Fractional helicity, Lorentz symmetry breaking, compactification and anyons

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Abstract

We construct the covariant, spinor sets of relativistic wave equations for a massless field on the basis of the two copies of the R-deformed Heisenberg algebra. For the finite-dimensional representations of the algebra they give a universal description of the states with integer and half-integer helicity. The infinite-dimensional representations correspond formally to the massless states with fractional (real) helicity. The solutions of the latter type, however, break down the (3+1)D Poincaré invariance to the (2+1)D Poincaré invariance, and via a compactification on a circle a consistent theory for massive anyons in $D=2+1$ is produced. A general analysis of the “helicity equation” shows that the (3+1)D Poincaré group has no massless irreducible representations with the trivial non-compact part of the little group constructed on the basis of the infinite-dimensional representations of $sl(2,\mathbb{C})$. This result is in contrast with the massive case where integer and half-integer spin states can be described on the basis of such representations, and means, in particular, that the (3+1)D Dirac positive energy covariant equations have no massless limit.
1 Introduction

It is generally accepted that the (2+1)-dimensional space-time reveals specific characteristics which are no more valid in higher dimensions. For instance, only in (2+1)D the states of arbitrary spin $\lambda \in \mathbb{R}$ and of corresponding anyonic statistics \cite{1, 2} different from the bosonic and fermionic ones do exist. There are several ways to introduce anyons in (2+1)D including the Chern-Simons \cite{8, 9, 10} and the group-theoretical \cite{11, 12, 13, 14, 15} approaches. In the latter, noticing that the $SO(2, 1)$ group is infinitely connected, the massive anyons of spin $\lambda \in \mathbb{R}$ are realized on the infinite-dimensional half-bounded representations of $SL(2, \mathbb{R})$ \cite{16}, the universal covering group of $SO(2, 1)$. Within that approach, in particular, the anyons can be described by the covariant vector \cite{17} and spinor \cite{18, 19} sets of linear differential equations for infinite-component fields.

The little group for the (2+1)D massive anyon is $SO(2)$ and it carries the same number of physical degrees of freedom as a (2+1)D massive scalar field \cite{20}. This specific feature is also valid for the massless states in (3+1)D. Indeed, in (3+1)D the little group in the massless case is $E(2)$, the group of rotations and translations in the 2D Euclidean space. Representing its non-compact part by zero, we reduce $E(2)$ to $SO(2)$. With this observation at hands, it was shown that some (2+1)D models for anyons can be obtained from the (3+1)D models of massless particles via the appropriate formal reduction \cite{21}. Further, since, unlike the massive (3+1)D case, the algebra of the little group gives no quantization restrictions, it seems that there is no strong reason for excluding the possibility to consider massless states with fractional (arbitrary real) helicity \cite{22}. Following this line of reasoning, some time ago it was claimed that, in fact, a massless analogue of the original Dirac spinor set of equations describing a massive spin-0 field in (3+1)D \cite{23, 24} gives rise to the (3+1)D massless states of helicity $\pm \frac{1}{4}$ called “quartions” \cite{23, 24}. Besides, it was noted that the non-covariant formal quantization of the massless superparticle preserving its classical $P$-invariance should result in the supermultiplet with helicity structure $(\frac{-1}{4}, \frac{+1}{4})$ \cite{27}. On the other hand, there exist the topological arguments (see, e.g., \cite{28, 29}) related to the two-connectedness of $SO(3, 1)$ which restrict the helicity of massless representations to integer and half-integer values. But then, appealing to the well-known relations between $D$-dimensional massive and $(D+1)$-dimensional massless representations, one can ask what corresponds to the (2+1)D massive representations with fractional spin. Formally, it could be the (3+1)D massless irreducible representations with fractional helicity constructed on the basis of infinite-dimensional representations of $sl(2, \mathbb{C})$. But since, due to the topological reasons, they cannot exist, what kind of defects have to appear in such a theory and how the (2+1)D massive anyons emerge from the corresponding (3+1)D massless theory?

In this paper we address in detail the problem of the description of fractional helicity massless fields in (3+1)D on the basis of the infinite-dimensional representations of the $sl(2, \mathbb{C})$ Lie algebra realized in terms of the two copies of the $R$-deformed Heisenberg algebra (RDHA) \cite{4, 30, 31}. The RDHA was first introduced by Yang \cite{32} in the context of Wigner generalized quantization schemes \cite{13} underlying the concept of parafields and parastatistics \cite{7} (in this context, see also Ref. \cite{34}). This algebra and its generalizations is nowadays ex-

\footnote{Strictly speaking, other generalizations of statistics called parafermions and parabosons exist in any space-time dimension \cite{3, 4, 5, 6, 7}, but via the so-called Green anzatz they can be represented in terms of ordinary bosons and fermions.}
exploited extensively in the mathematical and physical literature in different aspects (see Refs. [35, 18, 19, 36, 37]). It should be emphasized that the infinite-dimensional representations of $sl(2, \mathbb{C})$ we consider have nothing to do with those representations corresponding to the little group $E(2)$ with a non-trivial non-compact part, to which are usually referred as to representations with “continuous” or “infinite” spin.

We observe here that the corresponding irreducible infinite-dimensional representations of $sl(2, \mathbb{C})$ cannot be “exponentiated” to representations of the $SL(2, \mathbb{C})$ Lie group in the massless case. In other words, the fractional helicity representation of the little group $SO(2)$ cannot be promoted to a representation of the $(3+1)D$ Lorentz group being a subgroup of the corresponding massless representations of the $(3+1)$-dimensional Poincaré group. This is reflected in breaking of the Lorentz invariance at the level of solutions of the covariant spinor set of equations for fractional helicity massless fields. The symmetry breaking corresponds to the violation of the invariance with respect to the rotations in two directions and to the boosts in one direction. Consequently, the Lorentz group $SL(2, \mathbb{C})$ is broken down to $SL(2, \mathbb{R})$, and via a compactification and subsequent dimensional reduction from $(3+1)D$ to $(2+1)D$ a consistent theory for massive anyons in $(2+1)D$ is produced. At the same time, we show that the $(3+1)D$ Poincaré group has no massless irreducible representations characterized by the trivial non-compact part of the little group and which would be constructed on the basis of the infinite-dimensional representations of $sl(2, \mathbb{C})$. This results in the same restriction for helicity but not of a topological origin.

The paper is organized as follows. In Section 2 we realize representation of $sl(2, \mathbb{C})$ in terms of the two copies of the R-deformed Heisenberg algebra. The spinor set of relativistic equations based on this representation is considered in Section 3, where we show that for the finite-dimensional representations of RDHA such equations universally describe massless states with any integer and half-integer helicity. The infinite-dimensional representations correspond formally to the states with fractional helicities. We demonstrate that such solutions, however, break down the Poincaré invariance. In Section 4 we show that the compactification of the initial massless equations and subsequent reduction to $(2+1)$-dimensional space result in the consistent theory of massive anyons. The absence of the massless infinite-dimensional representations of the $(3+1)D$ Poincaré group is proved in Section 5 in a generic case, independently on the concrete form of equations. Section 6 is devoted to a brief discussion of the obtained results. Appendix reviews briefly the Dirac equations for massive spinless positve energy states.

2 RDHA and $sl(2, \mathbb{C})$: generalization of the Schwinger construction

The first $(3+1)D$ relativistic equation, due to which the infinite-dimensional unitary representations of $SL(2, \mathbb{C})$ were discovered, is the Majorana equation [39]. Its solutions, however, describe reducible representations of $ISO(3,1)$ characterized by the positive energy in the massive sector $p^2 < 0$. At the beginning of 70s Dirac [23, 24] (see also [40]) proposed a covariant spinor set of equations (see Appendix) from which the Majorana and Klein-Gordon equations appear in the form of integrability conditions. As a result, the Dirac spinor set of equations possesses a massive spin-0 positive energy solutions, whereas its vector modi-
fication considered by Staunton [40] describes massive spin-1/2 states. We are interested in analysing the possibility of constructing relativistic wave equations for a massless field carrying fractional helicity. For the purpose, the fields related to the infinite-dimensional representations of $sl(2,\mathbb{C})$ will be considered.

### 2.1 Fock representations of the R-deformed algebra

Having in mind the analogy with the $(2+1)D$ case of anyons, it is convenient to use the infinite-dimensional representations of $sl(2,\mathbb{C})$ realized by means of the two copies of RDHA \cite{32,30,31,19} with mutually commuting generators:

\[
[a^-, a^+] = \Pi + \nu R, \quad \{R, a^\pm\} = 0, \quad R^2 = \Pi,
\]

\[
[\bar{a}^-, \bar{a}^+] = \bar{\Pi} + \nu \bar{R}, \quad \{\bar{R}, \bar{a}^\pm\} = 0, \quad \bar{R}^2 = \bar{\Pi}.
\]

The operators $a^\pm$ will represent internal (spin) degrees of freedom. They generalize the two oscillator degrees of freedom (with $\nu = 0$) used by Dirac \cite{23,24}.

In the case of a direct sum of representations of the algebras with which we begin our analysis, $\Pi$ and $\bar{\Pi}$ are the projectors on the corresponding subspaces that in a matrix realization is reflected by the relations $\Pi = \frac{1}{2}(1 + \sigma_3)$, $\bar{\Pi} = \frac{1}{2}(1 - \sigma_3)$. The operators $R, \bar{R}$ have the sense of reflection operators for the internal variables and $\nu \in \mathbb{R}$ is the deformation parameter. As it was mentioned above, the RDHA and its representations were studied extensively in the literature. Here we will mainly refer to \cite{31}, where a universality of the RDHA was observed: when $\nu = -(2k + 1)$, $k \in \mathbb{N}$, its representations are finite-dimensional (parafermion-like), and are infinite-dimensional if not (being unitary paraboson-like for $\nu > -1$ \cite{30}). The choice $\nu = 0$ with a direct product of representations of the two algebras was used in the Dirac \cite{23,24} and Staunton \cite{40} sets of equations.

For the sake of clarity and self-contained presentation, we recall briefly the construction of representations of the algebra. The infinite-dimensional representations are built from the primitive vectors $|0\rangle$ and $|0\rangle^\ast$ (vacua), annihilated by $a^-$ and $\bar{a}^-$, respectively,

\[
\mathcal{R}_\nu = \{|n\rangle = \frac{(a^+)^n}{\sqrt{[n]_\nu!}}|0\rangle, \quad n \in \mathbb{N}\}, \quad \mathcal{R}_\nu^\ast = \{|\bar{n}\rangle = \frac{(\bar{a}^+)^n}{\sqrt{[\bar{n}]_\nu!}}|0\rangle, \quad n \in \mathbb{N}\},
\]

with $[n]_\nu! = [n]_\nu [n - 1]_\nu \cdots [1]_\nu$, $[0]_\nu! = 1$, $[n]_\nu = n + \frac{1}{2}(1 - (-1)^n)\nu$. The finite-dimensional representations of RDHA with $\nu = -(2k + 1)$, $k \in \mathbb{N}$, are built in the same vein \cite{2.2} but in this case there is another primitive vector $|2k\rangle$ annihilated by $a^+, a^+|2k\rangle = 0$ \cite{31}.

Let us consider the quadratic operators

\[
J_\pm = \frac{1}{2}(a^\pm)^2, \quad J_0 = \frac{1}{4}\{a^+, a^-\}, \quad \bar{J}_\pm = \frac{1}{2}(\bar{a}^\pm)^2, \quad \bar{J}_0 = \frac{1}{4}\{\bar{a}^+, \bar{a}^-\},
\]

\footnotetext[2]{More details on infinite component relativistic equations and corresponding $SL(2,\mathbb{C})$ representations can be found in Ref. \cite{41,42}.}

\footnotetext[3]{The necessary infinite-dimensional half-bounded representations of $sl(2,\mathbb{C})$ \cite{43,44} can be realized alternatively in terms of homogeneous monomials \cite{45} but the RDHA construction is more convenient for our purposes.}
forming the two copies of \( sl(2, \mathbb{R}) \) algebra,
\[
[J_0, J_{\pm}] = \pm J_{\pm}, \quad [J_-, J_+] = 2J_0,
\]
where \( J = J \) or \( \tilde{J} \), and \( [J, \tilde{J}] = 0 \). As a result, irreducible representations \( \mathcal{R}_\nu \) and \( \tilde{\mathcal{R}}_\nu \) are decomposed into the direct sums of the irreducible representations of \( sl(2, \mathbb{R}) \) bounded from below as \([31]\)
\[
\mathcal{R}_\nu = D_{+\mu}^{+} \oplus D_{-\mu}^{+}, \quad \tilde{\mathcal{R}}_\nu = \tilde{D}^{+\mu}_{+\mu} \oplus \tilde{D}^{+\mu}_{-\mu}, \tag{2.3}
\]
where \( j_0 = \kappa + l, \ l = 0, 1, \ldots \), are the eigenvalues of \( J_0 \) with \( \kappa = \frac{1+\nu}{4} \) and \( \kappa = \frac{3+\nu}{4} \), respectively. The “left”- and “right”-handed parts of the generators of the \( (3+1)D \) Lorentz algebra \( sl(2, \mathbb{C}) \), \( K_i \) and \( \bar{K}_i \), \( i = 1, 2, 3 \), obeying the relations
\[
[K_i, K_j] = i\epsilon_{ijk}K_k, \quad [K, \bar{K}] = 0,
\]
with \( K = K (\bar{K}) \), are defined in terms of the \( so(3, 1) \) generators \( J_{\mu\nu} \) as follows:
\[
K_i = \frac{1}{2}\epsilon_{ijk}J_{jk} + iJ_{0i}, \quad \bar{K}_i = \frac{1}{2}\epsilon_{ijk}J_{jk} - iJ_{0i}.
\]
Then the \( sl(2, \mathbb{C}) \) generators \( K_i \) can be identified with the \( sl(2, \mathbb{R}) \) generators \( J_0, J_{\pm} = J_1 \pm iJ_2 \) by means of the relations
\[
J_0 = -K_2, \quad J_1 = -iK_1, \quad J_2 = -iK_3. \tag{2.4}
\]
Such identification corresponds to the concrete choice of the \( \gamma \)-matrices in covariant spinor formalism (see below). Any other possible identifications between \( J \) and \( K \), e.g., those obtained from \((2.4)\) by a cyclic permutations of \( K_i \), lead to other realizations of the \( \gamma \)-matrices related by unitary transformations.

Note here that the Fock spaces of the usual oscillators corresponding to \( \nu = 0 \) are decomposed into the spin-1/4 and spin-3/4 representations of \( sl(2, \mathbb{R}) \) \([44]\). Another realization of \( sl(2, \mathbb{R}) \) generators, \( J_\pm = \frac{1}{2}(a^\pm)^2 \), \( J_0 = -\frac{1}{4}\{a^+, a^-\} \), results in the direct sum of bounded from above infinite-dimensional representations, \( \mathcal{R}_\nu = D_{1+\mu}^{\pm} \oplus D_{3+\mu}^{\pm} \).

Thus, we have generalized the Schwinger construction of \( su(2) \) to the case of the \( sl(2, \mathbb{C}) \) algebra.

### 2.2 Covariant spinor formalism

Introducing \( sl(2, \mathbb{C}) \) notations of dotted and undotted indices for two-dimensional spinors, all can be rewritten in covariant notations. The spinor conventions to raise/lower indices are as follow \([47]\):
\[
\psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta, \quad \psi_\alpha^* = \varepsilon^{\alpha\beta}\psi_\beta, \quad \bar{\psi}_\dot{\alpha} = \varepsilon_{\dot{\alpha}\dot{\beta}}\bar{\psi}^{\dot{\beta}}, \quad \bar{\psi}_\dot{\alpha}^* = \varepsilon^{\dot{\alpha}\dot{\beta}}\bar{\psi}_{\dot{\beta}} \quad \text{with} \quad (\psi_\alpha)^* = \bar{\psi}_{\dot{\alpha}},
\]
\( \varepsilon_{12} = \varepsilon_{\dot{1}\dot{2}} = -1, \varepsilon^{12} = \varepsilon^{\dot{1}\dot{2}} = 1 \). Defining the two spinor operators \( L_\alpha \) and \( \bar{L}_{\dot{\alpha}} \),
\[
L_1 = \frac{1}{\sqrt{2}}(a^+ + a^-), \quad L_2 = \frac{i}{\sqrt{2}}(a^+ - a^-), \quad \bar{L}_{\dot{1}} = \frac{1}{\sqrt{2}}(\pi^+ + \pi^-), \quad \bar{L}_{\dot{2}} = \frac{i}{\sqrt{2}}(\pi^+ - \pi^-), \tag{2.5}
\]
a direct calculation shows that they generate the $osp(4|1)$ superalgebra. Its bosonic part is

$$sp(4, \mathbb{R}) \sim so(3, 2) = AdS_4$$

with generators

$$L_{\alpha \beta} = \frac{1}{4} \{L_{\alpha}, L_{\beta}\}, \quad L_{\dot{\alpha} \dot{\beta}} = \frac{1}{4} \{L_{\dot{\alpha}}, L_{\dot{\beta}}\}, \quad M_{\alpha \dot{\alpha}} = \frac{1}{4} \{L_{\alpha}, L_{\dot{\alpha}}\},$$

but we shall be interested only in its $so(3, 1)$ part. Let us introduce the 4D Dirac matrices

$$\gamma_{\mu} = \begin{pmatrix} 0 & \sigma_{\mu} \\ \bar{\sigma}_{\mu} & 0 \end{pmatrix},$$

with

$$\sigma_{\mu \alpha \dot{\alpha}} = \begin{pmatrix} 1 & \sigma_i \\ -\sigma_i & 1 \end{pmatrix}, \quad \bar{\sigma}_{\mu \dot{\alpha} \dot{\alpha}} = \begin{pmatrix} 1 & -\sigma_i \\ \sigma_i & 1 \end{pmatrix}.$$

Then the $so(3, 1)$ spinor generators

$$\gamma_{\mu \nu} = \frac{i}{4} [\gamma_{\mu}, \gamma_{\nu}] = \text{diag} \left( i \sigma_{\mu \nu \alpha \beta}, i \bar{\sigma}_{\mu \nu \dot{\alpha} \dot{\beta}} \right)$$

allow us to present the $so(3, 1)$ generators in the form

$$J_{\mu \nu} = \frac{1}{2} \left( L_{\alpha} \sigma_{\mu \nu \alpha \beta} L_{\beta} - L_{\dot{\alpha}} \bar{\sigma}_{\mu \nu \dot{\alpha} \dot{\beta}} \bar{L}_{\dot{\beta}} \right).$$

The Pauli-Lubanski pseudo-vector is given by

$$W_{\mu} = \frac{1}{2} \varepsilon^{\mu \rho \sigma} P_{\nu} J_{\rho \sigma}$$

with $P_{\mu}$ being the generator of space-time translations.

### 2.3 Invariant scalar product

For any physically admissible representation of $ISO(3, 1)$, the generators of the Lorentz group should be Hermitian. This means that for any such a representation we have to construct an invariant scalar product with the necessary properties. In what follows we will discuss mainly the representation $D_{\lambda}^+ \oplus \bar{D}_{\lambda}^+$. Therefore, we consider in detail the construction of the invariant scalar product for this representation only. Though for free massless states the left- and right-handed sectors are uncoupled, the both are needed for the construction of the invariant scalar product [48]. We consider the vectors living on the reducible representation space, $\Psi \in \mathcal{R}_{\nu} \oplus \bar{\mathcal{R}}_{\nu}$, i.e. $\Psi = |\psi\rangle + |\bar{\chi}\rangle$ with $|\psi\rangle \in \mathcal{R}_{\nu}$ and $|\bar{\chi}\rangle \in \bar{\mathcal{R}}_{\nu}$.

The representations of (2.1) possess the natural involution

$$(a^+) = a^+, \quad |0\rangle = \langle 0|, \quad (\bar{a}^+) = \bar{a}^+, \quad |\bar{0}\rangle = \langle \bar{0}|.$$ 

This involution is not a covariant operation since it does not mix the left- and right-handed sectors. As a consequence, the state $\langle \psi^* | = |\psi\rangle^+$ is not contravariant while the original state $|\psi\rangle$ is a covariant vector of the representation space. The Lorentz generators are not
Hermitian with respect to such an involution. In order to obtain the covariant (Hermitian) conjugation, we introduce the intertwining operator $\Upsilon$ which permutes the left- and right-handed sectors,

$$\Upsilon a^\pm = \bar{a}^\pm \Upsilon, \quad \Upsilon R = \bar{R} \Upsilon, \quad \Upsilon |0\rangle = |\bar{0}\rangle, \quad \Upsilon |\bar{0}\rangle = |0\rangle, \quad \Upsilon^2 = 1.$$  

For the finite-dimensional representation $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$, the matrix elements of this operator correspond to the usual $\gamma^0$-matrix [18]. In general this operator can be represented as

$$\Upsilon = \sum_n (|\bar{n}\rangle \langle n| + |n\rangle \langle \bar{n}|).$$  

The state $\langle \bar{\psi} | \equiv \langle \psi^* | \Upsilon$ is a contravariant vector of the representation. Therefore, the $sl(2, \mathbb{C})$ invariant scalar product is $\Psi^\dagger \Psi$, where the Dirac-like conjugation is defined by $\Psi^\dagger = \Psi^+ \Upsilon$. One can verify that the Lorentz operators are Hermitian with respect to this scalar product,

$$J^\dagger_{\mu\nu} = \Upsilon J^+_{\mu\nu} \Upsilon = J_{\mu\nu}.$$  

For this conjugation we also have $(L_\alpha)^\dagger = \bar{L}_\dot{\alpha}$ and $R^\dagger = \bar{R}$.

The quantum mechanical (positively defined) probability density $\langle \psi^* | \psi \rangle = \langle \bar{\psi} | \Upsilon | \psi \rangle$ (the left-handed part here) is not a covariant object. For example, for the above mentioned finite-dimensional representation it corresponds to the zero component of the vector current: $\bar{\psi} \gamma^0 \psi$, where $\psi$ is the usual Dirac spinor.

3 Relativistic spinor equations for massless states

In this section we propose and investigate relativistic spinor equations of covariant form based on the representation of $sl(2, \mathbb{C})$ realised in terms of RDHA. It turns out that from the algebraic point of view they correspond to massless states of arbitrary real helicity. For the finite-dimensional representations of RDHA the equations universally describe states with any integer and half-integer helicity. However, the application of the infinite-dimensional representations of RDHA inevitably leads to a Lorentz symmetry breaking.

3.1 Algebraic aspect of equations

Using definitions and conventions of the previous section, let us introduce the fields $|\psi\rangle \in \mathcal{R}_\nu$, $|\bar{\psi}\rangle \in \bar{\mathcal{R}}_\nu$. By analogy with the $(2+1)D$ case [18, 19] we postulate the relativistic wave equations

$$P^\mu \tilde{\sigma}_\mu^\dagger \alpha \bar{L}_\dot{\alpha} |\psi\rangle = 0, \quad P^\mu \sigma_{\mu \alpha \dot{\alpha}} \tilde{L}^\dot{\alpha} |\bar{\psi}\rangle = 0. \quad (3.1)$$

Let us analyse the physical content of the equations (3.1) from the algebraic point of view. Using the identity

$$\left( P^\mu \tilde{\sigma}_{\mu}^\dagger \alpha L_\alpha \right) \left( P^\nu \sigma_{\nu}^\dagger \beta L_\beta \right) \varepsilon_{\dot{\alpha} \dot{\beta}} = iP^\mu P_\mu (1 + \nu R), \quad (3.2)$$

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and similar one for the right-handed part, we arrive at the equations

\[ P^\mu P_\mu |\psi\rangle = 0, \quad P^\mu P_\mu |\bar{\psi}\rangle = 0, \quad (3.3) \]

and, as a consequence, the proposed relativistic equations describe relativistic massless fields. Moreover, it is easy to see that the solutions to the equations \((3.1)\) obey also the relations

\[ (W^\mu + \frac{1+i\nu}{4} P^\mu) |\psi\rangle = 0, \quad (W^\mu - \frac{1+i\nu}{4} P^\mu) |\bar{\psi}\rangle = 0. \quad (3.4) \]

This means that they have the fixed helicity \(\lambda = -\frac{1+i\nu}{4}\) for the left-handed sector and \(\lambda = \frac{1+i\nu}{4}\) for the right-handed one. So, from the viewpoint of the Poincaré algebra the equations \((3.1)\) describe massless states with arbitrary helicity.

In what follows we mainly concentrate on the left-handed part since all the corresponding results for the right-handed sector can be reproduced straightforwardly. In components the first equations in \((3.1)\) read

\[ \left( a^+ (P^0 + iP_1 - P_2 + P_3) - a^- (P^0 - iP_1 + P_2 + P_3) \right) |\psi\rangle = 0, \]
\[ \left( a^+ (P^0 - iP_1 - P_2 - P_3) + a^- (P^0 + iP_1 + P_2 - P_3) \right) |\psi\rangle = 0. \quad (3.5) \]

Taking the sum and the difference of these equations, we get

\[ \left( a^+ (P^0 - P_2) + a^- (iP_1 - P_3) \right) |\psi\rangle = 0, \]
\[ \left( a^- (P^0 + P_2) - a^+ (iP_1 + P_3) \right) |\psi\rangle = 0. \quad (3.6) \]

### 3.2 Finite-dimensional representations: integer and half-integer helicities

Let us first consider a solution to the equations \((3.3)\) in the case of the finite-dimensional representations of RDHA \([31]\) with \(\nu = -(2k + 1), \; k \in \mathbb{N}\). As was noted above, the finite-dimensional representations are built similarly to \((2.2)\) and are characterised by the additional primitive vector \(|2k\rangle\) annihilated by \(a^+, \; a^+|2k\rangle = 0\). Such representations of RDHA are also reducible with respect to the \(\text{sl}(2, \mathbb{R})\) algebra, \(\mathcal{R}_\nu = D_{\frac{\nu}{2}} \oplus D_{\frac{\nu}{2}+1}\).

It is worth noting that formally the solutions to the equations \((3.6)\) have the structure similar to that of the Weyl equation \(p^\mu \sigma_\mu \psi = 0\) written in components as

\[ (p^0 + p_3)\psi_1 + (p_1 - ip_2)\psi_2 = 0, \]
\[ (p_1 + ip_2)\psi_1 + (p^0 - p_3)\psi_2 = 0. \]

Indeed, the cone \(\Gamma = \{ p_\mu : p_\mu p^\mu = 0, \; p^0 \neq 0 \}\) can be covered by the two charts \(U_\pm = \{ p_\mu : p^0 \pm p_3 \neq 0 \}\). In each chart the solution can be represented in the regular form

\[ \psi(p) \big|_{U_+} = \left( \frac{\omega(p)}{1} \right) \varphi_+(p)\delta(p^2), \quad \psi(p) \big|_{U_-} = \left( \frac{1}{\tilde{\omega}(p)} \right) \varphi_-(p)\delta(p^2), \quad (3.7) \]

where the functions

\[ \omega(p) = \frac{ip_2 - p_1}{p^0 + p_3}, \quad \tilde{\omega}(p) = \frac{p_1 + ip_2}{p_3 - p^0}. \]
obey on the cone the identity $\omega(p)\tilde{\omega}(p) = 1$. On the intersection $U_+ \cap U_-$, the functions $\varphi_\pm$ are related as $\varphi_-(p) = \omega(p)\varphi_+(p)$.

The solution to the equations (3.6) can be considered in the same way but with the covering charts $U_+ = \{ p_\mu : p^0 \pm p_2 \neq 0 \}$. The peculiarity of the $p_2$-direction is associated with the chosen representation of the $\sigma$-matrices and can be changed to any other direction by a unitary transformation. The solution to the first equation from (3.1) is

$$ \left| \psi \right|_{U_+} = \delta(p^2)\varphi_+(p) \sum_{n=0}^{k} C_n \Omega^n(p) |2n\rangle, \quad \left| \psi \right|_{U_-} = \delta(p^2)\varphi_-(p) \sum_{n=0}^{k} C_n \tilde{\Omega}^{k-n}(p) |2n\rangle, $$

with the functions $\varphi_\pm$ related on $U_+ \cap U_-$ as $\varphi_-(p) = \Omega^k(p)\varphi_+(p)$. Here $\varphi_\pm$ are the functions regular on $U_+$, and

$$ \Omega(p) = \frac{p_3 + ip_1}{p^0 + p_2}, \quad \tilde{\Omega}(p) = \frac{p_3 - ip_1}{p^0 - p_2}, \quad C_n = \frac{\sqrt{[2n]_\nu}!}{2^n n!}, $$

with the identity $\Omega(p)\tilde{\Omega}(p) = 1$ to be valid on the cone. From the explicit form of the solution one can see that the equations of motion (3.1) contain effectively the projector on the even invariant subspace $D_\frac{3}{2}$ (or $D_\frac{1}{2}$ for the right-handed sector). Here we imply that the parity is defined with respect to the action of the operator $R$ ($\tilde{R}$). The obtained solution describes a free left-handed massless particle with helicity $\frac{1}{2}$. For example, in the case of helicity $\frac{1}{2}$ the corresponding solution is given by

$$ \left| \lambda = \frac{1}{2} \right|_{U_+} = \delta(p^2)\varphi_+(p)(|0\rangle + \Omega(p)|2\rangle), \quad \left| \lambda = \frac{1}{2} \right|_{U_-} = \delta(p^2)\varphi_-(p)(|0\rangle + \tilde{\Omega}(p)|2\rangle). $$

This solution is in the exact correspondence with the solution (3.7) to the (unitarily transformed) Weyl equation. The solution to the second equation from (3.1) can be considered analogously.

Thus, in the case of the finite-dimensional representation, the equations (3.1) provide a consistent universal description of the free states with arbitrary integer and half-integer helicities. The equations have the same form for all such helicities and the information about the values of $\lambda$ is encoded in the parameter $\nu$.

### 3.3 Infinite-dimensional representations: Lorentz symmetry breaking

The equations (3.1) formally have covariant form and, as we have seen, give the consistent description of the massless finite-dimensional representations of the Poincaré group. But the situation for the infinite-dimensional representations turns out to be essentially different. A simple analysis of equations (3.1) reveals some contradictory properties of the solutions: they exist in some frames and do not exist in others. Indeed, the solution in the frame where $p^\mu = (E, 0, E, 0)$ is given by $\psi \propto |0\rangle$, with $|0\rangle$ the vacuum state of the RDHA. However, no normalized solutions can be found in the frames where $p^\mu = (E, 0, -E, 0)$, $p^\mu = (E, \pm E, 0, 0)$ or $p^\mu = (E, 0, 0, \pm E)$. This means that the solutions are not invariant under some transformations of the Lorentz group, and so, the Lorentz invariance is broken. Note,
that if another choice of the Dirac matrices would have been done, the Lorentz invariance
would be broken in other directions. We would like to note that this situation with the
Lorentz breaking has a formal analogy with the usual spontaneous breaking of a global
symmetry in field theory models, where the equations of motions are covariant with respect to
the corresponding symmetry group and the breaking occurs on the level of vacuum solutions.

To make the statement on the breaking of the Lorentz invariance more precise, we observe
that for the covering \( \{ U_+, U_- \} \) of the cone the formal solution to the first equation from (3.1)
exists on \( U_+ \) only,

\[
|\psi\rangle = \delta(p^2)\varphi(p) \sum_{n=0}^{\infty} C_n \Omega^n(p)|2n\rangle = \delta(p^2)\varphi(p) \exp \left( \frac{1}{2} \Omega(p)(a^+)^2 \right) |0\rangle,
\]

where \( \varphi(p) \) is a regular function, and \( C_n \) and \( \Omega(p) \) were defined in (3.8). Some care has to
be taken with infinite-dimensional representations since, generally, an infinite-dimensional
representation of the Poincaré algebra is not obligatory a representation of the Poincaré
group. Therefore, the solution (3.9) is proper one if its norm with respect to the internal
space scalar product,

\[
\langle \psi^* | \psi \rangle = \sum_{n=0}^{\infty} C_n^2 |\Omega(p)|^{2n},
\]

is finite. The radius of convergence of the series is equal to

\[
\lim_{n \to \infty} \frac{C_{n+1}}{C_n} = 1.
\]

Therefore, we arrive at the strict inequality \(|\Omega(p)|^2 < 1\), which can be rewritten as

\[
p_2 \left( p^0 + p_2 \right) > 0.
\]

(3.10)

All the formulas corresponding to the right-handed sector can be reproduced via the formal
substitution \( a^\pm \to \bar{a}^\pm, p_2 \to -p_2 \). Therefore, the corresponding convergence condition for
that sector is

\[
p_2 \left( p^0 - p_2 \right) < 0.
\]

(3.11)

Consequently, although the equations (3.1) look like \( SL(2, \mathbb{C}) \)-invariant equations, the
maximal invariance group of the solution is the one preserving the inequality (3.10) (or (3.11)
for the right-handed sector). This means that the equations are invariant under \( SL(2, \mathbb{R}) \)
subgroup of \( SL(2, \mathbb{C}) \) (the group generated by the rotations in the plane (1-3) and by the
boosts in the directions 1 and 3, which do not violate the relations (3.10), (3.11)). This is a
direct consequence of the fact that the used infinite-dimensional representation is ill defined
at the level of the Poincaré group, even if it is perfectly defined at the level of the Lie algebra.

The solution (3.9) illustrates an unusual type of Lorentz symmetry breaking. The mass-
less equations are formally covariant but in the case of infinite-dimensional representations
the Lorentz invariance is strongly broken on the level of the solutions. Formally, this breaking
is associated not with the infinite-dimensional representation of the \( SL(2, \mathbb{C}) \) group itself but
rather with the attempt to enclose it into the corresponding representation of the Poincaré group (cf. with massive case, see Appendix). Indeed, as follows from the inequalities \((3.10)\), \((3.11)\), the Lorentz violation is provoked by the transformations of the momentum, which is naturally associated with the Poincaré group.

Through the reduction \(sl(2, \mathbb{C}) \rightarrow sl(2, \mathbb{R})\), the infinite-dimensional representations \(R_\nu\) and \(\bar{R}_\nu\) are identified. In this case the representation \(D^\pm_\lambda\) can be exponentiated to a representation of the Lie group \(ISO(2, 1)\). So, the Poincaré invariance in \((3+1)D\), \(ISO(3, 1)\), is broken to \(ISO(2, 1)\), the Poincaré invariance in \((2+1)D\). As we shall see in the next section, this means that the dimensional reduction to the direction \(p_2\) gives rise to a consistent \((2+1)D\) theory describing a massive anyon of a mass \(m\) and spin \(\lambda = \pm \frac{1+\nu}{4}\).

One has to stress once again that for the case of the finite-dimensional representations, when \(2\lambda\) is an integer number, the problem encountered with infinite-dimensional representations is not present and, as we have seen, the consistent equations for the helicity \(\pm \lambda\) fields are obtained.

4 Compactification and reduction to \((2+1)D\) anyons

We have seen that in the case of infinite-dimensional representations the theory provided by the equations \((3.1)\) does not have a nontrivial content on the whole \((3+1)D\) Minkowski space \(M^4\) since the formal solution \((3.3)\) breaks the Lorentz invariance. But let us show that, in a sense, this problem can be “cured” by compactifying the singled out (in this case \(x_2\)) direction on a circle, \(M^4 \rightarrow M^3 \times S^1\).

Fixing the space geometry in the form of the three-dimensional Minkowski space times a circle of radius \(m^{-1}\) and denoting the compactified coordinate as \(\theta\), \(0 \leq \theta \leq 2\pi\), the state \(|\psi\rangle\) in coordinate representation can be expanded as

\[
|\psi\rangle = \sum_{n=-\infty}^{\infty} e^{in\theta} |\psi_n\rangle, \quad \text{with} \quad |\psi_n\rangle = \sum_{k=0}^{\infty} \psi_{nk}(y^a) |k\rangle
\]

depending only on the 3-dimensional coordinates \(y^a\), \(a = 0, 1, 2\), with \(y^0 = x^0\), \(y^1 = x^1\) and \(y^2 = x^3\).

So, starting with a \((3+1)D\) massless momentum \(P^a\), it is reduced to \((P^0, P^1, nm, P^3)\) for each state \(|\psi_n\rangle\). Denoting the three-dimensional momentum by \(p^a = (P^0, P^1, P^3)\), the mass-shell condition in \((2+1)D\) reads

\[
(p^a p_a + n^2 m^2) |\psi_n\rangle = 0,
\]

i.e. the initial massless system is formally reduced to an infinite tower of massive states plus a massless one. For \(sl(2, \mathbb{R})\) the left- and right-handed representations are equivalent, and so, the dotted and undotted indices are identical. The both equations in \((3.1)\), through the compactification process, lead to the equations

\[
(p^a (\gamma_\alpha)_\alpha^\beta + nm \delta^\alpha_\beta) L_\beta |\psi_n\rangle = 0.
\]

These equations can be obtained from \((3.1)\) by the formal substitution \(P_2 \rightarrow nm\) and multiplication by \(\sigma_2\) that, in fact, corresponds to lowering the free index. The \(\gamma\)-matrices are
given in the Majorana representation:

\[(\gamma_0)_{\alpha}^{\beta} = - (\sigma_2)_{\alpha}^{\beta}, \quad (\gamma_1)_{\alpha}^{\beta} = -i (\sigma_3)_{\alpha}^{\beta}, \quad (\gamma_2)_{\alpha}^{\beta} = i (\sigma_1)_{\alpha}^{\beta}.\]

Up to the spatial rotation \(y_1 \to -y_2, y_2 \to y_1\), the equations (1.1) actually coincide with the equations for anyons [18, 19] and, hence, for \(n \neq 0\) they consistently describe relativistic \((2+1)D\) anyons of the spin \(\lambda = -\text{sign}(n)\frac{1+n}{4}\). By analogy with (3.9), the formal solutions to these equations in the momentum representation are given by

\[|\psi_n\rangle = \delta(p^2 + n^2m^2)\varphi_n(p) \exp\left(\frac{1}{2}\Omega_n(p)(a^+)^2\right)|0\rangle,\]  \hfill (4.2)

with \(\Omega_n(p) = \frac{px + ip\nu}{p^0 + nm}\). The condition of normalizability (3.10) is transformed into the inequality

\[n(p^0 + nm) > 0.\]  \hfill (4.3)

This means that the solutions (4.2) are normalizable only in the case of energy with definite sign: \(\text{sign}(p^0) = \text{sign}(n)\). In other words, all the states \(|\psi_n\rangle\) with \(n > 0\) have the positive energy, those with \(n < 0\) have negative energy while the state \(|\psi_0\rangle\) does not belong to the spectrum of the theory. Such properties reveal the difference of the considered compactification from the usual procedure.

Let us discuss this issue from viewpoint of the symmetries usually associated with such a compactification on a circle. Following Ref. [19], one can infer that in the usual case an affine Kac-Moody algebra \(g\) is always associated with a compactified \(M^3 \times S^1\) theory. This infinite-dimensional algebra consists of the loop extension of \(iso(2,1)\),

\[\widetilde{iso}(2,1) = \{P^a_n = e^{i\theta}P^a, J^a_n = e^{i\theta}J^a\},\]

where \(P^a\) and \(J^a\) are the translation and Lorentz generators of \(iso(2,1)\), respectively, and of the additional set of operators \(Q_n = ie^{i\theta}\partial_\theta\) with the commutation relations

\[[Q_n, Q_m] = (n - m)Q_{n+m}, \quad [Q_n, P^a_m] = -mP^a_{n+m}, \quad [Q_n, J^a_m] = -mJ^a_{n+m}.\]

The affine Kac-Moody algebra \(g\) has the following natural triangular decomposition

\[g = g_+ \oplus g_0 \oplus g_-.,\]

where the subalgebras \(g_+\) and \(g_-\) are the sets of all the operators with \(n > 0\) and \(n < 0\), respectively, while \(g_0 = iso(2,1) \oplus u(1)\) with \(Q_0\) being the generator of the \(u(1)\) algebra.

In our case the symmetry corresponding to the affine algebra \(g\) is always partially broken. Indeed, the action of \(g_-\) is ill defined on the part of the spectrum with positive energy \((n > 0)\) while \(g_+\) is ill defined on the part of the spectrum with negative energy \((n < 0)\). Hence, the symmetry is always broken to \(g_+ \oplus g_0\) or to \(g_0 \oplus g_-\). The only symmetry of the whole spectrum is \(g_0\). Nevertheless, one can conclude that in spite of the partial breaking of the infinite-parametric symmetry, the compactified theory has the non-trivial content.

The considered compactification can be treated, in principle, as that “induced” by the breaking of the Lorentz invariance of the equations (3.1) for the infinite-dimensional representations case. The “induction” is understood here in the sense that the compactified theory,
unlike the initial one, is consistent and the direction of the compactification is defined by the Lorentz breaking.

The dimensional reduction emerges after choosing one level, say \( n = 1 \) or \( n = -1 \), and discarding all the others. Evidently, the reduction gives rise to a consistent \((2+1)D\) theory describing a massive anyon of a mass \( m \) and spin \( \lambda = \pm \frac{1+\nu}{4} \). Correspondingly, the formal \( ISO(3,1) \) invariance is reduced to the true \( ISO(2,1) \) invariance.

The equations (4.1) providing solutions with fixed sign energy have a close analogy with the massive Dirac \((3+1)D\) positive energy spinor equations [23, 24] (see Appendix). The construction of the positive energy massive Dirac equations is based, in fact, on the representation of the type \( \mathcal{R}_{\nu=0} \otimes \bar{\mathcal{R}}_{\nu=0} \), and in the massless limit they are reduced to the equations of the form (3.1) to be imposed on the same state. However, note that for \( p^0 > 0 \), the inequalities (3.10), (3.11) are incompatible. This means that there is no reference frame in which the normalizable solutions could exist in the left- and right-handed sectors simultaneously. Using this observation, one can assume that the Dirac equations [23, 24] have no proper solutions in the massless limit. In Appendix we demonstrate that this is indeed the case.

5 No-go theorem for massless infinite-dimensional representations of \( ISO(3,1) \)

We have seen that the proposed spinor sets of equations cannot be used to describe a massless field with fractional helicity in \((3+1)D\) because the maximal invariant group of the solution is broken down to the \((2+1)D\) Poincaré group. Moreover, the same problems arise under attempt to describe the massless field of integer or half-integer helicity by means of infinite-dimensional representations of \( sl(2,\mathbb{C}) \) associated with RDHA. So, we may wonder if this feature is general or it is specific to the equations (3.1) we have considered. In other words, formally we may address the general problem if fractional helicity states in \((3+1)D\) might exist. At this point, the natural question we should ask concerns the other type of equations that could be proposed. Following Ref. [40] and the results obtained in \((2+1)D\) case [17], a vector set of equations can be considered,

\[
(\alpha P^\mu + iJ^{\mu\nu} P_\nu)\Psi = 0,
\]

where \( \alpha \) is some constant that defines the helicity while the representation of \( \Psi \) is not fixed here. According to Refs. [18, 54], the equations of such a form describe all irreducible massless finite-dimensional representations of the Poincaré group. But we are going to consider these equations from the viewpoint of infinite-dimensional representations. In this sense the equations (5.1) are analogous to the massless limit of the equations proposed by Staunton in Ref. [10]. Contraction with \( P^\mu \) shows that we have a massless field. Then, choosing the

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4 The change of the sign before mass in the Dirac equations leads to solutions with negative energy.
frame where $P^\mu = E(1, 0, \epsilon, 0)$, we reduce the system of equations to the system

\[
\left(\alpha - \epsilon (J_0 - \bar{J}_0)\right) \Psi = 0,
\]
\[
(1 + \epsilon) (J_- + \bar{J}_+) + (1 - \epsilon) (J_+ + \bar{J}_-) \Psi = 0,
\]
\[
(1 + \epsilon) (J_- + \bar{J}_+) - (1 - \epsilon) (J_+ + \bar{J}_-) \Psi = 0.
\] (5.2)

For $\epsilon = +1$, the solution has to obey the relations

\[
J_- \Psi = J_+ \Psi = 0,
\] (5.3)

whereas for $\epsilon = -1$ they are changed for

\[
J_+ \Psi = \bar{J}_- \Psi = 0,
\] (5.4)

that is possible only for the finite-dimensional representations of $sl(2, \mathbb{C})$. In other words, we arrive at the same problems as before and Eqs. (5.1) are not consistent for the states carrying fractional helicity or, generally, with all the infinite-dimensional representations. On the other hand, one can show that the dimensional reduction of (5.1) leads to the (2+1)D vector set of equations that, as well as the spinor equations (4.1), consistently describes anyons [17].

On a general ground, any set of equations which can be proposed to describe a massless field of helicity $\lambda$ (fractional or not) has to give rise to the “helicity equation”

\[
\left(W^\mu - \lambda P^\mu\right) \Psi = 0,
\] (5.5)

with $W^\mu$ the Pauli-Lubanski vector [22]. The representation of $\Psi$ is also not specified providing the universality of the analysis.

The equations (5.5), like eqs. (5.1) and (5.1), are not compatible for infinite-dimensional representations. Indeed, e. g., in a frame where $P^\mu = E(1, 0, \epsilon, 0)$, the equations are simplified for

\[
\left(\lambda - \epsilon (J_0 + \bar{J}_0)\right) \Psi = 0,
\]
\[
(1 + \epsilon) (J_- + \bar{J}_+) + (1 - \epsilon) (J_+ + \bar{J}_-) \Psi = 0,
\]
\[
(1 + \epsilon) (J_- + \bar{J}_+) - (1 - \epsilon) (J_+ + \bar{J}_-) \Psi = 0.
\] (5.6)

One can see that these equations can be obtained from (5.2) by the substitution $\bar{J}_0 \to -\bar{J}_0$, $J_\pm \to -J_\pm$ and $\alpha \to \lambda$. The reason is that the equations (5.3) can be reproduced from (5.1) by the formal substitution $J^{\mu\nu} \to i \bar{J}^{\mu\nu}$, $\alpha \to \lambda$, where $\bar{J}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} J_{\rho\sigma}$. For $\epsilon = 1$ or $\epsilon = -1$ we arrive, correspondingly, at the equations (5.3) or (5.4). This means that the equations (5.5) are compatible for finite-dimensional representations (with integer or half-integer helicities) only. In other words, the (3+1)D Poincaré group has no massless irreducible representations of any (integer, half-integer or fractional) helicity with the trivial non-compact part of the little group constructed on the basis of infinite-dimensional representations of $sl(2, \mathbb{C})$. 

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6 Discussion and outlook

The specific properties of the (2+1)-dimensional space-time admit the existence of anyons. The little group of massive states in (2+1)\(D\) coincides with the compact part (which is the infinitely connected group \(SO(2)\)) of the little group of massless (3+1)\(D\) states. Consequently, the “charge” of its universal covering group \(\overline{SO(2)} = \mathbb{R}\) is not quantized. The group theory justification of anyons is related to the fact that the representations of the fractional charge describing relativistic anyons can be extended to representations of the whole (2+1)-dimensional Lorentz group which is also infinitely connected. For the purpose of describing the massless states with fractional \((\lambda \in \mathbb{R})\) helicity we have constructed representations of \(sl(2, \mathbb{C})\) in terms of two copies of the \(R\)-deformed Heisenberg algebra. In terms of such representations of the Poincaré algebra,

- The universal relativistic equations \((3.4)\) describing massless states of any integer and half-integer helicity have been proposed.

This possibility is related to the existence of the finite-dimensional representations of RDHA for \(\nu = - (2k + 1), \ k \in \mathbb{N}\). For \(\nu > -1\), the RDHA has infinite-dimensional unitary representations. In this case the equations \((3.1)\) formally describe the massless states with fractional helicity. Analyzing the solutions to the relativistic wave equations, we have traced out explicitly that the corresponding infinite-dimensional representations of the Lie algebra \(sl(2, \mathbb{C})\) cannot be exponentiated to representations of the Lie group \(SL(2, \mathbb{C})\) for such massless states. In other words, the (3+1)\(D\) Lorentz invariance is broken on the level of solutions. This resembles the Lorentz symmetry breaking in supersymmetric Yang-Mills theory considered in Ref. \([51]\), where the corresponding action is Lorentz invariant while the symmetry is broken at the field equation level. In spite of the Lorentz symmetry breaking we have shown that

- The dimensional reduction of the massless equations \((3.1)\) leads to the consistent (2+1)\(D\) theory of massive anyons with spin \(\lambda = \pm \frac{1+\nu}{4}\).

The dimensional reduction emerges from the compactification \(M^4 \to M^3 \times S^1\). We treat this compactification as that induced by the Lorentz symmetry breaking in the equations \((3.1)\) on the solution level. It would be interesting to find an example of such a compactification (induced by the Lorentz symmetry breaking on solution level) in other theories. We hope that the observed unusual symmetry breaking could be helpful in the context of the considerable activity looking for different mechanisms of the Lorentz symmetry violation \([51, 52, 53, 54, 55, 56, 57, 58]\) and compactification \([59, 60, 61, 62]\).

Concluding our investigation on existence of massless states with fractional helicity in the 4-dimensional space, we have analysed the fundamental equation \((5.5)\) defining massless irreducible representations of \(ISO(3,1)\), and have shown that

- The Poincaré group \(ISO(3,1)\) has no massless irreducible representations with the trivial non-compact part of the little group constructed on the basis of the infinite-dimensional representations of \(sl(2, \mathbb{C})\).

This means, in particular, that the massless (3+1)-dimensional fractional helicity states cannot be described in a consistent way and that the integer and half-integer helicity massless
particles can be described only in terms of finite-dimensional representations of \( \text{sl}(2, \mathbb{C}) \). It is worth also emphasizing that the obtained restriction on values of helicity is not of a topological origin. The topological arguments \([28, 29]\) do not forbid irreducible representations with integer and half-integer helicity constructed on the basis of the infinite-dimensional representations of \( \text{sl}(2, \mathbb{C}) \). Therefore, the topology provides only the necessary condition for massless representations to be the true representations of \( \text{ISO}(3, 1) \), which, as we see, is not the sufficient condition. This is quite an unexpected result since the massive irreducible representations of \( \text{ISO}(3, 1) \) with integer and half-integer spin can be constructed on the basis of the infinite-dimensional representations of \( \text{sl}(2, \mathbb{C}) \) \([23, 24, 40]\).

The situation with the equations (3.1) for the infinite-dimensional representation can be compared with the result obtained earlier in Ref. \([63]\) in a different context. Here the proposed equations look invariant under the \((3+1)D\) Poincaré transformations but relativistic invariance is broken at the level of the solutions. In the paper \([63]\) two of us have obtained similar results, which were related, however, to the fact that the considered there Lorentz automorphism of the underlying algebra is outer and not obligatory inner. Only when the outer automorphism becomes inner, a covariant equation can be obtained. All this means, in particular, that it is not enough to have the equation (or the set of equations) which looks invariant (or covariant) to obtain a covariant theory.

Acknowledgements

MP thanks L. Alvarez-Gaume, A. I. Oksak, V. I. Tkach, G. G. Volkov and A. A. Zheltukhin for early communications stimulated the present research and acknowledges the useful communications with L. N. Lipatov and L. Ryder. MRT acknowledges gratefully the useful discussions with G. Mennessier and M. J. Slupinski. One of us (MRT) would like to thank USACH for its hospitality, where the part of this work was realized. The work was supported in part by the grants 1980619, 7980044, 1010073 and 3000006 from FONDECYT (Chile) and by DICYT (USACH).

A Dirac positive-energy relativistic equations

The Dirac equations \([23, 24]\) corresponding to massive spinless particle with positive energy can be represented as

\[
(P^\mu \tilde{\gamma}_\mu - m \mathbb{1}) Q \Psi = 0, \quad \text{with} \quad Q = \begin{pmatrix} q_1 \\ q_2 \\ \eta_1 \\ \eta_2 \end{pmatrix},
\]

where \(q_1\) and \(q_2\) are commuting dynamical quantities and \(\eta_1\) and \(\eta_2\) denote the corresponding conjugate momenta, \([q_i, \eta_j] = i \delta_{ij}, i, j = 1, 2\). The matrices \(\tilde{\gamma}_\mu\) are related to those in the
Weyl representation (2.7) by the unitary transformation $\gamma_\mu = U\tilde{\gamma}_\mu U^\dagger$ with the matrix

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & 0 & 1 & i \\
1 & -i & 0 & 0 \\
i & -1 & 0 & 0 \\
0 & 0 & -i & -1
\end{pmatrix}.$$

Dirac showed that the equations (A.1) consistently describe massive spin-0 positive-energy states [23, 24]. In the “coordinate” representation, with the operators $q_i$ to be diagonal, the solution to the equations (A.1) has the form [23, 24]

$$\Psi \propto \delta(p^2 + m^2) \exp \left(-\frac{1}{2(p^0 - p_3)}(m(q_1^2 + q_2^2) - ip_1(q_1^2 - q_2^2) + 2ip_2q_1q_2) \right).$$  \hspace{1cm} (A.2)

One can verify that this solution is normalizable in “internal” variables for $m \neq 0$ only. Moreover, in the massless limit the solution (A.2) is singular on the cone $p^2 = 0$ due to the presence of the factor $(p^0 - p_3)^{-1}$. This means that the equations (A.1) have no proper solution in the massless limit.

It is worth noting that the formal change $m \rightarrow -m$ leads to the normalized solutions with negative energy.

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