Localization schemes

Simons Institute - Beyond the Boolean Cube Workshop

Yuansi Chen
joint work with Ronen Eldan

Duke University
1. Spectral independence via coordinate-by-coordinate localization
2. Glauber dynamics mixing in Ising model via Eldan’s stochastic localization
3. Glauber dynamics mixing in hardcore model via negative fields localization
Given a target measure $\mu$ (possibly unnormalized), on a state space $\mathcal{X} = \{-1, +1\}^n$ or $\mathbb{R}^n$, we want to draw samples $X \sim \mu$. 
Glauber dynamics for sampling $\mu$ on $\{-1, +1\}^n$

At current state $x \in \{-1, +1\}^n$, draw index $i$ uniformly from $[n]$

- move to $y = x \oplus e_i$ with probability $\frac{\mu(y)}{\mu(y) + \mu(x)}$
- otherwise, stay at $x$

Denote this transition kernel $P_{x \rightarrow y}$.
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**Mixing time:** starting from initial measure $\mu_{ini}$, let $\mu_{ini} P^k$ denote the measure at time $k$, how many iterations does it take so that

$$TV(\mu, \mu_{ini} P^k) \leq \epsilon?$$
Define the Dirichlet form

\[ \mathcal{E}_P(f, g) = \langle (1 - P)f, g \rangle_\mu \]

Poincaré inequality (or spectral gap)

\[ \lambda \text{Var}_\mu(f) \leq \mathcal{E}_P(f, f), \quad \forall f \]

For reversible lazy Markov chain, it implies variance decay:

\[ \text{Var}_\mu Pf \leq (1 - \lambda) \text{Var}_\mu f, \quad \forall f \]

Take \( f = \frac{\mu_{\text{ini}} P^k}{\mu} \), we can bound chi-squared divergence decay, leading to mixing time

\[ \frac{1}{\lambda} \left( \log \frac{1}{\mu_{\text{ini}, \text{min}}} + \log \frac{1}{\epsilon} \right) \]
Modified Log-Sobolev inequality (MLSI)

\[ \rho_{\text{MLSI}} \text{Ent}_\mu(f) \leq \mathcal{E}_P(f, \log f), \quad \forall f \geq 0 \]

We can bound KL-divergence decay, leading to mixing time

\[ \frac{1}{\rho_{\text{MLSI}}} \left( \log \log \frac{1}{\mu_{\text{ini},\min}} + \log \frac{1}{\epsilon} \right) \]
From now on, we focus on functional inequalities

- Target measure $\mu$
- $2^n \times 2^n$ Markov transition kernel $P$
- To prove mixing time, it suffice to prove

$$\lambda \text{Var}_\mu(f) \leq \mathcal{E}_P(f, f)$$

For product measure, it is easy.
Other than that, for what kind of target measure, can we prove spectral gap?
Coordinate-by-coordinate localization
Define the $n \times n$ pairwise influence matrix $\Psi_{\mu}$

$$\Psi_{\mu}[i, j] = \mathbb{P}_{x \sim \mu}(x_j = +1 \mid x_i = +1) - \mathbb{P}_{x \sim \mu}(x_j = +1 \mid x_i = -1)$$

$\mu$ is $\eta$-spectrally independent if

$$\|\Psi_{\mu}\|_2 \leq \eta$$
Spectral independence [Anari, Liu, Oveis Gharan ’20]

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\]

\( \mu \) is \( \eta \)-spectrally independent if

\[
\| \Psi_\mu \|_2 \leq \eta
\]

A sufficient condition for proving spectral gap: if all conditionals of \( \mu \) (the law of \( X | X_i = \pm 1 \) and \( X | X_i = \pm 1, X_j = \pm 1 \), etc.) are \( \eta \)-spectrally independent, then spectral gap

\[
\lambda \geq \prod_{i=0}^{n-2} \left(1 - \frac{\eta}{n - i}\right)
\]
Spectral independence is a condition on covariance

Since

\[ \text{Cov}_\mu = \text{diag}(\text{Cov}_\mu)(\Psi_\mu + I_n) \]

we have

\[ \text{Cov}_\mu \preceq (1 + \eta) \text{diag}(\text{Cov}_\mu) \iff \|\Psi_\mu + I_n\|_2 \leq 1 + \eta. \]
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Constraining the covariance makes sense, but

**Q1:** why do we have to put assumptions on all conditionals?

...trickling down, HDX, local-to-global

**Q2:** what are other ways to put assumptions to prove spectral gap, when direct proof is difficult?
What are localization schemes?

A localization scheme is a mapping from measure $\nu$ to a stochastic process $(\nu_t)_{t \geq 0}$ such that

- $\nu_0 = \nu$
- For any measurable $A$, $\nu_t(A)$ is a martingale (in other words, $\mathbb{E}[\nu_t(A) | \{\nu_\tau(A), \tau \leq s\}] = \nu_s(A), \forall 0 \leq s \leq t)$
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Our main standpoint:

- You pick a localization scheme
- Study the evolution of the variance $\text{Var}_{\nu_t}(f)$ along the process $(\nu_t)_t$
- Put assumptions to approximately conserve variance, then you can prove spectral gap!
Spectral independence assumption comes from coordinate-by-coordinate localization

Coordinate-by-coordinate localization

Start from $\nu$ on $\{-1, +1\}^n$. Let $(k_1, \ldots, k_n)$ be a random permutation of $[n]$, and $X$ is a random draw from $\nu$, independent of the rest. Define

$$\nu_i = \text{law of } \{X \mid X_{k_1}, \ldots, X_{k_i}\}$$
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Coordinate-by-coordinate localization

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$$
\nu_i = \text{law of } \{X \mid X_{k_1}, \ldots, X_{k_i}\}
$$

We claim that

In [Anari, Liu, Oveis Gharan ’20], $\eta$-spectrally independence for every conditional of $\nu$ is a condition to conserve variance along the coordinate-by-coordinate localization

$$
\left(1 - \frac{\eta}{n - i}\right) \mathbb{E}[\text{Var}_{\nu_i}(f)] \leq \mathbb{E}[\text{Var}_{\nu_{i+1}}(f)]
$$
Derivation: approximate conservation of variance
Similarly,

- Semi-log-concavity [Eldan, Shamir ’20]
- Fractional log-concavity [Alimohammadi, Anari, Shiragur, Vuong ’21]
- Entropic independence [Anari, Jain, Koehler, Pham, Vuong ’21]

which bounds covariance of all tilted measures,

are sufficient conditions to approximately conserve entropy
along the coordinate-by-coordinate localization
so that one could prove MLSI
Beyond coordinate-by-coordinate localization?
Let’s first take a tour \textbf{beyond the Boolean cube} to $\mathbb{R}^n$, where Eldan first introduced stochastic localization [Eldan ’13]
Eldan’s stochastic localization
Given an density $\nu$ on $\mathbb{R}^n$, the density at time $t$ is the solution of the SDE

$$d\nu_t(x) = (x - b(\nu_t))^{\top} C_t^{\frac{1}{2}} dW_t \cdot \nu_t(x), \quad \forall x \in \mathbb{R}^n$$

where $b(\nu_t)$ is the mean of $\nu_t$ and $W_t$ is the Brownian motion. Take $C_t = I_n$ to simplify explanation.
Explicit form of the random measure at time $t$

$\nu_t$ has an explicit form

$$
\nu_t(x) = \frac{1}{Z(c_t, t)} \exp \left( -\frac{t}{2} |x|^2 + c_t^T x \right) \nu(x)
$$

$$
dc_t = dW_t + b(\nu_t) dt
$$

At time $t$, the initial density is multiplied by a Gaussian with $1/t$ variance, while the center of the Gaussian is random.
Demonstration of Eldan’s stochastic localization in 2 dimension

Initialized with uniform distribution over a convex set \( n = 2 \)
Stochastic localization are used in high dimensional probability

Say we want to show a “property A” of the density $\nu$

- **Transform** via stochastic localization
- **Prove** “property A” for $\nu_t$ (usually easier)
- **Relate** “property A” of $\nu_t$ to that of $\nu$ (via SDE analysis)

See survey paper in 2022 ICM proceedings [Eldan], “property A” can be

- isoperimetric inequality (e.g. KLS conjecture [KLS ‘95])
- concentration of Lipschitz functions in Gaussian space
- noise stability inequality
- Poincaré inequality...
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Focus on sampling

1. The desired functional inequality is then our “property A”
2. Hopefully, this “property A” is easier to prove for the process at some time $t$
3. We put assumptions to make the approximate conservation of variance analysis go through
Use of localization schemes for sampling proofs

(a) $\mu_{\text{ini}} \xrightarrow{P} \mu_{\text{ini}} P \xrightarrow{p^{k-1}} \mu_{\text{ini}} p^{k} \xrightarrow{\text{Var}_\mu(f)} \approx \text{variance decay of } \mu_t$ 

(b) $\mu_t \xrightarrow{\mathbb{E}[\text{Var}_{\mu_t}(f)]} \supmarginale \geq \mathbb{E}[\mathcal{E}_P(f,f)]$
The probability measure on $\{-1, +1\}^n$ defined as

$$\mu(x) \propto \exp(\langle x, J x \rangle + \langle h, x \rangle)$$

is called Ising model with interaction matrix $J \in \mathbb{R}^{n \times n}$ and external field $h \in \mathbb{R}^n$. 
Theorem

Let $\nu_{\tau,v}(x) \propto \mu(x) \exp(-\tau \langle x, Jx \rangle + \langle v, x \rangle)$ if

$$\text{Cov}_{\nu_{\tau,v}} \preceq \alpha(\tau) I_n, \quad \forall \tau \in [0,1], \forall v$$

Then the MLSI constant of Glauber dynamics

$$\rho_{\text{MLSI}} \geq \frac{1}{n} \exp \left( -2 \|J\|_2 \int_0^1 \alpha(\tau) d\tau \right)$$
Glauber dynamics on Ising model

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For $J$ be a positive-definite matrix with $\|J\|_2 < \frac{1}{2}$ and $v \in \mathbb{R}^n$, adapting Bauerschmidt, Dagallier '22, we have

$$\|\text{Cov} (\nu_{\tau,v})\|_2 \leq \frac{1}{1 - 2(1 - \tau) \|J\|_2},$$

leading to $\rho_{\text{MLSI}} \geq \frac{1}{n} (1 - 2 \|J\|_2)$.
• The condition $\|J\|_2 \leq \frac{1}{2}$ is tight in general, as it is tight for Curie-Weiss model

• However, for the Sherrington-Kirkpatrick model, which assumes $J = \frac{\beta}{2}A$ where $A$ is drawn from GOE($n$). The above approach only gets fast mixing of Glauber dynamics for $\beta < \frac{1}{4}$, while the conjectured phase transition is at $\beta < 1$. 
What happens when we apply Eldan’s stochastic localization?

Take control matrix $C_t = (2J)$, for $t \in [0, 1], \nu_t(x) \propto \mu(x) \exp(-t \langle x, Jx \rangle + \langle c_t, x \rangle) \propto \exp((1 - t) \langle x, Jx \rangle + \langle h + c_t, x \rangle)$

where $c_t = C_t^2 dW_t + b(\nu_t)dt$. 
Take control matrix $C_t = (2J)$, for $t \in [0, 1]$,

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$$\propto \exp((1 - t) \langle x, Jx \rangle + \langle h + c_t, x \rangle)$$

where $c_t = C_t^2 dW_t + b(\nu_t)dt$.

At time $t = 1$, $\nu_t$ becomes a product measure (so easy to show MLSI).

Let’s take a look at the evolution of entropy
Evolution of entropy along Eldan’s SL

For \( f : \mathcal{X} \to \mathbb{R}_+ \)

\[
d\text{Ent}_{\nu_t}[f] = -\frac{1}{2} \mathbb{E}_{\nu_t}[f] \left| C_t^2 (b(\omega_t) - b(\nu_t)) \right|^2 dt + \text{martingale}
\]

where \( \omega_t \) is the probability measure \( \propto f \nu_t \).

Additionally, if \( \text{Cov}(\mathcal{T}_v \nu_t) \preceq A_t, \forall v \), then

\[
\frac{1}{2} \mathbb{E}_{\nu_t}[f] \left| C_t^2 (b(\omega_t) - b(\nu_t)) \right|^2 \leq \left\| C_t^2 A_t C_t^2 \right\|_2 \text{Ent}_{\nu_t}[f]
\]

Solving the equation, we obtain approximate conservation of entropy

\[
\mathbb{E}[\text{Ent}_{\nu_t}[f]] \geq e^{-2\|J\|_2} \int_0^t \alpha(\tau) d\tau \text{Ent}_{\nu_0}[f]
\]
Use of localization schemes for entropy decay

\[ \mu_{\text{ini}} \]
\[ P \]
\[ \mu_{\text{ini}} P \]
\[ p_{k-1} \]
\[ \mu_{\text{ini}} P^k \]
\[ \mu_{\text{ini}} P^k \]
\[ \mu_{\text{ini}} P^{k+1} \]
\[ \mathcal{E}_P(f, \log f) \]
\[ \mathbb{E}[\mathcal{E}_P(f, \log f)] \]

\[ \mu_t \]
\[ \mathbb{E}[\text{Ent}_{\mu_t}(f)] \]

localization

\approx

approximate conservation of entropy

\geq

entropy decay of \( \mu_t \)

supermarginale
Negative-fields localization
The hardcore model

Given a graph \( G = (V, E) \) with \(|V| = n\), a hardcore model with fugacity \( \lambda \) on \( \{-1, +1\}^n \) is

\[
\mu(\sigma) \propto \lambda^{|I_\sigma|},
\]

where \( \mu(\sigma) > 0 \) if the set \( I_\sigma = \{v \in V \mid \sigma_v = +1\} \) coorresponds to an independent set of \( G \).
Given a measure \( \nu \) on \( \{-1, 1\}^n \), the process \( \{\nu_t\}_{t\geq 0} \) evolves as

- For \( x \in \{-1, 1\}^n \), \( \nu_t \) solves the SDE

\[
d\nu_s(x) = \nu_s(x) \left\langle x - b(\nu_s), dJ_s \right\rangle,
\]

where

\[
dJ_{s,i} = -ds + \frac{1}{1 + b(\nu_s)_i} N_{s,i}
\]

where \( N_{s,i} \) is a Poisson point process with intensity \( 1 + b(\nu_s)_i \)

Inspired by field dynamics in Chen, Feng, Yin, and Zhang ’21
How does the measure $\nu_t$ look like?

- At time $t$, define $A_t = \{ i \in \{1, \ldots, n\} \mid N_{t,i} \geq 1 \}$. Since $N_{t,i}$ is non-decreasing, $A_t$ is an almost surely non-decreasing process of subsets of $\{1, \ldots, n\}$.
- We can write $\nu_t$ as

  $$\nu_t = \mathcal{T}_{-t1^\top} R_{A_t} \nu$$

  "$\nu_t$ is the density obtained by pinning all coordinates in $A_t$ to $+1$ and then tilt by $-t1^\top$"
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What is remaining?

- The mixing analysis on measures with large tilts are well-known in [Erbar, Henderson, Menz and Tetali ’17]
- We need to study the evolution of the process: this is where we use properties of the hardcore model to ensure approximate conservation of entropy.
Summary

• Introduced localization schemes to analyze mixing
• For each localization scheme,
  • we can study the evolution of variance (or entropy)
  • assumptions to ensure the approximate conservation of variance (or entropy) are usually the key assumptions
• Designing Localization schemes allows us to take advantage of our insights about target distributions
  • Recover results of spectral independence/fractional log-concavity
  • Optimal $O(n \log n)$ Glauber dynamics mixing bound for Ising models in the uniqueness regime under any external fields
  • $O(n \log n)$ Glauber dynamics mixing bound for the hardcore model in the tree-uniqueness regime
Thank you!
