2-Group Actions and Moduli Spaces of Higher Gauge Theory

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Abstract

A framework for higher gauge theory based on a 2-group is presented, by constructing a groupoid of connections on a manifold acted on by a 2-group of gauge transformations, following previous work by the authors where the general notion of the action of a 2-group on a category was defined. The connections are discretized, given by assignments of 2-group data to 1- and 2-cells coming from a given cell structure on the manifold, and likewise the gauge transformations are given by 2-group assignments to 0-cells. The 2-cells of the manifold are endowed with a bigon structure, matching the 2-dimensional algebra of squares which is used for calculating with 2-group data. Showing that the action of the 2-group of gauge transformations on the groupoid of connections is well-defined is the central result. The effect, on the groupoid of connections, of changing the discretization is studied, and partial results and conjectures are presented around this issue. The transformation double category that arises from the action of a 2-group on a category, as defined in previous work by the authors, is described for the case at hand, where it becomes a transformation double groupoid. Finally, examples of the construction are given for simple choices of manifold: the circle, the 2-sphere and the torus.
1 Introduction

1.1 General Background

The notion of symmetry plays a fundamental role throughout mathematics and its applications. In this paper, we investigate its role in higher gauge theory (HGT), a generalization of ordinary gauge theory of considerable recent interest in physics. For an introduction to HGT, see [1].

Gauge theory is the study of certain geometric structures on manifolds, namely connections. If $G$ is a group and $M$ a connected manifold, the moduli space of principal $G$-bundles over $M$ equipped with a connection is a space in which each point represents a choice of principal $G$-bundle with connection. These have a wide variety of applications. Physically, gauge theories are used to represent particles and fields on a background spacetime $M$, so that a connection on a principal bundle represents a (classical) state of such a physical system. In geometric topology, one can study a manifold by means of such geometric structures and their moduli spaces.

For such a theory on a given manifold, the symmetries of the moduli space are very important. In a physical situation modeled by a gauge theory, there may be many distinct connections on $G$-bundles which represent physically indistinguishable states. This situation can be described as a groupoid $\mathcal{A}(M, G)$. The objects of this groupoid are principal $G$-bundles over $M$ equipped with a connection. Other groupoids, such as $\mathcal{A}_0(M, G)$, whose objects are principal $G$-bundles with flat connection, also play important roles. In each case, the morphisms are symmetries of such configurations, which are the gauge transformations. One way of looking at this groupoid is in terms of global symmetry: each gauge transformation is a transformation of the bundle which fixes the base space, and transforms the connection in a compatible way. These form a group which acts on the space of connections. The
groupoid is, from this point of view, the transformation groupoid (also called the action groupoid) associated to this action.

More recently, it has become common, instead of a moduli space, to refer to a moduli stack. While we do not wish to enter into the general theory of stacks here, we merely note that they are associated to equivalence classes of groupoids, so that the groupoids described above may be seen to represent the same moduli stack. This bypasses the problems associated with both the approach of using the fine moduli space, in which all connections appear as distinct points, and the approach using the coarse moduli space, the quotient in which isomorphic connections are identified. The problem with the fine moduli space is precisely that such isomorphisms are not apparent, while the coarse moduli space may have inconvenient geometric features (as quotients of manifolds by group actions need not themselves be manifolds). However, if one takes these two spaces as the set of objects of two different groupoids (the second of which will contain only automorphisms of its objects), these will be equivalent as categories, and will represent the same moduli stack, which is one motivation for this approach.

Now, in higher gauge theory, the role played by the structure group $G$ in ordinary gauge theory is taken instead by a higher-categorical object, $G$, called a 2-group (or, more generally, an $n$-group). Yetter [20], and Martins and Porter [12] have discussed an extension, or “categorification” of the Dijkgraaf-Witten theory for $G$-bundles, which gives invariants of the type described above, replacing $G$ by a 2-group $G$. We will recall the definition and basic facts about 2-groups in Section 2. Here we only remark that there is a general program of reproducing the machinery of gauge theory for these higher-categorical groups. For a discussion of such invariants based on higher gauge theory, see for instance work by Baez and Schreiber [2], and Bartels [3].

To understand the symmetries of such a theory, we need to generalize the notion of symmetry which is relevant in the context of groups (and group actions in particular) to the context of 2-groups. This paper is the second in a series of three closely connected works which approach this question.

In the first paper of the series, we described the notion of a 2-group action on a category, and the categorified analog of the transformation groupoid associated to a group action [15]. This established that there is a double category associated to every such action, which we call the transformation double category.

Our aim in this paper is to describe the analog, in a higher gauge theory based on a 2-group $G$, of the symmetry group of all gauge transformations which acts on the moduli space of principal $G$-bundles with connection on a manifold $M$, in ordinary gauge theory. The analog will take the form of a 2-group $\text{Gauge}$ acting on a category $\text{Conn}$ of connections, which we may call the “moduli 2-space”.

In the third paper of the series, we will show a key result about the transformation double category $\text{Conn}/\text{Gauge}$ associated to this 2-group action. In particular, we demonstrate that it is equivalent to a double category which emerges naturally as a result of treating connections in terms of transport 2-functors. Indeed, by making suitable choices, we can arrange that the equivalence is a strict isomorphism. This result is an analog of a well-known fact in ordinary gauge theory showing the equivalence of two ways of describing the groupoid of connections [18].

To make sense of all this, we will need to establish the nature of what we have just referred to as the “moduli 2-space” on which a symmetry 2-group may act. Just as it makes sense to consider group actions on an object of any category, the natural context for 2-group actions is on objects of 2-categories. However, the familiar context for group actions is the category $\text{Set}$ of sets and functions - which can be refined in case the group and the set have additional structure, such as a Lie group acting on a manifold. Similarly, the natural situation for a 2-group action is in the 2-category $\text{Cat}$, of categories, functors, and natural transformations. This, again, may be refined in the case that these have extra structure, geometric or otherwise.
There are some new features that appear in moving from actions of groups to actions of 2-groups. As we will see, there are two intrinsically rather different kinds of symmetry relation between $G$-connections, which in the literature on higher gauge theory are often both described as “gauge transformations”. A distinction occurs here which cannot arise for ordinary group actions on sets and set-based structures, but will naturally occur in higher gauge theory. Namely, certain of the symmetries between connections occur naturally as morphisms in the 2-space (i.e. category) of connections itself, while others arise from the action. A similar distinction will occur in a higher gauge theory based on an $n$-group for any $n \geq 2$.

That is, two connections may be related, as objects of $\text{Conn}$, by a morphism in $\text{Conn}$ itself. Or, on the other hand, they may be related by the action of an object in the 2-group $\text{Gauge}$. In the construction we give here of the transformation double category of a 2-group action, these will be the horizontal and vertical morphisms respectively. We give advance notice here that we also refer to these occasionally as, respectively, costrict and strict gauge transformations. What are commonly called gauge transformations are in general mixtures of these two types.

Another layer of structure exists in this setting, also commonly included under the general umbrella term “gauge transformation”, which is represented by the morphisms of the symmetry 2-group $\text{Gauge}$. These we refer to simply as squares, or occasionally as gauge modifications. They may be thought of either as relating morphisms of $\text{Conn}$ (that is, costrict gauge transformations), or as relating objects of $\text{Gauge}$ (that is, strict gauge transformations), depending on the direction in the double category in which we take the source and target of a square.

The terminology of strict and costrict gauge transformations, and of gauge modifications, is chosen by analogy with that used in the context of 2-functors between 2-categories: a natural transformation is a map between 2-functors, and a modification is a map between natural transformations. In a forthcoming paper [16], we will show that this parallel is more than simply an analogy. There we will show that the transformation structure associated to the 2-group action constructed here is equivalent to a certain double category of transport functors, strict and costrict natural transformations, and modifications.

1.2 Overview

In section 2 we recall the notion of a 2-group $G$ in its guise as a crossed module, and then introduce a convenient 2D calculus based on squares labelled with crossed module data. In this context we note that in higher categories one is often faced with different choices for the shapes of higher morphisms, e.g. for 2-morphisms one can consider a simplicial shape (triangles), a cubical shape (squares), or a globular shape (bigons). The special squares of the 2D calculus combine features of squares and bigons, as we will see.

In section 3, we embark on our study of HGT based on a 2-group $G$ and with discretized connections. First, in subsection 3.1, we introduce the cell structure on the manifold $M$, which forms the basis for the discretization. The main feature of this cell structure is what we call a bigon structure on the 2-cells, which is set up to match the 2D algebra of squares. Discretized connections on $M$ are then defined to be suitable assignments of 2-group data to the 1- and 2- cells of $M$. The aim is to avoid the analytic issues associated with infinite-dimensional spaces of connections in general, which need to be addressed, for example, by the use of Frechet manifolds or diffeological spaces.

We construct a category of connections $\text{Conn}$ with discretized connections as its objects. The morphisms of this category can be thought of as certain types of gauge transformations, and this entire category, including the morphisms, constitutes the structure which corresponds to the fine moduli space. Thus we might describe $\text{Conn}$ as a moduli “2-space” to suggest this correspondence, with the understanding that a 2-space is merely a category in which both objects and morphisms
form spaces (e.g. topological spaces, manifolds, schemes, etc. depending on the context). We also show that the category $\text{Conn}$ has invertible morphisms, i.e. it is a groupoid.

Then in subsection 3.2, the 2-group of gauge transformations $\text{Gauge}$ is introduced, given by assignments of 2-group data to 0-cells of $M$. We recall the general notion of the action of a 2-group on a category from our previous work [15], and define the action of $\text{Gauge}$ on $\text{Conn}$. A main result is proving that this is indeed a valid action.

In subsection 3.3, we address the issue of what happens when the choice of discretization of $M$ is changed. We provide partial answers for how this affects the category $\text{Conn}$, along with some conjectures about more general statements and the continuum limit.

In section 4 we recall our construction [15] of a transformation double category from any action of a 2-group $G$ on a category $C$. When the category $C$ is a groupoid, the transformation double category becomes a transformation double groupoid. This is the categorification of the transformation groupoid that arises from the action of a group $G$ on a set. We then particularize to the case at hand and give details of the transformation double groupoid for the action of $\text{Gauge}$ on $\text{Conn}$. This explicit description is useful for two reasons. First, the double groupoid contains all the information in the action, packaged in a convenient way, and so is of intrinsic interest to understanding the symmetry of the 2-space $\text{Conn}$. Second, it will be the main object of study in our forthcoming third paper in this series [16], where we show it is equivalent to a double groupoid constructed using an approach based on transport 2-functors.

In section 5, we describe features of the action of $\text{Gauge}$ on $\text{Conn}$ for a number of elementary examples, when $M$ is the circle $S^1$, the sphere $S^2$, and the torus $T^2$. These are chosen because they are manifolds where only the first fundamental groupoid is nontrivial, where only the second fundamental groupoid is nontrivial, and where both are nontrivial. The examples are sufficient to illustrate the effect of each of these elements of the homotopy type of a manifold on the resulting construction. Moreover, they have special features that make them particularly interesting in their own right.

The example of the circle initially seems of limited importance, since the most salient feature of 2-group gauge theory is that it is possible to define 2-dimensional parallel transport over surfaces - hence, on manifolds of dimension at least 2. However, just as with 1-groups, the $G$-connections on the circle, and the various levels of gauge transformations between them, model the adjoint action of the 2-group on itself. Indeed, the structure of the circle as an oriented cell complex encodes precisely what we mean by this adjoint action by conjugation, since the edge representing the circle has its endpoints attached to each other with opposite orientations on the two endpoints.

As a final remark, we note that all calculations are carried out using ordinary algebra and the 2D algebra of squares introduced in section 2. However, along the way we also give various pointers to some underlying 3D and 4D algebraic structures, not pursued in depth here. See in particular Rem. 3.1.4, Rem. 3.2.5, Fig. 7 and Fig. 10.

1.3 Related Work

There are a number of closely related articles that we wish to draw attention to.

First, the use of double groupoids for approaching higher gauge theory is very much to the fore in an article by Soncini and Zucchini [19], although the perspective there is rather different to ours. These authors describe HGT transports on a manifold $M$ endowed with connection 1- and 2-forms, in terms of double functors from a double groupoid of rectangles in $\mathbb{R}^2$ to a double groupoid constructed from the chosen Lie crossed module. Gauge transformations are then given by double natural transformations and double modifications between these double functors.

A careful study of HGT along similar lines to ours was carried out by Bullivant, Calcada, Kádár,
Faria Martins and Martin [8] in the context of their investigation of topological phases of matter in 3+1 dimensions. They also discretize the higher connections using manifolds with an adapted cell structure, termed a 2-lattice structure. The 2-groups that they consider are finite, but this doesn’t prevent a comparison with our approach without this restriction. They focus on transports along 2-disks and holonomies along 2-spheres, whereas our examples include also the circle and the torus. The gauge transformations in [8] correspond to the morphisms in our category of connections and the objects of our 2-group of gauge transformations, as will be described in section 3 below. For their purpose they do not need the higher level of gauge transformations which in our approach are given by the morphisms of the gauge 2-group, and this is the most significant difference between the two approaches. However there are also many similarities, and we will return to more detailed comparisons at appropriate points in the main text.

We would also like to mention two articles by one of us with D. Bragança [4, 5], which concern the case where $M$ is a surface, possibly with boundary, and the group [4] or 2-group [5] is finite. There is an underlying cell structure on $M$ captured by the notion of “cut cellular surface”, and the counting invariants (which tie in with the Yetter invariant [20]) and TQFT’s that are the main focus in these articles call out for an interpretation as the counting measure of the moduli spaces of flat connections or flat higher connections that are the subject of the present paper.

Finally, in work by one of us together with J. Nelson, see [17] and references therein, models of quantum gravity in 2+1 dimensions are studied using traces of holonomies (Wilson loops). These exhibit area phases relating loops that are homotopic on the spatial surface, which strongly suggests an underlying HGT mechanism.

1.4 Considerations for Future Work

It is our hope that examining this 2-group symmetry action in higher gauge theory will be illuminating in its own right, and will serve as a practical illustration of our earlier work [15] on 2-group symmetry and transformation double categories. More specifically, however, in future work [16], we will show the equivalence result referred to above, between the double groupoid $\text{Conn}/\text{Gauge}$ and a transport double groupoid. This will reveal that a slightly unusual approach to categories of transport functors will be the relevant one for higher gauge theory. Beyond this, we expect that our double-category approach will be useful in extending other gauge theory constructions to higher gauge theories. In particular, the construction of extended topological field theories using 2-linearization [13], in contexts in which cobordism categories become double categories of cobordisms with corners, as previously studied by one of the authors [14], appears promising. Another natural direction in which to extend our results is to include manifolds with boundary, like the surfaces with boundary of [4, 5], and to study examples with $M$ of dimension 3 or more, and non-vanishing curvature on 3-cells.

2 Preliminaries on 2-Groups and Crossed Modules

There are several different manifestations of 2-groups, namely as a certain type of 2-categories, as categorical groups (a certain type of category), or as crossed modules (an algebraic definition with no explicit categorical content). See [15] for a detailed discussion of the relation between these different viewpoints. In this article we will mainly adopt the crossed module perspective and use a convenient 2D algebra based on squares labelled with crossed module data.
2.1 Crossed Modules and Calculus with Squares

**Definition 2.1.1** A crossed module \( \mathcal{G} \) consists of \((G,H,\triangleright,\partial)\), where \( G \) and \( H \) are groups, \( \triangleright \) is an action of \( G \) on \( H \) by automorphisms and \( \partial : H \to G \) is a homomorphism, satisfying the following two conditions:

\[
\partial(g \triangleright \eta) = g\partial(\eta)g^{-1} \quad (1)
\]

\[
\partial(\eta) \triangleright \zeta = \eta\zeta\eta^{-1} \quad (2)
\]

Nontrivial examples of crossed modules arise, e.g., from central extensions of groups. See [10, Examples 1.5-1.12] for more examples.

Given a crossed module \( \mathcal{G} \) we will be performing calculations using squares of the form

\[
\begin{array}{c}
g \\
\eta \\
g'
\end{array}
\]

where \( g, g' \in G \), \( \eta \in H \) and \( \partial(\eta) = g'g^{-1} \). These squares are special cases of the squares of the double groupoid \( \mathcal{D}(\mathcal{G}) \) of \( \mathcal{G} \) [6, 11, 15], having the side edges of the squares labelled by \( 1_G \in G \) (displayed by their being unlabelled), instead of a generic \( G \) element. Likewise omitting the label in the centre of the square denotes that it is labelled by \( 1_H \).

Horizontal and vertical composition of squares are given by:

\[
\begin{array}{c}
g_1 \\
\eta_1 \\
g'_1
\end{array}
\quad \begin{array}{c}
g_2 \\
\eta_2 \\
g'_2
\end{array}
= \begin{array}{c}
g_1g_2 \\
\eta_1(g_1 \triangleright \eta_2) \\
g'_1g'_2
\end{array}
\quad \begin{array}{c}
g \\
\eta \\
g'
\end{array}
= \begin{array}{c}
g \\
\eta' \eta \\
g''
\end{array}
\]

These operations are associative and satisfy the interchange law: given a 2 by 2 array of composable squares, the result of composing horizontally and then vertically is the same as composing vertically and then horizontally. Thus any rectangular array of squares has a unique evaluation as a single square.

Squares admit horizontal and vertical inverses, defined as follows:

\[
\begin{array}{c}
g^{-1} \\
\eta^{-h} \\
g'^{-1}
\end{array}
= \begin{array}{c}
g^{-1} \\
\eta^{-1} \\
g'^{-1}
\end{array}
\quad \begin{array}{c}
g' \\
\eta^{-v} \\
g
\end{array}
= \begin{array}{c}
g' \\
\eta^{-1} \\
g
\end{array}
\]

These inverses are appropriately both-sided (left/right for \( \eta^{-h} \) and up/down for \( \eta^{-v} \)). Furthermore
we will need the following properties for inverses of a horizontal or vertical composition:

\[(g_1 g_2)^{-1} = (\eta_1 (g_1 \triangleright \eta_2))^{-1} = \eta_1^{-1} g_2^{-1} g_1^{-1}\]  

\[(g'_1 g'_2)^{-1} = (\eta'_1 (g'_1 \triangleright \eta'_2))^{-1} = \eta'_1^{-1} g'_2^{-1} g'_1^{-1}\]  

\[g^{-1} = (\eta \eta)^{-h} = \eta^{-h} g^{-1} \eta^{-h} g^{-1}\]  

\[g'' = (\eta'' \eta')^{-v} = \eta''^{-v} g'' \eta''^{-v} g''\]

3 Categories of Connections in Higher Gauge Theory

Our objective is to describe higher connections and gauge transformations in relation to a given cell structure on \(M\), so that connections become assignments of group elements to the 1- and 2-cells, and gauge transformations become assignments of group elements to the 0-cells. The connection assignments arise, in a differential geometric framework, from parallel transports obtained by integrating local connection 1- and 2-forms along the cells, taking account of the transitions between the open sets covering \(M\). For a detailed account, see [11]. One of the underlying ideas is that these connection assignments constitute a discretization of a smooth connection, and the number of cells, although finite, can be as big as we like. On the other hand, for flat connections on simple manifolds, only a very small number of cells is required for a full characterization, and we will see in the examples of Section 5 how such minimal discretizations can give a very efficient description.

Because of the algebraic relations between transports along oriented 1-cells and 2-cells, we introduce, in subsection 3.1, an additional structure, a bigon structure, on the 2-cells of \(M\). This also reflects the fact that \(M\) is a manifold and not just a topological space, i.e. it rules out some of the pathologies which may occur for the attaching maps of a cell complex in general.

3.1 Discretized Higher Connections and the Category Conn

We take our manifold \(M\) to be endowed with a discretization \(\mathcal{D}\), consisting of a finite cellular decomposition, a choice of orientation \(O\) of the cells, and a choice of bigon structure \(B\) on the
2-cells, to be described below. The cellular decomposition is a CW-decomposition, and we will frequently refer to the 0-, 1- and 2-cells as vertices, edges and faces, respectively. The corresponding sets will be denoted $V$, $E$ and $F$.

Let $\psi : S^1 \to M$ be the attaching map of a 2-cell $f$ to the 1-skeleton of $M$, compatible with the positive orientation of the 2-cell. We say that the 2-cell has a bigon structure if the following conditions hold:

- the inverse image under $\psi$ of the 0-skeleton of $M$ consists of a finite subset $\{e^{i\theta_1}, \ldots, e^{i\theta_n}\} \subset S^1 \cong U(1)$, where $0 \leq \theta_1 < \theta_2 < \cdots < \theta_n < 2\pi$.
- $\psi$ is injective onto an open 1-cell of $M$ for each of the collection of open arcs which constitute the complement in $S^1$ of the finite subset above.
- one of the 0-cells in the boundary of the 2-cell is called the 0-source, denoted by $v$ in Figure 1, and one of the 0-cells is called the 0-target, denoted by $w$ in Figure 1. The 0-source and 0-target need not be distinct.
- assuming the positive orientation of $f$ in Figure 1, the concatenations of 1-cells connecting the 0-source and the 0-target along the upper, respectively lower, boundary, are called the 1-source and 1-target of $f$ respectively, denoted by $e$ and $d$ in Figure 1.

![Figure 1: Bigon structure on a 2-cell](image)

**Remark 3.1.1** Examples of 2-cells with bigon structure appear in Subsections 5.2 and 5.3. A related construction is the notion of a 2-lattice defined in [8, Def. 21], which is also a CW-decomposition of $M$ with additional conditions. In particular, for a 2-lattice all cells of any dimension are endowed with a (single) basepoint in their boundary, and amongst other conditions on the attaching maps, all attaching maps of 3-cells are embeddings.

Let $\mathcal{G} = (G,H,\triangleright,\partial)$ be a crossed module. A discretized connection is an assignment of $G$-elements to the edges of $M$ and $H$ elements to the faces of $M$, respecting the bigon structure on the 2-cells of $M$ as is made precise in the following definition.

**Definition 3.1.2** The category of connections, $\text{Conn} = \text{Conn}(M,\mathcal{G},D)$, is given as follows:

- **Objects** of $\text{Conn}$ consist of pairs of the form $(g,h)$, where $g : E \to G$, $h : F \to H$, subject to the condition:

  \[
  \begin{array}{ccc}
  g(e) & \rightarrow & h(f) \\
  h(f) & \rightarrow & g(d)
  \end{array}
  \]
for each face, i.e. \( g(e) \) and \( g(d) \) are not independent, but satisfy

\[
\partial(h(f)) = g(d)g(e)^{-1}.
\]

(7)

Since both \( d \) and \( e \) may be composed of more than one edge, and each of these may be oriented from left to right or from right to left, the value \( g(e) \), and likewise \( g(d) \), is determined by composing the \( G \) elements assigned to the oriented edges making up \( e \), taken in order from left to right, and replacing the \( G \) element by its inverse whenever the component edge is oriented from right to left.

- **Morphisms** of \( \text{Conn} \) consist of pairs \( ((g, h), \eta) \) where \( (g, h) \) is an object and \( \eta : E \to H \). The source of \( ((g, h), \eta) \) is \( (g, h) \) and the target of \( ((g, h), \eta) \) is \( (g', h') \) given by:

\[
\begin{array}{c}
g(e) \\
\eta(e) \\
g'(e)
\end{array}
\]

(8)

for each edge \( e \), and

\[
\begin{array}{c}
g(e) \\
h(f) \\
\eta(d) \\
g'(d)
\end{array}
\begin{array}{c}
g(e) \\
\eta(e) \\
h'(f) \\
g'(d)
\end{array}
= \begin{array}{c}
g(e) \\
\eta(e) \\
h'(f) \\
g'(d)
\end{array}
\]

(9)

for each face \( f \). Again, since both \( d \) and \( e \) may be composed of more than one edge, and each of these may be oriented from left to right or from right to left, the value \( \eta(e) \), and likewise \( \eta(d) \), is determined by multiplying horizontally the squares (8) assigned to the oriented edges making up \( e \), taken in order from left to right, and replacing the square by its horizontal inverse whenever the component edge is oriented from right to left.

- **Composition** of morphisms is defined by

\[
((g', h'), \eta') \circ ((g, h), \eta) = ((g, h), \eta'\eta)
\]

(10)

where \( (\eta'\eta)(e) = \eta'(e)\eta(e) \) for each edge.

- **Identities** are given, for each object \( (g, h) \), by

\[
id_{(g,h)} = ((g, h), 1)
\]

(11)

Remark 3.1.3 We note that (9) can be rewritten to give a formula for the square with \( h'(f) \), using the vertical inverse of the square above it in (9). We return to this in subsection 5.2.

Remark 3.1.4 We observe that (9) may be viewed from a 3D perspective as the equation for a commuting bigon cylinder - see Figure 2. We return to discussing such higher-dimensional perspectives in Remark 3.2.5 and in the examples of Section 5.
Remark 3.1.5 The morphisms in \textbf{Conn} correspond to a special case of the \textit{full gauge transformations of} \cite[Def. 86, Fig. 7]{8}). The latter are given in general by assignments of \(G\) elements to vertices and \(H\) elements to edges, and our morphisms correspond to trivializing the assignments to vertices (setting them all equal to \(1_G\)). We will return to this point in Remark 3.2.6, when we introduce our notion of gauge transformations.

\[
\begin{align*}
\eta(d) & \quad = \quad \eta(d) \\

\text{Figure 2: 3D perspective on (9) - the composition of the two visible faces of the bigon cylinder equals} \\
\text{the composition of the two hidden faces}
\end{align*}
\]

**Theorem 3.1.6** \textit{The category \textbf{Conn} is well-defined.}

**Proof:** The main point to verify is that composition is well-defined. This is shown by combining condition (9) for the morphism \(((g, h), \eta)\) with the corresponding condition for the morphism \(((g', h'), \eta')\):

\[
\begin{array}{c}
\begin{array}{c}
g'(e) \\
h'(f) \\
g'(d) \\
\eta'(d) \\
g''(d)
\end{array}
= \\
\begin{array}{c}
g'(e) \\
\eta'(e) \\
g''(e) \\
h''(f) \\
g''(d)
\end{array}
\end{array}
\]

and multiplying these equations vertically after inserting the square

\[
\begin{array}{c}
g'(d) \\
h'(f)^{v} \\
g'(e)
\end{array}
\]
between them. Cancelling the $h'(f)$ square with this vertical inverse on both sides of the equation, and composing the $\eta$ and $\eta'$ squares vertically, gives equation (9) for the composite morphism $((g, h), \eta' \eta)$, ensuring that it has the correct target:

\[
\begin{align*}
  \begin{array}{c}
g(e) \\
  h(f) \\
  g(d) \\
  (\eta' \eta)(d) \\
  g''(d)
\end{array}
  &=
  \begin{array}{c}
g(e) \\
  (\eta' \eta)(e) \\
  g''(e) \\
  h''(f) \\
  g''(d)
\end{array}
\end{align*}
\]

Composition is clearly associative and the identity morphisms obviously have the right properties.

□

To conclude this subsection, we note that the squares (8) have vertical inverses

\[
\begin{array}{c}
g'(e) \\
  \eta(e)^{-v} \\
  g(e)
\end{array}
\]

and hence we have:

**Lemma 3.1.7** *The category Conn is a groupoid.*

**Proof:** The properties of the vertical inverse ensure that each morphism of Conn has an inverse given by (12). This is well defined, since the condition corresponding to (9) is derived from (9) itself by composing vertically on both sides with (12) at the top, and composing with the corresponding square (12) for $d$ at the bottom. □

### 3.2 The 2-Group Gauge and its Action on Conn

With a 2-group $G$ and discretized manifold $(M, D)$ as before, the 2-group of gauge transformations is described by data assigned only to the 0-cells $V$, as follows.

**Definition 3.2.1** *The 2-group of gauge transformations, Gauge, regarded as a categorical group, is given as follows:*

- **Objects** are the set of maps $\gamma : V \to G$
- **Morphisms** are the set of pairs $(\gamma, \chi)$ where $\gamma$ is an object and $\chi : V \to H$. The source and target of $(\gamma, \chi)$ are $s(\gamma, \chi) = \gamma$ and $t(\gamma, \chi) = \gamma'$, where

\[
\begin{array}{c}
  \gamma(v) \\
  \chi(v) \\
  \gamma'(v)
\end{array}
\]


for each \( v \in V \).

- **Composition** of morphisms \((\gamma, \chi)\) and \((\gamma', \chi')\) is given pointwise by vertical composition of the squares (13).

- **Identities** are the morphisms with \( \chi(v) = 1, \forall v \).

- The **monoidal structure** of the categorical group is given pointwise by horizontal composition of the squares (13).

It is clear that \( \text{Gauge} \) is a well-defined categorical group, since it the product over \( V \) of the categorical group \( \mathcal{G} \).

In [15] we defined the general notion of the action of a 2-group on a category\(^1\). This can be done at the 2-categorical or at the categorical level; here we choose the categorical level, which is Definition 3.3 of [15]. In short, a (strict) action is a functor such that an action diagram commutes and a unit condition holds.

**Definition 3.2.2** A strict action of a categorical group \( \mathcal{G} \) on a category \( \mathcal{C} \) is a functor \( \hat{\Phi} : \mathcal{G} \times \mathcal{C} \to \mathcal{C} \) satisfying the action square diagram in \( \text{Cat} \) (strictly):

\[
\begin{array}{ccc}
\mathcal{G} \times \mathcal{G} \times \mathcal{C} & \xrightarrow{\circ \times 1\mathcal{C}} & \mathcal{G} \times \mathcal{C} \\
1\mathcal{D} \times \hat{\Phi} & & \phi \\
\mathcal{G} \times \mathcal{C} & \xrightarrow{\hat{\Phi}} & \mathcal{C}
\end{array}
\]  

(14)

and the unit condition

\[
\hat{\Phi}(1, x) = x, \quad \hat{\Phi}(id_1, f) = f
\]

for all objects \( x \) and morphisms \( f \) of \( \mathcal{C} \)

**Remark 3.2.3** Note that in [15] we omitted the unit condition on morphisms. Definition 3.2.2 may be regarded as defining a “categorified action”, since at the object level (14) and (15) translate to the usual conditions \( g_1.(g_2.x) = (g_1g_2).x \) and \( 1.x = x \) for a group action (where \( g.x \) denotes \( \hat{\Phi}(g, x) \)).

For the present case, we define the following action.

**Definition 3.2.4** The 2-group of gauge transformations acts on the groupoid of connections by:

- on objects \( \hat{\Phi}(\gamma, (g, h)) = (\gamma.g, \gamma.h) \) where, for any edge \( e \in E \) from \( v \) to \( w \),

\[
(\gamma.g)(e) := \gamma(v)g(e)\gamma(w)^{-1},
\]

(16)

and for any face with bigon structure \( f \in F \):

\[
\begin{array}{ccc}
(\gamma.g)(e) & \gamma(v) & \gamma(w)^{-1} \\
(\gamma.h)(f) & g(e) & h(f) \\
(\gamma.g)(d) & \gamma(v) & g(d) \gamma(w)^{-1}
\end{array}
\]

(17)

\(^1\)Note that the action of a 2-group \( \mathcal{G} \) on a category could also be regarded as the action of the associated double groupoid \([6, 11, 15] \mathcal{D}(\mathcal{G})\) on the category. In this context we would like to mention that there is a notion of action of a double groupoid on morphisms of groupoids \([7, \text{Def. 1.5}]\), but this is in the sense of a groupoid action on a map \([9, \text{Def. 2.1}]\), so a somewhat different perspective to ours. We are grateful to Ronnie Brown for drawing our attention to this point.
\begin{itemize}
  \item on morphisms $\hat{\Phi}((\gamma, \chi), ((g, h), \eta)) = ((\gamma.g, \gamma.h), (\gamma, \chi).\eta)$ where, for any edge $e \in E$ from $v$ to $w$,
  \begin{align}
  (\gamma.g)(e) & \quad \gamma(v) \quad g(e) \quad \gamma(w)^{-1} \\
  ((\gamma, \chi).\eta)(e) & \quad \chi(v) \quad \eta(e) \quad \chi(w)^{-1} \\
  (\gamma.g)'(e) & \quad \gamma'(v) \quad g'(e) \quad \gamma'(w)^{-1}
  \end{align}
\end{itemize}

Remark 3.2.5 Equation (18) could also be viewed from a 3D perspective as a commuting 3-cube. An instance of this is displayed in Figure 7 for the example when $M$ is a circle.

In [15] we used the symbol $\triangleright$ as a shorthand for the action in the general case (see Def. 3.5 of [15]). Thus for instance the unit condition (15) may be written: $1 \triangleright x = x$, $id_1 \triangleright f = f$. Here we are writing the action for our specific case using the symbol $\triangleright$, i.e. $(\gamma.g, \gamma.h)$ etc.

Remark 3.2.6 The relation between our 2-group of gauge transformations $\text{Gauge}$ and the full gauge transformations of [8, Def. 86, Fig. 7, Rem. 89] is as follows (recall Remark 3.1.5). Our action of $\text{Gauge}$ on $\text{Conn}$ at the object level is given by assignments of $G$ elements to the vertices of $M$, and is thus a special case of full gauge transformations, obtained by trivializing the assignments of $H$ elements to edges. Thus from our perspective, full gauge transformations combine the morphisms of $\text{Conn}$ and the action of $\text{Gauge}$ on $\text{Conn}$ at the object level. For the purpose of their study of topological phases of matter in 3+1 dimensions, Bullivant et al do not need to introduce the higher gauge transformations corresponding in our terms to the action of $\text{Gauge}$ on $\text{Conn}$ at the morphism level, but they do suggest an algebraic framework for these as 2-fold homotopies between crossed module homotopies [8, Rem. 94].

The central result of this article is:

Theorem 3.2.7 The action $\hat{\Phi}$ of the previous definition is a well-defined action of $\text{Gauge}$ on $\text{Conn}$.

We will divide the proof up into smaller lemmas. Apart from showing that $\hat{\Phi}$ is well-defined on morphisms, we need to show that it is a functor, that it satisfies the action square diagram and that it satisfies the unit condition.

Lemma 3.2.8 $\hat{\Phi}$ is well-defined on morphisms.

Proof: We need to show that $\hat{\Phi}((\gamma, \chi), ((g, h), \eta))$ is a well-defined morphism in $\text{Conn}$, satisfying (9) for each face, i.e.:

\begin{align}
\begin{array}{c|c|c}
(\gamma.g)(e) & \gamma(v) & g(e) \\
((\gamma, \chi).\eta)(e) & \chi(v) & \eta(e) \\
(\gamma.g)'(e) & \gamma'(v) & g'(e)
\end{array} = \begin{array}{c|c|c}
(\gamma.g)(e) & \gamma(v) & g(e) \\
((\gamma, \chi).\eta)(e) & \chi(v) & \eta(e) \\
(\gamma.g)'(e) & \gamma'(v) & g'(e)
\end{array}
\end{align}
This is the horizontal composition of the equation

\[
\begin{array}{ccc}
\gamma(v) & g(e) & \gamma(w)^{-1} \\
\gamma(v) & h(f) & \gamma(w)^{-1} \\
\chi(v) & g(d) & \chi(w)^{-h} \\
\gamma'(v) & g'(d) & \gamma'(w)^{-1} \\
\chi'(v) & \eta(e) & \chi'(w)^{-h} \\
\chi'(v) & g'(e) & \chi'(w)^{-1} \\
\gamma''(v) & g''(d) & \gamma''(w)^{-1} \\
\end{array}
\]

which holds since the three vertical compositions are equal on either side of the equation, the outer ones because of the properties of vertical identity squares, and the middle one since it is \((9)\) for the morphism \(((g,\eta),\eta)\).

**Lemma 3.2.9** \(\hat{\Phi}\) is functorial.

**Proof:** We need to show that \(\hat{\Phi}\) preserves compositions and identities. For compositions this means showing the equation:

\[
\hat{\Phi}((\gamma',\chi'),((g',h'),\eta')) \circ \hat{\Phi}((\gamma,\chi),((g,h),\eta)) = \hat{\Phi}((\gamma,\chi'),((g,h),\eta')).
\]

This holds, since the left hand side corresponds to:

\[
\begin{array}{ccc}
\gamma(v) & g(e) & \gamma(w)^{-1} \\
\chi(v) & \eta(e) & \chi(w)^{-h} \\
\gamma'(v) & g'(e) & \gamma'(w)^{-1} \\
\chi'(v) & \eta'(e) & \chi'(w)^{-h} \\
\gamma''(v) & g''(d) & \gamma''(w)^{-1} \\
\end{array}
\]

and multiplying vertically gives the right hand side.

For identities we need to show

\[
\hat{\Phi}((\gamma,1),((g,h),1)) = ((\gamma.g,\gamma.h),1)
\]

which is obvious, setting \(\chi(v),\eta(e)\) and \(\chi(w)\) equal to 1 in \((18)\).

**Lemma 3.2.10** \(\hat{\Phi}\) satisfies the action square diagram \((14)\).

**Proof:** The commutativity of the action diagram \((14)\) at the object level corresponds to the statement:

\[
((\tilde{\gamma}\gamma).g, (\tilde{\gamma}\gamma).h) = (\tilde{\gamma}.(\gamma.g), \tilde{\gamma}.(\gamma.h))
\]

for any \(\tilde{\gamma},\gamma : V \to G\). For functions of edges \(g : E \to G\) this is immediate from \((16)\). For functions
of faces $h : F \to H$, this follows from considering the array:

\[
\begin{array}{cccc}
\tilde{\gamma}(v) & \gamma(v) & g(e) & \gamma(w)^{-1} \tilde{\gamma}(w)^{-1} \\
\tilde{\gamma}(v) & \gamma(v) & g(d) & \gamma(w)^{-1} \tilde{\gamma}(w)^{-1} \\
\end{array}
\]

and either composing horizontally the two squares on the left and the two squares on the right, or composing horizontally the three squares in the middle.

To show the commutativity of the action diagram (14) at the morphism level, we consider the monoidal product of two morphisms in \textbf{Gauge}:

\[
\begin{array}{ccc}
\tilde{\gamma} & \gamma \\
\tilde{\chi} & \chi \\
\end{array}
\]

This commutativity then corresponds to the statement:

\[
(((\tilde{\gamma}\gamma).g, (\tilde{\gamma}\gamma).h), (\tilde{\chi}(\tilde{\gamma} \triangleright \chi)).\eta) = ((\tilde{\gamma}.(\gamma.g), \tilde{\gamma}.(\gamma.h)), (\tilde{\chi}, \chi).((\gamma, \chi).\eta))
\]

The equality of the first component has already been established. The equality of the second component follows from considering the array:

\[
\begin{array}{cccc}
\tilde{\gamma}(v) & \gamma(v) & g(e) & \gamma(w)^{-1} \tilde{\gamma}(w)^{-1} \\
\tilde{\chi}(v) & \chi(v) & \eta(e) & \chi(w)^{-h} \tilde{\chi}(w)^{-h} \\
\tilde{\gamma}'(v) & \gamma'(v) & g'(e) & \gamma'(w)^{-1} \tilde{\chi}'(w)^{-1} \\
\end{array}
\]

and either composing horizontally the two squares on the left and the two squares on the right, and using properties of the horizontal inverse, or composing horizontally the three squares in the middle.

\[\square\]

\textbf{Lemma 3.2.11} \(\hat{\Phi}\) satisfies the unit condition.

\textbf{Proof}: On objects, the unit condition is

\[(1.g, 1.h) = (g, h)\]

where \(1 : V \to G\) is such that \(1(v) = 1, \forall v \in V\). This clearly holds, putting \(\gamma = 1\) in (16) and (17).

On morphisms the unit condition is

\[\hat{\Phi}(\text{id}_1, ((g, h), \eta)) = (1, 1).((g, h), \eta) = ((g, h), \eta)\]

where \((1, 1) : V \to G \times H\) is such that \((1, 1)(v) = (1, 1), \forall v \in V\). Again this trivially holds putting \((\gamma, \chi) = (1, 1)\) in (18). \(\square\)
3.3 Effect of Changes in Discretization

It is worth noting that our definition of the category of connections makes sense only relative to a particular choice of discretization $D$. Nevertheless, this notion of a connection does capture some of the information contained in a connection in the sense of differential geometry. Indeed, for a flat connection, it will contain all the relevant information. The situation at hand is somewhat analogous to other situations where one makes arbitrary choices such as a choice of local coordinates, or fixing a gauge, in order to capture a geometric structure, and describes how the result transforms under a change in this choice. Aside from this section, we will continue to work in the setting where we have already made a specific choice, but in order to clarify the link to the usual differential-geometric picture, we will describe here how the construction of $\text{Conn}$ would vary with changes to $D$.

In particular, given $(M, D, G)$, the definition of $\text{Conn}$ depends in the first instance on the cell decomposition which is part of $D$, and we will observe that changing the choice of orientations $O$ on edges and faces, and changing the choice of bigon structures $B$ on faces, gives a straightforward, but non-identity, isomorphism between the respective categories $\text{Conn}$ associated to this change. We will formulate some conjectures concerning the effect of more substantial changes to $D$.

A connection in higher gauge theory based on a Lie 2-group $G$ given by the crossed module $(G, H, \triangleright, \partial)$ may be described locally in terms of a 1-form $A$, valued in $\text{Lie}(G)$, the Lie algebra of $G$, and a 2-form $B$, valued in $\text{Lie}(H)$, the Lie algebra of $H$. There are also $\text{Lie}(H)$-valued transition 1-forms and transition functions valued in $G$ and $H$, and a global description in terms of parallel transport, to which we will return in our forthcoming work [16]. For further discussion of higher gauge theory from this point of view, the reader may consult a variety of works on the subject, such as [2, 18, 11, 19].

The relation between such a situation and the discrete description given here is as follows. Recall that for ordinary gauge theory with Lie group $G$, a $\text{Lie}(G)$-valued connection 1-form gives $G$-valued holonomies or parallel transports along paths (for flat connections, this is determined by the homotopy class of the path). These tell how to transport a fibre $F$ which carries a $G$-action along a path $\gamma$ from $x$ to $y$ which, after fixing a basepoint in $F_x$ and $F_y$, determines a correspondence between the two fibres.

A 2-group-valued connection, on the other hand, gives parallel transports for both paths and surfaces. Given a homotopy of paths $\Gamma$, understood as a family of paths $\gamma_t$, with $t \in [0,1]$, which sweeps out a surface, the holonomy

$$h = \text{hol}(\Gamma)$$

for that surface can be seen as a 2-morphism $(g, h)$ in $\mathcal{G}$, regarded as a 2-category, relating

$$g = \text{hol}(\gamma_0)$$

and

$$\partial(h)g = \text{hol}(\gamma_1)$$

These holonomies are obtained from the local $G$-connection by integrating the connection forms over a path or surface, respectively, taking into account the transition 1-forms and transition functions - for a detailed description see [11].

Then, as in ordinary (group-valued) gauge theory, gauge transformations take one connection to another. Gauge transformations can be expressed locally as $G$-valued functions and $\text{Lie}(H)$-valued 1-forms. The latter give $H$-valued holonomies on paths after integration. A new feature of higher gauge theory is that there are higher gauge transformations as well, given by $H$-valued functions.

In our discrete setting, given a cell structure $D$, we can obtain assignments $g(e)$ by considering an oriented edge $e$ as an equivalence class of paths $\gamma_0$ consisting of all parametrizations of $e$, and
letting $g(e) = hol(\gamma_0)$. Similarly, to a face $f$ equipped with a bigon structure, there corresponds an equivalence class of homotopies of paths $\Gamma$, namely those taking its 1-source to its 1-target and having image $f$. Thus, we may take $h(f) = hol(\Gamma)$. Likewise for the 1-target we have $g(d) = hol(\gamma_1)$, and thus we have a match with the assignments in Def. 3.1.2 which are the objects of the category $\text{Conn}$.

Now consider a manifold $M$ with two choices of discretization $\mathcal{D}$ and $\mathcal{D}'$ which share the same cell structure and differ only by the choice of orientations $O$ and $O'$, and bigon structures $B$ and $B'$. Then we want to understand the relation between $\text{Conn}(M, \mathcal{D}, \mathcal{G})$ and $\text{Conn}(M, \mathcal{D}', \mathcal{G})$.

We consider the following four types of changes.

1) change of edge orientation

Suppose $\mathcal{D}'$ has edge $e_i$ of $\mathcal{D}$ replaced by $\overline{e}_i$, the oppositely oriented edge. Then the change in the objects of $\text{Conn}$ is given by $(g, h) \mapsto (\tilde{g}, \tilde{h})$ where on edges and faces respectively we have:

$$
\begin{align*}
  g(e_i) &\mapsto \tilde{g}(\overline{e}_i) = g(e_i)^{-1}, \\
  g(d) &\mapsto \tilde{g}(d) = g(d)
\end{align*}
$$

Indeed the assignments to faces are unchanged due to the orientation conventions given below (7) in Def. 3.1.2.

2) change of face orientation by vertical inversion

This change of orientation simultaneously affects the bigon structure and is given by the replacement of a 2-cell of $\mathcal{D}$ by a corresponding 2-cell of $\mathcal{D}'$ with the 1-source and 1-target exchanged:

This leads to the following changes in the objects of $\text{Conn}$, $(g, h) \mapsto (\tilde{g}, \tilde{h})$. The edges and the assignments to edges are unchanged, i.e. $g(e_i) \mapsto \tilde{g}(e_i) = g(e_i), \forall i$, and the assignments to faces are changed as follows:

$$
\begin{align*}
  h(f) &\mapsto \tilde{h}(\overline{f}) = h(f)^{-v}, \\
  g(d) &\mapsto \tilde{g}(d) = g(e)
\end{align*}
$$
3) change of face orientation by horizontal inversion

This change of orientation simultaneously affects the bigon structure and is given by the replacement of a 2-cell of $D$ by a corresponding 2-cell of $D'$ with the 0-source and 0-target exchanged, and the orientations of the 1-source and 1-target inverted:

![Diagram showing horizontal inversion of a face]

Figure 4: Horizontal inversion of a face

This leads to the following changes in the objects of $\text{Conn}$, $(g,h) \mapsto (\tilde{g},\tilde{h})$. The edges and the assignments to edges are unchanged, as in 2), and the assignments to faces are changed as follows:

$$
\begin{array}{c}
\begin{array}{c}
g(e) \\
g(d)
\end{array}
\end{array} 
\mapsto 
\begin{array}{c}
\begin{array}{c}
\tilde{g}(\bar{e}) \\
\tilde{g}(\bar{d})
\end{array}
\end{array} = 
\begin{array}{c}
\begin{array}{c}
g(e)^{-1} \\
g(d)^{-1}
\end{array}
\end{array}
$$

(25)

4) change of 0-source and 0-target

Consider the change in bigon structure on a face $f$ when we choose a different 0-source and 0-target, as in the following example, where $v, w$ are replaced by $v', w'$, the 1-source $e = e_1 e_2 e_3$ is replaced by $e' = e_2 e_3 d_3$, and the 1-target $d = d_1 d_2 d_3$ is replaced by $d' = \bar{\bar{e}_1} d_1 d_2$, as shown in Figure 5.

![Diagram showing two bigon structures on a face]

Figure 5: Two bigon structures on a face $f$

The face $f'$, regarded as a family of paths from the 1-source to the 1-target, is obtained from $f$ by “whiskering”:

$$
f' = \bar{\bar{e}_1} f \bar{d}_3
$$

This leads to the following changes in the objects of $\text{Conn}$, $(g,h) \mapsto (\tilde{g},\tilde{h})$. The edges and the assignments to edges are unchanged, as in 2), and the assignments to faces are changed as
Theorem 3.3.1 The correspondences (23), (24), (25) and (26) are functorial and yield isomorphisms of the categories $\text{Conn}(M, D, \mathcal{G})$ and $\text{Conn}(M, D', \mathcal{G})$.

Proof: Each of the correspondences between objects extends to a functor $\text{Conn}(M, D, \mathcal{G}) \to \text{Conn}(M, D', \mathcal{G})$ via the following maps of morphisms $((g, h), \eta) \mapsto ((\tilde{g}, \tilde{h}), \tilde{\eta})$:

1) For edge $e_i$ we have:

$$\begin{array}{c|c|c}
g(e_i) & \tilde{g}(e_i) & g(e_i) \mapsto g(e_i) \\
g' \mapsto \tilde{g}' & \tilde{g}'(e_i) & \tilde{g}' \mapsto \tilde{g}'(e_i) \\
\end{array}$$

On faces, equation (9) of Def. 3.1.2 is unchanged due to the orientation conventions given below (9).

2) We have $\tilde{g}(e_i) = g(e_i)$ and we set $\tilde{\eta}(e_i) = \eta(e_i)$ for all $i$, so that the squares (8) are unchanged. For the face of Figure 3, we have:

$$\begin{array}{c|c|c}
\tilde{g}(\tilde{e}) & g(d) & \tilde{g}(\tilde{d}) \\
\tilde{g}'(\tilde{e}) & \eta(d) & \tilde{g}'(\tilde{d}) \\
\end{array}, \quad \quad \begin{array}{c|c|c}
g(e) & \tilde{g}(e) & g(e) \mapsto g(e) \\
g' \mapsto \tilde{g}' & \tilde{g}'(e) & \tilde{g}' \mapsto \tilde{g}'(e) \\
\end{array}$$

(27)

It is a simple exercise to show that (9) for $((g, h), \eta)$ implies (9) for $((\tilde{g}, \tilde{h}), \tilde{\eta})$, using (24) and (28), together with property (6) of the vertical inverse.

3) As in the previous case, we have $\tilde{g}(e_i) = g(e_i)$ and we set $\tilde{\eta}(e_i) = \eta(e_i)$ for all $i$, so that the squares (8) are unchanged. For the face of Figure 4, we have:

$$\begin{array}{c|c|c|c}
\tilde{g}(\tilde{\tau}) & g(e) & \tilde{g}(\tilde{d}) & g(d) \\
\tilde{g}'(\tilde{\tau}) & g(e) \mapsto g(e) & \tilde{g}'(\tilde{d}) & g(d) \mapsto g(d) \\
\end{array}, \quad \quad \begin{array}{c|c|c|c}
g(e^{-1}) & \tilde{g}(e^{-1}) & g(d^{-1}) & \tilde{g}(d^{-1}) \\
g' \mapsto \tilde{g}' & \tilde{g}'(e^{-1}) & \tilde{g}' \mapsto \tilde{g}'(d^{-1}) \\
\end{array}$$

(29)

Equation (9) for $((\tilde{g}, \tilde{h}), \tilde{\eta})$ follows directly from (9) for $((g, h), \eta)$ by horizontal inversion, using property (6) of the horizontal inverse.
4) As in the two previous cases, we have \( \tilde{g}(e_i) = g(e_i) \) and we set \( \tilde{\eta}(e_i) = \eta(e_i) \) for all \( i \), so that the squares (8) are unchanged. For the face of Figure 5 we define \( \tilde{\eta} \) by

\[
\begin{array}{cccc}
\tilde{g}(e') & g(e)^{-1} & g(e) & g(d_3)^{-1} \\
\tilde{\eta}(e') & \eta(e) & & \\
g'(e') & g(e)^{-1} & g'(e) & g(d_3)^{-1}
\end{array}
\]

and an analogous equation with \( e', e \) replaced by \( d', d \). Equation (9) for \( ((\tilde{g}, \tilde{h}), \tilde{\eta}) \) then follows directly from (9) for \( ((g, h), \eta) \) by composing horizontally on the left and right with identity squares for \( g(e_1)^{-1} \) and \( g(d_3)^{-1} \) and using (26) and (30).

The functors we have obtained preserve identities and composition, as can be easily verified. With regards to composition and the first correspondence, this is based on

\[
(\tilde{\eta}' \tilde{\eta})(e_i) = (\eta' \eta)^{-h}(e_i) = \eta'^{-h}(e_i) \eta^{-h}(e_i) = \tilde{\eta}'(\tau_i) \tilde{\eta}(\tau_i)
\]

where the square notation is understood. For the other three correspondences \( \tilde{g} = g \) and \( \tilde{\eta} = \eta \) on edges, so composition is obviously preserved. Since the maps, both at the object and morphism level, are easily seen to be invertible, the correspondences yield isomorphisms of the categories \( \text{Conn}(M, D, G) \) and \( \text{Conn}(M, D', G) \).

\[ \square \]

**Remark 3.3.2** Theorem 3.3.1 is in the same spirit as [8, Def. 54, Lemma 55] concerning a change of the single basepoint in the boundary of a 2-cell.

The preceding theorem tells about how the groupoid \( \text{Conn} \) changes when we change the orientations and bigon structure of the cell structure. This shows that, up to isomorphism, \( \text{Conn} \) does not depend on the choice of \( (O, B) \), but only on the cells themselves and the associated attaching maps from the CW-structure. Since in general one is more interested in bare manifolds than ones equipped with a CW-structure, it is natural to ask whether analogous results hold when one changes this part of \( D \). This leads us toward connections in the usual smooth sense.

To address this question fully would require either or both of two bodies of theory which are more than we wish to engage with here: namely, a fuller category-theoretic treatment of double categories, and the analytic techniques used to handle the infinite-dimensional manifolds coming from spaces of functions and \( p \)-forms. Thus, we simply present two conjectures which suggest a possible line of inquiry.

First, consider the case where we restrict our attention to flat connections and the moduli spaces involved are finite-dimensional. Denote by \( \text{Conn}_0 \) the category of flat \( G \)-connections on \( (M, D) \). We hope to recover an analog of a result which holds for 1-groups: different discretizations yield transformation groupoids which are Morita equivalent.

Unfortunately, the notion of Morita equivalence, i.e. equivalence at the level of the representation categories, has not yet been sufficiently developed for double categories, as far as we know, and the issue of which higher category of double categories one works with could affect the representation category. Thus we are not in a position to state, much less prove, an analogous theorem. However, we conjecture that, for a suitable notion of Morita equivalence of double categories, the following will be true:
Conjecture 3.3.3 If \( D_1 \) and \( D_2 \) are two different discretizations of a manifold \( M \), then the double groupoids \( \text{Conn}_0/\!\!/\text{Gauge} \) for \((M, D_1, G)\) and \((M, D_2, G)\) are Morita equivalent.

For 1-groupoids, Morita equivalent groupoids describe equivalent physical situations. This is a key idea, for example, behind symplectic reduction. Providing that a suitable definition of Morita equivalence for double groupoids has the same property, we would then be able to say that, for flat connections, the choice of \( D \) is purely a convenience. Indeed, we propose this physical equivalence as a useful criterion for a suitable definition of Morita equivalence in this context.

If we are not considering flat connections, then of course we should not expect any such result. Rather, we are then treating \( D \) as a “probe” of a connection, which gives a finite approximation to the continuum theory by taking holonomies along particular edges and faces. At best, we may hope that there is a double groupoid \( \text{Conn}/\!\!/\text{Gauge} \) that is a limit over all discretizations as the probes are taken to be increasingly finer. This at least potentially makes sense, since there is a partial order relation on all CW-structures by refinement. In particular, for 2-dimensional manifolds \((V, E, F)\) is a refinement of \((V', E', F')\) if \( V' \) is a subset of \( V \), every edge in \( E \) lies within an edge of \( E' \), and every face in \( F \) lies within a single face of \( F' \). Clearly, knowing holonomies of a connection on \((M, D')\) is sufficient to determine them on \((M, D)\), and similarly for gauge transformations. So one can think of approaching the continuum limit through successive refinements of discretization.

This leads us to the following conjecture:

Conjecture 3.3.4 \( \text{Conn}/\!\!/\text{Gauge}(M, G) \) is the inductive limit of \( \text{Conn}/\!\!/\text{Gauge}(M, D, G) \) over all discretizations \( D \) of \( M \).

This conjecture might need to be improved in the light of analytic considerations about the presumably infinite-dimensional spaces of objects, morphisms, and squares in \( \text{Conn}/\!\!/\text{Gauge}(M, G) \), and how they arise as limits of finite-dimensional spaces, but does suggest the form of the hoped-for result.

In the case of flat connections, and in combination with Conjecture 3.3.3, it would imply that to find \( \text{Conn}_0/\!\!/\text{Gauge}(M, G) \) up to Morita equivalence, we need only find \( \text{Conn}_0/\!\!/\text{Gauge}(M, D, G) \) for any discretization \( D \) of \( M \).

4 The Transformation Double Groupoid for Higher Connections

It is well known that the action of a group \( G \) on a set \( X \) can be described by a (transformation) groupoid \( X/\!\!/G \), with objects \( X \) and morphisms of the form \( x \xrightarrow{(g,x)} g.x \). In [15] we showed that an analogous situation occurs for the action of a 2-group \( \mathcal{G} \) on a category \( C \), namely this “categorified action” can be described by a transformation double category, \( C/\!\!/\mathcal{G} \), which becomes a transformation double groupoid when the category \( C \) is a groupoid. Below we review the construction of [15] and present the transformation double groupoid \( \text{Conn}/\!\!/\text{Gauge} \) which arises in our case.

4.1 The transformation double category \( C/\!\!/\mathcal{G} \)

Given an action of a categorical group \( \mathcal{G} \) on a category \( C \), as in Def. 3.2.2, we showed in [15] that one can define a double category \( C/\!\!/\mathcal{G} \), with objects being the objects of \( C \), horizontal morphisms being the morphisms of \( C \), vertical morphisms being of the form \( x \xrightarrow{\gamma} \gamma \triangleright x \), where \( x \in \text{Ob} X \).
\( \gamma \in \text{Ob} \mathcal{G} \), and squares being \( \text{Mor} \mathcal{G} \times \text{Mor} \mathcal{C} \), with horizontal and vertical sources and targets given by:

\[
\begin{array}{ccc}
\gamma \triangleright x & \xrightarrow{f} & y \\
\downarrow \downarrow & & \downarrow \downarrow \\
\gamma \triangleright (\gamma, \chi) f & & (\partial(\chi) \gamma) \triangleright y
\end{array}
\]

Here \( \gamma \triangleright x \) denotes \( \hat{\Phi}(\gamma, x) \) and \( (\gamma, \chi) \triangleright f \) denotes \( \hat{\Phi}((\gamma, \chi), f) \), as in Def. 3.2.2. In [15] we prove that this is indeed a double category with suitable horizontal and vertical composition of the respective morphisms, and horizontal and vertical composition of squares. The double category \( \mathcal{C} \mathcal{G} \) captures in a single structure several different group actions that are at work simultaneously, namely the action of the objects of \( \mathcal{G} \) on the objects of \( \mathcal{C} \), the action of the objects of \( \mathcal{G} \) on the morphisms of \( \mathcal{C} \), and finally the action of the morphisms of \( \mathcal{G} \) on the morphisms of \( \mathcal{C} \). See [15, Def. 3.5] for a detailed exposition. The vertical morphisms of \( \mathcal{C} \mathcal{G} \) are invertible, and when the horizontal morphisms are also invertible, i.e. \( \mathcal{C} \) is a groupoid, \( \mathcal{C} \mathcal{G} \) becomes a double groupoid.

### 4.2 The Transformation Double Groupoid \( \text{Conn} \mathcal{G} \)

We now give a detailed definition of the transformation double groupoid \( \text{Conn} \mathcal{G} \) that arises in our specific case.

**Definition 4.2.1** Given an action of \( \mathcal{G} \) on \( \text{Conn} \), as defined in Def. 3.2.4, the transformation double groupoid \( \text{Conn} \mathcal{G} \) is given by:

- **Objects** are the objects \((g, h)\) of \( \text{Conn} \)
- **Horizontal morphisms** are the morphisms \(((g, h), \eta)\) of \( \text{Conn} \), with source maps, target maps and horizontal composition defined as in \( \text{Conn} \)
- **Vertical morphisms** are the set of pairs \((\gamma, (g, h))\), where \( \gamma : V \to G \) is an object of \( \mathcal{G} \) and \((g, h)\) is an object of \( \text{Conn} \). The source of \((\gamma, (g, h))\) is \((g, h)\) and the target is \((\gamma . g, \gamma . h)\). Composition of vertical morphisms is defined by:

\[
(\gamma, (\gamma . g, \gamma . h)) \circ (\gamma, (g, h)) = (\gamma \gamma, (g, h))
\]

- **Squares** are the set of pairs of morphisms of \( \mathcal{G} \) and \( \text{Conn} \), and are denoted

\[
(\gamma, \chi), ((g, h), \eta)
\]

Horizontal and vertical sources and targets are given as follows:

\[
\begin{array}{ccc}
(g, h) & \xrightarrow{((g, h), \eta)} & (g', h') \\
(\gamma, (g, h)) & \xrightarrow{((\gamma . g, \gamma . h), (\gamma, \chi), (g, h), \eta)} & (\gamma', (g', h'))
\end{array}
\]
Composition

Horizontal and vertical composition of squares are given by:

\[
(\gamma', \chi') \circ_{\alpha} (\gamma, \chi, ((g, h), \eta)) = (\gamma, \chi' \gamma \chi g \eta^{-h})
\]

and:

\[
(\tilde{\gamma}, \tilde{\chi}) \circ_{\alpha} (\gamma, \chi, ((g, h), \eta)) = (\tilde{\gamma} \gamma, \tilde{\chi} (\tilde{\gamma} \triangleright \chi))(g, h, \eta)
\]

Remark 4.2.2

As noted in the introduction, we remark here (because the terms will appear throughout our forthcoming paper [16]) that we also refer to the horizontal morphisms of this double groupoid as costrict gauge transformations, and the vertical morphisms as strict gauge transformations. The squares, in this usage, will be called gauge modifications between such transformations.

5 Geometrical Examples

The previous approach was set up to handle any finite number of cells in a tendentially local description of connections, but for simple manifolds it also allows an efficient global description of the action of \(\text{Gauge}\) on \(\text{Conn}\), employing a small number of cells.

The manifolds in our examples will all be of dimension less than 3, so that the connections will be automatically flat, since the curvature 3-form vanishes.

In each of the following examples we will highlight special features of the action.

5.1 The example of the circle

The circle can be endowed with a cell decomposition consisting of a single 0-cell \(v\) and a single 1-cell \(e\) - see Figure 6.

\[
\begin{array}{c}
\bullet^v \\
\circ^e \\
\bullet^v
\end{array}
\]

Figure 6: Cell decomposition for the circle \(S^1\)

Since there are no 2-cells, setting \(g(e) = g\) and \(\eta(e) = \eta\), the objects \(\{(g(e), 1)\}\) of \(\text{Conn}\) may be identified with \(G\) and the morphisms \(\text{Mor(Conn)}\) with \(G \times H\). Thus

\[\text{Conn}(S^1) \cong \mathcal{G}.
\]

Likewise setting \(\gamma(v) = \gamma\) and \(\chi(v) = \chi\), the 2-group \(\text{Gauge}\) may be identified with \(\mathcal{G}\), since we have a single 0-cell, i.e.

\[\text{Gauge}(S^1) \cong \mathcal{G}.
\]

The action of \(\text{Gauge}\) on \(\text{Conn}\) is then the adjoint action of \(\mathcal{G}\) on itself, as described in [15], given by particularizing (18):

\[
(\gamma \cdot g (\gamma, \chi), \eta) = (\chi, \eta) \gamma^{-1}.
\]

\[
(\gamma \cdot g)
\]

\[
\begin{array}{c}
\gamma \\
\chi \\
\eta \\
\gamma^{-h}
\end{array}
\]

\[
(\gamma \cdot g)
\]

\[
\begin{array}{c}
\gamma' \\
g' \\
\gamma'^{-1}
\end{array}
\]

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Thus for the case of $S^1$, the transformation double groupoid arising from this action is given by:

$$\text{Conn} \sslash \text{Gauge}(S^1) \cong \mathcal{G} \sslash \mathcal{G}$$

where the adjoint action is understood, in complete analogy with the gauge groupoid $G \sslash G$ for ordinary $G$-gauge theory on the circle. The squares of this transformation double groupoid are:

$$
\begin{array}{ccc}
\gamma \cdot g & \to & \gamma' \cdot g' \\
\downarrow & & \downarrow \\
(\gamma, g) \cdot (\gamma, \chi) \cdot \eta & \to & (\gamma', g') \cdot (\gamma', \chi') \cdot \eta \\
\end{array}
$$

(36)

We refer the reader to [15] for a detailed description of the transformation double groupoid $\mathcal{G} \sslash \mathcal{G}$.

Finally a suggestive 3-dimensional perspective on the action is given by taking a commuting 3-cube and identifying two opposite faces as in Figure 7. Note that to simplify this figure we have abbreviated:

$$
\begin{align*}
\eta_1 &= (\gamma, \chi) \cdot \eta \\
g_1 &= \gamma \cdot g \\
g'_1 &= \gamma' \cdot g'
\end{align*}
$$

5.2 The example of the sphere

The sphere $S^2$ can be realized as a single 2-cell $f$ with the bigon structure as in Figure 8, where the 1-cell edge $d$ is identified with $e$.

We start by describing $\text{Conn}(S^2)$. Setting $h(f) = h, g(e) = g$ one has $\partial(h) = gg^{-1} = 1$. Hence the objects of $\text{Conn}(S^2)$ are $\{(g, h) \mid \partial(h) = 1\} = G \times \ker(\partial)$. Setting $\eta(e) = \eta$, the morphisms of
**Figure 8:** Cell decomposition for the sphere $S^2$

$\text{Conn}(S^2)$ are $\{(g, h, \eta) \mid (g, h) \in G \times \ker(\partial), h \in H\}$ and the target of $((g, h, \eta))$ is $(g', h')$, where $g' = \partial(\eta)g$, from (8), and $h'$ is given, from (9), by vertical conjugation of $h$ by $\eta$ (see Remark 3.1.3):

\[
\begin{pmatrix}
g' \\
h' \\
g' \\
g'
\end{pmatrix}
= \begin{pmatrix}
g' \\
\eta^{-v} \\
g \\
h \\
g' \\
\eta \\
g'
\end{pmatrix}
\]

Since there are two 0-cells, $v$ and $w$, we have

\[\text{Gauge}(S^2) \cong G \times G\]

Setting $\gamma(v) = \gamma_1$, $\gamma(w) = \gamma_2$, $\chi(v) = \chi_1$, $\chi(w) = \chi_2$, the action of $(\gamma_1, \gamma_2)$ on $(g, h)$ is described by (17):

\[
\begin{pmatrix}
\gamma \cdot g \\
\gamma \cdot h
\end{pmatrix}
= \begin{pmatrix}
\gamma_1 \\
\gamma_1
\end{pmatrix}
\begin{pmatrix}
g \\
g
\end{pmatrix}
\begin{pmatrix}
\gamma_2^{-1} \\
\gamma_2^{-1}
\end{pmatrix}
= \begin{pmatrix}
\gamma_1 g \gamma_2^{-1} \\
\gamma_1 g \gamma_2^{-1}
\end{pmatrix}
\]

and the action of $(\gamma, \chi) = ((\gamma_1, \gamma_2), (\chi_1, \chi_2))$ on $(g, \eta)$ is described by (18):

\[
\begin{pmatrix}
\gamma \cdot g \\
(\gamma, \chi) \cdot \eta
\end{pmatrix}
= \begin{pmatrix}
\gamma_1 \\
\gamma_1
\end{pmatrix}
\begin{pmatrix}
g \\
\eta
\end{pmatrix}
\begin{pmatrix}
\gamma_2^{-1} \\
\gamma_2^{-1}
\end{pmatrix}
\begin{pmatrix}
\chi_1 \\
\chi_1
\end{pmatrix}
\begin{pmatrix}
\eta \\
\eta
\end{pmatrix}
\begin{pmatrix}
\chi_2^{-h} \\
\chi_2^{-h}
\end{pmatrix}
\]

Thus $\text{Gauge}(S^2)$ acts on $\text{Conn}(S^2)$ via two independent $\mathcal{G}$ actions from the left and from the right.
Remark 5.2.1 In [8, Examples 74, 75] the holonomy along $S^2$ is approached using two different cell structures. The first has a single vertex and a single 2-cell attached to it, and the second is a subdivision of our cell decomposition of Fig. 8, using four 1-cells from $v$ to $w$ to divide $f$ into four faces. In both cases the holonomy, corresponding to our $h(f)$, is likewise given in [8] by an element of ker($\partial$). Under full gauge transformations, corresponding to our action (37), the result [8, Thm. 97] is the same as ours, namely the holonomy $h(f)$ is transformed into $\gamma(v) \triangleright h(f)$, where $v$ is a vertex on the 2-sphere surface.

5.3 The example of the torus

The torus $T^2$ can be realized with a single 0-cell $v$, two 1-cells $e_1$ and $e_2$, and a single 2-cell $f$ with bigon structure as depicted in Figure 9. Note that this is an example of a 2-cell with bigon structure, as in Figure 1, where $e$ and $d$ are both concatenations of more than one 1-cell.

![Figure 9: Cell decomposition for the torus $T^2$](image)

We start by describing $\text{Conn}(T^2)$. Setting $g(e_1) = g_1$, $g(e_2) = g_2$, and $h(f) = h$, we have $\partial(h) = g_1g_2g_1^{-1}g_2^{-1}$, and hence the objects of $\text{Conn}(T^2)$ are of the form:

$$(g_1, g_2, h) \in G^2 \times \partial^{-1}([G, G]),$$

where $[G, G]$ denotes the commutator subgroup of $G$. Likewise setting $\eta(e_1) = \eta_1$, $\eta(e_2) = \eta_2$, the morphisms of $\text{Conn}(T^2)$ are given by:

$$(((g_1, g_2), h), (\eta_1, \eta_2)) \in G^2 \times \partial^{-1}([G, G]) \times H^2.$$ 

The target of a morphism $(((g_1, g_2), h), (\eta_1, \eta_2))$ is $((g'_1, g'_2), h')$, where $g'_i = \partial(\eta_i)g_i$, and $h'$ is given by

$$g'_2g'_1 = g'_2g'_1 = g'_2g'_1 = g_2g_1$$

$$\eta_2^{-v} \eta_1^{-v} \eta_1 \eta_2$$

where, on the right hand side, the horizontal compositions are performed first.

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Since there is a single 0-cell $v$, we have

$$\text{Gauge}(T^2) \cong G$$

Setting $\gamma(v) = \gamma$, $\chi(v) = \chi$, the action of $\gamma$ on $((g_1, g_2), h)$ is given by $\gamma.g_i = \gamma g_i \gamma^{-1}$, for $i = 1, 2$, and

$$
\begin{array}{c|c|c}
\gamma.g_1 & \gamma & \gamma^{-1} \\
\hline
\gamma.g_2 & g_2 & \gamma^{-1} \\
\hline
\gamma & h & \\
\hline
\end{array}
$$

The action of $(\gamma, \chi)$ on $(((g_1, g_2), h), \eta)$ is described by:

$$
\begin{array}{c|c|c}
\gamma.g_i & \gamma & \gamma^{-1} \\
\hline
\gamma.\chi.\eta_i & \chi & \chi^{-h} \\
\hline
\gamma & \eta_i & \\
\hline
\gamma' & \eta'_i & \gamma'^{-1} \\
\hline
\end{array}
$$

for $i = 1, 2$.

The whole action can be captured in a single 4D diagram, depicted in Figure 10, consisting of eight commuting 3-cubes (the inner 3-cube, its six adjacent 3-cubes and the outer 3-cube). In this figure, squares of a more general type appear, with possibly non-trivial labels on the side edges, as opposed to the squares (3). Focussing on the uppermost 3-cube, which describes the relation between the source and the target of the morphism $(((g_1, g_2), h), (\eta_1, \eta_2))$, we see that the target can be viewed as coming from the source by simultaneous horizontal and vertical conjugation by $\eta_1$ and $\eta_2$ - compare with [11, Thm. 5.17].

Figure 10: 4D perspective for the torus
Note that in this figure, we have used the following shorthand notation to maintain readability:

\[ k_i = \gamma g_i \gamma^{-1} \]
\[ g'_i = \partial(\eta_i) g_i \]
\[ \epsilon_i = \chi \eta_i \chi^{-1} \]
\[ k'_i = \partial(\epsilon_i) k_i \]

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