High-energy QCD amplitudes at two loops and beyond

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Abstract
Recent progress in our understanding of infrared singularities of multi-parton amplitudes has shown that the simplest form of Regge factorization for high-energy gauge-theory amplitudes fails starting at next-to-next-to-leading logarithmic accuracy. We provide a framework to organize the calculation of parton amplitudes at leading power in $t/s$, in terms of factorizing and non-factorizing contributions. This allows us to give explicit expressions for the leading Reggeization-breaking terms in two-loop and three-loop quark and gluon amplitudes in QCD. In particular, using only infrared information, we recover a known non-factorizing, non-logarithmic double-pole contribution at two-loops, and we compute the leading non-factorizing single-logarithmic contributions at three loops.

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1 Introduction

In the high-energy limit, in which the centre-of-mass energy $\sqrt{s}$ is much larger than the typical momentum transfer $\sqrt{-t}$, so that $|s/t| \to \infty$, with $t$ held fixed, gauge theory scattering amplitudes become very simple: they acquire a factorized structure, where the building blocks are given by a $t$-channel propagator, connecting two emission vertices, often called impact factors, characterizing the particles undergoing the scattering. This structure is often referred to as high-energy factorization: impact factors depend on the specific scattering process, but they have a simple coupling to the $t$-channel propagator, which is process independent.

Going from tree level to loop corrections, the picture remains the same, but the $t$-channel propagator gets dressed according to the schematic form [1],

$$\frac{1}{t} \to \frac{1}{t} \left( \frac{s}{-t} \right)^{\alpha(t)},$$

where $\alpha(t)$ is a function of the coupling constant, which in the weak coupling limit becomes a series expansion in the coupling. Because of the analytic structure of Eq. (1), which is typical of Regge theory, $\alpha(t)$ is called Regge trajectory.

Since the amplitude has a $t$-channel ladder-like structure, we can assume it to be even under $s \leftrightarrow u$ exchange. As a consequence, it must be composed of kinematic and color parts which are either both even or both odd under $s \leftrightarrow u$ exchange. If one considers $t$-channel gluon exchange, which is all that is needed at leading order and at leading logarithmic accuracy in $\ln(s/|t|)$, then one takes the amplitude to be composed of kinematic and color parts which are both odd under $s \leftrightarrow u$ exchange. To be definite, let us consider the amplitude for gluon-gluon scattering. In this case, for the process $g(k_1) + g(k_2) \to g(k_3) + g(k_4)$, one may write [2]

$$M_{a_1a_2a_3a_4}^{gg \to gg} \left( \frac{s}{\mu^2}, \frac{t}{\mu^2}, \alpha_s(\mu^2) \right) = 4\pi \alpha_s(\mu^2) \frac{s}{t} \left[ (T^b)_{a_1a_3}C_{\lambda_1\lambda_3}(k_1, k_3) \right] \times \left[ \left( \frac{s}{-t} \right)^{\alpha(t)} + \left( \frac{-s}{-t} \right)^{\alpha(t)} \right] \left[ (T^b)_{a_2a_4}C_{\lambda_2\lambda_4}(k_2, k_4) \right].$$

(2)

where $a_j$ and $k_j$ are the color index and momentum of gluon $j$, and $T^b$ is a color generator in the adjoint representation, so that $(T^a)_{bc} = -if_{abc}$. The impact factors, $C_{\lambda_i\lambda_j}(k_i, k_j)$, depend on the helicities of the gluons, but, as the notation suggests, carry no $s$ dependence. Both the impact factors and the Regge trajectory, in the weak coupling limit, can be expanded in powers of the renormalized coupling $\alpha_s(\mu^2)$: they are then affected by infrared and collinear divergences, which in Eq. (2) are (implicitly) regularized by dimensional regularization.

Beyond leading order, one should consider also the exchange of two or more reggeized gluons. Accordingly, one must include the contribution to the amplitude in which the kinematic and color parts are both even under $s \leftrightarrow u$ exchange, and in...
particular the case in which a color singlet is exchanged. Eq. \( \text{(2)} \), however, suffices to describe the amplitude at leading and at next-to-leading logarithmic (NLL) accuracy in \( \ln \left( \frac{s}{|t|} \right) \). \[3\]

By writing formulae similar to Eq. \( \text{(2)} \) for quark-quark and quark-gluon scattering, and considering them together with gluon-gluon scattering as given by Eq. \( \text{(2)} \), one obtains a system of three equations. Their expansion at one loop shows that each equation has a term proportional to \( \ln \left( \frac{s}{|t|} \right) \), which is the same for all three amplitudes. That term gives the one-loop Regge trajectory, and the fact that is the same for all three equations shows its universality, i.e. its independence of the particular scattering process under consideration. Conversely, the term independent of \( \ln \left( \frac{s}{|t|} \right) \) is different for each equation. Thus one gets an over-constrained system of three coefficients and two unknowns, the one-loop impact factors for quark and gluon scattering. One can use two of the coefficients to determine the one-loop impact factors, and the third to perform a consistency check on high-energy factorization. Repeating the same procedure at two loops, one can use the terms proportional to \( \ln \left( \frac{s}{|t|} \right) \) to determine the two-loop Regge trajectory and verify its universality, and the terms independent of \( \ln \left( \frac{s}{|t|} \right) \) to compute the two-loop impact factors and check that high-energy factorization holds. Such a check, however, fails \[4\], due to the presence of a term proportional to \( \alpha_s^2 \pi^2 / \epsilon^2 \), which therefore invalidates high-energy factorization, making the determination of the two-loop impact factors ambiguous.

A general approach to the high-energy limit of gauge theory amplitudes based on the universal properties of their infrared singularities, developed in \[5, 6\], following the earlier results of \[7, 8, 9\], suggests that the violation of high-energy factorization reported in \[4\] at order \( \alpha_s^2 \) and at next-to-next-to-leading logarithmic accuracy in \( \ln \left( \frac{s}{|t|} \right) \) is due to the amplitude becoming non-diagonal in the \( t \)-channel-exchange-basis. Such a violation iterates then at three loops in the \( \alpha_s^3 \) term proportional to \( \ln \left( \frac{s}{|t|} \right) \), invalidating the universality of the three-loop Regge trajectory. Thus, the eventual definition of a universal three-loop Regge trajectory requires additional conditions.

The goal of this letter is to pinpoint the origin of the high-energy factorization violation discovered in \[4\] at two loops, and to propose a way to isolate factorization-breaking terms at three loops and beyond, in order to be able to define unambiguously a universal Regge trajectory and the related impact factors. This implies the definition of a non-factorizing contribution to the amplitude, whose infrared and collinear divergent part can then be unambiguously predicted using the tools described in \[5, 6\]. We believe that a framework for consistently identifying factorizing and non-factorizing contributions to high-energy amplitudes can be useful both in practical finite-order calculations, to assess the reliability of high-energy resum-mations, and for theoretical developments. Indeed, a violation of naïve high-energy factorization, as given for example by Eq. \( \text{(2)} \), at NNLL accuracy and for non-planar contributions to the amplitude, could have been predicted in the context of Regge theory \[10\] by noting that at this level one may expect contributions to the ampli-
tude due to Regge cuts in the angular momentum plane, whereas expressions of the form of Eq. (2) arise under the assumption that the only singularities in the $l$ plane be isolated poles. A precise expression for the discrepancy between pole-based Regge factorization and the actual perturbative results for the amplitude may be useful at least as a boundary condition for future attempts to extend high-energy factorization to include the contributions of Regge cuts. Furthermore, our results are a first step in the direction of systematically combining informations on high-order amplitudes which arise from soft-collinear factorization, which is exact to all orders in perturbation theory for all singular contributions to the amplitudes, with those arising from Regge factorization, which apply to finite contributions to the amplitudes as well, but have limited validity in terms of logarithmic accuracy. The combination of the two approaches, within the framework discussed in the present letter, yields towers of constraints on real and imaginary parts of finite order amplitudes, which we will discuss in detail in a forthcoming publication [11].

In the following, we begin by briefly reviewing, in Section 2, the results of Ref. [5], in order to set up our notation in a general context. In Section 3, we provide a general parametrization of four-point quark and gluon amplitudes in the high-energy limit, which we then use in Section 4 to compare in detail the two factorizations. This allows us to recover the results of [4], and to provide a definite prediction for factorization-breaking terms at three loops. We conclude by briefly discussing the results and the prospects for future developments in Section 5.

### 2 Infrared divergences at high-energy

We consider a scattering process of $2 \to 2$ massless on-shell partons. Each parton carries a color index, and we may write the scattering amplitude as a vector in color space,

$$
\begin{align*}
\mathcal{M}^{a'd'b'}_{2\to2} \left( \frac{s}{\mu^2}, \frac{t}{\mu^2}, \alpha_s(\mu^2) \right) = \sum_j \mathcal{M}^{[j]}_{2\to2} \left( \frac{s}{\mu^2}, \frac{t}{\mu^2}, \alpha_s(\mu^2) \right) c^{a'd'b'}_{[j]},
\end{align*}
$$

where the index $[j] = 1, \ldots, r$ runs over the color representations which are allowed in a given channel exchange, and $c^{a'd'b'}_{[j]}$ is a suitable orthonormal basis of color tensors. For a detailed discussion of how such tensors can be enumerated and constructed when the external particles are in generic color representations, we refer the reader to [4]. As before, and as in the rest of the paper, in Eq. (3) we leave implicit the dependence on the infrared regulator $\epsilon = 2 - d/2 < 0$.

The structure of infrared and collinear singularities of multi-parton amplitudes can be described, at least to the accuracy required in the present paper, by means of the dipole formula [12, 13, 14, 15]. This result is based on the factorization theorem for soft singularities of fixed-angle multi-parton scattering amplitudes (see
for example [16] and references therein), which in this case can be written as
\[
\mathcal{M}_{2 \to 2} \left( \frac{s}{\mu^2}, \frac{t}{\mu^2}, \alpha_s \right) = \mathcal{Z}_{2 \to 2} \left( \frac{s}{\mu^2}, \frac{t}{\mu^2}, \alpha_s \right) \mathcal{H}_{2 \to 2} \left( \frac{s}{\mu^2}, \frac{t}{\mu^2}, \alpha_s \right). \tag{4}
\]
Here \( \mathcal{H} \) is a color vector, finite as \( \epsilon \to 0 \), and representing a matching condition to be determined order by order in perturbation theory after subtraction of all infrared divergent contributions. It can be expressed in the same color basis as the full amplitude, as
\[
\mathcal{H}_{2 \to 2}^{aa'bb'} \left( \frac{s}{\mu^2}, \frac{t}{\mu^2}, \alpha_s \right) = \sum_j \mathcal{H}_{2 \to 2}^{[j]} \left( \frac{s}{\mu^2}, \frac{t}{\mu^2}, \alpha_s \right) c_{[j]}^{aa'bb'}. \tag{5}
\]
On the other hand, \( \mathcal{Z} \) is an \( r \times r \) matrix in color space, with matrix elements \( \mathcal{Z}_{[j],[j']} \) and \( j, j' \) running over the \( r \) allowed color representations in the selected channel. \( \mathcal{Z} \) generates all the infrared and collinear singularities of the amplitude. As detailed in Refs. [13, 14], it can be written in full generality, for \( 2 \to n \) parton scattering, in terms of an anomalous dimension matrix \( \Gamma \) as
\[
\mathcal{Z}_{2 \to n} \left( \frac{p_i}{\lambda}, \alpha_s \right) = \mathcal{P} \exp \left[ \frac{1}{2} \int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2} \Gamma_{2 \to n} \left( \frac{p_i}{\lambda}, \alpha_s(\lambda^2) \right) \right], \tag{6}
\]
where \( \mathcal{P} \) denotes path ordering in color space, and all singularities in \( \epsilon \) are generated through the integration of the \( d \)-dimensional running coupling down to vanishing scale \( \lambda \to 0 \). The results of Refs. [17, 13, 14] show that, at least up to two loops, the anomalous dimension matrix takes the form
\[
\Gamma_{2 \to n}^{\text{dip}} \left( \frac{p_i}{\lambda}, \alpha_s(\lambda^2) \right) = \frac{1}{4} \tilde{\gamma}_K \left( \alpha_s(\lambda^2) \right) \sum_{(i,j)} \ln \left( \frac{-s_{ij}}{\lambda^2} \right) T_i \cdot T_j - \sum_{i=1}^{n+2} \gamma_J \left( \alpha_s(\lambda^2) \right). \tag{7}
\]
The basic feature of Eq. (7) is that the color structure, expressed in terms of color-insertion operators \( T^i \) appropriate to the color representation of hard parton \( i \), is simply expressed as a sum over color dipoles, with all higher-order multipoles vanishing exactly. Color degrees of freedom are tightly correlated with kinematics, through the invariants \( s_{ij} = (p_i + p_j)^2 \), where for the sake of simplicity we have taken all momenta as outgoing. Since the color structure in Eq. (7) is fixed at one loop, the path ordering symbol in Eq. (6) can be dropped when employing Eq. (7). All dependence on the coupling is confined to colorless anomalous dimensions: \( \tilde{\gamma}_K = \gamma_K^{[i]} / C_{[i]} \), where \( \gamma_K^{[i]} \) is the cusp anomalous dimension [18, 19] in representation \([i]\) and \( C_{[i]} \) is the corresponding quadratic Casimir eigenvalue, and the collinear anomalous dimensions \( \gamma_J \), which can be extracted from form factor data [14, 16, 20].

The dipole formula, Eq. (7), is exact up to two loops for massless partons. Possible corrections beyond two loops have been studied in detail in [14, 21, 22]: they can only take the form of tightly constrained conformal cross-ratios of kinematic invariants, starting at three loops and with at least four hard partons, or they can arise
as a consequence of violations of Casimir scaling for the cusp anomalous dimension, which can happen in principle starting at four loops. The exact calculation of the three-loop soft anomalous dimension matrix $\Gamma$ is a vastly challenging project, and recent progress to this end has very recently been summarized in [23]. Also very recently, evidence for a failure of the dipole formula at the four-loop level, and at NLL accuracy in the high-energy limit, was provided in [24]. While these are very interesting results, obtained with innovative techniques, they do not influence the outcome of our calculations, which only concern terms that are fully accounted for by the dipole ansatz.

In the high energy limit, $s/|t| \to \infty$, the four-point scattering amplitude is affected by large logarithms $\ln(s/(-t))$, which are the focus of our investigation. To leading power in $t/s$, the amplitude can then be organized as a double expansion, in the coupling constant and in the power of the large logarithm. For each color component of the vector $M$ we write

$$M^{[j]}_{2 \to 2} \left( \frac{s}{\mu^2}, \frac{t}{\mu^2}, \alpha_s \right) = 4\pi\alpha_s \sum_{n=0}^{\infty} \sum_{i=0}^{n} \left( \frac{\alpha_s}{\pi} \right)^n \ln^i \left( \frac{s}{s-t} \right) M^{(n),i,[j]} \left( \frac{t}{\mu^2} \right),$$

with corrections suppressed by powers of $t/s$. The components of the finite hard vector $H$ can be expanded likewise,

$$H^{[j]}_{2 \to 2} \left( \frac{s}{\mu^2}, \frac{t}{\mu^2}, \alpha_s \right) = 4\pi\alpha_s \sum_{n=0}^{\infty} \sum_{i=0}^{n} \left( \frac{\alpha_s}{\pi} \right)^n \ln^i \left( \frac{s}{s-t} \right) H^{(n),i,[j]} \left( \frac{t}{\mu^2} \right).$$

The matrix $Z$, on the other hand, was shown in [5, 6] to factorize, to leading power in $t/s$, according to

$$Z \left( \frac{s}{\mu^2}, \frac{t}{\mu^2}, \alpha_s \right) = Z_1 \left( \frac{t}{\mu^2}, \alpha_s \right) \tilde{Z} \left( \frac{s}{t}, \alpha_s \right) + O \left( \frac{t}{s} \right),$$

where

$$\tilde{Z} \left( \frac{s}{t}, \alpha_s \right) = \exp \left\{ K(\alpha_s) \left[ \log \left( \frac{s}{s-t} \right) T_s^2 + i\pi T_s^2 \right] \right\}$$

is a matrix in the same color space as $Z$, and is responsible for generating all the large logarithms of the amplitude which are accompanied by infrared poles. In Eq. (11) we have introduced the ‘Mandelstam’ combinations of color-insertion operators $T_s = T_1 + T_2$ and $T_t = T_1 + T_3$. The coefficients of the high-energy logarithms are determined by the function

$$K(\alpha_s) = -\frac{1}{4} \int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2} \tilde{\gamma}_K(\alpha_s(\lambda^2)),$$
which is a scale integral over the cusp anomalous dimension\(^6\). In Eq. (12) the (singular) \(\epsilon\) dependence is generated through integration of the \(d\)-dimensional version of the running coupling, so that the result is a pure counterterm, easily computed order by order in terms of the perturbative coefficients of the \(\beta\) function and of the cusp anomalous dimension. To two-loop order one finds for example

\[
K(\alpha_s) = \frac{\alpha_s}{\pi} \frac{\gamma_K^{(1)}}{4\epsilon} + \left( \frac{\alpha_s}{\pi} \right)^2 \left( \frac{\gamma_K^{(2)}}{8\epsilon} - \frac{b_0 \gamma_K^{(1)}}{32\epsilon^2} \right) + \mathcal{O}(\alpha_s^3). \tag{13}
\]

Note that the elements of the matrices \(Z\) and \(\tilde{Z}\) in Eqs. (10) and (11) may be written as double expansions in the coupling constant and in the large logarithms, as was done in Eqs. (8) and (9).

As shown explicitly in Ref. [5], Eq. (11) can be used as a starting point to analyze the all-order structure of high-energy logarithms accompanied by infrared poles. To leading logarithmic (LL) accuracy, one easily recovers the Reggeization of the parton exchanged in the \(t\) channel. Eq. (11) is however valid to all logarithmic orders at leading power, and one can use it to study Reggeization and its breaking beyond LL. For example, one finds that at NNLL non-Reggeizing logarithms must appear starting at three loops, with the leading effects arising from the operator

\[
\mathcal{E} \left( \frac{s}{t}, \alpha_s \right) \equiv -\frac{\pi^2}{3} K^3(\alpha_s) \ln \left( \frac{2}{s-t} \right) \left[ T^2_{s,t}, T^2_{s,s} \right], \tag{14}
\]

One of the goals of this letter is to evaluate explicitly the effect of this operator at three loops in quark and gluon amplitudes.

Turning back to Eq. (10), the remaining factor \(Z_1\) is a singlet in color space, and we write it explicitly here as

\[
Z_1 \left( \frac{t}{\mu^2}, \alpha_s \right) = Z_{1,\text{R}} \left( \frac{t}{\mu^2}, \alpha_s \right) \exp \left( -i \frac{\pi}{2} K(\alpha_s) C_{\text{tot}} \right), \tag{15}
\]

where we have isolated the phase factor, expressed in terms of the cusp and of the combined Casimir eigenvalue \(C_{\text{tot}} \equiv \sum_{i=1}^4 C_i\), leaving behind a function which is real in the physical region, and which in turn is given by

\[
Z_{1,\text{R}} \left( \frac{t}{\mu^2}, \alpha_s \right) = \exp \left\{ \frac{1}{2} \left[ K(\alpha_s) \log \left( \frac{-t}{\mu^2} \right) + D(\alpha_s) \right] C_{\text{tot}} + \sum_{i=1}^4 B_i(\alpha_s) \right\},
\]

\[
= \sum_{n=0}^{\infty} \left( \frac{\alpha_s}{\pi} \right)^n Z^{(n)}_{1,\text{R}} \left( \frac{t}{\mu^2} \right), \tag{16}
\]

\(^6\)This integral plays an important and ubiquitous role in perturbative QCD: the dimensionally regularized version in Eq. (12) emerged first in the resummation of infrared poles in the quark form factor in [20] and was recursively computed to all orders, in terms of the perturbative coefficients of \(\beta(\alpha_s)\) and \(\gamma_K(\alpha_s)\), in [25]. In the context of the high-energy limit a slightly different form of Eq. (12) was shown to give the all-order infrared part of the Regge trajectory in [9].
where in the second line we have written $Z_{1,R}$ as an expansion over the coupling constant. The functions $D(\alpha_s)$ and $B(\alpha_s)$ are given by scale integrals over the cusp and collinear anomalous dimensions, as in Eq. (12), and they similarly yield a perturbative series of pure counterterms, representing infrared and collinear divergences. Explicitly,

$$D(\alpha_s) = -\frac{1}{4} \int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2} \gamma_K(\alpha_s(\lambda^2)) \log \left( \frac{\mu^2}{\lambda^2} \right),$$

$$B_i(\alpha_s) = -\frac{1}{2} \int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2} \gamma_{J_i}(\alpha_s(\lambda^2)).$$

(17)

An important property of the operator $Z_{1,R}$, and indeed of $Z_1$, is that, to all orders, it is the product of four factors, each one associated with one of the external hard partons. One may write

$$Z_{1,R} \left( \frac{t}{\mu^2}, \alpha_s \right) = \prod_{i=1}^{4} Z_{1,R}^{(i)} \left( \frac{t}{\mu^2}, \alpha_s \right),$$

(18)

and similarly for $Z_1$. Each factor $Z_{1,R}^{(i)}$ is thus properly thought of as a ‘jet’ operator, and one may expect these jet operators to combine naturally to yield the divergent parts of the impact factors. We will see below that this is indeed the case.

3 The structure of high-energy parton amplitudes

In Eq. (2), we have displayed the Regge factorization formula for gluon-gluon scattering, with the $t$-channel exchange of a reggeized gluon. In order to include also quark-quark and quark-gluon scattering, we need to take into account the fact that the color factor for the quark-quark amplitude does not have a definite symmetry property under $s \leftrightarrow u$. In that case, therefore, the symmetric and the antisymmetric parts of the kinematic factor must have different weights. We write then, for the octet component of the matrix element,

$$\mathcal{M}_{ab}^{[8]} \left( \frac{s}{\mu^2}, \frac{t}{\mu^2}, \alpha_s \right) = 2\pi \alpha_s H_{ab}^{(0)[8]}$$

$$\times \left\{ C_a \left( \frac{t}{\mu^2}, \alpha_s \right) \left[ A_+ \left( \frac{s}{t}, \alpha_s \right) + \kappa_{ab} A_- \left( \frac{s}{t}, \alpha_s \right) \right] C_b \left( \frac{t}{\mu^2}, \alpha_s \right) 

+ R_{ab}^{[8]} \left( \frac{s}{\mu^2}, \frac{t}{\mu^2}, \alpha_s \right) + O \left( \frac{t}{s} \right) \right\},$$

(19)

where the indices $a, b$ label the parton species (quark or gluon), and

$$A_\pm \left( \frac{s}{t}, \alpha_s \right) = \left( \frac{t}{s} \right)^{\alpha(t)} \pm \left( \frac{s}{t} \right)^{\alpha(t)},$$

(20)
\[ \kappa_{gg} = \kappa_{qg} = 0, \quad \text{and} \quad \kappa_{qq} = \frac{(4 - N_c^2)}{N_c^2}. \]

In Eq. (19), we can expand the Regge trajectory and the impact factors in powers of the coupling constant, as

\[ \alpha(t) = \sum_{n=1}^{\infty} \left( \frac{\alpha_s}{\pi} \right)^n \alpha^{(n)}(t), \quad C_i \left( \frac{t}{\mu^2}, \alpha_s \right) = \sum_{n=0}^{\infty} \left( \frac{\alpha_s}{\pi} \right)^n C_i^{(n)} \left( \frac{t}{\mu^2} \right), \quad (21) \]

and we have chosen the prefactor (where \( H^{(0),[8]} = H^{(0,0),[8]} \) in the notations of Eq. (19)) so that \( C_i^{(0)} = 1 \). If one had included only the first line in braces in Eq. (19), the resulting expression would have been accurate only to NLL, and only for the real part of the amplitude. In order to promote the equality to leading power accuracy, we have included a non-factorizing remainder, \( R_{ab}^{[8]} \), collecting all terms in the matrix element which cannot be written in terms of a universal Regge trajectory with impact factors depending only on the parton species. We know from earlier results that the non-factorizing remainder starts at two loops and at NNLL level, therefore we expand it in powers of the coupling and of the high-energy logarithm as

\[ R_{ab}^{[8]} \left( \frac{s}{\mu^2}, \frac{t}{\mu^2}, \alpha_s \right) = \sum_{n=2}^{\infty} \sum_{k=0}^{n-2} \left( \frac{\alpha_s}{\pi} \right)^n \ln^k \left( \frac{s}{t} \right) R_{ab}^{(n,k),[8]} \left( \frac{t}{\mu^2} \right). \quad (22) \]

Clearly, as with any factorization which breaks down at some level of accuracy, there is a degree of ambiguity in the definition of the non-factorizing remainder \( R_{ab}^{[8]} \), as it may be possible to move some (non-logarithmic) terms from the remainder to the impact factors without invalidating Eq. (19). As we will see however, at least as far as infrared divergent contributions are concerned, the knowledge of the structure of the amplitude which comes from soft-collinear factorization provides us a very natural choice of ‘factorization scheme’, and therefore with a natural choice for the non-factorizing remainder.

### 4 Comparing soft-collinear and high-energy factorizations

We now have at our disposal two different factorizations: Eq. (11), with all the subsidiary information collected in Section 2, and Eq. (19). Soft-collinear factorization, embodied by Eq. (11), is exact to all orders in perturbation theory for infrared divergent contributions, and the high-energy limit of the \( Z \) matrix is accurate to leading power in \( t/s \). High-energy factorization as given in Eq. (19) applies also to finite contributions to the amplitude, but has a limited logarithmic accuracy. Our task is to intersect the informations from the two limits, extract the constraints that arise when both are applicable, and eventually make predictions based on one of them when the second one breaks down.

To illustrate our strategy, we briefly summarize what happens at one loop, where all ingredients are known and we are basically performing a consistency check.
Throughout this section we set $\mu^2 = -t$ so that all results for the trajectory and the impact factors are given by pure numbers. We begin by expanding the available expressions for the matrix elements to first order in $\alpha_s$. For simplicity, we will omit the parton indices $a, b$ whenever they are not specifically needed. Soft-collinear factorization yields the expressions

$$M^{(1),0} = \left[ Z_{1,R}^{(1)} + i\pi K^{(1)} \left( T_s^2 - \frac{1}{2} C_{\text{tot}} \right) \right] H^{(0)} + H^{(1),0},$$
$$M^{(1),1} = K^{(1)} T_i^2 H^{(0)} + H^{(1),1},$$

which are still vectors in color space, while for the octet component, high-energy factorization provides the expressions

$$M_{ab}^{(1),0,[8]} = \left[ C_a^{(1)} + C_b^{(1)} - \frac{i\pi}{2} (1 + \kappa_{ab}) \alpha^{(1)} \right] H_{ab}^{(0),[8]},$$
$$M_{ab}^{(1),1,[8]} = \alpha^{(1)} H_{ab}^{(0),[8]},$$

One of the constraints of Regge factorization is the fact that the Regge trajectory and the impact factors are required to be real: in other words, the imaginary part of the amplitude is completely determined by the ‘signature’ properties under the exchange $s \leftrightarrow u$, as given by Eq. (19) and by Eq. (20). There are therefore interesting informations to be extracted about the imaginary parts of the amplitude when comparing results such as Eq. (23) and Eq. (24). Detailed results for imaginary parts will be discussed in [11]: here we focus on the real part of the amplitude. Comparing first one-loop terms proportional to $\ln(s/(-t))$, we immediately see that we can write the one-loop Regge trajectory as

$$\alpha^{(1)} = \frac{K^{(1)} \left( T_i^2 H^{(0)} \right)^{[8]}}{H^{(0),[8]}} + \frac{H^{(1),1,[8]}}{H^{(0),[8]}},$$

In the high-energy limit, for all parton species, the tree-level amplitude is a pure color octet in the $t$-channel, and therefore it is an eigenvector of the $T^2_t$ operator, so that $T^2_t H^{(0)} = C_A H^{(0),[8]}$. Furthermore one easily verifies that $H^{(1),1,[8]} = O(\epsilon)$. As expected, the Regge trajectory then becomes

$$\alpha^{(1)} = C_A K^{(1)} + O(\epsilon),$$

which confirms the universality of the one-loop Regge trajectory [26, 27, 28, 29, 30, 31, 32] to $O(\epsilon)$.

Turning to non-logarithmic contributions to the matrix elements in Eqs. (23) and (24), we can consider separately the quark-quark and the gluon-gluon scattering amplitudes, and determine the respective impact factors. One finds that

$$C_a^{(1)} = \frac{1}{2} Z_{1,R,a}^{(1)} + \frac{1}{2} \hat{H}_{aa}^{(1),0,[8]},$$

9
where we defined $\hat{H}_{ab}^{(m),n,[J]} = H_{ab}^{(m),n,[J]} / H_{ab}^{(0),[8]}$, and where we have used the fact that, by virtue of Eq. (18),

$$Z_{1,R,gg}^{(1)} = \frac{1}{2} \left[ Z_{1,R,qg}^{(1)} + Z_{1,R,gg}^{(1)} \right].$$

(28)

Having determined both impact factors, one can finally verify the consistency of Regge factorization by constructing the high-energy quark-gluon scattering amplitude. One finds that requiring Regge factorization constrains the hard parts of the amplitudes to satisfy

$$\text{Re} \left( \hat{H}_{qq}^{(1),0,[8]} \right) = \frac{1}{2} \left[ \text{Re} \left( \hat{H}_{qq}^{(1),0,[8]} \right) + \text{Re} \left( \hat{H}_{qq}^{(1),0,[8]} \right) \right],$$

(29)

which is easily verified to be correct by using the explicit results listed, for example, in Ref. [33].

Repeating the procedure at two loops, one finds more interesting results, and, at the level of non-logarithmic terms, one begins to see the breakdown of the high-energy factorization as given in Eq. (19). Beginning at leading logarithms ($\ln^2(s/(-t))$ at two loops), one readily verifies that the coefficient of the highest power of the energy logarithm is determined by the one-loop result, as expected from high-energy resummation. At the level of single logarithms, comparing Eqs. (19) and (8) allows us to write the two-loop Regge trajectory [4, 34, 35, 36, 37]

$$\alpha^{(2)} = C A K^{(2)} + \text{Re} \left[ \hat{H}_{ab}^{(2),1,[8]} \right] + O(\varepsilon),$$

(30)

independently of the specific scattering process considered. This is again in perfect agreement with high-energy factorization. Turning to the terms which do not contain $\ln(s/(-t))$, however, we begin to see the effects of Reggeization breaking. In particular, deriving the two-loop quark and gluon impact factors from the factorized expression for the quark-quark and gluon-gluon scattering amplitudes respectively, we get, for the singular terms of the impact factors,

$$C_a^{(2)} = \frac{1}{2} Z_{1,R,aa}^{(2)} - \frac{1}{8} \left( Z_{1,R,aa}^{(1)} \right)^2 + \frac{1}{4} Z_{1,R,aa}^{(1)} \text{Re} \left[ \hat{H}_{aa}^{(1),0,[8]} \right] - \frac{1}{4} R_{aa}^{(2),0,[8]}$$

$$- \frac{\pi^2 (K^{(1)})^2}{4} \left\{ \left[ (T_{s,aa}^{2})^2 \right]_{[8],[8]} - C_{\text{tot,aa}} \left[ (T_{s,aa}^{2}) \right]_{[8],[8]} + \frac{1}{4} C_{\text{tot,aa}}^2 - \frac{(1 + \kappa_{aa}) C_A^2}{2} \right\},$$

(31)

with $a = q, g$, and where we have allowed for a non-vanishing non-factorizing remainder $R$, according to Eq. (19).

We observe that soft-collinear factorization has generated an expression for the impact factors which manifestly contains both universal and non-universal components. Indeed, the first line of Eq. (31), with the exception of the so-far undefined $R$ term, has all the characteristics of a proper impact factor: it is composed of terms that can be unambiguously assigned to each external leg of the amplitude, and it
is completely consistent with the interpretation of the impact factor as the action of two ‘jet operators’, as defined in Eq. (18), on the hard part of the amplitude $\hat{H}_{aa}$. The second line of Eq. (31), on the other hand, clearly does not admit an interpretation as an ‘impact factor’, which should be associated with pure color-octet exchange, and should depend only on the identity of the particles being scattered on either side of the $t$-channel Reggeized propagator. On the contrary, the second line of Eq. (31) contains the color operator $T_s^2$, which mixes the representations being exchanged in the $t$ channel, and depends on the identity of all the four particles participating in the scattering. Furthermore, although real, the second line in Eq. (31) originates from the phase factor in $Z_1$, which is difficult to reconcile with the reality properties required by high-energy factorization.

Armed with these considerations, we propose to define the impact factors precisely as the set of terms in Eq. (31) that arise from the action of the ‘jet operators’ in Eq. (18) on the hard coefficients. At two loops this gives

$$\tilde{C}_a^{(2)} = \frac{1}{2} Z_{1,R,aa}^{(2)} - \frac{1}{8} \left( Z_{1,R,aa}^{(1)} \right)^2 + \frac{1}{4} Z_{1,R,aa}^{(1)} \Re \left[ \hat{H}_{aa}^{(1),0,[8]} \right] + O (\epsilon^0).$$

(32)

Correspondingly, we propose to define the non-factorizing remainder $R$ at two loops as

$$\tilde{R}_{ab}^{(2),0,[8]} = -\frac{\pi^2 (K^{(1)})^2}{H_{ab}^{(0),[8]}} \left[ \left( (T_{s,ab}^2 H_{ab}^{(0)})^{[8]} - C_{tot,ab} \left( T_{s,ab}^2 H_{ab}^{(0)} \right)^{[8]} \right] - \left( \frac{1 + \kappa_{ab}}{2} N_c^2 - \frac{C_{tot,ab}^2}{4} \right) H_{ab}^{(0),[8]} \right] + O (\epsilon^0).$$

(33)

We note that Eq. (33) has no single pole terms, which is a consequence of the fact that it arises ultimately from the square of the phase factor in Eq. (15). The expression in Eq. (33) is still somewhat formal, but it can easily be made explicit, for each parton species, upon picking specific color bases for the various amplitudes. Working in the orthonormal bases described in detail in Ref. [38], we get

$$\tilde{R}_{qq}^{(2),0,[8]} = \frac{\pi^2}{4\epsilon^2} \left( 1 - \frac{3}{N_c^2} \right), \quad \tilde{R}_{gg}^{(2),0,[8]} = -\frac{3\pi^2}{2\epsilon^2}, \quad \tilde{R}_{qg}^{(2),0,[8]} = -\frac{\pi^2}{4\epsilon^2}.$$

In particular, one can verify, using the results of Ref. [39, 40], that $\tilde{R}_{qq}^{(2),0,[8]}$, together with the impact factors as defined in Eq. (32), accounts for all the poles of the two-loop quark-gluon scattering amplitude.

Note that, had we used the hypothesis of Regge factorization without a non-factorizing remainder, as was done in Ref. [4], we would have found a mismatch between the quark-gluon scattering amplitude and the one predicted by the Regge factorization formula, Eq. (19), without the remainder $R$. That mismatch may be
quantified by the function,

\[ \Delta(2,0,[8]) = \frac{M^{(2),0}_{gg}}{H^{(0)[8]}_{gg}} - \left[ C_q^{(2)} + C_g^{(2)} + C_q^{(1)} C_g^{(1)} - \frac{\pi^2}{4} (1 + \kappa) (\alpha^{(1)})^2 \right] \]

\[ = \frac{1}{2} \left( \tilde{R}^{(2),0,[8]}_{gg} - \frac{1}{2} \left( \tilde{R}^{(2),0,[8]}_{qg} + \tilde{R}^{(2),0,[8]}_{gg} \right) \right). \] (34)

Using data from our chosen color basis [38], we may evaluate explicitly Eq. (34), finding

\[ \Delta(2,0,[8]) = \pi^2 \left( K^{(1)} \right)^2 \left[ \frac{3}{2} \left( N_c^2 + 1 \right) \right] = \frac{\pi^2}{\varepsilon^2} \frac{3}{16} \left( \frac{N_c^2 + 1}{N_c^2} \right). \] (35)

Eq. (35) is in complete agreement with the discrepancy found in Ref. [4], and explains the origin of the problem, as arising from the mixing of color representations and the phase factors that are required by soft-collinear factorization.

Proceeding to three-loop order, one would expect that matching the single-logarithmic terms of Eqs. (19) and (8) should allow us to obtain a universal expression for the three-loop Regge trajectory \( \alpha(t) \). As predicted in [5, 6], however, a direct comparison yields a non-universal result, in agreement with Eq. (14). To illustrate the situation, we quote here the triple pole contribution, which is where the leading factorization-breaking effects arise, and which is completely determined by soft factors. A detailed discussion of the complete three-loop predictions for impact factors and for the Regge trajectory is left to Ref. [11]. The non-universal result for the three-loop Regge trajectory reads, at this level,

\[ \alpha^{(3)} = C_A K^{(3)} + \frac{\pi^2 (K^{(1)})^3}{2} \left[ C_{tot,ab} N_c \left( T_{s,ab}^{(2)} \right)_{[8],[8]} - \frac{C_{tot,ab}^2 N_c}{4} + \frac{1 + \kappa_{ab}}{2} N_c^3 \right. \]

\[ - \frac{1}{3} \sum_n \left( 2N_c + C_{[n]} \right) \left\| T_{s,ab}^{(2)} \right\|_{[8],[n]}^2 \right) - \frac{1}{2} R^{(3),1,[8]}_{ab} + O (\varepsilon^{-2}) , \] (36)

where the sum on the second line runs over all color representations that can be exchanged in the \( t \) channel. Once again, we recognize that the first term has the appropriate universality properties, and indeed corresponds to the all-order ansatz for infrared-singular contributions to \( \alpha(t) \) first given in [9] and then reproduced in [5, 6]. The other terms in Eq. (36) are clearly of a non-universal nature, and it is appropriate to attribute them to the non-factorizing remainder \( R \). We define then

\[ \tilde{\alpha}^{(3)} = K^{(3)} N_c + O (\varepsilon^0) , \]

\[ \tilde{R}^{(3),1,[8]}_{ij} = \pi^2 (K^{(1)})^3 \left[ C_{tot,ij} N_c \left( T_{s,ij}^{(2)} \right)_{[8],[8]} - \frac{C_{tot,ij}^2 N_c}{4} \right. \]

\[ + \frac{1 + \kappa}{2} N_c^3 - \frac{1}{3} \sum_n \left( 2N_c + C_{[n]} \right) \left\| T_{s,ij}^{(2)} \right\|_{[8],[n]}^2 \right) + O (\varepsilon^{-2}) . \] (37)

We emphasize that Eq. (37) is an absolute prediction for single-logarithmic terms of high-energy three-loop quark and gluon amplitudes, which is of purely infrared
origin and does not rely upon any input from lower-order finite contributions to the amplitudes. Similar results can be derived for double and single poles of $R^{(3),1,[8]}$, and will be described in [11], but they require progressively more detailed information from finite-order calculations. If we introduce the appropriate color factors in Eq. (37), working as before in the color bases if [38], we obtain the explicit results

$$
\tilde{R}^{(3),1,[8]}_{qq} = \left(\frac{\alpha_s}{\pi}\right)^3 \frac{\pi^2}{\epsilon^3} \frac{2N_c^2 - 5}{12N_c},
$$

$$
\tilde{R}^{(3),1,[8]}_{gg} = -\left(\frac{\alpha_s}{\pi}\right)^3 \frac{\pi^2}{\epsilon^3} \frac{2}{3} N_c,
$$

$$
\tilde{R}^{(3),1,[8]}_{qg} = -\left(\frac{\alpha_s}{\pi}\right)^3 \frac{\pi^2}{\epsilon^3} \frac{N_c}{24},
$$

which can be consistently used in Eq. (19), provided one also substitutes our new definitions of the impact factors and of the Regge trajectory, as given in Eqs. (32) and (37).

5 Perspective

High-energy factorization and soft-collinear factorization are often studied with different techniques, and applied to different kinematical domains. One might however argue that, in some sense, high-energy logarithms are a special class of infrared logarithms, arising when certain scales of the problem become much smaller than other ones. The wealth of techniques which are routinely applied to study the soft approximation becomes then available to study the high-energy limit as well. This viewpoint is of course well known: it was pioneered in [7, 8, 9], further developed in [5, 6] using more recent technical developments, and indeed it is a crucial ingredient of the methods recently proposed in [24].

In this paper, we have presented some preliminary results that follow from a detailed comparison of the two factorizations, order by order in perturbation theory. We have considered specifically quark and gluon amplitudes in QCD, though we emphasize that very similar results could easily be derived for other gauge theories (for example for the interesting case of $N = 4$ Super-Yang-Mills theory, where our results would concern contributions beyond the planar limit). Building upon the detailed factorization derived in [5, 6], we have used soft-collinear techniques in the high-energy limit to explore the limitations of the Reggeization picture, as realized under the assumption that only isolated poles arise in the complex angular momentum plane. As discussed most recently in [24], it is understood that this picture must break down, as Regge cuts arise at sufficiently high orders in perturbation theory. Soft-collinear factorization provides a powerful tool to explore the onset of these new effects, as it gives explicit expressions for the (infrared singular) contributions to the

\footnote{See also [11, 12, 13], for other analyses in a similar spirit.}
amplitudes that break the simplest form of Reggeization, starting at two loops for non-logarithmic terms, and continuing to higher orders at NNLL accuracy.

Comparing the two factorizations, we have noted that infrared constraints provide explicit expressions for the impact factors and for the Regge trajectory, which receive clearly non-universal contributions starting at two loops for the impact factors and at three loops for the Regge trajectory. We have proposed to collect the universal terms by properly redefining the impact factors and the Regge trajectory order by order, and to gather the non-universal contributions into a non-factorizing remainder function. Using our definitions, we have been able to reconstruct the origin of the discrepancy from high-energy factorization discovered in [4], which arises in our framework as a linear combination of the non-factorizing remainders of two-loop quark-quark, gluon-gluon and quark-gluon amplitudes. Furthermore, at the three-loop level, we have given a precise definition of the Reggeization-breaking factors which provide non-universal single-logarithmic contributions to the amplitudes, and we have explicitly computed these terms for all relevant QCD amplitudes.

We emphasize that, while in this letter we have given explicitly only the leading singular contributions to the non-factorizing remainders, similar expressions can be derived also for subleading terms, and they will be presented in detail in Ref. [11]. Similarly, we note that in the present paper we have focused on the real parts of the amplitudes, and our results mostly take the form of constraints on high-energy factorization arising from soft-collinear universality. In [11], we will also consider imaginary parts of amplitudes, and we will show that high-energy factorization, in turn, provides important constraints on the soft, collinear and hard functions entering the soft-collinear factorization formula. Finally, we note that we have concentrated here on four-point amplitudes, for the sake of simplicity. The results of Refs. [5, 6], however, apply also to multi-parton amplitudes in multi-Regge kinematics, where a high-energy factorized expression for the amplitude is also available. In view of the phenomenological relevance of this kinematical situation to LHC physics searches [44], it will be interesting to apply our techniques to explore the boundaries of high-energy factorization in this regime as well.

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