SYZYGY FILTRATIONS OF CYCLIC NAKAYAMA ALGEBRAS

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To Professors Kiyoshi Igusa and Gordana Todorov in celebration of seventieth birthdays

Abstract. For any cyclic Nakayama algebra $\Lambda$, we construct syzygy filtered algebra $\varepsilon(\Lambda)$ which corresponds to various syzygy modules as the name suggests. We prove that the category of modules over the syzygy filtered algebra $\varepsilon(\Lambda)$ is equivalent to the wide subcategory cogenerated by projective-injective modules of the original algebra $\Lambda$ along with other categorical equivalences.

In terms of this new algebra, we interpret the following homological invariants of $\Lambda$: left and right finitistic dimension, left and right $\varphi$-dimension, Gorenstein dimension, dominant dimension and their upper bounds. For all of them, we obtain a unified upper bound $2r$ where $r$ is the number of relations defining the algebra $\Lambda$. We show that the left finitistic and $\varphi$-dimensions are equal to the right finitistic and $\varphi$-dimensions respectively, as well as the difference between $\varphi$-dimension and the finitistic dimension is at most one. Furthermore, we recover various seemingly unrelated results in a uniform way.

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1. Introduction

For an artin algebra $A$ over the field $\mathbb{K}$, a major open question is the finitistic dimension conjecture. It states that supremum of finite projective dimensions of $A$-modules is finite. If the representation dimension of $A$ is less than or equal to three then the conjecture holds, the proof was given by K. Igusa and G. Todorov in [IT05]. In the proof, they introduced a function, which we denote by $\phi$, for each module $M$ and its value is always a finite number (see def. 3.1). Therefore we can take supremum of those values and define $\phi$-dimension of $A$ as

$$\phi \dim A := \sup \{ \phi(M) \mid M \text{ is an } A\text{-module} \}.$$ 

It turns out that this new homological tool has applications. For instance, an algebra is selfinjective if and only if $\phi$-dimension of algebra is zero [LM18]. If an algebra is Gorenstein then Gorenstein dimension and $\phi$-dimension are same [ES17]. Possible numerical values of $\phi$-dimension can be a useful homological measure for artin algebras.

An artin algebra is called Nakayama algebra if each left or right indecomposable projective module has a unique composition series. Nakayama algebras are either linear or cyclic depending on the underlying quiver. In [Sen21] we showed that for a cyclic Nakayama algebra of infinite global dimension, $\phi$-dimension is always an even number. In the same work, in order to study $\phi$-dimension, we considered modules having a particular filtration which is called $\Delta$-filtration. In this work, we develop the syzygy filtration method which is based on the $\Delta$-filtrations introduced at [Sen21]. In short, the core idea of the syzygy filtration method is constructing the algebra $\varepsilon(\Lambda)$ called syzygy filtered algebra whose modules are equivalent to the modules filtered by the second syzygies of the original algebra $\Lambda$. Definition and features of the method are stated in the first section. We put together all the results regarding the syzygy filtered algebra $\varepsilon(\Lambda)$ below where $\text{fin.dim } \Lambda$, $\text{gldim } \Lambda$, $\text{dom.dim } \Lambda$, $\text{gor.dim } \Lambda$ stands for finitistic dimension, global dimension, dominant dimension, Gorenstein dimension of $\Lambda$ respectively.

**Theorem A.** Let $\Lambda$ be a cyclic Nakayama algebra. Then,

i) $\Lambda$ is selfinjective if and only if $\Lambda \cong \varepsilon(\Lambda)$.

ii) $\phi \dim \Lambda = \phi \dim \varepsilon(\Lambda) + 2$, provided that $\phi \dim \Lambda \geq 2$ and $\text{gldim } \Lambda = \infty$.

iii) $\text{fin.dim } \Lambda = \text{fin.dim } \varepsilon(\Lambda) + 2$, provided that $\text{fin.dim } \Lambda \geq 2$ and $\text{gldim } \Lambda = \infty$.

iv) $\text{gldim } \Lambda = \text{gldim } \varepsilon(\Lambda) + 2$ provided that $\text{gldim } \Lambda \geq 2$ and $\text{gldim } \Lambda$ is finite.

v) If $\Lambda$ is Gorenstein then $\varepsilon(\Lambda)$ is Gorenstein. Moreover, $\text{gor.dim } \Lambda = \text{gor.dim } \varepsilon(\Lambda) + 2$ provided that $\text{gor.dim } \Lambda \geq 2$.

vi) $\text{dom.dim } \Lambda = \text{dom.dim } \varepsilon(\Lambda) + 2$ provided that $\text{dom.dim } \Lambda \geq 3$.

vii) Left and right finitistic dimensions are same i.e. $\text{fin.dim } \Lambda = \text{fin.dim } \Lambda^{op}$.

viii) Left and right $\phi$-dimensions are same i.e. $\phi \dim \Lambda = \phi \dim \Lambda^{op}$.
ix) Difference between \( \varphi \)-dimension and the finitistic dimension of \( \Lambda \) can be at most one i.e., \( \varphi \dim \Lambda - \text{fin} \dim \Lambda \leq 1 \).

By using the result v), we give an elementary proof of the equality of \( \varphi \)-dimension and Gorenstein dimension ([ES17], [LM18]) for Nakayama algebras.

Indeed, all homological dimensions we discussed in Theorem A share the same unified upper bound.

**Theorem B.** If \( \Lambda \) is a cyclic non-selfinjective Nakayama algebra which is defined by \( r \)-many irredundant relations, then \( \varphi \dim \Lambda, \text{fin} \dim \Lambda, \text{gor} \dim \Lambda, \text{dom} \dim \Lambda \) are bounded by \( 2r \).

The reductions ii), iii), iv), v) and vi) in Theorem A enable us to use mathematical induction on the homological dimensions, because the syzygy filtered algebra \( \varepsilon(\Lambda) \) is also a Nakayama algebra (see 2.44). Therefore we can apply the syzygy filtration method to \( \varepsilon(\Lambda) \) provided that it is cyclic and get higher syzygy filtered algebras \( \varepsilon^k(\Lambda) \) (see 2.49). The result concerning the higher syzygy filtered algebras is stated below.

**Theorem C.**

i) (Thm 3.8) If \( \Lambda \) is a cyclic non-selfinjective Nakayama algebra of infinite global dimension, then there exists a non-negative integer \( k \) such that \( \varepsilon^k(\Lambda) \) is not selfinjective and \( \varepsilon^{k+1}(\Lambda) \) is selfinjective Nakayama algebra.

ii) (Thm 4.1) If \( \Lambda \) is a cyclic connected Nakayama algebra of finite global dimension, then there exists a non-negative \( k \) such that \( \varepsilon^k(\Lambda) \) is a cyclic connected Nakayama algebra and \( \varepsilon^{k+1}(\Lambda) \) is not cyclic\(^1\). The algebra \( \varepsilon^{k+1}(\Lambda) \) can split into components that are either linear Nakayama algebras or semisimple components or both.

As an application of Theorem C part ii), we give a refinement of the main result of [MM18].

**Theorem D.** (Thm 4.14) If global dimension of cyclic Nakayama algebra \( \Lambda \) is finite, then it is bounded by \( 2m + r_{m-1} - C \) where \( \varepsilon^m(\Lambda) \) is linear Nakayama algebra and \( r_{m-1} \) is the minimal number of relations which define cyclic Nakayama algebra \( \varepsilon^{m-1}(\Lambda) \) and \( C \) is the number of connected components of \( \varepsilon^m(\Lambda) \).

In section 2, we define the algebra \( \varepsilon(\Lambda) \) and study its properties and show some categorical equivalences between certain module categories (see 2.5). Each of the statements of Theorem A is proved in section 3 separately. We give the proof and applications of Theorem C in section 4. Also, in the subsection 4.4 we list some results and open questions relying on the relationship between some syzygy modules and wide subcategories cogenerated by projective-injective modules (see 2.36) which we hope to extend results of this paper into broader context.

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\(^1\)We set \( \varepsilon^0(\Lambda) = \Lambda \)
First, we want to set up the scene and the notation. A module is called uniserial if its left and right composition series are unique. A bound quiver algebra is called Nakayama algebra if every indecomposable module is uniserial. There are two types of Nakayama algebras depending on the underlying quiver: either cyclic or linear.

Throughout the paper Λ is a cyclic Nakayama algebra with the underlying quiver $Q$ (see the figure 1) over the field $\mathbb{K}$. $S_i$ is the simple module at vertex $i$, $P_i = P(S_i)$ is the projective cover of $S_i$, $I_i = I(S_i)$ is the injective envelope of $S_i$. We use the term rank for the number of nonisomorphic simple modules. The socle and the top of an indecomposable $\Lambda$-module $M$ are the simple submodule of $M$ and the simple quotient of $M$ respectively. We denote them by $\text{soc}M$ and $\text{top}M$. The length of an indecomposable uniserial module $M$ is the number of simple composition factors of $M$ and denoted by $\ell(M)$. The radical of an indecomposable module $M$, denoted by $\text{rad}M$, is the longest submodule of $M$ i.e. $M/\text{rad}M \cong \text{top}M$ and $\ell(M) = \ell(\text{rad}M) + 1$. If $M$ is a (proper) submodule of $N$, we denote it by $M \subseteq N$ ($M \subset N$). We recall that a sequence of positive integers $(c_1, c_2, \ldots, c_N)$ satisfying $c_i \leq 1 + c_{i+1}$ for $1 \leq i \leq N - 1$ and $c_N \leq c_1 + 1$ is called Kupisch series where each $c_i$ is the length of projective module $P_i = P(S_i)$. Any Nakayama algebra can be described by Kupisch series up to cyclic permutation.

**Definition 2.1.** Two simple modules $S_i$, $S_j$ are called consecutive if $j = i + 1$ for $1 \leq i \leq N - 1$ or $S_i = S_N$ and $S_j = 1$ when the rank of $\Lambda$ is $N$. Equivalently, $S_i, S_j$ are called consecutive if one of them is Auslander-Reiten translate of the other, i.e. $\tau S_i \cong S_j$.

**Remark 2.2.** Let $\text{rank} \Lambda = N$. We order simple modules such that $\text{Ext}_\Lambda^1(S_i, S_j)$ is nontrivial if and only if $\tau S_i \cong S_j$. In other words, the set of the complete list of
representatives of $\Lambda$-modules of length two is
\[
\left\{ \begin{array}{c|c|c|c|c}
S_1 & S_2 & \cdots & S_{N-1} & S_N \\
S_2 & S_3 & \cdots & S_N & \end{array} \right\}.
\]

Here we collect some results about uniserial modules.

**Lemma 2.3.** [Sen21, lemma 2.5.7] If two indecomposable $\Lambda$-modules $M, N$ have the isomorphic socle, then either $M$ is a submodule of $N$ or $N$ is a submodule of $M$.

**Corollary 2.4.** If two indecomposable non-isomorphic $\Lambda$-modules $M, N$ have the isomorphic top, then either $M$ is quotient of $N$ or $N$ is quotient of $M$.

We refer to lemma 2.3 as the uniseriality lemma.

**Lemma 2.5.** [Sen21, lemma 2.5.11] If $A, B, C$ are indecomposable uniserial modules and
\[
0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0
\]
is nonsplit exact sequence, then the top of $A$ and the socle of $C$ are consecutive modules, i.e. $\tau \text{soc} C \cong \text{top} A$.

**Lemma 2.6.** [Sen21, lemma 2.5.9.] A projective module $P$ cannot be a proper subquotient of another indecomposable module $X$. Moreover, a projective module cannot be a submodule of a non-projective indecomposable module. An indecomposable projective-injective module cannot be a proper submodule of another projective-injective.

2.1. Nakayama algebras via relations. Before introducing the system of relations independently, first we want to discuss its relationship with Kupisch series which is more common tool for the community. The direct approach was given in [Sen21], but it brings its own burden, which is the cyclic ordering of particular simple and projective modules. To keep all the relevant data, intricate usage of indices is necessary and it is not easy to digest at first. To avoid these complications, it is convenient for us to interpret Kupisch series in terms of the structures of the modules. To perform this, the first step we take is the notion of minimal projective module.

**Definition 2.7.** An indecomposable projective $\Lambda$-module is called minimal if its radical is not projective. Dually, an indecomposable injective $\Lambda$-module is called minimal if its quotient is not injective.

**Proposition 2.8.** Let $\Lambda$ be given by Kupisch series $(c_1, \ldots, c_N)$. The projective module $P_i$ is minimal if and only if $c_i \leq c_{i+1}$ for $1 \leq i \leq N - 1$ and $c_N \leq c_1$ for $i = N$.

**Proof.** We work with the contrapositive of the statement.

$(\Rightarrow)$. Assume that $P_i$ is not minimal projective, hence $\text{rad} P_i \cong P_{i+1}$ is a projective module. If $\ell(P_i) = c_i$, then
\[
c_{i+1} = \ell(P_{i+1}) = \ell(\text{rad} P_i) = c_i - 1
\]
implies that $c_i > c_{i+1}$.

$(\Leftarrow)$. Assume that $c_i > c_{i+1}$. By the condition $c_i \leq c_{i+1} + 1$, we get $c_i = c_{i+1} + 1$. Simple
modules appearing in the composition series of projective modules are consecutive, therefore we can express the socles of $P_i$ and $P_{i+1}$ by their top modules, i.e.

\[
\soc P_i \cong \tau^{c_i-1} \top P_i \cong \tau^{c_i-1} S_i.
\]

\[
\soc P_{i+1} \cong \tau^{c_{i+1}-1} \top P_{i+1} \cong \tau^{c_{i+1}-1} S_{i+1}.
\]

Because $c_i = c_{i+1} + 1$, we get

\[
\tau^{c_i-1} S_i \cong \tau^{c_i-2} S_{i+1} \cong \tau^{c_{i+1}-1} S_{i+1}.
\]

This means $P_i$ and $P_{i+1}$ have isomorphic socle. Since rad $P_i$ is not projective module and top rad $P_i \cong \tau \top P_i = \tau S_i \cong S_{i+1}$. So rad $P_i$ is quotient of $P_{i+1}$, i.e.

\[
0 \to \ker f \to P_{i+1} \to \rad P_i \to 0
\]

is an exact sequence where $f : P_{i+1} \to \rad P_i$ is surjective. We need to show that $P_{i+1}$ is injective. Suppose not. So, there exists an injective envelope $I$ of $P_{i+1}$ where the first cosyzygy is nontrivial, i.e.

\[
0 \to P_{i+1} \to I \to \Sigma(P_{i+1}) \to 0.
\]

(2.1)

ker $f$ is submodule of $P_{i+1}$ and $P_{i+1}$ is submodule of $I$, we have an embedding of ker $f$ into $I$ which induces the exact sequence

\[
0 \to \ker f \to I \to I/\ker f \to 0.
\]

(2.2)

Now we will compare the lengths of certain modules. We have:

1) $\ell(I) = \ell(P_{i+1}) + \ell(\Sigma(P_{i+1}))$ by 2.1.
2) $\ell(I) = \ell(\ker f) + \ell(I/\ker f)$ by 2.2.

Since $P_i$ is minimal, by proposition 2.8, $c_i = \ell(P_i) \leq c_{i+1} = \ell(P_{i+1})$. Hence $\ell(I) > \ell(P_i)$. On the other hand $\soc \rad P_i \cong \soc I/\ker f$ by applying lemma 2.5 to the exact sequences 2.1 and 2.2. The sequence of the maps $\ker f \hookrightarrow P_{i+1} \hookrightarrow I$ implies that

\[
\ell(\rad P_i) = \ell(P_{i+1}/\ker f) < \ell(I/\ker f) \Rightarrow
\]

\[
\ell(P_i) = \ell(\rad P_i) + 1 \leq \ell(I/\ker f).
\]

This shows that $P_i$ is a subquotient of $I$ which is not possible by lemma 2.6. Therefore the injective envelope of $P_{i+1}$ is itself, $P_{i+1}$ is projective-injective module.

($\Leftarrow$) Assume that $P_{i+1}$ is projective-injective. $P_{i+1}$ and $P_i = P(S_i)$ cannot have the isomorphic socle. Since rad $P_i$ is the quotient of $P_{i+1}$, we have $\ell(\rad P_i) < \ell(P_{i+1})$. This implies

\[
c_i = \ell(P_i) = \ell(\rad P_i) + 1 \leq \ell(P_{i+1}) = c_{i+1}.
\]

By proposition 2.8, $P_i$ is minimal projective.

\[
\text{□}
\]

Proposition 2.10. If $c_i = c_{i+1}$, then we have

\[
\text{□}
\]
i) \( \tau \soc P_i \cong \soc P_{i+1} \).

ii) \( P_i \) is a minimal projective.

iii) \( P_{i+1} \) is projective-injective module which has no proper injective quotient.

Proof. i) Notice that top \( \rad P_i \cong \top P_{i+1} \). By the corollary to lemma 2.3, either \( \rad P_i \) is the quotient of \( P_{i+1} \) or vice versa. Because \( \ell(\rad P_i) = \ell(P_i) - 1 = c_i - 1 < c_i = c_{i+1} = \ell(P_{i+1}) \), the latter is not possible. We get the surjective map \( f : P_{i+1} \to \rad P_i \). The short exact sequence

\[
0 \to \ker f \to P_{i+1} \to \rad P_i \to 0
\]

implies that \( \ker f \) is simple module, because \( \ell(\ker f) = \ell(P_{i+1}) - \ell(\rad P_i) = c_{i+1} - c_i + 1 = 1 \). We conclude that \( \ker f \cong \soc P_{i+1} \) is consecutive to \( \soc \rad P_i \cong \soc P_i \). By the definition of consecutive modules 2.1, \( \tau \soc P_i \cong \soc P_{i+1} \).

ii) If \( P_i \) was not minimal projective, then \( \rad P_i \) would be projective module. However the sequence 2.3 shows that \( \rad P_i \) is the proper quotient of a projective module. Projective module cannot be a quotient of another module by 2.6. So, \( P_i \) is minimal projective.

iii) By ii), \( P_i \) is minimal projective. By the corollary 2.9, \( P_{i+1} \) is projective-injective. It is enough to show that \( I(\soc P_i) \not\cong P_{i+1} \) because if there exist another injective quotient \( I' \) of \( P_{i+1} \), then there has to be a sequence of surjective maps \( P_{i+1} \to I(\soc P_i) \to I' \). If we assume \( I(\soc P_i) \cong P_{i+1} \), then we get \( \soc P_i \cong \soc I(\soc P_i) \cong \soc P_{i+1} \). However by i), we know that \( \soc P_i \cong \tau^{-1} \soc P_{i+1} \), therefore \( I(\soc P_i) \not\cong P_{i+1} \) and \( P_{i+1} \) has no proper injective quotients.

\[\square\]

Proposition 2.11. If \( c_i < c_{i+1} \), we have

i) \( \tau^{c_{i+1} - c_i + 1} \soc P_i \cong \soc P_{i+1} \)

ii) \( P_i \) is a minimal projective

iii) \( P_{i+1} \) is a projective-injective module and there are \( c_{i+1} - c_i \) non-isomorphic proper injective quotients of \( P_{i+1} \).

Proof. We set

\[
d = c_{i+1} - c_i + 1.
\]

i) Notice that top \( \rad P_i \cong \top P_{i+1} \). By the corollary 2.4, either \( \rad P_i \) is the quotient of \( P_{i+1} \) or vice versa. Because \( \ell(\rad P_i) = \ell(P_i) - 1 = c_i - 1 < c_i < c_{i+1} = \ell(P_{i+1}) \), the latter is not possible. We get the onto map \( f : P_{i+1} \to \rad P_i \). The short exact sequence

\[
0 \to \ker f \to P_{i+1} \to \rad P_i \to 0
\]

and lemma 2.5 imply that top \( \ker f \cong \tau \soc \rad P_i \cong \tau \soc P_i \) and \( \ell(\ker f) = c_{i+1} - c_i + 1 = d \). If we apply Auslander-Reiten translate \( \tau^{d-1} \) to the isomorphism \( \tau \soc P_i \cong \top \ker f \), then we get \( \tau^d \soc P_i \cong \tau^{d-1} \top \ker f \). Since the length of \( \ker f \) is \( d \), we obtain \( \tau^{d-1} \top \ker f \cong \soc \ker f \cong \soc P_{i+1} \). Hence \( \tau^d \soc P_i \cong \soc P_{i+1} \).
ii) If $P_i$ was not minimal projective, then $\text{rad } P_i$ would be projective module. However the sequence 2.5 shows that $\text{rad } P_i$ is a proper quotient of a projective module. Projective modules cannot be quotient of another module. So, $P_i$ is minimal projective.

iii) It is enough to prove that the injective envelope $I$ of $\tau \text{soc } P_i$ is the quotient of $P_{i+1}$. First of all, this proves that $P_{i+1}$ is an injective module because non-projective but injective modules has to be quotients of projective-injectives by the dual of lemma 2.6. Secondly, when $I$ is an injective quotient, then the sequence of surjective maps

$$ P_{i+1} \to I \left( \frac{P_{i+1}}{\text{soc } P_{i+1}} \right) \to \cdots \to I = I(\tau \text{soc } P_i) $$

(2.6)

shows that each module are injective because $I$ is injective quotient of all of them.

Notice that $\text{soc } I / \text{soc } I \cong \text{soc } I / \tau S \cong S = \text{soc } P_i$. Relying on the uniseriality lemma 2.3, we get three cases:

Case i) $P_i \cong I / \tau S$ is not possible because a projective module cannot be a quotient of an injective by 2.6.

Case ii) $P_i \subset I / S$ is impossible because a projective module cannot be a subquotient of an injective by 2.6.

Case iii) We conclude that $I / \tau S$ is a proper submodule of $P_i$.

Now there are two possibilities, either $I / \tau S$ is isomorphic to $\text{rad } P_i$ or it is a proper submodule of $\text{rad } P_i$. We will show that the latter is impossible. If we suppose $I / \tau S$ is a proper submodule of $\text{rad } P_i$, we get

$$ \ell \left( \frac{I}{\tau S} \right) < \ell(\text{rad } P_i) = c_i - 1. $$

(2.7)

On the other hand, the exact sequence 2.5 gives

$$ \ell(\text{rad } P_i) = \ell(P_{i+1}) - \ell(\ker f) $$

(2.8)

Combining 2.7 and 2.8 we get

$$ \ell \left( \frac{P}{\tau S} \right) < \ell(\text{rad } P_i) = \ell(P_{i+1}) - \ell(\ker f). $$

(2.9)

If we substitute the lengths $\ell(P_i) = c_i$, $\ell(P_{i+1}) = c_{i+1}$, $\ell(\ker f) = d$ (see 2.4) to 2.9, then

$$ c_i - 1 < c_{i+1} - d $$

$$ c_i - 1 < c_{i+1} - (c_{i+1} - c_i + 1) $$

$$ -1 < -1 $$

which is not a true statement. Therefore $I / \tau S$ has to be isomorphic to the radical of $P_i$, and by the sequence 2.5, $P_{i+1}$ is the projective cover of $I$, which makes $P_{i+1}$ projective-injective. Moreover, this makes all the modules we considered in 2.6 injective. Their socles are $\tau \text{soc } P_i \cong \text{soc } I, \tau^2 \text{soc } P_i, \ldots, \tau^{d-1} \text{soc } P_i, \tau^d \text{soc } P_i \cong \text{soc } P_{i+1}$. There are $d - 1 = c_{i+1} - c_i$ proper injective quotients of $P_{i+1}$. \qed
**Proposition 2.12.** Let Λ be given by the Kupisch series \((c_1, \ldots, c_N)\). Projective modules \(P_{i+1}, P_{i+2}, \ldots, P_k, \ldots, P_{j-1}, P_j\) have isomorphic socle \(S\) and soc \(P_i \not\cong S\), soc \(P_{i+1} \not\cong S\) if and only if \(c_i \leq c_{i+1}\) and \(c_j \leq c_{j+1}\) and \(c_k > c_{k+1}\) for all \(i+1 \geq k \geq j\).

**Proof.** (\(\Rightarrow\)). If projective modules \(P_k\) have isomorphic socle \(S\) when \(i+1 \leq k \leq j\), by proposition 2.8 we get \(c_k > c_{k+1}\). Because soc \(P_j \not\cong S\), \(P_{i+1}\) is not a submodule of \(P_j\) which makes \(P_j\) minimal projective. Therefore, proposition 2.8 implies \(c_j \leq c_{j+1}\). Similarly, soc \(P_i \not\cong S\) implies that \(P_{i+1}\) is not submodule of \(P_i\), hence \(P_i\) is minimal projective and we get \(c_i \leq c_{i+1}\) by the same proposition 2.8.

(\(\Leftarrow\)) \(c_j \leq c_{j+1}\) and \(c_i \leq c_{i+1}\) imply that \(P_j\) and \(P_i\) are minimal projectives. The condition \(c_k > c_{k+1}\) for \(i+1 \leq k \leq j\) imply none of the projective modules from \(P_{i+1}\) to \(P_{j-1}\) are minimal. In other words, each \(P_k\) is isomorphic to (higher) radical of \(P_{i+1}\), i.e. \(P_k \cong \text{rad}^m P_{i+1}\) for some \(m\). Therefore they have isomorphic socle. \(\square\)

**Definition 2.13.** The collection of projective modules having isomorphic socle is called class of projective modules.

To find the relations, for each class of projective modules, we need to know the minimal projective and projective-injective of the class to express all the projective modules in it. By proposition 2.12, the information of projective-injective module of each class can be obtained from minimal projective of the previous class. Therefore, the important data is the relative positions and the length of minimal projectives. If we know the top and the length of a minimal projective, it produces the relation starting at the index of the top with the same length. We order each relation according to their relative position in the Kupisch series which describes classes of projective modules. Let \((c_1, \ldots, c_N)\) be a Kupisch series and \((c_{i_1}, c_{i_2}, \ldots, c_{i_r})\) be a subsequence of it such that each \(c_{i_m}\) is the length of minimal projective \(P_{i_m}\). Then the relations defining the algebra are

\[
\begin{align*}
\alpha_{i_1+c_{i_1}-1} \circ \alpha_{i_1+c_{i_1}-2} \circ \cdots \circ \alpha_{i_1+1} \circ \alpha_{i_1} &= 0 \\
\alpha_{i_2+c_{i_2}-1} \circ \alpha_{i_2+c_{i_2}-2} \circ \cdots \circ \alpha_{i_2+1} \circ \alpha_{i_2} &= 0 \\
&\vdots \\
\alpha_{i_r+c_{i_r}-1} \circ \alpha_{i_r+c_{i_r}-2} \circ \cdots \circ \alpha_{i_r+1} \circ \alpha_{i_r} &= 0
\end{align*}
\]

(2.10)

The converse is also true. We start with a system of relations which gives the index of minimal projectives. And the ordering of the relations gives us the ordering of the minimal projectives in the Kupisch series.

For our purposes, the socle of a projective module carries more information than the length of a projective module. Therefore, from now on we use relations (see 2.11 below) to define any cyclic Nakayama algebra.

### 2.2. Properties of systems of relations.

We describe cyclic Nakayama algebras in terms of the system of relations. Let Λ be Nakayama algebra of rank \(N\) with \(N \geq 2\) given by \(r \geq 1\) many relations \(\alpha_{k_1} \cdots \alpha_{k_{2r-1}} = 0\) where \(1 \leq i \leq r\) and \(k_f \in \{1, 2, \ldots, N\}\) for quiver \(Q\) as in the figure 1. Notice that each arrow \(\alpha_i\) starts at the vertex \(i\) and ends at the vertex \(i+1\) with the exception \(\alpha_N\) which starts at vertex \(N\) and ends at the vertex 1.
We assume that the algebra $\Lambda$ is given as the path algebra of the quiver $Q$ modulo the system of relations $\text{REL}$

$$
\alpha_{k_2} \cdots \alpha_{k_1+1} \alpha_{k_1} = 0 \\
\alpha_{k_4} \cdots \alpha_{k_3+1} \alpha_{k_3} = 0 \\
\vdots \\
\alpha_{k_{2r-2}} \cdots \alpha_{k_{2r-3}+1} \alpha_{k_{2r-3}} = 0 \\
\alpha_{k_{2r}} \cdots \alpha_{k_{2r-1}+1} \alpha_{k_{2r-1}} = 0.
$$

(2.11)

It is clear that $\Lambda$ is bound quiver algebra $\mathbb{K}Q/\langle \text{REL} \rangle$ where $\mathbb{K}$ is algebraically closed field. First of all, we assume that this system of relations is irredundant i.e. none of the relations is a consequence of the other relations. Secondly, we order them according to their starting index $1 \leq k_1 < k_3 < \ldots < k_{2r-1} \leq N$, which is equivalent to cyclic permutation of the corresponding Kupisch series as in 2.10.

**Remark 2.14.** The irredundant system of equations $\text{REL}$ 2.11 satisfies:

1) There exists at most one relation starting at each $k_j \in [1, N]$.
2) There exists at most one relation ending at each $k_j \in [1, N]$.
3) The number of relations $r$ is less than or equal to the rank $N$.
4) There is no restriction on the lengths of relations except that the system of relations has to be irredundant.
5) Each relation is composition of at least two arrows, because $\Lambda$ is a cyclic Nakayama algebra and there is no simple projective module.
6) As we stated in 2.13, projective modules can be described with respect to the socles. By using the relations 2.11, classes of projective modules are;

$$
P_{k_1} \leftrightarrow \ldots \leftrightarrow P_{(k_{2r-1})+1} \quad \text{have simple } S_{k_2} \text{ as their socle} \\
P_{k_3} \leftrightarrow \ldots \leftrightarrow P_{k_{1}+1} \quad \text{have simple } S_{k_4} \text{ as their socle} \\
\vdots \\
P_{k_{2r-1}} \leftrightarrow \ldots \leftrightarrow P_{(k_{2r-3})+1} \quad \text{have simple } S_{k_{2r}} \text{ as their socle}
$$

(2.12)

where $P_{k_1}, P_{k_3}, \ldots, P_{k_{2r-1}}$ are minimal projectives and $P_{k_{1}+1}, P_{k_{3}+1}, \ldots, P_{k_{2r-1}+1}$ are projective-injectives. We recall the relationship between Kupisch series and the REL stated in 2.11.

(a) $c_i > c_{i+1} \iff P(S_i)$ is not minimal projective, i.e. $\text{rad } P_i \cong P_{i+1}$.
(b) $c_i = c_{i+1} \iff P(S_i)$ is minimal projective and $P(S_{i+1})$ is projective-injective and has no proper injective quotients.
(c) $c_i < c_{i+1} \iff P(S_i)$ is minimal projective and $P(S_{i+1})$ has proper injective quotients.

**2.3. The socle set and the base set.** Let $S(\Lambda)$ be the complete set of representatives of socles of projective modules over $\Lambda$. By using the system of relations 2.11, it is

$$
S(\Lambda) = \{ S_{k_2}, S_{k_4}, \ldots, S_{k_{2r}} \}.
$$

(2.13)

$S(\Lambda)$ is called the *socle set*. We define the set $S'(\Lambda)$ which is the complete set of representatives of Auslander-Reiten translates of socles of indecomposable projective
modules. Hence \( S_i \in S(\Lambda) \) if and only if \( \tau S_i \in S'(\Lambda) \). Because \( \tau S_i \cong S_{i+1} \), we get
\[
S'(\Lambda) = \{ S_{k_2+1}, S_{k_4+1}, \ldots, S_{k_2r+1} \}.
\]

**Definition 2.15.** An indecomposable \( \Lambda \)-module \( M \) satisfying \( \text{top} M \in S'(\Lambda) \) and \( \text{soc} M \in S(\Lambda) \) is called *shortest* if the composition factors of \( M \) except \( \text{top} M \) and \( \text{soc} M \) are not elements of the socle and top sets.

**Definition 2.16.** Let \( \Lambda \) be a cyclic Nakayama algebra defined by the system of \( r \)-many relations. For each \( j \in \{1, 2, \ldots, r\} \), let \( \Delta_j \) be the shortest indecomposable uniserial module with \( \text{soc} \Delta_j \cong S_{k_2j} \) and \( \text{top} \Delta_j \cong S_{k_2(j-1)+1} \). The complete set of representatives of modules \( \Delta_j \)'s is called the *base set* and denoted by \( B(\Lambda) \). Explicitly we have
\[
B(\Lambda) := \left\{ \Delta_1 \cong \begin{array}{c} S_{k_2+1} \\ S_{k_2} \end{array}, \Delta_2 \cong \begin{array}{c} S_{k_4+1} \\ S_{k_4} \end{array}, \ldots, \Delta_j \cong \begin{array}{c} S_{k_2(j-1)+1} \\ S_{k_2j} \end{array}, \ldots, \Delta_r \cong \begin{array}{c} S_{k_2r-2+1} \\ S_{k_2r} \end{array} \right\}.
\]

**Proposition 2.17.** Regarding the base set \( B(\Lambda) \), we have:

1) The socle of each \( \Delta_i \) is an element of the socle set \( S(\Lambda) \), i.e. \( \text{soc} \Delta_i \in S(\Lambda) \).
   Any element of the socle set is an socle of an element of \( B(\Lambda) \).
2) The top of each \( \Delta_i \) is an element of the top set \( S'(\Lambda) \). Any element of the top set is a top of an element of \( B(\Lambda) \).
3) Any simple \( \Lambda \)-module \( S \) appears in the composition series of exactly one \( \Delta_i \).
   Equivalently, the simple composition factors of distinct \( \Delta_j \)'s are disjoint.
4) Distinct elements of the base set are Hom-orthogonal i.e. \( \text{Hom}_\Lambda (\Delta_i, \Delta_j) \cong 0 \) when \( i \neq j \) and \( \text{Hom}_\Lambda (\Delta_i, \Delta_i) \cong K \).
5) Each \( \Delta_i \) is a submodule of an indecomposable projective-injective module.
6) \( \Delta_i \) is simple \( \Lambda \)-module if and only if \( S \cong \Delta_i \) satisfies \( S \in S'(\Lambda) \cap S(\Lambda) \).

**Proof.**

1) By the definition of the base set 2.16, \( S \in S(\Lambda) \) if and only if \( S \subset \Delta_i \) for some \( i \).
2) By the definitions 2.13 and 2.14, \( S \in S'(\Lambda) \) if and only if \( S \cong \Delta_i \) for some \( i \).
3) Assume that \( S \) appears in the simple composition factors of both \( \Delta_i \) and \( \Delta_j \).
   Therefore for some positive integers \( a, b \) we have
   \[
   S \cong \tau^a \text{top} \Delta_i \cong \tau^b \text{top} \Delta_j.
   \]
   Without loss of generality, we assume that \( a \geq b \). This implies
   \[
   \tau^{a-b} \text{top} \Delta_i \cong \text{top} \Delta_j.
   \]
   \( \Delta_i \) is uniserial, so for some number \( c \), \( \tau^c \text{top} \Delta_i \cong \text{soc} \Delta_i \). Because \( S \) is a simple module of the composition factors of \( \Delta_i \), we get \( c \geq a \). If we combine this observation with 2.15, we conclude that \( \tau^{a-b} \text{top} \Delta_i \cong \text{top} \Delta_j \) is a simple module in the composition factors of \( \Delta_i \) which implies that \( \Delta_i \) is not shortest module according to the definition 2.15. Therefore any simple module \( S \) appears in the composition series of at most one \( \Delta_i \).
\[ \Lambda \text{ is cyclic, if } S \in \mathcal{S}(\Lambda), \text{ then } \tau^N S \cong S. \text{ By the result 2.17 1), let } S \cong \text{soc} \Delta_i \text{ for some } i. \text{ Therefore every simple } \Lambda\text{-module appears in the composition factors of at least one } \Delta_j \text{ for some } j. \]

4) By the result 3), the simple composition factors of distinct \( \Delta_j \)'s are disjoint, therefore \( \text{Hom}_{\Lambda}(\Delta_i, \Delta_j) \) is trivial when \( i \neq j \) and \( \mathbb{K} \) when \( i = j \).

5) Let \( P \) be a projective-injective module. Because \( \text{soc} \Delta_i \in \mathcal{S}(\Lambda) \) and \( \text{soc} P \in \mathcal{S}(\Lambda) \), by the uniseriality lemma 2.3, either \( \Delta_i \) is submodule of \( P \) or \( P \) is submodule of \( \Delta_i \). By lemma 2.6, \( \Delta_i \) is a submodule of some \( P \).

6) If \( S \in \mathcal{S}(\Lambda) \cap \mathcal{S}'(\Lambda) \), then \( S \) itself is the shortest module. Therefore \( S \in \mathcal{B}(\Lambda) \) by the definition 2.15. If \( S \in \mathcal{B}(\Lambda) \) and \( S \) is a simple module, then top \( S \in \mathcal{S}'(\Lambda) \) and \( \text{soc} S \in \mathcal{S}(\Lambda) \). Since \( S \) is simple we get \( S \cong \text{top} S \cong \text{soc} S \) which means \( S \in \mathcal{S}(\Lambda) \cap \mathcal{S}'(\Lambda) \).

\[ \square \]

**Remark 2.18.** [Sen21, remark 2.2.2] The following are equal:

1) the number of relations,
2) the number of minimal projectives,
3) the number of projective-injectives,
4) the number of minimal injectives,
5) the number of non-isomorphic socles of projective modules,
6) the number of non-isomorphic tops of injective modules,
7) the number of elements of the base set.

2.4. Other realizations of the base set. Here, we establish the link between the base set and particular syzygy modules.

**Lemma 2.19.** Elements of the base set cannot be projective-injective modules.

**Proof.** Assume that projective-injective module \( P_i \) is an element of the base set. Therefore top \( P_i \cong S_i \in \mathcal{S}'(\Lambda) \) which implies that \( \tau^{-1} S_i \cong S_{i-1} \) is an element of the socle set \( \mathcal{S}(\Lambda) \). Because algebra is cyclic, \( \ell(P_{i-1}) \geq 2 \) and there is a map \( f : P_i \to P_{i-1} \) which induces the short exact sequence

\[ 0 \to \ker f \to P_i \to \text{rad} P_{i-1} \to 0. \]

If \( \ker f \) is trivial, then \( P_i \cong \text{rad} P_{i-1} \), which makes \( P_i \) proper submodule of \( P_{i-1} \). This violates the projective-injectivity of \( P_i \).

If \( \ker f \) is nontrivial, then soc \( \ker f \cong \text{soc} P_i \) and top \( \ker f \cong \tau \text{soc} \text{rad} P_{i-1} \cong \tau \text{soc} P_i \) by lemma 2.5. Therefore the top of \( \ker f \) is an element of the top set \( \mathcal{S}'(\Lambda) \), socle of \( \ker f \) is an element of \( \mathcal{S}(\Lambda) \) and \( \ker f \subset P_i \). Therefore \( P_i \) is not the shortest module (see def. 2.15). This shows that \( P_i \) cannot be an element of the base set \( \mathcal{B}(\Lambda) \). \[ \square \]

**Proposition 2.20.** A module \( X \) is an element of the base set if and only if \( X \) is the first syzygy of the radical of a minimal projective.

**Proof.** (\( \Rightarrow \)). Assume that \( X \in \mathcal{B}(\Lambda) \). By lemma 2.19, \( X \) cannot be a projective-injective module. soc \( X \in \mathcal{S}(\Lambda) \) implies that \( X \) is a submodule of a projective-injective module \( PI \) by the uniseriality lemma 2.3. Let \( Q \) be the quotient i.e. \( Q = PI/X \). The socle of \( Q \) is \( \tau^{-1} \text{top} X \) by lemma 2.5, and \( \text{top} X \in \mathcal{S}'(\Lambda) \) implies that \( \text{soc} Q \in \mathcal{S}(\Lambda) \). Therefore \( Q \) is a submodule of a projective-injective module \( PI' \). Moreover, \( Q \) is a
submodule of a minimal projective module $P$ where $\text{soc } P \cong \text{soc } PI' \cong \text{soc } Q$. Such a $P$ exists since $Q$ is the quotient of a projective module and quotients cannot be projective.

We want to prove that $Q \cong \text{rad } P$. Assume to the contrary that let $Q \ncong \text{rad } P$. $Q$ is submodule of $P$, therefore there exists $i$ such that $Q \cong \text{rad}^i P$ where $i \geq 2$.

Let $P'$ be the indecomposable projective cover of $\text{rad}^{i-1} P$.

**Claim 2.21.** $\text{soc } P' \ncong \text{soc } P$.

*Proof.* If $\text{soc } P \cong \text{soc } P'$, then $P'$ becomes a projective submodule of $P$ which violates minimality of $P$. Hence $\text{soc } P \ncong \text{soc } P'$.

**Claim 2.22.** $\text{soc } P' \ncong \text{soc } PI$

*Proof.* If $\text{soc } P' \ncong \text{soc } PI$, then $PI$ becomes a submodule of $P'$ by the uniseriality lemma. This violates projective-injectivity of $PI$, i.e. if $S$ is socle of a projective module, then indecomposable projective-injective module is the longest module with the socle $S$.

**Claim 2.23.** $\text{top rad } P' \cong \text{top } PI \cong \text{top } Q$.

*Proof.* Since $P' = P(\text{rad}^{i-1} P)$, then the top of $P'$ is $\tau^{i-1} \text{top } P$. Therefore $\text{top rad } P' \cong \tau (\tau^{i-1} \text{top } P) \cong \tau^i \text{top } P$.

Since $Q \cong \text{rad}^i P$, then $\text{top } Q \cong \tau^i \text{top } P$. Moreover, $Q$ is the quotient of $PI$, hence $\text{top } Q \cong \text{top } PI$. As a result, we get the isomorphism.

**Claim 2.24.** $\text{rad } P'$ is a proper quotient of $PI$.

*Proof.* By the previous claim 2.23, $\text{top rad } P' \cong \text{top } PI$. By the corollary 2.4, $\text{rad } P'$ is the quotient of $PI$, since $PI$ is a projective-injective module. It is proper, because by the claim 2.21, $\text{soc rad } P' \cong \text{soc } P' \ncong \text{soc } PI$.

**Claim 2.25.** $Q$ is a proper quotient of $\text{rad } P'$.

*Proof.* By the claim 2.23, $\text{top } Q \cong \text{top rad } P'$. By the corollary 2.4, we have three possibilities.

Case 1) $Q \cong \text{rad } P'$ is impossible, because it makes $Q$ a proper submodule of $P'$ and $P'$ a proper submodule of $P$, which violates the minimality of $P$.

Case 2) Let $S \cong \text{top rad } P' \cong \text{top } Q$. Consider the following exact sequences

$$0 \to Q \cong \text{rad}^i P \to \text{rad}^{i-1} P \to \tau^{-1} S \to 0$$

$$0 \to \text{rad } P' \to P' \to \tau^{-1} S \to 0.$$

If $\text{rad } P'$ were quotient of $Q$, then $P'$ would be quotient of $\text{rad}^{i-1} P$. This violates projectivity of $P'$.

Case 3) Thus, $Q$ is a proper quotient of $\text{rad } P'$.

**Claim 2.26.** There exists a surjective map $f : PI \to \text{rad } P'$ such that $\ell(\ker f) < \ell(X)$. 


Proof. If we combine the previous claims 2.24 and 2.25, we get $PI \twoheadrightarrow \text{rad } P' \twoheadrightarrow Q$, therefore there exist surjective map $f : PI \twoheadrightarrow \text{rad } P'$, which induces the short exact sequence

\begin{equation}
0 \rightarrow \ker f \hookrightarrow PI \twoheadrightarrow \text{rad } P' \rightarrow 0.
\end{equation}

Since $Q$ is the quotient of $PI$, we have another exact sequence

\begin{equation}
0 \rightarrow X \rightarrow PI \rightarrow Q \rightarrow 0.
\end{equation}

The lengths of the modules satisfy $\ell(PI) = \ell(X) + \ell(Q) = \ell(\ker f) + \ell(\text{rad } P')$. Since $Q$ is a proper quotient of $\text{rad } P'$, then $\ell(Q) < \ell(\text{rad } P')$. Therefore, we get

$$\ell(X) > \ell(\ker f).$$

Both $X$ and $\ker f$ are submodules of $PI$. Moreover, by the uniseriality lemma 2.3 together with 2.17, $\ker f$ is a proper submodule of $X$. □

Claim 2.27. $\text{soc} \ker f \in S(\Lambda)$ and $\text{top} \ker f \in S'(\Lambda)$.

Proof. Since $\ker f$ is submodule of $PI$, then $\text{soc} \ker f \in S(\Lambda)$. Furthermore, the exact sequence 2.16 shows that $\tau \text{soc \ rad } P' \cong \text{top} \ker f$ by lemma 2.5. Since $\text{soc \ rad } P' \in S(\Lambda)$, we get $\text{top} \ker f \in S'(\Lambda)$. □

By the claims 2.26 and 2.27, $X$ is not the shortest module, because $\ker f$ is the proper submodule of $X$ and shorter than $X$. We get the contradiction, if the quotient $Q$ is not isomorphic to $\text{rad } P$, then $X$ cannot be an element of $B(\Lambda)$.

$(\Leftarrow)$ Let $Q \cong \text{rad } P$ where $P$ is minimal projective. By the corollary 2.9, the projective cover of $Q$ is projective-injective module. Let’s denote $P(Q) \cong P(\text{rad } P)$ by $PI$.

Assume to the contrary that module $X$ satisfying the exact sequence $0 \rightarrow X \rightarrow PI \rightarrow Q \rightarrow 0$ is not an element of $B(\Lambda)$. Since $\text{soc } X \in S(\Lambda)$, and $X$ is not shortest, then there exists a submodule $Y$ of $X$ such that $Y \in B(\Lambda)$. By lemma 2.5, the socle of the quotient $Q' = PI/Y$ is an element of $S(\Lambda)$. This forces that there has to be a projective module $P'$ such that $Q'$ is a proper submodule of $P'$ and its socle is not isomorphic to socles of $P$ and $PI$. We have

$$\text{soc } P' \cong \tau^i \text{soc } P, \text{ soc } PI \cong \tau^i \text{soc } P' \cong \tau^{i+j} \text{soc } P$$

and in particular $\text{top } P' \not\cong \text{top } P$ and $\text{top } P' \not\cong \text{top } PI$. However, because of $\text{top } PI \cong \tau \text{ top } P$, there is no projective module $P'$ satisfying the above conditions. Hence $X$ has to be the shortest module, hence $X \in B(\Lambda)$. □

Proposition 2.28. A module $X$ is an element of the base set if and only if $X$ is the second syzygy of a simple module which is the top of a minimal projective module.

Proof. We recall proposition 2.20, i.e. $X \in B(\Lambda)$ if and only if $X \cong \Omega^1(\text{rad } P)$ where $P$ is minimal projective. Therefore, the simple module $S = P_{/\text{rad } P} \cong \text{top } P$ proves the claim, i.e. $\Omega^2(S) \cong \Omega^1(\text{rad } P) \cong X$. □
2.5. The category of filtered modules. We want to draw attention to $\Lambda$-modules which are filtered by the base set $\mathcal{B}(\Lambda)$.

**Definition 2.29.** Let $\text{Filt}(\mathcal{B}(\Lambda))$ denote the category of $\Lambda$-modules which are filtered by the base set $\mathcal{B}(\Lambda)$, i.e. $M \in \text{mod-}\Lambda$ is an object of $\text{Filt}(\mathcal{B}(\Lambda))$ if and only if $\text{soc} M \in \mathcal{S}(\Lambda)$ and $\text{top} M \in \mathcal{S}'(\Lambda)$. All the maps in $\text{Filt}(\mathcal{B}(\Lambda))$ are induced from $\Lambda$-module morphisms.

**Proposition 2.30.** Let $M$ be an indecomposable $\Lambda$-module. Suppose $\text{top} M \in \mathcal{S}'(\Lambda)$ and $\text{soc} M \in \mathcal{S}(\Lambda)$ up to isomorphisms. Then

i) If $M$ is shortest among such modules, then $M \cong \Delta_i$ for some $i$.

ii) The module $M$ has a filtration by the modules in $\mathcal{B}(\Lambda)$.

*Proof.* See [Sen21, proposition 2.5.8]. □

**Remark 2.31.** Let $M$ be an indecomposable $\Lambda$-module. If $M \in \text{Filt}(\mathcal{B}(\Lambda))$, then there is a composition series of modules

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = M$$

where each $M_i \in \text{Filt}(\mathcal{B}(\Lambda))$ and $M_i / M_{i-1} \in \mathcal{B}(\Lambda)$ by the definition.

We will show equivalent formulations of this category.

**Proposition 2.32.** If $M$ is an indecomposable module with projective dimension greater or equal than two, then $\Omega^2(M)$ has $\mathcal{B}(\Lambda)$-filtration.

*Proof.* We consider the projective resolution of $M$:

$$\cdots \to P_2 \to P_1 \to M \to 0.$$  

The socles of $P_1$ and $P_2$ are in $\mathcal{S}(\Lambda)$. It is enough to show that $\text{top} \Omega^2(M)$ is in the top set $\mathcal{S}'(\Lambda)$. If we apply lemma 2.5 to the exact sequence

$$0 \to \Omega^2(M) \to P_2 \to \Omega^1(M) \to 0$$

we get $\text{top} \Omega^2(M) \cong \tau \text{soc} \Omega^1(M) \cong \tau \text{soc} P_1$. Since $\text{soc} P_1 \in \mathcal{S}(\Lambda)$, $\text{top} \Omega^2(M) \in \mathcal{S}'(\Lambda)$. Therefore $\Omega^2(M)$ has $\mathcal{B}(\Lambda)$-filtration. □

**Proposition 2.33.** If $M$ is a $\mathcal{B}(\Lambda)$-filtered indecomposable module, then all of the nontrivial higher syzygies of $M$ and their projective covers have $\mathcal{B}(\Lambda)$-filtration.

*Proof.* Socle of any projective module is in $\mathcal{S}(\Lambda)$. It is enough to show that the top of the projective covers of the syzygies are in $\mathcal{S}'(\Lambda)$ which implies all the projective covers are $\mathcal{B}(\Lambda)$-filtered by the definition 2.29.

Consider the projective resolution of $M$

$$\cdots \to P(\Omega^2(M)) \to P(\Omega^1(M)) \to P(M) \to M \to 0.$$  

$M \in \text{Filt}(\mathcal{B}(\Lambda))$ implies that $\text{top} P(M) \cong \text{top} M \in \mathcal{S}'(\Lambda)$, therefore $P(M)$ has $\mathcal{B}(\Lambda)$-filtration.

By lemma 2.5, $\text{top} \Omega^1(M) \cong \tau \text{soc} M$ provided that $\Omega^1(M)$ is non-trivial, therefore $\text{top} \Omega^1(M) \in \mathcal{S}'(\Lambda)$, which shows $\Omega^1(M) \in \text{Filt}(\mathcal{B}(\Lambda))$. Claim holds by induction. □

**Proposition 2.34.** If $X$ is an indecomposable $\mathcal{B}(\Lambda)$-filtered $\Lambda$-module which is not projective-injective, then there exist a module $M$ such that $X \cong \Omega^2(M)$. 
Proof. $X$ has $\mathcal{B}(\Lambda)$-filtration so $S = \text{soc } X \in \mathcal{S}(\Lambda)$. Therefore $X$ is a submodule of projective-injective $P$ with socle $S$. The socle of the quotient $Q = P/X$ is $\tau$-translate of the top of $X$, hence $Q$ is a submodule of another projective module $P'$. The exact sequences

$$0 \to X \to P \to Q = P/X \to 0$$
$$0 \to Q \to P' \to P'/Q \to 0$$

show that $\Omega^2(M) \cong X$ where $M = P'/Q$; since $Q$ is not projective. □

Remark 2.35. In [Sen21], we used the notation $\Delta$ instead of $\mathcal{B}(\Lambda)$.

Here we recall the definition of wide subcategory cogenerated by projective-injective $\Lambda$-module. Let $\hat{P}$ be the additive generator of projective-injective $\Lambda$-modules. Then $\mathcal{F} := \text{Cogen}(\hat{P}) = \{M \in \text{mod-} \Lambda \mid M \text{ is submodule of } \hat{P}\}$ is a torsion-free class. The wide subcategory associated to $\mathcal{F}$ is

$$\mathcal{W} := \{X \in \mathcal{F} \mid \forall (g : X \to Y) \in \mathcal{F}, \text{ then } \text{coker}(g) \in \mathcal{F}\}.$$ 

Proposition 2.36. The category $\text{Filt}(\mathcal{B}(\Lambda))$ is equivalent to the wide subcategory $\mathcal{W}$ cogenerated by projective-injective $\Lambda$-module $\hat{P}$.

Proof. ($\Rightarrow$) Assume that an indecomposable $\Lambda$-module $M$ has $\mathcal{B}(\Lambda)$-filtration. Therefore $\text{soc } M \in \mathcal{S}(\Lambda)$ and by the uniseriality lemma 2.3 together with 2.6, it is a submodule of a projective-injective $\Lambda$-module. Therefore $M \in \mathcal{F}$.

We need to show that for any module $Y \in \mathcal{F}$ and any map $g : M \to Y$ in $\mathcal{F}$, the coker$(g)$ is in $\mathcal{F}$. Since $M$ has $\mathcal{B}(\Lambda)$-filtration, the top of $M$ is in $\mathcal{S}'(\Lambda)$. This implies that $\text{soc coker}(g) \in \mathcal{S}(\Lambda)$, because $\text{soc coker}(g)$ and top $M$ are consecutive simple $\Lambda$-modules. By lemma 2.3, coker$(g)$ is a submodule of projective-injective module. Therefore coker$(g) \in \mathcal{F}$ for any $g : M \to Y$ in $\mathcal{F}$.

($\Leftarrow$) Let $M$ be an indecomposable $\Lambda$-module in $\mathcal{W}$. This implies that $M \in \mathcal{F}$ and for any map $g : M \to Y$ in $\mathcal{F}$, coker$(g)$ is in $\mathcal{F}$. The former implies that the socle of $M$ is an element of the set $\mathcal{S}(\Lambda)$, because $M$ is a submodule of projective-injective module. Similarly, the latter implies that the socle of coker$(g)$ is in $\mathcal{F}$, hence a submodule of projective-injective module. We need to show that top $M$ is in $\mathcal{S}'(\Lambda)$. Without loss of generality, let $Y$ be an indecomposable module in $\mathcal{F}$. By the short exact sequence

$$0 \to \text{Im } g \to Y \to \text{coker}(g) \to 0$$

the top of $\text{Im } g$ and the socle of coker$(g)$ are consecutive modules. Because $\text{soc coker}(g) \in \mathcal{S}(\Lambda)$, we get top $\text{Im } g \in \mathcal{S}'(\Lambda)$. This shows that $M$ has $\mathcal{B}(\Lambda)$-filtration since top $M \cong \text{top } \text{Im } g \in \mathcal{S}'(\Lambda)$. □

Corollary 2.37. The category $\text{Filt}(\mathcal{B}(\Lambda))$ is extension-closed, exact and abelian. Its simple objects are the elements of $\mathcal{B}(\Lambda)$.

Proof. By 2.36, $\text{Filt}(\mathcal{B}(\Lambda))$ is a wide subcategory of mod-$\Lambda$. Therefore it is exact abelian category which is closed under extensions. Elements of $\mathcal{B}(\Lambda)$ are simple objects because they are hom-orthogonal by the statement 4) of proposition 2.17. □
Now we are ready to construct new algebra which is filtered by the elements of $\mathcal{B}(\Lambda)$.

**Definition 2.38.** Let $\varepsilon(\Lambda)$ be the endomorphism algebra of the direct sum of projective $\Lambda$-modules which are projective covers of elements of the top set $S'(\Lambda)$, i.e.

$$\varepsilon(\Lambda) := \text{End}_\Lambda \left( \bigoplus_{S \in S'(\Lambda)} P(S) \right).$$

Since $\varepsilon(\Lambda)$ is the endomorphism algebra of $\mathcal{B}(\Lambda)$-filtered projective modules and the category $\text{Filt}(\mathcal{B}(\Lambda))$ is equivalent to the category of the second syzygies by propositions 2.33, 2.32 and 2.34, we give the name *syzygy filtered algebra* to $\varepsilon(\Lambda)$.

2.6. Modules of syzygy filtered algebras. Here we recall some definitions and statements from [ARS97] mainly chapter 2 to relate the module category of $\Lambda$ and the module category of the syzygy filtered algebra $\varepsilon(\Lambda)$. For any module $A$ in mod-$\Lambda$, the functor $\text{Hom}_\Lambda(A, -) : \text{mod-}\Lambda \to \text{mod-}\Gamma$ where $\Gamma = \text{End}_\Lambda(A)^{\text{op}}$ is called evaluation functor. Suppose $P$ is a projective $\Lambda$-module. Then we denote by mod-$P$ the full subcategory of mod-$\Lambda$ whose objects are those $A$ in mod-$\Lambda$ which have minimal projective presentations $P_1 \to P_0 \to A \to 0$ with $P_0, P_1 \in \text{add } P$.

**Proposition 2.39.** [ARS97, section II, proposition 2.5] Let $P$ be a projective $\Lambda$-module and let $\Gamma = \text{End}_\Lambda(P)^{\text{op}}$. Then the restriction $\text{Hom}(P, -)|_{\text{mod-}P} : \text{mod-}P \to \text{mod-}\Gamma$ of the evaluation functor $\text{Hom}(P, -) : \text{mod-}\Lambda \to \text{mod-}\Gamma$ is an equivalence of categories.

We apply this to our set up. Let $\mathcal{P}$ be $\mathcal{B}(\Lambda)$-filtered projective module, i.e.

$$\mathcal{P} = \bigoplus_{S \in S'(\Lambda)} P(S).$$

Therefore by proposition 2.39, we get the categorical equivalence

$$\text{Hom}_\Lambda(\mathcal{P}, -) : \text{mod-}\mathcal{P} \to \text{mod-\text{End}_\Lambda(\mathcal{P})^{op}}.$$

Now we need to express mod-$\mathcal{P}$ in terms of $\text{Filt}\mathcal{B}(\Lambda)$.

**Proposition 2.40.** $\text{Filt}(\mathcal{B}(\Lambda)) \cong \text{mod-}\mathcal{P}$ where $\mathcal{P} = \bigoplus_{S \in S'(\Lambda)} P(S)$.

**Proof.** $\mathcal{P}$ is the $\mathcal{B}(\Lambda)$-filtered projective module, so $\mathcal{P} \in \text{Filt}(\mathcal{B}(\Lambda))$. Also, $\mathcal{P}$ is in mod-$\mathcal{P}$ by definition of the full subcategory. Therefore, the projective module $\mathcal{P}$ is in both mod-$\mathcal{P}$ and $\text{Filt}(\mathcal{B}(\Lambda))$.

Let $M$ be an indecomposable non-projective $\mathcal{B}(\Lambda)$-filtered module. By proposition 2.33, $P(M)$ and $P(\Omega^1(M))$ are filtered by $\mathcal{B}(\Lambda)$, hence the sequence

$$\cdots \to P(\Omega^1(M)) \to P(M) \to M \to 0$$

is minimal projective presentation of $M$ which shows $M \in \text{mod-}\mathcal{P}$.

For the other direction, let $M$ have presentation by the modules in mod-$\mathcal{P}$, i.e.

$$P_1 \xrightarrow{f} P_0 \to M \to 0.$$ (2.18)
Notice that \( \text{soc Im } f \cong \text{soc } P_0 \in \mathcal{S}(\Lambda) \) and top Im \( f \cong \text{top } P_i \in \mathcal{S}'(\Lambda) \). By lemma 2.5, soc coker \( f \in \mathcal{S}(\Lambda) \). On the other hand top coker \( f \cong \text{top } P_0 \in \mathcal{S}'(\Lambda) \). After combining these, we conclude that \( M \cong \text{coker } f \) has \( \mathcal{B}(\Lambda) \)-filtration due to proposition 2.30.

We see that \( \text{mod-} \mathcal{P} \cong \text{Filt}(\mathcal{B}(\Lambda)) \) and by the definition \( \epsilon(\Lambda) \) is the endomorphism algebra of \( \mathcal{P} \). Therefore, we get the diagram

\[
\begin{array}{ccc}
\text{mod-} \Lambda & \xrightarrow{\text{Hom}(\mathcal{P}, -)} & \text{mod-} \epsilon(\Lambda) \\
\text{Filt}(\mathcal{B}(\Lambda)) & \searrow & \\
\end{array}
\]

where the restriction of \( \text{Hom}_\Lambda(\mathcal{P}, -) \) onto \( \text{Filt}(\mathcal{B}(\Lambda)) \) gives the equivalence of categories \( \text{mod-} \epsilon(\Lambda) \) and \( \text{Filt}(\mathcal{B}(\Lambda)) \). As we will explain in 2.7, we did not take the opposite of the endomorphism algebra wittingly. Nevertheless we can describe modules over the syzygy filtered algebra in the following way. \( \epsilon(\Lambda) = \text{End}_\Lambda \mathcal{P} \) is finite dimensional algebra and \( \mathcal{P} \) is a left \( \epsilon(\Lambda) \)-module. If \( X \) is a right \( \Lambda \)-module, then \( \text{Hom}_\Lambda(\mathcal{P}, X) \) is a right \( \epsilon(\Lambda) \)-module. The simple objects of \( \text{Filt}(\mathcal{B}(\Lambda)) \) are the elements of the base set \( \mathcal{B}(\Lambda) \), therefore simple \( \epsilon(\Lambda) \)-modules are of the form \( \text{Hom}_\Lambda(\mathcal{P}, \Delta_i) \) where \( \Delta_i \in \mathcal{B}(\Lambda) \). Projective \( \epsilon(\Lambda) \)-modules are of the form \( \text{Hom}_\Lambda(\mathcal{P}, P) \) where \( P \) is \( \mathcal{B}(\Lambda) \)-filtered projective. In general, any \( X \in \text{mod-} \epsilon(\Lambda) \) is isomorphic to \( \text{Hom}_\Lambda(\mathcal{P}, X') \) for some \( X' \in \text{Filt}(\mathcal{B}(\Lambda)) \).

**Remark 2.41.** The evaluation functor \( \text{Hom}_\Lambda(\mathcal{P}, -) \) is exact, because \( \mathcal{P} \) is projective module. If \( M \in \text{Filt}(\mathcal{B}(\Lambda)) \) have the projective resolution

\[
\cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0
\]

then

\[
\cdots \rightarrow \text{Hom}_\Lambda(\mathcal{P}, P_2) \rightarrow \text{Hom}_\Lambda(\mathcal{P}, P_1) \rightarrow \text{Hom}_\Lambda(\mathcal{P}, P_0) \rightarrow \text{Hom}_\Lambda(\mathcal{P}, M) \rightarrow 0
\]

is the projective resolution of \( X \in \text{mod-} \epsilon(\Lambda) \) where \( X = \text{Hom}_\Lambda(\mathcal{P}, M) \). Each \( \text{Hom}_\Lambda(\mathcal{P}, P_i) \), \( i \geq 0 \) are projective \( \epsilon(\Lambda) \)-modules. Any nonsplit exact sequence

\[
0 \rightarrow X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow 0
\]

in \( \text{mod-} \epsilon(\Lambda) \) can be carried into \( \text{mod-} \Lambda \) by the categorical equivalence, i.e.

\[
0 \rightarrow \Delta X_1 \rightarrow \Delta X_2 \rightarrow \Delta X_3 \rightarrow 0
\]

is exact in \( \text{mod-} \Lambda \) where each \( X_i \cong \text{Hom}_\Lambda(\mathcal{P}, \Delta X_i) \). By the abuse of terminology, we say \( M \in \text{mod-} \Lambda \) is the corresponding module to \( X \in \text{mod-} \epsilon(\Lambda) \) if \( \text{Hom}_\Lambda(\mathcal{P}, M) \cong X \).

To make distinction, we use the following notation, for any \( X \in \text{mod-} \epsilon(\Lambda) \), we denote the corresponding \( \mathcal{B}(\Lambda) \)-filtered \( \Lambda \)-module by \( \Delta X \) i.e. \( \text{Hom}_\Lambda(\mathcal{P}, \Delta X) \cong X \).

We exploit this idea when comparing various homological dimensions based on the projective dimensions of the modules.

**Proposition 2.42.** If \( M \in \text{mod-} \Lambda \) has finite projective dimension \( d \geq 2 \), then the projective dimension of \( M' = \text{Hom}_\Lambda(\mathcal{P}, \Omega^2(M)) \) in \( \text{mod-} \epsilon(\Lambda) \) is \( d - 2 \). If \( M \in \text{mod-} \Lambda \) has infinite projective dimension, then the projective dimension of \( M' = \text{Hom}_\Lambda(\mathcal{P}, \Omega^2(M)) \) in \( \text{mod-} \epsilon(\Lambda) \) is infinite.
Proof. We consider the projective resolution of $M$ in mod-$\Lambda$

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0.$$ 

By proposition 2.33 each $P_i$ and $\Omega^i(M)$ where $2 \leq i \leq d$ have $\mathcal{B}(\Lambda)$-filtration. The evaluation functor $\text{Hom}_\Lambda(\mathcal{P}, -)$ is exact, therefore the projective resolution of $\Omega^2(M)$ in mod-$\Lambda$

$$\cdots \to P_3 = P(\Omega^3(M)) \to P_2 = P(\Omega^2(M)) \to \Omega^2(M) \to 0$$

can be carried into mod-$\varepsilon(\Lambda)$ as

$$\cdots \to \text{Hom}_\Lambda(\mathcal{P}, P(\Omega^3(M))) \to \text{Hom}_\Lambda(\mathcal{P}, P(\Omega^2(M))) \to \text{Hom}_\Lambda(\mathcal{P}, \Omega^2(M)) \to 0 \quad (2.20)$$

By the categorical equivalence 2.19, $\text{Hom}_\Lambda(\mathcal{P}, P(\Omega^i(M)))$ where $i \geq 2$ is projective $\varepsilon(\Lambda)$-module. By the remark 2.41, 2.20 is the projective resolution of $M'$ where $\Omega^2(M)$ is the corresponding module, i.e. $M' \cong \text{Hom}_\Lambda(\mathcal{P}, \Omega^2(M))$. We conclude that

$$\text{p.dim}_{\varepsilon(\Lambda)} M' = \text{p.dim}_{\varepsilon(\Lambda)} \text{Hom}_\Lambda(\mathcal{P}, \Omega^2(M)) = \text{p.dim}_\Lambda \Omega^2(M) = \text{p.dim}_\Lambda M - 2 = d - 2.$$ 

If $d$ is infinite, then $\Omega^2(M)$ has also infinite projective dimension. By 2.20, we get $\text{p.dim}_\Lambda M = \text{p.dim}_\Lambda \Omega^2(M) = \text{p.dim}_{\varepsilon(\Lambda)} \text{Hom}_\Lambda(\mathcal{P}, \Omega^2(M)) = \infty$. □

Proposition 2.43. If $\Lambda$ is a cyclic Nakayama algebra with finite global dimension $d \geq 2$, then $\text{gldim } \Lambda = \text{gldim } \varepsilon(\Lambda) + 2$.

Proof. Because $\text{gldim } \Lambda \geq 2$, we can express it by the second syzygies, i.e.

$$\text{gldim } \Lambda = \sup \{ \text{p.dim } M | M \in \text{mod-}\Lambda \} = \sup \{ \text{p.dim } \Omega^2(M) | M \in \text{mod-}\Lambda \} + 2.$$ 

By proposition 2.32, $\Omega^2(M)$ is a $\mathcal{B}(\Lambda)$-filtered module, therefore

$$\text{gldim } \Lambda = 2 + \sup \{ \text{p.dim } \Delta X | \Delta X \in \text{Filt}(\mathcal{B}(\Lambda)), \Delta X \cong \Omega^2(M) \in \text{mod-}\Lambda \} .$$

By the categorical equivalence $\text{Hom}_\Lambda(\mathcal{P}, \Delta X) \cong X$, we get

$$\text{gldim } \Lambda = 2 + \sup \{ \text{p.dim}_{\varepsilon(\Lambda)} X | \Delta X \in \text{Filt}(\mathcal{B}(\Lambda)), \text{Hom}_\Lambda(\mathcal{P}, \Delta X) = X \} = 2 + \sup \{ \text{p.dim}_{\varepsilon(\Lambda)} X | X \in \text{mod-}\varepsilon(\Lambda) \} = 2 + \text{gldim } \varepsilon(\Lambda).$$

□

It turns out that the syzygy filtered algebra is again a Nakayama algebra which allows us to make the mathematical induction on the various homological dimensions of the original algebra by exploiting the idea in the proof of proposition 2.43 (see proposition 2.42).

Proposition 2.44. If $\text{gldim } \Lambda > 2$, then $\varepsilon(\Lambda)$ is a Nakayama algebra of rank $r$ where $r$ is the number of irredundant relations defining $\Lambda$.
Proof. When global dimension of $\Lambda$ is two, by the reduction 2.43, global dimension of $\varepsilon(\Lambda)$ is zero. In the last section, we show that it is a semisimple algebra (see proposition 4.2). So we need to exclude this case.

When the global dimension is greater than two, by the reduction 2.43, we get $\text{gldim} \, \varepsilon(\Lambda) \geq 1$ which shows the syzygy filtered algebra is not semisimple. By the construction 2.19, the category of $\varepsilon(\Lambda)$-modules is equivalent to the category of $\mathcal{B}(\Lambda)$-filtered modules. By proposition 2.36, $\text{Filt}(\mathcal{B}(\Lambda))$ is wide subcategory of mod-$\Lambda$. Since $\Lambda$ is Nakayama algebra, the indecomposable objects of $\text{Filt}(\mathcal{B}(\Lambda))$ are uniserial. By proposition 2.30, each $\mathcal{B}(\Lambda)$-filtered module has a unique filtration, therefore $\varepsilon(\Lambda)$ is a Nakayama algebra.

The number of simple objects of $\text{Filt}(\mathcal{B}(\Lambda))$ is equal to the cardinality of the base set which is $r$. Because $\varepsilon(\Lambda)$ is an endomorphism algebra of $r$-many $\mathcal{B}(\Lambda)$-filtered projective modules, we obtain $\text{rank} \, \varepsilon(\Lambda) = r$. □

Proposition 2.45. If global dimension of $\Lambda$ is infinite, then

i) global dimension of $\varepsilon(\Lambda)$ is infinite,

ii) $\varepsilon(\Lambda)$ is cyclic.

Proof. There exists $\Lambda$-module $M$ with $p \text{dim} \, M = \infty$ by the assumption, hence $p \text{dim}_{\varepsilon(\Lambda)} \text{Hom}_{\Lambda}(\mathcal{P}, \Omega^2(M))$ is infinite by proposition 2.42. So global dimension of the syzygy filtered algebra is unbounded. By proposition 2.44, it is a Nakayama algebra. Hence the underlying quiver has to be cyclic, because linear Nakayama algebras are representation-directed and have finite global dimension. □

We want to justify why our approach does not coincide with [CY14] in general. If $\Lambda$ is not selfinjective, then passing from $\Lambda$ to $\varepsilon(\Lambda)$ reduces projective dimensions of modules exactly by two. Also, by proposition 2.43, our construction works for the case of finite global dimension.

Remark 2.46. $\varepsilon(\Lambda)$ is Nakayama algebra so in principle we can express its defining relations. They are managed by $\mathcal{B}(\Lambda)$ in the following way. If $\text{Hom}_{\Lambda}(\mathcal{P}, \mathcal{P})$ is projective module which is minimal in mod-$\varepsilon(\Lambda)$, then it gives the relation

$$
\beta_{i_j + c_{i_j} - 1} \circ \beta_{i_j + c_{i_j} - 2} \circ \cdots \circ \beta_{i_j + 1} \circ \beta_{i_j} = 0
$$

where $i_j$ is the index of the top of $\text{Hom}_{\Lambda}(\mathcal{P}, \mathcal{P})$ and $c_{i_j}$ is the length of $\text{Hom}_{\Lambda}(\mathcal{P}, \mathcal{P})$ in mod-$\varepsilon(\Lambda)$ or equivalently by the remark 2.31, $c_{i_j}$ is the number of composition factors of $\mathcal{P}$ with respect to $\mathcal{B}(\Lambda)$-filtration because of the categorical equivalence 2.19.

Remark 2.47. [Sen21, remark 3.4.1] Let rank $\Lambda = N$. Then the following statements are equivalent:

1) $\Lambda$ is selfinjective.
2) All projective modules are injective.
3) All indecomposable projective modules have the same length.
4) Non-isomorphic projectives have non-isomorphic socle.
5) All radicals of indecomposable projective modules have the same length.
6) Every indecomposable projective $\Lambda$-module is minimal.
7) Every indecomposable projective module is projective-injective.
8) Each class of projectives (def 2.13) has exactly one object.

We prove the statement i) of Theorem A.

**Proposition 2.48.** For any cyclic Nakayama algebra $\Lambda$, $\Lambda \cong \varepsilon(\Lambda)$ if and only if $\Lambda$ is selfinjective.

**Proof.** Assume that $\Lambda$ is selfinjective. Then we get

$$\varepsilon(\Lambda) := \text{End} \left( \bigoplus_{S \in S'(\Lambda)} P(S) \right) = \text{End} \left( \bigoplus_{1 \leq i \leq N} P_i \right) \cong \Lambda,$$

since every projective $\Lambda$-module is minimal by the remark 2.47 6) which implies that the top set $S'(\Lambda)$ contains all simple $\Lambda$-modules.

For the other direction, assume that $\Lambda \cong \varepsilon(\Lambda)$. By the construction 2.19, $\text{Filt}(B(\Lambda)) \cong \text{mod-}\Lambda$. Hence the number of non-isomorphic simple $\Lambda$-modules has to be equal to the number of non-isomorphic $B(\Lambda)$-filtered simple modules. By the last statement 6) of proposition 2.17, we conclude that $S(\Lambda) = S'(\Lambda) = \{S_1, S_2, \ldots, S_N\}$. Therefore, each non-isomorphic projective module have non-isomorphic socle. By the remark 2.47 4), $\Lambda$ is selfinjective. □

Now we can add the equivalent statements below to the list given in the remark 2.47.

1) $\Lambda$ is selfinjective.
2) $\Lambda \cong \varepsilon(\Lambda)$.
3) $S(\Lambda) = S'(\Lambda) = B(\Lambda) = \{S_1, S_2, \ldots, S_N\}$.
4) $r = N = |S'(\Lambda)| = |S(\Lambda)| = |B(\Lambda)|$.

### 2.7. Higher syzygy filtered algebras.
Based on the construction of the syzygy filtered algebras 2.38, we can construct higher syzygy filtered algebras recursively.

**Definition 2.49.** Let $\Lambda$ be a cyclic Nakayama algebra and $\varepsilon(\Lambda)$ be its syzygy filtered algebra. $n$-th syzygy filtered algebra $\varepsilon^n(\Lambda)$ is the endomorphism algebra of direct sum of projective covers of simple $\varepsilon^{n-1}(\Lambda)$-modules which are elements of $S'(\varepsilon^{n-1}(\Lambda))$, i.e.

$$\varepsilon^n(\Lambda) := \text{End}_{\varepsilon^{n-1}(\Lambda)} \left( \bigoplus_{S \in S'(\varepsilon^{n-1}(\Lambda))} P(S) \right)$$

provided that $\varepsilon^{n-1}(\Lambda)$ is a cyclic Nakayama algebra.

We want to explain the reason behind the requirement that $\varepsilon^{n-1}(\Lambda)$ is cyclic. To apply the construction in the definition 2.38, the socle, top and base sets of $\varepsilon^{n-1}(\Lambda)$ have to be well-defined, which is true only for cyclic Nakayama algebras. The statements 3) and 5) of proposition 2.17 is not true for linear Nakayama algebras because projective-injective module $P_1$ cannot have $B(\Lambda)$-filtered submodule which should arise as the second syzgy of the simple projective module. Obviously it is not possible.

We can interpret $\text{mod-}\varepsilon^n(\Lambda)$ in terms of $B(\varepsilon^{n-1}(\Lambda))$-filtered $\varepsilon^{n-1}(\Lambda)$ modules as we discussed in the subsection 2.5 where $n = 1$ and $\varepsilon^0(\Lambda) = \Lambda$. Because of the cyclicity assumption on $\varepsilon^{n-1}(\Lambda)$, all $S(\varepsilon^{n-1}(\Lambda)), S'(\varepsilon^{n-1}(\Lambda)), B(\varepsilon^{n-1}(\Lambda)), \text{Filt}(B(\varepsilon^{n-1}(\Lambda)))$ are well-defined and the results can be carried in this set up. In other words, the $n$-th
syzygy filtered algebra \( \varepsilon^n(\Lambda) \) is the syzygy filtered algebra of the cyclic Nakayama algebra \( \varepsilon^{n-1}(\Lambda) \), that is

\[
\varepsilon^n(\Lambda) \cong \varepsilon(\varepsilon^{n-1}(\Lambda)).
\]  
(2.21)

If we denote projective modules in \( \mathcal{F}ilt(\varepsilon^{n-1}(\Lambda)) \) by \( \mathcal{P}_{\varepsilon^{n-1}(\Lambda)} \), i.e.

\[
\mathcal{P}_{\varepsilon^{n-1}(\Lambda)} = \bigoplus_{S \in \mathcal{S}(\varepsilon^{n-1}(\Lambda))} P(S),
\]

then we see that \( \text{mod-} \mathcal{P}_{\varepsilon^{n-1}(\Lambda)} \cong \mathcal{F}ilt(\mathcal{B}(\varepsilon^{n-1}(\Lambda))) \) by proposition 2.40, and by the definition of syzygy filtered algebra, \( \varepsilon^n(\Lambda) \) is the endomorphism algebra of \( \mathcal{P}_{\varepsilon^{n-1}(\Lambda)} \). Therefore, we get the diagram

\[
\text{mod-} \varepsilon^{n-1}(\Lambda) \xrightarrow{\text{Hom}(\mathcal{P}_{\varepsilon^{n-1}(\Lambda)}, -)} \text{mod-} \varepsilon^n(\Lambda)
\]

\[
\sim \xrightarrow{\cong} \mathcal{F}ilt(\mathcal{B}(\varepsilon^{n-1}(\Lambda)))
\]

(2.22)

The restriction of \( \text{Hom}_{\varepsilon^{n-1}(\Lambda)}(\mathcal{P}_{\varepsilon^{n-1}(\Lambda)}, -) \) onto \( \mathcal{F}ilt(\mathcal{B}(\varepsilon^{n-1}(\Lambda))) \) gives the equivalence of categories \( \text{mod-} \varepsilon^n(\Lambda) \) and \( \mathcal{F}ilt(\mathcal{B}(\varepsilon^{n-1}(\Lambda))) \). Because of the recursive nature of the construction (see 2.21) we do not want to take the opposite algebra back and forth. We can view all the modules as right modules. We can describe modules over the higher syzygy filtered algebras in the following way. \( \varepsilon^n(\Lambda) = \text{End}_{\varepsilon^{n-1}(\Lambda)} \mathcal{P}_{\varepsilon^{n-1}(\Lambda)} \) is finite dimensional algebra and \( \mathcal{P}_{\varepsilon^{n-1}(\Lambda)} \) is left \( \varepsilon^n(\Lambda) \)-module. If \( X \) is right \( \varepsilon^{n-1}(\Lambda) \)-module, then \( \text{Hom}_{\varepsilon^{n-1}(\Lambda)}(\mathcal{P}_{\varepsilon^{n-1}(\Lambda)}, X) \) is right \( \varepsilon^n(\Lambda) \)-module. Briefly we have:

i) The simple objects of \( \mathcal{F}ilt(\mathcal{B}(\varepsilon^{n-1}(\Lambda))) \) are the elements of the base set \( \mathcal{B}(\varepsilon^{n-1}(\Lambda)) \), therefore simple \( \varepsilon^n(\Lambda) \)-modules are of the form \( \text{Hom}_{\varepsilon^{n-1}(\Lambda)}(\mathcal{P}_{\varepsilon^{n-1}(\Lambda)}, \Delta_i) \) where \( \Delta_i \in \mathcal{B}(\varepsilon^{n-1}(\Lambda)) \).

ii) Projective \( \varepsilon^n(\Lambda) \)-modules are of the form \( \text{Hom}_{\varepsilon^{n-1}(\Lambda)}(\mathcal{P}_{\varepsilon^{n-1}(\Lambda)}, P) \) where \( P \) is \( \mathcal{B}(\varepsilon^{n-1}(\Lambda)) \)-filtered projective.

iii) In general, any \( X \in \text{mod-} \varepsilon^{n-1}(\Lambda) \) is isomorphic to \( \text{Hom}_{\varepsilon^{n-1}(\Lambda)}(\mathcal{P}_{\varepsilon^{n-1}(\Lambda)}, X') \) for some \( X' \in \mathcal{F}ilt(\mathcal{B}(\varepsilon^{n-1}(\Lambda))) \).

2.8. **Relationship between \( \Lambda \) and \( \varepsilon^n(\Lambda) \)**. One can construct the higher syzygy filtered algebras directly from the original algebra. Just to illustrate the idea, we focus on the case \( n = 2 \). Let \( \Lambda \) and \( \varepsilon(\Lambda) \) be cyclic Nakayama algebras (see 2.45, 4.11 when \( \varepsilon(\Lambda) \) is cyclic). Consider the following diagram:

\[
\text{mod-} \Lambda \xrightarrow{\text{Hom}(\mathcal{P}_{\Lambda}, -)} \text{mod-} \varepsilon(\Lambda) \xrightarrow{\text{Hom}(\mathcal{P}_{\varepsilon(\Lambda)}, -)} \text{mod-} \varepsilon^2(\Lambda)
\]

\[
\sim \xrightarrow{\cong} \mathcal{F}ilt(\mathcal{B}(\Lambda)) \xrightarrow{\sim} \mathcal{F}ilt(\mathcal{B}(\varepsilon(\Lambda)))
\]

(2.23)
**Proposition 2.50.** The composition \( \text{Hom}_{\varepsilon(\Lambda)}(\mathcal{P}_{\varepsilon(\Lambda)}, -) \circ \text{Hom}_{\Lambda}(\mathcal{P}, -) : \text{mod-} \varepsilon^2(\Lambda) \rightarrow \text{mod-} \varepsilon^2(\Lambda) \) is equivalent to
\[
\text{(2.24)} \quad \text{Hom}_{\Lambda}(\mathcal{P}', -) : \text{mod-} \Lambda \rightarrow \text{mod-} \varepsilon^2(\Lambda)
\]
where
\[
\text{(2.25)} \quad \mathcal{P}' = \bigoplus_{S \in S'(\Lambda)} P(\Omega^2(S)) = \bigoplus_{S \in \text{top-}P \atop P \in \text{mod-} \Lambda, \text{minimal}} P(\Omega^4(S))
\]
and \( S \) is top of minimal projective \( \Lambda \)-module in the right hand side.

**Proof.** We proved that \( \text{Filt}(\mathcal{B}(\Lambda)) \) is equivalent to \( \text{mod-} \varepsilon(\Lambda) \) via the evaluation functor \( \text{Hom}_{\Lambda}(\mathcal{P}_{\Lambda}, -) \) and by the same reason \( \text{Filt}(\mathcal{B}(\varepsilon(\Lambda))) \) is equivalent to \( \text{mod-} \varepsilon^2(\Lambda) \) via the evaluation functor \( \text{Hom}_{\varepsilon(\Lambda)}(\mathcal{P}_{\varepsilon(\Lambda)}, -) \). We claim that any element of the base set \( \mathcal{B}(\varepsilon(\Lambda)) \) can be realized as the fourth syzygies of simple \( \Lambda \)-modules which can be interpreted as the generalization of proposition 2.28.

\( M \in \mathcal{B}(\varepsilon(\Lambda)) \) if and only if \( M \cong \Omega^2(S') \) in \( \text{mod-} \varepsilon(\Lambda) \) and \( S' \) is top of minimal projective module \( P \) by propositions 2.20 and 2.28. Since \( M \cong \Omega^2(S') \), we have the exact sequences
\[
0 \rightarrow M \rightarrow P(\Omega^1(S')) \rightarrow \Omega^1(S') \rightarrow 0 \\
0 \rightarrow \Omega^1(S') \rightarrow P(S') \rightarrow S' \rightarrow 0
\]
in \( \text{mod-} \varepsilon(\Lambda) \). By the categorical equivalence between \( \text{Filt}(\mathcal{B}(\Lambda)) \) and \( \text{mod-} \varepsilon(\Lambda) \) via the evaluation functor, the following sequences are exact in \( \text{mod-} \Lambda \)
\[
0 \rightarrow \Delta M \rightarrow \Delta P(\Omega^1(S')) \rightarrow \Delta \Omega^1(S') \rightarrow 0 \\
0 \rightarrow \Delta \Omega^1(S') \rightarrow \Delta P(S') \rightarrow \Delta S' \rightarrow 0
\]
where \( \text{Hom}_{\Lambda}(\mathcal{P}, \Delta M) \cong M, \text{Hom}_{\Lambda}(\mathcal{P}, \Delta S') \cong S', \text{Hom}_{\Lambda}(\mathcal{P}, \Delta \Omega^1(S')) \cong \Omega^1(S'), \text{Hom}_{\Lambda}(\mathcal{P}, \Delta P(\Omega^1(S'))) \cong P(\Omega^1(S')) \) and \( \text{Hom}_{\Lambda}(\mathcal{P}, \Delta P(S')) \cong P(S') \).

\( S' \) is a simple \( \varepsilon(\Lambda) \)-module, so the corresponding module \( \Delta S' \in \text{mod-} \Lambda \) is an element of the base set \( \mathcal{B}(\Lambda) \) by the categorical equivalence 2.19 and the remark 2.41. On the other hand, there exists a simple \( \Lambda \)-module \( S \) which is a top of a minimal projective \( \Lambda \)-module such that \( \Omega^2(S) \cong \Delta S' \) by the same argument. As a result \( \Omega^4(S) \cong \Omega^2(\Delta S') \cong \Delta M \) in \( \text{mod-} \Lambda \). By the functor \( \text{Hom}_{\Lambda}(\mathcal{P}', -) \) introduced in 2.24, simple \( \varepsilon^2(\Lambda) \)-modules are of the form
\[
\text{(2.26)} \quad \text{Hom}_{\Lambda}(\mathcal{P}', \Omega^4(S)) = \text{Hom}_{\Lambda}(\mathcal{P}', \Delta M).
\]

Let \( \text{Filt}(\mathcal{B}^2(\Lambda)) \) denote the category of \( \Lambda \)-modules which are filtered by the fourth syzygies of simple \( \Lambda \)-modules. By the definition 2.49 we have
\[
\varepsilon^2(\Lambda) = \text{End}_{\varepsilon(\Lambda)} \left( \bigoplus_{S \in \mathcal{S}^2(\varepsilon(\Lambda))} P(S) \right)
\]
which is equivalent to
\[
\varepsilon^2(\Lambda) = \End_{\varepsilon(\Lambda)} \left( \bigoplus_{M \in \mathcal{B}(\varepsilon(\Lambda))} P(M) \right)
\]
where top \( M \in S'(\varepsilon(\Lambda)) \). By the remark 2.41 we can lift mod-\(\varepsilon(\Lambda)\) to mod-\(\Lambda\), i.e.
\[
\varepsilon^2(\Lambda) = \End_{\Lambda} \left( \bigoplus_{\Delta M \in \text{mod-} \Lambda, \text{Hom}_{\Lambda}(\mathcal{P}, \Delta M) \cong M} P(\Delta M) \right)
\]
where \( \text{Hom}_{\Lambda}(\mathcal{P}, \Delta M) \cong M \in \mathcal{B}(\varepsilon(\Lambda)) \). By the result 2.26, any \( \Delta M \) is the fourth syzygy of simple \( \Lambda \)-module, so
\[
\varepsilon^2(\Lambda) = \End_{\Lambda} \left( \bigoplus_{S \in \text{mod-} \Lambda, \Omega^4(S) \cong \Delta M} P(\Omega^4(S)) \right)
\]
By the definition of \( \mathcal{B}^2(\Lambda) \)-filtered modules, we conclude that
\[
\varepsilon^2(\Lambda) = \End_{\Lambda} \left( \bigoplus_{N \in \mathcal{B}^2(\Lambda)} P(N) \right).
\]
\[\square\]

Therefore the diagram 2.23 can be extended into

where \( \text{Filt}(\mathcal{B}^2(\Lambda)) \cong \text{mod-} \varepsilon^2(\Lambda) \) via the functor \( \text{Hom}_{\Lambda}(\mathcal{P}', -) \).

2.9. Examples. We want to give examples to make the concepts we have introduced so far concrete.
Example 2.51. Let $\Lambda$ be cyclic Nakayama algebra with rank $\Lambda = 5$ given by the relations

$$
\begin{align*}
\alpha_5\alpha_4\alpha_3\alpha_2\alpha_1 &= 0 \\
\alpha_1\alpha_5\alpha_4\alpha_3\alpha_2 &= 0 \\
\alpha_3\alpha_2\alpha_1\alpha_5\alpha_4 &= 0 \\
\alpha_4\alpha_3\alpha_2\alpha_1\alpha_5 &= 0.
\end{align*}
$$

The corresponding Kupisch series are $(5, 5, 6, 6, 6)$. All projective modules are minimal except $P_5$. The socle set is $\{S_1, S_3, S_4, S_5\}$, the top set is $\{S_1, S_2, S_4, S_5\}$. The base set is

$$
B(\Lambda) = \left\{S_1, S_2, S_3, S_4, S_5\right\}.
$$

$P_1$, $P_2$, $P_4$ and $P_5$ are $B(\Lambda)$-filtered. Therefore the Kupisch series of $\varepsilon(\Lambda)$ is $(4, 4, 5, 5)$ where the simple modules are $\Delta_1 = \text{Hom}(P, S_1)$, $\Delta_2 = \text{Hom}(P, S_2)$, $\Delta_3 = \text{Hom}(P, S_4)$, $\Delta_4 = \text{Hom}(P, S_5)$ and the rank is 4. We can apply the construction again. Algebra with Kupisch series $(4, 4, 5, 5)$ have the relations

$$
\begin{align*}
\alpha_4\alpha_3\alpha_2\alpha_1 &= 0 \\
\alpha_1\alpha_4\alpha_3\alpha_2 &= 0 \\
\alpha_3\alpha_2\alpha_1\alpha_4 &= 0
\end{align*}
$$

The corresponding base set is $\left\{\Delta_1, \Delta_2, \Delta_3, \Delta_4\right\}$. The filtered projective modules are $P_1$, $P_2$ and $P_4$. Therefore the Kupisch series of $\varepsilon^2(\Lambda)$ is $(3, 3, 4)$ where the simple modules are of the form

$$
\begin{align*}
\mathcal{E}_1 &= \text{Hom}(P, \Delta_1), \\
\mathcal{E}_2 &= \text{Hom}(P, \Delta_2), \\
\mathcal{E}_3 &= \text{Hom}(P, \Delta_3).
\end{align*}
$$

The relations are

$$
\begin{align*}
\alpha_3\alpha_2\alpha_1 &= 0 \\
\alpha_1\alpha_3\alpha_2 &= 0
\end{align*}
$$

so the base set is $\left\{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3\right\}$. The Kupisch series of $\varepsilon^3(\Lambda)$ is $(2, 2)$ which is selfinjective.

We want to illustrate the characterization of selfinjective algebras 2.47 here, the relations of selfinjective algebra given by $(2, 2)$ is $\alpha_2\alpha_1 = 0$ and $\alpha_1\alpha_2 = 0$. The base set becomes $\{S_1, S_2\}$ which is the simple modules of $\varepsilon^3(\Lambda)$. Therefore all projective modules have $B(\Lambda)$-filtration and $\varepsilon^3(\Lambda) \cong \varepsilon^4(\Lambda)$. 

Example 2.52. We point out that our method is different than the one given in [CY14]. Consider the algebra with Kupisch series $(4, 3, 3, 4, 3, 3)$. The relations are
\[
\begin{align*}
\alpha_4\alpha_3\alpha_2 &= 0 \\
\alpha_5\alpha_4\alpha_3 &= 0 \\
\alpha_1\alpha_6\alpha_5 &= 0 \\
\alpha_2\alpha_1\alpha_6 &= 0
\end{align*}
\]
The base set is
\[
\{S_5, S_6, S_7, S_3\}.
\]
Therefore $P_2, P_3, P_5, P_6$ have $B(\Lambda)$-filtration, and the Kupisch series of $\varepsilon(\Lambda)$ is $(2, 2, 2, 2)$ which we get in the first step as opposed to the one in [CY14].

Example 2.53. Let $N = 5$ and the relations are $\alpha_3\alpha_2 = 0, \alpha_1\alpha_5 = 0$. This determines minimal projectives directly which are $P_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $P_5 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$. $P_3$ and $P_4$ contain $P_5$ as a submodule, and $P_1$ contain $P_2$ as a submodule. The complete list of representatives of projective modules is
\[
P_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, P_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, P_3 = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 1 \end{bmatrix}, P_4 = \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix}, P_5 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}
\].
The corresponding Kupisch series is $(3, 2, 4, 3, 2)$. Notice that there are two different socles of projective modules $S_3$ and $S_1$. The base set is $\{P_2, P_4\}$, therefore $\varepsilon(\Lambda)$ is $A_1 \oplus A_1$.

Example 2.54. We want to give examples with finite global dimension. Consider algebra given by $(4, 3, 2, 4, 3, 2, 4, 3, 2)$. The relations are $\alpha_4\alpha_3 = 0, \alpha_7\alpha_6 = 0, \alpha_1\alpha_9 = 0$. The base set is $\{P_5, P_8, P_2\}$, hence $\varepsilon(\Lambda)$ is semisimple algebra $A_1 \oplus A_1 \oplus A_1$.

We view hereditary algebra of type $A_n$ as linear Nakayama algebra, given by the single relation $\alpha_n = 0$.

Example 2.55. Let $\Lambda$ be given by Kupisch series $(3, 3, 3, 4, 3, 3, 3, 3, 2)$. Relations are
\[
\begin{align*}
\alpha_3\alpha_2\alpha_1 &= 0 \\
\alpha_4\alpha_3\alpha_2 &= 0 \\
\alpha_5\alpha_4\alpha_3 &= 0 \\
\alpha_7\alpha_6\alpha_5 &= 0 \\
\alpha_8\alpha_7\alpha_6 &= 0 \\
\alpha_9\alpha_8\alpha_1 &= 0 \\
\alpha_1\alpha_9 &= 0
\end{align*}
\]
The base set is
\[
\{S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8, S_9\}.
\]
The syzygy filtered algebra is given by Kupisch series \((2, 2, 3, 2, 2, 3, 2)\). If we apply the construction once more, \(\varepsilon^2(\Lambda)\) is isomorphic to linear Nakayama algebra \((2, 2, 1) \oplus A_2\).

3. Applications of Syzygy Filtered Algebras

In each subsection, we will define a homological dimension and prove Theorem A in parts. We start with \(\varphi\)-dimension.

3.1. Results about \(\varphi\)-dimension. Let \(A\) be an artin algebra. Let \(K_0\) be the abelian group generated by all symbols \([X]\), where \(X\) is a finitely generated \(A\)-module, modulo the relations

1) \([A_1] = [A_2] + [A_3]\) if \(A_1 \cong A_2 \oplus A_3\).
2) \([P] = 0\) if \(P\) is projective.

Then \(K_0\) is the free abelian group generated by the isomorphism classes of indecomposable finitely generated non-projective \(A\)-modules. For any finitely generated \(A\)-module \(M\), let \(L(M) := [\Omega M]\) where \(\Omega\) is the first syzygy of \(M\). Since \(\Omega\) commutes with direct sums and takes projective modules to zero this gives a homomorphism \(L : K_0 \to K_0\).

For every finitely generated \(A\)-module \(M\), let \(\langle \text{add}\ M \rangle\) denote the subgroup of \(K_0\) generated by additive generator of \(M\), i.e. all the indecomposable summands of \(M\). \(\langle \text{add}\ M \rangle\) is a free abelian group because it is a subgroup of free abelian group \(K_0\). In other words, if \(M = \bigoplus_{i=1}^m M_i^{n_i}\) then \(\langle \text{add}\ M \rangle = \langle \{[M_i]\}_{i=1}^m \rangle\) and \(L(\langle \text{add}\ M \rangle) = \langle \{\Omega^i M_i\}_{i=1}^m \rangle\).

Definition 3.1. For a given module \(M\) in \(\text{mod-}A\), \(\varphi\)-dimension of \(M\) is

\[
\varphi(M) := \min \{t \mid \text{rank } (L^t(\langle \text{add}\ M \rangle)) = \text{rank } (L^{t+j}(\langle \text{add}\ M \rangle)) \text{ for } j \geq 1\}.
\]

Notice that \(\varphi(M)\) is finite for each module \(M\), since the rank has to become stable at some step. For example, if projective dimension of \(M\) is finite, then \(\varphi(M) = \dim \text{proj } (M)\).

Definition 3.2. If \(A\) is an artin algebra, then \(\varphi\)-dimension of \(A\) is

\[
\varphi \dim A := \sup \{\varphi(M) \mid M \in \text{mod-}A\}.
\]

We collect some statements about \(\varphi\)-dimension, proofs and other properties can be found in [Sen21].

Remark 3.3. i) Let \(A\) be an artin algebra of finite representation type. Let \(\{M_1, \ldots, M_m\}\) be a complete set of representatives of isomorphism classes of indecomposable \(\Lambda\)-modules. Then \(\varphi \dim A = \varphi(\bigoplus_{i=1}^m M_i)\).

ii) If global dimension of algebra \(A\) is finite, then \(\text{gldim } A = \varphi \dim A\). Recall that \(\text{fin.dim } A = \sup \{p \dim X \mid p \dim X < \infty, X \in \text{mod-}A\}\), which implies \(\text{fin.dim } A \leq \varphi \dim A\).

We recall the result about small values of \(\varphi\)-dimension as stated in [Sen21].

Theorem 3.4. [Sen21, Thm 3.5.1] Let \(\Lambda\) be a cyclic Nakayama algebra of infinite global dimension. Then,

1) \(\varphi \dim \Lambda = 0\) if and only if \(\Lambda\) is selfinjective algebra.
2) \(\varphi \dim \Lambda \neq 1\).
3) \(\varphi \dim \Lambda = 2\) if and only if each element of the base set \(\mathcal{B}(\Lambda)\) are periodic modules, i.e. \(\Delta_i \cong \Omega^j(\Delta_i)\) for any \(1 \leq i \leq r\).
Now, we want to translate it into the syzygy filtered algebra set up. By proposition 2.48, we can replace the first statement by “\( \varphi \dim \Lambda = 0 \) if and only if \( \Lambda \cong \varepsilon(\Lambda) \).” For the third statement, we can apply the construction 2.19 and conclude that any simple \( \varepsilon(\Lambda) \)-module \( S_i \cong \text{Hom}_\Lambda(\mathcal{P}, \Delta_i) \) is a periodic module. Therefore for any simple \( \varepsilon(\Lambda) \)-module, \( P(S) \) is minimal projective, otherwise \( \text{rad} P(S) \) would be a projective module which is not periodic module. By the characterization 2.47, we conclude that \( \varepsilon(\Lambda) \) is selfinjective. We restate the result in the new terminology.

**Theorem 3.5.** Let \( \Lambda \) be a cyclic Nakayama algebra of infinite global dimension. Then,

1) \( \varphi \dim \Lambda = 0 \) if and only if \( \Lambda \) is selfinjective algebra if and only if \( \Lambda \cong \varepsilon(\Lambda) \).
2) \( \varphi \dim \Lambda \neq 1 \).
3) \( \varphi \dim \Lambda = 2 \) if and only if \( \varepsilon(\Lambda) \) is selfinjective and \( \Lambda \) is not.

Here we give some results on Nakayama algebras by using the syzygy filtered algebras. First we give the proof of the statement ii) of Theorem A.

**Theorem 3.6.** If \( \Lambda \) is a cyclic Nakayama algebra, then \( \varphi \dim \Lambda = \varphi \dim \varepsilon(\Lambda) + 2 \) provided that \( \varphi \dim \Lambda \geq 2 \) and \( \text{gldim} \Lambda = \infty \).

**Proof.** For any indecomposable \( \varepsilon(\Lambda) \)-module \( X \), there exists \( B(\Lambda) \)-filtered \( \Lambda \)-module \( \Delta X \) such that \( \text{Hom}_\Lambda(\mathcal{P}, \Delta X) = X \) by the remark 2.41. By the categorical equivalence 2.19 we get

\[
\varphi_{\varepsilon(\Lambda)}(X) = \varphi_\Lambda(\Delta X).
\]

By proposition 2.34, for any non-projective \( \Delta X \) there exists \( M \in \text{mod-} \Lambda \) such that \( \Omega^2(M) \cong \Delta X \). Therefore

\[
\varphi(M) = 2 + \varphi(\Omega^2(M)) = 2 + \varphi(\Delta X) = 2 + \varphi_{\varepsilon(\Lambda)}(X).
\]

We can take the supremum,

\[
\varphi \dim \Lambda = \sup \{ \varphi(M) | M \in \text{mod-} \Lambda \}
= \sup \{ \varphi(\Omega^2(M)) | M \in \text{mod-} \Lambda \} + 2
= \sup \{ \varphi(\Delta M) | \Delta M \in \text{Filt}(\mathcal{B}(\Lambda)) \} + 2
= \sup \{ \varphi(M) | M \in \text{mod-} \varepsilon(\Lambda) \} + 2
= \varphi \dim \varepsilon(\Lambda) + 2
\]

which proves the statement. \( \square \)

**Corollary 3.7.** If \( \Lambda \) is a cyclic Nakayama algebra, then

\[
\varphi \dim \Lambda = 2d + \varphi \dim \varepsilon^d(\Lambda)
\]

for some \( d \geq 1 \), provided that \( \text{gldim} \Lambda = \infty \) and \( \varphi \dim \varepsilon^{d-1}(\Lambda) \geq 2 \).

**Proof.** By proposition 2.44 each syzygy filtered algebra is Nakayama algebra. Moreover, by Theorem 3.6, each of them have infinite global dimension hence cyclic. Therefore
we can apply Theorem 3.6 iteratively to get reduction two in each step, i.e.
\[ \varphi \dim \Lambda = 2 + \varphi \dim \epsilon(\Lambda) = 4 + \varphi \dim \epsilon^2(\Lambda) = \ldots = 2(d - 1) + \varphi \dim \epsilon^{d-1}(\Lambda) \]

By the assumption \( \varphi \dim \epsilon^{d-1}(\Lambda) \geq 2 \), we can make one more reduction to get
\[ \varphi \dim \Lambda = 2d + \varphi \dim \epsilon^d(\Lambda). \]

\( \square \)

Now, we give the proof of Theorem C part i).

**Theorem 3.8.** If \( \Lambda \) is a cyclic Nakayama algebra of infinite global dimension, then there exist \( d \) such that \( \varphi \dim \epsilon^{d-1}(\Lambda) = 2 \) and \( \epsilon^d(\Lambda) \) is selfinjective.

**Proof.** By the corollary 3.7, we can apply the syzygy filtered algebra construction upto positive integer \( d \) such that \( 0 \leq \varphi \dim \epsilon^d(\Lambda) \leq 2 \), and \( \varphi \dim \epsilon^{d-1}(\Lambda) = 2 + \varphi \dim \epsilon^d(\Lambda) \geq 2 \). Therefore
\[ 0 \leq \varphi \dim \Lambda - 2d = \varphi \dim \epsilon^d(\Lambda) \leq 2. \]

We have three cases to examine.

Case i) \( \varphi \dim \Lambda = 2d \), which makes \( \varphi \dim \epsilon^d(\Lambda) = 0 \). By the characterization 2.47, \( \epsilon^d(\Lambda) \) is selfinjective algebra and \( \varphi \dim \epsilon^{d-1}(\Lambda) = 2 \).

Case ii) \( \varphi \dim \Lambda = 2d + 2 \), which makes \( \varphi \dim \epsilon^d(\Lambda) = 2 \). We can apply Theorem 3.6 to get \( \varphi \dim \epsilon^{d+1}(\Lambda) = 0 \), so it is selfinjective algebra by the remark 2.47.

Case iii) By the result 3.5 2), \( \varphi \)-dimension of cyclic Nakayama algebra cannot be one.

\( \square \)

The following result was proved in [Sen21]. We state it in terms of the syzygy filtered algebras and give a different proof based on the syzygy filtered algebra construction.

**Theorem 3.9.** [Sen21, Thm 5.1.1] Assume that global dimension of cyclic Nakayama algebra \( \Lambda \) is infinite. Then \( \varphi \)-dimension of \( \Lambda \) is even. Furthermore, the upper bound of \( \varphi \dim \Lambda \) is \( 2r \), where \( r = |\mathcal{B}(\Lambda)| \).

**Proof.** By Theorem 3.6, in each step \( \varphi \)-dimension reduces by two. By Theorem 3.8, there exist \( d \) such that \( \varphi \dim \epsilon^d(\Lambda) = 2 \), which makes \( \varphi \dim \Lambda = 2d + 2 \), an even number.

To prove the second part, observe that number of simple modules of \( \epsilon^i(\Lambda) \) is given by \( |\mathcal{B}(\epsilon^{i-1}(\Lambda))| \) which has the same cardinality with the irredundant system of relations defining \( \epsilon^{i-1}(\Lambda) \). If we set \( \varphi \dim \Lambda \) to \( 2d \), then by Theorem 3.8 \( \epsilon^d(\Lambda) \) is selfinjective and \( \varphi \dim \epsilon^{d-1}(\Lambda) = 2 \). Let \( r_i \) denote the rank of \( \epsilon^{i+1}(\Lambda) \) for \( i \geq 1 \). Therefore rank \( \epsilon^d(\Lambda) = r_{d-1} \) and each rank satisfies \( r_{i+1} \leq r_i - 1 \). If we add the inequalities we get \( r_{d-1} \leq r - (d - 1) \). The minimal possible value of \( r_{d-1} \) is one, therefore \( d \leq r \), which implies \( \varphi \dim \Lambda = 2d \leq 2r \). In particular, \( r \leq N - 1 \), in terms of the rank of \( \Lambda \), the upper bound is \( 2N - 2 \).
3.2. Results about finitistic dimension. First we state and prove results regarding possible small values of the finitistic dimension and φ-dimension. Then we state the reduction method for the finitistic dimension. We recall definition of the finitistic dimension of artin algebra A,

\[ \text{fin.dim } A := \sup \{ \text{p.dim } M | M \text{ is an } A\text{-module with } \text{p.dim } M < \infty \} . \]

We start with the following observation.

**Lemma 3.10.** If \( \varphi \text{dim } \Lambda = 2 \), then \( \varphi \text{dim } \Lambda - \text{fin.dim } \Lambda \leq 1 \).

**Proof.** Since \( \varphi \text{dim } \Lambda = 2 \), by the remark 2.47 \( \Lambda \) is not selfinjective and there exists a projective module \( P \) which is not minimal. Therefore, the exact sequence

\[ 0 \rightarrow \text{rad } P \rightarrow P \rightarrow \text{top } P \rightarrow 0 \]

implies that \( \text{p.dim } \text{top } P \geq 1 \) and as a result \( \text{fin.dim } \Lambda \geq 1 \). On the other hand, \( \text{fin.dim } \Lambda \leq \varphi \text{dim } \Lambda \) which is stated in the remark 3.3 part ii), we conclude that \( \varphi \text{dim } \Lambda - \text{fin.dim } \Lambda \leq 1 \). \( \square \)

Now we give the proof of the statement iii) of Theorem A.

**Proposition 3.11.** If \( \Lambda \) is a cyclic Nakayama algebra, then

\[ \text{fin.dim } \Lambda = \text{fin.dim } \text{ε}(\Lambda) + 2, \]

provided that \( \text{fin.dim } \Lambda \geq 2 \) and \( \text{gldim } \Lambda = \infty \).

**Proof.** Both of \( \Lambda \) and the syzygy filtered algebra are finite representation type, so there exists \( \text{ε}(\Lambda)\)-module \( X \) such that \( \text{p.dim } X = \text{fin.dim } \text{ε}(\Lambda) \). If \( \Delta X \in \text{mod- } \Lambda \) is the corresponding module i.e. \( \text{Hom}_\Lambda (\mathcal{P}, \Delta X) \cong X \), then \( \text{p.dim } \text{ε}(\Lambda) X = \text{p.dim } \Delta X \) by proposition 2.42. Therefore we get

\[ \text{fin.dim } \lambda = \sup \{ \text{p.dim } M | M \in \text{mod- } \Lambda, \text{p.dim } M < \infty \} \]

(3.1)

\[ = \sup \{ \text{p.dim } \Omega^2(M) | M \in \text{mod- } \Lambda, \text{p.dim } M < \infty \} + 2. \]

(3.2)

Any \( \Omega^2(M) \) has \( \text{B}(\Lambda)\)-filtration by proposition 2.34 which implies

\[ \text{fin.dim } \Lambda = 2 + \sup \{ \text{p.dim } \Delta X | \Delta X \in \text{Filt}(\mathcal{B}(\Lambda)), \text{p.dim } \Delta X < \infty \} \]

\[ = 2 + \sup \{ \text{p.dim } X | X \in \text{mod- } \text{ε}(\Lambda), \text{p.dim } X < \infty \} \]

\[ = 2 + \text{fin.dim } \text{ε}(\Lambda) \]

where \( \text{Hom}_\Lambda (\mathcal{P}, \Delta X) \cong X \).

We want to analyze the case of \( \text{fin.dim } \Lambda = 2 \). The reduction above implies that \( \text{fin.dim } \text{ε}(\Lambda) = 0 \). This forces that all the simple \( \text{ε}(\Lambda)\)-modules have periodic resolution i.e \( \Omega^i(S) \cong \Omega^{i+j}(S) \) for some \( i, j \) where \( S \) is simple module. This implies that the radical of any projective module \( P \)

\[ 0 \rightarrow \text{rad } P(S) \rightarrow P(S) \rightarrow S \rightarrow 0 \]

is non-projective. Therefore, every projective \( \text{ε}(\Lambda)\)-module is minimal which is equivalent to \( \text{ε}(\Lambda) \) is selfinjective by the remark 2.47. \( \square \)

We placed the condition on the lower bound of the finitistic dimension, otherwise we could not pass from 3.1 to 3.2. For instance, if \( \alpha_2 \alpha_1 = 0, \alpha_1 \alpha_3 \alpha_2 = 0 \) are relations of rank 3 algebra \( \Lambda \), then \( \text{fin.dim } \Lambda = 1 \) and \( \text{fin.dim } \text{ε}(\Lambda) = 0 \), so the reduction is not two.
Corollary 3.12. If \( \Lambda \) is a cyclic Nakayama algebra, then
\[
\text{fin.dim } \Lambda = 2d + \text{fin.dim } \varepsilon^d(\Lambda)
\]
for any \( d \geq 1 \), provided that \( \text{gldim } \Lambda = \infty \) and \( \text{fin.dim } \varepsilon^{d-1}(\Lambda) \geq 2 \).

Proof. By proposition 2.44, the syzygy filtered algebra is cyclic Nakayama algebra and of infinite global dimension. Therefore we can apply propositions 2.44 and 3.11 iteratively to get reduction two in each step, i.e.
\[
\begin{align*}
\text{fin.dim } \Lambda &= 2 + \text{fin.dim } \varepsilon(\Lambda) \\
&= 4 + \text{fin.dim } \varepsilon^2(\Lambda) \\
&\vdots \\
&= 2(d-1) + \text{fin.dim } \varepsilon^{d-1}(\Lambda)
\end{align*}
\]
By the assumption \( \text{fin.dim } \varepsilon^{d-1}(\Lambda) \geq 2 \), we can make one more reduction to get
\[
\text{fin.dim } \Lambda = 2d + \text{fin.dim } \varepsilon^d(\Lambda).
\]
\( \square \)

Corollary 3.13. If \( \Lambda \) is cyclic Nakayama algebra of infinite global dimension, then there exists \( d \) such that
\[
1 \leq \text{fin.dim } \varepsilon^d(\Lambda) \leq 2.
\]

Proof. By Theorem 3.8, there exists \( d \) such that \( \varphi \text{ dim } \varepsilon^d(\Lambda) = 2 \) and \( \varepsilon^{d+1}(\Lambda) \) is selfinjective. If we apply lemma 3.10 to \( \varepsilon^d(\Lambda) \), then
\[
1 \leq \text{fin.dim } \varepsilon^d(\Lambda) \leq 2.
\]
\( \square \)

Now we can prove the statement \( \text{ix) } \) of Theorem A.

Theorem 3.14. If \( \Lambda \) is a cyclic Nakayama algebra then \( \varphi \text{ dim } \Lambda - \text{fin.dim } \Lambda \leq 1 \)

Proof. If global dimension of \( \Lambda \) is finite, statement is true, since \( \text{gl dim } \Lambda = \varphi \text{ dim } \Lambda = \text{fin.dim } \Lambda \). So, we assume that global dimension is infinite.

By Theorem 3.8, there exists \( d \) such that \( \varepsilon^d(\Lambda) \) is a selfinjective algebra but \( \varepsilon^{d-1}(\Lambda) \) is not. By the reduction 3.6 we get \( \varphi \text{ dim } \Lambda = 2d \), \( \varphi \text{ dim } \varepsilon^d(\Lambda) = 0 \) and \( \varepsilon^{d-1}(\Lambda) = 2 \).

By the corollary 3.13 we get
\[
1 \leq \text{fin.dim } \varepsilon^{d-1}(\Lambda) \leq 2.
\]
On the other hand we have \( 2 + \text{fin.dim } \varepsilon^k(\Lambda) = \text{fin.dim } \varepsilon^{k-1}(\Lambda) \) for any \( 1 \leq k \leq d - 1 \) because of the corollary 3.12. This implies
\[
1 + 2d - 2 \leq \text{fin.dim } \Lambda \leq 2 + 2d - 2 \Rightarrow \varphi \text{ dim } \Lambda - 1 \leq \text{fin.dim } \Lambda \leq \varphi \text{ dim } \Lambda.
\]
We obtain the desired inequality \( \varphi \text{ dim } \Lambda - \text{fin.dim } \Lambda \leq 1 \). \( \square \)
3.3. Results about Gorenstein dimension. We begin with definition of Gorenstein dimension. By using duality and representation type of Nakayama algebras, instead of injective dimension of $\Lambda$, we prefer to study projective dimensions of injective modules. Then we state and prove results regarding Gorenstein homological properties.

**Definition 3.15.** If projective resolution of injective modules are finite, algebra is called **Gorenstein**. Supremum of projective dimensions of injective modules is called **Gorenstein dimension**, and denoted by $\text{gor.dim.}$ The formulation is

$$\text{gor.dim.} \Lambda = \sup \{ \text{p.dim.} I \mid I \in \text{mod-}\Lambda \text{ is injective} \} < \infty.$$ 

First, we need possible small values of Gorenstein dimension. Then we show the reduction method.

**Lemma 3.16.** If $\Lambda$ is cyclic Nakayama algebra which is Gorenstein, then Gorenstein dimension cannot be one.

**Proof.** If $\Lambda$ is selfinjective, by definition, $\text{gor.dim.} \Lambda = 0$. If $\Lambda$ is not selfinjective, there exists a projective-injective module $PI$ such that its quotient is injective $I$ and the sequence

$$0 \to \text{soc} PI \to PI \to I \to 0$$

is exact. Since $\Lambda$ is cyclic, there is no simple projective module. Therefore $\text{p.dim.} I = 1 + \text{p.dim.} \text{soc} PI \geq 2$ which implies $\text{gor.dim.} \Lambda \neq 1$. $\square$

**Lemma 3.17.** Let $\Delta_i$ be an element of $\mathcal{B}(\Lambda)$ such that it is submodule of an indecomposable projective-injective $\Lambda$-module $PI$ and it has a proper submodule $X$, i.e.

$$X \hookrightarrow \Delta_i \hookrightarrow PI$$

Then the quotient $PI/X$ is an injective $\Lambda$-module.

**Proof.** $PI$ is projective-injective module, therefore $P = P(\tau^{-1} \text{top} PI)$ is minimal projective by the corollary 2.9. By proposition 2.20, $\Delta_i$ is the first syzygy of $\text{rad} P$. The difference of the lengths of $PI$ and $P$ is

$$\ell(PI) - \ell(P) = \ell(\text{rad} P) + (\ell(\Delta_i) - (\ell(\text{rad} P) + 1) = \ell(\Delta) - 1.$$ 

Since $X$ is proper submodule of $\Delta_i$, $\ell(\Delta) \geq 2$. By proposition 2.11, there are $\ell(\Delta_i) - 1 \geq 1$ proper injective quotients of $PI$ and it is clear that the quotient $PI/X$ is one of them. $\square$

**Proposition 3.18.** If $I$ is an injective but non-projective $\mathcal{E}(\Lambda)$-module, then there exists an injective but non-projective $\Lambda$-module $I_\Lambda$ such that $\text{Hom}_\Lambda (\mathcal{P}, \Omega^2(I_\Lambda)) \cong I$.

**Proof.** $I$ is an injective $\mathcal{E}(\Lambda)$-module, by the categorical equivalence 2.19 and the remark 2.41, there exists $\Lambda$-module $\Delta I$ such that $\text{Hom}_\Lambda (\mathcal{P}, \Delta I) \cong I$. We will show that $\Delta I$ is isomorphic to a second syzygy of an injective $\Lambda$-module $I_\Lambda$.

**Claim 3.19.** There exist a projective $\Lambda$-module $P$ such that $\Delta I$ is submodule of $P$. 
Proof. \( \Delta I \) has \( \mathcal{B}(\Lambda) \)-filtration, therefore \( \text{soc} \; \Delta I \in \mathcal{S}(\Lambda) \). By the categorical equivalence 2.19 there is a projective module \( P \) such that \( \text{soc} \; \Delta I \cong \text{soc} \; P \) (see remark 2.41). By the uniseriality lemma 2.3, either \( P \subseteq \Delta I \) or \( \Delta I \subset P \). The former is impossible because it makes \( \Delta I \) a projective module by 2.6 and \( \text{Hom}_\Lambda(\mathcal{P}, \Delta I) \) would be projective \( \mathcal{E}(\Lambda) \)-module which violates the assumption of proposition. Therefore \( \Delta I \) is a proper submodule of projective \( P \). \( \square \)

Claim 3.20. \( P \) is not \( \mathcal{B}(\Lambda) \)-filtered module.

Proof. Assume that \( P \) has \( \mathcal{B}(\Lambda) \)-filtration. Then \( I \cong \text{Hom}_\Lambda(\mathcal{P}, \Delta I) \) becomes proper submodule of projective \( \mathcal{E}(\Lambda) \)-module \( \text{Hom}_\Lambda(\mathcal{P}, \mathcal{P}) \) which violates the assumption on the injectivity of \( I \) in \( \text{mod-}\mathcal{E}(\Lambda) \). \( \square \)

Let’s denote the quotient \( P / \Delta I \) by \( Q \), i.e.

\[
0 \to \Delta I \to P \to Q = P / \Delta I \to 0
\]

Claim 3.21. \( Q \) is a proper submodule of an element of the base set \( \mathcal{B}(\Lambda) \).

Proof. Since \( \text{soc} \; Q \cong \tau^{-1} \text{top} \; \Delta I \in \mathcal{S}(\Lambda) \), there exist \( \Delta_i \in \mathcal{B}(\Lambda) \) such that \( \text{soc} \; \Delta_i \cong \text{soc} \; Q \). By the uniseriality lemma we have three cases to examine.

Case 1) \( \Delta_i \cong Q \) is not possible, otherwise we would conclude that \( P \) has \( \mathcal{B}(\Lambda) \)-filtration which violates the claim 3.20.

Case 2) \( \Delta_i \subset Q \) is not possible. Assume to the contrary that \( \Delta_i \) is a proper submodule of \( Q \). Then, the extension \( \Delta I' = \frac{\Delta_i}{\Delta I} \) of \( \Delta_i \) by \( \Delta I \) becomes \( \mathcal{B}(\Lambda) \)-filtered indecomposable proper submodule of \( P \). If we apply the evaluation functor \( \text{Hom}_\Lambda(\mathcal{P}, -) \) to \( \Delta I \) and \( \Delta I' \) we get \( \text{Hom}_\Lambda(\mathcal{P}, \Delta I) \hookrightarrow \text{Hom}_\Lambda(\mathcal{P}, I' \Delta) \) in \( \text{mod-}\mathcal{E}(\Lambda) \). This makes \( I \) a proper submodule of \( I' \). However, an injective module cannot be a proper submodule of any module by definition.

Case 3) Hence \( Q \) is a proper submodule of \( \Delta_i \).

\( \square \)

Because \( Q \) is proper submodule of \( \Delta_i \) and \( \Delta_i \) is a submodule of projective-injective module \( P' \) by proposition 2.17, we get the exact sequence

\[
0 \to Q \to P' \to P'/Q \to 0.
\]

If we apply proposition 3.18 to \( Q \hookrightarrow \Delta_i \hookrightarrow P' \), then the quotient \( P'/Q \) is an injective \( \Lambda \)-module. We can choose \( I_\Lambda \) as

\[
I_\Lambda \cong \frac{P'}{\left(P / \Delta I\right)}
\]

which satisfies \( \Omega^2(I_\Lambda) \cong \Delta I \) and \( \text{Hom}_\Lambda(\mathcal{P}, \Delta I) \cong I \) in \( \text{mod-}\mathcal{E}(\Lambda) \). \( \square \)

This result is useful, in the sense that for any injective but non-projective \( \mathcal{E}(\Lambda) \)-module \( I \), there exists at least one injective \( \Lambda \)-module \( I_\Lambda \) such that \( \text{Hom}_\Lambda(\mathcal{P}, \Omega^2(I_\Lambda)) \cong \)
However, the converse of this statement is not true in general. First of all, the second syzygy of injective \( \Lambda \)-modules can be either projective or periodic modules or even trivial. For example, if \( \Lambda \) is given by the Kupisch series \((4, 3, 4, 3)\), then \( \varepsilon(\Lambda) \) is selfinjective algebra with the Kupisch series \((2, 2)\) and its simple modules are \( \text{Hom}_\Lambda(\mathcal{P}, \Omega^2(I_1)) \) and \( \text{Hom}_\Lambda(\mathcal{P}, \Omega^2(I_3)) \). For the algebra given by the Kupisch series \((5, 4, 3, 2, 2)\), we get \( p\text{dim }I_2 = p\text{dim }I_3 = 1 \). Second observation is that \( I_\Lambda \) might not be the unique module verifying proposition 3.18. For example, algebra given by the Kupisch series \((5, 4, 4, 4, 3)\), we have \( \Omega^2(I_3) \cong \Omega^2(I_4) \). Nevertheless, the corollary below follows from propositions 3.18 and 2.42.

**Corollary 3.22.** If \( I \) is an injective non-projective \( \varepsilon(\Lambda) \)-module of finite projective dimension, then \( p\text{dim}_\Lambda I_\Lambda = p\text{dim}_\varepsilon(\Lambda) I + 2 \) where \( I_\Lambda \) is an injective module satisfying \( \text{Hom}_\Lambda(\mathcal{P}, \Omega^2(I_\Lambda)) \cong I \).

We give the proof of the statement \( v) \) of Theorem \( A \).

**Theorem 3.23.** If \( \Lambda \) is a cyclic Nakayama algebra which is Gorenstein then \( \text{gor}\text{dim }\Lambda = \text{gor}\text{dim }\varepsilon(\Lambda) + 2 \) provided that \( \text{gor}\text{dim }\Lambda \geq 2 \), and \( \varepsilon(\Lambda) \) is also Gorenstein.

**Proof.** If \( \Lambda \) is Gorenstein, then

\[
\text{gor}\text{dim }\Lambda = \sup \{p\text{dim }I | I \in \text{mod-}\Lambda \text{ is injective}\} = 2 + \sup \{p\text{dim }\Omega^2(I) | I \in \text{mod-}\Lambda \text{ is injective}\}
\]

(3.4)

\[
= 2 + \sup \{p\text{dim }\Delta I | \Delta I \cong \Omega^2(I) \in \text{Filt}(\mathcal{B}(\Lambda)), I \text{ is injective}\}
\]

(3.5)

\[
= 2 + \sup \{p\text{dim}_\varepsilon(\Lambda) I' | I' \in \text{mod-}\varepsilon(\Lambda), \text{Hom}_\Lambda(\mathcal{P}, \Delta I) \cong I' \text{ is injective}\}
\]

\[
= 2 + \text{gor}\text{dim }\varepsilon(\Lambda).
\]

where we used proposition 3.18 to pass from 3.4 to 3.5. In particular, the finiteness of \( \text{gor}\text{dim }\Lambda \) implies \( \text{gor}\text{dim }\varepsilon(\Lambda) \) is finite, so \( \varepsilon(\Lambda) \) is Gorenstein. \( \square \)

**Proposition 3.24.** If \( \Lambda \) is Gorenstein and \( \varphi\text{dim }\Lambda = 2 \), then \( \text{gor}\text{dim }\Lambda = 2 \).

**Proof.** Since \( \Lambda \) is Gorenstein, projective dimension of any injective module is finite. Therefore \( \text{gor}\text{dim }\Lambda \leq \text{fin}\text{dim }\Lambda \), and by Theorem 3.14, we get \( \text{gor}\text{dim }\Lambda \leq \varphi\text{dim }\Lambda = 2 \).

Because \( \varphi\text{dim }\Lambda = 2 \), \( \Lambda \) is not selfinjective, hence \( \text{gor}\text{dim }\Lambda \neq 0 \). We need to show that one is not in the range. Assume to the contrary that let \( \text{gor}\text{dim }\Lambda = 1 \). We can choose an injective module such that it is the quotient of a projective-injective module, i.e. \( \Omega^1(I) \) is simple module. Since \( \Lambda \) is cyclic, there is no simple projective module, we derive a contradiction. Hence \( \text{gor}\text{dim }\Lambda \neq 1 \). We obtain \( \text{gor}\text{dim }\Lambda = \varphi\text{dim }\Lambda = 2 \). \( \square \)

Based on the reduction stated in 3.23, we give another proof of equality about \( \varphi \)-dimension and Gorenstein dimension \([\text{ES17}]\).

**Theorem 3.25.** Let \( \Lambda \) be a cyclic Nakayama algebra which is Gorenstein. Then, \( \varphi \)-dimension and Gorenstein dimension of \( \Lambda \) match.

**Proof.** If global dimension of \( \Lambda \) is finite, then it is equal to both \( \varphi\text{dim }\Lambda \) and \( \text{gor}\text{dim }\Lambda \), which is trivial case. Also, if \( \Lambda \) is selfinjective then \( \varphi\text{dim }\Lambda = \text{gor}\text{dim }\Lambda = 0 \). So, we analyze the case of infinite global dimension with nonzero \( \varphi \)-dimension.
By Theorem 3.8, there exists $d$ such that $\varepsilon^d(\Lambda)$ is selfinjective but $\varepsilon^{d-1}(\Lambda)$ is not. If we combine this observation with proposition 3.24 we get
\[
\text{gor. dim } \varepsilon^{d-1}(\Lambda) = \varphi \text{ dim } \varepsilon^{d-1}(\Lambda) = 2.
\]
By the reductions 3.6 and 3.23, we conclude that
\[
\begin{align*}
\text{gor. dim } \varepsilon^{d-1}(\Lambda) &= \varphi \text{ dim } \varepsilon^{d-1}(\Lambda) = 2 \\
\iff \text{gor. dim } \varepsilon^{d-2}(\Lambda) &= \varphi \text{ dim } \varepsilon^{d-2}(\Lambda) = 4 \\
&\quad \vdots \\
&\iff \text{gor. dim } \varepsilon(\Lambda) = \varphi \text{ dim } \varepsilon(\Lambda) = 2d - 2 \\
&\iff \text{gor. dim } \Lambda = \varphi \text{ dim } \Lambda = 2d.
\end{align*}
\]

In Theorem 3.25, we showed the equality of $\varphi$-dimension and Gorenstein dimension of cyclic Nakayama algebras. We combine it with the result 3.9 to deduce possible values of Gorenstein dimension.

**Corollary 3.26.** If $\Lambda$ is Gorenstein with infinite global dimension, then $\text{gor. dim } \Lambda$ is even.

Gorenstein properties of Nakayama algebras were studied by Ringel in [Rin13]. In proposition 6 of the same work, it is stated that Gorenstein dimension is an even number under some conditions on the lengths of modules. The corollary above extends this result.

### 3.4. Results about dominant dimension

We study dominant dimension in terms of the syzygy filtered algebras.

**Definition 3.27.** [ARS97] Dominant dimension of algebra $A$ is:
\[
\text{dom. dim } A = \sup \{m | I_i \text{ is projective for } i = 0, 1, \ldots m\} + 1
\]
where $0 \to_A A \to I_0 \to I_1 \ldots$ is the minimal injective resolution of $A$. Dominant dimension of semisimple algebra is set to be infinity. We can define dominant dimension of any module $M$ by
\[
\text{dom. dim } M = \sup \{m | I_i \text{ is projective for } i = 0, 1, \ldots m\} + 1
\]
where $0 \to_A M \to I_0 \to I_1 \ldots$ is the minimal injective resolution of $M$. If we restrict $M$ to projective modules we get the following characterization of dominant dimension
\[
(3.6) \quad \text{dom. dim } A = \sup \{\text{dom. dim } P | P \text{ is projective non-injective } A \text{ module}\}.
\]

It is convenient for us to study projective dimensions of injective modules, because for a Nakayama algebra $\Lambda$, $\text{dom. dim } \Lambda = \text{dom. dim } \Lambda^{op}$ holds. Therefore the statement 3.6 is equivalent to
\[
(3.7) \quad \text{dom. dim } A = \sup \{\text{dom. dim } I | I \text{ is injective non-projective } A \text{ module}\}.
\]

We always use the characterization of dominant dimension 3.7 in the proofs.

We begin with a lemma which shows that the first syzygy of an injective module has to be a submodule of an element of the base set.
Lemma 3.28. If $I$ is an injective non-projective $\Lambda$-module, then $\Omega^1(I)$ is proper a submodule of an element $\Delta_i$ of the base set $\mathcal{B}(\Lambda)$.

Proof. Let $P_i$ be the injective envelope of $I$. The short exact sequence

$$0 \to \Omega^1(I) \to P_i \to I \to 0$$

implies that

(3.8) \[ \ell(P_i) - \ell(I) = \ell(\Omega^1(I)) \neq 0, \]

because $I$ is not projective.

On the other hand, $P_{i-1}$ is minimal injective by the corollary 2.9 which implies $\ell(P_{i-1}) \leq \ell(P_i)$. Moreover by the result 2.28, $\Omega^2(\text{top } P_{i-1}) \cong \Delta_i$ is an element of the base set $\mathcal{B}(\Lambda)$ and it satisfies

(3.9) \[ \ell(\Delta_i) = \ell(P_i) - \ell(\text{rad } P_{i-1}). \]

Notice that $\ell(\Delta_i) \neq 1$, otherwise by the result 2.10 $P_i$ would not have a proper injective quotient. If we subtract 3.8 from 3.9, we get

(3.10) \[ \ell(\Delta_i) - \ell(\Omega^1(I)) = \ell(I) - \ell(\text{rad } P_{i-1}). \]

Because top $P_{i-1} \cong \text{top } I$ we can apply the corollary 2.4 and find the only possible case. Case 1) $\text{rad } P_{i-1} \cong I$ is not possible, it makes $I$ a submodule of $P_{i-1}$ which violates dual of lemma 2.6.

Case 2) $I$ cannot be quotient of rad $P_{i-1}$, otherwise $I$ would become subquotient of $P_{i-1}$ and injective modules cannot be subquotients.

As a result, rad $P_{i-1}$ is the quotient of $I$ which implies $\ell(\Delta_i) > \ell(\Omega^1(I))$. They share the isomorphic socle, by the uniseriality lemma $\Omega^1(I)$ is a submodule of $\Delta_i$. \qed

Lemma 3.29. If $PI$ is an indecomposable projective-injective $\Lambda$-module which is filtered by $\mathcal{B}(\Lambda)$, then the module $\text{Hom}_\Lambda(P, PI)$ is projective-injective $\varepsilon(\Lambda)$-module.

Proof. Since $PI$ is projective module and $PI \in \text{Filt}(\mathcal{B}(\Lambda))$, $\text{Hom}_\Lambda(P, PI)$ is projective $\varepsilon(\Lambda)$-module. Therefore it is enough to show that $\text{Hom}_\Lambda(\mathcal{P}, PI) \in \text{mod-} \varepsilon(\Lambda)$ is injective. To emphasize its filtration, let’s denote $PI$ by $\Delta X$ and the corresponding module by $X$, i.e. $\text{Hom}_\Lambda(P, PI) = \text{Hom}_\Lambda(P, \Delta X) \cong X$.

Suppose that $X$ is not injective, so there has to be an injective envelope $I(X)$ which gives the nonsplit exact sequence

(3.11) \[ 0 \to X \to I(X) \to Q = \frac{I(X)}{X} \to 0 \]

in $\text{mod-} \varepsilon(\Lambda)$ with nontrivial $Q$. By the categorical equivalence, there are $\mathcal{B}(\Lambda)$-filtered modules $\Delta X$, $\Delta I$, $\Delta Q$ such that $\text{Hom}_\Lambda(\mathcal{P}, \Delta X) \cong X$, $\text{Hom}_\Lambda(\mathcal{P}, \Delta I) \cong I$, $\text{Hom}_\Lambda(\mathcal{P}, \Delta Q) \cong Q$ and the sequence

(3.12) \[ 0 \to \Delta X \to \Delta I \to \Delta Q \to 0 \]

is nonsplit and exact in $\text{mod-} \Lambda$. However $\Delta X$ is the indecomposable projective-injective module $PI$ and it cannot be a proper submodule of another module. Therefore $\Delta Q$ and $Q$ have to be trivial in 3.12 and 3.11 respectively, which contradicts to assumption that $Q$ is not trivial. As a result, $X = \text{Hom}_\Lambda(P, PI)$ is injective $\varepsilon(\Lambda)$-module with the corresponding module $PI$. \qed
The converse of the above statement is not true in general, a projective-injective $\epsilon(\Lambda)$-module has to have the corresponding $\Lambda$-module which is a projective but not necessarily a projective-injective.

The converse of proposition 3.18 is not true in general. However, under some conditions it can be valid. The key observation is stated below.

**Proposition 3.30.** Let $\Lambda$ be a cyclic Nakayama algebra such that $\text{dom.dim} \Lambda \geq 3$. Then, there is a bijection between indecomposable injective $\Lambda$-modules and indecomposable injective $\epsilon(\Lambda)$-modules via the functor $\text{Hom}_\Lambda(\mathcal{P}, \Omega^2(-))$.

**Proof.** Consider the projective resolution of an injective module $I$

$$
\cdots P_3 \to P_2 \to P_1 \to P_0 \to I
$$

in $\text{mod-}\Lambda$. Since $\text{dom.dim} \Lambda \geq 3$, we deduce that $P_0, P_1, P_2$ are projective-injective modules. Also, $\Omega^3(I)$ is nontrivial module, otherwise $P_2 \cong \Omega^2(I)$ implies that a syzygy module is projective-injective which is not possible by lemma 2.6. By proposition 2.33, we conclude that $P_2, \Omega^2(I) \in \text{Filt}(\mathcal{B}(\Lambda))$.

Assume that $\text{Hom}_\Lambda(\mathcal{P}, \Omega^2(I))$ is not an injective module in $\text{mod-}\epsilon(\Lambda)$. $\text{Hom}_\Lambda(\mathcal{P}, P_2)$ is projective-injective $\epsilon(\Lambda)$-module by lemma 3.29, therefore $\text{Hom}_\Lambda(\mathcal{P}, \Omega^2(I))$ is a quotient of $\text{Hom}_\Lambda(\mathcal{P}, P_2)$. We divide the proof into steps.

**Claim 3.31.** For any injective non-projective $\Lambda$-module $I$, the projective $\epsilon(\Lambda)$-module $\text{Hom}_\Lambda(\mathcal{P}, P_2)$ which covers $\text{Hom}_\Lambda(\mathcal{P}, \Omega^2(I))$ has injective quotients in $\text{mod-}\epsilon(\Lambda)$.

**Proof.** Assume to the contrary that $PI = \text{Hom}_\Lambda(\mathcal{P}, P_2)$ does not have any injective quotient in $\text{mod-}\epsilon(\Lambda)$. In particular $\text{Hom}_\Lambda(\mathcal{P}, \Omega^2(I))$ is not injective module, therefore there exists a projective module $\text{Hom}_\Lambda(\mathcal{P}, \Delta P)$ such that $\text{Hom}_\Lambda(\mathcal{P}, \Omega^2(I)) \subset \text{Hom}_\Lambda(\mathcal{P}, \Delta P)$ by the uniseriality lemma 2.3. By the construction 2.19 and the remark 2.41, $\Delta P$ is $\mathcal{B}(\Lambda)$-filtered projective $\Lambda$-module such that $\Omega^2(I) \subset \Delta P$ in $\text{mod-}\Lambda$. If we combine this observation with the resolution 3.13, we get the short exact sequences

$$
0 \to \Omega^2(I) \to \Delta P \to Q = \Delta P/\Omega^2(I) \to 0
$$

$$
0 \to \Omega^2(I) \to P_1 \to \Omega^1(I) \to 0
$$

where the quotient $Q$ has $\mathcal{B}(\Lambda)$-filtration by lemma 2.5.

Notice that $P_1$ and $\Delta P$ have isomorphic socles. By the uniseriality lemma 2.3 we have either $\Delta P \subset P_1$ or $P_1 \subset \Delta P$. The latter is not possible, because $P_1$ is projective-injective module. Therefore either $\Delta P \cong P_1$ or $\Delta P$ is a proper submodule of $P_1$. This implies that either $Q \cong \Omega^1(I)$ or $Q \subset \Omega^1(I)$ respectively which is deduced from the exact sequences 3.14. However this is not possible because of lemma 3.28, $\Omega^1(I)$ cannot have $\mathcal{B}(\Lambda)$-filtered submodules. \qed

**Claim 3.32.** For any injective non-projective $\Lambda$-module $I$, $\text{Hom}_\Lambda(\mathcal{P}, \Omega^2(I))$ is injective $\epsilon(\Lambda)$-module.

**Proof.** Assume that $\text{Hom}_\Lambda(\mathcal{P}, \Omega^2(I))$ is not an injective $\epsilon(\Lambda)$-module. Also, it is not a projective module by the assumption, i.e. $\Omega^3(I)$ is nontrivial. Let $I'$ be its injective envelope in $\text{mod-}\epsilon(\Lambda)$ and $\Delta I'$ be the corresponding module, i.e. $\text{Hom}_\Lambda(\mathcal{P}, \Delta I') \cong I'$. Their existence follows from the fact that each of these modules are $\mathcal{B}(\Lambda)$-filtered.
The exact sequences
\[ 0 \to \Omega^2(I) \to \Delta' \to Q = \Delta'/\Omega^2(I) \to 0 \]
\[ 0 \to \Omega^2(I) \to P_1 \to \Omega^1(I) \to 0 \]
(3.15)

imply that \( P_1 \) and \( \Delta' \) have isomorphic socles. By the uniseriality lemma 2.3 we have either \( \Delta' \subseteq P_1 \) or \( P_1 \subseteq \Delta P \). The latter is not possible, because \( P_1 \) is projective-injective module. Therefore either \( \Delta' \cong P_1 \) or \( \Delta' \) is a proper submodule of \( P_1 \). This implies that either \( Q \cong \Omega^1(I) \) or \( Q \subset \Omega^1(I) \) respectively. This contradicts with lemma 3.28.

**Claim 3.33.** If there exist at least two injective \( \Lambda \)-modules \( I_1, I_2 \) such that \( \Omega^2(I_1) \cong \Omega^2(I_2) \), then the projective covers are isomorphic, i.e. \( P(I_1) \cong P(I_2) \).

**Proof.** Consider the projective resolutions of \( I_i \)

\[ \cdots \to P(\Omega^1(I_i)) \to P(I_i) \to I_i \]

\[ \Omega^2(I_i) \to \Omega^1(I_i) \]

where \( 1 \leq i \leq 2 \). Because \( \Omega^2(I_i) \) is a submodule of \( P(\Omega^1(I_i)) \), by lemma 2.5 \( \text{soc} \, \Omega^1(I_1) \cong \text{soc} \, \Omega^1(I_2) \). In particular it is isomorphic to socles of \( P(I_i) \). Notice that \( P(I_i) \)'s are projective-injective modules since they are projective covers of injective modules. Indecomposable projective-injective modules with the same socle is unique upto isomorphism, therefore \( P(I_1) \cong P(I_2) \).

**Claim 3.34.** If there exists at least two injective \( \Lambda \)-modules \( I_1, I_2 \) satisfying \( \Omega^2(I_1) \cong \Omega^2(I_2) \), then the dominant dimension of \( \Lambda \) is one.

**Proof.** By the claim 3.33, the projective covers of \( I_1 \) and \( I_2 \) are isomorphic, however \( \Omega^1(I_1) \not\cong \Omega^1(I_2) \) because of \( I_1 \not\cong I_2 \). Therefore \( P(\Omega^1(I_1)) \not\cong P(\Omega^1(I_2)) \). Notice that their socles are isomorphic by the uniserialty lemma 2.3, since \( P(\Omega^1(I_1)) \supset \Omega^2(I_1) \cong \Omega^2(I_2) \subset P(\Omega^1(I_2)) \). They are uniserial modules, without loss of generality let \( P(\Omega^1(I_1)) \) be a proper submodule of \( P(\Omega^1(I_2)) \). Therefore \( P(\Omega^1(I_1)) \) cannot be a projective-injective module, by the definition 3.7, the dominant dimension of \( I_1 \) is one which makes \( \text{dom.dim} \, \Lambda = 1 \).

We have all the ingredients for the proof. We remind that \( \text{dom.dim} \, \Lambda \geq 3 \). By the claims 3.31 and 3.32, the second syzygies of injective \( \Lambda \)-modules are the corresponding modules for the injective \( \epsilon(\Lambda) \)-modules, i.e. \( \text{Hom}_\Lambda(\mathcal{P}, \Omega^2(I)) \) is injective \( \epsilon(\Lambda) \)-module for any injective non-projective \( I \in \text{mod-} \Lambda \). On the other hand, by the claim 3.34, \( \text{Hom}_\Lambda(\mathcal{P}, \Omega^2(-)) \) is a bijection, because non-isomorphic injective non-projective \( \Lambda \)-modules is equivalent to non-isomorphic injective non-projective \( \epsilon(\Lambda) \)-modules.

We state and prove the part vi) of Theorem A.

**Theorem 3.35.** Assume that dominant dimension of cyclic Nakayama algebra \( \Lambda \) is greater or equal than three. Then, we have the reduction

\[ \text{dom.dim} \, \Lambda = \text{dom.dim} \, \epsilon(\Lambda) + 2. \]


Proof. By the assumption $\text{dom.dim} \Lambda \geq 3$, there exists an injective $\Lambda$-module $I$ such that in the projective resolution
\[
\cdots \to P_3 \to P_2 \to P_1 \to P_0 \to I \to 0
\]
$P_0$, $P_1$ and $P_2$ are certainly projective-injective $\Lambda$-modules. Moreover, if $\text{dom.dim} I = d$, $d \geq 3$, then each $P_i$, $2 \leq i \leq d$ are projective-injective. By proposition 2.33, every $P_i$ and $\Omega^i(I)$, $i \geq 2$ are filtered by $B(\Lambda)$, therefore we can carry the resolution of $\Omega^2(I)$ into mod-$\varepsilon(\Lambda)$ as
\[
\cdots \to \text{Hom}_\Lambda(\mathcal{P}, P_3) \to \text{Hom}_\Lambda(\mathcal{P}, P_2) \to \text{Hom}_\Lambda(\mathcal{P}, \Omega^2(I)) \to 0.
\]
By proposition 3.30, $\text{Hom}_\Lambda(\mathcal{P}, \Omega^2(I)) \in \text{mod-}\varepsilon(\Lambda)$ is injective and by lemma 3.29, $\text{Hom}_\Lambda(\mathcal{P}, P_i)$ $d \geq i \geq 2$ are projective-injective. Therefore
\[
\text{dom.dim}_\Lambda I = 2 + \text{dom.dim} \Omega^2(I)
\]
\[
= 2 + \text{dom.dim}_{\varepsilon(\Lambda)} \text{Hom}_\Lambda(\mathcal{P}, \Omega^2(I))
\]
\[
= 2 + \text{dom.dim}_{\varepsilon(\Lambda)} I'
\]
where $I' = \text{Hom}_\Lambda(\mathcal{P}, \Omega^2(I))$ is an injective $\varepsilon(\Lambda)$-module. By the characterization 3.7 of the dominant dimension, we can take the supremum,
\[
\text{dom.dim} \Lambda = \sup \{ \text{dom.dim} I \mid I \in \text{mod-}\Lambda \text{ is injective non-projective} \}
\]
\[
= \sup \{ 2 + \text{dom.dim} \Omega^2(I) \mid \Omega^2(I) \in \text{Filt}(B(\Lambda)), I \in \text{mod-}\Lambda \}
\]
\[
= 2 + \sup \{ \text{dom.dim}_{\varepsilon(\Lambda)} I' \mid I' \cong \text{Hom}_\Lambda(\mathcal{P}, \Omega^2(I)) \in \text{mod-}\varepsilon(\Lambda) \}
\]
\[
= 2 + \text{dom.dim} \varepsilon(\Lambda).
\]

\[\Box\]

Remark 3.36. In Theorem 3.35, we assume that $\text{dom.dim} \Lambda \geq 3$. We will explain the mechanism behind it in section 4. Briefly, when global dimension is finite, semisimple components might arise in the syzygy filtered algebra and this is the obstacle. Let $\text{gldim} \Lambda < \infty$ and $\text{dom.dim} \Lambda = 2$ with an injective module $I$ satisfying $\text{p.dim} I = 2$. Then the syzygy filtered algebra has semisimple components. By the definition 3.27, the dominant dimension of semisimple algebra is infinity, and the dominant dimension of connected components can take any value. For example, let $(3, 3, 3, 2, 2, 2, 2, 2)$ be Kupisch series of $\Lambda$. Then $\text{gldim} \Lambda = 6$ and $\text{dom.dim} \Lambda = 2$. $\varepsilon(\Lambda)$ splits into linear Nakayama algebra $L$ given by Kupisch series $(2, 2, 2, 2, 1)$ and a semisimple component $A_1$. We get $\text{gldim} \varepsilon(\Lambda) = \max \{4, 0\} = 4$, however $\text{dom.dim} \varepsilon(\Lambda) = \min \{\text{dom.dim} L, \text{dom.dim} A_1\} = \min \{4, \infty\} = 4$ which is greater than $\text{dom.dim} \Lambda = 2$.

Proposition 3.37. If $\varphi \text{dim} \Lambda = 2$, then $1 \leq \text{dom.dim} \Lambda \leq 2$.

Proof. Any injective module is quotient of a projective-injective $\Lambda$-module, therefore dominant dimension has always lower bound one.

If $\Lambda$ is of finite global dimension or Gorenstein, then it is clear that dominant dimension is bounded by 2. So, we focus on the remaining case.

Assume to the contrary, let $\text{dom.dim} \Lambda \geq 3$. Therefore the projective resolution
\[
\cdots \to P_2 \to P_1 \to P_0 \to I \to 0
\]
of any injective $\Lambda$-module $I$ implies that $\Omega^2(I)$ is not projective and at least $P_0, P_1, P_2$ are projective-injective modules by the definition 3.7.

We can choose an injective module $I$ such that $\Omega^1(I)$ is simple module. Since $\Omega^2(I)$ is not projective, $P_1 = P(\Omega^1(I))$ is minimal projective module, i.e. $\text{rad } P_1 \cong \Omega^2(I)$ is not projective. This forces that $\Omega^1(I)$ is an element of the top set $S'(\Lambda)$. On the other hand $\Omega^1(I)$ is the socle of $P_0$, therefore $\Omega^1(I)$ is also an element of the socle set $S(\Lambda)$. By proposition 2.17 6), $\Omega^1(I) \in S(\Lambda) \cap S'(\Lambda)$ implies that $\Omega^1(I)$ is an element of the base set $B(\Lambda)$.

$P_0$ is projective-injective module, so $P = P(\text{top } P_0)$ is minimal projective where $\tau \text{ top } P \cong \text{top } P_0 \cong \text{top } I$. By propositions 2.20 and 2.28, $\Omega^2(\text{top } P) \cong \Omega^1(\text{rad } P)$ is isomorphic to $\Omega^1(I)$, which shows that $I \cong \text{rad } P$. By lemma 2.6, $I$ cannot be injective module, which creates the contradiction. Therefore $\text{dom.dim } \Lambda \leq 2$. □

Theorem 3.38. If $\Lambda$ is cyclic non-selfinjective Nakayama algebra then $\varphi \dim \Lambda \geq \text{dom.dim } \Lambda$.

Proof. If global dimension of $\Lambda$ is finite, then $\text{gldim } \Lambda = \varphi \dim \Lambda \geq \text{dom.dim } \Lambda$ is clear. When global dimension is infinite, but algebra is Gorenstein, then dominant dimension is again bounded by $\varphi$-dimension. Now we assume that algebra is not Gorenstein and is of infinite global dimension. By proposition 2.45, any higher filtered algebra is cyclic, hence there is no semisimple components.

Let $\varphi \dim \Lambda = 2d$. Therefore $\varphi \dim \varepsilon^{d-1}(\Lambda) = 2$, and by proposition 3.37

$$\text{dom.dim } \varepsilon^{d-1}(\Lambda) \leq \varphi \dim \varepsilon^{d-1}(\Lambda) = 2.$$ 

By the reductions 3.6 and 3.35 we get

$$\text{dom.dim } \varepsilon^{d-2}(\Lambda) \leq \varphi \dim \varepsilon^{d-2}(\Lambda) = 4.$$ 

$$\vdots$$

$$\text{dom.dim } \varepsilon(\Lambda) \leq \varphi \dim \varepsilon(\Lambda) = 2d - 2$$

$$\text{dom.dim } \Lambda \leq \varphi \dim \Lambda = 2d.$$ 

which is the upper bound for dominant dimension. □

This result appears in [Mar18]. We give another proof.

Corollary 3.39. Dominant dimension of cyclic non-selfinjective Nakayama algebra $\Lambda$ is bounded by $2r$, where $r$ is the number of relations defining the algebra.

Proof. By the results 3.38 and 3.9, $\text{dom.dim } \Lambda \leq \varphi \dim \Lambda \leq 2r$. □

3.5. Results about right finitistic and $\varphi$-dimensions. We define $\text{fin.dim } \Lambda^{\text{op}}$ as finitistic dimension of the algebra $\Lambda$ with respect to injective resolutions i.e.

$$\text{fin.dim } \Lambda^{\text{op}} := \sup \{ \text{in.dim}_\Lambda M | \text{ in.dim}_\Lambda M < \infty, M \in \text{mod-} \Lambda \}$$

In general left and right finitistic dimensions of an algebra can be different. But in the case of Nakayama algebras, we prove that they are same. To perform this, we need dual constructions stated in section 2.
Injective \( \Lambda \) modules are characterized by their tops using the relations given in 2.11:

\[
P_{(k_2r-1)+1} = I_{k_2} \rightarrow \ldots \rightarrow I_{k_4+1} \quad \text{have simple} \ S_{k_2r-1+1} \ \text{as their top}
\]

\[
P_{k_1+1} = I_{k_1} \rightarrow \ldots \rightarrow I_{k_2+1} \quad \text{have simple} \ S_{k_1+1} \ \text{as their top}
\]

\[
\vdots
\]

\[
P_{(k_2r-3)+1} = I_{k_2r} \rightarrow \ldots \rightarrow I_{k_4+1} \quad \text{have simple} \ S_{k_2r-3+1} \ \text{as their top}
\]

Let \( \mathcal{T}(\Lambda) \) be the complete set of representatives of tops of injective modules over \( \Lambda \).

By using the system of relations \( 2.11 \), it is

\[
\mathcal{T}(\Lambda) = \{ S_{k_1+1}, S_{k_3+1}, \ldots, S_{k_2r-1+1} \}.
\]

We call \( \mathcal{T}(\Lambda) \) as **opposite top set**. We define the set \( \mathcal{T}'(\Lambda) \) which is the complete set of representatives of inverse Auslander-Reiten translates of the tops of indecomposable injective modules. Hence \( S_i \in \mathcal{T}(\Lambda) \) if and only if \( \tau^{-1} S_i \in \mathcal{T}'(\Lambda) \). Because \( \tau^{-1} S_i \cong S_{i-1} \), we get

\[
\mathcal{T}'(\Lambda) = \{ S_{k_1}, S_{k_3}, \ldots, S_{k_2r-1} \}.
\]

We call \( \mathcal{T}'(\Lambda) \) as **opposite socle set**.

**Definition 3.40.** An indecomposable \( \Lambda \)-module \( M \) satisfying top \( M \in \mathcal{T}(\Lambda) \) and soc \( M \in \mathcal{T}'(\Lambda) \) is called **shortest** if the composition factors of \( M \) except top \( M \) and soc \( M \) are not elements of \( \mathcal{T}(\Lambda) \) and \( \mathcal{T}'(\Lambda) \).

**Definition 3.41.** Let \( \Lambda \) be a cyclic Nakayama algebra defined by the irredundant system of \( r \)-relations 2.11. For each \( j \in \{ 1, \ldots, r \} \) let \( \nabla_j \) be a shortest indecomposable uniserial module with soc \( \nabla_j \cong S_{k_2j+1} \) and top \( \nabla_j \cong S_{k_2j-1+1} \). The complete set of representatives of modules \( \nabla_j \)'s is called **the opposite base set** and denoted by \( \nabla(\Lambda) \).

Explicitly we have

\[
\nabla(\Lambda) := \left\{ \nabla_1 \cong \begin{array}{c|c}
S_{k_1+1} & \vdots \\
\vdots & \ddots \\
S_{k_3} & \vdots \\
\end{array}, \nabla_2 \cong \begin{array}{c|c}
S_{k_3+1} & \vdots \\
\vdots & \ddots \\
S_{k_5} & \vdots \\
\end{array}, \ldots, \nabla_j \cong \begin{array}{c|c}
S_{k_2j-1+1} & \vdots \\
\vdots & \ddots \\
S_{k_2j+1} & \vdots \\
\end{array}, \ldots, \nabla_r \cong \begin{array}{c|c}
S_{k_2r-1+1} & \vdots \\
\vdots & \ddots \\
S_{k_r} & \vdots \\
\end{array} \right\}
\]

**Proposition 3.42.** Regarding the opposite base set \( \nabla(\Lambda) \), we have

1) the top of each \( \nabla_i \) is an element of the opposite top set \( \mathcal{T}(\Lambda) \), i.e. top \( \nabla_i \in \mathcal{T}(\Lambda) \).

2) Any element \( S \) of the opposite socle set \( \mathcal{T}'(\Lambda) \) is a socle of an element of \( \nabla(\Lambda) \)

3) Any simple \( \Lambda \)-module \( S \) appears in the composition series of exactly one \( \nabla_i \).

   Equivalently, the simple composition factors of distinct \( \nabla_i \)'s are disjoint.

4) Distinct elements of the opposite base set are Hom-orthogonal i.e. \( \text{Hom}_\Lambda(\nabla_i, \nabla_j) \cong 0 \) when \( i \neq j \).

5) Each \( \nabla_i \) is a submodule of projective-injective module.

6) \( \nabla_i \) is simple \( \Lambda \)-module if and only if \( S \cong \nabla_i \) satisfies \( S \in \mathcal{T}'(\Lambda) \cap \mathcal{T}(\Lambda) \).

One can repeat all the constructions we discussed for projective resolutions and \( B(\Lambda) \)-filtered modules for injective resolutions and \( \nabla \)-filtered modules. We do not want to progress in those directions however the following statements can be concluded by using duality between projective \( \Lambda \)-modules and injective \( \Lambda^{op} \)-modules:
Remarks 3.43.  
i) The cosyzygy filtered algebra is the endomorphism algebra of injective envelopes of simple modules in $T'\Lambda$, i.e.

$$\eta(\Lambda) := \text{End}_\Lambda \left( \bigoplus_{S \in T'(\Lambda)} I(S) \right)$$

ii) If the second cosyzygy $\Sigma^2(M)$ of $M$ is not trivial, then it has $\nabla(\Lambda)$-filtration.

iii) Dual $\varphi$ function can be constructed as:

$$\varphi_R(M) := \min \{ t \mid \text{rank} (DL^t(\text{add } M)) = \text{rank} (DL^{t+j}(\text{add } M)) \text{ for } \forall j \geq 1 \}.$$

where $DL^t[M] := [\Sigma M]$ and gives map $DK_0 \mapsto DK_0$, where $DK_0$ is abelian group generated by all symbols $[X]$ modulo relations:

- $[A_1] = [A_2] + [A_3]$ if $A_1 \cong A_2 \oplus A_3$
- $[I] = 0$ if $I$ is injective.

iv) Therefore we can define $\varphi \dim \Lambda^{op}$ as:

$$\varphi \dim \Lambda^{op} := \sup \{ \varphi_R(M) \mid \text{for all } M \in \text{mod-}\Lambda \}$$

v) Let $\Lambda$ be cyclic non-selfinjective Nakayama algebra. $\varphi \dim \Lambda^{op} = 2$ if and only if $\eta(\Lambda)$ is selfinjective.

vi) Another result we need is dual of Theorem 3.14: If $\Lambda$ is Nakayama algebra then $\varphi \dim \Lambda^{op} - \text{fin} \dim \Lambda^{op} \leq 1$.

**Proposition 3.44.** \( \text{fin}. \dim \Lambda = 1 \) if and only if $S'(\Lambda) = T'(\Lambda)$.

Recall that we say module $M$ is periodic if there is number $i$ such that $M \cong \Omega^i(M)$.

**Proof.** ($\Rightarrow$). Let $\text{fin}. \dim \Lambda$ be one. Therefore either a simple module is of projective dimension one or its first syzygy is periodic. Therefore $\Omega^1(S)$ is periodic if and only if $S$ is top of minimal projective $P$. By using the relations 2.11, $S \in \{ S_{k_1}, S_{k_2}, \ldots, S_{k_{2r-1}} \}$, and the top of $\Omega^1(S)$ is in the set $\{ S_{k_{1+1}}, \ldots, S_{k_{2r-1+1}} \}$ by lemma 2.5. Since $\Omega^1(S)$ is periodic, $\Omega^1(S) \cong \Omega^i(S)$ for some $i$, therefore it has $\mathcal{B}(\Lambda)$-filtration by proposition 2.33. This means that the top of $\Omega^1(S) \in S'(\Lambda)$. This is true for all the tops of minimal projectives which is the set of size $r$, hence we get $S'(\Lambda) = T'(\Lambda)$.

($\Leftarrow$) If $S'(\Lambda) = T'(\Lambda)$, then $S(\Lambda) = T(\Lambda)$ and $\mathcal{B}(\Lambda) = \nabla(\Lambda)$. By using the system of relations 2.11, we see that each projective-injective module is $\mathcal{B}(\Lambda)$-filtered. Moreover, they have the same $\mathcal{B}(\Lambda)$-length. We get $\varphi \dim \Lambda = 2$ ([Sen21]). By Theorem 3.14, it is enough to show that there is no module of projective dimension two. If there was a module $M$ such that $\Omega^2(M)$ is projective, then the exact sequence

$$0 \to P_2 = \Omega^2(M) \to P_1 \to \Omega^1(M) \to 0$$

would imply that $\Omega^1(M)$ is quotient of $\Delta_i \in \mathcal{B}(\Lambda)$ for some $i$. Hence it cannot be a submodule of any projective module, so there is no module of projective dimension two.

**Proposition 3.45.** \( \text{fin}. \dim \Lambda = 1 \) if and only if $\text{fin}. \dim \Lambda^{op} = 1$. 

\[\square\]
Proof. By proposition 3.44, fin.\dim \Lambda = 1 if and only if \mathcal{S}(\Lambda) = \mathcal{T}(\Lambda) if and only if every projective-injective module is in \text{Filt}(\mathcal{B}(\Lambda)). If there exists \( M \) such that \( \Sigma^2(M) \) is injective module, then \( \Sigma^2(M) \) has \( \nabla(\Lambda) = \mathcal{B}(\Lambda) \)-filtration. This makes \( \Sigma^2(M) \) projective-injective module which is not possible. Therefore fin.\dim \Lambda^{op} = 1. \square

Theorem 3.46. For cyclic Nakayama algebras left and right finitistic dimensions are same, i.e. fin.\dim \Lambda = \dim \Lambda^{op}.

Proof. If global dimension is finite, it is a well known result. Therefore we consider the case of infinite global dimension.

We will prove it by induction on the dimension. The previous proposition 3.45 verifies that if \( n = 1 \) and \( n = 2 \), fin.\dim \Lambda = 1 if and only if fin.\dim \Lambda^{op} = 1 and fin.\dim \Lambda = 2 if and only if fin.\dim \Lambda^{op} = 2 \). Assume that fin.\dim \Lambda = fin.\dim \Lambda^{op} = d for all \( d \leq n \). Now, let fin.\dim \Lambda = n + 1. We get:

\[
\begin{align*}
\text{fin.}\dim \Lambda &= n + 1 \iff \text{fin.}\dim \varepsilon(\Lambda) = n - 1, \text{by prop 3.11} \\
\text{fin.}\dim \varepsilon(\Lambda) = n - 1 &\iff \text{fin.}\dim \varepsilon(\Lambda)^{op} = n - 1 \text{ by induction hypothesis} \\
\text{fin.}\dim \varepsilon(\Lambda)^{op} = n - 1 &\iff \text{fin.}\dim \Lambda^{op} = n + 1 \text{ by dual prop 3.11}
\end{align*}
\]

\square

Theorem 3.47. For cyclic Nakayama algebras left and right \( \varphi \)-dimensions are same i.e. \( \varphi \dim \Lambda = \varphi \dim \Lambda^{op} \).

Proof. It is enough to consider global dimension is infinite.

By Theorem 3.14, there are two possibilities:

\[
\varphi \dim \Lambda = \text{fin.}\dim \Lambda \\
\varphi \dim \Lambda = 1 + \text{fin.}\dim \Lambda
\]

The Theorem 3.46 together with the equalities of left and right finitistic dimensions imply

\[\varphi \dim \Lambda = \varphi \dim \Lambda^{op}.\]

\square

In [BMR], authors show equalities of left and right \( \varphi \)-dimension for truncated path algebras.

We can put together all the results on the upper bounds.

Theorem 3.48. Let \( \Lambda \) be cyclic non-selfinjective Nakayama algebra defined by \( r \) many irredundant system of relations over \( N \) vertices. Then, \( \varphi \dim \Lambda, \text{fin.}\dim \Lambda, \text{gor.}\dim \Lambda \), \( \text{dom.}\dim \Lambda \) are bounded by \( 2r \) where \( r = |\mathcal{B}(\Lambda)| \).

Corollary 3.49. The upper bound in terms of the rank of the algebra is \( 2N - 2 \).

Proof. By the previous result, each dimension is bounded by \( 2r \). By the characterization of selfinjective algebras 2.47, \( r = N \) if and only if algebra is selfinjective. Hence the greatest value that \( r \) can take is \( N - 1 \), therefore \( 2r \leq 2N - 2 \). \square
Example 3.50. We want to show that the upper bound is sharp. We reconsider the example given in [Sen21]. Let \( \Lambda \) be cyclic Nakayama algebra on \( N \) vertices with Kupisch series \((2N+1, 2N+1, \ldots, 2N+1, 2N)\). This has \( r = N-1 \) and \( \varphi \dim(\Lambda) = 2r = 2N-2 \).

Indeed, one can check that, it is Gorenstein and \( \dim \Lambda = \varphi \dim \Lambda = \text{gor.dim} \Lambda = \text{dom.dim} \Lambda = \text{fin.dim} \Lambda^{op} = \varphi \dim \Lambda^{op} = 2r = 2N-2 \).

4. The Structure of Syzygy Filtered algebras

The proof of the first part of Theorem C is stated in 3.8. Now we give the proof for the second part ii) which we recall below.

Theorem 4.1. If \( \Lambda \) is a cyclic connected Nakayama algebra of finite global dimension, then there exists a positive integer \( k \) such that \( \varepsilon^k(\Lambda) \) is a cyclic connected Nakayama algebra and \( \varepsilon^{k+1}(\Lambda) \) is not cyclic. \( \varepsilon^{k+1}(\Lambda) \) can split into components that are either linear Nakayama algebras or semisimple components or both.

4.1. When is the syzygy filtered algebra non-cyclic.

Proposition 4.2. If \( \Lambda \) is a cyclic Nakayama algebra of global dimension two, then the syzygy filtered algebra is semisimple.

Proof. Any simple module which is top of minimal projective have projective dimension two by the assumption on the global dimension. Therefore \( \Omega^2(S) \) is projective module and an element of the base set by proposition 2.28. Therefore the corresponding module \( \text{Hom}_\Lambda(\mathcal{P}, \Omega^2(S)) \) is simple projective \( \varepsilon(\Lambda) \)-module by the categorical equivalence 2.19.

We need to show that \( P = \Omega^2(S) \) cannot be a submodule of another projective \( \Lambda \)-module \( P' \) which has \( \mathcal{B}(\Lambda) \)-filtration. Assume to the contrary that \( P' \) is in \( \text{Filt}(\mathcal{B}(\Lambda)) \) and \( P \) is proper submodule of \( P' \). We get the exact sequence

\[
0 \to P \to P' \to Q = P'/P \to 0
\]

where \( Q \) is not trivial. Since \( P, P' \) have \( \mathcal{B}(\Lambda) \)-filtration, \( Q \) has \( \mathcal{B}(\Lambda) \)-filtration, because \( \text{Filt}(\mathcal{B}(\Lambda)) \) is exact category by the corollary 2.37. Moreover, the result 2.34 implies that there exists \( \Lambda \)-module \( M \) such that \( \Omega^2(M) \cong Q \). This means \( \Omega^3(M) \cong \Omega^1(S) \cong P \) and \( p \dim M = 3 \) which contradicts to \( \text{gldim} \Lambda = 2 \). Therefore \( \mathcal{B}(\Lambda) \)-filtered projective module \( P \) cannot be submodule of another \( P' \in \text{Filt}(\mathcal{B}(\Lambda)) \) which makes the category \( \text{Filt}(\mathcal{B}(\Lambda)) \) semisimple category. By the categorical equivalence 2.19, \( \text{mod-}\varepsilon(\Lambda) \) is semisimple algebra.

Example 4.3. Let \( \Lambda \) be given by the Kupisch series \((7, 6, 5, 4, 3, 6, 5, 4, 3)\). Then the base set is \( \{P_3, P_8\} \) and it is clear that the other projective modules are not filtered by it. Therefore \( \varepsilon(\Lambda) \cong A_1 \oplus A_1 \).

Proposition 4.4. Let \( \Lambda \) be cyclic Nakayama algebra with a simple module \( S \) satisfying \( p \dim S = 2 \). Then \( \varepsilon(\Lambda) \) is not cyclic Nakayama algebra.

Proof. Since \( p \dim S = 2 \), it has to be top of minimal projective \( P \). Otherwise, \( \Omega^1(S) \) would be isomorphic to \( \text{rad} P' \cong P'' \) where \( P' \) is not minimal, and makes \( p \dim S = 1 \). In 2.28, we showed that the second syzygy of top of a minimal projective is an element of the base set \( \mathcal{B}(\Lambda) \). Moreover by the assumption \( p \dim S = 2 \), \( \Omega^2(S) \) is projective module. By the categorical equivalence, the corresponding module \( \text{Hom}_\Lambda(\mathcal{P}, \Omega^2(S)) \)
is projective, because $\Omega^2(S)$ is projective

is simple, because $\Omega^2(S) \in \mathcal{B}(\Lambda)$.

Therefore $\varepsilon(\Lambda)$ has at least one simple projective module which shows it is not cyclic.

\begin{proposition}
\end{proposition}

\begin{proof}
By the remark 2.41, any nonsplit exact sequence

$$0 \to A \to B \to C \to 0$$

in $\text{mod-}\varepsilon(\Lambda)$ is equivalent to the exact sequence

$$0 \to \Delta A \to \Delta B \to \Delta C \to 0$$

in $\text{Filt}(\mathcal{B}(\Lambda))$. Therefore, a simple $\varepsilon(\Lambda)$-module $S$ has no extension if and only if the corresponding module $\Delta S$ has no extension in $\text{Filt}(\mathcal{B}(\Lambda))$. Therefore none of the terms (except $\Delta S$) of any nonsplit short exact sequence involving $\Delta S$ in $\text{mod-}\varepsilon(\Lambda)$ has $\mathcal{B}(\Lambda)$-filtration.

\end{proof}

\begin{corollary}
\end{corollary}

Let $\Lambda$ be cyclic Nakayama algebra with a simple module $S$ satisfying $p \dim S = 2$. Then $\text{Hom}_\Lambda(\mathcal{P}, \Omega^2(S))$ is semisimple if and only if none of the projective modules $P$ having $\Omega^2(S)$ as a proper submodule have $\mathcal{B}(\Lambda)$-filtration.

\begin{proof}
p \dim S = 2$ implies that $\Omega^2(S)$ is projective and by proposition 2.20 $\Omega^2(S)$ is an element of $\mathcal{B}(\Lambda)$. Therefore $\text{Hom}_\Lambda(\mathcal{P}, \Omega^2(S)) = S'$ is simple projective $\varepsilon(\Lambda)$-module.

By proposition 4.5, $S'$ is simple module of the semisimple component if and only if $\Omega^2(S)$ has no extension in $\text{Filt}(\mathcal{B}(\Lambda))$. Therefore any indecomposable projective $\Lambda$-module $P$ having the projective module $\Omega^2(P)$ as a proper submodule cannot have $\mathcal{B}(\Lambda)$-filtration.

\end{proof}

\begin{proposition}
\end{proposition}

If there exists a simple module $S$ satisfying $p \dim S = 2$ minimally, i.e. there is at least one simple module $S'$ such that $S' \not\cong S$ and $p \dim S' > 2$, then $\varepsilon(\Lambda)$ has at least one connected component which is linear Nakayama algebra.

\begin{proof}
By the result 4.4, $\varepsilon(\Lambda)$ is not cyclic. By the assumption of the statement, there is a simple module $S'$ such that $p \dim S' \geq 3$. Therefore in the projective resolution

$$\cdots \to P_3 \to P_2 \to P_1 \to P_0 \to S' \to 0$$

of $S'$, the exact sequence

$$0 \to \Omega^3(S') \to P_2 \to \Omega^2(S') \to 0$$

is nonsplit exact. By 2.33, the sequence

$$0 \to \text{Hom}_\Lambda(\mathcal{P}, \Omega^3(S')) \to \text{Hom}_\Lambda(\mathcal{P}, P_2) \to \text{Hom}_\Lambda(\mathcal{P}, \Omega^2(S')) \to 0$$

in $\text{mod-}\varepsilon(\Lambda)$ is exact by the categorical equivalence. Therefore the component having the exact sequence cannot be semisimple which means it is linear Nakayama algebra.

\end{proof}

\begin{example}
We will compare the algebras given by the Kupisch series $(3,2,2)$ and $(3,2,3,2,2)$. Both of them have global dimension 3 and unique simple modules with projective dimension two. However the syzygy filtered algebras are given by $(2,1)$ and $(2,1) \oplus A_1$.

\end{example}
4.2. When is the higher syzygy filtered algebra non-cyclic.

**Proposition 4.9.** If all simple modules which are tops of the minimal projective \(\Lambda\)-modules have the same projective dimension \(2k\), then \(\varepsilon^k(\Lambda)\) is semisimple.

*Proof.* We prove it by induction. The case \(k = 1\) is proved in 4.2.

**Claim 4.10.** If \(k = 2\), then \(\varepsilon(\Lambda)\) is cyclic and \(\varepsilon^2(\Lambda)\) is semisimple.

*Proof.* \(\Lambda\) is cyclic Nakayama algebra so we can construct \(\varepsilon(\Lambda)\) by the definition 2.38. By the categorical equivalence 2.19 and the remark 2.41, simple \(\varepsilon(\Lambda)\)-modules are of the form \(\text{Hom}_\Lambda(P, \Delta S)\). \(\Delta S\) is the second syzygy of simple \(\Lambda\)-module \(S\) which is the top of a minimal projective by propositions 2.34 and 2.20. By the assumption \(p.\dim S \neq 2\), therefore simple \(\varepsilon(\Lambda)\)-modules are not projective. This shows that the syzygy filtered algebra is cyclic, therefore we can construct \(\varepsilon^2(\Lambda)\). Moreover, by the reduction 2.43, \(\text{gldim } \Lambda = \text{gldim } \varepsilon(\Lambda) + 2\), which makes \(\text{gldim } \varepsilon(\Lambda) = 2\). By proposition 4.2, \(\varepsilon^2(\Lambda)\) is semisimple. \(\square\)

Now we can look arbitrary \(k\). Assume that the claim holds for all \(k = 1, 2, \ldots, m\). Let \(k = m + 1\). Since \(\varepsilon(\Lambda)\) is cyclic and of global dimension \(2k\) which satisfies the induction hypothesis, claim follows. \(\square\)

**Proposition 4.11.** Let \(\Lambda\) be a cyclic Nakayama algebra with a simple module \(S\) satisfying \(p.\dim S = 4\) minimally, i.e. there is no other simple module \(S'\) such that \(p.\dim S' = 2\) and there exists at least one simple module \(S''\) with \(p.\dim S'' > 4\). Then \(\varepsilon(\Lambda)\) is cyclic Nakayama algebra and \(\varepsilon^2(\Lambda)\) has linear component.

*Proof.* Since \(p.\dim S = 4\), its projective cover has to be a minimal projective i.e. \(\text{rad } P(S)\) is not projective. Therefore \(\Omega^2(S)\) is an element of \(\mathcal{B}(\Lambda)\). Moreover, by the assumption on the minimality, elements of the base set \(\mathcal{B}(\Lambda)\) are not projective. Therefore the corresponding \(\varepsilon(\Lambda)\)-module \(\text{Hom}(P, \Omega^2(S))\) is simple by the categorical equivalence 2.19 but not projective in mod-\(\varepsilon(\Lambda)\), hence \(\varepsilon(\Lambda)\) is cyclic.

Let’s denote \(\text{Hom}_\Lambda(P, \Omega^2(S))\) by \(S'\). In mod-\(\varepsilon(\Lambda)\), \(p.\dim S'\) is two, by the reduction 2.42. Now we can apply proposition 4.7 in order to conclude that \(\varepsilon^2(\Lambda)\) has component which is linear Nakayama algebra. \(\square\)

**Proposition 4.12.** Let \(\Lambda\) be a cyclic Nakayama algebra with a simple module \(S\) satisfying \(p.\dim S = 2k\) minimally, i.e. there is no other simple module \(S\) such that \(p.\dim S = 2k'\) with \(k' < k\) and there exists at least one simple module \(S''\) with \(p.\dim S'' > 2k\). Then \(\varepsilon^{k-1}(\Lambda)\) is cyclic Nakayama algebra and \(\varepsilon^k(\Lambda)\) has component which is linear Nakayama algebra.

*Proof.* Proof by induction. We gave the proofs of \(m = 1\) and \(m = 2\) in 4.7 and 4.11 respectively. We assume that the statement is true for all \(m = 1, \ldots, k\). We need to analyze the case \(m = k + 1\). Assume that there is a simple module \(S\) with projective dimension \(2k + 2\) and it is minimal in the sense that there is no other simple module with projective dimension \(2k'\) where \(k' < k + 1\). \(S\) has to be top of minimal projective module, otherwise \(p.\dim S\) would be one. By proposition 2.28, \(\Omega^2(S)\) is an element of the base set \(\mathcal{B}(\Lambda)\), and it is not projective, i.e. \(p.\dim \Omega^2(S) = 2k\). If we apply the
syzygy filtered algebra construction, the corresponding \( \varepsilon(\Lambda) \)-module \( \text{Hom}_\Lambda(\mathcal{P}, \Omega^2(S)) \) is simple and not projective. Therefore \( \varepsilon(\Lambda) \) is cyclic Nakayama algebra. If we denote the corresponding module \( \text{Hom}_\Lambda(\mathcal{P}, \Omega^2(S)) \) by \( S' \), then \( p \text{dim}_{\varepsilon(\Lambda)} S' = p \text{dim}_\Lambda S - 2 = 2k \). This means \( S' \) is top of a minimal \( \varepsilon(\Lambda) \)-projective module. Since the reduction is two, therefore \( p \text{dim} S' = 2k \) is minimal, i.e. there is no other simple \( \varepsilon(\Lambda) \)-module with smaller even projective dimension. By the induction hypothesis claim follows. \( \square \)

4.3. Applications of the splitting results. Based on the syzygy filtration construction, we give the proof of the Madsen’s result [Mad05].

**Proposition 4.13.** Global dimension of \( \Lambda \) is infinite if and only if there is no simple module of even projective dimension.

**Proof.** By propositions 4.7, 4.11 and 4.12 if there exists a simple module with even projective dimension, then we reach linear Nakayama algebra, so global dimension is finite. Assume that there is no simple module with even projective dimension. Therefore there is no simple projective module in \( \varepsilon(\Lambda) \) which makes it cyclic and \( \text{rank} \Lambda \geq \text{rank} \varepsilon(\Lambda) \). Furthermore there is no simple module in \( \text{mod-} \varepsilon(\Lambda) \) with even projective dimension, otherwise we can lift the resolution to \( \text{mod-} \Lambda \) and get \( p \text{dim} S \) even, not possible by the assumption. We can construct \( \varepsilon^2(\Lambda) \), and the same arguments are true, so there is no simple module of even projective dimension, and therefore there is no simple projective module. In each step we get nontrivial cyclic algebras \( \varepsilon^k(\Lambda) \). On the other hand in each step we get \( \text{rank} \varepsilon^k(\Lambda) \geq \text{rank} \varepsilon^{k+1}(\Lambda) \). Since the rank cannot reach to zero, each algebra is nontrivial, \( \lim_{i \to \infty} \text{rank} \varepsilon^i(\Lambda) \) has to stabilize, which means that we reach selfinjective algebra, so global dimension of \( \Lambda \) is infinite. \( \square \)

**Theorem 4.14.** If global dimension of cyclic Nakayama algebra \( \Lambda \) is finite, then there exists \( m \) such that \( \varepsilon^m(\Lambda) \) is not cyclic, and

\[
gldim \Lambda \leq 2m + r_{m-1} - C
\]

where \( r_{m-1} \) is the number of relations of the cyclic algebra \( \varepsilon^{m-1}(\Lambda) \) and \( C \) is the number of connected components of \( \varepsilon^m(\Lambda) \).

**Proof.** By proposition 4.13, global dimension of \( \Lambda \) is finite if and only if there exists at least one simple module \( S \) with even projective dimension. Therefore we can use propositions 4.12 or 4.9 depending on the conditions on the other simple modules. Assume that \( \Lambda \) satisfies the conditions in proposition 4.12. Therefore there exists \( m \) such that \( \varepsilon^{m-1}(\Lambda) \) is cyclic, \( \varepsilon^m(\Lambda) \) is not cyclic and

\[
gldim \Lambda = 2m + \text{gldim} \varepsilon^m(\Lambda).
\]

On the other hand, the rank of \( \varepsilon^m(\Lambda) \) is the number of relations of the cyclic algebra \( \varepsilon^{m-1}(\Lambda) \) which is denoted by \( r_{m-1} \). We recall a well-known result on linear Nakayama algebras (or more generally presentation directed algebras), global dimension of linear Nakayama algebra \( L \) is bounded by \( \text{rank} L - 1 \). If \( \varepsilon^m(\Lambda) \) splits into components \( L_1 \oplus L_2 \oplus \cdots \oplus L_C \), then \( \text{gldim} \varepsilon^m(\Lambda) \leq \max_i \{ \text{gldim} L_i \} \). Therefore global dimension of \( \varepsilon^m(\Lambda) \) can be at most \( r_{m-1} - C \). We conclude that

\[
\text{gldim} \Lambda = 2m + \text{gldim} \varepsilon^m(\Lambda) \\
\leq 2m + r_{m-1} - C.
\]
Now we assume that $\Lambda$ satisfies the conditions in proposition 4.9. In this case $\text{gldim } \varepsilon^{m-1}(\Lambda) = 2$, and $r_{m-1} = C$ gives the number of simple summands of the semisimple algebra $\varepsilon^m(\Lambda)$. There exists at least one component, so $r_{m-1} \geq 1$. This implies the desired inequality

$$\text{gldim } \Lambda = 2m \leq 2m + r_{m-1} - C.$$  

\[\square\]

**Remark 4.15.** We give another proof of the main theorem of [MM18] by using syzygy filtration method.

**Theorem 4.16.** [MM18] Let $\Lambda$ be a Nakayama algebra with a simple module $S$ of even projective dimension. Choose $m$ minimal such that a simple $\Lambda$ module has projective dimension equal to $2m$. Then the global dimension of $\Lambda$ is bounded by $N + m - 1$ where $N$ is the number of vertices of $\Lambda$.

**Proof.** By the result 4.14, we have the inequality

$$\text{gldim } \Lambda \leq 2m + r_{m-1} - C.$$  

We need to find the possible minimal value of $C$ and maximal value of $r_{m-1}$. It is clear that $C = 1$ is the minimum. Let $r_j$ denote the number of relations of higher syzygy filtered algebras $\varepsilon^j(\Lambda)$ where $1 \leq j \leq m - 1$ and $r$ denotes the number of relations of $\Lambda$. Therefore the result 2.44 applied to higher syzygy filtered algebras implies $\text{rank } \varepsilon(\Lambda) = r$ and $\text{rank } \varepsilon^j(\Lambda) = r_{j-1}$ for $2 \leq j \leq m$. In each reduction, ranks of the algebras reduce, therefore $r_{m-1} < r_{m-2} < \cdots < r_1 < r$ which is equivalent to the system of inequalities

$$r + 1 \leq N$$  

$$r_1 + 1 \leq r$$  

$$r_2 + 1 \leq r_1$$  

$$\vdots$$  

$$r_{m-1} + 1 \leq r_{m-2}.$$  

If we add all the terms, we get $r_{m-1} + m \leq N$. Therefore

$$\text{gldim } \Lambda \leq 2m + r_{m-1} - 1 \leq 2m + (N - m) - 1 = N + m - 1.$$  

\[\square\]

**Corollary 4.17.** [Gus85] Let $\Lambda$ be cyclic Nakayama algebra of finite global dimension. Then $\text{gldim } \Lambda \leq 2N - 2$ where $N$ is the number of vertices.

**Proof.** By the previous result, global dimension of $\Lambda$ where $\text{rank } \Lambda = N$ is bounded by $N + m - 1$ where $2m$ is projective dimension of a simple $\Lambda$-module which is minimal. On the other hand, $m$ is the number of reductions from $\Lambda$ to $\varepsilon^m(\Lambda)$. Since the minimal value of $\text{rank } \varepsilon^m(\Lambda) = 1$, $m$ can be at most $N - 1$. Therefore we get the upper bound $2N - 2$ for global dimension. \[\square\]

**Remark 4.18.** Indeed the Gustafson’s example is the unique algebra such that $\text{gldim } \Lambda = 2N - 2$ and $\text{rank } \Lambda = N$. We give a proof of this. When $N = 2$, the only algebra with finite global dimension 2 is given by the Kupisch series $(3, 2)$. Assume that there exist
\( \Lambda \) with rank \( \Lambda = 3 \) and \( \text{gldim} \, \Lambda = 4 \). Therefore the syzygy filtered algebra satisfies \( \text{gldim} \, \varepsilon(\Lambda) = 2 \), and the rank has to satisfy \( \text{rank} \, \varepsilon(\Lambda) \leq 2 \). The rank cannot be one, because rank one Nakayama algebra is either semisimple or selfinjective, which forces that the Kupisch series is \((3, 2)\). It has one injective module, by proposition 3.18 together with rank \( \Lambda = 3 \), there can be at most one injective \( \Lambda \)-module. Therefore the Kupisch series of \( \Lambda \) is either of the form \((n, n, n - 1)\) or \((n, n - 1, n - 1)\). Among them only \((4, 4, 3)\) satisfies all the conditions. The sequence of Kupisch series is \((3, 2), (4, 4, 3), (5, 5, 4), (6, 6, 6, 6, 5)\) etc. By induction on the rank, claim follows.

We want to emphasis that number of relations helps to lower the bound obtained in [MM18]. Notice that global dimension attains the bound if and only if \( r = N - 1 \) and \( r_i = r_{i-1} - 1 \) for all \( 1 \leq i \leq m - 1 \). If we take \( r = N \), \( \Lambda \) becomes selfinjective algebra.

**Example 4.19.** Let \( \Lambda \) be cyclic Nakayama algebra of \( N = 8 \) vertices with relations:

\[
\alpha_4 \alpha_3 = 0, \quad \alpha_6 \alpha_5 = 0, \quad \alpha_2 \alpha_1 \alpha_8 = 0
\]

By simple computation, \( m = 1 \), and global dimension is 2. So it is smaller than \( r = 3 \).

4.4. **Future Directions.** Proposition 2.36 suggests that the syzygy filtered algebra construction can be carried into other classes of algebras. In a series of forthcoming papers [STZ], we give complete classifications Nakayama algebras which are Auslander-Gorenstein and finitistic Auslander and linear Nakayama algebras which are higher Auslander algebras. We show that algebra \( A \) is selfinjective if and only if \( A \) is equivalent to its wide subcategory \( \mathcal{W}(A) \) cogenerated by projective-injective \( A \)-module, so the result 2.48 holds in general. If global dimension of \( A \) is \( d \) and dominant dimension is at least one, we show that global dimension of the wide subcategory \( \mathcal{W}(A) \) is at most \( d - 2 \). This suggests that we can use the wide subcategory approach to give upper bounds for global dimension in general.

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