Some engineering applications of new trigonometric cubic Bézier-like curves to free-form complex curve modeling

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Abstract

The construction of free-form complex curves is very hot topic in engineering and computer aided design/computer aided manufacturing (CAD/CAM). The trigonometric Bézier-like curves got more attention in the fields of mathematics and engineering in recent years because of their useful geometric properties as compared to classical Bézier curves as well as ordinary Bernstein basis functions. The new trigonometric cubic Bézier-like (or TC Bézier-like, for short) curves along with new trigonometric cubic Bernstein-like (or TC Bernstein-like, for short) basis functions with single shape parameter with continuity conditions are presented in this paper. The necessary and sufficient constraints for $C^2$ and $G^2$ between two contiguous TC Bézier-like curves are constructed in order to remove the difficulty to composite curves which they cannot often be constructed by means of single curve. The role of shape parameter on connected curves with detailed smooth continuity steps is also part of this study. The ellipses and parabolas can also be represented exactly by using the TC Bézier-like curves. Some important applications of TC Bézier-like curves as: approximate some conic curves, font designing and free form complex curves are discussed. Some modelling examples show that the proposed TC Bézier-like curve technique is time saving, very effective and efficient which they can easily be modelled and give a powerful tool to design engineering complex curves.

Keywords: Bernstein polynomials, Bézier curve, Trigonometric Bernstein-like basis functions, Trigonometric Bézier-like curves, Parametric and geometric continuity, Shape parameter, Conic curves

1. Introduction

Shape design of various products with geometric properties can be achieved by means of free-form curves which is an important research area and forceful tool in CAD/CAM systems (see e.g. Delgado and Peña, 2007, Linn, et al. (2014)). Bézier curves with Bernstein basis functions are well known; especially the cubic Bézier curves because of their simple representation, definition and very good geometric properties as well as have extensive application in science, engineering and computer based engineering design (see e.g. Rababah and Mann, 2011, Jaklič, et al. (2012)). However, there are two mainly disadvantages of the Bézier curves which are: (1) the control points are used to change their shapes of curves, and (2) they cannot represent conic curves precisely (Farin, 2002). Furthermore, the rational Bézier curves can be adjusted by changing the values of weights with keeping their control points fixed (Coelho, et al. (2017)). But they have many shortcomings, such as difficult calculations,
unwieldy integrals, and repetitive differentiation (Mamar, 2011). Some Bézier-like methods of constructing curves were presented by using shape parameters in order to improve the shape of curve in product shape design and modify the level of curve where it approaches to control polygon (see e.g. Han, 2006, Juhász and Hoffmann, 2009). The basic idea of these methods is to incorporate parameters into the classical basis functions, where the parameters can adjust the shape of the curves without changing the control points (Han and Zhu, 2012). For this purpose, the polynomial basis and trigonometric basis functions with shape parameters have constructed. These shape parameters help us and have better grip on the shape and position of curves and surfaces. The advantage of shape parameters is that we can generate curves which can move freely according to designer requirements. Therefore constructing trigonometric Bézier-like curves with shape parameter got more attention in recent years.

In the recent years, many researchers and scientists defined the Bézier curves especially trigonometric Bézier curves with variety of shape parameters. In the paper (Li, 2013), a class of cubic trigonometric Bézier curves with single shape parameter is constructed and it used to approximate the some conic curves. Li and Zhao, 2013 presented a cubic trigonometric Bézier curves with one shape parameter. They investigated image compression by using cubic trigonometric Bézier curve approximation method. Xei and Li, 2018 proposed a class of cubic trigonometric B-spline curves with single shape parameter and constructed arcs of ellipse and parabola. Chen and Li, 2016 defined $\alpha$-catmull Rom spline basis functions with single shape parameter $\alpha$ by extending definition of standard cubic catmull Rom spline. An automatic mathematical model established by Liu and Li, 2016 to get the optimal parameters using fairing criterion as well as they introduced the particle swarm optimization (PSO) algorithm for solution of optimal model. The cubic trigonometric Bézier-like s-shaped and c-shaped spiral curves with shape parameter presented by Misro, et al. (2017a). Furthermore, Misro, et al. (2017b) presented a Quintic trigonometric Bézier curve with two shape parameters. Liu, et al. (2012) defined a cubic trigonometric polynomial B-spline $C^2$ curves with a shape parameter and became closer to control points when value of shape parameter was increased. Li, (2018a,2018b and 2018c) defined $\alpha$-Bézier curves of degree $n$ with shape parameter $\alpha$ by extending definition of Bézier curve, planar T-Bézier curve with approximate minimum curvature variation and curvature variation minimizing cardanl spline curve with different shape parameters, respectively. Wen-tao and Guo-Zhao, 2005 presented Bézier-like curves having shape parameter by an integral approach. Moreover, Dube and Mishra, 2016 presented a kind of quasi-cubic Bézier curves by the blending of algebraic polynomials and trigonometric polynomials using weight method

Han, et al. (2009) developed a cubic trigonometric Bézier curves with two shape parameters and showed that they can be made closer to the control polygon as compared to the classical Bézier curves by altering the values of shape parameters. Yan and Liang, 2011 presented $xy$-Bézier curves with two shape parameters $x$ and $y$ and they constructed most of the conics, transcendental curves, helix, cycloid and asteroid. Li, et al. (2017) presented a class of cubic trigonometric interpolation spline curves having shape parameters. Furthermore, Li, 2016 presented a cubic trigonometric interpolation curves with two shape parameters. The proposed curves are $C^2$ and adjust their shape automatically by changing shape parameters. Li, 2012 defined trigonometric hermite parametric curves and surfaces with two shape parameters. Bashir, et al. (2013a and 2013b) presented $G^2$ and $C^2$ rational quadratic trigonometric Bézier curve with two shape parameters with its some applications and quasi-quintic trigonometric Bézier curves with two shape parameters, respectively. Juhász, 2018 defined Bézier-like curves with two shape parameters based on modifying the factorization of Bernstein polynomials. A novel shape adjustable generalized Bézier curves with multiple shape parameters constructed by Hu, et al. (2018a) and used them for surface modeling in engineering. Hu, et al. (2018b) proposed a novel technique to generate free-form complex curves using shape-adjustable generalized curves with some geometric continuity. They constructed the necessary and sufficient conditions for $G^1$ and $G^2$ continuity between two adjacent SG-Bézier curves with some suitable applications of them.

A new TC Bernstein-like basis functions analogous to standard Bernstein basis functions with single shape parameter is presented in this work. In this work, the second and third TC Bernstein-like functions are same
but the first and fourth TC Bernstein-like functions are different as well as more simpler than (Li, 2013) because they provide a straightforward computation for the derivation of parametric and geometric continuities constraints. Furthermore, they are used to construct the new TC Bézier-like curve similar to the classical cubic Bézier curve. The shape of the curve can be attuned by changing the values of shape parameter by keeping the control polygon fixed. The TC Bézier-like curve approximate exactly or closer to the given control polygon than the cubic Bézier curves. Some conic curves can be represented exactly using TC Bézier-like curve. The necessary and sufficient constraints for \( C^2 \) and \( G^2 \) continuity for two adjacent TC Bézier-like curves segments are constructed. The role of shape parameter on connected curves with detailed smooth continuity steps is also part of this study. The designing applications of TC Bézier-like curves like font designing and free form complex curves are also discussed.

This paper is laid out as: In section 2, a new TC Bernstein-like basis functions and their geometric properties with shape parameter are constructed. The new TC Bézier-like curves and their geometric properties are presented in section 3. The effect of shape parameter on the proposed curve is studied in section 4. Some conic curves are presented in section 5. The necessary and sufficient conditions for \( C^1, C^2 \) and \( G^1, G^2 \) continuity between two TC Bézier-like curves segments are constructed in section 6. Some engineering applications of proposed curve scheme are presented in section 7. Shape adjustment of TC Bézier-like curves with parametric and geometric continuities is constructed in section 8 and conclusion of proposed study is given in section 9.

2. The Trigonometric Cubic Bernstein-like Basis Functions

In this section, we define TC Bernstein-like basis functions with single shape parameter and discuss their some geometric properties.

**Definition 1**: The TC Bernstein-like basis functions with single shape parameter \( \alpha \) are defined as:

\[
a_0(\eta) = \alpha S^2 - \alpha S + U^2, a_1(\eta) = \alpha (S - S^2), a_2(\eta) = \alpha (S^2 + U - 1), a_3(\eta) = (1 - \alpha) S^2 - \alpha U + \alpha, (1)
\]

where \( S = \sin(\frac{\eta}{\alpha}), U = \cos(\frac{\eta}{\alpha}), \alpha \in (0, 2) \) and \( \eta \in [0, 1] \).

Figure 1 shows graph of TC Bernstein-like basis functions for different values of shape parameter. Dashed, dotted, thin and thick graphs are corresponding to \( \alpha = 0.5, 1, 1.5 \) and \( 1.9 \), respectively.

![Figure 1 The Trigonometric Cubic Bernstein-like basis functions for different values of \( \alpha \)](fig1.png)

**Theorem 1**: The TC Bernstein-like basis functions hold the following properties:

1. **Non-negativity**: For \( \alpha \in (0, 2) \), \( a_k(\eta) \geq 0 \), \( \forall k = 0, 1, 2, 3 \).
2. **Partition of Unity**: \( \sum_{k=0}^{3} a_k(\eta) = 1 \).
3. **Symmetry**: \( a_k(\eta) = a_{3-k}(1 - \eta), \forall k = 0, 1, 2, 3 \).
4. **Linearly Independent**: \( \sum_{k=0}^{3} \lambda_k a_k(\eta) = 0 \), gives \( \lambda_k = 0 \), \( \forall k = 0, 1, 2, 3 \).
Thus TC Bernstein-like basis functions are linearly independent.

Proof: It can be seen that

(1) It is obvious from the TC Bernstein-like basis functions that they are non negative for \( \eta \in [0, 1] \) and \( \alpha \in (0, 2) \) as shown in Figure 1.

(2) Partition of unity is obvious from the definition.

(3) For \( k = 0 \), \( a_0(\eta) = \alpha(S(\eta))^2 - \alpha S(\eta) + (U(\eta))^2 \). Replacing \( \eta \) by \( 1 - \eta \)

\[ a_0(1 - \eta) = \alpha(S(1 - \eta))^2 - \alpha S(1 - \eta) + (U(1 - \eta))^2 \]

\[ = \alpha(U(\eta))^2 - \alpha U(\eta) + (S(\eta))^2 \]

\[ = \alpha(1 - (S(\eta))^2) - \alpha U(\eta) + (S(\eta))^2 \]

\[ = (1 - \alpha)(S(\eta))^2 - \alpha U(\eta) + \alpha \]

\[ = a_3(\eta) \]

\[ = a_{3-k}(\eta). \]

Now for \( k = 1 \), \( a_1(\eta) = \alpha(S(\eta) - (S(\eta))^2) \). Replacing \( \eta \) by \( 1 - \eta \)

\[ a_1(1 - \eta) = \alpha(S(1 - \eta) - (S(1 - \eta))^2) \]

\[ = \alpha(U(\eta) - (U(\eta))^2) \]

\[ = \alpha(U(\eta) - (1 - (S(\eta))^2)) \]

\[ = \alpha(U(\eta) - 1 + (S(\eta))^2)) \]

\[ = a_2(\eta) \]

\[ = a_{3-k}(\eta). \]

Similarly for \( k = 2, 3 \).

(4) Let \( \lambda_0a_0(\eta) + \lambda_1a_1(\eta) + \lambda_2a_2(\eta) + \lambda_3a_3(\eta) = 0 \). This gives

\[ \lambda_0 - \lambda_1 \alpha + \lambda_2(\alpha + \lambda_3)(S(\eta))^2 + (-\lambda_0 \alpha + \lambda_1 \alpha)(S(\eta)) + \lambda_0(U(\eta))^2 \]

\[ + (\lambda_2 \alpha - \lambda_3 \alpha)(U(\eta)) + (-\lambda_2 \alpha + \lambda_3 \alpha) = 0. \]

Constant term yields \(-\lambda_2 \alpha + \lambda_3 \alpha = 0\). This implies

\[ \lambda_2 = \lambda_3. \]

By putting \( \eta = 0 \) in equation (2), \( \lambda_0 + \lambda_2 \alpha - \lambda_3 \alpha = -\lambda_2 \alpha + \lambda_3 \alpha = 0 \). This implies

\[ \lambda_0 = 0. \]

By putting \( \eta = 1 \) in equation (2), \( \lambda_0 \alpha - \lambda_1 \alpha + \lambda_2 \alpha - \lambda_3 \alpha = \lambda_2 - \lambda_3 \alpha = -\lambda_2 \alpha + \lambda_3 \alpha = 0 \), it gives

\[ \lambda_3 = 0. \]

Then equation (3) yields

\[ \lambda_2 = 0. \]

Derivative of equation (2) along with \( \eta = 0 \) gives \( S(\lambda_0 \alpha - \lambda_1 \alpha) = 0 \), it gives \( \lambda_0 = \lambda_1 \), so \( \lambda_1 = 0 \).

Thus TC Bernstein-like basis functions are linearly independent.

(5) Derivatives of TC Bernstein-like basis functions gives

\[ \frac{da_0(\eta)}{d\alpha} = -S(\eta) + (S(\eta))^2 \leq 0 \]  

\[ \frac{da_1(\eta)}{d\alpha} = S(\eta) - (S(\eta))^2 \geq 0 \]  

\[ \frac{da_2(\eta)}{d\alpha} = U(\eta) + (S(\eta))^2 - 1 \geq 0 \]  

\[ \frac{da_3(\eta)}{d\alpha} = 1 - ((S(\eta))^2 + U(\eta)) \leq 0 \]

Thus equations (4) and (7) show \( a_0(\eta), a_3(\eta) \) are monotonically decreasing and equations (5) and (6) show that \( a_1(\eta), a_2(\eta) \) are monotonically increasing as shown in Fig. 1.
3. The Trigonometric Cubic Bézier-like Curves

The TC Bézier-like curves with single shape parameter are defined as:

\[ B(\eta) = \sum_{k=0}^{3} P_k a_k(\eta), \]

where \( P_k \)'s are control points and \( a_k \)'s are TC Bernstein-like basis functions defined in equation (1).

These TC Bézier-like curves satisfy most of the properties similar to classical Bézier curves which described in following theorem.

**Theorem 2:** The TC Bézier-like curves satisfy the following properties:

(1) **Terminal Properties:** \( B(0) = P_0, B(1) = P_3, B'(0) = \frac{\pi}{2}(P_1 - P_0), B'(1) = \frac{\pi}{2}(P_3 - P_2), B''(0) = \frac{\pi^2}{4}(P_3 - 2P_1 + P_2), B''(1) = \frac{\pi^2}{4}(P_0 - 2P_2 + P_1). \)

(2) **Symmetry:**

\[ B(\eta : \alpha, P_0, P_1, P_2, P_3) = B(1 - \eta : \alpha, P_3, P_2, P_1, P_0), \quad \eta \in [0, 1], \alpha \in (0, 2). \]

(3) **Geometric Invariance:**

The TC Bézier-like curves defined in equation (8) are independent of the choice of coordinates i.e.

\[ B(\eta : \alpha, P_0 \times O, P_1, P_2, P_3) = B(\eta : \alpha, P_0, P_1, P_2, P_3) + O \]

where \( O \) is any vector in \( \mathbb{R}^n \) and \( Y \) is any \( n \times n \) matrix.

(4) **Convex-hull Property:**

The TC Bézier-like curves always lie in the convex hull defined by control points.

**Proof:**

(1) It is straightforward from the definition of TC Bézier-like curves.

(2) As TC Bernstein-like basis functions are symmetric so TC Bézier-like curves are trivially symmetric.

(3) Consider \( P_0 = (a_1, a_2); P_1 = (b_1, b_2); P_2 = (c_1, c_2); P_3 = (d_1, d_2); O = (e_1, e_2). \) Then

\[
B(\eta : \alpha, P_0 + O, P_1 + O, P_2 + O, P_3 + O)
\]

\[
= \left( (\alpha S^2 - \alpha S + U^2)(a_1 + e_1) + (\alpha(S - S^2))(b_1 + e_1) + (\alpha(S^2 + U - 1))(c_1 + e_1) 
+ ((1 - \alpha)S^2 - \alpha U + \alpha)(d_1 + e_1), (\alpha S^2 - \alpha S + U^2)(a_2 + e_2) + (\alpha(S - S^2))(b_2 + e_2) 
+ (\alpha(S^2 + U - 1))(c_2 + e_2) + ((1 - \alpha)S^2 - \alpha U + \alpha)(d_2 + e_2) \right)
\]

\[
= \left( (\alpha S^2 - \alpha S + U^2)a_1 + (\alpha(S - S^2))b_1 + (\alpha(S^2 + U - 1))c_1 + ((1 - \alpha)S^2 - \alpha U + \alpha)d_1 
+ (\alpha S^2 - \alpha S + U^2 + \alpha(S - S^2) + \alpha(S^2 + U - 1) + (1 - \alpha)S^2 - \alpha U + \alpha)e_1, (\alpha S^2 - \alpha S + \alpha U + \alpha) + U^2)a_2 + (\alpha(S^2 + U - 1))c_2 + ((1 - \alpha)S^2 - \alpha U + \alpha)d_2 + (\alpha S^2 - \alpha S + \alpha U + \alpha) + U^2 + \alpha(S - S^2) + \alpha(S^2 + U - 1) + (1 - \alpha)S^2 - \alpha U + \alpha)e_2 \right)
\]

\[
= \left( (\alpha S^2 - \alpha S + U^2)a_1 + (\alpha(S - S^2))b_1 + (\alpha(S^2 + U - 1))c_1 + ((1 - \alpha)S^2 - \alpha U + \alpha)d_1 + e_1, 
(\alpha S^2 - \alpha S + U^2)a_2 + (\alpha(S - S^2))b_2 + (\alpha(S^2 + U - 1))c_2 + ((1 - \alpha)S^2 - \alpha U + \alpha)d_2 + e_2 \right)
\]

\[
= \left( (\alpha S^2 - \alpha S + U^2)a_1 + (\alpha(S - S^2))b_1 + (\alpha(S^2 + U - 1))c_1 + ((1 - \alpha)S^2 - \alpha U + \alpha)d_1, 
(\alpha S^2 - \alpha S + U^2)a_2 + (\alpha(S - S^2))b_2 + (\alpha(S^2 + U - 1))c_2 + ((1 - \alpha)S^2 - \alpha U + \alpha)d_2 \right) + (e_1, e_2)
\]

\[ = B(\eta : \alpha, P_0, P_1, P_2, P_3) + O \]
Now let \( Y = \begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix} \), and consider
\[
B(q : a, P_0 \times Y, P_1 \times Y, P_2 \times Y, P_3 \times Y)
\]
\[
= \left( (aS^2 - aS + U^2)(a_1 q_1 + 2a_2 q_2) + (a(S^2 - aS + U - 1))((c_1 q_1 + c_2 q_2) + (a(S^2 + U - 1))(c_1 q_1 + c_2 q_2)
\right.
\]
\[
+ ((1 - a)S^2 - aU + a)(d_1 q_1 + d_2 q_2) + (a(S^2 + U - 1))(c_1 q_2 + c_2 q_2))
\]
\[
= \left( ((aS^2 - aS + U^2)a_1 + (a(S^2 - aS + U - 1))c_1 + ((1 - a)S^2 - aU + a)(d_1 q_1 + d_2 q_2)
\right.
\]
\[
+ ((aS^2 - aS + U^2)a_2 + (a(S^2 - aS + U - 1))c_2 + ((1 - a)S^2 - aU + a)(d_2 q_2 + d_3 q_3)
\right.
\]
\[
+ ((aS^2 - aS + U^2)a_3 + (a(S^2 - aS + U - 1))c_3 + ((1 - a)S^2 - aU + a)(d_3 q_3 + d_4 q_4)
\right.
\]
\[
= \left( ((aS^2 - aS + U^2)a_1 + (a(S^2 - aS + U - 1))c_1 + ((1 - a)S^2 - aU + a)(d_1 q_1 + d_2 q_2)
\right.
\]
\[
+ ((aS^2 - aS + U^2)a_2 + (a(S^2 - aS + U - 1))c_2 + ((1 - a)S^2 - aU + a)(d_2 q_2 + d_3 q_3)
\right.
\]
\[
+ ((aS^2 - aS + U^2)a_3 + (a(S^2 - aS + U - 1))c_3 + ((1 - a)S^2 - aU + a)(d_3 q_3 + d_4 q_4)
\right.
\]
\[
= \left( ((aS^2 - aS + U^2)a_1 + (a(S^2 - aS + U - 1))c_1 + ((1 - a)S^2 - aU + a)(d_1 q_1 + d_2 q_2)
\right.
\]
\[
+ ((aS^2 - aS + U^2)a_2 + (a(S^2 - aS + U - 1))c_2 + ((1 - a)S^2 - aU + a)(d_2 q_2 + d_3 q_3)
\right.
\]
\[
+ ((aS^2 - aS + U^2)a_3 + (a(S^2 - aS + U - 1))c_3 + ((1 - a)S^2 - aU + a)(d_3 q_3 + d_4 q_4)
\right.
\]
\[
= \left( ((aS^2 - aS + U^2)a_1 + (a(S^2 - aS + U - 1))c_1 + ((1 - a)S^2 - aU + a)(d_1 q_1 + d_2 q_2)
\right.
\]
\[
+ ((aS^2 - aS + U^2)a_2 + (a(S^2 - aS + U - 1))c_2 + ((1 - a)S^2 - aU + a)(d_2 q_2 + d_3 q_3)
\right.
\]
\[
+ ((aS^2 - aS + U^2)a_3 + (a(S^2 - aS + U - 1))c_3 + ((1 - a)S^2 - aU + a)(d_3 q_3 + d_4 q_4)
\right.
\]
\[
= B(q : a, P_0, P_1, P_2, P_3) \times Y
\]

This property can also be visualized by the Fig. 2. Figure 2(a) shows that black is first curve \( B(q) \) with control points \( P_k, k = 0, 1, 2, 3 \). Blue curve is obtained by adding a point \( O = (3, 9) \) in \( B(q) \). Yellow dotted circle on the blue circle is obtained by the control points \( P_k, k = 0, 1, 2, 3 \). Similarly, Fig. 2(b) shows that black is first curve \( B(q) \) with control points \( P_k, k = 0, 1, 2, 3 \). Blue curve is represented by the control points \( P_k \ast Y, k = 0, 1, 2, 3 \). Yellow dotted curve printed on the blue curve is obtained by multiplying \( B(q) \) with a matrix \( Y = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \).

(4) As TC Bernstein-like basis functions are non-negative and add up to one. So TC Bézier-like curves always lie in convex-hull defined by their polygon. Figure 3 shows TC Bézier-like curves lie in convex-hull of their control polygon.

**Theorem 3:** Suppose \( P_k, k = 0, 1, 2, 3 \) are non-collinear and \( P' = \frac{P_1 + P_3}{2} \), the relationship between TC-Bézier-like curve \( B(q) \) and classical Bézier curve \( C(q) \) is given as

\[
(1) B(0) = C(0),
\]
\[
(2) B(1) = C(1),
\]
\[
(3) \text{if } \frac{1}{2}(1 + \sqrt{2}) \leq a < 2, \text{ then } B(\frac{1}{2} - P') \leq C(\frac{1}{2}) - P'.
\]

**Proof:** It is simple from the definition that \( B(0) = P_0 = C(0) \) and \( B(1) = P_3 = C(1) \).

Let \( B(\frac{1}{2} - P') \leq C(\frac{1}{2}) - P' \), it yields

\[
\left( \frac{1}{2} + \frac{a}{2 \sqrt{2}} \right)(P_0 - P_1 - P_2 - P_3) \leq \frac{1}{8}(P_0 - P_1 - P_2 - P_3).
\]

(12)
Solving Eq. (12) for $\alpha$, we obtain

$$2 > \alpha \geq \frac{3}{4}(1 + \sqrt{2}).$$

Figure 4 shows that TC Bézier-like curves (green) are more nearer to control points $P_1$ and $P_2$ as compared to classical Bézier curve (blue). Note that when $\alpha = \frac{3}{4}(1 + \sqrt{2})$ it becomes $B(\frac{1}{2}) = C(\frac{1}{2})$.

4. Effects of Shape Parameter

As shape parameter $\alpha$ has different effects on TC Bernstein-like basis functions i.e. $a_0(\eta)$, $a_3(\eta)$ are monotonically decreasing and $a_1(\eta)$, $a_2(\eta)$ are monotonically increasing with respect to $\alpha$. In case of TC Bézier-like curves, we can get curves of different types by keeping the control points fixed and taking variations only in shape parameter $\alpha$. In Fig. 5, we have shown some effects of $\alpha$ on TC Bézier-like curve. Here black, green, yellow and red curves are constructed by using $\alpha = 0.5, 1, 1.5$ and $1.9$, respectively.
**Theorem 4:** For $\alpha = 0$, the TC Bézier-like curve defined in Eq. (8) becomes a straight line.

**Proof:** For $\alpha = 0$, the TC Bernstein-like basis functions become
\[
a_0(\eta) = U^2, \quad a_1(\eta) = 0, \quad a_2(\eta) = 0, \quad a_3(\eta) = S^2.
\] (13)

Thus, the curve (8) reduces to
\[
B(\eta) = a_0(\eta)P_0 + a_3(\eta)P_3.
\] (14)

This gives a straight line between point $P_0$ and $P_3$.

**Remark 1:** For collinear points $P_0, P_1, P_2$ and $P_3$ the TC Bézier-like curve gives a straight line from $P_0$ to $P_3$.

\[\begin{align*}
\text{Figure 5} & \quad \text{Effects of shape parameter to TC-Bézier-like curve}
\end{align*}\]

### 5. Representation of Some Conics

**Theorem 5:** For control points $P_0 = (a, 0), P_1 = (a, a) = P_2, P_3 = (0, a)$ and $\alpha = 1$, TC Bézier-like curve represents a circular arc.

**Proof:** Taking $\alpha = 1$ and $P_0 = (a, 0), P_1 = (a, a) = P_2, P_3 = (0, a)$, Eq. (8) becomes
\[
B(\eta) = (x(\eta), y(\eta)) = [a \cos(\frac{\pi \eta}{2}), a \sin(\frac{\pi \eta}{2})].
\] (15)

**Theorem 6:** Let the control points $P_0 = (0, 0), P_1 = (2a, 0), P_2 = P_3 = (2a, a)$ and $\alpha = 1$, the TC Bézier-like curve represents a parabolic arc.

**Proof:** Taking $\alpha = 1$ and $P_0 = (0, 0), P_1 = (2a, 0), P_2 = P_3 = (2a, a)$, Eq. (8) yields
\[
B(\eta) = (x(\eta), y(\eta)) = [2a \sin(\frac{\pi \eta}{2}), a \sin^2(\frac{\pi \eta}{2})].
\] (16)

**Theorem 7:** For control points $P_0 = (a, 0), P_1 = (a, b) = P_2, P_3 = (0, b)$ and $\alpha = 1$, the TC Bézier-like curve represents an elliptic arc.

**Proof:** Taking $\alpha = 1$ and $P_0 = (a, 0), P_1 = (a, b) = P_2, P_3 = (0, b)$, Eq. (8) becomes
\[
B(\eta) = (x(\eta), y(\eta)) = [a \cos(\frac{\pi \eta}{2}), b \sin(\frac{\pi \eta}{2})].
\] (17)

Figure 6 shows some conic TC Bézier-like curves with shape parameter. Fig.6(a), Fig.6(b) and Fig.6(c) depict the some conic arcs like circle, parabola and ellipse using TC Bézier-like curves, respectively.
6. The Continuity of TC Bézier-like Curves

In practice, some complex curve cannot be designed by a single curve. Method of piecewise helps us to construct such kind of curves. In this section, discussion on parametric and geometric continuity between two adjacent segments of TC Bézier-like curves is given.

**Theorem 8:** For two segments of TC Bézier-like curves \( B(\eta, \alpha) = \sum_{k=0}^{3} P_k a_k(\eta) \) and \( B_1(\eta, \alpha_1) = \sum_{k=0}^{3} Q_k a_k(\eta) \) with the control points \( P_k, k = 0, 1, 2, 3 \) and \( Q_k, k = 0, 1, 2, 3 \), respectively, the necessary and sufficient conditions of parametric continuities are given by

\[
\begin{align*}
(1) & \quad P_3 = Q_0, \text{ for } C^0 \text{ continuity.} \\
(2) & \quad P_3 = Q_0, Q_1 = \frac{-aP_2 + \alpha P_3 + \alpha_1 P_1}{\alpha_1}. \\
(3) & \quad Q_2 = \frac{1}{\alpha_1} \left( 2P_0 - \alpha P_0 + \alpha P_1 - 4\alpha P_2 + 4\alpha P_3 - 2Q_3 + \alpha_1 Q_1 \right).
\end{align*}
\]

where \( \alpha, \alpha_1 \) are shape parameters of first and second curves, respectively.

**Proof:** Given two curve segments \( B(\eta, \alpha) \) and \( B_1(\eta, \alpha_1) \) with control points \( P_k, k = 0, 1, 2, 3 \) and \( Q_k, k = 0, 1, 2, 3 \), respectively. Then

\[
\begin{align*}
(1) & \quad \text{For } C^0 \text{ continuity condition } B(1) = B_1(0), \text{ it gives } P_3 = Q_0. \\
(2) & \quad \text{For } C^1 \text{ continuity condition } B'(1) = B_1'(0), \text{ it yields}
\end{align*}
\]

\[
Q_1 = \frac{-aP_2 + \alpha P_3 + \alpha_1 P_1}{\alpha_1}.
\]

Solving it for \( Q_1 \)

\[
\begin{align*}
(3) & \quad \text{For } C^2 \text{ continuity condition } B''(1) = B_1''(0), \text{ it gives}
\end{align*}
\]

\[
\begin{align*}
\left( \frac{\pi^2}{2} + \frac{\pi^2 a}{4} \right) P_0 + \frac{\pi^2 a}{4} P_1 - \frac{\pi^2 a}{2} P_2 - \frac{1}{2} \pi^2 (1-\alpha) P_3 = \left( -\frac{\pi^2}{2} + \frac{\pi^2 a}{4} \right) Q_0 - \frac{\pi^2 a}{2} Q_1 + \frac{\pi^2 a}{4} Q_2 + \left( \frac{1}{2} \pi^2 (1-\alpha_1) + \frac{\pi^2 a_1}{4} \right) Q_3.
\end{align*}
\]

Simplification for \( Q_2 \),

\[
Q_2 = \frac{1}{\alpha_1} \left( 2P_0 - \alpha P_0 + \alpha P_1 - 4\alpha P_2 + 4\alpha P_3 - 2Q_3 + \alpha_1 Q_1 \right).
\]

**Theorem 9:** For two segments of TC Bézier-like curves \( B(\eta, \alpha) = \sum_{k=0}^{3} P_k a_k(\eta) \) and \( B_1(\eta, \alpha_1) = \sum_{k=0}^{3} Q_k a_k(\eta) \) with the control points \( P_k, k = 0, 1, 2, 3 \) and \( Q_k, k = 0, 1, 2, 3 \), respectively, the necessary and sufficient conditions of geometric continuities are given by...
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(1) \( P_3 = Q_0 \), for \( G^0 \) continuity.

(2) For \( G^1 \) continuity

\[
P_3 = Q_0, \quad Q_1 = \frac{-aP_2 + aP_3 + \alpha_1 P_3}{\mu\alpha_1}.
\]

(3) For \( G^2 \) continuity

\[
Q_2 = \frac{1}{\pi\alpha_1 \mu^2} \left( 2\pi P_0 - \pi\alpha P_0 + \pi\alpha P_1 + 2\alpha P_2 - 2\pi\alpha P_2 - 2\pi\alpha\mu P_2 - 2\pi P_3 - 2\alpha P_3 + 2\pi\alpha P_3 + 2\mu P_3 \right)
\]

\[
- 2\pi\mu^2 Q_2 + \pi\alpha^2 \alpha_1 Q_3 \right).
\]

where \( a, \alpha_1 \) are shape parameters of first and second curve respectively and \( \mu \) is a scale factor introduced in \( G^1 \) and \( G^2 \) continuities.

Proof: Given two curve segments \( B(\eta, \alpha) \) and \( B_1(\eta, \alpha_1) \) with control points \( P_k, k = 0, 1, 2, 3 \) and \( Q_k, k = 0, 1, 2, 3 \), respectively. Then

(1) For \( G^0 \) continuity condition \( B(1) = B_1(0) \) yields \( P_3 = Q_0 \).

(2) \( G^1 \) continuity condition \( B(1) = \mu B_1(0) \) yields \( \frac{1}{2}\pi\alpha P_2 + \frac{1}{2}\pi\alpha P_3 = \mu(\frac{1}{2}\pi\alpha_1 Q_0 + \frac{1}{2}\pi\alpha_1 Q_1) \).

Solving it for \( Q_1 \)

\[
Q_1 = \frac{-aP_2 + aP_3 + \alpha_1 P_3}{\alpha_1 \mu}.
\]

(3) If both curves \( B(\eta, \alpha) \) and \( B_1(\eta, \alpha_1) \) connect by \( G^2 \)-continuity, they need to connect first by \( G^1 \)-continuity which means

\[
P_3 = Q_0, \quad B_1(1) = \mu B_1(0), \quad \mu > 0
\]

Suppose that the vice normal vector \( N_1 = B_1(1) \times B_1(1) \) for \( B(\eta, \alpha) \) at \( \eta = 1 \) and reverse normal vector \( N_2 = B_1(0) \times B_1(0) \) for \( B_1(\eta, \alpha_1) \) at \( \eta = 1 \), then we have,

\[
N_1 = B_1(1) \times B_1(1), \quad \text{and} \quad N_2 = B_1(0) \times B_1(0).
\]

Both vice normal vectors \( N_1 \) and \( N_2 \) must have same direction at joint to achieve the \( G^2 \)-continuity. Four vectors \( B_1(1), \ B_1(1), \ B_1(0), \ B_1(0) \) become coplanar by combining equations (22) and (23). Thus we obtain,

\[
B_1'(1) = \delta B_1'(0) + \mu B_1'(0), \ \delta > 0.
\]

As \( G^2 \)-continuity is achieved if both curvatures \( \kappa(\eta) \) and \( \kappa_1(\eta) \) for the curves \( B(\eta, \alpha) \) and \( B_1(\eta, \alpha_1) \) have same value at joint point, i.e,

\[
\kappa(1) = \kappa_1(0).
\]

Using equations (22-24), we obtain

\[
\kappa(1) = \frac{|B_1(1) \times B_1'(1)|}{|B_1(1)|^3} = \frac{|\delta B_1(0) \times (\delta B_1'(0) + \mu B_1'(0))|}{(\delta)^3|B_1'(0)|^3} = \frac{|\delta B_1(0) \times B_1'(0)|}{(\delta)^2|B_1'(0)|^3} = \kappa_1(0),
\]

for \( \delta = \mu^2 \). Substituting this value in equation (24), we have

\[
B_1'(1) = \mu^2 B_1'(0) + \mu B_1'(0).
\]

Equation(25) yields

\[
\left( \frac{\pi^2}{2} - \frac{\pi^2 \alpha}{4} \right) P_0 + \left( \frac{\pi^2 \alpha}{4} - \frac{\pi^2 \alpha}{2} \right) P_1 - \frac{1}{2} \pi^2 (1 - \alpha) P_3
\]

\[
= \mu \left( \left( \frac{\pi^2}{2} + \frac{\pi^2 \alpha}{4} \right) Q_0 + \frac{\pi^2 \alpha}{2} Q_1 + \frac{\pi^2 \alpha}{4} Q_2 + \left( \frac{1}{2} \pi^2 (1 - \alpha) + \frac{\pi^2 \alpha}{4} \right) Q_3 \right) + \mu \left( \frac{1}{2} \pi^2 \alpha_1 Q_0 + \frac{1}{2} \pi^2 \alpha_1 Q_1 \right).
\]

Solving it for \( Q_2 \),

\[
Q_2 = \frac{1}{\pi\alpha_1 \mu^2} \left( 2\pi P_0 - \pi\alpha P_0 + \pi\alpha P_1 + 2\alpha P_2 - 2\pi\alpha P_2 - 2\alpha P_3 + 2\pi\alpha P_3 + 2\mu^2 P_3 - 2\mu^2 Q_3 + \pi^2 Q_1 \right).
\]

Figure 7 shows the TC Bézier continuity curves using \( C^2 \) and \( G^2 \) constraints. In Fig. 7(a), the continuity curves are shown as dotted and thick curves using \( C^2 \)-continuity conditions. Similarly, Fig. 7(b) is constructed with \( G^2 \)-continuity conditions.
7. Some Engineering Applications

In this section, some complex curves are constructed by using parametric and geometric continuity conditions of TC Bézier-like curves. However main focus is to construct curves by $G^2$-continuity with suitable control points and shape parameter. Note down that $\alpha$, $\alpha_1$ and $\mu$ show shape parameter for initial curve, continuity conditioned curve and scale factor throughout the section.

7.1. Applications of $C^1$ and $G^1$ continuities

Figure 8(a) shows first example of curve modeling, Arabic alphabet Bay is constructed by using $G^1$-continuity of TC Bézier-like curves. Black dotted curves are the initial curves and red dotted curves are constructed by using continuity conditions. Here shape parameter $\alpha = \alpha_1 = 1.99$ and $\mu = 3$ in both upper and lower curves of Bay. Figure 8(b) shows an example of font designing, in which we have modeled an alphabet S by using $C^1$-continuity. The right face of S is constructed by 4 continuity curves as red, yellow, green and blue with shape parameters 1.99, 0.50, 1.99 and 1.80, respectively. The left face of S is constructed by 5 continuity curves named as red, green, blue, green and red (from bottom) with shape parameter 1.73, 1, 1.99, 1.99 and 1.99, respectively

7.2. Applications of $G^2$ continuities

Figure 9 shows another example of font designing, where English alphabet B is constructed by applying $G^2$-continuity of TC Bézier-like curves twice with shape parameter $\alpha = \alpha_1 = 1.99$ and $\mu = 0.7$. First and second
curves are shown as green and red respectively in both upper and lower segments of English alphabet B. In Fig. 10, we designed a flower by using $G^2$-continuity twice, once in the right half with $\alpha = \alpha_1 = 1.99$ and $\mu = 1.07$, secondly in the left half with $\alpha = \alpha_1 = 1.99$ and $\mu = 1$. Figure 10(b) shows black as first curve and red as continuity curves in both parts. Figure 11(b) shows an example of mouse modeling. Mouse is constructed by using $G^2$-continuity with shape parameter $\alpha = 1.05$, $\alpha_1 = 1.99$ and continuity constant $\mu = 0.8$.

Following two figures show examples of free form complex curve modeling. In Fig. 12 we have modeled a cetacean by using $G^2$-continuity with shape parameters $\alpha = 1.8$, $\alpha_1 = 1.99$ and $\mu = 1.64$. Figure 13 shows modeling of fish by using $G^2$-continuity twice. Upper half is constructed with the help of shape parameters $\alpha = 1.8$, $\alpha_1 = 1.99$ and $\mu = 3$, where as lower half used $\alpha = 1.75$, $\alpha_1 = 0.7$ and $\mu = 2.1$. In both upper and lower half initial curves are shown as black and continuity curves are shown as red.
8. Shape Adjustment of Trigonometric Cubic Bézier-like Curves with Parametric and Geometric Continuities

Some influence of shape parameter on continuity conditions and shape of curve are given in the following propositions.

**Proposition 1**: For a composite TC Bézier-like curve with continuity conditions, some significant effects (Locally) can be obtained by changing scale factor $\mu$ and keeping control points fixed along with continuity conditions.

Figure 14(a) shows some influences of scale factor on shapes of rat. Green, Red and Blue curves are modeled with scale factor $\mu = 0.7, 0.8$ and $0.9$, respectively.

**Proposition 2**: For a composite TC Bézier-like curve with continuity conditions, some significant effects (Globally) can be obtained by changing shape parameter $\alpha$ and scale factor $\mu$ while keeping control points fixed along with continuity conditions.

Figure 14(b) shows some influences of scale factor $\mu$ and shape parameter $\alpha$ on shape of cetacean. Blue, Green and Red curves are constructed with $\alpha = 1.7, 1.8, 1.9$ scale factor $\mu = 1.9, 1.64$ and $1.94$, respectively.

9. Conclusion

As complex curves can not be constructed by classical Bézier curves, therefore, TC Bézier-like curves with single shape parameter have been constructed to overcome this problem. Geometric properties of TC Bernstein-like basis functions and TC Bézier-like curve have been discussed. Smooth continuity conditions between two adjacent segments of TC Bézier-like curves have also been derived. As applications of these curves, some conic curves have been constructed by choosing appropriate points and some complex curves by using continuity conditions.
Figure 14  Shape adjustment and $G^2$ continuities

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