Equational noethericity for graphs and hypergraphs

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Abstract. In 1999th G. Baumslag, A. Miasnikov and V. Remeslennikov posted the following problem. Let $G$ be a group. Does a noethericity by equations in one variable imply a noethericity by equations for the group $G$ in the language with constants from $G$? Same question may be asked for graphs and hypergraphs. It is shown in the article that the answer to the question above is positive for graphs and is negative for hypergraphs.

1. Introduction

Universal algebraic geometry is a relatively new mathematical direction that study equations in various algebraic structures. See [1, 2] for history and details about it. The property to be an equationally noetherian algebraic structure is very important from the point of view of universal algebraic geometry. In this pleasant case there exist so-called Unification theorems which give description of algebraic sets from different points of view. Therefore the knowledge what algebraic structures are equationally noetherian is important for the study of algebraic geometry over algebraic structures.

Definition 1. An algebraic structure $A$ is called equationally noetherian if for any system of equations $S$ in finitely many unknowns there exists a finite subsystem $S_0 \subset S$ such that their solution sets are the same.

There exists a weakened version of the equationally noetherian property:

Definition 2. Let $n$ be a natural number. We call that algebraic structure $A$ is $n$-equationally noetherian if any system of equations over $n$ unknowns is equivalent to its finite subsystem.

It easily follows from the definitions that if an algebraic structure is 1-equationally non-noetherian then it is equationally non-noetherian. The converse statement is not true in general and in 1999th in the article [2] G.Baumslag, A.Miasnikov and V.Remeslennikov posted the following question:

Question Let $G$ be a group and $L$ be the standard language of group theory with constants from $G$. If the group $G$ is noetherian by equations in one variable does it imply that $G$ is equationally noetherian?
The original question is still unresolved yet, but we may post the same problem for different categories of algebraic structures, such as graphs, orders, monoids, etc. A positive answer to the question above for a particular category of algebraic structures means that the problem of searching of equationally noetherian algebraic structures may be bounded by considering just one variable equations.

In this paper it is shown for undirected graphs that the question of determining which undirected graph is equationally noetherian is equivalent to the same question just for one variable equations.

In contrast, there exists a 3-hypergraph $H$ (3-hypergraphs means that any edge join only three different vertices) such that the notions of 1-equationally noethericity and equationally noethericity do not coincide. In addition, the author want to mention that he tried to construct an infinitely generated 3-step nilpotent group based on the hypergraph $H$ (in some sense a partially commutative group over hypergraph) to negatively solve the problem of Baumslag, Miasnikov and Remeslennikov but it didn’t lead to success.

2. Preliminaries and examples

Denote by $\mathcal{L} = \{ E(x, y) \}$ the standard language of undirected simple graphs where $E(x, y)$ is a binary joining predicate. Let $\Gamma = (V, E)$ be a graph with a vertex set $V$ and edges $E$. We will consider equations in the language $\mathcal{L}_{gr} = \mathcal{L} \cup \{ V(\Gamma) \}$, also known as Diophantine equations over graphs. The graph language $\mathcal{L}_{gr}$ is not very reach and we have just six kinds of equations: $E(x, y)$, $E(x, v)$, $E(u, v)$, $x = y$, $x = v$, $u = v$, where $x, y$ are variables or unknowns and $u, v$ are some constant vertices.

Also we introduce equations for undirected 3-hypergraphs in a similar way. Let $H = (V, F)$ be a hypergraph where $V$ is a vertex set and $F$ is a set of edges. Let $\mathcal{L} = \{ F(x, y, z) \}$ be the language of 3-hypergraphs. The 3-place predicate $F$ serves as analog for the binary joining predicate $E$ for graphs in the category of hypergraphs. $F(x, y, z)$ is true if and only if the vertices $x, y$ and $z$ are joined in $H$. Denote by $\mathcal{L}_{hyp} = \mathcal{L} \cup \{ V(H) \}$ the language of 3-hypergraphs extended by the vertex set of $H$. There are seven different kinds of equations here: $E(x, y, z)$, $E(x, y, v)$, $E(x, u, v), E(u, v, t)$, $x = y$, $x = v$, $u = v$, where $x, y, z$ are unknowns and $u, v, t$ are some constant vertices.

Below are examples of equationally noetherian algebraic structures:

- Abelian groups, linear groups, f.g. nilpotent group, \ldots;
- Finite algebraic structures;
- Locally finite simple graphs;

There are equationally nonnoetherian algebraic structures:

- $A \wr B$, $A$ – nonabelian, $B$ – infinite.
- $G^{\omega}$, where $G$ – nonabelian group.
- Some infinite graphs, orders, hypergraphs.
The reasoning for the facts above can be found in [2, 3, 1, 4]. The following lemma was proved in [5]. It is a powerful tool to discover non-Noetherian algebraic structures and we will intensively use this lemma in the paper.

**Lemma 1.** An algebraic structure \( A = \langle A, \mathcal{L} \rangle \) is not equationally Noetherian if and only if, there exist series \((a_i)_{i \in \mathbb{N}}, a_i \in A^n\) and series of equations \((s_i(X))_{i \in \mathbb{N}}\) of the language \( \mathcal{L} \) such that \( A \not\models s_i(a_i) \) for any \( i \), and \( A \models s_j(a_i) \) for any \( j < i \).

The picture below is a demonstration of the relations between the equations \((s_i(x))_{i \in \mathbb{N}}\) and the elements \((a_i)_{i \in \mathbb{N}}\) from the lemma above. The solid edge between \( s_i \) and \( a_j \) means that \( A \models s_i(a_j) \) and the dashed edge means that \( A \not\models s_i(a_i) \).

Let \( Q \) be an infinite clique with a vertex set \( V = \{a_1, a_2, \ldots\} \). Then the following system of equations and elements satisfy Lemma 1.

\[
\begin{align*}
E(x, a_1) & \quad a_1 \\
E(x, a_2) & \quad a_2 \\
\vdots & \quad \vdots \\
E(x, a_i) & \quad a_i \\
\vdots & \quad \vdots 
\end{align*}
\]

Therefore an infinite (countable or not) clique is not equationally Noetherian.

### 3. Equational noethericity for simple graphs

Notice that from the proof of Lemma 1, the following statement follows:

**Corollary 1.** For any non-Noetherian system of equations \( S(X) \) in \( n \) variables \( X = \{x_1, \ldots, x_n\} \) over an algebraic structure \( A \). Then the following statements are true:

(i) There exists an infinite subsystem \( S' = \{s_1(X), \ldots, s_i(X), \ldots\} \subset S \) and a sequence of elements \((a_i)_{i \in \mathbb{N}}, a_i \in A^n\) such that the conditions of the Lemma 1 are satisfied, that is, the system \( S' \) is not equationally Noetherian.

(ii) Any infinite subsystem \( S'' \subset S' \) and the corresponding subsequence of elements \((a'_i)_{i \in \mathbb{N}} \subset (a_i)_{i \in \mathbb{N}}\) also satisfies Lemma 1, and \( S'' \) is also equationally non-Noetherian.

(iii) Any equation from \( S' \) has an infinite number of solutions.
The following statement is true for the reason that in the language of graph theory \( \mathcal{L}(\Gamma) \) there is a finite number of types of equations.

**Proposition 1.** Let \( \Gamma \) be a simple graph and \( \mathcal{L}(\Gamma) \) be a language of simple graphs with constants \( V(\Gamma) \). Let \( S \) be an infinite system of equations in the language \( \mathcal{L}(\Gamma) \) in a finite set of variables \( X \) that does not contain infinite subsystems of equations, consisting only of constants. Then there is an infinite subsystem of equations \( S^1 \subset S \) in one variable from the set \( X \)

**Proof.** Let \( X = \{x_1, \ldots, x_n\} \). Then \( S = S^c \cup S' \cup S^{x_1} \cup \ldots \cup S^{x_n} \), where the subsystem \( S^{x_i} \) contains equations in only one variable \( x_i \), \( i = 1, \ldots, n \), \( S' \) contains equations without constants, \( S^c \) contains equations only with constants.

Since the number of unknowns is finite, the subsystem \( S' \) is finite. \( S^c \) is finite by condition. Hence it follows that at least one of the subsystems \( S^{x_i} \), \( i = 1, \ldots, n \) is infinite.

The following theorem gives an affirmative answer to the Problem 1 in [2], formulated for graphs.

**Theorem 1.** Let \( \Gamma \) be a simple graph that is not noetherian in the language of simple graphs with constants \( \mathcal{L}(\Gamma) = \{E^{(2)}\} \cup V(\Gamma) \). Then there is a sequence of equations \( S = (E(x, b_i))_{i \in \mathbb{N}} \) in one variable for which the conditions of the lemma 1 are satisfied. Therefore, the graph \( \Gamma \) is not equationally noetherian in one variable.

**Proof.** By the corollary 1, there exists an infinite system of equations \( S(X) \) and a sequence of elements \( (a_i)_{i \in \mathbb{N}} \) satisfying the lemma 1. According to the 1, there is a subsystem of equations \( S' \subseteq S \) in one variable \( x \). By the corollary 1, the subsystem \( S' \) is also non-noetherian. Then \( S'(x) = \{\{E(x, b_i), i \in I_1\} \cup \{x = c_i, i \in I_2\}\} \). Since the system \( S' \) is non-noetherian, then it does not contain equations of the form \( x = c \).

### 4. Equational noethericiicty for hypergraphs

In contrast with simple graphs, the notions of 1-equational noethericity and equational noethericity are not equivalent for hypergraphs. In this section we will construct an example of a 3-hypergraph which is 1-equationally noetherian, but it is not equationally noetherian in general.

We start with an infinite simple graph \( \Gamma \) with a vertex set \( V = \{a_1, a_2, \ldots, b_1, b_2, \ldots\} \). Edges defined by the rule: \( a_i \) and \( b_j \) are joined if and only if \( i < j \). The graph \( \Gamma \) is not equationally noetherian since equations \( E(x, b_i), i \in \mathbb{N} \) and vertices \( a_i, i \in \mathbb{N} \) satisfy the lemma 1:
At the second step we bifurcate all the vertices $b_i$ to $b_i$ and $b'_i$ keeping edges as for $b_i$ and joining $b_i$ with $b'_i$. Denote the new graph as $\Gamma'$. Now we may assume that $\Gamma'$ is a 3-hypergraph since each edge between $a_i$ and $b_j$ from the first step is mapped to the triangle joining $a_i$, $b_j$ and $b'_j$. Below is the $\Gamma'$ where green edges are gotten from the second step:

Consider the following system of equations over $\Gamma'$ and elements $(b_i, b'_i) \in V^2(\Gamma')$:

$$
\begin{align*}
E(x, y, a_1) & (b_1, b'_1) \\
E(x, y, a_2) & (b_2, b'_2) \\
\vdots & \vdots \\
E(x, y, a_i) & (b_i, b'_i) \\
\vdots & \vdots
\end{align*}
$$

These equations and elements obviously satisfy the lemma 1 and therefore $\Gamma'$ is not equationally noetherian hypergraph.

It remains to consider one variable equations of the type $H(x, u, v)$ where $x$ is an unknown and $u, v$ are constants from $V(\Gamma')$. If $H(x, u, v)$ holds then the following cases are possible:

- $u = b_j, v = b'_j$ then $x = a_i, i < j$. The number of solutions is equals to $j - 1$.
- $u = b_j, v = a_i, i < j$ then $x = b'_j$. There is a single solution in this case.
- $u = b'_j, v = a_i, i < j$ then $x = b_j$. There is a single solution in this case.

Therefore any one variable equation over $\Gamma'$ has just finite number of different solutions. Then by corollary 1 $\Gamma'$ is one variable equationally noetherian hypergraph. By the above reasoning the following theorem holds:
Theorem 2. The notions of equational noethericity and 1-equational noethericity are not equivalent for hypergraphs.

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