Some Conditions for P-Solubility of Finite Groups

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Abstract: A subgroup $H$ of a group $G$ is c-subnormal in $G$ if $G$ has a subnormal subgroup $T$ such that $HT=G$ and $3H \subseteq HG$. Using this concept, in Jaraden obtain some new conditions for solubility of a finite group are given. Here we obtain local versions of these results.

Key words: Finite group, p-soluble group, maximal subgroup, normal index, c-subnormal subgroup

INTRODUCTION

All groups that we consider are finite. Let $M$ be a maximal subgroup of a group $G$. Then normal index $|G|$: Min of $M$ in $G$ is equal to $|H/K|$ where $H/K$ is a chief factor of $G$ such that $K \subseteq M$ and $H \not\subseteq M$ (we note that every two chief factors with such property are isomorphic). This concept was introduced by Deskins where the following nice result was proved: A group $G$ is soluble if and only if for every its maximal subgroup $M$ it is true that $|G|$: $M_l = |G|$: $Min$. Local versions of this result were obtained by many researchers. In Wang, analyzing the concept of normal index, introduced the following important concept: A subgroup $H$ of a group $G$ is said to be c-normal if there exists a normal subgroup $T$ such that $HT = G$ and $3H \subseteq HG$ (where $HG$ is the intersection of all $G$-conjugates of $H$, i.e., the unique largest normal subgroup of $G$ contained in $H$). Using this concept Wang obtained several new interesting results on soluble and supersoluble groups. The concept of c-normal subgroup was used and analyzed. In particular, by Jaraden the following its generalization was considered.

Definition: A subgroup $H$ of a group $G$ is said to be c-subnormal in $G$ if there exists a subnormal subgroup $T$ such that $HT = G$ and $3H \subseteq HG$.

Using this concept, by Jaraden obtained some new conditions for solubility of a group were obtained. Here we prove the following theorems.

Theorem 1: A group $G$ is p-soluble if and only if every maximal subgroup $M$ with $p \mid |G|$: $Min$ is c-subnormal in $G$.

Theorem 2: A group $G$ is p-soluble if and only if it has a p-soluble maximal subgroup $M$ such that either $p \mid |G|$: $Min$ or $M$ is c-subnormal in $G$.

PRELIMINARIES

Notation is standard. We shall need the following well known facts about subnormal subgroups.

Lemma 1: Let $G$ be a group, $H$ be a subgroup of $G$. Then the following statements hold:
- If $H$ is subnormal in $G$ and $M \subseteq G$, then $H \subseteq M$.
- If $K \trianglelefteq G$ and $H$ is subnormal in $G$, then $HK/K$ is subnormal in $G/K$.

Lemma 2: Let $L$ be a minimal normal subgroup of a group $G$ and $T$ be a subnormal subgroup of $G$. Then $L \subseteq NG(T)$.

The following useful lemma was proved by Beidleman and Spencser.

Lemma 3: Let $M$ be a maximal in $G$ subgroup, $N/G$ and $N_M$. Then $|G/M| = |G/N: M/N| = |G/N: M/N|.$

Lemma 4: (Frattini argument). Let $N$ be a normal subgroup of a group $G$ and $N_p$ be a Sylow p-subgroup of $N$. Then $G = NNG(Np)$.

Recall that a primitive group is a group $G$ such that for some maximal subgroup $U$ of $G$, $UG = 1$.

A primitive group is of one of the following types (see [8; A.,(15.2) ]):
- $Soc(G)$, the socle of $G$ is an abelian minimal normal subgroup of $G$, complemented by $U$.
- $Soc(G)$ is a non-abelian minimal normal subgroup of $G$.
- $Soc(G)$ is the direct product of the two minimal normal subgroups of $G$ which are both non-abelian and complemented by $U$. 

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Lemma 5: Let \( M \) be a maximal subgroup of \( G \) with \( MG = 1 \), where \( G \) is a primitive group of type 2. Let \( R = \text{Soc}(G) \) be the socle of \( G \). If \( R \setminus M = 1 \), then \( M \) is a primitive group of type 2 and the simple component of \( R \) is isomorphic to a section of a simple component of \( \text{Soc}(M) \).

We shall also need the following observations on c-subnormal subgroups.

Lemma 6: Let \( G \) be a group and \( H \) a subgroup of \( G \). Then the following statements are true:

- If \( H \) is c-subnormal in \( G \) and \( H \trianglelefteq K \trianglelefteq G \), then \( H \) is c-subnormal in \( K \);
- Let \( K/G \) and \( K \trianglelefteq H \). Then \( H \) is c-subnormal in \( G \) if and only if \( H/K \) is c-subnormal in \( G/K \);
- If \( K/G \trianglelefteq H \) is c-subnormal in \( G \), then \( HK/K \) is c-subnormal in \( G/K \).

Proofs of Theorem 1 and 2

Proof of Theorem 1: First assume that \( G \) is a p-soluble group. Let \( M \) be a maximal subgroup of \( G \). Assume that \( p \mid |G: M| \). Let \( H/MG \) be a chief factor of \( G \). Then \( p \mid |H/MG| \) and so \( H/MG \) is an abelian p-group. Hence \( H \trianglelefteq M = MG \). Thus \( M \) is c-subnormal in \( G \).

Now assume that every maximal subgroup \( M \) of \( G \) with \( p \mid |G: M| \) is c-subnormal in \( G \). We shall show that \( G \) is p-soluble. Assume that it is false and let \( G \) be a counterexample with minimal order. Then

- \( p \mid |G| \) (it is evident)
- \( G \) is not simple. Indeed, assume that \( G \) is simple and let \( M \) be a maximal in \( G \) subgroup. Then \( p \mid |G: M| \) and so by hypothesis \( M \) is c-subnormal in \( G \). Let \( T \) be a subnormal subgroup of \( G \) such that \( MT = G \) and \( T \trianglelefteq MG = 1 \). Then \( |T| = |G: M| \). Hence \( G \) is not simple.
- If \( R \) is a minimal normal subgroup of \( G \), then \( R = \text{Soc}(G) \) is the unique minimal normal subgroup of \( G \), \( R \) is not abelian and \( p \mid |R| \).

Let \( H \) be a non-indentity normal subgroup of \( G \). And let \( M/H \) be a maximal subgroup of \( G/H \). Assume \( p \mid |G/H: M/H| \) and so by hypothesis \( M \) is c-subnormal in \( G \). Now using Lemma 6, we see that \( M/H \) is c-subnormal in \( G/H \). Thus the hypothesis holds for \( G/H \). But \( |G/H| < |G| \) and so by the choice of \( G \) we conclude that \( G/H \) is p-soluble. Since the class of all p-soluble groups is a formation we see that \( R = \text{Soc}(G) \) is the unique minimal normal subgroup of \( G \). It is clear also that \( p \mid |R| \) and that \( R \) is not abelian.

- \( G \) has a maximal subgroup \( M \) such that \( R \trianglelefteq M \) and \( p \mid |G: M| \).
- Indeed, let \( R_p \) be a Sylow p-subgroup of \( R \), \( P \) be a Sylow p-subgroup of \( G \) such that \( R_p \trianglelefteq P \). Let \( N = \text{NG}(R_p) \) be the normalizer of \( R_p \) in \( G \). Then \( |R_p| = |R| \cdot |P| \cdot |P| \). Besides since \( R \) is not abelian, we have \( N \neq G \). Now let as choose a maximal subgroup \( M \) of \( G \) such that \( N \trianglelefteq M \). Then of course \( p \mid |G: M| \). We note also that \( R \trianglelefteq M \). Indeed, by Frattini argument, \( G = RN \). But \( N \trianglelefteq M \) and so \( R \trianglelefteq M \).
- \( M \) has a subnormal complement \( T \) in \( G \).

Since by (4) \( R \trianglelefteq M \), we have \( MG = 1 \) and so \( p \mid |R| \). Hence by hypothesis \( M \) is c-subnormal in \( G \). Therefore \( G \) has a subnormal subgroup \( T \) such that \( TM = G \) and \( T \triangleright M \).

- Final contradiction.

Let \( L \) be a minimal subnormal subgroup of \( G \) contained in \( T \). Let \( L^n \) be the normal closure of \( L \) in \( G \). Then \( L^n \neq 1 \) and so \( R \trianglelefteq L^n \). Assume that \( L \trianglelefteq R \). Then by Lemma 1,

\[
L \triangleright R \text{ is a subnormal subgroup of } G \text{ and } 1 \trianglelefteq L \trianglelefteq R \trianglelefteq L^n. \text{ Hence } L = 1, \text{ since } L \text{ is a minimal subnormal subgroup of } G. \text{ By Lemma 2, } R \trianglelefteq \text{NG}(L). \text{ Hence } L = 1, \text{ since } L \trianglelefteq \text{CG}(R). \text{ Since } \text{CG}(R) \trianglelefteq G \text{ and } R \trianglelefteq \text{CG}(R). \text{ Then } R \text{ is an abelian group. This contradiction shows that } L \trianglelefteq R. \text{ Since } R \text{ is a minimal normal subgroup of } G,
\]

- \( R = A_1 \times \cdots \times A_t \), where \( A_1 = A_2 = \cdots = A_t = A \) and \( A \) is a non-abelian simple group. Hence \( L = A \). Clearly \( p \) divides the order \(|A| \) of the group \( A \). Hence \( p \) divides the order \(|L| \) of the group \( L \). By Lagrange’s theorem the order \(|L| \) of the group \( L \) divides the order \( T \) of the group \( T \). Hence the prime \( p \) divides \(|T| \). We have known that \( G = TM \) and \( T \triangleright M \). Hence \( |G| = |T| = |M| \mid |G| \mid |M| \mid |M| \). But the prime \( p \) does not divide the index \(|G: M| \) of \( M \) in \( G \). Hence \( p \) does not divide \(|T| \). This contradiction shows that \( G \) is a p-soluble group.

The theorem is proved.

Proof of Theorem 2: In view of Theorem 1 we have only to prove the sufficiency. Assume that it is false and let \( G \) be a counterexample with minimal order. Then
• G/N is p-soluble for every non-identity normal subgroup N of G.

Indeed, if N \not\subseteq M, then G/N = MN/N = M/N M 3 M is p-soluble. Let N \subseteq M. Then M/N is a p-soluble maximal subgroup of G/N such that either M/N is c-subnormal in G or p \mid |G/N|: M/Nn = |G|: Mn. Hence the hypothesis holds for G/N and so G/N is p-soluble by the choice of G since |G/N| < |G|.

• G has unique minimal normal subgroup H which is non-abelian and p \mid |H| (it directly follows from (1)).

• G has a subnormal subgroup T such that G = TM and T \cap M = 1.

Since by hypothesis M is p-soluble, then in view of (2) H \not\subseteq M. Now it is clear that |H| = |G|: Mn and so by (2), p \mid |G|: Mn. Hence by hypothesis M is c-subnormal in G. Let T be a subnormal in G subgroup such that TM = G and T 3 M \subseteq MG. But H \subseteq M and so MG = 1. Hence T 3 M = 1. (4) If

1 = H0 \leq H1 \leq \ldots \leq Hn = T = T0 \leq T1 \leq \ldots \leq Tm = G (1)

is a composition series of G, then every factor T1/T0, T2/T1, \ldots, Tm/Tm−1 is either a group of order p or a p’-group.

It is clear |G|: T = |T1/T0||T2/T1|\ldots |Tm/Tm−1|. Now we consider the following series

1=T03M\leq T13M \leq \ldots \leq Tm−1 3 M \leq Tm 3 M = M (2)

Evidently Ti−1 3 M \lhd Ti 3 M for all i = 1, 2, \ldots, m. Note also that

|T1 3 M)/(T0 3 M)||T2 3 M)/(T1 3 M)|\ldots |Tm 3 M)/(Tm−1 3 M)| = |M| = |G|: T = |T1/T0||T2/T1|\ldots |Tm/Tm−1|.

Since

(Ti 3 M)/(Ti−1 3 M) = (Ti 3 M)/(Ti 3 M)|Ti−1 = T1−(Ti 3 M)/Ti−1 \leq T/Ti−1, |(Ti 3 M)/(Ti−13 M)| \leq |T/Ti−1|

for all i = 1, 2, \ldots, m, a so (Ti 3 M)/(Ti−1 3 M) | Ti/Ti−1 is a simple group for all i = 1, 2, \ldots, m. Thus series (2) is a composition series of the group M. By hypothesis M is p-soluble. Hence every factor of the series (2) is either a group of order p or a p’-group and so every factor T1/T0, T2/T1, \ldots, Tm/Tm−1 is too.

\[ H\cap M = 1. \]

Let H = A1 \times \ldots \times At where A1 = \ldots = At = A is a non-abelian simple group. Let us consider the following composition series of G:

\[ 1 \leq A1 \leq A1A2 \leq \ldots \leq A1A2\ldots At−1 \leq H = K0 \leq K1 \leq \ldots \leq Kr = G \quad (3) \]

By Jordan-Holder Theorem [8; I,11.5] there exist indices i1, i2, \ldots, it such that

A1 = Hi1/Hi1−1, A1A2/A1 = Hi2/Hi2−1 \ldots ,A1A2\ldots At−1 = Hit/Hit−1. Hence |H| \leq |T| = |G|: Mn. But |G|: Mn = |H|: H 3 M 1 and so H\cap M = 1.

Final contradiction.

Let A be a composition factor of H. In view of (2), the group G is primitive of type 2 and so by (5) and Lemma 5, A is isomorphic to some section D/L where D \leq \text{Soc}(M). But by hypothesis M is p-soluble and so A is p-soluble. Then H is a p-soluble group and therefore H is a p-group, contrary to (2). The theorem is proved.

**SOME APPLICATIONS**

**Theorems 1:** and 2 have many corollaries. The most important of them we consider in this section.

**Corollary 1:** A group G is soluble if every its maximal subgroup M is c-subnormal in G\[^{[1]}\].

**Corollary 2:** A group G is soluble if it has a soluble maximal subgroup M which is c-subnormal in G\[^{[12]}\].

**Corollary 3:** A group G is soluble if every its maximal subgroup M is c-normal in G\[^{[7]}\].

**Corollary 4:** A group G is soluble if it has a soluble maximal subgroup M which is c-normal in G\[^{[7]}\].

It was proved that for a maximal subgroup M of a group G the following conditions are equivalent\[^{[7]}\]:

- M is c-normal in G;
- |G: M| = |G: Mn|.

Thus one can obtain from Theorem 1,2 the following known results.
Corollary 5: (W.E. Deskins\textsuperscript{[2]}) A group $G$ is soluble if for every its maximal subgroup $M$ we have $|G:M| = |G:M_n|$.

Corollary 6: (A. Ballester-Bolinches\textsuperscript{[5]}) A group $G$ is $p$-soluble if for every its maximal subgroup $M$ we have either $p | |G:M_n|$ or $|G:M| = |G:M_n|$.

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