A technique for generating Feynman diagrams

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Abstract

We present a simple technique that allows to generate Feynman diagrams for vector models with interactions of order $2^n$ and similar models (Gross-Neveu, Thirring model) using a bootstrap equation that uses only the free field value of the energy as an input. The method allows to find the diagrams to, in principle, arbitrarily high order and applies to both energy and correlation functions. It automatically generates the correct symmetry factor (as a function of the number of components of the field) and the correct sign for any diagram in the case of fermion loops. We briefly discuss the possibility of treating QED as a Thirring model with non-local interaction.

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1 Introduction

In field theory the calculation of physical quantities is commonly given as a sum of Feynman diagrams. Each diagram is calculated with a certain weight, dependent on the number of symmetries of the diagram. A rule to find this symmetry factor is given in [2, 3]. Furthermore expressions were found for the total number of diagrams that contribute in any order to various quantities of interest [4, 5]. In order to find the number and type of diagrams that contribute to a physical quantity, numerous programs were developed [6, 7]. In this article we want to present a technique for generating Feynman diagrams that is a generalisation of a similar technique previously used in the study of the vector model with quartic interaction in 0+0 dimensions [1].

The results obtained in this paper are valid for any number of dimensions. We will find all diagrams that contribute to the energy and the correlation functions (to, in principle, arbitrarily high order) with the correct symmetry factor. This factor can be a function of the number of components of the field. The technique is essentially different from the ones that are usually applied to generate diagrams, since it does not involve any combinatorics. It is a recursive method that requires only the free field value of the energy as an input.

In [1] it was observed that a particular relation between the partial derivatives of the energy of the vector model could be used to obtain a perturbation series for the energy. In the simple case of the 0+0 dimensional vector model it also allowed to find explicit expressions for the coefficients of the large $N$ expansion of the model. A general form for this coefficients could be found and proven to all orders. Unfortunately, a straightforward application of the method used in 0+0 dimensions to higher dimensions is not possible. However, if we slightly adapt the original model a similar technique can be used in higher dimensions. We will consider here the vector model with quartic self-interaction. Instead of considering a model with constant mass, we will allow the mass to depend on the coordinates (the mass is therefore an external field). The model is defined by the partition function:

$$Z = e^{-E} = \int \mathcal{D}\phi \exp \left[ -\frac{1}{2} \int dx \partial_{\mu} \phi \partial_{\mu} \phi + m(x)\phi^2(x) - g \int (\phi^2)^2 \right], \quad (1)$$

where $\phi$ is a $N$-component field. If we now consider the first and second
functional derivative of $E$ with respect to $m(x)$ we find:

$$\frac{\partial E}{\partial m(x)} = \frac{1}{2} < \phi^2(x) >$$

and:

$$\frac{\partial^2 E}{\partial m(x)^2} = -\frac{1}{4} < (\phi^2(x))^2 > + (\frac{\partial E}{\partial m(x)})^2,$$

(2)

where $< ... >$ denotes the expectation value of an operator. Integrating the last equation with respect to $x$ we find:

$$< \int (\phi^2)^2 >= 4 \int ((\frac{\partial E}{\partial m(x)})^2 - \frac{\partial^2 E}{\partial m(x)^2})$$

(3)

We rewrite this equation in the form that will be of interest to us:

$$\frac{\partial E}{\partial g} = 4 \int ((\frac{\partial E}{\partial m(x)})^2 - \frac{\partial^2 E}{\partial m(x)^2})$$

(4)

This is the generalisation of the formula found in [1]. It is clear how this formula can be used to generate a perturbation series for the vector model. If we start with the zeroth order approximation of the energy (i.e.: the value of the energy when there is no interaction), formula (4) gives us the derivative of the energy (with respect to the coupling constant) to the same order. Upon integration, we therefore find the energy to first order. This again can be used in (4) to find the energy to second order. By continuing this way we generate a perturbation series solely from the knowledge of the free theory and the fact that the energy obeys (4). Of course, since the perturbation series is known to be given by a sum of Feynman-diagrams it is not unexpected to find that we will simply generate the diagrams to any desired order, with the exact symmetry factors.

2 Generating Feynman diagrams

In order to be able to carry out the procedure described in the previous section, we must know the value of the energy when there is no interaction. Without any difficulty this is found to be:

$$E(g = 0) = \frac{N}{2} \text{Tr} \log(-\partial^2 + m)$$

(5)
(remember that \(m\) is a field, rather than a constant) If we denote by \(\{|x\rangle\}\) a complete set of eigenstates of the position operator, then (5) can be written as:

\[
E(g = 0) = \frac{N}{2} \int dy <y| \log(1 - \frac{m}{\partial^2})|y> \quad (6)
\]

For later convenience we have subtracted a quantity that is independent of \(m\). This will not change the result. We will also denote \(-\partial^{-2}\) as \(G\). The expression (6) is defined by its series expansion, and so is the functional derivative with respect to \(m(x)\). If we consider a general term in the series expansion, the derivative will work successively on each of the \(m\)’s that occur in the term. We find that the derivative of such a general term is given by:

\[
\frac{\partial}{\partial m(x)} <y|(mG)^n|y> = \sum_{j=0}^{n-1} <y|(mG)^j|x> <x|G(mG)^{n-1-j}|y> \quad (7)
\]

And when we integrate with respect to \(y\):

\[
\frac{\partial}{\partial m(x)} \int dy <y|(mG)^n|y> = n <x|G(mG)^{n-1}|x> \quad (8)
\]

We therefore find:

\[
\frac{\partial E}{\partial m(x)} = -\frac{N}{2} <x|G \frac{1}{1 + mG}|x> \quad (9)
\]

We will also need higher derivatives of the energy. Let us therefore consider the following derivative:

\[
\frac{\partial}{\partial m(x)} <y|G \frac{1}{1 + mG}|z> = \frac{\partial}{\partial m(x)} <y|G \sum_{n=0}^{\infty} (-mG)^n|z> \quad (10)
\]

Again using (7), we find that this is equal to:

\[
- \sum_{n=0}^{\infty} \sum_{j=0}^{n} <y|G(-mG)^j|x> <x|G(-mG)^{n-j}|z> \quad (11)
\]

By rearranging the double sum we can write this in the form:

\[
- \int dy <y|G \frac{1}{1 + mG}|x> <x|G \frac{1}{1 + mG}|z> \quad (12)
\]
This formula has a simple interpretation. Since \(< y \mid G_{1+mg} \mid z >\) is nothing but the propagator of the field between points \(y\) and \(z\), taking the derivative with respect to \(m(x)\) is seen to double this propagator by inserting an extra point \(x\) between the two endpoints (apart from a change of sign). We can denote this graphically as:

\[ \frac{\partial}{\partial m(x)}(y \rightarrow z) = -y \rightarrow x \rightarrow z \]

We will now illustrate the technique by generating the first orders in the series expansion of the energy. Starting from the lowest order approximation (i.e. the value of the energy without interaction), we find from the formulas given above:

\[ \frac{\partial E}{\partial m(x)} = \frac{N}{2} < x \mid G \frac{1}{1 + mg} \mid x > + \mathcal{O}(g) \]  

(13)

\[ \frac{\partial^2 E}{\partial m(x)^2} = -\frac{N}{2} < x \mid G \frac{1}{1 + mg} \mid x > < x \mid G \frac{1}{1 + mg} \mid x > + \mathcal{O}(g) \]  

(14)

Using (13) and (14) in (4) and integrating with respect to \(g\) (taking into account the correct integration constant), we find:

\[ E = \frac{N}{2} Tr \log(1 + mg) + N(N + 2)g \int dx < x \mid G \frac{1}{1 + mg} \mid x >^2 + \mathcal{O}(g^2) \]  

(15)

As long as \(m\) is a function of \(x\) the integral can not be carried out. Unfortunately, we can not put \(m\) equal to a constant at this point of the calculation, because we need the full expression to find the higher order terms. It is clear however that the first order expression we have found corresponds to the diagram:

\[ E = \frac{N}{2} \bigcirc + N(N + 2) g \bigcirc \bigcirc + \mathcal{O}(g^2) \]

as can be seen by drawing the propagators that occur in the term or by considering the limit of constant \(m\), in which case the term reduces to \((\int \frac{dp}{p^2 + m})^2\), which is indeed the value of the depicted diagram. This procedure can be continued straightforwardly to obtain higher order diagrams. Let us
illustrate this for the second order term. In order not to make the formulas too lengthy we will denote the propagator by:

$$ G(x, y) = \langle x|G\frac{1}{1 + mG}|y \rangle $$

The first and second derivative of the energy are now given by:

$$ \frac{\partial E}{\partial m(x)} = \frac{N}{2}G(x, x) + 2N(N + 2)g \int dyG(y, y)G(y, x)G(x, y) + O(g^2) \quad (16) $$

$$ \frac{\partial^2 E}{\partial m(x)^2} = \frac{\partial^2 E(g = 0)}{m(x)^2} + 2N(N + 2)g \int dyG(y, x)^2G(x, y)^2 + $$

$$ 4N(N + 2)g \int dyG(y, x)G(x, y)G(y, y)G(x, x) + O(g^2) \quad (17) $$

Again using this in (14) we find after integration:

$$ E = \frac{N}{2} Tr \log(1 + mG) + N(N + 2)g \int dxG(x, x)^2 + $$

$$ 4g^2N(N + 2) \int \int dxdyG(y, x)^2G(x, y)^2 + $$

$$ 4g^2N(N + 2)^2 \int \int dxdyG(y, x)G(x, y)G(y, y)G(x, x) + O(g^3) \quad (18) $$

Again by drawing the propagators or by considering the limit of constant $m$ we find that this corresponds to the following diagramatic expression:

$$ 4N(N + 2)^2 \ g^2 \ 
\overset{\bullet}{\bullet}\overset{\bullet}{\bullet} + 4N(N + 2) \ g^2 \ 
\overset{\bullet}{\bullet}\overset{\bullet}{\bullet} $$

which are indeed the correct diagrams with the correct symmetry factors. It is clear that this procedure can be continued to any order one desires. Of course the expressions will become rather elaborate as one advances towards higher orders. For this reason we will present in the next section a completely diagrammatic approach to the same problem, in which one does not work with mathematical expressions, but with Feynman diagrams that are combined together according to a set of simple rules.
3 Diagrammatic approach

The most important formula in developing this approach is formula (12) for which we have already given a graphical expression. We will mainly repeat the steps followed in the preceding section, but this time from a diagrammatic point of view. The lowest order approximation of the energy is now given by:

\[ E = \frac{N}{2} + \mathcal{O}(g) \]

and the first functional derivative with respect to \( m(x) \) is given by (see formula (9)):

\[ \frac{\partial E}{\partial m(x)} = \frac{N}{2} + \mathcal{O}(g) \]

From now on we use formula (12). In the above diagram we have one propagator. Since the derivative doubles each propagator in turn (and changes the sign) we find the following diagrammatic expression for the second functional derivative:

\[ \frac{\partial^2 E}{\partial m(x)^2} = -\frac{N}{2} + \mathcal{O}(g) \]

Taking the integral with respect to \( x \) in the diagrammatic approach corresponds to identifying the different points that are labelled with "\( x \)" and dropping the label. By doing this we create a vertex. Doing so and using formula (4) we find:

\[ \frac{\partial E}{\partial g} = N(N + 2) \]

recovering the result found in the preceding section. We can now proceed to higher orders by taking the functional derivative of the first order approximation of the energy. This involves taking the derivative of the the two loop
diagram shown above. In this case we have two propagators, so the derivative
will be the sum of diagrams, in which every line is doubled in turn (hereby
changing the sign of the diagram). Diagrammatically, this gives:

\[ \frac{\partial}{\partial m(x)} \left( \begin{array}{c}
\circ \circ \\
\end{array} \right) = -x \begin{array}{c}
\circ \circ \\
\end{array} - \begin{array}{c}
\circ \circ \rightarrow x
\end{array} = -2 \begin{array}{c}
\circ \circ \rightarrow x
\end{array} \]

The second order derivative is taken in the same way. We now have three
propagators in each diagram, so doubling each of these in turn we arrive at
the expression:

\[ \frac{\partial^2}{\partial m(x)^2} \left( \begin{array}{c}
\circ \circ \circ \\
\end{array} \right) = 2 x \begin{array}{c}
\circ \circ \circ \\
\end{array} x + 4 \begin{array}{c}
\circ \circ \circ \circ \rightarrow x \\
\end{array} \]

After integration with respect to \( x \) this gives:

\[ \int dx \frac{\partial^2}{\partial m(x)^2} \left( \begin{array}{c}
\circ \circ \circ \\
\end{array} \right) = 2 \begin{array}{c}
\circ \circ \circ \\
\end{array} + 4 \begin{array}{c}
\circ \circ \circ \circ \circ \\
\end{array} \]

Using the bootstrap formula (4) once more we find the diagrammatic equiva-
talent of the expression found earlier in this paper:

\[ \frac{\partial E}{\partial g} = N(N + 2) \begin{array}{c}
\circ \circ \\
\end{array} + 8N(N + 2)^2 g \begin{array}{c}
\circ \circ \circ \circ \circ \\
\end{array} + \mathcal{O}(g^2) \]

It is clear how this can be continued to higher orders. Both taking the
functional derivative and integrating with respect to \( x \) are translated into
simple diagrammatic manipulations, which allow a much simpler treatment
than the one presented in the preceding section. Carrying on straightforwardly,
one can easily generate Feynman diagrams to, in principle, an arbitrary number of loops. We give here all diagrams up to five loops with the
correct symmetry factor for general \( N \):
4 Correlation functions

So far we have shown how one can generate the diagrams that contribute to the energy of the model. If one were to use these diagrams to calculate the perturbation series, one would inevitably be faced with the problem of renormalisation. In order to be able to carry out the renormalisation program, it is necessary to know the divergences of the two- and four-point functions. Therefore the technique presented in the preceding section would be of little use if it would not allow the generation of these diagrams also. For this reason (and for the sake of completeness) it is useful to generalise the method to the generation of any kind of diagram (vacuum, two-point, four-point,...).
As a matter of fact this can be easily accomplished in a straightforward way by coupling a source term to the $\phi$-field. The model is then defined by:

$$Z = e^{-E} = \int \mathcal{D}\phi \exp \left(-\frac{1}{2} \int dx \partial_\mu \phi \partial_\mu \phi + m(x) \phi^2(x) - g \int (\phi^2)^2 - \sum_{i=1}^{N} \int J_i \phi_i \right),$$

(19)

This extra term does not break the validity of (4), therefore the recursive procedure can be used without any change. The only change we have to deal with is the free field value of the energy. Instead of (4) we now find that $E(g = 0)$ is given by:

$$E(g = 0) = \frac{N}{2} Tr \log(-\partial^2 + m) - \frac{1}{2} \sum_{i=1}^{N} J_i \frac{1}{(-\partial^2 + m)} J_i,$$

(20)

where the "scalar product" in the last term is taken in coordinate space. Since the diagrammatic approach is far simpler and neater we will use it here from the very beginning. The above formula can be written graphically as:

$$E(g = 0) = \frac{N}{2} \bigcirc - \frac{1}{2} J \bigcirc J + \mathcal{O}(g)$$

The same rules for derivation and integration over a point in space apply here. Taking the first and second derivative with respect to $m(x)$ we find:

$$\frac{\partial E}{\partial m(x)} = \frac{N}{2} \bigcirc_x + \frac{1}{2} J \bigcirc_x J + \mathcal{O}(g)$$

$$\frac{\partial^2 E}{\partial m(x)^2} = -\frac{N}{2} \bigcirc \bigcirc_x - J \bigcirc_x J + \mathcal{O}(g)$$

Putting this in (4) and integrating with respect to $g$ we find the first order value of the energy:

$$E = E(g = 0) + (N(N + 2) \bigcirc \bigcirc + (N + 2) J \bigcirc J \bigcirc J \bigcirc \bigcirc J + \mathcal{O}(g^2)$$
It is quite clear how the recursion can be continued towards higher orders. Doing this one generates $E$ as a functional of the source $J$ to an arbitrary number of loops. All correlation functions can then be obtained from this by taking successive functional derivatives with respect to the source. As an example we give here all one particle irreducible diagrams that contribute to the two point function up to four loops:

\[
\begin{align*}
\frac{N+2}{2}g & + \frac{(N+2)^2}{2}g^2 & + (N+2)g^2 \\
\frac{(N+2)^3}{2}g^3 & + (N+2)^2g^3 & + 3(N+2)^2g^3 \\
(N + 2)(N + 8)g^3 & + (N+2)^3g^3 & + \frac{(N+2)^3}{2}g^3 \\
\frac{(N+2)^4}{2}g^4 & + (N+2)^4g^4 & + \frac{(N+2)^4}{2}g^4 \\
(N + 2)^3g^4 & + 3(N+2)^3g^4 & + 2(N+2)^3g^4
\end{align*}
\]
5 Gross Neveu model

In this section we will present an analogous treatment of the Gross-Neveu model \[8\]. This well known model is defined by the partition function:

\[
Z = e^{-E} = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-\int \bar{\psi} \partial \psi - g(\bar{\psi}\psi)^2} \tag{21}
\]

In order to obtain the perturbation series for this massless model we will consider the case where the field is allowed to have a mass \(m\), which again depends on the coordinates. Instead of working with (21), we will investigate the theory with partition function:

\[
Z = e^{-E} = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-\int \bar{\psi}(\partial + m(x))\psi - g(\bar{\psi}\psi)^2} \tag{22}
\]

Once again we find a relation between the partial (functional) derivatives of the energy:

\[
\frac{\partial E}{\partial g} = \int \left( \left( \frac{\partial E}{\partial m(x)} \right)^2 - \frac{\partial^2 E}{\partial m(x)^2} \right) \tag{23}
\]

The only missing ingredient is the value of the energy at zero coupling. Again without difficulty one finds that this is given by:

\[
E(g = 0) = -NTr \log(\partial + m) = -NTr \log(1 + m \partial^{-1}) + cte \tag{24}
\]
In this case "Tr" means the trace taken in coordinate space as well as the trace of the $\gamma$-matrices that occur in the expression (one could write it as: $Tr = Tr^{(\gamma)} \otimes Tr^{(coord)}$, where $Tr^{(\gamma)}$ denotes the trace of $\gamma$-matrices and $Tr^{(coord)}$ is the trace in coordinate space). This is an important difference with the first model considered in this paper. We can also derive the equivalent formulae of (9) and (12), namely:

$$\frac{\partial E}{\partial m(x)} = -NTr^{(\gamma)} <x|G \frac{1}{1 + mG}|x>$$

and:

$$\frac{\partial}{\partial m(x)} <y|G \frac{1}{1 + mG}|z> = - <y|G \frac{1}{1 + mG}|x><x|G \frac{1}{1 + mG}|z>$$

The last formula is in fact the same as (12), except for the fact that $G$ is now $\beta^{-1}$.

Instead of working with the rather cumbersome mathematical expressions, we will immediately turn to the diagrammatic approach proposed in the preceding section (with a few minor changes to make it better adapted to this model). The lowest order approximation of the energy could be written as:

$$E = -N \bigcirc + O(g)$$

It is however more convenient to attribute a minus sign and a factor of $N$ to the closed loop itself, so that the energy in diagrammatic notation would simply be:

$$E = \bigcirc + O(g)$$

We now proceed to generate the diagrams with this "initial value" and remember to assign to a certain diagram a power of $N$ equal to the number of loops in the diagram and a minus sign if this number is odd. Moreover, around every loop one would have to take a trace of the $\gamma$-matrices in the loop. Let us make this clear in the lowest orders. The first and second functional derivative are given by:
\[ \frac{\partial E}{\partial m(x)} = \bigcirc \bigcirc \bigcirc + \mathcal{O}(g) \]

\[ \frac{\partial^2 E}{\partial m(x)^2} = - x \bigcirc \bigcirc \bigcirc x + \mathcal{O}(g) \]

We still have the property that taking the functional derivative doubles a propagator in the diagram and changes the sign. The main difference is that the propagator is now the propagator of a fermion field. Using the bootstrap formula, we find for the energy of the Gross-Neveu model in first order:

\[ E = \bigcirc + (\bigcirc \bigcirc \bigcirc + \bigcirc \bigcirc \bigcirc) g + \mathcal{O}(g^2) \]

In the last formula we have slightly changed the convention used in the preceding section. Instead of merging two points with the same label, we now connect them with a dotted line, reminiscent of the auxiliary field notation. We do this in order to be able to keep track of the number of (index-)loops in each diagram, which is important for the "weight" of factors of \( N \) that the diagram will receive, as well as for the sign of the diagram. If one thinks of the dotted line as a delta-function it is easy to see that this is indeed the same as the convention in the preceding section. The next order in the coupling constant is found in much the same way:

\[ \frac{\partial E}{\partial m(x)} = \bigcirc \bigcirc \bigcirc x - 2g \bigcirc \bigcirc \bigcirc x - 2g \bigcirc \bigcirc \bigcirc x + \mathcal{O}(g^2) \]

and:

14
\[ \frac{\partial^2 E}{\partial m(x)^2} = -2g x \xrightarrow{\cdots} x + 2g x \xrightarrow{\cdots} x + 4g x \xrightarrow{\cdots} x + \mathcal{O}(g^2) \]

which leads to the energy approximated in second order:

\[ 8g^2 \xrightarrow{\cdots} \xrightarrow{\cdots} + 16g^2 \xrightarrow{\cdots} \xrightarrow{\cdots} + 4g^2 \xrightarrow{\cdots} \xrightarrow{\cdots} \]

\[ 4g^2 \xrightarrow{\cdots} + 4g^2 \xrightarrow{\cdots} \]

The notation with the "auxiliary field" is clearly more convenient when it comes to taking traces of \( \gamma \)-matrices. The more conventional diagrams would obscure the way these traces have to be taken. When one wants to consider the massless case (as the Gross-Neveu model is originally defined), a number of diagrams will automatically yield zero. Indeed, in this case the propagator is given by \( \frac{2}{m^2} \) and therefore every propagator contributes one \( \gamma \)-matrix. This means that loops with an odd number of propagators automatically vanish. (this includes for instance all tadpole diagrams). Note that one cannot discard these diagrams before one has arrived at the end of the calculation. To obtain higher order diagrams correctly, one has to take into account all diagrams of lower order, also those that would vanish in the limit when \( m \) is taken to zero. Correlation functions for the GN-model can likewise be generated by adding a source term. In the fermionic case however, one would have to introduce two source terms:

\[ Z = e^{-E} = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{-\int \bar{\psi}(\partial + m(x))\psi - g(\bar{\psi}\psi)^2 - \bar{\eta}\psi - \bar{\eta}\psi} \]  

The lowest order approximation to the energy would then read:
and one could use (23) to generate the energy as a functional of $\eta$ and $\bar{\eta}$. Correlation functions are easily found by taking successive derivat ions with respect to the source terms.

6 Thirring model

The third model which can be treated in a way similar to the vector model and the Gross-Neveu model is the Thirring model [9]. This model also is a theory of interacting fermions, but this time with an interaction term $-g \sum_\mu \int (\bar{\psi} \gamma_\mu \psi)^2$. In order to be able to carry out the same techniques as for the previous model, it is not sufficient to consider an $x$-dependent mass. In this case we must take a mass that also depends on the Dirac indices. We will define this ”matrix-mass” as: $m = \gamma^\mu m_\mu(x)$. The generalised partition function that we consider is now given by:

$$Z = e^{-E} = \int D\bar{\psi} D\psi e^{-\int \bar{\psi}(\partial + \gamma^\mu m_\mu(x))\psi - g(\bar{\psi} \gamma_\mu \psi)^2}$$

The formulas for the functional partial derivatives are given by:

$$\frac{\partial E}{\partial m_\mu(x)} = <\bar{\psi} \gamma_\mu \psi(x)>$$

and:

$$\frac{\partial^2 E}{\partial m_\mu(x)^2} = - <(\bar{\psi} \gamma_\mu \psi(x))^2> + \left( \frac{\partial E}{\partial m_\mu(x)} \right)^2,$$

The bootstrap formula is easily generalised to:

$$\frac{\partial E}{\partial g} = \sum_\mu \int dx \left( \left( \frac{\partial E}{\partial m_\mu(x)} \right)^2 - \frac{\partial^2 E}{\partial m_\mu(x)^2} \right)$$

We also need to know the value of the energy when the interaction is switched of. Once more, this poses no serious problems and we find:

$$E = -NTr \log(\partial + fn) = -NTr \log(1 + fn \partial^{-1}) + cte$$
Just as in the Gross-Neveu model the trace is to be understood as a direct product of the trace in coordinate space and the trace of the $\gamma$-matrices in the expression. The formula for the functional derivative of an propagator is slightly different from the forms encountered above. We find without problems:

$$\frac{\partial E}{\partial m_\mu(x)} < y|G \frac{1}{1 + mG}|z >= - < y|G \frac{1}{1 + mG}|x > \gamma_\mu < x|G \frac{1}{1 + mG}|z >$$

(32)

This means that, apart from a doubling of the propagator and a changing of sign, the derivative inserts a $\gamma$-matrix at the point $x$. This poses no serious difficulties however in the diagrammatic expansion. As a matter of fact, one obtains the same diagrams as in the Gross-Neveu model (again using the notation with dotted lines to connect identical points). The only difference is the value that has to be assigned to each of these diagrams. The trace of $\gamma$-matrices around a loop now contains not only the matrices from the propagators in the loop, but also those inserted at the ”vertices” of the fermion field and the ”auxiliary field”. Because of this, the number of $\gamma$-matrices in a loop is always even. Therefore, unlike the massless Gross-Neveu model, all diagrams will contribute. The problem of generating diagrams that contribute to the various correlation functions can be treated by the addition of two extra source terms as in the case of the GN model.

7 Non-local interactions - QED

Another interesting generalisation of the method presented in this paper is the case of non-local interactions. Instead of an interaction term $\int dx (\phi^2(x))^2$ (or the analogon for the other models considered here) one could work with an interaction term:

$$\int dx \ dy \phi^2(x) f(x, y) \phi^2(y)$$

(33)

where $f(x, y)$ is some function of $x$ and $y$. The diagrams that contribute to the energy of this model are not very much different from the diagrams of the models we have considered in the preceding sections. As a matter of fact, when we use the auxiliary field notation they are exactly the same! The only difference is that up to now the propagator of the auxiliary field was merely regarded as a notation for the $\delta$-function and that in the end one
would have to identify the points that are connected by such a propagator. In the case of non-local interactions the propagator is a non-local function and the dotted lines now denote a real field propagating. This becomes clear when one considers the bootstrap equation for non-local interactions. We easily find:

$$\frac{\partial E}{\partial g} = 4 \int dx \, dy \, f(x, y) \left( \frac{\partial E}{\partial m(x)} \left( \frac{\partial E}{\partial m(y)} \right) - \frac{\partial^2 E}{\partial m(x) \partial m(y)} \right)$$

(34)

If one uses the diagrammatic approach, one easily finds that the recursive method now comes down to marking two different points \((x, y)\) in the diagrams and connecting them with a line which, in this case, represents a real field propagating between the two points. The functions \(f(x, y)\) can be seen as the propagator of the field in question.

The most important example of a theory that has this kind of interaction is of course QED. QED is defined by the partition function:

$$Z = e^{-E} = \int D\psi \bar{\psi} e^{-\int \frac{1}{2} \partial_\mu A_\mu \partial^\mu A^\mu + \bar{\psi}(i\gamma^\mu D_\mu + m + g\gamma^\mu A_\mu) \psi}$$

(35)

Since the action is quadratic in the photon field we can easily integrate this out and we end up with:

$$Z = e^{-E} = \int D\psi \bar{\psi} e^{-\int \bar{\psi}(i\gamma^\mu D_\mu \psi + \int dx \, dy \, \bar{\psi}(i\gamma^\mu D_\mu \psi)(x-y) \bar{\psi}(i\gamma^\mu D_\mu \psi)(x)}$$

(36)

where \(\Delta\) is nothing but the photon propagator. This action is just a Thirring model with non-local interaction. Therefore the technique we have presented also allows one to generate Feynman diagrams that contribute to the energy and the various correlation functions of QED.

8 Conclusion and outlook

We have presented in this paper a method that allows to generate Feynman diagrams for theories that involve fields with \(N\) components, both bosonic and fermionic. By applying a simple algorithm one obtains diagrams of a certain order from the knowledge of the diagrams of lower order. A simple, purely diagrammatic, method for this can be constructed. In this paper we have restricted ourselves to quartic interactions, but one clearly sees that
interactions of higher order can also be treated in a similar way. In this case
the equations (4), (23) and (30) will contain higher order derivatives with
respect to $m(x)$. For a interaction of order $2n$ one would have derivatives
up to $n^{th}$ order. One is however always restricted to even order interaction
terms (which was to be expected, since these are the only ones that make
sense for general vector models with $O(N)$-symmetry).

The techniques presented in this paper apply only to vector models so far.
It would of course be interesting to have a similar technique for the matrix
models [10]. Other interesting extensions are the generation of diagrams
in theories with odd interaction terms and theories with several interaction
terms. The most important example of such a theory is of course QCD.

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