Soliton solutions and their (in)stability for the focusing Davey–Stewartson II equation

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Received 1 October 2017, revised 8 May 2018
Accepted for publication 13 June 2018
Published 2 August 2018

Recommended by Dr Jean-Claude Saut

Abstract
We give a rigorous mathematical analysis of the one-soliton solution of the focusing Davey–Stewartson II equation and a proof of its instability under perturbation. Building on the fundamental perturbation analysis of Gadyl’shin and Kiselev, we verify their Assumption 1 and use Fredholm determinants to globalize their perturbation analysis.

Keywords: solitons, inverse scattering, Davey–Stewartson equation
Mathematics Subject Classification numbers: 37K10, 35C08, 35P25, 35Q35, 37L50, 32W05

(Some figures may appear in colour only in the online journal)

1. Introduction

In this paper we will give a rigorous proof that the one-soliton solution for the focusing Davey–Stewartson II (fDSI) equation in two-dimensions is spectrally unstable under smooth, compactly supported perturbations of the initial data. Our proof uses the inverse scattering method and sharp asymptotic analysis for a renormalized Fredholm determinant whose zeros signal the presence of soliton solutions. As we will explain, our approach builds on previous work of Gadyl’shin and Kiselev [16].

We will study the fDSI equation in the form

¹ Brown supported in part by Simons Collaboration Grant 422756.
² Perry supported in part by NSF grant DMS-1208778 and Simons Collaboration Grant 359431.
³ Appendix B written by Russell M Brown and Peter Perry.
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\[
\begin{aligned}
\left\{ \begin{array}{l}
\partial_t u = \left( \partial_x^2 + \partial_y^2 \right) u + \left( g + g \right) u \\
\partial_t g = -\frac{1}{2} \partial \left( |u|^2 \right)
\end{array} \right.
\end{aligned}
\]

Here \( u = u(x_1, x_2, t) \) is the amplitude of a monochromatic, weakly nonlinear surface wave in shallow water, while \( \partial \) and \( \overline{\partial} \) are the operators

\[
\partial = (1/2) \left( \partial_{x_1} - i \partial_{x_2} \right), \quad \overline{\partial} = (1/2) \left( \partial_{x_1} + i \partial_{x_2} \right).
\]

The fDSII equation is the shallow-water limit of the two-dimensional, dispersive nonlinear PDE derived by Davey–Stewartson [12] to describe the propagation of weakly nonlinear, monochromatic surface waves in an incompressible, irrotational, inviscid fluid with no surface tension. Djordjevic and Redekopp [13] and Ablowitz and Segur [4] generalized the DS equations to include surface tension; Craig, Schanz, and Sulem [11] proved that the DS approximation is consistent with the water wave equations on the time scale for which validity is expected.

Following [18, section 1.1] we summarize the essential steps in the derivation of the DS equation from the full water wave equations; see Lannes [26, section 8.2.6] for a thorough discussion and justification of the approximation to the full water wave equations. Introduce Cartesian coordinates \( x_1, x_2, x_3 \) and write \( X = (x_1, x_2) \). Placing the flat bed at \( x_3 = -h \), and parameterizing the free surface by a function \( \zeta(X, t) \), the velocity potential \( \phi \) obeys the free boundary problem

\[
\left\{ \begin{array}{l}
\Delta \phi = 0, \quad -h \leq x_3 \leq \zeta(X, t) \\
\frac{\partial \phi}{\partial x_3} \bigg|_{x_3 = -h} = 0
\end{array} \right. \tag{1.1}
\]

Conservation of mass at the free surface dictates that

\[
\zeta_t + \nabla_X \phi \cdot \nabla_X \zeta = \frac{\partial \phi}{\partial x_3} \tag{1.2}
\]

while conservation of energy dictates that

\[
g \zeta_t + \phi_t + \frac{1}{2} |\nabla_X \phi|^2 = TH(\zeta)(X, t) \tag{1.3}
\]

where \( H \) is the mean curvature of the fluid surface and \( T \) is the surface tension.

We seek an evolution equation for the amplitude envelope \( A \) of a monochromatic wave with small-amplitude, weakly nonlinear, slowly-varying, and ‘essentially one-dimensional’ modulation, assuming initial data of the form

\[
\zeta(X, 0) = i \omega \epsilon \left[ A(\epsilon x_1, \epsilon x_2) e^{i k x} - A(\epsilon x, \epsilon y) e^{-i k x} \right].
\]

Here \( k = 2\pi/\lambda \) is the wave number, \( \ell \) is a characteristic wave number for transverse oscillations, \( \kappa = \sqrt{k^2 + \ell^2} \), and \( \omega = \omega(\kappa) \) is computed from the linear dispersion relation for (1.1) and (1.2). From a multiscale analysis [4, 12, 13] one obtains the following system of equations for the amplitude \( A \) and mean velocity potential \( \phi \) as a function of ‘slow’ spatial variables \((\zeta, \eta)\) and a ‘slow’ time variable \( \tau \):

\[
\left\{ \begin{array}{l}
i \partial_{\zeta} A + A_{\zeta \zeta} + \mu A_{\eta \eta} = \chi_0 |A|^2 A + \chi_1 A \partial_{\zeta} \phi, \\
\alpha \phi_{\zeta \zeta} + \phi_{\eta \eta} = -\beta \left( |A|^2 \right)_{\zeta}.
\end{array} \right. \tag{1.4}
\]
Here the parameters \( \mu, \chi_0, \chi_1, \alpha, \beta \) are determined by the \( k, \kappa \), the fluid depth \( h \), the surface tension \( T \), and the acceleration \( g \) of gravity.

In general, the DS equation (1.4) is not completely integrable; however, in the shallow water limit (\( kh \to 0 \) and \( \varepsilon \ll (kh)^3 \)), one obtains a completely integrable system of the form

\[
\begin{align*}
  i \partial_t A - \sigma A_{xx} + A_{yy} &= \sigma \|A\|^2 A + \phi_x \\
  \sigma \phi_{xx} + \phi_{yy} &= -2 \left( \|A\|^2 \right)_x
\end{align*}
\]

for \( \sigma = \pm 1 \). The focussing DS II equation (1.1) corresponds (after a change of dependent variable) to (1.5) with \( \sigma = -1 \).

Ablowitz and Fokas [1–3] and Beals and Coifman [7–9] showed that the DS II equation (which also has a ‘defocussing’ version) is completely integrable. To formulate the complete integrability of (1.1), let \( u \in C^1(\mathbb{C} \times \mathbb{R}) \) be given and let \( g \in C^1(\mathbb{C} \times \mathbb{R}) \) with

\[
\partial_x g = -\left( \frac{1}{2} \right) \partial_x (\|u\|^2).
\]

Let

\[
J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & u \\ -\pi & 0 \end{pmatrix}, \quad B = \begin{pmatrix} i g & -i \partial_x g \\ -i \partial_x g & i g \end{pmatrix}.
\]

A function \( u \in C^1(\mathbb{C} \times \mathbb{R}) \) is a solution of the fDSII equation if and only if the relation

\[
\dot{L} = [A, L]
\]

holds as operators, where

\[
L = -\partial_x - iJ\partial_y + Q, \quad A = B - Q \partial_y + iJ \partial_y^2.
\]

The scattering data used to linearize the fDSII flow are determined by \( 2 \times 2 \) matrix-valued solutions \( \Psi(z, k) \) of the problem

\[
\begin{align*}
  (\partial t - ik) \Psi &= 0, \\
  \lim_{|z| \to \infty} \Psi(z, k) \begin{pmatrix} e^{-ikz} & 0 \\ 0 & e^{iz} \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{align*}
\]

where we now take for \( u \) a generic function \( u \in L^2(\mathbb{C}) \). Setting

\[
M(z, k) = \Psi(z, k) \begin{pmatrix} e^{-ikz} & 0 \\ 0 & e^{iz} \end{pmatrix},
\]

we obtain the spectral problem

\[
\begin{pmatrix}
  \partial M_{11} \\ \partial M_{21} \\ (\partial + ik) M_{21} \\
  (\partial + ik) M_{22}
\end{pmatrix} = \frac{1}{2} QM.
\]

Fix \( p > 2 \). We say that \( k \in \mathbb{C} \) is a regular point for the problem (1.6) if there exists a unique matrix-valued solution \( M(\cdot, k) \) with

\[
M(\cdot, k) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in L^p(\mathbb{C}),
\]

and an exceptional point if there is a solution \( M(\cdot, k) \) of (1.6) with \( M(z, k) \in L^p(\mathbb{C}) \). We denote by \( Z \) the set of all exceptional points, called the exceptional set. We will show that the
exceptional set is closed and bounded. The explicit solution for the one-soliton potential discussed in section 4 gives an example of solutions to (1.6) that shows their behavior at regular points and the one singular point.

We will say that an initial datum \( u_0 \) for (1.1) supports solitons if \( Z = \emptyset \) (for a detailed analysis see Sung [31]). On the other hand, Arkadiev, Progrebkov, and Polivanov [5] (see also Doctorov and Leble [14] for a clear textbook presentation) derived an explicit family of initial data

\[
\hat{e}_k(z) = \exp(i(kz + k_0 z)).
\]

(1.8)

Gadyl’shin and Kiselev showed that the solutions (1.7) are unstable in the sense that, for a set of \( C_0^\infty(\mathbb{C}) \) perturbations with finite codimension, the exceptional set \( Z \) for \( u_0 + \varepsilon \varphi \) does not contain \( k_0 \). A similar perturbative analysis was carried out by Pelinovsky and Sulem [27]. Their proof is perturbative in nature and relies on an unproven assumption about the spectrum of a Fredholm operator associated to the problem (1.6). Here we give a global analysis and, along the way, prove the spectral assumption made by Gadyl’shin and Kiselev. The formulation of (ii) below is due to Gadyl’shin and Kiselev but we globalize their result through the use of a Fredholm determinant associated to the direct scattering problem. We will prove:

Theorem 1.1. (i) The potential \( u_0(z) \) has \( Z = \{k_0\} \). (ii) Choose \( \varphi \in C_0^\infty(\mathbb{C}) \) so that

\[
\int_{\mathbb{C}} \left( \chi + |z|^2 \chi \right) \left( 1 + |z|^2 \right)^{-2} d\mu(z)
\]

is nonzero, where \( \chi = e^{-k_0 \varphi} \). Then, for all sufficiently small \( \varepsilon > 0 \), the potential \( u_\varepsilon = u_0 + \varepsilon \varphi \) has empty exceptional set.

Remark 1.2. Villarroel and Ablowitz [35] studied solutions to the linear system (1.6) with isolated singularities including simple poles and multiple poles. Their analysis shows that, at least on a formal level, isolated singularities always lead to soliton-like solutions.

Remark 1.3. That the exceptional set is empty for arbitrarily small perturbations of the one-soliton solution suggests that the solution to (1.1) with such Cauchy data should exist globally in time and exhibit dispersive behavior. For initial data of small norm (an hypothesis which excludes soliton solutions), Sung [32] proved that solutions exhibit \( O(t^{-1}) \) decay in accordance with dispersion for the corresponding linear equation. However, there is not yet a rigorous theorem which shows that absence of exceptional points implies dispersive behavior. Moreover, numerical studies of Klein, Muite, and Roidot discussed in remark 2 below suggest a dichotomy between perturbations that subtract and add energy to the soliton solution. Gadyl’shin and Kiselev [17] showed that, for compactly supported perturbations of the one-soliton solution, soliton-like behavior persists on a time scale of order \( \varepsilon^{-1} \).

Remark 1.4. Klein, Muite, and Roidot’s [23, section 4] numerical study of lump solutions to the focussing DS II equations suggests that the lump solutions should be unstable in the sense that subtracting energy leads to a dispersive solution and adding energy leads to blow-
up in finite time. The illuminating numerical studies of Klein and Saut [25] suggest that lump solutions do not persist for non-integrable perturbations of the focussing DS II equation. See also the comments in Klein and Saut’s survey [24].

To prove theorem 1.1, we reduce the study of the exceptional set to a renormalized determinant of a Fredholm operator associated to problem (1.6) through a series of (standard) symmetry reductions. We make use of the generalized determinant defined by Gohberg, Goldberg, and Krein [20, 21] to define the determinant and show that it solves a \( \partial \)-problem determined by scattering data. Using this equation we are able to compute the determinant associated to the soliton solution to show that \( Z = \{ k_0 \} \). We then study the behavior of the determinant under perturbations in order to show that \( Z = \emptyset \) for perturbations obeying the hypotheses of theorem 1.1.

The paper is organized as follows. In section 2 we fix notation, recall useful estimates on the \( \partial \)-problem, and summarize relevant results of perturbation theory. In section 3 we study the direct scattering problem (1.6), define the scattering data, and define the Fredholm determinant. In section 4, we compute the determinant of the one-soliton solution and prove the spectral assumption of Gadyl’shin and Kiselev. Finally, in section 5, we study perturbations of the determinant and prove theorem 1.1.

2. Preliminaries

2.1. Notation

If \( X \) and \( Y \) are Banach spaces with \( X \cap Y \) dense in \( X \) and \( Y \), we norm \( X \cap Y \) with the norm \( \| f \|_{X \cap Y} = \| f \|_X + \| f \|_Y \).

For any Banach spaces \( X \) and \( Y \), we denote by \( \mathcal{B}(X, Y) \) the Banach space of bounded linear operators from \( X \) to \( Y \), and by \( \mathcal{B}(X) \) the Banach space \( \mathcal{B}(X, X) \).

In what follows, it will be useful to define Fourier transforms adapted to the \( \partial \) and \( \bar{\partial} \) operators:

\[
(\mathcal{F}f)(k) = \frac{1}{\pi} \int_C e^{-k(z)} f(z) \, dm(z), \tag{2.1}
\]

\[
(\mathcal{F}^{-1}g)(z) = \frac{1}{\pi} \int_C e_k(z) g(k) \, dm(z) \tag{2.2}
\]

where \( dm(\cdot) \) denotes Lebesgue measure on \( \mathbb{C} \) and \( e_k \) was defined in (1.8).

For any complex-valued measurable function \( a \) on \( \mathbb{C} \times \mathbb{C} \), we denote by \( a^* \) the measurable function

\[
a^*(z, w) = a(w, z).
\]

We write \( f \lesssim g \) to indicate an upper bound up to absolute numerical constants, and \( f \lesssim_p g \) to indicate an upper bound up to positive constants depending on \( p \).

2.2. Cauchy and Beurling transforms

For \( q \in (1, 2) \), denote by \( \tilde{q} \) the Sobolev conjugate given by \( \tilde{q}^{-1} = q^{-1} - 1/2 \). The solid Cauchy transform

\[
\mathcal{C} [f](z) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{1}{z - \zeta} f(\zeta) \, dm(\zeta)
\]
satisfies $\partial \circ C = C \circ \partial = I$ on $C^\infty_0(C)$. The following standard estimates extend $C$ to larger function spaces and quantify the regularity of $C[f]$.

**Lemma 2.1.** Suppose that $1 < p < 2 < q < \infty$. For any $f \in L^{2q/(q+2)}(C)$,

$$
\|Cf\|_q \lesssim q \|f\|_{2q/(q+2)}.
$$

(2.3)

For any $f \in L^q(C)$,

$$
|\langle Cf \rangle(z) - \langle Cf \rangle(w)| \lesssim_q \|f\|_q |z - w|^{(q-2)/q}.
$$

(2.4)

Finally, for any $f \in L^p(C) \cap L^q(C)$,

$$
\|Cf\|_\infty \lesssim_{p,q} \|f\|_{L^p \cap L^q}.
$$

(2.5)

These estimates are proved, for example, in Vekua [34, chapter I.6] or Astala, Iwaniec, and Martin [6, section 4.3].

**Remark 2.2.** The estimates (2.3)–(2.5) are valid for the integral

$$
I_1(f)(x) = \int \frac{1}{|z - \xi|^\alpha} f(\xi) \, dm(\xi).
$$

The estimate (2.3) in this instance is the case $\alpha = 1$, $n = 2$ of the Hardy–Littlewood–Sobolev inequality $\|x|^{-\alpha} \ast f\|_q \lesssim_p \|f\|_p$ for $\frac{1}{q} = \frac{\alpha}{n} + \frac{1}{p} - 1$.

The following lemma is standard (see for example, [28, lemma 2.2]).

**Lemma 2.3.** Suppose that $p \in (2, \infty)$, that $u \in L^p(\mathbb{R}^2)$, that $f \in L^{2p/(p+2)}(C)$, and that $\partial u = f$ in distribution sense. Then $u = Cf$. Conversely, if $f \in L^{2p/(p+2)}(C)$ and $u = Cf$, then $\partial u = f$ in distribution sense.

Similarly, to solve the equation $\partial u = f$, we introduce the operator

$$
[Cf](z) = \frac{1}{\pi} \int \frac{1}{\overline{z} - \zeta} f(\zeta) \, dm(\zeta)
$$

which obeys analogous estimates. We do not state the obvious analogue of lemmas 2.1 and 2.3 for the operator $\overline{C}$.

The following formulas will help find a basis for the nullspace of the integral operator that describes the one-soliton solution for fDSII. Let

$$
\rho(z) = (1 + |z|^2)^{1/2}.
$$

(2.6)

From the trivial identities

$$
\partial (\overline{\rho}^{-2}) = \rho^{-4}, \quad \partial (\rho^{-2}) = -\overline{\rho}^{-4},
$$

their complex conjugates, and lemma 2.3, we easily deduce

$$
C [\rho^{-4}] = \overline{\rho}^{-2} \quad \overline{C} [\rho^{-4}] = \rho^{-2}
$$

$$
C [\overline{\rho}^{-2}] = -\rho^{-2} \quad \overline{C} [\overline{\rho}^{-2}] = -\overline{\rho}^{-4}.
$$

(2.7)
2.3. Beurling operator

Let
\[(Sf)(z) = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \left( \int_{|z-z'| > \varepsilon} \frac{1}{(z-z')^2} f(z') \, dm(z') \right)\]
initially defined on \(C_0^\infty(\mathbb{C})\). The operator \(S\) is the Beurling transform; see, for example, [6, section 4.5.2] for proofs and discussion.

**Lemma 2.4.** Suppose that \(p \in (1, \infty)\). The operator \(S\) extends to a bounded linear operator from \(L^p(\mathbb{C})\) to itself, unitary if \(p = 2\). Moreover, if \(\nabla f \in L^p(\mathbb{C})\) for some \(p \in (1, \infty)\), \(S(\partial \phi) = \partial \phi\).

2.4. Mixed \(L^p\) spaces

We review some basic facts about mixed \(L^p\) spaces; a standard reference is the paper of Benedek and Panzone [10]. Suppose that \(a\) is a measurable function on \(\mathbb{C} \times \mathbb{C}\). For \(1 < p, q < \infty\), we define
\[\|a\|_{L^p(L^q)} = \left( \int_{\mathbb{C}} \left( \int_{\mathbb{C}} |a(z, w)|^q \, dw \right)^{p/q} \, dz \right)^{1/p}\]
and
\[\|a\|_{L^p(L^\infty)} = \left( \int_{\mathbb{C}} \|a(z, \cdot)\|_\infty^p \, dz \right)^{1/p}.
\]

We denote by \(L^p(L^q)\) (resp. \(L^p(L^\infty)\)) the Banach space of complex-valued measurable functions \(a\) on \(\mathbb{C} \times \mathbb{C}\) with \(\|a\|_{L^p(L^q)}\) (resp. \(\|a\|_{L^p(L^\infty)}\)) finite. Note that \(L^1(L^1)\) is the space \(L^1(\mathbb{C} \times \mathbb{C})\). We have Hölder’s inequality
\[\|ab\|_{L^1} \leq \|a\|_{L^p(L^q)} \|b\|_{L^p(L^\infty)}\]
and
\[\|a\|_{L^p(L^q)} = \sup_{\|b\|_{L^p(L^\infty)} = 1} \left| \int_{\mathbb{C} \times \mathbb{C}} g(z, w)a(z, w) \, dw \, dz \right|.
\] (2.8)

We denote by \(L^p \cap L^{p'}(L^q)\) (resp. \(L^p \cap L^{p'}(L^\infty)\)) the space \(L^p(L^q) \cap L^{p'}(L^q)\) with norm
\[\|a\|_{L^p \cap L^{p'}(L^q)} = \|a\|_{L^p(L^q)} + \|a\|_{L^{p'}(L^q)}\]
while \(L^p \left( L^q \cap L^{p'} \right)\) denotes the space \(L^p(L^q) \cap L^p(L^{p'})\) with norm
\[\|a\|_{L^p \left( L^q \cap L^{p'} \right)} = \|a\|_{L^p(L^q)} + \|a\|_{L^p(L^{p'})}.
\]

Denote by \(S(\mathbb{C} \times \mathbb{C})\) the Schwarz class functions on \(\mathbb{C} \times \mathbb{C}\). If \(g \in S(\mathbb{C} \times \mathbb{C})\) and \(g_\Delta (\zeta) = g (\zeta, \bar{\zeta})\), then
\[\|g_\Delta\|_{L^p} \leq \|g\|_{L^p(L^\infty)}.\] (2.9)
2.5. Perturbation theory

In this subsection we recall some elements of Kato–Rellich perturbation theory as they apply to the perturbation of soliton solutions studied in sections 4 and 5. We consider a norm-continuous mapping \( t \mapsto A(t) \) from an open neighborhood \( U \) of \( 0 \) in \( \mathbb{R}^n \) to the compact operators on a Banach space \( X \).

Let us suppose that \( A(0) \) has the isolated eigenvalue 1. There is a \( \delta > 0 \) so that the circle \( |\lambda - 1| = \delta \) divides the spectrum of \( A(0) \) into disjoint sets, and there is an \( \varepsilon > 0 \) so that for all \( t \) with \( |t| < \varepsilon \), the same circle divides the spectrum of \( A(t) \) into two parts. We may form the projections

\[
P(t) = \frac{1}{2\pi i} \oint_{|\lambda - 1| = \delta} (A(t) - \lambda I)^{-1} \, d\lambda
\]

and

\[
Q(t) = I - P(t).
\]

The projections \( P(t) \) and \( Q(t) \) are continuous operator-valued functions for \( t \) with \( |t| < \varepsilon \). For each fixed \( t \), \( P(t) \) and \( Q(t) \) commute with \( A(t) \). Since \( P(t)^2 = P(t) \) it follows that \( P(t)Q(t) = 0 \). By decreasing \( \varepsilon \) if necessary we may assume that \( \|P(t) - P(0)\| < 1/2 \) for all \( t \) with \( |t| < \varepsilon \), so that \( \dim \text{Ran} P(t) = \dim \text{Ran} P(0) \).

**Lemma 2.5.** For all \( t \) sufficiently small, the operator \( (I - A(t)) \) is invertible if and only the operator \( (I - P(t)A(t)P(t)) \) is invertible.

**Proof.** Let us write \( A \), \( P \), \( Q \) for \( A(t) \), \( P(t) \), and \( Q(t) \). The operator \( QAQ \) has no spectrum in the region \( |\lambda - 1| < \delta \) so the inverse \( (I - QAQ)^{-1} \) exists for all \( t \) with \( |t| < \varepsilon \). Computing

\[
(I - A)(I - QAQ)^{-1} = (I - PAP - QAQ)(I - QAQ)^{-1} = I - PAP
\]

we see that \( (I - A)^{-1} \) exists if and only if \( (I - PAP)^{-1} \) exists. \( \square \)

We can make a further reduction using an observation of Nagy [33] already used by Gadyl’shin and Kiselev in their analysis of the one-soliton perturbation. Write \( P_0 \) for \( P(0) \).

**Lemma 2.6.** For sufficiently small \( t \), there is an invertible operator \( V(t) \) so that \( PAP \) is similar to \( P_0 V(t)^{-1} A(t) V(t) P_0 \), and \( \lambda I - PAP \) is invertible if and only if \( \lambda I - P_0 V(t)^{-1} A(t) V(t) P_0 \) is invertible.

**Proof.** We set

\[
V = (I - (P - P_0)^2)^{-1/2} [PP_0 + (1 - P)(1 - P_0)]. \tag{2.10}
\]

It is not difficult to see that, if \( \|P - P_0\| < 1/2 \),

\[
\|V - I\| \leq 2\|P - P_0\|
\]

and that \( PAP \) is similar to

\[
P_0 V^{-1} A V P_0 \tag{2.11}
\]

(see [29], notes to section XII.2 and problem 19 of chapter XII, and see also the classic paper of Nagy [33]). It now follows that \( (\lambda I - A) \) is invertible if and only if \( \lambda I - P_0 V^{-1} A V P_0 \) is invertible. \( \square \)
2.6. Eigenvalue multiplicities

In what follows we denote by $\det(I + \cdot)$ a generalized determinant defined on an algebra $\mathcal{E}$ of compact operators on a Banach space $X$, having the following properties:

(i) $(I + A)$ is invertible if and only if $\det (I + A) \neq 0$, and
(ii) $\det (I + F) = \det (I + F) \cdot \text{tr}(F)$ for finite-rank operators $F$.

In applications, $X = L^p$, $\mathcal{E}$ is the Mikhlin–Itskovich algebra, and $\det (I + \cdot)$ is the generalized determinant described in appendix A.

Recall that the geometric multiplicity of the eigenvalue $\lambda$ of a matrix is the dimension of its eigenspace, while the algebraic multiplicity is the order of the corresponding zero of the characteristic polynomial. An eigenvalue is called semisimple if the algebraic and geometric multiplicities are equal. Since the eigenspaces of compact operators corresponding to nonzero eigenvalues are finite-dimensional, these definitions also make sense for nonzero eigenvalues of compact operators. In this subsection, we prove:

**Lemma 2.7.** Suppose that $A(\kappa)$ is a $C^1$ compact operator-valued function defined on an open neighborhood of 0 in $\mathbb{C}$. Suppose further that the eigenvalue $\lambda = 1$ of $A(0)$ is semisimple, and that $\det (I - A(\kappa)) = c|\kappa|^m (1 + o(1))$ as $k \to 0$. Then $\ker (I - A(0))$ has dimension at most $m$.

**Proof.** The operator $A(0)$ is compact so $\ker (I - A(0))$ is at most finite-dimensional. Moreover, for $\kappa$ small, there is a $\delta > 0$ so that the circle $|\lambda - 1| = \delta$ divided the spectrum of $A(\kappa)$ into two disjoint parts. Let $P, Q, P_0, Q_0$, $V$ be as in section 2.5 above. We analyze $\det (I - A(\kappa))$ for $\kappa$ small by splitting $I - A = I - PAP - QAQ$. Using the determinant formula (A.2), we factor

$$(I - PAP - QAQ) = (I - PAP)(I - QAQ)$$

(since $PQ = QP = 0$) and conclude from (A.2) that

$$\det (I - A) = \det (I - PAP) \det (I - QAQ)$$

since $PAP \cdot QAQ = 0$. Moreover, from the discussion in the previous section, $PAP$ is similar to $P_0V^{-1}AVP_0$ so

$$\det (I - A) = \det (I - P_0V^{-1}AVP_0) \det (I - QAQ).$$

The second factor is nonvanishing and has a finite nonzero limit as $\kappa \to 0$, so the leading asymptotics are determined by the first factor. Since $A(\kappa) - A(0) = \mathcal{O}(|\kappa|)$ in operator norm as $\kappa \to 0$, it follows that, also, $\|P - P_0\| = \mathcal{O}(|\kappa|)$, $V - I = \mathcal{O}(|\kappa|)$. Since $A(0)$ has semisimple eigenvalues and $P_0V^{-1}AVP_0$ is a rank $N$ operator, we may choose a basis of eigenvectors $\{\psi_i\}_{i=1}^N$ for $A(0)$ in $\text{Im} P_0$ and a dual basis $\{\chi_i\}_{i=1}^N$ in $X^*$ so that

$$\langle \chi_i, \psi_j \rangle = \delta_{ij}$$

where $\langle \cdot, \cdot \rangle$ is the usual dual pairing. It follows that

$$P_0 = \sum_{i=1}^N \langle \chi_i, \cdot \rangle \psi_i.$$ 

Hence $\det (I - P_0V^{-1}AVP_0)$ is, up to strictly nonzero factors, the determinant of the $N \times N$ matrix $M$ with
\[ M_{ij} = \langle \chi_i, [I - V^{-1}AV] \psi_j \rangle. \]

Since \( V^{-1}A(\kappa)V = A(0) + \mathcal{O}(|\kappa|) \), it follows that \( M_{ij} = \mathcal{O}(|\kappa|^N) \) and so
\[
\det M = \mathcal{O}(|\kappa|^N).
\]

Hence \( m \geq N \).

**Remark 2.8.** The conclusion of lemma 2.7 is false if the eigenvalue \( \lambda = 1 \) is not semisimple. To see this, consider the matrix
\[
A(\varepsilon) = \begin{pmatrix}
1 & 1 & \varepsilon \\
\varepsilon & 1 & 1 \\
\varepsilon & \varepsilon & 1
\end{pmatrix}
\]
for which \( \det(I - A(\varepsilon)) = \varepsilon^3 + \varepsilon \), so \( N = 3 \) but \( m = 1 \).

### 3. A Fredholm determinant for direct scattering

In this section we characterize the exceptional set \( Z \) as the zero set of a renormalized Fredholm determinant associated to the scattering problem (1.6).

**3.1. Reduction by symmetries**

For \( p > 2 \), \( u \in L^2 \), and \( k \in \mathbb{C} \), define an operator \( S_{k,u} \in B(L^p) \) by
\[
S_{k,u}h = -\frac{1}{4} C(ue^{-iC}(e_k \bar{u}h)).
\]

In this subsection, we prove:

**Proposition 3.1.** Suppose that \( u \in L^2(\mathbb{C}) \) and \( p > 2 \). Then \( k \in \mathbb{C} \) is an exceptional point for the problem (1.6) if and only if \( \ker L(p)(I - S_{k,u}) \) is nontrivial.

Recall that \( k \) is an exceptional point for (1.6) if the problem (1.6a) has a nontrivial solution with \( M(z,k) \in L^p(\mathbb{C}) \). We reduce to a single integral equation involving the integral operator (3.1) in several steps.

**Lemma 3.2.** Fix \( p > 2 \). Suppose that \( u \in L^2(\mathbb{C}) \) and \( k \in \mathbb{C} \setminus Z \). Then, the unique solution \( M \) of (1.6) with \( M - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in L^p(\mathbb{C}) \) takes the form
\[
M(z,k) = \begin{pmatrix}
m_1(z,k) & -m_2(z,k) \\
m_2(z,k) & m_1(z,k)
\end{pmatrix}
\]
where
\[ \begin{align*}
\partial m_1 &= \frac{1}{2} m_2 \\
(\partial + ik)m_2 &= -\frac{1}{2} \overline{m_1} \\
(m_1(z,k) - 1, m_2(z,k)) &\in L^p(\mathbb{C})
\end{align*} \tag{3.3} \]

\textbf{Proof.} A straightforward computation shows that the function (3.2) solves (1.6), so the result now follows by unicity. \hfill \Box

Thus, to compute the exceptional set, it suffices to study the system (3.3). By lemma 2.3, we can reduce the system (3.3) to a system of integral equations using the Cauchy transform.

In what follows, the condition \( u \in L^{2p/(p+2)}(\mathbb{C}) \) insures that expressions such as \( \mathcal{C}(eku) \) define functions in \( L^p(\mathbb{C}) \).

\textbf{Lemma 3.3.} Fix \( p > 2, u \in L^2(\mathbb{C}) \cap L^{2p/(p+2)}(\mathbb{C}), \) and \( k \in \mathbb{C} \setminus \mathbb{Z} \). A vector-valued function \( m = (m_1, m_2) \) with \( (m_1 - 1, m_2) \in L^p(\mathbb{C}) \) solves (3.3) if and only if

\[ \begin{align*}
m_1 &= 1 + C \left( \frac{1}{4} um_2 \right) \\
m_2 &= -\frac{1}{2} e^{-k} \overline{C}(ek \overline{um_1})
\end{align*} \tag{3.4} \]

Finally, we can iterate to a scalar integral equation (3.5):

\textbf{Lemma 3.4.} Fix \( p > 2, u \in L^2(\mathbb{C}) \cap L^{2p/(p+2)}(\mathbb{C}), \) and \( k \in \mathbb{C} \setminus \mathbb{Z} \). The vector-valued function \( m \) with \( (m_1 - 1, m_2) \in L^p(\mathbb{C}) \) solves (3.4) if and only if

\[ m_1 = 1 - \frac{1}{4} \mathcal{C}(ue^{-k} \mathcal{C}(ek \overline{um_1})) \tag{3.5} \]

and

\[ m_2 = -\frac{1}{2} e^{-k} \overline{\mathcal{C}(ek \overline{um_1})}. \tag{3.6} \]

We omit the (easy) proofs of lemmas 3.3 and 3.4. The compositions with \( \mathcal{C} \) and \( \overline{\mathcal{C}} \) make sense since \( \mathcal{C}, \overline{\mathcal{C}} : L^{2p/(p+2)} \to L^p \) by (2.3) and \( uf \) in \( L^{2p/(p+2)} \) by Hölder’s inequality provided \( u \in L^2 \) and \( f \in L^p \). Note that the equation (3.5) is equivalent to

\[ m_1 - 1 = S_{k,u} 1 + S_{k,u}(m_1 - 1) \]

so that, if \( (I - S_{k,u}) \) is invertible,

\[ m_1 = 1 + (I - S_{k,u})^{-1} (S_{k,u} 1). \]

Since this formula determines \( m_1 \) and \( m_2 \) is determined by \( m_1 \), this reduces existence and uniqueness theory for the system (3.4) to the existence of the resolvent \( (I - S_{k,u})^{-1} \).

We apply Fredholm theory to the operator \( (I - S_{k,u}) \). We will sometimes decompose (3.1) as

\[ S_{k,u} = W_{k,u} \circ V_{k,u}, \tag{3.7} \]

where

\[ W_{k,u} h = \frac{1}{2} \mathcal{C}(ue^{-k} h), \quad V_{k,u} h = -\frac{1}{2} \overline{\mathcal{C}(ek \overline{h})}. \tag{3.8} \]
Lemma 3.5. Fix $p > 2$ and $k \in \mathbb{C}$. Suppose that $u \in L^2 \cap L^{2p/(p+2)}$. Then, the operator $S_{k,u}$ is compact as an operator from $L^p(\mathbb{C})$ to itself.

Proof. It follows from lemma 2.1 and its analogue for $\mathbb{C}$ that $S_{k,u}$ is a bounded operator on $L^p$ provided $u \in L^2 \cap L^{2p/(p+2)}$. Moreover, since $S_{k,u}$ is bilinear in $u$, it is easy to see that the map $L^p \cap L^{2p/(p+2)} \ni u \mapsto S_{k,u} \in \mathcal{B}(L^p)$ is continuous. Hence, to prove that $S_{k,u}$ is compact, it suffices to do so for $u \in C_0^\infty(\mathbb{C})$ and appeal to density. We can argue as in the first paragraph of [28, proof of lemma 3.1] that $W_{k,u}$ is compact, while $V_{k,u}$ is bounded by lemma 2.1 again. Hence $S_{k,u}$ is compact. $\square$

Proof of proposition 3.1. It follows from lemmas 3.2, 3.4 and 3.5 and the Fredholm alternative that the problem (1.6) has a unique solution if and only if $\ker(I - S_{k,u})$ is trivial. $\square$

3.2. Renormalized determinant

We will now define and study a renormalized determinant of $I - S_{k,u}$. In proposition B.1, it is shown that the operator $S_{k,u}$ belongs to the Miklhlin–Itskovich algebra $\mathcal{E}_p$ provided $p > 2$ and $u \in L^p(\mathbb{C}) \cap L^p(\mathbb{C})$ where

$$\frac{1}{2} + \frac{1}{p} < \frac{1}{t}, \quad \frac{1}{p} + \frac{1}{t} > 1.$$  \hfill (3.9)

Definition 3.6. We say that $(p, t)$ is an admissible pair if $p > 2$, $t \in [1, 2)$, and (3.9) holds.

Remark 3.7. The two constraints (3.9) together with $p > 2$ and $t > 1$ imply that $(1/p, 1/t)$ belong to the interior of the triangle with vertices $(0, 1)$, $(1/2, 1)$ and $(1/4, 3/4)$ in the $(1/p, 1/t)$-plane (see figure B1 in appendix B). If $(p, t)$ is an admissible pair and $u \in L^p(\mathbb{C}) \cap L^p(\mathbb{C})$, it is easy to see that $u \in L^{2p/(p+2)}$ since, by (3.9), the inequalities

$$\frac{1}{t} < \frac{1}{2} + \frac{1}{p} < \frac{1}{t}$$

hold.

For an admissible pair $(p, t)$, the renormalized determinant of theorem A.2

$$D(k, u) = \text{Det}(I - S_{k,u})$$  \hfill (3.10)

is a well-defined, bounded continuous function of $(k, u)$ with $D(k, u) \to 1$ as $k \to \infty$ and $D(k, u)$ continuous in $u \in L^p(\mathbb{C}) \cap L^p(\mathbb{C})$ uniformly in $k \in \mathbb{C}$.

We will define the determinant in Banach space of potentials large enough to include $C_0^\infty(\mathbb{C})$ perturbations of the soliton solution (1.7), and sufficiently restrictive that the $\partial \mathcal{E}$ equation stated in theorem 3.12 holds. For $\alpha \in (1/2, 1)$ let

$$X_\alpha = W^{1,2}(\mathbb{C}) \cap L^{2,\alpha}(\mathbb{C})$$

where $W^{1,2}(\mathbb{C})$ consists of $L^2$ functions with one weak derivative in $L^2$ and

$$L^{2,\alpha}(\mathbb{C}) = \left\{ f \in L^2(\mathbb{C}) : (1 + |z|^2)^{\alpha/2}f \in L^2(\mathbb{C}) \right\}.$$
Note that $X_t$ is the space $H^{1,1} (\mathbb{C})$ considered in [28]. We need $\alpha < 1$ to include the soliton solution (1.7), but $\alpha > 1/2$ for later estimates. It is easy to see that if $u \in X_\alpha$, then $u \in L^q (\mathbb{C})$ for $q \in \left( \frac{2}{1+\alpha}, \infty \right)$, so that $u \in L^1 (\mathbb{C}) \cap L^\prime (\mathbb{C})$ for $\frac{2}{1+\alpha} < t < \frac{2}{1-\alpha}$. To find admissible $(p,t)$ with $t \in (1, 4/3)$, we require $\alpha > 1/2$.

We also note:

**Lemma 3.8.** The inclusion $X_\alpha \to L^2 (\mathbb{C})$ is a compact embedding.

**Proof.** For any bounded set $\Omega \subset \mathbb{C}$ with smooth boundary, the compact embedding $W^{1,2} (\Omega) \hookrightarrow L^2 (\Omega)$ holds. Let $\chi \in C_0^\infty (\mathbb{C})$ with $\chi (z) = 1$ if $|z| \leq 1$ and $\chi (z) = 0$ if $|z| \geq 2$. Let $\chi_R (z) = \chi (z/R)$. For each $R > 0$, the map $f \mapsto \chi_R f$ is compact from $W^{1,2} (\mathbb{C})$ into $L^2 (\Omega)$. For $u \in X_\alpha$, $\| (1 - \chi_R) u \|_{L^1 (\mathbb{C})} \leq R^{-\alpha} \| u \|_{L^2 (\mathbb{C})}$ so taking $R \to \infty$ we see that the embedding into $L^2 (\mathbb{C})$ is compact. □

From proposition B.1 and the remarks above, we have:

**Proposition 3.9.** Suppose that $u \in X_\alpha$ for some $\alpha \in (1/2, 1)$. Then the renormalized determinant

$$D(k, u) = \text{Det} (I - S_{k,u})$$

is continuous in $k$ and $u$, and satisfies the asymptotic condition

$$\lim_{|k| \to \infty} D(k, u) = 1.$$

Clearly, $Z = \{ k \in \mathbb{C} : D(k) = 0 \}$. As an immediate corollary, we have:

**Corollary 3.10.** Suppose that $u \in X_\alpha$ for some $\alpha \in (1/2, 1)$. Then the exceptional set $Z$ is closed and bounded.

**Remark 3.11.** By remark A.3, if $S_{k,u}$ belongs to $E_p$ and $E_q$ for distinct $p$ and $q$, the determinants in $E_p$ and $E_q$ coincide.

### 3.3. A $\bar{\partial}$-equation for the determinant

We will now derive a $\bar{\partial}$-equation for $D(k, u)$ in terms of scattering data for $u$. In what follows we write $D(k, u)$ for $D(k, u)$ for brevity. We first define the scattering data for $u \in C_0^\infty (\mathbb{C})$ and then use continuity to pass to $u \in X_\alpha$.

For $\delta > 0$, let

$$\Omega_\delta = \{ k \in \mathbb{C} : \text{dist} (k, Z) > \delta \}.$$

By corollary 3.10, $\Omega_\delta$ is an unbounded open set that contains a neighborhood of infinity. On this set, the solution of (3.4) is unique, and we define scattering data $r$ and $s$, functions of $k \in \Omega_\delta$, by the asymptotic formulas

$$s(k) = 2 \lim_{|z| \to \infty} z (m_1 (z, k) - 1) \quad (3.11)$$

$$r(k) = -2 \lim_{|z| \to \infty} \left( e_{-k} (z) \frac{m_2 (z, k)}{z} \right) \quad (3.12)$$

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The existence of these limits is a simple consequence of the formula
\[ \lim_{|z| \to \infty} z C[f](z) = \frac{1}{\pi} \int f(z) \, dm(z) \] 
valid if \( f \in L^1(\mathbb{C}) \cap L^p(\mathbb{C}) \) for some \( p > 2 \) (see (2.5)). From (3.4) and (3.13), we deduce that
\[ s(k) = \frac{1}{\pi} \int u(z)m_2(z,k) \, dm(z) \]
\[ r(k) = -\frac{1}{\pi} \int e^{-k(z)}u(z)m_1(z,k) \, dm(z). \]

In this section, we will prove:

**Theorem 3.12.** Suppose that \( u \in X_\alpha \) for some \( \alpha > 1/2 \) and that \( \delta > 0 \). Then \( D(k) \) defined by (3.10) obeys the \( \bar{\partial} \)-equation
\[ \bar{\partial} \log D(k) = \frac{i}{2}s(k) - c(k) \]
for all \( k \in \Omega_\delta \), where
\[ c(k) = -\frac{i}{4\pi^2} \int \frac{e^{-k(w)}u(w)e_k(z)u(z)}{z-w} \, dm(w) \, dm(z). \]

**Remark 3.13.** Differentiating (3.17) with respect to \( k \) and using the analogue of lemma 2.3 for the \( \bar{\partial} \)-operator, we conclude that
\[ c(k) = \frac{1}{4\pi} \int \frac{1}{k-\zeta} |(F \cdot u)(\zeta)|^2 \, dm(\zeta). \]

We begin by considering \( u \in C_0^\infty(\mathbb{C}) \).

**Proposition 3.14.** Suppose that \( u \in C_0^\infty(\mathbb{C}) \) and \( \delta > 0 \). Then, the conclusion of theorem 3.12 holds.

**Proof.** We apply lemma A.4. Compute
\[ \bar{\partial}_k \log \text{Det} (I - S_{k,u}) = \text{Tr} \left( (-1)(I - S_{k,u})^{-1} \bar{\partial}_k S_{k,u} \right) + \text{Tr} \left( \bar{\partial}_k S_{k,u} \right). \]

Each of the two right-hand terms in (3.18) is the trace of a rank-one operator. To evaluate the traces, we begin with the second term. From (3.1), we easily compute that, for any \( f \in L^p \),
\[ (\bar{\partial}_k S_{k,u}f)(z) = \frac{i}{4} \mathcal{F}^{-1} \left( \mathcal{F}^{-1}(e_{-k}u)(z) \cdot \mathcal{F}^{-1}(\mathcal{F} \mu)(k) \right) \]
where \( \mathcal{F}^{-1} \) was defined in (2.2). This expression defines a rank-one integral operator with integral kernel
\[ K_2(z,w;k) = \frac{i}{4\pi^2} \int \frac{1}{z-\zeta} \frac{e^{-k(z)}u(z) \, dm(z)}{z-\zeta} \cdot e_k(w)u(w). \]
It follows that
\[
\text{Tr} \left( \overline{\partial}_k S_{k,u} \right) = \frac{i}{4\pi^2} \int \int \frac{1}{z-w} e^{-k(w)} u(w) \overline{e_k(z) u(z)} \, dw \, dz. \tag{3.20}
\]

To analyze the first right-hand term in (3.18), we first claim that
\[
\left( I - S_{k,u} \right)^{-1} \left( \frac{1}{2} \nabla^{-1} (e_{-k} u) \right) (z) = -e_{-k} m_2(z,k).
\tag{3.21}
\]

If so, it follows from (3.19) that for any \( f \in L^p \),
\[
-(I - S_{k,u})^{-1} (\overline{\partial}_k S_{k,u} f) (z) = \frac{i}{2} e_{-k}(z) m_2(z,k) \cdot F^{-1} (\overline{f}) (k).
\tag{3.22}
\]

This operator has integral kernel
\[
K_1(z,w;k) = \frac{i}{2\pi} e_{-k}(z) m_2(z,k) e_k(w) \overline{u(w)}
\]
and hence trace
\[
-\text{Tr} \left[ (I - S_{k,u})^{-1} (\overline{\partial}_k S_{k,u}) \right] = \frac{i}{2\pi} \int \overline{u(z)} m_2(z,k) \, dz. \tag{3.23}
\]

Combining (3.20) and (3.23) and using (3.14) then proves (3.16) in case \( u \in C_0^\infty(\Omega) \). The identity (3.17) is a consequence of (3.20).

To complete the proof, we need to show that (3.21) holds. From equations (3.5) and (3.6) and (3.7) and (3.8) it is easy to compute that
\[
e_k m_2 = V_{k,u} (I - W_{k,u} V_{k,u})^{-1} 1 = (I - V_{k,u} W_{k,u})^{-1} V_{k,u} 1.
\]

Taking complex conjugates we recover (3.21). \( \square \)

Now we would like to prove that \( D(k) \) solves the same \( \overline{\partial} \)-equation weakly if \( u \in X_\alpha \) for some \( \alpha \in (1/2, 1) \). The following proposition gives a ‘to-do list’.

**Proposition 3.15.** Fix \( \alpha \in (1/2, 1) \) and \( \delta > 0 \). Suppose that the map \( u \mapsto m_2(\cdot, k; u) \) defined by (3.5) and (3.6) has the following property: if \( u \in X_\alpha \) and \( \{u_n\} \) is a sequence from \( C_0^\infty(\Omega) \) converging to \( u \), and if \( Z_n \) (resp. \( Z \)) denotes the exceptional set of \( u_n \) (resp. \( u \)), then

(i) For the given \( \delta > 0 \) and all \( n \) sufficiently large depending on \( \delta \), the condition \( \text{dist}(k, Z_n) > 2\delta \) implies that \( \text{dist}(k, Z_n) > \delta \), and

(ii) \( m_2(\cdot, \delta; u_n) \rightarrow m_2(\cdot, \delta, u) \) in \( L^\infty(\Omega_\delta; L^p(\Omega)) \).

Then (3.16) holds weakly on \( \Omega_\delta \).

**Proof.** Assuming the continuity, let \( u \in X_\alpha \) and let \( \{u_n\} \) be a sequence from \( C_0^\infty(\Omega) \) with \( u_n \rightarrow u \) in \( X_\alpha \). Since \( \alpha > 1/2 \), \( u \in L^t \cap L^{t'} \) for some \( t \in (1, 4/3) \) and there is a \( p \) with \( \tilde{t} < p < t' \) so that \( (p, t) \) is an admissible pair of exponents. A short computation shows that \( p' \in (t, t') \) so that, also, \( u \in L^{p'}(\Omega) \) with norm bounded by \( \|u\|_{X_\alpha} \). Denoting by \( s_n \) the scattering data corresponding to \( u_n \), we have from (3.14) that
\[ \|s_n - s\|_{L^\infty(\Omega_\lambda)} \leq \|u - u_n\|_p \|m_2(\cdot, \diamond; u_n)\|_{L^\infty(\Omega_\lambda; L^p(\mathbb{C}))} + \|u_n\|_p \|m_2(\cdot, \diamond; u_n) - m_2(\cdot, \diamond; u)\|_{L^\infty(\Omega_\lambda; L^p(\mathbb{C}))} \]

so that, if the hypothesis holds, \( s_n \to s \) in \( L^\infty(\Omega_\lambda) \) as \( n \to \infty \).

Next, let
\[ c_n = -\frac{i}{4\pi^2} \int \int \frac{e^{-k(w)u_n(w)}e_k(z)u_n(z)}{z - w} \, dm(w) \, dm(z). \]

From (2.3) we easily see that
\[ |c_n| \lesssim \|u_n\|_p \|u_n\|_{2p/(p+2)} \]
so by bilinearity
\[ |c_n - c| \lesssim \|u_n - u\|_p \|u_n\|_{2p/(p+2)} + \|u\|_p \|u_n - u\|_{2p/(p+2)}. \]

The \( X_n \) norm dominates the \( L' \) and \( L^{2p/(p+2)} \) norms so that \( |c_n - c|_{L^\infty(\mathbb{C})} \to 0 \) as \( n \to \infty \).

Finally, let \( \varphi \in C_0^\infty(\Omega_\delta) \). For sufficiently large \( n \), \( \text{Det}(k; u_n) \) is defined for all \( k \in \Omega_\delta \) and we may compute
\[
\int_{\Omega_\delta} \left[ (-\vec{\partial}\varphi) \log D(k, u) - \varphi \left( \frac{1}{2\pi} \log |k| - c(k) \right) \right] \, dm(k) = \lim_{n \to \infty} \int_{\Omega_\delta} \left[ (-\vec{\partial}\varphi) D(k, u_n) - \varphi \left( \frac{1}{2\pi} \log |k| - c_n(k) \right) \right] \, dm(k) = \lim_{n \to \infty} \varphi \left[ \vec{\partial} \log D(k, u_n) - \left( \frac{1}{2\pi} \log |k| - c_n(k) \right) \right] \, dm(k) = 0.
\]

It remains to show that the hypothesis of proposition 3.15 holds. First:

\[ \Box \]

**Lemma 3.16.** Fix \( \delta > 0 \). Suppose that \( u \in X_\alpha \) for some \( \alpha \in (1/2, 1) \), and that \( \{u_n\} \) is a sequence from \( X_\alpha \) with \( u_n \to u \). Finally, let \( Z_\alpha \) and \( Z \) be the respective exceptional sets for \( u_n \) and \( u \). There is an \( N \) so that for any \( n > N \), \( \text{dist}(z, Z_\alpha) > \delta \) provided \( \text{dist}(z, Z) > 2\delta \).

**Proof.** Since \( u_n \to u \) in \( X_\alpha \), it follows that there is an admissible pair of exponents \((p, t)\) so that \( u_n \to u \) in \( L' \cap L^t \) and so \( S_{k,u_n} \to S_{k,u} \) is the Mikhlin–Itosovich algebra \( \mathcal{E}_p \). It then follows from theorem A.3 that \( D(k; u_n) \to D(k; u) \) uniformly in \( k \in C \). Choosing \( N \) so that \( \sup_{k \in C, n \geq N} |D(k; u_n) - D(k; u)| < \delta \) gives the desired conclusion. \( \Box \)

Next, we study continuity of the map \( u \mapsto m_2(\cdot, \diamond; u) \). As always we fix \( u \in X_\alpha \) for some \( \alpha \in (1/2, 1) \) and an admissible pair \((p, t)\). It follows from (3.5) and (3.6) that
\[ m_2 = e_{-k} V_{k,u} 1 + e_{-k} V_{k,u} \left( (I - S_{k,u})^{-1} S_{k,u} 1 \right) \]
so, to prove the continuity, it suffices to prove that
(i) \( e_{-k} V_{k,u} 1 \to e_{-k} V_{k,u} 1 \) in \( L^\infty(\Omega_\delta; L^p(\mathbb{C})) \) as \( n \to \infty \),
(ii) \( S_{k,u} 1 \to S_{k,u} 1 \) in \( L^\infty(\Omega_\delta; L^p(\mathbb{C})) \) as \( n \to \infty \),
(iii) \( (I - S_{k,u})^{-1} \to (I - S_{k,u})^{-1} \) in \( \mathcal{B}(L^p) \) as \( n \to \infty \), uniformly \( k \in \Omega_\delta \).
As before, we may always assume that \( k \in \Omega_\delta \) belongs to \( C \setminus Z_n \) if \( n \) is large enough.

First, we show:

**Lemma 3.17.** Fix \( u \in X_\alpha \) for some \( \alpha \in (1/2, 1) \), and let \((p, t)\) be an admissible pair. Let \( \{u_n\} \) be a sequence from \( X_\alpha \), converging to \( u \).

(i) \( e^{-t} V_{k,u_n} 1 \rightarrow e^{-t} V_{k,u} 1 \) in \( L^\infty(C; L^p(C)) \), and

(ii) \( S_{k,u_n} 1 \rightarrow S_{k,u} 1 \) in \( L^\infty(C; L^p(C)) \) as \( n \rightarrow \infty \).

**Proof.** Since \( u_n \rightarrow u \) in \( X_\alpha \), we also have \( u_n \rightarrow u \) in \( L^r \cap L^{r'} \), and hence in \( L^2 \). From (2.3) and Hölder’s inequality we have

\[
\| C[u] \|_p \lesssim_p \| |u|_2 \|_f \|_p. \tag{3.24}
\]

The conclusions (i) and (ii) follow from this estimate. \( \Box \)

The resolvents \( R_n = (I - S_{k,u_n})^{-1} \) and \( R = (I - S_{k,u})^{-1} \) exist for \( k \in \Omega_\delta \) by lemma 3.16. Observe that

\[
R_n - R = R_n (W_{k,u_n} \circ V_{k,u_n} - W_{k,u} \circ V_{k,u}) R
\]

so that continuity of the resolvent will follow from (i) estimates on \( \| R \|_{B(L^r)} \) and \( \| R_n \|_{L^r} \) uniform in \( k \in \Omega_\delta \), (ii) uniform estimates on \( \| W_{k,u_n} \|_{B(L^r)} \), \( \| V_{k,u_n} \|_{B(L^r)} \), and (iii) norm estimates on \( \| W_{k,u_n} - W_{k,u} \|_{B(L^r)} \) and \( \| V_{k,u_n} - V_{k,u} \|_{B(L^r)} \) which vanish as \( n \rightarrow \infty \). The uniform estimates (ii) and the norm estimates (iii) follow from (3.24). Thus, it remains to prove uniform estimates on the resolvents \( R \) and \( R_n \). For this, the following estimate will suffice.

**Lemma 3.18.** Suppose that \( u \in X_\alpha \) that \((t, p)\) is an admissible pair, and \( \delta > 0 \). Then

\[
\sup_{k \in \Omega_\delta} \| (I - S_{k,u})^{-1} \|_{L^r \rightarrow L^p} \lesssim \delta/2
\]

**Proof.** We will show first that \( (I - S_{k,u})^{-1} \) has norm bounded by 2 for all \( k \in C \) with \( |k| \geq R \) for some constant \( R \) depending on \( \|u\|_{X_\alpha} \). We will then use a continuity-compactness argument to show that

\[
\sup_{k \in C: |k| \in R, \text{dist}(k, Z) \geq \delta/2} \| (I - S_{k,u})^{-1} \|_{B(L^r)} \lesssim 1 \tag{3.25}
\]

where the implied constant depends on \( \delta \), \( R \), and \( u \).

First, we recall from [28, equation (3.13)] the estimate

\[
\| W_{k,u} \|_p \lesssim_p (k)^{-1} \left( \| u \|_2 \| \psi \|_p + \| \nabla u \|_2 \| \psi \|_p + \| u \|_p \| \nabla \psi \|_2 \right).
\]

Putting \( \psi = V_{k,u} h \) we recover

\[
\| S_{k,u} h \|_p \lesssim_p (k)^{-1} \left( \| u \|_2^2 + \| u \|_2 \| \nabla u \|_2 + \| u \|_p \| u \|_{2p/(p-2)} \right) \| h \|_p.
\]

(Recall that, for an admissible pair, \( \| u \|_{2p/(p+2)} \) is bounded by \( \| u \|_{L^r \cap L^{r'}} \) This shows that \( \| S_{k,u} \|_{B(L^r)} < 1/2 \) for \( k \) sufficiently large depending on \( \| u \|_{X_\alpha} \).

It remains to prove (3.25). The set

\[
U_{k,R} = \{ k \in C : |k| \leq R, \text{dist}(k, Z) \geq \delta/2 \}
\]
is a compact subset of $\mathbb{C}$, while the map $k \to (I - S_{k,u})^{-1}$ is continuous from $U_{\delta,R}$ into $B(L^p)$. It follows that
\[
\sup_{k \in U_{\delta,R}} \| (I - S_{k,u})^{-1} \|_{L^p \to L^p} \lesssim \delta R 1.
\]

**Proof of theorem 3.11.** An immediate consequence of propositions 3.14 and 3.15 together with lemmas 3.16–3.18.

4. The one-soliton solution

We now consider the one-soliton potential [5]
\[
u_0(z) = \frac{2e^{ik_0}}{\rho(z)^2}
\]
(recall (2.6)). With this choice of $u$, (3.3) admits the formal solution
\[
\begin{align*}
m_1(z,k) &= 1 + \frac{1}{k - k_0} \frac{i}{1 + |z|^2} \overline{z} \\
m_2(z,k) &= \frac{1}{k - k_0} \frac{i}{1 + |z|^2} e^{-k_0(z)}
\end{align*}
\]  

(4.2)

Using (3.11) and (3.12), we read off
\[
s(k) = \frac{2i}{k - k_0}
\]
\[
r(k) = 0.
\]

(4.3)  

(4.4)

The formal solution (4.2) is correct for large $|k|$ since equation (1.6) has a unique solution for $|k|$ sufficiently large. To conclude that this equation holds for all $k \neq k_0$ we must show that $k_0$ is the only exceptional point.

In what follows, we will set $\kappa = k - k_0$ and define
\[
T(\kappa) = S_{k_0 + \kappa,u_0} = -\frac{1}{4} C e^{-\kappa \rho} \overline{e}_\rho e_{\rho} \rho^{-2}.
\]

(4.5)

We will prove:

**Theorem 4.1.** For $u_0$ given by (4.1), the operator $I - T(\kappa)$ has a nontrivial nullspace if and only if $\kappa = 0$. Moreover, the zero eigenvalue of $I - T(0)$ is semisimple and of multiplicity two. Finally, if $P(0)$ projects onto the nullspace of $I - T(0)$, then
\[
P(0)T(\kappa)P(0) = \begin{pmatrix} 1 & i\kappa \\ -i\kappa & 1 \end{pmatrix} + \mathcal{O}_\delta (|\kappa|^{2-\delta})
\]

(4.6)

for any $\delta > 0$.

**Remark 4.2.** The error estimate in (4.6) can be improved to $\mathcal{O} (|\kappa|^2 \log |\kappa|)$ but we will not need this. We will need (4.6) for the perturbation calculations in section 5.
Proof. The theorem is an immediate consequence of propositions 4.3, 4.4 and 4.8 below.

We will prove theorem 4.1 in three steps. First, we show that $T(\kappa)$ is a differentiable operator-valued function in the $\mathcal{E}_p$ norm (which is stronger than the operator norm on $L^p$). Next, we use the determinant $\text{Det}(I - T(\kappa))$ to prove that there is a unique singular point, and compute the determinant explicitly. Combining this explicit formula together and the fact that $T(0)$ is conjugate to a self-adjoint operator, we show that the zero eigenvalue of $I - T(0)$ is semisimple and of multiplicity two. Finally, we use perturbation theory to obtain the formula (4.6).

4.1. Smooth dependence on $\kappa$

The operator $T(\kappa)$ defined in (4.5) belongs to the algebra $\mathcal{E}_p$ of integral operators on $L^p(\mathbb{C})$ for any $p > 2$ and each $\kappa \in \mathbb{C}$. The operator $T(0)$ has an eigenvalue $\lambda = 1$ since the functions

$$\psi_1(z) = z^2 \rho(z)^{-2}, \quad \psi_2(z) = \rho(z)^{-2}$$

(4.7)

are eigenvectors by (2.7). We will show that the eigenvalue $\lambda = 1$ is a semisimple eigenvalue with multiplicity two, so the functions (4.7) span the eigenspace. First, we note:

Proposition 4.3. The map $\kappa \to T(\kappa)$ is differentiable as a map from $\mathbb{C}$ to $\mathcal{E}_p$ for any $p > 2$.

Proof. To prove that the map is differentiable, we need to show that the formally obvious derivatives with respect to $\kappa$ and $\pi$ exist in $\mathcal{E}_p$. First note that the operator $T(\kappa)$ has integral kernel

$$K(z, w; \kappa) = -\frac{1}{4\pi^2} \int \frac{1}{z - z'} \rho(z')^{-2} \frac{e_{\kappa}(w - z')}{z - w} \rho(w)^{-2} \, dm(z').$$

To show that $K(z, w; \kappa)$ is differentiable in $\pi$, we need to show that the function

$$W(z, w, \kappa, h) = \int \frac{1}{z - z'} \rho(z')^{-2} \left( G(z' - w, h) \right) e_{\kappa}(w - z') \rho(w)^{-2} \, dm(z')$$

is $o(|h|)$ in $\mathcal{E}_p$, where, for $z = x_1 + ix_2$ and $h = h_1 + ih_2$,

$$G(z, h) = \frac{1}{2} \left( e^{ih_1 x_1} + e^{-ih_2 x_2} \right) - 1 - (ih_1 x_1 - ih_2 x_2)$$

converges to zero in $\mathcal{E}_p$ as $h \to 0$. From the trivial estimate

$$\left| \frac{G(z, h)}{z} \right| \leq 2^{1-2\theta} \left| h \right|^{1+\theta} \left| z \right|^\theta$$

and the inequality $|z' - w|^\theta \leq 2^\theta (|z'|^\theta + |w|^\theta)$ we have

$$|W(z, w, \kappa, h)| \leq 2|h|^{1+\theta} \int \frac{1}{|z - z'|} \rho(z')^{-2} \left(|z'|^\theta + |w|^\theta \right) \rho(w)^{-2} \, dm(z').$$

Fix $p > 2$ and choose $\theta$ so that $|z|^\theta \rho(z)^{-2} \in L^{2p/(p+2)} \cap L^p$. The right-hand side is a sum of $|h|^{1+\theta}$ times two terms of the form $(Cf)(z)g(w)$ where $Cf \in L^p$ and $g \in L^p$. It is now immediate that $W(z, w, \kappa, h)$ is $o(|h|)$ in $\mathcal{E}_p$-norm as $h \to 0$. 


The proof that $T(\kappa)$ is differentiable with respect to $\kappa$ is similar and is omitted. □

Since $T(\kappa)$ is differentiable, it follows that $\text{Det}(I - T(\kappa))$ is also differentiable and we may use the $\overline{\partial}_\kappa$-equation for the determinant to study the behavior of $\text{Det}(I - T(\kappa))$, compute the dimension of $\text{ker}(I - T(0))$, and study the splitting of eigenvalues for $\kappa \neq 0$.

4.2. Determinant, eigenvalue multiplicity

In this subsection, we prove:

**Proposition 4.4.** The operator $T(0)$ has a semi-simple eigenvalue of multiplicity 2 at $\lambda = 1$.

**Remark 4.5.** This proves ‘assumption 1’ in Gadyl’shin–Kiselev’s analysis of the one-soliton solution (see the remarks in [22, p 6091]).

The proof is in several steps.

First, we show that the space $\text{ker}_L(I - T(0))$ has dimension exactly two by computing the determinant $\text{Det}(I - T(\kappa))$. Formally we can integrate the $\partial$-equation (3.16) which, in our case, reads

$$\overline{\partial}_k \log D(k) = \frac{1}{k - k_0} - c(k).$$

From remark 3.13 we have

$$c(k) = \frac{1}{4\pi} \int \frac{1}{k - \zeta} |g(\zeta)|^2 \, dm(\zeta)$$

where

$$g(k) = \frac{1}{\pi} \int e^{-k(z)} u_0(w) \, dm(w).$$

From (4.1) we have

$$|g(k)|^2 = G(k - k_0)$$

where $G(\kappa)$ is a rapidly decreasing, radial function of $\kappa$ since $(1 + |z|^2)^{-1}$ is a radial, smooth decaying function with integrable derivatives of all orders. Letting $c(k) = \gamma(k - k_0)$, it follows that $\gamma$ admits a large-$\kappa$ asymptotic expansion of the form

$$\gamma(\kappa) \sim \sum_{j=0}^{\infty} c_j \kappa^{-j-1}$$

where $c_j = (4\pi)^{-1} \int \zeta^j G(\zeta) \, dm(\zeta)$. On the other hand $\gamma(\kappa)$ has a finite limit as $\kappa \to 0$. Moreover, since $G(\kappa)$ is radial, all of the $c_j$ with $j \geq 1$ vanish. By unitarity of the transform $F$,

$$c_0 = \frac{1}{4\pi} \int |u_0(z)|^2 \, dm(z) = 1.$$

If we now let $D(k) = \Delta(k - k_0)$, it follows that $\Delta(\kappa)$ obeys the $\overline{\partial}$-problem.
\[ \frac{\partial}{\partial \kappa} \log \Delta(\kappa) = \frac{1}{\kappa} - \gamma(\kappa) \]

\[ \lim_{|\kappa| \to \infty} \Delta(\kappa) = 1 \tag{4.8} \]

and for any positive integer \( N \),

\[ \frac{\partial}{\partial \kappa} \log \Delta(\kappa) = \begin{cases} \mathcal{O}\left(|\kappa|^{-N}\right) & |\kappa| \to \infty, \\ \pi^{-1} + \mathcal{O}\left(1\right) & \kappa \to 0, \end{cases} \]

presuming that the expression (4.3) remains correct. If this is so, we can integrate formally to find that

\[ \log \Delta(\kappa) = \log |\kappa|^2 + \mathcal{O}\left(1\right) \tag{4.9} \]

as \( \kappa \to 0 \), and conclude that \( \kappa = 0 \) is a zero of multiplicity two for \( \Delta(\kappa) \).

To prove that this is the case, we must know that the solution (4.2) is correct for all \( k \neq k_0 \), which will be the case provided \( \Delta(\kappa) \neq 0 \) for all nonzero \( \kappa \). Observe that \( \Delta(\kappa) \) is radial: if \( U(\theta) \) is the isometry \( (U(\theta)f)(\kappa) = f(e^{i\theta} \kappa) \) then

\[ U(\theta) T(\kappa) U(\theta)^{-1} = T(e^{i\theta} \kappa) \]

so that

\[ \Delta(\kappa) = \Delta(e^{i\theta} \kappa). \]

Now let \( \alpha \) be the modulus of the first zero of \( \Delta(\kappa) \). We claim that \( \alpha = 0 \). Writing \( \Delta(\kappa) = H(|\kappa|^2) \) for \( H : (0, \infty) \to \mathbb{C} \), it follows from (4.8) and (4.9) that

\[ \frac{d}{dt} \log H(t) = t^{-1} - h(t), \]

\[ \lim_{t \to \infty} H(t) = 1 \]

where

\[ h(t) = \begin{cases} t^{-1} + \mathcal{O}\left(t^{-N}\right) & t \uparrow \infty, \\ \mathcal{O}\left(1\right) & t \downarrow 0. \end{cases} \]

If \( \alpha \neq 0 \), we can integrate from \( \alpha \) to \( \infty \) to obtain

\[ \log H(\alpha) = \log \alpha + \int_{\alpha}^{1} h(t) \, dt - \int_{1}^{\infty} \left(t^{-1} - h(t)\right) \, dt \]

a contradiction since then \( \log H(\alpha) \) is finite and hence \( H(\alpha) \neq 0 \). We conclude that \( H(t) \) has no zeros in \((0, \infty)\), so \( D(k) \) has no zeros for \( |k - k_0| > 0 \). We also have the formula

\[ H(t) = ct \exp \left( - \int_{0}^{t} h(s) \, ds \right) \]

for \( t \in (0, 1) \), where

\[ c = \exp \left( \int_{0}^{1} h(t) \, dt - \int_{1}^{\infty} \left(t^{-1} - h(t)\right) \, dt \right). \tag{4.10} \]

We have proved:
Lemma 4.6. The determinant \( \text{Det}(I - T(k, u_0)) \) has no zeros for \( k \neq k_0 \), and
\[
\text{Det}(I - T(k, u_0)) = c|k - k_0|^2 \left( 1 + \mathcal{O}\left(|k - k_0|^2\right) \right)
\]
as \( k \to k_0 \), where \( c \) is given by (4.10).

We will now use this fact to show that the nullspace of \( I - T(k_0, u_0) \) is two-dimensional. Let \( T_0 = T(k_0, u_0) \) and recall (2.6). A short computation shows that
\[
\rho^{-1}T_0 \rho = B^*B
\]
where
\[
B = \frac{1}{2} \rho^{-1} \mathcal{C} \left[ \rho^{-1}(\cdot) \right].
\]
The operator \( B^*B \) is positive and compact as an operator from \( L^2(\mathbb{C}) \) to itself. We now apply lemma 2.7 to the family \( T^\sharp(\kappa) = \rho^{-1}T(k_0 + \kappa, u_0) \rho \) viewed as operators on \( L^2 \). Note that \( T^\sharp(0) \) has a semisimple eigenvalue at \( \lambda = 1 \). From lemma 4.6 we have
\[
\text{Det}(I - T^\sharp(\kappa)) = c|\kappa|^2 \left( 1 + \mathcal{O}\left(|\kappa|^2\right) \right)
\]
for a positive constant \( c \) and hence, by lemma 2.7, the kernel of \( I - T^\sharp(0) \) is at most two-dimensional. On the other hand, using the identities (2.7), it is easy to see that the orthonormal vectors (4.7) belong to \( \ker(I - T^\sharp(0)) \). Hence, we have proved:

Lemma 4.7. The operator \( I - T(0) \) has a two-dimensional kernel.

Proof of proposition 4.4. An immediate consequence of lemma 4.7. \( \square \)

4.3. Eigenvalue splitting

In this subsection, we prove:

Proposition 4.8. The asymptotic formula (4.6) holds for small \( \kappa \).

We begin by computing the Laurent expansion of \( T(\kappa) \) about \( \kappa = 0 \) and the splitting of the eigenvalue \( \lambda = 1 \) at \( \kappa = 0 \). Denote by \( T(\kappa)' \) the Banach space adjoint of \( T(\kappa) \) with respect to the dual pairing
\[
\langle f, g \rangle = \int_{\mathbb{C}} f(z)g(z) \, dm(z)
\]
of \( L^{\psi'} \) and \( L^\psi \). Then
\[
T(\kappa)' = -\frac{1}{4} \rho^{-2} \mathcal{C} e_{-\kappa} \rho^{-2} \mathcal{C}.
\]
Using (2.7), it is not difficult to see that the \( \lambda = 1 \) eigenspace of \( T(0)' \) is spanned by the vectors
\[
\chi_1(z) = \frac{2}{\pi} z \rho(z)^{-4}, \quad \chi_2(z) = \frac{2}{\pi} \rho(z)^{-4}.
\]
(4.11)
It is easy to check that $\langle \chi_i, \psi_j \rangle = \delta_{ij}$. It now follows that for $\kappa$ and $\lambda - 1$ small,

$$(\lambda I - T(\kappa))^{-1} = \frac{F}{\lambda - 1} + O(1)$$

as bounded operators on $L^p$, where

$$F = \langle \chi_1, \cdot \rangle \psi_1 + \langle \chi_2, \cdot \rangle \psi_2.$$ 

(4.12)

To compute the splitting of the eigenvalue $\lambda = 1$ for $\kappa$ small and nonzero, we first note that there is an $r > 0$ with the property that $\| (\lambda I - T(\kappa))^{-1} \|$ is bounded for all $\lambda$ on the circle $| \lambda - 1 | = r$ and all $\kappa$ sufficiently small. The projection

$$P(\kappa) = \frac{1}{2\pi i} \oint_{\Gamma_r} (\lambda I - T(\kappa))^{-1} d\lambda$$

has rank two for $\kappa$ small. Moreover, since $T(\kappa)$ is differentiable as a $B(L^p)$ operator-valued function, it follows that $P(\kappa)$ is also differentiable as an operator-valued function. We wish to compute the eigenvalues of the rank-two operator $P(\kappa)T(\kappa)P(\kappa)$ using ideas of section 2.5.

Let $W(\kappa) = P(\kappa) - P(0)$. Since $P(\kappa)$ is differentiable it follows that

$$\| W(\kappa) \| = O(|\kappa|) \text{ as } \kappa \to 0.$$ 

Now let

$$V(\kappa) = (I - W(\kappa)^2)^{-1/2} [I + P(\kappa)W(\kappa) + W(\kappa)P(0)]$$

(compare (2.10)). As an operator on $B(L^p)$,

$$V(\kappa) = I + P(0)W(\kappa) + W(\kappa)P(0) + O(|\kappa|^2)$$

(4.13)

so that

$$V(\kappa)^{-1} = I - P(0)W(\kappa) - W(\kappa)P(0) + O(|\kappa|^2).$$ 

(4.14)

We will now compute the eigenvalues of $P(\kappa)T(\kappa)P(\kappa)$ by computing those of the operator $P(0)V(\kappa)^{-1}T(\kappa)V(\kappa)P(0)$ (see (2.11)). Since $P(0)T(0)P(0)$ is diagonal, the commutators of $P(0)T(0)P(0)$ with $P(0)W(\kappa)P(0)$ vanish. Using this fact, the differentiation of $T(\kappa)$, and the asymptotic formulas (4.13) and (4.14), it is not difficult to see that

$$P(0)V(\kappa)^{-1}T(\kappa)V(\kappa)P(0) = P(0)T(\kappa)P(0) + O(|\kappa|^2).$$

We now compute the matrix of $P(0)T(\kappa)P(0)$ using $\{ \psi_1, \psi_2 \}$ (see (4.7)) and $\{ \chi_1, \chi_2 \}$ (see (4.11)) as respective basis sets for the domain and range. This entails evaluating the integrals

$$M_{11}(\kappa) = -\frac{2}{\pi^3} \int \bar{z} \rho(z)^{-4} \frac{1}{\bar{z} - z'} \rho(z')^{-2} \frac{e_\kappa(w - z')}{\bar{w} - w} \overline{\rho(w)^{-4}} dm(w, z', z),$$

$$M_{12}(\kappa) = -\frac{2}{\pi^3} \int \bar{z} \rho(z)^{-4} \frac{1}{\bar{z} - z'} \rho(z')^{-2} \frac{e_\kappa(w - z')}{\bar{w} - w} \rho(w)^{-4} dm(w, z', z),$$

$$M_{21}(\kappa) = -\frac{2}{\pi^3} \int \rho(z)^{-4} \frac{1}{\bar{z} - z'} \rho(z')^{-2} \frac{e_\kappa(w - z')}{\bar{w} - w} \overline{\rho(w)^{-4}} dm(w, z', z),$$

$$M_{22}(\kappa) = -\frac{2}{\pi^3} \int \rho(z)^{-4} \frac{1}{\bar{z} - z'} \rho(z')^{-2} \frac{e_\kappa(w - z')}{\bar{w} - w} \rho(w)^{-4} dm(w, z', z).$$
We will give hints to evaluate $M_{11}(\kappa)$; the others are similar. Since the integral is absolutely convergent we may carry out the $z'$-integration first using (2.7). The result is

$$M_{11}(\kappa) = \frac{2}{\pi^2} \int \rho(z')^{-4} \overline{\rho(w)}^{-4} \frac{\epsilon_\kappa(w - z')}{\overline{w}} \, dm(w, z').$$

Using the estimate

$$|e_\kappa(w) - 1 - i\kappa w - i\overline{\kappa} w| \leq C_\delta |w|^{2-\delta} |\kappa|^{2-\delta}$$

we see that

$$M_{11}(\kappa) = a + b\kappa + c\kappa + O(\delta) \left(|\kappa|^{2-\delta}\right)$$

where $a = 1$ and $b = c = 0$ by direct computation, using (2.7). Similar calculations for the remaining integrals show that (4.6) holds.

**Proof of proposition 4.8.** An immediate consequence of the computations above. \qed

5. Perturbation of the one-soliton solution

We now show that for $\varphi \in \mathcal{C}_0^\infty(C)$ satisfying a Fourier transform condition, and $\varepsilon$ small, $u_0 + \varepsilon \varphi$ has no soliton. This result is originally due to Gadyl’shin and Kiselev [15, 16] although we achieve some simplification of the proof and remove their Assumption 1.

We consider perturbations of the form $u = u_0 + \varepsilon \varphi$ for $\varphi \in \mathcal{C}_0^\infty(C)$. For computational convenience, we will set $\chi = \varphi e^{-k_0}$.

To study the perturbations, we study the spectrum of the operator

$$T(\kappa, \varepsilon) = S_{k_0 + \kappa, \mu + \varepsilon \varphi}$$

$$= \frac{C e^{-\kappa \rho^{-2} \overline{\rho}^{-2} \epsilon_\kappa}}{2}$$

$$+ \frac{\varepsilon}{2} \left( C e^{-\kappa \rho^{-2} \overline{\rho}^{-2} \epsilon_\kappa} - C e^{-\kappa \chi \overline{\rho}^{-2} \rho^{-2} \epsilon_\kappa} \right)$$

$$- \frac{\varepsilon^2}{4} C e^{-\kappa \chi \overline{\rho}^{-2} \rho^{-2} \epsilon_\kappa}.$$  (5.1)

Note that

$$T(\kappa, \varepsilon) = T(\kappa, 0) + O(\varepsilon)$$

in the $\mathcal{B}(L^p)$ operator norm, and note that $T(\kappa, 0)$ is the operator $T(\kappa)$ from the preceding section. Let us denote by $P$ and $P_0$ the respective projections

$$P(\kappa, \varepsilon) = \frac{1}{2\pi i} \oint_{\Gamma_1} (\lambda I - T(k, \varepsilon))^{-1} \, d\lambda$$

and

$$P(0, 0) = \frac{1}{2\pi i} \oint_{\Gamma_1} (\lambda I - T(0, 0))^{-1} \, d\lambda.$$

It is easy to see that, as operators from $L^p$ to itself,

$$\|P - P_0\| = O(\varepsilon + |\kappa|)$$

since $\kappa \to T(\kappa, 0)$ is differentiable at $\kappa = 0$. In what follows, we will write $T$ for $T(\kappa, \varepsilon)$ and $T_0$ for $T(0, 0)$.

We will prove:
Theorem 5.1. Let $u_0$ be the one-soliton solution (4.1) and let $\varphi \in C_0^\infty(\mathbb{C})$.

(i) For $\varepsilon$ and $\kappa = k - k_0$ small, the asymptotic formula

$$\det(I - T(\kappa, \varepsilon)) = |i\pi + \varepsilon\beta|^2 + \varepsilon^2|\alpha|^2 + o(\varepsilon^2 + \varepsilon|\kappa| + |\kappa|^2)$$

holds, where, setting $\chi = e^{-k_0}\varphi$,

$$\alpha = -\frac{1}{\pi} \int (\chi + \overline{\chi}|z|^2) \rho^{-4} \, dm(z)$$

$$\beta = -\frac{1}{\pi} \int (\chi - \overline{\chi})z^2 \rho^{-4} \, dm(z).$$

In particular, let $C > 0$ be given. If $\varepsilon \neq 0$ is sufficiently small and $\alpha \neq 0$, then $I - T(\kappa, \varepsilon)$ has trivial kernel for $|\kappa| < Ce$.

(ii) There is a $C > 0$ so that $(I - T(\kappa, \varepsilon))$ is invertible for all sufficiently small $\varepsilon > 0$ and $\kappa$ with $\kappa > Ce$.

Proof.

(i) We wish to show that the rank-two operator $P(T(\kappa, \varepsilon) - T(0, 0))P$ has nonzero eigenvalues for all small $\kappa, \varepsilon$. By lemma 2.6, this operator is similar to the rank-two operator $P_0V^{-1}(T - T_0)V_0P_0$

where

$$V = (I - (P - P_0)^2)^{-1/2} (I - (P - P_0)^2 + [P, P_0]).$$

We will show that $\det P_0V^{-1}(T - T_0)V_0P_0$ is nonvanishing for all sufficiently small $\varepsilon, \kappa$.

First, we note some reductions. Let $\delta T = T - T_0$. Note that

$$\|\delta T\| = O(\varepsilon + |\kappa|) \text{,} \quad \|P - P_0\| = O(\varepsilon + |\kappa|).$$

Then

$$P_0V^{-1}\delta TVP_0 = P_0(I + [P, P_0])\delta T(I - [P, P_0])P_0 = A + B$$

modulo terms of order $O((\varepsilon + |\kappa|)^2)$, where

$$A = P_0\delta TP_0, \quad B = P_0[[P, P_0], \delta T]P_0.$$ 

From the identity

$$\det(A + B) = \det A + \det B + \begin{vmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{vmatrix} + \begin{vmatrix} b_{11} & b_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

the estimate $\|A\| = O(\varepsilon + |\kappa|)$, and the estimate $\|B\| = O(\varepsilon^2 + \varepsilon|\kappa| + |\kappa|^2)$, it follows that

$$\det(P_0V^{-1}\delta TVP_0) = \det(A) + o(\varepsilon^2 + \varepsilon|\kappa| + |\kappa|^2).$$

(5.2)
We will now calculate \( \det A \) using the fact that \( P_0 = F \) (see (4.12)). Note that
\[
\delta T = [T(\kappa, 0) - T(0, 0)] + \varepsilon T^{(1)}(\kappa) + \mathcal{O} (\varepsilon^2 + \varepsilon |\kappa|) \tag{5.3}
\]
where (see (5.1))
\[
T^{(1)}(\kappa) = -\frac{1}{2} \left( Ce_{-\kappa} \bar{e} \rho^{-2} + Ce_{-\kappa} \rho^{-2} \bar{e} \bar{\tau} \right).
\]
We have already computed the matrix of \( P_0 T(\kappa, 0) P_0 \) (see (4.6)), while \( P_0 T(0, 0) P_0 \) is the identity matrix. Hence
\[
P_0 [T(\kappa, 0) - T(0, 0)] P_0 = \begin{pmatrix} 0 & i\kappa \\ -i\kappa & 0 \end{pmatrix} + \mathcal{O} (|\kappa|^{2-\beta}) \tag{5.4}
\]
by (4.6).

Next, observe that
\[
\varepsilon P_0 T^{(1)}(\kappa) P_0 = \varepsilon P_0 T^{(1)}(0) P_0 + \mathcal{O} (\varepsilon |\kappa|)
\]
and
\[
T^{(1)}(0) = -\frac{1}{2} (C \chi \bar{C} \rho^{-2} + C \rho^{-2} \bar{C} \bar{\chi}).
\]
We may compute the matrix of \( P_0 T^{(1)}(0) P_0 \) with respect to the basis \( \{\psi_1, \psi_2\} \) as
\[
M^{(1)} = \begin{pmatrix} \langle \chi_1, T^{(1)}(0) \psi_1 \rangle & \langle \chi_1, T^{(1)}(0) \psi_2 \rangle \\ \langle \chi_2, T^{(1)}(0) \psi_1 \rangle & \langle \chi_2, T^{(1)}(0) \psi_2 \rangle \end{pmatrix}.
\]
To carry out this computation, observe first that for any functions \( f_1 \) and \( f_2 \),
\[
\langle \chi, C f_1 \bar{C} f_2 \psi \rangle = -\langle C \chi, f_1 \bar{C} f_2 \psi \rangle
\]
while, by (2.7),
\[
(C \chi_1)(z) = -\frac{2}{\pi} \rho(z)^{-2}, \quad (C \chi_2)(z) = \frac{2}{\pi} z \rho(z)^{-2}.
\]
Using these identities, and using (2.7) to help compute the integrals, we find that
\[
M^{(1)} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \tag{5.5}
\]
where
\[
\alpha = -\frac{1}{\pi} \int (\chi + \bar{\chi}|z|^2) \rho^{-4} \, dm(z)
\]
\[
\beta = -\frac{1}{\pi} \int (\chi - \bar{\chi}) z \rho^{-4} \, dm(z).
\]
Combining (4.6), (5.3)–(5.5), we conclude that
\[
A = \begin{pmatrix} \varepsilon \alpha & \varepsilon \beta + i\kappa \\ -\varepsilon \beta - i\kappa & \varepsilon \alpha \end{pmatrix} + o(\varepsilon + |\kappa|)\]
so that
\[ \det(A) = |i\beta + \varepsilon^2\alpha|^2 + o(\varepsilon^2 + \varepsilon|\kappa| + |\kappa|^2). \] (5.6)

It now follows from (5.6) and (5.2) that
\[ \det(P_0V^{-1}\delta TVP_0) = |i\beta + \varepsilon^2\alpha|^2 + o(\varepsilon^2 + \varepsilon|\kappa| + |\kappa|^2). \]

Hence, if at least one of \(\alpha\) and \(\beta\) is nonzero, then the determinant is nonzero for all sufficiently small \(\varepsilon\) and \(|\kappa|\), including \(\kappa = 0\), so that \(I - T(\kappa, \varepsilon)\) has trivial kernel for such \(\varepsilon\) and \(\kappa\).

(ii) This is a simple perturbation argument. In what follows, \(\| \cdot \|\) denotes the \(B(L^p)\) operator norm. It follows from theorem 4.1 and (4.6) that
\[ \| (I - T(\kappa))^{-1} \| \leq C_1|\kappa|^{-1} \]
for a constant \(C_1\) independent of \(\kappa\). From this estimate and the second resolvent identity it is easy to see that
\[ |\kappa|\| (I - T(\kappa, \varepsilon))^{-1} \| \leq C_1 + C_1C_2|\kappa|^{-1} \left[ |\kappa|\| (I - T(\kappa, \varepsilon))^{-1} \| \right] \]
where \(C_2\) bounds \(\varepsilon^{-1}(T(\kappa, \varepsilon) - T(\kappa))\). It follows that for \(|\kappa| \geq 2C_1C_2\varepsilon\), the estimate
\[ |\kappa|\| (I - T(\kappa, \varepsilon))^{-1} \| \leq 2C_1 \]
holds, which shows that \((I - T(\kappa, \varepsilon))\) is invertible for \(\varepsilon\) sufficiently small and all \(\kappa\) with \(|\kappa| \geq 2C_1C_2\varepsilon\).

\[ \square \]

**Proof of theorem 1.1.** First, using theorem 5.1(ii), pick \(C_1 > 0\) and \(\varepsilon_0\) so that \((I - T(\kappa, \varepsilon))\) is invertible for all \(\varepsilon < \varepsilon_0\) and all \(\kappa\) with \(|\kappa| > C_1\varepsilon\). Next, by decreasing \(\varepsilon\) if needed, use theorem 5.1(i) with \(C = 2C_1\) to conclude that \((I - T(\kappa, \varepsilon))\) is also invertible for \(\kappa\) with \(|\kappa| < 2C_1\varepsilon\).

We now conclude that \((I - T(\kappa, \varepsilon))\) is invertible for every \(\kappa \in \mathbb{C}\) and all sufficiently small \(\varepsilon\), so that the exceptional set is empty. \[ \square \]

**Acknowledgments**

It is a pleasure to thank Ken McLaughlin, Peter Miller, Michael Music, Katharine Ott, and Brad Schwer for helpful discussions. We are grateful to the referees for a very careful reading of the manuscript and a number of suggestions which improved the paper. This work was supported in part by grants from the National Science Foundation (DMS-1208778, PAP) and from the Simons Foundation/SFARI (359431, PAP and 422756, RMB).

**Appendix A. Renormalized determinants**

In this subsection we recall results of Gohberg, Goldberg, and Krupp (see their paper [20] and the monograph [21]) which will allow us to define a Hilbert–Carlne determinant for certain integral operators on \(L^p(\mathbb{C})\).

We begin by recalling that, if \(F\) is a finite-rank operator acting on a Banach space \(X\),
\[ \det(I + F) = \prod_j (1 + \lambda_j(F)) \]
where $\{\lambda_j(F)\}$ are the finitely many eigenvalues of $F$. This determinant is multiplicative, i.e. $\det((I + A)(I + B)) = \det(I + A) \det(I + B)$, and obeys the identity
\[
\log \det(I + F) = \text{Tr} \log(I + F)
\]
when $F$ has small norm, where
\[
\text{Tr} \log(I + F) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \text{Tr}(F^n).
\]
A related, modified determinant is
\[
\text{Det}(I + F) = \det((I + F)e^{-F})
\]
where $e^F$ is defined by Taylor’s series for the exponential function. Under certain circumstances, both $\det(I \cdot + \cdot)$ and $\text{Det}(I \cdot + \cdot)$ can be extended to larger classes of compact operators acting on $X$. For example, if $X$ is a Hilbert space $\mathcal{H}$, $\det(I \cdot + \cdot)$ extends to the trace-class operators on $\mathcal{H}$, and $\text{Det}(I \cdot + \cdot)$ extends to the Hilbert–Schmidt operators on $\mathcal{H}$ (see, for example [30, chapters 3 and 9] or [19, chapter 4].

Next, we recall the Mikhlin–Itskovich algebra of integral operators on $L^p(M, \mu)$ for a measure space $(M, \mu)$, following [20, section 5] (see also the monograph [21] for a detailed exposition). Let $p \in (1, \infty)$, $p^{-1} + q^{-1} = 1$, and denote by $L^{p,q}(M \times M)$ the Banach space of measurable functions $a : M \times M \to \mathbb{C}$ with the norm
\[
\|a\|_{p,q} = \left( \int_{M} \left( \int_{M} |a(x,y)|^q \, d\mu(y) \right)^{p/q} \, d\mu(x) \right)^{1/p}.
\]

**Definition A.1.** We denote by $\mathcal{E}_p$ the linear space of integral operators
\[
(Af)(x) = \int_{M} a(x,y)f(y) \, d\mu(y)
\]
with $a \in L^{p,q}(M \times M)$ and $a^* \in L^{q,p}(M \times M)$, where
\[
a^*(x,y) = \overline{a(y,x)}.
\]
We norm $\mathcal{E}_p$ by
\[
\|A\|_{\mathcal{E}_p} = \max(\|a\|_{p,q}, \|a^*\|_{q,p}).
\]
In [20], it is shown that $\mathcal{E}_p$ is an embedded subalgebra of the bounded linear operators on $L^p(M, d\mu)$, that
\[
\|AB\|_{\mathcal{E}_p} \leq \|A\|_{\mathcal{E}_p} \|B\|_{\mathcal{E}_p},
\]
and that finite-rank operators $F_{\mathcal{E}_p}$ are norm-dense in $\mathcal{E}_p$. Gohberg, Goldberg, and Krein prove:

**Theorem A.2 ([20, section 5]).**

(i) The trace maps $F \mapsto \text{Tr}(F^n)$ have continuous extensions from $F_{\mathcal{E}_p}$ to $\mathcal{E}_p$ for every $n \geq 2$.
(ii) The determinant $\text{Det}(I + F)$ has a continuous extension to $\mathcal{E}_p$.
(iii) For $F \in F_{\mathcal{E}_p}$, we have
\[
\text{Det}(I + F) = \det(I + F) \exp(-\text{Tr}(F))
\]
where \( \det(I + (-\cdot)) \) is the usual trace-class determinant.

Note that when \( p = 2 \), the Mikhlin–Itskovich algebra consists of the Hilbert–Schmidt operators with the usual norm, and the determinant \( \det(I + (-\cdot)) \) is the renormalized determinant \( \det_2(I + (-\cdot)) \) (see for example [19, 30]).

**Remark A.3.** Observe that the finite-rank operators \( \mathcal{F}_E \) take the form \( \sum_{i=1}^{n} \langle \psi_i, \cdot \rangle \varphi_i \) where \( \psi_i \in L^p(M, \mu) \) and \( \varphi_i \in L^p(M, \mu) \). Supposing that \( (M, \mu) \) is a \( \sigma \)-finite measure space, the set \( \mathcal{D} \) of finite linear combinations of characteristic functions for sets of finite measure is dense in each \( L^p(M, \mu) \). The set of finite-rank operators with integral kernels of the form \( \sum_{i=1}^{n} \psi_i(x) \varphi_i(y) \) for \( \psi_i, \varphi_i \in \mathcal{D} \) is therefore dense in \( \mathcal{E}_p \) for any \( p \). This implies that if \( A \in \mathcal{E}_p \cap \mathcal{E}_{p'} \), the determinants \( \det(I + A) \) defined on \( \mathcal{E}_p \) and \( \mathcal{E}_{p'} \) coincide.

Using (A.1) and the multiplicative property of the ordinary determinant, we may easily show that

\[
\det([I - B](I - C)) = \det(I - B) \det(I - C) \exp(-\tr(BC)).
\]  
(A.2)

The following variant of the standard formula for differentiation of determinants is used to derive the \( \hat{\partial} \)-equation (3.16).

**Lemma A.4.** Suppose that \( t \mapsto A(t) \) is a differentiable map from \( (-\varepsilon, \varepsilon) \) into \( \mathcal{E}_p \) with the property that \( t \mapsto A'(t) \) is a continuous finite-rank operator-valued function. Then

\[
\frac{d}{dt} \log \det(I + A(t)) = \tr\left( (I + A(t))^{-1} A'(t) \right) - \tr(A'(t)).
\]  
(A.3)

**Proof.** First, if \( t \mapsto F(t) \) is a differentiable family of finite-rank operators, we have

\[
\frac{d}{dt} \log \det(I + F(t)) = \tr\left( (I + F(t))^{-1} F'(t) \right).
\]  
(A.3)

Now consider the operator \( A(t) \) and its determinant. Writing

\[ A(t) = A(0) + \int_0^t A'(s) \, ds \]

we can decompose \( A(t) \) into a fixed operator \( B = A(0) \) and a finite-rank operator-valued function \( F(t) = \int_0^t A'(s) \, ds \) of small norm for \( |t| \) small. Suppose first that \( (I + B) \) is invertible. Using (A.2) we compute

\[
\det(I + A(t)) = \det(I + B) \det(I + (I + B)^{-1} F(t)) \exp(-\tr(F(t))).
\]

Differentiating and using (A.3) we have

\[
\frac{d}{dt} \log \det(I + A(t)) = \tr\left( [I + (I + B)^{-1} F(t)]^{-1} (I + B)^{-1} F'(t) \right) - \tr(F'(t))
\]

\[
= \tr\left( [I + B + F(t)]^{-1} F'(t) \right) - \tr(F'(t))
\]

\[
= \tr\left( [I + A(t)]^{-1} A'(t) \right) - \tr(A'(t))
\]

as was to be proved.
Now consider the case where \((I + B)\) is not invertible. Since \(B\) is compact, \((I + zB)^{-1}\) has isolated singularities and so, for some \(\epsilon \neq 0\), \((I + (1 + \epsilon) B)\) is invertible. Write \((I + A(t)) = I + (1 + \epsilon) B + (F - \epsilon B)\) and further decompose \(F - \epsilon B = G + C\) where \(G\) is finite rank and \(C\) has small enough norm that \((1 + (1 + \epsilon)B + C)\) is invertible. We then replace \(B\) by \((1 + \epsilon)B + C\) and \(F\) by \(G\) and repeat the argument.

\[\Box\]

**Appendix B. Estimates on an integral operator for the direct scattering problem**

Recall that the operator \(S_{k,u}\) is a compact operator on \(L^p\) for any \(p > 2\) provided \(u \in L^p \cap L^n\).

We will show that, for suitable \(p\), \(S_{k,u}\) belongs to the Mikhlin–Itskovich algebra \(\mathcal{E}_p\) of integral operators on \(L^p(\mathbb{C})\) (definition A.1), so that, by theorem A.2, we may define a determinant \(\text{Det}(I - S_{k,u})\) whose zeros are the points of the exceptional set. We will prove:

**Proposition B.1.** Suppose that \(u \in L^t(\mathbb{C}) \cap L^n(\mathbb{C})\) for some \(t \in (1, 4/3)\). For any \(p > 2\) with

\[
\frac{1}{2} + \frac{1}{p} < \frac{1}{t}
\]

and

\[
\frac{1}{p} + \frac{1}{t} > 1,
\]

the operator \(S_{k,u}\) belongs to \(\mathcal{E}_p\) and the determinant

\[D(k, u) = \text{Det}(I - S_{k,u})\]

is a well-defined, bounded continuous function of \(k \in \mathbb{C}\) and \(u \in L^t(\mathbb{C}) \cap L^n(\mathbb{C})\). Moreover, \(D(k, u) \to 1\) as \(|k| \to \infty\) and

\[
\sup_{k \in \mathbb{C}} |D(k, u) - D(k, u')| \lesssim_{p,t} \|u - u'\|_{L^t L^n}
\]

where the implied constant depends on \(p\), \(t\), and a bound on \(\|u\|_{L^t L^n}\) and \(\|u'\|_{L^t L^n}\).

**Remark B.2.** The conditions (B.1) and (B.2) dictate that \(t \in (1, 4/3)\). The figure below shows the region of admissible \(p\) and \(t\).

The integral kernel of \(S_{k,u}\) is

\[a(z, w) = \left[\frac{1}{4\pi^2} \int \frac{1}{z - \zeta} e^{-\frac{1}{2} (\zeta - \frac{1}{w}) \frac{1}{\zeta - w} \ dm(\zeta)} \right] e_k(w)\bar{a}(w).
\]

In order to show that \(S_{k,u} \in \mathcal{E}_p\), we need to bound \(\|a\|_{L^p (L^n)}\) and \(\|a^*\|_{L^{n'} (L^t)}\). Note that

\[
|a(z, w)| \leq \frac{1}{4\pi^2} \int \frac{1}{|z - \zeta|} \left| \frac{1}{|\zeta - w|} |u(\zeta)| |u(w)| \ dm(\zeta)
\]

and

\[
|a^*(z, w)| \leq \frac{1}{4\pi^2} \int \frac{1}{|w - \zeta|} \left| \frac{1}{|\zeta - z|} |u(\zeta)| |u(z)| \ dm(\zeta)\right.
\]

\[
\frac{1}{4\pi^2} \int \frac{1}{|w - \zeta|} \left| \frac{1}{|\zeta - z|} |u(\zeta)| |u(z)| \ dm(\zeta)\right.\]
We will find conditions on $u$ so that $S_{k,u}$ belongs to the Mikhlin–Itskovich algebra. Before doing so we collect some preliminary estimates. For a measurable function $g$ on $\mathbb{C} \times \mathbb{C}$, define

$$I_1(g)(z,w) = \hat{g}(\zeta,w) \big|_{\zeta = z} \, dm(\zeta),$$

$$I_2(g)(z,w) = \hat{g}(z,\zeta) \big|_{\zeta = w} \, dm(\zeta).$$

(The ‘1’ and ‘2’ refer to integration with respect to the first or second argument of $g$).

**Lemma B.3.** The estimates

$$\|I_1(g)\|_{L^t(L^p)} \lesssim_{p,q} \|g\|_{L^q(L^t)} \cdot \quad \text{if} \quad p \in (1,2), \quad q \in (1,\infty) \quad (B.6)$$

$$\|I_1(g)\|_{L^\infty(L^p)} \lesssim_{p,q} \|g\|_{L^{p'\cap L^{p'}}(L^t)} \cdot \quad \text{if} \quad p \in (1,2), \quad q \in (1,\infty) \quad (B.7)$$

$$\|I_2(g)\|_{L^t(L^p)} \lesssim_{p,q} \|g\|_{L^p(L^t)} \cdot \quad \text{if} \quad p \in (1,\infty), \quad q \in (1,2) \quad (B.8)$$

$$\|I_2(g)\|_{L^\infty(L^p)} \lesssim_{p,q} \|g\|_{L^{p'\cap L^{p'}}(L^t)} \cdot \quad \text{if} \quad p \in (1,\infty) \quad (B.9)$$

hold.

**Proof.** To prove (B.6), we use the Hardy–Littlewood–Sobolev inequality and Minkowski’s integral inequality to estimate
\[
\left( \int \left( \int \left| \frac{g(\zeta, w)}{|\zeta - z|} \right|^q \, dm(\zeta) \right)^{\frac{p}{q}} \, dm(z) \right)^{1/p} \nleq \left( \int \left( \int \|g(\zeta, \cdot)\|_q \, dm(\zeta) \right)^{\frac{p}{q}} \, dm(z) \right)^{1/p} \nleq_p \left( \int \|g(\zeta, \cdot)\|_q^p \, dm(\zeta) \right)^{1/p} = C_p \|g\|_{L_p(L^p)}.
\]

To prove (B.7), we use Minkowski’s integral inequality and remark 2.2 to estimate
\[
\left( \int \left( \int \left| \frac{g(\zeta, w)}{|\zeta - z|} \right|^q \, dm(\zeta) \right)^{\frac{p}{q}} \, dm(z) \right)^{1/p} \nleq \left( \int \frac{1}{|\zeta - z|} \|g(\zeta, \cdot)\|_q \, dm(\zeta) \right)^{1/p} \nleq_q \|g\|_{L^\infty(L^\infty(L))}.
\]

To prove (B.8), we use the Hardy–Littlewood–Sobolev inequality to estimate
\[
\|I_2(g)\|_{L^p(L^\infty)} \nleq q \left( \int \|g(z, \cdot)\|_q^p \, dm(z) \right)^{1/p} = \|g\|_{L^p(L^p)}.
\]

To prove (B.9), we use remark 2.2 to estimate
\[
\|I_2(g)\|_{L^p(L^\infty)} \nleq q \left( \int \|g(z, \cdot)\|_{L^\infty(L^\infty)} \, dm(z) \right)^{1/p} = \|g\|_{L^p(L^p)}.
\]

**Lemma B.4.** Suppose that \( u \in L^t \cap L^{t'} \) where \( 1 \leq t < 2 \). Then, for any \( p > 2 \) satisfying (B.1) and (B.2), we have
\[
a \in L^p \left( L^{t'} \right), \quad a^\ast \in L^{p'} \left( L^p \right)
\]
with
\[
\max \left( \|a\|_{L^p(L^{t'})}, \|a^\ast\|_{L^{p'}(L^p)} \right) \nleq_p \|u\|_{L^t(L^t)}^2
\]
for a constant \( C \) independent of \( k \). Moreover,
\[
\lim_{|k| \to \infty} \|a\|_{L^p(L^{t'})} = \lim_{|k| \to \infty} \|a^\ast\|_{L^{p'}(L^p)} = 0.
\]

**Proof.** To estimate \( \|a\|_{L^p(L^{t'})} \), we use (2.8). Let \( g \in L^{p'} \left( L^{t'} \right) \) with \( \|g\|_{L^{p'}(L^{t'})} \leq 1 \). A short computation using (B.4) shows that, up to absolute numerical constants,
\[
\left| \int a(z,w)g(z,w) \, dm(z) \, dm(w) \right|
\leq \int |u(\zeta)| \int \left| \frac{|g(z,w)|}{|\zeta - w|} \right| \, dm(z) \, dm(\zeta)
\leq \int |u(\zeta)| \left[ I_2(u \cdot I_1(g)) \right](\zeta, \zeta) \, dm(\zeta)
\]  
(B.11)

where

\[(u \cdot I_1(g))(z,w) = u(w) \cdot I_1(g)(z,w).\]

By (B.6) we have \(I_1(g) \in L^s[L^p]\) with \(\|I_1(g)\|_{L^s[L^p]} \leq C \|g\|_{L^s[L^p]}\) where

\[
\frac{1}{s} = \frac{1}{2} - \frac{1}{p}.
\]

By Hölder’s inequality, using the fact that \(u \in L^\infty(L' \cap L')\) (viewed as a function of two variables depending only on \(w\)) we then have \(\|u(\cdot)I_1(g)\|_{L^s[L^p]} \leq C \|u\|_{\mathcal{L}'[L']}, \|g\|_{L^s[L^p]}\) provided \(\frac{1}{s}\) belongs to the interval \(J_1 = \left(\frac{1}{s} + \frac{1}{p}, \frac{1}{2} \right)\). We claim that there is an \(r \in (1, 2)\) with \(\left(\frac{1}{r}, \frac{1}{2}\right) \subset J_1\). Such an \(r\) exists provided

\[
\frac{1}{p} + \frac{1}{r} < \frac{1}{2}, \quad \frac{1}{p} + \frac{1}{r} > \frac{1}{2}.
\]

The second inequality is trivial since \(t < 2\) and the first is equivalent to (B.1). Choosing such an \(r\) we now have \(u(\cdot)I_1(g) \in L^r(L' \cap L')\). Now we use (B.9) to conclude that

\[
I_2(u \cdot I_1(g)) \in L^t(L^\infty)\] so that, by (2.9), \(\|I_2(u \cdot I_1(g))(\zeta, \zeta) \|_{L^t} \leq C \|u\|_{\mathcal{L}'[L']}, \|g\|_{L^s[L^p]}\).

Hence, we can bound the right-hand side of (B.11) by \(\|u\|_{\mathcal{L}'[L']}, \|I_2(u \cdot I_1(g))\|_{L^t(L^\infty)}\) provided \(\frac{1}{s} \in (\frac{1}{r}, \frac{1}{2})\). As \(\frac{1}{r} = \frac{1}{2} + \frac{1}{p}\) we need the two inequalities

\[
\frac{1}{2} + \frac{1}{p} < \frac{1}{t}, \quad \frac{1}{2} + \frac{1}{p} > 1 - \frac{1}{t}
\]

to hold. The first is (B.1) and the second is equivalent to \(\frac{1}{t} > \frac{1}{2} - \frac{1}{p}\) which is trivial since \(t < 2\).

Hence

\[
\|u\|_{\mathcal{L}'[L^t]} \leq C \|u\|_{\mathcal{L}'[L^t]}^{\frac{1}{t} - \frac{1}{2}}.
\]

Next, to estimate \(\|a^s\|_{L^p[L^s]}\) we choose \(g \in L^p(L^p)\) with \(\|g\|_{L^p[L^s]} \leq 1\). We then use (2.8) and (B.5) to bound (again up to numerical constants)

\[
\left| \int a^s g \, dm(z) \, dm(w) \right|
\leq \int |u(z)| \int \left| \frac{|g(z,w)|}{|\zeta - w|} \right| \, dm(w) \, dm(\zeta)
\leq \int |u(z)| \left[ I_2(u \cdot I_2(g)) \right](z, z) \, dm(z)
\]  
(B.12)
where
\[(u \cdot I_2(g))(z, w) = u(w)I_2(g)(z, w).\]

First, by (B.8), we have \(I_2(g) \in L^p(L')\) with \(\|I_2(g)\|_{L^p(L')} \leq C \|g\|_{L^r(L')}\), where \(\frac{1}{r} = \frac{1}{s} - \frac{1}{p}\). Since \(u\) (viewed as a function of two variables depending only on the second variable) belongs to \(L^\infty(L' \cap L')\), it follows that \(u \cdot I_2(g)\) belongs to \(L^p(L')\) for any \(r\) with \(\frac{1}{t} > \frac{1}{r}\) belonging to the interval \(J_2 = \left(\frac{1}{s} + \frac{1}{s}, \frac{1}{s} + \frac{1}{t}\right)\) and \(\|u \cdot I_2(g)\|_{L^p(L')} \lesssim_p \|u\|_{L^r(L')} \|g\|_{L^r(L')}\). We claim that there is an \(r \in (1, 2)\) with \(\left(\frac{1}{s}, \frac{1}{r}\right) \in J_2\). This is the case provided the two inequalities
\[
\frac{1}{s} + \frac{1}{t} < \frac{1}{2}, \quad \frac{1}{s} + \frac{1}{r} > \frac{1}{2}
\]
hold. The first is equivalent to (B.2). The second inequality is trivial since \(t < 2\). We can now use (B.9) to estimate
\[
\|I_2(u \cdot I_2(g))\|_{L^r(L')} \leq C \|u \cdot I_2(g)\|_{L^p(L')} \lesssim_p \|u\|_{L^r(L')} \|g\|_{L^r(L')}.
\]
Finally, using (2.9) and Hölder’s inequality, we can bound the right-hand side of (B.12) by
\[
C \|u\|_{L^r(L')} \|g\|_{L^r(L')} \|L^r(L')\|
\]
which is in turn bounded by \(C \|u\|_{L^\infty(L')}^2\) provided \(\frac{1}{p} \in \left(\frac{1}{r}, \frac{1}{r}\right)\). This is true provided \(\frac{1}{p} > \frac{1}{t}\) and \(\frac{1}{t} < \frac{1}{r}\). The first of these inequalities is trivial since \(p > t\) and the second is equivalent to (B.2).

To prove (B.10), it suffices to show that the limits are zero in case \(u \in C_0^\infty(\mathbb{C})\). Emphasizing the dependence of \(a\) on \(k\), write
\[
a(z, w, k) = \left[\frac{1}{\pi^2} \int \frac{1}{z - \zeta} e^{-k (\zeta)} u(\zeta) \frac{1}{\zeta - \bar{w}} \, dm(\zeta)\right] e_k(w) \mathfrak{F}(w).
\]
For each fixed \(z, w\) it follows from the Riemann–Lebesgue lemma that
\[
\lim_{|k| \to \infty} a(z, w, k) = 0
\]
for almost every \((z, w)\). Since \(|a(z, w, k)|\) is dominated by a fixed \(L^p(L^p)\) function, it follows that \(\|a(\cdot, \cdot, k)\|_{L^p(L^p)} \to 0\) as \(|k| \to \infty\). A similar argument shows that \(\|a^*(\cdot, \cdot, k)\|_{L^p(L^p)} \to 0\) as \(|k| \to \infty\).

**Proof of proposition B.1.** The continuity follows from the fact that the maps
\[
(k, u) \to a(z, w, k) \quad (k, u) \to a^*(z, w, k)
\]
respectively from \(L^r(\mathbb{C}) \times L^t(\mathbb{C}) \times \mathbb{C}\) to \(L^p(L^p)\) and to \(L^p(L^p)\) are continuous. The fact that \(D(k) \to 1\) follows from the fact that \(\|S_{k,a}\|_{L^p} \to 0\) as \(|k| \to \infty\), as follows from (B.10).
The estimate (B.3) follows from the bilinearity of $u \mapsto S_{k,u}$ and the fact that estimates on $\|a(\cdot,\cdot,k)\|_{L^p(L^q)}$ and $\|\partial^a(\cdot,\cdot,k)\|_{L^p(L^q)}$ are independent of $k$.

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