Time Scales in Probabilistic Models of Wireless Sensor Networks

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Abstract

We consider a stochastic model of clock synchronization in a wireless network consisting of \( N \) sensors interacting with one dedicated accurate time server. For large \( N \) we find an estimate of the final time synchronization error for global and relative synchronization. Main results concern a behavior of the network on different time scales \( t_N \to \infty, N \to \infty \). We discuss existence of phase transitions and find exact time scales on which an effective clock synchronization of the system takes place.

Keywords: Clock synchronization, time to synchronization, timeliness in real-time systems, wireless sensor networks, multi-dimensional Markov process, self-organization, phase transitions

1 Introduction

For many years distributed systems are a constant source of challenging research problems. Clock synchronization is one of the most well-known and widely discussed subjects. In distributed systems there is no global clock. Clocks of different components tick at different rates. But to achieve an efficient parallelization in their common job all components of the distributed system need a common notion of time. This is very important for real-time systems where scheduling and

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timeliness play a crucial role (see, for example, [7, 32]). Real-time applications must guarantee response within strict time constraints. Similarly, Wireless Sensor Networks (WSNs), having their own specific character among distributed systems, usually meet many real-time requirements. In particular, energy saving algorithms in WSNs would be impossible without a global consistent timescale [27]. Nowadays complex real-time embedded systems are distributed and have many active components. Modern WSNs also consist of very large number of sensors ([8, 24]). Hence an important requirement for clock synchronization in distributed systems is the scalability of corresponding algorithms.

In this paper we consider a mathematical model of a large WSN communicating with one dedicated server of accurate time. The model is based on a multi-dimensional stochastic process. The network consists of $N$ sensors equipped with non-perfect clocks. By using message timestamps sensors share information about their local times and receive accurate time from the dedicated server. It is assumed that all messages are sent at random time moments. Main focus of the paper is the asymptotical behavior of the network in the limit when the number $N$ of sensors tends to infinity.

This article is organized as follows. The model is defined in Section 2. It depends on a few parameters. The network is assumed to be symmetric. Here we do not touch many problems (synchronization protocols, energy saving optimization etc.) which are very important for practical use of WSNs. Such questions were widely discussed by many authors [30, 9, 27, 8, 25]. The present article is a mathematical paper. We intentionally make our model as transparent as possible to keep this paper short and to illuminate main results about existence of different phases in the evolution of the network (Theorems 1-3 in Section 3). It appears that the proposed model has almost explicit solution: in all asymptotical expressions we can play with parameters and discover many interesting phenomena related with large networks. In Section 6 we discuss possible generalization of the present results to more general models. We do not seek for direct practical implementation of our results. But we believe that such results are interesting not only from theoretical viewpoint.

Our probabilistic technique is similar to that was recently used to study mathematical models of multi-processor parallel computing [21, 11, 14, 13] and collective behavior in abstract particle systems with synchronization-like interaction [12, 10, 15, 11, 16, 18, 19]. Our stochastic models can be considered also as special classes of self-organizing systems [26, 5, 23, 29].
2 Mathematical model

There is a wireless network with \( N + 1 \) nodes. Each node \( i \) has its own clock. Let \( x_i(t) \) be the value of clock \( i \) (the local time at node \( i \)). The physical time is denoted by \( t \in \mathbb{R}_+ \). Nodes 2, 3, \ldots, \( N + 1 \) correspond to sensors. We assume that clocks of the sensors are identical but not perfect and progress in the following way:

\[
x_j(t) = x_j(0) + vt + \sigma B_j(t), \quad j = 2, \ldots, N + 1.
\]

Node 1 is a time server, we assume that its clock is perfect. Clock 1 reports a local time \( x_1(t) \) which is a linear function of the physical time:

\[
x_1(t) = x_1(0) + rt.
\]

The constants \( r \) and \( v \) are positive, they are called frequencies of the corresponding clocks (see, for example, [30]). In general, these frequencies are not equal: \( r \neq v \).

The parameter \( \sigma > 0 \) in (1) corresponds to the strength of a random noise related with imperfect clock of a sensor. For simplicity the random noise in (1) is modelled with independent standard Brownian motions

\[
(B_j(t), \ t \geq 0), \quad j = 2, \ldots, N + 1.
\]

This white noise assumption is usual for many oscillator clock models [32, 24].

As it was mentioned in Section 1 (see also [28]) the problem of clock synchronization is critical for the proper work of WSNs. To synchronize local times of a pair of sensors the most natural solution is to send time-stamped messages between them. After reading a new coming message the receiver adjusts its clock to the local time of the sender recorded in this message. Once being synchronized the clocks of the pair will diverge immediately due to assumptions made in (1) and (2).

We continue with the formal definition of the model. We shall call (1–2) a free dynamics. The free dynamics generates independent evolutions at nodes of the network. Now we add some special interaction between nodes. Namely, at random time moments each node sends messages to other nodes. Our assumptions are

- with the rate \( \alpha > 0 \) the server node 1 generates a message containing information about the current value of \( x_1 \) and sends this message to one of the sensors which is chosen randomly with probability \( \frac{1}{N} \);

- independently each sensor, with the rate \( \beta > 0 \), generates a message about its local time and sends it to another sensor which is chosen randomly with probability \( \frac{1}{N - 1} \);
messages reach their destinations instantly (there are no transmission delays);

- if sensor $j$ receives a message from sensor $i$ at some (random) time $\tau$ then
clock $j$ is immediately adjusted to the value of clock $i$: $x_j(\tau + 0) = x_i(\tau)$;

- between receiving of subsequent messages sensor nodes evolve according to
the free dynamics \[1\];

- server node 1 always follows the free dynamics \[2\].

Let us remind some terminology used above: a sequence of events $0 = \tau_0 < \cdots < \tau_n < \cdots$ generated with rate $\delta > 0$ is called also a Poisson flow of intensity $\delta$. It means that intervals between events $\tau_{n+1} - \tau_n$ are independent exponentially distributed random variables with mean $\gamma^{-1}$:

$$P(\tau_{n+1} - \tau_n > s) = e^{-\delta s}, \quad E(\tau_{n+1} - \tau_n) = \delta^{-1}. \quad (3)$$

Hence we defined the multi-dimensional stochastic process

$$x(t) = (x_1(t), x_2(t), \ldots, x_{N+1}(t)), \quad t \geq 0,$$

which appears to be a continuous time Markov process with values in $\mathbb{R}^{N+1}$. This process is an idealized mathematical model of the homogeneous wireless sensor network with dedicated accurate time server.

It should be noted that non-Markovian models of WSNs can be also considered in the framework of the present paper. We postpone discussion on possible generalizations to Subsection 4.2.

To estimate desynchronization in the network it is convenient to consider the following functions on the configuration space $\mathbb{R}^{N+1}$:

$$R(x) := \frac{1}{N} \sum_{j=2}^{N+1} (x_j - x_1)^2, \quad (4)$$

$$D(x) := \frac{1}{(N-1)N} \sum_{2 \leq j_1 < j_2} (x_{j_2} - x_{j_1})^2. \quad (5)$$

The first function corresponds to deviations of sensor’s clocks from the accurate time $x_1$. The second function is related only to internal inconsistency between the sensor’s local times $x_2, \ldots, x_{N+1}$. This function $D(x)$ may be more useful for causality-based real-time models where the right ordering of events is more significant than the global synchronization to physical time (see, for example, \[31\]). Since the both functions $R(x)$ and $D(x)$ are averagings of $(x_k - x_j)^2$, the squares of offsets between clocks, the true sense of a time synchronization error have their
square roots $\sqrt{R(x)}$ and $\sqrt{D(x)}$. The physical dimension of $\sqrt{R(x)}$ and $\sqrt{D(x)}$ is time, i.e., they are measured in seconds.

Note that for any $t$ values $R(x(t))$ and $D(x(t))$ are random. It is more convenient to deal with their expectations:

$$R_N(t) := \mathbb{E} R(x(t)) \quad \text{and} \quad D_N(t) := \mathbb{E} D(x(t)).$$

3 Main results

Let $x(t)$ be the Markov process introduced in Section 2. In Subsections 3.1–3.3 we present various results on asymptotical behavior of the functions $R_N(t)$ and $D_N(t)$. These results will provide us with detailed information on collective behavior of the network in the limit when $N \to \infty$. Here we give one particular corollary of our theorems which is a good illustration to an approach followed in this paper.

Corollary. Assume that $v \neq r$. For simplicity take $x_1(0) = \cdots = x_N(0) = 0$ as initial state of the network. Let $s > 0$ and $\gamma > 0$ be parameters of a new time scale. Namely, we put $t = s N^\gamma$ and look after the function $D_N(t)$ when $N \to \infty$. It appears that

$$D_N(s N^\gamma) \sim C(s, \gamma) N^{\phi(\gamma)}$$

where

$$\phi(\gamma) = \begin{cases} \gamma, & \gamma \leq \frac{1}{2}, \\ 3\gamma - 1, & \frac{1}{2} < \gamma \leq 1, \\ 2, & \gamma > 1. \end{cases}$$

The function $C(s, \gamma) = C(s, \gamma; \sigma, \alpha, \beta) > 0$ depends on parameters $\sigma, \alpha, \beta$ and is increasing in $s$ for fixed $\gamma$.

The choice $t = s N^\gamma$ means that we consider a new time unit $N^\gamma$ and the “size” of configuration $x_2, \ldots, x_N$ on this scale is of order $N^{\phi(\gamma)/2}$. The conclusion is that on different time scales the large network shows very different types of behavior. Note that the function $\phi(\gamma)$ is not smooth. This situation is very like the phase transitions phenomena in models of statistical physics. We give more details on time scales in Subsection 4.6.

To understand in what sense the algorithm of Section 2 drives the network to synchronization one should answer several questions.
3.1 \textit{N is fixed, }t \to \infty

The first question is a long-time behavior of the stochastic process

\[ x(t) = (x_1(t), \ldots, x_{N+1}) . \]

Since \( r > 0 \) and \( v > 0 \) it is quite clear from definition of the model that \( x_j(t) \to \infty \) \((t \to \infty)\) for each node \( j \). To have more detailed information on the process we put a moving observer at the point \( x_1(t) \). From the viewpoint of this observer states of the sensor nodes \( 2, \ldots, N+1 \) are given by the vector

\[ y(t) = (y_2(t), \ldots, y_{N+1}(t)) , \quad y_j = x_j - x_1. \]  \hfill (6)

It follows from the general theory of Markov processes (see Doeblin condition in \cite{3}) that \( y(t) \) has a limiting distribution as \( t \to \infty \). Similar situations were discussed in \cite{15, 18} for different synchronization models. Hence our first conclusion is: for fixed parameters \( \sigma, \alpha > 0, \beta > 0 \) and \( N \) the Markov process \( y(t) \) is ergodic and converges to its equilibrium as \( t \to \infty \). This limiting distribution (which is a probability measure on \( \mathbb{R}^N \)) can not be obtained in an explicit form. Nevertheless, some important functionals of the limiting distribution can be found explicitly. Note that \( R(x) = R(y) \) and \( D(x) = D(y) \) where \( x \) and \( y \) are related by \( (6) \). Thus the clocks \( x_i(t) \) of sensor nodes are synchronized in the following sense: mean values of time synchronization errors stabilize and do not change in time anymore. Namely, assuming that \( N \) is fixed and \( t \to \infty \) we can prove the following statement.

\textbf{Theorem 1.} We distinguish two cases: the skew \( v - r \) of a sensor’s clock relative to the accurate time is \textbf{zero} or \textbf{nonzero}.

\textbf{Case 1:} Assume that \( v = r \). Then

\[ R_N(t) \to R_N^{C_1}(\infty) := \sigma^2 / \alpha N , \]
\[ D_N(t) \to D_N^{C_1}(\infty) \sim 2 \sigma^2 / (\alpha + \beta) N . \]

\textbf{Case 2:} Assume that \( v \neq r \). Then

\[ R_N(t) \to R_N^{C_2}(\infty) := (v - r)^2 / \alpha^2 N^2 . \]
\[ D_N(t) \to D_N^{C_2}(\infty) \sim (v - r)^2 / (\alpha (\alpha + \beta)) N^2 . \]
As usual we write here and below  
\[ f_N \sim g_N \iff \lim_{N \to \infty} \frac{f_N}{g_N} = 1. \]

Now we come to a natural and very important question: how many time we should wait until the networks of “size” \( N \) will be synchronized? We devote to this question the next Subsections 3.2–3.4.

Comparing Case 1 and Case 2 in the above theorem we see that result of the global synchronization of sensor’s clocks is worse in the biased case \( v \neq r \). Indeed, condition \( v \neq r \) means that there is a systematic error in client’s clocks which appears to be more essential (a resulting “clock offset” is of order \( \sqrt{R_N^{C1}(\infty)} = O(N) \) ) than in situation of a pure random noise errors under condition \( v = r \) (when the “clock offset” is of order \( \sqrt{R_N^{C2}(\infty)} = O(N^{1/2}) \) ).

Note also that in Case 2 the limiting values \( R_N^{C2}(\infty) \) and \( D_N^{C2}(\infty) \) do not depend on \( \sigma \), the noise parameter of the sensor’s clocks. Moreover, these values give the right asymptotics even for \( \sigma = 0 \). This means that the so called locked synchronization [28] is not possible in our model due to the stochastic nature of the message-passing algorithm.

3.2 \( N \to \infty, t \to \infty \). Phases of synchronization

Synchronization in networks with large (or growing) number of nodes is of special interest. Our approach is to consider limits when both the number of client nodes \( N \) and the time \( t \) grow to infinity. More precisely, we shall consider sequences \((N, t_N)\), where physical time \( t = t_N \) is some increasing function of \( N \). We shall see below that for different choices of \( t_N \) network exhibits different asymptotical behavior. One can say that a large network passes different phases on its road to synchronization. For our model there exist several time scales \( t = t_N \) of qualitatively different behavior. We always assume that \( N \to \infty \).

To begin we assume in the present subsection that initial distribution of the local clocks is such that sequences \( \{R_N(0)\} \) and \( \{D_N(0)\} \) are bounded in \( N \):

\[
\sup_N R_N(0) < +\infty, \quad \sup_N D_N(0) < +\infty. \tag{7}
\]

In Subsec. 3.3 we shall discuss the situation when \( \{R_N(0)\} \) and \( \{D_N(0)\} \) are unbounded. As in Subsect. 3.1 we present results separately for Case 1 and Case 2.

Theorem 2. Let assumption (7) hold.

Case 1: \( v = r \) — zero skew.

P1. \( t_N/N \to 0 \) (phase of initial desynchronization):

\[
\left( \begin{array}{c} R_N(t_N) \\ D_N(t_N) \end{array} \right) \sim \left( \begin{array}{c} \sigma^2 \\ 2\sigma^2 \end{array} \right) t_N
\]
P2. \( t_N/N \to c > 0 \) (phase of effective synchronization):

\[
\begin{pmatrix}
R_N(t_N) \\
D_N(t_N)
\end{pmatrix} \sim (-cM)^{-1} (Id - e^{cM}) \begin{pmatrix}
\sigma^2 \\
2\sigma^2
\end{pmatrix} t_N,
\]

where \( Id \) is the identity map in \( \mathbb{R}^2 \),

\[
M := \begin{pmatrix}
-\alpha & 0 \\
2\alpha & -2(\alpha + \beta)
\end{pmatrix}.
\] (8)

P3. \( t_N/N \to +\infty \) (phase of final stabilization):

\[
\begin{pmatrix}
R_N(t_N) \\
D_N(t_N)
\end{pmatrix} \sim \begin{pmatrix}
\sigma^2/\alpha \\
2\sigma^2/(\alpha + \beta)
\end{pmatrix} N \sim \begin{pmatrix}
R_N^{C1}(\infty) \\
D_N^{C1}(\infty)
\end{pmatrix}.
\]

Remark 1. By using definition (8) of the matrix \( M \) it is easy to check that \( M \) has two distinct negative eigenvalues: \( \lambda_1 = -\alpha, \lambda_2 = -2(\alpha + \beta) \). Moreover, \((-cM)^{-1} (Id - e^{cM}) \to Id \) as \( c \to +0 \), \( e^{cM} \to 0 \) as \( c \to +\infty \) and

\[
(-M)^{-1} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix}
1/\alpha \\
2/(\alpha + \beta)
\end{pmatrix}.
\]

From this remark we see that phase P2 joins smoothly asymptotics of phases P1 and P3.

Remark 2. One can easily calculate \( R_N(t) \) and \( D_N(t) \) in the simplest case \( v = r \), \( \alpha = \beta = 0 \) (no bias, no synchronizing interaction). It appears that \( R_N(t) = \sigma^2 t \) and \( D_N(t) = 2\sigma^2 t \). This observation brings us to the following explanation of the phase 1: on time intervals \([0, o(N)]\) the influence of synchronizing jumps (clock adjustments) is negligible with respect to the impact of the random noise of the free dynamics. On the time scale of phase P2 \((t_N = cN)\) the cumulative effect of individual synchronizing adjustments become of the same order as the effect of random noise, so we obtain the non-trivial dependence of \( R_N(t_N) \) and \( D_N(t_N) \) on \( c \). On phase P3 the values of \( R_N(t_N) \) and \( D_N(t_N) \) correspond to the synchronized network. It means that the effective synchronization takes place on the times of order \( N \).

Theorem 3. Assume that condition (7) holds.

Case 2: \( v \neq r \) — nonzero skew.

P1. \( t_N/N \to 0 \) (phase of initial desynchronization):

\[
R_N(t_N) \sim \frac{1}{2} (v - r)^2 t_N^2.
\]
P1a. If \( \frac{t_N}{\sqrt{N}} \to 0 \), then \( D_N(t_N) \sim 2\sigma^2 t_N \).

P1b. If \( \frac{t_N}{\sqrt{N}} \to c_1, \quad c_1 > 0 \), then
\[
D_N(t_N) \sim \left( 2\sigma^2 + \frac{1}{3} \alpha(v-r)^2 c_1^2 \right) t_N.
\]

P1c. If \( \frac{t_N}{\sqrt{N}} \to \infty \) but \( \frac{t_N}{N} \to 0 \), then
\[
D_N(t_N) \sim \frac{1}{3} \alpha(v-r)^2 t_N^3 / N.
\]

P2. \( t_N/N \to c > 0 \) (phase of effective synchronization): there exist functions \( h_R \) and \( h_D \) not depending on \( N \) such that
\[
\left( \begin{array}{c} R_N(t_N) \\ D_N(t_N) \end{array} \right) \sim \left( \begin{array}{c} h_R(c) \\ h_D(c) \end{array} \right) (v-r)^2 t_N^2.
\]

P3. \( t_N/N \to +\infty \) (phase of final stabilization):
\[
\left( \begin{array}{c} R_N(t_N) \\ D_N(t_N) \end{array} \right) \sim \left( \begin{array}{c} (v-r)^2 \\ \alpha^2 \\ (v-r)^2 \\ \alpha (\alpha + \beta) \end{array} \right) N^2 \sim \left( \begin{array}{c} R_N^{C_2}(\infty) \\ D_N^{C_2}(\infty) \end{array} \right).
\]

Remark 3. Explicit form of the function \( h_R \) and \( h_D \) is given in (36), see Section 4.5.

It appears that
\[
\left( \begin{array}{c} h_R(c) \\ h_D(c) \end{array} \right) \sim \left( \begin{array}{c} 1/2 \\ \alpha c / 3 \end{array} \right) \quad (c \to +0),
\]
\[
\left( \begin{array}{c} h_R(c) c^2 \\ h_D(c) c^2 \end{array} \right) \to \left( \begin{array}{c} \alpha^{-2} \\ \alpha^{-1}(\alpha + \beta)^{-1} \end{array} \right) \quad (c \to +\infty).
\]

Hence the phase P2 “continuously” joins asymptotics of phases P1c and P3.

### 3.3 Situation of a “big initial disorder”

As in preceding subsection we consider here different time scales \( t = t_N \) assuming that \( t_N \to \infty \) as \( N \to \infty \). Our goal is to discuss what happens with synchronization phases if the sequences \( R_N(0) \) and \( D_N(0) \) grow to infinity as \( N \to \infty \). This problem is not too complicated and can be solved explicitly for any concrete assumption about initial disorder, i.e., about increasing rate of \( R_N(0) \) and \( D_N(0) \).
In this paper we shall not give universal and general answer to this question since it cannot be presented in short and transparent expressions. To avoid cumbersome formulae we just describe the answer in a few general words. There exist a so called initial disorder decay interval \((0, t^\circ_N)\). The function \(t^\circ_N\) depends on \(R_N(0), D_N(0)\) and all parameters of the model. Under above assumptions \(t^\circ_N \to \infty\) as \(N \to \infty\). If \(t_N = o(t^\circ_N)\), then the values \(R_N(0)\) and \(D_N(0)\) enter in asymptotics of \(R_N(t_N)\) and \(D_N(t_N)\), but they don’t enter in these asymptotics for \(t_N\) such that \(t_N/t^\circ_N \to \infty\).

Given a function \(t^\circ_N\), one should compare orders of \(t^\circ_N, N^{1/2}\) and \(N\) to see intersections of the initial disorder decay phase (IDDP) with the phases P1, P2 and P3 of Subsection 3.2. The IDDP dominates over any of the phases P1 (P1a–P1c), P2 and P3. So depending on the initial disorder in the network one can see the following different sequences of consecutive phases in evolution of the network: IDDP–P1–P2–P3, IDDP–P1b–P1c–P2–P3, IDDP–P1c–P2–P3, IDDP–P2–P3 or IDDP–P3.

3.4 Collective displacement from the etalon time

In Subsections 3.1, 3.3 we studied \(R(x)\) and \(D(x)\) which are quadratic functions of the clock configuration \(x \in \mathbb{R}^{N+1}\). For completeness we consider here a linear function \(d(x)\) which gives some information on a collective displacement of the sensors clocks \(x_2, \ldots, x_N\) from the clock of the server \(x_1\). Define functions

\[
d(x) = \frac{1}{N} \sum_{j=2}^{N+1} x_j - x_1, \quad d : \mathbb{R}^{N+1} \to \mathbb{R}^1,\tag{9}
\]

and

\[
d_N(t) = E d(x(t)), \quad d_N : \mathbb{R}_+ \to \mathbb{R}.
\]

**Theorem 4.** Assume that \(\sup_N d_N(0) < +\infty\).

**Case 1:** \(v = r\) — zero skew. It appears that \(d_N(t) \to 0\) as \(t \to \infty\) for any fixed \(N\).

**Case 2:** \(v \neq r\) — nonzero skew. For fixed \(N\)

\[
d_N(t) \to (v - r) \alpha^{-1} N \quad \text{as} \quad t \to \infty.
\]

For time scales \(t_N \to \infty\) \((N \to \infty)\) the following statements hold:

- if \(t_N/N \to 0\), then \(d_N(t_N) \sim (v - r)t_N\),
• if \( t_N/N \to c > 0 \), then

\[
d_N(t_N) \sim (1 - \exp(-\alpha c)) (v - r) \alpha^{-1} N \sim \frac{1 - \exp(-\alpha c)}{\alpha c} \cdot (v - r) t_N ,
\]

• if \( t_N/N \to +\infty \), then \( d_N(t_N) \sim (v - r) \alpha^{-1} N \).

The proof of this theorem is similar to proofs of Theorems 1–3 (see Section 4) but technically it is much easier. So we omit it.

**Remark 4.** Let us consider the degenerated model with \( \alpha = 0 \). It is easy to prove that

\[
d_N(t_N) - d_N(0) = (v - r) t_N .
\]

Comparing this with Theorem 4, we see that the formal limit \( \alpha \to 0 \) does not turn results of Theorem 4 into (10). Hence we conclude that limits \( N \to \infty \) and \( \alpha \to 0 \) do not commute.

### 4 Proofs

#### 4.1 Conditional averaging

Let \( 0 = \tau_0 < \ldots < \tau_n < \ldots \) be sequence of time moments when messages are sent (received). It follows from definition of the model (Section 2) that \( \{\tau_m - \tau_{m-1}\}_{m=0}^{\infty} \) are independent i.d. r.v. having exponential distribution with mean \((\alpha + N \beta)^{-1}\). In the sequel we will refer to this condition on \( \{\tau_n\} \) as to a **Markovian assumption**.

Let \( \Pi_t = \max \{m : \tau_m \leq t\} \). To get \( R_N(t) \) and \( D_N(t) \) we will calculate the chain of conditional expectations as follows

\[
E(\cdot) = E \left( E \left( E \left( \cdot \mid \{\tau_j\}_{j=1}^{\infty} \right) \mid \Pi_t \right) \right)
\]

Let \( f = f(x) \) be some function on the configuration space \( \mathbb{R}^{N+1} \). Introduce notation

\[
f^{(n)} = E \left( f(x(\tau_n)) \mid \{\tau_j\}_{j=1}^{\infty} \right) , \quad n = 1, 2, \ldots
\]

Hence \( f^{(n)} \) is a random variable functionally depending on the sequence \( \{\tau_j\}_{j=1}^{\infty} \).

In other words, to any function \( f = f(x) \) we put in correspondence \( f \mapsto \{f^{(n)}\} \) the sequence of random variables \( \{f^{(n)}\} \).
Denote
\[ V(x) = \begin{pmatrix} R(x) \\ D(x) \end{pmatrix}, \quad V : \mathbb{R}^{N+1} \to \mathbb{R}^2, \]
where functions \( R(x) \) and \( D(x) \) are defined in (4)–(5). Recall definition of the function \( d(x) \) from (9). In the next subsection we study random sequences \( \{V(n)\} \) and \( \{d(n)\} \) corresponding to the above functions \( V(x) \) and \( d(x) \) according to (11).

4.2 Recurrent equations

In this subsection we obtain difference equations for \( V(n) = \begin{pmatrix} R(n) \\ D(n) \end{pmatrix} \) and \( d(n) \).

To present this result we need some convenient notation which will be used till the end of the paper. Namely, we introduce the following reals
\[
\begin{align*}
  b &= v - r, \quad \alpha_N = \frac{\alpha}{N}, \quad \beta_N = \frac{\beta}{N-1}, \\
  u &= 2b, \quad \delta_N = \alpha + N\beta, \quad k_N = 1 - \delta_N^{-1}\alpha_N,
\end{align*}
\]
and two-dimensional vectors
\[
q_1 = \sigma^2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad q_2 = (v - r)^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad q_0 = u \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]

We consider also a 2 × 2-matrix
\[
L_N = \begin{pmatrix} -\alpha_N & 0 \\ 2\alpha_N & -2(\alpha_N + \beta_N) \end{pmatrix}
\]
and define a linear operator \( K_N = \text{Id} + \delta_N^{-1}L_N \), where \( \text{Id} \) is the identity map in \( \mathbb{R}^2 \).

Lemma 1. The following recurrent equations for \( V(n) \) and \( d(n) \) hold
\[
\begin{align*}
d(n) &= k_N \left( d(n-1) + b\Delta_n \right) \\
V(n) &= K_N \left( V(n-1) + \Delta_n^2 q_2 + \Delta_n q_1 + \Delta_n d(n-1) q_0 \right)
\end{align*}
\]
where \( \Delta_n = \tau_n - \tau_{n-1} \).

Proof of Lemma 1. The dynamics of the process \( x(t) \) consists of two parts: free motion and pairwise interaction between nodes of the network. Namely, the interaction is possible only at random time moments \( 0 < \tau_1 < \tau_2 < \cdots \) and has the form of synchronizing jumps: at time \( \tau_n \) a pair of nodes
\[
(i, j), \quad i = 1, 2, \ldots, N, \quad j = 2, \ldots, N, \quad i \neq j,
\]
is randomly chosen, and the value of clock $j$ jumps to the value of clock $i$:

$$(x_i, x_j) \rightarrow (x_i, x_i).$$

(14)

Inside intervals $(\tau_k, \tau_{k+1})$ components of the process $x(t)$ move according to the free evolutions (1) and (2) driven by independent Brownian motions. Recall that the pair $(i, j)$ mentioned above corresponds to a time-stamped message sent by clock $i$ to clock $j$ at time $\tau_n$. It is easy to see that any pair $(1, j)$ is chosen with probability $\alpha N \delta^{-1} N$ and any pair $(i, j), i > 1$, is chosen with probability $\beta N \delta^{-1} N$.

Keeping in mind (14) we introduce a family of maps $S(i,j)$:

$$S(i,j): \mathbb{R}^{N+1} \rightarrow \mathbb{R}^{N+1},$$

where $x'_j = x_i, \quad x'_k = x_k, \quad k \neq j,$ and define a map-valued random variable $S$ such that

$$P\{S = S(i,j)\} = \begin{cases} \alpha N \delta^{-1} N, & i = 1, \\ \beta N \delta^{-1} N, & i > 1. \end{cases}$$

(15)

So Lemma 1 will immediately follow from the next two lemmas.

**Lemma 2.** For any $x \in \mathbb{R}^{N+1}$

$$E V(Sx) = K_N V(x), \quad E d(Sx) = k_N d(x).$$

Denote an auxiliary stochastic process $z = z(t)$ evolving according to the free dynamics (1)–(2).

**Lemma 3.** For $s < t$

$$E (d(z(t)) \mid z(s)) = d(z(s)) + b\Delta$$

$$E (V(z(t)) \mid z(s)) = V(z(s)) + \Delta^2 \cdot q_2 + \Delta \cdot q_1 + \Delta \cdot d(z(s)) q_0$$

where $\Delta = t - s$.

The proofs of Lemmas 2 and 3 are quite straightforward and very similar to proofs of corresponding lemmas in [16, 18, 19]. They are omitted here.

It should be noted that, in fact, Lemmas 2 and 3 are valid under weaker conditions than the Markovian assumption. Namely, these lemma are true also for the following semi-Markovian assumption: messages are sent at epochs $\{\tau_n\}_{n=0}^\infty$ where intervals $\{\tau_m - \tau_{m-1}\}_{m=0}^\infty$ are independent random variables with identical continuous distribution; at time moment $\tau_n$ the pair $(i, j)=$(sender,receiver) is chosen independently according to the map-valued random variable $S$ defined in (15).

Briefly speaking, under Markovian assumption the sequence $\{\tau_n\}_{n=0}^\infty$ is a Poisson flow but under semi-Markovian assumption $\{\tau_n\}_{n=0}^\infty$ is a general renewal process [2, 4].
4.3 Decomposition into sum over diagrams

Having equations of Lemma 1 we can use them to express $V^{(n)}$ via $V^{(n-2)}$ and so on. Proceeding recursively we get a sum with large number of summands. We want to organize these summands in some proper way to be able to evaluate $V^{(n)}$ as an explicit function of $V^{(0)}$. To do this we use a general approach known as decomposition into sum over diagrams. We start from description of a set of diagrams corresponding to our specific task.

**Admissible diagrams.** Fix some $n$. Let us define a set of admissible diagrams $\mathcal{G}(n)$ of order $n$. We say that an oriented graph $G$ belongs to the set $\mathcal{G}(n)$ iff $G$ is a path

$$G = (v_0, v_1, \ldots, v_r), \quad 1 \leq r \leq n + 1,$$

with vertices $v_0, v_1, \ldots, v_r \in \mathcal{M}_n$ and oriented edges $v_{i-1} \to v_i, i = 1, \ldots, r$, satisfying to the following conditions.

a) Vertices of $G$ are labeled by pairs $(c, d)$ belonging to

$$\mathcal{M}_n = \left\{ v = (c, d) : (c, d) \in S_C^{(n)} \times S_D \cup \{("t", 2)\} \right\}$$

where $c \in S_C^{(n)} := \{0, 1, \ldots, n - 1, n\}$ and $d \in S_D := \{0, 1, 2\}$.

b) The initial vertex $v_0$ of any path $G$ is ("t", 2). For $i \geq 1$ the vertex $v_i$ has the form $(n + 1 - i, l_i), l_i \in S_D$. The final vertex may be

$$v_r = \begin{cases} 
(n + 1 - r, 0), & \text{if } r \leq n, \\
(0, d), & \text{if } r = n + 1 \text{ where } d \in S_D.
\end{cases}$$

c) The first edge $v_0 \to v_1$ of $G$ has the form ("t", 2) → $(n, l'), l' \in S_D$, and other edges $(v_i \to v_{i+1}), 1 \leq i \leq r - 1$, have the form $(n + 1 - i, l_i) \to (n - i, l_{i+1})$ where $l_{i+1} \leq l_i$. There are no edges of the form $(m, 0) \to (m - 1, 0)$.

For the path (16) we use notation

$$\mathcal{V}_G = \{v_0, v_1, \ldots, v_r\}, \quad \mathcal{E}_G = \{v_{i-1} \to v_i \mid i = 1, \ldots, r\}$$

for the set of vertices and the set of edges of $G$. For reasons which will be clear below, sometimes we will remap elements of the set $S_D$ as follows

$$0 \leftrightarrow "1", \ 1 \leftrightarrow "d", \ 2 \leftrightarrow "V".$$

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A. Manita

**Time scales in WSNs**

**Contribution of a path.** Let \( \vec{\tau} = (\tau_1, \ldots, \tau_n) \in \mathbb{R}_+^n \) and \( t > 0 \) are such that \( 0 < \tau_1 < \cdots < \tau_n < t \). We define \( J_G = J_G(\vec{\tau}, t) \in \mathbb{R}^2 \), a *contribution* of the path \( G \) (see (16)), as follows

\[
J_G(\vec{\tau}, t) = \prod_{a \in \mathcal{V}_G} \rho_a \prod_{b \in \mathcal{E}_G} \pi_b(b) \] ordered along the path,

where functions \( \rho_a = \rho_a(\vec{\tau}, t) \) and \( \pi_b = \pi_b(\vec{\tau}, t) \) are contributions of vertices and edges defined below. Namely, we put

\[
\rho_{v_0} = 1, \\
\rho_v = \begin{cases} 
V(0), & \text{if } v = (0, 2), \\
d(0), & \text{if } v = (0, 1), \\
1, & \text{if } v = (0, 0), 
\end{cases} \\
\rho_v = \begin{cases} 
K_N, & \text{if } v = (m, 2), \\
k_N, & \text{if } v = (m, 1), \\
1, & \text{if } v = (m, 0), 
\end{cases} \\
m = 1, \ldots, n.
\]

and

\[
\pi_b = f_{l,l'}(\tau_m - \tau_{m-1}), \quad m = 1, \ldots n + 1, \quad \tau_{n+1} \equiv t
\]

for the edge \( b = (m, l) \to (m - 1, l') \). The functions \( f_{l,l'} \) are defined as follows:

\[
\begin{align*}
        f_{22}(s) &= 1d, \\
    f_{20}(s) &= s^2q_2 + sq_1, \\
    f_{21}(s) &= sq_0, \\
    f_{11}(s) &= 1, \\
    f_{10}(s) &= sb.
\end{align*}
\]

**Proposition 1.** The following expansion holds

\[
E \left( V(x(t) \mid \{\tau_j\}_{j=1}^\infty \right) = \sum_{G \in \mathcal{G}(n)} J_G,
\]

where \( \mathcal{G}(n) \) is the set of admissible diagrams of order \( n \) (defined above) and \( J_G \in \mathbb{R}^2 \) is the contribution of a diagram \( G \).

The proof of Proposition 1 is just a careful development of the recurrent equations of Lemma 1. The only thing one should pay attention is the last interval \( [\tau_{n+1}, t] \) of the total time segment \( [0, t] \). There is no synchronization jump at time \( t \). Hence we need to apply for this interval only Lemma 3. This explains assignment \( \rho_{v_0} = 1 \) for the root vertex \( v_0 = (\cdot \cdot \cdot \cdot^2, 2) \).
4.4 Evaluation of the functions

Below we use agreement [17]. For any path $G \in \mathcal{G}(n)$ denote by $n_1 = n_1(G)$ the number of vertices of the form $(m, "d")$, $m \leq n$, and by $n_2 = n_2(G)$ the number of vertices of the form $(m, "V")$, $m \leq n$. It is clear that $n_1, n_2 \geq 0$ and $n_1 + n_2 \leq n + 1$. Now we decompose the set of admissible diagrams $\mathcal{G}(n)$ into the following nonintersecting subsets

$$
\mathcal{G}(n) = \mathcal{G}_0(n) \cup \mathcal{G}_2(n) \cup \mathcal{G}_{10}(n) \cup \mathcal{G}_{11}(n),
$$

$$
\mathcal{G}_0(n) = \{ G \in \mathcal{G}(n) : n_1 = 0, n_2 = n + 1 \} = \{ G_0 \}
$$

$$
\mathcal{G}_2(n) = \{ G \in \mathcal{G}(n) : n_1 = 0, 0 \leq n_2 \leq n \}
$$

$$
\mathcal{G}_{10}(n) = \{ G \in \mathcal{G}(n) : 1 \leq n_1 \leq n + 1, n_2 = n + 1 - n_1 \}
$$

$$
\mathcal{G}_{11}(n) = \{ G \in \mathcal{G}(n) : 1 \leq n_1 \leq n, n_2 \geq 0, n_1 + n_2 < n + 1 \}
$$

The subset $\mathcal{G}_0(n)$ consists of a single path

$$
G_0 = (("e", "V"), (n, "V"), (n - 1, "V"), \ldots, (0, "V"))
$$

and contribution of this path is

$$
J_{G_0}(\vec{\tau}, t) = \prod_{b \in \mathcal{E}_G} \pi_b \prod_{a \in \mathcal{V}_G} \rho_a \bigg|_{G = G_0} = K_N^n V^{(0)}.
$$

For $G \in \mathcal{G}_2(n)$ the final vertex $v_r$ is $(n - n_2, "1")$ and

$$
J_G(\vec{\tau}, t) = K_N^n \Delta_{n+1-n_2} q_2 + K_N^{n_2} \Delta_{n+1-n_2} q_1,
$$

where $\Delta_k = \tau_k - \tau_{k-1}$. If $G \in \mathcal{G}_{10}(n)$ then $v_r = (0, "d")$, $r = n + 1$, and

$$
J_G(\vec{\tau}, t) = K_N^{n+1-n_1} K_N^{n_1-1} \Delta_n d^{(0)} q_0.
$$

For $G \in \mathcal{G}_{11}(n)$ the final vertex $v_r$ is $(n - n_1 - n_2, "1")$ and

$$
J_G(\vec{\tau}, t) = K_N^{n_1} K_N^{n_1} \Delta_{n+1-n_2} \Delta_{n_1} b q_0.
$$

So we have

$$
E \left( V(x(t) \mid \{ \tau_j \}_{j=1}^{\infty} \right) = \left( \sum_{G \in \mathcal{G}_0(\Pi_1)} + \sum_{G \in \mathcal{G}_2(\Pi_1)} + \sum_{G \in \mathcal{G}_{10}(\Pi_1)} + \sum_{G \in \mathcal{G}_{11}(\Pi_1)} \right) J_G,
$$

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taking conditional expectation $E (\cdot | \Pi_t)$ of the both sides of this decomposition we obtain

$$
\mathbb{E} (V(x(t) | \Pi_t = n) = \left( \sum_{G \in \mathcal{G}_0(n)} + \sum_{G \in \mathcal{G}_2(m)} + \sum_{G \in \mathcal{G}_{10}(n)} + \sum_{G \in \mathcal{G}_{11}(n)} \right) J_G =
$$

$$
= K_N^n V^{(0)} + \sum_{n_2=0}^{n} \left( K_N^n s_{n+1-n_2}^{(2)} q_2 + K_N^n s_{n+1-n_2}^{(1)} q_1 \right) +
$$

$$
+ \left( \sum_{n_2=0}^{n} K_N^n k_N^{n-n_2} s_{n+1-n_2}^{(1)} \right) d^{(0)} q_0 +
$$

$$
+ \left( \sum_{n_2=0}^{n-1} \sum_{n_1=1}^{n-n_2} K_N^n k_N^{n_1} s_{n+1-n_2, n+1-n_1-n_2}^{(1,1)} \right) b q_0
$$

where

$$
s_k^{(m)} = s_k^{(m)}(t,n) = \mathbb{E} (\Delta_k^m | \Pi_t = n), \quad m = 1, 2,
$$

$$
s_{i,j}^{(1,1)} = s_{i,j}^{(1,1)}(t,n) = \mathbb{E} (\Delta_i \Delta_j | \Pi_t = n), \quad i \neq j,
$$

$$
\Delta_j = \tau_j - \tau_{j-1}.
$$

It is easy to check that in general semi-Markov case (Subsection 4.2)

$$
s_{k_1}^{(m)}(t,n) = s_{k_2}^{(m)}(t,n), \quad 1 \leq k_1, k_2 \leq n,
$$

$$
s_{i,j}^{(1,1)}(t,n) = s_{i_2,j_2}^{(1,1)}(t,n), \quad i_1, j_1, i_2, j_2 \in \{1, \ldots, n\}, \quad i_1 \neq j_1, i_2 \neq j_2.
$$

Under Markovian assumption we have much stronger result:

$$
s_k^{(1)}(t,n) = \frac{t}{n+1}, \quad 1 \leq k \leq n+1,
$$

$$
s_k^{(2)}(t,n) = \frac{2t^2}{(n+1)(n+2)}, \quad 1 \leq k \leq n+1,
$$

$$
s_{i,j}^{(1,1)}(t,n) = \frac{t^2}{(n+1)(n+2)}, \quad i, j \in \{1, \ldots, n+1\}, \quad i \neq j.
$$

Note that $2q_2 = b q_0$ and

$$
\mathcal{G}_2(n) \cup \mathcal{G}_{11}(n) = \{ G \in \mathcal{G}(n) : n_1 \geq 0, n_2 \geq 0, n_1 + n_2 \leq n \}.
$$

Hence we just proved the following statement.
Lemma 4. In the Markovian case

\[
E(V(x(t) \mid \Pi_t = n)) = K^n_N V^{(0)} + \frac{t}{n+1} \left( \sum_{n_2=0}^{n} K^n_{n_2} \right) q_1 + \frac{t}{n+1} \left( \sum_{n_2=0}^{n} K^n_{n_2} k_{n-n_2} \right) d^{(0)} q_0 + \frac{t^2}{(n+1)(n+2)} \left( \sum_{n_1+n_2 \leq n} K^n_{n_2} k_{n_1} k_{n-n_1} \right) b q_0. \tag{18}
\]

Now we are going to evaluate \(E V(x(t))\) by averaging on \(\Pi_t\):

\[
E V(x(t)) = E(E(V(x(t) \mid \Pi_t)) = \sum_{n=0}^{\infty} \frac{(\delta_N t)^n}{n!} e^{-\delta_N t} E(V(x(t) \mid \Pi_t = n)). \tag{21}
\]

To do this we need two technical lemmas.

Lemma 5. For any finite-dimensional matrix \(A\) the following identities hold

\[
E A^{\Pi_t} = e^{-\delta_N t(Id-A)} \tag{22}
\]

\[
E \frac{t A^{\Pi_{t+1}}}{\Pi_{t} + 1} = \frac{1}{\delta_N} \left( e^{-\delta_N t(Id-A)} - e^{-\delta_N t} Id \right) \tag{23}
\]

\[
E \frac{t^2 A^{\Pi_{t+2}}}{(\Pi_{t} + 1)(\Pi_{t} + 2)} = \frac{1}{\delta_N^2} \left( e^{-\delta_N t(Id-A)} - \frac{Id + \delta_N t A}{e^{\delta_N t}} \right) \tag{24}
\]

where \((\Pi_t, t \geq 0)\) is the Poissonian process of intensity \(\delta_N\).

Proof of Lemma 5 is very easy. We omit it.

Lemma 6. Let \(a_1, a_2 \in (0, 1)\). Consider

\[
U_1(a_1, a_2) := E \frac{t}{\Pi_{t} + 1} \sum_{n_2=0}^{\Pi_t} a_1^{n_2} a_2^{\Pi_t-n_2},
\]

\[
U_2(a_1, a_2) := E \frac{t^2}{(\Pi_{t} + 1)(\Pi_{t} + 2)} \sum_{n_1+n_2 \leq \Pi_t} a_1^{n_2} a_2^{n_1},
\]

where \((\Pi_t, t \geq 0)\) is a Poissonian process of intensity \(\delta_N > 0\). Then for \(a_1 \neq a_2\)

\[
U_1(a_1, a_2) = \delta_N^{-1} (a_1 - a_2)^{-1} \left( e^{-(1-a_1)\delta_N t} - e^{-(1-a_2)\delta_N t} \right)
\]

\[
\delta_N^2 \cdot U_2(a_1, a_2) = \frac{1}{(1-a_1)(1-a_2)} - \frac{1 - a_1}{a_1 - a_2}
\]

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and for $a_1 = a_2 = a$

$$U_1(a,a) = te^{-(1-a)\delta_N t},$$

$$\delta_N^2 \cdot U_2(a,a) = \frac{1}{(1-a)^2} - \left( \frac{1}{(1-a)^2} + \frac{\delta_N t}{1-a} \right) e^{-(1-a)\delta_N t}.$$

Proof of Lemma 6 will be given in Section 5.

Now we proceed with evaluation of (21). Putting $A = K_N$ in (22) and applying averaging (21) to the first summand in (18) we get

$$E(K_N \Pi_t V(0)) = e^{-\delta_N t(Id - K_N)} E V(0) = e^{tL_N} E V(0),$$

since $K_N = Id + \delta_N^{-1} L_N$ (see notation (12)–(13) at the beginning of Subsection 4.2).

Similarly, using (23) for $A = K_N$ and $A = Id$, we obtain

$$E \frac{t}{\Pi_t + 1} \left( \sum_{n=0}^{\Pi_t} K_N^n q_1 \right) q_1 = E \frac{t}{\Pi_t + 1} (Id - K_N)^{-1} (Id - K_N^{\Pi_t+1}) q_1$$

$$= (-L_N)^{-1} (Id - e^{L_N t}) q_1.$$

To find expectation of summands (19) and (20) we shall analyze spectrum of the operator $L_N : \mathbb{R}^2 \to \mathbb{R}^2$ and then apply Lemma 6. It is easy to check that operator $L_N$ has two different eigenvalues $\lambda_{1,N}$ and $\lambda_{2,N}$, corresponding to eigenvectors $e_{1,N}$ and $e_{2,N}$,

$$\lambda_{1,N} = -\alpha_N, \quad \lambda_{2,N} = -2(\alpha_N + \beta_N),$$

$$e_{1,N} = \begin{pmatrix} 1 \\ 2\alpha_N \\ \alpha_N + 2\beta_N \end{pmatrix} = \begin{pmatrix} 1 \\ 2\lambda_{1,N} \\ \lambda_{2,N} - \lambda_{1,N} \end{pmatrix}, \quad e_{2,N} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

where $\alpha_N$ and $\beta_N$ are the same as in (12). Hence actions of the operators $k_N Id$ and $K_N$ of the vector $e_{1,N}$ are identical,

$$k_N Id e_{1,N} = K_N e_{1,N} = a e_{1,N}, \quad a = 1 - \delta_N^{-1} \alpha_N = 1 + \delta_N^{-1} \lambda_{1,N},$$

but their actions on the vector $e_{2,N}$ are different:

$$k_N Id e_{2,N} = a_1 e_{2,N}, \quad a_1 = 1 - \delta_N^{-1} \alpha_N = 1 + \delta_N^{-1} \lambda_{1,N},$$

$$K_N e_{2,N} = a_2 e_{2,N}, \quad a_2 = 1 - 2\delta_N^{-1} (\alpha_N + \beta_N) = 1 + \delta_N^{-1} \lambda_{2,N}, \quad a_1 \neq a_2.$$
In the basis $e_{1,N}$, $e_{2,N}$ we have \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} = e_{1,N} + w_{2,N} e_{2,N} \), where
\[
  w_{2,N} = -\frac{2\alpha_N}{\alpha_N + 2\beta_N} = -\frac{2\lambda_{1,N}}{\lambda_{2,N} - \lambda_{1,N}},
\]
Substituting this decomposition into (19)–(20), we apply Lemma 6 to calculate expectations of (19) and (20) separately on each linear subspace \( \langle e_{1,N} \rangle \) and \( \langle e_{2,N} \rangle \). Finally, noting that \( (1 - a_i) \delta_N = -\lambda_{i,N} \) we come to the following statement.

**Proposition 2.** The functions \( R_N(t) \) and \( D_N(t) \) can be given in the following explicit form:
\[
\begin{pmatrix} R_N(t) \\ D_N(t) \end{pmatrix} = e^{L_N t} \begin{pmatrix} R_N(0) \\ D_N(0) \end{pmatrix} + (-L_N)^{-1} (Id - e^{L_N t}) \begin{pmatrix} \sigma^2 \\ 2\sigma^2 \end{pmatrix} +
+ (v - r) d_N(0) \left( t e^{\lambda_{1,N} t} e_{1,N} + \frac{e^{\lambda_{2,N} t} - e^{\lambda_{1,N} t}}{\lambda_{2,N} - \lambda_{1,N}} w_{2,N} e_{2,N} \right) +
+ \left( \frac{1}{\lambda_{1,N}} \right) \left( \frac{1}{\lambda_{2,N} - \lambda_{1,N}} \right) w_{2,N} (v - r)^2 e_{2,N},
\]
where \( d_N(t) := \mathbb{E} d(x(t)) \) with \( d(\cdot) \) defined in (9).

**Remark 5.** Note that many parameters in the above formula depend on \( N \). But in the limit \( N \to \infty \) these dependencies become rather simple:
\[
L_N \sim \frac{1}{N} M = \frac{1}{N} \begin{pmatrix} -\alpha & 0 \\ 2\alpha & -(\alpha + \beta) \end{pmatrix}, \quad e_{1,N} \sim \begin{pmatrix} 1 \\ \frac{2\alpha}{\alpha + 2\beta} \end{pmatrix}, \quad e_{2,N} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]
\[
\lambda_{1,N} \sim \frac{-\alpha}{N}, \quad \lambda_{2,N} \sim -\frac{2(\alpha + \beta)}{N}, \quad w_{2,N} \sim -\frac{2\alpha}{\alpha + 2\beta}.
\]

**Remark 6.** Proposition 2 can be rewritten in the following way:
\[
\begin{pmatrix} R_N(t) \\ D_N(t) \end{pmatrix} = e^{L_N t} \begin{pmatrix} R_N(0) \\ D_N(0) \end{pmatrix} + (-L_N)^{-1} (Id - e^{L_N t}) \begin{pmatrix} \sigma^2 \\ 2\sigma^2 \end{pmatrix} +
+ (v - r) t d_N(0) \left( g_1 (\lambda_{1,N} t) e_{1,N} + \frac{g_1 (\lambda_{2,N} t) - g_1 (\lambda_{1,N} t)}{\lambda_{2,N} - \lambda_{1,N}} t w_{2,N} e_{2,N} \right) +
+ (v - r)^2 t^2 \left( g_2 (\lambda_{1,N} t) e_{1,N} + \frac{g_2 (\lambda_{2,N} t) - g_2 (\lambda_{1,N} t)}{\lambda_{2,N} - \lambda_{1,N}} t w_{2,N} e_{2,N} \right),
\]
where \( g_1(y) = e^y \), \( g_2(y) = \frac{e^y - 1}{y} \). Using (27) we easily transform this representation as follows

\[
\left( \begin{array}{c} R_N(t) \\ D_N(t) \end{array} \right) = e^{L_N t} \left( \begin{array}{c} R_N(0) \\ D_N(0) \end{array} \right) + \left(-L_N\right)^{-1} \left( \text{Id} - e^{L_N t} \right) \left( \begin{array}{c} \sigma^2 \\ 2\sigma^2 \end{array} \right) + \\
+ (v - r) t d_N(0) \left( g_1'(\lambda_{1,N} t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + F_N(g_1, t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) + \\
+ (v - r)^2 t^2 \left( g_2'(\lambda_{1,N} t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + F_N(g_2, t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right),
\]

where

\[
F_N(g, t) = \left( \frac{g(\lambda_{2,N} t) - g(\lambda_{1,N} t)}{(\lambda_{2,N} - \lambda_{1,N}) t} - g'(\lambda_{1,N} t) \right) w_{2,N}.
\]

For the function \( d_N(t) \) we also have explicit formula.

**Proposition 3.** The function \( d_N(t) \) has the following form

\[
d_N(t) = d_N(0) \exp \left(-\frac{\alpha}{N} t\right) + \left(1 - \exp \left(-\frac{\alpha}{N} t\right)\right) \frac{(v - r)N}{\alpha} \tag{28}
\]

and hence is a solution to the following equation

\[
\frac{d}{dt} d_N(t) = -\frac{\alpha}{N} d_N(t) + (v - r) \tag{29}
\]

Proof of this proposition can be obtained by the same method as the proof of Proposition 2 but the corresponding reasonings are much more shorter and simpler. So we omit the proof of Proposition 3 and simply refer to [13] where similar statement was presented in full details.

The following statement follows from Proposition 2 by direct calculation.

**Corollary 1.** \( (R_N(t), D_N(t)) \) is a solution of the following system of differential equations:

\[
\frac{d}{dt} \left( \begin{array}{c} R_N(t) \\ D_N(t) \end{array} \right) = \left( \begin{array}{cc} -\alpha/N & 0 \\ 2\alpha/N & -2(\alpha/N + \beta/(N - 1)) \end{array} \right) \left( \begin{array}{c} R_N(t) \\ D_N(t) \end{array} \right) + \\
+ \left( \begin{array}{c} (v - r)d_N(t) \\ 0 \end{array} \right) + \left( \begin{array}{c} \sigma^2 \\ 2\sigma^2 \end{array} \right)
\]

where the function \( d_N(t) \) is the same as in (28) and (29).
4.5 Study of the asymptotical behavior

In this subsection we obtain all results on asymptotic behavior which were presented in Section 3.

Proof of Theorem 1

Here $N$ is fixed and $t$ tends to infinity. Since matrix $L_N$ has two different negative eigenvalues one easily concludes that in explicit representation of Proposition 2 all terms containing $e^{L_N t}$ or $e^{\lambda_{N} t}$ go to zero as $t \to \infty$. So the limit is equal to

$$
\begin{pmatrix}
R_N(\infty) \\
D_N(\infty)
\end{pmatrix} = (-L_N)^{-1} \left( \frac{\sigma^2}{2\alpha^2} + (v-r)^2 \left( \frac{1}{\lambda_{1,N}^2} e_{1,N} + \frac{1}{\lambda_{1,N} \lambda_{2,N}} w_{2,N} e_{2,N} \right) \right).
$$

Matrix $(-L_N)^{-1}$ can be calculated explicitly:

$$
(-L_N)^{-1} = \begin{pmatrix}
\alpha_N^{-1} & 0 \\
(\alpha_N + \beta_N)^{-1} & \frac{1}{\alpha} (\alpha_N + \beta_N)^{-1}
\end{pmatrix}.
$$

Using (26) and (27) after some algebraic transformation we get

$$
R_N(t) \to R_N(\infty) = \sigma^2 \alpha_N^{-1} + \frac{(v-r)^2}{\lambda_{1,N}^2} = \frac{\sigma^2}{\alpha} N + \frac{(v-r)^2}{\alpha^2} N^2,
$$

$$
D_N(t) \to D_N(\infty) = \frac{2 \sigma^2}{\alpha_N + \beta_N} + \frac{2 (v-r)^2}{\lambda_{1,N} \lambda_{2,N}} \sim \frac{2 \sigma^2}{\alpha + \beta} N + \frac{(v-r)^2}{\alpha \cdot (\alpha + \beta)} N^2.
$$

This proves Theorem 1.

Proofs of Theorems 2 and 3

At the beginning we study the first time scale, namely, we assume that $t_N \to \infty$ but $t_N/N \to 0$ as $N \to \infty$.

To do this we shall use the next lemma.

Lemma 7. Consider a function $g = g(y)$ which assumed to be analytical in some neighborhood of the point $y = 0$. Assume also that $g''(0) \neq 0$. Then for any sequences $\{y_N\}$ and $\{z_N\}$ tending to 0 as $N \to \infty$ we have

$$
g'(y_N) e_{1,N} + \frac{g(z_N) - g(y_N)}{z_N - y_N} w_{2,N} e_{2,N} =
$$

$$
g'(y_N) \begin{pmatrix}
1 \\
0
\end{pmatrix} + \left( \frac{1}{2} g''(0) (z_N - y_N) w_{2,N} + (z_N - y_N) O(|z_N| + |y_N|) \right) \begin{pmatrix}
0 \\
1
\end{pmatrix},
$$

where the notation $a_N = O(b_N)$ means, as usual, that $\limsup_{N} \left| \frac{a_N}{b_N} \right| < +\infty$. 

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Proof of this lemma is a straightforward calculation.
Since the both \( \lambda_{1,N} \) and \( \lambda_{2,N} \) are negative and condition (17) holds we remark that the first summand in the r.h.s. of Proposition 2 is bounded. Under assumption (30)

\[
\lambda_{1,N} t_N \to 0 \quad \text{and} \quad \lambda_{2,N} t_N \to 0.
\]

So the second summand has a rather simple behavior:

\[
(-L_N)^{-1} \left( \text{Id} - e^{L_N t_N} \right) \left( \frac{\sigma^2}{2\sigma^2} \right) \sim \left( \frac{\sigma^2}{2\sigma^2} \right) t_N + \left( O(t_N/N) \right) t_N.
\]

The next summands of the representation in Proposition 2 demands very careful analysis. We use Remark 6 and Lemma 7 for \( y_N = \lambda_{1,N} t_N \) and \( z_N = \lambda_{2,N} t_N \):

\[
g'_1(\lambda_{1,N} t) \, e_{1,N} + \frac{g_1(\lambda_{2,N} t) - g_1(\lambda_{1,N} t)}{(\lambda_{2,N} - \lambda_{1,N}) \, t} \, w_{2,N} \, e_{2,N} =
\]

\[
= e^{y_N} \left( \begin{array}{c} 1 \\ 0 \end{array} \right) + \left( \frac{1}{2} (z_N - y_N) \, w_{2,N} + (z_N - y_N) \, O(t_N/N) \right) \left( \begin{array}{c} 0 \\ 1 \end{array} \right) =
\]

\[
= (1 + O(\lambda_{1,N} t_N)) \left( \begin{array}{c} 1 \\ 0 \end{array} \right) + (\left| \lambda_{1,N} \right| \, t_N + O(\frac{t_N^2}{N^2})) \left( \begin{array}{c} 0 \\ 1 \end{array} \right) =
\]

\[
= \left( \begin{array}{c} 1 + O(t_N/N) \\ \left| \lambda_{1,N} \right| \, t_N + O(\frac{t_N^2}{N^2}) \end{array} \right)
\]

where we have use identity \((z_N - y_N) \, w_{2,N} = 2 \left| \lambda_{1,N} \right| \, t_N \) (see (25) and (27)). Similarly,

\[
g'_2(\lambda_{1,N} t) \, e_{1,N} + \frac{g_2(\lambda_{2,N} t) - g_2(\lambda_{1,N} t)}{(\lambda_{2,N} - \lambda_{1,N}) \, t} \, w_{2,N} \, e_{2,N} =
\]

\[
= \frac{1 - (1 - y_N)e^{y_N}}{y_N^2} \left( \begin{array}{c} 1 \\ 0 \end{array} \right) + \left( \frac{1}{2} \cdot \frac{1}{3} (z_N - y_N) \, w_{2,N} + (z_N - y_N) \, O(t_N/N) \right) \left( \begin{array}{c} 0 \\ 1 \end{array} \right) =
\]

\[
= \left( \frac{1}{2} + O(\lambda_{1,N} t_N) \right) \left( \begin{array}{c} 1 \\ 0 \end{array} \right) + \left( \frac{1}{3} \left| \lambda_{1,N} \right| \, t_N + O(\frac{t_N^2}{N^2}) \right) \left( \begin{array}{c} 0 \\ 1 \end{array} \right) =
\]

\[
= \left( \frac{1}{2} + O(t_N/N) \right) \left( \frac{1}{3} \left| \lambda_{1,N} \right| \, t_N + O(\frac{t_N^2}{N^2}) \right)
\]

Putting (31)–(33) into formula of Remark 6 we get

\[
R_N(t_N) = \sigma^2 t_N + O(t_N/N) + (v - r) \, d_N(0) t_N + (v - r) t_N O(t_N/N) + \frac{1}{2} (v - r)^2 t_N^2 + (v - r)^2 t_N^2 O(t_N/N)
\]
\[ D_N(t_N) = 2\sigma^2 t_N + O(t_N/N) + \\
+ (v - r) d_N(0) |\lambda_{1,N}| t_N^2 + (v - r) t_N O\left(t_N^2/N^2\right) + \\
+ \frac{1}{3} (v - r)^2 |\lambda_{1,N}| t_N^3 + (v - r)^2 t_N^2 O\left(t_N^2/N^2\right) \]

In the case \( v - r = 0 \) these formulae immediately imply item P1 of the Theorem 2.

Consider the case \( v - r \neq 0 \). Since \( t_N \to \infty \) terms containing \( (v - r)t_N \) are asymptotically smaller than corresponding terms containing \( (v - r)^2 t_N^2 \). Moreover, \( |\lambda_{1,N}| t_N^2 = t_N O(t_N/N) \). Hence

\[ R_N(t_N) = \sigma^2 t_N + O(t_N/N) + (v - r) d_N(0) t_N + \\
\frac{1}{2} (v - r)^2 t_N^2 + (v - r)^2 t_N^2 O(t_N/N) \]

\[ D_N(t_N) = 2\sigma^2 t_N + O(t_N/N) + (v - r) d_N(0) t_N O(t_N/N) + \\
\frac{1}{3} (v - r)^2 |\lambda_{1,N}| t_N^3 + (v - r)^2 t_N^2 O\left(t_N^2/N^2\right). \quad (34) \]

So our next conclusion is that under assumptions (30) and \( v - r \neq 0 \)

\[ R_N(t_N) \sim \frac{1}{2} (v - r)^2 t_N^2. \]

To analyze \( D_N(t_N) \) we should compare \( t_N \) and \( |\lambda_{1,N}| t_N^3 \). Recalling that \( |\lambda_{1,N}| = \alpha_N = \alpha/N \) we split the time scale (30) into three subscales

\[ \frac{t_N}{\sqrt{N}} \to 0, \quad \frac{t_N}{\sqrt{N}} \to c_1, \quad c_1 > 0, \quad \text{and} \quad \frac{t_N}{\sqrt{N}} \to \infty \text{ but} \quad \frac{t_N}{N} \to 0. \]

Considering (34) on each subscale we get that

- if \( \frac{t_N}{\sqrt{N}} \to 0 \), then \( D_N(t_N) \sim 2\sigma^2 t_N \),

- if \( \frac{t_N}{\sqrt{N}} \to c_1, \quad c_1 > 0 \), then

\[ D_N(t_N) \sim 2\sigma^2 t_N + \frac{1}{3} \alpha (v - r)^2 t_N^3 / N \sim \]

\[ \sim 2\sigma^2 c_1 \sqrt{N} + \frac{1}{3} \alpha (v - r)^2 c_1^3 \sqrt{N} \sim \left(2\sigma^2 + \frac{1}{3} \alpha (v - r)^2 c_1^3\right) t_N \]

- if \( \frac{t_N}{\sqrt{N}} \to \infty \) but \( \frac{t_N}{N} \to 0 \), then \( D_N(t_N) \sim \frac{1}{3} \alpha (v - r)^2 t_N^3 / N \).
Next let us prove Theorems 2 and 3 for the time scale

\[ t_N/N \to c > 0, \quad N \to \infty. \] (35)

Taking into account Remark 5 and assumption (35) we get

\[ t_N L_N \to cM, \quad \lambda_{1,N} t_N \to -\alpha c, \quad \lambda_{2,N} t_N \to -2(\alpha + \beta)c \]

where \( M \) is defined in (8). If \( g \) is an analytic function then \( g'(\lambda_{1,N} t_N) = g'(-\alpha c) + o(1) \) and \( F_N(g, t_N) = H(g, c) + o(1) \), where

\[ H(g, c) := \left( \frac{g(-2(\alpha + \beta)c) - g(-\alpha c)}{-2(\alpha + \beta)c + \alpha c} \right) \cdot \left( -\frac{2\alpha}{\alpha + 2\beta} \right). \]

Using Remark 6 we have

\[ \left( \begin{array}{c} \mathbf{R}_N(t_N) \\ \mathbf{D}_N(t_N) \end{array} \right) \sim (-cM)^{-1} (\text{Id} - e^{cM}) \left( \begin{array}{c} \sigma^2 \\ 2\sigma^2 \end{array} \right) t_N + o(t_N) + \]

\[ + (v - r) t_N \mathbf{E} \mathbf{d}_N(0) \left( g_1'(-\alpha c) \left( \begin{array}{c} 1 \\ 0 \end{array} \right) + H(g_1, c) \left( \begin{array}{c} 0 \\ 1 \end{array} \right) + \left( \begin{array}{c} o(1) \\ o(1) \end{array} \right) \right) + \]

\[ + (v - r)^2 t_N^2 \left( \left( g_2'(-\alpha c) + o(1) \right) \left( \begin{array}{c} 1 \\ 0 \end{array} \right) + \left( H(g_2, c) + o(1) \right) \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \right). \]

Now if \( v = r \) we easily get item P2 of Theorem 2. In the case \( v \neq r \) we obtain item P2 of Theorem 3 with

\[ h_R(c) = g_2'(-\alpha c), \]

\[ h_D(c) = H(g_2, c), \]

where \( g_2 \) is the same as in Remark 6. Now we are able to get explicit forms of the functions \( h_R(c) \) and \( h_D(c) \):

\[ h_R(c) = \frac{1 - (1 + \alpha c)e^{-\alpha c}}{\alpha^2 c^2}, \]

\[ h_D(c) = 2e^{-2} \cdot \frac{1 - (1 + \alpha c)e^{-\alpha c}}{\alpha (\alpha + 2\beta)} - \]

\[ \frac{2e^{-2}\alpha}{\alpha + 2\beta} \left( \frac{1}{2(\alpha + \beta)\alpha} - \frac{1}{\alpha + 2\beta} \left( \frac{e^{-\alpha c}}{\alpha} - \frac{e^{-2(\alpha + \beta)c}}{2(\alpha + \beta)} \right) \right). \] (36)

The study of the time scale \( t_N/N \to \infty \) is very similar to the proof of Theorem 4 (see above). So we omit details.
4.6 Time scale analysis

In this subsection we derive some corollaries from our main theorems. We examine behavior of the functions $R_N(t_N)$ and $D_N(t_N)$ on special time scales $t_N = sN^\gamma$ where $\gamma > 0$ and $s > 0$. Summing up Theorems 1–3 and Subsection 4.5 we get the following general statement: Assume that $N \to \infty$ and assumption (7) holds. Then

$$R_N(sN^\gamma) \sim C_R(s, \gamma) N^{\psi_R(\gamma)},$$
$$D_N(sN^\gamma) \sim C_D(s, \gamma) N^{\psi_D(\gamma)},$$

where all functions $C_R, C_D, \psi_R, \psi_D$ are positive. Moreover, these functions can be calculated explicitly:

**Case 1: $v = r$ — zero skew.**

\[
\psi_R(\gamma) = \min(\gamma, 1), \quad C_R(s, \gamma) = \begin{cases} 
\sigma^2 s, & \gamma < 1, \\
\sigma^2 l_R(s), & \gamma = 1, \\
\sigma^2 / \alpha, & \gamma > 1,
\end{cases} \]

\[
\psi_D(\gamma) = \min(\gamma, 1), \quad C_D(s, \gamma) = \begin{cases} 
2\sigma^2 s, & \gamma < 1, \\
2\sigma^2 l_D(s), & \gamma = 1, \\
2\sigma^2 / (\alpha + \beta), & \gamma > 1,
\end{cases}
\]

where

\[
l_R(s) = g_2(-\alpha s), \quad l_D(s) = \frac{\alpha g_2(-\alpha s) + 2\beta g_1(-2(\alpha + \beta)s)}{\alpha + 2\beta}
\]

and the function $g_2$ is the same as Remark [6].

**Case 2: $v \neq r$ — nonzero skew.**

\[
\psi_R(\gamma) = \min(2\gamma, 2), \quad C_R(s, \gamma) = \begin{cases} 
\frac{1}{2} (v-r)^2 s^2, & \gamma < 1, \\
(v-r)^2 s^2 h_R(s), & \gamma = 1, \\
(v-r)^2 \sigma^2, & \gamma > 1,
\end{cases} \]

\[
\psi_D(\gamma) = \begin{cases} 
\gamma, & \gamma \leq \frac{1}{2}, \\
3\gamma - 1, & \frac{1}{2} < \gamma < 1, \\
2, & \gamma \geq 1,
\end{cases} \quad C_D(s, \gamma) = \begin{cases} 
2\sigma^2 s, & \gamma < \frac{1}{2}, \\
2\sigma^2 s + \frac{1}{3} \alpha (v-r)^2 s^3, & \gamma = \frac{1}{2}, \\
\frac{1}{3} \alpha (v-r)^2 s^3, & \frac{1}{2} < \gamma < 1, \\
(v-r)^2 s^2 h_D(s), & \gamma = 1, \\
(v-r)^2 \sigma^{-1} (\alpha + \beta)^{-1}, & \gamma > 1.
\end{cases}
\]

where $h_R$ and $h_D$ are the same as in [35].
How to interpret and to explain the above results? The choice of the time scale $t_N = sN^\gamma$ means that we consider a new time unit which is equal to $N^\gamma$ units of the physical time $t$. Then parameter $s$ is the time measured in new units. We see that behavior of the stochastic system $x(t) = (x_1(t), \ldots, x_{N+1}(t))$ for large $N$ is different for different values of $\gamma$. We recall (Section 2) that $R_N(sN^\gamma)$ and $D_N(sN^\gamma)$ are squares of clock synchronization errors. The above corollaries show existence of several phases in the evolution of the network. In the case $v = r$ there are three such phases, they were discussed in Remark 2.

In the case $v \neq r$ on “small times” ($\gamma < 1$) the function $R_N(sN^\gamma)$ is proportional to the square of $t_N$. We can imagine that on these times effect of clock synchronization is negligible and the asymptotical value $R_N(sN^\gamma)$ depends only on summands of the form

$$(x_1(t) - x_j(t))^2 = (x_1(0) - x_j(0) + (r - v)t)^2 \sim (v - r)^2t^2.$$ (37)

For the scale $\gamma = 1$ the total number of synchronization messages from the server 1 is of order $N$, their influence become important and we observe a slowdown: for $s \to \infty$

$$s^2h_R(s) \uparrow \alpha^{-2} < +\infty.$$  

On the scales $\gamma > 1$ we see the result of full strength competition between synchronization jumps and the desynchronization generated by motions of the client nodes $2, \ldots, N+1$. This is the final synchronization phase.

In the same case $v \neq r$ the behavior of the function $D_N(sN^\gamma)$ is much more interesting. It appears that the “small times” ($\gamma < 1$) are splitted into three subphases: $0 < \gamma < \frac{1}{2}$, $\gamma = \frac{1}{2}$ and $\frac{1}{2} < \gamma < 1$. Recall that the function $D_N$ describes the internal inconsistency of the client’s clocks $x_2, \ldots, x_{N+1}$. On the first subphase ($0 < \gamma < \frac{1}{2}$) influence of the server node 1 is negligible in comparison with a “noise” produced by identical client nodes, hence we see the same asymptotics $2\sigma^2sN^\gamma$ as in the Case 1 (see also asymptotics in [18]). On the scale $\gamma = \frac{1}{2}$ these two forces are of the same order $N^{1/2}$ and $C_D(s, \frac{1}{2}) = 2\sigma^2s + \frac{1}{2}\alpha(v - r)^2s^3$. On the third subphase ($\frac{1}{2} < \gamma < 1$) the server node 1 dominates over the free dynamics of the clients. Surprisingly, that for scales $\frac{1}{2} < \gamma < 1$ the influence of the server of the accurate time produces a desynchronization (but not synchronization) of the client’s clocks. The explanations is the following one: on these scales only small part of client nodes had interacted with the node 1 till the time $sN^\gamma$. The most part of client nodes (of order $N$) still have no idea about the server’s clock $x_1$. Adjusting of a client’s clock to the value $x_1$ brings a large number of summands of the form (37) to the function $D_N$. On the scale $\gamma = 1$ the number of nodes interacted with the server 1 is of order $N$ and again we observe a slowdown of
desynchronization. The time scales $\gamma > 1$ correspond to the final synchronization phase.

5 Appendix

Proof of Lemma 6. Let $a_1 \neq a_2$. It is straightforward to check that

$$\sum_{n_2=0}^{n} a_1^{n_2} a_2^{n-n_2} = (a_1 - a_2)^{-1} (a_1^{n+1} - a_2^{n+1}).$$

Using (23) for $A = a_1$ and $A = a_2$ we find $U_1(a_1, a_2)$.

Now let us calculate $U_2(a_1, a_2)$. After some simple algebra we have identity

$$\sum_{n_1 \geq 0, n_2 \geq 0} a_1^{n_1} a_2^{n-n_2} = \frac{1}{a_1 - a_2} \left( \frac{1}{1 - a_1} - \frac{1}{1 - a_2} \right) - \frac{a_1^{n+2} - a_2^{n+2}}{a_1 - a_2},$$

Applying (24) for $A = 1$, $A = a_1$ and $A = a_2$, we get that $U_2(a_1, a_2)$ multiplied by $\delta_N^2$ is equal to

$$\frac{1}{1 - a_1} - \frac{1}{1 - a_2} \left( 1 - \frac{1 + \delta_N t}{e^{\delta_N t}} \right) - \frac{1}{a_1 - a_2} \left( \frac{e^{-\delta_N t(1-a_1)} - 1 + \delta_N t a_1}{e^{\delta_N t}} \right) \left( \frac{e^{-\delta_N t(1-a_2)} - 1 + \delta_N t a_2}{e^{\delta_N t}} \right).$$

By direct transformations and cancellation of terms this form can be reduced to the following one

$$\frac{e^{-\delta_N t(1-a_1)} - 1}{1 - a_1} - \frac{e^{-\delta_N t(1-a_2)} - 1}{1 - a_2} = \frac{1}{(1 - a_1)(1 - a_2)} - \frac{e^{-\delta_N t(1-a_1)} - 1}{a_1 - a_2} - \frac{e^{-\delta_N t(1-a_2)} - 1}{a_1 - a_2}. \tag{28}$$

This proves the statement of Lemma 6 for $a_1 \neq a_2$.

The case $a_1 = a_2$ is simpler than just considered case $a_1 \neq a_2$. So we omit details here. Note, that explicit expressions for $U_i(a, a)$, $i = 1, 2$, correspond to formal limits of $U_i(a_1, a_2)$ as $a_1 \to a, a_2 \to a$. □
6 Conclusion and future work

We proposed a basic probabilistic model of clock synchronization in large WSNs interacting with an accurate time server. It was shown that in large networks \((N \to \infty)\) there exists several time scales \(t = t_N\) of qualitatively different collective behavior of the network. In other words, a large network passes different phases on its road to synchronization. The \textit{phase of effective synchronization} is the most interesting among them. For our basic network we give a detailed description of this phase. Moreover, explicit formulae obtained in Theorems 1–3 provide keys to future analytical study of various optimization and performance evaluation problems related to WSNs.

We believe that obtained results about existence of several different phases in evolution should take place for more general classes of large networks. Mathematical tools used in the present study will work also for general nonhomogeneous time synchronization model. Of course, in that case we cannot expect such short and explicit results as in Section 3. Successful examples of synchronization studies for some special nonhomogeneous (or weakly nonhomogeneous) systems can be found in [12, 10, 13, 15].

Our basic model can be developed also for more general and realistic assumptions about message sending algorithms. We plan to get rid of condition (3) to be able to consider arbitrary distributed intervals between messages. The theory of general random flows [2] can be useful here but, unfortunately, the corresponding probabilistic model will be non-Markovian (similarly to [19]). The white noise assumption in the clock model (1) is confusing for the following reason: it contradicts to a general clock modelling principle that time never run backward (see page 313 in [30]). To justify our choice we note, first, that from the mathematical viewpoint this assumption is convenient but not necessary and it will be removed in future papers, and, in the second place, this is an usual assumption for many modern clock models (see [9, 32, 6]).

We hope that presented approach combined with other methods will be useful for analysis of many practical distributed systems.

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