Extremal graphs for the suspension of edge-critical graphs

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Abstract

The Turán number of a graph $H$, $ex(n, H)$, is the maximum number of edges in an $n$-vertex graph that does not contain $H$ as a subgraph. For a vertex $v$ and a multi-set $F$ of graphs, the suspension $F + v$ of $F$ is the graph obtained by connecting the vertex $v$ to all vertices of $F$ for each $F \in F$. For two integers $k \geq 1$ and $r \geq 2$, let $H_i$ be a graph containing a critical edge with chromatic number $r$ for any $i \in \{1, \ldots, k\}$, and let $H = \{H_1, \ldots, H_k\} + v$. In this paper, we determine $ex(n, H)$ and characterize all the extremal graphs for sufficiently large $n$. This generalizes a result of Chen, Gould, Pfender and Wei on intersecting cliques. We also obtain a stability theorem for $H$, extending a result of Roberts and Scott on graphs containing a critical edge.

Keywords: Turán number, extremal graph, edge-critical graph, $r$-partite

1 Introduction

Given a graph $H$, a graph $G$ is called $H$-free if it contains no copy of $H$ as a subgraph. The Turán number $ex(n, H)$ of $H$ is the maximum number of edges in an $H$-free graph on $n$ vertices. Determining $ex(n, H)$ is one of most important problems in extremal graph theory and the Turán graph plays a key role. For two integers $n$ and $r$ with $n \geq r \geq 2$, the Turán graph $T_r(n)$ is an $n$-vertex complete $r$-partite graph with parts of size $\lceil n/r \rceil$ or $\lfloor n/r \rfloor$. Let $t_r(n)$ denote the number of edges in $T_r(n)$. The classical Turán’s Theorem [18] shows that $ex(n, K_{r+1}) = t_r(n) = (1 - \frac{1}{r} + o(1))(\frac{n^2}{2})$ and the only extremal graph is $T_r(n)$.

Let $\chi(H)$ denote the chromatic number of $H$. If there is an edge $e$ of $H$ such that $\chi(H - e) = \chi(H) - 1$, then we say that $H$ is edge-critical and $e$ is a critical edge. The celebrated Erdős-Stone-Simonovits Theorem [7,8] states that $ex(n, H) = (1 - \frac{1}{\chi(H)-1} + o(1))(\frac{n^2}{2})$. For an edge-critical graph $H$ with $\chi(H) = r + 1$, Simonovits [17] proved that $T_r(n)$ is also the unique extremal graph for sufficiently large $n$.

Theorem 1.1 (Simonovits [17]). Let $H$ be an edge-critical graph with $\chi(H) = r + 1 \geq 3$. Then there exists some $n_0$ such that $ex(n, H) = t_r(n)$ for all $n \geq n_0$, and the unique extremal graph is $T_r(n)$.

Although the Turán numbers of non-bipartite graphs are asymptotically determined by the Erdős-Stone-Simonovits theorem, it is still a challenge to determine the exact Turán numbers for many non-bipartite graphs. There are only a few graphs whose Turán numbers are determined exactly, including edge-critical graphs and some other specific graphs (e.g. see [9,11,13,14,20–23]). Among all the existing results, the Turán number of the graph consisting of some specific graphs that intersect in exactly one common vertex is widely studied (e.g. see [4,10,12,15,19]).

In this paper, we mainly consider edge-critical graphs intersecting in a special vertex. For a vertex $v$ and a multi-set $F$ of graphs, the suspension $F + v$ of $F$ is the graph obtained by connecting the

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vertex $v$ to all vertices of $F$ for each $F \in \mathcal{F}$. If $\mathcal{F} = \{F\}$, then we simply write $F + v$ instead of $\mathcal{F} + v$.

We call the vertex $v$ the center vertex of $\mathcal{F} + v$. If $\mathcal{F}$ is a multi-set consisting of $k$ copies of $K_{r-1}$, then the graph $\mathcal{F} + v$ is known as a $(k,r)$-fan, denoted by $F_{k,r}$. Erdős, Füredi, Gould and Gunderson [6] first considered the Turán number of $F_{k,3}$ (also known as the friendship graph), and established the following result.

**Theorem 1.2** (Erdős, Füredi, Gould and Gunderson [6]). For every $k \geq 1$, and for every $n \geq 50k^2$,

$$\text{ex}(n, F_{k,3}) = \left\lceil \frac{n^2}{4} \right\rceil + \begin{cases} k^2 - k & \text{if } k \text{ is odd}, \\
2k^2 - \frac{3}{2}k & \text{if } k \text{ is even.} \end{cases}$$

For general $r$, Chen, Gould, Pfender and Wei [2] determined $\text{ex}(n, F_{k,r})$ for sufficiently large $n$.

**Theorem 1.3** (Chen, Gould, Pfender and Wei [2]). For every $k \geq 1$ and $r \geq 2$, and for every $n \geq 16k^3r^8$,

$$\text{ex}(n, F_{k,r}) = t_{r-1}(n) + \begin{cases} k^2 - k & \text{if } k \text{ is odd}, \\
2k^2 - \frac{3}{2}k & \text{if } k \text{ is even.} \end{cases}$$

We further extend this result and determine $\text{ex}(n, H)$ for $H := \mathcal{F} + v$, where $\mathcal{F}$ consists of $k$ edge-critical graphs $H_1, H_2, \ldots, H_k$ with $\chi(H_i) = r$ for each $1 \leq i \leq k$. Let $\mathcal{G}_{n,k,r}$ be a family of graphs, each of which is obtained from Turán graph $T_r(n)$ by embedding two vertex disjoint copies of $K_t$ in one partite set if $k$ is odd and embedding a graph with $2k - 1$ vertices, $k^2 - 3k/2$ edges with maximum degree $k - 1$ in one partite set if $k$ is even. Our main result is as follows.

**Theorem 1.4.** Suppose that $k \geq 1$ and $r \geq 2$ are integers. Let $H_i$ be an edge-critical graph with $\chi(H_i) = r$ for each $i \in \{1, \ldots, k\}$, and let $H := \{H_1, H_2, \ldots, H_k\} + v$. Then, for sufficiently large $n$,

$$\text{ex}(n, H) = t_r(n) + \begin{cases} k^2 - k & \text{if } k \text{ is odd}, \\
2k^2 - \frac{3}{2}k & \text{if } k \text{ is even.} \end{cases}$$

Moreover, the $\mathcal{G}_{n,k,r}$ is the family of extremal graphs for $H$.

We also obtain the following stability theorem for $H$ defined as above, extending a result of Roberts and Scott [16] on edge-critical graphs (see Lemma 3.1 for more details).

**Theorem 1.5.** Let $f(n) = o(n^2)$ be a function and let $H$ be defined as in Theorem 1.4. If $G$ is an $H$-free graph with $n$ vertices and at least $t_r(n) - f(n)$ edges, then $G$ can be made $r$-partite graph by deleting $O(n^{-1}f(n)^{3/2})$ edges.

This paper is organized as follows. In the remainder of this section, we describe notations and terminologies used in our proofs. In Section 2, we make a reduction of Theorem 1.4 and prove it assuming Theorem 2.4. We prove Theorem 2.4 in Section 3. In Section 4, we prove Theorem 1.5.

**Notation.** Let $G = (V(G), E(G))$ be a graph. We use $e(G)$ denote $|E(G)|$. We use $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degrees in $G$, respectively. For $S, T \subseteq V(G)$, we use $G[S]$ denote the graph induced by $S$. For $v \in V(G)$, let $N_S(v)$ denote the set of vertices in $S$ adjacent to $v$ and $d_S(v) = |N_S(v)|$. Let $N_S(T) = \cap_{v \in T} N_S(v)$. Let $V - S = \{v \in V : v \notin S\}$. In particular, if $S = V(G)$, then we substitute $N_G(v)$ and $d_G(v)$ for $N_{V(G)}(v)$ and $d_{V(G)}(v)$, respectively. A matching in $G$ is a set of edges from $E(G)$, no two of which share a common vertex. The matching number of $G$, denoted by $\nu(G)$, is the maximum number of edges in a matching in $G$. An $r$-partition of $G$ is a partition of $V(G)$ into $r$ pairwise disjoint nonempty subsets $V_1, V_2, \ldots, V_r$. For an integer $t$, let $[t] = \{1, 2, \ldots, t\}$. 


2 Reduction to $H$-free graphs with large minimum degree

In this section, we make a reduction in preparation for the proof of Theorem 1.4. We first introduce a function related to the number of edges in a graph with bounded matching number and maximum degree.

Let $G$ be a graph with its matching number $\nu(G)$ and maximum degree $\Delta(G)$. Define

$$f(\nu, \Delta) = \max\{e(G) \mid \nu(G) \leq \nu, \Delta(G) \leq \Delta\}.$$ 

Abbott, Hanson and Sauer [1] studied this function for $\nu = \Delta = k - 1$, and proved that

$$f(k - 1, k - 1) = \begin{cases} k^2 - k & \text{if } k \text{ is odd}, \\ k^2 - \frac{3k}{2} & \text{if } k \text{ is even}. \end{cases}$$

The extremal graphs are graphs with $2k - 1$ vertices, $k^2 - 3k/2$ edges with maximum degree $k - 1$ if $k$ is even, or two vertex disjoint copies of $K_k$ if $k$ is odd. For general $\nu$ and $\Delta$, Chvátal and Hanson [3] established the following theorem.

**Theorem 2.1** (Chvátal and Hanson [3]). For every $\nu \geq 1$ and $\Delta \geq 1$,

$$f(\nu, \Delta) = \nu \Delta + \left\lfloor \frac{\Delta}{2} \right\rfloor \left\lfloor \frac{\nu}{\Delta/2} \right\rfloor \leq \nu \Delta + \nu. \quad (1)$$

**Definition 2.2** (Good partition). For two integers $k, r \geq 2$, call a partition $V_1, V_2, \ldots, V_r$ of $G$ $k$-good, if the following properties hold for each $i \in [r]$:

(i) $V_i \neq \emptyset$ and $\Delta(G[V_i]) \leq k - 1$.

(ii) $\sum_{j \in [r] \setminus \{i\}} \nu(G[V_j]) \leq k - 1$ and

(iii) $d_{V_i}(u) + \sum_{j \in [r] \setminus \{i\}} \nu\left(G[N_G(u) \cap V_j]\right) \leq k - 1$ for each $u \in V_i$.

Chen, Gould, Pfender and Wei [2] characterised the properties of a $k$-good partition of $G$ by showing the following lemma.

**Lemma 2.3** (Chen, Gould, Pfender and Wei [2]). Suppose that $G$ has a $k$-good partition $V_1, V_2, \ldots, V_r$. Let $G'$ be the minimal induced subgraph of $G$ such that $e(G') - \sum_{1 \leq i < j \leq r} |V_i||V_j|$ is maximal, where $V_i' = V(G') \cap V_i$ for each $i \in [r]$. Then the following properties hold:

(i) $e(G') - \sum_{1 \leq i < j \leq r} |V_i'||V_j'| \leq f(k - 1, k - 1)$;

(ii) For each $i \in [r]$ and $x \in V_i'$, we have $0 < d_{G'}(x) - |V(G') \setminus V_i'| \leq k - 1 - \sum_{j \in [r] \setminus \{i\}} \nu(G'[V_j'])$;

(iii) If $\nu(G'[V_i']) \geq 2$ for each $i \in [r]$, then $e(G') - \sum_{1 \leq i < j \leq r} |V_i'||V_j'| < f(k - 1, k - 1)$.

Now, we begin with a reduction of our main theorem via the existence of a $k$-good partition, and prove Theorem 1.4 assuming Theorem 2.4. We leave the proof of Theorem 2.4 in the next section.

**Theorem 2.4.** For two integers $k, r \geq 2$, let $H_i$ be an edge-critical graph with $\chi(H_i) = r$ for each $i \in [k]$, and let $H := \{H_1, H_2, \ldots, H_k\} + u$. If $G$ is an $H$-free graph with $n$ vertices and $\delta(G) \geq \frac{1}{r}n - k$, then $G$ contains a $k$-good partition for sufficiently large $n$. 


Proof of Theorem 1.4 given Theorem 2.4. Let $G_{n,k,r}$ be the family of graphs defined in Section 1. We first show that $G$ is $H$-free for each $G \in G_{n,k,r}$. Otherwise, we consider an embedding of $H$ with center vertex $v$ into $G$. Without loss of generality, we may assume that $e(G[V_1]) = f(k-1,k-1)$. Note that $E(H_j + v) \cap E(G[V_1]) \neq 0$ for each $j \in [k]$ in view of $\chi(H_j) = r$. It follows that $v \notin V_1$ as $\Delta(G[V_1]) < k$ by the construction of $G_{n,k,r}$. Suppose that $v \in V_s$ for some $s \in [r] \setminus \{1\}$. In this situation, we have $E(H_j + v) \cap E(G[V_1])$ are pairwise disjoint for any $j \in [k]$. This means that $v(G[V_1]) \geq k$, a contradiction. Thus, $G$ is $H$-free and $e(G) = t_r(n) + f(k-1,k-1)$, implying the lower bound.

In what follows, we prove that $e(G) \leq t_r(n) + f(k-1,k-1)$ for any $H$-free graph $G$ on $n$ vertices. We first show that this is true if $\delta(G) \geq \frac{\sqrt{kn}}{r} - n - k$. By Theorem 2.4 and Lemma 2.3, there is a $k$-good partition $V_1, V_2, \ldots, V_r$ of $G$ such that

$$e(G) \leq \sum_{1 \leq i < j \leq r} |V_i||V_j| + f(k-1,k-1) \leq t_r(n) + f(k-1,k-1),$$

as desired. Next, we aim to deal with small vertices. For a graph $F$ and $f \in V(F)$, we say $f$ is a small vertex of $F$ if $d(f) < \frac{\sqrt{kn}}{r} - n - k$. We first delete a small vertex in $G$. As long as there is a small vertex in the resulting graph, we delete it. We keep doing this until the remaining graph $G^*$ ($G^*$ may be empty) has no small vertices. If $n^* := |V(G^*)| < \sqrt{kn}/4$, then

$$e(G) < e(G^*) + \sum_{i=n+1}^{n} \left( \frac{r-1}{r} i - k \right)$$

$$< \frac{kn}{8} + \frac{r-1}{r} (n + \sqrt{kn}/4)(n - \sqrt{kn}/4) - \left( n - \sqrt{\frac{kn}{4}} \right) k$$

$$= \frac{r-1}{r} n(n-1) \leq t_r(n),$$

as required. Thus, we may assume that $n^*$ is sufficiently large and $\delta(G^*) \geq \frac{\sqrt{kn}}{r} - n - k$. This implies that $e(G^*) \leq t_r(n^*) + f(k-1,k-1)$ as $G^*$ is also $H$-free. It follows that

$$e(G) < e(G^*) + \sum_{i=n+1}^{n} \left( \frac{r-1}{r} i - k \right)$$

$$\leq t_r(n^*) + f(k-1,k-1) + \sum_{i=n+1}^{n} \left( \frac{r-1}{r} i - k \right)$$

$$\leq t_r(n) + f(k-1,k-1),$$

(2)

where the last inequality holds as $t_r(s-1) + \frac{\sqrt{kn}}{r} (s-1) \leq t_r(s)$. Thus, $e(G) < t_r(n) + f(k-1,k-1)$.

Now, we prove the uniqueness of the extremal graph. Let $G$ be an $H$-free graph with

$$e(G) = t_r(n) + f(k-1,k-1).$$

(3)

Then $\delta(G) \geq \frac{\sqrt{kn}}{r} - n - k$ by the above argument. It follows from Theorem 2.4 that $G$ has a $k$-good partition $V_1, V_2, \ldots, V_r$. By Lemma 2.3, there exists an induced subgraph $G'$ of $G$ such that

$$e(G) - \sum_{1 \leq i < j \leq r} |V_i||V_j| = e(G') - \sum_{1 \leq i < j \leq r} |V'_i||V'_j| = f(k-1,k-1).$$

(4)

Without loss of generality, suppose that $|V'_i| > 0$ for $1 \leq i \leq s$ and $|V'_i| = 0$ for $s+1 \leq i \leq r$. Then, for each $i \in [s]$ and $x \in V'_i$

$$0 < d_{G'}(x) - |V(G')\setminus V'_i| \leq k - 1 - \sum_{j \in [x]|i|} v(G'[V'_j]),$$
implying that \(\nu(G'[V'_i]) \geq 1\) and \(\sum_{j=1}^{r} \nu(G'[V'_j]) \leq k - 2\). In addition, by (3) and (4), we have

\[
\sum_{i=1}^{r} e(G[V_i]) = \sum_{i=1}^{s} e(G'[V'_i]) = f(k-1,k-1).
\]  

(5)

Case 1. \(\sum_{j=1}^{r} \nu(G'[V'_j]) = 0\) for some \(i_0 \in [s]\). This implies that \(V'_j = \emptyset\) for each \(j \in [r] \setminus \{i_0\}\) and \(G' = G[V'_0]\). Thus, \(e(G[V'_0]) \geq e(G') = f(k-1,k-1)\). It follows from (5) that \(G \cong G_{n,k,r} \subset G_{n,k,r}\).

Case 2. \(1 \leq \sum_{j=1}^{r} \nu(G'[V'_j]) \leq k - 2\) for each \(i \in [s]\). Clearly, there exists an \(i_0 \in [r]\) such that \(\nu(G'[V'_0]) = 1\); otherwise, we get a contradiction by Lemma 2.3(iii). Without loss of generality, suppose that \(\nu(G'[V'_0]) = 1\). Then, we have

\[
\sum_{j=1}^{s} e(G'[V'_j]) \leq \sum_{2 \leq i \leq s} f(\nu(G'[V'_i]), k-1) + f(1,k-1)
\]
\[
\leq f\left(\sum_{2 \leq i \leq s} \nu(G'[V'_i]), k-1\right) + f(1,k-1)
\]
\[
\leq f(k-2,k-1) + f(1,k-1)
\]
\[
\leq f(k-1,k-1),
\]

where the last inequality is strictly true for \(k \geq 5\). This leads to a contradiction in view of (5). It remains to consider the situation for \(k \leq 4\).

If \(k = 3\), then \(s = 2\) and \(G[V'_0] \cong G[V'_2] \cong K_3\). This together with (5) yields that \(G\) is a graph formed by the complete \(r\)-partite graph \(G[V'_1] \cong G[V'_2] \cong K_3\). By the same argument of the case \(k = 3\), we can find a vertex \(x \in V_1\) such that \(d_{G[V'_1]}(x) + \sum_{i=2}^{s} \nu(G[N_{V_i}(x)]) = 5 > k - 1\), a contradiction to Definition 2.2(iii). Thus, we complete the proof of Theorem 1.4. □

3 \(H\)-free graphs with large minimum degree

In this section, we give a proof of Theorem 2.4. We first present the following useful lemmas given by Roberts and Scott [16].

Lemma 3.1 (Roberts and Scott [16]). Let \(F\) be a graph with a critical edge and \(\chi(F) = r + 1 \geq 3\), and let \(f(n) = o(n^2)\) be a function. If \(G\) is an \(F\)-free graph with \(n\) vertices and \(e(G) \geq t(n) - f(n)\), then \(G\) can be made \(r\)-partite by deleting \(O(n^2 f(n)^{3/2})\) edges.

Lemma 3.2 (Roberts and Scott [16]). Let \(r \geq 2\) and \(t \geq 1\) be integers. Suppose that the graph \(G \subseteq T_r(n)\) is \(T_r(rt)\)-free. Then \(e(G) \leq t(n) - n^2/2\) for sufficiently large \(n\).

We also need another easy lemma about edge-critical graphs.

Lemma 3.3. Let \(G\) be an edge-critical graph with \(\chi(G) = r \geq 2\), and let \(G^* = G + u\). If \(v_1v_2\) is a critical edge of \(G\), then both \(uv_1\) and \(uv_2\) are critical edges of \(G^*\).

Proof. By symmetry, it suffices to show that \(uv_1\) is a critical edge of \(G^*\). Since \(\chi(G) = r\) and \(v_1v_2\) is a critical edge of \(G\), there is a partition \((V_1, \ldots, V_r)\) of \(G\) such that \(V_i\) is an independent set of \(G\) for each \(i \in [r-1]\) and \(V_r = \{v_1\}\). Let \(G'\) be the graph obtained from \(G^*\) by deleting the edge \(uv_1\). Clearly, \((V_1, \ldots, V_{r-1}, V_r \cup \{u\})\) is an \(r\)-coloring of \(G'\). This means that \(uv_1\) is a critical edge of \(G^*\) in view of \(\chi(G^*) = r + 1\). □
Now, we are in a position to prove Theorem 2.4.

**Proof of Theorem 2.4.** Suppose that $H_j$ is an edge-critical graph with a critical edge $u_jv_j$ and $\chi(H_j) = r$ for each $j \in [k]$. Let $H := (H_1, H_2, \ldots, H_k) + u$ and $H' = H_j + u$ for $j \in [k]$. By Lemma 3.3, we have $uv_j$ is a critical edge of $H'_j$. Then, by Theorem 1.1 there exists a constant $p_j$ (or $p'_j$) such that $H_j$ can be embedded in $T_{r-1}((r-1)p_j) + e$ with $u_jv_j = e$ (or $H'_j$ can be embedded in $T_r(rp'_j) + e$ with $uv_j = e$) for each $j \in [k]$, where $e$ is any edge inside a vertex class of $T_{r-1}((r-1)p_j)$ (or $T_r(rp'_j)$).

Let $G$ be an $H$-free with maximum number of edges. This means that $G + e$ contains a copy of $H$ for any $e \notin E(G)$. It follows that $G$ contains a subgraph $D$ which is a copy of $H - e_0$ for some $e_0 \in H$. Without loss of generality, we may assume that $e_0 \in H'_k$ and $v_0$ is the center vertex of $D$. Let $D'$ denote the subgraph which is a copy of $(H_1, \ldots, H_{k-1}) + u$ in $D$. Note that $\delta(G) \geq \frac{r-1}{4}n - k$. Let $\ell = |V(D')|$. Choose a subset $S \subseteq N_G(v_0) - V(D')$ such that

$$|S| = \frac{r-1}{r}n - k - \ell.$$  

Clearly, $G[S]$ is $H_k$-free as $G$ is $H$-free. We show that $G[S]$ is close to $T_{r-1}(|S|)$. Note that

$$\delta(G[S]) \geq \delta(G) - (n - |S|) \geq \frac{r-2}{r}n - 2k - \ell = |S| - \frac{n}{r} - k.$$  

This implies that

$$e(G[S]) \geq \frac{|S|(|S| - \frac{n}{r} - k)}{2} \geq \frac{|S|(|S| + 1)}{2} - \frac{|S|(|S| + 1)}{2} - \frac{r-2}{r}n - 2k - \ell = \frac{r-2}{r}n - (k + \ell - 1) + (r-1)(k+1)$$

$$= \frac{r-2}{r}n - \frac{r-1}{2}(k + \ell - 1) + (r-1)(k+1) - |S|.$$  

For simple, let $C_{k,\ell} = \frac{(k+\ell-1)+(r-1)(k+1)}{2(r-1)}$. Since $(1 - \frac{1}{r})\frac{|S|}{r} \leq t_{r-1}(|S|)$, we have

$$e(G[S]) \geq t_{r-1}(|S|) - C_{k,\ell}|S|.$$  

For a partition $(S_1, S_2, \ldots, S_{r-1})$ of $G[S]$, we define an $(r-1)$-partite graph

$$G_S(S_1, S_2, \ldots, S_{r-1}) = (S, \{v_iv_j \in E(G) : v_i \in S_i, v_j \in S_j, 1 \leq i < i' \leq r - 1\}).$$  

Now we partition $S$ into $(S_1, S_2, \ldots, S_{r-1})$ such that $e(G[S_1, S_2, \ldots, S_{r-1}])$ is maximum. By Lemma 3.1 for some constant $c$

$$\sum_{1 \leq i \leq r-1} e(G[S_i]) \leq e^r|S|^{1/2} \leq cn^{1/2}.$$  

This together with (8) implies that

$$|S| - \frac{n}{r} \leq \epsilon_1 \frac{n}{r}$$

for some $\epsilon_1 \in (0, 10^{-6})$. Fix $i \in [r-1]$. For $x_i \in S_i$ and $i' \in [r-1]$ with $i' \neq i$, we have

$$d_{S,r}(x_i) \geq \delta(G[S_i]) - \sum_{q \in \{r-1\} \setminus \{i,i'\}} |S_q| - cn^r \geq \frac{1}{r}n - 2k - \ell - \frac{r-3}{r}n - cn^r \geq (1 - \epsilon_2) \frac{n}{r}$$

for some $\epsilon_2 \in (2\epsilon_1, 10^{-5})$.

Now, we show that $E(G[S_i])$ is empty for each $i \in [r-1]$. Suppose that there exists an edge $uv \in E(G[S_i])$. Pick $B_1 \subseteq (S_1 - \{u, v\})$ with $|B_1| = \frac{n}{2r}$. By (10) and (11), we have

$$|N_S(u) \cap N_S(v)| \geq (1 - 2(\epsilon_1 + \epsilon_2)) \frac{n}{r} \geq \frac{n}{2r}$$

for some $\epsilon_2 \in (2\epsilon_1, 10^{-5})$. 

However, this is a contradiction.
for each $2 \leq i \leq r - 1$. We can pick a subset $B_i \subseteq (N_S(u) \cap N_S(v))$ with $|B_i| = \frac{n}{r}$ for each $2 \leq i \leq r - 1$. Let $B = \bigcup_{i\in[r-1]} B_i$. Recall that $G[S]$ is $H_k$-free. It follows from Theorem 1.1 that $G_B[B_1, B_2, \ldots, B_{r-1}]$ is $T_{r-1}((r-1)p_i)$-free for some constant $p_i > 0$. By Lemma 3.2 we have

$$e(G_B[B_1, B_2, \ldots, B_{r-1}]) \leq t_{r-1} \left( \frac{(r-1)n}{2r} \right) - \frac{n^2}{8r^2},$$

implying that there are at least $\frac{n^2}{8r^2}$ edges missing between vertex classes $S_i$ for $i \in [r-1]$. This together with (9) shows that

$$e(G[S]) \leq t_{r-1}(|S|) + cn^{\frac{1}{2}} - \frac{n^2}{8r^2},$$

a contradiction to (8). Therefore $E(G[S_i]) = 0$ for each $i \in [r-1]$.

Since $e(G[S_i]) = 0$, we can further improve $|S_i|$ for each $i \in [r-1]$ by showing that

$$|S_i| \leq |S| - \delta(G[S]) \leq \frac{n}{r} + k$$

in view of (6) and (7). This further implies that

$$|S_i| = |S| - \sum_{i'\in[r-1]\setminus{i}} |S_{i'}| \geq \frac{n}{r} - k(r-1) - \ell.$$  \hfill (13)

Moreover, for $x \in S_i$ and $i' \in [r-1] \setminus \{i\}$, it follows from (7) and (12) that

$$d_{S_i}(x) \geq \delta(G[S]) - \sum_{q \in [r-1]\setminus\{i,i'\}} |S_q| \geq \frac{n}{r} - k(r-1) - \ell.$$ \hfill (14)

Recall that $D'$ denote the subgraph which is a copy of $(H_1, \ldots, H_{k-1}) + u$ in $D$. We consider the vertices not in $S \cup V(D')$. Let $S_0 = V(G) - S - V(D')$. Then

$$|S_0| = n - \ell - |S| = \frac{n}{r} + k. \hfill (15)$$

For $x_i \in S_i$ with $i \in [r-1]$, we have

$$d_{S_0}(x_i) \geq d_G(x_i) - (|S \cup V(D')| - |S_i|) \geq |S_i| \geq \frac{n}{r} - k(r-1) - \ell = |S_0| - kr - \ell.$$ \hfill (16)

Let

$$a = kr + \ell, \quad p^* = \sum_{i \in [k]} p_i^*,$$ \hfill (17)

and

$$S^*_0 = \{ x \in S_0 : d_{S_0}(x) \geq p^*(r-1)a + k \}.$$ \hfill (18)

**Claim 3.3.1.** $|S^*_0| \leq a(r-1)$.

**Proof.** Suppose that $|S^*_0| \geq a(r-1) + 1$. For each $i \in [r-1]$, let

$$S^*_i = \left\{ v \in S^*_0 : d_{S_i}(v) \geq \frac{|S_i|}{a + 1} \right\}.$$ 

Notice that $d_{S_0}(x_i) \geq |S_0| - a$ for $x_i \in S_i$ by (16) and (17). If $X \subseteq S^*_0$ with $|X| = a + 1$, then $S_i \subseteq \bigcup_{x \in X} N(x)$ for $i \in [r-1]$. This implies that $|S^*_0| \geq |S^*_0| - a$. Thus, $|\bigcap_{i\in[r-1]} S^*_i| \geq |S^*_0| - a(r-1) \geq 1$. We can choose a vertex $v \in S^*_0$ such that for each $i \in [r-1]$

$$d_{S_i}(v) \geq \frac{|S_i|}{a + 1}.$$
In the following, we aim to find a copy of $H$ with a center vertex $v$.

For $i \neq i' \in [r - 1]$ and $x \in S_i$, recall that $d_{S_i}(x) \geq n/r - k(r - 1) - \ell \geq |S_i| - a$ by (12), (14) and (17). This together with (18) shows that
\[
N_{S_i}(Y \cup \{v\}) \geq N_{S_i}(v) - a|Y| \geq \frac{|S_i| - aC_Y}{a + 1},
\]
where $Y \subseteq S - S_i$ and $|Y| = C_Y$ is a constant. For sufficiently large $n$, by (19), we can pick $Y^1_i \subseteq N_{S_i}(v)$ with $|Y_i^1| = p_i^1$, $Y_i \subseteq N_{S_i}(v) \cap (\cap_{r \in [i-1]} N_{S_i}(Y_i^r))$ satisfying $|Y_i^j| = p_i^j$ successively for $2 \leq i \leq r - 1$. Fix $i$, we choose
\[
Y_i^j \subseteq \left( N_{S_i}(v) \cap \left( \bigcap_{r \in [i-1]} N_{S_i}(Y_i^r) \right) \right) - \bigcap_{j \in [j-1]} Y_i^j
\]
satisfying $|Y_i^j| = p_i^j$ successively for $2 \leq j \leq k$. Fix $j \in [k]$, we let $Y_j = \bigcup_{r \in [j-1]} Y_i^j$. Then $|Y_j| = (r - 1)p_i^j$.

Now, we find an $H^*_j$ in $G\{Y_j \cup S_o\}$. It follows from (16) that
\[
|N(Y_j) \cap N_{S_o}(v)| \geq p^*(r - 1)a + k - p^*_j(r - 1)a \geq (p^* - p^*_j)(r - 1)a + k.
\]
Thus, for $j \in [k]$, we can pick $x_j \in N(Y_j) \cap N_{S_o}(v)$ such that $x_1, x_2, \ldots, x_k$ are pairwise distinct. Again by (16), for $j \in [k]$, we have
\[
|N_{S_o}(x_j)| \geq |S_o| - p^*_j(r - 1)a = \frac{n}{r} + k - p^*_j(r - 1)a.
\]
This means that we can pick $Y_0^j \subseteq N_{S_o}(x_j) = \{v, x_1, x_2, \ldots, x_k\}$, with $|Y_0^j| = p_i^j$, and pick
\[
Y_0^j \subseteq N_{S_o}(Y_j) - \left( \bigcup_{j \in [j-1]} Y_0^j \cup \{v, x_1, x_2, \ldots, x_k\} \right)
\]
with $|Y_0^j| = p_i^j$ for $j = 2, 3, \ldots, k$ successively.

For $j \in [k]$, we have chosen $x_j \in N_{S_o}(v)$ and subsets $Y_j, Y_0^j$. It is easy to see that $G\{Y_j \cup Y_0^j\}$ contains a copy of $T_s(rp^*_j)$. This together with the choice of the edge $vx_j$ and Theorem 3.3.1 shows that $H^*_j = H_j + u$ can be embedded in $G\{Y_j \cup Y_0^j, Y_1, \ldots, Y_k\} + vx_j$ such that $u = v$. Thus, we can get a copy of $H$ in $G$, a contradiction. \hfill \Box

Now, we consider the vertices in $S_0$ with small degree. Let $Z_0 = S_0 - S_0^*$, $Z_i = S_i$ for $i \in [r - 1]$ and $Z = Z_0 \cup Z_1 \cup \ldots \cup Z_{r-1}$. It follows from Claim 3.3.1 that
\[
\frac{n}{r} + k = |S_0| \geq |Z_0| \geq |S_0| - a(r - 1) \geq \frac{n}{r} + k - a(r - 1).
\]
By (12), (13) and (17), we have
\[
\frac{n}{r} + k \geq |Z_i| = |S_i| \geq \frac{n}{r} - a + k
\]
for $i \in [r - 1]$, and then
\[
|V(G) - Z| \leq \left| V(G) - \bigcup_{i=0}^{r-1} S_i \right| + a(r - 1) \leq \ell + a(r - 1).
\]
Recall that $\delta(G) \geq \frac{\ell}{r}n - k$ and $d_{S_i}(x) \leq p(r - 1)a + k - 1$ for $x \in Z_0$. This together with (21) and (22) shows that for $x \in Z_0$ and $i \in [r - 1],$
\[
d_{Z_i}(x) \geq d_G(x) - |V(G) - Z| - d_{Z_0}(x) - \sum_{i' \in [r-1]\{i\}} |Z_{i'}|
\]
\[ \frac{r - 1}{r} n - k - \ell - a(r - 1) - (p(r - 1)a + k - 1) - (r - 2) \left( \frac{n}{r} + k \right) \]
\[ \geq \frac{n}{r} - ((p + 1)(r - 1) + 1) a - p(r - 1)k \]
\[ \geq \frac{n}{r} - 2 ((p + 1)(r - 1) + 1) a. \]

(23)

For every \( i \in [r - 1] \) and every \( x \in \mathbb{Z} \), by Claim 3.3.1 and (16), we have

\[ d_{Z_0}(x) \geq d_{S_0}(x) - |S_0| \geq \frac{n}{r} - a + k - a(r - 1) = \frac{n}{r} - ar + k. \]

(24)

Claim 3.3.2. For every \( x \in V(G) - Z \), there exists an \( i = i(x) \) such that \( d_{Z_0}(x) < k \). Moreover, such an \( i \) is unique.

Proof. Suppose that there exists a vertex \( v \in V(G) - Z \) such that \( d_{Z_0}(v) \geq k \) for each \( i \in \{0\} \cup [r - 1] \). Without loss of generality, let \( d_{Z_0}(v) = \min\{d_{Z_0}(v) : 0 \leq i \leq r - 1\} \). By the pigeonhole principle, we have \( d_{Z_0}(v) \leq d_{G}(v)/r \). Thus, for \( i \in [r - 1] \), we have

\[ d_{Z_0}(v) \geq d_{G}(v) - |V(G) - Z| - d_{Z_0}(v) - \sum_{r \in [r - 1]} d_{Z_0}(v) \]
\[ \geq d_{G}(v) - (\ell + a(r - 1)) - d_{G}(v) - \sum_{r \in [r - 1]} |Z_r| \]
\[ \geq \frac{r - 1}{r} d_{G}(v) - \frac{r - 2}{r} n - ar \geq \frac{n}{2r^2}. \]

(25)

Now, we construct \( k \)-partite graphs. Recall that \( |N_{Z_0}(v)| \geq k \). We can pick \( k \) distinct vertices \( x_1, x_2, \ldots, x_k \) in \( N_{Z_0}(v) \) and choose \( k \) pairwise disjoint subsets \( Y^{1}_0, Y^{2}_0, \ldots, Y^{k}_0 \) in \( Z_0 \) such that \( Y^{j}_0 \subseteq N_{Z_0}(v) \cap N_{Z_0}(x_j) \) with \( |Y^{j}_0| = \frac{n}{4kr} \) for \( j \in [k] \). By (21), (23) and (25), we have

\[ |N_{Z_0}(v) \cap N_{Z_0}(x_j)| \geq d_{Z_0}(v) - (|Z| - d_{Z_0}(x_j)) \geq \frac{n}{4r^2}. \]

(26)

For \( i \in [r - 1] \), we can choose \( k \) pairwise disjoint \( Y^{i}_1, Y^{i}_2, \ldots, Y^{i}_k \) such that \( Y^{j}_0 \subseteq N_{Z_0}(v) \cap N_{Z_0}(x_j) \) and \( |Y^{j}_0| = \frac{n}{4kr} \) for \( j \in [k] \). This is possible. Since we can choose \( Y^{i}_1 \subseteq N_{Z_0}(v) \cap N_{Z_0}(x_i) \). Suppose that \( Y^{i}_1, \ldots, Y^{i}_{r-1} \) have been chosen. Due to (26), we choose

\[ Y^{i}_{r} \subseteq (N_{Z_0}(v) \cap N_{Z_0}(x'_j)) \setminus \bigcup_{j \in [r-1]} Y^{j}_{i}. \]

Let \( Y^{j} = \cup_{i=0}^{r-1} Y^{i}_{j} \) for \( j \in [k] \). Then, we obtain \( k \)-partite graphs \( G_{Y} [Y^{0}_0, Y^{1}_1, \ldots, Y^{r-1}_r] \).

Since \( G \) is \( H \)-free, there exists \( j_0 \in [k] \) such that \( G_{Y^{j}_0} [Y^{j}_0, Y^{j}_{1}, \ldots, Y^{j}_{r-1}] \) is \( T(rp) \)-free by Theorem 1.1. Thus, by Lemma 3.2, we have

\[ e(G_{Y^{j}_0} [Y^{j}_0, Y^{j}_{1}, \ldots, Y^{j}_{r-1}]) \leq t_r \left( \frac{n}{4kr} \right) - \frac{n^2}{32k^2r^4}. \]

This means that

\[ e(G_{Z_0} [Z_0 \cup \{v\}, Z_1, \ldots, Z_{r-1}]) \leq t_r(n) - \frac{n^2}{32k^2r^4}. \]

(27)

On the other hand, by (14), (21) and (24), we have

\[ e(G_{Z_0} [Z_0 \cup \{v\}, Z_1, \ldots, Z_{r-1}]) \geq e(G_{Z} [Z_0, Z_1, \ldots, Z_{r-1}]) = \sum_{x \in S} d_{Z_0}(x) + \sum_{i=1}^{r-2} \sum_{x \in S_{r'}} d_{S}(x) \]

(28)
\[ n - (|Z_i| + |Z_{j'}| - 2k) < (1 - 1/r)n - k \]

a contradiction to (27).

Now, we prove the uniqueness of \( i = i(x) \) for \( x \in V(G) - Z \). Suppose that there exists \( x \in V(G) - Z \) and \( i, i' \in [0] \cup [r - 1] \) such that both \( d_{Z_i}(x) \) and \( d_{Z_{i'}}(x) \) are less than \( k \). This means \( Z_i \cup Z_{i'} \) has at least \( |Z_i| + |Z_{i'}| - 2k + 2 \) vertices that are not adjacent to \( x \). Thus \( d_G(x) \leq n - (|Z_i| + |Z_{i'}| - 2k) < (1 - 1/r)n - k \) in view of (21), a contradiction. \( \square \)

By Claim 3.3.2, for each \( x \in V(G) - Z \), there is a unique \( i = i(x) \) such that \( d_{Z_i}(x) < k \). We can put \( x \) in \( Z_{i(x)} \). Then, we get an \( r \)-partition \( (V_0, V_1, \ldots, V_{r-1}) \) of \( G \) with \( Z_i \subseteq V_i \) for \( i \in [0] \cup [r - 1] \). For \( x \in V(G) \), we consider the degree of \( x \) in \( V_i \) with \( x \not\in V_i \). Suppose first that \( x \in V_{i(x)} \) for \( x \in V(G) - Z \). For \( 0 \leq i \leq r - 1 \) with \( i \neq i(x) \), by (20), (21) and (22),

\[ d_{V_i}(x) \geq d_G(x) - d_{Z_{i(x)}}(x) - |V(G) - Z| - \sum_{0 \leq j < r - 1 \atop j \neq i} |Z_j| \geq \frac{n}{r} - ar - k. \]  

Then, we bound \( d_{V_{i'}}(x) \) for \( x \in V_i \) and \( i \neq i' \). For \( i \in [0] \cup [r - 1] \), it follows from (20) and (21) that

\[ \frac{n}{r} + k + \ell + a(r - 1) \geq |Z_i| + |V(G) - Z| \geq |V_i| \geq |Z_i| \geq \frac{n}{r} + k - a(r - 1). \]  

Let \( i \in V_i \) and \( i' \neq i \), \( 0 \leq i' \leq r - 1 \). Combining (23), (24) and (28),

\[ d_{V_{i'}}(x) \geq d_{Z_{i'}}(x) \geq \frac{n}{r} - 2 ((p + 1)(r - 1) + 1) a. \]  

Let \( b_1 = k + \ell + a(r - 1) \) and \( b_2 = 2 ((p + 1)(r - 1) + 1) a. \) Fixing \( i \in [0] \cup [r - 1] \), (29) and (30) can be reduced to

\[ \frac{n}{r} + b_1 \geq |V_i| \geq \frac{n}{r} + k - a(r - 1) \]  

and

\[ d_{V_{i'}}(x) \geq \frac{n}{r} - b_2 \geq |V_{i'}| - (b_1 + b_2) \]  

for \( x \in V_i \) and \( i' \neq i \), \( 0 \leq i' \leq r - 1 \). By (31) and (32), we have

\[ e(G[V_0, V_1, \ldots, V_{r-1}]) = \sum_{i=0}^{r-2} \sum_{i' \in \{0 \cup [r - 1]\}} d_{V_{i'}}(x) \geq \left( \frac{n}{r} - b_2 \right) \sum_{i=1}^{r-1} |V_i| \]

\[ = t_r(n) - 2(a(r - 1) - k + b_2)n. \]  

In what follows, we prove that \( (V_0, V_1, \ldots, V_{r-1}) \) is a \( k \)-good partition of \( G \).

First, we show that \( (V_0, V_1, \ldots, V_{r-1}) \) satisfies (1) of Definition 2.2. Note that \( V_i \neq \emptyset \) for \( 0 \leq i \leq r - 1 \) by (29). If \( \Delta(G[V_i]) \geq k \) for some \( i \in [0, 1, \ldots, r - 1] \), then we can choose \( x \in V_i \) and \( x_1, x_2, \ldots, x_k \in N_{V_i}(x) \). As in the proof of Claim 3.3.2, we can find \( k \)-partite graphs and one of them is \( T_r(rp^*j) \)-free for some \( j \in [k] \) by Theorem 1.1. Thus, by Lemma 3.2, we have

\[ e(G[V_0, V_1, \ldots, V_{r-1}]) \leq t_r(n) - \varepsilon n^2 \]  

for some \( \varepsilon > 0 \), a contradiction to (33).

Then, we show that \( (V_0, V_1, \ldots, V_{r-1}) \) satisfies (2) of Definition 2.2. Otherwise, by symmetry, suppose that \( \sum_{i \in [r - 1]} \nu(G[V_i]) \geq k \). Let \( x_1, y_1, x_2, y_2, \ldots, x_k, y_k \) be the matching \( M \) of \( G \) with \( x_i y_j \in E_{i(x_i y_j)} \) for some \( i(x_i y_j) \in [r - 1] \). We use \( M \) to find \( k \)-partite graphs. By (31) and (32), we have

\[ \left| \bigcap_{j \in [k]} (N_{V_0}(x_j) \cap N_{V_0}(y_j)) \right| \geq |V_0| - 2k(b_1 + b_2) \geq 0. \]
Choose a vertex \( v \in \cap_{j \in [k]}(N_{V_0}(x_j) \cap N_{V_0}(x_j)) \). For \( j \in [k] \) and \( q \in [r-1] \) with \( q \neq i(x_jy_j) \), let \( X'_q = N_{V_0}(v) \cap N_{V_0}(u_j) \cap N_{V_0}(v) \). Clearly, \(|X'_q| \geq n/r - 3(b_1 + b_2) \geq n/(2r) + 2k \) by (32). We first choose \( r-2 \) subsets \( Y'_j \) with \( q \in [r-1] \) and \( q \neq i(x_jy_j) \) such that \( Y'_q \subseteq X'_q \) and \(|Y'_q| = \frac{n}{2r} \). Then, for \( j \in \{2, 3, \ldots, k\} \), we choose \( r-2 \) subsets \( Y'_j \) with \( q \in [r-1] \) and \( q \neq i(x_jy_j) \) such that \( Y_q' \subseteq X_q' \cup \bigcup_{s \in [j-1]} Y'_q \) and \(|Y_q'| = \frac{n}{2r} \). Finally, we choose

\[
Y_j' \cap (x_jy_j) \subseteq V_i(x_jy_j)(v) \setminus \bigcup_{s \in [j]} Y'_j(\forall x_jy_j)
\]

with \(|Y_j'| = \frac{n}{2r} \) for \( j \in [k] \). Let \( Y_j = \bigcup_{i \in [r-1]} Y'_j \) for \( j \in [k] \). Then, we obtain \( k \) \((r-1)\)-partite graphs \( G_j = G[V'_1, V'_2, \ldots, V'_{r-1}] \). Since \( G \) is \( H \)-free, there exists some \( j_0 \in [k] \) such that \( G_{j_0} \) is \( T_r(rp_{j_0}) \)-free by Theorem [1.1]. Thus, by Lemma [3.2] we have

\[
e(G_{j_0}) \leq t_r\left(\frac{n}{2k}\right) - \frac{n^2}{8k^2r^2}.
\]

This means that

\[
e(G[V_0, V_1, \ldots, V_{r-1}]) \leq t_r(n) - \frac{n^2}{8k^2r^2},
\]
a contradiction to (33).

In the end, we show that \((V_0, V_1, \ldots, V_{r-1})\) satisfies (3) of Definition [2.2]. Otherwise, by symmetry, suppose that there exists \( v \in V_0 \) such that \( d(v) + \sum_{i \in [r-1]} (G[N_i(v)]) \geq k \). Thus, we can pick \( z_1, z_2, \ldots, z_r \) in \( N_0(v) \) and \((k-s)\)-matching \( x_{s+1}y_{s+1}, x_{s+2}y_{s+2}, \ldots, x_ky_k \) in \( \sum_{i \in [r-1]} G[N_i(v)] \) such that \( x_j \) and \( y_j \) are in the same vertex class \( V_i(x_jy_j) \) for \( s+1 \leq j \leq k \). As the same methods used to verify (1) and (2) of Definition [2.2], we can show that \( e(G[V_0, V_1, \ldots, V_{r-1}]) \leq t_r(n) - \varepsilon n^2 \) for some \( \varepsilon > 0 \) by finding \( s \) \( k \)-partite graphs \( Y_1, Y_2, \ldots, Y_s \) with \( v_j \in V(Y) \), and \((k-s) \( (r-1)\)-partite graphs \( Y_{k-s+1}, \ldots, Y_k \) with \( x_j, y_j \in V(Y) \), a contradiction to (33). Thus, we conclude that \((V_0, V_1, \ldots, V_{r-1})\) is a \( k \)-good partition of \( G \), completing the proof of Theorem [2.4].

4 Stability results for the suspension of edge-critical graphs

In this section, we prove Theorem [1.5]. We first present the following fundamental result, the Erdős-Simonovits Stability Theorem, used in our proof.

**Theorem 4.1** (Erdős-Simonovits Stability Theorem [5, 17]). Let \( r \geq 2 \) and suppose that \( F \) is a graph with \( \chi(F) = r + 1 \). If \( G \) is an \( F \)-free graph with \( e(G) \geq t_r(n) - o(n^2) \), then \( G \) can be formed from \( T_r(n) \) by adding and deleting \( o(n^2) \) edges.

Now, we prove a weaker result as a stepping stone to Theorem [1.5].

**Lemma 4.2.** Let \( f(n) = o(n^2) \) be a function. Suppose that \( G \) is an \( H \)-free graph on \( n \) vertices such that \( e(G) \geq t_r(n) - f(n) \). Then \( G \) can be made \( r \)-partite by deleting \( O(f(n)) \) edges.

**Proof.** Let \( \delta, \varepsilon, \eta \in [0, 10^{-d}k^{-4}] \) with \( \varepsilon \leq \eta^2/5 \), \( \gamma = 100r^2k(e^{1/2} + \delta + \eta) \) and \( \theta > 100r^3k(e^{1/2} + \delta + \eta) \). Suppose that \( G \) is an \( H \)-free graph on \( n \) vertices with \( e(G) \geq t_r(n) - f(n) \). By Theorem [4.1] there exists some \( N_0 \) such that \( G \) can be formed from \( T_r(n) \) by adding and deleting \( o(n^2) \) edges for \( n \geq N_0 \). Suppose that \( n \geq 2N_0 \). We say that \( f \) is a “smaller” vertex of \( F \) if \( d(f) \leq (1 - \delta)\epsilon^{-1/2} |V(F)| - 1 \). We first delete a “smaller” vertex of \( G \). As long as there is a “smaller” vertex in the resulting graph, we delete it. We keep doing this until the remaining graph \( G' \) (\( G' \) maybe empty) has no such vertices. Let \( L = V(G) - V(G') \).

**Claim 4.2.1.** \( |L| \leq (k^2 + f(n))(\frac{2}{(2r-1)n} - 1) = o(n) \).
Proof. Note that $G'$ is $H$-free. Thus, $e(G') \leq t_r(n - |L|) + k^2$ by Theorem 4.2. It follows that

\[
e(G) \leq t_r(n - |L|) + k^2 + \sum_{i=|V'(G)|+1}^{n} \left( (1 - \delta)\frac{r-1}{r}i - 1 \right)
\]

\[
= t_r(n - |L|) + k^2 + \sum_{i=|V'(G)|+1}^{n} \left( \frac{r-1}{r}i - 1 \right) - \sum_{i=|V'(G)|+1}^{n} \left( \frac{r-1}{r}i \right)
\]

\[
\leq t_r(n) + k^2 - \sum_{i=|V'(G)|+1}^{n} \left( \frac{r-1}{r}i \right)
\]

\[
= t_r(n) + k^2 - \frac{\delta}{r}n(n-1) \left( \frac{r}{2} \right)
\]

\[
= t_r(n) + k^2 - \frac{\delta}{r}n(n-1) \left( \frac{r}{2} \right)
\]

Note that the function $f(x) = (1/2 + n)x - x^2/2$ is strictly monotone increasing on $[0, n]$. If $|L| \leq n/2$, then $f(|L|) \leq f(n/2)$. It follows from (36) that

\[
e(G) < t_r(n) + k^2 - \frac{\delta}{r}n(n-1) \left( \frac{r}{2} \right)
\]

This is a contradiction to $e(G) \geq t_r(n) - f(n)$ for sufficiently large $n$. Thus, $|L| < n/2$. In view of (35)

\[
e(G) \leq t_r(n) + k^2 - \frac{\delta}{r}n(n-1) \left( \frac{r}{2} \right)
\]

This together with $e(G) \geq t_r(n) - f(n)$ yields that $|L| \leq (k^2 + f(n)) \left( \frac{2\delta}{\delta r - 1}n^{-1} \right)$.

By Claim 4.2.1, we have

\[
e(G') \geq e(G) - |L| \left( (1 - \delta)\frac{r}{r + 1}n \right) = e(G) - O(f(n)) \geq t_{r+1}(n) - O(f(n))
\]

for sufficiently large $n$. So, it suffices to show that $G'$ can be formed from an $r$-partite graph by deleting at most $(r + 1)f(k - 1, k - 1) = o(n)$ edges.

Note that $|V(G')| = n - |L| = (1 - o(1))n \geq N_t$. Thus, we can partite $V(G')$ to $V'_i, V'_2, \ldots, V'_r$ such that there are at most $en^2$ edges within the vertex classes by Theorem 4.1 and (37). This implies that there are at most $3en^2/2$ edges missing between vertex classes $V'_1, V'_2, \ldots, V'_r$, i.e.,

\[
e(G'[V'_1, V'_2, \ldots, V'_r]) \geq t_r(n) - \frac{3}{2}en^2 \geq t_r(|V(G')|) - \frac{3}{2}en^2.
\]

For every $v \in V(G')$, we have

\[
d_{G'}(v) \geq (1 - \delta)\frac{r-1}{r}n - |L|
\]

\[
\geq (1 - \delta)\frac{r-1}{r}n - (1 - \delta)\frac{r-1}{r}(k^2 + f(n)) \left( \frac{2\delta}{\delta r - 1}n^{-1} \right)
\]

\[
= (1 - \delta)\frac{r-1}{r}n - o(n) \geq (1 - 2\delta)\frac{r-1}{r}n.
\]

(39)

Suppose that there exists some vertex $v$ having at least $km$ neighbours in every $V'_i$ for $i \in [r]$. For each $i \in [r]$, pick $F'_i \subset N_{V'_i}(v)$ with $|F'_i| = \eta n$, and pick $F'_j \subset N_{V'_i}(v) \setminus \cup_{j \neq i} F'_j$ for each $j \in [k] \setminus \{1\}$ with $|F'_j| = \eta n$. Let $Q_j = \cup_{j=1}^{k} F'_j$. Recall that $G'$ is $H$-free. Thus, there exists $j_0$ such that $G'_q [F'_{j_0}, F'_{j_0}, \ldots, F'_{j_0}]$ is $T_r(rp_{j_0})$-free. By Lemma 3.2 we have

\[
e(G'_{q_{j_0}} [F'_{j_0}, F'_{j_0}, \ldots, F'_{j_0}]) \leq t_r(rp) - \frac{(\eta n)^2}{2}.
\]
implying that there are at least \((\eta n)^2/2\) edges missing between \(V'_i, V'_2, \ldots, V'_r\), a contradiction to (38).

Without loss of generality, we can suppose that every vertex in \(G'\) has at most \(k\eta n\) neighbours inside its own vertex class. This together with (39) shows that for each \(v \in V'_i\) with \(i \in [r]\)

\[
d_{V'_i(V'_i \cup V'_j)}(v) \geq (1 - 2\delta) \frac{r - 1}{r} n - \eta n.
\]

By (38), it is easy to see that

\[
|V'_i| \geq \frac{|V(G')|}{r} - (r - 1) \sqrt{\frac{3}{2}} \varepsilon n \geq (1 - 2re^{1/2}) \frac{n}{r}
\]

for each \(i \in [r]\). On the other hand

\[
|V'_i| \leq n - (r - 1)(1 - 2re^{1/2}) \frac{n}{r} = (1 + 2r^2e^{1/2}) \frac{n}{r}.
\]

If \(e(G'[V_i]) \geq f(k - 1, k - 1)\) for some \(i \in [r]\), say \(i = 1\), then we can find a \(k\)-matching in \(V'_i\) or find a vertex \(v \in V'_i\) such that \(d_{V'_i}(v) \geq k\). In the following, we divide our proof into the following two cases.

Case 1. \(V'_i\) contains a vertex \(v\) such that \(d_{V'_i}(v) \geq k\). Suppose that \(v_1, v_2, \ldots, v_k\) are \(k\) distinct vertices in \(N_{V'_i}(v)\). By (40) and (41), we have

\[
|N_{V'_i}(v) \cap N_{V'_i}(v_i)| \geq |V'_i| - 2 \left( (n - |V'_i|) - \left( 1 - 2\delta \frac{r - 1}{r} n - \eta n \right) \right)
\geq \frac{n}{r} - \left( \frac{6re^{1/2} + 4\delta(r - 1)}{r} n + 2\eta n \right)
\geq (1 - \gamma) \frac{n}{r}.
\]

Pick \(F'_1 \subseteq V'_i \setminus \{v, v_1, v_2, \ldots, v_k\}\) with \(|F'_1| = (1 - \gamma) \frac{n}{2r}\), and pick \(F'_i \subseteq N_{V'_i}(v) \cap N_{V'_i}(v_i)\) with \(|F'_i| = (1 - \gamma) \frac{n}{2r}\) for \(i \in [r] \setminus \{1\}\). Then, pick

\[
F'_i \subseteq V'_i \setminus \left( \bigcup_{j \neq i} F'_j \cup \{v, v_1, v_2, \ldots, v_k\} \right)
\]

and \(F'_i \subseteq V'_i \setminus \cup_{j \neq i} F'_i\) with \(|F'_i| = (1 - \gamma) \frac{n}{2r}\) for each \(i \in [r] \setminus \{1\}\) and each \(j \in [k] \setminus \{1\}\). Let \(Q_j = \bigcup_{i=1}^{r} F'_i\). We obtain \(k\) \(r\)-partite graphs \(G_{Q_0}^{F_{1}, F_{2}, \ldots, F_{r}}\) for \(j \in [k]\). Since \(G\) is \(H\)-free, there exists \(j_0\) such that \(G_{Q_0}^{F_{1}, F_{2}, \ldots, F_{r}}\) is \(T_i(rp_{h})\)-free. Thus, by Lemma 3.2

\[
e(G_{Q_0}^{F_{1}, F_{2}, \ldots, F_{r}}) \leq tr \left( \frac{(1 - \gamma)n}{2k} \right) - \frac{(1 - \gamma)^2 n^2}{8k^2 r^2},
\]

a contradiction to (38).

Case 2. \(V'_i\) contains a \(k\)-matching, say \(\{u_1v_1, u_2v_2, \ldots, u_kv_k\}\). For \(i \in [r] \setminus \{1\}\), by (40) and (41)

\[
\left| \bigcap_{j=1}^{k} (N_{V'_i}(u_j) \cap N_{V'_i}(v_j)) \right| \geq |V'_i| - 2k \left( (n - |V'_i|) - \left( 1 - 2\delta \frac{r - 1}{r} n - \eta n \right) \right)
\geq \frac{n}{r} - \left( \frac{4k + 2re^{1/2} + 4k\delta(r - 1)}{r} n + 2kn \right)
\geq (1 - k\gamma) \frac{n}{r}.
\]
Note that \((1 - k\gamma)^2 > 1\). Choose \(v \in \cap_{i=1}^{k} (N_{V_i}(u_i) \cap N_{V_i}(v_j))\). By (40) and (42), we have
\[
d_{V}^i(v) \geq d_{V(G) \setminus V_i}(v) - \sum_{q \in [r-1] \setminus \{j\}} |V_q'| \geq (1 - \theta) \frac{n}{r}
\]
for \(j \in [r-1]\). This together with (42) and (43) shows that for \(i \in [r-1] \setminus \{1\}\)
\[
\left|N_{V_i}(v) \cap \left( \bigcap_{j=1}^{k} (N_{V_i}(u_j) \cap N_{V_i}(v_j)) \right) \right| \geq \left| \bigcap_{j=1}^{k} (N_{V_i}(u_j) \cap N_{V_i}(v_j)) \right| - (|V_i| - d_{V_i}(v))
\]
\[
\geq (1 - k\gamma) \frac{n}{r} - \left( (1 + 2r^2 e^{1/2}) \frac{n}{r} - (1 - \theta) \frac{n}{r} \right)
\]
\[
\geq (1 - 10k\theta) \frac{n}{r}.
\]
Pick \(F_1^i \subseteq N_{V_i}(v) \setminus \{u_1, v_1, u_2, v_2, \ldots, u_k, v_k\}\) with \(|F_1^i| = (1 - 10k\theta) \frac{n}{2kr}\), and pick \(F_j^i \subseteq N_{V_i}(v) \cap \left( \bigcap_{j=1}^{k} (N_{V_i}(u_j) \cap N_{V_i}(v_j)) \right)\) with \(|F_j^i| = (1 - 10k\theta) \frac{n}{2kr}\) for \(i \in [r] \setminus \{1\}\). Then, pick
\[
F_j^i \subseteq N_{V_i}(v) \setminus \left( \bigcup_{j \neq j} F_j^i \cup \{u_1, v_1, u_2, v_2, \ldots, u_k, v_k\} \right)
\]
with \(|F_j^i| = (1 - 10k\theta) \frac{n}{2kr}\), and \(F_j^i \subseteq V_i \setminus \cup_{j < j} F_j^i\) with \(|F_j^i| = (1 - 10k\theta) \frac{n}{2kr}\) for each \(i \in [r] \setminus \{1\}\) and each \(j \in [k] \setminus \{1\}\). Let \(Q_i^j = F_1^i \cup F_2^i \cup \ldots \cup F_{r-1}^i\) for each \(j \in [k]\). Since \(G\) is \(H\)-free, there exists \(j_0\) such that \(G_{Q_0}^{j_0} \cap \{F_1^{j_0}, F_2^{j_0}, \ldots, F_{r-1}^{j_0}\} \) is \(T_{r-1}(p_{j_0}^2(r-1))\)-free. Thus, by Lemma 3.2, we have
\[
e(G_{Q_0}^{j_0} \cap \{F_1^{j_0}, F_2^{j_0}, \ldots, F_{r-1}^{j_0}\}) \leq t_{r-1} \left( \frac{(1 - 10k\theta)(r-1)n}{2kr} - (1 - 10k\theta)^2 n^2 \right).
\]
This is a contradiction to (38).

Thus, \(e(G'[V_i]) \leq f(k-1, k-1)\) for each \(i \in [r]\). This implies that \(G'\) can be formed from \(r\)-partite graph by deleting at most \(r(f(k-1, k-1) = o(n)\) edges, completing the proof of Lemma 4.2. □

**Proof of Theorem 1.5** We can take a partition \((V_1, V_2, \ldots, V_r)\) of \(V(G)\) which minimises the number of edges inside vertex classes. By Lemma 4.2, there are \(c_0^i f(n) = O(f(n))\) edges within vertex classes and at most \(c_1^i f(n) = O(f(n))\) edges between vertex classes are not present in \(G\).

**Claim 4.2.2.** For \(i \in [r]\) and \(v \in V_i\), \(d_{V_i}(v) = O(f(n)^{1/2})\).

*Proof.* Suppose that there exists some \(i_0\) such that \(d_{V_i}(v) > c_0 f(n)^{1/2}\) for some \(v \in V_i\). Without loss of generality, let \(i_0 = 1\) and \(v \in V_1\) such that \(d_{V_1}(v) > 2k(c_1^1 f(n))^{1/2}\). Then, \(|N_{V_i}(v)| > 2k(c_1^1 f(n))^{1/2}\) for each \(i \in [r]\). Pick \(F_1 \subseteq N_{V_i}(v)\) and \(F_j \subseteq N_{V_i}(v) \setminus \cup_{j < j} F_j^i\) with \(|F_j^i| = 2(c_1^j f(n))^{1/2}\) for each \(i \in [r]\) and each \(j \in [k]\). Let \(Q_i^j = \cup_{j < j} F_j^i\). Since \(G\) is \(H\)-free, there exists \(j_0\) such that \(G[F_1^{j_0}, F_2^{j_0}, \ldots, F_r^{j_0}]\) is \(T_r(p_{j_0}^2(r))\)-free. Thus, by Lemma 3.2, we have
\[
e(G[F_1^{j_0}, F_2^{j_0}, \ldots, F_r^{j_0}]) \leq t_r \left( 2r(c_1^j f(n))^{1/2} - 2r c_1^j f(n) \right).
\]
implying that there are at least \(2r c_1^j f(n)\) edges missing between \(V_1, V_2, \ldots, V_r\), a contradiction. □

By Claim 4.2.2 and the proof of Lemma 4.2, \(G\) can be made \(r\)-partite by deleting at most \(O(f(n)^{1/2})\) edges. This completes the proof. □
References

[1] H. Abbott, D. Hanson, H. Sauer, Intersection theorems for systems of sets, J. Combin. Theory Ser. A 12 (1972) 381–389.

[2] G. Chen, R. Gould, F. Pfender, B. Wei, Extremal graphs for intersecting cliques, J. Combin. Theory Ser. B 89 (2003) 159–171.

[3] V. Chvátal, D. Hanson, Degrees and matchings, J. Combin. Theory Ser. B 20 (1976) 128–138.

[4] D. Desai, L. Kang, Y. Li, Z. Ni, M. Tait, J. Wang, Spectral extremal graphs for intersecting cliques. Linear Algebra Appl. 644 (2022) 234–258.

[5] P. Erdős, Some recent results on extremal problems in graph theory (Results), Theory of Graphs(Internl. Symp. Rome) (1966) 118–123.

[6] P. Erdős, Z. Füredi, R. Gould, D. Gunderson, Extremal graphs for intersecting triangles, J. Combin. Theory Ser. B 64 (1995) 89–100.

[7] P. Erdős, M. Simonovits, A limit theorem in graph theory, Studia Sci. Math. Hungar. 1 (1966) 51–57.

[8] P. Erdős, A. Stone, On the structure of linear graphs, Bull. Amer. Math. Soc. 52 (1946) 1087–1091.

[9] Z. Füredi, D. Gunderson, Extremal numbers for odd cycles, Combin. Probab. Comput. 24 (2015) 641–645.

[10] R. Glebov, Extremal graphs for clique-paths, arXiv:1111.7029v1.

[11] J. He, J. Ma, T. Yang, Some extremal results on 4-cycles, J. Combin. Theory Ser. B 149 (2021) 92–108.

[12] X. Hou, Y. Qiu, B. Liu, Decomposition of graphs into \((k, r)\)-fans and single edges, J. Graph Theory 87 (2018) 46–60.

[13] Y. Lan, T. Li, Y. Shi, J. Tu, The Turán number of star forests, Appl. Math. Comput. 348 (2019) 270–274.

[14] B. Lidický, H. Liu, C. Palmer, On the Turán number of forests, Electron. J. Comb. 20 (2013) P62.

[15] H. Liu, Extremal graphs for blow-ups of cycles and trees, Electron. J. Combin. 20 (2013) P65.

[16] A. Roberts, A. Scott, Stability results for graphs with a critical edge, European J. Combin. 74 (2018) 27–38.

[17] M. Simonovits, Extremal graph problems with symmetrical extremal graphs. Additional chromatic conditions, Discrete Math. 7 (1974) 349–376.

[18] P. Turán, Eine Extremalaufgabe aus der Graphentheorie, Mat. Fiz. Lapok 48 (1941) 436–452.

[19] L. Yuan, Extremal graphs for the \(k\)-flower, J. Graph Theory 89 (2018) 26–39.

[20] L. Yuan, Extremal graphs for odd wheels, J. Graph Theory 98 (2021) 691–707.
[21] L. Yuan, Extremal graphs for edge blow-up of graphs, J. Combin. Theory Ser. B 152 (2022) 379–398.

[22] L. Yuan, Extremal graphs of the pth power of paths, European J. Combin. 104 (2022) Paper No. 103548, 12 pp.

[23] L. Yuan, X. Zhang, Turán numbers for disjoint paths, J. Graph Theory 98 (2021) 499–524.