Abstract

In the current article, we would give some results related to concave function and introduce the definition of concaviable function and new notion of concaviable functions and obtain the new results in which involved concaviable function and we would obtain new majorization type results for weighted concaviable function. This article also recaptures the similar results for concave function as well as for convex function.

Keywords: Concave Function, Convex Function, Concaviable Function, Majorization.

I. Introduction and Preliminaries

Concepts of generalised concavity have been introduced and investigated by different authors in different articles, e.g., Hanson, Mangasarian, Ponstein, Karamardian, Greenberg and Pierskalla (see articles in [XI]).

A word of motivation is in order. Generalised concavity is important for several reasons. First, generalised concave functions arise naturally in applications. For example, unimodal probability density functions (like the gamma). Second, generalised concavity proves that this is useful in extreme value problems. Particularly in the theory of nonlinear programming, authors: Mangasarian and Zangwill have presented the case for the utility of generalised concavity in different articles (for this see [XI]).

Some other applications, in microeconomic theory and production functions that are usually assumed to be concave over some or all of their domains, resulting in diminishing returns to input factors [XIII]. Further that concavity of a function replaces the second derivative test to separate local maximum, minimum or saddle,
moreover, for a concave function a critical point which is local maximum (minimum) is global [XVIII].

Therefore, we say that theory of concave function has become a special domain of inequality’s theory it means they have closed relationship.

While convex theory plays an important role in several fields of physical sciences. This theory attracts many engineers and economists including mathematicians due to number of applications and important results in the above respective fields [VIII]. The theory of convex functions is almost implemented in every area of mathematics such as; differential equations, operations research, functional analysis, geometry, control theory, optimization, probability theory, operator theory, information theory, integral operator theory, numerical integration etc. Further that the convex function’s theory also acts as precious role in other several fields as: mechanics, statistics, finance, engineering, economics, management sciences and physics.

Here, we state useful definition which is extracted from [IX, XV]. Throughout the article, $L \subset \mathbb{R}$.

**Definition 1.1.** A function $\Psi: L \rightarrow \mathbb{R}$ is said to be concave if the following inequality holds

$$\Psi(\sigma u_1 + (1-\sigma)u_2) \geq \sigma \Psi(u_1) + (1-\sigma)\Psi(u_2)$$ (1.1)

$\forall \ u_1, u_2 \in L$ and $\sigma \in [0,1]$.

**Remark 1.2.** The following are the remarks about strictly concave, convex and strictly convex functions and recalled from [II, VI].

(1) If the inequality (1.1) is strict for each $u_1 \neq u_2$ and $\sigma \in (0,1)$, then $\Psi$ is known as strictly concave.

(2) If the inequality (1.1) is reversed, then $\Psi$ is known as convex and if it is strict for each $u_1 \neq u_2$ and $\sigma \in (0,1)$, then $\Psi$ is known as strictly convex.

For more study about higher order convex and concave functions (see [III, XVI]).

**1.3. Majorization.** The basic idea of majorization has come from measure of variety of m-tuple (m-dimensional) components of vector and it is nearly linked to convexity and concavity. The main contributors are Hardy, Littlewood & Polya, who discussed interesting basic ideas about the majorization in their book “Inequality”. Questions related to majorization were worked on by the comparatively few research scholars who were inspired by the book “Theory of Majorization and Its Application”, they put effort in order to rearrange ideas and to separate the literature valiantly. They have also given proofs on fundamental consequences and references to multiple points of view with respect to the wide range of applied discipline.

The application of theory of majorization is present in many fields such as pure and applied mathematics and engineering as well.
Here we state some definitions and results that would be used in sequel manner.

For fixed $m \geq 2$, $\mathbf{u} = (u_1, \ldots, u_m)$ and $\mathbf{v} = (v_1, \ldots, v_m)$ denote two $m$-tuples and $u[i] \leq \cdots \leq u[2] \leq u[1]$, $v[i] \leq \cdots \leq v[2] \leq v[1]$ be the ordered components.

**Definition 1.4.** For all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^m$,

$$\mathbf{u} \prec \mathbf{v} \text{ if } \sum_{i=1}^{k} v[i] \geq \sum_{i=1}^{k} u[i], \quad k \in \{1, \ldots, m-1\},$$

where $\mathbf{u} \prec \mathbf{v}$, $\mathbf{u}$ is majorized by $\mathbf{v}$ or $\mathbf{v}$ majorizes $\mathbf{u}$.

In [VII] Hardy et.al. has introduced this above notation for majorization.

The definition of majorization, stated in above involves the comparison of diversity of components of two $m$-tuples. J. E. Pečarić [IX, p.324] gave a similar definition for integrable functions. Let $\mathbf{u}$ and $\mathbf{v}$ be real-valued functions, stated in the interval $[p, q]$.

**Definition 1.5.** $\mathbf{u} \prec \mathbf{v}$ for $\mathbf{t} \in [p, q]$, if they are decreasing in $[p, q]$ and

$$\int_{p}^{s} u(t) \, dt \leq \int_{p}^{s} v(t) \, dt, \quad \text{for } s \in [p, q] \quad (1.2)$$

and equality in(1.2) holds for $s = q$.

**II. Results Related to Concave Function**

We give following theorem of majorization involving concave function.

**Theorem 2.1.** Let continuous function $\Psi: L \to \mathbb{R}$ be concave and $\mathbf{u} = (u_1, \ldots, u_m)$ and $\mathbf{v} = (v_1, \ldots, v_m)$ be two $m$-tuples, such that $u[i], v[i] \in L (i = 1, \ldots, m)$. If $\mathbf{u}$ majorizes $\mathbf{v}$, then we have the following inequality

$$\sum_{i=1}^{m} \Psi(u[i]) \geq \sum_{i=1}^{m} \Psi(v[i]) \quad (2.1)$$

**Proof.** If we put $\Psi = -\Psi$ in Proposition 4.B.1 of [XIV] then we get desired result.

**Remark 2.2.** (1) If function $\Psi$ be concave then $-\Psi$ be convex and vice versa.

(2) In the literature, the reverse of inequality (2.1) is said to be Karamata’s inequality [XI](also see [I,XIX]).

Now we introduce the following theorem for concave function $\Psi$.

**Theorem 2.3.** Suppose $\mathbf{u}$ and $\mathbf{v}$ are real-valued functions in the interval $[p, q]$ and $\mathbf{v} \prec \mathbf{u}$ for $\mathbf{t} \in [p, q]$ iff they are decreasing in $[p, q]$ and

$$\int_{p}^{q} \Psi(u(t)) \, dt \geq \int_{p}^{q} \Psi(v(t)) \, dt, \quad \text{for } t \in [p, q] \quad (2.2)$$

holds for each $\Psi$ that’s continuous and concave in $[p, q]$, provided integrals exist.

**Proof.** If we put $\Psi = -\Psi$ in Theorem 12.15 of [IX] then we get desired result.
Remark 2.4. If the inequality is reversed in above theorem, then obtain Theorem 12.15 of article [IX].

We give the following theorem which is concave form of Lemma 2 of [XII].

Theorem 2.5. Let \( \mathcal{W} \) be weighted integrable function and let \( u, v \) be non-negative integrable functions are defined on \([p, q]\) and suppose function \( \Psi : [0, \infty) \rightarrow \mathbb{R} \) is concave and that

\[
\int_p^r u(t) \mathcal{W}(t) \, dt \geq \int_p^r v(t) \mathcal{W}(t) \, dt,
\]

for all \( r \in [p, q] \) and

\[
\int_p^q u(t) \mathcal{W}(t) \, dt = \int_p^q v(t) \mathcal{W}(t) \, dt.
\]

(i) If \( u \) is decreasing on \([p, q]\), then

\[
\int_p^q \Psi(u(t)) \mathcal{W}(t) \, dt \geq \int_p^q \Psi(v(t)) \mathcal{W}(t) \, dt. \tag{2.3}
\]

(ii) If \( v \) is increasing on \([p, q]\), then

\[
\int_p^q \Psi(u(t)) \mathcal{W}(t) \, dt \leq \int_p^q \Psi(v(t)) \mathcal{W}(t) \, dt. \tag{2.4}
\]

Proof. If we show the inequalities for \( \varphi \in C^1(0, \infty) \) (i.e. \( \varphi' \) is continuous), then by smooth functions, the general case follows from point-wise approximation of \( \varphi \).

Because \( \varphi \) is a concave function in the interval \([0, \infty)\) it follows

\[\varphi(u(t)) - \varphi(v(t)) \geq (u(t) - v(t))\varphi'(u(t)); \quad u, v \geq 0.\]

(i) If we set \( F(r) = \int_p^r (u(t) - v(t)) \mathcal{W}(t) \, dt \), then by hypothesis \( F(1) \geq 0 \) for \( r \in [p, q] \) and \( F(p) = F(q) = 0 \).

If \( u \) is decreasing on \([p, q]\), then

\[
\int_p^q \mathcal{W}(t)[\Psi(u(t)) - \Psi(v(t))] \, dt \\
\geq \int_p^q \mathcal{W}(t)(u(t) - v(t))\Psi'(u(t)) \, dt \\
= \int_p^q \Psi'(u(t))dF(t).
\]

Using integration by parts in the Stieltjes integral, obtain

\[
\int_p^q \mathcal{W}(t)[\Psi(u(t)) - \Psi(v(t))] \, dt \\
\geq \Psi'(u(t))F(t)|_p^q - \int_p^q F(t) d\{\Psi'(u(t))\} \\
= - \int_p^q F(t) d\{\Psi'(u(t))\} \geq 0.
\]

(ii) Similarly, if \( v \) is increasing, then

\[
\int_p^q \mathcal{W}(t)[\Psi(v(t)) - \Psi(u(t))] \, dt
\]
\[
\int_p^q \mathcal{W}(\tau) \left( \Psi(\tau) - \Psi'(\tau) \right) \, d\tau \\
= \int_p^q \Psi'(\tau) \, d[-F(\tau)].
\]

Using integration by parts in the Stieltjes integral, obtain
\[
\int_p^q \mathcal{W}(\tau) \left( \Psi(\tau) - \Psi'(\tau) \right) \, d\tau \\
\geq -\Psi'(\tau) F(\tau) |^q_p + \int_p^q F(\tau) \, d[\Psi'(\tau)] \\
= \int_p^q F(\tau) \, d[\Psi'(\tau)] \geq 0.
\]

Hence inequalities are proved.

**Remark 2.6.** If the inequalities are reversed in above theorem, then obtain Lemma 2 of article [XII].

If we put \( \mathcal{W} = 1 \) in Theorem 2.5 then we obtain the following corollary.

**Corollary 2.7.** Let \( u, \nu \) be as in Theorem 2.5 and suppose function \( \Psi: [0, \infty) \rightarrow \mathbb{R} \) is concave and that
\[
\int_p^q u(\tau) \, d\tau \leq \int_p^q \nu(\tau) \, d\tau,
\]
for all \( r \in [p, q] \) and
\[
\int_p^q u(\tau) \, d\tau = \int_p^q \nu(\tau) \, d\tau.
\]

(i) If \( u \) is decreasing on \([p, q]\), then
\[
\int_p^q \Psi(u(\tau)) \, d\tau \geq \int_p^q \Psi(\nu(\tau)) \, d\tau.
\]

(ii) If \( \nu \) is increasing on \([p, q]\), then
\[
\int_p^q \Psi(u(\tau)) \, d\tau \leq \int_p^q \Psi(\nu(\tau)) \, d\tau.
\]

**III. Results Related to Concavifiable Functions**

3.1. **Concavifiable Functions.** In the following we introduce the definition of \textit{Concavifiable function} as S. Zlobec discussed \textit{Convexifiable function} in his article “Characterization of convexifiable function” [XVII](see also [I]).

**Definition 3.2.** Let a continuous function \( \Psi: L \rightarrow \mathbb{R} \) be define on compact interval \( L \subset \mathbb{R} \), assume another function \( \mathcal{F}: L \times \mathbb{R} \rightarrow \mathbb{R} \), stated as
\[
\mathcal{F}(u, \sigma) = \Psi(u) - \frac{\sigma}{2} u^2.
\]

If \( \mathcal{F}(u, \sigma) \) is concave function in the interval \( L \) for some \( \sigma = \sigma^* \), then \( \mathcal{F}(u, \sigma) \) is said to be concavification of \( \Psi \) and \( \sigma^* \) is its concavifier on \( L \). Function \( \Psi \) is concavifiable if it has a concavification.
A remark about concavifiable functions is given by us in the following as Muhammad Adil Khan had given in his article “Majorization theorem for convexifiable functions”.

**Remark 3.3.** If \( \sigma^* \) is a concavifier of \( \Psi \), then for each \( \sigma \geq \sigma^* \).

Concavifiable functions have been studied on \( \mathbb{R} \). The class of concavifiable functions is large: beside concave and twice continuous differentiable function.

Now we would obtain new version of Lemma 2 of [XII] in concavifiable function.

**Theorem 3.4.** Let integrable functions \( u \) and \( v \) be both non-negative and defined on \([p, q]\) and suppose function \( \varphi: [0, \infty) \rightarrow \mathbb{R} \) is a concavifiable and that

\[
\int_p^r u(t) \, dt \geq \int_p^r v(t) \, dt
\]

for all \( r \in [p, q] \) and \( \int_p^q u(t) \, dt = \int_p^q v(t) \, dt \).

(i) If function \( u \) is decreasing on \([p, q]\), then

\[
\int_p^q \Psi(u(t)) \, dt \geq \int_p^q \Psi(v(t)) \, dt + \frac{\sigma}{2} \int_p^q (u^2(t) - v^2(t)) \, dt. \tag{3.1}
\]

(ii) If function \( v \) is increasing on \([p, q]\), then

\[
\int_p^q \Psi(v(t)) \, dt \geq \int_p^q \Psi(u(t)) \, dt + \frac{\sigma}{2} \int_p^q (v^2(t) - u^2(t)) \, dt, \tag{3.2}
\]

where \( \sigma \in [0, 1] \).

**Proof.** From the proof of Theorem 2.5 we know that

\[
\varphi(u(t)) - \varphi(v(t)) \geq (u(t) - v(t)) \varphi'(u(t)); \quad u, v \geq 0.
\]

If \( \varphi \) function is concavifiable on \([0, \infty)\), then

\[
\Psi(u(t)) - \Psi(v(t)) - \frac{\sigma}{2} (u^2(t) - v^2(t)) \geq (u(t) - v(t)) [\varphi'(u(t)) - \sigma u(t)].
\]

(i) If we set \( F(r) = \int_p^r (u(t) - v(t)) \, dt \), then by hypothesis \( F(t) \geq 0 \) for \( r \in [p, q] \) and \( F(p) = F(q) = 0 \).

If \( u \) is decreasing on \([p, q]\), then

\[
\int_p^q \{ \Psi(u(t)) - \Psi(v(t)) - \frac{\sigma}{2} (u^2(t) - v^2(t)) \} \, dt \geq \int_p^q (u(t) - v(t)) [\varphi'(u(t)) - \sigma u(t)] \, dt
\]

\[
= \int_p^q [\varphi'(u(t)) - \sigma u(t)] F(t) \, dt.
\]
Using integration by parts in the Stieltjes integral, obtain
\[
\int_p^q \left( \Psi(u(t)) - \Psi'(u(t)) - \frac{\sigma}{2} (u^2(t) - \sigma^2(t)) \right) dt \\
\geq \left[ \Psi'(u(t)) - \sigma u(t) \right] F(t) \bigg|_p^q - \int_p^q F(t) d \left[ \Psi'(u(t)) - \sigma u(t) \right] \\
= - \int_p^q F(t) d \left[ \Psi'(u(t)) - \sigma u(t) \right] \geq 0.
\]

(ii) Similarly, if \( u \) is increasing, then
\[
\int_p^q \left( \Psi(u(t)) - \Psi'(u(t)) - \frac{\sigma}{2} (u^2(t) - \sigma^2(t)) \right) dt \\
\geq \int_p^q \left[ \Psi'(u(t)) - \sigma u(t) \right] F(t) dt \\
= \int_p^q F(t) d \left[ \Psi'(u(t)) - \sigma u(t) \right] \geq 0.
\]

Hence inequalities are proved.

**Remark 3.5.** (1) If put \( \sigma = 0 \) in Theorem 3.4, then recapture the Corollary 2.7.

(2) Theorem 3.4 also holds for convexifiable function if there exists reversed inequalities in above theorem (for this you can see our article [IV]).

Now, the following theorem is the weighted version of above theorem.

**Theorem 3.6.** Let \( W \) be weighted integrable function and let integrable functions \( u \) and \( v \) be both non-negative are defined on \([p, q]\) and suppose function \( \phi : [0, \infty) \rightarrow \mathbb{R} \) is concavifiable and that
\[
\int_p^q u(t)W(t)dt \geq \int_p^q \phi(u(t))W(t)dt,
\]
for all \( r \in [p, q] \) and
\[
\int_p^q u(t)W(t)dt = \int_p^q \phi(u(t))W(t)dt.
\]

(i) If function \( u \) is decreasing on \([p, q]\), then
\[
\int_p^q \Psi(u(t)) W(t) dt \geq \int_p^q \Psi(u(t)) W(t) dt + \int_p^q \left[ u^2(t) - \sigma^2(t) \right] W(t) dt \tag{3.3}
\]

(ii) If function \( v \) is increasing on \([p, q]\), then
\[
\int_p^q \Psi(v(t)) W(t) dt \geq \int_p^q \Psi(v(t)) W(t) dt + \int_p^q \left[ v^2(t) - \sigma^2(t) \right] W(t) dt, \tag{3.4}
\]
where $\sigma \in [0,1]$.

**Proof.** This result has same proof as we have done in Theorem 3.4 but involving weights $\mathcal{W}$.

**Remark 3.7.** (1) If put $\sigma = 0$ in Theorem 3.6, then recapture the Theorem 2.5 and then if we put $\mathcal{W} = 1$ in obtained inequalities, we recapture the Corollary 2.7.

(2) Theorem 3.6 also holds for convexifiable function involving weights if there exists reversed inequalities in above theorem (for this you can see our article [IV]).

**IV. New Majorization Type Results for Weighted Concavifiable Function**

We give integral version of result of article [V] with concavifiable function in the form of following theorem.

**Theorem 4.1.** Let function $\varphi$ be decreasing from $(p,q)$ to $(r,s)$. Also function $\zeta \circ \varphi^{-1}$ be decreasing on $(r,s)$. Let functions $u$ and $\sigma$ be non-negative for all $\mathbf{T} \in [p,q]$ and $\mathcal{W}$ be weighted integrable function and suppose function $\varphi: [0,\infty) \rightarrow \mathbb{R}$ is concavifiable and that

$$f_p^q \mathcal{W}(\mathbf{T}) \varphi(u(\mathbf{T})) \, d \mathbf{T} \geq f_p^q \mathcal{W}(\mathbf{T}) \varphi(\sigma(\mathbf{T})) \, d \mathbf{T}.$$

(a) If (i) $\zeta$ is decreasing and

(ii) The function $u$ is decreasing, then

$$f_p^q \mathcal{W}(\mathbf{T}) \zeta(u(\mathbf{T})) \, d \mathbf{T} \geq f_p^q \mathcal{W}(\mathbf{T}) \zeta(\sigma(\mathbf{T})) \, d \mathbf{T} + \frac{\sigma}{2} f_p^q \mathcal{W}(\mathbf{T})(u^2(\mathbf{T}) - \sigma^2(\mathbf{T})) \, d \mathbf{T}.$$

(b) If (i) $f_p^q \mathcal{W}(\mathbf{T}) \varphi(u(\mathbf{T})) \, d \mathbf{T} = f_p^q \mathcal{W}(\mathbf{T}) \varphi(\sigma(\mathbf{T})) \, d \mathbf{T}$ and

(ii) The function $u$ is decreasing, then

$$f_p^q \mathcal{W}(\mathbf{T}) \zeta(u(\mathbf{T})) \, d \mathbf{T} = f_p^q \mathcal{W}(\mathbf{T}) \zeta(\sigma(\mathbf{T})) \, d \mathbf{T} + \frac{\sigma}{2} f_p^q \mathcal{W}(\mathbf{T})(u^2(\mathbf{T}) - \sigma^2(\mathbf{T})) \, d \mathbf{T}.$$

(c) If (i) $\zeta$ is increasing and

(ii) The function $\sigma$ is increasing, then

$$f_p^q \mathcal{W}(\mathbf{T}) \zeta(u(\mathbf{T})) \, d \mathbf{T} \geq f_p^q \mathcal{W}(\mathbf{T}) \zeta(\sigma(\mathbf{T})) \, d \mathbf{T} + \frac{\sigma}{2} f_p^q \mathcal{W}(\mathbf{T})(\sigma^2(\mathbf{T}) - u^2(\mathbf{T})) \, d \mathbf{T}.$$

(d) If (i) $f_p^q \mathcal{W}(\mathbf{T}) \varphi(u(\mathbf{T})) \, d \mathbf{T} \leq f_p^q \mathcal{W}(\mathbf{T}) \varphi(\sigma(\mathbf{T})) \, d \mathbf{T}$ and

(ii) The function $\sigma$ is increasing, then

$$f_p^q \mathcal{W}(\mathbf{T}) \zeta(u(\mathbf{T})) \, d \mathbf{T} \leq f_p^q \mathcal{W}(\mathbf{T}) \zeta(\sigma(\mathbf{T})) \, d \mathbf{T} + \frac{\sigma}{2} f_p^q \mathcal{W}(\mathbf{T})(\sigma^2(\mathbf{T}) - u^2(\mathbf{T})) \, d \mathbf{T}.$$
where $\sigma \in [0, 1]$. 

**Proof.** From the proof of Theorem 2.5 we know that

$$
\phi(u(t)) - \phi(\nu(t)) \geq (u(t) - \nu(t))\phi'(u(t)); \quad u, \nu \geq 0.
$$

If $\phi$ function is a concaveifiable on $[0, \infty)$, then

$$
\zeta(u(t)) - \zeta(\nu(t)) - \frac{\sigma}{2}(u^2(t) - \nu^2(t)) \geq (u(t) - \nu(t))[\zeta'(u(t)) - \sigma u(t)].
$$

(a) If we set $F(r) = \int_{p}^{q} (u(t) - \nu(t)) \mathcal{W}(t) d \tau$, then by hypothesis $F(t) \geq 0$ for $r \in [p, q]$ and $F(p) = F(q) = 0$.

If $u$ and $\zeta$ are decreasing on $[p, q]$, then

$$
\int_{p}^{q} [\zeta(u(t)) - \zeta(\nu(t)) - \frac{\sigma}{2}(u^2(t) - \nu^2(t))] \mathcal{W}(t) d \tau \\
\geq \int_{p}^{q} (u(t) - \nu(t)) [\zeta'(u(t)) - \sigma u(t)] \mathcal{W}(t) d \tau \\
= \int_{p}^{q} [\zeta'(u(t)) - \sigma u(t)] dF(t).
$$

Using integration by parts in the Stieltjes integral, obtain

$$
\int_{p}^{q} [\zeta(u(t)) - \zeta(\nu(t)) - \frac{\sigma}{2}(u^2(t) - \nu^2(t))] \mathcal{W}(t) d \tau \\
\geq [\zeta'(u(t)) - \sigma u(t)]F(t)]_{p}^{q} - \int_{p}^{q} F(t) d [\zeta'(u(t)) - \sigma u(t)] \\
= - \int_{p}^{q} F(t) d [\zeta'(u(t)) - \sigma u(t)] \geq 0.
$$

(b) If we set $F(r) = \int_{p}^{q} (u(t) - \nu(t)) \mathcal{W}(t) d \tau$, then by hypothesis $F(t) = 0$ for $r \in [p, q]$ and $F(p) = F(q) = 0$.

If $u$ is decreasing on $[p, q]$, then

$$
\int_{p}^{q} [\zeta(u(t)) - \zeta(\nu(t)) - \frac{\sigma}{2}(u^2(t) - \nu^2(t))] \mathcal{W}(t) d \tau \\
= \int_{p}^{q} (u(t) - \nu(t)) [\zeta'(u(t)) - \sigma u(t)] \mathcal{W}(t) d \tau \\
= \int_{p}^{q} [\zeta'(u(t)) - \sigma u(t)] dF(t) = 0.
$$

(c) If we set $F(r) = \int_{p}^{q} (u(t) - \nu(t)) \mathcal{W}(t) d \tau$, then by hypothesis $F(t) \geq 0$ for $r \in [p, q]$ and $F(p) = F(q) = 0$.

If $\nu$ and $\zeta$ are increasing on $[p, q]$, then

$$
\int_{p}^{q} [\zeta(\nu(t)) - \zeta(u(t)) - \frac{\sigma}{2}(\nu^2(t) - u^2(t))] \mathcal{W}(t) d \tau \\
\geq \int_{p}^{q} (\nu(t) - u(t)) [\zeta'(\nu(t)) - \sigma \nu(t)] \mathcal{W}(t) d \tau
$$
Using integration by parts in the Stieltjes integral, obtain
\[
\int_p^q \left[ \zeta' \left( \nu(1) \right) - \sigma \nu(1) \right] d[F(1)] \geq -\int_p^q \left[ \zeta' \left( \nu(1) \right) - \sigma \nu(1) \right] d[F(1)] + \int_p^q \left[ \zeta' \left( \nu(1) \right) - \sigma \nu(1) \right] W(1) d1
\]
\[
= \int_p^q \left[ \zeta' \left( \nu(1) \right) - \sigma \nu(1) \right] W(1) d1 \geq 0.
\]
(d) If we set \( F(r) = \int_p^r \left( u(1) - \nu(1) \right) W(1) d1 \), then by hypothesis \( F(1) \leq 0 \) for \( r \in [p, q] \) and \( F(p) = F(q) = 0 \). If \( \sigma \) is increasing on \( [p, q] \), then
\[
\int_p^q \left[ \zeta' \left( \nu(1) \right) - \sigma \nu(1) \right] W(1) d1 \leq \int_p^q \left[ \zeta' \left( \nu(1) \right) - \sigma \nu(1) \right] W(1) d1
\]
\[
= \int_p^q \left[ \zeta' \left( \nu(1) \right) - \sigma \nu(1) \right] d[F(1)].
\]
Hence inequalities are proved.

**Remark 4.2.**

1. If put \( \sigma = 0 \) in Theorem 4.1, then we get similar results for concave function.
2. Theorem 4.1 also holds for convexifiable function involving weights if there exists reversed inequalities in above theorem (for this you can see our article [IV]).

**V. Conclusion**

In the last section, we have given some results related to concave function and introduced the class of convexifiable functions and given its results related to the convexifiable function and weighted convexifiable function. We have also obtained majorization type results for weighted convexifiable function. By using the results of this article, we can recapture the similar results of proved theorems for concave function and for convex function.
References

I. Adil Khan, Majorization theorems for convexifiable functions, Math. Commun., 18 (2013), 61–65.

II. Asif R. Khan and Faraz Mehmood, Some Remarks on Functions with Non-decreasing Increments, Journal of Mathematical Analysis, 11 (1) (2020), 1–16.

III. Asif R. Khan, Faraz Mehmood, Faisal Nawaz and Aamna Nazir, Some Remarks on Results Related to $V$–Convex Function, Submitted.

IV. Asif R. Khan, Faraz Mehmood and M. Azeem Ullah Siddique, Some Results Related to Convexifiable Functions, to appear.

V. Ehtisham Karim, Asif R. Khan and Syeda Sadia Zia, On Majorization Type Results, Commun. Optim. Theory, 2015, 2015:5, 1–17.

VI. Faraz Mehmood, On Function with Nondecreasing Increments, (Unpublished doctoral dissertation), Department of Mathematics, University of Karachi, Karachi, Pakistan, 2019.

VII. G. H. Hardy, J. E. Littlewood, G. Pólya, Inequalities, 2nd Ed. Cambridge University Press, England, (1952).

VIII. I. C. P. Niculescu and L. E. Persson, Convex functions and their applications, A contemporary approach, Springer, New York, (2006).

IX. J. E. Pečarić, F. Proschan and Y. L. Tong, Convex functions, partial orderings and statistical applications, Academic Press, New York, 187(1992).

X. J. Karamata, Sur une inégalité rélatitive aux fonctions convexes, Publ. Math. Univ. Belgrade, 1 (1932), 145–148.

XI. Jr. W. A. Thompson and Darrel W. Parke, Some Properties of Generalized Concave Functions, Operations Research 21 1 (Jan. - Feb., 1973), 305–313.

XII. L. Maligranda, J. E. Pečarić and L. E. Persson, Weighted Favard and Berwald inequalities, J. Math. Anal. Appl, (1995), pp.248–262.

XIII. Malcolm Pemberton and Nicholas Rau, Mathematics for Economists: An Introductory Textbook, Oxford University Press (2015) 363–364. ISBN 978-1-78499-148-7.

XIV. Marshall, A. W., and Olkin, I., Inequalities: Theory of Majorization and Its Applications, Academic Press, New York (1979).

XV. Martin J. Osborne, Mathematical methods for economic theory, version 2019-09-05, site built on the core of the OJS system.
XVI. Muhammad Adnan, A. R. Khan and Faraz Mehmood, Positivity of sums and integrals for higher order $\nabla$-convex and completely monotonic functions, arXiv:1710.07182v1, [math.CA], 13 Oct 2017.

XVII. S. Zlobec, Characterization of convexiable function, Optimization 55(2006), 251–261.

XVIII. Simon, Convex and concave function, Chapter 21, p. 505–522.

XIX. Z. Kadelburg, D. Dukić, M. Lukić, I. Matić, Inequalities of Karamata, Schur and Muirhead, and some application, The Teaching of Mathematics VIII (2005), 31–45.