New results on the existence of open loop Nash equilibria in discrete time dynamic games via generalized Nash games

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Abstract
We address the problem of finding conditions which guarantee the existence of open-loop Nash equilibria in discrete time dynamic games (DTDGs). A classical approach to DTDGs involves analyzing the problem using optimal control theory. Sufficient conditions for the existence of open-loop Nash equilibria obtained from this approach are mainly limited to linear-quadratic games (Başar and Olsder in Dynamic noncooperative game theory, 2nd edn, SIAM, Philadelphia, 1999). Another approach of analysis is to substitute the dynamics and transform the game into a static game. But the substitution of state dynamics makes the objective functions of the resulting static problems extremely hard to analyze. We introduce a third approach in which the dynamics are not substituted, but retained as constraints in the optimization problem of each player, resulting thereby in a generalized Nash game. Using this, we give sufficient conditions for the existence of open-loop Nash equilibria for a class of DTDGs where the cost functions of players admit a quasi-potential function. Our results apply with non-linear dynamics and without stage additive cost functions, and allow constraints on state and actions spaces, and in some cases, yield a generalization of similar results from linear-quadratic games.

Keywords Discrete time dynamic games · Open-loop Nash equilibrium · Potential games · Nonconvex optimization · Generalized Nash games
1 Introduction

Many scenarios which involve more than one player with decisions to be made over finitely many stages can be modeled as Discrete Time Dynamic Games (DTDGs). A player in a DTDG has to decide an action for each stage in order to optimize his overall cost function. A variable called the state of the game evolves in stages according to the state equation as a function of the decisions of players. The cost functions of players depend on the state. Many situations in power markets, robotics, network security, environmental economics, natural resource economics, industrial organization and so on are known to come under the framework of DTDGs (Van Long 2010; Kannan and Zavala 2011).

The information structure of the dynamic game declares what each player knows while making a decision. In an open-loop information structure, players only have the information of the initial state of the game while making their decisions. This paper derives new results on the existence of pure strategy Nash equilibria under the assumption of an open-loop information structure. The resulting equilibria are termed as open-loop Nash equilibria.

A classical approach for analyzing DTDGs consists of viewing each players’ problem as an optimal control problem and drawing up results from optimal control theory to show the existence of open-loop Nash equilibria ( Başar and Olsder 1999). Sufficient conditions for the existence of an open-loop Nash equilibrium that result from this approach are mostly confined to linear-quadratic games (Engwerda 2005; Jank and Abou-Kandil 2003; Reddy and Zaccour 2015). Another approach is to transform the dynamic game into a static game by substituting the state in the cost function and thus writing the cost function only in terms of the control variables and the initial state. This approach of analysis becomes extremely hard in practice because this substitution makes the cost function very complex. In this paper, we introduce a new approach for analyzing DTDGs using a concept of the conjecture of state by the players which leads to new results on the existence of open-loop Nash equilibria.

Instead of substituting the state into the cost function, our approach incorporates dynamics as constraints. To achieve this, we introduce a new set of variables which are dependent on players which can be interpreted as the conjecture of state by the players. This allows us to introduce dynamics independently as constraints in each player’s optimization problem. We thus derive a new and equivalent formulation of DTDGs that lies within the framework of generalized Nash games (Facchinei and Kanzow 2007). However, standard results from generalized Nash games do not apply due to the nonconvexity of the problem and the constraint structure. To remedy this, we define a class of games called quasi-potential DTDGs where the cost functions of players have a special structure: the cost function is a sum of two terms, one of which admits a potential function and the other is identical for all players. Our main result shows that any solution of a certain optimization problem is an open-loop Nash equilibrium of a quasi-potential DTDG. This result extends beyond linear-quadratic games and in some cases, generalizes earlier results from linear-quadratic games. It allows for nonlinear dynamics, constraints on actions and states in a DTDG, and cost functions that are not stagewise additive. We motivate the class of quasi-potential
games by considering a transboundary pollution game and we note that such games also arise in areas such as communication and transportation.

The term *conjecture* is used in the literature of dynamic games (see Figuières et al. 2004; Fershtman and Kamien 1985; Battigalli et al. 2015) as a supposition of a player about the behaviour of other players. In our approach, the conjecture is not about the decision of other players, but the state of the game. In Wiszniewska-Matyszkiel (2016), Wiszniewska-Matyszkiel (2017), the authors considered belief distorted Nash equilibria (BDNE) for a dynamic game using the concept of belief of a player, which is either a probability distribution of the future decisions of the other players and the future states (Wiszniewska-Matyszkiel 2017), or a multivalued correspondence that defines which future decisions of the other players and states are regarded as possible (Wiszniewska-Matyszkiel 2016), given the history of the game. The concept of conjecture of the state by the players that we use in this paper is closely related to the belief of a player with perfect foresight. However, the existence of an open-loop Nash equilibrium for quasi-potential games, which is the main focus of our paper, is not shown in any of the above mentioned papers.

The rest of the paper is organized as follows. The next section provides necessary preliminaries and background for the paper with an example from transboundary economics. Section 3 details the equivalent generalized Nash game formulation of DTDGs with the concept of conjectured state. Section 4 defines a class of games called the quasi-potential DTDGs and provide conditions for the existence of an equilibrium in such games. The paper ends with a conclusion in Sect. 5.

### 2 Preliminaries and background

We first present the definition of DTDGs and the open-loop Nash equilibria in DTDGs (Başar and Olsder 1999). We then discuss a transboundary pollution game as an example of such a game.

**Definition 1** A DTDG with open-loop information structure consists of the following (Başar and Olsder 1999).

1. An index set, $N = \{1, 2, \ldots, N\}$ called the **player set**, where $N$ denote the number of players.
2. An index set, $K = \{1, 2, \ldots, K\}$ called the **stage set** of the game, where $K$ is the maximum number of actions a player can make in a game.
3. A set $U^i_k$, $\forall i \in N$ and $\forall k \in K$ called the **action / control set** to which the action of player $i$ at stage $k$ belongs. The Cartesian set, $U^i \triangleq U^i_1 \times U^i_2 \times \cdots \times U^i_K$ is the action set of player $i$ and $U \triangleq U^1 \times \cdots \times U^N$ is the action set of the game. The action sets of adversaries of player $i$ at stage $k$ and for the game are defined as $U^{-i}_k \triangleq \prod_{j \in N\setminus\{i\}} U^j_k$ and $U^{-i} \triangleq U^{-i}_1 \times U^{-i}_2 \times \cdots \times U^{-i}_K$ respectively. Here $\prod$ denotes the Cartesian product of sets.
4. A set $X_k$, $\forall k \in K \cup \{K + 1\}$ called the **state space** of the game at stage $k$ to which the state of the game belongs. Since we are considering the open-loop information structure, the initial state of the game is assumed to be known to all players and is denoted by $x_1 \in X_1$.  

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A mapping \( f_k : X_k \times U_k^1 \times U_k^2 \times \cdots \times U_k^N \longrightarrow X_{k+1} \) is defined for each \( k \in \mathcal{K} \) such that
\[
x_{k+1} = f_k \left( x_k, u_k^1, u_k^2, \ldots, u_k^N \right)
\]  \hspace{1cm} (1)

is the state equation of the DTDG, where \( u_k^1 \in U_k^1, u_k^2 \in U_k^2, \ldots, u_k^N \in U_k^N \) are the actions of the players at stage \( k \) and \( x_k \in X_k \) is the state at stage \( k \).

A class of mappings denoted by \( \Gamma_k^i, i \in \mathcal{N}, k \in \mathcal{K} \) called the strategy set of player \( i \) at stage \( k \). A mapping \( \gamma_k^i \in \Gamma_k^i \), given by \( \gamma_k^i : X_1 \rightarrow U_k^i \) is the strategy of player \( i \) at stage \( k \). The aggregate mapping \( \gamma^i = \{ \gamma_1^i, \gamma_2^i, \ldots, \gamma_K^i \} \in \Gamma^i = \prod_{k=1}^{K} \Gamma_k^i \) is the strategy of player \( i \) in the game.

A function \( J^i : U^i \times X_1 \times X_2 \times \cdots \times X_{K+1} \longrightarrow \mathbb{R} \) is defined for each \( i \in \mathcal{N} \) called the cost function or the objective function of player \( i \) in the game of \( K \) stages.

For each fixed initial state \( x_1 \) and for a fixed \( N \)-tuple of permissible strategies \( \{ \gamma^i \in \Gamma^i; i \in \mathcal{N} \} \), there exist a unique set of actions \( \{ u_1^i, \gamma_1^i(x_1); i \in \mathcal{N}, k \in \mathcal{K} \} \) and the state evolves according to the state equation (1). Substituting these quantities into \( J^i, i \in \mathcal{N} \) leads to a unique \( N \)-tuple of numbers reflecting corresponding costs to the players. This implies the existence of \( L^i : \Gamma^1 \times \Gamma^2 \times \cdots \times \Gamma^N \rightarrow \mathbb{R} \), the cost function in the strategy space. That is \( L^i(\gamma^1, \ldots, \gamma^N) = J^i(u_1^1, \ldots, u^N_k, x_1, x_2, \ldots, x_{K+1}) \), where \( \gamma^i = \{ \gamma_1^i, \gamma_2^i, \ldots, \gamma_K^i \} \in \Gamma^i, i \in \mathcal{N} \) and \( u^i = \{ u_1^i, u^2_i, \ldots, u^N_i \} \in U^i, i \in \mathcal{N} \) and each \( u^i_k = \gamma_k^i(x_1) \) and \( x_{k+1} = f_k(x_k, u_k^1, u_k^2, \ldots, u_k^N) \), for \( k \in \mathcal{K} \).

In a DTDG with open-loop information structure, a player’s problem is to decide his strategies \( \gamma^i \in \Gamma^i \) which minimize his cost function \( L^i \).

**Definition 2** In an \( N \)-person DTDG of \( K \)-stages, player \( i \)’s cost function is said to be stage-additive if there exist functions \( g_k^i : X_{k+1} \times U_k^1 \times \cdots \times U_k^N \times X_k \rightarrow \mathbb{R}, \forall k \in \mathcal{K} \) such that,
\[
J^i(u^1, \ldots, u^N, x_1, \ldots, x_{K+1}) = \sum_{k=1}^{K} g_k^i \left( x_{k+1}, u_k^1, \ldots, u_k^N, x_k \right), \hspace{1cm} (2)
\]
where \( u^i = (u_1^i, \ldots, u_k^i) \in U^i, \forall i \in \mathcal{N} \) and \( x_k \in X_k, \forall k \in \mathcal{K} \cup \{ K + 1 \} \).

We define the open-loop Nash equilibrium in a DTDG as follows (Başar and Olsder 1999).

**Definition 3** An \( N \)-tuple of strategies \( \{ \gamma^i* \in \Gamma^i; i \in \mathcal{N} \} \) constitute an open-loop Nash equilibrium if and only if the following inequalities are satisfied \( \forall i \in \mathcal{N} \),
\[
L^i(\gamma^{1*}, \gamma^{2*}, \ldots, \gamma^{i*}, \ldots, \gamma^{N*}) \leq L^i(\gamma^{1*}, \ldots, \gamma^{(i-1)*}, \gamma^i, \gamma^{(i+1)*}, \ldots, \gamma^{N*}), \hspace{1cm} \forall \gamma^i \in \Gamma^i. \hspace{1cm} (3)
\]

Let \( \{ u^i* \in U^i; i \in \mathcal{N} \} \) be the actions of players corresponding to the equilibrium strategies \( (\gamma^{1*}, \gamma^{2*}, \ldots, \gamma^{N*}) \). In the open-loop information structure, the strategy
\[ \gamma^i_k \in \Gamma^i_k, \ i \in \mathcal{N}, \ k \in \mathcal{K} \] is a constant function (depending only on \( x_1 \)). Hence, we can write that the \( N \)-tuple of actions \( \{u^i_1 \in U^i; \ i \in \mathcal{N}\} \) corresponding to the equilibrium strategies constitutes an open-loop Nash equilibrium if and only if the following inequalities are satisfied \( \forall i \in \mathcal{N} \).

\[
J^i \left( u^{1*}, \ldots, u^{i*}, \ldots, u^{N*}, x_1, x_2^*, \ldots, x_{K+1}^* \right) \\
\leq J^i \left( u^{1*}, \ldots, u^{(i-1)*}, u^i, u^{(i+1)*}, \ldots, u^{N*}, x_1, x_2, \ldots, x_{K+1}^* \right), \ \forall u^i \in U^i,
\]

(4)

where \( x_{k+1}^* = f_k(x_k^*, u_k^1, u_k^2, \ldots, u_k^N), \ k \in \mathcal{K} \) and \( x_{k+1} = f_k(x_k, u_k^1, \ldots, u_k^{(i-1)*}, u_k^i, u_k^{(i+1)*}, \ldots, u_k^N), \ k \in \mathcal{K} \).

### 2.1 An example: transboundary pollution game

We adopt a transboundary pollution game (Van Long 2010) as an illustration of a DTDG model. In this game, the players have to choose their levels of CO\(_2\) emissions. The emission of CO\(_2\) is directly linked to the profit of the player, so for a higher profit, the emission of CO\(_2\) is higher. But the level of CO\(_2\) in the atmosphere contributed by all the players identically. Let \( \mathcal{N} \) be the set of players (e.g. countries) and \( \mathcal{K} \) be the set of stages (e.g. days) as we defined in Definition 1. Let \( u_k^i \in U_k^i \) be the amount of CO\(_2\) emission of player \( i \) at stage \( k \). Suppose \( x_k \in X_k \) is the amount of CO\(_2\) in the atmosphere at stage \( k \). The amount of CO\(_2\) in the atmosphere at any stage \( k \) is governed by the equation \( x_{k+1} = x_k + \sum_{i=1}^{N} u_k^i \) which forms the state equation with \( x_1 \) being the initial amount of CO\(_2\) in the atmosphere. Let \( b_k^i : U_k^i \rightarrow \mathbb{R} \) be the benefit of player \( i \) at stage \( k \) due to CO\(_2\) emissions at stage \( k \) and \( d_k : X_{k+1} \times U_k^1 \times \cdots \times U_k^N \rightarrow \mathbb{R} \) be the common environmental damage due to CO\(_2\) emissions at stage \( k \) which affects all the players identically. The payoff of player \( i \) at stage \( k \) is denoted by \( J_k^i \equiv b_k^i - d_k \).

Given other players decisions on CO\(_2\) emissions \( u^{-i} \in U^{-i} \) and initial amount of CO\(_2\) in the atmosphere \( x_1 \), player \( i \)'s problem is to maximize the objective function \( \sum_{k=1}^{K} [b_k^j(u_k^i) - d_k(u_k^i, x_k, x_{k+1})] \) using the decision variables \( u_k^i, k \in \mathcal{K} \) subject to the condition that \( x_{k+1} = x_k + \sum_{i=1}^{N} u_k^i, k \in \mathcal{K} \).

In this paper, we provide sufficient conditions for the existence of open-loop Nash equilibria for a class of DTDGs where the cost functions take this form. That is, the cost function of a player is written as a summation of two terms. One term is like the benefit term in the example which depends on the actions of the players and the other term indicates the identical effect of the actions of all players and depends on the state as well. In this example, the state equation is linear, but our result allows the state equation to be nonlinear. Moreover, our result also allows costs that are not stage-additive. Cost functions like these appear in other areas as well, e.g. in transportation and communication. In transportation, the congestion cost in a path forms the identical term in the objective function of a player. Similarly, in a communication network, this term arises from the delay in a particular link (Başar 2007).
A commonly used line of analysis for DTDGs is given as follows. A necessary condition for the existence of an open-loop Nash equilibrium based on discrete-time Pontryagin principle can be derived as in Başar and Olsder (1999). A sufficient condition for linear-quadratic DTDGs can also be obtained by solving a recursive Riccati equation (Başar and Olsder 1999), which at times is more difficult than solving a set of linear equations, which we obtain if we do not use this method. Under some additional assumptions, it is shown in Jank and Abou-Kandil (2003) that the two-point boundary value problem is uniquely solvable for a linear-quadratic case (see also Reddy and Zaccour 2015). Another potential line of analysis is as follows. Since an open-loop Nash equilibrium is equivalently given by a sequence of actions $u_i^* \in U_i, i \in \mathcal{N}$, the open-loop Nash equilibrium is also the Nash equilibrium of a static game obtained by substituting the state equation into the cost function of each player. Fixed point theory is then used to claim the existence of a Nash equilibrium. Evidently, this line of analysis succeeds only if the cost functions and the dynamics take a simple form, so that the hypothesis of fixed point theorems are satisfied (even mild nonlinearities in the dynamics lead to nonconvex static formulations thereby ruling out arguments based on Kakutani’s fixed point theorem). The novelty in our analysis is that we do not substitute the state equation in the cost function, but rather keep it as a constraint itself. We can do so because of the reformulation of DTDGs with the introduction of new decision variables which is explained in the next section.

3 Generalized Nash game formulation

In this section, we introduce a new formulation where we define a new variable and consider a player’s problem in a game with the newly defined variable taken as a decision variable. The resulting game is a generalized Nash game (Facchinei and Kanzow 2007) in the space of actions $u$ and the new variables.

Before going further, we recall the definition of generalized Nash games. In a Nash game, players are coupled through the objective functions of players. Specifically, the objective function of a player in a game is dependent not only on the decisions of that player but also on the decisions of the other players. If there exists a coupling across the players through the constraint set, then the game is called a generalized Nash game. Given the decisions of other players, denoted by $v^{-i}$, player $i$’s problem in a Nash game is as follows, given by the problem denoted by $\text{NG}(v^{-i})$.

$$\text{NG}(v^{-i}) \quad \text{minimize} \quad J^i(v^1, v^2, \ldots, v^N)$$
subject to $v^i \in \mathcal{V}^i$.

Here $\mathcal{V}^i$ denotes the set of strategies of player $i$ and $J^i : \mathcal{V}^1 \times \mathcal{V}^2 \times \cdots \times \mathcal{V}^N \to \mathbb{R}$ denotes the objective function of player $i$. Observe that in a Nash game, $\mathcal{V}^i$ is independent of $v^{-i}$. In a generalized Nash game (Facchinei and Kanzow 2007), given $v^{-i}$, player $i$’s problem is as follows, denoted by $\text{GenNG}(v^{-i})$.
Here the decision set of player $i$ is given by $\mathcal{V}^i(v^{i-})$, where $\mathcal{V}^i$ is a set valued function of the decisions of players other than player $i$. A (generalized) Nash equilibrium for the generalized Nash game is a tuple $(v^1*, \ldots, v^N*)$ such that for all $i$,

$$v^i* \in \arg\min \text{GenNG}(v^{i-})$$

In this section, we derive a generalized Nash game formulation of the DTDG, in which, the player $i$’s problem is given by the problem denoted by $P_1(u^{-i}, x^{-i}; x_1)$. We show that a Nash equilibrium of the generalized Nash game formulation is equivalent to an open-loop Nash equilibrium of the original DTDG. We use this relation to derive the result on the existence of open-loop Nash equilibria in DTDGs.

For deriving the generalized Nash game formulation of the DTDG, we define a new set $X^i_k$, $\forall i \in \mathcal{N}$ and $\forall k \in \mathcal{K} \cup \{K + 1\}$. The new set is defined such that $X^i_1 = X_k$, $\forall i \in \mathcal{N}$, $\forall k \in \mathcal{K} \cup \{K + 1\}$. Let $X^i = \prod_{k=2}^{K+1} X^i_k$ and $X \triangleq \prod_{i=1}^{N} X^i$. Further, let $X^i_{k-1} \triangleq \prod_{j \in \mathcal{N}\setminus\{i\}} X^j_k$ for a player $i$ at stage $k$, and $X^{-i} \triangleq \prod_{k=2}^{K+1} X^{i-}_k$. Since $X^i_{k+1} \equiv X_{k+1}$, $\forall i \in \mathcal{N}$, $\forall k \in \mathcal{K}$, the function $f_k$ can be equivalently defined for each $k \in \mathcal{K}$ as $f_k : X^i_k \times U^1_k \times U^2_k \times \cdots \times U^N_k \rightarrow X^i_{k+1}$. We now define a variable $x^i_k \in X^i_k$ for each $k \in \mathcal{K}$ and $i \in \mathcal{N}$, and require that $x^i_k$ satisfy the state equation.

$$x^i_{k+1} = f_k(x^i_k, u^1_k, u^2_k, \ldots, u^N_k) \quad (5)$$

Since the initial state $x_1$ is known to all players, we consider it as a given parameter of the game and we take $x^i_1 \in X^i_1$ equal to $x_1 \in X_1$, $\forall i \in \mathcal{N}$. The cost function of player $i$ with the new formulation is defined as a mapping given by $J^i : U \times X^1 \times X^2 \times \cdots \times X^{i+1}_{K+1} \rightarrow \mathbb{R}$ which is defined for each $i \in \mathcal{N}$. Given $u^{-i} \in U^{-i}, x^{-i} \in X^{-i}$ and $x_1$, consider the player $i$’s problem denoted by $P_1(u^{-i}, x^{-i}; x_1)$.

$$P_1(u^{-i}, x^{-i}; x_1)
\begin{align*}
\text{minimize} & \quad J^i(u^i, x^i, u^{-i}; x_1) \\
\text{subject to} & \quad (u^i, x^i) \in \Omega_i(u^{-i}, x^{-i}; x_1),
\end{align*}$$

where,

$$\Omega_i(u^{-i}, x^{-i}; x_1) = \{\hat{u}^1_1, \ldots, \hat{u}^i_K, \hat{x}^2_1, \ldots, \hat{x}^i_{K+1} | \hat{u}^i_k \in U^i_k, \forall k \in \mathcal{K}, \hat{x}^i_{k+1} \in X^i_{k+1}, \forall k \in \mathcal{K}, \hat{x}^i_2 = f_1(x_1, \hat{u}^1_1, u^{-i}_{-1}), \hat{x}^i_{k+1} = f_k(\hat{x}^i_k, \hat{u}^i_k, u^i_k), \forall k \in \mathcal{K}\setminus\{1\}\}.$$
game with player $i$’s problem denoted by $P_i(u^{-i}, x^{-i}; x_1)$ is a generalized Nash game formulation if and only if $u^* \in \Omega_i(u, x) \triangleq \prod_{i \in \mathcal{N}} \Omega_i(u^{-i}, x^{-i}; x_1)$, where $u = (u^1, u^2, \ldots, u^N) \in U$ and $x = (x^1, x^2, \ldots, x^N) \in X$. The fixed points of the set value map $\Omega_i(u, x)$ is denoted by $\mathcal{F}$ which is given by $\mathcal{F} \triangleq \{(u, x)|(u, x) \in \Omega(u, x)\}$. We can write the set $\mathcal{F}$ as follows.

$$\mathcal{F} = \{(u, x)|u \in U, x^i \in X^i, \forall i \in \mathcal{N}, x^1 = f_1(x_1, u^1_1, \ldots, u^N_1), \forall i \in \mathcal{N}, \forall k \in \mathcal{K}\setminus\{1\}\}.$$  

(6)

Notice that the set $\mathcal{F}$ depends on the initial state $x_1$. So in the set $\mathcal{F}$, for a fixed $x_1$, the new variable $x^i$ is consistent for all players, that is, for all $i \in \mathcal{N}$, $x^i = x^j, \forall j \in \mathcal{N}\setminus\{i\}$. For the generalized Nash game formulation described above, we define a Nash equilibrium as follows.

Definition 4 A tuple of strategies $\{(u^i, x^i); i \in \mathcal{N}\} \in \mathcal{F}$ is a Nash equilibrium of the generalized Nash game formulation if $\forall i \in \mathcal{N}$,

$$J_i(u^i, x^i, u^{-i}; x_1) \leq J_i(u^i, x^i, u^{-i*}; x_1), \quad \forall (u^i, x^i) \in \Omega_i(u^{-i*}, x^{-i*}; x_1).$$

Next, we analyse the Nash equilibrium of the generalized Nash game formulation defined above and relate it to the open-loop Nash equilibrium of the DTDG.

3.1 Equivalence to the classical formulation

The following proposition shows that the set of Nash equilibria in the generalized Nash game defined in the previous section is equivalent to the set of open-loop Nash equilibria in the original DTDG. That is, there exist a one-to-one relation between an open-loop Nash equilibrium of DTDG in the classical formulation to a Nash equilibrium in the generalized Nash game formulation.

Proposition 1 Consider a DTDG as defined in Definition 1 with open-loop information structure. Consider the generalized Nash game formulation with $X^1_{k+1} = X_{k+1}, \forall i \in \mathcal{N}, k \in \mathcal{K}$. Then for a given initial state $x_1$, $(u^*, x^*) \in \mathcal{F}$ is a Nash equilibrium of the generalized Nash game formulation if and only if $u^* \in U$ is an open-loop Nash equilibrium in the classical formulation with the state trajectory at the equilibrium given by any component $x^i* \in X^i$ of $x^*$.

Proof Let $(u^*, x^*) \in \mathcal{F}$ be a Nash equilibrium of the generalized Nash game formulation. Consider the problem of player $i$ in the generalized Nash game formulation.
given by \( P_i(u^{-i*}, x^{-i*}; x_1) \), where \( u^{-i*} \) is the equilibrium actions of other players and \( x_1 \), the initial state.

\[
P_i(u^{-i*}, x^{-i*}; x_1) \quad \text{minimize} \quad J_i^1(u^i, x_2^i, \ldots, x_{K+1}^i, u^{-i*}; x_1) \\
\text{subject to} \quad (u^i, x_2^i, \ldots, x_{K+1}^i) \in \Omega_i(u^{-i*}, x^{-i*}; x_1).
\]

By substituting the state equation (5) from the feasible set in the cost function, we can rewrite the problem only in the space of actions as follows:

\[
\text{minimize} \quad J_i^1(u^i, f_1(x_1, u^i_1, u^{-i*}_1), f_2(f_1(x_1, u^i_1, u^{-i*}_1), u^i_2, u^{-i*}_2), \ldots, u^{-i*}; x_1) \\
\text{subject to} \quad u^i_k \in U^i_k, \forall k \in K.
\]

However, this is precisely the form one would get if one substituted the dynamics into the cost function for the classical formulation of a DTDG. Hence, if \((u^*, x^*) \in \mathcal{F}\) is a Nash equilibrium in the generalized Nash game formulation, then \(u^* \in U\) an open-loop Nash equilibrium in the classical formulation. Conversely, if \(u^* \in U\) is an open-loop Nash equilibrium of the classical formulation, \((u^*, x^*) \in \mathcal{F}\) is an equivalent Nash equilibrium in the generalized Nash game formulation with \(x^{i*} = \tilde{x}^*, \forall i \in \mathcal{N}\), where \(\tilde{x}^*\) is the state trajectory in the classical formulation corresponding to the equilibrium actions \(u^*\). Therefore \((u^*, x^*) \in \mathcal{F}\) is a Nash equilibrium of the generalized Nash game formulation if and only if \(u^* \in U\) is an open-loop Nash equilibrium of the original formulation.

The new variable \(x^i\) introduced in the generalized Nash game formulation can be interpreted as the conjecture of the state by player \(i\). In the classical definition of DTDGs, the state of the game is defined as common for players and it evolves “in the background” according to the state equation but not within the direct control of any particular player. The conjectured state \(x^i\) satisfies the state equation, however since it is a player-dependent variable, it can be introduced through the constraints \(\Omega_i\) in the decision problem of the player. The new formulation has the advantage of being equivalent to the classical one, but admits easier analysis than the classical formulation.

In this formulation, it is seen that the game takes the form of a generalized Nash game rather than a dynamic game in the standard sense. In the example of the transboundary pollution game discussed in Sect. 2.1, the problem of player \(i\) in the generalized Nash game formulation is given by the following problem denoted by TPG_i.\(^1\)

\[
\text{TPG}_i \quad \text{minimize} \quad \sum_{k=1}^{K} \left[ d_k(u^i_k, x^{-i*}_k, x^i_{k+1}) - b^i_k(u^i_k) \right] \\
\text{subject to} \quad u^i_k \in U^i_k, x^i_{k+1} = x^i_k + \sum_{i=1}^{N} u^i_k, k \in K.
\]

\(^1\) TPG_i denotes player \(i\)’s problem in the transboundary pollution game in the generalized Nash game formulation.
Observe the extension of the decision space in the problem of a player in the generalized Nash game formulation. In the classical formulation, the decision space of player \( i \) is the actions of player \( i \) over the stages \( u_k^i, k \in K \). In the generalized Nash game formulation, we have the conjecture of the state by player \( i \), that is \( x_{k+1}^i, \forall k \in K \) also as decision variables. Proposition 1 shows that a Nash equilibrium of a game in the generalized Nash game formulation is equivalent to an open-loop Nash equilibrium in the classical formulation. Thus, we have a new generalized Nash game formulation equivalent to the classical one. We will exploit this new formulation to derive the new existence results in the following section.

4 Existence of an equilibrium for quasi-potential DTDGs

In this section, we present our results for the existence of Nash equilibria for a class of DTDGs called quasi-potential DTDGs. The problem \( P_i \) takes the form of a generalized Nash equilibrium problem (Facchinei and Kanzow 2007), with coupled but not shared constraints (Kulkarni and Shanbhag 2012). For \( \Omega \) to be a shared constraint, it must be true that \((u_i, x_i) \in \Omega_i(u_i^{-i}, x_i^{-i}; x_1) \iff (u, x) \in F\). It can be verified that this is not true. Hence Rosen’s result (Rosen 1965) on shared constraint games is not applicable. Moreover, since we allow dynamics to be nonlinear, the problems \( P_i \) are individually nonconvex. In this generalized setting, we utilize the structure of a class of games called the quasi-potential games to provide a result on the existence of equilibria. A quasi-potential game is a special case of a potential game (Monderer and Shapley 1996), introduced in Kulkarni and Shanbhag (2015) for multi-leader multi-follower games. A quasi-potential DTDG is defined as follows:

**Definition 5** Consider a DTDG defined in the generalized Nash game formulation with the cost function of players given by \( \{J_1, J_2, \ldots, J_N\} \). The DTDG is said to be quasi-potential if the cost function of players admits a quasi-potential function for which the following has to hold:

1. There exist functions \( \Phi^1, \Phi^2, \ldots, \Phi^N \) with \( \Phi^i : U \to \mathbb{R} \) and \( h : U \times X_1 \times X_2 \times \cdots \times X_{K+1} \to \mathbb{R} \) such that for all \( i \in N \), player \( i \)'s cost function can be written as \( J^i(u, x^i; x_1) = \Phi^i(u) + h(u, x^i, x_1), \forall u \in U, \forall x^i \in X_2 \times \cdots \times X_{K+1} \),
2. There exist a (potential) function \( \pi : U \to \mathbb{R} \) such that for all \( i \in N \),
   \[ \Phi^i(\tilde{u}^i, u^{-i}) - \Phi^i(\hat{u}^i, u^{-i}) = \pi(\tilde{u}^i, u^{-i}) - \pi(\hat{u}^i, u^{-i}), \forall \tilde{u}^i, \hat{u}^i \in U^i, \forall u^{-i} \in U^{-i}. \]

According to the above definition, for each player \( i \), the objective \( J^i \) must be a sum of two terms. The first term \( \Phi^i \) must be dependent only on actions \( u \) of all players and the second term must be of the form \( h(u, x^i, x_1) \) where the function \( h \) is identical for all players. The functions \( \Phi^1, \ldots, \Phi^N \) must admit a potential function. Since \( x^i \) lies in \( X^i \), and \( X^i_k \) was defined to be equal to \( X_k \) for all \( i, k \), such a definition for \( h \) is valid. Sufficient conditions for the existence of a potential function can be found in Monderer and Shapley (1996). We refer to the function \( \pi + h \) as the quasi-potential function of a quasi-potential DTDG.
Observe that the objective functions of the players in the transboundary pollution game in the generalized Nash formulation given by the problems $TPG_i, i \in \mathcal{N}$ admit a quasi-potential function. The objective function of player $i$ is given by $\sum_{k=1}^{K} d_k(u^1_k, u^2_k, \ldots, u^N_k, x^i_{k+1}) = b^i_k(u^i_k)$. Here the term $\sum_{k=1}^{K} b^i_k(u^i_k)$ is independent of the state of the game. Thus we take $\Phi^i(u) \equiv \sum_{k=1}^{K} b^i_k(u^i_k)$ in Definition 5. It is easy to see that $\Phi^1, \ldots, \Phi^N$ admit a potential function $\pi$ given by $\pi(u) \equiv \sum_{i=1}^{N} \sum_{k=1}^{K} -b^i_k(u^i_k)$. Notice that the other term in the objective function of player $i$, $\sum_{k=1}^{K} d_k(u^1_k, u^2_k, \ldots, u^N_k, x^i_{k+1})$, which denotes the common environmental damage due to CO2 emissions, can be written as $h(u, x^i; x_1)$ where the function $h$ is identical for all players. Thus, a quasi-potential function for the transboundary pollution game in the generalized Nash game formulation is given by $\sum_{k=1}^{K} d_k(u^1_k, u^2_k, \ldots, u^N_k, x^i_{k+1}) - \sum_{i=1}^{N} \sum_{k=1}^{K} b^i_k(u^i_k)$. A similar problem structure can be observed in other areas such as transportation with a common congestion cost (Aashtiani and Magnanti 1981) and communication network with a common delay (Başar 2007) and so on.

We need a lemma in order to prove the existence of Nash equilibria in quasi-potential DTDGs. For that, consider a set $\mathcal{F}^q$ which is defined as follows.

$$\mathcal{F}^q \triangleq \{(u^1, \ldots, u^N, x_1, w_2, w_3, \ldots, w_K, w_{K+1})|(u^1, \ldots, u^N) \in U, w_{k+1} \in X_{k+1}, \forall k \in \mathcal{K}, w_2 = f_1(x_1, u^1_1, \ldots, u^1_N), w_{k+1} = f_k(w_k, u^1_k, \ldots, u^N_k), \forall k \in \mathcal{K}\setminus\{1\}\}.$$ 

Note that the initial state $x_1$ is a parameter on which the set $\mathcal{F}^q$ depends. Now we consider the following lemma.

**Lemma 1** Consider a DTDG in the generalized Nash game formulation. For some $i \in \mathcal{N}$, consider the problem $P_i(u^{-i}, x^{-i}; x_1)$. A point $(u^i, w_2, w_3, \ldots, w_{K+1})$ is feasible for $P_i(u^{-i}, x^{-i}; x_1)$ if and only if $(u, x_1, w_2, w_3, \ldots, w_{K+1}) \in \mathcal{F}^q$. That is,

$$(u^i, w_2, w_3, \ldots, w_{K+1}) \in \Omega_i(u^{-i}, x^{-i}; x_1) \iff (u, x_1, w_2, w_3, \ldots, w_{K+1}) \in \mathcal{F}^q.$$ 

**Proof** “⇒” For given $u^{-i} \in U^{-i}$ and initial state $x_1$, consider a point $(u^i, w_2, w_3, \ldots, w_{K+1}) \in \Omega_i(u^{-i}; x_1)$. That means, $u^i \in U^i, w_2 = f_1(x_1, u^i_1, u^{-i})$, $w_{k+1} = f_k(w_k, u^i_k, u^{-i}), \forall k \in \mathcal{K}\setminus\{1\}$. Now by combining $u^i \in U^i$ and $u^{-i} \in U^{-i}$ we can rewrite the above equations as $u \in U, w_2 = f_1(x_1, u^i_1, u^{-i}), w_{k+1} = f_k(w_k, u^i_k, u^{-i}), \forall k \in \mathcal{K}\setminus\{1\}$. Hence, $(u, x_1, w_2, w_3, \ldots, w_{K+1}) \in \mathcal{F}^q$.

“⇐” Suppose $(u, x_1, w_2, w_3, \ldots, w_{K+1}) \in \mathcal{F}^q$. Hence, $u \in U, w_2 = f_1(x_1, u^i_1, u^{-i}), w_{k+1} = f_k(w_k, u^i_k, u^{-i}), \forall k \in \mathcal{K}\setminus\{1\}$. For some $i \in \mathcal{N}$, we separate the actions of player $i$ and adversaries as $u^i \in U^i$ and $u^{-i} \in U^{-i}$. For some fixed $u^{-i} \in U^{-i}$ and $x_1$, the equations can be rewritten as $u^i \in U^i, w_2 = f_1(x_1, u^i_1, u^{-i}), w_{k+1} = f_k(w_k, u^i_k, u^{-i}), \forall k \in \mathcal{K}\setminus\{1\}$. Hence, $(u^i, w_2, w_3, \ldots, w_{K+1}) \in \Omega_i(u^{-i}; x_1)$. $\square$
For a quasi-potential DTDG with quasi-potential function \( \pi + h \), consider an optimization problem denoted by \( P_q(x_1) \).

\[
P_q(x_1) \quad \text{minimize} \quad \pi(u) + h(u, w_2, w_3, \ldots, w_{K+1}; x_1) \\
\text{subject to} \quad (u, x_1, w_2, w_3, \ldots, w_{K+1}) \in \mathcal{F}^q
\]

Note that the objective function of the problem \( P_q(x_1) \) is the quasi-potential function \( \pi + h \) and the feasible set is \( \mathcal{F}^q \). We relate the solution of \( P_q(x_1) \) to a Nash equilibrium of a quasi-potential DTDG which is given in the following theorem.

**Theorem 1** Consider a quasi-potential DTDG with a quasi-potential function \( \pi + h \). Consider the optimization problem given by \( P_q(x_1) \) for the quasi-potential DTDG. If \((u, w_2, w_3, \ldots, w_{K+1}) \in \mathcal{F}^q \) is a global minimizer of the problem \( P_q(x_1) \), then \((u, x) \in \mathcal{F} \) is a Nash equilibrium of the quasi-potential DTDG with \( x_{k+1} = w_{k+1} \), \( \forall k \in K \), \( \forall i \in N \).

**Proof** Let \((u, x_1, w_2, w_3, \ldots, w_{K+1}) \in \mathcal{F}^q \) be the minimizer of the problem \( P_q(x_1) \). Then,

\[
\pi(u) + h(u, w_2, w_3, \ldots, w_{K+1}; x_1) \leq \pi(\tilde{u}) + h(\tilde{u}, \tilde{w}_2, \ldots, \tilde{w}_{K+1}; x_1), \quad \forall (\tilde{u}, x_1, \tilde{w}_2, \tilde{w}_3, \ldots, \tilde{w}_{K+1}) \in \mathcal{F}^q.
\]

By splitting the actions \( u, \tilde{u} \in U \) over player \( i \) as \( u = (u^i, u^{-i}) \) and \( \tilde{u} = (\tilde{u}^i, \tilde{u}^{-i}) \) respectively, we can rewrite as,

\[
\pi(u^i, u^{-i}) + h(u^i, u^{-i}, w_2, \ldots, w_{K+1}; x_1) \leq \pi(\tilde{u}^i, \tilde{u}^{-i}) + h(\tilde{u}^i, \tilde{u}^{-i}, \\
\tilde{w}_2, \ldots, \tilde{w}_{K+1}; x_1), \forall (\tilde{u}^i, \tilde{u}^{-i}, x_1, \tilde{w}_2, \tilde{w}_3, \ldots, \tilde{w}_{K+1}) \in \mathcal{F}^q.
\]

The inequality still holds even if we replace \( \tilde{u}^{-i} \) in the above relation by \( u^{-i} \in U^{-i} \), the minimizer components. Also, by using Lemma 1, the above inequality can be rewritten as,

\[
\pi(u^i, u^{-i}) + h(u^i, u^{-i}, w_2, \ldots, w_{K+1}; x_1) \leq \pi(\tilde{u}^i, u^{-i}) + h(\tilde{u}^i, u^{-i}, \\
\tilde{w}_2, \ldots, \tilde{w}_{K+1}; x_1), \forall (\tilde{u}^i, \tilde{w}_2, \tilde{w}_3, \ldots, \tilde{w}_{K+1}) \in \Omega_i(u^{-i}, x^{-i}; x_1), \forall i \in N.
\]

Since the game is a quasi-potential DTDG and \((w_2, w_3, \ldots, w_{K+1}) = x^i \in X^i, \forall i \in N, \Phi^i(u^i, u^{-i}) + h(u^i, u^{-i}, x^i; x_1) \leq \Phi^i(\tilde{u}^i, u^{-i}) + h(\tilde{u}^i, u^{-i}, \tilde{w}_2, \tilde{w}_3, \ldots, \\
\tilde{w}_{K+1}; x_1), \forall (\tilde{u}^i, \tilde{w}_2, \tilde{w}_3, \ldots, \tilde{w}_{K+1}) \in \Omega_i(u^{-i}, x^{-i}; x_1), \forall i \in N.
Hence,
\[
J^i(u, x^i; x_1) \leq J^i(\tilde{u}^i, u^{-i}, \tilde{w}_2, \tilde{w}_3, \ldots, \tilde{w}_{K+1}; x_1),
\forall (\tilde{u}^i, \tilde{w}_2, \tilde{w}_3, \ldots, \tilde{w}_{K+1}) \in \Omega_i(u^{-i}, x^{-i}; x_1), \forall i \in \mathcal{N}.
\]

Thus \((u, x) \in \mathcal{F}\) is a Nash equilibrium of the quasi-potential DTDG.

Thus, if the optimization problem \(P_q(x_1)\) has a minimizer then there exists a Nash equilibrium for the quasi-potential DTDG. The following result gives conditions which guarantee the existence of a minimum for the problem \(P_q(x_1)\).

**Corollary 1** A quasi-potential DTDG admits a Nash equilibrium if the quasi-potential function, \(\pi + h\) is continuous and the set \(\mathcal{F}^q\) is non empty and compact. The set \(\mathcal{F}^q\) is compact if the sets \(U, X\) are compact and \(f_k\) is continuous for all \(k \in \mathcal{K}\).

The Corollary 1 can be easily proved using Weierstrass Theorem. The continuity of \(f_k\) ensures that \(\mathcal{F}^q\) is closed. The requirement of compactness of the state space in Corollary 1 may be restrictive in some cases. The following result gives the existence of Nash equilibria without the condition of compactness of state or action spaces, instead imposing the condition of coercivity of quasi-potential function. The coercivity definition is given as follows.

**Definition 6** A quasi-potential function \(\pi + h\) is coercive if
\[
\liminf_{||u, w_2, \ldots, w_{K+1}|| \to \infty} \pi(u, x_1) + h(u, x_1, w_2, \ldots, w_{K+1}) = \infty.
\]

**Corollary 2** A quasi-potential DTDG admits a Nash equilibrium if the quasi-potential function, \(\pi + h\) is continuous and coercive and \(f_k\) is continuous for each \(k \in \mathcal{K}\).

**Proof** It is known that there exists a global minimizer for an optimization problem with a coercive and continuous objective function over a non-empty closed set (Beck 2014). Hence from Theorem 1, the result follows.

Corollaries 1 and 2 give conditions for the existence of Nash equilibria for quasi-potential DTDGs. It can be observed that stage additivity is not required for the results that we discussed so far. Next, we consider the case of stage-additive DTDGs.

### 4.1 Stage-additive quasi-potential DTDG

We consider the case of stage-additive DTDG as defined in Definition 2. For such games the stage-wise cost functions \(g_1^1, g_2^1, \ldots, g_N^N\) admit a quasi-potential function if the following conditions are satisfied for each \(k \in \mathcal{K}\).

1. There exist functions \(\Psi_k^1, \Psi_k^2, \ldots, \Psi_k^N\) and \(h_k\) such that \(\forall i \in \mathcal{N}, g_k^i\) is given by,
\[
g_k^i\left(x_{k+1}, u_1^i, \ldots, u_N^i, x_i^i\right) = \Psi_k^i(u_1^i, \ldots, u_N^i) + h_k(x_k, u_1^i, \ldots, u_N^i, x_{k+1}^i),
\forall u_k^j \in U_k^j, j \in \mathcal{N}, \forall x_k^i \in X_k, x_{k+1}^i \in X_{k+1}, \forall k \in \mathcal{K},
\]
2. and there exist functions $\pi_k, \forall k \in K$ such that $\forall i \in \mathcal{N}$ and $\forall u_k^{-i} \in U_k^{-i}$, 

$$
\Psi^i_k(u_k^i, u_k^{-i}) - \Psi^i_k(\hat{u}_k^i, u_k^{-i}) = \pi_k(\hat{u}_k^i, u_k^{-i}) - \pi_k(u_k^i, u_k^{-i}), \forall \hat{u}_k^i, u_k^i \in U_k^i.
$$

The function $\pi_k + h_k$ is a stage-wise quasi-potential function for the stage-additive quasi-potential DTDG. One can see that the transboundary pollution example that we considered in Sect. 2.1 is a stage-additive quasi-potential DTDG. In that example, the potential function part $\pi_k$ is given by $\sum_{i=1}^{N} b^i_k$ and the part of the payoff which is identical for all players is $d_k$. Thus the stage-wise quasi-potential function becomes $\sum_{i=1}^{N} b^i_k + d_k$. The following proposition shows that if the stage-wise cost function of players admit a stage-wise quasi-potential function, then the overall cost functions also admit a quasi-potential function.

**Proposition 2** Consider a stage-additive DTDG defined in Definition 2. If the stage-wise cost functions $g^i_k, i \in \mathcal{N}$ admit a stage-wise quasi-potential function, then the stage-additive cost functions $J^1, J^2, \ldots, J^N$ also admit a quasi-potential function.

**Proof** The stage-additive cost function of player $i$ in a stage-additive quasi-potential DTDG is given by,

$$
J^i(u^1, u^2, \ldots, u^N, x_1, x_2, \ldots, x_{K+1}) = \sum_{k=1}^{K} \Psi^i_k(u_k^1, \ldots, u_k^N) + \sum_{k=1}^{K} h_k(x_k^i, u_k^1, \ldots, u_k^N, x_{k+1}^i),
$$

where $u_k^i \in U_k^i, u^i \in U^i, x_{k+1}^i \in X_{k+1}^i, \forall i \in \mathcal{N}$. The first sum admits a potential function given by $\sum_{k=1}^{K} \pi_k$. The second sum is independent of players though the argument depends on. Hence, the stage-additive cost function of players admits a quasi-potential function.  

For a stage-additive quasi-potential DTDG with stage-wise quasi-potential function $\pi_k + h_k$, consider the following optimization problem denoted by $P^s_q$ with $w_1 = x_1$.

$$
P^s_q \begin{array}{l}
\text{minimize} \sum_{k=1}^{K} \pi_k(u_k^1, \ldots, u_k^N) + h_k(u_k, w_k, w_{k+1}) \\
\text{subject to} (u, x_1, w_2, w_3, \ldots, w_{K+1}) \in F^q.
\end{array}
$$

It can be seen from Theorem 1 that the minimizer of the problem $P^s_q$ is a Nash equilibrium of the stage-additive quasi-potential DTDG with $x_{k+1}^i = w_{k+1}, \forall i \in \mathcal{N}, \forall k \in K$. The problem $P^s_q$ has the structure of a standard control problem for which a vast theory on dynamic programming is available which can be utilized to compute the equilibria. We now show that the linear-quadratic DTDGs under certain assumptions comes under the class of stage-additive quasi-potential DTDGs. We first provide the definition of linear-quadratic DTDGs.
Definition 7  An $N$-person DTDG is of linear-quadratic type if $U_k^i = \mathbb{R}^{m_i}(i \in \mathcal{N}, k \in \mathcal{K})$, and

$$f_k(x_k^i, u_k^1, \ldots, u_k^N) \equiv A_k x_k^i + \sum_{j \in \mathcal{N}} B_k^i u_k^j,$$

$$g_k^i(x_{k+1}^i, u_k^1, \ldots, u_k^N, x_k^i) = \frac{1}{2} \left( x_{k+1}^i Q_{k+1}^i x_{k+1}^i + \sum_{j \in \mathcal{N}} u_j^i R_{k}^{ij} u_j^i \right),$$

where $A_k, B_k^i, Q_k^i, R_{k}^{ij}$ are matrices of appropriate dimensions, $Q_k^i + 1$ is symmetric, $R_{k}^{ii} \succ 0$ and $u_k^i \in U_k^i, x_{k+1}^i \in X_{k+1}^i, \forall i \in \mathcal{N}, \forall k \in \mathcal{K}$.

Lemma 2  For a linear-quadratic DTDG with $Q_k^{i+1} \triangleq = Q_k^{i+1}, \forall i, j \in \mathcal{N}, \forall k \in \mathcal{K}$, the stage-wise cost function of players, $g_k^i$ admit a quasi-potential function.

Proof  For a linear-quadratic DTDG, the stage-wise cost function is given by,

$$g_k^i(x_{k+1}^i, u_k^1, \ldots, u_k^N, x_k^i) = \frac{1}{2} \left( x_{k+1}^i Q_{k+1}^i x_{k+1}^i + \sum_{j \in \mathcal{N}} u_j^i R_{k}^{ij} u_j^i \right).$$

It can be easily verified that the second summation term is a quadratic function which admits a potential function with stage-wise potential function $\pi_k(u_k^1, \ldots, u_k^N) = \frac{1}{2} \sum_{i=1}^N u_k^i \Pi_{k}^{ij} u_k^i$. Hence, in the definition of quasi-potential DTDG, $\Psi_k^i(u_k) = \frac{1}{2} \sum_{j \in \mathcal{N}} u_j^i \Pi_{k}^{ij} u_j^i$. Also, the first function is identical for all players and the argument depends on the state conjectured by players. Hence, $h_k^i(x_{k+1}^i) \equiv \frac{1}{2} x_{k+1}^i Q_{k+1}^i x_{k+1}^i$, which implies that $g_k^i$ admits a quasi-potential function.

The fact that the stage-wise cost function of the linear-quadratic DTDGs under certain assumptions admit a stage-wise quasi-potential function can be utilized to provide the existence of Nash equilibria in such cases which is given in the following corollary.

Corollary 3  Consider a linear-quadratic DTDG as defined in Definition 7. Suppose the assumptions of Lemma 2 hold with $Q_{k+1}^i \succ 0, \forall k \in \mathcal{K}$. Then there exists a Nash equilibrium for the linear-quadratic DTDG which is given by the minimizer of the problem $P^{s}_{q}$.

Proof  Since $Q_{k+1}^i \succ 0, \forall k \in \mathcal{K}$ and $R_{k}^{ij} \succ 0, \forall i \in \mathcal{N}, \forall k \in \mathcal{K}$, the quasi-potential function $\sum_{k \in \mathcal{K}} (\pi_k + h_k)$ as defined in the proof of Lemma 2 for the linear-quadratic DTDG is coercive. Then the result follows directly from Corollary 2.

Thus, Corollary 3 generalizes classical results for a subclass of linear-quadratic DTDGs.
5 Conclusion

This paper provides new results on conditions for the existence of open-loop Nash equilibria for a class of games called the quasi-potential DTDGs. A new and equivalent generalized Nash game formulation based on the concept of the conjecture of state by players is introduced for the analysis of DTDGs. Using this formulation, a solution of a suitably defined optimization problem is shown to be an equilibrium of the quasi-potential DTDG. The main advantages of the new formulation are that it includes games with nonlinear dynamics, constraints on actions and states and cost functions which are not stage additive. Many games including the linear-quadratic games under certain conditions come under the class of quasi-potential DTDGs.

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