Exact Solutions of Schrödinger Equation in Cylindrical Coordinates for Double Ring-Shaped Coulomb Oscillator Potential Using SUSY QM Method

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Abstract. Schrödinger equation for a doubled ring-shaped coulomb oscillator potential is investigated using supersymmetric quantum mechanics approach. The three dimensional Schrödinger equation for a doubled ring-shaped coulomb oscillator potential in cylindrical coordinates is separated into three parts that contains one dimensional Schrödinger type equation which are solved using supersymmetric operator and the idea of shape invariant potential. The energy spectrum and the total wave functions are obtained.

Keywords : Schrödinger equation, Doubled Ring-Shaped Oscillator, Coulomb, Supersymmetry

1. Introduction

The Schrödinger equation is the most fundamental equation in non-relativistic quantum mechanics and the closely related Heisenberg equation, playing the same role as Hamilton’s laws of motion in non-relativistic classical mechanics. In pure mathematics, the Schrödinger equation and its variants are one of the basic equations studied in the field of partial differential equations, and has application to geometry, spectral and scattering theory, and to integrable systems [1]. Exact analytical solutions of the Schrödinger equation for some physical potential are important to obtain the wave function and the energy spectrum.

One of exactly solvable quantum systems that has been applied to various fields is the spherical oscillator [2,3]. Many authors have carried out significant investigations on non-spherical oscillator [4], ring-shaped oscillator [5-8], and ring-shaped non-spherical oscillator [9-12]. These studies became interesting in mathematics itself and have potential applications in theoretical chemistry. DRSO potential that proposed by Carpio-Bernido in 1989 has been investigated using path integral method [13-14], algebraic method [15], AIM method [16], SUSY QM and shape invariance approach [17].

The potential can be described as follows

\[ V(r, \theta) = \frac{1}{2} M \omega^2 r^2 + \frac{\hbar^2}{2m} \left( \frac{b}{r^2 \sin^2 \theta} + \frac{c}{r^2 \cos^2 \theta} \right) \] (1)

In this paper, a doubled ring-shaped oscillator potential have modified by adding coulomb potential in axial part for cartesian coordinates system becomes

\[ V(x, y, z) = \frac{1}{2} M \omega^2 (x^2 + y^2) + \frac{\hbar^2}{2m} \left( \frac{b}{x^2} + \frac{c}{y^2} \right) + \frac{\alpha}{z^2} + \frac{\beta}{z} \] (2)

Equation (2) is called doubled ring-shaped oscillator potential by applying cylindrical coordinates system in Schrodinger equation and using separation variables method, so each part can be solved to get the wave functions and energy spectra by using supersymmetric approach.
2. Review of Supersymmetric Quantum Mechanics Approach Using Operator

The exact analytical solutions of Schrödinger equations for some physical potentials are very important. According to the definition proposed by Witten, the Hamiltonian in a SUSY QM is expressed as [18]

\[ H_n = \begin{pmatrix} \frac{-\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{h}{\sqrt{2m}} \frac{d\varphi(x)}{dx} + \varphi'(x) & 0 \\ 0 & -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} - \frac{h}{\sqrt{2m}} \frac{d\varphi(x)}{dx} + \varphi'(x) \end{pmatrix} \]

With

\[ V_\pm(x, a_0) = \varphi^2(x) \mp \frac{\hbar}{\sqrt{2m}} \varphi'(x); \quad V_\pm(x, a_0) = \varphi^2(x) \pm \frac{\hbar}{\sqrt{2m}} \varphi'(x) \]  

Here \( H_- \) and \( H_+ \) are supersymmetry partner of the Hamiltonian, \( V_-(x) \) and \( V_+(x) \) are the supersymmetry partner potential. To simplify the determination of the energy spectrum and the wave functions, the new operators, raising and lowering operators, are introduced as

\[ A^+ = -\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + \phi(x) \quad \text{and} \quad A = \frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + \phi(x) \]

By inserting equation (5) into equation (3) we get the SUSY Hamiltonian as

\[ H_-(x) = A^+ A \quad \text{and} \quad H_+(x) = AA^+ \]

It is always possible to factorize the usual Hamiltonian as

\[ H = H_- + H_+ = E_0 = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_-(x; a_0) + E_0 \]

By using equations (4) and (7) it is obtained that

\[ V(x) = V_-(x; a_0) + E_0 = \phi^2(x; a_0) - \frac{\hbar}{\sqrt{2m}} \phi'(x; a_0) + E_0 \]

where \( V(x) \) is the effective potential, while \( \phi(x) \) is determined hypothetically from equation (8) based on the shape of effective potential from the associated system. In addition to the supersymmetry that gives the relationship between the eigenvalues and eigenfunctions between the two Hamiltonian partners but does not provide the actual spectrum [21], Gendenshtein [19] proposed the condition of shape invariant to determine the energy spectrum. A pair of potentials \( V_\pm(x) \) in equations (5) are said to be shape invariant if they are similar in shape but different in the parameters and this condition is given as,

\[ V_\pm(x; a_j) = V_\pm(x; a_{j+1}) + R(a_{j+1}) \]

where \( j = 0, 1, 2, \ldots \) and \( a \) is a parameter in our original potential, \( V_\pm \), whose ground state energy is zero, \( a_j = f_j(a_0) \) where \( f_j \) is a function applied \( j \) times, the remainder \( R(a_j) \) is \( a \)'s dependence but is independent of \( x \), then \( V_\pm(x; a_j) \) is said to be shaped invariant. The energy eigenvalue of the Hamiltonian \( H_- \) is given by [18]

\[ E_n^{(-)} = \sum_{k=1}^{n} R(a_k) \]

and by using equations (8) and (10) we get the energy spectra of the system given as,

\[ E_n = E_n^{(-)} + E_0 \]

Based on the characteristics of lowering operator, the ground state wave function of \( H_- \), whose ground state energy is zero, is obtained from condition that,

\[ H_- \psi_0^{(-)} = 0 \rightarrow A \psi_0^{(-)} = 0 \]
Subsequently, the excited wave function, $\psi_n(x, \alpha_n)\ldots \psi_1(x, \alpha_1)$ of $H$ are obtained by using raising operator operated to the lower wave function [19], given as

$$\psi_n(x, \alpha_n) = A^\dagger(x,\alpha_n)A^\dagger(x,\alpha_{n-1})\ldots A^\dagger(x,\alpha_1)\psi_0(x,\alpha_1)$$  \hspace{1cm} (13)

The SUSY QM and the idea of shape invariance potential are suitable to be used in solving one dimensional Schrödinger equation problems. By obtaining the super-potential, the potential partners $V_+(x,\alpha_n)$ and $V_-(x,\alpha_n)$, and the SUSY operators, $A^\dagger$ and $A$ are obtained and so the energy spectra and the wave functions.

3. Separation Variables Method for The Case

Schrödinger equation is defined by

$$-\frac{\hbar^2}{2m} \nabla^2 \psi(r) + V(r)\psi(r) = E\psi(r)$$ \hspace{1cm} (14)

Doubled ring-shaped coulomb oscillator potential is changed by applying cylindrical coordinates so it becomes

$$V(r,\theta,z) = \frac{1}{2} M \omega^2 r^2 + \frac{\hbar^2}{2m} \left( \frac{b}{r^2 \cos^2 \theta} + \frac{c}{r^2 \sin^2 \theta} \right) + \frac{\alpha}{r^2} + \frac{\beta}{z}$$ \hspace{1cm} (15)

By substituting this potential into Schrödinger equation, given as

$$-\frac{\hbar^2}{2m} \nabla^2 \psi(r) + \left[ \frac{1}{2} M \omega^2 r^2 + \frac{\hbar^2}{2m} \left( \frac{b}{r^2 \cos^2 \theta} + \frac{c}{r^2 \sin^2 \theta} \right) + \frac{\alpha}{r^2} + \frac{\beta}{z} \right] \psi(r) = E\psi(r)$$ \hspace{1cm} (16)

Then apply $\nabla^2$ for cylindrical coordinate given by

$$\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$ \hspace{1cm} (17)

By substituting equation (18) to equation (17) and the wave function is $\psi(r) = R(r)P(\theta)\Phi(z)$ so the axial part could be separated

$$-\frac{\hbar^2}{2m} \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial R}{\partial r} \right) + \frac{1}{2} \frac{\partial^2 P}{\partial \theta^2} \right) + \frac{1}{2} M \omega^2 r^2 + \frac{\hbar^2}{2m} \left( \frac{b}{r^2 \cos^2 \theta} + \frac{c}{r^2 \sin^2 \theta} \right) - E = \frac{\hbar^2}{2m} \frac{\partial^2 Z}{\partial z^2} - \left( \frac{\alpha}{z^2} + \frac{\beta}{z} \right) = \lambda_1$$ \hspace{1cm} (18)

The left side of equation (18) can be separated if it is multiplied with $r^2$, so we get

$$-\frac{\hbar^2}{2m} \frac{\partial}{\partial r} \left( r \frac{\partial R}{\partial r} \right) + \frac{1}{2} M \omega^2 r^4 - (E - \lambda_1) r^2 = \frac{\hbar^2}{2m} \frac{\partial^2 P}{\partial \theta^2} - \frac{\hbar^2}{2m} \left( \frac{b}{\cos^2 \theta} + \frac{c}{\sin^2 \theta} \right) = \lambda_2$$ \hspace{1cm} (19)

So the Schrödinger equation in Eq.(17) that has been separated may be written by

- **Radial part**

  $$-\frac{\hbar^2}{2m} \frac{\partial}{\partial r} \left( r \frac{\partial R}{\partial r} \right) + \frac{1}{2} M \omega^2 r^4 - (E - \lambda_1) r^2 = \lambda_2$$ \hspace{1cm} (20)

- **Angular part**

  $$-\frac{\hbar^2}{2m} \frac{\partial^2 P}{\partial \theta^2} + \frac{\hbar^2}{2m} \left( \frac{b}{\cos^2 \theta} + \frac{c}{\sin^2 \theta} \right) P = \lambda_2 P$$ \hspace{1cm} (21)

- **Axial part**
\[-\frac{\hbar^2}{2m} \frac{\partial^2 Z}{\partial z^2} + \left( \frac{\alpha}{z^2} + \frac{\beta}{z} \right) Z = \lambda Z \tag{22}\]

4. Solutions to The Each Part of The Three Dimensional Schrodinger Equation

Supersymmetric quantum mechanics is used to solve one dimensional Schr"{o}dinger equation with any shape invariance potentials. After has been separated into three parts, each part is solved by using SUSY QM method.

4.1. Solution of Axial Part

Assume that \( \alpha = \frac{\hbar^2}{2m} l(l+1) \), \( \beta = -e^2 \) and \( \lambda = E \zeta \). So equation (22) can be written as follows

\[-\frac{\hbar^2}{2m} \frac{\partial^2 Z}{\partial z^2} + \left( \frac{\hbar^2}{2m} \frac{l(l+1)}{z^2} - \frac{e^2}{z} \right) Z = E \zeta Z \tag{23}\]

Equation (23) is Schrodinger equation for axial part which have the effective potential is coloumb potential. We hypothesise that the superpotential is given by \( \phi_o(z,a_0) = \frac{A}{z} + \frac{B}{A} \), by applying equation (8) and equation (23) we get the superpotential, super-partner potentials and ground state energy are

\[\phi_o(z,a_0) = -\frac{\hbar^2}{2m} \frac{(l+1)}{z} - \frac{\sqrt{2me^2}}{2h(l+1)} \tag{24} \]

\[V_c(z,a_0) = \frac{\hbar^2}{2m} \frac{l(l+1)}{z^2} - \frac{e^2}{z} + \frac{me^4}{2h(l+1)^2} \tag{25} \]

\[V_c(z,a_0) = \frac{\hbar^2}{2m} \frac{(l+1)(l+2)}{z^2} - \frac{e^2}{z} + \frac{me^4}{2h(l+1)^2} \tag{26} \]

\[E_0 \zeta = -\frac{me^4}{2h(l+1)^2} \tag{27} \]

The raising and lowering operator of axial part are given by

\[A^+ = -\frac{\hbar}{2m} \frac{d}{dz} - \frac{\hbar}{\sqrt{2m}} \frac{l(l+1)}{z} - \frac{\sqrt{2me^2}}{2h(l+1)} \tag{28} \]

\[A = \frac{\hbar}{\sqrt{2m}} \frac{d}{dz} - \frac{\hbar}{2m} \frac{l(l+1)}{z} - \frac{\sqrt{2me^2}}{2h(l+1)} \tag{29} \]

Using the lowering operator equation (29) and applying equation (12) we get the ground state of wave function is

\[Z_0 = C \zeta^{(l+1)} e^{\frac{\sqrt{2me^2}}{2h(l+1)}} \tag{30} \]

And the first excited wave function is given by

\[Z_1(z,a_0) = C(l+1) \zeta e^{\frac{\sqrt{2me^2}}{2h(l+1)}} \tag{31} \]

With \( a_0 = 1, a_1 = l+1, ..., a_n = l+n \) so we can get the next excited wave functions of axial part by using raising equation (28) and lowering operator equation (29) and applying equation (14).
The eigen value of Schrödinger equation for axial part can be obtained by using equations (9) and (10) is given by

$$E_n = \frac{me^4}{2h(l+n_e+1)^2} \quad (32)$$

So we get the first separation variable constant $\lambda_1$ is like equation (33).

4.2. Solution of Angular Part

Assume that $b = v(v-1)$, $c = \kappa(\kappa-1)$ and $E''_0 = \lambda_2$. So equation (22) can be written as follows

$$-\frac{\hbar^2}{2m} \frac{\partial^2 P}{\partial \theta^2} + \frac{\hbar^2}{2m} \left( \frac{v(v-1)}{\cos^2 \theta} + \frac{\kappa(\kappa-1)}{\sin^2 \theta} \right) P = E'' P \quad (33)$$

Like in the subsection 4.1. We have to define the superpotential, by using equation (21) we hypothesizes that the superpotential is given by

$$\phi_0(\theta) = \frac{\hbar}{\sqrt{2m}} (v-1) \cot \theta - \frac{\hbar}{\sqrt{2m}} (\kappa-1) \tan \theta \quad (34)$$

And we get the ground state energy for angular part is given by

$$E''_0 = \frac{\hbar^2}{2m} (v-\kappa)^2 \quad (35)$$

The raising and lowering operator for angular part are given by

$$A^+(\theta, a_0) = -\frac{\hbar}{\sqrt{2m}} \frac{d}{d\theta} + \frac{\hbar}{\sqrt{2m}} (v-1) \cot \theta - \frac{\hbar}{\sqrt{2m}} (\kappa-1) \tan \theta \quad (36)$$

$$A(\theta, a_0) = \frac{\hbar}{\sqrt{2m}} \frac{d}{d\theta} + \frac{\hbar}{\sqrt{2m}} (v-1) \cot \theta - \frac{\hbar}{\sqrt{2m}} (\kappa-1) \tan \theta \quad (37)$$

Using the lowering operator equation (38) and applying equation (12) we get the ground state of wave function is

$$P_0 = D \left[ (\cos^{(v-1)} \theta) \left( \sin^{(\kappa-1)} \theta \right) \right] \quad (38)$$

And the first excited wave function for angular part is

$$P_1 = D \left[ (v-\kappa) \sin^{(v+\kappa-3)} \theta + (v-1) \cos^k \theta \sin^{(v-2)} \theta - (\kappa-1) \cos^{(\kappa-2)} \theta \sin^v \theta \right] \quad (39)$$

With $a_0 = v; \kappa, a_1 = (v-1); (\kappa-1), ..., a_n = (v-n); (\kappa-n)$ so we can get the next excited wave functions by using raising and lowering operator equations (37) and (38) and applying equation (14). The super-partner potentials for angular part are given as

$$V_-(\theta, a_0) = \frac{\hbar^2}{2m} \left( \frac{v(v-1)}{\cos^2 \theta} + \frac{\kappa(\kappa-1)}{\sin^2 \theta} \right) - \frac{\hbar^2}{2m} (v-\kappa)^2 \quad (40)$$

$$V_+(\theta, a_0) = \frac{\hbar^2}{2m} \left( \frac{(v-1)(v-2)}{\cos^2 \theta} + \frac{(\kappa-1)(\kappa-2)}{\sin^2 \theta} \right) - \frac{\hbar^2}{2m} (v-\kappa)^2 \quad (41)$$

The eigen value of Schrödinger equation for angular part can be obtained by using Eq.(9) and Eq.(10) is given by

$$E''_n = -\frac{\hbar^2}{2m} (v-\kappa)^2 \quad (42)$$

So the second variable constant is as like equation (43).
4.3. Solution of Radial Part

Radial part in Eq.(18) have to be reduced into Schrodinger equation by assumed that $R = \frac{U}{\sqrt{r}}$, $M = m$ and $E^m = E + \lambda_i$. Then the radial part has been reduced into Schrödinger equation, given as

$$-\frac{\hbar^2}{2m} \frac{\partial^2 U}{\partial r^2} + \left(\frac{1}{2} m\omega^2 r^2 - \frac{\hbar^2}{2m} \frac{\lambda_i + \frac{1}{2}}{r^2}\right) U = E^m U \quad (43)$$

We hypothesize that the superpotential for the radial part is $\phi_0(r) = Ar + \frac{B}{r}$, by using equation (21) we get the superpotential is

$$\phi_0(r) = \omega \sqrt{\frac{m}{2} r} - \frac{\hbar}{\sqrt{2m}} \sqrt{-\lambda_2 + \frac{1}{2}} r \quad (44)$$

And we get the ground state energy is given by

$$E_0^m = \hbar \omega \left(\sqrt{-\lambda_2 + \frac{1}{2}}\right) \quad (45)$$

The raising and lowering operator for radial part are given by

$$A^+(r,a_0) = -\frac{\hbar}{\sqrt{2m}} \frac{d}{dr} + \omega \sqrt{\frac{m}{2} r} - \frac{\hbar}{\sqrt{2m}} \sqrt{-\lambda_2 + \frac{1}{2}} \quad (46)$$

$$A(r,a_0) = \frac{\hbar}{\sqrt{2m}} \frac{d}{dr} + \omega \sqrt{\frac{m}{2} r} - \frac{\hbar}{\sqrt{2m}} \sqrt{-\lambda_2 + \frac{1}{2}} \quad (47)$$

Using the lowering operator equation (48) and applying equation (12) we get the ground state of wave function is

$$U_0 = F' r^{\frac{1}{2}} e^{-\omega r} \quad (48)$$

And the first excited wave function for radial part is given by

$$U_1 = F \left[ \omega \sqrt{\frac{m}{\hbar}} r^{\frac{1}{2}} e^{-\omega r} + \frac{2\hbar}{\sqrt{2m}} r^{\frac{1}{2}} e^{-\omega r} - \frac{\omega m}{\hbar} r^{\frac{1}{2}} e^{-\omega r} \right] \quad (49)$$

With $a_0 = -\sqrt{-\lambda_2}, a_1 = -\sqrt{-\lambda_2} + 1, ..., a_n = -\sqrt{-\lambda_2} + n$ so we can get the next excited wave functions by using raising and lowering operator equations (47) and (48) and applying equation (14).

The super-partner potentials for radial part are given as

$$V_-(r,a_0) = \frac{1}{2} m\omega^2 r^2 - \frac{\hbar^2}{2m} \lambda_2 + \frac{1}{2} \sqrt{-\lambda_2 + \frac{1}{2}} \quad (50)$$

$$V_+(r,a_0) = \frac{1}{2} m\omega^2 r^2 - \frac{\hbar^2}{2m} \left(\sqrt{-\lambda_2 + \frac{1}{2}} + \frac{3}{2}\sqrt{-\lambda_2 + \frac{1}{2}}\right) - \hbar \omega \left(\sqrt{-\lambda_2 + \frac{1}{2}} + \frac{1}{2}\right) \quad (51)$$

The eigen value of Schrodinger equation for radial part can be obtained by using equations (9) and (10) is given by
The energy spectra of Schrödinger equation for DRSCO potential that expressed in equation (2) is given in equation (54). There are three parts of the energy spectra, the first part is associated with the axial part of the potential, and the second part is associated with radial and polar parts of potential.

The total wave functions, the un-normalized ground state and first excited state ones, are obtained from equations (31), (32), (38), (39), (48), and (49) given as

\[
\psi_0(r, \theta, z) = G' r^{\lambda_z} e^{\frac{amr}{h}} \left[ \cos^{(v-1)} \theta \left( \sin^{(k-1)} \theta \right)^{l+1} \right] e^{\frac{\sqrt{m} z^3}{2h}} \]

\[
\psi_1(r, \theta, z) \cong \left[ \frac{\omega \sqrt{m}}{h} r^{\lambda_z} \left( \frac{2h}{\sqrt{2m}} r^{\lambda_z} \frac{1}{2} - \frac{amr}{h} r^{\lambda_z} \frac{1}{2} \right) e^{\frac{amr}{h}} \right] \times \left[ (v-\kappa) \sin^{(v-\kappa+1)} \theta + (v-1) \cos^4 \theta \sin^{(v-2)} \theta - (k-1) \cos^{(k-2)} \theta \sin^4 \theta \right] \times (l+1) \frac{z^3}{2h} e^{\frac{\sqrt{m} z^3}{2h}} \]

with \( \lambda_z \) expressed in equation (43) respectively.

5. Conclusion

By applying a cylindrical coordinate system, the Schrödinger equation for DRSCO potential reduces to a perfectly variable separable Schrödinger equation. Three dimensional Schrödinger equation reduces to one radial Schrodinger equation, one angular Schrodinger equation, and one axial Schrodinger equation. These three one dimensional Schrodinger equation are exactly solvable since each of Schrodinger equation with shape invariant potential. By using SUSY Quantum Mechanics method and the idea of shape invariance the energy spectra are calculated and the total wave function is obtained which are combination of axial, angular and radial parts.

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