Spontaneous symmetry breaking in loop quantum gravity

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Abstract
In this paper we investigate the question of how spontaneous symmetry breaking works in the framework of loop quantum gravity and compare it to the results obtained in the case of the Proca field, where we were able to quantize the theory in loop quantum gravity without introducing a Higgs field. We obtained that the Hamiltonian of the two systems is very similar, the only difference is an extra scalar field in the case of spontaneous symmetry breaking. This field can be identified as the field that carries the mass of the vector field. In the quantum regime this becomes a well-defined operator, which turns out to be a self-adjoint operator with continuous spectrum. To calculate the spectrum we used a new representation in the case of scalar fields, which in addition enabled us to rewrite the constraint equations to a finite system of linear partial differential equations. This made it possible to solve part of the constraints explicitly.

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1. Introduction

Spontaneous symmetry breaking is currently the most accepted tool to define mass of particles. Its success can be observed especially in the case of vector fields since their original Lagrangian—the Proca Lagrangian—is non-renormalizable. In [1] we showed how one can quantize the massive vector field in loop quantum gravity (LQG) without spontaneous symmetry breaking. The main problem was that the Proca field had a second-class constraint algebra which made it almost impossible to apply the framework of LQG. But with the help of symplectical embedding one could eliminate these difficulties. Now the question arises of what is the difference between the two theories. To study this one first has to apply the framework of LQG to a system where spontaneous symmetry breaking is used to generate mass for a $U(1)$ vector field. This is done in sections 2 (classical theory and 3+1 decomposition)
and 3 (regularization and quantization). In section 4, we introduce a new basis for scalar fields which is motivated by the fact that these are eigenstates of the configuration variables. It turns out that with the help of this new basis we are able to (partially) solve the constraints of the theory—this is done in section 5. Here we also analyze special solutions in order to understand the role of the scalar field. In particular we find that some of these are almost identical to the solutions obtained for the Proca field, thus we are able to relate the two theories. Section 6 deals with the ‘mass operator’ and its properties, concentrating especially on those cases where the eigenvalues of this operator can be identified with the mass parameter of the Proca field. We obtain that in a sense the case of the symmetry breaking is a linear combination of infinite Proca theories.

2. Classical theory

In this section we will analyze the general framework of spontaneous symmetry breaking from a Hamiltonian perspective. In the first subsection we derive the 3+1 decomposition of the theory, while in the second we first decompose the scalar field to its absolute value and argument and then perform the decomposition (this is useful because the similarities between the Proca field and spontaneous symmetry breaking become more transparent).

2.1. Symmetric theory

For simplicity we use the Lagrangian of a $U(1)$ vector field (electromagnetic field) coupled to gravity and a $U(1)$ complex scalar field on a spacetime manifold $M$. The Lagrangian of the matter part is

$$L^{\text{mat}} = \int_M d^4x \sqrt{-g} \left( -\frac{1}{4} F_{ab}^{(4)} E^a E^b - \frac{1}{2} D^a \Phi^* D^a \Phi - \frac{1}{4} \mu (\Phi^* \Phi - a^2)^2 \right),$$

(1)

where

$$\Phi = \text{Re} \Phi + i \text{Im} \Phi, \quad D^a \Phi = (\partial^a + i e A^a) \Phi$$

and $\mu$ and $a$ are positive constants. To distinguish between the electromagnetic field and the gravitational field the variables of the former are underlined.

First, let us make the 3+1 decomposition. Introduce on the spacetime manifold $M$ a smooth function $t$ whose gradient is nowhere vanishing and a vector field $t^a$ with affine parameter $t$ satisfying $t^a \nabla_a t = 1$. This gives a foliation of spacetime, i.e. each $t$ defines a three-dimensional hypersurface $\Sigma_t$. Let us decompose $t^a$ into its normal and tangential parts

$$t^a = N n_a + N_a,$$

(2)

where $n_a$ is the unit normal of the hypersurface $\Sigma_t$, $N$ is the lapse function and $N_a$ is the shift vector. Define the induced, positive-definite metric on $\Sigma_t$ via

$$q_{ab} = g_{ab} + n_a n_b.$$  

(3)

As was done in [1, 2] we define the pull-backs $A^a = q^a_b A^b_t$, $D_a = q^b_a D^b_t$ and define $A_0 = t^a A^a_t$, $A_0^t = t^a A^a_t$. Substituting these into the Lagrangian one obtains

$$L^{\text{mat}} = \int dt \int d^3x \left( \frac{N}{\sqrt{q}} q_{ab} \frac{E^a E^b - B^a B^b}{2} - \frac{1}{4} N \sqrt{q} \mu (\Phi^* \Phi - a^2)^2 - \frac{1}{2} N \sqrt{q} \right.$$  

$$\times \left[ q^{cd} D_c \Phi^* D_d \Phi - (L_c \Phi^* - i e A^0 \Phi^* - N_c(D_0 \Phi^*))(L_c \Phi + i e A^0 \Phi - N_c D_0 \Phi) \right].$$

(4)
where

\[ E_a = \sqrt{q} \left( \mathcal{L}_a \Delta_a - D_a \Delta^0 - \epsilon_{abc} B^b N^c \right) \]  

is the electric field and \( B_a \) is the magnetic field. We now define the canonical momenta

\[ \Pi^a = \frac{\delta \mathcal{L}}{\delta \dot{A}_a} = E_a, \]

\[ \pi = \frac{\delta \mathcal{L}}{\delta \dot{\Phi}} = \sqrt{q} \frac{\mathcal{L}_a \Phi^* - i e A^b \Phi^* - N^a (D_a \Phi)^*}{N}, \]

\[ \pi^* = \frac{\delta \mathcal{L}}{\delta \dot{\Phi}^*} = \sqrt{q} \frac{\mathcal{L}_a \Phi^* + i e A^b \Phi^* - N^b D_a \Phi}{N}. \]

Finally, we perform the Legendre transformation to arrive at the Hamiltonian

\[ H_{\text{mat}} = \int d^3 x \left( N \mathcal{H}_{\text{mat}}^a + N^a \mathcal{H}_{\text{mat}}^a + A_0 G \right), \]

\[ \mathcal{H}_{\text{mat}}^a = q_{ab} \left( E^a E^b + B^a B^b \right) + \frac{\pi^* \pi}{2} \left( \frac{d}{N} \right)^2 + \frac{1}{2} \sqrt{q} N \left( D_a \Phi \right)^* D_a \Phi + \frac{1}{4} \sqrt{q} \left( \Phi^* \Phi - a^2 \right)^2, \]

\[ G = D_a E_a + i e (\pi^* \Phi^* - \pi \Phi), \]

where \( \mathcal{H}_{\text{mat}}^a, \mathcal{H}_{\text{mat}}^a \) and \( G \) are the matter contributions to the Hamiltonian and diffeomorphism (modulo gauge transformations) constraints, and the electromagnetic Gauss constraint.

The (non-smeared, non-trivial) Poisson brackets are

\[ \{ \pi, \Phi \} = \delta(x, y), \]

\[ \{ \pi^*, \Phi^* \} = \delta(x, y), \]

\[ \{ E_a, A^b \} = \delta^b_a \delta(x, y). \]

Since we are going to do symmetry breaking with the help of the scalar field, we write here the transformation rules for the scalar fields and their canonical momenta with respect to the infinitesimal gauge transformation:

\[ \{ \tilde{G}(\Lambda), \Phi \} = -i e \Delta \Phi, \]

\[ \{ \tilde{G}(\Lambda), \Phi^* \} = i e \Delta \Phi^*, \]

\[ \{ \tilde{G}(\Lambda), \pi \} = i e \Lambda \pi, \]

\[ \{ \tilde{G}(\Lambda), \pi^* \} = -i e \Lambda \pi^*. \]

Before we continue, there are a few interesting observations that should be mentioned here:

- All the constraints are real and only the scalar fields and their canonical momenta are represented by complex variables (note that in the Hamiltonian picture \( \Phi \) and \( \Phi^* \) are independent variables).
- The transformations \( \Phi \leftrightarrow \Phi^*, \pi \leftrightarrow \pi^* \) are canonical transformations.
- The true diffeomorphism constraint \( \mathcal{H}_{\text{mat}}^a + A_0 G \) is independent of the coupling constant \( e \) (it contains partial derivatives only).
- This system has a first-class constraint algebra, furthermore all the components of \( \mathcal{H}_{\text{mat}} \) are gauge invariant, respectively.
2.2. New variables

In spontaneous symmetry breaking first we introduce new fields $\eta$ and $\Theta$ in the following way:

$$\Phi(x) := (a + \eta(x)) \exp\left(\frac{i\Theta(x)}{a}\right).$$  \hfill (15)

These variables are useful because the $U(1)$ symmetry of the theory becomes more transparent. If we substitute this into the Lagrangian, we obtain

$$L_{\text{mat}} = \int d^4x \sqrt{\mathcal{g}} \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \left( \frac{\partial^{(4)}\Theta}{a} + eA_\mu \right) \left( \frac{\partial^{(4)}\Theta}{a} + eA_\mu \right) - \frac{1}{4} \mu \eta^2 (2a + \eta)^2 \right].$$  \hfill (16)

If we compare this with the action of the symplectically embedded Proca field we immediately recognize the similarities between the two theories. The main difference is that where the Proca theory had a parameter ($m$), now we have a field $(\eta + a)$. We wish to see what the Hamiltonian looks like in terms of the new variables. To do this, first we do the 3+1 decomposition of the above Lagrangian. Repeating the steps of the previous section first we define the canonical momenta

$$\Pi^a = \frac{\delta L}{\delta \dot{A}_a}, \quad \frac{\pi_\Theta}{\delta \dot{\Theta}} = \frac{E_a}{a},$$  \hfill (17)

$$\pi_\eta = \frac{\delta L}{\delta \dot{\eta}} = \sqrt{q} \left( \frac{\mathcal{L}_\eta - N^a \partial_a \eta}{N} \right),$$  \hfill (18)

$$\pi_\Theta = \frac{\delta L}{\delta \dot{\Theta}} = \left( \frac{a + \eta}{a} \right)^2 \sqrt{q} \left( \frac{\mathcal{L}_\Theta - N^a \partial_a \Theta + aeA_0 - aeN^a A_a}{N} \right)$$  \hfill (19)

and after the Legendre transformation we obtain the Hamiltonian

$$H_{\text{mat}} = \int d^3x (N\mathcal{H}_{\text{mat}} + N^a H_{a}^{\text{mat}} + A_0 G).$$  \hfill (20)

$$\mathcal{H}_{\text{mat}} = q_{ab} \left( \frac{E^a E^b + B^a B^b}{2\sqrt{q}} + \frac{\pi_\Theta}{2\sqrt{q}} + \frac{\pi_\eta}{2\sqrt{q}} \frac{\partial_a \eta \partial_a \eta}{a} \right) + \frac{a}{a + \eta} \frac{\pi_\Theta}{2\sqrt{q}} + \frac{a + \eta}{a} \frac{\pi_\eta}{a} \left( \partial_a \Theta + aeA_0 \right)^2 + \frac{1}{4} \mu \eta^2 (2a + \eta)^2,$$  \hfill (21)

$$H_{a}^{\text{mat}} = \epsilon_{abc} E^c B^b + \pi_\eta \partial_a \eta + \pi_\Theta \partial_a \Theta + aeA_a \pi_\Theta,$$  \hfill (22)

$$G = D_a E_a - ae \pi_\Theta.$$  \hfill (23)

The (non-smeared, non-trivial) Poisson brackets:

$$\{\pi_\eta, \eta\} = \frac{1}{2} \delta(x, y),$$  \hfill (24)

$$\{\pi_\Theta, \Theta\} = \frac{1}{2} \delta(x, y),$$  \hfill (25)

$$\{E_a, A^b\} = \delta^b_a \delta(x, y).$$  \hfill (26)
Now if one compares the Hamiltonian (20) with the original (9), it is easy to see that the two are connected with the help of the following canonical transformation:

\[ \Phi := (a + \eta) \exp \left( \frac{i\Theta}{a} \right), \]

\[ \Phi^* := (a + \eta) \exp \left( -\frac{i\Theta}{a} \right), \]

\[ \pi := \left( \pi_\eta - \frac{ia}{a + \eta} \pi_\Theta \right) \exp \left( -\frac{i\Theta}{a} \right), \]

\[ \pi^* := \left( \pi_\eta + \frac{ia}{a + \eta} \pi_\Theta \right) \exp \left( \frac{i\Theta}{a} \right). \]

Furthermore, the above system is very similar to the case of the symplectically embedded Proca field. To see this, let us introduce the canonical transformation

\[ \pi_\Theta \rightarrow \pi_\Theta e^{a\Theta/a}, \]

and define

\[ m^2 = e^2(a + \eta)^2. \]

Then we will obtain exactly the Hamiltonian of [1], with the exception of a potential term. There are two major differences: there is an extra dynamical scalar field in the theory and the ‘mass’ is constructed from the field \( \eta \). The latter will be quite important since after quantization all the fields will become operators so we can define a ‘mass operator’, whose spectrum can be identified as the mass spectrum (in [1] the mass was a parameter of the theory).

2.3. Classical symmetry breaking

In quantum field theory we use the unitary gauge to do gauge fixing. In the \( U(1) \) case this means we introduce the gauge-fixed vector field

\[ \tilde{\Lambda}^{(4)}(x) := \Lambda^{(4)}(x) - \frac{1}{ea} \partial^{(4)} \Theta(x). \] (27)

Substituting this into the Lagrangian we get

\[ \tilde{L}_{\text{mat}} = \int d^4x \sqrt{-g} \left[ \frac{1}{4} \tilde{F}^{(4)ab} \tilde{F}^{(4)ab} - \frac{1}{2} \partial^{(4)} a \eta \partial^{(4)} a \eta - \frac{1}{2} e^2 (a + \eta)^2 \tilde{A}^{(4)a} \tilde{A}^{(4)a} \right. \]

\[ \left. - \frac{1}{4} \mu \eta^2 (2a + \eta)^2 \right]. \] (28)

Again we want to see how the Hamiltonian changes, so we do the 3+1 decomposition as we did in the previous sections. The canonical momenta will be

\[ \tilde{\Pi}^a = \frac{\delta L}{\delta \dot{\Lambda}^a} = \tilde{E}^a, \quad \pi_\eta = \frac{\delta L}{\delta \dot{\eta}} = \sqrt{q} \Lambda_{\eta \eta} - \frac{N^a}{N} \partial_a \eta. \]

and the constraints will be

\[ H_{\text{mat}} = q_{ab} \tilde{E}^a \tilde{E}^b + \frac{\tilde{B}^a \tilde{B}^b}{2\sqrt{q}} + \frac{\pi^2}{2\sqrt{q}} + \frac{1}{2} \mu \sqrt{q} \partial_a \eta \partial_a \eta \]

\[ + \frac{1}{2} \sqrt{q} \mu \eta^2 (2a + \eta)^2 + e^2 \frac{\sqrt{q} (a + \eta)^2}{2} \left( \frac{\tilde{A}_a \tilde{A}^a}{N} + \left( \frac{\tilde{A}_0 - N^a \Lambda_a}{N} \right)^2 \right) \] (29)

\[ H_a^{\text{mat}} = \epsilon_{abc} \tilde{B}^c \tilde{E}^b + \pi_\eta \partial_a \eta + e^2 \sqrt{q} (a + \eta)^2 \frac{\tilde{A}_0 - N^a \Lambda_a}{N}, \] (30)

\[ G = D_a \tilde{E}^a - e^2 \sqrt{q} (a + \eta)^2 \frac{\tilde{A}_0 - N^a \Lambda_a}{N}. \] (31)
The (non-smeared, non-trivial) Poisson brackets remain the same:

\[ \{ \pi_r, \eta \} = \frac{1}{2} \delta(x, y), \]  
\[ \{ \tilde{E}_a, \tilde{A}^b \} = \delta_b^a \delta(x, y). \]  

If we compare this gauge-fixed Hamiltonian with (20), we can see that in the Hamiltonian formalism the gauge fixing is equivalent to the introduction of the following two constraints:

\[ C_a \, := \, \partial_0 \text{Arg}(\Phi) = 0, \]  
\[ C \, := \, a\pi_f - e\sqrt{q(a + \eta)^2 \frac{\tilde{A}_0 - N^u \tilde{A}_u}{N} = 0} \]  

(the second is equivalent to \( L_\gamma \Theta = 0 \)). This is precisely the gauge we used in the case of the symplectically embedded Proca field. There we showed that in LQG gauge fixing is not necessary, in fact it makes the quantization extremely difficult if not impossible. So we will not fix the gauge, instead we will try to solve the constraint related to it.

To conclude we summarize the most important results of this section.

We checked how one can implement spontaneous symmetry breaking in the Hamiltonian formalism. It turned out that introducing new variables means a canonical transformation, while gauge fixing (as was shown earlier e.g. in [13]) can be done by introducing new constraints. Interestingly, these are exactly the same conditions which were introduced in the case of the Proca field in [1]. Furthermore, the Hamiltonian (20) is very similar to the symplectically embedded Proca Hamiltonian [[1], p 5], the only two exceptions are that we have an extra scalar field and the mass is not a parameter but defined with the help of the field \( \eta \).

3. Quantization

3.1. Gauge fields

Quantization of the gravitational and electromagnetic fields can be treated on the same footing since both are gauge fields—the gauge group of the former is, in the Ashtekar variables [2], \( SU(2) \) while the latter is a \( U(1) \) field. The detailed analysis of the method can be found in [3–8], here we just sketch the main idea and the notations.

Let us consider a Yang–Mills gauge field with a compact gauge group \( G \). The Hilbert space can be constructed in the following way: let \( \gamma \) be an oriented graph in \( \Sigma \) with \( e_1, \ldots, e_E \) edges and \( v_1, \ldots, v_V \) vertices. Let \( h_{e_i} \) be the holonomy of the \( G \)-valued connection of the field evaluated along the \( e_i \) edge. Let us define a cylindrical function with respect to a \( \gamma \) graph in the following way:

\[ f_\gamma(A) := f(h_{e_1}, \ldots, h_{e_E}). \]  

where \( f_\gamma \) is a complex-valued function mapping from \( G^E \). The Hilbert space of the Yang–Mills field is defined as the set of all cylindrical functions which are square-integrable with respect to a suitable measure (the Ashtekar–Lewandowski measure):

\[ \mathcal{H} := L_2(\tilde{A}, d\mu_{A,L,G}). \]  

In our case, \( G = SU(2) \times U(1) \), so

\[ \mathcal{H}_{G,YM} := L_2(\tilde{A}_{SU(2)}, d\mu_{SU(2)}) \otimes L_2(\tilde{A}_{U(1)}, d\mu_{U(1)}). \]
In order to analyze the action of the Hamiltonian and to compute its kernel, it is convenient to introduce a complete orthonormal basis on the Hilbert space (38).

On the space of \( L^2(\mathcal{A}_{SU(2)}, d\mu_{SU(2)}) \) these are called spin network functions and defined as follows: let \( \gamma \in \Sigma \) be a graph and denote its edges and vertices, respectively by \( (e_1, \ldots, e_N) \) and \( (v_1, \ldots, v_V) \). Associate a coloring with each edge defined by a set of irreducible representations \( (j_1, \ldots, j_N) \) of \( SU(2) \) (half-integers) and contractors \( (\rho_1, \ldots, \rho_V) \) to the vertices where \( \rho_l \) is an intertwiner which maps from the tensor product of representations of the incoming edges at the vertex \( v_l \) to the tensor product of representations of the outgoing edges. A spin network state is then defined as

\[
|T(A)\rangle_{\gamma, \vec{j}, \vec{\rho}} := \bigotimes_{i=1}^{N} j_i(h_{\gamma_i}(A)) \cdot \bigotimes_{k=1}^{V} \rho_k,
\]

where \( \cdot \) stands for contracting at each vertex \( v_k \) the upper indices of the matrices corresponding to all the incoming edges and the lower indices of the matrices assigned to the outgoing edges with all the indices of \( \rho_k \).

In the case of \( L^2(\mathcal{A}_{U(1)}, d\mu_{U(1)}) \) one must simply replace \( SU(2) \) with \( U(1) \) in the above definition—these are called flux network functions [9, 10]. Since \( U(1) \) is a commutative group, we will have the following definition: for each edge \( e_i \) of the graph associate an integer \( l_i \).

Then the flux network function is defined as

\[
|F(A)\rangle_{\gamma, \vec{l}} := \prod_{i=1}^{N} (h_{l_i}(A))^{l_i}.
\]

What remains is to define the operators corresponding to the connection and the electric field on the Hilbert space. If we want to implement the Poisson brackets in the quantum theory in a diffeomorphism covariant way, we have to use smeared versions of these fields. In the case of gauge fields the natural candidates are the holonomy and the electric flux, respectively:

\[
h_{\gamma}(A) = \mathcal{P} \exp \int_{\gamma} A
\]

\[
E(S) = \int_{S} \ast E,
\]

where \( e \) is a path and \( S \) is a surface in \( \Sigma \), and \( \ast E \) is the dual of the electric field. Then the action of the corresponding operators will be defined as

\[
\hat{h}_{\gamma}(A)f(A) := h_{\gamma}(A)f,
\]

\[
\hat{E}(S)f(A) := i\hbar \{E(S), f(A)\}.
\]

3.2. Scalar field

The crucial point of quantizing the scalar field is (see [5] or [11, 12]) that the field should be real valued. In our case the original variables are complex, but this does not cause significant difficulties since we can introduce new fields which are real, thus the usual techniques can be applied on them. The only non-trivial problem is an additional ambiguity which arises because this can be done in more than one way. What we are going to do is introduce two kinds of different choices for the configuration variables and the momentum operators.

Case A. The most natural choice is to define the operators with the help of the real and imaginary parts of the fields. Let \( v \) be a vertex of a graph \( \gamma \) with coordinates \( x_v \). Then let

\[
U(\lambda, v) := \exp(i\lambda \Re(\Phi(x_v)));
\]
\[ \hat{U}(\delta, v) := \exp(i\delta \text{ Im}(\Phi(x_v))), \]  

where \( \lambda \) and \( \delta \) are arbitrary real numbers which are required because otherwise the quantization would not be general enough (see [11, 12] for details). The variables for the momentum operator should be \((B \text{ is an open ball in } \Sigma)\)

\[ \Pi(B) = \int_B d^3x \text{ Re}(\pi), \quad \hat{\Pi}(B) = \int_B d^3x \text{ Im}(\pi), \]

and thus the Poisson brackets of the variables will be

\[ \{ \Pi(B), U(\lambda, v) \} = \delta_{v \cap B, 0} \frac{i\lambda}{2} U(\lambda, v), \]  
\[ \{ \Pi(B), \hat{U}(\delta, v) \} = -\delta_{v \cap B, 0} \frac{i\delta}{2} \hat{U}(\delta, v), \]  
\[ \{ \Pi(B), \hat{U}(\delta, v) \} = \{ \hat{\Pi}(B), U(\lambda, v) \} = 0. \]

The transformation rules of these quantities with respect to the (smeared) gauge transformation

\[ \{ G(\Lambda), U(\lambda, v) \} = i\hbar \Lambda \frac{\text{ Im}(\Phi(x_v))}{\text{ Re}(\Phi(x_v))} U(\lambda, v), \]  
\[ \{ G(\Lambda), \hat{U}(\delta, v) \} = -i\hbar \Lambda \frac{\text{ Re}(\Phi(x_v))}{\text{ Im}(\Phi(x_v))} \hat{U}(\delta, v). \]

Since we have two fields, the Hilbert space for the scalar field is a tensor product

\[ \mathcal{H}_\text{sc} = \mathcal{H}(U) \otimes \mathcal{H}(\hat{U}), \]

where the Hilbert spaces \( \mathcal{H}(U) \) and \( \mathcal{H}(\hat{U}) \) are the linear combination of the following monomials: let \( v = v_1, \ldots, v_N \) be the set of vertices for some \( \gamma \) graph and let \( \lambda \) and \( \bar{\delta} \) be two sets of real numbers, each pair associated with a vertex. Then a basic element of \( \mathcal{H}(U) \) is constructed as follows:

\[ |\lambda\rangle_\gamma = \prod_{k=1}^{N} U(\lambda_k, v_k). \]  

In a similar fashion,

\[ |\bar{\delta}\rangle_\gamma = \prod_{k=1}^{N} \hat{U}(\bar{\delta}_k, v_k) \]

will be a basic element in \( \mathcal{H}(\hat{U}) \). Both \( |\lambda\rangle_\gamma \) and \( |\bar{\delta}\rangle_\gamma \) form a complete orthonormal basis, that is \( \langle \lambda | \lambda\rangle_\gamma = \delta_{\lambda, \lambda\gamma} \) and the same is true for \( |\bar{\delta}\rangle_\gamma \).

Thus elements of \( \mathcal{H}_\text{sc} \) are linear combinations of monomials \( |\lambda\rangle_\gamma |\bar{\delta}\rangle_\gamma \).

The operators are defined in the same way as in the case of gauge fields:

\[ \hat{U}(\lambda, v) |\lambda\rangle_\gamma := U(\lambda, v) |\lambda\rangle_\gamma \]  
\[ \hat{\Pi}(B) |\lambda\rangle_\gamma := i\hbar \{ \Pi(B), |\lambda\rangle_\gamma \} \]  
\[ \hat{U}(\delta, v) |\bar{\delta}\rangle_\gamma := U(\delta, v) |\bar{\delta}\rangle_\gamma \]  
\[ \hat{\Pi}(B) |\bar{\delta}\rangle_\gamma := i\hbar \{ \hat{\Pi}(B), |\bar{\delta}\rangle_\gamma \}. \]

Because the \( U(1) \) group is commutative, the action of the operators is very simple:

\[ \hat{\Pi}(B) |\lambda\rangle_\gamma = -\frac{\hbar}{2} \sum_{v_j \in B} \lambda_j |\lambda\rangle_\gamma \]
\[ \hat{U}(\lambda, v)|\lambda'_\gamma\rangle = |\lambda'_\gamma\rangle, \quad \lambda'_i = \lambda_i + \delta_{v_i, \lambda_i} \]

and similar expressions hold for \( \hat{U}(\delta, v) \) and \( \hat{\Pi}(B) \). Also, because of (49) \( \hat{\Pi}(B)|\lambda'_\gamma\rangle = \hat{\Pi}(B)|\lambda'_\gamma\rangle = 0 \).

Case B. Another way is to use the absolute value and the argument of \( \Phi \). Actually these are equal (up to constant factors) to the fields \( \eta \) and \( \Theta \), respectively, so we suggest the following operators for the multiplication operators:

\[ U_\eta(\lambda, v) := \exp(i\lambda \eta(x_v)) \quad (58) \]
\[ U_\Theta(\delta, v) := \exp \left( i\delta \Theta(x_v) \right). \quad (59) \]

For the momentum operators it is plausible to use the quantities \( \pi_\eta \) and \( \pi_\Theta \) instead of \( \text{Re}(\Pi) \) and \( \text{Im}(\Pi) \):

\[ \Pi_\eta(B) = \int_B d^3x \pi_\eta, \quad \Pi_\Theta(B) = a \int_B d^3x \pi_\Theta. \]

The Poisson brackets of these variables are a bit different than in case A:

\[ \{ \Pi_\eta(B), U_\eta(\lambda, v) \} = i \frac{1}{2} \lambda \delta v \cap B, v U_\eta(\lambda, v), \quad (60) \]
\[ \{ \Pi_\Theta(B), U_\Theta(\delta, v) \} = i \frac{1}{2} \delta \delta v \cap B, v U_\Theta(\delta, v), \quad (61) \]
\[ \{ \Pi_\eta(B), U_\Theta(\delta, v) \} = \{ \Pi_\Theta(B), U_\eta(\lambda, v) \} = 0. \quad (62) \]

The transformation rule for these variables with respect to gauge transformations are

\[ \{ G(\Lambda), U_\eta(\lambda, v) \} = 0 \]
\[ \{ G(\Lambda), U_\Theta(\delta, v) \} = -\frac{1}{2} i e \Lambda \delta \Theta(x_v) U_\Theta(\delta, v), \]

which means that \( U_\eta(\lambda, v) \) is gauge invariant and the transformation rule for \( U_\Theta(\delta, v) \) is

\[ U_\Theta(\delta, v) \mapsto U_\Theta(\delta, v) U_\Theta \left( \frac{ae \Lambda \delta}{2}, v \right)^{-1}. \quad (63) \]

The construction of the phase space is completely identical to the construction in case A, the only difference is that one has to replace the old variables with the new ones. To avoid confusion, \( \mathcal{H}^{\text{new}} = \mathcal{H}(U_\eta) \otimes \mathcal{H}(U_\Theta) \) will stand for the new phase space,

\[ |\lambda^\eta\rangle_\gamma = \prod_{k=1}^N U_\eta(\lambda_k^\eta, v_k) \quad (64) \]

will label an element of \( \mathcal{H}(U_\eta) \) and

\[ |\delta^\Theta\rangle_\gamma = \prod_{k=1}^N U_\Theta(\delta_k^\Theta, v_k) \quad (65) \]

will be an element of \( \mathcal{H}(U_\Theta) \). The action of these operators are completely the same as in case A:

\[ \hat{\Pi}_\eta(B)|\lambda^\eta\rangle_\gamma = \frac{\hbar}{2} \sum_{v_j \in B} \lambda_j^\eta |\lambda^\eta\rangle_\gamma, \quad (66) \]
\[ \hat{U}_\eta(\lambda, v)|\lambda^\eta\rangle_\gamma = |\lambda'^\eta\rangle_\gamma, \quad \lambda'^\eta_i = \lambda_i^\eta + \delta_{v_i, \lambda_i} \lambda. \quad (67) \]
3.3. Regularization

In order to quantize this system, one first has to rewrite the Hamiltonian in terms of the variables defined in the previous section—this is called the regularization procedure. The key observation is [5] that if the gravitational field is dynamical, one can construct a well-defined, diffeomorphism covariant Hamiltonian operator. In the article mentioned above the reader will find the detailed analysis of the gravitational, Yang–Mills, scalar and fermion fields. Since the method is quite lengthy, we are going to concentrate only on those terms that are different to those mentioned above. Specifically these are the terms that contain the scalar field. We will deal with the two kinds of description (the original case with $\Phi$ and $\Phi^*$ and the case with new variables $\eta$ and $\Theta$) separately.

**Case A.** Although later we will use the formulae involving $\eta$ and $\Theta$ we shall provide the regularization of the original Hamiltonian, since it has some non-trivial steps. The potential term is the simplest: since $\Phi^* \Phi = \text{Re}(\Phi)^2 + \text{Im}(\Phi)^2$ and $\text{Re}(\Phi) = \arccos\left(\frac{U(\lambda, v) + U^{-1}(\lambda, v)}{2}\right)$, we can write this (using the notations of [5]) in the following form:

$$\hat{H}_{\text{pot}} = \frac{1}{4} \sum_v N(v) \mu \hat{V}\left(\frac{1}{X^2} \arccos\left(\frac{U(\lambda, v) + U^{-1}(\lambda, v)}{2}\right)^2 \right. \left. + \frac{1}{\delta^2} \arccos\left(\frac{U(\delta, v) + U^{-1}(\delta, v)}{2}\right)^2 - a^2\right)^2. \quad (68)$$

One may wonder why we used the arccos function instead of, for example, the logarithm. The main reason is that since spontaneous symmetry breaking requires the ground state of the potential, we are forced to regularize the potential term to be self-adjoint. It is easy to see that the above operator is self-adjoint, but this would not be the case if we used the logarithm function. Of course there are still ambiguities in the regularization, but this certainly narrows down the possibilities.

In a similar fashion, one replaces $\pi^* = \text{Re}(\pi)^2 + \text{Im}(\pi)^2$ in the kinetic term to obtain

$$\hat{H}_p = \frac{1}{2} \sum_v N(v) \frac{X(v)^2 + \tilde{X}(v)^2}{E(v)^2} \hat{G}_1(v), \quad (69)$$

where $X(v)$ and $\tilde{X}(v)$ are the invariant vector fields on $U(1)$ and $\hat{G}_1(v)$ contains only gravitational variables and is the same as in [5]:

$$\hat{G}_1(v) = \frac{8}{81m^2h^4\kappa^6} \sum_{\epsilon(\Delta) = v} e^{ijkL} e^{LMN} \epsilon_{ij} \epsilon_{lnm}$$

$$\times \hat{Q}_{s J(\Delta)}(v, \frac{1}{2}) \hat{Q}_{s J(\Delta')}^{\prime}(v, \frac{1}{2}) \hat{Q}_{s K(\Delta)}(v, \frac{1}{2}) \hat{Q}_{s K(\Delta')}^{\prime}(v, \frac{1}{2}) \hat{Q}_{s L(\Delta)}^{\prime}(v, \frac{1}{2}) \hat{Q}_{s L(\Delta')}^{\prime}(v, \frac{1}{2}). \quad (70)$$

The derivative term needs a more careful treatment. First we have to rewrite it in terms of $\text{Re}(\Phi)$ and $\text{Im}(\Phi)$

$$D_a \Phi (D_b \Phi)^* = (\partial_a + i e A_a)(\text{Re}(\Phi) + i \text{Im}(\Phi))(\partial_b - i e A_b)(\text{Re}(\Phi) - i \text{Im}(\Phi)).$$

From this we can see that we need to regularize the expression $(\partial_a \pm i e A_a)\text{Re}(\Phi)$. This is quite similar to the derivative term $\partial_a \Phi \pm A_a$ in [1], the only difference is that we have
a $iA_a \text{Re}(\Phi)$ term instead of $A_a$. Though this seems a minor change, it turns out that the regularized expression for this covariant derivative is more complicated, which is due to the fact that it contains the multiplication of the two fields. We can overcome this difficulty by doing the regularization in a step-by-step way. First we note that for small $\Delta t$

$$h_\delta = 1 + i e \Delta t \delta a A_a + o(\Delta t^2)$$

for an edge $s$. This means that ($v$ is the beginning of the edge $s$)

$$(h_\delta - 1) \arccos \left( \frac{U(\lambda, v) + U(\lambda, v)^{-1}}{2} \right) = ie \lambda \Delta t \delta a A_a \text{Re}(\Phi) + o(\Delta t^2),$$

so if we take into account that

$$U(\lambda, s(\Delta t)) = 1 + i \lambda(\text{Re}(\Phi) + \Delta t \delta a \partial_a \text{Re}(\Phi)) + o(\Delta t^2),$$

we arrive at the following regularized expression:

$$U(\lambda, s(\Delta t)) \left[ 1 + i(h_\delta - 1) \arccos \left( \frac{U(\lambda, v) + U(\lambda, v)^{-1}}{2} \right) \right] = \left[ 1 + i \lambda(\text{Re}(\Phi) + \Delta t \delta a \partial_a \text{Re}(\Phi)) \right]$$

$$\times (1 - e \lambda \Delta t \delta a A_a \text{Re}(\Phi))[1 - i \lambda(\text{Re}(\Phi))] + o(\Delta t^2)$$

$$= 1 + i \lambda \Delta t \delta a (\partial_a \text{Re}(\Phi)) + i e A_a \text{Re}(\Phi)) + o(\Delta t^2).$$

(71)

We obtain the same result for $(\partial_a - ie A_a) \text{Im}(\Phi)$ if we replace $U$ with $\bar{U}$. Also the term $(\partial_a - ie A_a) \text{Re}(\Phi)$ is obtained by replacing $h_\delta$ with $h_\delta^{-1}$. To simplify the result let us introduce a notation:

$$W(v, s, \lambda) = \frac{1}{\lambda} U(\lambda, s(\Delta t)) \left[ 1 + i(h_\delta - 1) \arccos \left( \frac{U(\lambda, v) + U(\lambda, v)^{-1}}{2} \right) \right] U(\lambda, v)^{-1} - \frac{1}{\lambda},$$

$$\bar{W}(v, s, \delta) = \frac{1}{\delta} \bar{U}(\delta, s(\Delta t)) \left[ 1 + i(h_\delta - 1) \arccos \left( \frac{\bar{U}(\delta, v) + \bar{U}(\delta, v)^{-1}}{2} \right) \right] \bar{U}(\delta, v)^{-1} - \frac{1}{\delta}.$$

Using the fact that $h_\delta^{-1} = h_\delta^{-1}$, the regulated expressions are the following:

$$W(v, s, \lambda) = ie t \delta a (\partial_a \text{Re}(\Phi)) + i e A_a \text{Re}(\Phi)) + o(\Delta t^2),$$

$$W(v, s^{-1}, \lambda) = ie t \delta a (\partial_a \text{Re}(\Phi)) - i e A_a \text{Re}(\Phi)) + o(\Delta t^2),$$

$$\bar{W}(v, s, \delta) = ie t \delta a (\partial_a \text{Re}(\Phi)) + i e A_a \text{Re}(\Phi)) + o(\Delta t^2),$$

$$\bar{W}(v, s^{-1}, \delta) = ie t \delta a (\partial_a \text{Re}(\Phi)) - i e A_a \text{Re}(\Phi)) + o(\Delta t^2).$$

This way the derivative term will be the limit of

$$\hat{H}_{\text{der}} = \frac{1}{2} \sum_v N(v) \sum_{v(\lambda) = v(\lambda') = v} \left[ W(\lambda, s, \lambda) + i \bar{W}(\lambda, s, \delta) \right]$$

$$\times \left[ W(\lambda, s^{-1}, \lambda) - i \bar{W}(\lambda, s^{-1}, \delta) \right] \hat{G}_{2r}^v (v),$$

where

$$\hat{G}_{2r}^v (v) = \frac{1}{2 \hbar^4 k^4} \left( \frac{4}{3} \right)^6 e^{i j k} e^{i l m} e_{a p q r} e_{t s t}$$

$$\times \left( v, \frac{3}{4} \right) \hat{\mathcal{Q}}_{s(\lambda)}^t \left( v, \frac{3}{4} \right) \hat{\mathcal{Q}}_{s(\lambda')}^t \left( v, \frac{3}{4} \right) \hat{\mathcal{Q}}_{s(\lambda)}^m \left( v, \frac{3}{4} \right) \hat{\mathcal{Q}}_{s(\lambda')}^m \left( v, \frac{3}{4} \right).$$
and the $\Delta$ and $\Delta'$ subscripts represent the tetrahedra where the holonomies and point holonomies should be calculated.

Case B. In this case one should be careful since the two scalar fields do not appear in a symmetric way. For instance, the potential term contains only $\eta$, so one simply replaces $\eta = \frac{1}{2} \arccos \left( \frac{U_\eta(\lambda, v) + U^{-1}_\eta(\lambda, v)}{2} \right)$ to obtain

$$\hat{H}_{\text{pot}} = \frac{1}{4} \sum_v N(v) \mu \hat{V} \frac{1}{\lambda^2} \arccos \left( \frac{U_\eta(\lambda, v) + U^{-1}_\eta(\lambda, v)}{2} \right)^2 \times \left[ \frac{1}{\lambda} \arccos \left( \frac{U_\eta(\lambda, v) + U^{-1}_\eta(\lambda, v)}{2} \right) + 2a \right]^2.$$  \hspace{1cm} (72)

The terms containing $\pi_\eta$ and $\pi_{\Theta_1}$ can be treated in the same way as in the previous case, one just has to be careful since the latter contains the expression $\frac{1}{(a + \eta)^2}$. But it is easy to see that if one carries out the regularization procedure as in [5] the only difference will be a term which is the above fraction expressed with the variables $U_\eta$. Thus the result for the two kinetic terms will be

$$\hat{H}_p = \frac{1}{2} \sum_v N(v) X_\eta(v)^2 + \left[ \frac{1}{2} \arccos \left( \frac{U_\eta(\lambda, s(\Delta t)) + U^{-1}_\eta(\lambda, v)}{2} \right) + a \right]^2 X_{\Theta_1}(v)^2 \hat{G}_1(v).$$  \hspace{1cm} (73)

The derivative terms for the two scalar fields are also different. In the case of $\eta$, one only needs the expression (for small $\Delta t$):

$$U_\eta(\lambda, s(\Delta t))U^{-1}_\eta(\lambda, v) - 1 = i\lambda \Delta t s^a \partial_a \eta + o(\Delta t^2),$$  \hspace{1cm} (74)

thus it has a contribution to the Hamiltonian

$$\hat{H}_{\text{der}}(\eta) = \frac{1}{2} \sum_v N(v) \sum_{s(\Delta) = s(\Delta')} \frac{1}{\lambda^2} [U_\eta(\lambda, s(\Delta))U^{-1}_\eta(\lambda, v) - 1] \times \left[ U_\eta(\lambda, s(\Delta'))U^{-1}_\eta(\lambda, v) - 1 \right] \hat{G}_{nr}^2(v).$$  \hspace{1cm} (75)

For the field $\Theta$ we remind the reader that in [1] the same kind of coupling appeared between the scalar field and the Maxwell field. Thus we may use the approximation mentioned there:

$$U_\Theta(\delta, s(\Delta t))U^{-1}_\Theta(\delta, v) - 1 = i\delta \Delta t s^b \left( \frac{\partial_b \Theta}{a} + eA_b \right).$$  \hspace{1cm} (76)

Treating the term $(a + \eta)^2$ as before, the regulated expression of this term will be

$$\hat{H}_{\text{der}}(\Theta) = \frac{1}{2} \sum_v N(v) \sum_{s(\Delta) = s(\Delta')} \frac{1}{\lambda^2} \arccos \left( \frac{U_\Theta(\lambda, v) + U^{-1}_\Theta(\lambda, v)}{2} \right)^2 \times \sum_{s(\Delta) = s(\Delta')} \frac{1}{\delta^2} [U_\Theta(\delta, s(\Delta))U^{-1}_\Theta(\delta, v) - 1][U_\Theta(\delta, s(\Delta'))U^{-1}_\Theta(\delta, v) - 1] \hat{G}_{nr}^2(v).$$  \hspace{1cm} (77)

4. New basis

In contrast to the Proca field, the mass here is represented by an operator, namely

$$\hat{m}(v) = e \left[ \frac{1}{\lambda} \arccos \left( \frac{U_\eta(\lambda, v) + U^{-1}_\eta(\lambda, v)}{2} \right) + a \right].$$  \hspace{1cm} (78)

Since we want to compare the two theories, it would be useful to work in a basis where $U_\eta(\lambda, v)$ is diagonal and this is what we are going to do in this section.
4.1. The spectrum of $U_\eta(\lambda, v)$

Let $|\phi\rangle := |\lambda_\eta^0, \ldots, \lambda_\eta^0\rangle$ be a base element for a $\gamma$ graph which has $N$ vertices. The action of $U_\eta(\lambda, v)$ on this state is

$$U_\eta(\lambda, v)|\phi\rangle = |\lambda_\eta^{0,1}, \ldots, \lambda_\eta^{0,1} + \lambda, \ldots, \lambda_\eta^{0,1}\rangle\delta(v, v_k).$$

(79)

This action suggests that we should look for eigenstates in the form

$$|\Lambda^\eta(\lambda, v), \tilde{\lambda}^\eta\rangle := \sum_{i=-\infty}^{\infty} \frac{U_\eta(\lambda, v)}{\Lambda^\eta(\lambda, v)} |\tilde{\lambda}^\eta_i\rangle,$$

(80)

where $|\tilde{\lambda}^\eta_i\rangle$ is an arbitrary state and $\Lambda^\eta(\lambda, v)$ is a (yet) arbitrary number (this will be the eigenvalue for a given $\lambda$ at a vertex $v$). It is easy to verify that

$$U_\eta(\lambda, v)|\Lambda^\eta(\lambda, v), \tilde{\lambda}^\eta\rangle = \Lambda(\lambda, v)|\Lambda^\eta(\lambda, v), \tilde{\lambda}^\eta\rangle.$$  

(81)

We shall call these one vertex eigenstates because $|\Lambda^\eta(\lambda, v), \tilde{\lambda}^\eta\rangle$ is the eigenstate of only those $U_\eta(\lambda, v')$ where $v' = v$. Since $U_\eta(\lambda, v)$ is unitary, we can write $\Lambda^\eta(\lambda, v)$ in the following form: $\Lambda^\eta(\lambda, v) = \exp(i\Delta^\eta(\lambda, v))$, where $\Delta^\eta$ is real. In fact, since $U_\eta(0, v) = 1$ and $U_\eta(\lambda_1, v)U_\eta(\lambda_2, v) = U_\eta(\lambda_1 + \lambda_2, v)$, we obtain that $\Delta^\eta(\lambda, v)$ is of the form $\Delta^\eta(\lambda, v) = \Gamma^\eta(\lambda, v)$. In summary, the spectrum of $U_\eta(\lambda, v)$ is of the form $\exp(i\Gamma^\eta(\lambda, v))$.

Furthermore, if we restrict the values of $\Gamma^\eta(\lambda, v)$ so that $0 \leq \Gamma^\eta(\lambda, v)$, these states will form a complete orthonormal basis in the sense

$$
\langle \Gamma^\eta(\lambda, v), \tilde{\lambda}^\eta_1 | \Gamma^\eta(\lambda, v), \tilde{\lambda}^\eta_2 \rangle = \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \langle \tilde{\lambda}^\eta_{1,k,j}, \tilde{\lambda}^\eta_{2,k,j} \rangle \exp \left(i\lambda \left( j \Gamma^\eta(\lambda, v) - k \Gamma^\eta(\lambda, v) \right) \right) = \delta(\tilde{\lambda}^\eta_1, \tilde{\lambda}^\eta_2),
$$

(82)

where $\delta(\tilde{\lambda}^\eta_1, \tilde{\lambda}^\eta_2) = \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \langle \lambda^{(1)}, \lambda^{(2)} \rangle$. To see that this is a complete orthonormal basis, one only has to check whether each original basis element can be expressed as the linear combination of the eigenstates.

Let us suppose then that there exist complex numbers $C_\lambda(\Gamma^\eta(\lambda, v))$ such that

$$
\sum_{\tilde{\lambda}^\eta} \int d\Gamma^\eta(\lambda, v) C_\lambda(\Gamma^\eta(\lambda, v)) |\Gamma^\eta(\lambda, v), \tilde{\lambda}^\eta\rangle = \langle \tilde{\lambda}^\eta |,
$$

(83)

for each $|\tilde{\lambda}^\eta\rangle$. Because of orthogonality we obtain for the coefficients the following:

$$
C_\lambda(\Gamma^\eta(\lambda, v)) = \langle \tilde{\lambda}^\eta | |\Gamma^\eta(\lambda, v), \tilde{\lambda}^\eta\rangle = \langle \lambda^\eta \rangle \sum_{k=-\infty}^{\infty} \exp(-i\lambda \lambda^{k}) U_\eta(\lambda, v)^k |\lambda^\eta\rangle.
$$

(84)

Now if for a $|\tilde{\lambda}^\eta\rangle$ there exists an integer $n$ such that $|\tilde{\lambda}^\eta\rangle = U(\lambda, v)^n |\lambda^\eta\rangle$ then the corresponding coefficient will be

$$
C_\lambda(\Gamma^\eta(\lambda, v)) = \exp(-in\lambda \Gamma^\eta(\lambda, v)),
$$
otherwise it is zero. It is easy to see that this correspondence is unique and since the original basis is complete, we verified our statement.

We define the graph eigenstate in a similar fashion. Let $\gamma$ be a graph and $|\lambda\eta\rangle$ be an arbitrary state on that graph. For each vertex let $\Gamma^\eta(v_i)i = 1, \ldots, N$ be a real number satisfying $0 \leq \lambda^\eta(v_i)\leq 2\pi$. Then the graph eigenstate will be the following:

$$|\Gamma_1^\eta,\lambda\eta\rangle := \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_N=-\infty}^{\infty} (e^{ik_1\lambda^\eta(v_i)}U_\eta(\lambda, v_1)^{k_1}) \cdots (e^{ik_N\lambda^\eta(v_N)}U_\eta(\lambda, v_N)^{k_N}) |\lambda\eta\rangle. \quad (85)$$

Using the results obtained for the one vertex eigenstates we can find an orthonormal basis in the case of the graph eigenstates: if $0 \leq \lambda^\eta(v_i)$ for all $v_i$ we get

$$\langle \Gamma_1^\eta,\lambda\eta | \Gamma_2^\eta,\lambda\eta \rangle = \delta(\lambda_1^\eta, \lambda_2^\eta) \prod_{k=1}^{N} \delta (\Gamma_1^\eta(v_1) - \Gamma_2^\eta(v_1)) \cdots \delta (\Gamma_1^\eta(v_N) - \Gamma_2^\eta(v_N)). \quad (86)$$

Also we can express any state in terms of graph eigenstates with the help of the following expression:

$$\sum_{\lambda^\eta} \int d\Gamma^\eta C_\lambda^{in} (\Gamma^\eta, \lambda^\eta) |\Gamma^\eta,\lambda^\eta\rangle = |\lambda^\eta\rangle, \quad (87)$$

where

$$\int d\Gamma^\eta = \int d\Gamma^\eta(v_1) \cdots \int d\Gamma^\eta(v_N).$$

What remains is the action of the momentum operators on an eigenstate. This is easy because of the following:

$$X(v)|\Gamma^\eta(v'),\lambda^\eta\rangle = X(v) \sum_{k=-\infty}^{\infty} \left( \frac{U_\eta(\lambda, v')}{\exp(i\lambda \Gamma^\eta(v'))} \right)^k |\lambda^\eta\rangle = \delta_{v, v'} \lambda_k |\Gamma^\eta(v'),\lambda^\eta\rangle - i \delta_{v, v'} \sum_{k=-\infty}^{\infty} ik\lambda \left( \frac{U_\eta(\lambda, v')}{\exp(i\lambda \Gamma^\eta(v'))} \right)^k |\lambda^\eta\rangle = \left( \delta_{v, v'} \lambda_k + i \delta_{v, v'} \frac{\partial}{\partial \Gamma^\eta(v')} \right) |\Gamma^\eta(v'),\lambda^\eta\rangle. \quad (88)$$

With a completely similar analysis one can show that

$$X(v)|\Gamma^\eta,\lambda^\eta\rangle = \left( \delta_{v, v'} \lambda_k + i \delta_{v, v'} \frac{\partial}{\partial \Gamma^\eta(v')} \right) |\Gamma^\eta,\lambda^\eta\rangle. \quad (89)$$

5. Solution to the constraints

In [1] we sketched how one could solve the constraints of the theory. In this section we will follow the same procedures mentioned there—especially in the case of the scalar constraint. This method can also be used in this case, with one difference, namely that for the fields $\eta$ and $\Theta$ we do not work in the usual basis, rather in the Fock space. Since for the other fields the algorithm remains the same, we will concentrate only on the scalar fields. Solving the diffeomorphism and gauge constraints will be rather simple, so we start with them. Then—in order to simplify things—we will introduce a compact notation where we separate the scalar fields from the others, which is described in appendix A. This is motivated by the fact that the scalar constraint is quite complicated, but with the new notation the structure of the equation will be much easier to examine.
Let us start with the diffeomorphism constraint. As was pointed out in [6] the infinitesimal
generator of the diffeomorphism constraint cannot be implemented in the quantum theory, thus
the techniques used to solve the Gauss or scalar constraint cannot be applied here. The strategy
is to use group averaging to solve the constraint, which can be generalized to the case where
matter fields also appear (see [5] for details). Since these are applied only to graphs not to
labels means that it is independent whether we use the Fock space or the dust network space.

The gravitational Gauss constraint is the same as in [6], so we can solve it by restricting
ourselves to gauge-invariant spin network states.

The $U(1)$ Gauss constraint contains variables of the electromagnetic field and the scalar
field $\Theta$ so we analyze it in detail. The (smeared) integrated constraint

$$
\int_G GA = \int_D \Lambda(D_vE_a - a\epsilon \pi_\alpha)
$$

(90)
can be regulated in the following way: let us look for solutions in the form

$$
\Psi = \sum_{s, f, \lambda} \int d\Gamma \int d\bar{\Gamma} C_{s, f, \lambda}^{\bar{\lambda}}(\Gamma, \bar{\Gamma})|\bar{f}||\bar{\Gamma}, \bar{\lambda}\rangle|\Gamma, \lambda\rangle
$$

(91)

It can be verified that the quantum version of the above constraint is the following:

$$
\langle \Psi | \sum_v \Lambda_v \sum_{e \in \Gamma(v)} l_e - \left( \delta_v + \frac{i}{\delta \Gamma(v)} \right)|\Phi\rangle = 0
$$

(92)

for all spin color network state $|\Phi\rangle$. Here $l_e$ is the integer on the edge $e$ (this comes from the
flux network). Since $\Lambda_v$ is arbitrary the above equation is equivalent to

$$
\int d\Gamma \langle \phi | C_{s, f, \lambda}^{\bar{\lambda}}(\Gamma, \bar{\Gamma})|\bar{f}||\bar{\Gamma}, \bar{\lambda}\rangle|\Gamma, \lambda\rangle = 0,
$$

(93)

where we inserted (91) into the constraint equation and used orthogonality of spin color
network states. Now after partial integration we obtain a (functional) differential equation on
the coefficients $C_{s, f, \lambda}^{\bar{\lambda}}(\Gamma, \bar{\Gamma})$:

$$
\left[ \sum_{e \in \Gamma(v)} l_e - \left( \delta_v - \frac{i}{\delta \Gamma(v)} \right) \right] C_{s, f, \lambda}^{\bar{\lambda}}(\Gamma, \bar{\Gamma}) = 0.
$$

(94)

Since we have a similar equation for all $v$, the solution to this constraint is

$$
C_{s, f, \lambda}^{\bar{\lambda}}(\Gamma, \bar{\Gamma}) = C_{s, f, \lambda}^{\bar{\lambda}}(\Gamma) \prod_v \exp \left[ -i \left( \sum_{e \in \Gamma(v)} l_e - \delta_v \right) \Gamma(v) \right].
$$

(95)

where the coefficients $C_{s, f, \lambda}^{\bar{\lambda}}(\Gamma)$ are arbitrary.

What remains is the scalar constraint. If we look at the Hamiltonian, it is clear that the
constraint equation will be a differential equation with respect to the variable $\Gamma$. First we write
down this equation. The condition we have to solve is

$$
\langle \Psi | \hat{H} |\phi\rangle = 0
$$

(96)

for arbitrary $|\phi\rangle$. Again we can say that the support of $N$ is at only one vertex $v$. Substituting
(91) into the above equation we obtain

$$
\sum_{s', f', \lambda'} \int d\Gamma'(v) \int d\bar{\Gamma}'(v) C_{s', f', \lambda'}^{\bar{\lambda}'}(\Gamma', \bar{\Gamma}')

\times \langle s'| \langle f'| |\bar{f}'||\bar{\Gamma}', \bar{\lambda}'\rangle |\bar{\Gamma}, \bar{\lambda}\rangle |\Gamma, \lambda\rangle |\hat{H} |\bar{\Gamma}, \bar{\lambda}\rangle |\Gamma', \lambda'\rangle |f\rangle |s\rangle = 0.
$$
In [1] we have shown a method (generalizing the results of [3]) which simplified the above equation by turning it into a finite number of equations. The main idea is that we take a basis element $|s⟩⟨f|$ (discarding the scalar field for the moment) and we create a set $S^{(1)}$ containing basis elements appearing in $H|s⟩⟨f|$. We continue this procedure and construct $S^{(n)}$ recursively from $S^{(n-1)}$. After this we search for solutions of the form

$$
|Ψ⟩ = \sum_i \sum_{(f)|s⟩|s⟩} C_i^{(s)}|s⟩⟨f|,
$$

where $s$ is a basis element. One can show that if we substitute this into the constraint equation we arrive at a finite number of conditions (details can be found in the mentioned articles).

Now we apply these results to the gravitational and electromagnetic fields and we arrive at a finite number of conditions (details can be found in the mentioned articles).

To simplify this term we look for solutions of the form

$$
\sum_I H^p_I (\lambda_v - \frac{i}{\delta \Gamma(v)}) \tilde{C}_I(\Gamma) = -\sum_I H_I(\Gamma) \tilde{C}_I(\Gamma),
$$

where

$$
H_I(\Gamma) = \frac{L(v)^2 H_P(v)}{\Gamma(v) + a^2} + H^2_{P} M(v) + H^2_{P} + \frac{\delta}{\delta \Gamma(v)} \sum_v H^2_{P} + \frac{\delta}{\delta \Gamma(v)} (\Gamma(v) + a)^2 + H^2_{P} (v) \Gamma(v) (2a)^2 + \frac{\delta}{\delta \Gamma(v)} H^2_{P} (v).
$$

To simplify this term we look for solutions of the form

$$
C_I(\Gamma) := \tilde{C}_I(\Gamma) \exp \left( -i \sum_v \lambda_v \Gamma(v) \right),
$$

since

$$
\left( \lambda_v - \frac{i}{\delta \Gamma(v)} \right) C_I(\Gamma) = -i \left( \frac{\delta}{\delta \Gamma(v)} \tilde{C}_I(\Gamma) \right) \exp \left( -i \sum_v \lambda_v \Gamma(v) \right).
$$

The other terms will also contain a factor $\exp(-i \sum_v \lambda_v \Gamma(v))$ so this drops out of the differential equation, leaving us with the following formula for $\tilde{C}_I(\Gamma)$:

$$
\sum_I H^p_I \frac{\delta^2}{\delta \Gamma(v)^2} \tilde{C}_I(\Gamma) = \sum_I H_I(\Gamma) \tilde{C}_I(\Gamma).
$$

5.1. Solving the scalar constraint

This system of linear differential equations can be solved using the method we have shown in appendix B if the matrix $H^p_I$ is invertible. If it is not invertible then let us diagonalize the left-hand side, i.e. find a unitary $U$ such that $U H^p U^{-1} = \text{diag}(k_1, \ldots, k_M)$ where $k_1, \ldots, k_M$ are the eigenvalues of $H^p$. Let us order the eigenvalues in a way that $k_1, \ldots, k_M$ ($M\leq N$) be all the zero eigenvalues. This means that the first $M$ equation in this case is not a differential equation but only an algebraic equation. Since in this case the left-hand side is zero, the right-hand side is zero if and only $\sum_{I=1}^{M} H^p_I(v) \tilde{C}_I = 0$, etc for all matrices appearing in $H_I(\Gamma)$ ($H^p_I(v) = (U H^p U^{-1})_I(v)$, etc), which means that after solving the algebraic equations we again arrive at a system of linear differential equations but with an invertible matrix on the left-hand side. So from now on we consider $H^p_I$ to be invertible.
To have a correct solution we must specify the initial condition on $\tilde{C}_I(\Gamma)$ and $\frac{\partial}{\partial \Gamma(v)} \tilde{C}_I(\Gamma)$. The fact that $\Gamma(v) = -a$ can be interpreted as the disappearance of the field $\eta$ implies that

$$\sum_I H_{I\Gamma}(\Gamma) \tilde{C}_I(\Gamma) = 0 \quad (100)$$

is the first condition. With the same reasoning the second condition is that the momentum of the field should disappear. In this case (since all $\lambda_v$ are zero) we arrive at the condition

$$\left. \frac{\partial}{\partial \Gamma(v)} \tilde{C}_I(\Gamma) \right|_{\Gamma(v) = -a} = 0. \quad (101)$$

Since $H_{I\Gamma}$ has a complicated structure, the differential equation cannot be solved explicitly. However we can solve it in some special cases.

First let us consider the case when $\Gamma(v) \approx -a$. In this case the system of differential equations takes the form

$$\sum_I H_{I\Gamma}(\Gamma) \tilde{C}_I(\Gamma) = \sum_I L(v)^2 H_{I\Gamma}(v) \tilde{C}_I(\Gamma) + \sum_I H_{I\Gamma}^{G+YM} \tilde{C}_I(-a). \quad (102)$$

Now if we multiply both sides with $(H^P)^{-1}$ and define $b_I = ((H^P)^{-1} H^{G+YM} \tilde{C}(-a))_I$ we get

$$\frac{\delta^2}{\delta \Gamma(v)^2} \tilde{C}_I(\Gamma) - \frac{L(v)^2}{(\Gamma(v) + a)^2} \tilde{C}_I(\Gamma) = b_I. \quad (103)$$

The general solution of this differential equation is the following:

$$\tilde{C}_I(\Gamma) = \frac{(\Gamma(v) + a)^2}{2 - L(v)^2} b_I + C_{I1}(\Gamma(v) + a)^{n1} + C_{I2}(\Gamma(v) + a)^{n2}, \quad (104)$$

where

$$n_1 = \frac{1 \pm \sqrt{1 + 4L(v)^2}}{2}$$

and $C_{I1}, C_{I2}$ are arbitrary constants. From $L(v)^2 \geq 0$ follows that $n_1 \geq 1$ and $n_2 \leq 0$, which means that if $L(v) \neq 0$ then $\tilde{C}_I(\Gamma)$ is singular in $\Gamma(v) = -a$.

Now let us consider the initial conditions. If $L(v) = 0$ then

$$\tilde{C}_I(\Gamma) = \frac{(\Gamma(v) + a)^2}{2} b_I + C_{I1}(\Gamma(v) + a) + C_{I2}. \quad (105)$$

Substituting into (100) and (101) implies that $C_{I1} = 0$ and $b_I = 0$. Furthermore from the definition of $b_I$ comes that $b_I = ((H^P)^{-1} H^{G+YM} \tilde{C}(-a))_I$, so the solution is

$$\tilde{C}_I(\Gamma) = C_{I2}, \quad (106)$$

where $C_{I2}$ must satisfy the condition

$$H_{I\Gamma}^{G+YM} C_{I2}^2 = 0. \quad (107)$$

This is not a surprising result since if $L(v) = 0$ then substituting this into the constraints we obtain a theory completely equivalent to the electromagnetic field coupled to gravity. If we rewrite the scalar constraint of this theory in terms of the notation used in appendix A, we obtain the above condition.

What happens if $L(v) \neq 0$. In this case $C_{I2}^2 = 0$ so that the solution does not become singular at $\Gamma = -a$. Substituting into (101) will yield the identity $0 = 0$, so we must check (100). For $L(v) = \pm 1$ this will be singular so in this case $C_{I1} = 0$ and only the first term
survives, but it will be zero too. Thus in this case the solution near \( \Gamma = -a \) is zero in first order. For \(|L(v)|\)' the condition (100) is also an identity. But in this case \( \tilde{C}_I(\Gamma) = 0 \), so 

\[ \tilde{C}_I(\Gamma) = \tilde{C}_I^I(\Gamma(v) + a)^m. \]  

This solution tends rapidly to zero as \( \Gamma \to -a \) (especially if \( L(v) \) is large), so as we reach this limit, the amplitude of the solution coming from the \( L(v) = 0 \) case will become significantly larger. In fact the larger \( L(v) \) is, the amplitude becomes much smaller in this region. So we can say that if \( \Gamma(v) + a \approx 0 \) (which—as we will see later—can be interpreted as the masses about zero) states that for which \( L(v) = 0 \) have the highest probability while the larger \( |L(v)| \) is, the smaller this probability will get.

These results show that in contrast to the Proca field, this theory provides us with the different amplitudes for different masses. However because the two theories are—in some aspect—very similar, it would be desirable to provide the solutions of this theory which can be identified as the solutions to the Proca field. The basic idea is very simple: we compare the two Hamiltonians. If we look at the matrix (98) in our differential equation, in the case \( \Gamma = 0 \) it will be the same as the Hamiltonian of the Proca field. So one just needs to imply the conditions

\[ \frac{\delta^2}{\delta \Gamma(v)^2} \tilde{C}_I(\Gamma) = 0, \]  

\[ \left( \sum_{I} H_{I,I}(\Gamma) \tilde{C}_I(\Gamma) \right) \bigg|_{\Gamma=0} = 0. \]  

The problem is that in this theory this will provide a distributional solution in the following sense. In the case of the Proca field the mass is fixed, which means that we are interested in solutions where \( \Gamma \) is constant. But now we have a differential equation so \( \Gamma \) is continuous. The way out of this is to say that in the interval \((-\epsilon, \epsilon)\) we solve (109), and outside this interval \( \tilde{C}_I(\Gamma) \) is zero. The required solution will be in the limit \( \epsilon \to 0 \). The reason for this strange behavior is that the equation we gained looks not like the Proca, but the linear combination of all the Procas.

6. Mass

In quantum field theory the mass is the coefficient of the term in the Hamiltonian which is quadratic in the boson field. However in this case we shall define the mass as an operator corresponding to the classical expression \( \eta + a \). The reasons for us to do so are the following: first—as was shown at the end of section 2.2—the expression \( \eta + a \) corresponds exactly to the mass parameter of the Proca field (the term \((\eta + a)^2\) not only appears in front of the quadratic term of the bosonic field but also appears in the denominator of the kinetic term of the other scalar field). The second reason is that in this case we can simplify our analysis regarding the scalar–boson interaction. This new interpretation—as we will see—gives a better understanding of the mass generation in the Hamiltonian framework. Note also that the substitution \( \eta = 0 \) gives back the ‘original’ mass.

Let \(|\Psi\rangle := \sum_{\lambda} \int d\Gamma \Gamma C_\lambda(\Gamma) |\Gamma, \lambda\rangle^0\) be a solution of the constraints. Then we can define the ‘mass operator’ as

\[ \tilde{\hat{m}} |\Psi\rangle = \frac{1}{\lambda} \arccos \left( \frac{U_\eta(\lambda, v) + U^{-1}_\eta(\lambda, v)}{2} \right) |\Psi\rangle = \sum_{\lambda} \int d\Gamma \Gamma C_\lambda(\Gamma) |\Gamma(\alpha \eta(v), \lambda\rangle^0). \]  

(111)
It is clear from the definition that this operator is self-adjoint, thus it has real eigenvalues. Furthermore its spectrum is continuous. Its expectation value is

$$m(\Psi, v) = \sum_{\lambda,_{\eta}} \int \sum_{\lambda} \sum_{\eta} C_{\lambda_{\eta}}(\Gamma_{\eta}) \left| C_{\lambda_{\eta}}(\Gamma_{\eta}) e(\Gamma_{\eta} + a) \right|^2 \left| \langle \Gamma_{\eta} | \langle \Gamma_{\eta} \right|$$

thus for a graph $\gamma$ we may define the mass as

$$m(\Psi, \gamma) = \sum_{\gamma} m(\Psi, v) = \sum_{\lambda} \int \sum_{\eta} C_{\lambda_{\eta}}(\Gamma_{\eta}) \left| C_{\lambda_{\eta}}(\Gamma_{\eta}) \right|^2 \sum_{v} e^2(\Gamma_{\eta} + a). \quad (113)$$

This means that $\Gamma(v) + a$ can be interpreted as ‘mass in a vertex’. If we look at a state where all $\Gamma$ are zero—the ‘vacuum’ (note that it is not the usual vacuum since we are not in the Fock space representation)—we obtain states with mass $ea$. But one may ask whether this is an observable or not. If one checks the commutator of the constraints and $\hat{m}$ the only non-vanishing term will be the $[\hat{H}_P, \hat{m}]$ commutator, which is proportional to $X(v)$. This means that if we take the subset of the solutions where $X(v)\Psi = 0$, the mass operator will be an observable. But if we look at the action of $X(v)$ in our new basis in (88) one will find that this is equivalent to condition (109). So $\hat{m}$ is an observable if the solutions are those which are equivalent to the solutions of the Proca field. But one may say that there are other solutions as well, since one does not have to impose (110). The answer is that these states are special cases which are contained in the Proca solution. This is because in this case one has to solve $\sum_{\Gamma} H_{P,_{\Gamma}}(\Gamma) C_{_{\Gamma}}(\Gamma)$ for all $\Gamma$, which means that the solution will have to be in the kernel of all matrices appearing in $H_{P,_{\Gamma}}$.

All in all the mass operator is an observable if the solutions are those which are equivalent to the solutions of the Proca field. Since the Proca field did not have a potential term, the correspondence is correct only if we consider states where all $\Gamma$ are zero (note that in other states the mass operator is also an observable, but it describes interactions).

7. Summary and open questions

In this paper we analyzed the mass generation to a $U(1)$ vector field via spontaneous symmetry breaking in LQG and compared the results obtained for the Proca field. Even at the classical level—after introducing new variables $\eta$ and $\Theta$—the two theories had many similarities. The main difference was the extra scalar field and the potential term in the case of the spontaneous symmetry breaking, and where the Proca field had a mass parameter, we obtained a field. Thus it was not a surprise that the quantized theories were also similar, and in the case of spontaneous symmetry breaking mass became an operator. We defined a new basis in the quantum region, where the motivation was to find the eigenstates of the configuration variable of the scalar field. By choosing this new basis we were also able to rewrite the constraints to finite linear systems of differential equations, thus we were able to analyze only the scalar field dependence of the theory. We were able to (partially) solve the constraints and describe the behavior of the states when $\Gamma + a$ (i.e. the mass) tends to zero. We found that there exist states which are non-degenerate in this region, and furthermore at $\Gamma + a = 0$ there is only one non-zero amplitude, the one which belongs to a solution to the ‘gravity coupled to the electromagnetic field’ case.
The eigenvalues of the mass operator are continuous (though they have a discrete structure due to the discreteness appearing in the coefficient matrices) and real, but not necessarily positive (one needs an extra input for this). A very interesting result is that the mass operator is an observable if the states are in the kernel of the corresponding momentum operator. This extra condition implies that in this case the other scalar field also becomes a gauge. This means that if we want a physically relevant mass operator, the scalar fields will not be real particles.

In the light of our results we can claim that—though the two theories are very similar—the spontaneous symmetry breaking has more advantages: (1) we are able to calculate the mass dependence of the states without solving the entire theory and (2) we can produce the limit \( m \to 0 \) without difficulty (the states have non-singular solutions). (3) In the case of the Proca field for different \( m \) we have different theories, while the spontaneous symmetry breaking deals with all values. This is important if we want to calculate transition amplitudes between states that have different masses. (4) The mass is an eigenvalue of an operator which can be an observable, while in the case of the Proca field mass is a parameter.

In our analysis it was crucial that the scalar field had a commutative group, otherwise the eigenvalues of the configuration operator would have been hard to find. Thus it is an interesting question that in the case of non-commutative groups how can we generalize these results? But the commutative case also provides a few questions, like the complete analysis of the differential equation (98). The main question is: what kind of restrictions do we have to make on the coefficient matrices to have a well-defined, non-singular square-integrable solution?

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Appendix A

Here we introduce a notation which simplifies the scalar constraint. Let us introduce a multi-index \( I \) for the indices \( s, f, \Gamma, \delta \) so that a type of expression \( \langle s'|f'|\Gamma|\delta\rangle|\hat{X}|\langle f|\Gamma|\delta\rangle|s\rangle \) will be denoted as \( X_{I'} \). Now consider those terms that do not contain \( U_\eta \) or \( \hat{X}_\eta \). These are \( \hat{H}_{\text{grav}} \) and \( \hat{H}_{YM} \), the Hamilton operator of the gravitational and Maxwell fields. So in our new notation the contribution of these terms to the constraint equations will be the following:

\[
\sum_{s',f',\Lambda,\Omega} \int d\Gamma(v)' \int d\hat{\Gamma}(v)' \left( H_{s'f'}^{G}\delta_{\Lambda}\delta_{\Omega} \delta(\Gamma(v)', \Gamma(v)) \delta(\hat{\Gamma}(v)', \hat{\Gamma}(v)) \right) \\
+ \sum_{f',\Lambda,\Omega} \int d\Gamma(v)' \int d\hat{\Gamma}(v)' \left( H_{s'f'}^{YM}\delta_{\Lambda}\delta_{\Omega} \delta(\Gamma(v)', \Gamma(v)) \delta(\hat{\Gamma}(v)', \hat{\Gamma}(v)) \right) C_{s'f',\Lambda,\Omega}(\Gamma', \hat{\Gamma}') \equiv \sum_{I} H_{1I}^{GYM}(v) C_{I}(\Gamma),
\]

where we performed the integration and sum on the variables related to the scalar fields.

The terms containing \( U_\eta \) or \( \hat{X}_{\eta} \) will be treated as follows: we shall write the dependence of these fields explicitly, while other expressions will be denoted (using the short notation) as \( O_{1I} \), etc. For example the contribution of the potential term \( \hat{H}_{\text{pot}} \) will be denoted as
one can simplify this expression in the following way:

\[
\sum_{s',f',\lambda',\lambda} \int d\Gamma(v) \int d\Gamma(v') \left[ \frac{1}{4} N(v) \mu(s'|s') \delta_{ff'} \delta(\Gamma', \Gamma') \delta_{\lambda,\lambda} \right] \frac{1}{\lambda^2} 
\times C_{s',f',\lambda,\lambda}(\Gamma', \Gamma)(\Gamma', \lambda) \arccos \left( \frac{U_\eta(\lambda, v) + U_\eta^{-1}(\lambda, v)}{2} \right) 
\times \left[ \frac{1}{\lambda} \arccos \left( \frac{U_\eta(\lambda, v) + U_\eta^{-1}(\lambda, v)}{2} \right) + 2a \right]^2 |\Gamma', \lambda'\rangle
\]

\[
= \sum_f \tilde{H}_{\text{det}}^\text{pe}(v) C_f(\Gamma(\nu)) (\Gamma(v) + 2a)^2. \quad (A.2)
\]

The other derivative term is a bit trickier since the \( U_\eta \) is evaluated in different vertices. We have

\[
\sum_{s',f',\lambda',\lambda} \int d\Gamma(v) \int d\Gamma(v') \left[ \frac{1}{2} \frac{N(v)}{E(v)^2} \right] \sum_{\nu(\Delta)=v} \frac{1}{\delta^2 (\Gamma', \lambda') [U_\eta (\Delta)] U_\eta^{-1} (\lambda, v) - 1]}
\times \left[ U_\eta (\lambda, s_\eta (\Delta)) U_\eta^{-1} (\lambda, v) - 1 \right] |\Gamma', \lambda'\rangle \langle s'|G_2^{\text{mr}}(v) |s'\rangle C_{s',f',\lambda',\lambda}(\Gamma', \Gamma')
\]

\[
= \sum_f \tilde{H}_{\text{det}}^\text{pe}(v) C_f(\Gamma(\nu)) (\Gamma(v) + a)^2. \quad (A.3)
\]

where the values \( \exp(\lambda \Gamma(s_\eta(\Delta))) \), etc have been assimilated in the coefficients \( H_{\text{det}}^\text{pe} \), etc since our differential equation will depend only on variables in vertex \( v \).

The last contribution is the momentum term. It is convenient to use equation (94) so that one can simplify this expression in the following way:

\[
\sum_{s',f',\lambda',\lambda} \int d\Gamma(v) \int d\Gamma(v') \left[ \frac{1}{2} \frac{N(v)}{E(v)^2} \right] (\Gamma', \lambda') \delta_{\lambda,\lambda} [\delta(f, f') \delta(\Gamma', \Gamma')]
\times \langle \Gamma', \lambda' | X_\eta(v^2) | \Gamma', \lambda' \rangle \arccos \left( \frac{U_\eta(\lambda, v) + U_\eta^{-1}(\lambda, v)}{2} \right) + a \right]^2 |\Gamma', \lambda'\rangle
\]

\[
= \sum_f \tilde{H}_{\text{det}}(v) \int d\Gamma(v) \int d\Gamma(v') \left[ \frac{1}{2} \frac{N(v)}{E(v)^2} \right] (\Gamma', \lambda') \delta_{\lambda,\lambda} [\delta(f, f') \delta(\Gamma', \Gamma')]
\times \langle \Gamma', \lambda' | X_\eta(v^2) | \Gamma', \lambda' \rangle + \frac{L(v)^2}{(\Gamma(v) + a)^2}
\]
\[ \times \delta(\Gamma(v'), \Gamma(v))\delta(E) \langle s | \hat{G}_1(v) | x' \rangle \delta(\Gamma(x'), \Gamma(x)) \langle x' | \hat{C}_{s', f', t', \delta} (\Gamma', \Gamma) \rangle \]

\[ = \sum_{\ell} \left( \left( \lambda_{\ell} - i \frac{\delta}{\delta \Gamma(v)} \right)^2 + \frac{L(v)^2}{(\Gamma(v) + a)^2} \right) H_{I' I}^p C_{I, I'} (\Gamma), \]  

(A.5)

where \( L(v) = \sum_v \ell_v \) is the sum.

### Appendix B

Here we describe the method to solve a system of differential equation of the form

\[ \ddot{\vec{c}}(t) = \vec{H}(t)\vec{c}(t), \]  

(B.1)

where \( \vec{H}(t) \) is a \( N \times N \) matrix.

The method is similar to that used in cases of path ordered integration. First we integrate the equation:

\[ \vec{c}(t) = \int_0^t dt_1 \vec{H}(t_1)\vec{c}(t_1) + \vec{c}_1, \]  

(B.2)

where

\[ \vec{c}_1 = \dot{\vec{c}}(0). \]

Another integration yields

\[ \vec{c}(t) = \int_0^t dt_1 \int_0^{t_1} dt_2 \vec{H}(t_2)\vec{c}(t_2) + \vec{c}_1 t + \vec{c}_0, \]  

(B.3)

where

\[ \vec{c}_0 = \vec{c}(0). \]

Now we iterate this equation and arrive at the result

\[ \vec{c}(t) = \left( 1 + \sum_{j=1}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{2j-1}} dt_{2j} \vec{H}(t_{2j}) \vec{H}(t_{2j-1}) \cdots \vec{H}(t_4) \vec{H}(t_3) \vec{H}(t_2) \right) \vec{c}_0 \]

\[ + \left( t + \sum_{j=1}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{2j-1}} dt_{2j} \vec{H}(t_{2j}) \vec{H}(t_{2j-1}) \cdots \vec{H}(t_4) \vec{H}(t_3) \right) \vec{c}_1. \]  

(B.4)

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