Field theory of the Eulerian perfect fluid

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Abstract
The Eulerian perfect-fluid theory is reformulated from its action principle in a pure field-theoretic manner. Conservation of the convective current is no longer imposed by Lin’s constraints, but rather adopted as the central idea of the theory. Our formulation, for the first time, successfully reduces redundant degrees of freedom promoting one half of the Clebsch variables to true dynamical fields. Interactions on these fields allow for the exchange of the convective current of quantities such as mass and charge, which are uniformly understood as the breaking of the underlying symmetry of the force-free fluid. The Clebsch fields play the essential role of exchanging angular momentum with the force field producing vorticity.

Keywords: perfect fluid, action principle, canonical formalism, gauge theory

1. Introduction
The local nature of the laws governing fluid motion demands a field theoretic formulation. While the Lagrangian-coordinate framework focuses on fluid particle dynamics, the Eulerian approach describes the spacetime behavior of fluid properties, representing a prototypical example of a classical field theory. Yet, in spite of its long history, fluid mechanics still lacks a complete action principle acceptable as a field theory. Incompleteness arises in its description of rotational flow; the action in the Eulerian description requires the Clebsch parametrization [1–8] of the fluid velocity $v$, represented as a linear combination of scalar gradients

$$v = -\nabla \phi - \frac{1}{\rho} \alpha \nabla \psi - \frac{1}{\rho} \beta \nabla \alpha,$$  

where $\phi$ is the velocity potential, $s$ is the entropy density, $\psi$ is the thermasy [4] and $\rho$ is the mass density. The remaining $\alpha$ and $\beta$ are the Clebsch potentials which are essential for...
rotational flow. This parametrization closely relates to the Lin constraints [6] expressed by the last three terms in the Lagrangian density:

\[ \mathcal{L} = \mathcal{L}_0 + \rho \mathcal{D}\phi + s \mathcal{D}\psi + \beta \mathcal{D}\alpha, \]

(2)

where \( \mathcal{D} \equiv \partial_t + \mathbf{v} \cdot \nabla \) is the convective derivative. The free part \( \mathcal{L}_0 \equiv \frac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} - U(\rho, s) \) (\( U \); internal energy) is constrained by conservation of \( \rho \), \( s \), and \( \beta \) [3, 6–11]. In other words, the conventional action takes \( \mathcal{L}_0 \) as its core regarding the fluid as a particle assembly, while the convection of \( \rho \), \( s \), and \( \beta \) is added as subsidiary constraints. In spite of its field-theoretic setup in coordinates, the conventional approach still relies on constructions built upon the particle substratum, and thus the Lin constraint is unavoidable. Here one may recall the Lagrangian-coordinate framework, where convection laws are trivially imposed using fluid-particle trajectories [12]; i.e. the Eulerian framework should account for these convection laws in addition to the equation of motion, which makes the problem more challenging than the Lagrangian-coordinate counterpart [7, 8].

Furthermore, the parametrization introduces several aspects of the potential variables themselves that are yet to be understood. For instance, (I) the Clebsch parametrization includes extra degrees of freedom of the Lagrange multipliers (\( \phi \), \( \psi \), and \( \alpha \)) of added constraints, which Schutz described as ‘too many’ and pointed out the necessity of an action principle with a minimum number of variables [8]. (II) The physical interpretation of the Clebsch potentials themselves is still controversial; some identify them as the Lagrangian-coordinate variables [6, 7, 11], and some relate them with the Chern–Simons theory [9, 10]. Yet, no conclusive agreement on the nature of the potentials has been reached. In any event, any candidate formulation of the action principle in fluid mechanics should naturally account for the points described above. This implies perhaps that the very basic set of ideas defining what we understand as fluid should be reconsidered.

Indeed, in revisiting the fundamental concepts of its field theoretical formulation, fluid dynamics has benefited from modern ideas developed in gauge theories. Within the Lagrangian coordinate approach, [13] addresses the infrared dynamics of non-dissipative fluid recast in the language of effective field theory from symmetry considerations, allowing to explore viscous correction by means of a derivative expansion or anomalous hydrodynamics [14] among others. In a rather mixed framework, in [9, 15–17] the Clebsch parametrization is realized by the introduction of a \( SU(n) \) group element \( g \) as potential variables. These novel approaches allowed the study of phenomena such as a fluid with non-Abelian charges or a spin–orbit interaction [18], we refer the reader to [19] for a recent review. In particular, a thorough description of non-Abelian fluids can have far-reaching consequences in our understanding at the early stages of relativistic heavy ion collisions. An inclusive treatment dealing with interactions becomes crucial, and although appealing, their focus on single particle dynamics brings about critical caveats; it still relies on a classical-particle substratum and an incomplete canonical formalism based on artificial symplectic structure. The fact that the dynamics governing fluid motion can be obtained from increasingly general principles is a reflection of the deeper and universal character of fluid theory.

In this paper, we go a step further and claim that a continuum’s convection property is the core concept behind the field-theoretic formulation of fluid dynamics; we depart from the usual particle-based formulation and choose the convection effect as the guiding principle of fluid field theory, now for the first time, free of the Lin constraints. Reduction of such constraints naturally removes the redundant degrees of freedom from action, successfully minimizing the number of potentials. The physical picture provided by our formulation promotes the Clebsch

\[ \text{Different notations are employed for the Clebsch potentials so that} (\alpha, \beta) \text{ form canonical–conjugate pairs.} \]
variables as dynamical fields instead of a mere clever parametrization for the vector velocity. We discuss interactions and in the context of gauge symmetry apply to the case of the non-Abelian charged fluid. Our findings reveal that not only may the assumptions discussed above be avoided, but also a minimal number of fields are enough to capture dynamics of the non-Abelian (Abelian) charged fluid. We will show that, in such field-theoretic framework, the Clebsch potentials play an essential role in coupling with the force field, which allows the exchange of angular momentum with the external system.

2. Formulation

2.1. Action of convection

Unlike the other physical fields, a fluid (or, generally speaking, continuum) is based on the velocity field by which various physical properties are transported; mass, electric charge, and other thermodynamic quantities are convected by this vector field with their current in the form of \( \rho_c \mathbf{v} \), where \( \rho_c \) is the density of some convective quantity. In general, convection is expressed in the form

\[
\frac{\partial \rho_c}{\partial t} + \nabla \cdot (\rho_c \mathbf{v}) = \text{source term},
\]

whose conservation holds in the absence of the source. In this section we search for the most general Lagrangian \( \mathcal{L}_c \) yielding equation (3) as the Euler–Lagrange equation from its internal symmetry. This means that \( \rho_c \) (a scalar) is obtained as a Noether charge, so \( \mathcal{L}_c \) should have a symmetry under a 1-dimensional transformation group; the associated symmetry is the shift of a scalar field, i.e. \( \phi_c \mapsto \phi_c + \text{const.} \) (cyclic variable), and thus, we may at least write \( \mathcal{L}_c(\partial_t \phi_c, \partial_\xi \phi_c) \). Under a general-coordinate transformation in space \( \{ \mathbf{x} \} \mapsto \{ \tilde{\mathbf{x}} \} \), \( \partial_t \phi \mapsto \partial_t \tilde{\phi} + x^i \partial_i \tilde{\phi} \), so \( \partial_t \phi \) acquires 3D diffeomorphism invariance if accompanied by \( + \mathbf{v} \cdot \partial \tilde{\phi} \), where a vector \( \mathbf{v} \) transforms as \( \mathbf{v}^i = x^j \partial_j \mathbf{v}^i \), which allows us to identify \( \mathbf{v} \) as the velocity field [20]. Using the invariant \( \mathcal{D}_c \equiv (\partial_t + \mathbf{v} \cdot \partial_\xi) \phi_c \), we naturally obtain a Lagrangian density \( \mathcal{L}_c(\mathcal{D}_c) \). Indeed, in this way

\[
\frac{\delta \mathcal{S}_c}{\delta \phi_c} = -\partial_t \frac{\partial \mathcal{L}_c}{\partial (\mathcal{D}_c)} - \partial_i \left( \frac{\partial \mathcal{L}_c}{\partial (\mathcal{D}_c)} \mathbf{v}^i \right) = 0,
\]

\[
\frac{\delta \mathcal{S}_c}{\delta \mathbf{v}} = \frac{\partial \mathcal{L}_c}{\partial (\mathcal{D}_c)} \phi_c; i = 0,
\]

where \( \partial \mathcal{L}_c/\partial (\mathcal{D}_c) \) serves as the convective quantity \( \rho_c \) in comparison with equation (3). The relation with the Lin’s constraint in equation (2) will be clarified in the Hamiltonian formalism. The canonical conjugate of \( \phi_c \) is \( \pi_c = \partial \mathcal{L}_c/\partial (\partial_\xi \phi_c) = \partial \mathcal{L}_c/\partial (\mathcal{D}_c)(\equiv \mathcal{F}_c(\mathcal{D}_c)) = \rho_c \), where we shall choose the functional form of \( \mathcal{L}_c \) so that the inverse function \( \mathcal{F}_c(\equiv \mathcal{F}_c^{-1}) ; \rho_c \mapsto \mathcal{D}_c \) exists, namely, \( \mathcal{D}_c = \mathcal{F}_c(\rho_c) \). Then the Hamiltonian density reads:

\[
\mathcal{H}_c = \rho_c \partial_t \phi_c - \mathcal{L}_c \circ \mathcal{F}_c(\rho_c) = -\rho_c \mathbf{v}^i \phi_j + \int^{\rho_c} F_c(\xi) d\xi,
\]

then

\[
\mathcal{S}_c = \int dt \int d^3x \left( \rho_c \mathcal{D}_c \phi_c - \int^{\rho_c} F_c(\xi) d\xi \right),
\]

which adds a non-trivial modification to the simple Lin constraint \( S_{\text{Lin}} = \int dt \int d^3x \rho_c \mathcal{D}_c \phi_c \) (see appendix A for its derivation). Here we soon recognize that the potential pair \((\phi_c, \rho_c)\)
in the Lin constraint is reproduced as the canonical–conjugate pair. One should not, however, mistake $S_c[\phi_c, v]$ as the phase-space counterpart of $S_{\text{Lin}}[\phi_c, \rho_c v]$; the Lin constraint alone cannot be a complete canonical action of $(\phi_c, \rho_c)$, precisely due to the presence of $\int_\mathbb{R} F_c(\xi) d\xi$ in equation (6). Although both $S_c$ and $S_{\text{Lin}}$ result in the same convective equation $(\partial_t \rho_c + (\rho_c v)^j)_j = 0)$, the equation for $\phi_c$ changes due to this modification; namely $S_c$ yields $\mathcal{D} \phi_c = F_c(\rho_c)$ while $S_{\text{Lin}}$ yields $\mathcal{D} \phi_c = 0$. Thus, the proposed $S_c[\phi_c, v]$ successfully reduces one degree of freedom from $S_{\text{Lin}}[\rho_c, \phi_c, v]$ by modifying a non-observed $\phi_c$ equation. Therefore, the phase space $\{\phi_c, \rho_c\}$ of conventional Lin constraint based formulations cannot be reduced to $\{\phi_c\}$.

Following the same strategy, the fluid model of equation (2) written by 6 scalars will be reformulated by only 3 potentials (see section 2.3). If one does not consider the entropy, the minimal model of rotational-barotropic fluid is written with only 2 potentials (see section 2.2). Although some pioneers in [7–9] have found that the potentials in the Clebsch parametrization form canonical pairs, it was not not possible to reduce the phase space $\{\phi, \psi, \alpha; \rho, s, \beta\}$ down to the configuration space $\{\phi, \psi, \alpha\}$. In order to obtain the real action in the configuration space, we shall abandon the Lin constraint of the standard formalisms [3, 6–11, 15, 16].

2.2. Kernel formalism

Following the logic behind the previously obtained action of convection $S_c$, we establish the element and formalism that will lead us to the most general fluid Lagrangian density. We start with $\phi$ and $v$ as dynamical variables. As seen in equation (4b), Lagrangian density $\mathcal{L}_F(\mathcal{D} \phi)$ alone does not contain the dynamics of $v$. Thus, a dependence on the kinetic term $V = \frac{1}{2} v^a v_a$ is imposed, $\mathcal{L}_F(\mathcal{D} \phi, V)$. Equation (4b) changes accordingly

$$\frac{\delta S}{\delta v^j} = \frac{\partial \mathcal{L}_F}{\partial (\mathcal{D} \phi)} \phi_j + \frac{\partial \mathcal{L}_F}{\partial V} v_j = 0,$$

(7)

where $\phi$ serves as a potential function. This becomes exactly the potential relation $v = -\nabla \phi$ when the Lagrangian density shares the same dependence in both $V$ and $\mathcal{D} \phi$, i.e. $\partial \mathcal{L}_F/\partial V = \partial \mathcal{L}_F/\partial (\mathcal{D} \phi)$. One could relax this condition so that they are proportional to each other, in this case by rescaling $\phi$ in the equations above we keep the potential relation intact. Alternatively, we introduce a new quantity $\mathcal{D} \phi$ as

$$\mathcal{D} \phi = \mathcal{D} \phi + \frac{1}{2} M v^j v_j,$$

(8)

and rewrite the Lagrangian density as $\mathcal{L}_F(\mathcal{D} \phi)$, with $M$ a constant. $\mathcal{D} \phi$, to be referred as the kernel hereafter, plays a central role in the later discussions. The corresponding Euler–Lagrange equations read

$$\frac{\delta S}{\delta \phi} = -\partial_t \frac{\partial \mathcal{L}_F}{\partial (\mathcal{D} \phi)} - \left( \frac{\partial \mathcal{L}_F}{\partial (\mathcal{D} \phi)} v^j \right)_j = 0,$$

(9a)

$$\frac{\delta S}{\delta v^j} = \frac{\partial \mathcal{L}_F}{\partial (\mathcal{D} \phi)} (\phi_j + M v_j) = 0,$$

(9b)

where convection of $\partial \mathcal{L}_F/\partial \mathcal{D} \phi$ and the potential relation are obtained. Note that Lagrangian density $\mathcal{L}_F(\mathcal{D} \phi)$ with the kernel (8) provides a minimal model for the perfect fluid. The spatial gradient of $\mathcal{L}_F(\mathcal{D} \phi)$ yields
\[
\partial_i \mathcal{L}_F(\tilde{\phi}) = \frac{\partial \mathcal{L}_F}{\partial (\tilde{\phi})}(\phi_{ji} + v^i_j \phi_j + v^i_j \phi_{ji} + \mathcal{M}v^i_j v^l_j) \\
= -M \frac{\partial \mathcal{L}_F}{\partial (\tilde{\phi})}(v_{ij} + v^i_j v^l_j) \\
\Leftrightarrow M \frac{\partial \mathcal{L}_F}{\partial (\tilde{\phi})}(v_{ij} + v^i_j v^l_j) = -\mathcal{L}_F, \tag{10}
\]

where we used equation (9). Equation (10) is the Euler equation where \( M \partial \mathcal{L}_F / \partial (\tilde{\phi}) \) and \( \mathcal{L}_F \) behave as the mass density and the pressure respectively. The pressure-mass relation is better appreciated once rewritten in terms of the canonical momentum of \( \phi \), \( \sigma \) defined as

\[
\sigma = \frac{\partial \mathcal{L}_F}{\partial \phi_j} = \frac{\partial \mathcal{L}_F}{\partial (\tilde{\phi})}(\equiv f(\tilde{\phi})).
\]

Then, the equation of state between pressure \( \mathcal{L}_F \) and mass density \( \rho(\equiv M \sigma) \) may be denoted as \( \mathcal{L}_F = p(\rho) \), where \( p(\rho) \) is a composite function \( p(\rho) = (\mathcal{L}_F \circ F)(\rho / M) \), \( F \colon \sigma \mapsto \tilde{\phi} \) is the inverse of \( f \). Thus \( \mathcal{L}_F(\tilde{\phi}) \) with the kernel (8) describes an irrotational barotropic fluid whose pressure is determined only by mass density \( M \sigma \).

In describing rotational flow, a reasonable guess would be to impose the Clebsch potential \( \alpha \) into the Lagrangian as \( \mathcal{L}_F(\tilde{\phi}, \tilde{\alpha}) \), then

\[
\mathbf{v} = -\mathbf{\nabla} \phi / M - \beta \mathbf{\nabla} \alpha / M \sigma \quad (\beta \equiv \partial \mathcal{L}_F / \partial \mathcal{D} \alpha).
\]

However, such approach fails as the resulting pressure \( \mathcal{L}_F \) would depend on both \( \sigma \) and \( \beta \), which contradicts the nature of the Clebsch potentials. In other words, \( \tilde{\phi} \) and \( \alpha \) are not independent variables in the Lagrangian density which can now be rewritten as a composite function \( \mathcal{L}_F \circ A(\tilde{\phi}, \tilde{\alpha}) \), where \( A(\tilde{\phi}, \tilde{\alpha}) \) is some real function. The canonical momentum \( \sigma \) now reads,

\[
\sigma = \frac{\partial \mathcal{L}_F}{\partial A}(A) \frac{\partial A}{\partial \tilde{\phi}}(\tilde{\phi}, \tilde{\alpha}). \tag{11}
\]

Only when \( \partial A / \partial \tilde{\phi} \) becomes a constant, a function \( F \colon \sigma \mapsto A \) exists so that pressure \( \mathcal{L}_F \) is determined by a function \( (\mathcal{L}_F \circ A \circ F)(\sigma) \) free from \( \beta \). This requires \( A \propto \tilde{\phi} + \mathcal{C}(\tilde{\alpha}) \) with a real function \( \mathcal{C}(\tilde{\alpha}) \). The demanded Lagrangian density is most simply written as \( \mathcal{L}_F(\tilde{\phi}) \) with the modified kernel

\[
\tilde{\phi} = \mathcal{D} \phi + \frac{1}{2} M v^i_j v^j_i + \mathcal{C}(\mathcal{D} \alpha), \tag{12}
\]

which gives the minimal model of rotational barotropic fluid, this time described only by 2 potentials. The equation of state is again given by \( \mathcal{L}_F = p(\rho) = (\mathcal{L}_F \circ F)((\rho / M) \mathcal{C}(\tilde{\alpha})) \) (see appendix B.1 for concrete examples of \( p(\rho) \)).

In our formulation, the kernel of equation (12) is the most fundamental ingredient in the description of a fluid, since any extension \( \mathcal{L}_F(\tilde{\phi}, \tilde{\psi}, \ldots) \) containing arbitrary numbers of kernels will always yield the same fluid equations. This function constitutes the most general Lagrangian density for the perfect fluid, which may be rewritten in a more compact form \( \mathcal{L}_F(\tilde{\phi}) \) by taking the kernel as a multiple-component object:

\[
\tilde{\phi} = \mathcal{D} \phi + \frac{1}{2} M v^i_j v^j_i + \mathcal{C}(\mathcal{D} \alpha). \tag{13}
\]
Here $\phi$ and $\alpha$ are $m$- and $n$-component real scalars respectively, $C$ is a $m$-component real function of $D\alpha$, $M$ is $m$-component constant ($m \neq n$ in general):

\[
\phi = (\phi_1, \phi_2, \cdots, \phi_m), \quad \alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n),
\]

\[
C = (C_1, C_2, \cdots, C_m), \quad M = (M_1, M_2, \cdots, M_m),
\]

where $T$ attached on the right-top side means the transposition of matrix components. Hereafter, vectors such as $\phi$ ($\phi^T$) are regarded as column (row) vectors, while the derivative operations such as $\partial/\partial \phi$ ($\partial/\partial \phi^T$) as rows (columns). We have two canonical conjugates $\sigma$ and $\beta$ given by

\[
\sigma^T = \frac{\partial L_F}{\partial \phi^T} = f^*(\tilde{D}\phi),
\]

\[
\beta^T = \frac{\partial L_F}{\partial \alpha^T} = \frac{\partial L_F}{\partial (\tilde{D}\phi)} \frac{\partial C}{\partial (\tilde{D}\alpha)} = \sigma^T h(\tilde{D}\alpha),
\]

where $h(\equiv \partial C/\partial (\tilde{D}\alpha))$ is $m \times n$ matrix. The Euler–Lagrange equations are given by

\[
\frac{\delta S}{\delta \phi^j} = -\sigma_j - (\sigma^v)^j = 0, \quad (14a)
\]

\[
\frac{\delta S}{\delta \alpha^j} = -\beta_j - (\beta^v)^j = 0, \quad (14b)
\]

\[
\frac{\delta S}{\delta v^j} = \sigma_j \phi_j + \beta_j \alpha_j + M^T \sigma v_j = 0, \quad (14c)
\]

where equation (14c) is the generalization of equation (1). Taking the spatial derivative of $L_F(\tilde{D}\phi)$ and combined with equation (14c) yield

\[
M^T \sigma \left( v_j + v_j v_j \right) = -L_{F,j}. \quad (15)
\]

Now $M^T \sigma$ behaves as the mass density. Indeed, (14a) multiplied by $M$ yields the conservation law as follows:

\[
\delta_M (M^T \sigma) = 0. \quad (16)
\]

As evident from the Euler–Lagrange equations, the Lagrangian density $L_F(\tilde{D}\phi)$ has as its foundation the convection property, and not momentum conservation. As a consequence, the Euler equation (15) and mass conservation (16) are automatically obeyed; in this sense they are universal features of the fluid rather than conditions to be imposed. In other words, in deriving equations (15) and (16) alone, the known thermodynamic variables and their relations are not necessary. Then, the perfect fluid is not only an idealized model of known materials, but rather a universal feature lying behind the convection. Needless to say, both hold for arbitrary values of $m$ and $n$, which can be thought as a discrete symmetry of our Lagrangian.

2.3. Hamiltonian formalism

Let us turn to the Hamiltonian formalism, here we can already see some clear differences between the conventional and present theories. The canonical Hamiltonian density reads,
\[ H_F = \sigma^j \phi_j + \beta^i \alpha_j - L_F \]
\[ = U(\sigma) + U_C(\beta, \sigma) - \frac{1}{2} \mathbf{M}^T \sigma \mathbf{v}^\sigma - \nabla^i (\sigma^j \phi_j + \beta^i \alpha_j), \tag{17} \]
where \( U(\sigma) = \int \mathbf{F}^I(\sigma^j) d\sigma^j \) (\( F; \sigma \mapsto \tilde{D} \phi \) is the inverse of \( f \)) is the internal energy, and
\[ U_C(\beta, \sigma) = \int \mathbf{H}^I(\beta, \sigma) d\beta^j (H; \beta \mapsto \tilde{D} \alpha) \]
are obtained as the inverse of \( h \) for fixed \( \sigma \), both of which correspond to \( \int \mathbf{F}_c(\xi) d\xi \) in equation (6). For a special case \((m, n) = (2, 1) \) with \( \mathbf{M} = (1, 0)^T \), equation (17) is reduced to
\[ H_F = U(p, s) + U_C(p, s, \beta) - \frac{1}{2} \rho_0 v_0^2 = v^i (\rho \phi_j + s \psi_j + \beta \alpha_j), \tag{18} \]
where \( \phi = (\phi, \psi)^T \) and \( \sigma = (p, s)^T \). Now the canonical momenta \( \rho \) and \( s \) can be interpreted as the mass and entropy densities. The equation of state may be denoted as \( L_F = p(\rho, s) \), where \( p(\rho, s) = (L_F \circ \mathbf{F})(\rho, s) \) (see appendix B.2 for a concrete example of \( p(\rho, s) \)). Note that our derivation of the Hamiltonian (18) does not trivially follow from the fluid energy \( H = \frac{1}{2} \rho v^2 + U \) nor does rely on a given set of Poisson brackets as in [15, 19, 24]. In particular, \( U_C(p, s, \beta) \) does not appear in the conventional formulation. And, although it does not affect the conservation of \( p, s, \) and \( \beta, \phi, \psi \) modifies the dynamics of \( \phi, \psi, \) and \( \alpha \):
\[ \partial_t \phi = \frac{\delta H_F}{\delta \rho} = \partial U(p, s) \partial \rho + \partial U_C(p, s, \beta) \partial \rho - \frac{1}{2} \frac{\partial^2 v^2}{\partial \phi_j}, \tag{19a} \]
\[ \partial_t \psi = \frac{\delta H_F}{\delta s} = T + \partial U_C(p, s, \beta) \partial s - \nabla^i \psi_j, \tag{19b} \]
\[ \partial_t \alpha = \frac{\delta H_F}{\delta \beta} = \partial U_C(p, s, \beta) \partial \beta - \nabla^i \alpha_j, \tag{19c} \]
\[ \partial_t \rho = - \frac{\delta H_F}{\delta \phi} = - \rho \psi^i j, \tag{19d} \]
\[ \partial_t s = - \frac{\delta H_F}{\delta \psi} = - s \psi^i j, \tag{19e} \]
\[ \partial_t \beta = - \frac{\delta H_F}{\delta \alpha} = - (\beta \psi^i j), \tag{19f} \]
where \( H_F = \int H_F d^3x \) is the total Hamiltonian, \( T = \partial U / \partial s \) is the temperature. Equations (19b) and (19c) yield \( \tilde{D} \psi = T = \partial U_C / \partial s \) and \( \tilde{D} \alpha = \partial U_C / \partial \beta \), while the conventional approaches give \( \tilde{D} \psi = T \) and \( \tilde{D} \alpha = 0 \) (see equation (2.22) in [8], equations (32)–(33) of [7], and equation (7) of [17], where their \( \alpha \) and \( \beta \) correspond respectively to our \( \beta / \rho \) and \( -\alpha \)). Then our \( \psi \) is no more the conventional thermasy but rather its extension. Also note that the presence of \( U_C \) leads to the extension of the known Bernoulli theorem to the non-barotropic, rotational, and unsteady flow; \( \rho \times (19a) + s \times (19b) + \beta \times (19c) \) yield
\[ \rho \phi_j + s \psi_j + \beta \alpha_j - \frac{1}{2} \rho v^2 = \left( \rho \frac{\partial}{\partial \rho} + s \frac{\partial}{\partial s} + \beta \frac{\partial}{\partial \beta} \right) (U + U_C)(p, s, \beta), \tag{20} \]
\[ \delta \textit{Since } U_C \textit{contains only } p, s, \textit{and } \beta, \textit{it only changes the canonical equations of } \phi, \psi \textit{and } \alpha \textit{neither of which is observed.} \]
which gives the first integral of the Euler equation by using $U_C$. Likewise, $U_C$ does not work as the usual potential energy, but as one sector of the total Hamiltonian generating the motions of $\phi$, $\psi$, and $\alpha$, which alters the physical interpretation of the velocity potentials.

The presence of $U_C$ modifies the physical interpretation of $H_F$ in a non-trivial manner. Using the canonical equations, we soon realize $H_F = \frac{1}{2} \rho v^2 + U + U_C$, and so in the phase-space trajectory, the Hamiltonian takes the value of the fluid energy plus the Clebsch-potential energy $U_C$. Thus the fluid-energy conservation does not trivially hold from the conservation of the total Hamiltonian. Considering $\partial_t U_C = \{U_C, H_F\}_{PB} = -\partial_j (U_C v^j)$, meaning $U_C$ also forms convective current conserving in the whole space, we reach the conservation of $\frac{1}{2} \rho v^2 + U$.

Finally, let us see the relation with equation (2). In the canonical formalism, the Lagrangian density reads

$$L_F = \rho \phi_t + s \psi_t + \beta \alpha_t - H_F = \frac{1}{2} \rho v^2 + U(\rho, s) - U_C(\rho, s, \beta) + \rho \mathcal{D} \phi + s \mathcal{D} \psi + \beta \mathcal{D} \alpha.$$ (21)

Although (21) looks similar to equation (2), the presence of Clebsch-potential energy $U_C$ results in different Euler–Lagrange equations. Also it is highly non-trivial that the last three terms at equation (21) are not added constraints as in equation (2) but naturally emerge through the Legendre transformation $L_F[\phi, \alpha] \to H_F[\phi, \alpha; \sigma, \beta]$ at equation (17); namely these three are no longer constraints. These differences make a critical gap between the present and conventional formalisms in terms of the dimensional reduction. On the basis of the Lagrangian density (2), due to $U_C$’s absence and different equations for potentials, the phase space $\{\phi, \psi, \alpha; \rho, s, \beta\}$ cannot be reduced to the configuration space $\{\phi, \psi, \alpha\}$ like we just have. As pointed out in section 2.1, the Lin’s constraint and the conventional potential equations are to be discarded in order to obtain the true Lagrangian density formulated in the configuration space.

3. Relativistic fluid

3.1. Relativistic kernel

The relativistic-fluid theory follows from requiring the Lorentz invariance of the kernel. For simplicity of discussion, we seek a fluid action in a flat spacetime, i.e. $S = \int L_F(\mathcal{D} \phi) d^4x$ (we adopt the metric signature: $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$). Instead of a three-dimensional vector $v$, the relativistic theory requires a four-dimensional time-like vector $w$. Satifying the conditions above, the simplest extention to the non-relativistic kernel (13) one can think of is

$$\mathcal{D} \phi = \mathcal{D} \phi - \frac{1}{2} M w_\alpha w^\alpha + C(\mathcal{D} \alpha) (\mathcal{D} \equiv w_\alpha \partial_{\alpha}).$$ (22)

Here we use the Greek alphabets to represent the indices of spacetime coordinates. The Euler–Lagrange equations read

$$\frac{\delta S}{\delta \phi^\gamma} = - (\sigma w^\mu)_{,\mu} = 0,$$ (23a)

$$\frac{\delta S}{\delta \alpha^\gamma} = - (\beta w^\mu)_{,\mu} = 0,$$ (23b)
\[
\frac{\delta S}{\delta \sigma_{\mu}} = \sigma^i \phi_{,\mu} + \beta^i \alpha_{,\mu} - M^i \sigma_{,\mu} = 0,
\]
where \( \sigma^i \equiv \partial L_x / \partial (\partial \phi / \partial x^i) \), \( \beta^i \equiv \partial L_x / \partial (\partial \alpha / \partial x^i) \). Taking the four-gradient of \( L_x (\partial \phi) \) and combined with equation (23) yields
\[
M^i \sigma_{,\mu} \phi_{,\nu} = L_{x,\mu},
\]
which describes the dynamics of \( w \). Using a normalized vector \( u(\equiv w/|w|) (u^\mu u_\mu = 1) \), the energy-momentum tensor is given by
\[
T^\mu_\nu = \frac{\partial L_x}{\partial \phi_{,\mu}} \phi_{,\nu} + \frac{\partial L_x}{\partial \alpha_{,\mu}} \alpha_{,\nu} - L_x \delta^\mu_\nu = M^i \sigma |w|^2 u^\mu u_\nu - L_x \delta^\mu_\nu.
\]
Now \( M^i \sigma |w|^2 \) and \( L_x \) act as the inertial mass density and the pressure respectively; namely the total mass density is to be defined by \( \rho \equiv M^i \sigma |w|^2 - L_x \). Multiplying equation (23a) by \( \rho \) yields
\[
(M^i \sigma w^\mu)_,\mu = (\rho u^\mu)_,\mu = 0,
\]
which may be interpreted as the conservation of the rest-mass density \( \rho \equiv M^i \sigma |w| \). Equation (24) is rewritten as the equation for \( u \):
\[
\{ (\rho + L_x) u_{\mu} u^\mu \}_,\mu = L_{x,\mu},
\]
which also directly follows from \( T^\mu_\nu ,\mu = 0 \). Then, our action based on the kernel (22) properly defines the relativistic perfect fluid, yielding equations (26) and (27), i.e. the conservation laws of four-momentum and rest mass.

Physical interpretation of \( w \) follows from \( |w| \):
\[
|w| = \frac{M^i \sigma |w|^2}{M^i \sigma |w|} = \frac{\rho + L_x}{\rho_0} (\equiv \mu)
\]
which exactly coincides with the specific inertial mass (\( \mu \) in [8]). Then \( w(= \mu u) \) is identical to Taub’s current vector \( \{ V \} \) in [21]). In generalization from non-relativistic (13) to relativistic (22), the current vector \( w \) naturally replaces the non-relativistic velocity \( v \) as a relativistic extension, which may be consistent with Taub’s finding [8, 21]. It is noticeable that Lagrangian-coordinates frameworks, in contrast, choose the four velocity \( u \) as a natural dynamical variable, focusing on particle motion rather than field dynamics [22, 23]. Finally we mention that \( \mu \) does not give any new degrees of freedom, but it is determined together with \( w \) by independent field variables \( (\phi, \alpha) \) via Euler–Lagrange equation (23).

### 3.2. Symmetry breaking and interactions

As in any well established classical field theory, interactions are incorporated, in principle, at the action level. Here we follow the same principle and discuss some possible interactions which are allowed in our framework. As previously mentioned, the free fluid \( L_x (\partial \phi) \) possesses shift symmetries of \( \phi \) and \( \alpha \) associated with its convective currents. Therefore, interactions may appear as the non-conservation of convective currents caused by symmetry breaking.

A simple example of such interaction on \( \phi \) may be given by \( L = L_x (\partial \phi) + \mathcal{U}(\phi) \), which breaks the shift symmetry of \( \phi \) and rest-mass conservation as well. A Yukawa-type interaction \( \mathcal{U} \propto \psi \psi \phi \) (\( \phi \) is a single scalar for simplicity) is also possible, which converts the rest-mass
energy of the fluid to energy of the Dirac field $\phi$. Another interesting example may be the rest-mass exchange between fluids; in a 2-fluid system given by $\mathcal{L} = \mathcal{L}_a(\bar{\nabla}_a \phi_a) + \mathcal{L}_b(\bar{\nabla}_b \phi_b) + \mathcal{U}(\phi_a, \phi_b)$ (each fluid sector $\mathcal{L}_I$ ($I = a, b$) have a single-component kernel $\bar{\nabla}_a$ containing $w_I$, $\alpha_I$, and $M_I$), one can observe the exchanging of their rest-mass density. Especially when $\mathcal{U}(\phi_a, \phi_b) = \mathcal{U}(\phi_a - \phi_b)$ with $M_a = M_b = M$, the total rest-mass current $\rho_u u_a + \rho_w u_b$ is conserved.

A gauge interaction on $\alpha$ is also considered, where we see the exchange of non-Abelian charge via gauge interaction. Let $\alpha$ be a $SU(n)$ multiplet, and impose invariance of the Lagrangian density under $SU(n)$ transformations of $\alpha$, while $\phi$ and $M$ are just multiple-component real scalars. In order to incorporate gauge symmetry into the theory, the associated kernel (22) itself must be invariant under local-gauge transformations. This implies that on the one hand, quadratic dependance on $\bar{\nabla} \alpha$, i.e. $\mathcal{C}(\gamma)$ with $\gamma \equiv (\bar{\nabla} \alpha) \bar{\nabla} \alpha$ (The conjugate $\beta$ becomes also a multiplet; $\beta = \partial \mathcal{C}/\partial \gamma \bar{\nabla} \alpha \bar{\nabla} \alpha$). Whereas on the other, that derivatives at kernel are to be modified according to $\bar{\nabla} = w^\nu \partial_\nu \rightarrow w^\nu \nu \partial_\nu$, where $D_{\mu} \equiv \partial_{\mu} - i e a_{\mu} A_\mu$ is the gauge-covariant derivative ($e$, $\tau$., and $A_\mu$ are the coupling constant, generator, and gauge field, $a = 1, 2, \cdots, n^2 - 1$) breaking the shift symmetry of $\alpha$. The total fluid-gauge coupled system is given by

$$S = \int L_\mathcal{F} (\bar{\nabla} \phi) d^4 x + S_{\mathcal{M}}$$

(28)

with $\bar{\nabla} \phi = \bar{\nabla} \phi - \frac{1}{2} M w_{\nu} w^\alpha + \mathcal{C}(\gamma)$. The gauge coupling in the Clebsch term does not affect the conservation law (23), so that the rest-mass conservation (26) again holds. $SU(n)$-charge current arises from the gauge symmetry:

$$J_\mu^a = -i e \left( \beta^\dagger t_a \alpha - \alpha^\dagger t_a \beta \right) w^\mu$$

$$= -i e \left( \bar{\nabla} \alpha \right) (\bar{\nabla} \alpha)^\dagger (t_a \alpha - \alpha^\dagger t_a \mathcal{D} \alpha) u^\mu.$$  

(29)

Taking the four-gradient of $L_\mathcal{F}(\bar{\nabla} \phi)$ yields

$$\{ (\rho + L_\mathcal{F}) u_\mu u^\mu \} \nu = L_{\mathcal{F}, \mu \nu} - F^a_{\mu \nu} F^\nu_{ a \mu},$$

(30)

where $F_{\mu \nu} \equiv A_{\mu \nu} - A_{\nu \mu} + e f_{abc} A^b_{\mu} A^c_{\nu}$ is the field strength, $f_{abc}$ is the structure constants of the Lie algebra: $[t_a, t_b] = i f_{abc} c^c$.

To appreciate the reach and consequences of our formalism, it is instructive to contrast our picture with pioneering, and current, works [15, 17–19, 24].

(i) Our $\phi$ ($n$ real scalars) and $\alpha$ ($SU(n)$ complex-scalar multiplet) amount to $m + 2n$ real scalars, which reduces to $1 + 2n$ ($m = 1$) degrees of freedom when the fluid carries only mass and charge. In contrast, the previous works include the group element $g \in SU(n)$ and mass which amount to $n^2$ real scalars$^6$. Let us also note that the present formalism enables us to consider the entropy current when $m = 2$, while it is not obvious how such extension could be implemented from these works.

(ii) It is remarkable that (30) is obtained solely based on convection and gauge symmetry, whereas previous works rely on further assumptions. One being, the dynamics of $SU(n)$ charge on the basis of Wong’s equation [25], which is an estimated model of $SU(n)$-charged classical particle. And another, the continuum limit of Wong’s particle by

$^6$In case of $SU(2)$ coupling, requires $n^2 = 4$ which is less than our $2n + 1 = 5$. However, $n = 2$ without gauge coupling requires 4 scalars, while our formalism needs only 2. Apart from the $SU(2)$-gauge coupling case, the present formalism requires less degrees of freedom.
replacing particle label as the Lagrangian coordinate, guaranteeing the covariant conservation of charge current. The charge conservation relies on Wong’s model and continuum limit; the framework is not a self-consistent one, as it has roots in a particular model of classical particle mechanics.

(iii) In the present formalism, the natural symplectic structure is derived via the co-tangent bundle of the configuration space \{φ, α\}. This constitutes an issue for previous approaches which start from the particle-substratum Lagrangian density \( \frac{1}{2} \rho \nabla^2 - U \) (or Hamiltonian density \( \frac{1}{2} \rho \nabla^2 + U \)) provided Poisson brackets; the symplectic structure is artificially introduced so that the Hamiltonian density \( \frac{1}{2} \rho \nabla^2 + U \) yields the Euler equation and mass conservation [15, 19, 24].

(iv) As a final remark, a similar treatment for the \( U(1) \)-charged fluid trivially follows by imposing \( U(1) \) symmetry on a complex scalar α. On the other hand, such a \( U(1) \)-gauge coupling model does not seem feasible in these formulations where \( g = e^{iθ} \) and \( A_μ \) are the only free variables which are not enough to describe electrically-charged fluid and the gauge field in its usual sense. Indeed the resultant Clebsch parametrization reads \( u_μ \propto \theta_μ - eA_μ \), where the vortical motion is completely constrained by \( A_μ \) due to the defect of the degrees of freedom\(^7\).

4. Conclusion

In this paper, we have reformulated the theory of the perfect fluid based on the symmetry behind the convective current. Peculiarly, its Lagrangian density \( L = (Dφ)^2 \) is uniquely characterized by the quantity \( Dφ \), which we termed as the kernel and is composed by dynamical fields \( (φ, α) \), velocity \( v \) and constants \( M \). A minimal model for rotational flow follows once provided the kernel (13) (or kernel (22) in the relativistic case) with \( m = n = 1 \), so the action contains only \( (m + n =)2 \) fields \( φ \) and \( α \). By discarding the particle-substratum approach built upon \( \mathcal{L}_0 = \frac{1}{2} \rho \nabla^2 - U \) and thereby Lin constraints of conventional formulations, we succeed at obtaining an, impossible otherwise, Lagrangian density in configuration space. This is actually equivalent to abandoning the conventional equations for non-observable potentials such as \( φ, ψ \), and \( α \) due to the presence of the Clebsch-potential energy \( U_C \). In addition to such dimensional reduction, the configuration-space formalism has also an advantage of much wider applicability. In case when the inverse functions \( F; σ \mapsto Dφ \) and \( H; β \mapsto Dα \) do not exist, the symplectic structure cannot be formulated, and the canonical phase-space formalism fails. The simplest example of this may be an incompressible fluid, where the Poisson brackets do not exist in the regular manner of [15, 19, 24]. Even in such case, the configuration-space formalism does hold, yielding the proper fluid equations consistently.

In our formulation, \( φ \) and \( α \) are not just secondary quantities parametrizing velocity \( u \), but dynamical fields capable of interacting with itself as well as other fundamental fields, playing a more active role in describing observable physics. As an example of the latter, we mention a possible scenario with a Yukawa coupling on \( φ \), causing the exchange of the energy contained in the rest-mass density \( ρ_0 \) with the Dirac field. Regarding models with self-interactions, we discussed scenarios where \( φ \)'s actively trigger the direct exchange of the rest mass. The non-Abelian (Abelian) gauge symmetry on \( α \) is also possible naturally resulting in a non-Abelian

\(^7\) In recent [18, 19] direct application is avoided, adding rest-mass current \( (J_m) \) and charge current \( (J_e) \) with their constitutive model \( J_m = J_e \), containing rest-mass density, \( ρ \), \( α \), \( β \), and \( g \in U(1) \), equal to 4 real scalars if the gauge freedom is unified into \( θ \), while ours contain only a real \( φ \) and a complex \( α \), which amounts to 3 real scalars.
(Abelian) Lorentz force on fluid, which provides a pure gauge theory of the perfect fluid for the first time. Discarding the particle-substratum paradigm our construction does not require any further assumptions other than gauge symmetry as in the usual field theories. In an analogous way, the diffeomorphism covariance imposed on $\alpha$ leads to a pure diffeomorphism-gauge theory of the spin fluid, again, free from extra assumptions. Also, $\alpha$ of the diffeomorphism covariance gives minimum degrees of freedom to describe the spin-gravity coupling in a completely novel way. This will be reported in a follow-up publication [26].

The field theoretic picture enables us to reconsider an underlying relation between rotational flow and the Clebsch potentials. According to Helmholtz’s vortex theorem [27], a vortex can be neither created nor destroyed without an external force; namely the creation and annihilation of a vortex is finally attributed to a force imposed on the fluid. As long as we take the action principle the starting point, the external force is introduced as an interaction term in the Lagrangian; among various interactions of this field theory, the electromagnetic field and its extension (such as Yang–Mills field) may be naturally considered as an external force on the fluid. These gauge fields have spin, so their emission and absorption are accompanied by the exchange of angular momentum, causing the vortex to be a localized orbital angular momentum of fluid. Thus, the presence of the Clebsch potentials allows the fluid to couple with these gauge forces producing vorticity. The Clebsch potentials serve as the mediator to the external field causing vorticity.

Finally, let us mention the prospects for a quantum field theory of the fluid. [28, 29] discusses this idea by considering excitations of the Lagrangian position of a fluid element as the target of quantization. Note that the Lagrangian picture of the fluid implies the rest-mass conservation; each world line of a fluid element running from the past to future does not terminate at some point. The present formalism, on the contrary, is formulated without such picture and is even able to describe the creation and annihilation of the rest mass. This suggests the possibility of another quantum field theory of very different type, where the fluid itself may be obtained as an excitation from the vacuum due to interactions with matter and gauge particles.

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Appendix A. Derivation of equation (6)

We write the time derivative of the first term $\rho_c \partial_t \phi$ in terms of convective derivative:

\[ \mathcal{H}_c = \rho_c \mathcal{D}_c \phi_c - \rho_c \omega^i \phi_j - \mathcal{L}_c \circ F_c(\rho_c) \]

where the first term may be rewritten as $\rho_c F_c(\rho_c)$. Differentiation of $\rho_c F_c(\rho_c) - \mathcal{L}_c \circ F_c(\rho_c)$ by $\rho_c$ yields

\[ 1 \times F_c(\rho_c) + \rho_c F'_c(\rho_c) - \rho_c F'_c(\rho_c) = F_c(\rho_c). \]

Thus $\rho_c F_c(\rho_c) - \mathcal{L}_c \circ F_c(\rho_c)$ can be expressed as $\int f^\infty F_c(\xi) d\xi$, where the constant is not written explicitly.
Appendix B. Equation of state

B.1. Polytropic fluid

Any Lagrangian density which can be written as a continuous function of the single-component kernel \( \mathcal{D} \phi \) defines a barotropic fluid. One straightforward way to see this, is to expand such a function in its canonical basis and look at the pressure-density relation. For this purpose, it is enough to consider an arbitrary power of the kernel, i.e. \( \mathcal{L}_\mathcal{F} \propto (\mathcal{D} \phi)^N \) (with \( N \) a constant). In this way, the mass density \( \rho(= M \sigma) \) becomes \( \rho \propto (\mathcal{D} \phi)^{N} \) whose inversion is \( \mathcal{D} \phi = F(\sigma) \propto \rho^{1/N} \). Since the Lagrangian density \( \mathcal{L}_\mathcal{F} \) gives the pressure value, the equation of state, \( \mathcal{L}_\mathcal{F} = p(\rho) \), is given by \( p(\rho) = (\mathcal{L}_\mathcal{F} \circ F)(\rho/M) \propto \rho^{1+1/N} \) which provides a barotropic (polytropic) relation with \( N \) being the polytropic index. Another example is \( \mathcal{L}_\mathcal{F} \propto e^{\mathcal{D} \phi/B} \), where \( B \) is a positive constant. The mass density is given by \( \rho \propto e^{\mathcal{D} \phi/B} \), which means \( p(\rho) \propto \rho \). This is also a barotropic relation \( p(\rho) \propto \rho^{1+1/N} \) of \( N = \infty \).

B.2. Ideal gas

A Lagrangian density composed of multiple kernels was found to describe the non-barotropic fluid. Combining the simple functions mentioned above, the ideal gas can be defined by the following Lagrangian density:

\[
\mathcal{L}_\mathcal{F}(\mathcal{D} \phi, \mathcal{D} \psi) = G(\mathcal{D} \psi)^\gamma e^{(\mathcal{D} \phi/\mathcal{D} \psi)/R},
\]

where \( \phi = (\phi, \psi)^T \), \( \sigma = (\rho, s)^T \), and \( M = (1, 0)^T \) as in section 2.3. Here \( R \) is the specific gas constant, \( \gamma \) is the Poisson constant defined by \( \gamma = C_p/C_v \) with \( C_v (C_p) \) being the specific heat capacity at constant volume (pressure) and \( G \) a positive constant. Thermodynamic variables \( (\rho, s) \) are given by

\[
\rho = \frac{\partial \mathcal{L}_\mathcal{F}}{\partial (\mathcal{D} \phi)} = \frac{G}{R} (\mathcal{D} \psi)^\gamma e^{(\mathcal{D} \phi/\mathcal{D} \psi)/R} (\gamma p - \mathcal{D} \phi/\mathcal{D} \psi),
\]

which gives the mapping \( f : (\mathcal{D} \phi, \mathcal{D} \psi) \mapsto (\rho, s) \). Its inverse \( F : (\rho, s) \mapsto (\mathcal{D} \phi, \mathcal{D} \psi) \) reads

\[
\mathcal{D} \phi = (R \rho/G)^{\gamma - 1} e^{(s/\rho)/C_v} (C_p - s/\rho),
\]

\[
\mathcal{D} \psi = (R \rho/G)^{\gamma - 1} e^{(s/\rho)/C_v}.
\]

Then the pressure value \( \mathcal{L}_\mathcal{F} \) is given by \( \mathcal{L}_\mathcal{F} = p(\rho, s) \), where

\[
p(\rho, s) = (\mathcal{L}_\mathcal{F} \circ F)(\rho, s) = G(R/eG)^\gamma \rho^{\gamma} e^{(s/\rho)/C_v}.
\]

Following equations (17) and (18), the internal energy is given by

\[
U(\rho, s) = \int \sigma F(\sigma')d\sigma' = \frac{C_v}{R} G(R/eG)^\gamma \rho^{\gamma} e^{(s/\rho)/C_v}.
\]

The Maxwell relation defines the temperature \( T \) as

\[
T = \frac{\partial U(\rho, s)}{\partial s} = (R \rho/G)^{\gamma - 1} e^{(s/\rho)/C_v}. \tag{B.6}
\]

Equations (B.3a) and (B.6) yield the known ideal gas law \( p(\rho, T) = R \rho T \).
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