Multiple Parameter Function Approaches to the Equations of Dynamic Convection in a Sea

Xiaoping Xu

Institute of Mathematics, Academy of Mathematics & System Sciences Chinese Academy of Sciences, Beijing 100190, P.R. China

Abstract

One of the most important topics in geophysics is to study convection in a sea. Based on the algebraic characteristics of the equations of dynamic convection in a sea, we introduce various schemes with multiple parameter functions to solve these equations and obtain families of new explicit exact solutions with multiple parameter functions. Moreover, symmetry transformations are used to simplify our arguments.

1 Introduction

Both the atmospheric and oceanic flows are influenced by the rotation of the earth. In fact, the fast rotation and small aspect ratio are two main characteristics of the large scale atmospheric and oceanic flows. The small aspect ratio characteristic leads to the primitive equations, and the fast rotation leads to the quasi-geostrophic equations (cf. [2], [7], [8], [10]). A main objective in climate dynamics and in geophysical fluid dynamics is to understand and predict the periodic, quasi-periodic, aperiodic, and fully turbulent characteristics of the large scale atmospheric and oceanic flows (e.g., cf. [4], [6]).

The general model of atmospheric and oceanic flows is very complicated. Various simplified models had been established and studied. For instance, Boussinesq equations are simpler models in atmospheric sciences (e.g., cf. [9]). Chae [1] proved the global regularity, and Hou and Li [3] obtained the well-posedness of the two-dimensional equations. Hsia, Ma and Wang [4] studied the bifurcation and periodic solutions of the three-dimensional equations.

The following equations in geophysics

\begin{align}
& u_x + v_y + w_z = 0, \quad \rho = p_z, \quad (1.1) \\
& \rho_t + u\rho_x + v\rho_y + w\rho_z = 0, \quad (1.2) \\
& u_t + uu_x + vv_y + wu_z + v = -\frac{1}{\rho}p_x, \quad (1.3)
\end{align}

1\textsuperscript{1}2000 Mathematical Subject Classification. Primary 35Q35, 35C05; Secondary 35L60.

2\textsuperscript{2}Research supported by China NSF 10871193
\[ v_t + w_x + vv_y + vw_z - u = -\frac{1}{\rho} p_y, \]  
\[ \text{(1.4)} \]

are used to describe the dynamic convection in a sea, where \( u, v \) and \( w \) are components of velocity vector of relative motion of fluid in Cartesian coordinates \((x, y, z)\), \(\rho = \rho(x, y, z, t)\) is the density of fluid and \( p \) is the pressure (e.g., cf. Page 203 in [5]). Ovsiannikov determined the Lie point symmetries of the above equations and found two very special solutions (cf. [5]).

In [11], we used the stable range of nonlinear term to solve the equation of nonstationary transonic gas flow. Moreover, we [12] solved the three-dimensional Navier-Stokes equations by asymmetric techniques and moving frames. Based on the algebraic characteristics of the equations (1.1)-(1.4) of dynamic convection in a sea, we introduce various schemes with multiple parameter functions to solve these equations and obtain families of new explicit exact solutions with multiple parameter functions. Moreover, symmetry transformations are used to simplify our arguments. By specifying these parameter functions, one can obtain the solutions of certain practical models.

For convenience, we always assume that all the involved partial derivatives of related functions always exist and we can change orders of taking partial derivatives. We also use prime `' to denote the derivative of any one-variable function. We will use the following symmetry transformations \(T_1-T_4\) due to Ovsiannikov (cf. Page 204 of [5]) of the equation (1.1)-(1.4) to simplify our solutions:

\[
T_1(u) = u(t, x + \alpha, y, z + \alpha''x - \alpha'y) - \alpha', \quad T_1(v) = v(t, x + \alpha, y, z + \alpha''x - \alpha'y),
\]
\[
T_1(w) = w(t, x + \alpha, y, z + \alpha''x - \alpha'y) - \alpha''u + \alpha'v - \alpha'''x + \alpha'''y,
\]
\[
T_2(u) = u(t, x + \alpha, y + \alpha', z + \alpha'x + \alpha''y), \quad T_2(v) = v(t, x + \alpha, y + \alpha', z + \alpha'x + \alpha''y) - \alpha',
\]
\[
T_2(w) = w(t, x + \alpha, y + \alpha', z + \alpha'x + \alpha''y) - \alpha''u - \alpha'''v - \alpha''''x - \alpha''''y,
\]
\[
T_1(p) = p(t, x + \alpha, y + \alpha'x - \alpha'y), \quad T_2(p) = p(t, x + \alpha, y + \alpha'x + \alpha''y),
\]
\[
T_3(u) = u(t, x, y + \alpha), \quad T_3(v) = v(t, x, y + \alpha),
\]
\[
T_3(w) = w(t, x, y + \alpha) - \alpha', \quad T_3(p) = p(t, x, y + \alpha),
\]
\[
T_4(p) = p + \alpha, \quad T_4(F) = F \quad \text{for } F = u, v, w,
\]

where \(\alpha\) is an arbitrary function of \(t\). The above transformations transform one solution of the equations (1.1)-(1.4) into another solution. Applying the above transformations to any solution found in this paper will yield another solution with four extra parameter functions.

In Section 2, we use a new variable of moving line to solve the equations (1.1)-(1.4). An approach of using the product of cylindrical invariant function with \(z\) is introduced in Section 3. In Section 4, we reduce the three-dimensional (spacial) equations (1.1)-(1.4) into a two-dimensional problem and then solve it with three different ansatzes.
2 Moving-Line Approach

Let $\alpha$ and $\beta$ be given functions of $t$. Denote

$$\varpi = \alpha'x + \beta'y + z.$$  

(2.1)

Suppose that $f, g, h$ are functions in $t, x, y, z$ that are linear in $x, y, z$ such that

$$f_x + g_y + h_z = 0.$$  

(2.2)

We assume

$$u = \phi(t, \varpi) + f, \quad v = \psi(t, \varpi) + g,$$  

(2.3)

and

$$w = h - \alpha'\phi(t, \varpi) - \beta'\psi(t, \varpi), \quad p = \zeta(t, \varpi),$$  

(2.4)

where $\phi, \psi, \zeta$ are two-variable functions. Note that the first equation in (1.1) naturally holds and $p = p_z = \zeta\varpi$ by the second equation in (1.1). Moreover, (1.2)-(1.4) become

$$\zeta_{\varpi t} + \zeta\varpi(\alpha''x + \beta''y + \alpha'f + \beta'g + h) = 0,$$  

(2.5)

$$f_t + g + f f_x + g f_y + h f_z + \alpha' + \phi_t + (f_x - \alpha' f_z)\phi + (f_y - \beta' f_z + 1)\psi \phi_{\varpi}$$  

$$+ \phi_{\varpi}(\alpha'' x + \beta'' y + \alpha' f + \beta' g + h) = 0,$$  

(2.6)

$$g_t - f + f g_x + g g_y + h g_z + \beta' + \psi_t + (g_x - \alpha' g_z - 1)\phi + (g_y - \beta' g_z)\psi$$  

$$+ \psi_{\varpi}(\alpha'' x + \beta'' y + \alpha' f + \beta' g + h) = 0.$$  

(2.7)

In order to solve the above system of partial differential equations, we assume

$$\alpha''x + \beta''y + \alpha'f + \beta'g + h = -\gamma'\varpi = -\gamma'(\alpha'x + \beta'y + z)$$  

(2.8)

for some function $\gamma$ of $t$, and

$$f_t + g + f f_x + g f_y + h f_z + \alpha' = 0,$$  

(2.9)

$$g_t - f + f g_x + g g_y + h g_z + \beta' = 0.$$  

(2.10)

Then (2.5)-(2.7) become

$$\zeta_{\varpi t} - \gamma'\varpi\zeta_{\varpi\varpi} = 0,$$  

(2.11)

$$\phi_t + (f_x - \alpha' f_z)\phi + (f_y - \beta' f_z + 1)\psi - \gamma'\varpi\phi_{\varpi} = 0,$$  

(2.12)

$$\psi_t + (g_x - \alpha' g_z - 1)\phi + (g_y - \beta' g_z)\psi - \gamma'\varpi\psi_{\varpi} = 0.$$  

(2.13)

According to (2.8),

$$h = -\alpha'' x - \beta'' y - \alpha' f - \beta' g - \gamma'\varpi.$$  

(2.14)

Substituting the above equation into (2.9) and (2.10), we have:

$$f_t + f(f_x - \alpha' f_z) + g(f_y - \beta' f_z + 1) - f_z(\alpha'' x + \beta'' y + \gamma'\varpi) + \alpha' = 0,$$  

(2.15)
\( g_t + f(g_x - \alpha' g_z - 1) + g(g_y - \beta' g_z) - g_z(\alpha'' x + \beta'' y + \gamma') + \beta' = 0. \) (2.16)

Our linearity assumption implies that
\[
A = \begin{pmatrix} f_x - \alpha' f_z & f_y - \beta' f_z + 1 \\ g_x - \alpha' g_z - 1 & g_y - \beta' g_z \end{pmatrix}
\] (2.17)
is a matrix function of \( t \). In order to solve the system (2.12) and (2.13), and the system (2.15) and (2.16), we need the commutativity of \( A \) with \( dA/dt \). For simplicity, we assume
\[
f_y - \beta' f_z + 1 = g_x - \alpha' g_z - 1 = 0.
\] (2.18)
So
\[
f_y = \beta' f_z - 1, \quad g_x = \alpha' g_z + 1.
\] (2.19)
Moreover, (2.15) and (2.16) become
\[
f_t + f(f_x - \alpha' f_z) - f_z(\alpha'' x + \beta'' y + \gamma') + \alpha' = 0,
\] (2.20)
\[
g_t + g(g_y - \beta' g_z) - g_z(\alpha'' x + \beta'' y + \gamma') + \beta' = 0.
\] (2.21)
Write
\[
f = \alpha_1 x + (\beta' \alpha_2 - 1) y + \alpha_2 z + \alpha_3,
\] (2.22)
\[
g = (\alpha' \beta_2 + 1) x + \beta_1 y + \beta_2 z + \beta_3
\] (2.23)
by our linearity assumption and (2.19), where \( \alpha_i \) and \( \beta_j \) are functions of \( t \).

Now (2.20) is equivalent to the following system of ordinary differential equations:
\[
\alpha'_1 + \alpha_1(\alpha_1 - \alpha' \alpha_2) - \alpha_2(\alpha'' + \gamma' \alpha') = 0,
\] (2.24)
\[
(\beta' \alpha_2)' + (\beta' \alpha_2 - 1)(\alpha_1 - \alpha' \alpha_2) - \alpha_2(\beta'' + \gamma' \beta') = 0,
\] (2.25)
\[
\alpha'_2 + \alpha_2(\alpha_1 - \alpha' \alpha_2 - \gamma') = 0,
\] (2.26)
\[
\alpha'_3 + \alpha_3(\alpha_1 - \alpha' \alpha_2) + \alpha' = 0.
\] (2.27)
Observe that (2.25) - \( \beta' \times (2.26) \) becomes
\[-\alpha_1 + \alpha' \alpha_2 = 0.
\] (2.28)
So (2.26) becomes
\[
\alpha'_2 - \gamma' \alpha_2 = 0 \implies \alpha_2 = b_1 e^\gamma, \quad b_1 \in \mathbb{R}.
\] (2.29)
According to (2.28),
\[
\alpha_1 = b_1 \alpha' e^\gamma.
\] (2.30)
With the data (2.29) and (2.30), (2.24) naturally holds. By (2.27), we take
\[
\alpha_3 = -\alpha.
\] (2.31)
Note that (2.21) is equivalent to the following system of ordinary differential equations:

\[
\begin{align*}
\alpha'\beta_2' + (\alpha'\beta_2 + 1)(\beta_1 - \beta'\beta_2) - \alpha'\beta_2\gamma' &= 0, \\
\beta_1' + \beta_1(\beta_1 - \beta'\beta_2) - \beta_2(\beta'' + \beta'\gamma') &= 0, \\
\beta_2' + \beta_2(\beta_1 - \beta'\beta_2 - \gamma') &= 0, \\
\beta_3' + \beta_3(\beta_1 - \beta'\beta_2) - \beta' &= 0.
\end{align*}
\]  

(2.32)  
(2.33)  
(2.34)  
(2.35)

Similarly, we have:

\[
\beta_1 = b_2\beta'e^\gamma, \quad \beta_2 = b_2e^\gamma, \quad \beta_3 = \beta
\]  

(2.36)

with \(b_2 \in \mathbb{R}\). Moreover, (2.2) gives \(\gamma' = 0\) by (2.14), (2.28) and (2.36). We take \(\gamma = 0\). Therefore, \(\phi = \Re(\varpi)\) and \(\psi = \im(\varpi)\) by (2.12) and (2.13) for some one-variable functions \(\Re\) and \(\im\). Furthermore, we take \(\zeta = \sigma(\varpi)\) by (2.11) for another one-variable function \(\sigma\). In summary, we have:

**Theorem 2.1.** Let \(\alpha, \beta\) be functions of \(t\) and let \(b_1, b_2 \in \mathbb{R}\). Suppose that \(\Re, \im\) and \(\sigma\) are arbitrary one-variable functions. The following is a solution of the equations (1.1)-(1.4) of dynamic convection in a sea:

\[
\begin{align*}
u &= b_1\alpha'x + (b_1\beta' - 1)y + b_1z - \alpha + \Re(\alpha'x + \beta'y + z), \\
v &= (b_2\alpha' + 1)x + b_2\beta'y + b_2z + \beta + \im(\alpha'x + \beta'y + z), \\
w &= -(\alpha'' + b_1(\alpha')^2 + (b_2\alpha' + 1)\beta'x - (\beta'' + \alpha'(b_1\beta' - 1) + b_2(\beta')^2)y - (b_1\alpha' + b_2\beta')z \\
&\quad + \alpha\alpha' - \beta\beta' - \alpha'\Re(\alpha'x + \beta'y + z) - \beta'\im(\alpha'x + \beta'y + z), \\
p &= \sigma(\alpha'x + \beta'y + z), \\
\rho &= \sigma'(\alpha'x + \beta'y + z).
\end{align*}
\]  

(2.37)  
(2.38)  
(2.39)  
(2.40)

We remark that we have tried some other forms of the matrix \(A\) in (2.17) such that \(A\) and \(dA/dt\) commute, but we have failed to get new solutions.

## 3 Approach of Cylindrical Product

Let \(\sigma\) be a fixed one-variable function and set

\[
\varpi = \sigma(x^2 + y^2)z.
\]  

(3.1)

Suppose that \(f\) and \(g\) are functions in \(t, x, z\) that are linear homogeneous in \(x, y\) and

\[
h = \frac{\gamma}{\sigma} - z(f_x + g_y),
\]  

(3.2)

where \(\gamma\) is a function of \(t\). Assume

\[
u = f + y\psi(t, \varpi), \quad v = g - x\psi(t, \varpi), \quad w = h, \quad p = \phi(t, \varpi)
\]  

(3.3)
where $\psi$ and $\phi$ are two-variable functions. Note

$$u_t = f_t + y\psi_t, \quad u_x = f_x + 2xyz\sigma'\psi_\infty,$$

$$u_y = f_y + \psi + 2y^2z\sigma'\psi_\infty, \quad u_z = f_z + y\sigma\psi_\infty,$$

$$v_t = g_t - x\psi_t, \quad v_x = g_x - \psi - 2x^2z\sigma'\psi_\infty,$$

$$v_y = g_y - 2xyz\sigma'\psi_\infty, \quad v_z = g_z - x\sigma\psi_\infty.$$

Hence (1.3) becomes

$$u_t + uu_x + vu_y + wu_z + v = f_t + y\psi_t + (f + y\psi)(f_x + 2xyz\sigma'\psi_\infty) + (g - x\psi)(f_y + 1 + \psi + 2y^2z\sigma'\psi_\infty) + y\sigma h\psi_\infty$$

and (1.4) gives

$$v_t + vw_x + vv_y + vw_z - u = g_t - x\psi_t + (f + y\psi)(g_x - 1 - \psi - 2x^2z\sigma'\psi_\infty) + (g - x\psi)(g_y - 2xyz\sigma'\psi_\infty) - x\sigma h\psi_\infty$$

In order to solve the above system of differential equations, we assume

$$f = \alpha'x - \frac{y}{2}, \quad g = \frac{x}{2} + \alpha'y, \quad \sigma(x^2 + y^2) = \frac{1}{x^2 + y^2}$$

for some function $\alpha$ of $t$. According to (3.2),

$$h = \frac{\gamma}{\sigma} - 2\alpha'z.$$

Now (3.8) becomes

$$(\alpha'' + (\alpha')^2 + 4^{-1} - \psi^2)x + y[\psi_t + 2\alpha'\psi + (\gamma - 4\alpha'\infty)\psi_\infty] = 2x\infty$$

and (3.9) yields

$$(\alpha'' + (\alpha')^2 + 4^{-1} - \psi^2)y - x[\psi_t + 2\alpha'\psi + (\gamma - 4\alpha'\infty)\psi_\infty] = 2y\infty.$$

The above system is equivalent to

$$\alpha'' + (\alpha')^2 + 4^{-1} - \psi^2 = 2\infty,$$

$$\psi_t + 2\alpha'\psi + (\gamma - 4\alpha'\infty)\psi_\infty = 0.$$
By (3.14), we take
\[ \psi = \sqrt{\alpha'' + (\alpha')^2 + 4^{-1} - 2\varpi}, \] (3.16)
due to the skew-symmetry of \((u, x)\) and \((v, y)\). Substituting (3.16) into (3.15), we get
\[ \alpha'' + 2\alpha'\alpha'' + 4\alpha'(\alpha'' + (\alpha')^2 + 4^{-1} - 2\varpi) - 2(\gamma - 4\alpha'\varpi) = 0, \] (3.17)
equivalently,
\[ \gamma = 2(\alpha')^3 + 3\alpha'\alpha'' + \frac{\alpha'''}{2}. \] (3.18)
According to the second equation in (1.1), we have \(\rho = \sigma \phi \varpi\). Note
\[ \rho_t = \sigma \phi \varpi t, \quad \rho_x = 2x\sigma'(\phi \varpi + \varpi \phi \varpi), \] (3.19)
\[ \rho_y = 2y\sigma'(\phi \varpi + \varpi \phi \varpi), \quad \rho_z = \sigma^2 \phi \varpi. \] (3.20)
So (1.2) becomes
\[ \phi_t - 2\alpha'\phi + (\gamma - 4\alpha'\varpi)\phi \varpi = 0. \] (3.21)
Modulo \(T_4\) in (1.12), the above equation is equivalent to:
\[ \tilde{\phi}_t + (\gamma - 4\alpha'\varpi)\tilde{\psi}_\varpi = 0, \] (3.22)
Set
\[ \tilde{\psi} = e^{2\alpha} \psi, \quad \tilde{\phi} = e^{2\alpha} \phi. \] (3.23)
Then (3.15) and (3.22) are equivalent to the equations:
\[ \tilde{\psi}_t + (\gamma - 4\alpha'\varpi)\tilde{\psi}_\varpi = 0, \quad \tilde{\phi}_t + (\gamma - 4\alpha'\varpi)\tilde{\phi}_\varpi = 0, \] (3.24)
respectively. So we have the solution
\[ \tilde{\phi} = \Im(\tilde{\psi}) \implies \phi = e^{-2\alpha} \Im \left( e^{2\alpha} \sqrt{\alpha'' + (\alpha')^2 + 4^{-1} - 2\varpi} \right) \] (3.25)
for some one-variable function \(\Im\). Thus we have:

**Theorem 3.1.** Let \(\alpha\) be any function of \(t\) and let \(\Im\) be arbitrary one-variable function. The following is a solution of the equations (1.1)-(1.4) of dynamic convection in a sea:
\[ u = \alpha' x - \frac{y}{2} + y\sqrt{\alpha'' + (\alpha')^2 + \frac{1}{4} - \frac{2z}{x^2 + y^2}}, \] (3.26)
\[ v = \alpha' y + \frac{x}{2} - x\sqrt{\alpha'' + (\alpha')^2 + \frac{1}{4} - \frac{2z}{x^2 + y^2}}, \] (3.27)
\[ w = \left( 2(\alpha')^3 + 3\alpha'\alpha'' + \frac{\alpha'''}{2} \right) (x^2 + y^2) - 2\alpha' z, \] (3.28)
\[ p = e^{-2\alpha} \Im \left( e^{2\alpha} \sqrt{\alpha'' + (\alpha')^2 + \frac{1}{4} - \frac{2z}{x^2 + y^2}} \right), \] (3.29)
\[ \rho = -\Im' \left( e^{2\alpha} \sqrt{\alpha'' + (\alpha')^2 + \frac{1}{4} - \frac{2z}{x^2 + y^2}} \right) \left( x^2 + y^2 \right) \left( \alpha' + \alpha^2 + \frac{1}{4} - \frac{2z}{x^2 + y^2} \right). \] (3.30)
4 Dimensional Reduction

Suppose that \( u, v, \zeta \) and \( \eta \) are functions in \( t, x, y \). Assume
\[
    w = \zeta - (u_x + v_y)z, \quad p = z + \eta, \quad \rho = 1. \tag{4.1}
\]
Then the equations (1.1)-(1.4) are equivalent to the following two-dimensional problem:
\[
    u_t + uu_x + vv_y + v = -\eta_x, \tag{4.2}
\]
\[
    v_t + uv_x + vv_y - u = -\eta_y. \tag{4.3}
\]
The compatibility \( \eta_{xy} = \eta_{yx} \) gives
\[
    (u_y - v_x)t + u(u_y - v_x)x + v(u_y - v_x)y + (u_x + v_y)(u_y - v_x + 1) = 0. \tag{4.4}
\]
Suppose that \( \vartheta \) is a function in \( t, x, y \) such that \( \vartheta_{xx} + \vartheta_{yy} = 0 \) (4.5) holds. We assume
\[
    u = \vartheta_{xx}, \quad v = \vartheta_{xy}. \tag{4.6}
\]
Then (4.4) naturally holds. Indeed,
\[
    u_t + uu_x + vv_y + v = \left( \vartheta_{xt} + 2^{-1}(\vartheta_{xx}^2 + \vartheta_{xy}^2) + \vartheta_y \right)_x, \tag{4.7}
\]
\[
    v_t + uv_x + vv_y - u = \left( \vartheta_{xt} + 2^{-1}(\vartheta_{xx}^2 + \vartheta_{xy}^2) + \vartheta_y \right)_y. \tag{4.8}
\]
By (4.2) and (4.3), we take
\[
    \eta = -\vartheta_{xt} - \vartheta_y - \frac{1}{2}(\vartheta_{xx}^2 + \vartheta_{xy}^2). \tag{4.8}
\]
Hence we have the following easy result:

**Proposition 4.1.** Let \( \vartheta \) and \( \zeta \) be functions in \( t, x, y \) such that (4.5) holds. The following is a solution of the equations (1.1)-(1.4) of dynamic convection in a sea:
\[
    u = \vartheta_{xx}, \quad v = \vartheta_{xy}, \quad w = \zeta, \tag{4.9}
\]
\[
    \rho = 1, \quad p = z - \vartheta_{xt} - \vartheta_y - \frac{1}{2}(\vartheta_{xx}^2 + \vartheta_{xy}^2). \tag{4.10}
\]

The above approach is the well-known rotation-free approach. We are more interested in the approaches that the rotation may not be zero. Let \( f \) and \( g \) be functions in \( t, x, y \) that are linear in \( x, y \). Denote
\[
    \omega = x^2 + y^2. \tag{4.11}
\]
Consider
\[ u = f + y\phi(t, \omega), \quad v = g - x\phi(t, \omega), \] (4.12)
where \( \phi \) is a two-variable function to be determined. Then
\[ u_x = f_x + 2xy\phi_t, \quad u_y = f_y + \phi + 2y^2\phi_\omega, \] (4.13)
\[ v_x = g_x - \phi - 2x^2\phi_\omega, \quad u_y = g_y - 2xy\phi_\omega. \] (4.14)
Thus
\[ u_x + v_y = f_x + g_y, \quad u_y - v_x = f_y - g_x + 2(\omega\phi)_\omega. \] (4.15)

For simplicity, we assume
\[ f = -\frac{\alpha'}{2\alpha} x - \frac{y}{2}, \quad g = \frac{x}{2} - \frac{\alpha' y}{2\alpha} \] (4.16)
for some functions \( \alpha \) and \( \beta \) of \( t \). Then (4.4) becomes
\[ (\omega\phi)_{\omega t} - \frac{\alpha'}{\alpha} (\omega\phi)_{\omega \omega} - \frac{\alpha'}{\alpha} (\omega\phi)_\omega = 0. \] (4.17)

Hence
\[ \phi = \frac{\gamma + \Im(\alpha\omega)}{\omega} \] (4.18)
for some function \( \gamma \) of \( t \) and one-variable function \( \Im \).

Now (4.12), (4.16) and (4.18) imply
\[ u = -\frac{\alpha' x}{2\alpha} - \frac{y}{2} + \frac{(\gamma + \Im(\alpha\omega))y}{\omega}, \] (4.19)
\[ v = \frac{x}{2} - \frac{\alpha' y}{2\alpha} - \frac{(\gamma + \Im(\alpha\omega))x}{\omega}. \] (4.20)

Moreover, (4.2) and (4.3) yield
\[ \left( \frac{(\alpha')^2 - 2\alpha\alpha''}{4\alpha^2} + \frac{1}{4} \right) x + \frac{\gamma' y}{\omega} - x\phi^2 = -\eta_x, \] (4.21)
\[ \left( \frac{(\alpha')^2 - 2\alpha\alpha''}{4\alpha^2} + \frac{1}{4} \right) y - \frac{\gamma' x}{\omega} - y\phi^2 = -\eta_y. \] (4.22)

Thus
\[ \eta = \frac{1}{2} \int \frac{(\gamma + \Im(\alpha\omega))^2}{\omega^2} d\omega - \frac{1}{2} \left( \frac{(\alpha')^2 - 2\alpha\alpha''}{4\alpha^2} + \frac{1}{4} \right) \omega + \gamma' \arctan \frac{y}{x}. \] (4.23)

**Theorem 4.2.** Let \( \alpha, \gamma \) be any functions of \( t \). Suppose that \( \Im \) is an arbitrary one-variable function and \( \zeta \) is any function in \( t, x, y \). The following is a solution of the equations (1.1)-(1.4) of dynamic convection in a sea:
\[ u = -\frac{\alpha' x}{2\alpha} - \frac{y}{2} + \frac{(\gamma + \Im((x^2 + y^2)\alpha))y}{x^2 + y^2}, \] (4.24)
\[ v = \frac{x}{2} - \frac{\alpha' y}{2\alpha} - \frac{(\gamma + \Im((x^2 + y^2)\alpha))x}{x^2 + y^2}. \] (4.25)
\[ w = \frac{\alpha'}{\alpha} z + \zeta, \quad \rho = 1, \]  
\[ p = z + \frac{1}{2} \int \frac{(\gamma + \Im(\alpha w))^2 d\omega}{\omega^2} - \frac{1}{2} \left( \frac{(\alpha')^2 - 2\alpha\alpha''}{4\alpha^2} + \frac{1}{4} \right) (x^2 + y^2) + \gamma' \arctan \frac{y}{x} \]

with \( \omega = x^2 + y^2 \).

Next we assume
\[ u = \varepsilon(t, x), \quad v = \phi(t, x) + \psi(t, x)y, \]  
where \( \varepsilon, \phi \) and \( \psi \) are functions in \( t, x \) to be determined. Substituting (4.28) into (4.4), we get
\[ \phi_{tx} + \psi_{tx}y + \varepsilon(\phi_{xx} + \psi_{xx}y) + (\phi + \psi y)\psi_x + (\varepsilon + \psi) (\phi_x + \psi xy - 1) = 0, \]
equivalently,
\[ (\phi_t + \varepsilon \phi_x + \phi \psi - \varepsilon)_x - \psi = 0, \]
\[ (\psi_t + \varepsilon \psi_x + \psi_x^2)_x = 0. \]

For simplicity, we take
\[ \psi = -\alpha', \]
a function of \( t \).

Denote
\[ \phi = \hat{\phi} + x. \]

Then (4.30) becomes
\[ (\hat{\phi}_t + \varepsilon \hat{\phi}_x - \alpha' \hat{\phi})_x = 0. \]

To solve the above equation, we assume
\[ \varepsilon = \frac{\beta}{\phi_x} - \frac{\vartheta(t, x)}{\phi_x(t, x)} \]
for some functions \( \beta \) of \( t \), and \( \vartheta \) of \( t \) and \( x \). We have the following solution of (4.34):
\[ \hat{\phi} = e^{\alpha \Im(\vartheta)} \implies \phi = e^{\alpha \Im(\vartheta)} + x \implies v = e^{\alpha \Im(\vartheta)} + x - \alpha' y \]
for another one-variable function \( \Im \). Moreover,
\[ \varepsilon = \frac{\beta e^{-\alpha}}{\vartheta x \Im'(\vartheta)} - \frac{\vartheta_t}{\vartheta_x}. \]

Note
\[ u_t + uu_x + uv_y + v = \left( \frac{\beta e^{-\alpha}}{\vartheta x \Im'(\vartheta)} - \frac{\vartheta_t}{\vartheta_x} \right) - \left( \frac{\beta e^{-\alpha}}{\vartheta x \Im'(\vartheta)} - \frac{\vartheta_t}{\vartheta_x} \right)_x + e^{\alpha \Im(\vartheta)} + x, \]
\[ v_t + uv_x + vv_y - u = ((\alpha')^2 - \alpha'')y + \beta - \alpha' x. \]
By (4.2) and (4.3),

\[
\eta = \int \left( \frac{\beta e^{-\alpha} \vartheta \vartheta' + \vartheta_t \vartheta''(\vartheta)}{(\vartheta_x \vartheta'(\vartheta))^2} + \frac{\vartheta tt \vartheta_x - \vartheta t \vartheta_{xt}}{\vartheta_x^2} - (\beta e^{-\alpha})' \frac{\vartheta}{\vartheta_x \vartheta'(\vartheta)} - e^\alpha \vartheta \vartheta' \right) dx \\
+ \alpha' xy - \beta y + \frac{(\alpha'' - (\alpha')^2)y^2 - x^2}{2} - \frac{1}{2} \left( \frac{\beta e^{-\alpha}}{\vartheta_x \vartheta'(\vartheta)} - \frac{\vartheta_t}{\vartheta_x} \right)^2.
\] 

(4.40)

**Theorem 4.3.** Let \(\alpha, \beta\) be functions of \(t\) and let \(\vartheta\) be a one-variable function. Suppose that \(\vartheta\) and \(\zeta\) are functions in \(t, x, y\). The following is a solution of the equations (1.1)-(1.4) of dynamic convection in a sea:

\[
u = \frac{\beta e^{-\alpha}}{\vartheta_x \vartheta'(\vartheta)} - \frac{\vartheta_t}{\vartheta_x}, \quad v = e^\alpha \vartheta(t) + x - \alpha' y,
\]

(4.41)

\[
w = \left( \alpha' + \frac{\beta e^{-\alpha} (\vartheta_{xx} \vartheta'(\vartheta) + \vartheta_{x} \vartheta''(\vartheta))}{(\vartheta_x \vartheta'(\vartheta))^2} + \frac{\vartheta tt \vartheta_x - \vartheta t \vartheta_{xx}}{\vartheta_x^2} \right) z + \zeta, \quad \rho = 1,
\]

(4.42)

\[
p = z + \int \left( \frac{\beta e^{-\alpha} (\vartheta_{tt} \vartheta'(\vartheta) + \vartheta_t \vartheta_{x} \vartheta''(\vartheta))}{(\vartheta_x \vartheta'(\vartheta))^2} + \frac{\vartheta tt \vartheta_x - \vartheta t \vartheta_{xt}}{\vartheta_x^2} - (\beta e^{-\alpha})' \frac{\vartheta}{\vartheta_x \vartheta'(\vartheta)} - e^\alpha \vartheta \vartheta' \right) dx \\
+ \alpha' xy - \beta y + \frac{(\alpha'' - (\alpha')^2)y^2 - x^2}{2} - \frac{1}{2} \left( \frac{\beta e^{-\alpha}}{\vartheta_x \vartheta'(\vartheta)} - \frac{\vartheta_t}{\vartheta_x} \right)^2.
\]

(4.43)

Finally, we suppose that \(\alpha, \beta\) are functions of \(t\) and \(f, g\) are functions of \(t, x, y\) that are linear homogeneous in \(x\) and \(y\). Denote \(\varpi = \alpha x + \beta y\). Assume

\[
u = f + \beta \phi(t, \varpi), \quad v = g - \alpha \phi(t, \varpi).
\]

(4.44)

Then

\[
u_y - v_x = f_y - g_x + (\alpha^2 + \beta^2) \phi_{\varpi}, \quad u_x + v_y = f_x + g_y.
\]

(4.45)

Now (4.4) becomes

\[
f_{yt} - g_{xt} + (\alpha^2 + \beta^2)' \phi_{\varpi} + (\alpha^2 + \beta^2)(\phi_{\varpi t} + (\alpha' x + \beta' y + \alpha f + \beta g) \phi_{\varpi \varpi}) \\
+(f_x + g_y)(f_y - g_x + 1 + (\alpha^2 + \beta^2) \phi_{\varpi}) = 0.
\]

(4.46)

In order to solve the above equation, we assume

\[
g_x = \varphi, \quad f_y = \varphi - 1,
\]

\[
(\alpha' x + \beta' y + \alpha f + \beta g) = 0
\]

(4.47)

(4.48)

for some function \(\varphi\) of \(t\). The equation (4.48) is equivalent to:

\[
\alpha' + \alpha f_x + \varphi \beta = 0 \implies f_x = -\frac{\alpha' + \varphi \beta}{\alpha},
\]

\[
\beta' + \beta g_y + \alpha(\varphi - 1) = 0 \implies g_y = -\frac{\beta' + \alpha(\varphi - 1)}{\beta}.
\]

(4.49)

(4.50)
Thus we have the following solution:

\[
\phi = \frac{\alpha + \beta}{\alpha^2 + \beta^2} e^{\int (\alpha \beta^{-1} (\varphi - 1) + \alpha^{-1} \beta \varphi) dt} \Im'(\varphi),
\]  
(4.52)

where \( \Im \) is an arbitrary one-variable function. Note

\[
u + uu_x + vu_y + v = \frac{\alpha(\alpha + \beta)}{\alpha^2 + \beta^2} \left( \frac{\alpha^2 - \alpha'^2}{\alpha^2 + b^2} - 1 \right) e^{\int (\alpha \beta^{-1} (\varphi - 1) + \alpha^{-1} \beta \varphi) dt} \Im'(\varphi) + \frac{(\varphi - 1)^2}{\beta^2} \alpha + (\varphi - 1) \alpha' \beta' - 2(\beta')^2 - (\varphi - 1)^2 \alpha \beta, \]
(4.53)

By (4.2) and (4.3),

\[
\eta = \frac{y^2}{2} \left( \frac{\beta(\varphi - 1) \alpha' + \beta \beta' - 2(\beta')^2 - ((\varphi - 1) \alpha)^2 - 3 \alpha \beta'(\varphi - 1)}{\beta^2} - (\varphi - 1)^2 \right)
- \frac{x^2}{2} \left( \frac{2(\alpha')^2 + (\varphi \beta)^2 + 3 \alpha' \beta \varphi - \alpha(\varphi \beta') - \alpha \alpha''}{\alpha^2} + \varphi^2 \right) + \frac{(\varphi - 1) (\alpha' + \varphi \beta)}{\alpha} - \varphi' + \frac{\varphi(\beta' + \alpha(\varphi - 1))}{\beta} xy + \frac{\alpha + \beta}{\alpha^2 + \beta^2} \left( 1 - \frac{\alpha \beta' - \alpha' \beta}{\alpha^2 + b^2} \right) e^{\int (\alpha \beta^{-1} (\varphi - 1) + \alpha^{-1} \beta \varphi) dt} \Im(\varphi). \]
(4.54)

**Theorem 4.4.** Let \( \alpha, \beta, \varphi \) be functions of \( t \) and let \( \Im \) be a one-variable function. Suppose that \( \zeta \) is functions in \( t, x, y \). The following is a solution of the equations (1.1)-(1.4) of dynamic convection in a sea:

\[
u = (\varphi - 1) y - \frac{(\alpha' + \varphi \beta) x}{\alpha} + \frac{(\alpha' + \varphi \beta) x}{\alpha^2 + \beta^2} e^{\int (\alpha \beta^{-1} (\varphi - 1) + \alpha^{-1} \beta \varphi) dt} \Im'(\varphi x + \beta y), \]
(4.56)

\[
u = \varphi x - \frac{(\beta' + (\varphi - 1) \alpha) y}{\beta} - \frac{(\alpha + \beta)}{\alpha^2 + \beta^2} e^{\int (\alpha \beta^{-1} (\varphi - 1) + \alpha^{-1} \beta \varphi) dt} \Im'(\varphi x + \beta y), \]
(4.57)

\[
w = \left( \frac{\alpha' + \varphi \beta}{\alpha} + \frac{(\beta' + (\varphi - 1) \alpha)}{\beta} \right) z + \zeta, \quad \rho = 1, \]
(4.58)

\[
p = z + \frac{y^2}{2} \left( \frac{\beta(\varphi - 1) \alpha' + \beta \beta' - 2(\beta')^2 - ((\varphi - 1) \alpha)^2 - 3 \alpha \beta'(\varphi - 1)}{\beta^2} - (\varphi - 1)^2 \right)
- \frac{x^2}{2} \left( \frac{2(\alpha')^2 + (\varphi \beta)^2 + 3 \alpha' \beta \varphi - \alpha(\varphi \beta') - \alpha \alpha''}{\alpha^2} + \varphi^2 \right) + xy \frac{(\varphi - 1) (\alpha' + \varphi \beta)}{\alpha} - \varphi' + \frac{\varphi(\beta' + \alpha(\varphi - 1))}{\beta} xy + \frac{\alpha + \beta}{\alpha^2 + \beta^2} \left( 1 - \frac{\alpha \beta' - \alpha' \beta}{\alpha^2 + b^2} \right) e^{\int (\alpha \beta^{-1} (\varphi - 1) + \alpha^{-1} \beta \varphi) dt} \Im(\varphi x + \beta y). \]
(4.59)
References

[1] D. Chae, Global regularity for the 2D Boussinesq equations with partial viscosity, *Adv. Math.* 203 (2006), 497-513.

[2] M. Gill and S. Childress, *Topics in Geophysical Fluid Dynamics, Atmospheric Dynamics, Dynamo Theory, and Climate Dynamics*, Springer-verlag, New York, 1987.

[3] T. Hou and C. Li, Global well-posedness of the viscous Boussinesq equations, *Discrete Contin. Dyn. Syst.* 12 (2005), 1-12.

[4] C. Hsia, T. Ma and S. Wang, Stratified rotating Boussinesq equations in geophysical fluid dynamics: dynamic bifurcation and periodic solutions, *J. Math. Phys.* 48 (2007), no. 6, 06560.

[5] N. H. Ibragimov, *Lie Group Analysis of Differential Equations*, Volume 2, CRC Handbook, CRC Press, 1995.

[6] E. N. Lorenz, Deterministic nonperiodic flow, *J. Atmos. Sci.* 20 (1963), 130-141.

[7] J. Lions, R. Teman and S. Wang, New formulations of the primitive equations of the atmosphere and applications, *Nonlinearity* 5 (1992), 237-288.

[8] J. Lions, R. Teman and S. Wang, On the equations of large-scale ocean, *Nonlinearity* 5 (1992), 1007-1053.

[9] A. Majda, *Introduction to PDEs and Waves for the Atmosphere and Ocean*, Courant Lecture Note in Mathematics, Vol. 9, AMS and CIMS, 2003.

[10] J. Pedlosky, *Geophysical Fluid Dynamics*, 2rd Edition, Springer-verlag, New York, 1987.

[11] X. Xu, Stable-Range approach to the equation of nonstationary transonic gas flows, *Quart. Appl. Math.* 65 (2007), 529-547.

[12] X. Xu, Asymmetric and moving-frame approaches to Navier-Stokes equations, *Quart. Appl. Math.*, in press, [arXiv:0706.1861](http://arxiv.org/abs/0706.1861)