Percolative Properties of Brownian Interlacements and Its Vacant Set

Xinyi Li

Received: 27 July 2018 / Revised: 23 August 2019 / Published online: 21 September 2019
© Springer Science+Business Media, LLC, part of Springer Nature 2019

Abstract
In this article, we investigate the percolative properties of Brownian interlacements, a model introduced by Sznitman (Bull Braz Math Soc New Ser 44(4):555–592, 2013), and show that: the interlacement set is “well-connected”, i.e., any two “sausages” in \(d\)-dimensional Brownian interlacements, \(d \geq 3\), can be connected via no more than \([ (d - 4)/2 ]\) intermediate sausages almost surely; while the vacant set undergoes a non-trivial percolation phase transition when the level parameter varies.

Keywords Brownian interlacements · Random interlacements · Wiener sausage · Percolation

Mathematics Subject Classification (2010) 60J65 · 60K35 · 82B43

1 Introduction
In this article, we investigate various aspects of the percolative properties of Brownian interlacements and show that the interlacements are well-connected and that the vacant set undergoes a non-trivial phase transition.

The model of Brownian interlacements, recently introduced by Sznitman in [29], is the continuous counterpart of random interlacements, a model that has already attracted a lot of attention and has been relatively thoroughly studied (see [27] for the seminal paper on this model and see [4,8] for a comprehensive introduction). Roughly speaking, Brownian interlacements can be described as a certain Poissonian cloud of doubly infinite continuous trajectories in the \(d\)-dimensional Euclidean space, \(d \geq 3\), with the intensity measure governed by a parameter \(\alpha > 0\). We are interested in both the interlacement set, which is an \(r\)-enlargement (sometimes colloquially referred to as...
“the sausages”) of the union of the trace in the aforementioned cloud (of trajectories), for some \( r > 0 \), and the \textit{vacant set}, which is the complement of the interlacement set.

Brownian interlacements bear similar properties, for instance long-range dependence, to random interlacements, due to similarities in the construction. Moreover, this model plays a crucial role in both the study of the limiting behaviors of various aspects of random interlacements (see for example [13,29]), and the interconnection of random interlacements, loop soups, and Gaussian free fields. Brownian interlacements, as a model of continuous percolation, could also shed light on the study of other models. For example, the visibility in the vacant set of Brownian interlacements is studied and compared with that of the Brownian excursion process in [9].

We now describe the model and our results in a more precise fashion. Readers are referred to Sect. 2 for notations and definitions. We consider Brownian interlacements on \( \mathbb{R}^d \), \( d \geq 3 \). We denote by \( \mathbb{P} \) the canonical law of Brownian interlacements and by \( I_{\alpha}^r \) (resp. \( V_{\alpha}^r \)) the corresponding interlacement set (resp. vacant set) at level \( \alpha \geq 0 \) with radius \( r \geq 0 \), which is \( \mathbb{P} \)-a.s. closed (resp. open).

Let us look at the interlacement set first.

As pointed out in (2.36) in [29], it is presently known that for any \( \alpha > 0 \) the Brownian fabric (i.e., \( I_{\alpha}^0 \)) is connected when \( d = 3 \) and disconnected when \( d \geq 4 \). However, despite its name “Brownian interlacements”, not much is a priori known about how the trajectories are actually interlaced: for example, the connectedness of \( I_{\alpha}^r \) for \( \alpha, r > 0 \) in dimension 4 and higher does not follow trivially from the definition (however, it is a simple corollary of Theorem 1.1).

In this work, we show that, for \( d \geq 3 \), and for all \( \alpha, r > 0 \), Brownian interlacements are well-connected in the following sense: two sausages in the interlacements can be connected via no more than \( s_d - 1 \) intermediate sausages, where

\[
s_d = \left\lceil \frac{d - 2}{2} \right\rceil \quad (1.1)
\]

(\( \lceil \frac{d - 2}{2} \rceil \) stands for the smallest integer greater or equal to \( \frac{d - 2}{2} \)).

We now phrase this result in a more precise fashion. Let \( W \) stand for the collection of doubly infinite continuous trajectories modulo time shift of Brownian interlacements at level \( \alpha \). We associate with \( W \) and \( r > 0 \) a graph \( G_{\alpha, r} = (V, E) \) where each vertex in \( V \) corresponds to a trajectory in \( W \), and the set of edges \( E \) consists of pairs of vertices whose corresponding sausages of radius \( r \) intersect with each other. Let \( \text{diam}(G_{\alpha, r}) \) stand for the diameter of \( G_{\alpha, r} \). The result regarding the connectivity of interlacements is summarized in the following theorem.

**Theorem 1.1** Let \( G_{\alpha, r} \) be the graph defined as above. For all \( \alpha, r > 0 \),

\[
\mathbb{P} \left[ \text{diam}(G_{\alpha, r}) = s_d \right] = 1. \quad (1.2)
\]

As a corollary, we obtain the connectedness of Brownian interlacements.

**Corollary 1.2** For all \( \alpha, r > 0 \), the interlacement set \( I_{\alpha}^r \) is \( \mathbb{P} \)-a.s. connected.

We now make a few comments on Theorem 1.1. The formulation of this problem and the strategy of proof are inspired by [23], which treated graph distance problem
on random interlacements and obtained the same graph diameter as (1.1). It is worth
mentioning that some of the methods and techniques used in [23] can also be adapted
to serve as the backbone in the solution to other problems, such as [3,22,24]. It is also
worth mentioning that in the case of random interlacements, the same result can be
proved through an essentially different approach, see [20], which involves the notion
of “stochastic dimensions.” However, this notion is only defined on the discrete lattice
and there is no adequate continuous equivalent.

We then turn to the vacant set. In this work, we show that for any $r > 0$, the vacant
set $V_α^r$ undergoes a non-trivial percolation phase transition. More precisely, we have
the following theorem [notice that the vacant set is “monotonously decreasing” with
respect to $α$, i.e., it is possible to construct Brownian interlacements simultaneously
for levels $α_1 > α_2$ and $r > 0$ in such a way that $V_α^r \subset V_β^r$, see (2.30)].

**Theorem 1.3** There exists $0 < α_1^*(d) < ∞$, such that

\[
\begin{align*}
V_α^r & \text{ percolates } \mathbb{P}\text{-a.s., when } α < α_1^* r^{2-d}; \text{ and } \\
V_α^r & \text{ does not percolate } \mathbb{P}\text{-a.s., when } α > α_1^* .
\end{align*}
\]

We refer to the case $0 < α < α_1^*$ the **supercritical regime** and the case $α > α_1^*$ the
**subcritical regime**, which is in line with random interlacements.

We now make a few comments about this theorem.

The precise relation between $α_1^*$’s for different values of $r$, given in (1.3), is due to
the scaling property of Brownian interlacements [see (2.37)], which also implies that
it suffices to study the phase transition with regard to one parameter only.

The critical percolation threshold $α_1^*$ could be related to some of the questions
concerning the complement of the Wiener sausage wrapping on a unit $d$-dimensional
torus discussed in [10], relevant in the local scale $t^{-1/(d-2)}$ [i.e., the local scale $φ_{local}(t)$
in the terminology of (1.5) in [10], see also Section 1.6.3, ibid., especially (1.43)] when
the Brownian motion on the unit torus runs over time $τ$. It is plausible, yet not known
at the moment, that $α_1^*$ enters into play in the following way: when one runs Brownian
motion on the unit $d$-dimensional torus for time $ατ$ and looks at the complement of
the $t^{-1/(d-2)}$-neighborhood of the trajectory, then

- when $α > α_1^*$, for large $t$, there are only “small” components, but
- when $α < α_1^*$, for large $t$, there is a “giant component,”

in analogy with what happens in the case of the discrete $d$-dimensional torus of large
side-length $n$, see [5,31]. Taking $t/α$ to play the role of $t$, the same applies to the
complement of the sausage of radius $(α/τ)^{1/(d-2)}$ of the Brownian motion in time $t$,
on the unit torus, for large $t$ (depending on $α > α_1^*$ or $α < α_1^*$).

In the course of proving Theorem 1.3, we are also able to show that $V_α^r$ undergoes
another phase transition with respect to connectivity. More precisely, let

\[
α_2^* = \inf \{ α > 0 : \lim_{L \to ∞} \mathbb{P}[∃ \text{ continuous path in } V_α^r \text{ connecting } B_∞(0, L) \text{ and } \partial B_∞(0, 2L)] = 0 \} \tag{1.4}
\]

stand for the critical level of sharp connectivity decay for $V_α^r$, where $B_∞(0, L)$ stands
for a ball centered at the origin of size $L$ under $l^∞$-norm, then

\[ Springer \]
\( \alpha_r^* \leq \alpha_r^{**} < \infty. \) (1.5)

It is hence a very natural question whether \( \alpha_r^* \) actually coincides with \( \alpha_r^{**} \), which would imply that the phase transition is sharp. Notice the similarity between \( \alpha_r^{**} \) and the critical parameter \( u^{**} \) for random interlacements, whose definition first appeared in [26] and was later improved subsequently in [18,28]. As the corresponding conjecture for random interlacements has been open for a long time, we do not expect a quick answer to this question here. See Remark 4.10 for more discussions.

We also refer to Remark 4.10 for further discussions, such as the open question of the uniqueness of percolation cluster in the supercritical regime and the existence of a critical threshold for percolation on a plane.

We now give some comments on the proofs.

For Theorem 1.1, the strategy we employ is inspired by [23] and involves developing parallel estimates for Brownian interlacements. The lower bound of the graph diameter is essentially a convolution estimate of Green’s function. The more involved upper bound is more technical than the discrete version, but generally follows from the same idea with similar capacity estimates. Pick a certain trajectory from the interlacement process, look at the corresponding vertex in the connectivity graph, and consider the union of sausages of trajectories that correspond to vertices of distance at most \( s \) from this vertex. We use induction to show that this union is \("(2s+2)\)-dimensional" in terms of capacity and when \( s = s_d - 1 \) it saturates the space in the sense that it will almost surely be hit by another trajectory from the interlacement process. This in turn gives the upper bound on the graph diameter. The above lines are of course mainly heuristic, and to make sense of the above heuristics, a multi-scale analysis is employed. For more details, see the beginning of Sect. 3.3.

We now turn to Theorem 1.3. In this work, we have chosen the combinatorial approach of [21] instead of the standard route map in proving non-trivial phase transitions for interlacements, namely via the “sprinkling” technique and decoupling inequalities (see [28] or Chapter 8 of [8]), for the latter is lengthier and more involving (but yields more quantitative controls, for instance in the region corresponding in our setup to \( \alpha > \alpha_r^{**} \)). A more detailed explanation on the proof strategy can be found at the beginning of Sect. 4.

We will now explain how this article is organized. In Sect. 2, we introduce notation and make a brief review on results concerning Brownian motion and its potential theory, the definition and basic properties of Brownian interlacements, renewal theory as well as other useful facts and tools. Section 3 is devoted to the proof of Theorem 1.1. The lower bound on the graph distance is proved in Proposition 3.3 and the upper bound on graph distance is proved in Proposition 3.11. In Sect. 4, we prove Theorem 1.3. The dyadic trees are defined in Sect. 4.1, where some preliminary results are also stated. In Sects. 4.2 and 4.3, we prove some preparatory results for the finiteness and positiveness of the percolation threshold, respectively, and the proof of Theorem 1.3 shall be completed in Sect. 4.4.

Finally, we explain the convention in this work. We denote by \( c, c', c'', \tilde{c}, \ldots \) positive constants with values changing from place to place. Throughout the article, the constants depend on the dimension \( d \). Unless otherwise stated, throughout the article we assume \( d \geq 3 \).
2 Some Useful Facts

In this section, we introduce various notation and recall useful facts concerning Brownian motion, its potential theory, Brownian interlacements and renewal theory.

2.1 Basic Notations

In this subsection, we introduce some useful notation. We write \( \mathbb{N} = \{0, 1, 2, \ldots \} \) for the set of natural numbers and write \( B(\mathbb{R}^d) \) for the collection of Borel sets in \( \mathbb{R}^d \). We write \( | \cdot | \) and \( | \cdot |_{\infty} \) for the Euclidean and \( l^\infty \)-norms on \( \mathbb{R}^d \). We denote by \( B(x, r) = \{ y \in \mathbb{R}^d ; |x - y| \leq r \} \) (resp. \( B^\circ(x, r) = \{ y \in \mathbb{R}^d ; |x - y| < r \} \) the closed (resp. open) Euclidean ball of center \( x \) and radius \( r \geq 0 \), and when \( A \) is a subset of \( \mathbb{R}^d \), we write \( B(A, r) = \bigcup_{x \in A} B(x, r) \) for the union of all closed balls of radius \( r \) and with center in \( A \) and call it the \( r \)-sausage of \( A \). We also write \( B_{\infty}(x, r) = \{ y \in \mathbb{R}^d ; |x - y|_{\infty} \leq r \} \) for the closed \( l^\infty \)-ball of center \( x \) and radius \( r \). In particular, for the sake of convenience we write \( B(R) = B(0, R) \) for short. When \( U \) is a subset of \( \mathbb{R}^d \), we denote by \( \partial U \) the boundary of \( U \) and we denote by Volume(\( U \)) the volume of \( U \).

We will make repeated use of the following basic observation:

\[
\text{for every } a \in \mathbb{R}^d, \text{ there exists } \tilde{a} \in \mathbb{Z}^d \text{ such that } |a - \tilde{a}| \leq \frac{\sqrt{d}}{2}. \quad (2.1)
\]

We call \( \gamma : [0, 1] \to \mathbb{R}^d \) (resp. \( \tilde{\gamma} : [0, \infty) \to \mathbb{R}^d \)) a continuous path from \( A \subset \mathbb{R}^d \) to \( B \subset \mathbb{R}^d \) (resp. infinity), if \( \gamma \) is continuous, \( \gamma(0) \in A \) and \( \gamma(1) \in B \) (resp. \( \lim \sup_{t \to \infty} |\gamma(t)| = \infty \)), and also say that \( \gamma \) connects \( A \) and \( B \) (resp. infinity). With slight abuse of preciseness, when we mention a continuous path we sometimes actually mean its trace, i.e., \( \gamma([0, 1]) \) or \( \gamma([0, \infty)) \) as a subset of \( \mathbb{R}^d \). We say that \( A \subset \mathbb{R}^d \) percolates, if \( A \) contains an unbounded connected subset. If in addition \( A \) is open, then \( A \) percolates if and only if there exists a continuous \( \gamma : [0, \infty) \to \mathbb{R}^d \) such that \( \gamma([0, \infty)) \subset A \) and \( \lim \sup_{t \to \infty} |\gamma(t)| = \infty \).

We now turn to discrete paths. We call \( \gamma : \{0, 1, \ldots, n\} \to \mathbb{Z}^d \) a nearest-neighbor path (resp. \( * \)-path) for all \( k \in \{0, 1, \ldots, n\} \), \( |\gamma(k + 1) - \gamma(k)| = 1 \) (resp. \( |\gamma(k + 1) - \gamma(k)|_{\infty} = 1 \)). By definition a nearest-neighbor path is also a \( * \)-path. Again, we do not distinguish a discrete path from its trace as a subset of \( \mathbb{Z}^d \).

2.2 Brownian Motion and Its Potential Theory

In this subsection, we introduce our notation for Brownian motion and state some useful results on the potential theory of Brownian motion.

We denote by \( W \) the subspace of \( C(\mathbb{R}, \mathbb{R}^d) \), which consists of continuous trajectories from \( \mathbb{R} \) into \( \mathbb{R}^d \) tending to infinity at both plus and minus infinite times. Similarly, we denote by \( W_+ \) the subspace of \( C(\mathbb{R}^+, \mathbb{R}^d) \) of continuous trajectories from \( \mathbb{R}^+ \) to \( \mathbb{R}^d \), tending to infinity at infinite time. We write \( X_t, t \in \mathbb{R} \) (resp. \( X_t, t \geq 0 \)) for the canonical process and denote by \( \theta_t, t \in \mathbb{R} \) (resp. \( \theta_t, t \geq 0 \)) the canonical shifts. The spaces \( W \) and \( W_+ \) are endowed with respective \( \sigma \)-algebras \( \mathcal{W} \) and \( \mathcal{W}_+ \) generated by
the canonical processes. For the convenience of notation, we sometimes write $X(t)$ instead of $X_t$. For an index set $I \subset \mathbb{R}$, we write
\[ X_I = \bigcup_{i \in I} \{X_i\} \quad (2.2) \]
for the trace of $X_t$ on $I$.

When $F$ is a closed subset of $\mathbb{R}^d$ and $w$ is in $W_+$, we write $H_F(w) = \inf\{s \geq 0, X_s(w) \in F\}$ and $\tilde{H}_F(w) = \inf\{s > 0, X_s(w) \in F\}$ for the respective entrance time and hitting time of $F$. When $U$ is an open subset of $\mathbb{R}^d$, we define $T_U(w)$ similarly, replacing the condition $s \geq 0$ by $s \in \mathbb{R}$.

Now, we turn to Brownian motion and its potential theory.

Since $d \geq 3$, and in this case Brownian motion on $\mathbb{R}^d$ is transient, we view $P_y$, the Wiener measure starting from $y \in \mathbb{R}^d$, as defined on $(\mathbb{W}^+, \mathbb{W}^+)$ and denote by $E_y$ for the corresponding expectation. Moreover, if $\rho$ is a finite measure (not necessarily a probability measure) on $\mathbb{R}^d$, we denote by $P_\rho$ and $E_\rho$ the measure $\int_{x \in \mathbb{R}^d} \rho(dx) P_x$ (not necessarily a probability measure) and its corresponding “expectation” (i.e., the integral with respect to the measure $P_\rho$).

We write
\[ p_t(x, x') = \frac{1}{(\sqrt{2\pi t})^d} \exp\left(-\frac{|x - x'|^2}{2t}\right) \quad \text{for} \quad t > 0, x, x' \in \mathbb{R}^d, \quad (2.3) \]
for the Brownian transition density. Accordingly, we denote the Green function of Brownian motion by
\[ g(y, y') = \int_0^\infty p_t(y, y') dt, \quad \text{for} \quad y, y' \in \mathbb{R}^d. \quad (2.4) \]
It is a classical result that
\[ g(y, y') = c |y - y'|^{2-d} \quad \text{for} \quad y, y' \in \mathbb{R}^d. \quad (2.5) \]

For $t \geq 0$, we write $P^t$ for the Brownian semi-group operator on $L^1(\mathbb{R}^d)$. More precisely, for all $f \in L^1(\mathbb{R}^d)$, we define $P^t f : \mathbb{R}^d \to \mathbb{R}$ in the following manner:
\[ P^t f(x) = \int_{\mathbb{R}^d} f(y) p_t(x, y) dy. \quad (2.6) \]
We write $G$ for the respective Green operator:
\[ Gf(x) = \int_0^\infty P^t f(x) dt. \quad (2.7) \]

We now derive in Lemma 2.1 an upper bound on the $L^\infty$-norm of $P^t f$ in terms of the $L^1$ and $L^\infty$-norms of $f$, and in Lemma 2.2 an estimate on Green function.
and Wiener sausages which is somewhat similar in flavor to Lemma 5.5 in [17] but tailor-made for the proof of Proposition 3.7.

**Lemma 2.1** For all \( f \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \), one has

\[
||P^t f||_\infty \leq C \max(||f||_1, ||f||_\infty) \left( t \vee 1 \right)^{d/2}.
\]

(2.8)

**Proof** When \( t \geq 1 \), one has

\[
|P^t f(x)| \leq \frac{1}{\sqrt{2\pi t}^d} e^{-\frac{|x-y|^2}{2t}} f(y) dy \leq \frac{1}{(\sqrt{2\pi t})^d} \int_{\mathbb{R}^d} |f(y)| dy = C t^{-d/2} ||f||_1.
\]

(2.9)

When \( t < 1 \), we have

\[
|P^t f(x)| \leq \frac{1}{\sqrt{2\pi t}^d} e^{-\frac{|x-y|^2}{2t}} |f(y)| dy \leq ||f||_\infty \int_{\mathbb{R}^d} \frac{1}{(\sqrt{2\pi t})^d} e^{-\frac{|x-y|^2}{2t}} dy = ||f||_\infty.
\]

(2.10)

The claim (2.8) then follows from combining (2.9) and (2.10). \( \square \)

**Lemma 2.2** Let \( d \geq 5 \) and \((z_i)_{i \geq 1}\) be a sequence of points in \( \mathbb{R}^d \). We consider \((X^i_t)_{t \geq 0}, i \geq 1\), a sequence of independent Brownian motions on \( \mathbb{R}^d \) with \( X^i(0) = z_i, i \geq 1\), and write \( E \) for the expectation with respect to their joint law. For all \( z \in \mathbb{R}^d \), let \( f_z(\cdot) = 1_{B(z,1)}(\cdot) \). For \( i, j = 1, \ldots, M \), we write

\[
F_L(i, j) = \int_{L/2}^{L} \int_{L/2}^{L} \int_{\mathbb{R}^d \times \mathbb{R}^d} g(x, y) f_{X^i_t}(x) f_{X^j_t}(y) dx dy ds dt.
\]

(2.11)

Then, for all positive integers \( M \) and for all \( L \geq 2 \),

\[
E \left[ \sum_{i,j=1}^{M} F_L(i, j) \right] \leq C (ML + M^2 L^{3-d/2}).
\]

(2.12)

**Proof** We divide the summation into two cases, namely \( i = j \) and \( i \neq j \). To prove (2.12), it suffices to prove that for all \( i = 1, \ldots, M \),

\[
E[F_L(i, i)] \leq c L
\]

(2.13)

and for all \( i, j = 1, \ldots, M, i \neq j \),

\[
E[F_L(i, j)] \leq c' L^{3-d/2}.
\]

(2.14)

We first prove (2.13). For \( f, g : \mathbb{R}^d \rightarrow \mathbb{R} \), let \( \langle f, g \rangle \) stand for the inner product of \( f \) and \( g \). We then rewrite \( E[F_L(i, i)] \) in the form of semi-group operators:
\[
E[F_L(i, i)] = 2E \left[ \int_{L/2}^{L} \int_{0}^{L-s} \int_{\mathbb{R}^d \times \mathbb{R}^d} g(x, y) f_{X_i^t}(x) f_{X_i^t}(y) dx dy ds \right] \\
= 2E \left[ \int_{L/2}^{L} E^* \left[ \int_{0}^{L-s} \int_{\mathbb{R}^d \times \mathbb{R}^d} g(x, y) f_{X_i^t}(x) f_{X_i+t^t}(y) dy dx ds \right] \right] \\
= 2E \left[ E^* \left[ \int_{L/2}^{L} \int_{0}^{L-s} \langle f_{X_i^t}, G f_{X_i+t^t} \rangle ds \right] \right],
\]

(2.15)

where we denote by \( X_i^t \) a Brownian motion started from \( 0 \in \mathbb{R}^d \) which is independent from \((X_i^t)_{t \geq 0}, i \geq 1\) and write \( E^* \) for its respective expectation. We are now ready to show (2.13) with the help of (2.8) from Lemma 2.1. Notice that by the observation \( f_{a} = f_{a-b}(0) \) it is straightforward that for \( t' \geq 0, \)

\[
E^*[f_{X_i^t + X_i+t^t}(x)] = E[f_{X_i^t}(-X_i^t + x)] = P^{t'} f_{X_i^t}(x),
\]

(2.16)

hence, we obtain that

\[
E[F_L(i, i)] = 2E \left[ \int_{L/2}^{L} \int_{0}^{L-s} \int_{\mathbb{R}^d \times \mathbb{R}^d} g(x, y) f_{X_i^t}(x) f_{X_i^t}(y) dx dy ds \right] \\
\leq 2E \left[ \int_{L/2}^{L} \int_{0}^{L-s} \int_{t'}^{\infty} \langle f_{X_i^t}, P^u f_{X_i^t} \rangle du ds \right] \\
\leq \int_{L/2}^{L} \int_{0}^{L-s} c' (1 \vee t')^{1-d/2} ds \leq c'' L.
\]

The claim (2.13) hence follows.

Now, we prove (2.14). Similarly, we know that for all \( i, j = 1, \ldots, M, i \neq j, \)

\[
E[F_L(i, j)] = E \left[ \int_{\{L/2, L\}^2} \langle f_{X_i^t}, G f_{X_j^t} \rangle ds \right] = E \left[ \int_{\{L/2, L\}^2} \langle f_{X_i^t}, G P^t f_{z_j} \rangle ds \right] \\
\leq \int_{\{L/2, L\}^2} \int_{t'}^{\infty} \frac{C'}{u^{d/2}} du ds \leq C L^{3-d/2}.
\]

This confirms (2.14) as well as (2.12) and finishes the proof of Lemma 2.2.

We now give a very brief introduction to Brownian capacity. We refer readers to [19] or Chapter 2 of [25] for more details.

Let \( K \) be a compact subset of \( \mathbb{R}^d \). We denote by \( e_K \) the equilibrium measure of \( K \) (see Theorem 1.10, p. 58 of [19]), which is supported on the boundary of \( K \). We denote by \( \tilde{e}_K \) the normalized equilibrium measure and denote by \( \text{cap}(K) \) the (Brownian)
capacity of $K$ which is equal to the total mass of $e_K$. There is a basic property relating equilibrium measure and the hitting probabilities which we will make repeated use of later in this work (see e.g., the proof of Theorem 1.10, p.58 in [19]):

$$P_z(H_K < \infty) = \int_{\mathbb{R}^d} g(z, y)e_K(dy). \quad (2.17)$$

The following lemma, which is a simple corollary of (2.17), is also a useful tool for estimating the Brownian capacity of a set.

**Lemma 2.3** For a compact $K \subset \mathbb{R}^d$ with positive volume, one has

$$\frac{\text{Volume}(K)}{\sup_{z \in K} \int_K g(z, y)dy} \leq \text{cap}(K) \leq \frac{\text{Volume}(K)}{\inf_{z \in K} \int_K g(z, y)dy}. \quad (2.18)$$

**Proof** Note that for $z \in K$, $P_z(H_K < \infty) = 1$. This implies that

$$\int_K \int_K g(z, y)e_K(dy)dz = \text{Volume}(K).$$

The claim (2.18) follows from (2.17) and the observation that $g(y, z) = g(z, y)$. \qed

The Brownian capacity satisfies sub-additivity and monotonicity. It is also invariant under translations and rotations. More precisely, for compact $A, B \subset \mathbb{R}^d$, one has

$$\text{cap}(A \cup B) \leq \text{cap}(A) + \text{cap}(B). \quad (2.19)$$

And if $A \subset B$, then

$$\text{cap}(A) \leq \text{cap}(B). \quad (2.20)$$

For $x \in \mathbb{R}^d$ and $\tilde{\rho}$ some rotation in $\mathbb{R}^d$, if $A' = A + x$ and $A'' = \tilde{\rho}(A)$ then

$$\text{cap}(A') = \text{cap}(A) = \text{cap}(A''). \quad (2.21)$$

See (4.15), Chap. 2, p. 70 and (4.17), Chap. 2, p. 71, in [25] for more details.

It is a classical result (see e.g., (3.55), p. 63 in [25]) that for $R \geq 0$

$$\text{cap}(B(0, R)) = CR^{d-2}. \quad (2.22)$$

We now state a classical variational characterization of the Brownian capacity (see Theorem 4.9, Chap 2, p. 76 in [25]).

**Theorem 2.4** Given $K$, a compact subset of $\mathbb{R}^d$, we denote by $M_1(K)$ the space of probability measures on $K$. Then,

$$\text{cap}(K) = (\inf \{\mathcal{E}(\mu), \mu \in M_1(K)\})^{-1} \quad (2.23)$$

where for $\mu \in M_1(K)$, $\mathcal{E}(\mu) = \int_{K \times K} g(x, y)\mu(dx)\mu(dy)$. 

\( \Box \) Springer
2.3 Brownian Interlacements

We now turn to the definition and basic properties of Brownian interlacements. The readers are referred to Section 2 of [29] for a complete description of the definition of this model.

We first remind readers the definition of the path space $W$ at the beginning of Sect. 2.2. We consider $W^*$ the set of equivalence classes of trajectories in $W$ modulo time shift, i.e.,

$$W^* = W / \sim,$$

(2.24)

where $w \sim w'$, if $w(\cdot) = w'(\cdot + t)$ for some $t \in \mathbb{R}$. Without loss of preciseness, we still refer to elements of $W^*$ as “trajectories.” We denote by $\pi^*$ the canonical projection on $W^*$ and introduce the $\sigma$-algebra

$$W^* = \{ A \subset W^*; (\pi^*)^{-1}(A) \in \mathcal{W} \},$$

(2.25)

which is the largest $\sigma$-algebra on $W^*$ such that $(W, \mathcal{W}) \overset{\pi^*}{\to} (W^*, \mathcal{W}^*)$ is measurable.

Given a compact subset $K$ of $\mathbb{R}^d$, we write $W_K$ for the subset of trajectories of $W$ that enter $K$, and $W^*_K$ for its image under $\pi^*$. We can now introduce the measurable map $\pi_K$ from $W^*_K$ into $W_+$ defined by

$$\pi_K: w^* \in W^*_K \mapsto (w(H_K + t))_{t \geq 0},$$

(2.26)

for any $w \in W_K$ such that $\pi^*(w) = w^*$.

We now introduce the canonical space for the Brownian interlacement point process, namely the space of point measures on $W^* \times \mathbb{R}^+$,

$$\Omega = \left\{ \tilde{\omega} = \sum_{i \geq 0} \delta_{(w^*_i, \alpha_i)}, \text{ with } (w^*_i, \alpha_i) \in W^* \times [0, \infty) \text{ and } \tilde{\omega}(W^*_K \times [0, \alpha]) < \infty, \right. \left. \text{ for any compact subset } K \text{ of } \mathbb{R}^d \text{ and } \alpha \geq 0 \right\}.$$

(2.27)

By Theorem 2.2 and (2.22) in [29], there exists a unique $\sigma$-finite measure $\nu$ on $(W^*, \mathcal{W}^*)$ such that for each compact subset $K$ of $\mathbb{R}^d$,

$$\text{the image of } 1_{W^*_K} \nu \text{ under } w^* \mapsto \pi_K(w^*) \text{ equals } P_{eK}. \quad (2.28)$$

We endow $\Omega$ with the $\sigma$-algebra $A$ generated by the evaluation maps $\tilde{\omega} \mapsto \tilde{\omega}(B)$, for $B \in \mathcal{W}^* \otimes B(\mathbb{R}^+)$ and denote by $\mathbb{P}$ the law on $(\Omega, A)$ of the Poisson point measure with intensity measure $\nu \times d\alpha$ on $W^* \times \mathbb{R}^+$.

When $\tilde{\omega} \in \Omega$, $\alpha \geq 0$, $r \geq 0$, we define Brownian interlacements at level $\alpha$ with radius $r$ through the following formula

$$I^\alpha_r(\tilde{\omega}) = \bigcup_{i \geq 0; \alpha_i \leq \alpha} \bigcup_{s \in \mathbb{R}} B(w_i(s), r), \text{ where } \tilde{\omega} = \sum_{i \geq 0} \delta_{(w^*_i, \alpha_i)} \text{ and } \pi^*(w_i) = w^*_i \text{ for } i \geq 0. \quad (2.29)$$
By definition, $I_\alpha^r$ is almost surely a closed subset of $\mathbb{R}^d$.

We easily see that for $\alpha \geq \alpha' \geq 0$ and $r \geq r' \geq 0$, under the measure $P$, $I_\alpha^r$ is monotonously increasing with respect to both $\alpha$ and $r$, i.e.,

$$I_\alpha^r \supseteq I_{\alpha'}^{r'} \quad \text{and} \quad I_{\alpha}^{r} \supseteq I_{\alpha'}^{r}.$$  \hspace{1cm} (2.30)

We also immediately see that for all $\alpha_1, \ldots, \alpha_n \geq 0$, $r \geq 0$ and independent Brownian interlacements $I_i \sim I_{\alpha_i}^r$,

$$\bigcup_{i=1}^n I_i \sim I_\alpha^r \text{ where } \alpha = \alpha_1 + \cdots + \alpha_n.$$  \hspace{1cm} (2.31)

We now give a local picture of Brownian interlacements. In fact, given $K$, a compact subset of $\mathbb{R}^d$ and $\alpha \geq 0$, the function from $\Omega$ to the set of finite point measures on $W_+$

$$\mu_{K, \alpha}(\tilde{\omega}) = \sum_{i \geq 0} 1_{\{\alpha_i \leq \alpha, w_i^* \in W_K^+\}} \delta_{\pi_K(w_i^*)}, \text{ where } \tilde{\omega} = \sum_{i \geq 0} \delta(w_i^*, \alpha_i) \in \Omega,$$  \hspace{1cm} (2.32)

satisfies, by (2.28), that

$$\mu_{K, \alpha} \text{ is a Poisson point process on } W_+ \text{ with intensity measure } \alpha P_{\epsilon K}.$$  \hspace{1cm} (2.33)

It follows from (2.33) that we can give a simple characterization of the law of $I_\alpha^r$ (see (2.32) in [29]): for all compact $K \subset \mathbb{R}^d$,

$$\mathbb{P}[I_\alpha^r \cap K = \emptyset] = e^{-\alpha \cdot \text{cap}(B(K,r))}.$$  \hspace{1cm} (2.34)

We call the complement of $I_\alpha^r$, the vacant set of Brownian interlacements:

$$V_\alpha^r(\tilde{\omega}) = \mathbb{R}^d \setminus I_\alpha^r(\tilde{\omega}), \text{ for } \tilde{\omega} \in \Omega, \alpha > 0, r \geq 0.$$  \hspace{1cm} (2.35)

Note that $V_\alpha^r$ is almost surely an open subset of $\mathbb{R}^d$. Thanks to this, showing whether $V_\alpha^r$ percolates is equivalent to showing whether it contains a continuous path to infinity, see the paragraph below (2.1). See also the second-to-last paragraph of Sect. 2.1.

Now, we recall some useful properties of Brownian interlacements. For all $\alpha > 0$, $r > 0$, $y \in \mathbb{R}^d$, $\lambda > 0$, under $P$, we know that (see (2.33), (2.35) and (2.36) in [29])

$$I_\alpha^r + y \text{ has the same law as } I_\lambda^r,$$  \hspace{1cm} (translation invariance);  \hspace{1cm} (2.36)

$$\lambda I_\alpha^r \text{ has the same law as } I_{\alpha}^{\lambda^{-2-d}} \text{ (scaling); }$$  \hspace{1cm} (2.37)

$I_0^r$ is a.s. connected, when $d = 3$ \hspace{1cm} (connectedness),

and a.s. disconnected, when $d \geq 4$ \hspace{1cm} (disconnectedness);  \hspace{1cm} (2.38)

$I_\alpha^r$ is rotational invariant \hspace{1cm} (rotational invariance).  \hspace{1cm} (2.39)
We also regard $\mathcal{I}_r^\alpha$ itself as a random closed set in the space $(\Sigma, \sigma, Q_r^\alpha)$ where $\Sigma$ stands for the set of closed (and possibly empty) subsets of $\mathbb{R}^d$, endowed with $\sigma$-algebra $\sigma$, which is generated by the sets $\{F \in \Sigma: F \cap K = \emptyset\}$, where $K$ varies over the compact subsets of $\mathbb{R}^d$ (see Section 2.1, p.27 in [14]) and $Q_r^\alpha$ stands for its law. See below (2.31), [29] for more details.

We end this subsection by the ergodicity of Brownian interlacements.

**Proposition 2.5** Let $(t_x)_{x \in \mathbb{R}^d}$ stand for the translations in $\mathbb{R}^d$. For all $\alpha, r \geq 0$,

$$(t_x)_{x \in \mathbb{R}^d}$$

is a measure preserving flow on $(\Sigma, \sigma, Q_r^\alpha)$ which is ergodic. (2.40)

Moreover,

$$P[\mathcal{I}_r^\alpha \text{ percolates}] \in \{0, 1\}. \quad (2.41)$$

**Proof** We start with (2.40). It follows from (2.36) that $(t_x)_{x \in \mathbb{R}^d}$ is a measure preserving flow on $(\Sigma, \sigma, Q_r^\alpha)$. Ergodicity immediately follows if we prove that for any compact $K \in \mathbb{R}^d$,

$$\lim_{|x| \to \infty} \mathbb{E}[F(\mu_{K, x}) F(\mu_{K, x}) \circ \tau_x] = \mathbb{E}[F(\mu_{K, x})]^2 \quad (2.42)$$

where $(\tau_x)_{x \in \mathbb{R}^d}$ stands for the translation on $\tilde{\omega} \in \Omega$ by $-x$, i.e., if $\tilde{\omega} = \sum_{i \geq 0} \delta(w_i, a_i) \in \Omega$,

$$\tau_x \tilde{\omega} = \sum_{i \geq 0} \delta(w_i - x, a_i) \quad \text{for } x \in \mathbb{R}^d, \quad (2.43)$$

for any $[0, 1]$-valued measurable function $F$ on the set of finite point measures on $W_+$, endowed with its canonical $\sigma$-algebra. By the translation invariance of Brownian interlacements, we can find $G$ (depending on $x$), with similar properties as $F$, such that the expectation in the left-hand side of (2.42) equals $\mathbb{E}[F(\mu_{K, x}) G(\mu_{K+x, x})]$, while

$$\mathbb{E}[G(\mu_{K+x, x})] = \mathbb{E}[F(\mu_{K, x})]. \quad (2.44)$$

By an argument similar to that between (2.11) and (2.15) in the proof of Theorem 2.1 in [27] we see that for $\alpha \geq 0$, $K$ compact, and $x \in \mathbb{R}^d$, $F, G$-measurable functions on the set of finite point measures on $W_+$ with values in $[0, 1]$, when $|x|$ is sufficiently large (we assume $\text{dist}(K, K+x) > 0$),

$$|\text{cov}_P(F(\mu_{K, x}), G(\mu_{K+x, x}))| \leq c \alpha \frac{\text{cap}(K)^2}{\text{dist}(K, K+x)}. \quad (2.45)$$

This implies (2.42) and thus concludes the proof of (2.40).

Since the event $\{\mathcal{I}_r^\alpha \text{ percolates}\}$ is translation invariant, (2.41) readily follows. \qed
2.4 Miscellaneous

We start with some basic but useful facts on the renewal theory of Brownian motion. Let \((X_t)_{t \geq 0}\) be a \(d\)-dimensional Brownian motion. We define a sequence of stopping times \(\tau^N\) inductively in the following way:

\[
\tau^1 = \inf \{ s > 0, |X_s - X_0| \geq 1 \}
\]

and when \(N \geq 1\)

\[
\tau^{N+1} = \tau^1 \circ \theta_{\tau^N} + \tau^N.
\]

In other words, \(\tau^N\) is the exit time after \(\tau^{N-1}\) from a ball of radius 1 centered at \(X_{\tau^{N-1}}\).

For \(t > 0\), we write \(N^t\) for the smallest integer \(n\) such that \(\tau^n\) is no less than \(t\), i.e.,

\[
N^t = \min\{n \in \mathbb{N}; \tau^n \geq t\}.
\]

From standard renewal theory, see for example (3) in Section 4.1, p. 47 and (17) in Section 4.5, p. 58 of [6], it is known that

\[
E[N^t] \leq C' t \quad \text{and} \quad \text{Var}(N^t) \leq C'' t.
\]

We end this section by stating a generalized version of Borel–Cantelli lemma (see [15] for more details).

**Lemma 2.6** Consider a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and a sequence of events \(\Delta_n \in \mathcal{F}\). Let \(\delta_n = 1_{\Delta_n}\) be the indicator function of the event \(\Delta_n\). If there exists a sequence \(b_n\) such that \(\sum b_n = \infty\) and for any \(d_i \in \{0, 1\}, i = 1, \ldots, n-1,\)

\[
\mathbb{P}[\Delta_n | \delta_1 = d_1, \ldots, \delta_{n-1} = d_{n-1}] \geq b_n > 0,
\]

then

\[
\mathbb{P}[\limsup_{k} \Delta_k] = 1.
\]

3 Graph Distance Between Trajectories of Brownian Interlacements

This section is entirely dedicated to the proof of Theorem 1.1 on the graph distance of trajectories of Brownian interlacements. The lower and upper bounds are proved separately in Proposition 3.3 of Sect. 3.1, and Proposition 3.11 of Sect. 3.3.

Throughout this section, we pick a fixed

\[
\alpha > 0
\]

except for the proof of Theorem 1.1. From now on, in this section, we omit the dependence of constants on \(\alpha\) in notation.
3.1 The Lower Bound

In this subsection, we prove in Proposition 3.3 that for all \( r > 0 \), almost surely \( \text{diam}(G_{\alpha, r}) \geq s_d \), which constitutes the lower bound in Theorem 1.1.

We start with notations. For \( l > 0 \), we write \( S_l(x, y) \subset W^* \) for the collection of all trajectories that intersects both \( B(x, l) \) and \( B(y, l) \). In the next lemma, we show that \( \nu(S_r(x, y)) \) [recall that \( \nu \) was defined in (2.28)] decays (as \( |x - y| \to \infty \)) at least as fast as \( c|x - y|^{2-d} \).

**Lemma 3.1** Let \( x, y \in \mathbb{R}^d \). One has

\[
\nu(S_r(x, y)) \leq C(r) \min(|x - y|^{2-d}, 1).
\] (3.2)

From here onward, we set

\[
\rho = \sqrt{d}/2 + 1,
\] (3.3)

and always apply this lemma with \( r = 2\rho \). Hence, the constant from (3.2) actually depends only on the dimension \( d \).

**Proof** Note that, if \( w^* \in S_r(x, y) \), then it must either first pass through \( B_1 = B(x, r) \) and then pass through \( B_2 = B(y, r) \), or vice versa. By (2.28), we obtain that

\[
\nu(S_{2\rho}(x, y)) \leq Pe_{B_1}[H_{B_2} < \infty] + Pe_{B_2}[H_{B_1} < \infty] \quad \text{Symmetry} = 2Pe_{B_1}[H_{B_2} < \infty]
\]

\[
\begin{aligned}
&\leq C(r) & \text{if } |x - y| < 4r \\
&2\int_{x \in \partial B_1} \int_{y \in \partial B_2} g(x, y)e_{B_1}(dx)e_{B_2}(dy) & \text{otherwise.}
\end{aligned}
\] (2.17)

\[
\begin{aligned}
&\leq C(r) & |x - y| < 4r \\
&C'(r)g(|x - y| - 2r) & |x - y| \geq 4r
\end{aligned}
\] (2.22)

\[
\leq C''(r) \min(|x - y|^{2-d}, 1).
\] (2.5)

This finishes the proof of (3.2). \( \square \)

Let \( \tilde{\omega} = \sum_{i \geq 0} \delta_{(w^*_{t_i}, \alpha_i)} \) be the interlacement process defined in Sect. 2.3 and write \( \omega = \sum_{i \geq 0, \alpha_i \leq \alpha} \delta_{w^*_t} \). By definition, \( \omega \) has the law of a Poisson point process with intensity measure \( \mu = \alpha v \). For \( r > 0 \), we write

\[
D_r(x, y) = \{\omega(S_r(x, y)) \neq 0\}
\] (3.4)

for the event that there exists a trajectory in the intersection of the support of the interlacement process at level \( \alpha \) and \( S_r(x, y) \). We then write

\[
E_r = \{\text{diam}(G_{\alpha, r}) \leq s_d - 1\}
\] (3.5)

for the event that the diameter of \( G_{\alpha, r} \) is no more than \( s_d - 1 \).
In the next proposition, we prove that the probability that \( x, y \in T^\alpha_\rho \) when \( E_\rho \) takes place, decays as \( |x - y| \) tends to infinity. For convenience of argument, we require that \( x, y \in \mathbb{Z}^d \), which is sufficient for the proof by contradiction we will conduct later.

**Proposition 3.2** For all \( x \neq y \in \mathbb{Z}^d \), one has

\[
\mathbb{P}[(x, y) \in T^\alpha_\rho \cap E_\rho] \leq C|x - y|^{-1}.
\]

**Proof** On \( (x, y) \in T^\alpha_\rho \cap E_\rho \), there exists \( n \in \{1, \ldots, s_d - 1\} \) with \( \zeta_0 = x, \zeta_{n+1} = y \) and \( \zeta_i \in \mathbb{R}^d \) such that \( D_\rho(\zeta_i, \zeta_{i+1}) \) happens for all \( i = 0, \ldots, n \) on different trajectories in \( \text{Supp}(\omega) \), the support of \( \omega \). By (2.1), in this case, there also exist \( z_0 = x, z_{n+1} = y, z_i \in \mathbb{Z}^d, i = 1, \ldots, n \) with \( z_i \neq z_{i+1}, i = 0, \ldots, n \) and such that \( D_{2\rho}(z_i, z_{i+1}), i = 0, \ldots, n \) happens on different trajectories in Supp(\( \omega \)). We denote this event by \( F_{z_1, \ldots, z_n} \) and by \( \sum_n \) the sum over all \( (n + 1) \)-tuples of pairwise different trajectories \( w_0^*, \ldots, w_n^* \in \text{Supp}(\omega) \). By definition of \( F_{z_1, \ldots, z_n} \), we have

\[
\mathbb{P}[F_{z_1, \ldots, z_n}] \leq \mathbb{E}\left[\sum_n \prod_{i=0}^n 1_{w_i^* \in S_{2\rho}(z_i, z_{i+1})}\right]
\]

Writing \( \# \) as a shorthand for \( \{z_1, \ldots, z_n \in \mathbb{Z}^d, z_i \neq z_{i+1} \text{ for } i = 0, \ldots, n\} \), we have

\[
\mathbb{P}[(x, y) \in T^\alpha_\rho \cap E_\rho] \leq \sum_{n=1}^{s_d-1} \sum_{\#} \mathbb{P}[F_{z_1, \ldots, z_n}]
\]

\[
\leq \sum_{n=1}^{s_d-1} \sum_{\#} \mathbb{E}\left[\sum_n \prod_{i=0}^n 1_{w_i^* \in S_{2\rho}(z_i, z_{i+1})}\right]
\]

\[
\overset{(*)}{=} \sum_{n=0}^{s_d-1} \sum_{\#} \prod_{i=0}^n \mathbb{E}\left[\mu(S_{2\rho}(z_i, z_{i+1}))\right] \overset{\Delta}{=} 1,
\]

where we obtain \( (*) \) from Slivnyak-Mecke theorem, see Chapter 13.1, especially Proposition 13.1.VII, in [7], see also the last paragraph of the proof of Lemma 3.1 in [23]. Moreover, by (1.38) of Proposition 1.7 in [12],

\[
\sum_{\#} \prod_{i=0}^n \min\left(1, |z_i - z_{i+1}|^{2-d}\right) \leq C(n)|x - y|^{2n - 2d - d} \text{ if } n < s_d,
\]

\[
eq \infty \text{ otherwise.}
\]

We hence obtain that

\[
\mathbb{P}[(x, y) \in T^\alpha_\rho \cap E_\rho] \leq \sum_{n=1}^{s_d-1} \sum_{\#} \prod_{i=0}^n \min\left(1, |z_i - z_{i+1}|^{2-d}\right)
\]

\[
\leq \sum_{n=1}^{s_d-1} C(n)|x - y|^{2n - 2d - d} \leq C|x - y|^{2s_d - d} \overset{(1.1)}{=} C|x - y|^{-1}.
\]

\( \square \) Springer
This ends the proof of (3.6).

Now, we rephrase and prove the main claim of this subsection.

**Proposition 3.3** For $d \geq 3$, $\alpha > 0$ and $r > 0$, one has

$$
P[\text{diam}(G_{\alpha,r}) \geq s_d] = 1. \tag{3.11}
$$

**Proof** It is a simple fact that $\text{diam}(G_{\alpha,r}) \geq 1$ because $G_{\alpha,1}$ has more than one vertex. When $d = 3, 4$, $s_d = 1$, hence the claim (3.11) follows directly. Therefore, it suffices to prove (3.11) for $d \geq 5$.

Thanks to the scaling property of Brownian interlacements [see (2.37)], we can assume without loss of generality that $r = 1$.

We assume by contradiction that for some $\delta > 0$ [recall the definition of $E_1$ in (3.5)],

$$
P[E_1] \geq \delta. \tag{3.12}
$$

On the one hand, by (2.34), we can take large $R$ such that

$$
P[I_{\alpha}^0 \cap B(0, R) \neq \emptyset] \geq 1 - \delta/3. \tag{3.13}
$$

By the translation invariance of Brownian interlacements, (3.13) also holds if one replaces $0$ by any $x \in \mathbb{R}$, hence with the assumption (3.12), we obtain that uniformly for any $x \in \mathbb{R}^d$,

$$
P\left[\left\{I_{\alpha}^0 \cap B(0, R) \neq \emptyset\right\} \cap \left\{I_{\alpha}^0 \cap B(x, R) \neq \emptyset\right\} \cap E_1\right] \geq \delta/3. \tag{3.14}
$$

On the other hand, we now show that

$$
\lim_{|x| \to \infty} P\left[\left\{I_{\alpha}^0 \cap B(0, R) \neq \emptyset\right\} \cap \left\{I_{\alpha}^0 \cap B(x, R) \neq \emptyset\right\} \cap E_1\right] = 0. \tag{3.15}
$$

By (2.1), we know that with our choice of $\rho$ [see (3.3)],

$$
\left\{I_{\alpha}^0 \cap B(0, R) \neq \emptyset\right\} \cap \left\{I_{\alpha}^0 \cap B(x, R) \neq \emptyset\right\} \subset \bigcup_{y \in B(0,R)\cap \mathbb{Z}^d} \bigcup_{z \in B(x,R)\cap \mathbb{Z}^d} \{y, z \in I_{\alpha}^\rho\}. \tag{3.16}
$$

Hence, by (3.6) and the fact that $E_1 \subseteq E_{\rho}$, we obtain that

$$
P\left[\left\{I_{\alpha}^0 \cap B(R) \neq \emptyset\right\} \cap \left\{I_{\alpha}^0 \cap B(x, R) \neq \emptyset\right\} \cap E_1\right] \leq \sum_{y \in B(0,R)\cap \mathbb{Z}^d} \sum_{z \in B(x,R)\cap \mathbb{Z}^d} P\left[\{y, z \in I_{\alpha}^\rho\} \cap E_{\rho}\right] \leq CR^{2d}|x|^{-1}. \tag{3.17}
$$

The right-most term in (3.17) converges to 0 as $x \to \infty$. This implies (3.15), which creates a contradiction with (3.14), concluding the proof of (3.11). \qed

© Springer
3.2 Some Preparatory Capacity Estimates for the Upper Bound

This subsection is dedicated to some preliminary results for the proof of the upper bound in Theorem 1.1. The central result in this subsection is Proposition 3.9. As noted in Proposition 3.12, the cases \( d = 3 \) and \( d = 4 \) are classical results, hence throughout this subsection, we will always assume \( d \geq 5 \).

We start with the following basic property of Brownian motion which follows from integrating (2.3) the transition density of Brownian motion.

**Lemma 3.4** There exists \( c_0 \in \mathbb{R}_+ \), such that for all \( R > 0 \)

\[
P_0[\tau_{B^{c}(0,R/2)} > c_0 R^2] \geq 0.99. \tag{3.18}
\]

Now, we define the central object of this subsection.

**Definition 3.5** Let \( R > 1 \) be a positive real number. Let \( w_1, \ldots, w_N \) be a series of \( N \) trajectories in \( \mathbb{R}^d \), with \( w_i(0) = x_i \in \mathbb{R}^d \). We denote by \( W_N = (w_1, \ldots, w_N) \) the collection of these trajectories. We denote by

\[
\Phi(W_N, R) = \bigcup_{i=1}^{N} B\left(w_i(0), \tau_{B^{c}(x_i,R/2)} \wedge c_0 R^2, 1\right) \tag{3.19}
\]

the union of the sausages of these trajectories stopped at the smaller of \( c_0 R^2 \) and the exiting time of the ball of radius \( R/2 \) centered at the respective starting point.

Let \( X^{(1)}, \ldots, X^{(N)} \) be \( N \) independent Brownian motions, with \( X_0^{(i)} = x_i \). Let \( \bar{X}_N = (X^{(1)}, \ldots, X^{(N)}) \). In the rest of this section, we are going to study the capacity of \( \Phi(\bar{X}_N, R) \).

We start with the upper bounds on its first and second moments.

**Lemma 3.6** Let \( \bar{X}_N \) be defined as above. We denote its joint law by \( E \). One has

\[
E[\text{cap}(\Phi(\bar{X}_N, R))] \leq CN R^2, \tag{3.20}
\]

and

\[
E[\text{cap}(\Phi(\bar{X}_N, R))^2] \leq CN^2 R^4. \tag{3.21}
\]

**Proof** By the definition of \( N^t \) [See (2.48)], we know that for each \( i = 1, \ldots, N \), \( \Phi((X^{(i)}), R) \) is covered by no more than \( N^{c_0 R^2} \) balls of radius 2, \( P_{x_i} \)-almost surely.

\[
\phi((X^{(i)}), R) \subadd \leq N^{c_0 R^2} \text{ balls of radius 2, } P_{x_i} \text{-almost surely.} \tag{3.22}
\]

Thanks to the independence of \( X^{(i)} \), \( i = 1, \ldots, N \), the sub-additivity of Brownian capacity, see (2.19), and in the case of (3.21) also by discrete Hölder inequality, to prove (3.20) and (3.21) it suffices to verify that

\[
E_{x_i}[N^{c_0 R^2}] \leq c R^2 \text{ and } E_{x_i}[(N^{c_0 R^2})^2] \leq c' R^4. \tag{3.23}
\]
In fact, one can easily check (3.23) by (2.49). This finishes the proof of (3.20) and (3.21).

In the next Proposition, we derive a lower bound on the first moment of \( \text{cap}(\Phi(X_N, R)) \).

**Proposition 3.7** With the same setup as in Lemma 3.6, one has

\[
E[\text{cap}(\Phi(X_N, R))] \geq C \min(NR^2, Rd^{-2}).
\] (3.24)

The key method used in the proof of this proposition is Theorem 2.4.

**Proof** In this proof, we use superscripts to distinguish stopping times with respect to different Brownian motions, i.e., \( T^{(i)} \) stands for the stopping times with respect to \( X^{(i)} \). Write

\[
J = \{ i \in \{1, \ldots, N\}, T^{(i)}_{B(x_i, R/2)} \geq c_0 R^2/2 \}. \tag{3.25}
\]

See (3.18) for the definition of \( c_0 \) in Lemma 3.4.

We then define a probability measure \( m \) on \( \Phi(X_N, R) \) through the density function \( h_m \) (recall the definition of \( f_x(\cdot) \) in the statement of Lemma 2.2)

\[
h_m(x) = \begin{cases} 
    c(|J|R^2)^{-1} \sum_{i \in J} \int_{c_0 R^2/4}^{c_0 R^2/2} f_{X^{(i)}}(x)dt & \text{if } J \neq \emptyset; \\
    \left( \text{Volume}(\Phi(X_N, R)) \right)^{-1} & \text{if } J = \emptyset,
\end{cases}
\] (3.26)

where \( c \) is a constant that makes \( m \) a probability measure. Let \( A = \{|J| \geq \frac{1}{2} N\} \). By (3.18), we know that

\[
P[A] \geq c_1 \tag{3.27}
\]

for some \( c_1 > 0 \).

Since \( m \in M_1(\Phi(X_N, R)) \), by Theorem 2.4, we see that [recall the definition of \( E(m) \) below (2.23)]

\[
E[\text{cap}(\Phi(X_N, R))] \geq E[E(m)^{-1}] \geq E[E(m)^{-1}; A]. \tag{3.28}
\]

And on \( A \) (note that in this case \( J \neq \emptyset \)) we obtain that [recall the definition of \( F_L(i, j) \) in (2.11)]

\[
E(m) = \frac{C}{|J|^2 R^4} \sum_{i, j \in J} F_{c_0 R^2}(i, j). \tag{3.29}
\]
Therefore, by Cauchy–Schwarz inequality, it follows that

\[
E[\mathcal{E}(m)^{-1}; A] \geq E \left[ \frac{C}{|J|^2 R^4} \sum_{i,j \in J} F_{\frac{c}{2} R^2(i,j)}; A \right]^{-1} \cdot P[A]^2
\]

(3.27)

\[
\geq C' N^2 R^4 E \left[ \sum_{i,j \in J} F_{\frac{c}{2} R^2(i,j)}; A \right]^{-1}
\]

(3.30)

\[
\geq C' N^2 R^4 E \left[ \sum_{i,j=1,\ldots,N} F_{\frac{c}{2} R^2(i,j)} \right]^{-1}
\]

(2.12)

\[
\geq C'' N^2 R^4 \left( c''(N R^2 + N^2 R^{6-d}) \right) \geq C''' \min(N R^2, R^{d-2}).
\]

The claim (3.24) hence follows by putting (3.28) and (3.30) together. \( \Box \)

Now, we turn to the construction of the central object of this and the next subsection. Let \( A \) be a compact set and let \( \omega \) be a point measure on \( \mathbb{W}^* \) such that \( \omega = \sum_{i \geq 0} \delta_{w_i^*} \).

Let \( N = \omega(W^*_A) \), we denote by [recall the definition of \( \pi_A \) in (2.26)]

\[
\overline{W}(\omega, A) = (\pi_A(w^*_1), \ldots, \pi_A(w^*_N)), \quad \text{where } \{i_1, \ldots, i_N\} = \{i \geq 0; \, w^*_i \in W^*_A\}
\]

(3.31)

the collection of trajectories in the support of \( \omega \) that touch \( A \). We then write

\[
\Psi(\omega, A, R) = \Phi(\overline{W}(\omega, A), R).
\]

(3.32)

Now, we introduce some notation on the restriction of point measures on \( \mathbb{W}^* \).

**Definition 3.8** Let \( r, R \) be real numbers such that \( 1 < r < R \), and let \( \omega \) be a point measure on \( \mathbb{W}^* \) such that \( \omega = \sum_{i \geq 0} \delta_{w_i^*} \) with \( w_i^* \in \mathbb{W}^* \). We write

\[
\omega_r = \sum_{i \geq 0, \, w_i^* \in \mathbb{W}^*_{B(r)}} \delta_{w_i^*}
\]

(3.33)

for a new point measure which is the restriction of \( \omega \) to the set of trajectories that intersect \( B(r) \), write

\[
\omega_{r,\infty} = \omega - \omega_r
\]

(3.34)

for the restriction of \( \omega \) to the set of trajectories that do not intersect \( B(r) \), and write

\[
\omega_{r,R} = \omega_R - \omega_r
\]

(3.35)

for the restriction of \( \omega \) to the set of trajectories that intersect \( B(R) \) but do not intersect \( B(r) \).
We now let \( \tilde{\omega} = \sum_{i \geq 0} \delta_{(w_i^*, \alpha_i)} \) be the interlacement process and apply
\[
\omega = \sum_{i \geq 0, \alpha_i \leq \alpha} \delta_{w_i^*}
\] (3.36)
to Definition 3.8 above. Note that point processes \( \omega_r \) and \( \omega_{r, \infty} \) are independent from each other.

We now give an estimate on the capacity of a random cactus constructed from \( \omega_{r, \infty} \) and a compact subset of \( \mathbb{R}^d \).

**Proposition 3.9** Let \( A \) be a compact subset of \( \mathbb{R}^d \), and let \( r, R \) be real numbers such that \( 1 < r < R \), one has
\[
\mathbb{E}[\text{cap}(\Psi(\omega_{r, \infty}, A, R))] \geq c \min(\text{cap}(A) R^2, R^{d-2}) - C r^{d-2} R^2. \tag{3.37}
\]

To prove Proposition 3.9, we need the following lemma. We omit its proof because it is identical to that of Lemma 4.3 in [23], with (3.24) playing the role of (4.5) in Lemma 4.2 in [23].

**Lemma 3.10** For all \( A \) compact subset of \( \mathbb{R}^d \), one has
\[
\mathbb{E}[\text{cap}(\Psi(\omega, A, R))] \geq c \min(\text{cap}(A) R^2, R^{d-2}). \tag{3.38}
\]

**Proof of Proposition 3.9** We start with decomposing \( \omega \):
\[
\omega = \omega_r + \omega_{r, \infty}. \tag{3.39}
\]

By the definition of \( \Psi \),
\[
\Psi(\omega, A, R) = \Psi(\omega_{r, \infty}, A, R) \cup \Psi(\omega_r, A, R). \tag{3.40}
\]

By the sub-additivity of capacities, we obtain that
\[
\mathbb{E}[\text{cap}(\Psi(\omega_{r, \infty}, A, R))] \geq \mathbb{E}[\text{cap}(\Psi(\omega, A, R))] - \mathbb{E}[\text{cap}(\Psi(\omega_r, A, R))]. \tag{3.41}
\]

Thanks to (3.38), to prove (3.37), it suffices to show that
\[
\mathbb{E}[\text{cap}(\Psi(\omega_r, A, R))] \leq C R^2 r^{d-2}. \tag{3.42}
\]

To this end, suppose that
\[
\omega_r = \sum_{i=1}^{N} \delta_{w_i^*} \tag{3.43}
\]

where \( N = \omega(W_{B(r)}^*) = \omega_r(W_{B(r)}^*), w_1^*, \ldots, w_N^* \in W^* \). By (2.33) and the definition of \( \omega \) [see (3.36)], we know that \( N \sim \text{Pois}(\alpha \text{cap}(B(r))) \) and conditioned on \( N \) and the
starting points \( X_0^{(i)} \), \( X^{(i)} \)'s are independent Brownian motions. Hence from (3.20), we obtain that

\[
E[\text{cap}(\Psi_{\omega_r, A, R})] \leq E\left[\sum_{n=0}^{\infty} \sup_{x_1, \ldots, x_n \in \partial A} \Phi(\omega, R) | N = n, (w_1(0), \ldots, w_n(0)) = (x_1, \ldots, x_n) \right] P[N = n]
\]

(3.20)

\[
\leq E[N] e R^2 \leq C R^2 r^{d-2}.
\]

(3.44)

This confirms (3.42) as well as (3.37).

3.3 The Upper Bound

In this subsection, we prove the more difficult part of Theorem 1.1, namely the upper bound on the diameter of \( G_{\alpha, 1} \). As mentioned in Sect. 1, the proof scheme is parallel to that of [23] except a few simplifications. Let us also point out some technical issues we encounter when dealing with Brownian motion. First, the energy estimate (see Proposition 3.7) that gives the lower bound of the expected capacity of the truncated sausages requires a version of Lemma 4.1 of [23] for Brownian motion. This is archived in Lemmas 2.1 and 2.2. Second, when estimating capacities of the upper bound, we need an estimate on number of balls needed to cover a Brownian motion path, see (3.22).

We start by rephrasing our main goal in this subsection.

**Proposition 3.11** For all \( d \geq 3 \), \( \alpha > 0 \)

\[
P[\text{diam}(G_{\alpha, 1}) \leq s_d] = 1.
\]

(3.45)

We first show that we only need to prove (3.45) for \( d \geq 5 \).

**Proposition 3.12** The claim (3.45) is true for \( d = 3 \) and \( d = 4 \).

**Proof** When \( d = 3 \) or \( d = 4 \), \( s_d = 1 \), hence it suffices to show that two Wiener sausages, regardless of their starting points, will hit each other almost surely. The case \( d = 3 \) is classical. The case \( d = 4 \) follows by Theorem 6.2 of [2].

From now on in this subsection, we only consider \( d \geq 5 \). Let \((X_t)_{t \geq 0}\) be a Brownian motion in \( \mathbb{R}^d \) with \( X_0 = x \). We denote its law and the respective expectation by \( P^X_x \) and \( E^X_x \). Let \( \omega^{(2)}, \omega^{(3)}, \ldots \in \Omega \) be i.i.d. interlacement processes at level \( \alpha \), which are also independent from \((X_t)_{t \geq 0}\). We denote by \( P^{(2)}, P^{(3)}, \ldots \) their laws and by \( E^{(2)}, E^{(3)}, \ldots \) the respective expectations. For \( s \geq 1 \), we write \( P^{(s)}_x \) for the joint law \( P^X_x \otimes P^{(2)} \otimes P^{(3)} \otimes \cdots \otimes P^{(s)} \), and \( E^{(s)}_x \) for the respective expectation.

Now let \( r \) and \( R \) be positive reals such that \( 1 < r < R \) and \( |x| < R \). We define a sequence of random subsets of \( \mathbb{R}^d \) associated with \( X \) and \( \omega^{(i)} \) in the following inductive manner. We write

\[
A^1(r, R) = \Phi(X(\cdot + T_{B^c(r)}), R),
\]

(3.46)
and for $s \geq 2$, we build $A^s(r, R)$ upon the measure $\omega_{r, \infty}^{(s)}$ (recall the notation in Definition 3.8) and $A^{s-1}(r, R)$ in the following manner:

$$A^s(r, R) = \Phi \left( \omega_{r, \infty}^{(s)}, A^{s-1}(r, R), R \right).$$  \hspace{1cm} (3.47)

It is worth noting that

$$A^{s-1}(r, R) \text{ is independent from } \omega^{(s)}. \hspace{1cm} (3.48)$$

**Remark 3.13** One can inductively prove that

$$A^{s-1}(r, R) \subset B(sR) \hspace{1cm} (3.49)$$

and this implies that the definition of $A^s(r, R)$ is not changed if we replace $\omega_{r, \infty}^{(s)}$ by $\omega_{r,sR}^{(s)}$, i.e.,

$$A^s(r, R) = \Phi \left( \omega_{r,sR}^{(s)}, A^{s-1}(r, R), R \right). \hspace{1cm} (3.50)$$

Now, we derive an upper bound on the second moment of cap$(A^s(r, R))$.

**Proposition 3.14** For all $s \leq s_d$, one has

$$\mathbb{E}^{(s)} x \left[ \text{cap}(A^s(r, R))^2 \right] \leq CR^{2 \min(d-2,2s)}. \hspace{1cm} (3.51)$$

Notice that the constants in the statement above and the proof below do depend on $s$. However, since we only look at $s \leq s_d$, we can safely drop this dependence in the notation.

**Proof** In this proof, we write $A^s$ as a shorthand for $A^s(r, R)$. By the monotonicity of the capacity and (3.49), we know that

$$\text{cap}(A^s) \leq \text{cap}(B((s + 1)R)) \leq cR^{d-2}. \hspace{1cm} (3.52)$$

Hence to prove (3.51), it suffices to show that for all $1 \leq s \leq s_d$, one has

$$\mathbb{E}^{(s)} x \left[ \text{cap}(A^s)^2 \right] \leq c''R^{4s}. \hspace{1cm} (3.53)$$

We now prove (3.53) by induction. When $s = 1$, Lemma 3.6 implies that

$$\mathbb{E}^{(1)} x \left[ \text{cap}(A^1)^2 \right] \leq c''R^4. \hspace{1cm} (3.54)$$

Suppose now that (3.53) is true for some $s - 1 \geq 1$. By the definition of $\Psi$, $A^s$ consists of $\omega_{r, \infty}^{(s)}(W^*_{A^{s-1}})$ Wiener sausages. Moreover, we know that conditioned on $A^{s-1}$

$$\omega_{r, \infty}^{(s)}(W^*_{A^{s-1}}) \leq \omega^{(s)}(W^*_{A^{s-1}}) \sim \text{Pois} \left( \alpha \text{cap}(A^{s-1}) \right). \hspace{1cm} (3.55)$$
Hence,

\[
\mathbb{E}_x^{(s)} \left[ (\text{cap}(A^s))^2 \right] \overset{(3.21)}{\leq} cR^4 \mathbb{E}_x^{(s)} \left[ \omega^{(s)}(W_{A^{s-1}}^*)^2 \right] \overset{(3.55)}{\leq} cR^4 \mathbb{E}_x^{(s)} \left[ \omega^{(s)}(W_{A^{s-1}}^*)^2 \right] \\
\overset{(3.55)}{\leq} cR^4 \left( \mathbb{E}_x^{(s-1)} \left[ (\alpha \text{cap}(A^{s-1}))^2 \right] + \mathbb{E}_x^{(s-1)} \left[ \alpha \text{cap}(A^{s-1}) \right] \right) \\
\overset{(3.53)}{\leq} c' R^4 \left( R^{4s-4} + R^{2s-2} \right) \leq c'' R^{4s}.
\] (3.56)

The claim (3.53) thus follows thus by induction. This finishes the proof of (3.51). \(\square\)

We now inductively derive a lower bound on the expectation of the capacity of \(A^s(r, R)\).

**Proposition 3.15** There exists an \(\bar{\epsilon} \in (0, 1)\), such that for all \(1 \leq s \leq s_d\), and for all positive reals \(r\) and \(R\) that satisfy

\[
1 < r^{d-2} \leq \bar{\epsilon} R,
\] (3.57)

one has

\[
\mathbb{E}_x^{(s)} [\text{cap}(A^s(r, R))] \geq c(s) R^{\min(d-2, 2s)}.
\] (3.58)

for a sequence of positive constants \(c(1), \ldots, c(s)\). In particular,

\[
\mathbb{E}_x^{(s_d)} [\text{cap}(A^{s_d}(r, R))] \geq c R^{d-2}.
\] (3.59)

**Proof** As in the proof of Proposition 3.14, we still write \(A^s\) for \(A^s(r, R)\). We postpone our choice of \(\bar{\epsilon}\) until the end of this proof. We prove (3.58) by induction on \(s\). By (3.24), we know that

\[
\mathbb{E}_x^{(s)} [\text{cap}(A^1)] \geq c(1) R^2.
\] (3.60)

Set \(\epsilon(1) = 1/2\). Let \(2 \leq s \leq s_d\), and assume the induction hypothesis holds for \(s - 1\):

\[
\mathbb{E}_x^{(s-1)} [\text{cap}(A^{s-1})] \geq c(s - 1) R^{\min(d-2, 2s-2)}
\] (3.61)

for some \(c(s - 1) > 0\), for all \(r, R\) such that \(1 < r^{d-2} \leq \epsilon(s - 1) R\). By (3.61) above and the upper bound on its second moment [see (3.51)], the Paley–Zygmund inequality (see e.g., [16]) implies that there exists a positive constant \(c_1(s - 1)\) and \(c_2(s - 1)\) such that

\[
\mathbb{E}_x^{(s-1)} [\text{cap}(A^{s-1})] \geq c_1(s - 1) R^{\min(d-2, 2s-2)} \geq c_2(s - 1).
\] (3.62)

\(\square\) Springer
Then, it follows from Proposition 3.9 and the definition of \( \Psi \) that

\[
\mathbb{E}_x^{(s)} [\text{cap}(A^s)] \stackrel{(3.37)}{=} \mathbb{E}_x^{(s-1)} \left[ c \min(\text{cap}(A^{s-1}), R^{d-2}) - C r^{d-2} R^2 \right] \\
\geq \mathbb{E}_x^{(s-1)} \left[ c' \min(R^{\min(d-2,2s-2)} \times R^2, R^{d-2}) \right] \geq c R^{d-2} R^2
\]

(3.62)

\[
= c''(s-1) \min(R^2 \times R^{\min(d-2,2s-2)}, R^{d-2}) - C r^{d-2} R^2
\]

(3.57)

\[
= c''(s-1) R^{\min(d-2,2s)} - c' \epsilon R^3 \geq \frac{c''(s-1)}{2} R^{\min(d-2,2s)},
\]

(3.63)

with a choice of sufficiently small \( \epsilon' \) which makes inequality marked with \( (\ast) \) valid. Letting \( c(s) = c''(s-1)/2 \) and \( \epsilon(s) = \min(\epsilon(s-1), \epsilon') \), we thus confirm the induction step for \( s \). Hence, (3.58) is verified for all \( s = 1, \ldots, s_d \). Let \( \bar{\epsilon} = \epsilon(s_d) \). The claim (3.59) follows from the definition of \( s_d \) [see (1.1)].

\[ \square \]

**Remark 3.16** (1) Note that in the proof above, in order to proceed with induction, one needs to have a bound like (3.62) for \( \text{cap}(A^s) \). This is obtained through Paley–Zygmund inequality, for which the lower bound on its first moment and the upper bound on the second moment are prerequisites.

(2) The combination of Propositions 3.14 and 3.15 indicates that the \( c R^{\min(d-2,2s)} \) is the right order for \( \mathbb{E}_x^{(s)} [\text{cap}(A^s(r, R))] \). Most importantly, \( A^{s_d} \), a subset of \( B((s_d + 1)R) \), has capacity of order \( R^{d-2} \). This means, in terms of capacity, \( A^{s_d} \) “saturates” the ball \( B((s_d + 1)R) \).

One can use the proposition above to study the probability of another independent Brownian motion hitting \( A^s(r, R) \).

**Definition 3.17** Consider \( d \geq 5 \). Let \( (Z_t)_{t \geq 0} \) be a Brownian motion on \( \mathbb{R}^d \) with \( Z_0 = z \in B(R) \), independent from \( X \) and \( \omega^{(2)}, \ldots, \omega^{(s)} \). In the following part of this section, we denote its law by \( P^Z \) and the respective expectation by \( E^Z \). We write \( \widehat{P} \) as a shorthand for \( P^Z \otimes P^{(s_d)}_x \).

**Proposition 3.18** \((d \geq 5)\) With the above definition, and the \( \bar{\epsilon} \) chosen in Proposition 3.15, there exist positive constants \( c_2 = c_2(\alpha, d) \) and \( c_3 = c(\alpha, d) \), \( \overline{R} > 0 \) such that for all positive reals \( r \) and \( R \) that satisfy

\[
r > 1 \text{ and } R \geq \max(r^{d-2}/\bar{\epsilon}, \overline{R})
\]

(3.64)

one has

\[
\widehat{P} \left[ H^Z(A^{s_d}(r, R)) < T^{(Z)}_{B^c(R^2)} \right] \geq c_3.
\]

(3.65)

**Proof** We write \( A \) for \( A^{s_d}(r, R) \) throughout this proof. On the one hand, by (2.17),

\[
\widehat{P} \left[ H^Z_A < \infty \right] = \mathbb{E}_x^{(s_d)} \left[ \int g(z, y)e_A(dy) \right].
\]

(3.66)
By (3.49), for any \( y \in A \),

\[
g(z, y) \geq CR^{2-d}.
\]  
(3.67)

By (3.59) [note that condition (3.64) is stronger than (3.57)], we obtain that

\[
\tilde{P} \left[ H^{(Z)}(A) < \infty \right] \geq CR^{2-d} E \left[ \text{cap}(A) \right] \geq c'' R^{d-2} R^{2-d} = c_4.
\]  
(3.68)

On the other hand, by the strong Markov property of \((Z_t)_{t \geq 0}\),

\[
\tilde{P} \left[ \infty > H^{(Z)}(\mathbb{Z}) \geq T^{(Z)}_{B^c(R^2)} \right] \leq \sup_{z' \in \partial B(R^2)} P^Z_{z'} \otimes \mathbb{P}^x \left[ H^{(Z)}_A < \infty \right] \leq \sup_{z' \in \partial B(R^2)} P^Z_{z'} [H_B(cR) < \infty],
\]  
(3.69)

moreover,

\[
\sup_{z' \in \partial B(R^2)} P^Z_{z'} [H_B(cR) < \infty] \leq \sup_{z' \in \partial B(R^2), y \in B(cR)} g(z', y) \text{cap}(B(cR)) \leq c R^{4-2d} \cdot c' R^{d-2} \leq c_5 R^{2-d}.
\]  
(3.70)

Hence, the claim (3.65) follows by combining (3.68), (3.69) and (3.70), choosing appropriate \( c_3 \) and \( R \).

\[\square\]

As the last step before the final theorem, we set up a sequence of scales and show that almost surely, at some scale, \((Z_t)_{t \geq 0}\) hits the “cactus-shaped” set \( A \) (in fact, we are able to show that such hitting happens at infinitely many scales).

For \( x, z \in \mathbb{R}^d \), we define two sequences of positive real numbers \((r_k)_{k \geq 0}\) and \((R_k)_{k \geq 0}\) through (see Propositions 3.15 and 3.18, respectively, for the definition of \( \epsilon \) and \( R_0 \))

\[
r_0 = \max(|x|, |z|) + 1 + 2(s_d + 1) R, \quad R_0 = \max(r_0, \epsilon^{-1} r_0^{d-2}),
\]  
(3.71)

and for \( k \geq 1 \)

\[
r_{k+1} = \max(R_k^2, 2(s_d + 1) R_k), \quad R_{k+1} = \epsilon^{-1} r_{k+1}^{d-2}.
\]  
(3.72)

Notice that for all \( k \geq 0 \), conditions (3.57) and (3.64) are satisfied (when \( r \) and \( R \) are replaced by \( r_k \) and \( R_k \)).

**Proposition 3.19** \((d \geq 5)\) For all \( x, z \in \mathbb{R}^d \) and for \((r_k)_{k \geq 0}\) and \((R_k)_{k \geq 0}\) defined inductively in (3.71) and (3.72) (note that they depend implicitly on \( x \) and \( z \)), one has

\[\square\] Springer
\begin{equation}
\hat{\mathbb{P}}\left[ \lim_{k} \sup \{ H(Z) (A^{sd}(r_k, R_k)) < \infty \} \right] = 1. \tag{3.73}
\end{equation}

**Proof** We fix $x$ and $z$ throughout this proof. For the convenience of notation, we write $A_k$ for $A^{sd}(r_k, R_k)$. We first claim that, to prove (3.73), it suffices to prove for any $g_1, \ldots, g_{k-1} \in \{0, 1\},$
\[\hat{\mathbb{P}}[\Gamma_k | \gamma_1 = g_1, \ldots, \gamma_{k-1} = g_{k-1}] \geq c > 0, \tag{3.74}\]
where
\[\Gamma_k = \left\{ H^{(Z)}(A^{sd}(r_k, R_k)) \circ \theta^{(Z)}_U < T^{(Z)}(B^o(R^2_k)) \circ \theta^{(Z)}_U \right\}, \tag{3.75}\]
(in which we write and $\mathbb{U} = T^{(Z)}(B^o(r_k)))$, and $\gamma_i = 1_{\Gamma_i}, i \in \mathbb{Z}^+$. In fact, by Borel–Cantelli lemma (see Lemma 2.6), (3.74) implies that
\[\hat{\mathbb{P}}\left[ \lim_{k} \Gamma_k \right] = 1. \tag{3.76}\]
Since the Brownian motion is transient when $d \geq 3$, $T^{(Z)}(B^o(R^2_k)) < \infty$, $\hat{\mathbb{P}}$-almost surely, (3.76) implies that
\[\lim_{k} \sup H^{(Z)}(A_k) < \infty, \quad \hat{\mathbb{P}}\text{-a.s.} \tag{3.77}\]
This finishes the proof of (3.73) once we confirm (3.74).

Now, we prove (3.74). Pick $k \in \mathbb{Z}^+$, and write
\[\mathcal{F}_k = \sigma \left( Z_t, 0 \leq t \leq T_{B^o(r_{k+1})}^{(Z)}; X_t, 0 \leq t \leq T_{B^o(r_{k+1})}^{(X)}; \omega_{r_{k+1}}, i = 1, \ldots, s_d \right). \tag{3.78}\]
$(\mathcal{F}_k)_{k \geq 1}$ forms a filtration. By (3.72) and (3.49), $\Gamma_k \in \mathcal{F}_k$.

It is straightforward that for any $g_1, \ldots, g_{k-1} \in \{0, 1\}$,
\[\{ \gamma_1 = g_1, \ldots, \gamma_{k-1} = g_{k-1} \} \in \mathcal{F}_{k-1}. \tag{3.79}\]
Hence to prove (3.74), it suffices to prove that
\[\hat{\mathbb{P}}[\Gamma_k | \mathcal{F}_{k-1}] \geq c > 0. \tag{3.80}\]
In fact, to benefit from Proposition 3.18, in the following calculation we are going to integrate out $\omega_{r_k}^i$, since $\omega_{r_k}^i, i = 1, \ldots, s_d - 1$ are independent from $\omega_{r_k, \infty}^i, i = 1, \ldots, s_d - 1$, and then apply twice the strong Markov property, first to $\left( Z_t \right)_{t \geq 0}$ at time
$U = T^{(Z)}_{B^o(r_k)}$ then to $\left( X_t \right)_{t \geq 0}$ at time $V = T^{(X)}_{B^o(r_k)}$. We write $x' = X_V$ and $z' = Z_V$.
and denote by $P_{\omega^j_{rk, \infty}}$ the law of $\omega^j_{rk, \infty}$ seen as the law of $\omega^j$. Now, with the properties above, we know that

$$
\tilde{P} \left[ \Gamma_k | \mathcal{F}_{k-1} \right] = P^Z \otimes P^X \otimes P^{(sd)} \left[ \Gamma_k | \mathcal{F}_{k-1} \right]
$$

(3.80)

$$
\equiv P^Z \otimes P^X \bigotimes_{i=2}^{sd} P_{\omega^j_{rk, \infty}} \left[ \Gamma_k | \sigma(Z_t, 0 \leq t \leq U; X_t, 0 \leq t \leq V) \right]
$$

Markov

$$
\equiv P^Z \otimes P^X \bigotimes_{i=2}^{sd} P_{\omega^j_{rk, \infty}} \left[ H_{A_k}^{(Z)} < T_{B^c(R^2_k)} \right] \geq c > 0,
$$

(3.48)

wherein (s) we use the fact that $A_k$ is independent from $\sigma(X_t, 0 \leq t \leq V)$. This confirms (3.74) and finishes the proof of Proposition 3.19.

Remark 3.20 From (3.73), we can conclude that there exist $w^2, w^3, \ldots, w^{sd} \in \text{supp}(\omega^2 + \cdots + \omega^{sd})$, such that $X(\mathbb{R}_+) \cap B(w^2(\mathbb{R}), 1) \neq \emptyset$, $w^2(\mathbb{R}) \cap B(w^3(\mathbb{R}), 1) \neq \emptyset$, $\ldots$, $w^{sd}(\mathbb{R}) \cap B(Z(\mathbb{R}_+), 1) \neq \emptyset$. Colloquially, this means one can connect $X$ and $Z$ via $sd - 1$ “intermediate sausages”.

Once we have proved Propositions 3.11 and 3.19 follows from repeating the same argument as Lemma 4.13 and Theorem 1.1 from [23], hence we omit the proof here.

Finally, Theorem 1.1 follows by combining Propositions 3.3 and 3.11 and applying scaling property of Brownian interlacements.

### 4 Existence of Non-trivial Phase Transition for the Vacant Set

This section is dedicated to Theorem 1.3, namely the existence of a non-trivial phase transition in percolation for the vacant set of Brownian interlacements. We start with some comments on the strategy. The central object in this approach is the dyadic renormalization tree, introduced in Sect. 4.1, with a set of specific rules on how vertices of the tree should be embedded in $\mathbb{Z}^d$ so that the image of all leaves is in some sense “well-separated”.

The finiteness of $\alpha_1^*$ follows from the following energy–entropy competition. To be more specific, if the vacant set crosses an annulus with the size of order $6^n$, then by a argument similar to [21], there exists $T$, an embedding of the dyadic tree of depth $n$, such that the crossing path passes through the image of all $2^n$ leaves of $T$, which are well-separated. On the one hand, we can show that the probability cost of the interlacement set avoiding all leaves is of order $\exp(-c\alpha 2^n)$, thanks to a capacity lower bound relying on the well-separation of leaves in the embedding. On the other hand, the number of possible embeddings is bounded by $C2^n$ according to the rules.
Hence, when $\alpha$ is sufficiently large, the crossing probability decays to 0 very quickly as $n$ tends to infinity, implying that there is no chance of percolation.

To show the positiveness of $\alpha_1^*$, we focus on a plane in the Euclidean space and prove that when $\alpha$ is sufficiently small, $\mathcal{V}_1$ percolates on this plane. Using planar duality, to this end, we only need to show that for a given large $L_0$, when $\alpha$ is small, the probability for the interlacement set to cross a planar annulus at scale $L_0 \cdot 6^n$ decays rapidly as $n$ tends to infinity. We are able to show that if such crossing takes place, the interlacement set will touch $2^n$ planar “frames” of size $L_0$, all of which are centered at the image of leaves of a planar embedding of the dyadic tree of depth $n$. The calculation of the probability of such event can be reduced to a large deviation estimate on the number of hitting of the frames by the trajectories from Brownian interlacements. Again, thanks to the well-separation of the leaves of dyadic renormalization tree, if $L_0$ is chosen sufficiently large, such probability can be arbitrarily small by letting $\alpha$ tend to 0, giving a bound strong enough to beat the combinatorial complexity. This shows that the crossing within the interlacement set is unlikely when $\alpha$ is small. Thus, it follows the positiveness of $\alpha_1^*$.

We now record notations we need later in this section. We denote by $F$ the plane in $\mathbb{R}^d$ passing through the origin:

$$F = \mathbb{R}^2 \times \{0\}^{d-2} \subset \mathbb{R}^d,$$

and by $F_Z$ the discrete plane passing through the origin:

$$F_Z = \mathbb{Z}^2 \times \{0\}^{d-2} = F \cap \mathbb{Z}^d.$$

We also denote the “stick” of length $L$ by

$$J_L = [0, L] \times \{0\}^{d-1}.$$

For $x \in \mathbb{Z}^d$, $R \in \mathbb{N}$, we denote by

$$S(x, R) = \{y \in \mathbb{Z}^d : |y - x|_\infty = R\}$$

the $l^\infty$-sphere (boundary of a discrete $l^\infty$-ball) centered at $x$ of radius $R$, and for $x \in F_Z$, by

$$S^{(2)}(x, R) = \{y \in F_Z : |y - x|_\infty = R\}$$

the two dimensional discrete “square” centered at $x$ with “size” $R$. From now on, we fix

$$\beta = 2\sqrt{d} + 4.$$

The next lemma gives an upper bound on the capacity of the inflation of $S^{(2)}(x, L)$.  

$\square$ Springer
Lemma 4.1  For all $L > 3\beta$,

$$\text{cap}(B(S(L^2)(x, L), \beta)) \leq \begin{cases} cL & d \geq 4 \\ c'L \ln L & d = 3. \end{cases} \quad (4.7)$$

Proof  Since $B(S(L^2)(x, L), \beta)$ can be covered by four “tubes” which are $B(J_L, \beta)$ after translation and rotation, to prove the claim (4.7) it suffices to show that for all $L > 3\beta$,

$$\text{cap}(B(J_L, \beta)) \leq \begin{cases} cL & d \geq 4 \\ c'L \ln L & d = 3. \end{cases} \quad (4.8)$$

Now, we prove (4.8). We consider a point $x = (x_1, x_2, \ldots, x_d) \in B(J_L, \beta)$. Without loss of generality, we assume that $x_1 \geq L/2$. For all $y = (y_1, y_2, \ldots, y_d) \in B(J_L, \beta)$ such that $y_1 \leq L/3$, we know that

$$|x - y| \leq \sqrt{|x_1 - y_1|^2 + c\beta^2} \leq c'|x_1 - y_1|. \quad (4.9)$$

By (2.18), to bound the capacity from above, it suffices to establish a lower bound on an integral regarding the Green function. We write $D$ for ball of radius $\beta$ centered at $0$ in $(d - 1)$ dimensions. We know that

$$\int_{y \in B(J_L, \beta)} g(x, y) dy \geq \int_{[0, L/3] \times D} g(x, y) dy = \int_{[0, L/3] \times D} c(|x - y|)^{2-d})dy \geq \int_{[0, L/3]} c''(|x_1 - y_1|)^{2-d})dy_1 \geq \begin{cases} c'' & d \geq 4 \\ \tilde{c}' \ln L & d = 3. \end{cases} \quad (4.10)$$

By (2.18), this readily implies (4.8). $\square$

4.1 The Dyadic Renormalization Tree

In this section, we construct the dyadic renormalization tree and state some of its useful properties. For readers’ convenience, we keep the same notation as in [21].

For $n \geq 0$, we write $T_{(n)} = \{1, 2\}^n$ (in particular, $T_{(0)} = \emptyset$) for the index set of the $n$-th generation on the tree. We denote by

$$T_n = \bigcup_{k=0}^n T_{(k)} \quad (4.10)$$

the dyadic tree of depth $n$. For $0 \leq k \leq n$, and a node $m = (\xi_1, \ldots, \xi_k) \in T_{(k)}$, we write

$$m_1 = (\xi_1, \ldots, \xi_k, 1) \text{ and } m_2 = (\xi_1, \ldots, \xi_k, 2), \quad (4.11)$$
for the two children of $m$ which lie in $T_{(k+1)}$. Given an integer $L_0 \geq 1$ (we will specify our choice of $L_0$ at the beginning of each subsection), we write down a sequence of scales

$$L_n = L_0 \cdot 6^n, \quad n \geq 0. \quad (4.12)$$

For $n \geq 0$, we also denote by $L_n = L_n \mathbb{Z}^d$ the integer lattice renormalized by $L_n$.

We call $T : T_n \to \mathbb{Z}^d$ (resp. $F_Z$) a proper embedding of $T_n$ into $\mathbb{Z}^d$ (resp. $F_Z$) rooted at $x \in L_n$ (resp. $L_n \cap F_Z$), if one has

$$T(\emptyset) = x, \quad (4.13)$$

plus for all $0 \leq k \leq n$, and $m \in T_{(k)}$,

$$T(m) \in L_{n-k} \ (\text{resp. } L_{n-k} \cap F_Z), \quad (4.14)$$

and moreover for all $0 \leq k < n$, and $m \in T_{(k)}$,

$$|T(m_1) - T(m)|_\infty = L_{n-k} \quad \text{and} \quad |T(m_2) - T(m)|_\infty = 2L_{n-k}. \quad (4.15)$$

For $x \in L_n$, one writes $\Lambda_{n,x}$ (resp. $\Lambda_{n,x}^F$) the set of proper embeddings of $T_n$ into $\mathbb{Z}^d$ (resp. $F_Z$) with root $x$.

Now, for the sake of completeness, we quote three lemmas without proof (Lemmas 3.2–3.4 in [21]) on dyadic trees and their embeddings.

The following lemma counts the total number of embeddings into of $T_n$ into $\mathbb{Z}^d$ and $F_Z$.

**Lemma 4.2** For $d \geq 2$, $L_0 \geq 1$, $n \geq 0$ and $x \in L_n$, there exists $C = C(d) > 0$ (which does not depend on $x$) such that

$$|\Lambda_{n,x}| = C2^n - 1. \quad (4.16)$$

In particular, for any $L_0 \geq 1$, $n \geq 0$ and $x \in L_n \cap F_Z$, $|\Lambda_{n,x}^F| = C(2)^{2^n - 1}$.

The following lemma shows that if for all $n \geq 0$ a *-path (see the beginning of Sect. 2 for the precise definition) goes through $S(x, L_n)$ and $S(x, 2L_n)$, then there is a proper embedding of $T_n$ on $\mathbb{Z}^d$ (for all $d \geq 2$) such that every leaf of this tree sits “on” the path. Note that this lemma applies to $F_Z$ as well for it is equivalent to the case of $d = 2$.

**Lemma 4.3** For any $L_0 \geq 1$, $n \geq 0$ and $x \in L_n$, if $\gamma$ is a *-path (and in particular a nearest-neighbor path) in $\mathbb{Z}^d$, $d \geq 2$, such that

$$\gamma \cap S(x, L_n - 1) \neq \emptyset \quad \text{and} \quad \gamma \cap S(x, 2L_n) \neq \emptyset, \quad (4.17)$$

then there exists $T \in \Lambda_{n,x}$ such that

$$\gamma \cap S(T(m), L_0 - 1) \neq \emptyset \quad \text{for all } m \in T_{(n)}. \quad (4.18)$$

\[ \square \] Springer
To state the next lemma, we need extra notations. For $0 \leq k \leq n$ and $m = (\xi_1, \ldots, \xi_n) \in T(n)$, we denote by $m|k$ the projection $(\xi_1, \ldots, \xi_k) \in T(k)$. For $m, m' \in T(n)$, we define the lexical distance between $m$ and $m'$ through

$$\rho(m, m') = \min\{k \geq 0 : m|n-k = m'|n-k\}. \quad (4.19)$$

For any $m \in T(n)$, we denote by

$$T^{m,k}(n) = \{m' \in T(n) : \rho(m, m') = k\} \quad (4.20)$$

all the leaves with lexical distance $k$ from $m$. Note that for all $1 \leq k \leq n$,

$$|T^{m,k}(n)| = 2^{k-1}. \quad (4.21)$$

The following lemma shows that a proper embedding is relatively “spread-out” on all scales and will be used in the proofs of Propositions 4.6 and 4.9.

**Lemma 4.4** For all $n \geq 1, x \in \mathcal{L}_n, T \in \Lambda_{n,x}, m \in T(n), k \geq 1$, and for all $m' \in T^{m,k}(n)$, $y \in S(T(m), L_0 - 1)$, $z \in S(T(m'), L_0 - 1)$, one has

$$|y - z| \geq L_{k-1}. \quad (4.22)$$

### 4.2 Preliminary Results for the Upper Bound on the Threshold

In this subsection, we prepare all the ingredients for the proof of the first part of (1.3), namely the finiteness of the percolation threshold $\alpha_1^*$. We fix, only in this subsection,

$$L_0 = 1. \quad (4.23)$$

We now assign symbols to the crossing events that we consider in this subsection. For all $\alpha > 0$, and $n \in \mathbb{N}$, we write (see the beginning of Sect. 2 for the definition of a continuous path and (4.6) for the definition of $\beta$)

$$A_{n,\alpha} = \{\exists \text{ a continuous path in } \mathcal{V}_{\beta}^{\alpha} \text{ connecting } B_\infty(0, L_n - 1) \text{ and } \partial B_\infty(0, 2L_n)\}. \quad (4.24)$$

and

$$D_{n,\alpha} = \{\exists \text{ nearest-neighbor path in } \mathcal{V}_1^{\alpha} \cap \mathbb{Z}^d \text{ connecting } S(0, L_n - 1) \text{ and } S(0, 2L_n)\}. \quad (4.25)$$

The following lemma shows that actually $A_{n,\alpha}$ is almost surely contained in $D_{n,\alpha}$. As it is almost obvious, we omit its proof.
Lemma 4.5 For all \( \alpha > 0 \), and \( n \in \mathbb{N} \),
\[
A_{\alpha n} \subseteq D_{\alpha n},
\]
and hence
\[
P[A_{\alpha n}] \leq P[D_{\alpha n}].
\]

We now state and prove the main result in this subsection.

Proposition 4.6 There exists \( \alpha^# > 0 \), such that for all \( \alpha > \alpha^# \), and for any \( n \geq 1 \),
\[
P[A_{\alpha n}] \leq \left( \frac{1}{2} \right)^2^n.
\]

Proof Recall that \( L_0 = 1 \). By Lemma 4.4, on the event \( D_{\alpha n}^\alpha \), there is at least one proper embedding of \( T_n \), which we denote by \( \hat{T} \in \Lambda_{n,0} \), such that there is a nearest-neighbor path \( \tilde{\gamma} : [0, \ldots, M] \to \mathbb{Z}^d \), for some \( M \in \mathbb{N} \), which lies in \( V_1^{\alpha} \), i.e., \( \tilde{\gamma}([0, \ldots, M]) \in V_1^{\alpha} \cap \mathbb{Z}^d \), that connects \( S(0, L_n - 1) \) to \( S(0, 2L_n) \) [see the definition of \( D_{\alpha n}^\alpha \) in (4.25)] and passes through all the points in \( \hat{T} \). This implies that \( \bigcup_{m \in T \cap \Lambda_{n,0}} \hat{T}(m) \) is not touched by \( \mathcal{T}_1^\alpha \). By (2.34), one has that
\[
P[\bigcup_{m \in T \cap \Lambda_{n,0}} \hat{T}(m) \cap \mathcal{T}_1^\alpha = \emptyset] = \exp(-\alpha \text{cap}(\mathcal{X}_{\hat{T}}))
\]
where for \( T \in \Lambda_{n,0} \) we define
\[
\mathcal{X}_T = \bigcup_{m \in T \cap \Lambda_{n,0}} B(T(m), 1).
\]

By Lemma 4.2, there are at most \( C2^n \) possible embeddings of \( T_n \); hence, we know that
\[
P[A_{\alpha n}] \leq P[D_{\alpha n}^\alpha] \leq C2^n \max_{T \in \Lambda_{n,0}} \exp(-\alpha \text{cap}(\mathcal{X}_T)).
\]

Now, we claim that (4.28) is proved if we can prove that for all \( T \in \Lambda_{n,0} \),
\[
\text{cap}(\mathcal{X}_T) \geq c2^n,
\]
uniformly for some \( c > 0 \). This is because with (4.32), if one chooses a sufficiently large \( \alpha^# > 0 \), then for all \( \alpha > \alpha^# \),
\[
P[A_{\alpha n}] \leq (C \exp(-c\alpha))2^n \leq (1/2)^2^n.
\]

Now, we prove (4.32). Thanks to (2.18) and the fact that \( |\mathcal{X}_T| = c2^n \), to give a lower bound on the capacity of \( \mathcal{X}_T \) it suffices to bound from above the denominator

\[\Box\] Springer
of the fraction in the first term of (2.18), i.e., an integral of the Green function. In fact, for all \( m \in T(n) \) and \( x \in B(T(m), 1) \) (note that such \( x \) runs over \( \mathcal{X}_T \)), we have

\[
\int_{y \in \mathcal{X}_T} g(x, y) \, dy = \sum_{m' \in T(n)} \int_{B(0,1)} g(T(m) + x, T(m') + y) \, dy
\]

\[
= \sum_{k=0}^{n} \sum_{m' \in T(m,k)} \int_{B(0,1)} g(T(m) + x, T(m') + y) \, dy
\]

\[
\leq c + c' \sum_{k=1}^{n} L_{k-1}^{2-d} \sum_{m' \in T(m,k)} \leq c''.
\] (4.34)

This finishes the proof of (4.32) as well as (4.28).

\[\Box\]

### 4.3 Preliminary Results for the Lower Bound on the Threshold

In this subsection, we prove some preparatory results for proof of the second half of (1.3), namely the positiveness of \( \alpha^* \). We postpone the choice of \( L_0 \) until the end of the proof of Proposition 4.9.

We now assign symbols to the crossing events we consider in this subsection. For \( \alpha > 0 \) and \( n \in \mathbb{N} \), we note the following event [recall the definition of \( L_n \) in (4.12)]

\[
\hat{A}_n^\alpha = \{ \text{there is a continuous path in } V_1^\alpha \text{ connecting } B_{\infty}(0, L_n) \text{ and infinity} \}.
\] (4.35)

Here by “a continuous path in \( V_1^\alpha \) connecting \( B_{\infty}(0, L_n) \) and infinity” we mean a continuous \( \gamma : [0, \infty) \to \mathbb{R}^d \) such that \( \gamma[0, \infty) \subseteq V_1^\alpha \) and \( \limsup_{t \to \infty} |\gamma(t)| = \infty \).

We also define the following events for \( x \in \mathbb{R}^d, \alpha > 0 \) and \( k \in \mathbb{N} \), [recall the definition of \( F \) in (4.1)],

\[
B_k^\alpha, x = \{ \text{there is a continuous path in } I_1^\alpha \cap F \text{ connecting } B_{\infty}(x, L_k) \text{ and } \partial B_{\infty}(x, 2L_k) \}.
\] (4.36)

and for all \( x \in F_Z \) [recall the definition of \( S^{(2)} \) in (4.5)],

\[
\overline{B}_k^\alpha, x = \exists \text{ nearest-neighbor path in } I_1^\alpha \cap F_Z \text{ connecting } S^{(2)}(x, L_k - 1) \text{ and } S^{(2)}(x, 2L_k).
\] (4.37)

The next lemma shows that almost surely \( B_k^\alpha, x \) is contained in \( \overline{B}_k^\alpha, x \). We omit its proof due to similarity with the proof of Lemma 4.5.

**Lemma 4.7** *For all \( \alpha > 0, k \in \mathbb{N}, x \in F_Z, \) one has that*

\[
B_k^\alpha, x \subset \overline{B}_k^\alpha, x.
\] (4.38)
The following lemma relates $\hat{A}_n^\alpha$ with $B_{k,x}^\alpha$ by asserting that on the event $(\hat{A}_n^\alpha)^c$, $B_{k,x}^\alpha$ must happen for a fixed number of choices of $x$.

**Lemma 4.8** For all $n \geq 2$,

$$(\hat{A}_n^\alpha)^c \subset \bigcup_{k \geq n-1} \bigcup_{x \in L_{k-1} \cap F_Z, |x| \leq 4L_{k+1}} B_{k,x}^\alpha. \tag{4.39}$$

**Proof** On the event $(\hat{A}_n^\alpha)^c$, $S(0, L_n)$ is completely surrounded by a continuous path $\gamma : [0, 1] \to F$, such that

$$\gamma(0) = \gamma(1) \quad \text{and} \quad \gamma([0, 1]) \subset \mathcal{T}_1^\alpha \cap F. \tag{4.40}$$

Now suppose $\gamma \subseteq B_\infty(0, 4L_{k+1})$ but $\gamma \not\subseteq B_\infty(0, 4L_k)$ for some $k \geq n - 1$. We then consider

$$Q_k = \{x \in L_{k-1} : \gamma \cap S(x, L_k) \neq \emptyset\}. \tag{4.41}$$

Pick $x_0 \in Q_k$. We claim that

$$\gamma \not\subseteq B_\infty(x_0, 2L_k) \tag{4.42}$$

hence there is a continuous path from $S(x_0, L_k)$ to $S(x_0, 2L_k)$. Now, suppose (4.42) is not true. Then, the $l_\infty$-diameter of $\gamma$ must be smaller or equal to $4L_k$. But, if this is the case, since on $F$ the origin is surrounded by $\gamma$, we would have $\gamma \subseteq B_\infty(0, 4L_k)$, a contradiction! This finishes the proof of (4.39). \hfill \square

This subsection culminates in the following proposition, which almost immediately implies the second half of Theorem 1.3, as we will see in Sect. 4.4.

**Proposition 4.9** There exist $L_0, \hat{\alpha} > 0$, such that for all $\alpha \in (0, \hat{\alpha})$ and all $x \in F_Z$,

$$\mathbb{P}\left[B_{n,x}^\alpha\right] \leq (1/4)^{2n} \tag{4.43}$$

holds for large $n$.

**Proof** We postpone the choice of $L_0$ till the end of this proof.

For our convenience, we write $\square_x$ for $S^{(2)}(x, L_0 - 1)$. Without loss of generality, we take $x = 0$. For $T_F \in \Lambda^F_{n,0}$ (see below (4.15) for the definition of $\Lambda^F_{n,0}$), we write

$$K(T_F) = \bigcup_{m \in T(n)} \square_{T(m)}, \tag{4.44}$$

which is a subset of $F$, and

$$K'(T_F) = B(K(T_F), \beta), \tag{4.45}$$
which is a subset of \( \mathbb{R}^d \). By Lemmas 4.2 and 4.3, one has that

\[
\mathbb{P}(\mathcal{B}_{n,0}^\alpha) \leq \mathbb{P}\left[ \bigcup_{T_F \in \Lambda_{n,0}^F} \bigcap_{m \in T_n} \{ \square_{T_F(m)} \cap T_{\beta}^\alpha \neq \emptyset \} \right] \leq C^{2^n} \max_{T_F \in \Lambda_{n,0}^F} \mathbb{P}\left[ \bigcap_{m \in T_n} \{ \square_{T_F(m)} \cap T_{\beta}^\alpha \neq \emptyset \} \right].
\]

(4.46)

Hence, to prove (4.43), it suffices to derive an adequate upper bound on

\[
\max_{T_F \in \Lambda_{n,0}^F} \mathbb{P}\left[ \bigcap_{m \in T_n} \{ \square_{T_F(m)} \cap T_{\beta}^\alpha \neq \emptyset \} \right].
\]

(4.47)

Let \( T_F \in \Lambda_{n,0}^F \) be an embedding of \( T_n \) into \( F_Z \). We drop the dependence on \( T_F \) in notation whenever there is no confusion arising. For \( w \in W^+ \), we write by \( \mathcal{N}(w) \) the number of frames (i.e., \( \square \)) with centers in \( T_F(T_n) \) (“leaves” of the embedding of dyadic tree) which are hit by the sausage of \( w \) with radius \( \beta \), i.e,

\[
\mathcal{N}(w) = \left| \{ m \in T_n, \ B(w([0, \infty)), \beta) \cap \square_{T_F(m)} \neq \emptyset \} \right|.
\]

(4.48)

Now, if \( T_{\beta}^\alpha \) intersects with all the frames on the leaf level \( T_n \), then the total count of hits must be at least \( 2^n \), i.e,

\[
\mathbb{P}\left[ \bigcap_{m \in T_n} \{ \square_{T_F(m)} \cap T_{\beta}^\alpha \neq \emptyset \} \right] \leq \mathbb{P}\left[ \sum_{j=1}^{N} \mathcal{N}(w_j) \geq 2^n \right],
\]

(4.49)

where \( N \) is determined through

\[
\mu_{K',\alpha}(\omega) = \sum_{j=1}^{N} \delta_{(w_j,\alpha_j)}
\]

(4.50)

[see (2.32) and (4.45) for notation]. Now, we write

\[
p = \max_{m \in T_n, \ y \in B(\square_{T_F(m)}, \beta)} P_y\left[ X([0, \infty)) \cap B(K \setminus \square_{T_F(m)}, \beta) \neq \emptyset \right].
\]

(4.51)

By the strong Markov property of Brownian motion, one knows that

\[
P_{\bar{\xi}_K}[\mathcal{N}(X) \geq k] \leq p^{k-1}
\]

(4.52)
for any \( k \geq 1 \). We are also able to show that \( p \) can be taken arbitrarily small if we take \( L_0 \) sufficiently large. In fact, for any \( m \in T(n) \) and \( y \in B(\square_{TF(m)}, \beta) \),

\[
p \leq \max_{m \in T(n), y \in B(\square_{TF(m)}, \beta)} \sum_{k=1}^{n} \sum_{m' \in T^{m,k}(n)} P_y[X[0, \infty) \cap B(\square_{TF(m')}, \beta) \neq \emptyset]
\]

(4.22),(2.5)\leq (2.17) \sum_{k=1}^{n} \sum_{m' \in T^{m,k}(n)} cL_{k-1}^{2-d} \text{cap}(B(\square_{TF(m')}, \beta))

(4.21) \leq (6^{2-d} \cdot 2)^{k-1} L_0^{2-d} c \text{cap}(B(\square_0, \beta)) \leq \left\{ \begin{array}{ll} c'L_0^{3-d} & \text{if } d \geq 4 \triangle \equiv q_d(L_0) \\ c''(\ln L_0)^{-1} & \text{if } d = 3 \end{array} \right.

(4.52)

With the help from (4.52) and (4.53), by the same argument (involving exponential Chebyshev inequality) as in the proof of Proposition 5.1 in [21] (see below (5.7) in [21]), we know that

\[
\mathbb{P}[\mathcal{B}_{n, \alpha}^{\#}] \leq \left( Cq_d(L_0) \right)^{2^m} \left[ \exp\left( \frac{\alpha \text{cap}(B(\square_0, \beta))}{q_d(L_0)} \right) \right]^{2^m}.
\]

(4.54)

We first fix \( L_0 \) sufficient large, such that \( Cq_d(L_0) < 1/8 \), and then choose \( \hat{\alpha} \) sufficiently small such that for all \( \alpha \in (0, \hat{\alpha}) \), \( \exp\left( \frac{\alpha \text{cap}(B(\square_0, \beta))}{q_d(L_0)} \right) < 2 \). With this choice of \( L_0 \) and \( \hat{\alpha} \), we know that the right hand side of (4.54) is bounded above from \((1/4)^{2^m}\). This finishes the proof of (4.43). \( \square \)

4.4 Denouement

In this subsection, we use Propositions 4.6 and 4.9 to prove Theorem 1.3.

Proof of Theorem 1.3 We start by showing that when \( \alpha \) is sufficiently large, the vacant set \( V_{\alpha}^{\#} \) does not percolate. By the definition of \( A_n^{\alpha} \) [see (4.24)], one knows that for \( \alpha > \alpha^{\#} \) and for all \( M \geq 0 \),

\[
\mathbb{P}[V_{\beta}^{\alpha} \text{ percolates}] \leq \mathbb{P}\left[ \bigcup_{n \geq M} A_n^{\alpha} \right]
\]

(4.55)

and by Proposition 4.6,

\[
\mathbb{P}\left[ \bigcup_{n \geq M} A_n^{\alpha} \right] \leq \sum_{n \geq M} (1/2)^{2^n} \to 0 \text{ as } M \text{ tends to infinity}.
\]

(4.56)
This means that
\[
\inf \left\{ \alpha \geq 0 : V_1^\alpha \text{ does not percolate a.s.} \right\} \leq \alpha^#.
\] (4.57)

By the scaling property [see (2.37)] of Brownian interlacements, we obtain that
\[
\alpha^*_1 \triangleq \inf \left\{ \alpha \geq 0 : V_1^\alpha \text{ does not percolate a.s.} \right\} \leq \alpha^# \beta^{d-2} < \infty,
\] (4.58)
and that for all \( r > 0 \),
\[
\alpha^*_r = \alpha^*_1 r^{2-d} = \inf \{ \alpha \geq 0 : V_r^\alpha \text{ does not percolate a.s.} \},
\] (4.59)
proving the first part of (1.3).

We then claim that
\[
V_1^\alpha \text{ percolates almost surely, when } \alpha < \hat{\alpha}
\] (4.60)
(see the statement of Proposition 4.9 for the definition of \( \hat{\alpha} \)), which readily implies the second part of (1.3).

We now prove (4.60). For any \( M \geq 0 \)
\[
P[V_{\beta}^\alpha \text{ does not percolate}] \leq P \left[ \bigcap_{n \geq M} (\hat{A}_n^\alpha)^c \right].
\] (4.61)

Since \( (\hat{A}_n^\alpha)_{n \geq 0} \) is a sequence of increasing events, to prove (4.60) it suffices to show that for all \( \alpha < \hat{\alpha} \),
\[
\lim_{n \to \infty} P [(\hat{A}_n^\alpha)^c] = 0.
\] (4.62)

Now, we prove (4.62). One first notices that by Lemma 4.8,
\[
P [ (\hat{A}_n^\alpha)^c ] \leq P \left[ \bigcup_{k=n-1}^{\infty} \bigcup_{x \in \mathcal{L}_{k-1} \cap F_Z, |x|_\infty \leq 4L_{k+1}} B_{k,x}^{\alpha} \right].
\] (4.63)

By Lemma 4.7, the fact that \( |\mathcal{L}_{k-1} \cap S^{(2)}(0, 4L_{k+1})| \leq C = C(d) \) and Proposition 4.9, one obtains that for \( \alpha < \hat{\alpha} \),
\[
P [ (\hat{A}_n^\alpha)^c ] \leq P \left[ \bigcup_{k=n-1}^{\infty} \bigcup_{x \in \mathcal{L}_{k-1} \cap F_Z, |x|_\infty \leq 4L_{k+1},} \overline{B}_{k,x}^{\alpha} \right] \leq C (1/4)^{2k-1} \to 0 \text{ as } n \to \infty.
\] (4.64)

This completes the proof of (4.62) and finishes the proof of (1.3).
Remark 4.10  (1) Recall the definition of $\alpha_r^{**}$, $\alpha_r^#$ and $\beta$, respectively, in (1.4), Proposition 4.6 and (4.6). It follows from Proposition 4.6, the definition of events $A_n^\varphi$ [see (4.24)], and scaling property of Brownian interlacements that

$$
(0 < )\alpha_r^* \leq \alpha_r^{**} \leq \alpha_r^# (\beta/r)^{d-2} ( < \infty).
$$

(4.65)

It is a natural question whether the two thresholds $\alpha_r^*$ and $\alpha_r^{**}$ coincide, or, in other words, whether the phase transition for the vacant set is sharp. As the corresponding conjecture in the case of random interlacements still remains open, we speculate that in our case, this question is also not easy to answer. It might be even harder to answer what happens at these critical values, e.g., whether $V_1^{\alpha_r}$ percolates or not.

(2) Note that as a byproduct of the proof of Theorem 1.3, we obtain that when $\alpha < \tilde{\alpha}$, $V_1^{\alpha_r}$ percolates not only in the whole space, but in a plane and in a slab $F_R = \mathbb{R}^2 \times [0, R]^{d-2}$ (where $R > 0$ stands for the “thickness”) as well. With this observation in mind, we define for $r > 0$

$$
\tilde{\alpha}_r = \sup \left\{ \alpha \geq 0 : \mathbb{P}[V_1^{\alpha_r} \text{ percolates in a plane}] = 1 \right\}
$$

(4.66)

and

$$
\tilde{\alpha}_{R,r} = \sup \left\{ \alpha \geq 0 : \mathbb{P}[V_1^{\alpha_r} \text{ percolates in } F_R] = 1 \right\}.
$$

(4.67)

It follows that

$$
(\tilde{\alpha} r^{2-d}) \leq \tilde{\alpha}_r \leq \tilde{\alpha}_{R,r} \leq \alpha_r^*,
$$

(4.68)

By an intuitive analogy to the case of Bernoulli percolation (see e.g., Sections 7.1 and 7.2 in [11]), we think it is very plausible that $\tilde{\alpha} r^{2-d} = \tilde{\alpha}_r$, while $\tilde{\alpha}_r < \alpha_r^*$ but $\lim_{R \to \infty} \tilde{\alpha}_{R,r} = \alpha_r^*$. Techniques developed in [1] may be helpful for answering the first conjecture.

(3) We are also prompted to ask whether the unbounded cluster in $V_1^{\alpha_r}$ is unique in the supercritical regime and wonder if it is possible to adapt the proof of the uniqueness of infinite cluster in the vacant set of random interlacements in [30] to tackle this problem.

Acknowledgements  The author wishes to express his gratitude to Alain-Sol Sznitman for suggesting these problems and for numerous valuable discussions and to thank Artëm Sapozhnikov and Ron Rosenthal for various useful discussions.

References

1. Ahlberg, D., Tassion, V., Teixeira, A.: Sharpness of the phase transition for continuum percolation in $\mathbb{R}^2$. Probab. Theory Relat. Fields 172, 525–581 (2018)
2. Albeverio, S., Zhou, X.Y.: Intersections of random walks and Wiener sausages in four dimensions. Acta Appl. Math. 45(2), 195–237 (1996)
3. Černý, J., Popov, S.: On the internal distance in the interlacement set. Electron. J. Probab. 17(29), 1–25 (2012)

Springer
4. Černý, J., Teixeira, A.: From random walk trajectories to random interlacements. Ens. Mat. 23, 1–78 (2012)
5. Černý, J., Teixeira, A.: Random walks on torus and random interlacements: macroscopic coupling and phase transition. Ann. Appl. Probab. 26(5), 2883–2914 (2016)
6. Cox, D.R.: Renewal Theory. Methuen & Co., London (1962)
7. Daley, D.J., Vere-Jones, D.: An Introduction to the Theory of Point Processes, vol. II, 2nd edn. Springer, Berlin (2008)
8. Drewitz, A., Ráth, B., Sapozhnikov, A.: An Introduction to Random Interlacements. Springer, Berlin (2014)
9. Elias, O., Tykesson, J.: Visibility in the vacant set of the Brownian interlacements and the Brownian excursion process. arXiv:1709.09052
10. Goodman, J., den Hollander, F.: Extremal geometry of a Brownian porous medium. Probab. Theory Relat. Fields 160(1–2), 127–174 (2014)
11. Grimmett, G.: Percolation, 2nd edn. Springer, Berlin (1999)
12. Hara, T., van der Hofstad, R., Slade, G.: Critical two-point functions and the lace expansion for spread-out high-dimensional percolation and related models. Ann. Probab. 31(1), 349–408 (2003)
13. Li, X., Sznitman, A.-S.: Large deviations for occupation time profiles of random interlacements. Probab. Theory Relat. Fields 161(1), 309–350 (2015)
14. Matheron, G.: Random Sets and Integral Geometry. Wiley, New York (1975)
15. Nash, S.W.: An extension of the Borel–Cantelli lemma. Ann. Math. Stat. 25, 165–167 (1954)
16. Paley, R.E.A.C., Zygmund, A.: A note on analytic functions in the unit circle. Proc. Camb. Philos. Soc. 28, 266–272 (1932)
17. Ráth, B.: A short proof of the phase transition for the vacant set of random interlacements. Electron. Commun. Probab. 20(3), 1–11 (2015)
18. Sznitman, A.-S.: Vacant set of random interlacements and percolation. Ann. Math. 171, 2039–2087 (2010)
19. Teixeira, A.: On the uniqueness of the infinite cluster of the vacant set of random interlacements. Ann. Appl. Probab. 19(1), 454–466 (2009)
20. Teixeira, A., Windisch, D.: On the fragmentation of a torus by random walk. Commun. Pure Appl. Math. 64(12), 1599–1646 (2011)