We consider supersymmetry algebras in arbitrary spacetime dimension and signature. Minimal and maximal superalgebras are given for single and extended supersymmetry. It is seen that the supersymmetric extensions are uniquely determined by the properties of the spinor representation, which depend on the dimension $D \mod 8$ and the signature $|\rho| \mod 8$ of spacetime.

1 Spacetime Symmetry Algebras and the Super Lie Algebras Containing Them

The idea that one can have a new type of algebraic structure generating symmetries between bosonic and fermionic states goes back to the 1970’s when several physicists discovered what are nowadays known as super Lie algebras. The super Lie algebra contains the usual spacetime symmetry algebra (Poincaré or conformal algebra) and its representations combine several representations of the spacetime algebra into a single module irreducible under the super algebra, thus giving rise to the concept of a supermultiplet. The central problem is therefore the following: given a spacetime symmetry algebra, to obtain the super Lie algebras that contain the given Lie algebra in its even part.

By spacetime we mean a real vector space with a nondegenerate quadratic form of type $(s, t)$. It is denoted by $V(s, t)$. The dimension of spacetime is denoted by $D = s + t$, and the signature of the quadratic form by $\rho = s - t$. The group of coordinate transformations in $V$ that leaves the metric invariant up to scale transformations is the simple group $SO(s + 1, t + 1)$ (for $D = s + t \geq 3$), being the Poincaré group $ISO(s, t)$ a subgroup of it. The Poincaré group $ISO(s, t)$ can also be obtained as a contraction of the conformal group in one dimension less $SO(s, t + 1)$. 
As an example, we can take a spacetime of type \((s, t) = (10, 1)\). We have

\[ \text{so}(10, 2) \xrightarrow{\text{contraction}} \text{iso}(10, 1) \xrightarrow{\text{subalgebra}} \text{so}(11, 2). \]

The supersymmetric extension of the conformal algebra in \(D = 10\) is \(\mathfrak{osp}(1|32)\), which gives as a contraction the M-theory superalgebra, that is, the super Poincaré algebra with five and two-brane charges. This can be seen as a subalgebra of \(\mathfrak{osp}(1|64)\), the superconformal algebra in \(D = 11\).

For a fixed dimension \(D\), theories defined on spacetimes with the same signature \(|\rho| \mod 8\) may be related by some duality since the superalgebras are the same. As an example, the theories introduced by Hull [6], M, M* and M’, are formulated in 11 dimensional spacetimes with signatures \((s, t) = (10, 1), (9, 2), (6, 5)\) respectively. In all these theories the signature of spacetime is \(\rho = s - t = \pm 1 \mod 8\).

In this paper we will see how the supersymmetric extensions of spacetime superalgebras are determined by \(D\) and \(|\rho| \mod 8\). All these results are contained in [6].

2 Spin Groups and Spin Modules.

For each spacetime \(V(s, t)\) there are two spin groups, \(\text{Spin}(s, t)\) and \(\text{Spin}_C(s, t)\), the real and complex spin groups, the former being a real form of the latter. As it is well known, when \(D\) is odd there is only one irreducible spin module of the complex group. It will be denoted by \(S\). When \(D\) is even there are two different irreducible spin modules denoted by \(S^\pm\) (chiral spin modules). For constructing superspacetimes and superfields we need real forms of the complex spin modules. These do not always exist and sometimes we have to combine several copies of them to get a real irreducible module for \(\text{Spin}(s, t)\).

The reality issues depend only on \(|\rho| \mod 8\). The type of a real irreducible module is the isomorphism class of the commutant of the module. Since this commutant is a division algebra, it has to be one of \(\mathbb{R}, \mathbb{C}, \mathbb{H}\). For \(\rho \equiv 0, 1, 7\) the irreducible real spin modules are actually real forms of the complex irreducible spin modules. For \(\rho \equiv 3, 4, 5\) they are quaternionic and on complexification become \(2S^\pm\) or \(2S\) (in the even and odd \(D\) cases respectively). For \(\rho \equiv 2, 6\) they are of complex type and their complexifications are \(S^+ \oplus S^-\).

The spin module may admit an invariant form. If this form is symmetric (it is unique up to multiplication by a scalar), then the spin module is said to be orthogonal. If it is antisymmetric the spin module is symplectic and if it has no invariant form it is called linear. The complex spin groups are embedded into the complex orthogonal, symplectic and linear groups respectively. The existence and symmetry properties of this form depend only on \(D \mod 8\).
These mod 8 results have been known for a long time both to physicists and mathematicians, but there is now a unified approach based on the super Brauer group.

3 Non extended supersymmetry

3.1 Super Poincaré Algebras

A non extended ($N=1$) super Poincaré algebra is a super Lie algebra whose even part is the Poincaré algebra, the odd part consists of one real irreducible spin representation and the anticommutator of two spinors (elements of the spin module) is proportional to the momentum. If we denote a spinor in the form $Q_\alpha$ we have that the anticommutator $\{Q_\alpha, Q_\beta\}$ is symmetric in the indices $\alpha, \beta$. The biggest Lie algebra that one can put in the right hand side of the anticommutator is the Lie algebra of symmetric matrices, $\mathfrak{sp}(2n)$ ($2n$ is the real dimension of the spin module). In fact, there is a superalgebra which has $\mathfrak{sp}(2n)$ as bosonic part, it is the orthosymplectic algebra $\mathfrak{osp}(1|2n)$.

In order to see if the required extension exists, we have to determine if there is an abelian subalgebra of $\mathfrak{sp}(2n)$ with the commutation rules of a vector with the generators of the orthogonal algebra. In other words, we have to determine if there is a morphism of $\text{Spin}(s,t)$ modules

$$S \otimes S \rightarrow V.$$ 

(It is easy to see that this condition is also sufficient).

In fact one can study the question for existence of maps $S \otimes S \rightarrow \Lambda^k V$ for all $k$. As we will see in the next subsection, the case $k = 2$ is important for constructing super conformal algebras, namely super Lie algebras extending $\mathfrak{so}(s+1,t+1)$.

The maps $S \otimes S \rightarrow \Lambda^k V$ are unique (projectively) if they exist and they are either symmetric (+) or antisymmetric (-). The full details can be worked out without difficulty using the formalism of Deligne and are listed in Table 1.

We denote $\rho = \rho_0 + n8$, $D = D_0 + m8$, with $0 \leq \rho_0, D_0 < 7$, $m$ and $n$ integers.

Real Case. There is a conjugation $\sigma$ on $S$ commuting with the action of the spin group, so there is a real form of $S$ which is a real $\text{Spin}(s,t)$-module. The anticommutator of two spinors is

$$\{Q_\alpha, Q_\beta\} = \sum_k \gamma_{\langle \alpha\beta \rangle}^{[\mu_1\ldots\mu_k]} Z_{[\mu_1\ldots\mu_k]},$$
so the morphism $\gamma^\mu_{\alpha\beta}$ must be symmetric in $\alpha, \beta$. This happens for

$$\rho_0 = 0, \quad D_0 = 2$$

$$\rho_0 = 1, 7, \quad D_0 = 1, 3.$$ 

If we consider also non chiral superalgebras (with $S^\pm$ both present), the we find that for $\rho_0 = 0$ and $D_0 = 0, 4$ there is a morphism $S^+ \otimes S^- \to V$, and the extension is possible

*Quaternionic case.* There is a pseudoconjugation $\sigma$ on $S$ commuting with $\text{Spin}(s, t)$. We need two copies $S \oplus S = S \otimes \mathbb{C}^2$ to be able to construct a conjugation $\hat{\sigma} = \sigma \otimes \sigma_0$ and then to impose a reality condition. The factor $\mathbb{C}^2$ is an internal index space. The biggest group that commutes with $\sigma_0$ is $\text{SU}(2) \simeq \text{USp}(2) \simeq \text{SU}^*(2)$.

In the anticommutator the internal index appears,

$$\{Q^i, Q^j\} = \sum_k \gamma^{[\mu_1 \cdots \mu_k]}_{(\alpha\beta)} Z^{0}^{[\mu_1 \cdots \mu_k]} \Omega^{ij} + \sum_k \gamma^{[\mu_1 \cdots \mu_k]}_{(\alpha\beta)} Z^{I}^{[\mu_1 \cdots \mu_k]} \sigma_I^{ij},$$

where $\Omega$ is the symplectic matrix and $\sigma_I$ are the symmetric Pauli matrices, $I = 1, 2, 3$.

If we impose invariance under $\text{SU}(2)$ (R-symmetry group) only the first term (SU(2) singlet) may appear and the morphism $\gamma^\mu_{\alpha\beta}$ must be antisymmetric. This happens for

$$\rho_0 = 3, 5, \quad D_0 = 5, 7$$

$$\rho_0 = 4 \quad D_0 = 6$$

$$\rho_0 = 4 \quad D_0 = 0, 4 \quad \text{(non chiral).}$$
If we want to take the generators in the second term, then the $R$-symmetry is broken. For example, taking $\delta^{ij}$ breaks the symmetry to the subgroup $\text{SO}^*(2)$. Then we have superalgebras for

$$
\rho_0 = 3, 5, \quad D_0 = 1, 3 \\
\rho_0 = 4, \quad D_0 = 2 \\
\rho_0 = 4, \quad D_0 = 0, 4 \quad (\text{non chiral}).
$$

Both types of $R$-symmetry will find an interpretation later. We note that non chiral algebras appear with both, SU(2) symmetry and $\text{SO}^*(2)$ for the same space times. In fact, one can easily see that $\text{SO}^*(2) \subset \text{SU}(2)$ and that they are isomorphic super algebras.

**Complex case.** There is no conjugation nor pseudoconjugation. We take the direct sum $S^+ \oplus S^-$, and there is a conjugation with the property $\sigma(S^\pm) = S^\mp$. Again we have an $R$-symmetry group, $U(1)$, with $S^\pm$ having charges $\pm 1$. If we want invariance under $U(1)$ the orthogonal generators must appear in the anticommutator $\{Q^+, Q^-\}$, so the morphism should be of type

$$
S^\pm \otimes S^\mp \rightarrow V.
$$

This happens for $\rho_0 = 2, 6, \quad D_0 = 0, 4$.

Otherwise, the morphisms can be of type $S^\pm \otimes S^\pm \rightarrow \Lambda^2$, (symmetric) which happens for $\rho_0 = 2, 6, \quad D_0 = 2$.

We note that for physical spacetimes there is always the possibility of supersymmetric extension.

### 4 Super Conformal Algebras

A super conformal algebra is a real, simple superalgebra whose even part contains the conformal group. The odd part of the super conformal algebra is a direct sum of $N$ real spinor modules.

A simple superalgebra $\mathfrak{a} = \mathfrak{a}_0 \oplus \mathfrak{a}_1$ has the propriety

$$
\{a_1, a_1\} = a_0.
$$

For a fixed $N$, depending on how big is $\mathfrak{a}_0$ we can have from minimal to maximal superconformal algebras.

As in the case of Poincaré, a necessary condition is that

$$
\mathfrak{o}(s, t) \subset \mathfrak{sp}(2n).
$$

(1)
The adjoint of $o(s, t)$ is the two fold antisymmetric representation, so one should look for the symmetries of morphisms

$$S \otimes S \rightarrow \Lambda^2.$$ 

$osp(1|2n)$ is a simple super algebra; it is in fact the maximal super conformal algebra in the cases where condition (1) is satisfied. This happens for

| Case        | $\rho_0$ | $D_0$ |
|-------------|----------|-------|
| Real case:  | $\rho_0 = 0$ | $D_0 = 4$ |
|             | $\rho_0 = 1, 7$ | $D_0 = 3, 5$ |
|             | $\rho_0 = 0$ | $D_0 = 2, 6$ (non chiral) |
| Quaternionic case: | $\rho_0 = 4$ | $D_0 = 8$ |
|             | $\rho_0 = 3, 5$ | $D_0 = 1, 7$ |
|             | $\rho_0 = 4$ | $D_0 = 2, 6$ (non chiral) |
|             | $\rho_0 = 4$ | $D_0 = 4$ (SO$^*(2)$) |
| Complex case: | $\rho_0 = 3, 5$ | $D_0 = 3, 5$ (SO$^*(2)$) |
|             | $\rho_0 = 2, 6$ | $D_0 = 2, 6$ |
|             | $\rho_0 = 2, 6$ | $D_0 = 4$ (no U(1)). |

As we have mentioned, the complex group is embedded into a larger group, depending whether the spinor representation is orthogonal, symplectic or linear. The spin representation goes into the fundamental representation of the corresponding group (orthogonal, symplectic or linear). The morphisms

$$S \otimes S \rightarrow \Lambda^0,$$

being the invariant bilinear forms, their existence is governed by $D \pmod{8}$.

The interesting question is to see if there is a real form of these groups containing the real Spin($s, t$) group. We have to play with the reality properties of the spinors, which depend on $\rho$, so we have all the possibilities for $D$ and $\rho$. The result is that the real form is uniquely determined. It is listed in Table 3.

To make an example, let $\rho_0 = 1, 7$ and $D_0 = 1, 7$. dim($S$) = $2^{D-1/2}$ and one can show that

$$\text{Spin}(V) \rightarrow \text{SO}(2^{(D-3)/2}, 2^{(D-3)/2}) \quad (\min(s, t) \geq 1).$$

If $D_0 = 3, 5$, the spin group goes inside a symplectic group and in fact we have

$$\text{Spin}(V) \rightarrow \text{Sp}(2^{(D-1)/2}, \mathbb{R}).$$
Some low dimensional examples are

\begin{align*}
\text{Spin}(4, 3) &\subset \text{SO}(4, 4), & \text{Spin}(8, 1) &\subset \text{SO}(8, 8) \\
\text{Spin}(2, 1) &\subset \text{SL}(2, \mathbb{R}), & \text{Spin}(3, 2) &\subset \text{Sp}(4, \mathbb{R}) \\
\text{Spin}(3) &\subset \text{SU}(2), & \text{Spin}(5) &\subset \text{USP}(4), \\
\text{Spin}(4, 1) &\subset \text{USp}(2, 2), & \text{Spin}(4, 3) &\subset \text{SO}(4, 4), \\
& & \text{Spin}(8, 1) &\subset \text{SO}(8, 8).
\end{align*}

The Lie algebra of this group is called the Spin\((s, t)\)-algebra. It happens
that for each combination of \(\rho\) and \(D\) from the ones listed above there is a
simple superalgebra whose even part is

\[\text{Spin}(s, t) - \text{ algebra } \otimes R - \text{ symmetry}.\]

This superalgebra is minimal inside the classical series, and the list of superalgebras is given in Table 2. There is an exceptional superalgebra that is smaller for dimension 5, with bosonic part \(\mathfrak{so}(5, 2) \oplus \mathfrak{su}(2)\), it is the exceptional super algebra \(\mathfrak{f}(4)\). For spacetimes of dimensions 3, 4 and 6 we also have the conformal algebra as a factor in the bosonic part of the superalgebra. We have

\[
\begin{align*}
\mathfrak{sp}(4, \mathbb{R}) &\approx \mathfrak{so}(3, 2) \\
\mathfrak{su}(2, 2) &\approx \mathfrak{so}(4, 2) \\
\mathfrak{so}^*(4, \mathbb{R}) &\approx \mathfrak{so}(6, 2).
\end{align*}
\]

5 Extended supersymmetry.

We want to construct now a super Lie algebra whose odd part consists of \(N\) copies of the spin module. There is again a maximal super Lie algebra, \(\mathfrak{osp}(1/\mathbb{N}2n)\), but the presence of the internal index space \(i = 1, \ldots N\) allows extra possibilities in the combinations of \(\rho\) and \(D\). For example, in the real case we can have

\[
\begin{align*}
\{Q^i_\alpha, Q^j_\beta\} &= \delta^{ij} \sum_k \gamma^{[\mu_1 \ldots \mu_k]}_{(\alpha \beta)} Z_{[\mu_1 \ldots \mu_k]}, \\
\{Q^i_\alpha, Q^j_\beta\} &= \Omega^{ij} \sum_k \gamma^{[\mu_1 \ldots \mu_k]}_{(\alpha \beta)} Z_{[\mu_1 \ldots \mu_k]},
\end{align*}
\]

and an orthogonal \(R\)-symmetry group, or

\[
\begin{align*}
\{Q^i_\alpha, Q^j_\beta\} &= \Lambda^{ij} \sum_k \gamma^{[\mu_1 \ldots \mu_k]}_{(\alpha \beta)} Z_{[\mu_1 \ldots \mu_k]},
\end{align*}
\]

and symplectic \(R\)-symmetry group.
These new possibilities fill in the gaps in Table 2. The minimal super Lie
algebras have an even part that is again of the form
\[ \text{Spin}(s, t) - \text{algebra} \otimes R - \text{symmetry}. \]

A conjugation must exist in the space \( S \otimes W \) (\( R \) is a module for the \( R \)-symmetry group). If there is a conjugation in \( S \), then there is a conjugation in \( W \) and a pseudoconjugation in \( S \) implies a pseudoconjugation in \( W \). If the spacetime group is unitary, then the \( R \)-symmetry group is also unitary and if the spacetime group is complex, then the \( R \)-symmetry is also complex.

The list is given in Table 3. We mark with a symbol ‘\( \star \)’ the cases that allow the possibility of a compact \( R \)-symmetry group. They correspond to \( \mathfrak{so}(s, 2) \), that is, the physical conformal groups. We mark with the symbol ‘\( \circ \)’ the cases that do not arise in the non extended case.

Non chiral superalgebras. In Tables 2 we have written a chiral algebra whenever it is possible and a non chiral one for the rest. Nevertheless, one can ask if there is a non chiral algebra in the cases where the chiral one already exists. As an example, consider a spacetime of type \((5, 1)\), that is, \( D = 6 \) and physical signature. The conformal group is \( \text{SO}(6,2) \), the orthogonal group of a spacetime of dimension 8 and signature 4. There exists a chiral superalgebra, namely \( \mathfrak{osp}(8^*|2) \). A non chiral superalgebra can be constructed if we go to one dimension more and then we make a dimensional reduction. We can consider the orthogonal groups \( \text{SO}(7,2) \) or \( \text{SO}(6,3) \). In both cases, the associated super conformal algebra is \( \mathfrak{osp}(16^*|2) \). We have the decomposition

\[
\begin{align*}
\mathfrak{so}^*(16) & \leftarrow \mathfrak{so}(7,2) \overset{\sim}{\rightarrow} \mathfrak{so}(6,2) \\
16 & \rightarrow 16 \rightarrow 8_L + 8_R.
\end{align*}
\]

So the super conformal algebra in dimension 7 can be seen as a non chiral superconformal algebra in dimension 6.

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