SOME APPLICATIONS OF DEGENERATE POLY-BERNOULLI NUMBERS AND POLYNOMIALS

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Abstract. In this paper, we consider degenerate poly-Bernoulli numbers and polynomials associated with polylogarithmic function and p-adic invariant integral on $\mathbb{Z}_p$. By using umbral calculus, we derive some identities of those numbers and polynomials.

1. Introduction

Let $p$ be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of $p$-adic integers, the field of $p$-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. The $p$-adic norm is normalized as $|p|_p = \frac{1}{p}$. For $k \in \mathbb{Z}$, the polylogarithmic function $\text{Li}_k(x)$ is defined by $\text{Li}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k}$. For $k = 1$, we have $\text{Li}_1(x) = -\log(1-x)$.

In [4], L. Carlitz considered the degenerate Bernoulli polynomials which are given by the generating function

$$\frac{t}{(1 + \lambda t)^k - 1} = \sum_{n=0}^{\infty} \beta_{n,\lambda}(x) \frac{t^n}{n!}.$$  

Note that $\lim_{\lambda \to 0} \beta_{n,\lambda}(x) = B_n(x)$, where $B_n(x)$ are the ordinary Bernoulli polynomials. When $x = 0$, $\beta_{n,\lambda} = \beta_{n,\lambda}(0)$ are called the degenerate Bernoulli numbers.

It is known that the poly-Bernoulli polynomials are defined by the generating function

$$\frac{\text{Li}_k(1-e^{-t})}{e^t - 1} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!}, \quad (\text{see } [3]).$$

When $x = 0$, $B_n^{(k)} = B_n^{(k)}(0)$ are called the poly-Bernoulli numbers.

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the $p$-adic invariant integral on $\mathbb{Z}_p$ is defined by

$$\int_{\mathbb{Z}_p} f(x) \, d\mu_0(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x) \mu_0(x + p^N \mathbb{Z}_p) = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \quad (\text{see } [13]).$$

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From (1.3), we have
\[
\int_{Z_p} f(x + 1) d\mu_0(x) - \int_{Z_p} f(x) d\mu_0(x) = f'(0),
\]
where \( f'(0) = \frac{df(x)}{dx} \bigg|_{x=0} \) (see [17]).

By (1.4), we get
\[
\int_{Z_p} (1 + \lambda t)^{(x+n)/\lambda} d\mu_0(y) = \frac{\log{(1 + \lambda t)^{\frac{x}{\lambda}}}}{(1 + \lambda t)^{\frac{x}{\lambda}} - 1} (1 + \lambda t)^{\frac{x}{\lambda}} = \frac{\log{(1 + \lambda t)^{\frac{x}{\lambda}}}}{\lambda t} \frac{t}{(1 + \lambda t)^{\frac{x}{\lambda}} - 1} (1 + \lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \sum_{l=0}^{n} \binom{n}{l} \lambda^{n-l} D_{n-l} \beta_{l,\lambda}(x) \frac{t^n}{n!},
\]
where \( D_n \) are the Daehee numbers of the first kind given by the generating function
\[
\frac{\log{(1 + t)}}{t} = \sum_{n=0}^{\infty} D_n \frac{t^n}{n!}, \quad \text{(see [9])}.
\]

Let \( \mathcal{F} = \{ f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!}, a_k \in \mathbb{C}_p \} \) be the algebra of formal power series in a single variable \( t \). Let \( \mathbb{P} \) be the algebra of polynomials in a single variable \( x \) over \( \mathbb{C}_p \). We denote the action of the linear functional \( L \in \mathbb{P}^* \) on a polynomial \( p(x) \) by \( \langle L | p(x) \rangle \), which is linearly extended as \( \langle cL + c'L | p(x) \rangle = c(L | p(x)) + c'(L' | p(x)) \), where \( c, c' \in \mathbb{C}_p \). We define a linear functional on \( \mathbb{P} \) by setting
\[
\langle f(t) | x^n \rangle = a_n, \quad \text{for all } n \geq 0 \text{ and } f(t) \in \mathcal{F}.
\]

By (1.7), we easily get
\[
\langle t^k | x^n \rangle = n!\delta_{n,k}, \quad (n, k \geq 0),
\]
where \( \delta_{n,k} \) is the Kronecker’s symbol (see [15]).

For \( f_L(t) = \sum_{k=0}^{\infty} \frac{L(x)^k}{k!} \), we have \( \langle f_L(t) | x^n \rangle = \langle L | x^n \rangle \). The map \( L \mapsto f_L(t) \) is vector space isomorphism from \( \mathbb{P}^* \) onto \( \mathcal{F} \). Henceforth \( \mathcal{F} \) denotes both the algebra of formal power series in \( t \) and the vector space of all linear functionals on \( \mathbb{P} \), and so an element \( f(t) \) of \( \mathcal{F} \) is thought of as both a formal power series and a linear functional. We call \( \mathcal{F} \) the umbral algebra. The umbral calculus is the study of umbral algebra.

The order \( o(f(t)) \) of the non-zero power series \( f(t) \) is the smallest integer \( k \) for which the coefficient of \( t^k \) does not vanish (see [10, 15]). If \( o(f(t)) = 1 \) (respectively, \( o(f(t)) = 0 \)), then \( f(t) \) is called a delta (respectively, an invertible) series.

For \( o(f(t)) = 1 \) and \( o(g(t)) = 0 \), there exists a unique sequence \( s_n(x) \) of polynomials such that \( \langle g(t) f(t)^k | s_n(x) \rangle = n!\delta_{n,k}(n, k \geq 0) \). The sequence \( s_n(x) \) is called the Sheffer sequence for \( (g(t), f(t)) \), and we write \( s_n(x) \sim (g(t), f(t)) \) (see [15]).

For \( f(t) \in \mathcal{F} \) and \( p(x) \in \mathbb{P} \), by (1.8), we get
\[
\langle e^{yt} | p(x) \rangle = p(y), \quad \langle f(t) g(t) p(x) \rangle = \langle g(t) f(t) p(x) \rangle = \langle f(t) | g(t) p(x) \rangle
\]
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(1.10) \( f(t) = \sum_{k=0}^{\infty} \langle f(t) \mid x^k \rangle \frac{t^k}{k!}, \ p(x) = \sum_{k=0}^{\infty} \langle t^k \mid p(x) \rangle \frac{x^k}{k!}, \) (see [13]).

From (1.10), we note that

(1.11) \( p^{(k)}(0) = \langle t^k \mid p(x) \rangle = \langle 1 \mid p^{(k)}(x) \rangle, \ (k \geq 0), \)

where \( p^{(k)}(0) \) denotes the \( k \)-th derivative of \( p(x) \) with respect to \( x \) at \( x = 0 \).

By (1.11), we get

(1.12) \( t^k p(x) = p^{(k)}(x) = \frac{d^k}{dx^k} p(x), \ (k \geq 0). \)

In [15], it is known that

(1.13) \( s_n(x) \sim (g(t), f(t)) \iff \frac{1}{g(\overline{f}(t))} e^{\overline{f}(t)} = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!}, \ (x \in \mathbb{C}_p), \)

where \( \overline{f}(t) \) is the compositional inverse of \( f(t) \) such that \( f(\overline{f}(t)) = \overline{f}(f(t)) = t \).

From (1.12), we can easily derive the following equation:

(1.14) \( e^{yt} p(x) = p(x+y), \ \text{where} \ p(x) \in P = \mathbb{C}_p[x]. \)

In this paper, we study degenerate poly-Bernoulli numbers and polynomials associated with polylogarithm function and \( p \)-adic invariant integral on \( \mathbb{Z}_p \). Finally, we give some identities of those numbers and polynomials which are derived from umbral calculus.

2. SOME APPLICATIONS OF DEGENERATE POLY-BERNOULLI NUMBERS

Now, we consider the degenerate poly-Bernoulli polynomials which are given by the generating function

(2.1) \( \frac{\text{Li}_k \left( 1 - (1 + \lambda t) - \frac{t^k}{k!} \right)}{(1 + \lambda t)^\frac{t}{\overline{f}} - 1} e^{xt} = \sum_{n=0}^{\infty} \beta^{(k)}_{n,\lambda}(x) \frac{t^n}{n!}, \ (k \in \mathbb{Z}). \)

From (1.13) and (2.1), we have

(2.2) \( \beta^{(k)}_{n,\lambda}(x) \sim \left( \frac{(1 + \lambda t)^\frac{t}{\overline{f}} - 1}{\text{Li}_k \left( 1 - (1 + \lambda t) - \frac{t^k}{k!} \right)} \right) (\overline{f}(t)), \)

and

(2.3) \( \beta^{(k)}_{n,\lambda}(x) = \sum_{i=0}^{n} \left( \frac{n}{i} \right) \beta^{(k)}_{i,\lambda} x^{n-i}, \)

where \( \beta^{(k)}_{i,\lambda} = \beta^{(k)}_{i,\lambda}(0) \) are called the degenerate poly-Bernoulli numbers.

Thus, by (2.2), we get

(2.4) \( \int_{x}^{x+y} \beta^{(k)}_{n,\lambda}(u) \, du = \frac{1}{n+1} \left\{ \beta^{(k)}_{n+1,\lambda}(x+y) - \beta^{(k)}_{n+1,\lambda}(x) \right\} = e^{yt} - \frac{1}{t} \beta^{(k)}_{n,\lambda}(x). \)
Let \( f(t) \) be the linear functional such that

\[
\langle f(t) | p(x) \rangle = \int_{\mathbb{R}} \frac{(e^t - 1) \text{Li}_k \left(1 - (1 + \lambda t)^{-\frac{1}{\lambda}}\right)}{t \left(1 + \lambda t\right)^{\frac{1}{\lambda}} - 1} p(x) \, d\mu_0(x)
\]

for all polynomials \( p(x) \). Then it can be determined as follows: for any \( p(x) \in \mathbb{P} \),

\[
\left\langle \frac{t}{e^t - 1} | p(x) \right\rangle = \int_{\mathbb{R}} p(x) \, d\mu_0(x).
\]

Replacing \( p(x) \) by \( \frac{e^t - 1}{t} h(t)p(x) \), for \( h(t) \in \mathcal{F} \), we get

(2.5)

\[
\langle h(t) | p(x) \rangle = \int_{\mathbb{R}} \frac{e^t - 1}{t} h(t)p(x) \, d\mu_0(x).
\]

In particular, for \( h(t) = 1 \), we obtain

(2.6)

\[
\int_{\mathbb{R}} \frac{e^t - 1}{t} p(x) \, d\mu_0(x) = p(0).
\]

Therefore, by (2.5) and (2.6), we obtain the following theorem as a special case.

**Theorem 1.** For \( p(x) \in \mathbb{P} \), we have

\[
\left\langle \text{Li}_k \left(1 - (1 + \lambda t)^{-\frac{1}{\lambda}}\right) | p(x) \right\rangle = \int_{\mathbb{R}} \frac{(e^t - 1) \text{Li}_k \left(1 - (1 + \lambda t)^{-\frac{1}{\lambda}}\right)}{t \left(1 + \lambda t\right)^{\frac{1}{\lambda}} - 1} p(x) \, d\mu_0(x),
\]

and

\[
\left\langle \frac{(e^t - 1) \text{Li}_k \left(1 - (1 + \lambda t)^{-\frac{1}{\lambda}}\right)}{t \left(1 + \lambda t\right)^{\frac{1}{\lambda}} - 1} \int_{\mathbb{R}} e^{yt} \, d\mu_0(y) \right\rangle p(x) = \int_{\mathbb{R}} \frac{(e^t - 1) \text{Li}_k \left(1 - (1 + \lambda t)^{-\frac{1}{\lambda}}\right)}{t \left(1 + \lambda t\right)^{\frac{1}{\lambda}} - 1} p(x) \, d\mu_0(x).
\]

In particular,

\[
\beta^{(k)}_{n, \lambda} = \left\langle \frac{(e^t - 1) \text{Li}_k \left(1 - (1 + \lambda t)^{-\frac{1}{\lambda}}\right)}{t \left(1 + \lambda t\right)^{\frac{1}{\lambda}} - 1} \int_{\mathbb{R}} e^{yt} \, d\mu_0(y) \right\rangle x^n, \quad (n \geq 0).
\]

Note that

\[
\left\langle \int_{\mathbb{R}} e^{yt} \, d\mu_0(y) \left| \frac{e^t - 1}{t} \beta^{(k)}_{n, \lambda} (x) \right\rangle = \frac{1}{n + 1} \left\langle \frac{t}{e^t - 1} \beta^{(k)}_{n+1, \lambda} (x + 1) - \beta^{(k)}_{n+1, \lambda} (x) \right\rangle.
\]
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\[ = \frac{1}{n+1} \sum_{l=0}^{n+1} \binom{n+1}{l} B_l \left( \beta_{n+1-l, \lambda}^{(k)} (1) - \beta_{n+1-l, \lambda}^{(k)} \right) = \beta_{n, \lambda}^{(k)}. \]

It is easy to show that

\[
\left( e^t - 1 \right) \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} (x+y)^n d\mu_0(y) \frac{t^n}{n!} = \beta_{n, \lambda}^{(k)} \sum_{n=0}^{\infty} \beta_{n, \lambda}^{(k)} \frac{t^n}{n!}.
\]

Thus, by (2.7), we get

\[
\beta_{n, \lambda}^{(k)}(x) = \frac{\left( e^t - 1 \right) \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} (x+y)^n d\mu_0(y)}{t \left( (1 + \lambda t)^{\frac{k}{2^2}} - 1 \right)} \cdot \sum_{n=0}^{\infty} \beta_{n, \lambda}^{(k)} \frac{t^n}{n!}.
\]

Therefore, by (2.8), we obtain the following theorem.

**Theorem 2.** For \( p(x) \in \mathbb{P} \), we have

\[
\sum_{n=0}^{\infty} \frac{\left( e^t - 1 \right) \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} (x+y)^n d\mu_0(y)}{t \left( (1 + \lambda t)^{\frac{k}{2^2}} - 1 \right)} e^{xt} = \sum_{n=0}^{\infty} \beta_{n, \lambda}^{(k)}(x) \frac{t^n}{n!}.
\]

For \( r \in \mathbb{N} \), let us consider the higher-order degenerate poly-Bernoulli polynomials as follows:

\[
\left( \frac{\left( e^t - 1 \right) \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} (x+y)^n d\mu_0(y)}{t \left( (1 + \lambda t)^{\frac{k}{2^2}} - 1 \right)} \right)^r \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e(x_1 + \cdots + x_r + x) d\mu_0(x_1) \cdots d\mu_0(x_r) = \left( \frac{\sum_{n=0}^{\infty} \beta_{n, \lambda}^{(k)}(x) \frac{t^n}{n!}}{(1 + \lambda t)^{\frac{k}{2^2}} - 1} \right)^r e^{xt} = \sum_{n=0}^{\infty} \beta_{n, \lambda}^{(k,r)}(x) \frac{t^n}{n!}.
\]
Thus, we obtain

\[
\beta_{n,\lambda}^{(k,r)}(x) = \left( \frac{\text{Li}_k \left(1 - (1 + \lambda t)^{-\frac{1}{\lambda}}\right)}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} \right)^x n, \lambda \times \int_{z_p} \cdots \int_{z_p} (x_1 + \cdots + x_r + x)^n d\mu_0(x_1) \cdots d\mu_0(x_r),
\]

where \( n \geq 0 \).

Here, for \( x = 0 \), \( \beta_{n,\lambda}^{(k,r)} = \beta_{n,\lambda}^{(k,r)}(0) \) are called the degenerate poly-Bernoulli numbers of order \( r \). From (2.9), we note that

\[
\beta_{n,\lambda}^{(k)}(x) \sim \left( \frac{(1 + \lambda t)^{-\frac{1}{\lambda}} - 1}{\text{Li}_k \left(1 - (1 + \lambda t)^{-\frac{1}{\lambda}}\right)} \right)^x t, \lambda.
\]

Therefore, by (2.10), we obtain the following theorem.

**Theorem 3.** For \( p(x) \in \mathbb{P} \) and \( r \in \mathbb{N} \), we have

\[
\beta_{n,\lambda}^{(k,r)}(x) = \left( \frac{(1 + \lambda t)^{-\frac{1}{\lambda}} - 1}{\text{Li}_k \left(1 - (1 + \lambda t)^{-\frac{1}{\lambda}}\right)} \right)^x t, \lambda.
\]

Let us consider the linear functional \( f_r(t) \) such that

\[
\langle f_r(t) | p(x) \rangle = \int_{z_p} \cdots \int_{z_p} p(x_1 + \cdots + x_r + x)^n d\mu_0(x_1) \cdots d\mu_0(x_r)
\]

for all polynomials \( p(x) \). Then it can be determined in the following way: for \( p(x) \in \mathbb{P} \),

\[
\langle (\frac{t}{e^t - 1})^r | p(x) \rangle = \int_{z_p} \cdots \int_{z_p} p(x_1 + \cdots + x_r)^n d\mu_0(x_1) \cdots d\mu_0(x_r).
\]
Replacing $p(x)$ by $\left(\frac{t}{e^{\frac{1}{t}} - 1}h(t)\right)^r p(x)$, for $h(t) \in \mathcal{F}$, we have
\begin{equation}
(2.13)
(\langle h(t)^r \mid p(x) \rangle) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left(\frac{e^t - 1}{t}h(t)\right)^r p(x)|_{x=x_1+\cdots+x_r} \, d\mu_0(x_1) \cdots d\mu_0(x_r).
\end{equation}
In particular, for $h(t) = 1$, we get
\begin{equation}
(2.14)
\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left(\frac{e^t - 1}{t}\right)^r p(x)|_{x=x_1+\cdots+x_r} \, d\mu_0(x_1) \cdots d\mu_0(x_r) = p(0).
\end{equation}
Therefore, by (2.13) and (2.14), we obtain the following theorem.

**Theorem 4.** For $p(x) \in \mathbb{P}$, we have
\[
\left\langle \left(\frac{\text{Li}_k \left(1 - (1 + \lambda t)^{-\frac{1}{t}}\right)}{(1 + \lambda t)^{\frac{1}{t}} - 1}\right)^r \mid p(x) \right\rangle = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left(\frac{e^t - 1}{t} \text{Li}_k \left(1 - (1 + \lambda t)^{-\frac{1}{t}}\right)\right)^r p(x)|_{x=x_1+\cdots+x_r} \, d\mu_0(x_1) \cdots d\mu_0(x_r),
\]
and
\[
\left\langle \left(\frac{e^t - 1}{t} \text{Li}_k \left(1 - (1 + \lambda t)^{-\frac{1}{t}}\right)\right)^r \right\rangle \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(x_1+\cdots+x_r)t} \, d\mu_0(x_1) \cdots d\mu_0(x_r) \left\mid p(x) \right\rangle = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left(\frac{e^t - 1}{t} \text{Li}_k \left(1 - (1 + \lambda t)^{-\frac{1}{t}}\right)\right)^r p(x)|_{x=x_1+\cdots+x_r} \, d\mu_0(x_1) \cdots d\mu_0(x_r).
\]
In particular,
\[
\beta^{(k,r)}_{n,\lambda} = \left\langle \left(\frac{e^t - 1}{t} \text{Li}_k \left(1 - (1 + \lambda t)^{-\frac{1}{t}}\right)\right)^r \right\rangle \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(x_1+\cdots+x_r)t} \, d\mu_0(x_1) \cdots d\mu_0(x_r) \left\mid x^n \right\rangle.
\]

**Remark.** It is not difficult to show that
\[
\left\langle \left(\frac{e^t - 1}{t} \text{Li}_k \left(1 - (1 + \lambda t)^{-\frac{1}{t}}\right)\right)^r \right\rangle \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(x_1+\cdots+x_r)t} \, d\mu_0(x_1) \cdots d\mu_0(x_r) \left\mid x^n \right\rangle
= \sum_{n_{n_1+\cdots+n_r} = n} \binom{n}{n_1, \ldots, n_r} \left(\frac{e^t - 1}{t} \text{Li}_k \left(1 - (1 + \lambda t)^{-\frac{1}{t}}\right)\right) \int_{\mathbb{Z}_p} e^{x_1 t} \, d\mu_0(x_1) \left\mid x^{m_1} \right\rangle \times \cdots \times \int_{\mathbb{Z}_p} e^{x_{n_r} t} \, d\mu_0(x_{n_r}) \left\mid x^{n_r} \right\rangle.
\]
Thus, we get
\[
\beta^{(k,r)}_{n,\lambda} = \sum_{n_{n_1+\cdots+n_r} = n} \binom{n}{n_1, \ldots, n_r} \beta^{(k)}_{n_1,\lambda} \cdots \beta^{(k)}_{n_r,\lambda}.
\]
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