Role of the brane curvature scalar in the brane world cosmology

N. J. Kim*, H. W. Lee†, and Y.S. Myung‡
Department of Physics, Graduate School, Inje University, Kimhae 621-749, Korea

Abstract

We include the brane curvature scalar to study its cosmological implication in the brane world cosmology. This term is usually introduced to obtain the well-defined stress-energy tensor on the boundary of anti de Sitter-Schwarzschild space. Here we treat this as the perturbed term for cosmological purpose. We find corrections to the well-known equation of the brane cosmology. It contains new interesting terms which may play the important role in the brane cosmology.

*E-mail address: dtpnjk@ijnc.inje.ac.kr
†E-mail address: hwlee@physics.inje.ac.kr
‡E-mail address: ysmymung@physics.inje.ac.kr
Recently there has been much interest in the phenomenon of localization of gravity proposed by Randall and Sundrum (RS) \[1,2\]. RS assumed a single positive tension 3-brane and a negative bulk cosmological constant in the 5D spacetime \[2\]. They have obtained a 4D localized gravity by fine-tuning the tension of the brane to the cosmological constant. The introduction of branes in a 5D anti de Sitter (AdS$_5$) space usually gives rise to a non-compact extra dimension.

More recently, several authors have studied its cosmological implications. The brane cosmology contains some important deviations from the Friedmann-Robertson-Walker (FRW) cosmology. One approach is first to assume the 5D dynamic metric (that is, BDL-metric \[3,4\]) which is manifestly $Z_2$-symmetric, $\hat{\omega} \rightarrow -\hat{\omega}$. Then one solves the Israel junction condition \[5\] and Einstein equation to find the behavior of the scale factor. We call its solution as the BDL cosmological solution.

The other approach starts with a static configuration which is two 5D anti de Sitter-Schwarzschild (AdSS$_5$) spaces joined by the domain wall. In this case the embedding into the moving domain wall is possible by choosing an appropriate normal vector $n_M$ \[6–8\]. The domain wall separating two such bulk spaces is taken to be located at $r = a(\tau)$, where $a(\tau)$ will determined by solving the Israel junction condition. Then observers on the wall will interpret their motion through the static bulk background as cosmological expansion or contraction. Mukhoyama et al. \[9\] performed a coordinate transformation \{$\hat{\tau}, \hat{x}_i, \hat{\omega}$\} → \{$t, r, \chi, \theta, \phi$\} in order to bring the BDL metric into the AdSS$_5$-metric. In this approach if two masses of Schwarzschild black holes are different, one can easily find a situation that does not possess a $Z_2$-symmetry manifestly \[9\]. We follow this approach. This gives us an asymmetric, cosmological evolution.

On the other hand, in the AdSS$_5$-space, its boundary metric acquires a divergent energy-momentum tensor $\Pi^{\mu\nu}$ as we take $r$ to infinity. In order to have a well-defined asymptotically AdSS$_5$ space ($R \times S^3$) on the boundary near infinity, we need the counter terms to cancel the divergence from the bulk \[12\]. This is $\mathcal{L}_{ct} = -\frac{3}{2} \sqrt{-h} (1 - \frac{\ell^2}{12} \mathcal{R})$, where $\mathcal{R}$ is the intrinsic curvature term on the boundary (domain wall). In view of the AdS/CFT correspondence, this inclusion of counter terms might be interpreted as the expectation value of the energy-momentum tensor in quantum conformal field theory.

In this letter, we investigate the cosmological implication of the curvature scalar on the brane. After the fine-tuning, this scalar can be of the same order as the terms that remain in the field equations for gravity on the brane \[13\]. Including this curvature scalar with the perfect fluid as the localized matter on the brane, one finds the fourth-order equation for $k + \dot{a}(\tau)^2$. Although the general solution to this equation is known, it is very hard to obtain a reliable equation like the FRW equation. Hence we consider the brane curvature scalar as the perturbation term. Then we find corrections to the known brane cosmology. This can be considered as one way to see the effect of brane curvature scalar on the brane cosmology.

---

1 $\hat{\omega}$ is the fifth coordinate and the metric is expressed in terms of the Gaussian normal coordinates.

2 In ref. \[10\], the authors derived a non-$Z_2$ symmetric term within the BDL approach. However, it is hard to find the origin of non-$Z_2$ symmetric term in this approach. Alternatives for non-$Z_2$ symmetric case in the brane physics, see ref. \[11\]
We will derive the Einstein equations on a 4D hypersurface embedded in an 5D bulk space. We divide the bulk space as two regions, $M_+$ and $M_-$ separated by the domain wall (brane), $B$. Here "+(-)" denote the right (left)-hand sides. We want to choose the different metric on both sides of the brane but with the same cosmological constant and $k$ for simplicity. Also we include the brane curvature scalar $R$ and the localized matter $\mathcal{L}_m$ including the brane tension as the brane action. At each point on the brane, we introduce a normal vector $n_M$ to the hypersurface such that $g^{MN}n_Mn_N = 1$, where $g^{MN}$ is the bulk metric and the capital indices $M, N, \cdots$ run over all bulk coordinates. Then the induced metric on the brane is given by the tangential components of the projection tensor

$$ h_{MN} = g_{MN} - n_Mn_N. \quad (1) $$

We start by combining two 5D bulk actions and two Gibbons-Hawking boundary terms \[13\]

$$ S_1 = \frac{1}{16\pi G} \int_{M_+} d^5x \sqrt{-g} \left[ R + \frac{12}{\ell^2} \right] + \frac{1}{8\pi G} \int_B d^4x \sqrt{-h} K^+, $$

$$ S_2 = \frac{1}{16\pi G} \int_{M_-} d^5x \sqrt{-g} \left[ R + \frac{12}{\ell^2} \right] + \frac{1}{8\pi G} \int_B d^4x \sqrt{-h} K^- \quad (2) $$

and the brane action\[14\]

$$ S_b = \frac{1}{16\pi G} \int_B d^4x \sqrt{-h} \left[ b\frac{\ell}{2} R + 16\pi GL_m \right], \quad (3) $$

where the brane tension term is included at $\mathcal{L}_m = -\sigma + \cdots$. The parameter $b$ is introduced for our cosmological purpose. For a counter term, we choose $\sigma = \frac{3}{8\pi G\ell^2}, b = 1$ \[12\]. In order to obtain the RS Minkowski brane \[2\], one has to choose $\sigma = \frac{6}{8\pi G\ell^2}, b = 0$. But we regard $b$ here as an arbitrary, small parameter. Here $G$ is the 5D Newtonian constant and the trace of the extrinsic curvature $K$ is introduced to obtain the well-defined variation on the brane (boundary). The extrinsic curvature is defined by

$$ K_{MN} = h_M^P \nabla_P n_N. \quad (4) $$

The Einstein equation for both sides leads to

$$ G_{MN} = R_{MN} - \frac{1}{2} R g^+_Mn_N = \frac{12}{\ell^2} g^+_Mn_N, \quad (5) $$

where we choose the same cosmological constants on the two sides. Further, from the boundary variation, one takes the form

---

3 Two papers in \[14,15\] have dealt with the case of different cosmological constants. But this case does not give rise to a non-$\mathbb{Z}_2$ symmetric evolution at $a^{-6}$-order.

4 The cosmological application of the brane curvature scalar $R$ discussed in refs. \[14,17\]. Deffayet in \[16\] considered this term to derive the Friedmann-like equation in 5D Minkowski spacetime, while authors in \[17\] pointed out the existence of $R$ within the AdS/CFT correspondence.
\[ \triangle K_{MN} - h_{MN} \triangle K = b \frac{\ell}{2} \left[ \mathcal{R}_{MN} - \frac{1}{2} \mathcal{R} h_{MN} \right] - 8 \pi G T_{MN}, \tag{6} \]

where
\[ \triangle K_{MN} \equiv K_{MN}^+ - K_{MN}^-, \quad T_{MN} = h_{MN} \mathcal{L}_m - 2 \frac{\delta \mathcal{L}_m}{\delta h_{MN}}. \tag{7} \]

From Eq. (6), we obtain the Israel junction condition
\[ \triangle K_{MN} = -\kappa^2 \left[ T_{MN} - \frac{1}{3} T_P^P h_{MN} \right] + b \frac{\ell}{2} \left[ \mathcal{R}_{MN} - \frac{1}{6} \mathcal{R} h_{MN} \right] \tag{8} \]

with \( \kappa^2 = 8 \pi G \). This describes the relation of the discontinuity of the extrinsic curvature across the surface (brane) to the surface stress-energy tensor, when the last term is absent.

Now we are in a position to discuss the static bulk solution to Eq. (5) [18]. For the cosmological embedding, the relevant solution is chosen as the AdS5-spacetime for both sides,
\[ ds^2_{\pm} = -h_{\pm}(r)dt^2 + \frac{1}{h_{\pm}(r)} dr^2 + r^2 \left[ d\chi^2 + f_k(\chi)(d\theta^2 + \sin^2 \theta d\phi^2) \right], \quad (k = 0, \pm 1) \tag{9} \]

where
\[ h_{\pm}(r) = k - \frac{\alpha_{\pm}}{r^2} + \frac{r^2}{\ell^2}, \quad f_0(\chi) = \chi, \quad f_1(\chi) = \sin \chi, \quad f_{-1}(\chi) = \sinh \chi. \tag{10} \]

In the case of \( \alpha_{\pm} = 0 \), we have two AdS5-spaces. However, \( \alpha_{\pm} \neq 0 \) generates the electric part of the Weyl tensor \( C^\pm_{MN} \) on each side. Its presence means that the bulk spacetime has the black hole horizon at small \( r = r_{\pm h} = \ell^2 (-k + \sqrt{k^2 + 4\alpha_{\pm}/\ell^2})/2 \). On the other hand, we assume that the domain wall (brane) is located at large \( r \). For large \( r \), its asymptote to AdS5-space is \( R \times S^3 \). In this case, the boundary energy-momentum tensor \( \Pi^{\mu\nu} \) diverges as \( r \) approaches infinity. It is due to the presence of \( r^2/\ell^2 \)-term in \( h_{\pm}(r) \). This corresponds to the intrinsic property of AdS5. Hence, to have a finite boundary energy-momentum tensor, one needs the boundary counter term such as \( \mathcal{L}_{ct} = -\frac{3}{2} \sqrt{-h} (1 - \frac{\ell^2}{12} \mathcal{R}) \). However, the mass-term of \( \alpha_{\pm}/r^2 \) has the extrinsic property from the boundary point of view. Because of its extrinsic property, the \( \alpha \)-term cannot be subtracted off. Hence we have to keep this term for the brane cosmology. We expect that its interaction with the brane curvature scalar plays an important role. Furthermore, if \( \alpha_{+} = \alpha_{-} \), we find a \( Z_2 \) symmetrical evolution of the domain wall [4]. Different \( \alpha \) (\( \alpha_{+} \neq \alpha_{-} \)) induces a non-\( Z_2 \) symmetric cosmology [7,8,13].

Now we consider the location of brane (domain wall) in the form of \( t = t(\tau), r = a(\tau) \) parametrized by the proper time \( \tau \) on the brane: \( (t, r, \chi, \theta, \phi) \rightarrow (t(\tau), a(\tau), \chi, \theta, \phi) \). Then the induced metric of dynamical domain wall will be given by the conventional FRW-type. Here \( \tau \) and \( a(\tau) \) mean the cosmic time and scale factor of the FRW universe, respectively. The tangent vectors of this brane can be expressed as
\[ u_\pm = \dot{t}_\pm \frac{\partial}{\partial t_\pm} + \dot{a} \frac{\partial}{\partial a}, \tag{11} \]
where the overdot means the differentiation with respect to $\tau$. These satisfy $u_{\pm M} u_{\pm N} g^{MN} = -1$. To find a dynamical solution, we need the normal 1-forms directed toward to each side: $n_{\pm M} n_{\pm N} g^{MN} = 1$. Here we choose these as

$$n_\pm = \pm \dot{a} dt_\pm \mp i_\pm da.$$  \hfill (12)

This case is consistent with the Randall-Sundrum case in the limit of $\alpha_\pm = 0$. Using this, we can express $i$ in terms of $\dot{a}$ as

$$i_\pm = \left( \dot{a}^2 + h_\pm(a) \right)^{1/2}. \hfill (13)$$

From the bulk metric Eq.(9) together with Eq.(13), we can derive the 4D induced metric

$$ds_4^2 = -d\tau^2 + a(\tau)^2 \left[ d\chi^2 + f_k(\chi)^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] \equiv h_{\mu\nu} dx^\mu dx^\nu. \hfill (14)$$

Hereafter we use the Greek indices for physics of the brane. The extrinsic curvatures for cosmological purpose are given by

$$(K_\pm)_{\tau\tau} = (K_\pm)_{MN} u_\pm^M u_\pm^N = \pm (h_\pm \dot{t}_\pm)^{-1} (\ddot{a} + h_\pm^4/2), \hfill (15)$$

$$(K_\pm)^{\chi}_{\chi} = (K_\pm)^{\theta}_{\theta} = (K_\pm)^{\phi}_{\phi} = \mp h_\pm i_\pm/a, \hfill (16)$$

where the prime stands for the derivative with respect to $a$. The above equation implies that the extrinsic curvature jumps across the brane. This jump is already realized through the Israel condition of Eq.(8). Here we need its four-dimensional version

$$\triangle K_{\mu\nu} = -\kappa^2 \left( T_{\mu\nu} - \frac{1}{3} T^{\lambda}_{\mu} h_{\lambda\nu} \right) + \frac{b\ell}{2} \left( R_{\mu\nu} - \frac{1}{6} R h_{\mu\nu} \right). \hfill (17)$$

In order to have a cosmological solution, let us choose the localized stress-energy tensor on the brane as the 4D perfect fluid

$$T_{\mu\nu} = (\varrho + p) u_\mu u_\nu + p h_{\mu\nu}. \hfill (18)$$

Here $\varrho = \rho + \sigma (p = P - \sigma)$, where $\rho(P)$ is the energy density (pressure) of the localized matter and $\sigma$ is the brane tension. In the absence of a localized matter, the first term of Eq.(17) takes the form of the RS case as $-\frac{\alpha^2}{3} h_{\mu\nu}$. We stress again that $u_\mu, h_{\mu\nu}$ are defined through the 4D induced metric of Eq.(14). In addition, we need the Gauss-Codazzi equations

$$\frac{\kappa^2}{2} \left[ (K_+)_{\mu\nu} + (K_-)_{\mu\nu} \right] T^{\mu\nu} = \triangle G_{\mu\nu} n^\mu n^\nu, \hfill (19)$$

$$\kappa^2 h_{\mu} \lambda \nabla_\nu T^{\lambda}_{\nu} = h_{\mu} \lambda \triangle G^{\lambda}_{\nu} n^\nu, \hfill (20)$$

where the last one is nothing but the conservation law

$$\frac{d}{d\tau} (\varrho a^3) + p \frac{d}{d\tau} (a^3) = 0. \hfill (21)$$
On the other hand, Eq. (19) denotes the average of the values of extrinsic curvature on the two sides. In Eqs. (13) and (20), the right-hand sides are zero because we choose the same cosmological constant for the two sides. From Eqs. (17), (18) and (19), one finds

\[(h_+ \dot{t}_+)^{-1}(\ddot{a} + h'_+/2) + (h_- \dot{t}_-)^{-1}(\ddot{a} + h'_-/2) = -\kappa^2 \left( p + \frac{2}{3} \right) + \frac{b \ell}{2a} \left( \frac{2 \ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right), \tag{22}\]

\[h_+ \dot{t}_+ + h_- \dot{t}_- = \frac{\kappa^2}{3} \varrho a - \frac{b \ell}{2a} \left( \dot{a}^2 + k \right), \tag{23}\]

\[(h_+ \dot{t}_+)^{-1}(\ddot{a} + h'_+/2) - (h_- \dot{t}_-)^{-1}(\ddot{a} + h'_-/2) = \frac{3p}{\varrho a} \left( h_+ \dot{t}_+ - h_- \dot{t}_- \right). \tag{24}\]

Equation (22) corresponds to the acceleration part of the moving domain wall, while Eq. (23) corresponds to the velocity of the moving domain wall. In the case of \(b = 0\), we can easily solve the above equations simultaneously \[8\]. However, it is very hard to solve the above equations as they stand. It is very interesting to derive the Friedmann-like equation. For this purpose, we use only the space component of the junction condition Eq. (23) which can be rewritten as

\[\sqrt{h_+ + \ddot{a}^2} + \sqrt{h_- + \ddot{a}^2} = \frac{\kappa^2}{3} \varrho a - \frac{b \ell}{2a} \left( \dot{a}^2 + k \right). \tag{25}\]

Let us introduce \(X = \ddot{a}^2 + k\) to derive the Friedmann-like equation from Eq. (25). Then this leads to the fourth-order equation as

\[AX^4 + BX^3 + CX^2 + DX + E = 0 \tag{26}\]

whose coefficients are given by

\[A = \frac{1}{64} \frac{b^4 \ell^4}{a^6}, \]

\[B = -\frac{b^2 \ell^2}{4a^2} \left( 1 + \frac{1}{6} b \kappa^2 \varrho \ell \right), \]

\[C = \frac{b^2 \ell^2}{4a^2} \left( -\frac{a^2}{\ell^2} + \frac{\alpha_+ + \alpha_-}{2a^2} + \frac{1}{6} \kappa^2 \varrho a^2 \right) + \frac{1}{3} b \kappa^2 \varrho \ell, \]

\[D = -\frac{1}{9} \kappa^4 \varrho^2 a^2 + \frac{1}{3} b \kappa^2 \varrho \ell \left( \frac{a^2}{\ell^2} - \frac{\alpha_+ + \alpha_-}{2a^2} - \frac{1}{18} \kappa^4 \varrho a^2 \right), \]

\[E = \frac{\alpha_+ - \alpha_-}{4a^2} + \frac{\alpha_+ + \alpha_-}{18} \kappa^4 \varrho^2 a^2 + \frac{1}{324} \kappa^8 \varrho^4 a^4 - \frac{1}{9} \kappa^4 \varrho a^4. \tag{27}\]

In the case of \(b = 0\), the above equation reduces to the first-order equation and its solution is

\[X_0 = \ddot{a}^2 + k = -\frac{a^2}{\ell^2} + \frac{\alpha_+ + \alpha_-}{2a^2} + \frac{1}{36} \kappa^4 \varrho^2 a^2 + \frac{9}{4} \frac{(\alpha_+ - \alpha_-)^2}{\kappa^4 \varrho^2 a^6}. \tag{28}\]

This is the well-known equation for the brane cosmology, which has derived in refs. \[7, 8, 13, 14\]. In the case of \(b \neq 0\), we have to solve the fourth-order equation. Although its general solution is known, it is a formidable task to extract a meaningful Friedmann-like equation. Here we wish to treat all \(b\)-terms as the perturbations around the background (brane cosmology).
equation of Eq.(28). In other words, \( b \) is considered as a small parameter. For simplicity we keep the first-order \( b \)-terms by considering \( A \to 0, B \to 0, C \to \frac{1}{3}\kappa^2\varrho\ell \) as
\[
\frac{1}{3}b\kappa^2\varrho\ell X^2 + \left(-\frac{1}{9}\kappa^4\varrho^2 a^2 + b\tilde{\varrho}\right) X + E = 0
\] (29)
with
\[
\tilde{\varrho} = \varrho \left(\frac{a^2}{\ell^2} - \frac{\alpha_+ + \alpha_-}{2a^2} + \frac{1}{18}\kappa^4\varrho^2 a^2\right).
\] (30)

Let us propose the perturbed solution as \( X = X_0 + b\beta \), then \( \beta \) is determined by
\[
\beta = \frac{3\kappa^2\varrho\ell + 9\tilde{\varrho}}{\kappa^4\varrho^2 a^2}X_0.
\] (31)

If we express it in terms of the conventional form
\[
\frac{1}{2}\dot{a}^2 + V(a) = -\frac{1}{2}k
\] (32)
the potential is given by
\[
V(a) = b \left(\frac{3}{2\kappa^4\varrho^2 \ell^2} - \frac{\kappa^2\varrho\ell}{24}\right)
+ \left[\frac{1}{2} - \frac{\kappa^4\varrho^2 \ell^2}{72}\right] + b \left(\frac{3}{2\kappa^2\varrho\ell} - \frac{\kappa^2\varrho\ell}{8} + \frac{\kappa^6\varrho^3 \ell^3}{432}\right) \frac{a^2}{\ell^2}
+ \left[-\frac{1}{2} + b \left(-\frac{3}{2\kappa^2\varrho\ell} + \frac{\kappa^2\varrho\ell}{18}\right)\right] \frac{\alpha_+ + \alpha_-}{a^2}
- b \frac{3}{4} \frac{\left(\alpha_+ + \alpha_-\right)^2}{a^4}
+ \left[\frac{9}{8} \frac{\left(\alpha_+ - \alpha_-\right)^2}{\kappa^4\varrho^2 \ell^2}\right] + b \left(-\frac{27}{8} \frac{\left(\alpha_+ - \alpha_-\right)^2}{\kappa^6\varrho^3 \ell^3} + \frac{1}{8} \frac{\left(\alpha_+ + \alpha_-\right)^2}{\kappa^2\varrho\ell} + \frac{3}{16} \frac{\left(\alpha_+ - \alpha_-\right)^2}{\kappa^2\varrho\ell}\right) \frac{\ell^2}{a^6}
- b \frac{27}{8} \frac{\left(\alpha_+ - \alpha_-\right)^2}{\kappa^6\varrho^3 \ell^3} \frac{\ell^4}{a^8}
+ b \frac{27}{16} \frac{\left(\alpha_+ - \alpha_-\right)^2 \left(\alpha_+ + \alpha_-\right)}{\kappa^6\varrho^3 \ell^3} \frac{\ell^4}{a^{10}},
\] (33)
where \( \kappa^2\varrho\ell \) is a dimensionless quantity. For an extremal wall with a fine-tuned tension, we have \( \kappa^2\varrho\ell = \kappa^2\sigma\ell = \frac{6}{7}\ell = 6 \). Then its potential reduces to
\[
V^{EXT}(a) = \left(-\frac{1}{2} + b \frac{1}{12}\right) \frac{\alpha_+ + \alpha_-}{a^2} - b \frac{\left(\alpha_+ + \alpha_-\right)^2}{12} \frac{\ell^2}{a^4}
+ \left[\frac{\left(\alpha_+ + \alpha_-\right)^2}{32} + b \left(\frac{\left(\alpha_+ - \alpha_-\right)^2}{64} + \frac{\left(\alpha_+ + \alpha_-\right)^2}{48}\right)\right] \frac{\ell^2}{a^6}
- b \frac{1}{8} \frac{\ell^4}{a^8} + b \frac{\left(\alpha_+ - \alpha_-\right)^2 \left(\alpha_+ + \alpha_-\right)}{128} \frac{\ell^4}{a^{10}}.
\] (34)

In the case of \( b = 0 \), we recover the motion of extremal wall with a fine-tuned tension Eq.(32) with [7]
\begin{equation}
V_{b=0}^{EXT}(a) = -\frac{\alpha_+ + \alpha_-}{2a^2} - \frac{(\alpha_+ - \alpha_-)^2 t^2}{32a^6}.
\end{equation}

The RS static configuration \([2]\) can easily be recovered from Eq.\((32)\) when \(k = \alpha_+ = \alpha_- = 0\).

By the similar way, we can obtain higher-order equation including \(b^2, b^3\) and \(b^4\) for small \(b\). Although we do not derive the Friedmann-like equation for the finite \(b\) case (for example, \(b = 1\) for a counter term), our perturbed equation \((32)\) with \((33)\) is valuable for investigating the role of the brane curvature scalar in the brane cosmology. This equation corresponds to corrections to the known brane cosmology of Eq.\((28)\).

We observe from Eq.\((33)\) that the non-zero curvature scalar on the brane induces some interesting consequences. In the first line, we find new constant terms \(b/\rho, b/\rho^2\) which are never found in any (brane) cosmology. In the second line, \(\rho^2 a^2\) corresponds to the famous term for the brane cosmology, but we have additional terms of \(b(1/\rho, \rho, \rho^3)a^2\). Here we find a standard term of \(\rho a^2\) in the Friedmann equation. This term never appears if one includes a brane curvature scalar. The third line represents mass-like terms of \(\alpha/a^2\), which are derived from the the electric part of the Weyl tensor. Here one obtains new \(b\)-dependent mass-like terms. The fourth line is totally a new term. We cannot find any term of \(\alpha^2/a^4\) in the existing literature. In the fifth one, we point out that \(\alpha_+ = \alpha_-\) induces a new term such as \(b\alpha^2/a^6\) which never appears in the case of \(b = 0, Z_2\)-symmetric evolution. In the case of \(\alpha_+ \neq \alpha_-\), we get the last two terms of Eq.\((33)\) which are new higher-order terms for \(a\) like \(b\alpha^2/a^8, b\alpha^3/a^{10}\) as well as for \(\rho\) like \(b/\rho^3\). These may enhance the non-\(Z_2\) symmetric evolution.

Finally let us compare our results with the existing literature. Collins and Holdom in \([13]\) have derived cosmological equations for a vacuum bubble expanding into a AdSS\(_5\) bulk space and the edge of a single AdS\(_5\) space. This calculation has been done without any condition for \(b\). As was explained before, Eq.\((24)\) is a quartic polynomial in \(k + \dot{a^2}\) which becomes manageable only for special cases. Hence they studied two : a vacuum bubble and an edge of a single AdS\(_5\) apace. But in the strict sense, these two cases do not belong to the brane cosmology. Even for an edge universe, they failed to derive a realistic cosmological evolution for a counter term of \(\sigma = \frac{a}{8\sqrt{G}b}\), \(b = 1\) because they found a complex potential. On the other hand, Deffayet \([16]\) investigated a role of the brane curvature scalar in Minkowski bulk space. He considered only the \(Z_2\)-symmetric evolution within the BDL approach. In this case he found also the right hand sides of Eqs.\((22)\) and \((23)\). But he took a view that the contribution from the intrinsic curvature is regarded as a cosmic fluid of density \(\rho_{\text{curv}}\) and pressure \(P_{\text{curv}}\). In addition he recovered a standard cosmology in a certain limit. However this picture is slightly different from a genuine brane cosmology. Although our result has a limitation such that it is valid only for small \(b\), it provides a new insight to see the role of brane scalar curvature in the brane cosmology.

We conclude that our equation \((32)\) corresponds to corrections the known brane cosmology Eq.\((28)\). This implies that brane curvature scalar induces new interesting implications for the brane cosmology.

\textbf{ACKNOWLEDGMENTS}

This work was supported in part by the Brain Korea 21 Program, Ministry of Education, Project No. D-0025 and KOSEF, Project No. 2000-1-11200-001-3.
REFERENCES

[1] L. Randall and R. Sundrum, Phys. Rev. Lett. 83 (1999) 3370 [hep-ph/9905221].
[2] L. Randall and R. Sundrum, Phys. Rev. Lett. 83 (1999) 4690 [hep-th/9906064].
[3] P. Binetruy, C. Deffayet and D. Langlois, Nucl. Phys. B 565 (2000) 269 [hep-th/9905012].
[4] P. Binetruy, C. Deffayet, U. Ellwanger and D. Langlois, Phys. Lett. B 477 (2000) [hep-th/9910219].
[5] W. Israel, Nuovo Cim. B44 (1966) 1; ibid. B48 (1967) 463.
[6] H. A. Chamblin and H. S. Reall, Nucl. Phys. B562 (1999) 133 [hep-th/9903225].
[7] P. Kraus, JHEP 9912 (1999) 011 [hep-th/9910149].
[8] D. Ida, JHEP 0009 (2000) 014.
[9] S. Mukhoyama, T. Shiromizu and K. Maeda, [hep-th/9912287].
[10] A. C. Davis, I. Vernon, S. C. Davis and W. B. Perkins, [hep-ph/0008132].
[11] Y. S. Myung and H. W. Lee, [hep-th/0001211]; G. Kang and Y.S. Myung, [hep-th/0007197]; Y. S. Myung, [hep-th/0009117]; Y. S. Myung, [hep-th/0010208].
[12] V. Balasubramanian and P. Kraus, Commun. Math. Phys. 208(1999) 413 [hep-th/9902123]; P. Kraus, F. Larsen and R. Siebelink, Nucl. Phys. B 563 (1999) 259 [hep-th/9906127].
[13] H. Collins and B. Holdom, [hep-ph/0003173].
[14] H. Stoica, S. H. H. Tye and I. Wasserman, [hep-th/0004126].
[15] N. Deruelle and T. Dolezel, [gr-qc/0004021].
[16] C. Deffayet, [hep-th/0010186].
[17] L. Anchordoqui, C. Nuñez, and K. Olsen, [hep-th/0007064].
[18] D. Birmingham, Class. and Quant. Grav. 16 (1999) 1197 [hep-th/9808032].