Group of Canonical Diffeomorphisms
and the Poisson-Vlasov Equations

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Abstract: Dynamics of collisionless plasma described by the Poisson-Vlasov equations is connected with the Hamiltonian motions of particles and their symmetries. The Poisson equation is obtained as a constraint arising from the gauge symmetries of particle dynamics. Variational derivative constrained by the Poisson equation is used to obtain reduced dynamical equations. Lie-Poisson reduction for the group of canonical diffeomorphisms gives the momentum-Vlasov equations. Plasma density is defined as the divergence of symplectic dual of momentum variables. This definition is also given a momentum map description. An alternative formulation in momentum variables as a canonical Hamiltonian system with a quadratic Hamiltonian functional is described. A comparison of one-dimensional plasma and two-dimensional incompressible fluid is presented.
1 Introduction

The purpose of this series of papers is to study the geometric structures underlying both kinematical descriptions and dynamical formulations of plasma motion, and to provide a geometrical framework for the Poisson-Vlasov equations

\[ \nabla^2 \phi_f (q) = -e \int f(z) \, d^3 p \]  

\[ \frac{\partial f}{\partial t} + \frac{p}{m} \cdot \nabla_q f - e \nabla_q \phi_f \cdot \nabla_p f = 0 \]

of plasma dynamics. These works are initiated from a detailed study of an unpublished review on plasma dynamics by Marsden and Ratiu [1] and inspired from the problems and ideas in Marsden and Morrison [2]. We refer to [3]-[5] for background materials and for references on early works such as [6]-[10]. The Poisson-Vlasov system was first written in Hamiltonian form by Morrison [11]-[16]. The Poisson bracket was then shown to be the Lie-Poisson bracket on the space of plasma densities [17]-[19].

The kinematical or Lagrangian description will provide us with the configuration space \( G = Diff_{can}(T^* Q) \); the group of canonical diffeomorphisms on the phase space \( T^* Q \) of motions of individual plasma particles. We shall elaborate, in the next section, the geometry of \( Diff_{can}(T^* Q) \) which is the framework for the Hamiltonian (Lie-Poisson) structure of the Poisson-Vlasov equations.

In section three, we describe the Poisson equation as a kinematical constraint on the dynamics of Eulerian variables. We show that the Poisson equation characterizes the set of zero values of the momentum map associated with the action of additive group of functions \( \mathcal{F}(Q) \) on the position space \( Q \) of particles. This is the gauge group of particle motion on the canonical phase space \( T^* Q \).

Momentum map realization of the Poisson equation implies that the true configuration space for the Poisson-Vlasov dynamics must be the semi-direct product space \( \mathcal{F}(Q) \ltimes Diff_{can}(T^* Q) \) with the action of the additive group \( \mathcal{F}(Q) \) of functions given by fiber translation on \( T^* Q \) and by composition on right with the canonical transformations. In order to adopt the configuration space \( Diff_{can}(T^* Q) \), which has been customary in earlier treatments of the subject [5],[17]-[19], we rather proceed by adapting a constraint variational derivative. In doing so, we implicitly take the advantage of the facts that the Lie-Poisson structure on \( \mathcal{F}^*(Q) \) is trivial and that the constraint is of first class. The dual vector space \( \mathfrak{g}^* \) of the Lie algebra \( \mathfrak{g} = \mathcal{X}_{ham}(T^* Q) \) of Hamiltonian vector fields turns
out to be the space of non-closed one-form densities on $T^*\mathcal{Q}$. By symplectic duality, this can be identified with the space of non-Hamiltonian vector fields on $T^*\mathcal{Q}$. This gives the decomposition of the tangent space $TT^*\mathcal{Q} = \mathfrak{g} \oplus (\mathfrak{g}^*)^2$ on which one can start a geometric treatment for Legendre transformation [20]. One of the main themes of the present work is to introduce the dynamical equations on $(\mathfrak{g}^*)^2$—part of $TT^*\mathcal{Q}$ and make their relations with the Vlasov-Poisson equations (1) and (2) precise.

In section four, we shall present the kinematical reduction of the dynamics on $T^*\mathcal{G}$. In the momentum coordinates $(\Pi_i, \Pi^i)$ of $\mathfrak{g}^*$ the Poisson-Vlasov equations take the form

$$\nabla^2 \phi_{\Pi} (q) = e \int \nabla_q \cdot \Pi_p (z) \, d^3p$$  \hfill (3)

$$\frac{d\Pi_i (z)}{dt} = -X_h (\Pi_i (z)) + e \frac{\partial^2 \phi_f (q)}{\partial q^i \partial q^j} \Pi^j (z)$$  \hfill (4)

$$\frac{d\Pi^i (z)}{dt} = -X_h (\Pi^i (z)) - \frac{1}{m} \delta^{ij} \Pi_j (z)$$  \hfill (5)

and they admit Lie-Poisson Hamiltonian structure with a Hamiltonian function linear in momenta.

Section five will be devoted to the usual density formulation of the Poisson-Vlasov equations. The reduced dynamics on $\mathfrak{g}^*$ has a further symmetry given by the action of the additive group $\mathcal{F}(T^*\mathcal{Q})$ of functions on $T^*\mathcal{Q}$. The momentum map $\mathfrak{g}^* \to \mathcal{F}^*(T^*\mathcal{Q}) = Den(T^*\mathcal{Q})$ defines the plasma density function $f$.

In section six, we remark that Eqs.(4) and (5) are not the only equations in momentum variables leading to the Vlasov equation. We present another set of equations described by a canonical Hamiltonian structure with a quadratic Hamiltonian functional.

In section seven, we compare one dimensional plasma with two-dimensional incompressible fluid.

2 Motion of Collisionless Plasma

2.1 Kinematical description

We let $\mathcal{Q} \subset \mathbb{R}^3$ denote the configuration space in which the plasma particles move. The cotangent bundle $T^*\mathcal{Q}$ is the corresponding momentum phase space. This has a natural symplectic structure given by the canonical two-form $\Omega_{T^*\mathcal{Q}} = dq^i \wedge dp_i$, where we employ the summation over the repeated indices. In the sequel, $\Omega^2_{T^*\mathcal{Q}}$ will denote the natural isomorphism $T^*T^*\mathcal{Q} \to TT^*\mathcal{Q}$ taking uniquely a one-form to a vector field on $T^*\mathcal{Q}$ [3],[4]. We let $t \to \varphi_t$ be a curve such that for each $t$, $\varphi_t$ is a
canonically transformed particle phase space $T^*Q$ preserving the canonical symplectic two form $\Omega_{T^*Q}$. Given a point $Z \in T^*Q$ regarded as a reference point or, a Lagrangian label, we let $z = \varphi_t(Z)$, also written as $z = (q, p) = \varphi_t(Z, t)$, denote the current phase space point or, the Eulerian coordinates of plasma particles. The phase space velocity is given by the time dependent vector

$$
\dot{z} = \frac{d}{dt}\varphi_t(Z) = X_t(\varphi_t(Z)) = X(z, t)
$$

(6)

and it generates the flow $\varphi_t$. Since $\varphi_t$ is canonical, $X$ is infinitesimally Hamiltonian. We shall assume that it is globally Hamiltonian and write $h(z, t)$ for the corresponding Hamiltonian function so that at each $t$, $X = X_h$.

2.2 Group of canonical diffeomorphisms

The flow $\varphi_t$ of the Hamiltonian vector field $X_h$ on $T^*Q$ is a one parameter family of elements of the group $G = Diff_{can}(T^*Q)$ of all transformations of $T^*Q$ preserving the symplectic two-form $\Omega_{T^*Q}$. In this work, we restrict the discussion to the canonical transformations connected to the identity $[3][21]$. $\varphi_t$ acts on left by evaluation on the space $T^*Q$ of reference plasma configuration to produce the motion of particles. Thus, a configuration of plasma can be specified by an element of $G$. The right action of $G$ commutes with the particle motions and constitute an infinite dimensional symmetry group of the kinematical description. This is the particle relabelling symmetry $[22]$. For the motion of particles on $T^*Q$ described by the left action

$$
G \times T^*Q \longrightarrow T^*Q : (\varphi, Z) \rightarrow L_\varphi(Z) = \varphi(Z) = z
$$

(7)

the velocity field on $T^*Q$ is the vector $X_\varphi$ lying in the tangent space

$$
T_\varphi G = \{X_\varphi : T^*Q \to TT^*Q \mid \tau_{T^*Q} \circ X_\varphi = \varphi\}
$$

(8)

of $G$ at $\varphi$. Here, $\tau_{T^*Q} : TT^*Q \to T^*Q$ is the natural projection of the tangent bundle of particle phase space. That means, $X_\varphi$ is a map over the element $\varphi : T^*Q \to T^*Q$ of the configuration space. Vector fields on $G$ are then defined to be the maps $X : G \to TG$ whose value at $\varphi \in G$ is given by $X_\varphi \in T_\varphi G$. Since the motion is associated with the left action, the velocity field $X$ is invariant under the right action $R_\psi : G \to G$ of $G$ for all $\psi \in G$.

The space $\{X \mid (R_\psi)_*X = TR_\psi \circ X \circ \psi^{-1} = X, \ \forall \psi \in G\}$ of right invariant vector fields on $G$ is isomorphic to the tangent space over the identity mapping of $T^*Q$ with the isomorphism given at each $\varphi \in G$ by

$$
X_h \to X_h \circ \varphi \ \forall X_h \in T_{id}G
$$

(9)
where $X_h$ is the Hamiltonian vector field $\Omega^\sharp_{T^\ast Q}(dh)$ on $T^\ast Q$ whose flow is $\varphi$. Then, the Lie algebra
\[ g = (\mathfrak{x}_{ham}(T^\ast Q) ; -[,]) \] (10)
of $G$ consists of Hamiltonian vector fields on $T^\ast Q$ and $[,]$ denotes the standard Jacobi-Lie bracket, with conventions as in [4]. The isomorphism in Eq.(9) is just the relation
\[ X_\varphi(Z, t) = X_h(z, t) = (X_h \circ \varphi_t)(Z) \] (11)
between the Lagrangian and the Eulerian velocities at the point $z$. In coordinates, if we decompose a canonical transformation $\varphi$ into position-momentum pairs as $\varphi = (\xi, \eta)$, then, we have
\[ X_\varphi(Z, t) = \xi^i (Z) \frac{\partial}{\partial q^i} + \eta_i (Z) \frac{\partial}{\partial p_i} = \dot{q}^i \frac{\partial}{\partial q^i} + \dot{p}_i \frac{\partial}{\partial p_i} = X_h(z, t). \] (12)

By the identification $X_h \to h$ modulo constants, and the homomorphism
\[ [X_h, X_k] = -X_{\{h,k\}T^\ast Q}, \] (13)
we have the identification $g \simeq (\mathcal{F}(T^\ast Q); \{,\}T^\ast Q)$ of the Lie algebra with the space of functions on $T^\ast Q$ endowed with the canonical Poisson bracket.

We define the dual vector space $g^\ast$ of the Lie algebra $g=\mathfrak{x}_{ham}(T^\ast Q)$ to be the non-closed one-form densities on $T^\ast Q$
\[ g^\ast = \{\Pi_{id} \otimes d\mu \in \Lambda^1(T^\ast Q) \otimes \text{Den}(T^\ast Q) : d\Pi_{id} \neq 0\} \] (14)
where $d\mu \in \text{Den}(T^\ast Q)$ is a volume six-form on $T^\ast Q$. At each point $z \in T^\ast Q$ the space $\Lambda^6(T^\ast Q)$ of six-forms is one-dimensional whose basis can be chosen to be the symplectic volume $d\mu = \Omega^3_{T^\ast Q} = \Omega_{T^\ast Q} \wedge \Omega_{T^\ast Q} \wedge \Omega_{T^\ast Q}$.

With these definitions we have the non-degenerate pairing
\[ \langle X_h, \Pi_{id} \otimes d\mu \rangle = \int_{T^\ast Q} <X_h(z), \Pi_{id}(z)> d\mu(z) = \int_{T^\ast Q} h(z) \nabla_z \cdot \Pi_{id}(z) d\mu(z) = \langle h, \nabla_z \cdot \Pi_{id} \otimes d\mu \rangle \] (15)
of $g$ and $g^\ast$ with respect to the $L^2$-norm. Here, $\Pi_{id}^z$ denotes the components of the vector $\Pi_{id}^z = \Omega^3_{T^\ast Q}(\Pi_{id}) = \Pi_{id}^z(z) \cdot \nabla_z$ and the pairing in the integrand is defined over the finite dimensional space $T^\ast Q$. From the last line of the above equations we conclude that the pairing of algebra
with its dual is nondegenerate if \( \nabla_z \cdot \Pi_{id}^z(z) \neq 0 \). Since, \( \Omega_{T^*Q} \) is nondegenerate, this is equivalent to the condition \( d\Pi_{id} \neq 0 \). The definition of dual space \( g^* \) of the Lie algebra \( g \) already implies that the identification of vector space \( g = \mathfrak{X}_{ham}(T^*Q) \) with the space of functions \( F(T^*Q) \) can be extended to the identification of dual space with the space \( \text{Den}(T^*Q) \) of densities on \( T^*Q \) via the definition of density

\[
\Pi_{id} \otimes d\mu \in g^* \longrightarrow f \ d\mu \in \text{Den}(T^*Q)
\]

which does not vanish due to nondegeneracy restriction in the definition of \( g^* \). That is, we have the one-sided correspondence

\[
\Pi_{id} \otimes d\mu \in g^* \longrightarrow f \ d\mu \in \text{Den}(T^*Q)
\]

with the definition of \( f \) given by Eq. (16) (see also the internet supplement to [3]). We shall show later the precise way of obtaining density in the context of a momentum map. By the above identifications with function spaces the pairing between Lie algebra \( g \equiv F(T^*Q) \) and its dual \( g^* \equiv \text{Den}(T^*Q) \) takes the form of multiply-and-integrate

\[
\langle h, f \otimes d\mu \rangle = \int_{T^*Q} h(z) f(z) \ d\mu(z).
\]

**Remark 1** The configuration space \( \text{Diff}_{can}(T^*Q) \) can also be given a description in terms of sections of the trivial bundle \( T^*Q_0 \times T^*Q \rightarrow T^*Q_0 \) where \( T^*Q_0 \) is the particle phase space with Lagrangian coordinates \( Z \) and \( T^*Q \) carries the Eulerian coordinates \( z \). The total space \( T^*Q_0 \times T^*Q \) is symplectic with the two-form \( \Omega_- = \Omega_{T^*Q_0} - \Omega_{T^*Q} \). A diffeomorphism \( \varphi : T^*Q_0 \rightarrow T^*Q \) is canonical if \( \Omega_{T^*Q_0} - \varphi^*\Omega_{T^*Q} = 0 \). It follows that \( \Omega_- \) vanishes when restricted to the graphs of canonical diffeomorphisms. Graphs are elements of the space \( \Gamma(T^*Q_0 \times T^*Q) \) of sections of the trivial bundle \( T^*Q_0 \times T^*Q \rightarrow T^*Q_0 \). For a base point \( Z \in T^*Q_0 \), the total space is twelve dimensional and the graph \( (Z, \varphi(Z)) \) of a diffeomorphism is a six dimensional subspace. When \( \varphi \) is canonical, \( \Omega_- \) vanishes on graphs and such graphs are called Lagrangian subspaces. If we denote the space of all sections of the trivial bundle on which the restriction of \( \Omega_- \) vanishes, namely the space of all Lagrangian sections, by \( \text{Lag}(\Gamma(T^*Q_0 \times T^*Q)) \), then we have the identification

\[
\text{Diff}_{can}(T^*Q) \cong \text{Lag}(\Gamma(T^*Q_0 \times T^*Q)).
\]

To have the corresponding description for the Lie algebra of Hamiltonian vector fields, first observe that we can identify the dual space \( g^* \) with the space of vector fields

\[
(g^*)^z = \{ \Pi_{id}^z = \Omega_{T^*Q}^z(\Pi_{id}) \in TT^*Q \mid d\Pi_{id} \neq 0 \}
\]
which are not Hamiltonian by definition. Then, we have the decomposition of the tangent space $TT^*Q = g \oplus (g^*)^2$ into the underlying vector spaces of the Lie algebra and its dual. It turns out that the Lie algebra of Hamiltonian vector fields can be identified with the space of all Lagrangian submanifolds of $\Gamma(TT^*Q)$ with respect to the Tulczyjew symplectic structure on $TT^*Q$ [23].

To extend the pairing of $g$ and $g^*$ to a pairing of tangent and cotangent bundles we define an element $\Pi_\varphi$ of the covector space $\Lambda^1_\varphi G$ of one-forms as a map over $\varphi$

$$\Lambda^1_\varphi G = \{\Pi_\varphi : T^*Q \mapsto T^*T^*Q, \pi_{T^*Q} \circ \Pi_\varphi = \varphi\} \tag{18}$$

where $\pi_{T^*Q} : T^*T^*Q \to T^*Q$ is the natural projection, and a 1-form field $\Pi$ on $G$ to be a section $\Pi : G \to \Lambda^1G$ such that for each $\varphi \in G$ we have $\Pi_\varphi \in \Lambda^1_\varphi G$. The cotangent space at $\varphi$ is then the space

$$T^*_\varphi G = \{\Pi_\varphi \otimes d\mu \in \Lambda^1_\varphi G \otimes \text{Den}(T^*Q)\} \tag{19}$$

of one-form densities on $G = Diff_{can}(T^*Q)$. The pairing of tangent and cotangent spaces at $\varphi \in G$ becomes

$$\langle X_\varphi, \Pi_\varphi \otimes d\mu \rangle = \int_{T^*Q} <X_\varphi(Z), \Pi_\varphi(Z)> d\mu(Z). \tag{20}$$

Following diagram summarizes the mapping properties with reference to particle phase space of elements of $T^*_\varphi G, T^*^*_\varphi G$, $g$ and $g^*$

$$
\begin{array}{cccc}
TT^*Q & \xleftarrow{\Omega^*_T \cdot Q, \Omega^*_T \cdot Q} & T^*T^*Q \\
\uparrow X_\varphi & & \downarrow \pi_{T^*Q} \uparrow \Pi_\varphi \\
X_h \uparrow \uparrow \tau_{T^*Q} & & T^*_\varphi G \\
T^*Q & \xrightarrow{\varphi} & T^*Q & \equiv & T^*Q \\
\downarrow \varphi & & \downarrow \varphi & \leftarrow \varphi & T^*Q
\end{array}
$$

The adjoint action $Ad_\varphi : g \to g$ of $G$ on its Lie algebra is given by push forward of Hamiltonian vector fields on $T^*Q$

$$Ad_\varphi(X_h) = T\varphi \circ X_h \circ \varphi^{-1} = \varphi_*(X_h). \tag{21}$$

The coadjoint action $(Ad_\varphi)^* : g^* \to g^*$ of $G$ on $g^*$ is the dual $Ad_{\varphi^{-1}}^*$ with respect to the pairing in Eq.(15) of the map $Ad_{\varphi^{-1}}$ and is given by push forward as well

$$Ad_{\varphi^{-1}}^*(\Pi_{id}) = \varphi_*(\Pi_{id}) = T^*_\varphi(R_{\varphi} \circ L_{\varphi^{-1}}) \circ \Pi_{id}. \tag{22}$$
Taking the derivatives at the identity one finds that the adjoint action $ad_{X_h}$ of $g$ on itself and the coadjoint action $ad^{\ast}_{X_h}$ of $g$ on $g^\ast$ are generated by the Lie derivative with respect to the Hamiltonian vector field $X_h \in T T^\ast Q$. In other words, given $X_h, X_k \in g$ and $\Pi_{id} \in g^\ast$, the Lie derivative $\mathcal{L}_{X_h}$ may be regarded as tangent vectors $ad_{X_h} \in T X_k g$ or $ad^{\ast}_{X_h} \in T \Pi_{id} g^\ast$ depending on whether it is associated with adjoint or coadjoint actions, respectively.

The cotangent lift to $T^\ast \psi G$ of the right action is $T^\ast \phi \circ \psi^{-1} R_\psi (\Pi_\varphi) = \Pi_\varphi \circ \psi^{-1}$. For $\psi = \varphi$ this gives the translation of the one form $\Pi_{id} = \Pi_\varphi \circ \varphi^{-1}$. By definition, this is an element of the dual space $g^\ast$ of the Lie algebra $g$ of Hamiltonian vector fields. The right invariant momentum map $J_L : T^\ast G \to g^\ast$ for the lifted left action (i.e. the plasma motion) is defined by

$$
\langle J_L (\Pi_\varphi) \otimes d\mu, X_h \rangle = \langle \Pi_\varphi \otimes d\mu, T_{id} R_\varphi \circ X_h \rangle
$$

$$
= \langle \Pi_{id} \otimes d\mu, X_h \rangle
$$

(23)

so that we have $J_L (\Pi_\varphi) = \Pi_{id} \in g^\ast$.

3 Poisson Equation as a Constraint

3.1 Momentum map description of Poisson equation

The canonical symplectic structure on $T^\ast Q$ is invariant under translation of fiber variable by an exact one-form over $Q$. This is the gauge transformation of canonical Hamiltonian formalism. If we identify $T^\ast Q$ with the space of one-forms $\Lambda^1(Q)$ on $Q$, this invariance may be described as the Hamiltonian action on $T^\ast Q$

$$
\Lambda^0(Q) \times \Lambda^1(Q) \to \Lambda^1(Q) : (\phi(q), p \cdot dq) \to p \cdot dq + d\phi(q)
$$

(24)

of the space $\Lambda^0(Q) \equiv \mathcal{F}(Q)$ of zero-forms. From an algebraic point of view, the exterior derivative $d : \Lambda^0(Q) \to \Lambda^1(Q)$ can be interpreted as a map describing a Lie algebra isomorphism of the additive algebra of functions $\mathcal{F}(Q)$ into the additive algebra of one-forms $\Lambda^1(Q)$ [24]. As the dual of any Lie algebra isomorphism is a momentum map, we have

$$
\mathbb{J}_{\mathcal{F}(Q)} : \Lambda^2(Q) \to \text{Den}(Q)
$$

(25)

where the space of two forms $\Lambda^2(Q)$ on $Q$ and the space of densities (three-forms) on $Q$ are the duals, with respect to the $L^2$-norm, of $\Lambda^1(Q)$ and $\Lambda^0(Q)$, respectively. We can identify $\Lambda^1(Q)$ and its dual space $\Lambda^2(Q)$.
by the Hodge duality operator $*$ associated with a Riemannian metric on $Q$. Then

$$\langle \mathbb{J}_{\mathcal{F}(Q)}(p \cdot dq, *d\phi(q)), \phi(q) \rangle = -\int_Q \phi(q) d* d\phi(q) \quad (26)$$

where $*d\phi(q)$ is considered as a two-form over the one-form $p \cdot dq$, and we used the pairing of $\Lambda^1(Q)$ and $\Lambda^2(Q)$ given by integration over $Q$. Thus, the momentum map is

$$\mathbb{J}_{\mathcal{F}(Q)}(p \cdot dq, *d\phi(q)) = -d* d\phi(q) \in \text{Den}(Q).$$

For the Euclidean metric on $Q$, the operator $d* d$ is the usual Laplacian $\nabla^2_q$ in Cartesian coordinates.

The action of the additive group $\mathcal{F}(Q)$ of functions can be carried over objects defined on $T^* Q$ such as the group of canonical diffeomorphisms, its Lie algebra and the dual of the Lie algebra. For our purpose of obtaining the Poisson equation as a constraint described by a momentum map, we restrict ourselves to the action of $\mathcal{F}(Q)$ on the space $\mathcal{F}(T^* Q)$ of functions on $T^* Q$. It will be convenient to describe the momentum map as the dual of some Lie algebra isomorphism into. We think of $\mathcal{F}(T^* Q)$ equipped with the Poisson bracket to be an algebra isomorphic to the Lie algebra of Hamiltonian vector fields (c.f. section 2.2). Then, $\mathcal{F}(Q)$ is a commutative subalgebra of $(\mathcal{F}(T^* Q), \{,\}_{T^* Q})$ corresponding to the generators of the action on $T^* Q$ by fiber translation $p \mapsto p - \nabla_q \phi(q)$. The Hamiltonian function is $-\phi(q) \in \mathcal{F}(Q) \subset \mathcal{F}(T^* Q)$. Thus, the required Lie algebra isomorphism is from the additive algebra of functions $\mathcal{F}(Q)$ into the Poisson bracket algebra on $\mathcal{F}(T^* Q)$. Together with the dualization we have

$$\begin{array}{c}
\left( \mathcal{F}(Q), + \right) \longrightarrow \left( \mathcal{F}(T^* Q), \{,\}_{T^* Q} \right) \\
\uparrow \quad \uparrow \\
\text{Den}(Q) \quad \mathbb{J}_{\mathcal{F}(Q)} \quad \text{Den}(T^* Q)
\end{array}$$

and the momentum map $\mathbb{J}_{\mathcal{F}(Q)} : \text{Den}(T^* Q) \rightarrow \text{Den}(Q)$ is computed from

$$\langle \mathbb{J}_{\mathcal{F}(Q)}(f(z) d\mu(z)), \phi(q) \rangle = -\int_{T^* Q} f(z) \phi(q) d\mu(z) \quad (27)$$

to be the volume density

$$\mathbb{J}_{\mathcal{F}(Q)}(f(z) d\mu(z)) = -\left( \int_{T^* Q} f(z) \ d^3 p \right) d^3 q \quad (28)$$
Combining the actions of $F(Q)$ on $\Lambda^1(Q)$ and $F(T^*Q)$ we have the momentum map

$$J_{F(Q)} : \Lambda^2(Q) \times \text{Den}(T^*Q) \longrightarrow \text{Den}(Q)$$

given by

$$J_{F(Q)}(*d\phi(q), ef(z) d\mu(z) ; \phi(q)) = -(\nabla^2 \phi(q) + e \int f(z) \ d^3p) \ d^3q$$

whose zero value is the Poisson equation. This constraints the region in the product space $\Lambda^2(Q) \times \text{Den}(T^*Q)$ for consideration of dynamics in the Eulerian variables $(\phi, f)$, namely

$$J_{F(Q)}^{-1}(0)/F(Q) = \{(*d\phi, f d\mu) \in \Lambda^2(Q) \times \text{Den}(T^*Q) \mid \nabla^2 \phi(q) + e \int f(z) \ d^3p = 0\}$$

is the reduced space for the Lie-Poisson description. This corresponds to a subset of the dual space $\Lambda^2(Q) \times \mathfrak{g}^*$ of the Lie algebra $\Lambda^1(Q) \times \mathfrak{g}$ with trivial bracket on the first factor. The underlying Lie group is $F(Q) \times Diff_{can}(T^*Q)$ with the first factor acting on canonical diffeomorphisms by composition with fiber translations.

### 3.2 Constraint variational derivative

The momentum map description of Poisson equation implies that we have to consider the configuration variables of collisionless plasma motion to be $(\phi, \varphi)$ where the electrostatic potential $\phi$ is a function on $Q$ and $\varphi$ is a canonical diffeomorphism of $T^*Q$ generating the particle motion. Hence, the configuration space of plasma motion must be $F(Q) \circledast Diff_{can}(T^*Q)$ where $\circledast$ denotes the semidirect product of groups with the additive group $F(Q)$ of functions acting on the second factor by composition on right. Here, we want to adopt an approach allowing the possibility to use much simpler configuration space $G = Diff_{can}(T^*Q)$. More precisely, we want to use the Poisson equation as a constraint for the variational derivatives of Eulerian variables, in particular, the plasma density function.

To this end, we consider the Green’s function solution

$$\phi_f(q, t) = e \int_{T^*Q} K(q|\dot{q}) f(\dot{z}) \ d\mu(\dot{z})$$

(29)

of the Poisson equation [11], which relates the plasma density $f$ and the electrostatic potential $\phi_f$ [11], [14], [25]. As an example of Eulerian quantities, we take the Hamiltonian function
\[ H_{LP}(f) = \int_{T^*Q} f(z) h_f(z) \, d\mu(z) \]  

of the Lie-Poisson formulation \([11],[14],[18],[25]\). Here, the density dependent function

\[ h_f(z) = \frac{p^2}{2m} + \frac{1}{2} e\phi_f(q) \]  

is related to the Hamiltonian function \(h\) governing the particle dynamics up to a multiplicative factor in potential term. The Hamiltonian functional in Eq.(30) is the total energy of the plasma in Eulerian coordinates.

**Lemma 2** \([14]\) For the functional in Eq.(30) with Eq.(31) we have

\[ \frac{\delta H_{LP}(f)}{\delta f} = \frac{1}{2m} p^2 + e\phi_f(q) = h(z). \]

**Proof.** Using the Poisson equation and the Green’s function solution, \(H_{LP}(f)\) can be put into the form

\[ H_{LP}(f) = \int_{T^*Q} \frac{1}{2m} p^2 f(z) \, d\mu(z) \]

\[ + \frac{e^2}{2} \int_{T^*Q} \int_{T^*Q} f(z) K(q|\dot{q}) f(\dot{z}) \, d\mu(z) \, d\mu(\dot{z}) \]  

up to the integral of the divergence term \(\nabla q \cdot (\phi_f(q) \nabla q \phi_f(q))\). It is now easy to obtain the lemma where a factor of 2 comes from the symmetry of the Green’s function \([11],[14],[25]\). ■

The constraint imposed by the Poisson equation is, in the language of Dirac formalism, first class and hence does not affect the Poisson bracket on the reduced space \([26]\). Thus, in obtaining equivalent dynamical formulations in alternative Eulerian variables we must use the same constraint. The foremost example of such a variable arises from the identification of the dual space \(g^*\) of the algebra of Hamiltonian vector fields with the space of densities \(Den(T^*Q)\). In the more basic formulation of dynamics with the momentum variables \(\Pi_{id} \in g^*\) the Hamiltonian functional turns out to be

\[ H_{LP}(\Pi_{id}) = \int_{T^*Q} \langle \Pi_{id}(z) , X_{h_f}(z) \rangle \, d\mu(z) \]  

which is equivalent to the functional \(H_{LP}(f)\) under the identification \([16]\).

**Lemma 3** For the Hamiltonian functional in Eq.(33) we have

\[ \frac{\delta H_{LP}(\Pi_{id})}{\delta \Pi_{id}} = X_h(z). \]
4 Momentum Formulation of Dynamics

4.1 Lie-Poisson dynamics

We apply the standard Lie-Poisson reduction to $G$ [17,18,27]. The right invariant extensions to $T^\ast G$ of elements of $g^\ast$ are obtained through composition with the momentum map $\mathcal{J}_L(\Pi_\varphi) = \Pi_{id}$. In particular, for a functional $H : g^\ast \to \mathbb{R}$ the right invariant extension is the functional $H^R : T^\ast G \to \mathbb{R}$ defined by

$$H^R = H \circ \mathcal{J}_L, \quad H^R(\varphi, \Pi_\varphi) = H(\Pi_{id}) = H (\Pi_\varphi \circ \varphi^{-1})$$

and applying the chain rule with $\Pi_\varphi = \Pi_{id} \circ \varphi$ we have the differential

$$\delta H^R = \delta H \frac{\delta H}{\delta \Pi_\varphi} \delta \varphi + \frac{\delta H}{\delta \Pi_{id}} \delta \Pi_{id} \delta \varphi + \frac{\delta H}{\delta \Pi_{id}} \frac{\delta H}{\delta \Pi_\varphi} \delta \Pi_\varphi.$$

If $\Pi_{id}(z) = \Pi_a(z)dz^a = \Pi_i(z)dz_i + \Pi^i(z)dp_i$ we have $(\delta \Pi_{id}/\delta \Pi_\varphi) |_{id}= 1$ and

$$\delta \Pi_{id} \bigg|_{id} = \delta \Pi_{a}(z)dz^a \bigg|_{id} = -d \Pi_{id}(z)$$

with $\Pi_{id}$ denoting the components of the one-form $\Pi_{id}$. Thus, for the differentiation of the right-invariant functionals we find

$$\frac{\delta H^R}{\delta \Pi_\varphi} \bigg|_{id} = \frac{\delta H}{\delta \Pi_{id}}, \quad \frac{\delta H^R}{\delta \Pi_{id}} \bigg|_{id} = -\left( \frac{\delta H}{\delta \Pi_{id}} \cdot \nabla z \right) \Pi_{id}$$

(34)

where the overhead arrow denotes the components of Lie algebra element. One can now evaluate the canonical Poisson bracket on $T^\ast G$ at the identity using the above relations. This gives (+)Lie-Poisson bracket on $g^\ast$, that is, $\{F^R, G^R\}_{T^*G} \mapsto \{F, G\}_{LP}$.

**Proposition 4** Let $\Pi_{id} \in g^*$ and $[ , ]$ be the Jacobi-Lie bracket on $g$. Then the Lie-Poisson bracket on $g^*$ is given by

$$\{ H(\Pi_{id}), K(\Pi_{id}) \}_{LP} = \int_{T^*Q} \Pi_{id}(z) \cdot \left[ \frac{\delta H}{\delta \Pi_{id}(z)}, \frac{\delta K}{\delta \Pi_{id}(z)} \right] \, d\mu(z)$$

(35)

where $\delta H/\delta \Pi_{id}$ is regarded to be an element of $g$.

The derivation can also be found in the internet supplement to [3]. The Lie-Poisson dynamics on $g^\ast$ may be written as follows:
Proposition 5 The Hamiltonian vector fields on $\mathfrak{g}^*$ for the Lie-Poisson structure defined by the bracket in Eq. (35) have the form

$$\frac{d\Pi_{id}}{dt} = -\text{ad}^*_h/\delta\Pi_{id}(\Pi_{id}) = \mathcal{L}_{\delta H/\delta\Pi_{id}}(\Pi_{id}) = J_{LP}(\Pi_{id}) \frac{\delta H}{\delta\Pi_{id}}$$

(36)

where the Hamiltonian operator defining the bracket (35) is given by

$$J_{LP}(\Pi_{id}) = \left( \begin{array}{ccc} \Pi_{id} & \frac{d}{dq} & \frac{d}{dp} \\ \frac{d}{dq} & \Pi_{id} & \frac{d}{dq} \\ \frac{d}{dp} & \frac{d}{dq} & \Pi_{id} \end{array} \right)$$

(37)

with $\frac{d}{dq} \cdot \Pi_j = \frac{d\Pi_{id}}{dq} + \Pi_j \frac{d}{dq}$ etc.

The operator in Eq. (37) is apparently skew adjoint with respect to the $L^2$-norm and satisfies the Jacobi identity by construction [28], [29]. The relation of $J_{LP}(\Pi_{id})$ to the Lie-Poisson bracket in Eq. (35) is

$$\Pi_{id} \cdot \left[ \frac{\delta H}{\delta\Pi_{id}} , \frac{\delta K}{\delta\Pi_{id}} \right] = \frac{\delta H}{\delta\Pi_{id}} \cdot J(\Pi_{id})(\frac{\delta K}{\delta\Pi_{id}}) - \nabla_z \cdot \frac{\delta K}{\delta\Pi_{id}} (\Pi_{id} \cdot \frac{\delta H}{\delta\Pi_{id}})$$

where the divergence term on the right disappears upon integration.

Remark 6 The Hamiltonian operator $J_{LP}(\Pi_{id})$ may be considered to be a map taking a Lie algebra element $X_h$ in $\mathfrak{g}$ to the corresponding generator $\text{ad}^*_{X_h}$ of the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$. That is,

$$J_{LP}(\Pi_{id}) : \mathfrak{g} \longrightarrow (-\text{ad}_{\mathfrak{g}}^* : \mathfrak{g}^* \rightarrow \mathfrak{g}^*)$$

and with reference to particle phase space we have

$$J_{LP} : T^*Q \longrightarrow T^*Q$$

where $\mathcal{O}(\Pi_{id}) = \{ \text{Ad}_{\varphi}^*(\Pi_{id}) = \varphi_*(\Pi_{id}) \mid \varphi \in G \}$ is the coadjoint orbit through $\Pi_{id} \in \mathfrak{g}^*$.

4.2 Vlasov equations in momentum variables

In the Lie-Poisson setting, the Poisson-Vlasov equations arise from the Hamiltonian functional that generates the kinematical symmetries. This is the momentum function defined by means of the momentum map

$$\mathbb{J}_L(X_k)(\Pi_{id}) = \int_{T^*Q} \langle \Pi_{id}(z), X_k(z) \rangle \, d\mu(z)$$

(38)

for the lifted left action of $G$. Here the generator $X_k \in \mathfrak{g}$ is yet to be specified. Due to the constraint imposed by the Poisson equation, that is the constraint variational derivative, the relevant Hamiltonian vector field $X_k$ in Eq. (38) turns out to be the one associated with the function $h_f$ given in Eq. (31).
Proposition 7  For the right invariant Hamiltonian functional in Eq.(33) the Lie-Poisson equations on $\mathfrak{g}^*$ are

$$
\frac{d\Pi_i(z)}{dt} = -X_h(\Pi_i(z)) + e\frac{\partial^2 \phi_f(q)}{\partial q^i \partial q^j} \Pi^j(z) \tag{39}
$$

$$
\frac{d\Pi^i(z)}{dt} = -X_h(\Pi^i(z)) - \frac{1}{m} \delta^{ij} \Pi_j(z) \tag{40}
$$

with the constraint

$$
\nabla_q^2 \phi_{\Pi}(q) = e \int \nabla_q \cdot \Pi_p(z) \, d^3p. \tag{41}
$$

Eqs.(39) and (40), which we call the momentum-Vlasov equations, follow from the Lie-Poisson structure expressed in coordinates of the momentum one-form $\Pi_{id}$ and will be shown, in the next section, to give rise to the Vlasov equation in the density variable. The proof of proposition follows from the constraint variational derivative of $H_{LP}(\Pi_{id})$ obtained before and from Eq.(36) by computing, for example, the Lie derivative of $\Pi_{id}$ with respect to $X_h$.

5 Density Formulation of Dynamics

The definition of plasma density is motivated by the following observation. Regarding the variational derivative of functions on $\mathfrak{g}^*$ as elements of $\mathfrak{g}$ means that there are functions $h, k$ on $T^*Q$ such that the Jacobi-Lie bracket in Eq.(35) can be written as canonical Poisson bracket of $h, k$. Then the Lie-Poisson bracket on $\mathfrak{g}^*$ becomes

$$
\int_{T^*Q} \left( \frac{\partial \Pi_i(z)}{\partial p_i} - \frac{\partial \Pi^i(z)}{\partial q^i} \right) \{h(z), k(z)\}_{T^*Q} \, d\mu(z) \tag{42}
$$

which requires, as in the definition of dual algebra, the divergence of the vector $\Pi^i_{id}$ to be non-zero.

5.1 Introducing the plasma density function

We first show that the definition in Eq.(16) of plasma density function leads to the correct Lie-Poisson structure in this variable. Then we show that the relation between formulations of plasma dynamics in the momentum variables $\Pi_{id}$ and the plasma density function $f$ can be made precise in terms of a momentum map. We shall remark, in the next section, by presenting a canonical Hamiltonian system in momentum variables, that Eqs.(39) and (40) are not unique in the sense that they yield the Vlasov equation in density variable.
Proposition 8 The Hamiltonian operator $J_{LP}(\Pi_{id})$ on $\mathfrak{g}^*$ transforms into the Hamiltonian operator

$$J_{LP}(f) = \nabla_p f \cdot \nabla_q - \nabla_q f \cdot \nabla_p$$  \hspace{1cm} (43)

on the space $\text{Den}(T^*Q)$ of densities under the correspondence in Eq. (16).

Proof. Regarding the definition (16) as a transformation of Eulerian variables we can compute the transformation of the Hamiltonian operator $J_{LP}(\Pi_{id})$ as follows. The derivative of $f$ in the direction of $\Pi_{id}$ is a $3 \times 6$ matrix of differential operators

$$D_f(\Pi_{id}(z)) = [(d\frac{d}{dp_j}) - (d\frac{d}{dq_i})] = [\nabla_p - \nabla_q]$$  \hspace{1cm} (44)

which transforms the Hamiltonian operator $J_{LP}(\Pi_{id})$ according to

$$J_{LP}(f) = D_f(\Pi_{id}) \cdot J_{LP}(\Pi_{id}) \cdot D^*_f(\Pi_{id})$$  \hspace{1cm} (45)

where $D^*_f$ is the adjoint of $D_f$ with respect to the $L^2$-norm [29]. A direct computation from Eq. (45) gives the operator $J_{LP}(f)$. \hfill \blacksquare

Remark 9 $D^*_f$ transforms the Lie algebra elements, that is if $H$ is a functional on $\text{Den}(T^*Q)$, then $\delta H/\delta f$ is a function on $T^*Q$ and $D^*_f(\delta H/\delta f)$ gives the components of the Hamiltonian vector field in $\mathfrak{g}$ corresponding to the Hamiltonian function $\delta H/\delta f$.

Remark 10 It can also be verified that the momentum-Vlasov equations (39) and (40) yield the Vlasov equation (2) for $f$ from the definition of the density.

We recall that the Lie-Poisson structure of the Vlasov equation in density variable is defined by the bracket

$$\{H(f), K(f)\}_{LP} = \int_{T^*Q} f(z) \left\{ \frac{\delta H}{\delta f(z)}, \frac{\delta K}{\delta f(z)} \right\}_{T^*Q} d\mu(z)$$  \hspace{1cm} (46)

associated with the operator $J_{LP}(f)$ and the Hamiltonian function $H_{LP}(f)$ [31].

5.2 Momentum map description of density

Proposition 11 The momentum map for the action of the additive group $\mathcal{F}(T^*Q)$ of functions on $\mathfrak{g}^*$ defines the plasma density.
Proof. By definition in Eq.(14) of \( g^* \), the one-form \( \Pi_{id} \) is invariant under the addition of an exact one-form on \( T^*Q \). So, \( \mathcal{F}(T^*Q) \) is the gauge group in the definition of \( g^* \). With reference to the identification of \( g \) with \( \mathcal{F}(T^*Q) \), the action of \( \mathcal{F}(T^*Q) \) on \( g^* \) by translation can be regarded as the action of \( g \) on \( g^* \) by

\[(X_k, \Pi_{id}) \mapsto \Pi_{id} + \Omega^{\mathcal{F}_T^*Q}_k(X_k) \quad (47)\]

where \( \Omega^{\mathcal{F}_T^*Q}_k = (\Omega^2_{T^*Q})^{-1} \). This can be interpreted as the action of the underlying vector space of \( g \). Thus, we consider the Lie algebra isomorphism \( \mathcal{F}(T^*Q) \to g: k \mapsto X_k \) for the gauge equivalent classes of one-forms in \( g^* \). The dual of this is the required momentum map

\[\mathbb{J}_{Tr} : g^* \to \mathcal{F}^*(T^*Q) \equiv Den(T^*Q) \quad (48)\]

for the definition of the plasma density from the momentum variables. From definitions, we have

\[\langle \mathbb{J}_{Tr}(\Pi_{id}), k \rangle = \langle \Pi_{id}, X_k \rangle = \langle \nabla_z \circ \Omega^2_{T^*Q} \circ \Pi_{id}, k \rangle \quad (49)\]

where \( \nabla_z \) is taken to be the dual of the exterior derivative \( d \). When evaluated in Eulerian coordinates \( z \) the momentum map (49) gives exactly the definition (16) of the plasma density function. \( \blacksquare \)

Remark 12 It has been argued that the physical initial conditions must satisfy \( f(z,0) > 0 \) [14],[30]. This requires the description of density by elements \( \Pi_{id} \in g^* \) with \( \nabla_z \cdot \Omega^2_{id}(z) > 0 \). Equivalently, in the language of differential forms, we have \( d(\Pi_{id} \wedge \Omega^2_{T^*Q}) > 0 \). Consider a six dimensional domain \( D \) in \( T^*Q \) with boundary \( \partial D \). Then, the positive divergence implies

\[\int_{\partial D} \Pi_{id}(z) \wedge \Omega^2_{T^*Q}(z) > 0 \quad (50)\]

so that we have a volume element or an orientation for the five dimensional boundary of the region \( D \). This can now be related to the non-degeneracy of the orbit symplectic structure on \( g^* \). An element of the orbit through \( \Pi_{id} \) will be of the form \( \mathcal{L}_{X_k}(\Pi_{id}) \). By definition, the orbit symplectic structure is

\[\Omega_{\Pi_{id}}(\mathcal{L}_{X_k}(\Pi_{id}), \mathcal{L}_{X_g}(\Pi_{id})) = \int_{\partial D} \{g(z), k(z)\} \Pi_{id}(z) \wedge \Omega^2_{T^*Q}(z)\]

which, by Eq.(50) does not vanish for arbitrary functions \( g \) and \( k \).
6 Equivalence of Momentum and Density Formulations

Proposition 13 \( H_{LP}(f) = H_{LP}(\Pi_{id}) \)

Proof. Replace \( f \) in Eq. (32) by its definition to get

\[
H_{LP}(f) = \int_{T^*Q} \frac{1}{2m} p^2 (\nabla_p \cdot \Pi_q(z) - \nabla_q \cdot \Pi_p(z)) \, d\mu(z)
+ \frac{e^2}{2} \int_{T^*Q} \nabla_q \cdot \Pi_p(z) K(q|\dot{q}) \nabla_{q'} \cdot \Pi_{p'}(\dot{z}) \, d\mu(z) d\mu(\dot{z})
\]

upto divergence. The first and the second integrals are equivalent to

\[- \frac{p}{m} \cdot \Pi_q(z) , \quad - \frac{e^2}{2} \Pi_p(z) \cdot \nabla_q \left(K(q|\dot{q}) \nabla_{q'} \cdot \Pi_{p'}(\dot{z}) \right), \quad (51)\]

respectively. Using Green’s function solution we obtain \( H_{LP}(\Pi_{id}) \). Conversely, starting from the function \( H_{LP}(\Pi_{id}) \) we compute

\[
H_{LP}(\Pi_{id}) = \int_{T^*Q} (\nabla_p h_f(z) \cdot \Pi_q(z) + \nabla_q h_f(z) \cdot \Pi_p(z)) \, d\mu(z)
\]

which verifies the equivalence of the Hamiltonian functionals of the Lie-Poisson structures in momentum and density formulations.

Proposition 14 The canonical Hamiltonian system with respect to the symplectic two form

\[
\omega(\Pi_i, \Pi^j) = \int_{T^*Q} \delta \Pi_i(z) \wedge \delta \Pi^j(z) \, d\mu(z) \quad (52)
\]

and for the Hamiltonian functional

\[
H_0(\Pi_{id}) = \int_{T^*Q} \left( \Pi_i X_h(\Pi^i) + \frac{1}{2m} \delta^{ij} \Pi_i \Pi_j + \frac{e}{2} \frac{\partial^2 \phi_f}{\partial q^i q^j} \Pi^i \Pi^j \right)(z) \, d\mu(z) \quad (53)
\]

which is quadratic in the fiber coordinates of \( T^*T^*Q \) gives the Vlasov equation.

Proof. First of all, the density \( \mathcal{H}_0(z) \) of the Hamiltonian functional \( H_0 \) satisfies the divergence equation

\[
\frac{\partial \mathcal{H}_0(z)}{\partial t} - \nabla_z \cdot \left( (\Pi_i (X_h(\Pi^i) + \frac{1}{m} \delta^{ij} \Pi_j)) X_h \right)(z) = 0 \quad (54)
\]
which is the conservation law in Eulerian form. To obtain the canonical equations for the Hamiltonian functional in Eq. (53) we substitute the expressions for $X_h$ and $\phi_f$ into Hamiltonian functional $H_0$ and rearrange the terms to obtain

$$H_0 (\Pi_j, \Pi^j) = \int_{T^*Q} \Pi_j (z) \left( \delta^{ik} \frac{p_k \partial \Pi^j}{\partial q^i} + \frac{1}{2m} \delta^{ij} \Pi_i \right) (z) \, d\mu (z)$$

$$-e^2 \int_{T^*Q} \Pi_j (z) \Pi^i (\dot{z}) \frac{\partial \Pi^j (z)}{\partial p_i} \frac{\partial^2 K (q, \dot{q})}{\partial q^i \partial q^j} \, d\mu (\dot{z}) \, d\mu (z)$$

$$+ \frac{e^2}{2} \int_{T^*Q} \Pi^i (\dot{z}) \Pi^j (z) \Pi^j (z) \frac{\partial^3 K (q, \dot{q})}{\partial q^i \partial q^j \partial q^j} \, d\mu (\dot{z}) \, d\mu (z).$$

The variation with respect to the components $\Pi_j$ of $\Pi_q$ can easily be computed to give

$$\frac{\delta H_0}{\delta \Pi_j} (z) = X_h (\Pi^j (z)) + \frac{1}{m} \delta^{ij} \Pi_i (z) = - \frac{d\Pi_j (z)}{dt}$$

so that the first set of equations holds. For the other set of Hamilton’s equations, we first note that the second integral in $H_0$ may be written, up to a divergence, as

$$e^2 \int_{T^*Q} \frac{\partial \Pi_j (z)}{\partial p_i} \Pi^i (\dot{z}) \Pi^j (z) \frac{\partial^2 K (q, \dot{q})}{\partial q^i \partial q^j} \, d\mu (\dot{z}) \, d\mu (z)$$

and the derivative of this with respect to $\Pi^j (z)$ gives

$$e \frac{\partial \phi_f (q)}{\partial q^i} \frac{\partial \Pi_j (z)}{\partial p_i} + \frac{\partial}{\partial q^j} \left( e^2 \int_{T^*Q} \frac{\partial \Pi_k (\dot{z})}{\partial p_i} \Pi^k (\dot{z}) \frac{\partial K (q, \dot{q})}{\partial q^i} \, d\mu (\dot{z}) \right). \quad (55)$$

Similarly, we compute the derivative of the last term in $H_0$ with respect to the components $\Pi^j (z)$ of $\Pi_p$ to be

$$e \Pi^i (z) \frac{\partial^2 \phi_f (q)}{\partial q^i \partial q^j} + \frac{\partial}{\partial q^i} \left( \frac{e^2}{2} \int_{T^*Q} \Pi^i (\dot{z}) \Pi^j (\dot{z}) \frac{\partial^3 K (q, \dot{q})}{\partial q^i \partial q^j \partial q^j} \, d\mu (\dot{z}) \right). \quad (56)$$

Collecting these results we find

$$\frac{\delta H_0}{\delta \Pi^j} (z) = -X_h (\Pi_j (z)) + e \frac{\partial^2 \phi_f (q)}{\partial q^i \partial q^j} \Pi^j (z) + \frac{\partial \Phi (q)}{\partial q^j}$$

$$\quad = \frac{d\Pi_j (z)}{dt} + \frac{\partial \Phi (q)}{\partial q^j}.$$
where $\Phi(q)$ is the function consisting of two gradient terms in Eqs. (55) and (56). In passing to the density formulation the additional gradient term is acted on by $p$-divergence which identically vanishes because it is a function of $q$ only. Thus, the Vlasov equations resulting from the canonical flow of the quadratic Hamiltonian in Eq. (53) and the momentum Vlasov equations are the same.

The canonical Hamiltonian system in proposition (14) can be regarded as a manifestation of the fact that, the addition of an exact one-form to $\Pi_{id}$ does not affect the Vlasov equation in $f$. In fact, there are infinitely many such flows in momentum formulation due to symmetries in the definition of momentum variables. To this end, we observe the following symmetries associated with the momentum formulations. The quadratic Hamiltonian $H_0(\Pi_{id})$ is invariant under shifts in velocity together with a reflection in position

$$\Pi_i \mapsto \Pi_i + 2mX_h(\Pi_i), \quad \Pi^i \mapsto -\Pi^i.$$  

Both the Hamiltonian and the canonical symplectic structure are invariant under arbitrary diffeomorphisms in the variables $\Pi^i$. The definition (16) of the plasma density $f$ is also invariant under

$$\Pi_i \mapsto \Pi_i + (\nabla_p \times A_p)_i, \quad \Pi^i \mapsto \Pi^i + (\nabla_q \times A_q)^i$$

for arbitrary vector functions $A_q(z)$ and $A_p(z)$.

Regarding the components $\Pi_i$ of $\Pi_q$ as momentum variables in the canonical structure of proposition (14), Eqs. (40) becomes inverse Legendre transformations to be solved for the momenta. Then, the Lagrangian functional

$$L_0[\Pi_p] = \int_{T^*Q} \left( \frac{m}{2} |X_h(\Pi_p)|^2 - \frac{e}{2} \frac{\partial^2 \phi_f}{\partial q^i \partial q^j} \Pi^i \Pi^j \right) (z) \ d\mu(z)$$

involving the velocity $d\Pi_p/dt$ shifted by the term $-X_h(\Pi_p)$, gives the Euler-Lagrange equations

$$\ddot{\Pi}^i(z) + 2X_h(\dot{\Pi}^i(z)) + X^2_h(\Pi^i(z)) + \frac{\delta^{ij}}{m} \frac{\partial^2 \phi_f(q)}{\partial q^i \partial q^j} \Pi^k(z) = 0$$

which can also be obtained from Eqs. (39) and (40) by eliminating the variables $\Pi_i$.

7 Comparison with 2D Incompressible Fluid

The motion of an incompressible fluid in a two dimensional region $\mathcal{M} \subseteq \mathbb{R}^2$ is described by volume (area) preserving diffeomorphisms. The generators are divergence-free vectors. Since the dimension is two, these are
equivalent to canonical diffeomorphisms and the generators are canonical Hamiltonian vectors of the form \( \mathbf{v} = -X_\psi \) where the Hamiltonian function \( \psi \) is the stream function. The curl of \( \mathbf{v} \) or, equivalently, the Laplacian of the stream function \( \psi \) is the vorticity. Geometrically, it is a two-form given by

\[
\omega = d(\mathbf{v} \cdot d\mathbf{z}) = \nabla_z^2 \psi dq \wedge dp, \quad \mathbf{z} = (q,p)
\]

or, as a function \( \omega = \nabla_z \cdot \Omega_M^\sharp (\mathbf{v} \cdot d\mathbf{z}) \).

The dynamics is governed by the Euler equation in vorticity form

\[
\frac{\partial \omega}{\partial t} = \{ \omega, \psi \}_M
\]

where \( \{ , \}_M \) is the canonical Poisson bracket on \( \mathcal{M} \). This is the Lie-Poisson equation on the dual of the Lie algebra of the group of volume preserving diffeomorphisms for the Lie-Poisson structure

\[
\{ H(\omega), K(\omega) \}_{LP} = \int_{\mathcal{M}} \omega(\mathbf{z}) \left\{ \frac{\delta H}{\delta \omega(\mathbf{z})}, \frac{\delta K}{\delta \omega(\mathbf{z})} \right\}_\mathcal{M} d^2\mathbf{z}
\]

with the Hamiltonian functional

\[
H = \frac{1}{2} \int \mathbf{v}^2 d^2\mathbf{z} = \frac{1}{2} \int (\nabla_z \psi)^2 d^2\mathbf{z} = -\frac{1}{2} \int \psi \omega d^2\mathbf{z}
\]

where a divergence term in the last expression is omitted. Alternatively, defining the Green’s function solution

\[
\psi(\mathbf{z}) = -\int K(\mathbf{z}|\mathbf{z}') \omega(\mathbf{z}') d^2\mathbf{z}'
\]

for the equation \( \omega = \nabla_z^2 \psi \) we have

\[
H = \frac{1}{2} \int \int \omega(\mathbf{z}) K(\mathbf{z}|\mathbf{z}') \omega(\mathbf{z}') d^2\mathbf{z} d^2\mathbf{z}'.
\]

See references [16, 28] from which we extract the above summary, and [31-35] for more on fluid motions.

We observe that the quadratic Hamiltonian functional for incompressible fluid is a direct consequence of the definition of the dual of the Lie algebra by a metric. In this case, the (weak) non-degeneracy of the pairing is the same as the non-degeneracy of the metric and the Lie algebra can be identified with its metric dual. On the other hand, the Lie algebra \( \mathfrak{g} \) of Hamiltonian vector fields and its dual \( \mathfrak{g}^* \) are \( L^2 \)-orthogonal in \( TT^*Q \).
The momentum-Vlasov equations in components of $\Pi_{id}$ expresses the evolution of a volume cell, that is the density $f$, in the phase space $T^*Q$ in terms of its boundaries, that is, surfaces of the momenta $\Pi_{id}$. This interpretation was first given by Ye and Morrison in [36] for the Clebsch variables $(\alpha, \beta)$ defined by $\{\alpha, \beta\}_{T^*Q} = f$. In the present context, they form a non-closed one-form $\alpha d\beta$ and can be identified with $\Pi_{id}$.

|                       | 2D – Fluid | 1D – Plasma |
|-----------------------|------------|-------------|
| **Configuration space** | $\text{Diff}_{\text{vol}}(\mathcal{M})$ | $\text{Diff}_{\text{can}}(T^*Q)$ |
| **particle motion**    | volume preserving diffeomorphisms | Hamiltonian diffeomorphisms |
| **Lie algebra (generators of motion)** | divergence – free vector fields | Hamiltonian vector fields |
| **identification of Lie algebra with functions** | $\psi : \text{stream functions}$ | $h : \text{Hamiltonian functions}$ |
| **dual of Lie algebra** | $g^\flat(v) : \text{metric dual of velocity}$ | non – closed one – forms |
| **identification of dual with function spaces** | $\omega = \nabla \circ \Omega_\Lambda^2 \circ g^\flat(v) = \nabla^2 \psi$ | $f = \nabla \circ \Omega_{T^*Q}^{2} \circ \Pi_{id}$ |
| **$L^2$ – dual**       | two – forms | non – closed one – form densities |
| **Clebsch variables**  | $g^\flat(v) = \alpha d\beta$ | $\Pi_{id} = \alpha d\beta$ |
|                        | $\omega = \{\alpha, \beta\}$ | $f = \{\alpha, \beta\}$ |
| **Hamiltonian functionals** | $H = -\frac{1}{2} \int \omega(z)\psi(z)d^2z$ | $H_{LP} = \frac{1}{2} \int f(z)h(z)d^2z$ |
|                        | $= \frac{1}{2} \int v^2(z)d^2z$ | $= \frac{1}{2} \int <X_h, \Pi_{id}> d^2z$ |
| **dynamical equations** | $\frac{\partial}{\partial t} = \{\psi, \omega\}$ | $\frac{\partial f}{\partial t} = \{h, f\}$ |
| **Poisson equation**   | $\nabla^2 \phi = \omega$ | $\nabla^2 \phi = -\int f(z)d^4p$ |
|                        | as definition | as constraint |

Inspired from the relation $\omega = \nabla^2 \psi$ between the vorticity and Hamiltonian functions of 2D fluid, we can establish a similar relation between the plasma density function $f$ and the Hamiltonian function $h$. 
of particle motion. The Hessian of $h$ can be considered to be a map $\text{Hess}(h): TT^*Q \to T^*T^*Q$ which is non-degenerate if the potential function $\phi_f$ is non-degenerate. Let $X \in TT^*Q$, and define the vector field

$$Y = \Omega^2_{TT^*Q} \circ \text{Hess}(h) \circ X.$$  \hfill (63)

If $X$ is a Hamiltonian vector field with a Hamiltonian function which is at least quadratic in momenta, then $Y$ is not Hamiltonian. In particular, if we choose the Hamiltonian function to be $h$ and identify $Y$ with $\Pi^i_m$, then we get the Poisson equation. In other words, the Poisson equation in plasma resembles the relation between vorticity and stream functions of 2D fluid. This relation may also be described by assuming a non-degenerate Lagrangian functional on $TT^*Q$ which is yet to be found.

Suppose we have a Lagrangian $l$ on $TT^*Q$ quadratic in the velocities $(\dot{q}, \dot{p})$. We can introduce the momenta which reads

$$\Pi_i = \frac{\delta l}{\delta \dot{q}_i} = \frac{1}{m} \dot{p}_i, \quad \Pi^i = \frac{\delta l}{\delta \dot{p}_i} = -e\delta^{ij} \frac{\partial \phi_f}{\partial q_j}$$

for the special choice $h$ of the Hamiltonian function. Then, the definition of plasma density in terms of momentum variables gives $f = 1/m + e \nabla^2_q \phi_f$. Note also that with a rescaling of $m$ and a redefinition of $f$ we can write $f = \text{tr}(\text{Hess}(h))$.

8 Conclusions

Gauge symmetries of the Hamiltonian motion of the plasma particles leads to the kinematical constraint described by the Poisson equation. Thus, the Poisson type equations naturally arise in kinetic theories of particles moving in accordance with a canonical Hamiltonian formulation. Moreover, this implies that the true configuration space appropriate for the dynamical formulation of the collisionless plasma motion in Eulerian variables is the semi-direct product space $\mathcal{F}(Q) \bowtie \text{Diff}_{can}(T^*Q)$. This is well suited for a geometric understanding of the limit $c \to \infty$ of the Maxwell-Vlasov equations.

The formulation of dynamics in density variable is obtained by further reduction of momentum-Vlasov equations by the symmetry defining the gauge equivalence classes of momentum variables. The gauge algebra is, as a vector space, shown to be the same as $g$ but with an action different from the coadjoint action. The Eulerian velocity and momenta are complementary in the vector space $TT^*Q$. Obviously, this and other geometric properties disappear upon identification of $g$ and $g^*$ with function spaces $\mathcal{F}(T^*Q)$ and $\text{Den}(T^*Q)$, respectively. As an example, the function $h_f$ and the density $f$ appear symmetrically in the Hamiltonian
functional of the Lie-Poisson structure whereas the corresponding variables $X_{h_f}$ and $\Pi_{id}$ are complementary in the sense that $\Omega^f_{T^*Q}(g)$ and $g^*$ decompose the space of one-forms on $T^*Q$ into spaces of exact and non-closed one-forms, respectively. The momentum formulation clarifies the geometric relation between the motions of plasma particles and the Lie-Poisson description of dynamics \[23\]. We expect the space $TT^*Q$ be important for Euler-Poincaré formulation of dynamics \[7\],[37],[38\], and for application of Tulczyjew construction for Legendre transformation \[20\] from Lie-Poisson formulation.

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