Complex Effective Potentials and Critical Bubbles*

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Abstract

The Higgs contribution to the effective potential appears to be complex. How do we interpret this, and how should we modify the calculation to calculate physical quantities such as the critical bubble free energy?

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1. A Toy Model with a Complex Effective Potential

Suppose we wish to calculate the 1-loop finite-temperature effective potential \( V \) for a theory with a single scalar (I’ll call it the Higgs), whose tree potential \( V_0 \) (Fig. 1) is of the form

\[
V_0(\phi) = \frac{\mu^2}{2\sigma^2}\phi^2(\phi - \sigma)^2 - \frac{\epsilon\phi^2}{\sigma^2}
\]  

(1)

The effective Higgs mass (at zero external momentum) is

\[
m^2(\phi, T) = V_0''(\phi) + \frac{\mu^2}{2\sigma^2}T^2
\]  

(2)

The last term of eq. (2) is the Higgs’s “self-plasma-mass” (SPM); let us choose our parameters to make the SPM small, so \( m^2 < 0 \) over roughly \( (1-1/\sqrt{3})/2 < \phi/\sigma < (1+1/\sqrt{3})/2 \).

The 1-loop contribution of the Higgs to \( V \) can be calculated from the vacuum-to-vacuum graph of Fig. 2a:

\[
V = V_0 + V_1 + \frac{T^4}{2\pi^2}I(m/T) \approx V_0 + \frac{T^2}{24}(V_0'') - \frac{T}{12\pi}(V_0'')^{3/2} + \cdots
\]  

(3)

or from the tadpole graph of Fig. 2b:

\[
V' = V_0' + V_1' + (V_0''')\frac{T^2}{24}F(m/T) \approx V_0 + \frac{T^2}{24}(V_0'') - \frac{T}{8\pi}(V_0'')(V_0'')^{1/2} + \cdots
\]  

(4)

where \( V_1 \) is the \( T \)-independent 1-loop result

\[
V_1 = \frac{1}{4\pi^2} \int dk \ k^2\sqrt{k^2 + m^2} = \frac{m^4}{64\pi^2} \left[ \ln \left( \frac{m^2}{\Lambda^2} \right) - \frac{3}{2} \right]
\]  

(5)

and (writing \( x = k/T, \ y = m/T \))

\[
I(y) \equiv \int_0^\infty dx \ x^2 \ln \left( 1 - e^{-\sqrt{x^2+y^2}} \right), \quad F(y) \equiv 6I'(y)/(\pi^2 y)
\]  

(6)

I have expanded in small \( m/T \). Eqs. 3 and 4 give identical results.

For \( m^2 < 0 \) the potential appears to be complex. How are we to interpret the imaginary part of the potential? How should we modify \( V \) to get a real quantity to plot and use in calculations? The naive answer, which I’ll call Method A, is simply to take the real part of \( V \). Several much fancier methods can be found in the literature[1].
2. Relation to the Standard Model

The toy model can, with minor modifications, represent the Standard Model after integrating out gauge bosons and fermions. Now \( \{\mu, \sigma, \epsilon\} \) depend on \( T \) (and are simply related to the usual \( \{\lambda_T, E, D\}^{[2]} \)). The contribution of the gauge bosons and fermions to the Higgs plasma mass is still given correctly by eq. (2), as can be verified by direct calculation of Feynman diagrams. Goldstone bosons double the SPM of eq. (2), and the themselves have a squared mass \( m^2 = V_0'/\phi + \text{SPM} \) which becomes negative over \( \frac{1}{2} \leq \phi/\sigma < 1 \) (for small SPM). The tadpole calculation eq. (4) must be used, using the 3-Higgs coupling of the original theory \( (6\lambda \phi^4/4) \) in place of \( (V''_0) \), to avoid overcounting diagrams. None of these modifications seem relevant to the questions about imaginary parts.

3. Homogeneous and Inhomogeneous Fields

At \( T = 0 \), Weinberg and Wu showed that the imaginary part represents the rate of decay of an unstable homogeneous field configuration to an inhomogeneous state\(^{[3]} \). Whether this is true at finite \( T \) remains to be shown.

For calculating percolation rates, however, we are more often interested in the free energy \( E_c \) of the critical bubble, an extremal configuration stable against any fluctuation in \( \phi(x) \) except overall growth or shrinkage (the “breathing mode”):

\[
E_c = \int d^3x \left[ V(\phi(x)) + \frac{1}{2}(\nabla \phi)^2 + \frac{AT}{m^3} \left( \frac{dm^2}{d\phi} \nabla \phi \right)^2 + \frac{BT}{m^9} \left( \frac{dm^2}{d\phi} \nabla \phi \right)^4 + \cdots \right]
\]

(7)

Here \( A, B \cdots \) come from derivative corrections to the action\(^{[4,5]} \).

4. \( \text{Im}\{V\} \) Does Not Represent Bubble Growth/Shrinkage

One might suppose that the contribution of \( \text{Im}\{V\} \) to \( E_c \) represents the instability of the breathing mode. We can disprove this hypothesis by examining a thin-wall bubble \( [\epsilon \ll \mu^2\sigma^2/4 \text{ in eq. (1)}] \), for which\(^{[6]} \)

\[
R = \frac{2S_1}{\epsilon}, \quad S_1 = \int d\phi \sqrt{2V} = \frac{2\mu\sigma^2}{9\sqrt{3}}, \quad \delta = 1/\mu
\]

(8)

where \( R \) is the bubble radius and \( \delta \) is its thickness. The contribution of \( \text{Im}\{V\} \) to \( E_c \) is \( \sim R^2 \), since \( V \) is only complex in the bubble wall, and the wall profile is independent of \( \epsilon \) for \( \epsilon \to 0 \). The breathing mode imaginary contribution to \( E_c \), on the other hand, is independent of \( R \), as can be seen by calculating\(^{[5,7,8]} \)

\[
E_c = E_0 + \sum_{n,l} (2l + 1) \left[ \frac{\omega_{n,l}}{2} + T \ln \left( 1 - e^{-\omega_{n,l}/T} \right) \right]
\]

(9)
where $E_0$ is the tree-level energy, and $\omega_{n,l}^2$ is the eigenvalue of $[-\nabla^2 + V_0''(\phi(x))]$ whose eigenfunction has $n$ radial nodes and angular dependence $Y_{m}^{l}(\theta, \phi)$. Eq. 9 is just the standard thermodynamic result for the free energy of a system of harmonic oscillators. The radial part $\chi(r)/r$ of the eigenfunction satisfies

$$\left[-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + V_0''(\phi(r)) - \omega_{n,l}^2\right] \chi_{n,l}(r) = 0$$

and we see that states bound to the wall approximately satisfy

$$\omega_{n,l}^2 = \omega_{n,0}^2 + \frac{l(l+1)}{R^2}$$

The breathing mode eigenvalue $\omega_{0,0}^2$ can thus be found from the translational mode eigenvalue $\omega_{0,1}^2 = 0$:

$$\omega_{0,0}^2 = -\frac{2}{R^2}, \quad [E_c]_{0,0} \approx \frac{i}{\sqrt{2}R} + \frac{i\pi T}{2} + T \ln \left(\frac{\sqrt{2}}{RT}\right)$$

One can argue that eq. (9) breaks down for unstable fluctuations (inverted harmonic oscillators), but even so it does not appear that $\text{Im}\{E_c\}$ grows as $R^2$. Thus the contribution of $\text{Im}\{V\}$ to $E_c$ must be canceled by the derivative corrections of eq. (7).

Such cancellation is plausible, since odd powers of $m$ in the derivative expansion give complex terms, and a similar cancellation is known to occur to restore gauge invariance. However, the divergences as $m \to 0$ get increasingly worse, so this expansion seems inappropriate for finding $\text{Im}\{E_c\}$.

5. Removal of Long-Wavelength Modes

The integral in eq. (6) comes from a sum over Fourier modes of Higgs field fluctuations, and the integrand is only complex for long wavelength modes ($x < |y|$, or $k < |m|$). At the hump ($\phi = \sigma/2$) for instance, $m^2 = -\mu^2/2$, so only modes of wavelength $\lambda > 2\sqrt{2}\pi/\mu$ contribute to $\text{Im}\{V\}$. This is several times the bubble wall thickness $\delta = 1/\mu$.

This suggests Method B for altering the calculation of $V$, namely changing the lower limit of integration in eq. (6) to $\text{Im}\{y\}$. Several schemes discussed in ref. [1] are in a similar spirit. Note that Methods A and B are equivalent for the $T$-independent part eq. (5), but not for eq. (6), since the integrand of the latter in the region $0 < x < \text{Im}\{y\}$ is complex, not pure imaginary.
6. What’s the Best Method?

To decide on a “best” method of calculating physical quantities from a complex $V$, we must decide what “best” means. It could mean that when we put our modified $V$ into eq. (7) and set $A = B = 0$, we reproduce the correct $E_c$. Alternately, it could mean the method by which eq. (4) gives the correct new degenerate minimum of $V$, as determined by eq. (3) (Method A satisfies this criterion).

We usually bury our heads in the sand at this point, claiming the Higgs sector contribution to $V$ is small in the Standard Model anyway. As experimental limits on the Higgs mass creep upward, however, it becomes increasingly important to address these questions.

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