Monopole Excitation to Cluster States

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Abstract. Strength of monopole excitation to cluster states is discussed in light nuclei. A detail analysis of $^{12}$C, typical in light nuclei, leads to the fact that the monopole excitation to cluster states is in general so strong to be comparable to the single particle strength and shares an appreciable portion of the sum rule value, in spite of large difference of the structure between the cluster state and the shell-model-like ground state. The present results imply that the measurement of strong monopole transitions or excitations is generally very useful for the study of cluster states.

1. Introduction

The monopole transitions from cluster states to ground states are rather strong in light nuclei. For example in $^{12}$C the monopole strength $M(E0)$ between the ground state and the first excited $0^+$ state at $E_x = 7.66$ MeV (so-called Hoyle state) which has a $3\alpha$ cluster structure \cite{1,2,3} is $5.4 \pm 0.2$ fm\textsuperscript{2}, the value of which shares about 16 % of the energy weighted sum rule value. Also in $^{16}$O the $M(E0)$ value between the ground state and the first and second excited $0^+$ states at $E_x = 6.05$ MeV and 12.05 MeV which have $^{12}$C+$\alpha$ cluster structure \cite{1,4} are $3.55 \pm 0.21$ fm\textsuperscript{2} and $4.03 \pm 0.09$ fm\textsuperscript{2}, respectively, which share about 3 % and 8 % of the energy weighted sum rule value. Recently the third $3/2^-$ state ($E_x = 8.56$ MeV) of $^{11}$B observed at the $^{11}$B($d,d'$) reaction was concluded to have a $2\alpha+t$ cluster structure \cite{5}. Among many reasons for this conclusion, one is a large monopole strength for the state which is similar to the $M(E0)$ value for the Hoyle state in $^{12}$C, and another is theoretical supports with the AMD calculation as well as the OCM one \cite{6}.

A rough estimate of the single particle monopole strength $\langle u_f(r)|r^2|u_i(r)\rangle$ is $(3/5)R^2 \sim 5.4$ fm\textsuperscript{2} for $p$- and $sd$-shell nuclei ($R \sim 3.0$ fm), in which the formula is obtained under the uniform-density approximation for $u_f(r)$ and $u_i(r)$ with $R$ standing for the nuclear radius. The experimental monopole strengths for $^{12}$C and $^{16}$O as mentioned above, thus, are comparable to the single particle strength. The single particle estimate is based on the assumption that the excited state has a one-particle one-hole excitation from the ground state. However, the cluster structure is very different from the shell-model-like structure of the ground state, and its state is described as a superposition of many-particle many-hole configurations when it is expanded in terms of shell model configurations. This means that the component of a one-particle one-hole excitation from the ground-state configuration is expected to be very small in the excited state with a cluster structure. Therefore the fact of rather large monopole strengths for the cluster
states looks not to be easy to explain. The microscopic cluster model calculations [1, 2, 3, 4], however, have reproduced rather well the experimental data. No explicit explanations on the reason why the cluster models reproduce plausibly the experimental data have been presented at all so far as long as we know. There should exist underlying physics in the monopole transition strengths in the light nuclei.

The purpose of this paper is to demonstrate the reason why monopole transition strength between a cluster state and the ground state in light nuclei is generally rather large in comparison with the single particle strength, and shares an appreciable portion of the sum rule value, although the large difference of structure exists between the initial and final states. We shall clarify it for $^{12}\text{C}$ as a typical example, as semi-quantitatively and intuitively as possible, without making huge numerical calculations like solving many-body Schrödinger equation, because qualitative treatment is very useful to understand compactly the reason. The present approach is applicable to other light nuclei.

2. Clustering degrees of freedom embedded in the ground-state wave functions

The nuclear SU(3) model [7] is known to describe rather well the ground state of light nuclei. This was confirmed recently by the symplectic no-core shell model calculation [8] with modern realistic $NN$ interactions. Bayman-Bohr theorem [9] tells us that the SU(3) shell model wave function describing the ground state is in most cases equivalent to the cluster-model wave function discussed by Wildermuth et al. [10]. Thus we can see what kinds of clustering degrees of freedom are embedded in the ground state and hence can be excited by the monopole operator. Here, we demonstrate them in case of $^{16}\text{O}$ and $^{12}\text{C}$.

The ground state of $^{16}\text{O}$ has a doubly closed shell structure of $0s$ and $0p$ orbits, belonging to the SU(3) irreducible representation $(\lambda, \mu) = (0, 0)$. According to the Bayman-Bohr theorem, this wave function is equivalent to a cluster wave function of $^{12}\text{C}$ and $\alpha$,

$$|0^+_1\rangle = \frac{1}{\sqrt{16!}} \det |(0s)^4(0p)^{12}| \times |\phi_G(r_G)|^{-1}$$

$$= N_{g0} \sqrt{\frac{12!4!}{16!}} A\{[\mathcal{R}_{40}(r, 3\nu_N)\phi_{L=0}(^{12}\text{C})]_{J=0}\phi(\alpha)\} \quad (2)$$

where $N_{g0}$ ($\nu_N$) denotes the normalization constant (nucleon size parameter: $\nu_N = M\omega/2\hbar$ ), and $\mathcal{R}_{NL}(u, \beta)$ stands for the harmonic oscillator function of the size parameter $\beta$ with the oscillator quantum number $N$ and angular momentum $L$. Equation (2) means that the ground state wave function of $^{16}\text{O}$ has originally a clustering degree of freedom, $^{12}\text{C}+\alpha$. Thus, the ground state of $^{16}\text{O}$ can be excited not only through single particle degrees of freedom by promoting nucleons from $0s$ and $0p$ orbits to higher orbits, but also through cluster degrees of freedom by exciting the $^{12}\text{C}−\alpha$ relative motion from $\mathcal{R}_{4L}(r, 3\nu_N)$ state to higher nodal states. The latter characteristic is an essential point to understand why the monopole transition matrix elements to cluster states are in general large in $^{16}\text{O}$, which will be discussed in detail elsewhere.

On the other hand, the ground state $(0^+_1)$ of $^{12}\text{C}$ is described by the SU(3) shell model wave function $\phi_{L=0}(^{12}\text{C})$ [see Eqs. (2)] belonging to the SU(3) irreducible representation $(\lambda, \mu) = (0, 4)$. According to the Bayman-Bohr theorem, the SU(3) shell model wave function is equivalent to a $3\alpha$ cluster-model wave function with the total quanta $N_{TOT} = 8$,

$$|0^+_1\rangle = |(0s)^4(0p)^8(0, 4)J = 0\rangle_{\text{internal}}$$

$$= \tilde{N}_{g0} \sqrt{\frac{4!4!4!}{12!}} A\{[\mathcal{R}_{44,4L=0}(s, t)\phi(\alpha_1)\phi(\alpha_2)\phi(\alpha_3)]\} \quad (3)$$

$$\mathcal{R}_{N_1, N_2, L}(s, t) \equiv [\mathcal{R}_{N_1L}(s, 2\nu_N)\mathcal{R}_{N_2L}(t, \nu_N)]_{J=0}, \quad (N_1 + N_2 = N). \quad (4)$$
where \( s \) and \( t \) denote the Jacobi coordinates among the \( 3\alpha \) clusters. Equation (3) tells that the ground-state wave function of \( ^{12}\text{C} \) has originally a \( 3\alpha \) clustering degree of freedom. This character plays an important role in producing rather large monopole strength between the ground state and Hoyle state as shown in next section.

3. Monopole transition to Hoyle state in \( ^{12}\text{C} \)

In this section, we calculate monopole transition strengths of \( ^{12}\text{C} \) from the ground state (0\(^+_2\)) to the Hoyle state (0\(^+_1\)). First, we use a simple SU(3) wave function for the ground state, which has no ground state correlations. Then, the effect of the ground state correlations to the monopole transition strength is investigated from the viewpoint of the microscopic cluster model theory.

3.1. Simple estimation with SU(3) wave function for the ground state

The SU(3) wave function in Eq. (3) is taken as the 0\(^+_1\) state to calculate the monopole transition strength to the Hoyle state. The Hoyle state (0\(^+_2\)) is known to have 3\(\alpha \) structure [1], and so we express its wave function as

\[
|0^+_2\rangle = \tilde{N}_H \sqrt{\frac{4!4!4!}{12!}} A \{ \tilde{\chi}_H(s, t) \phi(\alpha_1) \phi(\alpha_2) \phi(\alpha_3) \}. \tag{5}
\]

Expanding the relative wave function \( \tilde{\chi}_H(s, t) \) in terms of the harmonic oscillator wave functions, the number of the total oscillator quanta of these oscillator wave functions should be larger than \( N_{TOT} = 8 \), since the Hoyle state has to be orthogonal to the SU(3)-model ground-state wave function in Eq. (3). The monopole transition matrix element \( M(0^+_0, 0^+_2 - 0^+_1) \) is expressed as follows,

\[
M(0^+_0, 0^+_2 - 0^+_1) = \langle 0^+_1 \| \frac{1}{2} \sum_{i=1}^{12} (r_i - r_G)^2 \| 0^+_2 \rangle \\
= \frac{1}{2} \sum_{L} \tilde{N}_H \langle r^8_{4J,L}(s, t) \rangle \langle 2s^2 + \frac{8}{3}t^2 \| \tilde{\chi}_H(s, t) \rangle. \tag{6}
\]

Here we used the following relation,

\[
\sum_{i=1}^{12} (r_i - r_G)^2 = \sum_{k=1}^{3} \sum_{i \in \alpha_k} (r_i - X_k)^2 + 2s^2 + \frac{8}{3}t^2. \tag{7}
\]

where \( r_G \) (\( X_i \)) denotes the center-of-mass coordinate of \( ^{12}\text{C} \) (\( i\)-th \( \alpha \) cluster). It is noted that the first term in Eq. (7) does not contribute exactly to the monopole transition matrix element in the present study.

Recently the structure of the Hoyle state has been studied from a new point of view that this state is the Bose-condensed state of \( 3\alpha \) particles [11, 12, 13]. It has been demonstrated that the full solutions of the \( 3\alpha \) RGM (Resonating Group Method) equation of motion [2, 3] have large overlaps close to 100% with the \( 3\alpha \) Bose-condensed wave functions [13]. Therefore we here adopt as \( \tilde{\chi}_H(s, t) \) the following form

\[
\tilde{\chi}_H(s, t) = (1 - P) \tilde{\chi}_{HG}(s, t), \tag{8}
\]

\[
\tilde{\chi}_{HG}(s, t) = \left( \frac{8\gamma}{\sqrt{3}\pi} \right)^{\frac{3}{4}} \exp\left\{ -4\gamma \sum_{k=1}^{3} (X_k - r_G)^2 \right\}

= \left( \frac{4\gamma}{\pi} \right)^{\frac{3}{4}} \left( \frac{16\gamma}{3\pi} \right)^{\frac{3}{4}} \exp\left\{ -\gamma (2s^2 + \frac{8}{3}t^2) \right\}, \tag{9}
\]

\[
P = \sum_{N \leq 8} \sum_{N_1 + N_2 = N} \sum_{L} |\tilde{R}_{N_1, N_2, L}(s, t)\rangle \langle \tilde{R}_{N_1, N_2, L}(s, t)|, \tag{10}
\]
Figure 1. Integrated values of the harmonic-oscillator (h.o.) basis components ($\hbar \Omega = 22$ MeV) when the ground state [(a) $0_1^+$: square mark] and Hoyle state [(b) $0_2^+$: circle one] of $^{12}\text{C}$ obtained with the semi-microscopic $3\alpha$ cluster model [12] are expanded in terms of the h.o. basis with quanta $N$. The values are zero in case of $N < 8$.

where $\gamma$ denotes the width parameter characterizing the $3\alpha$ condensate wave function. $P$ is the projection operator onto the state of $SU(3)$ relative motion of the ground state and the states forbidden by the antisymmetrization. Then, the analytical expression of the monopole transition matrix element in Eq. (6) is given as follows:

$$M(E_0, 0_2^+ - 0_1^+) = \sqrt{\frac{7}{6}} \sqrt{\left\langle F_4 \right\rangle \left\langle F_5 \right\rangle} \times \xi_5(\nu_N, \gamma) \times \left\langle R_{40}(r, \nu_N)|r^2|R_{60}(r, \nu_N)\right\rangle,$$  \hspace{1cm} (11)

where $\left\langle F_n \right\rangle$ represents something like expectation values of the antisymmetrization operator acting among nucleons [14], and $\left\langle R_{40}(r, \nu_N)|r^2|R_{60}(r, \nu_N)\right\rangle = \sqrt{21}/8/\nu_N$. It is noted that the dependence of $M(E_0, 0_2^+ - 0_1^+)$ on the parameter $\gamma$ is contained only in the factor $\xi_5$.

In the present study we take the value $\nu_N = 0.168$ fm$^{-2}$, which reproduces the observed rms radius of $^{12}\text{C}$ with the $SU(3)$ shell model wave function in Eq. (3). Then, the monopole matrix element in Eq. (11) is expressed as $M(E_0, 0_2^+ - 0_1^+) = \xi_5 \times 0.882/\nu_N$ fm$^2$, in which $\xi_5$ depends on $\gamma$. According to Ref. [13], we should use the value of $\gamma \approx 0.018$, for the Hoyle state the use of which gives the rms radius close to 3.8 fm and very large overlap between the $3\alpha$ Bose-condensed wave function and the full solution of the $3\alpha$ RGM equation of motion. For this $\gamma$ value, we obtain $\xi_5 = 0.191$. This leads to $M(E_0, 0_2^+ - 0_1^+) \approx 1.3$ fm$^2$, which is the same order as the observed value ($5.4 \pm 0.2$ fm$^2$) but reproduces only about 25 % in comparison with that.

We should note that the description of the ground state here adopted for $^{12}\text{C}$ using the $SU(3)$ shell model is not necessarily good and a deviation from the $SU(3)$ shell model representation should be taken into account [1]. Figure 1(a) demonstrates the distribution of the component harmonic oscillator basis in the ground state obtained with the $3\alpha$ orthogonality condition model (OCM) [12]. We see that the $SU(3)\lambda, \mu = (0,4)$ component with the lowest quantum ($N_{TOT} = 8$) is only about 60 % in the ground state of $^{12}\text{C}$ in the microscopic $3\alpha$ cluster model. The better description of the ground state for $^{12}\text{C}$ should make the monopole matrix element be larger than the present value and comparable to the observed one. We will discuss it in next section.
3.2. Effect of the ground-state correlation

Here we demonstrate the effect of the ground state correlation to the monopole matrix element by adopting the following wave function as the ground state [13]:

\[ \psi_G(\tilde{\gamma}, \nu_N) = N_G \sqrt{\frac{4! \cdot 4!}{12!}} A \exp \{-\tilde{\gamma}(2s^2 + \frac{8}{3}t^2)\} \phi(\alpha_1)\phi(\alpha_2)\phi(\alpha_3), \]  

(12)

where \( N_G \) is normalization constant. In the present study, the rms radius of the ground-state wave function is fixed to the experimental one (2.47 fm). Then, the ratio \( \tilde{\gamma}/\nu_N \) is only the parameter which describes the property of the ground state. It is noted that \( \psi_G(\tilde{\gamma}, \nu_N) \) with \( \tilde{\gamma}/\nu_N = 1 \) agrees with the SU(3) \( (\lambda, \mu) = (0, 4) \) shell model wave function Eq. (3). Taking the \( \tilde{\gamma} \) value a little smaller than \( \nu_N \), \( \psi_G(\tilde{\gamma}, \nu_N) \) deviates from the SU(3) shell model wave function and the spatial localization of the 3\( \alpha \) correlation taking into account here and in Ref. [13]. The amount of the 3\( \alpha \) correlation decreases from unity, namely as the 3\( \alpha \)-like ground state correlation occurs and becomes stronger in the ground state. This can be reasonably understood from the fact that the ground state wave function \( \psi_G \) with stronger 3\( \alpha \)-like correlation has a significant amount of the 3\( \alpha \) wave function (3.8 fm). Then, the monopole matrix element is given as follows:

\[ M(E0, 0^+_1 - 0^+_2) = (\psi_G(\tilde{\gamma}, \nu_N))^\dagger \sum_{i=1}^{12} (\mathbf{r}_i - \mathbf{r}_G)^2 |\psi_H(\gamma, \tilde{\gamma}, \nu_N)|, \]  

(15)

which depends on only the parameter \( \tilde{\gamma}/\nu_N \).

Table 1 shows the monopole matrix elements [Eq. (15)] calculated at several \( \tilde{\gamma}/\nu_N \) values. The monopole matrix element increases as the ratio \( \tilde{\gamma}/\nu_N \) decreases from unity, namely as the 3\( \alpha \)-like correlation occurs and becomes stronger in the ground state. This can be reasonably understood from the fact that the ground state wave function \( \psi_G \) with stronger 3\( \alpha \)-like correlation has larger 3\( \alpha \)-cluster component which makes larger the overlap with the Hoyle state wave function \( \psi_H \) with the dilute 3\( \alpha \) cluster structure, and then the monopole matrix element becomes larger. At
Table 1. Dependence of the monopole matrix element in $^{12}$C on the amount of 3α-like correlation involved in the ground state, which is characterized by $\tilde{\gamma}/\nu_N$.

| $\tilde{\gamma}/\nu_N$ | $M(E0, 0^+_N - 0^+_N) [\text{fm}^2]$ |
|------------------------|-------------------------------------|
| 1.000                  | 1.326                               |
| 0.705                  | 1.810                               |
| 0.498                  | 2.473                               |
| 0.309                  | 3.597                               |
| 0.274                  | 4.035                               |

the value of $\tilde{\gamma}/\nu_N \sim 0.27$, the monopole matrix element is about 4.0 fm$^2$, which is about three times larger than that for $\tilde{\gamma}/\nu_N = 1$, and is closer to the observed value 5.4 ± 0.2 fm$^2$. It is noted that $\tilde{\gamma}/\nu_N \sim 0.27$ gives the nucleon size parameter $\nu_N \sim 0.26$ fm$^{-2}$, the value of which corresponds to that used usually in the microscopic 3α cluster model calculations [1, 2, 3, 12, 13]. As mentioned above, the SU(3)$\lambda, \mu = (0, 4)$ component with the lowest quantum ($N_{TOT} = 8$) is about 60% in the ground state of $^{12}$C in the microscopic 3α cluster model. The other component of about 40% is nothing but the 3α-cluster correlation discussed here. The 3α-like ground state correlation, thus, plays an important role in reproducing the monopole matrix element in $^{12}$C.

4. Summary

Demonstrating the case of $^{12}$C, we discussed the reason why the monopole strength between ground state and cluster states in light nuclei is generally strong in comparison with the single particle strength, although large difference of the structure exists between the cluster state and the shell-model-like ground state. According to Bayman-Bohr theorem, the SU(3)-model ground-state wave function of $^{12}$C ($^{16}$O) has originally the 3α ($^{12}$C-α) clustering degree of freedom. The ground-state correlation, i.e. spatial localization of the 3α clusters, which is induced by the NN interactions and is naturally taken into account in the microscopic cluster model, deviates significantly the character of the ground-state wave function from that of the SU(3)-model wave function. The nature of the monopole operator, acting the relative motions among the 3α clusters, thus, excites the 3α clustering degree of freedom in the ground state to populate rather strongly the 3α cluster state, i.e. Hoyle state. This story is expected to be realized in other light nuclei, which will be discussed elsewhere.

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