On convergence of dynamics of hopping particles
to a birth-and-death process in continuum

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Abstract
We show that some classes of birth-and-death processes in continuum
(Glauber dynamics) may be derived as a scaling limit of a dynamics of
interacting hopping particles (Kawasaki dynamics)

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1 Preliminaries
This letter deals with two classes of stochastic dynamics of infinite particle
systems in continuum. Let Γ denote the space of all locally finite subsets of
\( \mathbb{R}^d \), \( d \in \mathbb{N} \). This space is called the configuration space. Elements of Γ are
called configurations, and each point of a configuration represents position of a
particle. We endow Γ with the vague topology, i.e., the weakest topology in Γ
with respect to which every mapping of the form \( \Gamma \ni \gamma \mapsto \langle f, \gamma \rangle := \sum_{x \in \gamma} f(x) \),
with \( f \in C_0(\mathbb{R}^d) \), is continuous. Here \( C_0(\mathbb{R}^d) \) is the space of all real-valued functions on \( \mathbb{R}^d \) with compact support. We denote by \( \mathcal{B}(\Gamma) \) the Borel \( \sigma \)-algebra
in \( \Gamma \).

A dynamics of hopping particles (Kawasaki dynamics) is a Markov process
on \( \Gamma \) whose generator is given (on an appropriate set of functions on \( \Gamma \)) by

\[
(L_K F)(\gamma) = \sum_{x \in \gamma} \int_{\mathbb{R}^d} dy c(x, y, \gamma \setminus x) (F(\gamma \setminus x \cup y) - F(\gamma)).
\]

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Here and below, for simplicity of notations, we just write \(x, y\) instead of \(\{x\}, \{y\}\). The function \(c(x, y, \gamma \setminus x)\) describes the rate at which a particle \(x\) of configuration \(\gamma\) jumps to \(y\), taking into account the rest of configuration, \(\gamma \setminus x\).

A birth-and-death process in continuum (Glauber dynamics) is a Markov process on \(\Gamma\) with generator

\[
(L_G F)(\gamma) = \sum_{x \in \gamma} d(x, \gamma \setminus x)(F(\gamma \setminus x) - F(\gamma)) + \int_{\mathbb{R}^d} dy b(y, \gamma)(F(\gamma \cup y) - F(\gamma)).
\]

Here \(d(x, \gamma \setminus x)\) describes the rate at which a particle \(x\) of configuration \(\gamma\) dies, whereas \(b(x, \gamma)\) describes the rate at which, given configuration \(\gamma\), a new particle is born at \(y\). For some constructions and discussions of Glauber and Kawasaki dynamics in continuum, see [13, 11, 12, 13, 14, 16] and the references therein.

The aim of this letter it to show that, in many cases, a birth-and-death process may be interpreted as a limiting dynamics of hopping particles. We will restrict our attention to the case where the rate \(c\) of the Kawasaki dynamics is given by

\[
c(x, y, \gamma \setminus x) = a(x - y) \exp[-E\phi^-(x, \gamma \setminus x) - E\phi^+ (y, \gamma \setminus x)].
\]

Here \(a\) and \(\phi^\pm\) are even functions on \(\mathbb{R}^d\) (e.g. \(a(-x) = a(x)\)), \(a\) is bounded, \(a \geq 0\), \(\int_{\mathbb{R}^d} a(x) \, dx = 1\), and for \(x \in \mathbb{R}^d\) and \(\gamma \in \Gamma\),

\[
E\phi^\pm (x, \gamma) := \sum_{y \in \gamma} \phi^\pm (x - y),
\]

provided the sum converges absolutely. Thus, \(c(x, y, \gamma \setminus x)\) is a product of three terms: the term \(e^{E\phi^- (x, \gamma \setminus x)}\) describes the rate at which a particle \(x \in \gamma\) jumps, the term \(e^{-E\phi^+ (y, \gamma \setminus x)}\) describes the rate at which this particle lands at \(y\), and finally the term \(a(x - y)\) gives the distribution of an individual jump.

We now produce the following scaling of this dynamics. For each \(\varepsilon > 0\), we define \(a_\varepsilon(x) := \varepsilon^d a(\varepsilon x)\). We clearly have that \(\int_{\mathbb{R}^d} a_\varepsilon(x) \, dx = 1\). Let \(c_\varepsilon\) denote the \(c\) coefficient in which function \(a\) is replaced by \(a_\varepsilon\), and let \(L_\varepsilon\) denote the corresponding \(L_K\) generator. Letting \(\varepsilon \rightarrow 0\), we may suggest that only jumps of infinite length will survive, i.e., jumps from a point to ‘infinity’, and jumps from ‘infinity’ to a point. Thus, we expect to arrive at a birth-and-death process. To make our suggestion more explicit, we proceed as follows.

2 Convergence of the generator of the scaled evolution of correlation functions

For simplicity, we assume, in this section, that the functions \(\phi^\pm\) are from \(C_0(\mathbb{R}^d)\). Then \(E\phi^\pm (x, \gamma)\) are well defined for each \(x \in \mathbb{R}^d\) and \(\gamma \in \Gamma\).
Let us briefly recall some basic facts of harmonic analysis on the configuration space, see [8, 10] for further detail. Let Γ₀ denote the space of all finite configurations in \( \mathbb{R}^d \), i.e., \( \Gamma_0 = \bigcup_{n=0}^\infty \Gamma^{(n)} \), where \( \Gamma^{(n)} \) is the space of all \( n \)-point configurations in \( \mathbb{R}^d \). Clearly, \( \Gamma_0 \subset \Gamma \), and we define \( B(\Gamma_0) \) and \( B(\Gamma^{(n)}) \) as the trace \( \sigma \)-algebra of \( \Gamma \) on \( \Gamma_0 \) and \( \Gamma^{(n)} \), respectively. For a function \( G : \Gamma_0 \to \mathbb{R} \), we define a function \( (KG)(\gamma) := \sum_{\eta \in \gamma} G(\eta), \gamma \in \Gamma \), provided the summation makes sense. Here \( \eta \in \gamma \) means that \( \eta \) is a finite subset of \( \gamma \).

Let \( \mu \) be a probability measure on \((\Gamma, B(\Gamma))\). Then there exists a unique measure \( \rho \) on \((\Gamma_0, B(\Gamma_0))\) satisfying

\[
\int_{\Gamma} (KG)(\gamma) \mu(d\gamma) = \int_{\Gamma_0} G(\eta) \rho(d\eta)
\]

for each measurable function \( G : \Gamma_0 \to [0, \infty) \). The measure \( \rho \) is called the correlation measure of \( \mu \). Further, denote by \( \lambda \) the Lebesgue–Poisson measure on \( \Gamma_0 \), i.e.,

\[
\lambda = \delta_{\varnothing} + \sum_{n=1}^{\infty} \frac{1}{n!} dx_1 \cdots dx_n.
\]

Here \( \delta_{\varnothing} \) is the Dirac measure with mass at \( \varnothing \), and \( dx_1 \cdots dx_n \) is the Lebesgue measure on \( \Gamma^{(n)} \), which is naturally defined on this space. Assume that the correlation measure \( \rho \) of \( \mu \) is absolutely continuous with respect to \( \lambda \). Then \( \frac{d\rho}{d\lambda} \) is called the correlation functional of \( \mu \). For a given correlation functional \( k \), the corresponding Ursell functional \( u : \Gamma_0 \to \mathbb{R} \) is defined through the formula

\[
k(\eta) = \sum_{\pi \in P(\eta)} u_\pi(\eta),
\]

where \( P(\eta) \) denotes the set of all partitions of \( \eta \), and given a partition \( \pi = \{\eta_1, \ldots, \eta_k\} \) of \( \eta \), \( u_\pi(\eta) := u(\eta_1) \cdots u(\eta_k) \). Recall also that a function \( G : \Gamma_0 \to \mathbb{R} \) is called translation invariant if, for each \( x \in \mathbb{R}^d \), \( G(\eta_x) = G(\eta) \) for all \( \eta \in \Gamma_0 \), where \( \eta_x \) denotes the configuration \( \eta \) shifted by vector \( x \), i.e., \( \eta_x := \{ y + x \mid y \in \eta \} \). Clearly, the correlation functional \( k \) is translation invariant if and only if the corresponding Ursell functional \( u \) is translation invariant.

If \( k \) is the correlation functional of a probability measure \( \mu \) on \( \Gamma \), we denote

\[
k^{(n)}(x_1, \ldots, x_n) := k(\{x_1, \ldots, x_n\}), \quad n \in \mathbb{N},
\]

and analogously we define \( u^{(n)} \). The \( (k^{(n)})_{n=1}^\infty \) and \( (u^{(n)})_{n=1}^\infty \) are called the correlation and Ursell functions of \( \mu \), respectively. Note that, if \( k \) is translation invariant, then \( k^{(1)} = u^{(1)} \) is a constant.

For a function \( f : \mathbb{R}^d \to \mathbb{R} \), we define \( e_\varphi(f, \eta) := \prod_{x \in \eta} f(x), \eta \in \Gamma_0 \), where \( \prod_{x \in \varnothing} f(x) := 1 \). Further, let \( \varphi : \mathbb{R}^d \to \mathbb{R} \). Then

\[
(Ke_\varphi(e^{\varphi - 1}, \cdot))(\gamma) = e^{\langle \varphi, \gamma \rangle},
\]

so that

\[
\int_{\Gamma} e^{\langle \varphi, \gamma \rangle} \mu(d\gamma) = \int_{\Gamma_0} e_\varphi(e^{\varphi - 1}, \eta)k(\eta) \lambda(d\eta), \tag{1}
\]

under some proper conditions on \( \varphi \) and \( k \), see e.g. [10].
Assume that $L$ is a Markov generator on $\Gamma$. Denote $\hat{L} := K^{-1}L_{K}$, i.e., $\hat{L}$ is the operator acting on functions on $\Gamma_{0}$ which satisfies $K L G = L K G$. Denote by $L^{*}$ the dual operator of $L$ with respect to the Lebesgue–Poisson measure $\lambda$:

$$\int_{\Gamma_{0}} (\hat{L}G)(\eta)k(\eta)\lambda(d\eta) = \int_{\Gamma_{0}} G(\eta)(\hat{L}^{*}k)(\eta)\lambda(d\eta).$$

Assume now that a Markov process on $\Gamma$ with generator $L$ has initial distribution $\mu_{0}$. Denote by $\mu_{t}$ the distribution of this process at time $t > 0$. Assume that, for each $t \geq 0$, $\mu_{t}$ has correlation functional $k_{t}$. Then, at least at an informal level, one sees that the evolution of $k_{t}$ is described by the equation $\partial k_{t}/\partial t = \hat{L}^{*}k_{t}$, so that $\hat{L}^{*}$ is the generator of evolution of correlation functionals.

In the case where $L = L_{\varepsilon}$, we proceed as follows. First we write $L_{\varepsilon} = L_{\varepsilon}^{-} + L_{\varepsilon}^{+}$, where

$$L_{\varepsilon}^{-}(F)(\gamma) = \sum_{x \in \Gamma} \int_{\mathbb{R}^{d}} dy \alpha_{\varepsilon}(x-y)r(x,y,\gamma \backslash x)\{F(\gamma \backslash x) - F(\gamma)\},$$

$$L_{\varepsilon}^{+}(F)(\gamma) = \sum_{x \in \Gamma} \int_{\mathbb{R}^{d}} dy \alpha_{\varepsilon}(x-y)r(x,y,\gamma \backslash x)\{F(\gamma \backslash x) - F(\gamma \backslash y)\}.$$

Here, $r(x,y,\gamma \backslash x) := \exp\{E^{\phi^{-}}(x,\gamma \backslash x) - E^{\phi^{+}}(y,\gamma \backslash x)\}$. We also set

$$(L_{0}^{-}F)(\gamma) = \sum_{x \in \Gamma} \exp\{E^{\phi^{-}}(x,\gamma \backslash x)\}(F(\gamma \backslash x) - F(\gamma)),$$

$$(L_{0}^{+}F)(\gamma) = \int_{\mathbb{R}^{d}} dy \exp\{-E^{\phi^{+}}(y,\gamma)\}(F(\gamma \cup y) - F(\gamma)).$$

**Theorem 1.** Let $k$ be the correlation functional of a probability measure $\mu$ on $(\Gamma,B(\Gamma))$, and let $u$ be the corresponding Ursell functional. Assume that the following conditions are satisfied:

i) $k$ fulfills the bound $k(\eta) \leq (||\eta||)^{s}C^{||\eta||}$, $\eta \in \Gamma_{0}$, for some $0 \leq s < 1$ and $C > 0$. Here $||\eta||$ denotes the cardinality of set $\eta$.

ii) $k$ is translation invariant.

iii) The measure $\mu$ has a decay of correlations in the sense that, for any $n,m \in \mathbb{N}$, $a \in \mathbb{R}^{d}$, $a \neq 0$, and $\{x_{1},\ldots,x_{n+m}\} \in \Gamma^{(n+m)}$,

$$u(\{x_{1},\ldots,x_{n},x_{n+1} + (a/\varepsilon),\ldots,x_{n+m} + (a/\varepsilon)\}) \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

Then, for each $\eta \in \Gamma_{0}$,

$$(\hat{L}_{\varepsilon}^{-}k)(\eta) \rightarrow c^{-}(k)(\hat{L}_{0}^{-}k)(\eta), \quad (\hat{L}_{\varepsilon}^{+}k)(\eta) \rightarrow c^{+}(k)(\hat{L}_{0}^{+}k)(\eta),$$

where

$$c^{-}(k) := \int_{\Gamma_{0}} \lambda(d\xi) e_{\lambda}(e^{-\phi^{+}} - 1,\xi)k(\xi),$$

$$c^{+}(k) := \int_{\Gamma_{0}} \lambda(d\xi) e_{\lambda}(e^{\phi^{-}} - 1,\xi)k(\xi \cup 0).$$

(2)
Proof. A straightforward calculation (see [8]) shows that
\[
(\hat{L}_\varepsilon^* k)(\eta) = - \sum_{x \in \eta} \int_{\mathbb{R}^d} dy \ a_\varepsilon(x - y) r(x, y, \eta \setminus x) \\
\times \int_{\Gamma_0} \lambda(d\xi) k(\xi \cup \eta) e_\lambda(e^{\phi^-}(x^-) - e^{\phi^+}(y^-) - 1, \xi),
\] (3)
\[
(\hat{L}_\varepsilon^+ k)(\eta) = \sum_{y \in \eta} \int_{\mathbb{R}^d} dx \ a_\varepsilon(x - y) r(x, y, \eta \setminus y) \\
\times \int_{\Gamma_0} \lambda(d\xi) k(\xi \cup (\eta \setminus y) \cup x) e_\lambda(e^{\phi^-}(x^-) - e^{\phi^+}(y^-) - 1, \xi),
\]
\[
(\hat{L}_0^* k)(\eta) = - \sum_{x \in \eta} \exp[E^{\phi^-}(x, \eta \setminus x)] \\
\times \int_{\Gamma_0} \lambda(d\xi) e_\lambda(e^{\phi^-}(x^-) - 1, \xi) k(\eta \cup \xi),
\]
\[
(\hat{L}_0^+ k)(\eta) = \sum_{y \in \eta} \exp[-E^{\phi^-}(y, \eta \setminus y)] \\
\times \int_{\Gamma_0} \lambda(d\xi) e_\lambda(e^{-\phi^+}(y^-) - 1, \xi) k((\eta \setminus y) \cup \xi).
\]

We will now briefly explain the convergence of \((\hat{L}_\varepsilon^* k)(\eta)\) (the case of \((\hat{L}_\varepsilon^+ k)(\eta)\) can be dealt with analogously). From (3) and the definition of \(\lambda\), by making a change of variable, we easily have:
\[
(\hat{L}_\varepsilon^* k)(\eta) = - \sum_{x \in \eta} \int_{\mathbb{R}^d} dy \ a(y) r(x, (y/\varepsilon) + x, \eta \setminus x) \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \\
\times \int_{(\mathbb{R}^d)^n} du_1 \cdots du_n \prod_{i=1}^{k} (e^{-\phi^+((y/\varepsilon) + x - u_i)}(e^{\phi^-}(x - u_i) - 1)) \\
\times \prod_{j=k+1}^{n} (e^{-\phi^+((y/\varepsilon) + x - u_j)} - 1) k(\xi \cup \{u_1, \ldots, u_n\})
\]
\[
= - \sum_{x \in \eta} \int_{\mathbb{R}^d} dy \ a(y) r(x, (y/\varepsilon) + x, \eta \setminus x) \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{k!(n-k)!} \\
\times \int_{(\mathbb{R}^d)^n} du_1 \cdots du_n \prod_{i=1}^{k} (e^{-\phi^+((y/\varepsilon) - u_i)}(e^{\phi^-}(u_i) - 1)) \prod_{j=k+1}^{n} (e^{-\phi^+(u_j)} - 1) \\
\times k(\xi \cup \{u_1 + x, \ldots, u_k + x, u_{k+1} + x + (y/\varepsilon), \ldots, u_n + x + (y/\varepsilon)\}).
\]

Next, represent the correlation functionals in the above expression through a sum of Ursell functionals. Using the dominated convergence theorem and conditions i) and iii), we see that, in the limit, all the Ursell functionals containing at least one point from \(\xi \cup \{u_1 + x, \ldots, u_k + x\}\) and at least one point from
\{u_{k+1} + x + (y/\varepsilon), \ldots, u_n + x + (y/\varepsilon)\} \text{ will vanish, and by virtue of ii), we conclude that } (L_\varepsilon^{-k})(\eta) \text{ converges to }

\begin{align*}
- \sum_{x \in \eta} \int_{\mathbb{R}^d} dy \ a(y) \ \exp[E^\phi^-(x, \eta \setminus x)] \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \int_{\mathbb{R}^d} dy \ \prod_{i=1}^{k} (e^{\phi^-(x-u_i)} - 1) \ \prod_{j=k+1}^{n} (e^{-\phi^+(u_j)} - 1) \\
\times k(\xi \cup \{u_1, \ldots, u_k\}) k(\{u_{k+1}, \ldots, u_n\}),
\end{align*}

from where the statement follows. □

From Theorem 1, we can make the following conclusion. Assume that a dynamics of hopping particle with Markov generator \(L_K\) has initial distribution \(\mu_0\).

Let \(\mu_t\) be the distribution of this process at time \(t > 0\). Assume that, for each \(t \geq 0\), \(\mu_t\) has correlation functional \(k_t\) which satisfies conditions i)--iii) of Theorem 1. Further assume that \(c^\pm(k_t), t \geq 0\), given through (2) remain constant.

Then, we can expect that the scaled dynamics of hopping particles converges to a birth-and-death process with generator \(L_0 := c^-(k_0)L_0^- + c^+(k_0)L_0^+\) and initial distribution \(\mu_0\). We will discuss below two cases where this statement can be proven rigorously (at least in the sense of convergence of the generators).

3 Convergence of non-equilibrium free dynamics

This case has been discussed in [14], so here we will explain its connection with Theorem 1.

Let \(\Theta \in B(\Gamma)\) be the set of those configurations \(\gamma \in \Gamma\) for which there exist \(\alpha \geq d\) and \(K > 0\) such that

\[|\gamma \cap B(n)| \leq K n^\alpha, \quad \text{for all } n \in \mathbb{N},\]

where \(B(n)\) denotes the ball in \(\mathbb{R}^d\) centered at 0 and of radius \(n\). Note that the estimate (4) controls the growth of the number of particles of \(\gamma\) at infinity.

Let \(a \in S(\mathbb{R}^d)\) (the Schwartz space of rapidly decreasing, infinitely differentiable functions on \(\mathbb{R}^d\)). Consider a random walk in \(\mathbb{R}^d\) with transition kernel \(Q(x,dy) := a(x-y) dy\). This is a Markov process in \(\mathbb{R}^d\) with generator

\[(L^{(1)}f)(x) = \int_{\mathbb{R}^d} (f(y) - f(x)) a(x-y) dy.\]

The corresponding Markov semigroup on \(L^2(\mathbb{R}^d, dx)\) is then given by

\[(p_t f)(x) = e^{-t} f(x) + \int_{\mathbb{R}^d} G(x-y) f(y) dy,\]

where \(G\) is the inverse Fourier transform of \(e^{-t}(\exp[t(2\pi)^d/2\hat{a}]) - 1\), where \(\hat{a}\) is the Fourier transform of \(a\). (Note that we have normalized the direct and inverse
Fourier transforms so that they are unitary operators in $L^2(\mathbb{R}^d \to \mathbb{C}, dx).$ For any $\gamma \in \Theta,$ consider a dynamics of independent particles which starts at $\gamma$ and such that each separate particle moves according to the semigroup $p_t$ (i.e., independent random walks in $\mathbb{R}^d$). Then, this process has càdlàg paths on $\Gamma$ and a.s. it never leaves $\Theta,$ cf. [14]. The generator of the obtained Markov process on $\Theta$ is then given by

$$
(L_{\Theta}F)(\gamma) = \sum_{x \in \gamma} \int_{\mathbb{R}^d} dy \, a(x-y)(F(\gamma \setminus x \cup y) - F(\gamma)),
$$

so that now $\phi^\pm = 0.$

**Proposition 1.** Let $\mu_0$ be a probability measure on $\Gamma$ whose correlation functional $k_0$ satisfies conditions i)--iii) of Theorem 1 and $\mu_0(\Theta) = 1.$ Consider the Markov process on $\Theta$ with the generator $L_{\Theta}$ given by (6) and with the initial distribution $\mu_0.$ Denote by $\mu_t$ the distribution of this process at time $t > 0.$ Then, for each $t > 0,$ $\mu_t$ has correlation functional $k_t$ which satisfies conditions i)--iii) of Theorem 1 and furthermore $c^- (k_t) = 1$ and $c^+ (k_t) = k_0^{(1)},$ $t \geq 0.$

**Proof.** For each $f \in C_0(\mathbb{R}^d)$ and $t > 0,$ we have, by (1) and the construction of the process:

$$
\int_\Theta \mu_t(d\gamma) e^{(f, \gamma)} = \int_\Theta \mu_0(d\gamma) \prod_{x \in \gamma} (p_t e^f)(x)
$$

$$
= \int_{\Gamma_0} \lambda(d\eta) k_0(\eta) \prod_{x \in \eta} (p_t (e^f - 1))(x)
$$

$$
= 1 + \sum_{n=1}^\infty \frac{1}{n!} \int_{(\mathbb{R}^d)^n} dx_1 \cdots dx_n \, k^{(n)}(x_1, \ldots, x_n) \prod_{i=1}^n (p_t (e^f - 1))(x_i)
$$

$$
= 1 + \sum_{n=1}^\infty \frac{1}{n!} \int_{(\mathbb{R}^d)^n} dx_1 \cdots dx_n \, (p_t^{\otimes n} k^{(n)})(x_1, \ldots, x_n) \prod_{i=1}^n (e^f(x_i) - 1).
$$

Therefore, $\mu_t$ has correlation functional $k_t,$ and furthermore $k_t^{(n)} = p_t^{\otimes n} k_0^{(n)}.$ The latter equality, in turn, implies that $u_t^{(n)} = p_t^{\otimes n} u_0^{(n)}.$ From here it easily follows that, for each $t > 0,$ $\mu_t$ satisfies assumptions i)--iii) of Theorem 1. Furthermore, by (2),

$$
c^- (k_t) = k_t(\emptyset) = 1,
$$

$$
c^+ (k_t) = k_t(\{0\}) = k_t^{(1)} = p_t k_0^{(1)} = k_0^{(1)}. \quad \square
$$

Thus, according to Section 2, we expect that the scaled free dynamics with initial distribution $\mu_0$ converges to the birth-and-death process with generator

$$
(L_0F)(\gamma) = \sum_{x \in \gamma} (F(\gamma \setminus x) - F(\gamma)) + k_0^{(1)} \int_{\mathbb{R}^d} dy \, (F(\gamma \cup y) - F(\gamma)) \quad (7)
$$
Convergence of dynamics in continuum and initial distribution $\mu_0$. This dynamics can be constructed as follows, cf. [14, 19]. For each $\gamma \in \Theta$, denote by $P_\gamma$ the law of a process on $\Theta$ which is at $\gamma$ at time zero, and after this, points of $\gamma$ randomly die, independently of each other, so that the probability that at time $t > 0$ a particle $x \in \gamma$ is still alive is equal to $e^{-t}$. Next, let $\pi$ denote the Poisson point process in $\mathbb{R}^d \times (0, \infty)$ with the intensity measure $k^{(1)}_0 dx dt$. The measure $\pi$ is concentrated on configurations $\hat{\gamma} = \{(x_n, t_n)\}_{n=1}^{\infty}$ in $\mathbb{R}^d \times (0, \infty)$ such that $\{x_n\}_{n=1}^{\infty} \in \Theta$, $0 < t_1 < t_2 < \cdots$, and $t_n \to \infty$ as $n \to \infty$. For any such configuration, we denote by $P_{\hat{\gamma}}$ the law of a process on $\Theta$ such that at time $t = 0$, the configuration is empty, and then at each time $t_n$, $n \in \mathbb{N}$, a new particle is born at $x_n$, and after time $t_n$ this particle randomly dies, independently of the other particles, so that at time $s > t_n$ the probability that the particle is still alive is $e^{-(s-t_n)}$. Finally, the law of the process with generator (7) and initial distribution $\mu_0$ is given by

$$\int_\Theta \mu_0(d\gamma) P_\gamma \ast \int_\pi \pi(d\hat{\gamma}) P_{\hat{\gamma}}.$$

Here $\ast$ stays for convolution of measures, see [14] for details.

We will use $\tilde{\Gamma}$ to denote the space of multiple configurations over $\mathbb{R}^d$ equipped with the vague topology, see e.g. [9] for details. Note that $\Gamma \subset \tilde{\Gamma}$, and the trace $\sigma$-algebra of $B(\tilde{\Gamma})$ on $\Gamma$ is $B(\Gamma)$.

**Theorem 2** ([14]). Consider the stochastic process from Proposition 1 as taking values in $\tilde{\Gamma}$. Then, after scaling, this process converges, in the sense of weak convergence of finite-dimensional distributions, to the Markov process with the generator $L_0$ given by (7) and with the initial distribution $\mu_0$.

Note that the limiting process also lives in $\Theta$, and we used the $\tilde{\Gamma}$ space only to identify the type of convergence.

For reader’s convenience, let us explain the idea of the proof of Theorem 2. Fix arbitrary $0 = t_0 < t_1 < t_2 < \cdots < t_n$, $n \in \mathbb{N}$, and denote by $\mu_{t_0, t_1, \ldots, t_n}^\varepsilon$, $\varepsilon \geq 0$, the corresponding finite-dimensional distribution of the initial process scaled by $\varepsilon > 0$, and that of the limiting process if $\varepsilon = 0$, respectively. Then, by [9], the statement of the theorem is equivalent to staying that, for any non-negative $f_0, f_1, \ldots, f_n \in C_0(\mathbb{R}^d)$,

$$\int_{\Theta^n} \exp \left[ \sum_{i=0}^{n} \langle f_i, \gamma \rangle \right] d\mu_{t_0, t_1, \ldots, t_n}^\varepsilon (\gamma_0, \gamma_1, \ldots, \gamma_n) \to \int_{\Theta^n} \exp \left[ \sum_{i=0}^{n} \langle f_i, \gamma \rangle \right] d\mu_{t_0, t_1, \ldots, t_n}^0 (\gamma_0, \gamma_1, \ldots, \gamma_n) \quad \text{as} \quad \varepsilon \to 0. \quad (8)$$

For $\varepsilon > 0$, denote by $p_{\hat{\gamma}}(x, dy)$ the transition probability of the Markov semi-
group (4) scaled by $\varepsilon$. Set

$$g^\varepsilon(x) := e^{f_0(x)} \int_{R^d} p_{t_1}^\varepsilon(x, dx_1) \int_{R^d} p_{t_2-t_1}^\varepsilon(x_1, dx_2) \times \cdots \times \int_{R^d} p_{t_n-t_{n-1}}^\varepsilon(x_{n-1}, dx_n) \prod_{i=1}^n e^{f_i(x_i)}, \quad x \in R^d.$$ 

Then, by (1) and the construction of the process, the first integral in (8) (with $\varepsilon > 0$) is equal to

$$\int_{\Theta} \prod_{x \in \gamma} g^\varepsilon(x) \mu_0(d\gamma) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(R^d)^n} \prod_{i=1}^n (g^\varepsilon(x_i) - 1) \kappa_0^{(n)}(x_1, \ldots, x_n) dx_1 \cdots dx_n.$$ 

In the above integrals, one represents the correlation functions through the Ursell functions, makes a change of variables under the sign of integral, and after a careful analysis of the obtained expression, one takes its limit as $\varepsilon \to 0$. Finally, one shows that the obtained limit is indeed equal to the second integral in (8).

4 Convergence of equilibrium Kawasaki dynamics of interacting particles

In this section, we will consider equilibrium dynamics of interacting particles having a Gibbs measure as an equilibrium measure. Our result will extend that of [7], where just one special case of such a dynamics was considered (see also [15]). We start with a description of the class of Gibbs measures we are going to use.

A pair potential is a Borel-measurable function $\phi : R^d \to R \cup \{+\infty\}$ such that $\phi(-x) = \phi(x) \in R$ for all $x \in R^d \setminus \{0\}$. For $\gamma \in \Gamma$ and $x \in R^d \setminus \gamma$, we define a relative energy of interaction between a particle at $x$ and the configuration $\gamma$ as $E(x, \gamma) := \sum_{y \in \gamma} \phi(x-y)$, provided that the latter sum converges absolutely, and otherwise it is set to be $= \infty$. A (grand canonical) Gibbs measure corresponding to the pair potential $\phi$ and activity $z > 0$ is a probability measure $\mu$ on $(\Gamma, B(\Gamma))$ which satisfies the Georgii–Nguyen–Zessin identity:

$$\int_{\Gamma} \mu(d\gamma) \sum_{x \in \gamma} F(\gamma, x) = \int_{\Gamma} \mu(d\gamma) \int_{R^d} z \exp[-E(x, \gamma)] F(\gamma \cup x, x) \quad (9)$$

for any measurable function $F : \Gamma \times R^d \to [0, +\infty)$. A pair potential $\phi$ is said to be stable if there exists $B \geq 0$ such that, for any $\eta \in \Gamma_0$,

$$\sum_{\{x, y\} \subset \eta} \phi(x-y) \geq -B|\eta|. \quad (10)$$
In particular, we then have \( \phi(x) \geq -2B, x \in \mathbb{R}^d \). Next, we say that the condition of low activity–high temperature regime is fulfilled if

\[
\int_{\mathbb{R}^d} |e^{-\phi(x)} - 1| dx < (2e^{1+2B})^{-1},
\]

where \( B \) is as in (10). A classical result of Ruelle [17, 18] says that, under the assumption of stability and low activity–high temperature regime, there exists a Gibbs measure \( \mu \) corresponding to \( \phi \) and \( z \), and this measure has correlation functional which satisfies conditions i)–iii) of Theorem 1, with \( s = 0 \) in condition i) (which is then called the Ruelle bound). Furthermore, the corresponding Ursell functions satisfy \( u^{(n)}(0, \ldots, \cdot) \in L^1((\mathbb{R}^d)^{n-1}, dx_1 \cdots dx_n) \) for each \( n \geq 2 \).

In what follows, we will assume that the potential \( \phi \) is also bounded from above outside some finite ball in \( \mathbb{R}^d \) (which is always true for any realistic potential, since it should converge to zero at infinity).

We now fix arbitrary parameters \( u, v \in [0, 1] \), and assume that

\[
\int_{\mathbb{R}^d} |\exp[(2(u \vee v) - 1)\phi(x)] - 1| dx < \infty.
\]

It can be easily shown that, if \( u, v \in [0, 1/2] \), then (12) is a corollary of (11) and the condition that \( \phi \) be bounded outside some finite ball. Note that, even if \( u \vee v \in (1/2, 1] \), condition (12) still admits potentials which have ‘weak’ singularity at zero.

We introduce the set \( \mathcal{F}C_b(C_0(\mathbb{R}^d), \Gamma) \) of all functions of the form

\[
\Gamma \ni \gamma \mapsto F(\gamma) = g((f_1, \gamma), \ldots, (f_N, \gamma)),
\]

where \( N \in \mathbb{N}, f_1, \ldots, f_N \in C_0(\mathbb{R}^d), \) and \( g \in C_b(\mathbb{R}^N) \). Here \( C_b(\mathbb{R}^N) \) denotes the set of all continuous bounded functions on \( \mathbb{R}^N \). For each \( F \in \mathcal{F}C_b(C_0(\mathbb{R}^d), \Gamma) \), we define

\[
(L_K F)(\gamma) = \frac{1}{2} \sum_{x \in \gamma} \int_{\mathbb{R}^d} dy a(x - y) \left( \exp[uE(x, \gamma \setminus x) - (1 - v)E(y, \gamma \setminus x)] - \exp[vE(x, \gamma \setminus x) - (1 - u)E(y, \gamma \setminus x)] \right) (F(\gamma \setminus x \cup y) - F(\gamma)).
\]

Note that the first addend in (13) corresponds to the choice of \( \phi^- = u\phi, \phi^+ = (1 - v)\phi \), whereas the second addend corresponds to \( \phi^- = v\phi, \phi^+ = (1 - u)\phi \).

In the special case where \( u = v \), we get

\[
(L_K F)(\gamma) = \sum_{x \in \gamma} \int_{\mathbb{R}^d} dy a(x - y) \exp[uE(x, \gamma \setminus x) - (1 - u)E(y, \gamma \setminus x)] \times (F(\gamma \setminus x \cup y) - F(\gamma)).
\]

By [13], \( (L_K, \mathcal{F}C_b(C_0(\mathbb{R}^d), \Gamma)) \) is a Hermitian, non-negative operator in \( L^2(\Gamma, \mu) \), and we denote by \( (L_K, D(L_K)) \) its Friedrichs’ extension. As shown in [13] by using the theory of Dirichlet forms, there exists a Markov process on \( \Gamma \) with cádlág
paths whose generator is \((L_K, D(L_K))\). If we consider this process with initial distribution \(\mu\), then it is an equilibrium process, i.e., it has distribution \(\mu_t = \mu\) at any moment of time \(t \geq 0\). Thus, for each \(t \geq 0\), \(\mu_t = \mu\) has correlation function which satisfies conditions i)–iii) of Theorem 1.

**Lemma 1.** Let \(k\) denote the correlation function of the Gibbs measure \(\mu\) under consideration. Denote

\[
C_u := \int_{\Gamma} \mu(d\gamma) \exp[-(1-u)(\phi, \gamma)].
\]

Then we have:

\[
\int_{\Gamma_0} \lambda(d\xi) e_\lambda(e^{u(1-u)}\phi - 1, \xi) k(\xi) = C_u,
\]

\[
(14)
\]

\[
\int_{\Gamma_0} \lambda(d\xi) e_\lambda(e^{u\phi} - 1, \xi) k(\xi \cup 0) = z C_u.
\]

\[
(15)
\]

**Proof.** Equality (14) follows from (1). Next, using (1), (9), and translation invariance of \(k\), we have, for each \(f \in C_0(\mathbb{R}^d)\):

\[
\int_{\mathbb{R}^d} dx f(x) \int_{\Gamma} \mu(d\gamma) \exp[-(1-u)(\phi, \gamma)]
= \int_{\mathbb{R}^d} dx f(x) \int_{\Gamma} \mu(d\gamma) \exp[-(1-u)E(x, \gamma)]
= z^{-1} \int_{\Gamma} \mu(d\gamma) \sum_{x \in \gamma} f(x) \exp[uE(x, \gamma \setminus x)]
= z^{-1} \int_{\Gamma} \mu(d\gamma) \sum_{x \in \gamma} f(x) \sum_{\xi \subseteq \gamma \setminus x} e_\lambda(e^{u\phi} - 1, \xi)
= z^{-1} \int_{\Gamma} \mu(d\gamma) \sum_{\xi \subseteq \gamma} \sum_{x \in \xi} f(x) e_\lambda(e^{u\phi} - 1, \xi \setminus x)
= z^{-1} \int_{\Gamma_0} \lambda(d\xi) k(\xi) \sum_{x \in \xi} f(x) e_\lambda(e^{u\phi} - 1, \xi \setminus x)
= z^{-1} \int_{\Gamma_0} \lambda(d\xi) \int_{\mathbb{R}^d} dx k(\xi \cup x) f(x) e_\lambda(e^{u\phi} - 1, \xi)
= z^{-1} \int_{\mathbb{R}^d} f(x) \int_{\Gamma_0} \lambda(d\xi) k(\xi - x) e_\lambda(e^{u\phi} - 1, \xi)
= z^{-1} \int_{\mathbb{R}^d} f(x) \int_{\Gamma_0} \lambda(d\xi) k(\xi \cup 0) e_\lambda(e^{u\phi} - 1, \xi),
\]

from where equality (15) follows. □

Thus, by Lemma 1, according to Section 2, we expect that the scaled equilibrium dynamics (with initial distribution \(\mu\)) converges to the birth-and-death process.
process with generator
\[
(L_0 F)(\gamma) = \sum_{x \in \gamma} \frac{1}{2} \left( C_v \exp[uE(x, \gamma \setminus x)] + C_u \exp[vE(x, \gamma \setminus x)] \right)
\times (F(\gamma \setminus x) - F(\gamma))
+ \int_{\mathbb{R}^d} z \, dy \, \frac{1}{2} \left( C_u \exp[-(1 - v)E(y, \gamma)] + C_v \exp[-(1 - u)E(y, \gamma)] \right)
\times (F(\gamma \cup y) - F(\gamma))
\] (16)
and the initial distribution \( \mu \). In fact, by \cite{13}, \((L_0, \mathcal{F}C_0(\mathbb{R}^d), \Gamma))\) is a Hermitian, non-negative operator in \( L^2(\Gamma, \mu) \), and its Friedrichs’ extension \((L_0, D(L_0))\) is the generator of a Markov process on \( \Gamma \) with càdlàg paths.

We recall that \( L \) denotes the \( L_K \) generator (given by (13)) scaled by \( \varepsilon \). The following theorem states that, at least on an appropriate set of test functions, the operator \( L_\varepsilon \) converges to \( L_0 \) in the \( L^2 \)-norm.

**Theorem 3.** For each \( f \in C_0(\mathbb{R}^d) \), we have \( e^{(f, \cdot)} \in D(L_\varepsilon) \) for all \( \varepsilon \geq 0 \), and
\[
L_\varepsilon e^{(f, \cdot)} \to L_0 e^{(f, \cdot)} \quad \text{in} \quad L^2(\Gamma, \mu) \quad \text{as} \quad \varepsilon \to 0.
\]

**Proof.** We will only sketch the proof of the theorem. Let \( f \in C_0(\mathbb{R}^d) \). By approximation, one easily shows that, for each \( \varepsilon \geq 0 \), the function \( F(\gamma) = e^{(f, \gamma)} \) belongs to \( D(L_\varepsilon) \), and that the action of \( L_\varepsilon \) onto \( F \) is given, for \( \varepsilon > 0 \) by the right hand side of (13) in which \( a \) is replaced by \( a_\varepsilon \), and for \( \varepsilon = 0 \) by (16), respectively.

Denote
\[
(L_\varepsilon^- F)(\gamma) = \sum_{x \in \gamma} \int_{\mathbb{R}^d} dy \, a_\varepsilon(x-y)
\times \exp[uE(x, \gamma \setminus x) - (1-v)E(y, \gamma \setminus x)] e^{(f, \gamma \setminus x)} (1 - e^{f(x)}),
\]
\[
(L_\varepsilon^+ F)(\gamma) = \sum_{x \in \gamma} \int_{\mathbb{R}^d} dy \, a_\varepsilon(x-y)
\times \exp[uE(x, \gamma \setminus x) - (1-v)E(y, \gamma \setminus x)] e^{(f, \gamma \setminus x)} (e^{f(y)} - 1),
\]
\[
(L_0^- F)(\gamma) = C_v \sum_{x \in \gamma} \exp[uE(x, \gamma \setminus x)] e^{(f, \gamma \setminus x)} (1 - e^{f(x)}),
\]
\[
(L_0^+ F)(\gamma) = C_u \int_{\mathbb{R}^d} dy \, \exp[-(1-v)E(y, \gamma)] e^{(f, \gamma)} (e^{f(y)} - 1).
\]

To prove the theorem, it suffices to show that
\[
\|L_\varepsilon^\pm F\|_{L^2(\Gamma, \mu)}^2 \to \|L_0^\pm F\|_{L^2(\Gamma, \mu)}^2,
\]
\[
(L_\varepsilon^\pm F, L_0^\pm F)_{L^2(\Gamma, \mu)} \to \|L_0^\pm F\|_{L^2(\Gamma, \mu)}^2
\] (17)
as \( \varepsilon \to 0 \). To this end, one proceeds as follows. By using (9), one represents each of the expressions appearing in (17) in terms of integrals over \( \Gamma \) with
respect to $\mu$, as well as integrals over $\mathbb{R}^d$ with respect to Lebesgue measure. As a result one gets rid of all summations $\sum_{x \in \gamma}$. Then, one makes a change of variables, so that instead of $a_\varepsilon(x - y)$ one gets $a(x)$, and in $y$ variable one gets a function which is dominated by an integrable function of $y$. Next, one replaces integration $\int_\Gamma \mu(d\gamma) \cdots$ by corresponding integration $\int_\Gamma^0 \lambda(d\eta) k(\eta) \cdots$. In the obtained expression, one represents the correlation functional through a sum of Ursell functionals. Finally, one takes the limit as $\varepsilon \to 0$ by analogy with the final part of the proof of Theorem 1.

By using the well-known result of the theory of semigroups (see e.g. [2]), we get the following corollary of Theorem 3.

**Corollary 1.** Assume that the set of finite linear combinations of exponential functions $e^{\langle f, \cdot \rangle}$, $f \in C_0(\mathbb{R}^d)$, is a core for the limiting generator $(L_0, D(L_0))$. Then, we have the weak convergence of finite-dimensional distributions of the scaled Markov process in $\Gamma$ with the generator $(L_\varepsilon, D(L_\varepsilon))$ and with the initial distribution $\mu$ to the Markov process in $\Gamma$ with the generator $(L_0, D(L_0))$ and with the initial distribution $\mu$. In particular, if additionally $\phi \geq 0$, then this kind of convergence holds when $u = v = 0$.

We note that the final statement of Corollary 1 holds due to a result of [2] on essential self-adjointness of the generator of Glauber dynamics in the case $\phi \geq 0$ and $u = v = 0$ (see also [7]). In the latter case, we even expect that the weak convergence of laws holds. To this end, one needs to consider all processes as taking values in a negative Sobolev space. The tightness of the laws of scaled processes may be proven by analogy with the proof of [5, Theorem 7.1]. Next, one shows that this set of laws has, in fact, a unique limiting point—the law of the Markov process with generator $(L_0, D(L_0))$ and initial distribution $\mu_0$. This is done by identifying the limit via the martingale problem, and using convergence of the generators (compare with the proof of [6, Theorem 6.7] and that of [5, Theorem 7.5]).

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