BASE STOCK LIST PRICE POLICY IN CONTINUOUS TIME

ALAIN BENSOUSSAN* AND SONNY SKAANING
International Center for Decision and Risk Analysis
Jindal School of Management, University of Texas - Dallas
Richardson, TX 75080-5298, USA

Abstract. We study the problem of inventory control, with simultaneous pricing optimization in continuous time. For the classical inventory control problem in continuous time, see [5], as a recent reference. We incorporate pricing decisions together with inventory decisions. We consider the situation without fixed cost for an infinite horizon. Without pricing, under very natural assumptions, the optimal ordering policy is given by a Base stock, which we review briefly. With pricing, the natural generalization is the so called “Base Stock list price” (BSLP) term coined by E. Porteus, see [26], and was shown in discrete time by A. Federgruen and A. Herching to be the optimal strategy, see [14]. We extend the concept to continuous time which not only complicates the dynamics of the problem, which has never been considered before.

1. Introduction. Pricing and replenishment strategies are core areas of inventory management which in many instances are studied separately. Attempts have been made to consider the two problems jointly. However, this is done only in discrete time, see [14]. Since without pricing a complete theory exists in discrete, as well as in continuous time, it is natural to study the extension of the continuous time theory in order to incorporate pricing.

We assume in this paper no fixed cost, only a variable cost of ordering. We consider a single product problem. Without pricing in discrete time, under very natural assumptions, the optimal ordering policy is given by a Base Stock, see [5] for a complete review.

Extensions have also been made to incorporate pricing in discrete time. This natural extension is known as the Base stock list price policy, see [26] for complete explanation. This policy is characterized as follows. There is a Base Stock $S$. When the stock $x$ is below $S$, the optimal ordering is $S - x$, and the optimal price is a function of $S$, $\pi(S)$. When the stock $x$ is larger than $S$, the optimal order is 0, and the optimal price is $\pi(x)$. Moreover, $\pi(x)$ is decreasing in $x$, and $\pi(x) \leq \pi(S)$. This property means that when the stock is too large, it is optimal to offer a rebate depending on the inventory level $x$. Therefore, the larger the stock, the larger the rebate offered. Our objective in this work is to extend this idea to continuous time situations, which has not been considered before. We adopt, for modeling

2010 Mathematics Subject Classification. Primary: 58F15, 58F17; Secondary: 53C35.
Key words and phrases. Base stock list price, stochastic inventory control, continuous time, quasi-variational inequalities, stochastic dynamic programming.

Alain Bensoussan is also with the department of Systems Engineering and Engineering Management, the City University of Hong Kong :Research supported by a grant from the Research Grants Council of the Hong Kong Special Administrative Region (City U 500111).
* Corresponding author: Alain Bensoussan.
the demand, the classical situation of a deterministic rate of demand per unit of time, with a Gaussian uncertainty described by a Wiener process. However, the rate is influenced by the price which is set by the manager. The price policy must be decided together with the inventory control. We show that the Base Stock list price policy concept can be extended to this situation under some assumptions which couple the demand characteristics and the various costs. We cover the case of an average demand which decreases with price as a power function.

1.1. Literature Review. Optimizing inventory behavior has been a problem seeking a lot of attention since K.J. Arrow, T. Harris, and J. Marshak published their modern formulation of classical inventory problems, such as the News vendor problem, in [2]. They provided optimal inventory strategies that minimized the overall cost to cases where the demand was either deterministic or stochastic. They showed that the optimal inventory strategy followed a Base Stock format for the various cases.

The News vendor problem was later extended to consider the continuous time case by J.A. Bather, see [3]. Bather shows how the continuous analog to the problem statement in [2] reduces to a linear two point boundary value problem. He uses the solution to the linear two point boundary value problem to express the format of the optimal inventory policy.

The News vendor problem has been studied in literature in great detail for various business scenarios, both deterministic and stochastic. In [28], we have a thorough summary of the various scenarios with corresponding results.

In literature, considering the continuous time analog to existing discrete time cases is natural, as we see in [5], [8], [10], [12], [13], [19] and [29]. These papers consider different business applications, and show that the optimal inventory policy follows a Base Stock or (s,S) format.

The technique of concavity/convexity of the value function is very popular in literature, see for example [14]. A different technique, which is relevant to this paper, is the use of Quasi-Variational Inequalities. This technique was used to show optimality in [8] and [29].

Next we discuss the case of simultaneous pricing optimization. One of the earliest considerations was by Whitin, see [31]. Here he considered the classical EOQ problem and shows the combined optimal strategy for the case where price affects the demand linearly. For this case, he shows that the optimal inventory policy still follows the EOQ formula and that the optimal price is the solution to a cubic polynomial.

In literature, it is shown that the BSLP policy is the underlying structure of the optimal strategy, even when perturbations of the general setup have been made to fit various business scenarios, see for example [1], [14], [15], [20], [24] and [25].

In [25], a discount is considered for the company when high quantities are ordered. They show how a quantity based pricing scheme affects the pricing policy given by a BSLP methodology. They established regions where a uniform, as well as quantity, based pricing strategy is optimal.

In [24], the authors consider the format of selling units via an auction. They show how considering this pricing scheme affects the optimal pricing format, such that the optimal strategy for the non-limiting cases are a Base Stock reverse-auction price policy. However, for the limiting cases (only one buyer or large number of buyers) they show that the optimal strategy follows a BSLP format.
In [10], a two parameter demand structure is considered, and conditions for optimality of a BSLP policy is established. They show that the price does not need to be monotonically decreasing, which is the case for one parameter models. For a thorough history of the discrete cases we refer to [32].

Lastly, in [20] we see the case where pricing history has an effect on the customers. If the consumer sees the price below the historical level, a positive affect on demand happens, and vice versa. This phenomenon is known as a reference price. They show that the optimal inventory policy follows a Base Stock format and the optimal pricing policy follows a reference price strategy. In addition to establishing the optimal strategy, they also conclude that considering a pricing format following a reference pricing scheme is beneficial to the overall profit margin.

2. General Presentation of the Model.

2.1. Model and Assumptions. We consider a probability space \((\Omega, \mathcal{A}, P)\) on which there exists a standard Wiener process \(w(t)\) and we call \(\mathcal{F}^t = \sigma(w(s), s \leq t)\). The demand rate is described by

\[
dD(t) = \nu(\varpi(t))dt + \sigma dw(t)
\]

in which \(\sigma\) is the standard deviation of the random part. The average rate per unit of time is generally taken as a constant \(\nu > 0\). In our case, we introduce the decision \(\varpi(t)\), which is the price. It is a control variable, depending on the information. We assume full information on the past and present time. Therefore \(\varpi(t)\) is a stochastic process adapted to the filtration \(\mathcal{F}^t\), which is positive. The function \(\varpi \rightarrow \nu(\varpi)\) is defined from \(\mathbb{R}^+\) to \(\mathbb{R}^+\) and satisfies natural assumptions for a demand function. More specifically, we assume

\[
\begin{align*}
\varpi \rightarrow \nu(\varpi) & \text{ is decreasing, } \nu(0) = +\infty, \nu(\infty) = 0 \\
\varpi \rightarrow \varpi \nu(\varpi) & \text{ is decreasing, } \varpi \nu(\varpi) \big|_{\varpi=0} = +\infty, \varpi \nu(\varpi) \big|_{\varpi=\infty} = 0
\end{align*}
\]

The second condition expresses the fact, that not only the demand decreases with price, but also the sales. We next assume that

\[
\varpi \rightarrow \nu(\varpi) \text{ is continuously differentiable}
\]

\[
\varpi + \frac{\nu(\varpi)}{\nu'(\varpi)} \text{ is monotone increasing, takes the value 0 at 0 and the value } +\infty \text{ at } +\infty
\]

As a template of function \(\nu(\varpi)\) we shall use

\[
\nu(\varpi) = \frac{1}{\varpi^{\gamma+1}}, \gamma > 0
\]

for which

\[
\varpi + \frac{\nu(\varpi)}{\nu'(\varpi)} = \frac{\gamma}{\gamma + 1} \varpi
\]

Besides the price \(\varpi(t)\) there is a second control called \(v(t)\) which represents the accumulated orders up to time \(t\). It is a positive monotone increasing stochastic process, adapted to the filtration \(\mathcal{F}^t\), and \(v(0) = 0\). The state of the dynamic system \(x(t)\) is the inventory level at time \(t\), then the evolution is

\[
\begin{align*}
dx &= dv - \nu(\varpi(t))dt - \sigma dw(t) \\
x(0) &= x
\end{align*}
\]

The process \(v(t)\) is not necessarily continuous, so jumps in the inventory are possible. There is no delay in the delivery.
2.2. The Value Function. The initial value of the state is a parameter, denoted by $x$. We define the profit of the policy $\varpi(t)$, $v(t)$ by

$$J_x(\varpi(\cdot), v(\cdot)) = E \left[ \int_0^{+\infty} \exp(-\alpha t) \left( \varpi(t)\nu(\varpi(t)) - hx^+(t) - px^-(t)\right) dt - cdv(t) \right]$$ (7)

This expression is easy to figure out: per unit of time at time $t$, $\varpi(t)\nu(\varpi(t))$ represents the sales, $hx^+(t)$ the holding cost, $px^-(t)$ the shortage cost and $cdv(t)$ the ordering cost. The parameter $\alpha$ is the discount factor. A control is admissible if

$$E \int_0^{+\infty} \exp(-\alpha t) |x(t)| dt < +\infty$$ (8)

$E \int_0^{T} \exp(-\alpha t) (\varpi(t)\nu(\varpi(t)) dt - cdv(t))$ is convergent to a finite number as $T \to +\infty$

The profit $J_x(\varpi(\cdot), v(\cdot))$ is well defined for admissible controls. Admissible controls exist of course. For example take $v(t) = 0$ and $\varpi(t) = \varpi_0$. We next define the value function by

$$u(x) = \sup_{\varpi(\cdot), v(\cdot)} J_x(\varpi(\cdot), v(\cdot))$$ (9)

The sup is taken over the set of admissible controls. This will be implicit from now on.

2.3. Properties of the Value Function. We state the

**Proposition 1.** We assume (4) then $u(x)$ is increasing and $u(x) - cx$ is decreasing

**Proof.** To simplify the proof, we assume that there exists for any $x$ an optimal policy $\hat{\varpi}_x(\cdot)$, $\hat{v}_x(\cdot)$ such that

$$u(x) = J_x(\hat{\varpi}_x(\cdot), \hat{v}_x(\cdot))$$

We call $\hat{\varpi}_x(t)$ the corresponding optimal trajectory, with initial state $x$. We consider now a control which modifies only the inventory policy as follows

$$d\hat{v}_x(t) \Rightarrow d\hat{v}_{x+\xi}(t) + \xi \delta(t)$$

$$\hat{\varpi}_x(\cdot) \Rightarrow \hat{\varpi}_{x+\xi}(\cdot)$$

where $\xi > 0$ and $\delta(t)$ is the Dirac measure in 0. It is easy to check that the effect is the same as changing the state into $x + \xi$, then applying the optimal control for this new state. We give simply an impulse to the stock at time 0, and apply the optimal policy afterwards. Therefore

$$J_x(\hat{\varpi}_{x+\xi}(\cdot), \hat{v}_{x+\xi}(\cdot) + \xi \mathbb{1}_{t>0}) = J_{x+\xi}(\hat{\varpi}_{x+\xi}(\cdot), \hat{v}_{x+\xi}(\cdot)) - c\xi$$

Necessarily

$$u(x) \geq J_x(\hat{\varpi}_{x+\xi}(\cdot), \hat{v}_{x+\xi}(\cdot) + \xi \mathbb{1}_{t>0})$$

$$= J_{x+\xi}(\hat{\varpi}_{x+\xi}(\cdot), \hat{v}_{x+\xi}(\cdot)) - c\xi$$

$$= u(x + \xi) - c\xi$$

and since $\xi > 0$ arbitrary we obtain the second assertion.

Let us prove the first one. We write simply $\hat{\varpi}(\cdot)$, $\hat{v}(\cdot)$ instead of $\hat{\varpi}_x(\cdot)$, $\hat{v}_x(\cdot)$. We define for any $t$, $\varpi(t) > 0$ by the condition

$$\nu(\varpi(t)) - \nu(\hat{\varpi}(t)) = 1$$
The price $\omega(t)$ is uniquely defined from the assumption on the function $\nu(.)$ and $\omega(t) < \hat{\omega}(t)$. We next consider a policy $\hat{\nu}(.)$, $\hat{\omega}(.)$ defined by

$$\hat{\nu}(.) = \hat{\nu}(t), \quad \hat{\omega}(t) = \begin{cases} \omega(t), & 0 < t < \epsilon \\ \hat{\omega}(t), & t > \epsilon \end{cases}$$

We consider the state $\tilde{x}(t)$ corresponding to the policy $\hat{\nu}(.)$, $\hat{\omega}(.)$ and the initial state $\tilde{x}(0) = x + \epsilon$. We see easily that

$$\tilde{x}(t) = \hat{x}(t) + \epsilon - t, \quad 0 < t < \epsilon$$

$$\tilde{x}(t) = \hat{x}(t), \quad \epsilon \leq t < \epsilon$$

Therefore

$$J_{x+\epsilon}(\hat{\nu}(.), \hat{\omega}(.)) - J_{x}(\hat{\nu}(.), \hat{\omega}(.)) = E \int_{0}^{\epsilon} \exp(-at)[\omega(t)\nu(\omega(t)) - \hat{\omega}(t)\nu(\hat{\omega}(t)) - h(\hat{x}(t) - \hat{x}^{-}(t)) - p(\hat{x}^{-}(t)) - \hat{x}^{-}(t))]dt =$$

$$= E \int_{0}^{\epsilon} \exp(-at)[\omega(t)\nu(\omega(t)) - \hat{\omega}(t)\nu(\hat{\omega}(t)) - h(\hat{x}(t) - \hat{x}^{-}(t)) - (p + h)(\hat{x}^{-}(t)) - \hat{x}^{-}(t))]dt$$

Since $\tilde{x}(t) > \hat{x}(t)$ we have $\hat{x}^{-}(t) - \hat{x}^{-}(t) < 0$. Also $\omega(t)\nu(\omega(t)) > \hat{\omega}(t)\nu(\hat{\omega}(t))$. Therefore

$$J_{x+\epsilon}(\hat{\nu}(.), \hat{\omega}(.)) - J_{x}(\hat{\nu}(.), \hat{\omega}(.)) \geq -hE \int_{0}^{\epsilon} \exp(-at)(\hat{x}(t) - \hat{x}^{-}(t))dt$$

$$\geq -h \int_{0}^{\epsilon} \exp(-at)(\epsilon - t)dt \geq -h \frac{\epsilon^2}{2}$$

which implies

$$u(x + \epsilon) - u(x) \geq -h \frac{\epsilon^2}{2}$$

It follows that $u(x)$ is monotone increasing. Indeed for $\xi > 0$, we can write

$$u(x + \xi) = u(x) + \sum_{k=1}^{N} (u(x + \frac{k}{N}\xi) - u(x + \frac{k-1}{N}\xi))$$

$$\geq u(x) - \frac{h\xi^2}{2N}$$

and since $N$ is arbitrary, we get $u(x + \xi) \geq u(x)$, which is the monotonicity announced.

3. Dynamic Programming.

3.1. Optimality Principle. Following the standard methodology of Dynamic Programming and optimality principle, we consider the consequence of not ordering on a small interval of time $(0, \epsilon)$ and fixing during this period the price at the level $\omega$. After $\epsilon$, we apply the optimal ordering and pricing policy corresponding to the state attained at time $\epsilon$, which is $x - \nu(\omega)\epsilon - \sigma w(\epsilon)$. Noting that the profit during the interval of time $(0, \epsilon)$ is approximately $\epsilon(\omega(\omega) - hx^+ - px^-)$, we can write the inequality

$$u(x) \geq \epsilon(\omega(\omega) - hx^+ - px^-)+$$

$$+(1 - \alpha \epsilon)E u(x - \nu(\omega)\epsilon - \sigma w(\epsilon))$$
Assuming that $u(x)$ is smooth, and expanding on $\epsilon$, we obtain the differential inequality
\[-\frac{1}{2} \sigma^2 \frac{\partial^2 u}{\partial x^2} + \alpha u + \left( \frac{\partial u}{\partial x} - \varpi(x) \right) \nu(\varpi) + hx^+ + px^- \geq 0\]
and since $\varpi$ is an arbitrary positive number we can summarize all these inequalities for any $\varpi \geq 0$, by a single one
\[-\frac{1}{2} \sigma^2 \frac{\partial^2 u}{\partial x^2} + \alpha u + \min_{\varpi \geq 0} \left( \varpi - \nu(\varpi) \right) + hx^+ + px^- \geq 0\]
Before we proceed, we need to introduce the function
\[\Phi(\lambda) = \min_{\varpi \geq 0} \left( \lambda - \varpi \right) \nu(\varpi)\] (10)
We state the
**Lemma 3.1.** We assume (4), (5), (6). The function $\Phi(\lambda)$ is monotone increasing, concave, continuously differentiable, and satisfies
\[\Phi(\lambda) = -\infty, \text{ if } \lambda \leq 0\] (11)
\[-\infty < \Phi(\lambda) < 0, \text{ if } \lambda > 0\]
\[\Phi(+\infty) = 0\]
Moreover if $\lambda > 0$, the minimum $\hat{\varpi} = \hat{\varpi}(\lambda)$ is uniquely defined. It is a monotone increasing function of $\lambda$ and $\hat{\varpi}(0) = 0$, $\hat{\varpi}(+\infty) = +\infty$.
**Proof.** The first property is clear from the assumptions. Assume now $\lambda > 0$. Since $\varpi = 2\lambda$ is admissible, we have
\[\Phi(\lambda) \leq -\lambda \nu(2\lambda) < 0\]
Next we if we set
\[\Psi_\lambda(\varpi) = (\lambda - \varpi) \nu(\varpi)\]
we have
\[\frac{d}{d\varpi} \Psi_\lambda(\varpi) = \nu'(\varpi)(\lambda - \varpi - \frac{\nu(\varpi)}{\nu'(\varpi)})\]
and from the assumptions, there is a single value $\hat{\varpi} = \hat{\varpi}(\lambda)$, such that
\[\lambda - \hat{\varpi} - \frac{\nu(\hat{\varpi})}{\nu'(\hat{\varpi})} = 0\] (12)
and from the assumptions
\[\frac{d}{d\varpi} \Psi_\lambda(\varpi) \begin{cases} < 0 & \text{for } \varpi < \hat{\varpi} \\ > 0 & \text{for } \varpi > \hat{\varpi} \end{cases}\]
therefore $\hat{\varpi}$ is the single minimum of $\Psi_\lambda(\varpi)$. We have $\Phi'(\lambda) = \nu(\hat{\varpi}(\lambda))$. The function $\Phi(\lambda)$ is concave since
\[\Psi_{\theta\lambda^1 + (1-\theta)\lambda^2}(\varpi) = \theta \Psi_{\lambda^1}(\varpi) + (1-\theta) \Psi_{\lambda^2}(\varpi)\]
hence
\[\Phi(\theta\lambda^1 + (1-\theta)\lambda^2) \geq \theta \Phi(\lambda^1) + (1-\theta) \Phi(\lambda^2)\]
From (12) and the second assumption (4) we see that $\varpi(\lambda)$ is monotone increasing and $\varpi(0) = 0$, $\varpi(+\infty) = +\infty$. It follows that $\nu(\varpi(+\infty)) = 0$, $\varpi(+\infty)\nu(\varpi(+\infty)) = 0$, hence $\Phi(+\infty) = 0$ and
\[\Phi(+\infty) = \lim_{\lambda \to +\infty} \lambda \nu(\varpi(\lambda)) \geq 0\]
and since \( \Phi(+\infty) \leq 0 \), we get \( \Phi(+\infty) = 0 \). This concludes the proof.

In the example, we have \( \hat{\varpi}(\lambda) = \frac{\gamma+1}{\gamma} \lambda \), \( \Phi'(\lambda) = \frac{\gamma}{\gamma+1} \frac{1}{\lambda} \gamma+1 \), \( \Phi(\lambda) = -\frac{\gamma}{(\gamma+1)\gamma+1} \lambda \).

3.2. Quasi-Variational Inequality. We now consider an analytic problem, whose intuition is related to the preceding considerations. It is a quasi variational inequality, written in a strong sense, which means a set of differential inequalities, complemented with a complementarity slackness condition. The solution is a function \( u(x) \), which is \( C^1 \), has bounded derivatives, and is a.e. twice differentiable. Eventually, it will be the value function, given by (9), but at this stage it is introduced by itself, with no interpretation. The problem is the following

\[
-\frac{1}{2} \sigma^2 \frac{\partial^2 u}{\partial x^2} + \alpha u + \Phi(\partial u/\partial x) + hx^+ + px^- \geq 0, \text{ a.e. } x
\]

\[
0 < \frac{\partial u}{\partial x} \leq c
\]

\[
(\frac{\partial u}{\partial x} - c)\left(-\frac{1}{2} \sigma^2 \frac{\partial^2 u}{\partial x^2} + \alpha u + \Phi(\frac{\partial u}{\partial x}) + hx^+ + px^-\right) = 0, \text{ a.e. } x
\]

which we call the Bellman Q.V.I. of problem (9). For the general theory of Q.V.I., see [7].

3.3. Transformations. We first begin with a simple transformation, setting \( G(x) = u(x) - cx \). We obtain immediately

\[
-\frac{1}{2} \sigma^2 G'' + \alpha G + \Phi(G' + c) + (h + \alpha c)x^+ + (p - \alpha c)x^- \geq 0, \text{ a.e. } x
\]

\[
-c < G' \leq 0
\]

\[
G' \left(-\frac{1}{2} \sigma^2 G'' + \alpha G + \Phi(G' + c) + (h + \alpha c)x^+ + (p - \alpha c)x^-\right) = 0, \text{ a.e. } x
\]

We look for a solution of (14) as follows: Find \( S \), and \( G_S(x) \) such that

\[
-\frac{1}{2} \sigma^2 G'' S(x) + \alpha G_S(x) + \Phi(G'_S(x) + c) + (h + \alpha c)x^+ + (p - \alpha c)x^- = 0, x > S
\]

\[
G'_S(S) = 0, G'_S(x) \text{ bounded}
\]

For fixed \( S \), this is a second order differential equation on \( (S, +\infty) \), with two-point boundary conditions, of Neumann type (the derivative is given at \( S \), and the boundedness acts as a condition at \( +\infty \)). We then define \( S \) by adding a condition, namely

\[
G''_S(S) = 0
\]

This condition fixes automatically the value of \( G_S(S) \), by

\[
\alpha G_S(S) + \Phi(c) + (h + \alpha c)S^+ + (p - \alpha c)S^- = 0
\]

and setting \( G_S(x) = G_S(S) \), for \( x < S \), we get a \( C^2 \) function on \( R \). Because of the nature of the function \( \Phi \), we need to find a function \( G_S(x) \), such that

\[
G'_S(x) + c > 0
\]

How does this problem (15), (16), (17), (18) solves the original problem (14). Of course we define \( G(x) = G_S(x) \). We note that the complementarity slackness condition is satisfied. The first inequality is satisfied when \( x > S \) (it is an equality), and the
Therefore we can assert that assumption \( G' \leq 0 \) is satisfied when \( x < S \) (it is also an equality). Therefore we need to check that the solution of (15) satisfies \( G'_S \leq 0 \). In that case the second condition (14) is satisfied. There remains to satisfy the first inequality, when \( x < S \). Since for \( x < S \), we have \( G(x) = G_S(x) = G_S(S) \), we get
\[
\frac{1}{2} \sigma^2 G''(x) + \alpha G(x) + \Phi(G'(x) + c) + (h + \alpha c)x^+ + (p - \alpha c)x^- = \\
\alpha G_S(S) + \Phi(c) + (h + \alpha c)x^+ + (p - \alpha c)x^- = \\
(h + \alpha c)(x^+ - S^+) + (p - \alpha c)(x^- - S^-)
\]
and if we take \( S \leq 0 \) and assume \( p - \alpha c > 0 \), this expression, which is equal to \(- (p - \alpha c)(x - S)\) is positive for \( x \leq S \). We can summarize the results as follows, setting \( H_S(x) = G'_S(x) \), and differentiating (15). We look for a pair \( S \leq 0, H_S(x), x \geq S \) such that
\[
-\frac{1}{2} \sigma^2 H''_S(x) + \alpha H_S(x) + \frac{d}{dx} \Phi(H_S(x) + c) + (h + \alpha c)I_{x>0} - (p - \alpha c)I_{x<0} = 0, \ x > S
\]
\( H_S(S) = 0, \ H'_S(S) = 0, \ -c < H_S(x) \leq 0, \ H_S(+\infty) = -c \). \( H_S(x) \) is \( C^1 \) (19)
If we find a pair \( S \leq 0, H_S(x), x \geq S \) satisfying (19) then we define
\[
G(x) = \frac{(p - \alpha c)S - \Phi(c)}{\alpha}, \ x \leq S
\]
\[
G(x) = G(S) + \int_{S}^{x} H_S(\xi)d\xi, \ x > S
\]
then \( G(x) \) is \( C^2 \) and satisfies the Q.V.I. (14).

4. Solution of the Inventory Control and Pricing Problem.

4.1. Pricing feedback. Suppose we find a pair \( S \leq 0, H_S(x), x \geq S \) satisfying (19) and we define \( G(x) \) by (20), then we claim that \( u(x) = G(x) + cx \) is the value function (9) and that that there exists an optimal inventory control and pricing policy. We first define the feedback (it will be the optimal pricing feedback)
\[
\hat{\pi}(x) = \hat{\sigma}(H_S(x) + c), \ x \geq S
\]
\[
\hat{\pi}(x) = \hat{\sigma}(c), \ x \leq S
\]
This is a decreasing function (as expected, since a rebate is considered when the inventory is high). It takes the value 0, when \( x = +\infty (H_S(+\infty) = -c) \). In our example, \( \hat{\pi}(x) = \frac{\gamma + 1}{\gamma} (H_S(x) + c) \). To simplify technicalities, we make the following assumption
\[
\nu''(\varpi) \geq 0, \ 2 - \frac{\nu''}{(\nu')^2}(\varpi) \geq c_0 > 0, \ \forall \varpi \geq 0
\]
Since, from (12) we have
\[
1 = \hat{\sigma}'(\lambda)|2 - \frac{\nu''}{(\nu')^2}(\hat{\sigma}(\lambda))|, \ \lambda > 0
\]
we get immediately \( \frac{1}{c_0} \leq \hat{\sigma}'(\lambda) \leq \frac{1}{c_0} \). Therefore
\[
\hat{\pi}'(x) = \hat{\sigma}'(H_S(x) + c)H'_S(x),
\]
In fact, we shall check in the study of the problem (19) that \( H'_S(x) \geq -\frac{2}{\sigma} \sqrt{c(h + \alpha c)} \). Therefore we can assert that
\[
0 \geq \hat{\pi}'(x) \geq -\frac{2}{c_0}\sigma \sqrt{c(h + \alpha c)} \]
In the example, we have $c_0 = \frac{\gamma}{n+1}$. In fact $\dot{\pi}'(x) = \frac{n+1}{n} H_S(x)$.  

4.2. Inventory Ordering Policy. To define the optimal inventory policy, we need to solve a reflected (at $S$) diffusion equation. Namely, we consider the following problem: For a given initial condition $x \geq S$, find two stochastic processes $\hat{x}(t)$ and $\hat{v}(t)$ which are adapted to $\mathcal{F}^t$ and continuous such that

\begin{equation}
\hat{v}(t) \text{ is monotone increasing, } \hat{x}(0) = x, \hat{v}(0) = 0 \tag{25}
\end{equation}

\begin{equation}
\dot{\hat{x}}(t) \geq S, \forall \hat{t} \quad d\hat{x}(t) + \nu(\hat{\pi}(\hat{x}))dt + \sigma dw(t) = I_{\hat{x}(t) = S} \quad d\hat{v}(t) \tag{26}
\end{equation}

Suppose that we can solve this problem, for any $x \geq S$ and that the solution is unique. We consider now our control problem (6), (7). The initial inventory is $x$. We define a pair $\hat{\varpi}_x(t), \hat{v}_x(t)$ as follows

\begin{equation}
\text{if } x < S, \quad \hat{v}_x(0) = S - x
\end{equation}

This means that there is an initial jump of the inventory to bring it to $S$. So, it is sufficient to consider the case of an initial inventory larger or equal to $S$. Assume then $x \geq S$ and define the processes $\hat{x}(t)$ and $\hat{v}(t)$ by (25) (they depend of course of $x$). We set

\begin{equation}
\hat{\varpi}_x(t) = \hat{\pi}(\hat{x}(t)), \hat{v}_x(t) = \hat{v}(t) \tag{27}
\end{equation}

Then this is the optimal joint inventory and pricing policy. The process $\hat{x}(t)$ is the optimal inventory. We can interpret the policy in practical terms. First, the inventory cannot be strictly less than $S$, unless this is the case at the beginning. In that case, we immediately make a procurement, which brings the inventory to $S$. There is an ordering cost of value $c(S - x)$. But up to this case, we can always consider that the inventory is larger or equal to $S$. It can be strictly larger than $S$, either because this is the case at the initial time, or because the Wiener process brings naturally the inventory to a value which is strictly larger than $S$. Suppose that the inventory $\hat{x}(t)$ is strictly larger than $S$, at some time, then we do not order anything. Since we cannot reduce it by a jump, we have to wait until it decreases naturally to $S$. This can come by the drift term $\nu(\hat{\pi}(\hat{x}))$ which is strictly positive, or by the Wiener process, which can increase the depletion of the inventory coming from the drift term. The only action we can have concerns the price. Since $\varpi \to \nu(\varpi)$ is decreasing and $x \to \hat{\pi}(x)$ is also decreasing, the composite $x \to \nu(\hat{\pi}(x))$ is increasing. So the higher the inventory, the higher is the drift term to reduce it.

When the inventory is $S$, we use the price $\hat{\pi}(S)$, which is the minimum price. The drift term still tends to deplete the inventory below $S$, which we do not want. The Wiener process can either increase the effect of the drift or reduce it. On the average, the effect of the Wiener process is 0, so we put orders to keep the inventory at $S$. However, we do not want to put big orders, since we like to stay at $S$. We respond to small variations of the demand with small orders. Since there is no fixed cost of ordering, there is no penalty for small orders. We obtain what is called a shattering policy. It is not implementable in practice, but it is the solution of our problem, as it has been defined.

Note that the inventory and pricing policy which has been defined is indeed a Base Stock list price policy, as described in the discrete time case, in [13] for instance.
4.3. Verification Theorem. In this section, we provide a verification theorem to check that the policy defined before is indeed optimal. We first study the stochastic equation \((\ref{25})\). We have the

**Proposition 2.** We assume \((\ref{2}), (\ref{3}), (\ref{4})\) and \((\ref{22})\). There exists one and only one solution of \((\ref{25})\) such that

\[
E \sup_{0 \leq t \leq T} |\dot{x}(t)|^2 < +\infty, \forall T
\]

**Proof.** To simplify notation, we set \(y(t) = \dot{x}(t), \xi(t) = \dot{v}(t)\) and \(a(x) = \nu(\dot{x}(x))\). So we have

\[
dy + a(y)dt + \sigma dw(t) = 1_{y(t) = S}d\xi
y(t) \geq S, y(0) = x, \xi(0) = 0
\]

and \(y(t), \xi(t)\) are continuous, and \(\xi(t)\) is monotone increasing. We also have \(E \sup_{0 \leq t \leq T} |y(t)|^2 < +\infty\). We note that \(a'(x) \geq 0\). We can write

\[
(y(t) - S)(dy + a(y)dt + \sigma dw(t)) = 0
\]

Suppose there are 2 solutions denoted \(y^1(t), y^2(t)\). We get easily

\[
(y^2(t) - y^1(t))(dy^1 + a(y^1(t))dt + \sigma dw(t)) \geq 0
(y^1(t) - y^2(t))(dy^2 + a(y^2(t))dt + \sigma dw(t)) \geq 0
\]

These inequalities follows from the following calculations

\[
(y^2(t) - y^1(t))(dy^1 + a(y^1(t))dt + \sigma dw(t)) =
(y^2(t) - S - (y^1(t) - S))(dy^1 + a(y^1(t))dt + \sigma dw(t)) =
(y^2(t) - S)(dy^1 + a(y^1(t))dt + \sigma dw(t)) - (y^1(t) - S)(dy^1 + a(y^1(t))dt + \sigma dw(t)) =
(y^2(t) - S)(1_{y^1(t) = S}d\xi) \geq 0
\]

and by addition of the above inequalities we reach

\[
(y^2(t) - y^1(t))dy^1 + a(y^1(t))dt + \sigma dw(t) - \int 0 (a(y(\theta))d\theta + \sigma w(s)) \geq 0
\]

hence \((y^2(t) - y^1(t))dy^1 + a(y^1(t)) - a(y^2(t)) \geq 0\), which implies \(y^1(t) = y^2(t)\). From \((\ref{29})\) it follows immediately that \(\xi(t)\) is uniquely defined. To prove existence, one first notices that the process \(\xi(t)\) can be written in terms of \(y(t)\). Indeed, we have \(d\xi(t) = 1_{y(t) = S}d\xi(t)\) and

\[
\xi(t) = \max_{0 \leq s \leq t} \left( S - x + \int_0^s a(y(\theta))d\theta + \sigma w(s) \right)
\]

So in fact \(y(t)\) is the solution of the functional equation

\[
y(t) + \int_0^t a(y(s))ds + \sigma w(t) = x + \max_{0 \leq s \leq t} \left( S - x + \int_0^s a(y(\theta))d\theta + \sigma w(s) \right)
\]

As it is standard for differential equations, deterministic or stochastic, the existence of a solution of \((\ref{30})\) is done by proving the existence of a fixed point. The difficulty we have here is that the nonlinear function \(a(x)\) is not globally Lipchitz. The derivative \(a'(x)\) exists but it is not bounded. However, it is continuous, so bounded on any interval \([S, M]\). If we replace \(a(x)\) by \(a_M(x) = a(x \wedge M)\), then we get a
globally Lipschitz function. We can then rely on the theory of reflected diffusions, see for instance [21], to prove the existence and uniqueness of a solution of (30) in the functional space (28). We then proceed as follows. Consider the function \( a_1(x) \) and solve
\[
y_1(t) + \int_0^t a_1(y_1(s))ds + \sigma w(t) = x + \max_{0 \leq s \leq t} \left( S - x + \int_0^s a_1(y_1(\theta))d\theta + \sigma w(s) \right)^+
\]
We then define the stopping time \( \tau_1 = \inf_{t \geq 0} \{ y_1(t) \geq 1 \} \). Clearly \( y_1(\tau_1) = 1 \). We next consider \( a_2(x) \) and solve for \( t \geq \tau_1 \), the equation
\[
y_2(t) + \int_{\tau_1}^{t} a_2(y_2(s))ds + \sigma(w(t) - w(\tau_1)) = 1 + \max_{\tau_1 \leq s \leq t} \left( S - 1 + \int_{\tau_1}^{s} a_2(y_2(\theta))d\theta + \sigma(w(s) - w(\tau_1)) \right)^+
\]
We define \( \tau_2 = \inf_{t \geq \tau_1} \{ y_2(t) \geq 2 \} \). In this way we define a sequence of stopping times \( \tau_M \) and of stochastic processes \( y_M(t), t \geq \tau_M \), with \( y_M(\tau_M) = M \). Note that \( \tau_M < +\infty \text{ a.s.} \). We then set \( y(t) = y_M(t), \tau_M \leq t \leq \tau_{M+1} \). We also define, for \( \tau_M \leq t \leq \tau_{M+1} \)
\[
\xi(t) = \max_{0 \leq s \leq \tau_1} \left( S - x + \int_0^s a(y(\theta))d\theta + \sigma w(s) \right)^+ + \sum_{m=1}^{M-1} \max_{\tau_m \leq s \leq \tau_{m+1}} \left( S - m + \int_{\tau_m}^{s} a(y(\theta))d\theta + \sigma(w(s) - w(\tau_m)) \right)^+ + \max_{\tau_M \leq s \leq t} \left( S - M + \int_{\tau_M}^{s} a(y(\theta))d\theta + \sigma(w(s) - w(\tau_M)) \right)^+
\]
The second term vanishes when \( M = 1 \). We can write, for \( \tau_M \leq t \leq \tau_{M+1} \)
\[
y(t) + \int_{\tau_M}^{t} a(y(s))ds + \sigma(w(t) - w(\tau_M)) = y(\tau_M) + \xi(t) - \xi(\tau_M) \quad (31)
\]
also for \( t \leq \tau_M \)
\[
y(t) + \int_0^{t} a(y(s))ds + \sigma w(t) = x + \xi(t), \quad S \leq y(t) \leq M \quad (32)
\]
Let \( \tau_M \leq t \leq \tau_{M+1} \), and assume that \( y(t) > S \). From (31) we get
\[
\max_{\tau_M \leq s \leq t} \left( S - y(\tau_M) + \int_{\tau_M}^{s} a(y(\theta))d\theta + \sigma(w(s) - w(\tau_M)) \right)^+ > S - y(\tau_M) + \int_{\tau_M}^{t} a(y(\theta))d\theta + \sigma(w(t) - w(\tau_M))
\]
so we can find \( \eta > 0 \) such that
\[
\max_{\tau_M \leq s \leq t} \left( S - y(\tau_M) + \int_{\tau_M}^{s} a(y(\theta))d\theta + \sigma(w(s) - w(\tau_M)) \right)^+ > S - y(\tau_M) + \int_{\tau_M}^{t} a(y(\theta))d\theta + \sigma(w(t) - w(\tau_M)) + \eta
\]
Since, as $\varepsilon \to 0$,
\[
\max_{t \leq s \leq t + \varepsilon} \left( S - y(\tau_M) + \int_{\tau_M}^{t} a(y(\theta))d\theta + \sigma(w(s) - w(\tau_M)) \right) ^{+} \to \\
\left( S - y(\tau_M) + \int_{\tau_M}^{t} a(y(\theta))d\theta + \sigma(w(t) - w(\tau_M)) \right) ^{+}
\]
we can find $\varepsilon(\eta)$, such that for $\varepsilon \leq \varepsilon(\eta)$, we have
\[
\max_{t \leq s \leq t + \varepsilon} \left( S - y(\tau_M) + \int_{\tau_M}^{t} a(y(\theta))d\theta + \sigma(w(s) - w(\tau_M)) \right) ^{+} \leq \\
S - y(\tau_M) + \int_{\tau_M}^{t} a(y(\theta))d\theta + \sigma(w(t) - w(\tau_M)) + \eta <
\]
\[
\max_{\tau_M \leq s \leq t} \left( S - y(\tau_M) + \int_{\tau_M}^{s} a(y(\theta))d\theta + \sigma(w(s) - w(\tau_M)) \right) ^{+}
\]
Therefore, for $\varepsilon \leq \varepsilon(\eta)$,
\[
\max_{\tau_M \leq s \leq t + \varepsilon} \left( S - y(\tau_M) + \int_{\tau_M}^{s} a(y(\theta))d\theta + \sigma(w(s) - w(\tau_M)) \right) ^{+} = \\
\max_{\tau_M \leq s \leq t} \left( S - y(\tau_M) + \int_{\tau_M}^{s} a(y(\theta))d\theta + \sigma(w(s) - w(\tau_M)) \right) ^{+}
\]
which means that $\xi(t + \varepsilon) = \xi(t)$, for $\varepsilon \leq \varepsilon(\eta)$. This implies $d\xi(t) = 1_{y(t)=S}d\xi(t)$. This proves that we have constructed a solution of (29) on the interval $[0, \tau_M]$ for any $M$, and $0 \leq y(t) \leq M$, $y(\tau_M) = M$. The sequence $\tau_M$ is increasing. Now, we have
\[
(y(t) - S)(dy + a(S)dt + \sigma dw(t)) = -(a(y) - a(S))(y(t) - S) \leq 0
\]
Consider the following calculation
\[
\frac{1}{2} d[|y(t) - S|^2 \exp(-2a(S)t)] \\
= -a(S)|y(t) - S|^2 \exp(-2a(S)t)dt \\
+ \exp(-2a(S)t)|(y(t) - S)dy + \frac{\sigma^2}{2}dt \\
\leq \exp(-2a(S)t)|(y(t) - S)(dy + a(S)dt + \sigma dw) \\
- (y(t) - S)(a(S)dt + \sigma dw) + \frac{\sigma^2}{2}dt \\
= - \exp(-2a(S)t)\sigma dw + \frac{\sigma^2}{2} \exp(-2a(S)t)dt \\
+ \exp(-2a(S)t)|(y(t) - S)(dy + a(S)dt + \sigma dw) - (y(t) - S)a(S)dt \\
\leq - \exp(-2a(S)t)\sigma dw + \frac{\sigma^2}{2} \exp(-2a(S)t)dt
\]
(33)
By integration from $t = 0$ to $t = T \wedge \tau_M$ and taking the mathematical expectation we get
\[
\frac{1}{2} E\left(|y(T \wedge \tau_M) - S|^2 \exp(-2a(S)(T \wedge \tau_M))\right) \leq \frac{1}{2} |x - S|^2 + \frac{1}{2} \sigma^2 T
\]
By distributing the expectation recalling \( y(\tau_M) = M \) we get

\[
\frac{1}{2}(M - S)^2 E\left(\exp(-2a(S)(\tau_M \mathbf{1}_{\tau_M < \tau}))\right) \leq \frac{1}{2}|x - S|^2 + \frac{1}{2}\sigma^2 T
\]

We see that as \( M \to \infty \) we have \( \tau_M \to \tau^* \). We therefore know

\[
E(1_{\tau^* < T}) \leq E(\mathbf{1}_{\tau_M < T})
\]

hence

\[
E(\exp(-2a(S)(\tau_M \mathbf{1}_{\tau_M < \tau})) \leq \frac{1}{(M - S)^2}(|x - S|^2 + \sigma^2 T) \to 0 \tag{34}
\]

But

\[
E(\exp(-2a(S)(\tau^* \mathbf{1}_{\tau^* < T}))) \to E(\exp(-2a(S)(\tau^* \mathbf{1}_{\tau < T})))
\]

and therefore

\[
E(\exp(-2a(S)(\tau^* \mathbf{1}_{\tau^* < T})) = 0 \tag{35}
\]

This will give us that

\[
\exp(-2a(S)(\tau^* \mathbf{1}_{\tau^* < T})) = 0 \text{ a.s.}
\]

hence also \( \mathbf{1}^*_T = 0 \text{ a.s.} \) Since \( T \) is arbitrary \( \tau^* = \infty \text{ a.s.} \) We have thus constructed a solution of (29) on any finite interval of time. By similar calculations as in the preceding lines, we can check the estimate

\[
E \sup_{0 \leq t \leq T} |y(t) - S|^2 \leq C_T(S)
\]

The proof has been completed. \( \square \)

We then can state the verification theorem

**Theorem 4.1.** We make the assumptions of Proposition 2. We suppose that we can solve the equation (19). We construct the pricing and inventory ordering policy \( \hat{\nu}_x(t), \hat{v}_x(t) \) as explained in (21), (26), (27). It is optimal for the problem (9).

**Proof.** We have constructed a number \( S \leq 0 \) and a function \( u(x) \) which is \( C^2 \), satisfies (13) and moreover

\[
u(x) = cx + \frac{(p - \alpha c)S - \Phi(c)}{\alpha}, \quad x \leq S
\]

\[
-\frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2} + \alpha u + \Phi\left(\frac{\partial u}{\partial x}\right) + hx^+ + px^- = 0, \quad x \geq S, \quad \frac{\partial u}{\partial x}(S) = c \tag{36}
\]

From the definition of \( \Phi \), we can write

\[
-\frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2} + \alpha u + \left(\frac{\partial u}{\partial x} - \omega\right)v(\omega) + hx^+ + px^- \geq 0, \quad \forall x, \forall \omega \geq 0 \tag{37}
\]

Consider any admissible policy \( \overline{\nu}(t), \overline{v}(t) \) and the corresponding inventory \( x(t) \) defined by (6). We use Ito’s formula to compute

\[
d(u(x(t))\exp(-\alpha t)) = \exp(-\alpha t)[-\alpha u(x(t)) + \frac{\partial u}{\partial x}(x(t))(dv(t) - \nu(\omega(t))dt - \sigma dw(t) + \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2}(x(t))]
\]
Applying (37) with \( x = x(t) \) and \( \overline{w} = \overline{w}(t) \) we obtain
\[
d(u(x(t)) \exp(-\alpha t)) \leq \exp(-\alpha t)\left[\frac{\partial u}{\partial x}(x(t))dv(t) + (h\dot{x}^+(t) + px^-(t) - \overline{w}(t)\nu(\overline{w}(t)))dt - \sigma dw(t)\right] 
\]
Integrating between 0 and \( T \) and taking the mathematical expectation yields
\[
u(x) \geq Eu(x(T)) \exp(-\alpha T) + \int_0^T \mathbb{E}\left[\exp(-\alpha t) \left((\overline{w}(t)\nu(\overline{w}(t)) - h\dot{x}^+(t) - px^-(t))dt - cdv(t)\right)\right] 
\]
However, we know that \( \frac{\partial u}{\partial x} \geq c1_{x < S} \), therefore \( u(x) - u(S) \geq -c(x - S)^- \geq -cx^- \). Therefore, from (38)
\[
u(x) \geq -cE\exp(-\alpha T) + u(S)\exp(-\alpha T) + \int_0^T \mathbb{E}\left[\exp(-\alpha t) \left((\overline{w}(t)\nu(\overline{w}(t)) - h\dot{x}^+(t) - px^-(t))dt - cdv(t)\right)\right] 
\]
Since the control is admissible \( E\exp(-\alpha T) \rightarrow 0 \), as \( T \rightarrow +\infty \). The last term converges to \( J_x(\overline{w}(.), \nu(\cdot)) \), since the control is admissible. Therefore, we have obtained \( \nu(x) \geq J_x(\overline{w}(.), \nu(\cdot)) \) for any admissible control. To prove the optimality of the pair \( \hat{\omega}_x(t), \hat{v}_x(t) \), it is thus sufficient to check that
\[
u(x) = J_x(\hat{\omega}_x(\cdot), \hat{v}_x(\cdot)) \tag{39} 
\]
We can immediately reduce the problem to the case when \( x \geq S \). Indeed, if \( x < S \), the policy consists in an immediate jump of the inventory to the level \( S \), by incurring a cost \( c(S - x) \). This is captured by the relation \( \nu(x) = u(S) + c(S - x) \). So we may assume \( x \geq S \). For \( x \geq S \) the function \( \nu(x) \) satisfies (36), hence also
\[
-\frac{1}{2} \sigma^2 \frac{\partial^2 \nu}{\partial x^2} + \alpha \nu + \left(\frac{\partial u}{\partial x} - \nu(\hat{\pi}(x))\right) + h\dot{x}^+ + px^- = 0 
\]
from the definition of \( \hat{\pi}(x) \), see (21). Consider then the process \( \dot{x}(t) \) defined by (25). Applying Ito’s formula, we get
\[
d(u(\dot{x}(t)) \exp(-\alpha t)) = \exp(-\alpha t)\left[\frac{\partial u}{\partial x}(\dot{x}(t))d\dot{v}(t) + (h\dot{x}^+(t) + px^-(t) - \overline{w}(t)\nu(\overline{w}(t)))dt - \sigma dw(t)\right] 
\]
Therefore, also
\[
u(x) = Eu(\dot{x}(T)) \exp(-\alpha T) + \int_0^T \mathbb{E}\left[\exp(-\alpha t) \left((\overline{w}(t)\nu(\overline{w}(t)) - h\dot{x}^+(t) - px^-(t))dt - cd\dot{v}(t)\right)\right] 
\]
Next
\[
u(S) \exp(-\alpha T) \leq Eu(\dot{x}(T)) \exp(-\alpha T) \leq u(S) \exp(-\alpha T) + cE(\dot{x}(T) - S) \exp(-\alpha T) \tag{41} 
\]
Similarly let \( W \) be the solution of \( \text{(44)} \) in the functional space \( L^2(\mathbb{R}, S, \epsilon) \). Proposition 3.

Study of the Analytic Problem.

5.1. Approximation. Our objective is now to study problem \( \text{(19)} \), which allows to define an optimal control, as seen before. The major difficulty lies in the function \( \Phi \). It is increasing, but the derivative is not bounded. We will need to proceed with an approximation. Define

\[
\Phi_\epsilon(\lambda) = \Phi(\lambda^+ + \epsilon)
\]

which is defined on \( \mathbb{R} \), and not just on \( \mathbb{R}^+ \). We have \( \Phi'_\epsilon(\lambda) = \Phi'(\lambda + \epsilon) \mathbf{1}_{\lambda > 0} \). Hence \( 0 \leq \Phi'_\epsilon(\lambda) \leq \Phi'(\epsilon) \). Note that the function \( \Phi_\epsilon(\lambda) \) remains concave on \( \mathbb{R}^+ \), but is not globally concave. For any \( S \leq 0 \), given, we consider the problem

\[
-\frac{1}{2} \sigma^2 H''(x) + \alpha H(x) + \frac{d}{dx} \Phi_\epsilon(H(x) + c) + (h + \alpha c) \mathbf{1}_{x > 0} - (p - \alpha c) \mathbf{1}_{x < 0} = 0, \quad x > S
\]

\[
H(S) = 0, \quad H \text{ bounded}
\]

5.2. Approximation. Our objective is now to study problem \( \text{(19)} \), which allows to define an optimal control, as seen before. The major difficulty lies in the function \( \Phi \). It is increasing, but the derivative is not bounded. We will need to proceed with an approximation. Define

\[
\Phi_\epsilon(\lambda) = \Phi(\lambda^+ + \epsilon)
\]

which is defined on \( \mathbb{R} \), and not just on \( \mathbb{R}^+ \). We have \( \Phi'_\epsilon(\lambda) = \Phi'(\lambda + \epsilon) \mathbf{1}_{\lambda > 0} \). Hence \( 0 \leq \Phi'_\epsilon(\lambda) \leq \Phi'(\epsilon) \). Note that the function \( \Phi_\epsilon(\lambda) \) remains concave on \( \mathbb{R}^+ \), but is not globally concave. For any \( S \leq 0 \), given, we consider the problem

\[
-\frac{1}{2} \sigma^2 H''(x) + \alpha H(x) + \frac{d}{dx} \Phi_\epsilon(H(x) + c) + (h + \alpha c) \mathbf{1}_{x > 0} - (p - \alpha c) \mathbf{1}_{x < 0} = 0, \quad x > S
\]

\[
H(S) = 0, \quad H \text{ bounded}
\]

5. Study of the Analytic Problem.

5.1. Approximation. Our objective is now to study problem \( \text{(19)} \), which allows to define an optimal control, as seen before. The major difficulty lies in the function \( \Phi \). It is increasing, but the derivative is not bounded. We will need to proceed with an approximation. Define

\[
\Phi_\epsilon(\lambda) = \Phi(\lambda^+ + \epsilon)
\]

which is defined on \( \mathbb{R} \), and not just on \( \mathbb{R}^+ \). We have \( \Phi'_\epsilon(\lambda) = \Phi'(\lambda + \epsilon) \mathbf{1}_{\lambda > 0} \). Hence \( 0 \leq \Phi'_\epsilon(\lambda) \leq \Phi'(\epsilon) \). Note that the function \( \Phi_\epsilon(\lambda) \) remains concave on \( \mathbb{R}^+ \), but is not globally concave. For any \( S \leq 0 \), given, we consider the problem

\[
-\frac{1}{2} \sigma^2 H''(x) + \alpha H(x) + \frac{d}{dx} \Phi_\epsilon(H(x) + c) + (h + \alpha c) \mathbf{1}_{x > 0} - (p - \alpha c) \mathbf{1}_{x < 0} = 0, \quad x > S
\]

\[
H(S) = 0, \quad H \text{ bounded}
\]

Of course, the solution depends on \( S, \epsilon \) and we shall consider this dependence, and write \( H_{S,\epsilon}(x) \) or \( H_\epsilon(x) \). Presently, they are fixed and we save notation, omitting to write them explicitly. Since the domain is unbounded, we use functional spaces with weights. We introduce, for \( m \) positive integer

\[
L^2_m(R) = \{ \varphi(.) \mid \int_{-\infty}^{+\infty} \frac{|\varphi(x)|^2}{(1 + x^2)^m} dx < +\infty \}
\]

\[
W^{1,2}_m(R) = \{ \varphi(.) \mid \varphi, \varphi' \in L^2_m(R) \}
\]

Let \( L^2_m(S, +\infty) \) be the subspace of \( L^2_m(R) \) of functions which vanish on \( (-\infty, S) \). Similarly let \( W^{1,2}_m(S, +\infty) \) be the subspace of \( W^{1,2}_m(R) \) of functions which vanish on \( (-\infty, S) \).

We state the following result

**Proposition 3.** There exists a solution of \( \text{(44)} \) in the functional space \( W^{1,2}_m(S, +\infty) \). Moreover

\[
-c - \frac{h}{\alpha} \leq H(x) \leq -c + \frac{p}{\alpha}
\]
Proof. We use a fixed point approach. Take a measurable function \( z(x) \) and set \( b(x) = \Phi'_c(z(x) + x) \). The function \( b(x) \) is measurable and bounded, with \( 0 \leq b(x) \leq \Phi'(c) \). We begin by solving

\[
- \frac{1}{2} \sigma^2 H''(x) + \alpha H(x) + b(x) H'(x) + (h + \alpha c) 1_{x>0} - (p - \alpha c) 1_{x<0} = 0, \quad x > S
\]

in the functional space \( W^{1,2}_{1,0}(S,+\infty) \). We claim that there is one and only one solution. This follows from the linearity of the equation. We first check the uniqueness, by considering

\[
- \frac{1}{2} \sigma^2 H''(x) + \alpha H(x) + b(x) H'(x) = 0, \quad x > S
\]

This solution is a solution of (49). We next claim that the map \( f \) defined by (49) is a contraction. Indeed, considering the problem (49) in a weak form, by multiplying the equation with a test function \( \varphi \) and integrating by parts. We obtain

\[
\frac{\sigma^2}{2} \int_S^{+\infty} F' \varphi' dx - \sigma^2 \int_S^{+\infty} \frac{xF' \varphi}{1 + x^2} dx + (\alpha + \lambda) \int_S^{+\infty} \frac{F \varphi}{1 + x^2} dx +
\]

\[
\int_S^{+\infty} \frac{b(x) F'(x) \varphi(x)}{1 + x^2} dx = -(h + \alpha c) \int_S^{+\infty} \varphi(x) \frac{dx}{1 + x^2} dx + \lambda \int_S^{+\infty} \frac{f(x) \varphi(x)}{1 + x^2} dx
\]

The left hand side is a bilinear form in the pair \( F, \varphi \) so a bilinear continuous form on the Hilbert space \( W^{1,2}_{1,0}(S,+\infty) \). Since \( \lambda \) is large, it is coercive. The right hand side is a linear form on \( \varphi \) in the same space. So the solution \( F \) exists and is unique. This solution is a solution of (49). We next claim that the map \( f \to F \) defined by (49) is a contraction. Indeed, considering the problem

\[
- \frac{1}{2} \sigma^2 F''(x) + (\alpha + \lambda) F(x) + b(x) F'(x) = \lambda f
\]
\[ F(S) = 0, \ F \text{ bounded} \]

then it is standard to check that
\[ -\frac{\lambda}{\alpha + \lambda} \| f \|_{L^\infty} \leq F(x) \leq \frac{\lambda}{\alpha + \lambda} \| f \|_{L^\infty} \]

The contraction mapping property follows immediately from this estimate. The fixed point of this map satisfies (48). Similarly we can consider
\[ -\frac{1}{2} \sigma^2 H_2''(x) + \alpha H_2(x) + b(x) H_2'(x) - (p - \alpha c) \mathbb{I}_{x < 0} = 0 \quad (51) \]

and the sum \( H(x) = H_1(x) + H_2(x) \) is a solution of (47). Let us check (46). From the equation (47) we see that the function cannot have a local maximum on \( x > 0 \). But it cannot have a local minimum either, because otherwise it will have a local maximum later on, since it cannot grow indefinitely, being bounded. Therefore the function can only decrease for \( x > 0 \). But the derivative cannot have a strictly negative accumulation point, since otherwise the function cannot remain bounded. So \( H'( + \infty) = 0 \). The limit \( H( + \infty) \) cannot be strictly larger than \(- c - \frac{h}{\alpha}\). Otherwise \( H''( + \infty) > 0 \), which is impossible. So we have proved that the solution of (47) satisfies (46). Also the norm of the solution \( H \) in the Hilbert space \( W^{1,2}_1(S, + \infty) \) can be bounded by a constant which does not depend of the particular function \( z(x) \), since the sup norm of \( b(x) \) is bounded by \( \Phi'(c) \), independently of \( z \).

Consider now the space \( L^1_2(S, + \infty) \) and the subset
\[ \mathcal{K} = \left\{ \varphi : \| \varphi \|_{W^{1,2}_1(S, + \infty)} \leq C, -c - \frac{h}{\alpha} \leq H(x) \leq -c - \frac{p}{\alpha} \right\} \]

where \( C \) is the constant which bounds the norm of \( H \) in \( W^{1,2}_1(S, + \infty) \), as indicated above. It is easy to check that \( \mathcal{K} \) is a compact subset of \( L^1_2(S, + \infty) \). If we call \( T(z) \) the map \( z \to H \), defined by (47), then we have shown that \( T(z) \in \mathcal{K} \). So, in particular \( T \) maps \( \mathcal{K} \) into itself. It is also to check that \( T \) is continuous from \( L^1_2(S, + \infty) \) to itself. The application of the Leray-Schauder theorem implies that \( T \) has a fixed point. Such a fixed point is solution of (44) and (45), (46) are satisfied. The proof has been completed.

5.2. Integral Equation. In this section, we are going to show that a solution of (44) satisfies an integral equation. We start with considering the 2nd order differential equation
\[ -\frac{1}{2} \sigma^2 \chi''(x) + \alpha \chi(x) = g(x), \ x > S \]
\[ \chi(S) = 0, \ \chi \text{ bounded} \]

with a right hand side \( g \) bounded. If we look at (44) we can consider that \( H \) satisfies (52) with
\[ g(x) = -(h + \alpha c) \mathbb{I}_{x > 0} + (p - \alpha c) \mathbb{I}_{x < 0} - \frac{d}{dx} \Phi(x)(H(x) + c) \]

Now we use the fact that we can write an explicit formula for \( \chi(x) \). Indeed, if we introduce \( \beta = \frac{\sqrt{2\alpha}}{\sigma} \) then it is easy to check that \( \chi(x) \) is given by the following formula
\[ \chi(x) = \frac{1}{\beta \sigma^2} \int_S^x g(\xi) \exp(-\beta(x - \xi))(1 - \exp(-2\beta(\xi - S)))d\xi \quad (53) \]
After some easy but tedious calculations, one obtains

\[ + (1 - \exp(-2\beta(x - S))) \int_x^{+\infty} g(\xi) \exp(-\beta(\xi - x)) d\xi \]

We apply then this formula for the \( g \) above. We first define

\[ L(x) = \frac{1}{\beta \sigma^2} \left[ \int_x^S ((p - \alpha c) \mathbb{1}_{\xi < 0} - (h + \alpha c) \mathbb{1}_{\xi > 0}) \exp(-\beta(x - \xi))(1 - \exp(-2\beta(x - S))) d\xi \right] \]

\[ + (1 - \exp(-2\beta(x - S))) \int_x^{+\infty} ((p - \alpha c) \mathbb{1}_{\xi < 0} - (h + \alpha c) \mathbb{1}_{\xi > 0}) \exp(-\beta(\xi - x)) d\xi \]

After some easy but tedious calculations, one obtains

\[ L(x) = \frac{1}{\beta^2 \sigma^2} [(p - \alpha c)(2 - 2 \exp(-\beta(x - S)) - \exp(\beta x) + \exp(-\beta(x - 2S)))] \]

\[ -(h + \alpha c)(2 - \exp(-\beta x(1 + \exp 2\beta S))), \ \forall x > 0 \]

and

\[ L(x) = \frac{1}{\beta^2 \sigma^2} [(p - \alpha c)(2 - 2 \exp(-\beta(x - S)) - \exp(\beta x) + \exp(-\beta(x - 2S)))] \]

\[ -(h + \alpha c)(2 - \exp(-\beta x(1 + \exp 2\beta S))), \ \forall x < 0 \]

So we have

\[ H(x) = L(x) - \frac{1}{\beta \sigma^2} \left[ \int_S^x \frac{d}{dx} \Phi_\epsilon(H(\xi) + c) (1 - \exp(-2\beta(x - S))) d\xi \right] \]

\[ + (1 - \exp(-2\beta(x - S))) \int_x^{+\infty} \frac{d}{dx} \Phi_\epsilon(H(\xi) + c) \exp(-\beta(\xi - x)) d\xi \]

After integration by parts, we obtain the formula

\[ H(x) = L(x) + \frac{1}{\beta \sigma^2} \left| \int_S^x \Phi_\epsilon(H(\xi) + c) \exp(-\beta(x - \xi))(1 + \exp(-2\beta(x - S))) d\xi \right| \]

\[ -(1 - \exp(-2\beta(x - S))) \int_x^{+\infty} \Phi_\epsilon(H(\xi) + c) \exp(-\beta(\xi - x)) d\xi \]

This is an integral equation, of which \( H \) is a solution.

5.3. Interval for \( S \). We first compute the derivatives

\[ L'(x) = -\frac{1}{\beta \sigma^2} [(p - \alpha c)(2 - 2 \exp(-\beta(x - S)) - \exp(\beta x) + \exp(-\beta(x - 2S)))] \]

\[ -(h + \alpha c)(2 - \exp(-\beta x(1 + \exp 2\beta S))), \ \forall x > 0 \]

and

\[ L'(x) = \frac{1}{\beta \sigma^2} [(p - \alpha c)(2 - 2 \exp(-\beta(x - S)) - \exp(\beta x) - \exp(-\beta(x - 2S)))] \]

\[ -(h + \alpha c)(2 - \exp(-\beta x(1 + \exp(-2\beta(x - S)))))], \ \forall x < 0 \]

We obtain next

\[ H'(x) = L'(x) + \frac{1}{\beta \sigma^2} \left[ -\beta \int_S^x \Phi_\epsilon(H(\xi) + c) \exp(-\beta(x - \xi))(1 + \exp(-2\beta(x - S))) d\xi \right] \]

\[ -(1 + \exp(-2\beta(x - S))) \int_x^{+\infty} \Phi_\epsilon(H(\xi) + c) \exp(-\beta(\xi - x)) d\xi + 2\Phi_\epsilon(H(x) + c) \]
We can then compute
\[ H'(S) = L'(S) + \frac{2}{\sigma^2}[\Phi(c + \epsilon) - \beta \int_0^{+\infty} \Phi((H(S + \xi) + c)^+) \exp(-\beta \xi) d\xi] \] (61)
We know that \( H(S + \xi) + c \leq \frac{p}{\alpha} \). Since \( \Phi \) is monotone increasing we can assert that
\[ \Phi((H(S + \xi) + c)^+) \leq \Phi\left(\frac{p}{\alpha} + \epsilon\right) \]
hence
\[ H'(S) \geq L'(S) + \frac{2}{\sigma^2}[\Phi(c + \epsilon) - \Phi\left(\frac{p}{\alpha} + \epsilon\right)] \]
and by concavity of \( \Phi \), we get
\[ H'(S) \geq L'(S) + \frac{2}{\sigma^2}[\Phi(c) - \Phi\left(\frac{p}{\alpha}\right)] \] (62)
and from (59) it follows (reinstating the parameters \( S, \epsilon \))
\[ H'_{\epsilon, S}(S) \geq \frac{2}{\sigma^2}\left[\frac{p - \alpha c - (h + p) \exp(\beta S)}{\beta} + \Phi(c) - \Phi\left(\frac{p}{\alpha}\right)\right] \] (63)
We note that the right hand side does not depend on \( \epsilon \). Consider now \( H_{0, \epsilon}(x) \), which corresponds to the choice \( S = 0 \), so \( x > 0 \). We know that \( H_{0, \epsilon}(x) \) can only decrease, so \( H'_{0, \epsilon}(x) \leq 0 \). We must have \( H'_{0, \epsilon}(0) < 0 \), otherwise \( H_{0, \epsilon}(0) > 0 \), which is impossible. We now make the assumption
\[ \frac{p - \alpha c}{\beta} + \Phi(c) - \Phi\left(\frac{p}{\alpha}\right) > 0 \] (64)
We can then define \( S^* < 0 \), such that
\[ \frac{p - \alpha c + \beta(\Phi(c) - \Phi\left(\frac{p}{\alpha}\right))}{p + h} = \exp(\beta S^*) \] (65)
From the estimate (63), we can assert that
\[ H'_{\epsilon, S}(S) > 0, \forall S < S^* \]
and we know that \( H'_{0, \epsilon}(0) < 0 \). Since we can check easily, from formula (61), that \( H'_{\epsilon, S}(S) \) is a continuous function of \( S \) (\( \epsilon \) fixed), there exists a value \( S_\epsilon \), such that
\[ H'_{\epsilon, S}(S_\epsilon) = 0, \ S^* \leq S_\epsilon < 0 \] (66)
5.4. **Main Result.** Our objective is to prove the existence of the pair \( S, H_S(x) \) solution of (19). We begin by collecting results. We set \( H^*(x) = H_{S_\epsilon, \epsilon}(x) \). So we have obtained a pair \( S_\epsilon, H^*(x) \) such that
\[ S^* \leq S_\epsilon < 0, \ H^*(x) \text{ is in } C^1 \cap W_2^{1,2}(R), \ H^*(x) = 0, \ \forall x \leq S_\epsilon \] (67)
\[ -\frac{1}{2}\sigma^2(x) + \alpha H^*(x) + \frac{d}{dx} \Phi_\epsilon(H^*(x) + c) + (h + \alpha c) I_{x > 0} - (p - \alpha c) I_{x < 0} = 0, \ x > S_\epsilon \]
with the properties
\[ -c - \frac{h}{\alpha} \leq H^*(x) \leq -c + \frac{p}{\alpha} \] (68)
\[ H^*(+\infty) = -c - \frac{h}{\alpha}, \ (H^*)'(+\infty) = 0 \]
In fact, we are going to prove additional properties, due to the fact that \( (H^*)'(S_\epsilon) = 0 \). We have
Lemma 5.1. The function $H^*(x)$ satisfies

$$H^*(x) < 0, \quad -\frac{2(h + \alpha c)}{\sigma \sqrt{\alpha}} \leq (H^*)'(x) < 0, \quad \forall x > S_\epsilon$$  \hspace{1cm} (69)

Proof. We first notice that $\frac{1}{2} \sigma^2 (H^*)''(x) = -(p - \alpha c)$. Therefore $(H^*)'(x)$ and $H^*(x)$ are strictly negative for $x > S_\epsilon$, close to $S_\epsilon$. But from the equation , $H^*(x)$ cannot have a local minimum on $(S_\epsilon,0)$ which is negative. So $(H^*)'(x) < 0, \quad H^*(x) < 0$, on $(S_\epsilon,0]$. For $x > 0$, the function can only decrease, as seen before. Therefore $H^*(x) < 0, \quad (H^*)'(x) < 0, \quad \forall x > S_\epsilon$. It remains to prove the left inequality for $(H^*)'(x)$ in (69). We multiply the differential equation (67) by $(H^*)'(x)$ and obtain

$$\frac{1}{4} \sigma^2 \frac{d}{dx}((H^*)'(x))^2 + \frac{\alpha}{2} \frac{d}{dx}(H^*(x))^2 + \Phi_\epsilon(H^*(x) + c)((H^*)'(x))^2 + \Phi_\epsilon(H^*(x) + c) = 0$$

We notice that $\frac{d}{dx}(H^*(x))^2 > 0$, hence

$$-\frac{1}{4} \sigma^2 \frac{d}{dx}((H^*)'(x))^2 \leq -(h + \alpha c)(H^*)'(x)\mathbb{I}_{x > 0}$$

and integrating between $x > S_\epsilon$ and $+\infty$, we obtain

$$\frac{1}{4} \sigma^2((H^*)'(x))^2 \leq -(h + \alpha c) \int_{x^+}^{+\infty} (H^*)'(\xi)d\xi \leq -(h + \alpha c)H^*(+\infty)$$

$$= \frac{(h + \alpha c)^2}{\alpha}$$

and the result follows immediately. \hfill \Box

By integrating the equation (67) between $S_\epsilon$ and $x$ we next obtain

$$-\frac{1}{2} \sigma^2 (H^*)'(x) + \alpha \int_{S_\epsilon}^{x} (H^*(\xi) + c)d\xi + \Phi_\epsilon(H^*(x) + c) - \Phi_\epsilon(c) + h x^+ + p(x^- + S_\epsilon) = 0$$

Using previous estimates, we get easily

$$-\Phi_\epsilon(H^*(x) + c) \leq h x^+ + \alpha c(x - S_\epsilon) + \frac{h + \alpha c}{\sqrt{\alpha}} - \Phi_\epsilon(c) \hspace{1cm} (70)$$

$\forall x > S_\epsilon$

We can now state the main result

Theorem 5.2. We assume (4), (3), (4), (22), (64). There exists a solution $S, H_S(x)$ of (19), and moreover

$$-\frac{2}{\sigma} \sqrt{c(h + \alpha c)} < H_S'(x) < 0$$  \hspace{1cm} (71)

We have $S^* \leq S \leq 0$. For each $S$, the solution $H_S(x)$ is uniquely defined. By taking the smallest $S$ in the interval $[S^*,0]$, we identify a single $S$.

Proof. We prove the existence, by considering the sequence $S_\epsilon, H^*(x)$ and letting $\epsilon \to 0$. We first extract a subsequence of $S_\epsilon$ such that $S_\epsilon \to S$, with $S^* \leq S \leq 0$. Define next a subset of $L_1^2(R)$ as follows

$$\Gamma = \{\varphi(x)| -\frac{h}{\alpha} - c \leq \varphi(x) \leq 0, \quad -\frac{2(h + \alpha c)}{\sigma \sqrt{\alpha}} \leq \varphi'(x) \leq 0\}$$
then $\Gamma$ is a compact subset of $L^2_1(R)$, and $H'(x)$ belongs to $\Gamma$. So we can extract a subsequence, still denoted $H'(x)$ such that

$$H'(x) \to H(x), \text{ pointwise}$$

$$(H')'(x) \to H'(x) \text{ weakly in } L^2_1(R)$$

and the limit $H(x)$ belongs to $\Gamma$. Consider $S - \delta, \delta > 0$ fixed. We can find $\epsilon(\delta)$ such that for $\epsilon < \epsilon(\delta)$, $S^\prime > S - \delta$. So for $x < S - \delta$, we have $H'(x) = 0$, for $\epsilon < \epsilon(\delta)$. Therefore, necessarily $H(x) = 0$, for $x < S - \delta$. But $H(x)$ is continuous and $\delta$ is arbitrarily small. This implies $H(x) = 0$, $\forall x \leq S$. Consider next $S + \delta$ and $\epsilon(\delta)$ such that for $\epsilon < \epsilon(\delta)$, $S^\prime < S + \delta$. We take $x > S + \delta$, the inequality (70) is valid for $\epsilon < \epsilon(\delta)$. Passing to the limit, we obtain

$$-\Phi((H(x) + c)^+) \leq hx^+ + \alpha c(x - S) + \frac{h + \alpha c}{\sqrt{\alpha}}\sigma - \Phi(c)$$

Since the right hand side is finite, necessarily $H(x) + c > 0$, for $x > S + \delta$. Since there is continuity at $S$, we have necessarily

$$H(x) + c > 0, \quad -\Phi(H(x) + c) \leq hx^+ + \alpha c(x - S) +$$

$$+ \frac{h + \alpha c}{\sqrt{\alpha}}\sigma - \Phi(c), \quad \forall x \geq S, \quad x < +\infty$$

Take again $x > S + \delta$, and $\epsilon < \epsilon(\delta)$ so that $S^\prime < S + \delta$. We can write

$$-\frac{1}{2} \alpha^2 (H')'(x) + \alpha \int_{S^\prime}^x H'(\xi)d\xi + \Phi(H'(x) + c) - \Phi(c)$$

$$+ (h + \alpha c)x^+ + (p - \alpha c)(x^- + S_{\epsilon}) = 0$$

from which we can assert that

$$(H')'(x) \to L(x), \quad \forall x > S + \delta$$

with

$$L(x) = \frac{2}{\sigma^2}[\alpha \int_{S^\prime}^x H(\xi)d\xi + \Phi(H(x) + c) - \Phi(c) +$$

$$+ (h + \alpha c)x^+ + (p - \alpha c)(x^- + S)]$$

If we take a function $\varphi(x)$ which is smooth with compact support on $(S + \delta, +\infty)$, we can in particular state that

$$\int_{-\infty}^{+\infty} (H')'(x)\varphi(x)dx \to \int_{-\infty}^{+\infty} L(x)\varphi(x)dx$$

On the other hand, since $(H')'(x)$ converges weakly in $L^2_1(R)$ towards $H'(x)$, we must have $H'(x) = L(x)$, on $(S + \delta, +\infty)$. Since $\delta$ is arbitrarily small, this equality takes place on $(S, +\infty)$. So we have for $x > S$

$$H'(x) = \frac{2}{\sigma^2}[\alpha \int_{S^\prime}^x H(\xi)d\xi + \Phi(H(x) + c) - \Phi(c) +$$

$$+ (h + \alpha c)x^+ + (p - \alpha c)(x^- + S)]$$

As $x \downarrow S$, we see immediately that $H'(S + 0) = 0$. However, similar reasonings show that $H'(S - 0) = 0$. Hence $H(x)$ is $C^1$ and $H(x) = H'(x) = 0$, $\forall x \leq S$. Now, as $x \to +\infty$, $H'(x) \to 0$, and $H(x) + c \downarrow \varrho \geq 0$. Necessarily $\varrho = 0$, otherwise $\Phi(\varrho)$ is finite, which leads to a contradiction, in view of equation (74). Of course by differentiating (74), we obtain the differential equation (19).
We thus have completed the proof of existence of a pair \((S, H_S(x))\) solution of (19). Note that the estimate on the right hand side of (71) is better than the bound derived from the appatenance of \(H(x)\) to \(\Gamma\). This is due to the fact that \(H(+\infty) = -c\), whereas \(H'(+\infty) = -c - \frac{h}{\alpha}\).

It remains to prove the uniqueness, once we have chosen \(S\). Suppose we have two solutions, denoted by \(H_1, H_2\). We set \(\bar{H} = H_1 - H_2\). We have

\[
\bar{H}(S) = 0, \quad \bar{H}'(S) = 0, \quad \bar{H}''(S) = 0, \quad \bar{H}'(+\infty) = 0
\]

and

\[
-\frac{1}{2} \frac{\sigma^2}{\alpha} \bar{H}''(x) + \alpha \bar{H}(x) + \Phi'(H_1(x) + c)\bar{H}'(x) + \\
+ (\Phi'(H_1(x) + c) - \Phi'(H_2(x) + c))(H_2'(x)) = 0
\]

Let us check that necessarily \(\bar{H}(x) \leq 0, \forall x > S\). Indeed, if \(\bar{H}(x)\) becomes strictly positive, it must have a local positive maximum, in view of the fact that \(\bar{H}'(+\infty) = 0\). If \(x^*\) is such a local maximum, then \(H_1'(x^*) > H_2'(x^*)\). By the concavity of \(\Phi\), it follows that \(\Phi'(H_1(x^*) + c) < \Phi'(H_2(x^*) + c)\), and since \((H_2')'(x) \leq 0\), we have

\[
(\Phi'(H_1(x^*) + c) - \Phi'(H_2(x^*) + c))(H_2'(x^*)) > 0
\]

but this leads to a contradiction with equation (75). Hence the assertion \(\bar{H}(x) \leq 0, \forall x > S\). But the role of \(H_1, H_2\) can be interchanged. So the opposite is also true. Hence \(H_1(x) = H_2(x)\). The proof has been completed.

6. Numerical Work.

6.1. Introduction. It is our objective to give numerical results of the theoretical conclusions found in this paper. We saw in (19) the need to implement the approximation problem considered in (44). This was due to \(\frac{d}{dx} \Phi\) being unbounded when \(H_S(x) \to -c\). We will apply similar techniques to reach our numerical conclusions.

In this chapter, we give a brief summary of the analysis used when studying our case study given by (5). We discuss how solving the epsilon problem enables us to solve the base problem. It is our objective to provide the pair \((S_\epsilon, H_\epsilon(x))\), and show that for decreasing values of \(\epsilon\) we have \(H_\epsilon(x) \to H_S(x)\) and \(S_\epsilon \to S\). By using our convergence results we can provide the solution pair \((S, H_S(x))\) which constitutes our optimal strategy.

The MATLAB boundary value solver (bvp5c), see [23], was used to achieve our numerical solutions.

6.2. Setup and Numerical Methodology.

6.2.1. Epsilon Problem. For the epsilon problem, when we apply our case study, we seek to find the solution pair \((S_\epsilon, H_\epsilon(x))\) to the following nonlinear two point boundary value problem

\[
H_\epsilon''(x) = 2 \sigma^2 (h + \alpha c) 1_{x > 0} - 2 \sigma^2 (p - \alpha c) 1_{x < 0} + 2 \alpha \sigma^2 H_\epsilon(x) \\
+ 2 \sigma^2 \left( \frac{\gamma}{\gamma + 1} \right)^{\gamma + 1} \left( \frac{1}{(H_\epsilon(x) + c)^{\gamma} + \epsilon} \right)^{\gamma + 1} H_\epsilon'(x)
\]

\[
H_\epsilon(S_\epsilon) = 0, \quad H_\epsilon'(S_\epsilon) = 0, \quad H_\epsilon(+\infty) = -c - \frac{h}{\alpha}
\]

(76)
6.2.2. Base Problem. For the base problem, when we apply our case study, we seek to find the solution pair \((S, H_S(x))\) to the following nonlinear two point boundary value problem

\[
H_S''(x) = \frac{2}{\sigma^2}(h + \alpha c)1_{x>0} - \frac{2}{\sigma^2}(p - \alpha c)1_{x<0} + \frac{2\alpha}{\sigma^2} H_S(x) + 2\frac{\alpha}{\sigma^2} \left(\frac{\gamma}{\gamma + 1}\right)^{\gamma+1} \left(\frac{1}{H_S(x) + c}\right)^{\gamma+1} H_S'(x)
\]

\[H_S(S) = 0, \quad H_S'(S) = 0, \quad -c < H_S(x) \leq 0, \quad H_S(+\infty) = -c \quad H_S(x) \in C^1 \quad (77)\]

6.2.3. Methodology for Solution. We solve the epsilon problem first and use convergence results to achieve the solution to the base problem. We will briefly describe how (76) was solved using the MATLAB solver bvp5c.

The boundary conditions \(H_\epsilon(S_\epsilon) = -c - \frac{h}{\alpha}\) are imposed, and by iterating on the condition \(H_\epsilon'(S_\epsilon) = 0\) we fix \(S_\epsilon\). We know from Section 5.3 that \(S_\epsilon \in [S^*, 0]\) and can therefore use a searching algorithm to find \(S_\epsilon\). First, we pick a random finite truncation point \(b\) and find the corresponding \(S_\epsilon\) such that (76) is satisfied on \([S_\epsilon, b]\). Next, we increase the value of \(b\) to establish a suitable value in which we have stability of our scheme, numerical efficiency and a solution curve that shows essential information. The solver uses an initial guess from which it modifies the mesh based on the residual. It is therefore important that we apply a good guess function as was noted by Gökhan in [17]. We start with a simple guess function and let the solver calculate a solution. To improve on our guess function we use this solution as the new guess function and reiterate until error tolerance is met. This is shown to be beneficial for small epsilon.

Below we see figures showing our searching scheme and the solution pair \((S_\epsilon, H_\epsilon(x))\) for various values of \(b\). We note that the value of \(S_\epsilon\) did not change when we increased the value of \(b\).

![Figure 1. Finding the value of S](image-url)
Figure 2. $H_e$ with value of $S_e$ for $b = 20$

Figure 3. $H_e$ with value of $S_e$ for $b = 30$

Figure 4. $H_e$ with value of $S_e$ for $b = 100$
Based on these figures we picked the finite truncation point to be $b = 30$.

6.3. Results. For the case study results, the following constants, $\gamma = 2$, $\alpha = 0.9$, $c = 3$, $p = 10$, $\sigma = 2$, $h = 1$ were used with the understanding that (64) had to be satisfied. For specific application purposes these constants can be modified to fit the company’s needs. The lower bound $S^*$ is given by

$$exp(\beta S^*) = \frac{p - \alpha c + \beta (\Phi(c) - \Phi(p))}{p + h}$$  \hspace{1cm} (78)

which yields $S^* = -\frac{1345}{2193}$.

Next, we show the structure of $H_\epsilon(x)$ for decreasing epsilon and the base solution. We locate the fast convergence, and when $\epsilon = 0.1$ we have a good approximation of the solution pair $(S, H_S(x))$.  

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{$H_\epsilon$ with value of $S_\epsilon$ for $b = 250$}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure6.png}
\caption{$H_\epsilon(x)$ for decreasing epsilon}
\end{figure}
The solution pair \((S, H_S(x))\) gives us the Base Stock value \(S\) and the associated pricing policy \(H_S(x)\). For our case study, the pricing strategy is given by \(\hat{\pi}(x) = \frac{2x+1}{7} (H_S(x) + c)\). We see that the optimal price is at \(x = S\). When \(x > S\) the price decreases, with the price limiting to zero when \(x \to \infty\).

7. Conclusion. In this paper, we have shown the Base Stock list price policy can be extended to the continuous time case. We provide numerical results, which confirm our theoretical findings as well as provide the solution pairs \((S, H_e(x))\) and \((S, H_S(x))\). From the solution pair \((S, H_S(x))\) we construct the optimal strategy which will provide maximum profit. Future projects include fixed cost case, lead time associated with replenishment, different demand structure and no backlog for the continuous time case.

REFERENCES

[1] G. Allon and A. Zeevi, A Note on the Relationship Among Capacity, Pricing and Inventory in a Make-to-Stock System, Production and Operations Management, 20 (2011), 143–151.
[2] K. J. Arrow, T. Harris and J. Marshak, Optimal inventory policy, Econometrica, 19 (1951), 250–272.
[2] J. A. Bather, A continuous time inventory model, *Journal of Applied Probability*, 3 (1966), 538–549.
[3] R. Bellman, *Dynamic Programming*, Dover Books on Computer Science, 2003.
[4] A. Bensoussan, *Dynamic Programming and Inventory Control*, IOS Press, Studies in Probability, Optimization and Statistics, 2011.
[5] C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers*, Springer Science & Business Media, 1999.
[6] A. Bensoussan and J. L. Lions, *Impulse Control and Quasi-Variational Inequalities*, Dunod, 1982.
[7] A. Bensoussan, R. H. Liu and S. Sethi, Optimality of and (s, S) policy with compound poisson and diffusion demands: A QVI approach, *SIAM J. Control Optim.*, 44 (2005), 1650–1676.
[8] A. Bensoussan and Y. Houmin, *Inventory Control with Pricing Optimization*, 2013.
[9] S. Browne and P. Zipkin, Inventory models with continuous, stochastic demands, *The Annals of Applied Probability*, 1 (1991), 419–435.
[10] X. Chen and D. Simchi-Levi, Pricing and inventory management, *The Oxford Handbook of Pricing Management*, eds. R. Phillips and O. Ozalp, Oxford University Press, (2012), 784–822.
[11] X. Chen and J. Zhang, Production control and supplier selection under demand, *Journal of Industrial Engineering and Management*, 3 (2010), 421–446.
[12] T. Dohi, N. Kaio and S. Osaki, A continuous time inventory control for wiener process demand, *Computers Math. Appl.*, 26 (1993), 11–22.
[13] A. Federgruen and A. Heching, Combined pricing and inventory control under uncertainty, *Operations Research*, 47 (1999), 454–475.
[14] Q. Feng, G. Gallego, S. Sethi, H. Yan and H. Zhang, Are base-stock policies optimal in inventory problems with multiple delivery modes?, *Operations Research*, 54 (2006), 801–807.
[15] L. Gimpl-Heersink, C. Rudloff, M. Fleischmann and A. Taudes, Integrating pricing and inventory control: Is it worth the effort?, *Business Research Official Open Access Journal of VHB*, 1 (2008), 106–123.
[16] E. L. Porteus, *Stochastic Inventory Theory*, in *Handbooks in O.R. and M.S.*, (eds. D. Heyman, M.J. Sobel), Elsevier, 2 (1990), 605–652.
[17] K. Sato and K. Sawaki, A continuous-time inventory model with procurement from spot market, *Journal of the Operations Research Society of Japan*, 53 (2010), 136–148.
[18] L. F. Shampine, I. Gladwell and S. Thompson, *Solving ODEs with MATLAB*, Cambridge University Press, 2013.
[19] T. M. Whitin, *Inventory control and price theory*, *Management Science*, 2 (1955), 61–68.
[32] R. Zhang, *An Introduction to Joint Pricing and Inventory Management under Stochastic Demand*, 2013.

Received January 2016; revised June 2016.

E-mail address: axb046100@utdallas.edu
E-mail address: sonny.skaaning@utdallas.edu