ESTIMATES ON THE KODAIRA DIMENSION FOR FIBRATIONS OVER ABELIAN VARIETIES

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ABSTRACT. We give estimates on the Kodaira dimension for fibrations over abelian varieties, and give some applications. One of the results strengthens the subadditivity of Kodaira dimension of fibrations over abelian varieties.

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1. INTRODUCTION

In this paper, we give estimates on the Kodaira dimension for fibrations over abelian varieties over $\mathbb{C}$, and give some applications.

Theorem 1.1. Let $f : X \to A$ be a fibration from a smooth projective variety $X$ to an abelian variety $A$ where $f$ is smooth over an open set $V \subseteq A$, and $m$ a positive integer. Then

$$\kappa(V) \geq \kappa(A, \widetilde{\det f_*\omega_X^m}) \geq \dim V^0(A, f_*\omega_X^m).$$

If $m > 1$, then $\kappa(A, \widetilde{\det f_*\omega_X^m}) = \dim V^0(A, f_*\omega_X^m)$.

The line bundle $\widetilde{\det f_*\omega_X^m}$ is the reflexive hull of $\det f_*\omega_X^m$. Given a smooth quasi-projective variety $V$, $\kappa(V)$ denotes the log Kodaira dimension, defined as follows: for any smooth projective compactification $Y$ of $V$ such that $D = Y \setminus V$ is a divisor with simple normal crossing support, we have $\kappa(V) = \kappa(Y, K_Y + D)$. Let $\mathcal{F}$ be a coherent sheaf on an abelian variety $A$. The cohomological support locus $V^0(A, \mathcal{F})$ is defined by

$$V^0(A, \mathcal{F}) = \{ \alpha \in \text{Pic}^0(A) \mid \dim H^0(A, \mathcal{F} \otimes \alpha) > 0 \},$$

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see also Definition 2.2. If \( F \) admits a finite direct sum decomposition

\[
F \cong \bigoplus_{i \in I} (\alpha_i \otimes p_i^* F_i),
\]

where each \( A_i \) is an abelian variety, each \( p_i : A \to A_i \) is a fibration, each \( F_i \) is a nonzero \( M \)-regular coherent sheaf on \( A_i \), and each \( \alpha_i \in \text{Pic}^0(A) \) is a torsion line bundle, then we can characterize \( V^0(A,F) \) using this decomposition, and we have

\[ \dim V^0(A,F) = \max_{i \in I} \dim A_i. \]

See Definition 2.3 for the definition of \( M \)-regular coherent sheaves. We will use this observation in the proofs of our main theorems. The decomposition above is called the Chen–Jiang decomposition of \( F \). It is known that pushforwards of pluricanonical bundles under morphisms to abelian varieties have the Chen–Jiang decomposition by [CJ18, PPS17, LPS20] in increasing generality, and pushforwards of klt pairs under morphisms to abelian varieties have the Chen–Jiang decomposition, as proved independently in [Jia21] and [Men21].

If there exists a positive integer \( m \) such that \( \kappa(A, \widehat{\text{det}}_f \omega_X^{\otimes m}) = \dim A \), it is known that \( \kappa(V) = \dim A \) by [MP21, Theorem 2.4] (see Theorem 2.7) which is a consequence of [PS17] Theorems 4.1 and 3.5. The first part of the inequality in Theorem 1.1 is a generalization of this fact when \( \widehat{\text{det}}_f \omega_X^{\otimes m} \) is not necessarily big. In a different direction, by letting \( \kappa(V) = 0 \) in Theorem 1.1 we can recover [MP21, Theorem B] which gives the structures of pushforwards of pluricanonical bundles of smooth projective varieties under surjective morphisms to abelian varieties when the morphisms are smooth away from a closed set of codimension at least 2 in the abelian varieties.

By estimating the dimension of \( V^0(A, f^! \omega_X^{\otimes m}) \), we have the following corollary of Theorem 1.1.

**Corollary 1.2.** Let \( g : X \to Y \) be a smooth model of the Iitaka fibration of a smooth projective variety \( X \) with general fiber \( G \) where \( Y \) is a smooth projective variety, and \( f : X \to A \) a fibration to an abelian variety \( A \) where \( f \) is smooth over an open set \( V \subseteq A \). Then

\[ \kappa(V) \geq \dim A - q(G). \]

Given a projective variety \( G \), \( q(G) \) denotes the irregularity of \( G \), see Definition 2.1. A projective variety \( G \) is said to be regular if \( q(G) = 0 \). If the Iitaka fibration \( g \) has regular general fiber, then \( \kappa(V) = \dim A \) by Corollary 1.2 and thus \( f \) is not smooth unless \( A \) is a point. Thus we have the following corollary.

**Corollary 1.3.** Let \( g : X \to Y \) be a smooth model of the Iitaka fibration of a smooth projective variety \( X \) with general fiber \( G \) where \( Y \) is a smooth projective variety. If \( G \) is regular, then \( X \) has no nontrivial smooth morphisms to an abelian variety.
Corollary 1.3 implies that there are no nontrivial smooth morphisms from a projective variety of general type to an abelian variety which was proved in [VZ01] when the base is an elliptic curve and in [HK05] and [PS14] in general, see also [MP21] for related results.

We also have a quick corollary of Theorem 1.1 if the Albanese morphism of $X$ is a fibration. In Corollary 1.2 we let $f$ be the Albanese morphism of $X$. Then we deduce that

$$\kappa(V) \geq q(Y)$$

by Theorem 1.1 and [LPS20, Theorem D].

By [Kaw85, Theorem 1.1], Theorem 1.1 implies a special case of the Kebekus–Kovács conjecture when the base $V$ compactifies to an abelian variety. This conjecture bounds $\text{Var}(f)$ from above by the log Kodaira dimension $\kappa(V)$ assuming that the general fiber of $f$ has a good minimal model and has recently been proved in [Taj20]. For the definition of the variation $\text{Var}(f)$, see [Kaw85, Section 1]. In private communication from Mihnea Popa, he proposed the following conjecture.

**Conjecture 1.4 ([Pop22]).** Let $f: X \to A$ be a fibration from a smooth projective variety $X$ to an abelian variety $A$. Then

$$\dim V^0(A, f^*\omega_X^{\otimes m}) \geq \text{Var}(f)$$

for every integer $m > 1$ such that $f^*\omega_X^{\otimes m} \neq 0$.

In the case when $\kappa(X) = 0$ and $f$ is the Albanese morphism of $X$, this conjecture is essentially equivalent to Ueno’s Conjecture K, predicting that up to birational equivalence $f$ becomes a projection onto a factor after an étale base change. This is due to the fact that $f^*\omega_X^{\otimes m}$ is a torsion line bundle on $A$ for every positive integer $m$ such that $f^*\omega_X^{\otimes m} \neq 0$ by [HPS18, Theorem 5.2]. In the following corollary, we give an answer to his conjecture assuming that the general fiber of $f$ has a good minimal model.

**Corollary 1.5.** Let $f: X \to A$ be a fibration from a smooth projective variety $X$ to an abelian variety $A$. Assume that the general fiber of $f$ has a good minimal model. Then

$$\dim V^0(A, f^*\omega_X^{\otimes m}) \geq \text{Var}(f)$$

for every integer $m > 1$ such that $f^*\omega_X^{\otimes m} \neq 0$. Moreover, if $g: X \to Y$ is a smooth model of the Iitaka fibration of $X$ where $Y$ is a smooth projective variety, then

$$q(Y) \geq \dim V^0(A, f^*\omega_X^{\otimes m}) \geq \text{Var}(f)$$

for every integer $m > 1$ such that $f^*\omega_X^{\otimes m} \neq 0$.

In a different but related direction, the next theorem strengthens the result on the subadditivity of Kodaira dimension of fibrations over abelian varieties by [CP17] (see also [HPS18]).
**Theorem 1.6.** Let $f: X \to A$ be a fibration from a klt pair $(X, \Delta)$ to an abelian variety $A$, $F$ the general fiber of $f$, $m \geq 1$ a rational number, and $D$ a Cartier divisor on $X$ such that $D \sim_{\mathbb{Q}} m(K_X + \Delta)$. Then

$$\kappa(X, K_X + \Delta) \geq \kappa(F, K_F + \Delta|_F) + \dim V^0(A, f_* O_X(D)).$$

If $m > 1$, then

$$\kappa(X, K_X + \Delta) \geq \kappa(F, K_F + \Delta|_F) + \kappa(A, \widehat{\det} f_* O_X(D)).$$

If $f_* O_X(D) \neq 0$, $V^0(A, f_* O_X(D))$ is not empty since $f_* O_X(D)$ is a GV-sheaf by [Men21, Corollary 4.1]. We have the following corollary of Theorem 1.6.

**Corollary 1.7.** Let $g: X \to Y$ be a smooth model of the Iitaka fibration associated to $K_X + \Delta$ with general fiber $G$ where $(X, \Delta)$ is a klt pair and $Y$ is a smooth projective variety, and $f: X \to A$ a fibration to an abelian variety $A$ with general fiber $F$. Then

$$\kappa(X, K_X + \Delta) \geq \kappa(F, K_F + \Delta|_F) + \dim A - q(G).$$

We can rewrite the inequality in Corollary 1.7 as

$$\dim F - \kappa(F, K_F + \Delta|_F) \geq \dim G - q(G)$$

where $\dim G - q(G)$ is nonnegative since the Albanese morphism of $G$ is a fibration by $\kappa(G, K_G + \Delta|_G) = 0$ and [Wan16, Theorem B]. We immediately have the following corollary.

**Corollary 1.8.** Let $g: X \to Y$ be a smooth model of the Iitaka fibration associated to $K_X + \Delta$ with general fiber $G$ where $(X, \Delta)$ is a klt pair and $Y$ is a smooth projective variety, and $f: X \to A$ a fibration to an abelian variety $A$ with general fiber $F$. If $(F, \Delta|_F)$ is of log general type, then $G$ is birational to its Albanese variety.

Under the hypotheses of Corollary 1.8 the klt pair $(G, \Delta|_G)$ has a good minimal model by [Fuj13, Theorem 1.1] since $G$ is birational to its Albanese variety. Thus the klt pair $(X, \Delta)$ has a good minimal model over $Y$ by [HX13, Theorem 2.12]. Since $g: X \to Y$ is a smooth model of the Iitaka fibration associated to $K_X + \Delta$, we can deduce that $(X, \Delta)$ has a good minimal model by running a $(K_X + \Delta)$-MMP over $Y$ and applying the canonical bundle formula. The main result of [BC15] says that klt pairs fibered over normal projective varieties of maximal Albanese dimension with general fibers of log general type have good minimal models. By the discussion above, we give an intuitive explanation of why their result should be true.
We also have a quick corollary of Theorem 1.6 if the Albanese morphism of $X$ is a fibration. In Corollary 1.7 we let $f$ be the Albanese morphism of $X$. Then we deduce that

$$\kappa(X, K_X + \Delta) \geq \kappa(F, K_F + \Delta|_F) + q(Y)$$

by Theorem 1.6 and [Men21, Theorem 1.5].

For the proofs of the main theorems, we employ results from [Men21], techniques from [MP21], a hyperbolicity-type result from [PS17], and arguments on positivity properties of coherent sheaves.

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2. Preliminaries

We work over $\mathbb{C}$. A fibration is a projective surjective morphism with connected fibers. Let $\mathcal{F}$ be a coherent sheaf on a projective variety $X$, we denote $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ by $\mathcal{F}^\vee$.

We recall several definitions first.

**Definition 2.1.** Let $X$ be a smooth projective variety. The **irregularity** $q(X)$ is defined as $h^1(X, \mathcal{O}_X)$. If $X$ is a projective variety, the **irregularity** $q(X)$ is defined as the irregularity of any resolution of $X$.

If $X$ is a normal projective variety of rational singularities, then the irregularity $q(X)$ is equal to the dimension of its Albanese variety $\text{Alb}(X)$ since its Albanese variety coincides with the Albanese variety of any of its resolution by [Rei83, Proposition 2.3] and [Kaw85, Lemma 8.1].

**Definition 2.2.** Let $\mathcal{F}$ be a coherent sheaf on an abelian variety $A$. The **cohomological support loci** $V^i_l(A, \mathcal{F})$ for $i \in \mathbb{N}$ and $l \in \mathbb{N}$ are defined by

$$V^i_l(A, \mathcal{F}) = \{ \alpha \in \text{Pic}^0(A) \mid \dim H^i(A, \mathcal{F} \otimes \alpha) \geq l \}.$$  

We use $V(A, \mathcal{F})$ to denote $V^1_1(A, \mathcal{F})$.

**Definition 2.3.** A coherent sheaf $\mathcal{F}$ on an abelian variety $A$

(i) is a **GV-sheaf** if $\text{codim}_{\text{Pic}^0(A)} V^i(A, \mathcal{F}) \geq i$ for every $i > 0$.

(ii) is **$M$-regular** if $\text{codim}_{\text{Pic}^0(A)} V^i(A, \mathcal{F}) > i$ for every $i > 0$.

(iii) satisfies **IT$_0$** if $V^i(A, \mathcal{F}) = \emptyset$ for every $i > 0$.

It is known that $M$-regular sheaves are ample by [Deb06, Corollary 3.2], and GV-sheaves are nef by [PP11, Theorem 4.1]. We prove a useful lemma here by a similar method as in the proof of [PP11, Theorem 4.1].

**Lemma 2.4.** Let $\mathcal{F}$ be a torsion-free GV-sheaf on an abelian variety $A$. Then $\hat{\text{det}} \mathcal{F}$ is nef.
Proof. We denote by \( m_A : A \to A \) the multiplication by \( m \) where \( m \) is an integer. We can take an ample line bundle \( \mathcal{H} \) on \( A \) such that \((-1)_A^* \mathcal{H} \cong \mathcal{H} \) and thus we have \( m_A^* \mathcal{H} \cong \mathcal{H}^{\otimes m} \). Since \( \mathcal{F} \) is a GV-sheaf and \( m_A \) is an isogeny, \( m_A^* \mathcal{F} \) is a torsion-free GV-sheaf. We choose \( m \) to be positive now. We deduce that \( m_A^* \mathcal{F} \otimes \mathcal{H}^{\otimes m} \) satisfies IT\( _0 \) by [PP11, Proposition 3.1], and it is ample by [Deb06, Corollary 3.2]. Since an ample sheaf is big and \( m_A \) is an isogeny, we deduce that 

\[
\hat{\det}(m_A^* \mathcal{F} \otimes \mathcal{H}^{\otimes m}) \cong \hat{\det}m_A^* \mathcal{F} \otimes \mathcal{H}^{\otimes m \cdot \text{rank} \mathcal{F}} \cong m_A^* \hat{\det} \mathcal{F} \otimes \mathcal{H}^{\otimes m \cdot \text{rank} \mathcal{F}}
\]

is big by [Vie83, Lemma 3.2(iii)] (see also [Mor87, Properties 5.1.1]). We deduce that 

\[
m_A^*((\hat{\det} \mathcal{F})^{\otimes m} \otimes \mathcal{H}^{\otimes \text{rank} \mathcal{F}}) \cong (m_A^* \hat{\det} \mathcal{F} \otimes \mathcal{H}^{\otimes m \cdot \text{rank} \mathcal{F}})^{\otimes m}
\]

is big and thus ample since \( A \) is an abelian variety. Thus \((\hat{\det} \mathcal{F})^{\otimes m} \otimes \mathcal{H}^{\otimes \text{rank} \mathcal{F}}\) is ample for every \( m > 0 \) and we deduce that \( \hat{\det} \mathcal{F} \) is nef. \( \square \)

We now give the definition of the Chen–Jiang decomposition.

**Definition 2.5.** Let \( \mathcal{F} \) be a coherent sheaf on an abelian variety \( A \). The sheaf \( \mathcal{F} \) is said to have the Chen–Jiang decomposition if \( \mathcal{F} \) admits a finite direct sum decomposition 

\[
\mathcal{F} \cong \bigoplus_{i \in I} (\alpha_i \otimes p_i^* \mathcal{F}_i),
\]

where each \( A_i \) is an abelian variety, each \( p_i : A \to A_i \) is a fibration, each \( \mathcal{F}_i \) is a nonzero M-regular coherent sheaf on \( A_i \), and each \( \alpha_i \in \text{Pic}^0(A) \) is a torsion line bundle.

We state the following theorem which is a direct corollary of [Men21, Theorems 1.3 and 1.4] and omit the proof, see also [Jia21, Theorem 1.3] for the case when \( m > 1 \) is an integer.

**Theorem 2.6.** Let \( f : X \to A \) be a morphism from a klt pair \((X, \Delta)\) to an abelian variety \( A \), \( m > 1 \) a rational number, and \( D \) a Cartier divisor on \( X \) such that \( D \sim m(K_X + \Delta) \). Then there exists a fibration \( p : A \to B \) to an abelian variety \( B \) such that \( f_* \mathcal{O}_X(lD) \) admits, for every positive integer \( l \), a finite direct sum decomposition 

\[
f_* \mathcal{O}_X(lD) \cong \bigoplus_{i \in I} (\alpha_i \otimes p^* \mathcal{F}_i),
\]

where each \( \mathcal{F}_i \) is a nonzero coherent sheaf on \( B \) satisfying IT\( _0 \), and each \( \alpha_i \in \text{Pic}^0(A) \) is a torsion line bundle whose order can be bounded independently of \( l \). If \( g : X \to Y \) is a smooth model of the Iitaka fibration associated to the Cartier divisor \( D \) with general fiber \( G \) where \( Y \) is a smooth projective variety, then 

\[
\dim B \geq \dim A - q(G).
\]

Moreover, if \( f \) is surjective, then 

\[
q(Y) \geq \dim B.
\]
We will need the following hyperbolicity-type result proved in [PS17]. It relies on important ideas and results of Viehweg–Zuo and Campana–Păun, and on the theory of Hodge modules.

**Theorem 2.7** ([PS17 Theorem 4.1 and Theorem 3.5]). Let $f: X \to Y$ be a fibration between smooth projective varieties where $Y$ is not uniruled. Assume that $f$ is smooth over the complement of a closed subset $Z \subseteq Y$, and there exists a positive integer $m$ such that $\det f_* \omega_{X/Y}^m$ is big. Denote by $D$ the union of the divisorial components of $Z$. Then the line bundle $\omega_Y(D)$ is big.

The theorem above is stated in [PS17] only when $Z = D$, but the proof shows more generally the statement above, since all the objects it involves can be constructed from $Y$ with any closed subset of codimension at least 2 removed.

We include a useful lemma about the log Kodaira dimension on ambient varieties of nonnegative Kodaira dimension which is [MP21] Lemma 2.6.

**Lemma 2.8.** Let $X$ be a smooth projective variety with $\kappa(X) \geq 0$, $Z \subseteq X$ a closed reduced subscheme, and $V = X \setminus Z$. Assume that $Z = W \cup D$ where $\text{codim}_X W \geq 2$ and $D$ is a divisor. Then

$$\kappa(V) = \kappa(X, K_X + D).$$

### 3. Main results

We prove several useful lemmas first.

**Lemma 3.1.** Let $f: X \to Y$ be a surjective morphism between normal projective varieties, and $\varphi: Y' \to Y$ an étale morphism from a normal projective variety $Y'$. Consider the following base change diagram.

$$\begin{array}{ccc}
X' & \xrightarrow{\varphi'} & X \\
\downarrow f' & & \downarrow f \\
Y' & \xrightarrow{\varphi} & Y
\end{array}$$

Let $\mathcal{F}$ be a torsion-free coherent sheaf on $X$, then

$$\varphi'^* \det f_* \mathcal{F} \cong \det \varphi'^* f_* \mathcal{F} \cong \det f'_* \varphi'^* \mathcal{F}.$$

**Proof.** The coherent sheaves $f_* \mathcal{F}$ and $f'_* \varphi'^* \mathcal{F}$ are torsion-free since $\varphi'$ is étale. Since $\varphi$ is flat, we deduce that

$$\varphi'^* \det f_* \mathcal{F} \cong \varphi'^* \text{Hom}_{\mathcal{O}_Y}(\text{Hom}_{\mathcal{O}_Y}(\det f_* \mathcal{F}, \mathcal{O}_Y), \mathcal{O}_Y)$$

$$\cong \text{Hom}_{\mathcal{O}_{Y'}}(\text{Hom}_{\mathcal{O}_{Y'}}(\varphi'^* \det f_* \mathcal{F}, \varphi'^* \mathcal{O}_Y), \varphi'^* \mathcal{O}_Y)$$

$$\cong \text{Hom}_{\mathcal{O}_{Y'}}(\text{Hom}_{\mathcal{O}_{Y'}}(\det \varphi'^* f_* \mathcal{F}, \mathcal{O}_{Y'}), \mathcal{O}_{Y'}) \cong \det \varphi'^* f_* \mathcal{F} \cong \det f'_* \varphi'^* \mathcal{F}.$$  

$\square$
Lemma 3.2. Let \( f: X \to Y \) and \( g: Y \to Z \) be surjective morphisms where \( Y \) is a smooth projective variety, and \( X \) and \( Z \) are normal projective varieties. Consider the following base change diagram.

\[
\begin{array}{ccc}
X & \longrightarrow & X \\
\downarrow f & & \downarrow f \\
Y & \longrightarrow & Y \\
\downarrow g & & \downarrow g \\
\{z\} & \longrightarrow & Z
\end{array}
\]

Let \( F \) be a locally free sheaf of finite rank on \( X \). If \( z \) is a general point of \( Z \), then

\[
(\hat{\det} f_\ast F)|_{Y_z} \cong \hat{\det} f_{z\ast}(F|_{X_z}).
\]

Proof. Choose an open set \( V \subseteq Y \) such that \((f_\ast F)|_V\) is locally free and \( \text{codim}_Y Y \setminus V \geq 2 \). Consider the following base change diagram.

\[
\begin{array}{ccc}
V & \longrightarrow & V \\
\downarrow i_z & & \downarrow i \\
Y & \longrightarrow & Y
\end{array}
\]

We can choose \( z \) sufficiently general such that \( X_z \) is normal, \( Y_z \) is smooth, \( \text{codim}_{Y_z} Y_z \setminus V \geq 2 \), and

\[
(f_\ast F)|_{Y_z} \cong f_{z\ast}(F|_{X_z})
\]

by \([\text{LPS20]}\) Proposition 4.1. Thus \((f_{z\ast}(F|_{X_z}))|_{V_z}\) is locally free. By the property of reflexive sheaves, we have

\[
(\hat{\det} f_\ast F)|_{Y_z} \cong (i_\ast \det i^\ast f_\ast F)|_{Y_z} \quad \text{and} \quad \hat{\det} f_{z\ast}(F|_{X_z}) \cong i_{z\ast} \det i_{z\ast}^\ast f_{z\ast}(F|_{X_z}).
\]

We have the natural morphism

\[
(\hat{\det} f_\ast F)|_{Y_z} \cong (i_\ast \det i^\ast f_\ast F)|_{Y_z} \to i_{z\ast}((\det i^\ast f_\ast F)|_{V_z})
\]

\[
\cong i_{z\ast} i_{z\ast}^\ast (\det (f_\ast F)|_{Y_z}) \cong i_{z\ast} i_{z\ast}^\ast \det f_{z\ast}(F|_{X_z}) \cong \hat{\det} f_{z\ast}(F|_{X_z}).
\]

The morphism above is an isomorphism over the open set \( V_z \). Thus it is an isomorphism over \( Y_z \) since \((\hat{\det} f_\ast F)|_{Y_z}\) and \( \hat{\det} f_{z\ast}(F|_{X_z}) \) are line bundles, and \( \text{codim}_{Y_z} Y_z \setminus V \geq 2 \). \( \square \)

Lemma 3.3. Let \( f: X \to A \) be a surjective morphism from a klt pair \((X, \Delta)\) to an abelian variety \( A \), \( m \geq 1 \) a rational number, and \( D \) a Cartier divisor on \( X \) such that \( D \sim_Q m(K_X + \Delta) \). If \( f_\ast \mathcal{O}_X(D) \neq 0 \), then

\[
\kappa(A, \hat{\det} f_\ast \mathcal{O}_X(D)) \geq 0.
\]
Proof. By [Men21, Theorem 1.1], there exists an isogeny $\varphi: A' \to A$ such that $\varphi^* f_* \mathcal{O}_X(D)$ is globally generated. We deduce that $\det \varphi^* f_* \mathcal{O}_X(D)$ is globally generated and thus $\widehat{\det} \varphi^* f_* \mathcal{O}_X(D)$ is generically globally generated. In particular, the line bundle $\widehat{\det} \varphi^* f_* \mathcal{O}_X(D)$ has nonzero sections. By Lemma 3.1, we deduce

$\kappa(A, \widehat{\det} f_* \mathcal{O}_X(D)) = \kappa(A', \varphi^* \widehat{\det} f_* \mathcal{O}_X(D)) = \kappa(A', \widehat{\det} \varphi^* f_* \mathcal{O}_X(D)) \geq 0$.

□

Lemma 3.4. Let $f: X \to Y$ be a fibration between normal projective varieties, $F$ the very general fiber of $f$, and $L$ a line bundle on $X$. If $f_* L \cong B \oplus T$ where $B$ is an ample sheaf on $Y$, then

$\kappa(X, L) = \kappa(F, L|_F) + \dim Y$.

Proof. Let $\mathcal{H}$ be an ample line bundle on $Y$. Since $B$ is ample, there exists a positive integer $k$ such that $S^k B \otimes \mathcal{H}^{-1}$ is globally generated where $S^k B$ is the $k$-th symmetric product of $B$ (see e.g. [Deb06, Section 2]). We have the following morphism

$S^k B \hookrightarrow S^k f_* L \to f_* L^{\otimes k}$

which is the following nonzero multiplication homomorphism between $k(y)$-linear spaces when restricted at the general point $y$ of $Y$

$S^k (B_y \otimes \mathcal{O}_{Y,y} k(y)) \hookrightarrow S^k H^0(X_y, L|_{X_y}) \to H^0(X_y, L^{\otimes k}|_{X_y})$

by the base change theorem and generic flatness. Thus we deduce that the following homomorphism

$H^0(Y, S^k B \otimes \mathcal{H}^{-1}) \to H^0(Y, f_* L^{\otimes k} \otimes \mathcal{H}^{-1})$

is nonzero since $S^k B \otimes \mathcal{H}^{-1}$ is globally generated. We deduce that $f_* L^{\otimes k} \otimes \mathcal{H}^{-1}$ has a nonzero global section and thus $L^{\otimes k} \otimes (f^* \mathcal{H})^{-1}$ has a nonzero global section. Thus we have an injective morphism

$f^* \mathcal{H} \to L^{\otimes k}$.

By [Mor87, Proposition 1.14], we deduce that

$\kappa(X, L) = \kappa(F, L|_F) + \dim Y$.

□

We are ready to prove our main theorems now. We prove the first part of the inequality in Theorem 1.1 first.

Theorem 3.5. Let $f: X \to A$ be a fibration from a smooth projective variety $X$ to an abelian variety $A$ where $f$ is smooth over an open set $V \subseteq A$, and $m$ a positive integer. Then

$\kappa(V) \geq \kappa(A, \widehat{\det} f_* \omega_X^{\otimes m})$. 

Proof. If \( f_{*} \omega^{\otimes m}_{X} = 0 \), then the statement is trivial. Thus we can assume \( f_{*} \omega^{\otimes m}_{X} \neq 0 \). By Lemma 3.3, we have that

\[
\kappa(A, \hat{\det f_{*} \omega^{\otimes m}_{X}}) \geq 0.
\]

If \( \kappa(A, \hat{\det f_{*} \omega^{\otimes m}_{X}}) = 0 \), then the statement is trivial since \( \kappa(V) \geq 0 \). Thus we can assume \( \kappa(A, \hat{\det f_{*} \omega^{\otimes m}_{X}}) > 0 \). Denote \( Z = A \setminus V \) and assume that \( Z = W \cup D \) where \( \text{codim}_{A} W \geq 2 \) and \( D \) is an effective divisor. If \( \hat{\det f_{*} \omega^{\otimes m}_{X}} \) is big, we deduce that

\[
\kappa(V) = \kappa(A, K_{A} + D) = \dim A = \kappa(A, \hat{\det f_{*} \omega^{\otimes m}_{X}})
\]

by Theorem 2.7 and Lemma 2.8. Assume now \( \hat{\det f_{*} \omega^{\otimes m}_{X}} \) is not big. We can choose a positive integer \( N \) which is sufficiently big and divisible such that

\[
(\hat{\det f_{*} \omega^{\otimes m}_{X}})^{\otimes N} \cong O_{A}(E)
\]

where \( E \) is an effective divisor. By a well-known structural theorem, there exist a fibration \( p : A \rightarrow B \) between abelian varieties and an ample effective divisor \( H \) on \( B \) such that \( \dim A > \dim B > 0 \) and \( E = p^{*}H \). Denote the kernel of \( p \) by \( K \) which is an abelian subvariety of \( A \). By Poincaré’s complete reducibility theorem, there exists an abelian variety \( C \subseteq A \) such that \( C + K = A \) and \( C \cap K \) is finite, so that the natural morphism \( \varphi : C \times K \rightarrow A \) is an isogeny. We consider the following commutative diagram, \( q \) is the projection onto \( K \), \( k \in K \) is a general point, and \( f' \) and \( f'_{k} \) are obtained by base change from \( f \) via \( \varphi \) and the inclusion \( i_{k} \) of the fiber \( C_{k} \) of \( q \) over \( k \) respectively.

\[
\begin{array}{ccc}
X'_{k} & \longrightarrow & X' \\
\downarrow f'_{k} & & \downarrow f \\
C_{k} & \longrightarrow & C \times K \\
\downarrow & & \downarrow q \\
\{k\} & \longrightarrow & K \\
\end{array}
\]

By construction, the composition

\[
\psi_{k} = p \circ \varphi \circ i_{k} : C_{k} \rightarrow B
\]

is an isogeny. Since \( \varphi \) is étale, \( X' \) is smooth. If \( W' := \varphi^{-1}(W) \), then \( \text{codim}_{C \times K} W' \geq 2 \), and \( f' \) is smooth over \( V' := \varphi^{-1}(V) \). We can choose \( k \) sufficiently general such that \( X'_{k} \) is smooth, \( \text{codim}_{C_{k}} i_{k}^{-1}(W') \geq 2 \), and

\[
i_{k}^{*} \hat{\det f_{*} \omega^{\otimes m}_{X}} \cong \hat{\det f'_{k}}(\omega^{\otimes m}_{X'} |_{X'_{k}}) \cong \hat{\det f'_{k}^{*} \omega^{\otimes m}_{X'_{k}}}
\]

by Lemma 3.2. By Lemma 3.1, we deduce that

\[
\psi_{k}^{*}O_{B}(H) \cong i_{k}^{*} \varphi^{*}O_{A}(E) \cong i_{k}^{*} \varphi^{*}(\hat{\det f_{*} \omega^{\otimes m}_{X}})^{\otimes N} \\
\cong i_{k}^{*}(\hat{\det f'_{k}^{*} \omega^{\otimes m}_{X'_{k}}})^{\otimes N} \cong (\hat{\det f'_{k}^{*} \omega^{\otimes m}_{X'_{k}}})^{\otimes N}.
\]
Since $\psi_k$ is an isogeny, $\psi_k^*\mathcal{O}_B(H)$ is ample and thus $\det f_k^*\omega_{X_k}^{\otimes m}$ is ample. Since $\text{codim} c_h i_k^{-1}(W') \geq 2$ and $f_k'$ is smooth over $i_k^{-1}(V')$, we deduce that

$$\kappa(C, K_C + (i_k^*\varphi^*D)_{\text{red}}) = \dim C_k = \dim B = \kappa(A, E) = \kappa(A, \det f_*\omega_X^{\otimes m})$$

by Theorem 2.7. We can choose a rational number $\varepsilon > 0$ small enough such that $(C \times K, \varepsilon \varphi^*D)$ is a klt pair. By [CP17 Theorem 1.1] and choosing $k$ sufficiently general, we deduce that

$$\kappa(V) = \kappa(A, D) = \kappa(A, \varepsilon D) = \kappa(C \times K, \varepsilon \varphi^*D)$$

$$\geq \kappa(C, \varepsilon i_k^*\varphi^*D) = \kappa(C, (i_k^*\varphi^*D)_{\text{red}}) = \kappa(A, \det f_*\omega_X^{\otimes m}).$$

$\square$

Next, we prove the second part of the inequality in Theorem 1.1 in a more general setting which allows klt singularities. We also prove the first inequality in Theorem 1.6 along the way.

**Theorem 3.6.** Let $f: X \to A$ be a fibration from a klt pair $(X, \Delta)$ to an abelian variety $A$, $F$ the general fiber of $f$, $m \geq 1$ a rational number, and $D$ a Cartier divisor on $X$ such that $D \sim \mathbb{Q} m(K_X + \Delta)$. Then

$$\kappa(X, K_X + \Delta) \geq \kappa(F, K_F + \Delta|_F) + \dim V^0(A, f_*\mathcal{O}_X(D)),$$

and

$$\kappa(A, \det f_*\mathcal{O}_X(D)) \geq \dim V^0(A, f_*\mathcal{O}_X(D)).$$

**Proof.** If $f_*\mathcal{O}_X(D) = 0$, then the statement is trivial. Thus we can assume $f_*\mathcal{O}_X(D) \neq 0$. By [Men21 Theorem 1.3], $f_*\mathcal{O}_X(D)$ has the Chen–Jiang decomposition

$$f_*\mathcal{O}_X(D) \cong \bigoplus_{i \in I} (\alpha_i \otimes p_i^*\mathcal{F}_i),$$

where each $A_i$ is an abelian variety, each $p_i: A \to A_i$ is a fibration, each $\mathcal{F}_i$ is a nonzero M-regular coherent sheaf on $A_i$, and each $\alpha_i \in \text{Pic}^0(A)$ is a torsion line bundle. By [LPS20 Lemma 3.3], we deduce that

$$\dim V^0(A, f_*\mathcal{O}_X(D)) = \max_{i \in I} \dim A_i.$$

We consider the fibration $p_j: A \to A_j$ for a fixed $j \in I$. Denote the kernel of $p_j$ by $K$ which is an abelian subvariety of $A$. By Poincaré’s complete reducibility theorem, there exists an abelian variety $C \subseteq A$ such that $C + K = A$ and $C \cap K$ is finite, so that the natural morphism $\varphi: C \times K \to A$ is an isogeny. We consider the following commutative diagram, $q$ is the projection onto $K$, $k \in K$ is a general point, and $f'$ and $f_k'$ are obtained by base change from $f$ via $\varphi$ and the inclusion $i_k$ of the fiber $C_k$ of $q$ over $k$ respectively. We define a $\mathbb{Q}$-divisor $\Delta'$ by $K_X + \Delta' = \varphi^*(K_X + \Delta)$. Since $\varphi'$ is an étale morphism, the new pair $(X', \Delta')$ is klt and $\Delta'$ is effective. Define
Thus we deduce that $\det \kappa$ by Lemma 3.4. By [HMX18, Theorem 4.2], by construction, the composition

$$
(X'_k, \Delta'|_{X'_k}) \xrightarrow{\varphi'} (X', \Delta') \xrightarrow{\varphi} (X, \Delta)
$$

we have that $f_*\mathcal{O}_X(D') \cong \varphi^* f_*\mathcal{O}_X(D)$.

By [LPS20, Proposition 4.1]. We deduce that

$$f'_k \mathcal{O}_{X'_k}(D'|_{X'_k}) \cong i_k^* f_* \mathcal{O}_X(D) \cong (i_k^* \varphi^* \alpha_j \otimes \psi_k^* \mathcal{F}_j) \oplus i_k^* \varphi^* \mathcal{T}.
$$

We can choose $k$ sufficiently general such that $(X'_k, \Delta'|_{X'_k})$ is klt and $f''_{k*} \mathcal{O}_{X'_k}(D'|_{X'_k}) \cong i_k^* f''_* \mathcal{O}_{X'}(D') \cong i_k^* \varphi^* f_* \mathcal{O}_X(D)$ by [LPS20] Proposition 4.1. We deduce that

$$f''_{k*} \mathcal{O}_{X'_k}(D'|_{X'_k}) \cong \mathcal{H}_1 \oplus \mathcal{H}_2,$

with $\mathcal{H}_1$ ample and $\mathcal{H}_2$ a GV-sheaf. The very general fiber of $f'_k$ is $F$. We deduce that

$$\kappa(X'_k, D'|_{X'_k}) = \kappa(F, D|_F) + \dim C_k = \kappa(F, D|_F) + \dim A_j$$

by Lemma 3.4. By [HMX18] Theorem 4.2, $\kappa(F, K_F + \Delta_F)$ is constant for general fiber $F$ of $f$. By [CP17] Theorem 1.1 and choosing $k$ sufficiently general, we deduce that

$$\kappa(X, K_X + \Delta) = \kappa(X', K_X' + \Delta') \geq \kappa(X'_k, K_{X'_k} + \Delta'|_{X'_k})$$

$$= \kappa(X'_k, D'|_{X'_k}) = \kappa(F, D|_F) + \dim A_j = \kappa(F, K_F + \Delta_F) + \dim A_j.
$$

Thus we deduce that

$$\kappa(X, K_X + \Delta) \geq \kappa(F, K_F + \Delta_F) + \max_{i \in I} \dim A_i$$

$$= \kappa(F, K_F + \Delta_F) + \dim V^0(A, f_* \mathcal{O}_X(D)).$$

We next prove the second inequality in the theorem. Since $\mathcal{H}_1$ is big, $\det \mathcal{H}_1$ is a big line bundle by [Vie83, Lemma 3.2(iii)] (see also [Mor87].
Properties 5.1.1]). We deduce that \( \widehat{\det} \mathcal{H}_2 \) is a nef line bundle by Lemma 2.4. Thus their tensor product \( \widehat{\det} f'_k \mathcal{O}_{X'_k}(D' | X'_k) \) is big. We can choose \( k \) sufficiently general such that

\[
i_k^* \widehat{\det} f'_k \mathcal{O}_{X'_k}(D') \cong \widehat{\det} f'_k \mathcal{O}_{X'_k}(D' | X'_k)
\]

by Lemma 3.2. By Lemma 3.3, we can choose a positive integer \( N \) which is sufficiently big and divisible such that

\[
(\widehat{\det} f_* \mathcal{O}_X(D))^{\otimes N} \cong \mathcal{O}_A(E)
\]

where \( E \) is an effective divisor. By Lemma 3.1, we deduce that

\[
i_k^* \varphi^* \mathcal{O}_A(E) \cong i_k^* \varphi^* (\widehat{\det} f_* \mathcal{O}_X(D))^{\otimes N} \cong i_k^* (\widehat{\det} f'_k \mathcal{O}_{X'_k}(D' | X'_k))^{\otimes N}.
\]

By the same argument used at the end of the proof of Theorem 3.5 and choosing \( k \) sufficiently general, we deduce that

\[
\kappa(A, \widehat{\det} f_* \mathcal{O}_X(D)) = \kappa(A, \mathcal{O}_A(E)) \geq \kappa(C_k, i_k^* \varphi^* \mathcal{O}_A(E)) = \kappa(C_k, \widehat{\det} f'_k \mathcal{O}_{X'_k}(D' | X'_k)) = \dim C_k = \dim A_j.
\]

Thus we deduce that

\[
\kappa(A, \widehat{\det} f_* \mathcal{O}_X(D)) \geq \max_{i \in I} \dim A_i = \dim V^0(A, f_* \mathcal{O}_X(D)).
\]

\[ \square \]

**Remark 3.7.** We still have \( \kappa(A, \widehat{\det} f_* \mathcal{O}_X(D)) \geq \dim V^0(A, f_* \mathcal{O}_X(D)) \) if \( f \) is only assumed to be surjective by the same proof as above.

**Lemma 3.8.** Let \( f : X \to A \) be a surjective morphism from a klt pair \( (X, \Delta) \) to an abelian variety \( A \), \( m > 1 \) a rational number, and \( D \) a Cartier divisor on \( X \) such that \( D \sim_Q m(K_X + \Delta) \). Then

\[
\kappa(A, \widehat{\det} f_* \mathcal{O}_X(D)) = \dim V^0(A, f_* \mathcal{O}_X(D)).
\]

**Proof.** If \( f_* \mathcal{O}_X(D) = 0 \), then the statement is trivial. Thus we can assume \( f_* \mathcal{O}_X(D) \neq 0 \). By Theorem 2.6, there exists a fibration \( p : A \to B \) to an abelian variety \( B \) such that \( f_* \mathcal{O}_X(D) \) admits a finite direct sum decomposition

\[
f_* \mathcal{O}_X(D) \cong \bigoplus_{i \in I} (\alpha_i \otimes p^* \mathcal{F}_i),
\]

where each \( \mathcal{F}_i \) is a nonzero coherent sheaf on \( B \) satisfying IT\(_0\), and each \( \alpha_i \in \text{Pic}^0(A) \) is a torsion line bundle. Each \( \mathcal{F}_i \) is torsion-free. By [LPS20, Lemma 3.3], we deduce that

\[
\dim V^0(A, f_* \mathcal{O}_X(D)) = \dim B.
\]

Since \( p \) is flat and \( \widehat{\det}(\alpha_i \otimes p^* \mathcal{F}_i) \) is a line bundle, we deduce that

\[
\widehat{\det} f_* \mathcal{O}_X(D) \cong \bigotimes_{i \in I} \widehat{\det}(\alpha_i \otimes p^* \mathcal{F}_i) \cong \bigotimes_{i \in I} (\widehat{\det} p^* \mathcal{F}_i \otimes \alpha_i \otimes \text{rank} \mathcal{F}_i)
\]
\[
\approx \bigotimes_{i \in I} (p^* \hat{\det} F_i \otimes \alpha_i^{\otimes \text{rank} F_i}) \approx p^* (\bigotimes_{i \in I} \hat{\det} F_i) \otimes \bigotimes_{i \in I} \alpha_i^{\otimes \text{rank} F_i}.
\]

Since \( F_i \) satisfies IT\(_0\), it is ample by [PP03, Proposition 2.13] and [Deb06, Corollary 3.2]. The sheaf \( F_i \) is big since an ample sheaf is big (see e.g. [Deb06, Section 2] and [Mor87, Section 5]). Thus \( \hat{\det} F_i \) is a big line bundle by [Vie83, Lemma 3.2(iii)] (see also [Mor87, Properties 5.1.1]). We deduce that
\[
\kappa(A, \hat{\det} f_* \mathcal{O}_X(D)) = \kappa(A, p^* (\bigotimes_{i \in I} \hat{\det} F_i)) = \kappa(B, \bigotimes_{i \in I} \hat{\det} F_i) = \dim B = \dim V^0(A, f_* \mathcal{O}_X(D)).
\]

Proof of Theorem 1.1. It is a direct corollary of Theorem 3.5, Theorem 3.6 and Lemma 3.8.

Proof of Corollary 1.2. It is a direct corollary of Theorems 1.1 and 2.6.

Proof of Corollary 1.3. Let \( f: X \to A \) be a smooth morphism to an abelian variety \( A \). Then \( f \) is surjective. We consider its Stein factorization \( f = \varphi \circ h \) where \( B \) is a normal projective variety, \( h: X \to B \) is a fibration, and \( \varphi: B \to A \) is a finite surjective morphism. Since \( f \) is smooth, we deduce that \( h \) is smooth and \( \varphi \) is étale. Thus \( B \) is also an abelian variety. Since \( q(G) = 0 \) and \( h \) is a smooth fibration, we deduce that \( \dim A = \dim B = 0 \) by Corollary 1.2.

Proof of Corollary 1.5. By Theorem 2.6 and [LPS20, Lemma 3.3], there exists an abelian variety \( B \) such that
\[
\dim V^0(A, f_* \omega_X^m) = \dim B
\]
for every integer \( m > 1 \) such that \( f_* \omega_X^m \neq 0 \). By Theorem 1.1, we have
\[
\kappa(A, \hat{\det} f_* \omega_X^m) = \dim V^0(A, f_* \omega_X^m)
\]
for every integer \( m > 1 \). By [Kaw85, Theorem 1.1], there exists an integer \( k > 1 \) such that
\[
\kappa(A, \hat{\det} f_* \omega_X^k) \geq \text{Var}(f)
\]
since the general fiber of \( f \) has a good minimal model. In particular, \( f_* \omega_X^k \neq 0 \). Thus we deduce that
\[
\dim V^0(A, f_* \omega_X^m) = \dim V^0(A, f_* \omega_X^k) = \kappa(A, \hat{\det} f_* \omega_X^k) \geq \text{Var}(f)
\]
for every integer \( m > 1 \) such that \( f_* \omega_X^m \neq 0 \). If \( g: X \to Y \) is a smooth model of the Iitaka fibration of \( X \) where \( Y \) is a smooth projective variety, then
\[
q(Y) \geq \dim B \geq \text{Var}(f)
\]
by Theorem 2.6.
**Proof of Theorem 1.6.** It is a direct corollary of Theorem 3.6 and Lemma 3.8.

**Proof of Corollary 1.7.** It is a direct corollary of Theorems 1.6 and 2.6.

**Proof of Corollary 1.8.** It is a direct corollary of Corollary 1.7 and [Wan16, Theorem B].

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