Sparse halves in dense triangle-free graphs

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Abstract

Erdős [2] conjectured that every triangle-free graph $G$ on $n$ vertices contains a set of $\lfloor n/2 \rfloor$ vertices that spans at most $n^2/50$ edges. Krivelevich proved the conjecture for graphs with minimum degree at least $\frac{2}{5}n$ [7]. In [6] Keevash and Sudakov improved this result to graphs with average degree at least $\frac{2}{5}n$. We strengthen these results by showing that the conjecture holds for graphs with minimum degree at least $\frac{5}{14}n$ and for graphs with average degree at least $(\frac{2}{5} - \varepsilon)n$ for some absolute $\varepsilon > 0$. Moreover, we show that the conjecture is true for graphs which are close to the Petersen graph in edit distance.

Keywords: Triangle-free graph, sparse half, minimum degree, Petersen graph, edit distance, blow-up

1. Introduction

In this paper we consider the edge distribution in triangle-free graphs. A fundamental result in extremal graph theory, Turán’s theorem implies that every graph on $n$ vertices with more than $n^2/4$ edges contains a triangle. One can consider the following generalization of this problem first studied by Erdős, Faudree, Rousseau and Schelp in [4].

Suppose for given $0 < \alpha \leq 1$ every set of $\alpha n$ vertices of graph $G$ spans more than $\beta n^2$ edges. A natural question arises - what is the smallest $\beta = \beta(\alpha)$ such that every such graph $G$ necessarily contains a triangle? In particular, one of the Erdős’ old and favorite conjectures is on $\beta(\frac{1}{2})$ that he first proposed in [2] and offered a $250 prize for its solution later in [3].

Conjecture 1.1. Any triangle-free graph $G$ on $n$ vertices must contain a set of $\lfloor n/2 \rfloor$ vertices that spans at most $n^2/50$ edges.

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In [2] the authors also conjectured that $\beta(\alpha)$ is determined by some family of extremal triangle-free graphs. The bound for $\beta(\frac{1}{2})$ is obtained on the uniform blow-up of $C_5$ which is obtained from the 5-cycle by replacing each vertex $i$ by an independent set $V_i$ of size $n/5$ (for simplicity assume $n$ is divisible by 5) and each edge $ij$ by a complete bipartite graph joining $V_i$ and $V_j$ (note that the blow-up of Petersen graph also achieves this bound tightly, see Figure 1).

![Figure 1: A sparse half in the uniform blow-up of Petersen graph.](image)

In his paper [7] Krivelevich proved that the conjecture holds if $\frac{n^2}{50}$ is replaced by $\frac{n^2}{36}$. He also showed that it is true for triangle-free graphs with minimum degree $\frac{2}{5}n$. In Section 3 we improve this result by proving the following theorem.

**Theorem 1.1.** Every triangle-free graph on $n$ vertices with minimum degree $\frac{5}{14}n$ contains a set of $\left\lfloor \frac{n}{2} \right\rfloor$ vertices that spans at most $\frac{n^2}{50}$ edges.

Our proof of Theorem 1.1 is mainly based on the structural characterization of these graphs established by Jin, Chen and Koh in [1, 5]. We also use some averaging arguments similar to the ones used in [6, 7].

Sudakov and Keevash [6] improved the result of [7] showing that the conjecture holds for graphs with average degree $\frac{2}{5}n$. In Section 5 we extend their result as follows.

**Theorem 1.2.** There exists $\gamma > 0$ such that every triangle-free graph on $n$ vertices with at least $(\frac{1}{2} - \gamma)n^2$ edges contains a set of $\left\lfloor \frac{n}{2} \right\rfloor$ vertices that spans at most $\frac{n^2}{50}$ edges.

Finally, we study the validity of the conjecture in the neighborhood of the known extremal examples, in the following sense. In the uniform blow-up of the Petersen graph every set of $\left\lfloor \frac{n}{2} \right\rfloor$ has at least $\frac{n^2}{50}$ edges (see Figure 1). To the best of our knowledge, the uniform blow-ups of the Petersen graph and $C_5$, as defined in the beginning of the introduction, are the only known examples for which Conjecture 1.1 is tight. In Section 4 we develop a set of tools which allow us to prove the Conjecture 1.1 for the classes of
graphs which are close to a fixed graph in edit distance. In Section 6 we use these tools to verify the conjecture for graphs which are close to the Petersen graph, while Theorem 1.2 shows that it also holds for graphs close to the 5-cycle. These result can be considered as a proof of a local version of Conjecture 1.1, in the spirit of recent results of Lovász [8], which proves the Sidorenko conjecture locally in the neighborhood of the conjectured extremal example, and Razborov [10], which accomplishes a similar goal for the Caccetta-Häggkvist conjecture.

2. Notation and Preliminary Results

In a graph $G$, we denote by $N(v)$ the neighborhood of a vertex $v \in V(G)$ and by $d_G(v)$ (or just $d(v)$) the degree of $v$. The maximum and the minimum degrees of the graph are denoted by $\Delta(G)$ and $\delta(G)$, respectively.

A graph $G$ is called triangle-free if it does not contain a triangle. We say that $G$ contains a sparse half if there exists a set of $\lfloor n/2 \rfloor$ vertices in $G$ that spans at most $n^2/50$ edges. Conjecture 1.1 says that there must exist a sparse half in every triangle-free graph.

We say that $\omega : V(G) \to (0,1)$ is a weight function on $G$ if $\sum_{v \in V(G)} \omega(v) = 1$. A pair $(G,\omega)$, where $\omega$ is a weight function on $G$ is called a weighted graph. The weight $\omega(e)$ of an edge $e = (u,v)$ in $(G,\omega)$ is defined as $\omega(u) \cdot \omega(v)$. For a set $X$ of vertices or edges of $G$ let $\omega(X) := \sum_{x \in X} \omega(x)$. The degree of a vertex $v$ in a weighted graph $(G,\omega)$ is defined as $\omega(N(v))$. The minimum degree of the weighted graph $(G,\omega)$ we simply denote by $\delta(G,\omega)$.

We call a real function $s : V(G) \to \mathbb{R}^+$ a half of $G$ if $s(v) \leq \omega(v)$ for every $v \in V(G)$, and $s(V(G)) = 1/2$. If in addition to these conditions also $s(E(G)) \leq 1/50$ then $s$ is called a sparse half.

There is a natural way of associating a weighted graph to a graph $G$ of order $n$; that is to assign to each vertex $v$ weight equal to $\frac{1}{n}$. The resulting weighted graph we call a uniformly weighted $G$ and denote by $(G,\omega_u)$.

**Lemma 2.1.** If $(G,\omega_u)$ has a sparse half, then so does $G$.

**Proof.** We claim that, if $(G,\omega_u)$ has a sparse half, then it has a sparse half $s$ such that $s(v) = 0$ or $s(v) = \frac{1}{n}$ for all $v \in V(G)$ except for possible one vertex.

Indeed, let us choose a sparse half $s$ of $G$ such that the number of vertices $v \in V(G)$ such that either $s(v) = 0$ or $s(v) = \frac{1}{n}$ is maximum. We show that there exists at most one vertex $u$ such that $0 < s(u) < \frac{1}{n}$.

Suppose not and there exist $u, v \in V(G)$ such that $0 < s(u), s(v) < \frac{1}{n}$. We define a new half $s' : V(G) \to \mathbb{R}^+$. Let $s'$ be the same as $s$ on all vertices of $G$, except $u$ and $v$. Without loss of generality, suppose $s(N(u)) \leq s(N(v))$. Let $\delta = \min\{s(u), \frac{1}{n} - s(v)\}$ and define $s'(u) = s(u) - \delta$ and $s'(v) = s(v) + \delta$. We will show that $s'$ is a sparse half.
Suppose that \( u \) and \( v \) are adjacent, then
\[
s'(E(G)) = s(E(G)) - \sum_{x \in N(u)} \delta s(x) + \sum_{y \in N(v), y \neq v} \delta s(y) + s'(u)s'(v) - s(u)s(v)
\]
\[
= s(E(G)) - \sum_{x \in N(u)} \delta s(x) + \sum_{y \in N(v), y \neq v} \delta s(y) - \delta s(v) + \delta s(u) - \delta^2
\]
\[
= s(E(G)) - \delta (s(N(u)) - s(N(v))) - \delta^2 < s(E(G)).
\]
The calculation in the case when \( u \) and \( v \) are non-adjacent is similar.

It follows that \( s' \) contradicts the choice of \( s \). Hence for all vertices \( v \) of the graph except maybe one vertex either \( s(v) = 0 \) or \( s(v) = \frac{1}{n} \). Let \( S = \{ v \in V(G) \mid s(v) = 1/n \} \). It follows from the above that \( |S| \geq \lceil n/2 \rceil \). It is easy to see that \( E(G[S]) \leq n^2 s(E(G)) \leq n^2/50 \). Hence \( S \) is a sparse half in \( G \), as desired. \( \square \)

Lemma 2.1 allows us to work with weighted graphs, which proves to be convenient. We prove that every weighted triangle-free graph with minimum degree at least 5/14 contains a sparse half. Our proof uses structural characterization of these graphs found by Jin, Chen and Koh in [1, 5]. To state their result we need a few additional definitions.

A mapping \( \varphi : V(G) \to V(H) \) for graphs \( G \) and \( H \) is a homomorphism from \( G \) to \( H \), if for any pair of adjacent vertices \( u,v \in V(G) \), \( \varphi(u) \) and \( \varphi(v) \) are adjacent in \( H \). We say that \( G \) is of \( H \)-type if there exists a homomorphism from \( G \) to \( H \). Let \( \varphi : V(G) \to V(H) \) be a surjective homomorphism and let \( \omega \) be a weight function on \( G \). We define a weight function \( \omega_\varphi \) on \( H \) in the following way. For every vertex \( v \in V(H) \), let \( \omega_\varphi(v) := \omega(\varphi^{-1}(v)) \). The next lemma shows that a sparse half in a homomorphic image of the graph \( G \) can be lifted to a sparse half in the graph \( G \).

**Lemma 2.2.** Let \( G, H \) be graphs and let \( \varphi : V(G) \to V(H) \) be a surjective homomorphism. Then for any weight function \( \omega \), if \((H,\omega_\varphi)\) has a sparse half, then so does \((G,\omega)\).

**Proof.** Let \( \omega \) be a weight function on \( G \) and let \( s_H \) be a sparse half on \((H,\omega_\varphi)\). Define
\[
s_G(u) := \frac{\omega(u)}{\omega_\varphi(\varphi(u))} s_H(\varphi(u))
\]
for every \( u \in V(G) \). It is easy to check that \( s \) is a sparse half of \( G \). \( \square \)

Now let us introduce the family of the graphs that plays a key role in our results. For an integer \( d \geq 1 \), let \( F_d \) be the graph with
\[
V(F_d) = \{ v_1, v_2, \ldots, v_{3d-1} \},
\]
such that the vertex \( v_j \) has neighbors \( v_{j+d}, \ldots, v_{j+2d-1} \), these values taken modulo \( 3d-1 \). Throughout this paper whenever we deal with \( F_d \) graphs, we always take the indices of the vertices modulo \( 3d-1 \). In [5] it is shown that every triangle-free graph \( G \) of order \( n \) with minimum degree \( \delta(G) > 10n/29 \) is of \( F_9 \)-type and hence is 3-colorable. In [1] Chen, Jin and Koh proved that every triangle-free graph of order \( n \), with chromatic number \( \chi(G) \leq 3 \) and minimum degree \( \delta > \left\lceil \frac{(d+1)n}{3d+2} \right\rceil \) is of \( F_{d'} \)-type. Therefore, the following theorem holds.
Theorem 2.3 (Chen, Jin, Koh [1], [5]). Every triangle-free graphs of order $n$ with minimum degree at least $\frac{5}{14}n$ admits a homomorphism to $F_4$.

To use Lemma 2.1 and Theorem 2.3 together we need to make sure that the homomorphism in the statement of Theorem 2.3 is surjective. Fortunately, this is not difficult if we allow ourselves to change the target graph.

Lemma 2.4. Let $\varphi$ be a homomorphism from graph $G$ to $F_d$, $d \geq 2$. Then either $\varphi$ is surjective or there exists a homomorphism $\varphi'$ from $G$ to $F_{d-1}$.

Proof. Suppose $\varphi$ is a homomorphism from graph $G$ to $F_d$ that is not surjective. Let $V_i = \varphi^{-1}(v_i)$ for all $i = 1, 2, \ldots, 3d - 1$. Without loss of generality suppose $V_1$ is empty. Define a mapping $\varphi': V(G) \to V(F_{d-1})$ as follows,

$$
\varphi'(v) = \begin{cases} 
  v_{i+1}, & \text{if } v \in V_i \text{ and } 2 \leq i \leq d - 1 \\
  v_{d+1}, & \text{if } v \in V_d \cup V_{d+1} \\
  v_{i+2}, & \text{if } v \in V_i \text{ and } d + 2 \leq i \leq 2d - 2 \\
  v_{2d-1}, & \text{if } v \in V_{2d-1} \cup V_{2d} \\
  v_{i+3}, & \text{if } v \in V_i \text{ and } 2d + 1 \leq i \leq 3d - 1,
\end{cases}
$$

It is easy to check that $\varphi'$ is a homomorphism from $G$ to $F_{d-1}$.

In the next section we show that for $1 \leq d \leq 4$ the weighted graph $(F_d, \omega)$ with minimum degree at least $5/14$, has a sparse half for any positive weight function $\omega$. In particular, if $\varphi : V(G) \to F_d$ is a surjective homomorphism, then the weighted graph $(F_d, \omega \varphi)$ has a sparse half. By Lemma 2.1 this implies that graph $G$ has a sparse half. Hence the Theorem 1.1 will follow from the results of the next section and the results we have introduced.
3. Weighted triangle-free graphs with minimum degree $\geq \frac{5}{14}$

Theorem 3.1. Let $d \leq 4$ be a positive integer and let $(F_d, \omega)$ be a weighted graph with the minimum degree at least $5/14$. Then $(F_d, \omega)$ has a sparse half.

Proof. The argument is separated into cases based on the value of $d$.

$d=1$: Suppose $V(F_1) = \{v_1, v_2\}$ then since $\omega(v_1) + \omega(v_2) = 1$, either $\omega(v_1) \geq \frac{1}{2}$ or $\omega(v_2) \geq \frac{1}{2}$, therefore $v_1$ or $v_2$ supports a sparse half in $(F_1, \omega)$.

$d=2$: Let $V(F_2) = \{v_1, v_2, \ldots, v_5\}$. If any two consecutive vertices together have total weight at least $1/2$, then they induce an independent set, which means that they support a sparse half.

Now suppose that no two consecutive vertices have total weight at least $1/2$, then any three consecutive vertices have total weight at least $1/2$. We define the following halves $s_i$ for each $i = 1, 2, \ldots, 5$ on the vertices of the graph and prove that there is at least one sparse half among them.

$$s_i(v) = \begin{cases} \omega(v), & \text{if } v = v_i \text{ or } v = v_{i+1}, \\ \frac{1}{2} - (\omega(v_i) + \omega(v_{i+1})), & \text{if } v = v_{i+2}, \\ 0, & \text{otherwise.} \end{cases}$$

![Figure 3: A sparse half in uniformly weighted $C_5$](image)

Note that

$$s_i(E(G)) = \omega(v_i) \left(1/2 - (\omega(v_i) + \omega(v_{i+1}))\right).$$

(1)

Summing up the equations (1) over all $i = 1, \ldots, 5$, we get

$$\sum_{i=1}^{5} s_i(E(G)) = 1/2 - \sum_{i=1}^{5} \omega(v_i) (\omega(v_i) + \omega(v_{i+1}))$$

$$= \frac{1}{2} - \frac{1}{2} \sum_{i=1}^{5} (\omega(v_i) + \omega(v_{i+1}))^2$$

$$\leq \frac{1}{2} - \frac{1}{2} \cdot 5 \cdot \frac{4}{25} = \frac{1}{10}.$$
using Jensen’s inequality for the function \( x^2 \). Thus one of the functions \( s_i \) is a sparse half.

Note that the proof above did not use the minimum degree condition, while we use it in the other two cases.

**d=3:** Let \( F_3 = \{ v_1, v_2, \ldots, v_8 \} \). As the minimum degree of \( (F_3, \omega) \) is at least 5/14, we have

\[
\omega(v_j) + \omega(v_{j+1}) + \omega(v_{j+2}) \geq \delta(F_3, \omega) \geq 5/14, \tag{2}
\]

for all \( j = 1, 2, \ldots, 8 \). Summing these inequalities for \( j = i, i+1 \) and \( j = i+2 \), we obtain

\[
\omega(v_j) \geq 1/14 \text{ for } j = 1, 2, \ldots, 8.
\]

As in the case of \( d = 2 \), if there exist three consecutive vertices of total weight at least 1/2, then we are done, since they induce an independent set. Suppose not, then every five consecutive vertices have total weight more than 1/2. We define the following halves \( s_i \), for each \( i = 1, 2, \ldots, 8 \) on the vertices of the graph and prove that there is at least one sparse half among them.

\[
s_i(v) = \begin{cases} 
\omega(v), & \text{if } v = v_{i+1}, v_{i+2}, v_{i+3} \\
\frac{1}{2} \left( \frac{1}{2} - (\omega(v_{i+1}) + \omega(v_{i+2}) + \omega(v_{i+3})) \right), & \text{if } v = v_i \text{ or } v = v_{i+4}, \\
0, & \text{otherwise.}
\end{cases}
\]

**Claim 3.2.** For each \( i = 1, 2, \ldots, 8 \), \( s_i \) is a half.

**Proof.** It suffices to show that

\[
\frac{1}{2} \left( \frac{1}{2} - (\omega(v_{i+1}) + \omega(v_{i+2}) + \omega(v_{i+3})) \right) \leq \omega(v_i),
\]

\[
\frac{1}{2} \left( \frac{1}{2} - (\omega(v_{i+1}) + \omega(v_{i+2}) + \omega(v_{i+3})) \right) \leq \omega(v_{i+4}).
\]

By symmetry it suffices to prove the first inequality. By (2) we get that

\[
\frac{1}{2} \left( \frac{1}{2} - (\omega(v_{i+1}) + \omega(v_{i+2}) + \omega(v_{i+3})) \right) \leq \frac{1}{2} \cdot \left( \frac{1}{2} - \frac{5}{14} \right) = \frac{1}{14} \leq \omega(v_i).
\]

This finishes the proof of the claim. \( \square \)

We relegate the proof of the following lemma, which ensures that one of the halves \( s_i \) is sparse, to the Appendix.

**Lemma 3.3.** Let \( 1/14 \leq x_i \leq 1 \) for \( i = 1, 2, \ldots, 8 \). If \( \sum_{i=1}^{8} x_i = 1 \) and \( x_i + x_{i+1} + x_{i+2} \geq 5/14 \) for every \( i \), then there exists an \( i \) such that

\[
\frac{1}{2} \left( \frac{1}{2} - (x_i + x_{i+1} + x_{i+2}) \right) (x_i + x_{i+2}) + \frac{1}{4} \left( \frac{1}{2} - (x_i + x_{i+1} + x_{i+2}) \right)^2 \leq \frac{1}{50} \tag{3}
\]
**d=4:** Let $F_d = \{v_1, v_2, \ldots, v_{11}\}$. The minimum degree condition gives us the following inequality
\[
\omega(v_i) + \omega(v_{i+1}) + \omega(v_{i+2}) + \omega(v_{i+3}) \geq 5/14,
\]
for all $i = 1, 2, \ldots, 11$. It follows, as in the case $d = 3$, that $\omega(v_i) \geq 1/14$ for all $i$. If any of four consecutive vertices have total weight at least $1/2$, then we are done, since they induce an independent set.

For every $i = 1, 2, \ldots, 11$ by (4), we have $\omega(v_{i+5}) + \omega(v_{i+6}) + \omega(v_{i+7}) + \omega(v_{i+8}) \geq 5/14$, $\omega(v_{i+9}) \geq 1/14$ and $\omega(v_{i+10}) \geq 1/14$, therefore
\[
\omega(v_i) + \omega(v_{i+1}) + \cdots + \omega(v_{i+5}) \leq 1/2.
\]
It follows that
\[
\omega(v_i) + \omega(v_{i+1}) + \cdots + \omega(v_{i+6}) \geq 1/2,
\]
for all $i = 1, 2, \ldots, 11$. This allows us to define halves $s_i$ in the following way:

\[
s_i(v) = \begin{cases} 
\omega(v), & \text{if } v = v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}, \\
\frac{1}{2} \left( \frac{1}{2} - (\omega(v_{i+1}) + \omega(v_{i+2}) + \omega(v_{i+3}) + \omega(v_{i+4})) \right), & \text{if } v = v_i \text{ or } v = v_{i+5}, \\
0, & \text{otherwise.}
\end{cases}
\]

As in the previous case, it is easy to verify that each $s_i$ is a half, and the following lemma proved in the Appendix implies that at least one of these halves is sparse.

**Lemma 3.4.** Suppose given are $x_1, x_2, \ldots, x_{11}$ reals such that $1/14 \leq x_i \leq 1$ for each $i = 1, 2, \ldots, 11$ and $\sum_{i=1}^{11} x_i = 1$. If $x_i + x_{i+1} + x_{i+2} + x_{i+3} > 5/14$ for every $i$, then there exists an $i$ such that
\[
\frac{1}{2} \left( \frac{1}{2} - (x_{i+1} + x_{i+2} + x_{i+3} + x_{i+4}) \right) (x_{i+1} + x_{i+4}) \\
+ \frac{1}{4} \left( \frac{1}{2} - (x_{i+1} + x_{i+2} + x_{i+3} + x_{i+4}) \right)^2 \leq 1/50.
\]

**Proof of Theorem 1.1.** Let $G$ be a triangle-free graph on $n$ vertices with minimum degree $\geq 5n/14$. By Lemma 2.1 it suffices to prove that the uniformly weighted graph $(G, \omega)$ has a sparse half. By Theorem 2.3 and Lemma 2.4, the graph $G$ admits a surjective homomorphism $\phi$ to $F_d$ for $1 \leq d \leq 4$. Clearly, $(F_d, \omega_\phi)$ has minimum degree $\geq 5/14$ and thus has a sparse half by Theorem 3.1. Theorem 1.1 now follows from Lemma 2.2.

**4. Uniform sparse halves, balanced weights and disturbed graphs**

The purpose of this section is to develop a set of tools, which under fairly general circumstances allow us to show that graphs “close” to a fixed graph have sparse halves. A number of technical definitions will be necessary. We start by a variant of the definition of edit distance (see e.g. [9]).
Definition 4.1. Given a graph $G$ of order $n$ and a graph $H$, $V(H) = \{1, 2, \ldots, k\}$, we say that $G$ is $\varepsilon$-close to $H$, if there exists a partition $V = \{V_1, V_2, \ldots, V_k\}$ of $V(G)$ such that
\[
|V_i| - \frac{n}{k} \leq \varepsilon n, \text{ for all } i = 1, 2, \ldots, k; \text{ and } |E(G \triangle H_V)| \leq \varepsilon n^2,
\]
where $H_V$ is a graph on vertex set $V(H_V) := V(G)$, each $H_V[V_i]$ is an independent set and $(V_i, V_j)$ induces a complete bipartite graph in $H_V$ if $ij \in E(H)$ and an empty graph, otherwise.

We will also need a stronger notion of distance defined as follows.

Definition 4.2. Given a graph $G$ and $F \subseteq E(G)$, we say that $D \subseteq V(G)$ is an $\varepsilon$-controlling set for $F$, if $|D| \leq \varepsilon |V(G)|$ and every edge in $F$ has at least one end in $D$. We say that the graph $G'$ is an $\varepsilon$-disturbed graph of $G$ for some $0 < \varepsilon < 1$, if the following conditions hold:

1. $V(G') = V(G)$,
2. there exists an $\varepsilon$-controlling set for $E(G') - E(G)$ in $G'$,
3. $|N_G(v) - N_{G'}(v)| \leq \varepsilon |V(G)|$ for every vertex $v \in V(G)$.

Let $H$ be a graph. Let $\mathcal{N}(H)$ be the set of neighborhoods of vertices of $H$, and let $\mathcal{I}(H)$ be the set of maximum independent sets of $H$. Let $\mathcal{I}^*(H) = \mathcal{I}(H) - \mathcal{N}(H)$. We construct a graph $H^*$ as follows. Let $V(H^*) = V(H) \cup \mathcal{I}^*(H)$, let $H$ be an induced subgraph of $H^*$, and let every $I \in \mathcal{I}^*(H)$ be adjacent to every $v \in I$ and no other vertex of $H^*$. We say that a weighted graph $(H^*, \omega)$ is $\varepsilon$-balanced if $|\omega(v) - 1/|V(H)|| \leq \varepsilon$ for every $v \in V(H)$ and $\omega(v) \leq \varepsilon$ for every $v \in V(H^*) - V(H)$. We are now ready for our first technical result.

Theorem 4.3. Let $H$ be a maximal triangle-free graph. For every $\varepsilon > 0$ there exists $\delta$ such that if $G$ is a triangle-free graph which is $\delta$-close to $H$ then there exists a graph $G'$ and a homomorphism $\varphi$ from $G'$ to $H^*$ with the following properties

(i) $G$ is $\varepsilon$-disturbed graph of $G'$,

(ii) $(H^*, \omega_{\varphi})$ is $\varepsilon$-balanced,

(iii) $\varphi$ is a strong homomorphism, that is $uv \notin E(G')$ implies $\varphi(u)\varphi(v) \notin E(H^*)$ for every pair of vertices $u, v \in V(G')$.

Proof. We assume that $V(H) = \{1, 2, \ldots, k\}$. We show that $\delta > 0$ satisfies the theorem if $(k + 2)\sqrt{3} \leq \min(\varepsilon, 1/k)$. Let $V = (V_1, V_2, \ldots, V_k)$ be the partition of $V(G)$, and the graph $H_V$ be as in Definition 4.1. Let $n := |V(G)|$, $F := E(G) \triangle E(H_n)$ and let $J$ be the set of all vertices of $G$ incident to at least $\sqrt{3n}$ edges in $F$. We have $\delta n^2 \geq |F| \geq \frac{1}{2\sqrt{3}} |J|$. It follows that $|J| \leq 2\sqrt{3n}$.

We define a map $\varphi : V(G) \to V(H^*)$, as follows. If $v \in V_i - J$ for some $i \in V(H)$ then $\varphi(v) := i$. Now consider $v \in J$ and let
\[
I_0(v) := \{i \in V(G) \mid |N(v) \cap V_i| > \sqrt{3n}\}.
\]
Then $I_0(v)$ is independent, as otherwise there exist $i, j \in [k]$, such that $ij \in E(H)$, $|N(v) \cap V_i| > \sqrt{\delta n}$ and $|N(v) \cap V_j| > \sqrt{\delta n}$. As $G$ is triangle-free it follows that $|E(H \cap E(G))| > \varepsilon n^2$, contradicting the choice of $V$. Let $I(v)$ be a maximal independent set containing $I_0(v)$, chosen arbitrarily. Let $\varphi(v) = i$, if $I(v) = N_H(i)$ for some $i \in V(H)$, and let $\varphi(v) = I(v)$, otherwise.

Let $G'$ be the graph with $V(G') = V(G)$ and the vertices $uv \in E(G')$ if and only if $\varphi(u)\varphi(v) \in E(H^*)$. Then $\varphi$ is a strong homomorphism from $G'$ to $H^*$. For $i \in V(H)$ we have $V_i - J \subseteq \varphi^{-1}(i) \subseteq V_i \cup J$ and, therefore, $||\varphi^{-1}(i)||/n - 1/|k| \leq \delta + 2\sqrt{\delta}$. For $I \in V(H^*) - V(H)$ we have $|\varphi^{-1}(I)| \leq |J| \leq 2\sqrt{\delta}n$. Thus (ii) holds, as $\varepsilon \geq \delta + 2\sqrt{\delta}$. It remains to verify (i). We will show that $J$ is an $\varepsilon$-controlling set for $F' := E(G) - E(G')$ for $\delta$ sufficiently small. First, we show that every edge of $F'$ has an end in $J$. Indeed suppose that $uv \in E(G)$ for some $u \in V_i, v \in V_j$ with $i, j \in V(H)$ not necessarily distinct and $ij \not\in E(H)$. Then there exists $h \in V(H)$ adjacent to both $i$ and $j$, as $H$ is maximally triangle-free. It follows that both $v$ and $u$ have at least $(1/k - \delta - \sqrt{\delta})n$ neighbors in $V_h$ and share a common neighbor if $2(1/k - \delta - \sqrt{\delta}) > 1/k$. Thus our first claim holds, as $\delta + \sqrt{\delta} < 1/k$.

Consider now $v \in J$ and let $N'(v)$ be the set of neighbors of $v$ in $V(G') - V(J)$ joined to $v$ by edges of $F'$. Then $|N'(v)| \leq \sqrt{\delta}n$ for every $v \in V(H)$ by the choice of $\varphi(v)$. It follows that $|N'(v)| \leq k\sqrt{\delta}n$. Therefore $v$ is incident to at most $(k + 2)\sqrt{\delta}n$ edges in $F$, as $|J| \leq 2\sqrt{\delta}n$. Thus $J$ is an $\varepsilon$-controlling set for $F'$, as $(k + 2)\sqrt{\delta}n \leq \varepsilon$.

We say that $H$ is entwined if $\mathcal{I}^*(H)$ is intersecting. Note that the graph $F_i$ is entwined for every $i$, as $\mathcal{I}^*(H)$ is empty. It is routine to check that the Petersen graph is entwined. We say that the graph $G$ of order $n$ is $c$-maximal triangle-free if it is triangle-free and adding any new edge to $G$ creates at least $cn$ triangles. The following lemma follows immediately from definitions.

**Lemma 4.4.** Let $H$ be an entwined triangle-free graph, and let $0 < \varepsilon < 1/|V(H)|$. If $(H^*, \omega)$ is $\varepsilon$-balanced then it is $(1/|V(H)| - \varepsilon)$-maximal triangle-free.

**Definition 4.5.** For a weighted graph $(G, \omega)$ we call a distribution $\mathbf{s}$ defined on the set of halves of $(G, \omega)$ a $c$-uniform sparse half for some $0 < c \leq 1$, if

1. for every edge $e \in E(G)$ $\mathbb{E}[s(e)] \geq c\omega(e)$,
2. $\mathbb{E}[s(E(G))] \leq \frac{1}{50}$.

Whenever we refer to $c$-uniform sparse halves in unweighted graphs, they are understood as the $c$-uniform sparse halves in the corresponding uniformly weighted graphs.

**Theorem 4.6.** Let $0 < c < 1$ be real, let $G'$ be a $c$-maximal triangle-free graph and let $G$ be a triangle-free $\frac{c^2}{2(1+c)}$-disturbed graph of $G'$. If $G'$ has a $c$-uniform sparse half then $G$ has a sparse half.

**Proof.** Let $\varepsilon = \frac{c^2}{2(1+c)}$. Let $F = E(G) - E(G')$ and let $M$ be a maximal matching in $F$. Let $|M| = \delta n$ for some $0 \leq \delta \leq 1/2$. Since $G$ is an $\varepsilon$-disturbed graph of $G'$, it has an $\varepsilon$-controlling set for $F$. Let $D$ be the minimum one. Let $V(M)$ be the set of ends of the edges in $M$. By the choice of $M$ every edge of $F$ has at least one end in

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by linearity of the expectation. We have
\[
\mathbb{E} [s(E(G)) - s(E(G'))] = \mathbb{E} [s(F) - s(F')] = \mathbb{E} [s(F)] - \mathbb{E} [s(F')],
\]
by linearity of the expectation. We have
\[
\mathbb{E} [s(F')] = \sum_{e \in F'} \mathbb{E} [s(e)] \geq \sum_{e \in F'} c \omega(e) = c \sum_{e \in F'} \frac{1}{n^2} \geq \frac{c|F'|}{n^2}.
\]
On the other hand,
\[
\mathbb{E} [s(F)] = \sum_{e \in F} \mathbb{E} [s(e)] \leq \sum_{e \in F} \omega(e) = \frac{|F|}{n^2}.
\]
Finally, note that \(\delta \leq \varepsilon\), since every edge in \(M\) has at least one of its ends in \(D\). Hence,
\[
\mathbb{E} [s(E(G)) - s(E(G'))] \leq \frac{|F|}{n^2} - \frac{c|F'|}{n^2} \leq 2\delta\varepsilon - c(\varepsilon - 2\delta)\delta
\]
\[
= \delta(2\delta c + 2\varepsilon - c^2) \leq 0,
\]
where the last inequality holds, as \(2\delta c + 2\varepsilon - c^2 \leq 2(1+c)\varepsilon - c^2 = 0\). Therefore,
\[
\mathbb{E} [s(E(G))] \leq \mathbb{E} [s(E(G'))] \leq \frac{1}{30},
\]
and the graph \(G\) has a sparse half by Lemma 2.1.

We are now ready to prove the main result of this section.

**Theorem 4.7.** Let \(H\) be an entwined maximal triangle-free graph. Suppose that there exists \(\alpha > 0\) such that, if \((H^*, \omega)\) is \(\alpha\)-balanced, then \((H^*, \omega)\) has an \(\alpha\)-uniform sparse half. Then there exists \(\delta > 0\) such that every triangle-free graph \(G\) which is \(\delta\)-close to \(H\) has a sparse half.

**Proof.** Let \(\varepsilon := \min\left(\frac{1}{3|V(H)|^2}, \frac{n^2}{2(1+\alpha)}\right)\), and let \(\delta\) be chosen so that there exist \(\varphi\) and \(G'\) satisfying the conclusion of Theorem 4.3. The weighted graph \((H^*, \omega_\varphi)\) has an \(\alpha\)-uniform sparse half, as \(\varepsilon \leq \alpha\). Therefore, the graph \(G'\) has an \(\alpha\)-uniform sparse half. By Lemma 4.4, the graph \((H^*, \omega_\varphi)\) is \((1/|V(H)| - \varepsilon)\)-triangle-free. Therefore, the graph \(G'\) is \(\varepsilon\)-triangle-free, because \(\varphi\) is a strong homomorphism. Let \(c := \min(\alpha, 1/2|V(H)|)\). Then \(\varepsilon \leq c^2/(2(1+c))\) and \(c \leq 1/|V(H)| - \varepsilon\), by the choice of \(\varepsilon\). It follows that the conditions of Theorem 4.6 are satisfied for \(G'\) and \(G\). Thus \(G\) has a sparse half, as desired. \(\square\)
5. Triangle-free graphs with at least \((1/5 - \gamma)n^2\) edges

In order to establish the conjecture for the triangle-free graphs with average degree \(\left(\frac{2}{5} - \gamma\right)n\) we separate the cases when the graphs under consideration are close in the sense of Definition 4.1 to the blow-up of \(C_5\) and when they are not. In the second case, we use the result of Sudakov and Keevash [6] that can be rephrased in the following way.

**Theorem 5.1.** Let \(G\) be a triangle-free graph on \(n\) vertices such that one of the following conditions holds

(a) either \(\frac{1}{n} \sum_{v \in V(G)} d^2(v) \geq \left(\frac{2}{5} n\right)^2\) and \(\Delta(G) < \left(\frac{2}{5} + \frac{1}{135}\right)n\), or

(b) \(\Delta(G) \geq \left(\frac{2}{5} + \frac{1}{135}\right)n\) and \(\frac{1}{n} \sum_{v \in V(G)} d(v) \geq \left(\frac{2}{5} - \frac{1}{125}\right)n\).

Then \(G\) has a sparse half.

The next theorem represents the main technical step in the proof of Theorem 1.2.

**Theorem 5.2.** For every \(\varepsilon > 0\) there exists \(\delta > 0\) such that the following holds. If \(G\) is a triangle-free graph with \(|V(G)| = n\) and \(|E(G)| \geq \left(\frac{1}{5} - \delta\right)n^2\) then either

(i) \(G\) is \(\varepsilon\)-close to \(C_5\),

(ii) or at least \(\delta n\) vertices of \(G\) have degree at least \(\left(\frac{2}{5} + \delta\right)n\),

(iii) or there exists an \(F \subseteq E(G)\) with \(|F| \leq \varepsilon n^2\) such that the graph \(G - F\) is bipartite.

The proof of Theorem 5.2 uses the following technical lemmas.

**Lemma 5.3.** For \(\delta > 0\) let \(G\) be a graph with \(|V(G)| = n\) and \(|E(G)| \geq \left(\frac{1}{5} - \delta\right)n^2\). Then either

(1) at least \(\delta n\) vertices have degree at least \(\left(\frac{2}{5} + \delta\right)n\),

(2) or at most \(2\sqrt{\delta n}\) vertices have degree at most \(\left(\frac{2}{5} - 2\sqrt{\delta}\right)n\).

**Proof.** Suppose that the outcome (1) does not hold. Let \(sn\) be the number of vertices that have degree at most \(\left(\frac{2}{5} - 2\sqrt{\delta}\right)n\). Then

\[
2\left(\frac{1}{5} - \delta\right) n^2 \leq 2|E(G)| = \sum_{v \in V} d(v) \leq \sum_{v \in V, d(v) \leq \left(\frac{2}{5} - 2\sqrt{\delta}\right)n} d(v) + \sum_{v \in V, \left(\frac{2}{5} - 2\sqrt{\delta}\right)n < d(v) < \left(\frac{2}{5} + \delta\right)n} d(v) + \sum_{v \in V, d(v) \geq \left(\frac{2}{5} + \delta\right)n} d(v) < sn \left(\frac{2}{5} - 2\sqrt{\delta}\right)n + (1 - s)n \left(\frac{2}{5} + \delta\right)n + \delta n^2 = \left(\frac{2}{5} - \sqrt{\delta}s + 2\delta - s\delta\right).
\]
Thus \(-2\delta < 2\delta - 2s\sqrt{\delta} - s\delta\), and

\[
s < \frac{4\delta}{2\sqrt{\delta} + \delta} < 2\sqrt{\delta},
\]

as desired.

**Lemma 5.4.** Let \(H\) be a graph on \(n\) vertices with minimum degree at least \((\frac{2}{5} - \delta)\) \(n\). Let \(\varphi\) be a surjective homomorphism from \(H\) to \(C_5\).

\[
\left(\frac{1}{5} - 3\delta\right) n \leq |\varphi^{-1}(v)| \leq \left(\frac{1}{5} + 2\delta\right) n,
\]

for every \(v \in V(C_5)\).

**Proof.** Let the vertices of \(C_5\) be labelled \(v_1, v_2, \ldots, v_5\) in order and let \(V_i := \varphi^{-1}(v_i)\) for \(i = 1, 2, \ldots, 5\). Then

\[
|V_i| + |V_{i+1}| \geq \left(\frac{2}{5} - \delta\right) n
\]  
\[
|V_{i+2}| + |V_{i+3}| \geq \left(\frac{2}{5} - \delta\right) n
\]  
\[
|V_{i+4}| + |V_i| \geq \left(\frac{2}{5} - \delta\right) n,
\]

therefore

\[
|V_i| + \left(\frac{2}{5} - \delta\right) n + \left(\frac{2}{5} - \delta\right) n \leq |V_i| + (|V_{i+1}| + |V_{i+2}|) + (|V_{i+3}| + |V_{i+4}|) \leq n,
\]

which gives us the desired upper bound.

For the lower bound summing inequalities (6)-(8) we get

\[
n + |V_i| \geq \sum_{j=1}^{5} |V_j| + |V_i| \geq 3 \left(\frac{2}{5} - \delta\right) n,
\]

which gives \(|V_i| \geq \left(\frac{1}{5} - 3\delta\right) n\), as desired.

**Proof of Theorem 5.2.** We show that \(\delta := (\varepsilon/40)^2\) satisfies the theorem for \(\varepsilon \leq 1\). We apply Lemma 5.3. The first outcome of Lemma 5.3 corresponds to the outcome (ii) of the theorem. Therefore we assume that the second outcome holds: There exists \(|S| \leq \frac{\varepsilon}{20} n\) such that every vertex in \(V(G) - S\) has degree at least \((\frac{2}{5} - \frac{\varepsilon}{10})\) \(n\) in \(G\). It follows that \(G' := G - S\) has the minimum degree at least \((\frac{2}{5} - \frac{\varepsilon}{10})\) \(n\).

Since \((\frac{2}{5} - \frac{\varepsilon}{10})\) \(n \geq \frac{3}{10} |V(G')|\), the result of Chen, Jin and Koh quoted before Theorem 2.3 implies that there exists a homomorphism \(\varphi\) from \(G'\) to \(C_5\). Let \(V(C_5) = \{v_1, v_2, \ldots, v_5\}\) and \(V_i = \varphi^{-1}(v_i)\) for each \(i = 1, 2, \ldots, 5\).

If the homomorphism \(\varphi\) is not surjective then by Lemma 2.4 the graph \(G'\) is bipartite and therefore the graph \(G^* = (V(G), E(G'))\) is also bipartite. We have \(|E(G)| - |E(G')| \leq |S| n \leq \varepsilon n^2\). Thus outcome (iii) holds. Hence we can suppose that the
homomorphism \( \varphi \) is surjective. Applying Lemma 5.4 to the graph \( G' \) with \( \delta = \varepsilon / 10 \), we get that
\[
\left( \frac{1}{5} - \frac{2\varepsilon}{5} \right) n \leq |V_i| \leq \left( \frac{1}{5} + \frac{\varepsilon}{5} \right) n. \tag{9}
\]
Let \( V := (V_1 \cup S, V_2, \ldots, V_5) \) be a partition of \( V(G) \). From (9) we have \(|V_i| - \frac{n}{5} \leq \varepsilon n\).
Let \( H_V \) be as in Definition 4.1. Then
\[
|E(G \triangle H_V)| \leq |S| n + \left( |E(H_V)| - |E(G')| \right) \leq \frac{\varepsilon}{20} n^2 + \frac{n^2}{5} - \frac{1}{2} \left( \frac{2}{5} - \frac{\varepsilon}{10} \right) \left( 1 - \frac{\varepsilon}{20} \right) n^2 
\leq \varepsilon n^2.
\]
Thus outcome (i) holds.

If outcome (i) of Theorem 5.2 holds our goal is to apply Theorem 4.7. To do that we need to ensure that a \( c \)-balanced weighting of \( C_5 \) has a \( c \)-uniform sparse half for some \( c > 0 \).

**Theorem 5.5.** A \((1/50)\)-balanced weighted graph \((C_5, \omega)\) has a \((1/30)\)-uniform sparse half.

**Proof.** Let \( \delta := 1/50 \), let \( V(C_5) = \{v_1, v_2, \ldots, v_5\} \) and let \( E(C_5) = \{v_i v_{i+2}\}_{i=1}^5 \), as in the proof of Theorem 3.1. We define a distribution on the set of halves of the graph \( C_5 \). Recall the halves \( s_i, 1 \leq i \leq 5 \) that we have defined earlier in the proof of the Theorem 3.1.

Let the probability mass of the distribution \( s \) be \( \frac{1}{5} \) on every \( s_i, i = 1, 2, \ldots, 5 \). We show that \( s \) is a \( \frac{1}{30} \)-uniform sparse half. Let us begin by showing that \( \mathbb{E}[s(e)] \geq \frac{1}{30} \omega(e) \) for every \( e \in E(C_5) \). Let \( e = (v_i, v_{i+2}) \), then

\[
\mathbb{E}[s(e)] = \frac{1}{5} \cdot \omega(v_i) \left( \frac{1}{2} - \omega(v_i) + \omega(v_{i+1}) \right).
\]

hence

\[
\mathbb{E}[s(e)] \geq \frac{1}{5} \left( \frac{1}{5} - \delta \right) \left( \frac{1}{2} - 2 \left( \frac{1}{5} + \delta \right) \right) = \frac{1}{5} \cdot \left( \frac{1}{5} - \delta \right) \left( \frac{1}{10} - 2\delta \right).
\]

On the other hand,
\[
\omega(e) = \omega(v_i) \cdot \omega(v_{i+2}) \leq \left( \frac{1}{5} + \delta \right)^2.
\]

Thus it suffices to show that
\[
\frac{1}{5} \left( \frac{1}{5} - \delta \right) \left( \frac{1}{10} - 2\delta \right) \geq \frac{1}{30} \left( \frac{1}{5} + \delta \right)^2,
\]
which can be easily verified. It is shown in the proof of Theorem 3.1 that \( \mathbb{E}[s(E(G))] \leq \frac{1}{30} \). Thus \( s \) is a \( \frac{1}{30} \)-uniform sparse half of \( G \), as claimed.

We need a final technical lemma.
Lemma 5.6. For every \( \delta > 0 \) there exists a \( \gamma > 0 \) such that if \( G \) is a graph on \( n \) vertices with at least \( \left( \frac{1}{5} - \gamma \right) n^2 \) edges and at least \( \delta n \) vertices of degree at least \( \left( \frac{2}{5} + \delta \right)n \) then

\[
\frac{1}{n} \sum_{v \in V(G)} d^2(v) \geq \left( \frac{2}{5} \right)^2.
\]

Proof. Suppose that \( G \) contains \( \alpha n \) vertices of degree at most \( \frac{2}{5}n \) and \( \beta n \) vertices of degree at least \( \left( \frac{2}{5} + \delta \right)n \). We may assume that the average degree of \( G \) is less than \( \frac{2}{5}n \), as otherwise lemma clearly holds. Thus

\[
\frac{2}{5}n > (1 - \alpha - \beta) \frac{2}{5}n + \beta \left( \frac{2}{5} + \delta \right)n,
\]

hence \( \alpha > \frac{5}{2} \beta \delta \geq \frac{5}{2} \delta^2 \). We have

\[
n \sum_{v \in V(G)} d^2(v) - \left( \sum_{v \in V(G)} d(v) \right)^2 = \frac{1}{2} \sum_{u \neq v} (d(u) - d(v))^2 \geq \alpha \delta \left( \frac{2}{5} + \delta - \frac{2}{5} \right)^2 n^4 > \frac{5}{2} \delta^5 n^4.
\]

Hence

\[
\frac{1}{n} \sum_{v \in V(G)} d^2(v) \geq \frac{1}{n^2} \left( \sum_{v \in V(G)} d(v) \right)^2 + \frac{5}{2} \delta^5 n^2 \geq 4 \left( \frac{1}{5} - \gamma \right)^2 n^2 + \frac{5}{2} \delta^5 n^2 \geq \frac{4}{25} n^2 + \left( \frac{5}{2} \delta^3 - \frac{8}{5} \gamma \right) n^2 \geq \frac{4}{25} n^2,
\]

if we choose \( \gamma = \frac{25}{16} \delta^3 \). \( \square \)

Proof of Theorem 1.2. By Theorem 5.5, \( \alpha = 1/50 \) satisfies the conditions in the statement of Theorem 4.7 for \( H := C_5 \). Thus by Theorem 4.7 there exists \( 0 < \varepsilon \leq 1/50 \) such that every triangle-free graph \( G \) that is \( \varepsilon \)-close to \( C_5 \) has a sparse half. Let \( \delta \) be such that Theorem 5.2 holds, and finally let \( \gamma \) be such that Lemma 5.6 holds. We show that Theorem 1.2 holds for this choice of \( \gamma \).

We distinguish cases based on the outcome of Theorem 5.2 which holds for \( G \).

Case (i): If \( G \) is \( \varepsilon \)-close to \( C_5 \) then the theorem holds by the choice of \( \varepsilon \).

Case (ii): Now suppose that at least \( \delta n \) vertices of \( G \) have degree at least \( \left( \frac{2}{5} + \delta \right)n \). If \( \Delta(G) \geq \left( \frac{2}{5} + \frac{1}{135} \right)n \) then Theorem 5.1 (b) implies that there is a sparse half in \( G \). Therefore we assume that \( \Delta(G) < \left( \frac{2}{5} + \frac{1}{135} \right)n \). By Lemma 5.6 and the choice of \( \gamma \) we have \( \frac{1}{n} \sum_{v \in V(G)} d^2(v) \geq \left( \frac{2}{5} \delta^3 \right)^2 \). Hence Theorem 5.1 (a) implies that there is a sparse half in \( G \).

Case (iii): Lastly, suppose there exists an \( F \subseteq E(G) \) with \( |F| \leq \varepsilon n^2 \) such that the graph \( G' = (V(G), E(G) - F) \) is bipartite with bipartition \( (U, V) \). Then either \( |U| \geq \frac{n}{2} \) or \( |V| \geq \frac{n}{2} \). Without loss of generality suppose \( |U| \geq \frac{n}{2} \). The set \( U \) is independent in \( G' \), while in \( G \) it might not be, but \( |E(G[U])| \leq |F| \leq \varepsilon n^2 \leq \frac{n^2}{50} \). Hence \( U \) supports a sparse half in graph \( G \). \( \square \)
6. Neighbourhood of the Petersen Graph

The uniform blow-up of the Petersen graph, is an extremal example for Conjecture 1.1, that is every set of \( \lfloor n/2 \rfloor \) vertices spans at least \( n^2/50 \) edges. Here we show that the Conjecture 1.1 holds for any graph that is close to a uniform blow-up of Petersen graph in the sense of Definition 4.1. By Theorem 4.7 it is enough to show that sufficiently balanced blow-ups of \( P^* \) have uniform sparse halves.

![Graph](image)

Figure 4: The graph \( P^* \).

**Lemma 6.1.** Let the weighted graph \((P^*, \omega)\) be \((1/500)\)-balanced then it has a \( \frac{1}{80} \)-uniform sparse half.

**Proof.** Let \( \delta := 1/500 \) and the vertices of \( P^* \) be labeled as on Figure 4, where \( V := V(P) = \{v_1, v_2, \ldots, v_{10}\} \) and \( W := V(P^*) - V(P) = \{w_1, w_2, \ldots, w_5\} \). We define a collection \( \{s_{i,j}\}_{i \in [5], j \in [4]} \) of halves of \((P^*, \omega)\). Fix \( i \in [5] \) and consider the vertex \( w_i \). Let \( M_i := V - N(w_i) \) and not that \( M_i \) induces a matching of size three. Choose vertices \( \{v_{i1}, v_{i2}, v_{i3}\} \subseteq V(M_i) \) such that they are independent and there exist unique \( w_{ij} \neq w_i \) such that every \( v \in V(M_i) \setminus \{v_{i1}, v_{i2}, v_{i3}\} \) is adjacent to \( w_{ij} \). Note that for every \( i \) there...
exist four such choices of \( \{v_{i_1}, v_{i_2}, v_{i_3}\} \), fix one of them and assign

\[
s_{i,j}(w_q) = \begin{cases} 
\omega(w_q), & \text{if } q = i, \\
\frac{1}{4}\omega(w_q), & \text{if } q = i_j, \\
0, & \text{otherwise.}
\end{cases}
\]

and

\[
s_{i,j}(v_k) = \begin{cases} 
0, & \text{if } v_k \notin V(M_i) \\
\omega(v_k), & \text{if } k = j_1, j_2, j_3, \\
\left(\frac{1}{3} - \left(\frac{1}{2} - (\omega(v_{i_1}) + \omega(v_{i_2}) + \omega(v_{i_3}) + \omega(w_i) + \frac{1}{4}\omega(w_{i_3}))\right)\right) = \frac{1}{3}\omega(v_k), & \text{otherwise.}
\end{cases}
\]

It is easy to check that every \( s_{i,j} \) is a half. Let \( s \) be a distribution concentrated on \( \{s_{i,j}\}_{i,j} \in [5] \times [5] \) with each of the halves having the same probability \( 1/20 \). We show that \( s \) is a (1/80)-uniform sparse half for \( \omega \).

First, we show that \( \mathbb{E}[s(e)] \geq \frac{1}{80} \cdot \omega(e) \) for every edge \( e \in E(P) \). It can be routinely checked that for our choice of \( \delta \) one has

\[
\frac{1}{3}\left(\frac{1}{2} - (\omega(v_{i_1}) + \omega(v_{i_2}) + \omega(v_{i_3}) + \omega(w_i) + \frac{1}{4}\omega(w_{i_3}))\right) \geq \frac{1}{3}\omega(v_k),
\]

for every \( v_k \in V(M_i) \setminus \{v_{i_1}, v_{i_2}, v_{i_3}\} \). If both ends \( v, v' \) of \( e \in E(P) \) lie in \( V \) then, using (10), we have

\[
\mathbb{E}[s(e)] \geq \frac{4}{20} \cdot \frac{1}{3}\omega(v)\omega(v') = \frac{1}{15}\omega(e) \geq \frac{1}{80}\omega(e).
\]

If \( e \) joins \( v \in V \) and \( w \in W \) then

\[
\mathbb{E}[s(e)] \geq \frac{3}{20} \cdot \frac{1}{4}\omega(w_i) \cdot \frac{1}{3}\omega(v_j) = \frac{1}{80}\omega(e).
\]

It remains to prove that \( \mathbb{E}[s(E(G))] \leq \frac{1}{50} \). Note that,

\[
s_{i,j}(E(G)) = \left(\frac{1}{2} - (\omega(v_{i_1}) + \omega(v_{i_2}) + \omega(v_{i_3}) + \omega(w_i) + \frac{1}{4}\omega(w_{i_3}))\right) \times \left(\frac{1}{4}\omega(w_{i_3}) + \frac{1}{3}(\omega(v_{i_1}) + \omega(v_{i_2}) + \omega(v_{i_3}))\right).
\]

We finish the proof using the following technical lemma, the proof of which is included in the appendix.

**Lemma 6.2.** Suppose given are \( x_1, x_2, \ldots, x_{10}, y_1, y_2, \ldots, y_5 \) reals and

\[
L(y_i) = \{x_{i+1}, x_{i+2}, x_{i+4}, x_{i+5}, x_{i+8}, x_{i+9}\},
\]

for each \( 1 \leq i \leq 5 \) such that \( 0 \leq x_i \leq 1 \), \( 0 \leq y_j \leq 1 \) and \( \sum_{i=1}^{10} x_i + \sum_{j=1}^{5} y_j = 1 \). If there exists some \( 0 < \delta \leq \frac{1}{50} \) such that \( x_i \geq \frac{1}{10} - \delta \) for each \( i = 1, 2, \ldots, 10 \) then

\[
\sum_{x_{i_1}, x_{i_2}, x_{i_3} \in L(y_{j_1}), x_{i_1}, x_{i_2}, x_{i_3} \in L(y_{j_2})} \left(\frac{1}{2} - x_{i_1} - x_{i_2} - x_{i_3} - y_i - \frac{1}{4}y_j\right) \left(\frac{1}{4}y_j + \frac{1}{3}(x_{i_1} + x_{i_2} + x_{i_3})\right) \leq \frac{2}{5}.
\]

\[
(11)
\]
It is easy to see that $E[s(E(G))] \leq \frac{1}{50}$ follows from Lemma 6.2 applied with $x_i := \omega(v_i)$ and $y_j := \omega(w_j)$. Thus $s$ is a $1/80$-uniform sparse half, as claimed. □

As promised, Lemma 6.1 implies the main theorem of this section.

**Theorem 6.3.** There exists $\delta > 0$ such that any triangle-free graph $G$ on $n$ vertices which is $\delta$-close to the Petersen graph $P$ has a sparse half.

**Proof.** The theorem follows from Theorem 4.7, as the Petersen graph satisfies the requirements of that theorem with $\alpha = 1/500$ by Lemma 6.1. □

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7. Appendix

Proof of Lemma 3.3

Suppose the Lemma is false. Then for each \( i = 1, 2, \ldots, 8 \)
\[
\frac{1}{2} \left( \frac{1}{2} - x_i - x_{i+1} - x_{i+2} \right) \left( x_i + x_{i+2} \right) + \frac{1}{4} \left( \frac{1}{2} - x_i - x_{i+1} - x_{i+2} \right)^2 > \frac{1}{50}. \quad (12)
\]

Summing up these inequalities over all \( i = 1, 2, \ldots, 8 \) we get that
\[
\frac{8}{50} < \sum_{i=1}^{8} \frac{1}{2} \left( \frac{1}{2} - x_i - x_{i+1} - x_{i+2} \right) \left( x_i + x_{i+2} \right) + \sum_{i=1}^{8} \frac{1}{4} \left( \frac{1}{2} - x_i - x_{i+1} - x_{i+2} \right)^2
\]
\[
= \frac{1}{2} \sum_{i=1}^{8} x_i - \sum_{i=1}^{8} (x_i + x_{i+1} + x_{i+2}) x_i + \frac{1}{4} \cdot \frac{1}{4} \cdot 8 - \frac{1}{4} \sum_{i=1}^{8} (x_i + x_{i+1} + x_{i+2})^2
\]
\[
+ \frac{1}{4} \sum_{i=1}^{8} (x_i + x_{i+1} + x_{i+2})^2
\]
\[
= \frac{1}{2} - \sum_{i=1}^{8} (x_i + x_{i+1} + x_{i+2}) x_i + \frac{1}{2} - \frac{3}{4} + \frac{1}{4} \sum_{i=1}^{8} (x_i + x_{i+1} + x_{i+2})^2
\]
\[
= \frac{1}{4} - \frac{1}{4} \sum_{i=1}^{8} x_i^2 - \frac{1}{2} \sum_{i=1}^{8} x_i x_{i+2}. \quad (13)
\]

Let us find the maximum value under the conditions of the lemma. To find the maximum value of the expression in (13), we need to find the minimum value of
\[
S := \frac{1}{4} \sum_{i=1}^{8} x_i^2 + \frac{1}{2} \sum_{i=1}^{8} x_i x_{i+2}.
\]

Claim 7.1. For every \( 1 \leq i \leq 8 \)
\[
x_{i+1} + x_{i+2} + x_{i+3} < 0.394
\]

Proof. By inequality (12)
\[
\frac{1}{2} \left( \frac{1}{2} - (x_{i+1} + x_{i+2} + x_{i+3}) \right) \left( x_{i+1} + x_{i+3} \right) + \frac{1}{4} \left( \frac{1}{2} - (x_{i+1} + x_{i+2} + x_{i+3}) \right)^2 > \frac{1}{50}.
\]

Let \( \alpha = x_{i+1} + x_{i+2} + x_{i+3} \). Then \( x_{i+1} + x_{i+3} = \alpha - x_{i+2} \leq \alpha - \frac{1}{14} \). Therefore
\[
\frac{1}{2} \left( \frac{1}{2} - \alpha \right) \left( \alpha - \frac{1}{14} \right) + \frac{1}{4} \left( \frac{1}{2} - \alpha \right)^2 = -\frac{\alpha^2}{4} + \frac{\alpha}{28} + \frac{5}{112} > \frac{1}{50},
\]
which reduces to the inequality
\[
\frac{\alpha^2}{4} - \frac{\alpha}{28} - \frac{69}{2800} < 0.
\]

This quadratic inequality gives us the desired \( \alpha < 0.393412 \) bound. \( \square \)
It is easy to check that the following claim is true.

Claim 7.2.

\[ S \geq \frac{1}{2} \sum_{i=1}^{4} z_i^2 + \sum_{i=1}^{4} z_i z_{i+2}, \]

where \( z_i = (x_i + x_{i+4})/2 \) for all \( 1 \leq i \leq 4 \).

Claim 7.3. For every \( 1 \leq i \leq 4, z_i > 0.106 \).

Proof. For every \( 1 \leq i \leq 4 \) we have

\[ 1 = (x_i + x_{i+4}) + (x_{i+1} + x_{i+2} + x_{i+3}) + (x_{i+5} + x_{i+6} + x_{i+7}) \overset{(7.1)}{<} 2z_i + 2 \cdot 0.394, \]

therefore \( z_i > 0.106 \).

Let \( \beta = z_1 + z_3 \). Then

\[
S \geq \frac{1}{2}(z_1 + z_3)^2 + \frac{1}{2}(z_2 + z_4)^2 + z_1 z_3 + z_2 z_4
\]

\[
= \frac{1}{2} \beta^2 + \frac{1}{2} \left( \frac{1}{2} - \beta \right)^2 + z_1 z_3 + z_2 z_4
\]

\[
\geq \frac{1}{2} \beta^2 + \frac{1}{2} \left( \frac{1}{2} - \beta \right)^2 + 0.106 \cdot (\beta - 0.106) + 0.106 \left( \frac{1}{2} - \beta - 0.106 \right)
\]

\[
= \beta^2 - \frac{1}{2} \beta + 0.155528
\]

\[
> 0.093
\]

The last two inequalities hold because for fixed \( \beta \) the expression \( z_1 z_3 + z_2 z_4 \) achieves its minimum value when \( z_1 = z_2 = 0.106, z_3 = \beta - 0.106 \) and \( z_4 = \frac{1}{2} - \beta - 0.106 \). The expression \( \beta^2 - \frac{1}{2} \beta + 0.155528 \) achieves its minimum for \( \beta = \frac{1}{4} \). It follows from the inequality above that

\[
\frac{1}{4} - \frac{1}{4} \sum_{i=1}^{8} x_i^2 - \frac{1}{2} \sum_{i=1}^{8} x_i x_{i+2} = \frac{1}{4} - S < \frac{8}{50},
\]

a contradiction that finishes the proof of the lemma.

Proof of Lemma 3.4

Suppose the lemma is false. Then for all \( 1 \leq i \leq 11 \) we have

\[
\frac{1}{2} \left( \frac{1}{2} - (x_{i+1} + x_{i+2} + x_{i+3} + x_{i+4}) \right) (x_{i+1} + x_{i+4})
\]

\[
+ \frac{1}{4} \left( \frac{1}{2} - (x_{i+1} + x_{i+2} + x_{i+3} + x_{i+4}) \right)^2 > 1/50.
\]
Summing these inequalities for $1 \leq i \leq 11$ we obtain

$$\frac{11}{50} < \sum_{i=1}^{11} \frac{1}{2} \left( \frac{1}{2} - (x_{i+1} + x_{i+2} + x_{i+3} + x_{i+4}) \right) (x_{i+1} + x_{i+4})$$

$$+ \sum_{i=1}^{11} \frac{1}{4} \left( \frac{1}{2} - (x_{i+1} + x_{i+2} + x_{i+3} + x_{i+4}) \right)^2$$

$$= \frac{1}{4} \sum_{i=1}^{11} (x_{i+1} + x_{i+4}) - \sum_{i=1}^{11} (x_{i+1} + x_{i+2} + x_{i+3} + x_{i+4})x_{i+1}$$

$$+ \frac{11}{16} - \frac{1}{4} \sum_{i=1}^{11} (x_{i+1} + x_{i+2} + x_{i+3} + x_{i+4}) + \frac{1}{4} \sum_{i=1}^{11} (x_{i+1} + x_{i+2} + x_{i+3} + x_{i+4})^2$$

$$= \frac{3}{16} - \sum_{i=1}^{11} (x_{i+1} + x_{i+2} + x_{i+3} + x_{i+4})x_{i+1} + \frac{1}{4} \sum_{i=1}^{11} (x_{i+1} + x_{i+2} + x_{i+3} + x_{i+4})^2$$

$$= \frac{3}{16} + \frac{1}{2} \sum_{i=1}^{11} x_i(x_{i+1} - x_{i+3})$$

$$\leq \frac{3}{16} + \frac{1}{2} \left( \sum_{i=1}^{11} x_ix_{i+1} - \frac{11}{196} \right). \quad (14)$$

We claim that

$$f(x_1, \ldots, x_{11}) := \sum_{i=1}^{11} x_ix_{i+1} \leq \frac{77}{784}.$$ 

Indeed, consider any pair $(x_i, x_j)$, such that $j \not= i \pm 1$, let $\alpha = x_i + x_j$ is fixed. Note that $f$ is linear as a function of $(x_i, x_j)$. Therefore $f(x_i, x_j)$ achieves its maximum value on the region $R := \{0 \leq x_i \leq 1/14 \text{ for } 1 \leq i \leq 14, \sum_{i=1}^{11} 1x_i = 1\}$, when $x_i = \frac{1}{14}$ and $x_j = \alpha - \frac{1}{14}$, or when $x_j = \frac{1}{14}$ and $x_i = \alpha - \frac{1}{14}$. Thus $f$ attains its maximums on $R$ when all variables are equal $\frac{1}{14}$ except possibly two of them whose indices are consecutive, without loss of generality, say $x_{10}$ and $x_{11}$. It is easy to see that the maximum is achieved for $x_{10} = x_{11} = \frac{5}{28}$ and is equal to $\frac{77}{784}$, as claimed.

Thus (14) implies

$$\frac{11}{50} < \frac{3}{16} + \frac{1}{2} \left( \sum_{i=1}^{11} x_ix_{i+1} - \frac{11}{196} \right) \leq \frac{3}{16} + \frac{1}{2} \cdot \left( \frac{77}{784} - \frac{11}{196} \right) = \frac{327}{1568} < \frac{11}{50},$$

a contradiction that finishes the proof.
Proof of Lemma 6.2

Let $Y := \sum_{i=1}^{5} y_i$. Summing inequalities (11) over all $i, j$ such that $i \neq j$ and $x_{i,j_1}, x_{i,j_2}, x_{i,j_3} \in L(y_i) \cap L(y_j)$. We get

$$\sum_{i,j \neq i} \left( \frac{1}{2} - (x_{i,j_1} + x_{i,j_2} + x_{i,j_3} + y_i + \frac{1}{4} y_j) \right) \left( \frac{1}{4} y_j + \frac{1}{3} (x_{i,j_1} + x_{i,j_2} + x_{i,j_3}) \right)$$

$$= \frac{1}{8} \sum_{i,j \neq i} y_j - \frac{1}{4} \sum_{i,j \neq i} y_j (x_{i,j_1} + x_{i,j_2} + x_{i,j_3}) + \frac{1}{6} \sum_i (x_{i,j_1} + x_{i,j_2} + x_{i,j_3})$$

$$- \frac{1}{3} \sum_{i,j} (x_{i,j_1} + x_{i,j_2} + x_{i,j_3})^2 - \frac{1}{3} \sum_{i,j} \left( y_i + \frac{1}{4} y_j \right) (x_{i,j_1} + x_{i,j_2} + x_{i,j_3}) - \frac{1}{4} \sum_{i,j} y_j \left( y_i + \frac{1}{4} y_j \right)$$

$$\leq \frac{1}{2} Y - 3Y \left( \frac{1}{10} - \delta \right) + (1 - Y) - \frac{36(1 - Y)^2}{60} - 5Y \left( \frac{1}{10} - \delta \right)$$

$$= \frac{2}{5} - \frac{Y}{10} - \frac{3}{5} Y^2 + 8\delta Y$$

$$\leq \frac{2}{5},$$

since $\delta \leq \frac{1}{90}$. 

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