TIME-FRACTIONAL DIFFUSION
OF DISTRIBUTED ORDER

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Abstract
The partial differential equation of Gaussian diffusion is generalized by using the time-fractional derivative of distributed order between 0 and 1, in both the Riemann-Liouville (R-L) and the Caputo (C) sense. For a general distribution of time orders we provide the fundamental solution, that is still a probability density, in terms of an integral of Laplace type. The kernel depends on the type of the assumed fractional derivative except for the single order case where the two approaches turn to be equivalent. We consider with some detail two cases of order distribution: the double-order and the uniformly distributed order. For these cases we exhibit plots of the corresponding fundamental solutions and their variance, pointing out the remarkable difference between the two approaches for small and large times.

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1 Introduction

It is well known that the fundamental solution (or Green function) for the Cauchy problem of the linear diffusion equation can be interpreted as a Gaussian (normal) probability density function (pdf) in space, evolving in time. All the moments of this pdf are finite; in particular, its variance is proportional to the first power of time, a noteworthy property of the standard diffusion.

In this paper we illustrate via fractional calculus two types of generalization of this Cauchy problem. One type uses the fractional derivative in the sense of Riemann and Liouville (R-L), the other in the sense of Caputo (C). In its uses we distinguish between single and distributed orders of fractional derivatives. Specifically, we work out how to express their fundamental solutions in terms of an integral of Laplace type suitable for a numerical evaluation. Particular attention is devoted to the time evolution of the variance for the R-L and C cases. It is known that for large times the variance characterizes the type of anomalous diffusion.

The plan of the paper is as follows. In Section 2 we write down the two general forms of the time-fractional diffusion equation with distributed-order with R-L and C derivatives and the Fourier-Laplace representation of the corresponding fundamental solution. For this purpose we need to introduce a positive function \( p(\beta) \) that acts as a discrete or continuous distribution of orders. In addition to the particular case of a single order \( \beta_0 \) with \( 0 < \beta_0 \leq 1 \), we consider two case-studies for the fractional diffusion of distributed order: as a discrete distribution we take two distinct orders \( \beta_1, \beta_2 \) with \( 0 < \beta_1 < \beta_2 \leq 1 \); as continuous distribution we take the uniform density of orders between zero and 1.

Section 3 is devoted to the time evolution of the variance which is obtained from the Fourier-Laplace representation of the corresponding fundamental solution, by inverting only the Laplace transform. In the single order case we recover the sub-diffusion power-law common to the R-L and C forms; for the cases of distributed order we find a remarkable difference between the two forms, well visible from their asymptotic expressions for small and large times. In section 4 we illustrate our method to get the fundamental solutions from their Fourier-Laplace transforms, following the strategy of carrying out at first the Fourier inversion and then the Laplace inversion. We find instructive to show the graphical representation of the fundamental
solutions (in space at fixed times). For the case of fractional diffusion of single order, because of the self-similarity of the solutions we limit ourselves to show plots of the corresponding solutions at a fixed time $t = 1$ versus $x$. For the two case-studies of fractional relaxation of distributed order, because the self-similarity of the solutions is lost, we provide plots of the corresponding solutions versus $x$ at three fixed times, selected as $t = 0.1$, $t = 1$ and $t = 10$, contrasting the different evolution for the R-L and the C form, in a moderate space-range. We can note how the time evolution of the solution in the considered spatial range depends on the different time-asymptotic behaviour of the variance for the two forms.

Finally, concluding remarks are given in Section 5.

In order to have a self-contained mathematical treatment we have added three Appendices: the Appendix A is devoted to the basic notions of fractional calculus, whereas Appendices B and C deal with functions of Mittag-Leffler and Exponential Integral type, respectively, in view of their relevance for our treatment.

2 The equations for time-fractional diffusion of distributed order

2.1 The R-L and the C forms in space-time domain

The standard diffusion equation, that in re-scaled non-dimensional variables reads

$$\frac{\partial}{\partial t} u(x, t) = \frac{\partial^2}{\partial x^2} u(x, t), \quad x \in \mathbb{R}, \quad t \in \mathbb{R}_0^+, \quad (2.1)$$

with $u(x, t)$ as the field variable, can be generalized by using the notion of fractional derivative of distributed order in time\(^1\).

For this purpose we need to consider a function $p(\beta)$ that acts as weight for the order of differentiation $\beta \in (0, 1]$ such that

$$p(\beta) \geq 0, \quad \text{and} \quad \int_0^1 p(\beta) \, d\beta = c > 0. \quad (2.2)$$

\(^1\)We find an earlier idea of fractional derivative of distributed order in time in the 1969 book by Caputo \[5\], that was later developed by Caputo himself, see \[6\] \[7\] and by Bagley & Torvik, see \[3\] A basic framework for the numerical solution of distributed-order differential equations has been recently introduced by Diethelm & Ford \[12\], Diethelm & Luchko \[13\] and by Hartley & Lorenzo \[26\] \[31\].
The positive constant $c$ can be taken as 1 if we like to assume the normalization condition for the integral. Clearly, some special conditions of regularity and behaviour near the boundaries will be required for the weight function $p(\beta)$. Such function, that can be referred to as the order density if $c = 1$, is allowed to have $\delta$-components if we are interested in a discrete distribution of orders.

There are two possible forms of generalization depending if we use fractional derivatives intended in the R-L or C sense. Correspondingly we obtain the time-fractional diffusion equation of distributed order in the two forms:

$$\frac{\partial}{\partial t} u(x, t) = \int_0^1 p(\beta) t^{1-\beta} \left[ \frac{\partial^2}{\partial x^2} u(x, t) \right] d\beta, \quad x \in \mathbb{R}, \ t \geq 0, \quad (2.3a)$$

and

$$\int_0^1 p(\beta) \left[ t D^\beta u_*(x, t) \right] d\beta = \frac{\partial^2}{\partial x^2} u_*(x, t), \quad x \in \mathbb{R}, \ t \geq 0. \quad (2.3b)$$

From now on we shall restrict our attention on the fundamental solutions of Eqs. (2.3a)-(2.3b) so we understand that these equations are subjected to the initial condition $u(x, 0^+) = u_*(x, 0^+) = \delta(x)$. Since for distributed order the solution depends on the selected form (as we shall show hereafter), we now distinguish the two fractional equations and their fundamental solutions by decorating in the Caputo case the variable $u(x, t)$ with subscript * as it is customary for the notation of the corresponding derivative.

Diffusion equations of distributed order of both types have been recently discussed by several authors: in particular we find the C form e.g. in [7, 9, 10, 11, 46, 51] whereas the R-L form e.g. in [30, 51, 52]. In some papers the authors have referred to the C and R-L forms as to normal and modified forms of the time-fractional diffusion equation of distributed order, respectively.

For a thorough general study of fractional pseudo-differential equations of distributed order let us cite the paper by Umarov and Gorenflo [53]. For a relationship with the Continuous Random Walk models we may refer to the paper by Gorenflo and Mainardi [24].

\footnote{For the weight function $p(\beta)$ we conveniently require that its primitive $P(\beta) = \int_0^\beta p(\beta') d\beta'$ vanishes at $\beta = 0$ and is there continuous from the right, attains the value $c$ at $\beta = 1$ and has at most finitely many (upwards) jump points in the half-open interval $0 < \beta \leq 1$, these jump points allowing delta contributions to $p(\beta)$ (particularly relevant for discrete distributions of orders).}
2.2 The RL and C forms in Fourier-Laplace domain

The fundamental solutions for the time-fractional diffusion equations (2.3a)-(2.3b) can be obtained by applying in sequence the Fourier and Laplace transforms to them. We write, for generic functions \( v(x) \) and \( w(t) \), these transforms as follows:

\[
\mathcal{F} \{ v(x); \kappa \} = \hat{v}(\kappa) := \int_{-\infty}^{+\infty} e^{i\kappa x} v(x) \, dx, \quad \kappa \in \mathbb{R},
\]

\[
\mathcal{L} \{ w(t); s \} = \hat{w}(s) := \int_{0}^{+\infty} e^{-st} w(t) \, dt, \quad s \in \mathbb{C}.
\]

Then, in the Fourier-Laplace domain our Cauchy problems [with \( u(x,0^+) = u_*(x,0^+) = \delta(x) \)], after applying formulas for the Laplace transform appropriate to the R-L and C fractional derivatives, see (A.8') and (A.9), and observing \( \hat{\delta}(\kappa) \equiv 1 \), see e.g. [18], appear in the two forms

\[
\hat{s}\hat{u}(\kappa, s) - 1 = -\kappa^2 \left[ \int_{0}^{\infty} p(\beta) s^{1-\beta} d\beta \right] \hat{u}(\kappa, s), \quad (2.4a)
\]

\[
\left[ \int_{0}^{\infty} p(\beta) s^{\beta} d\beta \right] \hat{u}(\kappa, s) - \int_{0}^{\infty} p(\beta) s^{\beta-1} d\beta = -\kappa^2 \hat{u}(\kappa, s). \quad (2.4b)
\]

Then, introducing the relevant functions

\[
A(s) = \int_{0}^{1} p(\beta) s^{1-\beta} d\beta, \quad (2.5a)
\]

and

\[
B(s) = \int_{0}^{1} p(\beta) s^{\beta} d\beta, \quad (2.5b)
\]

we then get for the R-L and C cases the Fourier-Laplace representation of the corresponding fundamental solutions:

\[
\hat{u}(\kappa, s) = \frac{1}{s + \kappa^2 A(s)} = \frac{1/A(s)}{\kappa^2 + s/A(s)}, \quad (2.6a)
\]

and

\[
\hat{u}_*(\kappa, s) = \frac{B(s)/s}{\kappa^2 + B(s)}. \quad (2.6b)
\]

From Eqs. (2.6a)-(2.6b) we recognize that the passage between the R-L and the C form can be carried out by the transformation

\[
\{ C : B(s) \} \iff \left\{ \text{R-L : } \frac{s}{A(s)} \right\}. \quad (2.7)
\]
We note that in the particular case of time fractional diffusion of single order \( \beta_0 \) \((0 < \beta_0 \leq 1)\) we have \( p(\beta) = \delta(\beta - \beta_0) \) hence in (2.5a): \( A(s) = s^{1-\beta_0} \), in (2.5b): \( B(s) = s^{\beta_0} \), so that \( B(s) \equiv s/A(s) \). Then, Eqs. (2.6a) and (2.6b) provide the same result

\[
\hat{u}(\kappa, s) \equiv \hat{u}_*(\kappa, s) = \frac{s^{\beta_0-1}}{\kappa^2 + s_0^2}.
\]  

(2.8)

This is consistent with the well-known result according to which the two forms are equivalent for the single order case. However, for a generic order distribution, the Fourier-Laplace representations (2.6a) (2.6b) are different so they produce in the space-time domain different fundamental solutions, that however are interrelated in some way in view of the transformation (2.7).

3 The variance of the fundamental solutions

3.1 General considerations

Before trying to get the fundamental solutions in the space-time domain to be obtained by a double inversion of the Fourier-Laplace transforms, it is worth to outline the expressions of their second moment (the variance) since these can be derived from Eqs. (2.6a)-(2.6b) by a single Laplace inversion, as it is shown hereafter. We recall that the time evolution of the variance is relevant for classifying the type of diffusion.

Denoting for the two forms

\[
\text{R-L : } \sigma^2(t) := \int_{-\infty}^{+\infty} x^2 u(x, t) \, dx, \quad \text{C : } \sigma^2_*(t) := \int_{-\infty}^{+\infty} x^2 u_*(x, t) \, dx, \quad (3.1)
\]

we easily recognize that

\[
\text{R-L : } \sigma^2(t) = -\frac{\partial^2}{\partial \kappa^2} \hat{u}(\kappa = 0, t), \quad \text{C : } \sigma^2_*(t) = -\frac{\partial^2}{\partial \kappa^2} \hat{u}_*(\kappa = 0, t). \quad (3.2)
\]

As a consequence we need to invert only Laplace transforms taking into account the behaviour of the Fourier transform for \( \kappa \) near zero.

For the R-L case we get from Eq. (2.6a),

\[
\hat{u}(\kappa, s) = \frac{1}{s} \left( 1 - \kappa^2 \frac{A(s)}{s} + \ldots \right),
\]

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so we obtain
\[ \tilde{\sigma}^2(s) = -\frac{\partial^2}{\partial \kappa^2} \tilde{u}(\kappa = 0, s) = \frac{2A(s)}{s^2}, \] (3.3a)

For the C-case we get from Eq. (2.6b)
\[ \tilde{u}_*(\kappa, s) = \frac{1}{s} \left( 1 - \frac{\kappa^2}{B(s)} + \ldots \right), \]
so we obtain
\[ \tilde{\sigma}^2_*(s) = -\frac{\partial^2}{\partial \kappa^2} \tilde{u}_*(\kappa = 0, s) = \frac{2}{s B(s)}. \] (3.3b)

Except for the single order diffusion, were we recover the well-know result
\[ \sigma^2(t) \equiv \sigma^2_*(t) = 2 \frac{t^{\beta_0}}{\Gamma(\beta_0 + 1)}, \quad 0 < \beta_0 \leq 1, \] (3.4)

for a generic order distribution, we expect that the time evolution of the variance substantially depends on the chosen (R-L or C) form.

We shall now concentrate our interest to some typical choices for the weight function \( p(\beta) \) that characterizes the time-fractional diffusion equations of distributed order (2.3a) and (2.3b). This will allow us to compare the results for the R-L form and for the C form.

### 3.2 Fractional diffusion of double-order

First, we consider the choice
\[ p(\beta) = p_1 \delta(\beta - \beta_1) + p_2 \delta(\beta - \beta_2), \quad 0 < \beta_1 < \beta_2 \leq 1, \] (3.5)
where the constants \( p_1 \) and \( p_2 \) are both positive, conveniently restricted to the normalization condition \( p_1 + p_2 = 1 \).

Then for the R-L case we have
\[ A(s) = p_1 s^{1-\beta_1} + p_2 s^{1-\beta_2}, \] (3.6a)
so, in virtue of (3.3a), we have
\[ \tilde{\sigma}^2(s) = 2 p_1 s^{-(1+\beta_1)} + 2 p_2 s^{-(1+\beta_2)}. \] (3.7a)
Finally the Laplace inversion yields, see and compare [30, 51],

\[ \sigma^2(t) = 2p_1 \frac{t^{\beta_1}}{\Gamma(\beta_1 + 1)} + 2p_2 \frac{t^{\beta_2}}{\Gamma(\beta_2 + 1)} \sim \begin{cases} 
2p_1 \frac{t^{\beta_1}}{\Gamma(1 + \beta_1)}, & t \to 0^+, \\
2p_2 \frac{t^{\beta_2}}{\Gamma(1 + \beta_2)}, & t \to +\infty. 
\end{cases} \]  

(3.8a)

Similarly, for the C case we have

\[ B(s) = p_1 s^{\beta_1} + p_2 s^{\beta_2}, \]  

(3.6b)

so, in virtue of (3.3b),

\[ \tilde{\sigma}^2(s) = \frac{2}{p_1 s^{(1+\beta_1)} + p_2 s^{(1+\beta_2)}}. \]  

(3.7b)

Finally the Laplace inversion yields, see and compare [9],

\[ \sigma^2_*(t) = \frac{2}{p_2} t^{\beta_2} E_{\beta_2-\beta_1, \beta_2+1} \left(-\frac{p_1}{p_2} t^{\beta_2-\beta_1}\right) \sim \begin{cases} 
\frac{2}{p_2} \frac{t^{\beta_2}}{\Gamma(1 + \beta_2)}, & t \to 0^+, \\
\frac{2}{p_1} \frac{t^{\beta_1}}{\Gamma(1 + \beta_1)}, & t \to +\infty. 
\end{cases} \]  

(3.8b)

Then we see that for the R-L case we have an explicit combination of two power laws: the smallest exponent($\beta_1$) dominates at small times whereas the largest exponent ($\beta_2$) dominates at large times. For the C case we have a Mittag-Leffler function in two parameters so we have a combination of two power laws only asymptotically for small and large times; precisely we get a behaviour opposite to the previous one, so the largest exponent($\beta_2$) dominates at small times whereas the smallest exponent ($\beta_1$) dominates at large times.

We can derive the above asymptotic behaviours directly from the Laplace transforms (3.7a)-(3.7b) by applying the Tauberian theory for Laplace transforms\[^3\]. In fact for the R-L case we note that for $A(s)$ in (3.6a) $s^{1-\beta_1}$ is negligibly small in comparison with $s^{1-\beta_2}$ for $s \to 0^+$ and, viceversa, $s^{1-\beta_2}$ is negligibly small in comparison to $s^{1-\beta_1}$ for $s \to +\infty$. Similarly for the C case we note that for $B(s)$ in (3.6b) $s^{\beta_2}$ is negligibly small in comparison to $s^{\beta_1}$ for $s \to 0^+$ and, viceversa, $s^{\beta_1}$ is negligibly small in comparison $s^{\beta_2}$ for $s \to +\infty$.

\[^3\]According to this theory the asymptotic behaviour of a function $f(t)$ near $t = \infty$ and $t = 0$ is (formally) obtained from the asymptotic behaviour of its Laplace transform $\tilde{f}(s)$ for $s \to 0^+$ and for $s \to +\infty$, respectively.
3.3 Fractional diffusion of uniformly distributed order

Second, we consider the choice

\[ p(\beta) = 1, \quad 0 < \beta < 1. \quad (3.9) \]

For the R-L case we have

\[ A(s) = s \int_0^1 s^{-\beta} d\beta = \frac{s - 1}{\log s}, \quad (3.10a) \]

hence, in virtue of (3.3a),

\[ \tilde{\sigma}^2(s) = 2 \left[ \frac{1}{s \log s} - \frac{1}{s^2 \log s} \right]. \quad (3.11a) \]

Then, by inversion, see Appendix C, Eqs (C.15)-(C.16), we get

\[ \sigma^2(t) = 2 \left[ \nu(t, 0) - \nu(t, 1) \right] \sim \begin{cases} \frac{2 \log (1/t)}{t}, & t \to 0, \\ 2t / t \log t, & t \to \infty, \end{cases} \quad (3.12a) \]

where

\[ \nu(t, a) := \int_0^\infty \frac{t^{a+\tau}}{\Gamma(a + \tau + 1)} d\tau, \quad a > -1, \]

denotes a special function just introduced in Appendix C along with its Laplace transform.

For the C case we have

\[ B(s) = s \int_0^1 s^\beta d\beta = \frac{s - 1}{\log s}, \quad (3.10b) \]

hence, in virtue of (3.3b),

\[ \tilde{\sigma}^2(s) = 2 \frac{\log s}{s s - 1}. \quad (3.11b) \]

Then, by inversion, see Appendix C, Eqs. (C.11), (C.14), and compare with Eqs. (23)-(27) in [9]

\[ \sigma^2(t) = 2 \left[ \log t + \gamma + e^t \mathcal{E}_1(t) \right] \sim \begin{cases} \frac{2t \log (1/t)}{t}, & t \to 0, \\ 2 \log (t), & t \to \infty, \end{cases} \quad (3.12b) \]
where

$$\mathcal{E}_1(t) := \int_t^\infty \frac{e^{-u}}{u} \, du = e^{-t} \int_0^\infty \frac{e^{-u}}{u + t} \, du$$
denotes the exponential integral function recalled in Appendix C, and \( \gamma = 0.57721... \) is the so-called Euler-Mascheroni constant.

For the uniform distribution we find it instructive to compare the time evolution of the variance for the R-L and C forms with that corresponding to a few of single orders. In Fig. 1-Top we consider moderate times \( (0 \leq t \leq 10) \) using linear scales, whereas in Fig. 1-Bottom large times \( (10^1 \leq t \leq 10^7) \) using logarithmic scales.

## 4 Evaluation of the fundamental solutions

### 4.1 The two strategies

In order to determine the fundamental solutions \( u(x, t) \) and \( u_*(x, t) \) in the space-time domain we can follow two alternative strategies related to the order in carrying out the Fourier-Laplace in (2.6a) and (2.6b)

(S1) : invert the Fourier transforms getting \( \tilde{u}(x, s) \), \( \tilde{u}_*(x, s) \) and then invert the remaining Laplace transforms;

(S2) : invert the Laplace transform getting \( \hat{u}(\kappa, t) \), \( \hat{u}_*(\kappa, t) \) and then invert the remaining Fourier transform.

Before considering the general case of time-fractional diffusion of distributed order, we prefer to briefly recall the determination of the fundamental solution \( u(x, t) \) (common for both the R-L and C forms) for the single order case.

### 4.2 The single order diffusion

For the time-fractional diffusion equation of single order \( \beta_0 \) the strategy (S1) yields the Laplace transform

\[
\tilde{u}(x, s) = \frac{s^{\beta_0/2 - 1}}{2} e^{-|x| s^{\beta_0/2}}, \quad 0 < \beta_0 \leq 1.
\]  

(4.1)

Such strategy was adopted by Mainardi \[33, 34, 35\] to obtain the Green function in the form

\[
u(x, t) = t^{-\beta_0/2} U\left(\frac{|x|}{t^{\beta_0/2}}\right), \quad -\infty < x < +\infty, \quad t \geq 0,
\]

(4.2)

where the variable \( X := x / t^{\beta_0/2} \) acts as similarity variable and the function \( U(x) := u(x, 1) \) denotes the reduced Green function. Restricting from now
on our attention to $x \geq 0$, the solution turns out as

$$U(x) = \frac{1}{2} M_{\beta_0/2}(x) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} \Gamma[-\beta_0 k/2 + (1 - \beta_0/2)]$$

$$= \frac{1}{2\pi} \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} \Gamma[(\beta_0(k + 1)/2] \sin[\pi \beta_0(k + 1)/2],$$

(4.3)

where $M_{\beta_0/2}(x)$ is an entire transcendental function (of order $1/(1 - \beta_0/2)$) of the Wright type, see also [21, 22, 39] and [47].

Figure 2: The reduced Green function $U(x) = \frac{1}{2} M_{\beta_0/2}(x)$ versus $x$ (in the interval $|x| \leq 5$), for $\beta_0 = 0$, $1/4$, $1/2$, $3/4$, $1$.

Since the fundamental solution has the peculiar property to be self-similar it is sufficient to consider the reduced Green function $U(x)$. In Fig. 2 we show the graphical representations of $U(x)$ for different orders ranging from $\beta_0 = 0$, for which we recover the Laplace density

$$U(x) = \frac{1}{2} e^{-|x|},$$

(4.4)

to $\beta_0 = 1$, for which we recover the Gaussian density (of variance $\sigma^2 = 2$)

$$U(x) = \frac{1}{2\sqrt{\pi}} e^{-x^2/4}.$$
The Strategy (S2): yields the Fourier transform.

\[ \hat{u}(\kappa, t) = E_{\beta_0} \left( -\kappa^2 t^{\beta_0} \right) , \quad 0 < \beta_0 \leq 1 , \]  

(4.6)

where \( E_{\beta_0} \) denotes the Mittag-Leffler function, see Appendix B. The strategy (S2) has been followed by Gorenflo et al. [20] and by Mainardi et al. [37, 42] to obtain the Green functions of the more general space-time-fractional diffusion equations (of single order), and requires to invert the Fourier transform by using the machinery of the Mellin convolution and the Mellin-Barnes integrals. Restricting ourselves here to recall the final results, the reduced Green function for the time-fractional diffusion equation now appears, for \( x \geq 0 \), in the form:

\[ U(x) = \frac{1}{\pi} \int_0^\infty \cos(\kappa x) E_{\beta_0} \left( -\kappa^2 \right) d\kappa = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\Gamma(1-s)}{\Gamma(1-\beta_0 s/2)} x^s ds , \]  

(4.7)

with \( 0 < \sigma < 1 \). By solving the Mellin-Barnes integrals using the residue theorem, we arrive at the same power series (4.3) of the \( M \)-Wright function. Both strategies allow us to prove that the Green function is non-negative and normalized, so it can be interpreted as a spatial probability density evolving in time with the similarity law (4.2).

For readers’ convenience we like to mention other papers dealing with the fundamental solutions of fractional diffusion equations (of single order); a non-exhaustive list includes [1, 15, 25, 27, 29, 43, 44, 48, 50, 55] and references therein.

### 4.3 The distributed order diffusion

Similarly with the single order diffusion, also for the cases of distributed order we can follow either strategy (S1) or strategy (S2). In contrast with previous papers of our group where we have followed the strategy (S2), see [38, 40, 41], here we follow the strategy (S1). This choice implies to recall the Fourier transform pair (a straightforward exercise in complex analysis based on residue theorem and Jordan’s lemma)

\[ \frac{c}{d + \kappa^2} \leftrightarrow \frac{c}{2d^{1/2}} e^{-|x|d^{1/2}} , \quad d > 0 . \]  

(4.8)
In fact we recognize by comparing (4.8) with (2.6a)-(2.6b) that for the RL and C forms we have

\[
\begin{align*}
\text{R-L} & : \quad \begin{cases} 
    c = c(s) := 1/A(s) \\
    d = d(s) := s/A(s)
\end{cases} & \quad \text{C} & : \quad \begin{cases} 
    c = c(s) := B(s)/s \\
    d = d(s) := B(s)
\end{cases}
\end{align*}
\]

(4.9)

Now we have to invert the Laplace transforms obtained inserting (4.9) in the R.H.S of (4.8).

For the R-L case we have:

\[
\tilde{u}(x, s) = \frac{1}{2[sA(s)]^{1/2}} \exp \left\{ -|x|[s/A(s)]^{1/2} \right\} .
\]

(4.10a)

For the C case we have:

\[
\tilde{u}_*(x, s) = \frac{[B(s)]^{1/2}}{2s} \exp \left\{ -|x|[B(s)]^{1/2} \right\} .
\]

(4.10b)

Following a standard procedure in complex analysis, the Laplace inversion requires the integration along the borders of the negative real semi-axis in the \( s \)-complex cut plain; in fact this semi-axis, defined by \( s = re^{i\pi} \) with \( r > 0 \) turns out the branch-cut common for the functions \( s^{1-\beta} \) (present in \( A(s) \) for the RL form) and \( s^\beta \) (present in \( B(s) \) for the C form). Then, in virtue of the Titchmarsh theorem on Laplace inversion, we get the representations in terms of real integrals of Laplace type.

For the R-L case we get

\[
u(x, t) = -\frac{1}{\pi} \int_0^\infty e^{-rt} \text{Im} \left\{ \tilde{u}(x, re^{i\pi}) \right\} \, dr,
\]

(4.11a)

where, in virtue of (4.10a), we must know \( A(s) \) along the ray \( s = re^{i\pi} \) with \( r > 0 \). We write

\[
A \left( re^{i\pi} \right) = \rho \cos(\pi\gamma) + i\rho \sin(\pi\gamma),
\]

(4.12a)

where

\[
\begin{align*}
\rho &= \rho(r) = \left| A \left( re^{i\pi} \right) \right|, \\
\gamma &= \gamma(r) = \frac{1}{\pi} \text{arg} \left[ A \left( re^{i\pi} \right) \right].
\end{align*}
\]

(4.13a)

Similarly for the C case we obtain

\[
u_*(x, t) = -\frac{1}{\pi} \int_0^\infty e^{-rt} \text{Im} \left\{ \tilde{u}_*(x, re^{i\pi}) \right\} \, dr,
\]

(4.11b)
where, in virtue of (4.10b), we must know $B(s)$ along the ray $s = r e^{i\pi}$ with $r > 0$. We write

$$B \left( r e^{i\pi} \right) = \rho_\ast \cos(\pi \gamma_\ast) + i \rho_\ast \sin(\pi \gamma_\ast), \quad (4.12b)$$

where

$$\begin{cases}
\rho_\ast = \rho_\ast(r) = |B \left( r e^{i\pi} \right)|,
\gamma_\ast = \gamma_\ast(r) = \frac{1}{\pi} \text{arg} \left[ B \left( r e^{i\pi} \right) \right].
\end{cases} \quad (4.13b)$$

As a consequence we formally write the required fundamental solutions as

$$u(x, t) = \int_0^\infty e^{-rt} P(x, r) \, dr, \quad P(x, r) = -\frac{1}{\pi} \text{Im} \left\{ \tilde{u} \left( x, re^{i\pi} \right) \right\}, \quad (4.14a)$$

and

$$u_\ast(x, t) = \int_0^\infty e^{-rt} P_\ast(x, r) \, dr, \quad P_\ast(x, r) = -\frac{1}{\pi} \text{Im} \left\{ \tilde{u}_\ast \left( x, re^{i\pi} \right) \right\}, \quad (4.14b)$$

where the functions $P(x, r)$ and $P_\ast(x, r)$ must be derived by using Eqs. (4.10a)-(4.14a) and Eqs. (4.10b)-(4.14b), respectively. We recognize that, in view of the transformation (2.7), the expressions of $P$ and $P_\ast$ are related to each other by the transformation

$$\rho_\ast(r) \leftrightarrow r/\rho(r), \quad \gamma_\ast(r) \leftrightarrow 1 - \gamma(r). \quad (4.15)$$

We then limit ourselves to provide the explicit expression for the C form

$$P_\ast(x, r) = \frac{1}{2\pi r} \text{Im} \left\{ \rho_\ast^{1/2} e^{i\pi \gamma_\ast/2} \exp \left\{ -i\pi \gamma_\ast/2 \rho_\ast^{1/2} x \right\} \right\} \nonumber$$

$$= \frac{1}{2\pi r} \rho_\ast^{1/2} e^{-\rho_\ast^{1/2} x \cos(\pi \gamma_\ast/2)} \sin \left[ \pi \gamma_\ast/2 - \rho_\ast^{1/2} x \sin(\pi \gamma_\ast/2) \right]. \quad (4.16)$$

For the R-L form the corresponding expression of $P(x; r)$ is obtained from (4.16) by applying the transformation (4.15).

We note that the fundamental solutions found in this subsection are equivalent to those obtained by the Authors in [38] by following the strategy (S2) after a lengthy manipulation of Mellin-Barnes integrals.

Hereafter we exhibit some plots of the fundamental solutions for the two case studies considered in subsection 3.2 in order to point out the remarkable difference between the R-L and the C forms.
4.4 Plots of the fundamental solutions

For the case of two orders, we chose \( \{\beta_1 = 1/4, \beta_2 = 1\} \) in order to contrast the evolution of the fundamental solution for the R-L and the C forms.

Figure 3: The fundamental solution versus \( x \) (in the interval \( |x| \leq 5 \)), for the double-order distribution \( \{\beta_1 = 1/4, \beta_2 = 1\} \) at times \( t = 0.1, 1, 10 \). Top: R-L form; Bottom: C form.
Figure 4: The fundamental solutions versus $x$ (in the interval $|x| \leq 5$), for the uniform order distribution in R-L and C forms compared with the solutions for some cases of single order. Top: $t = 1$; Bottom: $t = 10$.

In Fig. 3 we exhibit the plots of the corresponding solution versus $x$ (in the interval $|x| \leq 5$), at different times, selected as $t = 0.1$, $t = 1$ and $t = 10$. In this limited spatial range we can note how the time evolution of
the pdf depends on the different time-asymptotic behaviour of the variance, for the two forms, as stated in Eqs. (3.12a)-(3.12b), respectively. For the uniform distribution, we find it instructive to compare in Fig. 4 the solutions corresponding to R-L and C forms with the solutions of the fractional diffusion of a single order $\beta_0 = 1/4, 3/4, 1$ at fixed times, selected as $t = 1, 10$. We have skipped $\beta_0 = 1/2$ for a better view of the plots.

5 Conclusions

We have investigated the time fractional diffusion equation with (discretely or continuously) distributed order between 0 and 1 in the Riemann-Liouville and in the Caputo forms, providing the Fourier-Laplace representation of the corresponding fundamental solutions. Except for the case of a single order, for which the two forms are equivalent with a self-similar fundamental solution, for a general order distribution the equivalence and the self-similarity are lost. In particular the asymptotic behaviour of the fundamental solution and its variance at small and large times strongly depends on the selected approach. We have considered two simple but noteworthy case-studies of distributed order, namely the case of a superposition of two different orders $\beta_1$ and $\beta_2$ and the case of a uniform order distribution. In the first case one of the orders dominates the time-asymptotics near zero, the other near infinity, but $\beta_1$ and $\beta_2$ change their roles when switching from the R-L form to the C form of the time-fractional diffusion. The asymptotics for uniform order density is remarkably different, the extreme orders now being (roughly speaking) 0 and 1. We now meet super-slow and slightly super-fast time behaviours of the variance near zero and near infinity, again with the interchange of behaviours between the R-L and C form. We have clearly pointed out the above effects with the figures in sub-section 3.3, in particular the extremely slow growth of the variance as $t \to \infty$ for the C form. After the analysis of the variance, that in practice requires only the inversion of a Laplace transform, we have considered the task of the double inversion of the Laplace-Fourier representation. For a general order distribution we were able to express the fundamental solution in terms of a Laplace integral in time with a kernel which depends on space and order distribution in a simple form, see Eqs. (4.14)-(4.16). For the two case studies the plots of the fundamental solutions (reported in sub-section 4.4) have shown their dependence on the different asymptotic behavior of the corresponding variance.
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Appendix A: The two fractional derivatives

The purpose of this Appendix is to clarify for the interested reader the main differences between the Riemann-Liouville (R-L) fractional derivative and the Caputo (C) fractional derivative for well-behaved functions \( f(t) \) with \( t \geq 0 \), exhibiting a finite limit \( f(0^+) \) as \( t \to 0^+ \). Denoting with \( \mu \in (0, 1] \) the order, the R-L derivative is defined as

\[
_t D^\mu f(t) := \begin{cases} 
\frac{1}{\Gamma(1-\mu)} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^\mu} d\tau, & 0 < \mu < 1, \\
\frac{d f(t)}{dt}, & \mu = 1.
\end{cases} \quad (A.1)
\]

and the C derivative as

\[
_t D^\mu_* f(t) := \begin{cases} 
\frac{1}{\Gamma(1-\mu)} \int_0^t \frac{f'(\tau)}{(t-\tau)^\mu} d\tau, & 0 < \mu < 1, \\
\frac{d f(t)}{dt}, & \mu = 1.
\end{cases} \quad (A.2)
\]

The two fractional derivatives are related to the Riemann-Liouville fractional integral as follows. The Riemann-Liouville fractional integral is

\[
_t J^\mu f(t) := \frac{1}{\Gamma(\mu)} \int_0^t f(\tau) (t-\tau)^{\mu-1} d\tau, \quad \mu > 0, \quad (A.3)
\]

This form is referred to Caputo who used it in the late sixties of the past century, see [4, 5]. Soon later this derivative was adopted by Caputo and Mainardi in the framework of the theory of Linear Viscoelasticity, see [8]. It was mainly with the 1997 CISM chapter by Gorenflo and Mainardi [23] and with the 1999 book by Podlubny [47] that such form was popularized.
(with the convention \( t^0 f(t) = f(t) \)) and is known to satisfy the semigroup property \( t^{\mu + \nu} = t^\mu t^\nu \), with \( \mu, \nu > 0 \). For any \( \mu > 0 \) the Riemann-Liouville fractional derivative is defined as the left inverse of the corresponding fractional integral (like the derivative of any integer order), namely \( t^\mu t^\nu f(t) = f(t) \). Then for \( \mu \in (0, 1] \) we have

\[
t^\mu \Delta^\mu f(t) := t^\mu D_1 J^{1-\mu} f(t), \quad t^\mu \Delta^\mu D_1 f(t) := t^\mu J^{1-\mu} D_1 f(t), \quad (A.4)
\]

Recalling the rule

\[
t^\mu t^\gamma = \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + 1 - \mu)} t^{\gamma - \mu}, \quad \mu \geq 0, \quad \gamma > -1, \quad (A.6)
\]

it turns out for \( 0 < \mu < 1 \),

\[
t^\mu \Delta^\mu D_\star f(t) = t^\mu [f(t) - f(0^+)] = t^\mu D_\mu f(t) - f(0^+) \frac{t^{-\mu}}{\Gamma(1 - \mu)}. \quad (A.7)
\]

Note that for \( \mu = 1 \) the two types of fractional derivative coincide, the constant \( f(0^+) \) playing no role.

The Caputo fractional derivative represents a sort of regularization in the time origin for the Riemann-Liouville fractional derivative and satisfies the relevant property of being zero when applied to a constant.

Let us now consider the behaviour of the above derivatives of non-integer order with respect to the Laplace transformation\[5\].

For the Riemann-Liouville derivative of non-integer order \( \mu \) we have

\[
\mathcal{L} \{ t^\mu D_\mu f(t); s \} = s^\mu \tilde{f}(s) - g(0^+) \quad g(0^+) := \lim_{t \to 0^+} t^{1-\mu} f(t), \quad 0 < \mu < 1. \quad (A.8)
\]

\[5\)The Laplace transform of a well-behaved function \( f(t) \) is defined as

\[
\tilde{f}(s) = \mathcal{L} \{ f(t); s \} := \int_0^\infty e^{-st} f(t) \, dt, \quad s \in \mathbb{C}.
\]

We recall that under suitable conditions the Laplace transform of the first derivative of \( f(t) \) is given by

\[
\mathcal{L} \{ t D_1 f(t); s \} = s \tilde{f}(s) - f(0^+), \quad f(0^+) := \lim_{t \to 0^+} f(t).
\]

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For the Caputo derivative of non-integer order $\mu$ we have

$$L \{ t^\mu f(t); s \} = s^\mu \tilde{f}(s) - s^{\mu-1} f(0^+) , \quad f(0^+) := \lim_{t \to 0^+} f(t) , \quad 0 < \mu < 1. \quad (A.9)$$

Thus the rule (A.8) is more cumbersome to be used than (A.9) since it requires initial values concerning an extra function $g(t)$ related to the given $f(t)$ through a fractional integral. However, when the limiting value $f(0^+)$ is finite we can see that $g(0^+)$ is vanishing so the formula (A.8) simplifies into

$$L \{ t^\mu f(t); s \} = s^\mu \tilde{f}(s) , \quad 0 < \mu < 1. \quad (A.8')$$

For further reading on the theory and applications of fractional calculus we recommend to consult in addition to the well-known books by Samko, Kilbas & Marichev [49], by Miller & Ross [45], by Podlubny [47], those appeared in the last few years, by Kilbas, Srivastava & Trujillo [28], by Magin [32], by West, Bologna & Grigolini [54], and by Zaslavsky [56].

Appendix B: The Mittag-Leffler functions

B.1 The classical Mittag-Leffler function

Let us recall that the Mittag-Leffler function $E_\mu(z)$ (with $\mu > 0$) is an entire transcendental function of order $1/\mu$, defined in the complex plane by the power series

$$E_\mu(z) := \sum_{k=0}^\infty \frac{z^k}{\Gamma(\mu k + 1)} , \quad z \in \mathbb{C}. \quad (B.1)$$

It was introduced and studied by the Swedish mathematician Mittag-Leffler at the beginning of the XX century to provide a noteworthy example of entire function that generalizes the exponential (to which it reduces for $\mu = 1$). For details on this function we refer e.g. to [14, 16, 23, 28, 36, 47, 49]. In particular we note that the function $E_\mu(-x)$ ($x \geq 0$) turns a completely monotonic function of $x$ if $0 < \mu \leq 1$. This property is still valid if we consider the variable $x = \lambda t^\mu$ where $\lambda$ is a positive constant. Thus the function $E_\mu(-\lambda t^\mu)$ preserves the complete monotonicity of the exponential $\exp(-\lambda t)$: indeed, for $0 < \mu < 1$ it is represented in terms of a real Laplace transform (of a
real parameter \( r \) of a non-negative function (that we refer to as the spectral function)

\[
E_\mu(-\lambda t^\mu) = \frac{1}{\pi} \int_0^\infty e^{-rt} \frac{\lambda t^{\mu-1} \sin(\mu \pi)}{\lambda^2 + 2\lambda r^\mu \cos(\mu \pi) + r^{2\mu}} \, dr. \quad (B.2)
\]

We note that as \( \mu \to 1 \) the spectral function tends to the generalized Dirac function \( \delta(r - \lambda) \).

We note that the Mittag-Leffler function (B.2) starts at \( t = 0 \) as a stretched exponential and decreases for \( t \to \infty \) like a power with exponent \(-\mu\):

\[
E_\mu(-\lambda t^\mu) \sim \begin{cases} 
1 - \lambda t^\mu / \Gamma(1 + \mu) & t \to 0^+ , \\
\frac{t^\mu}{\lambda \Gamma(1 - \mu)} & t \to \infty .
\end{cases} \quad (B.3)
\]

The noteworthy results (B.2) and (B.3) can also be derived from the Laplace transform pair

\[
\mathcal{L}\{E_\mu(-\lambda t^\mu); s\} = \frac{s^{\mu-1}}{s^\mu + \lambda}. \quad (B.4)
\]

In fact it is sufficient to apply the Titchmarsh theorem \((s = re^{i\pi})\) for deriving (B.2) and the Tauberian theory \( (s \to \infty \text{ and } s \to 0) \) for deriving (B.3).

If \( \mu = 1/2 \) we have for \( t \geq 0 \):

\[
E_{1/2}(-\lambda \sqrt{t}) = e^{\lambda^2 t} \text{erfc}(\lambda \sqrt{t}) \sim 1/(\lambda \sqrt{\pi \pi t}) , \quad t \to \infty , \quad (B.5)
\]

where \( \text{erfc} \) denotes the \text{complementary error function}, see e.g. [2].

**B.2 The generalized Mittag-Leffler function**

The Mittag-Leffler function in two parameters \( E_{\mu,\nu}(z) \) \((\Re\{\mu\} > 0, \nu \in \mathbb{C})\) is defined by the power series

\[
E_{\mu,\nu}(z) := \sum_{k=0}^\infty \frac{z^k}{\Gamma(\mu k + \nu)} , \quad z \in \mathbb{C} . \quad (B.6)
\]

It generalizes the classical Mittag-Leffler function to which it reduces for \( \nu = 1 \). It is an entire transcendental function of order \( 1/\Re\{\mu\} \) on which the
reader can inform himself by again consulting the references before outlined for the classical Mittag-Leffler function

The function $E_{\mu,\nu}(x)$ ($x \geq 0$) is completely monotonic if $0 < \mu \leq 1$ and $\nu \geq \mu$. Again this property is still valid if we consider the variable $x = \lambda t^\mu$ where $\lambda$ is a positive constant. In this case the asymptotic representations as $t \to 0^+$ and $t \to +\infty$ read

$$E_{\mu,\nu}(-\lambda t^\mu) \sim \begin{cases} 
\frac{1}{\Gamma(\nu)} - \lambda \frac{t^\mu}{\Gamma(\nu + \mu)}, & t \to 0^+, \\
\frac{1}{\lambda \Gamma(\nu - \mu)}, & t \to \infty.
\end{cases} \quad (B.7)$$

We point out the Laplace transform pair, see [47],

$$\mathcal{L}\{t^{\nu-1} E_{\mu,\nu}(-\lambda t^\mu); s\} = \frac{s^{\mu-\nu}}{s^\mu + \lambda}, \quad (B.8)$$

with $\mu, \nu \in \mathbb{R}^+$. For $0 < \mu = \nu \leq 1$ this Laplace transform pair can be used to derive the noteworthy identity

$$t^{-(1-\mu)} E_{\mu,\mu}(-\lambda t^\mu) = -\frac{1}{\lambda} \frac{d}{dt} E_{\mu}(-\lambda t^\mu), \quad 0 < \mu \leq 1. \quad (B.9)$$

**Appendix C: The Exponential integral and related functions**

The exponential integral function, that we denote by $\mathcal{E}_1(z)$, is defined as

$$\mathcal{E}_1(z) = \int_{z}^{\infty} \frac{e^{-t}}{t} dt = \int_{1}^{\infty} \frac{e^{-zt}}{t} dt. \quad (C.1)$$

We have used the letter $\mathcal{E}$ instead of $E$ (commonly adopted in the literature) in order to avoid confusion with the Mittag-Leffler functions that play a more relevant role in fractional calculus. This function exhibits a branch cut along the negative real semi-axis and admits the representation

$$\mathcal{E}_1(z) = -\gamma - \log z - \sum_{n=1}^{\infty} \frac{z^n}{n n!}, \quad |\arg z| < \pi, \quad (C.2)$$

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where \( \gamma = 0.57721... \) is the so-called Euler-Mascheroni constant. The power series in the R.H.S. is absolutely convergent in all of \( \mathbb{C} \) and represents the entire function called the \textit{modified exponential integral}

\[
\text{Ein}(z) := \int_0^z \frac{1 - e^{-\zeta}}{\zeta} \, d\zeta = -\sum_{n=1}^{\infty} \frac{z^n}{n n!},
\] (C.3)

Thus, in view of (C.2) and (C.3), we write

\[
E_1(z) = -\gamma - \log z + \text{Ein}(z), \quad |\arg z| < \pi. \] (C.4)

This relation is important for understanding the analytic properties of the classical exponential integral function in that it isolates the multi-valued part represented by the logarithmic function from the regular part represented by the entire function \( \text{Ein}(z) \). In \( \mathbb{R}^+ \) the function \( \text{Ein}(x) \) turns out to be a \textit{Bernstein function}, which means that is positive, increasing, with the first derivative \textit{completely monotonic}.

The asymptotic behaviour as \( z \to \infty \) of the exponential integrals can be obtained from the integral representation (C.1) noticing that

\[
E_1(z) := \int_z^{\infty} \frac{e^{-t}}{t} \, dt = e^{-z} \int_0^{\infty} \frac{e^{-u}}{u + z} \, du.
\] (C.5)

In fact, by repeated partial integrations in the R.H.S., we get for \( |\arg z| \leq \pi - \delta \),

\[
E_1(z) \sim \frac{e^{-z}}{z} \sum_{n=0}^{\infty} (-1)^n \frac{n!}{z^n}, \quad z \to \infty.
\] (C.6)

We now report a number of relevant Laplace transform pairs in which logarithmic and exponential integral functions are involved.

Taking \( t > 0 \), the basic Laplace transforms pairs are

\[
\mathcal{L}\{\log t; s\} = -\frac{\gamma + \log s}{s}, \quad \Re s > 0, \tag{C.7}
\]

\[
\mathcal{L}\{E_1(t); s\} = \frac{\log (s + 1)}{s}, \quad \Re s > -1. \tag{C.8}
\]

The proof of (C.7) and (C.8) is found, for example, in the treatise by Ghizzetti and Ossicini [19], see Eqs. [4.6.15-16], pp. 104-105. We then easily derive for \( \Re s > 0 \)

\[
\mathcal{L}\{\gamma + \log t; s\} = -\frac{\log s}{s}, \tag{C.9}
\]
We outline the different asymptotic behaviour of the three functions $\mathcal{E}_1(t)$, $\text{Ein}(t)$ and $\gamma + \log t + e^t \mathcal{E}_1(t)$, for small argument ($t \to 0^+$) and large argument ($t \to +\infty$). By using Eqs (C.2), (C.4) and (C.6), we have

$$\mathcal{E}_1(t) \sim \begin{cases} \log (1/t), & t \to 0^+, \\ e^{-t}/t, & t \to +\infty. \end{cases}$$

$$\text{Ein}(t) \sim \begin{cases} t, & t \to 0^+, \\ \log t, & t \to +\infty. \end{cases}$$

$$\gamma + \log t + e^t \mathcal{E}_1(t) \sim \begin{cases} t \log (1/t), & t \to 0^+, \\ \log t, & t \to +\infty. \end{cases}$$

We note that all the above asymptotic representations can be obtained from the Laplace transforms of the corresponding functions by invoking the Tauberian theory for regularly varying functions (power functions multiplied by slowly varying functions), a topic not so well known which is adequately treated in the treatise on Probability by Feller [17], see Chapter XIII.5.

We conclude this subsection pointing out the Laplace transform pair

$$\mathcal{L}\{\nu(t, a); s\} = \frac{1}{s^{a+1} \log s}, \quad \Re s > 0.$$  \hfill (C.15)

where

$$\nu(t, a) := \int_0^\infty \frac{t^{a+\tau}}{\Gamma(a + \tau + 1)} d\tau, \quad a > -1.$$  \hfill (C.16)

For details on this transcendental function the reader is referred to the third volume of the Handbook of the Bateman Project [16], see in Chapter XVIII (devoted to the Miscellaneous functions) §18.3, pp. 217-224.

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6Definition: We call a (measurable) positive function $a(y)$, defined in a right neighbourhood of zero, slowly varying at zero if $a(y) > 0$ and $a(cy)/a(y) \to 1$ with $y \to 0$ for every $c > 0$. We call a (measurable) positive function $b(y)$, defined in a neighbourhood of infinity, slowly varying at infinity if $b(cy)/b(y) \to 1$ with $y \to \infty$ for every $c > 0$. Examples: $(\log y)^\gamma$ with $\gamma \in \mathbb{R}$ and $\exp(\log y/\log \log y)$. 

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