Winding Hopfions on $\mathbb{R}^2 \times S^1$

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(Dated: June 20, 2013)

Abstract

We study Hopfions in the Faddeev-Skyrme model with potential terms on $\mathbb{R}^2 \times S^1$. Apart from the conventional Hopfions, there exist winding Hopfions, that is, the lump (baby Skyrmion) strings with the lump charge $Q$ with the $U(1)$ modulus twisted $P$ times along $S^1$, having the Hopf charge $PQ$. We consider two kinds of potential terms, that is, the potential linear in the field and the ferromagnetic potential with two easy axes, and present stable solutions numerically. We also point out that a Q-lump carries the unit Hopf charge per the period in $d = 2 + 1$.

An American liquorice (left) and a winding Hopfion (right).
I. INTRODUCTION

Knot structures are one of exotic field configurations with a non-trivial topology attracting considerable attentions in condensed matter physics, optics, fluid dynamics, and high energy physics. The Faddeev-Skyrme (FS) model \[1–3\] was proposed as a model admitting knot structures as stable topological solitons (Hopfions) associated with the Hopf number \(\pi_3(S^2) \simeq \mathbb{Z}\) [4]. It is an \(O(3)\) sigma model with a four derivative (Skyrme) term in \(d = 3 + 1\) dimensions. In the early trials, one constructed a Hopfion as a twisted closed lump string [2, 4]. A lump solution associated with \(\pi_2(S^2) \simeq \mathbb{Z}\) [5] is string-like in \(d = 3 + 1\). The lump has the size and phase moduli corresponding to the spontaneously broken scale invariance the \(U(1)\) rotation in the \(n_1-n_2\) plane in the target space, respectively. When the phase modulus is twisted along the closed lump string, it carries a Hopf number [2, 4]. Hopfions with higher charges were numerically studied, and, in particular, configurations with the Hopf charge seven or higher were in fact shown to exhibit knot structures [6–8]. Hopfions in the FS model with potential terms were also studied [10–15]. With the ferromagnetic potential term [11, 14, 15], Hopfions are toroidal domain walls rather than knots if the mass is sufficiently large [14, 15]. (Un)stable Hopfions are also studied in condensed matter systems [16] such as helium superfluids, exotic superconductors, ferromagnets, and Bose-Einstein condensates.

Other topological solitons have been studied in various geometries [17]. In particular, flat spaces with one or more directions compactified as circles are useful in T-duality between topological solitons in different dimensions. For instance, Yang-Mills instantons on \(\mathbb{R}^3 \times S^1\) and \(T^4\), known as Carolons, are related to monopoles, while lumps on \(\mathbb{R} \times S^1\) [18] and \(T^2\) [19] and vortices on \(\mathbb{R} \times S^1\) [20] and \(T^2\) [21] are related to domain walls. Hopfions on different geometry have not been studied much except for some outstanding works. First, Hopfions on \(S^3\) were considered on [22]. Hopfions on general three dimensional manifolds were first studied in Ref. [23]. Subsequently, numerical simulations for Hopfions on one compact direction \(\mathbb{R}^2 \times S^1\) and three compact directions \(T^3\) were performed in Ref. [24]. For instance, when spatial infinities are identified for geometry \(\mathbb{R}^2 \times S^1\) with one compact direction, one obtains a map from \(S^2 \times S^1\) to the target space \(S^2\). In this case, topology is rather complicated compared with conventional Hopfions on \(\mathbb{R}^3\) or \(S^3\) classified by \(\pi_3(S^2) \simeq \mathbb{Z}\); the map is classified by the lump charge \(Q\), characterized by \(\pi_2(S^2) \simeq \mathbb{Z}\), and the secondary
invariant $\mathbb{Z}_{2Q}$ [23, 24]. More physically, if we recall that a usual Hopfion can be made as
a twisted closed lump string, we consider a lump string with the lump charge $Q$ winding
around $S^1$ along which the phase modulus is twisted $P$ times. While such a winding Hopfion
configuration carries the Hopf charge $PQ$, neither of the twisting $P$ nor the Hopf charge $PQ$
is a homotopy invariant, but only $Q$ and $P$ modulo $2Q$ are homotopy invariants [23, 24]. In
Refs. [24, 25], winding Hopfions on $\mathbb{R}^2 \times S^1$ are shown to be dynamically unstable to become
tangled states, although the Hopf charge is conserved. More radically, the Hopf charge is
not conserved anymore for Hopfions on $T^3$ which were shown to be completely unstable
and to end up with states without zero Hopf charge. It has been shown in Ref. [26] that
a straight winding Hopfion spontaneously breaks the rotational symmetry and the helical
buckling instability appears depending on the compactification radius and the charge of the
winding Hopfion.

A similar topological structure has been experimentally studied in superfluid $^3$He-A in a
rotating container [27]. In this system, there are transverse soliton plane and vortices parallel
to the container intersecting the soliton plane. Vortices wind inside the soliton plane and
give a Hopf charge. The topological structure has been discussed by the torus homotopy of
the mapping: $S^1 \times S^2 \rightarrow S^2$ [28].

In this paper, we propose to introduce potential terms in order to stabilize winding
Hopfions on $\mathbb{R}^2 \times S^1$. The idea is that winding Hopfions are twisted lumps, and lumps are
unstable to expand in the presence of the Skyrme term without potentials. The size of the
lumps is fixed in the presence of the Skyrme and potential terms, in which case the lumps
are called baby Skyrmions in $d = 2 + 1$ [29, 30] and baby Skyrmion strings in $d = 3 + 1$
[31]. The two kinds of potential terms are often considered, that is, a potential term linear
in the field $V = m^2(1 - n_3)$ [10, 12, 29], sometimes called an old baby Skyrme potential,
and the ferromagnetic or Ising-type potential term quadratic in the field $V = m^2(1 - n_3^2)$
[11, 13, 14, 32, 33], sometimes called a new baby Skyrme potential. The potential term
$V = m^2(1 - n_3)$ admits the unique vacuum $n_3 = 1$, while the potential term $V = m^2(1 - n_3^2)$
admits two discrete vacua $n_3 = \pm 1$ and a domain wall solution interpolating these vacua
[11, 14, 30, 32, 33]. With the ferromagnetic potential, the baby Skyrmion is in a ring shape
in $d = 2 + 1$ [30, 34] and in a tube shape in $d = 3 + 1$. For these potentials, a baby Skyrmion
string still has a $U(1)$ phase modulus while the size is fixed. This mode can be interpreted
as a Nambu-Goldstone mode which is associated with the $U(1)$ symmetry rotating $n_1$ and
$n_2$ spontaneously broken in the presence of the string. This mode is localized in the core of string and propagates along the string. If we consider a phase kink of the $U(1)$ modulus, it is unstable against the expansion from the Derrick’s scaling argument [35]. It is then diluted along the string and eventually disappears, if the string extends to infinity. However, if the string winds around $S^1$, the length of the string is finite, and the expansion of the phase kink stops at the size of $S^1$, resulting in a winding Hopfion. This configuration is similar to twisted strings [36].

We numerically give stable solutions of $(P,Q)$ winding Hopfions, that is, lumps (baby Skyrmions) with the lump charge $Q$ winding around $S^1$ where the $U(1)$ modulus is twisted $P$ times. We first consider the linear potential $V = m^2(1 - n_3)$. We then consider the potential of the anti-ferromagnets with two easy axes, given by $V = m^2(1 - n_3^2) + \beta^2 n_1$ in the regime $\beta \ll m$, where the second term explicitly breaks the $U(1)$ symmetry rotating in the $n_1$-$n_2$ plane in the target space [34, 37–39]. This deformation is not necessary for the stability of solutions, but it adds some interesting feature; in $d = 2 + 1$, there appear sine-Gordon kinks on a domain wall ring as a lump by this deformation [34]. One unexpected feature is that the lump solution still posses a $U(1)$ modulus although the term $\beta^2 n_1$ explicitly breaks the $U(1)$ symmetry in the $n_1$-$n_2$ plane. This $U(1)$ modulus is a Nambu-Goldstone mode associated with the spontaneously broken $U(1)$ rotation in the $x^1$-$x^2$ plane in real space. In fact, this solution is non-axisymmetric and it spontaneously breaks the rotation of the $x^1$-$x^2$ plane in real space. In $d = 3 + 1$, sine-Gordon kinks become kink strings on a domain wall tube. In our Hopfions winding around $S^1$, these kink strings wrap the domain wall tube along $S^1$, constituting a braid. This configuration looks like an American liquorice.

This paper is organized as follows. In Sec. II, after our model is explained, we construct baby Skyrmions with the two kinds of the potentials in $d = 2 + 1$ dimensions (they are twisted domain wall rings for $V_2$). They can be linearly extended to baby Skyrmion strings (domain wall tubes for $V_2$) in $d = 3 + 1$. In Sec. III we construct winding Hopfions for the both kinds of the potentials. In Sec. IV we point out that a Q-lump carries the unit Hopf charge per a unit period in $d = 2 + 1$ space-time, as a Hopf instanton. Section V is devoted to summary and discussions.
II. THE BABY SKYRME MODELS AND SOLUTIONS

We consider the Faddeev-Skyrme model with potential terms. Let \( n(x) = (n_1(x), n_2(x), n_3(x)) \) be a unit three vector of scalar fields with a constraint \( n \cdot n = 1 \). The Lagrangian of our model is given by (\( \mu = 0, 1, 2, 3 \))

\[
\mathcal{L} = \frac{1}{2} \partial_\mu n \cdot \partial^\mu n - \mathcal{L}_4(n) - V(n),
\]

with the four-derivative (Faddeev-Skyrme) term

\[
\mathcal{L}_4(n) = \kappa F^2_{\mu\nu} = \kappa [n \cdot (\partial_\mu n \times \partial_\nu n)]^2 = \kappa (\partial_\mu n \times \partial_\nu n)^2
\]

\[
F_{\mu\nu} = n \cdot (\partial_\mu n \times \partial_\nu n).
\]

We consider either the conventional linear potential \( [10, 12, 29] \)

\[
V_1(n) = m^2(1 - n_3),
\]

or the ferromagnetic potential term with two easy axes \( [34, 37, 39] \)

\[
V_2(n) = m^2(1 - n_3^2) + \beta^2 n_1, \quad \beta \ll m.
\]

The potential \( V_1 \) admits the unique vacuum \( n_3 = 1 \), and the potential \( V_2 \) admits the two discrete vacua \( (n_1, n_2, n_3) = (-\beta^2/2m, 0, \pm \sqrt{1 - (\beta^2/2m)^2}) \), reducing to \( n_3 = \pm 1 \) for \( \beta = 0 \).

The energy density of static configurations is

\[
\mathcal{E} = \frac{1}{2} (\partial_a n \cdot \partial^a n) + \mathcal{L}_4(n) + V_{1,2}(n)
\]

with \( a = 1, 2, 3 \).

Let us first consider the linear potential \( V_1 \) followed by the ferromagnetic potential \( V_2 \). In \( d = 2 + 1 \) dimensions, the static baby Skyrmion solution for the linear potential has been discussed in \( [29] \). In order to compare the twisted string later, we numerically construct baby Skyrmion solutions with \( Q = 1, 2, 3 \), where \( Q \) is the topological lump charge of \( \pi_2(S^2) \approx \mathbb{Z} \),
FIG. 1: The $r = \sqrt{(x^1)^2 + (x^2)^2}$ dependence of (a) $n_3$ and (b) $\mathcal{E}$ of baby Skyrmions for the linear potential $V_1$ in Eq. (4). We fix $\kappa = 0.002$ and $m^2 = 80$.

given by

$$Q = \frac{1}{4\pi} \int d^2 x \, F_{12} = \frac{1}{4\pi} \int d^2 x \, n \cdot (\partial_1 n \times \partial_2 n) = \frac{1}{4\pi} \int d^2 x \, \epsilon_{ijk} n_i \partial_1 n_j \partial_2 n_k, \quad (7)$$

and plot the profiles of $n_3$ and $\mathcal{E}$ in Fig. 1. Since the distributions of $\mathcal{E}$ and $n_3$ are isotropic in the $\theta = \tan^{-1}(x^2/x^1)$-direction, we just plot the $r = \sqrt{(x^1)^2 + (x^2)^2}$-dependence of the both values. The massive region of $n_3 \sim -1$ localizes at the center for all $Q$. On the other hand, the energy density $\mathcal{E}$ has a hole of a small $\mathcal{E}$ region for higher $Q$. The configuration is axisymmetric in the sense that the solutions is invariant under a combination of the rotation of $n_1$-$n_2$ in the target space and the rotation in the $x_1$-$x_2$ plane in the real space, while the profile $n_3$ and the energy density $\mathcal{E}$ are invariant under either of these two transformations.

The orthogonal combination of these $U(1)$ symmetries is spontaneously broken, resulting in the presence of a $U(1)$ Nambu-Goldstone mode.

Next, we consider the ferromagnetic potential $V_2$. The potential $V_2$ with $\beta = 0$ admits the two discrete vacua $n_3 = \pm 1$ and a domain wall solution interpolating between them \[32, 33, 39\]

$$\theta(x^1) = 2 \arctan \exp \left\{ \pm \sqrt{2} m (x^1 - X) \right\}, \quad 0 \leq \theta \leq \pi,$$
FIG. 2: A twisted domain wall ring as a baby Skyrmion for the ferromagnetic potential $V_2$ in Eq. (5) with $\beta = 0$. (a) The textures $n(x)$. The color of each arrow shows the value of $n_3$. (b) The total energy density $\mathcal{E}$. (c) The topological lump charge density: $c \equiv \{n \cdot (\partial_1 n \times \partial_2 n)\}/(4\pi)$. The topological charges are $Q = 1, 2, 3$ from left to right. We fix $\kappa = 0.02$ and $m^2 = 20000$, and plot the values in the region $-0.29 \leq x_a \leq 0.29$.

\[
n_1 = \cos \alpha \sin \theta(x^1), \quad n_2 = \sin \alpha \sin \theta(x^1), \quad n_3 = \cos \theta(x^1),
\]

with a phase modulus $\alpha$ ($0 \leq \alpha < 2\pi$) and the translational modulus $X \in \mathbf{R}$ of the domain wall. The phase modulus is a Nambu-Goldstone mode associated with the spontaneously broken $U(1)$ symmetry in the $n_1$-$n_2$ plane in the target space. For $\beta \neq 0$, the vacua are shifted $(n_1, n_2, n_3) = (-\beta^2/2m, 0, \pm \sqrt{1 - (\beta^2/2m)^2})$, and the domain wall solutions is also modified for which $\alpha$ is fixed to be $\pi$. 

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FIG. 3: A twisted domain wall ring for the ferromagnetic potential $V_2$ in Eq. (5) with nonzero $\beta$. (a) The textures $n(x)$. The color of each arrow shows the value of $n_3$. (b) The energies $E_2$. (c) The total energies $E$. (d) The topological charge densities $c$. The topological charges are $Q = 1, 2, 3$ from left to right. We fix $\kappa = 0.02$, $m^2 = 20000$ and $\beta^2 = 2000$, and plot the values in the region $-0.29 \leq x_a \leq 0.29$.

In $d = 2 + 1$ dimensions, a baby Skyrmion can be interpreted as a closed domain line for $V_2$, where the $U(1)$ modulus $\alpha$ winds $Q$ times along the wall ring [34, 40]. It carries the topological lump charge $Q$ defined in Eq. (7) [40]. For $\beta = 0$, a baby Skyrmion is a twisted domain wall ring in Fig. 2. As the case of the linear potential, the configuration is axisymmetric in the sense that the solutions is invariant under a combination of the rotation
of $n_1-n_2$ in the target space and the rotation in the $x_1-x_2$ plane in the real space. The Nambu-Goldstone mode appears as a result of the spontaneously broken $U(1)$ symmetry of the other combination of these two $U(1)$ symmetries.

In the presence of nonzero $\beta$, the phase gradient along the domain wall ring is localized to become sine-Gordon kinks, and the baby Skyrmion looks like jewels on a domain wall ring \cite{34}, as shown in Fig. 3. The configurations are non-axisymmetric. The term $\beta^2 n_1$ explicitly breaks the $U(1)$ symmetry in the $n_1-n_2$ plane in the target space, unlike the cases of $V_1$ and $V_2$ with $\beta = 0$. Even in the absence of the internal $U(1)$ symmetry, the configurations have a $U(1)$ modulus, which is a Nambu-Goldstone mode associated with the spontaneously broken rotational symmetry in the $x_1-x_2$ plane in real space.

For either of $V_1$ or $V_2$ or no potentials, one can consider the Hopf charge of $\pi_3(S^2) \simeq \mathbb{Z}$ in $d = 3 + 1$, defined by

$$ C = \frac{1}{4\pi^2} \int d^3 x \epsilon^{\mu\nu\rho} F_{\mu\nu} A_{\rho}, \quad (9) $$

with a “gauge field” $A_{\mu}$ for the field strength in Eq. (3) satisfying $\partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}$ \cite{4}. Hopfions carry this charge. A preimage of a point of the $S^2$ target space is a closed line in real three dimensional space with a suitable identification at spatial infinity. When two preimages of two arbitrary points on the target space make a link, there is a Hopf charge. The linking number count the Hopf charge.

\section*{III. WINDING HOPFIONS}

In $d = 2 + 1$, baby Skyrmions have been constructed for the linear potential $V_1$ in Fig. 1 and for the ferromagnetic potential with $\beta = 0$ in Fig. 2 and with $\beta \neq 0$ in Fig. 3, as twisted domain wall rings. In $d = 3 + 1$, the baby Skyrmion is linearly extended to one direction to become a baby Skyrmion string, which is a tube or a cylinder for $V_2$. We place the straight string along the $z$-direction. We now compactify the $z$-direction to $S^1$ so that the string winds around $S^1$.

The baby Skyrmion has the $U(1)$ modulus corresponding to the rotation around the $n_3$ axis in the $S^2$ target space. Let us rotate the $U(1)$ modulus clockwise along the string as in Fig. 4. We show the examples with the lump charges $Q = 1$ and 2 in Fig. 4 (a) and (b),
FIG. 4: Twisted baby Skyrmion string. The gray regions are the $n_3 = 0$ surfaces, which represent, for $V_2$, domain wall tubes that separate the two vacua $n = (0, 0, 1)$ denoted by $\circ$ and $n = (0, 0, -1)$ denoted by $\otimes$. Along the $n_3 = 0$ surface, there are sequences of baby Skyrmions with the charges of (a) $Q = 1$ and (b) $Q = 2$. From the bottom to the top, $n$ rotates by $2\pi P$ ($P = 1$) in the $n_1-n_2$ plane, defining the number of twists $P$. The yellow and green lines indicate the locus of the $n_1 = 1$ and $n_1 = -1$ states along the tubes, respectively. The linking numbers are 1 and 2 for the left and right figures, respectively.

respectively.

The spatial infinities in transverse directions $r = \sqrt{x^2 + y^2} \to \infty$ are identified (one point compactification). Namely we consider

$$\mathbb{R}^3(x, y, z) \Rightarrow S^1(z) \times S^2(x, y),$$

instead of $S^3$ for the usual boundary condition without strings.

With introducing two complex scalar fields $\phi^T = (\phi_1, \phi_2)$ satisfying $|\phi_1|^2 + |\phi_2|^2 = 1$, the
three-vector scalar fields $n_i$ can be written by the Hopf map

$$n_i = \phi^\dagger \sigma_i \phi$$  \hspace{1cm} (11)$$

by using the Pauli matrices $\sigma_i$. Note that $\phi$ parametrizes $S^3 \simeq SU(2)$ because of the constraint. As an initial configuration, we consider an ansatz

$$\phi = \begin{pmatrix}
e^{iPg(z)} \cos\{q_r f(r)\} \\
e^{-iq\theta} \sin\{q_r f_r(r)\}
\end{pmatrix}, \hspace{1cm} 0 \leq r \leq \infty, \hspace{0.5cm} 0 \leq \theta \leq 2\pi, \hspace{0.5cm} 0 \leq z < L_z,  \hspace{1cm} (12)$$

with the integers $P, q_r, q_\theta \in \mathbb{Z}$ and the cylindrical coordinates $(r, \theta, z)$. The monotonically decreasing function $f(r)$ satisfies

$$f(r \to 0) \to \frac{\pi}{2}, \hspace{0.5cm} f(r \to \infty) \to 0,  \hspace{1cm} (13)$$

and the monotonically increasing function $g(r)$ satisfies

$$g(r \to 0) \to 0, \hspace{0.5cm} f(r \to L_z) \to 2\pi.  \hspace{1cm} (14)$$

From the Hopf map in Eq. (11), we have

$$n_1 = \cos\{Pg(z) - q_\theta \theta\} \sin\{2q_r f(r)\},$$

$$n_2 = -\sin\{Pg(z) - q_\theta \theta\} \sin\{2q_r f(r)\}, \hspace{0.5cm} n_3 = \cos\{2q_r f(r)\}.  \hspace{1cm} (15)$$

Let us first calculate the lump charge $Q$ at each slice of $z = \text{constant}$. We obtain

$$Q = \frac{1}{4\pi} \int d^2x \epsilon_{ijk} n_i \partial_x n_j \partial_y n_k = -\frac{q_r q_\theta}{2\pi} \int_{0}^{\infty} dr \int_{0}^{2\pi} d\theta \sin\{2q_r f(r)\} f'(r)$$

$$= \frac{q_\theta \{1 - (-1)^{q_r}\}}{2}.  \hspace{1cm} (16)$$

From this, we see that $q_r$ is not needed. We may take $q_r = 1$ and $Q = q_\theta$, but we leave $q_r$ general to the end of calculation.

We now calculate the Hopf number $\pi_3(S^2) \simeq \mathbb{Z}$ for $n_i$ parameterizing $S^2$ as the Skyrme charge (baryon number) $\pi_3(S^3) \simeq \mathbb{Z}$ for $\phi$ parameterizing $S^3$. To this end, real scalar fields
defined by $\phi_1 = m_1 + im_2$, $\phi_2 = m_3 + im_4$ can be read

$$
\begin{align*}
    m_1 &= \cos\{q_r f(r)\} \cos\{P g(z)\}, \\
    m_2 &= \cos\{q_r f(r)\} \sin\{P g(z)\}, \\
    m_3 &= \sin\{q_r f(r)\} \cos(q_\theta \theta), \\
    m_4 &= -\sin\{q_r f(r)\} \sin(q_\theta \theta). 
\end{align*}
$$

(17)

Since the complex scalar fields $\phi$ parametrize $S^3 \simeq SU(2)$, the Skyrmion charge (the baryon number) in $\pi_3(S^3)$ can be calculated from $\phi$ as

$$
C = \frac{1}{2\pi^2} \int d^3 x \epsilon_{stu} m_s \partial_1 m_t \partial_2 m_u \partial_3 m_v \\
= -\frac{Pq_r q_\theta}{4\pi^2} \int_0^\infty dr \int_0^{2\pi} d\theta \int_0^{L_z} dz \sin(2q_r f(r)) f'(r) g'(z) \\
= \frac{Pq_\theta (1 - (-1)^{q_\theta})}{2} = PQ.
$$

(18)

This precisely yields the Hopf charge in Eq. (9) through the Hopf map in Eq. (11).

The Hopf charge can be obtained from the linking number of the preimages of two generic points, as shown in Fig. 4.

To obtain numerical solutions, we use the relaxation method starting from the ansatz given in Eq. (12) with $q_r = 1$ as the initial configurations. During the calculation,
FIG. 6: Winding Hopfions with the ferromagnetic potential $V_2$ in Eq. (5). The green and blue surfaces in top panels are the isosurface of $n_3 = 0$ and $n_1 = -0.97$, respectively. The bottom panels show $E$ at the $z = L_z/2$ plane and the isosurface of 95% of the maximum of $E$. The topological charges are (a) $(P,Q) = (1,1)$, (b) $(P,Q) = (2,1)$, (c) $(P,Q) = (1,2)$, and (d) $(P,Q) = (2,2)$. We fix $\kappa = 0.002$, $m^2 = 800$, $\beta^2 = 80$, and $L_z = 1$, and plot the values in the region $-0.29 \leq x_a \leq 0.29$.

The topological charges do not change. Here, we give numerical solutions for $(P,Q) = (1,1), (1,2), (2,1)$ and $(2,2)$ with the two potentials $V_1$ in Eq. (4) and $V_2$ in Eq. (5). The solutions are shown in Fig. 5 for the linear potential $V_1$. Since the distributions of $E$ and $n_3$ are isotropic in the $\theta$-direction and uniform in the $z$-direction, we just plot the $r$-dependence of the both values with fixed $z$. Compared to the baby Skyrmion in $d = 2 + 1$, which corresponds to an unwinding Hopfion (untwisted string) with $P = 0$, the massive region of $n_3 \sim -1$ localizes more with $P$ and $E$ becomes larger because $\partial_3 n_3$ becomes finite.

Our numerical solutions are shown in Fig. 6 for the ferromagnetic potential $V_2$ in Eq. (5). There are domain wall tubes separating the two vacua $n_3 = \pm 1$, along which sine-Gordon kink strings localized around $n_1 \sim -1$ wind. The number of sine-Gordon kink strings is $Q$.
FIG. 7: The two preimages of $n_1 = 0.97$ (red surface) and $n_1 = -0.97$ (blue surface) of winding Hopfions with the linear potential $V_1$ for Fig. 5 (the upper panels) and the ferromagnetic potential $V_2$ for Fig. 6 (the lower panels). The topological charges are (a) $(P,Q) = (1,1)$, (b) $(P,Q) = (2,1)$, (c) $(P,Q) = (1,2)$, and (d) $(P,Q) = (2,2)$. Linking numbers of two preimages are 1, 2, 2, and 4 for (a), (b), (c), and (d) respectively for both potentials.

at each slice of constant $z$. They constitute a braid structure by rotating $2\pi P/Q$ along the $z$-axis with connected to themselves or others depending on the lump charge $Q$. Domain wall tube slightly buckles to the opposite direction of the sine-Gordon kink ($n_1 = -1$), because the sine-Gordon kinks pushes the domain wall tube. The curvature of the domain wall tube becomes larger as larger $P$ or smaller $Q$. This buckling of the domain wall tube is a consequence of the explicit breaking of the rotational symmetry in the $n^1-n^2$ plane due to the term $\beta^2 n_1$ in the potential $V_2$. On the other hand, we have not seen that the lump string buckles for the linear potential $V_1$ for our choice of the size $L_z$ of $S^1$, but we expect the buckling instability for a smaller $L_z$ even in the absence of the linear term breaking the rotational symmetry in the $n^1-n^2$ plane, as the case of Ref. [26]. The energy density distributions also exhibit braid structures as those of sine-Gordon kink strings. The configurations are spatially twisted because of non-axisymmetry of baby Skyrmions, which
is a unique feature of the ferromagnetic potential with two easy axes and is absent for the linear potential $V_1$ and $V_2$ with $\beta = 0$. They look like American Liquorices.

In Fig. 7, we plot the two preimages of $n_1 = \pm 1$ (red and blue curves, respectively) for our solutions of winding Hopfions with both linear and ferromagnetic potentials. Two preimages make a braid structure and its linking number is consistent with the Hopf number $PQ$, being independent of two types of potential.

Now we make comments on the stability issue of these configurations. Unlike the case without the potential term [24, 25], our solutions are static and stable. Since they are converged solutions in the relaxation method, they are at least at local minima of the configuration space. Since we have potentials, constituent lumps are baby Skyrmion strings which are stable against the expansion. Let us move to the stability of the twisting. The low-energy effective theory of the $U(1)$ phase modulus $\alpha$ of the lump string can be constructed by the moduli approximation [41, 42]; one promotes the moduli to fields and integrates the original Lagrangian over the codimension of solitons. We thus obtain a free $U(1)$ theory of $\alpha(t, z)$, a sigma model with target space $S^1$. Therefore the phase kink is unstable to expand without a potential term along a straight string from the Derrick’s scaling argument [35]. However, when the string winds around $S^1$ as in our case, the expansion stops once the kink becomes the size of $S^1$ [43].

IV. HOPF INSTANTONS AND Q-LUMPS

Hopfions as instantons were studied in $d = 2 + 1$ space-time [44]. In order to discuss an analogue of these, we identify the vertical direction in Figs. 4, 6 and 7 as the time direction in $d = 2 + 1$ dimensions. Then, in our model in $d = 2 + 1$, Hopf instantons can appear locally in time along a lump world-line. In Ref. [44], they studied a pair creation of a lump and an anti-lump and subsequent pair annihilation after twisting, corresponding to a conventional (isolated) Hopfion in Euclidean space, and discussed a braiding statistics. In our case, we have a braiding of two identical particles for $Q = 2$ as in the right two figures in Fig. 6, and, even for $Q = 1$, two points are braided as can be seen in Fig. 7.

If the configuration is periodic in time in the model with $\beta = 0$, it is nothing but a Q-lump. The Q-lump is a lump [5] with the phase modulus depending linearly on time which
is a BPS state and is stable [45]. For instance, one lump solution is

\[ u \equiv \frac{n_1 + in_2}{1 - n_3} = c(w - w_0)\exp(itm) \]  

(19)

with \( w \equiv x^1 + ix^2 \). In the time interval with the period \( m^{-1} \), the phase rotates once. The configuration in this period has the unit Hopf charge and may be called the Hopf instanton. In this case, the configuration is periodic and the Hopf instanton is not localized. On the other hand, if we turn on \( \beta \) in the potential, the Hopf instantons still exist as braidings mentioned above.

V. SUMMARY AND DISCUSSION

We have studied winding Hopfions in the Faddeev-Skyrme model on \( \mathbb{R}^2 \times S^1 \), with the two kinds of the potential terms, that is, the linear potential \( V_1 = m^2(1 - n_3) \) and the anti-ferromagnetic potential with two easy axes \( V_2 = m^2(1 - n_2^2) + \beta^2 n_1 \). The winding Hopfions are the lump (baby Skyrmion) strings with the lump charge \( Q \) with the \( U(1) \) modulus twisted \( P \) times along the cycle \( S^1 \), and they carry the Hopf charge \( PQ \). We have presented stable numerical solutions with \( P, Q = 1, 2 \) for the both type of the potentials. The configurations are axisymmetric for the linear potential, and we have given the profile function of \( n_3 \). The configurations are non-axisymmetric for the anti-ferromagnetic potential with two easy axes.

We briefly discussed that Q-lumps in \( d = 2 + 1 \) carry the Hopf charge density per unit period and can be understood as Hopf instantons.

A winding \((P,Q)\) Hopfion carries the Hopf charge \( PQ \), but the Hopf charge is not a homotopy invariant in this geometry. The lump charge \( Q \) is a homotopy invariant and \( P \) is a homotopy invariant modulo \( 2Q \) [23, 24]. Therefore, for instance, a configuration with \((P,Q) = (2,1)\) can be continuously deformed into a configuration with \((P,Q) = (0,1)\). However, our solution with \((P,Q) = (2,1)\) is at least at local minimum, implying a potential barrier to \((P,Q) = (0,1)\).

If one makes a closed lump string with a twisted \( U(1) \) phase, one obtains an isolated Hopfion [1]. Such Hopfions have been studied recently for \( V_2 \) with \( m \neq 0, \beta = 0 \) [14] and \( m, \beta \neq 0 \) [15]. The interaction between an isolated Hopfion and a winding Hopfion is an important future work. A winding Hopfion may absorb or release isolated Hopfions with
changing its Hopf charge.

The Faddeev-Skyrme model was proposed as low-energy effective theory of pure $SU(2)$ Yang-Mills theory, where Hopfions are suggested to describe glueballs [46]. Our result implies that $SU(2)$ Yang-Mills theory on $\mathbb{R}^3 \times S^1$ may contain gluon string winding around the cycle $S^1$.

This model admits a variety of solitons; in addition to a domain wall, lump (baby Skyrmion) strings and Hopfions [14] as elementary solitons in dimensions one, two and three, respectively, it also admits a composite soliton of lump strings ending on a domain wall (a D-brane soliton) [47–49]. The geometry $\mathbb{R}^2 \times S^1$ is useful to study duality between these solitons. In particular, a domain wall and a lump (baby Skyrmion) is T-dual to each other [18, 20]. It will be interesting to investigate what is a dual object of a Hopfion.

Acknowledgements

We thank the organizers of the conference “Quantized Flux in Tightly Knotted and Linked Systems,” held in 3 - 7 December 2012 at Isaac Newton Institute for Mathematical Sciences, where this work was initiated. We would like to thank Paul Sutcliffe for useful comments. This work is supported in part by Grant-in-Aid for Scientific Research (Grant No. 22740219 (M.K.) and No. 23740198 and No. 25400268 (M.N.)) and the work of M. N. is also supported in part by the “Topological Quantum Phenomena” Grant-in-Aid for Scientific Research on Innovative Areas (No. 23103515 and No. 25103720) from the Ministry of Education, Culture, Sports, Science and Technology (MEXT) of Japan.

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We need a potential term to stabilize the phase kinks along a straight string by breaking the $U(1)$ symmetry explicitly. One may think that this can be achieved by turning on $\beta$ in the potential which induces the sine-Gordon potential on $\alpha(t, z)$ in the effective theory and that the phase kink becomes a sine-Gordon kink with the size $\beta^{-1}$. However, this does not work since the term with $\beta$ does not induce the potential on the $U(1)$ modulus unexpectedly, because the $U(1)$ Nambu-Goldstone zero mode is shifted to a Nambu-Goldstone zero mode corresponding to the rotational symmetry in the $x$-$y$ plane in real space. On the other hand, the same technique does work to construct a sine-Gordon kink on a domain wall, which is a lump in the bulk point of view [34, 37, 38].

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