Beyond the Chinese Restaurant and Pitman-Yor processes: Statistical Models with Double Power-law Behavior

Fadhel Ayed∗†, Juho Lee*1,2 and François Caron1

1Department of Statistics, University of Oxford
2AITRICS

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Abstract

Bayesian nonparametric approaches, in particular the Pitman-Yor process and the associated two-parameter Chinese Restaurant process, have been successfully used in applications where the data exhibit a power-law behavior. Examples include natural language processing, natural images or networks. There is also growing empirical evidence that some datasets exhibit a two-regime power-law behavior: one regime for small frequencies, and a second regime, with a different exponent, for high frequencies. In this paper, we introduce a class of completely random measures which are doubly regularly-varying. Contrary to the Pitman-Yor process, we show that when completely random measures in this class are normalized to obtain random probability measures and associated random partitions, such partitions exhibit a double power-law behavior. We discuss in particular three models within this class: the beta prime process (Broderick et al. (2015, 2018), a novel process called generalized BFRY process, and a mixture construction. We derive efficient Markov chain Monte Carlo algorithms to estimate the parameters of these models. Finally, we show that the proposed models provide a better fit than the Pitman-Yor process on various datasets.

1 Introduction

Power-law distributions appear to arise in a wide range of contexts, including natural languages, natural images or networks. For example, the empirical distribution of the word frequencies in natural languages is well approximated by a power-law distribution, an observation attributed to Zipf [1935]. That is, the frequency \( f(k) \) of the \( k \)'s most frequent word in a corpus verifies, within some range

\[
f(k) \asymp Ck^{-\xi}
\]

where \( C \) is some constant and \( \xi > 0 \) is the power-law exponent which is typically close to 1 for natural languages. These empirical findings have motivated the development of numerous generative models that can reproduce this power-law behavior; see the reviews of Mitzenmacher [2004] and Newman [2005].

∗Equal contribution
†Corresponding author, fadhel.ayed@stats.ox.ac.uk
Amongst these generative models, Bayesian nonparametric hierarchical models based on infinite-dimensional random measures have been successfully used to capture the power-law behavior of various datasets. Applications include natural language processing [Goldwater et al., 2006, Teh, 2006, Wood et al., 2009, Mochihashi et al., 2009, Sato and Nakagawa, 2010], natural image segmentation Sudderth and Jordan [2009] or network analysis Caron [2012], Caron and Fox [2017], Crane and Dempsey [2018], Cai et al. [2016]. A very popular model is the Pitman-Yor (PY) process [Pitman, 1995, Pitman and Yor, 1997, Pitman, 2006], an infinite-dimensional random probability measure whose properties induce a power-law behavior. It admits two parameters \((0 \leq \alpha < 1, \theta > -\alpha)\). The PY random probability measure is almost surely discrete, with weights \(\pi_1 \geq \pi_2 \geq \ldots\) following the so-called two-parameter Poisson-Dirichlet distribution \(PD(\alpha, \theta)\) [Pitman and Yor, 1997]. For \(\alpha > 0\), the random weights verify
\[
\pi(k) \sim k^{-1/\alpha} S \quad \text{almost surely as } k \to \infty
\]
where \(S\) is a random variable. That is, small weights asymptotically follow a power-law distribution whose exponent is controlled by the parameter \(\alpha\). The PY process also enjoys tractable alternative constructions via the two-parameter Chinese restaurant process or the stick-breaking construction which explains its great popularity amongst models with similar properties. Other popular infinite-dimensional random measures that have been used for their similar power-law properties include the stable Indian buffet process [Teh and Gorur, 2009] or the generalized gamma process Hougaard [1986], Brix [1999].

**Double power-law in empirical data.** There is a growing empirical evidence that some datasets may exhibit a double power-law regime when the sample size is large enough. Examples include word frequencies in natural languages Ferrer i Cancho and Solé [2001], Montemurro [2001], Gerlach and Altmann [2013], Font-Clos et al. [2013], Twitter rates and retweet distributions Bild et al. [2015], or degree distributions in social [Csányi and Szendró, 2004], communication [Seshadri et al., 2008] or transportation networks Paleari et al. [2010]. In the case of word frequencies for example, it is conjectured that high frequency words approximately follow a power-law with Zipfian exponent \(\approx 1\), while the low frequency words follow a power-law with a higher exponent. An illustration is given in Figure 1, which shows the word frequencies of about 300,000 words from the American National Corpus.

In this paper, we introduce a class of completely random measures (CRMs), named doubly regularly-varying CRMs. We show, that a random measure in this class is normalized to obtain a random probability measure \(P\), and one repeatedly samples from \(P\), the resulting frequencies exhibit a double power-law behavior. Informally, the ranked frequencies verify
\[
f(k) \approx \begin{cases} C_1 k^{-1/\tau} & \text{for small rank } k \\ C_2 k^{-1/\alpha} & \text{for large rank } k \end{cases}
\]
where \(\tau > 0, \alpha \in (0, 1)\) and \(C_1, C_2 > 0\). The above statement is made mathematically accurate later in the article. Within this class, we consider three models in particular: the beta prime process of Broderick et al. [2015, 2018], a novel process named generalized BFRY process, and a mixture construction. We show how all three constructions can be obtained from transformations of the generalized gamma and stable beta processes. We derive Markov chain Monte Carlo inference algorithms for these models, and show that such models provide a good fit compared to a Pitman-Yor process on text and network datasets.

\[1\]http://www.anc.org/data/anc-second-release/frequency-data/
2 Background on (normalized) completely random measures

CRMs, introduced by [Kingman, 1967], are important building blocks of Bayesian nonparametric models [Lijoi and Prünster, 2010]. A homogeneous CRM on a Polish space \( \Theta \), without deterministic component nor fixed atoms, is almost surely (a.s.) discrete and takes the form

\[
W = \sum_{k \geq 1} w_k \delta_{\theta_k}
\]

where \((w_k, \theta_k)_{k \geq 1}\) are the points of a Poisson point process with mean measure \( \rho(dw)H(d\theta) \). \( H \) is some probability distribution on \( \Theta \), and \( \rho \) is a Lévy measure on \((0, \infty)\). We write \( W \sim \text{CRM}(\rho, H) \). A popular CRM is the generalized gamma process (GGP) [Hougaard, 1986, Brix, 1999] with Lévy measure

\[
\rho_{\text{GGP}}(dw; \sigma, \zeta) = \frac{1}{\Gamma(1 - \sigma)} w^{1 - \sigma} e^{-\zeta w} dw
\]

where \( \sigma \in (0, 1) \) and \( \zeta \geq 0 \) or \( \sigma \leq 0 \) and \( \zeta > 0 \). The GGP admits as special case the gamma process (\( \sigma = 0, \zeta > 0 \)) and the stable process (\( \sigma \in (0, 1), \zeta = 0 \)).

If

\[
\int_0^\infty \rho(dw) = \infty
\]

then the CRM is said to be infinite-activity: it has an infinite number of atoms and the weights verify \( 0 < W(\Theta) = \sum_{k=1}^{\infty} w_k < \infty \) a.s. We can therefore construct a random probability measure \( P \) by normalizing the CRM [Regazzini et al., 2003, Lijoi et al., 2007]

\[
P = \frac{W}{W(\Theta)}.
\]

We call \( P \) a normalized CRM (NCRM) and write \( P \sim \text{NCRM}(\rho, H) \). The Pitman-Yor process with parameters \( \theta \geq 0 \) and \( \alpha \in [0, 1) \) and distribution \( H \), written \( P \sim \text{PY}(\alpha, \theta, H) \) admits a representation as a (mixture of) CRMs [Pitman and Yor, 1997, Proposition 21]. If \( \theta, \alpha > 0 \) it is a mixture of normalized generalized gamma
processes

\[ \eta \sim \text{Gamma} \left( \frac{\theta}{\alpha}, 1 \right) \]  
\[ P \mid \eta \sim \text{NCRM}(\eta \rho_{\text{GGP}}(:\sigma, 1), H) \]

and for \( \theta = 0 \), it is a normalized stable process

\[ P \sim \text{NCRM}(\rho_{\text{GGP}}(:, \sigma, 0), H) \]

Although this representation is more complicated than the usual stick-breaking or urn constructions of the PY, it will be useful later on when we will discuss its asymptotic properties. The above construction essentially tells us that the PY has the same asymptotic properties as the normalized GGP for \( \theta > 0 \) and the stable process for \( \theta = 0 \).

3 Doubly regularly varying CRMs

3.1 General definition

We first introduce a few definitions on regularly varying functions [Bingham et al. (1989)].

**Definition 3.1 (Slowly varying function)** A positive function \( \ell \) on \((0, \infty)\) is slowly varying at infinity if for all \( c > 0 \) \( \ell(ct)/\ell(t) \to 1 \). Examples of slowly varying functions are constant functions, functions converging to a strictly positive constant, \((\log t)^a\) for any real \( a \), etc.

**Definition 3.2 (Regularly varying function)** A positive function \( f \) on \((0, \infty)\) is said to be regularly varying at infinity with exponent \( \xi \in \mathbb{R} \) if \( f(x) = x^\xi \ell(x) \) where \( \ell \) is a slowly varying function. Similarly, a function \( f \) is said to be regularly varying at 0 if \( f(1/x) \) is regularly varying at infinity, that is \( f(x) = x^{-\xi} \ell(1/x) \) for some \( \xi \in \mathbb{R} \) and some slowly varying function \( \ell \).

Informally, regularly varying functions with exponent \( \xi \neq 0 \) behave asymptotically similarly to a “pure” power-law function \( g(x) = x^\xi \).

A homogeneous CRM \( W \) on \( \Theta \) with mean measure \( \rho(dw)H(d\theta) \) is said to be doubly regularly varying if its tail Lévy intensity

\[ \overline{\rho}(x) = \int_x^\infty \rho(dw) \]  
\[ \overline{\rho}(x) \sim \begin{cases} x^{-\alpha} \ell_1(1/x) & \text{as } x \to 0 \\ x^{-\tau} \ell_2(x) & \text{as } x \to \infty \end{cases} \]

where \( \alpha \in [0, 1] \), \( \tau \geq 0 \) and \( \ell_1 \) and \( \ell_2 \) are slowly varying functions. The CRM is said to be doubly power-law if it is doubly regularly varying with exponents \( \alpha > 0 \) and \( \tau > 0 \). Note that in this case, the CRM necessarily verifies condition (4) and is therefore infinite activity.
3.2 Properties

In the following, let \( w^{(1)} \geq w^{(2)} \geq \ldots \) denote the ordered weights of the CRM. The first proposition states that, if the CRM is regularly varying at 0 with exponent \( \alpha > 0 \), the weights asymptotically scale as a power-law (up to slowly varying function). Its proof is given in the appendix.

**Proposition 1** A CRM, regularly varying at 0 with exponent \( \alpha > 0 \), verifies

\[
 w^{(k)} \sim k^{-1/\alpha} \ell_1^*(k) \quad \text{as} \quad k \to \infty
\]

where \( \ell_1^* \) is a slowly varying function whose expression, which depends on \( \ell_1 \) and \( \alpha \), is given in the appendix.

The next proposition states that, if the CRM is regularly varying at infinity with \( \tau > 0 \) and the scaling factor of the Lévy measure is large, the CRM has a power-law behavior for large weights.

**Proposition 2** [Kevei and Mason [2014, Theorem 1.2]]

Consider a CRM with mean measure \( \eta \rho(dw) H(d\theta) \), regularly varying at \( \infty \) with \( \tau > 0 \). Then, for any \( k \geq 1 \)

\[
 \frac{w^{(k+1)}}{w^{(k)}} \to \text{Beta}(k\tau, 1) \quad \text{in distribution as} \quad \eta \to \infty.
\]

Note that Equation (12) indicates a power-law behavior with exponent \( 1/\tau \), that is for a fixed \( k \) and large \( \eta \),

\[ w^{(k)} \approx Ck^{-1/\tau} \]

Indeed, Equation (12) can be rewritten as

\[ -\tau k \log \frac{w^{(k+1)}}{w^{(k)}} \to E \quad \text{as} \quad \eta \to \infty \]

where \( E \) denotes a unit-rate exponential random variable. Therefore, for large \( \eta \)

\[
 \log \frac{w^{(k+1)}}{w^{(k)}} \approx -\frac{1}{\tau k} \cdot \frac{E}{\tau k}
\]

\[
 \approx -\frac{1}{\tau} \log \frac{k+1}{k} \quad \text{for} \quad k \gg 1.
\]

**GGP and stable process.** The GGP with parameter \( \zeta > 0 \) is regularly varying at 0 with exponent \( \alpha = \max(0, \sigma) \). Hence, it verifies Proposition 1. However, the exponential decay of the tails of the Lévy measure implies that it is not regularly varying at \( \infty \). Large weights therefore decay exponentially fast. The stable process, which is a GGP with parameter \( \zeta = 0 \) and \( \sigma \in (0, 1) \), is doubly regularly-varying with the same power-law exponent \( \sigma \) at 0 and \( \infty \). Hence, it verifies Proposition 1. Additionally, Pitman and Yor [1997, Proposition 8] showed that the result of Proposition 12 holds non-asymptotically for the stable process. For all \( k \geq 1 \)

\[
 \frac{w^{(k+1)}}{w^{(k)}} \sim \text{Beta}(k\sigma, 1).
\]

The GGP with \( \zeta > 0 \) can capture a power-law behavior at 0, but has exponentially decaying tails at infinity. The stable has a double power-law behavior but with the same exponent \( \sigma \in (0, 1) \). In the following subsections, we describe processes with doubly regularly varying tail Lévy measure where one can flexibly tune both exponents. In the rest of the paper, we assume that the Lévy measure \( \rho \) is absolutely continuous with respect to the Lebesgue measure, and use the same notation for its density \( \rho(dw) = \rho(w)dw \).
3.3 Generalized BFRTY process

Consider the Lévy density

\[ \rho(w) = \frac{1}{\Gamma(1-\sigma)} w^{-1-\tau} \gamma(\tau - \sigma, cw) \]  

(13)

where \( \gamma(\kappa, x) = \int_0^x u^{\kappa-1} e^{-u} du \) is the lower incomplete gamma function and the parameters verify \( \sigma \in (-\infty, 1) \), \( \tau > \max(0, \sigma) \) and \( c > 0 \). We have

\[ \overline{\rho}(x) \sim \frac{\Gamma(\tau - \sigma)}{\tau \Gamma(1 - \sigma)} x^{-\tau} \]  

(14)

as \( x \) tends to infinity and, for \( \sigma > 0 \),

\[ \rho(\gamma) = \frac{c^{\tau-\sigma}}{\sigma(\tau-\sigma) \Gamma(1-\sigma)} \gamma^{\tau-\sigma} \]  

(15)

as \( x \) tends to 0. When \( \sigma \leq 0 \), \( \overline{\rho}(x) \) is a slowly varying function, with \( \lim_{x \to 0} \overline{\rho}(x) = \infty \) if \( \sigma = 0 \) and \( \lim_{\tau \to 0} \overline{\rho}(x) < \infty \) if \( \sigma < 0 \). \( \overline{\rho}(x) \) therefore verifies Equation (10) with \( \alpha = \max(\sigma, 0) \). When \( \sigma > 0 \), it is doubly power-law with exponent \( \sigma \in (0, 1) \) and \( \tau > 0 \).

The Lévy density admits the following latent construction via a GGP. Note that

\[ \rho(w) = \frac{c^{\tau-\sigma}}{\Gamma(1-\sigma)} w^{-1-\sigma} \int_0^1 z^{\tau-\sigma-1} e^{-wz} dz \]

where \( f_Z(z) = \tau z^{\tau-1} \) is the probability density function of a Beta(\( \tau, 1 \)) random variable. We therefore have the hierarchical construction. For \( k \geq 1 \),

\[ w_k = \frac{w_{0k}}{\beta_k}, \beta_k \sim \text{Beta}(\tau, 1) \]

where \( (w_{0k})_{k \geq 1} \) are the points of a Poisson process with mean measure \( c^{\tau-\sigma}/\tau \rho_{\text{GGP}}(w_0; \sigma, c) dw_0 \).

The process is somewhat related to, and can be seen as a natural generalization of the BFRTY distribution [Pitman and Yor, 1997, Winkel, 2005, Bertoin et al., 2006]. The name was coined by Devroye and James [2014] after the work of Bertoin, Fujita, Roynette and Yor. This distribution has recently found various applications in machine learning [Lee et al., 2016, 2017]. Taking \( c = 1, \tau \in (0, 1) \) and \( \sigma = \tau - 1 < 0 \), we have

\[ \rho(w) \propto w^{-1} (1 - e^{-w}) \]

which corresponds to the unnormalized pdf of a BFRTY random variable. The BFRTY random variable admits a representation as the ratio of a gamma and beta random variables, and the stochastic process introduced in this section, which admits a similar construction, can be seen as a natural generalization of the BFRTY distribution, and we call this process a generalized BFRTY (GBFRY) process. In Section D of the appendix, we provide more details on the BFRTY distribution and its generalization.

3.4 Beta prime process

Consider the Lévy density

\[ \rho(w) = \frac{\Gamma(\tau - \sigma)}{\Gamma(1 - \sigma)} w^{-1-\sigma} (c + w)^{\sigma-\tau} \]

(16)
where \( \sigma \in (-\infty, 1) \), \( \tau > 0 \) and \( c > 0 \) and . This density is a four-parameter extension of the beta prime (BP) process. This process was introduced by [Broderick et al. 2015] and generalized by [Broderick et al. 2018], as a conjugate prior for odds Bernoulli process. We have

\[
\bar{p}(x) \sim \frac{\Gamma(\tau - \sigma)}{\tau \Gamma(1 - \sigma)} x^{-\tau}
\]

as \( x \) tends to infinity and, for \( \sigma > 0 \),

\[
\bar{p}(x) \sim \frac{c^\sigma \Gamma(\tau - \sigma)}{\sigma \Gamma(1 - \sigma)} x^{-\sigma}
\]

as \( x \) tends to 0. When \( \sigma \leq 0 \), \( \bar{p}(x) \) is a slowly varying function, with \( \lim_{x \to 0} \bar{p}(x) = \infty \) if \( \sigma = 0 \) and \( \lim_{x \to 0} \bar{p}(x) < \infty \) if \( \sigma < 0 \). \( \bar{p}(x) \) therefore verifies Equation (10) with \( \alpha = \max(\sigma, 0) \).

The BP process is related to the stable beta process [Teh and Gorur 2009] with Lévy density

\[
\frac{\alpha \Gamma(\tau - \sigma)}{c^\tau \Gamma(1 - \sigma)} u^{-1-\sigma} (1 - u)^{\tau-1} \mathbb{1}_{u \in (0, 1)},
\]

via the transformation \( w = \frac{cu}{1-u} \). Similarly to the generalized BFRY model, the beta process can also be obtained via a GGP. Note that

\[
\rho(w) = \Gamma(\tau) c^{-\tau} \int_0^\infty y \rho_{G^2}(wy; \sigma, c) f_Y(y) dy
\]

where \( f_Y(y) = \frac{c^\tau y^{\tau-1} e^{-cy}}{\Gamma(\tau)} \) is the density of a Gamma(\( \tau, c \)) random variable. We therefore have the following hierarchical construction, for \( k \geq 1 \)

\[
w_k = \frac{w_0 k}{\gamma_k}, \quad \gamma_k \sim \text{Gamma}(\tau, c)
\]

where \( (w_0 k)_{k \geq 1} \) are the points of a Poisson process with mean measure \( c^{-\tau} \Gamma(\tau) \rho_{G^2}(w_0; \sigma, 1) dw_0 \).

### 3.5 Mixture model

Consider the Lévy density

\[
\rho(w) = \rho_0(w) + \beta f(w)
\]

where \( \rho_0 \) is a Lévy measure, regularly varying at 0, and \( f \) is the probability density function of a random variable with power-law tails. That is

\[
\bar{p}_0(x) \sim x^{-\sigma} f_1(1/x) \quad \text{as } x \to 0
\]

\[
\int_x^\infty f(t) dt \sim x^{-\tau} f_2(x) \quad \text{as } x \to \infty
\]

If we additionally assume that \( \bar{p}_0(x) \) has light tails at infinity (e.g. exponentially decaying tails), then the resulting CRM \( \rho \) is then doubly regularly varying and verifies Equation (10). For example, one can take for \( \rho_0 \) the Lévy density (3) of a GGP, and for \( f \) the pdf of a Pareto, generalized Pareto or inverse gamma distribution.
Normalized CRMs with double power-law

For some probability distribution $H$, Lévy measure $\rho$ verifying Equation (4) and $\eta > 0$, let

$$P = \frac{W}{W(\Theta)}$$

where $W \sim \text{CRM}(\eta \rho, H)$

and for $i = 1, \ldots, n$, $X_i \mid P \overset{	ext{i.i.d.}}{\sim} P$. As $P$ is a.s. discrete, there will be repeated values within the sequence $(X_i)_{i \geq 1}$. Let $K_n \leq n$ be the number of unique values in $(X_1, \ldots, X_n)$, and $m_{n,(1)} \geq m_{n,(2)} \geq \ldots \geq m_{n,(K_n)}$ their ranked multiplicities. For $k = 1, \ldots, K_n$, denote $f_{n,(k)} = \frac{m_{n,(k)}}{n}$ the ranked frequencies.

### 4.1 Double power-law properties

The following theorem provides a precise formulation of Equation (1) and shows that the ranked frequencies have a double power-law regime when the CRM is doubly regularly varying with strictly positive exponents.

**Theorem 1** The ranked frequencies verify

$$\left( f_{n,(1)}, f_{n,(2)}, \ldots \right) \to \left( \frac{w_{(1)}}{W(\Theta)}, \frac{w_{(2)}}{W(\Theta)}, \ldots \right)$$

almost surely as $n$ tends to infinity. If the CRM is regularly varying at 0 with exponent $\alpha > 0$ we have

$$\frac{w_{(k)}}{W(\Theta)} \sim W(\Theta)^{-1/k^{1/\sigma}} \ell_1^*(k) \text{ as } k \to \infty.$$  \hspace{1cm} (21)

If the CRM is regularly varying at $\infty$ with exponent $\tau > 0$ we have, for any $k \geq 1$

$$\tau k \log \frac{w_{(k+1)}}{w_{(k)}} \to \text{Exp}(1) \text{ in distribution as } \eta \to \infty.$$  \hspace{1cm} (22)

Theorem follows from [Gnedin et al., 2007, Proposition 26] and Propositions 1and 2. Instead of expressing the power-law properties in terms of the ranked frequencies, we can alternatively look at the asymptotic behavior of the number $K_{n,j}$ of elements with multiplicity $j \geq 1$, defined by

$$K_{n,j} = \sum_{k=1}^{K_n} 1_{m_{n,(k)} = j}.$$  \hspace{1cm} (23)

Let $p_{n,j} = \frac{K_{n,j}}{K_n}$. Note that $\sum_{j \geq 1} p_{n,j} = 1$. The following is a corollary of Equation (21). It follows from Proposition 23 and Corollary 21 in [Gnedin et al., 2007].

**Corollary 2** If the CRM is regularly varying at 0 with exponent $\sigma$, we have

$$p_{n,j} \to p_j \text{ a.s. as } n \to \infty$$  \hspace{1cm} (24)

where

$$p_j = \frac{\sigma \Gamma(j - \sigma)}{j \Gamma(1 - \sigma)} \sim \frac{\sigma}{\Gamma(1 - \sigma)} \frac{1}{j^{1+\sigma}} \text{ for large } j.$$  

Figure 2 shows some illustration of these empirical results for the GBFRY model.
Remark 1. The GGP with parameter $\sigma > 0$ is regularly varying at 0, but not at infinity. Hence, the normalized GGP with $\zeta > 0$ and the related Pitman-Yor process with $\theta > 0$ satisfy Equation (21) and (24) but not (22), due to the exponentially decaying tails of the Lévy measure of the GGP. The normalized GGP with $\zeta = 0$, which is the same as the Pitman-Yor with $\theta = 0$, verifies both Equations, but with the same exponent $\sigma \in (0, 1)$, lacking the flexibility of the three models presented in Section 3.

4.2 Posterior Inference

In this subsection, we briefly discuss the inference procedure for estimating the parameters of the normalized CRMs we introduced in Section 3. Additional details are provided in the appendix. Assume that the Lévy measure $\rho$ is parameterised by some parameters $\phi$ we want to estimate, in particular the two power-law exponents. We write $\rho(w; \phi)$ to emphasize this, and let $p(\phi)$ be the prior density. The objective is to approximate the posterior density of the parameters given the ranked counts $p(\phi \mid (m_{n,(k)})_{k=1,\ldots,K_n})$.

Parametrisation. Since we are working with normalized CRMs, multiplying $W$ by any positive constant $\xi > 0$ gives the same random probability measure $P$. In particular, the normalized CRMs with Lévy densities
\( \rho(w) \) and \( \tilde{\rho}(w) = \xi \rho(\xi w) \) have the same distribution. To avoid overparameterisation we set the parameter \( c = 1 \) in the GBFRY and BP processes, and estimate the parameter \( \phi = (\sigma, \tau, \eta) \).

We introduce a latent variable \( U \mid W \sim \text{Gamma}(n, W(\Theta)) \). Using Proposition 3 of James et al. [2009] (see also Pitman [2003]) and Equation (2.2) of Pitman [2006], the joint density is written as

\[
p((m_{n,(k)})_{k=1,\ldots,K_n}, u, \phi) \\
\propto p(\phi) u^{n-1} e^{-\psi(u;\phi)} \prod_{k=1}^{K_n} \kappa(m_{n,(k)}, u; \phi) \tag{25}
\]

where the normalizing constant only depends on \( n \) and the ranked counts, and

\[
\psi(t; \phi) = \eta \int_0^\infty (1 - e^{-tw}) \rho(w; \phi) dw, \tag{26}
\]
\[
\kappa(m, t; \phi) = \eta \int_0^\infty w^m e^{-tw} \rho(w; \phi) dw. \tag{27}
\]

If \( \psi \) and \( \kappa \) have analytic forms, one can derive a MCMC sampler to approximate the posterior by successively updating \( U \) and \( \phi \). Unfortunately, this is not the case for our models. For instance, in the generalized BFry process case, we have

\[
\psi(t; \phi) = \frac{\eta}{\sigma} \int_0^c ((y + t)^\sigma - y^\sigma) y^{\tau - \sigma - 1} dy \tag{28}
\]
\[
\kappa(m, t; \phi) = \frac{\eta \Gamma(m - \sigma)}{\Gamma(1 - \sigma)} \int_0^c \frac{y^{\tau - \sigma - 1}}{(y + t)^{m - \sigma}} dy. \tag{29}
\]

We may resort to a numerical integration algorithm to approximate \( \psi \) as only one evaluation of this function is needed at each iteration. We could do the same for \( \kappa \). However, this would require \( K_n \) numerical integrations at each step of the MCMC sampler, which is computationally prohibitive for large \( K_n \). Instead, building on the construction of the generalized BFry as a scaled generalized gamma process described in Section D, we introduce a set of latent variables \( Y = (Y_k)_{j=1,\ldots,K_n} \) whose conditional density is written as

\[
p(y_k | u, (m_{n,(k)})_{k=1,\ldots,K_n}) \propto \frac{y_k^{\tau - \sigma - 1}}{(y_k + u)^{m_{n,(k)} - \sigma}} \mathbb{I}_{0 < y_k < c}, \tag{30}
\]

and this gives the joint density

\[
p((m_{n,(k)})_{k=1,\ldots,K_n}, u, y, \phi) \\
\propto p(\phi) u^{n-1} e^{-\psi(u;\phi)} \prod_{k=1}^{K_n} \frac{\eta \Gamma(m_{n,(k)} - \sigma) y_k^{\tau - \sigma - 1}}{\Gamma(1 - \sigma)(y_k + u)^{m_{n,(k)} - \sigma}}. \tag{31}
\]

where the normalizing constant only depends on \( n \) and the ranked counts. Then we can alternate between updating \( \phi \) and \( U \) via Metropolis-Hastings and updating \( Y \) via Hamiltonian Monte-Carlo (HMC) [Duane et al., 1987; Neal et al., 2011] to estimate the posterior. See the appendix for more details. A similar strategy can be used for the beta prime process.
5 Experiments

We run the algorithms described in Section 4.2 for the GBFRY and BP models. We fix \( c = 1 \) as explained previously. We use standard normal prior on \( \log \eta, \log \tau \) and \( \logit \sigma \). The proposed model are compared to the normalized GGP with the same priors on \( \eta \) and \( \sigma \) and fixed \( \zeta = 1 \), and the PY process with standard normal prior on \( \log \theta \) and \( \logit \alpha \). We stress that the objective is to show that the proposed models provide a better fit than alternative models, not to test the double power-law assumption. The codes for the experiments are available at [https://github.com/OxCSML-BayesNP/doublepowerlaw](https://github.com/OxCSML-BayesNP/doublepowerlaw).

5.1 Synthetic data

We first sample simulated datasets from the normalized GBFRY and the BP models with parameters \( \sigma = 0.1, \tau = 2, c = 1 \) and \( \eta = 4000 \). We run the MCMC algorithm described in Section 4.2 with 100,000 iterations. The 95% credible intervals are \( \sigma \in (0.09, 0.12) \), \( \tau \in (1.6, 2.2) \) for the BFRY and \( \sigma \in (0.08, 0.11) \), \( \tau \in (1.8, 2.3) \) for the BP model. The MCMC algorithm is therefore able to recover the true parameters. Trace plots are reported in the appendix.

5.2 Real data

We then consider five real datasets, four of which are word frequencies in natural languages, and the last is the out-degree distribution of a Twitter network. We first provide a description of the different datasets.

**Word frequencies.** In this setting, each dataset is composed of \( n \) words \( X_1, \ldots, X_n \), with \( K_n \leq n \) unique words. The counts \( m_n(k) \) represent therefore the number of occurrences of the \( k \)th most frequent word in the dataset.

We consider the written dataset of the American National Corpus\(^2\) (ANC), composed of about 18 million word occurrences and 300,000 unique words. We also consider the words of a collection of most popular English books and French books, downloaded from the Project Gutenberg\(^3\). The first dataset is composed of about 3 million words and 71,000 unique words, the second of about 7 million words and around 135,000 unique words. The fourth dataset represents the words of a thousand papers from the NIPS conference. It contains about 2 million word occurrences and 68,000 unique words.

**Twitter network.** We consider a rank-1 edge-exchangeable model for directed multigraphs [Crane and Dempsey, 2018][Cai et al., 2016]. In this case, the atoms of \( W \) represent the nodes of the graph, and each directed edge \((X_i, Y_i)\) from node \( X_i \) to node \( Y_i \) is sampled independently from \( P \times P \). Note that when \( P \) is a Pitman-Yor process, the associated model corresponds to the urn-based Hollywood model of [Crane and Dempsey, 2018]. Here, we only consider the out-degree distribution. Therefore, \( n \) represents the number of directed edges and \( X_1, \ldots, X_n \) the source nodes of the directed edges sampled from the normalized CRM \( P \). \( m_n(k) \) corresponds to the \( k \)'s largest out-degree in the network. We consider a subset of 25 millions tweets of August 2009 from Twitter [Yang and Leskovec, 2011]. We construct a directed multigraph by adding an edge \((X_i, Y_i)\) whenever user \( X_i \) mentions user \( Y_i \) (with @) in tweet \( i \). The resulting graph contains about 4 millions edges and 300,000 source nodes.

\(^2\)http://www.anc.org/data/anc-second-release/frequency-data/
\(^3\)http://www.gutenberg.org/
| Dataset     | GBFRY | Beta Prime | GGP | PY  |
|-------------|-------|------------|-----|-----|
| Englishbooks| 0.072 | 0.041      | 0.12| 0.12|
| Frenchbooks | 0.064 | 0.032      | 0.11| 0.11|
| NIPS1000    | 0.041 | 0.081      | 0.08| 0.059|
| ANC         | 0.033 | 0.034      | 0.082| 0.081|
| Twitter     | 0.10  | 0.047      | 0.25| 0.26|

Table 1: Average Kolmogorov-Smirnov divergence between the data and the posterior predictive. Lower is better.

| Dataset     | GBFRY | Beta Prime | GGP | PY  |
|-------------|-------|------------|-----|-----|
| Englishbooks| (0.351, 0.362) | (0.912, 0.980) | (0.345, 0.358) | (0.974, 1.078) | (0.416, 0.423) | (0.416, 0.423) |
| Frenchbooks | (0.368, 0.375) | (0.967, 1.039) | (0.363, 0.371) | (1.04, 1.175)  | (0.407, 0.412) | (0.407, 0.412) |
| NIPS1000    | (0.538, 0.545) | (1.338, 1.906) | (0.538, 0.545) | (1.541, 2.286) | (0.542, 0.548) | (0.542, 0.549) |
| ANC         | (0.433, 0.438) | (0.998, 1.055) | (0.431, 0.436) | (1.09, 1.17)   | (0.461, 0.465) | (0.461, 0.465) |
| Twitter     | (0.282, 0.287) | (1.590, 1.600) | (0.099, 0.116) | (1.336, 1.411) | (0.272, 0.277) | (0.272, 0.277) |

Table 2: 95% posterior credible intervals of the power-law exponents.

Results

For each of the four models and each dataset, we approximate the posterior distribution of the parameters $\phi$ of the Lévy measure, and sample new datasets from the posterior predictive. The 95% credible intervals of the posterior predictive for the proportion of occurrences and ranked frequencies are reported in Figure 3 for the ANC dataset (plots for the other datasets are given in the appendix). As the results for the normalized GGP and PY are almost identical, we only show the plot for the PY model. As can clearly be seen from the posterior predictive plots, all models provide a good fit for low frequencies. However, the PY model (and similar the normalized GGP) fail to capture the power-law behavior for large frequencies. This behavior is better captured by the GBFRY and BP models. To illustrate quantitatively the comparison, we compute the average reweighted Kolmogorov-Smirnov divergence [Clauset et al., 2009] between the true data and the posterior predictive for each model, and report the results in Table 1. Finally, we report in Table 2 the 95% credible intervals of the parameters for each model and dataset. We can remark that to the exception of the NIPS dataset, we recover the Zipfian exponent $\tau = 1$ for large frequencies in the natural language datasets.

6 Conclusion

In this paper we presented a novel class of random measures with double power-law behavior. We focused on the case of iid sampling from a normalized completely random measure. More generally, one could build on this class of models for other CRM-based constructions. In particular, it would be interesting to explore the asymptotic degree distribution when such models are used for random graph models based on exchangeable point processes [Caron and Fox, 2017]. Building hierarchical versions of such models as for the hierarchical Pitman-Yor process [Teh, 2006] would also be of interest. Finally, it would be useful to explore the connections between the models presented here and the two-stage urn process suggested by Gerlach and Altmann [2013] and investigate if other urn schemes could be derived that provably exhibit a double power-law behavior.
Figure 3: Results on the ANC dataset: 95% credible interval of the posterior predictive in blue, data in red. (Top) Proportion of occurrences of a given size. (Bottom) Ranked occurrences.

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Appendices

A Background on regular variation

See [Bingham et al., 1989]. In the following, $U$ denotes a regularly varying function and $\ell$ denotes a slowly varying function, locally bounded on $(0, \infty)$.

**Theorem 3 (Karamata’s theorem)** [Bingham et al., 1989, Propositions 1.5.8 and 1.5.10]. Suppose $\rho > -1$ and $U(t) \sim t^\rho \ell(t)$ as $t$ tends to infinity. Then

$$\int_0^x U(t)dt \sim \frac{1}{\rho + 1} x^{\rho+1} \ell(x)$$

as $x$ tends to infinity.

Suppose $\rho < -1$. Then $U(t) \sim t^\rho \ell(t)$ as $t$ tends to infinity implies

$$\int_x^\infty U(t)dt \sim -\frac{1}{\rho + 1} x^{\rho+1} \ell(x)$$

as $x$ tends to infinity.

**Corollary 4** Suppose $\rho < -1$ and $U(y) \sim y^\rho \ell(1/y)$ as $y$ tends to 0. Then

$$\int_x^\infty U(y)dy \sim -\frac{1}{\rho + 1} x^{\rho+1} \ell(1/x)$$

as $x$ tends to 0.

**Proof.** $U(t) = t^\rho \ell_1(1/t)$ where $\ell_1(t) \sim \ell(t)$ as $t \to \infty$.

$$\int_x^\infty U(y)dy = \int_0^{1/x} t^{-2-\rho} \ell_1(t)dt$$

$$\sim -\frac{1}{1 + \rho} x^{1+\rho} \ell(1/x)$$

as $x$ tends to 0 by Theorem 3.

B Proofs

B.1 Proof of Equations (14) and (15)

For any $s$, the function $x \to \gamma(s, x)$ is both regularly varying at 0 and infinity with

$$\gamma(s, x) \sim \begin{cases} \frac{x^s}{\Gamma(s)} & \text{as } x \to 0 \\ \frac{e^{-x}}{\Gamma(1-s)} & \text{as } x \to \infty \end{cases}$$

we have therefore for the generalized BFYR process

$$\rho(w) \sim \begin{cases} \frac{e^{-w}}{\Gamma(1-s)} w^{-1-s} & \text{as } x \to 0 \\ \frac{1}{\Gamma(-\sigma) w^{-1-\tau}} & \text{as } x \to \infty \end{cases}$$

and Equations (14) and (15) follow from Theorem 3 and Corollary 4.
B.2 Proof of Equations (17) and (18)

We have for the beta prime process

\[ \rho(w) \sim \begin{cases} 
\frac{e^{-\tau} \Gamma(\tau - \sigma)}{\Gamma(1 - \sigma)} w^{1 - \sigma} & \text{as } x \to 0 \\
\frac{\Gamma(\tau - \sigma)}{\Gamma(1 - \sigma)} w^{1 - \tau} & \text{as } x \to \infty 
\end{cases} \]

Equations (17) and (18) then follow from Theorem 3 and Corollary 4.

B.3 Proof of Proposition 1

Lemma 1 Let \((X_k)_k\) Poisson random variables such that \(\frac{\log k}{k} \to 0\), then

\[ \frac{X_k}{E X_k} \to 1 \text{ a.s.} \]

**Proof.** Let \(X\) be a Poisson random variable with parameter \(\lambda\). Using the Chernoff bound, it comes that for any \(t > 0\)

\[ \P(|X - \lambda| \geq \lambda t) \leq 2e^{-\frac{\lambda t^2}{2(\lambda + t)}}. \]

Let \(0 < \epsilon < 1/2\). We deduce from previous inequality that

\[ \P\left(\left|\frac{X_k}{E X_k} - 1\right| \geq \epsilon\right) \leq 2e^{-\frac{\epsilon^2 E X_k}{4}} = 2k^{-\frac{\epsilon^2 E X_k}{4}} \]

Using the assumption, we have that \(\frac{\epsilon^2 E X_k}{4} \to -\infty\). Therefore, the RHS is summable. The almost sure result follows from Borel-Cantelli lemma. \(\square\)

Now we can prove Proposition 1. Let \(\nu = \sum_k \delta_{w_k}\). Then, for all \(x > 0\), \(\nu([x, +\infty))\) is a Poisson random variable with mean \(\rho(x)\). Let us show that,

\[ \nu((1/x, +\infty)) \overset{\chi}{\to} x^{\alpha} \ell(x) \text{ a.s.} \]

Using Lemma 1 on the sequence \((\nu((1/k, +\infty)))_{k \geq 1}\), we find that

\[ \nu((1/k, +\infty)) \overset{k \to +\infty}{\sim} k^{\alpha} \ell_1(k) \text{ a.s.} \]

Now, since \(x \mapsto \nu((1/x, +\infty))\) is almost surely non decreasing, it comes that

\[ \nu((1/|x|, +\infty)) \leq \nu((1/|x| + 1, +\infty)) \leq \nu((1/|x + 1|, +\infty)) \]

We get the desired result by noticing that

\[ (|x + 1|)^{\alpha} \ell_1(|x + 1|) \sim (|x|)^{\alpha} \ell_1(|x|) \sim x^{\alpha} \ell_1(x) \]

and using Proposition 23 of [Gnedin et al. 2007], leading to

\[ w(k) \sim k^{-1/\alpha} \ell_1^*(k) \]

where \(\ell_1^*\) is a slowly varying function defined by

\[ \ell_1^*(x) = \frac{1}{\{\ell_1^{1/\alpha}(x^{1/\alpha})\}^\#} \]

where \(\ell^\#\) is a de Bruijn conjugate of the slowly varying function \(\ell\).
C Useful properties

\[ \gamma(1, x) = 1 - e^{-x} \]

\[
\gamma(s, x) = \int_0^x u^{s-1} e^{-u} du \\
= x^s \int_0^1 u^{s-1} e^{-ux} du \\
\gamma(s, x) \sim \frac{x^s}{s}
\]
as \( x \to 0 \). We have

\[
\int_{w_0}^{\infty} w^{m-1} e^{-wt} dw = t^{m-1} \Gamma(m - \tau, tw_0)
\]

D Generalized BFry distribution

BFry random variable [Bertoin et al., 2006, Devroye and James, 2014] is a positive random variable \( W \) with density

\[
f_W(w) = \frac{\sigma}{\Gamma(1 - \sigma)} w^{1-\sigma} (1 - e^{-w}), \quad \sigma \in (0, 1).
\]

\( W \) is a heavy tailed random variable with infinite mean, and is known to have a close connection to the stable and generalized gamma processes [Lee et al., 2016]. \( W \) can be simulated as \( W = X/Y \) where \( X \sim \text{Gamma}(1-\sigma, 1) \) and \( Y \sim \text{Beta}(\sigma, 1) \).

Now let \( W = X/Y, X \sim \text{Gamma}(\tau - \sigma, 1) \) and \( Y \sim \text{Beta}(\sigma, 1) \), with additional parameter \( \tau > \sigma \). Then the density of \( W \) is computed as

\[
f_W(w) = \int_0^1 yf_X(wy) f_Y(y) dy \\
= \frac{\sigma}{\Gamma(\tau - \sigma)} \int_0^1 y(wy)^{\tau-\sigma-1} e^{-wy} y^{\sigma-1} dy \\
= \frac{\sigma}{\Gamma(\tau - \sigma)} w^{\tau-\sigma-1} \int_0^1 y^{\tau-1} e^{-wy} dy \\
= \frac{\sigma}{\Gamma(\tau - \sigma)} w^{\tau-\sigma-1} \gamma(\tau, w). \quad (32)
\]

The resulting distribution, which we call as the generalized BFry distribution, contains the BFry as its special case when \( \tau = 1 \) and \( \sigma \in (0, 1) \) (note that we don’t restrict \( \sigma \in (0, 1) \) in general), and has potentially heavier tail than the BFry distribution. Like the BFry distribution has a close connection with the stable and generalized gamma process, the generalized BFry distribution has a close connection with the generalized BFry process we described in the main text. Indeed, the name generalized BFry process can be thought as a process version of the generalized BFry random variable, and the name generalized BFry process was coined after this connection.

For \( m < \sigma \), the moments are computed as

\[
\mathbb{E}(W^m) = \frac{\sigma}{\sigma - m}, \quad (33)
\]

and \( \mathbb{E}(W^m) = \infty \) for \( m \geq \sigma \).
E Additional details on the inference

Here we describe detailed inference procedures for Generalized BFry process and Beta-prime process.

E.1 Generalized BFry process

The Lévy density of generalized BFry process is written as

$$\rho(w) = \frac{1}{\Gamma(1-\sigma)} w^{-1-\gamma} \gamma(\tau-\sigma, w),$$  

(34)

where we fixed $c = 1$. The quantities required for the evaluation of joint likelihood is

$$\psi(t) = \eta \frac{\gamma}{\sigma} \int_{0}^{1} ((y + t)^\sigma - y^\sigma) y^{\gamma-\sigma-1} dy$$  

(35)

$$\kappa(m, t) = \eta \frac{\gamma}{\sigma} \int_{0}^{1} \frac{y^{\gamma-\sigma-1}}{(y + t)^{m-\sigma}} dy.$$  

(36)

As explained in the main text, we introduce a set of latent variables $(y_j)_{j=1}^{K_n}$ with

$$p(y_j) \propto \frac{y_j^{\gamma-\sigma-1}}{(y_j + t)^{m_j-\sigma}} \mathbb{1}_{0 < y_j < 1}.$$  

(37)

The joint log-likelihood is then written as

$$p((m_j)_{j=1}^{K_n}, y, u | \eta, \sigma, \tau) \propto u^{n-1} e^{-\psi(u)}$$

$$\times \prod_{j=1}^{K_n} \frac{\eta \gamma}{\sigma} \frac{(y_j^{\gamma-\sigma-1})}{(y_j + u)^{m_j-\sigma}}.$$  

(38)

Since $y_j \in (0, 1)$, we take a transformation

$$y_j = \frac{1}{1 + e^{-\tilde{y}_j}},$$  

(39)

which yields

$$p((m_j)_{j}, \tilde{y}, u | \eta, \sigma, \tau) \propto u^{n-1} e^{-\psi(u)}$$

$$\times \prod_{j=1}^{K_n} \frac{\eta \gamma}{\sigma} \frac{(y_j^{\gamma-\sigma-1})}{(y_j + u)^{m_j-\sigma}}.$$  

(40)

Sampling $\tilde{y}$ We update $\tilde{y}$ via HMC [Duane et al., 1987, Neal et al., 2011]. The gradient of $\log p((m_j)_{j}, \tilde{y}, u | \eta, \sigma, \tau)$ w.r.t. $\tilde{y}_j$ is given as

$$\left(\frac{\tau - \sigma}{y_j} - \frac{1}{1 - y_j} - \frac{m_j - \sigma}{y_j + u}\right) \cdot y_j (1 - y_j).$$  

(41)

For all experiments, we used step size $\epsilon = 0.05$ and number of leapfrog steps $L = 30$. 

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**Sampling** $u$  We take a transform $u = e^\tilde{u}$ and update $\tilde{u}$ via Metropolis-Hastings with proposal distribution $q(\tilde{u}'|\tilde{u}) = \text{Normal}(\tilde{u}, 0.05)$.

**Sampling** $\eta$  We place a prior $\eta \sim \text{Lognormal}(0, 1)$, and updated $\eta$ via Metropolis-Hastings with proposal distribution $q(\hat{\eta}|\eta) = \text{Lognormal}(\log \eta, 0.05)$.

**Sampling** $\sigma$  We place a prior $\sigma \sim \text{Logitnormal}(0, 1)$, and updated $\sigma$ via Metropolis-Hastings with proposal distribution $q(\hat{\sigma}|\sigma) = \text{Logitnormal}(\logit(\sigma), 0.05)$.

**Sampling** $\tau$  Since $\tau > \sigma$, instead of directly sampling $\tau$, we sampled $\delta = \tau - \sigma > 0$. Then we place a prior $\delta \sim \text{Lognormal}(0, 1)$ and update $\delta$ via Metropolis-Hastings with proposal distribution $q(\hat{\delta}|\delta) = \text{Lognormal}(\log \delta, 0.05)$.

### E.2 Beta prime process

The Lévy density of Beta prime process is

$$\rho(w) = \frac{\Gamma(\tau - \sigma)}{\Gamma(1 - \sigma)} w^{-1-\sigma} (1 + w)^{\sigma - \tau}, \quad (42)$$

where we fixed $c = 1$. Then we have

$$\psi(t) = \frac{\eta}{\sigma} \int_0^\infty ((y + t)^\sigma - y^\sigma) y^{\tau - \sigma - 1} e^{-y} dy, \quad (43)$$

$$\kappa(m, t) = \frac{\eta \Gamma(m - \sigma)}{\Gamma(1 - \sigma)} \int_0^\infty \frac{y^{\tau - \sigma - 1} e^{-y}}{(y + t)^{m - \sigma}} dy. \quad (44)$$

As for the generalized BFry process, we augment the joint likelihood with a set of latent variables $(Y_j)_{j=1}^{K_n}$ with density

$$p(y_j) \propto \frac{y_j^{\tau - \sigma - 1}}{(y_j + u)^{m_j - \sigma}} \mathbb{1}_{y_j < 0}, \quad (45)$$

which yields

$$p((m_j)_j, y, u|\eta, \sigma, \tau) \propto u^{n-1} e^{-\psi(u)}$$

$$\times \prod_{j=1}^{K_n} \eta \frac{\Gamma(m_j - \sigma)}{\Gamma(1 - \sigma)} \frac{y_j^{\tau - \sigma - 1}}{(y_j + u)^{m_j - \sigma}}. \quad (46)$$

Since $y_j > 0$, we take a transformation $y_j = e^{\tilde{y}_j}$ to have

$$p((m_j)_j, y, u|\eta, \sigma, \tau) = \frac{u^{n-1} e^{-\psi(u)}}{\Gamma(n)}$$

$$\times \prod_{j=1}^{K_n} \eta \frac{\Gamma(m_j - \sigma)}{\Gamma(1 - \sigma)} \frac{y_j^{\tau - \sigma}}{(y_j + u)^{m_j - \sigma}}. \quad (47)$$
Figure 4: Trace plots of the parameter samples for the Generalized BFRY model. Dashed line represents true value of the parameter.

Figure 5: Trace plots of the parameter samples for the Beta prime process model. Dashed line represents true value of the parameter.

Sampling $\tilde{y}$  We update $\tilde{y}$ via HMC. The gradient required for $\tilde{y}$ is computed as

$$\tau - \sigma - y_j - (m_j - \sigma) \frac{y_j}{y_j + u}.$$ 

Sampling $u, \eta, \sigma, \tau$  Same as for the generalized BFRY process.

F  Results of experiments

F.1  Synthetic data

As explained in the main text, we sample simulated datasets from the GFRY and the BP models with parameters $\sigma = 0.1$, $\tau = 2$, $c = 1$ and $\eta = 4000$. We run the MCMC algorithm described in Section 4.2 with 100000 iterations. The 95% credible intervals are $\sigma \in (0.09, 0.12)$, $\tau \in (1.6, 2.2)$ for the BFRY and $\sigma \in (0.08, 0.11)$, $\tau \in (1.8, 2.3)$ for the BP model. The MCMC algorithm is therefore able to recover the true parameters. Trace plots are reported in Figures 4 and 5.
F.2 Real data

Here we report the results for the 5 datasets described in the main text. We report the 95% credible intervals of the posterior predictive for the proportion of occurrences and ranked frequencies of the Generalized BFRY, BP, normalized GGP and PY models for each dataset in Figures 6 to 10. We can see that as predicted the GGP and PY do not manage to capture the behavior of the large clusters (which are on the right of the figures displaying the proportion of clusters of a given size, and on the left on the figures displaying the ordered sizes of the clusters).
Figure 7: (Top) proportion of clusters of a given size in the English books dataset: 95% credible interval of the posterior predictive in blue, real values in red. (Bottom) ordered size of the clusters in the English books dataset: 95% credible interval of the posterior predictive in blue, real values in red.

Figure 8: (Top) proportion of clusters of a given size in the French books dataset: 95% credible interval of the posterior predictive in blue, real values in red. (Bottom) ordered size of the clusters in the French books dataset: 95% credible interval of the posterior predictive in blue, real values in red.
Figure 9: (Top) proportion of clusters of a given size in the NIPS dataset: 95\% credible interval of the posterior predictive in blue, real values in red. (Bottom) ordered size of the clusters in the NIPS dataset: 95\% credible interval of the posterior predictive in blue, real values in red.

Figure 10: (Top) proportion of nodes with a given degree in the Twitter dataset: 95\% credible interval of the posterior predictive in blue, real values in red. (Bottom) ordered size of the clusters in the Twitter dataset: 95\% credible interval of the posterior predictive in blue, real values in red.