"Non-1/r" Newtonian Gravitation and Stellar Structure

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Some analytical solutions for spherical-symmetrical equilibrium configurations (planets/stars) in Newtonian Theory of Gravitatio (NTG) with deviations from 1/r law are discussed. Stability of star against the first-order phase transition is particularly influenced by deviations of degree in power-law potential.

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I. INTRODUCTION

In the classical NTG, with the point-mass potential

$$U(r) = \frac{Gm}{r},$$

the equation of the spherical-symmetric equilibrium states reads:

$$\frac{1}{\rho(r)} \frac{dP(r)}{dr} = -\frac{Gm(r)}{r^2}.$$  \hspace{1cm} (2)

Here, as usual, \(G\) stands for constant of NTG, \(P(r), \rho(r)\) are local pressure and density of stellar matter, at distance \(r\) from the center of star, and \(m(r)\) is mass inside the sphere of radius \(r\) ("r-sphere"). Provided EOS \(P(\rho)\) is known, the Eq. (2) is ODE and is degenerate in the sense, that any deviation from standard law (1) spoils the fundamental characteristic of spherical-symmetric stars - dependence of local gravitational force, at radius \(r\), only on the mass inside "r-sphere". That is any deviation of 1/r law leads, in principle, to the enormous complexity of problem of the spherically-symmetric equilibrium states. Instead of ODE one has to deal with integro-differential equation (IDE), see next section.

In [1] authors discuss some recent schemes of compactification leading to the generalized form of the point-mass potential,

$$U(r) = \frac{Gm}{r}(1 + \alpha e^{-r/\lambda}),$$

where \(\alpha\) and \(\lambda\) are constants. In the celestial mechanics, the various "non-Newtonian" or "non-gravitational" forces have been considered quite a long ago [2]. In this note I consider only the problem of stellar structure in NNG (Non-Newtonian Gravitation). As nobody knows the exact form of "non-Newtonian" law, it is reasonable to consider first the various simple cases of potential \(U(r) = Gm \ast f(r)\). Here I consider "power-law case",

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\[ f(r) = 1 / r^{1+\alpha}, \]  

where \( \alpha \) is constant. Some manifestations of this law in stellar structure are rather dramatic that may help to judge in favor or against it. Despite this "applicational" aspect it’s certainly worth by itself considering the problem of stellar structure with non-1/r gravitational potential.

At first, I present the hydrostatic spherical-symmetric equilibrium equation in general case.

**II. EQUATIONS OF STELLAR STRUCTURE IN NNG**

If \( f(x) \) is "generalized Newtonian" law, then the potential of thin spherical shell is as follows:

\[ u_{shell}(r, a) \, da = 2 \pi G \rho(a) a^2 \, da \int_{|r-a|}^{a+r} \frac{f(x)x}{r \, a} \, dx. \]  

(5)

Here, \( a \) and \( da \) are radius and "thickness" of infinitesimally thin shell, \( \rho(a) \) is matter density at radius \( a \), and \( r \) is distance from the center of spherical shell.

Note, that only at \( f(x) = 1/x^1 \) (and, of course at \( f(x) = 1/x^0 \), when point-mass potential itself is constant!), the potential inside the spherical shell is constant (and gravitational force is zero). It’s worth mentioning that both laws (3) and (4) give the first-order correction leading to necessity of IDE, while the zeroth-order term coincides with 1/r law.

The potential \( U_{sphere}(\rho, r, R) \), at distance \( r \), of a sphere of radius \( R \) is:

\[ U_{sphere}(\rho, r, R) = \int_{0}^{R} u_{shell}(r, a) \, da. \]  

(6)

The Eq. (6) is valid both inside and outside the sphere.

The equation of the hydrostatic equilibrium of spherical-symmetric star reads:

\[ \frac{1}{\rho(r)} \frac{dP(r)}{dr} = \frac{dU_{sphere}(\rho(r), r, R)}{dr}. \]  

(7)

The right side of this equation is a double integral, and in general is of much more complicated form than in the standard \( f(r) = 1/r \) case, Eq. (2). Provided EOS \( P(\rho) \) is known, Eq. (7) is IDE for density function \( \rho(r) \).

As at any arbitrary EOS \( P(\rho) \) and non-1/r law it’s apparently impossible to solve the problem (5-7), it’s a good idea to consider some simple cases first. In the next section, I present the spherically-symmetric model of a homogeneous incompressible liquid, that corresponds to a polytropic index \( n = 0 \) in the polytropic EOS of the form \( P(\rho) = K \rho^{1+n} \).

**III. HOMOGENEOUS SPHERE**

At the point-mass potential of form (4), the gravitational potential of homogeneous sphere of radius \( R \) and density \( \rho = \text{const} \) is as follows:

\[ aN \equiv N - \alpha; \quad t \equiv r/R; \]
Here upper and lower signs correspond to $t < 1$ and $t > 1$ respectively. Values of potential at the center and the surface of the homogeneous sphere are:

$$U(t = 0) = 4\pi G \rho R a^2 / a 2; \quad U(t = 1) = 2\pi G \rho (2R)^a / (a 2 a 3).$$

(9)

The total gravitational potential energy of homogeneous sphere is:

$$W = -\pi^2 G \rho^2 (2R)^a / (a 2 a 3 a 5).$$

(10)

Solution of equation (7) in this case leads to the following relation for central pressure:

$$P_c = \rho * [U(t = 0) - U(t = 1)] = 2\pi G \rho^2 R^a (2 - 2a^2 / a 3) / a 2.$$

(11)

In Fig. 1, some relevant curves are presented. Of course, in "real" cases it appears that $|\alpha| << 1$, so the Fig. 1 is presented here only for pure illustrative purposes. We note that nothing extraordinary interesting happens with the gravitational potential of homogeneous sphere by broken $1/r^{1+\alpha}$ point-mass potential. Vertical scales are somewhat arbitrary as for different values of $\alpha$ dimensionality of constants are different, see Eqs. (8-11).

It may be noted, that in spite of "enormous", in principle, complexity of the spherical-symmetric equilibrium equations in NNG, in reality it is difficult to point out any macro-effect of "micro"-values of $\alpha$.

By the way, we notice a rather common opinion that in astronomical ("cosmic") scales, even small deviations from $1/r$ law may lead to some macro effects. For example, external parts
(outside the "R-sphere") of homogeneous dark-matter or any other background matter are usually assumed as non-influencing the equilibrium and stability of spherical structure with given radius $R$ [3].

It’s seems worth reconsidering this problem with account of deviations from $1/r$ law. Two potentially important cases may be the models with adiabatic index of EOS close to 4/3, suggested sometimes as model of supermassive stars (quasars), and the isothermic configurations suggested as model of inter- (or proto-) stellar clouds or other cosmic structures [9].

Here I consider one rather interesting effect of NNG. In NTG (as well as in GR) the problem of equilibrium and stability of a star (planet) with first-order phase transition (PT1) exists apparently since pioneer works by Ramsey [4].

In "classical" NTG, the elegant and quite surprising result is that if PT1 occurs at the star’s center with $q > 3/2$, $q = \rho_2/\rho_1$ being the ratio of new-to-old phase densities, the star undergoes a stability loss irrespective of EOS!

This "exact" value, $3/2$, has apparently its origin from another two exact values: 1, power of $r$ in point-mass potential law, and 3, dimension of space. It’s the place here to illustrate the first part of this assertion, for the simple model.

The simplest model of star (planet) with PT1 is a two-zone model with constant densities $\rho_1$ and $\rho_2$ in envelope and core respectively.

**IV. TWO-ZONE MODEL WITH PT1**

In this model density distribution is a piece-wise function: $\rho(0 < r < r_n) = \rho_2 = \text{const}$, and $\rho(R > r > r_n) = \rho_1 = \text{const}$, with $q = \rho_2/\rho_1 > 1$; here $r_n$ and $R$ are radii of core and of total star. At the core-envelope boundary, the pressure is constant, $P(r_n) = P_0 = \text{const}$.

The potential in envelope may be considered as a sum of internal potential of homogeneous sphere of density $\rho_1$ and radius $R$ and of external potential of homogeneous sphere of density $\rho_2 - \rho_1$ and radius $r_n$:

$$U_{\text{env}}(\rho_1, r, R) = U_{\text{int, sphere}}(\rho_1, r, R) + U_{\text{out, sphere}}(\rho_2 - \rho_1, r, r_n),$$

where two functions in the right site are given in Eq. (8). Hydrostatic equilibrium equation (7) has the solution, in the envelope, $r > r_n$:

$$P(r) = \rho_1 [U_{\text{env}}(\rho_1, r, R) - U_{\text{env}}(\rho_1, R, R)];$$

$$P_0 = \rho_1 [U_{\text{env}}(\rho_1, r_n, R) - U_{\text{env}}(\rho_1, R, R)].$$

The last expression is "initial condition" for the hydrostatic equilibrium equation (7) and determines the total radius $R$ as function of radius of core $r_n$. Total mass of model is:

$$M = 4 \pi G [\rho_1 R^3 + (\rho_2 - \rho_1)r_n^3]/3.$$  

On the other hand, $P_0$ is central pressure for "initial" homogeneous configuration with "initial values" of total mass $M_0$, total radius $R_0$, which are defined as in Eq. (11) with $R = R_0$.
\[ P_0 = 2\pi G\rho^2 R_0^2 (2 - 2a^2/a3)/a2; M_0 = 4\pi G \rho_1 R_0^3/3. \] (16)

These relations allow to exclude \( P_0 \) from Eqs. (13-15) and to express \( R \) and \( r_n \) in units of \( R_0 \), and also \( M \) in units of \( M_0 \).

According to the static criterion [5], the dependence of a total mass of equilibrium states on central pressure determines the stability of models. That is the branch of \( M_{eq}(P_c) \) with positive derivative \( dM_{eq}/dP_c \) presents the stable equilibrium states while the branch with negative \( dM_{eq}/dP_c \) presents unstable equilibrium states.

In our case we can use \( r_n \) (or even \( r_n/R \)) as independent variables as they both are monotonic functions of \( P_c \).

A. Classic NTG Case

For convenience, I present here, very briefly, some relevant formulae (and results) in NTG. Although they are known since quite a long ago [3] they have been rederived in literature sometimes, e.g. [6].

Instead of (13-16) we have (all variables are expressed in relevant ”initial” values):

\[ M = ((q - 1)x^3 + 1)R^3, \quad x = r_n/R; \] (17)

\[ R^3 - [1 - (2q - 3)r_n^2]R - 2(q - 1)r_n^3 = 0; \] (18)

\[ R = [1 + (2q - 3)x^2 - 2(q - 1)x^3]^{-1/2}; \] (19)

These equations allow a full analytical treatment of the problem. In particular:
\( dM(x)/dx \) may be negative only at \( q > 3/2 \);
\( dM(x)/dx = 0 \) at \( x \) determined by the equation
\( (q - 1)^2x^4 + 4(q - 1)x - (2q - 3) = 0; \)
the last equation has solutions \( x < \sqrt{2} - 1; \)
at mass minima, \( M = R^2 \).

B. The \( 1/r^{1+\alpha} \) Case

As relevant formulae are rather cumbersome in general case I’ll not rewrite them from Eqs. (8-16), and only consider a situation with small cores. For small values of a relative radius of core, \( x = r_n/R << 1 \), we have expansion up to \( x^3 \):

\[ R = \left( 1 + \frac{(3 - \alpha)(2 - \alpha)}{6 - 2\alpha - 2^{2-\alpha}} \frac{2^{2-\alpha}(q - 1)}{(2 - \alpha)(3 - \alpha)} x^{2-\alpha} - \frac{(1 - \alpha)}{3} x^2 - \frac{2(q - 1)}{3} x^3 \right)^{-1/(2-\alpha)}. \] (20)

This equation is valid for arbitrary \( \alpha \) and for small \( x \). At the limit \( \alpha \to 0 \) this equation coincides with exact expression (19) which is valid for any \( x \), in NTG case, providing \( q_{cri} = 3/2 \). However the case of non-zero \( \alpha \)'s differs from the classic case qualitatively. That is we don’t have, at small values of \( \alpha \), some small correction to critical value of PT1 in NTG \( q_{cr} = 3/2 \).
Instead, at $\alpha > 0$, the smallest power of $x$ in Eq. (20) is $a^2 < 2$, the corresponding coefficient is negative for any $q > 1$, therefore we have $dR/dx < 0$. As evident from (15), in $M$ ”additional” term $\propto x^3$, that is $\Delta M \propto \Delta R \propto (x^{2-\alpha} \text{ or } x^2)$, that provides the continuity of derivative $dM/dR$ along the curve of equilibrium states [7].

In other words, at $\alpha > 0$ (more steeper dependence of potential as compared with $1/r$ law of NTG), we have $q_{cr} = 1$, that is for small cores there is a situation of the absolute instability of star against PT1; while at $\alpha < 0$, there is reverse situation of the absolute stability of star against PT1.

For the larger (but still small enough) dimensions of core there takes place a restoration of the normal behavior of $M(x)$ curves, that is, $dM/dx > 0$ at $q < 3/2$ and $dM/dx < 0$ at $q > 3/2$, see Fig. 2.

FIG. 2. Dependence of the total mass of star with first-order phase transition on the relative radius of new-phase core, in NNG. Abscissae are $10^6 r_n/R$, ordinates are $10^{13} (M - 1)$. The upper panel corresponds to the case of positive $\alpha = .01$. Curves (starting from upper one) are shown for three values of $q = 1.44, 1.445$, and 1.45, respectively. Note, that, in principle, at $\alpha > 0$, all curves at $x = 0$ point have negative derivative, that is there is a minor region of small-core instability for any $q > 1$.

The reverse situation of negative $\alpha = -.01$ is shown in the lower panel. Three curves (starting from upper one) correspond to values of $q = 1.555, 1.554$, and 1.553. Here, all curves at $x = 0$ point have positive derivative, that is there is a minor region of small-core stability for any $q > 1$.

The region of ”abnormal” behavior of $dM/dx$ may be very small, a value of $x$, at which
again $R = 1$ and $M = 1$ is:

$$x_1 = \left( \frac{32^{\alpha^2}(q-1)}{(2-\alpha)(3-\alpha)(1+\alpha)} \right)^{1/\alpha}. $$

(21)

It’s worth mentioning that $x_1$ is very sensitive function of $q$ at small enough values of $\alpha$. It seems that despite the smallness of real values of $\alpha$, and therefore the smallness of the "abnormal" stability region, this influence of the power-law potential on stability of a star is potentially of some major interest.

V. CONCLUSION

The deviations from $1/r$ law of NTG revealing itself in changing the value power of $r$ lead to a dramatic change of the stellar stability against the first-order phase-transition in the center of star, with small new-phase cores. The steeper dependence of point-mass potential on $r$ leads to the absolute instability of star at smaller core dimensions, while the lesser (than reciprocal) dependence of the point-mass potential leads to the "absolute stability" of star. Though for "real" values of $\alpha$ this phenomenon takes place only at the very small dimensions of new-phase cores, nevertheless it may, in principle, serve as some means for proving or rejecting such form of deviations from $1/r$-law. One may note, for example, the microcollapses of neutron star revealing itself in pulsar timing "faults" [8]. Another possibility may occur in the larger astronomic structures, such as IMC [9] and supermassive stars [3, 5].

[1] A. Kehagias and K. Sfetsos, astro-ph/9905417; E.G. Floratos and G.K. Leontaris, hep-ph/9906238; E. Fischbach and D.K Krause, hep-ph/9906240.
[2] G.N. Duboshin. Nebesnaya mekhanika. Moskva, Nauka, 1964.
[3] G. McLaughlin and G. Fuller, Ap. J. 456 (1996) 71; G.S. Bisnovatyi-Kogan, astro-ph/9712323.
[4] W.H. Ramsey, MNRAS 110 (1950) 325; 113 (1951) 427; M.J. Lighthill, MNRAS 110 (1950) 339; see also Z.F. Seidov, Izv. Akad. Nauk Azerb. SSR, ser. fiz-tekh. matem. no. 5 (1968) 93; Space Research Inte Preprint (1984) Pr-889.
[5] Ya.B. Zeldovich, I.D. Novikov, Relativistic Astrophysics, Univ. Chicago Press, Chicago, 1971.
[6] H. Heiselberg and M. Hjorth-Jensen, nucl-th/9902033.
[7] L. Lindblom, Phys. Rev. D 58, 024008 (1998); gr-qc/9802072.
[8] N.K. Glendenning, S. Pei and F. Weber, Phys. Rev. Lett. 79 (1997) 1603; astro-ph/9705235.
[9] P.R. Shapiro, I.T. Ivlev and A.C. Paga, astro-ph/9810164; J. Madsen, astro-ph/9601123; J. Sommer-Larsen, H. Vedel and U. Helsen, astro-ph/9610085.