A BILINEAR FORM RELATING TWO LEONARD SYSTEMS

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Dedicated to Professor Tatsuro Ito on the occasion of his 60th birthday

Abstract. Let $\Phi, \Phi'$ be Leonard systems over a field $K$, and $V, V'$ the vector spaces underlying $\Phi, \Phi'$, respectively. In this paper, we introduce and discuss a balanced bilinear form on $V \times V'$. Such a form naturally arises in the study of $Q$-polynomial distance-regular graphs. We characterize a balanced bilinear form from several points of view.

1. Introduction

Leonard systems naturally arise in representation theory, combinatorics, and the theory of orthogonal polynomials (see e.g. [18, 21]). Hence they are receiving considerable attention. Indeed, the use of the name ‘Leonard system’ is motivated by a connection to a theorem of Leonard [10], [2, p. 260], which involves the $q$-Racah polynomials [1] and some related polynomials of the Askey scheme [9].

Let $\Phi, \Phi'$ be Leonard systems over a field $K$, and $V, V'$ the vector spaces underlying $\Phi, \Phi'$, respectively (§2). Suppose $\dim V' \leq \dim V$. We consider a situation where $\Phi, \Phi'$ are related by means of a bilinear form $(\cdot | \cdot) : V \times V' \to K$ satisfying certain orthogonality conditions (see §3 for the precise definition). In this case we say that $(\cdot | \cdot)$ is balanced with respect to $\Phi, \Phi'$, and call $\Phi'$ a descendant of $\Phi$. The notion of a balanced bilinear form originates in the theory of $Q$-polynomial distance-regular graphs [2, 3, 6]. Specifically, such a form arises in the context of subsets having minimal width and dual width [11, 12, 7] and in the context of certain irreducible modules for the Terwilliger algebra [14, 15, 16]. For example, let $V$ be the primary module of the hypercube $Q_d$ with respect to a base vertex $x$ (where $K = \mathbb{R}$, say). For the former context, let $V'$ be the primary module of an induced subgraph $Q_{d'}$ ($d' \leq d$) containing $x$. For the latter, let $V'$ be an irreducible module with respect to another base vertex $y$, and suppose that $V'$ has endpoint $\partial(x, y)$ and is not orthogonal to $V$. In each case, let $\Phi, \Phi'$ be the Leonard systems associated with $V, V'$ (cf. [15, 8, Example 1.4]), and for the latter we further replace $\Phi, \Phi'$ by their ‘duals’. Then the restriction of the standard inner product onto $V \times V'$ turns out to be balanced with respect to $\Phi, \Phi'$. See [13] for details. We believe that the study of a balanced bilinear form will lead to a unification of these two approaches at a certain level and thus help better understand the structure of $Q$-polynomial distance-regular graphs.

The contents of the paper are as follows. §2 reviews basic terminology, notation and facts concerning Leonard systems. In §3 we introduce a balanced bilinear form...
as well as a descendent. §§4 and 5 are devoted to its properties and a characterization in terms of the parameter arrays of $\Phi$, $\Phi'$ (Theorem (5.5)). It should be remarked that the isomorphism class of a Leonard system is determined by its parameter array (17 Theorem 1.9)). §6 establishes a classification of the descendents of Leonard systems (Theorem (6.9)). §7 deals with a 'converse' problem: given the Leonard system $\Phi$ and a bilinear form $\langle \cdot \rangle : V \times V' \to \mathbb{K}$, we ask whether there is a descendent $\Phi'$ defined on $V'$ so that $\langle \cdot \rangle$ is balanced with respect to $\Phi$, $\Phi'$. Theorem (7.3) is the main result on this topic. §8 discusses an interpretation of a balanced bilinear form as an orthogonality of some polynomials of the Askey scheme. The paper ends with an appendix containing a list of the parameter arrays of the Leonard systems. We shall apply these results to the study of $Q$-polynomial distance-regular graphs in future papers.

2. Leonard systems

Let $\mathbb{K}$ be a field, $d$ a positive integer, $\mathcal{A}$ a $\mathbb{K}$-algebra isomorphic to the full matrix algebra $\text{Mat}_{d+1}(\mathbb{K})$, and $V$ an irreducible left $\mathcal{A}$-module. We remark that $V$ is unique up to isomorphism, and that $V$ has dimension $d+1$. An element $A$ of $\mathcal{A}$ is said to be multiplicity-free if it has $d+1$ mutually distinct eigenvalues in $\mathbb{K}$. Let $A$ be a multiplicity-free element of $\mathcal{A}$ and $\{\theta_i\}_{i=0}^d$ an ordering of the eigenvalues of $A$. Then by elementary linear algebra there is a sequence of elements $\{E_i\}_{i=0}^d$ in $\mathcal{A}$ such that (i) $AE_i = \theta_i E_i$; (ii) $E_i E_j = \delta_{ij} E_i$; (iii) $\sum_{i=0}^d E_i = I$ where $I$ is the identity of $\mathcal{A}$. We call $E_i$ the primitive idempotent of $A$ associated with $\theta_i$.

A Leonard system in $\mathcal{A}$ (17 Definition 1.4)) is a sequence $\Phi = (A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ satisfying the following axioms (LS1)–(LS5):

- (LS1) Each of $A, A^*$ is a multiplicity-free element in $\mathcal{A}$.
- (LS2) $\{E_i\}_{i=0}^d$ is an ordering of the primitive idempotents of $A$.
- (LS3) $\{E_i^*\}_{i=0}^d$ is an ordering of the primitive idempotents of $A^*$.
- (LS4) $E_i^* A E_j^* = \begin{cases} 0 & \text{if } |i-j| > 1 \\ \neq 0 & \text{if } |i-j| = 1 \end{cases}$ $(0 \leq i, j \leq d)$.
- (LS5) $E_i A^* E_j = \begin{cases} 0 & \text{if } |i-j| > 1 \\ \neq 0 & \text{if } |i-j| = 1 \end{cases}$ $(0 \leq i, j \leq d)$.

We call $d$ the diameter of $\Phi$, and say that $\Phi$ is over $\mathbb{K}$. For notational convenience, we define $E_i = E_i^* = 0$ if $i < 0$ or $i > d$. We refer the reader to [13, 17, 19, 20, 21] for background on Leonard systems.

A Leonard system $\Psi$ in a $\mathbb{K}$-algebra $\mathcal{B}$ is isomorphic to $\Phi$ if there is a $\mathbb{K}$-algebra isomorphism $\gamma : \mathcal{A} \to \mathcal{B}$ such that $\Psi = \Phi \gamma := (A^\gamma; A^{*\gamma}; \{E_i^\gamma\}_{i=0}^d; \{E_i^{*\gamma}\}_{i=0}^d)$. Let $\xi, \xi^*, \zeta, \zeta^*$ be scalars in $\mathbb{K}$ such that $\xi \neq 0$, $\xi^* \neq 0$. Then

$$\Phi^* = (A^*; A; \{E_i^*\}_{i=0}^d; \{E_i\}_{i=0}^d),$$

$$\Phi^1 = (A; A^*; \{E_i\}_{i=0}^d; \{E_{d-i}^*\}_{i=0}^d),$$

$$\Phi^0 = (A; A^*; \{E_{d-i}\}_{i=0}^d; \{E_i^*\}_{i=0}^d).$$
are Leonard systems in $\mathcal{A}$. Viewing $\ast, \downarrow, \downarrow \downarrow$ as permutations on all Leonard systems,

$$
\ast^2 = \downarrow^2 = 1, \quad \downarrow \ast = \ast \downarrow, \quad \downarrow \ast \downarrow = \downarrow \downarrow \downarrow.
$$

The group generated by the symbols $\ast, \downarrow, \downarrow \downarrow$ subject to the above relations is the dihedral group $D_4$ with 8 elements. For the rest of this paper we shall use the following notational convention:

(2.6) For any $g \in D_4$ and for any object $f$ associated with $\Phi$, we let $f^g$ denote the corresponding object for $\Phi^g$; an example is $E_i^g(\Phi) = E_i(\Phi^g)$.

For $0 \leq i \leq d$ let $\theta_i$ (resp. $\theta_i^*$) be the eigenvalue of $A$ (resp. $A^*$) associated with $E_i$ (resp. $E_i^*$). By [17, Theorem 3.2] there are scalars $\phi_i (1 \leq i \leq d)$ in $K$ and a $K$-algebra isomorphism $\sharp: \mathcal{A} \to \text{Mat}_{d+1}(K)$ such that

$$
A^i = \begin{pmatrix}
\theta_0 \\
1 \\
\theta_1 \\
1 \\
\vdots \\
1 \\
\theta_d
\end{pmatrix},
A^{*i} = \begin{pmatrix}
\theta_0^* \\
\phi_1 \\
\theta_1^* \\
\phi_2 \\
\vdots \\
\phi_d \\
\theta_d^*
\end{pmatrix}.
$$

We define $\phi_i = \phi_i^0 (1 \leq i \leq d)$. The parameter array of $\Phi$ is the sequence

(2.7) $p(\Phi) = \{(\theta_i)_{i=0}^d; (\theta_i^*)_{i=0}^d; \{\phi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d\}$.

Terwilliger [17, Theorem 1.9] showed that the isomorphism class of $\Phi$ is determined by $p(\Phi)$, and that the set of parameter arrays of Leonard systems over $K$ with diameter $d$ is characterized by the following properties (2.8–2.12):

(2.8) $\varphi_i \neq 0, \phi_i \neq 0 (1 \leq i \leq d)$.

(2.9) $\theta_i \neq \theta_j, \theta_i^* \neq \theta_j^* \text{ if } i \neq j \text{ (0} \leq i, j \leq d)\text{).}$

(2.10) $\varphi_i = \sum_{\ell=0}^{i-1} \frac{\theta_\ell - \theta_{d-\ell}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)(\theta_{i-1} - \theta_d) (1 \leq i \leq d)$.

(2.11) $\phi_i = \sum_{\ell=0}^{i-1} \frac{\theta_\ell - \theta_{d-\ell}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)(\theta_{d-i+1} - \theta_0) (1 \leq i \leq d)$.

(2.12) $\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i} \text{ are equal and independent of } i \text{ (2} \leq i \leq d - 1\text{).}$

It is known [19, Theorem 6.1] that there is a unique antiautomorphism $\dagger$ of $\mathcal{A}$ such that $A^\dagger = A$ and $A^{i\dagger} = A^i$. For the rest of this paper let $\langle \cdot, \cdot \rangle: V \times V \to K$ denote a nondegenerate bilinear form on $V$ such that [19, §15])

(2.13) $\langle Xu, v \rangle = \langle u, X^\dagger v \rangle$ (u, v $\in V$, X $\in \mathcal{A}$).

We shall write

(2.14) $||u||^2 = \langle u, u \rangle$ (u $\in V$).

For convenience, let $\mathcal{D}$ be the subalgebra of $\mathcal{A}$ generated by $A$. Observe that

(2.15) $\mathcal{D} = \text{span}\{I, A, A^2, \ldots, A^d\} = \text{span}\{E_0, E_1, \ldots, E_d\}$.

Obviously $\dagger$ fixes each element of $\mathcal{D} \cup \mathcal{D}^*$, so that we have

(2.16) $\langle Xu, v \rangle = \langle u, Xv \rangle$ (u, v $\in V$, X $\in \mathcal{D} \cup \mathcal{D}^*$),
from which it follows that
\[(2.17) \quad \langle E_i V, E_j V \rangle = \langle E^*_i V, E^*_j V \rangle = 0 \quad \text{if} \quad i \neq j \quad (0 \leq i, j \leq d).\]

From now on let \(u\) be a nonzero vector in \(E_0 V\). Then \(E^*_i u \neq 0\) for \(0 \leq i \leq d\) ([19, Lemma 10.2]), so that \(\{E^*_i u\}_{i=0}^d\) is a basis of \(V\). Let
\[(2.18) \quad \nu = \text{trace}(E^*_0 E_0)^{-1}, \quad k_i = \text{trace}(E^*_i E_0)\nu \quad (0 \leq i \leq d).
\]
Indeed, by [19, Lemma 9.2] \(\text{trace}(E^*_i E_0) \neq 0 (0 \leq i \leq d)\). With this notation we have ([19, Theorem 15.3])
\[(2.19) \quad \langle E^*_i u, E^*_j u \rangle = \delta_{ij} k_i \nu^{-1} ||u||^2 \quad (0 \leq i, j \leq d).
\]

The dual switching element for \(\Phi\) ([11, Note 5.2]) is
\[(2.20) \quad S^* = \sum_{\ell=0}^d \phi_1 \phi_2 \cdots \phi_\ell \nu E^*_\ell.
\]
It follows that
\[(2.21) \quad S^* E_0 V = E_0 V,
\]
and moreover that any element \(X \in \mathcal{D}\) satisfying \(XE_0 V \subseteq E_0 V\) is a scalar multiple of \(S^*\) ([11, Theorem 6.7]).

Finally, we remark that
\[(2.22) \quad E_0^* V + E_1^* V + \cdots + E_i^* V = E_0^* V + AE_0^* V + \cdots + A^i E_0^* V \quad (0 \leq i \leq d).
\]
In particular:
\[(2.23) \quad V = \mathcal{D}E_0^* V.
\]

3. A balanced bilinear form

For the rest of this paper, we shall retain the notation of the previous section. Except in \(\S\) we shall always refer to the following set-up:

(3.1) Let \(\Phi' = (A'; A'^+; \{E'\}_{i=0}^d; \{E'^*_i\}_{i=0}^d)\) be a Leonard system over \(\mathbb{K}\) with diameter \(d' \leq d\). For any object \(f\) associated with \(\Phi\), we let \(f'\) denote the corresponding object for \(\Phi'\); an example is \(V' = V(\Phi')\).

A nonzero bilinear form \((\cdot | \cdot) : V \times V \to \mathbb{K}\) is balanced with respect to \(\Phi, \Phi'\) if (B1), (B2) hold below:

(B1) There is an integer \(\rho\) \((0 \leq \rho \leq d - d')\) such that \((E^*_i V | E'^*_j V') = 0\) if \(i - \rho \neq j \quad (0 \leq i \leq d, 0 \leq j \leq d')\).

(B2) \((E_i V | E'_j V') = 0\) if \(i < j\) or \(i > j + d - d'\) \((0 \leq i \leq d, 0 \leq j \leq d')\).

We call \(\rho\) the endpoint of \((\cdot | \cdot)\) (with respect to \(\Phi, \Phi'\)), and refer to the form also as \(\rho\)-balanced. We say that \(\Phi'\) is a \(\rho\)-descendent (or simply a descendent) of \(\Phi\) whenever such a form exists.

(3.2) Remark. A bilinear form \((\cdot | \cdot) : V \times V' \to \mathbb{K}\) which is balanced with respect to \(\Phi, \Phi'\) is in fact balanced with respect to the following pairs of Leonard systems:

| pair       | endpoint |
|------------|----------|
| \(\Phi, \Phi'\) | \(\rho\)  |
| \(\Phi^\downarrow, \Phi'^\downarrow\) | \(d - d' - \rho\) |
| \(\Phi^\uparrow, \Phi'^\uparrow\) | \(\rho\)  |
| \(\Phi^\downarrow, \Phi'^\downarrow\) | \(d - d' - \rho\) |
(3.3) Remark. If \( \Phi' \) is a \( \rho \)-descendent of \( \Phi \), then any \( \rho' \)-descendent \( \Phi'' \) of \( \Phi' \) is a \((\rho + \rho')\)-descendent of \( \Phi \). Indeed, let \((\cdot | \cdot) : V \times V' \to \mathbb{K} \) and \((\cdot | \cdot)' : V' \times V'' \to \mathbb{K} \) be corresponding balanced bilinear forms, where \( V'' = V(\Phi'') \). Let \( \text{proj}'' : V'' \to V' \) be a unique linear map such that \((v | v'') = (v' \text{proj}' v'')'\) for all \( v' \in V', v'' \in V'' \). Then the bilinear form \( V \times V'' \to \mathbb{K} \) defined by \((v, v'') \to (v \text{proj}' v'') (v \in V, v'' \in V'')\) is \((\rho + \rho')\)-balanced with respect to \( \Phi, \Phi'' \). (That this form is nonzero follows e.g. from (4.5) below.)

(3.4) Remark. If \( \Phi' \) is a descendent of \( \Phi \), then any Leonard system affine isomorphic to \( \Phi' \) is a descendent of any Leonard system affine isomorphic to \( \Phi \). Later we shall show that a balanced bilinear form has full-rank (cf. (4.5)), from which it follows that two Leonard systems are descendents of each other if and only if they are affine isomorphic (cf. (3.3)). Now, let \([\Phi], [\Phi']\) denote the affine-isomorphism classes of \( \Phi, \Phi' \), respectively, and write \([\Phi'] \preceq [\Phi] \) if \( \Phi' \) is a descendent of \( \Phi \). Then, in view of (3.3) and the above comments, \( \preceq \) defines a well-defined poset structure on the set of affine-isomorphism classes of Leonard systems over \( \mathbb{K} \).

4. Properties of a balanced bilinear form

In this section, we shall study the basic properties of a balanced bilinear form. With reference to (3.1), we shall assume that there is a bilinear form \((\cdot | \cdot) : V \times V' \to \mathbb{K} \) which is \( \rho \)-balanced with respect to \( \Phi, \Phi' \) \((0 \leq \rho \leq d - d')\).

We define a \( \mathbb{K} \)-algebra homomorphism \( \sigma : \mathcal{D}^* \to \mathcal{D}'^* \) by

\[
E_i^\sigma = E_{i-\rho}^\sigma \quad (0 \leq i \leq d).
\]

Clearly \( \sigma \) is surjective. By (B1) it follows that

\[
(X v | v') = (v | X^\sigma v') \quad (v \in V, v' \in V', X \in \mathcal{D}^*).
\]

Let \( \text{proj} : V \to V' \) and \( \text{proj}' : V' \to V \) be unique linear maps satisfying

\[
(v | v') = \langle \text{proj} v, v' \rangle = \langle v, \text{proj}' v' \rangle \quad (v \in V, v' \in V').
\]

It follows from (B2) that \( \text{proj} E_0 V \subseteq E_0' V' \). Furthermore, by (2.23) we have

\[
\langle E_0 V | V' \rangle = \langle E_0 | \mathcal{D}^* V' \rangle = \langle \mathcal{D}^* E_0 V | V' \rangle = (V | V') \neq 0,
\]

so that \( \text{proj} E_0 V \neq 0 \). Hence

\[
\text{proj} E_0 V = E_0' V'.
\]

With this explained, we have

(4.5) Proposition. Let \( u \) (resp. \( u' \)) be a nonzero vector in \( E_0 V \) (resp. \( E_0' V' \)). Then there is a nonzero scalar \( \epsilon \in \mathbb{K} \) such that

\[
\langle E_i^\sigma u | E_i^\sigma u' \rangle = \epsilon \delta_{i-\rho, j} k_j^i \nu^i-1 \|u\|^2 \quad (0 \leq i \leq d, 0 \leq j \leq d').
\]

In particular, \((\cdot | \cdot)\) has full-rank, i.e., \( \text{proj}' : V' \to V \) is injective.

Proof. By (4.1) there is a nonzero scalar \( \epsilon \in \mathbb{K} \) such that \( \text{proj} u = \epsilon u' \). Hence from (2.19) it follows that

\[
\langle E_i^\sigma u | E_i^\sigma u' \rangle = \epsilon \delta_{i-\rho, j} \langle u', E_j^\sigma u' \rangle' = \epsilon \delta_{i-\rho, j} k_j^i \nu^i-1 \|u\|^2.
\]

(4.6) Proposition. The following (i), (ii) hold:

(i) There are scalars \( \xi^*, \zeta^* \in \mathbb{K} \) such that \( A^* = \xi^* A^* + \zeta^* I' \), where \( I' \) is the identity of \( \mathcal{D}' \).
(ii) \( S^{*'} = \frac{\varphi_1 \varphi_2 \cdots \varphi_{\rho}}{\phi_1 \phi_2 \cdots \phi_{\rho}} S^{*\sigma} \).

Proof. (i): For \( 2 \leq i \leq d' \) we have
\[
\langle A^{* \sigma} E_0' V', E_i' V' \rangle = \langle E_0' V', A^{* \sigma} E_i' V' \rangle = (E_0 V) A^{* \sigma} E_i' V' \] by (4.3)
\[
= (A^{*} E_0 V) E_i' V' = 0
\]
since \( A^{*} E_0 V \subseteq E_0 V + E_1 V \); it follows that \( A^{* \sigma} E_0' V' \subseteq E_0' V' + E_1' V' \). Now it is clear from (2.22) that \( A^{* \sigma} \in \text{span}\{A^{*}, I'\} \).

(ii): For \( 0 \leq i \leq d' - 1 \) we have
\[
\langle S^{* \sigma} E_0' V', E_i' V' \rangle = \langle E_0' V', S^{* \sigma} E_i' V' \rangle = (E_0 V) S^{* \sigma} E_i' V' \] by (4.3)
\[
= (S^{*} E_0 V) E_i' V' = (E_0 V) E_i' V' \] by (2.21)
\[
= 0,
\]
so that \( S^{* \sigma} E_0' V' \subseteq E_0' V' \). Hence ([11, Theorem 6.7]) \( S^{* \sigma} \) is a scalar multiple of \( S^{*'} \), and the scalar factor is given by comparing the coefficient of \( E_0' \) in \( S^{*'} \), \( S^{* \sigma} \). □

5. Reconstruction of the balanced bilinear form

In this section, we shall see that (4.6) turns out to give a necessary and sufficient condition on the existence of a balanced bilinear form. With reference to (3.1), let \( \rho \) be an integer such that \( 0 \leq \rho \leq d - d' \). As in ([11] define a \( \mathbb{K} \)-algebra homomorphism \( \sigma : \mathcal{B}^{\ast} \rightarrow \mathcal{B}^{\ast'} \) by

\[(5.1) \quad E_i^{* \sigma} = E_i^{* \rho} \quad (0 \leq i \leq d). \]

We shall assume (i), (ii) below:

(i) There are scalars \( \xi^*, \zeta^* \in \mathbb{K} \) such that \( A^{* \sigma} = \xi^* A^{* \rho} + \zeta^* I' \).

(ii) \( S^{*'} = \frac{\varphi_1 \varphi_2 \cdots \varphi_{\rho}}{\phi_1 \phi_2 \cdots \phi_{\rho}} S^{* \sigma} \).

Let \( u \) (resp. \( u' \)) be a nonzero vector in \( E_0 V \) (resp. \( E_0' V' \)). We define a bilinear form \( \langle \cdot, \cdot \rangle : V \times V' \rightarrow \mathbb{K} \) by

\[(5.2) \quad \langle E_i^* u | E_i^* u' \rangle = \delta_{i,-j} k_j^j' \| u' \|^2 \quad (0 \leq i \leq d, \ 0 \leq j \leq d'). \]

Clearly we have

\[(5.3) \quad \langle X v' | v \rangle = \langle v | X^\sigma v' \rangle \quad (v \in V, \ v' \in V', \ X \in \mathcal{B}^{\ast}). \]

It follows that

\[(5.4) \quad \text{The following (i), (ii) hold:} \]

(i) \( (E_0 V + \cdots + E_{i-1} V | E_i^* V' + \cdots + E_{d'}^* V') = 0 \quad (1 \leq i \leq d'). \)

(ii) \( (E_{i+d'-1} V + \cdots + E_{d} V | E_0^* V' + \cdots + E_{i-1}^* V') = 0 \quad (1 \leq i \leq d'). \)

In other words, \( \langle \cdot, \cdot \rangle \) satisfies (B2) and is therefore balanced with respect to \( \Phi, \Phi' \).
Theorem. Hence the result follows from (i). □

\[ \Phi \]

the parameter arrays of Φ. Likewise we have

This is just a restatement of (4.6) and (5.4) in terms of the parameter arrays of Φ, Φ' -balanced with respect to Φ.

Finally, we may conveniently summarize (4.6) and (5.4) in the following form:

(5.5) Theorem. With reference to (3.1), Φ' is a ρ-descendent of Φ if and only if the parameter arrays of Φ, Φ' satisfy (i), (ii) below:

(i) There are scalars \( \xi^*, \zeta^* \in K \) such that \( \theta^* \sim \xi^* \theta^* \xi + \zeta^* \) (0 ≤ \( i \leq d' \)).

(ii) \( \frac{\phi_{i}}{\phi_{i}'} = \frac{\phi_{i+1}}{\phi_{i+1}'} \) (1 ≤ \( i \leq d' \)).

Moreover, if (i), (ii) hold above then a bilinear form \( (|:\cdot|) : V \times V' \rightarrow K \) which is ρ-balanced with respect to Φ, Φ' is unique up to scalar multiplication.

Proof. This is just a restatement of (4.6) and (5.4) in terms of the parameter arrays of Φ, Φ'. The uniqueness follows from (5.5). □

(5.6) Remark. The endpoint ρ is not necessarily uniquely determined by the parameter arrays of Φ, Φ'. Indeed, with the notation of (4.1) suppose that

\[ p(\Phi) = p(IIC; r, s, s^*, \theta_0, \theta_0^*), \quad p(\Phi') = p(IIC; r, s, s^*, \theta'_0, \theta_0^*, d'). \]

Then conditions (i), (ii) in (5.5) are satisfied for all 0 ≤ ρ ≤ d - d'.
6. Characterization of a balanced bilinear form in parametric form

In this section, we shall classify all the descendents of $\Phi$. With reference to (3.11), we shall assume that $\Phi'$ is a $\rho$-descendent of $\Phi$ $(0 \leq \rho \leq d - d')$, unless otherwise stated. Let

$$\vartheta_i = \sum_{t=0}^{i-1} \frac{\theta_{d-t} - \theta_{d-t}}{\theta_0 - \theta_d} \quad (1 \leq i \leq d).$$

Clearly, $\vartheta_1 = \vartheta_d = 1$. Moreover ([17, Lemma 6.5]):

(6.1) $\varphi_i - \phi_i = (\theta_i^* - \theta_{i-1}^*)(\theta_0 - \theta_d)\vartheta_i \quad (1 \leq i \leq d)$.

(6.2) With the notation of (5.5)(i), the following (i)–(iii) hold:

(i) $(\theta_d - \theta_0)\vartheta_{p+i}\varphi'_i = \xi^*(\theta_{d'} - \theta'_0)\vartheta_{d'}\varphi_{p+i} \quad (1 \leq i \leq d')$. 

(ii) $(\theta_d - \theta_0)\vartheta_{p+i}\varphi'_i = \xi^*(\theta_{d'} - \theta'_0)\vartheta_{d'}\varphi_{p+i} \quad (1 \leq i \leq d')$. 

(iii) $(\theta_d - \theta_0)\vartheta_{p+1}\varphi_{p+i}(\theta_{p+i}^* - \theta_p^*)(\theta_{d'-i+1}^* - \theta_0') = (\theta_{d'} - \theta'_0)\vartheta_{d'}(\theta_{p+1}\varphi_{p+i} - \vartheta_{p+i}\varphi_{p+i}) \quad (1 \leq i \leq d')$.

Proof. (i), (ii): Evaluate (5.5)(ii) in two ways using (6.1) and (5.5)(i).

(iii): From (2.11) and (5.5)(i) it follows that

$$\vartheta_i' = \varphi_i'\vartheta_i' + \xi^*(\theta_{p+i}^* - \theta_p^*)(\theta_{d'-i+1}^* - \theta_0').$$

On multiplying both sides above by $(\theta_d - \theta_0)\vartheta_{p+1}\varphi_{p+i}$ and simplifying the result using (i) and (ii), we obtain (iii).

We shall need the explicit values of the $\vartheta_i$. With the notation of (A.1) we have

$$\vartheta_i = \begin{cases} 
(q^i - 1)(q^{d-i+1} - 1) & \text{for Cases I, IA}, \\
(q-1)(q^d-1) & \text{for Cases II, II, III, IIIC}, \\
i(d-i+1)/d & \text{for Case III, d even, i even}, \\
i/d & \text{for Case III, d even, i odd}, \\
0 & \text{(1 \leq i \leq d)}. \\
(d-i+1)/d & \text{for Case III, d odd, i even;}
\\1 & \text{for Case III, d odd, i odd;}
\\0 & \text{or Case IV, i even;}
\\1 & \text{or Case IV, i odd}
\end{cases}$$

(6.3) 

(See e.g. [17, Lemma 10.2].) It should be remarked that from (2.8) and (2.9) we obtain restrictions on the scalar $q$ for Cases I, IA, and on the characteristic of $\mathbb{K}$ for Cases II, III, II, IIIC and III. We can then see that

(6.4) We have $\vartheta_i = 0$ precisely for Case III, d odd, i even; or Case IV, i even.

Henceforth let $\beta + 1$ denote the common value of (2.12). By convention, if $d < 3$ then $\beta$ can be taken to be any scalar in $\mathbb{K}$. We may remark that

$$\beta = \begin{cases} 
q + q^{-1} & \text{for Cases I, IA}, \\
2 & \text{for Cases II, II, II, IIIC}, \\
-2 & \text{for Case III}, \\
0 & \text{for Case IV}.
\end{cases}$$

(6.5)

It follows from (6.3)(i) that

$$\beta' = \beta.$$
For Case III, we have

(6.7) If \( p(\Phi) \) satisfies Case III, then the following (i), (ii) hold:

(i) If \( d \) is even, then either \( d' = 1 \) or \( d' \) is even.

(ii) If \( d \) is odd, then \( d' \) is odd and \( \rho \) is even.

Proof. From (2.8) and (6.2) (i) it follows that \( \vartheta_{\rho+i} = 0 \), for \( 1 \leq i \leq d' \). Hence the result follows from (6.4)–(6.6). \( \square \)

Likewise,

(6.8) If \( p(\Phi) \) satisfies Case IV, then \( (d', \rho) \in \{(1, 0), (1, 2), (3, 0)\} \).

(6.9) Theorem. With reference to (3.1), let the parameter array of \( \Phi \) be given as in \((A.1)\). Then \( \Phi' \) is a \( \rho \)-descendent of \( \Phi \) precisely when the parameter array of \( \Phi' \) takes the following form:

Case I:

\[ p(\Phi') = p(I; q, h', h^{\ast'}, r_1q^p, r_2q^p, sq^{d-d'}, s^*q^2, \theta'_0, \theta^{\ast'}_0, d'). \]

Case IA:

\[ p(\Phi') = p(IA; q, h^{\ast'}, r', s', \theta'_0, \theta^{\ast'}_0, d') \quad \text{where} \quad s'/r' = q^{d-d'} - r_{s/r}. \]

Case II:

\[ p(\Phi') = p(II; h', h^{\ast'}, r_1 + \rho, r_2 + \rho, s + d - d', s^* + 2\rho, \theta'_0, \theta^{\ast'}_0, d'). \]

Case IIA:

\[ p(\Phi') = p(IIA; h', r + \rho, s + d - d', s^{\ast'} + \theta'_0, \theta^{\ast'}_0, d'). \]

Case IIIB:

\[ p(\Phi') = p(IIIB; h^{\ast'}, r + \rho, s', s^* + 2\rho, \theta'_0, \theta^{\ast'}_0, d'). \]

Case IIC:

\[ p(\Phi') = p(IIIC; r', s', s^{\ast'}, \theta'_0, \theta^{\ast'}_0, d') \quad \text{where} \quad s' s^{\ast'}/r' = s s^*/r. \]

Case III, \( d \) even, \( d' \) even, \( \rho \) even; or Case III, \( d \) odd, \( d' \) odd, \( \rho \) even:

\[ p(\Phi') = p(III; h', h^{\ast'}, r_1 + \rho, r_2 + \rho, s - d + d', s^* - 2\rho, \theta'_0, \theta^{\ast'}_0, d'). \]

Case III, \( d \) even, \( d' \) even, \( \rho \) odd:

\[ p(\Phi') = p(III; h', h^{\ast'}, r_2 + \rho, r_1 + \rho, s - d + d', s^* - 2\rho, \theta'_0, \theta^{\ast'}_0, d'). \]

Case III, \( d \) even, \( d' = 1 \); or Case IV, \( (d', \rho) \in \{(1, 0), (1, 2)\} \):

\[ p(\Phi') = p(IIIC; r', s', s^{\ast'}, \theta'_0, \theta^{\ast'}_0, 1) \quad \text{where} \quad s' s^{\ast'}/r' = 1 - \phi_{\rho+1}/\varphi_{\rho+1}. \]

Case IV, \( (d', \rho) = (3, 0) \):

\[ p(\Phi') = p(IV; h', h^{\ast'}, r, s, s^*, \theta'_0, \theta^{\ast'}_0). \]

Case III, \( d \) even, \( d' \) odd \( \geq 3 \); or Case III, \( d \) odd, either \( d' \) even or \( \rho \) odd; or Case IV, \( (d', \rho) \in \{(1, 1), (2, 0), (2, 1)\} \): Does not occur.
Proof. Suppose first that $\Phi'$ is a $p$-descendent of $\Phi$. The last line follows from (6.7) and (6.3). For Cases I, IA, II, IIA, III, IIC; or for Case III with $d$ even and $d'$ even, it is a straightforward matter to show that $p(\Phi')$ is given as in (6.9) by evaluating (5.5)(i) and (6.2)(i)–(iii) using (6.3)–(6.6) and (A.1). For example, $h' = hq^{d-d'}(q^d-1)(\theta_d' - \theta_0'/q^d - 1)(\theta_d - \theta_0)$, $h' = \xi^*h^*q^{-p}$ for Case I. For Case III with $d$ odd, $d'$ odd, $\rho$ even, likewise we have
\[
\begin{align*}
\theta_i' &= \theta_0' + h^*(s^* - 2\rho - 1 + (1 - s^* + 2\rho + 2i)(-1)^i) \quad (0 \leq i \leq d'), \\
\theta_0' &= \theta_0' + 2h'(s - d + d' - 1 - i) \quad (0 \leq i \leq d', \ i \ odd), \\
\varphi_i' &= -4h'h^*(i + r_1 + \rho)(i + r_2 + \rho) \quad (0 \leq i \leq d', \ i \ odd), \\
\phi_i' &= -4h'h^*(i - s^* + \rho - r_1)(i - s^* + \rho - r_2) \quad (0 \leq i \leq d', \ i \ odd),
\end{align*}
\]
where $h' = h(\theta_0' - \theta_0)(\theta_d - \theta_0)^{-1}$, $h^* = \xi^*h^*$. Since $\beta' = -2$ by (6.6) we see that
\[
\begin{align*}
\theta_i' &= \theta_0' + 2h'i \quad (0 \leq i \leq d', \ i \ even)
\end{align*}
\]
by induction on even $i$. Hence from (2.10) and (2.11) it follows that $p(\Phi')$ is given as in (6.9). The same argument applies to Case IV with $(d', \rho) = (3, 0)$. For Case III with $d$ even and $d' = 1$; or for Case IV with $(d', \rho) \in \{(1, 0), (1, 2)\}$, we define $s' = \theta_1' - \theta_0$, $s'' = \theta_1' - \theta_0'$, $r' = -\varphi_1'$. Then by (2.11) we have $\phi_1' = -(r' - s''s'^{-1})$, so that $p(\Phi') = p(\text{IC}; r', s'', \theta_0', \theta_0')$. Furthermore, from (6.5)(ii) it is clear that $s's'^{-1}/r' = 1 - \varphi_1'/\phi_1' = 1 - \varphi_{p+1}/\phi_{p+1}$.

Conversely, suppose that $p(\Phi')$ is of the form in (6.9). Then it is easy to check (5.5)(i), (ii) and therefore $\Phi'$ is a $p$-descendent of $\Phi$. This completes the proof. 

\section{7. Characterization of $\Phi'$ in Terms of a Balanced Bilinear Form}

The goal of this section is to characterize the Leonard system $\Phi'$ in terms of the balanced bilinear form $\langle \cdot | \cdot \rangle$. We shall refer to the following set-up:

(7.1) Let $\Phi$ be the Leonard system (2.1) and let the parameter array of $\Phi$ be given as in (A.1). Let $d'$ be a positive integer such that $d' \leq d$, $\mathcal{A}'$ a $\mathbb{K}$-algebra isomorphic to $\text{Mat}_{d'+1}(\mathbb{K})$, and $V'$ an irreducible left $\mathcal{A}'$-module. Let $\rho$ be an integer such that $0 \leq \rho \leq d - d'$. We shall assume (i), (ii) below:

(i) For Case III, if $d$ is even then either $d' = 1$ or $d'$ is even; if $d$ is odd then $d'$ is odd and $\rho$ is even.

(ii) For Case IV, $(d', \rho) \in \{(1, 0), (1, 2), (3, 0)\}$.

(7.2) There is a $p$-descendent of $\Phi$ with diameter $d'$.

Proof. We shall invoke (6.9). Suppose first that we are in Case III with $d$ even and $d' = 1$; or in Case IV with $(d', \rho) \in \{(1, 0), (1, 2)\}$. From (6.1) and (6.4) it follows that $\varphi_{p+1} \neq \phi_{p+1}$ so that $s's'^{-1}/r' = 1 - \phi_{p+1}/\varphi_{p+1} \notin \{0, 1\}$. Hence $p(\text{IC}; r', s'', \theta_0', \theta_0', 1)$ satisfies (2.3) (2.12). For the other cases, the feasibility of the parameter array in (6.9) can be directly checked from [20] Examples 5.3–5.15. 

A decomposition of $V'$ shall mean a sequence $\{U_i\}_{i=0}^{d'}$ of one-dimensional subspaces of $V'$ such that $V' = U_0' + U_1' + \cdots + U_d'$ (direct sum). The following theorem will prove useful in the study of $Q$-polynomial distance-regular graphs (cf. [13]):
Then there is a \( \rho \)-descendent \( \Phi^\prime \) such that
\[
\Phi^\prime = (\Phi^\prime; E_i^\prime)^{d^\prime}_{i=0}; (\Phi^\prime_i)^{d^\prime}_{i=0}
\]
for \( 0 \leq i \leq d^\prime \) if \( \Phi \) is \( \rho \)-balanced with respect to \( \Phi \). Moreover, \( \Phi^\prime \) is unique up to affine transformation.

Proof. The uniqueness is easily verified, e.g. from (2.22). To show the existence, let \( \Phi^\prime = (\Phi^\prime; E_i^\prime)^{d^\prime}_{i=0}; (\Phi^\prime_i)^{d^\prime}_{i=0} \) be a \( \rho \)-descendent of \( \Phi \) in \( \mathcal{A}^\prime \) and let \( (\cdot|\cdot)_0 : V \times V' \to K \) be a bilinear form which is \( \rho \)-balanced with respect to \( \Phi, \Phi^\prime \). Let \( \pi^\prime : V \to V \) (resp. \( \pi^\prime'' : V' \to V \)) be a unique linear map satisfying \( (v|v') = \langle v, \pi^\prime v' \rangle \) (resp. \( (v|v')_0 = \langle v, \pi^\prime'' v' \rangle \) for all \( v \in V \), \( v' \in V' \). Since \( \pi^\prime, \pi^\prime'' \) are injective (cf. (1.5)) we have
\[
\pi^\prime V = \pi^\prime'' V = E_{\rho}^* V + \cdots + E_{\rho+d'}^* V,
\]
from which it follows that there is a unique invertible element \( C \) in \( \mathcal{A}^\prime \) such that \( \pi^\prime C \pi^\prime = \pi^\prime'' \pi^\prime \) for all \( v \in V \). Let \( \gamma : \mathcal{A} \to \mathcal{A}^\prime \) be the automorphism of \( \mathcal{A}^\prime \) defined by \( X^\gamma = C^* X C^{-1} \) \( (X \in \mathcal{A}) \) and define \( \Phi^\prime = \Phi^\prime \gamma \). It remains to show that \( E_i^\prime V' = U_i^\prime, \ E_i'' V' = U_i^\prime \) for \( 0 \leq i \leq d^\prime \). On one hand, from (i) it is clear that \( \pi^\prime U_i^\prime = E_{\rho+i}^* V \). On the other hand, we have
\[
\pi^\prime E_i^\prime V' = \pi^\prime C E_i^\prime C^{-1} V' = \pi^\prime E_i'' V' = E_{\rho+i}^* V.
\]
Hence \( E_i^\prime V' = U_i^\prime \). Likewise we have \( \pi^\prime E_i^\prime V' = \pi^\prime E_i'' V' \). Furthermore,
\[
\pi^\prime E_i^\prime V', \ \pi^\prime U_i^\prime \subseteq (E_{\rho}^* V + \cdots + E_{\rho+d'}^* V) \cap (E_i V + \cdots + E_{i+d-d'} V).
\]
Consequently, it is enough to prove the following:
\[
\dim E_{\rho}^* V + \cdots + E_{\rho+d'}^* V \cap (E_i V + \cdots + E_{i+d-d'} V) = 1 \quad (0 \leq i \leq d'),
\]
for it would imply that \( \pi^\prime E_i^\prime V' = \pi^\prime E_i'' V' = \pi^\prime U_i^\prime \) and hence that \( E_i^\prime V' = U_i^\prime \). For this purpose, as in the proof of (5.4)(i) we observe that
\[
\langle E_i V, \pi^\prime E_i^\prime V' \rangle = (E_i V|E_i^\prime V')' = (E_0 V + \cdots + E_i | E_i^\prime V' + \cdots + E_{d'}^* V')' = (E_0 V|E_0^* V' + \cdots + E_{d'}^* V')' \neq 0 \quad (0 \leq i \leq d'),
\]
by (1.4). Since \( (\cdot|\cdot)_0 \) is also balanced with respect to \( \Phi^\prime, \Phi^\prime'' \) (cf. (2.22)) we have
\[
\langle E_{i+d-d'} V, \pi^\prime E_i^\prime V' \rangle = (E_{i+d-d'} V|E_i^\prime V') \Phi^\prime'' E_{i+d-d'} V, \pi^\prime E_{i+d-d'} V \Phi^\prime'' V' \neq 0.
\]
Now let \( v_i \) be a nonzero vector in \( \pi^\prime E_i^\prime V' \) for \( 0 \leq i \leq d' \). Then \( \{v_i\}^d_i=0 \) is a basis of \( E_{\rho}^* V + \cdots + E_{\rho+d'}^* V \). Let \( v \in (E_{\rho}^* V + \cdots + E_{\rho+d'}^* V) \cap (E_i V + \cdots + E_{i+d-d'} V) \). Then \( v \) is a linear combination of \( v_{0}, v_{1}, \ldots, v_{d'} \), but no \( v_{0} \) with \( 0 \leq j < i \) can be involved since \( v_{0} \) is not orthogonal to \( E_{j} V \). Likewise no \( v_{0} \) with \( i < j \leq d' \) can be involved since \( v_{j} \) is not orthogonal to \( E_{j+d-d'} V \). It follows that \( v \) is a scalar multiple of \( v_{i} \), and therefore the proof is complete.
8. Remarks: balanced bilinear forms and polynomials from the Askey scheme

In this section, we return to the situation of (3.1). Let \( x \) be an indeterminate. For \( 0 \leq i \leq d \) we define a polynomial \( u_i \in \mathbb{K}[x] \) by

\[
(8.1) \quad u_i = \sum_{\ell=0}^i (\theta_i^\tau - \theta_{i-\ell}^\tau)(x - \theta_{i-\ell+1}^\tau) \prod_{j=0}^{\ell-1} (x - \theta_j^\tau).
\]

The \( u_i \) belong to the terminating branch of the Askey scheme \([11]\), consisting of the \( q \)-Racah, \( q \)-Hahn, dual \( q \)-Hahn, \( q \)-Krawtchouk, dual \( q \)-Krawtchouk, quantum \( q \)-Krawtchouk, affine \( q \)-Krawtchouk, Racah, Hahn, dual Hahn, Krawtchouk, Bannai/Ito and orphan polynomials. See \([20]\, \text{Examples 5.3–5.15}]\).

Let \( u, u', v, v' \) be nonzero vectors in \( E_0V, E_0'V', E_0''V, E_0''V' \), respectively. It is known \([19]\, \text{Theorem 15.8}]\) that

\[
(8.2) \quad E_i v = k^*_i \frac{(u, v)}{|u|^2} \sum_{j=0}^d u_j^*(\theta_j^\tau)E_j^\tau u, \quad E_i' v' = k^*_i \frac{(u', v')'}{|u'|^2} \sum_{j=0}^{d'} u_j'^*(\theta_j'^\tau)E_j'^\tau u'.
\]

Suppose now that \( \Phi' \) is a \( \rho \)-descendent of \( \Phi \) and let \( (\cdot, \cdot) : V \times V \to \mathbb{K} \) be a corresponding balanced bilinear form. Then from (1.5) it follows that

\[
(E_i v | E_j'^\tau v') = k^*_i k'^*_j \frac{(u, v)}{|u|^2} \frac{(u', v')'}{|u'|^2} \sum_{\ell=0}^{d'} u^*_\ell(\theta^*_\ell) \frac{(u^*_\ell u^*_\ell)(\theta^*_\ell)}{|E_\ell|^2} = \frac{\epsilon k^*_i k'^*_j}{\nu^*} \frac{(u, v)}{|u|^2} \sum_{\ell=0}^{d'} u^*_\ell(\theta^*_\ell) \frac{(u^*_\ell u^*_\ell)(\theta^*_\ell)}{|E_\ell|^2}.
\]

Consequently,

\[
(8.3) \quad \sum_{\ell=0}^{d'} u^*_\ell(\theta^*_\ell) \frac{(u^*_\ell u^*_\ell)(\theta^*_\ell)}{|E_\ell|^2} \leq 0 \quad \text{if } i < j \text{ or } i > j + d - d'.
\]

Conversely, it is easy to show that the orthogonality \((8.3)\) in turn characterizes the existence of a balanced bilinear form. We may remark that \((8.3)\) was previously observed by Hosoya and Suzuki \([7]\, \text{Proposition 1.3}]\) for Leonard systems arising from certain pairs of \(Q\-\text{polynomial distance-regular graphs.}\)

We illustrate \((8.3)\) with an example. With the notation of \([A.1]\), assume that

\[
p(\Phi) = p(\text{IIIC}; r, s, s^*, \theta_0, \theta_0^*), \quad p(\Phi') = p(\text{IIIC}; r, s, s^*, \theta_0^*, \theta_0').
\]

It follows that the \( u_i^* \), \( u_i'^* \) are Krawtchouk polynomials:

\[
u_i^*(\theta_i^\tau) = 2F1\left(\begin{array}{c} -i, -j \& \frac{1}{p} \\ -d \end{array} \right), \quad u_i'^*(\theta_i'^\tau) = 2F1\left(\begin{array}{c} -i, -j \& \frac{1}{p} \\ -d' \end{array} \right)
\]

where \( p = r/ss^* \). By \((6.9)\), \( \Phi' \) is a \( \rho \)-descendent of \( \Phi \) for all \( 0 \leq \rho \leq d - d' \). Using \([19]\, \text{Theorem 17.11}]\) we obtain

\[
\sum_{\ell=0}^{d'} \binom{d'}{\ell} \frac{(p + 1)}{(1 - p)^\ell} 2F1\left(\begin{array}{c} -i, -j - \rho - \ell \& \frac{1}{p} \\ -d \end{array} \right) 2F1\left(\begin{array}{c} -j, -\ell \& \frac{1}{p} \\ -d' \end{array} \right) = 0
\]

whenever \( i < j \) or \( i > j + d - d' \). This orthogonality for Krawtchouk polynomials can also be easily derived from \([5]\, \text{Proposition 2.1}]\). However, it should be remarked
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that (5.9) gives a complete classification of those polynomials satisfying (8.3) within the terminating branch of the Askey scheme.

**Appendix A. The list of parameter arrays**

In this appendix we display the parameter arrays of the Leonard systems. The data in (A.1) is taken from [20], but for future use (cf. [13]) the presentation is changed so as to be consistent with the notation in [2, 14, 15, 16]. In (A.1) the following implicit assumptions apply: the scalars \( \theta, \theta^*, \varphi, \phi \) are contained in \( K \), and the scalars \( q, h, h^*, \ldots \) are contained in the algebraic closure of \( K \).

(A.1) **Theorem** ([20 Theorem 5.16]). Let \( \Phi \) be the Leonard system from (2.1) and let \( p(\Phi) = (\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d) \) be the parameter array of \( \Phi \). Then at least one of the following cases I, IA, II, IIA, IIB, III, IV hold:

(I) \[ p(\Phi) = p(I; q, h, h^*, r_1, r_2, s, \theta_0, \theta_0^*, d) \text{ where } r_1r_2 = ss^*q^{d+1}, \]
\[ \theta_i = \theta_0 + h(1 - q^i)(1 - sq^{i+1})q^{-i}, \]
\[ \theta^*_i = \theta_0^* + h^*(1 - q^i)(1 - s^*q^{i+1})q^{-i}, \]

for \( 0 \leq i \leq d \), and
\[ \varphi_i = hh^*q^{1-2i}(1 - q^i)(1 - q^{i-d-1})(1 - r_1q^i)(1 - r_2q^i), \]
\[ \phi_i = \begin{cases} 
hh^*q^{1-2i}(1 - q^i)(1 - q^{i-d-1})(r_1 - s^*q^i)(r_2 - s^*q^i)/s^* & \text{if } s^* \neq 0, \\
hh^*q^{d+2-2i}(1 - q^i)(1 - q^{i-d-1})(s - r_1q^{i-d-1} - r_2q^{i-d-1}) & \text{if } s^* = 0 
\end{cases} \]

for \( 1 \leq i \leq d \).

(IA) \[ p(\Phi) = p(IA; q, h^*, r, s, \theta_0, \theta_0^*, d) \text{ where } \]
\[ \theta_i = \theta_0 - sq(1 - q^i), \]
\[ \theta^*_i = \theta_0^* + h^*(1 - q^i)q^{-i} \]

for \( 0 \leq i \leq d \), and
\[ \varphi_i = -rh^*q^{1-i}(1 - q^i)(1 - q^{i-d-1}), \]
\[ \phi_i = h^*q^{d+2-2i}(1 - q^i)(1 - q^{i-d-1})(s - rq^{i-d-1}) \]

for \( 1 \leq i \leq d \).

(II) \[ p(\Phi) = p(II; h, h^*, r_1, r_2, s, s^*, \theta_0, \theta_0^*, d) \text{ where } r_1 + r_2 = s + s^* + d + 1, \]
\[ \theta_i = \theta_0 + hi(i + 1 + s), \]
\[ \theta^*_i = \theta_0^* + h^*(i + 1 + s^*) \]

for \( 0 \leq i \leq d \), and
\[ \varphi_i = hh^*i(i - d - 1)(i + r_1)(i + r_2), \]
\[ \phi_i = hh^*i(i - d - 1)(i + s^* - r_1)(i + s^* - r_2) \]

for \( 1 \leq i \leq d \).

(IIA) \[ p(\Phi) = p(IIA; h, r, s, s^*, \theta_0, \theta_0^*, d) \text{ where } \]
\[ \theta_i = \theta_0 + hi(i + 1 + s), \]
\[ \theta^*_i = \theta_0^* + s^*i \]

for \( 0 \leq i \leq d \), and
\[ \varphi_i = hs^*i(i - d - 1)(i + r), \]
\[ \phi_i = hs^*i(i - d - 1)(i + r - s - d - 1) \]

for \( 1 \leq i \leq d \).

(II)

\[ p(\Phi) = p(\text{II}; h^*, r, s, s^*, \theta_0, \theta_0^*, d) \text{ where} \]

\[ \theta_i = \theta_0 + si, \]
\[ \theta_i^* = \theta_0^* + h^*i(i + 1 + s^*) \]

for \( 0 \leq i \leq d \), and
\[ \varphi_i = h^*si(i - d - 1)(i + r), \]
\[ \phi_i = -h^*si(i - d - 1)(i + s^* - r) \]

for \( 1 \leq i \leq d \).

(II)

\[ p(\Phi) = p(\text{II}; r, s, s^*, \theta_0, \theta_0^*, d) \text{ where} \]

\[ \theta_i = \theta_0 + si, \]
\[ \theta_i^* = \theta_0^* + s^*i \]

for \( 0 \leq i \leq d \), and
\[ \varphi_i = ri(i - d - 1), \]
\[ \phi_i = (r - ss^*)i(i - d - 1) \]

for \( 1 \leq i \leq d \).

(III)

\[ p(\Phi) = p(\text{III}; h, h^*, r_1, r_2, s, s^*, \theta_0, \theta_0^*, d) \text{ where} r_1 + r_2 = -s - s^* + d + 1, \]
\[ \theta_i = \theta_0 + h(s - 1 + (1 - s + 2i)(-1)^i), \]
\[ \theta_i^* = \theta_0^* + h^*(s^* - 1 + (1 - s^* + 2i)(-1)^i) \]

for \( 0 \leq i \leq d \), and
\[ \varphi_i = -4hh^*i(i + r_1) \]
\[ -4hh^*(i - d - 1)(i + r_2) \]
\[ -4hh^*i(i - d - 1) \]
\[ -4hh^*(i + r_1)(i + r_2) \]

if \( i \) even, \( d \) even,
if \( i \) odd, \( d \) even,
if \( i \) even, \( d \) odd,
if \( i \) odd, \( d \) odd,

\[ \phi_i = 4hh^*(i - s^* - r_1) \]
\[ 4hh^*(i - d - 1)(i - s^* - r_2) \]
\[ -4hh^*i(i - d - 1) \]
\[ -4hh^*(i - s^* - r_1)(i - s^* - r_2) \]

if \( i \) even, \( d \) even,
if \( i \) odd, \( d \) even,
if \( i \) even, \( d \) odd,
if \( i \) odd, \( d \) odd,

for \( 1 \leq i \leq d \).

(IV)

\[ p(\Phi) = p(\text{IV}; h, h^*, r, s, s^*, \theta_0, \theta_0^*) \text{ where} \text{char}(\mathbb{K}) = 2, \]
\[ d = 3, \text{ and} \]
\[ \theta_i = \theta_0 + h(s + 1), \quad \theta_2 = \theta_0 + h, \quad \theta_3 = \theta_0 + hs, \]
\[ \theta_i^* = \theta_0^* + h^*(s^* + 1), \quad \theta_2^* = \theta_0^* + h^*, \quad \theta_3^* = \theta_0^* + h^*s^*, \]
\[ \varphi_1 = hh^*r, \quad \varphi_2 = hh^*, \quad \varphi_3 = hh^*(r + s + s^*), \]
\[ \phi_1 = hh^*(r + s(1 + s^*)), \quad \phi_2 = hh^*, \quad \phi_3 = hh^*(r + s^*(1 + s)). \]
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