A MULTIPLICITY RESULT FOR A FRACTIONAL KIRCHHOFF EQUATION IN $\mathbb{R}^N$ WITH A GENERAL NONLINEARITY

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Abstract. In this paper we deal with the following fractional Kirchhoff equation

$$\left( p + q (1 - s) \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \right) (-\Delta)^s u = g(u) \text{ in } \mathbb{R}^N,$$

where $s \in (0, 1)$, $N \geq 2$, $p > 0$, $q$ is a small positive parameter and $g : \mathbb{R} \to \mathbb{R}$ is an odd function satisfying Berestycki-Lions type assumptions. By using minimax arguments, we establish a multiplicity result for the above equation, provided that $q$ is sufficiently small.

1. Introduction

In this paper we study the multiplicity of weak solutions to the following nonlinear fractional Kirchhoff equation

$$\left( p + q (1 - s) \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \right) (-\Delta)^s u = g(u) \text{ in } \mathbb{R}^N \quad (1.1)$$

where $s \in (0, 1)$, $N \geq 2$, $p > 0$, $q$ is a small positive parameter and $g$ is a nonlinearity which satisfies suitable assumptions. The operator $(-\Delta)^s$ is the fractional Laplacian which may be defined for a function $u$ belonging to the Schwartz space $\mathcal{S}(\mathbb{R}^N)$ of rapidly decaying functions as

$$(-\Delta)^s u(x) = C_{N,s} \text{ P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} \, dy, \quad x \in \mathbb{R}^N.$$

The symbol P.V. stands for the Cauchy principal value and $C_{N,s}$ is a normalizing constant; see [15] for more details.

When $s \to 1^-$ in (1.1), from Theorem 2 (and Corollary 2) in [12], we can see that (1.1) becomes the following Kirchhoff equation

$$- \left( p + q \int_{\mathbb{R}^N} |\nabla u(x)|^2 \, dx \right) \Delta u = g(u) \text{ in } \mathbb{R}^N, \quad (1.2)$$

which has been extensively studied in the last decade.

The equation (1.2) is related to the stationary analogue of the Kirchhoff equation

$$u(t) - \left( p + q \int_{\Omega} |\nabla u(x)|^2 \, dx \right) \Delta u = g(x, u)$$

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with $\Omega \subset \mathbb{R}^N$ bounded domain, which was proposed by Kirchhoff in 1883 [21] as an extension of the classical D’Alembert’s wave equation

$$\rho u_{tt} - \left( \frac{P_0}{h} + \frac{E}{2L} \int_0^L |u_x|^2 \, dx \right) u_{xx} = g(x, u)$$  \hspace{1cm} (1.3)$$

for free vibrations of elastic strings. Kirchhoff’s model takes into account the changes in length of the string produced by transverse vibrations. Here, $L$ is the length of the string, $h$ is the area of the cross section, $E$ is the Young modulus of the material, $\rho$ is the mass density and $P_0$ is the initial tension. The early classical studies dedicated to Kirchhoff equations were given by Bernstein [11] and Pohozaev [31]. However, equation (1.2) received much attention only after the paper by Lions [23], where a functional analysis approach was proposed to attack it. For more recent results concerning Kirchhoff-type equations we refer to [1, 2, 5, 7, 16, 22, 25, 30].

On the other hand, a great attention has been recently focused on the study of nonlinear fractional Kirchhoff problem. In [18], Fiscella and Valdinoci proposed an interesting interpretation of Kirchhoff’s equation in the fractional setting, by proving the existence of nonnegative solutions for a critical Kirchhoff type problem in a bounded domain of $\mathbb{R}^N$. Subsequently, in [4] the authors investigated the existence and asymptotic behavior of nonnegative solutions for a class of stationary Kirchhoff problems driven by a fractional integro-differential operator and involving a critical nonlinearity. Pucci and Saldi in [32] established the existence and multiplicity of nontrivial nonnegative entire solutions for a Kirchhoff type eigenvalue problem in $\mathbb{R}^N$ involving a critical nonlinearity and the fractional Laplacian. More recently, in [17] has been proved the existence of infinitely many weak solutions for a Cauchy problem for a fractional Kirchhoff-type equation by using the genus theory of Krasnosel’skii; see also [6, 26, 27, 28, 29, 33] for related problems.

Motivated by the above papers, in this work we aim to study the multiplicity of weak solutions to the fractional Kirchhoff equation (1.1) with $q$ small parameter and $g$ is a general subcritical nonlinearity. More precisely, we suppose that $g : \mathbb{R} \to \mathbb{R}$ satisfies Berestycki-Lions type assumptions [9, 10], that is:

1. $g \in C^1(\mathbb{R}, \mathbb{R})$ and odd;
2. $-\infty < \liminf_{t \to 0^+} \frac{g(t)}{t} \leq \limsup_{t \to 0^+} \frac{g(t)}{t} = -m < 0$;
3. $\lim_{t \to \pm \infty} \frac{|g(t)|}{|t|^{2s-1}} = 0$, where $2_s^* = \frac{2N}{N - 2s}$;

4. there exists $\zeta > 0$ such that $G(\zeta) := \int_0^\zeta g(t) \, dt > 0$.

Let us recall that when $q = 0$ and $p = 1$ in (1.1), in [3, 14] has been established the existence and multiplicity of radially symmetric solutions to the fractional scalar field problem

$$(-\Delta)^s u = g(u) \text{ in } \mathbb{R}^N.$$  \hspace{1cm} (1.4)$$

Now, we aim to study a generalization of (1.4), and we look for weak solutions to (1.1) with $q > 0$ sufficiently small.

Our main result is the following:
Theorem 1.1. Let us suppose that \((g_1), (g_2), (g_3)\) and \((g_4)\) are satisfied. Then, for any \(h \in \mathbb{N}\) there exists \(q(h) > 0\) such that for any \(0 < q < q(h)\) equation (1.1) admits at least \(h\) couples of solutions in \(H^s(\mathbb{R}^N)\) with radial symmetry.

A common approach to deal with nonlinear problems involving the fractional Laplacian, has been proposed by Caffarelli and Silvestre in [13]. It consists to realize \((-\Delta)^s\) as an operator that maps a Dirichlet boundary condition to a Neumann-type condition via an extension problem on the upper half-space \(\mathbb{R}_+^{N+1}\). More precisely, for \(u \in H^s(\mathbb{R}^N)\) one considers the problem

\[
\begin{cases}
-\text{div}(y^{1-2s}\nabla v) = 0 & \text{in } \mathbb{R}_+^{N+1} \\
v(x, 0) = u(x) & \text{on } \partial \mathbb{R}_+^{N+1}
\end{cases}
\]

from where the fractional Laplacian is obtained as

\[(-\Delta)^s u(x) = -\kappa_s \lim_{y \to 0^+} y^{1-2s} v_y(x, y),\]

where \(\kappa_s\) is a suitable constant and the equality holds in distributional sense.

In this paper we investigate the problem (1.1) directly in \(H^s(\mathbb{R}^N)\) in order to adapt the techniques developed in the classical case \(s = 1\).

More precisely, we follow the ideas in [8], and by combining the Mountain Pass approach introduced in [19] with the truncation argument of [20], we prove the multiplicity result above stated.

The paper is organized as follows: in Sec. 2 some notations and preliminaries are given, including lemmas that are required to obtain our main Theorem; in Sec. 3 we establish an abstract critical point result and finally in Sec. 4 we provide the proof of Theorem 1.1.

2. Preliminaries

For any \(s \in (0, 1)\) we define the fractional Sobolev spaces

\[H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \frac{|u(x) - u(y)|}{|x - y|^{N+2s}} \in L^2(\mathbb{R}^{2N}) \right\}\]

endowed with the natural norm

\[\|u\|_{H^s(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy\right)^{1/2}\]

where the so-called Gagliardo seminorm of \(u\) is given by

\[|u|^2_{H^s(\mathbb{R}^N)} = \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy.\]

For the reader’s convenience, we review the main embedding result for this class of fractional Sobolev spaces.

Theorem 2.1. Let \(s \in (0, 1)\) and \(N > 2s\). Then \(H^s(\mathbb{R}^N)\) is continuously embedded in \(L^q(\mathbb{R}^N)\) for any \(q \in [2, 2^*_s]\) and compactly in \(L^q_{loc}(\mathbb{R}^N)\) for any \(q \in [2, 2^*_s]\).
Let us introduce
\[ H^s_{\text{rad}}(\mathbb{R}^N) = \{ u \in H^s(\mathbb{R}^N) : u(x) = u(|x|) \} \]
the space of radial functions in \( H^s(\mathbb{R}^N) \). For this space it holds the following compactness result due to Lions [24]:

**Theorem 2.2.** [24] Let \( s \in (0, 1) \) and \( N \geq 2 \). Then \( H^s_{\text{rad}}(\mathbb{R}^N) \) is compactly embedded in \( L^q(\mathbb{R}^N) \) for any \( q \in (2, 2^*_s) \).

Finally, we recall the following fundamental compactness results:

**Lemma 2.1.** [8, 9] Let \( P \) and \( Q : \mathbb{R} \rightarrow \mathbb{R} \) be a continuous functions satisfying
\[ \lim_{t \to +\infty} \frac{P(t)}{Q(t)} = 0, \]
\[ \sup_{j \in \mathbb{N}} \int_{\mathbb{R}^N} |Q(v_j(x)) w| \, dx < +\infty, \]
\[ P(v_j(x)) \to v(x) \text{ a.e. in } \mathbb{R}^N. \]
Then \( \| (P(v_j) - v)w \|_{L^1(\mathcal{B})} \to 0 \), for any bounded Borel set \( \mathcal{B} \).

Moreover, if we have also
\[ \lim_{t \to 0} \frac{P(t)}{Q(t)} = 0, \]
and
\[ \lim_{|x| \to \infty} \sup_{j \in \mathbb{N}} |v_j(x)| = 0, \]
then \( \| (P(v_j) - v)w \|_{L^1(\mathbb{R}^N)} \to 0 \).

**Lemma 2.2.** [14] Let \((X, \| \cdot \|)\) be a Banach space such that \( X \) is embedded respectively continuously and compactly into \( L^q(\mathbb{R}^N) \) for \( q \in [q_1, q_2] \) and \( q \in (q_1, q_2) \), where \( q_1, q_2 \in (0, \infty) \). Assume that \( \{v_j\}_{j \in \mathbb{N}} \subset X \), \( v : \mathbb{R}^N \rightarrow \mathbb{R} \) is a measurable function and \( P \in C(\mathbb{R}, \mathbb{R}) \) is such that
\[ (i) \lim_{|t| \to 0} \frac{P(t)}{|t|^{q_1}} = 0, \]
\[ (ii) \lim_{|t| \to \infty} \frac{P(t)}{|t|^{q_2}} = 0, \]
\[ (iii) \sup_{j \in \mathbb{N}} \|v_j\| < \infty, \]
\[ (iv) \lim_{j \to \infty} P(v_j(x)) = v(x) \text{ for a.e. } x \in \mathbb{R}^N. \]
Then, up to a subsequence, we have
\[ \lim_{j \to \infty} \|P(v_j) - v\|_{L^1(\mathbb{R}^N)} = 0. \]
3. A critical point result

In this section we provide an abstract multiplicity result which allows us to prove Theorem 1.1. Let us introduce the following functional defined for \( u \in H^s(\mathbb{R}^N) \)

\[
F_q(u) = \frac{1}{2} [u]_{2}(\mathbb{R}^N) + q R(u) - \int_{\mathbb{R}^N} G(u) \, dx,
\]

where \( q > 0 \) is a small parameter and \( R : H^s(\mathbb{R}^N) \rightarrow \mathbb{R} \).

We suppose that

\( R = \sum_{i=1}^{k} R_i \)

and, for each \( i = 1, \ldots, k \) the functional \( R_i \) satisfies

\( (R_1) \) \( R_i \in C^1(H^s(\mathbb{R}^N), \mathbb{R}) \) is nonnegative and even;

\( (R_2) \) there exists \( \delta_i > 0 \) such that \( \langle R_i'(u), u \rangle \leq C \|u\|_{H^s(\mathbb{R}^N)}^{\delta_i} \) for any \( u \in H^s(\mathbb{R}^N) \);

\( (R_3) \) if \( \{u_j\}_{j \in \mathbb{N}} \subset H^s(\mathbb{R}^N) \) is weakly convergent to \( u \in H^s(\mathbb{R}^N) \), then

\[
\limsup_{j \to \infty} \langle R_i'(u_j), u - u_j \rangle \leq 0;
\]

\( (R_4) \) there exist \( \alpha_i, \beta_i \geq 0 \) such that if \( u \in H^s(\mathbb{R}^N), t > 0 \) and \( u_t = u(\cdot t) \), then

\[
R_i(u_t) = t^{\alpha_i} R_i(t^{\beta_i} u);
\]

\( (R_5) \) \( R_i \) is invariant under the action of the \( N \)-dimensional orthogonal group, i.e.

\[
R_i(u(g \cdot)) = R_i(u(\cdot)) \text{ for every } g \in O(N).
\]

Let us observe that for any \( u \in H^s(\mathbb{R}^N) \), \( R_i(u) - R_i(0) = \int_0^1 \frac{d}{dt} R_i(tu) \, dt \), so by the assumption \( (R_2) \) we have

\[
R_i(u) \leq C_1 + C_2 \|u\|_{H^s(\mathbb{R}^N)}^{\delta_i}.
\]

(3.2)

The main result of this section is the following

**Theorem 3.1.** Let us suppose \((g_1) - (g_4)\) and \((R_1) - (R_5)\). Then, for any \( h \in \mathbb{N} \) there exists \( q(h) > 0 \) such that for any \( 0 < q < q(h) \) the functional \( F_q \) admits at least \( h \) couples of critical points in \( H^s_{rad}(\mathbb{R}^N) \).

Let us define, for any \( t \geq 0 \),

\[
g_1(t) := (g(t) + mt)^+ \\
g_2(t) := g_1(t) - g(t),
\]

and we extend them as odd functions for \( t \leq 0 \). Observing that

\[
\lim_{t \to 0} \frac{g_1(t)}{t} = 0, \quad (3.3) \\
\lim_{t \to \infty} \frac{g_1(t)}{t^{2s-1}} = 0, \quad (3.4) \\
g_2(t) \geq mt \ \forall t \geq 0, \quad (3.5)
\]
we deduce that for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that
\begin{equation}
    g_1(t) \leq C_\varepsilon t^{2s-1} + \varepsilon g_2(t) \quad \forall t \geq 0.
\end{equation}

Setting
\begin{equation}
    G_i(t) := \int_0^t g_i(\tau) \, d\tau \quad i = 1, 2,
\end{equation}
by (3.5) immediately follows that
\begin{equation}
    G_2(t) \geq \frac{m}{2} t^2 \quad \forall t \in \mathbb{R},
\end{equation}
and, by (3.6) we can see that for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that
\begin{equation}
    G_1(t) \leq C_\varepsilon |t|^{2s} + \varepsilon G_2(t) \quad \forall t \in \mathbb{R}.
\end{equation}

In view of (R$_5$), all functionals that we will consider along the paper are invariant under rotations, so, from now on, we will directly define our functionals in $H^s_{rad}(\mathbb{R}^N)$. Following [20], let $\chi \in C^\infty([0, +\infty), \mathbb{R})$ be a cut-off function such that
\begin{align*}
    \chi(t) &= 1 \quad \text{for } t \in [0, 1] \\
    0 \leq \chi(t) &\leq 1 \quad \text{for } t \in (1, 2) \\
    \chi(t) &= 0 \quad \text{for } t \in [2, +\infty) \\
    \|\chi\|_\infty &\leq 2,
\end{align*}
and we set
\begin{equation}
    \xi_\Lambda(u) = \chi \left( \frac{\|u\|^2_{H^s(\mathbb{R}^N)}}{\Lambda^2} \right).
\end{equation}

Then we introduce the truncated functional $F^\Lambda_q : H^s_{rad}(\mathbb{R}^N) \to \mathbb{R}$ defined as
\begin{equation}
    F^\Lambda_q(u) = \frac{1}{2} [u]_{H^s(\mathbb{R}^N)}^2 + q \xi_\Lambda(u) R(u) - \int_{\mathbb{R}^N} G(u) \, dx.
\end{equation}

Clearly, a critical point $u$ of $F^\Lambda_q$ with $\|u\|_{H^s(\mathbb{R}^N)} \leq \Lambda$ is a critical point of $F_q$.

Our first aim is to prove that the truncated functional $F^\Lambda_q$ has a symmetric mountain pass geometry:

**Lemma 3.1.** There exist $r_0 > 0$ and $\rho_0 > 0$ such that
\begin{equation}
    F^\Lambda_q(u) \geq 0, \quad \text{for } \|u\|_{H^s(\mathbb{R}^N)} \leq r_0
\end{equation}
\begin{equation}
    F^\Lambda_q(u) \geq \rho_0, \quad \text{for } \|u\|_{H^s(\mathbb{R}^N)} = r_0.
\end{equation}
Moreover, for any $n \in \mathbb{N}$ there exists an odd continuous map
\begin{equation}
    \gamma_n : S^{n-1} \to H^s_{rad}(\mathbb{R}^N)
\end{equation}
such that
\begin{equation}
    F^\Lambda_q(\gamma_n(\sigma)) < 0 \quad \text{for all } \sigma \in S^{n-1},
\end{equation}
where
\begin{equation}
    S^{n-1} = \{\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathbb{R}^n : |\sigma| = 1\}. 
\end{equation}
Proof. Taking $\varepsilon = \frac{1}{2}$ in (3.8), and by using (3.7), the positivity of $R$, and Theorem 2.1, we have

$$F^\Lambda(u) = \frac{1}{2}[u]^2_{H^s(\mathbb{R}^N)} + \int_{\mathbb{R}^N} G_2(u) \, dx + q \xi_\Lambda(u) R(u) - \int_{\mathbb{R}^N} G_1(u) \, dx$$

$$\geq \frac{1}{2}[u]^2_{H^s(\mathbb{R}^N)} + \frac{m}{4}\|u\|^2_{L^2(\mathbb{R}^N)} - C_1 \|u\|^2_{L^2_s(\mathbb{R}^N)}$$

$$\geq \min\left\{\frac{1}{2}, \frac{m}{4}\right\} \|u\|^2_{H^s(\mathbb{R}^N)} - C_1 C^s \|u\|^2_{H^s(\mathbb{R}^N)}$$

from which easily follows (3.9).

Proceeding similarly to Theorem 10 in [10], for any $n \in \mathbb{N}$, there exists an odd continuous map $\pi_n : S^{n-1} \to H^s_{rad}(\mathbb{R}^N)$ such that

$$0 \notin \pi_n(S^{n-1})$$

$$\int_{\mathbb{R}^N} G(\pi_n(\sigma)) \, dx \geq 1 \text{ for all } \sigma \in S^{n-1}.$$

Let us define

$$\psi_n^t(\sigma) = \pi_n(\sigma)\left(\frac{\cdot}{t}\right) \text{ with } t \geq 1.$$

Then, for $t$ sufficiently large, we get

$$F^\Lambda_q(\psi_n^t(\sigma)) = \frac{t^{N-2s}}{2}[\pi_n(\sigma)]^2_{H^s(\mathbb{R}^N)}$$

$$+ q \chi \left(\frac{t^{N-2s}[\pi_n(\sigma)]^2_{H^s(\mathbb{R}^N)} + t^N\|\pi_n(\sigma)\|^2_{L^2(\mathbb{R}^N)}}{\Lambda^2}\right) R(\psi_n^t(\sigma))$$

$$- t^N \int_{\mathbb{R}^N} G(\pi_n(\sigma)) \, dx$$

$$\leq t^{N-2s} \left\{\frac{[\pi_n(\sigma)]^2_{H^s(\mathbb{R}^N)}}{2} - t^{2s}\right\} < 0.$$

Therefore, we can choose $\bar{t}$ such that $F^\Lambda_q(\psi_n^\bar{t}(\sigma)) < 0$ for all $\sigma \in S^{n-1}$, and by setting $\gamma_n(\sigma)(x) := \psi_n^\bar{t}(\sigma)(x)$, we can see that $\gamma_n$ satisfies the required properties.

Now we define the minimax value of $F^\Lambda_q$ by using the maps $\gamma_n : \partial D_n \to H^s_{rad}(\mathbb{R}^N)$ obtained in Lemma 3.1. For any $n \in \mathbb{N}$, let

$$b_n = b_n(q, \Lambda) = \inf_{\gamma \in \Gamma_n} \max_{\sigma \in D_n} F^\Lambda_q(\gamma(\sigma)),$$

where $D_n = \{\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathbb{R}^n : |\sigma| \leq 1\}$ and

$$\Gamma_n = \left\{\gamma \in C(D_n, H^s_{rad}(\mathbb{R}^N)) : \begin{array}{ll}
\gamma(-\sigma) = -\gamma(\sigma) & \text{for all } \sigma \in D_n \\
\gamma(\sigma) = \gamma_n(\sigma) & \text{for all } \sigma \in \partial D_n
\end{array}\right\}.$$
Let us introduce the following modified functionals
\[ \tilde{F}_q^\Lambda(\theta, u) = F_q^\Lambda(u(\cdot/e^\theta)) \]
for \((\theta, u) \in \mathbb{R} \times H^s_{\text{rad}}(\mathbb{R}^N)\).

We set
\[ \tilde{F}_q(\theta, u) = \frac{\partial}{\partial u} \tilde{F}_q(\theta, u), \]
\[ (\tilde{F}_q^\Lambda)'(\theta, u) = \frac{\partial}{\partial u} \tilde{F}_q^\Lambda(\theta, u), \]
\[ \tilde{b}_n = \tilde{b}_n(q, \Lambda) = \inf_{\tilde{\gamma} \in \tilde{\Gamma}_n} \max_{\sigma \in D_n} \tilde{F}_q^\Lambda(\tilde{\gamma}(\sigma)) , \]
where
\[ \tilde{\Gamma}_n = \left\{ \tilde{\gamma} \in C(D_n, \mathbb{R} \times H^s_{\text{rad}}(\mathbb{R}^N)) : \begin{align*}
\tilde{\gamma}(\sigma) &= (\theta(\sigma), \eta(\sigma)) \\
(\theta(\sigma), \eta(\sigma)) &= (\theta(\sigma), -\eta(\sigma)) & \text{for all } \sigma \in D_n \\
(\theta(\sigma), \eta(\sigma)) &= (0, \gamma_n(\sigma)) & \text{for all } \sigma \in \partial D_n
\end{align*} \right\} . \]

By the assumption \((R_4)\) we get
\[ \tilde{F}_q^\Lambda(\theta, u) = e^{(N-2s)\theta} \frac{[u]_{H^s(\mathbb{R}^N)}}{2} + q \sum_{i=1}^k e^{\alpha_i \theta} R_4(e^{\beta_i \theta} u) - e^{N\theta} \int_{\mathbb{R}^N} G(u) \, dx, \]
and
\[ \tilde{F}_q(\theta, u) = e^{(N-2s)\theta} \frac{[u]_{H^s(\mathbb{R}^N)}}{2} + q \chi \left( \frac{e^{(N-2s)\theta} [u]_{H^s(\mathbb{R}^N)}}{2} + e^{N\theta} \|u\|_{L^2(\mathbb{R}^N)}^2 \right) \sum_{i=1}^k e^{\alpha_i \theta} R_4(e^{\beta_i \theta} u) - e^{N\theta} \int_{\mathbb{R}^N} G(u) \, dx. \]

Proceeding as in [3, 19, 34], we can see that the following results hold.

**Lemma 3.2.** We have
\begin{enumerate}
\item there exists \( \bar{b} > 0 \) such that \( b_n \geq \bar{b} \) for any \( n \in \mathbb{N} \),
\item \( b_n \to +\infty \),
\item \( b_n = \tilde{b}_n \) for any \( n \in \mathbb{N} \).
\end{enumerate}

**Lemma 3.3.** For any \( n \in \mathbb{N} \) there exists a sequence \( \{(\theta_j, u_j)\} \subseteq \mathbb{R} \times H^s_{\text{rad}}(\mathbb{R}^N) \)
such that
\begin{enumerate}
\item \( \theta_j \to 0 \),
\item \( F_{q}^\Lambda(\theta_j, u_j) \to b_n \),
\item \( (F_{q}^\Lambda)'(\theta_j, u_j) \to 0 \) strongly in \( H^s_{\text{rad}}(\mathbb{R}^N) \),
\item \( \frac{\partial}{\partial \theta} \tilde{F}_{q}(\theta_j, u_j) \to 0 \).
\end{enumerate}
Our goal is to prove that, for a suitable choice of $\Lambda$ and $q$, the sequence $\{(\theta_j, u_j)\}_{j \in \mathbb{N}}$ given by Lemma 3.3 is a bounded Palais-Smale sequence for $F_q$. We begin proving the boundedness of $\{u_j\}_{j \in \mathbb{N}}$ in $H^s(\mathbb{R}^N)$.

**Proposition 3.1.** Let $n \in \mathbb{N}$ and $\Lambda_n > 0$ sufficiently large. There exists $q_n$, depending on $\Lambda_n$, such that for any $0 < q < q_n$, if $\{(\theta_j, u_j)\}_{j \in \mathbb{N}} \subset \mathbb{R} \times H^s_{\text{rad}}(\mathbb{R}^N)$ is the sequence given in Lemma 3.3, then, up to a subsequence, $\|u_j\|_{H^s(\mathbb{R}^N)} \leq \Lambda_n$, for any $j \in \mathbb{N}$.

**Proof.** Taking into account Lemma 3.2 and Lemma 3.3 we have

$$\begin{align*}
N \tilde{F}_q^\Lambda (\theta_j, u_j) - \frac{\partial}{\partial \theta} \tilde{F}_q^\Lambda (\theta_j, u_j) &= Nb_n + o_j(1),
\end{align*}$$

which can be written as

$$\begin{align*}
&= q \chi \left( \frac{\|u_j(\cdot/e^{\theta_j})\|_{H^s(\mathbb{R}^N)}^2}{\Lambda^2} \right) \sum_{i=1}^k (\alpha_i - N) R_i(u_j(\cdot/e^{\theta_j})) \\
&+ q \chi \left( \frac{\|u_j(\cdot/e^{\theta_j})\|_{H^s(\mathbb{R}^N)}^2}{\Lambda^2} \right) \sum_{i=1}^k e^{\alpha_i \theta_j} \langle R_i'(e^{\beta_i \theta_j} u_j), \beta_i e^{\beta_i \theta_j} u_j \rangle \\
&+ q \chi' \left( \frac{\|u_j(\cdot/e^{\theta_j})\|_{H^s(\mathbb{R}^N)}^2}{\Lambda^2} \right) \left( (N - 2s)e^{(N-2s)\theta_j} [u_j]_{H^s(\mathbb{R}^N)}^2 + Ne^{N\theta_j} \|u_j\|_{L^2(\mathbb{R}^N)} \right) \\
&+ N b_n + o_j(1) \\
&=: I_j + II_j + III_j + Nb_n + o_j(1). \tag{3.11}
\end{align*}$$

By the definition of $b_n$, if $\gamma \in \Gamma_n$, we deduce that

$$b_n \leq \max_{\sigma \in \mathcal{D}_n} F_q^\Lambda (\gamma(\sigma))$$

$$\leq \max_{\sigma \in \mathcal{D}_n} \left\{ \frac{1}{2} [\gamma(\sigma)]_{H^s(\mathbb{R}^N)}^2 - \int_{\mathbb{R}^N} G(\gamma(\sigma)) \, dx \right\} + \max_{\sigma \in \mathcal{D}_n} (q \xi_\Lambda(\gamma(\sigma)) \mathcal{R}(\gamma(\sigma)))$$

$$=: A_1 + A_2(\Lambda). \tag{3.12}$$

Now, if $[\gamma(\sigma)]_{H^s(\mathbb{R}^N)}^2 \geq 2\Lambda^2$ then $A_2(\Lambda) = 0$, otherwise, by (3.2), we can find $\delta > 0$ such that

$$A_2(\Lambda) \leq q \left( C_1 + C_2 \|\gamma(\sigma)\|_{H^s(\mathbb{R}^N)}^2 \right) \leq q \left( C_1 + C_2^2 \Lambda^\delta \right).$$

In addition we have the following estimates:

$$|I_j| \leq q \left( C_3 + C_4 \Lambda^\delta \right), \tag{3.13}$$

$$|II_j| \leq C_5 q \Lambda^\delta, \tag{3.14}$$

$$|III_j| \leq q \left( C_6 + C_7 \Lambda^\delta \right). \tag{3.15}$$

Putting together (3.12), (3.13), (3.14) and (3.15), from (3.11) we obtain

$$[u_j]_{H^s(\mathbb{R}^N)}^2 \leq C' + q \left( C_8 + C_9 \Lambda^\delta \right). \tag{3.16}$$
On the other hand, by \((iv)\) of Lemma 3.3 and 3.8, we deduce that

\[
\begin{align*}
\frac{(N - 2s)e^{(N-2s)\theta_j}}{2} [u_j]^{2}_{H^s(\mathbb{R}^N)} + q \chi \left( \frac{\|u_j(\cdot/e^{\theta_j})\|_{H^s(\mathbb{R}^N)}^2}{\Lambda^2} \right) \sum_{i=1}^{k} \alpha_i \mathcal{R}_i(u_j(\cdot/e^{\theta_j})) \\
+ q \chi \left( \frac{\|u_j(\cdot/e^{\theta_j})\|_{H^s(\mathbb{R}^N)}^2}{\Lambda^2} \right) \sum_{i=1}^{k} e^{\alpha_i \theta_j} \langle \mathcal{R}'_i(e^{\beta_i \theta_j} u_j), \beta_i e^{\beta_i \theta_j} u_j \rangle \\
+ q \chi' \left( \frac{\|u_j(\cdot/e^{\theta_j})\|_{H^s(\mathbb{R}^N)}^2}{\Lambda^2} \right) \left\{ (N - 2s)e^{(N-2s)\theta_j} [u_j]^{2}_{H^s(\mathbb{R}^N)} + N e^{N\theta_j} \|u_j\|^2_{L^2(\mathbb{R}^N)} \right\} \mathcal{R}(u_j(\cdot/e^{\theta_j})) \\
+ N e^{N\theta_j} \int_{\mathbb{R}^N} G_2(u_j) \, dx
\end{align*}
\]

\[
= N e^{N\theta_j} \int_{\mathbb{R}^N} G_1(u_j) \, dx + o_j(1)
\]

\[
\leq N e^{N\theta_j} \left( C_\varepsilon \int_{\mathbb{R}^N} |u_j|^2 \, dx + \varepsilon \int_{\mathbb{R}^N} G_2(u_j) \, dx \right) + o_j(1).
\]

Then, by using (3.7), (3.14), (3.15), (3.16), (3.17) and Theorem 2.1 we can infer

\[
\begin{align*}
& \frac{N e^{N\theta_j} m (1 - \varepsilon)}{2} \int_{\mathbb{R}^N} u_j^2 \, dx \\
& \leq (1 - \varepsilon) N e^{N\theta_j} \int_{\mathbb{R}^N} G_2(u_j) \, dx \\
& \leq N e^{N\theta_j} C_\varepsilon \int_{\mathbb{R}^N} |u_j|^2 \, dx - q \chi \left( \frac{\|u_j(\cdot/e^{\theta_j})\|_{H^s(\mathbb{R}^N)}^2}{\Lambda^2} \right) \sum_{i=1}^{k} e^{\alpha_i \theta_j} \langle \mathcal{R}'_i(e^{\beta_i \theta_j} u_j), \beta_i e^{\beta_i \theta_j} u_j \rangle \\
& \quad - q \chi' \left( \frac{\|u_j(\cdot/e^{\theta_j})\|_{H^s(\mathbb{R}^N)}^2}{\Lambda^2} \right) \left\{ (N - 2s)e^{(N-2s)\theta_j} [u_j]^{2}_{H^s(\mathbb{R}^N)} + N e^{N\theta_j} \|u_j\|^2_{L^2(\mathbb{R}^N)} \right\} \mathcal{R}(u_j(\cdot/e^{\theta_j})) + o_j(1) \\
& \leq C_{10} \left( [u_j]^{2}_{H^s(\mathbb{R}^N)} \right)^{\frac{2^*}{2}} + q \left( C_{11} + C_{12} \Lambda^\delta \right) + o_j(1) \\
& \leq C_{10} \left( C' + q \left( C_8 + C_9 \Lambda^\delta \right) \right)^{\frac{2^*}{2}} + q \left( C_{11} + C_{12} \Lambda^\delta \right) + o_j(1).
\end{align*}
\]

Now, we argue by contradiction. If we suppose that there exists no subsequence \(\{u_j\}_{j \in \mathbb{N}}\) which is uniformly bounded by \(\Lambda\) in the \(H^s\)-norm, we can find \(j_0 \in \mathbb{N}\) such that

\[
\|u_j\|_{H^s(\mathbb{R}^N)} > \Lambda \quad \text{for all} \quad j \geq j_0.
\]

Without any loss of generality, we can assume that (3.19) is true for any \(u_j\). As a consequence, by using (3.16), (3.18) and (3.19), we can deduce that

\[
\Lambda^2 < \|u_j\|^2_{H^s(\mathbb{R}^N)} \leq C_{13} + C_{14} q \Lambda^{\frac{2^*}{2}} \Lambda^\delta
\]
which is impossible for $\Lambda$ large and $q$ small enough. Indeed, to see this, we can observe that it is possible to find $\Lambda_0$ such that $\Lambda_0^2 > C_{13} + 1$ and $q_0 = q_0(\Lambda_0)$ such that $C_{14}q \Lambda^{\frac{2s}{4-2s}} < 1$, for any $q < q_0$, and this gives a contradiction. 

At this point, we prove the following compactness result:

**Lemma 3.4.** Let $n \in \mathbb{N}$, $\Lambda_n, q_n > 0$ as in Proposition 3.1 and $\{(\theta_j, u_j)\}_{j \in \mathbb{N}} \subset \mathbb{R} \times H^s_{\text{rad}}(\mathbb{R}^N)$ be the sequence given in Lemma 3.3. Then $\{u_j\}_{j \in \mathbb{N}}$ admits a subsequence which converges in $H^s_{\text{rad}}(\mathbb{R}^N)$ to a nontrivial critical point of $\mathcal{F}_q$ at the level $b_n$.

**Proof.** By Proposition 3.1 we know that $\{u_j\}_{j \in \mathbb{N}}$ is bounded, so, by using Theorem 2.2 we can suppose, up to a subsequence, that there exists $u \in H^s_{\text{rad}}(\mathbb{R}^N)$ such that

$$
\begin{align*}
&u_j \rightharpoonup u \text{ weakly in } H^s_{\text{rad}}(\mathbb{R}^N), \\
&u_j \to u \text{ in } L^p(\mathbb{R}^N), \ 2 < p < 2^*_s, \\
&u_j \to u \text{ a.e. in } \mathbb{R}^N.
\end{align*}
$$

By the weak lower semicontinuity we know that

$$
[u]_{H^s(\mathbb{R}^N)}^2 \leq \liminf_{j \to \infty} [u_j]_{H^s(\mathbb{R}^N)}^2.
$$

Recalling that $\|u_j\|_{H^s(\mathbb{R}^N)} \leq \Lambda_n$ for any $j \in \mathbb{N}$, we can see that, for every $v \in H^s_{\text{rad}}(\mathbb{R}^N)$,

$$
\langle \mathcal{F}'_q(\theta_j, u_j), v \rangle = \langle (\mathcal{F}'_{q_n})'(\theta_j, u_j), v \rangle \\
= e^{(N-2s)\theta_j} \int_{\mathbb{R}^{2N}} \frac{u_j(x) - u_j(y)}{|x - y|^{N+2s}} (v(x) - v(y)) \, dx \, dy \\
+ q \sum_{i=1}^k e^{(\alpha_i + \beta_i)\theta_j} \langle \mathcal{R}'_i(e^{\beta_i} \theta_j u_j), v \rangle \\
+ e^{N\theta_j} \int_{\mathbb{R}^N} g_2(u_j)v \, dx - e^{N\theta_j} \int_{\mathbb{R}^N} g_1(u_j)v \, dx.
$$

Taking into account (3.22) and (iii) of Lemma 3.3 we have

$$
\begin{align*}
o_j(1) &= \langle \mathcal{F}'_q(\theta_j, u_j), u_j \rangle - \langle \mathcal{F}'_q(\theta_j, u_j), u_j \rangle \\
&= e^{(N-2s)\theta_j} \int_{\mathbb{R}^{2N}} \frac{u_j(x) - u_j(y)}{|x - y|^{N+2s}} [(u(x) - u(y)) - (u_j(x) - u_j(y))] \, dx \, dy \\
&\quad + q \sum_{i=1}^k e^{(\alpha_i + \beta_i)\theta_j} \langle \mathcal{R}'_i(e^{\beta_i} \theta_j u_j), u - u_j \rangle \\
&\quad + e^{N\theta_j} \int_{\mathbb{R}^N} g_2(u_j)(u - u_j) \, dx - e^{N\theta_j} \int_{\mathbb{R}^N} g_1(u_j)(u - u_j) \, dx.
\end{align*}
$$

Now, by applying the first part of Lemma 2.1 for $P(t) = g_i(t)$, $i = 1, 2, Q(t) = |t|^{2^*-1}$, $v_j = u_j$, $v = g_i(u)$, $i = 1, 2$ and $w \in C_{0}^{\infty}(\mathbb{R}^N)$, by (g3), (3.21) and (3.20) we can see,
as \( j \to \infty \)
\[
\int_{\mathbb{R}^N} g_i(u_j)w \, dx \to \int_{\mathbb{R}^N} g_i(u)w \, dx \quad i = 1, 2,
\]
so we obtain
\[
\int_{\mathbb{R}^N} g_i(u_j)u \, dx \to \int_{\mathbb{R}^N} g_i(u)u \, dx \quad i = 1, 2.
\] (3.24)
Taking \( X = H^s(\mathbb{R}^N), \), \( q_1 = 2, \) \( q_2 = 2^* s, \) \( v_j = u_j, \) \( v = g_1(u)u \) and \( P(t) = g_1(t)t \) in Lemma 2.2 by (3.3), (3.4) and (3.20) we deduce
\[
\int_{\mathbb{R}^N} g_1(u_j)u_j \, dx \rightarrow \int_{\mathbb{R}^N} g_1(u)u \, dx.
\] (3.25)
On the other hand, (3.20) and Fatou’s Lemma yield
\[
\int_{\mathbb{R}^N} g_2(u)u \, dx \leq \liminf_{j \to \infty} \int_{\mathbb{R}^N} g_2(u_j)u_j \, dx.
\] (3.26)
Putting together (3.23), (3.24), (3.25), (3.26), and by using (R3) we get
\[
\limsup_{j \to \infty} [u_j]^2_{H^s(\mathbb{R}^N)} = \limsup_{j \to \infty} e^{(N-2s)\theta_j}[u_j]^2_{H^s(\mathbb{R}^N)}
\]
\[
= \limsup_{j \to \infty} \left[ e^{(N-2s)\theta_j} \int_{\mathbb{R}^{2N}} \frac{u_j(x) - u_j(y)}{|x-y|^{N+2s}}(u(x) - u(y)) \, dx \, dy + q \sum_{i=1}^k e^{(\alpha_i+\beta_i)\theta_j} \langle R_i(e^{\beta_i\theta_j}u_j), u - u_j \rangle 
\right.
\]
\[
+ e^{N\theta_j} \int_{\mathbb{R}^N} g_2(u_j)(u - u_j) \, dx - e^{N\theta_j} \int_{\mathbb{R}^N} g_1(u_j)(u - u_j) \, dx \right]
\leq [u]^2_{H^s(\mathbb{R}^N)},
\] (3.27)
Therefore (3.21) and (3.27) give
\[
\lim_{j \to \infty} [u_j]^2_{H^s(\mathbb{R}^N)} = [u]^2_{H^s(\mathbb{R}^N)},
\] (3.28)
which, in view of (3.23), yields
\[
\lim_{j \to \infty} \int_{\mathbb{R}^N} g_2(u_j)u_j \, dx = \int_{\mathbb{R}^N} g_2(u)u \, dx.
\] (3.29)
Since \( g_2(t) = mt^2 + h(t), \) with \( h \) a positive and continuous function, by Fatou’s Lemma follows that
\[
\int_{\mathbb{R}^N} h(u) \, dx \leq \liminf_{j \to \infty} \int_{\mathbb{R}^N} h(u_j) \, dx \quad (3.30)
\]
\[
\int_{\mathbb{R}^N} u^2 \, dx \leq \liminf_{j \to \infty} \int_{\mathbb{R}^N} u_j^2 \, dx.
\] (3.31)
By using (3.29) and (3.30) we can see that
\[
\limsup_{j \to \infty} \int_{\mathbb{R}^N} m u_j^2 \, dx = \limsup_{j \to \infty} \int_{\mathbb{R}^N} (g_2(u_j) u_j - h(u_j)) \, dx
\]
\[
= \int_{\mathbb{R}^N} g_2(u) u \, dx + \limsup_{j \to \infty} \left( -\int_{\mathbb{R}^N} h(u_j) \, dx \right)
\]
\[
= \int_{\mathbb{R}^N} (m u^2 + h(u)) \, dx - \liminf_{j \to \infty} \int_{\mathbb{R}^N} h(u_j) \, dx
\]
\[
= \int_{\mathbb{R}^N} m u^2 \, dx + \int_{\mathbb{R}^N} h(u) \, dx - \liminf_{j \to \infty} \int_{\mathbb{R}^N} h(u_j) \, dx
\]
\[
\leq \int_{\mathbb{R}^N} m u^2 \, dx
\]
which, together with (3.31), implies that \( u_j \to u \) strongly in \( L^2(\mathbb{R}^N) \). Then, we have proved that \( u_j \to u \) strongly in \( H^s_{\text{rad}}(\mathbb{R}^N) \). Since \( b_n > 0 \), \( u \) is a nontrivial critical point of \( F_q \) at the level \( b_n \).

Now, we are ready to prove the main result of this Section:

**Proof of Theorem 3.1.** Let \( h \geq 1 \). Since \( b_n \to +\infty \) (see (2) of Lemma 3.2), up to a subsequence, we can consider \( b_1 < b_2 < \cdots < b_h \). Then, in view of Lemma 3.4 we define \( q(h) = q_b > 0 \) and we get the thesis.

**4. Proof of Theorem 1.1**

In this Section we give the proof of Theorem 1.1. Let us introduce the following functional
\[
F_q(u) = \frac{1}{2} \left( p + \frac{q}{2} (1 - s) [u^2_{H^s(\mathbb{R}^N)}] \right) [u^2_{H^s(\mathbb{R}^N)}] - \int_{\mathbb{R}^N} G(u) \, dx.
\]
In view of Theorem 3.1 it is enough to verify that
\[
\mathcal{R}(u) = \frac{1 - s}{4} \, [u^4_{H^s(\mathbb{R}^N)}]
\]
satisfies the assumptions (\( \mathcal{R}_1 \))-\( (\mathcal{R}_5) \).
Clearly \( \mathcal{R} \) is an even and nonnegative \( C^1 \)-functional in \( H^s(\mathbb{R}^N) \). Since \( [u^2_{H^s(\mathbb{R}^N)}] \leq ||u||^2_{H^s(\mathbb{R}^N)} \), we can see that the assumptions (\( \mathcal{R}_1 \)) and (\( \mathcal{R}_2 \)) are satisfied.
Regarding (\( \mathcal{R}_3 \)), suppose that \( u_j \rightharpoonup u \) weakly in \( H^s_{\text{rad}}(\mathbb{R}^N) \) and \( [u^2_{H^s(\mathbb{R}^N)}] \rightarrow \ell \geq 0 \). If \( \ell = 0 \), then we have finished. Let us assume \( \ell > 0 \). From the weak lower semicontinuity, we have
\[
[u^2_{H^s(\mathbb{R}^N)}] \leq \liminf_{j \to \infty} [u^2_{H^s(\mathbb{R}^N)}],
\]

(4.2)
Finally, we prove the condition \((\mathcal{R})\) of critical points in \(H^{s}_{\text{rad}}(\mathbb{R}^N)\ldots\)

By using the following properties of \(\limsup\) and \(\liminf\) for sequences of real numbers:

\[
\limsup_{j \to \infty} a_j b_j = a \limsup_{j \to \infty} b_j \quad \text{if} \quad a_j \to a > 0,
\]

\[
\limsup_{j \to \infty} (a_j + b_j) = a + \limsup_{j \to \infty} b_j \quad \text{if} \quad a_j \to a,
\]

\[
\limsup_{j \to \infty} (-a_j) = - \liminf_{j \to \infty} a_j,
\]

and by applying (4.2), we obtain

\[
\limsup_{j \to \infty} \langle \mathcal{R}'(u_j), u - u_j \rangle = (1 - s) \ell \limsup_{j \to \infty} \int_{\mathbb{R}^{2N}} \frac{(u_j(x) - u_j(y))(u(x) - u(y)) - (u_j(x) - u_j(y))}{|x - y|^{N + 2s}} \, dx \, dy
\]

\[
= (1 - s) \ell \limsup_{j \to \infty} \int_{\mathbb{R}^{2N}} \frac{u_j(x) - u_j(y)}{|x - y|^{N + 2s}} \, dx \, dy - \liminf_{j \to \infty} [u_j]^2_{\mathcal{H}^s(\mathbb{R}^N)}
\]

\[
= (1 - s) \ell \left( [u]^2_{\mathcal{H}^s(\mathbb{R}^N)} - \liminf_{j \to \infty} [u_j]^2_{\mathcal{H}^s(\mathbb{R}^N)} \right) \leq 0,
\]

which gives \((\mathcal{R}_3)\).

Now, recalling the definition of \(u_t\) and by using (4.1), it follows that \((\mathcal{R}_4)\) is verified because of

\[
\mathcal{R}(u_t) = \frac{1 - s}{4} \left( \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N + 2s}} \, dx \, dy \right)^2
\]

\[
= \frac{(1 - s) \ell^2(N - 2s)}{4} \left( \int_{\mathbb{R}^{2N}} \frac{u(x) - u(y)^2}{|x - y|^{N + 2s}} \, dx \, dy \right)^2
\]

\[
= \ell^{2(N - 2s)} \mathcal{R}(u).
\]

Finally, we prove the condition \((\mathcal{R}_5)\). By using a change of variable, we can see that, for any \(g \in \mathcal{O}(N)\)

\[
\mathcal{R}(u(g \cdot)) = \frac{1 - s}{4} [u(g \cdot)]^4_{\mathcal{H}^s(\mathbb{R}^N)} = \frac{1 - s}{4} [u]^4_{\mathcal{H}^s(\mathbb{R}^N)} = \mathcal{R}(u).
\]

Then, by applying Theorem 3.1, we can infer that for any \(h \in \mathbb{N}\), there exists \(q(h) > 0\) such that for any \(0 < q < q(h)\) the functional \(\mathcal{F}_q\) admits at least \(h\) couples of critical points in \(H^s(\mathbb{R}^N)\) with radial symmetry. This means that (1.1) admits at least \(h\) couples of weak solutions in \(H^s_{\text{rad}}(\mathbb{R}^N)\).

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