RIGIDITY OF CONTRACTIONS ON HILBERT SPACES

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Abstract. We study the asymptotic behaviour of contractive operators and strongly continuous semigroups on separable Hilbert spaces using the notion of rigidity. In particular, we show that a “typical” contraction \( T \) contains the unit circle times the identity operator in the strong limit set of its powers, while \( T^{n_j} \) converges weakly to zero along a sequence \( \{n_j\} \) with density one. The continuous analogue is presented for isometric and unitary \( C_0 \)-semigroups.

1. Introduction

“Good” behaviour of the powers \( T^n \) of a linear operator \( T \) on a Banach space \( X \) has been studied intensively leading to many important applications. Here, “good” behaviour may mean “stability” in the sense that \( \lim_{n \to \infty} T^n = 0 \) with respect to one of the standard operator topologies. We refer to Müller [25], Chill, Tomilov [2] and [6] for a survey on these properties. Further “good” properties might be convergence to a projection, to a periodic or compact group. On the other hand, it is also well-known that linear operators and their powers may behave quite differently and, e.g., have a dense orbit \( \{T^n x, n \in \mathbb{N}\} \) for some \( x \) in \( X \). See the recent monograph of Bayart, Matheron [1] on such hypercyclic or chaotic operators.

In this paper we look at a still different type of asymptotic behaviour. In the context of contractions on separable Hilbert spaces we show that it can often happen that for some (necessarily very large) subsequence \( \{n_j\}_{j=1}^\infty \subset \mathbb{N} \) weak-\( \lim_{j \to \infty} T^{n_j} = 0 \) while for every \( \lambda \in \Gamma, \Gamma \) the unit circle, strong-\( \lim_{j \to \infty} T^{m_j} = \lambda I \) for some other sequence \( \{m_j\} \) (this second property is called \( \Gamma \)-rigidity and for \( \lambda = 1 \) rigidity). Analogous rigidity properties have been studied for measures on the unit circle (see Nadkarni [26, Chapter 7]) and for measure theoretical dynamical systems (see e.g. Katok [22], Nadkarni [26, Chapter 8], Goodson, Kwiatkowski, Lemanczyk, Liardet [20] and Ferenczi, Holton, Zamboni [14]). In our main results (Theorems 3.3 and 4.3) we show that for separable Hilbert spaces the set of all operators (\( C_0 \)-semigroups) having such behaviour is a residual set in an appropriate sense. This generalises [7], [10] and [12] and extends a result of Choksi, Nadkarni [3].

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for unitary operators, see Nadkarni [26, Chapter 7]. Connections to ergodic and measure theory are discussed.

2. Preliminaries

We now make precise what we mean by rigidity.

**Definition 2.1.** A bounded operator $T$ on a Banach space $X$ is called **rigid** if
\[
\text{strong-} \lim_{j \to \infty} T^{n_j} = I \quad \text{for some subsequence } \{n_j\}_{j=1}^\infty \subset \mathbb{N}.
\]

Note that one can assume in the above definition $\lim_{j \to \infty} n_j = \infty$. (Indeed, if $n_j$ does not converge to $\infty$, then $T^{n_0} = I$ for some $n_0$ implying $T^{n_0} = I$ for every $n \in \mathbb{N}$.) So, rigidity describes a certain asymptotic behaviour of the powers $T^n$.

**Remark 2.2.** Rigid operators have no non-trivial weakly stable orbit. In particular, by the Foiaş–Sz.-Nagy decomposition, see Sz.-Nagy, Foiaş [29] and Foguel [15], rigid contractions on Hilbert spaces are necessarily unitary.

As trivial examples of rigid operators take $T := \alpha I$ for $\alpha \in \Gamma$, $\Gamma$ the unit circle. Moreover, arbitrary (countable) combinations of such operators are rigid as well, as the following proposition shows.

**Proposition 2.3.** Let $X$ be a separable Banach space and let $T \in \mathcal{L}(X)$ be power bounded with discrete spectrum, i.e., satisfying
\[
H = \overline{\text{lin}} \{x \in X : T x = \lambda x \text{ for some } \lambda \in \Gamma\}.
\]

Then $T$ is rigid.

**Proof.** Since $X$ is separable, the strong operator topology is metrisable on bounded sets of $\mathcal{L}(H)$ (take for example the metric $d(T, S) := \sum_{j=0}^\infty \|T z_j - S z_j\|/(2^j \|z_j\|)$ for a dense sequence $\{z_j\}_{j=1}^\infty \subset X \setminus \{0\}$). So it suffices to show that $I$ belongs to the strong closure of $\{T^n\}_{n \in \mathbb{N}}$. For $\varepsilon > 0$, $m \in \mathbb{N}$ and $x_1, \ldots, x_m \in X$, we have to find $n \in \mathbb{N}$ such that $\|T^n x_j - x_j\| < \varepsilon$ for every $j = 1, \ldots, m$.

Assume first that each $x_j$ is an eigenvector corresponding to some unimodular eigenvalue $\lambda_j$, and hence $\|T^n x_j - x_j\| = |\lambda_j^n - 1| \|x_j\|$. Consider the compact group $\Gamma^m$ and the rotation $\varphi : \Gamma^m \to \Gamma^m$ given by $\varphi(z) := az$ for $a := (\lambda_1, \ldots, \lambda_m)$. By a classical recurrence theorem, see e.g. Furstenberg [18, Theorem 1.2], there exists $n$ such that $|\varphi^n(1) - 1| < \varepsilon_1 := \frac{\varepsilon}{\max_{j=1, \ldots, m} \|x_j\|}$, i.e.,
\[
|\lambda_j^n - 1| < \varepsilon_1 \quad \text{for every } j = 1, \ldots, m,
\]

implying $\|T^n x_j - x_j\| < \varepsilon$ for every $j = 1, \ldots, m$.

Assume now $0 \neq x_j \in \text{lin} \{x \in X : T x = \lambda x \text{ for some } \lambda \in \Gamma\}$ for every $j$. Then we have $x_j = \sum_{k=1}^K c_{j, k} y_k$ for $K \in \mathbb{N}$, eigenvectors $y_k \in X$, and $c_{j, k} \in \mathbb{C}$, $k = 1, \ldots, K$, $j = 1, \ldots, m$. Take $\varepsilon_2 := \frac{\varepsilon}{K \max_{j, k} |c_{j, k}|}$. By the above,
there exists \( n \in \mathbb{N} \) such that \( \|T^n y_k - y_k\| < \varepsilon \) for every \( k = 1, \ldots, K \) and therefore
\[
\|Tx_j - x_j\| \leq \sum_{k=1}^{K} |c_{jk}| \|Ty_k - y_k\| < \varepsilon
\]
for every \( j = 1, \ldots, m \). The standard density argument covers the case of arbitrary \( x_j \in X \). □

Analogously, one defines \( \lambda \)-rigid operators by replacing \( I \) by \( \lambda I \) in the above definition.

**Definition 2.4.** Let \( X \) be a Banach space, \( T \in \mathcal{L}(X) \) and \( \lambda \in \Gamma \). We call \( T \) \( \lambda \)-rigid if there exists a sequence \( \{n_j\}_{j=1}^{\infty} \subset \mathbb{N} \) such that
\[
\text{strong- lim }_{j \to \infty} T^{n_j} = \lambda I.
\]
Again one can choose the sequence \( \{n_j\}_{j=1}^{\infty} \) to converge to \( \infty \). We finally call \( T \) \( \Gamma \)-rigid if \( T \) is \( \lambda \)-rigid for every \( \lambda \in \Gamma \).

**Remark 2.5.** Since every \( \lambda \)-rigid operator is \( \lambda^n \)-rigid for every \( n \in \mathbb{N} \), we see that \( \lambda \)-rigidity implies rigidity. Moreover, \( \lambda \)-rigidity is equivalent to \( \Gamma \)-rigidity whenever \( \lambda \) is irrational, i.e., \( \lambda \notin 2\pi i \mathbb{Q} \). (We used the fact that for irrational \( \lambda \) the set \( \{\lambda^n\}_{n=1}^{\infty} \) is dense in \( \Gamma \) and that limit sets are always closed.)

The simplest examples are again operators of the form \( \lambda I \), \( |\lambda| = 1 \). Indeed, \( T = \lambda I \) is \( \lambda \)-rigid, and it is \( \Gamma \)-rigid if and only if \( \lambda \) is irrational.

Analogously, one can define rigidity for strongly continuous semigroups.

**Definition 2.6.** A \( C_0 \)-semigroup \( (T(t))_{t \geq 0} \) on a Banach space is called \( \lambda \)-rigid for \( \lambda \in \Gamma \) if there exists a sequence \( \{t_j\}_{j=1}^{\infty} \subset \mathbb{R}_+ \) with \( \lim_{j \to \infty} t_j = \infty \) such that
\[
\text{strong- lim }_{j \to \infty} T(t_j) = \lambda I.
\]
Semigroups which are 1-rigid are called rigid, and semigroups rigid for every \( \lambda \in \Gamma \) are called \( \Gamma \)-rigid.

As in the discrete case, \( \lambda \)-rigidity for some \( \lambda \) implies rigidity, and \( \lambda \)-rigidity for some irrational \( \lambda \) is equivalent to \( \Gamma \)-rigidity.

**Remark 2.7.** Again, rigid semigroups have no non-zero weakly stable orbit. This implies by the Foiaš–Sz.-Nagy–Foguel decomposition, see Sz.-Nagy, Foiaš [29] and Foguel [15] that every rigid semigroup on a Hilbert space is automatically unitary.

The simplest examples of rigid \( C_0 \)-semigroups are given by \( T(t) = e^{iat} I \) for some \( a \in \mathbb{R} \). In this case, \( T(\cdot) \) is automatically \( \Gamma \)-rigid whenever \( a \neq 0 \). Moreover, one has the following continuous analogue of Proposition 2.3.
Proposition 2.8. Let $X$ be a separable Banach space and let $T(\cdot)$ be a bounded $C_0$-semigroup with discrete spectrum, i.e., satisfying

$$H = \overline{\text{lin}}\{x \in X : T(t)x = e^{ita}x \text{ for some } a \in \mathbb{R} \text{ and all } t \geq 0\}.$$

Then $T$ is rigid.

So rigidity becomes non-trivial for operators ($C_0$-semigroups) having no point spectrum on the unit circle. By a version of the classical Jacobs–Glicksberg–de Leeuw theorem (see e.g. Krengel [23, pp. 108–110]) and for, i.e., power bounded operators on reflexive Banach spaces, the absence of point spectrum on $\Gamma$ is equivalent to

$$\text{weak- lim}_{j \to \infty} T^{n_j} = 0$$

for some subsequence $\{n_j\}$ with density 1, where the density of a set $M \subset \mathbb{N}$ is defined by

$$d(M) := \lim_{n \to \infty} \frac{\#(M \cap \{1, \ldots, n\})}{n}$$

whenever this limit exists. An analogous assertion holds for $C_0$-semigroups as well, see e.g. [9]. We call operators and semigroups with this property almost weakly stable. (For a survey on almost weak stability in the continuous case see Eisner, Farkas, Nagel, Serény [9]).

Restricting ourselves to Hilbert spaces, we will see that almost weakly stable and $\Gamma$-rigid operators and semigroups are the rule and not just an exception.

3. Discrete case: powers of operators

We now introduce the spaces of operators we will work with.

Take a separable infinite-dimensional Hilbert space $H$ and denote by $\mathcal{U}$ the space of all unitary operators on $H$ endowed with the strong* topology, i.e., the topology defined by the seminorms

$$p_x(T) := \sqrt{\|Tx\|^2 + \|T^*x\|^2}$$

(for details on this topology see e.g. Takesaki [30, p. 68]). Convergence in this topology is strong convergence of operators and their adjoints. Then $\mathcal{U}$ is a complete metric space with respect to the metric

$$d(U, V) := \sum_{j=1}^{\infty} \frac{\|Ux_j - Vx_j\|^2 + \|U^*x_j - V^*x_j\|^2}{2^j\|x_j\|^2}$$

for $U, V \in \mathcal{U}$, where $\{x_j\}_{j=1}^{\infty}$ is some dense subset of $H \setminus \{0\}$. Similarly, the space $\mathcal{I}$ of all isometric operators on $H$ will be endowed with the strong operator topology and then is a complete metric space with respect to

$$d(T, S) := \sum_{j=1}^{\infty} \frac{\|Tx_j - Sx_j\|^2}{2^j\|x_j\|^2}$$

for $T, S \in \mathcal{I}$. 
Finally, we denote by $\mathcal{C}$ the space of all contractions on $H$ endowed with the weak operator topology which is a complete metric space for the metric

$$d(T, S) := \sum_{j,k=1}^{\infty} \frac{|\langle (T - S)x_j, x_k \rangle|}{2^j \|x_j\| \|x_k\|}$$

for $T, S \in \mathcal{C}$.

The following is a basic step of our construction.

**Theorem 3.1.** Let $H$ be a separable infinite-dimensional Hilbert space. The set

$$M := \{ T : \lim_{j \to \infty} T^{n_j} = I \text{ strongly for some } n_j \to \infty \}$$

is residual for the weak operator topology in the set $\mathcal{C}$ of all contractions on $H$. This set is also residual for the strong operator topology in the set $I$ of all isometries and in the set $U$ of all unitary operators for the strong* operator topology.

**Proof.** We begin with the isometric case.

Let $\{x_l\}_{l=1}^{\infty}$ be a dense subset of $H \setminus \{0\}$. Since one can remove the assumption $\lim_{j \to \infty} n_j = \infty$ from the definition of $M$, we have

(1) $M = \{ T \in I : \exists \{n_j\}_{j=1}^{\infty} \subset \mathbb{N} \text{ with } \lim_{j \to \infty} T^{n_j}x_l = x_l \text{ } \forall l \in \mathbb{N} \}$.

Consider the sets

$$M_k := \{ T \in I : \sum_{l=1}^{\infty} \frac{1}{2^l \|x_l\|} \|T^{n_l}x_l - x_l\| < \frac{1}{k} \text{ for some } n \}$$

which are open in the strong operator topology. Therefore,

$$M = \bigcap_{k=1}^{\infty} M_k$$

implies that $M$ is a $G_\delta$-set. To show that $M$ is residual it just remains to prove that $M$ is dense. Since $M$ contains all periodic unitary operators which are dense in $I$, see e.g. Eisner, Sereny [10], the assertion follows.

While for unitary operators the above arguments work as well, we need more delicate arguments for the space $\mathcal{C}$ of all contractions.

We first show that

(2) $M = \{ T \in \mathcal{C} : \exists \{n_j\} \subset \mathbb{N} \text{ with } \lim_{j \to \infty} \langle T^{n_j}x_l, x_l \rangle = \|x_l\|^2 \text{ } \forall l \in \mathbb{N} \}$.

The inclusion “$\subset$” is clear. To prove the converse inclusion, assume that $\lim_{j \to \infty} \langle T^{n_j}x_l, x_l \rangle = \|x_l\|^2$ for each $l \in \mathbb{N}$. By the standard density argument we have $\lim_{j \to \infty} \langle T^{n_j}x, x \rangle = \|x\|^2$ for every $x \in H$. Strong convergence of $T^{n_j}$ to $I$ now follows from

$$\|(T^{n_j} - I)x\|^2 = \|T^{n_j}x\|^2 - 2\text{Re} \langle T^{n_j}x, x \rangle + \|x\|^2 \leq 2(\|x\|^2 - \langle T^{n_j}x, x \rangle).$$
We now define
\[ M_k := \{ T \in \mathcal{C} : \sum_{l=1}^{\infty} \frac{1}{2^l \|x_l\|^2} |\langle (T^n - I)x_l, x_l \rangle| < \frac{1}{k} \text{ for some } n \}, \]
and observe again that \( M = \bigcap_k M_k \).

It remains to show that the complement \( M_k^c \) of \( M_k \) is a nowhere dense set. Since the set of periodic unitary operators \( U_{per} \) on \( H \) is dense in the set of all contractions for the weak operator topology (see e.g. Eisner, Serény [10]), it suffices to show \( U_{per} \cap M_k^c = \emptyset \). Assume that this is not the case, i.e., that there exists a sequence \( \{ T_m \}_{m=1}^{\infty} \subset M_k^c \) converging weakly to a periodic unitary operator \( U \). Then by the standard argument (see, e.g., Eisner, Serény [10, Lemma 4.2]), \( \lim_{m \to \infty} T_m = U \) strongly, hence \( \lim_{m \to \infty} T_m^n = U^n \) strongly for every \( n \in \mathbb{N} \). However, since \( T_m \in A_k^c \) means that
\[ \sum_{l=1}^{\infty} \frac{1}{2^l \|x_l\|^2} |\langle (T_m^n - I)x_l, x_l \rangle| \geq \frac{1}{k} \text{ for every } n, m \in \mathbb{N}. \]

Since \( T_m^n \) converges strongly and hence weakly to \( U^n \) for every \( n \), and hence the expression on the left hand side of the above inequality is dominated by \( \{ \frac{1}{2^l} \} \) which is a sequence in \( \ell^1 \), we obtain by letting \( m \to \infty \) that
\[ \sum_{l=1}^{\infty} \frac{1}{2^l \|x_l\|^2} |\langle (U^n - I)x_l, x_l \rangle| \geq \frac{1}{k} \text{ for every } n, \]
contradicting the periodicity of \( U \). \( \square \)

We now observe that one can replace \( I \) in Theorem 3.1 by \( \lambda I \) for any \( \lambda \in \Gamma \). To show this we need the following lemma.

**Lemma 3.2.** Let \( H \) be a Hilbert space, \( \lambda \in \Gamma \) and \( N \in \mathbb{N} \). Then the set of all unitary operators \( U \) with \( U^n = \lambda I \) for some \( n \geq N \) is dense in the set of all unitary operators for the norm topology.

**Proof.** Let \( U \) be a unitary operator, \( \lambda = e^{i\alpha} \in \Gamma \), \( N \in \mathbb{N} \) and \( \varepsilon > 0 \). By the spectral theorem \( U \) is unitarily equivalent to a multiplication operator \( \tilde{U} \) on some \( L^2(\Omega, \mu) \) with
\[ (\tilde{U}f)(\omega) = \varphi(\omega)f(\omega), \quad \forall \omega \in \Omega, \]
for some measurable \( \varphi : \Omega \to \Gamma := \{ z \in \mathbb{C} : |z| = 1 \} \).

We now approximate the operator \( \tilde{U} \) as follows. Take \( n \geq N \) such that
\[ |1 - e^{i\frac{2\pi n}{N}}| \leq \varepsilon \]
and define for \( \alpha_j := e^{i\left(\frac{2\pi j}{N} + \frac{2\pi n}{N}\right)} \), \( j = 0, \ldots, n-1 \),
\[ \psi(\omega) := \alpha_{j-1}, \quad \forall \omega \in \varphi^{-1}(\{ z \in \Gamma : \arg(\alpha_j) \leq \arg(z) < \arg(\alpha_j) \}). \]
The multiplication operator \( \tilde{P} \) corresponding to \( \psi \) satisfies \( \tilde{P}^n = e^{i\alpha} \). Moreover,
\[ \| \tilde{U} - \tilde{P} \| = \sup_{\omega \in \Omega} |\varphi(\omega) - \psi(\omega)| \leq \varepsilon \]
proving the assertion. \( \square \)
We now describe the “typical” asymptotic behaviour of contractions (isometries, unitary operators) on separable Hilbert spaces. For an alternative proof in the unitary case based on the spectral theorem and an analogous result for measures on $\Gamma$ see Nadkarni [26, Chapter 7].

**Theorem 3.3.** Let $H$ be a separable infinite-dimensional Hilbert space and $\Lambda \subset \Gamma$ be countable. Then the set of all operators $T$ satisfying the following properties

1. there exists $\{n_j\}_{j=1}^{\infty} \subset \mathbb{N}$ with density 1 such that
   $$\lim_{j \to \infty} T^{n_j} = 0 \quad \text{weakly},$$
2. for every $\lambda \in \Lambda$ there exists $\{n_j(\lambda)\}_{j=1}^{\infty}$ with $\lim_{j \to \infty} n_j(\lambda) = \infty$ such that
   $$\lim_{j \to \infty} T^{n_j(\lambda)} = \lambda I \quad \text{strongly}$$

is residual for the weak operator topology in the set $C$ of all contractions. This set is also residual for the strong operator topology in the set $I$ of all isometries as well as for the strong* operator topology in the set $U$ of all unitary operators.

Recall that every contraction satisfying (2) is unitary, cf. Remark 2.2.

**Proof.** By Eisner, Serény [10], operators satisfying (1) are residual in $C$, $I$ and $U$.

We now show that for a fixed $\lambda \in \Gamma$, the set $M$ of all operators $T$ satisfying $\lim_{j \to \infty} T^{n_j} = \lambda I$ strongly for some sequence $\{n_j\}_{j=1}^{\infty}$ is residual. We again prove this first for isometries and the strong operator topology.

Take a dense set $\{x_l\}_{l=1}^{\infty}$ of $H \setminus \{0\}$ and observe that

$$M = \{T \in I : \exists \{n_j\} \text{ with } \lim_{j \to \infty} T^{n_j}x_l = \lambda x_l \forall l \in \mathbb{N}\}.$$

We see that $M = \bigcap_{k=1}^{\infty} M_k$ for the sets

$$M_k := \{T \in I : \sum_{l=1}^{\infty} \frac{1}{2^l\|x_l\|}\|T^{n_l}x_l - \lambda x_l\| < \frac{1}{k} \text{ for some } n\}$$

which are open for the strong operator topology. Therefore $M$ is a $G_\delta$-set which is dense by Lemma 3.2, and the residuality of $M$ follows.

The unitary case goes analogously, and we now prove the more delicate contraction case. To do so we first show that

$$M = \{T \in C : \exists \{n_j\}_{j=1}^{\infty} \text{ with } \lim_{j \to \infty} \langle T^{n_j}x_l, x_l \rangle = \lambda \|x_l\|^2 \forall l \in \mathbb{N}\}.$$

As in the proof of Theorem 3.1, the nontrivial inclusion follows from

$$\|(T^{n_j} - \lambda I)x\|^2 = \|T^{n_j}x\|^2 - 2\text{Re} \langle T^{n_j}x, \lambda x \rangle + \|x\|^2 \leq 2(\|x\|^2 - \langle T^{n_j}x, \lambda x \rangle) = 2\lambda(\langle \lambda I - T^{n_j}x, x \rangle).$$
For the sets
\[ M_k := \{ T \in C : \sum_{l=1}^{\infty} \frac{1}{2^l} |\langle (T^n - \lambda I)x_l, x_l \rangle| < \frac{1}{k} \text{ for some } n \}, \]
we have the equality \( M = \bigcap_{k=1}^{\infty} M_k \). Note again that it is not clear whether the sets \( M_k \) are open for the weak operator topology, so we use another argument to show that the complements \( M_k^c \) are nowhere dense. By Lemma 3.2 it suffices to show that \( M_k^c \cap U^\lambda = \emptyset \) for the complement \( M_k^c \) and the set \( U^\lambda \) of all unitary operators \( U \) satisfying \( U^n = \lambda I \) for some \( n \in \mathbb{N} \). This can be shown analogously to the proof of Theorem 3.1 by replacing \( I \) by \( \lambda I \).

Since \( \lambda \)-rigidity for an irrational \( \lambda \) already implies \( \Gamma \)-rigidity, the proof is complete. \( \square \)

We now present basic constructions leading to examples of operators with properties described in Theorem 3.3.

\[ \text{Example 3.4. a) A large class of abstract examples of } \Gamma \text{-rigid unitary operators which are almost weakly stable comes from harmonic analysis. There, a probability measure } \mu \text{ on } \Gamma \text{ is called } \lambda \text{-rigid if its Fourier coefficients satisfy } \lim_{j \to \infty} \hat{\mu}_{n_j} = \lambda \text{ for some } \{n_j\}_{j=1}^{\infty} \subset \mathbb{N}, \lim n_j = \infty. \]

A result of Choksi, Nadkarni [3], see Nadkarni [26, Chapter 7] states that \( \lambda \)-rigid continuous measures form a dense \( G_\delta \) (and hence a residual) set in the space of all probability measures with respect to the weak* topology.

Take a \( \lambda \)-rigid measure \( \mu \) for some irrational \( \lambda \). Note that the arguments used for the existence of such a measure are based on the Baire category theorem without yielding any concrete example. The unitary operator given by \( (Uf)(z) := zf(z) \) on \( L^2(\Gamma, \mu) \) satisfies conditions (1) and (2) of Theorem 3.3. To show (2), it again suffices to prove that \( U \) is \( \lambda \)-rigid since \( \lambda \) is irrational. By our assumption on \( \mu \), there exists a subsequence \( \{n_k\}_{k=1}^{\infty} \) such that the Fourier coefficients of \( \mu \) satisfy \( \lim_{k \to \infty} \hat{\mu}_{n_k} = \lambda \). A standard argument based on the fact that the characters \( \{z \mapsto z^n\}_{n=\infty}^{\infty} \) form an orthonormal basis in \( L^2(\Gamma, \mu) \) implies that \( \lim_{k \to \infty} U^{n_k} = \lambda I \) weakly, and hence strongly.

Conversely, if \( U \) satisfies conditions (1) and (2) of Theorem 3.3, then every spectral measure \( \mu \) of \( U \) is continuous and \( \lambda \)-rigid for every \( \lambda \in \Gamma \).

b) Another class of examples of rigid almost weakly stable unitary operators comes from ergodic theory. For a probability space \((\Omega, \mu)\) and a measureable, \( \mu \)-preserving transformation \( \varphi : \Omega \to \Omega \), the associated unitary operator \( U \) on \( H := L^2(\Omega, \mu) \) is given by \( (Uf)(\omega) := f(\varphi(\omega)) \). The transformation \( \varphi \) is called rigid if \( U \) is rigid, and \( \lambda \)-rigid or \( \lambda \)-weakly mixing if the restriction \( U_0 \) of \( U \) to the invariant subspace \( H_0 := \{ f : \int_{\Omega} f d\mu = 0 \} \) is \( \lambda \)-rigid. This restriction \( U_0 \) is almost weakly stable, i.e., satisfies (2) in Theorem 3.3 if and only if \( \varphi \) is weakly mixing. Thus, each weakly mixing
rigid (or $\lambda$-rigid) transformation corresponds to a unitary operator satisfying conditions (1) and (2) in Theorem 3.3.

Katok [22] proved that rigid transformations form a dense $G_\delta$-set in the set of all measure preserving transformations, and Choksi, Nadkarni [3] generalised this result to $\lambda$-rigid transformations. For more information we refer to Nadkarni [26, p. 59] and for concrete examples of rigid weakly mixing transformations using adding machines and interval exchange transformations see Goodson, Kwiatkowski, Lemanczyk, Liardet [20] and Ferenczi, Holton, Zamboni [14], respectively. For examples of rigid weakly mixing transformations given by Gaussian automorphisms see Cornfeld, Fomin, Sinai [4, Chapter 14].

Furthermore, there is an (abstract) method of constructing $\lambda$-rigid operators from a rigid one. The idea of this construction in the context of measures belongs to Nadkarni [26, Chapter 7].

Example 3.5. Let $T$ be a rigid contraction with $\lim_{j \to \infty} T^{n_j} = I$ strongly, and let $\lambda \in \Gamma$. We construct a class of $\lambda$-rigid operators from $T$. Note that if $\lambda$ is irrational and if $T$ is unitary with no point spectrum, this construction gives us a class of examples satisfying (1)-(2) of Theorem 3.3.

Take $\alpha \in \Gamma$ and consider the operator $T_\alpha := \alpha T$. Then we see that $T_\alpha$ is $\lambda$-rigid if $\lim_{j \to \infty} \alpha^{n_j} = \lambda$ for the above sequence $\{n_j\}$. Nadkarni [26, p. 49–50] showed that the set of all $\alpha$ such that the limit set of $\{\alpha^{n_j}\}_{j=1}^\infty$ contains an irrational number has full Lebesgue measure in $\Gamma$. Every such $\alpha$ leads to a $\Gamma$-rigid operator $\alpha T$.

We now show that one cannot replace the operators $\lambda I$ in Theorem 3.3 by any other operator.

Proposition 3.6. Let $V \in \mathcal{L}(H)$ be such that the set

$$M_V := \{T : \exists \{n_j\}_{j=1}^\infty \text{ such that } \lim_{j \to \infty} T^{n_j} = V \text{ strongly}\}$$

is dense in one of the spaces $\mathcal{U}$, $\mathcal{I}$ or $\mathcal{C}$. Then $V$ is a multiple of identity.

Proof. Consider the contraction case and assume that the set of all contractions $T$ such that weak-$\lim_{j \to \infty} T^{n_j} = V$ for some $\{n_j\}_{j=1}^\infty$ is dense in $\mathcal{C}$. Since every such operator $T$ commutes with $V$ by $TV = \lim_{j \to \infty} T^{n_j+1} = VT$, we obtain by assumption that $V$ commutes with every contraction. In particular, $V$ commutes with every one-dimensional projection implying that $V = \lambda I$ for some $\lambda \in \mathbb{C}$.

The same argument works for the spaces $\mathcal{I}$ and $\mathcal{U}$ using the density of unitary operators in the set of all contractions for the weak operator topology.

Remark 3.7. In the above proposition, one has $V = \lambda I$ for some $\lambda \in \Gamma$ in the unitary and isometric case. Moreover, the same holds in the contraction case if $M_V$ is residual. (This follows from the fact that the set of all non-unitary contractions is of first category in $\mathcal{C}$ by Remark 2.2 and Theorem 3.1, see also [7]).
Remark 3.8. It is not clear whether Theorem 3.3 remains valid under the additional requirement

\[ \{ \lambda \in \mathbb{C} : |\lambda| < 1 \} \cdot I \subset \{ T^n : n \in \mathbb{N} \} \]

where \( \sigma \) denotes the weak operator topology. Since countable intersections of residual sets are residual and the right hand side of (3) is closed, this question becomes whether, for a fixed \( \lambda \) with \( 0 < |\lambda| < 1 \), the set \( M_\lambda \) of all contractions \( T \) satisfying \( \lambda I \in \{ T^n : n \in \mathbb{N} \} \) is residual. Note that each \( M_\lambda \) is dense in \( C \) since, for \( \lambda = re^{is} \), it contains the set

\[ \{ cU : 0 < c < 1, \ c^n = r \text{ and } U^n = e^{is}I \text{ for some } n \} \]

which is dense in \( C \) for the norm topology by Lemma 3.2.

We finally mention that absence of rigidity does not imply weak stability, or, equivalently, absence of weak stability does not imply rigidity, as the following example shows.

Example 3.9. There exist unitary operators \( T \) with no non-trivial weakly stable orbit which are nowhere rigid, i.e., such that \( \lim_{j \to \infty} T^{n_j}x = x \) for some subsequence \( \{n_j\}_{j=1}^\infty \) implies \( x = 0 \). (Note that such operators are automatically almost weakly stable by Proposition 2.3.) A class of such examples comes from mildly mixing transformations which are not strongly mixing, see Furstenberg, Weiss [19], Frączek, Lemańczyk [16] and Frączek, Lemańczyk, Lesigne [17].

4. Continuous case: \( C_0 \)-semigroups

We now give the continuous analogue of the above results for unitary and isometric strongly continuous (semi)groups.

Let \( H \) be again a separable infinite-dimensional Hilbert space. We denote by \( \mathcal{U}^{\text{cont}} \) the set of all unitary \( C_0 \)-groups on \( H \) endowed with the topology of strong convergence of semigroups and their adjoints uniformly on compact time intervals. This is a complete metric and hence a Baire space for

\[
d(U(\cdot), V(\cdot)) := \sum_{n,j=1}^{\infty} \sup_{t \in [-n,n]} \frac{\|U(t)x_j - V(t)x_j\|}{2^n\|x_j\|} \quad \text{for } U(\cdot), V(\cdot) \in \mathcal{U}^{\text{cont}},
\]

where \( \{x_j\}_{j=1}^{\infty} \) is a fixed dense subset of \( H \setminus \{0\} \). We further denote by \( \mathcal{T}^{\text{cont}} \) the set of all isometric \( C_0 \)-semigroups on \( H \) endowed with the topology of strong convergence uniform on compact time intervals. Again, this is a complete metric space for

\[
d(T(\cdot), S(\cdot)) := \sum_{n,j=1}^{\infty} \sup_{t \in [0,n]} \frac{\|T(t)x_j - S(t)x_j\|}{2^n\|x_j\|} \quad \text{for } T(\cdot), S(\cdot) \in \mathcal{T}^{\text{cont}}.
\]

The proofs of the following results are similar to the discrete case, but require some additional technical details.
Theorem 4.1. Let $H$ be a separable infinite-dimensional Hilbert space. The set

$$M^{\text{cont}} := \{ T(\cdot) : \lim_{j \to \infty} T(t_j) = I \text{ strongly for some } t_j \to \infty \}$$

is residual in the set $I^{\text{cont}}$ of all isometric $C_0$-semigroups for the topology corresponding to strong convergence uniform on compact time intervals in $\mathbb{R}_+$. The same holds for unitary $C_0$-groups and the topology of strong convergence uniform on compact time intervals in $\mathbb{R}$.

Proof. We begin with the unitary case.

Choose $\{x_l\}_{l=1}^\infty$ as a dense subset of $H \setminus \{0\}$. Since one can replace $\lim_{j \to \infty} t_j = \infty$ in the definition of $M^{\text{cont}}$ by $\{t_j\}_{j=1}^\infty \subset [1, \infty)$ we have

(4) $M^{\text{cont}} = \{ T(\cdot) \in \mathcal{U}^{\text{cont}} : \exists \{t_j\} \in [1, \infty) : \lim_{j \to \infty} T(t_j)x_l = x_l \ \forall l \in \mathbb{N} \}.$

Consider now the open sets

$$M_{k,t} := \{ T(\cdot) \in \mathcal{U}^{\text{cont}} : \sum_{l=1}^\infty \frac{1}{2^l \|x_l\|} \|T(t)x_l - x_l\| < \frac{1}{k} \}$$

and $M^{\text{cont}}_k := \bigcup_{t \geq 1} M_{k,t}$. We have

$$M^{\text{cont}} = \bigcap_{k=1}^\infty M^{\text{cont}}_k,$$

and hence $M^{\text{cont}}$ is a $G_\delta$-set. Since periodic unitary $C_0$-groups are dense in $\mathcal{U}^{\text{cont}}$ by Eisner, Serény [11], and since they are contained in $M^{\text{cont}}$, we see that $M^{\text{cont}}$ is residual as a countable intersection of dense open sets.

The same arguments and the density of periodic unitary operators in $I^{\text{cont}}$ (see Eisner, Serény [11]) imply the assertion in the isometric case.

The following continuous analogue of Lemma 3.2 allows to replace $I$ by $\lambda I$.

Lemma 4.2. Let $H$ be a Hilbert space and fix $\lambda \in \Gamma$ and $N \in \mathbb{N}$. Then for every unitary $C_0$-group $U(\cdot)$ there exists a sequence $\{U_n(\cdot)\}_{n=1}^\infty$ of unitary $C_0$-groups such that

(a) For every $n \in \mathbb{N}$ there exists $\tau \geq N$ with $U_n(\tau) = \lambda I$,

(b) $\lim_{n \to \infty} \|U_n(t) - U(t)\| = 0$ uniformly on compact intervals in $\mathbb{R}$.

Proof. Let $U(\cdot)$ be a unitary $C_0$-group on $H$, $\lambda = e^{i\alpha} \in \Gamma$. By the spectral theorem, $H$ is isomorphic to $L^2(\Omega, \mu)$ for some finite measure space $(\Omega, \mu)$ and $U(\cdot)$ is unitarily equivalent to a multiplication group $\hat{U}(\cdot)$ given by

$$(\hat{U}(t)f)(\omega) = e^{itq(\omega)}f(\omega), \quad \omega \in \Omega,$$

for some measurable $q : \Omega \to \mathbb{R}$.

To approximate $\hat{U}(\cdot)$, let $N \in \mathbb{N}$, $\varepsilon > 0$, $t_0 > 0$ and take $m \geq N$, $m \in \mathbb{N}$, such that $\|1 - e^{2\pi i \frac{m}{m}}\| \leq \varepsilon/(2t_0)$. Define for $\alpha_j := e^{i(\pi + \frac{2\pi j}{m})}$, $j = 0, \ldots, m-1$, $p(\omega) := \alpha_{j-1}$ for all $\omega \in \varphi^{-1}(\{z \in \Gamma : \arg(\alpha_{j-1}) \leq \arg(z) < \arg(\alpha_j)\})$. 


The multiplication group $\tilde{V}(\cdot)$ defined by $\tilde{V}(t)f(\omega) := e^{itp(\omega)}f(\omega)$ satisfies $\tilde{V}(m) = e^{i\alpha}$. Moreover,

$$\|\tilde{U}(t)f - \tilde{V}(t)f\|^2 = \int_{\Omega} |e^{itq(\omega)} - e^{itp(\omega)}|^2 \|f(\omega)\|^2$$

$$\leq 2|t| \sup_{\omega \in \Omega} |q(\omega) - p(\omega)| \|f\|^2 < \varepsilon \|f\|^2$$

uniformly in $t \in [-t_0, t_0]$. 

We now obtain the following characterisation of the “typical” asymptotic behaviour of isometric and unitary $C_0$-(semi)groups on separable Hilbert spaces.

**Theorem 4.3.** Let $H$ be a separable infinite-dimensional Hilbert space. Then the set of all $C_0$-semigroups $T(\cdot)$ on $H$ satisfying the following properties

1. there exists a set $M \subset \mathbb{R}_+$ with density 1 such that
   $$\lim_{t \to \infty, t \in M} T(t) = 0 \text{ weakly},$$

2. for every $\lambda \in \Gamma$ there exists $(t_j^{(\lambda)})_{j=1}^\infty$ with $\lim_{j \to \infty} t_j^{(\lambda)} = \infty$ such that
   $$\lim_{j \to \infty} T(t_j^{(\lambda)}) = \lambda I \text{ strongly}$$

is residual in the set of all isometric $C_0$-semigroups for the topology of strong convergence uniform on compact time intervals in $\mathbb{R}_+$. The same holds for unitary $C_0$-groups for the topology of strong convergence uniform on compact time intervals in $\mathbb{R}$.

Recall that the density of a measurable set $M \subset \mathbb{R}_+$ is

$$d(M) := \lim_{t \to \infty} \frac{\mu(M \cap [0, t])}{t} \leq 1$$

whenever the limit exists.

**Proof.** By Eisner, Serény [11], $C_0$-(semi)groups satisfying (1) are residual in $I_\text{cont}$ and $U_\text{cont}$.

We show, for a fixed $\lambda \in \Gamma$, the residuality of the set $M^{(\lambda)}$ of all $C_0$-semigroups $T(\cdot)$ satisfying strong-$\lim_{j \to \infty} T(t_j) = \lambda I$ for some sequence $(t_j)_{j=1}^\infty$ converging to infinity. We prove this property for the space $I_\text{cont}$ of all isometric semigroups and the strong operator convergence uniform on compact intervals, the unitary case goes analogously.

Take $\lambda \in \Gamma$ and observe

$$M^{(\lambda)} = \{ T(\cdot) \in I_\text{cont} : \exists t_j \to \infty \text{ with } \lim_{j \to \infty} T(t_j)x_l = \lambda x_l \forall l \in \mathbb{N} \}$$
for a fixed dense sequence \( \{x_l\}_{l=1}^\infty \subset H \setminus \{0\} \). Consider now the open sets
\[
M_{k,t} := \left\{ T(\cdot) \in T^{\text{cont}} : \sum_{l=1}^\infty \frac{\| (T(t) - \lambda I)x_l \|}{2^l \| x_l \|} < \frac{1}{k} \right\}
\]
and their union \( M_k := \bigcup_{t \geq 1} M_{k,t} \) being open as well. The equality \( M^{(\lambda)} = \bigcap_{k=1}^\infty M_k \) follows as in the proof of Theorem 4.1. Since every \( M_k \) contains periodic unitary \( C_0 \)-groups and is therefore dense by Lemma 4.2, \( M^{(\lambda)} \) is residual as a dense countable intersection of open sets.

Since \( \lambda \)-rigidity for some irrational \( \lambda \) already implies \( \lambda \)-rigidity for every \( \lambda \in \Gamma \), the theorem is proved. \( \square \)

Note that every semigroup satisfying (1) and (2) above is a unitary group by Remark 2.7.

**Example 4.4.** a) There is the same correspondence between rigid (or \( \lambda \)-rigid) unitary \( C_0 \)-groups and rigid (or \( \lambda \)-rigid) probability measures on \( \mathbb{R} \) as in the discrete case, see Example 3.4. Here, to a probability measure \( \mu \) on \( \mathbb{R} \) one associates the multiplication group given by \( (T(t))f(s) := e^{ist} \) on \( H = L^2(\mathbb{R}, \mu) \). For \( \lambda \in \Gamma \), we call a measure \( \mu \) on \( \mathbb{R} \) \( \lambda \)-rigid if there exists \( t_j \to \infty \), \( t_j \in \mathbb{R} \), such that the Fourier transform of \( \mu \) satisfies
\[
\lim_{j \to \infty} \mathcal{F}\mu(t_j) = \lambda.
\]
Using exactly the same arguments as in Choksi, Nadkarni [3], see Nadkarni [26, Chapter 7], one shows that the set of all continuous \( \Gamma \)-rigid measures on \( \mathbb{R} \) is a dense \( G_\delta \) set in the set of all Radon measures with respect to the weak* topology. For each such measure, the associated unitary \( C_0 \)-group is \( \Gamma \)-rigid and almost weakly stable, and conversely, the spectral measures of an almost weakly stable \( \Gamma \)-rigid unitary group is continuous and \( \Gamma \)-rigid.

b) Again, another large class of examples comes from ergodic theory. Consider a *measure preserving semiflow*, i.e., a family of measure preserving transformations \( \{\varphi_t\}_{t \geq 0} \) on a probability space \((\Omega, \mu)\) such that the function \((t, \omega) \mapsto \varphi_t(\omega)\) is measurable on \( \mathbb{R}_+ \times \Omega \). On the Hilbert space \( H := L^2(\Omega, \mu) \) the semiflow induces an isometric semigroup by \((T(t))f(\omega) := f(\varphi_t(\omega))\) which is strongly continuous by Krengel [23, 6.16, Thm. 6.13]. If one/every \( \varphi_t \) is invertible, i.e., if we start by a *flow*, then \( T(\cdot) \) extends to a unitary group.

The semigroup \( T(\cdot) \) is almost weakly stable and \( \lambda \)-rigid if and only if the semiflow \( (\varphi_t) \) is weakly mixing and \( \lambda \)-rigid, where the last notion is defined analogously to the discrete case. However, such flows are not so well-studied and there seems to be no concrete construction for flows in ergodic theory as it was done for operators. So we apply the following abstract argument to present a large class of such flows.

We start from a probability space \((\Omega, \mu)\). As discussed above, a “typical” measure preserving transformation \( \varphi \) on \((\Omega, \mu)\) is weakly mixing and \( \Gamma \)-rigid. On the other hand, by de la Rue, de Sam Lazaro [5] a “typical” (for the same topology) measure preserving transformation \( \varphi \) is embeddable into a flow,
there exists a flow \((\varphi_t)_{t \in \mathbb{R}}\) such that \(\varphi = \varphi_1\). Therefore, a “typical”
transformation is weakly mixing, \(\Gamma\)-rigid and embeddable, and every such
transformation leads to an almost weakly stable \(\Gamma\)-rigid unitary group.

c) Analogously to b), we can construct a class of examples on arbitrary
Hilbert spaces. We use that every unitary operator \(T\) is embeddable into a
unitary \(C_0\)-group \(T(\cdot)\), i.e., \(T(1) = T\) for some \(T(\cdot)\), see e.g. [8]. Take now
any operator satisfying assertions of Theorem 3.3. Since such an operator
is automatically unitary by Remark 2.2, it is embeddable. Thus, every such
operator leads to an example of a \(C_0\)-group satisfying (1) and (2) of Theorem
4.3. (Note that condition (1) follows from the spectral mapping theorem for
the point spectrum, see e.g. Engel, Nagel [13, Theorem IV.3.7].)

We again show that limit operators \(\lambda I, |\lambda| = 1\), cannot be replaced in the
above theorem by any other operator.

**Proposition 4.5.** Let for some \(V \in \mathcal{L}(H)\) the set
\[
M^\text{cont}_V := \{T(\cdot) : \exists t_j \to \infty \text{ such that } \lim_{j \to \infty} T(t_j) = V \text{ strongly}\}
\]
be dense in \(\mathcal{I}^\text{cont}\) or \(\mathcal{U}^\text{cont}\). Then \(V = \lambda I\) for some \(\lambda \in \Gamma\).

**Proof.** We prove this assertion for \(\mathcal{I}\), the unitary case is analogous.

Observe that \(V\) commutes with every \(T(\cdot) \in M^\text{cont}_V\) by
\[
VT(t) = \text{strong-} \lim_{j \to \infty} T(t_j + t) = T(t)V
\]
implying that \(V\) commutes with every unitary \(C_0\)-group. Since unitary \(C_0\-
groups are dense in the set of all contractive \(C_0\)-semigroups for the topology
of weak operator convergence uniform on compact time intervals by Krol
[24], we see that \(V\) commutes with every contractive \(C_0\)-semigroup. We
observe now that orthogonal one-dimensional projections are embeddable
into a contractive \(C_0\)-semigroup by [8, Prop. 4.7] and its proof. So \(V\)
commutes with every orthogonal one-dimensional projection, which implies
\(V = \lambda I\) for some \(\lambda \in \mathbb{C}\). Moreover, \(|\lambda| = 1\) holds since \(V\) is the strong limit
of isometric operators. \(\square\)

**Remark 4.6.** Recall that the space of all contractive \(C_0\)-semigroups on \(H\)
is neither complete metric nor compact with respect to the topology of weak
operator convergence uniform on compact time intervals, see Eisner, Serény
[12]. So it is not clear whether one can formulate an analogue of the above
result for contractive \(C_0\)-semigroups as done in the discrete case.

5. Further remarks

We now consider some generalisations of the above results.

“Controlling” the sequences \(\{n_j\}\) and \(\{t_j\}\). We now take a closer look
at the sequences \(\{n_j\}\) and \(\{t_j\}\) occurring in Theorems 3.3(2) and 4.3(2).

Observe first that, by the same arguments as in the proofs of Theorems 3.1
and 3.3, we can replace \(T\) there by \(T^m\) for a fixed \(m\). Changing appropriately
the assertion and the proof of Lemma 3.2, we see that one can add the condition \( \{n_j\}_{j=1}^{\infty} \subset m\mathbb{N} \) to the sequence appearing in rigidity and \( \lambda \)-rigidity. More precisely, for every \( \lambda \in \Gamma \) and \( m \in \mathbb{N} \), the set of all operators \( T \) such that strong-lim \( j \to \infty \) \( T^{n_j} = I \) for some \( \{n_j\} \subset m\mathbb{N} \) is residual in \( \mathcal{U}, \mathcal{I} \) and \( \mathcal{C} \).

It is a hard problem to determine the sequences \( \{n_j\} \) and \( \{t_j\} \) exactly. However, one can generalise the above observation and “control” these sequences in the following sense. Let \( \Lambda \subset \mathbb{N} \) be an unbounded set. We call an operator \( T \) rigid along \( \Lambda \) if strong-lim \( j \to \infty \) \( T^{n_j} = I \) for some increasing sequence \( \{n_j\}_{j=1}^{\infty} \subset \Lambda \). Similarly, we define rigidity along an unbounded set \( \Lambda \subset \mathbb{R}_+ \) for \( \mathcal{C}_0 \)-semigroups, \( \lambda \)- and \( \Gamma \)-rigidity. It follows from a natural modification of Lemmas 3.2 and 4.2 that, for a fixed unbounded set \( \Lambda \) in \( \mathbb{N} \) and \( \mathbb{R}_+ \), respectively, one can assume \( \{n_j^{(\lambda)}\} \subset \Lambda \) and \( \{t_j^{(\lambda)}\} \subset \Lambda \) in Theorems 3.3(2) and 4.3(2). Thus, the set of all \( \Gamma \)-rigid operators (semigroups) along a fixed unbounded set is residual in \( \mathcal{U}, \mathcal{I} \) and \( \mathcal{C} \) (\( \mathcal{U}^{\text{cont}} \) and \( \mathcal{I}^{\text{cont}} \), respectively).

**Banach space case.** We finally discuss briefly the situation in Banach spaces.

Note first that Theorems 3.3 and 4.3 are not true in general separable Banach spaces. Indeed, since weak convergence in \( l^1 \) implies strong convergence, we see that (2) implies strong convergence to zero of \( T^n \) (or of \( T(t) \), respectively), making (3) or just rigidity impossible.

We now consider the question in which Banach spaces rigid and \( \Gamma \)-rigid operators are residual. Since in the contraction case our techniques heavily use Hilbert space methods, we only consider the isometric and unitary case. Let \( X \) be a separable infinite-dimensional Banach space, and \( \mathcal{I} \) be the set of all isometries on \( X \) endowed with the strong operator topology. Observe that the sets

\[
M_k := \{ T \in \mathcal{I} : \sum_{l=1}^{\infty} \frac{1}{2^l} \|x_l\| \|(T^n - I)x_l\| < \frac{1}{k} \text{ for some } n \}
\]

appearing in the proof of Theorem 3.1 for the isometric case are still open, and therefore \( M \) is a \( G_\delta \)-set containing periodic isometries. Thus Theorem 3.1 holds in all separable infinite-dimensional Banach spaces such that periodic isometries form a dense set of \( \mathcal{I} \). Analogously, the set of operators satisfying property (3) in Theorem 3.3 is residual in \( \mathcal{I} \) if and only if it is dense in \( \mathcal{I} \). This is the case whenever there exists an irrational \( \lambda \in \Gamma \) such that the set of all isometries \( T \) with strong-lim \( j \to \infty \) \( T^{n_j} = \lambda \) for some sequence \( \{n_j\} \) is dense in \( \mathcal{I} \). Analogously, the set of operators satisfying (1) and (2) of Theorem 3.3 is residual in \( \mathcal{I} \) if and only if it is dense in \( \mathcal{I} \). The same assertions hold for the set \( \mathcal{U} \) of all invertible isometric operators with the topology induced by the seminorms \( p_x(T) = \sqrt{\|Tx\|^2 + \|T^{-1}x\|^2} \), which is a complete metric space with respect to the metric

\[
d(T, S) = \sum_{j=1}^{\infty} \frac{\|Tx_j - Sx_j\| + \|T^{-1}x_j - S^{-1}x_j\|}{2^j\|x_j\|} \]
for a fixed dense sequence $\{x_j\}_{j=1}^\infty \subset X \setminus \{0\}$.

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