Black-Box Control for Linear Dynamical Systems

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Abstract

We consider the problem of controlling an unknown linear time-invariant dynamical system from a single chain of black-box interactions, and with no access to resets or offline simulation. Under the assumption that the system is controllable, we give the first efficient algorithm that is capable of attaining sublinear regret in a single trajectory under the setting of online nonstochastic control. We give finite-time regret bound of our algorithm, as well as a nearly-matching lower bound that shows this regret to be almost best-attainable by any algorithm.

1 Introduction

The ultimate goal in the field of adaptive control and reinforcement learning is to produce a truly independent learning agent. Such an agent can start in an unknown environment and follow one continuous and uninterrupted chain of experiences, until it performs as well as the optimal policy.

In this paper we consider this goal for the fundamental problem of controlling an unknown, linear time-invariant (LTI) dynamical system. This problem has received significant attention in the recent ML literature. However, all existing methods assume some knowledge about the environment, usually in the form of a stabilizing controller.

We henceforth describe a control algorithm that \textbf{only has black-box access} to an LTI system without any prior information. The algorithm is guaranteed to attain sublinear regret, converging on average to the performance of the best controller in hindsight from a set of reference policies. Furthermore, its guarantees apply to the setting of non-stochastic control, in which both the perturbations and cost functions can be adversarially chosen. Our results are accompanied by a novel lower bound on the start-up cost of computing a stabilizing controller. We show that this cost is inherently exponential in the natural parameters of the problem for any deterministic control method.

The question of controlling unknown systems under adversarial noise was posed in \cite{21}, our results quantify the difficulty of this task and provide an efficient solution. As far as we know, these results are the first finite-time sublinear regret bounds known for black-box, single-trajectory control in terms of both upper and lower bounds, in both the stochastic and nonstochastic models.

1.1 Statement of results

Consider a given LTI dynamical system with black-box access. The only interaction of the controller with the system is by sequentially observing a state $x_t$ and applying a control $u_t$. The evolution of the state is according to the dynamics equation

$$x_{t+1} = Ax_t + Bu_t + w_t,$$

where $x_t \in \mathbb{R}^{d_x}, u_t \in \mathbb{R}^{d_u}$. The system dynamics $A, B$ are unknown to the controller, and the disturbance $w_t$ can be adversarially chosen at the start of each round. An adversarially chosen convex cost function $c_t(x, u)$ is revealed after the controller’s action, and the controller suffers $c_t(x_t, u_t)$. In

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For a randomized control algorithm, we consider the expected cost. In the adversarial disturbances setting the optimal controller cannot be determined a priori, and depends on the disturbance realization. For this reason, we consider a comparative performance metric. The goal of the learning algorithm is to choose a sequence of controls \( \{u_t\}_{t=1}^T \) such that the total cost over \( T \) iterations is competitive with that of the best controller in a reference policy class \( \Pi \). Thus, the learner with only black-box access, and in a single trajectory, seeks to minimize regret defined as

\[
\text{Regret}_T(A) = J_T(A) - \min_{\pi \in \Pi} J_T(\pi).
\]

As a comparator class, we consider the set of all Disturbance-action Controllers (Definition 3), whose control is a linear function of past disturbances. This class is known to contain state of the art Linear Dynamical Controllers (Definition 4), and the \( H_2 \) and \( H_\infty \) optimal controllers.

Let \( L \) denote the upper bound on the system’s natural parameters, and \( \kappa^* \) be the controllability parameter of the stabilized system, see Section 2.1. The following statements summarize our main results in Theorem 6 and Theorem 8:

1. We give an efficient algorithm, whose regret in a single trajectory is with high probability at most

\[
\text{Regret}_T(A) \leq 2O(L \log L) + \tilde{O}(\text{poly}(L, \kappa^*)T^{2/3}).
\]

2. We show that any deterministic control algorithm must suffer a worst-case exponential startup cost in the regret, due to limited information. Formally, we show that for every deterministic controller \( A \), there exists an LTI where

\[
\text{Regret}_T(A) = 2^{\Omega(L)}.
\]

Further, this lower bound holds even without disturbances, and when the system-input matrix \( B \) is full rank.

To the best of our knowledge, these are the first finite-time regret bounds for control in a single trajectory with black-box access to the system. Our result quantifies the price of information for controlling unknown LTI systems to be exponential in the natural parameters of the problem, and show that this is inevitable.

One of the main technical hurdles to designing an efficient algorithm is obtaining a stabilizing controller from black-box interactions, in the presence of adversarial noise and changing convex costs. Our method consists of two phases. In the first phase, we identify the dynamics matrices up to some accuracy in the spectral norm by injecting large controls into the system. Previous works on system identification under adversarial noise either require non-explosive dynamics, or the knowledge of a strongly stable controller. However, our approach is not limited by these assumptions.

In the second phase, we use an SDP relaxation for the LQR proposed in (4) to solve for a strongly stable controller given the system estimates. The SDP was originally derived in the stochastic noise setting, but we show that it is applicable for our task. After we identify a strongly stable controller, we use the techniques of (8) for regret minimization.

For the lower bound, our approach is inspired by lower bounds for gradient-based methods from the optimization literature. Given a controller, we show a system construction that forces the states, and thus costs, to grow exponentially until all information about the system is learned. As far as we know, this is the first finite-time lower bound which is exponential in dimension for any online control setting.

### 1.2 Related work

The focus of our work is adaptive control, in which the controller does not have a priori knowledge of the underlying dynamics, and has to learn them as well as controlling the system according to given
convex costs. This task, recently put forth in the machine learning literature, differs substantially from the classical literature on control theory that we survey below in the following aspects:

1. The system is unknown ahead of time to the learner, nor is a stabilizing controller given.
2. The learner does not know the cost functions in advance. They can be adversarially chosen.
3. The disturbances are not assumed to be stochastic, and can be adversarially chosen.

To the best of our knowledge, our result is the only one thus far that holds under all three conditions.

**Robust and Optimal Control.** When the underlying system is known to the learner, the noise is stochastic, and costs are known, it is possible to compute the (closed or open loop) optimal controller a priori. In the LQR setting, \( x_{t+1} = Ax_t + Bu_t + w_t \) where \( w_t \) is i.i.d. Gaussian, and the learner incurs constant quadratic cost \( c(x, u) = \frac{1}{2} x^\top Q x + \frac{1}{2} u^\top R u \). The optimal policy for infinite horizon LQR is known to be linear: \( u_t = K x_t \), where \( K \) is the solution to the algebraic Ricatti equation (20, 22). Robust control was formulated in the framework of \( H_\infty \) control, which solves for the best controller under worst-case noise.

**LDS with Stochastic Noise:** The problem of controlling an LDS with known dynamics is extensively studied in the classical control literature, see (20, 22). In the online LQR setting (11, 12, 14), the noise remains i.i.d. Gaussian, but the performance metric is regret instead of cost. Recent algorithms in (12, 15, 4) attain \( \sqrt{T} \) regret, with polynomial runtime and polynomial dependence on relevant problem parameters in the regret. This was improved to \( O(\text{poly}(\log T)) \) in (8) for strongly convex costs. Regret bounds for partially observed systems were studied in (10, 11, 11), and the most recent bounds are in (15). In contrast to these results, our setting permits non-i.i.d., even adversarial noise sequences. Further, all of these results assume the learner is given a stabilizing controller.

**Nonstochastic Control:** The nonstochastic control problem for linear dynamical systems (2, 8) is the most relevant setting to our work. Under this setting, the controller has no knowledge of the system dynamics or the noise sequence. The controller generates controls \( u_t \) at each iteration to minimize regret over sequentially revealed adversarial convex cost functions, against all disturbance-action control policies. If a strongly stable controller is known, (8) gives an algorithm that achieves \( O(\text{poly}(L, \kappa^*) T^{2/3}) \) regret, where \( L \) is an upper bound on the system’s natural parameters and \( \kappa^* \) is the controllability parameter of the stabilized system, see Section 4.1. This was recently extended in (19) to partially observed systems, and better bounds for certain families of loss functions. However, all of these works assume that a stabilizing controller is given to the learner, and are not black-box as per our definition.

**Identification and Stabilization of Linear Systems:** If the system has stochastic noise, least squares can be used to identify the dynamics in the partially observable and fully observable settings (14, 18, 15). Using this method of system identification, recent work by (7) finds a stabilizing controller in finite time. However, no explicit bounds were given on the cost or the number of total iterations required to identify the system to sufficient accuracy. In contrast, our paper provides explicit bounds for optimally controlling the system. Moreover, least squares can lead to inconsistent solutions under adversarial noise. The algorithm in (10) tolerates adversarial noise, but the guarantees only hold for non-explosive systems. On the other hand, our results do not assume stability of the system (spectral radius bounded by 1), but only controllability.

2 Setting and Background

To enable the analysis of non-asymptotic regret bounds, we consider regret minimization against the class of strongly stable linear controllers. The notion of strong stability was formalized in (4) to quantify the rate of convergence to the steady-state distribution.

**Definition 1 (Strong Stability).** \( K \) is a \((\kappa, \gamma)\) strongly stable controller for a system \((A, B)\) if \( \|K\| \leq \kappa \), and there exist matrices \( H, L \) such that \( A + B K = H L H^{-1} \), and \( \|H\| \|H^{-1}\| \leq \kappa, \|L\| \leq 1 - \gamma \).

The regret definition in Section 1.1 is meaningful only when the comparator set \( K \) is non-empty. As shown in (4), a system \((A, B)\) has a strongly stable controller if it is strongly controllable. This notion is formalized in the next definition.
Assumption 2. The noise sequence is bounded such that \( x \). Under Assumption 1, the noiseless dynamical system \( k \) controllable system’s controllability index \( \kappa \) is at most \( d_x \).

Then \( (A, B) \) is \((k, \kappa)\) strongly controllable if \( C_k \) has full row-rank, and \( \|(C_k C_k^\top)^{-1}\| \leq \kappa \).

Assumption 1. The system \((A, B)\) is \((k, \kappa)\) strongly controllable for \( \kappa \geq 1 \), and \( \|A\|, \|B\| \leq \beta \) for some \( \beta \geq 1 \).

Under Assumption 1, the noiseless dynamical system \( x_{t+1} = Ax_t + Bu_t \) starting from \( x_1 \) can be driven to the zero state in \( k \) steps. Furthermore, Lemma B.4 in (8) gives an upper bound on the reset cost, defined as \( \sum_{t=1}^k \|x_t\|^2 + \|u_t\|^2 \). In Section 2.3 we show that a bounded reset cost implies the existence of a strongly stable controller. As a consequence of the Cayley–Hamilton theorem, a controllable system’s controllability index \( k \) is at most \( d_x \).

Finally we make the following mild assumptions on the noise sequence and the cost functions.

Assumption 2. The noise sequence is bounded such that \( \|u_t\| \leq 1 \) for all \( t \).

Assumption 3. The cost functions are convex, and for all \( x, u \) such that \( \|x\|, \|u\| \leq D \), \( \|\nabla_{x,u} c_t(x, u)\| \leq GD \). Without loss of generality, assume \( c_t(0, 0) = 0 \).

2.1 Notations

Using the convention from the theory of Linear Programming (13), we henceforth use \( \mathcal{L} \) to denote an upper bound on the natural parameters, which we interpret as the complexity of the system, i.e.

\[
\mathcal{L} = kd_u + d_x + G + \beta + \kappa,
\]

where

- \( \kappa \) = controllability parameter of the true system.
- \( k \) = controllability index of the true system.
- \( d_x \) = dimension of the states \( x_t \in \mathbb{R}^{d_x} \).
- \( d_u \) = dimension of the controls \( u_t \in \mathbb{R}^{d_u} \).
- \( G \) = upper bound on the Lipschitz constant of the cost functions \( c_t \).
- \( \beta \) = upper bound on the spectral norm of system dynamics \( A, B \).

Given a \((\tilde{\kappa}, \tilde{\gamma})\) strongly stable controller \( K \), we denote \( \kappa^* \) as the upper bound on the controllability parameter of the stabilized system \((A + BK, B)\), and \( \tilde{\kappa} \). We henceforth prove an upper bound on \( \kappa^* \) for the controller we recover, and show in Section 4.3 that \( \kappa^* \leq \text{poly}(\kappa, \beta^k, d_x) \). We use \( \tilde{O} \) to denote bounds that hold with probability at least \( 1 - \delta \), and omit the \( \log^k(\delta^{-1}) \) factor.

2.2 Disturbance-action Controller

In the canonical parameterization of the nonstochastic control problem, the total cost of a linear controller \( J(K) \) is not convex in \( K \). This problem is solved by considering a class of controllers called Disturbance-action Controllers (DACs) (2), which executes controls that are linear in past noises. The total cost of DACs is convex with respect to their parameters, and the cost of any strongly stable controller can be approximated by this class of controllers. Techniques in online convex optimization can then be used on this convex re-parameterization of the nonstochastic control problem. It is shown in (8) that for an unknown system \((A, B)\) and a known \((\kappa, \gamma)\) strongly stable controller \( K \), a DAC can achieve sublinear regret against all such controllers parametrized by \((K', M)\) where \( K' \) is \((\kappa, \gamma)\) strongly stable.

Definition 3 (Disturbance-action Controllers). A disturbance-action controller with parameters \((K, M)\) where \( M = [M^0, M^1, \ldots, M^{H-1}] \) outputs control \( u_t \) at state \( x_t \),

\[
u_t = Kx_t + \sum_{i=1}^{H} M^{i-1}u_{t-i}.\]

DACs also include the class of Linear Dynamic Controllers (LDCs), which is a generalization of static feedback controllers. Both \( \mathcal{H}_2 \) and \( \mathcal{H}_\infty \) optimal controllers under partial observation can be well-approximated by LDCs.
Indeed, a strongly stable controller can be extracted from any feasible solution to the SDP, as guaranteed by the following lemma.

\[ \nu > 0 \] for minimizing steady-state cost, proposed in (4). For this is composed of \((K, W)\) solution for the SDP, then the controller in Linear Quadratic control the cost functions are known ahead of time and fixed, and the noise is i.i.d., \(w_t \sim N(0, W)\). Given an instance of the LQ control problem defined by \((A, B, Q, R, W)\), the learner can obtain a strongly stable controller by solving the SDP relaxation for minimizing steady-state cost, proposed in (4). For \(\nu > 0\), the SDP is given by

\[
\text{minimize } J(\Sigma) = \left( \begin{array}{cc} Q & 0 \\ 0 & R \end{array} \right) \Sigma \\
\text{subject to } \Sigma_{xx} = (A \ B) \Sigma (A \ B)^\top + W, \ \Sigma = \left( \begin{array}{cc} \Sigma_{xx} & \Sigma_{xu} \\ \Sigma_{ux} & \Sigma_{uu} \end{array} \right), \\
\Sigma \succeq 0, \ \text{Tr}(\Sigma) \leq \nu.
\]

Indeed, a strongly stable controller can be extracted from any feasible solution to the SDP, as guaranteed by the following lemma.

**Lemma 5** (Lemma 4.3 in (4)). Assume that \(W \succeq \sigma^2 I\) and let \(\kappa = \sqrt{\nu/\sigma}\). Let \(\Sigma\) be any feasible solution for the SDP, then the controller \(K = \Sigma_{xx}^{-1/2} \Sigma_{xu} \Sigma_{xx}^{-1/2}\) is \((\kappa, 1/2\kappa^2)\) strongly stable.

**Existence of Strongly Stable Controllers** Suppose a noiseless system \(x_{t+1} = Ax_t + Bu_t\) can be driven to zero in \(k\) steps with resetting cost \(C\|x_1\|^2\). Theorem B.5 in (4) suggests that the SDP for the noisy system \(x_{t+1} = Ax_t + Bu_t + w_t\) with \(w_t \sim N(0, W)\) and \(\nu = C \cdot \text{Tr}(W)\) is feasible. Taking \(W = I\), the system \((A, B)\) has a \((\sqrt{Cd_x}, 1/(2Cd_x))\) strongly stable controller. Lemma B.4 in (4) shows that under Assumption 1, \(C = 3\kappa^2 k^2 \beta^k\).

### 3 Algorithm and main theorem

Now we describe our main algorithm for the black-box control problem, Algorithm 1. Overall we use the explore-then-commit strategy, and split the algorithm into three phases. In phase 1, we identify the underlying system dynamics to within some accuracy with large controls. In phase 2, we extract a strongly stable controller for the estimated system using the SDP in Section 2.3, and show that it is also strongly stable for the true system. We then alleviate the effects of using large controls by decaying the system to a state with constant magnitude. Finally in phase 3, we invoke Algorithm 1 in (8), which uses a DAC together with a known strongly stable controller to achieve sublinear regret.

Our main theorem below is stated using asymptotic notation that hides constants independent of the system parameters, and uses \(\mathcal{L}\) for an upper bound on the system parameters as defined in section 2.1.

**Theorem 6.** Under Assumptions 1, 2, 3, with high probability the regret of Algorithm 1 is at most

\[
\text{Regret}_T(A_1) \leq 2^{O(\mathcal{L} \log \mathcal{L})} + \tilde{O}(\text{poly}(\mathcal{L}, \kappa^* T^{2/3})).
\]

This is composed of

1. Phase 1: after \(T_1\) rounds we have \(\|x_{T_1}\|^2 \leq 2^{O(\mathcal{L} \log \mathcal{L})}\). The total cost is at most \(2^{O(\mathcal{L} \log \mathcal{L})}\).

2. Phase 2: Computing \(\hat{K}\) has zero cost. Decaying the system has total cost at most \(\tilde{O}(G\kappa^4 \|x_{T_1}\|^2 \gamma^{-3})\), where \(\kappa, \gamma\) are as defined in the algorithm. By the bound on \(\|x_{T_1}\|\), this phase has total cost \(2^{O(\mathcal{L} \log \mathcal{L})}\).

3. Phase 3: Nonstochastic control with a known strongly stable controller incurs regret at most \(\tilde{O}(\text{poly}(\mathcal{L}, \kappa^*)(T - T_1 - T_2)^{2/3})\), with high probability.
Algorithm 1 Nonstochastic Control with Black-box Access

1: Input: horizon $T$, $k$, $\kappa$ such that the system $(A, B)$ is $(k, \kappa)$ strongly controllable, $\beta \geq 1$ such that $\|A\|, \|B\| \leq \beta$.
2: Set $\varepsilon' = \sqrt{O(d_x \varepsilon')}, \gamma' = 1/(2\kappa'^2)$, where $C = 3\kappa'^2k^2\beta^6$.
3: **Phase 1: Black-box System Identification**
4: Set $\varepsilon = \frac{\gamma'^2}{10 \cdot d_x \varepsilon'}, \lambda = 8\beta$.
5: $(\hat{A}, \hat{B}) \leftarrow \text{AdvSysId}(\varepsilon, \lambda, x_1, k, \kappa)$ for $T_1 = d_u(k + 1) + 1$ rounds.
6: **Phase 2: Stable Controller Recovery**
7: $\hat{K} \leftarrow \text{ControllerRecovery}(\hat{A}, \hat{B}, \varepsilon', k', \gamma')$, set $\hat{\kappa} = \frac{2\kappa'^2d_i^{1/2}}{10d_x \varepsilon'}, \hat{\gamma} = \frac{\gamma'}{16d_x \varepsilon'}$.
8: Execute $\hat{K}$ for $T_2 = \max\{\frac{\ln(\hat{\gamma})}{x_1}, 0\}$ rounds.
9: **Phase 3: Nonstochastic Control**
10: Set $\kappa^* = 4\kappa'^2k^2\beta^2k^2, W = 2\kappa^*/\hat{\gamma}$.
11: Call Algorithm 1 in (3) with inputs $\hat{K}, \kappa^*, \hat{\gamma}, W$ for $T - T_1 - T_2$ rounds.

4 Analysis Outline

We provide an outline of our regret analysis in this section. All formal statements are in the appendix.

4.1 Black-box system identification

In this phase we obtain estimates of the system $\hat{A}, \hat{B}$ without knowing a stabilizing controller. Recall the definition of $C_k = [B, AB, \ldots , A^{k-1}B]$, and let $Y = [AB, A^2B, \ldots , A^kB]$. The procedure AdvSysId (Algorithm 2) consists of two steps. In the first step, we estimate each $A^jB$ for $j = 0, \ldots , k$ (in particular we obtain $\hat{B}$ close to $B$), and guarantee that $\|C_k - C_0\|_F, \|Y - C_1\|_F$ are small. In the second step, we take $A$ to be the solution to the system of equations in $X$: $XC_0 = C_1$. Note that Algorithm 2 is deterministic, and we bound the total cost in this phase instead of regret, matching the setting in our lower bound.

For the first step, the algorithm estimates matrices $A^jB$ by using controls that are scaled standard basis vectors once every $k + 1$ iterations, and using zero controls for the iterations in between. The state evolution satisfies

$$x_{t+1} = A^t x_t + \sum_{i=1}^t (A^{t-i} Bu_i + A^{t-i} w_i).$$

Intuitively, we choose scaling factors $\xi_i$ such that $j$ iterations after a non-zero control $\xi_i \cdot e_i$ is used, the state is dominated by $\xi_i A^{j-i} Be_i$, the scaled $i$-th column of $A^{j-i} B$. In the algorithm $\hat{M}_j$ is the concatenation of estimates for $A^j B e_i$, and we concatenate the $\hat{M}_j$’s to obtain $C_0, C_1$. We show in Lemma 2 that $\|\hat{M}_j - A^j B\|_F \leq O(d_x^2k^2\beta^2k^2\varepsilon_0)$, which implies the closeness of $C_0, C_1$ to $\hat{C}_k, Y$, respectively.

Under the assumption that $(A, B)$ is $(k, \kappa)$ strongly controllable, $A$ is the unique solution to the system of equations in $X$: $XC_0 = Y$. By perturbation analysis of linear systems, the solution to the system of equations $XC_0 = C_1$ is close to $A$, as long as $\|C_0 - C_k\|_F, \|C_1 - Y\|_F$ are sufficiently small. By our choice of $\varepsilon_0$, we conclude that $\|A - A\|_F \leq \varepsilon, \|B - B\|_F \leq \varepsilon$. Lemma 4 shows that the total cost of this phase is bounded by $O(\varepsilon \log \varepsilon)$.

4.2 Computing a stabilizing controller

The goal of phase 2 is to recover a strongly stable controller from system estimates obtained in phase 1 by solving the SDP presented in Section 2.3. The key to our task is setting the trace upper bound $\nu$ appropriately, so that the SDP is feasible and the recovered controller is strongly stable even for the original system. We justify our choice of $\nu$ in Lemma 18 and show that by our choice of $\varepsilon, \hat{A}, \hat{B}$ are sufficiently accurate and $\hat{K}$ is $(\hat{\kappa}, \hat{\gamma})$ strongly stable for the true system. We remark that (17) is an alternative procedure for recovering $K$, given system estimates.
Algorithm 2 AdvSysId

1: Input: accuracy parameter \( \varepsilon < 1/2, \| x_1 \| \leq 1 \). Let \( \lambda \geq 1 \) be such that \( \| A \|, \| B \| \leq \frac{1}{4} \lambda - 1 \), \( (k, \kappa) \) such that the system \((A, B)\) is \((k, \kappa)\) strongly controllable.
2: Set \( \varepsilon_0 = \frac{10^2 d_k^2 k^2 \lambda^2 \kappa d_k^2 \kappa^2 \varepsilon^2}{2} \).
3: for \( t = 1, \ldots, (k+1)d_u \) do
4: observe \( x_t \).
5: if \( t = 1 \mod (k+1) \) then
6: Let \( i = (t-1)/(k+1) + 1 \).
7: control with \( u_t = \xi_i \cdot e_i \) for \( \xi_i = \lambda^{t-1} \varepsilon_0^{-i} \), where \( e_i \) is the \( i \)-th standard basis vector.
8: else
9: control with \( u_t = 0 \).
10: end if
11: end for
12: For \( 0 \leq j \leq k, 1 \leq i \leq d_u \), define \( l(i, j) = (i-1)(k+1) + j + 2 \). Let \( x_t^j = x_{l(i,j)} \). Construct

\[
\hat{M}_j = [\begin{bmatrix} x_1^j & x_2^j & \cdots & x_{d_u}^j \end{bmatrix}] \in \mathbb{R}^{d_x \times d_u}.
\]
13: Define \( C_0 = [\hat{M}_0 \hat{M}_1 \cdots \hat{M}_{k-1}], C_1 = [\hat{M}_1 \hat{M}_2 \cdots \hat{M}_k] \in \mathbb{R}^{d_x \times d_u k} \).
14: Output \( \hat{A} = C_1 C_0^T (C_0 C_0^T)^{-1}, \hat{B} = \hat{M}_0 \).

Algorithm 3 ControllerRecovery

1: Input: \( \kappa', \gamma' \) such that there exists \( \hat{K} \) that is \((\kappa', \gamma')\) strongly stable for \((A, B)\); accuracy parameter \( \varepsilon \), and \( \hat{A}, \hat{B} \) such that \( \| A - \hat{A} \| \leq \varepsilon, \| B - \hat{B} \| \leq \varepsilon \).
2: Set \( \nu = \frac{2\kappa^4 d_k}{\gamma^2 \kappa} \).
3: Solve the following SDP:

\[
\begin{align*}
\text{minimize} & \quad 0 \\
\text{subject to} & \quad \Sigma_{xx} = (\hat{A} \hat{B}) \Sigma (\hat{A} \hat{B})^T + I, \text{ where} \\
& \quad \Sigma = \begin{bmatrix} \Sigma_{xx} & \Sigma_{xu} \\ \Sigma_{xu}^T & \Sigma_{uu} \end{bmatrix}, \Sigma \succeq 0, \text{Tr}(\Sigma) \leq \nu.
\end{align*}
\]
4: Denote a feasible solution as \( \hat{\Sigma} = \begin{bmatrix} \Sigma_{xx} & \Sigma_{xu} \\ \Sigma_{xu}^T & \Sigma_{uu} \end{bmatrix}, \) return \( \hat{K} = \hat{\Sigma}_{uu}^{-1} \).

4.2.1 Decaying the system

In phase 1 the algorithm uses large controls to estimated the system, and after \( T_1 \) iterations the state might have an exponentially large magnitude. Equipped with a strongly stable controller, we decay the system so that the state has a constant magnitude before starting phase 3. We show in Lemma 19 that following the policy \( u_t = K x_t \) for \( T_2 \) iterations decays the state to at most \( 2\kappa / \gamma \) in magnitude.

4.3 Nonstochastic control

Given a strongly stable controller \( \hat{K} \) for the underlying system, we run Algorithm 1 in (8) (Algorithm 4 in the appendix) which achieves sublinear regret. By Lemma 21, the system \((A + B\hat{K}, B)\) is \((\hat{k}, 4\hat{k}^2 \beta^2 \kappa)\) strongly controllable.

If we start Algorithm 4 from \( t = T_1 + T_2 \), the setting is consistent with the non-stochastic control setting where the noise is bounded by \( \| x_{T_1 + T_2} \| \), and with total iteration number \( T = T_1 + T_2 \). By Theorem 12 in (8) (Theorem 22 in the appendix) setting \( \kappa^* = 4\hat{k}^2 \beta^2 \kappa, W = 2\kappa^*/\gamma \), and
noticing that $\tilde{\gamma}^{-1} = \text{poly}(\kappa^*)$, with high probability, our total regret is at most

$$\tilde{O}(\text{poly}(\kappa^*, k, d_x, d_u, G)T^{2/3}).$$

5 Lower Bound on Black-box Control

In this section we prove that any deterministic black-box control algorithm incurs a loss which is exponential in the system dimension, even for stochastic control of LTI systems. We note that this lower bound is only consistent with the setting in phases 1 and 2, where the algorithms are deterministic.

Definition 7 (Black-box Control Algorithm). A (deterministic) black-box control algorithm $A$ outputs a control $u_t$ at each iteration $t$, where $u_t$ is a function of past information, i.e. $u_t = A(x_1, \ldots, x_t, c_1, \ldots, c_t)$. The next lemma justifies this iterative construction of $V$.

Lemma 8. Let $A$ be a control algorithm as per Definition 7. Then there exists a stabilizable system that is also $(1, 1)$-strongly controllable, and a sequence of oblivious perturbations and costs, such that with $x_1 = e_1$, and $T = d_x$, we have

$$\text{Regret}_T(A) = \tilde{O}(\kappa^*).$$

Let $c_t(x, u) = \|x\|^2 + \|u\|^2$ for all $t$. Consider the noiseless system $x_{t+1} = Q^TVx_t + u_t$ for some $Q$ and orthogonal $V$. Under this system $w_t = 0$ for all $t$, and a stabilizing controller is $K = -Q^TV$. Observe that $J(K)$ is constant. The system is also $(1, 1)$ strongly controllable because $B = I$. Let $V_i, Q_i \in \mathbb{R}^{d_x \times d_x}$ denote the rows of $V$ and $Q$, respectively. Fix a deterministic algorithm $A$, and let $u_t = A(x_1, x_2, \ldots, x_t, c_1, \ldots, c_t)$ be the control produced by $A$ at time $t$. There exists $Q, V$ such that under this system, $A$ outputs controls such that $\|x_{d_x}\| \geq 2^{d_x-1}$.

The construction. Set $x_1 = e_1$. We construct $Q$ and $V$ as follows: let $y_0 = e_1$, set $V_1 = y_0^T = e_1^T$; for $i = 1, \ldots, d_x - 1$, define

$$z_i = \begin{cases} u_i & \text{if } u_i \not\in \text{span}(V_1^T, \ldots, V_{i-1}^T) \\ v & \text{s.t. } v \in \text{span}(V_1^T, \ldots, V_{i-1}^T)^\perp, \|v\| = 1 & \text{otherwise} \end{cases}$$

Let $y_i$ be the component of $z_i$ that is independent of $V_1^T, \ldots, V_{i-1}^T$,

$$y_i = \frac{z_i - \sum_{j=1}^{i} \Pi_{V_j^T}(z_i)V_j^T}{\|z_i - \sum_{j=1}^{i} \Pi_{V_j^T}(z_i)V_j^T\|},$$

where $\Pi_v(z)$ denotes the projection of $z$ onto vector $v$. Set $Q_i = d_i y_i^T$ for some $d_i \neq 0$ to be specified later, and set $V_{i+1} = y_i^T$.

The next lemma justifies this iterative construction of $V$ by showing that the trajectory $x_1, \ldots, x_t$ is not affected by the choice of $V_i, Q_i$ for $i \geq t$. As a result, without loss of generality we can set $V_t$ after obtaining $x_t$, and set $Q_t$ after receiving $u_t$.

Lemma 9. As long as $V$ is orthogonal, the states satisfy $x_1 = \sum_{i=1}^{t-1} c_i V_i^T + \sum_{i=1}^{t-1} c_i y_{t-1}$ for some constants $c_i$ that only depend on $A$ and $\{Q_i\}_{i=1}^{t-1}$.

Proof. We prove the lemma by induction. For our base case, $x_1$ is trivially $c_1^T e_1$ and it is fixed for all choices of $Q, V$. Set $V_1 = e_1^T$. Assume the lemma is true for $x_i$, and we have specified $V_i$ for $i \leq t$, $Q_i$ for $i < t$. The specified rows of $V$ are orthonormal by construction. Note that by our construction, $x_i$ is obtained first, and then we set $V_i = y_{t-1}^T$. Since $u_0$ only depends on the current trajectory up to $x_i$, it is well-defined, and we can obtain $z_i$. By definition of $y_i$, we can write $u_t = \sum_{i=1}^{t} a_t^i V_i^T + a_t^i y_{t-1}$ for some coefficients $a_t^i$. Set $Q_t = d_t y_t^T$ as in the lemma. The next state
is then
\[
x_{t+1} = Q^T V x_t + u_t = Q^T V \sum_{i=1}^t c_i^t V_i^T + \sum_{i=1}^t a_i^t V_i^T + a_{t+1}^t y_t
\]
\[
V_i = y_{t-1}
\]
\[
= \sum_{i=1}^t c_i^t Q_i^T e_i + \sum_{i=1}^t a_i^t V_i^T + a_{t+1}^t y_t
\]
\[
= \sum_{i=1}^{t-1} c_i^t Q_i^T + \sum_{i=1}^t a_i^t V_i^T + a_{t+1}^t y_t
\]
\[
= \sum_{i=1}^t c_i^{t+1} V_i^T + c_i^t d_t y_t + a_{t+1}^t y_t
\]
\[
= \sum_{i=1}^t c_i^{t+1} V_i^T + c_i^t d_t y_t + a_{t+1}^t y_t
\]
\[
Q_i = d_i y_i^T = d_i V_i^T
\]
\[
V_i = y_{t-1}
\]
\[
V is orthogonal\]
\[
(1)
\]
We have shown in the inductive step that \(x_{t+1}\) does not depend on the choice of \(V_{t+1}\) as long as \(V\) is orthogonal, hence we can set \(V_{t+1} = y_{t-1}^T\). Moreover, \(x_{t+1}\) is not affected by \(Q_i\) for \(i \geq t + 1\) by inspection.

The magnitude of the state. In this section we specify the constants \(d_i\) in the construction to ensure that the state has an exponentially increasing magnitude. Let \(u_t = \sum_{j=1}^t a_j^t V_j^T + a_{t+1}^t y_t\), \(x_t = \sum_{j=1}^{t-1} c_j^t V_j^T + c_t^t y_{t-1}\). Set \(d_i = \text{sign}(c_i^t) \cdot \text{sign}(a_{i+1}^t) \cdot 2\). The quantities \(c_i^t\) and \(a_{i+1}^t\) are well-defined when we set \(Q_i\) after obtaining \(x_t\) and \(u_t\). Intuitively, \(Q^T V\) applied to \(x_t\) aligns \(y_{t-1}\) to \(y_t\), and we grow the magnitude of the \(y_t\) component in \(x_{t+1}\) multiplicatively.

**Lemma 10.** The states satisfy \(x_t = \sum_{i=1}^t c_i^t V_i^T\), and \(|c_i^t| \geq 2|c_{i-1}^t|\).

**Proof.** By equation (1) in Lemma 9 we can express \(x_{t+1} = \sum_{i=1}^t c_i^{t+1} V_i^T + c_t^t d_t y_t + a_{t+1}^t y_t\). As we claimed before, since \(x_{t+1}\) does not depend on the choice of \(V_{t+1}\), we set \(V_{t+1} = y_t\), and write \(x_{t+1} = \sum_{i=1}^{t+1} c_i^{t+1} V_i^T\). By our choice of \(d_t\), we have
\[
c_{t+1}^{t+1} = c_t^t d_t + a_{t+1}^t = \text{sign}(c_t^t) \cdot \text{sign}(a_{t+1}^t) \cdot 2c_t^t + a_{t+1}^t = a_{t+1}^t(2|c_t^t| + |a_{t+1}^t|)\]
We conclude that \(|c_{t+1}^{t+1}| = 2|c_t^t| + |a_{t+1}^t| \geq 2|c_t^t|\).

We observe that \(x_1 = c_1^1 e_1\) where \(|c_1^1| = 1\); therefore we have \(\|x_2\| \geq |c_2^{d_2}| \geq 2^{d_2-1}\).

Size of the system. Our construction only requires \(Q_1, \ldots, Q_{d_t-1}\) to be specified, and without loss of generality we take \(Q_{d_t} = d_t V_1 = 2V_1\). By inspection, \(Q\) can be written as \(Q = DPV\), where \(D = \text{diag}(d_1, d_2, \ldots, d_{d_t})\) and \(P\) is a permutation matrix that satisfies \((PV)_i = V_{i+1} (\text{mod } d_j)\). Therefore the spectral norm of \(Q^T V\) is at most \(\|Q\| \|V\| \leq 2\). We conclude that for this system, \(\mathcal{L} = d_u + d_x + 4\), and the total cost is at least \(2^{O(\mathcal{L})}\).

**6 Conclusion**

We present the first end-to-end, efficient black-box control algorithm for unknown linear dynamical systems in the nonstochastic control setting. With high probability the regret of our algorithm is sublinear in \(T\) with an exponential startup cost. Further, we show that this startup cost is nearly optimal by giving a lower bound for any deterministic black-box control algorithm.

As far as we know, this is the first explicit regret upper and lower bounds for black-box online control even in the stochastic setting with quadratic losses, for general systems. Our bounds hold, however, under the more difficult setting of adversarial noise and general convex cost functions.
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A Proofs for Section 4.1

In this section we present proofs for phase 1 of Algorithm 1. We show that the estimates \( \hat{A}, \hat{B} \) satisfy \( \|\hat{A} - A\| \leq \varepsilon, \|\hat{B} - B\| \leq \varepsilon, \) and bound the total cost of this phase. We first bound the magnitude of states in each iteration to guide our choice of scaling factors \( \xi \).

**Claim 11.** In Algorithm 2 for all \( t = 2, \ldots, (k + 1)d_w, \) let \( j = t - 2 \pmod{k + 1}, \) \( i = (t - 2 - j)/(k + 1) + 1, \) we have \( \|x_t\| \leq \lambda^{t-1}\varepsilon_0^{-i} \).

**Proof.** This can be seen by induction. For our base case, consider \( x_2, \) where \( i = 1, j = 0. \) \( \|x_2\| \leq \|A\| + \|B\|\|u_1\| + 1 \leq \frac{1}{4}\lambda(1 + \varepsilon_0^{-1}) + 1 \leq \lambda\varepsilon_0^{-i}. \) Assume \( \|x_t\| \leq \lambda^{t-1}\varepsilon_0^{-i} \) for \( t = (i - 1)(k + 1) + j + 2. \) If \( j = k, \) then \( t = i(k + 1) + 1, \) \( \|u_t\| = \lambda^{t-1}\varepsilon_0^{-i-1}, \) and \( t + 1 = i(k + 1) + 2. \) Therefore

\[
\|x_{t+1}\| \leq \|A\|\|x_t\| + \|B\|\|u_t\| + \|w_t\| \leq \frac{1}{4}\lambda(\lambda^{t-1}\varepsilon_0^{-i} + \lambda^{t-1}\varepsilon_0^{-i-1}) + 1 \leq \lambda^{t-1}\varepsilon_0^{-i}.
\]

With appropriate choice of \( \xi, \) we ensure that \( \hat{M}_j \) and \( A^jB \) are close in the Frobenius norm.

**Lemma 12.** For \( j = 0, \ldots, k, \) \( \hat{M}_j \) satisfies

\[
\|\hat{M}_j - A^jB\|_F \leq 3d_w^2 \kappa\lambda 2^k\varepsilon_0.
\]

In particular, \( \|\hat{M}_0 - B\| \leq 3d_w^2 \kappa\lambda 2^k\varepsilon_0 \leq \varepsilon. \)

**Proof.** Observe that by definition,

\[
x_{t+1} = A^t x_1 + \sum_{s=1}^t A^{t-s}Bu_s + A^{t-s}w_s.
\]

We have \( \|A^t x_1 + \sum_{s=1}^t A^{t-s}w_s\| \leq \lambda^t + \sum_{s=1}^t \lambda^{t-s} \leq (t + 1)\lambda^t. \) Note that the magnitude of this term should be small once we normalize by \( \xi. \) Let \( j = t - 2 \pmod{k + 1}, \) \( i = (t - 2 - j)/(k + 1) + 1, \) then \( t = (i - 1)(k + 1) + j + 2. \) We proceed to bound \( \|x_t/\xi - (A^jB)\|, \) where \( (A^jB)_i \) is the \( i \)th column of \( A^jB. \) The largest sum in \( x_t \) can be analyzed as follows,

\[
\sum_{s=1}^{t-1} A^{t-1-s}Bu_s = \sum_{s=1}^{i-1} A^{(i-1)(k+1)+j+1-s}Bu_s.
\]

Normalizing by the scaling factor,

\[
\frac{1}{\xi^i} \sum_{s=1}^{t-1} A^{t-1-s}Bu_s = \xi_0^i \lambda^{(i-1)(k+1)} \sum_{r=1}^{i} \xi_0^{-r} \lambda^{(r-1)(k+1)} A^{(i-r)(k+1)+j}Be_r.
\]

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The second term can be bounded as
\[
\| \sum_{r=1}^{i-1} \varepsilon_0^{-r} \lambda^{(r-i)(k+1)} A^{(i-r)(k+1)+j} B e_r \| \leq \sum_{r=1}^{i-1} \varepsilon_0^{-r} \lambda^{(r-i)(k+1)} \| A^{(i-r)(k+1)+j} B \|
\]
\[
\leq \lambda^{i+1} \sum_{r=1}^{i-1} \varepsilon_0^{-r} \leq (d_u - 1) \lambda^{2k} \varepsilon_0.
\]

Let \((\hat{M}_j)_i\) denote the \(i\)-th column of \(\hat{M}_j\), then we have
\[
\| (\hat{M}_j)_i - (A^j B)_i \| = \left\| \frac{x_j^i}{\xi_i} - (A^j B)_i \right\|
\]
\[
\leq \frac{1}{\xi_i} \| A^{t-1} x_1 + \sum_{s=1}^{t-1} A^{t-1-s} w_s \| + \frac{1}{\xi_i} \| \sum_{s=1}^{t-1} A^{t-1-s} B u_s - (A^j B)_i \|
\]
\[
\leq \frac{1}{\xi_i} t \lambda^{t-1} + (d_u - 1) \lambda^{2k} \varepsilon_0
\]
\[
\leq t \xi_0 \lambda^{2k} + (d_u - 1) \lambda^{2k} \varepsilon_0 \leq 3d_u k \lambda^{2k} \varepsilon_0.
\]

Thus we can bound the Frobenius norm of \(\hat{M}_j - A^j B\) by
\[
\| \hat{M}_j - A^j B \|_F^2 = \sum_{i=1}^{d_u} \| (\hat{M}_j)_i - (A^j B)_i \|^2 \leq 9d_u^2 k^2 \lambda^{4k} \varepsilon_0^2.
\]

We show that by our choice of \(\varepsilon_0\), \(\|C_0 - C_k\|_F\), \(\|C_1 - Y\|_F\) are sufficiently small to guarantee \(\hat{A}\) and \(A\) are close.

**Lemma 13.** Algorithm \([2]\) outputs \(\hat{A}\) such that \(\| \hat{A} - A \| \leq \varepsilon\).

**Proof.** By Lemma \([12]\) for all \(j\), \(\| \hat{M}_j - A^j B \|_F \leq 3d_u^2 k \lambda^{2k} \varepsilon_0\). Let \(C_k = [B \ A B^2 \ A B^3 \cdots A^{k-1} B]\), and \(Y = [AB \ A^2 B \cdots A^k B]\). We have
\[
\|C_0 - C_k\|_F^2 = \sum_{j=0}^{k-1} \| \hat{M}_j - A^j B \|_F^2 \leq 9d_u^4 k^2 \lambda^{4k} \varepsilon_0^2.
\]

Similarly, \(\|C_1 - Y\|_F^2 \leq 9d_u^4 k^3 \lambda^{4k} \varepsilon_0^2\).

Recall that \(A\) is the unique solution to the system of equations in \(X\): \(X C_k = Y\). Let \(A_i\) denote the \(i\)-th row of \(A\), and let \(\hat{A}_i\) denote the \(i\)-th row of \(\hat{A}\). By Lemma 22 in \([3]\), as long as \(\|C_0 - C_k\|_F \leq \sigma_{\text{min}}(C_k)\),
\[
\| A_i - \hat{A}_i \| \leq \frac{\| C_1 - Y \|_F + \| C_0 - C_k \|_F \| A_i \|}{\sigma_{\text{min}}(C_k)}
\]

By our assumption, \(\| (C_k C_k^T)^{-1} \| \leq \kappa\), so \(\sigma_{\text{min}}(C_k) \geq \kappa^{-1/2}\). We have
\[
\|C_0 - C_k\|_F \leq 3d_u^4 k^2 \lambda^{2k} \varepsilon_0 \leq \frac{\varepsilon}{2 \lambda^k d_x \sqrt{\kappa}} \leq \kappa^{-1/2} \leq \sigma_{\text{min}}(C_k).
\]

Further notice that \(\| A \| \leq \lambda\) implies \(\| A_i \| \leq \| A \|_F \leq \lambda \sqrt{d_x}\),
\[
\| A_i - \hat{A}_i \| \leq \frac{3d_u^2 k^2 \lambda^{2k} \varepsilon_0 (1 + \lambda \sqrt{d_x})}{\kappa^{-1/2} - 3d_u^2 k^2 \lambda^{2k} \varepsilon_0} \leq \frac{\varepsilon}{\sqrt{d_x}}.
\]

Finally, we have
\[
\| A - \hat{A} \| \leq \| A - A\|_F = \sqrt{\sum_{i=1}^{d_u} \| A_i - \hat{A}_i \|^2} \leq \varepsilon.
\]
Lemma 14. The total cost of estimating $A, B$ starting from $\|x_1\| \leq 1$ is bounded by

$$G(10^5 \lambda^{10k} \varepsilon^{-2} \kappa d_x^2 k^5 d_u^5) d_u.$$

Proof. The magnitude of the state and control is bounded by

$$\|x_t\|^2 + \|u_t\|^2 \leq 2 \lambda^{2\varepsilon-2} \varepsilon^{-2} d_u = 2 \lambda^{4d} k \varepsilon^{-2} = 2(4 \varepsilon^{-2} d_u k^4 \lambda^{10k} d_u^2) d_u$$

By Assumption taking $D^2 = 2(10^4 \varepsilon^{-2} d_u k^4 \lambda^{10k} d_u^2) d_u$, 

$$c_t(x_t, u_t) \leq \|\nabla_{(x,u)} c_t(x_t, u_t)\| \leq 2GD^2.$$ 

Summing over $(k+1)d_u \leq 2kd_u$ iterations, the total cost is upper bounded by 

$$8Gkd_u(10^4 \varepsilon^{-2} d_u k^4 \lambda^{10k} d_u^2) d_u \leq G(10^5 \lambda^{10k} \varepsilon^{-2} \kappa d_x^2 k^5 d_u^5) d_u.$$

Using our choice of $\varepsilon$ and $\lambda$, the total cost is bounded by 

$$G(10^5 \lambda^{10k} \varepsilon^{-2} \kappa d_x^2 k^5 d_u^5) d_u \leq G(10^{25k} \beta^{10k} \gamma^2 - 4 \kappa d_x^5 k^5 d_u^5 \kappa^{16}) d_u$$

$$\leq G(10^{30k} \beta^{10k} \kappa d_x^5 k^5 d_u^5 \kappa^{24}) d_u$$

$$\leq G(10^{30k} \beta^{10k} \kappa d_x^5 k^5 d_u^5 \kappa^{12}) d_u$$

$$\leq G(10^{40k} \beta^{25k} \kappa d_x^5 k^5 d_u^5 \kappa^{25}) d_u$$

$$= 2O(L \log L).$$

B Proofs for Section 4.2

In this section we prove that a $(\tilde{\kappa}, \tilde{\gamma})$ strongly stable controller can be obtained by solving the SDP in Algorithm. We first argue that for two systems close in spectral norm, a strongly stable controller for one system is also strongly stable for the other system.

Lemma 15. If $K$ is $(\kappa, \gamma)$ strongly stable for a system $(A, B)$ with $\kappa \geq 1$, and if $\tilde{A}, \tilde{B}$ satisfy $\|A - A\| \leq \varepsilon, \|B - B\| \leq \varepsilon$, then $K$ is $(\kappa, \gamma - 2\varepsilon\kappa^2)$ strongly stable for $(\tilde{A}, \tilde{B})$.

Proof. By definition, we have 

$$\tilde{A} + \tilde{B} K = A + BK - A - BK + \tilde{A} + \tilde{B} K = HLH^{-1} + (\tilde{A} - A) + (\tilde{B} - B) K$$

$$= H(L + H^{-1}(\tilde{A} - A + (\tilde{B} - B) K) H) H^{-1}$$

The lemma follows by observing that 

$$\|H^{-1}(\tilde{A} - A + (\tilde{B} - B) K) H\| \leq 1 - \gamma + \kappa \varepsilon(1 + \kappa) \leq 1 - \gamma + 2\varepsilon\kappa^2.$$ 

Now, we use Lemma twice to show that the recovered controller $\tilde{K}$ is strongly stable for the original system $(A, B)$. The following lemma computes $\tilde{\kappa}, \tilde{\gamma}$ in terms of $\varepsilon$.

Lemma 16. Algorithm returns $\tilde{K}$ that is $(\tilde{\kappa}, \tilde{\gamma})$ strongly stable for $A$ and $B$, where 

$$\tilde{\kappa} = \left(\kappa^{4d} d_x^2 \right)^{1/2}, \quad \tilde{\gamma} = \gamma' - \frac{2\varepsilon\kappa^2}{4d_x k^4} - 2\varepsilon \tilde{\kappa}^2.$$ 

Proof. We show in Section that a $(\kappa', \gamma')$ strongly stable controller exists for $(A, B)$. Let $K$ be a $(\kappa', \gamma')$ strongly stable controller. By Lemma and Lemma $K$ is $(\tilde{\kappa}, \tilde{\gamma})$ strongly stable for $\tilde{A}, \tilde{B}$, where $\tilde{\kappa} = \kappa', \tilde{\gamma} = \gamma' - 2\varepsilon\kappa^2$. With knowledge of $\tilde{\kappa}, \tilde{\gamma}$, we can set the trace upper bound
appropriately to extract a strongly stable controller from a feasible solution of the SDP. Specifically, we set

\[ \nu = \frac{2\kappa^4 d_u}{\gamma} \]

as in Lemma 18 and the SDP is feasible. We obtain \( \hat{K} \) that is \((\hat{\kappa}, \hat{\gamma})\) strongly stable for the system \( \hat{A}, \hat{B} \), where \( \hat{\kappa} = \frac{\hat{\gamma}^2 \sqrt{d_u}}{\sqrt{\gamma}} = \left( \frac{\kappa^4 d_u}{\gamma} \right)^{1/2} \), \( \hat{\gamma} = \frac{\gamma}{4d_u \kappa^{4/3}} = \frac{\gamma}{4d_u \kappa^{4/3}} \). We apply Lemma 18 again and conclude that \( \hat{K} \) is \((\hat{\kappa}, \hat{\gamma} - 2\hat{\kappa}^2)\) strongly stable for \( A, B \).

With our choice of \( \epsilon, \) we compute the final values of \( \hat{\kappa}, \hat{\gamma} \).

**Lemma 17.** Setting \( \epsilon = \frac{\sqrt{\gamma}}{10d_u \kappa^{4/3}} \), \( \hat{K} \) returned by Algorithm 3 is \((\frac{2\kappa^4 d_u}{\gamma}, \frac{\gamma}{10d_u \kappa^{4/3}})\) strongly stable for \((A, B)\).

**Proof.** With this choice of \( \epsilon \), we have \( 2\epsilon \kappa^2 = \frac{2\epsilon \gamma^2}{10d_u \kappa^{4/3}} \leq \frac{\gamma}{2} \). It follows that

\[ \hat{\kappa} = \left( \frac{\kappa^4 d_u}{\gamma} \right)^{1/2} \leq \frac{2\kappa^2 \sqrt{d_u}}{\sqrt{\gamma}}. \]

Therefore we have \( 2\epsilon \kappa^2 \leq \frac{\gamma}{10d_u \kappa^{4/3}} \). We obtain a lower bound on \( \hat{\gamma} \) as follows

\[ \hat{\gamma} = \frac{\gamma}{4d_u \kappa^{4/3}} - 2\epsilon \kappa^2 \geq \frac{\gamma}{4d_u \kappa^{4/3}} - \frac{\gamma}{10d_u \kappa^{4/3}} \geq \frac{\gamma}{8d_u \kappa^{4/3}} \geq \frac{\gamma}{16d_u \kappa^{4/3}}. \]

The following lemma details how we set the trace upper bound \( \nu \) in the SDP, and our application of results from (4) to extract \( \hat{K} \).

**Lemma 18.** For any system \( A, B \) with a \((\kappa, \gamma)\) strongly stable controller, the SDP in Algorithm 3 defined by \((A, B)\) with trace constraint \( \nu = \frac{2\kappa^2 d_u}{\gamma} \) is feasible. Moreover, a policy \( K \) such that \( \hat{K} \) is \((\frac{\kappa^4 d_u}{\gamma}, \frac{\gamma}{10d_u \kappa^{4/3}})\) strongly stable for \( A, B \) can be extracted from any feasible solution of the SDP.

**Proof.** We first show that the SDP is feasible. Let \( K \) be the \((\kappa, \gamma)\) strongly stable controller for \((A, B)\), and consider the system with Gaussian noise \( x_{t+1} = Ax_t + Bu_t + w_t, w_t \sim N(0, I) \). This system will converge to a steady state where the state covariance \( X = \mathbb{E}[xx^\top] \) satisfies

\[ X = (A + BK)X(A + BK)^\top + I. \]

Let \( KXK^\top \) be the steady-state covariance of \( u \) when following \( K \). By Lemma 3.3 in (4), \( \text{Tr}(X) \leq \frac{\kappa^2 d_u}{\gamma}, \text{Tr}(KXK^\top) \leq \frac{\gamma}{4d_u \kappa^{4/3}} \).

Consider the matrix

\[ \Sigma = \begin{pmatrix} X & XK^\top \\ KX & KK^\top \end{pmatrix}. \]

By Lemma 4.1 in (4), \( \Sigma \) is feasible for the SDP if \( \nu \geq \text{Tr}(X) + \text{Tr}(KXK^\top) \); since \( \nu = \frac{2\kappa^2 d_u}{\gamma}, \Sigma \) is feasible for the SDP. Now let \( \hat{\Sigma} \) be any feasible solution of the SDP, and write

\[ \hat{\Sigma} = \begin{pmatrix} \hat{\Sigma}_{xx} & \hat{\Sigma}_{xu} \\ \hat{\Sigma}_{ux}^\top & \hat{\Sigma}_{uu} \end{pmatrix}. \]

Consider \( \hat{K} = \hat{\Sigma}_{uu}^{-1} \hat{\Sigma}_{xu} \), which is well-defined because \( \hat{\Sigma}_{xx} \succeq I \) by the steady-state constraint. As shown in Lemma 4.3 in (4), \( \hat{K} \) is \((\sqrt{\nu}/(2\nu))\) strongly stable for \( A, B \). Under our choice of \( \nu \), \( \hat{K} \) is \((\frac{\kappa^4 d_u}{\gamma}, \frac{\gamma}{16d_u \kappa^{4/3}})\) strongly stable for \( A, B \).
B.1 Decaying the System

**Lemma 19.** Let \( K \) be a \((\check{\kappa}, \check{\gamma})\) strongly stable controller for the system, and \( x_1 \) be any starting state. Suppose \( \check{\kappa} \geq 1 \). After following \( K \) for \( T_2 = \max\{\frac{\ln(\check{\gamma}/\|x_1\|)}{\ln(\check{\gamma}/\|x_1\|)}, 0\} \) iterations, the final state \( x_{T_2+1} \) satisfies \( \|x_{T_2+1}\| \leq 2\check{\kappa}/\check{\gamma} \), and the total cost is bounded by

\[
O(G\check{\kappa}^4\|x_1\|^{3\check{\gamma}^{-3}}).
\]

**Proof.** Under the controller \( K \), the state evolution satisfies

\[
x_{t+1} = (A + BK)x_t + \sum_{i=1}^{t} (A + BK)^{t-i} u_i.
\]

By definition of strong stability, \( \|(A + BK)^t\| \leq \|H\|\|H^{-1}\|\|L\|^t \leq \check{\kappa}(1 - \check{\gamma})^t \). It follows that

\[
\|x_{t+1}\| \leq \check{\kappa}(1 - \check{\gamma})^t \|x_1\| + \check{\kappa}\sum_{i=1}^{t} (1 - \check{\gamma})^{t-i} \leq \check{\kappa}(1 - \check{\gamma})^t \|x_1\| + \frac{\check{\kappa}}{\check{\gamma}}.
\]

Let \( T_2 = \max\{\frac{\ln(\check{\gamma}/\|x_1\|)}{\ln(\check{\gamma}/\|x_1\|)}, 0\} \). If \( \ln(\check{\gamma}/\|x_1\|) \geq 0 \), we have \( T_2 \geq \frac{\ln(\check{\gamma}/\|x_1\|)}{\ln(\check{\gamma}/\|x_1\|)} \), hence \( (1 - \check{\gamma})^{T_2} \leq 1/(\check{\gamma}/\|x_1\|) \) and \( \|x_{T_2+1}\| \leq 2\check{\kappa}/\check{\gamma} \). Otherwise \( T_2 = 0 \) and \( \|x_1\| \leq 1/\check{\gamma} < 2\check{\kappa}/\check{\gamma} \). Notice that \( \|x_t\| \leq \check{\kappa}\|x_1\| + \check{\kappa}/\check{\gamma} \|u_t\| \leq \check{\kappa}^2\|x_1\| + \check{\kappa}^2/\check{\gamma} \) for all \( t \in [T_2+1] \). Taking \( D = \check{\kappa}^2\|x_1\| + \check{\kappa}^2/\check{\gamma} \) and assuming \( \ln(\check{\gamma}/\|x_1\|) \geq 0 \), the total cost of decaying the system is bounded by

\[
2(T_2 + 1)GD^2 = 2G\left(\frac{\ln(\check{\gamma}/\|x_1\|)}{\check{\gamma}} + 1\right)(\check{\kappa}^2\|x_1\| + \frac{\check{\kappa}^2}{\check{\gamma}})^2
\]

\[
\leq 4G\left(\frac{\ln(\check{\gamma}/\|x_1\|)}{\check{\gamma}} + 1\right)\check{\kappa}^4\left(\|x_1\|^2 + \frac{1}{\check{\gamma}^2}\right)
\]

\[
\leq 8G\left(\frac{\ln(\|x_1\|)}{\check{\gamma}} + 1\right)\check{\kappa}^4\|x_1\|^2\check{\gamma}^{-2}
\]

\[
\leq 8G\ln(\|x_1\|) + 1\check{\kappa}^4\|x_1\|^2\check{\gamma}^{-3}
\]

\[
\leq 16G\check{\kappa}^4\|x_1\|^3\check{\gamma}^{-3}.
\]

The same upper bound holds for \( T_2 = 0 \).

\[\square\]

C Proofs for Section 4.3

In this section we give an upper bound on quantities related to the controllability of the stabilized system \((A + BK, B)\), and include the main results in (8) for completeness. The following lemma is an equivalent characterization of strong controllability.

**Lemma 20.** A system defined by \( x_{t+1} = Ax_t + Bu_t \) is \((k, \kappa)\)-strongly controllable if and only if it can drive \( x_1 = 0 \) to any state \( x_f \) where \( \|x_f\| = 1 \) in \( k \) steps with control cost at most \( \kappa \). I.e., there exists \( u_1, \ldots, u_k, x_2, \ldots, x_{k+1} \) such that \( x_{k+1} = x_f, x_{t+1} = Ax_t + Bu_t, \) and

\[
\sum_{t=1}^{k} \|u_t\|^2 \leq \kappa.
\]

**Proof.** Consider the quadratic program:

\[
\min_{(u_t)_{t=1}^k} \sum_{t=1}^{k} \|u_t\|^2
\]

s.t. \( x_{t+1} = Ax_t + Bu_t \)

\[
x_{k+1} = x_f, x_1 = 0
\]

(2)
Recall $C_k = [B\ AB\ \cdots\ A^{k-1}B]$, and let $(v_1, v_2, \ldots, v_k) \in \mathbb{R}^{kn}$ denote the concatenation of $k$ $n$-dimensional vectors. Then this is equivalent to

$$\begin{equation}
\min_{(u_i)_{i=1}^k} \sum_{i=1}^k \|u_i\|^2 \\
\text{s.t.} \quad C_k(u_k, u_{k-1}, \ldots, u_1) = x_f
\end{equation}$$

(3)

Suppose the system is $(k, \kappa)$ strongly controllable, then $C_k$ has full row-rank, and $C_k C_k^T$ is invertible with $\|(C_k C_k^T)^{-1}\| \leq \kappa$. Therefore (3) is feasible for all unit vectors $x_f$. By Lemma B.6 in (4), an optimal solution to (3) is given by $C_k^T((C_k C_k^T)^{-1})^{-1} x_f$, and its value is at most

$$\sum_{i=1}^k \|u_i\|^2 = \|C_k^T((C_k C_k^T)^{-1})^{-1} x_f\|^2 = x_f^T (C_k C_k^T)^{-1} x_f \leq \|(C_k C_k^T)^{-1}\| = \kappa.$$

Now suppose for any unit vector $x_f$, there exists $u_1, u_2, \ldots, u_{k+1}$ such that $x_1 = x_f$, $x_{t+1} = Ax_t + Bu_t$, and $\sum_{t=1}^k \|u_t\|^2 \leq \kappa$. Then (3) is feasible for any unit vector $x_f$, implying that $C_k$ has full row-rank and $(C_k C_k^T)$ is invertible. Moreover, the optimal value is at most $\kappa$. Let $x_f$ be the eigenvector corresponding to the largest eigenvalue of $(C_k C_k^T)^{-1}$. Then an optimal solution to (3) is $C_k^T((C_k C_k^T)^{-1})^{-1} x_f$, and the value satisfies $\|C_k^T((C_k C_k^T)^{-1})^{-1} x_f\|^2 = x_f^T (C_k C_k^T)^{-1} x_f \leq \kappa$. We conclude that $\|(C_k C_k^T)^{-1}\| \leq \kappa$, and the system is $(k, \kappa)$ strongly controllable.

Using our characterization, we show an upper bound on the controllability parameter of $(A + BK, B)$ where $K$ is any linear controller with a bounded spectral norm.

**Lemma 21.** Suppose $(A, B)$ is $(k, \kappa)$ strongly controllable and $\|A\|, \|B\| \leq \beta$. Let $K$ be a linear controller with $\|K\| \leq \kappa'$, then the system $(A + BK, B)$ is $(k, \kappa_0)$ strongly controllable, with $\kappa_0 = 4\kappa'^2 k^2 \beta^2 k$.

**Proof.** Let $C_k = [B\ AB\ \cdots\ A^{k-1}B]$. By the definition of strong controllability, $C_k$ has full row-rank, and under the noiseless system $x_{t+1} = Ax_t + Bu_t$, any state is reachable by time $k + 1$ starting from $x_1 = 0$. We will show that any state is reachable at time $t + 1$ for the system $(A + BK, B)$ as well. Let $v \in \mathbb{R}^m$ be an arbitrary state, and the sequence of controls $(u_1, u_2, \ldots, u_k) = C_k^T((C_k C_k^T)^{-1}) v$ can be used to reach $v$ from initial state $x_1 = 0$, i.e.

$$x_{k+1} = \sum_{i=1}^k A^{k-i} Bu_i = C_k(u_1, u_2, \ldots, u_k) = v.$$

Let $\{x_t\}$ denote the state trajectory under controls $\{u_t\}$, where $x_{k+1} = v$. Consider the system $y_{t+1} = (A + BK) y_t + B z_t = A y_t + B(z_t + K y_t)$, where $y_t$’s are states and $z_t$’s are controls. We claim that the sequence of controls $z_t = u_t - K y_t$ can be used to drive the system to $v$ in $k + 1$ steps from initial state $y_1 = 0$. Let $\{y_t\}$ denote the system’s trajectory under controls $\{z_t\}$. For our base case, we have $y_2 = B(z_1 + K y_1) = Bu_1 = x_2$, since $y_1 = x_1 = 0$, $z_1 = u_1 - K y_1$. Assume $x_t = y_t$ for some $t \leq k$. For $t + 1$, $y_{t+1} = A y_t + B(z_t + K y_t) = A y_t + B u_t = A x_t + B u_t = x_{t+1}$. We conclude that the trajectories $\{x_t\}$ and $\{y_t\}$ are the same and $v = y_{k+1}$. Since we can write $y_{k+1} = \sum_{i=1}^k (A + BK)^{k-i} B z_i$, $y_{k+1}$ is in the range of the matrix $C_k = [B\ (A + BK) B\ \cdots\ (A + BK)^{k-1}B]$; therefore $C_k$ has full row-rank.

Now we show the controls $\{z_t\}$ satisfy $\sum_{t=1}^k \|z_t\|^2 \leq 4\kappa'^2 k^2 \beta^2 k \|v\|^2$. By our choice of $z_t$, we have $z_t = u_t - K y_t = u_t - K x_t$; therefore $\sum_{t=1}^k \|z_t\|^2 \leq 2 \sum_{t=1}^k (\|u_t\|^2 + \|K\|^2 \|x_t\|^2)$. By our choice of $u_t$, we have

$$\sum_{t=1}^k \|u_t\|^2 = \|C_k^T((C_k C_k^T)^{-1})^{-1} v\|^2 = v^T (C_k C_k^T)^{-1} v \leq \kappa \|v\|^2.$$

Further, the trajectory $\{x_t\}_{t=1}^k$ satisfies

$$\|x_t\|^2 \leq \sum_{i=1}^{t-1} \|A^{t-1-i} Bu_i\|^2 \leq k \sum_{i=1}^{t-1} \|A^{t-1-i} B\|^2 \|u_i\|^2 \leq k \beta^2 k \|v\|^2$$

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Hence we have
\[ \sum_{t=1}^{k} \|z_t\|^2 \leq 2\kappa \|v\|^2 + 2\kappa'^2 k^2 \beta^2 k \|v\|^2 \leq 4\kappa'^2 k^2 \beta^2 k \|v\|^2. \]

By Lemma 20, \((A + BK, B)\) is \((k, 4\kappa'^2 k^2 \beta^2 k)\) strongly controllable.

Algorithm 4 is the main algorithm (Algorithm 1) in (8), where \(T_0, \eta, H\) are internal parameters that can be set by the learner. In line 10, let \(\Pi_M\) denote projection onto the set \(M\), and let \(f_t\) denote the surrogate cost at time \(t\) as in Definition 11 of (8). Theorem 22 gives the regret bound for the algorithm when the internal parameters are set appropriately.

**Algorithm 4** Adversarial Control via System Identification

1: **Input:** Number of iterations \(T\), \(\tilde{\gamma}, \hat{K}\) such that \(\hat{K}\) is \((\tilde{\kappa}, \tilde{\gamma})\) strongly stable, \(\kappa^*, k\) such that \((A + B\hat{K}, B)\) is \((k, \kappa^*)\) strongly controllable, and \(\kappa^* \geq \tilde{\kappa}\).

2: **Phase 1: System Identification.**

3: Call Algorithm 2 in (8) with a budget of \(T_0\) rounds to obtain system estimates \(\tilde{A}, \tilde{B}\).

4: **Phase 2: Robust Control.**

Define the constraint set \(M = \{M = \{M^0 \ldots M^{H-1}\} : \|M^{i-1}\| \leq \kappa^4 (1 - \gamma)^i\}\).

5: Initialize \(\hat{w}_{T_0} = x_{T_0+1}\) and \(\hat{w}_t = 0\) for \(t < T_0\).

6: for \(t = T_0 + 1, \ldots, T\) do

7: Choose the action:

\[ u_t = \hat{K}x_t + \sum_{i=1}^{H} M^{i-1} \hat{w}_{t-i}. \]

8: Observe the new state \(x_{t+1}\) and cost \(c_t(x_t, u_t)\).

9: Record estimate \(\hat{w}_t = x_{t+1} - \tilde{A}x_t - \tilde{B}u_t\).

10: Update:

\[ M_{t+1} = \Pi_M(M_t - \eta \nabla f_t(M_t|\tilde{A}, \tilde{B}, \{\hat{w}\})) \]

11: end for

**Theorem 22.** [Theorem 12 in (8)] Suppose \(\hat{K}\) is \((\tilde{\kappa}, \tilde{\gamma})\) strongly stable for \((A, B)\), and the system \((A + B\hat{K}, B)\) is \((k, \kappa^*)\) strongly controllable. In addition, assume that the noise sequence \(w_t\) satisfies \(\|w_t\| \leq W\) for all \(t\). Then Algorithm 4 with \(H = \Theta(\tilde{\gamma}^{-1} \log((\kappa^*)^2 T))\), \(\eta = \Theta(GW\sqrt{T})^{-1}, T_0 = \Theta(T^{2/3} \log(1/\delta))\), incurs regret upper bounded by

\[ \text{Regret} = O(\text{poly}(\kappa^*, \tilde{\gamma}^{-1}, k, d_x, d_u, G, W)T^{2/3} \log(1/\delta)). \]

with probability at least \(1 - \delta\) for controlling an unknown LDS.