Fast $L_2$-approximation of integral-type functionals of Markov processes

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Abstract In this paper, we provide strong $L_2$-rates of approximation of the integral-type functionals of Markov processes by integral sums. We improve the method developed in [2]. Under assumptions on the process formulated only in terms of its transition probability density, we get the accuracy that coincides with that obtained in [3] for a one-dimensional diffusion process.

Keywords Markov processes, integral functional, rates of convergence, strong approximation

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1 Introduction

Let $X_t$, $t \geq 0$, be a Markov process with values in $\mathbb{R}^d$. Consider the following objects:

1) the integral functional

$$I_T(h) = \int_0^T h(X_t) \, dt$$

of this process;

2) the sequence of integral sums

$$I_{T,n}(h) = \frac{T}{n} \sum_{k=0}^{n-1} h(X_{(kT)/n}), \quad n \geq 1.$$
In this paper, we establish strong $L_2$-approximation rates, that is, the bounds for
$$E\left| I_T(h) - I_{T,n}(h) \right|^2.$$

The current research is mainly motivated by the recent papers [2] and [3].

In [3], strong $L_p$-approximation rates are considered for an important particular case where $X$ is a one-dimensional diffusion. The approach developed in this paper contains both the Malliavin calculus tools and the Gaussian bounds for the transition probability density of the process $X$, and relies substantially on the structure of the process.

Another approach to that problem has been developed in [2]. This approach is, in a sense, a modification of Dynkin’s theory of continuous additive functionals (see [1], Chap. 6) and also involves the technique similar to that used in the proof of the classical Khasminskii lemma (see, e.g., [4, Lemma 2.1]). This approach allows us to obtain strong $L_p$-approximation rates under assumptions on the process $X$ formulated only in terms of its transition probability density.

For a bounded function $h$, the strong $L_p$-rates of approximation of the integral functional $I_T(h)$ obtained in [2] essentially coincide with those established in [3]. However, under additional regularity assumptions on the function $h$ (e.g., when $h$ is Hölder continuous), the rates obtained in [3] are sharper (see [2, Thm. 2.2] and [3, Thm. 2.3]).

In this note, we improve the method developed in [2], so that under the assumption of the Hölder continuity of $h$, the strong $L_2$-approximation rates coincide with those obtained in [3], preserving at the same time the advantage of the method that the assumptions on the process $X$ are quite general and do not essentially rely on the structure of the process.

2 Main result

In what follows, $P_x$ denotes the law of the Markov process $X$ conditioned by $X_0 = x$, and $E_x$ denotes the expectation with respect to this law. Both the absolute value of a real number and the Euclidean norm in $\mathbb{R}^d$ are denoted by $| \cdot |$.

We make the following assumption on the process $X$.

A. The process $X$ possesses a transition probability density $p_t(x, y)$ that is differentiable with respect to $t$ and satisfies the following estimates:

$$p_t(x, y) \leq C_T t^{-d/\alpha} Q(t^{-1/\alpha}(x - y)), \quad t \leq T,$$
$$|\partial_t p_t(x, y)| \leq C_T t^{-1-d/\alpha} Q(t^{-1/\alpha}(x - y)), \quad t \leq T,$$
$$|\partial^2_{tt} p_t(x, y)| \leq C_T t^{-2-d/\alpha} Q(t^{-1/\alpha}(x - y)), \quad t \leq T,$$

for some fixed $\alpha \in (0, 2]$ and some distribution density $Q$ such that $\int_{\mathbb{R}^d} |z|^{2\gamma} Q(z) \, dz < \infty$. Without loss of generality, we assume that in (1)–(3) $C_T \geq 1$.

We assume that the function $h$ satisfies the Hölder condition with exponent $\gamma \in (0, \alpha/2]$, that is,

$$\|h\|_{\gamma} := \sup_{x \neq y} \frac{|h(x) - h(y)|}{|x - y|^{\gamma}} < \infty.$$
Now we formulate the main result of the paper.

**Theorem 1.** Suppose that Assumption A holds. Then

\[
\mathbb{E}_x |I_T(h) - I_{T,n}(h)|^2 \leq \left\{ \begin{array}{ll}
D_{T,\gamma,\alpha,Q} C_{\gamma,\alpha} \|h\|_{\gamma}^2 n^{-(1+2\gamma/\alpha)}, & \gamma \neq \alpha/2, \\
D_{T,\gamma,\alpha,Q} \|h\|_{\gamma}^2 n^{-2} \ln n, & \gamma = \alpha/2,
\end{array} \right.
\]

where

\[
D_{T,\gamma,\alpha,Q} = 8C_{T}^2 T^{2+2\gamma/\alpha} \int_{\mathbb{R}^d} |z|^{2\gamma} Q(z) \, dz,
\]

\[
C_{\gamma,\alpha} = \max \left\{ (1 - 2\gamma/\alpha)^{-1}(2\gamma/\alpha)^{-1}, \max_{n \geq 1} \left( \frac{(\ln n)^2}{n^{1-2\gamma/\alpha}} \right) \right\}.
\]

We provide the proof of Theorem 1 in Section 3.

**Remark 1.** Any diffusion process satisfies conditions (1)–(3) with \( \alpha = 2, Q(x) = c_1 e^{-c_2 |x|^2} \), and properly chosen \( c_1, c_2 \) (see [2]). In the case where \( X \) is a one-dimensional diffusion, Theorem 1 provides the same rates of convergence as those obtained in [3] (see Theorem 2.3 in [3]).

**Remark 2.** Similarly to [2], we formulate the assumption on the process \( X \) only in terms of its transition probability density. Condition A, compared with condition X (cf. [2]), contains the additional assumption (3).

### 3 Proof of Theorem 1

**Proof.** For \( t \in [kT/n, (k+1)T/n) \), denote

\[
\eta_n(t) = \frac{kT}{n}, \quad \zeta_n(t) = \frac{(k+1)T}{n},
\]

and put \( \Delta_n(s) := h(X_s) - h(X_{\eta_n(s)}) \), \( s \in [0, T] \).

By the Markov property of \( X \), for any \( r < s \), we have

\[
\mathbb{E}_x |X_s - X_r|^{2\gamma} = \mathbb{E}_x \int_{\mathbb{R}^d} p_{s-r}(X_r, z)|X_r - z|^{2\gamma} \, dz 
\leq C_T \mathbb{E}_x \int_{\mathbb{R}^d} (s-r)^{-d/\alpha} Q((s-r)^{-1/\alpha}(X_r - z))|X_r - z|^{2\gamma} \, dz 
= C_T (s-r)^{2\gamma/\alpha} \int_{\mathbb{R}^d} |z|^{2\gamma} Q(z) \, dz.
\]

Therefore, using the inequality \( s - \eta_n(s) \leq T/n, \ s \in [0, T] \) and the Hölder continuity of the function \( h \), we obtain:

\[
\mathbb{E}_x |\Delta_n(s)|^2 \leq C_T T^{2\gamma/\alpha} \left( \int_{\mathbb{R}^d} |z|^{2\gamma} Q(z) \, dz \right) \|h\|_{\gamma}^2 n^{-2\gamma/\alpha}.
\]

Split

\[
\mathbb{E}_x |I_T(h) - I_{T,n}(h)|^2 = 2 \int_0^T \int_s^T \Delta_n(s) \Delta_n(t) \, dt \, ds = J_1 + J_2 + J_3,
\]
where

\[ J_1 = 2\mathbb{E}_x \int_0^T \int_s^{T + T/n} \Delta_n(s) \Delta_n(t) \, dt \, ds, \]

\[ J_2 = 2\mathbb{E}_x \int_0^{T/n} \int_s^{T} \Delta_n(s) \Delta_n(t) \, dt \, ds, \]

\[ J_3 = 2\mathbb{E}_x \int_{T/n}^T \int_{\zeta_n(s) + T/n}^T \Delta_n(s) \Delta_n(t) \, dt \, ds. \]

For \(|J_1|\) and \(|J_2|\), the estimates can be obtained in the same way. Indeed, using the Cauchy inequality and (4), we get

\[
|J_1| \leq 2 \int_0^T \int_s^{\zeta_n(s) + T/n} (\mathbb{E}_x |\Delta_n(s)|)^{1/2} (\mathbb{E}_x |\Delta_n(t)|)^{1/2} \, dt \, ds
\leq 2C_T T^{2\gamma/\alpha} \|h\|_\gamma^2 \left( \int_{\mathbb{R}^d} |z|^{2\gamma} Q(z) \, dz \right) n^{-2\gamma/\alpha} \int_0^T (T/n + \zeta_n(s) - s) \, ds
\leq 4C_T T^{2+2\gamma/\alpha} \|h\|_\gamma^2 \left( \int_{\mathbb{R}^d} |z|^{2\gamma} Q(z) \, dz \right) n^{-(1+2\gamma/\alpha)}.
\]

In the last inequality, we have used the inequality \(\zeta_\alpha(s) - s \leq T/n, s \in [0, T]\). Similarly,

\[
|J_2| \leq 2C_T T^{2+2\gamma/\alpha} \|h\|_\gamma^2 \left( \int_{\mathbb{R}^d} |z|^{2\gamma} Q(z) \, dz \right) n^{-(1+2\gamma/\alpha)}.
\]

Now we proceed to the estimation of \(|J_3|\), which is the main part of the proof. Observe that the following identities hold:

\[
\int_{\mathbb{R}^d} \partial_{uv}^2 p_u(x, y) p_{v-u}(y, z) \, dz = \partial_{uv}^2 p_u(x, y) \int_{\mathbb{R}^d} p_{v-u}(y, z) \, dz
= \partial_{uv}^2 p_u(x, y) = 0, \quad y \in \mathbb{R}^d, \tag{6}
\]

\[
\int_{\mathbb{R}^d} \partial_{uv}^2 p_u(x, y) p_{v-u}(y, z) \, dy = \partial_{uv}^2 \int_{\mathbb{R}^d} p_u(x, y) p_{v-u}(y, z) \, dy
= \partial_{uv}^2 p_v(x, z) = 0, \quad z \in \mathbb{R}^d, \tag{7}
\]

where in (6) we used that \(\int_{\mathbb{R}^d} p_r(y, z) \, dz = 1, r > 0, y \in \mathbb{R}^d\), and in (7) we used the Chapman–Kolmogorov equation.

We have:

\[
J_3 = 2 \int_{T/n}^T \int_{\zeta_n(s) + T/n}^T \int_{\mathbb{R}^d} h(y) h(z) \left[ p_s(x, y) p_{t-s}(y, z) - p_{\eta_n(s)}(x, y) p_{t-\eta_n(s)}(y, z) - p_s(x, y) p_{\eta_n(t) - s}(y, z) + p_{\eta_n(s)}(x, y) p_{\eta_n(t) - \eta_n(s)}(y, z) \right] \, dz \, dy \, dt \, ds
\]
\[
\begin{align*}
&= 2 \int_{T/n}^{T} \int_{\zeta_n(s) + T/n}^{T} \int_{\eta_n(s)}^{T} \int_{\eta_n(t)}^{s} h(y)h(z) \partial_{uv}^2 (p_u(x, y) \\
&\quad \times p_{v-u}(y, z)) \, dv \, du \, dz \, dy \, dt \, ds \\
&= - \int_{T/n}^{T} \int_{\zeta_n(s) + T/n}^{T} \int_{\eta_n(s)}^{T} \int_{\eta_n(t)}^{s} (h(y) - h(z))^2 \partial_{uv}^2 (p_u(x, y) \\
&\quad \times p_{v-u}(y, z)) \, dv \, du \, dz \, dy \, dt \, ds,
\end{align*}
\]

where in the last identity we have used (6) and (7).

Further, we have
\[
\partial_{uv}^2 p_u(x, y)p_{v-u}(y, z) = p_u(x, y)\partial_{rr}^2 p_r(y, z)|_{r=v-u} + \partial_u p_u(x, y)\partial_r p_r(y, z)|_{r=v-u}.
\]

Then, using condition A and the Hölder continuity of the function \( h \), we obtain
\[
\begin{align*}
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (h(y) - h(z))^2 |\partial_{uv}^2 (p_u(x, y)p_{v-u}(y, z))| \, dz \, dy &\leq C_T^2 \|h\|_{2\gamma}^2 \left( \int_{\mathbb{R}^d} |z|^{2\gamma}Q(z) \, dz \right) \left((v - u)^{2\gamma/\alpha - 2} + (v - u)^{2\gamma/\alpha - 1}u^{-1}\right).
\end{align*}
\]

Therefore, according to (8) and (9),
\[
|J_3| \leq C_T^2 \|h\|_{2\gamma}^2 \left( \int_{\mathbb{R}^d} |z|^{2\gamma}Q(z) \, dz \right) \times \int_{T/n}^{T} \int_{\zeta_n(s) + T/n}^{T} \int_{\eta_n(s)}^{T} \int_{\eta_n(t)}^{s} (v - u)^{2\gamma/\alpha - 2} + (v - u)^{2\gamma/\alpha - 1}u^{-1} \, dv \, du \, dt \, ds.
\]

Denote \( a_{\alpha, \gamma}(u, v) := (v - u)^{2\gamma/\alpha - 2} + (v - u)^{2\gamma/\alpha - 1}u^{-1} \). Then
\[
\int_{T/n}^{T} \int_{\zeta_n(s) + T/n}^{T} \int_{\eta_n(s)}^{T} \int_{\eta_n(t)}^{s} a_{\alpha, \gamma}(u, v) \, dv \, du \, dt \, ds
= \sum_{i=1}^{n-1} \sum_{j=i+2}^{n-1} \int_{iT/n}^{(i+1)T/n} \int_{jT/n}^{(j+1)T/n} \int_{iT/n}^{(i+1)T/n} \int_{jT/n}^{(j+1)T/n} a_{\alpha, \gamma}(u, v) \, dv \, du \, dt \, ds
\]
\[
= \sum_{i=1}^{n-1} \sum_{j=i+2}^{n-1} \int_{iT/n}^{(i+1)T/n} \int_{jT/n}^{(j+1)T/n} \int_{u}^{\eta_n(s)} \int_{v}^{\eta_n(t)} a_{\alpha, \gamma}(u, v) \, dt \, ds \, dv \, du
\]
\[
\leq T^2n^{-2} \sum_{i=1}^{n-1} \sum_{j=i+2}^{n-1} \int_{iT/n}^{(i+1)T/n} \int_{jT/n}^{(j+1)T/n} a_{\alpha, \gamma}(u, v) \, dv \, du
\]
\[
= T^2n^{-2} \sum_{i=1}^{n-1} \int_{iT/n}^{(i+1)T/n} \int_{(i+2)T/n}^{T} a_{\alpha, \gamma}(u, v) \, dv \, du,
\]

where in the fourth line we used that, for \( u \in [iT/n, (i + 1)T/n) \) and \( v \in [jT/n, (j + 1)T/n) \), we always have \((i + 1)T/n - u \leq T/n\) and \((j + 1)T/n - v \leq T/n\).
Thus, from (10) we obtain

\[ |J_3| \leq C_T^2 T^2 \|h\|_2^2 \left( \int_{\mathbb{R}^d} |z|^{2\gamma} Q(z) \, dz \right) n^{-2} (S_1 + S_2), \quad (11) \]

where

\[ S_1 = \sum_{i=1}^{n-1} \int_{iT/n}^{(i+1)T/n} \int_{iT/n}^{T} (v - u)^{2\gamma/\alpha - 2} \, dv \, du, \]
\[ S_2 = \sum_{i=1}^{n-1} \int_{iT/n}^{(i+1)T/n} \int_{iT/n}^{(i+2)T/n} (v - u)^{2\gamma/\alpha - 1} u^{-1} \, dv \, du. \]

We estimate each term separately. In what follows, we consider the case \( \gamma < \alpha/2; \) the case of \( \gamma = \alpha/2 \) is similar and therefore omitted. We have

\[ S_1 \leq (1 - 2\gamma/\alpha)^{-1} \sum_{i=1}^{n-1} (i+1)T/n - u)^{2\gamma/\alpha - 1} \, du \]
\[ = (1 - 2\gamma/\alpha)^{-1} (2\gamma/\alpha)^{-1} \sum_{i=1}^{n-1} ((i+1)T/n - iT/n)^{2\gamma/\alpha} \]
\[ \leq (1 - 2\gamma/\alpha)^{-1} (2\gamma/\alpha)^{-1} T^{2\gamma/\alpha} n^{1-2\gamma/\alpha} \leq C_{\gamma,\alpha} T^{2\gamma/\alpha} n^{1-2\gamma/\alpha}. \quad (12) \]

Finally, since \( v - u \leq T \) for \( 0 \leq u < v \leq T \), we have

\[ S_2 \leq T^{2\gamma/\alpha} \sum_{i=1}^{n-1} \int_{iT/n}^{(i+1)T/n} \int_{iT/n}^{(i+2)T/n} (v - u)^{-1} u^{-1} \, dv \, du \]
\[ \leq T^{2\gamma/\alpha} \sum_{i=1}^{n-1} \left( \int_{iT/n}^{(i+1)T/n} u^{-1} \, du \right) \left( \int_{iT/n}^{(i+2)T/n} (v - (i+1)T/n)^{-1} \, dv \right) \]
\[ \leq T^{2\gamma/\alpha} \ln n \sum_{i=1}^{n-1} \left( \int_{iT/n}^{(i+1)T/n} u^{-1} \, du \right) = T^{2\gamma/\alpha} (\ln n)^2 \]
\[ \leq C_{\gamma,\alpha} T^{2\gamma/\alpha} n^{1-2\gamma/\alpha}. \quad (13) \]

Combining inequality (11) with (12) and (13), we derive

\[ |J_3| \leq 2C_{\gamma,\alpha} C_T^2 T^{2+2\gamma/\alpha} \|h\|_2^2 \left( \int_{\mathbb{R}^d} |z|^{2\gamma} Q(z) \, dz \right) n^{-(1+2\gamma/\alpha)}. \quad \square \]

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