HARRINGTON’S PRINCIPLE IN HIGHER ORDER ARITHMETIC

YONG CHENG AND RALF SCHINDLER

Abstract. Let $Z_2$, $Z_3$, and $Z_4$ denote 2nd, 3rd, and 4th order arithmetic, respectively. We let Harrington’s Principle, $HP$, denote the statement that there is a real $x$ such that every $x$-admissible ordinal is a cardinal in $L$. The known proofs of Harrington’s theorem “$Det(\Sigma_1^1)$ implies $0^\sharp$ exists” are done in two steps: first show that $Det(\Sigma_1^1)$ implies $HP$, and then show that $HP$ implies $0^\sharp$ exists. The first step is provable in $Z_2$. In this paper we show that $Z_2 + HP$ is equiconsistent with ZFC and that $Z_3 + HP$ is equiconsistent with ZFC + there exists a remarkable cardinal. As a corollary, $Z_3 + HP$ does not imply $0^\sharp$ exists, whereas $Z_4 + HP$ does. We also study strengthenings of Harrington’s Principle over 2nd and 3rd order arithmetic.

§1. Introduction. Over the last four decades, much work has been done on the relationship between large cardinal and determinacy hypothesis, especially the large cardinal-determinacy correspondence. The first result in this line was proved by Martin and Harrington.

Theorem 1.1 (Martin–Harrington, [5]). In ZF, $Det(\Sigma_1^1)$ if and only if $0^\sharp$ exists.

Definition 1.2. We let Harrington’s Principle, $HP$ for short, denote the following statement:

$$\exists x \in 2^{\omega\omega} \forall \alpha (\alpha \text{ is } x\text{-admissible} \rightarrow \alpha \text{ is an } L\text{-cardinal}).$$

Theorem 1.3 (Silver, [5]). In ZF, HP implies $0^\sharp$ exists.

Definition 1.4.

(i) $Z_2 = ZFC^- +$ Every set is countable.
(ii) $Z_3 = ZFC^- + \mathcal{P}(\omega)$ exists + Every set is of cardinality $\leq \beth_1$.
(iii) $Z_4 = ZFC^- + \mathcal{P}(\mathcal{P}(\omega))$ exists + Every set is of cardinality $\leq \beth_2$.

$Z_2$, $Z_3$, and $Z_4$ are the corresponding axiomatic systems for second order arithmetic (SOA), third order arithmetic, and fourth order arithmetic, respectively. Note that $Z_3 \vdash H_{\omega_1} \models Z_2$ and $Z_4 \vdash H_{\beth_1} \models Z_3$.

The known proofs of Harrington’s theorem “$Det(\Sigma_1^1)$ implies $0^\sharp$ exists” are done in two steps: first show that $Det(\Sigma_1^1)$ implies $HP$, and then show that $HP$
implies $0^d$ exists. The first step is provable in $Z_2$. In this paper we prove that $Z_2 + \text{HP}$ is equiconsistent with ZFC and $Z_1 + \text{HP}$ is equiconsistent with ZFC + there exists a remarkable cardinal. As a corollary, we have $Z_3 + \text{HP}$ does not imply $0^d$ exists. In contrast, $Z_4 + \text{HP}$ implies $0^d$ exists.

We also investigate strengthenings of Harrington’s Principle, HP($\phi$), over higher order arithmetic.

**Definition 1.5.** Let $\phi(-)$ be a $\Sigma_2$–formula in the language of set theory such that, provably in ZFC: for all $\alpha$, if $\phi(\alpha)$, then $\alpha$ is an inaccessible cardinal and $L \models \phi(\alpha)$. Let HP($\phi$) denote the statement:

$$\exists x \in 2^\omega \forall \alpha (\alpha \text{ is } x\text{-admissible} \rightarrow L \models \phi(\alpha)).$$

We show that $Z_2 + \text{HP}(\phi)$ is equiconsistent with ZFC + $\{\alpha | \phi(\alpha)\}$ is stationary and that $Z_3 + \text{HP}(\phi)$ is equiconsistent with

$$ZFC + \text{ there exists a remarkable cardinal } \kappa \text{ with } \phi(\kappa) + \{\alpha | \phi(\alpha) \land \{\beta < \alpha | \phi(\beta)\} \text{ is stationary in } \alpha\} \text{ is stationary.}$$

As a corollary, $Z_4$ is the minimal system of higher order arithmetic to show that HP, HP($\phi$), and $0^d$ exists are pairwise equivalent with each other.

§2. Definitions and preliminaries. Our definitions and notations are standard.

We refer to the textbooks [7], [10], [11], or [16] for the definitions and notations we use. For the definition of admissible sets, admissible ordinals, and $x$-admissible ordinals for $x \in 2^\omega$, see [1], [12], and [4]. Our classes will always be definable ones. Our notations about forcing are standard (see [7] and [6]). For the general theory of forcing, see [11], and for Jensen’s theory of subcomplete forcing, see [9]. For Revised Countable Support (RCS) iteration, see [17] and also [8]. For notions of large cardinals, see [10] or [16]. We say that $0^d$ exists if there exists an iterable premouse of the form $(L_n, \in, U)$ where $U \neq \emptyset$, see e.g. [16]. We can define $0^d$ in $Z_2$. In $Z_2$, $0^d$ exists if and only if

$$\exists x \in \omega^\omega (x \text{ codes a countable iterable premouse}),$$

which is a $\Sigma_1^1$ statement.

The notion of remarkable cardinals was introduced by the second author in [15].

**Definition 2.1 ([15]).** A cardinal $\kappa$ is remarkable if and only if for all regular cardinals $\theta > \kappa$ there are $\pi, M, \bar{\kappa}, \sigma, N$, and $\bar{\theta}$ such that the following hold: $\pi: M \rightarrow H_\theta$ is an elementary embedding, $M$ is countable and transitive, $\pi(\bar{\kappa}) = \kappa$, $\sigma: M \rightarrow N$ is an elementary embedding with critical point $\bar{\kappa}$. $N$ is countable and transitive, $\bar{\theta} = M \cap \text{Ord}$ is a regular cardinal in $N$, $\sigma(\bar{\kappa}) > \bar{\theta}$, and $M = H^N_{\bar{\theta}}$. i.e., $M \in N$ and $N \models M$ is the set of all sets which are hereditarily smaller than $\bar{\theta}$.

**Definition 2.2 ([15]).** Let $\kappa$ be an inaccessible cardinal. Let $G$ be $\text{Col}(\omega, < \kappa)$-generic over $V$, let $\theta > \kappa$ be a cardinal, and let $X \in [H^V_\theta]^{\omega_1} \cap V[G]$. We say that $X$ condenses remarkably if $X = \text{ran}(\pi)$ for some elementary

$$\pi : (H^V_\beta, \in, H^V_\beta \cap G) \rightarrow (H^V_\theta, \in, H^V_\theta \cap G),$$

where $\alpha = \text{crit}(\pi) < \beta < \kappa$ and $\beta$ is a regular cardinal in $V$. 
Lemma 2.3 ([15]). A cardinal \( \kappa \) is remarkable if and only if for all regular cardinals \( \theta > \kappa \) we have that

\[ \|X \in [H_\theta^{V[G]}]^{\omega} \cap V[\dot{G}]: X \text{ condenses remarkably} \| \text{ is stationary.} \]

From Lemma 2.3, \( \kappa \) is remarkable in \( L \) if and only if for any \( L \)-cardinal \( \mu \geq \kappa \), for any \( G \) which is \( \text{Col}(\omega, < \kappa) \)-generic over \( L \), we have \( L[G] \models \|S_\mu = \{ X < L_\mu | X \text{ is countable and o.t.}(X \cap \mu) \text{ is an } L \text{-cardinal} \| \) is stationary.

All the following facts on remarkable cardinals are from [15]: every remarkable cardinal is remarkable in \( L \); every remarkable cardinal \( \kappa \) is \( n \)-ineffable for every \( n < \omega \); if \( 0^1 \) exists, then every Silver indiscernible is remarkable in \( L \); if there exists a \( \omega \)-Erdős cardinal, then there exist \( \alpha < \beta < \omega_1 \) such that \( L_\beta \models \|ZFC + \alpha \| \) is remarkable.

§3. The strength of Harrington’s Principle over higher order arithmetic.

3.1. The strength of \( Z_2 + \text{Harrington’s Principle} \).

Theorem 3.1. \( Z_2 + \text{HP} \) is equiconsistent with \( \text{ZFC} \).

Proof. It is easy to see that \( Z_2 + \text{HP} \) implies \( L \models \text{ZFC} \).

We now show that \( \text{Con}(\text{ZFC}) \) implies \( \text{Con}(Z_2 + \text{HP}) \). We assume that \( L \) is a minimal model of \( \text{ZFC} \), i.e.,

\[ \text{there is no } \alpha \text{ such that } L_\alpha \models \text{ZFC}. \quad (3.1) \]

Let \( G \) be \( \text{Col}(\omega, < \text{Ord}) \)-generic over \( L \). Then \( L[G] \models Z_2 \). In \( L[G] \), we may pick some \( A \subseteq \text{Ord} \) such that \( V = L[A] \) and if \( \lambda \geq \omega \) is an \( L \)-cardinal, then \( A \cap (\lambda, \lambda + \omega) \) codes a well-ordering of \( (\lambda^+)^L \). By (3.1) we will then have that for all \( \alpha \geq \omega \),

\[ L_{\alpha+1}[A \cap \alpha] \models \alpha \text{ is countable.} \quad (3.2) \]

By (3.2) there exists then a canonical sequence \( (c_\alpha | \alpha \in \text{Ord}) \) of pairwise almost disjoint subset of \( \omega \) such that \( c_\alpha \) is the \( L_{\alpha+1}[A \cap \alpha] \)-least subset of \( \omega \) such that \( c_\alpha \) is almost disjoint from every member of \( \{ c_\beta | \beta < \alpha \} \). Do almost disjoint forcing to code \( A \) by a real (i.e., a subset of \( \omega \) \( x \)) such that for any \( \alpha \in \text{Ord}, \alpha \in A \iff |x \cap c_\alpha| < \omega \) (cf. e.g. [2, Section 1.2]). This forcing is c.c.c. Note that \( L[A][x] = L[x] \) and \( L[x] \models Z_2 \).

We claim that \( \text{HP} \) holds in \( L[x] \). It suffices to show that if \( \alpha \) is \( x \)-admissible, then \( \alpha \) is an \( L \)-cardinal. Suppose \( \alpha \) is \( x \)-admissible but is not an \( L \)-cardinal. Let \( \lambda \) be the largest \( L \)-cardinal \( < \alpha \). Note that we can define \( A \cap \alpha \) over \( L_\alpha[x] \).

Since \( A \cap (\lambda, \lambda + \omega) \subseteq L_\alpha[x] \) and \( A \cap (\lambda, \lambda + \omega) \) codes a well-ordering of \( (\lambda^+)^L \), we have \( (\lambda^+)^L \subseteq L_\alpha[x] \), as \( \alpha \) is \( x \)-admissible. But \( (\lambda^+)^L > \alpha \). Contradiction! So \( L[x] \models Z_2 + \text{HP} \).

3.2. The strength of \( Z_3 + \text{Harrington’s Principle} \).

Theorem 3.2. The following two theories are equiconsistent:

(1) \( Z_3 + \text{HP} \).
(2) \( \text{ZFC} + \text{there exists a remarkable cardinal} \).

Proof. We first prove that \( Z_3 + \text{HP} \) implies \( L \models \text{ZFC} + \text{there exists a remarkable cardinal} \). Assume \( Z_3 + \text{HP} \). It is easy to verify that \( L \models \text{ZFC} \). We now want to show that \( \omega_1^L \) is remarkable in \( L \). Suppose \( L \models \theta > \omega_1^L \) is regular, and set
\[ \eta = \theta^+L. \] Let \( x \in 2^\omega \) witness \( HP \), and let \( G \) be \( Col(\omega, < \omega_1^L) \)-generic over \( V \).

Let \( f : [L_\eta[G]]^\omega \to L_\eta[G] \), \( f \in L[G] \), and let \( X \prec L_\eta[x][\{\omega_1, \theta, f\}] \subseteq X \). Let \( \tau : L_\eta[x][\{\theta, f\}] \cong X \) be the collapsing map, where \( \alpha = \text{crit}(\tau), \tau(\alpha) = \omega_1^\omega \), and \( \tau(f) = f \). As \( \tau \) is \( x \)-admissible, \( \tau \) is an \( L \)-cardinal by the choice of \( x \) as witnessing \( HP \), and hence \( \beta = o.t.(X \cap \theta) = \tau^{-1}(\theta) \) is a regular \( L \)-cardinal.

Therefore, \( X \cap L_\eta[G] \) condenses remarkably. By absoluteness, there is in \( L[G] \) some elementary \( \tilde{\tau} : L_\eta[G \cap L_\alpha] \to L_\eta[G] \) such that \( \tilde{\tau}(\beta) = \theta \) and \( \tilde{\tau}(\tilde{f}) = f \). That is, in \( L[G] \), there is some \( X \in [H^1_\eta L[G]]^\omega \cap L[G] \) which condenses remarkably and is closed under \( f \). Hence \( \omega_1^V \) is remarkable in \( L \) by Lemma 2.3.

We now prove that the consistency of (2) implies the consistency of (1).

We assume that \( L \models "ZFC + \kappa \text{ is a remarkable cardinal}" \)

there is no \( \alpha \) such that \( L_\alpha \models "ZFC + \kappa \text{ is a remarkable cardinal}" \) \hspace{1em} \hspace{1em} \hspace{1em} \hspace{1em} (3.3)

In what follows, we shall write \( S_\mu \) for

\[ \{ X \in [L_\mu]^\omega | X \prec L_\mu \text{ and o.t.}(X \cap \mu) \text{ is an } L \text{-cardinal} \} \]

as defined in the respective models of set theory which are to be considered.

Let \( G \) be \( Col(\omega, < \kappa) \)-generic over \( L \). Since \( \kappa \) is remarkable in \( L \), \( L[G] \models "S_\mu \text{ is stationary for any } L \text{-cardinal } \mu \geq \kappa." \) Let \( H \) be \( Col(\kappa, < \text{Ord}) \)-generic over \( L[G] \).

Note that \( Col(\kappa, < \text{Ord}) \) is countably closed. Standard arguments give that

\[ L[G][H] = Z_3 + S_\mu \text{ is stationary for all } L \text{-cardinals } \mu \in \text{Card}^L \setminus (\kappa + 1). \] \hspace{1em} \hspace{1em} \hspace{1em} \hspace{1em} (3.4)

In \( L[G][H] \), we may pick some \( B \subseteq \text{Ord} \) such that \( V = L[B] \) and if \( \lambda \geq \omega_1 \) is an \( L \)-cardinal, then \( B \cap [\lambda, \lambda + \omega_1) \) codes a well-ordering of \( (\lambda^+)^L \). By (3.3) we will then have that for all \( \alpha \geq \omega_1 \),

\[ L_{\alpha+1}[B \cap \alpha] \models \text{Card}(\alpha) \leq \aleph_1. \] \hspace{1em} \hspace{1em} \hspace{1em} \hspace{1em} (3.5)

By (3.5), there exists then a canonical sequence \( \{ C_\alpha | \alpha \in \text{Ord} \} \) of pairwise almost disjoint subsets of \( \omega_1 \) such that \( C_\alpha \) is the \( L_{\alpha+1}[B \cap \alpha] \)-least subset of \( \omega_1 \) such that \( C_\alpha \) is almost disjoint from every member of \( \{ C_\beta | \beta < \alpha \} \). Do almost disjoint forcing to code \( B \) by some \( A \subseteq \omega_1 \) such that for any \( \alpha \in \text{Ord}, \alpha \in B \iff |A \cap C_\alpha| < \omega_1 \).

This forcing is countably closed and has the \( \text{Ord-c.c.} \). Note that \( L[B][A] = L[A] \) and \( L[A] \models Z_3 \). Also,

\[ L[A] \models "S_\mu \text{ is stationary for any } L \text{-cardinal } \mu \geq \kappa." \] \hspace{1em} \hspace{1em} \hspace{1em} \hspace{1em} (3.6)

Suppose \( \alpha > \omega_1 \) is \( A \)-admissible, but \( \alpha \) is not an \( L \)-cardinal. Let \( \lambda \) be the largest \( L \)-cardinal \( < \alpha \). Note that \( \lambda + \omega_1 < \alpha \) and we can compute \( B \cap \alpha \) over \( L_\alpha[A] \).

Hence \( B \cap [\lambda, \lambda + \omega_1) \subseteq L_\alpha[A] \), and \( B \cap [\lambda, \lambda + \omega_1) \) codes a well-ordering of \( \lambda^+L \).

So \( \lambda^+L < \alpha \), as \( \alpha \) is \( A \)-admissible. Contradiction! We have shown that in \( L[A] \),

\[ \text{every } A \text{-admissible ordinal above } \omega_1 \text{ is an } L \text{-cardinal}. \] \hspace{1em} \hspace{1em} \hspace{1em} \hspace{1em} (3.7)

Now over \( L[A] \) we do reshaping as follows (cf. e.g. [2, Section 1.3] on the original reshaping forcing).

**Definition 3.3.** Define \( p \in P \) if and only if \( p : \alpha \to 2 \) for some \( \alpha < \omega_1 \) and \( \forall \xi \leq \alpha \exists \gamma (L_\gamma[A \cap \xi], p \restriction [\xi]) \models "\xi \text{ is countable}" \) and every \( (A \cap \xi) \)-admissible \( \gamma \in [\xi, \gamma] \) is an \( L \)-cardinal).
It is easy to check the extendability property of $\mathbb{P}$: $\forall p \in \mathbb{P} \forall \alpha < \omega_1 \exists q \leq p (\text{dom}(q) \geq \alpha)$. Note that $|\mathbb{P}| = \aleph_1$, as CH holds true in $L[A]$.

We now vary an argument from [18], cf. also [14], to show the following.

CLAIM 3.4. $\mathbb{P}$ is $\omega$-distributive.

PROOF. Let $p \in \mathbb{P}$ and $\vec{D} = (D_n| n \in \omega)$ be a sequence of open dense sets. Take $v > \omega_1$ such that $\vec{D} \in L_v[A]$ and $L_v[A]$ is a model of a reasonable fragment of $ZFC^-$. By (3.7) we have that

$$L_\mu[A] \models \text{“every } A\text{-admissible ordinal } \geq \omega_1 \text{ is an } L\text{-cardinal.”} \tag{3.8}$$

where $\mu = (v^+)^L$. By (3.6) we can pick $X$ such that $\pi : L_\mu[A \cap \delta] \cong X \prec L_\mu[A]$. $|X| = \omega$, $\{p, \mathbb{P}, A, \vec{D}, \omega_1, v\} \subseteq X$, $\vec{\mu}$ is an $L$-cardinal, and $\pi(\vec{\delta}) = \omega_1, \delta = \text{crit}(\pi)$. Note that (3.8) yields that $L_\mu[A \cap \delta] \models \text{“every } A \cap \delta\text{-admissible ordinal } \geq \delta \text{ is an } L\text{-cardinal.”}$ Since $\vec{\mu}$ is an $L$-cardinal, we have that

$$\text{every } A \cap \delta\text{-admissible } \lambda \in [\delta, \vec{\mu}] \text{ is an } L\text{-cardinal.} \tag{3.9}$$

This is the key point. Let $\pi(\vec{\nu}) = v, \pi(\vec{\mu}) = \mathbb{P}$ and $\pi(\vec{D}) = \vec{D}$ with $\vec{D} = (\vec{D}_n| n \in \omega)$. By (3.5) we may let $(E_i| i < \delta) \in L_\mu[A \cap \delta]$ be an enumeration of all clubs in $\delta$ which exist in $L_\nu[A \cap \delta]$. Let $E$ be the diagonal intersection of $(E_i| i < \delta)$. Note that $E \setminus E_i$ is bounded in $\delta$ for all $i < \delta$. In $L[A]$, let us pick a strictly increasing sequence $(\omega_n| n < \omega)$ such that $\omega_n| n < \omega \subseteq E$ and $(\omega_n| n < \omega)$ is cofinal in $\delta$.

We want to find a $q \in \mathbb{P}$ such that $q \leq p, \text{dom}(q) = \delta, L_\mu[A \cap \delta] \models \text{“}\delta \text{ is countable.”}$ and $q \subseteq \vec{D}_n$ for all $n \in \omega$. For this we construct a sequence $(p_n| n \in \omega)$ of conditions such that $p_0 = p, p_{n+1} \leq p_n$, and $p_{n+1} \in \vec{D}_n = D_n \cap L_\nu[A \cap \delta]$ for all $n \in \omega$. Also we construct a sequence $\{\delta_n| n \in \omega\}$ of ordinals. Suppose $p_n \in L_\nu[A \cap \delta]$. By extendability, for all $\zeta$ with $\gamma \leq \zeta < \delta$ we may pick some $p^\zeta \leq p_n$ such that $p^\zeta \in \vec{D}_n,$ $\text{dom}(p^\zeta) \geq \zeta$, and for all limit ordinals $\lambda$ with $\gamma \leq \lambda \leq \zeta$ we have $p^\zeta(\lambda) = 1$ if and only if $\lambda = \zeta$. There exists $C \in L_\nu[A \cap \delta]$ which is a club in $\delta$ such that for all $\eta \in C, \zeta < \eta$ implies $\text{dom}(p^\zeta) < \eta$.

Now we work in $L_\mu[A \cap \delta]$. We may pick some $\eta \in E, \eta \geq \epsilon_n$, such that $E \setminus C \subseteq \eta$. Let $p_{n+1} = p^\eta$ and $\delta_n = \eta$. Note that $p_{n+1} \leq p_n$ and $p_{n+1} \in \vec{D}_n \cap L_\nu[A \cap \delta]$ is given. Let $\gamma = \text{dom}(p_n)$. Note that $\gamma < \delta$ since $p_n \in L_\nu[A \cap \delta]$. Now we work in $L_\nu[A \cap \delta]$. By extendability, for all $\zeta$ with $\gamma \leq \zeta < \delta$ we may pick some $p^\zeta \leq p_n$ such that $p^\zeta \in \vec{D}_n, \text{dom}(p^\zeta) > \zeta$, and for all limit ordinals $\lambda$ with $\gamma \leq \lambda \leq \zeta$ we have $p^\zeta(\lambda) = 1$ if and only if $\lambda = \zeta$.

Now let $q = \bigcup_{n \in \omega} p_n$. We need to check that $q \in \mathbb{P}$. Note that $\text{dom}(q) = \delta$. By (3.9) it suffices to check that $L_\mu[A \cap \delta] \models \text{“}\delta \text{ is countable.”}$ From the construction of the $p_n$’s we have $\{\lambda \in E \cap (\text{dom}(p) \setminus \text{dom}(p_n))| \lambda \text{ is a limit ordinal and } q(\lambda) = 1\} = \{\delta_n| n \in \omega\}$, which is cofinal in $\delta$, as $\delta_n \geq \epsilon_n$ for all $n < \omega$. Recall that $E \in L_\mu[A \cap \delta]$. So $\{\delta_n| n \in \omega\} \subseteq L_\mu[A \cap \delta]$. This witnesses that $\delta$ is countable in $L_\mu[A \cap \delta]$. \hfill \dashv

The proof of Claim 3.4 can be adapted to show that $\mathbb{P}$ is stationary preserving, cf. [14].

Forcing with $\mathbb{P}$ adds some $F : \omega_1 \to 2$ such that for all $\alpha < \omega_1$ there exists $\gamma$ such that $L_\nu[A \cap \alpha, F \upharpoonright \alpha] \models \text{“}\alpha \text{ is countable and every } (A \cap \alpha)\text{-admissible } \lambda \in [\alpha, \gamma] \text{ is an } L\text{-cardinal.”}$ for each $\alpha < \omega_1$ let $\alpha^* \text{ be the least such } \gamma$. Let $D = A \oplus F$. We may assume that for any $L$-cardinal $\lambda < \omega_1^v$, $D$ restricted to odd ordinals in $[\lambda, \lambda + \omega)$ codes a well-ordering of the least $L$-cardinal $> \lambda$. By Claim 3.4, $L[A][F] = L[D] \models Z_3$.
Now we do almost disjoint forcing over $L[D]$ to code $D$ by a real $x$. There exists a canonical sequence $(x_{\alpha}|\alpha < \omega_1)$ of pairwise almost disjoint subset of $\omega$ such that $x_\alpha$ is the $L_\alpha[D \cap \alpha]$-least subset of $\omega$ such that $x_\alpha$ is almost disjoint from every member of $\{x_\beta|\beta < \alpha\}$. Almost disjoint forcing adds a real $x$ such that for all $\alpha < \omega_1$, $\alpha \in D$ if and only if $|x_\alpha \cap x| < \omega$. The forcing has the c.c.c., and thus $L[D][x] = L[x] = Z_3$.

We finally claim that $L[x] \models HP$. Suppose $\alpha$ is $x$-admissible. We show that $\alpha$ is an $L$-cardinal. If $\alpha \geq \omega_1$, then $\alpha$ is also $A$-admissible and hence is an $L$-cardinal by (3.7). Now we assume that $\alpha < \omega_1$ and $\alpha$ is not an $L$-cardinal. Let $\lambda$ be the largest $L$-cardinal $< \alpha$. Recall that for $\xi < \omega_1$, $\xi^* > \xi$ is least such that $L_{\xi^*}[A \cap \xi, F \upharpoonright \xi] \models \xi$ is countable. Every $(D \cap \xi)$-admissible $\lambda' \in [\xi, \xi^*]$ is an $L$-cardinal.

CASE 1: For all $\xi < \lambda + \omega$, $\xi^* < \alpha$. Then $D \cap (\lambda + \omega)$ can be computed inside $L_\alpha[x]$. But then, as $\alpha$ is $x$-admissible, the ordinal coded by $D$ restricted to the odd ordinals in $[\lambda, \lambda + \omega)$, namely the least $L$-cardinal $> \lambda$, is in $L_\alpha[x]$, so that $\lambda^+ < \omega$. Contradiction!

CASE 2: Not Case 1. Let $\xi < \lambda + \omega$ be least such that $\xi^* \geq \alpha$. Then $D \cap \xi$ can be computed inside $L_\alpha[x]$. As $\alpha$ is $x$-admissible, $\alpha$ is thus $(D \cap \xi)$-admissible also. But all $(D \cap \xi)$-admissibles $\lambda' \in [\xi, \xi^*]$ are $L$-cardinals, so that $\alpha$ is an $L$-cardinal by $\xi < \alpha \leq \xi^*$. Contradiction!

We have shown that $L[x] \models Z_3 + HP$.

**Corollary 3.5.** $Z_3 + HP$ does not imply $0^\#$ exists.

### 3.3. Z₄ + Harrington’s Principle implies $0^\#$ exists.

We construe the following as part of the folklore, cf. [5].

**Theorem 3.6 (Z₄).** HP implies $0^\#$ exists.

**Proof.** Let $x \in 2^\omega$ witness HP. Now we work in $L[x]$. Take $\beta > \omega_2$ big enough such that $\beta$ is $x$-admissible and $\omega L_\beta[x] \subseteq L_\beta[x]$. Take $X < L_\beta[x]$ such that $\omega_2 \in X$, $|X| = \omega_1$, and $X^{\omega_2} \subseteq X$. Let $j : L_\omega[x] \cong X \prec L_\beta[x]$ be the collapsing map. Note that $\omega_1 \leq \theta < \omega_2$, $\theta$ is $x$-admissible, and $L_\theta[x]$ is closed under $\omega$-sequences. Let $\kappa = crit(j)$. Define $U = \{A \subseteq \kappa | A \in L \wedge \kappa \in j(A)\}$. Since $\theta$ is an $L$-cardinal by the choice of $x$ as witnessing HP, $(\kappa^+)^L \leq \theta < \omega_2$. Therefore, $U$ is an $L$-ultrafilter on $\kappa$.

Let $\alpha = (\kappa^+)^L$. Consider the structure $(L_\alpha, \in, U)$ which is a premouse. Since $L_\theta[x]$ is closed under $\omega$-sequences from $L_\theta[x]$, $U$ is countably complete.² So $(L_\alpha, \in, U)$ is iterable. Hence $0^\#$ exists.

So in $Z_4$, HP is equivalent to $0^\#$ exists. In fact in $Z_2$, $0^\#$ exists implies HP. By Corollary 3.5 and Theorem 3.6, we have $Z_4$ is the minimal system in higher order arithmetic to show that HP and $0^\#$ exists are equivalent with each other.

### §4. Strengthenings of Harrington’s Principle over higher order arithmetic.

Recall the hypothesis on $\varphi(\_\_\_)$ as stated in Definition 1.5: $\varphi(\_\_\_)$ is a $\Sigma_2$-formula in the language of set theory such that, provably in ZFC: for all $\alpha$, if $\varphi(\alpha)$, then $\alpha$ is an inaccessible cardinal and $L \models \varphi(\alpha)$. Let us give some examples of such $\varphi(\_\_\_)$: $\kappa$ is inaccessible, Mahlo, weakly compact, $\Pi^m_{\omega}$-indescribable, totally indescribable.

²I.e., if $\{X_n|n \in \omega\} \subseteq U$, then $\bigcap_{n \in \omega} X_n \neq \emptyset$. 
$n$-subtle, $n$-ineffable, totally ineffable cardinal, $\alpha$-iterable ($\alpha < \omega_1^\omega$), and $\alpha$-Erdős cardinal ($\alpha < \omega_3^\omega$). However, $\kappa$ being reflecting, unfoldable, or remarkable cannot be expressed in a $\Sigma_2$ fashion.

**Definition 4.1.** Let $\varphi(-)$ be as in Definition 1.5. Let $\delta$ be an inaccessible cardinal or $\delta = \text{Ord}$. We say that $\delta$ is $\varphi$–Mahlo iff $\{\alpha < \delta|\varphi(\alpha)\}$ is stationary in $\delta$. We say that $\delta$ is $2$–$\varphi$–Mahlo iff $\{\alpha < \delta|\varphi(\alpha) \land \{\beta < \alpha|\varphi(\beta)\}\}$ is stationary in $\alpha$ is stationary in $\delta$.

Notice that we do not require a $\varphi$–Mahlo or a $2$–$\varphi$–Mahlo to satisfy $\varphi(-)$.

**4.1. The strength of $Z_2 + \text{HP}(\varphi)$.**

**Theorem 4.2.** Let $\varphi(-)$ be as in Definition 1.5. The following theories are equiconsistent.

1. $Z_2 + \text{HP}(\varphi)$, and
2. $ZFC + \text{Ord} \text{ is } \varphi$–Mahlo.

**Proof.** Let us first suppose (1), and let $x \in 2^\omega$ be as in $\text{HP}(\varphi)$. There is a club class of $x$-admissibles, so that $\{\alpha | L_\alpha \models \varphi(\alpha)\}$ contains a club. Hence $L \models \text{"ZFC + $\{\alpha \in \text{Ord} | \varphi(\alpha)\}$ is stationary."}$ This shows (2) in $L$.

Let us now suppose (2). We force over $L$. Let $S = \{\alpha \in \text{Ord} | \varphi(\alpha)\}$. Let $G$ be $Col(\omega, \text{Ord})$-generic over $L$. Then $L[G] \models Z_2$, and in $L[G]$, $S$ is still stationary, because $Col(\omega, \text{Ord})$ has the $\text{Ord}$–$\text{c.c.}$ We can thus shoot a club through $S$ via $\mathbb{P} = \{p | p$ is a closed set of ordinals and $p \subseteq S\}$. Let $H$ be $\mathbb{P}$-generic over $L[G]$. Standard arguments give that $\mathbb{P}$ is $\omega$-distributive, which implies that $L[G][H] \models Z_2$. Let $C \subseteq S$ be the club added by $H$. We may pick $A \subseteq \text{Ord}$ such that $L[G][H] = L[A]$.

We need to reshape $A$ as follows. Let $p \in \mathbb{R}$ iff $p : \alpha \rightarrow 2$ for some ordinal $\alpha$ such that for all $\xi \leq \alpha$,

$$L_{\xi+1}[A \cap \xi, p \upharpoonright \xi] \models \xi \text{ is countable.}$$

We claim that $\mathbb{R}$ is $\omega$–distributive. To see this, let $(D_n | n < \omega)$ be a, say, $\Sigma_m$–definable sequence of open dense classes, and let $p \in \mathbb{R}$. Let $E$ be the class of all $\beta$ such that $L_\beta[G][H] \models \varphi(s_{\alpha+5})$ and $p$ as well as the parameters defining $(D_n | n < \omega)$ are all in $L_\beta[G][H]$. $E$ is club, and we may let $\alpha$ be the $\alpha$th element of $E$. Then $E \cap \alpha$ is $\Sigma_{m+6}$–definable over $L_\alpha[G][H]$ and cofinal in $\alpha$, so that $\alpha$ has cofinality $\omega$ in $L_{\alpha+1}[G][H]$. A much simplified variant of the argument from Claim 3.4, which we will leave as an exercise to the reader, then produces some $q \in \mathbb{R}$ with $q \leq p$, $q : \alpha \rightarrow 2$, and $q \in \bigcap_{n<\omega} D_n$.

Let $K$ be $\mathbb{R}$-generic over $L[G][H]$. In $L[G][H][K]$, we may then pick some $B \subseteq \text{Ord}$ such that $L[G][H][K] = L[B]$, if $\lambda \in C \setminus (\omega + 1)$, then $B \cap [\lambda, \lambda + \omega)$, restricted to the odd ordinals, codes a well-ordering of $\min(C \setminus (\lambda + 1))$, and for all $\alpha \geq \omega$,

$$L_{\alpha+1}[B \cap \alpha] \models \alpha \text{ is countable.} \tag{4.1}$$

We may now continue as in the proof of Theorem 3.1.

---

3 In the proof of Theorem 3.1 there was no need for reshaping due to (3.2).
We do standard almost disjoint forcing to add a real \( x \) such that if \( (c_\alpha | \alpha \in \text{Ord}) \) is the canonical sequence of pairwise almost disjoint subsets of \( \omega \) given by (4.1), then for any \( \alpha \in \text{Ord}, \alpha \in B \iff |x \cap c_\alpha| < \omega \). In particular, \( L[B][x] = L[x] \). This forcing is c.c.c., so that also \( L[x] \models Z_2 \).

We claim that in \( L[x] \), HP(\( \varphi \)) holds true. It suffices to show that if \( \alpha \) is \( x \)-admissible, then \( \alpha \in C \). Suppose \( \alpha \) is \( x \)-admissible but \( \alpha \notin C \). Let \( \lambda \) be the largest element of \( C \) such that \( \lambda < \alpha \). Note that we can define \( B \cap \alpha \) over \( L_\alpha[x] \).

Since \( B \cap [\lambda, \lambda + \omega) \subseteq L_\alpha[x] \) and \( B \cap [\lambda, \lambda + \omega) \), restricted to the odd ordinals, codes a well-ordering of \( \text{min}(C \setminus (\lambda + 1)) \), we have \( \text{min}(C \setminus (\lambda + 1)) \subseteq L_\alpha[x] \), because \( \alpha \) is \( x \)-admissible. But \( \text{min}(C \setminus (\lambda + 1)) > \alpha \). Contradiction! So \( L[x] \models Z_2 + \text{HP}(\varphi) \).

4.2. The strength of \( Z_3 + \text{HP}(\varphi) \).

Definition 4.3 ([9]).

1. Let \( N \) be transitive. \( N \) is full if and only if \( \omega \in N \) and there is \( \gamma \) such that \( L_\gamma(N) \models \text{ZFC}^\text{c.c.} \) and \( N \) is regular in \( L_\gamma(N) \), that is, if \( f : x \rightarrow N, x \in N \), and \( f \in L_\gamma(N) \), then \( \text{ran}(f) \in N \).

2. Let \( B \) be a complete Boolean algebra. Let \( \delta(B) \) be the smallest cardinality of a set which lies dense in \( B \setminus \{0\} \).

3. Let \( N = L^A_\gamma \{A \} \in A \cap L_\gamma[A] \) be a model of \( \text{ZFC}^\text{c.c.} \). Let \( X \cup \{\delta\} \subseteq N \).

Define \( C^N_\delta(X) = \text{the smallest } Y < N \text{ such that } X \cup \{\delta\} \subseteq Y \).

Definition 4.4 ([9, p.31]). Let \( B \) be a complete Boolean algebra. \( B \) is a sub-complete forcing if and only if for sufficiently large cardinals \( \theta \) we have: \( B \in H_\theta \) and for any \( \text{ZFC}^\text{c.c.} \) model \( N = L^A_\gamma \) such that \( \theta < \tau \) and \( H_\theta \subseteq N \) we have:

Let \( \sigma : \bar{N} \rightarrow N \) where \( \bar{N} \) is countable and full. Let \( \sigma(\theta, \bar{s}, \bar{B}) = \theta, s, \bar{B} \) where \( \bar{s} \in \bar{N} \). Let \( \bar{G} \) be \( \bar{B} \)-generic over \( \bar{N} \). Then there is \( b \in \bar{B} \setminus \{0\} \) such that whenever \( G \) is \( \bar{B} \)-generic over \( \bar{V} \) with \( b \in G \), there is \( \sigma' \in \bar{V}[G] \) such that

(a) \( \sigma' : \bar{N} \rightarrow N \),
(b) \( \sigma'(\theta, \bar{s}, \bar{B}) = \theta, s, \bar{B} \),
(c) \( C^N_\delta(\text{ran}(\sigma')) = C^N_\delta(\text{ran}(\sigma)) \) where \( \delta = \delta(\bar{B}) \),
(d) \( \sigma''(\bar{G}) \subseteq G \).

By [9], cf. also [8], sub-complete forcings add no reals and are closed under Revised Countable Support (RCS) iterations subject to the usual constraints (see [9, Theorem 3, p. 56]). In the following, we give some examples of forcing notions which are sub-complete that will be used in this paper.

The set \( \omega^<_c \) of monotone finite sequences in \( \omega_2 \) is a tree ordered by inclusion. Namba forcing is the collection of all subtrees \( T \neq \emptyset \) of \( \omega^<_c \) with a unique stem, \( \text{stem}(T) \), such that every element of \( T \) is compatible with \( \text{stem}(T) \), and every element extending \( \text{stem}(T) \) has \( \omega_2 \) immediate successors in \( T \). The order is defined by: \( T \leq \bar{T} \) if and only if \( T \subseteq \bar{T} \). If \( G \) is generic for Namba forcing, then \( S = \bigcup T \cap G \) is a cofinal map of \( \omega \) into \( \omega^<_c \). We call any such \( S \) a Namba sequence. Namba forcing is stationary set preserving and adds no reals if CH holds.

Fact 4.5 ([9], Lemma 6.2). Assume CH. Then Namba forcing is sub-complete.

Definition 4.6. Suppose \( \kappa \) is a cardinal or \( \kappa = \text{Ord} \). Define \( \text{Club}(\kappa, S) = \{p | p : \alpha + 1 \rightarrow S \text{ for some } \alpha < \kappa \text{ and } p \text{ is increasing and continuous}\} \). The extension relation is defined by: \( p \leq q \) if and only if \( p \supseteq q \).
The forcing $\text{Club}(\omega_1, S)$ has been used in the proof of Theorem 3.1. If $G$ is $\text{Club}(\omega_1, S)$-generic, then $\bigcup G : \omega_1 \to S$ is increasing, continuous, and cofinal in $S$.

**FACT 4.7** ([9, Lemma 6.3]). Let $\kappa > \omega_1$ be a regular cardinal. Let $S \subseteq \kappa$ be a stationary set. Then $\text{Club}(\omega_1, S)$ is subcomplete.

**LEMMA 4.8** ([3, Lemma 18.6]). Suppose CH holds and $S \subseteq \omega_2$ is such that $\{\alpha \in S \cap cf(\omega_1) \mid \text{there exists } C \subseteq S \cap \alpha \text{ such that } C \text{ is a club in } \alpha \}$ is stationary. Then $\text{Club}(\omega_2, S)$ is $\omega_1$-distributive.

**THEOREM 4.9.** The following two theories are equiconsistent:

1. $\text{ZFC} + \exists \kappa \text{ such that } \kappa \text{ is a remarkable cardinal with } \varphi(\kappa) + \text{ Ord is } 2-\varphi$-Mahlo.
2. $\text{ZFC} + \exists \kappa \text{ is a regular cardinal }$ $\exists \kappa \text{ is a remarkable cardinal with } \varphi(\kappa) + \text{ Ord is } 2-\varphi$-Mahlo.

**Proof.** We first prove that (2) implies that (1) holds in $L$. As $\text{HP}(\varphi)$ implies $\text{HP}$, Theorem 3.2 gives that $\text{ZFC} + \text{ HP}(\varphi)$ implies $L \models \text{ZFC} + \omega_1^\kappa$ is remarkable. Let $x \in 2^\omega$ witness $\text{HP}(\varphi)$. As $\omega_1^\kappa$ is $x$-admissible, $\varphi(\omega_1^\kappa)$ holds true in $L$.

There is a club of $\kappa$-admissibles, so that we may pick some club $C \subseteq \{\alpha \in \text{Ord} \mid L \models \varphi(\alpha)\}$. Suppose $D$ is a club in $L$. Pick $\alpha$ in $C \cap D$ of cofinality $\omega_1$ such that $\alpha$ is a limit point of $C \cap D$. Since $\alpha \in C, L \models \varphi(\alpha)$. We want to see that $\{\beta < \alpha \mid L \models \varphi(\beta)\}$ is stationary in $L$. Let $E \subseteq \alpha$ in $L$ be a club in $\alpha$. Note that $E \cap C \cap \alpha \neq \emptyset$. If $\beta \in E \cap C \cap \alpha$, then $L \models \varphi(\beta)$. Hence $\text{Ord}$ is $2-\varphi$-Mahlo in $L$.

Now we show that consistency of (1) implies consistency of (2). We force over $L$. Suppose that (1) holds in $L$.

Let $H$ be $\text{Col}(\omega_1, < \kappa)$-generic over $L$.

**CLAIM 4.10.** $\{\alpha < \kappa \mid L \models \varphi(\alpha)\}$ is stationary in $L[H]$.

**Proof.** We work in $L[H]$. Let $C \subseteq \kappa = \omega_1^{L[H]}$ be club, and let $L_\theta \models \varphi(\kappa)$, where $\theta > \kappa$ is regular. As $\kappa$ is remarkable, there is some $\sigma : L_{\tilde{\theta}}[H \cap L_\alpha] \to L_{\tilde{\theta}}[H]$ such that $\alpha = \text{crit}(\sigma), \sigma(\alpha) = \kappa, C \in \text{ran}(\sigma)$, and $\tilde{\theta}$ is a regular cardinal in $L$. By elementarity, $L_{\tilde{\theta}} \models \varphi(\alpha)$, which implies that $L \models \varphi(\alpha)$, as $\varphi$ is $\Sigma_2$. But $\alpha \in C$.

Let $H$ be $\text{Col}(\omega_1, < \kappa)$-generic over $L$. Over $L[H]$, we define a class RCS-iteration $((P_\alpha, Q_\alpha) \mid \alpha \in \text{Ord})$ as follows. We let $P_0 = \emptyset, P_{\alpha+1} = P_\alpha * \hat{Q}_\alpha$ for $\alpha \in \text{Ord}$ and for limit ordinal $\alpha$ we let $P_\alpha$ be the revised limit ($\text{Rlim}$) of $((P_\beta, \hat{Q}_\beta) \mid \beta \in L \models \varphi(\alpha)$ as $\varphi$ is $\Sigma_2$.

Let $\alpha \in C$.

Let $H$ be $\text{Col}(\omega_1, < \kappa)$-generic over $L$. Over $L[H]$, we define a class RCS-iteration $((P_\alpha, Q_\alpha) \mid \alpha \in \text{Ord})$ as follows. We let $P_0 = \emptyset, P_{\alpha+1} = P_\alpha * \hat{Q}_\alpha$ for $\alpha \in \text{Ord}$ and for limit ordinal $\alpha$ we let $P_\alpha$ be the revised limit ($\text{Rlim}$) of $((P_\beta, \hat{Q}_\beta) \mid \beta \in L \models \varphi(\alpha)$ as $\varphi$ is $\Sigma_2$.

The definition of $Q_\alpha$ splits into three cases as follows.

Let

1. $S_0 = \{\alpha \mid L \models \neg \varphi(\alpha)\}$.
2. $S_1 = \{\alpha \mid L \models \varphi(\alpha), \beta < \alpha \models \varphi(\beta)\}$ is not stationary in $L\}$, and
3. $S_2 = \{\alpha \mid L \models \varphi(\alpha), \beta < \alpha \models \varphi(\beta)\}$ is stationary in $L\}$.

**CASE 0.** If $\alpha \in S_0$, then let $Q_\alpha = \text{Col}(\omega_1, 2^{\omega_1})$ which collapses $2^{\omega_1}$ to $\omega_1$ by countable conditions.

**CASE 1.** If $\alpha \in S_1$, then let $Q_\alpha = \text{Namba forcing}$.

**CASE 2.** If $\alpha \in S_2$, then let $Q_\alpha = \text{Club}(\omega_1, S_1 \cap \alpha)$.

Note that if $L \models \varphi(\alpha)$, then $L^{\text{Col}(\omega_1, < \kappa) * P_\alpha} \models \varphi = \omega_2$ since $\text{Col}(\omega_1, < \kappa) * P_\alpha$ has the $\alpha$-c.c. This also implies that $S_1 \cap \alpha$ is stationary in $L^{\text{Col}(\omega_1, < \kappa) * P_\alpha}$. Moreover, in $L^{\text{Col}(\omega_1, < \kappa) * P_\alpha}$, $S_1 \cap \alpha$ consists of points of cofinality of $\omega$. So it makes sense to shoot a club subset of $\alpha$ with order type $\omega_1$ through $S_1 \cap \alpha$. 

Finally let $\mathbb{P}$ be the revised limit of $((P_\alpha, \dot{Q}_\alpha) | \alpha \in \text{Ord})$. By Facts 4.5 and 4.7 and by [9, Theorem 3, p. 56], $P_\alpha$ is subcomplete for all $\alpha \in \text{Ord}$. Standard arguments give us that $\mathbb{P}$ has the $\text{Ord}$-c.c. Hence $\mathbb{P}$ does not add reals and $\omega_1$ is preserved. Let $G$ be $\mathbb{P}$-generic over $L[H]$. $L[H, G] \models Z_3$. The following is stated for the record.

**Claim 4.11.** In $L[H][G]$, if $\alpha \in S_1$, then $cf(\alpha) = \omega$, and if $\alpha \in S_2$, then $cf(\alpha) = \omega_1$ and there is a club in $\alpha$ of order type $\omega_1$ contained in $S_1 \cap \alpha$.

For each $\text{L}$-cardinal $\mu > \omega_1$, we again let $S_\mu = \{X \prec L_\mu | X$ is countable and $\text{o.t.}(X \cap \mu)$ is an $\text{L}$-cardinal$\}$, as being defined in the respective models of set theory which are to be considered.

The following proof shows that subcomplete forcings preserve the stationarity of $S_\mu$.

**Claim 4.12.** In $L[H, G]$, for each $\text{L}$-cardinal $\mu > \omega_1$, $S_\mu$ as defined in $L[H, G]$ is stationary.

**Proof.** Fix an $\text{L}$-cardinal $\mu > \omega_1$. Suppose $S_\mu$ is not stationary in $L[G, H]$. Then there are $p \in P_\alpha$ and $\tau \in L[H]^P_\alpha$ for some $\alpha$ such that $p \restriction_{L[H]} \tau : [\tilde{\mu}]^{\omega_1} \to \tilde{\mu}$ and there is no countable $X \subseteq \tilde{\mu}$ such that $X$ is closed under $\tau$ and $\text{o.t.}(X)$ is an $\text{L}$-cardinal.” Let $\mu^*$ be an $\text{L}$-cardinal which is bigger than $\mu$. Let $\sigma : N \to L_{\mu^*}[H]$ where $N$ is countable, transitive and full, such that $P_\alpha$, $\mu$, $\tau \in N$. Let $\sigma(\tilde{P}, \delta, \tilde{\rho}, \tilde{\mu}, \tilde{\tau}) = P_\alpha, \omega_1, p, \mu, \tau$. Let us write $N = L_\gamma[H \upharpoonright \delta]$.

Because $\kappa$ was remarkable in $L$, cf. Lemma 2.3, may assume that $N$ was picked in such a way that $\gamma$ is an $\text{L}$-cardinal. Let $\tilde{G}$ be $\tilde{P}$-generic over $L_\gamma[H \upharpoonright \delta]$ with $\tilde{\rho} \in \tilde{G}$. Since $P_\alpha$ is subcomplete, by the definition of subcompleteness, there is $p^* \in P_\alpha$, $p^* \leq p$, such that whenever $G^*$ is $P_\alpha$-generic over $L[H]$ with $p^* \in G^*$, then there is $\sigma^* \in L[H][G^*]$ such that $\sigma^* : L_\gamma[H \upharpoonright \delta][\tilde{G}] \to L_\mu[H][G^*]$ and $\sigma^*(\tilde{P}, \delta, \tilde{\rho}, \tilde{\mu}, \tilde{\tau}) = P_\alpha, \omega_1, p, \mu, \tau$.

Since $p \in G^*$, there is no countable $X \subseteq \mu$ such that $X$ is closed under $\tau^{G^*}$ and $\text{o.t.}(X)$ is an $\text{L}$-cardinal. But $\text{ran}(\sigma^*) \cap \mu$ is countable, closed under $\tau^{G^*}$ and $\text{o.t.}(\text{ran}(\sigma^*) \cap \mu) = \gamma$ is an $\text{L}$-cardinal. Contradiction! $\dashv$

We now let $\mathbb{Q} = \text{Club}(\text{Ord}, S_1 \cup S_2)$. The proof of the following Claim imitates the proof of Lemma 4.8.

**Claim 4.13.** $\mathbb{Q}$ is $\omega_1$-distributive.

**Proof.** In $L[H, G]$, $S_2$ is stationary and $\text{CH}$ holds. Suppose $\tilde{D} = (D_i | i < \omega_1)$ is a, say $\Sigma_{\omega + 5}$, definable sequence of open dense classes. Pick $M \subseteq_{\Sigma_{\omega + 5}} V$ such that $M$ contains the parameters needed in the definition of $\tilde{D}$, $M^{\omega_1} \subseteq M$, and $M \cap \text{Ord} \subseteq S_2$.

Let us write $\delta = M \cap \text{Ord}$. By Claim 4.11, we may pick some $C \subseteq S_1 \cap \delta$, a club in $\delta$. Now we can simultaneously build a descending sequence $(p_i | i \leq \omega_1)$ with $p_0 = p$ and a continuous tower $(M_i | i \leq \omega_1)$ of countable elementary substructures of $M$ with $M_{\omega_1} = M$ such that for all $i < \omega_1$ we have:

(a) $p_i \in M_{i+1}$,
(b) $p_{i+1} \in D_i$ and $p_{i+1}(\text{max}(\text{dom}(p_{i+1}))) > \text{sup}(M_i \cap \text{Ord})$,
(c) $\text{sup}(M_i \cap \text{Ord}) \in C$, and
(d) if $i < \omega_1$ is a limit ordinal, then $p_i \upharpoonright \text{max}(\text{dom}(p_i)) = \bigcup_{j<i} p_j$ and hence $p_i(\text{max}(\text{dom}(p_i))) = \text{sup}(M_i \cap \text{Ord}) \in C$.

Then $p_{\omega_1} \leq p$ and $p_{\omega_1} \in \bigcap_{i<\omega_1} D_i$. $\dashv$
Let \( I \) be \( \mathbb{Q} \)-generic over \( L[H, G] \), and let \( C \subseteq S_1 \cup S_2 \) be the club added by \( I \). By Claim 4.13, \( L[H, G, I] \models Z_3 \). As in the proof of Theorem 3.2, we can pick \( B \subseteq \text{Ord} \) such that \( L[H, G, I] = L[B] \) and for any \( \alpha \in C \), \( B \) restricted to the odd ordinals in \( [\alpha, \alpha + \omega_1) \) codes a well-ordering of \( \min(C \setminus (\alpha + 1)) \).

We now reshape as follows.\(^4\)

**Definition 4.14.** Define \( p \in S \) if and only if \( p : \alpha \rightarrow 2 \) for some \( \alpha \) and for any \( \xi \leq \alpha, L_{\xi+1}[B \cap \alpha, p \upharpoonright \xi] \models \| \xi \| \leq \omega_1 \).

**Claim 4.15.** \( S \) is \( \omega_1 \)-distributive.

**Proof.** Let \( \bar{D} = (D_i | i < \omega_1) \) be a sequence of open dense subclass of \( S \). Let \( p \in S \).

We want to find \( p_{\alpha_1} \) such that \( p_{\alpha_1} \in \bigcap_{i < \omega_1} D_i \) and \( p_{\omega_1} \leq p \). Say \( \bar{D} \) is \( \Sigma_n \)-definable in \( L[B] \) with parameters \( \bar{s} \). Let \( (\beta_i | i < \omega_1) \) the first \( \omega_1 + 1 \) many \( \beta \) such that \( L_{\beta_i} \models L[B] \) and \( \omega_1 + 1 \cup \{ \bar{s} \} \subseteq L_{\beta_i}[B] \).

For every \( i \), \( \beta_i \cap i < \omega_1 \) is \( \Sigma_{n+6} \)-definable over \( L_{\beta_i} \). Since \( L[B] \) is countable and \( o.t. \), \( \beta_i \cap i < \omega_1 \) is also countable. Therefore, \( \beta_i \cap i < \omega_1 \) is \( \Sigma_2 \)-definable over \( L[B] \).

Let \( \bar{s} = \bigcup_{i < \omega_1} p_i \). Note that \( \text{dom}(p_{\alpha_1}) = \beta_{\alpha_1} \), \( p_{\alpha_1} \in S \), in fact \( p_{\alpha_1} \in \bigcap_{i < \omega_1} D_i \), and \( p_{\alpha_1} \leq p \).

By forcing with \( S \) over \( L[H, G, I] \), we get \( \bar{B} \subseteq \text{Ord} \) such that for any \( \alpha \in \text{Ord} \), \( L_{\alpha+1}[B \cap \alpha, \bar{B} \cap \alpha] \models |\alpha| \leq \omega_1 \).

Let \( E = B \cup \bar{B} \). Of course, \( L[E] \models Z_3 \), and for any \( \alpha \in \text{Ord} \), \( L_{\alpha+1}[E \cap \alpha] \models |\alpha| \leq \omega_1 \).

We also have that for all \( \alpha \in C \), \( E \) restricted to the odd ordinals in \( [\alpha, \alpha + \omega_1) \) codes a well-ordering of \( \min(C \setminus (\alpha + 1)) \).

By Claims 4.13 and 4.15, \( L[H, G] \) and \( L[E] \) have the same sets. Therefore, trivially, Claim 4.12 is still true with \( L[E] \) replacing \( L[H, G] \).

**Claim 4.16.** Define \( p \in \mathbb{R} \) if and only if \( p : \alpha \rightarrow 2 \) for some \( \alpha \) and for any \( \xi \leq \alpha \), \( \exists \gamma \in [\xi, \gamma] \models \| \xi \| \leq \omega_1 \).

**Claim 4.17.** \( \mathbb{R} \) is \( \omega \)-distributive.

**Proof.** Recall that for each \( L \)-cardinal \( \mu \) \( \omega_1 \), we defined \( S_\mu = \{ X < L_\mu | X \) is countable and \( o.t.(X \cap \mu) \) is an \( L \)-cardinal \}. We shall use the fact that in \( L[A] \), \( S_\mu \) as defined in \( L[A] \) is stationary.

In fact, essentially the same argument as in the proof of Claim 3.4 shows that \( \mathbb{R} \) is \( \omega \)-distributive. In the following we only point out the place we use \( \varphi \) is \( \Sigma_2 \) in our argument.

\(^4\)In the proof of Theorem 3.2 there was no need for reshaping at this point due to (3.3).
Let $p \in \mathbb{R}$ and $\mathcal{D} = (\mathcal{D}_n | n \in \omega)$ be a sequence of open dense sets. Pick large enough $\text{L}$-cardinal $\check{\mu}$ such that $\mathcal{D} \in \text{L}[\mathcal{A}]$ and $\text{L}[\mathcal{A}] \models \text{“if } \alpha \geq \omega_1 \text{ is } \text{A}-\text{admissible, then } L \models \varphi(\alpha).\text{”}$. As $\mathcal{S}_\mu$ is stationary, we can pick $X$ such that $\pi : \check{\mathcal{D}}_\mu[\mathcal{A} \cap \delta] \cong X < \text{L}[\mathcal{A}]\{x\} = \omega, \{p, \mathcal{P}, \mathcal{A}, \check{\mathcal{D}}, \omega_1, v\} \subseteq X$ and $\check{\mu}$ is an $\text{L}$-cardinal where $\pi(\delta) = \omega_1(\delta = X \cap \omega_1)$. Note that by elementarity, $\check{\mathcal{D}}_\mu[\mathcal{A} \cap \delta] \models \text{“if } \alpha \geq \delta \text{ is } \text{A} \cap \delta\text{-admissible, then } L \models \varphi(\alpha).\text{”}$. Suppose $\alpha \in (\delta, \check{\mu})$ is $\text{A} \cap \delta\text{-admissible}. Then $\check{\mathcal{D}}_\mu \models \varphi(\alpha)$. Since $\check{\mu}$ is an $\text{L}$-cardinal and $\varphi$ is $\Sigma_2, L \models \varphi(\alpha)$. The rest of the arguments are the same as in the proof of Claim 3.4.

Using Claim 4.10, a simple variant of the previous proof also shows the following.

**Claim 4.18.** \{\alpha < \kappa : L \models \varphi(\alpha)\} is stationary in $\text{L}[\mathcal{A}]\mathbb{R}$.

Forcing with $\mathbb{R}$ adds $F : \omega_1 \rightarrow 2$ such that for all $\alpha < \omega_1$ there exists $\gamma$ such that $\mathcal{L}_\alpha[\mathcal{A} \cap \alpha, F \upharpoonright \alpha] \models \alpha$ is countable and every $(\mathcal{A} \cap \alpha)$-admissible $\lambda \in [\alpha, \gamma]$ satisfies that $L \models \varphi(\lambda)$. Using Claim 4.10, we may force over $\text{L}[\mathcal{A}, F]$ and shoot a club $C^*$ through $\{\alpha < \kappa : L \models \varphi(\alpha)\}$ in the standard way. Let $\text{D} = \text{A} \oplus \text{F} \oplus C^*$. We may assume that for $\lambda \in C^*$, $D$ restricted to odd ordinals in $[\lambda, \lambda + \omega]$ codes a well-ordering of $\min(C^* \setminus (\lambda + 1))$. Since $\mathbb{R}$ and the club shooting adding $C^*$ are $\omega$-distributive, it is easy to see that $\text{L}[\mathcal{D}] \models \text{Z}_3$.

Now we work in $\text{L}[\mathcal{D}]$. Do almost disjoint forcing to code $D$ by a real $x$. This forcing is $\text{c.c.c.}$ Note that $\text{L}[\mathcal{D}][x] = \text{L}[x], \text{and } \text{L}[x] \models \text{Z}_3$.

Now we work in $\text{L}[x]$. Suppose $\alpha$ is $x$-admissible. We show that $L \models \varphi(\alpha)$. If $\alpha \geq \omega_1$, then $\alpha$ is also $\text{A}$-admissible and hence $L \models \varphi(\alpha)$. Now we assume that $\alpha < \omega_1$ and $L \not\models \varphi(\alpha)$. Then $\alpha \notin C^*$. Let $\lambda < \alpha$ be the largest element of $C^*$ which is smaller than $\alpha$ and $\check{\lambda} = \min(C \setminus (\alpha + 1)) > \alpha$. For every $\xi < \omega_1$, let $\xi^* > \xi$ be least such that $\text{L}_\xi[\mathcal{A} \cap \xi, F \upharpoonright \xi] \models \xi$ is countable. By the properties of $F$, every $(\mathcal{D} \cap (\xi))$-admissible $\lambda' \in [\xi, \xi^*]$ satisfies $L \models \varphi(\lambda')$.

**Case 1:** For all $\xi < \lambda + \omega$, $\xi^* < \alpha$. Then $\mathcal{D} \cap (\lambda + \omega)$ can be computed inside $\text{L}_\alpha[x]$. But then, as $\alpha$ is $x$-admissible, the ordinal coded by $D$ restricted to the odd ordinals in $[\lambda, \lambda + \omega]$, namely $\check{\lambda}$, is in $\text{L}_\alpha[x]$, so that $\check{\lambda} < \alpha$. Contradiction!

**Case 2:** Not Case 1. Let $\xi < \lambda + \omega$ be least such that $\xi^* \geq \alpha$. Then $\mathcal{D} \cap \xi$ can be computed inside $\text{L}_\alpha[x]$. As $\alpha$ is $x$-admissible, $\alpha$ is thus $(\mathcal{D} \cap \xi)$-admissible also. But all $(\mathcal{D} \cap \xi)$-admissibles $\lambda' \in [\xi, \xi^*]$ satisfy $L \models \varphi(\lambda')$, so that $L \models \varphi(\alpha)$ by $\xi < \alpha \leq \xi^*$. Contradiction!

We have shown that $L[x] \models \text{Z}_3 + \text{HP}(\varphi)$.

**Corollary 4.19.** $\text{Z}_3 + \text{HP}(\varphi)$ does not imply $0^\sharp$ exists.

By Theorem 3.6, $\text{Z}_4 + \text{HP}(\varphi)$ implies $0^\sharp$ exists. As a corollary, $\text{Z}_4$ is the minimal system of higher order arithmetic to show that $\text{HP}$, $\text{HP}(\varphi)$, and $0^\sharp$ exist are equivalent with each other.

Hugh Woodin conjectures that “$\text{Det}(\Sigma_4^1)$ implies $0^\sharp$ exists” can be proven in $\text{Z}_2$.

§5. Acknowledgments. The results in this paper strengthen the results from the first author’s Ph.D. thesis written in 2012 at the National University of Singapore under the supervision of Chong Chi Tat and W. Hugh Woodin. The first author would like to express his deep gratitude to Hugh Woodin as well as to the members of his Ph.D. committee for all their support.
REFERENCES

[1] J. Barwise. *Admissible Sets and Structures*, Perspectives in Mathematical Logic, vol. 7, Springer Verlag, Berlin, 1976.

[2] A. Bella, R. B. Jensen, and P. Welch. *Coding the Universe*, Cambridge University Press, Cambridge, 1982.

[3] James Cummings. *Iterated Forcing and Elementary Embeddings*, Handbook of set theory (M. Foreman and A. Kanamori, editors). Springer Verlag, Berlin, 2010.

[4] Keith J. Devlin. *Consectibility*. Springer. Berlin, 1984.

[5] L. A. Harrington. Analytic determinacy and $\mathcal{O}^1$. this Journal, 43(1978), pp. 685–693.

[6] Thomas J. Jech. *Multiple forcing*. Cambridge University Press. Cambridge, 1986.

[7] ———. *Set Theory*, Third millennium edition, revised and expanded. Springer. Berlin, 2003.

[8] Ronald Jensen. Iteration Theorems for Subcomplete and Related Forcings. handwritten notes. available at http://www.mathematik.hu-berlin.de/~raesch/org/jensen.html.

[9] ———. Lecture note on subcomplete forcing and L-forcing. handwritten notes. available at http://www.mathematik.hu-berlin.de/~raesch/org/jensen.html.

[10] Akihiro Kanamori. *The Higher Infinite: Large Cardinals in Set Theory from Their Beginnings*. second ed., Springer Monographs in Mathematics. Springer. Berlin, 2003.

[11] Kenneth Kunen. *Set Theory: An Introduction to Independence Proofs*. North Holland, Amsterdam, 1980.

[12] R. Mansfield and G. Weitkamp. *Recursive Aspects of Descriptive Set Theory*. Oxford University Press. Oxford, 1985.

[13] Ramez L. Sami. Analytic determinacy and $\mathcal{O}^1$: A forcing-free proof of Harrington’s theorem. *Fundamenta Mathematicae*. vol. 160 (1999), pp. 153–159.

[14] Ralf Schindler. Coding into K by reasonable forcing. *Transactions of the American Mathematical Society*. vol. 353 (2000), pp. 479–489.

[15] ———. Proper forcing and remarkable cardinals II. this Journal, vol. 66 (2001), pp. 1481–1492.

[16] ———. Set theory: Exploring independence and truth. Springer-Verlag. Berlin. 2014.

[17] S. Shelah. *Proper and Improper Forcing*. Perspectives in Mathematical Logic. Springer Verlag. Berlin. 1998.

[18] S. Shelah and L. Stanley. Coding and reshaping when there are no sharps. *Set Theory of the Continuum* (H. Judah et al., editors). Springer-Verlag. New York, 1992. pp. 407–416.

INSTITUT FÜR MATHEMATISCHE LOGIK UND GRUNDLAGENFORSCHUNG
UNIVERSITÄT MÜNSTER
EINSTEINSTR. 62
48149 MÜNSTER, GERMANY
E-mail: world-cyr@hotmail.com
E-mail: rds@math.uni-muenster.de