UPPER BOUNDS FOR EIGENVALUES OF CONFORMAL LAPLACIAN ON SPHERES

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Abstract. In this paper, we introduce a new functional for the conformal spectrum of the conformal laplacian on a closed manifold $M$ of dimension at least 3. For this new functional we provide a Korevaar type result. The main body of the paper deals with the case of the sphere but a section is devoted to more general closed manifolds.

CONTENTS

1. Introduction 1
2. Geometry of metric measure space 2
3. construction of test functions 4
4. Proof of Theorem 1.1 7
5. Generalizations and Related Questions 9
5.1. Generalizations to any closed manifolds 9
5.2. Hersch Type Results 11
References 12

1. Introduction

Let $(M^n, g)$ with $n \geq 3$ be a compact boundaryless manifold with scalar curvature $R_g$. Let $[g]$ be the conformal class of $g$ and let $c_n$ denote the constant $\frac{n-2}{4(n-1)}$. Consider the conformal laplacian

$\Box_g = -\Delta_g + c_n R_g.$

Let $\tilde{g}$ be a conformal metric to $g$, i.e. $\tilde{g} = \mu^{4/(n-2)} g$ for some positive function $\mu \in C^\infty(M)$. We investigate the eigenvalue problem

$(-\Delta_{\tilde{g}} + c_n R_{\tilde{g}}) u = \lambda u \quad (1.1)$

Suppose $(S^n, g)$ is the $n$-sphere with the round metric and $\tilde{g} \in [g]$. Recall that a celebrated result by Korevaar [5] implies that the $k$th eigenvalue $\lambda_k(S^n, -\Delta_{\tilde{g}})$ of the laplacian satisfies

$\lambda_k(S^n, -\Delta_{\tilde{g}}) \cdot \text{Vol}(S^n, \tilde{g})^{2/n} \leq C(n) k^{2/n}.$
However, the same estimate is false for the conformal laplacian \([1]\). By changing the quantity 
\[
\text{Vol}((S^n, \tilde{g}))^{2/n} = \left( \int_{S^n} \mu^{\frac{2}{n-2}} dV_{\tilde{g}} \right)^{2/n}
\]
to a smaller one in the sense of Lebesgue space embeddings, we have

**Theorem 1.1.** Let \((S^n, g)\) be the \(n\)-sphere with the round metric. For any metric \(\tilde{g} \in [g]\), the \(k\)th eigenvalue \(\tilde{\lambda}_k = \lambda_k(S^n, \tilde{g})\) of the conformal laplacian \(\Box_{\tilde{g}}\) satisfies the inequality

\[
\tilde{\lambda}_k \int_{S^n} \mu^{\frac{4}{n-2}} dV_g \leq C(n)k^{2/n},
\]

where \(C(n)\) is a constant only depending on \(n\).

The proof follows in a similar way from [3, 4], in which the idea traces back to [2] and [5].

Upper bounds for the conformal spectrum of geometric differential operators play a crucial role in the geometry of manifolds. An extensive program initiated by S. T. Yau and collaborators (see [7, 11] for instance) has been instrumental in these aspects. The existence of extremal metrics on general surfaces has been proved by Nadirashvili and the first author [9, 10] (see also [8]).

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## 2. Geometry of metric measure space

In this section, we will show the existence a large number of disjoint annuli carrying a sufficient amount of the volume measure. Let \((X, d)\) be a metric space. For any \(p \in X\) and \(0 \leq r < R < +\infty\), we denote the ball \(B(p, r) = \{ x \in X : d(x, p) < r \}\) and the annulus \(A(p; r, R) = \{ x \in X : r \leq d(x, p) < R \}\). Recall the following definition.

**Definition 2.1.** Given a positive integer \(N\), we say that a metric space \((X, d)\) satisfies \(N\)-covering property if for any ball \(B(p, r)\) in \(X\), there exists a family of at most \(N\) balls of radii \(r/2\), which cover \(B(p, r)\).

The main tool we will use to construct disjoint annuli is the following theorem by Grigoryan, Netrusov and Yau [2] (see Theorem 3.5 and Lemma 3.10).

**Theorem 2.2** ([2]). Let \((X, d)\) be a metric space and \(m\) be a Borel measure. Assume that

- \((X, d)\) satisfies \(N\)-covering property;
all balls in $X$ are precompact;
measure $m$ is finite and non-atomic.

Then for any positive integer $k$, there exists annuli $A_j = A(p_j; r_j, R_j)$ for $1 \leq j \leq k$ such that

- $2A_j = A_j(p_j; \frac{r_j}{2}, 2R_j)$ for $1 \leq j \leq k$ are pairly disjoint;
- for each $1 \leq j \leq k$, $m(A_j) \geq c \frac{m(X)}{k}$.

Here $c = c(N)$ is a constant only depending on $N$.

In our case, we will set metric space $(X, d) = (S^n, d_g)$, where $d_g$ is the distance function induced from the round metric $g$. Since $(S^n, g)$ is positively curved, it satisfies the doubling property. That is to say, there exists some constant $C(n)$, such that $\text{Vol}_g(B(p, 2r)) \leq C(n) \text{Vol}_g(B(p, r))$ for any ball $B(p, r) \subset S^n$. In fact, we can take $C(n) = 2^n$ for $(S^n, d_g)$ by using the volume comparison theorem. And it follows that $(S^n, d_g)$ satisfies $N$-covering property for any $N \geq 4^n$ by using a packing argument. In particular, the constant $c$ here only depends on the dimension $n$ if we set $N = 4^n$.

Let $\nu, \tilde{\nu}$ be the volume measure induced by the metric $g, \tilde{g}$ respectively. Furthermore, we will set the Borel measure

$$m = \mu^{-2}d\nu = \mu^\frac{4}{n-2}d\nu.$$  

(2.1)

Applying Theorem 2.2 to $(S^n, d_g, \mu^\frac{4}{n-2}d\nu)$ and positive integer $2k$, we obtain a collection of annuli $\{A_j = A(p_j; r_j, R_j)\}_{j=1}^{2k}$ such that

1. $2A_j = A_j(p_j; \frac{r_j}{2}, 2R_j)$ for $1 \leq j \leq 2k$ are pairly disjoint;
2. for each $1 \leq j \leq 2k$,

$$\int_{A_j} \mu^\frac{4}{n-2}d\nu \geq \frac{c}{k} \int_{S^n} \mu^\frac{4}{n-2}d\nu = \frac{c}{k} m(S^n),$$  

(2.2)

where $c = c(n)$ is a constant only depending on $n$.

Furthermore, we can assume the first $k$ many annuli in the collection $\{A_j\}_{j=1}^{2k}$ satisfy

$$\nu(2A_j) \leq \frac{\nu(S^n)}{k}, \quad \text{for } 1 \leq j \leq k.$$  

(2.3)

This is because $\{2A_j\}_{j=1}^{2k}$ are pairly disjoint and thus we have

$$\sum_{j=1}^{2k} \nu(2A_j) \leq \nu(S^n).$$

By re-indexing the collection of annuli, we can assume $\nu(2A_1) \leq \nu(2A_2) \leq \cdots \leq \nu(2A_{2k})$ and (2.3) clearly follows.
3. CONSTRUCTION OF TEST FUNCTIONS

In this section, we will construct test functions supported in each annulus $2A_j$ for $1 \leq j \leq k$.

Fix an annulus $A = A(p; r, R) \subset S^n$. Let $x = (x^0, x^1, \cdots, x^n) \in \mathbb{R}^{n+1}$ be the Cartesian coordinates. Let $\sigma_p : S^n \setminus \{p\} \to \mathbb{R}^n$ be the stereographic projection with the base point $p$ to the subspace $L_p = \{x \in \mathbb{R}^{n+1} : x \cdot p = 0\}$.

For any $t > 0$, we denote $\delta_t : \mathbb{R}^n \to \mathbb{R}^n$ as the rescaling map by factor $t$, i.e., $\delta_t(y) = ty$ for any $y \in \mathbb{R}^n$. Define $\theta_{p,t} = \sigma_p^{-1} \circ \delta_t \circ \sigma_p$, which is a conformal diffeomorphism of $(S^n, g)$, since each map in the composition is conformal. It is not hard to see the following properties of $\theta_{p,t}$ are satisfied.

- $\theta_{p,t}$ fixes the points $\pm p$.
- $\theta_{p,t}$ preserves level sets of the distance function $d_g(p, \cdot)$.
- For any $x \in S^n \setminus \{\pm p\}$, $\theta_{p,t}(x) \to p$ as $t \to +\infty$, while $\theta_{p,t}(x) \to -p$ as $t \to 0$.

We start the construction with a special case that $r = 0$ and $A(p; r, R) = B(p, R)$. Given $R \in (0, \pi)$, we can choose the value $t = t(R) \in \mathbb{R}^+$ so that $\theta_{p,t}$ maps $B(p, 2R)$ to the hemisphere $B(p, \pi)$. Let $x_p = x \cdot p$. We define

$$\varphi_{p,R}(x) = \begin{cases} x_p \circ \theta_{p,t}(x), & \text{if } x \in B(p, 2R), \\ 0, & \text{if } x \notin B(p, 2R). \end{cases}$$

Let $\nabla$ be the Levi-Civita connection for the round metric $g$. Then we have the following lemma on the function $\varphi_{p,R}$.

**Lemma 3.1.** $\varphi_{p,R}$ is a Lipschitz function on $S^n$ satisfying

- $0 \leq \varphi_{p,R}(x) \leq 1$ for any $x \in S^n$;
- $\varphi_{p,R}(x) \geq \frac{3}{4}$ for any $x \in B(p, R)$;
- $\text{supp } \varphi_{p,R} \subset B(p, 2R)$;
- $$\int_{S^n} |\nabla \varphi_{p,R}|_g^2 d\nu \leq C(n),$$

where $C(n)$ is a constant only depending on $n$.

**Proof.** The first property follows immediately by noting $x_p = x \cdot p = \cos (d_g(p, x))$. The third property is also straightforward by the definition of $\varphi_{p,R}$. It remains to check the second one and the last one.

We need to compute $t$ in terms of $R$ explicitly. As $(S^n, g)$ is homogeneous under the action of $\text{SO}(n+1)$, we can assume that $p =$
We denote $x' = (x^1, x^2, \ldots, x^n)$ and $x = (x^0, x')$. Recall the formulas for the stereographic projection:

$$\sigma_p(x) = \frac{x'}{1-x^0} \quad \text{for any } x \in S^n,$$

and

$$\sigma_p^{-1}(y) = \left( \frac{|y|^2 - 1}{|y|^2 + 1}, \frac{2y}{|y|^2 + 1} \right) \quad \text{for any } y \in \mathbb{R}^n.$$

By a straightforward computation, we obtain that for any $x \in S^n$

$$\theta_{p,t}(x) = \left( \frac{t^2(1 + x^0) - (1 - x^0)}{t^2(1 + x^0) + (1 - x^0)}, \frac{2tx'}{t^2(1 + x^0) + (1 - x^0)} \right). \quad (3.1)$$

Recall that $t$ is chosen so that $\theta_{p,t}$ maps $B(p, 2R)$ onto $B(p, \frac{\pi}{2})$. In particular, $\theta_{p,t}$ maps $\partial B(p, 2R)$ to $\partial B(p, \frac{\pi}{2})$ since it preserves level sets of the distance function $d_g(p, \cdot)$. Take $x \in \partial B(p, 2R)$. Then

$$x_p = x^0 = \cos(d_g(x, p)) = \cos(2R).$$

And $\theta_{p,t}(x) \in \partial B(p, \frac{\pi}{2})$ implies that

$$x_p(\theta_{p,t}(x)) = \frac{t^2(1 + x^0) - (1 - x^0)}{t^2(1 + x^0) + (1 - x^0)} = \cos(\pi/2) = 0.$$

Combine these two equations and we can solve

$$t = \sqrt{\frac{1 - \cos(2R)}{1 + \cos(2R)}} = \tan R.$$

We are now ready to prove the second property. Since $\varphi_{p,R}(x) = \cos(d_g(p, \theta_{p,t}(x)))$ on the closed ball $x \in B(p, R)$, the minimum is attained on the boundary $\partial B(p, R)$. For any $x \in \partial B(p, R)$, we have $x^0 = \cos R$ and

$$\varphi_{p,R}(x) = \frac{\tan^2 R \cdot (1 + \cos R) - (1 - \cos R)}{\tan^2 R \cdot (1 + \cos R) + (1 - \cos R)} = \frac{2 \cos R + 1}{2 \cos^2 R + 2 \cos R + 1}.$$

We set $f(R) = \frac{2 \cos R + 1}{2 \cos^2 R + 2 \cos R + 1}$. A straightforward computation shows that $f(R)$ is increasing on $(0, \frac{\pi}{2})$. Therefore, the second property follows by the fact

$$\lim_{R \to 0} f(R) = \frac{3}{5}.$$
Next we prove the last property of $\varphi_{p,R}$. Using the spherical coordinates, we have

\[\begin{align*}
    x^0 &= \cos \phi_1, \\
    x^1 &= \sin \phi_1 \cos \phi_2, \\
    x^2 &= \sin \phi_1 \sin \phi_2 \cos \phi_3, \\
    \vdots \\
    x^{n-1} &= \sin \phi_1 \sin \phi_2 \cdots \sin \phi_{n-1} \cos \phi_n, \\
    x^n &= \sin \phi_1 \sin \phi_2 \cdots \sin \phi_{n-1} \sin \phi_n.
\end{align*}\]

Accordingly, the round metric writes into

\[
g = d\phi^2 + \sin^2 \phi_1 d\phi_2^2 + \sin^2 \phi_1 \sin^2 \phi_2 d\phi_3^2 + \cdots + \sin^2 \phi_1 \sin^2 \phi_2 \cdots \sin^2 \phi_{n-1} d\phi_n^2.
\]

Thus, using the expression in (3.1), we obtain

\[
\int_{S^n} |\nabla \varphi_{p,R}|^n d\nu 
\leq 4^n \int_0^\pi \frac{t^{2n} \sin^{2n-1} \phi_1}{(t^2(1 + \cos \phi_1) + 1 - \cos \phi_1)^{2n}} d\phi_1 
\cdot \int_0^\pi \cdots \int_0^\pi \sin^{n-2} \phi_2 \sin^{n-3} \phi_3 \cdots \sin \phi_{n-1} d\phi_2 d\phi_3 \cdots d\phi_n.
\]

As the second integral on the right-hand side is a constant only depending on $n$, we just need to estimate the first one. Note that $t^2(1 + \cos \phi_1) + 1 - \cos \phi_1 \geq 2t \sin \phi_1$ by Cauchy-Schwartz inequality. It follows that

\[
\int_0^\pi \frac{t^{2n} \sin^{2n-1} \phi_1}{(t^2(1 + \cos \phi_1) + 1 - \cos \phi_1)^{2n}} d\phi_1 \leq 4^{1-n} \int_0^\pi \frac{t^2 \sin \phi_1}{(t^2(1 + \cos \phi_1) + 1 - \cos \phi_1)^2} d\phi_1 
= 4^{1-n} \int_{-1}^1 \frac{t^2}{(t^2(1 + x_1) + 1 - x_1)^2} dx_1 
= 2^{1-2n}.
\]

Therefore, there exists some constant $C(n)$ only depending on $n$, such that

\[
\int_{S^n} |\nabla \varphi_j|^n d\nu \leq C(n).
\]

\[
\square
\]

Similarly, for a given point $p \in S^n$ and $r \in (0, \pi)$, we choose $\tau = \tau(r) \in \mathbb{R}^+$ such that $\theta_{p,\tau}$ maps $B(p, \frac{r}{2})$ onto $B(p, \frac{\pi}{2})$. And we define

\[
\bar{\varphi}_{p,r}(x) = \begin{cases} 
0, & \text{if } x \in B(p, \frac{r}{2}), \\
-x_p \circ \theta_{p,\tau}(x), & \text{if } x \notin B(p, \frac{r}{2}).
\end{cases}
\]

$\bar{\varphi}_{p,r}(x)$ has similar properties as in Lemma 3.1.
Lemma 3.2. \( \bar{\varphi}_{ p, r } \) is a Lipschitz function on \( S^n \) satisfying

- \( 0 \leq \bar{\varphi}_{ p, r } (x) \leq 1 \) for any \( x \in S^n \);
- \( \bar{\varphi}_{ p, r } (x) \geq \frac{3}{5} \) for any \( x \notin B(p, r) \);
- \( \text{supp} \ \bar{\varphi}_{ p, r } \subset S^n \setminus B(p, \frac{r}{2}) \).

\[
\int_{S^n} |\nabla \bar{\varphi}_{ p, r }|^n g d\nu \leq C(n),
\]

where \( C(n) \) is a constant only depending on \( n \).

Generally, for a given annulus \( A(p; r, R) \subset S^n \), we construct the test function as

\( \varphi_{ p, r, R } = \varphi_{ p, R } \cdot \bar{\varphi}_{ p, r } \).

Combining Lemma 3.1 and Lemma 3.2, we have

Lemma 3.3. \( \varphi_{ p, r, R } \) is a Lipschitz function on \( S^n \) satisfying

- \( 0 \leq \varphi_{ p, r, R } (x) \leq 1 \) for any \( x \in S^n \);
- \( \varphi_{ p, r, R } (x) \geq \frac{9}{25} \) for any \( x \in A(p; r, R) \);
- \( \text{supp} \ \varphi_{ p, r, R } \subset A(p; \frac{r}{2}, 2R) \).

\[
\int_{S^n} |\nabla \varphi_{ p, r, R }|^n g d\nu \leq C(n),
\]

where \( C(n) \) is a constant only depending on \( n \).

4. Proof of Theorem 1.1

Proof of Theorem 1.1. Recall the collection of annuli \( \{ A_j = A(p_j; r_j, R_j) \}_{j=1}^k \) satisfying properties (i), (ii) and (2.3) in Section 2. For each annulus \( A_j \) with \( 1 \leq j \leq k \), set \( \varphi_j := \varphi_{ p_j; r_j, R_j } \). We use \( \nabla, \tilde{\nabla} \) to denote the Levi-Civita connection for the metric \( g \) and \( \tilde{g} \) respectively. For any Lipschitz function \( f \) on \( S^n \), let \( R(f) \) be the Rayleigh quotient:

\[
R(f) = \frac{\int_{S^n} |\nabla f|^2 \tilde{g} + c_n R_\tilde{g} |f|^2 d\tilde{\nu}}{\int_{S^n} |f|^2 d\tilde{\nu}},
\]

where \( c_n = \frac{n-2}{4(n-1)} \) is the constant appearing in the conformal laplacian.

To prove Theorem 1.1, it is sufficient to show that for \( 1 \leq j \leq k \),

\[
R (\varphi_j / \mu) \leq C(n) k^{2/n} \left( \int_{S^n} \mu^{\frac{2}{n-2}} dV_g \right)^{-1}.
\]

(4.1)

We will estimate the numerator and denominator in the Rayleigh quotient respectively.
We begin with the denominator. Since \( \varphi_j \geq \frac{9}{25} \) on \( A_j \) and by (2.2), we have
\[
\int_{S^n} |\varphi_j/\mu|^2 \, d\tilde{\nu} \geq \left( \frac{3}{5} \right)^4 \int_{A_j} \mu^{-2} d\tilde{\nu} \geq c \left( \frac{3}{5} \right)^4 \frac{m(S^n)}{k},
\]
where \( c \) is a constant only depends on \( n \).

Now we consider the denominator. We can actually change the metric \( \tilde{g} \) to the round metric \( g \) by the following lemma.

**Lemma 4.1.** For any Lipschitz function \( f \) on \( S^n \), we have
\[
\int_{S^n} |\nabla f|^2_{\tilde{g}} + c_n R_{\tilde{g}} |f|^2 \, d\tilde{\nu} = \int_{S^n} |\nabla (\mu f)|^2_g + c_n R_g |\mu f|^2 \, d\nu
\]
(4.3)

**Proof.** It is sufficient to consider \( f \in C^\infty(M) \). The conformal law for the conformal laplacian gives for metrics related by \( \tilde{g} = \mu^{4/(n-2)} g \)
\[
(-\Delta_g + c_n R_g)(\mu f) = \mu^{(n+2)/(n-2)} (-\Delta_{\tilde{g}} + c_n R_{\tilde{g}})(f)
\]
(4.4)

And the volume measures are related by
\[
d\tilde{\nu} = \mu^{2n/(n-2)} \, d\nu.
\]

Therefore,
\[
(-\Delta_g + c_n R_g)(\mu f) \cdot (\mu f) \, d\nu = \mu^{(n+2)/(n-2)} (-\Delta_{\tilde{g}} + c_n R_{\tilde{g}})(f) \cdot (\mu f) \mu^{-2n/(n-2)} d\tilde{\nu}
\]
\[
= (-\Delta_{\tilde{g}} + c_n R_{\tilde{g}})(f) \cdot f \, d\tilde{\nu}.
\]

The result follows immediately by integration by parts. \( \square \)

Applying (4.3) to the numerator in \( R(\varphi_j/\mu) \), we obtain
\[
\int_{S^n} |\nabla (\varphi_j/\mu)|^2_g + c_n R_g |\varphi_j/\mu|^2 \, d\tilde{\nu} = \int_{S^n} |\nabla (\varphi_j)|^2_g + c_n R_g |\varphi_j|^2 \, d\nu
\]
(4.5)

By using the Hölder inequality, as \( \text{supp} \varphi_j \subset 2A_j \),
\[
\int_{S^n} |\nabla \varphi_j|^2_g \, d\nu \leq \left( \int_{S^n} |\nabla \varphi_j|^n \, d\nu \right)^{2/n} \cdot \nu(2A_j)^{1-2/n}.
\]

Recalling (2.3) and (3.2), it follows that
\[
\int_{S^n} |\nabla \varphi_j|^2_g \, d\nu \leq C(n)^{2/n} \cdot \left( \frac{\nu(S^n)}{k} \right)^{1-2/n}.
\]

What’s more,
\[
\int_{S^n} c_n R_g |\varphi_j|^2 \, d\nu \leq n(n-1)c_n \cdot \nu(2A_j) \leq n(n-1)c_n \frac{\nu(S^n)}{k}.
\]
Plugging the above two inequalities into (4.5) and enlarging the constant \( C(n) \) if necessary,

\[
\int_{S^n} |\tilde{\nabla}(\varphi_j/\mu)|^2_g + c_n R_g|\varphi_j/\mu|^2 d\tilde{V} \leq \frac{C(n)}{k^{1-2/n}}.
\]

(4.6)

Combining (4.2) and (4.6), it follows that for any \( 1 \leq j \leq k \),

\[
\mathcal{R}\left( \frac{\varphi_j}{\mu} \right) \leq \frac{C(n)}{c} \left( \frac{5}{3} \right)^4 k^{2/n} (m(S^n))^{-1}.
\]

The result therefore follows by renaming the constant \( \frac{C(n)}{c} \left( \frac{5}{3} \right)^4 \) to \( C(n) \). □

5. Generalizations and Related Questions

5.1. Generalizations to any closed manifolds. Let \((M^n, g)\) be a closed Riemannian manifold of dimension \( m \geq 3 \). Then we can isometrically embed \((M, g)\) into \((S^N, g_0)\) for some \( N \in \mathbb{N} \) depending on \( n \) with the round metric \( g_0 \). Let \( \phi : (M, g) \to (S^N, g_0) \) be such an embedding satisfying

\[
\phi^*g_0 = g.
\]

Let \( \tilde{g} \in [g] \) be a conformal metric, which is related to \( g \) as

\[
\tilde{g} = \mu^{4/(n-2)} g,
\]

for some \( \mu \in C^\infty(M) \).

Recall the \( N \)-conformal volume \( V_c(N, \phi) \) of \( \phi \) (5.1), defined as

\[
V_c(N, \phi) = \sup \{ Vol(M, (s \circ \phi)^* g_0) : s \text{ is a conformal diffeomorphism of } S^N \}.
\]

(5.1)

For closed Riemannian manifolds, we can bound the eigenvalues of the conformal laplacians by the \( N \)-conformal volume \( V_c(N, \phi) \).

**Theorem 5.1.** For any metric \( \tilde{g} \in [g] \), the \( k \)th eigenvalue \( \tilde{\lambda}_k = \lambda_k(M^n, \tilde{g}) \) of the conformal laplacian \( \Box \tilde{g} \) satisfies the inequality

\[
\tilde{\lambda}_k \int_M \mu^{4-2} dV g \leq C(n) \left( V_c(N, \phi) k^{2/n} + \sup_M R_g \right).
\]

(5.2)

**Proof.** Let \( m = \phi_* \left( \mu^{4/(n-2)} dV g \right) \) be the push-forward measure on \( S^N \). For the metric space \((S^N, d_{g_0})\) and measure \( m \) and any \( k \in \mathbb{Z}^+ \), we can similarly construct the collection of annuli \( \{A_j\}_{j=1}^k \) satisfying properties (i), (ii) and (2.3) in Section 2. And for each \( A_j \), we can construct test functions \( \varphi_j \) satisfying the first three properties as in Lemma 3.3.
We will consider the Rayleigh quotient \( R(\varphi_j \circ \phi) \). Using (2.2), the denominator gives
\[
\int_M \left| \frac{\varphi_j \circ \phi}{\mu} \right|^2 dV_\tilde{g} \geq \left( \frac{3}{5} \right)^4 \int_{\phi^{-1}(A_j)} \mu^{-2} dV_\tilde{g} \geq c \left( \frac{3}{5} \right)^4 \frac{m(S^N)}{k}. \tag{5.3}
\]
Let \( \nabla \) and \( \tilde{\nabla} \) be the Levi-Civita connection for \( g \) and \( \tilde{g} \) respectively. For the numerator of the Rayleigh quotient,
\[
\int_M \left| \tilde{\nabla} \left( \frac{\varphi_j \circ \phi}{\mu} \right) \right|^2 dV_\tilde{g} + c_n R_\tilde{g} \left| \frac{\varphi_j \circ \phi}{\mu} \right|^2 dV_\tilde{g} = \int_M \left| \nabla (\varphi_j \circ \phi) \right|^2 g + c_n R_g |\varphi_j \circ \phi|^2 dV_g. \tag{5.4}
\]
The first term on the right side can be estimated as
\[
\int_M \left| \nabla (\varphi_j \circ \phi) \right|^2 g dV_g \leq \left( \int_M \left| \nabla (\varphi_j \circ \phi) \right|^n g dV_g \right)^{2/n} \cdot \left( \text{Vol}_g(2A_j) \right)^{1-2/n}.
\]
Recall that \( \varphi_j = x_{p_j} \circ \theta_{p_j} \) for some conformal diffeomorphism \( \theta_{p_j} \) of \( S^N \). Since \( \theta_{p_j} \circ \phi \) is conformal, we have
\[
(\theta_{p_j} \circ \phi)^* g_0 = \frac{1}{n} \left| \nabla (\theta_{p_j} \circ \phi) \right|^2 g.
\]
Therefore,
\[
\int_M \left| \nabla (\varphi_j \circ \phi) \right|^n g dV_g \leq \int_M \left| \nabla (\theta_{p_j} \circ \phi) \right|^n g dV_g
\]
\[
= \frac{n}{2} \text{Vol}(M, (\theta_{p_j} \circ \phi)^* g_0)
\]
\[
\leq \frac{n}{2} \text{Vol}_c(N, \phi).
\]
And by (2.3),
\[
\text{Vol}_g(2A_j) \leq \frac{\text{Vol}_g(M)}{k} \leq \frac{\text{Vol}_c(N, \phi)}{k}.
\]
Therefore,
\[
\int_M \left| \nabla (\varphi_j \circ \phi) \right|^2 g dV_g \leq \frac{n \text{Vol}_c(N, \phi)}{k^{1-2/n}}.
\]
On the other hand, the second term on the right side of (5.4) satisfies
\[
\int_M c_n R_g |\varphi_j \circ \phi|^2 dV_g \leq c_n \sup_M R_g \cdot \text{Vol}_g(2A_j) \leq c_n \sup_M R_g \cdot \frac{\text{Vol}_c(N, \phi)}{k}.
\]
Combining the above two estimates, we obtain
\[
\int_M \left| \tilde{\nabla} \left( \frac{\varphi_j \circ \phi}{\mu} \right) \right|^2 \tilde{g} + c_n R_\tilde{g} \left| \frac{\varphi_j \circ \phi}{\mu} \right|^2 dV_\tilde{g} \leq \left( n + c_n \frac{\sup_M R_g}{k^{2/m}} \right) \frac{\text{Vol}_c(n, \phi)}{k^{1-2/m}}. \tag{5.5}
\]
Using (5.5) and (5.3), the desired estimates for the Rayleigh quotient \( R(\varphi \cdot \phi) \) is obtained and the result follows immediately. \( \square \)

**Remark 5.2.** Given an immersion \( \phi : M \to S^N \), Li and Yau \([7]\) gives the following condition for the conformal volume \( V_c(N, \phi) \) to be identical to the volume of \( M \).

**Theorem 5.3** ([7]). Let \( M \) be a homogeneous Riemannian manifold of dimension \( n \). Suppose \( \phi : M \to S^N \) is an immersion of \( M \) into \( S^N \) which satisfies the properties:

(i) \( \phi \) is an isometric minimal immersion.

(ii) The transitive subgroup \( H \) of the isometry group of \( M \) is induced by a subgroup, also denoted by \( H \), of \( O(N + 1) \) (i.e., \( \phi \) is equivariant).

(iii) \( \phi(M) \) does not lie on any hyperplane of \( \mathbb{R}^{N+1} \).

Then

\[
V_c(N, \phi) = \text{Vol}(M).
\]

In particular, when \( M \) is an irreducible homogeneous manifold, a theorem of Takahashi (see \([6]\)) says that one can minimally immerse \( M \) isometrically into \( S^N \subseteq \mathbb{R}^{N+1} \) by its first eigenspace of \( M \). If we denote this isometric immersion \( M \to S^N \) by \( \phi \), then \( V_c(N, \phi) = \text{Vol}(M) \) and (5.2) writes into

\[
\bar{\lambda}_k \int_M \mu^{n-2} dV_g \leq C(n) \left( \text{Vol}(M, g) k^{2/n} + \sup_M R_g \right).
\]

5.2. **Hersch Type Results.** Let \((M, g)\) be a closed Riemannian manifold. For any metric \( \tilde{g} \in [g] \), related by \( \tilde{g} = \mu^{4/(n-2)} g \), we define the functional

\[
\bar{\lambda}_k(M, g, \tilde{g}) = \left( \lambda_k(\tilde{g}) \cdot \int_M \mu^{n-2} dV_g \right).
\]

And we also define the supreme over the conformal class as

\[
\Lambda_k(M, g) = \sup_{\tilde{g} \in [g]} \bar{\lambda}_k(M, g, \tilde{g}).
\]

It is natural to ask what \( \Lambda_k(M, g) \) is and whether \( \Lambda_k(M, g) \) is achieved by certain Riemannian metric in general. If the maximal metric exist, it is defined up to multiplication by a positive constant due to the rescaling invariance of the functional. We can prove the following results in this direction.

**Theorem 5.4.** For the sphere with the standard round metric \((S^n, g_0)\),

\[
\Lambda_0(S^n, g_0) = \tilde{\lambda}_0(S^n, g_0, g_0).
\]
And $\Lambda_0(S^n, g_0) = \bar{\lambda}_0(S^n, g_0, \tilde{g})$ holds only when $\tilde{g}$ is the round metric up to scaling.

Proof. For any $\tilde{g} \in [g_0]$ related as $\tilde{g} = \mu^{4/(n-2)} g_0$, take $1/\mu$ as the test function for the Rayleigh quotient and we obtain

$$\lambda_0(S^n, \tilde{g}) \leq \frac{\int_{S^n} \left| \nabla_{\tilde{g}} \left( \frac{1}{\mu} \right) \right|^2 + c_n R_{\tilde{g}} \left| \frac{1}{\mu} \right|^2 dV_{\tilde{g}}}{\int_{S^n} \left| \frac{1}{\mu} \right|^2 dV_{\tilde{g}}}.$$  \hspace{1cm} (5.8)

By using (4.3), the above inequality writes into

$$\lambda_0(S^n, \tilde{g}) \leq \frac{\int_{S^n} c_n R_{g_0} dV_{g_0}}{\int_{S^n} \mu^{4/(n-2)} dV_{g_0}} = \frac{\bar{\lambda}_0(S^n, g_0, g_0)}{\int_{S^n} \mu^{4/(n-2)} dV_{g_0}}.$$  

If (5.8) holds with equality, then $1/\mu$ is an eigenfunction with eigenvalue $\lambda_0(S^n, \tilde{g})$. By (4.4), we have

$$\lambda_0(S^n, g_0) = (-\Delta_{g_0} + c_n R_{g_0})1 = \mu^{(n-2)/(n-2)} (-\Delta_{\tilde{g}} + c_n R_{\tilde{g}})(1/\mu) = \lambda_0(S^n, \tilde{g}) \mu^{4/(n-2)}.$$  

Therefore, $\mu$ is a constant function and the result follows. \hfill $\square$

References

[1] Bernd Ammann and Pierre Jammes. The supremum of first eigenvalues of conformally covariant operators in a conformal class. In Variational problems in differential geometry, volume 394 of London Math. Soc. Lecture Note Ser., pages 1–23. Cambridge Univ. Press, Cambridge, 2012.

[2] A. Grigor’yan and S.-T. Yau. Decomposition of a metric space by capacitors. In Differential equations: La Pietra 1996 (Florence), volume 65 of Proc. Sympos. Pure Math., pages 39–75. Amer. Math. Soc., Providence, RI, 1999.

[3] G. Kokarev. Conformal volume and eigenvalue problems. ArXiv e-prints, December 2017.

[4] G. Kokarev. Bounds for Laplace eigenvalues of Kaehler metrics. ArXiv e-prints, January 2018.

[5] Nicholas Korevaar. Upper bounds for eigenvalues of conformal metrics. J. Differential Geom., 37(1):73–93, 1993.

[6] H. Blaine Lawson, Jr. Lectures on minimal submanifolds. Vol. I, volume 9 of Mathematics Lecture Series. Publish or Perish, Inc., Wilmington, Del., second edition, 1980.

[7] Peter Li and Shing Tung Yau. A new conformal invariant and its applications to the Willmore conjecture and the first eigenvalue of compact surfaces. Invent. Math., 69(2):269–291, 1982.

[8] N. Nadirashvili. Berger’s isoperimetric problem and minimal immersions of surfaces. Geom. Funct. Anal., 6(5):877–897, 1996.

[9] Nikolai Nadirashvili and Yannick Sire. Conformal spectrum and harmonic maps. Mosc. Math. J., 15(1):123–140, 182, 2015.

[10] Nikolai Nadirashvili and Yannick Sire. Maximization of higher order eigenvalues and applications. Mosc. Math. J., 15(4):767–775, 2015.
[11] Paul C. Yang and Shing Tung Yau. Eigenvalues of the Laplacian of compact Riemann surfaces and minimal submanifolds. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 7(1):55–63, 1980.

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