Detection of Gaussianity using envelopes

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Abstract
Here we introduce a new approach for Gaussianity testing using the envelope of a signal. The coefficient of variation of the envelope (CVE) of zero-mean Gaussian noise is a universal constant equal to $m = \sqrt{(4 - \pi)/\pi} \approx 0.523$. Thus, for any signal the result $\text{CVE} = m$ can be used as a fingerprint to assess Gaussianity. Interestingly, the CVE is also unique for uniform noise time series or low density Filtered Poisson Processes. Here we summarize the mathematics and computer methods behind using the CVE for Gaussianity testing in time series. In particular, we describe how to perform Gaussianity testing in a step-by-step fashion for experimental data using the Hilbert transform, showing that the sampling rate as well as the duration and the filtering of the data stream affect the analysis. Additionally, through the use of the Fourier transform phase randomization, we reveal the interconnections among CVE, Gaussianity and temporal modulation profiles. Furthermore, we use the CVE to assess the degree of synchronization in Kuramoto and Matthews-Mirollo-Strogatz models and show that CVE is relevant to the study of coupled oscillators systems. CVE Gaussianity testing provides a new tool for signal classification.

Keywords: Gaussianity, Envelope, Rayleigh Fading, Coupled Oscillators

1 Introduction
Assessing Gaussianity is a classic problem in the field of signal analysis, and a variety of methods have been devised to infer it from time series. These methods span ideas from both the time and frequency domains. While time domain approaches rely on goodness-of-fit tests (e.g. Jarque–Bera, Kolmogorov-Smirnov) [1, 2], frequency domain methods have been developed based on bispectrums [3]. In general, these two classes of methods present different drawbacks and in practice are difficult to use to settle the question of Gaussianity in time series. Here we show a different approach by exploiting a singular property of Gaussian signals: the envelope of zero-mean Gaussian noise is distributed according to the Rayleigh distribution — a distribution with the special property that the ratio between its standard deviation ($\sigma$) and mean ($\mu$), known as the coefficient of variation ($CV = \sigma/\mu$), is constant and equal to $\sqrt{(4 - \pi)/\pi} \approx 0.523$ [4]. Thus, when the envelope of a given signal is considered as a random variable, its CV (called the coefficient of variation of the envelope, CVE), can be used as a fingerprint for Gaussianity. Here we show the basic mathematics needed to understand envelopes and their coefficient of variation (CVE). Then, we exemplify how the CVE can be used to infer the Gaussianity of discrete-time signals. Finally, we show how to assess synchronization in both Kuromoto and Matthews-Mirollo-Strogatz models of coupled oscillators using CVE.
2 Analytic representation and envelope function

We begin our exposition with the basic theory of the analytic representation of signals followed by the derivation of some properties of the envelope function. Let \( s(t) \) be a \( L^2 \) continuous time function (i.e. a continuous function with a finite number of discontinuities) and \( \hat{s}(f) \) its (complex) Fourier transform (FT). If one chooses to consider only non-negative frequencies \( f \geq 0 \), it is possible to represent \( s(t) \) as the complex function \( s_a(t) \), known as the analytic signal, creating an implicit de-facto relationship between the two functions as demonstrated in [5]:

\[
s(t) = \text{Real}(s_a(t)) = A(t) \cdot \cos(\phi(t))
\]

In this polar representation, \( s(t) \) can be written as an amplitude modulation (AM) signal, where \( A(t) \) is a non-negative function called the instantaneous amplitude or envelope and \( \phi(t) \) is called the instantaneous phase. Functions \( s_a(t), A(t), \) and \( \phi(t) \) can be obtained analytically by using the Hilbert transform [6] (HT), which is defined as the convolution between the real-valued function \( s(t) \) and the kernel \( \frac{1}{\pi t} \):

\[
\mathcal{H}\{s(t)\} = s(t) * \frac{1}{\pi t} = \frac{1}{\pi} \text{P.V.} \int_{-\infty}^{\infty} \frac{s(\tau)}{t-\tau} d\tau
\]

where the integral must be interpreted in the principal value (P.V.) sense to overcome the singularity at \( t = \tau \) [7]. The Fourier transform of the Hilbert kernel as shown in [6] is

\[
\mathcal{F}\{1/\pi t\}(f) = -i \cdot \text{sgn}(f) = \begin{cases} 
-i & \text{for } f > 0 \\
0 & \text{for } f = 0 \\
i & \text{for } f < 0 
\end{cases}
\]

It can be seen that \( \mathcal{H}\{s(t)\} \) and \( s(t) \) are in quadrature. The analytic signal \( s_a(t) \) can be expressed using the original signal and its Hilbert Transform, in an idea explained first by Dénes Gabor in 1946 (page 433) [8]:

\[
s_a(t) = s(t) + i \mathcal{H}\{s(t)\}
\]

By definition, an analytic signal has no negative frequency components, unlike real-valued signals whose Fourier Transform are Hermitian functions. Thus, the analytic representation of \( s(t) \) is defined in the frequency domain as in [9]:

\[
\hat{s}_a(f) = \begin{cases} 
2\hat{s}(f) & \text{for } f > 0 \\
\hat{s}(f) & \text{for } f = 0 \\
0 & \text{for } f < 0 
\end{cases}
\]

This mathematical fact (Eq. 5) allows \( s(t) \) to be represented as a phasor of variable amplitude, whose envelope function \( A(t) \) can be calculated as the norm of the analytic signal:

\[
A(t) = \sqrt{s^2(t) + \mathcal{H}\{s(t)\}^2}
\]

The envelope is a positive, bandlimited, slow-varying signal, and according to Parseval’s theorem, the energy of the signal \( s(t) \) and the energy of its envelope \( A(t) \) are proportional [6]:

\[
\int_{-\infty}^{\infty} s^2(t) dt = \frac{1}{2} \int_{-\infty}^{\infty} A^2(t) dt
\]

But more importantly, the envelope \( A(t) \) can be treated as a time series in itself and the ratio between its standard deviation, and its mean (the statistics known as Coefficient of Variation, CV) can be used to assess Gaussianity (Figure 1).

2
1. **Signal, envelope and coefficient of variation of the envelope (CVE).** The envelope $A(t)$ of signal $s(t)$ is its instantaneous amplitude and is calculated using the Hilbert transform as $A(t) = \sqrt{s^2(t) + H\{s(t)\}^2}$. Notice how the envelope follows all local maxima of $s(t)$ and is always non-negative. The envelope is always positive and has a mean $\mu_A$ and a standard deviation $\sigma_A$ (right inset). These two parameters characterize the amplitude modulation pattern of $s(t)$ through the Coefficient of Variation of the Envelope: $CVE = \frac{\sigma_A}{\mu_A}$.

2. **Envelopes and CVE for Gaussian Noise**

A well-known result from telecommunication theory states that a multipath propagation channel with no direct line of sight exhibits Rayleigh fading, and thus its envelope follows a Rayleigh distribution [10]:

$$x \alpha^2 \cdot e^{-x^2/(2\alpha^2)}$$

The core of the demonstration is that if $X$ and $Y$ are two zero-mean Gaussian-distributed random variables, then the random variable defined by $Z = \sqrt{X^2 + Y^2}$ follows a Rayleigh distribution. This probability distribution has a very special peculiarity, as its coefficient of variation ($CV = \sigma/\mu$) is the constant $m$ (Eq. 9) [4]:

$$m = \sqrt{(4 - \pi)/\pi} \approx 0.523$$

Thus, if a signal is Gaussian noise, its envelope should have a CVE equal to $m$ independently of its power, as the CVE is a scale-independent parameter. It is important to underline that $CVE = m$ is valid for wide-band Gaussian noise of arbitrary amplitude, as well as for any filtered sub-band [10]. Thus, the invariance of $m$ for filtered/unfiltered Gaussian noise of any amplitude is the core mathematical result needed to test the Gaussianity of times series.

3. **CVE for sampled Gaussian noise**

Experimentally obtained signals differ from the infinite and continuous signals used in theoretical analysis. Experimental traces are finite in duration, band-limited and sampled by the data acquisition hardware at various bit resolutions (8/12/16/24). How do these unavoidable technical restrictions manifest themselves when calculating the CVE of sampled signals? To answer this question, we calculated the CVE for $10^6$ instances of Gaussian sampled data and obtained its distribution for a variety of conditions such as sampling rates, segment duration, and filtering.

In modern data analysis environments (MATLAB, R, IGOR, ...), the calculation of envelopes is a simple step involving the use of the Hilbert Transform, a commonly available function. Thus if a signal $x$ is sampled into an $N$-points vector ($x = x_1, x_2, ..., x_N$), its Hilbert transform $hx = hx_1, hx_2, ..., hx_N$ is easily calculated. The envelope
Figure 2: Coefficient of variation of the envelope (CVE) for sampled Gaussian noise. (A) Epoch length influences CVE. The CVE for $10^6$ wide-band Gaussian noise epochs of 2, 4, 8, 16, 32 [s] sampled at 250 [Hz] was calculated and the resulting distributions were determined. All distributions are approximately Gaussian. Their mode tends to $m$ and their variance decreases as epoch length increases. (B) Bandwidth influences CVE. The CVE for $10^6$ Gaussian noise epochs (32 [s] sampled at 250 [Hz]) low-pass filtered at 25, 50, 75, 100, 125 [Hz] was calculated and the resulting distributions determined. All distributions are approximately Gaussian. Their mode tends to $m$ and their variance decreases as bandwidth increases. (C) Sampling rate influences CVE. The CVE for $10^6$ wide-band Gaussian noise epochs of 32 [s] sampled at 50, 100, 150, 200, 250 [Hz] was calculated and the resulting distributions were determined. All distributions are approximately Gaussian. Their mode tends to $m$ and their variance decreases as sampling rate increases. Notice that the narrowest distributions in all three panels (32s, 0-125 and 250Hz) are the same to facilitate comparison (i.e. epochs behind these 3 distributions were simulated with the same parameters). Also note that reducing the sampling rate (and thus the number of points) to a given fraction is equivalent to reducing the bandwidth by the same factor (e.g. the 200 [Hz] curve is the same as the 0-100 curve), showing the influence of the Nyquist frequency on the variance of CVE. The dashed line at $m = \sqrt{4 - \pi}/\pi \approx 0.523$ is the fingerprint of continuous Gaussian noise.

ex = ex₁, ex₂, ..., exₙ, also an N-points vector, is calculated by this relation: $ex_i = \sqrt{x_i^2 + hx_i^2}$. To further clarify details, we outline the steps used to calculate the CVE for a given data segment with Algorithm 1.

Algorithm 1 CVE estimation
1: Sample data into vector $x[n] = (x₁, x₂, x₃, ..., xₙ)$.
2: Calculate $hx[n]$ the Hilbert transform of $x[n]$: $hx[n] = (hx₁, hx₂, hx₃, ..., hxₙ)$.
3: Calculate the envelope $ex[n]$ as $ex_i = \sqrt{x_i^2 + hx_i^2}$, also an N-point vector.
4: Calculate the mean ($\mu_A$) and standard deviation ($\sigma_A$) of the set $A = \{ex₁, ex₂, ..., exₙ\}$.
5: $\text{CVE} = \frac{\sigma_A}{\mu_A}$.
cases the resulting CVE distribution is centered around \( m \) and the distribution width depends on the low-pass filter \( f_{3dB} \) corner frequency, with narrower filters resulting in wider CVE distributions. The sampling rate also influences the CVE (Figure 2 C). For 32 [s] data segments, the CVE distributions were calculated for sampling rates of 250, 200, 150, 100, and 50 [Hz]. The resulting CVE distributions are unimodal, centered around \( m \), and their width increases as sampling rates decrease. Notice how the wider distribution does not excessively depart from \( m \) because the combination of length and sampling rate is not strong enough to produce a noticeable deviation, as is the case for short wide-band 2 [s] epochs sampled at 250 [Hz] (Figure 2 A, widest distribution). Comparing the effects of different sampling rates and signal bandwidths on the CVE, it can be seen that reducing the number of points by one fifth is equivalent to reducing the bandwidth by the same factor.

Overall, the distribution for the CVE of discrete-time zero-mean Gaussian noise is a function of the number of points \( N \) available and the bandwidth \( BW \) of the signal. Although we lack a general proof to explain these relations, computer simulations allow us to test the hypothesis of Gaussianity. Thus, as a first approach, the CVE of sampled Gaussian noise can be used to generate probability models and, for a given \( \alpha \) value, create confidence intervals to test Gaussianity (Algorithm 2).

**Algorithm 2** CVE model based on sampled Gaussian noise

1. Define the number of points \( N \) and the bandwidth \( BW \) of the signal under analysis.
2. Sample data from \( \mathcal{N}(0, \sigma) \) into vector \( g[n] = (g_1, g_2, g_3, ..., g_N) \), matching the number of points of the target signal.
3. Filter \( g[n] \) with the same filter applied to the target signal, obtaining the same bandwidth.
4. Calculate the CVE as shown in Algorithm 1.
5. Repeat 2, 3 and 4 \( 10^6 \) times to obtain a probability model of Gaussianity based on sampled Gaussian noise.
6. Create a confidence interval to test the null hypothesis \( H_0: CVE = CVE_{Gaussian} \)

5 CVE for sampled non-Gaussian processes

5.1 CVE for uniform noise

Although the CVE for Gaussian noise approaches \( m \), the result changes drastically for the case of uniform noise. In Figure 3 we repeat the calculations done for Gaussian noise but for a zero-mean uniform noise process (U[-1,1]). Crucially, the CVE from uniform distributed noise differs from \( m \) and hovers around \( u \approx 0.430 \). This result holds regardless of the variance (power) of the uniform noise (i.e for any U[-a,a]), as long as it is zero-mean. If the uniform noise is low pass filtered (by a sliding window filter) the CVE approaches \( m \) as the window increases in duration.

5.2 CVE for Filtered Poisson Processes

Using the same calculations for Gaussian noise, we studied the cve of Filtered Poisson Processes (FPP). We used an alpha function \( f(t) = \frac{a}{b} e^{-(t-b)/b} \) as a simple kernel to produce instances of FPP with different densities (rates) of elementary pulses. When densities are low, the corresponding CVE takes values around 1.2; however, as the density of pulses increases, the overall signal becomes Gaussian, and its CVE decreases towards \( m \) (Figure 4).

5.3 CVE for Rhythms

We explore the behavior of CVE for noisy sinusoids. When the Signal-To-Noise Ratio (SNR) is low, signals behave like Gaussian noise (i.e. CVE near \( m \)), but as the signal power increases (i.e higher SNR) the CVE decreases lower values (Figure 5).
Figure 3: Coefficient of variation of the envelope (CVE) for sampled uniform noise. The CVE for $10^6$ wide-band uniform noise epochs of 4, 8, 16, 32, 64 [s] sampled at 250[Hz] was calculated and the resulting distributions were determined. All distributions are bell-shaped and their modes are close to $u \approx 0.430$, the fingerprint of zero-mean uniform noise, and their variance decreases as epoch length increases. Variance also decreases with higher bandwidth and sampling frequency (not shown). The dashed line at $u \approx 0.430$ is the fingerprint of continuous uniform noise.

Figure 4: Coefficient of variation of the envelope (CVE) for filtered Poisson processes (FPP). Each FPP was created by convolving a Poisson process, with rate $\lambda$, with an alpha function of the form $f(t) = \frac{at}{b} e^{-(t-b)/b}$ with $a$ a random variable in $\{-1, 1\}$. The CVE for $10^6$ 2500 points FPP epochs of rate 0.01, 0.025, 0.05, 0.1, 0.25, 0.5 was calculated and the resulting distributions were determined. White dots mark the mode of each distribution. Notice that the mode moves closer to $m$ as the rate of the underlying Poisson process increases. Upper inset: an instance of a FFP with rate $= 0.01$ shows a distinctive pulsating waveform. Lower inset: an instance of a FFP with rate $= 0.5$ displays the profile of Gaussian noise. Dashed line at $m = \sqrt{(4 - \pi)/\pi} \approx 0.523$, the fingerprint of continuous Gaussian noise.

6 CVE, Signal morphology, and Fourier phase profile

In many domains of Signal Analysis, it is customary to work with the temporal and frequency representations of a given signal. Using concepts of frequency analysis, a simple calculation of the power spectrum of a given signal is commonly
Figure 5: Coefficient of variation of the envelope (CVE) for noisy sinusoids (NS) with different Signal-to-Noise Ratios. The CVE for $10^6$ epochs (2500 points and variable SNR: -20,-3,0,3,6,9 (dB)) was calculated and the resulting distributions were determined. White dots mark the mode of each distribution. Notice that the mode moves closer to $m$ as the SNR of the underlying sinusoid decreases. Upper inset: an instance of an NS with SNR $= 9$ dB shows a distinctive rhythmic waveform. Lower inset: an instance of an NS with SNR $= -20$ dB displays the profile of Gaussian noise. The dashed line at $m = \sqrt{(4 - \pi)/\pi} \approx 0.523$ is the fingerprint of continuous Gaussian noise.

used to indicate the given power emitted in a small frequency sub-band. Typical uses of this approach can be found in EEG, where the energy in the bands known as $\alpha$, $\delta$, $\gamma$ are relevant clinical indicators or in Seismology, where spectra are important to understand quake geodynamics. However, the CVE parameter opens a new avenue for signal analysis as two signals with identical power spectra can differ in their CVE and, hence, in their temporal profile. Thus, for example when a signal representing the epoch of a neonate during convulsion (Figure 6 A, top trace) (data from [11], subject 44) is randomly permuted in the Fourier phase space so that the original and permuted signals have exactly the same power spectra, interestingly, their temporal profiles differ along with their CVE.

A more extreme case can be observed with an artificial signal made of a sine wave combined with a cosine wave of the same frequency. In Figure 7, the original signal (the upper trace) has a CVE equal to 0.113 indicating that it is almost a pure tone, but if we explore the space of its phase-randomized versions ([12], Algorithm 3), it is possible to find a signal (lower trace) with the same power spectrum but a CVE = 0.727 indicating a signal similar to a deformed Gabor temporal pulse. Taken together the results from Figure 6 and Figure 7, it is clear that the CVE is a parameter that gives an overall measure of the temporal profile of a given signal independently of the power spectrum. Of course, this result is far from new. In effect, for any signal and any of its phase-randomized versions, their power spectra are identical but their Fourier transforms are not. Unfortunately, in many analyses the complexities of Fourier analysis are subsumed into the simple evaluation of the power spectrum, thus discarding temporal profile variations [13, 14].

Algorithm 3 Fourier transform phase randomization

1. Sample experimental data into vector $s[n] = (x_1, x_2, x_3, ..., x_N)$.
2. Calculate for $s[n]$ its Fourier transform $\hat{s}[n]$.
3. Express $\hat{s}[n]$ in polar form and replace the phase of every positive component by $\phi$ sampled from $[-\pi, \pi)$.
4. Express the positive component of $\hat{s}[n]$ in rectangular form. Flip horizontally this vector and calculate the complex conjugate to obtain the negative components.
5. Calculate the inverse Fourier transform.
Figure 6: Relationship between signal profiles and their CVE. (A) (Top trace) a portion of a neonate EEG showing a strong 1 Hz rhythm with a CVE = 0.379. (Lower trace) the same data after phase-randomization. The CVE now is valued at 0.541 and the rhythm is somewhat deformed. Both signals (raw data and phase randomized) have exactly the same power spectrum (inset, as theory predicted both spectra are exactly superimposed). (B) Another example of how phase randomization changes the CVE (0.607 > 0.524) while leaving the power spectrum invariant.

Figure 7: Signals with identical power spectra can have different CVE. (Top trace) An artificial signal was constructed by joining, in the middle, a sine and a cosine of the same frequency. As the resulting profile is similar to a rhythm its CVE is very low (0.113). An exhaustive search of phase-randomized versions of the original signal produced a signal with a rather different temporal profile (lower trace) with a CVE = 0.727 but with identical power spectra (inset).

7 CVE of coupled oscillators systems

The order parameter’s CVE of a system of coupled oscillators is a measure of the internal dynamics of the population [15]. Using two important coupled oscillators systems, we show the relation between the systems’ parameters and the
amplitude modulation patterns of the order parameter as characterized by the CVE.

7.1 Kuramoto model

The Kuramoto model consists of a population of phase oscillators in an all-to-all coupling arrangement that can become self-entrained if a coupling parameter \( K \) is above a critical value \( K_c \) [16]. Thus, the Kuramoto model displays three different regimes. For \( K < K_c \), oscillators act independently of each other (incoherence), and while for \( K \approx K_c \) there is partial synchronization between them, at \( K > K_c \) most oscillators become synchronized. In a population of \( N \) Kuramoto oscillators, all driven by the same coupling constant \( K \), each oscillator produces an elementary output \( \theta_i(t) \) (the phase of the \( i \)th oscillator) governed by the following differential equation:

\[
\dot{\theta}_j(t) = \omega_j + K r \sin(\psi - \theta_j), \quad j = 1, \ldots, N
\]  

(10)

The superposition of all elementary sources produces a macroscopic quantity called the order parameter:

\[
r e^{i\psi} = \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j}
\]  

(11)

which can be interpreted as the collective signal produced by the population. Thus, the envelope of the order parameter is

\[
r = \frac{1}{N} \sqrt{\left( \sum_{j=1}^{N} \cos(\theta_j) \right)^2 + \left( \sum_{j=1}^{N} \sin(\theta_j) \right)^2}
\]  

(12)

For a population of 100 oscillators, we studied the envelope of the order parameter using 20 [s] epochs (Figure 8). The natural frequency of each oscillator was taken from \( \mathcal{N}(10, 1.5) \) and the initial phases follow a uniform distribution on \([−\pi, \pi]\). If \( K \) is sufficiently low (\( K < K_c \)), the CVE has an empirical distribution centered on \( m \) because the resulting wave is essentially a low-amplitude, Rayleigh fading wave. On the contrary, for values \( K \approx K_c \) the resulting wave resembles a sine wave, the envelope of the order parameter \( r \) increases, and CVE adopts much lower values as the system approaches complete synchronization when \( K > K_c \). For each \( K \), we have a distribution of CVE values (shown as whiskers in Figure 8c) which, of course, depends on the number of points and bandwidth. This result is important as it shows that the CVE is a measure of the degree of synchronization among oscillators, enabling a statistical mechanics approach for the study of coupled-oscillators systems.

7.2 Matthews-Mirollo-Strogatz model

The model of Matthews-Mirollo-Strogatz (MMS) [18] is a generalization of the Kuramoto model as it allows for both phase and amplitude variations. In this model, oscillators follow an all-to-all arrangement as well, and the population dynamics is governed by two parameters: the bandwidth of the natural frequencies distribution \( \Delta \) and the coupling constant \( K \). Each oscillator is described in the complex plane by the following equation:

\[
\dot{z}_j(t) = (1 - |z_j|^2 + iw_j)z_j + K(z - z_j), \quad j = 1, \ldots, N
\]  

(13)

and the order parameter is the centroid of all oscillators:

\[
r e^{i\psi} = \frac{1}{N} \sum_{j=1}^{N} z_j
\]  

(14)

The MMS model displays a rich repertoire of dynamics not present in the Kuramoto model. The envelope of the order parameter shows new, characteristic amplitude modulations patterns not found in the Kuramoto dynamics.
Figure 8: CVE variations for Kuramoto model dynamics. A system of 100 oscillators was used for simulations. The natural frequencies are normally distributed ($\mu=10$, $\sigma=1.5$), and initial phases are uniformly distributed on $[-\pi, \pi]$.

(A) The order parameter (grey trace) and its envelope (black trace) are shown for each of three regimes displayed by the model (incoherence, partial synchronization, full synchronization). The vectors in the second row correspond to representative snapshots of the oscillators’ phases for the respective traces above, at the middle point of the depicted time course. (B), (C) The envelope of the order parameter $r$ and the CVE were calculated for different values of $K$ (0, 1, 2, ..., 30). Dots indicate the median and whiskers portray Q1 and Q3 of 100 simulation instances. Note how CVE is an indicator of the overall synchronization of the ensemble. Numerical solutions were obtained using Runge-Kutta methods [17].

(Figure 9). During incoherence (Figure 9 A) the order parameter is, as in the Kuramoto model, a relatively feeble signal with random fluctuations. This behavior shows the robustness of Rayleigh fading, as it can also be obtained from a population of amplitude-varying elements. Synchronous MMS regimes (Figure 9 B) have high-amplitude, low-variance envelopes and the order parameter resembles a pure sinusoid. But beyond these regimes, there is a fourth stationary state known as amplitude death (Figure 9 C) that emerges when oscillators are sufficiently coupled and the distribution...
Figure 9: MMS Model Dynamics. The dynamics of the MMS model is governed by the bandwidth ($\Delta$) and the coupling constant ($K$). For each case, the time course of the envelope associated with the order parameter ($r$) is estimated. (A) Incoherence ($\Delta = 0.8, K = 0.2$). A low coupling strength allows oscillators to behave independently, resulting in a Rayleigh fading envelope with CVE close to $\sqrt{(4 - \pi)/\pi}$. (B) Phase Locking ($\Delta = 0.3, K = 1.0$). For $K$ sufficiently strong, oscillators are locked into a common frequency, and their phase differences remain constant. (C) Amplitude death ($\Delta = 1.3, K = 1.3$). For $K > 1$ and high bandwidth, oscillators pull each other to origin and the amplitude of the envelope decays to zero. (D) Large-Amplitude Oscillations ($\Delta = 0.45, K = 0.75$) At the edge between Incoherence and Phase Locking, oscillators undergo periodic oscillations, alternating between narrow and sparse phase distributions. Values in parentheses are the parameters used in each simulation. Phase snapshots represent the phase profile at points indicated by arrows.

of natural frequencies is wide enough [19]. The envelope in this case has a low variance and a low amplitude. Also many unsteady, chaotic states appear for medium combinations of $\Delta$ and $K$, and these regimes are characterized by high CVE profiles.

The fact that CVE can classify the rich diversity of MMS regimes (Figure 10, their Figure 17.A in [18]) reinforces the idea that CVE measures amplitude modulation patterns as well as signal morphology. The segregation of the MMS phase space induced by CVE suggests that using the envelope of signals (SHOULD SAY CVE) as part of our toolbox in Signal Analysis is an idea to be pursued.
Figure 10: MMS phase space categorized by CVE. The behavior of the MMS model is controlled by the oscillators’ bandwidth ($\Delta$) and their coupling constant ($K$) represented by the x and y axes respectively. In this image the range of both variables is [0.0:1.5] and divided in 1024 steps. For every pixel defined by a ($\Delta$, $K$) condition, the MMS equations were solved using fourth order Runge-Kutta methods, and one waveform was computed and its corresponding CVE calculated. For each pixel ($\Delta$, $K$), this procedure was repeated 25 times to obtain an average CVE, which was color-coded according to the color-scale. This CVE-based categorization produces the same boundaries (white dashed lines) qualitatively detected in (Matthews et al., 1991, Fig. 17). Thus the CVE by itself segregates the several regimes hidden in the MMS model.

8 Discussion

We have presented a method to assess Gaussianity that is especially suited for the long time series usually found in biomedical research (see examples in [20, 21]), opening a new avenue for signal classification. The core of the CVE method depends on a property of the envelope of Gaussian noise: the ratio $\text{stdev/mean}$ for an infinitely long envelope of an infinitely long Gaussian noise signal equals $m = \sqrt{(4 - \pi)/\pi} \approx 0.523$. For discrete-time signals, it is possible to build empirical distributions that are centered on $m$ but whose dispersion depends on sampling rate, signal duration and signal filtering. The CVE method allows precise confidence intervals to be constructed from these distributions according to the parameters of the experimental signal. Thus, for a single epoch of digitized data, its CVE acts as a discriminating statistic to test Gaussianity. Interestingly, this result holds for Gaussian noise or any of its sub-bands, but
not for other types of processes (uniform Noise or filtered Poisson process). Furthermore, the CVE captures aspects related to signal morphology. CVE values statistically lower than $m$ reflect rhythmic signals, while a CVE statistically higher than $m$ identifies pulsating or phasic signals, and a CVE near $m$ resembles Gaussian noise. As the CVE reflects signal morphology, its value is connected to the Fourier phase profile of the sampled epoch. Thus, two epochs with exactly the same power spectrum can have very different CVE values. The fact that coloring the phase space of the MMS model according to CVE recuperates the qualitative segmentation obtained in Matthews’ original paper attests to the classification power of the CVE parameter. Moreover, the CVE method is simple to implement and not computationally intense. Indeed, it can be applied online as an automatic signal classifier because it does not require the adjustment of parameters once the sampling rate, epoch duration, and filtering characteristics of data acquisition are defined.

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