An extended framework of continuous-stage Runge-Kutta methods

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Abstract

We propose an extended framework for continuous-stage Runge-Kutta methods which enables us to treat more complicated cases especially for the case weighting on infinite intervals. By doing this, various types of weighted orthogonal polynomials (e.g., Jacobi polynomials, Laguerre polynomials, Hermite polynomials etc.) can be used in the construction of Runge-Kutta-type methods. Particularly, families of Runge-Kutta-type methods with geometric properties can be constructed in this new framework. As examples, some new symmetric and symplectic integrators by using Legendre polynomials, Laguerre polynomials and Hermite polynomials are constructed.

Keywords: Ordinary differential equations; Continuous-stage Runge-Kutta methods; Legendre polynomials; Laguerre polynomials; Hermite polynomials; Hamiltonian systems; Symplectic methods.

1. Introduction

It is widely believed that numerical methods are a vital component in the discovery and study of natural phenomena and law, especially for solving various differential equations arising in the field of natural science, primarily due to the fact that in most cases we can not find the analytical or precise solutions. If we put aside those intricate and complicated mathematical theories, then it is conceived that a “good” numerical method should be simple, easily understood and conveniently implemented in the practical application. As is well known, polynomials are the most simple mathematical object for computers to treat, and thus they ought to be strongly suggested for use in the construction of numerical methods — indeed, they have attracted much attention and been extensively used in many fields, including finite element methods [2], spectral methods [14 15], numerical differentiation and integration (quadrature) [32], interpolation theory and numerical approximation [32 36] and so forth. Amongst multitudinous types of polynomials, orthogonal polynomials are of

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significant importance mainly owing to that they are more convenient to use and sometimes it may lead to surprising applications [6].

The well-known Runge-Kutta (RK) method, as one of the most popular classes of methods for solving initial value problems, has been a central topic in numerical ordinary differential equations (ODEs) since the pioneering work of Runge (1895) [29]. They have been highly developed for over a hundred and twenty years (see [7, 4, 8, 16, 17] and references therein). One of the most important points we have to speak out is that RK methods possess a close relationship with polynomials. A basic evidence for supporting such view is that, the RK order conditions corresponding to bushy trees (by B-series theory [8, 18]) are equivalent to those polynomial-based quadrature rules [16]. Another good case in evidence is the $W$-transformation [17] which is defined on the basis of Legendre orthogonal polynomials. By using $W$-transformation many special-purpose methods can be established, e.g., symplectic RK methods, symmetric RK methods, algebraically stable RK methods, stiffly accurate & $L$-stable RK methods etc [10, 11, 16, 17, 24, 33, 34].

A novel RK approach reflecting stronger relationship between RK-type methods and polynomials are developed in recent years. It is clear and easy to understand it by introducing the concept of “continuous-stage Runge-Kutta (csRK) methods”. Actually, the theory of csRK methods was initially launched by Butcher in 1972 [5] (see also [8]) and subsequently developed by Hairer [19, 20], Tang & Sun [37, 38, 39, 42], and Miyatake & Butcher [26, 27]. The coefficients of csRK methods are assumed to be “continuous” functions and thus it is a simple and natural choice to let them be polynomials. The first example of such methods was given by Hairer using Lagrangian interpolatory polynomials for deriving energy-preserving collocation (EPC) methods [19], whereas a low-order version of EPC methods was proposed earlier by Quispel & McLaren [28] with the name “averaged vector field (AVF) methods” but without being interpreted as csRK methods. A closely-related type of methods which can also be transformed into csRK methods attributes to Brugnano et al [3], called infinity Hamiltonian Boundary Value Methods (denoted by $\infty$-HBVMs), but also had not been explained within the framework of csRK methods at that time. It was firstly pointed out in [37] (and subsequently in [38]) that four kinds of Galerkin time finite element methods can be recast as csRK methods, and particularly, EPC methods, AVF methods as well as $\infty$-HBVMs can be explained as continuous finite element methods, and all of them can be unified in the context of csRK methods. Instead of using Lagrangian polynomials [19], Tang et al [37, 38, 39, 44] resort to Legendre orthogonal polynomials for constructing csRK methods, and some special-purpose methods including symplectic csRK methods [38, 39, 44], conjugate-symplectic (up to a finite order) csRK methods [39], symmetric csRK methods [38, 39, 44], energy-preserving csRK methods [38, 39, 44] are developed in use of Legendre polynomials. Besides, Some extensions of csRK methods are also obtained by Tang et al [40, 41, 43]. Another interesting related work are given by Li & Wu [25], showing that their energy-preserving methods can be explained as a class of csRK methods.

More recently, new ideas have been introduced for construction of csRK methods in [44], [45, 46, 47], where the Butcher coefficients are partially assumed to be polynomial functions by
leading weight functions (not necessarily polynomial) into the formalism of csRK methods. These new developments have greatly extended the previous studies from the simplest case with coefficient $B_\tau = 1$ to a much more general case with $B_\tau \neq 1$. As is shown in [44, 45, 46, 47], by using different weighted orthogonal polynomials including Legendre polynomials, Chebyshev polynomials and their superset “Jacobi polynomials”, new families of symplectic methods and symmetric methods can be derived. However, these investigations are following Hairer’s original formalism [19] with integrals restricted on a finite interval $[0, 1]$. In this paper, we are going to break through such limitation and further enlarge the primitive framework of csRK methods to a super new one which admits us to treat the case for infinite intervals.

This paper will be organized as follows. Section 2 is devoted to give some preliminaries about orthogonal polynomials and weighted interpolatory quadrature. This is followed by Section 3, in which we will speak out our new idea for extending the previous framework of csRK methods. After that, some discussions about geometric integration by csRK methods will be given in Section 4. Section 5 is devoted to showing a few numerical experiments for the sake of verifying our theoretical results. At last, we conclude this paper.

2. Orthogonal polynomials and weighted interpolatory quadrature

Assume $I$ is an interval, and we do not restrict it to be a finite interval. Particularly, it can be an infinite interval in the following three types: (1) $(-\infty, b]$ with $b$ a finite real number; (2) $[a, +\infty)$ with $a$ a finite real number; (3) $(-\infty, +\infty)$.

Definition 2.1. A non-negative (or positive almost everywhere) function $w(x)$ is called a weight function on $I$, if it satisfies the following two conditions:

(a) The $k$-th moment $\int_I x^k w(x) \, dx$, $k \in \mathbb{N}$ exists;
(b) For any non-negative function $u(x)$, $\int_I u(x) w(x) \, dx = 0$ implies $u(x) \equiv 0$.

It is known that for a given weight function $w(x)$, there exists a sequence of orthogonal polynomials in the weighted function space (Hilbert space) [36]

$$L^2_w(I) = \{ u \text{ is measurable on } I : \int_I |u(x)|^2 w(x) \, dx < +\infty \}$$

which is linked with the inner product

$$(u, v)_w = \int_I u(x) v(x) w(x) \, dx.$$ 

In this paper, we denote a sequence of weighted orthogonal polynomials by $\{P_n(x)\}_{n=0}^{\infty}$, which consists of a complete set in the Hilbert space $L^2_w(I)$. It is known that $P_n(x)$ has exactly $n$ real simple zeros in the interval $I$. For convenience, in what follows we always assume the orthogonal polynomials are normalized, i.e., satisfying

$$(P_i, P_j)_w = \delta_{ij}, \quad i, j = 0, 1, 2, \ldots .$$
It is known that orthogonal polynomials have important applications in numerical integration. An $s$-point weighted interpolatory quadrature formula is in the form

$$
\int_I \Phi(\tau) w(\tau) \, d\tau \approx \sum_{i=1}^{s} b_i \Phi(c_i), \quad c_i \in I,
$$

(2.1)

where

$$
b_i = \int_I \ell_i(\tau) w(\tau) \, d\tau, \quad \ell_i(\tau) = \prod_{j=1, j \neq i}^{s} \frac{\tau - c_j}{c_i - c_j}, \quad i = 1, \ldots, s.
$$

Here, remark that for the simplest case $s = 1$, the number of Lagrangian basis functions in interpolation is just one, i.e., $\ell_1(\tau) = \tau/c_1$.

As far as we know, the optimal quadrature technique is the well-known Gauss-Christoffel’s, which can be stated as follows (see, for example, [1, 32]).

**Theorem 2.1.** Consider $I$ is a finite interval. If $c_1, c_2, \ldots, c_s$ are chosen as the $s$ distinct zeros of the normalized orthogonal polynomial $P_s(x)$ of degree $s$ in $L^2_w(I)$, then the interpolatory quadrature formula (2.1) is exact for polynomials of degree $2s - 1$, i.e., of the optimal order $p = 2s$. If $\Phi \in C^{2s}$, then it has the following remainder

$$
\int_I \Phi(\tau) w(\tau) \, d\tau - \sum_{i=1}^{s} b_i \Phi(c_i) = \Phi^{(2s)}(\xi) \frac{1}{(2s)!} \mu_s^2,
$$

for some $\xi \in I$, where $\mu_s$ is the leading coefficient of $P_s(x)$, and $b_i = -\frac{\mu_{s+1}}{\mu_s P_s'(c_i)P_{s+1}(c_i)} > 0$.

**Remark 2.1.** If the weight functions are defined on an infinite interval, then we have similar results. For instance, if we take the weight functions as

$$
e^{-x}, \quad x \in [0, +\infty) \quad \text{and} \quad e^{-x^2}, \quad x \in (-\infty, +\infty),$$

respectively, then we have the well-known Gauss-Christoffel-Laguerre and Gauss-Christoffel-Hermite quadrature rules of order $p = 2s$ and the remainders of them are dependent on $\Phi^{(2s)}(\xi)$ (for more details, see [1]).

Other suboptimal quadrature rules, e.g., Gauss-Christoffel type quadrature with some fixed nodes (including Gauss-Radau, Gauss-Lobatto quadrature etc.) can be found in many literatures (see, for example, [32]). We find that numerical solution of ordinary differential equations can be closely related to quadrature techniques associated with suitable orthogonal polynomials. This will be clearly shown later by studying csRK methods.

### 3. Continuous-stage Runge-Kutta methods

Consider the following initial value problem

$$
\dot{z} = f(t, z), \quad z(t_0) = z_0 \in \mathbb{R}^d,
$$

(3.1)
where $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is assumed to be sufficiently differentiable. For such problem, we propose the following concept of csRK methods weighted on $I$, the initial version of which was proposed by Tang in [45].

**Definition 3.1.** Let $w(x)$ be a weight function defined on $I$, $A_{\tau, \sigma}$ be a function of two variables $\tau$, $\sigma$, and $B_{\tau}, C_{\tau}$ be functions of $\tau$. The one-step method $\Psi_h : z_0 \mapsto z_1$ given by

$$Z_{\tau} = z_0 + h \int_I A_{\tau, \sigma} w(\sigma) f(t_0 + C_{\sigma} h, Z_{\sigma}) \, d\sigma, \quad \tau \in I,$$

$$z_1 = z_0 + h \int_I B_{\tau} w(\tau) f(t_0 + C_{\tau} h, Z_{\tau}) \, d\tau,$$

is called a continuous-stage Runge-Kutta (csRK) method, where $Z_{\tau} \approx z(t_0 + C_{\tau} h)$. Here we often assume

$$C_{\tau} = \int_I A_{\tau, \sigma} w(\sigma) \, d\sigma,$$

and denote a csRK method by the triple $(A_{\tau, \sigma} w(\sigma), B_{\tau} w(\tau), C_{\tau})$.

**Remark 3.1.** For the case when $I$ is an infinite interval, we assume that the improper integrals of (3.2) satisfy some conditions (in terms of uniform convergence) such that differentiation under the integral sign with respect to parameter $h$ (step size) is legal. Consequently, the Taylor expansion of the numerical solution can be written as a B-series [18] and the standard order theory of B-series integrators can be applied.

**Remark 3.2.** If take $I$ as the standard interval $[0, 1]$, then we regain the methods developed in [45]. However, remark that the primitive framework of csRK methods given in [45] can not cover the case for weighting on an infinite interval and many classical orthogonal polynomials such as Laguerre polynomials, Hermite polynomials etc can not be applied in that framework.

Following Hairer’s idea [19], we introduce the following variant of simplifying assumptions:

$$\tilde{B}(\xi) : \int_I B_{\tau} w(\tau) C_{\tau}^{\kappa-1} \, d\tau = \frac{1}{\kappa}, \quad \kappa = 1, \ldots, \xi,$$

$$\tilde{C}(\eta) : \int_I A_{\tau, \sigma} w(\sigma) C_{\sigma}^{\kappa-1} \, d\sigma = \frac{1}{\kappa} C_{\tau}^{\kappa}, \quad \kappa = 1, \ldots, \eta,$$

$$\tilde{D}(\zeta) : \int_I B_{\tau} w(\tau) C_{\tau}^{\kappa-1} A_{\tau, \sigma} \, d\tau = \frac{1}{\kappa} B_{\sigma}(1 - C_{\sigma}^{\kappa}), \quad \kappa = 1, \ldots, \zeta.$$

Remark that the range of integration on the left-hand side is $I$ (finite or infinite) which has extended the original simplifying assumptions (with $I = [0, 1]$) given by Hairer [19]. The following result is similar to the classical result by Butcher in 1964 [1].

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1 It should be noticed that in $\tilde{D}(\zeta)$ we have removed “$w(\sigma)$” from both sides of the formula.
Theorem 3.1. If the coefficients \((A_{\tau,\sigma}w(\sigma), B_{\tau}w(\tau), C_{\tau})\) of method (3.2) satisfy \(\tilde{B}(\xi), \tilde{C}(\eta)\) and \(\tilde{D}(\zeta)\), then the method is of order at least \(\min(\xi, 2\eta + 2, \eta + \zeta + 1)\).

In this article, we will hold on a simple assumption abidingly as done in [39, 41, 45]

\[ C_{\tau} \equiv \tau, \quad \tau \in I. \tag{3.5} \]

The following results are presented for generalizing those available results given in [45].

Lemma 3.1. Under the assumption (3.5), \(\tilde{B}(\xi), \tilde{C}(\eta)\) and \(\tilde{D}(\zeta)\) are equivalent to respectively,

\[ \tilde{B}(\xi) : \int I B_{\tau}w(\tau)\phi(\tau) \, d\tau = \int_0^1 \phi(x) \, dx, \quad \text{for } \forall \phi \text{ with } \deg(\phi) \leq \xi - 1, \quad (3.6) \]

\[ \tilde{C}(\eta) : \int I A_{\tau,\sigma}w(\sigma)\phi(\sigma) \, d\sigma = \int_0^\tau \phi(x) \, dx, \quad \text{for } \forall \phi \text{ with } \deg(\phi) \leq \eta - 1, \quad (3.7) \]

\[ \tilde{D}(\zeta) : \int I B_{\tau}A_{\tau,\sigma}w(\tau)\phi(\tau) \, d\tau = B_{\sigma} \int_{\sigma}^1 \phi(x) \, dx, \quad \text{for } \forall \phi \text{ with } \deg(\phi) \leq \zeta - 1, \quad (3.8) \]

where \(\deg(\phi)\) stands for the degree of polynomial function \(\phi\).

Proof. Please refer to Lemma 2.1 of [45] to obtain a straightforward proof.

Theorem 3.2. Suppose \(B_{\tau}, A_{\tau,\sigma}, (B_{\tau} A_{\tau,\sigma}) \in L^2_w(I)\), then, under the assumption (3.5) we have

(a) \(\tilde{B}(\xi)\) holds \(\iff\) \(B_{\tau}\) has the following form in terms of the normalized orthogonal polynomials in \(L^2_w(I)\):

\[ B_{\tau} = \sum_{j=0}^{\xi - 1} \int_0^1 P_j(x) \, dx P_j(\tau) + \sum_{j=\xi}^{\xi} \lambda_j P_j(\tau), \tag{3.9} \]

where \(\lambda_j\) are any real parameters;

(b) \(\tilde{C}(\eta)\) holds \(\iff\) \(A_{\tau,\sigma}\) has the following form in terms of the normalized orthogonal polynomials in \(L^2_w(I)\):

\[ A_{\tau,\sigma} = \sum_{j=0}^{\eta - 1} \int_0^\tau P_j(x) \, dx P_j(\sigma) + \sum_{j=\eta}^{\eta} \phi_j(\tau) P_j(\sigma), \tag{3.10} \]

where \(\phi_j(\tau)\) are any real functions;

\(^2\)The notation \(A_{\tau,\sigma}\) stands for the one-variable function in terms of \(\sigma\), and \(A_{\tau,\star}\) can be understood likewise.
(c) \( \tilde{D}(\zeta) \) holds \( \iff \) \( B_\tau A_{\tau,\sigma} \) has the following form in terms of the normalized orthogonal polynomials in \( L_w^2(I) \):

\[
B_\tau A_{\tau,\sigma} = \sum_{j=0}^{\zeta-1} B_\sigma \int_\sigma^1 P_j(x) \, dx P_j(\tau) + \sum_{j \geq \zeta} \psi_j(\sigma) P_j(\tau), \tag{3.11}
\]

where \( \psi_j(\sigma) \) are any real functions.

Proof. This theorem can be proved in the same manner as Theorem 2.3 of \([45]\). \hfill \square

In general, we must truncate the series (3.9) and (3.10) suitably for practical use, and the integrals of (3.2) need to be approximated by using a weighted interpolatory quadrature formula (2.1). Thus, after applying the numerical quadrature, it results in an \( s \)-stage conventional RK method

\[
\tilde{Z}_i = z_0 + h \sum_{j=1}^{s} b_j A_{c_i, c_j} f(t_0 + c_j h, \tilde{Z}_j), \quad i = 1, \cdots, s, \tag{3.12}
\]

\[
z_1 = z_0 + h \sum_{i=1}^{s} b_i B_{c_i} f(t_0 + c_i h, \tilde{Z}_i),
\]

where \( \tilde{Z}_i \approx Z_{c_i} \). The following result is also an extension of the previous result by Tang et al \([39, 41]\).

**Theorem 3.3.** Assume the underlying quadrature formula (2.1) is of order \( p \), and \( A_{\tau,\sigma} \) is of degree \( \pi_A^\tau \) with respect to \( \tau \) and of degree \( \pi_A^\sigma \) with respect to \( \sigma \), and \( B_\tau \) is of degree \( \pi_B^\tau \). If all the simplifying assumptions \( \tilde{B}(\xi), \tilde{C}(\eta) \) and \( \tilde{D}(\zeta) \) in (3.4) are fulfilled, then the standard RK method (3.12) is at least of order

\[
\min(\rho, 2\alpha + 2, \alpha + \beta + 1),
\]

where \( \rho = \min(\xi, p - \pi_B^\tau) \), \( \alpha = \min(\eta, p - \pi_A^\sigma) \) and \( \beta = \min(\zeta, p - \pi_A^\tau - \pi_B^\tau) \).

Proof. Please refer to Theorem 2.4 of \([45]\), the proof of which can be adapted to the present case. \hfill \square

4. Geometric integration by csRK methods

In this section, we discuss the geometric integration by csRK methods within the new framework of csRK methods. It is well-known that symmetric methods and symplectic methods are of significant importance in numerical integration of time-reversible systems and Hamiltonian systems respectively, owing to that they have excellent discrete dynamic behaviors in long-time numerical simulation \([18]\). The definitions and relevant knowledge can be found in many literatures (see, for example, \([12, 13, 16, 18, 23, 31]\) and references therein).

\[^3\]That is, (2.1) exactly holds for any polynomial \( \Phi(x) \) of degree less than \( p \).
4.1. Symmetric methods

Definition 4.1. [18] A one-step method \( \phi_h \) is called symmetric if it satisfies
\[
\phi^*_h = \phi_h,
\]
where \( \phi^*_h = \phi^{-1}_{-h} \) is referred to as the adjoint method of \( \phi_h \).

By the definition, a method \( z_1 = \phi_h(z_0; t_0, t_1) \) is symmetric if exchanging \( h \leftrightarrow -h \), \( z_0 \leftrightarrow z_1 \) and \( t_0 \leftrightarrow t_1 \) leaves the original method unaltered. For the sake of deriving symmetric integrators, we now assume the interval \( I \) to be the following two cases:

(i) \( I = [a, b] \) (finite interval) with \( a + b = 1 \);

(ii) \( I = (-\infty, +\infty) \) (infinite interval).

Theorem 4.1. Under the assumption (3.3) and we suppose \( \Bar{B}(\xi) \) holds with \( \xi \geq 1 \) (which means the method is of order at least 1), then when \( I = [a, b] \) or \( I = (-\infty, +\infty) \) the csRK method (3.2) is symmetric, if
\[
A_{\tau, \sigma} w(\sigma) + A_{1-\tau, 1-\sigma} w(1-\sigma) \equiv B_{\sigma} w(\sigma), \quad \tau, \sigma \in I. \tag{4.1}
\]

Particularly, if \( w(\sigma) = w(1-\sigma) \), then the symmetric condition (4.1) becomes
\[
A_{\tau, \sigma} + A_{1-\tau, 1-\sigma} \equiv B_{\sigma}, \quad \tau, \sigma \in I. \tag{4.2}
\]

Proof. For the proof, we have referred to the technique given in [44]. Obviously, (4.1) implies \( B_{\sigma} w(\sigma) \equiv B_{1-\sigma} w(1-\sigma) \) in \( I \). Furthermore, by taking an integral on both sides of (4.1) with respect to \( \sigma \) (in use of (3.3)), we get \( C_{\tau} + C_{1-\tau} \equiv 1, \quad \tau \in I \).

Next, let us establish the adjoint method of a given csRK method. From (3.2), by interchanging \( t_0, z_0, h \) with \( t_1, z_1, -h \) respectively, we have
\[
Z_\tau = z_1 - h \int_I A_{\tau, \sigma} w(\sigma) f(t_1 - C_{\sigma} h, Z_\sigma) \, d\sigma, \quad \tau \in I,
\]
\[
z_0 = z_1 - h \int_I B_{\tau} w(\tau) f(t_1 - C_{\tau} h, Z_\tau) \, d\tau,
\]
Note that \( t_1 - C_{\tau} h = t_0 + (1 - C_{\tau}) h = t_0 + C_{1-\tau} h \), thus the second formula can be recast as
\[
z_1 = z_0 + h \int_I B_{\tau} w(\tau) f(t_0 + C_{1-\tau} h, Z_\tau) \, d\tau.
\]
By plugging it into the first formula, it ends up with
\[
Z_\tau = z_0 + h \int_I (B_{\sigma} - A_{\tau, \sigma}) w(\sigma) f(t_0 + C_{1-\sigma} h, Z_\sigma) \, d\sigma, \quad \tau \in I,
\]
\[
z_1 = z_0 + h \int_I B_{\tau} w(\tau) f(t_0 + C_{1-\tau} h, Z_\tau) \, d\tau,
\]
By change of integral variables, we obtain an equivalent scheme\footnote{Note that according to the assumption about $I$ the ranges of integration are invariant.}

$$Z^*_{\tau} = z_0 + h \int_I (B_{1-\tau} - A_{1-\tau,1-\sigma}) w(1-\sigma) f(t_0 + C_{\sigma} h, Z^*_\sigma) d\sigma, \quad \tau \in I,$$

$$z_1 = z_0 + h \int_I B_{1-\tau} w(1-\tau) f(t_0 + C_{\tau} h, Z^*_\tau) d\tau,$$

which is the adjoint method of the original method, where $Z^*_\tau = Z_{1-\tau}$. Take into account that

$$(B_{1-\sigma} - A_{1-\tau,1-\sigma}) w(1-\sigma) \equiv B_{\sigma} w(\sigma) - A_{1-\tau,1-\sigma} w(1-\sigma),$$

$$B_{1-\tau} w(1-\tau) \equiv B_{\tau} w(\tau),$$

and let the coefficients of the adjoint method match with that of the original method, i.e., imposing the condition (4.1), then by definition the original method is symmetric. The formula (4.2) is straightforward by removing $w(\sigma) (= w(1-\sigma))$ from both sides of (4.1).

**Theorem 4.2.** If the coefficients of the underlying csRK method (3.2) satisfy (4.2), then the RK scheme (3.12) derived by numerical quadrature is symmetric, provided that the quadrature weights and abscissae (see (2.1)) satisfy $b_{s+1-i} = b_i$ and $c_{s+1-i} = 1 - c_i$ for all $i$.

**Proof.** Please refer to the proof of Theorem 3.5 in \[44\].

**Theorem 4.3.** Suppose that $w(\sigma) \equiv w(1-\sigma)$ and $A_{\tau,\sigma}/B_{\sigma} \in L_w^2(I \times I)$, then symmetric condition (4.1) is equivalent to the fact that $A_{\tau,\sigma}$ has the following form in terms of the orthogonal polynomials $P_n(x)$ in $L_w^2(I)$

$$A_{\tau,\sigma} = B_{\sigma} \left( \frac{1}{2} + \sum_{\substack{i+j \text{ is odd} \atop 0 \leq i, j \leq n}} \alpha_{(i,j)} P_i(\tau) P_j(\sigma) \right), \quad \alpha_{(i,j)} \in \mathbb{R},$$

(4.3)

with $B_{\sigma} \equiv B_{1-\sigma}$, where $\alpha_{(i,j)}$ can be any real parameters for odd $i + j$, provided that the orthogonal polynomials $P_n(x)$ satisfy

$$P_n(1 - x) = (-1)^n P_n(x), \quad n \in \mathbb{Z}.$$

(4.4)

**Proof.** Please refer to the proof of Theorem 4.6 of \[45\].

**Theorem 4.4.** If $w(x)$ is an even function, i.e., satisfying $w(-x) \equiv w(x)$, then the shifted function defined by $\hat{w}(x) = w(2\mu x - \mu)$ satisfies the symmetry relation: $\hat{w}(x) \equiv \hat{w}(1 - x)$. Here $\mu$ is a non-zero constant and usually we take it as $\mu = 1$. 

The following results can be easily verified by using a change of variables.
Theorem 4.5. If a sequence of polynomials \( \{P_n(x)\}_{n=0}^{\infty} \) satisfy the symmetry relation
\[
P_n(-x) = (-1)^n P_n(x), \quad n \in \mathbb{Z},
\]
then the shifted polynomials defined by \( \hat{P}_n(x) = P_n(2\mu x - \mu) \) are bound to satisfy the property (4.4). Here \( \mu \) is a non-zero constant and usually we take it as \( \mu = 1 \).

We find that many classical (standard) orthogonal polynomials including Hermite polynomials, Legendre polynomials, Chebyshev polynomials of the first and second kind, and any other general Gegenbauer polynomials etc., do not satisfy (4.4), but they possess the symmetry relation (4.5). Theorem 4.5 helps us to find suitable orthogonal polynomials to satisfy (4.4).

Theorem 4.6. If a sequence of polynomials \( \{P_n(x)\}_{n=0}^{\infty} \) satisfy (4.4), then we have
\[
\int_0^1 P_j(x) \, dx = 0, \quad \text{for } j \text{ is odd},
\]
and the following function
\[
B_\sigma = \sum_{j=0}^{\xi-1} \int_0^1 P_j(x) \, dx P_j(\sigma) + \sum_{\text{even } j \geq \xi} \lambda_j P_j(\sigma), \quad \xi \geq 1,
\]
always satisfies \( B_\sigma \equiv B_{1-\sigma} \).

With the help of these results, we can conveniently construct symmetric methods by Theorem 4.3.

4.2. Symplectic methods

We can directly apply some available results of [44, 45] to csRK method (3.2), it then gives the following results.

Theorem 4.7. If the coefficients of a csRK method (3.2) satisfy
\[
B_\tau A_{\tau,\sigma} + B_\sigma A_{\sigma,\tau} \equiv B_\tau B_\sigma, \quad \tau, \sigma \in I,
\]
then it is symplectic.

Proof. This theorem can be proved in a similar manner as Theorem 3.1 of [44]. In fact, after conducting the same arguments as shown in [44], we find that if
\[
\left( B_\tau w(\tau) \right) \left( A_{\tau,\sigma} w(\sigma) \right) + \left( B_\sigma w(\sigma) \right) \left( A_{\sigma,\tau} w(\tau) \right) \equiv \left( B_\tau w(\tau) \right) \left( B_\sigma w(\sigma) \right),
\]
then the method (3.2) is symplectic. By removing the factor “\( w(\tau)w(\sigma) \)” from both sides of (4.8), it gives (4.7). And this in turn implies that if (4.7) is fulfilled, then it gives (4.8) which verifies the symplecticity of the method.
Remark 4.1. Theorem 4.7 implies that the symplecticity of the csRK methods (3.2) is independent of its weight function.

Theorem 4.8. The RK scheme (3.12) based on the underlying csRK method with coefficients satisfying (4.7) is always symplectic.

Proof. Please see \[44\] for getting a similar proof.

Theorem 4.9. Under the assumption (3.5), for a symplectic csRK method with coefficients satisfying (4.7), we have the following statements:

(a) $\tilde{B}(\xi)$ and $\tilde{C}(\eta)$ $\implies$ $\tilde{D}(\zeta)$, where $\zeta = \min\{\xi, \eta\}$;

(b) $\tilde{B}(\xi)$ and $\tilde{D}(\zeta)$ $\implies$ $\tilde{C}(\eta)$, where $\eta = \min\{\xi, \zeta\}$.

Proof. Please refer to the proof of Theorem 4.1 in \[45\].

Theorem 4.10. Suppose that $A_{\tau,\sigma}/B_\sigma \in L^2_w(I \times I)$, then symplectic condition (4.7) is equivalent to the fact that $A_{\tau,\sigma}$ has the following form in terms of the normalized orthogonal polynomials $P_n(x)$ in $L^2_w(I)$

$$A_{\tau,\sigma} = B_\sigma \left( \frac{1}{2} + \sum_{0<i+j \in \mathbb{Z}} \alpha(i,j) P_i(\tau) P_j(\sigma) \right), \quad \alpha(i,j) \in \mathbb{R},$$

where $\alpha(i,j)$ is skew-symmetric, i.e., $\alpha(i,j) = -\alpha(j,i)$, $i + j > 0$.

Proof. Please refer to the proof of Theorem 4.3 in \[45\].

On the basis of these preliminaries, we could introduce an operational procedure for establishing symplectic RK-type methods — actually the similar technique has been developed in \[45, 46, 47\]. The procedure is as follows:\footnote{For more details, we refer the readers to \[47\].
An alternative technique is to consider using $\tilde{D}(\zeta)$.}

Step 1. Make an ansatz for $B_\tau$ which satisfies $\tilde{B}(\xi)$ with $\xi \geq 1$ according to (3.9), and a finite number of $\lambda_i$ could be kept as parameters;

Step 2. Suppose $A_{\tau,\sigma}$ is in the form (by Theorem 4.10)

$$A_{\tau,\sigma} = B_\sigma \left( \frac{1}{2} + \sum_{0<i+j \in \mathbb{Z}} \alpha(i,j) P_i(\tau) P_j(\sigma) \right), \quad \alpha(i,j) = -\alpha(j,i),$$

where $\alpha(i,j)$ are kept as parameters with a finite number, and then substitute $A_{\tau,\sigma}$ into $\tilde{C}(\eta)$ (see (3.7), usually we let $\eta < \xi$) for determining $\alpha(i,j)$:

$$\int_I A_{\tau,\sigma} w(\sigma) \phi_k(\sigma) d\sigma = \int_0^T \phi_k(x) dx, \quad k = 0, 1, \cdots, \eta - 1,$$
Here, $\phi_k(x)$ stands for any polynomial of degree $k$, which performs very similarly as the “test function” used in general finite element analysis;

**Step 3.** Write down $B_\tau w(\tau)$ and $A_{r,\sigma} w(\sigma)$ (satisfy $\tilde{B}(\xi)$ and $\tilde{C}(\eta)$ automatically), which results in a symplectic csRK method of order at least $\min\{\xi, 2\eta + 2, \eta + \zeta + 1\}$ with $\zeta = \min\{\xi, \eta\}$ by Theorem 3.1 and 4.9. If needed, we then get symplectic RK methods by using quadrature rules (see Theorem 4.8).

Remark that the procedure above provides a general approach for establishing symplectic integrators. In view of Theorem 3.3 and 4.9, it is suggested to design RK coefficients with low-degree $A_{r,\sigma}$ and $B_\tau$. For the sake of easily calculating those integrals of $\tilde{C}(\eta)$ in the second step, the following ansatz may be advisable (with $C_\tau$ given by (3.5) and let $\rho \geq \eta$ and $\xi \geq 2\eta$)

$$B_\tau = \sum_{j=0}^{\xi-1} \int_0^1 P_j(x) \, dx P_j(\tau), \quad A_{r,\sigma} = B_\sigma \left( \frac{1}{2} + \sum_{0<i+j\geq Z, \rho,j \leq \xi-\eta} \alpha_{(i,j)} P_i(\tau) P_j(\sigma) \right),$$

where $\alpha_{(i,j)} = -\alpha_{(j,i)}$. Because of the index $j$ restricted by $j \leq \xi - \eta$ in the second formula of (4.11), we can use $\tilde{B}(\xi)$ to arrive at (please c.f. (3.6))

$$\int_I A_{r,\sigma} w(\sigma) \phi_k(\sigma) \, d\sigma = \int_I B_\sigma \left( \frac{1}{2} + \sum_{0<i+j\geq Z, \rho,j \leq \xi-\eta} \alpha_{(i,j)} P_i(\tau) P_j(\sigma) \right) w(\sigma) \phi_k(\sigma) \, d\sigma$$

$$= \frac{1}{2} \int_0^1 \phi_k(x) \, dx + \sum_{0<i+j\geq Z, \rho,j \leq \xi-\eta} \alpha_{(i,j)} P_i(\tau) \int_0^1 P_j(\sigma) \phi_k(\sigma) \, d\sigma, \quad 0 \leq k \leq \eta - 1.$$

Therefore, $\tilde{C}(\eta)$ implies that

$$\frac{1}{2} \int_0^1 \phi_k(x) \, dx + \sum_{0<i+j\geq Z, \rho,j \leq \xi-\eta} \alpha_{(i,j)} P_i(\tau) \int_0^1 P_j(\sigma) \phi_k(\sigma) \, d\sigma = \int_0^\tau \phi_k(x) \, dx, \quad 0 \leq k \leq \eta - 1.$$ (4.12)

Finally, it needs to settle $\alpha_{(i,j)}$ by transposing, comparing or merging similar items of (4.12) after the polynomial on right-hand side being represented by the basis $\{P_j(x)\}_{j=0}^\infty$. In view of the skew-symmetry of $\alpha_{(i,j)}$, if we let $r = \min\{\rho, \xi - \eta\}$, then actually the degrees of freedom of these parameters is $r(r + 1)/2$, by noticing that

$$\alpha_{(i,i)} = 0, \quad i \geq 1 \quad \text{and} \quad \alpha_{(i,j)} = 0, \quad \text{for} \quad i > r \quad \text{or} \quad j > r.$$

### 4.3. Some examples

In this part, we give some examples for illustrating the application of our theoretical results given in the preceding section. In view of the skew-symmetry of $\alpha_{(i,j)}$, we only provide the values of $\alpha_{(i,j)}$ with $i < j$, and the Gauss-Christoffel’s quadrature rule with optimal order (see Theorem 2.1 and Remark 2.1) will be used in these examples.
Example 4.1. Consider using the normalized Legendre polynomials which are orthogonal with respect to the weight function \( w(x) = 1 \) on \([-1, 1]\). These Legendre polynomials \( P_n(x) \) can be defined by Rodrigues’ formula

\[
P_0(x) = \frac{1}{\sqrt{2}}, \quad P_n(x) = \frac{\sqrt{2n + 1}}{2^{n+\frac{1}{2}} n!} \frac{d^n}{dx^n} (x^2 - 1)^n, \quad n = 1, 2, \cdots.
\]

Let \( \xi = 3, \eta = 1, \rho = 2 \), after some elementary calculations, it gives

\[
\alpha(0,1) = -\frac{2\sqrt{3}}{3}, \quad \alpha(1,2) = -\frac{2\sqrt{3}}{3} \alpha(0,2).
\]

Let \( \mu = \alpha(0,2) \) be a free parameter, then we get a family of \( \mu \)-parameter symplectic csRK methods of order 3. By using Gauss-Christoffel’s quadrature rules with 2 nodes and 3 nodes respectively, we get symplectic RK methods of order 3 which are shown in Tab. 4.1 and 4.2.

Remark 4.2. If we shift the Legendre polynomials from \([-1, 1]\) to the standard interval \([0, 1]\), then we can use Theorem 4.3 and 4.10 to derive many classical high-order symmetric and symplectic RK methods (e.g., Gauss-Legendre RK methods). The construction of symplectic RK methods with shifted Legendre polynomials has been investigated in the literature [38, 39, 41, 44].

Example 4.2. Consider using the normalized Laguerre polynomials which are orthogonal with respect to the weight function \( w(x) = e^{-x} \) on \([0, +\infty)\). These Laguerre polynomials \( L_n(x) \) can be defined by Rodrigues’ formula

\[
L_0(x) = 1, \quad L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n} (x^n e^{-x}), \quad n = 1, 2, \cdots.
\]
Table 4.3: A family of 2-stage 2-order symplectic RK methods, based on Laguerre polynomials defined on 
\([0, +\infty)\).

Let \(\xi = 3\), \(\eta = 1\), \(\rho = 2\), after some elementary calculations, it gives

\[
\alpha_{(0,1)} = 1 - \frac{1}{3} \alpha_{(0,2)}, \quad \alpha_{(1,2)} = -2 \alpha_{(0,2)}.
\]

Let \(\mu = \alpha_{(0,2)}\) be a free parameter, then we get a family of \(\mu\)-parameter symplectic csRK methods of order 3. A family of symplectic RK methods of order 2 are shown in Tab. 4.3.

If we use 3-point Gauss-Christoffel’s quadrature rule, then we get the family of 3-stage 3-order symplectic RK methods (we do not show it here since the exact zeros of \(L_3(x)\) is too complicated to be exhibited.).

**Example 4.3.** Consider using the normalized Hermite polynomials which are orthogonal with respect to the weight function \(w(x) = e^{-x^2}\) on \((-\infty, +\infty)\). These Hermite polynomials \(H_n(x)\) can be defined by Rodrigues’ formula

\[
H_0(x) = \frac{1}{\pi^{\frac{1}{4}}}, \quad H_n(x) = \frac{(-1)^n e^{x^2}}{\sqrt{2^nn!\pi^{\frac{1}{4}}}} \frac{d^n}{dx^n}(e^{-x^2}), \quad n = 1, 2, \ldots.
\]

Let \(\xi = 3\), \(\eta = 1\), \(\rho = 2\), after some elementary calculations, it gives

\[
\alpha_{(0,1)} = \frac{1}{3} \alpha_{(0,2)} - \frac{\sqrt{2\pi}}{2}, \quad \alpha_{(1,2)} = -\sqrt{2} \alpha_{(0,2)}.
\]

Let \(\mu = \alpha_{(0,2)}\) be a free parameter, then we get a family of \(\mu\)-parameter symmetric and symplectic csRK methods of order 4. We show the family of 3-stage 4-order symmetric and symplectic RK methods in Tab. 4.5.
Remark 4.3. From these examples, it is seen that the same type of orthogonal polynomials can be defined on different intervals (by shifting variables), and when we use their different versions, it may result in different symplectic schemes with different order accuracy.

5. Numerical experiments

For the sake of verifying the efficiency of our new integrators, we perform some numerical experiments with symplectic integrators of order 2, 3 and 4 respectively (see Tab. 4.1, 4.3 and 4.5). We consider the well-known Kepler’s problem determined by the Hamiltonian function \[ H(p, q) = \frac{p_1^2 + p_2^2}{2} - \frac{1}{\sqrt{q_1^2 + q_2^2}} \] with initial value conditions \((p_1(0), p_2(0), q_1(0), q_2(0)) = (0, 1, 1, 0)\). The exact solution is \(p_1(t) = -\sin(t), \ p_2(t) = \cos(t), \ q_1(t) = \cos(t), \ q_2(t) = \sin(t)\).

The numerical results are shown in Fig. 5.1-5.6. From the plots of energy error and solution error, we observe that all the symplectic methods nearly preserve the energy with a bounded error and they basically possess a linear growth of solution error. Besides, the exact trajectory of the Kepler’s problem is well simulated by our numerical methods. The symplecticity of our methods is therefore verified by these numerical behaviors.

6. Conclusions

This paper shows how to extend the primitive framework of continuous stage Runge-Kutta methods to a new larger one such that it enables us to treat more complicated cases.
Figure 5.1: Energy error and solution error by 2-order Laguerre symplectic method ($\mu = 0$) given in Tab. 4.3, step size $h = 0.1$.

Figure 5.2: Trajectories of exact and numerical solution, using 2-order Laguerre symplectic method ($\mu = 0$) given in Tab. 4.3, step size $h = 0.1$. 
Figure 5.3: Energy error and solution error by 3-order Legendre symplectic method given in Tab. 4.1, step size $h = 0.1$.

Figure 5.4: Trajectories of exact and numerical solution, using 3-order Legendre symplectic method given in Tab. 4.1, step size $h = 0.1$. 
Figure 5.5: Energy error and solution error by 4-order Hermite symplectic method ($\mu = \sqrt{2\pi}/14$) given in Tab. 4.5, step size $h = 0.1$.

Figure 5.6: Trajectories of exact and numerical solution, using 4-order Hermite symplectic method ($\mu = \sqrt{2\pi}/14$) given in Tab. 4.5, step size $h = 0.1$. 
By using the new theoretical framework, we can not only obtain more general-purpose RK-type methods for practical use, but also it allows us to discover new geometric integrators for special purpose e.g., symplectic-structure preserving integrators for Hamiltonian systems. There is a very interesting fact in the construction of symplectic methods, that is, by using the same type of orthogonal polynomials we can get different symplectic RK schemes if the polynomials are shifted to different intervals. It is shown that the orthogonal range $I$ of the polynomials could heavily impact the formalism of the resulting methods.

Although we only present three examples for deriving symmetric and symplectic integrators with order up to 4, essentially the same technique can be directly used to get more higher order methods. Moreover, other types of orthogonal polynomials can also be considered for use, regardless of weighting on a finite or infinite interval.

At last but not the least, we emphasize that our new symplectic methods may not be necessarily implemented via RK schemes. Recognize that for all our practical csRK methods (after truncating the series (3.9) and (3.10)), the Butcher coefficient $A_{\tau, \sigma}w(\sigma)$ appeared in (3.2) is a polynomial with respect to $\tau$ (but not to $\sigma$ in general, since $w(\sigma)$ may be not a polynomial), this implies that the stage value $Z_{\tau}$ of the csRK methods is also a (vector-valued) polynomial. This fact suggests that we can use a polynomial expansion of $Z_{\tau}$ in terms of unknown coefficients $\gamma_j$, say, let

$$Z_{\tau} = \sum_{j=1}^{k+1} \gamma_j \phi_j(\tau), \quad (6.1)$$

where $Z_{\tau}$ is assumed to be of degree $k$ and $\{\phi_j(\tau)\}$ is a suitable basis in the polynomial function space of degree $k$ at most. Therefore, by substituting the expansion (6.1) into (3.2), it results in an algebraic system in terms of $\gamma_j$ which can be solved by iteration. Compared with the classical RK scheme (3.12), such a technique may bring us more cost savings in computation, especially when we consider using a high-order quadrature formula with many nodes for approximating the integrals of (3.2). In a word, conventional RK scheme (see (3.12)) may be implemented in other ways and we can understand the same numerical method from a different perspective.

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References

[1] M. Abramowitz, I.A. Stegun, Eds., Handbook of Mathematical Functions, Dover, New York, 1965.
[2] S. C. Brenner, L. R. Scott, *The Mathematical Theory of Finite Element Methods, 3rd ed.*, vol. 15, Texts in Applied Mathematics, New York: Springer, (2008), doi: 10.1007/978-0-387-75934-0.

[3] L. Brugnano, F. Iavernaro, D. Trigiante, *Hamiltonian boundary value methods: energy preserving discrete line integral methods*, J. Numer. Anal., Indust. Appl. Math. 5 (1–2) (2010), 17–37.

[4] J.C. Butcher, *Implicit Runge-Kutta processes*, Math. Comput. 18 (1964), 50–64.

[5] J.C. Butcher, *An algebraic theory of integration methods*, Math. Comp., 26 (1972), 79-106.

[6] J.C. Butcher, *The role of orthogonal polynomials in numerical ordinary differential equations*, J Comput. Appl. Math., 43 (1992), 231–242.

[7] J.C. Butcher, *A history of Runge-Kutta methods*, Appl. numer. Math., 20 (1996), 247–260.

[8] J.C. Butcher, *The Numerical Analysis of Ordinary Differential Equations: Runge-Kutta and General Linear Methods*, John Wiley & Sons, 1987.

[9] M. Calvo, J.M. Franco, J.I. Montijano, L. Rández, *Sixth-order symmetric and symplectic exponentially fitted Runge-Kutta methods of the Gauss type*, J. Comput. Appl. Math., 223 (2009), 387–398.

[10] R.P.K. Chan, *On symmetric Runge–Kutta methods of high order*, Computing 45 (1990), 301–309.

[11] X. Ding, J. Tan, *Implicit RungeCKutta methods based on Radau quadrature formula*, Int. J. Comput. Math., 86 (8) (2009), 1394–1404.

[12] K. Feng, *K. Feng’s Collection of Works*, Vol. 2, Beijing: National Defence Industry Press, 1995.

[13] K. Feng, M. Qin, *Symplectic Geometric Algorithms for Hamiltonian Systems*, Spriger and Zhejiang Science and Technology Publishing House, Heidelberg, Hangzhou, First edition, 2010.

[14] D. Gottlieb, S.A. Orszag, *Numerical Analysis of Spectral Methods: Theory and Applications*, SIAM-CBMS, Philadelphia, 1977.

[15] B. Guo, *Spectral Methods and their Applications*, World Scientifc, Singapore, 1998.

[16] E. Hairer, S.P. Nørsett, G. Wanner, *Solving Ordairy Differential Equations I: Nonstiff Problems*, Springer Series in Computational Mathematics, 8, Springer-Verlag, Berlin, 1993.
[17] E. Hairer, G. Wanner, *Solving Ordinary Differential Equations II: Stiff and Differential-Algebraic Problems*, Second Edition, Springer Series in Computational Mathematics, 14, Springer-Verlag, Berlin, 1996.

[18] E. Hairer, C. Lubich, G. Wanner, *Geometric Numerical Integration: Structure-Preserving Algorithms For Ordinary Differential Equations*, Second edition, Springer Series in Computational Mathematics, 31, Springer-Verlag, Berlin, 2006.

[19] E. Hairer, *Energy-preserving variant of collocation methods*, JNAIAM J. Numer. Anal. Indust. Appl. Math. 5 (2010), 73–84.

[20] E. Hairer, C.J. Zbinden, *On conjugate-symplecticity of B-series integrators*, IMA J. Numer. Anal. 33 (2013), 57–79.

[21] K. L. Judd, “Quadrature Methods” presented at University of Chicago’s “Initiative for Computational Economics 2012”, 2012.

[22] F. Lasagni, *Canonical Runge-Kutta methods*, ZAMP 39 (1988), 952–953.

[23] B. Leimkuhler, S. Reich, *Simulating Hamiltonian dynamics*, Cambridge University Press, Cambridge, 2004.

[24] H. Liu and G. Sun, *Implicit Runge-Kutta methods based on Lobatto quadrature formula*, Int. J. Comput. Math., 82(1) (2005), 77–88.

[25] Y. Li, X. Wu, *Functionally fitted energy-preserving methods for solving oscillatory non-linear Hamiltonian systems*, SIAM J. Numer. Anal., 54 (4)(2016), 2036–2059.

[26] Y. Miyatake, *An energy-preserving exponentially-fitted continuous stage Runge-Kutta method for Hamiltonian systems*, BIT Numer. Math., (54)(2014), 777-799.

[27] Y. Miyatake, J. C. Butcher, *A characterization of energy-preserving methods and the construction of parallel integrators for Hamiltonian systems*, SIAM J. Numer. Anal., 54(3)(2016), 1993–2013.

[28] G.R.W. Quispel, D.I. McLaren, *A new class of energy-preserving numerical integration methods*, J. Phys. A: Math. Theor. 41 (2008) 045206.

[29] C. Runge, *Ueber die numerische Auflösung von Differentialgleichungen*, Math. Ann., 46 (1895), 167–178.

[30] J.M. Sanz-Serna, *Runge-Kutta methods for Hamiltonian systems*, BIT 28 (1988), 877–883.

[31] J.M. Sanz-Serna, M.P. Calvo, *Numerical Hamiltonian problems*, Chapman & Hall, 1994.
[32] E. Süli, D. F. Mayers, An Introduction to Numerical Analysis, Cambridge University Press, 2003.

[33] G. Sun, Construction of high order symplectic Runge-Kutta methods, J. Comput. Math., 11 (1993), 250–260.

[34] G. Sun, A simple way constructing symplectic Runge-Kutta methods, J. Comput. Math., 18 (2000), 61–68.

[35] Y.B. Suris, Canonical transformations generated by methods of Runge-Kutta type for the numerical integration of the system $x'' = -\frac{\partial U}{\partial x}$, Zh. Vychisl. Mat. iMat. FiZ. 29 (1989), 202–211.

[36] G. Szegő, Orthogonal Polynomials, vol. 23, Amer. Math. Soc., 1985.

[37] W. Tang, Y. Sun, Time finite element methods: A unified framework for numerical discretizations of ODEs, Appl. Math. Comput. 219 (2012), 2158–2179.

[38] W. Tang, Y. Sun, A new approach to construct Runge-Kutta type methods and geometric numerical integrators, AIP. Conf. Proc. 1479 (2012), 1291–1294.

[39] W. Tang, Y. Sun, Construction of Runge-Kutta type methods for solving ordinary differential equations, Appl. Math. Comput., 234 (2014), 179–191.

[40] W. Tang, Y. Sun, J. Zhang, High order symplectic integrators based on continuous-stage Runge-Kutta-Nyström methods, preprint, 2015.

[41] W. Tang, G. Lang, X. Luo, Construction of symplectic (partitioned) Runge-Kutta methods with continuous stage, Appl. Math. Comput., 286 (2016), 279–287.

[42] W. Tang, Y. Sun, W. Cai, Discontinuous Galerkin methods for Hamiltonian ODEs and PDEs, J. Comput. Phys., 330 (2017), 340–364.

[43] W. Tang, J. Zhang, Symplecticity-preserving continuous-stage Runge-Kutta-Nyström methods, Appl. Math. Comput., 323 (2018), 204–219.

[44] W. Tang, A note on continuous-stage Runge-Kutta methods, submitted, 2018.

[45] W. Tang, Continuous-stage Runge-Kutta methods based on weighted orthogonal polynomials, preprint, 2018.

[46] W. Tang, Chebyshev symplectic methods based on continuous-stage Runge-Kutta methods, preprint, 2018.

[47] W. Tang, Symplectic integration with Jacobi polynomials, preprint, 2018.