On $(p, q)$-Appell Polynomials

P. Njionou Sadjang*†

December 6, 2017

Abstract

We introduce polynomial sets of $(p, q)$-Appell type and give some of their characterizations. The algebraic properties of the set of all polynomial sequences of $(p, q)$-Appell type are studied. Next, we give a recurrence relation and a $(p, q)$-difference equation for those polynomials. Finally, some examples of polynomial sequences of $(p, q)$-Appell type are given, particularly, a set of $(p, q)$-Hermite polynomials is given and their three-term recurrence relation and a second order homogeneous $(p, q)$-difference equation are provided.

1 Introduction

Let $P_n(x)$, $n = 0, 1, 2, \cdots$ be a polynomial set, i.e. a sequence of polynomials with $P_n(x)$ of exact degree $n$. Assume that $\frac{dP_n(x)}{dx} = P'_n(x) = nP_{n-1}(x)$ for $n = 0, 1, 2, \cdots$. Such polynomial sets are called Appell sets and received considerable attention since P. Appell \[2\] introduced them in 1880.

Let $q$ and $p$ be two arbitrary real or complex numbers and define the $(p, q)$-derivative \[8\] of a function $f(x)$ by means of

$$D_{p,q}f(x) = \frac{f(px) - f(qx)}{(p - q)x},$$

which furnishes a generalization of the so-called $q$-derivative (or Hahn derivative)

$$D_qf(x) = \frac{f(x) - f(qx)}{(1 - q)x},$$

which itself is a generalization of the classical differential operator $\frac{d}{dx}$.

The purpose of this paper is to study the class of polynomial sequences $\{P_n(x)\}$ which satisfy

$$D_qP_n(x) = [n]_{p,q}P_{n-1}(px), \quad n = 0, 1, 2, 3, \cdots$$

where $[n]_{p,q}$ is defined below. We note that when $p = 1$, \[3\] reduces to $D_qP_n(x) = [n]_qP_{n-1}(x)$ so that we may think of $(p, q)$-Appell sets as a generalization of $q$-Appell sets (see [11]).

2 Preliminary results and definitions

2.1 $(p, q)$-number, $(p, q)$-factorial, $(p, q)$-binomial coefficients, $(p, q)$-power.

Let us introduce the following notations (see [5,6,8])

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}, \quad 0 < q < p$$

*Faculty of Industrial Engineering, University of Douala, Cameroon
†pnjionou@yahoo.fr
for any positive integer \( n \in \mathbb{N} \). The twin-basic number is a natural generalization of the \( q \)-number, that is
\[
\lim_{p \to 1} [n]_{p,q} = [n]_{q}.
\]
The \((p,q)\)-factorial is defined by (see \([6,8]\))
\[
[n]_{p,q}! = \prod_{k=1}^{n} [k]_{p,q}!, \quad n \geq 1, \quad [0]_{p,q}! = 1.
\]
Let us introduce also the so-called \((p,q)\)-binomial coefficients
\[
\binom{n}{k}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}! \cdot [n-k]_{p,q}!}, \quad 0 \leq k \leq n.
\]
Note that as \( p \to 1 \), the \((p,q)\)-binomial coefficients reduce to the \( q\)-binomial coefficients.

It is clear by definition that
\[
\binom{n}{k}_{p,q} = \binom{n}{n-k}_{p,q}.
\]
Let us introduce also the so-called falling and raising \((p,q)\)-powers, respectively \([8]\)
\[
(x \ominus a)^n_{p,q} = (x - a)(px - aq) \cdots (xp^{n-1} - aq^{n-1}),
\]
\[
(x \oplus a)^n_{p,q} = (x + a)(px + aq) \cdots (xp^{n-1} + aq^{n-1}).
\]
These definitions are extended to
\[
(a \ominus b)^\infty_{p,q} = \sum_{k=0}^{\infty} (ap^k - q^k b),
\]
\[
(a \oplus b)^\infty_{p,q} = \sum_{k=0}^{\infty} (ap^k + q^k b),
\]
where convergence is required.

### 2.2 The \((p,q)\)-derivative and the \((p,q)\)-integral
Let \( f \) be a function defined on the set of the complex numbers.

**Definition 1** (See \([8]\)). The \((p,q)\)-derivative of the function \( f \) is defined as
\[
D_{p,q} f(x) = \frac{f(px) - f(qx)}{(p - q)x}, \quad x \neq 0,
\]
and \((D_{p,q} f)(0) = f'(0)\), provided that \( f \) is differentiable at 0.

**Proposition 2** (See \([8]\)). The \((p,q)\)-derivative fulfills the following product and quotient rules
\[
D_{p,q} (f(x)g(x)) = f(px)D_{p,q} g(x) + g(qx)D_{p,q} f(x),
\]
\[
D_{p,q} (f(x)g(x)) = g(px)D_{p,q} f(x) + f(qx)D_{p,q} g(x).
\]

### 2.3 \((p,q)\)-exponential and \((p,q)\)-trigonometric functions.
As in the \( q\)-case, there are many definitions of the \((p,q)\)-exponential function. The following two \((p,q)\)-analogues of the exponential function (see \([5]\)) will be frequently used throughout this paper:
\[
e_{p,q}(z) = \sum_{n=0}^{\infty} p^{(2)}_{n} \frac{z^n}{[n]_{p,q}!}, \quad (4)
\]
\[
E_{p,q}(z) = \sum_{n=0}^{\infty} q^{(2)}_{n} \frac{z^n}{[n]_{p,q}!}.
\]
Proposition 3. The following equation applies:
\[ e_{p,q}(x)E_{p,q}(-x) = 1. \] (6)

Proof. The result is proved in [5] using \((p, q)\)-hypergeometric series. We provide here a direct proof. From (4) and (5), and the general identity (which is a direct consequence of the Cauchy product)
\[
\left( \sum_{n=0}^{\infty} \frac{a_n}{[n]_{p,q}} \right) \left( \sum_{n=0}^{\infty} \frac{b_n}{[n]_{p,q}} \right) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \binom{n}{k}[k]_{p,q} a_k b_{n-k} \right) \frac{t^n}{[n]_{p,q}},
\] (7)
it follows that
\[
e_{p,q}(x)E_{p,q}(-x) = \left( \sum_{n=0}^{\infty} \frac{p^{(n)}}{[n]_{p,q}!} x^n \right) \left( \sum_{n=0}^{\infty} \frac{q^{(n)}}{[n]_{p,q}!} (-x)^n \right) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \binom{n}{k}[k]_{p,q} (-1)^k q^{(k)} p^{(n-k)} \right) \frac{x^n}{[n]_{p,q}!}.
\]

It remains to prove that
\[
\sum_{k=0}^{n} \binom{n}{k}[k]_{p,q} (-1)^k q^{(k)} p^{(n-k)} = \delta_{n,0}.
\]

It is not difficult to prove that for every polynomial \(f_n(x)\) of degree \(n\), the Taylor formula
\[
f_n(x) = \sum_{k=0}^{n} \frac{(D_{p,q}f)(0)}{[k]_{p,q}!} x^n,
\]
holds true. Applying this formula to \(f_n(x) = (a \odot x)^n_{p,q}\) it follows that
\[
(a \odot x)^n_{p,q} = \sum_{k=0}^{n} \binom{n}{k}[k]_{p,q} q^{(k)} p^{(n-k)} (-x)^k a^{n-k}.
\]

Taking finally \(x = a = 1\), the result follows. \(\square\)

The next proposition gives the \(n\)-th derivative of the \((p, q)\)-exponential functions.

Proposition 4. Let \(n\) be a nonnegative integer, \(\lambda\) a complex number, then the following equations hold
\[
D_{p,q}^{n} e_{p,q}(\lambda x) = \lambda^n p^{(n)}_{p,q} e_{p,q}(\lambda p^n x), \quad (8)
\]
\[
D_{p,q}^{n} E_{p,q}(\lambda x) = \lambda^n q^{(n)}_{p,q} E_{p,q}(\lambda q^n x). \quad (9)
\]

Proof. The proof follows by induction from the definitions of the \((p, q)\)-exponentials and the \((p, q)\)-derivative. \(\square\)

3 \((p, q)\)-Appell polynomials

Definition 5. A polynomial sequence \(\{f_n(x)\}_{n=0}^{\infty}\) is called a \((p, q)\)-Appell polynomial sequence if and only if
\[
D_{p,q} f_{n+1}(x) = [n + 1]_{p,q} f_n(px), \quad n \geq 0.
\] (10)

It is not difficult to see that the polynomial sequence \(\{f_n(x)\}_{n=0}^{\infty}\) with \(f_n(x) = (x \odot a)^n_{p,q}\) is a \((p, q)\)-Appell polynomial sequence since (see [8])
\[
D_{p,q} (x \odot a)^n_{p,q} = [n]_{p,q} (px \odot a)^{n-1}_{p,q}, \quad n \geq 1.
\]
Remark 6. Note that when \( p = 1 \), we obtain the classical \( q \)-Appell polynomial sequences known in the literature \([1] \). When \( q = 1 \), we obtain the new basic Appell polynomial sequences of type II introduced and extensively studied in \([10] \).

Next, we give several characterizations of \( (p, q) \)-Appell polynomial sequences.

Theorem 7. Let \( \{f_n(x)\}_{n=0}^\infty \) be a sequence of polynomials. Then the following are all equivalent:

1. \( \{f_n(x)\}_{n=0}^\infty \) is a \( (p, q) \)-Appell polynomial sequence.

2. There exists a sequence \( (a_k)_{k \geq 0} \), independent of \( n \), with \( a_0 \neq 0 \) and such that

\[
f_n(x) = \sum_{k=0}^{n} \binom{n}{k} p^{(\frac{n-k}{2})} a_k x^{n-k}.
\]

3. \( \{f_n(x)\}_{n=0}^\infty \) is generated by

\[
A(t)e_{p,q}(xt) = \sum_{n=0}^{\infty} f_n(x) \frac{t^n}{[n]_{p,q}!},
\]

with the determining function

\[
A(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{[n]_{p,q}!}.
\]

4. There exists a sequence \( (a_k)_{k \geq 0} \), independent of \( n \) with \( a_0 \neq 0 \) and such that

\[
f_n(x) = \left( \sum_{k=0}^{\infty} a_k p^{(\frac{n-k}{2})} \frac{x^k}{[k]_{p,q}!} D_{p,q}^k \right) x^n.
\]

Proof. First, we prove that \( (1) \implies (2) \implies (3) \implies (1) \).

\( (1) \implies (2) \). Since \( \{f_n(x)\}_{n=0}^\infty \) is a polynomial set, it is possible to write

\[
f_n(x) = \sum_{k=0}^{n} a_{n,k} \binom{n}{k} p^{(\frac{n-k}{2})} x^{n-k}, \quad n = 1, 2, \ldots,
\]

where the coefficients \( a_{n,k} \) depend on \( n \) and \( k \) and \( a_{n,0} \neq 0 \). We need to prove that these coefficients are independent of \( n \). By applying the operator \( D_{p,q} \) to each member of \( (12) \) and taking into account that \( \{f_n(x)\}_{n=0}^\infty \) is a \( (p, q) \)-Appell polynomial set, we obtain

\[
f_{n-1}(px) = \sum_{k=0}^{n-1} a_{n,k} \binom{n-1}{k} p^{(\frac{n-k-1}{2})} (px)^{n-1-k}, \quad n = 1, 2, \ldots,
\]

since \( D_{p,q} x^0 = 0 \). Shifting the index \( n \to n + 1 \) in \( (13) \) and making the substitution \( x \to xp^{-1} \), we get

\[
f_n(x) = \sum_{k=0}^{n} a_{n+1,k} \binom{n}{k} p^{(\frac{n-k}{2})} x^{n-k}, \quad n = 0, 1, \ldots,
\]

Comparing \( (12) \) and \( (14) \), we have \( a_{n+1,k} = a_{n,k} \) for all \( k \) and \( n \), and therefore \( a_{n,k} = a_k \) is independent of \( n \).
(2) \implies (3). From (2), and the identity (7), we have

\[ \sum_{n=0}^{\infty} f_n(x) \frac{t^n}{[n]_{p,q}!} = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \binom{n}{k} p^{(n-k)}_{p,q} a_k x^{n-k} \right) \frac{t^n}{[n]_{p,q}!} = \left( \sum_{n=0}^{\infty} a_n \frac{t^n}{[n]_{p,q}!} \right) \left( \sum_{n=0}^{\infty} [p]_{p,q}^{(n)} t^n \right) = A(t)e_{p,q}(xt). \]

(3) \implies (1). Assume that \( \{f_n(x)\}_{n=0}^{\infty} \) is generated by

\[ A(t)e_{p,q}(xt) = \sum_{n=0}^{\infty} f_n(x) \frac{t^n}{[n]_{p,q}!}. \]

Then, applying the operator \( D_{p,q} \) (with respect to the variable \( x \)) to each side of this equation, we get

\[ tA(t)e_{p,q}(pxt) = \sum_{n=0}^{\infty} D_{p,q}f_n(x) \frac{t^n}{[n]_{p,q}!}. \]

Moreover, we have

\[ tA(t)e_{p,q}(pxt) = \sum_{n=0}^{\infty} f_n(px) \frac{t^{n+1}}{[n]_{p,q}!} = \sum_{n=0}^{\infty} [n]_{p,q} f_{n-1}(px) \frac{t^n}{[n]_{p,q}!}. \]

By comparing the coefficients of \( t^n \), we obtain (1).

Next, (2) \iff (4) is obvious since \( D_{p,q}^{k} t^n = 0 \) for \( k > n \). This ends the proof of the theorem. \( \square \)

### 4 Algebraic structure

We denote a given polynomial set \( \{f_n(x)\}_{n=0}^{\infty} \) by a single symbol \( f \) and refer to \( f_n(x) \) as the \( n \)-th component of \( f \). We define (as done in [2] [12]) on the set \( P \) of all polynomials sequences the following three operations +, ., and *. The first one is given by the rule that \( f + g \) is the polynomial sequence whose \( n \)-th component is \( f_n(x) + g_n(x) \) provided that the degree of \( f_n(x) + g_n(x) \) is exactly \( n \). On the other hand, if \( f \) and \( g \) are two sets whose \( n \)-th components are, respectively,

\[ f_n(x) = \sum_{k=0}^{n} \alpha(n,k) x^k, \quad g_n(x) = \sum_{k=0}^{n} \beta(n,k) x^k, \]

then \( f * g \) is the polynomial set whose \( n \)-th component is given by

\[ (f * g)_n(x) = \sum_{k=0}^{n} \alpha(n,k) p^{-\binom{k}{2}} g_k(x). \]

If \( \lambda \) is a real or complex number, then \( \lambda f \) is defined as the polynomial sequence whose \( n \)-th component is \( \lambda f_n(x) \). We obviously have

\[ f + g = g + f \quad \text{for all} \quad f, g \in P, \]
\[ \lambda f * g = (f * \lambda g) = \lambda (f * g). \]

Clearly, the operation * is not commutative (see [12]). One commutative subclass is the set \( A \) of all Appell polynomials (see [2]).

In what follows, \( A(p,q) \) denotes the class of all \((p,q)\)-Appell sets.

In \( A(p,q) \) the identity element (with respect to *) is the \((p,q)\)-Appell set \( I = \{ p^{(2)} x^n \} \). Note that \( I \) has the determining function \( A(t) = 1 \). This is due to identity (4). The following theorem is easy to prove.
Theorem 8. Let $f, g, h \in \mathcal{A}(p, q)$ with the determining functions $A(t), B(t)$ and $C(t)$, respectively. Then

1. $f + g \in \mathcal{A}(p, q)$ if $A(0) + B(0) \neq 0$,
2. $f + g$ belongs to the determining function $A(t) + B(t)$,
3. $f + (g + h) = (f + g) + h$.

The next theorem is less obvious.

Theorem 9. If $f, g, h \in \mathcal{A}(p, q)$ with the determining functions $A(t), B(t)$ and $C(t)$, respectively, then

1. $f \ast g \in \mathcal{A}(p, q)$
2. $f \ast g = g \ast f$,
3. $f \ast g$ belongs to the determining function $A(t)B(t)$,
4. $f \ast (g \ast h) = (f \ast g) \ast h$.

Proof. It is enough to prove the first part of the theorem. The rest follows directly. According to Theorem 7 we may put

$$f_n(x) = \sum_{k=0}^{n} \binom{n}{k} p^{(x)}_{k,p,q} a_k x^{n-k} = \sum_{k=0}^{n} \binom{n}{k} p^{(x)}_{k,p,q} a_{n-k} x^k,$$

so that

$$A(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{[n]_{p,q}^!}.$$

Hence

$$\sum_{n=0}^{\infty} (f \ast g)_n(x) \frac{t^n}{[n]_{p,q}^!} = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \binom{n}{k} a_{n-k}g_k(x) \right) \frac{t^n}{[n]_{p,q}^!} = \left( \sum_{n=0}^{\infty} a_n \frac{t^n}{[n]_{p,q}^!} \right) \left( \sum_{n=0}^{\infty} g_n(x) \frac{t^n}{[n]_{p,q}^!} \right) = A(t)B(t) e_{p,q}(xt).$$

This ends the proof of the theorem. 

Corollary 10. Let $f \in \mathcal{A}(p, q)$, then $f$ has an inverse with respect to $\ast$, i.e. there is a set $g \in \mathcal{A}(p, q)$ such that

$$f \ast g = g \ast f = I.$$

Indeed $g$ belongs to the determining function $(A(t))^{-1}$ where $A(t)$ is the determining function for $f$.

In view of Corollary 10 we shall denote this element $g$ by $f^{-1}$. We are further motivated by Theorem 9 and its corollary to define $f^0 = I$, $f^n = f \ast (f^{n-1})$ where $n$ is a non-negative integer, and $f^{-n} = f^{-1} \ast (f^{n-1})$. We note that we have proved that the system $(\mathcal{A}(p, q), \ast)$ is a commutative group. In particular this leads to the fact that if

$$f \ast g = h$$

and if any two of the elements $f, g, h$ are $(p, q)$-Appell then the third is also $(p, q)$-Appell.
Proposition 11. If $f$ is a $(p,q)$-Appell sequence with the determining function $A(t)$, and if we set
\[
A^{-1}(t) = \sum_{n=0}^{\infty} b_n \frac{t^n}{[n]_{p,q}!}
\]
then
\[
x^n = p^{-(2)} \sum_{k=0}^{n} \binom{n}{k} p_k f_{n-k}(x).
\]

Proof. Since $f$ is a $(p,q)$-Appell sequence, we have
\[
\sum_{n=0}^{\infty} p^{(2)} x^n \frac{t^n}{[n]_{p,q}!} = (A(t))^{-1} A(t) e_{p,q}(xt)
\]
\[= \left( \sum_{n=0}^{\infty} b_n \frac{t^n}{[n]_{p,q}!} \right) \left( \sum_{n=0}^{\infty} f_n(x) \frac{t^n}{[n]_{p,q}!} \right)
\]
\[= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \binom{n}{k} b_k f_{n-k}(x) \right) \frac{t^n}{[n]_{p,q}!}.
\]
The result follows by comparing the coefficients of $t^n$.

5 $(p,q)$-difference and $(p,q)$-recurrence relations for $(p,q)$-Appell polynomials

In this section, we derive a recurrence relation and a $(p,q)$-difference equation for the $(p,q)$-Appell polynomials.

Theorem 12. Let $\{f_n(x)\}_{n=0}^{\infty}$ be the $(p,q)$-Appell polynomial sequences generated by
\[
A(x,t) = A(t) e_{p,q}(xt) = \sum_{n=0}^{\infty} f_n(x) \frac{t^n}{[n]_{p,q}!}.
\]

Then the following linear homogeneous recurrence relation holds true:
\[
f_n(px/q) = \frac{1}{[n]_{p,q}} \sum_{k=0}^{n} \binom{n}{k} \alpha_k f_{n-k}(x) + p^n q^{-1} x f_{n-1}(x).
\]

where
\[
D_{p,q}^{(t)} A(t) = \sum_{n=0}^{\infty} \alpha_n \frac{t^n}{[n]_{p,q}!}.
\]

Proof. Applying the $(p,q)$-derivative $D_{p,q}$ (with respect to the variable $t$) to each
\[
A(px,t) = A(t) e_{p,q}(ptx) = \sum_{n=0}^{\infty} f_n(px) \frac{t^n}{[n]_{p,q}!}
\]
and multiplying the obtained equation by $t$, we get the following equations
\[
t D_{p,q}^{(t)} A(px,t) = t \sum_{n=0}^{\infty} [n]_{p,q} f_n(px) \frac{t^{n-1}}{[n]_{p,q}!} = \sum_{n=0}^{\infty} [n]_{q} f_n(px) \frac{t^n}{[n]_{p,q}!},
\]
From the assumption (16), it follows that

\[ tD_{p,q}^{(t)} A(px,t) = t \left[ D_{p,q}^{(t)} (A(t)e_{p,q}(pxt)) \right] \]

\[ = t \left[ A(pt)D_{p,q}^{(t)} e_{p,q}(pxt) + e_{p,q}(pqt)D_{p,q}^{(t)} A(t) \right] \]

\[ = tpxA(pt)e_{p,q}(pqt) + tD_{p,q}A(t)e_{p,q}(pqt) \]

\[ = A(pt)e_{p,q}(pqt) \left( tpx + t \frac{D_{p,q}^{(t)} A(t)}{A(pt)} \right) \]

\[ = A(qx,pt) \left( tpx + t \frac{D_{p,q}^{(t)} A(t)}{A(pt)} \right) \]

From Theorem 12, we know that the \( A \) is valued around the point \( t = 0 \).

Proof. From Theorem 12, we know that the \( f_n \)'s satisfy the recursion formula (15). Since \( \{f_n(x)\}_{n=0}^{\infty} \) is a \( (p,q) \)-Appell polynomial sequence, we have

\[ D_{p,q}^{(t)} A(t) \]

\[ = \sum_{n=0}^{\infty} \alpha_n \frac{t^n}{[n]_{p,q}!} \]

is valued around the point \( t = 0 \). Then the \( f_n \)'s satisfy the \( (p,q) \)-difference equation

\[ \left( \sum_{k=0}^{n} \frac{\alpha_k}{[k]_{p,q}!} L_p^{-k} D_{p,q}^k + p^n q^{-1} x L_p^{-1} D_{p,q} \right) f_n(x) - [n]_{p,q} f_n(px/q) = 0, \]

with

\[ L_p^k f_n(x) = f_n(p^k x), \quad k \in \mathbb{Z}. \]

Proof. From Theorem 12, we know that the \( f_n \)'s satisfy the recursion formula (15). Since \( \{f_n(x)\}_{n=0}^{\infty} \) is a \( (p,q) \)-Appell polynomial sequence, we have

\[ D_{p,q}^k f_n(x) = \frac{[n]_{p,q}!}{[n-k]_{p,q}!} f_{n-k}(p^k x), \quad 0 \leq k \leq n. \]

It follows that

\[ f_{n-k}(x) = \frac{[n-k]_{p,q}!}{[n]_{p,q}!} L_p^{-k} D_{p,q}^k f_n(x), \quad 0 \leq k \leq n. \]
Then (15) becomes

\[ f_n(px/q) = \frac{1}{[n]_{p,q}} \sum_{k=0}^{n} \binom{n}{k}_{p,q} \alpha_k \frac{[n-k]_{p,q}!}{[k]_{p,q}!} \mathcal{L}_{p,q}^{-k}D_{p,q}^k f_n(x) + p^n q^{-1} x f_{n-1}(x) \]

which, when simplified, yields

\[ = \frac{1}{[n]_{p,q}} \sum_{k=0}^{n} \frac{\alpha_k}{[k]_{p,q}!} \mathcal{L}_{p,q}^{-k}D_{p,q}^k f_n(x) + p^n q^{-1} x f_{n-1}(x) \]

\[ = \frac{1}{[n]_{p,q}} \left( \sum_{k=0}^{n} \frac{\alpha_k}{[k]_{p,q}!} \mathcal{L}_{p,q}^{-k}D_{p,q}^k + p^n q^{-1} x \mathcal{L}_{p,q}^{-1}D_{p,q} \right) f_n(x) \]

and the result follows \(\Box\)

6 Some \((p, q)\)-Appell polynomial sequences

In this section, we give four examples of \((p, q)\)-Appell polynomial sequences and prove some of their main structure relations. The bivariate \((p, q)\)-Bernoulli, the bivariate \((p, q)\)-Euler and the bivariate \((p, q)\)-Genocchi polynomials are introduced in [3] and some of their relevant properties are given. Without any lost of the generality, we will restrict ourselves to the case \(y = 0\). Also, we introduce a new generalization of the \((p, q)\)-Hermite polynomials.

6.1 The \((p, q)\)-Bernoulli polynomials

The \((p, q)\)-Bernoulli polynomials are \((p, q)\)-Appell polynomials for the determining function

\[ A(t) = \frac{t}{e_{p,q}(t)} \]

Thus, the \((p, q)\)-Bernoulli polynomials are defined by the generating function

\[ \frac{t}{e_{p,q}(t)} - 1 = \sum_{n=0}^{\infty} B_n(x; p, q) \frac{t^n}{[n]_{p,q}!} \]

Let us define the \((p, q)\)-Bernoulli numbers \(B_{n, p, q}\) by the generating function

\[ \frac{t}{e_{p,q}(t)} - 1 = \sum_{n=0}^{\infty} B_{n, p, q} \frac{t^n}{[n]_{p,q}!} \]

so that

\[ B_n(0; p, q) = B_{n, p, q} \quad (n \geq 0). \]

**Proposition 14.** The \((p, q)\)-Bernoulli polynomials \(B_n(x; p, q)\) have the representation

\[ B_n(x; p, q) = \sum_{k=0}^{n} \binom{n}{k}_{p,q} \mathcal{L}_{p,q}^{-k} \mathcal{E}_{p,q}^{-k} B_{k, p, q} x^{n-k}. \] (17)

**Proof.** The proof follows from Theorem [7] \(\Box\)

6.2 The \((p, q)\)-Euler polynomials

The \((p, q)\)-Euler polynomials are \((p, q)\)-Appell polynomials for the determining function

\[ A(t) = \frac{2}{e_{p,q}(t) + 1} \]

Thus, the \((p, q)\)-Euler polynomials are defined by the generating function

\[ \frac{2}{e_{p,q}(t) + 1} e_{p,q}(xt) = \sum_{n=0}^{\infty} E_n(x; p, q) \frac{t^n}{[n]_{p,q}!} \]

Let us define the \((p, q)\)-Euler numbers \(E_{n, p, q}\) by the generating function

\[ \frac{2}{e_{p,q}(t) + 1} = \sum_{n=0}^{\infty} E_{n, p, q} \frac{t^n}{[n]_{p,q}!} \]
so that
\[ \mathcal{E}_n(0; p, q) = \mathcal{E}_{n, p, q}, \quad (n \geq 0). \]

**Proposition 15.** The \((p, q)\)-Euler polynomials \(\mathcal{E}_n(x; p, q)\) have the representation
\[ \mathcal{E}_n(x; p, q) = \sum_{n=0}^{\infty} \left[ \begin{array}{c} n \\ k \end{array} \right]_{p, q} p^{(n-k)} \mathcal{E}_{k, p, q} x^{n-k}. \quad (18) \]

**Proof.** The proof follows from Theorem 7.

\[ \square \]

### 6.3 The \((p, q)\)-Genocchi polynomials

The \((p, q)\)-Genocchi polynomials are \((p, q)\)-Appell polynomials for the determining function
\[ A(t) = \frac{2t}{e_{p, q}(t) + 1}. \]
Thus, the \((p, q)\)-Genocchi polynomials are defined by the generating function
\[ \frac{2t}{e_{p, q}(t) + 1} e_{p, q}(xt) = \sum_{n=0}^{\infty} \mathcal{G}_n(x; p, q) \frac{t^n}{[n]_{p, q}!}. \]

Let us define the \((p, q)\)-Genocchi numbers \(\mathcal{G}_{n, p, q}\) by the generating function
\[ \frac{2t}{e_{p, q}(t) + 1} = \sum_{n=0}^{\infty} \mathcal{G}_{n, p, q} \frac{t^n}{[n]_{p, q}!} \]
so that
\[ \mathcal{G}_n(0; p, q) = \mathcal{G}_{n, p, q}, \quad (n \geq 0). \]

**Proposition 16.** The \((p, q)\)-Genocchi polynomials \(\mathcal{G}_n(x; p, q)\) have the representation
\[ \mathcal{G}_n(x; p, q) = \sum_{n=0}^{\infty} \left[ \begin{array}{c} n \\ k \end{array} \right]_{p, q} p^{(n-k)} \mathcal{G}_{k, p, q} x^{n-k}. \quad (19) \]

**Proof.** The proof follows from Theorem 7.

\[ \square \]

### 6.4 The \((p, q)\)-Hermite polynomials

In this section we construct \((p, q)\)-Hermite polynomials and give some of their properties. Also, we derive the three-term recurrence relation as well as the second-order differential equation satisfied by these polynomials.

We define \((p, q)\)-Hermite polynomials by means of the generating function
\[ F_{p, q}(x, t) := \frac{2t}{e_{p, q}(t) + 1} e_{p, q}(xt) = \sum_{n=0}^{\infty} H_n(x; p, q) \frac{t^n}{[n]_{p, q}!}. \quad (20) \]

where
\[ F_{p, q}(t) = \sum_{n=0}^{\infty} (-1)^n p_n^{(n-1)} \frac{t^{2n}}{[2n]_{p, q}!!}, \quad \text{with} \quad [2n]_{p, q}!! = \prod_{k=1}^{n} [k]_{p, q}, \quad [0]_{p, q}!! = 1. \quad (21) \]

It is clear that
\[ \lim_{p, q \to 1} F_{p, q}(x, t) = e^{xt} \lim_{p, q \to 1} \sum_{n=0}^{\infty} (-1)^n p_n^{(n-1)} \frac{t^{2n}}{[2n]_{p, q}!!} = e^{xt} \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n)(2n-2)\cdots 2} \]
\[ = e^{xt} \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{2^nn!} = \exp \left( tx - \frac{t^2}{2} \right). \]
Moreover,
\[ D_{p,q}^{(t)}F_{p,q}(t) = \sum_{n=1}^{\infty} (-1)^n p^n (n-1) \frac{t^{2n-1}}{(2n-2)!!} \frac{p^{n(n-1)+2n}}{(2n-2)!!} = tF_{p,q}(pt), \]
Hence
\[ \frac{D_{p,q}^{(t)}F_{p,q}(t)}{F_{p,q}(pt)} = -t. \]

**Theorem 17.** The \((p, q)\)-Hermite polynomials \(H_n(x; p, q)\) have the following representation
\[ H_n(x, p, q) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k p^{n-k} (n-k)!!}{(2k)^{n-k}!!} x^n. \]

**Proof.** Indeed, expanding the generating function \(H_{p,q}(x, t)\), we have
\[ H_{p,q}(x, t) = \left( \sum_{k=0}^{\infty} (-1)^k t^k \frac{x^k}{k!!} \right) \left( \sum_{n=0}^{\infty} \frac{p^n x^n}{[n]_{p,q}!!} \right) = \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k t^k (n-k)!!}{(2k)^{n-k}!!} x^n. \]
The result follows by using the series manipulation formula (7) of Lemma 11 in [11].

**Theorem 18.** The following linear homogeneous recurrence relation for the \((p, q)\)-Hermite polynomials holds true
\[ H_{n+1}(px, p, q) = p^{n+1}x H_n(qx, p, q) - p^{n-1} [n]_{p,q} H_{n-1}(qx, p, q), \quad (n \geq 1). \]

**Proof.** The result comes from Theorem 12 using the fact that \( \frac{D_{p,q}^{(t)}F_{p,q}(t)}{F_{p,q}(pt)} = -t. \)

**Theorem 19.** The \((p, q)\)-Hermite polynomials \(H_n(x; p, q)\) satisfy the \((p, q)\)-difference equation
\[ \mathcal{L}_p^{-2} D_{p,q}^2 H_n(x; p, q) - p^2 q^{-1} x \mathcal{L}_p^{-1} D_{p,q} H_n(x; p, q) + p^{2-n} [n]_{p,q} H_n(px/q) = 0. \quad (22) \]

**Proof.** The proof follows from Theorem 13.

Note that as \( p \) and \( q \) tend to 1, Equation (22) reduces to the second order differential equation satisfied by the Hermite polynomials.

**Concluding remarks**

In this work we have introduced \((p, q)\)-Appell sequences and have given several characterizations of these sequences. Also, by a suitable choose of the determining functions, we have recovered the \((p, q)\)-Bernoulli and the \((p, q)\)-Euler polynomials already given in [3]. It worth noting that we could set the problem of defining a new set of \((p, q)\)-Appell sequences by changing the small \((p, q)\)-exponential function \(e_{p,q}\) by the big \((p, q)\)-exponential function \(E_{p,q}\). But, this problem is useless since it is not difficult to see that \( e_{p,q} = E_{q,p} \). Note also that the \((p, q)\)-Appell defined here generalized both \(q\)-Appell functions of type I and of type II already found in the literature and can be viewed as a unified definition.

**Acknowledgements**

This work was supported by the Institute of Mathematics of the University of Kassel to whom I am very grateful.
References

[1] W. A. Al-Salam, *q-Appell polynomials*, Ann. Mat. Pura Appl., 4 (1967), pp. 31–45.

[2] P. Appell, *Une classe de polynomes*, Annalles scientifique, Ecole Normale Sup., ser. 2, vol. 9 (1880), pp. 119–144.

[3] U. Duran, M. Acikgoz, S. Araci, *On (p,q)-Bernoulli, (p,q)-Euler and (p,q)-Genocchi polynomials*, Preprint available at http://openaccess.hku.edu.tr/bitstream/handle/20.500.11782/86/Araci15.pdf?sequence=1

[4] R. Jagannathan, *P, Q-Special Functions*, arXiv:math/9803142v1, 1998.

[5] R. Jagannathan, K. Srinivasa Rao, *Two-parameter quantum algebras, twin-basic numbers, and associated generalized hypergeometric series*, in: Proceedings of the International Conference on Number Theory and Mathematical Physics, Srinivasa Ramanujan Centre, Kumbakonam, India, 20-21 December 2005

[6] R. Jagannathan, R. Sridhar, *p,q-Rogers-Szegö Polynomials and the (p,q)-Oscillator*, K. Alladi et al. (eds.), The Legacy of Alladi Ramakrishnan in the Mathematical Sciences, 2010.

[7] V. Kac, P. Cheung, *Quantum calculus*, Springer, (2001).

[8] P. Njionou Sadjang, *On the fundamental theorem of (p,q)-calculus and some (p,q)-Taylor theorems*, http://arxiv.org/abs/1309.3934 (2013), Submitted.

[9] P. Njionou Sadjang, *On two (p,q)-analogues of the Laplace transform*, J. Difference. Equ. Appl. DOI: 10.1080/10236198.2017.1340469, (2017)

[10] P. Njionou Sadjang, *On a new q-analogue of Appell polynomials*, Submitted.

[11] E. D. Rainville: *Special Functions*, The Macmillan Company, New York, (1960).

[12] I. M. Sheffer, *On sets of polynomials and associated linear functional operator and equations*, Amer. J. Math. 53 (1931), pp. 15–38