THE ASCENT-DESCENT PROPERTY FOR 2-TERM SILTING COMPLEXES

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Abstract. We will prove that over commutative rings the silting property of 2-term complexes induced by morphisms between projective modules is preserved and reflected by faithfully flat extensions.

1. Introduction

Let $\lambda : R \to S$ be a homomorphism of unital rings. We consider the extension of scalars functor $- \otimes_R S : \text{Mod-}R \to \text{Mod-}S$, and we will say that a property $P$ associated to a complex of modules ascends along $\lambda$ if the functor $- \otimes_R S$ preserves the property $P$, i.e. for every complex $C$ of right $R$-modules which satisfies $P$, the complex $C \otimes_R S$ satisfies $P$ in $\text{Mod-}S$. The property $P$ descends along $\lambda$ if a complex $C$ in $\text{Mod-}R$ satisfies $P$ provided that $C \otimes_R S$ satisfies $P$ as a complex of right $S$-modules. The above definitions are natural extension of the corresponding ascent/descent notions associated to module properties, e.g. [15, Definition 3.5]. The properties of modules which ascend along flat ring homomorphisms and descend along faithful flat ring homomorphisms (called ascent-descent properties) play an important role in commutative algebra since the corresponding properties associated to quasi-coherent sheaves have a local character, [15, Lemma 3.4]. For instance, for modules over commutative rings the properties “projective”, [19] and [23, Section 058B], and “1-tilting”, [20, Theorem 3.13], are ascent-descent. We mention here that the ascending property of tilting also plays an important role in the non-commutative case since they are used to characterize derived equivalences, [28]. We refer to [25] for a general approach of this case.

In this paper we will study the ascent and descent properties for 2-term silting complexes. These are complexes $\cdots \to 0 \to P^{-1} \otimes_R P^0 \to 0 \to \cdots$, concentrated in $-1$ and 0, which are silting objects in the unbounded derived category $D(R)$ of $R$. In order to simplify the presentation, we will identify, as in [5], every 2-term complex $\cdots \to 0 \to P^{-1} \otimes_R P^0 \to 0 \to \cdots$ with the homomorphism $\sigma : P^{-1} \to P^0$. If $\sigma$ is a 2-term silting complex we will say that $\text{Coker}(\sigma)$ is a silting module (with respect to the homomorphism $\sigma$). These notions were introduced in [5] as non-compact versions of the $\tau$-tilting modules, [1], and of the two term silting complexes, [21]. The role of silting modules in the study of module categories is described in [3]. For the finitely presented case we mention here the results proved in [11] and [16, Section 5]. If $R$ is a ring then (bounded) silting complexes play in the derived category $D(R)$ a similar role with that of tilting modules in module categories, [31]. In spite of this, correspondences or similarities between the influences of tilting modules

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and silting homomorphisms on the module category can be established only for particular instances (e.g. [13]). We refer to [2] for the general theory of silting objects in triangulated categories.

In Theorem 2.2 we provide, for the general case, a characterization for the ascending property of 2-term silting complexes. For the commutative case the ascending property of 2-term silting complexes is valid along all ring homomorphisms, Theorem 2.7. Moreover, we will prove in Theorem 3.16 that the silting property associated to 2-term complexes descends along faithfully flat ring homomorphisms of commutative rings.

In this paper all rings and all ring homomorphisms are unital. If $\lambda : R \to S$ is a homomorphism of rings then the extension of scalars functor is denoted by $- \otimes_R S : \text{Mod-}R \to \text{Mod-}S$. The restriction of scalars functors is denoted by $\lambda^* = \text{Hom}_S(S, -) : \text{Mod-}S \to \text{Mod-}R$. When there is no danger of confusion, we will consider every class of objects in $\text{Mod-}S$ as a subclass of $\text{Mod-}R$. Therefore, if $X$ is a class of $R$-modules, and $Y$ is a class of $S$-modules then $X \cap Y$ means the class of all modules from $N \in Y$ such that $\lambda^*(N) \in X$. In particular, we will identify a right $S$-module $N$ with its image $\lambda^*(N)$.

2. The ascent property

Let $R$ be a unital ring. We consider a homomorphism of projective right $R$-modules $\sigma : P^{-1} \to P^0$, and we denote by $T$ the cokernel of $\sigma$.

Then we can associate to $\sigma$ the class

$$D_\sigma = \{ X \in \text{Mod-}R \mid \text{Hom}_R(\sigma, X) \text{ is an epimorphism} \}. $$

Since $D_\sigma$ is the kernel of the functor $\text{Coker}(\text{Hom}_R(\sigma, -))$, by using the properties of this functor (e.g. [14] Proposition 4), it follows that $D_\sigma$ is closed with respect to epimorphic images, extensions, and direct products.

Following [5], we will say that $\sigma$ is partial silting or that the module $T = \text{Coker}(\sigma)$ is partial silting with respect to $\sigma$ if $D_\sigma$ is a torsion class (i.e. $D_\sigma$ is also closed under direct sums) and $T \in D_\sigma$. Then $\text{Gen}(T) \subseteq D_\sigma \subseteq T^\perp$ and $(\text{Gen}(T), T^\circ)$ is a torsion pair, where

$$\text{Gen}(T) = \{ M \in \text{Mod-}R \mid \text{there exists an epimorphism } T^{(I)} \to M \}$$

is the class of all $T$-generated right $R$-modules, $T^\perp = \text{KerExt}^1_T(T, -)$, and $T^\circ = \text{KerHom}_R(T, -)$. If $D_\sigma = \text{Gen}(T)$ then we will say that $\sigma$ is a 2-term silting complex and $T$ is called a silting module with respect to $\sigma$ (we recall that in [5] Theorem 4.9] it is proved that a homomorphism $\sigma$ satisfies the above condition if and only if it represents a silting complex in the associated unbounded derived category).

If $\lambda : R \to S$ is a ring homomorphism and $\sigma : P^{-1} \to P^0$ is a homomorphism of projective right $R$-modules then we will denote $\sigma \otimes_R S = \sigma \otimes_R 1_S$ the induced homomorphism of projective right $S$-modules.

**Lemma 2.1.** Let $\lambda : R \to S$ be a ring homomorphism. If $\sigma : P^{-1} \to P^0$ is a homomorphism between projective right $R$-modules then $D_{\sigma \otimes_R S} = D_\sigma \cap \text{Mod-}S$. 

Proof. Let $M$ be a right $S$-module. We have the commutative diagram
\[
\begin{array}{ccc}
\Hom_R(P^0, \Hom_S(S, M)) & \xrightarrow{\Hom_R(\sigma, \Hom_S(S, M))} & \Hom_R(P^{-1}, \Hom_S(S, M)) \\
\cong & & \cong \\
\Hom_S(P^0 \otimes_R S, M) & \xrightarrow{\Hom_S(\sigma \otimes_S S, M)} & \Hom_S(P^{-1} \otimes_R S, M)
\end{array}
\]
It follows that a right $S$-module $M$ belongs to $D_{\sigma \otimes_R S}$ if and only if the right $R$-module $\lambda^*(M) = \Hom_S(S, M)$ is in $D_{\sigma}$. □

**Theorem 2.2.** Suppose that $\sigma$ represents a 2-term silting complex. If $\lambda : R \to S$ is a ring homomorphism then the following are equivalent:
1. $\sigma \otimes_R S$ is a 2-term silting complex in $\Mod-S$;
2. $\lambda^*(T \otimes_R S)$ is $T$-generated.

**Proof.** (1)$\Rightarrow$(2) Since $\sigma \otimes_R S$ represents a 2-term silting complex, it follows that $T \otimes_R S \in D_{\sigma \otimes_R S}$. Then (2) is a consequence of Lemma 2.1.

(2)$\Rightarrow$(1) Since $D_{\sigma}$ is closed under direct sums, it is obvious, by Lemma 2.1, that $D_{\sigma \otimes_R S}$ has the same property.

For every $M \in D_{\sigma \otimes_R S}$ we have $\lambda^*(M) \in D_{\sigma}$. Since $\sigma$ is a 2-term silting complex, it follows that $\lambda^*(M)$ is $T$-generated. Then there exists an $R$-epimorphism $T(\otimes) \to \lambda^*(M)$ which induces an $S$-epimorphism $T \otimes_R S(\otimes) \to \lambda^*(M) \otimes_R S$. But it is well known that the canonical $S$-homomorphism $\lambda^*(M) \otimes_R S \to M, x \otimes s \mapsto x(s)$, is an epimorphism. It follows that $D_{\sigma \otimes_R S} \subseteq \Gen(T \otimes_R S)$.

Therefore, it is enough to prove that $T \otimes_R S \in D_{\sigma \otimes_R S}$. By Lemma 2.1 this is equivalent to $\lambda^*(T \otimes_R S) \in D_{\sigma}$. Since $\sigma$ is silting, we have $\lambda^*(T \otimes_R S) \in D_{\sigma}$ if and only if $\lambda^*(T \otimes_R S)$ is $T$-generated, and this last property is assumed in (2). □

**Remark 2.3.** A similar proof, using the isomorphism $\Hom_S(X, \Hom_R(S, I)) \cong \Hom_R(X \otimes_S S, I)$, can be used to obtain the dual of Theorem 2.2 for cosilting modules (we refer to [12] and [30] for the basic properties of these modules). Therefore, if $\lambda : R \to S$ is a unital ring homomorphism and $C \in \Mod-R$ is a cosilting module with respect to the injective copresentation $\beta : P^0 \to P^1$ then the coinduced $S$-module $\Hom_R(S, C)$ is a cosilting module with respect to $\Hom_R(S, \beta)$ if and only if $\Hom_R(S, C)$ is $C$-cogenerated.

In the case of surjective ring homomorphisms, the extension of scalars functor always preserves the 2-term silting complexes.

**Corollary 2.4.** Let $\lambda : R \to S$ be a surjective ring homomorphism. Then for every 2-term silting complex $\sigma$ the induced complex $\sigma \otimes_R S$ is a 2-term silting complex.

**Proof.** If $I = \Ker(\lambda)$ then for every right $R$-module $M$ we have a natural $R$-isomorphism $M \otimes_R S \cong M/IM$ (see [32] 12.11). □

However, even in the case of non-surjective ring epimorphisms the above corollary is not true. The following example was communicated to me by Lidia Angeleri-Hügel.

**Example 2.5.** Let $R$ be the (Kronecker) path algebra associated to the graph $2 \leftarrow 1$ over a field $K$. The silting modules over Kronecker algebras are described in [6] Examples 5.10, and 5.18 and in [22] Example 2.20. If $1$ is the simple injective $R$-module and $2$ is the simple projective module in $\Mod-R$, we denote by $P$ an
indecomposable preprojective module which is not isomorphic to \(2\). By [6, Example 5.18] there exists a universal localization \(\lambda : R \to S\) such that \(\lambda^*(\text{Mod-}S) = \text{Add}(P)\). Since \(\lambda\) is a universal localization, the natural homomorphism \(2 \otimes_R \lambda : 2 \to 2 \otimes_R S\) is a the \(\text{Add}(P)\)-reflection of \(2\), i.e. for every \(X \in \text{Add}(P)\), every homomorphism \(2 \to X\) factorizes through \(2 \otimes_R \lambda\). But \(\text{Hom}_R(2, \text{Add}(P)) \neq 0\), so it follows that \(2 \otimes_R S \neq 0\). Moreover, it is obvious that \(2\) cannot generate the modules from \(\text{Add}(P)\). Note that \(2\) is a silting module with respect to a homomorphism \(\sigma\).

By the above remarks it follows that the complex \(\sigma \otimes_R S\) is not silting.

In the following we will see an example of a tilting module such that the induced module with respect to a ring homomorphism is silting, but it is not 1-tilting. This is based on the example presented in [7, Example 4.2].

**Example 2.6.** Let \(R\) be the algebra associated to the quiver

\[
\begin{array}{ccc}
1 & \overset{\beta}{\leftarrow} & 2 & \overset{\alpha}{\leftarrow} & 3,
\end{array}
\]

and \(S\) algebra associated to the quiver

\[
\begin{array}{ccc}
1 & \overset{\beta}{\leftarrow} & 2 & \overset{\alpha}{\leftarrow} & 3
\end{array}
\]

such that \(\alpha \beta \gamma = 0\). Then \(S\) is a split by nilpotent extension of \(R\) induced by the module generated by \(\gamma\). Let \(\lambda : R \to S\) be the corresponding ring homomorphism. Then \(T = 1 \oplus 2 \oplus 2 \oplus 3\) is a tilting \(R\)-module. If \(0 \to P^{-1} \xrightarrow{\sigma} P^0 \to T \to 0\) is a minimal projective presentation for \(T\), it is proved in [7, Example 4.2], by using [7, Lemma 2.2], that \(T \otimes S\) is not of projective dimension at most 1, so it is not an 1-tilting \(S\)-module. However \(\lambda^*(T \otimes S) \in \text{Gen}(T)\), hence \(\sigma \otimes S\) is a 2-term silting complex.

In the commutative case the silting property associated to 2-term silting complexes ascends along all ring homomorphisms.

**Theorem 2.7.** If \(R\) and \(S\) are commutative rings, \(\lambda : R \to S\) is a unital ring homomorphism, and \(\sigma : P^{-1} \to P^0\) is a 2-term silting complex then the complex \(\sigma \otimes_R S\) is silting.

**Proof.** Let \(T = \text{Coker}(\sigma)\). In order to apply Theorem 2.2 we will prove that \(\lambda^*(T \otimes_R S)\) is \(T\)-generated. Let \(\alpha : R^{(1)} \to S \to 0\) be an epimorphism of \(R\)-modules. Since \(R\) and \(S\) are commutative, \(\alpha\) is a homomorphism of \(R\)-\(R\)-bimodules, and it follows that \(1_T \otimes_R \alpha : T \otimes_R R^{(1)} \to T \otimes_R S\) is an \(R\)-epimorphism. \(\square\)

### 3. The descent property for commutative rings

The main aim of this section is to prove that in the commutative case the property “2-term silting complex” descends along faithfully flat ring homomorphisms. We note that the restriction to faithfully flat ring homomorphisms is natural.

**Example 3.1.** Let \(\lambda : \mathbb{Z} \to \mathbb{Q}\) be the canonical embedding. Therefore, \(\lambda\) is a ring epimorphism, but it is not faithfully flat. Let \(0 \to F^{-1} \xrightarrow{\sigma} F^0 \to \mathbb{Q} \to 0\) be a projective presentation in \(\text{Mod-}\mathbb{Z}\) for the group of rational numbers. Then \(\mathcal{D}_\sigma = \text{KerExt}_\mathbb{Z}^1(\mathbb{Q}, -)\) contains all finite abelian groups, [17, Property 52 (D)]. But
for every finite group $G$, we have $\text{Hom}_\mathbb{Z}(\mathbb{Q}, G) = 0$, hence $D_\sigma \neq \text{Gen}(\mathbb{Q})$. It follows that $\sigma$ is not a 2-term silting complex. However, it is easy to see that $\sigma \otimes_{\mathbb{Z}} \mathbb{Q}$ is a 2-term silting complex of $\mathbb{Q}$-modules.

In this section all rings are commutative. If $R$ is a commutative ring then $\text{Spec}(R)$ will be the spectrum of $R$, and for every $p \in \text{Spec}(R)$ we will denote by $\kappa(p)$ the field of fractions of $R/p$. If $I$ is an ideal of $R$ then $V(I) = \{ p \in \text{Spec}(R) \mid I \subseteq p \}$. If $\lambda : R \to S$ is a faithfully flat ring homomorphism, then it is injective. Therefore, in order to simplify the presentation, we will often view $R$ as a subring of $S$. For instance, if $J$ is an subset of $S$ we will write $R \cap J$ instead of $R \cap \lambda^{-1}(J)$. We refer to [24] for other notations and for the basic properties which will be used here.

If $\sigma : P^{-1} \to P^0$ is a homomorphism between projective $R$-modules then we will associate to $\sigma$, as in [4], the class $T_\sigma = \{ M \in \text{Mod}_R \mid \sigma \otimes_R M$ is a monomorphism $\}$. Moreover, we also use the class $V_\sigma = \{ p \in \text{Spec}(R) \mid \sigma \otimes_R \kappa(p)$ is not a monomorphism $\}$.

The class $T_\sigma$ is closed under submodules and extensions (e.g. [11] Lemma 2.2.2). Moreover, using [4] Lemma 3.3 and Lemma 4.2, we observe that if $\sigma$ is a 2-term silting complex then $T_\sigma$ is the torsion-free class associated to a hereditary torsion theory of finite type $(A_\sigma, T_\sigma)$, i.e. $T_\sigma$ is also closed under direct products, injective envelopes, and direct limits.

A class $D$ of $R$-modules is called a silting class if there exists a 2-term silting complex $\sigma$ such that $D = D_\sigma$ (i.e. there exits a silting module $T$ such that $D = \text{Gen}(T)$). In [4] Theorem 4.7 it is proved that there exists a bijective correspondence between the silting classes of a commutative ring and the Gabriel filters of finite type. We recall from [18] Theorem 2.2] that if $R$ is a commutative ring then there exist $1 \sim 1$ correspondences between the class of Gabriel filters of finite type defined on $R$, the class of hereditary torsion theories of finite type on $\text{Mod}_R$ and the set of Thomason subsets of $\text{Spec}(R)$, i.e. unions of families of subsets of the the form $V(I)$ with $I$ finitely generated ideals (these are the open sets associated to Hochster’s topology defined on $\text{Spec}(R)$). Therefore, we can enunciate the following

**Theorem 3.2.** [4 Theorem 4.7], [18 Theorem 2.2] If $R$ is a commutative ring then there exist bijections between the following classes:

1. the class of hereditary torsion theories of finite type in $\text{Mod}_R$;
2. the class of Gabriel filters of finite type;
3. the class of Thomason subsets in $\text{Spec}(R)$;
4. the class of silting classes in $\text{Mod}_R$.

These bijections are described in the above mentioned papers. For reader’s convenience we list here the correspondences which will used in the following. We refer to [33] Section 10.39] for the basic properties of the support. In particular, we recall that for every finitely presented $R$-module $M$ we have, by Nakayama’s Lemma, $\text{supp}(M) = \{ p \mid M \otimes_R \kappa(p) \neq 0 \}$ (see the proof of [33 Lemma 10.39.8]).

If $G$ is a Gabriel filter of finite type on $R$ and $\mathcal{B} \subseteq G$ is a cofinal set of finitely generated ideals then
the torsion free class associated to $G$ is
$$F_G = \text{KerHom}_R(\bigoplus_{I \in B} R/I, -);$$

the Thomason subset associated to $G$ is
$$V_G = \{ p \in \text{Spec}(R) \mid \exists I \in G \text{ such that } p \in \text{supp} R/I \}.$$ We can use [33, Lemma 10.39.9] to conclude that
$$V_G = \{ p \in \text{Spec}(R) \mid \exists I \in B \text{ such that } R/I \otimes_R \kappa(p) \neq 0 \};$$

the silting class induced by $G$ is
$$D_G = \bigcap_{I \in G} \text{Ker}(- \otimes_R R/I) = \bigcap_{I \in B} \text{Ker}(- \otimes_R R/I).$$

If $\sigma$ is a 2-term silting complex then the Gabriel filter of finite type induced by $T_\sigma$ via [18, Theorem 2.2] is
$$G_\sigma = \{ I \leq R \mid \text{Hom}_R(R/I, T_\sigma) = 0 \}.$$ In the bijective correspondence constructed in [4, Theorem 4.7] the Gabriel filter associated to each silting class $D$ is defined as
$$G_D = \{ I \leq R \mid M = IM \text{ for all } M \in D \}.$$ 

Lemma 3.3. If $D$ is a silting class and $\sigma$ is a 2-term silting complex such that $D = D_\sigma$ then $G_D = G_\sigma$.

Proof. Since in [4, Lemma 3.3(2)] it is proved that the definable classes $D = D_\sigma$ and $T_\sigma$ are dual, it follows from [27, Corollary 3.4.21] that a module $X$ belongs to $T_\sigma$ if and only if $X^+ \in D$, where $X^+ = \text{Hom}_{\mathbb{Z}}(X, \mathbb{Q}/\mathbb{Z})$. Therefore, the correspondence $D \mapsto G_\sigma$ is independent of the choice of $\sigma$. Moreover, a module $M$ is in $D$ if and only if $X^+ \in T_\sigma$. By [18, Theorem 2.2] this is equivalent to $\text{Hom}(R/I, X^+) = 0$ for all $I \in G_\sigma$. Using the natural isomorphism induced by $\text{Hom}$ and $\otimes$, we obtain $M \in D$ if and only if $R/I \otimes_R M = 0$ for all $I \in G_\sigma$. This means that $G_\sigma$ is the Gabriel filter of finite type constructed in [4, Proposition 4.4], and the proof is complete. \qed

In the following we present some properties of the classes involved in Theorem 3.2. We start with a well-known lemma.

Lemma 3.4. If $p \in \text{Spec}(R)$ and $E(R/p)$ is the injective envelope for $R/p$ then
(i) $E(R/p)$ is an injective cogenerator for $\text{Mod-R}_p$;
(ii) there is a natural isomorphism $\text{Hom}_R(-, \kappa(p)) \cong \text{Hom}_{R_p}(- \otimes_R \kappa(p), E(R/p))$.

Proof. The first statement is well known. For the the second statement, we observe that there are the natural isomorphisms
$$\text{Hom}_R(-, \kappa(p)) \cong \text{Hom}_{R_p}(\kappa(p), E(R/p)) \cong \text{Hom}_{R_p}(- \otimes_R \kappa(p), E(R/p)),$$ so we have the required isomorphism. \qed

Corollary 3.5. Let $G$ be a Gabriel filter of finite type, and $p \in \text{Spec}(R)$. Then the following are equivalent:
(i) $p \notin V_G$;
(ii) $\kappa(p) \in F_G$;
(iii) $R/p \in F_G$. 



In particular, if \( \sigma \) is a 2-term silting complex, and \( \mathcal{G} \) is the induced Gabriel filter, then \( \mathcal{F}_\mathcal{G} = \mathcal{T}_\sigma \), and \( \mathcal{V}_\mathcal{G} = \mathcal{V}_\sigma \).

Proof. The equivalence (i)\iff (ii) follows from Lemma 3.3, while (ii)\iff (iii) is true since \( \mathcal{F}_\mathcal{G} \) is closed under submodules and injective envelopes.

If \( \sigma \) is a 2-term silting complex, the equality \( \mathcal{F}_\mathcal{G} = \mathcal{T}_\sigma \) is true since \( \mathcal{G} \) is the Gabriel filter associated to \( \mathcal{T}_\sigma \). The second equality follows from the equivalence (i)\iff (ii). \( \square \)

Theorem 3.6. Let \( \lambda : R \to S \) be a homomorphism of commutative rings. If \( P^{-1} \) and \( P^0 \) are projective \( R \)-modules and \( \sigma : P^{-1} \to P^0 \) is a homomorphism, we denote by \( \mathcal{T}_{\sigma \otimes_R S} \subseteq \text{Mod-}S \) the class associated to the \( S \)-homomorphism \( \sigma \otimes_R S \). The following are true:

(i) \( \mathcal{T}_{\sigma \otimes_R S} = \mathcal{T}_\sigma \cap \text{Mod-}S \).

(ii) Suppose that \( \mathcal{T}_\sigma \) is faithfully flat. For a module \( M \in \text{Mod-}R \) we have \( M \in \mathcal{T}_\sigma \) if and only if \( M \otimes_R S \in \mathcal{T}_{\sigma \otimes_R S} \).

Proof. (i) This follows by using the natural isomorphisms \( \sigma \otimes_R S \otimes_S M \cong \sigma \otimes_R M \) for all \( M \in \text{Mod-}S \).

(ii) Suppose that \( M \in \mathcal{T}_\sigma \). Then \( \sigma \otimes_R M \) is monic. Since \( S \) is flat, it follows that \( \sigma \otimes_R M \otimes_R S \) is monic, hence \( \lambda^*(M \otimes_R S) \in \mathcal{T}_\sigma \). By (i) we obtain \( M \otimes_R S \in \mathcal{T}_{\sigma \otimes_R S} \).

Conversely, suppose that \( M \otimes_R S \in \mathcal{T}_{\sigma \otimes_R S} \). Then \( \lambda^*(M \otimes_R S) \in \mathcal{T}_\sigma \). Since \( S \) is faithfully flat we know that \( M \) can be embedded as a submodule of \( \lambda^*(M \otimes S) \), hence \( M \in \mathcal{T}_\sigma \). \( \square \)

In the following, we will use, as in [14] the notation \( \text{Def}_\sigma = \text{Coker}(\text{Hom}_R(\sigma, -)) : \text{Mod-}R \to \text{Ab} \). Therefore, for every homomorphism \( \sigma \) we have \( \mathcal{D}_\sigma = \{ M \in \text{Mod-}R \mid \text{Def}_\sigma(M) = 0 \} \).

Lemma 3.7. Let \( \lambda : R \to S \) be a homomorphism of commutative rings. If \( L^{-1} \) and \( L^0 \) are finitely generated and projective \( R \)-modules and \( \sigma : L^{-1} \to L^0 \) is an \( R \)-morphism then we have a natural isomorphism

\[
\text{Def}_{\sigma \otimes_R S}(- \otimes_R S) \cong \text{Def}_\sigma(-) \otimes_R S.
\]

Proof. This is a consequence of the fact that for every finitely presented \( R \)-module \( L \), there exists a natural isomorphism between the functions \( \text{Hom}_S(L \otimes_R S, - \otimes_R S) \) and \( \text{Hom}_R(L, -) \otimes_R S \) (see [24] Theorem 7.11). \( \square \)

Lemma 3.8. Suppose that \( \lambda : R \to S \) is a homomorphism of commutative rings and that \( \sigma \) is a 2-term silting complex. Then

(a) \( \mathcal{D}_\sigma \otimes_R S \subseteq \mathcal{D}_{\sigma \otimes_R S} \);

(b) if \( \lambda \) is faithfully flat and \( N \in \text{Mod-}R \) then \( N \in \mathcal{D}_\sigma \) if and only if \( N \otimes_R S \in \mathcal{D}_{\sigma \otimes_R S} \).

Proof. (a) If \( N \in \mathcal{D}_\sigma \) then \( N \otimes_R S \) is \( T \)-generated. Since it is an \( S \)-module we obtain \( N \otimes_R S \in \mathcal{D}_{\sigma \otimes_R S} \) by Theorem 2.2.

(b) Suppose that \( N \otimes_R S \in \mathcal{D}_{\sigma \otimes_R S} \). It was proved in [3] Theorem 2.3] and in [23] Theorem 6.3] that every silting class is of finite type. Therefore, there exists a family \( \sigma_i, i \in I \), of homomorphisms between finitely generated projective \( R \)-modules such that \( \mathcal{D}_\sigma = \bigcap_{i \in I} \mathcal{D}_{\sigma_i} \). By Lemma 2.1 it follows that \( N \otimes_R S \in \mathcal{D}_\sigma \).
Using the proof of Lemma 2.1 and Lemma 3.7 we observe that for every $i \in I$ we have the isomorphisms
\[ 0 = \text{Def}_\sigma(N \otimes S) \cong \text{Def}(\otimes_R S)(N \otimes_R S) \cong \text{Def}_\sigma(N) \otimes_R S. \]
Since $S$ is faithfully flat, it follows that $N \in \mathcal{D}_{\sigma}$ for all $i \in I$, so $N \in \mathcal{D}_\sigma$. \hfill \Box

Remark 3.9. We recall from [5] that if $\sigma$ is 2-term silting complex then it is a generator in $\mathcal{D}(R)$, i.e. the smallest triangulated subcategory which contains $\sigma$ and is closed under direct sums is $\mathcal{D}(R)$.

Moreover, it was proved in [2, Theorem 3.14] (see also [5, Remark 2.7]) that an object in $\mathcal{D}(R)$ is a generator if and only if it has the property: for every $Y \in \mathcal{D}(R)$, from $\text{Hom}(X, Y[i]) = 0$ for all $i \in \mathbb{Z}$ it follows that $Y = 0$.

For the proof of the following lemma we use the same techniques as those used in the proof of [24, Lemma 4.12].

\textbf{Lemma 3.10.} Let $\sigma : P^{-} \rightarrow P^{0}$ be a homomorphism between projective $R$-modules with $\text{Coker}(\sigma) = T$. If the complex (concentrated in $-1$ and $0$) which is induced by $\sigma$ is a generator for $\mathcal{D}(R)$ then for every $p \in \text{Spec}(R)$ we have $T \otimes_R \kappa(p) \neq 0$ or $\text{Ker}(\sigma \otimes_R \kappa(p)) \neq 0$.

\textbf{Proof.} Since $\sigma$ is a generator for $\mathcal{D}(R)$, for every $R$-module $M$ there exists $i \in \{0, 1\}$ such that $\text{Hom}_{\mathcal{D}(R)}(\sigma, M[i]) = \text{Hom}_{\mathcal{K}(R)}(\sigma, M[i]) \neq 0$. It follows that $\text{Hom}(\sigma, M)$ is not an isomorphism for all $R$-modules $M$.

If $p \in \text{Spec}(R)$, we take $M = \kappa(p) = \text{Hom}_{\mathcal{K}(R)}(\kappa(p), E(R/p))$, where $E(R/p)$ is the injective envelope of $\kappa(p)$ in $\text{Mod}-R_p$. Using Lemma 3.4 we obtain that $\text{Hom}_{\mathcal{K}(R)}(\sigma, \kappa(p))$ is not an isomorphism if and only if $\sigma \otimes_R \kappa(p)$ is not an isomorphism. \hfill \Box

We recall from [29] and [5] some basic facts about ring epimorphisms.

\textbf{Lemma 3.11.} Let $R$ be a commutative ring and $\delta : R \rightarrow B$ a ring epimorphism. Then
\begin{enumerate}[(i)]  
  \item $B$ is commutative;
  \item the canonical map $\delta^* : \text{Spec}(B) \rightarrow \text{Spec}(R)$ is injective;
  \item for every $q \in \text{Spec}(B)$, $\kappa(q) = \kappa(\delta^*(q))$.
\end{enumerate}

\textbf{Proposition 3.12.} Let $\sigma : P^{-} \rightarrow P^{0}$ be a homomorphism between projective $R$-modules. The following are equivalent:
\begin{enumerate}[(1)]  
  \item $\sigma$ is a 2-term silting complex;
  \item (i) $\sigma$ is partial silting, 
    \begin{enumerate}[(i)]  
      \item for every $p \in \text{Spec}(R)$ we have $T \otimes_R \kappa(p) \neq 0$ or $\text{Ker}(\sigma \otimes_R \kappa(p)) \neq 0$.
    \end{enumerate}
\end{enumerate}

\textbf{Proof.} (1)⇒(2) This follows from Lemma 3.10.

Since $\sigma$ is partial silting, then $\text{Gen}(T) \subseteq \mathcal{D}_\sigma$ are torsion classes, and the torsion-free class corresponding to Gen$(T)$ is KerHom$_R(T, -)$, [5, Remark 3.8]. Therefore, in order to obtain $\text{Gen}(T) = \mathcal{D}_\sigma$, it is enough to prove that $\mathcal{D}_\sigma \cap \text{KerHom}_R(T, -) = 0$.

Let $\mathcal{Y} = \mathcal{D}_\sigma \cap \text{KerHom}_R(T, -)$. By [6, Proposition 3.3] the class $\mathcal{Y}$ is bireflective and extension closed. Therefore, it induces an epimorphism of rings $\delta : R \rightarrow B$, and $\mathcal{Y}$ is the essential image of the restriction of scalars $\delta^* : \text{Mod}-B \rightarrow \text{Mod}-R$.

Suppose that $B \neq 0$. Then Spec$(B) \neq \emptyset$. By Lemma 3.11 it follows that for every $q \in \text{Spec}(B)$ we can identify $\kappa(q) = \kappa(p)$ for some $p \in \text{Spec}(R)$. Therefore,
there exists \( p \in \text{Spec}(R) \) such that \( \kappa(p) \in \mathcal{Y} \). If \( p \) is such an ideal we obtain that \( \text{Hom}_R(\sigma, \kappa(p)) \) is an isomorphism. Using Lemma 3.11 it follows that \( \sigma \otimes_R \kappa(p) \) is an isomorphism, and this contradicts (ii). Then \( B = 0 \), and the proof is complete. \( \square \)

The following properties are well known.

**Lemma 3.13.** Let \( \lambda : R \to S \) be a faithfully flat homomorphism of commutative rings. If \( q \in \text{Spec}(S) \) and \( p = q \cap R \) then there is a natural isomorphism

\[
- \otimes_R S \otimes_S \kappa(q) \cong - \otimes_R \kappa(p) \otimes_{\kappa(p)} \kappa(q).
\]

Moreover \( \kappa(q) \) is faithfully flat as a \( \kappa(p) \)-module.

We recall that we use the notation

\[
V_\sigma = \{ p \in \text{Spec}R \mid \sigma \otimes_R \kappa(p) \text{ is not a monomorphism} \}
\]

for every homomorphism \( \sigma \) between projective modules.

**Lemma 3.14.** Suppose that \( \lambda : R \to S \) is faithfully flat, and \( \sigma : P^{-1} \to P^0 \) is a homomorphism between projective \( R \)-modules such that \( \sigma \otimes_R S \) is a 2-term silting complex. Let \( \lambda^* : \text{Spec}(S) \to \text{Spec}(R) \), \( \lambda^*(q) = q \cap R \), be the canonical map. Then

(i) for an ideal \( q \in \text{Spec}(S) \), we have \( \kappa(q) \in T_{\sigma \otimes S} \) if and only if \( \kappa(\lambda^*(q)) \in T_\sigma \);

(ii) \( \lambda^*(\text{Spec}(S) \setminus V_{\sigma \otimes S}) = \text{Spec}(R) \setminus V_\sigma \);

(iii) \( \lambda^*(V_{\sigma \otimes S}) = V_\sigma \).

**Proof.** (i) If \( q \in \text{Spec}(S) \) and \( p = q \cap R \), it follows from Lemma 3.13 that \( \sigma \otimes_R \kappa(p) \) is injective if and only if \( \sigma \otimes_R \kappa(p) \otimes_{\kappa(p)} \kappa(q) \) is injective, and this is equivalent to \( \sigma \otimes_R S \otimes_S \kappa(q) \) is injective.

(ii) We observe that

\[
V_\sigma = \{ p \in \text{Spec}(S) \mid \kappa(p) \notin T_\sigma \} \text{ and } V_{\sigma \otimes R S} = \{ q \in \text{Spec}(S) \mid \kappa(q) \notin T_{\sigma \otimes R S} \}.
\]

By (i) it follows that for an ideal \( q \in \text{Spec}(S) \) we have \( q \notin V_{\sigma \otimes R S} \) if and only if \( \lambda^*(q) \notin V_\sigma \).

Since \( \lambda^* \) is surjective, it follows that \( \lambda^*(\text{Spec}(S) \setminus V_{\sigma \otimes R S}) = \text{Spec}(R) \setminus V_\sigma \).

(iii) This follows by using (ii) and the surjectivity of \( \lambda^* \). \( \square \)

In the end of these preliminary considerations we recall the results obtained in [20] for the study of the descent of 1-tilting modules.

**Proposition 3.15.** [20] Section 4 Let \( \overline{\lambda} : \overline{R} \to \overline{S} \) be a faithfully flat homomorphism of rings. If \( T \) and \( V \) are \( \overline{R} \)-modules such that

(a) \( V \) is tilting,
(b) \( T \otimes_{\overline{R}} \overline{S} \) is a tilting \( \overline{S} \)-module, and
(c) \( \text{Gen}(T \otimes_{\overline{R}} \overline{S}) = \text{Gen}(V \otimes_{\overline{R}} \overline{S}) \)

then \( T \) is a tilting \( \overline{R} \)-module and \( \text{Gen}(T) = \text{Gen}(V) \).

We are ready to prove the descent property for 2-term silting complexes.

**Theorem 3.16.** Suppose that \( \lambda : R \to S \) is a faithfully flat ring homomorphism. If \( \sigma : P^{-1} \to P^0 \) is a homomorphism in \( \text{Mod}-R \) such that \( \sigma \otimes_R S \) is a 2-term silting complex of \( S \)-modules then \( \sigma \) is a 2-term silting complex of \( R \)-modules.
Proof. Since $\sigma \otimes_R S$ is a 2-term silting complex it follows that $P^{-1} \otimes_R S$ and $P^0 \otimes_R S$ are projective $S$-modules. Using the descend property of projective modules, [19], it follows that the $R$-modules $P^{-1}$ and $P^0$ are projective.

Let $\lambda^*: \text{Spec}(S) \to \text{Spec}(R)$, $\lambda^*(q) = q \cap R$ be the canonical map. If $V_{\sigma \otimes_R S} \subseteq \text{Spec}(S)$ is the Thomason set associated to $\sigma \otimes S$ then we use [20, Lemma 3.14(i)] together with Lemma 3.14(ii) to conclude that $\lambda^*(V_{\sigma \otimes_R S})$ is a Thomason subset in $\text{Spec}(R)$. By Theorem 3.2 and Corollary 3.3 there exists a 2-term silting complex $\rho: L^{-1} \to L^0$ in $\text{Mod-}R$ such that $V_{\rho} = \lambda^*(V_{\sigma \otimes_R S})$. Using the ascent property proved in Theorem 2.4 we conclude that $\rho \otimes_R S$ is silting in $\text{Mod-}S$.

We apply Lemma 3.14(iii) to $\rho$ and $\sigma$, and we obtain that

$$\lambda^*(V_{\rho \otimes_R S}) = V_{\rho} = \lambda^*(V_{\sigma \otimes_R S}) = V_{\sigma}.$$  

We use Lemma 3.14(i) and we obtain the equality $V_{\rho \otimes_R S} = V_{\sigma \otimes_R S}$. It follows that the homomorphisms $\rho \otimes_R S$ and $\sigma \otimes_R S$ induce the same silting class, i.e. $D_{\sigma \otimes_R S} = D_{\rho \otimes_R S}$.

Let us denote $T = \text{Coker}(\sigma)$ and $\mathcal{G}$ be the Gabriel filter associated to $\rho$ as in Theorem 3.2 and Lemma 3.6. It follows that

$$D_\rho = \bigcap_{I \in \mathcal{G}} \text{Ker}(- \otimes_R R/I).$$

Since $T \otimes S \in D_{\rho \otimes_R S}$, we can use Lemma 3.8 to conclude that $T \in D_\rho$. Therefore, for every $I \in \mathcal{G}$ we have $T \otimes_R R/I = 0$.

Claim 1. For every $I \in \mathcal{G}$ there exists a set $K$ and a pushout diagram

$$\begin{array}{ccc}
(P^{-1})^{(K)} & \xrightarrow{\sigma^{(K)}} & (P^0)^{(K)} \\
\downarrow{\delta} & & \downarrow{\alpha} \\
R & \xrightarrow{\alpha} & E \\
\end{array}$$

such that $E \otimes_R R/I = 0$.

Let $I \in \mathcal{G}$ be a fixed ideal. We will use the notation $\sigma \otimes_R R/I = \sigma_R/I$.

Applying $- \otimes_R R/I$ to $\sigma$ we obtain a short exact sequence of $R/I$-modules

$$0 \to \text{Ker}(\sigma_R/I) \xrightarrow{\alpha} P^{-1} \otimes_R R/I \xrightarrow{\sigma_R/I} P^0 \otimes_R R/I \to 0,$$

which splits since $P^0 \otimes_R R/I$ is projective. Therefore $\text{Ker}(\sigma_R/I)$ is projective.

Moreover, since this exact sequence splits, for every $p \in \text{Spec}(R/I)$ we obtain a commutative diagram

$$\begin{array}{ccc}
\text{Ker}(\sigma_R/I) \otimes_R R/I & \xrightarrow{\kappa(p)} & P^{-1} \otimes_R R/I \otimes_R R/I \kappa(p) \\
\downarrow{u \otimes_R R/I \kappa(p)} & & \downarrow{\kappa(p)} \\
\text{Ker}(\sigma \otimes_R \kappa(p)) & \xrightarrow{} & P^{-1} \otimes_R \kappa(p) \\
\end{array}$$

such that the horizontal lines are (split) short exact sequences and the vertical arrows are isomorphisms. Since $V_\sigma$ is the Thomason set corresponding to $\mathcal{G}$, we have $V(I) \subseteq V_\sigma$. It follows that $\text{Ker}(\sigma_R/I) \otimes_R R/I \kappa(p) \neq 0$ for all $p \in \text{Spec}(R/I)$, hence $\text{Ker}(\sigma_R/I)$ is a projective generator for $\text{Mod-}R/I$.  

Therefore, there exists a set $K$ and an $R/I$-epimorphism $\gamma : \text{Ker}(\tilde{\sigma}_{R/I})^{(K)} \to R/I$. We also fix a homomorphism $\nu : (P^{-1})^{(K)} \otimes_R R/I \to \text{Ker}(\tilde{\sigma}_{R/I})^{(K)}$ such that $\nu u^{(K)} = 1$. If $\pi : R \to R/I$ is the canonical surjective homomorphism, then there exists a homomorphism $\delta : (P^{-1})^{(K)} \otimes_R R \to R$ such that $\pi \delta = \gamma \nu (1 \otimes_R \pi)$. In order to simplify the presentation we identify $(P^{-1})^{(K)} \otimes_R R$ with $(P^{-1})^{(K)}$, and all these data are represented in the following commutative diagram:

\[\begin{array}{c}
\text{Ker}(\tilde{\sigma}_{R/I})^{(K)} \xrightarrow{u^{(K)}} (P^{-1})^{(K)} \otimes_R R/I \\
\downarrow \gamma \hspace{2cm} \downarrow \pi \hspace{2cm} \downarrow \gamma \nu \otimes_R \pi \\
R/I \otimes_R R/I \xrightarrow{\delta \otimes_R R/I} \pi \otimes R/I \\
\end{array}\]

where the dashed arrow is $\delta$. We apply the functor $- \otimes_R R/I$ to this diagram, and we obtain the commutative diagram

\[\begin{array}{c}
\text{Ker}(\sigma^{(K)} \otimes_R R/I) \xrightarrow{u^{(K)}} (P^{-1})^{(K)} \otimes_R R/I \\
\downarrow \alpha \hspace{2cm} \downarrow \nu \otimes_R R/I \\
\text{Ker}(\tilde{\sigma}_{R/I})^{(K)} \otimes_R R/I \xrightarrow{u^{(K)} \otimes_R R/I} (P^{-1})^{(K)} \otimes_R R/I \\
\downarrow \gamma \otimes_R R/I \\
R/I \otimes_R R/I \xrightarrow{\delta \otimes_R R/I} \pi \otimes_R R/I \\
\end{array}\]

where $\alpha$ is the canonical map. Using the obvious identifications and the natural isomorphisms $R/I \otimes_R R/I \cong R/I \otimes_R R/I \otimes_R R/I \cong R/I$, [10 Proposition II.2], it follows that $\alpha$ and $\pi \otimes R/I$ are isomorphisms. It is not hard to conclude that $(\delta \otimes_R R/I) u^{(K)}$ is an epimorphism.

We construct the pushout diagram

\[\begin{array}{c}
(P^{-1})^{(K)} \xrightarrow{\sigma^{(K)}} (P^0)^{(K)} \\
\downarrow \delta \hspace{2cm} \downarrow \alpha \hspace{2cm} \downarrow \delta \otimes_R R/I \\
R \xrightarrow{E} \text{T}^{(K)} \xrightarrow{0} \\
\end{array}\]

Applying the tensor product $- \otimes_R R/I$, and using the commuting property of the tensor product with respect to direct sums together with the well-known fact that the direct sums preserve exact sequences we obtain the commutative diagram

\[\begin{array}{c}
\text{Ker}(\tilde{\sigma}_{R/I})^{(K)} \xrightarrow{u^{(K)}} (P^{-1} \otimes_R R/I)^{(K)} \\
\downarrow \delta \otimes_R R/I \\
\text{Ker}(\alpha \otimes_R R/I) \xrightarrow{\delta \otimes_R R/I} R \otimes_R R/I \\
\end{array}\]

Since $(\delta \otimes_R R/I) u^{(K)}$ is an epimorphism, it follows that $E \otimes_R R/I = 0$.  

\[\text{THE ASCENT-DESCENT PROPERTY FOR 2-TERM SILTING COMPLEXES 1 1}\]
Claim 2. $D_\sigma \subseteq D_\rho$.

In order to prove this, let us fix a module $M \in D_\sigma$. For every $I \in \mathcal{G}$ we construct a pushout diagram as in Claim 1.

We consider an epimorphism $f : R^\ell \rightarrow M$. Then $f\delta^\ell : [(P^{-1})(K)]^\ell \rightarrow M$ can be extended to a homomorphism $[(P^0)(\sigma^0)]^\ell \rightarrow M$ through $[\sigma^0]^\ell$. It follows that there exists a homomorphism $g : E^\ell \rightarrow M$ such that $f = g\alpha^\ell$.

Since $f$ is an epimorphism, we also have that $g$ is an epimorphism. Using the property $E \otimes_R R/I = 0$, we obtain $M \otimes_R R/I = 0$.

It follows that $M \otimes_R R/I = 0$ for all $I \in \mathcal{G}$. Therefore, $M \in D_\rho$, and the proof for the inclusion $D_\sigma \subseteq D_\rho$ is complete.

Claim 3. If $V = \text{Coker}(\rho)$ then $\text{Ann}(T) = \text{Ann}(V)$.

Let $U = \text{Ann}(V)$. We have $US \subseteq \text{Ann}(V \otimes_R S) = \text{Ann}(T \otimes_R S)$. But $S$ is faithfully flat, hence $U \subseteq US$. We can view $T$ as a submodule of $T \otimes_R S$. Therefore, $UT = 0$, and it follows that $U \subseteq \text{Ann}(T)$.

Since in this proof we only used the equality $\text{Ann}(V \otimes_R S) = \text{Ann}(T \otimes_R S)$, it follows that the converse inclusion is valid, so the proof for Claim 3 is complete.

In the following we will use the notations $U = \text{Ann}(V) = \text{Ann}(T)$, $\overline{R} = R/U$, and $\overline{S} = S/US$.

Claim 4. The $\overline{R}$-modules $V$ and $T$ are tilting and $\text{Gen}(T) = \text{Gen}(V)$.

We view $\text{Mod-}\overline{R}$ as a full subcategory of $\text{Mod-}R$ via the canonical homomorphism $R \rightarrow R/U$. Then for every $M \in \text{Mod-}\overline{R}$ we have $UM = 0$, and it follows that the left side homomorphism in the exact sequence

$$M \otimes_R US \rightarrow M \otimes_R S \rightarrow M \otimes_R \overline{S} \rightarrow 0$$

is zero. Therefore, the restrictions of the functors $- \otimes_R S$ and $- \otimes_R \overline{S}$ to $\text{Mod-}\overline{R}$ are naturally isomorphic. Since $\overline{S}$ is also an $\overline{R}$-module, we also have a natural isomorphism $- \otimes_R \overline{S} \cong - \otimes_{\overline{R}} \overline{S}$ for the restrictions of these functors to $\text{Mod-}\overline{R}$.

[[10] Proposition II.2]. This shows that the canonical homomorphism $\lambda : \overline{R} \rightarrow \overline{S}$, induced by $\lambda$, is faithfully flat.

From [[5] Proposition 3.2 and Proposition 3.10] it follows that $V$ is a tilting $\overline{R}$-module. By [[20] Lemma 2.4] we have that $V \otimes_{\overline{R}} S = V \otimes_{\overline{R}} \overline{S}$ is a tilting $\overline{S}$-module. This implies that the annihilator of the $\overline{S}$-module $V \otimes_{\overline{R}} \overline{S}$ is zero, and it follows that $\text{Ann}_{\overline{R}}(T \otimes_{\overline{R}} \overline{S}) = 0$. Since $\sigma \otimes_{\overline{R}} \overline{S}$ is a 2-term silting complex, we can use [[3] Proposition 3.2 and Proposition 3.10] one more time to obtain that $T \otimes_{\overline{R}} \overline{S} = T \otimes_{\overline{R}} S$ is a tilting $\overline{S}$-module. By Proposition 3.13 we obtain that $T$ is tilting as an $\overline{R}$-module and $\text{Gen}(T) = \text{Gen}(V)$.

Claim 5. We have $D_\rho \subseteq D_\sigma$. In particular $T^{(I)} \in D_\sigma$ for all sets $I$.

In $\text{Mod-}\overline{R}$ we have the projective resolution

$$P^{-1} \otimes_{\overline{R}} \overline{R} \xrightarrow{\sigma \otimes_{\overline{R}} \overline{R}} P^0 \otimes_{\overline{R}} \overline{R} \rightarrow T \rightarrow 0.$$

But $T$ is tilting as an $\overline{R}$-module, hence it is of projective dimension at most 1. It follows that we can find in $\text{Mod-}\overline{R}$ a direct decomposition $P^{-1} \otimes_{\overline{R}} \overline{R} = \overline{P} \oplus \overline{K}$, where $\overline{K} = \text{Im}(\sigma \otimes_{\overline{R}} \overline{R})$ and $\overline{P} = \text{Ker}(\sigma \otimes_{\overline{R}} \overline{R})$. 


Since the complex $\sigma \otimes_R S$ is silting in $\text{Mod}-S$, we can apply Theorem 2.7 to conclude that

$$\sigma \otimes_R S \cong \sigma \otimes_R R \otimes_{\overline{R}} S \cong \sigma \otimes_R S \otimes_{\overline{R}} S$$

is 2-term silting complex in $\text{Mod}-S$. Moreover, since the induced ring homomorphism $\overline{X} : \overline{R} \rightarrow S$ is faithfully flat, and $T$ is a tilting $\overline{R}$-module, we obtain by using $[20]$ Lemma 2.4] that $T \otimes_{\overline{R}} S$ is tilting. It follows that

$$D_{\sigma \otimes_{\overline{R}} S} \cong \text{Gen}(T \otimes_{\overline{R}} S) = \operatorname{Ker} \operatorname{Ext}_S^1(T \otimes_{\overline{R}} S, -),$$

hence

$$\text{Hom}_S(P \otimes_R S, \text{Gen}(T \otimes_{\overline{R}} S)) = 0.$$

Since $S$ is faithfully flat, the last equality implies that $\text{Hom}_{\overline{R}}(P, \text{Gen}(T)) = 0$.

Let $X \in D_\rho = \text{Gen}(V) = \text{Gen}(T)$. Then $U \leq \text{Ann}(X)$. If $f : P^{-1} \rightarrow X$ is a homomorphism, we obtain the commutative diagram

$$
\begin{array}{cccccc}
0 & \rightarrow & P^{-1} \otimes_R U & \rightarrow & P^{-1} \otimes_R R & \rightarrow & P^{-1} \otimes_R \overline{R} & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & X \otimes_R U & \rightarrow & X \otimes_R R & \rightarrow & X \otimes_R \overline{R} & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & XU & \rightarrow & X & \rightarrow & X/XU & \rightarrow & 0,
\end{array}
$$

where the vertical arrows in the bottom rectangle are the natural ones. Since $XU = 0$, it follows that $f$ factorizes through $P^{-1} \otimes_R \overline{R}$. Therefore, there exists $\overline{f} : P^{-1} \otimes_R \overline{R} \rightarrow X$ such that $f = \overline{f} \circ h$, where $h : P^{-1} \rightarrow P^{-1} \otimes_R R \rightarrow P^{-1} \otimes_R \overline{R}$ is the canonical map.

But $T$ is tilting as an $\overline{R}$-module. It follows that there exists a homomorphism $g : P^0 \otimes_R \overline{R} \rightarrow X$ such that $\overline{f} \circ h = g(\sigma \otimes_R \overline{R})$. Since $\text{Hom}(\overline{P}, X) = 0$, we have $\overline{f}(\overline{P}) = 0$. Then $\overline{f} = g(\sigma \otimes_R \overline{R})$, and we obtain the commutative diagram

$$
\begin{array}{ccccccc}
P^{-1} & \rightarrow & \sigma & \rightarrow & P^0 & \rightarrow & T & \rightarrow & 0 \\
h & & \downarrow & & \downarrow & & \downarrow & & \\
P^{-1} \otimes_R \overline{R} & \rightarrow & \sigma \otimes R \overline{R} & \rightarrow & P^0 \otimes_R \overline{R} & \rightarrow & T & \rightarrow & 0, \\
\overline{f} & & \downarrow & & \downarrow & & \downarrow & & \\
& & X & & g & & \\
\end{array}
$$

where the composition of the vertical left side arrows is $f$. Then $f$ factorizes through $\sigma$, hence $X \in D_\rho$.

We conclude that $D_\rho \subseteq D_\sigma$.

Using Claim 2 and Claim 5 we obtain $D_\sigma = D_\rho$, and $T \in D_\sigma$. It follows that $\sigma$ is partial silting. By Proposition 3.12 in order to complete the proof, it is enough to prove

**Claim 6.** For every $p \in \text{Spec}(R)$ we have $T \otimes_R \kappa(p) \neq 0$ or $\operatorname{Ker}(\sigma \otimes_R \kappa(p)) \neq 0$. 

Let $p \in \text{Spec}(R)$. Since $\lambda$ is faithfully flat, the induced map of spectra
\[ \lambda^*: \text{Spec}(S) \to \text{Spec}(R), \quad \lambda^*(q) = q \cap R, \]
is surjective, so there exists $q \in \text{Spec}(S)$ such that $q \cap R = p$. By Lemma 3.13 we observe that $\kappa(q)$ is faithfully flat as a $\kappa(p)$-module (see also the proof of [20, Proposition 3.16]). Applying the functors from Lemma 3.13 to $\sigma$ it follows that
\[ \text{Ker}(\sigma \otimes_R S \otimes_S \kappa(q)) \cong \text{Ker}(\sigma \otimes_R \kappa(p) \otimes_{\kappa(p)} \kappa(q)), \]
and
\[ T \otimes_R S \otimes_S \kappa(q) \cong T \otimes_R \kappa(p) \otimes_{\kappa(p)} \kappa(q). \]
By Proposition 3.12 for every $q \in \text{Spec}(S)$ we have $T \otimes_R S \otimes_S \kappa(q) \neq 0$ or $\text{Ker}(\sigma \otimes_R S \otimes_S \kappa(q)) \neq 0$. Since $\kappa(q)$ is faithfully flat as a $\kappa(p)$-module, we obtain Claim 6, and the proof is complete. □

We close the paper with some comments on the proof of Theorem 3.16.

Remark 3.17. (a) From Claim 3 and Claim 4 it follows that $T$ is tilting as an $R/\text{Ann}(T)$-module. This is equivalent to the fact that the $R$-module $T$ is a finendo quasi-tilting module by [5, Proposition 3.2]. However, this is not enough to conclude that $T$ is a silting $R$-module, as it is proved in [4, Example 5.4] and [8, Example 5.12].

(b) In the noetherian case the proof can be done using the Claims 3–6 (i.e. the inclusion $D_\rho \subseteq D_\sigma$ and Claim 6) in the following way. We view $\sigma$ as a complex concentrated in $-1$ and $0$. It is easy to see that $\text{Hom}_{D(R)}(\sigma, \sigma^{[1]}[1]) = 0$ if and only if $T^{[1]} \in D_\sigma$. Therefore, we can use [5, Theorem 4.9] together with the inclusion $D_\rho \subseteq D_\sigma$ to observe that $\sigma$ is a 2-term silting complex if and only if it is a generator in $D(R)$. In the noetherian case the converse of Lemma 3.10 is also valid by using [9, Theorem 9.5] or [26, Theorem 2.8]. It follows that $\sigma$ is a generator if and only if $\sigma \otimes_R \kappa(p)$ is not an isomorphism for all $p \in \text{Spec}(R)$. Therefore, by using Claim 6 we obtain that $\sigma$ is a generator, and the proof is complete.

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