Diffusion of Innovation over Social Networks under Limited-trust Equilibrium

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Abstract—We consider the diffusion of innovation in social networks using a game-theoretic approach. Each individual plays a coordination game with its neighbors and decides what alternative product to adopt to maximize its payoff. As products are used in conjunction with others and through repeated interactions, individuals are more interested in their long-term benefits and tend to show trustworthiness to others to maximize their long-term payoffs. To capture such trustworthy behavior, we deviate from the expected utility theory and use a new notion of rationality based on limited-trust equilibrium (LTE). By incorporating such notion into the diffusion model, we analyze the convergence of emerging dynamics to their equilibrium points using a mean-field approximation. We study the equilibrium state and the convergence rate of the diffusion process using the absorption probability and the expected absorption time of a reduced-size absorbing Markov chain. We also show that the LTE diffusion model under the best-response strategy can be converted to the well-known linear threshold model. Simulations show that when agents behave trustworthy, their long-term payoffs will increase significantly compared to the case when they are solely self-interested. Moreover, the Markov chain analysis provides a good estimation of the convergence property over random networks.

Index Terms—Diffusion dynamics, limited-trust equilibrium, Markov chains, social networks.

I. INTRODUCTION

Imagine that a new product will be released soon, and the manufacturer would like to know whether the new product will become popular and dominate the market. Suppose that a few early adopters are willing to try the new product or a few individuals in the social network are initially provided with free samples. They interact with other individuals who have older versions or alternatives. Through interactions, the new product may gradually spread in the network as more individuals adopt it because they can benefit more by switching to using the new product. Alternatively, it is possible that the spread of the new product eventually dies out because it has poor compatibility with the older versions and hence cannot bring more value to the adopters when the majority of the society still uses the older versions.

An example of new products’ dominating the market or dying out would be the smartphone operation systems in the past 15 years. Android and iOS were released in the years when Symbian, which was predominantly adopted by Nokia, was the most popular smartphone system, and they overtook Symbian and became dominant soon after their release. By contrast, there were other competitors like Windows Phone, which emerged during the same period but ended up being discontinued. Products like cell phones, cameras, and computers are likely to be used over a long term. As individuals frequently interact with others and update the choice of their product in conjunction with others, the long-term benefit obtained from consuming a product becomes more important than the short-term benefit that one can derive in isolation. That motivates individuals to give up some of their immediate gains in order to benefit in the long term. Therefore, from a managerial perspective, a good understanding of the consumers’ long-term behavior can help companies better promote their products over the socioeconomic network.

The above scenario is one specific example of the diffusion of innovation over socioeconomic networks, and one can consider many other applications such as epidemics spread in biological networks [1], information spread in computer networks [2], and belief propagation in social networks [3]. The diffusion of innovation (e.g., new product, virus, opinion, etc.) has been extensively studied in the past literature, where broadly speaking, the goal is to propagate certain types of innovations or behaviors in a desired way through the network [4]–[17]. In general, there are two main approaches to analyzing the diffusion of innovation: i) epidemic-based modeling, in which individuals adopt an innovation based on the intensity of being exposed to that innovation [7], [18], and ii) game-theoretic modeling, in which individuals make a decision about adopting alternative innovations in order to maximize their utility functions [11], [19]. In this work, we follow the second approach to analyze the behavior of diffusion dynamics under a new equilibrium concept.

In this paper, we focus on a game-theoretic model for diffusion dynamics. More precisely, we consider the diffusion of innovation based on the notion of limited-trust equilibrium (LTE) [20] over social networks. The LTE is a novel notion of equilibrium based on an individual’s rationality on short-term and long-term utilities, as opposed to the Nash equilibrium (NE), which solely accounts for an individual’s short-term utility. On the notion of LTE, each individual is willing to sacrifice some amount of instantaneous utility for its neighbors in order to get more long-term benefits when it is its neighbors’ turn (see Section II). In each iteration, the individuals play a coordination game with their neighbors and show a certain degree of trustworthiness when making a decision. The main contributions of this paper are as follows:

• Developing an LTE diffusion model as opposed to the
conventional NE dynamics;
• Analyzing the convergence rate and behavior of the LTE diffusion dynamics on random graphs using Markov chains and mean-field approximation;
• Establishing a connection between the LTE diffusion model and the standard linear-threshold model.

Simulation results show that the LTE diffusion dynamics result in desirable convergence compared to the NE diffusion dynamics in the following two ways. When the preferred innovation alternative is the risk-averse option, both LTE and NE diffusion dynamics converge to the preferred innovation option, and LTE diffusion dynamics converges faster than NE dynamics. When the preferred innovation alternative and the risk-averse alternative do not coincide, the NE diffusion dynamics converges to the risk-averse and undesirable option, whereas the LTE diffusion dynamics converges to the preferred innovation alternative or converges to the risk-averse option at a slower rate than the NE dynamics. The average utility by following LTE diffusion dynamics increases drastically compared to the case of NE diffusion dynamics, which reveals that the prevalence of innovation is a result of the individuals’ desire for better long-term utility provided that they behave trustworthy. We also justify the theoretical bounds obtained from our Markov chain analysis using extensive simulation results. The analytical results from the Markov chain in terms of probability of innovation domination and expected time to equilibrium match the simulation results reasonably well, especially when the underlying structure of the Markov chain is 1-dimensional.

A. Related Work

There has been a long literature on the evolution of diffusion dynamics over social networks [7], [18], [19]. It captures the essence of “viral marketing” which takes advantage of the power of “word-of-mouth” among the interactions of individuals in order to promote a product, an innovation, a belief, etc. The epidemic-based models are based on the exposure of individuals to the innovation or influence. The two most notable epidemic-based diffusion models are the independent cascade (IC) model and the linear threshold (LT) model [7]. In the IC model, each newly activated individual can activate each of its inactive neighbors with some probability; in the LT model, each agent has a threshold, and an inactive agent will become active if the sum of weights from its activated neighbors exceeds the threshold.

The LT and IC models are closely related to the influence maximization problem in which the goal is to choose a fixed-size set of initial adopters in a social network to maximize the spread of influence. For the LT and IC models, the seed selection problem can be formulated as a combinatorial optimization problem [7], [8]. For these models, it was shown in [7] that the objective function of the influence maximization problem is a submodular function of the initial seed set, and a hill-climbing greedy algorithm can be used to get a $(1 - 1/e)$-approximation of the optimal seed set. Subsequent studies directly related to the original LT and IC models are [8], [21], which extended the IC and LT models to a more general setting and generalized the results about submodularity. Some variants of the LT model include the non-progressive LT model in which an individual can switch between active and inactive states [13], [15] and the deterministic LT model where threshold values are input to the model [14]. An extension of the IC model is the multi-cascaded diffusion model, where multiple types of cascade spread over the social network [17].

Apart from the diffusion model itself, seed selection is a dedicated topic and the central problem for various diffusion models. Early work on seed selection is centrality-based heuristics, including degree and distance centrality heuristics [22]–[24] that associate the influence of a node to its degree or its distance from other nodes. [25] designed Bayesian evidence-based parallel cascade model to analyze the diffusion dynamics of two competing products and to study the optimal seed selection strategy based on such dynamics. The parallel cascade model assumes that an individual can be inactive, positive active, or negative active. [25] investigates the temporal aspects of the diffusion process based on the parallel cascade model, which helps decision-makers set up time horizons for short-term and long-term goals. A generalization of the seed selection problem is the seed activation scheduling [27], [28], in which the decision-makers are allowed to activate the seeds in a sequence as diffusion processes rather than activating the seeds all at once.

On the other hand, most game-theoretic diffusion models are based on utility maximization under the concept of Nash equilibrium (NE). In each interaction, an individual plays a coordination game with its neighbors and makes a rational choice by choosing the product that maximizes its instantaneous utility. [6] investigated the equilibrium emerging from the long-term behavior of the noisy best-response dynamics and concluded that the long-term equilibrium coincides with the risk-dominant equilibrium introduced by [29]. The work [9] further characterized the long-run behavior of the noisy best response dynamics and the convergence properties of the same process in terms of the geometry of the underlying network. More recent results in that direction include [11], [12], [16], which provided bounds on the expected convergence time of the noisy best response dynamics that are either dependent or independent of the network structure.

Most literature on the game-theoretic models relies on individuals being rational but myopic, i.e., they make decisions based on the short-term benefit that they can obtain from a single interaction (rather than the long-term benefit from future interactions over the long run). However, an innovative product may be used continually or repetitively over a long time; a rational individual cares more about the long-term benefit it can gain while using the product over the entire period. That motivates studying the diffusion process based on a different notion of rationality. [20] has recently introduced a new concept of equilibrium named limited-trust equilibrium (LTE), where each individual is willing to sacrifice a certain amount of short-term utility in order to get benefited more over the long run, provided that its limited sacrifice improves the social utility by at least that much. Results in [30] show that the utility of individuals can indeed increase if the agents adopt such trustworthy behavior. However, their result has not
been studied in the diffusion setting.

**B. Organization and Notation**

The paper is organized as follows. In Section II, we provide some preliminary results on LTE and conventional NE diffusion dynamics. In Section III we formally introduce the LTE diffusion model. In Section IV we transform the LTE diffusion model to a reduced-size Markov chain and characterize its convergence time and behavior. In Section V we make a connection between the LTE diffusion model and the notable linear-threshold diffusion model and show their equivalence under certain regimes. Simulation results are provided in Section VI which are followed by conclusions in Section VII.

**Notation:** We use bold fonts of lower-case letters to denote vectors (e.g., $x$) and subscript indices to denote their components (e.g., $x_i$). We use bold fonts of upper-case letters to denote matrices (e.g., $P$) and subscript indices to denote their entries (e.g., $P_{ij}$). For any positive integer $k$, we let $[k] := \{1, \ldots, k\}$.

**II. PRELIMINARIES**

**A. Limited-Trust Equilibrium**

Under the conventional Nash equilibrium (NE), each individual behaves selfishly. However, in many cases, individuals make choices that do not appear to benefit them in the short term, showing some degree of altruism with the hope of a similar return when it is their turn. For example, people show concern for others and their well-being. If each individual only considers itself and makes a decision selfishly, no one will care for others, and no one will get support because one cannot get benefited instantly by spending time listening to and encouraging others. The NE could not provide a reasonable explanation for this everyday phenomenon. The authors in [20] have introduced a new concept of equilibrium named limited-trust equilibrium (LTE). As opposed to the NE, where individuals play rationally to maximize their short-term utilities, under the LTE, an agent aims to maximize its long-term utility by trusting the others and giving up a certain amount of utility so that the others gain considerably more than that agent loses, hoping that the others will return that favor in the future. The LTE provides a new answer to such situations, where individuals invest a certain amount of short-term utility for future returns.

Formally, in a social network, each individual (agent) $i$ has a hard trust limit (or trust level) $\delta_i > 0$. Under the concept of LTE, player $i$ chooses the strategy that maximizes the social welfare among all the strategies whose utilities are at most $\delta_i$ less than the maximum utility that player $i$ can obtain. Consider a finite $n$-player game. Let $\Sigma_i$ denote the strategy set of player $i \in [n]$ and $\sigma_i \in \Sigma_i$ be a strategy. Denote by $\sigma = (\sigma_1, \ldots, \sigma_n)$ the strategy of all players. Given a player $i$, denote by $\sigma_{-i} = (\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_n)$ the strategy of the players other than $i$, and let $u_i(\sigma_i, \sigma_{-i})$ and $u(\sigma_i, \sigma_{-i}) = \sum_{i=1}^n u_i(\sigma_i, \sigma_{-i})$ be the payoff of player $i$ and the social welfare, respectively. The set $\sigma_i^G(\sigma_{-i}) \triangleq \arg\max_{\sigma_i \in \Sigma_i} u_i(\sigma_i, \sigma_{-i})$ is defined as the greedy best response of player $i$ given $\sigma_{-i}$. Let $\sigma_i^G \in \sigma_i^G(\sigma_{-i})$, and denote the trust limits of all the players by $\delta = (\delta_1, \ldots, \delta_n)$, where $\delta_i > 0, \forall i \in [n]$.  

**Definition 1.** A strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a limited-trust equilibrium (LTE) if and only if $u_i(\sigma_i^G, \sigma_{-i}) - u_i(\sigma_i, \sigma_{-i}) \leq \delta_i$ and $u(\sigma_i, \sigma_{-i}) \geq u(\sigma_i^G, \sigma_{-i})$ for all $i \in [n]$ and any $\sigma_i' \in \sigma_i^G : u_i(\sigma_i', \sigma_{-i}) - u_i(\sigma_i^G, \sigma_{-i}) \leq \delta_i$.

Therefore, given fixed strategy of other players $\sigma_{-i}$, the limited-trust best response of player $i$, denoted by $\sigma_i^G(\sigma_{-i})$, can be obtained by solving the following program:

$$\sigma_i^G(\sigma_{-i}) = \arg\max_{\sigma_i \in \Sigma_i} u_i(\sigma_i, \sigma_{-i})$$  \hspace{1cm} (P1)

s.t. $u_i(\sigma_i^G, \sigma_{-i}) - u_i(\sigma_i, \sigma_{-i}) \leq \delta_i$.

Like NE, one can show that LTE exists in every $n$-player finite game with positive trust limits. In particular, computation of LTE in general is PPAD-hard [20].

**B. Game-theoretic Model for Diffusion under NE**

A social network of $n$ players is represented by a weighted undirected graph $G = ([n], E)$ with edge weights $\{w_{ij}\}_{(i,j) \in E}$. Each vertex represents one player in the social network, and each edge represents the interaction between two players. For an edge $(i, j) \in E$, the interaction between $i$ and $j$ is symmetric. The weights of edges describe the probability, strength or importance of the interaction. Each player $i$ has two choices: $A$ and $B$, i.e., $\Sigma_i = \{A, B\}$. The state of player $i$ is denoted by $x_i \in \{A, B\}$, and the state of the process is denoted by $x = (x_1, \ldots, x_n) \in \{A, B\}^n$.

The utility for player $i$ of choosing $A$ and $B$ is composed of two components, an individual component, denoted by $v_i(x_i)$, representing the player’s private preference for $A$ or $B$ irrespective of other players’ choices, and a social component resulted from the externalities due to other players’ choices. The social component for $i$ is expressed as $\sum_{j \in N(i)} w_{ij} v(x_i, x_j)$, which is the sum of utilities from the interaction with each of $i$’s neighbors $N(i)$. Here, $v(x_i, x_j)$ is the payoff function of a two-person coordination game in which each player has two pure strategies $A$ and $B$. The payoff matrix of the coordination game is shown in Table [I]. It is assumed that conformity is always better than non-conformity. That is, in the interaction with each of its neighbors, the player gets higher payoff when it chooses the same product as the neighbor than when it chooses the different product. This implies that $a > d$ and $b > c$. Therefore, the (total) utility to agent $i$ in state $x$ equals $u_i(x) = \sum_{j \in N(i)} w_{ij} v(x_i, x_j) + v_i(x_i)$.

**Example 1.** Assume that the players are users of iPhone and Android phone in a social network. Suppose that $A$ is iPhone and $B$ is Android phone. Each player has its own preference for iPhone or Android phone, which is reflected in $v_i(x_i)$ for all $i \in [n]$. The externality due to the interaction between two users is embodied in $v(x_i, x_j)$. Most likely, each type of phone has better compatibility with the same type, resulting in larger utility when two users are in conformity.
TABLE I: Coordination game payoff matrix with $a > d$ and $b > c$.  

|    | $A$ | $B$ |
|----|-----|-----|
| $A$| $a, a$| $c, d$ |
| $B$| $d, c$| $b, b$ |

In fact, one can show that the above $n$-person game is a potential game with the potential function $[9]$

$$\Phi(x) = (a - d)w_{AA}(x) + (b - c)w_{BB}(x) + \sum_{i=1}^{n} v_i(x_i), \quad (1)$$

where $w_{AA}(x)$ is the sum of the weights on all edges $(i, j)$ such that $x_i = x_j = A$, and $w_{BB}(x)$ is defined in a similar way. Therefore, one can obtain a pure-strategy Nash equilibrium of this game by maximizing the potential function $[1]$. As maximizing $[1]$ in general could be a difficult task, a popular method for obtaining its NE points is to analyze the long-term behavior of certain natural diffusion dynamics that may arise due to repeated interactions of the players over the social network.

The diffusion dynamics is a continuous-time process in which each player is assumed to have an independent Poisson clock that rings once per unit time in expectation. A player updates its strategy every time its Poisson clock rings. This can be naturally extended to a discrete-time process in which, at each time instant, one player is randomly selected to update its strategy. When a player updates its strategy, instead of deterministically playing its best response strategy, the player is assumed to choose a strategy with probability described by the following logit function:

**NE Diffusion Model**

$$P(x_i = A|x_{-i}) = \frac{e^{\beta u_i(A, x_{-i})}}{e^{\beta u_i(A, x_{-i})} + e^{\beta u_i(B, x_{-i})}}, \quad (2)$$

where $\beta$ is a parameter measuring the sensitivity of the agent to payoff differences. As $\beta \to \infty$, the stochastic logit dynamics $[2]$ degenerates to deterministic best response dynamics. Hence, on the top of rationality, Eq. $[2]$ introduces randomness and stochasticity into the diffusion process.

As is shown in $[4], [9]$, following the stochastic diffusion process $[2]$, a long run diffusion over the game state $x$, which is concentrated on the states that maximize the potential function $[1]$, i.e., the pure NE points. In particular, if $v_i(A) = v_i(B)$ for all $i \in [n]$, i.e., everyone is indifferent between $A$ and $B$, the potential function $[1]$ reduces to

$$\Phi(x) = (a - d)w_{AA}(x) + (b - c)w_{BB}(x), \quad (1b)$$

and the diffusion process $[2]$ will converge to the risk-dominant equilibrium $[6], [9], [29]$. If $a - d > b - c$, all-$A$ state is the risk-dominant equilibrium, and $A$ is the risk-dominant strategy. If $a - d < b - c$, all-$B$ state and $B$ are the risk-dominant equilibrium and risk-dominant strategy, respectively. The risk-dominant equilibrium and strategy reflect the riskaverse behavior in face of uncertainty.

### C. Linear Threshold Model

One of the notable models to study the influence spread and epidemic is the linear threshold (LT) model introduced in $[7]$. The social network is represented by a directed graph $G = ([n], E)$. Each player, represented by a vertex, has two states: active and inactive. There are two types of LT model: progressive and non-progressive models. In the progressive model, once a player becomes active, it will remain active forever and cannot be inactive again; in the non-progressive model, each individual can switch back and forth between active and inactive states. The progressive model is adopted for the conventional LT model. Each player $i \in [n]$ is assigned a threshold $\theta_i$ drawn uniformly at random from $[0, 1]$. Each player is influenced by its (out) neighbors $j \in N^+(i)$ according to a weight $b_{ij}$ such that $\sum_{j \in N^+(i)} b_{ij} \leq 1$. At the beginning, an initial set of active players is chosen. In the diffusion process, at step $t = 1, 2, \ldots$, each player $i \in [n]$ examines the states of its neighbors and becomes active if $\sum_{j \in N^+(i)} b_{ij} \geq \theta_i$. Given $\theta_i$ for all $i \in [n]$ and initial set of active players, the diffusion progresses deterministically over the network until no more activation takes place.

### III. DIFFUSION DYNAMICS UNDER LIMITED-TRUST EQUILIBRIUM

We consider the diffusion of innovation on an undirected social network $G = ([n], E)$ with two competing products $A$ and $B$. As before, each vertex represents one player, and each edge represents an interconnection between two players. For $i \in [n]$, we let $N(i)$ be the set of neighbors of player $i$ and $d(i)$ be the degree of vertex $i$. We assume that $G$ is a simple graph such that it does not contain self-loops or parallel edges. This is a realistic assumption because individuals obtain information about a product from the people they know and get utility through their interaction with them.

The diffusion process takes place over a fixed time horizon $T$ and is modeled as a discrete-time process. During the process, each player can choose between two strategies: $A$ and $B$. At time instant $t \in [T]$, one player is randomly picked to update its strategy, in which case that player plays a coordination game with each of its neighbors. The payoff matrix of the coordination game is the same as the one for the diffusion model under NE as shown in Table $[1]$ where without loss of generality, we may assume $a, b, c, d > 0$. As in the diffusion model under NE, we assume conformity is always better than non-conformity, i.e., $a > d$ and $b > c$. Furthermore, we assume that $A$ is the innovation that brings about more utility, i.e., $a > b$. We can express the state of all players at time $t$ using an $n$-dimensional vector $x(t) \in \{A, B\}^n$. Note that this discrete-time process can be naturally extended to a continuous-time process in which each player has an independent Poisson clock with unit rate, and the player updates its strategy every time its Poisson clock rings.

In the conventional NE diffusion model $[9], [10], [12]$, an individual’s decision-making is based on the single-stage utility. The player will play myopically and tend to adopt the product that maximizes its instantaneous utility. However, as pointed out in $[20]$, such a decision process could not explain the phenomenon where individuals aim to maximize their long-term utility and make choices that do not benefit their short-term payoff. Therefore, we take this higher level of
rationality into consideration and set up the diffusion dynamics based on the concept of LTE. Unlike the conventional diffusion model where players play according to the NE, in our LTE diffusion model, each player takes into account both its utility and its neighbors’ utilities when it makes a decision and evaluates the gain and loss of both factors. When the cost of selecting some product that may not be the best one for the player in terms of utility but maximizes the social welfare can be afforded, the player will choose that product to benefit its neighbors and hope that its neighbors will return the favor to it in the same way in future.

More precisely, let \( x_i \in \{A, B\} \) denote the state of player \( i \), and denote by \( v(x_i, x_j) \) the payoff of \( i \) in the coordination game between \( i \) and its neighbor \( j \). The total utility of player \( i \) by playing \( x_i \) equals \( u_i(x_i, x_{-i}) = \sum_{j \in N(i)} v(x_i, x_j) \), and the social welfare under state \( x \), denoted by \( u(x) \), is given by

\[
u(x) = \sum_{i \in V} u_i(x) = \sum_{(i,j) \in E} [v(x_i, x_j) + v(x_j, x_i)].\]

Given trust levels \( \delta = (\delta_1, \ldots, \delta_n) > 0 \), let

\[
S_i(x_{-i}) = \left\{ x_i \in \{A, B\} : \max_{x_i' \in \{A, B\}} u_i(x_i', x_{-i}) - u_i(x_i, x_{-i}) \leq \delta_i \right\}
\]

denote the set of strategies whose utilities are at least the utility of the best response minus player \( i \)'s trust level. This is equivalent to finding the strategy set satisfying the constraint in [P1]. Since the strategy set of player \( i \) contains only two elements, the strategies in \( \arg\max_{x_i \in \{A, B\}} u_i(x_i, x_{-i}) \) are in \( S_i(x_{-i}) \), and whether the strategies in \( \arg\min_{x_i \in \{A, B\}} u_i(x_i, x_{-i}) \) are in \( S_i(x_{-i}) \) depends on the difference between \( \max_{x_i \in \{A, B\}} u_i(x_i, x_{-i}) \) and \( \min_{x_i \in \{A, B\}} u_i(x_i, x_{-i}) \). If \( \min_{x_i \in \{A, B\}} u_i(x_i, x_{-i}) \geq \max_{x_i \in \{A, B\}} u_i(x_i, x_{-i}) - \delta_i \), then both \( A \) and \( B \) are in \( S_i(x_{-i}) \), satisfying the limited-trust constraint in [P1]. As a result, both strategies are under player \( i \)'s consideration, and the player will choose whichever maximizes the social welfare. On the contrary, if \( \min_{x_i \in \{A, B\}} u_i(x_i, x_{-i}) < \max_{x_i \in \{A, B\}} u_i(x_i, x_{-i}) - \delta_i \), \( \arg\max_{x_i \in \{A, B\}} u_i(x_i, x_{-i}) \) contains only one strategy, and this strategy is the only choice for the player because the player will lose too much utility if it chooses the strategy not in \( \arg\max_{x_i \in \{A, B\}} u_i(x_i, x_{-i}) \) regardless of the social welfare. By introducing randomness into the model in a similar manner and adopting the logit dynamics described by [3] in the NE diffusion model, we can formally introduce the LTE diffusion model. Let us define

\[
W_i(x_{-i}) = \arg\max_{x_i \in \{A, B\}} u_i(x_i, x_{-i}),
\]
\[
U_i(x_{-i}) = \arg\max_{x_i \in \{A, B\}} u_i(x_i, x_{-i}),
\]
\[
u_i^*(x_{-i}) = \max_{x_i \in \{A, B\}} u_i(x_i, x_{-i}),
\]
\[
u_i'(x_{-i}) = \min_{x_i \in \{A, B\}} u_i(x_i, x_{-i}).
\]

Then, the LTE diffusion dynamics can be described by

LTE Diffusion Model

1) If \( W_i(x_{-i}) \cap U_i(x_{-i}) \neq \emptyset \),

\[
P(x_i \in W_i(x_{-i}) \cap U_i(x_{-i}) | x_{-i}) = 1,
\]

2) If \( W_i(x_{-i}) \cap U_i(x_{-i}) = \emptyset \),

a) \( u_i'(x_{-i}) \geq u_i^*(x_{-i}) - \delta_i \),

\[
P(x_i = A | x_{-i}) = \frac{e^{\beta u(A, x_{-i})}}{e^{\beta u(A, x_{-i})} + e^{\beta u(B, x_{-i})}},
\]

b) \( u_i'(x_{-i}) < u_i^*(x_{-i}) - \delta_i \),

\[
P(x_i = A | x_{-i}) = \frac{e^{\beta u(A, x_{-i})}}{e^{\beta u(A, x_{-i})} + e^{\beta u(B, x_{-i})}},
\]

where \( \beta' \) and \( \beta \) are the parameters measuring the sensitivity of players to the difference between social welfare and utility, respectively. A player who maximizes the social welfare benefits its neighbors and hence aims to gain more utility in the long run as its neighbors return the favor to it similarly. In this regard, social welfare embodies a player’s long-term utility. Therefore, \( \beta' \) reflects the sensitivity of a player to its long-term utility, whereas \( \beta \) reflects the sensitivity to the short-term utility as in the NE model. When the social welfare and utility maximizers coincide (Case 1), the player chooses the maximizing product deterministically because it is beneficial to both short-term and long-term utilities. When the social welfare and utility maximizers differ (Case 2), the player will compare the gain and the loss. As in the NE diffusion model, instead of choosing the best-response strategy deterministically, the players are allowed to “make a mistake” and choose the “worse” alternative with a small probability. The probability function for decision-making is softened by a logistic function. A player may attach different importance to long-term and short-term utilities, reflecting different sensitivity to social welfare and utility.

Simulation results are provided in Section VI to compare the diffusion process under NE and LTE dynamics. Simulations show that, on average, players gain more utility when they behave trustworthy, which results in a rapid spread of innovation over the network. In Appendix A, we provide several numerical examples for various range of parameters to illustrate the essence of LTE diffusion dynamics.

IV. MEAN FIELD APPROXIMATION FOR DIFFUSION PROCESS AND MARKOV CHAIN ANALYSIS

In this section, we study the following problem: given a certain number of initial adopters in the social network, what is the probability that the innovation dominates the network or dies out eventually? And how rapidly will this take place? These questions are important from a managerial perspective as they allow the companies to know how many free samples they have to distribute to the network such that their product prevails with high confidence. A reasonable decision is to distribute free samples to influential individuals with high connectivity with others. However, the companies often may not have a complete picture of the network’s structure, especially if they expand into new markets. Even if the network structure is fully known, it is computationally difficult to determine the most influential individuals on a large-scale network, hence
justifying the use of approximation schemes. Thus, we study the problem in a pessimistic way and assume that the network is random. We first introduce the mean-field approximation of diffusion on random graphs and then convert the diffusion process to the evolution of a reduced-size Markov chain. By analyzing such a Markov chain, we study the outcome and convergence rate of the diffusion dynamics to answer the above questions.

A. Mean Field Approximation

The mean-field theory provides a powerful tool to study the dynamics on random graphs represented by degree distribution with a large population \([31], [32]\). To describe our mean-field analysis, we partition the set of players into groups based on their degrees. All interactions are approximated by an average interaction and represented by the mean-field parameter, same for all vertices independent of their connectivity and position. Moreover, it is assumed that there is no spatial correlation among the players choosing \(A\) or \(B\) across time. Hence, based on the assumptions, given an arbitrary player with degree \(k\), each of its neighbors is in state \(A\) with probability \(\theta\), and the number of neighbors in state \(A\) follows a binomial distribution \(\text{Bin}(k, \theta)\).

Suppose the network is represented by a random graph with \(n\) vertices. Without loss of generality, suppose that the graph is connected; otherwise, we can analyze each component separately. Let \((P(k))_{k=1}^{K}\) denote the degree distribution of the network where \(K\) is the maximum degree of the graph. Let \(\langle k \rangle = \sum_{k=1}^{K} k P(k)\) denote the average degree of graph \(G\). Denote by \(f(k)\) the proportion of players with degree \(k\) in state \(A\). Define the mean-field parameter \(\theta\) as follows:

\[
\theta = \frac{1}{\langle k \rangle} \sum_{k=1}^{K} k P(k) f(k).
\]

\(\theta\) is the probability that a given link is incident to a player in state \(A\). It also approximates all interactions in the network. Given a player with degree \(k\) and trust limit \(\delta\), define the normalized trust limit

\[
\delta' = \frac{\delta}{k},
\]

which represents the average trust level that the player distributes to each of its neighbors. It is assumed that \(\delta'\) is the same for all players. This is a realistic assumption as players with high social connectivity have higher tolerance of sacrifice of short-term utility than the players with low social connectivity.

Given the payoff matrix, a player with degree \(k\), \(k_A\) neighbors in state \(A\) and normalized trust limit \(\delta'\), the utility and social welfare can be computed, and hence the probability that the player chooses \(A\) can be calculated using the NE diffusion model [Eq. (2)] or the LTE diffusion model [Eqs. (3)–(5)] as stated in Section III. Denote this probability by \(P(A|k, k_A, \delta')\), where we recall that \(k_A \sim \text{Bin}(k, \theta)\). If we denote by \(Q(A|k, \theta, \delta')\) the overall probability that a player with degree \(k\) and normalized trust limit \(\delta'\) chooses \(A\), we can compute \(Q(A|k, \theta, \delta')\) by taking the expectation of \(P(A|k, k_A, \delta')\) with respect to \(k_A\) to get

\[
Q(A|k, \theta, \delta') = \mathbb{E}_{k_A \sim \text{Bin}(k, \theta)}[P(A|k, k_A, \delta')]
= \sum_{k_A=0}^{k} P(A|k, k_A, \delta') \frac{k}{k_A} \theta^{k_A} (1 - \theta)^{(k - k_A)}.
\]

(8)

As the probability term \(P(A|k, k_A, \delta')\) embodies diffusion dynamics and the binomial term \(\frac{k}{k_A} \theta^{k_A} (1 - \theta)^{(k - k_A)}\) embodies the network structure, Eq. (8) connects the diffusion model and the social network in a convenient fashion.

B. Markov Chain for Diffusion Process

Based on the mean-field approximation, the diffusion process can be modeled as a discrete-time Markov chain. Given a degree distribution \((P(k))_{k=1}^{K}\) and the number of vertices \(n\) of the graph, one can compute the degree sequence. Let \(N = (N_1, N_2, \ldots, N_K)\) be a \(K\)-dimensional tuple such that \(N_k\) is the number of vertices of degree \(k\), i.e., \(N_k = n P(k)\). Let \(S = \{0, 1, \ldots, N_1\} \times \{0, 1, \ldots, N_2\} \times \cdots \times \{0, 1, \ldots, N_K\}\), and define \(y = (y_1, y_2, \ldots, y_K)\), where \(y_k\) is the number of players with degree \(k\) in state \(A\). Clearly, \(S\) is the feasible set for \(y\). One can construct a Markov chain with state space \(S\) and states \(y \in S\). As \(y_k \in \{0, 1, \ldots, N_k\}, \forall k \in [K]\), the size of state space of the above Markov chain equals \(|S| = \prod_{k=1}^{K} (n_k + 1)\), and thus \(\Omega(n) \leq |S| \leq O(2^n)\) where \(\Omega\) and \(O\) are the notations for asymptotic lower and upper bounds respectively.

Given a state \(y = (y_1, \ldots, y_K)\), there are at most \((2K + 1)\) transitions from \(y\). At most \(K\) of them are forward transitions: 

\[
(y_1, \ldots, y_k, \ldots, y_K) \rightarrow (y_1, \ldots, y_{k-1}, y_k + 1, y_{k+1}, \ldots, y_K)
\]

for some \(k \in [K]\); at most \(K\) of them are backward transitions: 

\[
(y_1, \ldots, y_k, \ldots, y_K) \rightarrow (y_1, \ldots, y_{k-1}, y_k - 1, y_{k+1}, \ldots, y_K)
\]

for some \(k \in [K]\); one is a self-loop \((y_1, \ldots, y_k) \rightarrow (y_1, \ldots, y_K)\). A forward transition implies that one more player adopts the innovation \(A\), whereas a backward transition implies that one fewer player adopts the innovation \(A\). Using the mean-field approximation, one can derive the transition probabilities between different states of the Markov chain as:

\[
P((y_1, \ldots, y_K), (y_1, \ldots, y_{k-1}, y_k + 1, y_{k+1}, \ldots, y_K))
= \frac{N_k - y_k}{n} Q(A|k, \theta(y), \delta')
\]

(9)

\[
P((y_1, \ldots, y_K), (y_1, \ldots, y_{k-1}, y_k - 1, y_{k+1}, \ldots, y_K))
= \frac{y_k}{n} (1 - Q(A|k, \theta(y), \delta'))
\]

(10)

\[
P((y_1, \ldots, y_K), (y_1, \ldots, y_K))
= 1 - \frac{N_k - 2y_k}{n} Q(A|k, \theta(y), \delta') - \frac{y_k}{n}
\]

(11)

where the mean-field parameter \(\theta\) is now a function of state \(y\) and can be calculated as

\[
\theta(y) = \sum_{k=1}^{K} k y_k \frac{n}{\langle k \rangle}.
\]

(12)

Combining Eqs. (8)–(12), one can characterize the one-step transition probability matrix of the Markov chain. The Markov chain can be viewed as a random walk with heterogeneous
transition probabilities on a $K$-dimensional lattice graph with the length of the $k$th dimension being $(N_k + 1)$. This lattice graph is named the underlying graph of the Markov chain. A Markov chain is $k$-dimensional if its underlying graph is $k$-dimensional.

Two properties of the diffusion process are of interest: the probability of innovation domination and the expected time to equilibrium. The first one answers the question of how likely is the innovation to dominate the social network, and the second one answers the question of how fast, on average, the innovation dominates the market or dies out. We assume that the diffusion process terminates once either $A$ or $B$ completely dominates the network. In this regard, there are two absorbing states in the above Markov chain: the state $y = 0$ and the state $y = (N_1, \ldots, N_K)$. All other states are transient. Hence, given an arbitrary initial state, the Markov chain will eventually be absorbed into one of the absorbing states; that is, all individuals in the social network will adopt either $A$ or $B$. The absorption probability and the expected time to absorption are two important properties of absorbing Markov chains. The absorption probability to the innovation-dominating state $y = (N_1, \ldots, N_K)$ (i.e., all-$A$ state) reflects the probability of innovation domination; the expected time to absorption corresponds to the expected time to reach an equilibrium.

The absorption probability and the expected time to absorption can be calculated by solving a system of linear equations. Let $P$ denote the transition probability matrix of the Markov chain, and $Q$ and $R$ be the submatrices of $P$ which represent the transition among transients states and from transient states to absorbing states, respectively. The absorption probability $B$ and expected time to absorption $\pi$ can be calculated by solving $B = QB + R$ and $\pi = Q\pi + 1$. In Section VI, we present the results of solving these two systems of equations and the simulation results on regular random graphs to show that the mean-field approximation estimates the convergence property of the LTE diffusion dynamics reasonably well.

An important special case is when the underlying social network is a $k$-regular graph, in which case the above Markov chain becomes a random walk on the path and reduces to a 1-dimensional Markov chain. The state space is $S = \{0, 1, \ldots, n\}$, and each state $y \in S$ represents the number of innovation-adopting agents in the network. Given an arbitrary state $y \in S$, the mean-field parameter is $\theta(y) = y/n$. Absorption probability and expected time to absorption for such Markov chain can be characterized in a close form as follows. Let $a_y = P[y, y + 1]$ be the forward transition probability of the 1-dimensional Markov chain, and $b_y = P[y, y - 1]$ be the backward transition probability. For simplicity, we define $a_0 = b_0 = a_n = b_n = 0$ since states 0 and $n$ are absorbing states. The one-step transition probability matrix $P \in \mathbb{R}^{(n+1) \times (n+1)}$ can be represented as follows:

$$P_{i,j} = \begin{cases} a_i, & \text{if } j = i + 1; \\ b_i, & \text{if } j = i - 1; \\ 1 - a_i - b_i, & \text{if } i = j; \\ 0, & \text{otherwise.} \end{cases} \quad (13)$$

Denote by $P^n_0$ the probability that the chain is absorbed in state 0 given the initial state $y$ and by $P^n_y$ the probability that the chain is absorbed in state $n$ given the initial state $y$. Also, let $\tau_y$ be the expected time that the chain is absorbed given the initial state $y$. The following theorem gives a characterization of $P^n_0$, $P^n_y$, and $\tau_y$, the proof of which is in Appendix B.

**Theorem 1.** Let

$$\Delta = \sum_{i=1}^n \prod_{j=i}^{n-1} a_j \prod_{j=1}^{i-1} b_j. \quad (14)$$

Then $P^n_0$, $P^n_y$, and $\tau_y$ can be characterized by the equations:

$$P^n_y = \frac{1}{\Delta} \sum_{i=1}^n \prod_{j=i}^{n-1} a_j \prod_{j=1}^{i-1} b_j, \quad (15)$$

$$P^n_0 = 1 - P^n_y = \frac{1}{\Delta} \sum_{i=y+1}^n \prod_{j=i}^{n-1} a_j \prod_{j=1}^{i-1} b_j,$$

$$\pi_y = \frac{1}{\Delta} \left[ \sum_{j=1}^{n-1} \left( \sum_{l=k}^{\min(y,j)} \prod_{i=1}^{l-1} a_i \prod_{i=l+1}^{k-1} b_i \right) \right. \left( \sum_{k=\max(y,j)+1}^{n-1} \prod_{i=1}^{k-1} a_i \prod_{i=k+1}^{n} b_i \right). \quad (16)$$

**Remark 1.** For non-regular graphs, one cannot expect to find a closed-form expression for the absorption probability and the expected absorption time as, in general, they involve an exponential number of terms. Nevertheless, using the same approach to solving a system of linear equations $B = QB + R$ and $\pi = Q\pi + 1$, one can find these quantities for general chains in polynomial time in terms of the problem parameters.

One application of the Markov chain analysis for the LTE diffusion model is to address the seed placement problem. On the one hand, given an allocation of “free samples” to the players based on their degree, which is equivalent to the initial state of the Markov chain, one can compute the absorption probability starting from that initial state to estimate the likelihood of innovation domination. On the other hand, given a desired confidence level $\alpha$, one can compute how many initial seeds (adopters) are needed and how the free samples are allocated among the players by looking up the initial states in $B$. In a 1-dimensional Markov chain, the initial states from which the Markov chain is absorbed in state $n$ (all-$A$ state) with a probability of at least $\alpha$ are given by

$$\{y \in \{0, 1, \ldots, n\} : P^n_y \geq \alpha\}, \quad (17)$$

where $P^n_y$ is the probability that the chain is absorbed in state $n$ given initial state $y$, the expression of which is given by Eq. (14). To find the minimum number of free samples, one can find the minimum value of $y$ that belongs to the set (17). In a high-dimensional Markov chain, not only the number of free samples, but also the distribution among the players with different degrees affects the absorption probability. Denote by $P^n_{y'}$ the absorption probability to the state $(N_1, \ldots, N_K)$ starting from an arbitrary state $y = (y_1, \ldots, y_K) \in S$. Recall that $S$ is the state space and $S = \{0, 1, \ldots, N_1\}$.
{0, 1, \cdots, N_2} \times \cdots \times \{0, 1, \cdots, N_K\}$. In that case, the initial states from which the Markov chain is absorbed in all-$A$ state $(N_1, \cdots, N_K)$ with probability at least $\alpha$ are given by \( \{y \in S : P_y^N \geq \alpha\} \).

V. Threshold Diffusion Model under Limited-trust Equilibrium

As we saw in Section III the LT model is a well-studied diffusion model for influence maximization over social networks. In this section, we will show that the LTE diffusion model can be converted to a non-progressive LT model if the players update their states deterministically by following the best-response rule. Another advantage of such reduction to the LT model is that it allows us to study the convergence behavior of the LTE dynamics under certain assumptions using the rich set of results on the LT model.

A. Threshold LTE Model

Let us consider the case where the players strictly follow the best-response strategy on the notion of LTE. That being said, there is no randomness in the diffusion model. Using the same notations as in Section III the best-response LTE (BR-LTE) diffusion model can be described as follows:

**BR-LTE Diffusion Model**

a) $u_i'(x_{-i}) \geq u_i^*(x_{-i}) - \delta_i$,

\[
P(x_i \in W_i(x_{-i})|x_{-i}) = 1; \tag{18}
\]

b) $u_i'(x_{-i}) < u_i^*(x_{-i}) - \delta_i$,

\[
P(x_i \in U_i(x_{-i})|x_{-i}) = 1. \tag{19}
\]

where $u_i'(x_{-i})$, $u_i^*(x_{-i})$, $W_i(x_{-i})$ and $U_i(x_{-i})$ are defined as in Section III. In fact, the BR-LTE diffusion model is equivalent to the LTE diffusion model by setting $\beta \rightarrow \infty$ and $\beta' \rightarrow \infty$ such that the randomness represented by $1/\beta$ and $1/\beta'$ approaches zero.

The BR-LTE diffusion model can be converted to a non-progressive LT (NP-LT) model. In an NP-LT model, an active player is allowed to become inactive again if the proportion of the active neighbors is less than its threshold. This conversion establishes a connection between the epidemic and the game-theoretic diffusion models. Let product $A$ be the active state and product $B$ be the inactive state. We use the notations of $k^i$ and $k^i_A$ for the number of neighbors and the number of neighbors in state $A$ of player $i$, and $k^i_A/k^i$ to represent the proportion of the player’s neighbors in state $A$. Define $q^*_i$ as the threshold that player $i$ becomes active. If $k^i_A/k^i \geq q^*_i$, player $i$ will adopt $A$ (becomes active); if $k^i_A/k^i < q^*_i$, player $i$ will adopt $B$ (becomes inactive). $q^*_i$ depends on the utility of the products, which is embodied in the payoff matrix, and the player’s normalized trust limit $\delta_i'$. In general, the higher utility the innovation brings about to the player and its neighbors, the smaller the threshold will be; the more trustworthy the player is, the lower threshold it will have. Therefore, the BR-LTE model is equivalent to the following LT model named linear threshold LTE (LT-LTE) diffusion model:

**LT-LTE Diffusion Model**

\[
P(x_i = A|x_{-i}) = \begin{cases} 0 & \text{if } k^i_A/k^i < q^*_i \\ 1 & \text{if } k^i_A/k^i \geq q^*_i \end{cases} \tag{20}
\]

where

\[
q^*_i = \min \left\{ \max \left\{ 0, \frac{2b - c - d}{2(a + b - c - d)} \right\}, \frac{b - c + \delta'_i}{a + b - c - d} \right\}. \tag{21}
\]

Now, let us define

\[
q_w = \max \left\{ 0, \frac{2b - c - d}{2(a + b - c - d)} \right\}, \\
q_a = \frac{b - c}{a + b - c - d}, \\
\delta'_i = \frac{\delta'_i}{a + b - c - d}.
\]

Here, $q_w$ represents the proportion of neighbors in state $A$ such that $A$ and $B$ yield same social welfare. Similarly, $q_a$ represents the proportion of neighbors in state $A$ such that $A$ and $B$ yield same utility to player. Moreover, $\delta'$ represents the relative normalized trust limit of player $i$ with respect to the payoff matrix. Then, the threshold $q^*_i$ can be simplified as

\[
q^*_i = \min \left\{ \max \left\{ q_w, q_a - \delta'_i \right\}, q_a + \delta'_i \right\}. \tag{21a}
\]

It is easy to see that $q^*_i$ is monotonically nonincreasing with respect to $\delta'_i$ or $\delta_i'$, implying that players with high normalized trust limit have thresholds not greater than the players with low normalized trust limit. For all payoff matrices, the range of $q_w$ is $[0, 1/2)$, and the range of $q_a$ is $(0, 1)$. If $q_w < q_a$, $q^*_i \in [q_w, q_a)$ for all $\delta'_i > 0$; if $q_w > q_a$, $q^*_i \in \{q_w, q_a\}$ for all $\delta'_i > 0$; if $q_w = q_a$, $q^*_i$ is a constant and $q^*_i = q_w$ independent of the player’s trust limit. Note that in the BR-LTE diffusion model, one needs the information about the payoff matrix, the player’s trust limit, and the number of its neighbors to compute the best-response strategy numerically. In the LT-LTE model, this information is encoded in a single scalar parameter $q^*_i$.

B. Markov Chain with LT-LTE Model

The LT-LTE model and the probability function in Eq. (20) can be adopted for the Markov chain introduced in Section IV-B. The expression of $Q(A|k, \theta, \delta')$ can be reduced to the cumulative distribution function (CDF) of a binomial distribution by combining Eqs. (9) and (20) as

\[
Q(A|k, \theta, \delta') = \sum_{k_A=0}^{k} P(A|k, k_A, \delta') \left( \binom{k}{k_A} \theta^{k_A} (1-\theta)^{k-k_A} \right) \\
= \sum_{k_A = [q^* k]}^{k} \left( \binom{k}{k_A} \theta^{k_A} (1-\theta)^{k-k_A} \right) \\
= 1 - F_{q^* k, \theta}(k). \tag{22}
\]
Hence, Eqs. (9)–(11) can be rewritten as

\[
P[(y_1, \cdots, y_K), (y_1, \cdots, y_K + 1, y_{k+1}, \cdots, y_K)] = \frac{N_k - y_k}{n} \left(1 - F_{Y_{k,y}}(\lceil q^*k \rceil - 1)\right),
\]

\[
P[(y_1, \cdots, y_K), (y_1, \cdots, y_K + 1, y_{k+1}, \cdots, y_K)] = \frac{y_k}{n} F_{Y_{k,y}}(\lceil q^*k \rceil - 1),
\]

\[
P[(y_1, \cdots, y_K), (y_1, \cdots, y_K + 1, y_{k+1}, \cdots, y_K)] = 1 + \frac{N_k - 2y_k}{n} F_{Y_{k,y}}(\lceil q^*k \rceil - 1) - \frac{N_k - y_k}{n},
\]

where

\[
Y_{k,y} \sim Bin\left(k, \frac{\sum_{i=1}^{K} i y_i}{\sum_{i=1}^{K} i N_i}\right).
\]

In particular, for a \( k \)-regular graph, the transition probability functions can be calculated in a simpler form as

\[
P[y, y + 1] = \left(1 - \frac{y}{n}\right) \left(1 - F_{Y_{k,y} \sim Bin(k, \frac{y}{n})}(\lceil q^*k \rceil - 1)\right),
\]

\[
P[y, y - 1] = \frac{y}{n} F_{Y_{k,y} \sim Bin(k, \frac{y}{n})}(\lceil q^*k \rceil - 1),
\]

\[
P[y, y] = \left(1 - \frac{2y}{n}\right) F_{Y_{k,y} \sim Bin(k, \frac{y}{n})}(\lceil q^*k \rceil - 1) - \frac{y}{n}.
\]

For non-regular graphs, the Markov chain model is a lazy walk on a high-dimensional lattice graph. The dimension of the lattice can be at most \( K \), which is the maximum degree of the graph. The number of system states can be exponentially large. However, one can compress the \( K \)-dimensional lattice to a one-dimensional path and approximate the interaction using the one-dimensional Markov chain by assuming that at any state, all degree groups have the same proportion of players in state \( A \). Suppose that the total number of players in state \( A \) is \( y \). For all degree groups, the proportion of players in state \( A \) is \( y/n \). Under this assumption, it is sufficient to describe the state space as \( S = \{0, 1, \cdots, n\} \). A state \( y \in S \) represents the total number of players in state \( A \). As a consequence, at state \( y \in S \), the mean-field parameter is \( \theta(y) = y/n \). Since the heterogeneity of degree is preserved, the transition probabilities become

\[
P[y, y + 1] = \left(1 - \frac{y}{n}\right) \sum_{k=1}^{K} P(k) \left(1 - F_{Y_{k,y} \sim Bin(k, \frac{y}{n})}(\lceil q^*k \rceil - 1)\right),
\]

\[
P[y, y - 1] = \frac{y}{n} \sum_{k=1}^{K} P(k) F_{Y_{k,y} \sim Bin(k, \frac{y}{n})}(\lceil q^*k \rceil - 1),
\]

\[
P[y, y] = \frac{y}{n} + \left(1 - \frac{2y}{n}\right) \sum_{k=1}^{K} P(k) \left(1 - F_{Y_{k,y} \sim Bin(k, \frac{y}{n})}(\lceil q^*k \rceil - 1)\right),
\]

and the results for 1-dimensional Markov chain analysis in Theorem 1 will apply.

VI. SIMULATION RESULTS

In this section, we first provide simulation results to compare the evolution of NE and LTE diffusion dynamics on a small-world network. Then, we present some simulations to compare the results obtained from Markov chain analysis and the diffusion dynamics on regular random graphs.

First, we compare the diffusion process under NE and LTE models. We simulate the diffusion of innovation with payoff matrices in Table IIa and IIb. The sensitivity parameters are \( \beta = \beta' = 1 \). A randomly-generated Watts-Strogatz small-world network with 100 players is used. Each player is connected to 4 other agents, and the rewiring probability is 0.1. The small-world network is visually represented by Fig. 1. We consider various values of \( \delta' \) from 0.25 to 2 and assume that each player has the same \( \delta' \). We also consider various numbers of initial adopters of innovation (5%, 10%, 20%, and 50%). For each setting, the average result of 100 trials is taken. Fig. 2 shows the average proportion of innovation adoption as diffusion progresses; Fig. 3 shows the players’ average utility versus diffusion time.

As shown in Figs. 2a and 3a under the NE dynamics, the innovation dies out regardless of the number of initial adopters, because \( B \) is the risk-dominant strategy (\( b - c = 2 > 1 = a - d \)). This may happen when the new product \( A \) has poor compatibility with product \( B \). As the players start to behave trustworthily, though product \( B \) still prevails when \( \delta' \) is small, the convergence rate is much slower than the NE dynamics. As the players increase their trust limit, the innovation starts to dominate, and the diffusion converges to the all-\( A \) state. When \( \delta' \) becomes large, the innovation rapidly spreads over the network. By contrast, in Figs. 2b and 3b under the NE diffusion dynamics, the innovation gradually spreads over the social network since \( A \) is the risk-averse product according to the payoff matrix in Table IIb. Under the LTE diffusion dynamics, the innovation spreads faster than the case of NE dynamics, provided that there are a few initial adopters existing in the social network at the beginning. For both payoff matrices, no matter whether the innovation finally dominates the social network or dies out, the LTE model...
ensures a higher proportion of innovation adoption than the NE model, and the difference becomes larger as the players’ trust limit increases in a certain range.

The results from Fig. 3 show that when players behave trustworthy, the average utility is always greater than the case when players are solely self-interested as diffusion progresses. This difference becomes even more prominent when the trust limit increases. In Fig. 3a and for the payoff matrix in Table IIa, during the diffusion, the average utility increases when players show moderate to large trustworthiness (for example, when $\delta' = 1$ and 2), whereas it decreases when players are completely selfish (NE). In Fig. 3b and for the payoff matrix in Table IIb, the average utility keeps increasing for both NE and LTE models. Compared to the NE model, under the LTE dynamics, the average utility increases significantly when players show a small amount of trustworthiness (compare the curves for the NE model and for the LTE model with $\delta' = 0.25$). Compared to Fig. 2, these results suggest that the prevalence of innovation is the result of the players’ incentive to pursue higher long-term utility in a dynamic game setting.

Next, we present the simulation results of diffusion on random graphs and compare them with the results from the Markov chain analysis. We consider two types of random graphs: (a) regular graphs, which corresponds to a 1-dimensional Markov chain, and (b) random graphs with two types of degree, which correspond to a 2-dimensional Markov chain. Firstly, we simulate the diffusion on a random 4-regular graph with 100 players. The payoff matrix in Table IIa is used, and various numbers of initial adopters (5%, 10%, 20%, and 50%) and normalized trust limits ($\delta' = 0.25, 1$ and 2) are used. The sensitivity parameters are $\beta = \beta' = 2$. For each set of parameters, 100 trials are carried out, and a new random graph is generated in each trial. The average result of 100 trials is taken. The results are shown in Fig. 4.

In Fig. 4, each color represents a set of three curves under the same $\delta'$. The curve in the solid line represents the diffusion simulation result; the horizontal dashed line and the vertical dotted line represent the absorption probability to the innovation-dominating state (all-A state) and the expected time to absorption computed from the Markov chain analysis. Figure 4 shows that the results from the Markov chain analysis accord with the simulation results. The diffusion (solid lines) converges to the equilibrium state indicated by the absorption probability (horizontal dashed lines), and equilibrium is roughly achieved at the expected time to absorption (vertical dotted lines) as the diffusion curves become flat. This shows that the 1-dimensional Markov chain model gives a good estimation of the convergence property of the LTE dynamics on random regular graphs.

Next, we simulate the diffusion on random graphs with two types of degrees: 2 and 4. The corresponding Markov chain of such a random graph is 2-dimensional. There are 100 players in the network: 50 players with degree 2 and 50 players with degree 4. The payoff matrix in Table IIa is used. We consider the number of initial seeds to be 5 and 10. The free samples are allocated either completely to the players with degree 2 or completely to the players with degree 4. Various values of normalized trust limit ($\delta' = 0.25, 1$ and 2) are used. The sensitivity parameters are $\beta = \beta' = 2$. For each set of parameters, 100 trials are carried out, and a new random graph

![Fig. 2: Simulation results of diffusion on Watts-Strogatz small-world network: average proportion of innovation adoption vs. time.](image)

![Fig. 3: Simulation results of diffusion on Watts-Strogatz small-world network: average utility vs. time.](image)
is generated in each trial. The average result of 100 trials is taken. The results are shown in Fig. 5.

In Fig. 5, the diffusion simulation result, absorption probability to innovation-dominating state, and expected time to absorption are plotted in solid, dashed, and dotted lines, respectively. Each color represents a value of $\delta'$. We can see from the results that when the absorption probability to the innovation-dominating state (all-$A$ state) is close to 0 or 1 from Markov chain analysis (for example, when $\delta' = 0.25$ and 2), the Markov chain analysis matches the simulation result well. When the confidence level of innovation dominating the social network or dying out is not high (for example, when $\delta' = 1$), the Markov chain analysis gives an intermediate value of absorption probability to the innovation-dominating state), the prediction of the probability of innovation domination is less accurate. The results reveal that the Markov chain analysis overestimates the probability of innovation domination approximately by a factor of 2. However, the Markov chain is still beneficial considering the simple construction and accurate prediction of high-confidence levels about innovation domination or dying out.

VII. CONCLUSION

In this paper, we considered the diffusion of innovation in social networks. Each individual has a trust limit and behaves trustworthy towards its neighbors for long-term benefit. In each iteration, an individual plays a coordination game with its neighbors based on the notion of limited-trust equilibrium and makes a choice that maximizes its long-term utility. An LTE diffusion model is introduced to model the diffusion dynamics based on this new level of rationality.

To study whether the innovation prevails or dies out, the diffusion model is transformed into a reduced-size Markov chain based on mean-field approximation. The Markov chain represents a random walk on a lattice graph with heterogeneous transition probabilities and two absorbing states. The probability of innovation domination and expected time to equilibrium are represented by the absorption probability to the innovation-dominating state and the expected time to absorption of the absorbing Markov chain and can be computed by solving a system of linear equations. In particular, for regular graphs, a closed-form expression can be obtained. The best-response LTE diffusion model can be transformed into a linear threshold diffusion model, and therefore, a connection between game-theoretic and epidemic diffusion models is established. Simulation results show that when players behave trustworthy, their long-term utility is significantly greater than the case when they are solely self-interested. The incentive for long-term utility also drives the innovation to rapidly spread over the network when their trustworthiness is large. Further simulation results show that the Markov chain analysis estimates the convergence property of the original LTE dynamics reasonably well.

The Markov chain analysis is useful from a managerial perspective. On the one hand, it is helpful for the innovation companies to estimate the probability that the innovation dominates the social network or dies out and to estimate the time frame that such dominance or dying-out takes place. It is easy to use the Markov chain analysis to check whether the seed selection strategy can enable the innovation to spread over the social network with high confidence when the company has limited information about the social network structure. On the other hand, given the desired confidence level that the innovation dominates the social network, the Markov chain analysis can be used to design seed selection strategies. The simple form of the Markov chain analysis makes them useful for the companies to carry out a first-step estimation and strategy design.

This work is the first one to study the diffusion process under the novel notion of LTE rationality, and it opens several interesting future directions. For instance, how will diffusion progress when only a portion of agents behave trustworthy? or how can one obtain a high-probability bound for the expected convergence time of the LTE diffusion dynamics to their equilibrium points?

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APPENDIX A
NUMERICAL EXAMPLES

A. Numerical Examples for LTE Diffusion Model

Consider an arbitrary player $i \in [n]$. Let $k$ be the number of neighbors of $i$, and $k_A$ be the number of neighbors of $i$ in state $A$. Using the payoff matrix in Table IX, we can rewrite Eqs. (2) and (3) as follows:

$$P(x_i = A|x_{-i}) = \frac{e^{\beta[k_A(a-d)+(k-k_A)(c-b)]}}{e^{[k_A(a-d)+(k-k_A)(c-b)]}+1}.$$  \hspace{1cm} (27)

Eqs. (4):

$$P(x_i = A|x_{-i}) = \frac{e^{\beta'[k_A(2a-c-d)+(k-k_A)(c+d-2b)]}}{e^{[k_A(2a-c-d)+(k-k_A)(c+d-2b)]}+1}.$$ \hspace{1cm} (28)

As shown in (27) and (28), the structure of payoff matrix plays a significant role in the diffusion process. In this subsection, we consider a few special cases of the payoff matrix. We assume that the player has the same sensitivity to utility and social welfare, i.e., $\beta = \beta'$.

\textbf{Case 1:} Consider a player with $k$ neighbors in a network. Suppose that initially the player and its neighbors all have product $B$. If the player deviates from its neighbors and purchases $A$, it will suffer a loss of $k(b-c)$. Under the NE diffusion model, the probability that the player changes its state from $B$ to $A$ is small. However, if by doing so its neighbors’ utility increases, under the LTE diffusion model, the player will be incentivized to purchase $A$ provided that the decrease in utility is within its trust limit. If the decrease in utility is beyond its trust limit, the player will play according to NE, and the state-updating probability is the same as the NE diffusion model. Therefore, in this case, the probability that the player changes its state to $A$ under the LTE diffusion model is at least as large as that under the NE diffusion model. Such payoff matrix has a structure that $c + d > 2b$, resulting in a property that for any player in the network, choosing $A$ always yields higher social welfare regardless of the number and the states of its neighbors. This is because, along any edge connecting two interacting players, either player adopting $A$ yields a higher sum of utilities regardless of the other player’s state. The payoff matrix with $c + d > 2b$ is called payoff matrix without social welfare gap as any player’s deviation from $B$ to $A$ results in an increase in social welfare regardless of the others’ states. An example of the payoff matrix without social welfare gap is shown in Table IIIa.

When there is no social welfare gap in the payoff matrix, given any state of the system, the probability that a player chooses product $A$ under the LTE diffusion model is at least as large as that under the NE diffusion model. A brief analysis is provided in the following, where subscripts NE and LTE are used to distinguish the state-updating probability under the NE and LTE diffusion models, respectively.

1) If product $A$ maximizes both social welfare and utility:

$$1 = P_{LTE}(x_i = A|x_{-i}) > P_{NE}(x_i = A|x_{-i})$$

2) If $A$ maximizes social welfare and $B$ maximizes utility:

| $k_A$ | $P_{LTE}(x_i = A|x_{-i})$ | $P_{NE}(x_i = A|x_{-i})$ |
|-------|----------------------------|---------------------------|
| 0     | $0.0034$                   | $0.0034$                  |
| 1     | $0.00669$                  | $0.00669$                 |
| 2     | $0.11920$                  | $0.11920$                 |
| 3     | $0.73106$                  | $1.00$                    |
| 4     | $0.98201$                  | $1.00$                    |

| $P_{LTE}(x_i = A|x_{-i})$ | $P_{NE}(x_i = A|x_{-i})$ |
|---------------------------|---------------------------|
| 0                         | $0.0034$                  |
| 1                         | $0.04743$                 |
| 2                         | $0.88080$                 |
| 3                         | $0.99999$                 |
| 4                         | $0.99999$                 |

\textbf{TABLE III:} Examples of payoff matrices without and with social welfare gap.

(a) Payoff matrix without social welfare gap.

(b) Payoff matrix with social welfare gap.

\textbf{TABLE IV:} Examples of $P(x_i = A|x_{-i})$ under NE and LTE models with payoff matrices in Table IX.

(a) Computation results of $P(x_i = A|x_{-i})$ with payoff matrix in Table IX under NE and LTE models.

(b) Computation results of $P(x_i = A|x_{-i})$ with payoff matrix in Table IX under NE and LTE models.

| $k_A$ | $P_{LTE}(x_i = A|x_{-i})$ | $P_{NE}(x_i = A|x_{-i})$ |
|-------|----------------------------|---------------------------|
| 0     | $0.0034$                   | $0.0034$                  |
| 1     | $0.04743$                  | $0.4743$                  |
| 2     | $0.88080$                  | $1.00$                    |
| 3     | $0.99999$                  | $1.00$                    |
| 4     | $0.99999$                  | $1.00$                    |

\textbf{TABLE IVa:} Summary of computation results of $P(x_i = A|x_{-i})$ for a player with 4 neighbors, $k = 4$, $k_A \in \{0, 1, 2, 3, 4\}$, $\beta = 1$ and various values of $\delta_i$ using the payoff matrix in Table IX under NE and LTE diffusion models. Under the NE diffusion model, the player will select $A$ with a large probability only if three or more neighbors are in $A$ because the utility of $A$ surpasses the utility of $B$ when $k_A = 3$. By contrast, under the LTE model, with a large trust limit, the player will select $A$ with a high probability when only two neighbors are in $A$. By trusting its neighbors and choosing a product that benefits them, the player will benefit in the same manner when it is its neighbor’s turn to update the state. The innovation spreads faster under the LTE diffusion model than under the NE model, as the long-term return provides an additional incentive.

From the efficiency perspective, under certain circumstances, the LTE model can push the diffusion process to converge to the social welfare-maximizing equilibrium. Since the NE diffusion process only converges to the risk-dominant equilibrium $[6], [9], [29]$, this equilibrium is not necessarily the most efficient equilibrium that maximizes every player’s utility. Risk-dominance depends on the relative magnitude of $a - d$ and $b - c$ rather than $a$ and $b$. However, an LTE diffusion process can converge to the most efficient equilibrium different...
from the NE diffusion process if the risk-dominant equilibrium and the social-welfare-maximizing equilibrium differ. Simulation results in Section VI show that when players have moderate to large values of trust limit, the diffusion process under LTE dynamics can converge to the equilibrium where the innovative alternative dominates the social network rather than the risk-averse alternative.

Category 2: Contrary to the payoff matrices without social welfare gap, the payoff matrices in which \( c + d \leq 2b \) are termed payoff matrices with social welfare gap. An example of such payoff matrices is shown in Table III. For such payoff matrices, it is no longer guaranteed that the probability that an agent chooses A under the LTE diffusion model is greater than or equal to that under the NE. Under certain circumstances, when there are few players who adopt the innovation at the beginning of the diffusion, the innovation may rapidly die out in the system because purchasing \( A \) results in a decrease of utility to the player as well as its neighbors. The player will not adopt \( A \) under such circumstances, as per Case 1 in the LTE diffusion model. Therefore, a sufficient number of initial adopters must be ensured in the system. These initial users are usually the targets under the “viral marketing” strategy who are offered free “samples” or “incentives” to promote the adoption.

Consider a player with four neighbors in a star network. Table IV summarizes the computation results of \( P(x_i = A|x_{-i}) \) for a player with 4 neighbors, \( k = 4 \), various values of \( k_A \) and \( \delta_i \), and \( \beta = 1 \) using the payoff matrix in Table III under NE and LTE diffusion models. Consider a small star network with one central player connected with four leaves. Firstly, the results show that the center player will adopt \( A \) with positive probability only if at least one of its neighbors has done so. As an example, suppose that none of its neighbors is in state \( A \). Then, product \( A \) may spread in the social network only if the center player is selected as the “seed” at the beginning of the process. A further calculation for the leaves shows that the system will converge to the all-\( A \) state (the state where all players adopt the innovation) with positive probability only if at least one player (either center or leaf) is selected as the seed. Without initial adopters, the system will get stuck in the all-\( B \) equilibrium state, and product \( A \) will never diffuse. Secondly, the center player will adopt \( A \) with a probability close to or equal to 1 once one neighbor is in state \( A \) if it has a large trust limit and will choose \( A \) with a probability 1 once two neighbors are in state \( A \) if it behaves trustworthy. This implies that once a sufficient number of individuals have adopted the innovation, the innovation will spread much faster under the LTE dynamics than under the NE dynamics because, on top of the individual’s instantaneous benefit, the long-term benefit and potential return in the future provide an additional incentive driving the non-adopters to adopt the innovation.

### B. Numerical Examples for Linear Threshold LTE Diffusion Model

Consider a player with four neighbors in a star network. Table V shows the computation results of \( q_w \), \( q_u \), and \( q^* \) under various values of \( \delta' \) using payoff matrix in Table IIIa. Since the payoff matrix is without social welfare gap, \( 2b - c - d < 0 \) and \( q_w = 0 \). \( q^* \) decreases from 0.667 to 0 as \( \delta' \) increases from 0 to 2; as \( \delta' \) further increases, \( q^* \) remains 0. Note that the cases of \( \delta' = 0.25 \) and \( \delta' = 1 \) in Table Va correspond to the cases of \( \delta = 1 \) and \( \delta = 4 \) in Table IVa, respectively. The results of Table IVa and Va are consistent. For example, for \( \delta' = 1 \) (\( \delta = 4 \)), when at least two neighbors are in \( A \), the player will update its state to \( A \). This is in line with the result in Table IIIa that \( P(x_i = A|x_{-i}) \) is one or very close to one under the LTE diffusion model when \( k_A = 2, 3, 4 \). Table Vb shows the computation results using the payoff matrix in Table IIIb. Since this is a payoff matrix with social welfare gap, \( q_w > 0 \), \( q_u = 0.1 \) indicates that only when at least 10% of the neighbors are in state \( A \), choosing \( A \) brings about higher social welfare than \( B \). As \( \delta' \) increases from 0 to 1.5, \( q^* \) decreases from 0.4 to 0.1; it does not change with further increase of \( \delta' \). The results are again consistent with Table IVb. When \( \delta' = 0.25 \) (\( \delta = 1 \)), the player adopts \( A \) when at least two neighbors are in \( A \) according to the LTE threshold model. In Table Vb when \( k_A = 1 \), the player plays according to NE and chooses \( A \) with a small probability; when \( k_A = 2 \), this probability becomes 1.

We have seen that the range of \( q^* \) is between \( q_w \) and \( q_u \), which is determined by the payoff matrix. For payoff matrices without social welfare gap, \( q_w = 0 \) and \( q^* \) can be as small as zero. The players with zero \( q^* \) are those who always try the new products once they are released. One interesting question is how large \( q^* \) can be. For certain “bad” payoff matrices, the threshold \( q^* \) can be very large. This implies that the player will adopt the innovation only if almost all its neighbors have adopted it. Table Va shows a peculiar payoff matrix, and Table Vb shows the computation result of \( q^* \) under that payoff matrix. When \( \delta' \) is small, \( q^* \) is large and very close to \( q_u \), which is close to 1. In this payoff matrix, \( a \) is close to \( d \) whereas \( b \) is much greater than \( c \). This implies that when a few of its neighbors are in state \( A \), switching from \( B \) to \( A \) will cause a significant loss to the player; when the majority of its neighbors are in \( A \), changing its state from \( B \) to \( A \) will not bring much benefit to its utility. Therefore, if the player is more self-interested and has a small value of \( \delta' \), the NE dynamics will prevail, and the player will have very little incentive to adopt the innovation.
TABLE VI: Examples of \( q^* \) a peculiar payoff matrix.

(a) A peculiar payoff matrix with large value of \( q^* \):

\[
\begin{array}{c|cc}
 & A & B \\
\hline
A & 10 & 10 \\
B & 9 & 9 \\
\end{array}
\]

(b) Computation results of \( q_w \), \( q_u \), and \( q^* \) for various values of \( \delta' \) using payoff matrix in Table VIa.

| \( \delta' \) | \( q_w \) | \( q_u \) |
|---|---|---|
| \( \delta' = 1 \) | 0.361 | 0.835 |
| \( \delta' = 2 \) | 0.778 | 0.888 |

APPENDIX B
PROOF OF THEOREM [1]

The Markov chain with transition probability matrix \( P \) characterized by Eq. (13) has 2 absorbing states and \( (n-1) \) transient states. Write \( P \) in canonical form and partition it into four submatrices: \( P = \begin{bmatrix} Q & R \\ 0 & I \end{bmatrix} \), where \( Q \) and \( R \) represent the transition among transients states and from transient states to absorbing states, respectively. To prove Theorem [1] it is sufficient to characterize \( (I - Q)^{-1} \).

Lemma 1. The determinant of \( I - Q \) is \( \Delta \).

**Proof.** We use superscript to denote the number of players in the social network represented by the 1-dimensional Markov chain. \( P^{(n)} \in \mathbb{R}^{(n+1) \times (n+1)} \) denotes the transition probability matrix for a 1-dimensional Markov chain with \( n \) players in the social network. We prove this by induction. For the base step, we have

\[
\text{det}((I - Q)^{(2)}) = a_1 + b_1, \quad \text{det}((I - Q)^{(3)}) = (a_1 + b_1)(a_2 + b_2) - a_1 b_2 = a_1 a_2 + a_2 b_1 + b_1 b_2.
\]

Now, for the induction step, we can write

\[
\begin{align*}
\text{det}((I - Q)^{(n+1)}) &= \begin{vmatrix} (I - Q)^n & 0 \\ 0 & -a_{n-1} \end{vmatrix} \\
&= (a_n + b_n)\text{det}((I - Q)^n) - a_{n-1} b_n \text{det}((I - Q)^{n-1}) \\
&= \sum_{i=1}^{n} \prod_{j=i}^{n-1} a_j \prod_{j=1}^{i} b_j + b_n \sum_{i=1}^{n-1} \prod_{j=i}^{n-1} a_j \prod_{j=1}^{i} b_j \\
&\quad - b_n \sum_{i=1}^{n-1} \prod_{j=i}^{n-1} a_j \prod_{j=1}^{i} b_j \\
&= \prod_{i=1}^{n} a_j \prod_{j=1}^{i} b_j + \sum_{i=1}^{n} \prod_{j=i}^{n} a_j \prod_{j=1}^{i} b_j \\
&\quad - \sum_{i=1}^{n} \prod_{j=i}^{n} a_j \prod_{j=1}^{i} b_j \\
&= \prod_{i=1}^{n+1} a_j \prod_{j=1}^{i} b_j.
\end{align*}
\]

Lemma 2. The \((i, j)\)-th entry of \((I - Q)^{-1}\) equals

\[
(I - Q)^{-1}_{i,j} = \frac{1}{\Delta} \left( \sum_{k=1}^{\min\{i,j\}} \prod_{l=k}^{i-1} a_l \prod_{l=1}^{j-1} b_l \right) \\
\leq \left( \sum_{k=\max\{i,j\}+1}^{n} \prod_{l=k}^{i-1} a_l \prod_{l=1}^{j-1} b_l \right) .
\]

**Proof.** Let \( p_i \) denote the \( i \)-th row of \((I - Q)^{-1}\) and \( q_j \) denote the \( j \)-th column of \((I - Q)\). When \( i = j \), we have

\[
\begin{align*}
p_i \cdot q_j &= - \frac{a_i - 1}{\Delta} \left( \sum_{k=1}^{i-1} \prod_{l=k}^{i-2} a_l \prod_{l=1}^{j-1} b_l \right) \\
&\quad + \frac{a_i + b_i}{\Delta} \left( \sum_{k=1}^{i-1} \prod_{l=k}^{i-2} a_l \prod_{l=1}^{j-1} b_l \right) \\
&\quad - \frac{b_{i+1} - 1}{\Delta} \left( \sum_{k=1}^{i+1} \prod_{l=k}^{i+1} a_l \prod_{l=1}^{j-1} b_l \right) \\
&\quad + \frac{a_i + b_i}{\Delta} \left( \sum_{k=1}^{i+1} \prod_{l=k}^{i+1} a_l \prod_{l=1}^{j-1} b_l \right) \\
&\quad - \frac{b_{i+1} - 1}{\Delta} \left( \sum_{k=1}^{i+1} \prod_{l=k}^{i+1} a_l \prod_{l=1}^{j-1} b_l \right) \\
&\quad + \frac{a_i + b_i}{\Delta} \left( \sum_{k=1}^{i+1} \prod_{l=k}^{i+1} a_l \prod_{l=1}^{j-1} b_l \right) \\
&= 1
\end{align*}
\]

When \( i \neq j \), without loss of generality, let \( i < j \). We have

\[
\begin{align*}
p_i \cdot q_j &= - \frac{a_j - 1}{\Delta} \left( \sum_{k=1}^{i-1} \prod_{l=k}^{i-2} a_l \prod_{l=1}^{j-1} b_l \right) \\
&\quad + \frac{a_j + b_j}{\Delta} \left( \sum_{k=1}^{i-1} \prod_{l=k}^{i-2} a_l \prod_{l=1}^{j-1} b_l \right) \\
&\quad - \frac{b_{j+1} - 1}{\Delta} \left( \sum_{k=1}^{j+1} \prod_{l=k}^{j+1} a_l \prod_{l=1}^{j-1} b_l \right) \\
&\quad + \frac{a_j + b_j}{\Delta} \left( \sum_{k=1}^{j+1} \prod_{l=k}^{j+1} a_l \prod_{l=1}^{j-1} b_l \right) \\
&\quad - \frac{b_{j+1} - 1}{\Delta} \left( \sum_{k=1}^{j+1} \prod_{l=k}^{j+1} a_l \prod_{l=1}^{j-1} b_l \right) \\
&\quad + \frac{a_j + b_j}{\Delta} \left( \sum_{k=1}^{j+1} \prod_{l=k}^{j+1} a_l \prod_{l=1}^{j-1} b_l \right) \\
&= 0
\end{align*}
\]

Since \((I - Q)\) is invertible, Eq. (29) characterizes its inverse. \qed

Denote the absorption probability and expected time to absorption by \( B \) and \( \pi \) respectively. The theorem follows as

\[
B = (I - Q)^{-1} R \quad \text{and} \quad \pi = (I - Q)^{-1} 1
\]

where 1 denotes the one-vector.