Correspondence and Canonicity Theory of Quasi-Inequalities 
and $\Pi_2$-Statements in Modal Subordination Algebras

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Abstract

In the present paper, we study the correspondence and canonicity theory of modal subordination algebras and their dual Stone space with two relations, generalizing correspondence results for subordination algebras in [13, 14, 15, 25]. Due to the fact that the language of modal subordination algebras involves a binary subordination relation, we will find it convenient to use the so-called quasi-inequalities and $\Pi_2$-statements. We use an algorithm to transform (restricted) inductive quasi-inequalities and (restricted) inductive $\Pi_2$-statements to equivalent first-order correspondents on the dual Stone spaces with two relations with respect to arbitrary (resp. admissible) valuations.

Keywords: correspondence theory; canonicity; $\Pi_2$-statements; modal subordination algebras

1 Introduction

In the study of the relations among logic, algebra and topology, a key tool is the dualities between algebras and topological spaces. This line of research started from Stone [26], who showed the duality between Boolean algebras and compact Hausdorff zero-dimensional spaces (which are now called Stone spaces). Later on, many other dualities are studied, e.g. Priestley duality for distributive lattices and Priestley spaces [23, 24], Esakia duality for Heyting algebras and Esakia spaces [18], Jónsson-Tarski duality for modal algebras and modal spaces [19, 20], and de Vries duality for de Vries algebras and compact Hausdorff spaces [16].

De Vries duality connects compact Hausdorff spaces with de Vries algebras, which are complete Boolean algebras with a binary relation satisfying some additional properties. Subordination algebras [4] are Boolean algebras with a binary relation (called the subordination relation) which generalizes de Vries algebras. Subordinations on Boolean algebras are in 1-1 correspondence with quasi-modal operators [9]. It is proved in [4] that subordination algebras are dually equivalent to Stone spaces with a closed relation, which are called subordination spaces. Modal compact Hausdorff spaces are studied in [2] as a the compact Hausdorff generalization of modal spaces, whose counterpart of de Vries duality yields modal de Vries algebras. Therefore, it is natural to consider the generalizations of modal de Vries algebras, namely modal subordination algebras, as the background theory of studying modal compact Hausdorff spaces. Indeed, modal subordination algebras are dually equivalent to Stone spaces with two relations, one closed relation corresponding to the subordination relation, and one clopen relation corresponding to the modality.

It is natural to consider the correspondence theory of propositional formulas on subordination spaces with their first-order corresponding conditions on subordination spaces. Indeed, in the literature, there are works of correspondence theory for subordination algebras and subordination spaces. In [13, 14, 15], de Rudder et al. studied correspondence theory of subordination algebras in the perspective of quasi-modal operators. In [25], Santoli studied the topological correspondence theory between conditions on algebras and first-order conditions on the dual subordination spaces, in the language of a binary connective definable from the squigarrow associated with the subordination relation, using
the so-called ∀∃-statements [3, 5, 6] (which we call Π₂-statements in the present paper). Balbiani and Kikot [1] investigated the Sahlqvist theory in the language of region-based modal logics of space, which uses a contact relation.

Since the language of modal subordination algebras involves a binary subordination relation, and many natural conditions on subordination algebras and modal subordination algebras involves quasi-inequalities and Π₂-statements, we will follow the approach of Santoli in the sense of using Π₂-statements, and partly follow the approach of de Rudder in the sense of using quasi-modal operators which are not closed under the admissible subsets. We use an algorithm to transform (restricted) inductive quasi-inequalities and (restricted) inductive Π₂-statements to equivalent first-order correspondents on the dual Stone spaces with two relations with respect to arbitrary (resp. admissible) valuations.

The paper is organized as follows: Section 2 gives preliminaries on modal subordination algebras and Stone space with two relations. In Part I, we study the correspondence and canonicity theory for quasi-inequalities. Section 3 gives the syntax and semantics of the logic language we are considering. Section 4 gives preliminaries on algorithmic correspondence. We give the definition of inductive quasi-inequalities in Section 5 define a version of the algorithm ALBA in Section 6 show its soundness in Section 7 success on inductive quasi-inequalities in Section 8 and the canonicity of restricted inductive quasi-inequalities in Section 9. Section 10 gives some examples of the execution of ALBA on quasi-inequalities. In Part II, we study the correspondence and canonicity theory for Π₂-statements. Section 11 gives the syntax and semantics for Π₂-statements. Section 12 defines inductive Π₂-statements and restricted inductive Π₂-statements. Section 13 gives a version of the algorithm ALBAΠ₂ for Π₂-statements. We state its soundness with respect to arbitrary valuations in Section 14. We prove its success on inductive and restricted inductive Π₂-statements in Section 15 and canonicity of restricted inductive Π₂-statements in Section 16. Section 17 gives an example of the execution of ALBA on a Π₂-statement. We give some comparisons with existing works in Section 18.

2 Modal Subordination Algebras and Stone Space with Two Relations

In this section, we give the definitions of modal subordination algebras and special classes of modal subordination algebras, as well as their dual Stone space with two relations. Basically, we generalize upper continuous modal de Vries algebras (abbreviation: UMDVs, see [2]) to modal subordination algebras. The reason why we choose UMDVs is that the diamond there is finitely additive (see Proposition 9), which makes it easier to develop a correspondence theory.

2.1 Subordination Algebras, de Vries Algebras and Modal de Vries Algebras

Definition 1 (Subordination Algebra, Definition 2.1.1 in [25]). A subordination algebra is a pair \((B, <)\) where \(B\) is a Boolean algebra and \(<\) is a subordination, which is a binary relation on \(B\) satisfying the following properties:

- \(0 < 0 \) and \(1 < 1\);
- \(a < b\) and \(a < c\) implies \(a < b \land c\);
- \(a < c\) and \(b < c\) implies \(a \lor b < c\);
- \(a \leq b < c \leq d\) implies \(a < d\).

Equivalently, we can describe a subordination \(<\) on a Boolean algebra \(B\) by an operation \(\rightsquigarrow: B \times B \rightarrow \{0, 1\} \subseteq B\) such that
- \(a \rightsquigarrow b \in \{0, 1\}\);
• $0 \leadsto 0 = 1 \leadsto 1 = 1$;
• $a \leadsto b = 1$ and $a \leadsto c = 1$ implies $a \leadsto b \land c = 1$;
• $a \leadsto c = 1$ and $b \leadsto c = 1$ implies $a \lor b \leadsto c = 1$;
• $b \leadsto c = 1$, $a \leq b$ and $c \leq d$ implies $a \leadsto d = 1$.

Given a subordination $< \ B$, we can obtain an operation $\leadsto \ : B \times B \rightarrow \{0, 1\} \subseteq B$ satsying the properties above by defining $a \leadsto b = 1$ if $a < b$. Given an operation $\leadsto \ : B \times B \rightarrow \{0, 1\} \subseteq B$ satsying the properties above, we can obtain a subordination $< \ B$ by defining $< \ = \{(a, b) \in B \times B : a \leadsto b = 1\}$. Therefore, we have a 1-1 correspondence between subordinations $< \ B$ and operations $\leadsto \$, satisfying the properties above.

**Definition 2** (Contact Algebra, Definition 2.1.3 in [25]). A contact algebra is a subordination algebra $(B, <)$ where $<$ satisfies the following two additional properties:

• $a < b$ implies $a \leq b$;
• $a < b$ implies $\neg b < \neg a$.

**Definition 3** (Compingent Algebra, Definition 2.1.4 in [25]). A compingent algebra is a contact algebra $(B, <)$ where $<$ satisfies the following two additional properties:

• $a < b$ implies $\exists c : a < c < b$;
• $a \neq 0$ implies $\exists b \neq 0 : b < a$.

**Definition 4** (de Vries Algebra, Definition 2.2.1 in [25]). A de Vries algebra is a compingent algebra $(B, <)$ where $B$ is a complete Boolean algebra.

**Definition 5** (Modal de Vries Algebra, Definition 4.7 in [2]). A modal de Vries algebra is a tuple $(B, <, \Diamond)$ where $(B, <)$ is a de Vries algebra and $\Diamond$ is de Vries additive, i.e. it satisfies the following two properties:

• $\Diamond 0 = 0$;
• for all $a_1, a_2, b_1, b_2 \in B$, $a_1 < a_2$ and $b_1 < b_2$ implies that $\Diamond(a_1 \lor a_2) < \Diamond(b_1 \lor b_2)$.

**Proposition 6** (Proposition 4.8 in [2]). In a modal de Vries algebra $(B, <, \Diamond)$, $\Diamond$ is proximity preserving, i.e. $a < b$ implies $\Diamond a < \Diamond b$ for all $a, b \in B$.

**Proposition 7** (Proposition 4.10 in [2]). In a de Vries algebra $(B, <)$, if $\Diamond$ is finitely additive (i.e. $\Diamond(a \lor b) = \Diamond a \lor \Diamond b$ for all $a, b \in B$) and proximity preserving, then it is de Vries additive.

Notice that in a modal de Vries algebra $(B, <, \Diamond)$, the de Vries additive operation $\Diamond$ is not necessarily order-preserving or finitely additive. However, in upper continuous modal de Vries algebras, $\Diamond$ is finitely additive.

**Definition 8** (Upper continuity, Definition 4.14 in [2]). A modal de Vries algebra $(B, <, \Diamond)$ is upper continuous, if it satisfies the following property:

$\Diamond a = \bigwedge\{\Diamond b : a < b\}$ for any $a, b \in B$.

**Proposition 9** (Proposition 4.15 in [2]). In an upper continuous modal de Vries algebra $(B, <, \Diamond)$, $\Diamond$ is both order-preserving and finitely additive.
2.2 Modal Subordination Algebras, Stone Spaces with Two Relations and their Object-Level Duality

In this subsection we define the modal subordination algebras, and their dual Stone spaces with two relations, and give their object-level duality. The reason why we choose to make modal subordination algebras as subordination algebras with normal and finitely additive operations is that it is easier to develop the correspondence theory for normal and finitely additive operations.

**Definition 10** (Modal Subordination Algebra). A modal subordination algebra is a tuple \((B, \prec, \Diamond)\) where \((B, \prec)\) is a subordination algebra, \(\Diamond\) satisfies the following conditions: for all \(a, b \in B\),

- \(\Diamond\) is normal, namely \(\Diamond 0 = 0\);
- \(\Diamond\) is finitely additive, namely \(\Diamond(a \lor b) = \Diamond a \lor \Diamond b\).

**Definition 11.** A modal subordination algebra is

- a modal contact algebra, if \((B, \prec)\) is a contact algebra;
- a modal compingent algebra, if \((B, \prec)\) is a compingent algebra;
- proximity preserving, if for all \(a, b \in B, a \prec b\) implies that \(\Diamond a < \Diamond b\);
- an additive modal de Vries algebra, if \((B, \prec)\) is a de Vries algebra and \(\Diamond\) is proximity preserving;
- an upper continuous modal de Vries algebra, if \((B, \prec, \Diamond)\) is an additive modal de Vries algebra and \(\Diamond\) is upper continuous.

In what follows we will define the dual topological structure of a modal subordination algebra. Since there is the duality between subordination algebras and Stone spaces with closed relations (see [25, Section 2.1.1]), and there is the duality between modal algebras and modal spaces (see [7, Chapter 5]), we can put the two together to obtain the duality between modal subordination algebras and Stone spaces with two relations.

**Definition 12** (Definition 2.1.9 in [25]). Take any modal subordination algebra \((B, \prec, \Diamond)\) and any \(S \subseteq B\). We define \(\uparrow S\) to be the upset of \(S\) with respect to the relation \(\prec\), i.e.: \(\uparrow S := \{b \in B : \exists s \in S\text{ such that } s < b\}\).

**Definition 13** (Stone Spaces with two relations). A Stone space with two relations \((X, \tau, R, R')\) is defined as follows:

- \((X, \tau)\) is a Stone space;
- \(R\) is a closed relation on \(X\), i.e. for each closed subset \(F\) of \(X\), both \(R[F]\) and \(R^{-1}[F]\) are closed;
- \(R'[x]\) is closed for all \(x \in X\) and \(R'^{-1}[U]\) is clopen for all clopen \(U \subseteq X\).

Given a modal subordination algebra \((B, \prec, \Diamond)\), its dual Stone space with two relations \((X, \tau, R, R')\) is defined as follows:

- \((X, \tau)\) is the dual Stone space of \(B\);
- \(R\) is such that \(xRy\) iff \(\uparrow x \subseteq y\) (It is easy to check that \(R\) is a closed relation on \(X\));
- \(R'\) is such that \(xR'y\) iff for all \(a \in y, \Diamond a \in x\) (It is easy to check that \(R'[x]\) is closed for all \(x \in X\) and \(R'^{-1}[U]\) is clopen for all clopen \(U \subseteq X\)).
Given a Stone space with two relations \((X, \tau, R, R')\), its dual modal subordination algebra \((B, <, \Diamond)\) is defined as follows:

- \(B\) is the dual Boolean algebra of the Stone space \((X, \tau)\), i.e. \(B\) consists of the clopen subsets of \(X\);
- \(U < V\) iff \(R[U] \subseteq V\);
- \(\Diamond U = R'^{-1}[U]\).

### 2.3 Canonical Extensions

In this subsection, we define the canonical extensions of modal subordination algebras, as well as give the semantic environment of the correspondence and canonicity theory for modal subordination algebras.

#### 2.3.1 Canonical Extensions of Boolean Algebras

**Definition 14** (Canonical Extension of Boolean Algebras, cf. Chapter 6, Definition 104 in [8]). The canonical extension of a Boolean algebra \(B\) is a complete Boolean algebra \(B^\delta\) containing \(B\) as a sub-Boolean algebra such that the following two conditions hold:

- *(denseness)* each element of \(B^\delta\) can be represented both as a join of meets and as a meet of joins of elements from \(B\);
- *(compactness)* for all \(X, Y \subseteq B\) with \(\bigwedge X \leq \bigvee Y\) in \(B^\delta\), there are finite subsets \(X_0 \subseteq X\) and \(Y_0 \subseteq Y\) such that \(\bigwedge X_0 \leq \bigvee Y_0\).

An element \(x \in B^\delta\) is open (resp. closed) if it is the join (resp. meet) of some \(X \subseteq B\). We use \(O(B^\delta)\) (resp. \(K(B^\delta)\)) to denote the set of open (resp. closed) elements of \(B^\delta\). It is easy to see that elements in \(B\) are exactly the ones which are both closed and open (i.e., clopen).

It is well-known that for any given \(B\), its canonical extension is unique up to isomorphism and that assuming the axiom of choice, the canonical extension of a Boolean algebra is a perfect Boolean algebra, i.e., a complete and atomic Boolean algebra.

#### 2.3.2 Canonical Extensions of Maps

Let \(A, B\) be Boolean algebras. There are two canonical ways to extend an order-preserving map \(f : A \to B\) to a map \(A^\delta \to B^\delta\):

**Definition 15** (\(\sigma\)- and \(\pi\)-extension). ([8, page 375]) For any order-preserving map \(f : A \to B\) and \(u \in A^\delta\), we define

\[
\begin{align*}
\sigma^\delta(u) &= \bigvee \{ \bigwedge \{ f(a) : x \leq a \in A \} : u \geq x \in K(A^\delta) \} \\
\pi^\delta(u) &= \bigwedge \{ \bigvee \{ f(a) : y \geq a \in A \} : u \leq y \in O(A^\delta) \}
\end{align*}
\]

\[1\]In fact, this is an equivalent formulation of the definition in [8].
2.3.3 Canonical Extensions of Modal Subordination Algebras

Since in a modal subordination algebra \((B, <, \Diamond)\), \(\Diamond\) is normal and finitely additive, it is smooth, i.e., \(\Diamond^\Diamond = \Diamond^\pi\) (cf. [8] Proposition 11(3)). Therefore, in the canonical extension we can take either of the two extensions.

For the \(<\) relation, we define its canonical extension by defining the \(\pi\)-extension \(\rightsquigarrow^\pi_\prec\colon B^\Diamond \times B^\Diamond \to B^\Diamond\) of the associated strict implication \(\rightsquigarrow_\prec\) and then take the associated subordination \(<_{\rightsquigarrow^\pi_\prec}\) (which we also denote \(<^\pi\)) associated with \(\rightsquigarrow^\pi_\prec\).

**Proposition 16** (Folklore.)

- \(\Diamond^\pi\) is completely join-preserving, i.e. it preserves arbitrary (including empty) joins.
- \(\rightsquigarrow^\pi_\prec\) is completely join-reversing in the first coordinate and completely meet-preserving in the second coordinate, i.e.

\[
\begin{align*}
\bigvee \{a_i : i \in I\} \rightsquigarrow^\pi_\prec b &= \bigwedge \{a_i \rightsquigarrow^\pi_\prec b : i \in I\}; \\
\bigwedge \{b_i : i \in I\} \rightsquigarrow^\pi_\prec \left(\bigvee \{a_i \rightsquigarrow^\pi_\prec b_i : i \in I\}\right).
\end{align*}
\]

Indeed, there is another way of obtaining the canonical extension of the modal subordination algebras, namely first take its dual space \((X, \tau, R, R')\), and then drop off the topological structure to obtain a birelational Kripke frame \((X, R, R')\), and then define the canonical extension \((B^\Diamond, <^\Diamond, \Diamond^\Diamond)\) as follows:

- \(B^\Diamond\) is the power set Boolean algebra of \(X\), i.e. \(B^\Diamond\) consists of all subsets of \(X\);
- \(U <^\Diamond V\) iff \(R[U] \subseteq V\);
- \(\Diamond^\Diamond U = R'^{-1}[U]\).

**Proposition 17.** (Folklore.) The two definitions are equivalent, i.e. \((B^\Diamond, <^\Diamond, \Diamond^\Diamond)\) = \((B^\Diamond, <^\pi, \Diamond^\pi)\).

Therefore, the following diagram describes the relation between modal subordination algebras and their canonical extensions, as well as their dual Stone space with two relations, and the birelational Kripke frames obtained by dropping the topology:

\[
\begin{array}{ccc}
(B^\Diamond, <^\pi, \Diamond^\pi) & \overset{\cong^0}{\longrightarrow} & (X, R, R') \\
\downarrow & & \downarrow \\
(B, <, \Diamond) & \overset{\cong^0}{\longrightarrow} & (X, \tau, R, R')
\end{array}
\]

Here \(\cong^0\) means dual equivalence, \(U\) is the forgetful functor dropping the topology and replacing the clopen set Boolean algebra by the powerset Boolean algebra, and \((\cdot)^\Diamond\) is taking the canonical extension.

**Part I: Correspondence and Canonicity for Quasi-Inequalities**

### 3 Syntax and Semantics

In this section, we introduce the syntax and semantics of the language of modal subordination algebras.
3.1 Language and Syntax

Definition 18. Given a countably infinite set $\text{Prop}$ of propositional variables, the modal subordination language $\mathcal{L}$ is defined as follows:

$$\varphi ::= p \mid \bot \mid T \mid \neg \varphi \mid (\varphi \land \varphi) \mid (\varphi \lor \varphi) \mid (\varphi \rightarrow \varphi) \mid \Box \varphi \mid \Diamond \varphi \mid \Diamond \varphi \mid \Box \varphi,$$

where $p \in \text{Prop}$. We will follow the standard rules for omission of the parentheses. We also use $\text{Prop}(\varphi)$ to denote the propositional variables occurring in $\varphi$. We use the notation $\vec{p}$ to denote a set of propositional variables and use $\varphi(\vec{p})$ to indicate that the propositional variables occur in $\varphi$ are all in $\vec{p}$. We call a formula pure if it does not contain propositional variables. We use the notation $\overline{\theta}$ to indicate a finite list of formulas. We use the notation $\theta(\eta/p)$ to indicate uniformly substituting $p$ by $\eta$.

In the language of formulas, we use $\Diamond$ and $\Box$ as the syntax for the normal and finitely additive operation, and use $\Diamond$ and $\Box$ as the syntax for the subordination on the Boolean algebra. We will also use the following syntactic binary relations $\leq$ and $<$ to formalize the “less than or equal to” relation and the subordination relation, respectively. Therefore, we use both $\Diamond, \Box$ and $<$ to denote the subordination relation.

Definition 19.

- An inequality is of the form $\varphi \leq \psi$ or $\varphi < \psi$, where $\varphi$ and $\psi$ are formulas.
- A meta-conjunction of inequalities is of the form $\varphi_1 \preceq_1 \psi_1 \land \ldots \land \varphi_n \preceq_n \psi_n$, where $n \geq 1$, $\preceq_i \in \{\leq, <\}$. Typically a meta-conjunction of inequalities is abbreviated as $\varphi \triangleleft \psi$, and if all $\varphi$s are $\leq$ (resp. $<$), then it is written as $\varphi \preceq \psi$ (resp. $\varphi \prec \psi$).
- A quasi-inequality is of the form $\varphi \prec \psi \Rightarrow \overline{\varphi} \prec \overline{\psi}$.

3.2 Semantics

We interpret formulas on the dual Stone space with two relations, with two kinds of valuations, namely admissible valuations which interpret propositional variables as clopen subsets of the space (i.e. interpret them as elements of the dual modal subordination algebras), and arbitrary valuations which interpret propositional variables as arbitrary subsets of the space (i.e. interpret them as elements of the canonical extensions of the dual modal subordination algebras).

Definition 20. In a Stone space with two relations $(X, \tau, R, R')$ (we abuse notation to use $X$ to denote the space), we call $X$ the domain of the space, and $R$ is the relation corresponding to the subordination as well as $\Diamond$ and $\Box$, and $R'$ is the relation corresponding to the modalities $\Diamond$ and $\Box$.

- A pointed Stone space with two relations is a pair $(X, w)$ where $w \in X$.
- An admissible model is a pair $M = (X, V)$ where $V : \text{Prop} \rightarrow \text{Clop}(X)$ is an admissible valuation on $X$ such that for all propositional variables $p$, $V(p)$ is a clopen subset of $X$.
- An arbitrary model is a pair $M = (X, V)$ where $V : \text{Prop} \rightarrow \text{P}(X)$ is an arbitrary valuation on $X$ such that for all propositional variables $p$, $V(p)$ is an arbitrary subset of $X$.

Given a valuation $V$, a propositional variable $p \in \text{Prop}$, a subset $A \subseteq X$, we can define $V_A^\varphi$, the $p$-variant of $V$ as follows: $V_A^\varphi(q) = V(q)$ for all $q \neq p$ and $V_A^\varphi(p) = A$.

Now the satisfaction relation can be defined as follows: given any Stone space with two relations $(X, \tau, R, R')$, any (admissible or arbitrary) valuation $V$ on $X$, any $w \in X$, any
The definitions of validity are similar to formulas. Notice that here ⊡ to make ϕ as follows:

For the semantics of inequalities, meta-conjunctions of inequalities, quasi-inequalities, they are given for any formula ϕ we just use V(ϕ) interpreted as R(ϕ) for every arbitrary valuation V.

We say that ϕ is interpreted as follows:

A quasi-inequality is interpreted as follows:

The definitions of validity are similar to formulas.
4 Preliminaries on Algorithmic Correspondence

In this section, we give necessary preliminaries on the correspondence algorithm ALBA in the style of \[10, 11, 27\]. The algorithm ALBA transforms the input quasi-inequality

\[ \varphi_1 \leq \psi_1 \land \ldots \land \varphi_n \leq \psi_n \land \gamma_1 < \delta_1 \land \ldots \land \gamma_m < \delta_m \Rightarrow \alpha \preceq \beta \] (where \( \preceq \in \{<, \leq\} \))

into an equivalent set of pure quasi-inequalities which does not contain occurrences of propositional variables, and therefore can be translated into the first-order correspondence language via the standard translation of the expanded language (see page 10).

The ingredients for the algorithmic correspondence proof to go through can be listed as follows:

- An expanded language as the syntax of the algorithm, as well as its semantics;
- An algorithm ALBA which transforms a given quasi-inequality into equivalent pure quasi-inequalities;
- A soundness proof of the algorithm;
- A syntactically identified class of quasi-inequalities (namely the inductive quasi-inequalities) on which the algorithm is successful;
- A first-order correspondence language and first-order translation which transforms pure quasi-inequalities into their equivalent first-order correspondents.

In the remainder of this part, we will define an expanded language which the algorithm will manipulate (Section 4.1), define the first-order correspondence language of the expanded language and the standard translation (Section 4.2). We give the definition of inductive quasi-inequalities (Section 5), define a version of the algorithm ALBA (Section 6), and show its soundness (Section 7), success on inductive quasi-inequalities (Section 8) and the canonicity of restricted inductive quasi-inequalities (Section 9).

4.1 The expanded hybrid modal language

In the present subsection, we give the definition of the expanded language, which will be used in the execution of the algorithm:

\[ \varphi ::= p \mid i \mid T \mid F \mid \neg \varphi \mid (\varphi \land \varphi) \mid (\varphi \lor \varphi) \mid (\varphi \rightarrow \varphi) \mid \lozenge \varphi \mid \Box \varphi \mid \lozenge \Box \varphi \mid \Box \lozenge \varphi \mid \Box \varphi \mid \diamond \varphi \mid \Box \diamond \varphi \mid \diamond \Box \varphi \mid \diamond \varphi \mid \Box \varphi \]

where \( i \in \text{Nom} \) is called a nominal. For \( i \), it is interpreted as a singleton set. For \( \Box \) and \( \diamond \), they are interpreted as the box and diamond modality on the inverse relation \( R^{-1} \), and for \( \lozenge \) and \( \lozenge \), they are interpreted as the box and diamond modality on the relation \( R \).

For the semantics of the expanded language, the valuation \( V \) is extended to \( \text{Prop} \cup \text{Nom} \) such that \( V(i) \) is a singleton for each \( i \in \text{Nom} \).\(^2\)

The additional semantic clauses can be given as follows:

\[
\begin{align*}
X, V, w \vdash i & \iff V(i) = \{w\}; \\
X, V, w \vdash \Box \varphi & \iff \forall v(R^*vw \Rightarrow X, V, v \vdash \varphi); \\
X, V, w \vdash \lozenge \varphi & \iff \exists v(R^*vw \text{ and } X, V, v \vdash \varphi); \\
X, V, w \vdash \Box \lozenge \varphi & \iff \forall v(Rvw \Rightarrow X, V, v \vdash \varphi); \\
X, V, w \vdash \lozenge \Box \varphi & \iff \exists v(Rvw \text{ and } X, V, v \vdash \varphi).
\end{align*}
\]

\(^2\)Notice that we allow admissible valuations to interpret nominals as singletons, even if singletons might not be clopen. The admissibility restrictions are only for the propositional variables.
4.2 The first-order correspondence language and the standard translation

In the first-order correspondence language, we have two binary predicate symbols $R$ and $R'$ corresponding to the two binary relations in the Stone space with two relations, a set of unary predicate symbols $P$ corresponding to each propositional variable $p$.

**Definition 22.** The standard translation of the expanded language is defined as follows:

- $ST_x(p) := Px$;
- $ST_x(\bot) := \bot$;
- $ST_x(\top) := \top$;
- $ST_x(i) := x = i$;
- $ST_x(\neg \varphi) := \neg ST_x(\varphi)$;
- $ST_x(\varphi \land \psi) := ST_x(\varphi) \land ST_x(\psi)$;
- $ST_x(\varphi \lor \psi) := ST_x(\varphi) \lor ST_x(\psi)$;
- $ST_x(\varphi \rightarrow \psi) := ST_x(\varphi) \rightarrow ST_x(\psi)$;
- $ST_x(\Box \varphi) := \forall_y (R'xy \rightarrow ST_y(\varphi))$;
- $ST_x(\Diamond \varphi) := \exists_y (R'xy \land ST_y(\varphi))$;
- $ST_x(\square \varphi) := \forall_y (Ryx \rightarrow ST_y(\varphi))$;
- $ST_x(\lozenge \varphi) := \exists_y (Ryx \land ST_y(\varphi))$;
- $ST_x(\lozenge \varphi) := \forall_y (Ryx \rightarrow ST_y(\varphi))$;
- $ST_x(\lozenge \varphi) := \exists_y (Ryx \land ST_y(\varphi))$;
- $ST(\varphi \leq \psi) := \forall x (ST_x(\varphi) \rightarrow ST_x(\psi))$;
- $ST(\varphi \prec \psi) := \forall x (ST_x(\Diamond \varphi) \rightarrow ST_x(\psi))$;
- $ST(\varphi_1 \triangleleft_1 \psi_1 \& \ldots \& \varphi_n \triangleleft_n \psi_n) := ST(\varphi_1 \triangleleft_1 \psi_1) \ldots \land ST(\varphi_n \triangleleft_n \psi_n)$;
- $ST(\varphi \triangleleft \psi) \Rightarrow \varphi \triangleleft \psi) := ST(\varphi \triangleleft \psi) \rightarrow ST(\varphi \triangleleft \psi)$.

It is easy to see that this translation is correct:

**Proposition 23.** For any Stone space with two relations $X$, any valuation $V$ on $X$, any $w \in X$ and any expanded language formula $\varphi$,

$$X, V, w \models \varphi \iff X, V \models ST_x(\varphi)[w].$$

**Proposition 24.** For any Stone space with two relations $X$, any valuation $V$ on $X$, and inequality $\text{Ineq}$, meta-conjunction of inequalities $\text{MetaConIneq}$, quasi-inequality $\text{Quasi}$,

$$X, V \not\models \text{Ineq} \iff X, V \not\models ST(\text{Ineq});$$

$$X, V \not\models \text{MetaConIneq} \iff X, V \not\models ST(\text{MetaConIneq});$$

$$X, V \not\models \text{Quasi} \iff X, V \not\models ST(\text{Quasi}).$$
5 Inductive Quasi-Inequalities for Modal Subordination Algebras

In this section, we define inductive quasi-inequalities for modal subordination algebras and Stone spaces with two relations. Here we consider quasi-inequalities of the form

\[ \varphi_1 \leq \psi_1 \land \ldots \land \varphi_n \leq \psi_n \land \gamma_1 < \delta_1 \land \ldots \land \gamma_m < \delta_m \Rightarrow \alpha \ll \beta \quad (\text{where } \alpha \in \{<, \leq\}), \]

where \( n, m \geq 0 \). We follow the presentation of [12].

Definition 25 (Order-type of propositional variables). (cf. [11, page 346]) For an \( n \)-tuple \((p_1, \ldots, p_n)\) of propositional variables, an order-type \( \varepsilon \) of \((p_1, \ldots, p_n)\) is an element in \([1, \partial]^n\). We say that \( p_1 \) has order-type \( \partial \) if \( \varepsilon(i) = 1 \), and denote \( \varepsilon(p_1) = 1 \) or \( \varepsilon(i) = 1 \); we say that \( p_i \) has order-type \( \partial \) if \( \varepsilon(i) = \partial \), and denote \( \varepsilon(p_i) = \partial \) or \( \varepsilon(i) = \partial \).

Definition 26 (Signed generation tree). (cf. [12, Definition 4]) The positive (resp. negative) generation tree of \( \theta \) is defined by first labelling the root of the generation tree of \( \theta \) with + (resp. −) and then labelling the children nodes as follows:

- Assign the same sign to the children nodes of any node labelled with \( \Box, \Diamond, \lozenge, \blacklozenge, \blacklozenge, \lozenge, \lor, \land; \)
- Assign the opposite sign to the child node of any node labelled with −;
- Assign the opposite sign to the first child node and the same sign to the second child node of any node labelled with →.

Nodes in signed generation trees are positive (resp. negative) if they are signed + (resp. −).

Signed generation trees will be used in the quasi-inequalities

\[ \varphi_1 \leq \psi_1 \land \ldots \land \varphi_n \leq \psi_n \land \gamma_1 < \delta_1 \land \ldots \land \gamma_m < \delta_m \Rightarrow \alpha \ll \beta \quad (\text{where } \alpha \in \{<, \leq\}), \]

where the positive generation trees \(+\varphi_i, +\delta_j, +\alpha\) and the negative generation trees \(\neg\varphi_i, \neg\gamma_j, \neg\beta\) will be considered. We will also say that a quasi-inequality is uniform in a variable \( p_i \) if all occurrences of \( p_i \) in \(+\varphi_i, +\delta_j, +\alpha, \neg\varphi_i, \neg\gamma_j, \neg\beta\) have the same sign, and that a quasi-inequality is \( \varepsilon \)-uniform in an array \( \vec{\beta} \) if it is uniform in \( p_i \), occurring with the sign indicated by \( \varepsilon \) (i.e., \( p_i \) has the sign + if \( \varepsilon(p_i) = 1 \), and has the sign − if \( \varepsilon(p_i) = \partial \), for each propositional variable \( p_i \) in \( \vec{\beta} \).

For any given formula \( \theta(p_1, \ldots, p_n) \), any order-type \( \varepsilon \) over \( n \), and any \( 1 \leq i \leq n \), an \( \varepsilon \)-critical node in a signed generation tree of \( \theta \) is a leaf node \(+p_i\) when \( \varepsilon(i) = 1 \) or \( \neg p_i \) when \( \varepsilon(i) = \partial \). An \( \varepsilon \)-critical branch in a signed generation tree is a branch from an \( \varepsilon \)-critical nodes. The \( \varepsilon \)-critical occurrences are intended to be those which the algorithm ALBA will solve for. We say that \(+\theta\) (resp. \(\neg\theta\)) is \( \varepsilon \)-uniform, and write \( \varepsilon(+)\) (resp. \( \varepsilon(\neg)\)) if every leaf node in the signed generation tree of \(+\theta\) (resp. \(\neg\theta\)) is \( \varepsilon \)-critical.

We will also use the notation \( +\iota \ll +\theta \) (resp. \( +\neg \ll +\theta \)) to indicate that an occurrence of a subformula \( \iota \) inherits the positive (resp. negative) sign from the signed generation tree \(+\theta\), where \( * \in \{+, \neg\} \). We will write \( \varepsilon(i) \ll +\theta \) (resp. \( \varepsilon^\partial(i) \ll +\theta \)) to indicate that the signed generation subtree \( \iota \), with the sign inherited from \(+\theta\), is \( \varepsilon \)-uniform (resp. \( \varepsilon^\partial \)-uniform). We say that a propositional variable \( p \) is positive (resp. negative) in \( \theta \) if \(+p \ll +\theta\) (resp. \(-p \ll +\theta\)).

In what follows, we will use the following classification of nodes:

Definition 27. (Classification of nodes, cf. [12, Definition 5]) Nodes in signed generation trees are called \( \Delta \)-adjoints, syntactically left residual (SLR), syntactically right adjoint (SRA), and syntactically right residual (SRR), according to Table [11].

Definition 28 (Good/PIA/Skeleton branches). A branch in a signed generation tree is called a

\footnote{For a detailed explanation why these names are used, see [22, Remark 3.24].}
Table 1: Skeleton and PIA nodes.

| Skeleton | PIA |
|----------|-----|
| \(\Delta\)-adjoints | SRA |
| + \(\lor\) | + \(\land\) \(\square\) \(\Diamond\) |
| \(\land\) | \(\lor\) \(\land\) \(\Diamond\) |

- **good branch** if it is the concatenation of two paths \(P_1\) and \(P_2\), one of which might be of length 0, such that \(P_1\) is a path from the leaf consisting (apart from variable nodes) of PIA-nodes only, and \(P_2\) consists (apart from variable nodes) of Skeleton-nodes only;

- **PIA branch** if it is a good branch and \(P_2\) is of length 0;

- **Skeleton branch** if it is a good branch and \(P_1\) is of length 0;

**Definition 29** (Inductive signed generation trees). (cf. [12, Definition 6]) For any order-type \(\varepsilon\) and any irreflexive and transitive binary relation \(\prec_{\Omega}\) on \(p_1, \ldots, p_n\) (called dependence order on the variables), the signed generation tree \(*\theta\) (\(* \in \{-, +\}\)) of a formula \(\theta(p_1, \ldots p_n)\) is \((\Omega, \varepsilon)\)-inductive if

1. for all \(1 \leq i \leq n\), every \(\varepsilon\)-critical branch with leaf \(p_i\) is good;
2. every SRR-node in an \(\varepsilon\)-critical branch is either \(\star(i, \eta)\) or \(\star(\eta, i)\), where \(\star\) is a binary connective, the \(\varepsilon\)-critical branch goes through \(\eta\), and
   (a) \(\varepsilon^\beta(i) \ll \star\theta\) (cf. page 11), and
   (b) \(p_k \prec_{\Omega} p_i\) for every \(p_k\) that occurs in \(i\).

The definitions above are mostly straightforward adaptations from standard ALBA setting for inequalities to the present paper. The following two definitions are specific in the quasi-inequality setting. Intuitively, the receiving inequality \(\varphi \leq \psi\) (resp. \(\gamma \prec \delta\)) is used to receive minimal valuations, while the solvable inequality \(\varphi \leq \psi\) (resp. \(\gamma \prec \delta\)) will be transformed into inequalities with minimal valuation \(\theta \leq p\) (if \(\varepsilon(p) = 1\)) or \(p \leq \theta\) (if \(\varepsilon(p) = 0\)), such that all propositional variables \(q\) occurring in \(\theta\) has dependence order below \(p\), i.e. \(q \prec_{\Omega} p\).

**Definition 30** (Receiving inequality). An inequality \(\varphi \leq \psi\) (resp. \(\gamma \prec \delta\)) is said to be \((\Omega, \varepsilon)\)-receiving, if both of \(-\varphi, +\psi\) (resp. \(-\gamma, +\delta\)) are \(\varepsilon^\beta\)-uniform;

**Definition 31** (Solvable inequality). An inequality \(\varphi \leq \psi\) (resp. \(\gamma \prec \delta\)) is said to be \((\Omega, \varepsilon)\)-solvable, if

- exactly one of \(-\varphi, +\psi\) (resp. \(-\gamma, +\delta\)) is \(\varepsilon^\beta\)-uniform (without loss of generality we denote the \(\varepsilon^\beta\)-uniform one \(\star\theta\) and the other one \(\ast\iota\));
- \(\ast\iota\) is \((\Omega, \varepsilon)\)-inductive, and all \(\varepsilon\)-critical branches in \(\ast\iota\) are PIA branches;
- for all the \(\varepsilon\)-critical branches in \(\ast\iota\) ending with \(p\), all propositional variables \(q\) in \(\star\theta\), we have \(q \prec_{\Omega} p\).

**Definition 32**. A quasi-inequality

\[ \varphi_1 \leq \psi_1 \land \ldots \land \varphi_n \leq \psi_n \land \gamma_1 < \delta_1 \land \ldots \land \gamma_m < \delta_m \Rightarrow \alpha \ast \beta \text{ (where } \ast \in \prec, \leq) \]

is \((\Omega, \varepsilon)\)-inductive if
each inequality \( \varphi \leq \psi \) and \( \gamma < \delta \) is either \((\Omega, \varepsilon)\)-receiving or \((\Omega, \varepsilon)\)-solvable;

- \(+\alpha \) and \( -\beta \) are \((\Omega, \varepsilon)\)-inductive signed generation trees;

A quasi-inequality is inductive if it is \((\Omega, \varepsilon)\)-inductive for some \((\Omega, \varepsilon)\).

6 Algorithm

In the present section, we define the algorithm ALBA which compute the first-order correspondence of the input quasi-inequality in the style of [11]. The algorithm ALBA proceeds in three stages. Firstly, ALBA receives a quasi-inequality

\[
\varphi_1 \leq \psi_1 \land \ldots \land \varphi_n \leq \psi_n \land \gamma_1 < \delta_1 \land \ldots \land \gamma_m < \delta_m \Rightarrow \alpha \triangleleft \beta \quad \text{(where } \triangleleft \in \{\leq, <\} \text{)}
\]

as input, which do not contain nominals or black connectives \( \Diamond, \blacksquare, \square, \neg \).

1. Preprocessing and first approximation:

(a) In each inequality \( \theta \triangleleft \eta \) (where \( \triangleleft \in \{\leq, <\} \) ) in the quasi-inequality, consider the signed generation trees \( +\theta \) and \( -\eta \), apply the distribution rules:

i. Push down \( +\Diamond, +\Diamond \cdot, -\neg, +\land, -\rightarrow \) by distributing them over nodes labelled with \(+\lor\) which are Skeleton nodes (see Figure 1), and

ii. Push down \( -\square, -\square, +\neg, -\lor, -\rightarrow \) by distributing them over nodes labelled with \(-\land\) which are Skeleton nodes (see Figure 2).

\[
\begin{align*}
+\Diamond & \implies +\lor \\
+\lor & \implies -\neg \\
+\alpha +\beta & \implies +\alpha +\beta
\end{align*}
\]

Figure 1: Distribution rules for \(+\lor\)

(b) Apply the splitting rules to each inequality occurring in the quasi-inequality:

\[
\begin{align*}
\theta \leq \eta \land \iota & \quad \frac{\theta \leq \iota \land \eta \leq \iota}{\theta \leq \eta \land \theta \leq \iota} \\
\theta \lor \eta \leq \iota & \quad \frac{\theta \leq \iota \land \eta \leq \iota}{\theta \leq \iota \land \eta \leq \iota} \\
\theta < \eta \land \iota & \quad \frac{\theta < \iota \land \eta < \iota}{\theta < \eta \land \theta < \iota} \\
\theta \lor \eta < \iota & \quad \frac{\theta < \iota \land \eta < \iota}{\theta < \iota \land \eta < \iota}
\end{align*}
\]
Now we have a quasi-inequality of the form
\[ \varphi \leq \psi \Rightarrow \alpha \leq \beta, \]
which we split into a meta-conjunction of quasi-inequalities
\[ \varphi \leq \psi \Rightarrow \alpha \leq \beta, \]
where \( \alpha \leq \beta \) belong to \( \overline{\varphi} \leq \overline{\psi} \).

We denote Preprocess:=\{\( \varphi \leq \psi \Rightarrow \alpha \leq \beta : \alpha \leq \beta \) belong to \( \overline{\varphi} \leq \overline{\psi} \)\}.

Now for each quasi-inequality in Preprocess, we apply the following first-approximation rule:
\[ \varphi \leq \psi \Rightarrow \alpha \leq \beta \]
\[ \varphi \leq \psi \& i_0 \leq \alpha \& \beta \leq \neg i_1 \Rightarrow i_0 \leq \neg i_1 \]

Now for each quasi-inequality, we focus on the set of its antecedent inequalities \( \{ \varphi \leq \psi, i_0 \leq \alpha, \beta \leq \neg i_1 \} \), which we call a system.
2. The reduction-elimination cycle:

In this stage, for each \( \varphi \leq \psi, i_0 \leq \alpha, \beta \leq -i_1 \), we apply the following rules together with the splitting rules in the previous stage to eliminate all the propositional variables in the set of inequalities:

(a) Residuation rules:

\[
\frac{\diamond \theta \leq i}{\theta \leq \square i} \quad \text{(\(\diamond\)-Res)} \\
\frac{\neg \theta \leq i}{\neg \leq \theta} \quad \text{(-Res-Left)} \\
\frac{\theta \leq i}{\square \leq \theta} \quad \text{(\(\square\)-Res)} \\
\frac{\theta \leq -i}{\neg \leq \theta} \quad \text{(-Res-Right)} \\
\frac{\theta \land i \leq \eta}{\theta \leq i \to \eta} \quad \text{(\(\land\)-Res-1)} \\
\frac{\theta \land \neg \leq i}{\theta \land \neg \leq i} \quad \text{(\(\land\)-Res-2)} \\
\frac{\theta \leq i \to \eta}{\theta \land \eta \leq \eta} \quad \text{(-Res-1)} \\
\frac{\theta \leq i \to \eta}{\theta \land \eta \leq \eta} \quad \text{(-Res-2)} \\
\frac{\diamond \theta \leq i}{\theta \leq \square i} \quad \text{(\(\diamond\)-Res)} \\
\frac{\theta \leq \square i}{\square \leq \theta} \quad \text{(\(\square\)-Res)}
\]

(b) Approximation rules:

\[
\frac{i \leq \diamond \theta}{j \leq \theta \leq \diamond j} \quad \frac{\square \theta \leq -i}{\theta \leq -j \land \neg \leq -i} \\
\frac{i \leq \diamond \theta}{j \leq \theta \leq \diamond j} \quad \frac{\square \theta \leq -i}{\theta \leq -j \land \neg \leq -i} \\
\frac{\alpha \to \beta \leq -i}{j \leq \alpha \land \beta \leq \neg \leq \neg i} \quad \frac{j \to \neg \leq \neg i}{j \to \neg \leq \neg i}
\]

The nominals introduced by the approximation rules must not occur in the system before applying the rule.

(c) The Ackermann rules. These two rules are the core of ALBA, since their application eliminates propositional variables. In fact, all the preceding steps are aimed at reaching a shape in which the rules can be applied. Notice that an important feature of these rules is that they are executed on the whole set of inequalities, and not on a single inequality.

The right-handed Ackermann rule:

\[
\begin{align*}
\theta_1 \leq p \\
\vdots \\
\theta_n \leq p \\
\eta_1 \leq \tau_1 \\
\vdots \\
\eta_m \leq \tau_m
\end{align*}
\]

is replaced by

\[
\begin{align*}
\eta_1((\theta_1 \lor \ldots \lor \theta_n)/p) \leq \tau_1((\theta_1 \lor \ldots \lor \theta_n)/p) \\
\vdots \\
\eta_m((\theta_1 \lor \ldots \lor \theta_n)/p) \leq \tau_m((\theta_1 \lor \ldots \lor \theta_n)/p)
\end{align*}
\]

where:
i. $p$ does not occur in $\theta_1, \ldots, \theta_n$;
ii. Each $\eta_i$ is positive, and each $\iota_i$ negative in $p$, for $1 \leq i \leq m$.

The left-handed Ackermann rule:

\[
\begin{align*}
\text{The system } & \quad \begin{cases}
p \leq \theta_1 \\
p \leq \theta_n \\
\eta_1 \leq \iota_1 \\
\vdots \\
\eta_m \leq \iota_m
\end{cases} \\
is replaced by \quad \begin{cases}
\eta_1((\theta_1 \wedge \ldots \wedge \theta_n)/p) \leq \iota_1((\theta_1 \wedge \ldots \wedge \theta_n)/p) \\
\vdots \\
\eta_m((\theta_1 \wedge \ldots \wedge \theta_n)/p) \leq \iota_m((\theta_1 \wedge \ldots \wedge \theta_n)/p)
\end{cases}
\end{align*}
\]

where:

i. $p$ does not occur in $\theta_1, \ldots, \theta_n$;
ii. Each $\eta_i$ is negative, and each $\iota_i$ positive in $p$, for $1 \leq i \leq m$.

3. **Output**: If in the previous stage, for some set of inequalities, the algorithm gets stuck, i.e. some propositional variables cannot be eliminated by the application of the reduction rules, then the algorithm halts and output “failure”. Otherwise, each initial set of inequalities after the first approximation has been reduced to a set of pure inequalities \(\text{Reduce}(\varphi \leq \psi, i_0 \leq \alpha, \beta \leq \neg i_1)\), and then the output is a set of quasi-inequalities \(\{\&\text{Reduce}(\varphi \leq \psi, i_0 \leq \alpha, \beta \leq \neg i_1) \Rightarrow i_0 \leq \neg i_1, \varphi \leq \psi \Rightarrow \alpha \leq \beta \in \text{Preprocess}\}\). Then we can use the standard translation of the set of quasi-inequalities to obtain the first-order correspondence.

### 7 Soundness

In this section we show the soundness of the algorithm with respect to the arbitrary valuations. The soundness proof follows the style of [11]. For some of the rules, the soundness proofs are the same to existing literature and hence are omitted, so we only give details for the proofs which are different.

**Theorem 7.1** (Soundness). *If ALBA runs successfully on an input quasi-inequality Quasi and outputs a first-order formula FO(Quasi), then for any Stone space with two relations \((X, \tau, R, R')\),

\[X \vDash_p \text{Quasi} \iff X \vDash \text{FO(Quasi)}.\]

*Proof.* The proof goes similarly to [11] Theorem 8.1. Let

\[\varphi \leq \psi \Rightarrow \alpha \bowtie \beta\]

denote the input quasi-inequality Quasi, let

\[\varphi_i \leq \psi_i \Rightarrow \alpha_i \leq \beta_i, i \in I\]

denote the quasi-inequalities before the first-approximation rule, let

\[\varphi_i \leq \psi_i \& i_{i,0} \leq \alpha_i \& \beta_i \leq \neg i_{i,1} \Rightarrow i_{i,0} \leq \neg i_{i,1}, i \in I\]

denote the quasi-inequalities after the first-approximation rule, let

\[\text{Reduce}(\varphi_i \leq \psi_i, i_{i,0} \leq \alpha, \beta \leq \neg i_{i,1}), i \in I\]

denote the sets of inequalities after Stage 2, let

\[\text{FO(Quasi)}\]
denote the standard translation of the quasi-inequalities into first-order formulas, then it suffices to show the equivalence from (1) to (5) given below:

(1) \( X \vDash_P \overline{\varphi} \leq \overline{\psi} \Rightarrow \alpha \prec \beta \)
(2) \( X \vDash_P \overline{\varphi}_i \leq \overline{\psi}_i \Rightarrow \alpha_i \leq \beta_i, \text{ for all } i \in I \)
(3) \( X \vDash_P \overline{\varphi}_i \leq \overline{\psi}_i \& \overline{\iota}_{i,0} \leq \alpha_i \& \beta_i \leq \overline{\iota}_{i,1} \Rightarrow \overline{\iota}_{i,0} \leq \overline{\iota}_{i,1}, i \in I \)
(4) \( X \vDash_P \text{Reduce}(\overline{\varphi}_i \leq \overline{\psi}_i, \overline{\iota}_{i,0} \leq \alpha, \beta \leq \overline{\iota}_{i,1}) \Rightarrow \overline{\iota}_{i,0} \leq \overline{\iota}_{i,1}, i \in I \)
(5) \( X \vDash \text{FO(Quasi)} \)

- The equivalence between (1) and (2) follows from Proposition 33
- the equivalence between (2) and (3) follows from Proposition 34
- the equivalence between (3) and (4) follows from Propositions 35, 36
- the equivalence between (4) and (5) follows from Proposition 24

In the remainder of this section, we prove the soundness of the rules in Stage 1, 2 and 3.

**Proposition 33** (Soundness of the rules in Stage 1). *For the distribution rules, the splitting rules, the monotone and antitone variable-elimination rules and the subordination rewriting rule, they are sound in both directions in \( X \).*

**Proof.** 1. For the soundness of the distribution rules, it follows from the fact that the corresponding distribution laws are valid in \( X \), which can be found in [27, Proposition 6.2].

2. For the soundness of the splitting rules, the rules involving \( \leq \) are the same to the same rules in [27, Proposition 6.2]. For the rules involving \( \prec \), it follows from the following fact:

\[
X, V \vDash \theta < \eta \land t
\]
iff \( R[V(\theta)] \subseteq V(\eta) \land V(t) \)
iff \( R[V(\theta)] \subseteq V(\eta) \cap V(t) \)
iff \( X, V \vDash \theta < \eta \land X, V \vDash \theta < t. \)

\[
X, V \vDash \theta \lor \eta < t
\]
iff \( R[V(\theta) \cup V(\eta)] \subseteq V(t) \)
iff \( R[V(\theta)] \cup R[V(\eta)] \subseteq V(t) \)
iff \( R[V(\theta)] \subseteq V(t) \) and \( R[V(\eta)] \subseteq V(t) \)
iff \( X, V \vDash \theta < t \land X, V \vDash \eta < t. \)

3. For the soundness of the monotone and antitone variable elimination rules, we show the soundness for the first rule. Suppose \( p \) is positive in \( \overline{\varphi}, \overline{\psi}, \overline{\beta}, \overline{\chi} \) and negative in \( \overline{\varphi}, \overline{\delta}, \overline{\sigma}, \overline{\xi} \).

(\( \| \)): If
\( X \vDash_P \overline{\varphi}(p) \leq \overline{\psi}(p) \& \overline{\delta}(p) < \overline{\sigma}(p) \Rightarrow \overline{\alpha}(p) \leq \overline{\beta}(p) \& \overline{\xi}(p) < \overline{\chi}(p), \)
then for all valuations \( V \), we have
\( X, V \vDash \overline{\varphi}(p) \leq \overline{\psi}(p) \& \overline{\delta}(p) < \overline{\sigma}(p) \Rightarrow \overline{\alpha}(p) \leq \overline{\beta}(p) \& \overline{\xi}(p) < \overline{\chi}(p), \)
then for the valuation $V_p^\phi$ such that $V_p^\phi$ is the same as $V$ except that $V_p^\phi(p) = \emptyset$,
\[
X, V_p^\phi \models \overline{\psi}(p) \leq \overline{\psi}(\bot) \land \overline{\gamma}(p) < \overline{\delta}(p) \Rightarrow \overline{\alpha}(p) \leq \overline{\beta}(p) \land \overline{\xi}(p) < \overline{\chi}(p),
\]
therefore
\[
X, V \models \overline{\alpha}(\bot) \leq \overline{\psi}(\bot) \land \overline{\gamma}(\bot) < \overline{\delta}(\bot) \Rightarrow \overline{\alpha}(\bot) \leq \overline{\beta}(\bot) \land \overline{\xi}(\bot) < \overline{\chi}(\bot),
\]
so
\[
X \models_{p} \overline{\alpha}(\bot) \leq \overline{\psi}(\bot) \land \overline{\gamma}(\bot) < \overline{\delta}(\bot) \Rightarrow \overline{\alpha}(\bot) \leq \overline{\beta}(\bot) \land \overline{\xi}(\bot) < \overline{\chi}(\bot).
\]
(‡): For the other direction, consider any valuation $V$ on $X$, by the fact that $p$ is positive in $\overline{\gamma}, \overline{\beta}, \overline{\chi}$ and negative in $\overline{\psi}, \overline{\delta}, \overline{\alpha}, \overline{\xi}$, we have that
\[
X, V \models \overline{\psi}(p) \leq \overline{\psi}(\bot)
\]
\[
X, V \models \overline{\delta}(p) \leq \overline{\delta}(\bot)
\]
\[
X, V \models \overline{\alpha}(p) \leq \overline{\alpha}(\bot)
\]
\[
X, V \models \overline{\xi}(p) \leq \overline{\xi}(\bot),
\]
and
\[
X, V \models \overline{\alpha}(\bot) \leq \overline{\gamma}(p)
\]
\[
X, V \models \overline{\gamma}(\bot) \leq \overline{\gamma}(p)
\]
\[
X, V \models \overline{\beta}(\bot) \leq \overline{\beta}(p)
\]
\[
X, V \models \overline{\chi}(\bot) \leq \overline{\chi}(p).
\]
Suppose that
\[
X, V \models \overline{\alpha}(\bot) \leq \overline{\psi}(p) \land \overline{\gamma}(p) < \overline{\delta}(p),
\]
then
\[
X, V \models \overline{\alpha}(\bot) \leq \overline{\psi}(p) \leq \overline{\psi}(\bot) \land \overline{\gamma}(\bot) \leq \overline{\gamma}(p) < \overline{\delta}(p) \leq \overline{\delta}(\bot),
\]
so
\[
X, V \models \overline{\alpha}(\bot) \leq \overline{\psi}(\bot) \land \overline{\gamma}(\bot) < \overline{\delta}(\bot),
\]
therefore by assumption that
\[
X \models_{p} \overline{\alpha}(\bot) \leq \overline{\psi}(\bot) \land \overline{\gamma}(\bot) < \overline{\delta}(\bot) \Rightarrow \overline{\alpha}(\bot) \leq \overline{\beta}(\bot) \land \overline{\xi}(\bot) < \overline{\chi}(\bot),
\]
we have
\[
X, V \models \overline{\alpha}(\bot) \leq \overline{\beta}(\bot) \land \overline{\xi}(\bot) < \overline{\chi}(\bot),
\]
so
\[
X, V \models \overline{\alpha}(p) \leq \overline{\alpha}(\bot) \leq \overline{\beta}(\bot) \leq \overline{\beta}(p) \land \overline{\xi}(p) \leq \overline{\xi}(\bot) < \overline{\chi}(\bot) \leq \overline{\chi}(p),
\]
then
\[
X, V \models \overline{\alpha}(p) \leq \overline{\beta}(p) \land \overline{\xi}(p) < \overline{\chi}(p),
\]
so we get that for all $V$,
\[
X, V \models \overline{\alpha}(p) \leq \overline{\psi}(p) \land \overline{\gamma}(p) < \overline{\delta}(p) \Rightarrow \overline{\alpha}(p) \leq \overline{\beta}(p) \land \overline{\xi}(p) < \overline{\chi}(p),
\]
so
\[
X \models_{p} \overline{\alpha}(p) \leq \overline{\psi}(p) \land \overline{\gamma}(p) < \overline{\delta}(p) \Rightarrow \overline{\alpha}(p) \leq \overline{\beta}(p) \land \overline{\xi}(p) < \overline{\chi}(p).
\]
The soundness of the other variable elimination rule is similar.
4. For the subordination rewriting rule, its soundness follows from the following fact:

\[
X, V \models \theta < \eta \\
\text{iff } R[V(\theta)] \subseteq V(\eta) \\
\text{iff } V(\bigtriangleup \theta) \subseteq V(\eta) \\
\text{iff } X, V \models \bigtriangleup \theta \leq \eta.
\]

\[\square\]

**Proposition 34.** (2) and (3) are equivalent, i.e. the first-approximation rule is sound in \(\mathbb{F}\).

**Proof.** (2) \(\Rightarrow\) (3): Suppose \(X \vdash P \vec{\phi} \leq \psi \Rightarrow \alpha \leq \beta\). Then for any valuation \(V\), if \(V(\vec{\phi}) \subseteq V(\psi)\), \(X, V \vdash i_0 \leq \alpha\) and \(X, V \vdash \beta \leq -i_1\), then \(X, V, V(i_0) \vdash \alpha\) and \(X, V, V(i_1) \vdash \beta\), so by \(X \vdash P \vec{\phi} \leq \psi \Rightarrow \alpha \leq \beta\) we have \(X, V, V(i_0) \vdash \beta\), so \(V(i_0) \neq V(i_1)\), so \(X, V \vdash i_0 \leq -i_1\).

(3) \(\Rightarrow\) (2): Suppose \(X \vdash P \vec{\phi} \leq \psi \& i_0 \leq \alpha \& \beta \leq -i_1 \Rightarrow i_0 \leq -i_1\). If \(X \not\vdash P \vec{\phi} \leq \psi \Rightarrow \alpha \leq \beta\), there is a valuation \(V\) and a \(w \in X\) such that \(V(\vec{\phi}) \subseteq V(\psi)\), \(X, V, w \vdash \alpha\) and \(X, V, w \not\vdash \beta\). Then by taking \(V_{w,w}^{i_0,i_1}\) to be the valuation which is the same as \(V\) except that \(V_{w,w}^{i_0,i_1}(i_0) = V_{w,w}^{i_0,i_1}(i_1) = \{w\}\), since \(i_0, i_1\) do not occur in \(\alpha\) and \(\beta\), we have that \(X, V_{w,w}^{i_0,i_1}, w \vdash \alpha\) and \(X, V_{w,w}^{i_0,i_1}, w \not\vdash \beta\), therefore \(X, V_{w,w}^{i_0,i_1}, i_0 \leq \alpha\) and \(X, V_{w,w}^{i_0,i_1}, \beta \leq -i_1\). It is easy to see that \(V_{w,w}^{i_0,i_1}(\vec{\phi}) \subseteq V_{w,w}^{i_0,i_1}(\vec{\psi})\). By \(X \vdash P \vec{\phi} \leq \psi \& i_0 \leq \alpha \& \beta \leq -i_1 \Rightarrow i_0 \leq -i_1\), we have that \(X, V_{w,w}^{i_0,i_1}, i_0 \leq -i_1\), so \(X, V_{w,w}^{i_0,i_1}, w \vdash i_0\) implies that \(X, V_{w,w}^{i_0,i_1}, w \vdash -i_1\), a contradiction. So \(X \vdash P \vec{\phi} \leq \psi \Rightarrow \alpha \leq \beta\). \[\square\]

The next step is to show the soundness of each rule of Stage 2. For each rule, before the application of this rule we have a set of inequalities \(S\) (which we call the system), after applying the rule we get a set of inequalities \(S'\), the soundness of Stage 2 is then the equivalence of the following two conditions:

- \(X \vdash P \& S \Rightarrow i_0 \leq -i_1;\)
- \(X \vdash P \& S' \Rightarrow i_0 \leq -i_1;\)

where \(\& S\) denote the meta-conjunction of inequalities of \(S\). It suffices to show the following property:

- For any \(X\), any valuation \(V\), if \(X, V \not\vdash S\), then there is a valuation \(V'\) such that \(V'(i_0) = V(i_0)\), \(V'(i_1) = V(i_1)\) and \(X, V' \vdash S'\);
- For any \(X\), any valuation \(V'\), if \(X, V' \not\vdash S'\), then there is a valuation \(V\) such that \(V(i_0) = V'(i_0)\), \(V(i_1) = V'(i_1)\) and \(X, V \vdash S\).

**Proposition 35.** The splitting rules, the approximation rules and the residuation rules in Stage 2 are sound in both directions in \(X\).

**Proof.** The soundness proofs are the same to the soundness of the same rules in [27] Proposition 6.4-Lemma 6.8, Lemma 6.11-6.12]. \[\square\]

**Proposition 36.** The Ackermann rules are sound in \(X\).

**Proof.** The proof is similar to the soundness of the Ackermann rules in [11] Lemma 4.2, 4.3, 8.4]. We only prove it for the right-handed Ackermann rule, the left-handed Ackermann rule is similar. Without loss of generality we assume that \(n = 1\). By the discussion on page [19] it suffices to show the following right-handed Ackermann lemma:

**Lemma 37** (Right-Handed Ackermann Lemma). Let \(\theta\) be a formula which does not contain \(p\), let \(\eta_i(p)\) (resp. \(\iota_i(p)\)) be positive (resp. negative) in \(p\) for \(1 \leq i \leq m\), and let \(\bar{q}\) (resp. \(\bar{f}\)) be all the propositional variables (resp. nominals) occurring in \(\eta_1(p), \ldots, \eta_m(p), \iota_1(p), \ldots, \iota_m(p), \theta\) other than \(p\). Then for all \(\bar{a} \in P(X), \bar{x} \in X\), the following are equivalent:
1. \( V(\eta(\theta/p)) \subseteq V(\iota(\theta/p)) \) for \( 1 \leq i \leq m \), where \( V(\bar{d}) = \bar{a} \), \( V(\bar{J}) = \{\bar{x}\} \).

2. there exists \( a_0 \in P(X) \) such that \( V'(\theta) \leq a_0 \) and \( V'(\eta(p)) \subseteq V'(\iota(p)) \) for \( 1 \leq i \leq m \), where \( V' \) is the same as \( V \) except that \( V'(p) = a_0 \).

\( \Rightarrow \): Take \( V' \) such that \( V' \) is the same as \( V \) except that \( V'(p) = V(\theta) \). Since \( \theta \) does not contain \( p \), it is easy to see that \( V(\theta) = V'(\theta) \). Therefore we can take \( a_0 := V(\theta) \) and get 2.

\( \Leftarrow \): It is obvious by monotonicity.

8 Success

In this section, we show the success of the algorithm on inductive quasi-inequalities. Notice that here we do not allow the input quasi-inequality to contain any black connectives \( \textbullet, \text{■, ■, ◆} \), but we allow \( \text{◇} \) and \( \text{□} \) to occur in the input quasi-inequality.

Definition 38 (Definite \((\Omega, \varepsilon)\)-inductive signed generation tree). Given a dependence order \( \prec_{\Omega} \), an order type \( \varepsilon, * \in \{-, +\} \), the signed generation tree \(*\theta\) of the formula \( \theta(p_1, \ldots, p_n) \) is definite \((\Omega, \varepsilon)\)-inductive if there is no \(+\lor, -\land\) occurring in the Skeleton part on an \( \varepsilon \)-critical branch.

Lemma 39. Given an input \((\Omega, \varepsilon)\)-inductive quasi-inequality

\[ \varphi \leq \psi \land \gamma < \delta \Rightarrow \alpha \land \beta \text{ (where } \alpha \in \{\leq, <\}) \],

after the first stage, it is transformed into a quasi-inequality of the form

\[ \varphi \leq \psi \land \text{◇} \gamma \leq \delta \Rightarrow \alpha \leq \beta \land \text{□} \xi \leq \chi, \]

and further of the form

\[ \varphi \leq \psi \land \text{◇} \gamma \leq \delta \Rightarrow \text{◇} \xi \leq \chi, \]

or

\[ \varphi \leq \psi \land \text{◇} \gamma \leq \delta \Rightarrow \text{□} \xi \leq \chi, \]

where

- each \( \varphi \leq \psi \) or \( \text{◇} \gamma \leq \delta \) is either \((\Omega, \varepsilon)\)-receiving or \((\Omega, \varepsilon)\)-solvable;
- \(+\alpha\) and \(-\beta\) (resp. \(\text{◇} \xi\) and \(-\chi\)) are definite \((\Omega, \varepsilon)\)-inductive signed generation trees;
- each formula contains no black connectives.

Proof. It is easy to see that by applying the distribution rules, in each inequality \( \theta \leq \eta \) or \( \theta < \eta \), consider the signed generation trees \(+\theta\) and \(-\eta\), all occurrences of \(+\lor\) and \(-\land\) in the Skeleton part of an \( \varepsilon \)-critical branch have been pushed up towards the root of the signed generation trees \(+\theta\) and \(-\eta\). Then by exhaustively applying the splitting rules, all such \(+\lor\) and \(-\land\) are eliminated.

Since by applying the distribution rules, the splitting rules and the monotone/antitone variable elimination rules do not change the \((\Omega, \varepsilon)\)-inductiveness of a signed generation tree for \(+\alpha\), \(-\beta\) (resp. \(\text{◇} \xi\), \(-\chi\)), and do not change the property of being \((\Omega, \varepsilon)\)-receiving or \((\Omega, \varepsilon)\)-solvable for \( \varphi \leq \psi \) or \( \gamma < \delta \), so in

\[ \varphi \leq \psi \land \text{◇} \gamma \leq \delta \Rightarrow \text{◇} \xi \leq \chi, \]

- each \( \varphi \leq \psi \) or \( \text{◇} \gamma \leq \delta \) is either \((\Omega, \varepsilon)\)-receiving or \((\Omega, \varepsilon)\)-solvable;
- each pair of \(+\alpha\) and \(-\beta\) (resp. \(\text{◇} \xi\) and \(-\chi\)) are definite \((\Omega, \varepsilon)\)-inductive signed generation trees.
It is easy to see that no formula contains black connectives since the input quasi-inequality contains no black connective and no rule in Stage 1 introduces them. □

**Lemma 40.** After first approximation, the input \((\Omega, \varepsilon)\)-inductive inequality

\[ \varphi \leq \psi, \vartheta < \delta \Rightarrow \alpha \triangleleft \beta \] (where \(\triangleleft \in \{\leq, <\}\))

is rewritten into the form

\[ \varphi \leq \psi, \vartheta \leq \delta \& i_0 \leq \alpha \& \beta \leq i_1 \Rightarrow i_0 \leq i_1 \]

or

\[ \varphi \leq \psi, \vartheta \leq \delta \& i_0 \leq \emptyset \& \chi \leq i_1 \Rightarrow i_0 \leq i_1, \]

where

- each \(\varphi \leq \psi \) or \(\emptyset \leq \delta\) is either \((\Omega, \varepsilon)\)-receiving or \((\Omega, \varepsilon)\)-solvable;
- +\(\alpha\) and -\(\beta\) (resp. +\(\emptyset\xi\) and -\(\chi\)) are definite \((\Omega, \varepsilon)\)-inductive signed generation trees;
- each formula contains no black connectives.

**Proof.** Straightforward. □

Now we proceed to Stage 2 and work with set of inequalities (i.e. systems).

\[ \{\varphi \leq \psi, \emptyset \leq \delta, i_0 \leq \alpha, \beta \leq i_1\} \]

or

\[ \{\varphi \leq \psi, \emptyset \leq \delta, i_0 \leq \emptyset \& \chi \leq i_1\}. \]

**Lemma 41.** For each inequality \(i_0 \leq \alpha\) or \(\beta \leq i_1\) (here we merge the \(i_0 \leq \emptyset \xi\) case into the \(i_0 \leq \alpha\) case, and merge the \(\chi \leq i_1\) case into the \(\beta \leq i_1\) case), by applying the approximation rules, splitting rules and residuation rules involving \(\neg\) exhaustively, we obtain a set of inequalities, each inequality of which contains no black connectives and belong to one of the three classes:

- it is pure;
- it is \((\Omega, \varepsilon)\)-receiving;
- it is of the form \(i \leq \alpha'\) or \(\beta' \leq i\), where +\(\alpha'\) and -\(\beta'\) are \((\Omega, \varepsilon)\)-inductive and their \(\varepsilon\)-critical branches contain only PIA nodes.

**Proof.** First of all, since the approximation rules, the splitting rules and the residuation rules involving \(\neg\) do not introduce black connectives and \(\alpha, \beta\) do not contain black connectives, so it is easy to see that the obtained inequalities do not contain black connectives. Then we prove by induction on the complexity of +\(\alpha\) and -\(\beta\) to show that the obtained inequalities can be classified into three classes mentioned above:

- When \(\alpha\) is \(p\) and \(\varepsilon(p) = 1\), then it is obvious that +\(p\) is critical, \(i_0 \leq \alpha\) belongs to the third class;

when \(\alpha\) is \(p\) and \(\varepsilon(p) = \partial\), then it is obvious that +\(p\) is non-critical, \(i_0 \leq \alpha\) belongs to the second class;

when \(\beta\) is \(p\) and \(\varepsilon(p) = \partial\), then it is obvious that -\(p\) is critical, \(\beta \leq i_1\) belongs to the third class;

when \(\beta\) is \(p\) and \(\varepsilon(p) = 1\), then it is obvious that -\(p\) is non-critical, \(\beta \leq i_1\) belongs to the second class;
Lemma 42. For each $(\Omega, \varepsilon)$-solvable inequality $\varphi \leq \psi$ or $\Diamond \gamma \leq \delta$, by applying the residuation rules and splitting rules exhaustively, we obtain a set of inequalities, each inequality of which belong to one of the three classes:

- it is pure;
- it is $(\Omega, \varepsilon)$-receiving;
- it is of the form $\theta \leq p$ when $\varepsilon(p) = 1$, or of the form $p \leq \theta$ when $\varepsilon(p) = 0$. In $\theta \leq p$, $-\theta$ is $\varepsilon^\theta$-uniform, and in $p \leq \theta$, $+\theta$ is $\varepsilon^\theta$-uniform. In addition, for all propositional variables $q$ in $\theta$, we have $q <_\Omega p$. 

\[\Box]
Proof. Since $\Diamond \gamma \leq \delta$ is the special case of $\varphi \leq \psi$ when the outermost connective of $\varphi$ is $\Diamond$, we only consider $\varphi \leq \psi$ here.

Without loss of generality we consider the situation when

- $\neg \varphi$ is $\varepsilon^{\beta}$-uniform;
- $+\psi$ has $\varepsilon$-critical branches and is $(\Omega, \varepsilon)$-inductive, and all $\varepsilon$-critical branches in $+\psi$ are PIA branches;
- for all the $\varepsilon$-critical branches in $+\psi$ ending with $p$, all propositional variables $q$ in $\neg \varphi$, we have $q <_{\Omega} p$.

The situation where $+\psi$ is $\varepsilon^{\beta}$-uniform etc. is symmetric.

We prove by induction on the complexity of $\psi$:

- when $\psi$ is $p$: $\varphi \leq \psi$ belongs to the third class and $\varepsilon(p) = 1$;
- when $\psi$ is $\bot$ or $\top$: it cannot be the case, since $+\psi$ contains $\varepsilon$-critical branches;
- when $\psi$ is $\neg \theta$: we can first apply the residuation rule for $\neg$ to $\varphi \leq \neg \theta$ to obtain $\theta \leq \neg \varphi$, and then we can apply the induction hypothesis for the case where
  - $+\neg \varphi$ is $\varepsilon^{\beta}$-uniform;
  - $- \theta$ has $\varepsilon$-critical branches and is $(\Omega, \varepsilon)$-inductive, and all $\varepsilon$-critical branches in $- \theta$ are PIA branches;
  - for all the $\varepsilon$-critical branches in $- \theta$ ending with $p$, all propositional variables $q$ in $\neg \varphi$, we have $q <_{\Omega} p$;
- when $\psi$ is $\Diamond \theta$: it cannot be the case, since $\Diamond$ is on an $\varepsilon$-critical branch in $+\psi$ but it is not a PIA node;
- when $\psi$ is $\Diamond \theta$: similar to the $\Diamond \theta$ case;
- when $\psi$ is $\Box \theta$: we can first apply the residuation rule for $\Box$ to $\varphi \leq \Box \theta$ to obtain $\Diamond \varphi \leq \theta$, and then we can apply the induction hypothesis for the case where
  - $- \Diamond \varphi$ is $\varepsilon^{\beta}$-uniform;
  - $+\theta$ has $\varepsilon$-critical branches and is $(\Omega, \varepsilon)$-inductive, and all $\varepsilon$-critical branches in $+\theta$ are PIA branches;
  - for all the $\varepsilon$-critical branches in $+\theta$ ending with $p$, all propositional variables $q$ in $\neg \Diamond \varphi$, we have $q <_{\Omega} p$;
- when $\psi$ is $\Box \theta$, the situation is similar to the $\Box \theta$ case;
- when $\psi$ is $\theta \land \eta$, we first apply the splitting rule for $\land$ to $\varphi \leq \theta \land \eta$ to obtain $\Diamond \varphi \leq \theta$, and then there are two possibilities, namely both of $+\theta$ and $+\eta$ have $\varepsilon$-critical branches, and only one of $+\theta$ and $+\eta$ has $\varepsilon$-critical branches; for the first possibility, we can apply the induction hypothesis for the case where
  - $- \varphi$ is $\varepsilon^{\beta}$-uniform;
  - $+\theta$ and $+\eta$ have $\varepsilon$-critical branches and are $(\Omega, \varepsilon)$-inductive, and all $\varepsilon$-critical branches in $+\theta$ and $+\eta$ are PIA branches;
  - for all the $\varepsilon$-critical branches in $+\theta$ and $+\eta$ ending with $p$, all propositional variables $q$ in $\neg \varphi$, we have $q <_{\Omega} p$;
for the second possibility, without loss of generality we assume that $+\theta$ has $\varepsilon$-critical branches and $+\eta$ is $\varepsilon^\delta$-uniform, then $\varphi \leq \eta$ is $(\Omega, \varepsilon)$-receiving, and we can apply the induction hypothesis to $\varphi \leq \theta$ for the case where

- $-\varphi$ is $\varepsilon^\delta$-uniform;
- $+\theta$ have $\varepsilon$-critical branches and are $(\Omega, \varepsilon)$-inductive, and all $\varepsilon$-critical branches in $+\theta$ are PIA branches;
- for all the $\varepsilon$-critical branches in $+\theta$ ending with $p$, all propositional variables $q$ in $-\varphi$, we have $q \leq \Omega p$;

• when $\psi$ is $\theta \lor \eta$: since $\lor$ is an SRR node, only one of $\theta$ and $\eta$ contains $\varepsilon$-critical branches (without loss of generality we assume that it is $\eta$). Then

- $+\theta$ is $\varepsilon^\delta$-uniform;
- for each $p$ in an $\varepsilon$-critical branch in $+\eta$, each $q$ that occurs in $+\theta$, we have $q \leq \Omega p$;

Now we apply the residuation rule for $\lor$ to $\varphi \leq \theta \lor \eta$ to obtain $\varphi \land \neg \theta \leq \eta$, then we can apply the induction hypothesis for the case where

- $-(\varphi \land \neg \theta)$ is $\varepsilon^\delta$-uniform;
- $+\eta$ has $\varepsilon$-critical branches and is $(\Omega, \varepsilon)$-inductive, and all $\varepsilon$-critical branches in $+\eta$ are PIA branches;
- for all the $\varepsilon$-critical branches in $+\eta$ ending with $p$, all propositional variables $q$ in $-(\varphi \land \neg \theta)$, we have $q \leq \Omega p$;

• when $\psi$ is $\theta \rightarrow \eta$: the situation is similar to the $\theta \lor \eta$ case (in the sense of using one of the residuation rules for $\rightarrow$).

\[ \square \]

**Lemma 43.** Inequalities of the form $i \leq \alpha'$ or $\beta' \leq -i$, where $+\alpha'$ and $-\beta'$ are $(\Omega, \varepsilon)$-inductive and their $\varepsilon$-critical branches contain only PIA nodes are $(\Omega, \varepsilon)$-solvable, and therefore the previous lemma is applicable to them.

**Proof.** Straightforward. \[ \square \]

Now the set of inequalities have three kinds of inequalities, namely

- pure inequalities;
- $(\Omega, \varepsilon)$-receiving inequalities;
- $\theta \leq p$ when $\varepsilon(p) = 1$, or $p \leq \theta$ when $\varepsilon(p) = \partial$. In $\theta \leq p$, $-\theta$ is $\varepsilon^\delta$-uniform, and in $p \leq \theta$, $+\theta$ is $\varepsilon^\delta$-uniform. In addition, for all propositional variables $q$ in $\theta$, we have $q \leq \Omega p$.

It is ready to repeatedly apply the Ackermann lemmas, and then all propositional variables are eliminated. Since after the monotone/antitone variable elimination rules in the first stage, each variable has both critical branches and non-critical branches in the corresponding signed generation trees, therefore in Stage 2, they have occurrences in both $(\Omega, \varepsilon)$-receiving inequalities and inequalities of the form $\theta \leq p$ or $p \leq \theta$, so they all can be eliminated.

**Theorem 8.1 (Success Theorem).** For all $(\Omega, \varepsilon)$-inductive quasi-inequalities which do not contain black connectives, ALBA successfully reduce them to pure quasi-inequalities, and therefore can transform them into first-order correspondents.

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9 Canonicity

In the present section, we will prove that all inductive quasi-inequalities which do not contain black connectives or \(\Diamond, \Box\) (we call them restricted inductive quasi-inequalities) are canonical, in the sense that if they are admissibly valid on a Stone space with two relations, then they are valid on the same space. The proof style is similar to [11].

**Definition 44** (Restricted inductive quasi-inequality). Given an order-type \(\varepsilon\) and a dependence order \(<\Omega\), we say that a quasi-inequality is \((\Omega, \varepsilon)\)-restricted inductive, if it is \((\Omega, \varepsilon)\)-inductive and does not contain nominals, \(\Diamond, \Box\) or black connectives \(\Diamond, \Box, \Diamond\).

**Definition 45** (Canonicity). We say that a quasi-inequality is canonical if whenever it is admissibly valid on a Stone space with two relations, it is also valid on it.

By the duality theory of modal subordination algebras, the canonicity definition above is equivalent to the preservation under taking canonical extensions of modal subordination algebras.

9.1 U-Shaped Argument

To prove the canonicity of inductive quasi-inequalities, we use the canonicity-via-correspondence argument, which is a variation of the standard U-shaped argument (cf. [11]) represented in the diagram below. To go through the U-shaped argument, we show that topologically correct executions of ALBA is sound with respect to the Stone spaces with two relations, while we replace validity (with respect to arbitrary valuations) by admissible validity (with respect to clopen/admissible valuations).

\[
\begin{align*}
X \models_{\text{Clop}} \text{Quasi} & \quad & X \not\models_{\text{P}} \text{Quasi} \\
\not\models & \quad & \not\models \\
X \models \text{FO(Quasi)} & \iff & X \not\models \text{FO(Quasi)}
\end{align*}
\]

This argument starts from the top-left corner with the validity of the input quasi-inequality Quasi on \(X\), then uses topologically correct executions of ALBA to transform the quasi-inequality into an equivalent set of quasi-inequalities as well as its first-order translation FO(Quasi). Since the validity of the first-order formulas does not depend on the admissible set, the bottom equivalence is obvious. The right half of the argument goes on the side of arbitrary valuations, the soundness of which was already shown in Section 7.

We will focus on the equivalences of the left-arm, i.e., the soundness of topologically correct executions of ALBA with respect to admissible valuations.

9.2 Topological Correct Executions

In the present subsection, we show that ALBA can be executed in a topologically correct way on restricted inductive quasi-inequalities (see Definition 44), which paves the way for the use of the topological Ackermann lemmas in the next subsection.

We first define syntactically closed and syntactically open formulas as follows:

**Definition 46** (Syntactically closed and open formulas). 1. A formula is syntactically closed if all occurrences of nominals and \(\Diamond, \Box\) are positive, and all occurrences of \(\Diamond, \Box\) are negative;

2. A formula is syntactically open if all occurrences of nominals and \(\Diamond, \Box\) are negative, and all occurrences of \(\Diamond, \Box\) are positive.
As is discussed in [11, Section 9], the underlying idea of the definitions above is that given an admissible valuation, the truth set of a syntactically closed (resp. open) formula is always a closed (resp. open) subset in \( X \) (see Definition [13]).

We recall the right-handed and left-handed Ackermann rule:

The right-handed Ackermann rule:

\[
\begin{align*}
\theta_1 & \leq p \\
\vdots \\
\theta_n & \leq p \\
\eta_1 & \leq \iota_1 \\
\vdots \\
\eta_m & \leq \iota_m 
\end{align*}
\]

The system \( \eta_1 \leq \iota_1 \) is replaced by

\[
\begin{align*}
\eta_1((\theta_1 \lor \ldots \lor \theta_n)/p) & \leq \iota_1((\theta_1 \lor \ldots \lor \theta_n)/p) \\
\vdots \\
\eta_m((\theta_1 \lor \ldots \lor \theta_n)/p) & \leq \iota_m((\theta_1 \lor \ldots \lor \theta_n)/p)
\end{align*}
\]

where:

1. \( p \) does not occur in \( \theta_1, \ldots, \theta_n \);

2. Each \( \eta_i \) is positive, and each \( \iota_i \) negative in \( p \), for \( 1 \leq i \leq m \).

The left-handed Ackermann rule:

\[
\begin{align*}
p & \leq \theta_1 \\
\vdots \\
p & \leq \theta_n \\
\eta_1 & \leq \iota_1 \\
\vdots \\
\eta_m & \leq \iota_m 
\end{align*}
\]

The system \( \eta_1 \leq \iota_1 \) is replaced by

\[
\begin{align*}
\eta_1((\theta_1 \land \ldots \land \theta_n)/p) & \leq \iota_1((\theta_1 \land \ldots \land \theta_n)/p) \\
\vdots \\
\eta_m((\theta_1 \land \ldots \land \theta_n)/p) & \leq \iota_m((\theta_1 \land \ldots \land \theta_n)/p)
\end{align*}
\]

where:

1. \( p \) does not occur in \( \theta_1, \ldots, \theta_n \);

2. Each \( \eta_i \) is negative, and each \( \iota_i \) positive in \( p \), for \( 1 \leq i \leq m \).

**Definition 47 (Topologically Correct Executions).**

- We call an execution of the Ackermann rule topologically correct, if for each non-pure inequality in the system, the left-hand side is syntactically closed, and the right-hand side is syntactically open.

- We call an execution of ALBA topologically correct, if each execution of right-handed and left-handed Ackermann lemma is topologically correct.

**Theorem 9.1.** Given a restricted inductive quasi-inequality as input, ALBA can topologically correctly execute on it.

**Proof.** We basically follow the success proof in Section 8, while pay attention to some details (on the topological correctness) of the execution. From the proof of Lemma 39, given an \((\Omega, \varepsilon)\)-restricted inductive quasi-inequality

\[
\bar{\phi} \leq \bar{\psi} \land \bar{\gamma} \leq \delta \Rightarrow \alpha \ast \beta \text{ (where } \ast \in (\leq, <))
\]

after the first stage, it is transformed into a set of quasi-inequalities of the form

\[
\bar{\phi} \leq \bar{\psi} \land \bar{\gamma} \leq \delta \Rightarrow \alpha \leq \beta
\]

or

\[
\bar{\phi} \leq \bar{\psi} \land \bar{\gamma} \leq \delta \Rightarrow \diamond \xi \leq \chi
\]

where

- each \( \phi \leq \psi \) or \( \diamond \gamma \leq \delta \) is either \((\Omega, \varepsilon)\)-receiving or \((\Omega, \varepsilon)\)-solvable;
• $+\alpha$ and $-\beta$ (resp. $+\Diamond\xi$ and $-\chi$) are definite $(\Omega, \varepsilon)$-inductive signed generation trees;
• each formula contains no black connectives.

Since for the rules in the Stage 1, only the subordination rewriting rule introduce $\Diamond$ and $\Diamond$ only occurs in the outermost level, it is easy to see that $\varphi, \psi, \gamma, \delta, \alpha, \beta, \xi, \chi$ do not contain $\Diamond, \Box, \lozenge, \blacksquare$, so they are both syntactically closed and open.

Therefore, if we apply the first-approximation rule to the quasi-inequality, we obtain
\[
\varphi \leq \psi \land \Diamond \gamma \leq \delta \land \iota_0 \leq \alpha \land \beta \leq -\iota_1 \Rightarrow \iota_0 \leq -\iota_1
\]

or
\[
\varphi \leq \psi \land \Diamond \gamma \leq \delta \land \iota_0 \leq \Diamond \xi \land \chi \leq -\iota_1 \Rightarrow \iota_0 \leq -\iota_1,
\]

where
• each $\varphi \leq \psi$ or $\Diamond \gamma \leq \delta$ is either $(\Omega, \varepsilon)$-receiving or $(\Omega, \varepsilon)$-solvable;
• $+\alpha$ and $-\beta$ (resp. $+\Diamond\xi$ and $-\chi$) are definite $(\Omega, \varepsilon)$-inductive signed generation trees;
• each of $\varphi, \psi, \gamma, \delta, \alpha, \beta, \xi, \chi$ is both syntactically closed and open.

For the second case, we apply the approximation rule for $\Diamond$ and get
\[
\varphi \leq \psi \land \Diamond \gamma \leq \delta \land \iota_0 \leq \Diamond \iota \land \iota \leq \xi \land \chi \leq -\iota_1 \Rightarrow \iota_0 \leq -\iota_1.
\]

Now in both cases, we have a quasi-inequality of the following form:
\[
\varphi \leq \psi \land \iota \leq \alpha' \land \beta \leq -\iota_1 \land \text{Pure} \Rightarrow \iota_0 \leq -\iota_1
\]

where
• each $\varphi \leq \psi$ is either $(\Omega, \varepsilon)$-receiving or $(\Omega, \varepsilon)$-solvable;
• $+\alpha'$ and $-\beta$ are definite $(\Omega, \varepsilon)$-inductive signed generation trees;
• each of $\varphi, \beta$ is syntactically closed and each of $\psi, \alpha'$ is syntactically open.

Now we can easily check that the following property holds for the quasi-inequality:

In the antecedent of the quasi-inequality, in each non-pure inequality, the left-hand side is syntactically closed and the right-hand side is syntactically open.

It is easy to check that for each rule in Stage 2, it does not break this property, so for each execution of the Ackermann rule, it is topologically correct, and after the execution of the Ackermann rule, it still satisfies the property stated above. Therefore, the execution of ALBA is topologically correct. □

### 9.3 Topological Ackermann Lemmas

In the present section we will prove the soundness of the algorithm ALBA with respect to admissible validity on Stone spaces with two relations. Indeed, similar to other semantic settings (see e.g., [11]), the soundness proof on the admissible valuation side goes similar to that of the arbitrary valuation side (i.e., Theorem [7.1]), and for every rule except for the Ackermann rules, the proof goes without modification. Thus we will only focus on the Ackermann rules here, which are justified by the topological Ackermann lemmas given below. The proof is similar to [11], so we only state the lemmas without giving proof details of the lemmas.
For the Ackermann rules, the soundness proof with respect to arbitrary valuations is justified by the Ackermann lemmas (see Lemma 36). As discussed in the literature (e.g., [11, Section 9]), the soundness proof of the Ackermann rules, namely the Ackermann lemmas, cannot be straightforwardly adapted to the setting of admissible valuations, since formulas in the expanded language might be interpreted as non-clopen subsets even if all the propositional variables are interpreted as clopen subsets in $X$. Thus by taking $a_0 = \bigvee \{ V(\theta) \}$, we cannot guarantee that $a_0$ is a clopen subset of $X$. Therefore, some adaptations are necessary based on the syntactic shape of the formulas, which are defined in the previous subsection.

Now we state the topological Ackermann lemmas without proofs. Similar proofs can be found in [11, Section 9].

**Lemma 48** (Right-handed topological Ackermann lemma). Let $\theta$ be a syntactically closed formula which does not contain $p$, let $\eta_i(p)$ (resp. $\iota_i(p)$) be syntactically closed (resp. open) and positive (resp. negative) in $p$ for $1 \leq i \leq m$, and let $\vec{q}$ (resp. $\vec{j}$) be all the propositional variables (resp. nominals) occurring in $\eta_1(p), \ldots, \eta_m(p), \iota_1(p), \ldots, \iota_m(p), \theta$ other than $p$. Then for all $\vec{a} \in \text{Clop}(X), \vec{x} \in X$, the following are equivalent:

1. $V(\eta_i(\theta/p)) \subseteq V(\iota_i(\theta/p))$ for $1 \leq i \leq m$, where $V(\vec{q}) = \vec{a}$, $V(\vec{j}) = \{ \vec{x} \}$.
2. There exists $a_0 \in \text{Clop}(X)$ such that $V'(\theta) \leq a_0$ and $V'(\eta_i(p)) \subseteq V'(\iota_i(p))$ for $1 \leq i \leq m$, where $V'$ is the same as $V$ except that $V'(p) = a_0$.

**Lemma 49** (Left-handed topological Ackermann lemma). Let $\theta$ be a syntactically open formula which does not contain $p$, let $\eta_i(p)$ (resp. $\iota_i(p)$) be syntactically closed (resp. open) and negative (resp. positive) in $p$ for $1 \leq i \leq m$, and let $\vec{q}$ (resp. $\vec{j}$) be all the propositional variables (resp. nominals) occurring in $\eta_1(p), \ldots, \eta_m(p), \iota_1(p), \ldots, \iota_m(p), \theta$ other than $p$. Then for all $\vec{a} \in \text{Clop}(X), \vec{x} \in X$, the following are equivalent:

1. $V(\eta_i(\theta/p)) \subseteq V(\iota_i(\theta/p))$ for $1 \leq i \leq m$, where $V(\vec{q}) = \vec{a}$, $V(\vec{j}) = \{ \vec{x} \}$.
2. There exists $a_0 \in \text{Clop}(X)$ such that $a_0 \leq V'(\theta)$ and $V'(\eta_i(p)) \subseteq V'(\iota_i(p))$ for $1 \leq i \leq m$, where $V'$ is the same as $V$ except that $V'(p) = a_0$.

The proof of the topological Ackermann lemmas is similar to [21, Section B], and therefore is omitted. We only show how these lemmas justify the soundness of the topologically correct executions of the Ackermann rules.

**Proposition 50.** The topologically correct executions of the Ackermann rules are sound with respect to the admissible valuations.

**Proof.** Here we prove it for the right-handed Ackermann lemma, and the left-handed Ackermann lemma is similar. Without loss of generality we assume that $n = 1$, then for a topological correct execution of this rule, the system $S$ before execution is of the following shape:

$$
\begin{align*}
\theta &\leq p \\
\eta_1 &\leq \iota_1 \\
\vdots \\
\eta_m &\leq \iota_m \\
\text{Pure}
\end{align*}
$$

where:

1. $p$ does not occur in $\theta$ and $\theta$ is syntactically closed;
2. Each $\eta_i$ is syntactically closed and positive in $p$, and each $\iota_i$ is syntactically open and negative in $p$, for $1 \leq i \leq m$;
3. Pure is pure;
the system $S'$ after the execution is of the following shape:

$$\left\{ \begin{array}{l}
\eta_1(\theta/p) \leq \iota_1(\theta/p) \\
\vdots \\
\eta_m(\theta/p) \leq \iota_m(\theta/p) \\
\text{Pure}
\end{array} \right. $$

Similar to the discussion of page 19, it suffices to show the following:

1. For any $X$, any admissible valuation $V_0$, if $X, V_0 \not\models S$, then there is an admissible valuation $V_1$ such that $V_1(i_0) = V_0(i_0), V_1(i_1) = V_0(i_1)$ and $X, V_1 \not\models S'$.

2. For any $X$, any admissible valuation $V_1$, if $X, V_1 \not\models S'$, then there is an admissible valuation $V_0$ such that $V_0(i_0) = V_1(i_0), V_0(i_1) = V_1(i_1)$ and $X, V_0 \not\models S$.

Proof of 1. Consider any $X$ and admissible valuation $V_0$ on $X$. If $X, V_0 \not\models S$, then take $a_0 = V_0(p)$, we have:

$$\left\{ \begin{array}{l}
V_0(\theta) \leq V_0(p) = a_0 \\
V_0(\eta_i) \leq V_0(\iota_i) \text{ for } 1 \leq i \leq m \\
\text{Pure is true under } V_0,
\end{array} \right. $$

so condition 2 holds, by the right-handed Ackermann lemma, condition 1 holds, so $V_1(\eta_i(\theta/p)) \subseteq V_1(\iota_i(\theta/p))$ for $1 \leq i \leq m$ and some admissible valuation $V_1$ which differs from $V_0$ at most at $p$.

It is easy to see that Pure is true under $V_0$ iff it is true under $V_1$, so $X, V_1 \not\models S'$.

Proof of 2. Consider any $X$ and admissible valuation $V_1$ on $X$. If $X, V_1 \not\models S'$, then we have:

$$\left\{ \begin{array}{l}
V_1(\eta_i(\theta/p)) \leq V_1(\iota_i(\theta/p)) \text{ for } 1 \leq i \leq m \\
\text{Pure is true under } V_1,
\end{array} \right. $$

so condition 1 holds, by the right-handed Ackermann lemma, condition 2 holds, so there exists $a_0 \in Cl(X)$ such that $V_0(\theta) \leq a_0$ and $V_0(\eta_i(p)) \subseteq V_0(\iota_i(p))$ for $1 \leq i \leq m$, where $V_0$ is the same as $V_1$ except that $V_0(p) = a_0$. It is easy to see that Pure is true under $V_1$ iff it is true under $V_0$.

$$\left\{ \begin{array}{l}
V_0(\theta) \leq V_0(p) = a_0 \\
V_0(\eta_i) \leq V_0(\iota_i) \text{ for } 1 \leq i \leq m \\
\text{Pure is true under } V_0,
\end{array} \right. $$

so $X, V_0 \not\models S$.

\[ \Box \]

10 Examples

Example 10.1. Given a modal subordination algebra $(B, \prec, \Diamond)$, it is proximity preserving iff its dual Stone space with two relations $(X, \tau, R, R')$ satisfies the following interaction condition:

$$\forall w(R[R'^{-1}(w)] \subseteq R'^{-1}[R(w)], \text{ i.e.}$$

for all $u, v, w \in X$ such that $vR'w$ and $vRu$, there exists a $t \in X$ such that $wRt$ and $uR't$.  

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Proof. It suffices to show that
\[(B, <, \Diamond) \models (a < b) \Rightarrow (\Diamond a < \Diamond b)\]
iff its dual
\[(X, \tau, R, R') \models \forall w(R[R^{-1}(w)] \subseteq R'^{-1}[R(w)]).\]
Indeed,
\[(B, <, \Diamond) \vdash (a < b) \Rightarrow (\Diamond a < \Diamond b)\]
iff \[(X, \tau, R, R') \vdash_{\text{Clo}} \forall a \forall b((a < b) \Rightarrow (\Diamond a < \Diamond b))\]
iff \[(X, \tau, R, R') \vdash_{\text{Clo}} \forall a \forall b((\Diamond a \leq b) \Rightarrow (\Diamond \Diamond a \leq \Diamond b))\]
iff \[(X, \tau, R, R') \vdash_{\text{Clo}} \forall a \forall b \forall i \forall j ((a \leq b & i \leq \Diamond \Diamond a & \Diamond b \leq \neg j \Rightarrow i \leq \neg j))\]
iff \[(X, \tau, R, R') \vdash_{\text{Clo}} \forall a \forall b \forall i \forall j ((\Diamond a \leq b & i \leq \Diamond \Diamond k & k \leq a & \Diamond b \leq \neg j \Rightarrow i \leq \neg j))\]
iff \[(X, \tau, R, R') \vdash_{\text{Clo}} \forall k((\Diamond \Diamond \Diamond k \leq \Diamond \Diamond k))\]
iff \[(X, \tau, R, R') \vdash_{\text{Clo}} \forall w(R[R^{-1}(w)] \subseteq R'^{-1}[R(w)])\].

\[\square\]

Example 10.2 (See Lemma 2.1.12 in [25]). Given a modal subordination algebra \((B, <, \Diamond)\) and its dual Stone space with two relations \((X, \tau, R, R')\).

1. \((B, <, \Diamond) \models \forall a \forall b (a < b \Rightarrow a \leq b)\) iff \(R\) is reflexive;

2. \((B, <, \Diamond) \models \forall a \forall b (a < b \Rightarrow \neg b < \neg a)\) iff \(R\) is symmetric.

Proof. 1. It suffices to show that
\[(B, <, \Diamond) \models a < b \Rightarrow a \leq b\]
iff its dual
\[(X, \tau, R, R') \models \forall w R w w .\]
Indeed,
\[(B, <, \Diamond) \models a < b \Rightarrow a \leq b\]
iff \[(X, \tau, R, R') \vdash_{\text{Clo}} \forall a \forall b((a < b) \Rightarrow a \leq b)\]
iff \[(X, \tau, R, R') \vdash_{\text{Clo}} \forall a \forall b((\Diamond a \leq b) \Rightarrow a \leq b)\]
iff \[(X, \tau, R, R') \vdash_{\text{Clo}} \forall a \forall b \forall i \forall j ((\Diamond a \leq b & i \leq \Diamond \Diamond a & \Diamond b \leq \neg j \Rightarrow i \leq \neg j))\]
iff \[(X, \tau, R, R') \vdash_{\text{Clo}} \forall b \forall i \forall j ((\Diamond i \leq \Diamond b & \Diamond b \leq \neg j \Rightarrow i \leq \neg j))\]
iff \[(X, \tau, R, R') \vdash_{\text{Clo}} \forall i ((\Diamond i \leq \Diamond i))\]
iff \[(X, \tau, R, R') \vdash_{\text{Clo}} \forall w (w \in R[w])\]
iff \[(X, \tau, R, R') \models \forall w R w w .\]

2. It suffices to show that
\[(B, <, \Diamond) \models \forall a \forall b (a < b \Rightarrow \neg b < \neg a)\]
iff its dual
\[(X, \tau, R, R') \models \forall w \forall v (R v w \Rightarrow R w v).\]
Indeed,
Part II: Correspondence and Canonicity for $Π_2$-statements

In this part, we focus on $Π_2$-statements of the form $\overline{\varphi} \mathrel{\prec} \overline{\psi} \Rightarrow \exists \overline{q}(\overline{\varphi} \mathrel{\prec} \overline{\psi})$ and develop their correspondence and canonicity theory. We will define inductive $Π_2$-statements and restricted inductive $Π_2$-statements (Section 12), give a version of the algorithm ALBA$^{Π_2}$ for $Π_2$-statements (Section 13), state its soundness with respect to arbitrary valuations (Section 14), success on inductive and restricted inductive $Π_2$-statements (Section 15), and canonicity of restricted inductive $Π_2$-statements (Section 16).

11 Syntax and Semantics

For $Π_2$-statements, the special feature of its syntax is the propositional quantifiers of the form $\exists q$. The semantics of the propositional quantifiers is already different between admissible valuations and arbitrary valuations. The semantics of propositional quantifiers are given as follows: for any Stone space with two relations $X$,

- for any admissible valuation $V_0$, $(X, V_0) \vDash \exists q \exists \overline{\vartheta}(\overline{\varphi} \prec \overline{\psi})$ iff there exist $A \in Clop(X)$ such that $X, (V_0)^{\overline{\varphi}}_{\overline{\psi}} \vDash \exists \overline{q}(\overline{\vartheta} < \overline{\delta})$;

- for any arbitrary valuation $V_1$, $(X, V_1) \vDash \exists q \exists \overline{\vartheta}(\overline{\varphi} \prec \overline{\psi})$ iff there exist $A \in P(X)$ such that $X, (V_1)^{\overline{\varphi}}_{\overline{\psi}} \vDash \exists \overline{q}(\overline{\vartheta} < \overline{\delta})$.

For the semantics of $Π_2$-statements, the global satisfaction relation for $Π_2$-statements is defined differently between admissible valuations and arbitrary valuations. It is given as follows:

Definition 51. Given a $Π_2$-statement $\overline{\varphi} \mathrel{\prec} \overline{\psi} \Rightarrow \exists \overline{q}(\overline{\varphi} \mathrel{\prec} \overline{\psi})$, a Stone space with two relations $X$, an admissible valuation $V_0$, an arbitrary valuation $V_1$,

- $X, V_0 \vDash \overline{\varphi} \mathrel{\prec} \overline{\psi} \Rightarrow \exists \overline{q}(\overline{\varphi} \mathrel{\prec} \overline{\psi})$ iff whenever $X, V_0 \vDash \overline{\varphi} \mathrel{\prec} \overline{\psi}$, there exist $\bar{A} \in Clop(X)$ such that $X, (V_0)^{\overline{\varphi}}_{\overline{\psi}} \vDash \overline{\varphi} \mathrel{\prec} \overline{\psi}$;
• $X, V_1 \vDash \varphi \rightarrow \psi \Rightarrow \exists \vec{q} (\gamma \rightarrow \delta)$ iff whenever $X, V_1 \vDash \varphi \rightarrow \psi$, there exist $\vec{A} \in P(X)$ such that $X_i (V_1)_{\vec{q}} \vDash \vec{q} \rightarrow \vec{\delta}$.

• $\varphi \rightarrow \psi$ is admissibly valid in $X$, i.e. $X \vDash_{cl\lor p} \varphi \rightarrow \psi \Rightarrow \exists \vec{q} (\vec{q} \rightarrow \vec{\delta})$ iff $X, V_0 \vDash \varphi \rightarrow \psi \Rightarrow \exists \vec{q} (\vec{q} \rightarrow \vec{\delta})$ for all admissible valuations $V_0$;

• $\varphi \rightarrow \psi$ is valid in $X$, i.e. $X \vDash_{p} \varphi \rightarrow \psi \Rightarrow \exists \vec{q} (\vec{q} \rightarrow \vec{\delta})$ iff $X, V_1 \vDash \varphi \rightarrow \psi \Rightarrow \exists \vec{q} (\vec{q} \rightarrow \vec{\delta})$ for all arbitrary valuations $V_1$.

12 Inductive and Restricted Inductive $\Pi_2$-Statements

In the present section, we define inductive and restricted inductive $\Pi_2$-statements of the shape

$$\varphi \leq \psi \& \chi \prec \xi \Rightarrow \exists \vec{q} (\vec{p} \leq \vec{q} \& \vec{\delta} < \vec{\xi})$$

where each of $\varphi, \psi, \chi, \xi, \alpha, \beta, \gamma, \delta$ has certain syntactic restrictions.

12.1 Intuitive ideas behind the definition

1. The basic idea behind defining inductive $\Pi_2$-statements is as follows: We divide the algorithm into two parts, the first part of the algorithm equivalently transforms $\exists \vec{q} (\vec{p} \leq \vec{\beta} \& \vec{\gamma} < \vec{\delta})$ part into a meta-conjunction of inequalities $\text{MetaConIneq}$, such that

$$\varphi \leq \psi \& \chi \prec \xi \Rightarrow \exists \vec{q} (\vec{p} \leq \vec{\beta} \& \vec{\gamma} < \vec{\delta})$$

is an inductive quasi-inequality, and we can then use the second part of the algorithm (which is the same as the algorithm ALBA for quasi-inequalities) to transform it into a set of pure quasi-inequalities. The first part of the algorithm eliminate all the existential propositional quantifiers by application of Ackermann rules, but this time we do not have nominals in the elimination of existential propositional quantifiers like in the elimination of the (implicit) universal propositional quantifiers (which is what will be done in the second part of the algorithm).

2. For the elimination of the existential propositional quantifiers, we again divide the set of inequalities in

$$\exists \vec{q} (\vec{p} \leq \vec{\beta} \& \vec{\gamma} < \vec{\delta})$$

into two parts:

- One part is the solvable inequalities, which will finally be transformed into inequalities of the form $\theta \leq q$ or $q \leq \theta$. The solvable inequalities will be used to compute the minimal valuations of the propositional variables in $\exists \vec{q}$.
- The other part is the receiving inequalities, which will be used to receive minimal valuations.

3. For the canonicity proof, we again need to guarantee certain syntactic topological properties, which will be further explained in the canonicity proof. These restrictions lead to the definition of restricted inductive $\Pi_2$-statements.
12.2 The Definition of Inductive \( \Pi_2 \)-Statements

In this section, we will define inductive \( \Pi_2 \)-statements in the language without black connectives \( \& \),  \( \lor \),  \( \leftrightarrow \) or nominals, which will be shown to have first-order correspondents over arbitrary valuations on Stone spaces with two relations. This definition will not be given in a syntactic description, for a syntactic definition we will leave to future work.

Before defining inductive \( \Pi_2 \)-statements, we first define the so-called first-round good \( \exists \)-statements of the form \( \exists \overrightarrow{q}(\overrightarrow{r} \leq \overrightarrow{p} \& \overrightarrow{r} < \overrightarrow{d}) \), which are the ones that can be equivalently transformed into a “good” meta-conjunction of inequalities without existential propositional quantifiers:

**Definition 52** ((\( \Omega, \epsilon_{\overrightarrow{p}} \))-First-Round Good \( \exists \)-Statements). Given an order-type \( \epsilon_{\overrightarrow{p}} \) for \( \overrightarrow{p} \), and a dependence order \( \triangleleft_{\Omega} \) on \( \overrightarrow{p} \), an \( \exists \)-statement \( \exists \overrightarrow{q}(\overrightarrow{r} \leq \overrightarrow{p} \& \overrightarrow{r} < \overrightarrow{d}) \) is (\( \Omega, \epsilon_{\overrightarrow{p}} \))-first-round good, if by the first part of the algorithm ALBA\( \Pi_2 \) defined in Section 13 it can be transformed into a meta-conjunction of inequalities MetaConIneq, each inequality in which is (\( \Omega, \epsilon_{\overrightarrow{p}} \))-inductive (i.e. the positive tree of the left-hand side and the negative tree of the right-hand side are both (\( \Omega, \epsilon_{\overrightarrow{p}} \))-inductive, which might contain black connectives) for the same (\( \Omega, \epsilon_{\overrightarrow{p}} \)).

**Definition 53** (Inductive \( \Pi_2 \)-Statements). Given an order-type \( \epsilon_{\overrightarrow{p}} \) for \( \overrightarrow{p} \), and a dependence order \( \triangleleft_{\Omega} \) on \( \overrightarrow{p} \), a \( \Pi_2 \)-statement

\[ \overrightarrow{\varphi} \leq \overrightarrow{\psi} \& \overrightarrow{r} < \overrightarrow{c} \Rightarrow \exists \overrightarrow{q}(\overrightarrow{r} \leq \overrightarrow{p} \& \overrightarrow{r} < \overrightarrow{d}) \]

is (\( \Omega, \epsilon_{\overrightarrow{p}} \))-inductive, if

- it does not contain nominals or black connectives \( \& \),  \( \lor \),  \( \leftrightarrow \);
- \( \exists \overrightarrow{q}(\overrightarrow{r} \leq \overrightarrow{p} \& \overrightarrow{r} < \overrightarrow{d}) \) is (\( \Omega, \epsilon_{\overrightarrow{p}} \))-first-round good;
- \( \overrightarrow{\varphi} \leq \overrightarrow{\psi} \& \overrightarrow{r} < \overrightarrow{c} \Rightarrow \) MetaConIneq is an (\( \Omega, \epsilon_{\overrightarrow{p}} \))-inductive quasi-inequality (which might contain black connectives), where MetaConIneq is defined as in Definition 54.

A \( \Pi_2 \)-statement is inductive, if it is (\( \Omega, \epsilon_{\overrightarrow{p}} \))-inductive for some \( \epsilon_{\overrightarrow{p}} \) and \( \triangleleft_{\Omega} \).

12.3 The Definition of Restricted Inductive \( \Pi_2 \)-Statements

In this section, we will define restricted inductive \( \Pi_2 \)-statements, which will be shown to be canonical. For this class of \( \Pi_2 \)-statements, we will give a syntactic description.

Again, before defining restricted inductive \( \Pi_2 \)-statements, we first defined the restricted first-round good \( \exists \)-statements.

**Definition 54** ((\( \Omega, \epsilon_{\overrightarrow{q}} \))-Restricted First-Round Good \( \exists \)-Statements). Given order-types \( \epsilon_{\overrightarrow{p}} \) and \( \epsilon_{\overrightarrow{q}} \) for \( \overrightarrow{p} \) and \( \overrightarrow{q} \), and a dependence order \( \triangleleft_{\Omega} \) on \( \overrightarrow{p} \) and \( \overrightarrow{q} \) such that \( p \triangleleft_{\Omega} q \) for all \( p \) in \( \overrightarrow{p} \) and \( q \) in \( \overrightarrow{q} \), an \( \exists \)-statement \( \exists \overrightarrow{q}(\overrightarrow{r} \leq \overrightarrow{p} \& \overrightarrow{r} < \overrightarrow{d}) \) is (\( \Omega, \epsilon_{\overrightarrow{q}} \))-restricted first-round good, if inequalities in \( \overrightarrow{r} \leq \overrightarrow{p} \& \overrightarrow{r} < \overrightarrow{d} \) can be divided into the following two kinds:

- \( \theta \ast \eta \), which we call (\( \Omega, \epsilon_{\overrightarrow{q}} \))-restricted receiving inequality, where
  - all branches in \( -\theta \) and \( +\eta \) ending with propositional variables in \( \overrightarrow{q} \) are \( \epsilon_{\overrightarrow{q}} \)-critical (i.e. in the first half of the algorithm, the inequality \( \theta \ast \eta \) is a “receiving” inequality);
  - all branches of \( +\theta \) and \( -\eta \) are Skeleton branches (i.e. in the second half of the algorithm, their branches behave like Skeleton branches);

- \( \iota \ast \zeta \), which we call (\( \Omega, \epsilon_{\overrightarrow{q}} \))-restricted solvable inequality, where
– exactly one of $-\iota$, $+\zeta$ is such that all branches ending with propositional variables in $\vec{q}$ are $\varepsilon_{\vec{q}}^\beta$-critical (without loss of generality we denote this generation tree $\star\rho$ and the other one $\star\kappa$), and the other contains $\varepsilon_{\vec{q}}$-critical branches;

– $\star\kappa$ is $(\Omega, \varepsilon_{\vec{q}})$-inductive, and all $\varepsilon_{\vec{q}}$-critical branches in $\star\kappa$ are PIA branches;

– for all the $\varepsilon_{\vec{q}}$-critical branches in $\star\kappa$, all propositional variables $r$ in $\star\rho$, we have $r <_\Omega q$.

– for every branch in $-\iota$ and $+\zeta$ which is not $\varepsilon_{\vec{q}}$-critical, it is a Skeleton branch in $+\iota$ and $-\zeta$.

The basic idea behind the “solvable inequality” is that after the application of residuation rules exhaustively, the resulting inequality is of the form $\varphi \leq \psi$ (when $\varepsilon(r) = 1$) or of the form $\psi \leq \varphi$ (when $\varepsilon(r) = \partial$) where all branches in $+\psi$ (resp. $-\psi$) ending with $p$ in $\vec{p}$ is a Skeleton branch.

**Definition 55** (Restricted Inductive $\Pi_2$-Statements). Given order-types $\varepsilon_{\vec{p}}$, $\varepsilon_{\vec{q}}$ for $\vec{p}$, $\vec{q}$, and a dependence order $<_\Omega$ on $\vec{p}$ and $\vec{q}$ such that $p <_\Omega q$ for all $p$ in $\vec{p}$ and $q$ in $\vec{q}$, a $\Pi_2$-statement

$$\varphi \leq \psi & \chi < \xi \Rightarrow \exists \vec{q}[\overline{\alpha} \leq \overline{\beta} & \overline{\gamma} < \overline{\delta}]$$

is $(\Omega, \varepsilon_{\vec{p}})$-restricted inductive,

– it does not contain nominals or black connectives $\Diamond$, $\Box$, $\blacklozenge$, $\blacklozenge$ or $\Diamond$, $\Box$;

– $\exists \vec{q}[\overline{\alpha} \leq \overline{\beta} & \overline{\gamma} < \overline{\delta}]$ is $(\Omega, \varepsilon_{\vec{q}})$-restricted first-round good;

– each inequality in $\varphi \leq \psi$ and $\chi < \xi$ is either $(\Omega, \varepsilon_{\vec{p}})$-receiving or $(\Omega, \varepsilon_{\vec{p}})$-solvable.

A $\Pi_2$-statement is restricted inductive, if it is $(\Omega, \varepsilon_{\vec{p}})$-restricted inductive for some $\varepsilon_{\vec{p}}$ and $<_\Omega$.

**13 The Algorithm ALBA$^{\Pi_2}$**

The algorithm ALBA$^{\Pi_2}$ transforms the input $\Pi_2$-statement

$$\varphi \leq \psi & \chi < \xi \Rightarrow \exists \vec{q}[\overline{\alpha} \leq \overline{\beta} & \overline{\gamma} < \overline{\delta}]$$

into an equivalent set of pure quasi-inequalities which does not contain occurrences of propositional variables or propositional quantifiers, and therefore can be translated into the first-order correspondence language via the standard translation of the expanded language (see page 10). The language on which we manipulate the algorithm is almost the same as the algorithm for quasi-inequalities, except that we have additional existential propositional quantifiers $\exists q$.

Now we define the algorithm in two halves, the first half aims at reducing

$$\exists \vec{q}[\overline{\alpha} \leq \overline{\beta} & \overline{\gamma} < \overline{\delta}]$$

into a meta-conjunction of inequalities, and the second half is just the algorithm ALBA for quasi-inequalities.

**First Half** The algorithm receives a $\Pi_2$-statement

$$\varphi \leq \psi & \chi < \xi \Rightarrow \exists \vec{q}[\overline{\alpha} \leq \overline{\beta} & \overline{\gamma} < \overline{\delta}]$$

as input. The first half of the algorithm operates on the $\exists$-statement $\exists \vec{q}[\overline{\alpha} \leq \overline{\beta} & \overline{\gamma} < \overline{\delta}]$, and executes in three stages.
1. Preprocessing\textsuperscript{5}

(a) In each inequality $\theta \triangleright \eta$ (where $\triangleright \in \{\leq, \prec\}$) in the $\exists$-statement $\exists \vec{q}(\alpha \leq \beta \& \gamma \prec \delta)$, consider the signed generation trees $+\theta$ and $-\eta$, apply the distribution rules:

i. Push down $+\Box, +\Diamond, -\neg, +\wedge, -\to$ by distributing them over nodes labelled with $+\lor$ which are Skeleton nodes (see Figure 1 on page 13), and

ii. Push down $-\Box, -\Box, +\neg, -\lor, -\to$ by distributing them over nodes labelled with $-\land$ which are Skeleton nodes (see Figure 2 on page 14).

(b) Apply the splitting rules to each inequality occurring in the $\exists$-statement:

\[
\begin{align*}
\theta \leq \eta \land \iota & \quad \theta \land \eta \leq \iota \\
\theta \lor \eta \leq \iota & \quad \theta \leq \iota \& \eta \leq \iota
\end{align*}
\]

(c) Apply the monotone and antitone variable-elimination rules for variables in $\vec{q}$ to the whole $\exists$-statement:

\[
\begin{align*}
\exists \vec{q}.\exists \vec{q}(\alpha(q) \leq \beta(q) \& \gamma(q) \prec \delta(q)) \\
\exists \vec{q}(\alpha(\perp) \leq \beta(\perp) \& \gamma(\perp) \prec \delta(\perp))
\end{align*}
\]

if $q$ is positive in $\alpha, \gamma$ and negative in $\beta, \delta$;

\[
\begin{align*}
\exists \vec{q}.\exists \vec{q}(\alpha(q) \leq \beta(q) \& \gamma(q) \prec \delta(q)) \\
\exists \vec{q}(\alpha(\top) \leq \beta(\top) \& \gamma(\top) \prec \delta(\top))
\end{align*}
\]

if $q$ is negative in $\alpha, \gamma$ and positive in $\beta, \delta$;

(d) Apply the subordination rewriting rule to each inequality with $<$ in order to turn it into $\leq$:

\[
\frac{\theta < \eta}{\theta \leq \eta}
\]

Now we have a $\exists$-statement of the form $\exists \vec{q}(\alpha \leq \beta)$.

2. The reduction-elimination cycle:

In this stage, for each inequality $\alpha \leq \beta$ in the $\exists$-statement $\exists \vec{q}(\alpha \leq \beta)$, we apply the following rules together with the splitting rules in the previous stage to eliminate all the propositional variables in the set of inequalities\textsuperscript{6}.

(a) Residuation rules, as described on page 15.

(b) The Ackermann rules. These rules are executed on the whole $\exists$-statement $\exists \vec{q}(\alpha \leq \beta)$. We take all the inequalities in this $\exists$-statement.

The right-handed Ackermann rule:

\textsuperscript{5}Notice that here we do not have first-approximation anymore.

\textsuperscript{6}Notice that since we do not have nominals in this stage, we do not have approximation rules anymore.
The $\exists$-statement $\exists q \exists \vec{q}$ is replaced by

$$
\begin{cases}
\theta_1 \leq q \\
\vdots \\
\theta_n \leq q \\
\eta_1 \leq \iota_1 \\
\vdots \\
\eta_m \leq \iota_m
\end{cases}
$$

where:

i. $q$ does not occur in $\theta_1, \ldots, \theta_n$;

ii. Each $\eta_i$ is positive, and each $\iota_i$ negative in $q$, for $1 \leq i \leq m$.

The left-handed Ackermann rule:

The $\exists$-statement $\exists q \exists \vec{q}$ is replaced by

$$
\begin{cases}
\eta_1((\theta_1 \lor \ldots \lor \theta_n)/q) \leq \iota_1((\theta_1 \lor \ldots \lor \theta_n)/q) \\
\vdots \\
\eta_m((\theta_1 \lor \ldots \lor \theta_n)/q) \leq \iota_m((\theta_1 \lor \ldots \lor \theta_n)/q)
\end{cases}
$$

where:

i. $q$ does not occur in $\theta_1, \ldots, \theta_n$;

ii. Each $\eta_i$ is negative, and each $\iota_i$ positive in $q$, for $1 \leq i \leq m$.

3. **Output**: If in the previous stage, for some existential propositional quantifier $\exists q$, the algorithm gets stuck, i.e. some propositional quantifiers cannot be eliminated by the application of the reduction rules, then the algorithm halts and output “failure”. Otherwise, we get a meta-conjunction of inequalities of the form $\alpha_1 \leq \beta_1 \land \ldots \land \alpha_n \leq \beta_n$, and the algorithm proceeds in the second half of the algorithm with input quasi-inequality

$$
\overline{\varphi} \leq \overline{\psi} \land \overline{x} < \overline{z} \Rightarrow \alpha_1 \leq \beta_1 \land \ldots \land \alpha_n \leq \beta_n.
$$

**Second Half**

The second half of the algorithm receives the quasi-inequality

$$
\overline{\varphi} \leq \overline{\psi} \land \overline{x} < \overline{z} \Rightarrow \alpha_1 \leq \beta_1 \land \ldots \land \alpha_n \leq \beta_n.
$$

as input, and the proceeds as the algorithm ALBA defined in Section 6.

14 **Soundness**

The soundness proof with respect to arbitrary valuations of the algorithm ALBA$^{\Pi_2}$ in this section is similar to Section 7 and hence is omitted.
15 Success

15.1 Success of ALBA\(\Pi^2\) on Inductive \(\Pi^2\)-Statements

For inductive \(\Pi^2\)-statements, their definition is given by the execution result of the algorithm, therefore after the first half of the algorithm, the \(\exists\)-statement part is transformed into a meta-conjunction of inequalities, each of which is \((\Omega, \epsilon_p)\)-inductive, and therefore the input of the second half is an \((\Omega, \epsilon_p)\)-inductive quasi-inequality, by the success of ALBA on inductive quasi-inequalities, we have that the algorithm succeeds on inductive \(\Pi^2\)-statements.

15.2 Success of ALBA\(\Pi^2\) on Restricted Inductive \(\Pi^2\)-Statements

For restricted inductive \(\Pi^2\)-statements, what we need to show is that the algorithm ALBA\(\Pi^2\) succeeds on restricted inductive \(\Pi^2\)-statements.

Lemma 56. Given an \((\Omega, \epsilon_p)\)-restricted inductive \(\Pi^2\)-statement

\[\varphi \leq \psi \& \chi < \xi \Rightarrow \exists \varphi(\overline{\alpha} \leq \overline{\beta} \& \overline{\gamma} < \overline{\delta})\]

in which the \(\exists\)-statement \(\exists \varphi(\overline{\alpha} \leq \overline{\beta} \& \overline{\gamma} < \overline{\delta})\) part is \((\Omega, \epsilon_p)\)-restricted first-round good, after the first stage of the first half of the algorithm, the \(\exists\)-statement is transformed into another \((\Omega, \epsilon_p)\)-restricted first-round good \(\exists\)-statement of the shape \(\exists \varphi(\overline{\alpha} \leq \overline{\beta})\), which contains no black connective.

Proof. Straightforward checking. □

For each \((\Omega, \epsilon_p)\)-restricted first-round good \(\exists\)-statement \(\exists \varphi(\overline{\alpha} \leq \overline{\beta})\), it can be written in the form \(\exists \varphi(\theta \leq \eta \& \iota \leq \zeta)\), where each inequality \(\theta \leq \eta\) is \((\Omega, \epsilon_p)\)-restricted receiving, and each inequality \(\iota \leq \zeta\) is \((\Omega, \epsilon_p)\)-restricted solvable.

Lemma 57. For each \((\Omega, \epsilon_p)\)-restricted solvable inequality \(\iota \leq \zeta\) in the \(\exists\)-statement above, it can be transformed into a meta-conjunction of inequalities of the following kinds:

- \(\gamma \leq \delta\), where \(\gamma \leq \delta\) is a \((\Omega, \epsilon_p)\)-restricted receiving inequality;
- \(\kappa \leq q\) (if \(\epsilon(q) = 1\)) or \(q \leq \kappa\) (if \(\epsilon(q) = 0\)), where every branch in \(+\kappa\) (resp. \(-\kappa\)) is a Skeleton branch.

Proof. Without loss of generality we consider the situation when

- \(\neg \iota\) is such that all branches ending with propositional variables in \(\overline{q}\) are \(\epsilon_q^0\)-critical, and \(+\zeta\) contains \(\epsilon_q^0\)-critical branches;
- \(+\zeta\) is \((\Omega, \epsilon_p)\)-inductive, and all \(\epsilon_q^0\)-critical branches in \(+\zeta\) are PIA branches;
- for all the \(\epsilon_q^0\)-critical branches in \(+\zeta\) ending with \(q\), all propositional variables \(r\) in \(\neg \iota\), we have \(r <_{\Omega} q\);
- for every branch in \(\neg \iota\) and \(+\zeta\) which is not \(\epsilon_q^0\)-critical, it is a Skeleton branch in \(+\iota\) and \(-\zeta\).

The situation where \(+\zeta\) is such that all branches ending with \(\overline{q}\) are \(\epsilon_q^0\)-critical etc. is symmetric.

We prove by induction on the complexity of \(\zeta\). It is easy to see that \(\zeta\) does not contain any black connective.

- when \(\zeta = q\): \(\iota \leq \zeta\) belongs to the second class and \(\epsilon_q = 1\);
• when $\zeta$ is $p$ or $\bot$ or $T$: it cannot be the case, since $+\zeta$ contains $\varepsilon_q$-critical branches;

• when $\zeta$ is $-\gamma$: we can first apply the residuation rule for $-\iota$ to $\iota \leq -\gamma$ to obtain $\gamma \leq -\iota$, and then we can apply the induction hypothesis for the case where
  
  $-$ $+\neg\iota$ is such that all branches ending with propositional variables in $\bar{\gamma}$ are $\varepsilon_q^0$-critical;
  $-$ $-\gamma$ contains $\varepsilon_q$-critical branches, and is $(\Omega, \varepsilon_q)$-inductive, and all $\varepsilon_q$-critical branches in $-\gamma$ are PIA branches;
  $-$ for all the $\varepsilon_q$-critical branches in $-\gamma$ ending with $q$, all propositional variables $r$ in $+\neg\iota$, we have $r <_{\Omega} q$;
  $-$ for every branch in $+\neg\iota$ and $-\gamma$ which is not $\varepsilon_q$-critical, it is a Skeleton branch in $-\neg\iota$ and $+\gamma$;

• when $\zeta$ is $\Diamond\gamma$ or $\Diamond\gamma$: it cannot be the case, since $\Diamond$ or $\Diamond$ is on an $\varepsilon_q$-critical branch in $+\zeta$ but it is not a PIA node;

• when $\zeta$ is $\Box\gamma$: we can first apply the residuation rule for $\Box$ to $\iota \leq \Box\gamma$ to obtain $\Box\iota \leq \gamma$, and then we can apply the induction hypothesis for the case where
  
  $-$ $-\Box\iota$ is such that all branches ending with propositional variables in $\bar{\gamma}$ are $\varepsilon_q^0$-critical;
  $-$ $+\gamma$ contains $\varepsilon_q$-critical branches, and is $(\Omega, \varepsilon_q)$-inductive, and all $\varepsilon_q$-critical branches in $+\gamma$ are PIA branches;
  $-$ for all the $\varepsilon_q$-critical branches in $+\gamma$ ending with $q$, all propositional variables $r$ in $-\Box\iota$, we have $r <_{\Omega} q$;
  $-$ for every branch in $-\Box\iota$ and $+\gamma$ which is not $\varepsilon_q$-critical, it is a Skeleton branch in $+\Box\iota$ and $-\gamma$;

• when $\zeta$ is $\Box\gamma$, the situation is similar to the $\Box\gamma$ case;

• when $\zeta$ is $\gamma \wedge \delta$, we first apply the splitting rule for $\wedge$ to $\iota \leq \gamma \wedge \delta$ to obtain $\iota \leq \gamma$ and $\iota \leq \delta$, then there are two possibilities, namely both of $+\gamma$ and $+\delta$ have $\varepsilon_q$-critical branches, and only one of $+\gamma$ and $+\delta$ has $\varepsilon_q$-critical branches;

  for the first possibility, we can apply the induction hypothesis for the case where
  
  $-$ $-\iota$ is such that all branches ending with propositional variables in $\bar{\gamma}$ are $\varepsilon_q^0$-critical;
  $-$ $+\gamma$ and $+\delta$ contain $\varepsilon_q$-critical branches, and are $(\Omega, \varepsilon_q)$-inductive, and all $\varepsilon_q$-critical branches in $+\gamma$ and $+\delta$ are PIA branches;
  $-$ for all the $\varepsilon_q$-critical branches in $+\gamma$ and $+\delta$ ending with $q$, all propositional variables $r$ in $-\iota$, we have $r <_{\Omega} q$;
  $-$ for every branch in $-\iota$ and $+\gamma$ and $+\delta$ which is not $\varepsilon_q$-critical, it is a Skeleton branch in $+\iota$ and $-\gamma$ and $-\delta$;

  for the second possibility, without loss of generality we assume that $+\gamma$ has $\varepsilon_q$-critical branches and $+\delta$ contains no $\varepsilon_q$-critical branches, then $\iota \leq \delta$ is $(\Omega, \varepsilon_q)$-receiving (it is easy to check the Skeleton condition in $+\iota$ and $-\delta$), and we can apply the induction hypothesis to $\iota \leq \gamma$ for the case where
  
  $-$ $-\iota$ is such that all branches ending with propositional variables in $\bar{\gamma}$ are $\varepsilon_q^0$-critical;
  $-$ $+\gamma$ contains $\varepsilon_q$-critical branches, and is $(\Omega, \varepsilon_q)$-inductive, and all $\varepsilon_q$-critical branches in $+\gamma$ are PIA branches;
Lemma 58. Given a \( \exists \)-statement \( \exists(\gamma \leq \delta \& \bar{k} \leq \bar{q}) \) where each \( \gamma \leq \delta \) is \((\Omega, e_q)\)-inductive, and each \( k \leq q \) (when \( e(q) = 1 \)) or \( q \leq k \) (when \( e(q) = \delta \)) is such that every branch in +\( k \) (resp. −\( k \)) is a Skeleton branch, after the Ackermann rule, it is either again a \( \exists \)-statement divided into the two kinds of inequalities, or all inequalities of the form \( k \leq q \) or \( q \leq k \) are eliminated.

Proof. Without loss of generality we consider the application of the right-handed Ackermann rule

\[
\begin{align*}
\kappa_1 & \leq q \\
\vdots \\
\kappa_n & \leq q \\
\gamma_1 & \leq \delta_1 \\
\vdots \\
\gamma_m & \leq \delta_m \\
\iota_1 & \leq \zeta_1 \\
\vdots \\
\iota_k & \leq \zeta_k
\end{align*}
\]

where the \( \exists \)-statement \( \exists q \exists \bar{q} \) is replaced by

\[
\begin{align*}
\gamma_1((\kappa_1 \lor \ldots \lor \kappa_n)/q) & \leq \delta_1((\kappa_1 \lor \ldots \lor \kappa_n)/q) \\
\vdots \\
\gamma_m((\kappa_1 \lor \ldots \lor \kappa_n)/q) & \leq \delta_m((\kappa_1 \lor \ldots \lor \kappa_n)/q) \\
\iota_1 & \leq \zeta_1 \\
\vdots \\
\iota_k & \leq \zeta_k
\end{align*}
\]

where:

• for all the \( \varepsilon_q \)-critical branches in +\( \gamma \) ending with \( q \), all propositional variables \( r \) in −\( \iota \), we have \( r <_{\Omega} q \);
• for every branch in −\( \iota \) and +\( \gamma \) which is not \( \varepsilon_q \)-critical, it is a Skeleton branch in +\( \iota \) and −\( \gamma \);
• when \( \zeta \) is \( \gamma \lor \delta \): since \( \lor \) is an SRR node, only one of \( \gamma \) and \( \delta \) contains \( \varepsilon_q \)-critical branches (without loss of generality we assume that it is \( \delta \)). Then

\[
\begin{align*}
\text{1.} & \quad +\gamma \text{ is such that all branches ending with propositional variables in } \bar{q} \text{ are } \varepsilon_q^\beta \text{-critical;} \\
\text{2.} & \quad \text{for each } q \text{ in an } \varepsilon_q \text{-critical branch in } +\delta, \text{ each } r \text{ that occurs in } +\gamma, \text{ we have } r <_{\Omega} q; \\
\end{align*}
\]

Now we apply the residuation rule for \( \lor \) to \( \iota \leq \gamma \lor \delta \) to obtain \( \iota \land \neg \gamma \leq \delta \), then we can apply the induction hypothesis for the case where

\[
\begin{align*}
\text{1.} & \quad -(\iota \land \neg \gamma) \text{ is such that all branches ending with propositional variables in } \bar{q} \text{ are } \varepsilon_q^\beta \text{-critical;} \\
\text{2.} & \quad +\delta \text{ has } \varepsilon_q \text{-critical branches and is } (\Omega, e_q)\text{-inductive, and all } \varepsilon_q \text{-critical branches in } +\delta \text{ are PIA branches;} \\
\text{3.} & \quad \text{for all the } \varepsilon_q \text{-critical branches in } +\delta \text{ ending with } q, \text{ all propositional variables } r \text{ in } -(\iota \land \neg \gamma), \text{ we have } r <_{\Omega} q; \\
\text{4.} & \quad \text{for every branch in } -(\iota \land \neg \gamma) \text{ and } +\delta \text{ which is not } \varepsilon_q \text{-critical, it is a Skeleton branch in } +(\iota \land \neg \gamma) \text{ and } -\delta; \\
\end{align*}
\]

Lemma 58. Given a \( \exists \)-statement \( \exists(\gamma \leq \delta \& \bar{k} \leq \bar{q}) \) where each \( \gamma \leq \delta \) is \((\Omega, e_q)\)-inductive, and each \( k \leq q \) (when \( e(q) = 1 \)) or \( q \leq k \) (when \( e(q) = \delta \)) is such that every branch in +\( k \) (resp. −\( k \)) is a Skeleton branch, after the Ackermann rule, it is either again a \( \exists \)-statement divided into the two kinds of inequalities, or all inequalities of the form \( k \leq q \) or \( q \leq k \) are eliminated.

Proof. Without loss of generality we consider the application of the right-handed Ackermann rule
1. \( q \) does not occur in \( \kappa_1, \ldots, \kappa_n \);

2. Each \( \gamma_i \) is positive, and each \( \delta_i \) negative in \( q \), for \( 1 \leq i \leq m \);

3. Each \( \iota_i, \zeta_i \) does not contain \( q \).

Here each \( \gamma_i \leq \delta_i \) is \((\Omega, \varepsilon_{\vec{q}})\)-restricted receiving, \( \varepsilon(q) = 1 \), and each branch in \( +\kappa_i \) is a Skeleton branch. Since each branch in \( +\gamma_i \) and \( -\delta_i \) is Skeleton branch, and \( q \) is positive in each \( +\gamma_i \) and \( -\delta_i \), so replacing \( q \) by \( \kappa_1 \lor \cdots \lor \kappa_n \) makes \( +\gamma_i((\kappa_1 \lor \cdots \lor \kappa_n)/q) \) and \( -\delta_i((\kappa_1 \lor \cdots \lor \kappa_n)/q) \) again trees where each branch is Skeleton, and each \( \iota_i \leq \zeta_i \) is kept the same, so the \( \exists \)-statement can still be divided into the two kinds of inequalities, or all inequalities of the form \( \kappa \leq q \) or \( q \leq \kappa \) are eliminated.

By repeatedly applying Lemma 58, since after the monotone and antitone variable elimination rules, every propositional variable in \( \vec{q} \) has a critical branch in the \( \exists \)-statement, so there will be an inequality of the form \( \kappa \leq q \) or \( q \leq \kappa \) after application of the residuation rules, therefore all propositional variables appearing in the existential propositional quantifiers can be eliminated.

Summarizing the proofs above, we have the following result:

**Theorem 15.1.** The first half of \( \text{ALBA}^{\Pi_2} \) succeeds on \((\Omega, \varepsilon_{\vec{q}})\)-restricted first-round good \( \exists \)-statements, and output a meta-conjunction of the form \( \overline{\gamma} \leq \overline{\delta} \), where each branch of \( +\gamma_i \) and \( -\delta_i \) is Skeleton.

Since each branch of \( +\gamma_i \) and \( -\delta_i \) is Skeleton, we have that \( \gamma_i \leq \delta_i \) is \((\Omega, \varepsilon_{\vec{q}})\)-inductive for all order-types and all dependence orders, so the second half of the algorithm succeeds on the input quasi-inequality, so we have the following final result:

**Theorem 15.2.** \( \text{ALBA}^{\Pi_2} \) succeeds on all restricted inductive \( \Pi_2 \)-statements.

### 16 Canonicity

In this section we prove the canonicity for restricted inductive \( \Pi_2 \)-statements. The basic idea is again to check the topological correctness of the running of \( \text{ALBA}^{\Pi_2} \), and the focus here is again the Ackermann lemmas.

**Theorem 16.1.** Given a restricted inductive \( \Pi_2 \)-statement as input, \( \text{ALBA}^{\Pi_2} \) can topologically correctly execute on it.

**Proof.** We basically follow the success proof in Section 15, while pay attention to the topological correctness of the execution.

From Lemma 58, given a restricted inductive \((\Omega, \varepsilon_{\vec{q}})\)-\( \Pi_2 \)-statement

\[
\overline{\gamma} \leq \overline{\psi} & \overline{x} < \overline{\xi} \Rightarrow \exists \overline{q}(\overline{\alpha} \leq \overline{\beta} \& \overline{\gamma} < \overline{\delta})
\]

in which the \( \exists \)-statement \( \exists \overline{q}(\overline{\alpha} \leq \overline{\beta} \& \overline{\gamma} < \overline{\delta}) \) part is \((\Omega, \varepsilon_{\vec{q}})\)-restricted first-round good, after the first stage of the first half of the algorithm, the \( \exists \)-statement is transformed into another \((\Omega, \varepsilon_{\vec{q}})\)-restricted first-round good \( \exists \)-statement of the shape \( \exists \overline{q}(\overline{\alpha} \leq \overline{\beta}) \), which contains no black connective, and it is easy to check that each inequality is syntactically closed on the left-hand side, and syntactically open on the right-hand side.

Now we can easily check that the following property holds for the \( \exists \)-statement:

in each inequality inside the \( \exists \)-statement, the left-hand side is syntactically closed and the right-hand side is syntactically open.
It is easy to check that for each rule in Stage 2 of the first half, it does not break this property, so for each execution of the Ackermann rule, it is topologically correct, and after the execution of the Ackermann rule, it still satisfies the property stated above. Therefore, the execution of ALBA\(\Pi_2\) is topologically correct in the first half.

Now for the second half, the input quasi-inequality is of the form
\[ \varphi \leq \psi \land \chi < \xi \Rightarrow \alpha_1 \leq \beta_1 \land \ldots \land \alpha_n \leq \beta_n, \]
where
- each inequality in \(\varphi \leq \psi\) and \(\chi < \xi\) is either \((\Omega, \varepsilon_\beta)\)-receiving or \((\Omega, \varepsilon_\beta)\)-solvable;
- each branch in \(+\alpha_i\) and \(\neg\beta_i\) is a Skeleton branch.

Now by applying the subordination rewriting rule and then splitting the quasi-inequality into a meta-conjunction of quasi-inequalities of the form
\[ \varphi \leq \psi \Rightarrow \alpha_i \leq \beta_i. \]

By applying the first-approximation rule, we get
\[ \varphi \leq \psi \land \imath \leq \alpha_i \land \beta_i \leq \neg \jmath \Rightarrow \imath \leq \neg \jmath, \]
where each \(\varphi\) is syntactically closed, each \(\psi\) is syntactically open, every branch in \(+\alpha_i\) and \(\neg\beta_i\) are Skeleton branches.

Now by an argument similar to Lemma 41, we have that \(\imath \leq \alpha_i\) and \(\beta_i \leq \neg \jmath\) can be reduced to a meta-conjunction of the following kinds of inequalities:
- pure inequalities;
- inequalities of the form \(k \leq p\) or \(p \leq \neg k\) (notice that they are syntactically closed on the left-hand side and syntactically open on the right-hand side);

Now the antecedent system has two kinds of inequalities, one kind is pure, the other kind is syntactically closed on the left-hand side and syntactically open on the right-hand side.

It is easy to check that for each rule in Stage 2 of the second half, it does not break this property, so for each execution of the Ackermann rule, it is topologically correct, and after the execution of the Ackermann rule, it still satisfies the property stated above. Therefore, the execution of ALBA\(\Pi_2\) is topologically correct in the second half.

\[\square\]

**Corollary 16.2.** Restricted inductive \(\Pi_2\)-statements are canonical.

**17 Examples**

**Example 17.1** (See Lemma 2.1.12 in [25]). Given a modal subordination algebra \((B, \prec, \Diamond)\) and its dual Stone space with two relations \((X, \tau, R, R')\),

\[ (B, \prec, \Diamond) \vdash \forall a \forall b (a < b \Rightarrow \exists c (a < c < b)) \] iff \(R\) is transitive.
Proof. It suffices to show that

\[(B, <, \Diamond) \vdash \forall a \forall b (a < b \Rightarrow \exists c (a < c < b))\]

iff its dual

\[(X, \tau, R, R') \vdash \forall w \forall v \forall u (Rwv \land Rvu \rightarrow Rwu).\]

Indeed,

\[
\begin{align*}
(B, <, \Diamond) & \vdash \forall a \forall b (a < b \Rightarrow \exists c (a < c < b)) \\
\text{iff} & \quad (X, \tau, R, R') \vdash \text{Clop} \; \forall a \forall b (a < b \Rightarrow \exists c (a < c < b)) \\
\text{iff} & \quad (X, \tau, R, R') \vdash \text{Clop} \; \forall a \forall b (\Diamond a < b \Rightarrow \exists c (\Diamond a < c \land \Diamond c \leq b)) \\
\text{iff} & \quad (X, \tau, R, R') \vdash \text{Clop} \; \forall a \forall b (\Diamond a < b \Rightarrow \Diamond a < b) \\
\text{iff} & \quad (X, \tau, R, R') \vdash \text{Clop} \; \forall a \forall b (\Diamond a < b \Rightarrow a < b) \\
\text{iff} & \quad (X, \tau, R, R') \vdash \text{Clop} \; \forall a \forall b (\Diamond a < b \Rightarrow a < b) \\
\text{iff} & \quad (X, \tau, R, R') \vdash \text{Clop} \; \forall a \forall b (\Diamond a < b \Rightarrow a < b) \\
\text{iff} & \quad (X, \tau, R, R') \vdash \text{Clop} \; \forall a \forall b (\Diamond a < b \Rightarrow a < b) \\
\text{iff} & \quad (X, \tau, R, R') \vdash \forall w (R[R[w]] \subseteq R[w]) \\
\text{iff} & \quad (X, \tau, R, R') \vdash \forall w \forall v \forall u (Rwv \land Rvu \rightarrow Rwu).
\]

\[\square\]

18 Comparison with Existing Works

In this section, we compare the present paper’s method with existing works on the correspondence theory of subordination algebras and subordination spaces. In general, the three approaches we mention here treat subordination algebras and subordination spaces in different forms, while they do not treat modal subordination algebras and Stone spaces with two relations as we deal with.

18.1 de Rudder et al.’s approach

In [13] [14] [15], de Rudder et al. studied correspondence theory of subordination algebras in the perspective of quasi-modal operators and slanted algebras. The basic idea behind their work is as follows (we follow the notation in [15]):

**Definition 59** (Definition 6.4 in [15]). A tense slanted BAE is defined as \((B, \Diamond, \square)\), where \(B\) is a Boolean algebra and \(\Diamond : B \rightarrow K(B^0), \square : B \rightarrow O(B^0)\) are such that

- \(\Diamond \bot = \bot\) and \(\Diamond (a \lor b) = \Diamond a \lor \Diamond b\);
- \(\square \top = \top\) and \(\square (a \land b) = \square a \land \square b\);
- \(\Diamond a \leq b\) iff \(a \leq \square b\).

For any subordination algebra \((B, <)\), define \(\Diamond_<\) and \(\square_<\) as follows:

\[
\Diamond_< a := \bigwedge \{ b \in B \mid a < b \}
\]

\[
\square_< a := \bigvee \{ b \in B \mid b < a \}
\]

then the associated tense slanted BAE is defined as \((B, \Diamond_<, \square_<)\).

Given a tense slanted BAE \((B, \Diamond, \square)\), define \(<_\Diamond\) as

\(a <_\Diamond b\) iff \(\Diamond a \leq b\) iff \(a \leq \square b\),
then the associated subordination algebra is \((B, <_\diamondsuit)\).

Then it is easy to see that there is a 1-1 correspondence between subordination algebras and tense slanted BAEs (see Proposition 6.7 in [15]).

The Sahlqvist formulas are then defined in the language of tense slanted BAEs by using the bimodal language \(\diamondsuit, \blacksquare\) and there duals \(\Box := \neg \diamondsuit \neg\) and \(\blacklozenge := \neg \blacksquare \neg\):

**Definition 60** (Definition 3.3 in [14]).
- A bimodal formula \(\varphi\) is *closed* (resp. *open*) if it is obtained from \(T, \bot, \) propositional variables and their negations by applying \(\land, \lor, \diamondsuit, \blacklozenge\) (resp. \(\land, \lor, \Box, \blacksquare\)).
- A bimodal formula \(\varphi\) is *positive* (resp. *negative*) if it is obtained from \(T, \bot\) and propositional variables (resp. negations of propositional variables) by applying \(\land, \lor, \diamondsuit, \blacklozenge\).
- A bimodal formula \(\varphi\) is *s-positive* (resp. *s-negative*) if it is obtained from closed positive formulas (resp. open negative formulas) by applying \(\land, \lor, \Box, \blacksquare\) (resp. \(\land, \lor, \diamondsuit, \blacklozenge\)).
- A bimodal formula \(\varphi\) is *g-closed* (resp. *g-open*) if it is obtained from closed (resp. open) formulas by applying \(\land, \lor, \Box, \blacksquare\) (resp. \(\land, \lor, \diamondsuit, \blacklozenge\)).
- A strongly positive bimodal formula \(\varphi\) is a conjunction of formulas of the form \(\Box^{\mu_1} \blacksquare^{\mu_2} \ldots \Box^{\mu_k} p\) where \(\mu_i\) is a natural number for each \(i\).
- An s-untied bimodal formula \(\varphi\) is obtained from strongly positive and s-negative formulas by applying only \(\land, \lor, \diamondsuit, \blacklozenge\).
- An s-Sahlqvist bimodal formula \(\varphi\) is of the form \(\Box^{\mu_1} \blacksquare^{\mu_2} \ldots \Box^{\mu_k} (\varphi_1 \rightarrow \varphi_2)\) where \(\varphi_1\) is s-untied and \(\varphi_2\) is s-positive.

**Remark 61.** The difference and similarity between de Rudder et al.’s approach and our approach can be summarized as follows:

1. de Rudder et al. are using our \(\Box, \Diamond, \blacksquare, \blacklozenge\) as their signature, and they allow nested occurrences of these modalities, while in our approach, \(\Diamond\) only occurs in the form \(\Diamond \varphi \leq \psi\) which stands for \(\varphi < \psi\).
2. de Rudder et al.’s modalities are slanted, i.e. the diamond (resp. open) modalities are mapping clopen elements to closed (resp. open) elements since they correspond to the subordination relation, while we have two kinds of modalities, namely modalities corresponding to the subordination relation and ordinary modalities which map clopens to clopens.
3. In de Rudder et al.’s approach, they only treat formulas, while our first-class citizens are quasi-inequalities and \(\Pi_2\)-statements.

### 18.2 Santoli’s approach

In [25], Santoli studied the topological correspondence theory between conditions on algebras and first-order conditions on the dual subordination spaces, in the language of a binary connective definable from the squigarrow associated with the subordination relation, using the so-called \(\forall \exists\)-statements (which we call \(\Pi_2\)-statements in the present paper).

In [25], Santoli uses a binary modality \(\diamondsuit\) which is defined as \(\varphi \circ \psi := \neg (\varphi \leadsto \psi)\) (see page 2), and its dual \(\Box\) which is defined as \(\varphi \Box \psi := \neg (\neg \varphi \circ \neg \psi)\). His definition of Sahlqvist formulas and \(\forall \exists\)-statements are given as follows:

**Definition 62** (Definition 6.1.1 in [25]).
- A formula \(\varphi\) is *positive \(\circ\)-free* if it is obtained from \(T\) and propositional variables by applying \(\lor, \land\).
• A formula \( \theta \) is a Sahlqvist antecedent if it is obtained from \( \top \) and \( \varphi \diamond \psi \) using \( \lor, \land \) where \( \varphi \) and \( \psi \) are positive \( \diamond \)-free.

• A formula \( \chi \) is positive if it is obtained from \( \top \) and \( \varphi \diamond \psi \) and \( \varphi \Box \psi \) using \( \lor, \land \) where \( \varphi \) and \( \psi \) are positive \( \diamond \)-free.

• A non-separating formula \( S(p) \) is \( F(p) \lor G(\neg p) \) where there are positive \( \diamond \)-free formulas \( \varphi, \psi \) such that \( F(p) \) is equal to either \( \varphi \diamond p \) or \( p \diamond \varphi \) and \( G(\neg p) \) is equal to either \( \psi \diamond \neg p \) or \( \neg p \diamond \psi \).

• A general positive formula is a formula \( \chi(p) \) which is a conjunction of non-separating formulas \( S(p_1), \ldots, S(p_n) \) and positive formulas, where \( \overline{p} = p_1, \ldots, p_n \) are propositional variables.

• A Sahlqvist formula is a formula \( \theta \rightarrow \chi(p) \) where \( \theta \) is a Sahlqvist antecedent, \( \chi(p) \) is a general positive formula, and the propositional variables \( \overline{p} \) do not occur in \( \theta \).

**Definition 63** (Definition 6.1.1 in [25]). A Sahlqvist statement is \( \Psi \) in the signature of \( (\land, \neg, \top, \diamond) \) of the form

\[
\Psi := \forall \overline{q}(\forall \overline{p}(\theta \land (\bigwedge_{l=1}^{k} S_l(p_l)) = \top) \Rightarrow \forall \overline{r}(\chi(\overline{r}) = \top))
\]

where

• \( \theta \) is a Sahlqvist antecedent;

• \( \chi(p) \) is a general positive formula;

• the \( S_l(p_l) \)'s are non-separating formulas;

• \( \overline{q} \) are all propositional variables not among \( \overline{p} = p_1, \ldots, p_k \) and \( \overline{r} \) which occur in the formula;

• the proposition variables \( \overline{p} \) and \( \overline{r} \) do not occur anywhere but in their respective non-separating formulas.

**Remark 64.** The difference and similarity between Santoli’s approach and our approach can be summarized as follows:

1. Santoli treats both Sahlqvist formulas and Sahlqvist \( \forall \exists \)-statements, which is similar to our approach. However, our treatment of \( \forall \exists \)-statements (i.e. \( \Pi_2 \)-statements) are based on quasi-inequalities, which is different from Santoli’s approach.

2. Santoli uses a diamond-type binary modality as signature, while we use both the subordination relation and the dotted modalities to interpret the subordination relation and its corresponding closed relation.

3. Santoli uses the non-separating formulas, which is a special feature in his approach and not used anywhere else.

4. Santoli does not allow nested occurrences of modalities, which is similar to our usage of dotted modalities, while we allow nested occurrences of ordinary modalities.
18.3 Balbiani and Kikot’s approach

Balbiani and Kikot \[1\] investigated the Sahlqvist theory in the language of region-based modal logics of space, which uses a contact relation. The syntax of \[1\] is two-layered. The inner layer is the layer of Boolean terms, and is defined as follows:

\[
a ::= p \mid 0 \mid \neg a \mid (a \cup b)
\]

and 1 is defined as \(-0\), \((a \cap b)\) is defined as \(\neg(\neg a \cup \neg b)\). A term \(a\) is positive if \(a\) is obtained from variables and 1 by applying \(\cup, \cap\). The second layer is defined as follows:

\[
\varphi ::= a \equiv b \mid C(a, b) \mid \bot \mid \neg \varphi \mid (\varphi \lor \psi)
\]

and \(\top\) is defined as \(\neg \bot\), \((\varphi \land \psi)\) is defined as \(\neg(\neg \varphi \lor \neg \psi)\), \((\varphi \rightarrow \psi)\) is defined as \((\neg \varphi \lor \psi)\) and \((\varphi \leftrightarrow \psi)\) for \((\neg(\varphi \lor \psi) \lor \neg(\neg \varphi \lor \neg \psi))\), \(a \neq b\) is defined as \(\neg a \equiv b\), \(\bar{C}(a, b)\) is defined as \(\neg \bar{C}(a, b)\).

The semantics of the language is given as follows:

**Definition 65** (Section 3 in \[1\]). A Kripke frame is \(F = (W, R)\) where \(W\) is non-empty and \(R\) is a binary relation on \(W\). A valuation on \(F\) is a function \(V\) assigning to each Boolean variable \(p\) a subset \(V(p)\) of \(W\). The interpretation of Boolean terms are given as follows:

- \(V(0) = \emptyset\);
- \(V(\neg a) = W - V(a)\);
- \(V(a \cup b) = V(a) \cup V(b)\).

A Kripke model is \(M = (W, R, V)\) where \(F = (W, R)\) is a Kripke frame and \(V\) is a valuation on \(F\). The satisfaction relation is defined as follows:

- \(M \vDash a \equiv b\) iff \(V(a) = V(b)\);
- \(M \vDash C(a, b)\) iff there exist \(x, y \in W\) such that \(xRy, x \in V(a)\) and \(y \in V(b)\);
- \(M \vDash \bot\);
- \(M \vDash \neg \varphi\) iff \(M \nvDash \varphi\);
- \(M \vDash \varphi \lor \psi\) iff \(M \vDash \varphi\) or \(M \vDash \psi\).

The definition of Sahlqvist formulas is given as follows:

**Definition 66** (Section 2 in \[1\]).

- A formula \(\varphi\) is negation-free if it is obtained from \(\top\), \(a \neq 0\) and \(C(a, b)\) (where \(a\) and \(b\) are positive terms) by applying \(\land, \lor\).
- A formula \(\varphi\) is positive if it is obtained from \(\top, a \neq 0, \neg a \equiv 0, C(a, b)\) and \(\bar{C}(\neg a, \neg b)\) (where \(a\) and \(b\) are positive terms) by applying \(\land, \lor\).
- A formula \(\varphi\) is Sahlqvist if it is of the form \(\psi \rightarrow \chi\) where \(\psi\) is negation-free and \(\chi\) is positive.

According to \[1\], precontact logics contains all instances of the following formulas:

- \(C(a, b)\)
  \(\text{toa} \neq 0 \land b \neq 0\);
- \(C(a \cup b, c) \leftrightarrow C(a, c) \lor C(b, c)\);
\( C(a, b \cup c) \leftrightarrow C(a, b) \lor C(a, c) \).

which means that the relation \( C \) on a Boolean algebra is a \textit{precontact relation} or \textit{proximity} (see Definition 2.1.2 in [25] or [17]). Given a subordination \( \prec \), the relation \( arets b := a \nless \lnot b \) is a proximity. Given a proximity \( C \), the relation \( a \less C b := aC\lnot b \) is a subordination. Indeed, there is a 1-1 correspondence between proximities and subordinations.

\textbf{Remark 67.} The similarity and difference between Balbiani and Kikot’s approach and our approach can be summarized as follows:

- Balbiani and Kikot use a two-layered syntax, which is similar to our usage of formulas as one layer and inequalities of the form \( \varphi \leq \psi, \varphi < \psi \), quasi-inequalities and \( \Pi_2 \)-statements as another layer.
- Balbiani and Kikot uses the precontact relation, while we use the subordination relation.
- Balbiani and Kikot do not allow nested occurrences of \( C \), which is similar to our approach where the dotted modalities are not nested, although we allow nested occurrences of ordinary modalities.
- Balbiani and Kikot do not have \( \Pi_2 \)-statements, which is what we use substantially.

\textbf{Acknowledgement} The research of the author is supported by Taishan University Starting Grant “Studies on Algebraic Sahluqvist Theory” and the Taishan Young Scholars Program of the Government of Shandong Province, China (No.tsqn201909151). The author would like to thank Nick Bezhanishvili for his suggestions and comments on this project.

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