THE MINIMAL IDEAL IN MULTIPLIER ALGEBRAS

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Abstract. Let $A$ be a simple, $\sigma$-unital, non-unital, non-elementary C*-algebra and let $I_{\text{min}}$ be the intersection of all the ideals of $M(A)$ that properly contain $A$. $I_{\text{min}}$ coincides with the ideal defined by Lin [19] in terms of approximate units of $A$ and $I_{\text{min}}/A$ is purely infinite and simple. If $A$ is separable, or if $A$ has the (SP) property and its dimension semigroup $D(A)$ of Murray-von Neumann equivalence classes of projections of $A$ is order separable, or if $A$ has strict comparison of positive elements by traces, then $A \neq I_{\text{min}}$.

If the tracial simplex $T(A)$ is nonempty, let $I_{\text{cont}}$ be the closure of the linear span of the elements $A \in M(A)_+$ such that the evaluation map $A(\tau) = \tau(A)$ is continuous. If $A$ has strict comparison of positive element by traces then $I_{\text{min}} = I_{\text{cont}}$. Furthermore, $I_{\text{min}}$ too has strict comparison of positive elements in the sense that if $A, B \in (I_{\text{min}})_+$, $B \notin A$ and $d_\tau(A) < d_\tau(B)$ for all $\tau \in T(A)$ for which $d_\tau(B) < \infty$, then $A \preceq B$.

However if $A$ does not have strict comparison of positive elements by traces then $I_{\text{min}} \neq I_{\text{cont}}$ can occur: a counterexample is provided by Villadsen’s AH algebras without slow dimension growth. If the dimension growth is flat, $I_{\text{cont}}$ is the largest proper ideal of $M(A)$.

1. Introduction

The ideal structure of the multiplier algebra of a simple, $\sigma$-unital non-unital non-elementary, C*-algebra has received over the years a lot of attention. In this paper we will focus on the study of the smallest (closed) ideal properly containing $A$.

Lin [18, Lemma 2] gave a constructive proof of the existence of such a smallest ideal for AF algebras in terms of the tracial simplex of the algebra (see §2.2).

Then Lin and Zhang [22], proved that every simple, separable, non-unital, non-elementary C*-algebra with property (SP) and with an approximate identity of projections (such algebras do not need to have real rank zero) contains an $\ell^1$-sequence of projections (see Definition 3.8 for a generalization). Furthermore, all the principal ideals generated by projections associated to such sequences coincide with the minimal ideal properly containing $A$.

In [19] Lin defined for every simple $\sigma$-unital C*-algebra an ideal $I$ in terms of an approximate identity of positive elements and proved that $I$ is contained in any ideal properly containing $A$. If $A$ is separable, then $A \neq I$. This construction is instrumental in proving that $M(A)/A$ is simple if and only if it is purely infinite and simple if and only if $A$ has a continuous scale ([21 Theorems 2.4 and 3.2]).

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For simple C*-algebras with real rank zero, stable rank one, and weakly unperforated \(K_0\), (equivalently, strictly unperforated monoid \(V(A)\) of Murray-von Neumann equivalence classes of projections in \(A \otimes K\)) Perera proved that there is a lattice isomorphism between the ideals of \(M(A)\) and the order ideals of \(V(A) \cup \operatorname{Aff}^+(S_u)\) (see \cite{22} Theorems 2.1 and 3.9 and notations therein) and then proved \cite{25} Proposition 4.1 that \(V(A) \cup \operatorname{Aff}^+(S_u)\) is the smallest order ideal properly containing \(V(A)\), thus obtaining the smallest ideal properly containing \(A\). Here \(\operatorname{Aff}^+(S_u)\) is the space of strictly positive continuous affine functions on the state space \(S_u\); see also §2.2 and §4. This ideal, denoted by \(L(A)\), plays an important role in the study by Perera \cite{25} and Kucerovsky and Perera \cite{15} of the ideal structure of the multiplier algebra and the characterization of when the corona algebra \(M(A)/A\) is purely infinite.

The goal of this paper is to clarify the relations between the various constructions of the minimal ideal and to further investigate its properties. Throughout the paper, \(\mathcal{A}\) will denote a simple, \(\sigma\)-unital, non-unital and non-elementary C*-algebra.

We revisit Lin’s definition \cite[Lemma 2.1]{19} of a nonclosed ideal of \(M(A)\) defined in terms of an approximate unit \(\{e_n\}\) of positive elements, which we denote by \(L(K_o(\{e_n\}))\) and Lin did denote by \(I_0\). It is easy to see that \(L(K_o(\{e_n\}))\) is a left ideal. Lemma 2.1 \cite{19} states that it is also \(*\)-invariant, and hence, a two-sided ideal, however Example 3.5 shows a case when \(L(K_o(\{e_n\}))\) is not two-sided. Nevertheless, its closure, that we denote by \(I_{\min}\) (and Lin did denote by \(I\)), is proved to be indeed a two-sided ideal \(\text{(Corollary 3.4)}\), and thus all the further results in \cite{19} and \cite{21} are correct. The proof that \(I_{\min}\) is two-sided depends on a bidual decomposition result \(\text{(Theorem 4.4)}\) which is in the line of the tri-diagonal decomposition of elements in \(M(A)\), first introduced by Elliot in \cite{10} proof of Theorem 3.1. More background on bi-diagonal and tri-diagonal decompositions is presented before Theorem 4.4. As a consequence of the proof, one also sees that \(I_{\min}\) does not depend on the approximate identity chosen.

In \cite[Remark 2.9]{19} Lin did prove that \(I_{\min}\) is contained in the intersection \(J_o\) of all the ideals properly containing \(\mathcal{A}\). In Theorem 4.7 we prove that \(I_{\min} = J_o\) and in Theorem 4.8 we show that if \(I_{\min}/\mathcal{A}\) is nonzero (and necessarily simple), then it is purely infinite.

Furthermore, \(\mathcal{A} \neq I_{\min}\) if and only if there exists a thin sequence of positive elements for \(\mathcal{A}\) \(\text{(Definition 5.8, Theorem 5.14)}\). This notion can be seen as a generalization of the notion of \(\ell^1\) sequence of projections introduced for the (SP) case in \cite{22}, thus providing a bridge between the approaches in \cite{22} and \cite{19}.

If \(\mathcal{A}\) is separable, or if \(\mathcal{A}\) has the (SP) property and the dimension semigroup of Murray-von Neumann equivalence classes of projections is countable, or, more generally, is just order separable, then a thin sequence exists, and hence, \(\mathcal{A} \neq I_{\min}\). This includes the case of type II\(_1\) factors.

We do not have examples when \(\mathcal{A} = I_{\min}\). A natural test case we did consider is the nonseparable simple C*-algebra with both a nonzero finite and an infinite projection studied by Rordam in \cite{29}, but it still yields \(\mathcal{A} \neq I_{\min}\) \(\text{\text{(see last paragraph of Section 5)}}\).

In the case when \(\mathcal{A}\) has a nonempty tracial simplex \(\mathcal{T}(\mathcal{A})\), another natural ideal inspired by the approaches in \cite{18} and \cite{25} is \(I_{\cont}\), the ideal generated by positive elements with continuous evaluation function over \(\mathcal{T}(\mathcal{A})\) \(\text{(Definition 5.1)}\). We show that \(\mathcal{A} \subseteq I_{\cont}\) \(\text{(Proposition 5.4)}\). If in addition, \(\mathcal{A}\) has strict comparison of positive
elements by traces, then $I_{\text{min}} = I_{\text{cont}}$, and hence, $A \not\cong I_{\text{min}}$ (Theorem 5.3). This result can be seen as a generalization of Perera’s construction [25] of the minimal ideal in the case that all quasitraces of $A$ are traces (e.g., $A$ is exact), while the weak unperforation of the $K_0$ group is equivalent to strict comparison by quasitraces, and hence, to strict comparison by traces.

What happens when there is no strict comparison by traces? In the case of the AH-algebras without slow dimension growth studied by Villadsen, which are known to have perforation, we prove that $I_{\text{min}} \neq I_{\text{cont}}$ (Theorem 7.8). In addition, we show that if $A$ has flat dimension growth, every positive element not in $I_{\text{cont}}$ must be full (Theorem 7.10), and hence, $I_{\text{cont}}$ contains every other proper ideal of $\mathcal{M}(A)$. If however the dimension growth is very fast, then this is no longer true (Proposition 7.12).

Finally, we prove that if $A$ has strict comparison of positive elements, then so does $I_{\text{min}}$. This result extends a previous result obtained by us in the case when $A$ is separable and has real rank zero ([13, Proposition 3.1]). The methods used are inspired by the techniques used in [14, Theorem 6.6] to prove that $\mathcal{M}(A)$ has strict comparison of positive elements if so does $A$ and $A$ has quasicontinuous scale in the sense of [15].

2. Preliminaries

2.1. Cuntz subequivalence. Cuntz subequivalence in a $C^*$-algebra $B$ is denoted by $\preceq$, that is, if $a, b \in B_+$, then $a \preceq b$ if there is a sequence $x_n \in B$ such that $\|x_n b x_n^* - a\| \to 0$. If $a \preceq b$ and $b \preceq a$, then $a$ is said to be equivalent to $b$ ($a \sim b$). It is well known that for projections subequivalence in this sense coincides with Murray-von Neumann subequivalence.

We will use the following notation:

$$f_x(t) := \begin{cases} 0 & \text{for } t \in [0, \epsilon] \\ \frac{t}{\epsilon} & \text{for } t \in (\epsilon, 2\epsilon] \\ 1 & \text{for } t \in (2\epsilon, \infty). \end{cases}$$

For ease of reference we list here the following well known facts (see for instance [7,28]).

**Lemma 2.1.** Let $B$ be a $C^*$-algebra, $a, b \in B_+$, $x \in B$, $\delta > 0$. Then

(i) $xax^* \preceq a$;

(ii) $xx^* \sim x^*x$;

(iii) If $a \preceq b$ then $a \preceq b$;

(iv) If $\|a - b\| < \delta$ then $(a - \delta)_+ \preceq b$;

(v) If $a \preceq b$, then there is $r \in B$ and $\delta' > 0$ such that $(a - \delta)_+ = r(b - \delta')_+ r^*$; there is also $r' \in B$ such that $(a - \delta)_+ = r'br'^*$;

(vi) If $a \preceq a'$ and $b \preceq b'$ then $a + b \preceq a' + b'$;

(vii) If $ab = 0$, then $(a + b - \delta)_+ = (a - \delta)_+ + (b - \delta)_+$;

(viii) [14] Lemma 2.3 If $a \preceq b$, then $(a - \delta)_+ \preceq (b - \delta)_+$;

(ix) [14] Lemma 2.4. (iii) $(a + b - \delta_1 - \delta_2)_+ \preceq (a - \delta_1)_+ + (b - \delta_2)_+$ for $\delta_1, \delta_2 \geq 0$.

**Lemma 2.2.** Let $B$ be a $C^*$-algebra, $a, b \in B_+$, and $\|a - b\| < \delta$. Then for all $\epsilon \geq 0$, $(a - \epsilon - \delta)_+ \preceq (b - \epsilon)_+$.

**Proof.** Since $\|a - (b - \epsilon)_+\| \leq \|a - b\| + \|b - (b - \epsilon)_+\| < \epsilon + \delta$, the conclusion follows from Lemma 2.1 (iv).
Lemma 2.3. Let \( \mathcal{B} \) be a C*-algebra and \( a \in \mathcal{B}_+ \). For every \( \epsilon > 0 \) there is \( y \in \mathcal{B} \) such that \( \|a - a^{1/2}yay^*a^{1/2}\| < \epsilon \) and \( \|yay^*\| = 1 \).

Proof. Choose \( g_\epsilon(t) := \sqrt{f_\epsilon(t)} \) and set \( y = g_\epsilon(a) \). Then \( yay^* = f_\epsilon(a) \) and hence both conditions are satisfied. \( \square \)

Lemma 2.4. Let \( \mathcal{B} \) be a simple C*-algebra and \( 0 \neq a, b \in \mathcal{B}_+ \). Then there is \( 0 \neq c \in \mathcal{B}_+ \) such that \( c \leq a \) and \( c \leq b \).

Proof. Since \( \mathcal{B} \) is simple, there are elements \( x_k, y_k \in \mathcal{B} \) such that

\[
\sum_{j=1}^{n} x_k y_k b - b < \frac{\|b\|}{2}.
\]

Then \( \sum_{j=1}^{n} x_k y_k b \neq 0 \), and hence, there is some \( k \) such that \( x_k y_k b \neq 0 \). Then also

\[
c := (x_k y_k b)^* (x_k y_k b) \neq 0, \quad d := (x_k y_k b)(x_k y_k b)^* \neq 0.
\]

First notice that

\[
d \leq \|b\|^2 \|y_k\|^2 \|a\| \|x_k ax_k^* \| \leq a,
\]

whence \( d \leq a \). Since \( c \sim d \) by Lemma 2.1 (ii), it follows that \( c \leq a \). On the other hand,

\[
c \leq \|a\|^2 \|x_k\|^2 \|y_k\|^2 \|b\| \|b\|,
\]

hence \( c \leq b \), by scaling if necessary \( c \), which preserves the relation \( c \leq a \). \( \square \)

For the convenience of the readers, we give the proof of the following well known results.

Lemma 2.5. Let \( \mathcal{B} \) be a simple, non-elementary C*-algebra. Then for every element \( 0 \neq a \in \mathcal{B}_+ \) there is an infinite sequence of mutually orthogonal elements \( 0 \neq a_k \in \mathcal{B}_+ \) such that \( \sum_{j=1}^{n} a_k \leq a \) for all \( n \).

Proof. Choose \( \delta > 0 \) such that \( (a - \delta)_+ \neq 0 \). Then her((\( a - \delta \))_+) contains a positive element \( b \) with infinite spectrum (e.g., [20] 1.11.45). In fact it contains an element with spectrum \([0, 1]\) by [3] pg 67). Since \( b \leq \|a\| a \), to simplify notations, assume that \( b \leq a \). Now choose by compactness a converging sequence of distinct elements in \( t_k \in \sigma(b) \), and by passing to a subsequence assume that the \( t_k \) are monotone and that \( |t_k - t_{k+1}| \) is also monotone. Let \( \epsilon_k := \frac{1}{2} |t_k - t_{k+1}| \). Then the intervals \([t_k - 2\epsilon_k, t_k + 2\epsilon_k] \) are disjoint. Let \( g_k \) be the continuous function with

\[
g_k(t) := \begin{cases} 
0 & t \in \left[0, t_k - 2\epsilon_k\right] \cup \left[t_k + 2\epsilon_k, \infty\right) \\
t - \epsilon_k & t \in [t_k - \epsilon_k, t_k + \epsilon_k] \\
linear & t \in [t_k - 2\epsilon_k, t_k - \epsilon_k] \\
linear & t \in [t_k + \epsilon_k, t_k + 2\epsilon_k]
\end{cases}
\]

Let \( a_k := \frac{2g_k(t)}{2\pi} \). Then \( 0 \neq a_k \leq \frac{b}{2^j} \leq \frac{b}{2^{j+1}} \) and \( a_i a_k = 0 \) for \( i \neq j \). Thus we conclude that \( \sum_{j=1}^{\infty} a_k \leq a \). \( \square \)
Lemma 2.6. Let £ be a C*-algebra and let a, b, c ∈ £. Then
(i) $xa^2c^* \sim xa^2c^*$.
(ii) $b^{1/2}a^{1/2} \sim bab$
(iii) If $b \leq c$, then $bab \leq cac$.

Proof.
(i) First we see that $xa^2c^* \leq \|a\|xa^2c^*$ and hence $xa^2c^* \leq xa^2c^*$. For every $\delta > 0$, $0 \leq (a - \delta)^+ \leq \frac{1}{\delta^2}a^2$ and hence $x(a - \delta)^+c^* \leq xa^2c^*$. Thus

$$xax^* = \lim_{\delta \to 0} x(a - \delta)^+c^* \leq xa^2c^*,$$

which concludes the proof.
(ii) $b^{1/2}a^{1/2} \sim a^{1/2}b^{1/2}$ (by Lemma 2.1 (ii))
    $\sim a^{1/2}b^{1/2}a^{1/2}$ (by (i))
    $\sim bab$ (by Lemma 2.1 (ii))
(iii) $bab \sim b^{1/2}a^{1/2}$ (by (ii))
    $\sim a^{1/2}b^{1/2}$ (by Lemma 2.1 (ii))
    $\leq a^{1/2}c^{1/2}$ (by Lemma 2.1 (iii), since $a^{1/2}b^{1/2} \leq a^{1/2}c^{1/2}$)
    $\sim cac$ (by the same two equivalences above.)

\[\square\]

2.2. The tracial simplex and strict comparison. Given a simple σ-unital (possibly unital) C*-algebra $A$ and a nonzero positive element $e$ in the Pedersen ideal $Ped(A)$ of $A$, denote by $\mathcal{T}(A)$ the collection of the (norm) lower semicontinuous densely defined tracial weights $\tau$ on $A_+$, that are normalized on $e$. Explicitly, a trace $\tau$ is an additive and homogeneous map from $A_+$ into $[0, \infty]$ (a weight); satisfies the trace condition $\tau(x^*) = \tau(x^*)$ for all $x \in A$; the cone $\{x \in A_+ \mid \tau(x) < \infty\}$ is dense in $A_+$ ($\tau$ is also called densely finite, or semifinite); satisfies the lower semicontinuity condition $\tau(x) \leq \lim_{x_n \to x} \tau(x_n)$ for $x, x_n \in A_+$ and $\|x_n - x\| \to 0$, or equivalently, $\tau(x) = \lim \tau(x_n)$ for $0 \leq x_n \uparrow x$ in norm; and $(\tau(e) = 1$ (\tau$ is normalized on e). We will assume henceforth that $\mathcal{T}(A) \neq \emptyset$, and hence, $A$ is stably finite.

When equipped with the topology of pointwise convergence on $Ped(A)$, $\mathcal{T}(A)$ is a Choquet simplex (e.g., see [32, Proposition 3.4] and [11]). The collection of the extreme points of $\mathcal{T}(A)$ is denoted by $\partial_e(\mathcal{T}(A))$ and is called the extremal boundary of $\mathcal{T}(A)$. For simplicity’s sake we call the elements of $\mathcal{T}(A)$ (resp., $\partial_e(\mathcal{T}(A))$) traces (resp., extremal traces.) Tracial simplexes $\mathcal{T}(A)$ arising from different nonzero positive elements in $Ped(A)$ are homeomorphic; so we will not reference explicitly which element $e$ is used. A trace $\tau$ on $A$ is naturally extended to the trace $\tau \otimes Tr$ on $A \otimes K$, and so we can identify $\mathcal{T}(A \otimes K)$ with $\mathcal{T}(A)$. For more details, see [32], [11] and also [14] and [12].

Recall also that as remarked in [12, 5.3], by the work of F. Combes [6, Proposition 4.1, Proposition 4.4] and Ortega, Rordam, and Thiel [24, Proposition 5.2], every $\tau \in \mathcal{T}(A)$ has a unique extension, (which we will still denote by $\tau$) to a lower semicontinuous (i.e., normal) tracial weight (trace for short) on the enveloping von Neumann algebra $A''$, and hence to a trace on the multiplier algebra $M(A)$. 

Definition 2.7. Given a convex compact space $K$, 

(i) $\text{Aff}(K)$ denotes the Banach space of the continuous real-valued affine functions on $K$ with the uniform norm;

(ii) $\text{LAff}(K)$ denotes the collection of the lower semicontinuous affine functions on $K$ with values in $\mathbb{R} \cup \{+\infty\}$;

(iii) $\text{Aff}(K)_{++}$ (resp., $\text{LAff}(K)_{++}$) denotes the cone of the strictly positive functions (i.e., $f(x) > 0$ for all $x \in K$) in $\text{Aff}(K)$ (resp., in $\text{LAff}(K)$).

For every $A \in \mathcal{M}(\mathcal{A})_{+}$, denote by $\hat{A}$ the evaluation map

\[(2.2) \quad \mathcal{T}(\mathcal{A}) \ni \tau \rightarrow \hat{A}(\tau) := \tau(A) \in [0, \infty],\]

and denote by $[\hat{A}]$ the dimension map

\[(2.3) \quad \mathcal{T}(\mathcal{A}) \ni \tau \rightarrow [\hat{A}](\tau) := d_\tau(A) \in [0, \infty]\]

where

\[d_\tau(A) := \lim_n \tau(A^{1/n})\]

is the dimension function.

Then it is well known that $\hat{A} \in \text{LAff}(\mathcal{T}(\mathcal{A}))_{++}$ and $[\hat{A}] \in \text{LAff}(\mathcal{T}(\mathcal{A}))_{++}$ for every $A \neq 0$. By definition of the topology on $\mathcal{T}(\mathcal{A})$, if $a \in \text{Ped}(\mathcal{A})$, then $\hat{a} \in \text{Aff}(\mathcal{T}(\mathcal{A}))$.

As shown in [24] Remark 5.3,\n
\[(2.4) \quad d_\tau(A) = \tau(R_A) \quad \text{where } R_A \in \mathcal{A}^{**} \text{ is the range projection of } A.\]

We will also use frequently the following well known facts. If $A, B \in \mathcal{M}(\mathcal{A})_{+}$, and $\tau \in \mathcal{T}(\mathcal{A})$ then

\[(2.5) \quad A \leq B \Rightarrow \hat{A}(\tau) \leq \hat{B}(\tau)\]

\[(2.6) \quad A \leq B \Rightarrow d_\tau(A) \leq d_\tau(B)\]

\[(2.7) \quad AB = 0 \Rightarrow d_\tau(A + B) = d_\tau(A) + d_\tau(B)\]

\[(2.8) \quad \tau(A) \leq \|A\| d_\tau(A)\]

\[(2.9) \quad d_\tau((A - \delta)_{+}) < \frac{1}{\delta} \tau(A) \quad \forall \delta > 0.\]

We will use the following notions of strict comparison.

Definition 2.8. Let $\mathcal{A}$ be a simple $C^*$-algebra with $\mathcal{T}(\mathcal{A}) \neq \emptyset$. Then we say that $\mathcal{A}$ has strict comparison of positive elements by traces if $a \leq b$ for $a, b \in \mathcal{A}_{+}$ such that $d_\tau(a) < d_\tau(b)$ for all those $\tau \in \mathcal{T}(\mathcal{A})$ for which $d_\tau(b) < \infty$.

\[d_\tau((A - \delta)_{+}) < \frac{1}{\delta} \tau(A) \quad \forall \delta > 0.\]

Notice that strict comparison is often defined in terms of 2-quasitraces. In [12] Theorem 2.9] we proved that if a unital simple C*-algebra of real rank zero and stable rank one has strict comparison of positive elements by traces (equivalently, of projections, due to real rank zero) then all 2-quasitraces are traces. Recently this result was extended by showing that if a simple stable C*-algebra has strict comparison of positive elements by traces then all 2-quasitraces are traces [23].

Notice also that for $\mathcal{M}(\mathcal{A}) \neq \mathcal{A}$, which is not simple, we must add the obviously necessary hypothesis that $A \in I(B)$ as that condition does not follow in general from the comparison condition. Indeed if there is an element $B \in \mathcal{A}_{+}$ with $d_\tau(B) = \infty$
for all $\tau \in T(A)$ (and this is certainly the case when $A$ is stable) then the condition $d_+ (A) < d_+ (B)$ for all those $\tau \in T(A)$ for which $d_+ (B) < \infty$ is trivially satisfied for every $A \in M_+(A)$ and yet $A \not\sim B$.

2.3. Cones and ideals in C*-algebras. Let $\mathcal{B}$ be a C*-algebra and $K \subset \mathcal{B}_+$. Set

\begin{equation}
L(K) := \{ x \in \mathcal{B} \mid x^* x \in K \}.
\end{equation}

\begin{equation}
L(K)^* L(K) := \{ \sum_{j=1}^{n} x_j^* y_j \mid x_j, y_j \in L(K), n \in \mathbb{N} \}.
\end{equation}

**Definition 2.9.** Let $\mathcal{B}$ be a C*-algebra and $K \subset \mathcal{B}_+$.

(i) $K$ is a cone if $x + y \in K$ and $tx \in K$ whenever $x, y \in K$ and $0 \leq t \in \mathbb{R}$; $K$ is hereditary if $x \in K$ whenever $0 \leq x \leq y \in K$.

(ii) A subalgebra $\mathcal{C} \subset \mathcal{B}$ is hereditary if the cone $\mathcal{C}_+$ is hereditary.

(iii) A cone $K$ is

(a) invariant if $ax^* \in K$ whenever $x \in K$ and $a \in \mathcal{B}$;

(b) strongly invariant if $x^* x \in K$ whenever $x \in \mathcal{B}$ and $x^* x \in K$;

(c) weakly invariant if $ax^* \in K$ whenever $x \in K$ and $a \in \mathcal{B}$.

Hereditary cones are also called order ideals. It is well known and immediate to see that if $K$ is a hereditary cone, then $L(K)$ is a left ideal of $\mathcal{B}$, $L(K)^* L(K)$ and $L(K)^* \cap L(K)$ are *-subalgebras of $\mathcal{B}$, and $L(K)^* L(K) \subset L(K)^* \cap L(K)$. Furthermore, if $K$ is a hereditary cone, then

\begin{equation}
L(K) \text{ is two-sided if and only if } K \text{ is invariant.}
\end{equation}

\begin{equation}
L(K) = L(K)^* \text{ if and only if } K \text{ is strongly invariant.}
\end{equation}

**Theorem 2.10.** Let $\mathcal{B}$ be a C*-algebra and $K \subset \mathcal{B}_+$ be a hereditary cone. Then

(i) $\overline{K}$ is a hereditary cone ([9, Theorem 2.5]);

(ii) $L(K)^* L(K) = \text{span } K$ (the the collection of complex linear combinations of $K$) and $(L(K)^* L(K))_+ = K$ ([31], Proposition 3.21);

(iii) If $K$ is closed, then $L(K)^* L(K) = L(K)^* \cap L(K)$ and the mappings $\mathcal{B} \to \mathcal{B}_+$, $K \to L(K)$, and $L \to L^* \cap L$ define bijective, order preserving correspondences between the sets of hereditary C*-subalgebras of $\mathcal{B}$, closed hereditary cones of $\mathcal{B}_+$, and closed left ideals of $\mathcal{B}$ ([31, Theorem 2.4], [26, Theorem 1.5.2]).

We collect here some simple properties of hereditary cones in C*-algebras that we will use in this paper.

**Lemma 2.11.** Let $\mathcal{B}$ be a C*-algebra and $K \subset \mathcal{B}_+$ be a cone.

(i) The (norm) closure $\overline{K}$ of $K$ is a cone.

(ii) If $K$ is weakly invariant, then $\overline{K}$ is invariant.

(iii) If $K$ is invariant, then $\overline{K} = \{ x \in \mathcal{B}_+ \mid (x - \delta)_+ \in K \ \forall \ \delta > 0 \}$.

(iv) If $K$ is closed and invariant, then it is hereditary and strongly invariant.

**Proof:**

(i) Obvious.

(ii) Let $x \in \overline{K}$, $a \in \mathcal{B}$ and let $x_n \in K$ be a sequence converging (in norm) to $x$. Since $ax_n a^* \in \overline{K}$ for every $n$, it follows that $axa^* = \lim_n ax_n a^* \in \overline{K}$, that is, $\overline{K}$ is invariant.
(iii) Let \( K' := \{ x \in \mathcal{B}_+ \mid (x - \delta)_+ \in K \ \forall \ \delta > 0 \} \). Since \( \lim_{\delta \to 0} (x - \delta)_+ = x \) for all \( x \in \mathcal{B}_+ \), it follows that \( K' \subset \bar{K} \). Conversely, let \( x \in \bar{K}, \ \delta > 0 \), and choose \( y \in K \) such that \( |x - y| < \delta \). Then \( (x - \delta)_+ = ryr^* \in K \) for some \( r \in \mathcal{B} \) by Lemma [2.14] (iv) and (v). Thus \( x \in K' \), which proves that \( K' = \bar{K} \).

(iv) Let \( x \leq y \), with \( x \in \mathcal{B}_+ \) and \( y \in K \). By Lemma [2.14] (iii) and (v) for every \( \delta > 0 \) there is an \( r \in \mathcal{B} \) such that \( (x - \delta)_+ = ryr^* \in K \) (because \( K \) is invariant). Thus \( x = \lim_{\delta \to 0} (x - \delta)_+ \in K \) (because \( K \) is closed), which proves that \( K \) is hereditary.

Now let \( x^*x \in K \) and \( x = v|x| \) be the polar decomposition of \( x \). By [11] Lemma 2.1, \( v|x|^{1/n} \in \mathcal{B} \) for every \( n \in \mathbb{N} \), hence \( (v|x|^{1/n})x^*x(v|x|^{1/n})^* \in K \). Since \( |x|^{1/n}|x| \to |x| \) in norm, it follows that also

\[
(xx^* = vxx^*v^* = \lim_n (v|x|^{1/n})x^*x(v|x|^{1/n})^* \in K,
\]

which proves that \( K \) is strongly invariant. \( \square \)

In the course of the proof of (iv) we have shown that

\[
(2.14) \quad \text{If } K \text{ is invariant and } 0 \leq x \leq y \in K \text{ then } (x - \delta)_+ \in K \text{ for all } \delta > 0.
\]

From Example 3.5 and Corollary 4.3, we will see that the condition in (iii) that \( K \) is invariant cannot be replaced by condition that \( K \) is weakly invariant.

**Corollary 2.12.** Let \( \mathcal{B} \) be a \( C^* \)-algebra and \( K \subset \mathcal{B}_+ \) is a weakly invariant hereditary cone in a \( C^* \)-algebra \( \mathcal{B} \), then \( L(K) = L(\bar{K}) \), \( L(K)_* \) is a two-sided ideal, and \( L(K)_* = \bar{K} \).

**Proof.** By Lemma 2.11 (i), (ii), and (iv), \( \bar{K} \) is a strongly invariant hereditary cone. By (2.13), \( L(\bar{K}) = L(K)_* \) and by Theorem 2.10 (i) and (ii), span \( \bar{K} = L(\bar{K}) \). Since \( K \) is hereditary, \( L(K) \) is a left ideal, and hence, so is \( L(\bar{K}) \). Moreover, \( K \subset L(K), \) hence \( \text{span } \bar{K} \subset L(\bar{K}) \), and hence, \( L(\bar{K}) \subset L(K) \). On the other hand, \( L(K) \subset L(\bar{K}), \) and hence \( \overline{L(K)} \subset L(\bar{K}), \) and thus \( L(\bar{K}) = L(\bar{K}) \). \( \square \)

2.4. **Approximate identities.** When \( \mathcal{B} \) is a \( \sigma \)-unital \( C^* \)-algebra, and \( \{e_n\} \) is an approximate identity, we will always assume that

\[
(2.15) \quad \{e_n\} \text{ is strictly increasing } (0 \leq e_n \leq e_{n+1}) \quad \text{and that } \quad e_{n+1}e_n = e_n \quad \forall \ n.
\]

It is also convenient to define \( e_0 = 0 \). Notice that \( e_n \in \text{Ped}(\mathcal{A}) \) and \( \|e_n\| = 1 \) for all \( n \geq 1 \).

Notice that

\[
(e_{m+1} - e_{n-1})(e_m - e_n) = e_m - e_n \quad \forall \ m > n,
\]

and hence,

\[
(2.17) \quad e_m - e_n \leq Re_m - e_n \leq e_{m+1} - e_{n-1} \quad \forall \ m > n.
\]

**Remark 2.13.** We can always pass from an approximate identity satisfying the above conditions to a subsequence \( f_n \) satisfying the following two stronger conditions assumed in [19]:

(a) Let \( g_n := f_n - f_{n-1} \) (with \( f_0 := 0 \)), then \( \|g_n\| = 1 \) for all \( n \) and \( g_ng_m = 0 \) for \( |m - n| \geq 2 \).

(b) There are \( a_n \in \mathcal{B}_+ \) with \( \|a_n\| = 1 \) such that \( a_n \leq g_n, \ a_ng_n = g_na_n = a_n \) and \( a_ng_m = 0 \) for \( n \neq m \).
Proof. Let $f_n := e_{5n}$. Clearly, $(f_n - f_{n-1})(f_m - f_{m-1}) = 0$ for $|m - n| \geq 2$. Set $a_n := e_{5n-1} - e_{5n-4}$. Then $f_n - f_{n-1} \geq a_n$ by the monotonicity of $e_n$ and
\[
(f_n - f_{n-1})a_n = a_n(f_n - f_{n-1}) = a_n
\]
by (2.16). Furthermore, $\|a_n\| = 1$ since by (2.17), $a_n \geq R_{e_{5n-2}e_{5n-3}} \neq 0$; in particular, $\|f_n - f_{n-1}\| = 1$.

3. The minimal ideal and its hereditary cone

**Definition 3.1.** [13] Lemma 2.1] Let $\mathcal{A}$ be a simple, $\sigma$-unital, non-unital C*-algebra with an approximate identity $\{e_n\}$. Then we define the following set of positive elements in $\mathcal{M}(\mathcal{A})$:
\[
K_0(\{e_n\}) := \{X \in \mathcal{M}(\mathcal{A})_+ \mid \forall 0 \neq a \in \mathcal{A}_+ \exists N \in \mathbb{N}
\]
\[
\exists \ m > n \geq N \Rightarrow (e_m - e_n)X(e_m - e_n) \preceq a\}.
\]

**Remark 3.2.**
(i) By Lemma 2.6 (iii),
\[
K_0(\{e_n\}) := \{X \in \mathcal{M}(\mathcal{A})_+ \mid \forall 0 \neq a \in \mathcal{A}_+ \exists N \in \mathbb{N}
\]
\[
\exists \ m > n \geq N \Rightarrow (e_m - e_n)X(e_m - e_n) \preceq a\}.
\]

This equivalent formulation will also be used in the paper.
(ii) If $\mathcal{A}$ has the $(SP)$ property, (i.e., every nonzero hereditary subalgebra of $\mathcal{A}$ contains a nonzero projection), then for every $0 \neq a \in \mathcal{A}_+$ there is a projection $0 \neq p \leq a$. Thus in the defining property of $K_0(\{e_n\})$ we can replace “for all nonzero elements $a \in \mathcal{A}_+$” with “for all nonzero projections $p \in \mathcal{A}$.”

**Lemma 3.3.**
(i) $X \in K_0(\{e_n\})$ if and only if $X^{1/2} \in K_0(\{e_n\})$.
(ii) $K_0(\{e_n\})$ is a hereditary cone of $\mathcal{M}(\mathcal{A})$ if and only if $\mathcal{A}$ is non-elementary.

**Proof.**
(i) Immediate from the definition and Lemma 2.6 (i).
(ii) It is also immediate to verify that $K_0(\{e_n\})$ is always hereditary and that if $X \in K_0(\{e_n\})$ then $tX \in K_0(\{e_n\})$ for every $t \geq 0$. Assume first that $\mathcal{A}$ is non-elementary and that $X, Y \in K_0(\{e_n\})$. Let $0 \neq a \in \mathcal{A}_+$, then by Lemma 2.6 we can find two elements $0 \neq a', a'' \in \mathcal{A}_+$ with $a'a'' = 0$ and $a' + a'' \leq a$. Let $N'$ (resp., $N''$) be such that for all $m > n \geq N'$ (resp., $m > n \geq N''$), we have
\[
(e_m - e_n)X(e_m - e_n) \preceq a' \ (\text{resp., } (e_m - e_n)Y(e_m - e_n) \preceq a'').
\]
Hence, for all $m > n \geq N := \max(N', N'')$ we have by Lemma 2.1 vi
\[
(e_m - e_n)(X + Y)(e_m - e_n) = (e_m - e_n)X(e_m - e_n) + (e_m - e_n)Y(e_m - e_n) \preceq a' + a'' \leq a.
\]
Thus $K_0(\{e_n\})$ is a cone.

Assume now that $\mathcal{A} = \mathcal{K}$, and hence, $\mathcal{M}(\mathcal{A}) = B(\mathcal{H})$, and let $\{e_n\}$ be an increasing sequence of rank $n$ projections. Then it is easy to verify that
\[
K_0(\{e_n\}) = \{x \in B(\mathcal{H})_+ \mid \exists n \ \text{rank}(1 - e_n)x(1 - e_n) \leq 1\}.
\]
Let $\{\eta_n\}$ be an orthonormal basis of $\mathcal{H}$ such that span$\{\eta_1, \ldots, \eta_n\} = R_{e_n}$, and let $\xi := \sum_{j=1}^{n} \frac{1}{\sqrt{2}} \eta_{2j}$ and $\xi' := \sum_{j=1}^{\infty} \frac{1}{\sqrt{2j+1}} \eta_{2j+1}$. Then both $\xi \otimes \xi$ and $\xi' \otimes \xi'$ belong to $K_0(\{e_n\})$ since they have rank one, but $(1 - e_n)(\xi \otimes \xi + \xi' \otimes \xi')(1 - e_n)$ has rank two for every $n$, and hence, $\xi \otimes \xi + \xi' \otimes \xi' \notin K_0(\{e_n\})$.

□
Corollary 3.4. Let $\mathcal{A}$ be a simple, $\sigma$-unital, non-unital, non-elementary C*-algebra with an approximate identity $\{e_n\}$. Then $L(K_o(\{e_n\}))$ is a left ideal and

$$L(K_o(\{e_n\}))_+ = K_o(\{e_n\}).$$

That $L(K_o(\{e_n\}))$ is a left ideal follows immediately from the fact that $K_o(\{e_n\})$ is a hereditary cone. The equality $K_o(\{e_n\}) = L(K_o(\{e_n\}))_+$ was suggested by H. Lin (private communications). $L(K_o(\{e_n\}))$ was denoted by $I_o$ in [19]. Contrary to what was stated in [19 Lemma 2.1], the following example shows that $K_o(\{e_n\})$ is in general not invariant, i.e., the ideal $L(K_o(\{e_n\}))$ is not two-sided.

Example 3.5. Let $\mathcal{A}_o$ be a simple, unital, finite non-elementary C*-algebra and let $\mathcal{A} := \mathcal{A}_o \otimes K$. Let $\{e_{ij}\}$ be the standard matrix units in $K$, then $e_n := 1 \otimes \sum_{k=1}^{\infty} e_{kk}$ is an increasing approximate identity of projections of $\mathcal{A}$. Let

$$V := 1 \otimes \sum_{k=1}^{\infty} 2^{-k/2} e_{1,k}.\]$$

Then $VV^* = e_1 = 1 \otimes e_{11} \in K_o(\{e_n\})$, i.e., $V^* \in L(K_o(\{e_n\}))$. Let

$$P := V^*V = 1 \otimes \sum_{h,k=1}^{\infty} 2^{-(h+k)/2} e_{h,k}.\]$$

For every $n > 1$ and $0 \neq a \in (\mathcal{A}_o)_+$ with $a \neq 1$ we have

$$(e_n - e_{n-1})P(e_n - e_{n-1}) = 1 \otimes 2^{-n} e_{n,n} \sim 1 \otimes e_{11} \neq a \otimes e_{11}.\]$$

Thus $P \notin K_o(\{e_n\})$, i.e., $V \notin L(K_o(\{e_n\}))$. This example shows that the cone $K_o(\{e_n\})$ is not invariant, and, equivalently, that $L(K_o(\{e_n\}))$ is not a two-sided ideal. It also shows that $K_o(\{e_n\})$ does not satisfy the conclusion of Lemma 2.1 (iii) since $P \in K_o(\{e_n\})$ and yet $\frac{1}{2}P = (P - \frac{1}{2})_+ \notin K_o(\{e_n\})$. Furthermore, if we choose an approximate identity $f_n = 1 \otimes \sum_{k=1}^{n} f_{kk}$ with $f_{k,1} = \sum_{h,k=1}^{\infty} 2^{-(h+k)/2} e_{h,k}$, we see that $P \in K_o(\{f_n\})$, which shows that $K_o(\{e_n\}) \neq K_o(\{f_n\})$.

In Corollary 4.3 we will see that $K_o(\{e_n\})$ is always weakly invariant, and hence, $K_o(\{e_n\})$ is strongly invariant and that $K_o(\{e_n\})$ does not depend on the approximate identity $\{e_n\}$. Meanwhile, the next lemma shows that refinements of an approximate identity do not change the cone $K_o$.

Lemma 3.6. Let $\mathcal{A}$ be a simple $\sigma$-unital non-unital non-elementary C*-algebra with an approximate identity $\{e_n\}$. Then $K_o(\{e_n\}) = K_o(\{e_{n_k}\})$ for any strictly increasing sequence $n_k$ of integers.

Proof. Let $X \in K_o(\{e_{n_k}\})$ and $0 \neq a \in A_+$. Then there is an $L \in \mathbb{N}$ such that if $k > L$ then $(e_{n_k} - e_{n_L})X(e_{n_k} - e_{n_L}) \preceq a$. Let $m > n_L$ and choose $k$ such that $n_k \geq m$. Then $e_m - e_{n_L} \preceq e_{n_k} - e_{n_L}$, and hence, by Lemma 2.0 (iii)

$$(e_m - e_{n_L})X(e_m - e_{n_L}) \preceq (e_{n_k} - e_{n_L})X(e_{n_k} - e_{n_L}) \preceq a.$$

Thus $X \in K_o(\{e_n\})$. The opposite inclusion is obvious.

Given any approximate identity $\{e_n\}$ of $\mathcal{A}$, it is clear that $e_n a e_n \in K_o(\{e_n\})$ for every $a \in A_+$ and $n \in \mathbb{N}$. Since $e_n a e_n \to a$, it follows that

$$A_+ \subset K_o(\{e_n\}).$$
The inclusion $\mathcal{A}_+ \subset K_o(\{e_n\})$ is however equivalent to the condition that $\mathcal{A}$ has continuous scale. Recall that $\mathcal{A}$ is said to have continuous scale if for some (and hence, for every) approximate identity $\{e_n\}$ and for every $0 \neq a \in \mathcal{A}_+$ there is an $N \in \mathbb{N}$ such that $e_m - e_n \leq a$ for all $m > n \geq N$.

**Lemma 3.7.** Let $\mathcal{A}$ be a simple, $\sigma$-unital, non-unital, and non-elementary $C^*$-algebra with an approximate identity $\{e_n\}$. The following are equivalent.

(i) $\mathcal{A}$ has continuous scale;
(ii) $K_o(\{e_n\}) = \mathcal{M}(\mathcal{A})_+$;
(iii) $K_o(\{e_n\}) = \mathcal{M}(\mathcal{A})_+$;
(iv) $\mathcal{A}_+ \subset K_o(\{e_n\})$.

**Proof.** (i) $\Rightarrow$ (ii). For every $x \in \mathcal{M}(\mathcal{A})_+$ and every $m > n$ we have

$$(e_m - e_n)x(e_m - e_n) \leq \|x\|(e_m - e_n) \leq e_m - e_n.$$ 

(ii) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iv). Obvious

(iii) $\Rightarrow$ (ii). Since $1 \in K_o(\{e_n\})$, there is an $x \in K_o(\{e_n\})$ such that $\|x - 1\| < 1$. Thus $x$ is invertible, and hence, $\rho \leq x$ for some scalar $\rho > 0$. Since $K_o(\{e_n\})$ is a hereditary cone, it follows that $1 \in K_o(\{e_n\})$, hence $\mathcal{M}(\mathcal{A})_+ \subset K_o(\{e_n\})$ and thus (ii) holds.

(iv) $\Rightarrow$ (i). Let $b := \sum_{k=1}^{\infty} \frac{1}{k} (e_{k+1} - e_k)$ where the convergence is in norm, and hence, $b \in \mathcal{A}_+ \subset K_o$. Then for every $0 \neq a \in \mathcal{A}_+$ there is an $N \in \mathbb{N}$ such that if $m \geq n \geq N + 1$ then $(e_m + 1 - e_n)b(e_m + 1 - e_n) \leq a$. But then by (2.16) we also have

$$(e_m - e_n) \sim \frac{1}{m - 1} (e_m - e_n)$$

$$= \frac{1}{m - 1} (e_{m+1} - e_{n-1})(e_m - e_n)(e_{m+1} - e_{n-1})$$

$$= (e_{m+1} - e_{n-1}) \sum_{k=n}^{m-1} \frac{1}{m - 1} (e_{k+1} - e_k)(e_{m+1} - e_{n-1})$$

$$\leq (e_{m+1} - e_{n-1}) \sum_{k=n}^{m-1} \frac{1}{k} (e_{k+1} - e_k)(e_{m+1} - e_{n-1})$$

$$\leq (e_{m+1} - e_{n-1})b(e_{m+1} - e_{n-1})$$

$$\leq a.$$ 

Thus the scale is continuous.

The implication (iii) $\Rightarrow$ (ii) is essentially the “only if” part of [13] Theorem 2.10. The following notions have appeared in various forms and various names in the literature (e.g., [22], [5, 4.3.11]) and for ease of reference we present them by the following formal definition.

**Definition 3.8.** Let $\mathcal{B}$ be a $C^*$-algebra.

(i) A sequence $0 \neq s_i \in \mathcal{B}_+$ is called order dense for $\mathcal{B}$ if for every $0 \neq a \in \mathcal{B}_+$ there is an integer $n$ for which $s_n \leq a$.

(ii) A sequence of mutually orthogonal elements $0 \neq t_i \in \mathcal{B}_+$ is called thin for $\mathcal{B}$ if for every $0 \neq a \in \mathcal{B}_+$ there is an integer $N$ such that $\sum_{m=n}^{\infty} t_i \geq a$ for all $m \geq n \geq N$.

Recall that a thin sequence of projections is called an $\ell^1$ sequence in [22]. Clearly, thin sequences are order dense; also if $\{s_i\}$ is an order dense sequence for $\mathcal{B}$ and
0 \neq s_i \in \mathcal{B}_+, with s_i \preceq s'_i for every i, then \{s_i\} is also order dense for \mathcal{B}. Similarly, let 0 \neq s_i, s'_i \in \mathcal{B}_+.

\text{(3.2)} \quad \text{If } \{s'_i\} \text{ is thin, } s_is_j = 0 \text{ for } i \neq j \text{ and } s_i \preceq s'_i \forall i, \text{ then } \{s_i\} \text{ is thin.}

This follows from Lemma 2.4 (vi) since \sum_{i=n}^m s_i \preceq \sum_{i=n}^m s'_i \forall m \geq n. It is also immediate to see that

\text{(3.3)} \quad \text{If } \{s'_i\} \text{ is thin, } s_i = \alpha_is'_i \text{ for some } \alpha_i > 0, \text{ then } \{s_i\} \text{ is thin.}

In separable C*-algebras, it is easy to construct order dense sequences (see also the construction in [19, Lemma 2.4] and [36] for projections).

**Proposition 3.9.** Every separable C*-algebra has an order dense sequence.

**Proof.** Let \mathcal{B} be a separable C*-algebra and let \{b_m\} be a sequence of positive elements dense in the unit ball of \mathcal{B}_+. Let \{s_n\} be an enumeration of the nonzero elements in the collection \\{(b_m - \frac{1}{2})_+ \mid m \in \mathbb{N}\}. For every 0 \neq a \in \mathcal{B}_+ there is an m \in \mathbb{N} such that \|\frac{a}{n\|a\|} - b_m\| < \frac{1}{2}. Then \|b_m\| > \frac{1}{2}, hence \(b_m - \frac{1}{2})_+ \neq 0, \text{ and thus } (b_m - \frac{1}{2})_+ = s_n \text{ for some } n. \text{ Then } s_n \preceq \frac{a}{n\|a\|} \sim a \text{ by Lemma 2.1 (iv). Thus } \{s_n\} \text{ is an order dense sequence.} \Box

Another case when order dense sequences are immediate to obtain is the following. For every C*-algebra \mathcal{A}, denote by \text{D}(\mathcal{A}) the (possibly empty) dimension semigroup of Murray-von Neumann equivalence classes of projections. We say that \text{D}(\mathcal{A}) is order separable if there is a sequence p_n of nonzero projections of \mathcal{A} such that for every projection 0 \neq p \in \mathcal{A} there is a p_n \preceq p. Of course, if \text{D}(\mathcal{A}) is countable, it is also order separable, but type II_1 von Neumann factors are examples of (non-separable) C*-algebras with a dimension semigroup \text{D}(\mathcal{A}) that is order separable but not countable.

**Proposition 3.10.** Every C*-algebra \mathcal{B} with (SP) property and with order separable dimension semigroup \text{D}(\mathcal{B}) has an order dense sequence of projections.

**Proof.** By the (SP) property, for every 0 \neq a \in \mathcal{B}_+ there is a nonzero projection q \in \text{her}(a), and hence, q \preceq a. Since p_n \preceq q \text{ for some } n, \text{ we have } p_n \preceq a. \text{ Thus } \{p_n\} \text{ is order dense for } \mathcal{B}. \Box

The following construction permits to construct thin sequences starting with order dense sequences. For future use in this paper, we will prove a slightly stronger version than needed in this section. When s, t \in \mathcal{B}_+ and n \in \mathbb{N}, we will denote by ns \text{ an } n\text{-fold direct sum of } s \text{ with itself. Then } ns \in M_n(\mathcal{B}_+) \text{ and the subequivalence relation } ns \preceq t \text{ is understood to hold in } M_n(\mathcal{B}_+). \text{ In particular, if } s \preceq t_i \text{ for } 1 \leq i \leq n \text{ and } t_i \text{ are mutually orthogonal, then by Lemma 2.4 (vi), } ns \preceq \sum_{i=1}^n t_i.

**Lemma 3.11.** Let \mathcal{B} be a simple non-elementary C*-algebra. Then for every sequence s_i \text{ of elements } 0 \neq s_i \in \mathcal{B}_+, \text{ there is a sequence of mutually orthogonal elements } 0 \neq t_i \in \mathcal{B}_+ \text{ such that } n \sum_{i=n}^m t_i \preceq s_n \text{ for every pair of integers } m \geq n.

**Proof.** Let 0 \neq a_i \in \mathcal{B}_+ \text{ be a sequence of mutually orthogonal elements (e.g., see Lemma 2.4). By Lemma 2.4 \text{ there are elements } 0 \neq s'_i \in \mathcal{B}_+ \text{ with } s'_i \preceq a_i \text{ and } s'_i \preceq s_i. \text{ For every } i, \text{ use Lemma 2.5 to find an infinite sequence of mutually orthogonal nonzero elements } 0 \neq s'_{i,j} \in \mathcal{B} \text{ such that } \sum_{j=1}^n s'_{i,j} \preceq s'_i \text{ for every } n. \text{ For every } j, \text{ set } s_{1,j} = s'_{1,j}. \text{ Applying Lemma 2.4 \text{ find an element } 0 \neq s_{2,j} \preceq s'_{2,j}, \text{ such}
that $s_{2,j} \leq s_{1,j}$. By iterating the construction, find sequences $0 \neq s_{i,j} \leq s'_{i,j}$ such that

$$s_{i,j} \preceq s_{i-1,j} \preceq \cdots \preceq s_{1,j} \quad \forall i, j.$$ 

Now apply again Lemma 2.4 to find mutually orthogonal elements $0 \neq t_{i,j} \in A_+$ such that $\sum_{j=1}^n t_{i,j} \leq s_{i,i}$. By Lemma 2.4 we can assume again that for every $i$.

$$t_{i,i} \preceq t_{i,i-1} \preceq \cdots \preceq t_{i,1}.$$ 

Let $t_i := t_{i,i}$. Notice that the sequences $t_i \leq s_{i,i} \leq s'_{i,i} \leq a_i$ are mutually orthogonal. Thus for every $n \leq i \in \mathbb{N}$

$$n t_i \preceq \sum_{j=1}^i t_{i,j} \leq s_{i,i},$$

and hence,

$$n \sum_{i=n}^m t_i \preceq \sum_{i=n}^m s_{i,i} \preceq \sum_{i=n}^m s'_{i,i} \preceq \sum_{i=n}^m s'_n \preceq s_n.$$ 

The following consequence is now immediate.

**Corollary 3.12.** Let $B$ be a simple non-elementary $C^*$-algebra. If $B$ has an order dense sequence $s_i$, then it has a thin sequence $t_i$ with $t_i \preceq s_i$ for every $i$.

If $A$ is a simple, $\sigma$-unital, non-unital $C^*$-algebra and assume that $D := \sum_{j=1}^{\infty} d_j$ is diagonal with respect to an approximate identity $\{e_n\}$. Then $D \in K_0(\{e_n\})$ if and only if the sequence $\{d_j\}$ is thin.

**Lemma 3.13.** Let $A$ be a simple, $\sigma$-unital, non-unital $C^*$-algebra and assume that $D := \sum_{j=1}^{\infty} d_j$ is diagonal with respect to an approximate identity $\{e_n\}$. Then $D \in K_0(\{e_n\})$ if and only if the sequence $\{d_j\}$ is thin.

**Proof.** Let $n_j$ and $m_j$ be sequences of positive integers such that $n_j < m_j < n_{j+1}$ for every $j$ and $d_j \in A_+$ is a bounded sequence for which $d_j \leq M_j(e_{m_j} - e_{n_j})$ for some $M_j > 0$ and all $j$. Assume first that the sequence $\{d_j\}$ is thin. Since for every $p \geq L \in \mathbb{N}$ we have

$$(1 - e_{n_L}) \sum_{j=1}^{L-1} d_j = 0 \quad \text{and} \quad e_{m_p} \sum_{j=p+1}^{\infty} d_j = 0$$

it follows that

$$(e_{m_p} - e_{n_L}) D = (e_{m_p} - e_{n_L}) \sum_{j=L}^{p} d_j.$$ 

Now let $0 \neq a \in A_+$ and $L \in \mathbb{N}$ be such that if $p \geq L$, then $\sum_{j=L}^{p} d_j \leq a$. For every $m > N := n_L$ choose $p$ such that $m_p \geq m$. Then by Lemma 2.4 (iii) and Lemma
(i), we have
\[(e_m - e_N)D(e_m - e_N) \leq (e_{mp} - e_N)D(e_{mp} - e_N)\]
\[= (e_{mp} - e_N) \sum_{j=L}^{p} d_j (e_{mp} - e_N)\]
\[\leq \sum_{j=L}^{p} d_j \leq a.\]

This proves that \(D \in K_o(\{e_n\})\).

Assume now that \(D \in K_o(\{e_n\})\) and let \(0 \neq a \in A_+\). Then there is an integer \(N\) such that \((e_m - e_N)D(e_m - e_N) \leq a\) for every \(m \geq N\). Let \(L\) be such that \(n_L \geq N + 1\) and \(p \geq L\). Since \(\sum_{j=L}^{p} d_j \leq M(e_{mp} - e_{n_L})\) and
\[(e_{mp+1} - e_{n_L} - 1)(e_{mp} - e_{n_L}) = e_{mp} - e_{n_L},\]
it follows that \((e_{mp+1} - e_{n_L} - 1) \sum_{j=L}^{p} d_j = \sum_{j=L}^{p} d_j.\) But then \(mp + 1 > N\), and hence,
\[\sum_{j=L}^{p} d_j = (e_{mp+1} - e_{n_L} - 1) \sum_{j=L}^{p} d_j (e_{mp+1} - e_{n_L} - 1)\]
\[\leq (e_{mp+1} - e_{n_L} - 1)D(e_{mp+1} - e_{n_L} - 1)\]
\[\leq (e_{mp+1} - e_N)D(e_{mp+1} - e_N)\]
\[\leq a.\]

This proves that \(\{d_j\}\) is thin.

\[\square\]

**Theorem 3.14.** Let \(A\) be simple, \(\sigma\)-unital, non-unital, non-elementary \(C^*\)-algebra with an approximate identity \(\{e_n\}\). Then the following are equivalent.
(i) \(A_+ \neq K_o(\{e_n\})\);
(ii) \(A\) has an order dense sequence;
(iii) \(A\) has a thin sequence;
(iv) \(A\) has a thin sequence \(d_j\) such that \(D = \sum_{j=1}^{\infty} d_j\) converges strictly to an element \(D \in K_o(\{e_n\}) \setminus A\).

**Proof.** As usual, set \(K_o = K_o(\{e_n\})\).

(i) \(\Rightarrow\) (ii) \(A_+ \neq K_o\) if and only if there is an element \(X \in K_o \setminus A\). Then for every \(k\), \((1 - e_k)X(1 - e_k) \neq 0\), hence there is some integer \(m_k > k\) such that
\[s_k := (e_{m_k} - e_k)X(e_{m_k} - e_k) \neq 0.\]
By the defining property of \(K_o\), for every \(0 \neq a \in A_+\) there is an integer \(N\) such that \(s_N \leq a\).

(ii) \(\Rightarrow\) (iii) by Corollary 3.12

(iii) \(\Rightarrow\) (iv) Assume that \(t_j\) is a thin sequence for \(A_+\). By Lemma 2.4 for every \(j\) we can find \(0 \neq \tilde{d}_j \in A_+\) such that \(\tilde{d}_j \leq t_j\) and \(\tilde{d}_j \leq e_{2j} - e_{2j-1}\). Let \(d_j := \frac{\tilde{d}_j}{||d_j||}\) and \(D := \sum_{j=1}^{\infty} d_j\). The sequence \(\{d_j\}\) is mutually orthogonal and thin by (3.2) and (3.3), and by construction, \(D\) is diagonal with respect to \(\{e_n\}\). Then by Lemma 3.13 \(D \in K_o \setminus A\).

(iv) \(\Rightarrow\) (i) Obvious.

\[\square\]
Immediate consequences of Theorem 3.14 Proposition 3.9 and Proposition 3.10 and Lemma 3.7 are the following (ii) was obtained in [19, Lemma 2.4]).

Corollary 3.15. Let \( \mathcal{A} \) be a simple, \( \sigma \)-unital, non-unital, non-elementary \( \mathcal{C}^* \)-algebra with an approximate identity \( \{ e_n \} \). Then \( \mathcal{A}_+ \neq \mathcal{K}_o \) in any of the following cases:

(i) \( \mathcal{A} \) is separable;
(ii) the Cuntz semigroup is order separable;
(iii) \( \mathcal{A} \) has the (SP) property and its dimension semigroup \( D(\mathcal{A}) \) of Murray-von Neumann equivalence classes of projections is order separable;
(iv) \( \mathcal{A} \) has a continuous scale.

We will see in Section 4 that another case when \( \mathcal{A}_+ \neq \mathcal{K}_o \) is when \( \mathcal{A} \) has strict comparison of positive elements by traces (see Proposition 5.4 and Theorem 5.3).

4. The minimal ideal

We proceed now to prove that for every approximate identity \( \{ e_n \} \), as usual, satisfying (2.15), \( K_o(\{ e_n \}) \) is weakly invariant and to obtain properties of \( L(\mathcal{K}_o(\{ e_n \})) \). In order to do that, we need first to strengthen a result obtained in [14, Theorem 4.2]. Diagonal series have proven a very valuable tool in working with multiplier algebras, started with [10] and used by [17], [36], [27] among many others. It is well known that a Weyl-von Neumann decomposition of selfadjoint elements into the sum of a diagonal series plus a remainder in \( \mathcal{K}_o \) is separable; \( \mathcal{K}_o(\{ e_n \}) \) isfying (2.15), \( K_o(\{ e_n \}) \) is separable; (ii) the Cuntz semigroup is order separable;

\[ D \in M(\mathcal{A}_+) \]

A refinement of that construction, but with fewer hypotheses on \( \mathcal{A} \), was obtained in [14] where we proved that if \( \mathcal{K}_o(\{ e_n \}) \) is \( \sigma \)-unital, then every positive element \( T \in M(\mathcal{A}_+) \) can be decomposed into the sum of a selfadjoint element in \( \mathcal{A} \) of arbitrarily small norm and a \( \text{bidiagonal} \) series. A bidiagonal series \( D := \sum_{k=1}^{\infty} d_k \) is a strictly converging series with summands \( d_k \in \mathcal{A}_+ \) such that \( d_k d_{k'} = 0 \) for \( |k-k'| > 1 \). In particular, \( D = D + e + D_o \), where \( D_e := \sum_{k=1}^{\infty} d_{2k} \) and \( D_o := \sum_{k=1}^{\infty} d_{2k-1} \) are diagonal series.

If \( T \in \mathcal{K}_o(\{ e_n \}) \), the original proof in [14] can be modified to show that the bidiagonal series can be chosen in \( \mathcal{K}_o(\{ e_n \}) \). Also in order to obtain some further enhancements that will be needed later in this paper, and for the readers’ convenience, we will present here a self-contained proof.

Theorem 4.1. Let \( \mathcal{A} \) be a simple, \( \sigma \)-unital, non-unital, non-elementary \( \mathcal{C}^* \)-algebra with approximate identities \( \{ e_n \} \) and \( \{ f_m \} \), and let \( X^*X \in \mathcal{K}_o(\{ e_n \}) \) for some \( X \in M(\mathcal{A}) \). Then for every \( \epsilon > 0 \), there exist an element \( t = t^* \in \mathcal{A} \) with \( \| t \| < \epsilon \), and a bidiagonal series \( D := \sum_{k=1}^{\infty} d_k \) such that \( XX^* = D + t \) and \( D \in \mathcal{K}_o(\{ f_m \}) \).

Proof. Without loss of generality, assume that \( \| X \| = 1 \) and assume also that \( X^*X \notin \mathcal{A} \) as the conclusion is trivial when \( X^*X \in \mathcal{A} \) (e.g., see [3.14]). By Theorem 3.14 there exists a thin sequence \( t_k \). By the definition of \( \mathcal{K}_o(\{ e_n \}) \) there is an increasing sequence \( N_k \), such that

\[ (e_m - e_{N_k})X^*X(e_m - e_{N_k}) \leq t_{k+1} \quad \forall m > N_k. \]

Since \( \mathcal{K}_o(\{ e_n \}) = \mathcal{K}_o(\{ e_{N_k} \}) \) by Lemma 3.6 to simplify notations assume that

\[ (e_m - e_n)X^*X(e_m - e_n) \leq t_{n+1} \quad \forall m > n. \]
By the triangle inequality,

\[ \|e_n X^*(1 - f_m)\| < \frac{\epsilon}{4^3}. \]

Then choose \( n_2 > n_1 = 1 \) such that

\[ \|(1 - e_{n_2}) X^* f_m\| < \frac{\epsilon}{4^5}. \]

By iterating, construct strictly increasing sequences of integers \( m_k \) and \( n_k \) such that

\[ \|e_{n_k} X^*(1 - f_{m_k})\| < \frac{\epsilon}{4^{k+2}} \text{ for } k \geq 1 \]

\[ \|(1 - e_{n_k}) X^* f_{m_k - 2}\| < \frac{\epsilon}{4^{k+2}} \text{ for } k \geq 3. \]

When \( A, B, C \) are bounded operators, \( \|C\| \leq 1 \), and \( 0 \leq A \leq B \), then

\[ \|A^{1/2}C\|^2 = \|C^* AC\| \leq \|C^* BC\| \leq \|BC\|. \]

Using the fact that \( \|X\| = 1 \) and \( \|f_{m_k}\| = 1 \) for all \( k \), we can apply this inequality to \( A := (e_{n_k} - e_{n_k-1}) \) and

\[ B := e_{n_k} \quad \text{and} \quad C := X^*(1 - f_{m_k}) \]

and also to

\[ B := 1 - e_{n_k-1} \quad \text{and} \quad C := X^* f_{m_k - 2}. \]

Thus we obtain

\[ \|(e_{n_k} - e_{n_k-1})^{1/2} X^*(1 - f_{m_k})\| \leq \frac{\epsilon}{2^{k+2}} \text{ for } k \geq 1 \]

\[ \|(e_{n_k} - e_{n_k-1})^{1/2} X^* f_{m_k - 2}\| \leq \frac{\epsilon}{2^{k+2}} \text{ for } k \geq 3. \]

By the triangle inequality,

\[ \|(e_{n_k} - e_{n_k-1})^{1/2} X^* - (e_{n_k} - e_{n_k-1})^{1/2} X^*(f_{m_k} - f_{m_k - 2})\| \]

\[ = \|(e_{n_k} - e_{n_k-1})^{1/2} X^*(1 - f_{m_k}) + (e_{n_k} - e_{n_k-1})^{1/2} X^* f_{m_k - 2}\| \]

\[ < \frac{\epsilon}{2^{k+1}}. \]

From the inequality \( \|A^* A - B^* B\| \leq (\|A\| + \|B\|)(\|A - B\|) \) and again using the fact that \( \|X\| = 1 \) and \( \|e_{n_k}\| = \|f_{m_k}\| = 1 \), we thus have

(4.2) \[ \|X (e_{n_k} - e_{n_k-1}) X^* - (f_{m_k} - f_{m_k - 2}) X (e_{n_k} - e_{n_k-1}) X^* (f_{m_k} - f_{m_k - 2})\| \leq \frac{\epsilon}{2^k}. \]

Set

\[ c_k := (f_{m_k} - f_{m_k - 2}) X (e_{n_k} - e_{n_k-1}) X^* (f_{m_k} - f_{m_k - 2}) \]

\[ D := \sum_{k=1}^{\infty} c_k. \]

Since \( f_m \) is an approximate identity for \( A \) and the sequence

\[ c_k \leq \|X\|^2 (f_{m_k} - f_{m_k - 2})^2 \leq f_{m_k} - f_{m_k - 2} \]
is uniformly bounded, it is clear that the series converges strictly. Furthermore,

\[ XX^* = \sum_{k=1}^{\infty} X(e_{n_k} - e_{n_{k-1}})X^* \]

where the series also converges strictly. Set

\[ t := XX^* - D = \sum_{k=1}^{\infty} \left( X(e_{n_k} - e_{n_{k-1}})X^* - c_k \right). \]

It follows from (4.2) that this series converges in norm, hence \( t = t^* \in \mathcal{A} \). Moreover

\[ \|t\| \leq \sum_{k=1}^{\infty} \|X(e_{n_k} - e_{n_{k-1}})X^* - c_k\| < \epsilon. \]

Thus we have the decomposition \( XX^* = D + t \). We need to verify that \( D \) is a bidiagonal series and that \( D \in K_o(\{f_m\}) \). We will use now (4.1), which is a consequence of \( X^*X \in K_o(\{e_n\}) \) \( \setminus \mathcal{A} \). For every \( k > 1 \)

\[
\begin{align*}
  c_k &\leq X(e_{n_k} - e_{n_{k-1}})X^* \quad \text{(by Lemma 2.1 (i))} \\
  &\sim (e_{n_k} - e_{n_{k-1}})^{1/2}X^*X(e_{n_k} - e_{n_{k-1}})^{1/2} \quad \text{(by Lemma 2.1 (ii))} \\
  &\sim (e_{n_k} - e_{n_{k-1}})X^*X(e_{n_k} - e_{n_{k-1}}) \quad \text{(by Lemma 2.1 (ii))} \\
  &\leq t_k \quad \text{(by Lemma 2.1).}
\end{align*}
\]

Set \( d_k := c_{2k} + c_{2k-1} \). By Lemma 2.1 (vi),

(4.3)

\[ d_k \leq t_{2k} + t_{2k-1}. \]

Furthermore,

(4.4)

\[ d_k \leq 2\|X\|^2(f_{m_{2k}} - f_{m_{2k-3}}) \]

whence we see that \( D \) is bidiagonal. In particular, the even and odd sequences

\[
\begin{align*}
  d_{2k} &\leq 2\|X\|^2(f_{m_{4k}} - f_{m_{4k-3}}) \\
  d_{2k+1} &\leq 2\|X\|^2(f_{m_{4k+2}} - f_{m_{4k-1}})
\end{align*}
\]

are both mutually orthogonal, satisfy the intertwining condition of Lemma 3.13 and are thin by (4.3) and (4.2) since

\[
\begin{align*}
  d_{2k} &\leq t_{4k} + t_{4k-1} \\
  d_{2k+1} &\leq t_{4k+2} + t_{4k+1}
\end{align*}
\]

and both sequences \( \{t_{4k} + t_{4k-1}\} \) and \( \{t_{4k+2} + t_{4k+1}\} \) are thin. But then their sums

\[ D_e := \sum_{k=1}^{\infty} d_{2k} \quad \text{and} \quad D_o := \sum_{k=1}^{\infty} d_{2k-1} \]

are both in \( K_o(\{f_m\}) \), and hence, \( D = D_e + D_o \in K_o(\{f_m\}) \), which concludes the proof. \( \square \)

**Remark 4.2.** If in Theorem 4.1 we start with an element \( B \in \mathcal{M}(\mathcal{A})_+ \) and drop the hypothesis that \( B \in K_o(\{e_n\}) \), the same proof yields the decomposition \( B = D + t \) where \( D \) is a bidiagonal series. Furthermore, if \( \{f_m\} \) is an approximate identity, then we can choose \( D \) to be the sum \( D = D_e + D_o \) of two diagonal series with respect to \( \{f_m\} \). In fact to obtain this result we only need to require that \( \mathcal{A} \) is \( \sigma \)-unital (see [14, Theorem 4.2]).
Corollary 4.3. Let $\mathcal{A}$ be simple, $\sigma$-unital, non-unital, non-elementary and let $e_n, f_m$ be two approximate identities for $\mathcal{A}$. Then

(i) $K_o(\{e_n\}) = K_o(\{f_m\})$

(ii) $K_o(\{e_n\})$ is weakly invariant, hence $\overline{K_o(\{e_n\})}$ is hereditary and strongly invariant.

(iii) $L(K_o(\{e_n\}))$ is a two-sided ideal and
\[
L(K_o(\{e_n\})) = L(K_o(\{e_n\})) = \text{span}(K_o(\{e_n\})).
\]

Proof:

(i) If $T \in K_o(\{e_n\})$, then by applying Theorem 4.4 to $X := T^\perp$ we see that $T \in K_o(\{f_m\})$, that is, $K_o(\{e_n\}) \subset \overline{K_o(\{f_m\})}$. Thus $K_o(\{e_n\}) \subset \overline{K_o(\{f_m\})}$. By reversing the role of the approximate identities we obtain equality.

(ii) If $X \in K_o(\{e_n\})$ and $A \in \mathcal{M}(\mathcal{A})$ then
\[
(X^{1/2}A^*)^*(X^{1/2}A^*) = X^{1/2}A^*AX^{1/2} \leq \|A\|^2X \in K_o(\{e_n\}),
\]
and by Theorem 4.4
\[
AXA^*(X^{1/2}A^*)^*(X^{1/2}A^*) \in \overline{K_o(\{e_n\})}.
\]
Thus $K_o(\{e_n\})$ is weakly invariant, and hence, by Lemma 2.11 (ii) and (iv), we obtain that $\overline{K_o(\{e_n\})}$ is hereditary and strongly invariant.

(iii) Follows immediately from Corollary 2.12 and Theorem 2.10

The independence of $L(K_o(\{e_n\}))$ on the approximate identity was obtained in [19] Remark 2.9. From now on, we will denote
\[(4.5) \quad I_{\text{min}} := L(K_o(\{e_n\})).\]

The following result sheds some additional light on the relation between $I_{\text{min}}$ and $L(K_o(\{e_n\}))$.

Proposition 4.4. Let $\mathcal{A}$ be simple, $\sigma$-unital, non-unital, non-elementary and let $\{e_n\}$ be an approximate identity for $\mathcal{A}$. Then $I_{\text{min}} = A + L(K_o(\{e_n\}))$.

Proof. The inclusion $\mathcal{A} + L(K_o(\{e_n\})) \subset I_{\text{min}}$ is obvious, and to prove equality it is enough to verify that if $D \in (I_{\text{min}})_+ = K_o(\{e_n\})$, then $D \in \mathcal{A} + L(K_o(\{e_n\}))$.

Without loss of generality, $\|D\| \leq 1$ and by Remark 4.2 we can assume that $D$ is diagonal with respect to $\{e_n\}$. By further decomposing if necessary $D = \sum_{j=1}^{\infty} d_j$ into a sum of at most three diagonal series, we can assume that there is a sequence $m_k$ such that $(e_{m_k} - e_{m_{k-1}})d_k = d_k$ for all $k$. To simplify notations, assume that
\[(4.6) \quad (e_k - e_{k-1})d_k = d_k \quad \forall k\] (setting $e_0 = 0$). By Theorem 3.11 $\mathcal{A}$ has a thin sequence $\{t_j\}$. For every $k$ find $b_k \in K_o(\{e_n\})$ such that $\|D - b_k\| \leq \frac{1}{k}$ and an integer $n_k$ such that
\[
(e_m - e_{n_k})b_k(e_m - e_{n_k}) \leq t_k \quad \forall m > n_k.
\]
Since
\[
\|e_m - e_{n_k}\|D(e_m - e_{n_k}) - (e_m - e_{n_k})b_k(e_m - e_{n_k})\| \leq \|D - b_k\| < \frac{1}{k}
\]
it follows from Lemma 2.11 (iv) that for all $m > n_k$,
\[
((e_m - e_{n_k})D(e_m - e_{n_k}) - \frac{1}{k})_+ \leq (e_m - e_{n_k})b_k(e_m - e_{n_k}) \leq t_k.
\]
By (4.6),
\[
((e_m - e_n)D(e_m - e_n) - \frac{1}{k})_+ = \left( \sum_{j=n_k}^m d_j - \frac{1}{k} \right)_+ = \sum_{j=n_k}^m (d_j - \frac{1}{k})_+.
\]

Set \( \delta_j := \frac{1}{k} \) for \( n_k \leq j < n_{k+1} \). Thus for all \( k \in \mathbb{N} \), \( \sum_{j=n_k}^{n_{k+1}-1} (d_j - \delta_j)_+ \leq t_k \). Then for every \( 0 \neq a \in \mathcal{A}_+ \) there is \( K \in \mathbb{N} \) such that \( \sum_{j=K}^{m} t_j \leq a \) for all \( k > K \). For all \( m > n_K \), choose \( n_H \geq m \). Then
\[
(e_m - e_n) \left( \sum_{j=n}^{\infty} (d_j - \delta_j)_+ \right) (e_m - e_n) = \sum_{j=n_K}^{m} (d_j - \delta_k)_+ \\
\leq \sum_{j=n_K}^{n_H} (d_j - \delta_k)_+ \leq \sum_{k=K}^{H-1} \sum_{j=n_k}^{n_{k+1}-1} (d_j - \delta_j)_+ \leq \sum_{k=K}^{H-1} t_k \leq a
\]
which proves that
\[
\sum_{j=1}^{\infty} (d_j - \delta_j)_+ \in K_{\mathcal{O}}(\{e_n\}) \subset L(K_{\mathcal{O}}(\{e_n\})).
\]

Finally,
\[
D - \sum_{j=1}^{\infty} (d_j - \delta_j)_+ = \sum_{j=1}^{\infty} (d_j - (d_j - \delta_j)_+) \in \mathcal{A}_+
\]
since \( 0 \leq d_j - (d_j - \delta_j)_+ \leq \delta_j(e_{j+1} - e_j) \).

We proceed now to justify the notation \( I_{\text{min}} \). The natural “minimal ideal” is the intersection \( \mathcal{J}_o \) of all ideals (not necessarily proper) properly containing \( \mathcal{A} \), in symbols

\[
(4.7) \quad \mathcal{J}_o := \bigcap \{ \mathcal{J} \subset \mathcal{M}(\mathcal{A}) : \mathcal{A} \subset \mathcal{J} \}.
\]

Obviously \( \mathcal{A} \subset \mathcal{J}_o \), but we do not know whether \( \mathcal{A} \neq \mathcal{J}_o \) holds in general. However, we will prove now that \( I_{\text{min}} = \mathcal{J}_o \) (see Theorem 4.7). A key tool in that proof, and used also throughout this paper, is the following result obtained in [14].

**Proposition 4.5.** [14 Proposition 3.2] Let \( \mathcal{B} \) be a non-unital C*-algebra and let \( A = \sum_{n=1}^{\infty} A_n \), \( B = \sum_{n=1}^{\infty} B_n \) where \( A_n, B_n \in \mathcal{M}(\mathcal{B})_+ \), \( A_nA_m = 0 \), \( B_nB_m = 0 \) for \( n \neq m \) and the two series converge in the strict topology, and \( A_n \preceq (B_n - \delta)_+ \) for some \( \delta > 0 \) and for all \( n \). Then for every \( \epsilon > 0 \) and \( 0 < \delta' < \delta \) there is an \( X \in \mathcal{M}(\mathcal{B}) \) such that \( (A - \epsilon)_+ = X(B - \delta')_+X^* \), and hence, \( A \preceq (B - \delta')_+ \leq B \).

If the sum of a positive diagonal series in \( \mathcal{M}(\mathcal{B}) \) is subequivalent to another strictly converging series in \( \mathcal{M}(\mathcal{B}) \) (not necessarily diagonal) then we can deduce the following relations between the summands.

**Proposition 4.6.** Let \( \mathcal{B} \) be a non-unital C*-algebra, \( A = \sum_{k=1}^{\infty} a_k \), \( B = \sum_{k=1}^{\infty} b_k \), be two strictly converging series with \( a_k, b_k \in \mathcal{B}_+ \) and elements \( a_k \) mutually orthogonal. If \( A \preceq B \), then for every \( \delta > 0 \) and \( M \in \mathbb{N} \) there is an \( N \in \mathbb{N} \) such that for every \( n \geq N \) there is an \( m \geq M \) such that
\[
\sum_{k=N}^{n} (a_k - \delta)_+ \leq \sum_{k=M}^{m} b_k.
\]
Proof. By Lemma 2.4 (v), there is an \( X \in \mathcal{M}(\mathcal{B}) \) such that \( (A - \frac{\delta}{6})_+ = XB X^* \), and hence, by Lemma 2.3 there is a \( Y \in \mathcal{M}(\mathcal{B}) \) such that

\[
\|(A - \frac{\delta}{6})_+ - ((A - \frac{\delta}{6})_+)^{1/2}YXB X^* Y^* ((A - \frac{\delta}{6})_+)^{1/2}\| < \frac{\delta}{6}
\]

and \( \|YXB X^* Y^*\| \leq 1 \). Because of the mutual orthogonality of \( a_k \), and hence, of \( (a_k - \frac{\delta}{6})_+ \), we have for every \( n \)

\[
(A - \frac{\delta}{6})_+ = \sum_{k=1}^{\infty} (a_k - \frac{\delta}{6})_+ \geq \sum_{k=1}^{n} (a_k - \frac{\delta}{6})_+.
\]

If \( a, b, c \) are positive elements in a C*-algebra \( \mathcal{C} \) with \( a \leq b \) and \( \|c\| \leq 1 \), then

\[
\|a - a^{1/2}ca^{1/2}\| = \|a^{1/2}(1 - c)a^{1/2}\| = \|(1 - c)^{1/2}a(1 - c)^{1/2}\|
\]

\[
\leq \|(1 - c)^{1/2}b(1 - c)^{1/2}\| = \|b - b^{1/2}cb^{1/2}\|.
\]

Thus from (4.8), (4.9), and (4.10) we have for all \( n \)

\[
\|\sum_{k=n}^{\infty} (a_k - \frac{\delta}{6})_+ - (\sum_{k=n}^{\infty} (a_k - \frac{\delta}{6})_+)^{1/2}YXB X^* Y^* \left( \sum_{k=n}^{\infty} (a_k - \frac{\delta}{6})_+ \right)^{1/2}\| < \frac{\delta}{6}.
\]

Since \( YX \sum_{k=1}^{M-1} b_k X^* Y^* \in \mathcal{B} \) and the sequence \( \sum_{k=n}^{\infty} (a_k - \frac{\delta}{6})_+ \to 0 \) strictly, we can find an integer \( N \) such that

\[
\\|
\left( \sum_{k=N}^{\infty} (a_k - \frac{\delta}{6})_+ \right)^{1/2}YX \sum_{k=1}^{M-1} b_k X^* Y^* \left( \sum_{k=N}^{\infty} (a_k - \frac{\delta}{6})_+ \right)^{1/2}\| < \frac{\delta}{6}.
\]

As a consequence of (4.11) and (4.12) we thus obtain

\[
\|\sum_{k=N}^{\infty} (a_k - \frac{\delta}{6})_+ - (\sum_{k=N}^{\infty} (a_k - \frac{\delta}{6})_+)^{1/2}YX \sum_{k=M}^{\infty} b_k X^* Y^* \left( \sum_{k=N}^{\infty} (a_k - \frac{\delta}{6})_+ \right)^{1/2}\| < \frac{2\delta}{6},
\]

and hence,

\[
\sum_{k=N}^{\infty} (a_k - \frac{3\delta}{6})_+ \leq (\sum_{k=N}^{\infty} (a_k - \frac{\delta}{6})_+)^{1/2}YX \sum_{k=M}^{\infty} b_k X^* Y^* \left( \sum_{k=N}^{\infty} (a_k - \frac{\delta}{6})_+ \right)^{1/2} \leq \sum_{k=M}^{\infty} b_k.
\]

A fortiori, for every \( n \geq N \), we have \( \sum_{k=N}^{n} (a_k - \frac{3\delta}{6})_+ \leq \sum_{k=M}^{\infty} b_k \). Then again by Lemma 2.4 (v), there is a \( Z \in \mathcal{M}(\mathcal{B}) \) such that

\[
\sum_{k=N}^{n} (a_k - \frac{4\delta}{6})_+ = Z \sum_{k=M}^{\infty} b_k Z^*.
\]

Choose \( e \in \mathcal{B} \) such that \( \|Z \sum_{k=M}^{\infty} b_k Z^* - eZ \sum_{k=M}^{\infty} b_k Z^* e\| < \frac{\delta}{6} \), and then choose \( m \geq M \) such that \( \|eZ \sum_{k=m+1}^{\infty} b_k Z^* e\| < \frac{\delta}{6} \). Then

\[
\|\sum_{k=N}^{n} (a_k - \frac{4\delta}{6})_+ - eZ \sum_{k=M}^{m} b_k Z^* e\| < \frac{2\delta}{6},
\]

and hence

\[
\sum_{k=N}^{m} (a_k - \delta)_+ \leq eZ \sum_{k=M}^{m} b_k Z^* e \leq \sum_{k=M}^{m} b_k.
\]

\( \square \)
The inclusion $I_{\min} \subset J_o$ in the following theorem has been obtained in [19] Theorem 2.8, but for completeness’s sake we include its proof.

**Theorem 4.7.** Let $A$ be a simple, $\sigma$-unital, non-unital, non-elementary $C^*$-algebra. Then $I_{\min} = J_o$

**Proof.** To prove that $I_{\min} \subset J_o$, it is enough to show that given an approximate identity $\{e_n\}$, an element $D \in K_o(\{e_n\})$ and an element $C \in M(A)_+ \setminus A$, then $D \in I(C)$. By Theorem 4.11 and Remark 4.12 $C = C_e + C_0 + t$ for some $t = t^* \in A$ and two positive diagonal series $C_e$ and $C_0$ (with respect to $\{e_n\}$), at least one of which, say $C_e$, does not belong to $A$. Then, $I(C_e) \subset I(C_e + C_0) = I(C)$, thus it is enough to prove that $D \in I(C_e)$. To simplify notations, assume that $C = \sum_{k=1}^\infty c_k$ itself is diagonal with respect to $\{e_n\}$. By Theorem 4.11 and Remark 4.12 we can also assume that the series $D = \sum_{k=1}^\infty d_k$ is diagonal with respect to $\{e_n\}$. Since $\lim (C - \delta)_+ = C \notin A$, there is some $\delta > 0$ such that $(C - \delta)_+ \notin A$. Since $(C - \delta)_+ = \sum_{k=1}^\infty (c_k - \delta)_+$, we can assume without loss of generality that $(c_k - \delta)_+ \neq 0$ for every $k$.

By Lemma 4.13 the sequence $\{d_j\}$ is thin, hence for every $k$ there is an integer $n_k$ such that

$$\sum_{j=n_k+1}^m d_j \preceq (c_k - \delta)_+ \quad \forall m \geq n_k, \; k \in \mathbb{N}.$$  

Choose the sequence $n_k$ so to be strictly increasing. Then in particular

$$\sum_{j=n_k+1}^{n_k+1} d_j \preceq (c_k - \delta)_+ \quad \forall k \in \mathbb{N},$$

and hence, by Proposition 4.10

$$\sum_{j=n_k+1}^\infty d_j \preceq (C - \frac{\delta}{2})_+ \leq C.$$  

Thus $D \in I(C)$, which shows that $I_{\min} \subset J_o$.

Now to prove that $J_o = I_{\min}$, we need to consider only the case that $A \neq J_o$. We will prove that then $J_o$ contains a thin sequence, which by Theorem 4.14 implies that $A \neq I_{\min}$ and hence that $J_o \subset I_{\min}$. Equality then holds by the first part of the proof.

Choose $D \in (J_o)_+ \setminus A$ and by invoking Theorem 4.11 and Remark 4.12 as in the first part of the proof, assume that $D := \sum_{k=1}^\infty d_k$ is diagonal with respect to $\{e_n\}$. Let $\delta > 0$ be such that $(D - \delta)_+ \notin A$. We claim that the sequence $\{(d_k - \delta)_+\}$ is thin. Since $\sum_{k=1}^\infty (d_k - \delta)_+ = (D - \delta)_+ \notin A$, we can assume without loss of generality that $(d_k - \delta)_+ \neq 0$ for all $k$. Let $0 \neq a \in A_+$. By Lemma 3.11 applied to the stationary sequence $s_i = a$, there is a sequence of mutually orthogonal elements $0 \neq t_i \in A_+$ such that $n \sum_{i=n}^m t_i \preceq a$ for every pair of integers $m \geq n$. By Lemma 2.4 there are elements $0 \neq a'_i \leq e_{2i} - e_{2i-1}$ and $a'_i \preceq t_i$ for every $i$. Let $a_i := \frac{a'_i}{\|a'_i\|}$. Then the series converges strictly to an element $A := \sum_{i=1}^\infty a_i \in M(A) \setminus A$ because $a_i \leq \|a'_i\|(e_{2i} - e_{2i-1})$ and $\|a_i\| = 1$ for every $i$. Furthermore, for every $m \geq M \in \mathbb{N}$ we have

$$\sum_{i=M}^m a_i \sim M \sum_{i=M}^m a'_i \preceq M \sum_{i=M}^m t_i \preceq a.$$  

(4.14)
Thus for every \( \epsilon > 0 \), there is some \( M \) such that 
\[(D - \frac{\epsilon}{2})_+ \leq MA.\]
By Proposition 4.5, there is some \( N \) such that for every \( n \geq N \) there is \( m \geq M \) for which 
\[
\sum_{k=N}^{n} (d_k - \delta)_+ \leq \sum_{i=M}^{m} Ma_i \sim M \sum_{i=M}^{m} a_i \leq a.
\]
This proves that the sequence \((d_k - \delta)_+\) is thin and thus concludes the proof.

In [21], Lin proved that if \( \mathcal{M}(A)/A \) is simple, then it is purely infinite. Thus if \( I_{\min} = \mathcal{M}(A) \), and hence \( \mathcal{M}(A)/A \) is simple, then \( I_{\min}/A \) is purely infinite. We can relax the condition \( I_{\min} = \mathcal{M}(A) \).

**Theorem 4.8.** Let \( A \) be a simple, \( \sigma \)-unital, non-unital, non-elementary \( C^* \)-algebra and assume that \( I_{\min} \neq \mathcal{A} \). Then \( I_{\min}/A \) is purely infinite simple.

**Proof.** By Theorem 4.7 it is trivial to see that \( I_{\min}/A \) is simple. Denote by \( \pi : I_{\min} \rightarrow I_{\min}/A \) the canonical quotient map. Choose a positive element \( T \in I_{\min} \setminus A \). Given an approximate identity \( \{c_n\} \), by Theorem 4.1 and Remark 4.2 we can find a series \( D := \sum_{k=1}^{\infty} d_k \) diagonal with respect to \( \{c_n\} \) and with \( 0 \neq \pi(D) \leq \pi(T) \). Choose \( \delta > 0 \) such that \((D - \delta)_+ \notin A\). By the diagonality of \( D \), \((D - \delta)_+ = \sum_{k=1}^{\infty} (d_k - \delta)_+ \) and assume that \((d_k - \delta)_+ \neq 0 \) for every \( k \). Apply Lemma 4.11 to the sequence \( \{(d_k - \delta)_+\} \) to find a mutually orthogonal sequence \( 0 \neq c''_k \in A_+ \) such that \( n c''_k \leq (d_k - \delta)_+ \) for every \( n \in \mathbb{N} \) and \( k \geq n \), where \( n c''_k \) denotes as before the n-fold direct sum of \( c''_k \) with itself. Choose \( 0 \neq c'_k \leq c_{2k} - c_{2k-1} \) with \( c'_k \leq c''_k \) for every \( k \). Define \( c_k := \frac{c'_k}{\|c'_k\|} \) and \( C := \sum_{k=1}^{\infty} c_k \). Then the series converge strictly and \( C \notin A \). Moreover, 
\[
nc_k \leq n c''_k \geq (d_k - \delta)_+ \quad \forall \ k \geq n.
\]
By Proposition 4.3
\[
n\sum_{k=n}^{\infty} c_k \leq \sum_{k=n}^{\infty} d_k.
\]
But then 
\[
n\pi(C) = n \pi \left( \sum_{k=n}^{\infty} c_k \right) \leq \pi \left( \sum_{k=n}^{\infty} d_k \right) = \pi(D) \leq \pi(T) \quad \forall \ n \in \mathbb{N}.
\]
In particular, \( \pi(C) \leq \pi(T) \), that is, \( C \in (I_{\min})_+ \setminus A \).

On the other hand, \( I_{\min}/A = J_0/A \) by Theorem 4.4, and hence, it is simple. Thus for every \( \epsilon > 0 \) there is an \( m \) such that \( \pi((T - \epsilon)_+) \leq m \pi(C) \), and hence,
\[
\pi((T - \epsilon)_+) \oplus \pi((T - \epsilon)_+) \leq 2m \pi(C) \leq \pi(T).
\]
Since \( \epsilon \) is arbitrary, it follows that \( \pi(T) \oplus \pi(T) \leq \pi(T) \) which proves that \( I_{\min}/A \) is purely infinite.

\( \square \)
5. The minimal ideal when $\mathcal{A}$ has strict comparison

**Definition 5.1.** Let $\mathcal{A}$ be a simple, $\sigma$-unital, non-unital C*-algebra with nonempty tracial simplex $\mathcal{T}(\mathcal{A})$. Set:

(i) $K_c := \{X \in \mathcal{M}(\mathcal{A})_+ \mid \hat{X} \in \text{Aff}(\mathcal{T}(\mathcal{A}))\}$.

(ii) $I_{\text{cont}} := \overline{L(K_c)}$.

**Proposition 5.2.** Let $\mathcal{A}$ be a simple, $\sigma$-unital, non-unital C*-algebra with nonempty tracial simplex $\mathcal{T}(\mathcal{A})$. Then

(i) $K_c$ is a hereditary strongly invariant cone; $L(K_c)$ is a two-sided selfadjoint ideal and hence so is $\overline{L(K_c)} = L(K_c)$.

(ii) \[
(I_{\text{cont}})_+ = \overline{K_c} = \{X \in \mathcal{M}(\mathcal{A})_+ \mid \hat{X} \in \text{Aff}(\mathcal{T}(\mathcal{A}))\} = \{X \in \mathcal{M}(\mathcal{A})_+ \mid (X - \delta)_+ \in \text{Aff}(\mathcal{T}(\mathcal{A})) \text{ } \forall \delta > 0\}.
\]

(iii) For a projection $P \in \mathcal{M}(\mathcal{A})$, $P \in I_{\text{cont}}$ if and only if $\hat{P}$ is continuous.

(iv) $I_{\text{cont}} = \overline{\text{span}K_c}$.

**Proof.**

(i) Since the map $\mathcal{M}(\mathcal{A})_+ \ni X \to \hat{X} \in \text{LAff}(\mathcal{T}(\mathcal{A}))_+$ satisfies the conditions $\hat{X + Y} = \hat{X} + \hat{Y}$ and $t\hat{X} = \hat{tX}$ for $X, Y \in \mathcal{M}(\mathcal{A})_+$ and $t \in \mathbb{R}_+$, it is clear that $K_c$ is a cone. Moreover, if $0 \leq X \leq Y \leq Z$ in $K_c$, then $\hat{X + Y - X} = \hat{Y}$.

Since $\hat{Y}$ is affine and continuous and both $\hat{X}$ and $\hat{Y}$ are affine, lower semicontinuous, and non-negative, it is immediate to verify that both must be continuous. Thus $X \in K_c$, and hence, $K_c$ is hereditary. Since $\hat{X^*} = \hat{X}^*$ for all $X \in \mathcal{M}(\mathcal{A})$, $K_c$ is strongly invariant. Therefore, the rest of the conclusions in (i) follows from (2.13), Lemma 2.11, (2.12), and Corollary 2.12.

(ii) By Corollary 2.12 and Theorem 2.10 (i) and (ii) we have that

\[
(\overline{L(K_c)})_+ = L(K_c)_+ = \overline{K_c}.
\]

which is the first equality in (ii). The second equality is given by Lemma 2.11 (iii).

(iii) Since $(P - \delta)_+ = \begin{cases} (1 - \delta)P & 0 \leq \delta < 1 \\ 0 & \delta \geq 1 \end{cases}$, we have by (ii) that $P \in (I_{\text{cont}})_+$ if and only if $\hat{P} \in \text{Aff}(\mathcal{T}(\mathcal{A}))$.

(iv) Since by (i) and Theorem 2.10, $I_{\text{cont}} = L(K_c) = \overline{\text{span}K_c}$ is closed, it is immediate to see that $\overline{\text{span}K_c} = \overline{\text{span}K_c}$.

Notice that if $\mathcal{A} = \mathcal{K}$, then $K_c$ consist of the positive cone of the trace class operators, and hence, $I_{\text{cont}} = K_c$.

It is immediate to verify that $\mathcal{A} \subset I_{\text{cont}}$. Indeed $(a - \delta)_+ \in \text{Ped}(\mathcal{A})$ for every $\delta > 0$ and $a \in \mathcal{A}_+$, hence $(a - \delta)_+$ is continuous, that is, $(a - \delta)_+ \in K_c$. Thus $a \in \overline{K_c} \subset I_{\text{cont}}$. To further relate $I_{\text{cont}}$ to $\mathcal{A}$ and to $I_{\text{min}}$ we need first the following lemma.

**Lemma 5.3.** Let $\mathcal{A}$ be a simple, non-elementary C*-algebra with $\mathcal{T}(\mathcal{A}) \neq \emptyset$. Then for every $\epsilon > 0$ there is an element $0 \neq c \in \mathcal{A}_+$ such that $d_\tau(c) < \epsilon$ for all $\tau \in \mathcal{T}(\mathcal{A})$. Furthermore, the element $c$ can be chosen in $\text{Ped}(\mathcal{A})$. 

Proposition 5.5. Let 0 ≠ f ∈ Ped(\(\mathcal{A}\)) and recall that \(\hat{f} \in \text{Aff}(\mathcal{T}(\mathcal{A}))_+\). Choose \(\delta > 0\) such that \((f - \delta)_+ \neq 0\), and an integer \(M \geq \frac{\|f\|}{\epsilon}\). By Lemma 2.5 we can find nonzero positive mutually orthogonal elements \(a_j\) such that \(\sum_{j=1}^{M} a_j \leq (f - \delta)_+\). By Lemma 2.4 we can find a nonzero positive element \(c \leq a_j\) for \(1 \leq j \leq M\). By Lemma 2.1(vi) it follows that

\[
Mc \leq \sum_{j=1}^{M} a_j \leq (f - \delta)_+.
\]

Thus for every \(\tau \in \mathcal{T}(\mathcal{A})\)

\[
Md_\tau(c) = d_\tau(Mc) \leq d_\tau((f - \delta)_+) \leq \frac{1}{\delta} \tau(f) \leq \frac{1}{\delta} \|\hat{f}\|.
\]

Thus \(d_\tau(c) < \epsilon\). Finally, \((c - \delta)_+ \in \text{Ped}(\mathcal{A})\) for every \(\delta > 0\). Choose \(\delta > 0\) such that \((c - \delta)_+ \neq 0\). Then \(d_\tau((c - \delta)_+) \leq d_\tau(c) < \epsilon\) for all \(\tau \in \mathcal{T}(\mathcal{A})\). \(\square\)

**Proposition 5.4.** Let \(\mathcal{A}\) be a simple, \(\sigma\)-unital, non-unital, non-elementary \(C^*\)-algebra with nonempty tracial simplex \(\mathcal{T}(\mathcal{A})\). Then \(\mathcal{A} \subseteq I_{\text{cont}}\).

**Proof.** By Lemma 5.3 there is an infinite sequence of elements \(0 \neq \tilde{a}_k \in \mathcal{A}_+\) such that \(d_\tau((\tilde{a}_k)) \leq \frac{1}{2^k}\) for all \(k\) and all \(\tau \in \mathcal{T}(\mathcal{A})\). By Lemma 2.4 we can find \(0 \neq a'_k \leq e_{3k} - e_{3k-1}\) with \(a'_k \leq \tilde{a}_k\) for all \(k\). Let \(a_k := \frac{a'_k}{\|a'_k\|}.\) Then

\[
\tau(a_k) \leq d_\tau(a_k) \leq d_\tau(a'_k) \leq d_\tau(\tilde{a}_k) \leq \frac{1}{2^k} \forall k \text{ and } \forall \tau \in \mathcal{T}(\mathcal{A}).
\]

Furthermore, \(a_k \leq \|a'_k\|(e_{3k+1} - e_{3k-2}) \in \text{Ped}(\mathcal{A}),\) hence \(\tilde{a}_k \in \text{Aff}(\mathcal{T}(\mathcal{A}))_+\). Let \(A := \sum_{k=1}^{\infty} a_k\). Then the series converges strictly and since it is diagonal \((a_ka_k') = 0\) for \(k \neq k'\) and does not converge in norm, \(A \notin \mathcal{A}\). On the other hand, \(\hat{A} = \sum_{k=1}^{\infty} \tilde{a}_k\) is continuous since the series is uniformly convergent. Thus \(A \in (I_{\text{cont}})_+.\) \(\square\)

**Proposition 5.5.** Let \(\mathcal{A}\) be a simple, \(\sigma\)-unital, non-unital, non-elementary \(C^*\)-algebra with nonempty tracial simplex \(\mathcal{T}(\mathcal{A})\). Then \(K_o(\{e_n\}) \subseteq K_c\) for every approximate identity \(\{e_n\}\). Consequently, \(I_{\text{min}} \subseteq I_{\text{cont}}\).

**Proof.** Let \(0 \neq X \in K_o(\{e_n\})\) and \(\epsilon > 0\). By Lemma 5.3 we can find an element \(0 \neq c \in \mathcal{A}_+\) such that \(d_\tau(c) < \frac{\epsilon}{\|X\|}\) for every \(\tau \in \mathcal{T}(\mathcal{A})\). By the definition of \(K_o(\{e_n\})\) there is an \(N \in \mathbb{N}\) such that

\[
(e_n - e_m)X(e_n - e_m) \leq c \quad \forall n > m, n \geq N.
\]

Now \(X^{1/2}(e_n - e_m)X^{1/2} \in \text{Aff}(\mathcal{T}(\mathcal{A}))_+\) because \(X^{1/2}(e_n - e_m)X^{1/2} \in \text{Ped}(\mathcal{A})\). Moreover,

\[
X^{1/2}(e_n - e_m)X^{1/2}(\tau) = (e_n - e_m)^{1/2}X(e_n - e_m)^{1/2}(\tau)
\]

\[
\leq \|X\|d_\tau((e_n - e_m)^{1/2}X(e_n - e_m)^{1/2})
\]

\[
= \|X\|d_\tau((e_n - e_m)X(e_n - e_m))
\]

\[
\leq \|X\|d_\tau(c) < \epsilon.
\]
Thus the series $\hat{X} = \sum_{n=1}^{\infty} X^{1/2}(e_n - e_{n-1})X^{1/2}$ converges uniformly and hence $X \in K_c$. This proves that $K_o(\{e_n\}) \subset K_c$, and hence, $I_{\min} \subset I_{\text{cont}}$.

\[\square\]

In general, $I_{\min}$ may fail to coincide with $I_{\text{cont}}$ as we will see in section 7.

**Theorem 5.6.** Let $\mathcal{A}$ be a simple, $\sigma$-unital, non-unital, non-elementary C*-algebra with strict comparison of positive elements by traces. Then $I_{\min} = I_{\text{cont}}$.

**Proof.** By Proposition 5.5, we need to prove that $(I_{\text{cont}})_+ \subset (I_{\min})_+$. As in the proof of Theorem 4.7, it is enough to verify that if $\{e_n\}$ is an approximate identity for $\mathcal{A}$, $D = \sum_{k=1}^{\infty} d_k$ is diagonal with respect to $\{e_n\}$, and $D \in I_{\text{cont}}$, then $D \in I_{\min}$. Let $\delta > 0$, and by dropping if necessary the zero summands in the series $(D - \delta)_+ = \sum_{k=1}^{\infty} (d_k - \delta)_+$, assume that $(d_k - \delta)_+ \neq 0$ for all $k$. We claim that the sequence $\{(d_k - \delta)_+\}$ is thin.

Let $0 \neq a \in \mathcal{A}_+$. Recall that the function $d_\tau(a)$ is lower semicontinuous, and hence, $\min_{\tau \in \mathcal{T}(\mathcal{A})} d_\tau(a) > 0$. By Proposition 5.2, $(D - \delta)_+ \in \text{Aff}(\mathcal{T}(\mathcal{A}))$ and since $(d_k - \delta)_+ \in \text{Ped}(\mathcal{A})$ for all $k$, also $(d_k - \delta)_+ \in \text{Aff}(\mathcal{T}(\mathcal{A}))$. Since

$$\tau((D - \delta)_+) = \sum_{k=1}^{\infty} \tau((d_k - \delta)_+),$$

by Dini’s Theorem the series converges uniformly on $\mathcal{T}(\mathcal{A})$ for every $\delta > 0$. In particular, there is an $N$ such that if $j \geq i \geq N$ and $\tau \in \mathcal{T}(\mathcal{A})$, then

$$\sum_{k=i}^{j} \tau((d_k - \delta)_+) < \frac{\delta}{2} \min_{\tau \in \mathcal{T}(\mathcal{A})} d_\tau(a).$$

By 24, $d_\tau((d_k - \delta)_+) \leq \frac{2}{\delta} \tau((d_k - \delta)_+)$, and hence, by 24,

$$d_\tau(\sum_{k=i}^{j} (d_k - \delta)_+) \leq \sum_{k=i}^{j} d_\tau((d_k - \delta)_+) \leq \sum_{k=i}^{j} \frac{2}{\delta} \tau((d_k - \delta)_+) < \min_{\tau \in \mathcal{T}(\mathcal{A})} d_\tau(a).$$

By the hypothesis of strict comparison of positive elements by traces, we thus have that $\sum_{k=i}^{j} (d_k - \delta)_+ \preceq a$, which proves that the sequence $\{(d_k - \delta)_+\}$ is thin. But then $(D - \delta)_+ \in K_o(\{e_n\})$ by Lemma 3.13. Since $\delta$ is arbitrary, it follows that $D \in I_{\min} = K_o(\{e_n\})$ which concludes the proof.

\[\square\]

As a consequence of this theorem, any counterexample for $I_{\min} \neq \mathcal{A}$, could only be found among non-separable C*-algebras with no strict comparison of positive elements. Among such algebras is the C*-algebra $\mathcal{A}$ introduced by Rordam to provide an example of a simple unital C*-algebra with both infinite projections and nonzero finite projections (24 Theorem 5.6). Recall that $\mathcal{A}$ is the C*-inductive limit $\mathcal{A} = \lim_{n \to \infty} \mathcal{M}(\mathcal{B}_n)$, where all $\mathcal{B}_n$ are separable C*-algebras. So, while the algebras $\mathcal{M}(\mathcal{B}_n)$ are not separable and hence neither is $\mathcal{A}$, the order dense sequences for $\mathcal{B}_n$ are order dense also for $\mathcal{M}(\mathcal{B}_n)$ and therefore their union is order dense for $\mathcal{A}$. As a consequence $I_{\min} \neq \mathcal{A}$.  


6. Strict comparison in the minimal ideal.

In [14, Theorem 6.6] we proved that if $A$ is a $\sigma$-unital simple $C^*$-algebra with strict comparison of positive elements by traces and with quasicontinuous scale (e.g., with finite extremal boundary), then strict comparison of positive element by traces (see Definition 2.8) holds also in $\mathcal{M}(A)$. In this section we will show that if we restrict our attention to comparison between elements in $I_{cont}$, then strict comparison holds without requiring the scale to be quasicontinuous.

For the first step we list here a slightly modified version of [14, Lemma 6.2].

**Lemma 6.1.** Let $A$ be a simple, $\sigma$-unital, non-unital $C^*$-algebra with nonempty tracial simplex $T(A)$ and let $A \in (I_{cont})_+$, $B \in \mathcal{M}(A)_+$, and assume that $d_\tau(A) < d_\tau(B)$ for every $\tau \in T(A)$ for which $d_\tau(B) < \infty$. Then for every $\epsilon > 0$ there is a $\delta > 0$ and $\alpha > 0$ such that $d_\tau((A - \epsilon)_+) + \alpha \leq d_\tau((B - \delta)_+)$ for every $\tau \in T(A)$.

The proof being essentially the same, we refer the reader to [14, Lemma 6.2]. The only difference is that here we need to replace the condition used in [14, Lemma 6.2] by $I_{cont}$.

The next lemma extends the results of [14, Lemma 6.4].

**Lemma 6.2.** Let $A$ be a simple, $\sigma$-unital, non-unital $C^*$-algebra with nonempty tracial simplex $T(A)$ and let $B = \sum_{k=1}^{\infty} b_k$ be a strictly converging series with $b_k \in A_+$ for all $k$ and $b_nb_m = 0$ for $|n - m| \geq 2$. Assume that $B \in (I_{cont})_+$ and that $\delta > 0$. Then

(i) $d_\tau\left(\left(\sum_{n} b_k - \delta\right)_+\right) \downarrow 0$ uniformly on $T(A)$.

(ii) For every $\epsilon > 0$ and $0 < \delta' < \delta$ there is an $n$ such that for all $\tau \in T(A)$

$$d_\tau\left(\left(\sum_{k=1}^{n} b_k - \delta'\right)_+\right) > d_\tau\left(\left(\sum_{k=1}^{\infty} b_k - \delta\right)_+\right) - \epsilon.$$

**Proof.**

(i) The sequence $d_\tau\left(\left(\sum_{n} b_k - \delta\right)_+\right)$ is monotone decreasing by Lemma 2.1 (viii) and (2.6). Moreover, by Lemma 2.1 (ix)

$$d_\tau\left(\left(\sum_{n} b_k - \delta\right)_+\right) \leq d_\tau\left(\left(\sum_{k \geq n, k \text{ even}} b_k - \delta\right)_+\right) + d_\tau\left(\left(\sum_{k \geq n, k \text{ odd}} b_k - \delta\right)_+\right).$$

The series of the even and odd terms separately are diagonal and dominated by $B$, hence they still belong to $I_{cont}$. Thus it is enough to assume that $\sum_{k=1}^{\infty} b_k$ itself is diagonal.

Then $(B - \delta/2)_+ \in K_c$ by Proposition 5.2 (ii), hence

$$(B - \delta/2)_+ = \sum_{k=1}^{\infty} (b_k - \delta/2)_+ \in \text{Aff}(T(A))_+. $$
Since also \((b_k - \frac{\tau}{2})_+ \in \text{Aff}(\mathcal{T}(A))_+\) for every \(k\), by Dini’s Theorem this series converges uniformly. But then
\[
d_{\tau}\left(\left(\sum_{n} b_k - \delta\right)_+\right) = \sum_{n} d_{\tau}((b_k - \delta)_+) \leq \frac{2}{\delta} \sum_{n} \tau((b_k - \delta)_+) \to 0
\]
uniformly on \(\mathcal{T}(A)\).

(ii) Again, by Lemma 2.1 (ix), for every \(0 < \delta' < \delta\) we have
\[
d_{\tau}\left(\left(\sum_{k=1}^{n} b_k - \delta\right)_+\right) \leq d_{\tau}\left(\left(\sum_{k=1}^{n} b_k - \delta'\right)_+\right) + d_{\tau}\left(\left(\sum_{n+1}^{\infty} b_k - (\delta-\delta')\right)_+\right).
\]
By (i), we can choose \(n\) such that \(d_{\tau}\left(\left(\sum_{n+1}^{\infty} b_k - (\delta-\delta')\right)_+\right) < \epsilon\) for all \(\tau\). \(\Box\)

**Remark 6.3.** If \(B \leq \|B\|P\) for some projection \(P \in I_{\text{cont}}\), as [14, Lemma 6.4] shows, but as it is also easy to verify directly, the uniform convergence in (i) holds also for \(\delta = 0\), and hence, (ii) strengthens to the statement that
\[
d_{\tau}\left(\left(\sum_{k=1}^{n} b_k - \delta\right)_+\right) \to d_{\tau}\left(\left(\sum_{k=1}^{\infty} b_k - \delta\right)_+\right) \quad \text{uniformly on } \mathcal{T}(A).
\]

However, these stronger results do not hold in general as it is readily seen by considering \(B := \sum_{k=1}^{\infty} \frac{1}{k}(e_{k+1} - e_k)\) for some approximate identity \(\{e_n\}\) in a stable algebra \(A\). Indeed then \(B \in A \subset I_{\text{cont}}, \) but \(d_{\tau}\left(\sum_{k=1}^{\infty} b_k\right) = \infty\) for all \(n\).

We are ready now to prove that strict comparison holds for \(I_{\text{min}}\) provided that it holds for \(A\).

**Theorem 6.4.** Let \(A\) be a simple, \(\sigma\)-unital, non-elementary \(C^*\)-algebra with strict comparison of positive elements by traces, \(A, B \in (I_{\text{min}})_+\), and assume that \(B \not\in A\). If \(d_{\tau}(A) < d_{\tau}(B)\) for all \(\tau \in \mathcal{T}(A)\) for which \(d_{\tau}(B) < \infty\), then \(A \preceq B\).

**Proof.** Let \(\epsilon > 0\). By Theorem 3.6 \(I_{\text{min}} = I_{\text{cont}}\). Thus by Lemma 6.1 there is a \(\delta > 0\) and \(\alpha > 0\) such that
\[
d_{\tau}\left(\left(A - \epsilon\right)_+\right) + \alpha \leq d_{\tau}\left(\left(B - 4\delta\right)_+\right) \quad \forall \tau \in \mathcal{T}(A).
\]
By the assumption that \(B \not\in A\), we can reduce if necessary \(\delta\) so that \((B - 4\delta)_+ \not\in A\). By Theorem 4.1 and Remark 4.2 \(B = \sum_{k=1}^{\infty} b_k + t\) where \(\sum_{k=1}^{\infty} b_k\) is a strictly converging bi-diagonal series, \(t = t^* \in A\), and \(\|t\| < \delta\). Then by Lemma 2.2
\[
(B - 4\delta)_+ \leq \left(\sum_{k=1}^{\infty} b_k - 3\delta\right)_+ \leq B
\]
whence by 2.10 for all \(\tau\)
\[
d_{\tau}\left(\left(A - \epsilon\right)_+\right) + \alpha \leq d_{\tau}\left(\left(\sum_{k=1}^{\infty} b_k - 3\delta\right)_+\right).
\]
By Lemma 6.2 (ii), there is an \(n_1\) such that
\[
d_{\tau}\left(\left(A - \epsilon\right)_+\right) < d_{\tau}\left(\left(\sum_{k=1}^{n_1} b_k - 2\delta\right)_+\right) \quad \forall \tau \in \mathcal{T}(A).
\]
Since \((B - 4\delta) + \not\in A\), we have by (6.1) that \((\sum_{k=1}^{\infty} b_k - 2\delta)_+ \not\in A\). But then

\[
\forall \ n \exists \ m \geq n \text{ such that } \left(\sum_{n}^{m} b_k - 2\delta\right)_+ \neq 0.
\]

Otherwise if there were an \(n\) such that \((\sum_{n}^{m} b_k - 2\delta)_+ = 0\) for all \(m\), the strict convergence \((\sum_{n}^{m} b_k - 2\delta)_+ \to (\sum_{n}^{\infty} b_k - 2\delta)_+\) (see [14, Lemma 3.1]) would imply that \((\sum_{n}^{\infty} b_k - 2\delta)_+ = 0\), and hence, from Lemma 2.1(ix),

\[
\left(\sum_{k=1}^{\infty} b_k - 2\delta\right)_+ \leq \sum_{k=1}^{n-1} b_k + \left(\sum_{n}^{\infty} b_k - 2\delta\right)_+ \in A,
\]

a contradiction. Now starting with the integer \(n_1\) just constructed, and by the same argument, define inductively an increasing sequence of integers \(n_k \geq n_{j+1} + 2\) such that

\[
\left(\sum_{j=n_k+1}^{n_k+2} b_k - 2\delta\right)_+ \neq 0 \forall k.
\]

Let \(d_1 := \sum_{j=1}^{n_1} b_k\) and \(d_{k+1} := \sum_{j=n_k+1}^{n_k+2} b_k\). By construction, \(d_n d_m = 0\) for \(n \neq m\) and

\[
\sum_{k=1}^{\infty} d_k \leq \sum_{j=1}^{\infty} b_k.
\]

By construction \((d_k - 2\delta)_+ \neq 0\) for all \(k\) and the function \(d_\tau((d_k - 2\delta)_+)\) is lower semicontinuous and strictly positive. Let

\[
\beta_k := \min_{\tau \in \mathcal{T}(A)} d_\tau((d_k - 2\delta)_+).
\]

By (6.2) we also have

\[
(6.5) \quad d_\tau((A - \epsilon)_+) < d_\tau((d_1 - 2\delta)_+) \forall \tau.
\]

Now apply Theorem 4.1 and Remark 4.2 to decompose \(A\) into the strictly converging sum of a series \(\sum_{j=1}^{\infty} a_k\) and a selfadjoint remainder \(a \in A\) with \(a_k \in A_+\), \(a_k a_i = 0\) for \(|i - j| \geq 2\), and \(|a| \leq \epsilon\). By Lemma 6.2 (i) we can find a strictly increasing sequence of integers \(m_k\) such that

\[
d_\tau\left(\sum_{j=m_k+1}^{\infty} a_k - 2\epsilon\right)_+ < \beta_{k+1} \forall \tau \in \mathcal{T}(A).
\]

Set \(m_0 = 0\) and \(c_k := \sum_{j=m_k+1}^{m_k+1} a_k\). We claim that

\[
(6.6) \quad d_\tau((c_k - 2\epsilon)_+) < d_\tau((d_k - 2\delta)_+) \forall \tau \in \mathcal{T}(A), k \geq 1.
\]

For \(k = 1\) we have

\[
(c_1 - 2\epsilon)_+ = \left(\sum_{j=1}^{m_1} a_k - 2\epsilon\right)_+ \leq \left(\sum_{j=1}^{\infty} a_k - 2\epsilon\right)_+ \leq (A - \epsilon)_+.
\]

where the first sub-equivalence follows from Lemma 2.1 (viii) and the second one from Lemma 2.2. Then by (6.1) and (6.5),

\[
d_\tau((c_1 - 2\epsilon)_+) \leq d_\tau((A - \epsilon)_+) < d_\tau((d_1 - 2\delta)_+),
\]
that is, (6.6) holds for \( k = 1 \). For \( k \geq 2 \), by Lemma 2.1 (viii) and (2.6) we have for all \( \tau \in \mathcal{T}(A) \) that
\[
d_{\tau}((c_k - 2\epsilon)_+) \leq d_{\tau}\left(\sum_{j=m_{k-1}+1}^{\infty} a_k\right) < \beta_k,
\]
and hence, (6.6) also holds.

By the strict comparison of positive elements in \( A \), it follows that
\[
(c_k - 2\epsilon)_+ \preceq (d_k - 2\delta)_+ \quad \forall k \geq 1.
\]
By construction, \( \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} c_k \) with convergence in the strict topology and \( c_n c_m = 0 \) for \( |n - m| \geq 2 \). Thus \( C_e := \sum_{k=1}^{\infty} c_{2k} \) and \( C_o := \sum_{k=1}^{\infty} c_{2k-1} \) are two diagonal series also converging strictly and \( \sum_{k=1}^{\infty} a_k = C_e + C_o \). Furthermore,
\[
(C_e - 2\epsilon)_+ = \sum_{k=1}^{\infty} (c_{2k} - 2\epsilon)_+ \quad \text{and} \quad (C_o - 2\epsilon)_+ = \sum_{k=1}^{\infty} (c_{2k-1} - 2\epsilon)_+.
\]
By Proposition 4.5 we have
\[
(c_k - 3\epsilon)_+ \prec \left(\sum_{k=1}^{\infty} d_{2k} - \delta\right)_+ \quad \text{and} \quad (c_k - 3\epsilon)_+ \prec \left(\sum_{k=1}^{\infty} d_{2k-1} - \delta\right)_+.
\]
Therefore
\[
(A - 7\epsilon)_+ \preceq (C_e + C_o - 6\epsilon)_+ \quad \text{(by Lemma 2.4)}
\]
\[
\preceq (C_e - 3\epsilon)_+ + (C_o - 3\epsilon)_+ \quad \text{(by Lemma 2.1 (ix))}
\]
\[
\preceq \left(\sum_{k=1}^{\infty} d_{2k} - \delta\right)_+ \quad \oplus \quad \left(\sum_{k=1}^{\infty} d_{2k-1} - \delta\right)_+ \quad \text{(by Lemma 2.1 (vi))}
\]
\[
= \left(\sum_{k=1}^{\infty} d_k - \delta\right)_+ \quad \text{(by Lemma 2.1 (vii))}
\]
\[
\preceq \left(\sum_{k=1}^{\infty} b_k - \delta\right)_+ \quad \text{(by Lemma 2.1 (viii))}
\]
\[
\preceq B \quad \text{(by Lemma 2.1 (iv))}
\]
Since \( \epsilon \) is arbitrary, we conclude that \( A \preceq B \).

\[
\square
\]

7. An Example Where \( I_{\min} \neq I_{\text{cont}} \).

From Theorem 5.9, examples where \( I_{\min} \neq I_{\text{cont}} \) can be found only among "pathological" algebras that do not have strict comparison of positive elements. In this section we prove that the algebras constructed by Villadsen in [35] provide such examples. We will largely follow his notations. Let
\[
X_0 = \mathbb{D}^{n_0} \quad \text{and} \quad X_i = X_{i-1} \times \mathbb{C} P^{n_i} \quad \text{for} \ i \in \mathbb{N},
\]
that is,
\[
X_i = \mathbb{D}^{n_0} \times \mathbb{C} P^{n_1} \times \mathbb{C} P^{n_2} \times \cdots \times \mathbb{C} P^{n_i}.
\]
We will always assume that
\[
n_i \geq \sigma(i) := \begin{cases} 1 & \text{if} \ i = 0, \\ i(i!) & \text{if} \ i \geq 1, \end{cases}
\]
and hence,

(7.2) \[ \dim(X_i) = 2 \sum_{k=0}^{i} n_k \geq 2 \sum_{k=0}^{i} \sigma(k) = 2(i + 1)! \]

This condition, together with the appropriate connecting maps, will guarantee that the AH algebra \( \mathcal{A} \) constructed in this process will not have \textit{slow dimension growth}, which by [34, Corollary 4.6] would imply strict comparison of positive elements. We refer the reader to Villadsen’s definition ([35, pp. 1092-1093]) of the connecting maps

(7.3) \[ \Phi_{i,i+1} : C(X_i) \otimes K \to C(X_{i+1}) \otimes K \]

and their compositions

\[ \Phi_{i,j} = \Phi_{j-1,i} \circ \cdots \circ \Phi_{i,i+1} : C(X_i) \otimes K \to C(X_j) \otimes K. \]

Identifying as usual projections with complex vector bundles, given a complex vector bundle \( \eta \) over \( X_i \), \( \Phi_{i,i+1}(\eta) \) denotes a complex vector bundle over \( X_{i+1} \). Denoting by \( k\eta \) (resp., \( k\pi \)) the \( k \)-fold direct sum of the vector bundle \( \eta \) (resp., of the projection \( \pi \)) with itself, we then have

(7.4) \[ \Phi_{i,i+1}(\eta) \cong \eta \times (i + 1) \text{rank}(\eta) \gamma_{n_{i+1}}. \]

Here \( \gamma_k \) denotes the universal line bundle over the projective space \( \mathbb{C}P^k \) (see (7.9) below for a key property of \( \gamma_k \)). Iterating we have for every \( j > i \),

(7.5) \[ \Phi_{i,j}(\eta) \cong \eta \times \frac{\sigma(i+1)}{(i+1)!} \text{rank}(\eta) \gamma_{n_{i+1}} \times \frac{\sigma(i+2)}{(i+1)!} \text{rank}(\eta) \gamma_{n_{i+2}} \times \cdots \times \frac{\sigma(j)}{(i+1)!} \text{rank}(\eta) \gamma_{n_j}. \]

In particular, since for every \( i \) and \( j \), \( \text{rank}(\gamma_i) = 1 \), \( \sum_{k=0}^{j} \sigma(k) = (j + 1)! \), and

\[ \text{rank}(\Phi_{i,j}(\eta)) = \text{rank}(\eta) \left( 1 + \sum_{k=i+1}^{j} \frac{\sigma(k)}{(i+1)!} \text{rank}(\gamma_k) \right), \]

we then have

(7.6) \[ \text{rank}(\Phi_{i,j}(\eta)) = \frac{(j + 1)!}{(i+1)!} \text{rank}(\eta) \quad \forall j \geq i. \]

Let \( \theta \) be a trivial line bundle over \( X_0 \) and set

\[ p_i := \Phi_{0,i}(\theta) \text{ for } i > 0; \]
\[ A_i = p_i(C(X_i) \otimes K)p_i \text{ for } i \geq 0; \]
\[ A = \lim_{\rightarrow}(A_i, \Phi_{i,i+1}). \]

Here \( \Phi_{i,i+1} \) denotes the \textit{restriction} of \( \Phi_{i,i+1} \) to \( A_i \). Let \( \Phi_{i,\infty} : A_i \to A \) denote the unital embedding \( A_i \hookrightarrow A \). By Villadsen’s construction, these maps are injective and we denote by \( \Phi_{\infty,i} : \Phi_{i,\infty}(A_i) \to A_i \) the inverse map of \( \Phi_{i,\infty} \). As usual, we will identify \( A_i \) with their images in \( A \) and focus on the algebraic inductive limit \( \bigcup A_i \subset A \).

For ease of reference, notice that

(7.7) \[ \text{rank}(p_i) = (i + 1)! \quad \forall i. \]

By [8] (see also a short proof in [35]), \( A \) is a simple, unital, AH-algebra and it has a unique tracial state \( \tau \). Villadsen proved that if \( n_i = n \sigma(i) \) for a fixed \( n \in \mathbb{N} \), then \( A \) has stable rank \( n + 1 \). What interests us here is that by (7.2) and
If \( \dim(X_i) \geq 2 \) and hence \( \mathcal{A} \) does not have slow dimension growth, the group \( K_0(\mathcal{A}) \) has perforation, and \( \mathcal{A} \) does not have strict comparison of projections by its trace. The same holds for other choices of \( n_i \geq \sigma(i) \) as readily seen from Villadsen’s construction.

We will show that \( I_{\min} \neq I_{\text{cont}} \) for the underlying algebra \( \mathcal{A} \otimes \mathcal{K} \) and that every element outside \( I_{\text{cont}} \) is full if \( \sup_{\sigma(i)} \frac{\text{rank}(p_i)}{\sigma(i)} < \infty \) (\( \mathcal{A} \) has flat dimension growth), while this is not the case for an unbounded dimension growth as \( n_i = i!\sigma(i) \).

To prove these results, we will focus on diagonal projections of \( \mathcal{M}(\mathcal{A} \otimes \mathcal{K}) \), i.e., projections of the form \( S = \bigoplus_{k=1}^{\infty} t_k s_k \) where \( t_k \in \mathbb{N} \), \( s_k \) is a projection in \( \Phi_{k,\infty}(\mathcal{A}_k) \), and \( t_k s_k \) is the direct sum of \( t_k \) copies of \( s_k \).

To determine if the diagonal projection \( S \) is in \( I_{\text{cont}} \) is easy. Since \( \mathcal{A} \) has a unique tracial state \( \tau \), and hence, \( I_{\text{cont}} = I_\tau \), the projection \( S \) is in \( I_{\text{cont}} \) if and only if \( \tau(S) < \infty \), i.e., \( \sum_{k=1}^{\infty} t_k \tau(s_k) < \infty \). If \( \eta_k = \Phi_{\infty,k}(s_k) \) is the complex vector bundle over \( X_k \) corresponding to \( s_k \), i.e., \( s_k = \Phi_{k,\infty}(\eta_k) \), then by (7.6)

\[
\tau(s_k) = \frac{\text{rank}(\eta_k)}{\text{rank}(p_k)} = \frac{\text{rank}(\eta_k)}{(k+1)!},
\]

and hence,

\[
\tau(S) = \sum_{k=0}^{\infty} t_k \frac{\text{rank}(\eta_k)}{(k+1)!}.
\]

To construct a diagonal projection \( S \neq I_{\min} \) we will make use of algebraic topology tools, more precisely, properties of the Euler classes. For a complex vector bundle \( \eta \) on a compact metric space \( X \), \( e(\eta) \) will denote the Euler class in the cohomology ring \( H^*(X) \). To simplify notations, we will suppress explicit reference to the base space \( X \). We start by recalling that for the universal line bundles \( \gamma_n \), used in defining the connecting maps (7.4), we have

\[
e(\gamma_n) = \begin{cases} 0 & n \leq n_i, \\ 1 & n > n_i. \end{cases}
\]

**Lemma 7.1.** Let \( \eta \) be a vector bundle over \( X_i \) and let \( j > i \).

(i) If \( e(\eta) = 0 \), then \( e(\Phi_{i,j}(\eta)) = 0 \).

(ii) If \( e(\eta) \neq 0 \) and \( \text{rank}(\eta) \leq (i+1)! \), then \( e(\Phi_{i,j}(\eta)) \neq 0 \).

**Proof.** Recall the fact that the Euler class of \( \Phi_{i,j}(\eta) \) is the cup product of the Euler classes of its components in the Cartesian product in (7.5) (viewed as vector bundles over \( X_j \) via pullbacks of the relevant projection maps). That is,

\[
e(\Phi_{i,j}(\eta)) = e(\eta) \cdot e \left( \frac{\sigma(i+1)}{(i+1)!} \text{rank}(\eta) \gamma_{n_{i+1}} \right) \cdots e \left( \frac{\sigma(j)}{(j+1)!} \text{rank}(\eta) \gamma_{n_j} \right)
\]

Thus if \( e(\eta) \) vanishes, so does \( e(\Phi_{i,j}(\eta)) \). By the Kunneth formula, since the cohomology groups considered have no torsion, it follows that \( e(\Phi_{i,j}(\eta)) \neq 0 \) if and only if all the factors in the above decomposition do not vanish. By (7.3), a necessary and sufficient condition for that to happen is that \( n_k \geq \frac{\sigma(k)}{(k+1)!} \text{rank}(\eta) \) for all \( i < k \leq j \). By the assumption (7.1), a sufficient condition is that \( \text{rank}(\eta) \leq (i+1)! \).
the spaces $X_i$, and $\eta_j = \Phi_{ij}(\eta_i)$ for $j \geq i$. In view of Lemma 7.1, it is convenient to set the following definition.

**Definition 7.2.** Let $p \in \left( \bigcup_{j=0}^{\infty} A_j \right) \otimes K$ be a projection. We say that

(i) $e(p) = 0$ if $e(\eta_i) = 0$ for some $i$ for which $p \in A_i \otimes K$ (and hence $e(\eta_j) = 0$ for every $j \geq i$).

(ii) $e(p) \neq 0$ if $e(\eta_j) \neq 0$ for every $j$ for which $p \in A_j \otimes K$.

In order to verify that $e(p) \neq 0$, by Lemma 7.1 it is sufficient to show that $e(\eta_i) \neq 0$ and that $\text{rank}(\eta_i) \leq (i + 1)!$ for the smallest $i$ for which $p \in A_i \otimes K$.

**Corollary 7.3.** Let $q, r \in \left( \bigcup_{j=0}^{\infty} A_j \right) \otimes K$ be projections, $q \preceq r$ and $e(q) = 0$. Then $e(r) = 0$.

**Proof.** There is an $i$ such that $q, r \in A_i \otimes K$, $e(\Phi_{i,\infty,i} \otimes \text{id})(q) = 0$, and the subequivalence $q \preceq r$ holds within $A_i \otimes K$, i.e., $r = q' \oplus s$ for some projections $q', s \in A_i \otimes K$ with $q' = vv^*$ and $q = v^*v$ for some $v \in A_i \otimes K$. But then

\[
e(\Phi_{i,\infty,i} \otimes \text{id})(r) = e((\Phi_{i,\infty,i} \otimes \text{id})(q'))e((\Phi_{i,\infty,i} \otimes \text{id})(s))
\]

\[
= e((\Phi_{i,\infty,i} \otimes \text{id})(q))e((\Phi_{i,\infty,i} \otimes \text{id})(s)) = 0.
\]

By Definition 7.2, $e(r) = 0$. \qed

We will construct now two sequence of projections $\{q_i\}$ and $\{r_i\}$ in $A \otimes K$ for which $e(q_i) = 0$ and $e(r_i) \neq 0$ for all $i$.

By the definition of $p_i$ it is immediate to find a trivial complex line bundle $\theta_i \leq p_i$ over $X_i$. Let $q_i := \Phi_{i,\infty}(\theta_i) \otimes e_{ii} \in A \otimes K$, so that the projections $q_i$ are mutually orthogonal. Then it is clear that

\[
Q := \bigoplus_{i=1}^{\infty} q_i \in \mathcal{M}(A \otimes K) \setminus A \otimes K \quad \text{and} \quad \tau(Q) = \sum_{i=1}^{\infty} \tau(q_i) = \sum_{i=1}^{\infty} \frac{1}{(i + 1)!} < \infty,
\]

and hence

\[
(7.10) \quad Q \in I_{\text{cont}}.
\]

Furthermore, by construction,

\[
(7.11) \quad e(q_i) = 0 \quad \forall i.
\]

Next, from the definition of $p_i$ and the construction of the maps $\Phi_{i, i+1}$ in (7.3), we see that there is a complex line bundle $\rho_i \in C(X_i) \otimes K$ with $\rho_i \leq p_i$ and $\rho_i \sim \pi_i^{2*}(\gamma_n_i)$, where $\pi_i^{2*}$ denotes the pull back map from vector bundles on $\mathbb{C}P^n_i$ to those on the space $X_i$. Thus $e(\rho_i)^k = 0$ if and only if $e(\gamma_{n_i})^k = 0$, i.e., by (7.4), if and only if $k > n_i$. When there is no risk of confusion, we will write $\gamma_{n_i}$ for $\rho_i$ as well as for the pullbacks to vector bundles over $X_j$ for $j > i$.

Set

\[
\tau(R) = \sum_{i=1}^{\infty} \tau(r_i) = \sum_{i=1}^{\infty} \frac{1}{(i + 1)!} < \infty,
\]

By definition, the projections $r_i$ are mutually orthogonal. Set

\[
R := \sum_{i=1}^{\infty} r_i.
\]

It is the clear that $R \in \mathcal{M}(A \otimes K) \setminus A \otimes K$, and

\[
\tau(R) = \sum_{i=1}^{\infty} \tau(r_i) = \sum_{i=1}^{\infty} \frac{1}{(i + 1)!} < \infty.
\]
and hence

\begin{equation}
R \in I_{\text{cont}}.
\end{equation}

**Lemma 7.4.** \(e(nr_i) \neq 0\) for all \(n \leq \min(n_i, (i+1)!).\) In particular, \(e(r_i) \neq 0\) for all \(i.\)

**Proof.** By (7.4) and the assumption that \(n \leq n_i\) we have \(e(n\gamma_{n_i}) = (\gamma_{n_i})^n \neq 0.\) Moreover, \(\text{rank}(n\gamma_{n_i}) = n \leq (i+1)!\), hence \(e(nr_i) \neq 0\) by Lemma (7.4) and Definition 7.2. \(\Box\)

**Lemma 7.5.** For all integers \(j \geq i\)

\begin{enumerate}[(i)]
  \item \(\frac{1}{j!}r_j \leq r_i;\)
  \item \(\sum_{k=i+1}^{j} r_k \leq r_i;\)
  \item \(\sum_{k=i+1}^{j} r_k \leq 2r_i.\)
\end{enumerate}

**Proof.**

\begin{enumerate}[(i)]
  \item By (7.4) we have \(\Phi_{i+1}(\rho_i) \cong \rho_i \times (i+1)\gamma_{n_{i+1}},\) and hence, \((i+1)r_{i+1} \leq r_i.\) Then (i) follows immediately.
  \item The proof is by induction on \(j - i \geq 1.\) By (i), \(r_{i+1} \leq (i+1)r_{i+1} \leq r_i\) so the condition holds for \(j - i = 1.\) Assume condition (ii) holds for some \(j - i > 1\) and hence \(\sum_{k=i+1}^{j+1} r_k \leq r_{i+1}.\) Then

\[
\sum_{k=i+1}^{j+1} r_k \leq r_{i+1} + \sum_{k=i+2}^{j+1} r_k \leq 2r_{i+1} \leq (i+1)r_{i+1} \leq r_i
\]

where the last relation in the chain holds by (i).
  \item Obvious from (ii). \(\Box\)
\end{enumerate}

**Lemma 7.6.** The sequence \(\{r_k\}\) is order dense (see Definition 3.8).

**Proof.** In view of Lemma 7.4 (iv) and the density of \(\bigcup_{i=1}^{\infty} \Phi_{i,\infty}(A_i)\) in \(A,\) in order to show that \(\{r_k\}\) is order dense in \(A \otimes K,\) it is enough to show that for every \(i,h \in \mathbb{N}\) and \(0 \neq a \in p_i(C(X_i) \otimes M_h(\mathbb{C}))(\pi_h)\) there is some \(j \geq i\) such that \(r_j \leq \Phi_{i,j}(a)\).

To do that we need to examine more closely the construction of the connecting maps \(\Phi_{i,j+1}\) and their iterations \(\Phi_{i,j}\). We again refer the reader to the definition in Definition 3.1 and also to [16]. Disregarding the isomorphism between \(K \otimes K\) and \(K,\) we may view \(\Phi_{i,j+1}(a)\) to be in the following matrix form:

\[
\begin{pmatrix}
  a \circ \pi_{i+1},i & a(\pi_{i+1},(y_{i+1}^j)) \otimes r_{i+1}^j \\
  \vdots & \ddots \\
  a(\pi_{i+1},(y_{i+1}^j)) \otimes r_{i+1}^j
\end{pmatrix}
\]

where \(r_{i+1}^j\) are mutually orthogonal projections all equivalent to \(r_{i+1},\) \(\pi_{i+1},i\) denotes the projection from \(X_{i+1}\) onto \(X_i,\) and the points \(y_{j}^j \in X_j\) are chosen so that the collection of their projections \(\{\pi_{j,i}(y_{j}^j) \ | \ 1 \leq j \leq i, \ j \geq i\}\) is dense in \(X_i\) for every \(i.\)

Since \(a\) is a continuous, there is a \(j \geq i\) and a \(1 \leq k \leq j\) such that \(a(\pi_{j,i}(y_{j}^j)) \neq 0.\) But then, \(0 \leq a(\pi_{j,i}(y_{j}^j)) \otimes r_j^j \leq \Phi_{i,j+1}(a).\) By diagonalizing \(a(\pi_{j,i}(y_{j}^j))\), we can find a \(\lambda > 0\) and a rank one projection \(s\) such that \(\lambda s \otimes r_j^j \leq \Phi_{i,j+1}(a)\), and hence, \(r_j \leq \Phi_{i,j}(a).\) This proves the claim. \(\Box\)
Corollary 7.7. The projection \( R \) belongs to \( I_{\min} \), and hence, it generates \( I_{\min} \).

Proof. Let \( e_n := 1_A \otimes \sum_{k=1}^n e_{kk} \), then \( e_n \) is an approximate identity of \( A \otimes K \). By Lemma 7.6, the sequence \( \{e_n\} \) is order dense, and hence, by Lemma 7.4 it is thin. But then \( R \in K_0(\{e_n\}) \subset I_{\min} \) by Lemma 3.13. Since \( R \notin A \otimes K \) and \( I_{\min} \) is minimal among the ideals properly containing \( A \otimes K \), it follows that \( R \) generates \( I_{\min} \). \( \square \)

Theorem 7.8. The projection \( Q \) does not belong to \( I_{\min} \), and hence, \( I_{\min} \neq I_{\cont} \).

Proof. Assume by contradiction that \( Q \in I_{\min} \). By Corollary 7.7, \( I_{\min} = I(R) \), and hence there is an \( n \in \mathbb{N} \) such that \( Q \leq nR \), i.e., \( \bigoplus_{k=1}^\infty q_k \preceq \bigoplus_{k=1}^\infty nr_k \). Choose \( i \) such that \( n \leq \sigma(i-1) \). Then \( n \leq \min(n_{i-1}, i!) \) by the assumption (7.1) and hence \( e(nr_{i-1}) \neq 0 \) by Lemma 7.4. On the other hand, by Proposition 4.10, there are \( m, j \in \mathbb{N} \), \( j \geq i \), such that \( q_m \preceq \bigoplus_{k=1}^j nr_k \). By Lemma 7.5 (ii), \( q_m \preceq nr_{i-1} \) and since \( e(q_m) = 0 \), it follows from Corollary 7.2 that \( e(nr_{i-1}) = 0 \), a contradiction. \( \square \)

Remark 7.9.

(i) A consequence of Lemma 7.6 is the known fact that Villadsen’s algebras have the (SP) property (e.g., see the proof of the (SP) property for the Villadsen’s type algebras studied in [30].

(ii) The same argument in the proof of Theorem 7.5 shows that \( q_m \not\preceq r_i \) for every \( m, i \in \mathbb{N} \) which is an illustration of the well known fact that strict comparison of projections does not hold in \( A \otimes K \).

Notice that so far we have only assumed that \( n_i \geq \sigma(i) \). We can obtain more if we assume that \( A \) has flat dimension growth, that is \( \sup \frac{\dim(X_n)}{\text{rank}(p)} < \infty \), (see 33 Definition 1.2)), which are exactly Villadsen’s finite stable rank algebras studied in [35].

Theorem 7.10. Assume that \( A \) has flat dimension growth, then \( I_{\cont} \) is the largest proper ideal of \( \mathcal{M}(A \otimes K) \).

Proof. To prove that \( I_{\cont} \) contains every proper ideal of \( \mathcal{M}(A) \), it suffices to prove that if \( S \in \mathcal{M}(A)_+ \setminus I_{\cont} \) then \( S \) is full, namely \( I(S) = \mathcal{M}(A) \). Assume without loss of generality that \( \|S\| = 1 \). By Theorem 4.1 and Remark 4.2, \( S = D_e + D_o + a \) where \( a = a^* \in A \subset I_{\cont} \) and \( D_e \) and \( D_o \) are diagonal series. Then at least one of the two series must also not belong to \( I_{\cont} \). To simplify notations, assume that \( S \) itself is diagonal, namely \( S = \bigoplus_{k=1}^\infty s_k \) where \( s_k \in A \otimes K_+ \) for every \( k \) and the series converges strictly. Furthermore, find \( \delta > 0 \) for which \( \tau(S - \delta)_+ = \infty \). Let \( M := \sup \frac{\dim(X_n)}{\text{rank}(p)} \) and choose an increasing subsequence \( m_k \) such that \( \sum_{j=m_{k+1}}^{m_k+1} \tau((s_j - \delta)_+) > \frac{M}{2} + 2 \). To simplify notations, assume \( m_k = k \), i.e., \( \tau((s_k - \delta)_+) > \frac{M}{2} + 2 \) for every \( k \). It was proven in [16] Lemma 2.5 that for every \( 0 \neq c \in (A \otimes M_\tau)_+ \) and \( \epsilon > 0 \), there is a projection \( q \) with \( \|\tau(q) - \lim_{k \to \infty} \tau(c^{1/k})\| < \epsilon \) and \( q \in c(A \otimes M_n)c \), and hence, \( q \leq c \). While the standard assumption in [16] was that \( n_i = \sigma(i) \), no conditions on \( n_i \) were used in the proof of that lemma. Moreover, it is routine to extend that lemma to \( 0 \neq c \in (A \otimes K)_+ \). Thus we can find projections \( q_k \preceq (s_k - \delta)_+ \), such that for all \( k \)

\[ \tau(q_k) > \frac{1}{2^k} \geq \tau((s_k - \delta)_+) > \frac{1}{2^k} > \frac{M}{2} + \tau(1_A \otimes e_{kk}). \]
By [33 Definition 2.1], \( M \geq \text{drr}(A) \) (we refer the reader to [33] for the definition of the dimension-rank ratio of \( A \)) and by [33 Theorem 3.10] it follows that

\[
1_A \otimes e_{kk} \leq q_k \leq (s_k - \delta)_+ \quad \forall k.
\]

Then by Proposition [33] we have that

\[
1_{M(A)} = \bigoplus_{k=1}^{\infty} 1_A \otimes e_{kk} \leq \bigoplus_{k=1}^{\infty} s_k = S
\]

which proves that \( S \) is full. \( \Box \)

Without the flat dimension growth condition, the conclusion of Theorem [7.10] no longer necessarily holds. To show that, we first we need the following refinement of Lemma [7.5].

**Lemma 7.11.** Let \( \eta = \bigoplus_{k=1}^{\infty} \Phi_{k,j}(t_k \gamma_{n_k}) \), where \( i \leq j \) are integers and \( t_k \) is a monotone nondecreasing sequence of integers. For every \( j' \geq j \) we have

\[
\Phi_{j,j'}(\eta) \cong m_i \gamma_{n_i} \times m_{i+1} \gamma_{n_{i+1}} \times \cdots \times m_{j'} \gamma_{n_{j'}}
\]

where \( m_k \in \mathbb{N} \) and

\[
m_k \leq \begin{cases} t_i & k = i \\ t_k (1 + e^{\frac{\sigma(k)}{t_k+1}}) & i + 1 \leq k \leq j \\ t_j \frac{\sigma(k)}{t_k+1} & j + 1 \leq k \leq j'. \end{cases}
\]

**Proof.** From (7.3) we have for every \( j' \geq j \)

\[
\begin{align*}
\Phi_{i,j'}(t_i \gamma_{n_i}) & \cong t_i \gamma_{n_i} \times t_i \frac{\sigma(i+1)}{t_i+1} \gamma_{n_{i+1}} \times \cdots \times t_j \frac{\sigma(i)}{t_j+1} \gamma_{n_{i'}} \\
\Phi_{i+1,j'}(t_{i+1} \gamma_{n_{i+1}}) & \cong t_{i+1} \gamma_{n_{i+1}} \times t_{i+1} \frac{\sigma(i+2)}{t_{i+1}+2} \gamma_{n_{i+2}} \times \cdots \times t_j \frac{\sigma(i)}{t_j+1} \gamma_{n_{i'}} \\
\Phi_{i+2,j'}(t_{i+2} \gamma_{n_{i+2}}) & \cong t_{i+2} \gamma_{n_{i+2}} \times t_{i+2} \frac{\sigma(i+3)}{t_{i+2}+3} \gamma_{n_{i+3}} \times \cdots \times t_j \frac{\sigma(i)}{t_j+1} \gamma_{n_{i'}} \\
\vdots & \vdots \vdots \vdots \\
\Phi_{j,j'}(t_j \gamma_{n_j}) & \cong t_j \gamma_{n_j} \times \cdots \times t_j \frac{\sigma(j')}{t_j+1} \gamma_{n_{j'}}
\end{align*}
\]

Recall that if \( \rho_1 \cong s_1 \alpha \times t_1 \beta \) and \( \rho_2 \cong s_2 \alpha \times t_2 \beta \) for some complex vector bundles \( \alpha \) and \( \beta \) on spaces \( X \) and \( Y \), and integers \( s_1, s_2, t_1, t_2 \), then \( \rho_1 \oplus \rho_2 \cong (s_1 + s_2) \alpha \times (t_1 + t_2) \beta \).

Thus by summing the integer multipliers of the universal bundles \( \gamma_{n_k} \) we obtain that

\[
m_k = \begin{cases} t_i & k = i \\ t_k + \sigma(k) \sum_{h=i+1}^{k} \frac{t_h}{h!} & i + 1 \leq k \leq j \\ \sigma(k) \sum_{h=i+1}^{j} \frac{t_h}{h!} & j + 1 \leq k \leq j'. \end{cases}
\]

By using the Lagrange remainder of the Taylor series for the exponential function, we see that \( \sum_{h=i+1}^{k} \frac{t_h}{h!} \leq \frac{e^{t_k} - 1}{t_k} \). This inequality together with the monotonicity of the sequence \( t_k \) establishes the claim. \( \Box \)

**Proposition 7.12.** Let \( R_\infty := \bigoplus_{k=1}^{\infty} k! r_k \). Then \( R_\infty \not\in I_{\text{cont}} \). If \( n_k \geq k! \sigma(k) \), then \( Q \not\in I(R_\infty) \).
Proof. Clearly $R_\infty \in \mathcal{M}(\mathcal{A} \otimes \mathcal{K}) \setminus \mathcal{A} \otimes \mathcal{K}$ is a projection and $R_\infty \not\in I_{\text{cont}}$ follows from $\tau(R_\infty) = \sum_{k=1}^{\infty} \frac{k!}{(k+1)!} = \infty$. To show that $Q \not\in I(R_\infty)$ we reason as in the proof of Theorem 7.8. For every $n \in \mathbb{N}$, choose $i$ such that $(i+1)! \geq 2en$ and let $j \geq i$. Let $\eta$ be the complex vector bundle over $X_j$ corresponding to $\sum_{k=1}^{j} nk!r_k$.

Then $\eta \cong \bigoplus_{k=i} \Phi_{k,j}(nk!\gamma_{nk})$, and hence, by Lemma 7.11

$$\Phi_{j,j'}(\eta) \cong nm_i\gamma_{n_i} \times nm_{i+1}\gamma_{n_{i+1}} \times \cdots nm_j\gamma_{n_j}.$$ 

Since

$$nm_k \leq \begin{cases} n! & \text{for } k = i

nk!(1 + \frac{\sigma(k)}{(i+1)!}) & \text{for } i < k \leq j

\sigma(k) & \text{for } j + 1 < k \leq j'
\end{cases}$$

we see that $e(\Phi_{j,j'}(\eta)) \neq 0$ for every $j' \geq j$. Thus $e(n \bigoplus_{k=i} t_kr_k) \neq 0$ by Definition 7.2. Reasoning as in the proof of Theorem 7.8 we then conclude that $Q \not\in I(R_\infty).$

\[\square\]

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