Majorana edge modes protected by emergent symmetry in a one dimensional fermi gas

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We show that a one dimensional ultra-cold Fermi gas with Rashba-like spin orbit coupling, a Zeeman field and intrinsic attractive interactions exhibits a novel topological superfluid state, which forms in spite of total number conservation and the absence of a single particle gap. Majorana zero modes are localized to the interface between a topological region in the middle of the trap and trivial regions at its wings. Unlike the realization of a topological superconductor in proximity coupled nano-wires, the Majorana modes do not carry a quantum number associated with the total fermion parity. Instead, the topological degeneracy is protected by an emergent $\mathbb{Z}_2$ symmetry present only at low energies. We discuss the experimental implications of the novel zero modes, as manifest for example in the response to modulation of a local potential near the position of the Majorana bound states. For the range of interaction strength corresponding to $1 < K < 2$ the zero modes are unseparable from the gapless phonon continuum and therefore show up as an algebraic zero bias resonance in the response. For $K > 2$, on the other hand the zero-mode can be detected as a sharp low frequency response at an energy which generically scales with system size as $1/L^{K/2}$ and is therefore parametrically separated from the phonons.

Recent experiments with semiconductor nanowires have shown signatures of Majorana zero modes, the hallmarks of a topological superconducting state, localized at the ends of the wires [1, 2]. The crucial ingredients in the experimental system that enabled the establishment of the topological state were, on the one hand, a single particle dispersion affected by spin-orbit coupling and a Zeeman field, and on the other hand, pairing-correlations induced by coupling of the wire to a standard s-wave superconductor [3, 4]. Effective spin orbit coupling and Zeeman field can also be generated in systems of ultra-cold atoms confined to one dimension [5–8], however it is much more difficult, in this case, to induce pairing correlations externally. This naturally leads us to address the following basic question: can intrinsic attractive interactions lead to a topological phase and Majorana zero modes without externally induced pairing?

If the system were two or three dimensional then attractive interactions, naturally generated in atomic systems with Feshbach resonances, would give rise to a BCS pairing gap equivalent to that induced by proximity to a bulk superconductor. But this is not the case in the one-dimensional system in question, where neither long range order nor even a single-particle gap are established. Moreover, as pointed out in Refs. [9, 10], in a charge conserving wire the Majorana zero modes would represent degeneracy between states that differ in total particle number. Without fine tuning of an external potential such states would be split by an energy of order $1/(\kappa L)$ where $\kappa$ is the uniform compressibility and $L$ the length of the wire. For these reasons it was concluded that a single wire cannot support a protected Majorana degeneracy. The minimal model constructed in [9, 10] that supports a topological q-bit consists of two spin-orbit coupled wires coupled to one s-wave superconducting wire with power-law order. Hence, true long range superfluid order is not essential. But a configuration with multiple wires was deemed necessary to avoid the electrostatic energy splitting: the remaining double degeneracy can be associated with exchanging a particle between the two wires.

Notwithstanding the above arguments, we show in this paper that protected ground state degeneracy due to one pair of Majorana zero modes can in fact occur in an isolated one-dimensional Fermi gas with intrinsic attractive interactions. The Majorana modes are localized, not at the ends of the wire, but rather at the interfaces between

FIG. 1. (a) A one dimensional Fermi gas with synthetic spin-orbit coupling, a Zeeman field and attractive interactions in a one dimensional harmonic trap. Majorana zero modes are localized at the interface between a topological and trivial approximately where the chemical potential dips below the Zeeman gap at wave vector $q = 0$. (b) The Rashba-like dispersion and $\mu = 0$ showing our notations of the four modes crossing the Fermi energy.
a "topological" region in the middle of the wire and a "trivial" superconductor on the two sides, where the density is sufficiently depleted (see Fig. 1a). Our results are obtained within an exact low energy description of the interacting system.

The most interesting aspect of the topological degeneracy thus established concerns the way it overcomes the aforementioned charging energy obstacle. The Majorana zero modes are made of composite fermions that do not carry the global $U(1)$ charge. Instead, these fermions carry a $\mathbb{Z}_2$ charge associated with the relative fermion parity between the two pairs of modes crossing the Fermi energy (those at $q \sim 0$ and at $q \sim \pm 2k_0$, see Fig. 1a). Accordingly the topological state is protected by the relative fermion parity symmetry, rather than by the total fermion parity as usual. The zero modes are in this sense similar to that found in two chains models constructed to have only pair-hopping between the chains$^{[11, 12]}$. In our case however the protecting $\mathbb{Z}_2$ symmetry is not present in the microscopic model, but rather is emergent at low energies.

The fact that the topological degeneracy is protected by an emergent symmetry has implications on how it is split in a finite system. In presence of symmetry breaking perturbations, such as backscattering from impurities, the splitting of the degeneracy scales algebraically with the separation between the interfaces, with a power reflecting the (negative) scaling dimension of the perturbation in the Luttinger liquid. In principle, the harmonic trap can itself generate such a back-scattering at the interface. However the strength of this scattering is reduced exponentially with the inverse trap frequency and hence with system size.

The novel nature of the Majorana modes has important consequences for how they can be detected. Because the associated degeneracy does not involve states with different particle numbers a tunnel probe is not needed. Instead, the dynamic response to a local time dependent potential, which induces backscattering near the interface modes, will show a sharp power-law peak at zero frequency. In contrast, far from the interfaces the response will show a power-law suppression reflecting the fact that backscattering is irrelevant at low energy.

Model – As a concrete model we consider a one dimensional Fermi gas with spin-orbit coupling, a Zeeman field and short ranged attractive interactions described by the following Hamiltonian

$$
\mathcal{H} = \int dx \left[ \psi^\dagger \left( -\frac{\nabla^2}{2m} + V(x) - \mu + \alpha \sigma^x i \partial_x - \delta_z \sigma^z \right) \psi - U \psi^\dagger \psi^\dagger \psi_\uparrow \psi_\uparrow \right].
$$

Here $m$ is the particle mass, $\alpha$ the spin-orbit coupling strength, $\mu$ the chemical potential, $\delta_z$ is an effective Zeeman field and $U > 0$ is the interaction strength.

A parabolic trap potential is described within this model by a position dependent chemical potential. We consider filling the system to a point that the chemical potential in the middle of the trap is located in the Zeeman gap. In the usual case where there is a small proximity induced s-wave pairing field $\Delta < \delta_z$, the spatially dependent chemical potential tunes the system from a topological state in the middle of the trap to a trivial state in the flanks.

An alternative way to tune the system between the same two phases is by varying the ratio of $\Delta/\delta_z$, while keeping the chemical potential fixed in the middle of the gap. The topological phase is established in the region where $\Delta/\delta_z < 1$. This way of tuning would prove to be a convenient theoretical tool in analyzing the problem without an external pairing field.

Low energy theory – As a preparatory step consider an infinite homogenous wire described by the fully charge conserving Hamiltonian (1) with $\mu = 0$. It is convenient to formulate the long wavelength theory starting from the case with $\lambda_z = 0$. Then we have four fermion modes $R_a, L_a$ crossing the Fermi energy, as shown in Fig. 1(a). $a = 0, 2$ labels the modes at $k = 0$ and $k = \pm 2k_0$ respectively. Next, we bosonize the two modes at $2k_0$, while leaving the $k = 0$ sector in fermion representation: $R_2 \sim F_R \sqrt{\rho_0} e^{i(\theta - \phi)}$, $L_2 \sim F_L \sqrt{\rho_0} e^{i(\theta + \phi)}$. $F_{LR}$ are Klein factors and $\rho_0$ the average density. This leads to the following low energy Hamiltonian:

$$
\mathcal{H} = \frac{v_F}{2\pi} \int dx \left[ \frac{K(\partial_x \theta)^2}{R} + \frac{1}{\pi} (\partial_x \phi)^2 \right]
- \int dx \left[ -i v_F \left( R_2^0 \partial_x R_0 - L_0^0 \partial_x L_0 \right) - U R_0^0 L_0^0 R_0 \right]
- \int dx \left[ \delta_z R_0^0 L_0 + \Delta e^{i2\theta} R_0^1 L_0^1 + h.c. \right].
$$

where $u \approx v_F \sqrt{1 - \frac{g^2}{2}}, K \approx \sqrt{\frac{1 + g^2}{2}}, g \leq 1, \frac{\mu}{\pi v_F}$ and $\Delta \equiv U \rho_0 / \pi$. We note that in addition to the $U(1)$ symmetry, related to total charge conservation, this model also enjoys a $\mathbb{Z}_2$ symmetry associated with the parity of the relative fermion number between the two pairs of low energy modes. This is an emergent symmetry not present in the microscopic model (1). It would be broken by irrelevant terms, such as backscattering.

The last term in (2) has the form of a BCS pairing potential, but with a fluctuating phase $\theta$. We absorb these fluctuations into the fermion fields through the unitary transformation $^{[13]} U = \exp \left[ -i \int dx \theta \left( R_0^0 R_0 + L_0^0 L_0 \right) \right]$. $U$ takes the fermions $L_0^0, R_0^0$ to $L_1 = e^{i\theta} L_0^0$ and $R_1 = e^{i\theta} R_0^0$. While the original fields were charged under the global $U(1)$ the new ones are neutral, carrying only the relative parity quantum number. The simplification achieved in the transformation, namely that the fermions are affected by a non fluctuating BCS pairing potential, comes with the cost of generating new interactions be-
tween the fermions and the Luttinger liquid phonons. But, as we will see below, these interactions are tractable.

At this point, we introduce the Majorana representation of the fermion operator \( R = \frac{1}{\sqrt{2}} (\xi_R + i \eta_R) \) and \( L = \frac{1}{\sqrt{2}} (\eta_L + i \xi_L) \). The two Majorana species \( \xi \) and \( \eta \) acquire the quadratic mass terms \( -i \Delta + \eta_L \xi_R - i \Delta \xi_L \xi_R \), where \( \Delta \equiv \Delta - \delta_z \). Hence, by varying the effective Zeeman field \( \delta_z \), we tune the pair of lower energy Majorana modes \( \xi^T = (\xi_L, \xi_R) \) through a critical point at which their gap changes sign. The second pair of Majorana modes retains a higher gap throughout, which makes the quartic interaction between the four Majorana species highly irrelevant and allows to safely integrate out the fields \( \eta_L, R \).

This leads to a low energy Hamiltonian of the form:

\[
\mathcal{H} = \frac{1}{2} \int dx \xi^T \left[ v(-i \partial_x) \tau^z + \Delta_- \tau^y \right] \xi 
+ \frac{u}{2\pi} \int dx \left[ K(\partial_x \theta)^2 + \frac{1}{K}(\partial_x \phi)^2 \right] 
+ i \lambda \int dx \partial_x \phi \xi_R \xi_L 
\]  

(3)

This theory describes a transition between two different pairing phases in the fermion sector as the gap \( \Delta_- \) changes sign. The coupling term \( \lambda \) can affect the critical point in an interesting way [13], but it cannot change the two phases where the Majorana modes are gapped. We note that this coupling is not obtained directly from the unitary transformation applied to (2) above, but is rather generated at higher order upon integrating out the higher energy modes. Hence the coupling constant \( \lambda \) is aproiori small.

**Majorana zero modes** – Having derived the low energy theory we can now consider situations with spatial variation of the gap \( \Delta_- = \Delta - \delta_z \) in (3). This is naturally brought about, for example, by the spatial variation of the trapping potential relative to the chemical potential. Within the low energy theory the change in potential has the same effect as direct variation of the Zeeman field \( \delta_z \). Lowering of the chemical potential from the Dirac point shifts the Fermi points away from \( q = 0 \). This makes the uniform Zeeman field less relevant, reducing its effective value. For weak attractive interactions the gap \( \Delta_- \) changes sign near to the points where the chemical potential dips below the single particle Zeeman gap, as illustrated in Fig. 1(a).

The fermion sector of (3) is known to give rise to Majorana bound states localized around the points \( x_j \) at which the mass term \( \Delta_-(x) \) changes sign [14]. The zero mode operators are given explicitly by \( \gamma_j = \int dx \chi_j^T(x) \xi(x) \), with the wave function

\[
\chi_j(x) \approx \left( \begin{array}{c} 1 \\ \pm 1 \end{array} \right) \exp \left[ -\frac{1}{v} \int_0^x dx' \Delta_-(x' - x_j) \right],
\]  

(4)

obtained as an approximate solution of the Bogoliubov-de Gennes equations.

It is instructive to obtain a simple form of the bound state wave-function in the relevant case of a shallow harmonic trap \( V(x) = \frac{1}{2} m \Omega^2 x^2 \). We note that in the fermion sector alone this problem is equivalent to one with an externally induced BCS gap, such as was addressed in Refs. [15–17]. For simplicity we consider the case \( \Delta \sim \delta_z \ll \epsilon_F \). Linearizing the gap function \( \Delta_-(x) \) around the zero crossing points at \( x_j \equiv \pm L/2 \approx \pm \Omega^{-1} \sqrt{2 \sqrt{\delta_z} / m} (1 - (\Delta / \delta_z)^1/4 \) leads to a gaussian wave function \( \chi_j(r) \sim \exp(-(x - x_j)^2 / l_0^2) \) with a localization length \( l_0 \approx \sqrt{\nu_F L / 8(\delta_z - \Delta)} \). Hence the splitting of the ground state degeneracy due to the overlap of the bound states from the two sides is exponentially small with the distance \( L \) between the two edge modes.

The exponential scaling is not destroyed by the interaction \( \lambda \) in (3) of the fermions with the phonon modes. This term couples the localized Majorana zero modes to the phonons through \( V_j \sim \lambda \partial_x \phi(x_j) \gamma_j \bar{\gamma}_j \), that is through a gapped Majorana operator \( \gamma_j \equiv \int dx \gamma_j^T(x) \xi(x) \), where \( \xi = (\xi_L, -\xi_R)^T \). The interaction generated by this term between the two Majorana end modes will decay exponentially with \( L \) due to the decay of the correlation function \( \langle \xi(x_1) \bar{\xi}(x_2) \rangle \). At a more fundamental level, this interaction is ineffective in splitting the degeneracy because it does not break the (relative) \( Z_2 \) fermion parity symmetry, which protects the topological phase. The phonon operator \( \partial_x \phi \) is neutral under this symmetry and hence cannot couple linearly to the end modes \( \gamma_j \).

A different coupling between the Majorana and Luttinger liquid modes is generated by backscattering at wave-vector \( 2k_0 \), \( V_{2k_0} R^z L_0 + L^z R_0 + H.c. \), which can be induced by impurities, but is also naturally present due to the trap potential. Because this term transfers a single fermion between outer and inner modes it explicitly breaks the protecting relative \( Z_2 \) symmetry. But it is also highly irrelevant in the bulk, as it always couples to a gapped fermion. The situation is different near the interface, where the backscattering has a low energy component involving the Majorana zero mode, there it is given by (see S1):

\[
\mathcal{H}_{bs} = \sum_{j=1,2} V_{2k_0}(x_j) i \gamma_j \times \left[ F_R \cos \left( \phi(x_j) + \delta_\phi^j \right) - F_L \sin \left( \phi(x_j) + \delta_\phi^j \right) \right],
\]  

(5)

where \( \delta_\phi^j \) and \( \delta_\phi^j \) are unimportant phases which depend on the location of the interface. Note the operators \( F_R \cos \phi \) and \( F_L \sin \phi \) appearing here are in fact fermion (Majorana) operators, which anti-commute with \( \gamma_j \). This is to be contrasted with the \( 4k_0 \) term which features the more standard bosonic operator \( \cos(2 \phi) \sim R^z L_2 + H.c. \).

We can now obtain the effective coupling between the two Majorana end modes by second order perturbation theory. This involves integrating over the Luttinger liq-
uid modes, leading to the effective coupling

$$\mathcal{H}_{12} \approx \frac{V_{2k_0}}{u \rho_0} \langle e^{i \phi(x_1)} e^{-i \phi(x_2)} \rangle i \gamma_1 \gamma_2,$$

which splits the ground state degeneracy by \(\Delta E = V_{2k_0}^2 (u \rho_0)^{-1} (\rho_0 L)^{-K/2}\). Here \(L \equiv x_1 - x_2\) is the distance between the two boundaries (see Fig.1(a)). The splitting of the zero modes should be compared with the finite size gap of the Luttinger liquid phonons which scales as \(1/L\). Hence, to ensure that the Majorana modes are parametrically separated from the phonon continuum for arbitrary impurity induced backscattering requires the Luttinger parameter to be \(K > 2\), which necessitates at least moderately strong attractive interactions.

However, the situation is far better in a clean ultracold atom system, where the source of backscattering is the harmonic trap potential itself. In this case the same scale, the trap frequency \(\Omega\), determines the separation between the interface modes \(L \propto \Omega^{-1}\), the bound state wave functions \(\chi(x)\) and the backscattering amplitude associated with the slope of the potential at the interface points: \(V(x - x_i) \sim \frac{1}{2} m \Omega^2 L(x - x_i)\). Remarkably, as shown by direct calculation in the SI, the resulting effective backscattering potential at the interface is reduced exponentially with the separation \(L\) between the two interface points. We then find the energy splitting of the zero modes in a Harmonic trap to be

$$\Delta E \sim L^{-\frac{3+2K}{2}} e^{-L/L_0} \sim \Omega^{\frac{3+2K}{2}} e^{-\Omega_0/\Omega},$$

where \(L_0 \equiv (\delta_z - \Delta)/(k_0 \epsilon_0)\) and \(\Omega_0 \equiv 4 \epsilon_0 \sqrt{\epsilon_0/\delta_z}[1 - \Delta/\delta_z]^{-1/4}\).

**Probing** – Detection of the Majorana edge modes in proximity-coupled wires requires single particle tunneling into the edge. In our case, however, because the Majorana zero modes are not related to total fermion parity, a much simpler probe involving a local density modulation can be used. Specifically we consider the dynamic response to an oscillatory local impurity potential imposed for example by a highly focused blue detuned laser. The most important component of the impurity potential is scattering at momenta \(2k_0\), which connects the outer the outer and inner modes and thereby breaks the protecting \(\mathbb{Z}_2\) symmetry \([18]\). The low energy contribution of the \(2k_0\) scattering connects the Luttinger liquid modes to the Majorana zero modes at the edges. With a local impurity at \(x = x_p\), the low energy probe Hamiltonian is

$$\mathcal{H}_p = V_{2k_0} \cos(\omega t) i [F_R \cos \phi(x_p) - F_L \sin \phi(x_p)] \times [\chi_1(x_p) \gamma_1 + \chi_2(x_p) \gamma_2]$$

The linear-response to this probe is readily calculated within the low energy theory to be

$$A_{2k_0}(\omega, x) \sim e^{-\left(\frac{\omega}{\omega_0}\right)^2} |\omega|^\frac{K}{2} - 1.$$

We identify two phases. For \(K < 2\) the local \(2k_0\) scattering at the interface is relevant, effectively mixing the Majorana mode with the phonon continuum. The power-law zero-bias peak in the response function is a signature of the Majorana mode that has become a resonance due to this mixing. The exponential decay of the peak with increasing distance from the interface tracks the localized Majorana wave-function.

For \(K > 2\) the interface \(2k_0\) scattering is irrelevant and the zero mode is effectively decoupled from the phonons. In this case the scaling of the "zero" mode energy \(\Delta E(L) \sim L^{-K/2}\) parametrically separates it from the phonons. It is then possible, in principle, to observe the interface modes as a sharp peak, and manipulate its occupation coherently. Even then, the probe \((9)\) alone can only excite the zero mode together with a higher energy phonon (at energy of at least \(\Omega\)). Therefore, to this order, we would have vanishing response at frequencies around \(\Delta E\). However, in presence of a static impurity potential \(V_2\) at the other interface, the probe can excite an off-shell virtual excitation of the Luttinger liquid, which will be absorbed at the other edge. The spectral function at frequencies far below the trap frequency \(\Omega\) is then given by \(A(\omega) \sim (V_2^2/\Omega^2)(\Omega/\epsilon_F)^K \delta[\omega - \Delta E(\Omega)]\).

**Conclusions** – We predicted that an isolated one dimensional ultra-cold Fermi gas with Rashba-like spin orbit coupling, a Zeeman field and intrinsically attractive interactions will form a novel topological state. Majorana zero modes are established at the interfaces between "topological" and "trivial" regions along the trap. The Zero modes, which coexist with gapless phonons in the system, are of different nature than those realized in semiconducting wires with externally induced pairing\([1–4]\). In our case the ground state degeneracy does not involve a change in the total particle number or fermion parity. The Majorana modes are enabled by a \(\mathbb{Z}_2\) symmetry emergent at low energies, which is associated with the relative fermion number between the two pairs of modes crossing the Fermi surface.

Backscattering by impurities breaks the \(\mathbb{Z}_2\) symmetry and leads to a splitting of the degeneracy, which scales, for a generic local impurity, as \((1/L)^{K/2}\). Hence the zero mode is parametrically separated from the phonon continuum for \(K > 2\). The point \(K = 2\) is a critical point separating two different regimes. For \(K < 2\) the zero mode is inseparable from the Luttinger liquid, and the response exhibits a broad algebraic singularity (resonance) at zero bias. For \(K > 2\) the zero mode is decoupled from the phonons and shows up as a sharp excitation that can be manipulated as a q-bit. Interestingly backscattering due to the harmonic potential alone, without localized impurities, is ineffective in removing the degeneracy, leaving a splitting exponentially small in the system size.

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SUPPLEMENTARY INFORMATION FOR – MAJORANA EDGE MODES PROTECTED BY EMERGENT SYMMETRY IN A ONE DIMENSIONAL FERMI GAS

Solving the Bogoliubov-deGennes equations for the zero modes

We consider the model Hamiltonian given in Eq. (1) of the main text with a harmonic trap potential $V(x) = \frac{1}{2}m\Omega^2x^2$ and with $\mu = 0$ so that at the middle of the trap the chemical potential lies exactly in the middle of the $k = 0$ Zeeman gap. Within a low density approximation, we can think of the external potential as giving rise to a local chemical potential $\tilde{\mu}(x) = V(x)$, where of course the true electro-chemical potential is constant along the trap. Decreasing the local chemical potential can be viewed instead as lowering of the effective Zeeman field $\tilde{\delta}_z(x)$ in a constant chemical potential, as it becomes gradually less effective in polarizing the spin. The effective value of the Zeeman field is determined by the expectation value of the Zeeman energy of particles at the fermi energy. Using the single particle dispersion and assuming that throughout the region of interest $V(x) \ll \epsilon_F$, the effective Zeeman field is given by

$$\tilde{\delta}_z(x) \equiv \delta_z(\sigma^z) \approx \delta_z \left[ 1 - \left( \frac{V(x)}{\delta_z} \right)^2 \right]$$ (10)

Note that in this expression we have assumed that $\Delta$ is sufficiently large such that $V(x)$ at the position of the bound state is small compared to the full Zeeman gap $\delta_z$, the accurate expressions are found in Ref.[16].

The gap in the Majorana sector of the low energy theory (Eq. (3) in the main text) is determined by the difference between the pairing interaction and the effective Zeeman field $\Delta_- = \Delta - \delta_z(x)$. The points in the trap where this function changes sign can now be given in terms of the bare Zeeman field and pairing interaction (The number conserving interaction $\Delta$ appears as a pairing field in the anti-symmetric, i.e. relative mode, sector):

$$x_{1,2} = \pm \sqrt{\frac{2\delta_z}{m} \left[ 1 - \frac{\Delta}{\delta_z} \right]^{1/4} \frac{1}{\Omega}}.$$ (11)

This result is consistent with that derived using a mean field (Bogoliubov-DeGennes) treatment of the Fermi gas in a trap in Ref.[16]. The simpler result given in the main text was for very small $\Delta$ where it can be neglected compared to $\delta_z$. The crucial point is that in any case the distance between the two interface points scales inversely with the trap frequency, i.e. as $L = x_2 - x_1 \sim 1/\Omega$.

To obtain the explicit form of the zero mode wave function from the general formula Eq. (4) in the main text we
linearize $\Delta_-(x)$ around $x_1$ and $x_2$: $\Delta_-(x) \approx -v_F(x - x_j)/l_0^2$, with

$$l_0 \equiv \sqrt{\frac{v_F L}{8(\delta_z - \Delta)}}.$$ 

Using this gap function we get

$$\chi_j(x) \sim \exp \left( \frac{1}{v} \int_0^x dx' \Delta_-(x' - x_j) \right) \approx \left( \frac{1}{\pi l_0} \right)^{1/4} e^{-\left( \frac{v x}{\pi l_0} \right)^2}. \quad (12)$$

We see that the Majorana interface modes are localized on the length $l_0 \sim \sqrt{L} \sim 1/\sqrt{\Omega}$. Hence the two interface modes become increasingly better defined as we take the limit of a larger trap $\Omega \to 0$. The bound state operator $\gamma_i$ and the gapped modes orthogonal to it $\bar{\gamma}_i$ are given by the transformation

$$
\begin{pmatrix} \gamma_j \\ \bar{\gamma}_j \end{pmatrix} = \int dx \chi_j(x) \begin{pmatrix} 1 & \eta_j \\ -\eta_j & 1 \end{pmatrix} \begin{pmatrix} \xi_R(x) \\ \xi_L(x) \end{pmatrix}
$$

where $\eta_j = \pm 1$ for the states at the opposite ends $j = 1, 2$.

**Ground-state energy splitting in the harmonic trap**

Here we consider the backscattering induced by the harmonic trap in absence of any other impurity potential and derive the energy splitting of the zero modes in this case. The hamiltonian coupling the trap potential to the $2k_0$ component of the density is given by

$$\mathcal{H}_V = \int dx V(x) \left[ e^{2ik_0 x} R_2^\dagger L_0 + e^{-2ik_0 x} L_2^\dagger R_0 + \text{H.c.} \right] \quad (14)$$

with $V(x) = m\Omega^2 x^2/2$. The bosonised description of the outer modes and Majorana representation of the inner modes as defined in the main text are given by

$$R_2(x) \approx F_R \sqrt{\frac{\rho_0}{2\pi}} e^{i\theta(x) - i\phi(x)}; \quad L_2(x) \approx F_L \sqrt{\frac{\rho_0}{2\pi}} e^{i\theta(x) + i\phi(x)}$$

$$R_0 = \frac{e^{i\theta(x)}}{\sqrt{2}} (\xi_R(x) + i\eta_R(x)); \quad L_0 = \frac{e^{i\theta(x)}}{\sqrt{2}} (\eta_L(x) + i\xi_L(x)) \quad (15)$$

The Klein factors $F_{L,R}$ can be taken to be Majorana fermion operators. We now rewrite the backscattering Hamiltonian above using this representation and project it to the low energy Majorana sector $\xi_{L,R}$ to obtain

$$\mathcal{H}_V = \sqrt{\frac{\rho_0}{4\pi}} \int dx V(x) \left[ i e^{i(2k_0 x + \phi)} F_R \xi_L(x) + e^{-i(2k_0 x + \phi)} F_L \xi_R(x) + \text{H.c.} \right] \quad (16)$$

Because the Majorana modes $\xi_{L,R}(x)$ are gapped in the bulk, their only low energy contributions come from the zero modes localized near the interfaces. We therefore project out the gapped modes by making the replacement $\xi_{L,R}(x) \rightarrow \sum_{j=1,2} \chi_j(x) \gamma_j$. The wave functions $\chi_j$ of the Majorana modes localized near the two interfaces at $x_{1,2} = \pm L/2$ were given in Eq. (12) above. Plugging this expansion into (16) we have:

$$\mathcal{H}_V = i \sqrt{\frac{\rho_0}{\pi}} \int dx V(x) \left[ F_R \cos(2k_0 x + \phi) - F_L \sin(2k_0 x + \phi) \right] \sum_{j=1,2} \chi_j(x) \gamma_j \quad (17)$$

At this point we can derive an effective Hamiltonian acting only within the zero mode sector by integrating over the Luttinger liquid modes $\phi$. This will be a self consistent procedure, leading to a local in time interaction, as long as the resulting zero mode splitting $\Delta E(\Omega)$ is far below the lowest phonon mode $\sim \Omega$. Carrying out the integration over $\phi$ gives

$$\mathcal{H}_{12}^{\text{eff}} = \frac{4i}{\pi u} \int dx dx' V(x)V(x') \left< i F_L F_R \cos(2k_0 x + \phi(x)) \sin(2k_0 x' + \phi(x')) \right> \chi_1(x) \chi_2(x') \gamma_1 \gamma_2, \quad (18)$$
The wave-functions $\chi_1(x)$ and $\chi_2(x)$ are localized around the points $x_1 = -L/2$ and $x_2 = L/2$ respectively, thus we can linearize the potential $V(x) \approx -m\Omega^2 L(x + L/2)$ and $V(x') \approx m\Omega^2 L(x' - L/2)$. We next transform to relative coordinates $x = R + r/2$ and $x' = R - r/2$, compute the Luttinger liquid correlation function and use the form of the wavefunctions $\chi_j(x)$ (12) to obtain

$$H_{12} = \frac{i}{\pi u} \frac{m^2 \Omega^4 L^2}{\sqrt{\pi l_0}} \int drdR \sin 2k_0 r \left( R^2 - \frac{1}{2}(r - L)^2 \right) \exp \left[ -\frac{R^2}{l_0^2} - \frac{(r - L)^2}{4l_0^2} - \frac{K}{2} \log |\rho_0 r| \right] \gamma_1 \gamma_2$$

$$= \frac{im^2 \Omega^4 L^2}{8\pi u} \int dr \sin 2k_0 r \left( l_0^2 - \frac{1}{2}(r - L)^2 \right) \exp \left[ -\frac{(r - L)^2}{4l_0^2} - \frac{K}{2} \log |\rho_0 r| \right] \gamma_1 \gamma_2$$

Finally we expand the logarithm in the limit $\rho_0 L \gg 1$ and perform the integral which yields

$$H_{12} \approx \frac{m^2 \Omega^4 L^2}{8\pi u} \left( \frac{1}{\rho_0 L} \right)^{K/2} \int dz \sin 2k_0 (z + L) \left[ l_0^2 - \frac{z^2}{2} \right] e^{-\frac{z^2}{4l_0^2}} i\gamma_1 \gamma_2$$

Using the results of section (13), namely that $\Omega \sim L$ and $l_0 \sim \sqrt{L}$ we obtain the following scaling of the energy splitting

$$\Delta E \sim L^{-\frac{3+2K}{4}} e^{-\frac{k_0 \rho_0}{\pi\rho_0 L}}$$

Surprisingly, there is an exponential suppression which is a result of the special case where the localization length of the bound states $l_0$ depends on the distance between the Majorana modes. The factor $(1/L)^{K/2}$ is the only universal scaling factor which was also estimated in Refs.[9,11].