Recursive maximum likelihood identification of jump Markov nonlinear systems

Emre Özkan∗, Fredrik Lindsten∗, Carsten Fritsche†, and Fredrik Gustafsson∗

October 14, 2013

Abstract

In this contribution, we present an online method for joint state and parameter estimation in jump Markov non-linear systems (JMNLS). State inference is enabled via the use of particle filters which makes the method applicable to a wide range of non-linear models. To exploit the inherent structure of JMNLS, we design a Rao-Blackwellized particle filter (RBPF) where the discrete mode is marginalized out analytically. This results in an efficient implementation of the algorithm and reduces the estimation error variance. The proposed RBPF is then used to compute, recursively in time, smoothed estimates of complete data sufficient statistics. Together with the online expectation maximization algorithm, this enables recursive identification of unknown model parameters. The performance of the method is illustrated in simulations and on a localization problem in wireless networks using real data.

Keywords—Adaptive Filtering, Particle Filter, Rao-Blackwellization, Expectation Maximization, Parameter Estimation, Jump Markov Systems

1 Introduction

Jump Markov processes have been extensively used in control theory, signal processing, telecommunications and econometrics for modeling multi-modal behavior of systems (see e.g. [15, 29] for a brief review of applications). Most studies have focused on a special class of these models, jump Markov linear systems (JMLS), also known as conditionally linear Gaussian models. In these models, a finite-state Markov chain switches between different linear modes. The true posterior of a JMLS is a mixture of Gaussians with an exponentially increasing number of components, which is intractable to compute in any realistic scenario. However, many approximate state inference algorithms have been proposed for JMLS, most of which rely on Kalman filters for computing

∗Emre Özkan, Fredrik Lindsten and Fredrik Gustafsson are with the Division of Automatic Control, Linköping University, Linköping, Sweden,(emre,lindsten,fredrik)@isy.liu.se.
†Carsten Fritsche is with IFEN GmbH, Alte Gruber Str. 6, 85586 Poing, Germany, carsten@isy.liu.se
the conditional estimates for the linear Gaussian modes. These include the gen-
eralized pseudo Bayesian (GPB) approach, the interacting multiple model
(IMM) filter and the Rao-Blackwellized particle filter (RBPF).

In cases when the underlying model has unknown parameters the problem
becomes even more challenging, owing to the coupling between the unknown
model parameters and the latent states. There are a number of studies which fo-
cuses on the identification and/or adaptation of the JMLS. However, these algorithms are not applicable when the dynamic modes of the
system are nonlinear. Such jump Markov nonlinear systems (JMNLS) arise
in various applications including target tracking, localization, econometrics,
eand audio signal processing. Identification of JMNLS is a challenging problem. Indeed, addressing the state inference problem alone
is problematic as the various approximate algorithms mentioned above cannot
be used in this setting. Specially tailored sequential Monte Carlo (SMC) sam-
plers, i.e. particle filters, have been proposed in the literature during the last
decade. These methods can be used for state inference in JMNLS. However, there has not been much progress made on addressing the joint state and parameter estimation problem for JMNLS.

In this paper, we consider the problem of recursive (i.e. online) maximum
likelihood identification of JMNLS. The method that we propose is based on an
online expectation maximization (EM) algorithm. The (batch) EM algorithm
is one of the most popular methods for maximum likelihood identification of
latent variable models. It has been applied to a wide range of practical problems
in different fields such as statistics, biology and signal processing. Recent contributions have focused on using EM in an online setting, i.e.,
when the observations are processed only once and never stored. The online
EM algorithm was initially proposed for hidden Markov models (HMMs) with
a finite number of states and observations. This idea has then been ex-
tended to generalized HMMs with, possibly, continuous observations. In
a particle-filter-based online EM algorithm is proposed for joint state and
parameter estimation in general (possibly non-linear/non-Gaussian) state-space
models. This algorithm is further developed in by making use of forward-
only smoothing techniques. In online EM is used to solve the simultaneous
localization and mapping problem. The particle-filter-based online EM algorithm is used in for estimating the measurement noise distribution in a
general state-space models. The same approach is used specifically for JMNLS
in without making use of Rao-Blackwellization for the discrete mode vari-
ables.

While the algorithms by can be used also for JMNLS, they do not
exploit the inherent structure of these models. As we shall see, this can result
in poor performance. Any standard particle filter (see e.g., ) can be
used for state inference in JMNLS. However, this can lead to problems due to
severe particle degeneracy around mode changes. Different improvement
strategies have been proposed to address this issue, enabling efficient use of
SMC for JMNLS. In particle depletion is prevented by splitting the filtering
recursions for the discrete mode and the continuous state, resulting in the IMM
particle filter. In auxiliary particle filters are used to construct an efficient
sampling strategy. In a Markovian prior is assumed for the discrete modes
which allows the transition probabilities to evolve over time, resulting in more
robust estimators.

In this paper we propose an alternative modification, namely to make use of a RBPF. As mentioned above, the RBPF is most well known as an algorithm for state inference in conditionally linear Gaussian models, where one state component is marginalized by running conditional Kalman filters [9, 14, 41]. A general JMNLS is not conditionally linear Gaussian, so this approach is not directly applicable. However, we may still exploit the idea of Rao-Blackwellization, by marginalizing the mode variable using conditional HMM filters. This improves the performance over a standard particle filter as the asymptotic variance is reduced [27, 10]. Furthermore, by not using particles to represent the mode variable, we are less affected by the degeneracy problems around mode changes as reported in [17].

To the best of the authors’ knowledge, the proposed RBPF is a novel approach to state inference in JMNLS. However, the main contribution of this paper lies in the adaption of this RBPF to address the forward-only smoothing problem which lies at the core of online EM. This further extends the online EM algorithms by [3, 12] to general JMNLS. The resulting method can be used to estimate the unknown transition probabilities as well as the unknown model parameters jointly with the state in an online fashion.

2 Expectation Maximization

The EM algorithm [13] is an iterative method which is useful for computing ML estimates, \( \hat{\theta}^{\text{ML}} \), of unknown parameters \( \theta \) in probabilistic models involving latent variables. Consider the (batch) ML problem,

\[
\hat{\theta}^{\text{ML}} = \underset{\theta \in \Theta}{\arg \max} \log p(y_{1:n}; \theta),
\]

where \( y_{1:n} \) is a collection of \( n \) observations and \( \Theta \) is the feasible set of parameters. The idea of the EM algorithm is to separate the original ML estimation problem into two linked problems, denoted by the expectation (E) step and the maximization (M) step, each of which is hopefully easier to solve than the original problem. Let \( z_{1:n} \) denote the latent variables of the models (for a state-space model, these are typically given by the unobserved state variables). We then introduce the auxiliary quantity,

\[
Q(\theta, \theta') = \mathbb{E}_{\theta'} \left[ \log p(y_{1:n}, z_{1:n}; \theta) \mid y_{1:n} \right] = \int \log p(y_{1:n}, z_{1:n}; \theta) \log p(z_{1:n} \mid y_{1:n}; \theta') \, dz_{1:n}.
\]

The auxiliary quantity can be thought of as a proxy for the log-likelihood function. The EM algorithm is useful when maximization of \( \theta \mapsto Q(\theta, \theta') \), for fixed \( \theta' \), is simpler than direct maximization of the log-likelihood, \( \theta \mapsto \log p(y_{1:n}; \theta) \).

The procedure is initialized at some \( \theta^0 \in \Theta \) and then iterates between,

- **E-Step:** Compute \( Q(\theta, \theta^{m-1}) \).
- **M-Step:** Compute \( \theta^m = \arg \max_{\theta \in \Theta} Q(\theta, \theta^{m-1}) \).

At each iteration of the EM algorithm, the parameters are updated so that the value of the log-likelihood is non-decreasing. The EM algorithm is thus a monotone optimization algorithm. Furthermore, the resulting sequence \( \{\theta^m\}_{m \geq 0} \)
will, under weak assumptions, converge to a stationary point of the likelihood \( p(y_{1:n}; \theta) \).

Note that the auxiliary quantity (2) is given by the smoothed estimate of the so called complete data log-likelihood \( \log p(y_{1:n}, z_{1:n} ; \theta) \). This poses an apparent difficulty in using the EM algorithm for solving the online identification problem, as smoothing is typically an offline procedure. However, it has been recognized that this is indeed possible. The key enabler of the online EM algorithm \([34, 33, 32, 12]\) is to make use of forward-only smoothing techniques. This enables the computation of a stochastic approximation of the auxiliary quantity in an online fashion. This approximation can then be subsequently used to update the parameters in the M-step at each iteration. We will discuss how this is done specifically in the context of JMNLS in the subsequent sections.

3 Jump Markov Nonlinear Systems

We will derive an online EM algorithm for jump Markov nonlinear systems (JMNLS) in the form,

\[
\begin{align*}
    r_t &\sim \Pi(r_t | r_{t-1}), \\
    x_t &\sim f_{r_t}(x_t | x_{t-1}; \theta_{r_t}), \\
    y_t &\sim g_{r_t}(y_t | x_t; \theta_{r_t}).
\end{align*}
\]  

This is a hybrid system with a continuous state variable \( x_t \in X \) and a discrete mode variable \( r_t \in \{1, \ldots, K\} \), where \( K \) is the number of modes. The system states \( r_t \) and \( x_t \) are latent, but observed indirectly through the measurements \( y_t \), taking values in some set \( Y \). The mode variable follows a (finite state-space) hidden Markov model (HMM) with transition probabilities

\[
\pi_{k\ell} = \Pi(\ell | k) = \mathbb{P}(r_t = \ell | r_{t-1} = k). 
\]  

(4)

The system thus switches between different nonlinear dynamical modes. While in mode \( k \), the transition density function for the state \( x_t \) and the likelihood of the measurement \( y_t \) are given by \( f_k(x_t | x_{t-1}; \theta_k) \) and \( g_k(y_t | x_t; \theta_k) \), respectively. Each mode \( k \) is parameterized by its own set of parameters \( \theta_k \). Furthermore, the transition probabilities \( \pi_{k\ell} \) for the mode sequence \( \{r_t\} \) are assumed to be unknown parameters. By abuse of notation we let \( \Pi \) refer to both the transition kernel for \( r_t \), as in \([30]\), and the \( K \times K \) transition probability matrix with entries \( [\Pi]_{k\ell} = \pi_{k\ell} \). The unknown parameters of the model are thus given by

\[
\theta = (\{\theta_k\}_{k=1}^K, \Pi).
\]  

(5)

For notational convenience, we assume that the initial state of the system \((x_0, r_0)\) is known. The generalization to an unknown initial state, exogenous inputs and/or time-inhomogeneous dynamics is straightforward.

4 EM Algorithm for JMNLS

For JMNLS, direct optimization of \( (1) \) is typically not possible due to the intractability of computing the likelihood \( p(y_{1:n}; \theta) \). To address this difficulty, we
make use of the EM algorithm. We start the derivation of the online EM algorithm by considering batch EM for JMNLS, and then continue with the online formulation.

4.1 Complete data sufficient statistics

Let the latent variables \(z_{1:n}\) be given by the system states, i.e., \((x_{1:n}, r_{1:n})\). If follows that the complete data likelihood can be factorized as,

\[
p(x_{1:n}, r_{1:n}, y_{1:n}; \theta) = \prod_{t=1}^{n} p(x_t, r_t, y_t | x_{t-1}, r_{t-1}; \theta) = \prod_{t=1}^{n} p(x_t, r_t, y_t | x_{t-1}, r_{t-1}; \theta)
\]  

(6)

In the following, we focus on the complete data sufficient statistics formulation of the EM algorithm [3, 31]. It is assumed that the nonlinear dynamical system corresponding to each mode belongs to the curved exponential family. That is, for each \(k \in 1, \ldots, K\) we have

\[
g_k(y_t | x_t; \theta_k) f_k(x_t | x_{t-1}; \theta_k) = C_k \exp(\langle \psi_k(\theta_k), s_k(y_t, x_t, x_{t-1}) \rangle) - A_k(\theta_k),
\]  

(7)

where \(C_k\) may depend on \(y_t, x_t\) and \(x_{t-1}\), but is independent of \(\theta_k\). \(\langle \cdot, \cdot \rangle\) denotes inner product; \(\psi_k(\theta_k)\) is the natural parameter; \(s_k(y_t, x_t, x_{t-1})\) is the complete data sufficient statistic and \(A_k(\theta_k)\) denotes the log-partition function.

Furthermore, we assume that there exist unique maximizers of the complete data likelihoods. That is, for each \(k = 1, \ldots, K\), there exists a mapping \(\Lambda_k(S) : \theta_k \rightarrow \theta_k\) given by,

\[
\Lambda_k(S) = \arg \max_{\theta_k \in \Theta_k} \{ \langle \psi_k(\theta_k), S \rangle - A_k(\theta_k) \},
\]  

(8)

where \(\Theta_k\) is the feasible set for the parameter \(\theta_k\). In Appendix 7 we provide the explicit expressions for these mappings for the special case of jump Markov Gaussian systems with unknown noise parameters.

In order to compute the auxiliary quantity \(Q(\theta, \theta')\), we make use of the indicator function \(\mathbb{1}(\cdot)\) and write the logarithm of the complete data likelihood \(\log p(x_{1:n}, r_{1:n}, y_{1:n}; \theta)\) as (omitting constant terms),

\[
\log p(x_{1:n}, r_{1:n}, y_{1:n}; \theta) = \sum_{t=1}^{n} \log \mathbb{1}(r_t | r_{t-1}) + \sum_{t=1}^{n} \log (g_{r_t}(y_t | x_t, r_{t-1}) f_{r_t}(x_t | x_{t-1}, r_{t-1}))
\]

\[
= \sum_{k=1}^{K} \sum_{t=1}^{n} \log (\pi_{k,t}) \mathbb{1}(r_t = k, r_{t-1} = k) + \sum_{k=1}^{K} \sum_{t=1}^{n} \mathbb{1}(r_t = k) (\langle \psi_k(\theta_k), s_k(y_t, x_t, x_{t-1}) \rangle - A_k(\theta_k)).
\]  

(9)

The auxiliary quantity of the EM algorithm can be written as,

\[
Q(\theta, \theta') = \mathbb{E}_\theta \left[ \log p(x_{1:n}, r_{1:n}, y_{1:n}; \theta) | y_{1:n} \right] = \sum_{k=1}^{K} \sum_{t=1}^{n} S_{k,t,n}^{(1)} \log \pi_{k,t} + \sum_{k=1}^{K} \left( \langle \psi_k(\theta_k), s_k(y_t, x_t, x_{t-1}) \rangle - A_k(\theta_k) S_{k,n}^{(2)} \right),
\]  

(10)
where we have introduced the sufficient statistics,

$$S^{(1)}_{k\ell,n} = \sum_{t=1}^{n} \mathbb{E}_{\theta} [\mathbb{1}(r_t = \ell, r_{t-1} = k) | y_{1:n}], \quad (11a)$$

$$S^{(2)}_{k,n} = \sum_{t=1}^{n} \mathbb{E}_{\theta} [\mathbb{1}(r_t = k) | y_{1:n}], \quad (11b)$$

$$S^{(3)}_{k,n} = \sum_{t=1}^{n} \mathbb{E}_{\theta} [\mathbb{1}(r_t = k)s_{k,t}(y_t, x_t, x_{t-1}) | y_{1:n}], \quad (11c)$$

for $k, \ell = 1, \ldots, K$.

For the M-step, we need to maximize (10) w.r.t. $\theta$. However, we note that the objective function is separable across the modes. That is, we can maximize each term of the second sum independently w.r.t. $\theta_k$. Furthermore, the transition probability matrix $\Pi$ only enters the first sum in (10). This term can thus be maximized, under the constraints $\pi_{k\ell} \geq 0$ and $\sum_{\ell=1}^{K} \pi_{k\ell} = 1$, by using standard formulae for HMMs (see e.g. [5]). The M-step can be computed as follows:

$$\hat{\theta}_k = \Lambda_k (S^{(3)}_{k,n} / S^{(2)}_{k,n}), \quad k = 1, \ldots, K, \quad (12a)$$

$$\hat{\pi}_{k\ell} = \frac{S^{(1)}_{k\ell,n}}{\sum_{j=1}^{K} S^{(1)}_{kj,n}}, \quad k, \ell = 1, \ldots, K. \quad (12b)$$

Note that it is possible to extend the model to account for common but unknown parameters across different modes. Furthermore, constraints on the parameters in the original ML formulation carry over to the M-step of the EM algorithm. This makes the algorithm suitable for constrained parameter estimation problems.

### 4.2 Online-EM for JMNLS

A closer look at (11) reveals that the EM algorithm requires the computation of smoothed additive functionals. In an offline implementation, standard forward/backward or two-filter smoothers may be used to compute these smoothed estimates; see e.g. [26] for a recent survey. However, for online EM, the smoothed functionals need to be computed online. For the case of additive functionals, this is in fact possible by using so called forward-only smoothing techniques (see e.g. [6, 12]) which are based on dynamic programming.

For notational simplicity, we use the joint state variable $\xi_t = (r_t, x_t)$. Let,

$$s^{(1)}_t(\xi_t, \xi_{t-1}) = \text{vec}([\mathbb{1}(r_t = \ell, r_{t-1} = k)]_{k=1}^{K}), \quad (13a)$$

$$s^{(2)}_t(\xi_t, \xi_{t-1}) = \text{vec}([\mathbb{1}(r_t = k)]_{k=1}^{K}), \quad (13b)$$

$$s^{(3)}_t(\xi_t, \xi_{t-1}) = \text{vec}([\mathbb{1}(r_t = k)s_{k,t}(y_t, x_t, x_{t-1})]_{k=1}^{K}), \quad (13c)$$

where $\text{vec}(\cdot)$ is the vectorization operator, which stacks the elements of a set in a vector (using some convenient ordering) and where we have removed the dependence on $y_t$ in the notation for brevity. Furthermore, let

$$s_t(\xi_t, \xi_{t-1}) = \begin{pmatrix} s^{(1)}_t(\xi_t, \xi_{t-1}) \\ s^{(2)}_t(\xi_t, \xi_{t-1}) \\ s^{(3)}_t(\xi_t, \xi_{t-1}) \end{pmatrix}. \quad (14)$$
It follows that (11) can be written compactly as,

$$S_n = E_{\theta'}[S_n(\xi_{0:n}) \mid y_{1:n}].$$

(15)

with

$$S_n(\xi_{0:n}) = \sum_{t=1}^{n} s_t(\xi_{t-1}, \xi_t).$$

(16)

Consider the intermediate quantity $T_t(\xi_t) \triangleq E_{\theta'}[S_t(\xi_{0:t}) \mid \xi_t, y_{0:t}]$. Note that $T_t(\xi_t)$ is a function of the joint state $\xi_t$. From the tower property of conditional expectation, it follows that

$$S_t = E_{\theta'}[T_t(\xi_t) \mid y_{1:t}] = \int T_t(\xi_t)p(\xi_t \mid y_{1:t}; \theta') d\xi_t.$$  

(17)

That is, the smoothed additive functional (15) is given by the filtered estimate of $T_t(\xi_t)$. Furthermore, the additive form (16) allows us to express $T_t$ recursively,

$$T_t(\xi_t) = \int [T_{t-1}(\xi_{t-1}) + s_t(\xi_{t-1}, \xi_t)]$$

$$\times p(\xi_{t-1} \mid \xi_t, y_{1:t-1}; \theta') d\xi_{t-1},$$

(18)

with $T_0(\xi_t) \equiv 0$.

The online EM algorithm exploits the recursive form in (18). At each time step $t$, the intermediate quantity $T_t(\xi_t)$ is updated and a new parameter estimate $\hat{\theta}$ is computed according to (12). Since the parameters of the model are updated on the fly, a stochastic approximation type of forgetting is used to update the intermediate quantity. That is, we update $T_t(\xi_t)$ at each iteration according to,

$$T_t(\xi_t) \leftarrow \int [(1 - \gamma_t)T_{t-1}(\xi_{t-1}) + \gamma_t s_t(\xi_{t-1}, \xi_t)]$$

$$\times p(\xi_{t-1} \mid \xi_t, y_{1:t-1}; \hat{\theta}^{t-1}) d\xi_{t-1},$$

(19)

where $\hat{\theta}^{t-1}$ is the current parameter estimate and $\{\gamma_t\}_{t \geq 1}$ is a sequence of decreasing step-sizes, satisfying the stochastic approximation requirements $\sum_{t \geq 1} \gamma_t = \infty$ and $\sum_{t \geq 1} \gamma_t^2 < \infty$. See [3, 12, 4] for further discussion on the online EM algorithm.

5 SMC Implementation

Exact computation of the smoothed statistics in (17) and (19) is not possible in general for a JMNLS. We now turn to computational methods based on SMC to approximate these quantities.

5.1 Rao-Blackwellized particle filter for JMNLS

Rao-Blackwellization (or marginalization) is a key step in efficient implementation of particle filters. Much previous work has been focused on conditionally linear Gaussian models, where one state component is marginalized by running conditional Kalman filters [9, 13, 11]. This is not possible for the case of
JMNLS, as the dynamical modes are themselves nonlinear. Instead, we propose to utilize Rao-Blackwellization by marginalizing the discrete state variable using conditional HMM filters.

We start by considering the filtering problem, i.e., to compute the filtering densities \( p(x_t, r_t \mid y_{1:t}) \) for \( t = 1, \ldots, n \). For brevity, we drop the unknown parameter \( \theta \) from the notation throughout this section. To be able to marginalize the mode variable \( r_t \), we consider the extended target density,

\[
p(x_{1:t}, r_t \mid y_{1:t}) = p(r_t \mid x_{1:t}, y_{1:t})p(x_{1:t} \mid y_{1:t}).
\]

Note that the filtering density is given as a marginal of the above PDF. The second factor is approximated using a PF, which is represented by a set of \( N \) weighted particles \( \{x^i_{1:t}, w^i_t\}_{i=1}^N \), each being a state trajectory \( x_{1:t} \in X \). The particles define a point-mass approximation in the form,

\[
\hat{p}^N(x_{1:t} \mid y_{1:t}) = \sum_{i=1}^{N} w^i_t \delta_{x^i_{1:t}}(x_{1:t}),
\]

where \( \delta_z(x) \) is a Dirac point-mass located at the point \( z \). Conditionally on \( x_{1:t} \), the mode variable \( r_t \) follows a finite state-space HMM. Hence, the conditional density of \( r_t \) in (20) is available by running a conditional HMM filter. This allows us to compute the mode probabilities,

\[
\alpha^i_t(\ell) \triangleq P(r_t = \ell \mid x^i_{1:t}, y_{1:t}),
\]

for \( \ell = 1, \ldots, K \) and \( i = 1, \ldots, N \).

At time \( t = 0 \) we have \( \alpha^i_0(\ell) \equiv 1(\ell = 0 = \ell) \), since the initial state is assumed known (as before, the generalization to an unknown initial state is straightforward). Additionally, we set \( x^i_0 = x_0 \) and \( w^i_0 = 1/N \) for \( i = 1, \ldots, N \). Assume that we have obtained approximations according to (21) and (22) for time \( t - 1 \), represented by the particle system

\[
\{x^i_{1:t-1}, w^i_{t-1}, \alpha^i_{t-1|t-1}(\cdot)\}_{i=1}^N.
\]

Here, we have included the conditional filtering probabilities for the mode variable in the particle system. For notational convenience, \( \alpha^i_{t-1|t-1}(\cdot) \) refers to the set \( \{\alpha^i_{t-1|t-1}(\ell)\}_{\ell=1}^K \) (and similarly for the prediction probabilities). We will now derive the update equations for the RBPF, and see how to propagate this particle system to time \( t \).

First, as for any SMC sampler, resampling is conducted to rejuvenate the particles and reduce the effects of degeneracy [16]. Resampling does not have to be done at every iteration of the algorithm. Instead, we can choose to resample only when, say, the effective sample size drops below some user-defined threshold (see e.g. [6, 28]). In either case, let

\[
\{\tilde{x}^i_{1:t-1}, \tilde{w}^i_{t-1}, \tilde{\alpha}^i_{t-1|t-1}(\cdot)\}_{i=1}^N.
\]

refer to the weighted particle system obtained after the resampling step of the algorithm. Note that (24) is identical to (23) if no resampling is done at time \( t - 1 \).
Consider now the time update of the conditional HMM filter. Analogously to \cite{22}, we define the predictive mode probabilities (w.r.t. the resampled particle trajectories),
\begin{equation}
\bar{\alpha}^i_{t-1}(\ell) \triangleq \mathbb{P}(r_t = \ell \mid \tilde{x}^i_{1:t-1}, y_{1:t-1}),
\end{equation}
(25)
By using the Markov property of the mode sequence we can write
\begin{align*}
\mathbb{P}(r_t = \ell, r_{t-1} = k \mid x_{1:t-1}, y_{1:t-1}) & = \Pi(r_t = \ell \mid r_{t-1} = k)\mathbb{P}(r_{t-1} = k \mid x_{1:t-1}, y_{1:t-1}).
\end{align*}
(26)
By marginalizing the above expression over \(r_{t-1}\) we thus get,
\begin{equation}
\bar{\alpha}^i_{t|t-1}(\ell) = \sum_{k=1}^{K} \pi_k \bar{\alpha}^i_{t-1|t-1}(k),
\end{equation}
(27)
for \(\ell = 1, \ldots, K\) and \(i = 1, \ldots, N\).

Next, we consider updating the continuous state variable. To extend the particle trajectories to time \(t\), we draw new samples from some proposal kernel according to
\begin{equation}
x^i_t \sim q_t(x_t \mid \tilde{x}^i_{1:t-1}, y_{1:t}),
\end{equation}
(28)
and set \(x^i_{1:t} = (\tilde{x}^i_{1:t-1}, x^i_t)\) for \(i = 1, \ldots, N\). Given the new particles and the current measurement \(y_t\), we can compute the updated mode probabilities \cite{22}.

This constitutes the measurement update of the conditional HMM filter. Note, however, that the continuous state \(x_t\) carries information about \(r_t\), and thus serves as an “extra measurement”. Let us define the quantities,
\begin{align*}
\gamma^i_t(r_t) & \triangleq p(y_t, x^i_t, r_t \mid \tilde{x}^i_{1:t-1}, y_{1:t-1}) \\
& = g_{r_t}(y_t \mid x^i_t) f_{r_t}(x^i_t \mid \tilde{x}^i_{t-1}) \bar{\alpha}^i_{t|t-1}(r_t).
\end{align*}
(29)
Since \(p(r_t \mid x_{1:t}, y_{1:t}) \propto p(y_t, x_t, r_t \mid x_{1:t-1}, y_{1:t-1})\), it follows that,
\begin{equation}
\alpha^i_{t|t}(\ell) = \frac{\gamma^i_t(\ell)}{\sum_{k=1}^{K} \gamma^i_t(k)},
\end{equation}
(30)
for \(\ell = 1, \ldots, K\) and \(i = 1, \ldots, N\).

Finally, to account for the discrepancy between the target and the proposal distributions, the particles are assigned importance weights according to,
\begin{equation}
w^i_t \propto \frac{p(x^i_t, y_t \mid \tilde{x}^i_{1:t-1}, y_{1:t-1})}{q_t(x^i_t \mid \tilde{x}^i_{1:t-1}, y_{1:t})} \bar{w}^i_{t-1}.
\end{equation}
(31)

The numerator of this expression is given by marginalizing \cite{20} over \(r_t\), i.e., by the normalization constant \(\sum_{k=1}^{K} \gamma^i_t(k)\). As in a standard PF, the weights are then normalized to sum to one.

It is worth to emphasize that standard modifications from the SMC literature may be used together with the RBPF, e.g. resampling with adjustment weights \cite{40} or incorporating MCMC moves in the sampler \cite{20}. It can also be of interest, from an implementation point of view, to note that the “bootstrap proposal” for the RBPF is given by the mixture distribution,
\begin{equation}
p(x_t \mid \tilde{x}^i_{1:t-1}, y_{1:t-1}) = \sum_{k=1}^{K} f_k(x_t \mid \tilde{x}^i_{t-1}) \bar{\alpha}^i_{t|t-1}(k).
\end{equation}
(32)
We summarize the RBPF in Algorithm 1.
Algorithm 1 RBPF for JMNLS (at time $t$)

1. **Input:** \( \{x_{i,t-1}, w_{i,t-1}, \alpha_{i,t-1}(\cdot)\}_{i=1}^{N} \).

2. **Resampling:** Optionally resample the particles, or retain the previous particle system. Let \( \{\tilde{x}_{i,t-1}, \tilde{w}_{i,t-1}, \tilde{\alpha}_{i,t-1}(\cdot)\}_{i=1}^{N} \) denote the result.

3. **For** $i = 1$ **to** $N$ **do:**
   
   (a) Compute \( \{\alpha_{i,t}(\ell)\}_{\ell=1}^{K} \) according to (27).
   
   (b) Draw \( x_{i,t} \sim q_{t}(x_{t} \mid \tilde{x}_{1,t-1}, y_{1:t}) \) and set \( x_{i,t-1} = (\tilde{x}_{1,t-1}, x_{i,t}) \).
   
   (c) Compute \( \{\gamma_{i,t}(\ell)\}_{\ell=1}^{K} \) according to (29).
   
   (d) Compute \( \{\alpha_{i,t}(\ell)\}_{\ell=1}^{K} \) according to (30).
   
   (e) Set \( w_{i,t}^{i,j} = \sum_{k=1}^{K} \gamma_{i,t}(k) \frac{q_{t}(x_{i,t} \mid \tilde{x}_{1,t-1}, y_{1:t})}{\tilde{w}_{i,t-1}}. \)

End for

4. **Normalize:** \( w_{i,t} = w_{i,t}^{i,j} / \sum_{j=1}^{N} w_{i,t}^{j,j}, i = 1, \ldots, N. \)

5. **Output:** \( \{x_{i,t}, w_{i,t}, \alpha_{i,t}(\cdot)\}_{i=1}^{N} \).

5.2 RBPF-based online-EM

Using (17) and (19), we seek a recursive approximation of \( S_{t} \) based on the RBPF. For each particle in the system \( \{x_{i,t}, w_{i,t}\}_{i=1}^{N} \), we will compute an approximation \( \hat{T}_{i}(r_{t}) \) of the intermediate quantity \( T_{i}(x_{t}, r_{t}) \), i.e.,

\[
\hat{T}_{i}(\ell) \approx T_{i}(x_{i,t}, r_{t} = \ell)
\]

for \( i = 1, \ldots, N \) and \( \ell = 1, \ldots, K \). Given these quantities, it follows from (17) that we can approximate \( S_{t} \) by the RBPF, according to,

\[
S_{t} \approx \hat{S}_{t}^{N} \triangleq \sum_{i=1}^{N} \sum_{\ell=1}^{K} w_{i,t}^{i,j} \hat{T}_{i}(\ell).
\]

It remains to compute the intermediate quantities (33). From the updating equation (19), we note that this requires us to compute an expectation under the so-called backward kernel \( p(\xi_{t-1} \mid \xi_{t}, y_{1:t-1}) \). The key step in computing (33) is thus to find an approximation of the backward kernel based on the RBPF particles. We will consider two different approaches, leading to different algorithms. The first is more accurate, but its computational cost scales quadratically with the number of particles. The latter leads to a cruder approximation, but its computational cost scales only linearly with the number of particles.

Note that (19) can be written as

\[
T_{t}(\xi_{t}) \leftarrow \sum_{r_{t-1}} \int [(1 - \gamma_{t})T_{t-1}(\xi_{t-1}) + \gamma_{t} s_{t}(\xi_{t-1}, \xi_{t})] \]
where we have extended the integration to the complete trajectory $x_{t:t-1}$ (which does not alter the value of the integral). The extended backward kernel density can be written as,

\[ p(x_{1:t-1}, r_{t-1} \mid x_t, r_t, y_{1:t-1}) \propto f_{r_t}(x_t \mid x_{t-1}) \Pi(r_t \mid r_{t-1}) \]

\[ \times p(r_{t-1} \mid x_{t:t-1}, y_{1:t-1}) p(x_{1:t-1} \mid y_{1:t-1}). \]  

(36)

By plugging in the RBPF approximations (21) and (22) we get,

\[ \text{(35)} \]

\[ p(x_{1:t-1}, r_{t-1} = k \mid x_{t}^{i}, r_{t} = \ell, y_{1:t-1}) \]

\[ \approx \sum_{j=1}^{N} \sum_{m=1}^{K} \bar{w}_{t}^{i,j}(k, \ell) \sum_{u=1}^{N} \sum_{m=1}^{K} \bar{w}_{t}^{i,u}(m, \ell) \delta_{x_{t-1}^{i}}(x_{1:t-1}), \]  

(37)

with

\[ \bar{w}_{t}^{i,j}(k, \ell) = f_{t}(x_{t}^{i} \mid x_{t-1}^{j}) \pi_{k,t} \alpha_{i-1}^{j \mid t-1}(k) w_{t-1}^{i}. \]  

(38)

By using this approximation of the backward kernel, we obtain the following update equation for the intermediate quantities,

\[ \hat{T}_{t}^{i}(\ell) = \sum_{j=1}^{N} \sum_{k=1}^{K} \left( \sum_{u=1}^{N} \sum_{m=1}^{K} \bar{w}_{t}^{i,j}(k, \ell) \sum_{u=1}^{N} \sum_{m=1}^{K} \bar{w}_{t}^{i,u}(m, \ell) \right) \]

\[ \times \left[ (1 - \gamma_t) \hat{T}_{t-1}^{i}(k) + \gamma_t s_t(x_{t-1}^{i}, r_{t-1} = k, x_{t-1}^{i}, r_{t} = \ell) \right]. \]  

(39)

The recursion is initialized by $\hat{T}_{0}^{i}(\ell) \equiv 0$.

The computational complexity of computing (39) for $\ell = 1, \ldots, K$ and $i = 1, \ldots, N$ is $O(K^2N^2)$. One way to reduce the computational complexity of the forward smoother, is to rely on path-based smoothing. The backward kernel approximation (37) can be thought of as considering all particles at time $t-1$ as possible ancestors to each particle at time $t$. An alternative is to only consider the “actual” ancestors. Recall that $\tilde{x}_{t-1}$ are the resampled particles at time $t-1$ and that $x_{t}^{i}$ originates from $\tilde{x}_{t-1}^{i}$ as in (28). This suggests to approximate (36) according to,

\[ p(x_{1:t-1}, r_{t-1} = k \mid x_{t}^{i}, r_{t} = \ell, y_{1:t-1}) \]

\[ \approx \sum_{m=1}^{K} \frac{\pi_{k,t} \alpha_{i-1}^{i \mid t-1}(\ell)}{\sum_{m=1}^{K} \pi_{m,t} \alpha_{i-1}^{i \mid t-1}(m)} \delta_{x_{t-1}^{i}}(x_{1:t-1}). \]  

(40)

Note that the factors $f_{t}(x_{t}^{i} \mid \tilde{x}_{t-1}^{i})$ cancel when normalizing the distribution. Let $\{\tilde{T}_{t-1}^{i}(\ell)\}_{i=1}^{N}$ be the set of resampled intermediate statistics, arising from resampling $\{\tilde{T}_{t-1}^{i}(\ell)\}_{i=1}^{N}$ along with the particles at time $t-1$. By using (40) in (35) we get the alternative updating equation for the intermediate quantities,

\[ \hat{T}_{t}^{i}(\ell) = \sum_{k=1}^{K} \left( \frac{\pi_{k,t} \alpha_{i-1}^{i \mid t-1}(k)}{\sum_{m=1}^{K} \pi_{m,t} \alpha_{i-1}^{i \mid t-1}(m)} \right) \]

\[ \times \left[ (1 - \gamma_t) \hat{T}_{t-1}^{i}(k) + \gamma_t s_t(x_{t-1}^{i}, r_{t-1} = k, x_{t}^{i}, r_{t} = \ell) \right]. \]  

(41)
Algorithm 2 Online EM for JMNLS (at time $t$)

1. **Filter update:**
   (a) Parameterize the model with $\hat{\theta}^{t-1}$.
   (b) Run one step of the RBPF (Algorithm 1).

2. **Parameter update:**
   (a) Compute $\{\hat{T}_i^t(\cdot)\}_{i=1}^N$ according to
      - (Forward smoothing): (39)
      - (Path-based smoothing): (41)
   (b) Compute $\hat{S}_t^N = \sum_{i=1}^N \sum_{\ell=1}^K w_i \alpha_i^t(\ell) \hat{T}_i^t(\ell)$.
   (c) Update the parameter $\hat{\theta}^t$ according to (12).

As before, the recursion is initialized with $\hat{T}_i^0(\ell) \equiv 0$. The computational complexity of computing these quantities for $\ell = 1, \ldots, K$ and $i = 1, \ldots, N$ is $O(K^2N)$. The price we pay for the reduced computational complexity is a cruder approximation of the backward kernel. Indeed, since (40) relies on path-based smoothing, it will suffer from path degeneracy. However, it turns out that the effect of the degeneracy is not as bad as one might first think, due to the inherent forgetting factor in the online EM algorithm. Still, as we shall see in Section 6, (41) leads to a larger variance of the resulting parameter estimates than (39).

We summarize the RBPF-based online EM algorithm for JMNLS in Algorithm 2.

6 Experimental Results

6.1 Simulations

In this section we compare the performance of different implementations of the online EM algorithm on a benchmark model and illustrate the gain in Rao-Blackwellization. Consider the following modified benchmark model:

$$x_t = \frac{1}{2} x_{t-1} + 25 \frac{x_{t-1}}{1 + x_{t-1}^2} + 8 \cos(1.2t) + v_t,$$

$$y_t = \frac{x_t^2}{20} + e^{(r_t)},$$

where $v_t \sim N(0,1)$ and where the measurement noise is governed by a 2-state Markov chain $r_t \in \{1, 2\}$. The mode-dependent measurement noise is assumed to be Gaussian distributed, $e_t^{(k)} \sim N(\mu_{e,k}, \Sigma_{e,k})$, $k = 1, 2$. The mode-dependent mean and variance as well as the transition probabilities of the Markov chain are assumed unknown, i.e., the parameters of the model are $\theta = \{\{\theta_k\}_{k=1}^2, \Pi\}$. Here, $\Pi$ refers to the $2 \times 2$ transition probability matrix (TPM) with entries $[\Pi]_{kl} = \pi_{kl}$ and $\theta_k = \{\mu_{e,k}, \Sigma_{e,k}\}$. The model parameter values used in the simulations are summarized in Table 1.
We compare the estimation performance of four different online EM algorithms:

- **PF-Path**: Path-based particle filter [3].
- **PF-FS**: Forward-smoothing-based particle filter [12].
- **RBPF-Path**: Path-based RBPF (Algorithm 2).
- **RBPF-FS**: Forward-smoothing-based RBPF (Algorithm 2).

We simulate a batch of 10,000 measurements \( y_{1:n} \) and run all the algorithms 100 times on the same data to investigate the errors arising from the Monte Carlo approximations. All methods are bootstrap implementations with \( N = 150 \) particles and the step size sequence \( \gamma_t = t^{-0.7} \). See Appendix 7 for further details on the implementation. The results are shown in Figure 1. Table 2 reports the time averaged Monte Carlo variances for the different methods. The runtimes of the algorithms are given in Table 3. All simulations are run in Matlab(R) R2012b on a standard laptop Intel(R) Core(TM) i7-M640 2.80GHz platform with 8GB of RAM.

It can be seen that Rao-Blackwellization has a positive effect on reducing the Monte Carlo variance of the estimates. More specifically, the Monte Carlo variances for RBPF-Path and RBPF-FS are smaller than for PF-Path and PF-FS, respectively. Taking the computation times into account, RBPF-Path appears to provide a good trade-off between runtime and accuracy. The estimation performance of RBPF-Path is similar to RBPF-FS and PF-FS, but with a significantly lower computational time.

### Table 1: Benchmark model parameters

| Parameter | Value | Parameter | Value |
|-----------|-------|-----------|-------|
| \( \pi_{11} \) | 0.95  | \( \pi_{22} \) | 0.8   |
| \( \mu_{e,1} \) | 0     | \( \mu_{e,2} \) | 3     |
| \( \Sigma_{e,1} \) | 1     | \( \Sigma_{e,2} \) | 4     |

### Table 2: Time averaged Monte Carlo variances

|                | PF-Path | RBPF-Path | PF-FS | RBPF-FS |
|----------------|---------|-----------|-------|---------|
| \( \mu_{e,1} \) (\( \times 10^{-3} \)) | 21.8    | 4.16      | 3.46  | 1.23    |
| \( \mu_{e,2} \) (\( \times 10^{-2} \)) | 38.0    | 3.98      | 4.45  | 1.71    |
| \( \Sigma_{e,1} \) (\( \times 10^{-2} \)) | 7.66    | 1.77      | 2.05  | 1.54    |
| \( \Sigma_{e,2} \) (\( \times 10^{0} \)) | 2.42    | 1.74      | 1.87  | 1.32    |
| \( \pi_{11} \) (\( \times 10^{-4} \)) | 343     | 6.10      | 8.10  | 1.53    |
| \( \pi_{22} \) (\( \times 10^{-4} \)) | 268     | 3.17      | 12.6  | 1.62    |

### 6.2 Transition Probability Estimation for JMLS

In this section, we illustrate the performance of the algorithm on a jump Markov linear model. JMLS are studied thoroughly in the literature and many dedi-
Figure 1: Estimation results over 10,000 iterations. From left to right: PF-Path, RBPF-Path, PF-FS, and RBPF-FS. From top to bottom: \((\mu_{e1}, \mu_{e2}), (\Sigma_{e1}, \Sigma_{e2}), \) and \((\pi_{11}, \pi_{22})\). The lines show the averages and the transparent shaded areas show the upper and lower bounds over 100 independent runs on the same data batch.

Table 3: Average runtimes in milliseconds/time step

|                | PF-Path | RBPF-Path | PF-FS | RBPF-FS |
|----------------|---------|-----------|-------|---------|
|                | 0.42    | 1.08      | 2.90  | 6.71    |

cated algorithms are proposed for the estimation of the transition probabilities which exploit the linear Gaussian structure in the model [22, 37, 38]. Using inaccurate TPMs may lead to performance degradation of the state estimation, due to the sensitivity of the multiple model state estimators to the TPM used. The uncertainty regarding the TPM is a major issue in the application of multiple models to real-life problems [22]. The proposed online EM solution is naturally applicable to linear systems and does not involve any IMM filtering type approximations. In IMM type mixing approximations, many components in the posterior are systematically collapsed into a single Gaussian which deteriorates the statistics associated with the dominant modes and degrades the performance. In the simulation below, we have considered the benchmark model originally given in [22], and used in [37] and [38] for comparison of different TPM estimation algorithms:

\[
x_{t+1} = x_t + v_t \tag{43}
\]
\[
y_t = r_t x_t + (100 - 90r_t) e_t \tag{44}
\]

where \(x_0 \sim N(x_0; 0, 20^2)\), \(v_t \sim N(v_t; 0, 2^2)\), and \(e_t \sim N(e_t; 0, 1)\) with \(x_0, v_t\) and \(e_t\) being mutually independent for \(t = 1, 2, \ldots\). The mode sequence \(r_t \in \{0, 1\}\)
is a 2-state homogenous Markov process with TPM given as,

\[
\Pi = \begin{bmatrix}
0.6 & 0.4 \\
0.85 & 0.15
\end{bmatrix}.
\]  

(45)

This system corresponds to a system with frequent measurement failures with the modal state \( r_t = 0 \) corresponding to a failure. The online EM algorithm is run on the simulated measurements of this system with initial transition probabilities \( \pi_{00}^{(0)} = \pi_{11}^{(0)} = 0.5 \). We compare three algorithms in a single run using the same data set used in [38]. The first algorithm is a Kullback-Leibler-distance-based TPM estimation method, denoted as IMM-KL, which is proposed in [37]. The second algorithm is a maximum-likelihood-based method, denoted as IMM-ML, which is presented in [38]. These two algorithms rely on the aforementioned IMM approximations. The third algorithm is the proposed RBPF-Path method, using 500 particles and the step size sequence \( \gamma_t = t^{-0.95} \). In Figure 2, the estimated transition probabilities of the three algorithms are depicted. The RBPF-Path appears to provide satisfactory results, showing fast convergence to the vicinity of the true parameters at the beginning and providing smoother estimates towards the end. It is worth to note that, contrary to the special purpose algorithms IMM-KL and IMM-ML, this is accomplished without exploiting the linearity of the dynamic modes.

6.3 Mobile Terminal Positioning in Wireless Networks

The results in this section are provided to illustrate the validity of the proposed method on real data. We consider the mobile terminal (MT) positioning example in a wireless network, where time of arrival (ToA) measurements from three base stations are available to determine the position of the MT. The measurements have been collected during a field trial performed in Kista, Sweden; see [32] for more details on experimental setup. The scenario can be considered
as dense urban, where many multi-storey buildings prevent that the radio signal from the base stations (BSs) arrive via the direct line-of-sight (LOS) path at the MT. Due to multiple reflections from buildings, the radio signals often propagate via an indirect non-line-of-sight (NLOS) path to the MT. In the literature, these switching propagation conditions are often modeled with a two-state Markov chain affecting the noise distribution of the measurement; see for instance [18, 23]. This approach is also followed here, but with the assumption that the underlying measurement noise statistics as well as the parameters of the Markov chain are unknown and have to be estimated.

It is assumed that the MT motion can be modeled with a nearly constant velocity model, according to

$$x_t = F x_{t-1} + \Gamma v_t$$  \hspace{1cm} (46)

with

$$F = \begin{bmatrix} 1 & \Delta T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \Delta T \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} \Delta T^2/2 & 0 \\ \Delta T & 0 \\ 0 & \Delta T^2/2 \\ 0 & \Delta T \end{bmatrix}, \hspace{1cm} (47)$$

where $x_t = [x_{MT,t}, \dot{x}_{MT,t}, y_{MT,t}, \dot{y}_{MT,t}]^T$ is the MT position and velocity vector and $\Delta T = 0.53$ s is the sampling time. The noise is distributed according to $v_t \sim N(0, \Sigma_v)$ with $\Sigma_v = \sigma_v^2 I_2$, where $I_2$ denotes the $2 \times 2$ identity matrix.

The switching is modeled with a 2-state Markov chain $\{r_t\}$, where the state $r_t = 1$ is assigned to the event LOS and the state $r_t = 2$ is assigned to the event NLOS. The Markov chain is assumed to be time-homogeneous with transition probability matrix $\Pi$. In the following, the ToA measurements are expressed in terms of distance measurements (by multiplication with speed of light), so that the measurement at each BS can be described with

$$y_t = h(x_t) + e^{(r_t)}_t,$$  \hspace{1cm} (48)

where

$$h(x_t) = \sqrt{(x_{MT,t} - x_{BS,t})^2 + (y_{MT,t} - y_{BS,t})^2}$$  \hspace{1cm} (49)

and $(x_{BS,t}, y_{BS,t})$ are the BS position coordinates. The noise is distributed according to $e^{(r_t)}_t \sim N(\mu_{r_t}, \sigma_{r_t}^2)$, meaning that the LOS and NLOS errors are modeled with a Gaussian distribution with different means and variances according to

$$\mu_{r_t} = \begin{cases} \mu_{LOS}, & r_t = 1, \\ \mu_{NLOS}, & r_t = 2, \end{cases} \quad \sigma_{r_t}^2 = \begin{cases} \sigma_{LOS}^2, & r_t = 1, \\ \sigma_{NLOS}^2, & r_t = 2. \end{cases}$$  \hspace{1cm} (50)

For simplicity, only BS 3, which is severely affected by switching propagation conditions, is modeled according to (48). The measurement noise of the other two BSs is assumed to be Gaussian distributed, where the mean and variance have been determined prior to running the algorithm. Thus, the unknowns stemming from the measurement model of BS 3 can be collected in $\theta = (\mu_{LOS}, \mu_{NLOS}, \sigma_{LOS}^2, \sigma_{NLOS}^2, \Pi)$.  

16
In Figure 3, the estimation results for the MT coordinates are shown using the proposed RBPF-Path algorithm with 500 particles. The variances $\sigma_{\text{LOS}}^2, \sigma_{\text{NLOS}}^2$ are constrained to be less than 100. It can be observed that the estimated MT position coordinates follow the true ones. The mean terms $\mu_{\text{LOS}}$ and $\mu_{\text{NLOS}}$ together with the error in distance measurements of BS 3, obtained from the true distance measured by GPS is shown in Figure 4. It can be observed, that the (time varying) biases in the distance measurements can be generally well tracked by the algorithm. In Figure 5 the mode estimates of the algorithm is plotted. The switchings from LOS to NLOS modes are tracked successfully.

7 Conclusion

We have proposed a method based on the online EM algorithm for joint state estimation and identification of JMNLS. We use Rao-Blackwellization to exploit the structure of JMNLS, resulting in a significant improvement in the estimation accuracy. The algorithm is applicable to a large class of models which involve sudden regime changes, unknown parameters and heavy non-linearities as illustrated via simulations. The algorithm was also successfully tested on real data for localization in a wireless network.

[Sufficient Statistics for Noise Parameters in Jump Markov Gaussian Systems] Consider the jump Markov System given below.

\[
x_t = f_{r_t}(x_{t-1}) + v_t^{(r_t)}, \quad (51a)
\]
\[
y_t = h_{r_t}(x_t) + e_t^{(r_t)}, \quad (51b)
\]

where the noise is distributed according to $v_t^{(r_t)} \sim N(\mu_{v,r_t}, \Sigma_{v,r_t})$ and $e_t^{(r_t)} \sim N(\mu_{e,r_t}, \Sigma_{e,r_t})$. The unknowns parameters are $\theta = \{\theta_k\}_{k=1}^K, \Pi$, where $\Pi$ refers to the $K \times K$ transition matrix with entries $[\Pi]_{k\ell} = \pi_{k\ell}$ and $\theta_k = \{\mu_{v,k}, \Sigma_{v,k}, \mu_{e,k}, \Sigma_{e,k}\}$. Below, we provide the sufficient statistics $s_t(\xi_t, \xi_{t-1})$
Figure 4: Estimated mean terms $\mu_{LOS}, \mu_{NLOS}$ plotted on error in distance measurements of BS 3, calculated from the distance measured by GPS.

Figure 5: Mode estimate vs. time where '1' is for the LOS mode and '2' is for the NLOS mode.
as well as the closed-form expressions for the mappings $A_k(\cdot)$ appearing in [12]. These mappings can be found by explicitly evaluating the M-step. Similarly to [4], for jump Markov Gaussian systems, the parameters can be updated according to:

$$
\hat{\pi}_{k\ell} = \frac{S_{k\ell,t}^{(1)}}{\sum_{j=1}^{K} S_{kj,t}^{(1)}},
$$

$$
\hat{\mu}_{v,k} = \frac{S_{k,v}^{(1)}}{S_k^{(2)}}, \quad \hat{\Sigma}_{v,k} = \frac{S_{k,v}^{(3)(2)}}{S_k^{(2)}} - \hat{\mu}_{v,k}\hat{\mu}_{v,k}^T,
$$

$$
\hat{\mu}_{e,k} = \frac{S_{k,e}^{(1)}}{S_k^{(2)}}, \quad \hat{\Sigma}_{e,k} = \frac{S_{k,e}^{(3)(2)}}{S_k^{(2)}} - \hat{\mu}_{e,k}\hat{\mu}_{e,k}^T.
$$

The corresponding sufficient statistics are given by

$$
s_{k\ell,t}^{(1)} = \mathbb{I}(r_t = k, r_{t-1} = l),
$$

$$
s_{k,t}^{(2)} = \mathbb{I}(r_t = k),
$$

$$
s_{k,v,t}^{(3)(1)} = \mathbb{I}(r_t = k)[x_t - f_{r_t}(x_{t-1})],
$$

$$
s_{k,v,t}^{(3)(2)} = \mathbb{I}(r_t = k)[x_t - f_{r_t}(x_{t-1})][\cdot]^T,
$$

$$
s_{k,e,t}^{(3)(1)} = \mathbb{I}(r_t = k)[y_t - h_{r_t}(x_t)]
$$

$$
s_{k,e,t}^{(3)(2)} = \mathbb{I}(r_t = k)[y_t - h_{r_t}(x_t)][\cdot]^T.
$$

Acknowledgment

The authors gratefully acknowledge the financial support from the Swedish Research Council under the Linnaeus Center (CADICS) and would like to thank Umut Orguner for providing the code of his previous works for comparison in Section 6.2.

References

[1] C. Andrieu, M. Davy, and A. Doucet. Efficient particle filtering for jump Markov systems. application to time-varying autoregressions. *Signal Processing, IEEE Transactions on*, 51(7):1762–1770, 2003.

[2] HAP Blom and Y. Bar-Shalom. The interacting multiple model algorithm for systems with Markovian switching coefficients. *Automatic Control, IEEE Transactions on*, 33(8):780–783, 1988.

[3] O. Cappé. Online sequential Monte Carlo EM algorithm. In *Statistical Signal Processing, 2009. SSP ’09. IEEE/SP 15th Workshop on*, pages 37–40, Sept. 2009.

[4] O. Cappé. Online EM algorithm for hidden Markov models. *Journal of Computational and Graphical Statistics*, 20(3):728–749, 2011.
[5] O. Cappé, E. Moulines, and T. Rydén. Inference in Hidden Markov Models. Springer, 2005.

[6] O. Cappé, E. Moulines, and T. Rydén. Inference in Hidden Markov Models. Springer Series in Statistics. Springer Science + Business Media, LLC, New York, NY, USA, 2005.

[7] F. Caron, M. Davy, E. Duflot, and P. Vanheeghe. Particle filtering for multisensor data fusion with switching observation models: Application to land vehicle positioning. Signal Processing, IEEE Transactions on, 55(6):2703–2719, 2007.

[8] C. M. Carvalho and H. F. Lopes. Simulation-based sequential analysis of Markov switching stochastic volatility models. Computational Statistics & Data Analysis, 51:4526–4542, 2007.

[9] R. Chen and J. S. Liu. Mixture Kalman filters. Journal of the Royal Statistical Society: Series B, 62(3):493–508, 2000.

[10] N. Chopin. Central limit theorem for sequential Monte Carlo methods and its application to Bayesian inference. The Annals of Statistics, 32(6):2385–2411, 2004.

[11] E. Cinquemani, R. Porreca, G. Ferrari-Trecate, and John Lygeros. A general framework for the identification of jump Markov linear systems. In Decision and Control, 2007 46th IEEE Conference on, pages 5737–5742. IEEE, 2007.

[12] P. Del Moral, A. Doucet, and S. Singh. Forward smoothing using sequential Monte Carlo. arXiv preprint arXiv:1012.5390, 2010.

[13] A. P. Dempster, N. M. Laird, and D. B. Rubin. Maximum likelihood from incomplete data via the EM algorithm. Journal of the Royal Statistical Society. Series B (Methodological), 39(1):1–38, 1977.

[14] A. Doucet, S. J. Godsill, and C. Andrieu. On sequential Monte Carlo sampling methods for Bayesian filtering. Statistics and Computing, 10(3):197–208, 2000.

[15] A. Doucet, N. J. Gordon, and V. Krishnamurthy. Particle filters for state estimation of jump Markov linear systems. IEEE Transactions on Signal Processing, 49(3):613–624, 2001.

[16] A. Doucet and A. Johansen. A tutorial on particle filtering and smoothing: Fifteen years later. In D. Crisan and B. Rozovskii, editors, The Oxford Handbook of Nonlinear Filtering. Oxford University Press, 2011.

[17] H. Driessen and Y. Boers. Efficient particle filter for jump Markov nonlinear systems. IEE Proceedings-Radar, Sonar and Navigation, 152(5):323–326, 2005.

[18] C. Fritsche, U. Hammes, A. Klein, and A. Zoubir. Robust mobile terminal tracking in NLOS environments using interacting multiple model algorithm. In Proc. of IEEE International Conference on Acoustics, Speech and Signal Processing, pages 3049–3052, Taipei, Taiwan, Apr. 2009.
[19] C. Fritsche, E. Ozkan, and F. Gustafsson. Online EM algorithm for jump Markov systems. In Information Fusion (FUSION), 2012 15th International Conference on, pages 1941–1946. IEEE, 2012.

[20] W. R. Gilks and C. Berzuini. Following a moving target – Monte Carlo inference for dynamic Bayesian models. Journal of the Royal Statistical Society. Series B (Statistical Methodology), 63(1):127–146, 2001.

[21] F. Gustafsson. Particle filter theory and practice with positioning applications. IEEE Aerospace and Electronic Systems Magazine, 25(7):53–82, 2010.

[22] V.P. Jilkov and X.R. Li. Online Bayesian estimation of transition probabilities for Markovian jump systems. Signal Processing, IEEE Transactions on, 52(6):1620 – 1630, june 2004.

[23] L. Jung-Fen and C. Bor-Sen. Robust mobile location estimator with NLOS mitigation using IMM algorithm. Wireless Communications, IEEE Transactions on, 5(11):3002–3006, Nov. 2006.

[24] S. Le Corff, G. Fort, and E. Moulines. Online Expectation Maximization algorithm to solve the SLAM problem. In Statistical Signal Processing Workshop (SSP), 2011 IEEE, pages 225 –228, Nice, France, June 2011.

[25] X. R. Li and V. P. Jilkov. A survey of maneuvering target tracking part v: Multiple-model methods. In Conference on Signal and Data Processing of Small Targets, volume 4473, pages 559–581, 2003.

[26] F. Lindsten and T. B. Schön. Backward simulation methods for Monte Carlo statistical inference. Foundations and Trends in Machine Learning, 6(1):1–143, 2013.

[27] F. Lindsten, T. B. Schön, and J. Olsson. An explicit variance reduction expression for the Rao-Blackwellised particle filter. In Proceedings of the 18th IFAC World Congress, Milan, Italy, August 2011.

[28] J. S. Liu. Monte Carlo Strategies in Scientific Computing. Springer, 2001.

[29] A. Logothetis and V. Krishnamurthy. Expectation maximization algorithms for map estimation of jump Markov linear systems. Signal Processing, IEEE Transactions on, 47(8):2139 –2156, aug 1999.

[30] E. Mazor, A. Averbuch, Y. Bar-Shalom, and J. Dayan. Interacting multiple model methods in target tracking: A survey. Aerospace and Electronic Systems, IEEE Transactions on, 34(1):103–123, 1998.

[31] G. McLachlan and T. Krishnan. The EM algorithm and extensions, volume 382. John Wiley & Sons, 2007.

[32] J. Medbo, I. Siomina, A. Kangas, and J. Furuskog. Propagation channel impact on LTE positioning accuracy - A study based on real measurements of observed time difference of arrival. In Proc. of IEEE International Symposium on Personal, Indoor and Mobile Radio Communications, pages 2213 – 2217, Tokyo, Japan, Sep. 2009.
[33] L. Mihaylova, D. Angelova, S. Honary, D.R. Bull, C. N. Canagarajah, and B. Ristic. Mobility tracking in cellular networks using particle filtering. *IEEE Transactions on Wireless Communication*, 6:3589–3599, Oct 2007.

[34] G. Mongillo and S. Denève. Online learning with hidden Markov models. *Neural Computation*, 20(7):1706–1716, 2008.

[35] A. Munir and D.P. Atherton. Adaptive interacting multiple model algorithm for tracking a manoeuvring target. *IEE Proceedings-Radar, Sonar and Navigation*, 142(1):11–17, 1995.

[36] M. Nicoli, C. Morelli, and V. Rampa. A jump Markov particle filter for localization of moving terminals in multipath indoor scenarios. *Signal Processing, IEEE Transactions on*, 56(8):3801–3809, 2008.

[37] U. Orguner and M. Demirekler. An online sequential algorithm for the estimation of transition probabilities for jump Markov linear systems. *Automatica*, 42(10):1735–1744, 2006.

[38] U. Orguner and M. Demirekler. Maximum Likelihood estimation of transition probabilities of jump Markov linear systems. *IEEE Transactions on Signal Processing*, 56(10):5093–5108, October 2008.

[39] E. Ozkan, C. Fritsche, and F. Gustafsson. Online EM algorithm for joint state and mixture measurement noise estimation. In *Information Fusion (FUSION), 2012 15th International Conference on*, pages 1935–1940. IEEE, 2012.

[40] M. K. Pitt and N. Shephard. Filtering via simulation: Auxiliary particle filters. *Journal of the American Statistical Association*, 94(446):590–599, 1999.

[41] T. Schön, F. Gustafsson, and P.-J. Nordlund. Marginalized particle filters for mixed linear/nonlinear state-space models. *IEEE Transactions on Signal Processing*, 53(7):2279–2289, July 2005.

[42] RH Shumway and DS Stoffer. Dynamic linear models with switching. *Journal of the American Statistical Association*, 86(415):763–769, 1991.

[43] J. Tugnait. Adaptive estimation and identification for discrete systems with Markov jump parameters. *Automatic Control, IEEE Transactions on*, 27(5):1054–1065, 1982.

[44] Ba-Ngu Vo, A. Pasha, and H. D. Tuan. A gaussian mixture PHD filter for nonlinear jump Markov models. In *Decision and Control, 2006 45th IEEE Conference on*, pages 3162–3167. IEEE, 2006.

[45] N. Whiteley and A. Johansen. Recent developments in auxiliary particle filtering. In *Inference and Learning in Dynamic Models (to appear)*. Cambridge University Press, 2013.

[46] C. F. J. Wu. On the convergence properties of the EM algorithm. *The Annals of Statistics*, 11(1):95–103, 1983.