The Cosmological Origin of Inertia: Mach’s Principle

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The axes of gyroscopes experimentally define local non-rotating frames, i.e. the time-evolution of axes of inertial frames. But what physical cause governs the time-evolution of gyroscope axes? Starting from an unperturbed FRW cosmology with k = 0 we consider cosmological vorticity perturbations (i.e. vector perturbations) at the linear level, and we ask: Will cosmological rotational perturbations exactly drag the axes of a gyroscopes relative to the directions of geodesics to galaxies in the asymptotic FRW space? Using Cartan’s formalism with local orthonormal bases we cast the laws of gravitomagnetism into a form showing the close correspondence with the laws of ordinary magnetism. Our results, valid for any equation of state for cosmological matter, are: 1) The dragging of a gyroscope axis by rotational perturbations of matter beyond the H-dot radius (H = Hubble constant) is exponentially suppressed. 2) If the perturbation of matter is a homogeneous rotation inside a perturbation radius, then exact dragging of the gyroscope axis by the rotational perturbation is reached exponentially fast as the perturbation radius gets larger than the H-dot radius. 3) The time-evolution of a gyroscope axis exactly follows a specific average of the matter inside the H-dot radius. In this sense Mach’s Principle (that axes of local non-rotating frames precisely follow some average of the motion of cosmic matter) is a consequence of cosmology with Einstein Gravity.

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I. INTRODUCTION

The observational fact.— In tests of general relativity in the solar system, two type of things are compared. On the one hand measurements of the precession of perihelion shifts and the measurement undertaken by Newton’s bucket experiment) is given by the stars. Since stars and quasars have proper motions, this formulation cannot be exact. But the proper motions of quasars relative to the uniform Hubble flow are negligible for present-day tests of General Relativity.— We conclude that the measurements of perihelion shifts and the measurement undertaken by Gravity Probe B are tests of two things combined, on the one hand tests of Einstein’s equations in the solar system, on the other hand tests of the principle “Mach0”.

Classical mechanics, special relativity, and general relativity for isolated systems in asymptotic Minkowski space give no explanation of the observational fact of “Mach0”, except saying that this is an accident of initial conditions. In these theories one could have different initial conditions where all stars could be in rotational motion around us relative to our gyroscopes.— Within these three theories the local non-rotating frames (one aspect of inertial frames) can be experimentally determined by axes of gyroscopes, and conversely the time-evolution of gyroscopes axes is dictated by the laws of inertia, i.e. that the gyroscopes axes cannot rotate with respect to local inertial axes. Hence in these three theories things are fully consistent but circular, and the question remains: What physical cause governs the time-evolution of the axes of gyroscopes and inertial frames?

Mach’s Principle.— In the 1880’s Mach stated clearly and forcefully, as an alternative to Newtonian physics, the hypothesis that the axes of local non-rotating frames (i.e. axes of gyroscopes) in their time-evolution are determined by (are exactly dragged by, precisely follow) “some average” of the motion of matter in the universe. This is what we take as the formulation of Mach’s principle.— Many alternative formulations of Mach’s principle have been proposed by other authors later. We shall discuss Einstein’s proposal in a future paper. Many of the alternatives to Mach’s original formulation have been enumerated and briefly discussed by Bondi and Samuel. Quite a number of these alternatives have almost nothing in common with Mach’s ideas.

Gravitomagnetism.— At the time of Mach there was no known mechanism, by which matter in the universe could influence the motion of gyroscope axes. With General Relativity came the needed mechanism, gravitomagnetism. Thirring in 1918 analyzed the partial dragging of the axes of inertial frames inside a rotating in-
finely thin spherical shell with uniform surface mass density and total mass $\mathcal{M}$. In the weak field approximation, $G_N(M/R)_{\text{shell}} \ll 1$, he found that inside the shell the axes of local inertial systems at all points rotate relative to asymptotic Minkowski space with the same precession rate $\Omega = f_{\text{drag}}\Omega_{\text{shell}}$. For the dragging fraction $f_{\text{drag}}$ he obtained $f_{\text{drag}} = 4G_N(M/R)_{\text{shell}} \ll 1$. There is only a tiny dragging effect, unless $(M/R)_{\text{shell}}$ approaches the value for a black hole. To be relevant for Mach’s hypothesis, exact dragging by the masses in the universe, not a small influence, one must go to cosmological models.

In this paper we analyze realistic cosmological models (as opposed to toy models) with realistic cosmological matter (as opposed to the contrived energy-momentum tensors discussed in the literature). We start from an unperturbed Friedmann-Robertson-Walker (FRW) cosmology with $k=0$, we add the most general cosmological vorticity perturbations (i.e. vector perturbations) at the linear level, and we ask: Will cosmological rotational perturbations exactly drag the axes of any gyroscope relative to the directions of geodesics from the gyroscope to galaxies in the asymptotic FRW space? In our analysis we cast the laws of gravitomagnetism into a form showing clearly the close correspondence with the laws of ordinary magnetism. This is achieved by using Car- tan’s formalism with local orthonormal bases (LONBs) and fiducial observers (FIDOs). Our results, stated in the abstract and presented in sections IV and V, show that Mach’s Principle (that axes of local non-rotating frames precisely follow some average of the motion of cosmic matter) is a consequence of cosmology with Einstein Gravity. The crucial equation is Einstein’s $G\_\text{-equation}$, Ampère’s law for gravitomagnetism, Eq. (28) in a spatially flat FRW universe. The mathematical statement, what average of the energy flow out there in the universe determines the time-evolution of gyroscopes’ axes here, is given in Eq. (28). A short version of this work appeared in [8]. In a subsequent paper we shall present the analysis for vorticity perturbations on a FRW background with $k = \pm 1$. We use the conventions of Misner, Thorne, and Wheeler [3].

II. VORTICITY PERTURBATIONS, FIDUCIAL OBSERVERS, AND THE GRavitomagnetic FIELD

Cosmological Vorticity Perturbations.— Linear cosmological perturbations, see J. Bardeen [10], decouple in three sectors, 3-scalars (density perturbations), 3-vectors (vorticity perturbations), and 3-tensors (gravitational waves). In the vector sector all quantities must be constructed from a 3-vector field with vanishing divergence. Hence the 3-scalar $\delta g_{00}$ is zero, the lapse function is unperturbed, and the slicing of space-time in slices $\Sigma_t$ is unique, i.e. there is no gauge ambiguity about the time coordinate. For linear vector perturbations the intrinsic geometry of each $\Sigma_t$ remains unperturbed. This holds because the perturbations of the Ricci scalar, of the space-space components and the trace of $T_{\mu\nu}$, and of the Ricci tensor all must vanish, and for $d = 3$ the Riemann tensor can be built from the Ricci tensor. Our choices of spatial coordinates are Cartesian for $k = 0$ resp spherical FRW coordinates for $k = \pm 1$. The line element is

$$ds^2 = -dt^2 + (ah)^2(dx)^2 + 2(ah)^2\beta^i dx^i dt,$$

(1)

where $\beta^i$ is the shift 3-vector. Geodesics on $\Sigma_t$ are straight lines on our choice of chart. We consider vector perturbations in an asymptotic FRW universe, i.e. $\beta^i \to 0$ for $r \to \infty$. Our coordinates are fixed to “distant galaxies”, i.e. to galaxies in the asymptotic unperturbed FRW space, and the basis vectors in the coordinate basis, $\epsilon_i(P) = \partial/\partial x^i$, point along geodesics in $\Sigma_t$ from $P$ to fixed “distant galaxies”.

Fiducial Observers.— Our aim is to obtain the laws of linearized gravitomagnetism in a form analogous to electromagnetism in a $3 + 1$ formulation. What is the operational definition for $\vec{E}_g$ (gravitoelectric field) and $\vec{B}_g$ (gravitomagnetic field)? According to the equivalence principle for a free-falling, non-rotating observer there are no gravitational forces at his position, $\vec{E}_g = 0$, $\vec{B}_g = 0$. It all depends on the choice of fiducial observers, FIDOs, with their local ortho-normal bases, LONBs, see Thorne et al. [11]. Hence we work in the formalism of E. Car- tan [4]. Our choice of FIDOs: The world lines of our FIDOs are at fixed $x^i$ in our coordinates, which are fixed to distant galaxies in the asymptotic FRW universe, and $\vec{e}_i(P) = \vec{u}_\text{FIDO}(P)$. Hats refer to LONBs, and bars designate space-time vectors. We choose the spatial basis vectors of our FIDOs, $\vec{e}_i(P)$, fixed to directions of geodesics on $\Sigma_t$ from $P$ to distant galaxies in the asymptotic FRW universe. Specifically we fix $\vec{e}_i(P)$ in the same spatial directions as $\vec{\epsilon}_i(P) \equiv \partial/\partial x^i$, i.e. in 4-space the directions of $\vec{e}_i$ and $\vec{\epsilon}_i$ differ by a pure Lorentz boost, see Eq. (33) below. The 3-velocity of our FIDOS relative to the normals on $\Sigma_t$ is equal to the shift 3-vector $\beta^i$ in Eq. (1).

The operational definitions of $\vec{E}_g$ and $\vec{B}_g$.— These defin-itions are independent of perturbation theory. They involve FIDOs (of any given choice) measuring the first time-derivatives along their world lines, on the one hand of the momentum components $p_i$ of free-falling quasistatic test particles, and on the other hand of the spin components $S_i$ of gyroscopes carried along by the FIDOS,

$$\frac{d}{dt} p_i \equiv m \vec{E}_g^i$$

free-falling quasistatic test particle, (2)

$$\frac{d}{dt} S_i \equiv -\frac{1}{2}[\vec{B}_g \wedge \vec{S}_i]$$

gyro comoving with FIDO, (3)

where $t$ is the local time measured by the FIDO. Arrows denote 3-vectors in the tangent spaces spanned by the spatial legs of our LONBs. $\vec{E}_g = \vec{g}$ is the gravitational acceleration of free-falling quasistatic test particles relative to the FIDO. Eqs. (23) are the same as for a classical charged spinning test particle in an electromagnetic
field except that \( q \) is replaced by \( m \), and the gyromagnetic ratio \( q/(2m) \) is replaced by \( 1/2 \). Eq. 3 gives the angular velocity of precession of the gyroscope's spin axis relative to the axes of the FIDO,

\[
\Omega_{\text{gyro}} = -\frac{1}{2} \dot{B}_g,
\]

which is an equivalent operational definition of \( \dot{B}_g \).

### III. CONNECTION COEFFICIENTS AND THE EQUATIONS OF MOTION FOR MATTER IN GRAVITOMAGNETISM

**Connection 1-forms.**— The connection 1-forms \( \tilde{\omega}^a_b \) resp their components in LONBs (\( = \) Ricci rotation coefficients), \( (\omega^a_b)_\ell \), are defined by

\[
\nabla_\ell \tilde{e}_b = \tilde{e}_\ell (\omega^\ell_b)_a.
\]

In words: the Ricci rotation coefficients \( (\omega^\ell_b)_a \) give the rotation resp the Lorentz boost \( (\omega^\ell_b) \) of the LONBs relative to parallel transport along \( \tilde{e}_\ell \). Parallel transport is given by free fall for \( \tilde{u} = \tilde{e}_0 \) and gyroscope axes for \( \tilde{e}_\ell \). Relative to FIDOs the equations of motion for free-falling test particles (geodesic equation) and for spin axes of gyroscopes (Ferni transport) specialized to gyroscopes carried along by a FIDO are

\[
\frac{dp^a}{dt} + (\omega^a_b) p^b \frac{dx^c}{dt} = 0, \quad \frac{dS^i}{dt} + (\omega^i_j) \dot{S}^j = 0.
\]

With Eqs. 2 the operational definitions Eqs. 2 get translated into the equivalent definitions involving connection coefficients with a displacement index \( \dot{0} \), namely a Lorentz boost \( \omega^\dot{0}_\dot{i} \) per unit time (acceleration \( \ddot{g} = \ddot{E}_g \)) resp a rotation angle \( \omega^{ij} \) per unit time (angular velocity \( \Omega_{\text{gyro}} = -\frac{1}{2} \dot{B}_g \)),

\[
(\omega^\dot{0}_\dot{i})_0 = -E^\dot{0}_i, \quad (\omega^{ij})_0 = -\frac{1}{2} B^{ij}_0,
\]

where \( B^{ij} \equiv \epsilon_{ijk} B_k \).

**The computation of connection coefficients in Cartan’s formalism.**— The first step is to express our choice of LONBs \( \tilde{e}_a(P) \) in terms of the coordinate bases \( e^\alpha_a(P) = \partial/\partial x^\alpha \), i.e. \( \tilde{e}_a = (e^\alpha_a)^{\alpha} e_a \). To first order in \( (\beta^\ell/c) \)

\[
\tilde{e}_0 = \tilde{e}_0, \quad \tilde{e}_k = \frac{1}{ah_k} (e^\ell_k + \beta^\ell_0) e_0.
\]

For FRW with \( k = 0 \) in Cartesian spatial coordinates all \( h_i = 1 \). For FRW in spherical coordinates and \( k = 0, \pm 1 \) we have \( h_\chi = 1, h_\theta = R(\chi), h_\phi = R(\chi) \sin \theta \) with \( R(\chi) \equiv \{ \chi, \sin \chi, \sinh \chi \} \). Since the spatial LONBs point in the same spatial directions as the spatial coordinate bases, we use latin letters from the middle of the alphabet both for spatial LONBs (with hat) and for spatial coordinate bases (without hat). The dual bases (basis 1-forms) \( \tilde{\omega}^a_b \) for LONBs are defined by \( (\tilde{\theta}^a, \tilde{e}_b) = \delta^a_b \), where tildes designate space-time 1-forms. The LONB 1-forms \( \tilde{\theta}^a \) are expanded in the coordinate basis 1-forms, \( \tilde{\theta}^0 = dx^\ell \), i.e. \( \tilde{\theta}^0 = (\theta^\ell) a \tilde{\theta}^a \), by

\[
\tilde{\theta}^0 = \tilde{\theta}^0 - \beta^\ell \tilde{\theta}^\ell, \quad \tilde{\theta}^k = ah_k \tilde{\theta}^k.
\]

The coefficients of the inverse expansion, i.e. coordinate bases in terms of LONBs, are \( (e^\alpha_a)^{\alpha} = (\theta^\ell) a \) resp \( (\theta^\ell) a = (e^\alpha_a)^{\alpha} \).

**The second step** is computing the exterior derivative \( d \) of the basis 1-forms, \( (d\theta^\ell)_{\alpha\beta} = \partial_\alpha (\theta^\ell)_{\beta} - \partial_\beta (\theta^\ell)_{\alpha} \), where \( [\alpha\beta] \) must be in the coordinate basis. Then one converts to components \( [\alpha\beta] \) in the LONB,

\[
(d\theta^\ell)_{\alpha\beta} = -C_{a\dot{b}}^\ell = (e^\alpha_a)^{\alpha} [\partial_\alpha (\theta^\ell)_{\beta} - \partial_\beta (\theta^\ell)_{\alpha}] (e^\beta_b)^{\beta}.
\]

The coefficients \( C_{a\dot{b}}^\ell \) are identical to the commutation coefficients of the basis vectors, \( [\tilde{e}_a, \tilde{e}_b] \equiv C_{a\dot{b}}^\ell \tilde{e}_\ell \). This is easily shown by using \( \partial_\alpha (\tilde{\theta}^\ell, \tilde{e}_b) = 0 \equiv [\partial_\alpha (\theta^\ell)_{\beta} (e^\beta_b)^{\beta} + (\theta^\ell)_{b\beta} \partial_\beta (e^\beta_b)^{\beta}] \).

**The third step** is obtaining the connection coefficients from the commutation coefficients. The definition of the connection via basis 1-forms is \( (\nabla_\alpha \theta^\ell)_{\beta} = -(\omega^\ell_b)_{\alpha} (\theta^\ell)_{\beta} \). We take both the displacement index \( a \) and the equation’s component index \( \beta \) in the coordinate basis, and we antisymmetrize in \( [\alpha\beta] \). This makes the Christoffel symbols \( \Gamma^\alpha_{\beta\gamma} \) on the left-hand side disappear, since they are symmetric in \( \alpha, \beta \). Hence the left-hand side reduces to \( (d\theta^\ell)_{\alpha\beta} \). Dropping the equation’s component indices \( [\alpha\beta] \) gives

\[
\tilde{d} \tilde{\theta}^\ell = -\tilde{\omega}^{\ell\dot{a}} \wedge \tilde{\theta}^\dot{a}.
\]

This is Cartan’s first equation. The wedge product (exterior product) of two 1-forms \( \tilde{\sigma} \) and \( \tilde{\rho} \) is \( (\tilde{\sigma} \wedge \tilde{\rho})_{a\beta} \equiv \sigma_\alpha \rho_\beta - \sigma_\beta \rho_\alpha \). Taking Cartan’s first equation in LONB components gives the commutation coefficients, Eq. 10, on the left-hand side, and the right-hand side simplifies in LONB because \( (\theta^\ell)_{b\dot{b}} = \delta^\ell_b \). Hence Cartan’s first equation in LONB components is

\[
C_{a\dot{b}}^\ell = (\omega^\ell_b)_{\dot{a}} - (\omega^\ell_b)_{\dot{b}}.
\]

This equation is easily solved for the rotation coefficients,

\[
(\omega^\ell_b)_{\dot{a}} = \frac{1}{2} [C_{\dot{b}a} + C_{\dot{b}a} - C_{\dot{b}ae}].
\]

**Connection coefficients for vorticity perturbations on a Minkowski background.**— To first order in \( \beta^\ell \) and with Cartesian spatial coordinates on \( \Sigma_t \), the commutation coefficients are very simple to compute, because only \( (\theta^\ell)_{t\dot{b}} = -\beta^\ell_0 \) is space-time dependent, and the prefactors in Eq. 10, \( (e^\alpha_a)_{\alpha} \) and \( (e^\beta_b)^{\beta} \), can be set to 1. Hence
\[ C_{\bar{a} \bar{b}} = (d \beta)_{ab}, \] and
\[ (\omega_{i0})_0 = -E_i^a = \partial_t \beta_i, \]
\[ (\omega_{ij})_0 = -\frac{1}{2} \varepsilon_{ijk} B_k^\beta = -\frac{1}{2} (d \beta)_{ij}, = (\omega_{i0})_j, \]
\[ (\omega_{ij})_k = 0. \]
(14)

All components of connection coefficients with respect to LONB’s are directly measurable (in contrast to Christoffel symbols, which refer to coordinate bases).

**Connection coefficients for vorticity perturbations on FRW with \( k = 0, \pm 1 \).**— It is again straightforward to compute the commutation coefficients and the connection coefficients using Eqs. [15-18].

\[ (\omega_{i0})_0 = -E_i^a = \frac{1}{a} \partial_t \beta_i, \]
\[ (\omega_{ij})_0 = -\frac{1}{2} \varepsilon_{ijk} B_k^\beta = -\frac{1}{2a^2} \partial_t (d \beta)_{ij}, \]
\[ (\omega_{i0})_j = -\frac{1}{2} \varepsilon_{ijk} B_k^\beta \delta_{ij} H, \]
\[ (\omega_{ij})_k = \frac{\delta_{ik}}{a h_j} (H \beta_j + \partial_j L_i) - \frac{\delta_{jk}}{a h_i} (H \beta_i + \partial_i L_j), \]
(18)

where \( L_i \equiv \log h_i \). Since we work to first order in the vorticity perturbations (i.e. in \( \beta \)), we can identify \( \vec{E}_g \) and \( \vec{B}_g \) with vectors in \( \Sigma_t \). From Eqs. [15-18], we see that the shift vector \( \vec{\beta} \) must be identified with the gravitomagnetic vector potential \( \vec{A}_g \). From Eqs. [15-18] follow

\[ \vec{B}_g = \text{curl} \vec{A}_g, \quad \vec{E}_g = -\frac{1}{a} \partial_t (a \vec{A}_g), \]
(19)
\[ \text{curl} \vec{E}_g + \frac{1}{a^2} \partial_t (a^2 \vec{B}_g) = 0. \]
(20)

These equations are identical with the homogeneous equations for electromagnetism in FRW space-times with \( k = 0, \pm 1 \).

**Equation of motion for free-falling test particles.**— The equation of motion (geodesic equation) for test particles of arbitrary velocities \( v \leq c \) in linear vorticity perturbations on a Minkowski background reads
\[ \frac{d}{dt}(p_i) = \varepsilon[(\vec{E}_g + (\vec{v} \wedge \vec{B}_g))_i], \]
(21)

identical with the one for electromagnetism, except that the charge \( q \) is replaced by the energy \( \varepsilon \) of the test particle. With Eq. [4] and in a stationary gravitomagnetic field (\( \vec{E}_g = 0 \)) Eq. [21] becomes \( \frac{d}{dt}(p_i) = -2 \varepsilon [\vec{v} \wedge \Omega_{\text{gyro}}]_i \), the Coriolis force law. Note that \( \Omega_{\text{gyro}} \) is minus the rotation velocity of the FIDO relative to the gyroscopes’ axes. A homogeneous gravitomagnetic field can be transformed away completely by going to a rigidly rotating coordinate system, i.e. physics in a homogeneous gravitomagnetic field is equivalent to physics on a merry-go-round in Minkowski space.— Note that there are no terms bilinear in \( \vec{v} \) for test particles of arbitrary velocities \( v \leq c \).

For **FRW with \( k = 0 \)** we obtain
\[ \frac{d}{dt}(ap_t) = \varepsilon[(\vec{E}_g + (\vec{v} \wedge \vec{B}_g)) + H \vec{v} \wedge (\vec{\beta} \wedge \vec{v})]_i. \]
(22)

For **FRW with \( k \pm 1 \)** there are additional terms from \( \partial_t L_j \) in Eq. [18]. These terms are present even in the absence of vorticity perturbations and of Hubble expansion, because in the spherical basis spatial LONBs are not parallelized.

**IV. EINSTEIN’S EQUATIONS FOR GRAVITOMAGNETISM: AMPÈRE’S LAW**

**Curvature.**— The curvature 2-form \( \tilde{\mathcal{R}}^\alpha_{\beta c} \) has LONB components \( (\mathcal{R}^\alpha_{\beta c})_{\bar{d}} \), which are the LONB components of the Riemann tensor \( R^\alpha_{\beta c d} \). The Riemann tensor can be operationally defined by the action of \( (\nabla_\gamma \nabla_\delta - \nabla_\delta \nabla_\gamma) \) on the LONB 1-form \( \tilde{\theta}^\alpha \),
\[ (\nabla_\gamma \nabla_\delta - \nabla_\delta \nabla_\gamma) \tilde{\theta}^\alpha = -\tilde{\theta}^\beta (\mathcal{R}^\alpha_{\beta c})_{\gamma d}, \]
(23)

where the covariant derivatives \( \nabla_\gamma \) and \( \nabla_\delta \) must be in the coordinate basis. To compute the curvature 2-form we first use \( \nabla_\gamma \tilde{\theta}^\alpha = -\tilde{\omega}^\gamma_{\delta} \tilde{\theta}^\delta \). Then we let this right-hand side be acted on by \( \nabla_\gamma \). This gives two terms. One term comes from \( \nabla_\gamma \) acting on \( \tilde{\theta}^\hat{e} \), and after antisymmetrization in \( [\gamma \delta] \) it produces \( -\tilde{\omega}^\gamma_{\delta} \omega^\delta_{\epsilon} \gamma \tilde{\theta}^\hat{\epsilon} \). The other term comes from \( \nabla_\gamma \) acting on the expansion coefficient (number field) \( \omega^\gamma_{\delta} \gamma \), where it can be replaced by \( \partial_\gamma \), and after antisymmetrization in \( [\gamma \delta] \) it produces \( -\partial_\gamma \omega^\gamma_{\delta} \gamma \tilde{\theta}^\hat{\epsilon} \). Hence we obtain
\[ \tilde{\mathcal{R}}^\alpha_{\beta c} = \tilde{\omega}^\alpha_{\delta b} + \tilde{\omega}^\alpha_{\epsilon b} \wedge \tilde{\omega}^\epsilon_{\delta b}, \]
(24)

which is Cartan’s second equation.

**Cartan’s 2nd equation in LONB.**— To obtain the LONB components of the first term of the right-hand side, \( (d \omega^\alpha_{\delta b})_{\bar{c}d} \), we must first convert the connection components of Eqs. [15-18] from the LONB to the coordinate basis, \( (\omega^\alpha_{\delta b})_{\beta d} = (\omega^\alpha_{\delta b})_{\beta d}(\theta^d)_\delta \), then take the exterior derivative, \( \partial_\gamma \{ (\omega^\alpha_{\delta b})_{\gamma d}(\theta^d)_\delta \} - [\gamma \leftrightarrow \delta] \), and then convert the result back from coordinate components to LONB components. The partial derivative of the product gives two terms, one with a partial derivative of \( (\omega^\alpha_{\delta b})_{\gamma d} \), the other with \( \partial_\gamma (\theta^d)_\delta \), which produces another connection 1-form component. In the second term of Cartan’s second equation these conversions from LONB to coordinate basis and back again cancel, since there is no derivative in between. The result is Cartan’s 2nd equation in LONB components,
\[ (\mathcal{R}^\alpha_{\beta c})_{\bar{d}} = [(\varepsilon^d)\gamma \partial_\gamma (\omega^\alpha_{\delta b})_{\gamma d} - (\omega^\alpha_{\delta b})_{\beta d}] \varepsilon(\omega^\alpha_{\delta b})_{\gamma d}] \\
\[ - [\hat{e} \leftrightarrow \hat{d}]. \]
(25)
Einstein equations for vorticity perturbations of Minkowski space.— For linear vorticity perturbations of Minkowski space (with Cartesian coordinates for 3-space) all non-zero connection coefficients in Eqs. (14) - (18) are of first order in the perturbations. Therefore the second term of Cartan’s second equation can be neglected, and in the first term one need not distinguish components in LONB from components in the coordinate basis. For vorticity perturbations the important Einstein equation is the equation for $G_{0i} = R_{0i}$,

$$R_{0i} = (R_{0j})_{ij} = (dw_0)_{ij} = \frac{1}{2} \left( \text{curl} B \right)_{ij}. \quad (26)$$

Hence Einstein’s $G_{0i}$-equation for vorticity perturbations in Minkowski space reads

$$\text{curl} B_g = -16\pi G_N J_e. \quad (27)$$

$J_e$ is the energy current density, which is equal to the momentum density. Eq. (24) is identical to the original law of Ampère for magnetism, except that the charge current $J_e$ is replaced by the energy current $J_e$, and the prefactor $4\pi r$ is replaced by the prefactor $(-16\pi G_N)$. In contrast to the Ampère-Maxwell equation, the Maxwell term $(\partial_i \beta_j - \partial_j \beta_i)$ is absent in gravitomagnetodynamics. The $G_{ij}$ is an equation at fixed time, a constraint equation, called momentum constraint, since the momentum density appears on the right-hand side of Eq. (27).

To see the analogous structures of gravitomagnetism and electromagnetism, it is more instructive to formulate this constraint equation, as we have done in Eq. (24), via the connection 1-forms, which involves $\left( \partial_i \beta_j - \partial_j \beta_i \right)$, i.e. the gravitomagnetic field, than via the extrinsic curvature tensor $K_{ij}$, which involves $\left( \partial_i \beta_j + \partial_j \beta_i \right)$. Of course the resulting constraint, if written in terms of $\bar{A}_g = \beta$, is the same, $\Delta \bar{A}_g = 16\pi G_N J_e$.

The $G_{0i}$ equation with the source $T_{0i}$ is trivially fulfilled, since these objects are 3-scalars and therefore vanish in the vector sector. The source $T_{ij}$ vanishes, since it is of second order in the perturbation. The $G_{ij}$ equations give $\partial_i (\partial_i \beta_j + \partial_j \beta_i) = 0$, i.e. the shear of the field $\beta$ has vanishing time-derivative.

Einstein equations for vorticity perturbations of spatially flat FRW space.— With Cartesian comoving coordinates for flat 3-space there are two new terms in the connection coefficients, $(\omega_0^0)_{ij} = \delta_i \beta_j + \delta_j \beta_i$, and $(\omega_2^2)_{ij} = H (\delta_i \beta_j - \delta_j \beta_i)$. Computing $R_{0i} = (R_{0j})_{ij}$ with Eq. (25), we obtain the corresponding Einstein equation,

$$\text{curl} B_g - 4H \bar{A}_g = -16\pi G_N \bar{J}_e. \quad (28)$$

where we have used $\bar{\beta} = \bar{A}_g$. The scale factor $a$ of the spatially flat FRW universe does not appear in these equations. From $\bar{H} = -4\pi G_N (\rho + p)$ we see that $\bar{H} \leq 0$ for $p \geq -\rho$. Therefore we define the $H$-dot radius by $R_{\bar{H}}^2 = (-\bar{H})^{-1}$, and we define $\mu^2 = -4\bar{H} = (\frac{1}{2} R_{\bar{H}})^{-2}$.

We insert the vector potential $\bar{A}_g = \beta$, we use $\text{div} \bar{A}_g = 0$, hence curl $\bar{A}_g = -\Delta \bar{A}_g$. Therefore Eq. (28) becomes

$$(-\Delta + \mu^2) \bar{A}_g = -16\pi G_N \bar{J}_e. \quad (29)$$

The new term on the left-hand side, $-4\bar{H} \bar{\beta} = (\mu^2 \bar{A}_g)$, dominates for superhorizon perturbations.

V. MACH’S PRINCIPLE

Our first result.— The solution of Eq. (29) is the Yukawa potential for $\bar{A}_g = \bar{\beta}$ in terms of the sources $\bar{J}_e$ at the same fixed time,

$$\bar{A}_g(r, t) = -4G_N \int d^3 r’ \bar{J}_e(r’, t) \frac{\exp(-\mu |\bar{r} - \bar{r}'|)}{|\bar{r} - \bar{r}'|}. \quad (30)$$

This is analogous to the formula for ordinary magnetostatics except for the exponential cutoff. The Green function which is exponentially growing for $|r’| \rightarrow \infty$ is rejected on the standard grounds of field theory. The Yukawa potential in Eq. (30) has an exponential cutoff for $|\bar{r} - \bar{r}'| \geq 1/\mu$. This gives our first important conclusion: The contributions of vorticity perturbations beyond the $H$-dot radius are exponentially suppressed.

Our second result concerns the exact dragging of gyroscope axes by a homogeneous rotation of cosmological matter out to significantly beyond the $H$-dot radius (for the exponential cutoff to be effective). This holds for any equation of state. This is easily seen from Einstein’s $G_{0i}$ equation (29) in $k$-space for superhorizon perturbations, $k^2 \ll (-\bar{H})$, where the $\Delta$-term can be dropped. Using $\bar{J}_e = (\rho + p) \bar{v}_{\text{fluid}}$ and $\bar{H} = -4\pi G_N (\rho + p)$ we see that all the prefactors cancel, and we obtain $\bar{\beta}(\bar{x}) = -\bar{v}_{\text{fluid}}(\bar{x})$. With $\bar{\Omega}_{\text{gyro}} = -\frac{1}{3} (\nabla \times \bar{\beta})$ and with $\bar{\Omega}_{\text{fluid}} = \frac{1}{3} (\nabla \times \bar{v}_{\text{fluid}})$ we obtain the result that for $R_{\text{pert}} \gg R_{\bar{H}}$ there is an exponentially fast approach to

$$\bar{\Omega}_{\text{gyro}} = \bar{\Omega}_{\text{matter}}. \quad (31)$$

This proves exact dragging of gyroscope axes here by a homogeneous rotation of cosmological matter out to significantly beyond the $H$ radius.

Our third result concerns the most general vorticity perturbation in linear approximation, and it states what specific average of energy flow in the universe determines the motion of gyroscope axes here at $r = 0$. We take $\bar{B}_g = -\frac{1}{2} \bar{\Omega}_{\text{gyro}}$, and obtain the equation for $\bar{\Omega}_{\text{gyro}}$ in terms of the sources at the same fixed time,

$$\bar{\Omega}_{\text{gyro}} = 2G_N (\rho + p) \int d^3 r \frac{1}{r^3} \left[ (1 + \mu r) e^{-\mu r} \right] \bar{v} \wedge \bar{v}(\bar{r}). \quad (32)$$

This is the precise expression for Mach’s principle, it says exactly what average of the motion of energy in the universe determines $\bar{\Omega}_{\text{gyro}}$. Mach had asked: “What share
has every mass in the determination of direction in the
law of inertia? No definite answer can be given by our
experiences.” In the integrand we have the grav-
itomagnetic moment density \( \vec{\mu}_g = \frac{1}{2}\vec{r} \times \vec{J}_e \), analogous
to the magnetic moment density of an electric current
distribution. The expression \( \vec{r} \times \vec{J}_e = (\vec{L}_e) \) is the mea-
sured angular momentum. This is the lowest term, the
\( \ell = 1 \) term, in the multipole expansion of the source for
\( r_{\text{obs}} = r_{\text{gyro}} = 0 \) and \( r_{\text{source}} > 0 \). Higher multipoles can-
not contribute to the gravitomagnetic field \( \vec{B}_g \) at \( r = 0 \).
At each radius \( r \) only a term equivalent to a rigid rotation
with angular velocity \( \vec{\Omega}(r) \) contributes in Eq. (32). Using this we obtain
\[
\vec{\Omega}_{\text{gyro}} = \frac{4}{3} \int \frac{dr}{R_H} \vec{\Omega}_{\text{matter}}(r) C_\mu(r), \tag{33}
\]
where \( C_\mu(r) \) is the cutoff function, i.e. the first square
bracket in Eq. (32). For the special case \( \vec{\Omega}_{\text{matter}}(r) \) independent of \( r \) we recover Eq. (31).

The \( r \)-dependence of the weight function of the energy
current \( \vec{J}_e(\vec{r}) \) in the integral for \( \vec{B}_g(0) \) for \( r \ll R_H \) is the same \((1/r^3)\) law as in Ampère’s law. Hence the weight
function for the measured angular momentum \((\vec{r} \times \vec{J}_e) \) is \((1/r^4)\). For fixed \( \vec{\Omega} \) the energy current density increases
linearly with \( r \), and the weight function per unit \( r \) is the first power of \( r \), Eq. (33). Therefore a perturbation
which is a rigid rotation out to \( R_{\text{pert}} \) and zero outside has
a dragging fraction \( f_{\text{drag}} \), which grows quadratically with
\( R_{\text{pert}} \), until \( f_{\text{drag}} \) reaches a value near 1 for \( R_{\text{pert}} = R_H \).
For \( R_{\text{pert}} \) increasing beyond \( R_H \) the dragging fraction
approaches the value 1 exponentially fast.

Dark energy with \( p/\rho = -1 \), i.e. a cosmological con-
stant, does not contribute in Mach’s principle, Eq. (32),
since there is no flow of energy associated with it, its
energy current \( \vec{J}_e = (\rho + p)\vec{v} \) vanishes.

Measuring everything relative to axes of gyroscopes at
a given location (instead of relative to galaxies in the
asymptotic FRW space) makes the the left-hand side of
Eq. (32) vanish. Hence Eq. (32) reduces to the statement
that the angular momentum of matter, measured relative
to the gyroscopes at the given location, will vanish after
averaging with the weight \( r^{-3} \) and with the exponential
cutoff \( C_\mu(r) \). From the point of view of measurements
it is preferable to measure relative to gyroscopes at one given
location. But to see the structure of gravitomagnetism
most clearly, it is best to measure relative to galaxies in the
asymptotic FRW space.

Analogous dragging effects in magnetostatics (e.g. a
rotating charged spherical shell acting on magnetic dipole
moments inside), have the opposite sign in Ampère’s law
compared to gravitomagnetism, Eq. (27), and this causes
antidragging in the magnetostatic case.

Einstein’s objection to Mach’s principle in his autobi-
ographical notes of 1949.— Einstein wrote [12]: “Mach
conjectures that inertia would have to depend upon the
interaction of masses, precisely as was true for Newton’s
other forces, a conception which for a long time I consid-
ered as in principle the correct one. It presupposes im-
plcitly, however, that the basic theory should be of the
general type of Newton’s mechanics: masses and their
interactions as the original concepts. The attempt at
such a solution does not fit into a consistent field theory,
as will be immediately recognized.” We have shown, how
this apparent difficulty is resolved in General Relativ-
ity, specifically in weak Gravitomagnetism: The relevant
Einstein equation has the form of Ampère’s law with a
Yukawa term, Eq. (28), hence the measured mass-energy
flow out there in the universe does indeed determine the
precession of gyroscope axes here, Eq. (32).

VI. MEASURED MATTER INPUT FOR
EINSTEIN’S EQUATIONS AND FOR
MACH’S PRINCIPLE

The input on the right-hand side of Eq. (32), which
expresses Mach’s principle, is the measured angular ve-
locity of matter (stars, galaxies, etc), measured relative
to galaxies in the asymptotic FRW space (“asymptotic
galaxies”). No knowledge of the metric perturbation \( \beta \)
is needed when determining the input for Eq. (32). This
is a purely kinetic input, which means that it is directly
determined by the measured state of motion of matter
at a given time without knowing the metric perturbation
\( \beta \). Similarly the measured angular momentum is a purely
kinetic input, \((\rho + p)(\vec{r} \times \vec{v})r^2 dr = (\rho + p)r^2 \vec{v} \times \vec{r} \cdot dr \). The
dynamical output of solving Einstein’s \( G_{0i} \) equation is
\( \beta = \vec{A}_g \), the gravitomagnetic vector potential, in Eq. (29)
and the gravitomagnetic force \( \vec{B}_g \) in Eq. (32). The mea-
sured, kinetic angular momentum must be distinguished
from the canonical angular momentum, which we intro-
duce (review) in the following paragraphs.

Action and Lagrangian for Gravitomagnetism.— The
Einstein-Hilbert action for linear vorticity perturbations
on a Minkowski background and for point particles is
\[
S = (16\pi G_N)^{-1} \int d^4x \sqrt{g} (\text{curl} \vec{A}_g)^2 + \int dt \sum_n \left[ \frac{1}{2} m_n \vec{x}_n^2 + m_n \vec{\dot{x}}_n \vec{A}_g(\vec{x}_n, t) \right] \tag{34}
\]
Except for the prefactor \((16\pi G_N)^{-1}\) and the sign of the
first term, the action is the same as for electromagnetism
without the \((\partial_t \vec{A})^2\)-term.— Generally the equations of
motion are given by the Lagrangian via the standard
Euler-Lagrange equations (as in classical mechanics and
classical electrodynamics), if and only if the Lagrangian
is defined by
\[
S = \int dt L \tag{35}
\]
without any metric factors in the integrand. Hence the
Lagrangian for a point particle in a gravitomagnetic field
is given by the square bracket of the matter term in Eq. (34).

The canonical momentum is defined by \( (p_{\text{can}})_k = \partial L / \partial \dot{\mathbf{x}}^k \), where \( k = 1, 2, 3 \). From Eq. (34) we obtain in Cartesian coordinates

\[
\vec{p}_{\text{can}} = m(\dot{\mathbf{x}} + \vec{A}_g).
\] (36)

This is the same equation as in classical mechanics for point particles in an electromagnetic field, except that the electric charge \( q \) is replaced by \( m \).— The kinetic momentum is \( m\dot{\mathbf{x}} \). It is directly measured, it can be used as the input for solving Einstein’s equations, because it is independent of the gravitomagnetic vector potential \( \vec{A}_g = \vec{\beta} \), which is an output of solving Einstein’s equations. On the other hand the canonical momentum (for a given measured state of motion) depends on the gravitomagnetic vector potential \( \vec{A}_g \). Therefore the canonical momentum cannot be used as an input for solving Einstein’s equations. The canonical momentum cannot be determined by a FIDO from measurements before having solved Einstein’s equations.— In curvilinear coordinates of 3-space the general definition of the canonical momentum, \( (p_{\text{can}})_k = \partial L / \partial \dot{\mathbf{x}}^k \), gives

\[
(p_{\text{can}})_k = m g_{kn}(\dot{x}^n + A^n_g) \quad k, n = 1, 2, 3.
\] (37)

From its definition via the Lagrangian, the canonical momentum is a 1-form in 3-space, i.e. it has a lower 3-index.— On the other hand we can also start from the 4-velocity \( u^\mu \), which is the archetype of a 4-vector, multiply with the mass to obtain the 4-momentum \( p^\mu \), and pull down the 4-index with \( 4g_{\nu\mu} \). This gives \( p_\nu = g_{\nu\mu}(mu^\mu) = m g_{kn}(\dot{x}^n + A^n_g) \), which is the same as the canonical momentum in Eq. (37).— Going to spherical coordinates we note that \( p_\phi \) has the physical meaning of canonical angular momentum.

In the special case of azimuthal symmetry, i.e. when \( \vec{e}_\phi = \vec{\xi} \) is a rotational Killing vector, the canonical angular momentum \( p_\phi = \langle \vec{p}, \vec{\xi} \rangle \) is conserved, while the measured, kinetic angular momentum of matter, \( \langle rp_\nu \rangle \), is not conserved, because of the gravitoelectric induction field \( \vec{E}_g \), Eq. (20), which acts in the \( \phi \)-direction. The canonical angular momentum is relevant for the time-evolution, i.e. for the dynamics, and conservation laws, not for kinematics (measurements at a given time).

In the continuum description of matter (energy current density) the LONB components \( T^\mu_{\ 0k} \) can be used as input for solving Einstein’s equations, since they can be measured without knowing the output \( \beta_k \) of Einstein’s equations. On the other hand the coordinate-basis components \( T^0_{\ 0k} \) cannot be used as an input, because they cannot be determined by measurements without a knowledge of \( \beta_k \).

In Mach’s principle the input is the observations of the angular velocities of stars and galaxies, and from there the measured kinetic angular momentum \( \langle \rho + p \rangle [\vec{r} \wedge \vec{v}] \) as shown in Eq. (32).— Bičák et al [13] have proposed to use \( p_\phi \), the canonical angular momentum as the input on the right-hand side of the Einstein equation. For the reasons given above in this section, we consider this to be a fundamental mistake. If one does this, the \( HA_g \) term on the left-hand side of the Einstein equation (28) gets cancelled, and the exponential suppression factor disappears.

Einstein’s objection to Mach’s Principle in his letter to Felix Pirani of 2 February 1954 [14], which is quoted by Ehlers in [15]: “If you have a tensor \( T_{\mu\nu} \) and not a metric, then this does not meaningfully describe matter. There is no theory of physics so far, which can describe matter without already the metric as an ingredient of the description of matter. Therefore within existing theories the statement that the matter by itself determines the metric is neither wrong nor false, but it is meaningless.” We agree with this statement, as long as the components of \( T_{\mu\nu} \) are given in a coordinate basis, as e.g. \( T^0_\alpha \) in the proposal of Bičák et al [13]. But we disagree with Einstein’s objection, if the components are given in a LONB, \( T^\mu_{\ 0\alpha} \), because these components can be directly measured by the FIDOs. The FIDO only needs clocks, meter sticks, and markers, which give the directions of his spatial axes. The FIDO has only the metric of Special Relativity, \( \eta_{\alpha\beta} = \text{diag}(-1, +1, +1, +1) \), available in the tangent space at his space-time point. But the FIDO does not need any more structure, not a connection and not the metric potential functions \( g_{\mu\nu} \), which in our case are given by the gravitomagnetic vector potential \( \vec{A}_g \), the solution and output of Einstein’s equations. Therefore we disagree with Einstein’s objection quoted above (from which he concluded that one should no longer speak of Mach’s Principle at all), if the components of the energy-momentum tensor are given in a LONB.

VII. THE LOCAL VORTICITY MEASURED BY NON-ROTATING OBSERVERS

This quantity is defined as \( \epsilon_{\text{fluid}} \) measured in the local inertial coordinate system which is comoving with the fluid, i.e. measured relative to the axes of local gyroscopes (“local compass of inertia”). In general coordinates the local vorticity measured by non-rotating observers is

\[
\omega^\alpha = -\varepsilon^{\alpha\beta\gamma\delta} u_\beta \nabla_\gamma u_\delta,
\] (38)

where \( \varepsilon^{0123} = -1 \). Mach’s principle, as formulated as a general hypothesis by Mach and made precise in Eq. (32), states that the gyroscope axes here follow the rotational flow \( \vec{J}_r \) of matter in the universe averaged with a \( r^{-2} \) weight and an exponential cutoff at the \( H \) radius. In general the gyroscope axes here most definitely do not follow the motion of the local fluid here, i.e. relative to gyroscopes the local vorticity is nonzero according to Mach’s principle in general. There is a special case, a rigid rotation of a fluid out to a perturbation radius
In the limit $R_{\text{pert}}/R_H \to \infty$ the local vorticity measured by non-rotating observers vanishes. But this limit is uninteresting, since it produces an unperturbed FRW universe.— Unfortunately many authors have considered the vanishing of the vorticity relative to the local compass of inertia to be a test for Mach’s principle. See e.g. Ozsváth and Schücking’s solution and discussion of a Bianchi IX model for perturbations of the original closed and static Einstein universe [16]. In contrast we conclude that the vanishing of the vorticity relative to the local compass of inertia is not relevant as a test of Mach’s principle.

[1] http://einstein.stanford.edu
[2] H. Bondi and J. Samuel, gr-qc/9607009
[3] J.B. Barbour and H. Pfister, eds., Mach’s Principle: From Newton’s Bucket to Quantum Gravity, Birkhäuser (1995).
[4] E. Mach, Die Mechanik in Ihrer Entwicklung: Historisch-Kritisch Dargestellt, Brockhaus, Leipzig (1. Auflage 1883, 7., verbesserte und vermehrte Auflage 1912). English translation by T.J. McCormack, The Science of Mechanics: A Critical and Historical Account of Its Development, (6th ed., Open Court Publishing, La Salle, Ill., 1960). Chap. 2, Sec.6, Subsec. 7.
[5] E. Mach, Die Geschichte und die Wurzel des Satzes von der Erhaltung der Arbeit, (1872), History and Roots of the Principle of the Conservation of Energy, Chicago, Open Court, p. 77-80.
[6] A. Einstein, Ann. Physik, 55, 241 (1918).
[7] H. Thirring, Phys. Zeitschr. 19, 33 (1918).
[8] C. Schmid, gr-qc/0201095
[9] C.W. Misner, K.S. Thorne, and J.A. Wheeler, Gravitation, Freeman (1973).
[10] J.M. Bardeen, Phys. Rev. D 22, 1882 (1980).
[11] K.S. Thorne, R.H. Price, and D.A. MacDonald, editors, Black Holes, The Membrane Paradigm, Yale University Press (1986).
[12] A. Einstein, “Autobiographical Notes”. in Albert Einstein, Philosopher-Scientist, ed. P.A. Schlipp, Evanston, Illinois: The Library of Living Philosophers, Inc., Illinois (1949) 29.
[13] J. Bicák, D. Lynden-Bell, and J. Katz, Phys. Rev. D 69, 064011 (2004).
[14] Albert Einstein (Princeton) to Felix Pirani (Cambridge, Engl.) 2 February 1954, Albert Einstein Archives, The Hebrew University of Jerusalem, Call no. 17-447. [3 typed sheets]
[15] Ref. [3], p.93.
[16] I. Ozsváth, and E. Schücking, Nature 193, 1168 (1962), and Ann. Phys. (N.Y.), 55, 166 (1969).