SLIDING MODE ON TANGENTIAL SETS OF FILIPPOV SYSTEMS

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Abstract. We consider piecewise smooth vector fields $Z = (Z_+, Z_-)$ defined in $\mathbb{R}^n$ where both vector fields are tangent to the switching manifold $\Sigma$ along a submanifold $M \subset \Sigma$. We shall see that, under suitable assumptions, Filippov convention gives rise to a unique sliding mode on $M$, governed by what we call the tangential sliding vector field. Here, we will provide the necessary and sufficient conditions for characterizing such a vector field. Additionally, we prove that the tangential sliding vector field is conjugated to the reduced dynamics of a singular perturbation problem arising from the Sotomayor-Teixeira regularization of $Z$ around $M$. Finally, we analyze several examples where tangential sliding vector fields can be observed, including a model for intermittent treatment of HIV.

1. Introduction

In this paper, we consider piecewise smooth vector fields (PSVFs) in two zones defined on $U \subset \mathbb{R}^n$ for which there exists a switching codimension-one manifold $\Sigma$ separating $U$ into two parts $\Sigma_+$ and $\Sigma_-$, such that smooth vector fields $Z_+$ and $Z_-$ are defined in each one of these parts.

Nowadays, there is a vast literature about PSVFs and their applications. A non-exhaustive list of books concerning this theme includes [3, 11, 14, 21, 29, 36]. Also, papers like [4, 5, 9, 15, 17, 19, 27, 34, 35] deal with applications of such theory to real-world phenomena.

Certainly, Filippov’s book [14] is one of the most valuable texts concerning PSVFs, where the definition for solutions of discontinuous differential equations is established by means of differential inclusions (see Section 2). Several scenarios of low codimension were deeply investigated by Filippov, which gave rise to the notions of sliding, escaping, and crossing solutions. Along these solutions, the smooth vector fields $Z_+$ and $Z_-$ are transversal to the switching manifold $\Sigma$. Filippov also investigated the dynamics around isolated tangencies between $\Sigma$ and the vector fields $Z_+$ and $Z_-$. However, he has not further explored the differential inclusions when both vector fields $Z_+$ and $Z_-$ are simultaneously tangent to $\Sigma$ along a submanifold $M \subset \Sigma$ of dimension greater or equal to one. Consequently, much of the subsequent development also dealt with the dynamics around isolated tangencies (see, for instance, [6, 8, 16, 22, 25, 32]), while the dynamics on higher-dimensional tangential manifold remained unexplored.

2010 Mathematics Subject Classification. 34A36, 34A60, 34E15.

Key words and phrases. Filippov systems, sliding mode, tangential set, regularization, singular perturbation problem.
1.1. Main goals and results. In the present paper, we aim to explore the dynamics of Filippov systems on higher-dimensional tangential manifolds.

First, we shall show that, under suitable assumptions, the Filippov convention gives rise to a unique sliding mode on $M$, which is governed by what we call the tangential sliding vector field. In this direction, Theorem 1 provides the necessary and sufficient conditions for the existence of tangential sliding vector fields, which are then formalized in Definition 3. Theorem 2 establishes that the trajectories of a tangential sliding vector field correspond to solutions of the Filippov differential inclusion.

We shall also investigate how the sliding mode provided by the tangential sliding vector field behaves under Sotomayor-Teixeira regularization. Smoothing processes of PSVF are a good ally in understanding the dynamics and applicability of non-smooth models. Regarding Filippov systems, Sotomayor-Teixeira regularization, introduced in [31], is intrinsically related to Filippov convention for sliding solutions. Indeed, it was proven in [33] that the Sotomayor-Teixeira regularization of a Filippov system around a sliding set produces a singular perturbation problem with reduced dynamics conjugated to the sliding dynamics. Also, a similar relation for sliding dynamics with hidden terms was obtained in [23] by allowing regularizations with non-monotonic transition functions (see also [10]). Since Sotomayor-Teixeira’s paper [31], isolated tangencies have also been approached by means of regularizations (see, for instance, [1, 2, 24]), but, again, higher-dimensional tangential manifolds have not been considered. In this regard, Theorem 3 shows that the tangential sliding vector field is conjugated to the reduced dynamics of a singular perturbation problem arising from the Sotomayor-Teixeira regularization of $Z$ around $M$.

Finally, we analyze several examples where tangential sliding vector fields can be observed, including an applied Filippov model for intermittent treatment of HIV.

1.2. Structure of the paper. Section 2 provides the basic concepts and notions needed in this paper concerning Filippov systems, regularization processes, and singular perturbation problems. Section 3 presents our first main results, Theorems 1 and 2, regarding the definition of tangential sliding vector fields. Section 4 provides several examples where tangential sliding vector fields can be observed, including an applied Filippov model for intermittent treatment of HIV. Finally, Section 5 is dedicated to proving our third main result, Theorem 3, about the regularization of tangential sliding vector fields.

2. Basic theory

This section is devoted to discuss the basic notions needed in this paper.

2.1. Piecewise smooth vector fields and Filippov convention. Let $h : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$ be a smooth function with 0 as a regular value. Denote $\Sigma = h^{-1}(0)$, $\Sigma_+ = \{x \in U \mid h(x) \geq 0\}$, and $\Sigma_- = \{x \in U \mid h(x) \leq 0\}$. Consider $Z_+$ a smooth vector field defined in $\Sigma_+$ and $Z_-$ a smooth vector field defined in $\Sigma_-$. A PSVF
\( Z = (Z_+, Z_-) \) defined in \( U \subset \mathbb{R}^n \) is given by

\[
Z(x) = \begin{cases} 
Z_+(x), & h(x) \geq 0, \\
Z_-(x), & h(x) \leq 0.
\end{cases}
\]

For points in \( \Sigma_+ \setminus \Sigma \) and \( \Sigma_- \setminus \Sigma \), the local trajectories of (1) are given by \( Z_+ \) and \( Z_- \). For points in \( \Sigma \), we get that \( Z \) is multi-valued and a non-classical theory must be considered. In fact, the notion of local trajectories for PSVF (1) was stated by Filippov [14] as solutions of the following differential inclusion

\[
p \in F_Z(p) = \frac{Z_+(p) + Z_-(p)}{2} + \text{sign}(h(p)) \frac{Z_+(p) - Z_-(p)}{2},
\]

where

\[
\text{sign}(s) = \begin{cases} 
-1 & \text{if } s < 0, \\
[-1, 1] & \text{if } s = 0, \\
1 & \text{if } s > 0.
\end{cases}
\]

A solution of the differential inclusion (2) is defined as an absolutely continuous function \( \varphi(t) \) such that \( \varphi'(t) \in F_Z(\varphi(t)) \) for almost every \( t \). This approach is called Filippov convention. A PSVF (1) is called a Filippov system when it is governed by Filippov convention. For more information on differential inclusions, see, for instance, [30].

A class of solutions of the differential inclusion (2) was deeply explored by Filippov in [14] and, due to its wide range of application, continues to receive significant attention. In order to better understand the class of solutions of the differential inclusion (2) explored by Filippov, we must determine the contact of the trajectories of \( Z_+ \) and \( Z_- \) with \( \Sigma \). For this purpose, consider the Lie derivatives \( Z_+ h(p) = \langle \nabla h(p), Z_+(p) \rangle \) and \( Z_+^i h(p) = \langle \nabla Z_+^{i-1} h(p), Z_+(p) \rangle \) for \( i \geq 2 \), where \( \langle ., . \rangle \) is the usual inner product and \( \nabla h(p) \) denotes the gradient of the function \( h \) at \( p \). The same for \( Z_- \).

We distinguish the following regions on the discontinuity set \( \Sigma \):

- **Crossing Region**: \( \Sigma^c = \{ p \in \Sigma \mid Z_+ h(p) Z_- h(p) > 0 \} \). Moreover, we let \( \Sigma^{c+} = \{ p \in \Sigma \mid Z_+ h(p) > 0, Z_- h(p) > 0 \} \) and \( \Sigma^{c-} = \{ p \in \Sigma \mid Z_+ h(p) < 0, Z_- h(p) < 0 \} \).
- **Sliding Region**: \( \Sigma^s = \{ p \in \Sigma \mid Z_+ h(p) < 0, Z_- h(p) > 0 \} \).
- **Escaping Region**: \( \Sigma^e = \{ p \in \Sigma \mid Z_+ h(p) > 0, Z_- h(p) < 0 \} \).

A point \( q \in \Sigma \) satisfying \( (Z_+ h(q))(Z_- h(q)) = 0 \) and \( Z_+ h(q) \neq 0 \) is called a **tangential singularity** (or also a **tangency point**). We denote by \( \Sigma^{\tan} \) the set of tangential singularities. According to [37] (see also [7]), we say that a smooth vector field \( Z_\pm \) presents a generic contact of multiplicity \( k \) with \( \Sigma \) at \( p \) if \( Z_\pm^r h(p) = 0 \) for \( r = 1, \ldots, k - 1 \), and \( (Z_\pm)^k h(p) \neq 0 \).

Despite the richness of the book [14], Filippov does not further explore solutions of the differential inclusion (2) that may live on \( \Sigma^{\tan} \).
2.2. Regularization and singular perturbation problem. The concept of \( \phi \)-regularization of PSVFs was introduced by Sotomayor and Teixeira in [31]. It provides a 1-parameter family of smooth vector fields \( Z_\varepsilon \) such that, for each \( \varepsilon > 0 \), \( Z_\varepsilon \) is equal to \( Z_\pm \) in all points of \( \Sigma_\pm \) whose distance to \( \Sigma \) is greater than or equal to \( \varepsilon \). In what follows, we provide the formal definition.

**Definition 1.** A function \( \phi : \mathbb{R} \to \mathbb{R} \) is said to be a \( C^r \)-transition function, \( r \geq 0 \), if it is of class \( C^r \), \( \phi(x) = -1 \) for \( x \leq -1 \), \( \phi(x) = 1 \) for \( x \geq 1 \), and \( \phi'(x) > 0 \) if \( x \in (-1, 1) \). The \( \phi \)-regularization of \( Z = (X, Y) \) is the 1-parameter family \( Z_\varepsilon \) given by

\[
Z_\varepsilon(q) = \left( \frac{1}{2} + \frac{\phi_\varepsilon(h(q))}{2} \right) Z_+(q) + \left( \frac{1}{2} - \frac{\phi_\varepsilon(h(q))}{2} \right) Z_-(q),
\]

with \( q \in U \) and \( \phi_\varepsilon(x) = \phi(x/\varepsilon) \), for \( \varepsilon > 0 \).

The differential equation

\[
\dot{q} = Z_\varepsilon(q)
\]

can be studied in terms of geometric singular perturbation theory [13, 18].

**Definition 2.** Let \( U \subseteq \mathbb{R}^n \) be an open subset and take \( \varepsilon \geq 0 \). A singular perturbation problem in \( U \) (SP-Problem) is a differential system which can be written as

\[
x' = \frac{dx}{d\tau} = l(x, y, \varepsilon), \quad y' = \frac{dy}{d\tau} = \varepsilon m(x, y, \varepsilon)
\]

with \( x \in \mathbb{R}, \ y \in \mathbb{R}^{n-1} \) and \( l, m \) smooth in all variables, or equivalently, after the time rescaling \( t = \varepsilon \tau \),

\[
\varepsilon \dot{x} = \varepsilon \frac{dx}{dt} = l(x, y, \varepsilon), \quad \dot{y} = \frac{dy}{dt} = m(x, y, \varepsilon).
\]

The differential system (4) is called the fast system, and the differential system (5) is called the slow system of the SP-problem. Note that for \( \varepsilon > 0 \), the phase portraits of the fast and the slow systems coincide.

By taking \( \varepsilon = 0 \) in (4), we get the layer problem

\[
x' = \frac{dx}{d\tau} = l(x, y, 0), \quad y' = 0.
\]

Accordingly, the slow set \( S \) of the SP-problem is defined as the critical points of (6), that is,

\[
S = \{(x, y) : l(x, y, 0) = 0\}.
\]

By taking \( \varepsilon = 0 \) in (5), we get the reduced problem

\[
0 = l(x, y, \varepsilon), \quad \dot{y} = \frac{dy}{dt} = m(x, y, 0),
\]

which induces a dynamics on \( S \).

3. Tangential Sliding Vector Field

Let \( M \subset \Sigma^{\text{tan}} \) be described by

\[
M = \eta^{-1}(0),
\]

where \( \eta : V \subset \mathbb{R}^n \to \mathbb{R}^m \), with \( m < n \) and \( V \subset U \) open, is a smooth function with \( 0 \in \mathbb{R}^m \) as a regular value. The next result is a well-known fact concerning Differential Geometry (see [20, Corollary 5.24 and Lemma 5.29]):
Proposition 1. If $\eta : \mathbb{R}^n \to \mathbb{R}^m$, where $m < n$, is a smooth function with $0 \in \mathbb{R}^m$ as a regular value, then $M = \eta^{-1}(0)$ is a codimension $m$ submanifold, and

$$T_pM = \ker(d\eta(p)) \forall p \in M,$$

where $d\eta$ is the differential of $\eta$.

In what follows, we shall study necessary and sufficient conditions for the existence of a tangential sliding vector field on $M$. For each $p \in \Sigma$, define

$$C_p = \{C(p, \lambda) : \lambda \in [-1, 1]\},$$

where

$$C(p, \lambda) = \frac{1-\lambda}{2}Z_+(p) + \frac{1+\lambda}{2}Z_-(p).$$

For $p \in M$, the tangential sliding vector field will be constructed by means of the intersection of the set $C_p$ with the tangent space of $M$ at $p$, $T_pM$. Accordingly, the next result provides necessary and sufficient conditions for $C_p \cap T_pM \neq \emptyset$.

Theorem 1. Consider $p \in M$. Thus, $C_p \cap T_pM \neq \emptyset$ if, and only if,

$$\langle d\eta(p)Z_+(p), d\eta(p)Z_-(p) \rangle = -\|d\eta(p)Z_+(p)\|\|d\eta(p)Z_-(p)\|.$$

Moreover,

1. If $\|d\eta(p)Z_+(p)\|\|d\eta(p)Z_-(p)\| \neq 0$, then $C_p \cap T_pM = \{C(p, \lambda^*(p))\}$ with

$$\lambda^*(p) := \frac{\|d\eta(p)Z_+(p)\| - \|d\eta(p)Z_-(p)\|}{\|d\eta(p)Z_+(p)\| + \|d\eta(p)Z_-(p)\|} \in (-1, 1).$$

2. If $d\eta(p)Z_+(p) = 0$ and $d\eta(p)Z_+(p) \neq 0$, then $C_p \cap T_pM = \{Z_-(p)\}.$

3. If $d\eta(p)Z_+(p) = 0$ and $d\eta(p)Z_-(p) \neq 0$, then $C_p \cap T_pM = \{Z_+(p)\}.$

4. If $d\eta(p)Z_+(p) = d\eta(p)Z_-(p) = 0$, then $C_p \subset T_pM.$

Proof. Since, from Proposition 1, $T_pM = \ker(d\eta(p))$, we have

$$C(p, \lambda) \in T_pM \iff d\eta(p) \cdot C(p, \lambda) = 0,$$

replacing the expression (8) we obtain

$$\frac{(1-\lambda)}{2}d\eta(p)Z_+(p) = -\frac{(1+\lambda)}{2}d\eta(p)Z_-(p).$$

Taking into account that $\lambda \in [-1, 1]$, the relationship (11) means that the vectors $d\eta(p)Z_+(p)$ and $d\eta(p)Z_-(p)$ are proportional with a negative proportionality constant. Hence, $C_p \cap T_pM \neq \emptyset$ if, and only if, (9) holds.

Case 1. Assuming that $\|d\eta(p)Z_+(p)\|\|d\eta(p)Z_-(p)\| \neq 0$, we get that (11) holds if, and only if, $\lambda = \lambda^*(p)$, where $\lambda^*(p)$ is given by (10). Notice that, in this case, $\lambda^*(p) \in (-1, 1).$

Cases 2 and 3. Assuming that $d\eta(p)Z_-(p) = 0$ and $d\eta(p)Z_+(p) \neq 0$, the relationship (11) reduces to $(1-\lambda)d\eta(p)Z_+(p) = 0$, which holds if, and only if, $\lambda = 1$. In this case, $C_p \cap T_pM = \{Z_-(p)\}.$ In an analogous way, for Case 3 we conclude that

$$C_p \cap T_pM = \{Z_+(p)\}.$$
Cases 4. Assuming \(d\eta(p)Z_-(p) = d\eta(p)Z_+(p) = 0\), the relationship (11) holds for every \(\lambda \in [-1, 1]\). Hence, \(C_p \subset T_pM\).

In what follows, considering Theorem 1, we provide the definition of tangential sliding vector fields.

**Definition 3.** Assume that

\[
(d\eta(p)Z_+(p), d\eta(p)Z_-(p)) = -\|d\eta(p)Z_+(p)\|\|d\eta(p)Z_-(p)\| \neq 0
\]

for every \(p \in M\). The **tangential sliding vector field**, \(Z_{M}^{\tan} : M \to TM\), of \(Z\) on \(M\) is defined by

\[
Z_{M}^{\tan}(p) := C(p, \lambda^*(p)) = \frac{1 - \lambda^*(p)}{2}Z_+(p) + \frac{1 + \lambda^*(p)}{2}Z_-(p),
\]

where \(\lambda^*(p) \in (-1, 1)\) is given by (10).

**Remark 1.** Definition 3 provides a tangential sliding vector field \(Z_{M}^{\tan}\) on a submanifold \(M \subset \Sigma\) for which the conditions of Case 1 of Theorem 1 hold. Such a vector field can be extended for points on the boundary of \(M\) for which the conditions of Cases 2 and 3 of Theorem 1 hold, but not for Case 4. That is because, under condition (9) of Theorem 1, \(C_p \cap T_pM\) is not single-valued only in Case 4. In addition, Cases 2 and 3 are limiting scenarios of Case 1.

Notice that a trajectory \(\phi : I \to M\) of the tangential sliding vector field (13) satisfies

\[
\varphi'(s) = Z_{M}^{\tan}(\varphi(s)) = C(\varphi(s), \lambda^*(\varphi(s))) \in F_Z(\varphi(s)),
\]

for every \(s \in I\). Accordingly, we have proven the following result providing that trajectories of the tangential sliding vector field (13) constitute, indeed, a class of solutions of the differential inclusion (2) and, therefore, a class of Filippov trajectories of the PSVF (1):

**Theorem 2.** Any trajectory of the tangential sliding vector field (13) is a solution of the differential inclusion (2).

### 4. Some examples of tangential sliding vector fields

The function \(\eta\), which describes the manifold \(M\) contained in \(\Sigma^{\tan}\), and the tangential sliding vector field \(Z_{M}^{\tan}\), depend on the multiplicity of the tangential contact between \(Z_+\) and \(Z_-\) with \(\Sigma\) at points of \(M\). This section is devoted to exploring several examples where the function \(\eta\) can be obtained in terms of \(h\) and the Lie derivatives \(Z_i^\|h\).

#### 4.1. Example 1: Multiplicity 2 tangential manifold. In this first example, we investigate the sliding tangential vector field defined on a manifold of tangential points multiplicity 2.

Consider the PSVF \(Z = (Z_+, Z_-) : \mathbb{R}^4 \to \mathbb{R}^4\) where

\[
Z_+(x_1, x_2, x_3, x_4) = (a_1, a_2, a_3, a_4x_1),
\]

\[
Z_-(x_1, x_2, x_3, x_4) = (b_1, b_2, b_3, b_4x_1),
\]
and the switching manifold is given by \( h(x_1, \ldots, x_4) = x_4 \). Computing the Lie derivatives, we get
\[
Z_+ h(x_1, x_2, x_3, 0) = a_4 x_1, \quad Z_- h(x_1, x_2, x_3, 0) = b_4 x_1,
\]
\[
Z^2_+ h(x_1, x_2, x_3, 0) = a_1 a_4, \quad Z^2_- h(x_1, x_2, x_3, 0) = b_1 b_4.
\]
We assume that \( a_1 a_4 b_1 b_4 \neq 0 \). So, the manifold \( M \subset \Sigma^\text{tan} \subset \Sigma \subset \mathbb{R}^4 \) that corresponds to the tangential points is given by \( M = \eta^{-1}(0) \), where
\[
\eta(x_1, x_2, x_3, x_4) = (x_1, x_4).
\]
Notice that \( d\eta(p) Z_+(p) = (0, a_1) \) and \( d\eta(p) Z_-(p) = (0, b_1) \). Thus, condition (12) of Definition 3 is satisfied provided that \( a_1 b_1 < 0 \). Accordingly, the tangential vector field is given by
\[
(14) \quad Z^\text{tan}_m(x_2, x_3) = \left(0, \frac{a_2 |b_1| + b_2 |a_1|}{|b_1| + |a_1|}, \frac{a_3 |b_1| + b_3 |a_1|}{|b_1| + |a_1|}, 0 \right).
\]

4.2. Example 2: Higher order multiplicity tangential manifold. It is possible for a PSVF, \( Z = (Z_+, Z_-) \), to have tangential points with distinct multiplicities in such a way that, for each multiplicity \( i \), a manifold \( M_i \subset \Sigma^\text{tan} \), constituted by the tangential points of multiplicity \( i \), can be defined. In this case, a tangential sliding vector field can be defined on each manifold \( M_i \). Accordingly, in the next example we consider a class of PSVFs \( Z = (Z_+, Z_-) \) defined in \( \mathbb{R}^n \) for which both vector fields \( Z_\pm \) have a generic contact of multiplicity \( m, l \leq n \).

Consider the PSVF \( Z = (Z_+, Z_-) : \mathbb{R}^n \to \mathbb{R}^n \) where
\[
Z_+(x) = \left(a_1 x_2, a_2 x_3, \ldots, a_m x_{m+1}, \ldots, a_{l-2} x_{l-1}, a_{l-1}, \ldots, a_{n-1}, x_1\right)
\]
\[
Z_-(x) = \left(b_1 x_2, b_2 x_3, \ldots, b_m x_{m+1}, b_m, \ldots, b_{n-1}, x_1\right),
\]
with \( a_i b_j \neq 0, i = 1, \ldots, l-1, j = 1, \ldots, m-1 \) and the switching manifold given by \( h(x) = x_n \). Here, \( x = (x_1, \ldots, x_n) \). Notice that the origin is a contact point of \( Z_+ \) and \( Z_- \) with \( \Sigma \) of multiplicity \( l \) and \( m \), respectively. Assume \( l > m \). Computing the Lie derivatives, we get
\[
Z_+ h(x) = x_1, \quad Z_- h(x) = x_1,
\]
\[
Z^2_+ h(x) = a_1 x_2, \quad Z^2_- h(x) = b_1 x_2,
\]
\[
\vdots \quad \vdots
\]
\[
Z^{m-1}_+ h(x) = a_1 \ldots a_{m-2} x_{m-1}, \quad Z^{m-1}_- h(x) = b_1 \ldots b_{m-2} x_{m-1},
\]
\[
Z^m_+ h(x) = a_1 \ldots a_{m-1} x_m, \quad Z^m_- h(x) = b_1 \ldots b_{m-1} \neq 0,
\]
\[
Z^{m+1}_+ h(x) = a_1 \ldots a_m x_{m+1},
\]
\[
\vdots
\]
\[
Z^{l-1}_+ h(x) = a_1 \ldots a_{l-2} x_{l-1},
\]
\[
Z^l_+ h(x) = a_1 \ldots a_{l-1} \neq 0.
\]
The manifold $M_2 \subset \Sigma^{\text{tan}}$, composed of the tangential points of multiplicity 2 is given by $\eta_2^{-1}(0)$, where $\eta_2 : \mathbb{R}^n \setminus \{x : x_2 = 0\} \to \mathbb{R}^2$ is given by

$$\eta_2(x_1, \ldots, x_n) = (h(x), Z_+ h(x)) = (x_n, x_1).$$

Notice that for $p_2 \in M_2$, $d\eta_2(p_2) Z_+(p_2) = (0, a_1 x_2)$ and $d\eta_2(p_2) Z_-(p) = (0, b_1 x_2)$. Thus, condition (12) of Definition 3 is satisfied provided that $a_1 b_1 < 0$. Accordingly, the tangential vector field defined on $M_2$ is given by

$$Z^\text{tan}_{M_2}(p_2) = \frac{|b_1|}{|a_1| + |b_1|} Z_+(p_2) + \frac{|a_1|}{|a_1| + |b_1|} Z_-(p_2)$$

$$= \frac{1}{|a_1| + |b_1|} \left(0, x_3(a_2 |b_1| + b_2 |a_1|), \ldots, x_{m-1}(a_{m-2} |b_1| + b_{m-2} |a_1|),
\right.$$

$$\left.x_m a_{m-1} |b_1| + b_{m-1} |a_1|, \ldots, a_{n-1} |b_1| + b_n |a_1|, 0\right).$$

Analogously, the manifold $M_3 \subset \Sigma^{\text{tan}}$ composed of the tangential points of multiplicity 3 is given by $\eta_3^{-1}(0)$, where $\eta_3 : \mathbb{R}^n \setminus \{x : x_3 = 0\} \to \mathbb{R}^3$ is given by

$$\eta_3(x_1, \ldots, x_n) = (h(x), Z_+ h(x), Z^2_+ h(x)) = (x_n, x_1, a_1 x_2).$$

Notice that for $p_3 \in M_3$, $d\eta_3(p_3) Z_+(p_3) = (0, 0, a_1 a_2 x_3)$ and $d\eta_3(p_3) Z_-(p) = (0, 0, a_1 b_2 x_3)$. Thus, condition (12) of Definition 3 is satisfied provided that $a_2 b_2 < 0$. Accordingly, the tangential vector field defined on $M_3$ is given by

$$Z^\text{tan}_{M_3}(p_3) = \frac{|b_2|}{|a_2| + |b_2|} Z_+(p_2) + \frac{|a_2|}{|a_2| + |b_2|} Z_-(p_2)$$

$$= \frac{1}{|a_2| + |b_2|} \left(0, 0, x_4(a_3 |b_2| + b_3 |a_2|), \ldots, x_{m-1}(a_{m-2} |b_2| + b_{m-2} |a_2|),
\right.$$

$$\left.x_m a_{m-1} |b_2| + b_{m-1} |a_2|, \ldots, a_{n-1} |b_2| + b_n |a_2|, 0\right).$$

In general, the manifold $M_m \subset \Sigma^{\text{tan}}$ composed of the tangential points of multiplicity $m$ is given by $\eta^{-1}_m(0)$, where $\eta_m : \mathbb{R}^n \setminus \{x : x_m \geq 0\} \to \mathbb{R}^m$ is given by

$$\eta_m(x_1, \ldots, x_n) = (h(x), Z_+ h(x), Z^2_+ h(x), \ldots, Z^{m-1}_+ h(x))$$

$$= (x_n, x_1, a_1 x_2, \ldots, a_1 x_m, a_{m-1} x_m).$$

Notice that for $p_m \in M_m$, $d\eta_m(p_m) Z_+(p_m) = (0, \ldots, 0, a_1 a_2 \ldots a_{m-1} x_m)$ and $d\eta_m(p_m) Z_-(p_m) = (0, \ldots, 0, a_1 a_2 \ldots a_{m-2} b_{m-1})$. Thus, condition (12) of Definition 3 is satisfied provided that $a_{m-1} b_{m-1} > 0$. Accordingly, the tangential vector field defined
on $M_m$ is given by

$$Z_{M_m}^{\tan}(p_m) = \frac{|b_{m-1}|}{|a_{m-1}x_m| + |b_{m-1}|} Z_+(p_m) + \frac{|a_{m-1}x_m|}{|a_{m-1}x_m| + |b_{m-1}|} Z_-(p_m)$$

$$= \frac{1}{b_{m-1} - a_{m-1}x_m} \left( 0, 0, \ldots, 0, -a_{m-1}x_m b_m + b_{m-1} a_m x_{m+1}, \ldots, -a_{m-1}x_m b_{l-2} + b_{m-1} a_{l-2} x_{l-1}, -a_{m-1}x_m b_{l-1} + b_{m-1} a_{l-1}, 0 \right).$$

Finally, we notice that, although one can define a manifold composed of contact points of multiplicity higher than $m$, a tangential vector field cannot be defined on it. In fact, let $M_{m+1} \subset \Sigma^{\tan}$ be the submanifold composed of the tangential points of multiplicity $m + 1$, that is, $M_{m+1} = \eta_{m+1}(0)$, where $\eta_{m+1} : \mathbb{R}^n \setminus \{x : x_{m+1} = 0\} \to \mathbb{R}^{m+1}$ is given by

$$\eta_m(x_1, \ldots, x_n) = (h(x), Z_+ h(x), Z_+^2 h(x), \ldots, Z_+^{m-1} h(x), Z_+^m h(x)) = (x_n, x_1, a_1 x_2, \ldots, a_1 \ldots a_{m-2} x_{m-1}, a_1 \ldots a_{m-1} x_m).$$

Thus, for $p_{m+1} \in M_{m+1}$, one has $d\eta_{m+1}(p_{m+1}) Z_+(p_{m+1}) = (0, \ldots, 0, a_1 a_2 \ldots a_m x_{m+1})$ and $d\eta_{m+1}(p_{m+1}) Z_-(p_{m+1}) = (0, \ldots, 0, a_1 a_2 \ldots a_{m-2} b_{m-1}, a_1 a_2 \ldots a_{m-1} b_m)$. Thus, condition (12) of Definition 3 is satisfied provided that

$$a_1 a_2 \ldots a_{m-2} b_{m-1} = 0 \quad \text{and} \quad a_m b_m x_{m+1} < 0,$$

which contradicts the hypothesis $a_i b_j \neq 0, i = 1, \ldots, l - 1$ and $j = 1, \ldots, m - 1$ (recall that $l > m$).

4.3. Example 3: A model of intermittent treatment of HIV. Consider the PSVF $Z = (Z_+, Z_-) : U \subset \mathbb{R}^3 \to \mathbb{R}^3$ where

$$Z_+ = \left( -k x_1 x_3 + s - \alpha x_1, k x_1 x_3 - \delta x_2, \theta x_2 - c x_3 \right),$$

$$Z_- = \left( -(1 - \eta_{RT}) k x_1 x_3 + s - \alpha x_1, (1 - \eta_{RT}) k x_1 x_3 - \delta x_2, \right.$$

$$\left. (1 - \eta_{PI}) \theta x_2 - c x_3 \right).$$

The PSVF above models the intermittent treatment of human immunodeficiency virus. For more details about the formal mathematical analysis of this model, see [4]. Here, $x_2(t)$ denotes the infected cell population size at time $t$, $x_1(t)$ denotes the uninfected cell population size at time $t$, $x_3(t)$ denotes the concentration of infectious virus particles at time $t$, and the parameters $\eta_{RT}, \eta_{PI}, k, \alpha, \delta, \theta, c$ are positive real numbers related to the properties of the model and the treatment protocol of this disease. For biological reasons, it is assumed that

$$\alpha < \delta, \quad k \theta > c \delta \quad \text{and} \quad \frac{s}{\delta} + \frac{c(\delta - \alpha)}{k \theta} < C_T < \min \left\{ \frac{s}{\delta}, \frac{s}{\eta_{RT} - 1}(\eta_{PI} - 1) k \theta \right\}.$$
more precisely, the therapy is triggered below this value and stopped above this value. The first Lie derivatives are given by

\[ Z_+ h(C_T - x_2, x_2, x_3) = -\alpha C_T + s + x_2(\alpha - \delta), \]

\[ Z_- h(C_T - x_2, x_2, x_3) = -\alpha C_T + s + x_2(\alpha - \delta). \]

Solving \( Z_\pm h = 0 \) yields the curve

\[ \Sigma^{\text{tan}} = \left\{ \left( \frac{s - C_T \delta}{\alpha - \delta}, \frac{\alpha C_T - s}{\alpha - \delta}, x_3 \right); x_3 \geq 0 \right\} \]

and, over \( \Sigma^{\text{tan}} \), there exist two cusp points given by:

\[ p_+^c = \left( \frac{s - C_T \delta}{\alpha - \delta}, \frac{\alpha C_T - s}{\alpha - \delta}, \frac{\delta\alpha C_T - s}{k(s - C_T \delta)} \right), \]

\[ p_-^c = \left( \frac{s - C_T \delta}{\alpha - \delta}, \frac{\alpha C_T - s}{\alpha - \delta}, \frac{\delta\alpha C_T - s}{(\eta_{RT} - 1)k(s - C_T \delta)} \right). \]

The submanifold \( M_2 \subset \Sigma^{\text{tan}} \) of tangential points of multiplicity 2 is given by \( \eta_2^{-1}(0) \), where \( \eta: \mathbb{R}^3 \setminus \{ p_+^c, p_-^c \} \rightarrow \mathbb{R}^2 \) is given by

\[ \eta_2(x_1, x_2, x_3) = (h(x_1, x_2, x_3), Z_+ h(C_T - x_2, x_2, x_3)). \]

Let \( p = \left( \frac{s - C_T \delta}{\alpha - \delta}, \frac{\alpha C_T - s}{\alpha - \delta}, x_3 \right) \in \Sigma^{\text{tan}} \). It can be noted that

\[ d\eta_2(p) Z_+(p) = \left( 0, kx_3(s - C_T \delta) + \frac{\delta\alpha C_T - s}{\alpha - \delta} \right) \]

and

\[ d\eta_2(p) Z_-(p) = \left( 0, \frac{\delta\alpha C_T - s}{\alpha - \delta} - (\eta_{RT} - 1)kx_3(s - C_T \delta) \right). \]

Thus, under assumptions (16), the condition (12) of Definition 3 is satisfied. In this case,

\[ \lambda^*(p) = \frac{-2\alpha C_T \delta + C_T \delta \eta_{RT} kx_3 - 2C_T \delta kx_3 - \eta_{RT} ksx_3 + 2ksx_3 + 2\delta s}{\eta_{RT} kx_3(s - C_T \delta)}. \]

The tangential vector field (13) is given by

\[ Z^\text{tan}_{M_2}(x_3) = \left( 0, 0, -\frac{\alpha C_T}{\alpha - \delta} + \frac{c\delta x_3}{\alpha - \delta} - \frac{\alpha C_T \eta P_I \theta}{\alpha - \delta} + \frac{\alpha C_T \theta}{\alpha - \delta} \right. \]

\[ + \frac{\alpha C_T \eta P_I \theta}{\alpha - \delta} \left( 1 - \frac{\delta(s - C_T (\alpha + kx_3)) + ksx_3}{\delta(s - C_T (\alpha + kx_3)) - (\eta_{RT} - 1)kx_3(s - C_T \delta)} \right) \]

\[ - \frac{\eta P_I \theta s}{\alpha - \delta} \left( 1 - \frac{\delta(s - C_T (\alpha + kx_3)) + ksx_3}{\delta(s - C_T (\alpha + kx_3)) - (\eta_{RT} - 1)kx_3(s - C_T \delta)} \right) + \frac{\eta P_I \theta s}{\alpha - \delta} \frac{\theta s}{\alpha - \delta}. \]
This expression coincides with the expression of the tangential vector field given in [4]. Note that this tangential vector field possesses an equilibrium point

\[ p_2^* = \left( 0, 0, \frac{1}{2 \epsilon \eta \lambda (\alpha - \delta)(C_T \delta - \epsilon)} \left( -k \theta (\eta_{PI} - \eta_{RT})(s - \alpha C_T)(s - C_T \delta) + \sqrt{\Delta} \right) \right), \]

where

\[ \Delta = k \theta (s - \alpha C_T)^2 (s - C_T \delta)(4c \theta \eta_{PI} \eta_{RT} (\alpha - \delta) + k \theta (\eta_{PI} - \eta_{RT})^2 (s - C_T \delta)) \].

5. Regularized tangential vector field

Let \( Z = (Z_+, \ldots, Z_m) \) be a PSVF. Let \( \eta : V \subset U \to \mathbb{R}^n \), with \( m < n \) and \( V \subset U \) open, be a smooth function is defined by \( 0 \in \mathbb{R}^m \) as a regular value. Assume that \( M = \eta^{-1}(0) \subset \Sigma^\tan \) satisfies condition (12) of Definition 3. Following [23], the next result shows that the tangential sliding vector field \( Z_M^\tan \) is locally conjugated to the reduced dynamics of a SP-Problem arisen from the regularization of \( Z \), and restricted to a manifold contained in the slow set.

**Theorem 3.** For each \( p \in M \), there exists a neighbourhood \( D \subset \mathbb{R}^n \) of \( p \) such that, for any \( C^r \) (resp. continuous) transition function \( \phi \), the \( \phi \)-regularization \( Z^\epsilon \) of \( Z |_D \) can be written as a SP-Problem (according to Definition 2) such that the reduced system has an invariant manifold \( \Sigma^\tan \), which is contained in the slow set \( S \) and is \( C^r \)-diffeomorphic (resp. homeomorphically) to \( M \cap D \). In addition, the reduced system restricted to \( \Sigma^\tan \) is \( C^r \)-conjugated (resp. to be topologically conjugated) to the tangential sliding vector field \( Z^\tan_{M \cap D} \).

**Proof.** First, notice that there exists a neighborhood \( D \subset \mathbb{R}^n \) of \( p \) and a local coordinate system for which \( h(x_1, \ldots, x_n) = x_n \) and \( \eta(x_1, \ldots, x_n) = (x_{n-m+1}, \ldots, x_n) \). In the given coordinates, we get

\[ \Sigma^\tan = \{ p = (x_1, \ldots, x_{n-m}, 0, \ldots, 0) \} \subset \Sigma. \]

By denoting \( Z^\pm = (Z^+_1, \ldots, Z^+_m, Z^-_1, \ldots, Z^-_m) \), the tangential sliding vector field (13) is written as

\[ Z^\tan_M(p) = C(p; \lambda^*(p)) = \frac{1 - \lambda^*(p)}{2} Z^+(p) + \frac{1 + \lambda^*(p)}{2} Z^-(p). \]

In what follows, we denote \( Z^\tan_M = Z^\tan = (Z^\tan_1, \ldots, Z^\tan_m) \). Note that the trajectories of the tangential sliding vector field satisfy the differential system

\[ \begin{align*}
\dot{x}_i &= Z^\tan_i(p) = \frac{1 - \lambda^*(p)}{2} Z^+(p) + \frac{1 + \lambda^*(p)}{2} Z^-(p), \quad i = 1, 2, \ldots, n-m \\
\dot{x}_j &= Z^\tan_j(p) = 0, \quad j = n-m+1, \ldots, n.
\end{align*} \]

Now, consider the \( \phi \)-regularization of \( Z \), defined by (3) as

\[ Z^\epsilon(x) = \frac{1 - \phi^\epsilon(x_n)}{2} Z^+(x) + \frac{1 + \phi^\epsilon(x_n)}{2} Z^-(x). \]

Notice that the trajectories of \( Z^\epsilon \) satisfy the following differential system

\[ \begin{align*}
\dot{x}_i &= \frac{1 - \phi^\epsilon(x_n)}{2} Z^+_i(x) + \frac{1 + \phi^\epsilon(x_n)}{2} Z^-_i(x), \quad i = 1, \ldots, n.
\end{align*} \]
By applying the change of variables \( u = (x_1, x_2, \ldots, x_{n-m}), v = (x_{n-m+1}, \ldots, x_{n-1}), \) and \( w = x_n/\varepsilon, \) for \( \varepsilon > 0 \) small, the differential system (19) is written as the following singular perturbation SP-problem

\[
\begin{align*}
\dot{u}_i &= \frac{(1 - \phi(w))}{2} Z_i^+(u, v, \varepsilon w) + \frac{(1 + \phi(w))}{2} Z_i^-(u, v, \varepsilon w), \quad i = 1, \ldots, n - m, \\
\dot{v}_j &= \frac{(1 - \phi(w))}{2} Z_j^+(u, v, \varepsilon w) + \frac{(1 + \phi(w))}{2} Z_j^-(u, v, \varepsilon w), \quad j = n - m + 1, \ldots, n - 1, \\
\varepsilon \dot{w} &= \frac{(1 - \phi(w))}{2} Z_n^+(u, v, \varepsilon w) + \frac{(1 + \phi(w))}{2} Z_n^-(u, v, \varepsilon w),
\end{align*}
\]

where \( \varepsilon > 0 \) is the singular perturbation parameter. By taking \( \varepsilon = 0, \) we get the reduced problem

\[
\begin{align*}
\dot{u}_i &= \frac{(1 - \phi(w))}{2} Z_i^+(u, v, 0) + \frac{(1 + \phi(w))}{2} Z_i^-(u, v, 0), \quad i = 1, \ldots, n - m, \\
\dot{v}_j &= \frac{(1 - \phi(w))}{2} Z_j^+(u, v, 0) + \frac{(1 + \phi(w))}{2} Z_j^-(u, v, 0), \quad j = n - m + 1, \ldots, n - 1, \\
0 &= \frac{(1 - \phi(w))}{2} Z_n^+(u, v, 0) + \frac{(1 + \phi(w))}{2} Z_n^-(u, v, 0) =: K(u, v, w),
\end{align*}
\]

which describes the dynamics on the “slow” timescale \( t \) (for standard concepts of singularly perturbed or slow-fast systems see [13, 18]). This dynamics inhabits the slow set \( S := \{(u, v, w) : K(u, v, w) = 0\} \). Notice that, in the \( (u, v, w) \)-coordinate system, \( \Sigma^\tan = D \cap \{(u, 0, 0) : u \in \mathbb{R}^{n-m}\} \). Thus, by taking \( w^*(u) = \phi^{-1}(\lambda^*(u, 0, 0)) \), we have that

\[
K(u, 0, w^*(u)) = \frac{(1 - \lambda^*(u, 0, 0))}{2} Z_n^+(u, 0, 0) + \frac{(1 + \lambda^*(u, 0, 0))}{2} Z_n^-(u, 0, 0) = Z_n^\tan(u, 0, 0) = 0.
\]

Hence, \( \Sigma^\tan := \{(u, 0, w^*(u))\} \) is a manifold contained in the slow set \( S \). In addition, for \( j = n - m + 1, \ldots, n - 1 \), we have that

\[
\frac{(1 - \phi(w^*(u)))}{2} Z_j^+(u, 0, 0) + \frac{(1 + \phi(w^*(u)))}{2} Z_j^-(u, 0, 0) = Z_j^\tan(u, 0, 0) = 0,
\]

which implies that the flow of the reduced problem (21) is invariant over the manifold \( \mathcal{S}^\tan \) and is written as

\[
\begin{align*}
\dot{u}_i &= \frac{(1 - \lambda^*(u, 0, 0))}{2} Z_i^+(u, 0, 0) + \frac{(1 + \lambda^*(u, 0, 0))}{2} Z_i^-(u, 0, 0), \quad i = 1, \ldots, n - m, \\
\dot{v}_j &= 0, \quad j = n - m + 1, \ldots, n - 1, \\
w &= w^*(u).
\end{align*}
\]

Finally, by defining \( H : \Sigma^\tan \to \mathcal{S}^\tan \) as

\[
H(p) = (u, 0, w^*(u)) = (u, 0, \phi^{-1} \circ \lambda^*(p)), \quad p = (u, 0, 0),
\]

we conclude.
we can see that the manifold $S^{\text{tan}}$ is $C^r$–diffeomorphic (resp. homeomorphic) to $\Sigma^{\text{tan}}$ provided that $\phi$ is a $C^r$ (resp. continuous) transition function. This completes the proof of the first part of the theorem.

Finally, denote the solution of the differential system (18) starting at $p = (u, 0, 0) \in \Sigma^{\text{tan}}$ by $t \mapsto x_t(p) := (u(t, p), 0, 0)$. Similarly, denote the solution of the reduced problem (22) starting at $H(p)$ by $t \mapsto X_t(H(p))$. Since the first $n - 1$ equations of the differential systems (18) and (22) coincide and do not depend on the last one, we have that

$$X_t(H(p)) = \left( u(t, p), 0, w^*(u(t, p)) \right) = H(x_t(p)).$$

Consequently, the reduced dynamics restricted to $S^{\text{tan}}$ is $C^r$–conjugated (resp. topologically conjugated) to the tangential sliding vector field $Z_{M \cap D}^{\text{tan}}$ provided that $\phi$ is a $C^r$ (resp. continuous) transition function. This completes the proof of the second part of the theorem. □

In Figure 1 we illustrate the double tangential set and the manifold $S^{\text{tan}}$.

**Figure 1.** Regularization. In (a) we get a PSVF with a double tangency set and, in (b), its regularization.

**Remark 2.** The invariant manifold $S^{\text{tan}}$ of the reduced problem (21) is never normally hyperbolic. Indeed,

$$\frac{\partial K}{\partial w}(u, 0, w^*(u)) = \phi'(w^*(u))Z_n^-(u, 0, 0) - Z_n^+(u, 0, 0) = 0,$$

for every $u$ such that $(u, 0, 0) \in D$. Therefore, Fenichel’s theory cannot be applied to study the persistence of $S^{\text{tan}}$ as an invariant manifold of (20) for $\epsilon > 0$ small. Addressing this problem requires the use of blow-up methods (see, for instance, [12, 28] and also [24]), which is not in the scope of the present study.

**Remark 3.** In [26] the notion of solutions of Filippov systems has been extended as being limiting trajectories of some regularization. In this regard, Theorem 3 shows that the sliding mode provided by Definition 3, arising from the Filippov convention, also corresponds to solutions in the context considered by [26].
In what follows, we present an example illustrating the equivalence between the tangential vector field and the invariant dynamics on the slow manifold of the $\phi$-regularized vector field.

**Example 1.** Consider the PSVF presented in Example 4.1 given by
\[ Z = (Z_+, Z_-) : U \subset \mathbb{R}^4 \to \mathbb{R}^4 \]
where
\[ Z_+ = (a_1, a_2, a_3, a_4x_1) \]
\[ Z_- = (b_1, b_2, b_3, b_4x_1) \].

Recall that the switching manifold is given by $h(x_1, \ldots, x_4) = x_4$. By (19) and considering the notation presented in the proof of Theorem 3 given by $x_1 = v_1, x_2 = u_1, x_3 = u_2, x_4 = w\varepsilon$, the regularized vector field is
\[
\begin{align*}
\dot{v}_1 &= (a_1 - b_1)\phi(w) + b_1, \\
\dot{u}_1 &= (a_2 - b_2)\phi(w) + b_2, \\
\dot{u}_2 &= (a_3 - b_3)\phi(w) + b_3, \\
\varepsilon\dot{w} &= v_1((a_4 - b_4)\phi(w) + b_4).
\end{align*}
\]

In the limit $\varepsilon = 0$, considering the expression of $\lambda^*(p)$ provided in Theorem 1, we get
\[ \lambda^*(p) = -1 + \frac{2|a_1|}{|a_1| + |b_1|} \]
and the existence of a slow manifold $S$, given by the restriction $v_1 = 0$ and $w = \phi^{-1}(\lambda^*(p))$. Considering the restriction of (23) on $S$, we obtain the reduced problem which, in this case, coincides with the tangential vector field given in (14).

**Acknowledgements**

Tiago Carvalho is partially supported by São Paulo Research Foundation (FAPESP) grants # 2019/10269-3, # 2021/12395-6, and # 2022/02819-6 and by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), grants 305026/2020-8 and 309378/2023-0.

Douglas Duarte Novaes is partially supported by São Paulo Research Foundation (FAPESP) grants # 2018/13481-0, # 2019/10269-3, and # 2022/09633-5 and by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) grant 309110/2021-1.

Durval José Tonon is partially supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) grant # 310362/2021-0.

**References**

[1] C. Bonet-Revés, J. Larrosa, and T. M-Seara. Regularization around a generic codimension one fold-fold singularity. *J. Differential Equations*, 265(5):1761–1838, 2018.

[2] C. Bonet-Revés and T. M-Seara. Regularization of sliding global bifurcations derived from the local fold singularity of Filippov systems. *Discrete Contin. Dyn. Syst.*, 36(7):3545–3601, 2016.
[3] B. Brogliato. *Nonsmooth Mechanics: Models, Dynamics and Control*. Communications and Control Engineering Series. Springer-Verlag London, second edition, 1999.

[4] T. Carvalho, R. Cristiano, L. F. Gonçalves, and D. J. Tonon. Global analysis of the dynamics of a mathematical model to intermittent HIV treatment. *Nonlinear Dynamics*, 101:719–739, 2020.

[5] T. Carvalho, D. D. Novaes, and L. F. Gonçalves. Sliding shilnikov connection in filippov-type predator–prey model. *Nonlinear Dynamics*, 100(3):2973–2987, 2020.

[6] T. Carvalho, M. A. Teixeira, and D. J. Tonon. Asymptotic stability and bifurcations of 3D piecwise smooth vector fields. *Zeitschrift für angewandte Mathematik und Physik*, 67(2):31, 2016.

[7] M. M. Castro, R. M. Martins, and D. D. Novaes. A note on vishik’s normal form. *Journal of Differential Equations*, 281:442–458, 2021.

[8] A. Colombo, M. di Bernardo, E. Fossas, and M. R. Jeffrey. Teixeira singularities in 3D switched feedback control systems. *Systems and Control Letters*, 59(10):615–622, 2010.

[9] R. Cristiano, T. Carvalho, D. J. Tonon, and D. J. Pagano. Hopf and homoclinic bifurcations on the sliding vector field of switching systems in $\mathbb{R}^3$: A case study in power electronics. *Physica D: Nonlinear Phenomena*, 347:12–20, 2017.

[10] P. R. da Silva, I. S. Meza-Sarmiento, and D. D. Novaes. Nonlinear Sliding of Discontinuous Vector Fields and Singular Perturbation. *Differ. Equ. Dyn. Syst.*, 30(3):675–693, 2022.

[11] M. di Bernardo, C. J. Budd, A. R. Champneys, and P. Kowalczyk. *Piecewise-smooth Dynamical Systems: Theory and Applications*. Number 163 in Applied Mathematical Sciences. Springer-Verlag London, first edition, 2008.

[12] F. Dumortier. Singularities of vector fields on the plane. *J. Differential Equations*, 23(1):53–106, 1977.

[13] N. Fenichel. Geometric singular perturbation theory for ordinary differential equations. *J. Differential Equations*, 31(1):53–98, 1979.

[14] A. F. Filippov. *Differential Equations with Discontinuous Righthand Sides*, volume 18 of *Mathematics and its Applications*. Springer Netherlands, first edition, 1988.

[15] L. F. Gonçalves, D. S. Rodrigues, P. F. A. Mancera, and T. Carvalho. Sliding mode control in a mathematical model to chemoimmunotherapy: the occurrence of typical singularities. *Applied Mathematics and Computation*, 387:124782, 2020.

[16] A. Jacquemard, M. A. Teixeira, and D. J. Tonon. Piecewise smooth reversible dynamical systems at a two-fold singularity. *International Journal of Bifurcation and Chaos*, 22(8):1250192, 13, 2012.

[17] A. Jacquemard and D. J. Tonon. Coupled systems of non-smooth differential equations. *Bulletin des Sciences Mathématiques*, 136(3):239–255, 2012.

[18] C. K. R. T. Jones. Geometric singular perturbation theory. In *Dynamical systems* (Montecatini Terme, 1994), volume 1609 of *Lecture Notes in Math.*., pages 44–118. Springer, Berlin, 1995.

[19] T. Kousaka, T. Kido, T. Ueta, H. Kawakami, and M. Abe. Analysis of border-collision bifurcation in a simple circuit. In 2000 IEEE International Symposium on Circuits and Systems. Emerging Technologies for the 21st Century. *Proceedings (IEEE Cat No.00CH36353)*, volume 2, pages 481–484, 2000.

[20] J. M. Lee. *Introduction to Smooth Manifolds*. Version 3.0. University of Washington, Department of Mathematics, 2000.

[21] R. Leine and H. Nijmeijer. *Dynamics and Bifurcations of Non-Smooth Mechanical Systems*, volume 18 of *Lecture Notes in Applied and Computational Mechanics*. Springer-Verlag Berlin Heidelberg, first edition, 2004.

[22] D. Novaes and L. Silva. On the non-existence of isochronous tangential centers in filippov vector fields. *Proceedings of the American Mathematical Society*, 150:5349–5358, 2022.

[23] D. D. Novaes and M. R. Jeffrey. Regularization of hidden dynamics in piecewise smooth flows. *J. Differential Equations*, 259(9):4615–4633, 2015.

[24] D. D. Novaes and G. Rondón. Smoothing of nonsmooth differential systems near regular-tangential singularities and boundary limit cycles. *Nonlinearity*, 34(6):4202–4263, 2021.
[25] D. D. Novaes and L. A. Silva. Lyapunov coefficients for monodromic tangential singularities in Filippov vector fields. *Journal of Differential Equations*, 300:565–596, 2021.

[26] D. Panazzolo and P. R. da Silva. Regularization of discontinuous foliations: blowing up and sliding conditions via Fenichel theory. *J. Differential Equations*, 263(12):8362–8390, 2017.

[27] D. S. Rodrigues, P. F. A. Mancera, T. Carvalho, and L. F. Gonçalves. A mathematical model for chemoimmunotherapy of chronic lymphocytic leukemia. *Applied Mathematics and Computation*, 349:118–133, 2019.

[28] A. Seidenberg. Reduction of singularities of the differential equation $A\,dy = B\,dx$. *Amer. J. Math.*, 90:248–269, 1968.

[29] D. J. W. Simpson. *Bifurcations in Piecewise-Smooth Continuous Systems*, volume 70 of *World Scientific Series on Nonlinear Science, Series A*. World Scientific Publishing, 2010.

[30] G. V. Smirnov. *Introduction to the theory of differential inclusions*, volume 41 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2002.

[31] J. Sotomayor and M. A. Teixeira. Regularization of discontinuous vector fields. In *International Conference on Differential Equations (Lisboa, 1995)*, pages 207–223. World Sci. Publ., River Edge, NJ, 1998.

[32] M. A. Teixeira. Stability conditions for discontinuous vector fields. *Journal of Differential Equations*, 88(1):15–29, 1990.

[33] M. A. Teixeira and P. R. da Silva. Regularization and singular perturbation techniques for non-smooth systems. *Physica D: Nonlinear Phenomena*, 241(22):1948–1955, 2012. Dynamics and Bifurcations of Nonsmooth Systems.

[34] K. Tirok and U. Gaedke. Regulation of planktonic ciliate dynamics and functional composition during spring in lake constance. *Aquatic Microbial Ecology*, 49(1):87–100, 2007.

[35] K. Tirok and U. Gaedke. Internally driven alternation of functional traits in a multispecies predator–prey system. *Ecology*, 91(6):1748–1762, 2010.

[36] V. I. Utkin, J. Guldner, and J. Shi. *Sliding Mode Control in Electro-Mechanical Systems*. Automation and Control Engineering, CRC Press, 2009.

[37] S. M. Višik. Vector fields in the neighborhood of the boundary of a manifold. *Vestnik Moskov. Univ. Ser. I Mat. Meh.*., 27(1):21–28, 1972.

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