EMBEDDING THEOREMS FOR FLEXIBLE VARIETIES

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Abstract. Let $Z$ be an affine algebraic variety and $X$ be a smooth flexible variety. We develop some criteria under which $Z$ admits a closed embedding into $X$. In particular, we show that if $\dim X \geq \max(2 \dim Z + 1, \dim TZ)$ and $X$ is isomorphic (as an algebraic variety) to a special linear group or to a symplectic group, then $Z$ admits a closed embedding into $X$.

1. Introduction

All algebraic varieties which appear in this paper are considered over an algebraically closed field $k$ of characteristic zero. If $Z$ is an affine algebraic variety and $TZ$ is its Zariski tangent bundle then we call $ED(Z) = \max(2 \dim Z + 1, \dim TZ)$ the embedding dimension of $Z$. Holme’s theorem [Hol, Theorem 7.4] (later rediscovered in [Ka91] and [Sr]) states that $Z$ admits a closed embedding into any affine space $\mathbb{A}^n$ with $n \geq ED(X)$. In the smooth case (when $ED(Z) = 2 \dim Z + 1$) this fact was proven earlier by Swan [Swan, Theorem 2.1]. The latter result is sharp - examples of smooth irreducible $d$-dimensional affine algebraic varieties with $d \geq \frac{n}{2}$ such that they do not admit closed embeddings in $\mathbb{A}^n$ were constructed in [BMS]. Recently Feller and van Santen [FvS21] proved that if $X$ is an affine variety isomorphic to a simple linear algebraic group and $Z$ is smooth, then $Z$ admits a closed embedding into $X$, provided that $\dim X > ED(Z)$. They also proved that for every $n$-dimensional algebraic group $G$ (with $n > 0$) there exist smooth irreducible $d$-dimensional affine algebraic varieties with $d \geq \frac{n}{2}$ such that they do not admit closed embeddings in $G$ [FvS21, Corollary 4.4]. In particular, their embedding result is optimal if the dimension of $X$ is even. However, they did not know whether their result is sharp in the case the dimension of $X$ is odd and a specific question posed in [FvS21] asks whether a smooth affine algebraic variety of dimension 7 can be embedded properly into $SL_4(k)$. We consider a more general situation. Namely, starting from dimension 2 affine spaces and linear algebraic groups without nontrivial characters are examples of so-called flexible varieties. Recall that a normal quasi-affine variety $X$ of dimension at
least 2 is flexible if $\text{SAut}(X)$ acts transitively on the smooth part $X_{\text{reg}}$ of $X$ where $\text{SAut}(X)$ is the subgroup of the group $\text{Aut}(X)$ of algebraic automorphisms of $X$ generated by all one-parameter unipotent subgroups (in what follows one-parameter unipotent groups will be called $\mathbb{G}_a$-groups and $\mathbb{G}_a^m$ will stand for the $m$-th power of a $\mathbb{G}_a$-group). The main results of this paper are the following.

**Theorem 1.1.** Let $X$ be a smooth flexible variety equipped with a $\mathbb{G}_a^m$-action such that the minimal dimension of its orbits is $n$. Suppose that $Z$ is an affine variety such that $\dim Z \leq n$ and $\text{ED}(Z) \leq \dim X$. Then there exists a closed embedding of $Z$ into $X$.

**Theorem 1.2.** Let $X$ be isomorphic (as an algebraic variety) to a connected linear algebraic group $G \neq \mathbb{G}_a$ without nontrivial characters. Suppose that $G' \cong \mathbb{G}_a^{m'}$ and $G'' \cong \mathbb{G}_a^{m''}$ are subgroups of $G$ such that $G' \cap G''$ coincides with the identity element of $G$. Let $Z$ be an affine algebraic variety such that $\dim Z \leq m' + m''$ and $\text{ED}(Z) \leq \dim X$. Then there exists a closed embedding of $Z$ into $X$.

Theorems 1.1 and 1.2 imply the following.

**Corollary 1.3.** Let $X$ be a smooth flexible variety equipped with a free $\mathbb{G}_a^l$-action. Let $Z$ be an affine algebraic variety of dimension at most $n + l$ such $\dim X + n \geq \text{ED}(Z)$. Suppose that $\psi : X \times \mathbb{A}^n \to Y$ is a finite morphism onto a normal variety $Y$ and $S$ is a closed subvariety of $Y$ such that it contains $Y_{\text{sing}}$ and $\dim Z < \text{codim}_Y S$. Then $Z$ admits a closed embedding into $Y$ with the image contained in $Y \setminus S$.

**Corollary 1.4.** Let $X$ be isomorphic (as an algebraic variety) either to a special linear group $\text{SL}_n(k)$ or to a symplectic group $\text{Sp}_{2n}(k)$ and $Z$ be an affine algebraic variety such that $\text{ED}(Z) \leq \dim X$. Then there exists a closed embedding of $Z$ into $X$.

In particular, the question of Feller and van Santen has a positive answer. Corollary 1.4 can be extended to semi-simple Lie groups whose Lie algebras are direct sums of simple Lie algebras with Dynkin diagrams $A_n$ or $C_n$. In fact, we have more.

**Corollary 1.5.** Let $Z$ be an affine algebraic variety, $X$ be an algebraic variety of the form $\mathbb{A}^{n_0} \times G_1 \times G_2 \times \ldots \times G_l$ where each $G_i$ is either $\text{SL}_{m_i}(k)$ or $\text{Sp}_{2m_i}(k)$. Suppose that $\varphi : X \to Y$ is a finite morphism into a normal variety $Y$, $\text{ED}(Z) \leq \dim Y$ and $S$ is a closed subvariety of $Y$ containing $Y_{\text{sing}}$ such that $\dim Z < \text{codim}_Y S$. Then $Z$ admits a closed embedding into $Y$ with the image contained in $Y \setminus S$.

The proofs of Theorems 1.1 and 1.2 are heavily based on the theory of flexible varieties and the technique developed in [AFKKZ], [Ka20], [KaUd] and [Ka21] whose survey can be found in Section 2. As a part of this survey we describe injective immersions of affine algebraic varieties.
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into smooth flexible varieties. In section 3 we consider a surjective morphism \( \varphi : A^t \to X \) (every flexible variety \( X \) admits such morphism) and for a closed subvariety \( Z \) of \( A^t \) we develop a criterion of properness of the morphism \( \varphi|_Z : Z \to X \). Checking the validity of the criterion for injective immersions under the assumptions of Theorems 1.1 and 1.2 we prove these theorems in sections 4 and 5.

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2. Flexible varieties

Let us start with the main definitions for the theory of flexible varieties.

**Definition 2.1.** (1) Given an irreducible algebraic variety \( A \) and a map \( \varphi : A \to \text{Aut}(X) \) we say that \((A, \varphi)\) is an algebraic family of automorphisms of \( X \) if the induced map \( A \times X \to X \), \((\alpha, x) \mapsto \varphi(\alpha).x\) is a morphism (see [Ra]).

(2) If we want to emphasize additionally that \( \varphi(A) \) is contained in a subgroup \( G \) of \( \text{Aut}(X) \), then we say that \( A \) is an algebraic \( G \)-family of automorphisms of \( X \).

(3) In the case when \( A \) is a connected algebraic group and the induced map \( A \times X \to X \) is not only a morphism but also an action of \( A \) on \( X \) we call this family a connected algebraic subgroup of \( \text{Aut}(X) \).

(4) Following [AFKKZ, Definition 1.1] we call a subgroup \( G \) of \( \text{Aut}(X) \) algebraically generated if it is generated as an abstract group by a family \( G \) of connected algebraic subgroups of \( \text{Aut}(X) \).

**Definition 2.2.** (1) A nonzero derivation \( \delta \) on the ring \( A \) of regular functions on an affine algebraic variety \( X \) is called locally nilpotent if for every \( a \in A \) there exists a natural \( n \) for which \( \delta^n(a) = 0 \). This derivation can be viewed as a vector field on \( X \) which we also call locally nilpotent. The set of all locally nilpotent vector fields on \( X \) will be denoted by \( \text{LND}(X) \). The flow of \( \delta \in \text{LND}(X) \) is an algebraic \( \mathbb{G}_a \)-action on \( X \), i.e., the action of the group \((\mathbb{k}, +)\) which can be viewed as a one-parameter unipotent group \( U \) in the group \( \text{Aut}(X) \) of all algebraic automorphisms of \( X \). In fact, every \( \mathbb{G}_a \)-action is a flow of a locally nilpotent vector field (e.g., see [Fr, Proposition 1.28]).

(2) If \( X \) is a quasi-affine variety, then an algebraic vector field \( \delta \) on \( X \) is called locally nilpotent if \( \delta \) extends to a locally nilpotent vector field \( \tilde{\delta} \) on some affine algebraic variety \( Y \) containing \( X \) as an open subset such that \( \tilde{\delta} \) vanishes on \( Y \setminus X \) where \( \text{codim}_C(Y \setminus X) \geq 2 \). Note that under this assumption \( \delta \) generates a \( \mathbb{G}_a \)-action on \( X \) and we use again the notation \( \text{LND}(X) \) for the set of all locally nilpotent vector fields on \( X \).
Definition 2.3. (1) For every locally nilpotent vector fields $\delta$ and each function $f \in \text{Ker} \, \delta$ from its kernel the field $f\delta$ is called a replica of $\delta$. Recall that such a replica is automatically locally nilpotent.

(2) Let $\mathcal{N}$ be a set of locally nilpotent vector fields on $X$ and $G_N \subset \text{Aut}(X)$ denotes the group generated by all flows of elements of $\mathcal{N}$. We say that $G_N$ is generated by $\mathcal{N}$.

(3) A collection of locally nilpotent vector fields $\mathcal{N}$ is called saturated if $\mathcal{N}$ is closed under conjugation by elements in $G_N$ and for every $\delta \in \mathcal{N}$ each replica of $\delta$ is also contained in $\mathcal{N}$.

Definition 2.4. Let $X$ be a normal quasi-affine algebraic variety of dimension at least 2, $\mathcal{N}$ be a saturated set of locally nilpotent vector fields on $X$ and $G_N = G_N \subset \text{Aut}(X)$ denotes the group generated by all flows of elements of $\mathcal{N}$. We say that $G_N$ is generated by $\mathcal{N}$.

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Notation 2.5. Further in this paper $X$ is always a smooth quasi-affine variety and $G$ is a group acting transitively on $X$ such that $G$ is algebraically generated by a collection $\mathcal{G}$ of connected algebraic subgroups of $G$. Given a sequence $\mathcal{H} = (H_1, \ldots, H_s)$ of elements of $\mathcal{G}$ we consider the map

$$
(1) \quad \Phi_{\mathcal{H}} : H \times X \longrightarrow X \times X, \quad (h_s, \ldots, h_1, x) \mapsto ((h_s \cdot \ldots \cdot h_1).x, x)
$$

where $H = H_s \times \ldots \times H_1$. By $\varphi_{\mathcal{H}} : H \longrightarrow X$ we denote the restriction of $\Phi_{\mathcal{H}}$ to $H \times x_0$ where $x_0$ is a fixed point of $X$.

Proposition 2.6. Suppose that $\mathcal{G}$ is closed under conjugation by $G$.

Then a sequence $\mathcal{H} = (H_1, \ldots, H_s)$ can be chosen so that for a dense open subset $U$ of $H$ the morphism $\Phi_{\mathcal{H}}$ is smooth on $U \times X$ (in particular, $\varphi_{\mathcal{H}}$ is smooth on $U$).

(2) Let $\mathcal{H} = (H_1, \ldots, H_s)$ be as in (1) and $H$ be any element $\mathcal{G}$. Then the sequence $H_1, \ldots, H_m, H$ (resp. $H, H_1, \ldots, H_m$) satisfies the conclusions of (1) as well.

(3) Furthermore, increasing the number of elements in $\mathcal{H}$ one can suppose that the codimension of $H \setminus U$ in $H$ is arbitrarily large.

Proof. The first statement follows from [AFKKZ, Proposition 1.16], the second statement follows from [Ka20, Proposition 1.10]) and the third one from [AFKKZ, p. 778, footnote].

We shall use the notion of a perfect (algebraic) $G$-family of automorphisms of $X$ (see [Ka21, Definition 2.7]). Without stating the formal definition of such families we need to emphasize some of their properties.
Proposition 2.7. ([Ka21, Proposition 2.8]) Let $\mathcal{A}$ be a perfect $G$-family of automorphisms of a smooth $G$-flexible variety $X$ and $H_0 \in \mathcal{G}$. Then $H_0 \times \mathcal{A}$ and $\mathcal{A} \times H_0$ are also perfect $G$-families of automorphisms of $X$. Furthermore, $\mathcal{A}$ satisfies the transversality theorem ([AFKKZ, Theorem 1.15], see also [Ka21, Theorem 2.2]), e.g., if $Z$ and $W$ are subvarieties of $X$ with $\dim Z + \dim W < \dim X$, then one has $\alpha(Z) \cap W = \emptyset$ for a general $\alpha \in \mathcal{A}$.

Theorem 2.8. Let $X$ be a smooth quasi-affine $G$-flexible variety, $\mathcal{A}$ be a perfect $G$-family of automorphisms of $X$, $Q$ be a normal algebraic variety and $\varphi : X \to Q$ be a dominant morphism. Suppose that $Q_0$ is a smooth open dense subset of $Q$, $X_0$ is an open subset of $X$ contained in $\varphi^{-1}(Q_0)$ and

\[ X_0 \times_{Q_0} X_0 = 2 \dim X - \dim Q. \]

Let $Y$ be the closure of $\bigcup_{x \in X_0} \ker \{ \varphi_* : T_x X_0 \to T_{\varphi(x)} Q_0 \}$ in $TX$ and

\[ \dim Y = 2 \dim X - \dim Q. \]

Then $Y$ is an open subset of $X$ and $\dim Z < \codim_{\varphi^{-1}(Q_0)}(\varphi^{-1}(Q_0) \setminus X_0)$. Then for a general element $\alpha \in \mathcal{A}$ the morphism $\varphi|_{\alpha(Z) \cap X_0} : \alpha(Z) \cap X_0 \to Q_0$ is an injective immersion.

Proof. In the case of $X_0 = \varphi^{-1}(Q_0)$ the statement is the combination of [Ka21, Theorem 2.6] and [Ka21, Proposition 2.8(5)]. In the general case the proof goes without change if one observes that $\alpha(Z)$ does not meet $\varphi^{-1}(Q_0) \setminus X_0$ for a general $\alpha \in \mathcal{A}$ by the transversality theorem. \hfill \Box

Proposition 2.9. Let the assumptions and conclusions of Proposition 2.6 hold. Suppose that $H$ itself is an $F$-flexible variety. Let $Z$ be a locally closed reduced subvariety of $H$ with $\dim Z \leq \dim X$ (and by the conclusions of Proposition 2.6 with $\dim Z < \codim_H(H \setminus U)$). Then for a general element $\beta \in \mathcal{B}$ in any perfect $F$-family $\mathcal{B}$ of automorphisms of $H$ the morphism $\varphi_H|_{\beta(Z)} : \beta(Z) \to X$ is an injective immersion.

Proof. Since $\varphi_H|_U : U \to X$ is a smooth morphism Formulas (2) and (3) hold with $\varphi : X \to Q, Q_0$ and $X_0$ replaced by $\varphi_H : H \to X, X$ and $U$, respectively. Hence, the desired conclusion follows from Theorem 2.8. \hfill \Box

Corollary 2.10. Let the assumptions and conclusions of Proposition 2.6 hold and $Z$ be an affine algebraic variety with $\dim Z \leq \dim X$ (and by the conclusions of Proposition 2.6 with $\dim Z < \codim_H(H \setminus U)$). Suppose that each element of $\mathcal{G}$ is a unipotent group, i.e. $H \simeq \mathbb{G}_t^r$ where $t \geq \dim X$. Then $Z$ can be treated as a closed subvariety of $H$ and for a general element $\beta \in \mathcal{B}$ in any perfect $F$-family $\mathcal{B}$ of automorphisms of $H$ the morphism $\varphi_H|_{\beta(Z)} : \beta(Z) \to X$ is an injective immersion.
Proof. The first statement follows from Holme’s theorem and the second from Proposition 2.9.

Since every smooth flexible variety $X$ admits a morphism $\varphi_H : H \to X$ as in Corollary 2.10 we have the following.

**Theorem 2.11.** ([Ka21, Theorem 3.7]) Let $Z$ be an affine algebraic variety and $X$ be a smooth quasi-affine flexible variety of dimension at least $\text{ED}(Z)$. Then $Z$ admits an injective immersion into $X$.

**Remark 2.12.** It is worth mentioning that if $\varphi : Z \to X$ is an injective immersion, then it may happen that $Z$ is not isomorphic to $\varphi(Z)$. As an example one can consider the morphism $\mathbb{A}^1 \setminus \{1\} \to \mathbb{A}^2$, $t \mapsto (t^2 - 1, t(t^2 - 1))$. It maps $\mathbb{A}^1 \setminus \{1\}$ onto the polynomial curve given in $\mathbb{A}^2$ by the equation $y^2 = x^2(x + 1)$.

We have also in our disposal the following slightly improved version of ([Ka21, Theorem 3.2].

**Theorem 2.13.** Let $\psi : X \to Y$ be a finite morphism where $X$ is a smooth flexible variety and $Y$ is normal. Let $Z$ be a quasi-affine algebraic variety which admits a closed embedding in $X$ and has $\text{ED}(Z) \leq \dim X$. Suppose also that $S$ is a closed subvariety of $Y$ such that it contains $Y_{\text{sing}}$ and $\dim Z < \text{codim}_Y S$. Then $Z$ admits a closed embedding in $Y$ with the image contained in $Y \setminus S$.

**Proof.** One can treat $Z$ as a closed subvariety of $X$. By [AFKKZ, Theorem 1.15] there exists an algebraic family $A$ of automorphisms of $X$ such that for a general $\alpha \in A$ the variety $\alpha(Z)$ does not meet $\psi^{-1}(S)$. By Proposition 2.7 enlarging $A$ we can suppose that it is a perfect family. Theorem 2.8 and [Ka21, Proposition 2.9] imply now that $\psi|_{\alpha(Z)} : \alpha(Z) \to Y_{\text{reg}} \subset Y$ is an injective immersion. Since $\psi$ is finite $\psi|_{\alpha(Z)}$ is also proper. Hence, we are done.

### 3. Criterion of Properness

**Notation 3.1.** In this section an affine space $H = \mathbb{A}^1$ is equipped with a fixed coordinate system. This coordinate system defines an embedding $H \hookrightarrow \mathbb{P}^d = \bar{H}$ and we let $D = H \setminus H$. By $\varphi : H \to X$ we denote a surjective morphism onto a smooth quasi-affine algebraic variety $X$ (of positive dimension) with irreducible fibers and by $\psi : \bar{H} \dashrightarrow \bar{X}$ we denote the rational map into a completion $\bar{X}$ of $X$ extending $\varphi$.

**Proposition 3.2.** Let $\pi : Y \to \bar{H}$ be a resolution of the indeterminacy set of $\psi$, (i.e., $H$ is naturally contained as an open dense subset in $Y$ and $\chi := \psi \circ \pi : Y \to \bar{X}$ is a proper morphism). Let $V = \chi^{-1}(X) \setminus H$ and $W = \pi(V)$. Suppose that $Z$ is a closed subvariety of $H$ and $Z$ is its closure in $\bar{H}$. Then $\varphi|_Z : Z \to X$ is a proper morphism if and only if $\bar{Z} \cap W = \emptyset$. 


Proof. Let $\hat{Z} = \pi^{-1}(\bar{Z}) \cap V$. Note that $\varphi|_Z = \chi|_Z$ is proper if and only if $\hat{Z} = \emptyset$. Note also that $\pi(\hat{Z}) = \bar{Z} \cap W$. In particular, $\hat{Z} = \emptyset$ if and only if $\bar{Z} \cap W = \emptyset$. This yields the desired conclusion. □

Definition 3.3. We call the set $W$ as in Proposition 3.2 the improperness set of $\varphi$.

It is easy to see that if $\dim Z > \operatorname{codim}_D D$, then $\bar{Z} \cap W \neq \emptyset$. Hence, in the rest of this section we describe some conditions which guarantee that $\operatorname{codim}_D W$ is sufficiently large.

Proposition 3.4. Let Notation 3.1 hold and $G$ be a subgroup of the group of affine transformations of $H$ (in particular, the natural action of $G$ extends to $H$). Suppose that $G$ acts on $X$ so that the morphism $\varphi : H \to X$ is equivariant. Then $\bar{X}$ and a resolution $\pi : Y \to \bar{H}$ of the indeterminacy points of $\psi$ can be chosen such that $G$ acts on $Y$ and $\pi$ is equivariant.

Proof. By Sumihiro’s theorem [Su] we can suppose that the $G$-action on $X$ extends to a $G$-action on $\bar{H}$. Then $\psi$ is an equivariant rational map into a complete variety and the desired conclusion follows from the Reichstein-Youssin theorem [ReYo]. □

Proposition 3.5. Under the assumptions of Proposition 3.4 suppose that $G$ acts on $H$ by translations (in particular, the $G$-action on $D$ is trivial) and the minimal dimension of orbits of $G$ in $X$ is $m$. Then the codimension of the improperness set $W$ of $\varphi$ in $D$ is at least $m$.

Proof. Let $U$ be an irreducible component of $V$ where $V$ is as in Proposition 3.2. Since $\chi|_U : U \to X$ is equivariant the dimension of a general $G$-orbit in $U$ is at least $m$. Since the $G$-action on $D$ is trivial a general fiber of $\pi|_U : U \to \pi(U) \subset D$ contains a $G$-orbit. Hence $\dim \pi(U) \leq \dim U - m$. Since $\dim U \leq \dim D$ we have the desired conclusion. □

Proposition 3.6. Suppose that the assumptions of Proposition 3.4 hold, $G$ acts on $H$ by translations and the dimension of general orbits of $G$ in $X$ is $n$. Let $R \subset X$ be the union of non-general orbits of $G$. Suppose that $\chi(U)$ is not contained in $R$ for every irreducible component $U$ of $V$ where $V$ is as in Proposition 3.2. Then the codimension of the improperness set $W$ of $\varphi$ in $D$ is at least $n$.

Proof. Since $\chi|_U : U \to \chi(U) \subset X$ is equivariant the dimension of a general $G$-orbit in $U$ is at least the same as the dimension of general $G$-orbits in $\chi(U)$. By the assumption, the latter dimension is $n$. Since a general fiber of $\pi|_U : U \to \pi(U) \subset D$ contains a general $G$-orbit one has $\dim \pi(U) \leq \dim U - n \leq \dim D - n$ which concludes the proof. □

4. Main Theorem I

The aim of this section is the following.
Theorem 4.1. Let $X$ be a smooth flexible variety equipped with a $G_a^m$-action such that the minimal dimension of its orbits is $n$. Suppose that $Z$ is an affine variety such that $\dim Z \leq n$ and $\text{ED}(Z) \leq \dim X$. Then there exists a closed embedding of $Z$ into $X$.

Let us start with the following.

Lemma 4.2. Let $G'$ be a $G_a^m$-subgroup of $\text{SAut}(X)$ acting on $X$. Consider the natural $G'$-action on $X \times X$ given by $(g, x_1, x_2) \mapsto (g.x_1, x_2)$. Let $\Phi_H : H \times X \to H \times X, (h, x) \mapsto (h.x, x)$ be as in Proposition 2.6. Then $H$ can be chosen such that $H$ is an affine space equipped with a free $G'$-action for which $\Phi_H$ is $G'$-equivariant (where $G'$ acts on $H \times X$ by $(g, h, x) \mapsto (g.h, x)$). Furthermore, $H$ can be equipped with a coordinate system such that $G'$ acts on $H$ by translations.

Proof. We can suppose that $G$ in Notation 2.5 is the collection of all $G_a$-subgroups of $\text{SAut}(X)$ which implies that $H$ is an affine space. By Proposition 2.6(2) we can also suppose that

$$H = (H_1, \ldots, H_s, H_{s+1}, \ldots, H_{s+m})$$

where $H_{s+1}, \ldots, H_{s+m}$ are commuting $G_a$-groups generating $G'$. Let $g' = (h^0_{s+m}, \ldots, h^0_{s+1}) \in G' = H_{s+m} \times \cdots \times H_{s+1}$ and $h = (h_{s+m}, \ldots, h_1) \in H = H_{s+m} \times \cdots \times H_1$. Suppose that the $G'$-action on $H$ is given by

$$(g', h) \mapsto (h_{s+m}h^0_{s+m}, \ldots, h_{s+1}h^0_{s+1}, h_s, \ldots, h_1).$$

(4) Commutativity and Formula (1) imply that $\Phi_H(g'.h, x) = (g'.(h.x), x)$ which yields the first statement. One can equip each $H_i \cong H^1$ with a coordinate $\zeta_i$ (with the zero element of $H_i$ corresponding to $\zeta_i = 0$). This yields the coordinate system $(\zeta_{s+m}, \ldots, \zeta_1)$ on $H$. In this coordinate system the action of $g'$ given by Formula (4) is a translation and we are done.

$\square$

Proof of Theorem 4.1. Let the conclusions of Lemma 4.2 hold, $\varphi_H : H \to X$ be the restriction of $\Phi_H$ to $H \times x_0, x_0 \in X$ and $U$ be as in Proposition 2.6. By Holme’s theorem we can treat $Z$ as a closed subvariety of $H$ and by Proposition 2.6(3) we can suppose $\dim Z < \text{codim}_H(H \setminus U)$. By Proposition 3.5 and Lemma 4.2 the improperness set $W$ of $\varphi_H$ is of codimension at least $n$ in $D = \overline{H \setminus H = \mathbb{P}^t \setminus \mathbb{A}^t}$. For any perfect family $A$ of automorphisms on $H$ and a general $\alpha \in A$ the morphism $\varphi_H|_{\alpha(Z)} : \alpha(Z) \to X$ is an injective immersion by Corollary 2.10. Let $K = \text{SL}_{s+m}(k)$ where $t = s + m$. Then we have the natural $K$-action on $\overline{H}$ such that $D$ is invariant under it and the restriction of the action to $D$ is transitive. By Proposition 2.7 $K \times A$ is still a perfect $\text{SAut}(H)$-family of automorphisms of $H$. That is, for a general $(\beta, \alpha) \in K \times A$ the morphism $\varphi_H|_{\beta \circ \alpha(Z)} : \beta \circ \alpha(Z) \to X$ is still an injective immersion. Let $P$ be the intersection of $D$ with the closure of $\beta \circ \alpha(Z)$ in $\overline{H}$, i.e., $\dim P \leq n - 1$. Since the restriction of the $K$-action to $D$ is transitive, $P$ does not meet $W$ for a general $(\beta, \alpha) \in K \times A$ by
[AFKKZ, Theorem 1.15]. Hence, \( \varphi_H|_{\beta\alpha(Z)} : \beta \circ \alpha(Z) \to X \) is proper by Proposition 3.2 and we are done.

**Corollary 4.3.** Let \( X \) be a smooth flexible variety equipped with a free \( G_a \)-action. Let \( Z \) be an affine algebraic variety of dimension at most \( n + l \) such dim \( X + n \geq \text{ED}(Z) \). Suppose that \( \psi : X \times A^n \to Y \) is a finite morphism onto a normal variety \( Y \) and \( S \) is a closed subvariety of \( Y \) such that it contains \( Y_{\text{sing}} \) and dim \( Z < \text{codim}_Y S \). Then \( Z \) admits a closed embedding into \( Y \) with the image contained in \( Y \setminus S \).

**Proof.** Since \( X \times A^n \) admits a free \( G_{a}^{n+l} \)-action, by Theorem 4.1 there is a closed embedding of \( Z \) into \( X \times A^n \). Hence, the desired conclusion follows from Theorem 2.13. \( \square \)

**Corollary 4.4.** Let \( X \) be isomorphic (as an algebraic variety) to a special linear group \( \text{SL}_n(k) \) and \( Z \) be an affine variety with \( \text{ED}(Z) \leq \text{dim} X \). Suppose also that \( \text{dim} Z \leq m = \frac{n^2}{4} \) if \( n \) is even and \( \text{dim} Z \leq m = \frac{n^2-1}{4} \) if \( n \) is odd. Then \( Z \) admits a closed embedding into \( X \).

**Proof.** Let \( I \) be the identity matrix in \( \text{SL}_n(k) \). For even \( n \) consider the set \( G' \) of all matrices of the form \( I + A \) where \( A = [a_{ij}] \) is the matrix such that \( a_{ij} = 0 \) as soon as \( i \leq \frac{n}{2} \) or \( j > \frac{n}{2} \). If \( n \) is odd, then we require that \( a_{ij} = 0 \) as soon as \( i \leq \frac{n-1}{2} \) or \( j > \frac{n+1}{2} \). In both cases \( G' \) is a \( G_a^m \)-group acting freely on \( X \) with multiplication given by \( (I + A) \cdot (I + A') = I + (A + A') \). Thus, the desired conclusion follows from Theorem 4.1. \( \square \)

5. **Main Theorem II**

**Notation 5.1.** In this section \( X \) is always isomorphic (as an algebraic variety) to a connected linear algebraic group \( G \neq G_a \) without nontrivial characters. By \( \mathcal{G} \) we denote the collection of all \( G_a \)-subgroups of \( G \) (the absence of nontrivial characters implies that such subgroups generate \( G \)). In particular, if \( \mathcal{H} = (H_1, \ldots, H_s) \) is a sequence in \( \mathcal{G} \), then the affine space \( H = H_s \times \cdots \times H_1 \) is equipped with a natural coordinate system as in Lemma 4.2. Recall that we have a morphism \( \Phi_{\mathcal{H}} : H \times X \to X \times X \) given by \( \Phi_{\mathcal{H}}(h,x) = ((h_s \cdot \cdots \cdot h_1).x,x) \) for \( h = (h_s, \ldots, h_1) \in H_s \times \cdots \times H_1 \). Since we suppose that \( G \) acts on \( X \) naturally (i.e., \( g.x \) coincides with the product \( gx \)) \( \Phi_{\mathcal{H}}(h,x) = (hx,x) \) where \( h \) in the right-hand side is treated as the element \( h_s \cdot \cdots \cdot h_1 \) of \( G \). We also suppose that \( G' \) is a \( G_a^m \)-subgroup of \( G \) which acts on \( H \) in the manner described in Lemma 4.2.

Our aim is to strengthen Theorem 4.1 for such \( X \) and, in particular, to improve Corollary 4.4. Let us start with some technical facts.

**Lemma 5.2.** Let Notation 5.1 hold, \( \text{pr}_1 : X \times X \to X \) be the natural projection to the first factor and \( \Phi_{\mathcal{H}}^1 = \text{pr}_1 \circ \Phi_{\mathcal{H}} : H \times X \to X \). Let \( \Lambda : G' \times G \times H \times X \to H \times X, (g',g,h,x) \mapsto (g'.h,xg^{-1}) \), \( \Delta : G' \times
implies the following.

From now on we suppose that the conclusions of Lemma Notation 5.4.

Lemma 5.3. Let the assumptions of Lemma 5.2 hold, \( \bar{H} \) be a \( G' \times \) G-equivariant completion of \( H \times X \) and \( \Psi : \bar{H} \rightarrow \bar{X} \times \bar{X} \) (resp. \( \Psi_1 : H \rightarrow \bar{X} \times \bar{X} \)) be the rational extension of \( \Phi_\mathcal{H} \) (resp. \( \Phi_1 \)). Then a resolution \( \pi : Y \rightarrow \bar{H} \) of the indeterminacy points of \( \Psi \) can be chosen such that the \( G' \times G \)-action on \( H \times X \) extends to \( Y \) and the morphisms \( \lambda = \Psi \circ \pi : Y \rightarrow \bar{X} \times \bar{X} \) and \( \chi = \Psi_1 \circ \pi : Y \rightarrow \bar{X} \) are \( G' \times G \)-equivariant.

Notation 5.4. From now on we suppose that the conclusions of Lemma 5.3 hold and we denote the extension of the \( \Lambda \)-action on \( H \times X \) to \( Y \) by the same letter \( \Lambda \) and the extension of the \( \Delta_1 \)-action to \( \bar{X} \) by the same letter \( \Delta_1 \). For a \( G'' \)-subgroup \( G'' \) of \( G \) we consider the quotient morphism \( \gamma : G \rightarrow Q = G''/G \). The fiber of this morphism over a point \( q \in Q \) is a right coset of \( G'' \) denoted by \( C_q \). Fixing an isomorphism \( G \simeq X \) we treat \( C_q \) as a subset of \( X \) and let \( H_q = H \times C_q \). Finally, by \( Y_q \) we denote the closure of \( H_q \) in \( Y \).

Lemma 5.5. Let Notation 5.4 hold and \( \chi_q : Y_q \rightarrow \bar{X} \) be the restriction of \( \chi \). Suppose that \( V_q = \chi_q^{-1}(X) \setminus H_q \) and \( R \) is a proper closed subvariety of \( X \). Then for a general \( q \in Q \) there is no irreducible component \( U_q \) of \( V_q \) with \( \chi_q(U_q) \) contained in \( R \).

Proof. Note that \( V_q = (\chi^{-1}(X) \cap Y_q) \setminus H_q = (\chi^{-1}(X) \setminus (H \times X)) \cap Y_q = Y_q \cap V \) where \( V = \chi^{-1}(X) \setminus (H \times X) \). Since \( \bar{X} \times X \) is \( \Delta_1 \)-invariant \( \chi^{-1}(\bar{X} \setminus X) \) is \( \Delta_1 \)-invariant. Since \( H \times X \) is also \( \Lambda \)-invariant, so is \( V = Y \setminus (\chi^{-1}(\bar{X} \setminus X) \cup (H \times X)) \). Note that the \( \Lambda \)-action yields a transitive action on the collection \( \{H \times C_q\}_{q \in Q} \) and, therefore, on \( \{Y_q\}_{q \in Q} \) and, consequently, on \( \{V_q\}_{q \in Q} \). Thus, \( V = \bigcup_{q \in Q} V_q \) is a \( \Lambda \)-orbit of \( V_{q_0} \) where \( q_0 \) is any point in \( Q \). Let \( q_0 \) be the coset \( G'' \). Note that the action of any element of the subgroup \( G' \times G'' \subset G' \times G \) preserves \( H \times C_{q_0} \) and, therefore, \( V_{q_0} \). Hence, the image of \( V_{q_0} \) under the action of \( (g', g) \in G' \times G \) is completely determined by \( \hat{\gamma}(g) \) where \( \hat{\gamma} : G \rightarrow G/G'' =: \hat{Q} \) is the quotient morphism. Let \( q \) be the image of \( \hat{\gamma}(g) \) under the map \( \hat{Q} \rightarrow Q \) induced \( G \rightarrow G, g \mapsto g^{-1} \). The description
of the $\Lambda$-action in Lemma 5.2 implies that $(g', g)V_{g_0} = V_q$. Note also that every irreducible component $U_{g_0}$ of $V_{g_0}$ is preserved by the action of $G' \times G''$ since the latter subgroup is connected. Hence, $(g', g)U_{g_0}$ is a well-defined irreducible component $U_q$ of $V_q$ depending only on $\gamma(g)$. This implies that $\bigcup_{q \in Q} U_q$ is the $\Lambda$-orbit of $U_{g_0}$. Thus, $\chi(\bigcup_{q \in Q} U_q) = X$ because $\chi$ is equivariant and the $\Delta_1$-action is transitive on $X$. In particular, $\chi_q(U_q)$ is not contained in $R$ for a general $q \in Q$. This yields the desired conclusion. \hfill $\square$

**Lemma 5.6.** Let the assumptions of Lemma 5.5 hold, $q$ be a general point of $Q$ and $C_q = G''g_0$. Then $H_q$ is an affine space equipped with a coordinate system such that in this system the group $G' \times (g_0^{-1}G''g_0)$ acts on $H_q$ freely by translations.

**Proof.** The space $H_q$ is affine since it is isomorphic to $H \times G''$. Lemma 4.2 yields a free action of $G'$ on the first factor, while $g_0^{-1}G''g_0$ acts on the second by multiplications from the right. Note also that if $H$ is equipped with a coordinate system from Lemma 4.2 and $G''$ with a coordinate system induced by the structure of a $\mathbb{G}_m^*$-subgroup, then $G' \times g_0^{-1}G''g_0$ acts on $H_q$ by translations. Hence, we are done. \hfill $\square$

**Lemma 5.7.** A completion $\bar{H}$ of $H \times X$ in Lemma 5.3 can be chosen such that for every $q \in Q$ the closure $H_q$ of $H_q$ in $H$ is a projective space that is the completion of $H_q$ associated with the coordinate system from Lemma 5.6.

**Proof.** By [Gro58, Theorem 3] the quotient morphism $\gamma : G \to Q$ is a principal $G''$-bundle which is locally trivial in the Zariski topology. Let $\{Q_i\}$ be a cover of $Q$ by open subsets over which $\gamma$ admits sections $\sigma_i : Q_i \to G$. The coordinate system on $H$ (from Lemma 4.2) allows us to treat $H$ as $\mathbb{G}_m^*$-group. Thus, $\tau : H \times G \to Q$ is a principal $H \times G''$-bundle whose fiber $\tau^{-1}(q) = H_q$ and we have the trivialization isomorphisms

$$\eta_i : Q_i \times H \times G'' \to \tau^{-1}(Q_i), (q, h, g'') \mapsto (h, g''\sigma_i(q)) \in H_q$$

with the transition functions

$$\kappa_{ij} : Q_{ij} \times H \times G'' \to Q_{ij} \times H \times G'', (q, h, g'') \mapsto (q, h, g''\sigma_i(q)\sigma_j(q)^{-1}).$$

Consider the $G$-action on $Q$ such that $g \in G$ sends $q = G''g_0$ to $G''g_0g^{-1}$ and the set

$$S_{ij} = \{(g', g, q, h, g'') \in G' \times G \times Q_i \times H \times G'' | g.q \in Q_j\}.$$ 

Then $\eta_j^{-1} \circ \Lambda \circ (\text{id, } \eta_i) : S_{ij} \to Q_j \times H \times G''$ is given by

$$(5) \quad (g', g, q, h, g'') \mapsto \eta_j^{-1}((g', g) \eta_i(q, h, g'')) = (g.q, g'h, g''\tilde{y}_{ij}'''),$$

where $G'' \ni \tilde{y}_{ij}' = \sigma_i(q)g^{-1}(\sigma_j(g,q))^{-1}$. Equip $H \times G'' \simeq \mathbb{A}^t$ (where $t = s + m''$) with the coordinate system $\tilde{\zeta} = (\zeta_1, \ldots, \zeta_t)$ from Lemma 5.6.
If $\zeta \in \mathbb{A}^t$ are the coordinates of $(h, g'')$ and $\tilde{\zeta}^0(g, q)$ are the coordinates of $(\tilde{0}, \tilde{g}'_j) \in H \times G''$, then the coordinate form of Formula (5) is

$$ (g', g, q, \tilde{\zeta}) \mapsto \eta^{-1}(\tilde{g}', g, q, \tilde{\zeta}) = (g, q, \tilde{\zeta} + \tilde{\zeta}^0(g, q)). \quad (6) $$

There is the natural embedding $\mathbb{A}^t \hookrightarrow \mathbb{P}^t$ where $\mathbb{P}^t$ is equipped with the coordinate system $\tilde{\xi} = (\xi_0 : \xi_1 : \ldots : \xi_t)$ such that $\xi_i = \zeta_i \xi_0$ for $i \geq 1$ and $\xi_0 \neq 0$. Since $\kappa_{ij}$ are translations over $Q_{ij}$ the isomorphisms $\eta_{ij}$ extend to the trivialization isomorphisms $\tilde{\eta}_i : Q_i \times \mathbb{P}^t \to \tilde{\tau}_i^{-1}(Q_i)$ where $\tilde{\tau}_i : \widetilde{H \times G} \to Q_i$ is the proectivization of the bundle $\tau : H \times G \to Q$. For $\tilde{S}_{ij} = \{(g', g, q, \tilde{\xi}) \in G' \times G \times Q_i \times \mathbb{P}^t | g.q \in Q_j \}$ formula (6) admits the extension to the morphism $\tilde{S}_{ij} \to Q_j \times \mathbb{P}^t$ sending $((g', g, q, \tilde{\xi})$ to $(g.q, \xi + \zeta^0(g, q))$ where $\zeta^0(g, q) = (\xi_0 : \xi_1(g, q) : \ldots : \xi_t(g, q))$ with $\xi_i(g.q) = \xi_i(g, q)\xi_0$ for $i \geq 1$. Such morphisms yield the isomorphisms $(\text{id}, \tilde{\eta}_i)(\tilde{S}_{ij}) \to \tilde{\tau}_j^{-1}(Q_j)$ which are in turn the extensions of $\Lambda$ restricted to $(\text{id}, \eta_i)(S_{ij})$. Hence, we have a $(G' \times G)$-action on $\widetilde{H \times G}$ extending $\Lambda$. Thus, a $(G' \times G)$-equivariant completion of $H \times G$ yields $H$ which concludes the proof.

**Theorem 5.8.** Let $X$ be isomorphic (as an algebraic variety) to a connected linear algebraic group $G \neq \mathbb{G}_a$ without nontrivial characters. Suppose that $G' \simeq \mathbb{G}^m_a$ and $G'' \simeq \mathbb{G}^m_a$ are subgroups of $G$ such that $G' \cap G''$ coincides with the identity element of $G$. Let $Z$ be an affine variety such that $\dim Z \leq m' + m''$ and $\text{ED}(Z) \leq \dim X$. Then there exists a closed embedding of $Z$ into $X$.

**Proof.** Let $q \in Q$, $C_q = G''g_0$, $H_q$ and $Y_q$ be as in Notation 5.4 and Lemma 5.6 (i.e., $H_q \simeq \mathbb{A}^t$ is an affine space). Consider the group $F = G' \times (g_0^{-1}G''g_0)$ and the $F$-actions on $H_q$ and $X$ that are the restrictions of $\Lambda$ and $\Delta_t$ from Lemma 5.2, respectively. By Lemma 5.2 the morphism $\varphi_q = \Phi^1_{H_q} : H_q \to X$ is $F$-equivariant. By Lemma 5.6 $H_q$ is equipped with a coordinate system such that $F$ acts on $H_q$ by translations. Let $\psi_q : H_q \to X$ be the rational extension of $\varphi_q$ to the projective space $\mathbb{P}^t$ which is the completion of $H_q$ associated with the coordinate system. By Lemmas 5.3 and 5.7 we can suppose that $\pi_q = \pi |_{Y_q} : Y_q \to \widetilde{H_q}$ is a $F$-equivariant resolution of the indeterminacy points of $\psi_q$. Hence, by Proposition 3.6 and Lemma 5.5 we can suppose that the codimension of the improperness set $W_q$ of $\varphi_q$ in $D_q = H_q \setminus H_q$ is at least the dimension of general orbits of $F$ in $X$. Treating $g_0$ as a point in $X \simeq G$ we see that the $F$-orbit of $g_0$ has dimension $m' + m''$. Thus, the dimension of general $F$-orbits is at least $m' + m''$ and $\text{codim}_{D_q} W_q \geq m' + m''$.

Let $K = \text{SL}_t(\mathbf{k})$ and $\mathcal{A}$ be a perfect family $\mathcal{A}$ of automorphisms on $H_q$. By Holme’s theorem we can treat $Z$ as a closed subvariety of $H_q$. Arguing as in the proof of Theorem 4.1 we see that for a general $(\beta, \alpha) \in K \times \mathcal{A}$ the morphism $\varphi_q |_{\beta \circ \alpha(Z)} : \beta \circ \alpha(Z) \to X$ is an injective
immersion. Let $P$ be the intersection of $D_q$ with the closure of $\beta \circ \alpha(Z)$ in $\hat{H}_q$, i.e., $\dim P \leq m' + m'' - 1$. Since the natural $K$-action on $H_q$ extends to the action on $\hat{H}_q$ so that its restriction to $D_q$ is transitive, $P$ does not meet $W_q$ for a general $(\beta, \alpha) \in K \times \mathcal{A}$ by [AFKKZ, Theorem 1.15]. Hence, $\varphi_{q|\beta \circ \alpha(Z)} : \beta \circ \alpha(Z) \to X$ is proper by Proposition 3.2 and we are done.

**Corollary 5.9.** Let $X$ be isomorphic (as an algebraic variety) either to a special linear group $\text{SL}_n(k)$ or to a symplectic group $\text{Sp}_{2n}(k)$ and $Z$ be an affine algebraic variety such that $\text{ED}(Z) \leq \dim X$. Then there exists a closed embedding of $Z$ into $X$.

**Proof.** Suppose that $G'$ is the $G_a^n$-subgroup of $\text{SL}_n(k)$ (in particular, it is a unipotent abelian subgroup of a maximal dimension by [Ma45]) as in the proof of Corollary 4.4 and $G''$ is the subgroup that consists of the transposes of elements of $G'$. Note that $G' \cap G'' = e$ (where $e$ is the identity element of $G$) and $\dim G' = \dim G'' \geq \frac{\dim X}{2}$. Hence, $\dim Z \leq \dim G' + \dim G''$ since $\text{ED}(Z) \leq \dim X$ and, thus, $\dim Z \leq \frac{\dim X - 1}{2}$. Similarly, for $X \simeq \text{Sp}_{2n}(k)$ the maximal dimension of a unipotent abelian subgroup $G'$ is greater than $\frac{\dim X}{2}$ by [Ma45] (see also [Law]). Furthermore, $G'$ can be chosen so that in a root space decomposition its Lie algebra is generated by subspaces with positive roots [Law, page 7]. Replacing these positive roots by the corresponding negative roots we get the Lie algebra of a maximal unipotent abelian subgroup $G'$ such that $\dim G'' = \dim G'$ and $G' \cap G'' = e$. Hence, $\dim Z \leq \dim G' + \dim G''$ as before and Theorem 5.8 implies the desired conclusion. \qed

In a more general setting we have the following.

**Corollary 5.10.** Let $Z$ be an affine algebraic variety, $X$ be an algebraic variety of the form $\mathbb{A}^{n_0} \times G_1 \times G_2 \times \ldots \times G_l$ where each $G_i$ is either $\text{SL}_{n_i}(k)$ or $\text{Sp}_{2n_i}(k)$. Suppose that $\varphi : X \to Y$ is a finite morphism into a normal variety $Y$, $\text{ED}(Z) \leq \dim Y$ and $S$ is a closed subvariety of $Y$ containing $Y_{\text{sing}}$ such that $\dim Z < \text{codim}_Y S$. Then $Z$ admits a closed embedding into $Y$ with the image contained in $Y \setminus S$.

**Proof.** By Theorem 2.13 it suffices to consider the case of $Y = X$. Since $X$ is isomorphic as an algebraic variety to a linear algebraic group $G = \mathbb{G}_a^{n_0} \times G_1 \times G_2 \times \ldots \times G_l$ Theorem 5.8 implies that it is enough to construct $\mathbb{G}_a^{n_0}$-subgroups $G'$ and $G''$ of $G$ such that $G' \cap G'' = e$ and $\dim Z \leq \dim G' + \dim G''$. The proof of Corollary 5.9 implies that one can find similar subgroups $G_i'$ and $G_i''$ in each factor $G_i$ of $G$ such that $\dim G_i' + \dim G_i'' \geq \frac{\dim G_i}{2}$. Thus, letting $G_i' = \mathbb{G}_a^{n_0} \oplus \bigoplus_{r=1}^{n_0} G_i'$ and $G_i'' = \bigoplus_{r=1}^{n_0} G_i''$ we see that $\dim Z \leq \dim G_i' + \dim G_i''$ since $\dim Z \leq \frac{\dim G_i}{2}$. This yields the desired conclusion. \qed
Remark 5.11. If $G$ is a simple Lie group whose Dynkin diagram differs from $A_n$ or $C_n$, then there is no unipotent abelian subgroup of $G$ whose dimension is at least $\dim G - 1$ [Ma45]. Hence, for such groups and a smooth $Z$ our method is less effective than the one in [FvS21].

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