BIFURCATION OF LIMIT CYCLES IN A FAMILY OF PIECEWISE SMOOTH SYSTEMS VIA AVERAGING THEORY

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ABSTRACT. In this paper we study the maximal number of limit cycles for a class of piecewise smooth near-Hamiltonian systems under polynomial perturbations. Using the second order averaging method, we obtain the maximal number of limit cycles of two systems respectively. We also present an application.

1. Introduction. As we know, one of the main problems, in the qualitative theory of ODEs, is to study the number of limit cycles of planar differential systems, which is related to the Hilbert’s 16th problem. Many works have been done on the problem for smooth planar differential systems, see for instance [1, 3, 5, 9, 16] and the references therein. In recent years, a lot of papers have appeared to study periodic solutions by qualitative theory for the non-smooth systems, see [2, 4, 7, 12, 13, 15, 18].

In order to obtain the maximal number of periodic solutions or an upper bound of the number of periodic solutions for piecewise smooth differential equations, people have developed various methods. Among them, the Melnikov function method and the averaging method are the most widely used ones. For example, the authors in [11] developed the Melnikov function method for piecewise smooth planar systems, and established a formula for the first order Melnikov function. From [7, 18, 19], we know that one can consider the number of limit cycles for piecewise polynomial systems by using the method of first order Melnikov function in Hopf and generalized homoclinic bifurcations. Recently the authors of [17] applied the Melnikov function theory to high-dimensional piecewise smooth near-integrable systems and gave a formula for the first order Melnikov vector function. One can use the first and second order averaging methods for studying the periodic solutions of piecewise smooth periodic differential systems, see [4, 15, 13]. The authors [10] used the higher order averaging theory for studying a class of quartic polynomial differential

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systems. In [14], the authors obtained the formulas of averaged functions at any order for discontinuous piecewise differential systems with many zones.

It is worth noting that the averaging method is equivalent to the Melnikov function method for studying the number of limit cycles of planar analytic (or $C^\infty$) near-Hamiltonian systems, see [6].

X. Liu and M. Han [11] investigated the following piecewise polynomial system:

$$\begin{align*}
(\dot{x} & \quad \dot{y}) = \\
&= \begin{cases}
ay + \varepsilon f^+(x, y), & x > 0, \\
-ax + \varepsilon g^+(x, y), & x \leq 0,
\end{cases}
\end{align*}$$

(1)

where

$$a > 0, \quad b > 0,$$

$$f^\pm(x, y) = \sum_{i+j=0}^{n} a^\pm_{ij} x^i y^j, \quad g^\pm(x, y) = \sum_{i+j=0}^{n} b^\pm_{ij} x^i y^j,$$

and proved that system (1) has at most $n$ limit cycles bifurcating from the unperturbed period annulus by using the first order Melnikov function.

If further the constant terms $a^\pm_{00}, b^\pm_{00}$ do not appear, the system has the following form

$$\begin{align*}
(\dot{x} & \quad \dot{y}) = \\
&= \begin{cases}
ay + \varepsilon \sum_{i+j=1}^{n} a^+_{ij} x^i y^j, & x > 0, \\
-ax + \varepsilon \sum_{i+j=1}^{n} b^+_{ij} x^i y^j, & x \leq 0,
\end{cases}
\end{align*}$$

(2)

Inspired by [6], we study the maximal number of limit cycles for systems (1) and (2) via the second order averaging method.

For any given $\varepsilon_0 > 0$ sufficiently small and $N > \varepsilon_0$ sufficiently large, denote by $H_1(n)$ the maximal number of limit cycles of system (1) bifurcating from the region $\varepsilon_0 \leq x^2 + y^2 \leq N$, $H_2(n)$ the maximal number of limit cycles of system (2) bifurcating from the region $x^2 + y^2 \leq N$. Our main result is as follows:

**Theorem 1.1.** For $|\varepsilon| > 0$ sufficiently small, using the second order averaging method we have $H_1(n) \leq 2n - 1$ and $H_2(n) \leq 2n - 2$.

The paper is organized as follows. In section 2, we introduce the first order and the second order averaging methods for piecewise smooth systems. In section 3, we compute the averaged functions associated to systems (1) and (2). Then we study the maximal number of zeros of averaged functions and prove Theorem 1.1. Finally we give an application.

2. **Preliminary theorems.** In this section we present some known basic results from the averaging theory for discontinuous differential systems, see [4] for more details.

Consider a $T$-periodic differential equation of the form

$$x'(t) = \varepsilon F(t, x, \varepsilon, \delta),$$

(3)
with $T > 0$ constant, $F$ being given for $0 \leq t \leq T$ by

$$F(t, x, \varepsilon, \delta) = \begin{cases} F_1(t, x, \varepsilon, \delta), & (t, x) \in D_1, \\ F_2(t, x, \varepsilon, \delta), & (t, x) \in D_2, \\ \vdots \\ F_k(t, x, \varepsilon, \delta), & (t, x) \in D_k, \end{cases}$$

where $x \in J \subset R$ with $J$ an open interval, $|\varepsilon| < \varepsilon_0$, $\delta \in V \subset R^n$ with $V$ a compact set, $k$ $C^r$ functions $F_j(t, x, \varepsilon, \delta)$ are defined for all $(t, x) \in U(D_j)$ with $U(D_j)$ being an open set containing $\overline{D}_j$, $\overline{D}_j$ denoting the closure of the set $D_j$ which have the following form

$$D_j = \{(t, x)|h_j-1(x) \leq t < h_j(x), x \in J\}, \ j = 1, \ldots, k,$$

where $h_j(x)$ are $C^r$ functions defined on $J$ satisfying

$$h_0(x) = 0 < h_1(x) < \cdots < h_{k-1}(x) < T = h_k(x), \ x \in J, k \geq 2, r \geq 1.$$

Note that $F$ is periodic in $t$ with period $T$ and may not be continuous on the switch lines $l_1, \ldots, l_{k-1}$, where

$$l_j = \{(t, x)|t = h_j(x), x \in J\}, \ j = 0, \ldots, k.$$

The equation (3) is called a $k$-piecewise $C^r$ smooth periodic equation, as called in [4].

Let

$$f(x, \delta) = \int_0^T F(t, x, 0, \delta) dt = \sum_{j=1}^k \int_{h_{j-1}(x)}^{h_j(x)} F(t, x, 0, \delta) dt.$$

(4)

For $x_0 \in J$, define the solution of equation (3) satisfying $x(0) = x_0$ for $t \in [0, T]$ as $x(t, x_0, \varepsilon, \delta)$. The Poincaré map of (3) is given by

$$P(x_0, \varepsilon, \delta) = x(T, x_0, \varepsilon, \delta) = x_0 + \varepsilon \tilde{g}_k(x_0, \varepsilon, \delta).$$

(5)

In [4], the author developed the averaging theory and obtained the following results.

**Lemma 2.1.** Consider the periodic equation (3). We have

(I) For any given closed interval $I \subset J$, there exists $\varepsilon^* > 0$ such that the function $\tilde{g}_k(x_0, \varepsilon, \delta)$ is well defined and of $C^r$ in $(x_0, \varepsilon, \delta)$ for all $x_0 \in I$, $|\varepsilon| < \varepsilon^*$ and $\delta \in V$.

(II) If there exists an integer $m$, $1 \leq m \leq r$, such that the function $f$ defined in (4) has at most $m$ zeros in $x \in J$ for all $\delta \in V$, multiplicity taken into account, then for any closed interval $I \subset J$, there exists $\varepsilon_1 = \varepsilon_1(I) > 0$, such that for $0 < |\varepsilon| < \varepsilon_1, \delta \in V$ the periodic equation (3) has at most $m$ $T$-periodic solutions with the property that the range of each of them is a subset of $I$.

**Remark 1.** From the proof of Theorem 1.1 in [4], we see that if $F(t, 0, \varepsilon, \delta) = 0, J = (0, +\infty)$, and $f$ has at most $m$ zeros in $x \in J$ for all $\delta \in V$, multiplicity taken into account, then for any $N > 0$, there exists $\varepsilon_1 = \varepsilon_1(N) > 0$ such that for $0 < |\varepsilon| < \varepsilon_1, \delta \in V$, (3) has at most $m$ positive periodic solutions whose ranges are subsets of $(0, N]$.

Using the second order averaging theory, we can do further study to the maximal number of periodic solutions for the piecewise smooth periodic equations.

In fact, from (5) we know that if $f(x, \delta) = 0$, the Poincaré map of (3) can be written as

$$P(x_0, \varepsilon, \delta) = x(T, 0, x_0, \varepsilon, \delta) = x_0 + \varepsilon^2 \tilde{g}_k(x_0, \varepsilon, \delta),$$

(6)
where \( \varepsilon \tilde{g}_k(x_0, \varepsilon, \delta) = \tilde{g}_k(x_0, \varepsilon, \delta) \). Moreover, according to Lemma 2.1 and [8], for any given closed interval \( I \subset J \), there exists \( \varepsilon^* > 0 \) such that the function \( \tilde{g}_k(x_0, \varepsilon, \delta) \) is well defined and of \( C^{r-1} \) in \( (x_0, \varepsilon, \delta) \) for all \( x_0 \in I, |\varepsilon| < \varepsilon^* \) and \( \delta \in V \).

By lemma 9 (the fundamental lemma) of [13], we have
\[
\tilde{g}_k(x_0, 0, \delta) = f_2(x_0, \delta),
\]
where \( f_2(x_0, \delta) \) is given by
\[
f_2(x, \delta) = \int_0^T \left( D_x \tilde{F}_1(t, x, \delta) \int_0^t \tilde{F}_1(s, x, \delta) ds + \tilde{F}_2(t, x, \delta) \right) dt,
\]
satisfying
\[
\tilde{F}_1(t, x, \delta) = F(t, x, 0, \delta),
\]
\[
\tilde{F}_2(t, x, \delta) = \left\{ \begin{array}{ll}
\frac{\partial F_1(t, x, \delta)}{\partial x} |_{\varepsilon = 0}, & (t, x) \in D_1, \\
\frac{\partial F_2(t, x, \delta)}{\partial x} |_{\varepsilon = 0}, & (t, x) \in D_2, \\
\vdots & \\
\frac{\partial F_k(t, x, \delta)}{\partial x} |_{\varepsilon = 0}, & (t, x) \in D_k,
\end{array} \right.
\]
\[
D_x \tilde{F}_1(t, x, \delta) = \sum_{j=1}^{k} \chi_{D_j} D_x F_j(t, x, 0, \delta),
\]
with
\[
\chi_{D_j}(t, x) = \left\{ \begin{array}{ll}
1, & (t, x) \in D_j, \\
0, & (t, x) \notin D_j.
\end{array} \right.
\]

Clearly, \( f_2 \in C^{r-1} \). Similar to the proof of Theorem 1.1 in [4], we can obtain

**Lemma 2.2.** Consider the periodic equation (3). Suppose \( f(x, \delta) = 0 \). If there exists an integer \( m, 1 \leq m \leq r - 1 \), such that the function \( f_2 \) defined in (7) has at most \( m \) zeros in \( x \) for all \( \delta \in V \), multiplicity taken into account, then for any closed interval \( I \subset J \), there exists \( \varepsilon_1 = \varepsilon_1(I) > 0 \), such that for \( 0 < |\varepsilon| < \varepsilon_1, \delta \in V \), the periodic equation (3) has at most \( m \) \( T \)-periodic solutions with the property that the range of each of them is a subset of \( I \).

**Lemma 2.3.** If the condition of Lemma 2.2 is satisfied together with \( F(t, 0, \varepsilon, \delta) = 0 \) and \( J = (0, +\infty) \), then for any \( N > 0 \), there exists \( \varepsilon_1 = \varepsilon_1(N) > 0 \) such that for \( 0 < |\varepsilon| < \varepsilon_1, \delta \in V \), (3) has at most \( m \) positive periodic solutions whose ranges are subsets of \( (0, N] \).

3. **Proof of main results.** In this section, we proceed to prove our main theorem. The proof is divided into two steps. First we study the limit cycle bifurcations of system (1). For the purpose, we introduce polar coordinate transformation.

**Lemma 3.1.** The polar coordinate transformation
\[
(x, y) = (r \cos \theta, r \sin \theta)
\]
carries system (1) into
\[
r' = \left\{ \begin{array}{ll}
\varepsilon \sum_{i+j=1}^{n+1} w_{ij} r^{i+j-1} \cos^{i} \theta \sin^{j} \theta, & -\frac{\pi}{2} \leq \theta < \frac{\pi}{2}, \\
\varepsilon \sum_{i+j=1}^{n+1} w_{ij} r^{i+j-1} \cos^{i} \theta \sin^{j} \theta, & \frac{\pi}{2} \leq \theta < \frac{3\pi}{2},
\end{array} \right.
\]
(9)
\[
\theta' = \begin{cases} 
-a + \varepsilon \sum_{i+j=1}^{n+1} \tau_{ij}^+ r^{i+j-2} \cos^i \theta \sin^j \theta, & -\frac{\pi}{2} \leq \theta < \frac{\pi}{2}, \\
b + \varepsilon \sum_{i+j=1}^{n+1} \tau_{ij}^- r^{i+j-2} \cos^i \theta \sin^j \theta, & \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2},
\end{cases}
\]  
(10)

where \( w_{ij}^\pm = a_{i-1,j}^\pm + b_{i,j-1}^\pm, \tau_{ij}^\pm = b_{i-1,j}^\pm - a_{i,j-1}^\pm, a_{i-1,j}^\pm = b_{i-1,j}^\pm = a_{i,-1}^\pm = 0 \) for \( i, j = 0, 1, \ldots, n + 1 \).

**Proof.** System (1) has first integrals \( H^+(x, y) = \frac{a}{2}(x^2 + y^2) \) for \( x > 0 \) and \( H^-(x, y) = \frac{b}{2}(x^2 + y^2) \) for \( x \leq 0 \). From (8) we have
\[
\dot{x} = \cos \theta \dot{r} - r \sin \theta \dot{\theta}', \quad \dot{y} = \sin \theta \dot{r} + r \cos \theta \dot{\theta}',
\]
which yields that
\[
\dot{r}' = \cos \theta \dot{x} + \sin \theta \dot{y}, \quad \dot{\theta}' = \frac{1}{r}(\cos \theta \dot{y} - \sin \theta \dot{x}).
\]  
(11)

From (1) and (11) we obtain
\[
r' = \begin{cases} 
\varepsilon \sum_{i+j=0}^{n} r^{i+j}(a_{ij}^+ \cos^{i+1} \theta \sin^j \theta + b_{ij}^+ \cos^i \theta \sin^{j+1} \theta), & -\frac{\pi}{2} \leq \theta < \frac{\pi}{2}, \\
\varepsilon \sum_{i+j=0}^{n} r^{i+j}(a_{ij}^- \cos^{i+1} \theta \sin^j \theta + b_{ij}^- \cos^i \theta \sin^{j+1} \theta), & \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}.
\end{cases}
\]  
(12)

\[
\theta' = \begin{cases} 
-a + \varepsilon \sum_{i+j=0}^{n} r^{i+j-1}(-a_{ij}^+ \cos^i \theta \sin^{j+1} \theta + b_{ij}^+ \cos^{i+1} \theta \sin^j \theta), & -\frac{\pi}{2} \leq \theta < \frac{\pi}{2}, \\
b - \varepsilon \sum_{i+j=0}^{n} r^{i+j-1}(-a_{ij}^- \cos^i \theta \sin^{j+1} \theta + b_{ij}^- \cos^{i+1} \theta \sin^j \theta), & \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}.
\end{cases}
\]  
(13)

Denote \( w_{ij}^\pm = a_{i-1,j}^\pm + b_{i,j-1}^\pm, \tau_{ij}^\pm = b_{i-1,j}^\pm - a_{i,j-1}^\pm \) with \( a_{i,1,j}^\pm = b_{i,-1,j}^\pm = a_{i,-1}^\pm = b_{i,-1,1}^\pm = 0 \) for \( i, j = 0, 1, \ldots, n + 1 \). Then clearly (9) follows from (12) and (10) follows from (13). This completes the proof of the lemma.

From the equations (9) and (10) in Lemma 3.1, it yields the following \( 2\pi \)-periodic equation
\[
\frac{dx}{d\theta} = \begin{cases} 
\varepsilon F_1^+(\theta, r) + \varepsilon^2 F_2^+(\theta, r) + \varepsilon^3 R^+(\theta, r, \varepsilon), & -\frac{\pi}{2} \leq \theta < \frac{\pi}{2}, \\
\varepsilon F_1^-(\theta, r) + \varepsilon^2 F_2^-(\theta, r) + \varepsilon^3 R^-(\theta, r, \varepsilon), & \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2},
\end{cases}
\]  
(14)

where
\[
F_1^+(\theta, r) = -\frac{1}{a} \sum_{i+j=1}^{n+1} w_{ij}^+ r^{i+j-1} \cos^i \theta \sin^j \theta,
\]
\[
F_1^- (\theta, r) = -\frac{1}{b} \sum_{i+j=1}^{n+1} w_{ij}^- r^{i+j-1} \cos^i \theta \sin^j \theta,
\]
\[
F_2^+(\theta, r) = -\frac{1}{a^2} \left( \sum_{i+j=1}^{n+1} w_{ij}^+ r^{i+j-1} \cos^i \theta \sin^j \theta \right) \left( \sum_{i+j=1}^{n+1} \tau_{ij}^+ r^{i+j-2} \cos^i \theta \sin^j \theta \right),
\]
\[
F_2^- (\theta, r) = -\frac{1}{b^2} \left( \sum_{i+j=1}^{n+1} w_{ij}^- r^{i+j-1} \cos^i \theta \sin^j \theta \right) \left( \sum_{i+j=1}^{n+1} \tau_{ij}^- r^{i+j-2} \cos^i \theta \sin^j \theta \right).
\]  
(15)

It is obvious that seeking the limit cycles of system (1) bifurcated from the period annulus is equivalent to searching for the \( 2\pi \)-periodic solutions of (14). Let
\[
F(\theta, r) = \begin{cases} 
F_1^+(\theta, r), & -\frac{\pi}{2} \leq \theta < \frac{\pi}{2}, \\
F_1^- (\theta, r), & \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}.
\end{cases}
\]
Then from the formula (4) we have
\[
f_1(r) = -\frac{1}{a} \int_{-\pi/2}^{\pi/2} \sum_{i+j=1}^{n+1} w_{ij}^+ r^{i+j-1} \cos^i \theta \sin^j \theta d\theta \\
- \frac{1}{b} \int_{-\pi/2}^{\pi/2} \sum_{i+j=1}^{n+1} w_{ij}^- r^{i+j-1} \cos^i \theta \sin^j \theta d\theta \\
= \sum_{i+j=1}^{n+1} u_{ij} r^{i+j-1} \\
= \sum_{k=0}^{n} v_k r^k, \tag{16}
\]
where
\[
v_k = \sum_{i+j=k+1}^{i+j=1} u_{ij}, \\
u_{ij} = \frac{1}{a} w_{ij}^+ \int_{-\pi/2}^{\pi/2} \cos^i \theta \sin^j \theta d\theta - \frac{1}{b} w_{ij}^- \int_{-\pi/2}^{\pi/2} \cos^i \theta \sin^j \theta d\theta.
\]
We can easily see that (16) has at most \(n\) isolated positive zeros, as in [11]. If \(f_1(r) \equiv 0\), then we need to study the function \(f_2(r)\) by using formula (7). Through direct computation we get
\[
D_r F_1^+(\theta, r) = -\frac{1}{a} \sum_{i+j=2}^{n+1} w_{ij}^+ (i+j-1) r^{i+j-2} \cos^i \theta \sin^j \theta,
\]
\[
D_r F_1^- (\theta, r) = -\frac{1}{b} \sum_{i+j=2}^{n+1} w_{ij}^- (i+j-1) r^{i+j-2} \cos^i \theta \sin^j \theta. \tag{17}
\]
Then by (15) and (17), we obtain
\[
\int_{-\pi/2}^{\pi/2} F_2^+(\theta, r) d\theta = \sum_{k=0}^{2n-1} N_+^k r^k, \\
\int_{-\pi/2}^{\pi/2} F_2^- (\theta, r) d\theta = \sum_{k=0}^{2n-1} N_-^k r^k, \\
\int_{-\pi/2}^{\pi/2} D_r F_1^+(\theta, r) \int_{-\pi/2}^{\theta} F_1^+(t, r) dt d\theta = \sum_{k=0}^{2n-1} M_+^k r^k, \\
\int_{-\pi/2}^{\pi/2} D_r F_1^- (\theta, r) \int_{-\pi/2}^{\theta} F_1^- (t, r) dt d\theta = \sum_{k=0}^{2n-1} M_-^k r^k, \tag{18}
\]
where \(M_\pm^k, N_\pm^k\) are constants that depend on the coefficients of system (1) for \(k = 0, 1, \ldots, 2n - 1\).

Inserting (18) into (7) gives
\[
f_2(r) = \int_{-\pi/2}^{\pi/2} \left( D_r F_1^+(\theta, r) \int_{-\pi/2}^{\theta} F_1^+(t, r) dt + F_2^+(\theta, r) \right) d\theta
\]
\begin{equation*}
+ \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left( D, F_1(\theta, r) \int_{\frac{\pi}{2}}^{0} F^{-}_{1}(t, r) dt + F^{-}_{2}(\theta, r) \right) d\theta
= \sum_{k=0}^{2n-1} V_k r^k,
\end{equation*}

where \( V_k = M_k^+ + M_k^- + N_k^+ + N_k^- \) for \( k = 0, 1, \cdots , 2n - 1 \).

Clearly \( f_2(r) \) has at most \( 2n - 1 \) isolated positive zeros. From Lemma 2.2, we get the following result.

**Theorem 3.2.** For any given \( \varepsilon_0 > 0 \) sufficiently small and \( N > \varepsilon_0 \) sufficiently large, system (1) has at most \( 2n - 1 \) limit cycles bifurcating from the region \( \varepsilon_0 \leq x^2 + y^2 \leq N \) for \( |\varepsilon| > 0 \) sufficiently small by the second order averaging method.

**Remark 2.** The reason we require \( \varepsilon_0 \leq x^2 + y^2 \) is that equation (14) may not be well defined at \( r = 0 \).

Second we consider the limit cycle bifurcations of system (2). Note that \( a_{00}^+ = b_{00}^- = 0 \). This implies \( v_0 = V_0 = 0 \). Using the above results, we can easily see that the function \( f_1(r) \) has at most \( n - 1 \) isolated positive zeros. If \( f_1(r) \equiv 0 \), clearly \( f_2 \) has at most \( 2n - 2 \) isolated positive zeros.

Since equation (14) is well defined at the origin and \( r = 0 \) is its zero solution, we get the next result from Lemmas 2.2 and 2.3.

**Theorem 3.3.** For \( |\varepsilon| > 0 \) sufficiently small, system (2) has at most \( 2n - 2 \) limit cycles on any given compact set of the plane containing the origin using the second order averaging method.

Combining Theorems 3.2 and 3.3, we complete the proof of Theorem 1.1.

4. **Application.** Consider a continuous Hamiltonian system with piecewise perturbation of the form

\begin{equation}
\begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix}
= \begin{cases}
ay + \varepsilon \sum_{i+j=1}^{3} a_{ij}^+ x^i y^j, & x > 0, \\
-ax &,
\end{cases}
\end{equation}

Let (H) denote the set of conditions

\[ a_{10}^+ + a_{10}^- = 0, \quad a_{02}^+ + 2a_{20}^+ = 0, \quad 3a_{30}^+ + a_{12}^+ = 0. \]

**Proposition 1.** Under the condition (H), system (19) can have 4 limit cycles for \( |\varepsilon| > 0 \) sufficiently small by the second order averaging method.

**Proof.** Applying Theorem 1.1, it is easy to see that system (19) has at most 4 limit cycles by the averaging method of second order. Now we prove 4 limit cycles can appear.

From (16), the function \( f_1 \) is given by

\[ f_1(r) = -\pi a_0^+ (a_{10}^+ + a_{10}) r - \frac{2}{3a} (a_{02}^+ + 2a_{20}^+ ) r^2 - \frac{\pi}{8a} (3a_{30}^+ + a_{12}^+ ) r^3. \]

Noting that \( r^i, i = 1, 2, 3 \) are linearly independent, we have \( f_1(r) \equiv 0 \) if the condition (H) holds.
In order to use the second order averaging theory, we need to compute the function $f_2$. From the formula (7), we obtain
\[
f_2(r) = \frac{1}{a^2} \left[ \left( \frac{\pi^2}{4} a_{10}^+ + \frac{\pi}{4} a_{10}^+ a_{01}^+ \right) r + \left( \frac{8}{9} a_{11}^+ a_{10}^+ - \frac{4}{3} a_{20}^+ a_{01}^+ \right) r^2 + \left( \frac{11\pi}{32} a_{10}^+ a_{03}^+ \right) \right] - \left( \frac{3\pi}{8} a_{01}^+ a_{30}^+ + \frac{9\pi}{32} a_{21}^+ a_{10}^+ - \frac{\pi}{8} a_{11}^+ a_{03}^+ \right) r^3 - \left( \frac{8}{15} a_{11}^+ a_{30}^+ + \frac{4}{5} a_{20}^+ a_{03}^+ \right) r^4 - \left( \frac{3\pi}{16} a_{01}^+ a_{30}^+ + \frac{\pi}{16} a_{21}^+ a_{30}^+ \right) r^5 \right] \]
\[= \sum_{k=1}^5 \hat{V}_k r^k. \]

Let $a_{10}^+ = a_{20}^+ = a_{30}^+ = 1$. Denote
\[
\delta = (a_{01}^+, a_{11}^+, a_{21}^+, a_{03}^+), \quad \delta_0 = (-\pi, -\frac{3\pi}{2}, -\frac{29\pi}{9}, \pi).
\]
Then through direct calculation we have
\[
\hat{V}_k(\delta_0) = 0, k = 1, \ldots, 4, \quad \hat{V}_5(\delta_0) = \frac{\pi^2}{12a^2} \neq 0.
\]
Further,
\[
\det \left[ \frac{\partial (\hat{V}_1, \hat{V}_2, \hat{V}_3, \hat{V}_4)}{\partial (a_{01}^+, a_{11}^+, a_{21}^+, a_{03}^+)} \right](\delta_0) = \begin{vmatrix}
\frac{\pi}{4a^2} & 0 & 0 & 0 \\
0 & \frac{8\pi}{30a^2} & 0 & 0 \\
0 & 0 & \frac{\pi}{8a^2} & 0 \\
0 & 0 & -\frac{29\pi}{90a^2} & 0
\end{vmatrix} - \frac{\pi^2}{20a^3}.
\]

Similar to the proof of Corollary 2.4.1 in [3], it follows that $\hat{V}_k, k = 1, \ldots, 4$ can be taken as free parameters. Hence, we can vary $\delta$ such that
\[
0 < |\hat{V}_1| \ll |\hat{V}_2| \ll |\hat{V}_3| \ll |\hat{V}_4| \ll 1, \quad \hat{V}_i \hat{V}_{i+1} < 0, i = 1, 2, 3, 4,
\]
which ensure that $f_2(r)$ has 4 isolated positive zeros $r_1, r_2, r_3, r_4$ with $0 < r_1 < r_2 < r_3 < r_4 \ll 1$. Then by the implicit function theorem, the function $\hat{g}_k(x_0, \varepsilon, \delta)$ defined in (6) has 4 zeros $r_i + O(\varepsilon), i = 1, 2, 3, 4$. The proof is completed. \[\square\]

From this example we know that more limit cycles can be produced by using the second order averaging method than using the first order averaging method.

By the second order averaging method, we conjecture that $2n - 1$ limit cycles can appear for system (1) and $2n - 2$ limit cycles can appear for system (2) for $|\varepsilon| > 0$ sufficiently small. Take system (1) with $n = 1, 2$ for example.

For the case $n = 1$, suppose $a_{01}^- = b_{01}^+$ and $b_{01}^- = b_{01}^+$. From (16) we obtain
\[
f_1(r) = -\frac{2}{a} a_{01}^+ + \frac{2}{b} a_{01}^- - \frac{\pi}{2ab} (ba_{10}^+ + aa_{10}^-) r.
\]
Obviously $f_1(r) \equiv 0$ is equivalent to the following two conditions
\[
a_{01}^- = \frac{b}{a} a_{01}^+, \quad a_{10}^- = -\frac{b}{a} a_{10}^+.
\]
Under the conditions, the function $f_2(r)$ can be expressed as

$$f_2(r) = \frac{2}{a^2}a_{00}^+a_{01}^+ + \frac{\pi}{a^2}a_{00}^+a_{10}^+ + \frac{\pi}{4a}a_{10}^2 + \frac{1}{a}a_{01}a_{10}^+ r$$

$$= \sum_{k=0}^{1} V_k r^k.$$ 

Similar to the proof of Proposition 1, let $a_{00}^+ = 1, a_{01}^+ = -\frac{\pi}{2}$, denote $\delta = a_{10}^+, \delta_0 = 1$. Then we have

$$V_0(\delta_0) = 0, \quad V_1(\delta_0) = \frac{\pi^2}{8a^2} \neq 0, \quad \det \frac{\partial V_0}{\partial a_{10}^+}(\delta_0) = \frac{\pi}{a^2} \neq 0.$$ 

It follows that $V_0$ can be taken as free parameter. Hence, we can vary $\delta$ such that $f_2(r)$ has one isolated positive zero $r_1$. Through Lemma 2.2, system (1) can have one limit cycle for $n = 1$.

For the case $n = 2$, suppose $a_{02}^+ = a_{20}^+ = b_{1j}^+ = 0$. We obtain

$$f_1(r) = -\frac{2}{a}a_{00}^+ + \frac{2}{b}a_{00}^- - \pi \frac{1}{2ab} (ba_{10}^- + aa_{10}^+) r + \frac{2}{3b} (a_{02}^- + 2a_{20}^-) r^2.$$ 

Then $f_1(r) \equiv 0$ is equivalent to the following three conditions

$$a_{00}^- = b \frac{b_{a_{00}}}{a_{a_{00}}}, \quad a_{10}^- = b \frac{a_{10}}{a_{a_{00}}}, \quad a_{02}^- = -2a_{20}^-.$$ 

Under the conditions, the function $f_2(r)$ can be expressed as

$$f_2(r) = \frac{1}{2a^2 b^2} \left[ 72 \pi b^2 a_{00}^+ a_{10}^- + 144 \pi b a_{00}^+ a_{10}^- + 144 b^2 a_{00}^+ a_{01}^- + (18 \pi^2 b^2 a_{10}^-)^2 ight.$$ 

$$+ 36 \pi b a_{00}^+ a_{11}^- - 18 \pi b a_{10}^- a_{11}^- + 36 \pi b^2 a_{00}^+ a_{11}^- + 18 \pi b^2 a_{10}^- a_{11}^- + 9 \pi a^2 a_{10}^- a_{20}^- r^2 + 2 \pi a^2 a_{10}^- a_{20}^- r^3 

$$

$$= \sum_{k=0}^{3} V_k r^k.$$ 

Let $a_{00}^+ = a_{10}^+ = a_{20}^- = a_{11}^- = 1$, denote

$$\delta = (a_{01}, a_{01}^+, a_{11}^-), \quad \delta_0 = \left( \frac{\pi b^2}{6a^2}, -\frac{\pi(-b + 3a)}{6a}, -\frac{\pi a + 4a}{4b} \right).$$ 

Then we have

$$V_0(\delta_0) = V_1(\delta_0) = V_2(\delta_0) = 0, \quad V_3(\delta_0) = -\frac{\pi}{8a^2} \neq 0,$$

$$\det \frac{\partial (V_0, V_1, V_2)}{\partial (a_{01}, a_{01}^+, a_{11}^-)}(\delta_0) = -\frac{4(\pi b - 3a - b)}{9a^5 b^2} \neq 0.$$ 

It follows that $V_0, V_1, V_2$ can be taken as free parameters. Similarly we can prove system (1) has 3 limit cycles for $n = 2$.

REFERENCES

[1] R. Benterki and J. Llibre, Periodic solutions of the Duffing differential equation revisited via the averaging theory, Journal of Nonlinear Modeling and Analysis, 1 (2019), 11–26.

[2] X. L. Cen, S. M. Li and Y. L. Zhao, On the number of limit cycles for a class of discontinuous quadratic differential systems, Journal of Mathematical Analysis and Applications, 449 (2017), 314–342.

[3] M. A. Han, Bifurcation Theory of Limit Cycles, Science Press Beijing, Beijing, Alpha Science International Ltd., Oxford, 2017.
[4] M. A. Han, On the maximum number of periodic solution of piecewise smooth periodic equations by average method, *Journal of Applied Analysis and Computation*, 7 (2017), 788–794.

[5] M. A. Han, G. Chen and C. Sun, On the number of limit cycles in near-Hamiltonian polynomial systems, *International Journal of Bifurcation and Chaos*, 17 (2007), 2033–2047.

[6] M. A. Han, V. G. Romanovski and X. Zhang, Equivalence of the Melnikov function method and the averaging method, *Qualitative Theory of Dynamical Systems*, 15 (2016), 471–479.

[7] M. A. Han and L. J. Sheng, Bifurcation of limit cycles in piecewise smooth systems via Melnikov function, *Journal of Applied Analysis and Computation*, 5 (2015), 809–815.

[8] M. A. Han, L. J. Sheng and X. Zhang, Bifurcation theory for finitely smooth planar autonomous differential systems, *Journal of Differential Equations*, 264 (2018), 3596–3618.

[9] M. Han and P. Yu, *Normal Forms, Melnikov Functions and Bifurcations of Limit Cycles*, Springer, New York, 2012.

[10] J. Itikawa, J. Llibre and D. D. Novaes, A new result on averaging theory for a class of discontinuous planar differential systems with applications, *Revista Matematica Iberoamericana*, 33 (2017), 1247–1265.

[11] X. Liu and M. A. Han, Bifurcation of limit cycles by perturbing piecewise Hamiltonian systems, *International Journal of Bifurcation and Chaos*, 20 (2010), 1379–1390.

[12] J. Llibre and A. C. Mereu, Limit cycles for discontinuous quadratic differential systems with two zones, *Journal of Mathematical Analysis and Applications*, 413 (2014), 763–775.

[13] J. Llibre, A. C. Mereu and D. D. Novaes, Averaging theory for discontinuous piecewise differential systems, *Journal of Differential Equations*, 258 (2015), 4007–4032.

[14] J. Llibre, D. D. Novaes and C. A. B. Rodrigues, Averaging theory at any order for computing limit cycles of discontinuous piecewise differential systems with many zones, *Physica D Nonlinear Phenomena*, 353/354 (2017), 1–10.

[15] J. Llibre, D. D. Novaes and M. A. Teixeira, On the birth of limit cycles for non-smooth dynamical systems, *Bulletin Des Sciences Mathématiques*, 139 (2015), 229–244.

[16] S. Y. Sui and L. Q. Zhao, Bifurcation of limit cycles from the center of a family of cubic polynomial vector fields, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 28 (2018), 1850063, 11 pp.

[17] H. H. Tian and M. A. Han, Bifurcation of periodic orbits by perturbing high-dimensional piecewise smooth integrable systems, *Journal of Differential Equations*, 263 (2017), 7448–7474.

[18] Y. Q. Xiong and M. A. Han, Limit cycle bifurcations in a class of perturbed piecewise smooth systems, *Applied Mathematics and Computation*, 242 (2014), 47–64.

[19] J. H. Yang and L. Q. Zhao, Limit cycle bifurcations for piecewise smooth integrable differential systems, *Discrete and Continuous Dynamical Systems Serise B*, 22 (2017), 2417–2425.

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