Semiclassical Black Hole States and Entropy

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Abstract

We discuss semiclassical states in quantum gravity corresponding to Schwarzschild as well as Reissner-Nordström black holes. We show that reduced quantisation of these models is equivalent to Wheeler-DeWitt quantisation with a particular factor ordering. We then demonstrate how the entropy of black holes can be consistently calculated from these states. While this leads to the Bekenstein-Hawking entropy in the Schwarzschild and non-extreme Reissner-Nordström cases, the entropy for the extreme Reissner-Nordström case turns out to be zero.
The issues of black hole entropy and Hawking radiation play a key role in any attempt to quantise the gravitational field. For a deeper understanding it is of central importance to provide a satisfactory interpretation of black hole entropy from statistical mechanics. Recently, progress on this question has been achieved in the context of string theory [1], but the more conservative framework of quantum general relativity provides interesting insight into this question, too. In $2 + 1$ dimensions a statistical interpretation has been suggested using the Chern-Simons form [2], but in $3 + 1$ dimensions this remains still elusive.

On the level of the semiclassical approximation, black hole entropy has been discussed in the framework of path integrals [3,4]. Such a treatment exploits the formal analogy of euclidean path integrals in standard quantum field theory to partition sums in statistical mechanics. The entropy is then calculated as the logarithm of the density of states in the partition function which is found from an appropriate saddle point approximation to the path integral. To ensure thermodynamical stability, the black hole has to be enclosed in a spatially finite box (or, alternatively, has to be embedded in an anti-de Sitter spacetime [5]). If appropriate boundary conditions are imposed at the wall of the box and at the black hole horizon, the partition function can be evaluated, and the black hole entropy is found from the boundary term at the horizon to take the Bekenstein-Hawking value $A/4\hbar G$, where $A$ is the area of the horizon [4].

The purpose of the present paper is to investigate how the black hole entropy can be consistently found from semiclassical solutions to the Wheeler-DeWitt equation in quantum gravity. Since from a physical point of view there is a close connection between the WKB approximation for wave functionals and the saddle point approximation for path integrals, this should be possible to achieve. Some interesting new aspects will turn out in this discussion which thus complements the standard treatment in the path integral context.

In the following we shall consider spherically symmetric gravitational systems which include the important cases of the Schwarzschild and the Reissner-Nordström black holes. A WKB solution of the Wheeler-DeWitt equation for the former case was given in [6] (see also [7]). On the other hand, a reduced quantisation was performed in [8] with the mass
of the black hole as the only remaining configuration variable (see \cite{9} for an analogous
discussion in the framework of connection dynamics). We shall show the equivalence of
reduced quantisation to Wheeler-DeWitt quantisation in a particular factor ordering. We
extend the discussion to include the Reissner-Nordström case, where we present a careful
investigation into the notions of ‘classically allowed’ and ‘classically forbidden’ regions. Our
main point then will be the recovery of the black hole entropy from the Hamilton-Jacobi
functional in a consistent way. In the course of this discussion it will turn out in a natural
way that the entropy of the extreme Reissner-Nordström black hole vanishes.

We start from the ADM form of the general spherically symmetric spacetime metric on
the manifold $\mathbb{R} \times \mathbb{R} \times S^2$:

$$ds^2 = -N^2dt^2 + \Lambda^2(r,t)(dr + N^r dt)^2 + R^2(r,t)d\Omega^2,$$

(1)

where $d\Omega^2$ denotes the standard metric on $S^2$, and $N$ ($N^r$) is the lapse function (shift
function). Inserting the ansatz (1) into the Einstein-Hilbert action and varying with respect
to $N$ and $N^r$ leads to the Hamiltonian constraint and the radial momentum constraint \cite{6–8},

$$\mathcal{H}_G \equiv \frac{G \Lambda P^2}{2R^2} - \frac{G P^r P_R}{R} + \frac{V_G}{G} \approx 0,$$

(2)

$$\mathcal{H}_r \equiv P_R R^r - \Lambda P^r_\Lambda \approx 0,$$

(3)

where the gravitational potential term $V_G$ reads explicitly

$$V_G \equiv \frac{RR''}{\Lambda} - \frac{RR'N'}{\Lambda^2} + \frac{R^2}{2\Lambda} - \frac{\Lambda}{2}.$$  

(4)

The inclusion of the cosmological constant $\lambda$ is straightforward and would lead to an
additional term $\lambda\Lambda R^2/2$ in (4). In the following we shall include in addition a spherically
symmetric electromagnetic field \cite{5}. The corresponding vector potential is written in the
form

$$A = \phi(r,t)dt + \Gamma(r,t)dr.$$  

(5)

This, then, leads to the addition of the kinetic term
to (2), while (3) remains unchanged. Furthermore, variation with respect to the Lagrange multiplier $\phi$ in the action leads to the Gauss constraint

$$G \equiv P'_\Gamma \approx 0.$$  \hspace{1cm} (7)

Boundary conditions for all fields are assumed to hold such that all integrals are well-defined and such that the classical spacetime metric is nondegenerate $[5]$. Quantisation is then performed in the standard formal manner by replacing all momenta with $\hbar/i$ times functional derivatives and implementing all constraints by acting on wave functionals $\Psi[\Lambda(r), R(r), \Gamma(r)]$. At this point one normally has to rely on a particular factor ordering, but for the following results we do not need to fix this ambiguity. The electromagnetic part is trivially to solve: Eq. (7) becomes

$$\frac{d}{dr} \delta \Psi \delta \Gamma(r) = 0,$$  \hspace{1cm} (8)

which is solved by $\Psi = f(f_{-\infty}^{\infty} \Gamma(r) dr) \psi[\Lambda(r), R(r)]$, where $f$ is an arbitrary differentiable function. Note that the structure of (8) ensures that $\delta \Psi/\delta \Gamma(r)$ does not depend explicitly on $r$ and therefore guarantees that the second derivatives $\delta^2 \Psi/\delta \Gamma^2(r)$ are well defined. In fact, one immediately finds from $(\hat{H}_G + \hat{H}_E) \Psi = 0$ the solutions

$$\Psi = e^{i \frac{q}{\hbar} \int_{-\infty}^{\infty} \Gamma(r) dr} \psi[\Lambda(r), R(r)],$$  \hspace{1cm} (9)

where $\psi$ satisfies the Wheeler-DeWitt equation, which reads with ‘naive’ factor ordering,

$$\left\{ -\frac{G\hbar^2 \Lambda}{2R^2} \frac{\delta^2}{\delta \Lambda^2} + \frac{G\hbar^2}{R} \frac{\delta^2}{\delta \Lambda \delta R} + \frac{V_G}{G} + \frac{\Lambda q^2}{2R^2} \right\} \psi = 0.$$  \hspace{1cm} (10)

General solutions are found by performing superpositions of the states (9) with respect to $q$.

The form (9) is of course well known from the discussion of two-dimensional QED in the functional Schrödinger picture [10]. The role of the ‘charge’ $q$ is there played by the
background value of the electric field. In analogy to [10], one might also wish in our case to study the transformation of Ψ with respect to large gauge transformations, Ψ → Ψe−2πiqn, with a θ-parameter θ ≡ 2πq, but we shall not discuss this in the following.

It is convenient to consider the following functional [8]

\[ M(r) = \frac{P_\Lambda^2}{2R} + R \left[ 1 - \left( \frac{R'}{\Lambda} \right)^2 \right]. \tag{11} \]

Making use of the constraints, it is straightforward to show that

\[ \frac{dM(r)}{dr} = -\frac{R'q^2}{2R^2} \tag{12} \]

and thus

\[ M(r) = m - \frac{q^2}{2R(r)}. \tag{13} \]

It is evident that this is just the total energy, with m being the ADM mass and −q^2/2R the electrostatic energy, see e.g. [11].

We now assume the total state to be of the form (9) and solve (10) in a WKB approximation. Writing as usual \( \psi \approx Ce^{iS_0/\hbar} \) with a slowly varying prefactor C, one finds for \( S_0 \) the Hamilton-Jacobi equation

\[ \frac{G\Lambda}{2R^2} \left( \frac{\delta S_0}{\delta \Lambda} \right)^2 - \frac{G}{R} \frac{\delta S_0}{\delta \Lambda} \frac{\delta S_0}{\delta R} + \frac{V_G}{G} + \frac{\Lambda q^2}{2R^2} = 0 \tag{14} \]

and the momentum constraint equation

\[ R'\frac{\delta S_0}{\delta R} - \Lambda \frac{d}{dr} \frac{\delta S_0}{\delta \Lambda} = 0. \tag{15} \]

In generalisation of [6,7] one finds the solutions (up to a constant)

\[ S_0 = \pm G^{-1} \int_{-\infty}^{\infty} dr \left\{ \Lambda Q - RR'\text{arcosh} \frac{R'}{\Lambda \sqrt{1 - \frac{2M}{R}}} \right\} \]

\[ = \pm G^{-1} \int_{-\infty}^{\infty} dr \left\{ \Lambda Q - \frac{1}{2} RR' \ln \frac{R'}{R} + \frac{Q}{R} \right\}, \tag{16} \]

where Q is the functional
\[ Q \equiv R \sqrt{\frac{R^2}{\Lambda^2} + \frac{2M}{R}} - 1. \]  

(17)

An analogous solution is found for two-dimensional dilaton gravity \[12\] (see also \[13\]). Later we shall only consider the solution with the plus sign in front of the integral. One may of course consider also superpositions of WKB states, but they will decohere after a coupling to other quantum fields is taken into account \[13\]. We note that the classical momenta found from (16) read

\[ P_{\Lambda} \equiv \frac{\delta S_0}{\delta \Lambda} = \pm Q, \quad P_{R} \equiv \frac{\delta S_0}{\delta R} = \pm Q^{-1} \left[ \Lambda (m - R) + R \left( \frac{RR'}{\Lambda} \right)' \right]. \]  

(18)

Since \( S_0 \) can be considered as the generator of a canonical transformation, it is clear that spherically symmetric gravity can be classically reduced to a finite-dimensional system, since instead of arbitrary functions only the parameters \( m \) and \( q \) are contained in (16). This reduction has been explicitly done in \[8,9\]. The variables conjugate to \( m \) and \( q \) are obtained in the usual way from (16) according to

\[ p_m = \frac{\partial S_0}{\partial m} = \mp \int dr \left( 1 - \frac{2M}{R} \right)^{-1} \frac{\Lambda Q}{R}, \]  

(19)

\[ p_q = \frac{\partial}{\partial q} \left( S_0 + q \int dr \Gamma(r) \right) = \pm \int dr \left\{ \left( 1 - \frac{2M}{R} \right)^{-1} \frac{\Lambda q Q}{R^2} + \Gamma \right\}. \]  

(20)

While \( p_m \) describes the difference of parametrisation times at the left and right infinities of the Kruskal diagram \[8\], \( p_q \) is related to the electromagnetic gauge choice \[5\]. As can be easily seen, \( \psi \equiv \exp(iS_0/\hbar) \) is an exact solution for the Wheeler-DeWitt equation

\[ \left\{ -\frac{Gh^2}{2R^2} \frac{\Lambda}{\delta \Lambda} Q \frac{\delta}{\delta Q} Q^{-1} \frac{\delta}{\delta Q} + \frac{Gh^2}{R} Q \frac{\delta}{\delta R} Q^{-1} \frac{\delta}{\delta R} + \frac{V_G}{G} + \frac{\Lambda q^2}{2R^2} \right\} \psi = 0, \]  

(21)

where a particular factor ordering has been chosen (compare also \[12\] for a similar remark in the context of two-dimensional dilaton gravity). Wheeler-DeWitt quantisation of the

\[^1\] Compare our expressions with the Eqs. (157) and (159) in \[8\] by use of \( P_M = F^{-1}R^{-1}\Lambda\frac{\delta}{\delta \Lambda} \) and the Eqs. (4.3a) and (4.3b) in \[8\].
constraints (2) and (3) by use of this particular factor ordering is thus equivalent to the
quantisation of the reduced model. However, if another factor ordering in the Wheeler-
DeWitt equation is chosen, going beyond the first WKB level would amount to take into
account a much wider class of three-geometries for consideration in the wave functional. In
this case the Wheeler-DeWitt approach can thus not be assumed to be equivalent to the
reduced approach to quantisation. Explicit calculations for higher order WKB terms have
as yet been performed in most cases only in a formal sense [14].

We want to comment now on some interpretational issues with regard to (16). Since
classically \( Q = P_\Lambda \) is real, \( Q^2 \) must be positive, and therefore
\[
\left( \frac{R'}{\Lambda} \right)^2 \geq 1 - \frac{2M}{R}
\]  
(22)
is the condition for the ‘classically allowed region’. We note that for the classical spacetime
metric, the condition \( 1 - 2M/R < 0 \) describes the interior of the event horizon. One thus
recognises from (22) that this interior region is always classically allowed. We also note that
in this region the logarithm in (16) acquires a term \( i\pi \), since
\[
\left( \frac{R'}{\Lambda} + \frac{Q}{R} \right) \left( \frac{R'}{\Lambda} - \frac{Q}{R} \right) = 1 - \frac{2M}{R} < 0.
\]
(The first factor becomes zero upon crossing the leftgoing horizon of the Kruskal diagram,
while the second factor becomes zero upon crossing the rightgoing horizon [8].) In which
sense this imaginary part is related to entropy will be discussed below. What are the
classically forbidden regions? It is obvious that they must correspond to three-geometries
which cannot be embedded in a classical spacetime described by this model. Such three-
geometries can, however, be embedded in the euclideanised classical spacetime. Since then
\( (R'/\Lambda)^2 \leq 1 - 2M/R \), the euclidean spacetime can cover only regions outside the horizon,
which is well known from the standard treatment [3]. As \( \text{arcosh } z = \pm i \text{ arccos } z \), the solutions
(16) for the classically forbidden region read (note that the argument of arccos is smaller
than one)
\[
S_0 = \pm iG^{-1} \int_{-\infty}^{\infty} dr \left\{ \Lambda R \sqrt{1 - \frac{2M}{R}} - \frac{R'^2}{\Lambda^2} - RR' \text{ arccos} \frac{R'}{\Lambda \sqrt{1 - \frac{2M}{R}}} \right\}
\]
\[ = \pm iG^{-1} \int_{-\infty}^{\infty} dr \left\{ \Lambda R \sqrt{1 - \frac{2M}{R} - \frac{R'^2}{A^2} - RR' \arctan \frac{Q\Lambda}{RR'}} \right\}, \quad (23) \]

the latter being in agreement with the form of the generator with respect to the reduced euclidean model, see Eq. (6.10) in [15].

Since black hole entropy is recovered in the path integral formulation from a saddle point approximation [3,4], one should obtain it in the present framework from the Hamilton-Jacobi functional. In fact, it has been claimed that the entropy is related to the imaginary part of \( S_0 \) coming from the interior of the horizon in (16) [16]. Inspecting first a three-geometry which, when embedded in the classical Kruskal spacetime, crosses both horizons, one recognise that in (16) the contributions from the two horizon crossings to \( \text{Im}S_0 \) cancel each other. There would thus be no candidate for the entropy. Such a cancellation was noted in [13] for dilaton gravity, and is consistent with a similar observation made in [17] within the path integral framework.

Standard discussions of black hole thermodynamics often employ three-geometries which originate at the bifurcation two-sphere in the Kruskal diagram [15] or, more generally, the bifurcation surface of a Killing horizon [18]. Is there a contribution to the imaginary part of \( S_0 \) from such a boundary if the lower integration limit in (16) is chosen to be this region? The answer is no, since the three-geometry only covers the region outside the horizon where no \( i\pi \)-term emerges from the logarithm in (16).

The crucial point in the recovery of the entropy is, however, the fact that boundary conditions at the bifurcation point lead to additional degrees of freedom [19,20]. This is fully analogous to the asymptotically flat case where additional degrees of freedom (there the generators of the Poincaré group) are present at spatial infinity [21]. We assume in the following that the upper integration in (16) corresponds to this situation [8]. For simplicity, we shall concentrate on the situation at the bifurcation sphere (where we assume for \( r \) the value \( r = 0 \)) and will make use of the equations found by Kuchař in [8] for the upper
What are the boundary conditions at the bifurcation sphere? They are chosen in such a way that the classical solutions have a nondegenerate horizon, and that the hypersurfaces \( t = \text{constant} \) begin at \( r = 0 \) in a manner asymptotic to hypersurfaces of constant Killing time [3,13]. In particular, one has near \( r = 0 \):

\[
N(t, r) = N_1(t)r + O(r^3), \quad (24)
\]

\[
\Lambda(t, r) = \Lambda_0(t) + O(r^2), \quad (25)
\]

\[
R(t, r) = R_0(t) + R_2(t)r^2 + O(r^4), \quad (26)
\]

where \( R_0 \equiv R(0) \) is defined by \( (1 - 2M/R)|_{r=0} = 0 \). Note that in (24) \( N_1 \) is only non-vanishing if \( N \) has a single root at the origin. In the case of a double root, \( \partial N/\partial r = 0 = N_1 \). This will become important for the extreme Reissner-Nordström case. Employing the above boundary conditions (and similar ones for the conjugate momenta and the shift function [3,13]), one recognises that the variation of the classical action (in the following we shall neglect the \( \Gamma \)-part)

\[
S_\Sigma[\Lambda, P_\Lambda, R, P_R; N, N^r] = \int dt \int_0^\infty dr (P_\Lambda \dot{\Lambda} + P_R \dot{R} - N\mathcal{H}_G - N^r\mathcal{H}_r) \quad (27)
\]

leads to the following term at \( r = 0 \):

\[
\delta S_\Sigma|_{r=0} = -\left. \frac{\partial}{\partial r} \left( N\frac{\partial \mathcal{H}_G}{\partial R^r} \right) \right|_{r=0} \delta R \bigg|_{r=0} = -\frac{N_1 R_0}{\Lambda_0} \delta R_0. \quad (28)
\]

If \( N_1 \neq 0 \) one must subtract this boundary term from the original action (27). Otherwise the variation of the action with respect to \( R \) would lead to the unwanted conclusion that \( N_1/\Lambda_0 = 0 \). This is analogous to the situation at infinity where one has to add the ADM energy term [8,21]. One thus has to consider the classical action

\[2\text{ Generalisations to black holes embedded in a box [15] or in an anti-de Sitter spacetime [3] can easily be done.} \]
\begin{align*}
S[\Lambda, P_{\Lambda}, R, P_R; N, N'] &= \int dt \int_0^\infty dr (P_{\Lambda} \dot{\Lambda} + P_R \dot{R} - N \mathcal{H}_G - N' \mathcal{H}_r) \\
&\quad + \frac{1}{2} \int dt \frac{N_i R_0^2}{G \Lambda_0} - \int dt N_+ M_+,
\end{align*}

where $M_+$ denotes the ADM energy and $N_+$ the lapse function at infinity. The boundary term $\frac{1}{2G} \int dt R_0^2 \delta \left( \frac{N_i}{\Lambda_0} \right)$ vanishes if one assumes that $N_i / \Lambda_0 \equiv N_0$ is fixed at $r = 0$ [14]. If $N_i = 0$, no boundary term emerges and one is left with the original action (27). This happens in the case of extreme Reissner-Nordström black holes. A similar conclusion has been reached from a euclidean viewpoint in [20].

The necessity of fixing $N_0$ at $r = 0$ and $N_+$ at infinity can be removed by introducing the parametrisations

$$N_0(t) = \dot{\tau}_0(t), \quad N_+(t) = \dot{\tau}_+(t)$$

with $\tau_0$ and $\tau_+$ as additional variables. Instead of (29) one then considers

\begin{align*}
S[\Lambda, P_{\Lambda}, R, P_R; N, N'] &= \int dt \int_0^\infty dr (P_{\Lambda} \dot{\Lambda} + P_R \dot{R} - N \mathcal{H}_G - N' \mathcal{H}_r) \\
&\quad + \int dt \left( \frac{R_0^2}{2G} \dot{\tau}_0 - M_+ \dot{\tau}_+ \right).
\end{align*}

From the canonical point of view, this does not yet yield a satisfactory description, since there are no momenta canonically conjugate to $\tau_0$ and $\tau_+$. One can interpret the action (31) as representing a mixed Hamilton-Lagrangian form. In order to introduce the canonical momenta $\pi_0$ and $\pi_+$, one has to perform a standard Legendre transformation. But this can only be consistently done, if two new constraints are introduced:

$$C_0 \equiv \pi_0 - \frac{R_0^2}{2G} \approx 0,$$

$$C_+ \equiv \pi_+ + M_+ \approx 0.$$  

These constraints must be adjoined to the action by Lagrange multipliers $N_+$ and $N_0$:

$$S[\Lambda, P_{\Lambda}, R, P_R; \tau_0, \pi_0, \tau_+, \pi_+; N, N', N_0, N_+]$$

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\begin{align*}
&\int dt \int_0^\infty dr (P_\Lambda \dot{\Lambda} + P_\Gamma \dot{\Gamma} - N\mathcal{H}_G - N^r \mathcal{H}_r) \\
&+ \int dt \left( \pi_0 \dot{\tau}_0 + \pi_+ \dot{\tau}_+ - N_0 C_0 - N_+ C_+ \right).
\end{align*}

The new form (34) of the action gives rise to additional terms in the WKB approximation arising from implementing the constraints (32) and (33) on wave functions:

\begin{align}
\frac{\partial S_0}{\partial \tau_0} - \frac{R_0^2}{2G} &= 0, \\
\frac{\partial S_0}{\partial \tau_+} + M_+ &= 0,
\end{align}

which alter the above solution \( S_0 \) of the Hamilton-Jacobi equation by

\[ S_0 \rightarrow S_0 + \frac{R_0^2}{2G} \tau_0 - M_+ \tau_+. \]

To recover from this expression a term which can be interpreted as an entropy requires an appropriate euclideanisation in order to make contact with the formalism involving partition functions. The standard transition to the euclidean regime uses \( N_0 \equiv -iN_0^E \) to obtain a well-defined partition function from the path integral, where \( N_0^E \) is the euclidean lapse function in the line element

\[ ds^2_E = (N_0^E)^2 \tilde{r}^2 dt^2 + d\tilde{r}^2 + R^2 d\Omega^2 \]

with \( d\tilde{r} = \Lambda dr \) (and \( N^r = 0 \)). Thus, \( \tau_0 \rightarrow -i \int dt N_0^E \equiv -i\tau_0^E \), which means that one has to choose \( \tau_0^E = 2\pi \) because the ‘time’-integration (38) is around the circle \( S^1 \), and \( \tau_0^E \) is the angle. One thus arrives at the ‘euclidean’ WKB-state

\[ \psi[\Lambda(r), R(r); \beta, \tau_0 = 2\pi] = \exp \left\{ -\frac{1}{\hbar} S_0^E - \frac{\beta}{\hbar} M_+ + \frac{\pi R_0^2}{\hbar G} \right\}, \]

where \( S_0^E \) is \( (-i) \) times the expression (23). One can interpret \( \beta^{-1} = -i\mathcal{T}^{-1} \) with \( \mathcal{T} \equiv \int dt N_+ (t) \) as the renormalised temperature at infinity as in [3], and \( \pi R_0^2 / \hbar G \equiv A / 4\hbar G \) as the Bekenstein-Hawking entropy. Since no boundary term at the bifurcation two-sphere arises in the extreme Reissner-Nordström case, there is no entropy in that case. Note that
for the above derivation it was not necessary to perform explicit lapse-redefinitions such as in [15]. The result (39) is in full analogy to the result from the euclidean path integral, where the entropy arises from the surface term in the Einstein-Hilbert action [3].

The above derivation suggests the following viewpoint with regard to the recovery of the entropy from WKB quantum states: For three-geometries which in the classical spacetime correspond to slices through the full Kruskal diagram, the ‘information’ is maximal in the sense that data on such a slice allow one to recover the full spacetime. This point was also made in [17] to explain the vanishing of the entropy, which was found from the path integral for such slices. For slices which start at the bifurcation sphere, the information is less than maximal for the Schwarzschild as well as for the non-extreme \((q^2 < m^2)\) Reissner-Nordström black holes. Therefore they are attributed the entropy \(A/4\hbar G\). In the extreme Reissner-Nordström case \((q^2 = m^2)\) the maximum information is already available for such slices (compare the Penrose diagram of this case [22]). Therefore its entropy is zero, in accordance with results obtained from the euclidean formalism [20,23]. We note that the discontinuity of entropy in the extremal limit may be problematic in the framework of string theory [1], since the solution with vanishing entropy may be unstable.

An interesting result is obtained in the Schwarzschild case for a three-geometry that starts at the singularity, crosses one of the horizons and goes to infinity. In this case one cannot assume the euclidean viewpoint, since, as has been discussed above, there is no interior region in this case. However, \(S_0\) in Eq. (16) now acquires an imaginary part – leading to a real part in the exponential of the WKB state – from the interior region where the imaginary term of the logarithm is \(i\pi\). This yields

\[
\text{Im} S_0 = -\frac{1}{2G} \int dr RR' \pi = -\frac{\pi}{2G} \int_0^{R_0} dR \frac{1}{2} R^2 = -\frac{\pi R_0^2}{4G}.
\]  

(40)

Since \(\psi \approx \exp(i\text{Re} S_0 - \text{Im} S_0)\), this corresponds to an entropy of \(\pi R_0^2/4G\), one fourth of the Bekenstein-Hawking entropy. This lower value arises because now part of the interior regions can also be recovered from initial data on such slices in the classical theory, corresponding to ‘more information’.
In summary, we have shown that black hole entropy can be recovered from WKB quantum states in a natural way. We were able to deduce the Bekenstein-Hawking entropy by taking into account additional degrees of freedom. These additional degrees of freedom arise by inspecting the boundary conditions for particular three-geometries after they are interpreted as embeddings in the underlying Kruskal diagram of the classical theory. The connection between these degrees of freedom and the entropy of black holes was made through a euclideanisation of the classical line element. This seems to suggest that the Bekenstein-Hawking entropy can only be derived from WKB quantum states after a Wick rotation has been accomplished. For this reason we are not able to obtain a discrete mass spectrum of the black hole by using the periodicity of the Wick-rotated coordinate, in contrast to [24]. Note that in our approach this periodicity is needed to interpret part of the Hamilton-Jacobi functional as the entropy of the black hole in the first place. On the other hand, we could follow a recent suggestion [25] and use the periodicity of $\tau_0$ to demand the validity of Bohr-Sommerfeld conditions for the canonical pair $(\tau_0, \pi_0)$ in the euclideanised spacetime,

$$nh = \oint \pi_0 d\tau_0 = \int_0^{2\pi} \frac{R^2_0}{2G} d\tau_0 = \frac{4\pi M^2}{G}. \quad (41)$$

This leads to the same discrete spectrum for the mass as in [26]. Whether such a quantisation of the black hole entropy also holds in the physically relevant lorentzian case is an open issue and subject to future investigations.

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