Improved Perturbation Theory
and
Four-Dimensional Space-Time in the IIB Matrix Model

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Abstract

We have analyzed the IIB matrix model on the basis of the improved mean field approximation (IMFA) and have obtained evidence suggesting that the four-dimensional space-time appears as its most stable vacuum. This method is a systematic way to obtain an improved perturbation series and was first applied to the IIB matrix model by Nishimura and Sugino. In a previous paper, we reformulated this method and proposed a criterion for the convergence of the improved series, that is, the appearance of a “plateau.” In this paper, we carry out higher-order calculations, and find that our improved free energy tends to have a plateau, which shows that IMFA works well in the IIB matrix model.

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1 Introduction and summary

String theory has been proposed as a unified theory of fundamental interactions including gravity. If it is the well-defined quantum gravity theory which our nature adopts, it should be able to predict several properties of our universe, for instance, the gauge symmetry at low energy, the particle content and their masses, and the dimensionality of our space-time. The last of these is the property we would like to investigate in this paper.

As is well known, superstring theory has infinitely many vacua with various dimensionalities that arise perturbatively. One of the most promising scenarios to construct a realistic four-dimensional model from superstring theory is a compactification in which the ten-dimensional space-time consists of a flat four-dimensional space-time and a six-dimensional compact manifold that is small enough to be invisible to our experiments. There are several ways to realize this scenario, for example, Calabi-Yau compactification [3], fermionic construction [4], and so on, but they are all stable and there is no way to single out the true vacuum perturbatively. However, it has been revealed that this problem originates from perturbative formulations themselves, and to overcome this difficulty, we need a nonperturbative formulation and analysis of string theory.

In the mid-1990s, some models were presented as a constructive formulation of M-theory [5] and certain kinds of superstring theories, such as type IIA matrix string [6], type I superstring [7] and heterotic superstring [8]. Here we would like to analyze the model called the IKKT model [9], which is conjectured to be a nonperturbative definition of type IIB superstring theory. It seems to be most promising for nonperturbative analyses of superstring theory, and some kinds of extensions of this model have been proposed [10]. For a review of the IIB matrix model, see Ref. [11]. Also, there are some mechanisms proposed for dynamical breakdown of Lorentz symmetry in this model (see Ref. [16]).

In order to analyse the IIB matrix model, we use a method that we call the “improved mean field approximation” (IMFA) [1]. This approximation was applied to the large-\(N\) reduced Yang-Mills models by Oda and Sugino [12]. Then, Nishimura and Sugino applied it to the IIB matrix model in an excellent work [2]. They obtained a result that suggests the breakdown of Lorentz symmetry to the four-dimensional symmetry. In a previous paper [1], we analyzed how the IMFA scheme works and discovered a general structure that we call the “improved Taylor expansion.” Furthermore, we proposed a principle for choosing the mean fields, that is, the existence of a “plateau.” The emergence of a plateau indicates that the approximation scheme works well and, in fact, its existence can be confirmed in some exactly solvable models.
(See Ref. [1] for reference. There, many examples are given which show how good this scheme is.) Furthermore we developed a computational method using two-particle irreducible (2PI) graphs, which simplifies the calculation drastically. The 2PI free energy has a close relationship with the Schwinger-Dyson equation as discussed in Ref. [1].

Using this method, we calculated the free energy up to 5th order and obtained a preliminary result which suggests that the eigenvalue distribution of the matrices preserves only the four-dimensional rotational symmetry.

In this paper, we carry out a further calculation up to 7th order. We find evidence suggesting the emergence of a plateau, which was not clear at 5th order. In fact, we have evaluated the free energy and the extent of the eigenvalue distribution for various Lorentz symmetries by introducing corresponding mean fields. In the case of SO(4) symmetry, the number of extrema of the free energy increases as we go to higher orders, and it seems that they form a plateau. Furthermore, the eigenvalues are widely distributed in the four “non-compact” directions, while they tend to gather in the six compact directions. One the other hand, if we impose SO(7) symmetry, the number of extrema of the free energy does not grow enough to form a plateau, and the eigenvalues are distributed somewhat isotropically in the ten directions. As shown in Ref. [1], the other cases are reduced to the above two cases, or do not have a plateau at all.

We evaluate the 2738 graphs to obtain the free energy up to 7th order. All the calculations, including the generation and computation of the graphs, are totally automatized now, and we will be able to go further and find reliable results.

In §2, we provide a short review of the improved mean field approximation with a $\phi^4$ matrix model as an example. We examine a distribution of extrema of the free energy of this model, which provides a method of the search for a plateau in the IIB matrix model. In §3, we apply IMFA to the IIB matrix model. We obtain the free energy, investigate the distribution of its extrema to search for a plateau and examine the eigenvalue distributions. Section 4 is devoted to a conclusion and discussion. In Appendix A, we make use of a $\phi^4$-QED type matrix model to confirm our counting of the graphs.
2 Improved Mean Field Approximation for the Free Energy

In this section, we review the improved mean field approximation that was developed in our previous paper [1]. Although all the techniques we introduce here are the same as those used in Ref. [1], we explain them to make this paper self-contained and for reader’s convenience.

2.1 Ordinary vs. Improved perturbation theory

Suppose we have some action function $S(x) = S_0(x) + S_1(x)$ and its free energy

$$F = -\ln Z = -\ln \left( \int dx \ e^{-S(x)} \right), \quad (2.1)$$

where $S_0(x)$ is the unperturbed part, which can be integrated analytically, and $S_1(x)$ is difficult or impossible to integrate and is treated as a perturbation of $S_0$.

In ordinary perturbation theory, we keep the unperturbed part in the exponential and expand the perturbation into a series. For convenience, we introduce a formal coupling constant $g$ and rewrite the action as $S = S_0 + g S_1$. Then we obtain the ordinary perturbation series for the free energy with respect to $g$ as follows:

$$F = -\ln \left( \int dx \ \sum_{k=0}^{\infty} \frac{(-g)^k}{k!} S_1^k e^{-S_0(x)} \right). \quad (2.2)$$

We can obtain the free energy up to order $n$ in the perturbation theory by truncating the infinite series up to order $n$ and setting $g = 1$.

In general, the perturbation series has a finite (or maybe even zero) convergence radius with respect to the parameters appearing in the action, for example, the inverse mass squared $1/m_0^2$ in $S_0$. To see what happens and when the perturbation theory fails, let us consider a zero-dimensional $\phi^4$ one-matrix model.

2.2 The $\phi^4$ matrix model

The action of the zero dimensional $\phi^4$ matrix model is

$$S = \frac{m_0^2}{2} \text{Tr} \phi^2 + \frac{1}{4} \text{Tr} \phi^4, \quad (2.3)$$
where $\phi$ is an $N \times N$ hermitian matrix. In the ordinary perturbation theory, we treat the quadratic part, $m_0^2 \phi^2 / 2$, as the unperturbed action $S_0$ and the $\phi^4 / 4$ part as the perturbation $S_1$. Here, as an example, we consider the free energy. After introducing a formal coupling constant $g$ and expanding the exponential part with respect to it, we obtain

$$F = -\ln \left[ \sum_{k=0}^{\infty} \int d\phi \left( \frac{-g}{4} \right)^k \frac{(\text{Tr}\phi^4)^k}{k!} \exp \left( -\frac{m_0^2}{2} \text{Tr}\phi^2 \right) \right], \quad (2.4)$$

and we can estimate this series using the ordinary Feynman diagram technique. Here we consider the large-$N$ limit, where only planar graphs contribute to the free energy. Thus, in the ordinary perturbation theory, the free energy is given by a series with respect to $g/m_0^4$ as follows:

$$F = O(g^3)$$

On the other hand, one can evaluate this free energy exactly by analyzing the eigenvalue distribution [15], and it is known that the convergence radius of this series is $1/12$. Thus, after we set the formal coupling $g$ to 1, this series converges when $m_0^4 > 12$. This means that we cannot calculate the free energy for the massless case, i.e. $m_0^2 = 0$, as the limit of this perturbation series. Because the IIB matrix model of interest does not have a quadratic term, it corresponds to the massless case, in which the ordinary perturbation theory does not work.

To overcome this difficulty we introduce a new method, the improved mean field approximation, and obtain an improved perturbation series.

### 2.3 Improved mean field approximation

When we apply the improved mean field approximation scheme, we first introduce a “mean field” $S_m(x, a)$ which can be easily integrated, for instance, a quadratic term. Here, $a$ is a set of parameters in the mean field, and we will tune it later to make the approximation better. Then, we rewrite the original action as

$$S = S_0 + S_1 \Rightarrow S = S_m + (S_0 + S_1 - S_m), \quad (2.6)$$

and we take $S_m$ as an unperturbed action and the part within parenthesis as the perturbation. We introduce a formal coupling constant $g$ as before. Thus the action becomes $S = S_m + g(S_0+$
$S_1 - S_m$, and the expansion of the exponential with respect to $g$ yields another perturbation series, which we call the improved perturbation series. Finally, we tune the set of parameters $a$ to make the improved series converge, as we explain below. We call this procedure the “improved mean field approximation.” In particular, if we consider the first order of this approximation for the free energy defined in (2.5), and tune the parameter $a$ using the condition $dF/da = 0$, it is simply the ordinary mean field approximation.

As an example, we consider again the zero-dimensional $\phi^4$ matrix model.

2.4 IMFA for the $\phi^4$ matrix model

In this case, we consider a quadratic term as the mean field:

$$S_m = \frac{m^2}{2} \text{Tr} \phi^2. \quad (2.7)$$

Then, we construct a modified action, including a formal coupling $g$, as

$$S = \frac{m^2}{2} \text{Tr} \phi^2 + g \left( \frac{m_0^2}{2} \text{Tr} \phi^2 + \frac{1}{4} \text{Tr} \phi^4 - \frac{m^2}{2} \text{Tr} \phi^2 \right)$$

$$= \frac{g}{4} \text{Tr} \phi^4 + \frac{m^2 + g(m_0^2 - m^2)}{2} \text{Tr} \phi^2. \quad (2.8)$$

We emphasize that we do not need to calculate graphs again. By comparing (2.8) with the original action (2.3), we have only to substitute $m^2 + g(m_0^2 - m^2)$ for the mass squared $m_0^2$ appearing in the ordinary perturbation theory.

Therefore, by substituting $m^2 + g(m_0^2 - m^2)$ for $m_0^2$, and re-expanding with respect to $g$, (2.3) becomes

$$F_{\text{improved}} = -\frac{1}{2} \ln \left( \frac{1}{m^2 + g(m_0^2 - m^2)} \right) + \frac{g}{2} \frac{1}{(m^2 + g(m_0^2 - m^2))^2}$$

$$- \frac{9}{8} g^2 \frac{1}{(m^2 + g(m_0^2 - m^2))^2} + O(g^3)$$

$$= -\frac{1}{2} \ln \left( \frac{1}{m^2} \right) + \frac{g}{2} \left( \frac{1}{m^4} + \frac{m_0^2 - m^2}{m^2} \right)$$

$$+ g^2 \left( -\frac{9}{8} \frac{1}{m^8} \frac{m_0^2 - m^2}{m^6} - \frac{1}{4} \frac{(m_0^2 - m^2)^2}{m^4} \right) + O(g^3). \quad (2.9)$$

In this form, the massless limit is no longer singular and can be taken simply by setting $m_0^2 = 0$. Furthermore, from this form, we know how to calculate the improved perturbation series even for a massless theory, to which the ordinary perturbation theory cannot be applied.
First, we add a mass term $m^2 \phi^2/2$ to the original action by hand and calculate the free energy perturbatively. Then we substitute $m^2 - gm^2$ for $m^2$ and re-expand it with respect to $g$. Finally, by setting $g = 1$, we obtain the improved perturbation series for the massless theory. As we see, this prescription is useful for calculating the IIB matrix model, which also has no quadratic term.

Let us return to the case of a general value of $m_0^2$. The final step of this prescription is to tune the parameter $m^2$ so that the improved series converges. To elucidate this situation, in Fig. 1, we plot the improved free energies with respect to $m_0^2$ at various orders for $m^2 = 2$ and 4. It is clear that the improved series converges quite well in some domain. The position of the domain of the convergence depends on the choice of the parameter $m^2$, for example, around $m_0^2 = 0$ for $m^2 = 2$ and around $m_0^2 = 4$ for $m^2 = 4$. Here, we have first plotted the improved series with respect to $m_0^2$ for various values of $m^2$, and then determined the value of $m^2$ to make the series converge around the value of $m_0^2$ in question.

![Figure 1: Free energies with $m^2 = 2$ (left) and $m^2 = 4$ (right). The horizontal axis denotes $m_0^2$.](image)

### 2.5 The plateau and how we identify it

There is an easier way to find an appropriate value of the parameter $m^2$ than that considered above for a fixed value of the bare parameter $m_0^2$, for example, the massless case $m_0^2 = 0$. Fig. 2 displays the improved free energies as functions of $m^2$ with $m_0^2 = 0$. We see that a plateau develops at higher orders. Intuitively, this fact is very natural, because $m^2$ is an artificially
introduced parameter, and the true value of the free energy must be independent of its choice. Some years ago, Dhar and Stevenson advocated a “principle of minimal sensitivity” \[14\]. They stated that in the improved perturbation theory one should choose the set of parameters such that the improved quantity is stationary with respect to it, i.e. \( \partial F_{\text{improved}} / \partial m^2 = 0 \) in our case. However, we claim that the criterion for good approximation should be the existence of a plateau; that is, one should choose the set of parameters to be on the plateau. This claim was first made in a previous paper \[1\], and some people have been studying the properties of the plateau and seeking a better definition, especially in the case that there are many parameters in the mean fields. In such cases, it becomes difficult to identify a plateau, because we cannot visualize it easily.\[\]

To obtain a concrete criterion for the plateau, we plot the extrema of the improved free energy.

\[\text{Figure 2: Free energies with } m_0^2 = 0 \text{ at 7th, 8th, 27th order and from 0th to 29th order. The horizontal axis denotes } m^2.\]

\[\text{From 0th to 29th order} \quad \text{7th order} \quad \text{8th order} \quad \text{27th order}\]

\[\text{Nishimura, Okubo and Sugino proposed “the histogram prescription” to identify a plateau for the many parameter case \[13\].}\]
energy of the massless $\phi^4$ matrix model with respect to $m^2$ in Fig. 3. It can be seen from this figure that there are two characteristic extrema up to 10th order, except at the two lowest orders. At each order, one of the two extrema gives the highest free energy, which is quite stable, and the other gives the lowest free energy, which decreases as we go to higher orders. Hereafter we call the highest and the lowest extremum the “overshooting” and “undershooting” extremum, respectively. The other extrema tend to accumulate between these two extrema, and they are expected to form a plateau. In fact, if we take a closer look at a plateau, we find that it consists of many extrema. In other words, the accumulation of extrema leads to a plateau. In general, at much higher orders, a finite number of extrema yield higher or lower free energies. However, the situation remains unchanged in which many of the other extrema accumulate between these higher and lower extrema and contribute to the formation of the plateau.

![Figure 3](image)

**Figure 3:** Extrema of the improved free energy of the massless $\phi^4$ matrix model from 1st to 10th order.

This is the key observation for finding a plateau presented in this paper. In the next section, we explore the extrema of the free energy of the IIB matrix model and see whether they accumulate to form a plateau or not.
3 Free Energies of IIB Matrix Model

In this section, we would like to analyze the IIB matrix model \[9\]. The action is given by

\[
S_{\text{IIB}} = -\text{Tr} \left( \frac{g_0^2}{4} [A_\mu, A_\nu]^2 - \frac{g_0}{2} \bar{\psi} \Gamma^\mu [A_\mu, \psi] \right),
\]

where \( A_\mu (\mu = 1, \ldots, 10) \) and \( \psi^\alpha (\alpha = 1, \ldots, 16) \) are all \( N \times N \) hermitian matrices transforming as the vector and left-handed spinor representations under \( \text{SO}(10) \). \( g_0 \) is the only parameter in this model, which is dimensional, and we take the large \( N \) limit while fixing the 't Hooft coupling \( g_0^2 N \) to 1. This is defined as a zero-dimensional reduced model of a ten-dimensional super Yang-Mills theory. Because it has no quadratic term, we cannot apply the ordinary perturbation theory.

We use the improved mean field approximation to evaluate the free energy. Following the method explained in the last section, we add and subtract a quadratic term for bosonic and fermionic matrices as mean fields. Then our action becomes

\[
S = -N \text{Tr} \left( \frac{g}{4} [A_\mu, A_\nu]^2 - \frac{\sqrt{g}}{2} \bar{\psi} \Gamma^\mu [A_\mu, \psi] \right) + S_m - gS_m,
\]

\[
S_m = \frac{1}{2} \left( C^{-1}(\mu\nu) N \text{Tr}(A_\mu A_\nu) - \frac{i}{2} \left( C^{-1}(\mu\nu) \right)_{[\alpha\beta]} N \text{Tr}(\psi^\alpha \psi^\beta) \right),
\]

where \( S_m \) is the quadratic term introduced as the mean fields\[3\]. Here we introduce the formal coupling constant \( g \) as \( g_0^2 \) in (3.1). The coefficients \( C_{\mu\nu} \) and \( \gamma = u_{\mu\nu\rho} \Gamma^{\mu\nu\rho}/3! \) are the propagators for bosonic and fermionic matrices in the perturbation theory, respectively. \( C_{(\mu\nu)} \) is a second rank symmetric tensor, and \( u_{(\mu\nu\rho)} \) is a third rank antisymmetric tensor.

Here we should comment on how these mean fields are constructed. The symmetries of the original IIB matrix model are the matrix rotation \( U(N) \), ten-dimensional Lorentz symmetry \( \text{SO}(10) \), translational symmetry, which changes \( A_\mu \) to \( A_\mu + \text{const.} \times 1_N \), and the type IIB supersymmetry. Our mean field preserves \( U(N) \), while it breaks \( \text{SO}(10) \) and supersymmetry. One might worry that this leads to an inconsistency because the existence of the type IIB supersymmetry plays an important role in the IIB matrix model. However, if the supersymmetry is restored in the true ground state of our model, the parameters will go to \( C = \infty \) and \( u = \infty \) and our mean field should vanish. This is the standard story of models in which dynamical symmetry breakdown occurs, as in the Nambu–Jona-Lasinio model. In this case, even if we introduce a mass term that breaks the chiral symmetry, it vanishes in the phase where the symmetry is restored. If \( C \) and \( u \) are still finite after the large \( N \) limit is taken,

\[2 \quad C \text{ is the charge conjugation matrix defined as } tC = -C, \quad CT^\mu = -t \Gamma^\mu C \ (\mu = 1, \ldots, 10). \]
both the Lorentz symmetry and supersymmetry are spontaneously broken. This is the very scenario we expect.

Here we summarize the procedure we carry out below. Following the prescription for the massless case described in §2.4, we first calculate the free energy perturbatively for the action

$$S' = -N \text{Tr} \left( \frac{g}{4} [A_\mu, A_\nu]^2 - \frac{\sqrt{g}}{2} \bar{\psi} \Gamma^\mu [A_\mu, \psi] \right) + S_m. \tag{3.4}$$

Then we replace $C^{-1}$ and $u^{-1}$ with $(1 - g) C^{-1}$ and $(1 - g) u^{-1}$, respectively, in order to recover the contributions from the $-g S_m$ term. Finally, we obtain the improved free energy by expanding the result with respect to $g$.

3.1 The 2PI free energy and ansatz

To obtain the free energy, we only have to calculate planar connected vacuum graphs. However, in such calculations, we have to treat too many graphs at higher orders. In order to avoid this problem, we introduce a two-particle irreducible (2PI) free energy. The 2PI free energy is considered in Ref. [1] and it is deeply related to the Schwinger-Dyson equations, as discussed in Ref. [1]. In this paper, however, we use the 2PI free energy only as a tool to obtain the ordinary free energy easily. From this viewpoint, the definition and properties of the 2PI free energy are summarized as follows:

- The 2PI graph is the graph that contains no self-energy graphs as its subgraphs; that is, it is two-particle irreducible. This means that propagators in a 2PI graph can be regarded as the exact propagators.

- Suppose $G$ is a sum of planar, vacuum, connected 2PI graphs in some theory. Then the ordinary free energy in the planar limit is given by the Legendre transformation of $G$ with respect to the exact propagators.

In the IIB matrix model, we also force $G$ to contain no tadpole graphs, because all the one-point functions satisfy $\langle A_\mu \rangle = 0$, due to Lorentz invariance.

Once we obtain a 2PI free energy $G$ of (3.4) up to some order, the next task is to perform a Legendre transformation. At this stage, however, we face the new problem that there are too many parameters to carry out the transformation. Therefore we need some ansatz that reduces the number of parameters to a tractable one. We use the same ansatz as in Ref. [1]; that is, we assume several unbroken Lorentz symmetries that restrict the mean field parameters $C_{\mu\nu}$.
and $u_{\mu\nu\rho}$. Two peculiar examples, which we consider in this paper, are the $\text{SO}(7) \times \text{SO}(3)$ case and the $\text{SO}(4) \times \text{SO}(3) \times \text{SO}(3) \times \mathbb{Z}_2$ case. As we comment below, these two ansatz are of particular importance among other possibilities. When the unbroken symmetry is $\text{SO}(7) \times \text{SO}(3)$, which we call the $\text{SO}(7)$ ansatz, the parameters associated with the bosonic fields $A_\mu$ are limited as $C_{\mu\nu} = \text{diag}(V_1, V_1, V_1, V_1, V_1, V_2, V_2, V_2, V_2, V_2, V_2, V_2, V_2, V_2, V_2, V_2)$, and those with the fermionic fields $\psi_\alpha$ are restricted according to $u_{8,9,10} = -u_{9,8,10} = (\text{cyclic}) = u$, while the others are zero. The other one is called the $\text{SO}(4)$ ansatz, which preserves $\text{SO}(4) \times \text{SO}(3) \times \text{SO}(3) \times \mathbb{Z}_2$ symmetry. The $\text{SO}(4)$ symmetry acts on the indices $\mu = 1, \cdots, 4$, and the two $\text{SO}(3)$ symmetries act on $\mu = 5, 6, 7$ and $\mu = 8, 9, 10$, respectively. The $\mathbb{Z}_2$ symmetry exchanges $\mu = 5, 6, 7$ and $\mu = 8, 9, 10$ directions, that reverses the 1st direction so that it is an element of the $\text{SO}(10)$.

Then, in the $\text{SO}(4)$ ansatz, the parameters are limited as $C_{\mu\nu} = \text{diag}(V_1, V_1, V_1, V_2, V_2, V_2, V_2, V_2, V_2, V_2, V_2, V_2, V_2, V_2, V_2, V_2)$, $u_{5,6,7} = u_{8,9,10} = u/\sqrt{2}$, up to the cyclic permutation of indices with signature, while the other components are zero.

Here we should comment on the other ansatz. In Ref. [1], we consider, besides the ansatz mentioned above, $\text{SO}(1)$, $\text{SO}(2)$, $\text{SO}(3)$, $\text{SO}(5)$ and $\text{SO}(6)$ ansatz. For each ansatz, $\text{SO}(n)$ represents the residual symmetry for the expanded direction, which at the end should be understood as the space-time dimension. According to the analysis given in Ref. [1], the $\text{SO}(2)$ and $\text{SO}(3)$ ansatz restore $\text{SO}(4)$ symmetry, the $\text{SO}(5)$ and $\text{SO}(6)$ ansatz restore $\text{SO}(7)$ symmetry, and the $\text{SO}(1)$ ansatz has no extrema and, of course, no plateau. (See Ref. [1] for details.) After all, we concentrate on the $\text{SO}(4)$ and $\text{SO}(7)$ cases, which behave quite differently, as we now see.

### 3.2 Calculation of the free energy and the extent of space-time

Now we have all the tools needed to compute the improved free energy. Our first task is to calculate the 2PI free energy $G$ for the action (3.4). It can be computed as

$$
\frac{G}{N^2} = \underbrace{\text{circle}}_{\text{c}} + \underbrace{\text{circle}}_{\text{c}} + g \underbrace{\text{circle}}_{\text{c}} + g^2 \underbrace{\text{circle}}_{\text{c}} + g^2 \underbrace{\text{circle}}_{\text{c}} + O(g^3)
$$

$$
= -\frac{1}{2} \ln(\det C) + \frac{1}{2} \ln(\det \gamma) + g \left( -\frac{1}{2} (\text{tr}_\mu(C^2) - (\text{tr}_\mu C)^2) - \frac{1}{2} C_{\mu\nu} \text{tr}_\alpha(\gamma \Gamma^\mu \gamma \Gamma^\nu) \right)
$$

$$
+ g^2 \left( \frac{3}{4} (\text{tr}_\mu(C^4) - (\text{tr}_\mu C^2)^2) + \frac{1}{4} C_{\mu\nu} C_{\rho\lambda} \text{tr}_\alpha(\gamma \Gamma^\mu \gamma \Gamma^\rho \gamma \Gamma^\nu \gamma \Gamma^\lambda) \right) + O(g^3),
$$

(3.5)
where the solid and dashed lines represent fermion and boson propagators, respectively, $\text{tr}_\mu$ represents the trace in the vector representation of SO(10), and $\text{tr}_\alpha$ is the trace taken in the left-handed spinor representation of SO(10). In order to count the order, the formal coupling constant $g$ is explicitly inserted. We calculated this 2PI free energy to 7th order in $g$. The numbers of graphs we computed are 2 at 0th, 1st and 2nd orders, 4 at 3rd order, 12 at the 4th order and 49 at 5th order. All of these were calculated in Ref. [1] and all the explicit graphs are presented in that paper. Our new result is for 6th and 7th order. The numbers of the graphs at 6th and 7th orders are 321 and 2346, respectively. The generation and calculation of the planar 2PI graphs have been now totally automatized.

Next, we perform the Legendre transformation. We define the variables conjugate to $V_i$ and $u$ as

$$M^i = \frac{\partial}{\partial V_i} G(V, u),$$

$$m = \frac{\partial}{\partial u} G(V, u),$$

and obtain the ordinary free energy via the Legendre transformation as follows:

$$F(M^i, m) = G - \sum_i M^i V_i - mu.$$  

(3.8)

Finally, the improved free energy $F_{\text{improved}}$ to order $g^k$ can be obtained by subtraction, as in the case of the massless $\phi^4$ model discussed in §2:

$$F_{\text{improved}}^k = F(M^i - gM^i, m - gm)|_k.$$  

(3.9)

Here $|_k$ denotes expansion with respect to $g$, ignoring $O(g^{k+1})$ terms and setting $g = 1$.

Thus we obtain the improved free energies to 7th order. As we have discussed, in general it is very difficult to identify where plateaus are and how they grow.

In the next subsection, we find the extrema of these free energies at several orders, and we compare their distribution with that for the $\phi^4$ matrix model.

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3In order to check the algorithm for generating graphs, we have used a matrix $\phi^4$-QED model that provides the same 2PI graphs as those of the IIB matrix model and is easier to compute analytically. Then, we confirmed that the set of graphs and the symmetry factors are all correct. (See Appendix A.)
3.3 Extrema of the improved free energies and the extent of space-time

We now have the improved free energies of the SO(4) and SO(7) ansatz to 7th order in the improved perturbation. In order to search for a plateau, we list all the extrema of these free energies, as done in §2. The extrema we have found are listed in Table 1.

Now we consider the “extent” of space-time in two directions, which is defined by the moment of the eigenvalue distribution as follows:

\[
\begin{align*}
R^2 &= \langle \frac{1}{N} \text{Tr} A_1^2 \rangle = - \frac{\partial F}{\partial M_1}(M^i - gM^i, m - gm), \\
r^2 &= \langle \frac{1}{N} \text{Tr} A_{10}^2 \rangle = - \frac{\partial F}{\partial M_2}(M^i - gM^i, m - gm).
\end{align*}
\]

We call \( R \) the extent of “our” space-time and \( r \) that of the internal one. The values of \( R, r \) and their ratio, \( \rho = R/r \), are also shown in Table 1 for each extremum of the SO(4) and SO(7) ansatz.

In Figs. 4 and 5, these extrema and the ratios \( \rho \) are plotted for each ansatz.

We find that these two ansatz yield significantly different behavior. The SO(7) ansatz has fewer extrema than the SO(4) ansatz, and, in particular, it has no extrema at even orders. Therefore, we speculate that extrema for the SO(7) ansatz do not accumulate to form a plateau. This means that the SO(7) ansatz is not a realistic assumption for the eigenvalue distribution of the IIB matrix model, and thus a seven-dimensional flat space-time would not be realized as its stable vacuum. Even if the SO(7) ansatz does develop a plateau at higher order, it would not be regarded as a compactification, because the ratio of the extents seems to be stabilized around 2.

In contrast, the SO(4) ansatz exhibits interesting behavior. In this case, the number of extrema grows as the order increases. It seems that even orders are not stable compared to odd orders. We observe similar situations in various models. Actually, in the zero-dimensional \( \phi^4 \) model, even lower orders have no extrema, while odd orders develop a plateau even at lower orders \[1\]. Assuming this is the case, we ignore even orders. Then we find peculiar behavior: There are two characteristic extrema that can be taken as the counterparts of the overshooting and undershooting extrema of the \( \phi^4 \) matrix model, and they are unrelated to a plateau. It seems that the other extrema tend to accumulate and can be expected to form a plateau. We further observe that the ratio of the extents for the SO(4) ansatz are around 2 or 3 for these undershooting and overshooting extrema, whereas the other extrema have rather large ratios,
| ansatz | order |\( F \) | \( \rho = R/r \) | \( R^2 \) | \( r^2 \) |
|---|---|---|---|---|---|
| **SO(7)** | 1st | 5.52272 | 1.95530 | 0.410551 | 0.107383 |
| | 3rd | 5.62072 | 1.91133 | 0.463883 | 0.126981 |
| | 5th | 5.52146 | 1.92523 | 0.481755 | 0.129975 |
| | 7th | 5.45127 | 1.93442 | 0.491098 | 0.13124 |
| **SO(4)** | 1st | 6.1533 | 1.85728 | 0.562580 | 0.163090 |
| | 3rd | 6.34486 | 1.85336 | 0.650887 | 0.189489 |
| | 4th | 1.17141 | 4.42435 | 1.90023 | 0.097047 |
| | 5th | 6.26112 | 1.92138 | 0.70007 | 0.189617 |
| | 6th | 2.85746 | 5.00919 | 2.12045 | 0.084569 |
| | 7th | 5.16286 | 6.42722 | 2.84737 | 0.0689285 |

Table 1: Extrema of the free energy and the extent of space-time for the SO(4) and SO(7) ansatz.
Figure 4: Extrema of the free energy (upper) and ratio of the extents (lower) for the $\text{SO}(7)$ ansatz.
Figure 5: Extrema of the free energy (upper) and ratio of the extents (lower) for the SO(4) ansatz. The numbers assigned at 7th order display a correspondence between the extrema and the value of the ratio of the extents.
around 6 - 9. This implies that on the plateau, the ratio of the extents takes a large value. In short, we can conclude that the SO(4) ansatz develops a plateau, and it predicts a quite large value for the ratio of the sizes of the internal and external directions. This indicates that our scenario for spontaneous compactification to a flat four-dimensional space-time is promising.

4 Conclusions and Discussions

We have performed the improved mean field approximation for the IIB matrix model up to 7th order and obtained the following conclusions:

- We first conclude that the SO(7) and SO(4) ansatz exhibit different types of behavior, as described below.

- The SO(7) ansatz has fewer extrema than the SO(4) ansatz, and it does not have a tendency to form a plateau. The eigenvalue distribution of the SO(7) ansatz is rather isotropic, and will not be realized as a compactification vacuum even if it has a plateau.

- The SO(4) ansatz has many extrema at higher orders and, except for the two special extrema mentioned below, it yields a quite large ratio of the extents of the four-dimensional space-time and that of the internal one.

- The extrema of the SO(4) ansatz are distributed, as in the $\phi^4$ matrix model, which develops a plateau. The $\phi^4$ matrix model has an overshooting and an undershooting extremum, which are located over and under the plateau, and the other extrema tend to accumulate. Thus, we conjecture that the IIB matrix model exhibits similar behavior and develops a plateau under the SO(4) ansatz.

At this stage we do not have clear plateaus for any ansatz, and therefore we cannot definitely tell which vacuum is realized in the IIB matrix model. Indeed, in order to do this, we need to identify the plateau for each ansatz, if such exists, and compare the values of the free energies at the plateaus. If an ansatz has no plateaus, we conclude that it is not realized as a vacuum. We expect that our SO(4) ansatz is close to the true vacuum, which reproduces our universe, and it has a plateau where the free energy has the lowest value. In order to confirm this, we should analyze higher orders in the improved perturbation series. As mentioned above, our calculation is now totally automatized, and it seems possible to carry out further analysis with the help of supercomputers.
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A The 2PI free energy of $\phi^4$-QED type matrix model

In this appendix we consider a matrix model that has a $\phi^4$ type and a QED type interaction term. The action is given by

$$S = N \text{Tr} \left( \frac{m_B}{2} A^2 + \frac{g^2}{4} A^4 + J A + \frac{m_F}{2} \psi^2 + g A \psi^2 \right),$$

(A.1)

where $A$ and $\psi$ are $N \times N$ matrices and $\psi$ is assumed to have flavor $f$. As we discuss below, we have introduced a source $J$ for $A$ in order to cancel the tadpole graphs.

By comparing this model with the IIB matrix model with the mean field action (3.4), it is easy to see that (A.1) generates the same vacuum graphs with the same symmetry factors as the IIB matrix model. These two models have the same types of propagators and vertices, except for the source term $J A$, which we explain below. Here $m_B$ and $m_F$ play the roles of $C^{-1}$ and $\psi^{-1}$, respectively, and $f$ should be set to $-1$, because $\psi$ is fermionic in the IIB matrix model. However, we do not fix $f$, in order to classify graphs via the number of $\psi$-loops. It is worth noting that in (A.1) there is no symmetry that forces the one-point function $\langle A \rangle$ to vanish. This is in contrast to the situation in the IIB matrix model, where it cannot have a non-zero value, due to the Lorentz symmetry. In order to eliminate unwanted one-point functions from our model, we choose $J$ in such a way that $\langle \text{Tr} A \rangle = 0$ order by order in the perturbation theory with respect to $g$.

Our aim is to confirm that our list of the planar 2PI graphs of the IIB matrix model is complete. For each graph, we read off the number of boson propagators (B), fermion propagators (F), fermion loops (L), $A^4$ vertices (V) and $A \psi^2$ vertices (Y), and we deduce the 2PI free energy of (A.1) as

$$-\frac{1}{\text{symmetry factor}} \sum (-g^2)^V (-g)^Y m_B^{-B} m_F^{-F} f^L,$$

(A.2)
where the summation is taken over all possible planar 2PI graphs without tadpoles up to the
given order of $g$. Note that in this expression cancellation between different graphs never
occurs, because the symmetry factor is always positive. We compare this with the 2PI free
energy of (A.1) in the large-$N$ limit, computed with a completely different method that we
now explain.

## A.1 Loop equation

The field redefinition in (A.1) gives

$$S = N \text{Tr} \left( \frac{1}{2} A^2 + \frac{g^2}{4} A^4 + J A + \frac{1}{2} \psi^2 + \lambda g A \psi^2 \right).$$  \hfill (A.3)

In order to recover the original parameters in (A.1), we only need to do is to make the
replacements $g \rightarrow g/m_B$, $J \rightarrow J/\sqrt{m_B}$, $\lambda \rightarrow \sqrt{m_B}/m_F$.

For the purpose of computing the 2PI free energy of (A.3), we first compute the two-point
function \( \langle \text{Tr}(A^2) \rangle / N \) in the large-$N$ limit, retrieve the parameters $m_B$ and $m_F$, and integrate
it with respect to $m_B$ to obtain the ordinary free energy. Finally, we carry out its Legendre
transformation in terms of $m_B$ and $m_f$ to obtain the 2PI free energy.

Performing the Gaussian integration in (A.3) with respect to $\psi$, we obtain

$$S_{\text{eff}} = N \text{Tr} \left( \frac{1}{2} A^2 + \frac{g^2}{4} A^4 + J A \right) - \frac{f}{2} \sum_{p=1}^{\infty} \sum_{q=0}^{\infty} \frac{(-\lambda g)^{p+q}}{p+q} \text{Tr}(A^p)\text{Tr}(A^q).$$ \hfill (A.4)

In order to compute the two-point function \( \langle \text{Tr}(A^2) \rangle / N \) of this model, we start with the loop
equation (Schwinger-Dyson equation)

$$0 = \int dA \frac{\partial}{\partial A^\alpha} \text{Tr}(A^\alpha \psi^\beta) e^{-S_{\text{eff}}},$$ \hfill (A.5)

where $t^\alpha$ is the orthogonal basis of hermitian $N \times N$ matrices: $\text{Tr}(t^\alpha t^\beta) = \delta^{\alpha\beta}$. By defining

$$w_n = \frac{1}{N} \langle \text{Tr}(A^n) \rangle,$$ \hfill (A.6)

this equation gives us the relation between these correlation functions,

$$w_n = \sum_{m=0}^{n-2} w_m w_{n-2-m} - g^2 w_{n+2} + J w_{n-1} + f \sum_{p=1}^{\infty} \sum_{q=0}^{\infty} (-\lambda g)^{p+q} \text{Tr}(A^p)\text{Tr}(A^q),$$ \hfill (A.7)

where we have used the factorization property in the large-$N$ limit. Note that this equation
holds even in the $n = 1$ case if we ignore the first term on the right-hand side. The loop
equation enables us to determine $w_n$ order by order in the perturbation theory as follows. First we expand each $w_n$ in terms of $g$:

$$w_n = \sum_{k=0}^{\infty} g^k w_n^{(k)}.$$

(A.8)

Substituting this into (A.7) and comparing both sides to each order in $g$, we obtain the following equation:

$$w_n^{(k)} = \sum_{m=0}^{n-2} \sum_{l=0}^{k} w_m^{(l)} w_{n-2-m}^{(k-l)} - w_{n+2}^{(k-2)} - J w_n^{(k-1)}$$

$$+ f \sum_{p=1}^{k} \sum_{q=0}^{k-p} \sum_{l=0}^{k-p-q} (-\lambda)^{p+q} p_{q-1} C_q w_n^{(l)} w_{n+p-2}^{(k-p-q-l)}.$$

(A.9)

Because $w_0 = 1/N \langle \text{Tr}1 \rangle = 1$, we have the “boundary condition” $w_0^{(l)} = \delta^i_0$. Note that only the quantities $w_m^{(l)}$ with $l$ smaller than or equal to $k$ appear on the right-hand side and that when $w_n^{(k)}$ appears on the right-hand side, $m$ is always smaller than $n$. Because of this property, (A.9) together with the boundary condition determines all the $w_n^{(k)}$.

Once we determine $w_1^{(k)}$ up to a given order, for example, the 14th, corresponding to the 7th in the IIB matrix model, we can tune $J$ order by order in such a way that $w_1^{(k)}$ vanishes for each $k$. Substituting the resulting expression for $J$ into $w_2^{(k)}$, we can derive the two-point function up to the order we desire. Because of this procedure, graphs containing a tadpole as a subgraph no longer contribute to $w_2^{(k)}$, and therefore the free energy obtained by integrating $w_2 = \sum_k g^k w_2^{(k)}$ gives the sum of all the planar vacuum graphs without tadpoles. Once we have the free energy, it is easy to carry out the Legendre transformation to obtain the 2PI free energy.

We have compared the two forms of the 2PI free energy obtained using the two totally different methods and found complete agreement. This proves that our list of all planar 2PI graphs of the IIB matrix model is complete.

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