A First Approximation for Quantization of Singular Spaces
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February 26, 2008

Abstract

Many mathematical models of physical phenomena that have been proposed in recent years
require more general spaces than manifolds. When taking into account the symmetry group of the
model, we get a reduced model on the (singular) orbit space of the symmetry group action. We
investigate quantization of singular spaces obtained as leaf closure spaces of regular Riemannian
foliations on compact manifolds. These contain the orbit spaces of compact group actions and
orbits. Our method uses foliation theory as a desingularization technique for such singular
spaces. A quantization procedure on the orbit space of the symmetry group - that commutes with
reduction - can be obtained from constructions which combine different geometries associated
with foliations and new techniques originated in Equivariant Quantization. The present paper contains
the first of two steps needed to achieve these just detailed goals.

Mathematics Subject Classification (2000) : 53D50, 53C12, 53B10, 53D20

Key words : Quantization, singular space, reduction, foliation, equivariant symbol calculus

1 Introduction

Quantization of singular spaces is an emerging issue that has been addressed in an increasing num-
ber of recent works, see e.g. [BHP06], [Hue02], [Hue06], [HRS07], [Hui07], [Pfl02] ...

One of the reasons for this growing popularity originates from current developments in Theore-
tical Physics related with reduction of the number of degrees of freedom of a dynamical system with
symmetries. Explicitly, if a symmetry Lie group acts on the phase space or the configuration space of
a general mechanical system, the quotient space is usually a singular space, an orbifold or a stratified
space ... The challenge consists in the quest for a quantization procedure for these singular spaces that
in addition commutes with reduction.

In this work, we investigate quantization of singular spaces obtained as leaf closure spaces of reg-
ular Riemannian foliations of compact manifolds. These contain the orbit spaces of compact group
actions (see [Rich01]). We build a quantization that commutes by construction with projection onto
the quotient.

Our method uses the foliation as desingularization of the orbit space $M/\bar{F}$, where $\bar{F}$ is the singular
Riemannian foliation made up by the closures of the leaves of the regular Riemannian foliation $F$
on manifold $M$. More precisely, we combine Foliation Theory with recent techniques from Natural and
Equivariant Quantization. Close match can indeed be expected, as both topics are tightly connected

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supported by grant R1F105L10. This author also thanks the Erwin Schrödinger Institute in Vienna for hospitality and
support during his visits in 2006 and 2007.

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Luxembourg Ministry of Culture, Higher Education and Research, for grant BFR 06/077

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Quantization of Singular spaces

with natural bundles and natural operators.

Equivariant quantization, in the sense of C. Duval, P. Lecomte, and V. Ovsienko, developed as from 1996, see [LMT96], [LO99], [DLO99], [Lec00], [BM01], [DO01], [BHMP02], [BM06]. This procedure requires equivariance of the quantization map with respect to the action of a finite-dimensional Lie subgroup of the symmetry group Diff($\mathbb{R}^n$) of configuration space $\mathbb{R}^n$. Equivariant quantization has first been studied in Euclidean space, mainly for the projective and conformal subgroups, then extended in 2001 to arbitrary manifolds, see [Lec01]. An equivariant, or better, a natural quantization on a smooth manifold $M$ is a vector space isomorphism

$$Q[\nabla]: \text{Pol}(T^*M) \ni s \rightarrow Q[\nabla](s) \in \mathcal{D}(M)$$

that verifies some normalization condition and maps, in this paper, a smooth function $s \in \text{Pol}(T^*M)$ of “phase space” $T^*M$, which is polynomial along the fibers, to a differential operator $Q[\nabla]$ that acts on functions $f \in C^\infty(M)$ of “configuration space” $M$. The quantization map $Q[\nabla]$ depends on the projective class $[\nabla]$ of an arbitrary torsionless covariant derivative $\nabla$ on $M$, and it is natural with respect to all its arguments and for the action of the group Diff($M$) of all local diffeomorphisms of $M$, i.e.

$$Q[\phi^*\nabla](\phi^*s)(\phi^*f) = \phi^*(Q[\nabla](s))(f), \quad \forall s \in \text{Pol}(T^*M), \forall f \in C^\infty(M), \forall \phi \in \text{Diff}(M).$$

Existence of such natural and projectively invariant quantizations has been investigated in several works, see e.g. [Bor02], [MR05], [Han06].

In Foliation Theory, one distinguishes different geometries associated with a foliated manifold $(M, \mathcal{F})$ (defined by a Haefliger cocycle), namely adapted geometry, foliated geometry, and transverse geometry. We denote in this introduction objects of the adapted (resp. foliated, transverse) “world” by $O_3$ (resp. $O_2, O_1$), whereas objects of leaf closure space $M/\mathcal{F}$ are denoted by $O_0$. Ideally, geometric structures of level $i$ project onto geometric structures of level $i-1$, so that $p(O_i) = O_{i-1}$, if we agree to denote temporarily any of these projections by $p$. Let us also recall that, roughly, adapted objects are objects on $M$ with some special properties, foliated objects are locally constant along the leaves and live in the normal bundle of the foliation, and that transverse objects are objects on the transverse manifold $N$, which are $\mathcal{H}$-invariant, where transverse manifold $N$ and the holonomy pseudo-group $\mathcal{H}$ depend on the chosen defining cocycle of foliation $\mathcal{F}$. In order to build a quantization $Q_0$ on $M/\mathcal{F}$, which commutes with the projection onto this singular space, we construct adapted, foliated, and transverse quantizations $Q_3, Q_2,$ and $Q_1$, in such a way that

$$Q_{i-1}[p \nabla_i](p \, s_i)(p \, f_i) = p \, (Q_i[\nabla_i](s_i)(f_i)), \quad \forall i \in \{1, 2, 3\}. \tag{1}$$

Hence,

$$Q_0[\nabla_0](s_0)(f_0) = Q_0[p^3 \nabla_3](p^3 s_3)(p^3 f_3) = p^3 \, (Q_3[\nabla_3](s_3)(f_3)).$$

Observe that adapted quantization $Q_3$ quantizes objects on $M$, whereas singular quantization $Q_0$ only quantizes the objects of $M/\mathcal{F}$. Eventually, quantization actually commutes with projection onto the quotient.

The proofs of the three stages mentioned in Equation (1) are not equally hard. Since foliated geometric objects on a foliated manifold $(M, \mathcal{F})$ are in 1-to-1 correspondence with $\mathcal{H}$-invariant geometric objects on the transverse manifold $N$ associated with the chosen cocycle, it is clear that stage $Q_2 - Q_1$ is quite obvious. The passages $Q_3 - Q_2$ between the “big” adapted and “small” foliated quantizations, as well as transition $Q_1 - Q_0$ from transverse quantization to singular quantization are much more intricate.

The present paper should be accessible for readers who are not necessarily experts in both fields, Natural Quantization and Foliation Theory. In order to limit the length of the article, we publish the stages $Q_3 - Q_2$ and $Q_1 - Q_0$ in two different works. This publication deals with the first approximation $Q_3 - Q_2$ for quantization of singular spaces.
2 Natural and projectively invariant quantization

The constructions of $Q_1$, $Q_2$, and $Q_3$ are nontrivial extensions to the adapted, foliated, and transverse contexts, of the proof of existence of natural and projectively invariant quantization maps on an arbitrary smooth manifold, see [MR05]. In the present section, we concisely describe the basic ideas of this technique.

In the theory of star-products, A. Lichnerowicz extensively used the standard ordering prescription $Q_{\text{aff}}(\nabla)$ associated with a covariant derivative $\nabla$. More precisely, consider the space $D^k(\Gamma(E),\Gamma(F))$ of $k$th order differential operators between the spaces of sections $\Gamma(E)$ and $\Gamma(F)$ of two vector bundles $E, F \to M$ over a manifold $M$ as well as the corresponding symbol space $\Gamma(S^{k}TM \otimes E^{*} \otimes F)$. If $\nabla$ is a covariant derivative on $E$, denote by $\nabla^k : \Gamma(E) \ni f \to \nabla^k f \in \Gamma(S^{k}TM \otimes E \otimes F)$. $Q$ is a differential operator $Q_{\text{aff}}(\nabla)(s) \in D^k(\Gamma(E),\Gamma(F))$ defined on any section $f \in \Gamma(E)$ by

$$Q_{\text{aff}}(\nabla)(s)(f) := \hat{i}_{s}(\nabla^k f) \in \Gamma(F).$$

The following example allows understanding the idea, due to M. Bordemann, see [Bor02], underlying the construction of natural and projectively invariant quantizations $Q$ on a manifold $M$, see above. Set $M = S^n$, where $S^n$ is the $n$-dimensional sphere, and $G = \text{GL}(n+1, \mathbb{R})$. The elements $g \in G$ act on $\mathbb{R}^{n+1}$, $g : \mathbb{R}^{n+1} \ni x \to gx \in \mathbb{R}^{n+1}$, and on $S^n$, $\phi_g : S^n \ni x \to gx / ||gx|| \in S^n$, where notations are self-explaining. Observe that $M := \mathbb{R}^{n+1}\{0\} \to S^n = M$ is a bundle with typical fiber $\mathbb{R}^+_n$, and note that all $g$ preserve the canonical connection of $\mathbb{R}^{n+1}$, but that the induced $\phi_g$ do usually not preserve the canonical Levi-Civita connection on $S^n$. It seems therefore natural to lift the complex situation on $M$ to the simpler situation on $M$. Thus, in order to define $Q(\nabla)(s)(f)$, where, see above, $\nabla$ denotes a torsionless covariant derivative on $M$, $s$ a symbol in $\Gamma(STM) \simeq \text{Pol}(T^{*}M)$, and $f$ a function in $C^{\infty}(M)$, one constructs natural and projectively invariant lifts

$$\nabla \to \tilde{\nabla}, \ s \to \tilde{s}, \ f \to \tilde{f},$$

then sets

$$(Q(\nabla)(s)(f)) := Q_{\text{aff}}(\tilde{\nabla})(\tilde{s})(\tilde{f}),$$

where $Q_{\text{aff}}$ is the standard ordering. The point is that the normal ordering prescription, see its definition, is natural—but of course not projectively invariant—and that we require naturality and projective invariance for all the lifts. It immediately follows that $Q$ inherits these properties (if the projection onto the base behaves properly).

One of the proofs of existence of natural and projectively invariant quantizations on an arbitrary smooth manifold $M$ is based on the preceding example $M = S^n$ and consists of four stages. In order to ensure readability of this paper, we recall some concepts that are basic for further investigations and we briefly depict the mentioned four stages.

2.1 Basic concepts

Let $M'$ and $M''$ be two smooth manifolds, let $m'$ be a point in $M'$, and $U'$ a neighborhood of $m'$. Two smooth functions $f : U' \to M''$ and $g : U' \to M''$ have at $m'$ a contact of order $\geq r$, $r \in \mathbb{N}$, if and only if $f(m') = g(m') = m''$ and, for any chart of $M'$ around $m'$ and any chart of $M''$ around $m''$, the components of the local forms $F$ of $f$ and $G$ of $g$ have the same partial derivatives up to order $r$ at $m'$. It is well-known that it suffices that this condition be satisfied for one pair of charts. The classes of equivalence relation “contact of order $\geq r$ at $m''$” are the $r$-jets at $m'$. Clearly, if we denote the coordinates of $M'$ around $m'$ by $Z$, the $r$-jet $j^{m'}_{r} f$ of $f$ at $m'$ of a function $f$ is characterized by the package $(\partial^{\alpha} F)(Z(m'))$, $|\alpha| \leq r$, $i \in \{1, \ldots, n''\}$, $n'' = \dim M''$. Of course, a change of coordinates entails a change of the characterizing package of derivatives. If, for instance,
if we exchange current coordinates $X$ in target manifold $M'$ for new coordinates $Y$, the current and new local forms $F(Z) = X(f(Z))$ and $F'(Z) = Y(f(Z))$ are related by $F'(Z) = Y(f(Z))$, where, in order to simply, we used notations from Physics. It follows that

$$\partial_Z F^α = \partial_X Y^i \partial_Z F^a$$

(5)

and that

$$\partial_{Z \cdot 2^i} F^α = \partial_{X \cdot X} Y^i \partial_{Z \cdot 2} F^β \partial_{Z} F^α + \partial_{X \cdot Y} Y^i \partial_{Z, 2^i} F^α.$$  

(6)

These formulae will be needed below. Let us also recall that, for fixed charts, the characterizing package of derivatives of the jet $j^r_m(h \circ f)$ of a compound map, is obtained, roughly spoken, by composition of the limited Taylor expansions of the local forms of $h$ and $f$, if one agrees to suppress the terms that have order $> r$.

We denote by $P^r$, $r \in \mathbb{N}$, the natural functor of order $r$— between the category of $n$-dimensional smooth manifolds $M$ and immersions $\phi : M \to M'$ (or, equivalently, globally defined local diffeomorphisms) and the category of fiber bundles and bundle maps—the objects of which are the $r$th order frame bundles $P^r M = \{j^r_0(f) : f : 0 \in U \subset \mathbb{R}^n \to M, T_0 f \in \text{Isom}(\mathbb{R}^n, T_f(0) M)\}$, and the morphisms of which are the principal bundle morphisms $P^r \phi : P^r M \to P^r M'$ defined by $(P^r \phi)(j^r_0(f)) = j^r_0(\phi \circ f)$. The structure group of principal bundle $P^r M$ is $G^r_n = \{j^r_0(\phi) \mid \phi : 0 \in U \subset \mathbb{R}^n \to \mathbb{R}^n, \phi(0) = 0, T_0 \phi \in \text{GL}(n, \mathbb{R})\}$ and its action on $P^r M$, $j^r_0(f)$, $j^r_0(\phi) := j^r_0(f \circ \phi)$, is well-defined in view of the above remark on jets of compound maps. Note that structure group $G^r_n$ of the principal bundle of linear frames $P^1 M = LM$ is $G^1_n \simeq \text{GL}(n, \mathbb{R})$. Remark further that if $j^r_0(\phi) \in G^r_n$ is characterized by $(0^j, A^i_k, S^i_{kl})$ and $j^r_0(f) \in P^2 M$ is characterized in coordinates $X$ around $m := f(0)$ by $(X^i(m), B^i_k, T^i_{kl})_X$, then $j^r_0(f).j^r_0(\phi) = j^r_0(f \circ \phi)$ is characterized by

$$(X^i(m), B^i_k, T^i_{kl})_X \cdot (0^j, A^i_k, S^i_{kl}) = (X^i(m), B^i_k A^j_i, B^i_k S^i_{kl} + T^i_{ab} A^j_k A^l_i)_X$$

(7)

It is easily verified that the isotropy subgroup of $[e_{n+1}] := [(0, \ldots, 0, 1)] \in \mathbb{R} P^n$ for the canonical action of the projective group

$$\text{PGL}(n+1, \mathbb{R}) = \left\{ \begin{pmatrix} A & h \\ \alpha & a \end{pmatrix} : A \in \text{GL}(n, \mathbb{R}), \alpha \in \mathbb{R}^{n \times}, h \in \mathbb{R}^n, a \in \mathbb{R}_0 \right\} / \mathbb{R}_0 \text{id}$$

on the $n$-dimensional real projective space $\mathbb{R} P^n$, is

$$H(n+1, \mathbb{R}) = \left\{ \begin{pmatrix} A & 0 \\ \alpha & a \end{pmatrix} : A \in \text{GL}(n, \mathbb{R}), \alpha \in \mathbb{R}^{n \times}, a \in \mathbb{R}_0 \right\} / \mathbb{R}_0 \text{id},$$

and that $H(n+1, \mathbb{R})$ acts locally on $\mathbb{R}^n$ by affine fractional transformations that preserve the origin. Hence, $H(n+1, \mathbb{R})$ can be viewed as Lie subgroup of structure group $G_n^r$.

**Proposition 1.** The natural inclusion $I : H(n+1, \mathbb{R}) \to G_n^r$ reads $I : \left[ \begin{pmatrix} A & 0 \\ \alpha & 1 \end{pmatrix} \right] \mapsto (0, A^i_j, -A^j_i \alpha_k - A^i_k \alpha_j)$.

**Proof.** The natural action of an element $\left[ \begin{pmatrix} A & 0 \\ \alpha & 1 \end{pmatrix} \right] \in H(n+1, \mathbb{R}) \subset \text{PGL}(n+1, \mathbb{R})$ on $Z \in U \subset \mathbb{R}^n$, where $U$ is a sufficiently small neighborhood of $0$, is $\frac{AZ}{AZ+1} \in \mathbb{R}^n$. A short and easy computation then shows that the second jet at $0$ of map $\varphi : Z \mapsto \frac{AZ}{AZ+1}$ is characterized in canonical coordinates by $(0, A^i_j, -A^j_i \alpha_k - A^i_k \alpha_j)$. \hfill $\Box$

### 2.2 Stage 1: Cartan bundle

A projective structure on a smooth manifold $M$ is a class $[\nabla]$ of all torsion-free linear connections $\nabla'$ on $M$ that are projectively equivalent to $\nabla$, i.e. that have the same geometric geodesics as $\nabla$, or better still, that verify

$$\nabla' X - \nabla X = \alpha(X)Y + \alpha(Y)X,$$

(8)
for all $X, Y \in \text{Vect}(M)$ and some fixed $\alpha \in \Omega^1(M)$ \cite{H. Weyl}.

The next theorem contains the first of two essential observations, see \cite{MR05}, that allow solving the problem of the aforementioned projectively invariant lift $\nabla \rightarrow \tilde{\nabla}$, i.e. that allow associating a unique connection to each projective structure.

**Theorem 1.** Let $M$ be a smooth manifold. There is a canonical 1-to-1 correspondence between projective structures $[\nabla]$ on $M$ and reductions $P = P(M, H(n + 1, \mathbb{R}))$ to structure group $H(n + 1, \mathbb{R})$ of the principal bundle $P^2M = P^2M(M, G^2_n)$ of second order frames on $M$.

In the sequel, we refer to the bundles $P = P(M, H(n + 1, \mathbb{R}))$ as Cartan bundles.

### 2.3 Stage 2: Cartan connection

The second observation then settles the question of connection lift $[\nabla] \rightarrow \tilde{\nabla}$:

**Theorem 2.** A unique normal Cartan connection is associated with every Cartan bundle $P(M, H(n + 1, \mathbb{R}))$ of $M$.

Let $G$ be a Lie group, $H$ a closed subgroup, $\mathfrak{g}$ and $\mathfrak{h}$ the corresponding Lie algebras, and let $P = P(M, H)$ denote a principal $H$-bundle over a manifold $M$, such that $\dim M = \dim G/H$. In this setting, a Cartan connection on $P(M, H)$ is a differential 1-form $\omega \in \Omega^1(P) \otimes \mathfrak{g}$ valued (not in Lie algebra $\mathfrak{h}$, but) in Lie algebra $\mathfrak{g}$, which verifies the usual requirements for connection 1-forms, i.e.

$$\tau^*_s \omega = \text{Ad}(s^{-1})\omega \quad \text{and} \quad \omega(X^h) = h,$$

where $\tau_s$ denotes the right action by $s \in H$ and where $X^h$ is the fundamental vector field associated with $h \in \mathfrak{h}$. However, a third condition asks that $\omega_u : T_u P \rightarrow \mathfrak{g}$ be a vector space isomorphism for any $u \in P$. Hence, we have $\ker \omega_u = 0$, so that the basic difference with Ehresmann connections is the absence of a horizontal subbundle. For instance, if $H$ is a closed subgroup of a Lie group $G$, the canonical Maurer-Cartan form is a Cartan connection on the principal bundle $G(G/H, H)$.

### 2.4 Stage 3: Lifts of symbols and functions

In view of the preceding remarks, the role of connection lift $\tilde{\nabla}$ to bundle $\tilde{M}$, see Equation \eqref{equation:lift}, is played by the unique Cartan connection $\omega$ associated with the unique Cartan bundle $P = P(M, H)$, $H = H(n + 1, \mathbb{R})$, defined by the considered projective structure $[\nabla]$ on $M$. Lifting symbols $s \in \Gamma(S^2TM)$ and in particular functions $f \in C^\infty(M)$ to objects $\tilde{s}$ and $\tilde{f}$ of $\tilde{M} \simeq P$, is then quite obvious. Indeed, we have $\Gamma(S^kTM) = C^\infty(P^1M, S^k\mathbb{R}^n)_{\text{GL}(n, \mathbb{R})}$, where the RHS denotes the space of $\text{GL}(n, \mathbb{R})$-invariant $S^k\mathbb{R}^n$-valued functions of the linear frame bundle $P^1M$. Since, there are canonical projections $P \subset P^2M \rightarrow P^1M$ and $H \subset G^2_n \rightarrow G^1_n = \text{GL}(n, \mathbb{R})$, it is easily seen that

$$\Gamma(S^kTM) = C^\infty(P^1M, S^k\mathbb{R}^n)_{\text{GL}(n, \mathbb{R})} \simeq C^\infty(P, S^k\mathbb{R}^n)_H,$$

where the $H$-action on $S^k\mathbb{R}^n$ is induced by the corresponding $\text{GL}(n, \mathbb{R})$-action.

### 2.5 Stage 4: Construction of a natural and invariant quantization

Equations \eqref{equation:Q} and \eqref{equation:Omega} suggest defining a natural and projectively invariant quantization on a smooth manifold $M$, endowed with a projective structure $[\nabla] \simeq P$, by

$$(Q[\nabla](s)(f)) = Q_{\text{aff}}(\omega)(\tilde{s})(\tilde{f}) = i_\tilde{s}(\nabla^\omega)^k \tilde{f},$$

where $\nabla^\omega$ denotes a covariant derivative associated with connection 1-form $\omega$. Whereas $\nabla^\omega$ can easily be defined, it turns out that the RHS of Equation \eqref{equation:Newton} is a function of $P$ that is not $H$-invariant, so that it does not project onto a function $Q[\nabla](s)(f)$ of $M$, see Equation \eqref{equation:lift}, set $k = 0$, and note that $\sim$ is just the isomorphism $\simeq$. The solution consists in the substitution to $\tilde{s} \in C^\infty(P, S^k\mathbb{R}^n)_H$
of a linear combination of lower degree terms. These are obtained from tensor field \( s \) by means of a degree-lowering divergence operator \( \text{Div}^s = \sum_{i} i_{\omega_i} \nabla \omega_i \), where \((e_i)_j \) and \((\epsilon^i)_j \) are the canonical bases of \( \mathbb{R}^n \) and \( \mathbb{R}^{n^*} \) respectively. Eventually, it can be proven, see [MR05], that

\[
(Q[\nabla](s))(f) = \sum_{k} c_k \ell \cdot i_{(\text{Div}^s)^{k-\ell}}(\nabla^s)^{k-\ell} \tilde{f}
\]
defines a natural and projectively invariant quantization on \( M \), if the coefficients \( c_k \ell \in \mathbb{R} \) have some precise values.

In the following, we study extensions of the just detailed modus operandi to the adapted and foliated geometries associated with foliated manifolds.

## 3 Adapted and foliated projective structures

In this section, we investigate the link between adapted (resp. foliated) projective structures and reductions of the principal bundle of adapted (resp. foliated) second order frames.

### 3.1 Adapted and foliated connections

Let \((M, \mathcal{F})\) be a foliated manifold, more precisely, let \(M\) be an \( n \)-dimensional smooth manifold endowed with a regular foliation \( \mathcal{F} \) of dimension \( p \) (and codimension \( q = n - p \)). It is well-known that such a foliation can be defined as an involutive subbundle \( T \mathcal{F} \subset TM \) of constant rank \( p \).

Foliation \( \mathcal{F} \) can also be viewed as a partition into (maximal integral) \( p \)-dimensional smooth submanifolds or leaves, such that in appropriate or adapted charts \((U_i, \phi_i)\) the connected components of the traces on \( U_i \) of these leaves lie in \( M \) as \( \mathbb{R}^n \) [pages of a book], with transition diffeomorphisms of type \( \psi_{ij} = \phi_j \circ \phi_i^{-1} : \phi_i(U_{ij}) \ni (x, y) \mapsto (\psi_{ji,1}(x, y), \psi_{ji,2}(y)) \in \phi_j(U_{ij}), U_{ij} = U_i \cap U_j \) [the \( \psi_{ij} \) map a page onto a page]. The pages provide by transport to manifold \( M \) the so-called plaques or slices and these glue together from chart to chart—in the way specified by the transition diffeomorphisms—to give maximal connected injectively immersed submanifolds, precisely the leaves of the foliation.

Eventually, foliation \( \mathcal{F} \) can be described by means of a Haefliger cocycle \( \mathcal{U} = (U_{ij}, f_{ij}, g_{ij}) \) modelled on a \( q \)-dimensional smooth manifold \( N_0 \). The \( U_i \) form an open cover of \( M \) and the \( f_{ij} : U_i \rightarrow f_i(U_i) =: N_i \subset N_0 \) are submersions that have connected fibers [the connected components of the traces on the \( U_i \) of the leaves of \( \mathcal{F} \)] and are subject to the transition conditions \( g_{ij} f_i = f_j \), where the \( g_{ij} : f_i(U_{ij}) =: N_{ij} \rightarrow N_{ij} := f_j(U_{ij}) \) are diffeomorphisms that verify the usual cocycle condition \( g_{ij} g_{jk} = g_{ik} \). We refer to the disjoint union \( N = \bigsqcup \mathcal{N}_i \) as the (smooth, \( q \)-dimensional) transversal manifold and to \( \mathcal{H} := g_{ij} \) as the pseudogroup of (locally defined) diffeomorphisms or holonomy pseudogroup associated with the chosen cocycle \( \mathcal{U} \).

A vector field \( X \in \text{Vect}(M) \), such that \([X, Y] \in \Gamma(T\mathcal{F})\), for all \( Y \in \Gamma(T\mathcal{F}) \), is said to be adapted (to the foliation). The space \( \text{Vect}_\mathcal{F}(M) \) of adapted vector fields is obviously a Lie subalgebra of the Lie algebra \( \text{Vect}(M) \), and the space \( \Gamma(T\mathcal{F}) \) of tangent (to the foliation) vector fields is an ideal of \( \text{Vect}_\mathcal{F}(M) \). The quotient algebra \( \text{Vect}(M, \mathcal{F}) = \text{Vect}_\mathcal{F}(M)/\Gamma(T\mathcal{F}) \) is the algebra of foliated vector fields.

Let \((x, y)\) be local coordinates of \( M \) that are adapted to \( \mathcal{F} \), i.e. \( x = (x^1, \ldots, x^p) \) are leaf coordinates and \( y = (y^1, \ldots, y^q) \) are transverse coordinates. The local form of an arbitrary (resp. tangent, adapted, foliated) vector field is then \( X = \sum_{i=1}^{p} X^i(x, y) \partial_{x^i} + \sum_{i=1}^{q} X^i(y) \partial_{y^i} \) (resp. \( X = \sum_{i=1}^{p} X^i(x, y) \partial_{x^i} \)).

\[
X = \sum_{i=1}^{p} X^i(x, y) \partial_{x^i} + \sum_{i=1}^{q} X^i(y) \partial_{y^i}, \quad [X] = \sum_{i=1}^{q} X^i(y) \partial_{y^i} \tag{11}
\]

\[
[X] = [\sum_{i=1}^{q} X^i(y) \partial_{y^i}]. \tag{12}
\]
where \([\cdot]\) denotes the classes in the aforementioned quotient algebra).

In Foliation Theory, vocabulary is by no means uniform. Let us stress that adapted and foliated vector fields, see Equations (11) and (12), may be viewed as prototypes of all adapted and foliated structures used in this paper.

For instance, a smooth function \(f \in C^\infty(M)\) is foliated (or basic) if and only if \(L_Y f = 0, \forall Y \in \Gamma(TF)\). We denote by \(C^\infty(M,F)\) the space of all foliated functions of \((M,F)\). A differential k-form \(\omega \in \Omega^k(M)\) is foliated (or basic) if and only if \(i_Y \omega = i_Y \omega = 0, \forall Y \in \Gamma(TF)\), where notations are self-explaining. Again, we denote by \(\Omega^k(M,F)\) the space of all foliated differential k-forms of \((M,F)\).

It is easily checked that \(C^\infty(M,F) \times \text{Vect}(M,F) \ni (f,[X]) \mapsto [fX] := [fX] \in \text{Vect}(M,F)\) defines a \(C^\infty(M,F)\)-module structure on \(\text{Vect}(M,F)\). Furthermore, \(\text{Vect}(M,F) \times C^\infty(M,F) \ni ([X], f) \mapsto L_X f \in C^\infty(M,F)\) is the natural action of foliated vector fields on foliated functions. Eventually, the contraction of a foliated 1-form \(\alpha \in \Omega^1(M,F)\) and a foliated vector field \([X] \in \text{Vect}(M,F)\) is a foliated function \(\alpha([X]) := \alpha(X) \in C^\infty(M,F)\).

**Definition 1.** Let \((M,F)\) be a foliated manifold. An adapted connection \(\nabla_F\) is a linear torsion-free connection on \(M\), such that \(\nabla_F : \text{Vect}(M,F) \times \Gamma(TF) \rightarrow \Gamma(TF)\) and \(\nabla_F : \text{Vect}(M,F) \times \text{Vect}(M,F) \rightarrow \text{Vect}(M,F)\).

**Remark** In the following, we use the Einstein summation convention, and, as already adumbrated above, Latin indices \(i, k, l, \ldots\) (resp. Greek indices \(\iota, \kappa, \lambda, \ldots\), German indices \(i, \tau, l, \ldots\)) are systematically and implicitly assumed to vary in \(\{1, \ldots, n\}\) (resp. \(\{1, \ldots, p\}, \{1, \ldots, q\}\)).

As torsionlessness means that \(\nabla_{F,Y}X = \nabla_{F,X}Y + [Y,X]\), it follows that \(\nabla_F : \Gamma(TF) \times \text{Vect}(M,F) \rightarrow \Gamma(TF)\).

Further, locally, in adapted coordinates, we have \(\nabla_{F,X}Y = (X^i \partial_i Y^k + \Gamma^k_{il} X^i Y^l) \partial_k\), so that condition \(\nabla_F : \text{Vect}(M,F) \times \Gamma(TF) \rightarrow \Gamma(TF)\) means that

\[
\Gamma^k_{\iota \lambda} = \Gamma^k_{\lambda \iota} = 0,
\]

whereas condition \(\nabla_F : \text{Vect}(M,F) \times \text{Vect}(M,F) \rightarrow \text{Vect}(M,F)\) is then automatically verified provided that Christoffel’s symbols \(\Gamma^k_{il}\) are independent of \(x\), \(\Gamma^k_{il} = \Gamma^k_{il}(y)\).

**Definition 2.** Consider a foliated manifold \((M,F)\). A foliated torsion-free connection \(\nabla(F)\) on \((M,F)\) is a bilinear map \(\nabla(F) : \text{Vect}(M,F) \times \text{Vect}(M,F) \rightarrow \text{Vect}(M,F)\), such that, for all \(f \in C^\infty(M,F)\) and all \([X],[Y] \in \text{Vect}(M,F)\), the following conditions hold true:

- \(\nabla(F)_{[X]} [Y] = f \nabla(F)_{X} [Y]\),
- \(\nabla(F)_{[X]} (f[Y]) = (L_{[X]} f)[Y] + f \nabla(F)_{X} [Y]\),
- \(\nabla(F)_{[X]} Y = \nabla(F)_{Y} [X] + [[X],[Y]]\).

In view of the above definitions, the local form (in adapted coordinates \((x,y)\)) of a foliated vector field is \([X] = X^i [\partial_i]\), \(X^i = X^i(y)\), and a foliated connection reads

\[
\nabla(F)_{[X]} [Y] = X^i \left( L_{[\partial_i]} Y^l \right) [\partial_i] + X^i Y^l \Gamma(F)_{il} [\partial_i], \quad \Gamma(F)_{il} = \Gamma(F)_{il}(y).
\]

**Proposition 2.** If two adapted connections \(\nabla_F\) and \(\nabla'_F\) of a foliated manifold \((M,F)\) are projectively equivalent, the corresponding differential 1-form \(\alpha \in \Omega^1(M)\) is foliated, i.e. \(\alpha \in \Omega^1(M,F)\).

**Proof.** In adapted local coordinates \((x,y)\), projective equivalence of \(\nabla_F\) and \(\nabla'_F\) reads \((\Gamma^k_{il} - \Gamma^k_{il}') X^i Y^l = \alpha_i X^i Y^k + \alpha_k Y^i X^k, \forall k\). When writing this equation for \(X^i = \delta^i_\iota, Y^i = \delta^i_\iota\), and \(k = 1\), we get, in view of Equation (13), \(\alpha_i = \theta\). If we now choose \(X^i = \delta^i_\iota, Y^i = \delta^i_\iota\), and \(k = 1\), we finally see that \(\alpha_i\) is independent of \(x\). □

The following proposition is well-known:
Definition 3. Two foliated connections $\nabla(F)$ and $\nabla'(F)$ of a foliated manifold $(M,F)$ are projectively equivalent, if and only if there is a foliated 1-form $\alpha \in \Omega^1(M,F)$, such that, for all $[X],[Y] \in \text{Vect}(M,F)$, one has $\nabla'(F)[X][Y] - \nabla(F)[X][Y] = \alpha([X])[Y] + \alpha([Y])[X]$.

Eventually, adapted connections induce foliated connections.

Proposition 3. Let $(M,F)$ be a foliated manifold of codimension $q$. Any adapted connection $\nabla_F$ of $M$ induces a foliated connection $\nabla(F)$, defined by $\nabla(F)[X][Y] := [\nabla_F,X]Y$. In adapted coordinates, Christoffel’s symbols $\Gamma(F)^i_{jk}$ of the connection $\nabla(F)$ coincide with the corresponding Christoffel symbols $\Gamma^i_{jk}$ of $\nabla_F$. Eventually, projective classes of adapted connections induce projective classes of foliated connections.

Proof. It immediately follows from the definition of adapted connections that for any $[X],[Y] \in \text{Vect}(M,F)$, the class $\nabla(F)[X][Y] := [\nabla_F,X]Y$ is well-defined. All properties of foliated connections are obviously satisfied. If $(x,y)$ are adapted coordinates, we have $\Gamma(F)^0_0[\partial_k] = \nabla(F)[\partial_0][\partial_k] = [\nabla_F,\partial_0]\partial_k = [\Gamma^0_{x,0},\partial_k] = \Gamma^0_{x,0}[\partial_k]$, since $\Gamma^x_{x,0} = 0$ and $\Gamma^x_{x,0} = \Gamma^x_{x,0}(y)$. The remark on projective structures follows immediately from preceding observations.

3.2 Adapted and foliated frame bundles

3.2.1 Adapted frame bundles

Since an adapted linear frame is a frame $(v_1, \ldots, v_{p+1})$ of a fiber $T_mM$, $m \in M$, the first vectors $(v_1, \ldots, v_p)$ of which form a frame of $T_mF$, we denote by $P^*_F M$ the principal bundle $P^*_F M = \{j^*_0(f) \mid f : 0 \in U \subset \mathbb{R}^n \rightarrow M, T_0f \in \text{Isom}(\mathbb{R}^n, T_j(0)M), Tf(TF_0) = TF\}$, where $F_0$ is the canonical regular $p$-dimensional foliation of $\mathbb{R}^n$. The structure group of $P^*_F M$ is $G^r_{n,F_0} = \{j^*_0(\phi) \mid \phi : 0 \in U \subset \mathbb{R}^n \rightarrow \mathbb{R}^n, \phi(0) = 0, T_0\phi \in \text{GL}(n,\mathbb{R}), T_0 \phi(TF_0) = TF_0\}$, its action on $P^*_F M$ is canonical. We call $P^*_F M$ the principal bundle of adapted $r$-frames on $M$. For instance, $P^*_1 M = LxM$ is the bundle of adapted linear frames of $M$ with structure group

$$G^1_{n,F_0} \simeq \text{GL}(n,q,\mathbb{R}) = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} : A \in \text{GL}(p,\mathbb{R}), B \in \text{gl}(p \times q,\mathbb{R}), D \in \text{GL}(q,\mathbb{R}) \right\}. \quad (14)$$

Of course, the isotropy subgroup of $[e_{n+1}]$ for the natural action of

$$\text{PGL}(n+1, q+1, \mathbb{R}) = \left\{ \begin{pmatrix} A & B & h' \\ 0 & D & h'' \\ 0 & \alpha'' & a \end{pmatrix} : A \in \text{GL}(p,\mathbb{R}), B \in \text{gl}(p \times q,\mathbb{R}), \right\} \quad (15)$$

on $\mathbb{R}P^n$ is

$$H(n+1, q+1, \mathbb{R}) = \left\{ \begin{pmatrix} A & B & 0 \\ 0 & D & 0 \\ 0 & \alpha'' & a \end{pmatrix} : A \in \text{GL}(p,\mathbb{R}), B \in \text{gl}(p \times q,\mathbb{R}), \right\} \quad (16)$$

$D \in \text{GL}(q,\mathbb{R}), \alpha'' \in \mathbb{R}^{q*}, a \in \mathbb{R}_0 \} / \mathbb{R}_0 \text{id}$

Proposition 4. Inclusion $I : H(n+1, \mathbb{R}) \rightarrow G^2_n$ of Proposition restricts to an inclusion $I_{F_0} : H(n+1, q+1, \mathbb{R}) \rightarrow G^2_{n,F_0}$.

Proof. It follows from the proof of Proposition that the representative matrix of the tangent map at $0$ of the smooth map $\phi$ induced by an element of $H(n+1, q+1, \mathbb{R})$ is $A = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$. Hence the conclusion.

We are now prepared to word the adapted version of Theorem.
Projective classes of adapted connections

Theorem 3. For any foliated manifold \((M, F)\), there exists a canonical injection from the set of projective classes of adapted \(\nabla\) into the set of reductions \(P_{\alpha}\) of the principal bundle \(P_{F}^{2} M\) of \(F\)-adapted second order frames on \(M\) to structure group \(\mathcal{G}(n + q, q + 1, \mathbb{R}) \subset G_{n,F}^{2}\).

Proof. The proof consists of three stages.

1. Let \(\nabla\) be an adapted connection of a foliated manifold \((M, F)\). We will define the reduction \(P_{\alpha}\) of \(P_{F}^{2} M\) to \(H := \mathcal{G}(n + q, q + 1, \mathbb{R})\) by means of local sections \(\sigma_{\alpha}\) of \(P_{F}^{2} M\) over open domains \(W_{\alpha} \subset M\) of adapted coordinates \(X_{\alpha} = (x_{\alpha}, y_{\alpha})\) that form a cover \((W_{\alpha})\) of \(M\). Of course, the fiber \(P_{\alpha,F,M}\) of \(P_{\alpha}\) at \(m \in W_{\alpha}\) is then \(\sigma_{\alpha}(m) \cdot H\), where \(\cdot\) denotes the action of \(G_{n,F}^{2}\) on \(P_{F}^{2} M\). The reduction \(P_{\alpha,F,M}\) is well-defined if and only if the corresponding cocycle \(\sigma_{\alpha}: W_{\alpha} \cap W_{\beta} \rightarrow G_{n,F}^{2}\), which links the local sections, is \(\sigma_{\alpha} = \sigma_{\beta} \cdot \sigma_{\alpha,\beta}\), is valued in \(H\).

2. We now prove that the just constructed reduction \(P_{\alpha}\) of \(P_{F}^{2} M\) to \(H\) does not depend on the considered adapted connection \(\nabla\), but only on the projective class \([\nabla]\) of this connection. If \(\nabla\) is a projectively equivalent adapted connection, and if we set \(\partial X_{\alpha} = \partial_{i}\), Equation (8) entails that \([\nabla]\) defines a reduction \(P_{\alpha}\) of \(P_{F}^{2} M\) to \(H\).

3. If the images \(P_{\alpha}[\nabla]\) and \(P_{\alpha}[\nabla']\) coincide, their fibers over any domain \(W_{\alpha}\) of adapted coordinates \(X_{\alpha}\) coincide. In particular, for any \(m \in W_{\alpha}\), there is a unique \(h_{\alpha}(m) \in H\), such that \(\sigma_{\alpha}(m) = \sigma_{\alpha}(m) \cdot h_{\alpha}(m)\). Hence, \(h_{\alpha}(m) = (0, A_{\alpha,i}^{j}(m), -A_{\alpha,k}^{i}(m) \cdot \xi_{\alpha,j}(m) - A_{\alpha,i}^{j}(m) \cdot \xi_{\alpha,k}(m))\), see Proposition 4 where \(\xi_{\alpha,i}(m) = 0\), see Proposition 4. It easily follows that \(A_{\alpha,k}^{i}(m) = \delta_{k}^{i}\) and that \(\Gamma_{\alpha,k}^{i}(m) = \delta_{k}^{i} \xi_{\alpha,i}(m) + \delta_{i}^{j} \xi_{\alpha,k}(m)\). Thus, for any \(X, Y \in \text{Vect}(M)\), we have on \(W_{\alpha}\), \(\nabla_{X,Y} - \nabla_{X,Y} = \xi_{\alpha}(X) Y + \xi_{\alpha}(Y) X\). Hence, \(\xi_{\alpha} = \xi_{\beta}\) on \(W_{\alpha,\beta}\) and the \(\xi_{\alpha} \in C^\infty(W_{\alpha}, \mathbb{R}^{n})\) define a unique differential 1-form \(\xi_{\alpha} \in \Omega^{1}(M)\). Eventually, we get \([\nabla] = [\nabla']\).
3.2.2 Foliated frame bundles

We next prove existence of a similar injection from projective classes of foliated connections into reductions of the “foliated” second order frame bundle.

Consider a foliated manifold \((M, \mathcal{F})\) and let \(\mathcal{U} = (U_i, f_i, g_{ij})\) be a Hæfliger cocycle of \(\mathcal{F}\) with associated transverse manifold \(N\). As \(f_i\) is a submersion the fibers (preimages) of which are parts of the leaves of \(\mathcal{F}\), the kernel of \(T_m f_i : T_m U_i \to T_{f_i(m)} N_i, m \in U_i\), is \(\ker T_m f_i = T_m \mathcal{F}\), and \(N_m f_i : N_m U_i, f_i) := T_m M/T_m \mathcal{F} \ni v \mapsto (T_m f_i)(v) \in T_{f_i(m)} N = N_{f_i(m)} (N_i, 0)\) is a vector space isomorphism. Of course, \((N_i, 0)\) denotes the manifold \(N_i\) endowed with its canonical foliation by points. Actually, the normal functor \(N\) is a functor between the category \(\mathcal{FM}_q\) of codimension \(q\) foliated manifolds and smooth maps that preserve the foliations, on one hand, and the category \(\mathcal{FB}\) of foliated fiber bundles, i.e., fiber bundles whose total space is foliated by a foliation whose leaves are covering space of leaves on the base space and bundle maps, on the other (see [Wol89]). If confusion with the transverse manifold \(N\) is excluded, most authors denote the normal bundle \(N(M, \mathcal{F})\) simply by \(N\). Observe also that \(N f_i\) is just the tangent map \(T f_i\) viewed as map between normal bundles.

We now define the principal bundle \(P^r(M, \mathcal{F})\), \(r \in \mathbb{N}_0\), of normal \(r\)th order frames associated with any object \((M, \mathcal{F}) \in \text{Ob}(\mathcal{FM}_q)\). Remark first that each vector space isomorphism \(\mathbb{R}^q \to N_m = N_m(M, \mathcal{F}), m \in M\), implements a normal linear frame \((n_1, \ldots, n_q) \in N_m^q\). In order to obtain such isomorphisms, we consider the jets of transverse smooth maps \(f : 0 \in V \subset \mathbb{R}^q \to M\), such that \(\text{im} T f \cap T \mathcal{F} = T M\). Indeed, then \(N_v f : \mathbb{R}^q \ni v \to [(T z f)(v)] \in N_{f(v)}^v, z'' \in V\), is a vector space isomorphism. Hence, the set \(P^r(M, \mathcal{F})\) (the better notation \(P^r N(M, \mathcal{F})\) is not prevailing) of normal \(r\)-frames is defined by \(P^r(M, \mathcal{F}) = \{J^r_0(f)\} : 0 \in V \subset \mathbb{R}^q \to M, \text{im} T f \cap T \mathcal{F} = T M\}\). The \(r\)-jet \(J^r_0(f)\) at 0 of a transverse function \(f\) is the equivalence class of \(f\) for the following relation: two transverse functions \(f\) and \(g\) that map a neighborhood \(V \subset \mathbb{R}^q\) of 0 into \(M\) are equivalent if and only if \(f(0) = g(0) := m\) and, for any submersion \(\mathcal{X} : m \in W \subset M \to \mathbb{R}^q\) that is constant along the leaves of \(\mathcal{F}\), the components of the maps \(\hat{\mathcal{X}} := \mathcal{X} \circ f\) and \(\mathcal{G} := \mathcal{X} \circ g\) have the same partial derivatives at 0 up to order \(r\). Of course, it suffices that this condition be satisfied for one submersion. If \(\mathcal{X} = (x, y)\) is a system of adapted coordinates of \(M\) around \(m\), we can choose \(\mathcal{X} = y\). It is helpful to observe that (just as \(T f_i\), see above) \(T \mathcal{X}\) is a pointwise isomorphism of vector spaces from \(N\) onto \(\mathbb{R}^q\), so that \(T \mathcal{X} = T \mathcal{X} \circ T f\). The just defined space \(P^r(M, \mathcal{F})\) is a principal bundle over \(M\) with structure group \(G^r\) and projection \(\pi^r : J^r_0(f) \mapsto f(0)\). The right action is given by \(J^r_0(f) \cdot J^r_0(\varphi) = J^r_0(f \circ \varphi)\). Bundle \(P^1(M, \mathcal{F}) := L(M, \mathcal{F})\) for instance, is the principal bundle of normal linear frames. Just as \(N\) (see above), \(P^r\) (or better \(P^r N\)) is a functor between the categories \(\mathcal{FM}_q\) and \(\mathcal{FB}\). Let us mention that both functors are (prototypes of) foliated natural functors in the sense of [Wol89].

**Theorem 4.** For any foliated manifold \((M, \mathcal{F})\) of codimension \(q\), there exists an injection from the set of projective classes of foliated connections \([\nabla(\mathcal{F})]\) into the set of reductions \(P(\mathcal{F})\) of \(P^2(M, \mathcal{F})\) to structure group \(H(q + 1, \mathbb{R}) \subset G^2\).

**Proof.** The proof of this theorem is similar to that of Theorem 3. Hence, we put down only a sketch of this proof.

If \(\nabla(\mathcal{F})\) is a foliated connection of a foliated manifold \((M, \mathcal{F})\), the reduction \(P(\mathcal{F})\) of \(P^2(M, \mathcal{F})\) to \(H := H(q + 1, \mathbb{R})\) is defined, over an open domain \(W_\alpha \subset M\) of adapted coordinates \(x_\alpha = (x_\alpha, y_\alpha)\), by a local section

\[
\sigma_\alpha : W_\alpha \ni m \mapsto (y_\alpha(m), \partial_i, -\Gamma^i_{\alpha;it}(m))_{x_\alpha} \in P^2_m(M, \mathcal{F}),
\]

where the \(\Gamma^i_{\alpha;it} \in C^\infty(W_\alpha)\) are Christoffel’s symbols of \(\nabla(\mathcal{F})\). A similar argument than in the adapted case, again allows checking that the cocycle \(\sigma_{\beta \alpha}\), which links \(\sigma_\alpha\) and \(\sigma_\beta\), is valued in subgroup \(H\). Also invariance of the reduction for a change of foliated connection within the same projective class, as well as injectivity of the just defined mapping between projective classes and reductions, can be verified as above.

3.2.3 Projections

It is a well-known fact (see above, adapted and foliated vector fields, adapted and foliated connections) that adapted objects induce (usually) foliated objects. In this subsection, we describe canonical
projections from an adapted frame bundle $P_{\mathcal{F}} M$ (resp. adapted Cartan bundle $P_{\mathcal{F}}$) onto the corresponding foliated frame bundle $P^r(M, \mathcal{F})$ (resp. foliated Cartan bundle $P(\mathcal{F})$).

We denote by $p^r_{\mathcal{F}}$ (resp. $p^r(\mathcal{F})$), $r \geq 1$, the canonical projection $p^r_{\mathcal{F}} : P^r_{\mathcal{F}} M \ni j_0^r(f) \mapsto j_0^{r-1}(f) \in P^{r-1}_{\mathcal{F}} M$ (resp. $p^r(\mathcal{F}) : P^r(M, \mathcal{F}) \ni j_0^r(f) \mapsto j_0^{r-1}(f) \in P^{r-1}(M, \mathcal{F})$). Furthermore, if $f : 0 \in \mathbb{R}^n \to M$, $\alpha_0 f \in \text{Isom}(\mathbb{R}^n, T_{f(0)} M)$, $T_f(T_{\mathcal{F}0}) = T \mathcal{F}$, and if $i_q : \mathbb{R}^q \ni z'' \mapsto (0, z'') \in \mathbb{R}^n$, then, obviously, $f \circ i_q : 0 \in V \subset \mathbb{R}^n \to M$, $\text{im}(T(f \circ i_q) \oplus T \mathcal{F}) = TM$. Since two foliation preserving locally defined diffeomorphisms $f$ and $g$ that have the same jets $j_0^r(f) = j_0^r(g)$, induce two transverse maps $f \circ i_q$ and $g \circ i_q$, such that $J_0^r(f \circ i_q) = J_0^r(g \circ i_q)$, there is a canonical projection $p^r_{\mathcal{F}} : P^r_{\mathcal{F}} M \ni j_0^r(f) \mapsto j_0^r(f \circ i_q) \in P^r(M, \mathcal{F})$. Observe that if $Z = (z', z'') \in \mathbb{R}^p \times \mathbb{R}^q = \mathbb{R}^n$ (resp. $X = (x, y)$) are adapted coordinates in $(\mathbb{R}^n, \mathcal{F}_0)$ (resp. $(M, \mathcal{F})$ around $f(0)$), jet $j_0^r(f)$ is characterized by the derivatives

$$\partial^r_{\alpha_i}(X^i(f(Z)))(0), |\alpha| \leq r, i \in \{1, \ldots, n\},$$

whereas jet $J_0^r(f \circ i_q)$ is characterized by

$$\partial^r_{\alpha_i}(y^i(f(0, z'')))(0) = \partial^r_{\alpha_i}(y^i(f(Z)))(0, 0), |\alpha| \leq r, i \in \{1, \ldots, q\}.$$

**Proposition 5.** For any foliated manifold $(M, \mathcal{F})$ endowed with an adapted projective structure and the induced foliated projective structure, projection $\pi^r_{\mathcal{F}} : P^r_{\mathcal{F}} M \to P^2(M, \mathcal{F})$ restricts to a projection $\pi^r_{\mathcal{F}} : P_{\mathcal{F}} \to P(\mathcal{F})$, and the diagram

$$\begin{array}{ccc}
P_{\mathcal{F}} & \xrightarrow{\pi^r_{\mathcal{F}}} & L_{\mathcal{F}} M \\
\downarrow & & \downarrow \\
P(\mathcal{F}) & \xrightarrow{\pi^r(\mathcal{F})} & L(M, \mathcal{F})
\end{array}$$

is commutative.

**Proof.** It suffices to prove that $\pi^r_{\mathcal{F}}$ maps $P_{\mathcal{F}}$ into $P(\mathcal{F})$. Consider a point of $P_{\mathcal{F}, m}$, $m \in M$, i.e., in adapted coordinates $X = (x, y)$ around $m$, see Equation (14), a point

$$(X^i(m), \delta^i_k, -\Gamma^i_{jk}(m)X \cdot (0^i, A^i_k, -A^i_k \alpha_l - A^i_l \alpha_k),$$

where the element of $G^2_{n, \mathcal{F}_0}$ is induced by a member

$$\begin{pmatrix} A & 0 \\ \alpha & 1 \end{pmatrix}, A = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}, \alpha = (0, \alpha'').$$

of $H(n + 1, q + 1, \mathbb{R})$. When using Equation (14), the above description of $\pi^r_{\mathcal{F}}$ in terms of packages of derivatives, the local characterization of an adapted connection, see Equation (13), as well as Proposition 3 we see that the considered point of $P_{\mathcal{F}, m}$ is mapped by $\pi^r_{\mathcal{F}}$ to

$$(y^i(m), D^i_k, -D^i_k \alpha_l - D^i_l \alpha_k - \Gamma(\mathcal{F})^i_{jk}(m)D^i_k D^k_l \alpha_k)X = (y^i(m), \delta^i_k, -\Gamma(\mathcal{F})^i_{jk}(m)X \cdot (0^i, D^i_k, -D^i_k \alpha_l - D^i_l \alpha_k').$$

It then follows directly from Equation (18) and Proposition 1 that $\pi^r_{\mathcal{F}}$ is valued in $P(\mathcal{F})$. 

**4 Lift of adapted and foliated symbols**

Below, we study adapted and foliated symbols, as well as their lifts to the Cartan fiber bundles $P_{\mathcal{F}}$ and $P(\mathcal{F})$. Investigations are again similar in both settings. Whereas we detailed above the adapted situation, we describe below especially the foliated case.
4.1 Foliated differential operators and symbols

We have already mentioned, see Subsection 322.2, that $N$, $LN$, and more generally $P^rN$, $r \in \mathbb{N}_0$, are (covariant) foliated natural functors, i.e. (regular) functors $F : \mathcal{FM}_q \to FB$, such that for any morphism $f : (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2)$, morphism $F(f) : F(M_1, \mathcal{F}_1) \to F(M_2, \mathcal{F}_2)$ covers $f$ and is pointwise a diffeomorphism.

For instance, if $(M, \mathcal{F})$ is defined by means of a cocycle $\mathcal{U} = (U_i, f_i, g_{ij})$, submersion $f_i : (U_i, \mathcal{F}) \to (N_i, 0)$ (resp. diffeomorphism $g_{ij} : (N_{ij}, 0) \to (N_{ji}, 0)$) is a morphism of $\mathcal{FM}_q$, and the corresponding morphism $N(f_i)$ (resp. $N(g_{ji})$) is a pointwise isomorphism

$$N_m(f_i) : N_m(U_i, \mathcal{F}) \to N_{f_i(m)}(N_i, 0) = T_{f_{i(m)}}N_i,$$

$m \in U_i$ (resp. $N_m(g_{ji}) = T_m(g_{ji}) : N_m(N_{ij}, 0) = T_mN_{ij} \to N_{g_{ji}(m)}(N_{ji}, 0) = T_{g_{ji}(m)}N_{ji},$)

$m \in N_{ij}$). It is easily checked that $(N(U_i, \mathcal{F}), N(f_i), N(g_{ij}))$, and more generally $(F(U_i, \mathcal{F}), F(f_i), F(g_{ij}))$, is a cocycle that defines a foliation $\mathcal{F}_F$ on the total space $F(M, \mathcal{F})$ (and that $\mathcal{F}_F$ is independent of $\mathcal{U}$). Hence, the name “foliated natural bundle”. Further, it follows from Equation (19) that

$$N(U_i, \mathcal{F}) \simeq f_i^*N(N_i, 0) = f_i^*TN_i. \tag{21}$$

More generally, $F_{ij} = \Pi_i f_i^*F(N_i, 0) / \sim$, where $(i, m', v') \sim (j, m'', v'')$ if and only if $m' = m''$ and $v'' = F(g_{ji})(v')$, is a well-defined fibre bundle over $M$. The projections

$$f_i^*F(N_i, 0) \to F(N_i, 0) \tag{22}$$

define a foliation on $F_{ij}$. It is obvious, see preceding equations, that (the foliated) bundle $F(M, \mathcal{F})$ is isomorphic to (the foliated) bundle $F_{ij}$ and that this isomorphism is foliation preserving.

The “mental picture” of foliation $\mathcal{F}_F$ induced on $F(M, \mathcal{F})$ is clear from Equation (22). In particular, foliation $\mathcal{F}_F$ has the same dimension as foliation $\mathcal{F}$ and its leaves project onto the leaves of $\mathcal{F}$.

Let us also recall (see [Wol89]) that a foliated geometric structure is a foliated subbundle of a foliated natural bundle $F(M, \mathcal{F})$, i.e. a subbundle (in particular a section) the total space of which is saturated for foliation $\mathcal{F}_F$ (“it contains as many leaves as can reasonably be expected”).

**Definition 4.** A foliated differential operator of a foliated manifold $(M, \mathcal{F})$ (where $\mathcal{F}$ is of dimension $p$ and codimension $q$) is an endomorphism $D \in \text{End}_q(C^\infty(M, \mathcal{F}))$ that reads in any system of adapted coordinates $(x, y) = (x^1, \ldots, x^p, y^1, \ldots, y^q)$ over any open subset $U \subset M$,

$$D|_U = \sum_{|\gamma| \leq k} D_\gamma [\partial_{y^1}]^\gamma_1 \cdots [\partial_{y^q}]^\gamma_q,$$

where $k \in \mathbb{N}$ is independent of the considered adapted chart and where the coefficients $D_\gamma \in C^\infty(U, \mathcal{F})$ are locally defined foliated functions. The smallest possible integer $k$ is called the order of operator $D$.

We denote by $D(M, \mathcal{F})$ (resp. $D^k(M, \mathcal{F})$) the space of all foliated differential operators (resp. all foliated differential operators of order $\leq k$). Of course, the usual filtration

$$D(M, \mathcal{F}) = \cup_{k \in \mathbb{N}} D^k(M, \mathcal{F}) \tag{23}$$

holds true.

**Definition 5.** The graded space $S(M, \mathcal{F})$ associated with the filtered space $D(M, \mathcal{F})$,

$$S(M, \mathcal{F}) = \oplus_{k \in \mathbb{N}} S^k(M, \mathcal{F}) = \oplus_{k \in \mathbb{N}} D^k(M, \mathcal{F})/D^{k-1}(M, \mathcal{F}),$$

is the space of foliated symbols.
It is easily checked that the well-known vector space isomorphism between the spaces of symbols of degree $k$ and of symmetric contravariant $k$-tensor fields, extends to the foliated setting,

$$S^k(M, \mathcal{F}) \simeq \Gamma(S^kN(M, \mathcal{F}); \mathcal{F}_{S^kN}) ,$$

where the RHS denotes the space of foliated sections of the foliated natural bundle $S^kN(M, \mathcal{F})$ (see above, foliated geometric structures). Below, we identify these two spaces.

**Theorem 5.** Let $(M, \mathcal{F})$ be a foliated manifold of codimension $q$ endowed with a foliated projective structure $[\nabla(\mathcal{F})]$, and denote by $P(\mathcal{F})$ the corresponding reduction of $P^2(M, \mathcal{F})$ to $H(q + 1, \mathbb{R}) \subset G^2_q$. The following canonical vector space isomorphisms hold:

$$\sim : S^k(M, \mathcal{F}) = \Gamma(S^kN(M, \mathcal{F}); \mathcal{F}_{S^kN}) \ni s \mapsto \hat{s} \in C^{\infty}(LN(M, \mathcal{F}), S^k \mathbb{R}^q; \mathcal{F}_{LN})_{\text{GL}(q, \mathbb{R})} \quad (24)$$

$$\wedge : S^k(M, \mathcal{F}) = \Gamma(S^kN(M, \mathcal{F}); \mathcal{F}_{S^kN}) \ni s \mapsto \hat{s} \ast y^2(\mathcal{F}) \in C^{\infty}(P(\mathcal{F}), S^k \mathbb{R}^q; \mathcal{F}_{P^2N})_{H(q + 1, \mathbb{R})} . \quad (25)$$

**Proof.** 1. Observe first that the foliated natural vector bundle $S^kN(M, \mathcal{F})$, associated with the foliated natural principal bundle $LN(M, \mathcal{F})$ of normal linear frames: $S^kN(M, \mathcal{F}) = LN(M, \mathcal{F}) \times_{\text{GL}(q, \mathbb{R})} S^k \mathbb{R}^q$. Hence, only the foliated aspect of Isomorphism (24) has to be explained. Consider a section $s \in \Gamma(S^kN(M, \mathcal{F}); \mathcal{F}_{S^kN})$ and a normal linear frame $u_m = ([v_1], \ldots, [v_q]) \in LN_m(M, \mathcal{F})$. Isomorphism $\sim$ is of course defined by $\hat{s}(u_m) = (s^{i_1 \ldots i_k}(m)) \in S^k \mathbb{R}^q$, where the RHS is made up by the components of $s_m = \sum_{\ell_1 \leq \ldots \leq k}s^{i_1 \ldots \ell_k}(m)[v_{i_1}] \ldots [v_{i_k}]$ in the induced linear frame of $S^kN_m(M, \mathcal{F})$. Hence, the $\text{GL}(q, \mathbb{R})$-equivariance of $\hat{s}$ is obvious. But this function is also foliated, i.e., locally constant along the leaves of $\mathcal{F}_{LN}$. Indeed, let $(U_i, f_i, g_i)$ be a defining cocycle of $\mathcal{F}$, and let $u_{m'} \in LN_m(M, \mathcal{F})$ be a normal linear frame on the same local leaf of $\mathcal{F}_{LN}$ than $u_m$; the leaves of $\mathcal{F}_{LN}$ are locally defined by the projections

$$f_i^* LN(N_i, 0) = f_i^* LTN_i \to LTN_i . \quad (26)$$

Since section $s$ is foliated and as the local leaves of the corresponding foliation $\mathcal{F}_{S^kN}$ are defined by the projections $f_i^* S^kN(N_i, 0) = f_i^* S^kTN_i \to S^kTN_i$, it is clear that the tensors $s_m$ and $s_{m'}$ have the same components in the frames $u_m$ and $u_{m'}$ respectively, so that $\hat{s}(u_m) = \hat{s}(u_{m'})$. A similar argument shows that to any foliated equivariant function is associated a foliated section.

2. We will show in Point 3 that the spaces of foliated equivariant functions on $LN(M, \mathcal{F})$ and on $P(\mathcal{F})$, see Equations (24) and (25), are isomorphic. This is a foliated variant of a result that has already been proven in [MR05].

Let us recall that the action $\tilde{\rho}$ of $H(q + 1, \mathbb{R}) \simeq \text{GL}(q, \mathbb{R}) \times \mathbb{R}^q$ on $S^k \mathbb{R}^q$ is induced by the action $\rho$ of $\text{GL}(q, \mathbb{R})$ on $S^k \mathbb{R}^q$:

$$\tilde{\rho} \left( \begin{array}{cc} A & 0 \\ \alpha & a \end{array} \right) = \rho(A) . \quad (27)$$

In order to understand that the target space of Equation (25) makes sense, observe that $P(\mathcal{F})$ is a foliated subbundle of the foliated natural bundle $P^2N(M, \mathcal{F})$. Indeed, first it is clear that if $\phi : (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2)$ is a morphism of category $\mathcal{F}M_\mathbb{R}$, then the corresponding morphism $P^rN(\phi)$ of category $\mathcal{F}B$ is defined by $P^rN(\phi) : P^rN(M_1, \mathcal{F}_1) \ni J_0^r(f) \mapsto J_0^r(\phi \circ f) \in P^rN(M_2, \mathcal{F}_2)$; in particular, $P^rN(f_1) : P^rN(U_i, \mathcal{F}) \ni J_0^r(f_1) \mapsto J_0^r(f_1 \circ f) \in P^rTN_i$, with self-explaining notations. Furthermore, since, in adapted coordinates $X = (x, y)$ over an open subset $W \subset M$, we have $\sigma(x, y) = (y; \delta_1^k \Gamma(\mathcal{F})(y)) \in P_{(x, y)}(\mathcal{F})$, see Equation (13), the local section $\sigma$ of $P(\mathcal{F})$ is constant along any local leaf of $\mathcal{F}$ in $W$. Hence, $\sigma$ is valued in a leaf of $\mathcal{F}_{P^2N}$. Eventually, the action by an element $h = j_0^2(\phi) \in H(q + 1, \mathbb{R}) \subset G^2_q$ maps a local leaf of $\mathcal{F}_{P^2N}$ into another local leaf. As a matter of fact, if $u^2 = J_0^2(f)$ and $u^2 = J_0^2(f')$ belong to the same local leaf of $\mathcal{F}_{P^2N}$, we have $j_2^0(f_1 \circ f) = P^2N(f_1)(u^2) = P^2N(f_1)(u^2) = j_2^0(f_1 \circ f')$. But then, $P^2N(f_1)(u^2 \cdot h) = j_2^0(f_1 \circ f \cdot h) = j_2^0(f_1 \circ f' \cdot h) = P^2N(f_1)(u^2 \cdot h)$. Thus, $P(\mathcal{F})$ is actually a foliated subbundle.

3. Observe first that mapping $p : p^2(\mathcal{F}) : J_0^2(f) \in P(\mathcal{F}) \subset P^2N(M, \mathcal{F}) \to J_0^2(f) \in LN(M, \mathcal{F})$ is surjective. Indeed, the fiber $LN_m(M, \mathcal{F})$, $m \in M$, is equivalent to $\text{GL}(q, \mathbb{R})$. On the other hand, in
adapted coordinates $X = (x, y)$ around $m$, the projection of the corresponding fiber $P_m(F)$ of $P(F)$ is made up, see Equation (18), by the elements
\[
p(\sigma(m) \cdot h) = p((y'(m), \delta_1, -\Gamma(\mathcal{F})_1(m))_X \cdot (0^t, \mathcal{A}_1^2, -\mathcal{A}_1^1 \alpha_1 - \mathcal{A}_2^1 \alpha_2)) = (y'(m), \mathcal{A}_1^2)_X,
\]
where $h$ runs through $H(q + 1, \mathbb{R})$, so that $\mathcal{A}$ runs through $GL(q, \mathbb{R})$.

For any $\hat{f} \in C^\infty(LN(M, F), S^k\mathbb{R}^q; F_{LN})_{GL(q, \mathbb{R})}$, we now set $\hat{f} = f \circ \rho \in C^\infty(P(F), S^k\mathbb{R}^q)$. This map $\hat{f}$ is $H(q + 1, \mathbb{R})$-equivariant. Actually, for any $u^2_m \in P_m(F)$, $u^2_m = (y'(m), \mathcal{B}_1^2, T_1^1)_X$, and any $h = (0^t, \mathcal{A}_1^2, -\mathcal{A}_1^1 \alpha_1 - \mathcal{A}_2^1 \alpha_2) \in H(q + 1, \mathbb{R})$, we have
\[
f(p(u^2_m \cdot h)) = f((y'(m), \mathcal{B}_1^2, T_1^1)_X) = f((y'(m), \mathcal{B}_1^2)_X \cdot (0^t, \mathcal{A}_1^2))
\]
\[
= \rho(\mathcal{A}^{-1})(f(p(u^2_m))) = \hat{\rho}(h^{-1})(f(p(u^2_m))),
\]
in view of Equation (27). Eventually, $\hat{f} \in C^\infty(P(F), S^k\mathbb{R}^q; F_{P^2N})_{H(q+1, \mathbb{R})}$. Indeed, if $u^2 = J_0^2(f)$ and $u^2 = J_0^2(f')$ are two points on the same leaf of $F_{P^2N}$ in $P(F)$, see Point 2, then $f(p(u^2)) = f(J_0^2(f)) = f(J_0^2(f')) = f(p(u^2'))$, since $f$ is locally constant along the leaves of $F_{LN}$.

It is clear that $\wedge : f \mapsto \hat{f}$ is a bijection (since $p$ is surjective). Map $\wedge$ is also surjective. Indeed, any function $\varphi \in C^\infty(P(F), S^k\mathbb{R}^q; F_{P^2N})_{H(q+1, \mathbb{R})}$ factors through $LN(M, F)$, i.e. $\varphi = \varphi \circ \rho$. Note first that it follows from Equations (28) and (29) that for any $u^1 \in LN(M, F)$ and any $\mathcal{A} \in GL(q, \mathbb{R})$, there is $u^2 \in P(F)$ and $h \in H(q + 1, \mathbb{R})$, such that $p(u^2) = u^1$, $\mathcal{A}$ is the upper left submatrix of $h$, and $p(u^2 \cdot h) = u^1 \cdot \mathcal{A}$. Hence, $\varphi(u^1 \cdot \mathcal{A}) = \varphi(u^2 \cdot h) = \hat{\rho}(h^{-1}) \varphi(u^2) = \rho(\mathcal{A}^{-1}) \varphi(u^1)$, and $\varphi$ is GL($q, \mathbb{R}$)-equivariant. It is also well-defined, since, if $u^2, u^2' \in P(F)$ project both onto $u^1$, we have $u^2 = u^2 \cdot h, h \in H(q+1, \mathbb{R})$, and $u^1 = u^1 \cdot \mathcal{A}$. Thus, $\mathcal{A} = id, \varphi(u^2 \cdot h) = \varphi(u^2)$, and $\varphi \in C^\infty(LN(M, F), S^k\mathbb{R}^q)_{GL(q, \mathbb{R})}$. In order to prove that $\varphi$ is foliated for $F_{LN}$, observe that, as the leaves of $F_{P^2N}$, $r \in \mathbb{N}_0$, are locally defined as the fibers of submersion
\[
P''N(f_i) : P''N(U_i, F) \ni J_0^2(f) \mapsto j_0^2(f_i \circ f) \in P''TN_i,
\]
the local leaves are made up by the jets $J_0^2(f) \neq J_0^2(f')$ of those transverse functions $f$ and $f'$, the last $q$ adapted coordinates of which have the same derivatives at 0 up to order $r$, but that map 0 to $m := f(0) \neq f'(0) =: m'$ on the same local leaf of $F$ in $U_i$, see definitions of the jets $J_0^2$ and $j_0^2$.

Equation (28) then entails that two jets $u^2 = J_0^2(f)$ and $u^2 = J_0^2(f')$ on the same leaf of $F_{LN}$ are the projections of two jets $u^2$ and $u^2$ of $P(F)$ on the same leaf of $F_{P^2N}$. Hence, $\varphi$ is foliated.

It is interesting to observe that (the mental picture associated with) foliation $F_{LN}$ is of course the same, irrespective of the fact it is defined by Equation (20) or by Equation (30).

### 4.2 Adapted differential operators and symbols

As aforementioned, in order to limit the length of this paper, we confine ourselves in the adapted case to a description of the main points. Hence, we refrain for instance to give a general description of adapted natural functors, see Winkler, but provide examples of such functors.

Let $(M, \mathcal{F})$ be a manifold equipped with a dimension $p$ and codimension $q$. Denote by $(U_i, f_i, g_{ij})$ a cocycle of $\mathcal{F}$. Then, for any $r \in \mathbb{N}_0$, $P''U_i := P''U_i$,
\[
P''(f_i) : P''U_i \ni J_0^2(f_i) \mapsto j_0^2(f_i \circ f \circ i_q) \in P''N_i \ni P''N_i = P''N_i,
\]
and $P''(g_{ij}) : P''N_{j} \ni J_0^2(g_{ij}) \mapsto j_0^2(g_{ij} \circ g) \in P''N_{ij}$, form a cocycle that defines a foliation $F_{P''}$ on $P''M$.

Consider now an adapted atlas of $(M, \mathcal{F})$ and take an adapted chart $(U, \phi), \phi = (\phi_1, \phi_2) : U \rightarrow \mathbb{R}^p \times \mathbb{R}^q$. The map
\[
\phi_2^* : L\mathcal{F}U \ni (v_1, \ldots, v_p, v_{p+1}, \ldots, v_{p+q}) \mapsto (T\phi_2(v_{p+1}), \ldots, T\phi_2(v_{p+q})) \in L\mathbb{R}^q
\]
is a submersion that defines a foliation $F_{L}$ on $L\mathcal{F}M$. It is clear that foliation $F_{P''}$, defined by Equation (31), and foliation $F_{L}$, defined by Equation (32), coincide. Moreover, the leaves of $F_{P''} = F_{L}$ project onto the leaves of $\mathcal{F}$ (since $P''(f_i)$ and $\phi_2^*$ are bundle maps over $f_i$ and $\phi_2$, respectively) and the
dimension of foliation $\mathcal{F}_{L}$ is $p + np$ (no restrictions imposed, neither on $v_{1}, \ldots, v_{p} \in T\mathcal{F}$, nor on the tangential parts of $v_{p+1}, \ldots, v_{p+q}$). The “mental picture” of $\mathcal{F}_{L}$ follows.

Foliation $\mathcal{F}$ similarly induces a foliation $\mathcal{F}_{T}$ on $TM$ (defined for instance by means of $T\phi_{2} : TU \to \mathbb{R}^{q}$). Observe that adapted vector fields of $M$, see Equation $11$, coincide with sections of $TM$ that are foliated for $\mathcal{F}_{T}$: $\text{Vect}_{\mathcal{F}}(M) = \Gamma(TM; \mathcal{F}_{T})$.

Let us also mention that adapted functions are just foliated functions: $C_{\mathcal{F}}^{\infty}(M) := C^{\infty}(M, \mathcal{F})$.

**Definition 6.** An adapted differential operator of a foliated manifold $(M, F)$ (where $F$ is of dimension $p$ and codimension $q$) is an endomorphism $D \in \text{End}_{\mathbb{R}}(C^{\infty}(M)) \cap \text{End}_{\mathbb{R}}(C_{\mathcal{F}}^{\infty}(M))$ that reads in any system of adapted coordinates $(x, y) = (x^{1}, \ldots, x^{p}, y^{1}, \ldots, y^{q})$ over any open subset $U \subset M$,

$$D|_{U} = \sum_{|\gamma| \leq k} D_{\gamma} \partial_{x}^{\gamma_{1}} \cdots \partial_{x}^{\gamma_{p}} \partial_{y}^{\gamma_{q+1}} \cdots \partial_{y}^{\gamma_{p+q}},$$

where $k \in \mathbb{N}$ is independent of the considered adapted chart, where $D_{\gamma} \in C^{\infty}(U)$, and where the coefficients $D_{\gamma}$ with $\gamma^{1} = \ldots = \gamma^{p} = 0$ are locally defined adapted functions. The smallest possible integer $k$ is called the order of operator $D$.

We denote by $D_{F}(M)$ the filtered space of all adapted differential operators on $(M, F)$. The corresponding graded space $\mathcal{S}_{F}(M)$ is the space of adapted symbols on $(M, F)$. Of course, $\mathcal{S}_{F}^{k}(M) \simeq \Gamma_{F}(S^{k}TM)$, where the RHS denotes the space of adapted sections of $S^{k}TM$. The definition of adapted sections of $S^{k}TM$ is clear in view of the definitions of adapted vector fields and adapted differential operators. Furthermore, as in the case of the tangent bundle, foliation $\mathcal{F}$ induces a foliation $\mathcal{F}_{S^{k}T}$ on $S^{k}TM$ (defined by means of the extension $(T\phi_{2})^{\circ} : S^{k}TU \to S^{k}\mathbb{R}^{q}$ of $T\phi_{2} = TU \to \mathbb{R}^{q}$) and adapted sections of $S^{k}TM$ coincide with sections of this bundle that are foliated for $\mathcal{F}_{S^{k}T}$: $S_{\mathcal{F}}^{k}(M) \simeq \Gamma_{F}(S^{k}TM) \simeq \Gamma(S^{k}TM; \mathcal{F}_{S^{k}T})$. Below, we identify the first two of the preceding spaces.

**Theorem 6.** Let $(M, F)$ be a foliated manifold of codimension $q$ endowed with an adapted projective structure $[\nabla_{F}]$, and denote by $P_{F}$ the corresponding reduction of $P_{\mathcal{F}}^{n}M$ to $H(n + 1, q + 1, \mathbb{R}) \subset G_{n,F_{\mathcal{F}}^{\infty}}^{\infty}$. We then have the following canonical vector space isomorphisms:

$$\sim : \mathcal{S}_{F}^{k}(M) = \Gamma_{F}(S^{k}TM) \ni s \mapsto \bar{s} \in C_{\mathcal{F}}^{\infty}(L\mathcal{F}_{M}, S^{k}\mathbb{R}^{n}) \quad (33)$$

$$\land : \mathcal{S}_{F}^{k}(M) = \Gamma_{F}(S^{k}TM) \ni s \mapsto \bar{s} = \bar{s} \circ p_{F}^{2} \in C_{\mathcal{F}}^{\infty}(P_{F}, S^{k}\mathbb{R}^{n})_{H(n + 1, q + 1, \mathbb{R})} \quad (34)$$

The proof of this theorem is on the same lines than that of Theorem 5. Let us explain the meaning of $C_{\mathcal{F}}^{\infty}$ in Equations (33) and (34). In the following, we denote by $p_{n,q}$ canonical projections, such as $p_{n,q} : GL(n, q, \mathbb{R}) \to GL(q, \mathbb{R})$, $p_{n,q} : S^{k}\mathbb{R}^{n} \to S^{k}\mathbb{R}^{q}$, ... Adapted functions $f \in C_{\mathcal{F}}^{\infty}(P_{F}^{n}M, S^{k}\mathbb{R}^{n})$ are then the functions $f \in C^{\infty}(P_{F}^{n}M, S^{k}\mathbb{R}^{n})$, for which $p_{n,q} \circ f \in C^{\infty}(P_{F}^{n}M, S^{k}\mathbb{R}^{q}; \mathcal{F}_{L})$ is foliated for $\mathcal{F}_{L}$. If we set $k = 0$, we get that adapted functions coincide with foliated functions, see above.

### 4.3 Projections

Adapted symbols project onto foliated symbols. Indeed, we have the

**Proposition 6.** For any foliated manifold $(M, F)$, there is a canonical degree-preserving projection $\mathcal{F}_{\pi} : \mathcal{S}_{F}^{k}(M) \to S^{k}(M, F, k \in \mathbb{N}$). If the considered foliated manifold is endowed with an adapted and the corresponding foliated projective structures, and if $\mathcal{F}_{\pi}$ (resp. $\mathcal{F}_{\bar{\pi}}$) denotes projection $\mathcal{F}_{\pi}$ read through the isomorphisms $\sim$ (resp. $\land$) detailed in Theorems 3 and 4, we have, for any symbol $s \in \mathcal{S}_{F}^{k}(M)$, $k \in \mathbb{N}$,

$$\mathcal{F}_{\pi} \circ \mathcal{F}_{\pi}^{1} = p_{n,q} \circ \bar{s} \quad \text{(resp. } \mathcal{F}_{\bar{\pi}} \circ \mathcal{F}_{\bar{\pi}}^{2} = p_{n,q} \circ \bar{s} \text{).} \quad (35)$$

**Proof.** As usual, we denote by $n$ the dimension of $M$ and by $p$ (resp. $q$) the dimension (resp. codimension) of $F$. Since $\mathcal{F}_{*}^{p} : P_{*}^{n}M \ni J^{0}_{0}(f) \mapsto J^{0}_{0}(f \circ i_{q}) \in P^{*}(M, F)$, as $J^{1}_{0}(f), f(0) = m$, corresponds to the basis $(v_{1}, \ldots, v_{n}) \in (T_{m}M)^{\times n}$,

$$v_{i} = \sum_{j} \partial_{Z}(X^{j}(f(Z)))(0)\partial_{X_{j}},$$
where $Z = (z', z'')$ are canonical coordinates in $\mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^q$ and $X = (x, y)$ are adapted coordinates in $M$ around $m$, and as $J^1_0(f \circ i_q)$ corresponds to the basis $(n_1, \ldots, n_q) \in (N_m)^\times q$,

$$n_i = \sum_j \partial_{z^j}(y^j(f(Z)))(0)[\partial_{y^i}],$$

we see that $\mathcal{F}p^1(v_1, \ldots, v_n) = ([v_{p+1}], \ldots, [v_{p+q}])$.

Consider now $s \in S^b_F(M) = \Gamma_F(S^bTM) = \Gamma_F(L_F^sM \times_{GL(n,q,\mathbb{R})} S^b \mathbb{R}^n)$ and set

$$s(m) = [(v_1, \ldots, v_n), (s^{i_1 \ldots i_k}(m))]_{GL(n,q,\mathbb{R})}, \quad (36)$$

where $m \in M$, and where $(s^{i_1 \ldots i_k}(m))$ is the tuple of components of $s(m)$ in the basis induced by $(v_1, \ldots, v_n)$.

Define projection $\mathcal{F}_\pi$ by

$$(\mathcal{F}_\pi s)(m) := \{\mathcal{F}p^1(v_1, \ldots, v_n), p_{n,q}(s^{i_1 \ldots i_k}(m))\}_{GL(q,\mathbb{R})} = \{(v_{p+1}, \ldots, v_{p+q}), (s^{p+i_1 \ldots p+i_k}(m))\}_{GL(q,\mathbb{R})} \in \mathcal{L}m(M, F) \times GL(q,\mathbb{R}) S^b \mathbb{R}^q \quad (37)$$

Since $s$ is adapted, we have $\mathcal{F}_\pi s \in \Gamma(LN(M, F) \times GL(q,\mathbb{R}) S^b \mathbb{R}^q, \mathcal{F}_{S+}N) = \Gamma(S^bN(M, F); \mathcal{F}_{S+}N) = S^b(M, F)$. Of course, $\mathcal{F}_\pi s$ is well-defined. Indeed, if $A \in GL(n,q,\mathbb{R})$, and if we denote the submatrices of $A$ by $A, B, D$, we get

$$\mathcal{F}p^1((v_1, \ldots, v_n) \cdot A) = \mathcal{F}p^1(A_k v_k, \ldots, A^k_n v_n) = (A_{p+1}^{p+1}v_{p+1}, \ldots, A_{p+q}^{p+q}v_{p+q})$$

and

$$p_{n,q}(\rho(A^{-1})(s^{i_1 \ldots i_k}(m))) = (A_{p+1}^{-1}v_{p+1} \ldots A_{p+q}^{-1}v_{p+q}) = \rho(D^{-1})(s^{p+i_1 \ldots p+i_k}(m)). \quad (38)$$

Let us recall that the isomorphism between the space $\Gamma(B \times_G V)$ of sections of a vector bundle $B \times_G V \to M$ associated with a principal bundle $B(M, G, \pi)$ and the space $C^\infty(B, V)_G$ of $G$-equivariant functions, assigns to a section $s$ the function $\tilde{s}$ that maps a “basis” $b \in B$ to the “components” $v$ of the “vector” $s(\pi(b)) = [b, v]_G$ in “basis” $b$. It therefore follows from Equations $(36)$ and $(37)$, as well as from the definition $\mathcal{F}_\pi \tilde{s} = (\mathcal{F}_\pi s)$, that the first part of Equation $(35)$ holds true.

Eventually, $\tilde{s} = \tilde{s} \circ p_G^b(F)$ (resp. $\tilde{s} = \tilde{s} \circ p^2_F$) in the foliated (resp. adapted) case. The second part of Equation $(35)$ is then a consequence of the definition $\mathcal{F}_\pi \tilde{s} = (\mathcal{F}_\pi s)$, of Proposition $5$ and of the first part of Equation $(35)$. \qed

**Remark.** Natural projectively invariant and equivariant quantizations are often valued in differential operators between tensor densities of weights $\lambda$ and $\mu$. Symbols are then sections in $\Gamma(STM \otimes \Delta^\nu TM) = \Gamma(LM \times_{GL(n,\mathbb{R})} (\mathbb{R}^n \otimes \Delta^\nu \mathbb{R}^n))$, where $\nu = \mu - \lambda$, where $\Delta^\nu TM$ is the line bundle of $\nu$-densities on $M$ and $\Delta^\nu \mathbb{R}^n$ is its typical fiber. As the action of a change of basis, say $A \in GL(n,\mathbb{R})$, on the component $r$ of a $\nu$-density of $\mathbb{R}^n$ is, as easily checked, $\rho(A^{-1}) r = r \det A |^\nu$, we get, for $A \in GL(n, q, \mathbb{R})$,

$$p_{n,q}(\rho(A^{-1}) r) = r \det A |^\nu \det D |^\nu \neq r \det D |^\nu = \rho(D^{-1}) r,$$

see Equation $(38)$. Hence, differential operators between tensor densities, see [DLO99], [MR05], or even between sections of arbitrary vector bundles associated with the principal bundle of linear frames, see [BHMP02, Han06], are more intricate. Corresponding investigations are postponed to future work.

5 Construction of the normal Cartan connection

The method exposed in [MR05] in order to solve the problem of the natural and projectively equivariant quantization uses the notion of normal Cartan connection. We are going to adapt this object firstly to the adapted situation and secondly to the foliated situation. Finally, in a third step, we are going to analyze the link between the adapted normal Cartan connection and the foliated one.
5.1 Construction in the adapted case

First, recall the notion of Cartan connection on a principal fiber bundle:

**Definition 7.** Let $G$ be a Lie group and $H$ a closed subgroup. Denote by $\mathfrak{g}$ and $\mathfrak{h}$ the corresponding Lie algebras. Let $P \rightarrow M$ be a principal $H$-bundle over $M$, such that $\dim M = \dim G/H$. A Cartan connection on $P$ is a $\mathfrak{g}$-valued one-form $\omega$ on $P$ such that

- If $R_a$ denotes the right action of $a \in H$ on $P$, then $R_a^\ast \omega = \text{Ad}(a^{-1})\omega$,
- If $k^\ast$ is the vertical vector field associated to $k \in \mathfrak{h}$, then $\omega(k^\ast) = k$,
- $\forall u \in P$, $\omega_u : T_uP \rightarrow \mathfrak{g}$ is a linear bijection.

Recall too the definition of the curvature of a Cartan connection:

**Definition 8.** If $\omega$ is a Cartan connection defined on a $H$-principal bundle $P$, then its curvature $\Omega$ is defined as usual by

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega]. \quad (39)$$

Next, one adapts Theorem 4.2. cited in [Koba72] p.135 in the following way:

**Theorem 7.** Let $P\mathfrak{x}$ be an $H(n, 1, q, 1, \mathbb{R})$-principal fiber bundle on a manifold $M$. If one has a one-form $\omega_{-}$ with values in $\mathbb{R}^n$ of components $\omega^i$ and a one-form $\omega_0$ with values in $\mathfrak{gl}(n, q, \mathbb{R})$ (the Lie algebra of $\text{GL}(n, q, \mathbb{R})$) of components $\omega_j^i$ that satisfy the three following conditions:

- $\omega_{-}(h^\ast) = 0$, $\quad \omega_0(h^\ast) = h_0$, $\quad \forall h \in \mathfrak{gl}(n, q, \mathbb{R}) + \mathbb{R}^n$, where $h_0$ is the projection with respect to $\mathfrak{gl}(n, q, \mathbb{R})$ of $h$,
- $(R_a)^\ast(\omega_{-} + \omega_0) = (\text{Ad}a^{-1})(\omega_{-} + \omega_0)$, $\quad \forall a \in H(n, 1, q, 1, \mathbb{R})$, where $\text{Ad}a^{-1}$ is the application from $\mathbb{R}^n + \mathfrak{gl}(n, q, \mathbb{R}) + \mathbb{R}^n/\mathbb{R}^n$ into $\mathbb{R}^n + \mathfrak{gl}(n, q, \mathbb{R}) + \mathbb{R}^n$ in itself induced by the adjoint action $\text{Ad}a^{-1}$ from $\mathbb{R}^n + \mathfrak{gl}(n, q, \mathbb{R}) + \mathbb{R}^n$ into $\mathbb{R}^n + \mathfrak{gl}(n, q, \mathbb{R}) + \mathbb{R}^n$,
- If $\omega_{-}(X) = 0$, then $X$ is vertical,

and the following additional condition:

$$d\omega^i = -\sum \omega^j_k \wedge \omega^k, \quad (40)$$

then there is a unique Cartan connection $\omega = \omega_{-} + \omega_0 + \omega_1$ whose curvature $\Omega$ of components $(0; \Omega^i_j; \Omega^i_j)$ satisfies the following property:

$$\sum_{i=p+1}^{n} K^i_{jkl} = 0, \quad \forall j \in \{p + 1, \ldots, n\}, \forall \ell,$$

where

$$\Omega^i_j = \sum \frac{1}{2} K^i_{jkl} \omega^k \wedge \omega^l.$$ 

**Proof.** The proof goes as in [Koba72]. Let $\omega = (\omega^i; \omega^i_j; \omega^i_j)$ be a Cartan connection with the given $(\omega^i; \omega^i_j)$. Thanks to the definition of the curvature, we have

$$d\omega^i_j = -\sum \omega^j_k \wedge \omega^j_k - \omega^j_k \wedge \omega^j_j + \delta^i_j \sum \omega^j_k \wedge \omega^j + \Omega^i_j \quad (41)$$

and

$$d\omega_j = -\sum \omega^j_k \wedge \omega^j_k + \Omega_j.$$ 

Applying exterior differentiation $d$ to (40), making use of (40) and (41) and collecting the terms not involving $\omega^i_j$ and $\omega_j$, we obtain the first Bianchi identity:

$$\Omega^i_j \wedge \omega^j = 0,$$
or equivalently,
\[ K_{jkl}^i + K_{kjl}^i + K_{ikj}^i = 0. \]
Then the condition \( \sum_{i=p+1}^{n} K_{jil}^i = 0 \) implies also
\[ \sum_{i=p+1}^{n} K_{jil}^i = 0. \]

Now prove the uniqueness of a normal Cartan connection. Let \( \varpi = (\omega^i; \omega^j; \varpi) \) be another Cartan connection with the given \( (\omega^i; \omega^j) \). Thanks to the fact that \( \varpi_j - \omega_j \) vanishes on vertical vector fields, we can write
\[ \varpi_j - \omega_j = \sum A_{jk} \omega^k, \]
where the coefficients \( A_{jk} \) are functions on \( P \). Denoting the curvature of \( \varpi \) by \( \Omega = (0; \Omega_j; \Omega_j) \) and writing
\[ \Omega_j = \sum \frac{1}{2} K_{jkl}^i \omega^k \land \omega^l, \]
we obtain using (41) the following relations between \( K_{jkl}^i \) and \( K_{jkl}^i \):
\[ K_{jkl}^i - K_{jkl}^i = -\delta_{ik}^j A_{jk} + \delta_{jl}^i A_{jl} + \delta_{ij}^k A_{kl} - \delta_{ij}^k A_{lk}. \]
Hence,
\[ \sum_{i=p+1}^{n} (K_{jkl}^i - K_{jkl}^i) = (q+1)(A_{kl} - A_{lk}), \tag{42} \]
\[ \sum_{i=p+1}^{n} (K_{jil}^i - K_{jil}^i) = (q-1)A_{jl} + (A_{jl} - A_{ij}). \tag{43} \]

If \( \omega \) and \( \varpi \) are normal Cartan connections, i.e., \( \sum_{i=p+1}^{n} K_{jil}^i = \sum_{i=p+1}^{n} K_{jil}^i = 0 \), then \( A_{ij} = 0 \) and hence \( \omega = \varpi \). This proves the uniqueness of the normal Cartan connection.

To prove the existence, one assumes that there is a Cartan connection \( \omega = (\omega^i; \omega^j; \omega_j) \) with the given \( (\omega^i; \omega^j) \). The goal is then to find functions \( A_{jk} \) such that \( \varpi = (\omega^i; \omega^j; \varpi) \) becomes a normal Cartan connection. If \( 1 \leq j \leq p \), \( A_{jk} \) is of course equal to zero. If \( p+1 \leq j \leq n \) and if \( p+1 \leq k \leq n \), one can view thanks to (42) and (43) that it suffices to set
\[ A_{jk} = \frac{1}{(q+1)(q-1)} \sum_{i=p+1}^{n} K_{jik}^i - \frac{1}{q-1} \sum_{i=p+1}^{n} K_{jik}^i. \tag{44} \]
If \( 1 \leq k \leq p \), one sees thanks to (43) that it suffices to set
\[ A_{jk} = -\sum_{i=p+1}^{n} \frac{1}{q} K_{jik}^i. \tag{45} \]

The last step of the proof consists in showing that there is at least one Cartan connection \( \omega \) with the given \( (\omega^i; \omega^j) \). Let \( \{ U_\alpha \} \) be a locally finite open cover of \( M \) with a partition of unity \( \{ f_\alpha \} \). If \( \omega_\alpha \) is a Cartan connection in \( P_\varpi \mid U_\alpha \) with the given \( (\omega^i; \omega^j) \), then \( \sum_\alpha (f_\alpha \circ \pi) \omega_\alpha \) is a Cartan connection in \( P_\varpi \) with the given \( (\omega^i; \omega^j) \), where \( \pi : P_\varpi \rightarrow M \) is the projection. Hence, the problem is reduced to the case where \( P_\varpi \) is a product bundle. Fixing a cross section \( \sigma : M \rightarrow P_\varpi \), set \( \omega_j(X) = 0 \) for every vector \( X \) tangent to \( \sigma(M) \). If \( Y \) is an arbitrary tangent vector of \( P_\varpi \), we can write uniquely
\[ Y = R_\alpha(X) + W, \]
where $X$ is a vector tangent to $\sigma(M)$, $a$ is in $H(n+1,q+1,\mathbb{R})$ and $W$ is a vertical vector. Extend $W$ to a unique fundamental vector field $A^*$ of $P_F$ with $A \in \text{gl}(n,q,\mathbb{R}) \oplus \mathbb{R}^*$. Thanks to the properties of the Cartan connections, we have to set

$$\omega(Y) = \text{Ad}(a^{-1})(\omega(X)) + A.$$

This defines the desired $(\omega_J)$. Actually, $X$ is equal to $(\sigma \circ \pi_*^*)Y$ and one can take the section $\sigma$ equal to $\sigma_n$. 

One can remark that the codimension of the foliation $\mathcal{F}$ has to be different from 1.

One can define on $P_F$ an one-form in the following way:

**Definition 9.** If $u = j_0^2 f$ is a point belonging to $P_F$ and if $X$ is a tangent vector to $P_F$ at $u$, the canonical form $\theta_X$ of $P_F$ is the 1-form with values in $\mathbb{R}^n \oplus \text{gl}(n,q,\mathbb{R})$ defined at the point $u$ in the following way:

$$\theta_{X,u}(X) = (P^1 f)^{-1}_e (p^2_{p_*}X),$$

where $e$ is the frame at the origin of $\mathbb{R}^n$ represented by the identity matrix.

**Theorem 8.** One can associate to the projective class of an adapted connection $[\nabla_{\mathcal{F}}]$ a Cartan connection on $P_F$ in a natural way. We will denote by $\omega_{\mathcal{F}}$ this Cartan connection.

**Proof.** The canonical one-form defined above is the restriction to $P_F$ of the canonical one-form of $P^2(M)$ defined in [Koba72] p.140. It is too the restriction to $P_F$ of the restriction to $P$ of the canonical one-form of $P^2(M)$, where $P$ is the projective structure associated to $\nabla_{\mathcal{F}}$ defined in [Koba72]. Thanks to the fact that the canonical one-form on $P$ satisfies the properties of Theorem 4.2. mentioned in [Koba72], $\theta_{\mathcal{F}}$ satisfies the properties mentioned in Theorem 7. One defines then the adapted normal Cartan connection $\omega_{\mathcal{F}}$ as the unique Cartan connection on $P_F$ beginning by $\theta_{\mathcal{F}}$ and satisfying the property linked to the curvature cited in Theorem 7. Because of the naturality of this property, the naturality of $\theta_{\mathcal{F}}$ and the uniqueness of the Cartan connection mentioned in Theorem 7, $\omega_{\mathcal{F}}$ is a Cartan connection on $P_F$ associated naturally to the class $[\nabla_{\mathcal{F}}]$.

5.2 Construction in the foliated case

The reduction $P(\mathcal{F})$ is actually an example of a foliated bundle defined in [Blum84]. The Cartan connection that we are going to define on it is an example of a Cartan connection in a foliated bundle defined too in [Blum84]. It is the reason for which we are going first to recall the definitions of these notions.

**Definition 10.** Let $M$ be a manifold of dimension $m$ and let $\mathcal{F}$ be a codimension $q$ foliation of $M$. Let $T(M)$ be the tangent bundle of $M$ and let $T\mathcal{F}$ be the tangent bundle of $\mathcal{F}$. Let $H$ be a Lie group and let $\pi : P \rightarrow M$ be a principal $H$-bundle. We say $\pi : P \rightarrow M$ is a foliated bundle if there is a foliation $\mathcal{F}$ of $P$ satisfying

- $\mathcal{F}$ is $H$-invariant,
- $\bar{E}_u \cap V_u = \{ 0 \}$ for all $u \in P$,
- $\pi_{\ast u}(\bar{E}_u) = T\mathcal{F}_{\pi(u)}$ for all $u \in P$,

where $\bar{E}$ is the tangent bundle of $\mathcal{F}$ and $V$ is the bundle of vertical vectors.

**Definition 11.** Let $\mathcal{F}$ be a codimension $q$ foliation of $M$. Let $G$ be a Lie group and let $H$ be a closed subgroup of $G$ with dimension$(G/H) = q$. Let $\pi : P \rightarrow M$ be a foliated principal $H$-bundle. Let $\mathfrak{g}$ be the Lie algebra of $G$ and let $\mathfrak{h}$ be the Lie algebra of $H$. For each $A \in \mathfrak{h}$, let $A^*$ be the corresponding fundamental vector field on $P$.

A Cartan connection in the foliated bundle $\pi : P \rightarrow M$ is a $\mathfrak{g}$-valued one-form $\omega$ on $P$ satisfying

- $\omega(A^*) = A$ for all $A \in \mathfrak{h}$,
\[ (R_a)^* \omega = \text{Ad}(a^{-1}) \omega \] for all \( a \in H \) where \( R_a \) denotes the right translation by \( a \) acting on \( P \) and \( \text{Ad}(a^{-1}) \) is the adjoint action of \( a^{-1} \) on \( g \).

- For each \( u \in P \), \( \omega_u: T_u P \to g \) is onto and \( \omega_u(\dot{E}_u) = 0 \),

- \( L_X \omega = 0 \) for all \( X \in \Gamma(\dot{E}) \) where \( \Gamma(\dot{E}) \) denotes the smooth sections of \( \dot{E} \) and \( L_X \) is the Lie derivative.

**Theorem 9.** The reduction \( P(\mathcal{F}) \) is a foliated bundle.

**Proof.** One can easily view that \( F_{P^2 N} \) satisfies the properties of the definition of a foliated bundle: first, \( F_{P^2 N} \) is \( H(q + 1, \mathbb{R}) \)-invariant because, if \( (U_i, f_i, g_{ij}) \) is a cocycle corresponding to the foliation \( \mathcal{F} \), if \( J^2_0 f \in P(\mathcal{F}) \) and if \( J^2_0(f_i \circ f) \) is constant, then \( J^2_0(f_i \circ f \circ h) = J^2_0(f_i \circ f) \circ J^2_0(h) \) is constant, where \( J^2_0(h) \in H(q + 1, \mathbb{R}) \).

If \( X \in V_u \), then \( X = \frac{d}{dt} \exp(th)|_{t=0} \), where \( h \in h(q + 1, \mathbb{R}) \). If \( u = J^2_0(f) \) and if \( \exp(th) = J^2_0(g_i) \), then \( J^2_0(f_i \circ f \circ g_i) \) is constant if \( X \) is tangent to the foliation \( F_{P^2 N} \). One has then that \( J^2_0(f \circ g_i) \) is constant and then \( X = 0 \).

If \( X \) is tangent to the foliation \( F_{P^2 N} \), then \( X = \frac{d}{dt} \gamma(t)|_{t=0} \), where \( \gamma(t) \in F_{P^2 N} \). Then \( \frac{d}{dt} \pi^2_\omega(X) = \frac{d}{dt} \pi^2(\gamma(t))|_{t=0} \), that belongs to \( TF_{\pi(u)} \) because \( \pi^2(\gamma(t)) \) belongs to \( \mathcal{F} \). Indeed, if \( \gamma(t) = J^2_0(f_i), f_i \circ f \) is constant because \( J^2_0(f_i \circ f_i) \) is constant.

**Theorem 10.** One can associate to the class of a foliated connection \( [\nabla(\mathcal{F})] \) a Cartan connection on \( P(\mathcal{F}) \) in a natural way. We will denote this connection by \( \omega(\mathcal{F}) \).

**Proof.** If \( (U_i, f_i, g_{ij}) \) is a Haefliger cocycle of \( \mathcal{F} \), the image by \( P^2 N(f_i) \) of \( P(\mathcal{F})|_{U_i} \) is a reduction of \( P^2 N(f_i)(P^2 N(U_i, \mathcal{F})) \) to \( H(q + 1, \mathbb{R}) \). Indeed, if \( J^2_0(h) \in H(q + 1, \mathbb{R}) \), \( J^2_0(f_i \circ f \circ h) = J^2_0(f_i \circ f \circ h) \) is constant, where \( J^2_0(h) \in P(\mathcal{F}) \) if \( J^2_0(f \in P(\mathcal{F}) \). Moreover, if \( J^2_0(f_i \circ f') \) and \( J^2_0(f_i \circ f) \) belong to the same fiber, then \( f_i \circ f'(0) = f_i \circ f(0) \). If \( y \) denotes the passing to the transverse coordinates of an adapted coordinates system, one has then \( J^2_0(y \circ f') = (x^i, \delta^i_k, -\Gamma^i_{jk})H \) and \( J^2_0(y \circ f) = (x^i, \delta^i_k, -\Gamma^i_{jk})H \), with \( H \) and \( H' \) belonging to \( H(q + 1, \mathbb{R}) \) and with the \( \Gamma^i_{jk} \) equal to the Christoffel symbols of \( \nabla(\mathcal{F}) \). We have thus \( J^2_0(f_i \circ f') = J^2_0(f_i \circ f)H^{-1}H' \), with \( H^{-1}H' \in H(q + 1, \mathbb{R}) \).

We will denote by \( \overline{P} \) the reduction of \( P^2 N(f_i)(P^2 N(U_i, \mathcal{F})) \) to \( H(q + 1, \mathbb{R}) \).

One builds locally the normal Cartan connection \( \omega(\mathcal{F}) \) on \( P(\mathcal{F}) \) in the following way : if \( \overline{\omega} \) denotes the normal Cartan connection on \( \overline{P} \), then \( \omega(\mathcal{F})|_{P^2 N(U_i, \mathcal{F})} := (P^2 N(f_i))^* \overline{\omega} \).

One can show (see [Blum84]) that the connection \( \omega(\mathcal{F}) \) is a well-defined foliated Cartan connection.

Thanks to the naturality of the normal Cartan connection, \( \omega(\mathcal{F}) \) is associated naturally to the class of the foliated connection \( [\nabla(\mathcal{F})] \).

One can remark that, as the foliation \( F_{P^2 N} \) is of dimension \( p \), the third condition of the definition of a foliated Cartan connection implies that, in our case, the kernel of \( \omega(\mathcal{F})|_{\mathcal{U}} \) will be exactly equal to the tangent space to \( F_{P^2 N} \).

**5.3 Link between adapted and foliated Cartan connections**

**Remark.** If \( y \) denotes the passing to the transverse coordinates of an adapted coordinates system and if \( P^2 y \) denotes the following application:

\[ P^2 y : P^2(M, \mathcal{F}) \to P^2 \mathbb{R}^q : J^2_0 f \mapsto J^2_0(y \circ f), \]

the image by \( P^2 y \) of \( P(\mathcal{F}) \) is a reduction of \( P^2(U) \) to \( H(q + 1, \mathbb{R}) \), where \( U \) is an open set of \( \mathbb{R}^q \).

We will denote by \( P_U \) this reduction of \( P^2(U) \) to \( H(q + 1, \mathbb{R}) \). If \( \omega_U \) denotes the normal Cartan connection on \( P_U \), then \( \omega(\mathcal{F})(P^2 y)|_{P^2 y} - 1_{P_U} = (P^2 y)^* \omega_U \). Indeed, if \( \phi \) denotes the diffeomorphism such that \( \phi \circ y = f_i \), then \( P^2 \phi(P_U) = \overline{P} \). By naturality of the normal Cartan connection, \( \omega_U = (P^2 \phi)^* \overline{\omega} \) and then \( (P^2 y)^* \omega_U = (P^2 y)^* \overline{\omega} \).
Proposition 7. If $\theta_U$ denotes the canonical one-form on $P_U$, then

$$\left( P^2 y \circ (\tilde{\nabla}^2) \right)^* \theta_U = p_{n,q} \theta_F.$$ 

Proof. On one hand, if $u = j_2^* f_0 \in P_F$ and if $X = \frac{d}{dt} j_2^* (f_1)|_{t=0} \in T_u P_F$, one has $\theta_F u (X) = \left( (P^1 f_0)^{-1}\right)_* p_{n,q}^* \frac{d}{dt} j_2^* (f_1)|_{t=0} = \frac{d}{dt} (P^1 f_0)^{-1} j_2^* (f_1)|_{t=0} = \frac{d}{dt} j_0^* (f_1)|_{t=0}.$

On the other hand, $\left( P^2 y \circ (\tilde{\nabla}^2) \right)^* \theta_U = \theta_U \circ (\rho_{y \circ (\tilde{\nabla}^2)}(\omega) (P^2 y \circ (\tilde{\nabla}^2) X)) = \frac{d}{dt}((y \circ f_0 \circ i_q)^{-1} \circ \frac{d}{dt}((y \circ f_0 \circ i_q)|_{t=0}) = p_{n,q}^* \left( (P^1 f_0)^{-1}\right)_* \frac{d}{dt} (P^1 f_0)^{-1} j_2^* (f_1)|_{t=0} = p_{n,q}^* \frac{d}{dt} j_2^* (f_1)|_{t=0}.$

We first prove that $p_{n,q}^* \omega \circ (\tilde{\nabla}^2) X$ is equal to $\frac{d}{dt} ((y \circ f_0 \circ i_q)^{-1} \circ (y \circ f_0 \circ i_q)|_{t=0}$ if $p_{n,q}^* \left( (P^1 f_0)^{-1}\right)_* \frac{d}{dt} (P^1 f_0)^{-1} j_2^* (f_1)|_{t=0} = \frac{d}{dt} j_2^* (f_1)|_{t=0}.$

Theorem 11. The connections $\omega_F$ and $\omega(F)$ are linked by the following relation:

$$\tilde{\omega} \circ (\tilde{\nabla}^2) \omega(F) = p_{n,q} \omega_F.$$ 

Proof. To prove that, it suffices to prove that

$$\left( P^2 y \circ (\tilde{\nabla}^2) \right)^* \omega_U = p_{n,q} \omega_F.$$ 

If one denotes by $\nabla_U$ the connection on $U$ whose Christoffel symbols are the Christoffel symbols of $\nabla(F)$ (the $\Gamma_{jk}^i$, with $i, j, k$ between $p + 1$ and $n$), if $(e^1, \ldots, e^n)$ denotes the canonical basis of $\mathbb{R}^n$, $(\hat{e}^1, \ldots, \hat{e}^n)$ denotes the canonical basis of $\mathbb{R}^m$, one has

$$\omega_F = \hat{\omega} - \sum_{j=p+1}^n \sum_{k=1}^n \Gamma_{jk}^i (\theta^1_{F-1}) e^j$$

and

$$\omega_U = \hat{\omega} - \sum_{j=1}^q \sum_{k=1}^q \Gamma_{jk}^i (\theta^1_{U-1}) e^j,$$

where $\hat{\omega}$ is the Cartan connection induced by $\nabla(F)$ (resp. $\nabla_U$), $\Gamma$ is the deformation tensor corresponding to $\nabla(F)$ (resp. $\nabla_U$) (see [CSS97]).

One recall that $\hat{\omega}$ is the unique Cartan connection such that its component with respect to $\mathbb{R}^n$ vanishes on the section $(x^i, \delta_i, -\Gamma_{jk})$. If $\sigma_F$ (resp. $\sigma_U$) denotes the section $(x^i, \delta_i, -\Gamma_{jk})$, the connection $\hat{\omega}$ (resp. $\hat{\omega}_U$) is defined in this way:

$$\hat{\omega} u (X) = \text{Ad}(b^{-1}) \theta_F ((\sigma_F \circ \pi^2)_* X) + B,$$

and

$$\hat{\omega}_U u (X) = \text{Ad}(b^{-1}) \theta_U ((\sigma_U \circ \pi^2)_* X) + B,$$

where $\pi^2$ is the projection on $M$ (resp. $U$), $R_b(\sigma_F(\pi^2(u))) = u$ (resp. $R_b(\sigma_U(\pi^2(u))) = u$) and $B^* = X - R_b(\sigma_F, \pi^2) X$ (resp. $B^* = X - R_b(\sigma_U, \pi^2) X$).

The deformation tensor $\Gamma_F$ (resp. $\Gamma_U$) is defined in this way:

$$\Gamma_F (X) = (\hat{\omega} - \omega_F)(\omega_F^{-1}(X))$$

and

$$\Gamma_U (X) = (\hat{\omega}_U - \omega_U)(\omega_U^{-1}(X)).$$

In fact, the sections $(x^i, \delta_i, -\Gamma_{jk})$ correspond to the section $\sigma$ of the end of the theorem the connections $\hat{\omega}$ and $\hat{\omega}_U$ correspond to the connection $\omega$ of the proof of this theorem, the connections $\omega_F$ and $\omega_U$ correspond to the connection $\omega$ whereas the $\Gamma_{jk}$ correspond to the functions $\sigma_F$. We first prove that $\left( P^2 y \circ (\tilde{\nabla}^2) \right)^* \hat{\omega}_U = p_{n,q} \hat{\omega}_F$. 

Indeed, we have
\[ \tilde{\Gamma}_X^a(X) = \text{Ad}(b^{-1})\theta_\omega((\sigma_X \circ \pi^2)_X) + B, \]
where \( R_b(\sigma_X(\pi^2(u))) = u \) and \( B^* = X - R_{b*}\sigma_X\pi^2_X \) whereas
\[ (P^2 y \circ (\tilde{\mathbf{p}}^2)^*) \tilde{\Gamma}_U^a \,(\mathbf{X}) = \text{Ad}(a^{-1})\theta_\omega((\sigma_U \circ \pi^2)_X)(P^2 y \circ (\tilde{\mathbf{p}}^2)_X) + A, \]
where \( R_a(\sigma_U(\pi^2(P^2 y \circ (\tilde{\mathbf{p}}^2))(u))) = (P^2 y \circ (\tilde{\mathbf{p}}^2))u \) and \( A^* = (P^2 y \circ (\tilde{\mathbf{p}}^2))_X - R_{a*}\sigma_U\pi^2_X(P^2 y \circ (\tilde{\mathbf{p}}^2))_X. \)

One can see that \( a = p_{n,q}b \).

Moreover, as \((\sigma_U \circ \pi^2) \circ (P^2 y \circ (\tilde{\mathbf{p}}^2)) = (P^2 y \circ (\tilde{\mathbf{p}}^2)) \circ (\sigma_U \circ \pi^2) \) and \((P^2 y \circ (\tilde{\mathbf{p}}^2)) \circ \theta_U = p_{n,q}\theta_\omega \) (see proposition \( \Box \)), one has \( \text{Ad}((p_{n,q}b)^{-1})\theta_U((\sigma_U \circ \pi^2)_X(P^2 y \circ (\tilde{\mathbf{p}}^2))_X) = \text{Ad}((p_{n,q}b)^{-1})(p_{n,q}\theta_{\omega})(\sigma_U \circ \pi^2)_X = p_{n,q}(\text{Ad}(b^{-1})\theta_\omega((\sigma_X \circ \pi^2)_X)). \) One can see too that \( A^* = (P^2 y \circ (\tilde{\mathbf{p}}^2))^*_X, B^* \), thus \( A = p_{n,q}B \).

Now, prove that \((P^2 y \circ (\tilde{\mathbf{p}}^2))^* \sum_{j=1}^q \sum_{k=1}^r (\Gamma_U^j)(\theta_U^k)^{-1})^j = \sum_{j=p+1}^n \sum_{k=1}^r (\Gamma^j_k)(\theta^p_{j-1})^j. \)

One has \((P^2 y \circ (\tilde{\mathbf{p}}^2))^* \sum_{j=1}^p \sum_{k=1}^r (\Gamma_U^j)(\theta_U^k)^{-1})^j = \sum_{j=1}^q \sum_{k=1}^r (P^2 y \circ (\tilde{\mathbf{p}}^2))^*(\Gamma_U^j)(\theta^p_{j-1})^j. \)

It remains then to prove that \((P^2 y \circ (\tilde{\mathbf{p}}^2))^*(\Gamma_U^j) = \Gamma^j_{f+q,k+p} \) and that \( \Gamma^j_{f+q,k+p} = 0 \) if \( 1 \leq k \leq p \).

Indeed, if \( 1 \leq k \leq p \), \( \Gamma^j_{f+q,k+p} = \Gamma^j_{f+q,k+p} \) where \( R_{\omega_X} \) denotes the equivariant function on \( \omega_X \) representing the curvature tensor of \( \omega_X \) thanks to the equation \( \Box \) of the Theorem \( \Box \) and thanks to the fact that the \( K_{i,j} \) represent the components of \( R_{\omega_X} \) (see \( \Box \)). Thanks to the fact that \( \omega_X \) is adapted, one can see that if \( 1 \leq k \leq p, \Gamma^j_{f+q,k+p} = 0 \). Moreover, \( \Gamma^j_{f+q,k+p} = \Gamma^j_{f+q,k+p} \) where \( R_{\omega_X} \) denotes the equivariant function on \( \omega_X \) representing the curvature tensor of \( \omega_X \) whereas if \( p+1 \leq k \leq n, \Gamma^j_{f+q,k+p} = \Gamma^j_{f+q,k+p} \).

Thanks to the equation \( \Box \) of the Theorem \( \Box \) this allows to prove that \((P^2 y \circ (\tilde{\mathbf{p}}^2))^*(\Gamma_U^j) = \Gamma_{f+q,k+p}. \)

\[ \square \]

6 Construction of the quantization

In a first step, we are going to explain how to build the quantization in the adapted and foliated situations. In a second step, we are going to prove that the quantization commutes with the reduction. In other words, quantize adapted objects is equivalent to quantize the induced foliated objects.

6.1 Construction in the adapted situation

In the adapted situation, we can define the operator of invariant differentiation exactly in the same way as in the standard situation:

**Definition 12.** Let \( V \) be a vector space. If \( f \in C^\infty(P_{\omega_X}, V) \), then the invariant differential of \( f \) with respect to \( \omega_X \) is the function \( \nabla^\omega_X f \in C^\infty(P_{\omega_X}, \mathbb{R}^n \otimes V) \) defined by
\[ \nabla^\omega_X f(u)(X) = L_{\omega_X^{-1}(X)}f(u) \quad \forall u \in P_{\omega_X}, \quad \forall X \in \mathbb{R}^n. \]

We will also use an iterated and symmetrized version of the invariant differentiation

**Definition 13.** If \( f \in C^\infty(P_{\omega_X}, V) \) then \( \nabla^\omega_X{k} f \in C^\infty(P_{\omega_X}, \mathbb{R}^n \otimes V) \) is defined by
\[ (\nabla^\omega_X{k}) f(u)(X_1, \ldots, X_k) = \frac{1}{k!} \sum_{\nu} L_{\omega_X^{-1}(X_{\nu(1)})} \circ \cdots \circ L_{\omega_X^{-1}(X_{\nu(k)})}f(u) \]
for \( X_1, \ldots, X_k \in \mathbb{R}^n \).

**Proposition 8.** If \( v \in \mathbb{R}^n \) and if \( p_{n,q}(v) = 0 \), then \( \omega^\omega_X^{-1}(v) \) is tangent to \( F_{P_{\omega_X}} \).

*Proof.* Indeed, as \( p_{n,q}(v) = \tilde{\mathbf{p}}^2(v, \mathcal{F}) \), one has \( \omega(\mathcal{F})(\tilde{\mathbf{p}}^2(v, \omega^\omega_X^{-1}(v))) = 0 \). As \( \tilde{\mathbf{p}}^2, \omega^\omega_X^{-1}(v) \) is then tangent to \( F_{P_{\omega_X}} \), one can easily show that \( \omega^\omega_X^{-1}(v) \) is then tangent to \( F_{\omega_X} \). \( \square \)
In the adapted situation, the invariant differentiation has a particular property:

**Proposition 9.** If \( f \) is a foliated function on \( P_X \), then

\[
(\nabla^{\omega^*} f)(v_1, \ldots, v_k) = (\nabla^{\omega^*} f)((0, p_{n,q} v_1), \ldots, (0, p_{n,q} v_k)).
\]

**Proof.** Indeed, one can show that if \( f \) is constant along the leaves of \( \mathcal{F}_{P^2} \), then \( L_{\omega^*}(0, p_{n,q} v) f \) is a foliated function too if \( v \in \mathbb{R}^n \). Indeed, if \( X \) is tangent to \( \mathcal{F}_{P^2} \), then \( L_X L_{\omega^*}(0, p_{n,q} v) f = 0 \). To show that, it suffices to prove that \( L_{[X, \omega^*]}(0, p_{n,q} v) f = 0 \). The fact that \( i_X \omega(\mathcal{F}) = i_X d\omega(\mathcal{F}) = 0 \) if \( X \) is tangent to \( \mathcal{F}_{P^2} \), that \( p_{n,q} \omega = \bar{\omega}^2 \omega(\mathcal{F}) \) and that \( \bar{\omega}^2 X \) is tangent to \( \mathcal{F}_{P^2} \) if \( X \) is tangent to \( \mathcal{F}_{P^2} \) implies that \( i_X p_{n,q} \omega = i_X p_{n,q} d\omega = 0 \) if \( X \) is tangent to \( \mathcal{F}_{P^2} \).

Remark that as the kernel of \( p_{n,q} \omega \mathcal{F} \) has a dimension equal to the dimension of \( \mathcal{F}_{P^2} \) (i.e. \( p + n p \)), the kernel of \( p_{n,q} \omega \mathcal{F} \) is equal to the tangent space to \( \mathcal{F}_{P^2} \). One has then that

\[
L_{[X, \omega^*]}(0, p_{n,q} v) f = 0.
\]

One has then that \([X, \omega^*](0, p_{n,q} v)\) is tangent to \( \mathcal{F}_{P^2} \) and then \( L_{[X, \omega^*]}(0, p_{n,q} v) f = 0 \).

One concludes using the fact that \( \omega^* f \) is tangent to \( \mathcal{F}_{P^2} \) if \( p_{n,q} f = 0 \).

In the adapted situation, we define a divergence operator analogous to the divergence operator defined in [MR05].

We fix a basis \((e_1, \ldots, e_n) \in \mathbb{R}^n\) and we denote by \((e^1, \ldots, e^n)\) the dual basis in \( \mathbb{R}^{n*} \).

**Definition 14.** The Divergence operator with respect to the Cartan connection \( \omega \mathcal{F} \) is defined by

\[
\text{Div}^{\omega \mathcal{F}} : C^\infty(P_X, S^k(\mathbb{R}^n)) \to C^\infty(P_X, S^{k-1}(\mathbb{R}^n)) : S \mapsto \sum_{j=p+1}^{n} i(e^j) \nabla^{\omega \mathcal{F}} e_j S,
\]

where \( i \) denotes the inner product.

**Remark.** If \( S \in C^\infty(P_X, S^k(\mathbb{R}^n)) \) and if \( f \in C^\infty(P_X, \mathbb{R}; \mathcal{F}_{P^2}) \), thanks to Proposition [9] we have \( (\text{Div}^{\omega \mathcal{F}} S, \nabla^{\omega^*} f) = (p_{n,q} \text{Div}^{\omega \mathcal{F}} S, p_{n,q} \nabla^{\omega^*} f) \).

One can then easily adapt Proposition 4, Lemma 7, Lemma 8, Propositions 9 and 10 from [MR05]:

**Proposition 10.** Let \((V, \rho)\) be a representation of \( GL(n, q, \mathbb{R}) \) and \( \rho' \) the induced action on \( \mathbb{R}^{n*} \otimes V \). If \( f \) belongs to \( C^\infty(P_X, V)_{GL(n, q, \mathbb{R})} \), then \( \nabla^{\omega \mathcal{F}} f \in C^\infty(P_X, \mathbb{R}^{n*} \otimes V)_{GL(n, q, \mathbb{R})} \).

**Proof.** The result is a consequence of the Ad-invariance of the Cartan connection \( \omega \mathcal{F} \). Indeed:

\[
(\nabla^{\omega \mathcal{F}} f)(u g) = \rho'(g)^{-1}(\nabla^{\omega \mathcal{F}} f)(u) \quad \forall u \in P_X, \forall g \in GL(n, q, \mathbb{R})
\]

\[
(\nabla^{\omega \mathcal{F}} f)(u g)(X) = [\rho'(g)^{-1}(\nabla^{\omega \mathcal{F}} f)(u)](X) \quad \forall u \in P_X, \forall g \in GL(n, q, \mathbb{R}), \forall X \in \mathbb{R}^n
\]

\[
(L_{\omega^*}(0, p_{n,q} v) f)(u g) = \rho(g)^{-1}(L_{\omega^*}(0, p_{n,q} v) f)(u) \quad \forall u \in P_X, \forall g \in GL(n, q, \mathbb{R}), \forall X \in \mathbb{R}^n.
\]

If one denotes by \( \varphi_t \) the flow of \( \omega^* \) and by \( \varphi'_t \) the flow of \( \omega^* \), it suffices then to verify that

\[
\frac{d}{dt} f(\varphi_t(u g))|_{t=0} = \rho(g)^{-1} \frac{d}{dt} f(\varphi'_t(u))|_{t=0} \quad \forall u \in P_X, \forall g \in GL(n, q, \mathbb{R}),
\]

or that

\[
\varphi_t(u g) = \varphi'_t(u) g \quad \forall u \in P_X, \forall g \in GL(n, q, \mathbb{R}).
\]

This property is satisfied: indeed, the fields \( \omega^* \) (\( gX \)) and \( \omega^* (X) \) are \( R^2 \)-linked because of the Ad-invariance of \( \omega \mathcal{F} \).  

In the same way, we have the following result:

**Proposition 11.** Let $\rho$ be the action of $GL(q, \mathbb{R})$ on $S^k(\mathbb{R}^q)$ and $\rho'$ the induced action on $\mathbb{R}^q \otimes S^k(\mathbb{R}^q)$. If $S \in C^\infty(P_{\mathcal{F}}, S^k(\mathcal{F}^n))$ is such that $(p_{n,q}S)(ug) = \rho(p_{n,q}g^{-1})(p_{n,q}S(u)) \forall g \in GL(n, q, \mathbb{R})$, then

$$(p_{n,q}\nabla^\omega S)(ug) = \rho'(p_{n,q}g^{-1})(p_{n,q}\nabla^\omega S(u)).$$

**Proof.** The proof is analogous to the proof of the previous result.

$$(p_{n,q}\nabla^\omega S)(ug) = \rho'(p_{n,q}g)^{-1}(p_{n,q}\nabla^\omega S)(u) \forall u \in P_{\mathcal{F}}, \forall g \in GL(n, q, \mathbb{R})$$

$$(p_{n,q}\nabla^\omega S)(ug) = \rho'(p_{n,q}g)^{-1}(p_{n,q}\nabla^\omega S)(u) \forall u \in P_{\mathcal{F}}, \forall g \in GL(n, q, \mathbb{R}), \forall X \in \mathbb{R}^q$$

If one denotes by $\varphi_t$ the flow of $\omega_{\mathcal{F}}^{-1}(0, X)$ and by $\varphi'_t$ the flow of $\omega_{\mathcal{F}}^{-1}(0, (p_{n,q}g)X)$, it suffices then to verify that

$$\frac{d}{dt}p_{n,q}S(\varphi_t(ug))|_{t=0} = \rho(p_{n,q}g^{-1})\frac{d}{dt}p_{n,q}S(\varphi'_t(ug))|_{t=0} \forall u \in P_{\mathcal{F}}, \forall g \in GL(n, q, \mathbb{R}).$$

One concludes using the fact that $\varphi_t(u) = \varphi'_t(u)g'$

with $p_{n,q}g' = p_{n,q}g$ because the fields $\omega_{\mathcal{F}}^{-1}(0, (p_{n,q}g)X)$ and $\omega_{\mathcal{F}}^{-1}(0, X)$ are $Rg'$-linked by $g'$ such that $p_{n,q}g' = p_{n,q}g$. □

**Proposition 12.** Let $\rho$ be the action of $GL(q, \mathbb{R})$ on $S^k(\mathbb{R}^q)$ and $\rho'$ the action on $S^{k-1}(\mathbb{R}^q)$. If $S \in C^\infty(P_{\mathcal{F}}, S^k(\mathcal{F}^n))$ is such that $(p_{n,q}S)(ug) = \rho(p_{n,q}g^{-1})(p_{n,q}S(u)) \forall g \in GL(n, q, \mathbb{R})$, then

$$(p_{n,q}\text{Div}^\omega S)(ug) = \rho'(p_{n,q}g^{-1})(p_{n,q}\text{Div}^\omega S(u)).$$

**Proof.** This can be checked directly from the definition of the divergence and from the proposition [11]

We have successively:

$$(p_{n,q}\text{Div}^\omega S)(ug) = \sum_{j=p+1}^n (p_{n,q}\nabla^\omega S)(ug)(p_{n,q}e_j)(p_{n,q}e^j)$$

$$= \rho'(p_{n,q}g^{-1}) \sum_{j=p+1}^n (p_{n,q}\nabla^\omega S)(u)(p_{n,q}g)(p_{n,q}e_j)(p_{n,q}e^j)(p_{n,q}g^{-1})$$

$$= \rho'(p_{n,q}g^{-1}) \sum_{j=p+1}^n \sum_{i=p+1}^n \sum_{r=p+1}^n (p_{n,q}\nabla^\omega S)(u)(p_{n,q}e_i)(p_{n,q}e^r)g^1 g^{-1}_{ij}$$

$$= \rho'(p_{n,q}g^{-1})p_{n,q} \sum_{i=p+1}^n (\nabla^\omega S)(u)(e_i)(e^i).$$

□

In our computations, we will make use of the infinitesimal version of the equivariance relation: if $(V, \rho)$ is a representation of $H(n+1, q+1, \mathbb{R})$ and if $f \in C^\infty(P_{\mathcal{F}}, V_{H(n+1, q+1, \mathbb{R})})$ then one has

$$L_h \cdot f(u) + \hat{\rho}_*(h)f(u) = 0, \quad \forall h \in gl(n, q, \mathbb{R}) \otimes \mathbb{R}^q \subset \text{sl}(n+1, \mathbb{R}), \forall u \in P_{\mathcal{F}}. \quad (46)$$
Proposition 13. For every $S \in C^\infty(P_F, S^k(R^n))$ such that $(p_{n,q}S)(ug) = \rho(p_{n,q}g^{-1})(p_{n,q}S(u)) \forall g \in GL(n,q,R)$, we have

$$p_{n,q}[L_h, \text{Div}^{\omega_F} S] = (q + 2k - 1)p_{n,q} i(0, h) S,$$

for every $h \in R^{q*}$.

Proof. First we remark that the Lie derivative with respect to a vector field commutes with the evaluation : if $\eta^1, \ldots, \eta^{k-1} \in R^{q*}$, we have

$$(L_h, \text{Div}^{\omega_F} S)(\eta^1, \ldots, \eta^{k-1}) = L_h \text{Div}^{\omega_F} S(\eta^1, \ldots, \eta^{k-1})$$

Now, the definition of a Cartan connection implies the relation

$$[h^*, \omega_{F}^{-1}(X)] = \omega_{F}^{-1}([h, X]), \forall h \in gl(n, q, R) \oplus R^{q*}, X \in R^n,$$

where the bracket on the right is the one of $sl(n + 1, R)$. It follows that the expression we have to compute is equal to

$$\sum_{j=p+1}^{n} (L_{\omega_{F}^{-1}(e_j)}L_h, p_{n,q}S(p_{n,q}\epsilon^j, \eta^1, \ldots, \eta^{k-1}) + (L_{[h,e_j]}p_{n,q}S(p_{n,q}\epsilon^j, \eta^1, \ldots, \eta^{k-1})).$$

Finally, we obtain

$$p_{n,q} \text{Div}^{\omega_F} (L_h, S)(\eta^1, \ldots, \eta^{k-1}) - (\rho_* (p_{n,q}[h, e_j])p_{n,q}S(\eta^1, \ldots, \eta^{k-1})$$

$$= p_{n,q} \text{Div}^{\omega_F} (L_h, S)(\eta^1, \ldots, \eta^{k-1}) + (\rho_* (p_{n,q}(h \otimes e_j + [h, e_j]Id))p_{n,q}S(p_{n,q}\epsilon^j, \eta^1, \ldots, \eta^{k-1}).$$

The result then easily follows from the definition of $\rho$ on $S^k(R^n)$.

Proposition 14. If $S$ is an equivariant function on $P_F$ representing an adapted symbol, we have

$$p_{n,q}[L_h, (\text{Div}^{\omega_F})^l S] = (q + 2k - l)p_{n,q} i(h) (\text{Div}^{\omega_F})^{l-1} S,$$

for every $h \in R^{q*}$.

Proof. For $l = 1$, this is simply the proposition 13. Then the result follows by induction, using propositions 12 and 13. One has indeed, if one supposes the result true to $l - 1$, that

$$L_h, p_{n,q} \text{Div}^{\omega_F} (\text{Div}^{\omega_F})^{l-1} S = p_{n,q} \text{Div}^{\omega_F} L_h, S$$

is equal to

$$(q + 2(k - l + 1) - 1)p_{n,q} i(h) \text{Div}^{\omega_F} S + p_{n,q} \text{Div}^{\omega_F} L_h, \text{Div}^{\omega_F} S - p_{n,q} \text{Div}^{\omega_F} L_h, S,$$

i.e. to

$$l(q + 2k - l)p_{n,q} i(h) \text{Div}^{\omega_F} S.$$

Proposition 15. If $f \in C^\infty(P_F, R)_{GL(n,q,R)}$, then

$$L_h, (\nabla^{\omega_F})^k f = (\nabla^{\omega_F})^k L_h, f = -k(k - 1)(\nabla^{\omega_F})^{k-1} f \land h,$$

for every $h \in R^{q*}$. 
Proof. If \( k = 0 \), then the formula is obviously true. Then we proceed by induction. In view of the symmetry of the expressions that we have to compare, it is sufficient to check that they coincide when evaluated on the \( k \)-tuple \((X, \ldots, X)\) for every \( X \in \mathbb{R}^n \). The proof is similar to the one of proposition [13]: first the evaluation and the Lie derivative commute:
\[
(L_h^* (\nabla^{\omega_F})^k f)(X, \ldots, X) = L_{h^*}((\nabla^{\omega_F})^k f(X, \ldots, X)).
\]
Next, we use the definition of the iterated invariant differential and we let the operators \( L_h^* \) commute so that the latter expression becomes
\[
L_{\omega_F^k}(X) L_h^*(((\nabla^{\omega_F})^{k-1} f)(X, \ldots, X)) + (L_{[h,X]}^* (((\nabla^{\omega_F})^{k-1} f))(X, \ldots, X).
\]
By the induction, the first term is equal to
\[
(\nabla^{\omega_F})^k L_h^* f(X, \ldots, X) - (k-1)(k-2)((\nabla^{\omega_F})^{k-1} f \circ h)(X, \ldots, X).
\]
For the second term, we use proposition [10] and relation (46) and we obtain, if one denotes by \( \rho \) the action on \( S^{k-1} \mathbb{R}^n \),
\[
(\rho_{\ast}(h \otimes X) + \langle h, X \rangle \text{Id})((\nabla^{\omega_F})^{k-1} f))(X, \ldots, X).
\]
The result follows by the definition of \( \rho_{\ast} \).

**Theorem 12.** In the adapted situation, the formula giving the quantization \( Q_F \) is then the following:
\[
Q_F(\nabla_F, S)(f) = p_F^{2n+2} \sum_{l=0}^k C_{k,l}(\text{Div}^{\omega_F} p_F^{2n+2} S, \nabla^{\omega_F}_{\ast}^{k-1} p_F^{2n+2} f)) \tag{47}
\]
if
\[
C_{k,l} = \frac{(k-1) \cdots (k-l)}{(q+2k-1) \cdots (q+2k-l)} \binom{k}{l}, \forall l \geq 1, \quad C_{k,0} = 1.
\]

**Proof.** The proof goes as in [MR05]. First, we have to check that the formula makes sense: the function
\[
\sum_{l=0}^k C_{k,l}(\text{Div}^{\omega_F} p_F^{2n+2} S, \nabla^{\omega_F}_{\ast}^{k-1} p_F^{2n+2} f)
\]
has to be \( H(n+1, q+1, \mathbb{R}) \)-equivariant. It is obviously \( GL(n, q, \mathbb{R}) \)-equivariant by propositions [10] and [12]. It is then sufficient to check that it is \( \mathbb{R}^{q*} \)-equivariant. This follows directly from propositions [14] and [15] and from the relation
\[
C_{k,l}(q+2k-l) = C_{k,l-1}(k-1)(k-l).
\]
Next we see, using the results of [CSS97] p.47 that the principal symbol of \( Q_F(\nabla_F, S) \) is exactly \( S \), and formula (47) defines a quantization, that is projectively invariant, by the definition of \( \omega_F \). Next, the naturality of the quantization defined in this way is easy to understand: it follows from the naturality of the association of an adapted projective structure \( P_F \rightarrow M \) endowed with an adapted normal Cartan connection \( \omega_F \) to a class of projectively equivalent torsion-free adapted connections on \( M \) and from the naturality of the lift of the equivariant functions on \( P_F^1 \rightarrow M \) to equivariant functions on \( P_F \). \[ \square \]

### 6.2 Construction in the foliated situation

In the foliated situation, one can define the invariant differentiation in this way:

**Proposition 16.** The following definition makes sense: if \( f \) is a foliated function on \( P(F) \), then
\[
(\nabla^{\omega(F)} f)(v_1, \ldots, v_k) = \frac{1}{k!} \sum_{\nu} L_{\omega(F)^{-1}(v_{\nu_1})} \circ \cdots \circ L_{\omega(F)^{-1}(v_{\nu_k})} f(v),
\]
where \( \omega(F)^{-1}(v) \) is a vector field such that its image by \( \omega(F) \) is equal to \( v \).
Proof. One has to show that the definition is independent of the choice of the vector field. Indeed, two such vector fields differ by a vector field tangent to \(F_{P^2N}\) and one can show that if \(f\) is constant along the leaves of \(F_{P^2N}\), then \(L_{\omega(F)}^{-1}(v)\) is a foliated function too if \(v \in \mathbb{R}^q\). Indeed, if \(X\) is tangent to \(F_{P^2N}\), then \(L_X L_{\omega(F)}^{-1}(v) = 0\). To show that, it suffices to prove that \(L_{[X,\omega(F)]^{-1}(v)} = 0\).

One has \(0 = d\omega(F)(X,\omega(F)^{-1}(v)) = X.v - \omega(F)^{-1}(v).\omega(F)(X) - \omega(F)([X,\omega(F)^{-1}(v)])\). As the first two terms are equal to 0, the third term vanishes too. One has then that \([X,\omega(F)^{-1}(v)]\) is tangent to \(F_{P^2N}\) and then \(L_{[X,\omega(F)]^{-1}(v)} = 0\).

In the foliated situation, we define the divergence operator in this way:

**Definition 15.** The divergence operator with respect to the Cartan connection \(\omega(F)\) is defined by

\[
\text{Div}^{\omega(F)} : C^\infty(P(F), S^k(\mathbb{R}^q); F_{P^2N}) \to C^\infty(P(F), S^{k-1}(\mathbb{R}^q); F_{P^2N}) : S \mapsto \sum_{j=1}^q i(e^j)\nabla_{e^j}^{\omega(F)} S.
\]

One can then easily adapt the propositions \([10][12][13][14][15]\). The proofs of these propositions are completely similar to the proofs of the corresponding results in \([MR05]\).

**Proposition 17.** If \(f\) is a \(\text{GL}(q, \mathbb{R})\)-equivariant foliated function on \(P(F)\) then \(\nabla^{\omega(F)} f\) is \(\text{GL}(q, \mathbb{R})\)-equivariant too.

**Proposition 18.** If \(S \in C^\infty(P(F), S^k(\mathbb{R}^q); F_{P^2N})_{\text{GL}(q, \mathbb{R})}\),

\[
\text{Div}^{\omega(F)} S \in C^\infty(P(F), S^{k-1}(\mathbb{R}^q); F_{P^2N})_{\text{GL}(q, \mathbb{R})}.
\]

In our computations, we will make use of the infinitesimal version of the equivariance relation: if \((V, \hat{\rho})\) is a representation of \(H(q + 1, \mathbb{R})\), if \(f \in C^\infty(P(F), V)_{H(q+1, \mathbb{R})}\) then one has

\[
L_h^* f(u) + \hat{\rho}_s(h) f(u) = 0, \quad \forall h \in \text{gl}(q, \mathbb{R}) \oplus \mathbb{R}^{q*} \subset \text{sl}(q + 1, \mathbb{R}), \forall u \in P(F).
\]

**Proposition 19.** For every \(S \in C^\infty(P(F), S^k(\mathbb{R}^q); F_{P^2N})_{\text{GL}(q, \mathbb{R})}\) we have

\[
L_h^* \text{Div}^{\omega(F)} S - \text{Div}^{\omega(F)} L_h^* S = (q + 2k - 1)i(h)S,
\]

for every \(h \in \mathbb{R}^{q*}\).

**Theorem 13.** For every \(S \in C^\infty(P(F), S^k(\mathbb{R}^q); F_{P^2N})_{\text{GL}(q, \mathbb{R})}\), we have

\[
L_h^*(\text{Div}^{\omega(F)})^l S - (\text{Div}^{\omega(F)})^l L_h^* S = l(q + 2k - l)i(h)(\text{Div}^{\omega(F)})^{l-1} S,
\]

for every \(h \in \mathbb{R}^{q*}\).

**Theorem 14.** If \(f \in C^\infty(P(F), \mathbb{R}; F_{P^2N})_{\text{GL}(q, \mathbb{R})}\), then

\[
L_h^*(\nabla^{\omega(F)})^k f - (\nabla^{\omega(F)})^k L_h^* f = -k(k-1)(\nabla^{\omega(F)})^{k-1} f \forall h,
\]

for every \(h \in \mathbb{R}^{q*}\).

**Theorem 15.** In the foliated situation, the formula giving the quantization \(Q(F)\) is the following:

\[
Q(F)(\nabla(F), S)(f) = p^2(F)^{-1}(\sum_{l=0}^k C_{k,l} (\text{Div}^{\omega(F)})^l (p^2(F))^l S, \nabla^{\omega(F)} S)^{k-l}(p^2(F))^l f)
\]

if

\[
C_{k,l} = \frac{(k-1) \cdots (k-l)}{(q + 2k - 1) \cdots (q + 2k - l)} \binom{k}{l}, \forall l \geq 1, \quad C_{k,0} = 1.
\]
6.3 Quantization commutes with reduction

**Proposition 20.** If \( f \) is an equivariant function on \( P \) representing a basic function, then

\[
(\nabla^{\omega^k} f)(v_1, \ldots, v_k) = \mathfrak{p}^{2*}(\nabla^{\omega^k} (\mathcal{F}S))(p_{n,q}v_1, \ldots, p_{n,q}v_k).
\]

*Proof.* Indeed, one has first that \( \mathfrak{p}^{2*}(\omega^k(p)) \) is equal to \( \omega^k(p) \) modulo a vector field tangent to \( \mathcal{F}P \). By induction, if the proposition is true to \( k-1 \), it is true for \( k \) :

\[
(\nabla^{\omega^k} f)(v_1, \ldots, v_k) = L_{\omega_{\mathcal{F}}^k(v_k)}(\nabla^{\omega^{k-1}} f)(v_1, \ldots, v_{k-1})
= L_{\omega_{\mathcal{F}}^k(v_k)}\mathfrak{p}^{2*}(\nabla^{\omega^{k-1}}(\mathcal{F}S))(p_{n,q}v_1, \ldots, p_{n,q}v_{k-1})
= \mathfrak{p}^{2*}(L_{\omega_{\mathcal{F}}^k(v_k)}\nabla^{\omega^{k-1}}(\mathcal{F}S))(p_{n,q}v_1, \ldots, p_{n,q}v_{k-1})
= \mathfrak{p}^{2*}(\nabla^{\omega^{k-1}}(\mathcal{F}S))(p_{n,q}v_1, \ldots, p_{n,q}v_k).
\]

\[\square\]

In an other part,

**Proposition 21.** If \( S \) is an equivariant function on \( P \) representing an adapted symbol, then

\[
p_{n,q}(\text{Div}^{\omega^k} S) = \mathfrak{p}^{2*}(\text{Div}^{\omega(\mathcal{F})^{l-1}}(\mathcal{F}\pi S)).
\]

*Proof.* Indeed, by induction, if it is true to \( l-1 \), it is true for \( l \) :

\[
p_{n,q}(\text{Div}^{\omega^l} S) = p_{n,q} \sum_{j=p+1}^{n} i(\omega^l) L_{\omega_{\mathcal{F}}^l(v_j)}(\text{Div}^{\omega^{l-1}} S)
= \sum_{j=p+1}^{n} L_{\omega_{\mathcal{F}}^l(v_j)}p_{n,q}i(\omega^l)(\text{Div}^{\omega^{l-1}} S)
= \sum_{j=p+1}^{n} L_{\omega_{\mathcal{F}}^l(v_j)}i(p_{n,q}\omega^l)\mathfrak{p}^{2*}(\text{Div}^{\omega(\mathcal{F})^{l-1}}(\mathcal{F}\pi S))
= \sum_{j=p+1}^{n} \mathfrak{p}^{2*} L_{\omega(\mathcal{F})^{-1}(p_{n,q}v_j)}i(p_{n,q}\omega^l)(\text{Div}^{\omega(\mathcal{F})^{l-1}}(\mathcal{F}\pi S)).
\]

\[\square\]

**Theorem 16.** If \( S \) is an equivariant function on \( P \) representing an adapted symbol and if \( f \) is an equivariant function on \( P \) representing a basic function, then

\[
\langle \text{Div}^{\omega^k} S, \nabla^{\omega^{k-1}} f \rangle = \mathfrak{p}^{2*}(\text{Div}^{\omega(\mathcal{F})^{l-1}}(\mathcal{F}\pi S), \nabla^{\omega(\mathcal{F})^{k-1}}(\mathcal{F}\pi f))
\]

if \( S \) is of degree \( k \). The quantization commutes then with the reduction :

\[
Q_{\mathcal{F}}(\nabla S)(f) = Q(\mathcal{F})(\nabla S)))(\mathcal{F}\pi S)(\mathcal{F}\pi f).
\]

*Proof.* Indeed, if \( \text{Div}^{\omega^k} S = v_1 \vee \ldots \vee v_{k-1} \),

\[
\langle \text{Div}^{\omega^k} S, \nabla^{\omega^{k-1}} f \rangle = \langle \nabla^{\omega^{k-1}} (\mathcal{F}\pi f) \rangle(v_1, \ldots, v_{k-1})
= \mathfrak{p}^{2*}(\nabla^{\omega^{k-1}}(\mathcal{F}\pi f))(p_{n,q}v_1, \ldots, p_{n,q}v_{k-1})
= \langle p_{n,q} \text{Div}^{\omega^k} S, \mathfrak{p}^{2*} \nabla^{\omega(\mathcal{F})^{k-1}}(\mathcal{F}\pi f) \rangle.
\]

The conclusion follows then from Theorems \[\underline{12}\] and \[\underline{15}\].
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