Paper

Control of deterministic diffusion generated by chaotic dynamical systems through time delayed feedback control

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Received July 31, 2017; Revised December 7, 2017; Published April 1, 2018

Abstract: Time delayed feedback can control chaotic motion to periodic motion by stabilizing an unstable periodic orbit that is embedded in a chaotic set. The time delayed feedback control method can be applied to a non-stationary stochastic process with Gaussian noise, and it can control the diffusion processes in a stochastic system. In this study, we apply time delayed feedback control to the diffusion processes in systems with noise that is more complicated than Gaussian noise, e.g., diffusions induced by chaotic noise generated by chaotic dynamical systems, a logistic map and a one-dimensional piecewise-linear map.

Key Words: deterministic diffusion, time delayed feedback control, chaos

1. Introduction

The concept of irregular motions is important for understanding various phenomena such as chaos, turbulence, biological membranes, nano science and Brownian motion [1–7]. Irregular motions of a few state variables are typically undesirable because of unpredictability. Thus, to construct a control method for converting irregular motion to regular motion is important from the viewpoint of applications, particularly engineering and medicine. The origin of irregular motions can be divided into two kinds. The first is deterministic chaos, which is driven by nonlinear dynamical systems. The second is random effects, which are driven by stochastic systems, including quantum systems.

The control theory of chaotic motions as irregular motions is well studied in the field of nonlinear dynamical theory. One of the most efficient methods of converting chaotic irregular motion to regular motion is time delayed feedback control [8]. This method allows for the stabilization of unstable periodic orbits that are embedded in a chaotic invariant set. Chaotic irregular motion is converted to periodic regular motion by the time delayed feedback inputs with appropriate feedback gain and time delay. Time delayed feedback control can be applied to various chaotic systems [9–13].

In recent years, time delayed feedback control has been applied to a random walk driven by Gaussian noise, which is one of the typical stochastic systems and molecular dynamical systems in which each particle interacts by obeying the Lennard-Jones potential. The study focused on non-stationary stochastic systems [14] to which conventional analysis for stationary stochastic systems cannot be
applied [15, 16] and showed that time delayed feedback control can control the diffusion processes in these stochastic systems [17]. Let us consider the random walks driven by Gaussian noise.

\[ x_{n+1} = x_n + \alpha \xi_n, \]  

(1)

where \( x_n \) is the state vector of \( m \) random walkers for discrete time \( n \), \( \xi_n \) is an \( m \)-dimensional random variable vector that follows the normal distribution, \( N(0,1) \), for each entry, and \( \alpha \) is the amplitude vector of the noise. Without loss of generality, we can consider a case of a random walker for \( m = 1 \), \( x_{n+1} = x_n + \alpha \xi_n \), where \( x_n \) is the state variable of a random walker and \( \alpha \) is the amplitude of noise. It is well known that a state variable in the system (Eq. (1)) exhibits diffusive motion with a diffusion coefficient, \( \alpha^2 \).

To control the random walk in \( x_n \), we introduce time delayed feedback control input as follows:

\[ x_{n+1} = x_n + \alpha \xi_n + K(x_{n-\tau} - x_n), \]  

(2)

where \( K \) is the feedback gain with a value of \( 0 \leq K < 1 \) and \( \tau \) is the delay time. It is noted that any nonlinear stochastic dynamical system with neutral fixed points can be linearized around its neutral fixed point and described in the abovementioned form (Eq. (1)). Thus, we can apply the control method to nonlinear stochastic dynamical systems, \( x_{n+1} = f(x_n) + \alpha \xi_n \), for a nonlinear function, \( f(x) \) [17]. Furthermore, the method can be extended to continuous dynamical systems with noise straightforwardly.

We calculate the system (Eq. (2)) with \( \alpha = 1 \) and \( K = 0.2 \) numerically. Figure 1 (top) shows the time series of the system (Eq. (2)). It is observed that the time delayed feedback control suppresses the diffusion of a random walk. This result is counterintuitive because from the viewpoint of the theory of

![Graph showing time series and diffusion coefficient](image)

**Fig. 1.** Numerical calculations of the system (Eq. (2)) with \( \alpha = 1 \) and \( K = 0.2 \). (Top) Dynamics in random walks of the system (Eq. (2)). The green and red lines represent dynamics in the system without control, i.e. \( \tau = 0 \), and with control (\( \tau = 100 \)), respectively. (Bottom) Diffusion coefficient \( D \) vs delay time \( \tau \). As \( \tau \) increases, the diffusion of a random walk is suppressed.
differential equations with time delay, time delays make a system unstable. However, the opposite is observed in the system (Eq. (2)). Diffusion coefficient $D$ is calculated using $\lim_{T \to T_\infty} \langle (X_T - X_0)^2 \rangle / T$, where $\langle \rangle$ represents the average over the initial ensemble of values of $X_0$. In Fig. 1 (bottom), the diffusion coefficient is shown as a function of $\tau$. The result is averaged over 1000 realizations with $T_\infty = 100000$. It is evident that $D$ decays monotonically as $\tau$ increases. In principle, it is possible to control the amount of diffusion from zero to a specific value up to $\alpha^2$.

It is important to demonstrate the applicability of time delayed feedback control to systems with non-Gaussian and more complicated noise to extend the range of applications. As examples of diffusion processes that are more complex than random walks, we consider the diffusion process that is observed in chaotic dynamical systems [18–23]. For instance, in systems $y_{t+1} = y_t + v_t$, if quantity $v$ is generated by chaotic dynamical systems with the mixing property, quantity $y_t$ exhibits diffusive statistics. This phenomenon is referred to as deterministic diffusion or chaotic diffusion, and it can be found in various fields [24]. In this study, we show that this control method can be applied to chaotic dynamical systems that exhibit deterministic diffusion.

The rest of this paper is organized as follows: In §2, we discuss the application of the time delayed feedback control method to a system with chaotic noise that is generated by chaotic dynamical systems. The summary is provided in §3.

2. Applications

In this section, we consider the diffusion processes generated by chaotic dynamical systems, which is referred to as a deterministic diffusion. The properties of deterministic diffusion are almost the same as those of typical diffusive processes in stochastic systems. However, the mechanism of the diffusion processes is completely different because deterministic diffusion occurs through chaotic randomness and not through stochastic randomness.

We apply time delayed feedback control to two cases of deterministic diffusion. The first is random walkers with chaotic noise generated by a logistic map, which is a typical chaotic dynamical system, and the second is diffusions induced by the chains of a piecewise-linear map.

2.1 Chaotic random walkers with noise generated by the logistic map

Let us consider a chaotic random walk driven by chaos that is generated by a deterministic dynamical system, as follows:

$$x_{n+1} = x_n + \alpha w_n,$$

where $x_n$ is a state variable of a random walker for discrete time $n$, and $\alpha$ is the amplitude of noise. Variable $w_n$ is generated by a logistic map, and the time average of $w_n$ is zero. Namely, $w_n = \tilde{w}_n - \langle \tilde{w} \rangle$, $\tilde{w}_{n+1} = \alpha \tilde{w}_n (1 - \tilde{w}_n)$, where $\langle \rangle$ denotes the long time average defined as $\langle x \rangle = \sum_{n=1}^{T} x_n / T$ for large $T$. When the logistic map generates chaotic dynamics, the variable $w_n$ corresponds to a stochastic variable and can be considered as noise. Thus, we call the variable $w_n$ chaotic noise. The origin of chaotic noise $w_n$ in this system (Eq. (3)) is more complicated than that of the Gaussian noise (Eq. (1)).

Time delayed feedback control can be applied to this system as follows:

$$x_{n+1} = x_n + \alpha w_n + K (x_{n-\tau} - x_n),$$

where $K$ is the feedback gain with a value of $0 \leq K < 1$ and $\tau$ is the delay time. In this paper, we use the system parameter $\alpha = 1$ and the feedback gain $K = 0.5$.

First, we consider the system with chaotic noise that is generated by a logistic map with a control parameter of $a = 4$. The origin of this chaotic noise is essentially the same as a uniform distribution, and the correlation time of the chaotic noise is zero. The logistic map with $a = 4$ is phase conjugate with a shift map. In addition, the distribution of the chaotic noise does not possess singularities (see Fig. 2 (bottom)).

Figure 2 (top) shows the dynamics of the state variable, $x_n$, of this system (Eq. (4)) without and with control inputs ($\tau = 100$). It is observed that the state variables of a random walker without and with control inputs move diffusively with large and small diffusion coefficients, respectively.
Fig. 2. (Top) Dynamics in chaotic random walks of the system (4) with \(a = 4.0\). The green and red lines represent when \(\tau = 0\) and \(\tau = 100\), respectively. (Bottom) The distribution of chaotic noise that is generated by the logistic map with \(a = 4.0\). The form of the distribution is known as the distribution of \(w \propto \frac{1}{\sqrt{\pi w(1-w)}}\) analytically.

Fig. 3. Diffusion coefficient \(D\) vs delay time \(\tau\) in the system (Eq. (4)) with \(a = 4.0\). Diffusion coefficient \(D\) decreases monotonically as delay time \(\tau\) increases.

In Fig. 3, the diffusion coefficient \(D\), defined by \(\lim_{T \to T_{\infty}} \langle (x_T - x_0)^2 \rangle / T\), is shown as a function of \(\tau\). The result is averaged over 1000 realizations with \(T_{\infty} = 100000\). As is the case with a simple random walk, \(D\) decreases drastically with increase in \(\tau\).

Second, we consider the system with chaotic noise that is generated by the logistic map with a control parameter of \(a = 3.75\). It is found that the distribution of the chaotic noise possesses singularities and the correlation time of the chaotic noise is finite. As seen in Fig. 4 (bottom), the origin of this chaotic noise is more complex than a uniform distribution.

Figure 4 (top) shows the time series of this system without and with control inputs (\(\tau = 100\)).
Fig. 4. (Top) Dynamics in chaotic random walks of the system (Eq. (4)) with $a = 3.75$. The green and red lines represent when $\tau = 0$ and $\tau = 100$, respectively. (Bottom) The distribution of chaotic noise that is generated by the logistic map with $a = 3.75$. It is known that the form of the distribution of chaotic noise is not analytic.

Fig. 5. Diffusion coefficient $D$ vs delay time $\tau$ in the system (Eq. (4)) with $a = 3.75$. Diffusion coefficient $D$ decreases monotonically as delay time $\tau$ increases.

As in the system with $a = 4.0$, a random walker without and with control inputs moves diffusively with large and small diffusion coefficients, respectively. In Fig. 5, diffusion coefficient $D$ is shown as a function of $\tau$, and $D$ decreases with increase in $\tau$, as is the case in the system with $a = 4.0$.

The abovementioned results indicate that the time delayed feedback control method can be applied to random walkers with complicated noises that are generated by chaotic dynamical systems. We consider that the control method can be applied to systems with chaotic noise that is generated by other chaotic systems such as the Lorenz model.
2.2 Deterministic diffusion induced by chains of a piecewise-linear map

In this subsection, we consider a system that does not separate the term of randomness from the equation of motion of the state variable, i.e., a system with non-additional noise. Here, we consider the following map as an example of systems that exhibit the deterministic diffusion:  

$$x_{n+1} = f(x_n) = \begin{cases} a(x_n - N) + N, & (N < x_n \leq N + \frac{1}{2}) \\ a(x_n - N - 1) + N + 1, & (N + \frac{1}{2} < x_n \leq N + 1) \end{cases} \quad \text{for } \forall N \in \mathbb{N}. \quad (5)$$

This is a piecewise-linear map constructed using two lines with a uniform slope, $a$, within a $1 \times 1$ square and arranging them in a stepwise manner as sketched in Fig. 6. It is found that the state variable, $x_n$, of this system exhibits diffusive motion for any slope $a$ [22, 23]. Furthermore, the diffusion coefficient ($D$) in the case of the system with integer slope $a$ is given as follows [19–21]:  

$$D(a) = \begin{cases} \frac{1}{27}(a - 1)(a - 2), & a = 2k, k \in \mathbb{N} \\ \frac{1}{27}(a - 1)^2, & a = 2k - 1, k \in \mathbb{N} \end{cases}. \quad (6)$$

The diffusion coefficient for general parameters, $a \in \mathbb{R}$, has complex parameter-dependence, that is, the diffusion coefficient exhibits a fractal structure as a function of control parameter $a$ [22].

Time delayed feedback control can be applied to this system as follows:  

$$x_{n+1} = f(x_n) = \begin{cases} a(x_n - N) + N + K(x_{n-\tau} - x_n), & (N < x_n \leq N + \frac{1}{2}) \\ a(x_n - N - 1) + N + 1 + K(x_{n-\tau} - x_n), & (N + \frac{1}{2} < x_n \leq N + 1) \end{cases} \quad \text{for } \forall N \in \mathbb{N}. \quad (7)$$

where $K$ is the feedback gain with a value of $0 \leq K < 1$ and $\tau$ is the delay time. In this paper, we use the feedback gain $K = 0.5$.

In the system that is considered in the previous subsection, $x_{n+1} = x_n + \alpha w_n$, the deterministic term, $x_n$, and the noise-like term, $\alpha w_n$, separate completely, indicating that this system is a kind of stochastic system with additional noise, which is the same as simple random walks. Contrary, the deterministic and noise-like terms in this system (Eq. (5)) do not separate, indicating that this systems corresponds to a stochastic system with non-additional noise.

Figure 7 shows the state variable, $x_n$, of this system without and with control inputs ($\tau = 100$). The state variables without and with control inputs move diffusively with large and small diffusion coefficients, respectively. This is the same as the result described in the previous subsection.

We have checked that the control method can be applied to nonlinear maps in addition to piecewise-linear maps.

3. Summary

In this work, we have applied time delayed feedback control to systems with chaotic noise that is generated by a logistic map and the chains of a piecewise-linear map. The chaotic noise in these
Fig. 7. Dynamics in chaotic random walks of the system (Eq. (7)) with $a = 3.1415$. The green and red lines represent when $\tau = 0$ and $\tau = 100$, respectively.

systems is more complex than Gaussian noise. In the case of systems with chaotic noise generated by the logistic map (Eq. (4)), the origin of the chaotic noise is not a Gaussian distribution; it is a uniform distribution ($a = 4$) and a distribution with singularities ($a = 3.75$). Note that the time series of the chaotic noise generated by the logistic map with $a = 3.75$ has finite correlation time. The diffusion processes induced by the chains of a piecewise-linear map (Eq. (7)) correspond to stochastic systems with non-additional noise. Consequently, the diffusion coefficients in both systems can be controled by time delayed feedback control.

The control method in this study has been used for nonlinear dynamics. The mechanism of this control of diffusion driven by chaos is different from chaos control, which stabilizes an unstable periodic orbit, because the control of deterministic diffusion is not related to the stabilization of unstable periodic orbits. Furthermore, the mechanism is different from the control of simple random walks because the systems that are considered in this study are not stochastic; they are deterministic. Understanding the mechanism of the control method in this work and its applications to more complicated systems, such as systems with large degrees of freedom, will be reported in near future.

Acknowledgments
It is a pleasure to thank Professor H. Ando for valuable discussions and suggestions. This work is supported by JSPS KAKENHI Grant Number 15K17582. Computations are done by the aid of Collaborative Research Program for Young Scientists of ACCMS and IIMC, Kyoto University and the MEXT Joint Usage / Research Center “Center for Mathematical Modeling and Applications”, Meiji University, Meiji Institute for Advanced Study of Mathematical Sciences (MIMS).

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