THE HOWE DUALITY AND LIE SUPERALGEBRAS

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ABSTRACT. Howe’s duality is considered from a unifying point of view based on Lie superalgebras. New examples are offered. In particular, we construct several simplest spinor-oscillator representations and compute their highest weights for the “stringy” Lie superalgebras (i.e., Lie superalgebras of complex vector fields (or their nontrivial central extensions) on the supercircle $S^1|n$ and its two-sheeted cover associated with the Möbius bundle).

In our two lectures we briefly review, on the most elementary level, several results and problems unified by “Howe’s duality”. Details will be given elsewhere. The ground field in the lectures is $\mathbb{C}$.

§1. Introduction

In his famous preprint [24] R. Howe gave an inspiring explanation of what can be “dug out” from H. Weyl’s “wonderful and terrible” book [55], at least as far as invariant theory is concerned, from a certain unifying viewpoint. According to Howe, much is based on a remarkable correspondence between certain irreducible representations of Lie subalgebras $\Gamma$ and $\Gamma'$ of the Lie algebra $\mathfrak{o}(V)$ or $\mathfrak{sp}(V)$ provided $\Gamma$ and $\Gamma'$ are each other’s “commutants”, i.e., centralizers. This correspondence is known ever since as Howe’s correspondence or Howe’s duality. In [24] and subsequent papers Howe gave several examples of such a correspondence previously known, mostly, inadvertently. Let us remind some of them (omitting important Jacquet-Langlands-Shimizu correspondence, S. Gelbart’s contributions, etc.) :

1) decomposition of $\mathfrak{o}(V)$-module $S^*(V)$ into spherical harmonics;
2) Lefschetz decomposition of $\mathfrak{sp}(V)$-module $\Lambda^*(V)$ into primitive forms (sometimes this is called Hodge–Lépage decomposition);
3) a striking resemblance between spinor representation of $\mathfrak{o}(n)$ and oscillator (Shale–Segal–Weil–metaplectic–...) representation of $\mathfrak{sp}(2n)$.

As an aside Howe gives the “shortest possible” proof of the Poincaré lemma. (Recall that this lemma states that in any sufficiently small open star-shaped neighborhood of any point on any manifold any closed differential form is exact.) In this proof, Lie superalgebras, that lingered somewhere in the background in the previous discussion but were treated rather as a nuisance than help, are instrumental to reach the goal. This example shows also that the requirement of reductivity of $\Gamma$ and $\Gamma'$ to form a “dual pair” is extra. Elsewhere we will investigate what are the actual minimal restrictions on $\Gamma$ and $\Gamma'$ needed to reach one of the other problems usually solved by means of Howe duality: decompose the symmetric or exterior algebra of a module over $\Gamma \oplus \Gamma'$. Howe’s manuscript was written at the time when supersymmetry theory was being conceived. By the time [24] was typed, the definition of what is nowadays called superschemes ([34]) was not yet rewritten in terms to match

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physical papers (language of points was needed; now we can recommend [3]) nor translated
into English and, therefore, was unknown; the classification of simple finite dimensional Lie
superalgebras over $\mathbb{C}$ had just been announced. This was, perhaps, the reason for a cau-
tious tone with which Howe used Lie superalgebras, although he made transparent how
important they might be for a lucid presentation of his ideas and explicitly stated so.

Since [25], the published version of [24], though put aside to stew for 12 years, underwent
only censorial changes, we believe it is of interest to explore what do we gain by using Lie
superalgebras from the very beginning (an elaboration of other aspects of this idea [4] are not
published yet). Here we briefly elucidate some of Howe’s results and notions and give several
new examples of Howe’s dual pairs. In the lectures we will review the known examples 1) –
3) mentioned above but consider them in an appropriate “super” setting, and add to them:

4) a refinement of the Lefschetz decomposition — J. Bernstein’s decomposition (2) of the
space $\Omega^*_{\mathfrak{h}}$ of “twisted” differential forms on a symplectic manifold with values in a line bundle
with connection whose curvature form differs by a factor $\hbar$ from the canonical symplectic
form;

5) a decomposition of the space of differential forms on a hyper-Kählerian manifold sim-
ilar to the Lefschetz one (33) but with $\mathfrak{sp}(4)$ instead of $\mathfrak{sp}(2) = \mathfrak{sl}(2)$ and its refinement
associated with the $\mathfrak{osp}(1|4)$.

6) Apart from general clarification of the scenery and new examples even in the old setting,
i.e., on manifolds, the superalgebras introduced ab ovo make it manifest that there are at
least two types of Howe’s correspondence: the conventional one and several “ghost” ones
associated with quantization of the antibracket [40].

7) Obviously, if $\Gamma \oplus \Gamma'$ is a maximal subalgebra of $\mathfrak{osp}$, then $(\Gamma, \Gamma')$ is an example of Howe
dual pair. Section 6 gives some further examples, partly borrowed from 18, where more
examples can be found.

We consider here only finite dimensional Lie superalgebras with the invariant theory in
view. In another lecture (§§3,4) we consider spinor-oscillator representations in more detail.
In these elementary talks we do not touch other interesting applications such as Capelli iden-
tities (18, 19), or prime characteristic (17). Of dozens of papers with examples of Howe’s
duality in infinite dimensional cases and still other examples, we draw attention of the reader
to the following selected ones: 12, and various instances of bose-fermi correspondence, cf.
13 and 20. Observe also that the Howe duality often manifests itself for $q$-deformed alge-
bias, e.g., in Klimyk’s talk at our conference, or 23. To treat this $q$-Howe duality in a similar
way, we first have to explicitly $q$-quantize Poisson superalgebras $\mathfrak{po}(2n|m)$ (for $mn = 0$ this
is straightforward replacement of (super)commutators from 33 with $q$-(super)commutators.

§2. The Poisson superalgebra $\mathfrak{g} = \mathfrak{po}(2n|m)$

2.1. Certain $\mathbb{Z}$-gradings of $\mathfrak{g}$. Recall that $\mathfrak{g}$ is the Lie superalgebra whose superspace
is $\mathbb{C}[q,p,\Theta]$ and the bracket is the Poisson bracket $\{\cdot,\cdot\}_{PB}$ is given by the formula

$$
\{f,g\}_{PB} = \sum_{i \leq n} \left( \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right) - (-1)^{p(f)} \sum_{j \leq m} \frac{\partial f}{\partial \theta_j} \frac{\partial g}{\partial \theta_j} \quad \text{for } f, g \in \mathbb{C}[p,q,\Theta].
$$

(2.1)

Sometimes it is more convenient to redenote the $\Theta$’s and set

$$
\xi_j = \frac{1}{\sqrt{2}}(\Theta_j - i\Theta_{r+j}); \quad \eta_j = \frac{1}{\sqrt{2}}(\Theta_j + i\Theta_{r+j})
$$

for $j \leq r = [m/2]$ (here $i^2 = -1$), $\theta = \Theta_{2r+1}$
and accordingly modify the bracket (if \( m = 2r \), there is no term with \( \theta \)):

\[
\{ f, g \}_\text{P.h.} = \sum_{i \leq n} \left( \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right) - (-1)^{p(f)} \left[ \sum_{j \leq m} \left( \frac{\partial f}{\partial \xi_j} \frac{\partial g}{\partial \eta_j} + \frac{\partial f}{\partial \eta_j} \frac{\partial g}{\partial \xi_j} \right) + \frac{\partial f}{\partial \theta} \frac{\partial g}{\partial \theta} \right].
\]

Setting \( \deg_{\text{Lie}} f = \deg f - 2 \) for any monomial \( f \in \mathbb{C}[p, q, \Theta] \), where \( \deg p_i = \deg q_i = \deg \Theta_j = 1 \) for all \( i, j \), we obtain the standard \( \mathbb{Z} \)-grading of \( \mathfrak{g} \):

| degree of \( f \) | -2 | -1 | 0 | 1 | \ldots |
|-------------------|----|----|---|---|--------|
| \( f \)           | \( p, q, \theta \) | \( f : \deg f = 2 \) | \( f : \deg f = 3 \) | \ldots |

Clearly, \( \mathfrak{g} = \bigoplus_{i \geq -2} \mathfrak{g}_i \) with \( \mathfrak{g}_0 \simeq \mathfrak{osp}(m|2n) \). Consider now another, “rough”, grading of \( \mathfrak{g} \). To this end, introduce: \( Q = (q, \xi) \), \( P = (p, \eta) \) and set

\[
\deg Q_i = 0, \quad \deg \theta = 1, \quad \deg P_i = \begin{cases} 1 & \text{if } m = 2k \\ 2 & \text{if } m = 2k + 1. \end{cases} \tag{*}
\]

**Remark.** Physicists prefer to use half-integer values of \( \deg \) for \( m = 2k + 1 \) by setting \( \deg \theta = \frac{1}{2} \) and \( \deg P_i = 1 \) at all times.

The above grading (\( \ast \)) of the polynomial algebra induces the following *rough grading* of the Lie superalgebra \( \mathfrak{g} \). For \( m = 2k \) just delete the columns of odd degrees and delete the degrees by 2:

\[
\begin{array}{c|c|c|c|c|c}
\text{degree} & \ldots & 2 & 1 & 0 & -1 & -2 \\
\hline
\text{elements} & \ldots & \mathbb{C}[Q]^P_2 & \mathbb{C}[Q] \theta & \mathbb{C}[Q]^P & \mathbb{C}[Q] \theta & \mathbb{C}[Q] \\
\end{array}
\]

### 2.2. Quantization

We call the nontrivial deformation \( Q \) of the Lie superalgebra \( \mathfrak{po}(2n|m) \) quantization (for details see \( \cite{[1]} \)). There are many ways to quantize \( \mathfrak{g} \), but all of them are equivalent. Recall that we only consider \( \mathfrak{g} \) whose elements are represented by polynomials; for functions of other types (say, Laurent polynomials) the uniqueness of quantization may be violated.

Consider the following quantization, so-called \( QP \)-quantization, given on linear terms by the formulas:

\[
Q : Q \mapsto \hat{Q}, \quad P \mapsto \hbar \frac{\partial}{\partial Q}, \tag{*}
\]

where \( \hat{Q} \) is the operator of left multiplication by \( Q \); an arbitrary monomial should be first rearranged so that the \( Q \)'s stand first (normal form) and then apply (\( \ast \)) term-wise.

The deformed Lie superalgebra \( Q(\mathfrak{po}(2n|2k)) \) is the Lie superalgebra of differential operators with polynomial coefficients on \( \mathbb{R}^{n|k} \). Actually, it is an analog of \( \mathfrak{gl}(V) \). This is most clearly seen for \( n = 0 \). Indeed,

\[
Q(\mathfrak{po}(0|2k)) = \mathfrak{gl}(\Lambda^* \langle \xi \rangle) = \mathfrak{gl}(2^{k-1}|2^{k-1}).
\]

In general, for \( n \neq 0 \), we have

\[
Q(\mathfrak{po}(2n|2k)) = \left. \mathfrak{gl}(\mathcal{F}(Q)) = \mathfrak{diff}(\mathbb{R}^{n|k}) \right. \]

For \( m = 2k - 1 \) we consider \( \mathfrak{po}(0|2k - 1) \) as a subalgebra of \( \mathfrak{po}(0|2k) \); the quantization sends \( \mathfrak{po}(0|2k - 1) \) into \( \mathfrak{q}(2^{k-1}) \). For \( n \neq 0 \) the image of \( Q \) is an infinite dimensional analog of \( \mathfrak{q} \), indeed (for \( J = i(\theta + \frac{\partial}{\partial \theta}) \) with \( i^2 = -1 \)):

\[
Q(\mathfrak{po}(2n|2k - 1)) = \mathfrak{q} \mathfrak{diff}(\mathbb{R}^{n|k}) = \{ D \in \mathfrak{diff}(\mathbb{R}^{n|k}) : [d, J] = 0 \}. \]

**2.3. Fock spaces and spinor-oscillator representations.** The Lie superalgebras \( \mathfrak{diff}(\mathbb{R}^{n|k}) \) and \( \mathfrak{q} \mathfrak{diff}(\mathbb{R}^{n|k}) \) have indescribably many irreducible representations even for \( n = 0 \). But one of the representations, the identity one, in the superspace of functions on \( \mathbb{R}^{n|k} \), is
the “smallest” one. Moreover, if we consider the superspace of \( \text{diff}(\mathbb{R}^{n|k}) \) or \( \text{qdiff}(\mathbb{R}^{n|k}) \) as the associative superalgebra (denoted \( \text{Diff}(\mathbb{R}^{n|k}) \) or \( \text{QDiff}(\mathbb{R}^{n|k}) \)), this associative superalgebra has only one irreducible representation — the same identity one. This representation is called the Fock space.

As is known, the Lie superalgebras \( \mathfrak{osp}(m|2n) \) are rigid for \( (m|2n) \neq (4|2) \). Therefore, the through map

\[
\mathfrak{h} \longrightarrow \mathfrak{g}_0 = \mathfrak{osp}(m|2n) \subset \mathfrak{g} = \mathfrak{po}(2n|m) \rightarrow \mathfrak{diff}(\mathbb{R}^{n|k})
\]

sends any subsuperalgebra \( \mathfrak{h} \) of \( \mathfrak{osp}(m|2n) \) (for \( (m|2n) \neq (4|2) \)) into its isomorphic image. (One can also embed \( \mathfrak{h} \) into \( \mathfrak{diff}(\mathbb{R}^{n|k}) \) directly.) The irreducible subspace of the Fock space which contains the constants is called the spinor-oscillator representation of \( \mathfrak{h} \). In particular, cases, for \( m = 0 \) or \( n = 0 \) this subspace turns into the usual spinor or oscillator representation, respectively. We have just given a unified description of them. (A more detailed description follows.)

2.4. Primitive alias harmonic elements. The elements of \( \mathfrak{osp}(m|2n) \) (or its subalgebra \( \mathfrak{h} \)) act in the space of the spinor-oscillator representation by inhomogeneous differential operators of order \( \leq 2 \) (order is just the filtration associated with the “rough” grading):

\[
\begin{array}{c|c|c|c}
\text{degree} & \text{elements} & \text{degree} & \text{elements} \\
-1 & P^2 & -2 & P^2 \\
0 & PQ & -1 & Pθ \\
1 & Q^2 & 0 & PQ \\
\end{array}
\]

The elements from \((\mathbb{C}[Q])^{P^2}\) for \( m = 2k \) or \((\mathbb{C}[Q, θ])^{Pθ}\) for \( m = 2k + 1 \) are called primitive or harmonic ones. More generally, let \( \mathfrak{h} \subset \mathfrak{osp}(m|2n) \) be a \( \mathbb{Z} \)-graded Lie superalgebra embedded consistently with the rough grading of \( \mathfrak{osp}(m|2n) \). Then the elements from \((\mathbb{C}[Q])^{h-1}\) for \( m = 2k \) or \((\mathbb{C}[Q, θ])^{h-1}\) for \( m = 2k + 1 \) will be called \( \mathfrak{h} \)-primitive or \( \mathfrak{h} \)-harmonic.

2.4.1. Nonstandard \( \mathbb{Z} \)-gradings of \( \mathfrak{osp}(m|2n) \). It is well known that one simple Lie superalgebra can have several nonequivalent Cartan matrices and systems of Chevalley generators, cf. [20]. Accordingly, the corresponding divisions into positive and negative root vectors are distinct. The following problem arises: How the passage to nonstandard gradings affects the highest weight of the spinor-oscillator representation defined in sec. 3? (Cf. [44].)

2.5. Examples of dual pairs. Two subalgebras \( \Gamma, \Gamma' \) of \( \mathfrak{g}_0 = \mathfrak{osp}(m|2n) \) will be called a dual pair if one of them is the centralizer of the other in \( \mathfrak{g}_0 \).

If \( \Gamma \oplus \Gamma' \) is a maximal subalgebra in \( \mathfrak{g}_0 \), then, clearly, \( \Gamma, \Gamma' \) is a dual pair. A generalization: consider a pair of mutual centralizers \( \Gamma, \Gamma' \) in \( \mathfrak{gl}(V) \) and embed \( \mathfrak{gl}(V) \) into \( \mathfrak{osp}(V \oplus V^*) \). Then \( \Gamma, \Gamma' \) is a dual pair (in \( \mathfrak{osp}(V \oplus V^*) \)). For a number of such examples see [49]. Let us consider several of these examples in detail.

2.5.1. \( \Gamma = \mathfrak{sp}(2n) = \mathfrak{sp}(W) \) and \( \Gamma' = \mathfrak{sp}(2) = \mathfrak{sl}(2) = \mathfrak{sp}(V \oplus V^*) \). Clearly, \( \mathfrak{h} = \Gamma \oplus \Gamma' \) is a maximal subalgebra in \( \mathfrak{o}(W \otimes (V \oplus V^*)) \). The Fock space is just \( \Lambda'(W) \).

The following classical theorem and its analog 5.2 illustrate the importance of the above notions and constructions.

**Theorem.** The \( \Gamma' \)-primitive elements of \( \Lambda'(W) \) of each degree \( i \) constitute an irreducible \( \Gamma \)-module \( P_{\mathfrak{sp}}^i \), \( 0 \leq i \leq n \).

This action of \( \Gamma' \) in the superspace of differential forms on any symplectic manifold is well known: \( \Gamma' \) is generated (as a Lie algebra) by operators \( X_+ \) of left multiplication by the symplectic form \( \omega \) and \( X_- \), application of the bivector dual to \( \omega \).

2.5.2. \( \Gamma = \mathfrak{o}(2n) = \mathfrak{o}(W) \) and \( \Gamma' = \mathfrak{sp}(2) = \mathfrak{sl}(2) = \mathfrak{sp}(V \oplus V^*) \). Clearly, \( \mathfrak{h} = \Gamma \oplus \Gamma' \) is a maximal subalgebra in \( \mathfrak{sp}(W \otimes (V \oplus V^*)) \). The Fock space is just \( S'(W) \).

**Theorem.** The \( \Gamma' \)-primitive elements of \( S'(W) \) of each degree \( i \) constitute an irreducible \( \Gamma \)-module \( P_{\mathfrak{o}}^i \), \( i = 0, 1, \ldots \).
This action of $\Gamma'$ in the space of polynomial functions on any Riemann manifold is also well known: $\Gamma'$ is generated (as a Lie algebra) by operators $X_\pm$ of left multiplication by the quadratic polynomial representing the metric $g$ and $X_-$ is the corresponding Laplace operator.

Clearly, a mixture of Examples 2.5.1 and 2.5.2 corresponding to symmetric or skew-symmetric forms on a supermanifold is also possible: the space of $\Gamma'$-primitive elements of the space of twisted differential forms $\Omega = \ker D_\pm \cap \mathcal{P}_h$ is equal to $\mathfrak{g}_0$ (mod 2), for $\varphi \in \Omega$, $s \in \Omega_h^0$.

**Theorem.** (2) On $\Omega_h$, the operators $D_+$ and $D_-$ generate an action of the Lie superalgebra $\mathfrak{osp}(1|2)$ commuting with the action of the group $\hat{G}$ of $\nabla$-preserving automorphisms of the bundle $L$.

Bernstein studied the $\hat{G}$-action, more exactly, the action of the Lie algebra $\mathfrak{po}(2n|0)$ corresponding to $\hat{G}$; we are interested in the part of this action only: in $\mathfrak{sp}(2n) = \mathfrak{po}(2n|0)_0$-action.

In Example 2.5.1 the space $\mathcal{P}_h$ consisted of differential forms with constant coefficients. Denote by $\mathcal{P}^i = \mathcal{P}_h^i \otimes S^i(V)$ the space of primitive forms with polynomial coefficients. The elements of the space $\sqrt{\mathcal{P}_h^i} = \ker D_\pm \cap \mathcal{P}_h^i$ will be called $\nabla$-primitive forms of degree $i$ and twist $h$.

Bernstein showed that $\sqrt{\mathcal{P}_h^i}$ is an irreducible $\mathfrak{g}_0$-module. It could be that over subalgebra $\mathfrak{g}_0$ the module $\sqrt{\mathcal{P}_h^i}$ becomes reducible but the general theorem of Howe (which is true for $\mathfrak{osp}(1|2n)$) states that this is not the case, it remains irreducible. Shapovalov and Shmel’ev literally generalized Bernstein’s result for $(2n|m)$-dimensional supermanifolds, see review [31]. In particular, Shapovalov, who considered $n = 0$, “took a square root of Laplacian and the metric”.

**2.5.4.** Inspired by Bernstein’s construction, let us similarly define a “square root” of the hyper-Kähler structure. Namely, on a hyper-Kählerian manifold $(M, \omega_1, \omega_2)$ consider a line bundle $L$ with two connections: $\nabla_1$ and $\nabla_2$, whose curvature forms are equal to $h_1 \omega_1$ and $h_2 \omega_2$ for some $h_1, h_2 \in \mathbb{C}$. The pair $h = (h_1, h_2)$ will be called a twist; the space of tensor fields of type $\rho$ and twist $h$ will be denoted by $T_h(\rho)$. Verbitsky [39] defined the action of $\mathfrak{sp}(4)$ in the space $\Omega$ of differential forms on $M$. Let us naturally extend the action of the generators $X_j^\pm$ for $j = 1, 2$ of $\mathfrak{sp}(4)$ from $\Omega$ onto the space $\Omega_h$ of twisted differential forms using the isomorphism $T_h(\rho) \simeq T(\rho) \otimes \Gamma(L)$, where $\Gamma(L) = \Omega_h^0$ is the space of sections of the
line bundle $L$; here $X^+_j$ is the operator of multiplication by $\omega_j$ and $X^-_j$ is the operator of convolution with the dual bivector.

Define the space of primitive $i$-forms (with constant coefficients) on the hyper-Kählerian manifold $(M, \omega_1, \omega_2)$ by setting

$$P^i = \ker X^-_1 \cap \ker X^-_2 \cap \Omega^i.$$  \hfill (HK)

According to the general theorem \[25\] this space is an irreducible $\mathfrak{sp}(2n; \mathbb{H})$-module.

Set $D^-_i = [X^-_i, D^+_i]$. The promised square root of the decomposition (HK) is the space

$$\mathcal{P}^i_h = \ker D^-_1 \cap \ker D^-_2 \cap \Omega^i_h.$$ \hfill (\sqrt{HK})

The operators $D^\pm_i$, where $D^+_i = \nabla_i$, generate $\mathfrak{osp}(1|4)$.

### 2.6. Further examples of dual pairs.

The following subalgebras $\mathfrak{g}_1(V_1) \oplus \mathfrak{g}_2(V_2)$ are maximal in $\mathfrak{g}(V_1 \otimes V_2)$, hence, are dual pairs:

| $\mathfrak{g}_1$         | $\mathfrak{g}_2$         | $\mathfrak{g}$                        |
|------------------------|------------------------|-------------------------------------|
| $\mathfrak{osp}(n_1|2m_1)$ | $\mathfrak{osp}(n_2|2m_2)$ | $\mathfrak{osp}(n_1n_2 + 4m_1m_2|2n_1m_2 + 2n_2m_1)$ |
| $\mathfrak{o}(n)$      | $\mathfrak{osp}(n_2|2m_2)$ | $\mathfrak{osp}(nn_2|2nm_2)$, $n \neq 2, 4$ |
| $\mathfrak{sp}(2n)$   | $\mathfrak{pe}(n_2)$    | $\mathfrak{osp}(2nm_2|4nm_2)$       |
| $\mathfrak{pe}(n_1)$   | $\mathfrak{pe}(n_2)$    | $\mathfrak{osp}(2n_1n_2|2n_1n_2)$, $n_1, n_2 > 2$ |
| $\mathfrak{osp}(n_1|2m_1)$ | $\mathfrak{pe}(n_2)$    | $\mathfrak{pe}(n_1n_2 + 2m_1n_2)$ if $n_1 \neq 2m_1$ |
| $\mathfrak{o}(n)$      | $\mathfrak{pe}(m)$      | $\mathfrak{pe}(n_1n_2 + 2m_1n_2)$ if $n_1 = 2m_1$ |
| $\mathfrak{sp}(2n)$   | $\mathfrak{pe}(m)$      | $\mathfrak{pe}(nm)$                 |

In particular, on the superspace of polyvector fields, there is a natural $\mathfrak{pe}(n)$-module structure, and $\mathfrak{pe}(1)$, its dual partner in $\mathfrak{osp}(2n|2n)$, is spanned by the divergence operator $\Delta$ (“odd Laplacian”), called the BRST operator \[1\], the even operator of $\mathfrak{pe}(1)$ being $\deg_x - \deg_\theta$, where $\theta_i = \pi(\frac{\partial}{\partial x^i})$, $\pi$ being the shift of parity operator.

For further examples of maximal subalgebras in $\mathfrak{gl}$ and $\mathfrak{q}$ see \[49\]. These subalgebras give rise to other new examples of Howe dual pairs. For the decomposition of the tensor algebra corresponding to some of these examples see \[51, 52\]. Some of the latter are further elucidated in \[53\]. Some further examples of Howe’s duality, considered in a detailed version of our lectures, are: (1) over reals; (2) dual pairs in simple subalgebras of $\mathfrak{po}(2n|m)$ distinct from $\mathfrak{osp}(m|2n)$; in particular, (3) embeddings into $\mathfrak{po}(2n|m; r)$, the nonstandard regradings of the Poisson superalgebra, cf. \[51\]; (4) a “projective” version of the Howe duality associated with embeddings into the Lie superalgebra of Hamiltonian vector fields, the quotient of the Poisson superalgebra, in particular, the exceptional cases in dimension $(2|2)$, cf. \[50\]. It is also interesting to consider the prime characteristic and an “odd” Howe’s duality obtained from quantization of the antibracket (the main objective of \[1\]), to say nothing of $q$-quantized versions of the above.

### §3. Generalities on spinor and spinor-like representations

#### 3.1. The spinor and oscillator representations of Lie algebras

The importance of the spinor representation became clear very early. One of the reasons is the following. As is known from any textbook on representation theory, the fundamental representations $R(\varphi_1) = W$, $R(\varphi_2) = \Lambda^2(W)$, ..., $R(\varphi_{n-1}) = \Lambda^{n-1}(W)$ of $\mathfrak{sl}(W)$, where $\dim W = n$ and $\varphi_i$ is the highest weight of $\Lambda^i(W)$, are irreducible. Any finite dimensional irreducible $\mathfrak{sl}(n)$-module $L^\lambda$ is completely determined by its highest weight $\lambda = \sum \lambda_i \varphi_i$ with $\lambda_i \in \mathbb{Z}_+$. The module $L^\lambda$ can be realized as a submodule (or quotient) of $\otimes (R(\varphi_i)^{\otimes \lambda_i})$. 
Similarly, every irreducible $\mathfrak{gl}(n)$-module $L^\lambda$, where $\lambda = (\lambda_1, \ldots, \lambda_{n-1}; c)$ and $c$ is the eigenvalue of the unit matrix, is realized in the space of tensors, perhaps, twisted with the help of $c$-densities, namely in the space $\otimes_i (R(\varphi_i) \otimes \lambda_i) \otimes \text{tr}^c$, where $\text{tr}^c$ is the Lie algebraic version of the $c$th power of the determinant, i.e., infinitesimally, trace, given for any $c \in \mathbb{C}$ by the formula $X \mapsto c \cdot \text{tr}(X)$ for any matrix $X \in \mathfrak{gl}(W)$. Thus, all the irreducible finite dimensional representations of $\mathfrak{sl}(W)$ are naturally realized in the space of tensors, i.e., in the subspaces or quotient spaces of the space $T^p_q = W \otimes \cdots \otimes W \otimes W^* \otimes \cdots \otimes W^*$, where $W$ is the space of the identity representation. For $\mathfrak{gl}(W)$, we have to consider the space $T^p_q \otimes \text{tr}^c$.

For $\mathfrak{sp}(W)$, the construction is similar, except the fundamental module $R(\varphi_i)$ is now a part of the module $\Lambda^i(id)$ consisting of the primitive forms.

For $\mathfrak{o}(W)$, the situation is totally different: not all fundamental representations can be realized as (parts of) the modules $\Lambda^i(id)$. The exceptional one (or two, for $\mathfrak{o}(2n)$) of them is called the spinor representation; for $\mathfrak{o}(W)$, where $\dim W = 2n$, it is realized in the Grassmann algebra $E^*(V)$ of a “half” of $W$, where $W = V \oplus V^*$ is a decomposition into the direct sum of subspaces isotropic with respect to the form preserved by $\mathfrak{o}(W)$. For $\dim W = 2n + 1$, it is realized in the Grassmann algebra $E^*(V \oplus W_0)$, where $W = V \oplus V^* \oplus W_0$ and $W_0$ is the 1-dimensional space on which the orthogonal form is nondegenerate.

The quantization of the harmonic oscillator leads to an infinite dimensional analog of the spinor representation which after Howe we call oscillator representation of $\mathfrak{sp}(W)$. It is realized in $S^*(V)$, where as above, $V$ is a maximal isotropic subspace of $W$ (with respect to the skew form preserved by $\mathfrak{sp}(W)$). The remarkable likeness of the spinor and oscillator representations was underlined in a theory of dual Howe’s pairs, [23].

The importance of spinor-oscillator representations is different for distinct classes of Lie algebras and their representations. In the description of irreducible finite dimensional representations of classical matrix Lie algebras $\mathfrak{gl}(n)$, $\mathfrak{sl}(n)$ and $\mathfrak{sp}(2n)$ we can do without either spinor or oscillator representations. We can not do without spinor representation for $\mathfrak{o}(n)$, but a pessimist might say that spinor representation constitutes only $\frac{1}{n}$th of the building bricks. Our, optimistic, point of view identifies the spinor representations as one of the two possible types of the building bricks.

For the Witt algebra $\mathfrak{witt}$ and its central extension, the Virasoro algebra $\mathfrak{vir}$, every irreducible highest weight module is realized as a quotient of a spinor or, equivalently, oscillator representation, see [3], [4]. This miraculous equivalence is known in physics under the name of *bose-fermi correspondence*, see [18], [24].

For the list of generalizations of $\mathfrak{witt}$ and $\mathfrak{vir}$, i.e., simple (or close to simple) stringy Lie superalgebras or Lie superalgebras of vector fields on $N$-extended supercircles, often called by an unfortunate (as explained in [21]) name “superconformal algebras”, see [27]. The importance of spinor-oscillator representations diminishes as $N$ grows, but for the most interesting — distinguished ([21]) — stringy superalgebras it is high, cf. [14], [46].

3.2. Semi-infinite cohomology. An example of applications of spinor-oscillator representations: semi-infinite (or BRST) cohomology of Lie superalgebras. These cohomology were introduced by Feigin first for Lie algebras ([1]); then he extended the definition to Lie superalgebras via another construction, equivalent to the first one for Lie algebras ([1]). For an elucidation of Feigin’s construction see [14], [31] and [54]. Feigin rewrote in mathematical terms and generalized the constructions physicists used to determine the critical dimensions of string theories, i.e., the dimensions in which the quantization of the superstring is possible, see [12], [18]. These critical dimensions are the values of the central element (central charges) on the spin-or-oscillator representation constructed from the adjoint representation;
to this day not for every central element of all distinguished simple stringy superalgebras their values are computed on every spinor-oscillator representation, not even on the ones constructed from the adjoint representations.

§4. THE SPINOR-OSCILLATOR REPRESENTATIONS AND LIE SUPERALGEBRAS

4.1. Spinor (Clifford–Weil–wedge– . . . ) and oscillator representations. As we saw in [40], \( \mathfrak{po}(2n|m) \cong \mathfrak{osp}(m|2n) \), the superspace of elements of degree 0 in the standard \( \mathbb{Z} \)-grading of \( \mathfrak{po}(2n|m) \) or, which is the same, the superspace of quadratic elements in the representation by generating functions. At our first lecture we defined the spinor-oscillator representation as the through map (here \( k = \lceil \frac{m}{2} \rfloor \) and \( \mathcal{Q} \) is the quantization)

\[
\mathfrak{g} \rightarrow \mathfrak{po}(2n|m) \overset{\mathcal{Q}}{\longrightarrow} \begin{cases} \mathfrak{diff}(n|k) & \text{if } m = 2k \\ \mathfrak{qdiff}(n|k) & \text{if } m = 2k - 1, \end{cases}
\]

where \( \text{Im}(\mathfrak{g}) \subset \mathfrak{po}(2n|m) \cong \mathfrak{osp}(m|2n) \). Actually, such requirement is too restrictive, we only need that the image of \( \mathfrak{g} \) under embedding into \( \mathfrak{po}(2n|m) \) remains rigid under quantization. So various simple subalgebras of \( \mathfrak{po}(2n|m) \) will do as ambients of \( \mathfrak{g} \).

This spinor-oscillator representation is called the spinor representation of \( \mathfrak{g} \) if \( n = 0 \), or the oscillator representation if \( m = 0 \). We will denote this representation \( \text{Spin}(V) \) and set \( \text{Osc}(V) = \text{Spin}(\Pi(V)) \), where \( V \) is the standard representation of \( \mathfrak{osp}(m|2n) \). In other words, if \( \text{Spin}(V) \) is a representation of \( \mathfrak{osp}(m|2n) \), then \( \text{Osc}(V) \) is a representation of \( \mathfrak{osp}(2n|m) \), so \( \text{Osc}(V) \) only exists for \( m \) even.

If \( V \) is a \( \mathfrak{g} \)-module without any bilinear form, but we still want to construct a spinor-oscillator representation of \( \mathfrak{g} \), consider the module \( W = V \oplus V^\ast \) (where in the infinite dimensional case we replace \( V^\ast \) with the restricted dual of \( V \); roughly speaking, if \( V = \mathbb{C}[x] \), then \( V^\ast = \mathbb{C}[\frac{\partial}{\partial x}] \), whereas the restricted dual is \( \mathbb{C}[\frac{\partial}{\partial x}] \) endowed with the form (for \( v_1, w_1 \in V \), \( v_2, w_2 \in V^\ast \)) symmetric for the plus sign and skew-symmetric otherwise:

\[
B((v_1, v_2), (w_1, w_2)) = v_2(w_1) \pm (-1)^{p(v_1)p(w_2)}w_2(v_1).
\]

Now, in \( W \), select a maximal isotropic subspace \( U \) (not necessarily \( V \) or \( V^\ast \)) and realize the spinor-oscillator representation of \( \mathfrak{g} \) in the exterior algebra of \( U \).

Observe that the classical descriptions of spinor representations differ from ours, see, e.g., [17], where the embedding of \( \mathfrak{g} \) (in their case \( \mathfrak{g} = \mathfrak{o}(n) \)) into the quantized algebra (namely into \( \mathcal{Q}(\mathfrak{po}(0|n-1)) \)) is considered, not into \( \mathfrak{po}(0|m) \). The existence of this embedding is not so easy to see unless told, whereas our constructions are manifest and bring about the same result.

To illustrate our definitions and constructions, we realize the orthogonal Lie algebra \( \mathfrak{o}(n) \) as the subalgebra in the Lie superalgebra \( \mathfrak{po}(0|n) \).

Case \( \mathfrak{o}(2k) \): Basis:

\[
\begin{align*}
X_1^+ &= \xi_2\eta_1, & \cdots & & X_{k-1}^+ &= \xi_k\eta_{k-1}, & X_k^+ &= \eta_k\eta_{k-1}; \\
X_1^- &= \xi_1\eta_2, & \cdots & & X_{k-1}^- &= \xi_{k-1}\eta_k, & X_k^- &= \eta_{k-1}\xi_k; \\
H_1 &= \xi_1\eta_1 - \xi_2\eta_2, & \cdots & & H_{k-1} &= \xi_{k-1}\eta_{k-1} - \xi_k\eta_k, & H_k &= \xi_{k-1}\eta_{k-1} + \xi_k\eta_k.
\end{align*}
\]

For \( R(\varphi_k) \) take the subspace of space functions \( \mathbb{C}[\xi]_{ev} \) which contains the constants \( \mathbb{C} \cdot \hat{1} \), where \( \hat{1} \) is just the constant function 1; clearly, \( \hat{1} \) is the vacuum vector.

Quantization (see above) sends: \( \xi_i \) into \( \hat{\xi}_i \), and \( \eta_i \) into \( \hbar \frac{\partial}{\partial \xi_i} \), so \( X_i^+\hat{1} = 0 \) for \( i < k \), hence,

\[
H_i\hat{1} = [X_i^+, X_i^-]\hat{1} = 0 \text{ for } i < k. \quad \text{Contrariwise,}
\]

\[
H_k\hat{1} = [X_k^+, X_k^-]\hat{1} = [\partial_k\partial_{k-1}, \hat{\xi}_k\hat{\xi}_k]\hat{1} = \partial_k(-\hat{\xi}_{k-1}\partial_{k-1} + 1)\hat{\xi}_k\hat{1} = \hat{1}.
\]
So we see that the spinor representation is indeed a fundamental one.

Case $\mathfrak{o}(2k + 1)$. Basis:

$$X_1^+ = \xi_1 \eta_1, \quad \ldots, \quad X_{k-1}^+ = \xi_k \eta_{k-1}, \quad X_k^+ = \sqrt{2} \eta_k \theta;$$

$$X_1^- = \xi_1 \eta_2, \quad \ldots, \quad X_{k-1}^- = \xi_k \eta_k, \quad X_k^- = \sqrt{2} \theta \xi_k;$$

$$H_1 = \xi_1 \eta_1 - \xi_2 \eta_2, \quad \ldots, \quad H_{k-1} = \xi_{k-1} \eta_{k-1} - \xi_k \eta_k, \quad H_k = 2 \xi_k \eta_k.$$

For $R(\varphi_k)$ consider the space of even functions $\mathbb{C}[\xi_1, \ldots, \xi_k, \theta]_{ev}$ and realize $\mathfrak{o}(2k + 1)$ so that $\xi_i \mapsto \hat{\xi}_i$, $\eta_i \mapsto \hbar \frac{\partial}{\partial \xi_i}$, $\theta \mapsto \hbar (\hat{\theta} + \frac{\partial}{\partial \theta})$. As above for $\mathfrak{o}(2k)$, set $\hbar = 1$.

Then, as above, $H_i v = [X_i^+, X_i^-] \hat{1} = 0$ for $i < k$, whereas

$$H_k \hat{1} = [X_k^+, X_k^-] \hat{1} = \frac{2}{\hbar} \left( \partial_k (\hat{\theta} + \frac{\partial}{\partial \theta})^2 \hat{\xi}_k + \hat{\xi}_k \hat{\eta}_k + \frac{\partial}{\partial \theta} \partial_k \right) \hat{1} = \hat{1}.$$

So $\hat{1}$ is indeed the highest weight vector of the $k$th fundamental representation.

**4.2. Stringy superalgebras. Case $\mathfrak{vir}$.** For the basis of $\mathfrak{vir}$ take $e_i = t^{i + 1} \frac{d}{dt}$, $i \in \mathbb{Z}$, and the central element $z$; let the bracket be

$$[e_i, e_j] = (j - i) e_{i+j} - \frac{1}{12} \delta_{ij} (i^3 - i) z. \quad (*)$$

We advise the reader to refresh definitions of stringy superalgebras and various modules over them, see [21], where we also try to convince physicists not to use the term “superconformal algebra” (except, perhaps, for $\mathfrak{k}^+(\mathfrak{l} | 1)$ and $\mathfrak{k}^+(\mathfrak{m} | 1)$). In particular, recall that $\mathcal{F}_{\lambda, \mu} = \text{Span}(\varphi_i = t^{\mu+i}(dt)^{\lambda} | i \in \mathbb{Z})$.

**Statement.** The only instances when $\mathcal{F}_{\lambda, \mu}$ possesses an invariant symmetric nondegenerate bilinear form are the space of half-densities, $\sqrt{\text{Vol}} = \mathcal{F}_{1/2, 0}$, and its twisted version, $\mathcal{F}_{1/2, 1/2}$ and in both cases the form is:

$$(f \sqrt{dt}, g \sqrt{dt}) = \int f g \cdot dt;$$

the only instances when $\mathcal{F}_{\lambda, \mu}$ possesses an invariant skew-symmetric forms are the quotient space of functions modulo constants, $d \mathcal{F} = \mathcal{F}_{0,0}/\mathbb{C} \cdot 1$, and $\frac{1}{2}$-twisted functions, $\sqrt{t} \mathcal{F} = \mathcal{F}_{0,1/2}$ and in both cases the form is:

$$(f, g) = \int f \cdot dg.$$

Let $\partial_i = \frac{\partial}{\partial \varphi_i}$ (where $\varphi_i = t^i (dt)^{\lambda}$). Let $\text{osc}(\sqrt{\text{Vol}})$ be the $\mathfrak{vir}$- submodule of the exterior algebra on $\varphi_i$ for $i < 0$ containing the constant $\hat{1}$. Since the generators $e_i$ of $\mathfrak{vir}$ acts on $\mathcal{F}_{\lambda, \mu}$ as (sums over $i \in \mathbb{Z}$)

$$e_1 = \sum (\mu + i + 2 \lambda) \varphi_{i+1} \partial_i = \sum i \varphi_{i+1} \partial_i,$$

$$e_{-1} = \sum (\mu + i + 1) \varphi_i \partial_{i+1} = \sum (i + 1) \varphi_i \partial_{i+1};$$

$$e_2 = \sum (\mu + i - \lambda) \varphi_{i+1} \partial_i = \sum i \varphi_{i+1} \partial_i,$$

$$e_{-2} = \sum (\mu + i + 3 \lambda) \varphi_i \partial_{i+1} = \sum (i + 1) \varphi_i \partial_{i+1},$$

and representing $e_0$ and $z$ as brackets of $e_{\pm 1}$ and $e_{\pm 2}$ from (*) we immediately deduce that the highest weights $(c, h)$ of $\text{osc}(\sqrt{\text{Vol}})$ is $(-\frac{1}{3}, 0)$.

For the spinor representations $\text{spin}(\sqrt{t} \mathcal{F})$ and $\text{spin}(d \mathcal{F})$ (realized on the symmetric algebra of $\varphi_i$ for $i < 0$) we similarly obtain that the highest weights $(c, h)$ are $(\frac{1}{6}, \frac{1}{2})$ for $\text{spin}(\sqrt{t} \mathcal{F})$ and $(-\frac{1}{6}, 0)$ for $\text{spin}(d \mathcal{F})$.  


Observe that the representations \(\text{spin}(\sqrt{t}F), \text{spin}(dF)\) and \(\text{osc}(\sqrt{\text{Vol}})\) are constructed on a half of the generators used to construct \(\text{Spin}(\mathcal{F}_{\lambda,\mu})\).

4.3. The highest weights of the spinor representations of \(\mathfrak{t}^L(1|n)\) and \(\mathfrak{t}^M(1|n)\). In the following theorem we give the coordinates \((c, h; H_1, \ldots)\) of the highest weight of the spinor representations \(\text{Spin}(\mathcal{F}_{\lambda,\mu})\) of the contact superalgebra \(\mathfrak{t}^L(1|n)\) with respect to \(z\) (the central element), \(K_i\), and, after semicolon, on the elements of Cartan subalgebra, respectively. For \(\mathfrak{t}^M(1|n)\) we write \(\tilde{h}; \tilde{H}_i\). (Observe that for \(n > 4\) the Cartan subalgebra has more generators than just \(H_1 = K_{\xi_1\eta_1}, \ldots, H_k = K_{\xi_k\eta_k}\) which generate the Cartan subalgebra of \(\mathfrak{t}(1|2k)\), the algebra of contact vector fields with polynomial coefficients.)

| \(n\) | 0 | 1 | 2 | \(\geq 3\) |
|---|---|---|---|---|
| \(c\) | \(12\lambda^2 - 12\lambda + 2\) | \(-12\lambda + 3\) | 6 | 0 |
| \(h\) | \((\mu + 2\lambda)(\mu + 1)\) | \(\mu + 2\lambda\) | \(2\mu + 2\lambda + \nu\) | \(2n^{-1}(\mu + \lambda) + 2n^{-5}\) |
| \(\tilde{h}\) | \(-2\mu + 3\lambda - \frac{1}{2}\) | \(2\mu + 2\lambda - \frac{1}{2}\) | \(2n^{-1}(\mu + \lambda)\) |

Theorem. Let \((c, h; H_1, \ldots)\) be the highest weight of the spinor representation \(\text{Spin}(\mathcal{F}_{\lambda,\mu})\) of \(\mathfrak{t}^L(1|n)\). The highest weight of the oscillator representation \(\text{Osc}(\mathcal{F}_{\lambda,\mu}) = \text{Spin}(\Pi(\mathcal{F}_{\lambda,\mu}))\) is \((-c, h; H_1, \ldots)\) and similarly for \(\mathfrak{t}^M(1|n)\).

For \(n \neq 2\), all the coordinates of the highest weight other than \(c, h\) vanish. For \(n = 2\) the value of \(H\) on the highest weight vector from \(\text{Spin}(\mathcal{F}_{\lambda,\nu,\mu})\) is equal to \(\nu\).

The values of \(c\) and \(h\) (or \(\tilde{h}\)) on modules \(\text{Spin}(\mathcal{F}_{\lambda,\mu})\) are given in the above table.

Up to rescaling, these results are known for small \(n\), see [29, 28] and refs. therein.

Remark. For the contact superalgebras \(\mathfrak{g}\) on the \(1|n\)-dimensional supercircle our choice of \(\mathfrak{g}\)-modules \(V = \mathcal{F}_{\lambda,\mu}\) from which we constructed \(\text{Spin}(V \oplus V^*)\) is natural for small \(n\): there are no other modules! For larger \(n\) it is only justified if we are interested in semi-infinite cohomology of \(\mathfrak{g}\) and not in representation theory per se. For the superalgebras \(\mathfrak{g}\) of series \(\mathfrak{vect}\) and \(\mathfrak{s vect}\) the adjoint module \(\mathfrak{g}\) is of the form \(\mathcal{T}(\text{id}^*)\), i.e., it is either coinduced from multidimensional representation \((\mathfrak{vect})\), or is a submodule of such a coinduced module \((\mathfrak{s vect})\). Spinor-oscillator representations of this type were not studied yet, cf. sec. 5.

4.4. Other spinor representations. 1) Among various Lie superalgebras for which it is interesting to study spinor-oscillator representations, the simple (or close to them) maximal subsuperalgebras of \(\mathfrak{po}\) are most interesting. The list of such maximal subalgebras is being completed; various maximal subalgebras listed in [18] distinct from the sums of mutual centralizers also provide with spinor representations.

As an interesting example consider A. Sergeev’s Lie superalgebra \(\mathfrak{as}\), the nontrivial central extension of the Lie superalgebra \(\mathfrak{spe}(4)\) preserving the odd bilinear form and the volume on the \((4|4)\)-dimensional superspace, see [19, 20]. Namely, consider \(\mathfrak{po}(0|6)\), the Lie superalgebra(11,11),(992,988) whose superspace is the Grassmann superalgebra \(\Lambda(\xi, \eta)\) generated by \(\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3\) and the bracket is the Poisson bracket. Recall also that the quotient of \(\mathfrak{po}(0|6)\) modulo center is \(\mathfrak{h}(0|6) = \text{Span}(H_f \mid f \in \Lambda(\xi, \eta))\), where

\[
H_f = (-1)^{p(f)} \sum \left( \frac{\partial f}{\partial \xi_j} \frac{\partial}{\partial \eta_j} + \frac{\partial f}{\partial \eta_j} \frac{\partial}{\partial \xi_j} \right).
\]

Now, observe that \(\mathfrak{spe}(4)\) can be embedded into \(\mathfrak{h}(0|6)\). Indeed, setting \(\deg \xi_i = \deg \eta_i = 1\) for all \(i\) we introduce a \(\mathbb{Z}\)-grading on \(\Lambda(\xi, \eta)\) which, in turn, induces a \(\mathbb{Z}\)-grading on \(\mathfrak{h}(0|6)\) of the form \(\mathfrak{h}(0|6) = \bigoplus_{i \geq -1} \mathfrak{h}(0|6)_i\). Since \(\mathfrak{sl}(4) \cong \mathfrak{o}(6)\), we can identify \(\mathfrak{spe}(4)_0\) with \(\mathfrak{h}(0|6)_0\).

It is not difficult to see that the elements of degree \(-1\) in the standard gradings of \(\mathfrak{spe}(4)\) and \(\mathfrak{h}(0|6)\) constitute isomorphic \(\mathfrak{sl}(4) \cong \mathfrak{o}(6)\)-modules. It is subject to a direct verification that it is really possible to embed \(\mathfrak{spe}(4)_1\) into \(\mathfrak{h}(0|6)_1\).
A. Sergeev’s extension as is the result of the restriction onto $\mathfrak{osp}(4) \subset \mathfrak{h}(0|6)$ of the cocycle that turns $\mathfrak{h}(0|6)$ into $\mathfrak{po}(0|6)$. The quantization (with parameter $\lambda$) deforms $\mathfrak{po}(0|6)$ into $\mathfrak{gl}(\Lambda(\xi))$; the through maps $T_\lambda : \mathfrak{as} \to \mathfrak{po}(0|6) \to \mathfrak{gl}(\Lambda(\xi))$ are representations of $\mathfrak{as}$ in the $4|4$-dimensional modules $\text{Spin}_\lambda$. The explicit form of $T_\lambda$ is as follows:

$$T_\lambda : \begin{pmatrix} a & b \\ c & -a^t \end{pmatrix} + d \cdot z \mapsto \begin{pmatrix} a & b - \lambda \tilde{c} \\ c & -a^t \end{pmatrix} + \lambda d \cdot 1_{4|4},$$

where $1_{4|4}$ is the unit matrix and $\tilde{c}$ is extended via linearity from matrices $c_{ij} = E_{ij} - E_{ji}$ on which $\tilde{c}_{ij} = c_{kl}$ for any even permutation $(1234) \mapsto (ijkl)$. Clearly, $T_\lambda$ is an irreducible representation for any $\lambda$ and $T_\lambda \neq T_\mu$ for $\lambda \neq \mu$.

2) Maximal subalgebras (for further examples see [48]) and a conjecture. Let $V_1$ be a linear superspace of dimension $(r|s)$; let $\Lambda(n)$ be the Grassmann superalgebra with $n$ odd generators $\xi_1, \ldots, \xi_n$ and $\mathfrak{vect}(0|n) = \mathfrak{der}(\Lambda(n)$ the Lie superalgebra of vector fields on the $(0|n)$-dimensional supermanifold.

Let $\mathfrak{g} = \mathfrak{gl}(V_1) \otimes \Lambda(n) \subset \mathfrak{vect}(0|n)$ be the semidirect sum (the ideal at the closed part of $\otimes$) with the natural action of $\mathfrak{vect}(0|n)$ on the ideal $\mathfrak{gl}(V_1) \otimes \Lambda(n)$. The Lie superalgebra $\mathfrak{g}$ has a natural faithful representation $\rho$ in the space $V = V_1 \otimes \Lambda(n)$ defined by the formulas

$$\rho(X \otimes \varphi)(v \otimes \psi) = (-1)^{p(\varphi)p(\psi)}Xv \otimes \varphi\psi,$$

$$\rho(D)(v \otimes \psi) = (-1)^{p(D)p(v)}v \otimes D\psi$$

for any $X \in \mathfrak{gl}(V_1)$, $\varphi, \psi \in \Lambda(n)$, $v \in V_1$, $D \in \mathfrak{vect}(0|n)$.

Let us identify the elements from $\mathfrak{g}$ with their images under $\rho$, so we consider $\mathfrak{g}$ embedded into $\mathfrak{gl}(V)$.

**Theorem** ([48]) 1) The Lie superalgebra $\mathfrak{gl}(V_1) \otimes \Lambda(n) \subset \mathfrak{vect}(0|n)$ is maximal irreducible in $\mathfrak{sl}(V_1 \otimes \Lambda(n))$ unless a) $\dim V_1 = (1,1)$ or b) $n = 1$ and $\dim V_1 = (1,0)$ or $(0,1)$ for $(r|s)$ for $r \neq s$.

2) If $\dim V_1 = (1,1)$, then $\mathfrak{gl}(1|1) \cong \Lambda(1) \subset \mathfrak{vect}(0|1)$, so

$$\mathfrak{gl}(V_1) \otimes \Lambda(n) \subset \mathfrak{vect}(0|n) \subset \Lambda(n+1) \subset \mathfrak{vect}(0|n+1)$$

and it is the bigger superalgebra which is maximal irreducible in $\mathfrak{sl}(V)$.

3) If $n = 1$ and $\dim V_1 = (r|s)$ for $r > s > 0$, then $\mathfrak{g}$ is maximal irreducible in $\mathfrak{gl}(V)$.

**Conjecture.** Suppose $r + s = 2^N$. Then, $\dim V$ coincides with $\dim \Lambda(W)$ for some space $W$. We suspect that this coincidence is not accidental but is occasioned by the spinor representations of the maximal subalgebras described above. The same applies to $\mathfrak{q}(V_1) \otimes \Lambda(n) \subset \mathfrak{vect}(0|n)$, a maximal irreducible subalgebra in $\mathfrak{g}(V_1 \otimes \Lambda(n))$.

**4.5. Selected problems.** 1) The spinor and oscillator representations are realized in the symmetric (perhaps, supersymmetric) algebra of the maximal isotropic (at least for $\mathfrak{g} = \mathfrak{sp}(2k)$ and $\mathfrak{o}(2k)$) subspace $V$ of the identity $\mathfrak{g}$-module $\text{id} = V \oplus V^*$. But one could have equally well started from another $\mathfrak{g}$-module. For an interesting study of spinor representations constructed from $W \neq \text{id}$, see [49].

To consider in a way similar to sec. 2 contact stringy superalgebras $\mathfrak{g} = \mathfrak{f}^L(1|n)$ and $\mathfrak{f}^M(1|n)$, as well as other stringy superalgebras from the list [23], we have to replace $\mathfrak{F}_{\lambda,\mu}$ with modules $\mathcal{T}_\mu(W)$ of (twisted) tensor fields on the supercircle and investigate how does the highest weight of $\tilde{1} \in \text{Osc}(\mathcal{T}_\mu(W))$ or $\tilde{1} \in \text{Spin}(\mathcal{T}_\mu(W))$ constructed from an arbitrary irreducible $\mathfrak{co}(n)$-module $W = V \oplus V^*$ depend on the highest weight of $W$. (It seems that the new and absolutely remarkable spinor-like representation Polentaev recently constructed [10] is obtained in this way.)

To give the reader a feel of calculations, we consider here the simplest nontrivial case $\mathfrak{o}(3) = \mathfrak{sl}(2)$. The results may (and will) be used in calculations of $\text{Spin}(\mathcal{T}_\mu(W))$ for $\mathfrak{g} = \mathfrak{f}^L(1|n)$ and $\mathfrak{f}^M(1|n)$ for $n = 3, 4$. As is known, for every $N \in \mathbb{Z}_+$ there exists an
irreducible \((N+1)\)-dimensional \(g\)-module with highest weight \(N\). This module possesses a natural nondegenerate \(g\)-invariant bilinear form which is skew-symmetric for \(N=2k+1\) and symmetric for \(N=2k\). The corresponding embeddings \(g \to o(2k+1)\) and \(g \to sp(2k)\) are called principal, see \[14\] and references therein. Explicitly, the images of the Chevalley generators \(X^\pm\) of \(\mathfrak{sl}(2)\) are as follows: \(X^- \mapsto \sum X_i^-\),

\[
X^+ \mapsto \begin{cases} 
N(N+1)X_N^+ + \sum_{1 \leq i \leq N-1} i(N+1-i)X_i^+ & \text{for } N = 2k+1 \\
N^2X_N^+ + \sum_{1 \leq i \leq N-1} i(2N-i)X_i^+ & \text{for } N = 2k.
\end{cases}
\]

From the commutation relations between \(X^+\) and \(X^-\) we derive that only \(X_N^\pm\) give a nontrivial contribution to the highest weight \(HW\) of the \(\mathfrak{sl}(2)\)-module \(\text{Spin}(L^N)\); we have:

\[
HW = \begin{cases} 
N(N+1) & \text{if } N = 2k+1 \\
-\frac{1}{2}N^2 & \text{if } N = 2k.
\end{cases}
\]

2) Observe, that the notion of spinor-oscillator representation can be broadened to embrace the subalgebras of the Lie superalgebra \(h\) of Hamiltonian vector fields and their images under quantization; we call the through map the projective spinor-oscillator representation. Since the Lie superalgebra \(h\) has more deformations than \(po\) (\[14\]), and since the sets of maximal simple subalgebras of \(po\) and \(h\) are distinct, the set of examples of projective spinor-oscillator representations differs from that of spinor-oscillator representations.

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