ON THE COX RING OF DEL PEZZO SURFACES

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Abstract. Let $S_r$ be the blow-up of $\mathbb{P}^2$ in $r$ general points, i.e., a smooth Del Pezzo surface of degree $9 - r$. For $r \leq 7$, we determine the quadratic equations defining its Cox ring explicitly. The ideal of the relations in $\text{Cox}(S_8)$ is calculated up to radical. As conjectured by Batyrev and Popov, all the generating relations are quadratic.

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1. INTRODUCTION

Over an algebraically closed field $\mathbb{K}$, the blow-up $S_r$ of $\mathbb{P}^2$ in $r$ points in general position\(^1\) ($1 \leq r \leq 8$) is a Del Pezzo surface of degree $d = 9 - r$. Its Picard group is a free $\mathbb{Z}$-module of rank $r + 1$.

Once we have chosen representatives $\mathcal{L}_0, \ldots, \mathcal{L}_r$ for a basis of $\text{Pic}(S_r)$, we can define its Cox ring, or total coordinate ring, as

$$\text{Cox}(S_r) := \bigoplus_{(\nu_0, \ldots, \nu_r) \in \mathbb{Z}^{r+1}} \Gamma(S_r, \mathcal{L}_0^{\otimes \nu_0} \otimes \cdots \otimes \mathcal{L}_r^{\otimes \nu_r}).$$

The multiplication of sections induces the multiplication in $\text{Cox}(S_r)$. The Cox ring is graded by $\text{Pic}(S_r)$ and is independent of the choice of the basis.

The intersection form is a non-degenerate bilinear form on $\text{Pic}(S_r)$. We will write it as $(D_1, D_2)$ for $D_1, D_2 \in \text{Pic}(S_r)$. (We will often use the same notation for divisors and their class in $\text{Pic}(S_r)$. It will be clear from the context what is meant.) A prime divisor $D$ whose self-intersection number $(D, D)$ is negative is called a negative curve. On smooth Del Pezzo surfaces, every negative curve has self-intersection number $-1$.

For $r \in \{3, \ldots, 7\}$, $\text{Cox}(S_r)$ is generated by non-zero sections of the $N_r$ negative curves ([BP04 Theorem 3.2]), see Table I for the values of $N_r$. For

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\(^1\)I.e., no three points on one line, no six points on a conic, no eight points with one of them a double point on a cubic.
\( r = 8 \), we must add two independent sections of \( \Gamma(S_8, -K_{S_8}) \). Let \( R_r \) be the free polynomial ring whose variables correspond to these generators of \( \text{Cox}(S_r) \). We want to determine the relations between these generators.

For \( r \leq 3 \), the Cox ring is a polynomial ring in \( r + 3 \) generators. This is due to the fact that in these cases, \( S_r \) is toric (see [Cox95] for Cox rings of toric varieties).

**Definition 1.** For \( n \geq 1 \), a divisor class \( D \) is called an \((n)\)-ruling if \( D = D_1 + D_2 \) for two negative curves \( D_1, D_2 \) whose intersection number \( (D_1, D_2) \) is \( n \geq 1 \). A \((1)\)-ruling is also called a ruling.

Each \((n)\)-ruling defines quadratic relations between generators of \( \text{Cox}(S_r) \), see Lemma 4. Relations coming from \((1)\)-rulings define an ideal \( I_r \subset R_r \). For \( r \in \{4, 5, 6\} \), \( \text{Cox}(S_r) = R_r / \text{rad}(I_r) \) by [BP04, Theorem 4.9]. We extend this result to \( r \in \{7, 8\} \) as follows:

**Theorem 2.** For \( r \in \{4, \ldots, 8\} \), we have \( \text{Cox}(S_r) = R_r / \text{rad}(J_r) \), where
- for \( r \in \{4, 5, 6\} \), \( J_r := I_r \);
- the ideal \( J_7 \) is generated by the 504 quadratic relations coming from the 126 rulings, and 25 quadratic relations coming from the \((2)\)-ruling \(-K_{S_7}\);
- the ideal \( J_8 \) is generated by the 10800 quadratic relations coming from the 2160 rulings, 6480 quadratic relations coming from 240 \((2)\)-rulings, and 119 quadratic relations coming from the \((3)\)-ruling \( 2 \cdot (-K_{S_8}) \).

It is known that the ideal \( I_4 \) is radical (see [BP04]). Batyrev proved that the same holds for \( I_5 \) (unpublished). Here we prove:

**Theorem 3.** For \( r \in \{4, \ldots, 7\} \), the ideals \( J_r \) are radical, and

\[
\text{Cox}(S_r) = R_r / J_r.
\]

It was conjectured by Batyrev and Popov that the ideal of relations defining \( \text{Cox}(S_r) \) is generated by quadrics for \( r \in \{4, \ldots, 8\} \), see [BP04, Conjecture 4.3]. To prove this conjecture, it now remains to show that \( J_8 \) is radical.

After recalling some general results on Del Pezzo surfaces in Section 2, we will handle the cases \( r \in \{6, 7, 8\} \) separately.

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## 2. Smooth Del Pezzo surfaces

In this section, we summarize some facts on smooth Del Pezzo surfaces.

- Let \( E_1, \ldots, E_r \) be the exceptional divisors of the blow-up of \( \mathbb{P}^2 \) in \( r \) points \( p_1, \ldots, p_r \) in general position. A basis of \( \text{Pic}(S_r) \) is given by (the classes of) \( H, E_1, \ldots, E_r \), where \( H \) is the pullback of the hyperplane section in \( \mathbb{P}^2 \).
In terms of this basis, the intersection form is given by a diagonal matrix of size \( r + 1 \) whose diagonal is \((1, -1, \ldots, -1)\). The anticanonical divisor is \(-K_{S_r} = 3H - (E_1 + \cdots + E_r)\).

The curves with self-intersection number \(-1\) are described in [BP04, Theorem 2.1]. There are no curves whose self-intersection is \(- \leq -2\).

The Weyl group \( W_r \) acting on \( \text{Pic}(S_r) \) depends on \( r \):

\[
\begin{array}{c|cccccccc}
 r & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
 W_r & A_1 & A_2 & A_2 + A_1 & A_4 & D_5 & E_6 & E_7 & E_8 \\
\end{array}
\]

For more details, see [BP04, Section 2].

As explained in the introduction, for \( r \leq 6 \), all relations in the Cox ring are induced by rulings, and these relations also play an important role for \( r \in \{7, 8\} \). More precisely, by the discussion following [BP04, Remark 4.7], each ruling is represented in \( r - 1 \) different ways as the sum of two negative curves, giving \( r - 3 \) linearly independent quadratic relation in \( \text{Cox}(S_r) \). Therefore, if each of the \( N_r \) negative curves intersects \( n_r \) negative curves with intersection number 1, we have \( N'_r = (N_r \cdot n_r) / 2 \) pairs, the number of rulings is \( N''_r = N'_r / (r - 1) \), and the number of quadratic relations coming from rulings is \( N''_r \cdot (r - 3) \) (see Table 1).

```
| r | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|
| \( N_r \) | 6 | 10 | 16 | 27 | 56 | 240 |
| \( n_r \) | 2 | 3 | 5 | 10 | 27 | 126 |
| \( N''_r \) | 3 | 5 | 10 | 27 | 126 | 2160 |
| relations | 0 | 5 | 20 | 81 | 504 | 10800 |
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Table 1. The number of relations coming from rulings.

Now we describe how to obtain explicit equations for \( \text{Cox}(S_r) \) and how to prove Theorem 2 and Theorem 3. We isolate the steps that must be carried out for each of the degrees 3, 2, and 1 and complete the proofs in the following sections.

**Choice of coordinates.** Choose coordinates for \( p_1, \ldots, p_r \in \mathbb{P}^2 \). We may assume that the first four points are

\[ (1) \quad p_1 = (1 : 0 : 0), \quad p_2 = (0 : 1 : 0), \quad p_3 = (0 : 0 : 1), \quad p_4 = (1 : 1 : 1). \]

By the general position requirement, the other points must have non-zero coordinates, and we can write \( p_j = (1 : \alpha_j : \beta_j) \) for \( j \in \{5, \ldots, r\} \).

**Curves in \( \mathbb{P}^2 \).** As explained in the introduction, \( \text{Cox}(S_r) \) is generated by sections of the negative curves for \( r \leq 7 \). For a negative curve \( D \), we denote the corresponding section by \( \xi(D) \), and for a generating section \( \xi \), let \( D(\xi) \) be the corresponding divisor. For \( r = 8 \), we need two further generators: linearly independent sections \( \kappa_1, \kappa_2 \) of \(-K_{S_8}\). Let \( K_1 := D(\kappa_1) \), \( K_2 := D(\kappa_2) \) be the corresponding divisors in the divisor class \(-K_{S_8}\).

Let \( D_r \) be the set of divisors corresponding to sections generating \( \text{Cox}(S_r) \) (including \( K_1, K_2 \) if \( r = 8 \)).

We need an explicit description of the image of each generator \( D \) of \( \text{Cox}(S_r) \) under the projection \( \pi : S_r \to \mathbb{P}^2 \). According to the seven cases
in [BP04, Theorem 2.1], \(\pi(D)\) can be a curve, determined by a form \(f_D\) of degree \(d \in \{1, \ldots, 6\}\), or a point (if \(D = E_i\)). If \(\pi(D)\) is a point, the convention to choose \(f_D\) as a non-zero constant will be useful later.

For \(r = 8\), we have the following situation: The image of \(K_i\) is a cubic through the eight points \(p_1, \ldots, p_8\). The choice of two linearly independent sections \(\kappa_1, \kappa_2\) corresponds to the choice of two independent cubic forms \(f_{\kappa_1}, f_{\kappa_2}\) vanishing in the eight points. Every cubic through these points has the form \(a_1 f_{\kappa_1} + a_2 f_{\kappa_2}\) where \((a_1, a_2) \neq (0, 0)\), and the cubic does not change if we replace \((a_1, a_2)\) be a non-zero multiple. This gives a one-dimensional projective space of cubics through the eight points.

Let \(X_1, \ldots, X_n\) be the monomials of degree \(d\) in three variables \(x_0, x_1, x_2\). For \(D \in D_r\), we can write

\[
f_D = \sum_{i=1}^{n} a_i \cdot X_i
\]

for suitable coefficients \(a_i\), which we can calculate in the following way: If \(p_j\) lies on \(\pi(D)\), this gives a linear condition on the coefficients \(a_i\) by substituting the coordinates of \(p_j\) for \(x_0, x_1, x_2\). If \(p_j\) is a double point of \(\pi(D)\), all partial derivatives of \(f_D\) must vanish at this point, giving three more linear conditions. If \(p_j\) is a triple point, we get six more linear conditions from the second derivatives. With \(p_1, \ldots, p_r\) in general position, we check that these conditions determine \(f_D\) uniquely up to a non-zero constant.

**Relations corresponding to \((n)\)-rulings.** Suppose that an \((n)\)-ruling \(D\) can be written as \(D_j + D_j'\) for \(k\) different pairs \(D_j, D_j' \in D_r\) where \(j \in \{1, \ldots, k\}\). Then the products

\[
f_{D_j} \cdot f_{D_j'}, \ldots, f_{D_k} \cdot f_{D_k'}
\]

are \(k\) homogeneous forms of the same degree \(d\), and they span a vector space of dimension \(n + 1\) in the space of homogeneous polynomials of degree \(d\). Therefore, there are \(k - (n + 1)\) independent relations between them, which we write as

\[
\sum_{j=1}^{k} a_{j,i} \cdot f_{D_j} \cdot f_{D_j'} = 0 \quad \text{for } i \in \{1, \ldots, k - (n + 1)\}.
\]

for suitable constants \(a_{j,i}\). They give an explicit description of the quadric relations coming from \(D\):

**Lemma 4.** In this situation, the \((n)\)-ruling \(D\) gives the following \(k - (n + 1)\) quadratic relations in \(\text{Cox}(S_r)\):

\[
q_i := \sum_{j=1}^{k} a_{j,i} \cdot \xi(D_j) \cdot \xi(D_j') = 0 \quad \text{for } i \in \{1, \ldots, k - (n + 1)\}.
\]

We will describe the \((n)\)-rulings in more detail in the subsequent sections.

Let \(J_r\) be the ideal in \(R_r\) which is generated by the \((n)\)-rulings (where \(n = 1\) for \(r \leq 6\), \(n \in \{1, 2\}\) for \(r = 7\), and \(n \in \{1, 2, 3\}\) for \(r = 8\)).

**The proof of Theorem 2.** For \(r \in \{4, 5, 6\}\), this is [BP04, Theorem 4.9]. For \(r \in \{7, 8\}\), we use a refinement of its proof.
Let $Z_r = \text{Spec}(R_r / \text{rad}(J_r)) \subset \text{Spec}(R_r)$. We want to prove that $Z_r$ equals $\mathbb{A}(S_r) \subset \text{Spec}(R_r)$, where $\mathbb{A}(S_r) := \text{Spec}(\text{Cox}(S_r))$. Obviously, $0 \in \text{Spec}(R_r)$ is contained in both $Z_r$ and $\mathbb{A}(S_r)$. Its complement $\text{Spec}(R_r) \setminus \{0\}$ is covered by the open sets

$$U_D := \{\xi(D) \neq 0\}, \quad \text{where } D \in D_r.$$  

In the case $r = 8$, we will show that it suffices to consider the sets $U_D$ for $D \in D_8 \setminus \{K_1, K_2\}$.

We want to show

$$Z_r \cap U_D \cong Z_{r-1} \times (\mathbb{A}^1 \setminus \{0\}).$$

Note that we can identify the negative curves $D_{r-1}$ of $S_{r-1}$ with the subset $D'_r$ of $D_r$ containing the negative curves which do not intersect $D$. We define

$$\psi : Z_r \cap U_D \to Z_{r-1} \times (\mathbb{A}^1 \setminus \{0\}) \quad \text{where } (\xi(D') \mid D' \in D_r) \mapsto ((\xi(D') \mid D' \in D_{r-1}), \xi(D)).$$

For $r \in \{7, 8\}$, we will prove:

**Lemma 5.** Every $\xi(D'')$ for $D'' \in D_r$ intersecting $D$ is determined by

$$\xi(D') \quad \text{and} \quad \{\xi(D') \mid D' \in D_r \quad \text{with} \quad (D', D) = 0\},$$

provided that $\xi(D) \neq 0$ and using the relations generating $J_r$.

By the proof of [BP04 Prop. 4.4],

$$\mathbb{A}(S_r) \cap U_D \cong \mathbb{A}(S_{r-1}) \times (\mathbb{A}^1 \setminus \{0\}).$$

By induction, $Z_{r-1} = \mathbb{A}(S_{r-1})$. Therefore, $Z_r \cap U_D = \mathbb{A}(S_r) \cap U_D$ for every negative curve $D$, which implies $Z_r = \mathbb{A}(S_r)$, completing the proof of Theorem 2 once Lemma 5 is proved.

**Proof of Theorem 3.** We want to show that the ideal $J_r$ is radical.

**Lemma 6.** The Hilbert polynomial of $S_r/J_r$ has degree $r + 2$.

For $r = 5$, this was proved by Batyrev. We will prove it for $r \in \{6, 7\}$.

**Remark 7.** The problem of calculating the Hilbert polynomial of $J_8$ seems out of reach of the current computer algebra packages. It is the only step missing in the proof of Theorem 3 for Del Pezzo surfaces of degree 1.

Under the condition of the proof of Lemma 5 the depth of $R_r/J_r$ is $r + 3$. As $\text{Spec}(R_r/J_r)$ is irreducible by [BP04], and $\text{Cox}(S_r) = \text{Spec}(R_r / \text{rad}(J_r))$ by Theorem 2 the $R_r$-module $R_r/J_r$ is Cohen-Macaulay. Therefore, we need to check the following claim in order to prove that the ideal $J_r$ is radical:

**Lemma 8.** $R_r/J_r$ has a smooth point.

3. Degree 3

We consider the case $r = 6$, i.e., smooth cubic surfaces. The set $D_6$ of negative curves on $S_6$ consists of the following 27 divisors:

- exceptional divisors $E_1, \ldots, E_6$, preimages of $p_1, \ldots, p_6 \in \mathbb{P}^2$,
- transforms $m_{i,j} = H - E_i - E_j$ of the 15 lines $m_{i,j}$ through the points $p_i, p_j$ ($i \neq j \in \{1, \ldots, 6\}$, and
Lemma 9. The extended Dynkin diagram of negative curves has the following structure:

1. It has 27 vertices corresponding to the 27 lines $E_i, m_{i,j}, Q_i$. Each of them has self-intersection number $-1$.
2. Every line intersects exactly 10 other lines: $E_i$ intersects $m_{i,j}$ and $Q_j$ (for $j \neq i$); $m_{i,j}$ intersects $E_i, E_j, Q_i, Q_j$ and $m_{k,l}$ (for $\{i,j\} \cap \{k,l\} = \emptyset$); $Q_i$ intersects $m_{i,j}$ and $E_j$ (for $j \neq i$). Correspondingly, there are 135 edges in the Dynkin diagram.
3. There are 45 triangles, i.e., triples of lines which intersect pairwise: 30 triples $E_i, m_{i,j}, Q_j$ and 15 triples $m_{i_1,j_1}, m_{i_2,j_2}, m_{i_3,j_3}$ where $\{i_1, j_1, i_2, j_2, i_3, j_3\} = \{1, \ldots, 6\}$. This corresponds to 45 triangles in the Dynkin diagram, where each edge is contained in exactly one of the triangles, and each vertex belongs to exactly five triangles.

Proof. We can check directly that $D + D' + D'' = -K_{S_6} - D$ if and only if $D, D', D''$ form a triangle in the sense of Lemma 9(3). There are five such pairs for any given $D$.

On the other hand, by Table 1 the total number of rulings is 27, and each ruling can be expressed in exactly five ways as the sum of two negative curves. □

Let $D$ be one of the 27 lines of $S_6$, and consider the projection $\psi_D : S_6 \rightarrow \mathbb{P}^2$ from $D$. Then

$$\psi_D^*(O_{\mathbb{P}^2}(1)) = -K_{S_6} - D = \begin{cases} H - E_i, & D = Q_i, \\ 2H - (E_1 + \cdots + E_6) + E_i + E_j, & D = m_{i,j}, \\ 3H - (E_1 + \cdots + E_6) - E_i, & D = E_i. \end{cases}$$

These are exactly the rulings.

A generating set of $\text{Cox}(S_6)$ is given by section $\eta_i, \mu_{i,j}, \lambda_i$ corresponding to the 27 lines $E_i, m_{i,j}, Q_i$, respectively. Let

$$R_6 := \mathbb{K}[\eta_i, \mu_{i,j}, \lambda_i].$$

The quadratic monomials in $\Gamma(S_6, -K_{S_6} - D)$ corresponding to the five ways to express $-K_{S_6} - D$ as the sum of the negative curves are:

- $\mu_{i,j} \eta_j$ if $D = Q_i$
- $\eta_i \lambda_j, \eta_j \lambda_i, \mu_{k_1,k_2} \mu_{k_3,k_4}$ if $D = \mu_{i,j}$ (with $\{i, j, k_1, \ldots, k_4\} = \{1, \ldots, 6\}$)
- $\mu_{i,j} \lambda_j$ if $D = E_i$
Lemma 4, we use the coordinates of \((1)\) for \(p_1, \ldots, p_4\), and
\[
p_5 = (1: a:b), \quad p_6 = (1: c:d).
\]

We write
\[
E := (b-1)(c-1) - (a-1)(d-1) \quad \text{and} \quad F := bc - ad
\]
for simplicity. The three relations corresponding to a line \(D\) will be denoted by \(q_D, q_D', q_D''\).

\[
\begin{align*}
q_{Q_1} &= -\eta_2\mu_{1,2} - \eta_3\mu_{1,3} + \eta_4\mu_{1,4} \\
q_{Q_1}' &= -a\eta_2\mu_{1,2} - b\eta_3\mu_{1,3} + \eta_5\mu_{1,5} \\
q_{Q_1}'' &= -c\eta_2\mu_{1,2} - d\eta_3\mu_{1,3} + \eta_6\mu_{1,6} \\
q_{Q_2} &= \eta_1\mu_{1,2} - \eta_3\mu_{2,3} + \eta_4\mu_{2,4} \\
q_{Q_2}' &= \eta_1\mu_{1,2} - b\eta_3\mu_{2,3} + \eta_5\mu_{2,5} \\
q_{Q_2}'' &= \eta_1\mu_{1,2} - d\eta_3\mu_{2,3} + \eta_6\mu_{2,6} \\
q_{Q_3} &= \eta_1\mu_{1,3} + \eta_2\mu_{2,3} + \eta_4\mu_{3,4} \\
q_{Q_3}' &= \eta_1\mu_{1,3} + a\eta_2\mu_{2,3} + \eta_5\mu_{3,5} \\
q_{Q_3}'' &= \eta_1\mu_{1,3} + c\eta_2\mu_{2,3} + \eta_6\mu_{3,6} \\
q_{Q_4} &= \eta_1\mu_{1,4} + \eta_2\mu_{2,4} + \eta_3\mu_{3,4} \\
q_{Q_4}' &= (1-b)\eta_1\mu_{1,4} + (a-b)\eta_2\mu_{2,4} + \eta_5\mu_{4,5} \\
q_{Q_4}'' &= (1-d)\eta_1\mu_{1,4} + (c-d)\eta_2\mu_{2,4} + \eta_6\mu_{4,6} \\
q_{Q_5} &= 1/b\eta_1\mu_{1,5} + a/b\eta_2\mu_{2,5} + \eta_3\mu_{3,5} \\
q_{Q_5}' &= (1-b)/b\eta_1\mu_{1,5} + (a-b)/b\eta_2\mu_{2,5} + \eta_4\mu_{4,5} \\
q_{Q_5}'' &= (b-d)/b\eta_1\mu_{1,5} + F/b\eta_2\mu_{2,5} + \eta_6\mu_{5,6} \\
q_{Q_6} &= 1/d\eta_1\mu_{1,6} + c/d\eta_2\mu_{2,6} + \eta_3\mu_{3,6} \\
q_{Q_6}' &= (1-d)/d\eta_1\mu_{1,6} + (c-d)/d\eta_2\mu_{2,6} + \eta_4\mu_{4,6} \\
q_{Q_6}'' &= (b-d)/d\eta_1\mu_{1,6} + F/d\eta_2\mu_{2,6} + \eta_5\mu_{5,6} \\
q_{m_{1,2}} &= \mu_{4,5}\mu_{3,6} - \mu_{3,5}\mu_{4,6} + \mu_{3,4}\mu_{5,6} \\
q_{m_{1,2}}' &= (b-d)\mu_{3,5}\mu_{4,6} + (d-1)\mu_{3,4}\mu_{5,6} + \eta_2\lambda_1 \\
q_{m_{1,2}}'' &= F\mu_{3,5}\mu_{4,6} + a(d-c)\mu_{3,4}\mu_{5,6} + \eta_1\lambda_2
\end{align*}
\]
\[ q_{m,1,3} = \mu_{4,5}\mu_{2,6} - \mu_{2,5}\mu_{4,6} + \mu_{2,4}\mu_{5,6} \]
\[ q'_{m,1,3} = (c-a)\mu_{2,5}\mu_{4,6} + (1-c)\mu_{2,4}\mu_{5,6} + \eta_3\lambda_1 \]
\[ q''_{m,1,3} = -F\mu_{2,5}\mu_{4,6} + b(c-d)\mu_{2,4}\mu_{5,6} + \eta_1\lambda_3 \]

\[ q_{m,2,3} = \mu_{4,5}\mu_{1,6} - \mu_{1,5}\mu_{4,6} + \mu_{1,4}\mu_{5,6} \]
\[ q'_{m,2,3} = (a-c)\mu_{1,5}\mu_{4,6} + a(c-1)\mu_{1,4}\mu_{5,6} + \eta_3\lambda_2 \]
\[ q''_{m,2,3} = (b-d)\mu_{1,5}\mu_{4,6} + b(d-1)\mu_{1,4}\mu_{5,6} + \eta_2\lambda_3 \]

\[ q_{m,1,4} = \mu_{3,5}\mu_{2,6} - \mu_{2,5}\mu_{3,6} + \mu_{2,3}\mu_{5,6} \]
\[ q'_{m,1,4} = -E\mu_{2,5}\mu_{3,6} + (b-1)(c-1)\mu_{2,3}\mu_{5,6} + \eta_4\lambda_1 \]
\[ q''_{m,1,4} = -F\mu_{2,5}\mu_{3,6} + bc\mu_{2,3}\mu_{5,6} + \eta_1\lambda_4 \]

\[ q_{m,2,4} = \mu_{3,5}\mu_{1,6} - \mu_{1,5}\mu_{3,6} + \mu_{1,3}\mu_{5,6} \]
\[ q'_{m,2,4} = E\mu_{1,5}\mu_{3,6} + (a-b)(c-1)\mu_{1,3}\mu_{5,6} + \eta_4\lambda_2 \]
\[ q''_{m,2,4} = (b-d)\mu_{1,5}\mu_{3,6} - b\mu_{1,3}\mu_{5,6} + \eta_2\lambda_4 \]

\[ q_{m,3,4} = \mu_{2,5}\mu_{1,6} - \mu_{1,5}\mu_{2,6} + \mu_{1,2}\mu_{5,6} \]
\[ q'_{m,3,4} = -E\mu_{1,5}\mu_{2,6} + (a-b)(1-d)\mu_{1,2}\mu_{5,6} + \eta_4\lambda_3 \]
\[ q''_{m,3,4} = (c-a)\mu_{1,5}\mu_{2,6} + a\mu_{1,2}\mu_{5,6} + \eta_3\lambda_4 \]

\[ q_{m,1,5} = \mu_{3,4}\mu_{2,6} - \mu_{2,4}\mu_{3,6} + \mu_{2,3}\mu_{4,6} \]
\[ q'_{m,1,5} = -E\mu_{2,4}\mu_{3,6} + (a-c)(1-b)\mu_{2,3}\mu_{4,6} + \eta_5\lambda_1 \]
\[ q''_{m,1,5} = (d-c)\mu_{2,4}\mu_{3,6} + c\mu_{2,3}\mu_{4,6} + \eta_1\lambda_5 \]

\[ q_{m,2,5} = \mu_{3,4}\mu_{1,6} - \mu_{1,4}\mu_{3,6} + \mu_{1,3}\mu_{4,6} \]
\[ q'_{m,2,5} = aE\mu_{1,4}\mu_{3,6} + (a-b)(c-a)\mu_{1,3}\mu_{4,6} + \eta_5\lambda_2 \]
\[ q''_{m,2,5} = (1-d)\mu_{1,4}\mu_{3,6} - \mu_{1,3}\mu_{4,6} + \eta_2\lambda_5 \]

\[ q_{m,3,5} = \mu_{2,4}\mu_{1,6} - \mu_{1,4}\mu_{2,6} + \mu_{1,2}\mu_{4,6} \]
\[ q'_{m,3,5} = -bE\mu_{1,4}\mu_{2,6} + (a-b)(b-d)\mu_{1,2}\mu_{4,6} + \eta_5\lambda_3 \]
\[ q''_{m,3,5} = (c-1)\mu_{1,4}\mu_{2,6} + \mu_{1,2}\mu_{4,6} + \eta_3\lambda_5 \]

\[ q_{m,4,5} = \mu_{2,3}\mu_{1,6} - \mu_{1,3}\mu_{2,6} + \mu_{1,2}\mu_{3,6} \]
\[ q'_{m,4,5} = b(c-a)\mu_{1,3}\mu_{2,6} + a(b-d)\mu_{1,2}\mu_{3,6} + \eta_5\lambda_4 \]
\[ q''_{m,4,5} = (c-1)\mu_{1,3}\mu_{2,6} + (1-d)\mu_{1,2}\mu_{3,6} + \eta_4\lambda_5 \]
\[ q_{m,1.6} = \mu_{3,4}\mu_{2,5} - \mu_{2,4}\mu_{3,5} + \mu_{2,3}\mu_{4,5} \]
\[ q_{m,1.6}^\prime = -E\mu_{2,4}\mu_{3,5} + (a - c)(1 - d)\mu_{2,3}\mu_{4,5} + \eta_6\lambda_1 \]
\[ q_{m,1.6}^{\prime\prime} = (b - a)\mu_{2,4}\mu_{3,5} + a\mu_{2,3}\mu_{4,5} + \eta_1\lambda_6 \]
\[ q_{m,2.6} = \mu_{3,4}\mu_{1,5} - \mu_{1,4}\mu_{3,5} + \mu_{1,3}\mu_{4,5} \]
\[ q_{m,2.6}^\prime = cE\mu_{1,4}\mu_{3,5} + (a - c)(d - c)\mu_{1,3}\mu_{4,5} + \eta_6\lambda_2 \]
\[ q_{m,2.6}^{\prime\prime} = (1 - b)\mu_{1,4}\mu_{3,5} - \mu_{1,3}\mu_{4,5} + \eta_2\lambda_6 \]
\[ q_{m,3.6} = \mu_{2,4}\mu_{1,5} - \mu_{1,4}\mu_{2,5} + \mu_{1,2}\mu_{4,5} \]
\[ q_{m,3.6}^\prime = -dE\mu_{1,4}\mu_{2,5} + (d - b)(d - c)\mu_{1,2}\mu_{4,5} + \eta_6\lambda_3 \]
\[ q_{m,3.6}^{\prime\prime} = (a - 1)\mu_{1,4}\mu_{2,5} + \mu_{1,2}\mu_{4,5} + \eta_3\lambda_6 \]
\[ q_{m,4.6} = \mu_{2,3}\mu_{1,5} - \mu_{1,3}\mu_{2,5} + \mu_{1,2}\mu_{3,5} \]
\[ q_{m,4.6}^\prime = d(c - a)\mu_{1,3}\mu_{2,5} + c(b - d)\mu_{1,2}\mu_{3,5} + \eta_6\lambda_4 \]
\[ q_{m,4.6}^{\prime\prime} = (a - 1)\mu_{1,3}\mu_{2,5} + (1 - b)\mu_{1,2}\mu_{3,5} + \eta_4\lambda_6 \]
\[ q_{m,5.6} = \mu_{2,3}\mu_{1,4} - \mu_{1,3}\mu_{2,4} + \mu_{1,2}\mu_{3,4} \]
\[ q_{m,5.6}^\prime = d(c - 1)\mu_{1,2}\mu_{3,4} + c(1 - d)\mu_{1,2}\mu_{3,4} + \eta_6\lambda_5 \]
\[ q_{m,5.6}^{\prime\prime} = b(a - 1)\mu_{1,3}\mu_{2,4} + a(1 - b)\mu_{1,2}\mu_{3,4} + \eta_5\lambda_6 \]

\[ q_{E_1} = (d - b)/E\mu_{1,2}\lambda_2 + (c - a)/E\mu_{1,3}\lambda_3 + \mu_{1,4}\lambda_4 \]
\[ q_{E_1}^\prime = (d - 1)/E\mu_{1,2}\lambda_2 + (c - 1)/E\mu_{1,3}\lambda_3 + \mu_{1,5}\lambda_5 \]
\[ q_{E_1}^{\prime\prime} = (b - 1)/E\mu_{1,2}\lambda_2 + (a - 1)/E\mu_{1,3}\lambda_3 + \mu_{1,6}\lambda_6 \]
\[ q_{E_2} = F/E\mu_{1,2}\lambda_1 + (c - a)/E\mu_{2,3}\lambda_3 + \mu_{2,4}\lambda_4 \]
\[ q_{E_2}^\prime = (c - d)/E\mu_{1,2}\lambda_1 + (c - 1)/E\mu_{2,3}\lambda_3 + \mu_{2,5}\lambda_5 \]
\[ q_{E_2}^{\prime\prime} = (a - b)/E\mu_{1,2}\lambda_1 + (a - 1)/E\mu_{2,3}\lambda_3 + \mu_{2,6}\lambda_6 \]
\[ q_{E_3} = F/E\mu_{1,3}\lambda_1 + (b - d)/E\mu_{2,3}\lambda_2 + \mu_{3,4}\lambda_4 \]
\[ q_{E_3}^\prime = (c - d)/E\mu_{1,3}\lambda_1 + (1 - d)/E\mu_{2,3}\lambda_2 + \mu_{3,5}\lambda_5 \]
\[ q_{E_3}^{\prime\prime} = (a - b)/E\mu_{1,3}\lambda_1 + (1 - b)/E\mu_{2,3}\lambda_2 + \mu_{3,6}\lambda_6 \]
\[ q_{E_4} = F/(a - c)\mu_{1,4}\lambda_1 + (b - d)/(a - c)\mu_{2,4}\lambda_2 + \mu_{3,4}\lambda_3 \]
\[ q_{E_4}^\prime = c/(a - c)\mu_{1,4}\lambda_1 + 1/(a - c)\mu_{2,4}\lambda_2 + \mu_{4,5}\lambda_5 \]
\[ q_{E_4}^{\prime\prime} = a/(a - c)\mu_{1,4}\lambda_1 + 1/(a - c)\mu_{2,4}\lambda_2 + \mu_{4,6}\lambda_6 \]
Its degree is $r = 8$. We calculate the Hilbert polynomial of $R_6/J_6$ over the field of fractions of the polynomial ring $\mathbb{Q}[a, b, c, d]$ using Magma:

$$h(t) = \frac{1}{8!} (372t^8 + 4464t^7 + 25200t^6 + 86184t^5 + 193788t^4$$

$$+ 291816t^3 + 284640t^2 + 161856t + 40320).$$

Its degree is $r = 2 = 8$ as required.

**Proof of Lemma 8.** The point $p$ with coordinates

$$(\eta_5, \eta_6, \mu_{1,2}, \mu_{1,4}, \mu_{2,3}, \mu_{3,4}, \lambda_5, \lambda_6) = (c(d - 1), a(b - 1), 1, -1, 1, 1, 1, 1),$$

and all other coordinates zero is a smooth point of $R_6/J_6$. Indeed, we check that $p$ fulfills all the relations generating $J_r$ (which is obvious for all of them except $q_{m5,6}, q'_{m5,6}, q''_{m5,6}$), and we calculate directly that the $81 \times 27$ Jacobian matrix has full rank 18 at this point.

4. Degree 2

Let $S_7$ be a smooth Del Pezzo surface of degree $d = 2$, i.e., the blow-up of $\mathbb{P}^2$ in $r = 7$ points. The set $D_7$ contains 56 negatives curves which are the transforms of the following curves in $\mathbb{P}^2$:

- 21 lines $m'_{i,j}$ through $p_i, p_j$,
- 21 conics $Q'_{i,j}$ through five of the seven points, missing $p_i, p_j$,
- 7 singular cubics $C'_i$ through all seven points, where $p_i$ is a double point, and

$$C_i = 3H - (E_1 + \cdots + E_7) - E_i.$$

The Cox ring $\text{Cox}(S_7)$ is generated by the sections $\eta_i, \mu_{i,j}, \nu_{i,j}, \lambda_i$ corresponding to the 56 negative curves $E_i, m_{i,j}, Q_{i,j}, C_i$, respectively. Let

$$R_7 := \mathbb{K}[\eta_i, \mu_{i,j}, \nu_{i,j}, \lambda_i]$$

be the polynomial ring in 56 generators.

Consider the ideal $I_7 \subset R_7$ generated by the quadratic relations corresponding to rulings. In view of Lemma 4, we need to know the six different ways to write each of the 126 rulings as a sum of two negative curves in order...
to describe $I_7$ explicitly. Here, we do not write the resulting 504 relations down because of the length of this list.

**Lemma 11.** Each of the 126 rulings can be written in six ways as a sum of two negative curves:

1. For the seven rulings $H - E_i$:
   \[ \{E_j + m_{i,j} | j \neq i\}. \]

2. For the 35 rulings $2H - (E_1 + \cdots + E_7) + E_i + E_j + E_k$:
   \[ \{E_i + Q_{j,k}, E_j + Q_{i,k}, E_k + Q_{i,j}, m_{i,j,l_2} + m_{l_3,l_4} | \{i,j,k,l_1,l_2,l_3,l_4\} = \{1,\ldots,7\}\}. \]

3. For the 42 rulings $3H - (E_1 + \cdots + E_7) + E_i - E_j$:
   \[ \{E_i + C_j, Q_{i,k} + m_{j,k} | k \neq i,j\}. \]

4. For the 35 rulings $4H - (E_1 + \cdots + E_7) - E_i - E_j - E_k$:
   \[ \{C_i + m_{j,k}, C_j + m_{i,k}, C_k + m_{i,j}, Q_{l_1,l_2} + Q_{l_3,l_4} | \{i,j,k,l_1,l_2,l_3,l_4\} = \{1,\ldots,7\}\}. \]

5. For the seven rulings $5H - 2(E_1 + \cdots + E_7) + E_i$:
   \[ \{C_j + Q_{i,j} | j \neq i\}. \]

However, we have more quadratic relations in $\text{Cox}(S_7)$: Note that the point $q$, with $\eta_1 = \lambda_1 = 1$ and other coordinates zero, satisfies the 504 relations. Indeed, $(E_1,C_1) = 2$, but all quadratic monomials which occur in the relations correspond to pairs of divisors whose intersection number is 1. Therefore, all these monomials and all the relations vanish in $q$. On the other hand, we check that the 504 × 56 Jacobian matrix has rank 54 in this point, which means that $q$ is contained in a component of the variety defined by $I_7$ which has dimension 2. As $\mathbb{A}(S_7)$ is irreducible of dimension 10, we must find other relations to exclude such components.

As $E_1 + C_1 = -K_{S_7}$, we look for more relations in degree $-K_{S_7}$ of $\text{Cox}(S_7)$: We check that in this degree, we have exactly 28 monomials:

\[ \{\eta_i \lambda_i | 1 \leq i \leq 7\} \cup \{\mu_{j,k} \nu_{j,k} | 1 \leq j < k \leq 7\}, \]

corresponding to $-K_{S_7} = E_i + C_i = m_{j,k} + Q_{j,k}$. As $\dim \Gamma(S_7, -K_{S_7}) = 3$, and as none of the relations coming from rulings induces a relation in this degree, we obtain 25 independent relations. Note that $-K_{S_7}$ is the unique (2)-ruling of $S_7$.

We can calculate the relations explicitly as they correspond to the relations between the polynomials $f_{E_i} \cdot f_{C_i}$ and $f_{m_{i,j}} \cdot f_{Q_{i,j}}$, which are homogeneous of degree 3, as described in Lemma 4.

Let $J_7$ be the ideal generated by these 529 relations.

**Proof of Lemma 5** In order to show that $\text{Cox}(S_7)$ is described by $\text{rad}(J_7)$, we must prove Lemma 5 in the case $r = 7$.

For any $D \in \mathcal{D}_7$, consider a coordinate $\xi(D')$ where $(D,D') = 1$. This is determined by the ruling $D + D'$. Indeed, this ruling induces a relation of the form

\[ \xi(D')\xi(D') = \sum a_i \xi(D_i)\xi(D'_i), \]

where $D_i + D'_i = D + D'$. Therefore,

\[ (D, D_1 + D'_1) = (D, D + D') = (D, D) + (D, D') = -1 + 1 = 0, \]
which implies \((D, D_i) = (D, D_i') = 0\) since the only negative curve intersecting \(D\) negatively is \(D\) itself. Since \(\xi(D) \neq 0\), the only unknown variable \(\xi(D')\) is determined by this relation.

Furthermore, there is exactly one coordinate \(\xi(D'')\) where \((D, D'') = 2\). The unique \((2)\)-ruling \(D + D'' = -K_{S_7}\) induces a relation of the form

\[\xi(D)\xi(D'') = \sum a_i\xi(D_i)\xi(D_i'),\]

where \(\xi(D'')\) is the only unknown variable.

**Proof of Lemma 6.** In a special case, we can calculate the Hilbert polynomial:

**Example 12.** Over the field \(\mathbb{F}_{101}\) with \(p_1, \ldots, p_4\) as in (11) and
\[p_5 = (1 : 2 : 3), \quad p_6 = (1 : 5 : 7), \quad p_7 = (1 : 13 : 17)\]

in general position, we can use Macaulay2 to calculate the Hilbert polynomial of \(J_7\) as

\[(2) \quad h(t) = \frac{1}{9!} \cdot (95049^9 + 85536t^8 + 412992t^7 + 1294272t^6 + 2860704t^5 + 4554144t^4 + 5125248t^3 + 3863808t^2 + 1752192t + 362880).\]

The Hilbert polynomial does not depend on the choice of the field or the points. Therefore, \(h(t)\) is the Hilbert polynomial of \(R_7/J_7\). Its degree is \(r + 2 = 9\).

5. **Degree 1**

In this section, we consider blow-ups of \(\mathbb{P}^2\) in \(r = 8\) points in general position, i.e., Del Pezzo surfaces \(S_8\) of degree 1.

The set \(D_8\) contains the transforms of the following 242 curves:

- Blow-ups \(E_1, \ldots, E_8\) of \(p_1, \ldots, p_8\);
- 28 lines \(m_{i,j}^{'}\) through \(p_i, p_j\):
  \[m_{i,j} = H - E_i - E_j;\]
- 56 conics \(Q_{i,j,k}^{'}\) through 5 points, missing \(p_i, p_j, p_k\):\n  \[Q_{i,j,k} = 2H - (E_1 + \cdots + E_8) + E_i + E_j + E_k;\]
- 56 cubics \(C_{i,j}^{'}\) through 7 points missing \(p_j\), where \(p_i\) is a double point:
  \[C_{i,j} = 3H - (E_1 + \cdots + E_8) - E_i + E_j;\]
- 56 quartics \(V_{i,j,k}^{'}\) through all points, where \(p_i, p_j, p_k\) are double points:
  \[V_{i,j,k} = 4H - (E_1 + \cdots + E_8) - (E_i + E_j + E_k);\]
- 28 quintics \(F_{i,j}^{'}\) through all points, where \(p_i, p_j\) are simple points and the other six are double points:
  \[F_{i,j} = 5H - 2(E_1 + \cdots + E_8) + E_i + E_j;\]
- 8 sextics \(T_{i}^{'}\), where \(p_i\) a triple point and the other seven points are double points:
  \[T_i = 6H - 2(E_1 + \cdots + E_8) - E_i;\]
two independent cubics $K'_1, K'_2$ through the eight points:

$$[K_1] = [K_2] = -K_{S_8} = 3 - (E_1 + \cdots + E_8).$$

The Cox ring of $S_8$ is generated by the 242 sections

$$\eta_i, \mu_{i,j}, \nu_{i,j,k}, \lambda_{i,j}, \phi_{i,j,k}, \psi_{i,j}, \sigma_i, \kappa_i$$

of $E_i, m_{i,j}, Q_{i,j,k}, C_{i,j}, V_{i,j,k}, F_{i,j}, T_i, K_i$, respectively.

**Lemma 13.** Each of the 2160 rulings can be expressed in the following seven ways as a sum of two negative curves:

- 8 rulings of the form $H - E_i$:
  $$\{E_j + m_{i,j} \mid j \neq i\}.$$

- $8 \cdot \binom{8}{4} = 70$ rulings of the form $2H - (E_i + E_j + E_k + E_l)$:
  $$\left\{ \begin{array}{l}
m_{i,j} + m_{k,l}, m_{i,k} + m_{j,l}, \\
m_{i,l} + m_{j,k}, E_a + Q_{b,c,d} \end{array} \right\} \{a, b, c, d, i, j, k, l\} = \{1, \ldots, 8\}.$$

- $8 \cdot \binom{8}{2} = 168$ rulings of the form $3H - (E_1 + \cdots + E_8) - E_j - E_k$
  $$\{E_j + C_{i,k}, E_k + C_{i,j}, m_{i,l} + Q_{j,k,l} \mid l \notin \{i, j, k\}\}.$$

- $8 \cdot \binom{8}{3} = 280$ rulings $4H - (E_1 + \cdots + E_8) + E_i - (E_j + E_k + E_l)$:
  $$\left\{ \begin{array}{l}
E_i + V_{j,k,l}, Q_{i,a,b} + Q_{i,c,d} \\\nC_{j,i} + m_{k,l}, C_{k,i} + m_{j,k}, C_{i,l} + m_{j,k} \end{array} \right\} \{a, b, c, d, i, j, k, l\} = \{1, \ldots, 8\},$$
  and 8 rulings of the form $4H - (E_1 + \cdots + E_8) - 2E_i$:
  $$\{m_{i,j} + C_{i,j} \mid j \neq i\}.$$  

- $8 \cdot 7 = 56$ rulings of the form $5H - 2(E_1 + \cdots + E_8) + 2E_i + E_j$:
  $$\{E_i + F_{i,j}, C_{i,k} + Q_{i,j,k} \mid k \notin \{i, j\}\},$$
  and $8 \cdot \binom{7}{3} = 280$ rulings $5H - (E_1 + \cdots + E_8) - 2E_i - (E_j + E_k + E_l)$:
  $$\left\{ \begin{array}{l}
m_{i,j} + V_{i,k,l}, m_{i,k} + V_{i,j,l} \\\nm_{i,l} + V_{i,j,k}, C_{i,a} + Q_{b,c,d} \end{array} \right\} \{a, b, c, d, i, j, k, l\} = \{1, \ldots, 8\}.$$

- $8 \cdot \binom{8}{2} \cdot \binom{6}{2} = 420$ rulings $6H - 2(E_1 + \cdots + E_8) - (E_i + E_j) + E_k + E_l$:
  $$\{m_{i,j} + F_{i,k,l} + V_{j,i,m} + Q_{k,l,m}, C_{i,k} + C_{j,l}, C_{i,l} + C_{j,k} \mid m \notin \{i, j, k, l\}\}.$$

- $8 \cdot 7 = 56$ rulings of the form $7H - 2(E_1 + \cdots + E_8) - 2E_i - E_j$:
  $$\{m_{i,j} + T_i, C_{i,k} + V_{i,j,k} \mid k \notin \{i, j\}\},$$
  and $8 \cdot \binom{7}{3} = 280$ rulings $7H - 3(E_1 + \cdots + E_8) + 2E_i + E_j + E_k + E_l$:
  $$\left\{ \begin{array}{l}
F_{i,j} + Q_{i,k,l}, F_{i,k} + Q_{i,j,l} \\\nF_{i,l} + Q_{i,j,k}, C_{a,i} + V_{b,c,d} \end{array} \right\} \{a, b, c, d, i, j, k, l\} = \{1, \ldots, 8\}.$$

...
There is no way to write a ruling as the sum of \( D(2) \)-rulings coming from rulings explicitly.

\[ \begin{align*}
&\text{and 8 rulings of the form } 8H \equiv 3(E_1 + \cdots + E_8) + E_i + E_j + E_k + E_l : \\
&\{ C_{i,j} + F_{k,l}, C_{i,k} + F_{j,l}, C_{i,l} + F_{j,k}, T_i + Q_{j,k,l}, V_{i,a,b} + V_{i,c,d} \mid \{a, b, c, d, i, j, k, l\} = \{1, \ldots, 8\} \},
\end{align*} \]

There is no way to write a ruling as the sum of \(-K_{S_8}\) and negative curves.

**Proof.** Because of the Weyl group symmetry, we need to prove the last statement only in one case, say \( H - E_1\). In this case, it is obvious.

By Table 1, there can be no other rulings, and each ruling can be expressed in no further ways as the sum of two negative curves.

With this information, Lemma 11 allows us to determine the 10800 relations coming from rulings explicitly.

We can find more quadratic relations in the degrees corresponding to \((2)\)-rulings: Because of the Weyl group symmetry, it is enough to consider the \((2)\)-ruling \( D := E_2 + C_{2,1} \). This can also be written as \( E_j + C_{j,1} \) for any \( j \neq 1 \) and as \( m_{i,j} + Q_{1,i,j} \) for any \( i, j \neq 1 \), giving 28 section in \( \Gamma(S_8, D) \).

As \( D = -K_{S_8} + E_1 \), we get two further section \( \eta_1 \kappa_1, \eta_1 \kappa_2 \). As the previous quadratic relations do not induce relations in this degree of \( \text{Cox}(S_8) \), and because we calculate \( \dim \Gamma(S_8, D) = 3 \) for this nef degree, we obtain 27 relations, which can be calculated explicitly as before.

Every negative curve has intersection number 2 with exactly 56 other curves (e.g. \( (E_1, D) = 2 \) if and only if \( D \in \{ C_{1,i}, V_{1,i,j}, F_{i,j}, T_i \} \) for \( i, j \neq 1 \)), so it occurs in exactly 56 \((2)\)-rulings. On the other hand, as every \((2)\)-ruling can be written in 28 ways as the sum of two negative curves, the total number of \((2)\)-rulings is \( \frac{240 \cdot 56}{28} = 240 \). Therefore, we obtain another 6480 relations from the \((2)\)-rulings. To determine them explicitly, we need the following more detailed information:

**Lemma 14.** Each of the 240 \((2)\)-rulings can be written as a sum of two negative curves in the following 28 ways:

- 8 \((2)\)-rulings of the form
  \[-K_{S_8} + E_i = 3H - (E_1 + \cdots + E_8) + E_i : \]
  \[\{ E_j + C_{j,i}, m_{j,k} + Q_{i,j,k} \mid j, k \neq i \}.\]
• \((\frac{8}{2}) = 28\) \((2)\)-rulings of the form 
\[-K_{S_8} + m_{i,j} = 4H - (E_1 + \cdots + E_8) - (E_i + E_j) :\]
\[
\begin{align*}
E_k + V_{i,j,k}, m_{i,k} + C_{j,k}, & \quad k \notin \{i, j\}, \\
m_{j,k} + C_{i,k}, Q_{a,b,c} + Q_{d,e,f} & \quad \{i, j, a, b, c, d, e, f\} = \{1, \ldots, 8\}.
\end{align*}
\]

• \((\frac{6}{3}) = 56\) \((2)\)-rulings of the form 
\[-K_{S_8} + Q_{i,j,k} = 5H - 2(E_1 + \cdots + E_8) + E_i + E_j + E_k :\]
\[
\begin{align*}
E_i + F_{j,k}, E_j + F_{i,k}, E_k + F_{i,j}, m_{a,b} + V_{c,d,e} & \quad \{i, j, k, a, b, c, d, e\} \\
Q_{i,j,l} + C_{i,k}, Q_{i,k,l} + C_{l,j}, Q_{j,k,l} + C_{l,i} & \quad = \{1, \ldots, 8\}, l \notin \{i, j, k\}.
\end{align*}
\]

• \((\frac{7}{2}) = 28\) \((2)\)-rulings of the form 
\[-K_{S_8} + T_i = 8H - 3(E_1 + \cdots + E_8) + E_i + E_j :\]
\[
\begin{align*}
S_k + Q_{i,j,k}, F_{i,k} + C_{k,j}, & \quad k \notin \{i, j\}, \\
F_{j,k} + C_{i,k}, V_{a,b,c} + V_{d,e,f} & \quad \{i, j, a, b, c, d, e, f\} = \{1, \ldots, 8\}.
\end{align*}
\]

Furthermore, the 242 generators give the 123 quadratic monomials
\[
\eta \sigma_i, \quad \mu_{i,j} \psi_{i,j}, \quad \nu_{i,j,k} \phi_{i,j,k}, \quad \lambda_{i,j} \lambda_{j,i}, \quad \kappa_1^2, \kappa_1 \kappa_2, \kappa_2^2
\]
in the 4-dimensional subspace \(\Gamma(S_8, 2 \cdot (-K_{S_8}))\) of \(\text{Cox}(S_8)\). Note that \(2 \cdot (-K_{S_8})\) is the unique \((3)\)-ruling. As the relations coming from rulings and \((2)\)-rulings do not induce relations in \(\Gamma(S_8, 2 \cdot (-K_{S_8}))\), we obtain another 119 relations. Their equations can be calculated in the same way as before.

**Lemma 15.** There are exactly 17399 independent quadratic relations in \(\text{Cox}(S_8)\).

**Proof.** The relations in \(\text{Cox}(S_8)\) are generated by relations which are homogeneous with respect to the \(\text{Pic}(S_8)\)-grading. A quadratic relation involving a term \(\xi(D_1)\xi(D_2)\) has degree \(D = D_1 + D_2\). The relations of degree \(D_1 + D_2\) depend on the intersection number \(n = (D_1, D_2)\):

- If \(n = 1\), then \(D\) is a \((1)\)-ruling. As described above, we have exactly 10800 corresponding relations.
- If \(n = 2\), then \(D\) is a \((2)\)-ruling. We have described the 6480 resulting relations.
• If \( n = 3 \), then \( D = 2 \cdot (-K_{S_8}) \), which results in exactly 119 quadratic relations.

• If \( n = 0 \), then \( D = D_1 + D_2 \) is not nef since \( (D, D_1) = -1 \). However, by results of [HT04, Section 3], the relations in \( \text{Cox}(S_8) \) are generated by relations in nef degrees.

• If \( n = -1 \), then \( D_1 = D_2 \), and \( (D, D_1) = -2 \), so \( D \) is not nef, giving no generating relations as before.

There are no other quadratic relations involving \( \kappa_i \) because the 240 degrees \(-K_{S_8} + D_1 \) for some negative curve \( D_1 \) are exactly the (2)-rulings, and the degree \( 2 \cdot (-K_{S_8}) \) has also been considered. \( \square \)

Let \( J_8 \) be the ideal generated by these 17399 quadratic relations in

\[
R_8 = \mathbb{K}[\eta_i, \mu_{i,j}, \nu_{i,j,k}, \lambda_{i,j}, \phi_{i,j,k}, \psi_{i,j}, \sigma_i, \kappa_i].
\]

**Proof of Lemma 5.** Let \( D \in \mathcal{D}_8 \setminus \{K_1, K_2\} \) be any negative curve. We call a variable \( \xi(D') \) for a negative curve \( D' \in \mathcal{D}_8 \) an \((n)\)-variable if \( (D, D') = n \).

As for \( r = 7 \) in the previous section, we show that the rulings determine the \((1)\)-variables in terms of the \((0)\)-variables and \( \xi(D) \neq 0 \).

For the two variables \( \kappa_i = \xi(K_i) \) corresponding to \(-K_{S_8}\), we use the \((2)\)-ruling \(-K_{S_8} + D\): As \( (D, -K_{S_8} + D) = 0 \), we have \( (D, D_i) = (D, D'_i) = 0 \) for any other possibility to write \(-K_{S_8} + D\) as the sum of two negative curves \( D_i, D'_i \). Since \(-K_{S_8} + D, -K_{S_8}\) = 2, by [BP04, Prop. 3.4], the quadratic monomials \( \xi(D_i)\xi(D'_i) \) span \( \Gamma(S_8, -K_{S_8} + D) \), so this \((2)\)-ruling induces relations of the form

\[
\kappa_i \xi(D) = \sum a_i \xi(D_i) \xi(D'_i).
\]

Therefore, \( \kappa_1, \kappa_2 \) are determined by \( \xi(D) \) and the \((0)\)-variables.

Any \((2)\)-coordinate \( \xi(D') \) is determined by the \((2)\)-ruling \( D + D' \): As \( (D, D + D') = 1 \), we have \( (D, D_i) = 0 \) and \( (D, D'_i) = 1 \) for every other possibility to write \( D + D' \) as the sum of two negative curves \( D_i, D'_i \). Furthermore, if \( D + D' = -K_{S_8} + D'' \), then \( (D, D'') = 0 \). Therefore, the relations corresponding to this \((2)\)-ruling determine \( \xi(D') \) in terms of the \((0)\)- and \((1)\)-variables and \( \kappa_1, \kappa_2, \xi(D) \).

Finally, there is a unique \((3)\)-coordinate \( D' \), where \( D + D' = 2 \cdot (-K_{S_8}) \) is the \((3)\)-ruling. As all other variables are known at this point, the relations corresponding to \( 2 \cdot (-K_{S_8}) \) is spanned by the monomials \( \xi(D_i)\xi(D'_i) \) for \((3)\)-rulings \( D_i, D'_i \). Therefore, we have relations of the form

\[
\kappa_j^2 = \sum a_i \xi(D_i) \xi(D'_i),
\]

which shows that \( \xi(D_i) \neq 0 \) for some \( i \). This proves that \( Z_8 \setminus \{0\} \) is covered by the sets \( U_D \) for \( D \in \mathcal{D}_8 \setminus \{K_1, K_2\} \).
Proof of Lemma 8. Let $p \in \text{Spec } R_8$ be the point whose coordinates are zero, except that $\eta_8, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7, \mu_8$ are 1 and
\[
(\eta_1, \eta_2, \eta_4, \eta_5, \eta_6) = \left( \frac{\alpha_3 \alpha_4}{\alpha}, \frac{\alpha_4}{\alpha}, \frac{(1 - \alpha_3) \alpha_4}{\alpha}, \frac{(\alpha_1 - \alpha_3) \alpha_4}{\alpha}, \frac{(\alpha_2 - \alpha_3) \alpha_4}{\alpha}, \frac{\alpha_3}{\alpha} \right),
\]
where $\alpha := \alpha_4 - \alpha_3$. This point satisfies the five equations corresponding to the ruling $H - E_3$:

\[
\begin{align*}
\eta_1 \mu_1 & - \frac{\alpha_3 \alpha_4}{\alpha_3 - \alpha_4} \eta_1 \mu_3,7 + \frac{\alpha_3 \alpha_4}{\alpha_3 - \alpha_4} \eta_8 \mu_3,8 = 0, \\
\eta_2 \mu_2 & - \frac{\alpha_3}{\alpha_3 - \alpha_4} \eta_2 \mu_3,7 + \frac{\alpha_4}{\alpha_3 - \alpha_4} \eta_8 \mu_3,8 = 0, \\
\eta_4 \mu_4 & + \frac{\alpha_3 \alpha_4 - \alpha_3}{\alpha_3 - \alpha_4} \eta_4 \mu_3,7 + \frac{-\alpha_3 \alpha_4 + \alpha_4}{\alpha_3 - \alpha_4} \eta_8 \mu_3,8 = 0, \\
\eta_5 \mu_5 & + \frac{-\alpha_1 \alpha_3 + \alpha_3 \alpha_4}{\alpha_1 \alpha_3 - \alpha_1 \alpha_4} \eta_5 \mu_3,7 + \frac{\alpha_1 \alpha_4 - \alpha_3 \alpha_4}{\alpha_1 \alpha_3 - \alpha_1 \alpha_4} \eta_8 \mu_3,8 = 0, \\
\eta_6 \mu_6 & + \frac{-\alpha_2 \alpha_3 + \alpha_3 \alpha_4}{\alpha_2 \alpha_3 - \alpha_2 \alpha_4} \eta_6 \mu_3,7 + \frac{\alpha_2 \alpha_4 - \alpha_3 \alpha_4}{\alpha_2 \alpha_3 - \alpha_2 \alpha_4} \eta_8 \mu_3,8 = 0.
\end{align*}
\]

Consider intersection numbers between the negative curves corresponding to the twelve non-zero coordinates. They are zero except for the six pairs corresponding to the ruling $H - E_3$. Therefore, no pair of non-zero coordinates occurs in relations corresponding to other $(n)$-rulings, which shows that $p \in \mathcal{A}(S_8)$. We check directly that the Jacobian in $p$ has full rank 231.

References

[BP04] Victor V. Batyrev and Oleg N. Popov. The Cox ring of a del Pezzo surface. In Arithmetic of higher-dimensional algebraic varieties (Palo Alto, CA, 2002), volume 226 of Progr. Math., pages 85–103. Birkhäuser Boston, Boston, MA, 2004.

[Cox95] David A. Cox. The homogeneous coordinate ring of a toric variety. J. Algebraic Geom., 4(1):17–50, 1995.

[HT04] Brendan Hassett and Yuri Tschinkel. Universal torsors and Cox rings. In Arithmetic of higher-dimensional algebraic varieties (Palo Alto, CA, 2002), volume 226 of Progr. Math., pages 149–173. Birkhäuser Boston, Boston, MA, 2004.

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