A General Theory of Wightman Functions

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Abstract: One of the main open problems of mathematical physics is to consistently quantize Yang-Mills gauge theory. If such a consistent quantization were to exist, it is reasonable to expect that some version of the Wightman reconstruction theorem, by which a Hilbert space and quantum field operators are recovered from gauge invariant $n$-point functions, might hold. However, the original version of the Wightman theorem, and its algebraic formulation due to Borchers, are not equipped to deal with gauge fields or fields taking values in a noncommutative space. This paper explores a consistent generalization of the Borchers algebra, which has all of the nice mathematical features of the original model, and which appears to be the correct framework in which to formulate a generalized Wightman reconstruction theorem amenable to modern quantum theories such as gauge theories and matrix models.

Keywords: Yang-Mills Theory, Wightman Functions, Matrix Models
The Wightman axioms were formulated by Gårding and Wightman in the early 1950’s, but no nontrivial examples existed at that time, and consequently the axioms were not published until 1964 [5], at which time their publication had been motivated by the Haag-Ruelle scattering theory. The axioms are thoroughly discussed and many consequences are derived in the two excellent books [3] and [4]. We will also formulate the axioms below in Section 1.2 by way of introduction.

It is known that the Wightman axioms, in their original and unmodified form, describe only a small subset of the mathematical models used in elementary particle physics. Thus, many authors have considered modifications of the axioms which allow newer and more exotic physical theories to be formulated as rigorous mathematics. If we wish to perturb the axioms slightly, one obvious change with clear-cut physical implications is to relax the requirement that the test function space be $\mathcal{S}(\mathbb{R}^4)$. A large class of alternative test function spaces which still allow a formulation of the microscopic causality condition were proposed and developed by Jaffe [7]. The results of the present
paper are a more radical modification, in which the test functions in $\mathcal{S}(\mathbb{R}^4)$ are replaced by functions into a noncommuting $*$-algebra.

There are at least two types of equivalent reformulations of the Wightman axioms. One is due to Wightman, who wrote down a set of postulates governing a sequence of tempered distributions

$$\mathcal{W}_n \in \mathcal{S}(\mathbb{R}^{4n}), \quad n = 0, 1, \ldots$$

and proved that the postulates guarantee that the distributions $\mathcal{W}_n$ arise as the vacuum expectation values of a unique field theory satisfying the Wightman axioms, and conversely that the postulates hold in any Wightman field theory. This is what is known as the Wightman reconstruction theorem, and first appeared in the seminal paper [3]. The part of this construction relevant to representation theory is known in functional analysis as the GNS construction. A second reformulation in terms of the Schwinger functions, not directly used in the present work, was given by Osterwalder and Schrader [8, 9].

Borchers reformulated Wightman’s reconstruction theorem in several important papers [1, 2], with the result that a scalar boson quantum field theory is known to be characterized by a topological $*$-algebra $A$ (with unit element $1_A$) and a continuous positive form $\omega$ on $A$, satisfying

$$\omega(aa^*) \geq 0, \quad \omega(1_A) = 1, \quad a \in A.$$ 

Realistic models are generally described by tensor algebras, and the action of the state $\omega$ is computed from vacuum expectation values of products of fields. Although there is reason to believe the framework of states on tensor algebras could apply in general to a large class of quantum field theories, previous formulations of the Wightman reconstruction theorem have focused on scalar boson quantum field theories.

There are by now many known examples of a low-energy limit or compactification of string theory in which the theory is shown to be equivalent to a gauge theory with compact gauge group. It is also known that exact quantum string amplitudes can be computed from various flavors of matrix models. If a mathematically rigorous description using constructive field theory techniques is at all possible for these problems coming from string theory, then some generalization of the Wightman reconstruction theorem is needed.

The purpose of the present paper is to extend the work of Wightman and Borchers to include matrix-valued fields of the type required by gauge theory and matrix models. We first develop the mathematics, and then make contact with physical applications. The remainder of this introduction reviews the well-known Wightman procedure for commuting scalar fields; this serves to fix notation and set the context for the later sections. Section 2 explains how to generalize the Borchers construction and considers several important examples. Section 3 is concerned with the application of these ideas to two-dimensional Yang-Mills theory. In Section 4 we recall important recent work which applies matrix models to high energy physics and then show that, in the same sense in which scalar quantum field theories are Wightman states, matrix models are described by matrix states, and thus are special cases of the construction in Section 2. Finally, Section 5 provides a summary of the results obtained, in the form of a very general set of postulates for quantum field theory.

1.1 The Borchers Construction

Let

$$\mathcal{S}_0 = \mathbb{C}, \quad \mathcal{S}_n = \mathcal{S}(\mathbb{R}^{4n}), \quad \mathfrak{S} = \bigoplus_{n=0}^{\infty} \mathcal{S}_n.$$ 

The latter is a complete nuclear space under the direct sum topology. There is a natural map $\iota : \otimes^k \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^{kn})$ given by $\iota(f_1 \otimes \ldots \otimes f_k) = \prod_{j=1}^{k} f_j(x_j)$, where each $x_j \in \mathbb{R}^n$, and the image of $\iota$ is dense.

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Endow $\mathcal{S}$ with the noncommutative multiplication

$$
(f \times g)_l = \sum_{i+h=l} f_i \times g_h, \quad (1.1)
$$

and the involution $f^* = (f_0^*, f_1^*, \ldots)$, where $f_0^* = f_0^*$ and for $i \geq 1$,

$$
f_i^*(x_1, \ldots, x_i) = f_i(x_{i+1}, \ldots, x_i) \quad (1.2)
$$

The multiplication $\times$ and the unit $1 = (1, 0, 0, \ldots)$ make $\mathcal{S}$ into a unital $*$-algebra with no zero divisors. The center of $\mathcal{S}$ is $\{\lambda 1 : \lambda \in \mathbb{C}\}$. $1$ is the only nonzero idempotent, and the set of invertible elements equals the center. This implies the triviality of the radical

$$\text{rad}(\mathcal{S}) = \{g \in \mathcal{S} : 1 + f \times g \text{ has inverse } \forall f\}. $$

An element $g \in \mathcal{S}$ is called positive if $\exists f_i$ such that $g = \sum_i f_i^* \times f_i$. This induces a positive cone $\mathcal{S}^+$ and a semi-ordering. We define the set of Hermitian elements $\mathcal{S}_h = \{f \in \mathcal{S} : f^* = f\}$. $\mathcal{S}_h$ is a real vector space and we have $\mathcal{S} = \mathcal{S}_h + i\mathcal{S}_a$. Also, $\mathcal{S}_a^+$ is a convex cone with $\mathcal{S}^+ \cap (-\mathcal{S}^+) = \{0\}$. Moreover, we have $\mathcal{S}_a = \mathcal{S}^+ - \mathcal{S}^-$, which follows by polar decomposition: if $f = f^*$ then

$$
f = \frac{1}{4}(1+f) \times (1+f) - \frac{1}{4}(1-f) \times (1-f)
$$

1.2 The Wightman Axioms

**Axiom 1.** For every $f \in \mathcal{S}$, an operator $\varphi(f)$ exists, such that $D \subset \text{dom} (\varphi(f))$, $\varphi(f)D \subset D$,

$$
(\psi, \varphi(f)\chi) = (\varphi(f^*)\psi, \chi) \text{ for all } \psi, \chi \in D
$$

and $f \to (\psi, \varphi(f)\chi)$ is a continuous linear functional on $\mathcal{S}$.

**Axiom 2.** Let $f_a(x) = f(x-a)$. There exists a strongly continuous unitary representation $U : G \to \mathcal{H}$ of the translation group $G$, such that $U(a)D \subset D \forall a \in G$, and $U(a)A(f)U^{-1}(a)\psi = A(f_a)\psi$ for all $a \in G, f \in \mathcal{S}_1, \psi \in D$.

In standard constructive quantum field theory models, elements of $\mathcal{S}_1$ are test functions on spacetime, so there is a canonical action of $P_1^+$ on $\mathcal{S}_1$ for all $l$, in other words a representation $\alpha : P_1^+ \to \text{Aut}(\mathcal{S})$. We mention the representation $\alpha$ because in Section 3, we will write down a set of mathematical data which specify a quantum field theory, and invariance under a symmetry group can be expressed in terms of representations similar to $\alpha$.

**Axiom 3.** There exists $\Omega \in D$ such that $U(a)\Omega = \Omega \forall a \in G$, and the set of vectors of the form $\{\Omega, \phi(f)\Omega, \phi(f_1)\phi(f_2)\Omega, \ldots\}$ spans $\mathcal{H}$.

The GNS Construction Satisfies Axiom 1

Let $\mathcal{S}'$ be the space of continuous linear functionals $T : \mathcal{S} \to \mathbb{C}$. For $f \in \mathcal{S}$, denote the action of $T$ by $(T, f)$. The space $\mathcal{S}'$ also has a natural involution; define $T^*$ by $(T^*, f) := (\overline{T, f^*})$. We say a functional is real if $T = T^*$, and positive if $(T, p) \geq 0$ for all $p \in \mathcal{S}^+$ (the corresponding spaces are denoted $\mathcal{S}_r$, and $\mathcal{S}_p^+$).

The set of states is

$$E(\mathcal{S}) = \{T \in \mathcal{S}_r^+ : (T, 1) = 1\}.$$

For each state $T$, the left-kernel is defined to be

$$L(T) := \{f \in \mathcal{S} : (T, f^* \times f) = 0\}.$$

The left-kernel is so named because it is a left ideal in the Borchers algebra. The right-kernel $R(T)$, defined by the analogous relation $(T, f \times f^*) = 0$, is a right ideal.
Lemma 1. $\mathcal{S}' = \mathcal{S}'_h + i\mathcal{S}'_h$, and $\mathcal{S}'^+ \subset \mathcal{S}'_h$. If $T \in \mathcal{S}'^+$ then we have the Schwartz inequality,
\[ |(T, f \times g)|^2 \leq (T, f \times f)(T, g \times g) \]
and $(T, 1) = 0 \Rightarrow T = 0$. $E(\mathcal{S})$ is a base for the cone $\mathcal{S}'^+$.

Theorem 1. Each state $T \in E(\mathcal{S})$ canonically defines a representation $\phi_T$ of $\mathcal{S}$ in a Hilbert space $\mathcal{H}_T$ such that the restriction $\phi_T|_{S_1}$ satisfies Axiom 1. Conversely, if $\{\phi(f)\}$ are a set of fields satisfying Axiom 1, then every $\psi \in D$ s.t. $\|\psi\| = 1$ defines a state $T_\psi$ by linear extension of
\[ (T_\psi, f_1 \times f_2 \times \cdots \times f_n) = (\psi, \phi(f_1)\phi(f_2)\cdots\phi(f_n)\psi), \quad f_i \in S_1 \]
The field $A_{T_\psi}$ is unitarily equivalent to $\{A_\psi, D_\psi, \mathcal{H}_\psi\}$ where
\[ D_\psi = \text{Linear Span of } \psi, \phi(f)\psi, \phi(f_1)\phi(f_2)\psi, \text{ etc.} \]
$\mathcal{H}_\psi$ is the closure of $D_\psi$, and $\phi_\psi(f) = \phi(f)|_{D_\psi}$.

Proof. As a full proof can be found elsewhere $[1, 3]$, we merely recall the central idea for convenience, as it is used later. $T$ defines a non-degenerate positive definite bilinear form on $\mathcal{S}/L(T)$ by the relation
\[ ([f], [g]) = (T, f^* \times g), \quad [f], [g] \in \mathcal{S}/L(T) \]
Define $\mathcal{H}_T$ to be the completion of the pre-Hilbert space $\mathcal{S}/L(T)$, and define a representation of $\mathcal{S}$ by $\phi([g]) = [f \times g]$ for $f \in S_1, g \in \mathcal{S}$. The rest of the proof is straightforward. \hfill \Box

Translation Invariant States Satisfy Axioms 1-3

Let $a \in \mathbb{R}^4$. The map $\alpha_a$ defined by
\[ \alpha_a f_i(x_1, \ldots, x_i) = f_i(x_1 - a, \ldots, x_i - a) \]
is an element of $\text{Aut}(\mathcal{S})$. A state $T$ is translation-invariant if $(T, \alpha_a f) = (T, f)$ holds for all $f \in \mathcal{S}$ and for all $a \in \mathbb{R}^4$.

Theorem 2. Let $T$ be a translation invariant state. Then $A_T(f)$ satisfies Axioms 1-3. Conversely, assume that the system $\{A(f), D, \Omega \in D\}$ satisfies 1-3. Then $T_\Omega$ defined by $(T_\Omega, f) = (\Omega, A(f)\Omega)$ is translation invariant.

1.3 Tensor Products of States

Given two states $T_1, T_2 \in E(\mathcal{S})$, let $\{A_i(f), \mathcal{H}_i, D_i, \Omega_i\}$ be the associated GNS representations. Then the triplet
\[ \{A_1(f) \otimes I_2 + I_1 \otimes A_2(f), \mathcal{H}_1 \otimes \mathcal{H}_2, D_1 \times D_2\} \]
satisfies Axiom 1, and hence it corresponds to a new state, $T_1 \otimes_s T_2$ which is the same as the vector state $T_\psi$ with $\psi = \Omega_1 \times \Omega_2$. Let $P_{n,m}$ denote the set of all ordered splittings of $n + m$ elements into two subsets, of respective sizes $n$ and $m$. Let $T_n \in \mathcal{S}'_n, S_m \in \mathcal{S}'_m$, then $T_n \otimes_s S_m$ is given by
\[ (T_n \otimes_s S_m)(x_1, \ldots, x_{n+m}) = \sum_{P_{n,m}} T_n(x_{i_1}, \ldots, x_{i_n})S_m(x_{j_1}, \ldots, x_{j_m}) \]
For $T, S \in \mathcal{S}'_n$, we define $(T \otimes_s S)_n = \sum_{i+k=n} T_i \otimes S_k$. This coincides with our previous definition of the $\otimes_s$-product. It is clearly associative and abelian.
1.4 Real Scalar Fields

Before discussing more complicated generalizations, we briefly indicate how the above construction can describe the salient properties of the quantum theory of a one-component real scalar field.

In quantum theory of real scalar fields, the field algebra is the Borchers algebra $A_\Phi$ where $\Phi = \mathbb{R}S_1$, the real subspace of the nuclear Schwartz space $S(\mathbb{R}^m)$ of complex $C^\infty$ functions $\phi(x)$ on $\mathbb{R}^m$ such that

$$\|\phi\|_{k,l} = \max_{|\alpha| \leq l} \sup_{x \in \mathbb{R}^m} (1 + |x|)^k |\partial^{|\alpha|} \phi(x)|,$$

is finite for any collection $(\alpha_1, \ldots, \alpha_m)$ and all $l, k \in \mathbb{N}$. The space $\mathbb{R}S_m$ is endowed with the set of seminorms $\|\phi\|_{k,l}$ and the associated topology. It is reflexive.

Since $\otimes^n S(\mathbb{R}^n)$ is dense in $S(\mathbb{R}^{kn})$, every continuous form on the subspace has a unique continuous extension. Every bilinear functional $M(\phi_1, \phi_2)$ which is separately continuous in $\phi_1 \in S(\mathbb{R}^n)$ and $\phi_2 \in S(\mathbb{R}^m)$ may be expressed uniquely in the form

$$M(\phi_1, \phi_2) = \int F(x,y)\phi_1(x)\phi_2(y)d^mxd^my, \quad F \in S'(\mathbb{R}^{n+m}).$$

As a consequence, every state $\omega$ on the Borchers algebra $A_{\mathbb{R}S_1}$ is represented by a family of distributions $W_n \in S'(\mathbb{R}^{4n})$. For any $\omega$, there exists a sequence $\{W_n\}$ such that

$$\omega(\phi_1 \times \cdots \times \phi_n) = \int W_n(x_1, \ldots, x_n)\phi_1(x_1) \cdots \phi_n(x_n)d^4x_1 \cdots d^4x_n.$$

This is precisely why a state on the Borchers algebra contains the same information as a scalar quantum field theory; it is known that the latter is completely determined mathematically by its Wightman functions. If $\omega$ obeys the Wightman axioms, then we have a quantum field theory and $W_n$ are the familiar $n$-point functions of that theory. They calculate observable quantities such as cross-sections and decay rates.

1.5 Spectral Condition, Locality, and Uniqueness of the Vacuum

The remaining two essential defining properties of a quantum field theory (spectral condition and locality) are equivalent to $\ker \omega$ containing certain ideals.

The spectral condition is the statement that $\ker \omega \supset I_1$, where

$$I_1 = \left\{ \int d^4a F(a)\alpha_a f : f \in \mathcal{S}, f_0 = 0, F(a) \in \mathcal{F}[\mathcal{S}_1(CV^+)\] \right\}$$

where $\mathcal{F}$ is the Fourier transform, and $\mathcal{S}_1(CV^+)$ is the set of functions in $\mathcal{S}_1$ that vanish on the forward light cone $V^+$.

Spacetime locality is the statement that $\ker \omega \supset I_2$, where $I_2$ is the smallest closed two-sided ideal in $\mathcal{S}$ containing all elements of the form $f \times g - g \times f$ where $f$ and $g$ have spacelike-separated supports.

Uniqueness of the vacuum also has a simple interpretation in terms of Wightman functionals. We call a field theory reducible if the algebra of field operators acts reducibly on the Hilbert space. A Wightman functional is said to be decomposable if there exists $\alpha \in (0, 1)$ such that

$$W = \alpha W^{(1)} + (1 - \alpha)W^{(2)} \quad (1.3)$$

and $W^{(1)}$, $W^{(2)}$ are Wightman functionals different from $W$. We note that indecomposability of the Wightman functional is equivalent to uniqueness of the vacuum; details are to be found in [1].
2. Noncommutative Target Space Perspective

The field algebra with multiplication and involution given by (1.1)-(1.2) admit a natural generalization to the noncommutative setting. This generalization has many applications in physics, all of which come from interpreting elements of the Borchers algebra as gauge fields on a $d$-dimensional spacetime. For $d \leq 1$, this gives rise to matrix models and matrix quantum mechanics. For $d \geq 2$, it is Yang-Mills theory. It is not immediately obvious that a correct description of gauge theory is possible in terms of test functions valued in a noncommutative space, so we argue for this approach from first principles before continuing.

Quantum fields are operator-valued distributions. Consider a pure gauge theory with gauge group $G$ and Lie algebra $\mathfrak{g} = \text{Lie}(G)$. In a classical pure gauge theory, the fundamental fields are $\mathfrak{g}$-valued one-forms, with each such form determining a connection on a principal $G$-bundle. In a quantum version of the same gauge theory, these classical fields would be promoted to operator-valued distributions with the same algebraic structure.

For concreteness, let $\mathcal{S}(\mathbb{R}^4)$ denote the Schwartz space of rapidly decreasing functions on $\mathbb{R}^4$. An operator-valued distribution is a continuous map

$$\mathcal{S}(\mathbb{R}^4) \rightarrow \text{Op}(\mathcal{H})$$

where we consider $\mathcal{S}(\mathbb{R}^4)$ to be endowed with the Schwartz topology, $\mathcal{H}$ is a Hilbert space, and Op($\mathcal{H}$) denotes a suitable space of unbounded operators on $\mathcal{H}$. In the example of a free real scalar boson, $\mathcal{H}$ is the usual bosonic Fock space, and Op($\mathcal{H}$) would be a class of operators large enough to include all operators of the form $\phi(f)$, where $\phi$ is a quantum field and $f$ is any test function. The operators Op($\mathcal{H}$), in this example, have a common core including all smooth, compactly supported Fock states with finite particle number.

In this notation, then, an “operator-valued distribution with Lie algebra indices” is a continuous map

$$\mathcal{S}(\mathbb{R}^4) \rightarrow \text{Op}(\mathcal{H}) \otimes \mathfrak{g}$$

where $\mathfrak{g}$ is a Lie algebra. There is a natural transformation of categories, or “duality,” by which the space of continuous maps of the form (2.1) is naturally isomorphic to the space of continuous maps

$$\mathcal{S}(\mathbb{R}^4) \otimes \mathfrak{g}^* \rightarrow \text{Op}(\mathcal{H})$$

For concreteness, we exhibit the isomorphism between (2.1) and (2.2) explicitly. Let

$$\phi : \mathcal{S}(\mathbb{R}^4) \rightarrow \text{Op}(\mathcal{H}) \otimes \mathfrak{g}$$

This implies that for any $f \in \mathcal{S}(\mathbb{R}^4)$, $\phi(f)$ admits an expansion in terms of homogeneous elements

$$\phi(f) = \sum_{i=1}^{n} A_i \otimes x_i, \quad A_i \in \text{Op}(\mathcal{H}), \quad x_i \in \mathfrak{g}$$

In terms of $\phi$ we may define a new map

$$\tilde{\phi} : \mathcal{S}(\mathbb{R}^4) \times \mathfrak{g}^* \rightarrow \text{Op}(\mathcal{H})$$

by the formula

$$\tilde{\phi}(f, \tilde{y}) \equiv \sum_{i=1}^{n} \tilde{y}(x_i), \quad \tilde{y} \in \mathfrak{g}^*.$$ (2.3)

The map $\tilde{\phi}$ is obviously multilinear, and therefore factors through to a map on the tensor product. We have proved that $\phi \rightarrow \tilde{\phi}$ gives an explicit isomorphism between the two spaces of maps (2.1) and (2.2).
Again for concreteness, suppose that we have chosen a particular matrix representation of the Lie algebra $\mathfrak{g}$; i.e., a faithful embedding of $\mathfrak{g}$ into $GL(n)$ for some $n$, in such a way that the Lie bracket corresponds to the commutator of matrices. This also gives a matrix realization of the dual space $\mathfrak{g}^*$. We therefore interpret the isomorphism $\phi \rightarrow \tilde{\phi}$ in quantum field theory language, as the statement that it is entirely mathematically equivalent to consider either an operator-valued distribution which transforms in the adjoint representation of a Lie algebra (such as a quantized Yang-Mills field), or an operator-valued distribution which acts on test functions taking their values in the dual of the Lie algebra.

Quantum field theory with test functions taking values in $\mathfrak{g}^*$ is well-described by a noncommutative version of the Borchers construction, and the latter mathematical structure will occupy us for the rest of this section and, in some form or other, for the rest of the paper. We summarize the results of the previous paragraphs in a lemma.

**Lemma 2.** The following structures are equivalent:

1. An operator-valued distribution which transforms in the adjoint representation of a Lie algebra $\mathfrak{g}$ (i.e., a quantized Yang-Mills field)

2. An operator-valued distribution which acts on $\mathfrak{g}^*$-valued test functions.

### 2.1 Generalizing the Borchers Construction

For any space $\Sigma$ let $\mathcal{A} = \mathcal{A}(\Sigma, \mathcal{B})$ denote a vector space of “test functions” from $\Sigma$ to a (possibly noncommutative) star-algebra $\mathcal{B}$ with product $\cdot$. The notation $\Sigma$ is in anticipation of the application to gauge theory, where $\Sigma$ will denote a Riemann surface. We assume for convenience that the base field of $\mathcal{B}$ is $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. $\mathcal{A}$ is then naturally a left and right module over $\mathcal{B}$, and of course also over $\mathcal{F}$. Let

$$T\mathcal{A} = \mathbb{C} \oplus \mathcal{A} \oplus (\mathcal{A} \otimes \mathcal{A}) \oplus \ldots$$

(2.4)

We will abbreviate $\mathcal{A} \otimes \cdots \otimes \mathcal{A}$ ($n$ factors) by $\mathcal{A}^\otimes n$. Let $x$ and $y$ denote elements of $\Sigma$. We define a map $\iota$ which identifies $f \otimes g$ with the $\mathcal{B}$-valued function of two variables given by $f(x) \cdot g(y)$. This identification, and the natural extension of this map to higher tensor powers $\mathcal{A}^\otimes n$, determine an algebra homomorphism

$$\mathcal{A}^\otimes n \xrightarrow{\iota} \mathcal{A}(\Sigma^n, \mathcal{B})$$

(2.5)

where $\Sigma^n$ denotes the $n$-fold Cartesian product $\Sigma \times \Sigma \times \ldots \times \Sigma$. In other words,

$$\iota(f_1 \otimes \cdots \otimes f_n)(x_1, \ldots, x_n) = f_1(x_1) \cdot \ldots \cdot f_n(x_n).$$

Both the kernel and the image of this homomorphism are important.

The prototypical way in which the map $\iota$ can have a nonzero kernel is if $f(x)$ commutes with $g(y)$ for all $x, y \in \Sigma$, which implies $\iota(f \otimes g) = \iota(g \otimes f)$. For our purposes, we would like to assume that (2.5) is injective; to attain this injectivity it is necessary to quotient by the kernel of $\iota$, which is equivalent to working with the universal enveloping algebra.

Since $\mathcal{B}$ is defined to be an associative algebra, it is naturally also a Lie algebra. Therefore $\mathcal{A}$, the algebra of functions $\Sigma \rightarrow \mathcal{B}$, is also a Lie algebra. Let $\underline{\mathcal{A}}$ be the universal enveloping algebra $\mathcal{U}(\mathcal{A}) = T\mathcal{A}/I$ where $I$ is the two-sided ideal generated by elements of the form

$$f \otimes g - g \otimes f - [f, g], \quad f, g \in \mathcal{A}$$

Since $I = \ker(\iota)$, it follows that $\iota$ is injective on $\underline{\mathcal{A}}$.

If $\mathcal{B}$ is a real, abelian algebra, then the above construction reduces to the classic construction of Wightman. In particular, we may consider $\mathcal{B} = \mathbb{R}$ which corresponds to a single real scalar field.
In this case, \([f, g] = 0\) always, and \(I\) is the ideal generated by elements of the form \(f \otimes g - g \otimes f\). This identifies \(A\) with the symmetric tensor algebra over \(A\), which is precisely the algebra used in Wightman’s original construction [3]. Once again, \(\iota\) is injective on the space of interest; Wightman uses this injectivity without stating it explicitly in the proof of the reconstruction theorem.

When the interpretation is clear from context, as in (2.6), we will not explicitly write the map \(\iota\). With this convention, our notation becomes compatible with the notation of [3], in which (for example) one would write

\[(h \otimes f_k)(x_1, \ldots, x_{k+1}) = h(x_1)f_k(x_2, x_3, \ldots, x_{k+1})\]

Since \(A\) is generated as a vector space by homogeneous elements, we can define a cross product on \(A\) by the equation

\[(f_n \times g_m)(x_1, \ldots, x_{n+m}) = f_n(x_1, \ldots, x_n) \cdot g_m(x_{n+1}, \ldots, x_{n+m})\]  

(2.6)

For \(f_n \in A(\Sigma^n, \mathcal{B})\) and \(g_m \in A(\Sigma^m, \mathcal{B})\), this determines an element \(f_n \times g_m \in A(\Sigma^{n+m}, \mathcal{B})\), and this element is the same as the function \(\iota(f_n \otimes g_m)\). The cross product extends to all of \(A\) in a manner similar to eq. (1.1),

\[(f \times g)_l = \sum_{i+h=l} f_i \times g_h.\]  

(2.7)

The involution on \(A\) is defined in a manner similar to the Borchers construction. For \(f \in A\), we define \(f^* := f(x)^*\) where \(*_\mathcal{B}\) denotes the involution in \(\mathcal{B}\). We define the involution to satisfy the axioms of a *-algebra, so that \((f + \lambda h)^* = f^* + \lambda h^*\) and \((f \times g)^* = g^* \times f^*\). Since \(A\) generates \(A\) as an algebra, this determines the star operation on all of \(A\).

In the above, \(\mathcal{B}\) was defined to be an associative algebra, and we also used the natural Lie bracket coming from taking commutators in the associative algebra’s product. Finally, we remark that a completely analogous construction may be carried out even in the case where \(\mathcal{B}\) is an abstract Lie algebra on which no associative algebra structure is defined. To do this, the definition of \(A\) as the universal enveloping algebra is the same, and the definition of the cross product (2.6) must be modified, to become

\[(f_n \times g_m)(x_1, \ldots, x_{n+m}) = [f_n(x_1, \ldots, x_n), g_m(x_{n+1}, \ldots, x_{n+m})]\]  

(2.8)

However, this is unlikely to be a useful construction because the actual Hilbert space inner product which we will later construct (Theorem 3) will be identically zero, assuming the state \(t \in E(\mathcal{B})\) used there is a cyclic state, meaning that

\[t(ab\ldots c) = t(cab\ldots)\]

Of course, the trace on any matrix algebra or Hilbert space is a cyclic state.

2.2 States and Bilinear Forms

The cross product provides a mapping from states to the bilinear forms that arise in quantum physics. Explicitly, the advantage of the cross product (2.6)-(2.7) is that any state \(\omega\) on \(A\) determines a bilinear form \(\langle \cdot , \cdot \rangle_\omega\) by the relation

\[\langle f, g \rangle_\omega = \omega(f^* \times g),\]  

(2.9)

and the bilinear forms (2.9) are of the type that arise in the construction of the Fock-Hilbert space for a quantum field theory. Since a state necessarily satisfies the positivity axiom, the associated bilinear form \(\langle \cdot , \cdot \rangle_\omega\) is a positive semi-definite inner product. It is a positive definite inner product.
on the quotient by the kernel of the state \( \omega \), and the completion of \( \mathcal{A} \) modulo \( \text{Ker}(\omega) \) forms a Hilbert space \( \mathcal{H}_{\text{phys}} \) which we may identify as the physical Hilbert space of the theory.

The state \( \omega \) defines a non-degenerate positive definite sesquilinear form on \( \mathcal{A}/L(\omega) \) by the relation
\[
([f], [g]) = \omega(f^* \times g)
\]
(2.10)
Define field operators \( \Phi(f) \), \( f \in \mathcal{A} \) acting on \( \mathcal{A} \) by the formula
\[
\Phi(f)(a_0, a_1, a_2, \ldots) = (0, f a_0, f \otimes a_1, f \otimes a_2, \ldots)
\]
(2.11)
These are analogous to the \( \phi(f) \) operators encountered in Sec. 1.1. Define \( \mathcal{H}_{\text{phys}} \) to be the completion of the pre-Hilbert space \( \mathcal{A}/L(\omega) \), and define a representation of \( \mathcal{A} \) on \( \mathcal{H}_{\text{phys}} \) by
\[
\varphi(f)[g] = [f \times g]
\]
for \( f \in \mathcal{A} \), \( g \in \mathcal{A} \). A short proof shows that \( \Phi \), as defined by (2.11), is well-defined on equivalence classes and is the same as \( \varphi \) upon passing to the quotient.

### 2.3 Pulled Back States and the Hilbert-Schmidt Form

The generalized Borchers construction presented here differs from the usual quantum theory of scalar fields, because in the generalized Borchers construction, a sequence \( \mathcal{W} = \{W_n\} \) of tempered distributions does not directly define a (complex-valued) sesquilinear form. The sequence \( \mathcal{W} \) does naturally define an associated linear map \( \Omega_\mathcal{W} : \mathcal{A} \to \mathcal{B} \) by
\[
\Omega_\mathcal{W}(f = \{f_0, f_1, \ldots\}) = \sum_{n=0}^{\infty} \int f_n(x_1, \ldots, x_n)W_n(x_1, \ldots, x_n) \prod_{i=1}^{n} dx_i \in \mathcal{B}
\]
In the intended application, a scalar-valued functional \( \omega \) is recovered by composing the above map with a natural scalar-valued state \( \text{tr} : \mathcal{B} \to \mathbb{C} \), given by the trace. The functional \( \omega \), in turn defines a sesqui-linear form via (2.10). In general, we will call states which arise from composing \( \Omega_\mathcal{W} \) with a scalar-valued state **pulled-back states**. These are the type of states which necessarily arise in Wightman reconstruction for gauge theories.

**Theorem 3.** Let \( t \in E(\mathcal{B}) \) be a state. If the distributions \( \mathcal{W} \) come from vacuum expectation values of products of fields in a Hilbert space, then the pullback \( \omega = t \circ \Omega_\mathcal{W} \) is a state on \( \mathcal{A} \).

**Proof.** The quantity which we must prove to be positive is \( \omega(f^* \times f) = t(\Omega_\mathcal{W}(f^* \times f)) \). The latter is a sum of terms which all have the form
\[
\int t(f_n(x_n, \ldots, x_1)^* f_n(x_{n+1}, \ldots, x_{2n}))W_{2n}(x_1, \ldots, x_{2n}) \prod_i dx_i
\]
Since the distributions \( W_n \) come from expectation values of products of fields \( \varphi_{jk} \) in the ground state \( \Psi_0 \), we can re-write this integral as the norm of a vector which takes the form
\[
f_0 \Psi_0 + \varphi_{11}(f_1) \Psi_0 + \int \varphi_{21}(x_1) \varphi_{22}(x_2) f_2(x_1, x_2) dx_1 dx_2 \Psi_0 + \ldots
\]
where \( \Psi_0 \) is the Fock vacuum vector. This completes the proof. \( \square \)

**Remark 1.** Suppose that \( \varphi \) is a representation of a \( C^* \) algebra \( A \) on a Hilbert space \( \mathcal{H} \), and \( x \) is a unit vector in \( \mathcal{H} \). With \( \omega_x \) the corresponding vector state of \( \mathcal{B}(\mathcal{H}) \) the composite function \( \omega_x \circ \varphi \) is a state of \( A \), and it is known that each state of a \( C^* \)-algebra arises in this way, from a vector state in an appropriate representation. In Theorem 3 we happen to know the representation (Fock representation) and vector (vacuum vector) explicitly.
Every state \( t \) on \( \mathcal{B} \) determines a sesqui-linear form \( \langle , \rangle_{HS} \) defined by
\[
\langle a, b \rangle_{HS} = t(a^* \cdot b)
\] (2.12)
which we call the associated Hilbert-Schmidt form. In case \( \mathcal{B} \) is a \( C^* \)-algebra, we can consider it as a subalgebra of the algebra of bounded operators on some Hilbert space \( \mathcal{H} \). In this case, the trace \( \text{Tr}_\mathcal{H}(\ ) \) is a natural state on \( \mathcal{B} \), and (2.12) is the usual Hilbert-Schmidt inner product, making \( \mathcal{B} \) also into a Hilbert space.

The formula (2.12) may be extended in a natural way to the case where \( a \) and \( b \) represent continuous functions from a finite measure space \( \Sigma \) into \( \mathcal{B} \). The following inner product will arise in Example 0 below,
\[
\langle a, b \rangle_{HS} = t(A^* \cdot B)
\] (2.13)
where capital letters denote that we have taken the total integral of the function denoted by the corresponding lowercase character, i.e. \( A = \int_\Sigma a \, d\mu \). We refer to (2.13) as the integrated Hilbert-Schmidt inner product. More generally, there may be a kernel which couples the functions \( a \) and \( b \).

### 2.4 Positive Definiteness

A number of quantum field theory models are known which do not satisfy the Wightman axiom of positivity. The general properties of these models are summarized in the modified Wightman axioms of indefinite metric QFT, due to Morchio and Strocchi \[10\]. Albeverio, Gottschalk and Wu \[11\] investigated Euclidean random fields as generalized white noise and remarked that the Wightman functionals belonging to those fields do not generally satisfy positivity. Further, Albeverio et al. \[12\] showed that those nonpositive Wightman functionals satisfy the following weaker condition, known as the Hilbert space structure condition (HSSC).

**Axiom (Hilbert space structure condition).** There exist seminorms \( p_n \) on \( S_n \) such that
\[
|W_{n+m}(f_n^* \otimes g_m)| \leq p_n(f_n)p_m(g_m)
\] (2.14)
for all \( p_n \in S_n, g_m \in S_m \).

This axiom needs no modification in order to apply to the general Borchers construction of Section 2.1; the \( p_n \) are simply reinterpreted as seminorms on the image of \( \mathcal{A}^\otimes n \) in the universal enveloping algebra \( \mathcal{A} \equiv \mathcal{U}(\mathcal{A}) = T\mathcal{A}/\ker(\iota) \) discussed previously.

Jakobczyk and Strocchi also introduced a related condition known as the Krein structure condition, which is satisfied by the physically important Gupta-Bleuler formalism for free QED, and has many attractive features from a mathematical standpoint.

**Axiom (Krein positivity).** There exists a dense unital subalgebra \( \mathcal{A}_0 \) of the Borchers algebra, and a mapping \( \alpha : \mathcal{A}_0 \to \mathcal{A}_0 \), such that for all \( f, g \in \mathcal{A}_0 \),
1. \( \omega(\alpha^2(f)^* \times g) = \omega(f^* \times g) \);
2. \( \omega(\alpha(f)^* \times f) \geq 0 \);
3. \( \omega(\alpha(f)^* \times g) = \omega(f^* \times \alpha(g)) \); and
4. \( p_n(f) \equiv \omega(\alpha(f)^* \times f)^{1/2} \) is continuous in the topology of the Borchers algebra.

In the original paper \[13\], it is shown that the Krein positivity condition is stronger than the Hilbert space structure condition, is satisfied by free QED, and it guarantees the existence of
a majorizing Hilbert space structure associated to the Wightman functions with the property of being of Krein type.

It is once again easily seen that the Krein positivity condition may be applied to the generalized Borchers algebra, and state $\omega$ on that algebra, simply by re-interpreting the terminology within the new context.

Since the simple model of free QED is known not to satisfy Wightman positivity, it is likely that other more complicated gauge theories display similar behavior. Therefore, in order to recover a Hilbert space from such theories, a formalism similar to that of Jakobczyk and Strocchi may indeed be necessary. Therefore, it is of importance to write down a consistent framework in which the HSSC or the Krein condition can be formulated for nonabelian gauge theories. The present work fills this need, since both the Hilbert space structure condition and the Krein condition generalize immediately to the scenario of Section 2.1.

### 2.5 Complex-valued bilinear forms acting on matrix-valued fields

In the introduction, we mentioned the well-known result that every bilinear functional $M(\phi_1, \phi_2)$ which is separately continuous in $\phi_1 \in S(\mathbb{R}^n)$ and $\phi_2 \in S(\mathbb{R}^m)$ may be expressed uniquely in the form

$$M(\phi_1, \phi_2) = F(\phi_{12}), \quad F \in S'((\mathbb{R}^{n+m}),$$

where

$$\phi_{12}(x_1, \ldots, x_{n+m}) \equiv \phi_1(x_1, \ldots, x_n)\phi_2(x_{n+1}, \ldots, x_{n+m}). \quad (2.15)$$

This implies, then, that every state on the Borchers algebra is determined by a collection of distributions which, assuming the relevant axioms hold, may be identified with the $n$-point functions of some quantum field theory.

The proof goes as follows. $M$ naturally determines a linear functional $M\big|_\mathcal{S}$ on the subset of $S((\mathbb{R}^{n+m}))$ consisting of those Schwartz functions which are factorizable into two Schwartz functions on $\mathbb{R}^n$ and $\mathbb{R}^m$ as in (2.15). It is easily seen that the restricted functional $M\big|_\mathcal{S}$ is continuous in the relative topology induced from $S((\mathbb{R}^{n+m}))$, and therefore it may be extended to a continuous map on all of $S((\mathbb{R}^{n+m}))$; this map is precisely the distribution $F$ which we wanted to construct. □

The purpose of the present section is to prove the analogous result for the general Borchers algebra introduced in Section 2.1. On this note, suppose that $\varphi_1$ and $\varphi_2$ are Schwartz functions from $\mathbb{R}^n, \mathbb{R}^m$ respectively, into the space of $k \times k$ matrices over $\mathbb{C}$. Let $M(\varphi_1, \varphi_2)$ be a bilinear functional, separately continuous in both variables. We claim there exists a distribution $\mathcal{F} \in S'((\mathbb{R}^{n+m}), \text{Mat}_{k \times k}(\mathbb{C}))$ such that

$$M(\varphi_1, \varphi_2) = \mathcal{F}(\varphi_{12})$$

where

$$\varphi_{12}(x_1, \ldots, x_{n+m})^{ab} \equiv \sum_c \varphi_1(x_1, \ldots, x_n)^{ac}\varphi_2(x_{n+1}, \ldots, x_{n+m})^{cb}. \quad (2.16)$$

The proof of this slightly more general statement proceeds exactly as before, and we conclude that any state on the generalized Borchers algebra determines a set of generalized $n$-point functions.

### 2.6 Example 0: Constant Wightman Functions

Eq. 2.9 determines a bilinear form on $\mathcal{A}$ giving $\mathcal{A}/L(\omega)$ the structure of a pre-Hilbert space. In this section and the next, we compute this bilinear form for two cases in which the Wightman functions have a relatively simple structure.
Suppose the Wightman distributions are given by a sequence of non-negative real constants, \( \alpha_n \in \mathbb{R}_{\geq 0} \). We compute

\[
\Omega_W(f^* \times g) = \sum_{k=0}^{\infty} \sum_{n+m=k} \int_{x_1, \ldots, x_k} \alpha_n f_n(x_n, \ldots, x_1)^* g_m(y_1, \ldots, y_m) \prod_{i,j} dx_i dy_j
\]

\[
= \sum_{k=0}^{\infty} \alpha_k \sum_{n+m=k} \left( \int f_n^* \right) \left( \int g_m \right)
\]

The associated sesquilinear form is then given by

\[
(f, g) = \sum_{k=0}^{\infty} \alpha_k \sum_{n+m=k} \langle F_n, G_m \rangle_{HS}
\]  

(2.17)

where capital letters denote integration, i.e. \( F_n = \int f_n(x_1, \ldots, x_n) d^n x \), etc.

Let us make some remarks on the interpretation of (2.17). A direct sum of Hilbert spaces \( \mathcal{H}_0 \oplus \mathcal{H}_1 \) always refers to the orthogonal direct sum, in which we have \( \mathcal{H}_0 \perp \mathcal{H}_1 \) in the resulting Hilbert space, i.e.

\[
\langle (\psi_0, \psi_1), (\phi_0, \phi_1) \rangle = \langle \psi_0, \phi_0 \rangle + \langle \psi_1, \phi_1 \rangle
\]

However, under a different inner product on \( \mathcal{H}_0 \oplus \mathcal{H}_1 \), it might no longer be true that \( \mathcal{H}_0 \perp \mathcal{H}_1 \) and in general,

\[
\langle (\psi_0, \psi_1), (\phi_0, \phi_1) \rangle = \langle \psi_0, \phi_0 \rangle + \langle \psi_1, \phi_0 \rangle + \langle \psi_1, \phi_1 \rangle + \langle \psi_0, \phi_1 \rangle
\]

The corresponding formula for an infinite (non-orthogonal) direct sum is

\[
\langle (\psi_0, \psi_1, \ldots), (\phi_0, \phi_1, \ldots) \rangle = \sum_{k=0}^{\infty} \left( \sum_{n+m=k} \langle \psi_n, \phi_m \rangle \right)
\]

This leads to an interpretation of (2.17) as the \( \alpha \)-weighted, infinite, non-orthogonal sum of copies of the space of maps \( \Sigma \rightarrow \mathcal{B} \), where the latter is endowed with the integrated Hilbert-Schmidt norm (2.13). Clearly for the inner product which is derived from the sesquilinear form (2.17) to be positive definite, it is necessary that \( \alpha_n \geq 0 \).

2.7 Example 1: Products of Delta Functions = Generalized Hilbert-Schmidt

If \( \mathcal{B} \) is a \( C^* \)-algebra, then it can be realized as an algebra of bounded operators on some Hilbert space \( \mathcal{H} \). In this case, \( \mathcal{B} \) is itself a Hilbert space with the standard Hilbert-Schmidt inner product defined by

\[
\langle A, B \rangle_{HS} = \text{Tr}(A^* B), \quad A, B \in \mathcal{B}.
\]

This extends to tensor products of \( \mathcal{B} \) in the usual way,

\[
\langle A \otimes B, C \otimes D \rangle_{HS} = \text{Tr}(A^* C) \text{Tr}(B^* D).
\]  

(2.18)

This gives rise to a generalized Hilbert-Schmidt inner product on \( \mathcal{A}/L(\omega) \), and this is an example of the inner product arising from a state \( \omega \) on the generalized Borchers algebra. Since our fields are \( \mathcal{B} \)-valued functions, our inner product will be the integrated version of the Hilbert-Schmidt inner product for tensors given as (2.18). The state which generates this inner product is defined by considering all possible products of delta functions.
Explicitly, consider the Wightman functions

\[
W_2 = \delta(x_1 - x_2) \\
W_4 = \delta(x_1 - x_4)\delta(x_2 - x_3) \\
\vdots \\
W_{2n} = \prod_{i=1}^{n} \delta(x_{i+n} - x_{p(i)})
\]

where \( p \) is the permutation of \( \{1, \ldots, n\} \) that completely reverses the order. An example using \( W_4 \) is

\[
\langle f, g \rangle = \int \text{Tr}(f(x_2, \ldots, x_1)^* g(x_3, x_4)) W_4(x_1, \ldots, x_4) [\Pi_i dx_i]
\]

\[
= \int \text{Tr}(f(x_2, x_1)^* g(x_3, x_4)) \delta(x_1 - x_4)\delta(x_2 - x_3) [\Pi_i dx_i]
\]

\[
= \int_{\Sigma^2} \langle f, g \rangle_{HS}
\]

At first glance, one might worry that the integrand of this expression looks different from (2.18). In fact they are the same; to see this, let \( f = f_a \otimes f_b \) and \( g = g_a \otimes g_b \). Then we have

\[
\langle f, g \rangle = \int_{\Sigma^2} \langle f, g \rangle_{HS} = \int_{\Sigma^2} \text{Tr} \left[ (f_a \otimes f_b)^* \cdot (g_a \otimes g_b) \right]
\]

\[
= \int_{\Sigma^2} \text{Tr} \left[ (f_a^* \cdot g_a) \otimes (f_b^* \cdot g_b) \right]
\]

\[
= \int_{\Sigma^2} \text{Tr} (f_a^* \cdot g_a) \text{Tr}(f_b^* \cdot g_b)
\]

3. Application to Gauge Theory

We would like to use the structure developed above to express quantities of interest in gauge theory. The difficulty with this outlook is that there are different possible choices for complete sets of observables. It is known that Wilson loop functionals are a complete set of observables for Yang-Mills theory in any dimension, but as functionals on the loop space, they cannot be directly used to generate a state on the generalized Borchers algebra. Fortunately, in some cases, a complete set of gauge-invariant correlation functions is available, and they possess a mathematical structure which is convenient for our viewpoint in this paper.

3.1 Complete Sets of Observables

In any number of dimensions, the Yang-Mills field strength is a Lie algebra valued 2-form; a special feature of two dimensions is that in this case the field strength is mapped to a \( g \)-valued scalar field by the Hodge star. This field is denoted \( \xi(x) \), and given explicitly by

\[
F_{\mu\nu}(x) = \xi(x) \sqrt{g(x)} \varepsilon_{\mu\nu}
\]

The Yang-Mills action in two dimensions is

\[
S = \frac{1}{8\pi^2 \varepsilon} \int_{\Sigma} \text{Tr} F \wedge *F, \quad (3.1)
\]
with the trace taken in the fundamental representation for \( g \). In our convention, the gauge field \( A \) is anti-Hermitian. In terms of \( \xi \) the pure Yang-Mills action takes the form

\[
\int_{\Sigma} d\mu \, \text{Tr}(\xi^2)
\]

with the appropriate coupling constant prefactor inserted. Here \( d\mu = \sqrt{g(x)} d^2 x \) is the Riemannian volume measure on \( \Sigma \).

It is clear at this point that one class of correlators which arise in this theory is the class of field strength correlators, which are linear combinations of objects of the form

\[
\langle \xi^a(x_1)\xi^b(x_2)\xi^c(x_3)\ldots\xi^d(x_n) \rangle
\]

They become gauge invariant after taking the trace and contracting all internal indices. The rest of this subsection will be devoted to describing a second type of correlator, which are sometimes called \( \phi \)-field correlators.

In two dimensions there are no propagating degrees of freedom (i.e. no gluons) so the only degrees of freedom come from spacetimes of nontrivial topology or Wilson loops. Since there are so few degrees of freedom, there is a very large group of local symmetries. \( YM_2 \) is invariant under the group \( \text{SDiff}(\Sigma_T) \) of area preserving diffeomorphisms, which is a larger symmetry group than local gauge invariance.

The following equivalent action is called the “first-order formalism” because Gaussian integration over \( \phi \) gives back the original action (3.1).

\[
Z_{\Sigma}(\varepsilon) = \int DA \exp \left( \frac{1}{8\pi^2\varepsilon} \int_{\Sigma} \text{Tr} F \wedge * F \right) = \int DA D\phi \, e^{-S(A,\phi)}
\]

where

\[
S(A, \phi) = -\frac{i}{4\pi^2} \int_{\Sigma} \text{Tr}(\phi F) - \frac{\varepsilon}{8\pi^2} \int_{\Sigma} d\mu \, \text{Tr} \phi^2.
\]

(3.2)

Here \( \phi \) is a Lie-algebra valued 0-form; \([3.2] \) shows that the gauge coupling \( e^2 \) and total area \( a = \int_{\Sigma} d\mu \) always enter together, and gives rise to the natural generalization

\[
I = \int_{\Sigma} \left[ i \, \text{tr}(\phi F) + \mathcal{U}(\phi)d\mu \right]
\]

(3.3)

where \( \mathcal{U} \) is any invariant function on the Lie algebra \( g \). Thus, ordinary \( YM_2 \) is one example of a general class of theories parameterized by invariant functions on \( g \). It is natural to restrict to the ring of invariant polynomials on \( g \). For \( G = SU(N) \), this ring is generated by \( \text{tr} \phi^k \), so we may describe the general theory by coordinates \( t_k \), in terms of which

\[
\mathcal{U} = \sum t_k \prod_j (\text{tr} \phi^j)^{k_j}.
\]

The following material, in which we develop the Borchers formulation for some of the simplest Yang-Mills theories, applies equally well to the general case (3.3) with arbitrary \( \mathcal{U}(\phi) \).

A complete set of physical observables for Yang-Mills theory in any dimension are Wilson loops. These are, in particular, interesting observables for Yang-Mills theory in \( d = 2 \). However, gauge invariant polynomials of the field \( \phi \) form another complete set of observables naturally suited to evaluation of the partition function. These observables include products of \( \text{Tr} \phi^2(x_i) \) at various points \( x_i \), and more generally, traces of any homogeneous invariant polynomial defined on the Lie algebra \( g \). This is in marked contrast to \( d = 4 \) Yang-Mills where the only dimension four gauge invariant operators are \( \text{tr}(F \wedge * F) \) and \( \text{tr}(F^2) \), with the latter a topological term.
The most important property of expectation values of gauge-invariant observables in two dimensions is that they are almost topological. A sample calculation shows that

\[ d \left\langle \frac{1}{8\pi^2} \text{Tr} \phi^2(x) \right\rangle_\varepsilon = \left\langle \frac{1}{4\pi^2} \text{Tr} \phi(x) d_A \phi(x) \right\rangle_\varepsilon = \int [DAD\phi] i \text{Tr} \phi(x) \frac{\delta}{\delta A} \exp(-S(A,\phi)) = 0. \] (3.4)

The action

\[ S_{\text{top}} = -\frac{1}{2} \int i \text{tr}(\phi F) \]

describes a true topological field theory whose path integral is concentrated on flat connections \( F = 0 \). The field \( \phi \) is sometimes denoted by \( B \), in which case the Lagrangian is proportional to \( \text{Tr}(BF) \), and the terminology \( BF \) theory was introduced. In the small area limit (or the \( \mathcal{U} \to 0 \) limit) \( YM_2 \) reproduces the results of this topological field theory. We will adopt this terminology, and refer to expectations of products of \( \text{Tr} \phi^2(x_i) \) as \( \phi \)-field correlators.

### 3.2 Gauge Theory Wightman Functionals

The above discussion leads to the conclusion that there are essentially three types of gauge-invariant correlators which are easily computed from the two-dimensional Yang-Mills action:

1. Field Strength Correlators
2. \( \phi \)-field correlators
3. Wilson loops

We now discuss the application of our general Borchers construction (Section 2) to gauge theories.

In two-dimensional theories, the field strength correlators and the \( \phi \)-field correlators share in common that they are gauge-invariant combinations of a Lie algebra valued zero-form on \( \Sigma \). In these cases, the correlators bear a similar algebraic structure to the scalar field theory discussed previously, in the sense that the Wightman \( n \)-point functions are scalar-valued generalized functions of the coordinates \( x_i \). Their analytic properties may, of course, be quite different.

The result (3.4) implies that the gauge-invariant correlators of two-dimensional Yang-Mills theories are constants (which, however, may depend on the metric geometry and group-theoretic invariants of the gauge group). We infer that the inner product of 2d Yang-Mills theory is well described by the generalized Borchers construction with constant Wightman functions, as in Example Zero above (see Sec. 2.6 and in particular eq. (2.17)).

### 3.3 The Hilbert Space of YM_2

We consider quantization of \( YM_2 \) on the cylinder with periodic spatial coordinate of period \( L \). This model is well understood and we will make no attempt at exposition since several excellent references exist in the literature \[19\][20\][21\][22]. Our purpose here is to point out an unexpected mathematical relationship having to do with the space of class functions on a Lie group that is predicted by the generalized Wightman construction introduced earlier in the paper.

The Hilbert space of this model is known to be the space of \( L^2 \) class functions on \( G \) with inner product

\[ \langle f_1 | f_2 \rangle = \int_G dU f_1^\dagger(U) f_2(U) \] (3.5)
where $dU$ is the Haar measure normalized to give volume one. For compact gauge groups, the Peter-Weyl theorem implies the decomposition of $L^2(G)$ into unitary irreps,

$$L^2(G) = \oplus_R R \otimes \overline{R}$$

Consequently a natural basis for the Hilbert space of states is provided by the characters in the irreducible unitary representations. This is known as the representation basis. The states $|R\rangle$ have wavefunctions $\chi_R(U)$ defined by

$$\langle U | R \rangle \equiv \chi_R(U) \equiv \text{Tr}_R(U)$$

(3.6)

While eqns. (3.5) and (3.6) provide two different expressions for the inner product of YM$_2$, a third expression for the same inner product can be derived from the generalized Borchers construction, in the special case of constant Wightman functions. The sesquilinear form is given by (2.17), and the inner product of YM$_2$ therefore comes from (2.17) after taking the quotient by zero-norm states, and subsequently, taking the completion.

The constants $\alpha_n$ appearing in (2.17), i.e. the gauge-invariant correlators of the theory, are determined in YM$_2$ by representation-theoretic invariants of the gauge group such as higher-order Casimir operators, and by the integration measure defined by the Riemannian metric on the Riemann surface $\Sigma$. The expressions are quite complicated, but they have been written down explicitly by Nunes and Schnitzer [23], using the abelianization technique for path integrals developed by Blau and Thompson. For example, the gauge-invariant two-point function for 2d $SU(N)$ Yang-Mills theory on a Riemann surface is

$$\alpha_2 = \langle \xi^a(x)\xi^b(y) \rangle = \frac{e^4}{Z_{\Sigma_g}} \sum_l \dim(l)^{2-2g} e \left( -\frac{e^2 AC_2(l)}{2} \right) \times \left[ \frac{(\rho, \rho)\delta_{ab}}{N^2} \delta^2_{x,y} - (p^{ab}(l + \rho))^2 + m^{ab}n^2 \right]$$

(3.7)

where $\dim(l)$ and $C_2(l)$ denote respectively the dimension and quadratic Casimir of the irreducible representation of $SU(N)$ with highest weight $l$. $\rho$ is the half-sum of the positive roots, $A$ is the area of $\Sigma$,

$$p^{ab} = \begin{cases} \frac{1}{N(N-1)} & \text{if } a \neq b \\ \frac{1}{N} & \text{if } a = b \end{cases}$$

$$m^{ab} = \begin{cases} \frac{1}{N(N-1)} & \text{if } a \neq b \\ 0 & \text{if } a = b \end{cases}$$

and

$$n = \text{total number of boxes in the tableau defined by } l.$$
4.2 Hermitian Matrix Models

A Hermitian one-matrix integral (see [14] for a review) takes the form

\[ Z = \int [dN^2 M] \exp(N \text{Tr} S(M)) \]

where \( S(M) \) is an arbitrary function. The model is said to be solvable if the integral can be performed explicitly, at least in the large \( N \) limit. We briefly indicate how this can be done in the simplest case. Diagonalize \( M \) via the transformation \( M = O^+ xO \) where \( x \) is diagonal and \( O \in U(N) \). The corresponding measure can be written as:

\[ dN^2 M = d[O]U(N) \Delta^2(x) \prod dx_k \]  

where \( \Delta(x) = \prod_{i > j} (x_i - x_j) \) is the Vandermonde determinant. The integrand does not depend on \( O \), so integration over \( O \) produces a group volume factor. The remaining integral over the eigenvalues is \( Z = \int \prod_{k=1}^{N} dx_k e^{NS(x_k)} \Delta^2(x) \). In the large \( N \) limit the corresponding saddle point equation takes the form

\[ \frac{1}{N} \frac{\partial S}{\partial x_k} = S'(x_k) + \frac{1}{N} \sum_{j \neq k} \frac{1}{x_k - x_j} = 0 \] 

4.3 Dijkgraaf-Vafa Matrix Models

Now consider \( \mathcal{N} = 1 \) \( U(N) \) gauge theory coupled to an adjoint chiral superfield. Take the superpotential

\[ W(\phi) = \sum_{k=1}^{n+1} \frac{g_k}{k} \text{Tr}(\phi^k) \] 

for some \( n \). To get a supersymmetric vacuum we impose D- and F-term conditions. Taking \( \phi \) to be diagonal would satisfy the D-term condition, and the F-term condition is \( W'(\phi) = 0 \). This equation has \( n \) roots which we call \( a_i \), \( (i = 1 \ldots n) \) and hence \( W'(x) = g_{n+1} \prod_{i=1}^{n} (x - a_i) \). By taking \( \phi \) to have eigenvalue \( a_i \) with multiplicity \( N_i \), \( U(N) \) is broken down to \( \prod_{i=1}^{n} U(N_i) \) with \( \sum N_i = N \).

If the \( a_i \) are distinct, the chiral superfields are all massive and can be integrated out, yielding a low energy effective action. The chiral part of the low energy effective Lagrangian can be written as [14]

\[ L_{\text{eff}} = \int d^2 \theta \, W_{\text{eff}} + \text{c.c.}, \quad W_{\text{eff}} = f(S_k, g_k) + \sum_{i,j} \tau_{ij} \omega_{\alpha i} \omega_{\alpha j}^{\alpha}, \]

where \( S_k = -\frac{1}{2\pi} \text{Tr} W_{\alpha i} W^{\alpha i} \) and \( \omega_{\alpha i} = \frac{1}{4\pi} \text{Tr} W_{\alpha i} \), where \( W^{\alpha i} \) are the \( U(N_i) \) gauge superfields.

The Dijkgraaf-Vafa conjecture is that the chiral part of the effective action can be given by a holomorphic function \( F_G(S_k) \), such that

\[ W_{\text{eff}} = \sum_{i=1}^{n} N_i \frac{\partial F_G}{\partial S_i} + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 F_G}{\partial S_i \partial S_j} \omega_{\alpha i} \omega_{\alpha j}^{\alpha} \] 

The Dijkgraaf-Vafa conjecture further identifies \( F_G \) as the free energy of a nonsupersymmetric matrix model with potential given by \( W \), the superpotential of the 4d supersymmetric gauge theory. The matrix model free energy \( F_0 \) is defined by

\[ e^{-\beta F_0} = \int_{\phi \in U(M)} D\phi \, e^{-\frac{1}{\beta} W(\phi)} , \]

where \( \phi \in U(M) \). For the model we are considering, one needs also to assume that the \( U(M) \) symmetry is broken down to \( \prod_{i=1}^{n} U(M_i) \) where \( \sum_{i=1}^{n} M_i = M \). Moreover one should also identify \( S_i = g_i M_i \). In the large \( M \) limit one can compute \( F_0 \) order by order using planar diagrams.
The prescription is that, for example, the \( l \)th instanton contribution to the effective action can be reproduced from a perturbative contribution with \( l \) loops in the matrix model. The effective superpotential is obtained by

\[
W_{\text{eff}} = \sum_{i=1}^{n} \left( N_i \frac{\partial F_0}{\partial S_i} - 2\pi i \tau_0 S_i \right). \tag{4.7}
\]

where \( \tau_0 \) is the bare coupling of the theory.

### 4.4 Matrix Models as Matrix States

The interesting point we wish to make in this section is that the aforementioned matrix models (arising in string theory, condensed matter, and other branches of physics) are special cases of the noncommutative-target Borchers construction with a matrix state in the \(*\)-algebra sense (see Def. [1]). An acceptable mathematical terminology is define the term matrix model to simply be the noncommutative-target Borchers algebra with a matrix state, in the context of \(*\)-algebras.

We may suppose that \( V \) is a compact manifold, so the constant functions are in the Schwartz space. Consider the noncommutative Borchers algebra \( A = \mathcal{A}(V, \mathfrak{h}_N) \) into the space \( \mathfrak{h}_N \) of Hermitian matrices. Define a state \( f \) on \( A \) by the prescription

\[
f_2(a_1 \times a_2) = \begin{cases} 
0, & \text{otherwise} \\
\sum_{I_1, I_2} (a_1)_{I_1} (a_2)_{I_2} K_{I_1, I_2}, & (3i) \ a_i \text{ is nonconstant} 
\end{cases}
\]

where

\[
K_{I_1, I_2} = \int_{\mathfrak{h}_N} [Da] \ a_{I_1} a_{I_2} e^{-S(a)}
\]

with \( f_n \) for \( n > 2 \) defined analogously. The measure \([Da]\) runs over \( \mathfrak{h}_N \). Here, \( I_\alpha = (i_\alpha, j_\alpha) \) denotes a pair of indices which together specify a matrix element. \( S(a) \) denotes the action of the matrix model, which might be \( \text{Tr}(a^2) \) for example. We have chosen to use the notation of Section 4.2, which describes Hermitian matrix models; however, the framework is completely general.

After taking the quotient by the kernel of \( f = (f_0, f_1, f_2, \ldots) \), the one-particle space is simply \( \mathfrak{h}_N \), and we recover a simple Hilbert space and “operator formalism” for matrix models. The matrix model is called solvable if the \( N^2 \)-dimensional integrals in (4.8) can be reduced to \( N \)-dimensional integrals or evaluated by saddle-point techniques.

### 5. Conclusions

For standard constructive field theory models involving scalar bosons and Dirac or Majorana fermions, it is known that the Wightman axioms can be formulated in terms of the Wightman \( n \)-point functionals, which are tempered distributions. An equivalent formulation exists in terms of the Schwinger functions. A seminal result, due in its original form to Wightman [6], is the reconstruction of the Hilbert space, vacuum vector, and quantum field operators from the \( n \)-point distributions.

In previous sections, we have outlined one possible method of extending Wightman’s construction to theories with gauge symmetry, including nonabelian pure Yang-Mills models and matrix models. It was found that Yang-Mills theory (including matrix models) and constructive field theory models [17] possess the following common structure:

1. A \(*\)-algebra \( \mathcal{B} \) (not necessarily commutative), with the generalized Borchers algebra \( \mathcal{A} \) of functions into \( \mathcal{B} \).

\[\text{On this point, the author is particularly indebted to Daniel Jafferis for enlightening discussions.}\]
2. A functional $\omega \in \mathcal{A}'$, which satisfies $\omega(f^* \times f) \geq 0$ ($\forall f$) in theories with no gauge symmetry, or in gauge theories with no charged states. More generally, such as for Gupta-Bleuler QED, the functional $\omega$ is postulated to satisfy a Hilbert space structure condition as in Section 2.4.

3. A symmetry group $G$ and a representation $\alpha : G \rightarrow \text{Aut}(\mathcal{A})$, which is $\omega$-invariant in the sense that $\omega(\alpha_g(f)) = \omega(f)$ for all $f \in \mathcal{A}$ and $g \in G$.

4. A collection of ideals $I_1, \ldots, I_n$ of the algebra $\mathcal{A}$ which are required to lie in the kernel of $\omega$. (Each ideal represents a physical property satisfied by the $n$-point functions; in Wightman QFT, $n = 2$ and the two ideals represent locality and the positive light-cone spectral condition.)

5. A Hilbert space $\mathcal{H}_{\text{phys}}$ defined to be the completion of $\mathcal{A}/\ker(\omega)$

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