Regularity of the solutions for nonlinear biharmonic equations in $\mathbb{R}^N$ *

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Abstract

The purpose of this paper is to establish the regularity the weak solutions for the nonlinear biharmonic equation

$$\begin{aligned}
\Delta^2 u + a(x)u &= g(x, u), \\
    u &\in H^2(\mathbb{R}^N),
\end{aligned}$$

where the condition $u \in H^2(\mathbb{R}^N)$ plays the role of a boundary value condition, and as well expresses explicitly that the differential equation is to be satisfied in the weak sense.

Key words and phrases: Nonlinear biharmonic equation, regularity, fundamental solutions.

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1 Introduction

The purpose of this paper is to establish the regularity of the weak solutions for a certain nonlinear biharmonic equation in $\mathbb{R}^N$. We consider solutions $u: \mathbb{R}^N \to \mathbb{R}$ of the problem

$$\begin{aligned}
\Delta^2 u + a(x)u &= g(x, u), \\
    u &\in H^2(\mathbb{R}^N),
\end{aligned} \tag{1.1}$$

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where the condition \( u \in H^2(\mathbb{R}^N) \) plays the role of a boundary value condition, and as well expresses explicitly that the differential equation is to be satisfied in the weak sense. We assume that

\[
H_1) \quad g(x, u): \mathbb{R}^N \times \mathbb{R}^1 \to \mathbb{R}^1 \text{ is measurable in } x \text{ and continuous in } u, \quad \sup_{x \in \mathbb{R}^N} |g(x, u)| < \infty \text{ for every } M > 0;
\]

\[
H_2) \quad \text{there exist two constants } \sigma > \delta > 0 \text{ and two functions } b_1(x), b_2(x) \in L^\infty(\mathbb{R}^N) \text{ such that } |g(x, u)| \leq b_1(x)|u|^{\delta+1} + b_2(x)|u|^\sigma+1;
\]

\[
H_3) \quad \lim_{|x| \to \infty} a(x) = k^2 > 0 \text{ with } k > 0 \text{ and } (k^2 - a(x)) \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N).
\]

Then we have the following theorems:

**Theorem 1.1.** Assume that \( H_1) \) to \( H_3) \) hold with \( \sigma + 1 < \frac{N+4}{N-4} \) if \( N \geq 5 \). Let \( u \) be a weak solution of (1.1). Then \( u \in H^4(\mathbb{R}^N) \cap W^{2,s}(\mathbb{R}^N) \) for \( 2 \leq s \leq +\infty \). In particular \( u \in C^2(\mathbb{R}^N) \) and \( \lim_{|x| \to \infty} u(x) = 0, \lim_{|x| \to \infty} \Delta u(x) = 0 \).

Dealing with regularity of solutions is much more complicated for biharmonic equations than for problems that can be treated by well-developed standard methods, such as second-order elliptic problems. First of all, there is no maximum principle for the biharmonic problem. So we can’t get some estimates of the solutions by the methods used to deal with second-order elliptic problems. Secondly, we know little about the properties of the eigenfunctions of the biharmonic operator in \( \mathbb{R}^N \). To overcome these difficulties, we first introduce the fundamental solutions for the linear biharmonic operator \( \Delta^2 + k^2 \) for \( k > 0 \). By applying some properties of Hankel functions, which are the solutions of Bessel’s equation, we obtain the asymptotic representation of the fundamental solution of \( \Delta^2 + k^2 \) at \( \infty \) and \( 0 \). Then we prove that, for \( p > 1 \),

\[
\Delta^2 - \lambda: \quad W^{2,p}(\mathbb{R}^N) \to L^p(\mathbb{R}^N)
\]

is an isomorphism if \( \lambda < 0 \). Some estimates of the solutions of (1.1) can be obtained from the properties of the fundamental solutions of \( \Delta^2 - \lambda \). We also establish some \( L^p \) theory for the biharmonic problem (1.1) so that a bootstrap argument can be used to deduce the regularity of the solutions of the biharmonic problem (1.1). Please refer to Grunau [3], Jannelli [5], Noussair, Swanson and Yang [8], Peletier and Van der Vorst [9], Pucci and Serrin [10] for the early results on the existence and other properties of solutions associated with biharmonic operators.

The organization of this paper is as follows: In Section 2 we introduce the fundamental solutions of \( \Delta^2 - \lambda \) for \( \lambda < 0 \) and establish some properties of these fundamental solutions.
In Section 3, we show that a weak solution of the linear problem
\[
\begin{align*}
\Delta^2 u - \lambda u &= f(x), \\
u \in H^2(\mathbb{R}^N),
\end{align*}
\] belongs to $H^4(\mathbb{R}^N)$ whenever $f \in L^2(\mathbb{R}^N)$. In Section 4, we obtain a sharper relationship between the regularity of the weak solutions of the linear biharmonic problem (1.2) and the properties of the inhomogeneous term $f$ in (1.2). In Section 5, we establish the regularity of the weak solutions for the nonlinear problem (1.1).

2 Fundamental solutions

In this section, we give some properties of the fundamental solutions for the biharmonic operators $\Delta^2 + k^2$. The proof of these properties can be find in [2].

**Lemma 2.1.** Let $G_k^{(N)}(|x|)$ be the fundamental solutions of biharmonic operator $\Delta^2 + k^2$ for $k > 0$ and $g_\delta^{(N)}(|x|)$ be the fundamental solutions of Laplace operator $-\Delta + \delta$. Then we have

i) $G_k^{(N)}(x) \in C^\infty(\mathbb{R}^N) \setminus \{0\}$

and

$$\Delta^2 G_k^{(N)}(x) + k^2 G_k^{(N)}(x) = 0 \quad \text{for } x \neq 0. \quad (2.20)$$

ii) As $|x| \to \infty$,

$$e^{(\sqrt{k}/\sqrt{2})|x|}G_k^{(N)}(x) \to 0 \quad \text{and} \quad e^{(\sqrt{k}/\sqrt{2})|x|}\left|\nabla G_k^{(N)}(x)\right| \to 0. \quad (2.21)$$

iii) As $|x| \to 0$,

$$G_k^{(N)}(r) = \frac{2^{\nu-2}\Gamma(\nu - 1)}{2(2\pi)^{N/2}} r^{2-2\nu} + O(r^{4-2\nu}) \quad \text{if } \nu = \frac{N-2}{2} > 1 \text{ and } \nu \notin \mathcal{N};$$

$$G_k^{(N)}(r) = \frac{2^{\nu-2}\Gamma(\nu - 1)}{2(2\pi)^{N/2}} r^{2-2\nu} + O(r^{4-2\nu} + \ln r) \quad \text{if } \nu = \frac{N-2}{2} \geq 2 \text{ and } \nu \in \mathcal{N};$$

$$G_k^{(N)}(r) \approx O(\ln r) \quad \text{if } N = 4 \left(\nu = \frac{N-2}{2} = 1\right);$$

$$G_k^{(N)}(r) = O(1) \quad \text{if } N = 2, 3 \left(\nu = 0, \frac{1}{2}\right).$$
iv) \(|G_k^{(N)}(r)| \leq Cg_\delta^{(N)}(r)| for some positive constants \(C\) and \(0 < \delta < \frac{\sqrt{N}}{\sqrt{2}}\).

It follows from properties ii) and iii) of Lemma 2.1 that:

**Corollary 2.2.**

\[
\begin{align*}
G_k^{(N)}(x) &\in L^p(\mathbb{R}^N) \quad \text{for } 1 \leq p \leq +\infty, \quad \text{if } N = 2, 3, \\
G_k^{(N)}(x) &\in L^p(\mathbb{R}^N) \quad \text{for } 1 \leq p < +\infty, \quad \text{if } N = 4, \\
G_k^{(N)}(x) &\in L^p(\mathbb{R}^N) \quad \text{for } 1 \leq p < \frac{N}{N-4}, \quad \text{if } N \geq 5, \\
|\nabla G_k^{(N)}(x)| &\in L^p \quad \text{for } 1 \leq p < \frac{N}{N-3}, \quad \text{if } N > 3, \\
|\nabla G_k^{(N)}(x)| &\in L^p \quad \text{for } 1 \leq p < +\infty, \quad \text{if } N = 3, \\
|\nabla G_k^{(N)}(x)| &\in L^p \quad \text{for } 1 \leq p \leq +\infty, \quad \text{if } N = 2, \\
|\Delta G_k^{(N)}(x)| &\in L^p \quad \text{for } 1 \leq p < \frac{N}{N-2}, \quad \text{if } N \geq 3, \\
|\Delta G_k^{(N)}(x)| &\in L^p \quad \text{for } 1 \leq p < +\infty, \quad \text{if } N = 2.
\end{align*}
\] (2.24)

Using this information about \(G_k^{(N)}(x)\), we can express solutions of the inhomogeneous biharmonic equation as convolutions of fundamental solutions with the inhomogeneous term. The following Theorem can also be found in [2].

**Theorem 2.3.**

i) Let \(f \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)\) and

\[
u = \int_{\mathbb{R}^N} f(z)G_k^{(N)}(x-z) \, dz.
\]

Then

\[
\Delta^2 u + k^2 u = f(x).
\]

ii) Let \(u\) be a distribution such that

\[
\Delta^2 u + k^2 u = f
\]

and \(f \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)\). Then

\[
u = \int_{\mathbb{R}^N} f(z)G_k^{(N)}(x-z) \, dz.\] (2.25)

iii) There are no nontrivial distributions such that

\[
\begin{cases}
\Delta^2 u + k^2 u = 0, \\
u \in W^{2,2}(\mathbb{R}^N).
\end{cases}
\] (2.26)
3 \( H^4 \)-regularity

The main purpose of this section is to show that a weak solution of the linear problem

\[
\begin{align*}
\Delta^2 u - \lambda u &= f(x), \\
u &\in H^2(\mathbb{R}^N),
\end{align*}
\]

(3.1)

belongs to \( H^4(\mathbb{R}^N) \) whenever \( f \in L^2(\mathbb{R}^N) \). To this end, we recall a well-known result which can be found in [11].

**Lemma 3.1.** Let \( h \in L^p(\mathbb{R}^N) \) for some \( p \in [1, +\infty] \), and consider the equation

\[-\Delta u + u = h \quad (3.2)\]

in the sense of distributions.

a) There is a unique tempered distribution \( u = \Gamma(h) \) satisfying (3.2).

b) If \( h \in L^p(\mathbb{R}^N) \) for some \( p \in (1, +\infty) \), then \( \Gamma(h) \in W^{2,p}(\mathbb{R}^N) \) and there exists a constant \( C(N, p) \) such that

\[||\Gamma(h)||_{W^{2,p}} \leq C(N, p) ||h||_{L^p}\]

for all \( h \in L^p(\mathbb{R}^N) \).

c) For \( p \in (1, +\infty) \), \( -\Delta + 1 : W^{2,p}(\mathbb{R}^N) \to L^p(\mathbb{R}^N) \) is an isomorphism.

By applying this lemma, we can obtain the \( W^{4,p}(\mathbb{R}^N) \) regularity for the linear biharmonic problem.

**Lemma 3.2.** Let \( v \in W^{2,p}(\mathbb{R}^N) \), \( w \in L^p(\mathbb{R}^N) \) for some \( p \in (1, +\infty) \) be such that

\[\int_{\mathbb{R}^N} \Delta v \Delta z \, dx = \int_{\mathbb{R}^N} wz \, dx \quad \text{for all } z \in C^\infty_0(\mathbb{R}^N). \quad (3.3)\]

Then \( v \in W^{4,p}(\mathbb{R}^N) \) and \( \Delta^2 v = w \).

**Proof.** From (3.3), it follows that \( u = \Delta v \) is a distribution solution of

\[\Delta u = w \quad \text{and} \quad u, w \in L^p(\mathbb{R}^N). \quad (3.4)\]

Thus

\[(-\Delta + 1)u = u - w \in L^p(\mathbb{R}^N).\]
By applying Lemma 3.1 we find that \(-\Delta + 1: W^{2,p}(\mathbb{R}^N) \rightarrow L^p(\mathbb{R}^N)\) is an isomorphism. So there exists \(\varphi \in W^{2,p}(\mathbb{R}^N)\) such that

\[
(-\Delta + 1)\varphi = u - w,
\]

that is

\[
-\int_{\mathbb{R}^N} \varphi \Delta z \, dx + \int_{\mathbb{R}^N} \varphi z \, dx = \int_{\mathbb{R}^N} uz \, dx - \int_{\mathbb{R}^N} wz \, dx
\]

for all \(z \in C_0^\infty(\mathbb{R}^N)\). From (3.3) we have

\[
\int_{\mathbb{R}^N} (\varphi - u) \Delta z \, dx = \int_{\mathbb{R}^N} (\varphi - u) z \, dx
\]

for all \(z \in C_0^\infty(\mathbb{R}^N)\) and hence

\[
\int_{\mathbb{R}^N} (\varphi - u)(-\Delta z + z) \, dx = 0 \quad \text{for all} \quad z \in C_0^\infty(\mathbb{R}^N).
\]

Consider the equation

\[
-\Delta z + z = |\varphi - u|^{p-2}(\varphi - u).
\]

It follows from \(\varphi - u \in L^p\) that \(|\varphi - u|^{p-2}(\varphi - u) \in L^{p'}(\mathbb{R}^N)\) with \(\frac{1}{p} + \frac{1}{p'} = 1\). By Lemma 3.1, the problem (3.6) possesses a unique solution \(z \in W^{2,p}(\mathbb{R}^N)\). Since \(C_0^\infty(\mathbb{R}^N)\) is dense in \(W^{2,p}(\mathbb{R}^N)\), we can find a sequence \(\{z_n\} \subset C_0^\infty(\mathbb{R}^N)\) such that

\[
z_n \rightarrow z \quad \text{in} \quad W^{2,p}(\mathbb{R}^N) \quad \text{as} \quad n \rightarrow \infty.
\]

From (3.5) and (3.6), we have

\[
0 = \int_{\mathbb{R}^N} (\varphi - u)(-\Delta z_n + z_n) \, dx \rightarrow \int_{\mathbb{R}^N} (\varphi - u)(-\Delta z + z) \, dx
\]

\[
= \int_{\mathbb{R}^N} |\varphi - u|^p \, dx.
\]

This implies that \(\varphi - u \equiv 0\) and hence \(u \in W^{2,p}(\mathbb{R}^N)\). It follows that \(v \in W^{4,p}(\mathbb{R}^N)\).

To obtain the \(H^4\)-regularity of solutions of (3.1), we rewrite the problem (3.1) in the form

\[
\begin{cases}
\Delta^2 u = f + \lambda u, \\
u \in H^2(\mathbb{R}^N).
\end{cases}
\]

The \(H^4\)-regularity of solutions of (3.1) follows from Lemma 3.2. In fact, we can get a more general result:

**Lemma 3.3.** Let \(f \in L^p(\mathbb{R}^N)\) for some \(p \in (1, +\infty)\), and let \(u\) be the solution of the linear biharmonic problem

\[
\begin{cases}
(\Delta^2 - \lambda) u = f, \\
u \in W^{2,p}(\mathbb{R}^N).
\end{cases}
\]

Then \(u \in W^{4,p}(\mathbb{R}^N)\).
In the following lemma, we show that the problem (3.7) possesses a unique solution $u \in W^{2,p}(\mathbb{R}^N)$ for given $p \in [2, +\infty)$ if $f \in L^p(\mathbb{R}^N)$ and $\lambda < 0$.

**Lemma 3.4.** For $p \in [2, +\infty)$, $f \in L^p(\mathbb{R}^N)$, the problem (3.7) possesses a unique solution if $\lambda < 0$.

**Proof.** From Lemma 3.3, the solution of (3.7) must belong to $W^{4,p}(\mathbb{R}^N)$. Suppose that $u \in W^{4,p}(\mathbb{R}^N)$ is a solution of the homogeneous problem

$$
\begin{cases}
\Delta^2 u - \lambda u = 0, \\
u \in W^{4,p}(\mathbb{R}^N).
\end{cases}
$$

Rewrite (3.8) in the form

$$
\begin{cases}
-\Delta(-\Delta u) = \lambda u, \\
u \in W^{4,p}(\mathbb{R}^N), -\Delta u \in W^{2,p}(\mathbb{R}^N).
\end{cases}
$$

By using a bootstrap argument, it follows that

$$
u \in C^4(\mathbb{R}^N) \cap L^p(\mathbb{R}^N), \quad \Delta u \in C^2(\mathbb{R}^N) \cap L^p(\mathbb{R}^N),$$

and $\lim_{|x| \to \infty} u(x) = 0$, $\lim_{|x| \to \infty} \Delta u(x) = 0$. Define $u_1 = (\Delta - \sqrt{\lambda}) u$, $u_2 = (\Delta + \sqrt{\lambda}) u$. Then

$$
(\Delta + \sqrt{\lambda}) u_1 = 0, \quad (\Delta - \sqrt{\lambda}) u_2 = 0,
$$

and

$$u = \frac{1}{2\sqrt{\lambda}} (u_2 - u_1),$$

$$\lim_{|x| \to \infty} u_1(x) = 0, \quad \lim_{|x| \to \infty} u_2(x) = 0.$$

For $\lambda < 0$, the solution of (3.9) can be expressed in terms of Hankel functions. By the asymptotic behavior of Hankel functions (see (2.8)), we can deduce that

$$e^{\text{Im}(\lambda)^{1/4}|x|} u_i(x) \to 0, \quad i = 1, 2, \quad \text{as } |x| \to \infty.$$

Thus we have

$$e^{\text{Im}(\lambda)^{1/4}|x|} u(x) \to 0 \quad \text{as } |x| \to \infty;$$

it follows from (3.10) that $u \in L^r(\mathbb{R}^N)$ for all $r \in [2, +\infty)$. In particular, $u \in L^2(\mathbb{R}^N)$ and hence $u \in H^4(\mathbb{R}^N)$. Theorem 2.3 gives us that $u \equiv 0$.

This completes the proof of our lemma. \qed
4 $W^{2,p}(\mathbb{R}^N)$-regularity

In this section, we obtain a sharper relationship between the regularity of the weak solutions of the linear biharmonic problem (3.1) and the properties of the inhomogeneous term $f$ in (3.1).

Recalling the properties of the fundamental solution $G_k^{(N)}$ for $k > 0$ (see Corollary 2.2), Young’s inequality for convolutions [13] shows that the convolution $f * G_k^{(N)}$ defines an element of $L^s(\mathbb{R}^N)$ subject to the following restrictions:

\[
\begin{cases}
  p \leq s \leq +\infty & \text{if } p > \frac{N}{4}, \\
  p \leq s < +\infty & \text{if } p = \frac{N}{4}, \\
  p \leq s \leq \frac{Np}{N-4p} & \text{if } 1 \leq p < \frac{N}{4}.
\end{cases}
\] (4.1)

Setting $T_k f = f * G_k^{(N)}$, we see that

\[ T_k: L^p(\mathbb{R}^N) \to L^s(\mathbb{R}^N) \text{ is a bounded linear operator.} \] (4.2)

Referring again to Corollary 2.2 we can deduce that for $i = 1, 2, \ldots, N$, the convolution $f * \partial_i G_k^{(N)}$ defines an element of $L^s$ whenever $f \in L^p$ subject to the restrictions

\[
\begin{cases}
  p \leq s \leq +\infty & \text{if } p > \frac{N}{3}, \\
  p \leq s < +\infty & \text{if } p = \frac{N}{3}, \\
  p \leq s < \frac{Np}{N-3p} & \text{if } 1 \leq p < \frac{N}{3};
\end{cases}
\] (4.3)

and the convolution $f * \Delta G_k^{(N)}$ defines an element of $L^s$ whenever $f \in L^p$ subject to the restrictions

\[
\begin{cases}
  p \leq s \leq +\infty & \text{if } p > \frac{N}{2}, \\
  p \leq s < +\infty & \text{if } p = \frac{N}{2}, \\
  p \leq s < \frac{Np}{N-2p} & \text{if } 1 \leq p < \frac{N}{2}.
\end{cases}
\] (4.4)

Setting $S_k^i f = f * \partial_i G_K^{(N)}$, $i = 1, 2, \ldots, N$, and $S_k^\Delta f = f * \Delta G_k^{(N)}$, we see that

\[ S_k^i: L^p \to L^s \text{ is a bounded linear operator} \] (4.5)

under the restrictions (4.3) and

\[ S_k^\Delta: L^p \to L^s \text{ is a bounded linear operator} \] (4.6)

under the restrictions (4.4).
THEOREM 4.1. Given $k > 0$ and $f \in C^2_0(\mathbb{R}^N)$, set

$$T_k f(x) = f \ast G^{(N)}_k(x) \quad \text{for } x \in \mathbb{R}^N.$$  \hfill (4.7)

Then $T_k f \in C^4(\mathbb{R}^N)$, $\lim_{|x| \to \infty} T_k f(x) = 0$, and $u = T_k f$ satisfies the biharmonic equation

$$\Delta^2 u = \lambda u + f \quad \text{on } \mathbb{R}^N,$$  \hfill (4.8)

where $\lambda = -k^2$. Furthermore, for all $x \in \mathbb{R}^N$, the following formulae are valid for $i, j, l, m = 1, 2, \ldots, N$:

$$T_k f(x) = \int f(x - z)G^{(N)}_k(z) \, dz = \int G^{(N)}_k(x - z)f(z) \, dz,$$

$$\partial_i T_k f(x) = \int \partial_i f(x - z)G^{(N)}_k(z) \, dz = \int \partial_i G^{(N)}_k(x - z)f(z) \, dz,$$

$$\partial_j \partial_i T_k f(x) = \int \partial_j f(x - z)\partial_i G^{(N)}_k(z) \, dz,$$

$$\partial_m \partial_j \partial_i T_k f(x) = \int \partial_m \partial_i f(x - z)\partial_j G^{(N)}_k(z) \, dz = \int \partial_i f(x - z)\partial_m \partial_j G^{(N)}_k(z) \, dz,$$

$$\partial_i \partial_m \partial_j \partial_l T_k f(x) = \int \partial_m \partial_j f(x - z)\partial_l \partial_i G^{(N)}_k(z) \, dz.$$

Proof. Noting that

$$T_k f(x) = T_1 f_k(kx) \quad \text{where } f_k(y) = k^{-4} f \left( \frac{y}{k} \right),$$

we see that, by a change of scale, it is enough to treat the case $k = 1$. In the following, we take $k = 1$ and simplify the notation by setting $T_1 = T$, $G^{(N)}_1 = G^{(N)}$. Since $f$, $\partial_i f$ and $\partial_{ij} f \in C^0(\mathbb{R}^N)$ and $\partial_i G^{(N)}$, $\partial_{ij} G^{(N)} \in L^1(\mathbb{R}^N)$, it follows that the convolutions

$$T f = f \ast G^{(N)}, \quad \partial_i f \ast G^{(N)}, \quad f \ast G^{(N)},$$

$$\partial_i f \ast \partial_j G^{(N)}, \quad \partial_{ij} f \ast \partial_m G^{(N)}, \quad \partial_i f \ast \partial_{jm} G^{(N)}, \quad \text{and } \partial_{ij} f \ast \partial_{ml} G^{(N)}$$

are defined and are continuous on $\mathbb{R}^N$. They all tend to zero as $|x| \to \infty$. Hence to prove the theorem it is sufficient to establish the following statements.

1) $\partial_i T f$ exists and $\partial_i T f = \partial_i f \ast G^{(N)}$.

2) $\partial_i f \ast G^{(N)} = f \ast \partial_i G^{(N)}$.

3) $\partial_{ij} T f$ exists and $\partial_{ij} T f = \partial_{ij} f \ast \partial_i G^{(N)}$.

4) $\partial_{mij} T f$ exists and $\partial_{mij} T f = \partial_{mij} f \ast \partial_i G^{(N)} \ast \partial_i G^{(N)}$.
5) $\partial_{mji}Tf$ exists and $\partial_{mji}Tf = \partial_m\partial_jf*\partial_i G^{(N)}$.

6) $\Delta^2Tf + Tf = f$ on $\mathbb{R}^N$.

(1) Let $e_i$ be an element of the usual basis for $\mathbb{R}^N$ and $h$ a non-zero real number. Then

$$\frac{Tf(x + he_i) - Tf(x)}{h} = \int_{\mathbb{R}^N} \frac{f(x + he_i - z) - f(x - z)}{h} G^{(N)}(z) \, dz$$

and

$$\lim_{h \to 0} \frac{f(x + he_i - z) - f(x - z)}{h} = \partial_i f(x - z).$$

Also

$$\left| \frac{f(x + he_i - z) - f(x - z)}{h} \right| \leq \frac{1}{h} \int_0^1 \frac{d}{dt} f(x + the_i - z) \, dt$$

$$= \left| \int_0^1 \partial_i f(x + the_i - z) \, dt \right|$$

$$\leq \max_{z \in \mathbb{R}^N} |\partial_i f(z)| = |\partial_i f|_\infty.$$

Hence, by the dominated convergence theorem,

$$\lim_{h \to 0} \frac{Tf(x + he_i) - Tf(x)}{h} = \int_{\mathbb{R}^N} \partial_i f(x - z) G^{(N)}(z) \, dz.$$

(2) For $i = 1, 2, \ldots, N$,

$$\partial_i G^{(N)}(x) = \lim_{\epsilon \to 0} \int_{|z| \geq \epsilon} \partial_i f(x - z) G^{(N)}(z) \, dz$$

$$= -\lim_{\epsilon \to 0} \int_{|z| \geq \epsilon} \frac{\partial}{\partial z_i} f(x - z) G^{(N)}(z) \, dz$$

$$= \lim_{\epsilon \to 0} \left\{ \int_{|z| = \epsilon} \frac{z_i}{|z|^2} f(x - z) G^{(N)}(z) \, dz + \int_{|z| \geq \epsilon} f(x - z) \partial_i G^{(N)}(z) \, dz \right\}.$$

Now from Lemma 2.1

$$\left| \int_{|z| = \epsilon} \frac{z_i}{|z|^2} f(x - z) G^{(N)}(z) \, dz \right| \leq |f|_\infty \int_{|z| = \epsilon} |G^{(N)}(z)| \, dz$$

$$= |f|_\infty |G^{(N)}(\epsilon)| \int_{|z| = \epsilon} z \, dz = |f|_\infty |G^{(N)}(\epsilon)| \cdot w_N \epsilon^{N-1}$$

$$= \begin{cases} 2^{\nu-2} \Gamma(\nu - 1) / (2\pi)^{N/2} \epsilon^{4-N} w_N \epsilon^{N-1} |f|_\infty & \text{if } N \geq 5, \\ O(\ln \epsilon) \epsilon^{N-1} w_N |f|_\infty & \text{if } N = 4, \\ O(1) \epsilon^{N-1} w_N |f|_\infty & \text{if } N = 2, 3. \end{cases}$$
where \( \nu = \frac{N^2}{2} \). Hence

\[
\lim_{\epsilon \to 0} \int_{|z| = \epsilon} \frac{z_i}{|z|^2} f(x - z) G^{(N)}(z) \, dz = 0
\]

and

\[
\partial_i f \ast G^{(N)}(x) = \lim_{\epsilon \to 0} \int_{|z| \geq \epsilon} f(x - z) \partial_i G^{(N)}(z) \, dz = f \ast \partial_i G^{(N)}.
\]

(3) Repeat the proof of (1) with \( G^{(N)} \) replaced by \( \partial_i G^{(N)} \).

(4) Repeat the proof of (1) and (2) with \( G^{(N)} \) and \( f \) replaced by \( \partial_i G^{(N)} \) and \( \partial_j f \).

(5) Repeat the proof of (1) with \( G^{(N)} \) and \( f \) replaced by \( \partial_i \partial_l G^{(N)} \) and \( \partial_j f \).

(6)

\[
\Delta^2 T f(x) = \int \Delta f(x - z) \Delta G^{(N)}(z) \, dz
\]

\[
= \lim_{\epsilon \to 0} \int_{|z| \geq \epsilon} \Delta z f(x - z) \Delta G^{(N)}(z) \, dz
\]

\[
= \lim_{\epsilon \to 0} \left\{ \int_{|z| = \epsilon} f(x - z) \frac{\partial \Delta G^{(N)}}{\partial r} - \Delta G^{(N)} \frac{\partial f(x - z)}{\partial r} \, dz 
\right. 

\left. + \int_{|z| \geq \epsilon} f(x - z) \Delta^2 G^{(N)}(z) \, dz \right\}
\]

\[
= \lim_{\epsilon \to 0} \int_{|z| = \epsilon} \left( f(x - z) \frac{\partial G^{(N)}}{\partial r} - \frac{\partial \Delta G^{(N)}}{\partial r} \frac{\partial f(x - z)}{\partial r} \right) \, dz - T f(x).
\]

Since

\[
\Delta^2 G^{(N)}(z) = -G^{(N)}(z) \text{ for all } z \neq 0,
\]

we obtain that

\[
\lim_{\epsilon \to 0} \int_{|z| = \epsilon} \Delta G^{(N)} \frac{\partial f(x - z)}{\partial r} \, dz = 0, \quad (4.9)
\]

\[
\lim_{\epsilon \to 0} \int_{|z| = \epsilon} f(x - z) \frac{\partial}{\partial r} \left( \Delta G^{(N)}(z) \right) \, dz = f(x). \quad (4.10)
\]

In fact, from (2.13),

\[
(G^{(N)}(r))' = -2\pi r G^{(N+2)}(r).
\]

Thus

\[
(G^{(N)}(r))'' = 2^2 \pi^2 r^2 G^{(N+4)}(r) - 2\pi G^{(N+2)}(r),
\]

\[
\Delta G^{(N)}(x) = (G^{(N)}(r))'' + \frac{N - 1}{r} (G^{(N)}(r))'
\]

\[
= 4\pi^2 r^2 G^{(N+4)}(r) - 2\pi NG^{(N+2)}(r),
\]

\[
(\Delta G^{(N)}(r))' = -16\pi^3 r^3 G^{(N+6)}(r) + (8 + 4N)\pi^2 r G^{(N+4)}(r).
\]
By the asymptotic behavior of $G_k^{(N)}(r)$ (see Lemma 2.1) we deduce that, as $r = |x| \to 0$,

$$
\Delta G^{(N)}(x) \approx 4\pi^2 \tau^2 G^{(N+4)}(r) \approx \frac{2^{(N/2)-1} \Gamma \left( \frac{N}{2} \right)}{2(2\pi)^{N/2}} r^{2-N}, 
$$

(4.11)

$$
(\Delta G^{(N)}(r))'_r \approx -16\pi^3 \tau^3 G^{(N+6)}(r) \approx \frac{2^{N/2} \Gamma \left( \frac{N}{2} + 1 \right)}{2(2\pi)^{N/2}} r^{1-N}. 
$$

(4.12)

Thus

$$
\left| \int_{|z|=\epsilon} \Delta G^{(N)}(z) \partial f(x - z) \frac{\partial r}{\partial r} \, dz \right| \approx \left| \int_{|z|=\epsilon} \frac{2^{(N/2)-1} \Gamma \left( \frac{N}{2} \right)}{2(2\pi)^{N/2}} |z|^{2-N} \partial f(x - z) \frac{\partial r}{\partial r} \, dz \right|
$$

$$
\leq |\nabla f|_{L^\infty} \int_{|z|=\epsilon} \frac{2^{(N/2)-1} \Gamma \left( \frac{N}{2} \right)}{2(2\pi)^{N/2}} |z|^{2-N} \, dz
$$

$$
= |\nabla f|_{L^\infty} \frac{2^{(N/2)-1} \Gamma \left( \frac{N}{2} \right)}{2(2\pi)^{N/2}} \epsilon^{2-N} \epsilon^{-N} w_N \to 0 
$$
as $\epsilon \to 0$.

This gives (4.9).

Now we are going to prove (4.10). From (4.12) we have

$$
\epsilon^{N-1} (\Delta G^{(N)}(r))'_r \bigg|_{r=\epsilon} \to \frac{2^{N/2} \Gamma \left( \frac{N}{2} + 1 \right)}{(2\pi)^{N/2}} \quad \text{as } \epsilon \to 0.
$$

Thus

$$
\int_{|z|=\epsilon} f(x - z) (\Delta G^{(N)}(z))'_r \, dz = \int_{|z|=\epsilon} (f(x - z) - f(x)) (\Delta G^{(N)}(z))'_r \, dz
$$

$$
+ f(x) (\Delta G^{(N)}(r))'_r \bigg|_{r=\epsilon} \cdot \epsilon^{N-1} w_N
$$

$$
\approx \int_{|z|=\epsilon} (f(x - z) - f(x)) \cdot \frac{2^{N/2} \Gamma \left( \frac{N}{2} + 1 \right)}{(2\pi)^{N/2}} r^{1-N} \, dz
$$

$$
+ f(x) \frac{2^{N/2} \Gamma \left( \frac{N}{2} + 1 \right)}{(2\pi)^{N/2}} w_N
$$

$$
= \frac{2^{N/2} \Gamma \left( \frac{N}{2} + 1 \right)}{(2\pi)^{N/2}} \epsilon^{1-N} \int_{|z|=\epsilon} (f(x - z) - f(x)) \, dz
$$

$$
+ f(x) \cdot \frac{N \Gamma \left( \frac{N}{2} \right)}{\pi^{N/2}} \cdot \frac{2\pi^{N/2}}{N \Gamma \left( \frac{N}{2} \right)}
$$

$$
= \frac{2^{N/2} \Gamma \left( \frac{N}{2} + 1 \right)}{(2\pi)^{N/2}} \epsilon^{1-N} \int_{|z|=\epsilon} (f(x - z) - f(x)) \, dz + f(x).
$$

The limit (4.10) follows from the facts that

$$
|f(x - z) - f(x)| \leq |\nabla f|_{L^\infty} |z|
$$
for all \( z \in \mathbb{R}^N \) and hence
\[
\left| \epsilon^{1-N} \int_{|z| = \epsilon} (f(x - z) - f(x)) \, dz \right| \leq \epsilon^{1-N} |\nabla f|_{L^\infty} \epsilon \cdot \epsilon^{N-1} w_n \to 0.
\]
This completes the proof of Theorem 4.1. \( \square \)

**Theorem 4.2.** Let \( \lambda = -k^2 \) where \( k > 0 \) and let \( f \in L^p \) where \( p \in \left( \frac{2N}{N+4}, 2 \right]. \) Then \( T_k f \in H^2(\mathbb{R}^N) \) and \( \partial_i T_k f = S^i_k f \) and \( \Delta T_k f = S^\Delta_k f. \) Furthermore, \( T_k f \) is a weak solution of (3.1), where \( S^i_k \) and \( S^\Delta_k \) are given by (4.5) and (4.6).

**Proof.** Let \( \{f_n\} \subset C^2_0 \) be a sequence such that \( |f_n - f|_p \to 0 \) as \( n \to \infty. \) Since \( p > \frac{2N}{N+4}, \) we have \( \frac{2N}{N+4} < \frac{Np}{N-2p} \) when \( N > 4 \) and \( p < \frac{N}{4}. \) From (4.2) it follows that
\[
T_k f_n \text{ and } T_k f \in L^s
\]
and that
\[
|T_k f_n - T_k f|_{L^s} \to 0 \text{ as } n \to \infty,
\]
provided that \( p \leq s \leq \frac{2N}{N-4} \) for \( N \geq 5 \) and \( p \leq s < +\infty \) for \( N = 2, 3, 4. \) Similarly, from (4.3), (4.4), it follows that
\[
S^i_k f_n \text{ and } S^i_k f \in L^s,
\]
\[
|S^i_k f_n - S^i_k f|_{L^s} \to 0 \text{ as } n \to \infty;
\]
\[
S^\Delta_k f_n \text{ and } S^\Delta_k f \in L^s,
\]
\[
|S^\Delta_k f_n - S^\Delta_k f|_{L^s} \to 0 \text{ as } n \to \infty,
\]
provided that \( p \leq s \leq 2. \)

By Theorem 4.1 we know that \( T_k f \in C^4 \) and that \( \partial_i T_k f_n = S^i_k f_n \) for \( i = 1, 2, \ldots, N. \) Putting \( s = 2 \) in the preceding statements we deduce that \( T_k f \in H^2 \) with \( \partial_i T_k f = S^i_k f \) for \( i = 1, 2, \ldots, N, \) and \( \Delta T_k f = S^\Delta_k f. \) Furthermore, setting \( w = T_k f \) for any \( v \in C^\infty_0(\mathbb{R}^N), \) we have
\[
\int \Delta w \Delta v - (\lambda w + f) v \, dx = \int w \Delta^2 v - (\lambda w + f) v \, dx
\]
\[
= \lim_{n \to \infty} \int T_k f_n \Delta v - (\lambda T_k f_n + f_n) v \, dx
\]
\[
= \lim_{n \to \infty} \int \Delta(T_k f_n) \Delta v - (\lambda T_k f_n + f_n) v \, dx
\]
\[
= \lim_{n \to \infty} \int (\Delta^2(T_k f_n) + k^2 T_k f_n - f_n) v \, dx = 0
\]
by Theorem 4.1. This proves that \( w \) is a weak solution of (3.1). \( \square \)
Having established this relationship between weak solutions and convolutions with fundamental solutions, we now have a better understanding of the regularity of the weak solutions.

**Theorem 4.3.** Let $f \in L^p \cap L^q$ where $p \in \left(\frac{2N}{N+4}, 2\right]$ and $q \geq p$. Let $u$ be a solution of (3.1) for $f$ and some $\lambda \in \mathbb{R}$. Then

i) $u \in W^{2,s}(\mathbb{R}^N)$ where

\[
p \leq s \leq \infty \quad \text{if } q > \frac{N}{2},
p \leq s < \infty \quad \text{if } q = \frac{N}{2},\]

\[
p \leq s < \frac{Nq}{N-2q} \quad \text{if } q < \frac{N}{2};\]

ii) if $q > \frac{N}{4}$, then $u \in L^\infty \cap C$ and

\[
\lim_{|x| \to \infty} u(x) = 0;
\]

iii) $u \in L^s$ where

\[
p \leq s < \infty \quad \text{if } q = \frac{N}{4},
p \leq s < \frac{Nq}{N-4q} \quad \text{if } q < \frac{N}{4}.
\]

**Proof.** i) For all $v \in C_0^\infty(\mathbb{R}^N)$,

\[
\int \Delta u \Delta v \, dx = \int (\lambda u + f) \, v \, dx = \int (-u + g) \, v \, dx,
\]

where $g = (\lambda + 1)u + f$. Now since $u \in H^2(\mathbb{R}^N)$, $(\lambda + 1)u \in L^r$ for $2 \leq r < +\infty$ if $N = 2, 3, 4$ and $2 \leq r < \frac{2N}{N-4}$ for $N \geq 5$. Thus $u$ is a weak solution of

\[
\Delta^2 u = -u + g,
\]

and so $u = T_1(\lambda + 1)u + T_1f$ (from Theorem 1.2). Using Lemma 3.3 and a bootstrap argument to deal with the term $T_1(\lambda + 1)u$, the result now follows from (4.1) to (4.6).

ii) By i), $u \in W^{2,s}$ for some $s > \frac{N}{2}$ provided that $q > \frac{N}{4}$. For $s > \frac{N}{2}$, $W^{2,s} \hookrightarrow C \cap L^\infty$ and $\lim_{|x| \to \infty} u(x) = 0$ for all $u \in W^{2,s}$.

iii) This follows from i) and the Sobolev inclusions, or directly from (4.1). \qed

### 5 Regularity for nonlinear equations

In this section, we establish the regularity of weak solutions of (1.1).
Proof of Theorem 1.1. Let \( u(x) \) be a solution of (1.1). Set \( f = g(x, u(x)) - a(x)u(x) \). Then \( u(x) \) must be a solution of

\[
\begin{cases}
\Delta^2 u = f, \\
u \in H^2(\mathbb{R}^N).
\end{cases}
\]

From the assumptions \( H_2 \) and \( H_3 \) and \( 1 \leq \sigma + 1 < \frac{N+4}{N-4} \), it follows that for all \( u \in H^2(\mathbb{R}^N) \),

\[
f \in \mathcal{L}^p \quad \text{where} \quad \begin{cases} 2 \leq (\sigma + 1)p < \infty & \text{if } N = 2, 3, 4, \\ 2 \leq (\sigma + 1)p \leq \frac{2N}{N-4} & \text{if } N \geq 5. \end{cases}
\]

Now

\[
\left[ \frac{2}{\sigma+1}, \frac{2N}{N-4} \cdot \frac{1}{\sigma+1} \right] \cap \left( \frac{2N}{N+4}, 2 \right) \neq \emptyset,
\]

since the restrictions on \( \sigma \) ensure that \( \frac{2}{\sigma+1} \leq 2 \) and \( \frac{2N}{N-4} \cdot \frac{1}{\sigma+1} > \frac{2N}{N+4} \). Thus \( u \) is a weak solution of (3.1) for \( f \) and \( \lambda = 0 \) where \( f \in \mathcal{L}^p \) for some \( p \in \left( \frac{2N}{N+4}, 2 \right) \). From Theorem 4.3 \( u \in W^{2,s}(\mathbb{R}^N) \) for some \( s > 2 \), and so

\[
f \in \mathcal{L}^p \quad \text{where} \quad \begin{cases} 2 \leq (\sigma + 1)p \leq \infty & \text{if } s > \frac{N}{2}, \\ 2 \leq (\sigma + 1)p < \infty & \text{if } s = \frac{N}{2}, \\ 2 \leq (\sigma + 1)p \leq \frac{Ns}{N-2s} & \text{if } s < \frac{N}{2}. \end{cases}
\]

Noting that \( \frac{Ns}{N-2s} > \frac{2N}{N-4} \) for \( 2 < s < \frac{N}{2} \), we see by Lemma 3.2 and a bootstrap argument that \( u \in W^{2,s} \) for all \( 2 \leq s \leq \infty \). This implies that \( u \in \mathcal{L}^\infty \) and so \( f \in \mathcal{L}^2 \). Again by Lemma 3.2 we now also have \( u \in H^4 \). \( \square \)

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