space of $\mathcal{F}_s(\mathbb{R}^n)$ endowed with the $L_{p,s}$ summations introduced in Section 2. Therefore, we analyze the properties of projection for functions and $L_{p,s}$ supremal-convolution in Subsection 5.1 and for $L_{p,s}$ Asplund summation in Subsection 5.2, respectively. In conclusion, we obtain the integral representation of $L_{p,s}$ mixed quermassintegral for functions via variation formula of $L_{p,s}$ Asplund summation. This works as it is reasonable to take the first variation formula with the linear coefficients for $L_p$ mean of base functions and Legendre transformation similar to $L_p$ mean of support functions for convex bodies in $[2]$.

To begin with, recall the following classes of functions:

\[
 \mathcal{F}_s(\mathbb{R}^n) = \{ f : \mathbb{R}^n \to \mathbb{R}_+, f \text{ is } s\text{-concave, u.s.c, } f \in L^1(\mathbb{R}^n), f(o) = \|f\|_\infty > 0 \},
\]

\[
 C_s(\mathbb{R}^n) = \left\{ u : \mathbb{R}^n \to \mathbb{R}_+ \cup \{+\infty\}, u \text{ is convex, l.s.c, } u(o) = 0, \lim_{x \to \infty} \frac{u(x)}{\|x\|} = +\infty \right\}.
\]
5.1. **Projection for functions and $L_{p,s}$ supremal-convolution.** Using a geometry point of view—the epigraph and subgraph of a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, we can see the $L_{p,s}$ supremal-convolution satisfy elegant geometric properties for its related graphs. Consider two sets in $\mathbb{R}^{n+1}$

$$Epi f = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq t\}, \quad Sub f = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : f(x) \geq t\},$$

we have the following property by using $Epi f$ for convex function (open up) and $Sub f$ for concave function (open down) $f$ correspondingly.

**Proposition 5.1.** For $f, g \in \mathcal{F}_s(\mathbb{R}^n)$ and $s \in [-\infty, \infty]$, we have

1. $Epi (f \oplus g)^s = Epi (f^s) + Epi (g^s), \ s < 0$;
2. $Sub (f \oplus g)^s = Sub (f^s) + Sub (g^s), \ s \geq 0$.

Here $\oplus$ is the classic Minkowski sum for sets in $\mathbb{R}^{n+1}$.

**Proof.** (1) Note that for $s \geq 0$ and an $s$-concave function $f$, $Sub f^s$ is a convex set in $\mathbb{R}^{n+1}$ and $Epi (-f^s) = A_{(n+1)\times (n+1)} (Sub f^s)$, where $A_{(n+1)\times (n+1)}$ is the reflection matrix satisfying $A_{(n+1)\times (n+1)} (x_1, x_2, \ldots, x_n, x_{n+1}) = (x_1, x_2, \ldots, x_n, -x_{n+1})$ for any $(n+1)$-dimensional vector $(x_1, x_2, \ldots, x_n, x_{n+1}) \in \mathbb{R}^{n+1}$. That is,

$$A_{(n+1)\times (n+1)} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & -1 \end{pmatrix} \in O(n+1), \quad A^2_{(n+1)\times (n+1)} = I_{(n+1)\times (n+1)},$$

where $I_{(n+1)\times (n+1)}$ is the identity matrix. For $s < 0$, we have by the definition of supremal-convolution and formula (10) that

$$Epi ((f \oplus g)^s) = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : \sup_{x_1+x_2} [f^s(x_1) + g^s(x_2)]^{1/s} \leq t\}$$

$$= \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : \inf_{x_1+x_2} (f^s(x_1) + g^s(x_2)) \leq t\}$$

$$= \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : \left(\left(f^s \square (g^s)\right) (x) \leq t\right)$$

$$= Epi \left(\left(f^s \square (g^s)\right)\right)$$

$$= Epi (f^s) + Epi (g^s).$$

Then, for $s$-concave functions $f, g \geq 0$ and $s \geq 0$, one has

$$A_{(n+1)\times (n+1)} (Sub ((f \oplus g)^s)) = Epi (- (f \oplus g)^s)$$

$$= \left\{(x,t) \in \mathbb{R}^n \times \mathbb{R} : - \sup_{x_1+x_2} (f^s(x_1) + g^s(x_2)) \leq t\right\}$$
Hence, \( \text{Sub} (f + s g)^s = \text{Sub} (f^s) + \text{Sub} (g^s). \)

(2) The proofs for \( s \geq 0 \) and \( s < 0 \) follow naturally from (1) in similar lines. \( \square \)

Next, we consider the definition for the projection of \( s \)-concave functions \( f \in F_s(\mathbb{R}^n) \) onto the \((n - j)\)-dimensional subspace \( H \in G_{n,n-j} \) as

\[
f_H(z) = (P_H f)(z) := \sup_{y \in H^\perp} f(z + y), \quad f \in F_s(\mathbb{R}^n),
\]

and the projection of convex base function \( f \in C_s(\mathbb{R}^n) \) onto the \((n - j)\)-dimensional subspace \( H \) as

\[
u_H(x) = (\tilde{P}_H u)(x) = \inf_{y \in H^\perp} u(x + y), \quad u \in C_s(\mathbb{R}^n).
\]

Here we list some elegant properties for the above definitions of projections for functions with the supremal-convolution. Recall that in \([1]\), \( \text{Sub}(P_H f) = (\text{Sub} f)|\tilde{H} \) for \( s \geq 0 \) and \( \text{Epi}(P_H f) = (\text{Epi} f)|\tilde{H} \) for \( s < 0 \). Here \( \tilde{H} = \text{span}\{H, e_{n+1}\} \), where \( H \in G_{n,n-j} \) is the Grassmannian manifold on \( \mathbb{R}^n \) with the orthonormal basis \( \{e_1, \ldots, e_n\} \) and \( e_{n+1} \perp \mathbb{R}^n \) is a unit vector.

**Proposition 5.2.** For any functions \( f, g \in F_s(\mathbb{R}^n), \ j \in \{0, \ldots, n - 1\} \) and \( H \in G_{n,n-j} \), we have the following identities.

1. \( P_H f^s = (P_H f)^s, \quad s > 0; \)
   \[
   \tilde{P}_H f^s = (\tilde{P}_H f)^s, \quad s < 0; \]
   
   \( P_H (\log f) = \log(P_H f), \quad s = 0. \)

2. \( P_H (\alpha \times_s f) = \alpha \times_s (P_H f), s \in [-\infty, \infty]. \)

3. \( P_H (f \oplus_p g) = P_H f \oplus_p g, s \in [-\infty, \infty], \quad p \geq 1. \)

**Proof.** (1) It is easy to see that for \( s > 0 \), we have

\[
P_H(f^s)(z) = \sup_{y \in H^\perp} f^s(z + y) = \left[ \sup_{y \in H^\perp} f(z + y) \right]^s = [P_H(f)(z)]^s;
\]

for \( s < 0, \)

\[
\tilde{P}_H(f^s)(z) = \inf_{y \in H^\perp} f^s(z + y) = \left[ \sup_{y \in H^\perp} f(z + y) \right]^s = [P_H(f)(z)]^s;
\]
for $s = 0$,

$$P_H(\log f)(z) = \sup_{y \in H} \log f(z + y) = \log[\sup_{y \in H} f(z + y)] = \log[P_H(f)(z)].$$

(2) By the definition of supremal-convolution, we have

$$P_H(\alpha \times_s f) = P_H(\alpha^s f(\frac{x}{\alpha})) = \sup_{z \in H} \alpha^s f(\frac{x}{\alpha} + z) = \alpha^s P_H(f(\frac{x}{\alpha})) = \alpha \times_s P_H f(x),$$

as desired.

(3) For $p \geq 1$, $j \in \{0, \cdots, n-1\}$, and a subspace $H \subset G_{n,n-j}$, we denote $\bar{H} = \text{span}\{H, e_{n+1}\}$, where $e_{n+1} \perp H$. Then, for $s > 0$, we obtain

$$\text{Sub}(P_H f^s) = \text{Sub}(f^s)|\bar{H} = A_{(n-j+1) \times (n-j+1)}(\text{Epi}(\nabla f^s)|\bar{H}) = A_{(n-j+1) \times (n-j+1)}(\text{Epi}(\nabla f^s)|\bar{H}) = \text{Sub}(f^s)|\bar{H},$$

where

$$A_{(n-j+1) \times (n-j+1)} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & -1 \end{pmatrix} \in O(n-j+1).$$

In particular, Proposition 5.1 (1) and Proposition 5.2 (1) imply

$$\text{Sub}(P_H (f \oplus_s g)^s) = \text{Sub}((f \oplus_s g)^s)|\bar{H} = \text{Sub}(f^s) + \text{Sub}(g^s)|\bar{H} = \text{Sub}(f^s)|\bar{H} + \text{Sub}(g^s)|\bar{H} = \text{Sub}(P_H(f))^s + \text{Sub}(P_H(g))^s = \text{Sub}((P_H f)^s) + \text{Sub}((P_H g)^s).$$

Hence,

$$P_H(f \oplus_s g) = P_H f \oplus_s P_H g, \quad s > 0.$$
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$$= \bigcup_{0 \leq \lambda \leq 1} \text{Sub} (((1 - \lambda) \frac{1}{q} \times_s P_H f) \oplus_s (\lambda \frac{1}{q} \times_s P_H g)^s)$$

$$= \bigcup_{0 \leq \lambda \leq 1} \text{Sub} (((1 - \lambda) \frac{1}{q} \times_s P_H f)^s + (\lambda \frac{1}{q} \times_s P_H g)^s)$$

$$= \bigcup_{0 \leq \lambda \leq 1} ((1 - \lambda) \frac{1}{q} \cdot \text{Sub} (P_H f)^s + \lambda \frac{1}{q} \cdot \text{Sub} (P_H g)^s)$$

$$= \bigcup_{0 \leq \lambda \leq 1} (1 - \lambda) \frac{1}{q} \cdot (\text{Sub} f^s|\bar{H}) + \lambda \frac{1}{q} \cdot (\text{Sub} g^s|\bar{H}))$$

$$= \left( \bigcup_{0 \leq \lambda \leq 1} (1 - \lambda) \frac{1}{q} \cdot (\text{Sub} f^s) + \lambda \frac{1}{q} \cdot (\text{Sub} g^s) \right)|\bar{H}$$

$$= \left( \bigcup_{0 \leq \lambda \leq 1} \text{Sub} \{(1 - \lambda) \frac{1}{q} \times_s f \}^s + \text{Sub} \{\lambda \frac{1}{q} \times_s g \}^s \right)|\bar{H}$$

$$= \left( \bigcup_{0 \leq \lambda \leq 1} \text{Sub} \{(1 - \lambda) \frac{1}{q} \times_s f \oplus_s \lambda \frac{1}{q} \times_s g \}^s \right)|\bar{H}$$

$$= \text{Sub} \left( P_H \{f \oplus_{p,s} g \}^s \right),$$

as projection is distributive over set union operation.

For $s < 0$, we only need to replace “Sub” by “Epi”, then the proof follows in similar lines by Proposition 5.1 and Proposition 5.2. For $s = 0$, change $f^s = \log f$, and the formulas holds in a similar method. Therefore, one has

$$P_H \left( f \oplus_{p,s} g \right)^s = P_H \left( f \oplus_{p,s} g \right)^s = (P_H f \oplus_{p,s} P_H g)^s;$$

i.e.,

$$P_H \left( f \oplus_{p,s} g \right) = P_H f \oplus_{p,s} P_H g, \quad p \geq 1.$$

Moreover, it is easy to check that for $u \in C_s(\mathbb{R}^n)$, one has

$$P_H \left[ (1 - su(x)) \frac{1}{s} \right] = (1 - s \bar{P}_H u(x)) \frac{1}{s}, \quad s \in [-\infty, \infty].$$

5.2. Projection for function and $L_{p,s}$ Asplund summation. In this part, we examine the properties of projection functions and $L_{p,s}$ Asplund summation. We begin with the following proposition which demonstrates that the $L_p$ addition of convex functions for $p \geq 1$ is stable under projections given by (19). This paves the way to compute the variation formula for
quermassintegral for functions, i.e., the integral representation of $L_{p,s}$ mixed quermassintegral shown in Subsection 5.3.

**Proposition 5.3.** Let $p \geq 1$, $u, v \in C_s(\mathbb{R}^n)$, and $\alpha, \beta \geq 0$. Then, for any $H \in G_{n,n-j}$, $j \in \{0, 1, \ldots, n-1\}$, one has

\[
[(\alpha \otimes_p u) \boxdot_p (\beta \otimes_p v)]_H = [\alpha \otimes_p u_H] \boxdot_p [\beta \otimes_p v_H].
\]

**Proof.** To begin with, we consider the epigraphs of $u$ and $v$. Let \( \{e, \ldots, e_n, e_{n+1}\} \) be the canonical basis on \( \mathbb{R}^{n+1} \) and set \( \bar{H} = \text{span}(H, e_{n+1}) \) a \( (n-j+1) \)-dimensional space for \( H \in G_{n,n-j} \). Then by the fact that in $\mathbb{H}$, \( \text{Sub}(P_H f) = (\text{Sub} f)|\bar{H} \) for $s \geq 0$ and \( \text{Epi}(P_H f) = (\text{Epi} f)|\bar{H} \) for $s < 0$, we obtain by (10) that

\[
\text{Epi}([\alpha \times u \Box \beta \times v]|_H) = \text{Epi}(\alpha \times u \Box \beta \times v)|\bar{H} = [\alpha \text{Epi}(u) + \beta \text{Epi}(v)]|\bar{H} = \alpha \text{Epi}(u_H) + \beta \text{Epi}(v_H) = \text{Epi}(\alpha \times u_H \Box \beta \times v_H).
\]

(38)

Therefore, we have that

\[
[\alpha \times u \Box \beta \times v]|_H = \alpha \times u_H \Box \beta \times v_H.
\]

Finally, observe that by (38) and Lemma 2.5 (1), one has

\[
[(\alpha \otimes_p u) \boxdot_p (\beta \otimes_p v)]_H(x) = \inf_{y \in x + H^\perp} [(\alpha \otimes_p u) \boxdot_p (\beta \otimes_p v)](y)
\]

\[
= \inf_{y \in x + H^\perp} \left[ (\alpha(u^*(y))^p + (v^*(y))^p)^{\frac{1}{p}} \right]^*
\]

\[
= \inf_{y \in x + H^\perp} \left[ \sup_{0 \leq \lambda \leq 1} \left\{ \alpha^{\frac{1}{p}}(1-\lambda)^{\frac{1}{q}} u^*(y) + \beta^{\frac{1}{p}} \lambda^{\frac{1}{q}} v^*(y) \right\} \right]^*
\]

\[
= \inf_{y \in x + H^\perp} \inf_{0 \leq \lambda \leq 1} \left[ \alpha^{\frac{1}{p}}(1-\lambda)^{\frac{1}{q}} u^*(y) + \beta^{\frac{1}{p}} \lambda^{\frac{1}{q}} v^*(y) \right]^*
\]

\[
= \inf_{0 \leq \lambda \leq 1} \left[ \alpha^{\frac{1}{p}}(1-\lambda)^{\frac{1}{q}} \times u_H \Box \beta^{\frac{1}{p}} \lambda^{\frac{1}{q}} \times v_H \right](x)
\]

\[
= \inf_{0 \leq \lambda \leq 1} \left[ \alpha^{\frac{1}{p}}(1-\lambda)^{\frac{1}{q}} u_H^* + \beta^{\frac{1}{p}} \lambda^{\frac{1}{q}} v_H^* \right]^*(x)
\]

\[
= \left[ \sup_{0 \leq \lambda \leq 1} \alpha^{\frac{1}{p}}(1-\lambda)^{\frac{1}{q}} u_H^* + \beta^{\frac{1}{p}} \lambda^{\frac{1}{q}} v_H^* \right]^*(x)
\]

\[
= \left[ (\alpha(u_H^*(x))^p + (v_H^*(x))^p)^{\frac{1}{p}} \right]^*(x)
\]

completing the proof. \(\Box\)
5.3. **Variation formula of general quermassintegral for functions and** \( p \geq 1 \). Next we consider the “\( L_{p,q} \) mixed quermassintegral” of two functions \( f, g \in \mathcal{F}_s(\mathbb{R}^n) \). This is based on the \( p \)-mixed quermassintegral definition for convex bodies in Lutwak’s work \[39\]. First, we give the definition of quermassintegeral for functions.

**Definition 5.4.** The \( j \)-th quermassintegral of function \( f = (1 - su)_+ \in \mathcal{F}_s(\mathbb{R}^n) \) and \( u \in C_s(\mathbb{R}^n) \) for \( j \in \{0, \ldots, n-1\} \), is defined as

\[
W_j(f) := c_{n,j} \int_{G_{n,n-j}} \int_H P_H f(x) dx \, d\nu_{n,n-j}(H) = c_{n,j} \int_{G_{n,n-j}} J_s(\tilde{P}_H u) d\nu_{n,n-j}(H).
\]

For each function \( f \in \mathcal{F}_s(\mathbb{R}^n) \), and any \( j \in \{0, \ldots, n-1\} \), an application of Fubini’s theorem yields the following

\[
W_j(f) = c_{n,j} \int_{G_{n,n-j}} \int_H P_H f(x) dx \, d\nu_{n,n-j}(H)
\]

\[
= c_{n,j} \int_{G_{n,n-j}} \int_0^\infty \text{vol}_{n-j}(\{x: P_H f(x) \geq t\}) dt \, d\nu_{n,n-j}(H)
\]

\[
= c_{n,j} \int_{G_{n,n-j}} \int_0^\infty \text{vol}_{n-j}(\{x: f(x) \geq t\}) |H| dt \, d\nu_{n,n-j}(H)
\]

\[
= \int_{G_{n,n-j}} \int_0^\infty c_{n,j} \text{vol}_{n-j}(\{x: f(x) \geq t\}) |H| d\nu_{n,n-j}(H) \, dt
\]

\[
= \int_0^\infty W_j(\{f \geq t\}) dt.
\]

Therefore, the quantity \( W_j(f) \) can be interpreted in terms of the usual quermassintegrals of its super-level sets, which was originally considered in \[13\]. We remark that several works on quermassintegrals for functions have appeared in the literature, for example, see \[13, 17, 18, 44\].

Next, we may choose \( \Omega(K) = W_j(K) \) in Theorem \[3.1\] for \( K \in \mathcal{K}^n_{(o)} \) and \( j \in \{0, 1, \ldots, n-1\} \). The Brunn-Minkowski inequality for \( W_j(\cdot) \), together with Hölder’s inequality and homogeneity, asserts that \( W_j(\cdot) \) is \( \alpha \)-concave for any \( \alpha \in [-\infty, \frac{1}{n-j}] \). Therefore, Theorem \[3.1\] implies the following class of the \( L_p \) Borell-Brascamp-Lieb inequalities for the \( j \)-th quermassintegrals of elements of \( \mathcal{F}_s(\mathbb{R}^n) \).

**Theorem 5.5.** Let \( p, q \in [1, \infty] \) be such that \( 1/p + 1/q = 1 \), \( t \in [0, 1] \), and \( j \in \{0, 1, \ldots, n-1\} \).

Suppose that \( \alpha \in [-1, \frac{1}{n-j}] \) and let \( \gamma \in [-\alpha, \infty) \). Let \( f, g \in \mathcal{F}_\alpha(\mathbb{R}^n) \). Then we have

\[
W_j((1-t) \times_{p,\alpha} f \oplus_{p,\alpha} t \times_{p,\alpha} g) \geq [(1-t)W_j(f)^\beta + tW_j(g)^\beta]^{1/\beta}, \quad \beta = \frac{p\alpha\gamma}{\alpha + \gamma}.
\]
Definition 5.6. For any \( f, g \in F_s(\mathbb{R}^n) \), \( j \in \{0, \ldots, n-1\} \), \( s \in [-\infty, \infty] \), the \( L_{p,s} \) mixed quermassintegral of \( f, g \in \mathcal{F}_s(\mathbb{R}^n) \) with respect to the \( L_{p,s} \) Asplund summations is defined as

\[
W^s_{p,j}(f, g) := \lim_{\varepsilon \to 0^+} \frac{W_j(f *_{p,s} g) - W_j(f)}{\varepsilon},
\]

which is the first variation of the \( j \)-th quermassintegral of function \( f \).

In particular, if \( f = \chi_K \) for \( K \in \mathcal{K}_n \), \( W_j(\chi_K) \) recovers the quermassintegral for convex bodies \( K \), i.e., \( W_j(K) \). Moreover, let \( f = \chi_K \) and \( g = \chi_L \) for \( K, L \in \mathcal{K}^n(\omega) \), the \( L_{p,s} \) mixed quermassintegral goes back to \( p \)-mixed quermassintegral for convex bodies in [39].

More generally, containing the special cases of \( s \)-concave functions as special cases, we define for the generalized quermassintegral with functional \( \Omega : \mathbb{R}_+ \to \mathbb{R}_+ \) which is a bounded decreasing smooth function that decays faster than the exponential at infinity. Therefore, we further define the \( \Omega \)-\( L_{p,s} \) mixed quermassintegral for base functions on \( C_s(\mathbb{R}^n) \) as follows.

Definition 5.7. (General Quermassintegral for functions on \( C_s(\mathbb{R}^n) \))

1. The operator \( I_{\Omega} : C_s(\mathbb{R}^n) \to \mathbb{R}_+ \) defined for \( u \in C_s(\mathbb{R}^n) \) is the general \( \Omega \)-total mass

\[
I_{\Omega}(u) := \int_{\mathbb{R}^n} \Omega(u(x))dx.
\]

2. For \( j \in \{0, \ldots, n-1\} \), the \( \Omega \)-\( j \)th-quermassintegral is defined for \( u \in C_s(\mathbb{R}^n) \) by

\[
\mathcal{W}^\Omega_j(u) := c_{n,j} \int_{G_{n,n-j}} \int_H \Omega(u_H(x))dxd\nu_{n,n-j}(H).
\]

3. The \( \Omega \)-\( j \)th \( L_p \)-mixed quermassintegral of \( u, v \in C_s(\mathbb{R}^n) \) is defined as

\[
\mathcal{W}^\Omega_{p,j}(u, v) := \lim_{\varepsilon \to 0^+} \frac{\mathcal{W}^\Omega_j(u *_{p,\varepsilon} v) - \mathcal{W}^\Omega_j(u)}{\varepsilon},
\]

Our next goal is an integral representation for \( \mathcal{W}^\Omega_{p,j}(f, g) \) for functions \( f = (1 - su)^{1/s}_+ \), \( g = (1 - sv)^{1/s}_+ \) for \( u, v \in C^{2,+}(\mathbb{R}^n) \subset C_s(\mathbb{R}^n) \) where

\[
C^{2,+}(\mathbb{R}^n) = \{ u \in C_s(\mathbb{R}^n) : \text{Hess} \ u(x) > 0 \text{ for all } x \in \mathbb{R}^n \}.
\]

We need the following proposition which can be deduced from the Rockafeller’s book [50] and [211, Page 17].

Proposition 5.8. Let \( u \in C^{2,+}(\mathbb{R}^n) \) and set \( \varphi = u^* \). Then the following hold true:

1. \( \nabla u \) is a diffeomorphism;
2. \( \varphi \in C^2(\mathbb{R}^n) \);
3. \( (\nabla \varphi) = (\nabla u)^{-1} \);
4. for every \( y \in \mathbb{R}^n \), \( \text{Hess} \ \varphi(y) = [\text{Hess} \ u(\nabla \varphi(y))]^{-1} \) (here inverse is in the sense of matrices); in particular, \( \text{Hess} \ \varphi(y) > 0 \) for all \( y \in \mathbb{R}^n \);
(5) for every $y \in \mathbb{R}^n$
\[ \varphi(y) = \langle y, \nabla \varphi(y) \rangle - u(\nabla \varphi(y)). \]

Analogously, for every $x \in \mathbb{R}^n$,
\[ u(x) = \langle x, \nabla u(x) \rangle - \varphi(\nabla u(x)). \] (39)

Let $p \geq 1$, $u \in C^{2,+}({\mathbb{R}}^n)$, $\varphi = u^*$, and $\psi \in C_c^\infty({\mathbb{R}}^n)$. For $\varepsilon > 0$, we set $\varphi_\varepsilon = (\varphi^p + \varepsilon \psi^p)^{1/p}$. There exists some $\bar{\varepsilon} > 0$ such that $\varphi_\varepsilon \in C^{2,+}({\mathbb{R}}^n)$ for all $\varepsilon \leq \bar{\varepsilon}$. For such $\varepsilon > 0$, set $u_\varepsilon = (\varphi_\varepsilon)^*$. We require the following lemma with respect to the variation formula for the projection function of $u_\varepsilon$.

**Lemma 5.9.** Let $p \geq 1$, $u \in C^{2,+}({\mathbb{R}}^n)$, $\varphi = u^*$, and $\psi \in C_c^\infty({\mathbb{R}}^n)$, and fix $H \in G_{n,n-j}$ for $j \in \{0, \ldots, n-1\}$. Set $\varphi_\varepsilon = (\varphi^p + \varepsilon \psi^p)^{1/p}$ for all $\varepsilon \leq \bar{\varepsilon}$, and $u_\varepsilon = (\varphi_\varepsilon)^*$. Then, for every $x \in \text{int}(\text{dom}(u)|H)$, one has
\[ \frac{d}{d\varepsilon}[(u_\varepsilon)_H(x)] = -\frac{d}{d\varepsilon}[(\varphi_\varepsilon)_H(\nabla(u_\varepsilon)_H(x))]. \]

Moreover, for each $x \in \text{int}(\text{dom}(u)|H)$, one has
\[ \frac{d}{d\varepsilon}[(u_\varepsilon)_H(x)] \bigg|_{\varepsilon=0} = -\frac{1}{p} \psi_H(\nabla u_H(x))^p \varphi_H(\nabla u_H(x))^{1-p}. \]

**Proof.** Fix $x \in \text{int}(\text{dom}(u)|H)$ and $\varepsilon > 0$ sufficiently small. Using (39), we have
\[ (u_\varepsilon)_H(x) = \langle x, \nabla(u_\varepsilon)_H(x) \rangle - (\varphi_\varepsilon)_H(\nabla(u_\varepsilon)_H(x)). \]

Therefore, we obtain
\[ \frac{d}{d\varepsilon}[(u_\varepsilon)_H(x)] \bigg|_{\varepsilon=0} = \frac{d}{d\varepsilon} \left[ \langle x, \nabla(u_\varepsilon)_H(x) \rangle - (\varphi_\varepsilon)_H(\nabla(u_\varepsilon)_H(x)) \right] \bigg|_{\varepsilon=0} \]
\[ = \left[ \langle x, \frac{d}{d\varepsilon} \nabla(u_\varepsilon)_H(x) \rangle - \frac{d}{d\varepsilon}[(\varphi_\varepsilon)_H(\nabla(u_\varepsilon)_H(x))] \right. \]
\[ - \langle \nabla(\varphi_\varepsilon)_H(\nabla(u_\varepsilon)_H(x)), \frac{d}{d\varepsilon} \nabla(u_\varepsilon)_H(x) \rangle \bigg|_{\varepsilon=0} \]
\[ = -\frac{d}{d\varepsilon}[(\varphi_\varepsilon)_H(\nabla(u_\varepsilon)_H(x))] \bigg|_{\varepsilon=0} \]
\[ = -\frac{1}{p} \psi_H(\nabla u_H(x))^p \varphi_H(\nabla u_H(x))^{1-p}, \]
where we have used the fact that $\nabla(u_\varepsilon)_H$ and $\nabla(\varphi_\varepsilon)_H$ are inverse of one another (Proposition 5.8 (3)). The second assertion follows form the fact that all functions involved are of class $C^{2,+}(H)$. \qed

We require the following Blaschke-Petkantschin formula, which can be found in [58].
Lemma 5.10. Let \( H \in G_{n,n-j} \) for \( j \in \{1, \ldots, n-1\} \), and \( f: \mathbb{R}^n \to \mathbb{R}_+ \) be a bounded Borel measurable function. Then the following holds:

\[
\int_{\mathbb{R}^n} f(x)dx = c_{n,j} \int_{G_{n,n-j}} \int_{H} f(x)\|x\|^j dx d\nu_{n,n-j}(H).
\]

We are now prepared to establish the variational formula for the \( \Omega-L_{p,s} \) mixed quermassintegral of functions on \( C_s(\mathbb{R}^n) \) with the general quermassintegral in Definition 5.7 based on the lemmas above.

Theorem 5.11. Let \( j \in \{1, \ldots, n-1\} \) and \( H \in G_{n,n-j} \). Let \( \Omega: \mathbb{R}_+ \to \mathbb{R}_+ \) be a bounded smooth function such that \( \lim_{\|x\| \to \infty} \frac{\Omega(x)}{\|x\|^p} = 0 \). Let \( p \geq 1, j \in \{0, \ldots, n-1\} \). Then, for any \( u \in C^{2,1}(\mathbb{R}^n) \cap C^\infty_c(\mathbb{R}^n) \) and \( \psi \in C^\infty_c(\mathbb{R}^n) \), with \( \varphi = u^s \) and \( \psi = v^s \), the following holds:

\[
\mathcal{W}_{p,j}^{\Omega}(u,v) = -\frac{1}{p} \int_{\mathbb{R}^n} \frac{\Omega(u(x))\psi_H(\nabla u_H(x))^p \varphi_H(\nabla u_H(x))^{1-p}}{\|x\|^j} dx.
\]

Proof. By definition of \( \mathcal{W}_{p,j}^{\Omega}(u,v) \), we have

\[
\mathcal{W}_{p,j}^{\Omega}(u,v) = \lim_{\varepsilon \to 0^+} \mathcal{W}_{p,j}^{\Omega}(u \boxplus_p (\varepsilon \boxtimes_p v)) - \mathcal{W}_{p,j}^{\Omega}(u)
\]

\[= c_{n,j} \int_{G_{n,n-j}} \left( \lim_{\varepsilon \to 0^+} \int_{H} \frac{\Omega([u \boxplus_p (\varepsilon \boxtimes_p v)]_H(x) - \Omega(u_H(x))}{\varepsilon} dx \right) d\nu_{n,n-j}(H)
\]

\[= c_{n,j} \int_{G_{n,n-j}} \left( \lim_{\varepsilon \to 0^+} \int_{H} \frac{\Omega([u_H \boxplus_p (\varepsilon \boxtimes_p v_H)]_H(x) - \Omega(u_H(x))}{\varepsilon} dx \right) d\nu_{n,n-j}(H),
\]

where we have used the Proposition 5.3 and Lemma 5.10.

For \( \varepsilon > 0 \) sufficiently small, we see that \( u_H \boxplus_p \varepsilon \boxtimes_p v_H \in C^{2,1}(\mathbb{R}^n) \cap C^\infty_c(\mathbb{R}^n) \), \( \Omega(u_H) \) and \( \Omega([u_H \boxplus_p (\varepsilon \boxtimes_p v_H)]) \) are integrable on \( H \). Considering \( B_r := \{ x \in H : \|x\| \leq r \} = B_r \cap H, r > 0 \), from the dominated convergence theorem, we see that

\[
\mathcal{W}_{p,j}^{\Omega}(u,v) = c_{n,j} \int_{G_{n,n-j}} \left( \lim_{\varepsilon \to 0^+} \int_{H} \frac{\Omega([u_H \boxplus_p (\varepsilon \boxtimes_p v_H)]_H(x) - \Omega(u_H(x))}{\varepsilon} dx \right) d\nu_{n,n-j}(H)
\]

\[= c_{n,j} \int_{G_{n,n-j}} \left( \lim_{\varepsilon \to 0^+} \lim_{r \to \infty} \int_{B_r} \frac{\Omega([u_H \boxplus_p (\varepsilon \boxtimes_p v_H)]_H(x) - \Omega(u_H(x))}{\varepsilon} dx \right) d\nu_{n,n-j}(H)
\]

\[= c_{n,j} \int_{G_{n,n-j}} \lim_{r \to \infty} \int_{B_r} \left( \lim_{\varepsilon \to 0^+} \frac{\Omega([u_H \boxplus_p (\varepsilon \boxtimes_p v_H)]_H(x) - \Omega(u_H(x))}{\varepsilon} dx \right) d\nu_{n,n-j}(H).
\]

By applying Lemma 5.9, we see that

\[
\lim_{\varepsilon \to 0^+} \frac{\Omega([u_H \boxplus_p (\varepsilon \boxtimes_p v_H)]_H(x) - \Omega(u_H(x))}{\varepsilon} = -\frac{1}{p} \Omega'(u_H(x))\psi_H(\nabla u_H(x))^p \varphi_H(\nabla u_H(x))^{1-p}.
\]

Therefore,

\[
\mathcal{W}_{p,j}^{\Omega}(u,v) = c_{n,j} \int_{G_{n,n-j}} \lim_{r \to \infty} \int_{B_r} \left( \lim_{\varepsilon \to 0^+} \frac{\Omega([u_H \boxplus_p (\varepsilon \boxtimes_p v_H)]_H(x) - \Omega(u_H(x))}{\varepsilon} dx \right) d\nu_{n,n-j}(H)
\]
\[
= -\frac{1}{p} c_{n,j} \int_{G_{n,n-j}} \left( \lim_{r \to 0} \int_{B_{r}} \Omega'(u_H(x)) \psi_H(\nabla u_H(x))^p \varphi_H(\nabla u_H(x))^{1-p} dx \right) \, d\nu_{n-n-j}(H)
\]
\[
= -\frac{1}{p} c_{n,j} \int_{G_{n,n-j}} \int_{H} \Omega'(u_H(x)) \psi_H(\nabla u_H(x))^p \varphi_H(\nabla u_H(x))^{1-p} dx \, d\nu_{n-n-j}(H)
\]
\[
= -\frac{1}{p} c_{n,j} \int_{G_{n,n-j}} \int_{H} \frac{\Omega'(u_H(x)) \psi_H(\nabla u_H(x))^p \varphi_H(\nabla u_H(x))^{1-p}}{\|x\|^j} \|x\|^j dx \, d\nu_{n-n-j}(H)
\]
\[
= -\frac{1}{p} \int_{\mathbb{R}^n} \frac{\Omega'(u_H(x)) \psi_H(\nabla u_H(x))^p \varphi_H(\nabla u_H(x))^{1-p}}{\|x\|^j} \, dx,
\]
where in the last step we have used Lemma 5.10.

\[\square\]

**Remark 5.12.** We remark that the right-hand side of identity (40) may not be convergent. If we choose \(\Omega\) such that \(\lim_{\|x\| \to 0} \frac{\Omega'(u_H(x)) \psi_H(\nabla u_H(x))^{1-p}}{\|x\|^j} < \infty\), for example, in [28, Theorem 5.7], when \(\Omega(u) = e^{-u}\) and \(j = 0\), suppose that there exists a constant \(k > 0\) such that

\[(41) \quad \det \left( \nabla^2 (u^*)^p(y) \right) \leq k(u^*(y))^{n(p-1)} \det (\nabla^2 u^*(y))
\]

holds for all \(y \in \mathbb{R}^n \setminus \{0\}\), then the integral is finite.

Here we list some special cases for formula (40) with typical parameters. Let \(p \geq 1\), \(j \in \{0, 1, \ldots, n-1\}\), \(s \in (-\infty, \infty)\), and set \(\Omega_r(s) = (1 - sr)^{1/s}\). Let \(u, v \in C_s(\mathbb{R}^n)\). We denote

\[\mathbb{W}^s_{p,j}(u, v) := \frac{p}{n-j} \mathbb{W}^s_{p,j}(u, v).
\]

Consequently, we obtain the following corollary with respect to the \(L_{p,s}\) mixed quermassintegral \(W_{p,j}^{s}(f, g)\) based on the \(\Omega_{s}-L_{p,s}\) mixed quermassintegral of \(\mathbb{W}_{p,j}^{s}(u, v)\) above for base functions \(u, v \in F_s(\mathbb{R}^n)\). That is,

**Corollary 5.13.** For \(p \geq 1\), \(j \in \{0, \ldots, n-1\}\), and \(s \in (-\infty, \infty)\), let \(f = (1 - su)^{1/s} g = (1 - sv)^{1/s}\) such that \(u, v \in C_s(\mathbb{R}^n)\) and \(u \in C^2(\mathbb{R}^n)\), and \(\psi \in C^\infty(\mathbb{R}^n)\) with \(\psi = v^*\). Then the \(L_{p,s}\) mixed quermassintegral for \(f, g \in F_s(\mathbb{R}^n)\) has the following integral representation:

\[W_{p,j}^{s}(f, g) = \frac{1}{n-j} \int_{\mathbb{R}^n} \frac{\left[1 - su_H(x)\right]^{\frac{j-1}{j}} \psi_H(\nabla u_H(x))^p \varphi_H(\nabla u_H(x))^{1-p} dx}{\|x\|^j} \]

For \(s = 0\), the above becomes

\[W_{p,j}^{0}(f, g) = \frac{1}{n-j} \int_{\mathbb{R}^n} \frac{e^{-u_H(x)} \psi_H(\nabla u_H(x))^p \varphi_H(\nabla u_H(x))^{1-p} dx}{\|x\|^j}.
\]

Furthermore, when \(j = 0\) and \(p \geq 1\), it goes back to the results in [28] by Fang, Xing and Ye where the formula (41) holds. The author in [54] also present an integral formula for \(0 < p < 1\). If \(\varphi = h_K(u)\) and \(\psi = h_L(u)\) for \(u \in S^{n-1}\), the support functions of two convex
bodies $K, L \in \mathcal{K}^n_1$, $j = 0$ and $s = 1$, it recovers the $L_p$ mixed volume for convex bodies $V_p(K, L)$ \cite{39}, i.e.,

$$V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h^p_L(u)h^{1-p}_K dS(K, u).$$

6. ACKNOWLEDGMENT

The authors would like to thank Prof. Artem Zvavitch and Dr. Sergii Myroshnychenko for providing valuable suggestions and discussions during writing of this paper.

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