Fast robustness quantification with variational Bayes

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Abstract

Bayesian hierarchical models are increasing popular in economics. When using hierarchical models, it is useful not only to calculate posterior expectations, but also to measure the robustness of these expectations to reasonable alternative prior choices. We use variational Bayes and linear response methods to provide fast, accurate posterior means and robustness measures with an application to measuring the effectiveness of microcredit in the developing world.

1. Introduction

Researchers and policymakers in economics increasingly have access to results from several experimental studies of the same phenomenon. Particularly in development economics, following the recent proliferation of randomized controlled trials to study key anti-poverty interventions, the question of how to aggregate the results of multiple experiments across different contexts has now arisen. Recent attempts to perform this aggregation have noted that different studies of the same intervention often produce different results, but both the extent of the true variation in the underlying treatment effects and the source of such variation are often unclear (Meager, 2015; Vivalt, 2015; Burke et al., 2015). There is however a methodology which is ideally suited to aggregating evidence and assessing the extent of heterogeneity across contexts, and has been well developed by statisticians: Bayesian hierarchical models (Rubin, 1981; Gelman & Rubin, 1992). Bayesian analysis turns the data model likelihood and a distribution of prior beliefs into a posterior distribution over the model parameters through application of Bayes’ rule. Often, the posterior is summarized by certain moments of the model parameters (e.g. the mean or variance).

However, with moderately large datasets, Bayesian hierarchical model posteriors can be time-consuming to estimate even using cutting-edge Markov Chain Monte Carlo (MCMC) software such as Stan (Stan, 2015). Furthermore, lower levels of the hierarchical model, which often represent the quantities of practical interest, can be sensitive to the choice of priors, leading to non-robust posteriors. If different reasonable choices of the prior lead to substantially different posterior means, then the model is not robust, since different arbitrary prior choices could lead to different substantive conclusions. It is important for the modeler to be aware of non-robustness, either so that the model can be improved or the conclusions qualified. Though procedures exist to measure robustness with MCMC draws, an easy-to-use, general-purpose methodology is still lacking (Berger et al., 2012).

In this paper we take a step towards addressing the slow estimation times of hierarchical models and provide automated measures of robustness using variational Bayes (VB). VB is an optimization-based method for performing approximate Bayesian posterior inference (Wainwright & Jordan, 2008; Bishop, 2006). In contrast to MCMC, which produces draws from a distribution that approaches the true
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posterior asymptotically, VB finds the best approximation to the true posterior within a restricted class of distributions. Since VB is an optimization procedure rather than a Markov chain, it can often produce posterior approximations much more quickly than MCMC, though at the cost of providing only an approximation, even asymptotically.

In addition to being generally faster than MCMC, variational Bayes (VB) techniques are more readily amenable to robustness analysis. The derivative of a posterior expectation with respect to a perturbation of the prior or the data is a measure of local robustness to the prior or likelihood (Gustafson, 2012). Because VB casts posterior inference as an optimization problem, its methodology is built on the ability to calculate derivatives of posterior quantities with respect to model parameters, even in very complex models.

In order to provide fast estimates of local robustness, we use the machinery of Linear response variational Bayes (LRVB), previously developed by (Giordano et al., 2015). Giordano et al. (2015) show how perturbing the posterior distribution can yield improved estimates of posterior covariance over vanilla VB. In this work, we demonstrate that a similar idea can be applied to derive fast, easy-to-use robustness measures.

In the remainder of this work, we start by briefly describing VB and LRVB in Section 2. In Section 3 we describe how to measure local robustness using LRVB. Finally, in Section 4, we demonstrate our methods on a meta-analysis of data from seven randomized controlled trials of microcredit expansions.

2. Variational Bayes and linear response

Denote our \( N \) data points by \( x = (x_1, \ldots, x_N) \) with \( x_n \in \mathbb{R}^D \). Denote our parameter by the vector \( \theta \in \mathbb{R}^K \). We denote the prior parameters by \( \alpha \in \mathbb{R}^M \). Let \( p^\alpha \) denote the posterior distribution of \( \theta \), as given by Bayes’ Theorem:

\[
p^\alpha(\theta) := p(\theta|x, \alpha) = \frac{p(x|\theta) p(\theta|\alpha)}{p(x)}.
\]

VB approximates \( p^\alpha(\theta) \) by selecting the distribution, \( q^\alpha \), that is closest to \( p^\alpha(\theta) \) in Kullback-Leibler (KL) divergence within a restricted class \( Q \). We consider the case where \( Q \) is a class of products of exponential family distributions (Bishop, 2006):

\[
q^\alpha := \arg\min_{q \in Q} \{KL(q||p)\} \text{ for } \alpha \in \mathbb{R}^M\]

\[
Q = \left\{ q : q(\theta) = \prod_{k=1}^K q(\theta_k); q(\theta_k) \propto e^{\eta_k \theta_k}, \forall k \right\}
\]

\[
KL := -E_q[\log p(x|\theta)] + E_q[\log p(\theta|\alpha)] + E_q[\log q(\theta)] + \text{Constant}.
\]

We assume that \( q^\alpha \), the solution to Eq. (1), has interior exponential family parameter \( \eta_k \). In this case, \( q^\alpha \) can be completely characterized by its mean parameters, \( m := E_{q^\alpha}[\theta] \) (Wainwright & Jordan, 2008).

It is well-known that optimal \( q^\alpha \) chosen from this particular \( Q \) under-estimates the posterior variance of \( \theta \) (and provides no estimate of the covariance between distinct \( \theta_k \)) even when the posterior means are well-estimated (Turner & Sahani, 2011; Wang & Titterington, 2004). In order to improve the posterior covariance estimates, Giordano et al. (2015) estimates derivatives of the posterior cumulant generating function by perturbing the objective in Eq. (1), giving the LRVB covariance estimate:

\[
\Sigma := \left( \frac{\partial^2 KL}{\partial m \partial m^T} \right)^{-1}.
\]

The LRVB approximation Eq. (2) is exactly equal to \( \Sigma := \text{Cov}_p(\theta) \) when VB estimates the posterior means exactly. If the posterior mean estimates from VB are close to the truth, then \( \Sigma \) can form a good approximation of \( \Sigma \). In our experiments described in section Section 4 we find this to be the case. In Section 3, we will discuss how the idea of LRVB can be extended to provide local robustness measures.

3. Measuring robustness with LRVB

A typical end product of a Bayesian analysis might be a posterior expectation of some function \( g(\theta) \) (e.g., a mean or variance): \( E_{p^\alpha}[g(\theta)] \), which is a functional of \( g \). We suppose that we have determined that the prior parameter \( \alpha \) belongs to some set \( A \), perhaps after expert prior elicitation. Finding the extrema of \( E_{p^\alpha}[g(\theta)] \) as \( \alpha \) ranges over all of \( A \) is intractable or difficult except in special cases (Moreno, 2012). An alternative is to examine how much \( E_{p^\alpha}[g(\theta)] \) changes locally in response to small perturbations in the value of \( \alpha \):

\[
\frac{dE_{p^\alpha}[g(\theta)]}{d\alpha} \bigg|_{\Delta \alpha}.
\]

That is, we consider local robustness properties in lieu of global ones (Gustafson, 2012). By calculating Eq. (3) for all \( \Delta \alpha \in A - \alpha \), we can estimate the robustness of \( E_{p^\alpha}[g(\theta)] \) in a small neighborhood of \( \alpha \). For the rest of the paper we will take \( g(\theta) = \theta \) for simplicity.

Appendix A of Giordano et al. (2015) shows that the derivation of the approximate covariance given in Eq. (2) arise as a special case of a general perturbation formula,

\[
\frac{dE_{q^\alpha}[\theta]}{dt} \bigg|_{t=0} = \left( \frac{\partial^2 KL}{\partial m \partial m^T} \right)^{-1} \frac{\partial f(m)}{\partial m}.
\]

1See Giordano et al. (2015) for discussion of how to efficiently calculate and invert \( \frac{\partial^2 KL}{\partial m \partial m^T} \).
By choosing an appropriate \( f(m) \) in Eq. (4), we can calculate estimates of the local prior sensitivity from Eq. (3). Let \( \alpha_t := \alpha + \delta_\alpha t \) be the prior parameter perturbed in the direction \( \delta_\alpha \) by a small scalar amount \( t \), so that \( \delta_\alpha t \) plays the role of \( \Delta \alpha \) in Eq. (3). For example, we could measure the sensitivity of \( \mathbb{E}_q[\theta] \) to the \( i \)th component of \( \alpha \) by taking \( \delta_\alpha \) to be a vector of all zeros except with a 1 in the \( i \)th place.

To tidy up notation, define \( \ell(\alpha, m) := \mathbb{E}_q[\log p(\theta|\alpha)] \). Note that \( \ell(\alpha, m) \) is a smooth function of \( m \), since it is an expectation with respect to the exponential family \( q \), which is completely parameterized by \( m \). If we additionally assume that \( \log p(\theta|\alpha) \) is a smooth function of \( \alpha \), then by a Taylor expansion in \( \delta_\alpha t \),

\[
\ell(\alpha + t\delta_\alpha, m) = \ell(\alpha, m) + \frac{\partial \ell}{\partial \alpha} \delta_\alpha t + O(t^2).
\]

We can then estimate the sensitivity of \( \mathbb{E}_q[\theta] \) to the change \( \delta_\alpha t \) in the prior parameter:

\[
q_t := \arg\min_{\eta \in \mathcal{O}} \left\{ KL + \frac{\partial \ell}{\partial \alpha} \delta_\alpha t \right\}
\]

\[
\frac{d\mathbb{E}_q[\theta]}{dt}_{t=0} = \left( \frac{\partial^2 KL}{\partial m \partial \alpha^2} \right)^{-1} \frac{\partial^2 \ell}{\partial m \partial \alpha^2} \delta_\alpha.
\]

This easy-to-calculate closed-form expression is the LRVB approximation to the local prior sensitivity Eq. (3) to the change \( \delta_\alpha t \). As in Giordano et al. (2015), these derivatives are in fact the exact sensitivity of the variational posterior expectations to prior perturbation. The extent to which it represents the true prior sensitivity depends on the extent to which the VB means are good estimates of the true posterior means.

4. Microcredit experiment

We apply the methods of Section 2 and Section 3 to a hierarchical model from Meager (2015). Randomized controlled trials were run in seven different sites to try to measure the effect of access to microcredit on various measures of business success, household poverty indicators, and community welfare. However, it was unclear what if any generalizable information had been learned about microcredit that could be applied to other settings for policy purposes. Thus, Meager (2015) fit a series of Bayesian hierarchical models to estimate the general impact of microcredit on poor households and assess the heterogeneity in this impact across the studies. For the purposes of demonstrating robust Bayes techniques with VB, we will focus on the simpler of the two models in Meager (2015) and ignore covariate information.

We will index sites with \( k = 1, \ldots, K \) (here, \( K = 7 \)) and businesses within a site by \( n = 1, \ldots, N_k \) (\( N_k \) ranged from 961 to 16560). In site \( k \) and business \( n \) we observe whether the business was randomly selected for increased access to microcredit, denoted \( T_{nk} \), and the profit after intervention, \( y_{nk} \). We follow Rubin (1981) and assume that each site has an idiosyncratic average profit, \( \mu_k \), and average improvement in profit, \( \tau_k \), due to the intervention. Given \( \mu_k, \tau_k, \) and \( T_{nk} \), the observed profit is assumed to be generated with variance \( \sigma_{\tau k}^2 \) according to

\[
y_{nk}|\mu_k, \tau_k, T_{nk}, \sigma_k \overset{\text{indep}}{\sim} N(\mu_k + T_{nk} \tau_k, \sigma_k^2).
\]

The site effects, \((\mu_k, \tau_k)\), are assumed to be drawn independently from an overall pool of effects. For a given \( k \), \( \mu_k \) and \( \tau_k \) may be correlated.

\[
\begin{pmatrix} \mu_k \\ \tau_k \end{pmatrix} \overset{\text{indep}}{\sim} N\left( \begin{pmatrix} \mu \tau \end{pmatrix}, C \right).
\]

The effects \( \mu, \tau, \sigma_{\tau k}^2 \), and the covariance matrix \( C \) are unknown parameters that require priors. For the covariance matrix \( C \), we followed the recommended practice of the software package Stan (Stan, 2015) and used the non-conjugate LKJ prior (Lewandowski et al., 2009) with covariance parameter \( \eta = 15.01 \) and inverse scale prior \( \Gamma(20.01, 20.01) \). We used a conjugate gamma prior on \( \sigma_k^{-2} \overset{\text{iid}}{\sim} \Gamma(2.01, 2.01) \). Finally, for \((\mu, \tau)\) we used a bivariate normal prior:

\[
\begin{pmatrix} \mu \\ \tau \end{pmatrix} \overset{\text{indep}}{\sim} N\left( \begin{pmatrix} \mu_0 \tau_0 \end{pmatrix}, \Lambda^{-1} \right).
\]

We used \( \mu_0 = \tau_0 = 0 \), and \( \Lambda \) with entries 0.03 and 0.02 on the diagonals and zero off-diagonal.

To generate the MCMC samples, we used Stan (Stan, 2015). To calculate all the derivatives and Hessians necessary for VB and LRVB we implemented the objective function in C++ and then used the autodifferentiation library of Stan (Carpenter et al., 2015).

4.1. Speed and validity of LRVB

VB was over an order of magnitude faster than MCMC. Generating one set of 2500 MCMC draws took 45 minutes. Optimizing the VB objective and calculating the LRVB estimates, including all the reported sensitivity measures, took 58 seconds.

As can be seen in Fig. (1), the VB and MCMC posterior means are nearly identical, indicating that the necessary assumptions for LRVB hold. Next, Fig. (2) shows that the ordinary VB standard deviations underestimate the true posterior standard deviations (as measured by MCMC, which we take to be the ground truth), but that LRVB provides a good correction.

Next, we turn to the evaluation of robustness, with an emphasis on the prior on \((\mu, \tau)\). In Fig. (3) we compare
our LRVB robustness estimates to the (extremely time-consuming) effect of manually changing a prior parameter and re-running the MCMC chain. Specifically, we changed \( \Lambda_{11} \) from 0.03 to 0.04 and measured how the change in the posterior mean compared with the change predicted by LRVB. The results in Fig. (3) show that the LRVB sensitivity estimates match the actual sensitivity very closely. Since VB estimates the means reasonably well, as shown in Fig. (1), and Eq. (5) gives the exact sensitivity of the VB means to prior perturbations, Fig. (3) should not come as a surprise, but it is a reassuring sanity check.

4.2. Analysis results

Finally, we examine what our results tell us about microcredit within the context of this particular simple model and prior choice. We will focus on the parameter \( \tau \) in Eq. (6), which is intended to represent the overall “global” average microcredit effectiveness.

The VB posterior mean and LRVB standard deviation for \( \tau \) are

\[
E_q[\tau] = 3.08 \quad \text{StdDev}_q(\tau) = 1.83.
\]

Under a normal assumption on the posterior of \( \tau \), this does not provide strong evidence for the effectiveness of microcredit, since the mean is only 1.68 standard deviations from zero.

However, the standard deviation is not necessarily the full story. We might also ask whether, if our priors were different, we might have come to a different conclusion. The sensitivity of \( \tau \) to its prior, Eq. (7), is shown in Fig. (4). In this graph, we report the sensitivity in units of posterior standard deviations of \( \tau \) in order to show how one can affect posterior inference by changing the prior parameter. In particular, notice that \( E_q[\tau] \) is quite sensitive to the prior parameter \( \Lambda \). For example, the first bar in Fig. (4) shows that increasing \( \Lambda_{11} \) by 0.04 would increase \( E_q[\tau] \) by 0.04 \cdot 8.88 standard deviations, which would be enough to make \( \tau \) look significantly greater than zero. \( E_q[\tau] \) is even more sensitive to \( \Lambda_{12}(= \Lambda_{21}) \), the off-diagonal covariance terms. As seen in the second graph of Fig. (4), it is not particularly sensitive to its prior mean.

The meaning of non-robustness results like this depends on the modeler’s beliefs about the prior. In this case, the question is whether we think that a change of 0.04 in \( \Lambda_{11} \) or other similar influential perturbations indicated by Fig. (4) would be reasonable expressions of prior uncertainty. This decision must always depend on the context. If a reasonable range of priors could lead to a range of posterior means that greatly exceeds the spread of the original posterior, then the posterior standard deviation must represent an under-estimate of subjective uncertainty. In any case, as a prerequisite to making such decisions, the modeler needs to be able to measure the robustness, and this measurement is made easily available through LRVB.

5. Conclusion

Hierarchical models are a valuable tool for the social sciences, but they can be slow to fit with MCMC. Furthermore, they can suffer from non-robustness in the form of sensitivity to the choice of priors, and MCMC does not provide an easy-to-use, general-purpose robustness measure. VB, together with LRVB, can provide good approximations to Bayesian posteriors over an order of magnitude faster than MCMC. Furthermore, LRVB also provides easy-to-calculate measures of robustness that can alert the modeler to excessive prior sensitivity.
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Figure 4. Normalized sensitivity of tau

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