On N=2 SUSY gauge theories
and integrable systems

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Abstract: This note gives a brief review of the integrable structures presented in
the Seiberg-Witten approach to the N=2 SUSY gauge theories with emphasize on
the case of the gauge theories with matter hypermultiplets included (described by
spin chains). The web of different N=2 SUSY theories is discussed.

General remarks. Since the paper by N. Seiberg and E. Witten [1], there have
been a lot of attempts to get better understanding of the structures arising in the
low-energy sector of N=2 SUSY gauge theories. In a sense, the paper [1] pointed
out the importance of objects completely different from those typically dealt with
in quantum field theory. In particular, one of the main quantities in the Seiberg-
Witten (SW) approach is the prepotential giving the low-energy effective action of
the theory.

One of the constituent parts of arena where the low-energy effective theory lives
is, in accordance with [1], a Riemann surface, while a subspace of the moduli space
of Riemann surfaces gives the moduli space of vacua of the physical theory. The
whole world-sheet of the low-energy theory can be described in terms of 5-brane in
M-theory [2, 3, 4, 5].

To these counterparts of the field theory objects, one should also add the analog
of the symmetry principle which arises within the SW framework. Namely, it turns
out that the symmetry properties of theory in the low-energy limit are encoded in
the integrable system that underlines the low-energy dynamics.

The very fact of existence of an integrable system behind SW solution has been
first realized in the paper [6] that dealt with the pure gauge theory. Since then, many
more examples of the N=2 SUSY theories has been investigated, and corresponding
integrable systems have been revealed [7]-[11].

The goal of the present short review is to sketch the general scheme of connection
between the SW solutions and integrable systems. We also describe the correspon-
dence (SUSY theory $\leftrightarrow$ integrable system) in concrete examples discussing what deformations of integrable systems correspond to deformations of physical theories.

**SW anzatz, prepotential and integrable system.** For the $\mathcal{N} = 2$ SUSY gauge theory the SW anzatz can be *formulated* in the following way. One starts with two *bare* spectral curves. One of them, with a holomorphic 1-form $d\omega$, is elliptic curve (torus)

$$E_1(\tau) : y^2 = \prod_{a=1}^{3} (x - e_a(\tau)), \quad \sum_{a=1}^{3} e_a(\tau) = 0, \quad d\omega = \frac{dx}{y} \equiv d\xi, \quad (1)$$

when the YM theory contains the adjoint matter hypermultiplet, or its degeneration $\tau \rightarrow i\infty$ – the double-punctured sphere (“annulus”):

$$x \rightarrow w \pm \frac{1}{w}, \quad y \rightarrow w \mp \frac{1}{w}, \quad d\omega = \frac{dw}{w} \quad (2)$$

otherwise. In particular, this latter possibility is the case for the theory with the fundamental matter hypermultiplets.

The second bare spectral curve is also elliptic curve $E_2(\tau')$ or its degenerations depending on the dimension of the space-time.

In the integrable framework, the two bare spectral curves are related by the full spectral curve that is just the Riemann surface emerging within the Seiberg-Witten construction. There are two different types of integrable system with the corresponding associated full spectral curve.

Integrable systems of the first type which could be naturally called Hitchin type systems are described as follows. First, one introduces the Lax operator $\mathcal{L}(x, y)$ that is defined as a $N \times N$ matrix-valued function (1-differential) on the first bare spectral curves. Then, the *full* spectral curve $\mathcal{C}$ is given by the Lax-eigenvalue equation: $\det(\mathcal{L}(x, y) - \lambda) = 0$, where $\lambda$ is given on the second bare curve. As a result, $\mathcal{C}$ arises as a ramified covering over the bare spectral curves $E_1(\tau)$:

$$\mathcal{C} : \mathcal{P}(\lambda; x, y) = 0 \quad (3)$$

The typical system of this type is the Calogero-Moser system. The Lax operators in the systems of the first type satisfy linear Poisson brackets with generally speaking *dynamical* elliptic $r$-matrix [12].

On contrary, integrable systems of the second type are characterized by the quadratic Poisson brackets with the *numerical* $r$-matrix (certainly, quadratic Poisson relations can be easily rewritten as the linear ones with dynamical $r$-matrix [13]). The typical systems of this type are lattice systems and spin chains [14]. They are described by $2 \times 2$ matrix-valued transfer-matrices $T_N(\xi)$, and the full spectral curve is given by the equation $\det(T_N(\xi) - w) = 0$, where $w$ is given on the second bare curve. In fact, only the systems when at least one of the bare curves is degenerated
are investigated in detail. Therefore, either \( w \) is the coordinate on the cylinder, or \( \lambda \) is the coordinate on the sphere or cylinder.

The function \( P \) in (3) depends also on parameters (moduli) \( s_I \), parametrizing the moduli space \( M \). From the point of view of integrable system, the Hamiltonians (integrals of motion) are some specific co-ordinates on the moduli space. From the point of view of gauge theory, the co-ordinates \( s_I \) include \( s_i \) – (the Schur polynomials of) the adjoint-scalar expectation values \( h_k = \frac{1}{k} \langle \text{Tr} \phi^k \rangle \) of the vector \( N = 2 \) supermultiplet, as well as \( s_i = m_i \) – the masses of the hypermultiplets. One associates with the handle of \( C \) the gauge moduli and with punctures – massive hypermultiplets, masses being residues in the punctures.

The generating 1-form \( dS \cong \xi d\omega \) is meromorphic on \( C \) (the equality modulo total derivatives is denoted by “\( \cong \)”), where \( \xi \in E_2 \) is associated with one of the bare curve and \( d\omega \) – with another one \( E_1 \) (they are related via the spectral curve equation). In integrable system terms, this form is just the shorten action “\( pdq \)” along the non-contractible contours on the Hamiltonian tori, i.e. related to the symplectic form “\( d\xi \wedge d\omega \)”.

The prepotential is defined in terms of the cohomological class of \( dS \):

\[
\int_{A_I} dS = \frac{\partial F}{\partial a_I} = \int_{B_I} dS
\]

(4)

The cycles \( A_I \) include the \( A_i \)’s wrapping around the handles of \( C \) and \( A_i \)’s, going around the singularities of \( dS \). The conjugate contours \( B_I \) include the cycles \( B_i \) and the non-closed contours \( B_i \), ending at the singularities of \( dS \) (see [15] for more details). The integrals \( \int_{B_i} dS \) are actually divergent, but the coefficient of divergent part is equal to residue of \( dS \) at particular singularity, i.e. to \( a_i \). Thus, the divergent contribution to the prepotential is quadratic in \( a_i \), while the prepotential is normally defined modulo quadratic combination of its arguments (which just fixes the bare coupling constant). In particular models \( \int_{A_{k,l}} dS \) for some conjugate pairs of contours are identically zero on entire \( M \): such pairs are not included into our set of indices \{I\}.

Note that the data the period matrix of \( C \) \( T_{ij}(a_i) = \frac{\partial F}{\partial a_i \partial a_j} \) as a function of the action variables \( a_i \) gives the set of coupling constants in the effective theory.

The most important property of the differential \( dS \cong \xi d\omega \) is that its derivatives w.r.t. moduli gives holomorphic differentials on \( C \) (see [17]). The prepotential in the context of integrable systems was also discussed in [16].

**Spin chains: gauge theories with fundamental matter.** The crucial difference between integrable systems of the two types is in interpretation of the spectral curve determinant equation. It is the general corollary of existence of the linear Poisson bracket (even with dynamical \( r \)-matrix) that the spectral determinant equation generates the conserved quantities [17]. Therefore, the coefficients of the spectral
curve polynomial are the integrals of motion (and give some coordinates on the moduli space). However, in integrable system of the second type, there is some more direct meaning of the spectral determinant equation.

Namely, it can be described as periodicity condition that is imposed onto the transfer-matrix. In fact, the existence of the transfer-matrix describing the evolution into the discrete direction \( \text{see [14]} \) is the main peculiarity of this kind integrable systems.

The simplest example of the integrable system of the second type is the periodic Toda chain. This system is, at the same time, of the first (Hitchin) type. This surprising fact looks accidental and is due to the possibility of two different descriptions of the Toda chain \([7]\). The Lax operators in the first description satisfy linear Poisson brackets with the trigonometric numerical \( r \)-matrix, while in the second one – those with the rational numerical \( r \)-matrix. In this case, \( E_1 \) degenerates into the cylinder with coordinate \( w \), while \( E_2 \) – into the sphere with coordinate \( \lambda \).

Let us now switch to the SUSY gauge theories that are underlined by the above described integrable systems. The periodic Toda chain is associated with the pure gauge \( \mathcal{N} = 2 \) theory \([6, 8]\). Possible deformations of this physical theory is to add matter hypermultiplets. In fact, one can add either one matter hypermultiplet in adjoint representation which gives rise to the UV-finite theory, or several fundamental matter hypermultiplets so that the theory is still asymptotically free, or UV-finite\(^2\). This corresponds to the two natural deformations of the periodic Toda chain. The first deformation (by adding the adjoint matter) is associated with the elliptic Calogero-Moser system \([9]\) and is the deformation within the Hitchin-like approach to the Toda system. The other possible way to deform the Toda chain is to consider more general system admitting the transfer-matrix description. This system is the (inhomogeneous) periodic XXX spin chain and describes exactly the theory with fundamental matter \([7]\).

In fact, there are more general spin chains that can be also associated with some physical theories. We return to this question in the next paragraph. Now let us just note that all periodic inhomogeneous chains admit the general description so that the chain of length \( n \) is given by the Lax matrices \( L_i(\xi + \xi_i) \), \( \xi_i \) being the chain inhomogeneities, and periodic boundary conditions. Thus, integrable systems of the second type differ from each other only by different concrete Lax operators \( L_i(\xi) \) \([7, 18]\).

The linear problem in the spin chain has the following form

\[
L_i(\xi)\Psi_i(\xi) = \Psi_{i+1}(\xi)
\]

where \( \Psi_i(\xi) \) is the two-component Baker-Akhiezer function. The periodic boundary

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1. This latter degeneration of the torus \( E(x, y) \) is described as \( x \to 0, \ y \to \lambda \), or \( x \to \lambda, \ y \to 0 \) and \( d\omega \to d\lambda \).
2. For the \( SU(N_c) \) gauge group, there should be at most \( 2N_c \) fundamental hypermultiplets.
conditions are easily formulated in terms of the Baker-Akhiezer function and read as

\[ \Psi_{i+n}(\xi) = -w \Psi_i(\xi) \]  

(6)

where \( w \) is a free parameter (diagonal matrix). One can introduce the transfer matrix shifting \( i \) to \( i + n \)

\[ T(\xi) \equiv L_n(\xi) \ldots L_1(\xi) \]  

(7)

which provides the spectral curve equation

\[ \det(T(\xi) + w \cdot 1) = 0 \]  

(8)

and generates a complete set of integrals of motion. Note that in this approach the parameter \( w \) of the bare spectral curve \( E_1 \) just describes the periodicity conditions.

Integrability of the spin chain follows from quadratic \( r \)-matrix relations (see, e.g. [14])

\[ \{ L_i(\xi) \otimes L_j(\xi'), \delta_{ij} \} = \delta_{ij} [ r(\xi - \xi'), L_i(\xi) \otimes L_i(\xi') ] \]  

(9)

The crucial property of this relation is that it is multiplicative and any product like (7) satisfies the same relation

\[ \{ T(\xi) \otimes T(\xi') \} = [ r(\xi - \xi'), T(\xi) \otimes T(\xi') ] \]  

(10)

**Zoo of \( \mathcal{N} = 2 \) SUSY theories**  
Thus far, we mentioned two possible generalizations of the periodic Toda chain: to the elliptic Calogero-Moser system, and to the inhomogeneous XXX spin chain. In fact, these integrable systems admit further deformations. Indeed, the elliptic Calogero-Moser system is the system with coordinate variables living on the torus, but momentum variables – on the sphere. One can naturally deform this system to involve momenta living on the cylinder (elliptic Ruijsenaars model [19]) or even on another torus (the second elliptic bare curve, double elliptic system [20]).

At the same time, XXX spin chains described by the rational \( r \)-matrix can be deformed to the either XXZ or XYZ spin chains, described correspondingly by the trigonometric or elliptic \( r \)-matrices [14]. In these theories the second bare curve is the cylinder or torus respectively, and this is the manifold where momenta of the system lives. It implies that the momentum variables of integrable system get restricted values. One might think of this as of a sort of Kaluza-Klein mechanism. This interpretation, indeed, turns out to be correct so that the XXZ spin chain describes the 5 dimensional SUSY gauge theories with fundamental matter\(^3\) and one of the 5 dimensions compactified onto the circle [18, 22]. At the same time, the XYZ chain describes the 6 dimensional theory with fundamental matter and with 2 dimensions compactified onto the bare torus [11, 18, 22].

\(^3\)The pure gauge theory in 5d is described by degeneration of the XXZ chain, namely, by the relativistic Toda chain [21, 22].
Note that 6 dimensions exhaust the room for consistent theories. It perfectly matches the fact that the $XYZ$ chain seems not to admit further deformations. Still there is yet another possible deformation of the spin chain. Namely, one can consider, instead of $sl(2)$, (inhomogeneous) $sl(p)$ spin chains that are described by the $p \times p$ matrix-valued Lax operators $[14, 5, 18]$. Such systems are associated $[5]$ with the SUSY theory with the gauge group being the product of simple factors and with bi-fundamental matter hypermultiplets $[2]$.

After these identifications made, one can build the whole picture of the correspondence (integrable systems $\leftrightarrow$ SUSY gauge theories). As the starting point, one describes the pure gauge theory by the simple periodic Toda chain. Then, adding matter hypermultiplets leads to the spin chain and adding adjoint matter leads to the elliptic Calogero-Moser (Hitchin type) system. This latter procedure implies that the first spectral curve (target manifold of the coordinate variables) is elliptic. At the same time, increasing the dimension of the space-time leads to the cylindrical (5d) or elliptic (6d) second bare curve (target manifold of the momentum variables).

In 5 dimensions the corresponding pure gauge system is the relativistic Toda chain $[21, 15, 18]$ and the theory with adjoint matter is the elliptic Ruijsenaars model $[21]$. Analogous systems in 6 dimensions are described less manifestly (see, however, $[20]$), excluding the case of $XYZ$ spin chain associated with the fundamental matter $[11, 18, 22]$. At last, considering theories with the gauge group that is the product of simple factors, one should enlarge either the matrix dimensions of the Lax operator at the single site (within the spin chain framework) or the number of marked points (in the Hitchin-like approach) $[4, 18]$.

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$^4$Note that this system also admits two representations – as a spin chain and within the Hitchin-like approach $[23]$. 
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