Extension of the Hoff solutions framework to cover Navier-Stokes equations for a compressible fluid with anisotropic viscous-stress tensor

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Abstract

This paper deals with the Navier-Stokes system governing the evolution of a compressible barotropic fluid. We extend Hoff’s intermediate regularity solutions framework [Hof95a, Hof95b] by relaxing the integrability needed for the initial density which is usually assumed to be $L^\infty$. By achieving this, we are able to take into account general fourth order symmetric viscous-stress tensors with coefficients depending smoothly on the time-space variables. More precisely, in space dimensions $d = 2, 3$, under periodic boundary conditions, considering a pressure law $p(\rho) = a\rho^\gamma$ with $a > 0$ respectively $\gamma \geq d/(4 - d)$ and under the assumption that the norms of the initial data $(\rho_0 - M, u_0) \in L^2(\mathbb{T}^d) \times (H^1(\mathbb{T}^d))^d$ are sufficiently small, we are able to construct global weak solutions. Above, $M$ denotes the total mass of the fluid while $\mathbb{T}$ with $d = 2, 3$ stands for periodic box. When comparing to the results known for the global weak solutions à la Leray, i.e. constructed assuming only the basic energy bounds, we obtain a relaxed condition on the range of admissible adiabatic coefficients $\gamma$.

Keywords: Compressible fluids, Navier-Stokes Equations, Anisotropic Viscous-Stress Tensor, Hoff solutions, Intermediate regularity

MSC: 35Q35, 35B25, 76T20.

1 Introduction and main result

In this paper, we study the problem of existence of global solutions with intermediate regularity as pioneered by Hoff [Hof95a, Hof95b] for the Navier-Stokes equations governing the flow of a compressible fluid. Our aim is to extend the existence theory as to accommodate general smooth fourth order symmetric viscous-stress tensors. More precisely we consider the following system:

\[
\begin{align*}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla(a\rho^\gamma) &= \text{div}(\tilde{E}(\nabla u)),
\end{align*}
\]

where $(\rho, u)$ represent the density and the velocity field of the fluid. We assume that the pressure is given by $p(\rho) = a\rho^\gamma$ with $a > 0$ and $\gamma \geq d/(4 - d)$. We will restrict ourselves to the case of $d = 2, 3$ space-dimensions and periodic boundary conditions. The $d$-dimensional torus, $d \in 2, 3$ is denoted by $\mathbb{T}^d$. We consider a general fourth order symmetric viscous stress tensor:

$$\tilde{E} = (\tilde{\varepsilon}_{ijkl})_{i,j,k,l \in \mathbb{T}^d},$$

where we use the notation:

$$\tilde{E}(\nabla u) = \tilde{\varepsilon}_{ijkl}\partial_l u^k, \quad \text{div}(\tilde{E}(\nabla u)) = \partial_j (\tilde{\varepsilon}_{ijkl}\partial_l u^k).$$

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1$d$ stands for the set $1, 2, \cdots, d$
2All along this paper, we will use the Einstein summation over repeated indices convention.
The system is completed with the initial data
\[
\rho|_{t=0} = \rho_0 \geq 0, \quad \rho u|_{t=0} = m_0.
\] (1.2)

Most of the literature concerning compressible fluid mechanics deals with the classical isotropic tensor
\[
I = (\varepsilon_{ij\ell k}^{iso})_{i,j,k,\ell \in \mathbb{1},d}
\] (1.3)
which is given by
\[
\varepsilon_{ij\ell k}^{iso} = \varepsilon_{k\ell ij}^{iso} = \begin{cases} 
\mu & \text{if } (i,j) = (k,\ell) \text{ and } i \neq j, \\
2\mu + \lambda & \text{if } (i,j) = (k,\ell) \text{ and } i = j, \\
0 & \text{otherwise},
\end{cases}
\] (1.4)
and \(\mu, \lambda > 0\) are given constants. This implies in particular that one has
\[-\text{div}(I \nabla u) = -\mu \Delta u - (\mu + \lambda) \nabla \text{div} u.\]

In the following, we will recall some well known results concerning the existence of solutions for compressible Navier-Stokes equations for both isotropic viscous tensor and for anisotropic viscous tensors. In order to understand why we are required to work within the Hoff-solutions framework and, in particular, why it is necessary to relax the integrability of density to \(L^p\) with \(p < \infty\), we will briefly pass in review different notions of solutions (strong solutions, critical spaces, global weak solutions à la Leray, intermediate regularity à la Hoff). In this paper, we do not discuss density dependent viscosity for compressible Navier-Stokes equations.

1) A short review of known results for isotropic stress tensors. The study of system (1.1) – (1.2) with a given pressure law \(s \mapsto p(s)\) in the case of isotropic stress tensors goes back to the work of J. Nash [Nas62] where the author shows the existence of local-in-time strong solutions in Hölder spaces. Then local strong existence for initial data in Sobolev spaces was investigated by Solonnikov [Sol80] in the 80’s while the first global result is due to Matsumura and Nishida [MN+80] where they prove the existence of global-in-time solutions in \(3\text{D}\) if the initial data are sufficiently close to equilibrium in \(H^3\). In the 2000s, R. Danchin [Dan00] constructed global small solutions in the so-called critical spaces. Smooth solutions for the IBVP with Dirichlet boundary conditions in dimension \(d \geq 2\) are known to blow-up since the work of Vaigant [Vai94] where the author constructs an explicit solution for the NSC system and shows that the \(L^\infty\)-norm of the density blows-up in finite time. Very recently, [MRRS19] F. Merle, P. Raphaël, I. Rodnianski and J.Szeftel prove that in \(3\text{D}\), for small \(\gamma \leq 1 + 2/\sqrt{3}\) there exists local smooth solutions which explode in finite time: the \(L^\infty\)-norms of the density and the velocity blow-up. Thus, in some sense, the smallness condition, which express the fact that the initial configuration is sufficiently close to an constant equilibrium state, is necessary in order to insure global well-posedness. Let us mention the recent result of R. Danchin and P.B. Mucha [DM19] where the authors construct global solutions in the two-dimensional case requiring only that the divergence of the velocity field should be small.

Another category of results regarding the solvability of (1.1) – (1.2) concerns the so-called weak-solutions à la Leray: solutions in the sense of distributions satisfying the energy inequality for which one can guarantee their global existence for arbitrary large initial data. Of course, few things are known regarding the uniqueness of these solutions. We mention the, by now classical results of P.L. Lions [Lio96], E. Feireisl et al. [FNP01]. Recently, the first author and P.–E. Jabin extended these two results in order to cover on one hand some anisotropic stress tensors [BJ18] and on the other hand more general pressure functions [BJ18, BJW21] that could not be treated within the Lions-Feireisl theory. We will return back and comment a bit more on these results in the context of anisotropy.

A third category of results concerns an intermediate regularity functional framework which was pioneered in the works of D. Hoff [Hof95a, Hof95b, Hof02, HS08] (that we will called solutions à la Hoff) and B. Desjardins [Des97]. By intermediate regularity we mean of course between the regularity needed to construct strong solutions and weak solutions à la Leray (see [Lio96]). These solutions are interesting since they allow to work with discontinuous densities while granting some extra regularity for the velocity field which turns out to be sufficiently regular in order to generate a log-Lipschitz flow. These solutions were used by D. Hoff
to study the dynamics of a surface of discontinuities initially present in the density, see \[\text{Hof02} \] and found applications in the context of multifluid flows, see the work of the first author and X. Huang \[\text{BH11}\]. Since the present work deals with these kind of solutions, we will take the time to give more details. In \[\text{Hof95a} \] \[\text{Hof95b}\], for the case of isotropic stress tensors, D. Hoff introduced and studied the properties of two energy-type functionals

\[ A_1 (t) = \frac{\mu \sigma (t)}{2} \sum_{i=1}^{d} \sum_{k=1}^{d} \int |\partial_k u^i (t)|^2 + \frac{(\mu + \lambda) \sigma (t)}{2} \int |\text{div} \ u (t)|^2 + \int_0^t \int \sigma \rho |\dot{\sigma}|^2 \]  \[ (1.5) \]

and

\[ A_2 (t) = \sigma^{1+d} (t) \int \frac{\rho (t) |\dot{\sigma}|^2}{2} + \mu \sum_{i=1}^{d} \sum_{k=1}^{d} \int_0^t \int \sigma^{1+d} |\partial_k \dot{u}^i|^2 + (\mu + \lambda) \int_0^t \int \sigma^{1+d} |\text{div} \ \dot{u}|^2 \]  \[ (1.6) \]

where

\[ \dot{u} = u_t + u \cdot \nabla u \] and \( \sigma (t) = \min \{ 1, t \} \).

These functionals naturally appear: The first one when multiplying the momentum equation with \( \sigma \dot{u} \) and integrating while the other one appears when applying \( \partial_t + \text{div} (u \cdot \nabla) \) to the momentum equation and multiplying with \( \sigma^{1+d} \dot{u} \). D. Hoff shows that \( A_1, A_2 \) can be controlled globally in time if the initial data have suitably small energy and \( \rho_0 \) is close to a constant in \( L^\infty \). The fact that these two functionals can be controlled translate some fine smoothing properties due to the diffusion: it turns out that \( u \) is Hölder continuous in time-space, far from \( t = 0 \) and that curl \( u \) and the effective flux

\[ F = (2\mu + \lambda) \text{div} \ u - p (\rho) \]

are \( H^1 \) in space for a.e. \( t > 0 \). In particular, this later properties render mathematically clear the fact that discontinuities in the density are advected by the flow but in such a way that the so called effective flux, i.e. \( F \) stays fairly smooth. This property enjoyed by the effective flux, known and exploited in the \( 1d \) case in \[\text{HSS95} \], also turns out to be crucial when showing the stability of sequences of weak-solutions. In order to give a meaning to \( A_1, A_2 \) very little extra information is need when comparing to the energy level, by which we mean \( \rho_0 u_0^2 \in L^1, \ \rho_0 \in L^\gamma \), which essentially is that \( \rho_0 \in L^\infty \) and \( u_0 \in L^{2^\gamma} \) (in the whole space case). If more information is available for the initial data, modified versions of the two functionals can be used: for instance if \( u_0 \in H^1 \), one can control

\[ \tilde{A}_1 (t) = \frac{\mu}{2} \sum_{i=1}^{d} \sum_{k=1}^{d} \int |\partial_k u^i (t)|^2 + \frac{\mu + \lambda}{2} \int |\text{div} \ u (t)|^2 + \int_0^t \int \sigma |\dot{\sigma}|^2 , \]  \[ (1.7) \]

respectively

\[ \tilde{A}_2 (t) = \sigma (t) \int \frac{\rho (t) |\dot{\sigma}|^2}{2} + \mu \sum_{i=1}^{d} \sum_{k=1}^{d} \int_0^t \int \sigma |\partial_k \dot{u}^i|^2 + (\mu + \lambda) \int_0^t \int \sigma |\text{div} \ \dot{u}|^2 , \]  \[ (1.8) \]

which of course express the fact that due to the extra information the solution is better behaved close to the initial time layer \( t = 0 \). We also mention the related but independent work of B. Desjardins \[\text{Des97}\] where the author obtains local in time results showing that is possible to control a function which is essentially equivalent to \( A_1 \). We mention that in all the above cited papers, the assumption that \( \rho_0 \in L^\infty \) turns out to be crucial. The fact that one can propagate control of the \( L^\infty \)-norm of the density heavily depends on the algebraic structure of the isotropic Navier-Stokes system throughout the so called-effective flux \( F = (2\mu + \lambda) \text{div} \ u - p (\rho) \) defined above.

2) The case of anisotropic stress tensors. In this case, the mathematical results are in short supply. Let us mention that in the context of strong solutions \[\text{MN+80} \], \[\text{Dan00}\] where the results are proved by maximal
regularity results, at least if the stress tensor is "close enough to the isotropic tensor" then there should virtually be little change needed in order to accommodate these kind of solutions. However, as explained above, when dealing with classical solutions the density is a continuous function thus excluding many interesting situations in applications (for example, mixtures of fluids).

The first paper providing a result in this direction has been obtained by the first author and P.-E. Jabin in [BJ18] and concerns the existence of global weak solutions à la Leray with an anisotropic diffusion of the form:

\[- \text{div}(\hat{A}(t)\nabla u) - (\mu + \lambda)\nabla \text{div} u\]  \hspace{1cm} (1.9)

where

\[\hat{A}(t) = \mu \text{Id} + A(t) \quad \text{with} \quad \mu > 0.\]

The result proved in [BJ18] states that there exists an universal constant \(c > 0\) such that if

\[\|A(t)\|_{L^\infty} \leq c \left( \frac{2\mu}{d} + \lambda \right)\]

and if

\[\gamma > \frac{d}{2} \left[ \left( 1 + \frac{1}{d} \right) + \sqrt{1 + \frac{1}{d^2}} \right] \] \hspace{1cm} (1.10)

then, there exists global weak solutions à la Leray for the Navier-Stokes system \((1.1) - (1.2)\) with the viscous-stress tensor given by \((1.9)\). This result extended to the anisotropic case the global existence of weak solution à la Leray obtained for the isotropic case in [Lio96], [FNP01]. The result of the first author and P.-E. Jabin is based on new estimates for the transport equation. This result requires in a crucial manner some form of compactness in space for

\[(2\mu + \lambda) \text{div} u - L(\rho^\gamma)\]

where \(L\) is a non-local operator of order 0. It is at this level that the authors use the fact that \(A\) depends only on time has been used by the authors. The extension of this result to space dependent strain tensors represents a serious difficulty that remains an open problem. Moreover, the restriction for the adiabatic coefficient \(\gamma\) given by \((1.10)\) excludes most of the physically realistic values: monoatomic gases \(5/3\), ideal diatomic gases \(7/5\), viscous shallow–water \(\gamma = 2\).

Let us also mention our results concerning related models for compressible fluids, on the one hand, existence of global weak solutions à la Leray for the quasi-stationary compressible Stokes in [BB20] where an anisotropic diffusion \(-\text{div}(A\nabla u)\) is considered and where no smallness assumption on the anisotropic amplitude is needed in order to develop the existence theory and on the other hand our result regarding the stationary compressible Navier-Stokes equations in [BB21] where we treat a viscous diffusion operator given by \(-\mathcal{A}u\) (under some constraints) where \(\mathcal{A}\) is composed by a classical constant viscous part plus an anisotropic contribution and a possible nonlocal contribution.

3) Motivation to extend the framework of Hoff-type solutions and description of our main result. When dealing with weak solutions for non-linear PDE systems, one of the most delicate aspects is the stability analysis: given a sequence of weak solutions for some well-chosen approximated systems, show that this sequence converges to a solution for the initial system. The key ingredient in [BB20] and [BB21] is an identity that we found when comparing on the one hand, the limiting energy equation and on the other hand, the equation of the energy associated to the limit system. In order to justify such an identity, a crucial assumption seems to be the fact that the pressure is \(L^2\), an apriori estimate which is ensured by basic a-priori estimates in the case of the Stokes system or for the stationary Navier-Stokes system. However, in the case of system \((1.1) - (1.2)\) in the isotropic case, the best estimate for the density is due to P. Lions who showed for global weak solutions à la Leray that \(\rho \in L^{5\gamma/3-1}_{t,x}\). This makes it impossible to write the energy equation because, loosely speaking, the velocity cannot be used as a test function in a weak-formulation of \((1.1)\). Thus, it seems hopeless to justify the limiting passage as in [BB20] and [BB21] in the most general setting of weak-solutions à la Leray. Obviously, one may ask if we can work in an intermediate regularity setting. However, one learns

\[3d\] stands for the space dimension.
fast that we are faced with a serious problem when trying to propagate the $L^\infty$-norm for the density. In the isotropic case, this is achieved based on the fact that the second Hoff functional, namely (1.6) controls, at least far from the initial time layer $t = 0$, the $L^\infty$-norm of the effective viscous flux:

$$(2\mu + \lambda) \text{div} u - \left( a\rho^{\gamma} - \int_{T^d} a\rho^{\gamma} \right) = \Delta^{-1} \text{div}(\rho \dot{u}).$$

(1.11)

Of course, the situation is not the same in the anisotropic case, where

$$(2\mu + \lambda) \text{div} u - \left( a\rho^{\gamma} - \int_{T^d} a\rho^{\gamma} \right) = \Delta^{-1} \text{div}(\rho \dot{u}) + \Delta^{-1} \text{div} \left( \left( \tilde{\mathcal{E}} - \mathcal{I} \right)(\nabla u) \right)$$

(1.12)

and the term $\Delta^{-1} \text{div} \left( \left( \tilde{\mathcal{E}} - \mathcal{I} \right)(\nabla u) \right)$, being of the same order as $\nabla u$, we cannot expect it to be $L^\infty$.

Because of the lack of algebraic structure we are led to abandon any hope of propagating an $L^\infty$ bound for the density. A natural question then appears: is it possible to bound the Hoff functionals without working in an $L^\infty$ framework for the density? The main contribution of this paper is to show that this is indeed the case. Of course, this fact makes it possible to construct global weak solutions close to equilibrium for the Navier–Stokes system in an intermediate regularity setting. This program requires establishing $L^p$-estimates for the density that are compatible with the Hoff functionals. In order to avoid further technical difficulties, we will assume the best information possible for the velocity, namely $u_0 \in (H^1)^d$ such that we will rather work with the anisotropic equivalent of the functionals defined in (1.7)−(1.8).

Our result should be seen to be complementary to the work of the first author and P.-E. Jabin [BJ18]. In particular, this extended intermediate regularity framework allows us to:

- consider viscous-stress tensors depending on time and also on the space variable;
- consider a range of adiabatic coefficient namely

$$\gamma \geq \frac{d}{4 - d} \text{ for } d \in \{2, 3\}.$$  

In particular, in $2D$ we are able to treat all coefficients that are of practical interest $\gamma \geq 1$.

- this method could be adapted to bounded domains with Dirichlet boundary conditions for which the existence result of global weak solutions (à la Leray or intermediate regularity) with anisotropic tensors remains open.

Note that the range for the coefficient $\gamma$ is larger than the one in [BJ18] namely (1.10) and we cover strain tensors which may depend on the space variable. Of course, the price to pay is that the initial conditions are supposed to be close to equilibrium and that we require the initial velocity field to be in $(H^1_0)^d$.

**Assumptions and notations.** We will rather write $\tilde{\mathcal{E}} = \mathcal{I} + \mathcal{E}$ with $\mathcal{I}$, the usual isotropic tensor (1.3)−(1.4) and where $\mathcal{E}$ measures in some sense the anisotropic perturbation. With these new notations and setting the constant $a = 1$ in the pressure term[1], the system (1.1) becomes

$$\begin{cases}
\rho_t + \text{div}(\rho u) = 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla \rho^{\gamma} = \mu \Delta u + (\mu + \lambda) \nabla \text{div} u + \text{div}(\mathcal{E}(\nabla u)).
\end{cases}$$

(1.13)

We suppose that $\mu, \lambda \in \mathbb{R}$ such that

$$\mu > 0 \text{ and } \mu + \lambda \geq 0.$$  

(1.14)

We will assume that $\mathcal{E} = (\varepsilon_{ijk})_{i,j,k \in T^d}$ verifies the following properties:

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4Of course, this does not affect the generality of the result, the choice $a = 1$ is only a matter of simplifying the computations.
For all \(i, j, k, \ell \in \mathbb{T}_d\) we assume the following symmetry property:

\[
\varepsilon_{ijk\ell} = \varepsilon_{k\ell ij}.
\]  

(H1)

The latter property ensures that

\[
\varepsilon_{ijk\ell} a_{ij} b_{k\ell} = \varepsilon_{ijk\ell} b_{ij} a_{k\ell}.
\]

Strict coercivity of the diffusive part:

\[
\varepsilon |a_{ij}|^2 \geq \varepsilon_{ijk\ell} a_{ij} a_{k\ell} \geq -\varepsilon |a_{ij}|^2
\]

where \(\varepsilon > 0\) such that

\[0 < \mu - \varepsilon\]

Regularity: for all \(i, j, k, \ell \in \mathbb{T}_d\), \(\varepsilon_{ijk\ell} \in W^{1,\infty} (0, \infty)\times \mathbb{T}_d\) with

\[
\|\partial_t \varepsilon_{ijk\ell}\|_{L^\infty((0, +\infty)\times \mathbb{T}_d)} + \|\nabla \varepsilon_{ijk\ell}\|_{L^\infty((0, +\infty)\times \mathbb{T}_d)} < \infty.
\]  

(H3)

Main Result. Let us define the following:

\[
E (\rho/M, u) = \int_{\mathbb{T}_d} (H_1(\rho/M) + \frac{1}{2} \rho |u|^2)
\]

with \(0 < M < +\infty\) and where

\[
H_1(\rho/M) = H_1(\rho) - H_1(M) + H'_1(M)(\rho - M)
\]  

(1.15)

Also, we introduce

\[
H_\ell(\rho/M) = \rho \int_{M}^{\rho} \frac{|P(s) - P(M)|^{\ell-1}(P(s) - P(M))}{s^2} ds \text{ with } \ell \in \{2, 3\}.
\]  

(1.16)

We are now in the position of stating our main result:

**Theorem 1** Consider \(\mu, \lambda \in \mathbb{R}\) such that \(\mu > 0, \mu + \lambda > 0\). Let \(\hat{\mathcal{E}} = \mathcal{I} + \mathcal{E}\) with \(\mathcal{I}\) the isotropic viscous-stress tensor defined in (1.3) – (1.4) and \(\mathcal{E} = (\varepsilon_{ijk\ell})_{i,j,k,\ell \in \mathbb{T}_d}\) a fourth order tensor verifying the hypothesis (H1) – (H3). Then, there exists two positive constant \(\eta, c_0\) independent of \(\mu\) and \(\lambda\) such that the following holds true. Assume that

\[
\|E\|_{L^\infty((0, +\infty)\times \mathbb{T}_d)} = \sup_{i,j,k,\ell \in \mathbb{T}_d} \|\varepsilon_{ijk\ell}\|_{L^\infty((0, +\infty)\times \mathbb{T}_d)} \leq \eta \min \{\mu, 2\mu + \lambda\}.
\]  

(H4)

Then, for any \((\rho_0, u_0) \in L^{2\gamma} (\mathbb{T}_d) \times (H^1 (\mathbb{T}_d))^d\) with

\[
\int_{\mathbb{T}_d} \rho_0 = M, \quad \int_{\mathbb{T}_d} \rho_0 u_0 = \mathcal{P} \in \mathbb{R}^d,
\]

such that

\[
E (\rho_0/M, u_0) + \int_{\mathbb{T}_d} H_2 (\rho_0/M) + \|u_0\|_{(H^1 (\mathbb{T}_d))^d}^2 \leq c_0,
\]

there exists a global weak solution \((\rho, u)\) for (1.1) – (1.2) with

\[
(\rho - M, \rho u - \mathcal{P}) \in C([0, +\infty); H^{-1} (\mathbb{T}_d)) \times C([0, +\infty); (H^{-1} (\mathbb{T}_d))^d)
\]

\(\text{All along this paper, we will use the Einstein summation over repeated indices convention.}\)
and that for all $t \geq 0$ we have:

$$E(\rho(t)/M, u(t)) + (\mu - \frac{1}{2}) \sum_{i=1}^{d} \sum_{k=1}^{d} \int_{\mathbb{T}^d} |\partial_{k} u_i^i(t)|^2 + (\mu + \lambda) \int_{\mathbb{T}^d} |\text{div} u|^2 \leq E(\rho_0/M, u_0),$$

$$\frac{1}{2} \left\{ \mu \sum_{i=1}^{d} \sum_{k=1}^{d} \int_{\mathbb{T}^d} |\partial_{k} u_i^i(t)|^2 + (\mu + \lambda) \int_{\mathbb{T}^d} |\text{div} u|^2 \right\} + \int_{0}^{T} \int_{\mathbb{T}^d} |\dot{u}|^2 \leq C_0,$$

$$\sigma(t) \int_{\mathbb{T}^d} \frac{\rho(t)}{2} |\dot{u}(t)|^2 + \mu \sum_{i=1}^{d} \sum_{k=1}^{d} \int_{0}^{T} \sigma |\partial_{k} \dot{u}|^2 + (\mu + \lambda) \int_{0}^{T} \sigma |\text{div} \dot{u}|^2 \leq C_0,$$

$$\int_{\mathbb{T}^d} H_2(\rho(t)/M) + \sigma(t) \int_{\mathbb{T}^d} H_3(\rho(t)/M) + \int_{0}^{T} \int_{\mathbb{T}^d} |P(\rho) - P(M)|^3 + \int_{0}^{T} \int_{\mathbb{T}^d} \sigma |P(\rho) - P(M)|^4 \leq C_0,$$

where $\sigma(t) = \min\{1, t\}$ while $C = C(\mu, \lambda, \gamma, M, E_0, c_0)$ is a constant that depends on $\mu, \lambda, \gamma, M, E_0$.

**Remark 1.1** It is important to remark that it seems a difficult problem to propagate the $L^\infty$-norm for the density as it has been done by D. Hoff for the isotropic compressible Navier-Stokes equations with a barotropic pressure law.

**Remark 1.2** The uniform control of the Hoff functionals that we obtain in Theorem 1 plays a crucial role in the stability part of the proof. For instance, the fact that the pressure is bounded in $L^1_{t,x}$ allows us to justify the equation

$$\partial_t P(\rho) + \text{div} (P(\rho) u) + (\rho P'(\rho) - P(\rho)) \text{div} u = 0,$$

from the mass equation

$$\partial_t \rho + \text{div} (\rho u) = 0,$$

of the limit system. Another crucial aspect is that when considering a sequence of solutions of systems that approximate the Navier-Stokes system, controlling the second Hoff functional allows to obtain information for the time derivative of the velocities. As a consequence of the Aubin-Lions lemma, we obtain that the sequence of velocities converges strongly in $L^2_{t,x}$, at least far from $t = 0$ which is crucial in order to implement the idea from [BB20].

**Main steps and organization of the paper.** We detail below the main steps of the proof of Theorem 1. Inspired by the approximate system proposed by the first author and P.-E. Jabin in [BJIS], we will consider a regularized version of the Navier-Stokes system (1.1):

$$\begin{cases}
\rho_t + \text{div} (\rho u) = 0, \\
(\rho u)_t + \text{div} (\rho u \otimes u) + \nabla \rho \gamma = \mu \Delta u + (\mu + \lambda) \nabla \text{div} u + \omega_\delta \ast \text{div} (\mathcal{E} (\nabla \omega_\delta \ast u)),
\end{cases}$$

(1.17)

where

$$\omega_\delta (x) = \frac{1}{\delta^d} \omega \left( \frac{x}{\delta} \right),$$

(1.18)

with $\omega$ a smooth, nonnegative, radial function compactly supported in the unit ball centered at the origin and with integral equal to 1. Since system (1.17) can be seen as a regular perturbation of the Navier-Stokes system for a compressible barotropic fluid, classical results [Sol80, Des97, Dan10] can be invoked in order to ensure the existence of a local barotropic solution.

**Remark 1.3** *(Important remark on the anisotropy).* To simplify the writing of the paper, we will assume in the proof that

$$\int_{0}^{T} \int_{\mathbb{T}^d} \mathcal{E}(\nabla w) : \nabla w \geq 0 \quad \text{ for all } T \in (0, +\infty] \text{ and } w \in L^2((0,T); H^1(\mathbb{T}^d)).$$
This assumption is needed in order to treat the stability of weak-solutions of system (1.17) part of the proof. In order to avoid this assumption and treat the general case, it is sufficient to consider an approximate system with diffusion given by

\[(\mu - \varepsilon)\Delta u + (\mu + \lambda) \nabla \text{div } u + \varepsilon \Delta \omega_\delta * u + \omega_\delta * \text{div}(\mathcal{E}(\nabla \omega_\delta * u)).\]

change the coefficients \(\lambda\) and \(\mu\) in the isotropic part to allow to satisfy these assumptions.

We show that these solutions have the property that the two Hoff functionals associated are bounded independently of \(\delta\). This is one of the main contributions of this paper.

Following exactly the same steps as in R. Danchin and P.B. Mucha [DM19], shows that the local solutions of (1.17) can be prolonged to global ones. The fact that the Hoff functionals are independent of \(\delta\) is of course crucial in order to show that we can extract a subsequence converging to a weak-solution à la Hoff of (1.1) – (1.2). Here, we are faced with, let us say the classical difficulty in compressible fluid mechanics which is to be able to identify the pressure in the limit. More precisely,

\[
\lim (\rho_n)^\gamma = (\lim \rho_n)^\gamma.
\]

Of course, when dealing with weak solutions, the density is just a \(L^p\) function for some \(p < \infty\) and no gain of regularity is to be expected. Since weak limits, in general, do not commute with nonlinear functions, showing (1.19) has to take into consideration some algebraic properties of solutions of the NS system. Let us recall that classical techniques due to P.L. Lions [Lio96] and E. Feireisl [Fei01] do not apply in this context, see the discussions from the introductions of [BJ18], [BB20], [BB21] for more details. Moreover, the work by the first author and P.E. Jabin requires a relative large \(\gamma\) and, maybe more importantly, as it was explained above, it is not straightforward to extend it to heterogenous in space anisotropic tensors (the fact that \(\mathcal{E}\) can depend also on the space variable). Here, it is crucial to extend our idea from [BB20] that we successfully implemented in order to construct global weak solutions à la Leray for the Stokes-Brinkman system in [BB20] and for the stationary NS system in [BB21]. In these two papers, we did not need however to impose any restriction on the size of the initial data or the forcing terms. This is essentially due to the fact that, in the previous cases, the pressure turns out to be an \(L^2_{t,x}\) function (if \(\gamma\) is large for the stationary NS system) and that the convective term behaves better in the aforementioned cases.

The rest of the paper unfolds as follows. In the second section, we prove the main result: First we recall basic mass conservation and energy estimate, secondly we extend the Hoff estimates in a \(L^p\) framework, third we construct a sequence of approximate solutions and then finally we show the stability property. In an appendix, we present a tool box with Fourier multipliers properties, Sobolev inequality and Gronwall-Bihari inequality and finally we give the detailed computations for the Hoff functionals that we strongly use.

2 Proof of the main result

2.1 Basic mass conservation and energy estimate

The conservation of mass and momentum. The simplest \textit{a priori} estimate we have is given by the conservation of mass and momentum:

\[
\int_{\Omega} \rho (t) = \int_{\Omega} \rho_0 := M > 0, \tag{2.1}
\]

\[
\int_{\Omega} \rho (t) u (t) = \int_{\Omega} m_0 := \mathcal{P} \in \mathbb{R}^d. \tag{2.2}
\]

The energy estimate. From the continuity equation, we can also deduce the following equation

\[
\partial_t b (\rho) + \text{div} (b (\rho) u) + (\rho b' (\rho) - b (\rho)) \text{div } u = 0, \tag{2.3}
\]
Taking \( b ( \rho ) = \rho^\alpha \) in (2.3) yields
\[
\partial_t \rho^\alpha + \text{div} ( \rho^\alpha u ) + ( \alpha - 1 ) \rho^\alpha \text{div} u = 0.
\]

Also, we can write that
\[
u \cdot \nabla P ( \rho ) = \text{div} ( u ( P ( \rho ) - P ( M ) ) ) - ( P ( \rho ) - P ( M ) ) \text{div}_u
\]
\[
= \text{div} ( u ( P ( \rho ) - P ( M ) ) ) + \frac{d}{dt} H_1 ( \rho / M ) + \text{div} ( H_1 ( \rho / M ) u ) ,
\]
where \( H_1 ( \rho / M ) \) has been defined in (1.15). The function \( H_1 ( \rho / M ) \) is more appropriate in order to study densities that are close to some constant state. Thus we get the following energy estimate
\[
\int_{\mathbb{T}^d} \left( H_1 ( \rho / M ) + \frac{\rho |u|^2}{2} \right) + \mu \int_0^t \int_{\mathbb{T}^d} |\nabla u|^2 + ( \mu + \lambda ) \int_0^t \int_{\mathbb{T}^d} |\text{div} u|^2 + \frac{1}{\gamma} \int_0^t \int_{\mathbb{T}^d} \varepsilon_{ijkl} \partial_t u_k^i \partial_j u_k^j
\]
\[
\leq \int_{\mathbb{T}^d} \left( H_0 ( \rho_0 / M ) + \frac{\rho_0 |u_0|^2}{2} \right) := E_0 .
\]
Note that we assume that \( E_0 \) is small in Theorem 4.

### 2.2 Extension of Hoff’s estimates in a \( L^p \) framework

This part is the key of the paper: Assuming the initial velocity \( u_0 \in H^1 ( \mathbb{T}^d ) \) and \( \rho_0 \in L^2 ( \mathbb{T}^d ) \), instead of \( \rho_0 \in L^\infty ( \mathbb{T}^d ) \) as in Hoff95a, we allow more general densities than in Hoff95a. This \( L^p, p < \infty \) framework for the density is important when considering anisotropic viscous tensors for which it is not so straightforward to propagate \( L^\infty \)-information. Consider
\[
A_1 ( t ) = \frac{1}{2} \mu \int_{\mathbb{T}^d} |\partial_k u^i ( t )|^2 + ( \mu + \lambda ) \int_{\mathbb{T}^d} |\text{div} u ( t )|^2 + \int_{\mathbb{T}^d} \varepsilon_{ijkl} \partial_t u_k^i ( t ) \partial_j u_k^j ( t ) + \frac{1}{2} \int_{\mathbb{T}^d} \rho |\dot{u}^i|^2, 
\]
and
\[
A_2 ( t ) = \frac{\rho ( t ) |\dot{u} ( t )|^2}{2} + \mu \int_0^t \int_{\mathbb{T}^d} |\partial_k \dot{u}^i ( t )|^2 + ( \mu + \lambda ) \int_0^t \int_{\mathbb{T}^d} |\text{div} \dot{u}|^2 + \frac{1}{\gamma} \int_0^t \int_{\mathbb{T}^d} \varepsilon_{ijkl} \partial_t u_k^i \partial_j u_k^j .
\]

Multiplying the momentum equation with
\[
\dot{u} = u_t + u \cdot \nabla u
\]
we obtain (see the detailed computations in the appendix) that
\[
A_1 ( t ) = \frac{1}{2} \mu \int_{\mathbb{T}^d} |\partial_k u_0^i|^2 + ( \mu + \lambda ) \int_{\mathbb{T}^d} |\text{div} u_0|^2 + \int_{\mathbb{T}^d} \varepsilon_{ijkl} \partial_t u_0^i \partial_j u_0^j ( t )
\]
\[
- \mu \int_0^t \int_{\mathbb{T}^d} \partial_k u^i \partial_k \dot{u}^i \partial_t u^i + \mu + \frac{1}{2} \int_0^t \int_{\mathbb{T}^d} |\partial_k u^i|^2 \text{div} u
\]
\[
- ( \mu + \lambda ) \int_0^t \int_{\mathbb{T}^d} \text{div} u \partial_t u^i \partial_k u^i + \frac{\mu + \lambda}{2} \int_0^t \int_{\mathbb{T}^d} (\text{div} u)^3
\]
\[
+ \frac{1}{\gamma} \int_0^t \int_{\mathbb{T}^d} \{ \partial_t \varepsilon_{ijkl} + \partial_i ( \varepsilon_{ijkl} u^i ) \} \partial_j u_k^i \partial_k u_i^j - \int_0^t \int_{\mathbb{T}^d} \varepsilon_{ijkl} \partial_t u_k^i \omega_\delta \star ( \partial_j u^i \partial_k u_i^j )
\]
in order to close the estimates. The classical Calderón-Zygmund theory ensures that for
\[ \int_0^t \sigma \cdot (\partial_t \varepsilon_{ijkl} + \partial_q(u^q \varepsilon_{ijkl})) \partial_k u^i \partial_j \hat{u}^3 - \int_0^t \int_{\mathcal{T}^d} \sigma \varepsilon_{ijkl}(\omega^k + (\partial_t u^a \partial_q u^k)) \partial_j \hat{u}^3 \]
+ \int_0^t \int_{\mathcal{T}^d} \sigma \varepsilon_{ijkl}(u^q, \omega^k \partial_q u^k) \partial_j \hat{u}^3 + \int_0^t \int_{\mathcal{T}^d} \sigma \varepsilon_{ijkl}(u^q, \omega^k \partial_q u^k) \partial_j \hat{u}^3 \]
\[ - \int_0^t \int_{\mathcal{T}^d} \sigma \left\{ P(\rho) \partial_j u^i \partial_k \hat{u}^3 + (\rho P'(\rho) - P(\rho)) \right\} \div u \div \hat{u} \right). \quad (2.6) \]

Applying the operator \( \partial_t + \div(u) \) to the momentum equation we obtain (see the detailed computations in the appendix) that:
\[
A_2(t) = \int_0^t \int_{T^d} \sigma \frac{\hat{u}^2}{2} + \mu \int_0^t \int_{T^d} \sigma \partial_k u^a \partial_q u^i \partial_k \hat{u}^3 + \mu \int_0^t \int_{T^d} \sigma \partial_k u^a \partial_k u^i \partial_q \hat{u}^3 - \mu \int_0^t \int_{T^d} \sigma \div u \partial_k u^i \partial_k \hat{u}^3 \\
+ (\mu + \lambda) \int_0^t \int_{T^d} \sigma \partial_k u^a \partial_q u^i \div \hat{u} + (\mu + \lambda) \int_0^t \int_{T^d} \sigma \partial_i u^a \partial_q \hat{u}^3 \div u - (\mu + \lambda) \int_0^t \int_{T^d} \sigma \|u\|^2 \div \hat{u} \\
- \int_0^t \int_{T^d} \sigma \left( \partial_t \varepsilon_{ijkl} + \partial_q(u^q \varepsilon_{ijkl}) \right) \partial_k u^i \partial_j \hat{u}^3 - \int_0^t \int_{T^d} \sigma \varepsilon_{ijkl}(\omega^k + (\partial_t u^a \partial_q u^k)) \partial_j \hat{u}^3 \\
- \int_0^t \int_{T^d} \sigma \varepsilon_{ijkl}(u^q, \omega^k \partial_q u^k) \partial_j \hat{u}^3 + \int_0^t \int_{T^d} \sigma \varepsilon_{ijkl}(u^q, \omega^k \partial_q u^k) \partial_j \hat{u}^3 \\
- \int_0^t \int_{T^d} \sigma \left\{ P(\rho) \partial_j u^i \partial_k \hat{u}^3 + (\rho P'(\rho) - P(\rho)) \right\} \div u \div \hat{u} \right). \quad (2.7) \]

Let us introduce the effective flux
\[ F = (2\mu + \lambda) \div u - (P(\rho) - P(M)). \]

The details leading to these formulae are by now classic for the isotropic case and they were used by D. Hoff in the series of works with isotropic viscosities [Hof95a, Hof95b, Hof02, HS08]. Here, the added value is that these estimates are adapted for the anisotropic approximate system (1.17). As mentioned before, for the reader’s convenience we gather and detail these computations in the Appendix. One of the key difficulties is to recover information for the gradient of the velocity. A quick analysis of \( A_1 \) and \( A_2 \) reveals that we need to control
\[ \int_0^t \|\nabla u\|_{L^3(T^d)}^3 \text{ and } \int_0^t \sigma \| \nabla u \|_{L^4(T^d)}^4. \]
in order to close the estimates. The classical Calderón-Zygmund theory ensures that for \( p \in \{3, 4\} \) one has
\[ \| \nabla u \|_{L^p(T^d)}^p \leq C \left( \| \text{curl} u \|_{L^p(T^d)} + \| \div u \|_{L^p(T^d)}^p \right), \]
for some numerical constant \( C \). We deduce that
\[ \| \nabla u \|_{L^p(T^d)}^p \leq C \left( \frac{1}{\mu^p} \| \mu \curl u \|_{L^p(T^d)}^p + \frac{1}{(2\mu + \lambda)^p} \| (2\mu + \lambda) \div u \|_{L^p(T^d)}^p \right) \]
\[ \leq C \left( \frac{1}{\mu^p} \| \mu \curl u \|_{L^p(T^d)}^p + \frac{1}{(2\mu + \lambda)^p} \| F \|_{L^p(T^d)}^p + \frac{1}{(2\mu + \lambda)^p} \| P(\rho) - P(M) \|_{L^p(T^d)}^p \right), \]
from which we infer that
\[ \int_0^t \| \nabla u \|_{L^3(T^d)}^3 + \int_0^t \sigma \| \nabla u \|_{L^4(T^d)}^4 \]
\[ \leq C \left( \frac{1}{\mu^3} \int_0^t \| \mu \curl u \|_{L^3(T^d)}^3 + \frac{1}{(2\mu + \lambda)^3} \int_0^t \| F \|_{L^3(T^d)}^3 + \frac{1}{(2\mu + \lambda)^3} \int_0^t \| P(\rho) - P(M) \|_{L^3(T^d)}^3 \right) \]
\[ + C \left( \frac{1}{\mu^4} \int_0^t \sigma \| \mu \curl u \|_{L^4(T^d)}^4 + \frac{1}{(2\mu + \lambda)^4} \int_0^t \sigma \| F \|_{L^4(T^d)}^4 + \frac{1}{(2\mu + \lambda)^4} \int_0^t \sigma \| P(\rho) - P(M) \|_{L^4(T^d)}^4 \right). \quad (2.8) \]

Thus, in order to close the estimate, we have to recover control for the density.
Remark 2.1 This is where our approach starts to diverge from Hoff’s approach. In the isotropic setting, there is an extra algebraic structure which allows to recover an $L^\infty$-bound for the density. In the anisotropic case, we have to work with weaker norms, essentially because of the failure of homogeneous Fourier multipliers of order 0 to map $L^\infty$ to $L^\infty$. The idea is to try only to propagate what seems to be necessary to show that the two functionals $A_1$ and $A_2$ are bounded:

$$\int_0^t \| P(\rho) - P(M) \|^3_{L^3(\mathbb{T}^d)} , \quad \int_0^t \sigma \| P(\rho) - P(M) \|^3_{L^3(\mathbb{T}^d)} .$$

### 2.2.1 Bounds for the density

In the following lines we want to obtain estimates for the density. We begin by arranging (2.2) as

$$\partial_t b(\rho) + \text{div} (b(\rho) u) + (\rho b'(\rho) - b(\rho)) \frac{(P(\rho) - P(M))}{2\mu + \lambda} = - \frac{1}{2\mu + \lambda} (\rho b'(\rho) - b(\rho)) (2\mu + \lambda) \text{div} u - (P(\rho) - P(M)).$$

(2.9)

Recall the definitions of $H_2 (\cdot / M)$ and $H_3 (\cdot / M)$ given in (1.16). A $L^3$ control for the pressure. Let us take $b = H_2 (\cdot / M)$ in (2.9) with

$$\rho H_2' (\rho / M) - H_2 (\rho / M) = |P(\rho) - P(M)| (P(\rho) - P(M)),$$

and use Young’s inequality in order to obtain that

$$\int_{\mathbb{T}^d} H_2 (\rho(t)/M) + \frac{1}{2(2\mu + \lambda)} \int_0^t \int_{\mathbb{T}^d} |P(\rho) - P(M)|^3$$

$$\leq \int_{\mathbb{T}^d} H_2 (\rho_0/M) + \frac{1}{2\mu + \lambda} \int_0^t \int_{\mathbb{T}^d} (2\mu + \lambda) \text{div} u - (P(\rho) - P(M)) |^3.$$  

(2.10)

A $L^4$ control of the pressure. Finally, take $b = H_3 (\cdot / M)$ with

$$\rho H_3' (\rho/M) - H_3 (\rho/M) = (P(\rho) - P(M))^3$$

in order to obtain that

$$\sigma(t) \int_{\mathbb{T}^d} H_3 (\rho(t)/M) + \frac{1}{2(2\mu + \lambda)} \int_0^t \int_{\mathbb{T}^d} \sigma |P(\rho) - P(M)|^4$$

$$\leq \int_0^1 \int_{\mathbb{T}^d} \rho(\sigma) ds + \frac{1}{2\mu + \lambda} \int_0^t \int_{\mathbb{T}^d} \sigma |(2\mu + \lambda) \text{div} u - (P(\rho) - P(M))|^4.$$  

(2.11)

A simple computation gives us:

$$H_3 (\rho/M) = \frac{1}{3\gamma - 1} (\rho^{3\gamma} - M^{3\gamma - 1}) - \frac{3M^\gamma}{2\gamma - 1} (\rho^{2\gamma} - M^{2\gamma - 1})$$

$$+ \frac{3M^2}{\gamma - 1} (\rho^\gamma - M^\gamma - 1) + M^{3\gamma} - \rho M^{3\gamma - 1}.$$  

Then, we observe that

$$\int_0^1 \int_{\mathbb{T}^d} H_3 (\rho/M) = \frac{1}{3\gamma - 1} \int_0^1 \int_{\mathbb{T}^d} (\rho^{3\gamma} - M^{3\gamma}) - \frac{3M^\gamma}{2\gamma - 1} \int_0^1 \int_{\mathbb{T}^d} (\rho^{2\gamma} - M^{2\gamma}) + 3M^2 \int_0^1 \int_{\mathbb{T}^d} H_1 (\rho/M)$$

$$\leq \frac{1}{2\gamma - 1} \int_0^1 \int_{\mathbb{T}^d} (\rho^{3\gamma} - M^{3\gamma}) - \frac{3M^\gamma}{2\gamma - 1} \int_0^1 \int_{\mathbb{T}^d} (\rho^{2\gamma} - M^{2\gamma}) + 3M^2 E_0.$$  

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owing to the convexity of $s \to s^{3\gamma}$. Thus, since $\gamma \geq 1$ we obtain that
\[
\int_0^1 \int_{\mathbb{T}^4} (\rho^{3\gamma} - M^{3\gamma}) = \int_0^1 \int_{\mathbb{T}^4} \rho^{3\gamma} - M^{3\gamma} - 3\gamma M^{3\gamma - 1} (\rho - M) \geq 0,
\]
owing to the convexity of $s \to s^{3\gamma}$. Thus, since $\gamma \geq 1$ we obtain that
\[
\int_0^1 \int_{\mathbb{T}^4} H_3 (\rho/M) \leq 3M^{2\gamma} E_0 + \int_0^1 \int_{\mathbb{T}^4} |P(\rho) - P(M)|^3.
\] (2.13)
Using the above estimate, (2.11) and (2.10) we obtain that
\[
\sigma (t) \int_{\mathbb{T}^4} H_3 (\rho(t)/M) + \frac{1}{2(2\mu + \lambda)} \int_0^t \int_{\mathbb{T}^4} \sigma |P(\rho) - P(M)|^4
\leq 3M^{2\gamma} E_0 + \int_0^1 \int_{\mathbb{T}^4} H_2 (\rho_0/M) + \int_0^1 \int_{\mathbb{T}^4} |P(\rho) - P(M)|^3 + \frac{C}{2\mu + \lambda} \int_0^t \int_{\mathbb{T}^4} \sigma |F|^4.
\] (2.14)
Let us combine (2.14) with (2.11) to deduce that:
\[
B(t) := (1 + 2(2\mu + \lambda)) \int_{\mathbb{T}^4} H_2 (\rho(t)/M) + \sigma (t) \int_{\mathbb{T}^4} H_3 (\rho(t)/M)
+ \frac{1}{2(2\mu + \lambda)} \int_0^t \int_{\mathbb{T}^4} |P(\rho) - P(M)|^3 + \frac{1}{2(2\mu + \lambda)} \int_0^t \int_{\mathbb{T}^4} \sigma |P(\rho) - P(M)|^4
\leq (1 + 2(2\mu + \lambda)) \int_{\mathbb{T}^4} H_2 (\rho_0/M) + 3M^{2\gamma} E_0
+ \frac{1 + 2(2\mu + \lambda)}{2\mu + \lambda} \int_0^t \|F\|_{L^3 (\mathbb{T}^d)}^3 + \frac{1}{2\mu + \lambda} \int_0^t \sigma \|F\|_{L^4 (\mathbb{T}^d)}^4.
\] (2.15)

2.2.2 Bounds for the Hoff functionals

We recall that
\[
\left\{ \begin{array}{l}
\mu \text{curl } u = \Delta^{-1} \text{curl}(\rho \hat{u}) + \Delta^{-1} \text{curl} \left( \text{div}(\omega \ast (\mathcal{E} \nabla (\omega \ast u))) \right), \\
F = \Delta^{-1} \text{div}(\rho \hat{u}) + \Delta^{-1} \text{div} \left( \text{div}(\omega \ast (\mathcal{E} \nabla (\omega \ast u))) \right)
\end{array} \right.
\]
and therefore, using also (2.10) we have that
\[
\int_0^t \|\nabla u\|_{L^3(\mathbb{T}^d)}^3 \leq \frac{C}{(2\mu + \lambda)^3} \int_0^t \|P(\rho) - P(M)\|_{L^3(\mathbb{T}^d)}^3 + \frac{C}{(2\mu + \lambda)^3} \int_0^t \|\mu \text{curl } u\|_{L^3(\mathbb{T}^d)}^3 + \frac{C}{(2\mu + \lambda)^3} \int_0^t \|F\|_{L^3(\mathbb{T}^d)}^3
\] (2.16)
\[
\leq \frac{C}{(2\mu + \lambda)^3} \int_{\mathbb{T}^d} H_2 (\rho_0/M) + C \int_0^t \|\Delta^{-1} \text{div}(\rho \hat{u})\|_{L^3(\mathbb{T}^d)}^3 + C \int_0^t \|\Delta^{-1} \text{curl}(\rho \hat{u})\|_{L^3(\mathbb{T}^d)}^3
+ C \|\mathcal{E} - \mathcal{T}\|_{L^\infty(0,t) \times \mathbb{T}^d} \max \left\{ \frac{1}{(2\mu + \lambda)^3}, \frac{1}{\mu^3} \right\} \int_0^t \|\nabla u\|_{L^3(\mathbb{T}^d)}^3
\] (2.17)
where $C$ is a generic constant that depends only on the dimension $d \in \{2, 3\}$ whose exact value can change from one line to the other. Let us observe that using interpolation and Sobolev imbedding inequalities, we obtain that:
\[
\int_0^t \| \Delta^{-1} \operatorname{div}(\rho \dot{u}) \|_{L^3(T^d)}^3 + \int_0^t \| \Delta^{-1} \operatorname{curl}(\rho \dot{u}) \|_{L^3(T^d)}^3 \\
\leq \int_0^t \| \Delta^{-1} \operatorname{div}(\rho \dot{u}) \|_{L^2(T^d)}^2 \| \Delta^{-1} \operatorname{div}(\rho \dot{u}) \|_{L^2(T^d)}^2 + \int_0^t \| \Delta^{-1} \operatorname{curl}(\rho \dot{u}) \|_{L^2(T^d)}^2 \| \Delta^{-1} \operatorname{curl}(\rho \dot{u}) \|_{L^2(T^d)}^2 \\
\leq C \left( \sup_{t>0} \| \Delta^{-1} \operatorname{div}(\rho \dot{u}) \|_{L^2(T^d)} + \sup_t \| \Delta^{-1} \operatorname{curl}(\rho \dot{u}) \|_{L^2(T^d)} \right) \sup_{t>0} \left\| \sqrt{\rho(t)} \right\|_{L^{4/(4-d)}(T^d)}^2 \int_0^t \| \sqrt{\rho} \|_{L^2(T^d)}^2.
\] (2.19)

In order to close the estimate, we need the following:

**Lemma 2.1** For all \( \rho \geq 0 \) we have that:
\[
(P(\rho) - P(M))^2 \leq 2\gamma M^\gamma H_1(\rho/M) + (2\gamma - 1) H_2(\rho/M).
\] (2.20)

**Proof of Lemma 2.1** Consider \( g : (0, +\infty) \rightarrow \mathbb{R} \) defined by:
\[
g(\rho) = \alpha \frac{H_1(\rho/M)}{\rho} + \beta \frac{H_2(\rho/M)}{\rho} - \frac{(P(\rho) - P(M))^2}{\rho}
\]
and observe that
\[
g(M) = 0.
\]
We have that
\[
\rho^2 g'(\rho) = (\alpha - 2\gamma M^\gamma)(\rho^\gamma - M^\gamma) + \beta |\rho^\gamma - M^\gamma|(\rho^\gamma - M^\gamma) - (2\gamma - 1)(\rho^\gamma - M^\gamma)^2.
\]
For \( \alpha = 2\gamma M^\gamma \) and \( \beta = 2\gamma - 1 \) we see that
\[
g'(\rho) \leq 0 \text{ on } \rho \in [0, M] \text{ and } g'(\rho) \geq 0 \text{ if } \rho \geq M.
\]
This ends the proof of Lemma 2.1.

Using (2.20) in Lemma 2.1 we have that
\[
\| P(\rho) - P(M) \|_{L^\infty((0,T) \times L^2(T^d))} \leq 2\gamma M^\gamma E_0 + (2\gamma - 1) B(t)
\]
Also, remark that owing to \( \gamma \geq d/(4-d) \) we infer that
\[
\left\| \sqrt{\rho} \right\|_{L^{4/(4-d)}(T^d)}^4 = \left\| \rho \right\|_{L^{2d/(4-d)}(T^d)}^2 \leq \left\| P(\rho) \right\|_{L^3(T^d)}^{2/\gamma} + M^\gamma \left\| \right\|_{L^2(T^d)}^2 \leq C_{\mu,\lambda,\gamma,M}(M + B(t))\ (2.21)
\]
where \( C_{\mu,\lambda,\gamma,M} \) is a generic constant that depends only on the \( \mu, \lambda, \gamma, M \) and whose exact value can change from one line to the other. It is of course, different from the generic constant \( C \) appearing in (2.18) and (2.19) that only depends on the dimension. Combining (2.18) with (2.21) we obtain:
\[
\sup_{t>0} \left\| \Delta^{-1} \operatorname{div}(\rho \dot{u}) \right\|_{L^2(T^d)} + \sup_t \left\| \Delta^{-1} \operatorname{curl}(\rho \dot{u}) \right\|_{L^2(T^d)} \leq C_{\mu,\lambda,\gamma,M} \left( E_0 + B(t) + \sqrt{A_1(t)} \right).
\]

Thus, using (1.14), the last term of the RHS can be absorbed into the LHS thus giving
\[
\int_0^t \| \nabla u \|_{L^3(T^d)}^3 \leq \frac{C}{(2\mu + \lambda)^2} \int_{T^d} H_2(\rho_0/M) + C_{\mu,\lambda,\gamma,M} \left( E_0 + B(t) + \sqrt{A_1(t)} \right) A_1(t). \quad (2.22)
\]
Similarly, using (2.13) we have that

\[
\int_0^t \sigma \| \nabla u \|^4_{L^4(\mathbb{T}^4)} \leq \frac{C}{(2\mu + \lambda)^3} \int_0^t \sigma \| P(\rho) - P(M) \|^4_{L^4(\mathbb{T}^4)} + \frac{C}{(2\mu + \lambda)^2} \int_0^t \sigma \| \mu \text{curl} u \|^4_{L^4(\mathbb{T}^4)} + \frac{C}{(2\mu + \lambda)} \int_0^t \sigma \| F \|^4_{L^4(\mathbb{T}^4)}
\]

\[
\leq \frac{3M^{2\gamma}E_0}{(2\mu + \lambda)^3} + \frac{1}{(2\mu + \lambda)^2} \int_{\mathbb{T}^4} H_2(\rho_0/M) + \frac{C}{(2\mu + \lambda)} \int_0^1 \| P(\rho) - P(M) \|^3_{L^3(\mathbb{T}^4)}
\]

\[
+ \frac{C}{(2\mu + \lambda)^4} \int_0^t \sigma \| \mu \text{curl} u \|^4_{L^4(\mathbb{T}^4)} + \frac{C}{(2\mu + \lambda)^2} \int_0^t \sigma \| F \|^4_{L^4(\mathbb{T}^4)} .
\]  

(2.23)

Next, we see that

\[
\frac{C}{\mu^4} \int_0^t \sigma \| \mu \text{curl} u \|^4_{L^4(\mathbb{T}^4)} + \frac{C}{(2\mu + \lambda)} \int_0^t \sigma \| F \|^4_{L^4(\mathbb{T}^4)}
\]

\[
\leq \frac{C}{\mu^4} \int_0^t \sigma \| \sqrt{\mu} \|^4_{L^4(\mathbb{T}^4)} \int_0^t \| \mu \text{curl} u \|^4_{L^4(\mathbb{T}^4)} + C \| \mathcal{E} - \mathcal{I} \|^4_{L^\infty((0,t) \times \mathbb{T}^4)} \max \left\{ \frac{1}{(2\mu + \lambda)^4}, \frac{1}{\mu^4} \right\} \int_0^t \sigma \| \nabla u \|^4_{L^4(\mathbb{T}^4)}
\]

\[
\leq \frac{C}{\mu^4} \sup_t \| \sqrt{\mu} \|^4_{L^4(\mathbb{T}^4)} \sup_t \sigma \| \mu \text{curl} u \|^4_{L^4(\mathbb{T}^4)} \int_0^t \sigma \| \nabla u \|^4_{L^4(\mathbb{T}^4)}
\]

\[
+ C \| \mathcal{E} - \mathcal{I} \|^4_{L^\infty((0,t) \times \mathbb{T}^4)} \max \left\{ \frac{1}{(2\mu + \lambda)^4}, \frac{1}{\mu^4} \right\} \int_0^t \sigma \| \nabla u \|^4_{L^4(\mathbb{T}^4)}
\]

\[
\leq \frac{C}{\mu^4} \sup_t \| \sqrt{\mu} \|^4_{L^4(\mathbb{T}^4)} A_1(t) A_2(t) + C \| \mathcal{E} - \mathcal{I} \|^4_{L^\infty((0,t) \times \mathbb{T}^4)} \max \left\{ \frac{1}{(2\mu + \lambda)^4}, \frac{1}{\mu^4} \right\} \int_0^t \sigma \| \nabla u \|^4_{L^4(\mathbb{T}^4)}
\]  

(2.24)

Of course, under Hypothesis (14) we see that the last term from the RHS above inequality can be absorbed into the LHS. From the above estimates, it transpires that \( \| P(\rho) - P(M) \|_{L^3((0,1) \times \mathbb{T}^4)} \) verifies the same estimate (2.23). Combining the estimates (2.21), (2.23) and (2.24) we end up with:

\[
\int_0^t \sigma \| \nabla u \|^4_{L^4(\mathbb{T}^4)} \leq \frac{M^{2\gamma}E_0}{(2\mu + \lambda)} + \frac{C}{(2\mu + \lambda)^3} \int_{\mathbb{T}^4} H_2(\rho_0/M)
\]

\[
+ C_{\mu,\lambda,\gamma,M} (M + B(t)) \left( E_0 + B(t) + \sqrt{A_1(t)} + A_2(t) \right) A_1(t)
\]

\[
+ C \| \mathcal{E} - \mathcal{I} \|^4_{L^\infty((0,t) \times \mathbb{T}^4)} \max \left\{ \frac{1}{(2\mu + \lambda)^4}, \frac{1}{\mu^4} \right\} \int_0^t \sigma \| \nabla u \|^4_{L^4(\mathbb{T}^4)} ,
\]

where, we recall that here, \( C \) depends only on the dimension. Thus, under the hypothesis (14) we obtain that

\[
\int_0^t \| \nabla u \|^3_{L^3(\mathbb{T}^4)} + \int_0^t \sigma \| \nabla u \|^2_{L^4(\mathbb{T}^4)}
\]

\[
\leq C_{\mu,\lambda,\gamma,M} \left( E_0 + \int_{\mathbb{T}^4} H_2(\rho_0/M) \right) + C_{\mu,\lambda,\gamma,M} (M + B(t)) \left( E_0 + B(t) + \sqrt{A_1(t)} + A_2(t) \right) A_1(t) .
\]  

(2.25)

**Remark 2.2** In this section we used two generic constants, \( C \) and \( C_{\mu,\lambda,\gamma,M} \) in order to emphasize that the anisotropic amplitude is small only with respect \( \mu \) and \( 2\mu + \lambda \), see Hypothesis (14). In the estimates that follow in the next sections, \( C \) will denote a generic constant that depends on the parameters of the problem \( C = C(\mu,\lambda,\gamma,M) \), the exact value of which can change from a line to another.
Terms appearing in the RHS of (2.6). Recall that

\[
A_1(t) = \frac{\mu}{2} \int_{\mathbb{T}^d} |\partial_k u_0|^2 + \frac{\mu + \lambda}{2} \int_{\mathbb{T}^d} |\text{div} \ u_0|^2 + \int_{\mathbb{T}^d} \varepsilon_{ijkl} \partial_{\ell} u_{0,0}^k \partial_j u_{0,0}^l \nonumber
\]
\[
\int_{\mathbb{T}^d} P(\rho (t)) \text{div} \ u(t) - \int_{\mathbb{T}^d} P(\rho (0)) \text{div} \ u(0) \nonumber
\]
\[
- \mu \int_{0}^{t} \int_{\mathbb{T}^d} \partial_k u \partial_k u \partial_{\ell} u \partial_{\ell} u + \frac{\mu + \lambda}{2} \int_{0}^{t} \int_{\mathbb{T}^d} |\partial_k u|^2 \text{div} \ u \nonumber
\]
\[
- (\mu + \lambda) \int_{0}^{t} \int_{\mathbb{T}^d} \text{div} \ u \partial_{\ell} u \partial_{\ell} u + \frac{\mu + \lambda}{2} \int_{0}^{t} \int_{\mathbb{T}^d} (\text{div} \ u)^3 \nonumber
\]
\[
+ \frac{1}{2} \int_{0}^{t} \int_{\mathbb{T}^d} \{ \partial_i \varepsilon_{ijkl} + \partial_i (\varepsilon_{ijkl} u_j^l) \} \partial_j u_{0,0}^k \partial_k u_{0,0}^l \nonumber
\]
\[
- \int_{0}^{t} \int_{\mathbb{T}^d} \varepsilon_{ijkl} \partial_{\ell} u_{0,0}^l \partial_j u_{0,0}^l \{ \partial_j u_{0,0}^l \partial_{\ell} u_{0,0}^l \} - \int_{0}^{t} \int_{\mathbb{T}^d} \varepsilon_{ijkl} \partial_{\ell} u_{0,0}^l \partial_j u_{0,0}^l \partial_{\ell} u_{0,0}^l \partial_j u_{0,0}^l \nonumber
\]
\[
+ \int_{0}^{t} \int_{\mathbb{T}^d} \rho P'(\rho) \partial_{\ell} u_{0,0}^k \partial_k u_{0,0}^l + \int_{0}^{t} \int_{\mathbb{T}^d} (\rho P'(\rho) - P(\rho)) (\text{div} \ u)^2. \quad (2.26)
\]

First, using (2.1) we have that

\[
\int_{\mathbb{T}^d} P(\rho (t)) \text{div} \ u(t) = \int_{\mathbb{T}^d} (P(\rho (t)) - P(M)) \text{div} \ u(t) \nonumber
\]
\[
\leq C(\eta) \int_{\mathbb{T}^d} (P(\rho (t)) - P(M))^2 + \eta \int_{\mathbb{T}^d} |\text{div} \ u|^2(t) \nonumber
\]
\[
\leq C(\eta) \int_{\mathbb{T}^d} H_1(\rho) + H_2(\rho) + \eta \int_{\mathbb{T}^d} |\text{div} \ u|^2(t), \nonumber
\]

where \( \eta \) will be chosen later. Using the last estimate, we obtain that

\[
\int_{\mathbb{T}^d} P(\rho (t)) \text{div} \ u(t) - \int_{\mathbb{T}^d} P(\rho (0)) \text{div} \ u(0) \nonumber
\]
\[
+ \mu \int_{\mathbb{T}^d} |\partial_k u_0|^2 + \frac{\mu + \lambda}{2} \int_{\mathbb{T}^d} |\text{div} \ u_0|^2 + \frac{1}{2} \int_{\mathbb{T}^d} \varepsilon_{ijkl} \partial_{\ell} u_{0,0}^k \partial_j u_{0,0}^l + \frac{1}{2} \int_{\mathbb{T}^d} \partial_i \varepsilon_{ijkl} \partial_j u_{0,0}^l \partial_k u_{0,0}^l \nonumber
\]
\[
\leq C(\eta) \left\{ E_0 + \|\nabla u_0\|_{L^2(\mathbb{T}^d)} + B(t) \right\} + \eta \int_{\mathbb{T}^d} |\nabla u|^2(t) \quad (2.27)
\]

Using (2.22) and hypothesis (B3) we infer that

\[
\int_{0}^{t} \int_{\mathbb{T}^d} \rho P'(\rho) \partial_{\ell} u_{0,0}^k \partial_k u_{0,0}^l = \gamma P(M) \int_{0}^{t} \int_{\mathbb{T}^d} \partial_{\ell} u_{0,0}^k \partial_k u_{0,0}^l + \gamma \int_{0}^{t} \int_{\mathbb{T}^d} (P(\rho) - P(M)) \partial_{\ell} u_{0,0}^k \partial_k u_{0,0}^l \nonumber
\]
\[
\leq C E_0 + \int_{0}^{t} \int_{\mathbb{T}^d} (P(\rho) - P(M))^3 + \int_{0}^{t} \int_{\mathbb{T}^d} |\nabla u|^3. \quad (2.28)
\]

The term \( \int_{0}^{t} \int_{\mathbb{T}^d} (\rho P'(\rho) - P(\rho)) (\text{div} \ u)^2 \) is treated similarly.

Using Proposition 3.4.1 in order to treat the commutator term, we have that:

\[
- \mu \int_{0}^{t} \int_{\mathbb{T}^d} \partial_k u \partial_k u \partial_{\ell} u + \frac{\mu + \lambda}{2} \int_{0}^{t} \int_{\mathbb{T}^d} |\partial_k u|^2 \text{div} \ u + (\mu + \lambda) \int_{0}^{t} \int_{\mathbb{T}^d} \text{div} \ u \partial_{\ell} u \partial_{\ell} u + \frac{\mu + \lambda}{2} \int_{0}^{t} \int_{\mathbb{T}^d} (\text{div} \ u)^3 \nonumber
\]
\[
- \int_{0}^{t} \int_{\mathbb{T}^d} \varepsilon_{ijkl} \partial_{\ell} u_{0,0}^l \partial_j u_{0,0}^l \partial_{\ell} u_{0,0}^l \partial_j u_{0,0}^l - \int_{0}^{t} \int_{\mathbb{T}^d} \varepsilon_{ijkl} \partial_{\ell} u_{0,0}^l \partial_j u_{0,0}^l \partial_{\ell} u_{0,0}^l \partial_j u_{0,0}^l \leq C \int_{\mathbb{T}^d} \|\nabla u\|^3_{L^3(\mathbb{T}^d)}. \quad (2.29)
\]

Using Poincaré’s inequality, we obtain that

\[
\frac{1}{2} \int_{\mathbb{T}^d} \{ \partial_i \varepsilon_{ijkl} + \partial_j (\varepsilon_{ijkl} u^l) \} \partial_j u_{0,0}^l \partial_k u_{0,0}^l
\]
\[ \leq CE_0 + C \int_0^t \int \|\nabla u\|^3_{L^1(T^t)} + \int_0^t \int u^q(\tau) \left| \int_{T^t} \partial_{ij} u_i \partial_j u_i \right| \right. 
\]

In order to treat the mean of \( u^q \) we follow the idea of P.-L. Lions’s \( \text{[69]} \). Let us recall, after verifying that \( \rho^2 \) is controlled by \( B(t) \), that

\[ \left\| \rho(t) \left( u^q(t) - \int_{T^t} u^q(t) \right) \right\|_{L^1(T^t)} \leq C \left\| \rho(t) \right\|_{L^2(T^t)} \left\| \nabla u(t) \right\|_{L^2(T^t)}. \]

Thus, we have that

\[ \left\| \int_{T^t} \left( \rho(t,x) \int_{T^t} u^q(t,y) dy - \rho(t,x) u^q(t,x) \right) \right\|_{L^1(T^t)} \leq \left\| \rho(t) \right\|_{L^2(T^t)} \left\| \nabla u(t) \right\|_{L^2(T^t)} \]

from which it follows that

\[ M \left\| \int_{T^t} u^q(t,y) dy \right\| \leq M^{\frac{1}{2}} E_0^{\frac{1}{2}} + C \left\| \rho(t) \right\|_{L^2(T^t)} \left\| \nabla u(t) \right\|_{L^2(T^t)}. \]

Consequently

\[ \left\| \int_0^t \left\| \int_{T^t} u^q(\tau) \right\|_{L^1(T^t)} \right\|_{L^1(T^t)} \leq \left\| \partial_{ij} u_i \right\|_{L^\infty((0,t) \times T^t)} \left( M^{\frac{1}{2}} E_0^{\frac{1}{2}} + C \int_0^t \left\| \nabla u(s) \right\|_{L^2(T^t)} ds \right) \]

\[ \leq \left\| \partial_{ij} u_i \right\|_{L^\infty((0,t) \times T^t)} \left( M^{\frac{1}{2}} E_0^{\frac{1}{2}} + C \int_0^t \left\| \nabla u(s) \right\|_{L^2(T^t)} ds \right). \]

Taking \( \eta \) sufficiently small and summing up (2.27), (2.28) we obtain

\[ A_1(t) \leq C \left( E_0 + \int_{T^t} H_2(\rho_0/M) + \left\| \nabla u_0 \right\|_{L^2(T^t)}^2 + C_{u,\gamma,M} \left( E_0 + B(t) + \sqrt{A_1(t)} \right) A_1(t). \]

**Terms appearing in the RHS of (2.7).** We recall the definition of \( A_2(t) \) which is

\[ \sigma(t) \left( \int_{T^t} \frac{\rho(t) \dot{u}(t)}{2} + \mu \int_{T^t} \sigma \left| \partial_\alpha u_i \right|^2 + (\mu + \lambda) \int_{T^t} \sigma \left| \partial_\alpha \dot{u}_i \right|^2 + \int_{T^t} \sigma \epsilon_{ijkl} \partial_\alpha \dot{u}_i \partial_\beta \partial_\gamma \dot{u}_k \partial_\delta \dot{u}_l \right. \]

\[ = \int_0^t \int_{T^t} \sigma \left| \partial_\alpha \dot{u}_i \right|^2 \]

\[ + \mu \int_0^t \int_{T^t} \sigma \partial_\alpha u_i \partial_\beta \partial_\gamma \partial_\delta \dot{u}_i + \mu \int_0^t \int_{T^t} \sigma \partial_\alpha u_i \partial_\beta \partial_\gamma \partial_\delta \dot{u}_i - \mu \int_0^t \int_{T^t} \sigma \partial_\alpha \dot{u}_i \partial_\beta \partial_\gamma \dot{u}_i \]

\[ + (\mu + \lambda) \int_0^t \int_{T^t} \sigma \partial_\alpha u_i \partial_\beta \partial_\gamma \dot{u}_i \partial_\delta \dot{u}_i - (\mu + \lambda) \int_0^t \int_{T^t} \sigma \partial_\alpha \dot{u}_i \partial_\beta \partial_\gamma \dot{u}_i \]

\[ + \int_0^t \int_{T^t} \sigma (\partial_\alpha \epsilon_{ijkl} + \partial_\alpha (u^q \epsilon_{ijkl})) \partial_\beta \partial_\gamma \partial_\delta \dot{u}_i \dot{u}_l - \int_0^t \int_{T^t} \sigma \epsilon_{ijkl} (\omega_{ij} \partial_\alpha \partial_\beta \partial_\gamma \partial_\delta \dot{u}_i \dot{u}_l - \int_0^t \int_{T^t} \sigma \epsilon_{ijkl} \partial_\alpha \partial_\beta \partial_\gamma \partial_\delta \dot{u}_i \dot{u}_l)
\]

\[ + \int_0^t \int_{T^t} \sigma (\partial_\alpha \epsilon_{ijkl} [u^q, \omega_{ij}] \partial_\beta \partial_\gamma \partial_\delta \dot{u}_i \dot{u}_l + \int_0^t \int_{T^t} \sigma \epsilon_{ijkl} \partial_\alpha \partial_\beta \partial_\gamma \partial_\delta \dot{u}_i \dot{u}_l)
\]

\[ - \int_0^t \int_{T^t} \sigma \left\{ P(\rho) \partial_\alpha \partial_\beta \partial_\gamma \partial_\delta \dot{u}_i \partial_\gamma \dot{u}_j + (P'(\rho) - P(\rho)) \partial_\alpha \partial_\beta \partial_\gamma \partial_\delta \dot{u}_i \partial_\gamma \dot{u}_j \right\}.
\]

We observe that

\[ \mu \int_0^t \int_{T^t} \sigma \partial_\alpha u_i \partial_\beta \partial_\gamma \partial_\delta \dot{u}_i + \mu \int_0^t \int_{T^t} \sigma \partial_\alpha u_i \partial_\beta \partial_\gamma \partial_\delta \dot{u}_i - \mu \int_0^t \int_{T^t} \sigma \partial_\alpha \dot{u}_i \partial_\beta \partial_\gamma \dot{u}_i \]

\[ = \int_0^t \int_{T^t} \sigma \partial_\alpha u_i \partial_\beta \partial_\gamma \partial_\delta \dot{u}_i + \int_0^t \int_{T^t} \sigma \partial_\alpha u_i \partial_\beta \partial_\gamma \partial_\delta \dot{u}_i - \mu \int_0^t \int_{T^t} \sigma \partial_\alpha \dot{u}_i \partial_\beta \partial_\gamma \dot{u}_i \]
Moreover, using the Poincaré inequality along with (2.31) we get that
\[
- \int_0^t \int_{\mathbb{T}^d} \sigma (\partial_t \varepsilon_{ijkl} + \partial_q (u^q \varepsilon_{ijkl})) \partial_t u^k_j \partial_j u^l_k \leq C(\eta) E_0 + \eta \int_0^t \int_{\mathbb{T}^d} \sigma |\nabla u|^2.
\]

We thus see that for $\eta$ sufficiently small, we obtain that
\[
A_2(t) \leq C \left( E_0 + \int_{\mathbb{T}^d} H_2 (\rho_0/M) + \|\nabla u_0\|_{L^2(\mathbb{T}^d)^4}^2 \right) + C(M + B(t)) \left( E_0 + B(t) + {\sqrt{A_1(t) + A_2(t)}} \right) A_1(t).
\]

(2.33)

**The bootstrap argument.** Suppose that initially, $(\rho, u)$ are defined on a time interval $[0, T)$. Assume that
\[
E(\rho_0/M, u_0) + \int_{\mathbb{T}^d} H_2 (\rho_0/M) + \|\nabla u_0\|_{L^2(\mathbb{T}^d)}^2 \leq c_0
\]
for some $c_0$ to be fixed later. We want to show that there exists a constant $C = C(\mu, \lambda, \gamma, M, E_0)$ depending on $\mu, \lambda, \gamma, M, E_0$ such that for $c_0$ sufficiently small
\[
\forall t \in [0, T^*): \max_{[0, T^*]} \{A_1(t) + A_2(t) + B(t)\} \leq C (\mu, \lambda, \gamma, M, E_0) c_0.
\]

We will use a bootstrap argument. Recall that in the previous two sections, we showed, see (2.32) and (2.33), that there exists a constant $C$ such that
\[
A_1(t) + A_2(t) \leq C \left( E_0 + \int_{\mathbb{T}^d} H_2 (\rho_0/M) + \|\nabla u_0\|_{L^2}^2 \right) + C(M + B(t)) \left( E_0 + B(t) + {\sqrt{A_1(t) + A_2(t)}} \right) A_1(t).
\]

We observe that
\[
B(t) \leq C \left( E_0 + \int_{\mathbb{T}^d} H_2 (\rho_0/M) \right) + C(M + B(t)) \left( {\sqrt{A_1(t) + A_2(t)}} \right) A_1(t).
\]

We thus obtain that there exists a constant $\tilde{C} = \tilde{C}(\mu, \lambda, \gamma, M, E_0)$ such that
\[
A_1(t) + A_2(t) + B(t) \leq \tilde{C} \left( E_0 + \int_{\mathbb{T}^d} H_2 (\rho_0/M) + \|\nabla u_0\|_{L^2}^2 \right) + \tilde{C}(M + B(t)) \left( E_0 + B(t) + {\sqrt{A_1(t) + A_2(t)}} \right) A_1(t).
\]

(2.34)

Let us introduce $T^* \in (0, T]$ such that
\[
\max_{t \in [0, T^*)} \left\{ \frac{1}{2} A_1(t) + A_2(t) + B(t) \right\} \leq 2\tilde{C} c_0.
\]
If \( c_0 \) is sufficiently small such that
\[
\hat{C}(M + 2\hat{C}c_0)E_0 \leq \frac{1}{2}
\]
then the last two inequalities we observe that
\[
\frac{1}{2} A_1(t) + A_2(t) + B(t) \leq \hat{C}c_0 + \hat{C}(M + 2\hat{C}c_0)
\]
and thus for \( c_0 \) sufficiently small we have that
\[
\sqrt{\hat{C}}c_0 \hat{C}(M + 2\hat{C}c_0) \left( \sqrt{2\hat{C} + 2\hat{C}\sqrt{c_0}} \right) 2\hat{C} \leq \frac{1}{2}\hat{C},
\]
such that by a bootstrap argument we obtain that \( T^* = T \).

### 2.3 Construction of a sequence of approximate solution to (1.11)

This entire part of the proof, from the local to the global existence of strong solutions for the approximate system, can be achieved by repeating the same arguments as in the work of R. Danchin and P.B. Mucha [DM19], Section 3. It is for this reason that we only briefly recall the strategy from [DM19] and we comment on what is different in our case. The form of the approximate systems (1.17) is inspired from the one proposed by the first author and P.-E. Jabin in [BJ18] which we recopy here for the reader’s convenience:

\[
\begin{aligned}
\{ & \rho_t + \text{div}(\rho u) = 0, \\
& (\rho u)_t + \text{div}(\rho u \otimes u) + \nabla \rho^\gamma = \mu \Delta u + (\mu + \lambda) \nabla \text{div} u + \omega \ast \text{div}(\nabla \omega \ast u). \}
\end{aligned}
\]

(2.35)

It differs from the classical (isotropic) Navier-Stokes system by a smooth term. Therefore, we argue that the classical results regarding existence of local solutions, see for instance Theorem 3.1. from [DM19] remain true in our case with a proof that is essentially the same. Thus, we have that

**Theorem 2** Let \( \rho_0 \in W^{1,p}(\mathbb{T}^d) \) and \( u_0 \in W^{2-d,2}(\mathbb{T}^d) \) for some \( p > d \) with \( d \geq 2 \). Assume that \( \rho_0 > 0 \). Then there exists \( T_\ast > 0 \) depending only on the norms of the data and on \( \inf_{\mathbb{T}^d} \rho_0 \) such that (2.35) supplemented with data \( \rho_0 \) and \( u_0 \) has a unique solution \((\rho, u)\) on the time interval \([0, T_\ast]\), satisfying

\[
u \in W^{1,p}((0, T_\ast); L^p(\mathbb{T}^d)) \cap L^p((0, T_\ast); W^{2-p}(\mathbb{T}^d)) \text{ and } \rho \in C([0, T_\ast]; W^{1,p}(\mathbb{T}^d)).
\]

Consider

\[
\rho_0 \in L^{2^\gamma}(\mathbb{T}^d) \text{ and } u_0 \in (H^1(\mathbb{T}^d))^d,
\]

with

\[
\int_{\mathbb{T}^d} \rho_0(x) \, dx = M.
\]

Also, consider \( \omega_\delta = \frac{1}{\delta} \omega \left( \frac{\cdot}{\delta} \right) \) with \( \omega \) a smooth, nonnegative, radial function compactly supported in the unit ball centered at the origin and with integral equal to 1. To this end, for all \( \delta \in (0, M) \) there exists \( \xi_\delta > \delta \) such that

\[
\hat{\rho}_\delta^\delta(x) = \min \{ \xi_\delta, \rho_0(x) + \delta \} \text{ and } \int_{\mathbb{T}^d} \hat{\rho}_\delta^\delta(x) \, dx = M.
\]

We consider

\[
\rho_\delta^\delta(x) = \omega_\delta \ast \hat{\rho}_\delta^\delta \text{ and } u_\delta^\delta = \omega_\delta \ast u_0.
\]

Observe that for a subsequence that, which by slightly abusing the notation we still denote by the index \( \delta \), it holds that:

\[
\int_{\mathbb{T}^d} \rho_\delta^\delta(x) \, dx = M \text{ and } \lim_{\delta \to 0} \| \rho_\delta^\delta - \rho_0 \|_{L^{2^\gamma}(\mathbb{T}^d)} + \lim_{\delta \to 0} \| u_\delta^\delta - u_0 \|_{H^1(\mathbb{T}^d)} = 0.
\]

Consider \((\rho^\delta, u^\delta)\) the sequence of solutions for the Cauchy problem associated to system (1.17) with initial

\[
\begin{aligned}
\{ & \rho(t=0) = \rho_0^\delta, \\
u(t=0) = u_0^\delta,
\end{aligned}
\]

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the existence of which is granted by Theorem 2.2. A priori, each of \((\rho^\delta, u^\delta)\) is defined on its own maximal time interval \([0, T_\delta]\) with \(T_\delta \in (0, \infty]\). On these time intervals the solution has enough regularity such that the computations performed above make sense and, as a consequence, we have that \((\rho^\delta, u^\delta)\) have

\[
E\left( \frac{\rho^\delta}{M}, u^\delta \right), A^\delta_1(t), A^\delta_2(t), B^\delta(t)
\]

bounded independently w.r.t. \(\delta\) where \(A^\delta_1(t)\), \(A^\delta_2(t)\), \(B^\delta(t)\) are the expressions defined in (2.4), (2.5) respectively in \([4,14]\) with \((\rho^\delta, u^\delta)\) instead of \((\rho, u)\). From here on, the argument leading to the conclusion that \(T^\delta = +\infty\) continues as in \([4,19]\) due to the fact that the term \(\omega_\delta \ast \div (\nabla \omega_\delta \ast u)\) is regular.

In particular, taking also into account (2.3), we get that for all \(T > 0\) we have that

\[
\|\rho^\delta\|_{L^\infty((0, T); L^2(\mathbb{T}^d))} + \|u^\delta\|_{L^3((0, T); W^{1, 3}(\mathbb{T}^d))} + \|\sqrt{\sigma} \nabla \dot{u}^\delta\|_{L^2((0, T); \mathbb{T}^d)} \leq C(\mu, \lambda, \gamma, \mathcal{E}).
\]

**(Strong convergence of the sequence \((u^\delta)_{\delta > 0}\).** We want to prove that the uniform bounds verified by the solutions constructed above imply that up to a subsequence

\[
\lim_{\delta \to 0} u^\delta = u \text{ strongly in } L^2\left((\frac{1}{n}, T) \times \mathbb{T}^d\right),
\]

for all \(n \in \mathbb{N}^+\). This a consequence of the fact that the second Hoff functional is uniformly bounded w.r.t. \(\delta > 0\) which implies that for all \(T > 0\) and all \(\delta > 0\)

\[
\int_0^T \int_{\mathbb{T}^d} \sigma(t) \|\nabla u^\delta\|^2_{L^2(\mathbb{T}^d)} \leq c,
\]

for some \(c\). This implies that for any \(n \in \mathbb{N}^+\)

\[
\int_0^T \int_{\mathbb{T}^d} \|\nabla u^\delta\|^2_{L^2(\mathbb{T}^d)} \leq nc.
\]

We remark that since \(d \in \{2, 3\}\) we have that

\[
\|u^\delta\|_{L^3((0, T); L^p(\mathbb{T}^d))} + \|\nabla u^\delta\|_{L^3((0, T); L^p(\mathbb{T}^d))}
\]

is uniformly bounded for any \(p < \infty\). Taking into account that \(\partial_t u^\delta = \dot{u}^\delta - u^\delta \cdot \nabla u^\delta\), (the mean value of \(\dot{u}^\delta\) is controlled exactly as in 2.31) we obtain that for all \(\eta \in (0, 1]\) we have

\[
\partial_t u^\delta \text{ is uniformly bounded in } L^2\left((1/n, T) \times \mathbb{T}^d\right).
\]

For any \(n\), by the Aubin-Lions theorem \((u^\delta)_{\delta > 0}\) converges strongly in \(L^2\left((\frac{1}{n}, T) \times \mathbb{T}^d\right)\) while applying a Cantor’s diagonal type process provides us with a subsequence \((u^\delta)_{\delta > 0}\) converging for any \(n\) in \(L^2\left((\frac{1}{n}, T) \times \mathbb{T}^d\right)\).

**Remark 2.3** We use \(\delta\) as upperscript to designate the sequence of approximate solutions \((\rho^\delta, u^\delta)_{\delta > 0}\). This should not to be confused with the lower-script notation used in the previous section which denoted the regularisation with \(\omega_\delta\). In order to avoid possible confusion, all along this section we explicitly write \(\omega_\delta \ast (\cdot)\).

### 2.4 Stability of solutions for (2.35)

In this section we show that from the sequence \((\rho^\delta, u^\delta)_{\delta}\) constructed in the previous section and verifying uniformly the Hoff-type estimates (2.36), one can extract a subsequence converging weakly towards a solution of the system (1.13). We recall that this is not trivial given the fact that the pressure is a nonlinear function of the density. From the estimates we have gathered so far for \((\rho^\delta, u^\delta)_{\delta}\) we infer the existence of \((\rho, u, \rho^\gamma)\) such that modulo an extraction of a subsequence:
\[
\begin{aligned}
\rho^\delta &\to \rho \text{ weakly in } L^{3\gamma}((0,T) \times \mathbb{T}^d) \text{ and weakly- } * \text{ in } L^{\infty}((0,T); L^{2\gamma}(\mathbb{T}^d)), \\
\rho^\delta &\to \rho \text{ strongly in } C^0([0,T]; L^{2\gamma}_{weak}), \\
(\rho^\delta)^\gamma &\to \rho^\gamma \text{ weakly in } L^3((0,T) \times \mathbb{T}^d) \text{ and weakly- } * \text{ in } L^{\infty}((0,T); L^2(\mathbb{T}^d)), \\
(\rho^\delta)^\gamma &\to \rho^\gamma \text{ strongly in } C^0((0,T); L^{2\gamma}_{weak}), \\
u^\delta &\to \nu \text{ weakly in } (L^3(0,T; W^{1,3}(\mathbb{T}^d)))^d, \\
u^\delta &\to \nu \text{ strongly in } (L^2(\frac{1}{3}, T \times \mathbb{T}^d))^d,
\end{aligned}
\]

and, in view of (2.37) for any \( n \in \mathbb{N} \) and for any \( p \in [2,3) \) :

\[
\partial_t u \in L^\frac{7}{5}((1/n,T); L^p(\mathbb{T}^d)).
\] 

See the Appendix for a definition for \( C^0([0,T]; L^2_{weak}) \). In particular, since we work in dimension \( d \in \{2,3\} \), since \( u \in (L^3((0,T); W^{1,3}(\mathbb{T}^d)))^d \) and owing to (2.39) we obtain that for all \( n \in \mathbb{N} \) we have:

\[
\begin{aligned}
\rho \in L^2 T, \quad \rho u \in L^3((0,T); L^\frac{7}{5}(\mathbb{T}^d)), \\
\rho u \in L^2((0,T) \times \mathbb{T}^d), \\
\partial_t u \in L^\frac{7}{5}(1/n,T); L^\frac{7}{5}(\mathbb{T}^d)), \\
\partial_t \|u\|^2 \in L^1(1/n,T); L^\frac{7}{5}(\mathbb{T}^d)), \\
\nabla \|u\|^2 \in L^\frac{7}{5}(0,T); L^\frac{7}{5}(\mathbb{T}^d)).
\end{aligned}
\] 

All the relations of (2.38) are applications of classical results from functional analysis. The second and fourth relations are obtained using a weak variant of the Arzelà-Ascoli see Theorem 3 from the appendix, a proof of which can be found, for instance in Vrabie [Vra03]. We will give details for the proof of the fourth relation since the second one follows from similar arguments. Since the sequence \((\langle \rho^\delta \rangle)^\gamma_{\delta>0} \) is bounded in \( L^{\infty}((0,T); L^2(\mathbb{T}^d)) \) the second condition from Theorem 3 obviously holds true. It remains to prove that \((\langle \rho^\delta \rangle)^\gamma_{\delta>0} \) is weakly equicontinuous on \([0,T] \). Fix \( \varepsilon > 0 \) and a \( w \in L^2(\mathbb{T}^d) \) and consider \( \tilde{w} \in C_{per}(\mathbb{R}^d) \). We see that there exists a numerical constant \( C \) independent of \( \delta \) such that

\[
\begin{aligned}
\langle (\rho^\delta)^\gamma(t) - (\rho^\delta)^\gamma(s), w \rangle &\leq 2 \| (\rho^\delta)^\gamma\|_{L^\infty(0,T); L^\gamma(\mathbb{T}^d))} \|w - \tilde{w}\|_{L^\gamma(\mathbb{T}^d)} + C \int_s^t \| (\rho^\delta)^\gamma\|_{L^\gamma(\mathbb{T}^d))} \|u^\delta\|_{W^{1,3}(\mathbb{T}^d)} \|\nabla \tilde{w}\|_{L^1(\mathbb{T}^d)} \\
&\leq 2 \| (\rho^\delta)^\gamma\|_{L^\infty(0,T); L^\gamma(\mathbb{T}^d))} \|w - \tilde{w}\|_{L^\gamma(\mathbb{T}^d)} + C \|t - s\| \| (\rho^\delta)^\gamma\|_{L^\gamma((0,T) \times \mathbb{T}^d))} \|u^\delta\|_{L^\gamma((0,T); W^{1,3}(\mathbb{T}^d))} \|\nabla \tilde{w}\|_{L^1(\mathbb{T}^d)}.
\end{aligned}
\]

Using (2.40), the first term can be made arbitrarily small because we can approximate \( L^2(\mathbb{T}^d) \) functions with smooth periodic functions while the second term can be made arbitrarily small provided \( |t-s| \) is chosen appropriately.

Using the strong convergences of \( \rho^\delta \to \rho \) in \( C^0([0,T]; L^{2\gamma}_{weak}) \), of \( (\rho^\delta)^\gamma \to \rho^\gamma \) in \( C^0([0,T]; L^2_{weak}) \) and the fact that \( \rho^\delta_0 \to \rho_0 \) strongly in \( L^{2\gamma}(\mathbb{T}^d) \) we recover that

\[
\begin{aligned}
\lim_{t \to \tau} \int_{\tau}^{T} \rho(t,x) \psi(x) \, dx &= \int_{\tau}^{T} \rho_0(x) \psi(x) \, dx \\
\lim_{t \to \tau} \int_{\tau}^{T} \rho^\gamma(t,x) \psi(x) \, dx &= \int_{\tau}^{T} \rho_0^\gamma(x) \psi(x) \, dx,
\end{aligned}
\]

for all \( \psi \in C_{per}^\infty \).

It is by now well-understood that the assumptions (2.38) are sufficient in order to conclude that

\[
\begin{aligned}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) - \mu \Delta u - (\mu + \lambda) \text{div} u + \nabla \rho \nabla \text{div} u &= \text{div} (\mathcal{E}(\nabla u)),
\end{aligned}
\]

Moreover,

\[
\rho \in C([0,T]; L^p(\mathbb{T}^d)) \quad \text{for all } 1 \leq p < 2\gamma,
\]

see for instance Lemma 6.15, page 312 of [NS04]. In the same manner, using the fact that

\[
\partial_t \rho \nabla + \text{div}(\rho \nabla u) + (\gamma - 1) \rho \nabla \text{div} u = 0,
\]
which comes from passing to the limit in the equations verified by \((\rho^\delta)\gamma\) we obtain that:
\[
\overline{\rho}^\gamma \in C \left( [0, T); L^p (T^d) \right) \quad \text{for } 1 \leq p < 2. \quad (2.44)
\]

Of course, in order to finish the proof we must show that the function \(\overline{\rho}^\gamma\) coincides with the function \(\rho^\gamma\). To this end we will essentially mimic the proof from \([\text{III}], \text{IV}\) which consists of taking the difference between the limit of the energy equations with the energy equation of the limiting system and "multiplying" it with an appropriate quantity that yields a "conservative" identity.

Using once more \([\text{III}], \text{IV}\) we obtain the existence of positive functions \(\nabla u : \nabla u, (\text{div } u)^2, \mathcal{E}(\nabla u) : \nabla u \in L^2 \left( (0, T) \times T^d \right)\) such that up to a subsequence we have
\[
\begin{align*}
\nabla u^\delta : \nabla u^\delta &\to \nabla u : \nabla u \text{ in } L^2 \left( (0, T) \times T^d \right) \text{ and } \nabla u : \nabla u \leq \nabla u : \nabla u, \\
(\text{div } u^\delta)^2 &\to (\text{div } u)^2 \text{ in } L^2 \left( (0, T) \times T^d \right) \text{ and } (\text{div } u)^2 \leq (\text{div } u)^2, \\
\mathcal{E}(\nabla (\omega^\delta \ast u^\delta)) : \nabla (\omega^\delta \ast u^\delta) &\to \mathcal{E}(\nabla u) : \nabla u \text{ in } L^2 \left( (0, T) \times T^d \right) \text{ and } \mathcal{E}(\nabla u) : \nabla u \leq \mathcal{E}(\nabla u) : \nabla u.
\end{align*}
\]

(2.45)

It is in the proof of the last property, that we need to regularize a positive definite operator and the assumption made in Remark \([\text{I}], \text{III}\). See the remark to see that simple change of shear viscosity that may be made to satisfy such property starting with a viscosity tensor satisfying Hypothesis \([\text{III}], \text{IV}\).

**Lower semi-continuity.** Indeed, for any \(\phi \in L^3((0, T) \times T^d)\) with \(\phi \geq 0\), denoting \(u^\delta = (u^\delta_1, \ldots, u^\delta_d)\), we have that
\[
0 \leq \int_0^T \int_{T^d} \mathcal{E}(\nabla (\omega^\delta \ast u^\delta) - \nabla u) : (\nabla (\omega^\delta \ast u^\delta) - \nabla u) \phi \\
= \int_0^T \int_{T^d} \delta_{ijkl}(\partial_k \omega^\delta \ast u^\delta_i - \partial_k u^\delta_i)(\partial_j \omega^\delta \ast u^\delta_i - \partial_j u^\delta_i) \phi \\
= \int_0^T \int_{T^d} \varepsilon_{ijkl}(\partial_k \nabla \omega^\delta \ast u^\delta_i - \partial_k u^\delta_i)(\partial_j \nabla \omega^\delta \ast u^\delta_i - \partial_j u^\delta_i) - \int_0^T \int_{T^d} \varepsilon_{ijkl}(\partial_k u^\delta_i)(\partial_j \omega^\delta \ast u^\delta_i - \partial_j u^\delta_i) \\
+ \int_0^T \int_{T^d} \varepsilon_{ijkl}(\partial_k u^\delta_i)(\partial_j u^\delta_i) \phi.
\]

We obviously have
\[
\lim_{\delta \to 0} \int_0^T \int_{T^d} \omega^\delta \ast (\varepsilon_{ijkl}(\partial_i \nabla u^\delta_i) \partial_j u^\delta_i) = \int_0^T \int_{T^d} \varepsilon_{ijkl}(\partial_i \nabla u^\delta_i) \partial_j u^\delta_i
\]
and the same for the other similar term. Thus we obtain that
\[
0 \leq \lim_{\delta \to 0} \int_0^T \int_{T^d} \varepsilon_{ijkl}(\partial_k \nabla \omega^\delta \ast u^\delta_i - \partial_k u^\delta_i)(\partial_j \nabla \omega^\delta \ast u^\delta_i - \partial_j u^\delta_i) - \int_0^T \int_{T^d} \varepsilon_{ijkl}(\partial_k u^\delta_i)(\partial_j \omega^\delta \ast u^\delta_i - \partial_j u^\delta_i) \\
+ \int_0^T \int_{T^d} \mathcal{E}(\nabla u) : \nabla u - \mathcal{E}(\nabla u) : \nabla u) \phi.
\]

**Energy identities and conclusion.** On the one hand, for any \(\delta > 0\), the regularity of \((\rho^\delta, u^\delta)\), see Theorem \([\text{II}]\) and the remark that follows, we may write the following energy equation:
\[
\frac{1}{2} \partial_t \left\{ \rho^\delta |u^\delta|^2 + \frac{(\rho^\delta)^2}{\gamma - 1} \right\} + \text{div} \left( \left( \rho^\delta |u^\delta|^2 + \frac{(\rho^\delta)^2}{\gamma - 1} \right) u^\delta \right) + \mu \nabla u^\delta : \nabla u^\delta + (\mu + \lambda) \text{div} u^\delta
- \mu \Delta \frac{|u^\delta|^2}{2} - (\mu + \lambda) \text{div} (u^\delta \text{div} u^\delta) - \omega^\delta \ast \text{div} (\mathcal{E} \nabla \omega^\delta \ast u^\delta) u^\delta = 0,
\]
\[
(2.46)
\]

Let us observe that for all \(\phi \in C_c \left( (0, T); C_{\text{per}}(\mathbb{R}^d) \right)\) we have that
\[
- \int_{T^d} \omega^\delta \ast \text{div} (\mathcal{E} \nabla \omega^\delta \ast u^\delta) u^\delta \phi = \int_{T^d} \mathcal{E} \nabla (\omega^\delta \ast u^\delta) : \nabla \omega^\delta \ast \nabla (u^\delta \phi)
\]
\[
\begin{align*}
E_\delta = \int_{T^d} \mathcal{E} \nabla (\omega_\delta * u^\delta) : \omega_\delta * (\nabla u^\delta \phi) + \int_{T^d} \mathcal{E} \nabla (\omega_\delta * u^\delta) : \omega_\delta * (u^\delta \otimes \nabla \phi).
\end{align*}
\]

Owing to the fact that there exits some \( n \) such that

\[
\text{Supp} \, (\cdot, \cdot) \subset (1/n, T) \times \mathbb{R}^d,
\]

that \( u^\delta \to u \) strongly in \( L^2((1/n, T) \times \mathbb{T}^d)^d \) and that \( \nabla (\omega_\delta * u^\delta) \to \nabla u \) weakly in \( L^3((0, T) \times \mathbb{T}^d)^{d \times d} \) we obtain that

\[
\lim_{\delta \to 0} \int_0^T \int_{T^d} \mathcal{E} \nabla (\omega_\delta * u^\delta) : \omega_\delta * (u^\delta \otimes \nabla \phi) = \int_0^T \int_{T^d} \mathcal{E} (\nabla u) : (u \otimes \nabla \phi).
\]

Next, we observe that

\[
\int_0^T \int_{T^d} \mathcal{E} \nabla (\omega_\delta * u^\delta) : \omega_\delta * (\nabla u^\delta \phi)
\]

\[
\begin{align*}
= \int_0^T \int_{T^d} \mathcal{E} \nabla (\omega_\delta * u^\delta) : \nabla (\omega_\delta * u^\delta) \phi + \int_0^T \int_{T^d} \mathcal{E} \nabla (\omega_\delta * u^\delta) : [\omega_\delta *, \phi] \nabla u^\delta.
\end{align*}
\]

Now, for any \( j, q \in \overline{1, d} \) one has

\[
[\omega_\delta *, \phi] \partial_j u^{\delta, q} (t, x) = (\omega_\delta * (\phi \partial_j u^{\delta, q}) - \phi \omega_\delta * \partial_j u^{\delta, q}) (t, x)
\]

\[
= \int_{T^d} (\phi (t, x - y) - \phi (t, x)) \partial_j u^{\delta, q} (t, x - y) \omega_\delta (y) \, dy
\]

\[
= \int_{T^d} (\phi (t, x - \delta z) - \phi (t, x)) \partial_j u^{\delta, q} (t, x - \delta z) \omega (z) \, dz.
\]

Thus

\[
\left| \int_0^T \int_{T^d} \mathcal{E} \nabla (\omega_\delta * u^\delta) : [\omega_\delta *, \phi] \nabla u^\delta \right| \leq \delta \max_{i, j, k, l} \| \mathcal{E} \|_{L^\infty(T^d)} \| \nabla u^\delta \|_{L^2(T^d)}^2 \| \nabla \phi \|_{L^\infty} \to 0.
\]

Moreover, using the information of relation (2.45) we may pass to the limit in (2.46) such as to obtain

\[
\frac{1}{2} \frac{\partial}{\partial t} \left( \rho |u|^2 + \frac{\rho y}{\gamma - 1} \right) + \text{div} \left( \left( \rho |u|^2 + \frac{\rho y}{\gamma - 1} \right) u \right) + \mu \nabla u : \nabla u + (\mu + \lambda) \text{div} u^2 + \mathcal{E} (\nabla u) : \nabla u
\]

\[
- \mu \Delta \frac{|u|^2}{2} - (\mu + \lambda) \text{div} (u \text{div} u) - \text{div} (u \mathcal{E} (\nabla u)) = 0.
\]

(2.47)

On the other hand, let us observe that system (2.42) can be put under the form

\[
\begin{aligned}
\partial_t \rho + \text{div} (\rho u) &= 0, \\
\partial_t (\rho u) + \text{div} (\rho u \otimes u) - \text{div} (\mathcal{E} \nabla u) + \nabla \rho y &= \nabla (\rho y - \rho z).
\end{aligned}
\]

(2.48)

We observe that in view of the estimates (2.40) and by a density argument, the weak formulations for the transport equation holds true for any test function \( \psi \) such that

\[
\partial_t \psi \in L^1((0, T); L^2(T^d)), \quad \nabla \psi \in L^2((0, T); L^2(T^d))
\]

while the momentum equation holds true for vector fields with coefficients \( \psi \) such that

\[
\partial_t \psi \in L^2((0, T); L^2(T^d)) \text{ and } \nabla \psi \in L^3((0, T) \times \mathbb{T}^d).
\]

The same estimates (2.41) show that for any \( \phi \in C^\infty_c ((0, T); C^\infty_{\text{per}} (\mathbb{R}^d)) \) one can use \( |u|^2 \phi / 2 \) as a test function in the first equation of (2.48) while one can also use \( u \phi \) as a test function in the second equation of (2.48). By doing so, summing up the two relations that result, taking into account the chain-rule respectively the
derivation rule of products of functions in Sobolev spaces, Proposition 3.1 and finally the fact that \( \phi \) is chosen arbitrarily, we obtain that:

\[
\frac{1}{2} \frac{\partial}{\partial t} \left( \rho |u|^2 + \frac{\rho^\gamma}{\gamma - 1} \right) + \text{div} \left( \left( \rho |u|^2 + \frac{\rho^\gamma}{\gamma - 1} \right) u \right) + \mu \nabla u : \nabla u + (\mu + \lambda) (\text{div} u)^2 + \mathcal{E} (\nabla u) : \nabla u - \mu \Delta |u|^2 - (\mu + \lambda) \text{div} (u \text{div} u) - \text{div} (u \mathcal{E}(\nabla u)) = \text{div} (u (\rho^\gamma - \bar{\rho}^\gamma)) - (\rho^\gamma - \bar{\rho}^\gamma) \text{div} u,
\]

which holds true in \( \mathcal{D}'_{t,x} \left( (0, T) \times \mathbb{R}^d_{\text{per}} \right) \). Next, we take the difference between (2.43) and (2.44), we multiply it with \( \gamma - 1 \) in order to obtain that

\[
\begin{align*}
\partial_t \Theta + \text{div} (\Theta u) + (\gamma - 1) \Theta \text{div} u &= - (\gamma - 1) \Xi \text{ in } \mathcal{D}'_{t,x} \left( (0, T) \times \mathbb{R}^d_{\text{per}} \right),
\end{align*}
\]

where

\[
\begin{align*}
\Theta &\stackrel{\text{not.}}{=} \bar{\rho}^\gamma - \rho^\gamma, \\
\Xi &\stackrel{\text{not.}}{=} \left( \mu \nabla u : \nabla u + (\mu + \lambda) (\text{div} u)^2 + \mathcal{E} (\nabla u) : \nabla u \right) \\
&\quad - (\mu \nabla u : \nabla u + (\mu + \lambda) (\text{div} u)^2 + \mathcal{E} (\nabla u) : \nabla u).
\end{align*}
\]

Obviously,

\[
\Theta, \Xi \geq 0.
\]

We regularize the previous equation with the help of a sequence of approximations of the identity \( \omega_\varepsilon \):

\[
\begin{align*}
\partial_t \omega_\varepsilon * \Theta + \text{div} (\omega_\varepsilon * \Theta u) + (\gamma - 1) \omega_\varepsilon * (\Theta \text{div} u) &= r_\varepsilon (\Theta, u) - (\gamma - 1) \omega_\varepsilon * \Xi,
\end{align*}
\]

see the notations introduced in 3.1 and 3.2. Since the time derivative \( \partial_t \omega_\varepsilon * \Theta \) belongs to some Lebesgue space, we may multiply relation (2.51) with \( \frac{1}{\gamma}(h + \omega_\varepsilon * \Theta)^{\frac{1}{\gamma} - 1} \) where \( h > 0 \) is a fixed positive constant and apply the chain rule. We end up with

\[
\begin{align*}
\partial_t (h + \omega_\varepsilon * \Theta) &\stackrel{\text{not.}}{=} \frac{1}{\gamma}(h + \omega_\varepsilon * \Theta)^{\frac{1}{\gamma} - 1} r_\varepsilon (\Theta, u) \leq \frac{1}{\gamma}(h + \omega_\varepsilon * \Theta)^{\frac{1}{\gamma} - 1} (\gamma - 1) \omega_\varepsilon * \Xi,
\end{align*}
\]

Owing to (2.43) and (2.44), the application \( t \to \int_{\mathbb{T}^d} (h + \omega_\varepsilon * \Theta)^{\frac{1}{\gamma}} \) is continuous and since by integrating w.r.t. space in (2.52), its distributional time derivative belongs to some Lebesgue space, we deduce that it is absolutely continuous and that the distributional derivative coincides with the derivative a.e.. We may thus write that for any \( t \in (0, T) \) we have that

\[
\begin{align*}
\int_{\mathbb{T}^d} (h + \omega_\varepsilon * \Theta)^{\frac{1}{\gamma}} (t) &= \int_{\mathbb{T}^d} (h + \omega_\varepsilon * \Theta)^{\frac{1}{\gamma}} (0) - \int_0^t \int_{\mathbb{T}^d} \left( \frac{1}{\gamma} - 1 \right) (h + \omega_\varepsilon * \Theta)^{\frac{1}{\gamma} - 1} [\omega_\varepsilon, \text{div} u] \Theta + \int_0^t \int_{\mathbb{T}^d} (h + \omega_\varepsilon * \Theta)^{\frac{1}{\gamma} - 1} h \text{div} u \\
&\quad + \int_0^t \int_{\mathbb{T}^d} \left[ \frac{1}{\gamma}(h + \omega_\varepsilon * \Theta)^{\frac{1}{\gamma} - 1} r_\varepsilon (\Theta, u) - \frac{1}{\gamma}(h + \omega_\varepsilon * \Theta)^{\frac{1}{\gamma} - 1} (\gamma - 1) \omega_\varepsilon * \Xi \right] \\
&\leq \int_{\mathbb{T}^d} (h + \omega_\varepsilon * \Theta)^{\frac{1}{\gamma}} (0) - \int_0^t \left( \frac{1}{\gamma} - 1 \right) (h + \omega_\varepsilon * \Theta)^{\frac{1}{\gamma} - 1} [\omega_\varepsilon, \text{div} u] \Theta \\
&\quad + \int_0^t \int_{\mathbb{T}^d} \frac{1}{\gamma}(h + \omega_\varepsilon * \Theta)^{\frac{1}{\gamma} - 1} r_\varepsilon (\Theta, u).
\end{align*}
\]
where we used the positivity of $\Xi$. Using Proposition 3.1, we obtain that 

$$[\omega_\varepsilon, \text{div } u] \Theta \text{ and } r_\varepsilon (\Theta, u) \to 0 \text{ in } L^1 \left((0, T) \times \mathbb{T}^d\right).$$

Notice that since $\gamma > 1$ along with $\omega_\varepsilon * \Theta \geq 0$, we also have that 

$$(h + \omega_\varepsilon * \Theta)^{1/\gamma - 1} \leq h^{1/\gamma - 1}.$$ 

Taking into account the last observations, by making $\varepsilon \to 0$ we get that 

$$\int_{T^d} (h + \Theta)^{\frac{1}{\gamma}} (t) \leq \int_{T^d} (h + \Theta)^{\frac{1}{\gamma}} (0) + h^{1/\gamma} \int_0^t \int_{T^d} |\text{div } u|.$$

Letting $h$ go to zero and using that at initial time $\Theta(0)$ is 0, shows that 

$$\int_{T^d} \Theta^{\frac{1}{\gamma}} (t) = \int_{T^d} (\rho^{\gamma} (t) - \rho^{\gamma} (0))^{\frac{1}{\gamma}} = 0,$$

from which follows the conclusion that 

$$\rho^{\gamma} \equiv \rho^{\gamma} \text{ a.e. on } (0, T) \times \mathbb{T}^d.$$

This ends the proof of Theorem 1.

3 Appendix

3.1 Appendix A: tool box

In this section, we gather some classical results that are used throughout the text.

**Lemma 3.1 (Fourier Multipliers)** Consider $m : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ a function verifying 

$$|\partial^\alpha m (\xi)| \leq c_\alpha |\xi|^{-\alpha}$$

for all $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq d + 1$. Then, for all $p \in (1, \infty)$, there exists $C_p$ such that for any $u \in L^p$

$$\left\| \mathcal{F}^{-1} \left( m (\xi) \mathcal{F} \left( u - \int_{\mathbb{T}^d} u \right) \right) \right\|_{L^p(\mathbb{T}^d)} \leq C_p \|u\|_{L^p(\mathbb{T}^d)}.$$

**Lemma 3.2 (Sobolev’s Inequality)** For all $p \in (1, d)$ and $u \in L^p \cap D^{1,p*}$ we have that 

$$\left\| u - \int_{\mathbb{T}^d} u \right\|_{L^p(\mathbb{T}^d)} \leq \|\nabla u\|_{L^{p*}(\mathbb{T}^d)},$$

where $1/p + 1/d = 1/p^*$. 

Let $X$ be a Banach space. We consider $C^0([0, T]; X_{\text{weak}})$ the space of continuous functions from $[0, T]$ to $X$ endowed with the weak topology:

$$C^0([0, T]; X_{\text{weak}}) = \left\{ f : [0, T] \to X \text{ such that } \forall w \in X', t \to \langle w, f (t) \rangle_{X' \times X} \text{ is continuous} \right\}.$$

**Definition 3.1** A subset $\mathcal{F}$ of $C^0([0, T]; X_{\text{weak}})$ is called weakly equicontinuous on $[0, T]$ if for all $w \in X'$ and for all $\varepsilon > 0$ there exists a $\delta = \delta (w, \varepsilon) > 0$ such that for all $t, s \in [0, T]$

$$|t - s| \leq \delta \Rightarrow |\langle f (t) - f (s), w \rangle| \leq \varepsilon.$$
Theorem 3 Let $X$ be a reflexive Banach space. A subset $F$ of $C^0([0,T]; X_{\text{weak}})$ endowed with the uniformly weak topology is sequentially relatively compact if and only if

- $F$ is weakly equicontinuous on $[0,T]$.
- There exists $D \subset [0,T]$ dense such that for all $t \in D$ the set $F(t) := \{ f(t) : f \in F \}$ is bounded in $X$.

For a proof of a slightly more general result see Theorem A.3.1. from Vrabie [Vra02], page 302.

Let $g \in L^q((0,T); L^p(\mathbb{T}^d))$ with $p, q \geq 1$, introduce a new function

$$g_\delta(x) = g \ast \omega_\delta(x) \quad \text{with} \quad \omega_\delta(x) = \frac{1}{\delta^d} \omega\left(\frac{x}{\delta}\right) \quad (3.1)$$

with $\omega$ a smooth, nonnegative, even function compactly supported in the unit ball centered at the origin and with integral equal to 1. We recall the following classical analysis result

$$\lim_{\delta \to 0} \|g_\delta - g\|_{L^q(0,T; L^p(\mathbb{T}^d))} = 0.$$ 

Next let us recall the following commutator estimate which was obtained for the first time by DiPerna and Lions:

Proposition 3.1 Consider $\beta \in (1, \infty)$ and $(a,b)$ such that $a \in L^\beta((0,T) \times \mathbb{T}^d)$ and $b, \nabla b \in L^p((0,T) \times \mathbb{T}^d)$ where $\frac{1}{p} = \frac{1}{\beta} + \frac{1}{d} \leq 1$. Then, we have

$$\lim_{\delta \to 0} r_\delta^k(a,b) = 0 \text{ in } L^1((0,T) \times \mathbb{T}^d),$$

for $k \in \{1, 2\}$ where

$$r_\delta^1(a,b) = b\partial_t a_\delta - (b\partial_t a)_\delta \quad \text{and} \quad r_\delta^2(a,b) = \partial_t (a_b) - \partial_t ((ab)_\delta). \quad (3.2)$$

Moreover, the following commutator estimates hold true

$$\|b\partial_t a_\delta - (b\partial_t a)_\delta\|_{L^1_t L^\beta_x} \leq \|\nabla b\|_{L^p_t L^\beta_x} \|a\|_{L^\beta_t L^\beta_x} \quad (3.3)$$

$$\|\partial_t (a_b) - \partial_t ((ab)_\delta)\|_{L^1_t L^\beta_x} \leq \|\nabla b\|_{L^p_t L^\beta_x} \|a\|_{L^\beta_t L^\beta_x} \quad (3.4)$$

where $b\partial_t a$ should be understood as $b\partial_t a = \partial_t (ab) - a\partial_t b$.

Whenever we have a regular solution for the transport equation

$$\partial_t \rho + \text{div} (\rho u) = 0, \quad (3.5)$$

then, multiplying the former equation with $b'(\rho)$ gives

$$\partial_t b(\rho) + \text{div} (b(\rho) u) + \{\rho b'(\rho) - b(\rho)\} \text{div} u = 0. \quad (3.6)$$

The following proposition gives us a framework for justifying this computations when $\rho$ is just a Lebesgue function.

Proposition 3.2 Consider $2 \leq \beta < \infty$ and $\lambda_0, \lambda_1$ such that $\lambda_0 < 1$ and $-1 \leq \lambda_1 \leq \beta/2 - 1$. Also, consider $\rho \in L^3((0,T) \times \mathbb{T}^d)$, $\rho \geq 0$ a.e. and $u, \nabla u \in L^2((0,T) \times \mathbb{T}^d)$ verifying the transport equation in the sense of distributions. Then, for any function $b \in C^0([0,\infty)) \cap C^1((0,\infty))$ such that

$$\begin{cases} b'(t) \leq ct^{-\lambda_0} & \text{for } t \in (0,1], \\ |b'(t)| \leq ct^{\lambda_1} & \text{for } t \geq 1. \end{cases}$$

Then, equation (3.6) holds in the sense of distributions.
3.2 Appendix B: detailed computations for the Hoff functionals

3.2.1 Hoff’s first energy functional

The momentum equation reads:

$$\rho \dot{u} - \mu \Delta u - (\mu + \lambda) \nabla \text{div} u = \mu \omega_8 * \mathcal{E}(\nabla u_8) + \nabla P (\rho) = \rho f.$$ 

where

$$\dot{u} = \partial_t u + u \nabla u.$$ 

We multiply the above equation with $\dot{u}$ and integrate. Owing to the hypothesis

$$\varepsilon_{ijkl} a_{ij} b_{kl} = \varepsilon_{ijkl} a_{ij} b_{kl},$$

we can write that

$$\langle \text{div} \omega_8 * \mathcal{E}(\omega_8 * \nabla u), \dot{u} \rangle = - \int_{\mathbb{T}^d} \partial_j (\varepsilon_{ijkl} \partial_k u^i_8) u^j_8$$

$$= \int_{\mathbb{T}^d} \varepsilon_{ijkl} \partial_k u^i_8 \partial_j u^j_8 + \int_{\mathbb{T}^d} \varepsilon_{ijkl} \partial_k u^i_8 \omega_8 * (\partial_j u^j_8 \partial_k u^l_8) + \int_{\mathbb{T}^d} \varepsilon_{ijkl} \partial_k u^i_8 \omega_8 * (u^i_8 \partial^2_{ij} u^l_8)$$

$$= \frac{1}{2} \left\{ \int_{\mathbb{T}^d} \varepsilon_{ijkl} \partial_k u^i_8 \partial_j u^j_8 + \int_{\mathbb{T}^d} \varepsilon_{ijkl} \partial_k u^i_8 \partial_j u^j_8 \right\} + \int_{\mathbb{T}^d} \varepsilon_{ijkl} \partial_k u^i_8 \omega_8 * (\partial_j u^j_8 \partial_k u^l_8) + \int_{\mathbb{T}^d} \varepsilon_{ijkl} \partial_k u^i_8 \omega_8 * (u^i_8 \partial^2_{ij} u^l_8)$$

$$= \frac{1}{2} \int_{\mathbb{T}^d} \varepsilon_{ijkl} \partial_j u^j_8 \partial_k u^k_8 - \frac{1}{2} \int_{\mathbb{T}^d} \partial_i \varepsilon_{ijkl} \partial_k u^k_8 \partial_j u^j_8 + \int_{\mathbb{T}^d} \varepsilon_{ijkl} \partial_k u^i_8 \omega_8 * (\partial_j u^j_8 \partial_k u^l_8) + \int_{\mathbb{T}^d} \varepsilon_{ijkl} \partial_k u^i_8 \omega_8 * (u^i_8 \partial^2_{ij} u^l_8)$$

$$= \frac{1}{2} \int_{\mathbb{T}^d} \varepsilon_{ijkl} \partial_j u^j_8 \partial_k u^k_8 + \int_{\mathbb{T}^d} \varepsilon_{ijkl} \partial_k u^i_8 \omega_8 * (\partial_j u^j_8 \partial_k u^l_8) - \frac{1}{2} \int_{\mathbb{T}^d} \left( \partial_i \varepsilon_{ijkl} + \partial_i (\varepsilon_{ijkl} u^l) \right) \partial_j u^j_8 \partial_k u^k_8 $$

$$+ \int_{\mathbb{T}^d} \varepsilon_{ijkl} \partial_k u^i_8 \omega_8 * (u^i_8 \partial^2_{ij} u^l_8).$$

Similar computations show that

$$- \langle (\mu \Delta + (\mu + \lambda) \nabla \text{div}) u, \dot{u} \rangle = \frac{1}{2} d \int_{\mathbb{T}^d} \left\{ \mu \int_{\mathbb{T}^d} |\partial_k u^i|^2 + (\mu + \lambda) \int_{\mathbb{T}^d} |\text{div} u|^2 \right\}$$

$$+ \mu \int_{\mathbb{T}^d} \partial_i u^i \partial_k u^k \partial_i u^i - \frac{\mu}{2} \int_{\mathbb{T}^d} |\partial_k u^i|^2 \text{div} u$$

$$+ (\mu + \lambda) \int_{\mathbb{T}^d} \text{div} u \partial_i u^i \partial_k u^i - \frac{\mu + \lambda}{2} \int_{\mathbb{T}^d} (\text{div} u)^3$$

Next, we treat the pressure term

$$\int_{\mathbb{T}^d} \dot{u} \nabla P (\rho) = - \int_{\mathbb{T}^d} P (\rho) \text{div} \dot{u} = - \frac{d}{dt} \left\{ \int_{\mathbb{T}^d} P (\rho) \text{div} u \right\} + \int_{\mathbb{T}^d} \partial_t P (\rho) \text{div} u - \int_{\mathbb{T}^d} P (\rho) \text{div} (u \nabla u)$$

$$= - \frac{d}{dt} \left\{ \int_{\mathbb{T}^d} P (\rho) \text{div} u \right\} + \int_{\mathbb{T}^d} \partial_t P (\rho) \text{div} u - \int_{\mathbb{T}^d} P (\rho) \partial_t u^k \partial_k u^l$$

$$= - \frac{d}{dt} \left\{ \int_{\mathbb{T}^d} P (\rho) \text{div} u \right\} + \int_{\mathbb{T}^d} \partial_t P (\rho) \text{div} u - \int_{\mathbb{T}^d} P (\rho) \partial_t u^k \partial_k u^l$$

$$= - \frac{d}{dt} \left\{ \int_{\mathbb{T}^d} P (\rho) \text{div} u \right\} + \int_{\mathbb{T}^d} \partial_t P (\rho) + \text{div} (P(\rho) u) \text{div} u - \int_{\mathbb{T}^d} P (\rho) \partial_t u^k \partial_k u^l.$$
Putting together all the above computations, we end up with

\[
\frac{1}{2} \frac{d}{dt} \left\{ \mu \int_{\Omega} |\partial_t u|^2 + (\mu + \lambda) \int_{\Omega} |\nabla u|^2 + \int_{\Omega} \varepsilon_{ijkl} \partial_t u^i_k \partial_j u^j_k - \int_{\Omega} P(\rho) \nabla u \right\} + \int_{\Omega} \rho |\dot{u}|^2 \\
= -\mu \int_{\Omega} \partial_t u^i \partial_t u^i \nabla u^i + \frac{\mu + \lambda}{2} \int_{\Omega} |\partial_t u|^2 \nabla u \\
- (\mu + \lambda) \int_{\Omega} \partial_t u^i \partial_t u^i \nabla u^i + \frac{\mu + \lambda}{2} \int_{\Omega} (\nabla u)^3 \\
+ \frac{1}{2} \int_{\Omega} \left\{ \partial_t \varepsilon_{ijkl} + \partial_j (\varepsilon_{ijkl} u^q) \right\} \partial_j u^i_k \partial_i u^i_k - \int_{\Omega} \varepsilon_{ijkl} \partial_t u^i_k \omega^j_k \partial_j \omega^i_k - \int_{\Omega} \varepsilon_{ijkl} \partial_t u^i_k [u^q, \omega^j_k] \partial_j \omega^i_k \\
+ \int_0^t \int_{\Omega} \rho P'(\rho) \partial_t u^i_k \partial_i u^i_k + \int_{\Omega} (\rho P'(\rho) - P(\rho)) (\nabla u)^2 + \int_{\Omega} \rho \dot{u} f. \tag{3.7}
\]

### 3.2.2 Hoff's second energy functional

The idea leading to the construction of this second functional is to apply to the momentum equation the material time derivative \(\partial_t + \nabla (u \cdot)\), multiply with \(\dot{u}\) and integrate. The detailed computations are presented below. First, we obviously have that

\[
\int_{\Omega} (\partial_t (\rho \dot{u}^i) + \partial_k (u^k \rho \dot{u}^i)) \dot{u}^j = \frac{d}{dt} \int_{\Omega} \rho |\dot{u}|^2/2
\]

Next, let us deal with the pressure term. First of all, owing to the density equation we write that

\[
\partial_t P(\rho) + \nabla (P(\rho) u) + (P'(\rho) - P(\rho)) \nabla u = 0
\]

which implies that for all \(j \in \Omega\), it holds true that

\[
\partial_j \partial_j P(\rho) + \nabla (\partial_j P(\rho) u) + \nabla (P(\rho) \partial_j u) + \partial_j \left\{ (P'(\rho) - P(\rho)) \nabla u \right\} = 0.
\]

We use this relation in order to infer

\[
\int_{\Omega} (\partial_t \partial_j P(\rho) + \partial_k (u^k \partial_j P(\rho))) \dot{u}^j = -\int_{\Omega} \left\{ \nabla (P(\rho) \partial_j u) + \partial_j \left\{ (P'(\rho) - P(\rho)) \nabla u \right\} \right\} \dot{u}^j \\
= \int_{\Omega} \left\{ P(\rho) \partial_j u^k \partial_i u^i + (P'(\rho) - P(\rho)) \nabla u \nabla \dot{u} \right\}.
\]

Finally, let us treat the dissipative term. We observe that

\[
- \langle \partial_t \nabla \omega_{k} * \mathcal{E} (\nabla u_{j}), \dot{u} \rangle \\
- \int_{\Omega} \partial_j \left( \partial_t \varepsilon_{ijkl} \partial_t u^i_k \right) \dot{u}^j - \int_{\Omega} \partial_j \left( \varepsilon_{ijkl} \partial_t \nabla u^i_k \right) \dot{u}^j - \int_{\Omega} \partial_q \left( u^q \omega^j_k * \partial_j \left( \varepsilon_{ijkl} \partial_t u^i_k \right) \right) \dot{u}^j \\
= \int_{\Omega} \partial_t \varepsilon_{ijkl} \partial_t u^i_k \partial_j u^j_k + \int_{\Omega} \varepsilon_{ijkl} \partial_t \nabla u^i_k \partial_j u^j_k + \int_{\Omega} \partial_j \left( \varepsilon_{ijkl} \partial_t u^i_k \right) \omega^j_k * (u^q \partial_q \dot{u}^i) \\
= \int_{\Omega} \partial_t \varepsilon_{ijkl} \partial_t u^i_k \partial_j u^j_k \\
+ \int_{\Omega} \varepsilon_{ijkl} \partial_t (\partial_t u^i_k + \omega^j_k (u^q \partial_q u^i)) \partial_j u^j_k - \int_{\Omega} \varepsilon_{ijkl} \partial_t (\omega^j_k (u^q \partial_q u^i)) \partial_j u^j_k
\]
Again, integrating by parts leads to the following identity:

\[-(\partial_t \div \omega_\delta \star E(\nabla u_\delta) + \div (u \div \omega_\delta \star E(\nabla u_\delta)), \dot{u})\]

\[= \int_{T^d} \partial_t \varepsilon_{ijkl} \partial_t u_\delta^k \partial_j \dot{u}_\delta^i + \int_{T^d} \varepsilon_{ijkl} \partial_t u_\delta^k \partial_j \dot{u}_\delta^i \]

\[-\int_{T^d} \varepsilon_{ijkl}(\omega_\delta * (\partial_t u^q \partial_q u^k)) \partial_j \dot{u}_\delta^i - \int_{T^d} \varepsilon_{ijkl}(\omega_\delta * (\partial_j u^q \partial_q \dot{u}^i)) \]

\[-\int_{T^d} \varepsilon_{ijkl} ([u^q, \omega_\delta] \partial^2_{\partial_q u^k}) \partial_j \dot{u}_\delta^i - \int_{T^d} \varepsilon_{ijkl} \partial_t u_\delta^k ([u^q, \omega_\delta] \partial^2_{\partial_q \dot{u}^i}) \]

\[-\int_{T^d} \varepsilon_{ijkl} \partial_t u_\delta^k \partial_j \dot{u}_\delta^i \]

Similar computations lead to the identity

\[-\langle \partial_\lambda (\mu \Delta u + (\mu + \lambda) \nabla \div u) + \div (u (\mu \Delta u + (\mu + \lambda) \nabla \div u)), \dot{u} \rangle \]

\[= \mu \int_{T^d} |\partial_k \dot{u}^i|^2 + (\mu + \lambda) \int_{T^d} |\div \dot{u}|^2 \]

\[-\mu \int_{T^d} \partial_k u^q \partial_q \partial_k u^i \div \dot{u} - \mu \int_{T^d} \partial_k u^q \partial_k u^i \partial_q \dot{u}^i + \mu \int_{T^d} \div u \partial_k u^i \partial_k \dot{u}^i \]

\[-(\mu + \lambda) \int_{T^d} \partial_t u^q \partial_q u^i \div \dot{u} - (\mu + \lambda) \int_{T^d} \partial_t u^q \partial_q \dot{u}^i \div u + (\mu + \lambda) \int_{T^d} |\div u|^2 \div \dot{u} \]

Putting together all the above computations, we end up with

\[\frac{d}{dt} \int_{T^d} \frac{\rho |\dot{u}|^2}{2} + \mu \int_{T^d} |\partial_k \dot{u}^i|^2 + (\mu + \lambda) \int_{T^d} |\div \dot{u}|^2 + \int_{T^d} \varepsilon_{ijkl} \partial_t u_\delta^k \partial_j \dot{u}_\delta^i \]

\[= \mu \int_{T^d} \partial_k u^q \partial_q \partial_k \dot{u}^i + \mu \int_{T^d} \partial_k u^q \partial_k \dot{u}^i \partial_q \dot{u}^i - \mu \int_{T^d} \div u \partial_k u^i \partial_k \dot{u}^i \]

\[+ (\mu + \lambda) \int_{T^d} \partial_t u^q \partial_q u^i \div \dot{u} + (\mu + \lambda) \int_{T^d} \partial_t u^q \partial_q \dot{u}^i \div u - (\mu + \lambda) \int_{T^d} |\div u|^2 \div \dot{u} \]

\[-\int_{T^d} \partial_t \varepsilon_{ijkl} + \partial_q (u^q \varepsilon_{ijkl}) \partial_t u_\delta^k \partial_j \dot{u}_\delta^i \]
\[-\int_{T^d} \varepsilon_{ijkl} (\omega_\delta * (\partial_t u^a \partial_q u^k)) \partial_j \hat{u}_\delta^i - \int_{T^d} \varepsilon_{ijkl} \partial_t u^b_\delta \omega_\delta * (\partial_j u^a \partial_q \hat{u}^i) \]
\[+ \int_{T^d} \varepsilon_{ijkl} ([u^a, \omega_\delta] \partial_q^2 u^k) \partial_j \hat{u}_\delta^i + \int_{T^d} \varepsilon_{ijkl} \partial_t u^b_\delta ([u^a, \omega_\delta] \partial_q^2 \hat{u}^i) \]
\[- \int_{T^d} \{ P (\rho) \partial_j u^b \partial_k \hat{u}^j + (\rho P' (\rho) - P (\rho)) \text{div} u \text{div} \hat{u} \} . \]

We multiply the above with \( \sigma (t) \) such that we obtain

\[\sigma (t) \int_{\mathbb{T}^d} \frac{\rho (t) |\dot{u} (t)|^2}{2} + \mu \int_{\mathbb{T}^d} \sigma |\partial_k \hat{u}^i|^2 + (\mu + \lambda) \int_{\mathbb{T}^d} \sigma |\text{div} \hat{u}|^2 + \int_{\mathbb{T}^d} \sigma \varepsilon_{ijkl} \partial_t u^b_\delta \partial_j \hat{u}_\delta^i \]
\[= \int_0^1 \int_{\mathbb{T}^d} \frac{\rho (t) |\dot{u} (t)|^2}{2} + \mu \int_{\mathbb{T}^d} \sigma |\partial_k \hat{u}^i|^2 + (\mu + \lambda) \int_{\mathbb{T}^d} \sigma |\text{div} \hat{u}|^2 + \int_{\mathbb{T}^d} \sigma \varepsilon_{ijkl} \partial_t u^b_\delta \partial_j \hat{u}_\delta^i \]
\[+ \mu \int_0^t \int_{\mathbb{T}^d} \sigma \partial_t u^a \partial_q u^i \partial_k \hat{u}^i + \mu \int_0^t \int_{\mathbb{T}^d} \sigma \partial_k u^a \partial_k u^i \partial_q \hat{u}^i - \mu \int_0^t \int_{\mathbb{T}^d} \sigma \text{div} u \partial_k u^i \partial_q \hat{u}^i \]
\[+ (\mu + \lambda) \int_0^t \int_{\mathbb{T}^d} \sigma \partial_t u^a \partial_q u^i \text{div} \hat{u} + (\mu + \lambda) \int_0^t \int_{\mathbb{T}^d} \sigma \partial_t u^a \partial_q \hat{u}^i \text{div} u - (\mu + \lambda) \int_0^t \int_{\mathbb{T}^d} \sigma |\text{div} u|^2 \text{div} \hat{u} \]
\[- \int_0^t \int_{\mathbb{T}^d} \sigma (\partial_t \varepsilon_{ijkl} + \partial_q (u^a \varepsilon_{ijkl})) \partial_t u^b_\delta \partial_j \hat{u}_\delta^i - \int_0^t \int_{\mathbb{T}^d} \sigma \varepsilon_{ijkl} (\omega_\delta * (\partial_t u^a \partial_q u^k)) \partial_j \hat{u}_\delta^i - \int_0^t \int_{\mathbb{T}^d} \sigma \varepsilon_{ijkl} \partial_t u^b_\delta \omega_\delta * (\partial_j u^a \partial_q \hat{u}^i) \]
\[+ \int_0^t \int_{\mathbb{T}^d} \sigma \varepsilon_{ijkl} ([u^a, \omega_\delta] \partial_q^2 u^k) \partial_j \hat{u}_\delta^i + \int_0^t \int_{\mathbb{T}^d} \sigma \varepsilon_{ijkl} \partial_t u^b_\delta ([u^a, \omega_\delta] \partial_q^2 \hat{u}^i) \]
\[- \int_0^t \int_{\mathbb{T}^d} \sigma \{ P (\rho) \partial_j u^b \partial_k \hat{u}^j + (\rho P' (\rho) - P (\rho)) \text{div} u \text{div} \hat{u} \} . \]

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