Cointegration with Occasionally Binding Constraints

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Abstract

In the literature on nonlinear cointegration, a long-standing open problem relates to how a (nonlinear) vector autoregression, which provides a unified description of the short- and long-run dynamics of a collection of time series, can generate ‘nonlinear cointegration’ in the profound sense of those series sharing common nonlinear stochastic trends. We consider this problem in the setting of the censored and kinked structural VAR (CKSVAR), which provides a flexible yet tractable framework within which to model time series that are subject to threshold-type nonlinearities, such as those arising due to occasionally binding constraints, of which the zero lower bound (ZLB) on short-term nominal interest rates provides a leading example. We provide a complete characterisation of how common linear and nonlinear stochastic trends may be generated in this model, via unit roots and appropriate generalisations of the usual rank conditions, providing the first extension to date of the Granger–Johansen representation theorem to a nonlinearly cointegrated setting, and thereby giving the first successful treatment of the open problem. The limiting common trend processes include regulated, censored and kinked Brownian motions, none of which have previously appeared in the literature on cointegrated VARs. Our results and running examples illustrate that the CKSVAR is capable of supporting a far richer variety of long-run behaviour than is a linear VAR, in ways that may be particularly useful for the identification of structural parameters.

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1 Introduction

Nonstationarity, in the form of highly persistent, randomly wandering time series, is ubiquitous in macroeconomics and finance. It presents both a challenge for inference (Stock, 1994; Watson, 1994) and an opportunity for the identification of dynamic causal effects (Blanchard and Quah, 1989). The canonical framework for modelling such behaviour is as (common) stochastic trends, generated by a linear vector autoregression (VAR) with unit roots, underpinned by the powerful Granger–Johansen representation theorem (Johansen, 1995, Ch. 4). However, this framework is inadequate when even one of the variables is subject to an occasionally binding constraint, such as the zero lower bound (ZLB) constraint on short-term nominal interest rates, which has latterly gained particular prominence in macroeconomic policy analysis (e.g. Summers, 2014; Williams, 2014; Eggertsson, Mehrtra, and Robbins, 2019; Kocherlakota, 2019).

In an autoregressive model, an occasionally binding constraint naturally gives rise to nonlinearity in the form of multiple endogenously switching regimes, whose presence significantly complicates the links between unit roots and stochastic trends. While it has become increasingly common to introduce stochastic trends into (linear) empirical models of monetary policy by treating the ‘natural rate of interest’ or ‘trend inflation’ as latent random walks (Laubach and Williams, 2003; Cogley and Sbordone, 2008; Del Negro, Giannone, Giannoni, and Tambaletti, 2017; Andrade, Gali, Bihan, and Matheron, 2019; Bauer and Rudebusch, 2020; Schmitt-Grohé and Uribe, 2022), little is understood about what would happen to the implied time series properties of the observable series, such as the actual inflation rate and the nominal rate of interest, if nonlinearities were introduced into such models via the ZLB constraint. There is thus a pressing need to extend our understanding of how stochastic trends may be modelled beyond the linear VAR framework, to more general settings capable of accommodating such nonlinearities.

This paper addresses this problem in the setting of the censored and kinked structural VAR (CKSVAR) model (Mavroeidis, 2021; Aruoba, Mlikota, Schorfheide, and Villalvazo, 2022), which provides a flexible yet tractable framework within which to model time series that are subject to occasionally binding constraints, and more generally to threshold-type nonlinearities. In this model, which is otherwise like a linear structural VAR, one series is allowed to enter differently according to whether it lies above or below a threshold; e.g. in applications to monetary policy, this series may be taken to represent the stance of monetary policy in each period, which above zero coincides with the short-term policy rate, and below zero corresponds to the (unobserved) ‘shadow rate’. We provide a complete characterisation of how common linear and nonlinear stochastic trends may be generated in this model, via unit roots and appropriate generalisations of the usual rank conditions, providing the first extension to date of the Granger–Johansen representation theorem to a nonlinearly cointegrated setting. Our results, which describe the behaviour of both the short- and long-run components of the CKSVAR, are foundational for frequentist inference in this setting, in the presence of unit roots.

As in a linear VAR, unit roots are unavoidable if one wishes to apply the CKSVAR to series with stochastic trends. The usual criticism of simply estimating a model in differences – that this obliterates the identifying long-run information carried by the cointegrating relations – is here magnified by the threshold nonlinearity in the model, which dictates whether the affected variable (e.g. interest rates) should enter in levels or differences, and in turn prescribes appropriate
ate forms for the other variables. To put it another way, that nonlinearity prevents series from being simply ‘differenced to stationarity’. Moreover, we show that in our setting, unit roots are not a mere technical nuisance: rather, their presence may impart significant identifying power to the low frequency behaviour of the series. For instance, the possibility that cointegrating relations between series may change as one series crosses a threshold, something accommodated by the CKSVAR, may be utilised to test hypotheses about the relative effectiveness unconventional monetary policy (see Examples 2.1a and 2.1b below; this is a problem that has been studied econometrically by e.g. Gambacorta, Hofmann, and Peersman, 2014; Wu and Xia, 2016; Debortoli, Gali, and Gambetti, 2020; and Ikeda, Li, Mavroeidis, and Zanetti, 2022).

In analysing the CKSVAR with unit roots, we make a major contribution to the literature on nonlinear cointegration. Here a long-standing open problem relates to how a (nonlinear) vector autoregression, which provides a unified description of the short- and long-run dynamics of a collection of time series, can generate nonlinear cointegration between those series – where ‘nonlinear cointegration’ is understood in the profound sense of those series having common nonlinear stochastic trends with possibly nonlinear cointegrating relations between those trends. As discussed in the recent review by Tjøstheim (2020), despite the voluminous literature on the subject of ‘nonlinear cointegration’, this problem has yet to be addressed at any reasonable level of generality.\(^1\) Within the framework of the CKSVAR, we provide a resolution of this problem, showing that the model naturally gives rise to nonlinear cointegration, generating nonlinear common trend processes – censored, regulated, and kinked Brownian motions (see Definition 3.3 below) – that have not previously appeared in multivariate settings.

To clarify how our work relates to the existing literature on ‘nonlinear cointegration’, and to explain why we have been able to make progress in an area that has previously seemed intractable, we briefly recall the two main strands of that literature. One strand (Tjøstheim, 2020, p. 657) starts from the vector error correction model (VECM) representation of a cointegrated VAR, and introduces nonlinearity into the error correction mechanism; a prototypical model is

\[
\Delta z_t = g(\beta^T z_{t-1}) + \sum_{i=1}^{k-1} \Gamma_i \Delta z_{t-i} + u_t, \tag{1.1}
\]

in which the usually linear loadings \(\alpha[\beta^T z_{t-1}] = \alpha \xi_{t-1}\) on the equilibrium errors are replaced by a general nonlinear function. In the original ‘threshold cointegration’ conception of this model, due to Balke and Fomby (1997), \(g\) is piecewise linear, i.e. \(g(\xi_t) = \sum_{i=1}^{m} \alpha^{(i)} 1\{\xi_t \in \Xi_i\} \xi_t\), where each of the \(\alpha^{(i)}\)'s correspond to different ‘regimes’, and \(\{\Xi_i\}_{i=1}^{m}\) partitions the domain of \(\xi_t\); the values of \(\{\Gamma_i\}\) may also be dependent on (a nominated lag of) \(\xi_t\), or some other stationary component (such as lags of \(\Delta z_t\)). (For regime-switching versions, including of the smoothed variety, see e.g. Hansen and Seo, 2002; Saikkonen, 2005, 2008; and Seo, 2011; for versions in which \(g\) is allowed to be a more general nonlinear function, but the \(\{\Gamma_i\}\) matrices are fixed, see Escribano and Mira, 2002; and Kristensen and Rahbek, 2010, 2013). Notably, the nonlinearity in such models is wholly confined to the short-run dynamics: as in a linearly cointegrated VAR,

\(^1\)While Cai, Gao, and Tjøstheim (2017), make an important effort in this direction, their results are limited to a first-order bivariate VAR with two regimes, in which one of those regimes is delimited by a compact set, and so makes a negligible contribution to the long-run behaviour of the series generated by the model. Their results are thus markedly different from those obtained below.
there remains a globally defined cointegrating space spanned by the columns of $\beta$ (i.e. which is common to all ‘regimes’), and the limiting common trends remain a (vector) Brownian motion.

The other strand (Tjøstheim, 2020, pp. 658–666) takes as its starting point the triangular representation of a linearly cointegrated system, and introduces nonlinearity directly into the common trends, by specifying

$$y_t = f(x_t) + \varepsilon_{yt}, \quad x_t = x_{t-1} + \varepsilon_{xt}. \quad (1.2)$$

Here $f(x_t)$ replaces what would ordinarily be a linear function, with the consequence that the weak limit of $Y_n(\lambda) := n^{-1/2} y_{[n\lambda]}$ will now be a nonlinear transformation of the limiting Brownian motions associated with $X_n(\lambda) := n^{-1/2} x_{[n\lambda]}$. The errors $\varepsilon_t = (\varepsilon_{yt}, \varepsilon_{xt})$ may be weakly dependent and cross-correlated, permitting $\{x_t\}$ to be endogenous. The function $f$ is typically estimated via some sort of regression, either parametrically (Park and Phillips, 1999, 2001; Chan and Wang, 2015; Li, Tjøstheim, and Gao, 2016) or nonparametrically (Karlsen, Myklebust, and Tjøstheim, 2007; Wang and Phillips, 2009, 2016; Duffy, 2017; Duffy and Kasparis, 2021); there is also literature on specification testing in this setting (e.g. Wang and Phillips, 2012; Dong, Gao, Tjøstheim, and Yin, 2017; Wang, Wu, and Zhu, 2018; Berenguer-Rico and Nielsen, 2020). Notable variants have used $f$ to model transitions between regimes with distinct linear cointegrating relations (Saikkonen and Choi, 2004; Gonzalo and Pitarakis, 2006), or allowed it to take the ‘functional coefficient’ form $\beta(w_t)x_t$ (Cai, Li, and Park, 2009; Xiao, 2009).

In developing a VAR model that exhibits both nonlinearity in its short-run dynamics, as in (1.1), and in the implied (long-run) common trends, as in (1.2), this paper is the first to bridge the remarkably wide gulf that has existed between these two strands of the literature. The CKSVAR turns out to provide just enough flexibility to accommodate meaningful departures from linear cointegration, while retaining tractability. We show that depending on the rank conditions imposed on (submatrices of) the autoregressive polynomial evaluated at unity, the CKSVAR is capable of generating three distinct kinds of nonlinear cointegration, which we term: (i) regulated cointegration; (ii) kinked cointegration; and (iii) linear cointegration in a nonlinear VECM.

At a technical level, our contribution consists of identifying the alternative configurations of the model that give rise to cases (i)–(iii), which are essentially exhaustive of the possibilities here, and deriving analogues of the Granger–Johansen representation theorem in these three cases.\footnote{Our analysis is exhaustive with respect to the possibilities for generating series that are integrated of order one (in a suitably extended sense of the term; see Definition 3.1) within the CKSVAR; higher orders of integration are not considered here.} In analysing case (i), our work relates to that of Cavaliere (2005), Liu, Ling, and Shao (2011), Gao, Tjøstheim, and Yin (2013), and most closely to Bykhovskaya and Duffy (2022), all of whom obtain convergence to regulated Brownian motions in univariate models. Case (ii), while it specialises readily to the univariate case, does not appear to have been considered by any previous literature. Case (iii) holds for a configuration of the model that falls within the very broad class of nonlinear VECMs considered by Saikkonen (2008); because of its lesser novelty, a detailed development is deferred to Duffy, Mavroeidis, and Wycherley (2023, Sec. 5).

The remainder of the paper is organised as follows. Section 2 introduces the CKSVAR model
and the stylised structural macroeconomic models that we use as running examples. Section 3 develops the heuristics of the model with unit roots, outlining the tripartite classification noted above. Extensions of the Granger–Johansen representation theorem to cover cases (i) and (ii) are given in Section 4. Section 5 concludes. All proofs appear in the appendices.

Notation. $e_{m,i}$ denotes the $i$th column of an $m \times m$ identity matrix; when $m$ is clear from the context, we write this simply as $e_i$. In a statement such as $f(a^+, b^+) = 0$, the notation ‘$a^+$’ signifies that both $f(a^+, b^+) = 0$ and $f(a^-, b^-) = 0$ hold; similarly, ‘$a^\pm$’ denotes that both $a^+$ and $a^-$ are elements of $A$. All limits are taken as $n \to \infty$ unless otherwise stated. $\overset{p}{\to}$ and $\rightsquigarrow$ respectively denote convergence in probability and in distribution (weak convergence). We write ‘$X_n(\lambda) \rightsquigarrow X(\lambda)$ on $D_{\mathbb{R}^m}[0, 1]$’ to denote that $\{X_n\}$ converges weakly to $X$, where these are considered as random elements of $D_{\mathbb{R}^m}[0, 1]$, the space of cadlag functions $[0, 1] \to \mathbb{R}^m$, equipped with the uniform topology; we denote this as $D[0, 1]$ whenever the value of $m$ is clear from the context. $\|\|$ denotes the Euclidean norm on $\mathbb{R}^m$, and the matrix norm that it induces. For $X$ a random variable and $p \geq 1$, $\|X\|_p := (\mathbb{E}|X|^p)^{1/p}$.

2 Model: the censored and kinked SVAR

We consider a VAR($k$) model in which one series, $y_t$, enters differently according to whether it is above or below a time-invariant threshold $b$, with the other $p-1$ series, collected in $x_t$, enter linearly. Defining

$$y_t^+ := \max\{y_t, b\} \quad y_t^- := \min\{y_t, b\},$$

we specify that $(y_t, x_t)$ follow

$$\phi_0^+ y_t^+ + \phi_0^- y_t^- + \Phi_0^x x_t = c + \sum_{i=1}^k [\phi_i^+ y_{t-i}^+ + \phi_i^- y_{t-i}^- + \Phi_i^x x_{t-i}] + u_t$$

or, more compactly,

$$\phi^+(L)y_t^+ + \phi^-(L)y_t^- + \Phi^z(L)x_t = c + u_t, \quad (2.3)$$

where $\phi^+(\lambda) := \phi_0^+ - \sum_{i=1}^k \phi_i^+ \lambda^i$ and $\Phi^x(\lambda) := \Phi_0^x - \sum_{i=1}^k \Phi_i^x \lambda^i$, for $\phi_i \in \mathbb{R}^{p \times 1}$ and $\Phi_i \in \mathbb{R}^{p \times (p-1)}$, and $L$ denotes the lag operator. As in a linear SVAR, $\{u_t\}$ may be an i.i.d. sequence of mutually orthogonal structural shocks, but our results below permit them to be cross-correlated or weakly dependent. Through an appropriate redefinition of $y_t$ and $c$, we may take $b = 0$ without loss of generality, and will do so throughout what follows.\(^3\) In this case, $y_t^+$ and $y_t^-$ respectively equal the positive and negative parts of $y_t$, and $y_t = y_t^+ + y_t^-$. (Throughout the following, the notation ‘$a^\pm$’ denotes $a^+$ and $a^-$ as objects associated respectively with $y_t^+$ and $y_t^-$, or their lags. If we want to instead denote the positive and negative parts of some $a \in \mathbb{R}$, we shall do so by writing $\{a\}_+ := \max\{a, 0\}$ or $\{a\}_- := \min\{a, 0\}$.)

\(^3\)Defining $y_{b,t} := y_t - b$, $y_{b,t}^+ := \max\{y_{b,t}, 0\}$, $y_{b,t}^- := \min\{y_{b,t}, 0\}$ and $c_b := c - |\phi^+(1) + \phi^-(1)|b$, we can rewrite (2.3) as

$$\phi^+(L)y_{b,t}^+ + \phi^-(L)y_{b,t}^- + \Phi^x(L)x_t = c_b + u_t.$$
Models of the form of (2.3) have previously been employed in the literature to account for the dynamic effects of censoring, occasionally binding constraints, or endogenous regime switching. Mavroeidis (2021) proposed exactly this model, which he termed the *censored and kinked structural VAR* (CKSVAR) model, to describe the operation of monetary policy during periods when a zero lower bound may bind on the policy rate: in our notation, $y_t$ corresponds to his ‘shadow rate’, expressing the central bank’s desired policy stance, and $y_t^+$ to the actual policy rate. Aruoba et al. (2022) considered a model in which one variable is subject to an occasionally binding constraint, which although in its initial formulation is somewhat more general, reduces to an instance of the CKSVAR once the conditions necessary for the model to have a unique solution (for all values of $u_t$) have been imposed (see their Proposition 1(i)). This version of their model – i.e. that in which the ‘private sector regression functions’ are piecewise linear and continuous – is thus accommodated by (2.3).\footnote{See also Aruoba, Cuba-Borda, Higa-Flores, Schorfheide, and Villalvazo, 2021, for a DSGE model with an occasionally binding constraint, in which agents’ decision rules are approximated by functions with these properties.}

**Example 2.1.** Consider the following stylised structural model of monetary policy in the presence of a zero lower bound (ZLB) consisting of a composite IS and Phillips curve (PC) equation

\[
\pi_t - \pi_t = \chi(\pi_{t-1} - \pi_{t-1}) + \theta[i_t^+ + \mu i_t^- - (r_t^* + \pi_t)] + \varepsilon_t \tag{2.4}
\]

and a policy reaction function (Taylor rule)

\[
i_t = (r_t^* + \pi_t) + \gamma(\pi_t - \pi_t) \tag{2.5}
\]

where $r_t^*$ denotes the (real) natural rate of interest, $\pi_t$ the central bank’s inflation target, $\pi_t$ inflation, and $\varepsilon_t$ a mean zero, i.i.d. innovation. $i_t$ measures the stance of monetary policy; thus $i_t^+ := [i_t]$ gives the actual policy rate (constrained to be non-negative), and $i_t^- := [i_t]$ the desired stance of policy when the ZLB binds, to be effected via some form of ‘unconventional’ monetary policy, such as long-term asset purchases. We maintain that $\gamma > 0$, $\theta < 0$, and $\chi \in [0, 1)$. The parameter $\mu \in [0, 1]$ reflects the relative efficacy of unconventional policy, with $\mu = 1$ if this is as effective as conventional policy. When $\chi = 0$, the preceding corresponds to a simplified version of the model of Ikeda et al. (2022); a model with $\chi > 0$ emerges by augmenting their Phillips curve with a measure of backward-looking agents.

To ‘close’ the model, we consider two alternative specifications for the underlying processes followed by $\{r_t^*\}$ and $\{\pi_t\}$. In the first of these (henceforth, **Example 2.1a**) the inflation target is assumed to be constant and is normalised to zero (i.e. $\pi_t = \pi = 0$), while the natural real rate of interest follows an AR(1) process,

\[
r_t^* = \psi r_{t-1}^* + \eta_t \tag{2.6}
\]

where $\psi \in (-1, 1)$, and $\eta_t$ is an i.i.d. mean zero innovation, possibly correlated with $\varepsilon_t$. (When $\psi = 1$, $\{r_t^*\}$ is integrated, as in the model of Laubach and Williams, 2003.) By substituting
(2.5) into (2.4) and (2.6), we render the system as a CKSVAR for \((i_t, \pi_t)\) as

\[
\begin{bmatrix}
1 & 1 & -\gamma \\
0 & \theta(1-\mu) & 1-\theta\gamma
\end{bmatrix}
\begin{bmatrix}
i_t^+ \\
i_t^-
\end{bmatrix} =
\begin{bmatrix}
\psi & \psi & -\psi\gamma \\
0 & 0 & \chi
\end{bmatrix}
\begin{bmatrix}
i_{t-1}^+ \\
i_{t-1}^-
\end{bmatrix} +
\begin{bmatrix}
\eta_t \\
\varepsilon_t
\end{bmatrix}.
\]

This model (without the simplifying assumptions that \(\pi = 0\) and \(E_r^* = 0\)) is used in Duffy et al. (2023) to illustrate criteria for stationarity and ergodicity; here it will provide an example of our second kind of nonlinear cointegration.

In the second variant of the model (henceforth, Example 2.1b) the natural rate is assumed to be constant and, for simplicity of exposition, normalised to zero (i.e. \(r_t^* = r^* = 0\)), while the inflation target is allowed to be time-varying, according to

\[
\pi_t = \pi_{t-1} + \delta(\pi_{t-1} - \pi_{t-1}) + \eta_t \tag{2.8}
\]

where \(\delta \in (-1,0]\), and \(\eta_t\) is an i.i.d. innovation as above. When \(\delta = 0\), this corresponds to a model in which the inflation target follows a pure random walk, possibly reflecting the time-varying preferences of the central bank (cf. Cogley and Sbordone, 2008); when \(\delta < 0\), the model allows past deviations of inflation from target to feed back into the target, such that e.g. below-target inflation induces an upward revision of the inflation target. Motivation for this aspect of the model comes from the manner in which the ZLB may constrain policy to be excessively deflationary for sustained periods, something that has prompted the literature to consider the costs and benefits of adopting a higher inflation target (e.g. Blanchard, Dell’Ariccia, and Mauro, 2010, pp. 207f.; Coibion, Gorodnichenko, and Wieland, 2012). Supposing additionally that \(\gamma > 1\), we may put (2.4), (2.5) and (2.7) in the form of as CKSVAR as

\[
\begin{bmatrix}
-1 & -1 & \gamma \\
\varphi_1 & \varphi\mu & -\varphi_1
\end{bmatrix}
\begin{bmatrix}
i_t^+ \\
i_t^-
\end{bmatrix} =
\begin{bmatrix}
\delta - 1 & \delta - 1 & \gamma - \delta \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
i_{t-1}^+ \\
i_{t-1}^-
\end{bmatrix} +
\begin{bmatrix}
(\gamma - 1) \\
0
\end{bmatrix}
\begin{bmatrix}
\eta_t \\
\varepsilon_t
\end{bmatrix} \tag{2.9}
\]

where \(\varphi\mu := (1-\mu\theta\gamma) - \theta(1-\mu)\). Depending on the assumptions made on the model parameters (in particular \(\delta\)), this model is capable of generating either of the first two types of nonlinear cointegration developed in Section 3.

While both Mavroeidis (2021) and Aruoba et al. (2022) motivate and interpret (2.3) as a structural model, empirically motivated reduced-form models of this kind have also appeared in the literature, particularly in the univariate \((p = 1)\) case of (2.3), which encompasses both varieties of the dynamic Tobit model (Maddala 1983, p. 186; for applications, see e.g. Demiralp and Jordà, 2002; De Jong and Herrera, 2011; and Bykhovskaya, 2023).

Example 2.2 (univariate). Consider (2.3) with \(p = 1\) and \(\phi_0^+ = \phi_0^- = 1\), so that

\[
y_t = c + \sum_{i=1}^{k} (\phi_i^+ y_{t-i}^+ + \phi_i^- y_{t-i}^-) + u_t. \tag{2.10}
\]

Now suppose that only \(y_t^+\) is observed. In the nomenclature of Bykhovskaya and Duffy (2022,
Sec. 1), if \( \phi_i^- = 0 \) for all \( i \in \{1, \ldots, k\} \), so that only the positive part of \( y_{t-i} \) enters the r.h.s., then

\[
y_t^+ = \left[ c + \sum_{i=1}^{k} \phi_i^+ y_{t-i} + u_t \right]_+
\]

(2.11)

follows a censored dynamic Tobit; whereas if \( \phi_i^+ = \phi_i^- = \phi_i \) for all \( i \in \{1, \ldots, k\} \), then \( y_t^+ \) follows a latent dynamic Tobit, being simply the positive part of the linear autoregression

\[
y_t = c + \sum_{i=1}^{k} \phi_i y_{t-i} + u_t.
\]

We follow Mavroeidis (2021) and Aruoba et al. (2022) in maintaining the following, which are necessary and sufficient to ensure that (2.3) has a unique solution for \((y_t, x_t)\), for all possible values of \( u_t \). Define

\[
\Phi_0 := \begin{bmatrix}
\phi_0^+ & \phi_0^- & \Phi_0^x
\end{bmatrix}
= \begin{bmatrix}
\phi_{0,yy} & \phi_{0,yx} & \phi_{0,xx}^T
\phi_{0,xy} & \phi_{0,yy} & \Phi_{0,xx}
\end{bmatrix},
\]

\[
\Phi_0^+ := [\phi_0^+, \Phi_0^x] \quad \text{and} \quad \Phi_0^- := [\phi_0^-, \Phi_0^x].
\]

**Assumption DGP.**

1. \( \{(y_t, x_t)\} \) are generated according to (2.1)–(2.3) with \( b = 0 \), with possibly random initial values \((y_i, x_i)\), for \( i \in \{-k+1, \ldots, 0\} \);

2. \( \text{sgn}(\det \Phi_0^+) = \text{sgn}(\det \Phi_0^-) \neq 0 \).

3. \( \Phi_{0,xx} \) is invertible, and

\[
\text{sgn}\{\phi_{0,yy} - \phi_{0,yx}^T \Phi_{0,xx}^{-1} \phi_{0,xy}\} = \text{sgn}\{\phi_{0,yy} - \phi_{0,yx} \Phi_{0,xx}^{-1} \phi_{0,xy}\}> 0.
\]

For a further discussion of these conditions, including why DGP.3 may be maintained without loss of generality when DGP.2 holds, see Duffy et al. (2023, Sec. 2). As in that paper, we shall designate a CKSVAR as canonical if

\[
\Phi_0 = \begin{bmatrix}
1 & 1 & 0 \\
0 & 0 & I_{p-1}
\end{bmatrix} =: I_p^*.
\]

(2.12)

While it is not always the case that the reduced form of (2.3) corresponds directly to a canonical CKSVAR, by defining the canonical variables

\[
\begin{bmatrix}
\tilde{y}_t^+ \\
\tilde{y}_t^- \\
\tilde{x}_t
\end{bmatrix}
= \begin{bmatrix}
\phi_{0,yy} & 0 & 0 \\
0 & \phi_{0,yy} & 0 \\
\phi_{0,xy} & \phi_{0,xy} & \Phi_{0,xx}
\end{bmatrix}
\begin{bmatrix}
\tilde{y}_t^+ \\
\tilde{y}_t^- \\
\tilde{x}_t
\end{bmatrix}
= \begin{bmatrix}
y_t^+ \\
y_t^- \\
x_t
\end{bmatrix} =: P^{-1}
\begin{bmatrix}
y_t^+ \\
y_t^- \\
x_t
\end{bmatrix},
\]

(2.13)

where \( \phi_{0,yy}^+ := \phi_{0,yy} - \phi_{0,yx}^T \Phi_{0,xx}^{-1} \phi_{0,xy}^+ > 0 \) and \( P^{-1} \) is invertible under DGP; and setting

\[
\begin{bmatrix}
\phi^+(\lambda) & \phi^-(\lambda) & \Phi^x(\lambda)
\end{bmatrix}
= Q
\begin{bmatrix}
\phi^+(\lambda) & \phi^-(\lambda) & \Phi^x(\lambda)
\end{bmatrix} P,
\]

(2.14)
where
\[ Q := \begin{bmatrix} 1 & -\phi_{0,yy}^{-1} \\ 0 & I_{p-1} \end{bmatrix}. \tag{2.15} \]
we obtain a canonical CKSVAR for \((\tilde{y}_t, \tilde{x}_t)\). This is formalised by the following, which reproduces the first part of Proposition 2.1 in Duffy et al. (2023).

**Proposition 2.1.** Suppose DGP holds. Then there exist \((\tilde{y}_t, \tilde{x}_t)\) such that (2.13)-(2.14) hold,
\[
\tilde{y}_t^+ = \max\{\tilde{y}_t, 0\}, \quad \tilde{y}_t^- = \min\{\tilde{y}_t, 0\}
\]
and
\[
\tilde{\phi}^+(L)\tilde{y}_t^+ + \tilde{\phi}^-(L)\tilde{y}_t^- + \tilde{\Phi}(L)\tilde{x}_t = \tilde{c} + \tilde{u}_t, \tag{2.16}
\]
is a canonical CKSVAR, where \(\tilde{c} = Qc\) and \(\tilde{u}_t = Qu_t\).

Since the time series properties of the CKSVAR are inherited from its derived canonical form, we shall often work with this more convenient representation of the system, and indicate this as follows.

**Assumption DGP*.** \(\{(y_t, x_t)\}\) are generated by a canonical CKSVAR, i.e. DGP holds with \(\Phi_0 = [\phi_0^+, \phi_0^-, \Phi] = I_p\), so that (2.2) may be equivalently written as
\[
\begin{bmatrix} y_t \\ x_t \end{bmatrix} = c + \sum_{i=1}^{k} \begin{bmatrix} \phi_i^+ & \phi_i^- \\ \Phi_i \end{bmatrix} \begin{bmatrix} y_{t-i}^+ \\ y_{t-i}^- \\ x_{t-i} \end{bmatrix} + u_t. \tag{2.17}
\]

**Example 2.1a** (trending natural rate; ctd). Using (2.7) above, the canonical form of the model is derived by taking
\[
\begin{bmatrix} \tilde{y}_t \\ \tilde{x}_t \end{bmatrix} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 + \tau \mu & 0 \\ 0 & \theta(1 - \mu) & 1 - \theta \gamma \end{bmatrix} \begin{bmatrix} \tilde{y}_t^+ \\ \tilde{y}_t^- \\ \tilde{x}_t \end{bmatrix} + \begin{bmatrix} \tilde{\eta}_t \\ \tilde{\varepsilon}_t \end{bmatrix} \tag{2.18}
\]
where \(\tau \mu := \gamma \theta(1 - \mu)(1 - \theta \gamma)^{-1}\) is such that \(1 + \tau \mu > 0\), for which it holds that
\[
\begin{bmatrix} \tilde{y}_t \\ \tilde{x}_t \end{bmatrix} = \begin{bmatrix} \psi & \psi - \chi \tau \mu \kappa \mu & \gamma(\chi \kappa_4 - \psi) \kappa_1 \\ 0 & -\chi \theta(1 - \mu) \kappa \mu & \chi \kappa_1 \end{bmatrix} \begin{bmatrix} \tilde{y}_{t-1}^+ \\ \tilde{y}_{t-1}^- \\ \tilde{x}_{t-1} \end{bmatrix} + \begin{bmatrix} \tilde{\eta}_t \\ \tilde{\varepsilon}_t \end{bmatrix} \tag{2.19}
\]
where \(\kappa \mu := (1 - \mu \theta \gamma)^{-1}\). (In the special case where \(\chi = 0\), the canonical form is a linear system, since then both \(\tilde{y}_{t-1}^+ \) and \(\tilde{x}_{t-1}\) enter the r.h.s. with the same coefficients.)

3 Unit roots and nonlinear cointegration: heuristics

3.1 Nonlinearity and cointegration

It is well known that a linear VAR can faithfully replicate the high persistence, random wandering and long-run co-movement that is characteristic of a great many macroeconomic time series, via the imposition of unit autoregressive roots and the familiar rank conditions (Johansen, 1995).
The question thus arises as to whether, and how, such behaviour may also be generated within a CKSVAR, so that it might still be applied to series for which a stationary CKSVAR would be inappropriate. As we shall see, it is possible not only to accommodate linear cointegration within the CKSVAR, but also to generate a variety of nonlinear forms of cointegration, owing to the richer class of common trend processes that the model supports. Moreover, as our examples below illustrate, such departures from linear cointegration may also aid in the identification of structural parameters.

In developing the CKSVAR with unit roots, we shall find it necessary to depart from the familiar classification of processes according to their orders of integration, since the nonlinearity in the model generally prevents it from generating series that are difference stationary. This is an issue commonly encountered in regime-switching cointegration models: see e.g. the discussion in Gonzalo and Pitarakis (2006, pp. 816f.), where this motivates the definition of the ‘order of summability’ of a time series, and the allied notion of ‘co-summability’ as a generalisation of familiar classification of processes according to their orders of integration, since the nonlinearity in the model generally prevents it from generating series that are difference stationary. This is an issue commonly encountered in regime-switching cointegration models; see e.g. the discussion in Gonzalo and Pitarakis (2006, pp. 816f.), where this motivates the definition of the ‘order of summability’ of a time series, and the allied notion of ‘co-summability’ as a generalisation of linear cointegration (Berenguer-Rico and Gonzalo, 2014, pp. 335f.). While those concepts could well be applied to the CKSVAR, the following properties, which may be more easily verified, will suffice for our purposes.

**Definition 3.1.** Let \( \{w_t\}_{t \in \mathbb{N}} \) be a random sequence taking values in \( \mathbb{R}^p \). We say that \( \{w_t\} \) is:

(i) \(*\)-stationary, denoted \( w_t \sim I^*(0) \), if \( \sup_{1 \leq t \leq n} \|w_t\| = o_p(n^{1/2}) \); or

(ii) \(*\)-integrated (of order one), denoted \( w_t \sim I^*(1) \), if \( n^{-1/2}w_{[n\lambda]} \sim \ell(\lambda) \) on \( D_{\mathbb{R}^p}[0, 1] \), where \( \ell \) is a non-degenerate process;

and analogously for subvectors (and individual elements) of \( \{w_t\} \).

**Example 2.2** (univariate; ctd). A simple, non-trivial example of series that are \( I^*(d) \) but not \( I(d) \) is provided by the censored dynamic Tobit. When \( \sum_{i=1}^{k} \phi_i^+ = 1 \), this model has a unit root, and Bykhovskaya and Duffy (2022) show that \( n^{-1/2}y_{[n\lambda]}^+ \sim Y^+(\lambda) \) on \( D[0, 1] \), where \( Y^+ \) is a Brownian motion regulated (at zero) from below (their Theorem 3.2; see Definition 3.3 below), and \( \|\Delta y_t^+\|_{2+\delta_u} \) is uniformly bounded (their Lemma B.2). Thus \( \Delta y_t^+ \sim I^*(0) \) and \( y_t^+ \sim I^*(1) \), even though, due to the nonlinearity in the model, neither series are \( I(d) \) in the usual sense. □

We also need an enlarged notion of ‘cointegration’ that is sufficiently general to encompass the possibilities of the CKSVAR model, such as is provided by the following (cf. Gonzalo and Pitarakis, 2006, p. 817).

**Definition 3.2.** Let \( \mathcal{Z}^+ := \{(y, x) \in \mathbb{R}^p \mid y \geq 0\} \) and \( \mathcal{Z}^- := \{(y, x) \in \mathbb{R}^p \mid y \leq 0\} \), \( r^\pm \in \{0, \ldots, p\} \), and \( \beta^\pm \in \mathbb{R}^{p+r^\pm} \) have full column rank. Suppose \( z_t = (y_t, x_t^T)^T \sim I^*(1) \), but

\[
1\{z_t \in \mathcal{Z}^{(i)}\} \theta^T z_t \sim I^*(0) \iff \theta \in \text{sp} \beta^{(i)}
\]

for \( (i) \in \{+, -\} \). Then \( z_t \) is said to be cointegrated on \( \mathcal{Z}^{(i)} \), with \( r^{(i)} \) the cointegrating rank on \( \mathcal{Z}^{(i)} \), \( \text{sp} \beta^{(i)} \) the cointegrating space on \( \mathcal{Z}^{(i)} \), and any (nonzero) element of \( \text{sp} \beta^{(i)} \) a cointegrating vector on \( \mathcal{Z}^{(i)} \), for \( (i) \in \{+, -\} \). If \( \beta^{(i)} \) does not depend on \( (i) \), we drop the ‘on \( \mathcal{Z}^{(i)} \)’ qualifiers.
3.2 The CKSVAR with unit roots

Our next step is to rewrite the CKSVAR, as in (2.2) or (2.3) above, in the form of a vector error-correction model (VECM). Define the autoregressive polynomials

\[ \Phi^\pm(\lambda) := \begin{bmatrix} \phi^\pm(\lambda) & \Phi^x(\lambda) \end{bmatrix}, \]

and let \( \Gamma_i^\pm := -\sum_{j=i+1}^k \Phi_j^\pm =: [\gamma_i^\pm, \Gamma^x] \) for \( i \in \{1, \ldots, k - 1\} \), so that \( \Gamma^\pm(\lambda) := \Phi_0^\pm - \sum_{i=1}^{k-1} \Gamma_i^\pm \lambda^i \) is such that

\[ \Phi^\pm(\lambda) = \Phi^\pm(1)\lambda + \Gamma^\pm(\lambda)(1 - \lambda). \]

Set \( \pi^\pm := -\phi^\pm(1) \) and \( \Pi^x := -\Phi^x(1) \). Then

\[ \Phi_0 \begin{bmatrix} \Delta y_t^+ \\ \Delta y_t^- \\ \Delta x_t \end{bmatrix} = c + \begin{bmatrix} \pi^+ & \pi^- & \Pi^x \end{bmatrix} \begin{bmatrix} y_{t-1}^+ \\ y_{t-1}^- \\ x_{t-1} \end{bmatrix} + \sum_{i=1}^{k-1} \begin{bmatrix} \gamma_i^+ & \gamma_i^- & \Gamma_i^x \end{bmatrix} \begin{bmatrix} \Delta y_{t-i}^+ \\ \Delta y_{t-i}^- \\ \Delta x_{t-i} \end{bmatrix} + u_t, \quad (3.1) \]

where \( \Delta := 1 - L \) denotes the difference operator, and for clarity we note that \( \Delta y_t^+ = y_t^+ - y_{t-1}^+ \) (rather than being the positive part of \( \Delta y_t \)). In the case of a canonical CKSVAR, (3.1) helpfully reduces to

\[ \begin{bmatrix} \Delta y_t \\ \Delta x_t \end{bmatrix} = c + \begin{bmatrix} \pi^+ & \pi^- & \Pi^x \end{bmatrix} \begin{bmatrix} y_{t-1}^+ \\ y_{t-1}^- \\ x_{t-1} \end{bmatrix} + \sum_{i=1}^{k-1} \begin{bmatrix} \gamma_i^+ & \gamma_i^- & \Gamma_i^x \end{bmatrix} \begin{bmatrix} \Delta y_{t-i}^+ \\ \Delta y_{t-i}^- \\ \Delta x_{t-i} \end{bmatrix} + u_t. \quad (3.2) \]

While our main results apply to the general CKSVAR, the reader may find it helpful to work through the remainder of this section under the supposition that \( (y_t, x_t) \) are generated by a canonical CKSVAR.

Just as in a linear (cointegrated) VAR, which corresponds to the special case of (3.2) in which \( \pi^+ = \pi^- \) and \( \gamma_i^+ = \gamma_i^- \) for all \( i \in \{1, \ldots, k - 1\} \), the long-run dynamics will be governed by the matrix of coefficients on the lagged levels. More precisely, there are two such matrices, \( \Pi^+ \) and \( \Pi^- \), defined by

\[ \Pi^\pm := [\pi^\pm, \Pi^x] = -\Phi^\pm(1). \quad (3.3) \]

Although the canonical CKSVAR technically has \( 2^k \) distinct autoregressive ‘regimes’ (corresponding to the possible sign patterns of \( y_{t-1} := (y_{t-1}, \ldots, y_{t-k})^T \); see also Duffy et al., 2023, Sec. 4.2), the behaviour of the CKSVAR with unit roots depends largely on the two regimes in which the elements of \( y_{t-1} \) are all either positive or negative, which we shall loosely refer to as the ‘positive’ and ‘negative’ regimes, to which \( \Pi^+ \) and \( \Pi^- \) correspond. This simplification occurs because whenever \( y_t \sim I^*(1) \), it spends most of its time away from the origin, so that all elements of \( y_{t-1} \) will have the same sign almost all of the time.

Our baseline assumptions on the CKSVAR with unit roots may now be stated.

**Assumption CVAR.**

1. \( \det \Phi^\pm(\lambda) \) has \( q^\pm \) roots at unity, and all others outside the unit circle;
2. \( \text{rk} \Pi^\pm = r^\pm = p - q^\pm \); and
ing on the ranks of $\Pi^\pm$ the set of non-stochastic (i.e. $\Pi^\pm$ is defined such that it is possible to regard the system as having a cointegrating rank of at least $r$, and that there exists a full column rank matrix $\beta \in \mathbb{R}^{p \times r}$ such that $\beta^T z_t \sim I^*(0)$, where $r = r^+ = r^-$. Thus $\{z_t\}$ is cointegrated in the sense of Definition 3.2, with cointegrating space $\text{sp } \beta$.

While linear cointegration may occur in a CKSVAR, other phenomena are possible, depending on the ranks of $\Pi^+$, $\Pi^-$ and $\Pi^x$. Within the framework of $I^*(0)$ and $I^*(1)$ processes, as delimited by cvar, there are three possibilities, each of which generate profoundly different trajectories for $\{y_t\}$. These are characterised in Table 3.1: in each case $r$ (without a superscript) is defined such that it is possible to regard the system as having a cointegrating rank of at least $r$ in each ‘regime’. Since, $\Pi^+$ and $\Pi^-$ differ by only their first column, $r^+$ and $r^-$ may differ by at most one, so that the case where $r^+ \neq r^-$ may be identified with case (i) in the table without loss of generality.

### 3.3 Common trends and long-run trajectories

To develop some intuition for the properties of the model in these three cases, in advance of the representation theory developed in the next section, it is helpful to regard the long-run behaviour of the processes as being characterised by a space of common trends $\mathcal{M}$, defined as the set of non-stochastic (i.e. $u_t = 0$, $\forall t$) steady state solutions to (3.1):

$$\mathcal{M} := \{(y, x) \in \mathbb{R}^p \mid \pi^+[y]_+ + \pi^-[y]_- + \Pi^x x = 0\}$$

$$= \{(y, x) \in \mathcal{Z}^+ \mid \pi^+ y + \Pi^x x = 0\} \cup \{(y, x) \in \mathcal{Z}^- \mid \pi^- y + \Pi^x x = 0\}$$

$$=: \mathcal{M}^+ \cup \mathcal{M}^-.$$
that

\[ Z_n(\lambda) := \begin{bmatrix} Y_n(\lambda) \\ X_n(\lambda) \end{bmatrix} := \frac{1}{n^{1/2}} \begin{bmatrix} y_{[n\lambda]} \\ x_{[n\lambda]} \end{bmatrix} \rightsquigarrow \begin{bmatrix} Y(\lambda) \\ X(\lambda) \end{bmatrix} =: Z(\lambda). \quad (3.5) \]

In a linear VAR, \( M = \ker \Pi \) is a \( q \)-dimensional linear subspace of \( \mathbb{R}^p \), the orthogonal complement of which is the cointegrating space; and \( Z \) is a \( p \)-dimensional Brownian motion with rank \( q \) covariance matrix – i.e. it is a rank \( q \) linear function of \( U \) in (3.4) above – taking values in \( M \). Whereas in the CKSVAR, \( M \) no longer need be a linear subspace, but merely the (connected) union of two ‘half’ subspaces, \( M^+ \) and \( M^- \), with the vectors orthogonal to these spaces corresponding to the cointegrating vectors \( \beta^+ \) and \( \beta^- \) on the half spaces \( \mathcal{Z}^+ \) and \( \mathcal{Z}^- \) respectively (recall Definition 3.2 above). While \( Z \) remains a function of \( U \), that function need not be linear: indeed, in addition to (linear) Brownian motions (BM), any of the following nonlinear processes may also appear among the limiting stochastic trends generated by the CKSVAR.

**Definition 3.3.** Let \( W \) be a linear BM initialised from some \( W(0) \in \mathbb{R}^q \). Suppose \( q = 1 \); the scalar process \( V \) is said to be a

(i) **censored BM (from below)**, if \( V(\lambda) = \max\{W(\lambda), 0\} \)

(ii) **regulated BM (from below)**, if \( V(\lambda) = W(\lambda) + \sup_{\lambda \leq \lambda}[−W(\lambda')]_+ \).

If \( V \) is as in (i) or (ii), then \(-V\) is respectively censored or regulated from above. Suppose now that \( q \geq 1 \), and let \( G : \mathbb{R} \to \mathbb{R}^{p \times q} \) be a map that depends only on the sign of its argument, and is such that: (a) \( h := e_1^T G(1) = \mu e_1^T G(-1) \) for some \( \mu > 0 \); and (b) \( w \mapsto G(h^T w) w \) is continuous. Then the \( p \)-dimensional process \( V \) is said to be a

(iii) **kinked BM**, if \( V(\lambda) = G[h^T W(\lambda)] W(\lambda) = G[V_1(\lambda)] W(\lambda) \).

**Remark 3.2.** Trajectories of these processes (denoted by \( V \)) in the univariate case \( (p = q = 1) \), together with realisation of the standard Brownian motion \( W \) used to construct them, are plotted in Figure 3.1. While both censored and regulated BMs are constrained to be positive, it is evident from panels (b) and (c) that there are important differences between them. For the former, the censoring does not feed back into the underlying dynamics, and \( V \) spends long stretches at zero (while \( W \) is negative); whereas for the latter, the \( \sup_{\lambda \leq \lambda}[−W(\lambda')]_+ \) term continually reflects \( V \) away from zero, so that \( V \) spends relatively little time near zero. The kinked BM in panel (d) is constructed as

\[ V(\lambda) = \sigma_- W(\lambda) 1\{W(\lambda) < 0\} + \sigma_+ W(\lambda) 1\{W(\lambda) \geq 0\} \]

with \( \sigma_- = 1 \) and \( \sigma_+ = 2 \); hence it tracks \( W \) exactly when \( W(\lambda) < 0 \), but doubles the scale of \( W \) when \( W(\lambda) > 0 \). In the multivariate case, kinked BMs are linear combinations of the elements of \( W \), with weights that depend on the sign of \( V_1 \). Thus when plotted individually they appear similarly to panel (d), but if \( G(\pm 1) \) is rank deficient, then there will also be certain linear combinations of the elements of \( V \) that will be zero, with those combinations depending on the sign of \( V_1 \).
Figure 3.1: Linear Brownian motion and derived nonlinear processes

For \( p = 2 \), various possible shapes of \( \mathcal{M} \) are illustrated graphically in Figures 3.2–3.4, along with matching example trajectories for \( (y_t, x_t) \). In all figures, \( (y_t, x_t) \) are generated by a canonical CKSVAR with \( k = 1, c = 0, \ u_t \sim_{\text{i.i.d.}} N[0, I_2] \), and \( y_0 = x_0 = 0 \); the specification of the model is completed by specifying \( \pi^+, \pi^- \) and \( \Pi^x \) in (3.2), the values of which are given in each panel. The main qualitative features of the three cases are as follows.

(i) Regulated cointegration. The distinguishing characteristic of this case is that the common trends are restricted to the region where \( y \geq 0 \), so that \( Y \) will always be a Brownian motion regulated (from below) at zero, even though \( y_t \) itself may take negative values. In the case that \( q = 1 \), as e.g. when \( p = 2 \) and \( r = 1 \) as depicted in Figure 3.2, \( X \) will also be a regulated process, i.e. we have cointegration where both processes share a common regulated stochastic trend. (This may also occur when \( q \geq 2 \), though then \( X \) will also depend on \( q - 1 \) additional linear BMs.) Accordingly, a model configured as in case (i) would be most appropriate when \( y_t \) appears to wander randomly above a threshold, and makes only brief sojourns below that threshold.

Example 2.1b (trending inflation target; ctd). In this first-order model, \( \Pi^\pm = -\Phi^\pm(1) = \Phi_0^\pm - \Phi_0^\pm \), and thus it follows from (2.9) that

\[
\Pi^+ = \begin{bmatrix} \delta & -\delta \\ -\varphi_1 & \varphi_1 \end{bmatrix} \begin{bmatrix} \delta \\ -\varphi_1 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} \quad \Pi^- = \begin{bmatrix} \delta & -\delta \\ -\varphi_1 & \varphi_1 \end{bmatrix}. \tag{3.6}
\]
Figure 3.2: Case (i) configuration of $\mathcal{M}$ and trajectories of $(y_t, x_t)$

$\beta = (1, -1)^T$

$[\pi^+ \quad \pi^- \quad \Pi^*]$

$= \begin{bmatrix}
-0.5 & -0.4 & 0.5 \\
-0.1 & -0.3 & 0.1 \\
\end{bmatrix}$

Figure 3.3: Case (ii) configurations of $\mathcal{M}$ and trajectories of $(y_t, x_t)$

$\beta^+ = (-1, 1)^T$

$\beta^- = (-\frac{1}{2}, 1)^T$
Suppose $\delta < 0$. Then unless $\mu = 1$ (in which case the model is linear), $\varphi_\mu \neq \varphi_1$ and $\Pi^-$ has full rank. It follows that $\mathcal{M}^- = \{0\}$, whereas since $\text{rk} \Pi^+ = 1$, $\mathcal{M}^+$ is the ‘half’ subspace orthogonal to $\beta := (1, -1)^T$, as depicted in Figure 3.2. It follows (via Theorem 4.1 below) that $i_t$ and $\pi_t$ are cointegrated, with cointegrating vector $\beta$, when $i_t$ is positive; but $I^+(0)$ when $i_t$ is negative; their common limiting stochastic trend is a regulated BM.

For the economics underlying this, note that (2.4) and (2.5) (with $\chi = 0$) imply that when the solution to the model has $i_t > 0$, i.e. when the ZLB is not binding, monetary policy is able to fully achieve its objectives, in the sense that inflation is stabilised to within an i.i.d. error of its target, as $\pi_t = \pi_t + (1 - \gamma \theta)^{-1} \varepsilon_t$. As a consequence, (2.8) entails that $\pi_t$ has a stochastic trend, which is inherited by both $\pi_t$ and $i_t^+$ – in the latter case, because the equilibrium rate of interest implied by (2.5) is $r_t^+ + \pi_t = \pi_t$ (recall $r_t^+$ is constant and normalised to zero). Both $i_t^+$ and $\pi_t$ thus put the same loading on the common trend, whence $(1, -1)^T$ is the cointegrating vector when $i_t > 0$. On the other hand, when the solution to the model has $i_t < 0$, i.e. when the ZLB binds, the lesser effectiveness of unconventional monetary policy ($\mu < 1$) entails that policy is too contractionary, and $\pi_t$ begins to drift below $\pi_t$. However, via (2.8) this discrepancy raises the inflation target, and thereby raises the rate of interest required to achieve target inflation. This feedback actually renders $\pi_t$ as $I^+(0)$, and hence also $\pi_t$ and $i_t^-$. In a relatively short time, the solution to the model entails $i_t > 0$ again, and thus the economy tends to spend relatively little time in the vicinity of the ZLB.

It will be observed that the qualitative behaviour described above is wholly contingent on $\mu < 1$, i.e. on the ZLB as actually constraining the conduct of monetary policy. This manifests itself, quantitatively, as $\text{rk} \Pi^+ + 1 = \text{rk} \Pi^- = 2$ when $\mu < 1$, as opposed to $\text{rk} \Pi^+ = \text{rk} \Pi^- = 2$ when $\mu = 1$. Thus, in this setting, a test for $\Pi^+$ having reduced rank would amount to a test of the null that unconventional monetary policy is less effective than conventional policy, against the alternative that it is equally effective.

(ii) Kinked cointegration. This case is perhaps more reminiscent of linear cointegration, which it accommodates as a special case. Here $\mathcal{M}^+$ and $\mathcal{M}^-$ trace out ‘half’ subspaces of the same dimension $q$, but they need not be parallel, giving rise to a kink in $\mathcal{M}$ at the origin. In general, $(Y, X)$ will follow a kinked Brownian motion driven by $q$ linear BMs, whose loadings, and the associated $r$ cointegrating relations, vary with the sign of $Y$. In the top panel of Figure 3.3, $y_t$ and $x_t$ are cointegrated, but with distinct cointegrating vectors $\beta^+ = (1, -1)^T$ or $\beta^- = (\frac{1}{2}, -1)^T$ applying on $Z^+$ or $Z^-$, i.e. when $y_t$ is positive or negative. We have a kind of ‘threshold cointegration’, with the movement of $y_t$ across zero causing the model to switch between distinct cointegrating spaces (cf. Saikkonen and Choi, 2004). This switch is reflected in the trajectories of $y_t$ and $x_t$: when $y_t \geq 0$, the two series move together approximately one-for-one; whereas if $y_t \leq 0$, $y_t$ changes by two units for every one-unit change in $x_t$, in the long run. Relative to case (i), the trajectories of $\{y_t\}$ will now much more closely resemble those of unit root processes, and $\{y_t\}$ will accordingly tend to spend long stretches in both the positive and negative regimes, with no tendency to revert to one or the other.

The bottom panel of Figure 3.3 presents an important special case where the cointegrating relationships are such that $\{x_t\}$ behaves like an $I^+(0)$ process when $y_t < 0$, but cointegrates with $y_t$ when $y_t > 0$; the limiting process $X$ will thus be a censored Brownian motion.
Example 2.1a (trending natural rate; ctd). Suppose now that \( \chi = 0 \) and \( \psi = 1 \), so that (2.6) implies that the natural rate \( r_t^\ast \) follows a random walk. Then similarly to (3.6), we have from (2.7) that

\[
\Pi^+ = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \Pi^= \begin{bmatrix} 0 \\ -1 \end{bmatrix},
\]

where \( -1 \) is \((1 - \theta) -1 \leq 0, \) with strict inequality unless \( \mu = 1 \). Thus \( \text{rk} \Pi^+ = \text{rk} \Pi^= 1, \) and (by Theorem 4.2 below), there is a common stochastic trend present in both regimes – but when \( i_t > 0 \) this loads only on \( i_t, \) and the co-integrating vector is \( \beta^+= (0, 1)^T. \) If unconventional policy is as effective as conventional policy (\( \mu = 1 \)), this holds also when \( i_t < 0 \); otherwise the trend is shared by both \( i_t \) and \( \pi_t, \) and their co-integrating vector in the negative regime is \( \beta^- = (\gamma^{-1}\tau\mu, 1)^T. \) Qualitatively, the behaviour of the series is as plotted for \((y_t, x_t)\) in the second row of Figure 3.3 (with \( y_t = -i_t \) and \( x_t = -\pi_t, \) i.e. with the signs reversed); in the limit, \( n^{-1/2}\pi_{[n\lambda]} \) will be a censored BM (from above).

To account for this in economic terms, recall that (2.4)–(2.5) imply that when \( i_t > 0, \) the central bank is able to fully stabilise inflation, in the sense that \( \pi_t = \pi_t + (1 - \gamma\theta)^{-1}\varepsilon_t = (1 - \gamma\theta)^{-1}\varepsilon_t, \) since \( \pi_t \) is constant and normalised to zero. Thus \( \pi_t 1\{i_t > 0\} \sim I^*(0); \) whereas, \( i_t^* \sim I^*(1), \) since it inherits the stochastic trend in the natural rate of interest. However, if the model solution requires \( i_t < 0, \) then the ZLB constraint inhibits the operation of monetary policy (\( \mu < 1 \), and inflation begins to drift away from its target. That drift corresponds to the stochastic trend in \( r_t^\ast, \) which is thus present in both \( i_t^* \) and \( \pi_t, \) and hence these series co-integrate.

By contrast, if monetary policy is not effectively constrained by the ZLB, then \( \pi_t \) would remain \( I^*(0), \) irrespective of the sign of \( i_t. \) Thus the long run behaviour of \( \pi_t \) – whether it is \( I^*(0), \) or whether it follows a stochastic trend (and so is \( I^*(1) \)) when interest rates are at the ZLB – here provides identifying information on the relative effectiveness of unconventional monetary policy, i.e. on whether the ZLB is ever a truly binding constraint on the central bank, just as it did in Example 2.1b. \( \Box \)

Example 2.1b (trending inflation target; ctd). Suppose now that \( \delta = 0 \) in (2.8), i.e. that the inflation target is randomly wandering, but there is no feedback from past failures to hit that target. Then (3.6) simplifies to

\[
\Pi^+ = \begin{bmatrix} 0 \\ -\varphi_1 \end{bmatrix} \Pi^= \begin{bmatrix} 0 \\ -\varphi_1 \end{bmatrix},
\]

where \( \varphi^{-1}_1 \varphi_2 = 1 + \hat{\theta}\theta^{-1}(1 - \mu) \in (0, 1). \) Thus \( i_t \) and \( \pi_t \) are \( I^*(1) \) and co-integrated, but with co-integrating relations \( \beta^+ = (1, -1)^T \) and \( \beta^- = (\varphi^{-1}_1 \varphi_2, -1)^T \) that depend on the sign of \( i_t, \) unless \( \mu = 1. \) Even though \( i_t^* \) is unobserved, we can still distinguish between the cases \( \mu = 1 \) and \( \mu < 1 \) on the basis that, when \( \mu < 1, \) the (long-run) variance of \( \pi_t \) will differ depending on whether \( i_t > 0 \) or \( i_t = 0, \) as is evident in the behaviour of \( x_t \) in the top panel of Figure 3.3.\(^5\) \( \Box \)

\(^5\)Theorem 4.2 below implies that \( n^{-1/2}\pi_{[n\lambda]} \) converges weakly to \( \sigma_t (\gamma - 1)[1 + (\varphi^{-1}_1 \varphi_2 - 1)1\{W(\lambda) < 0\}]W(\lambda), \) for \( \sigma_t^2 = \mathbb{E}\varepsilon_t^2 \) and \( W \) a standard BM.
(iii) Linear cointegration in a nonlinear VECM. This case, which is depicted in the Figure 3.4, entails that no trends are present in \{y_t\}, which is in fact a stationary process. The common trends are loaded entirely on \{x_t\}, and the cointegrating relationships between the elements of \{x_t\} are unaffected by the sign of \(y_t\), exactly as in literature on nonlinear VECM models. Indeed, as discussed in Duffy et al. (2023, Sec. 5), this configuration of the model can be analysed via results from that literature, particularly those of Saikkonen (2008).

4 Representation theory

We now proceed to develop analogues of the Granger–Johansen representation theorem for cases (i) and (ii), referring the reader to Section 5 of Duffy et al. (2023) for a rigorous treatment of case (iii), which is of lesser novelty. These results are of interest in their own right, because by characterising the processes generated by the model, under alternative configurations of \(\Pi^\pm\), they delimit the classes of observed time series that the model might be fruitfully applied to. Indeed, they indicate precisely the restrictions on \(\Pi^\pm\) that might be appropriate for specific applications, according to whether \{y_t\} is observed to either: wander randomly above a threshold, but spend only brief periods below it (case (i)); wander randomly on both sides of a threshold (case (ii)); or behave like a stationary process (case (iii)). Beyond such guidance, our results also lay the groundwork for the development of the asymptotics of likelihood-based estimators of the CKSVAR model in the presence of unit roots.

To facilitate the exposition, we shall initially suppose the data to be generated by a canonical CKSVAR, i.e. that \(DGP^*\) holds. The theorems below are stated under this assumption, with the minor modifications required when \(DGP^*\) is replaced by \(DGP\) given in the subsequent remarks. To avoid the need to specify how some of the quantities below should be defined when \(k = 1\), we shall suppose throughout that this case is treated as a special case of the model with \(k = 2\), in which \(\phi_2^+ = \phi_2^- = \phi_2 = 0\) and \(\Phi_2^\pm = 0\); and thus henceforth \(k \geq 2\), unless otherwise stated. Because of the overlap between the arguments used to analyse the CKSVAR in each case, it will be occasionally necessary to redefine objects that have already appeared in the discussion of another case. While we have endeavoured to keep such notational conflicts to a minimum, and
explicitly indicated wherever these arise, the reader is advised to treat each of the following two
subsections, and the accompanying subsections of Appendix A where the proofs of the theorems
appear, as independent of each other.

Notation. For $A \in \mathbb{R}^{m \times n}$ having full column rank, $A_\perp \in \mathbb{R}^{m \times (m-n)}$ denotes a full column rank
matrix such that $A_\perp^t A = 0$; we refer to $A_\perp$ (which is unique only up to its column span) as
‘the orthocomplement of $A$’. (Note that it is not implied that the columns of $A_\perp$ should be
orthogonal vectors; any further normalisation of $A_\perp$ will be noted in the text if required.) $A_{i:j}$
denotes the submatrix formed from rows $\{i, i+1, \ldots, j\}$ of $A$.

4.1 Case (i): regulated cointegration

Recalling Table 3.1, we have the familiar factorisation

$$\Pi^+ = \alpha^+\beta^+T,$$

where $\alpha^+, \beta^+ \in \mathbb{R}^{p \times r}$ have rank $r$. As we show below, $\beta^+$ spans the cointegrating space on
$\mathcal{Z}^+ = \mathbb{R}_+ \times \mathbb{R}^{p-1}$ (recall Definition 3.2), so that $(y^+_t, x_t) \sim I^*(1)$ but $\beta^+T(y^+_t, x_t) \sim I^*(0)$. 
Moreover, since $y^+_t \sim I^*(0)$, it follows that $\beta^+T(y_t, x_t) \sim I^*(0)$, so that the columns of $\beta^+$ are
globally cointegrating vectors.

In a linear cointegrating system, no assumptions additional to CVAR and ERR are needed to pin
down the behaviour of the system, but the nonlinearity of the CKSVAR prevents our assumptions
on the roots of $\Phi^\pm(\lambda)$ from being sufficient to ensure that the short-memory components (i.e.
the equilibrium errors and the first differences) are indeed $I^*(0)$. For this reason, two further
regularity conditions are required. To state these, let

$$F_\delta := \begin{bmatrix} I_{p(k-1)+r} + \beta^+T\alpha^+ & \beta^+T(\phi^- - \phi^+) & \beta^+T(\varphi^- - \varphi^+) \\ e_1^T\alpha^+ & [1 + e_1^T(\phi^- - \phi^+)]\delta & e_1^T(\varphi^- - \varphi^+) \\ 0_{1 \times [p(k-1)+r]} & \delta & 0_{1 \times (k-1)} \\ 0_{(k-2) \times [p(k-1)+r]} & 0_{(k-2) \times 1} & D \end{bmatrix} \quad (4.1)$$

where $\varphi^\pm := [\phi_2^\pm, \ldots, \phi_k^\pm], D := [I_{k-2}, 0_{(k-2) \times 1}] \in \mathbb{R}^{kp \times [p(k-1)+r]}$ with

$$\alpha^+ := \begin{bmatrix} \alpha^+ & \Gamma^+_{1 \times \cdots \times \Gamma^+_{k-1}} & I_p \\ I_p & \cdots & I_p \end{bmatrix}, \quad \beta^+T := \begin{bmatrix} \beta^T & -I_p & \cdots & -I_p \\ I_p & \cdots & I_p \end{bmatrix} \quad (4.2)$$

and $\beta^+_{1:p}$ denotes the first $p$ rows of $\beta^+$.

\footnote{To avoid any ambiguity, the definition of $\beta^+T$ in (4.2) should be read ‘row-wise’, with the ‘\cdots’ signifying that
successive rows of the matrix are formed by replicating the relevant block (i.e. that to the upper left of the ‘\cdots’),
shifting it to the right by the width of the block; all other entries are zeros. Thus for $m \in \{1, \ldots, k-1\}$, rows $r + mp + 1$ to $r + mp$ of $\beta^+T$ are given by $[0_{p \times (m-1)p}, I_p, -I_p, 0_{p \times (k-m-1)}]$. The definitions given in e.g. (4.8)
should be interpreted similarly.}
for $\alpha_1^+$ and $\beta_1^+$ the orthocomplements of $\alpha^+$ and $\beta^+$. For $A \subset \mathbb{R}^{n \times m}$ a bounded collection of matrices, let $\rho_{\text{JSR}}(A) := \limsup_{t \to \infty} \sup_{B \in A} \rho(B)^{1/t}$ denote its joint spectral radius (JSR; e.g. Jungers, 2009, Defn. 1.1), for $A^t := \{ \prod_{s=1}^t M_s \mid M_s \in A \}$ the set of $t$-fold products of matrices in $A$.\footnote{For a further discussion of the JSR, and references to the literature on methods for numerically approximating it, see Duffy et al. (2023, Sec. 4.1).}

Let $z_t := (y_t, x_t^T)^T$.

**Assumption CO(i).**

1. $r^+ = \text{rk} \Pi^2 = r$ and $r^- = r + 1$, for some $r \in \{0, 1, \ldots, p - 1\}$.
2. $\rho_{\text{JSR}}(\{F_0, F_1\}) < 1$.
3. $\kappa_1 < 0$, where $\kappa_1$ denotes the first element of $\kappa := P_{\beta_1^+} \pi^-$.
4. a. $\beta^+ z_t, y_t^-$, and $\Delta z_t$ have uniformly bounded $2 + \delta_u$ moments, for $t \in \{-k + 1, \ldots, 0\}$.
   b. $n^{-1/2} z_0 \overset{p}{\to} Z_0 = [y_0^T] \in \mathcal{M}$, where $Z_0$ is non-random.

**Theorem 4.1.** Suppose $\text{DGP}^*$, $\text{ERR}$, $\text{CVAR}$ and $\text{CO(i)}$ hold, and let $U_0(\lambda) := \Gamma^+(1) Z_0 + U(\lambda)$. Then:

(i) $\xi_t^+ := \beta^+ z_t \sim I^*(0), \Delta z_t \sim I^*(0)$, and $y_t^- \sim I^*(0)$; and

(ii) on $D[0, 1]$,

\[
\begin{bmatrix}
Y_n(\lambda) \\
X_n(\lambda)
\end{bmatrix} \overset{\text{as}}{\to} \begin{bmatrix}
Y(\lambda) \\
X(\lambda)
\end{bmatrix} = P_{\beta_1^+} U_0(\lambda) + \kappa_1^{-1} \kappa \sup_{\lambda' \leq \lambda} [-e_1^T P_{\beta_1^+} U_0(\lambda')]_+,
\]

where in particular $Y(\lambda) = e_1^T P_{\beta_1^+} U_0(\lambda) + \sup_{\lambda' \leq \lambda} [-e_1^T P_{\beta_1^+} U_0(\lambda')]_+$.

**Remark 4.1.** (i). If $\text{DGP}^*$ is replaced by $\text{DGP}$, then the theorem continues to hold as stated, except that $\text{CO(i.2)}$ should be replaced by

\[
\text{CO(i.2') } \rho_{\text{JSR}}(\{\tilde{F}_0, \tilde{F}_1\}) < 1;
\]

where the tildes refer to the parameters of the canonical CKSVAR derived from the structural form via Proposition 2.1. This follows by straightforward calculations, which are given in Online Appendix D.

(ii). The contrast with the behaviour of a linear cointegrating system is marked. $Y$ is now a regulated Brownian motion, which also enters into other components of $X$. Indeed, as noted in Section 3.3 above, some components of $X$ may themselves be regulated BMs.

(iii). Part (i) of the theorem is proved by obtaining a nonlinear VAR representation for $\xi_t^+, \Delta z_t$ and $y_t^-$, whose companion form can be expressed in terms of the matrices $\{F_\delta \mid \delta \in [0, 1]\}$ (Lemma A.3). Since the parameters of that VAR depend on $y_t^-$, these processes cannot be stationary, but $\text{CO(i.2)}$ ensures that the system is sufficiently ‘constrained’ that they will be $I^*(0)$.

A necessary but not sufficient condition for $\text{CO(i.2)}$ is that $F_0$ and $F_1$ have all their eigenvalues inside the unit circle. (This is implied by $\text{CVAR}$; see Online Appendix C.)

(iv). It will be seen from the proof that part (ii) holds if $\text{CO(i.2)}$ is replaced by any condition sufficient to ensure $(\xi_t^+, \Delta z_t, y_t^-) \sim I^*(0)$. There may thus be some scope for relaxing this
assumption, which takes essentially a worst-case approach to the behaviour of the nonlinear VAR governing the evolution of these processes. However, even in the much more tractable univariate set-up of Bykhovskaya and Duffy (2022), it is far from obvious what this condition – which corresponds to their Assumption A4 – might be replaced by.

(v). In deriving the weak limit of \( n^{-1/2} y_{n\lambda} \), a key step is to obtain a univariate representation for \( y_t^+ \) as a regulated process. The proof of Theorem 4.1 shows that it is possible to write \( y_t^+ - \kappa_1 y_t^- = w_t \) for a certain series \( \{w_t\} \): the role of CO(i).3 is to ensure that this equation is solved uniquely by taking \( y_t^+ = [w_t]_+ \). It is possible that this assumption, as stated, may be redundant: as shown in Online Appendix C, this condition is implied by our other assumptions if either \( k = 1 \) or \( p = 1 \).

Example 2.2 (univariate; ctd). In the univariate (\( p = 1 \)) model (2.10), case (i) with \( r = 0 \) corresponds to a model in which \( c = 0, \sum_{i=1}^k \phi_i^+ = 1 \) and \( \phi^- (\lambda) \) has all its roots outside the unit circle, so that

\[
y_t = \sum_{i=1}^k (\phi_i^+ y_{t-i} + \phi_i^- y_{t-i}) + u_t = (1 + \pi^+) y_{t-1} + \sum_{i=1}^{k-1} \gamma_i^+ \Delta y_{t-i} + \sum_{i=1}^k \phi_i^- y_{t-i} + u_t \quad (4.3)
\]

may be loosely regarded as an autoregressive model with a unit root regime (since \( \pi^+ = 0 \)) and a stationary regime (though the model technically has \( 2^k \) distinct autoregressive regimes). Theorem 4.1 implies that \( y_t^+ \sim I^*(0) \), and

\[
Y_n(\lambda) \sim Y(\lambda) = K(\lambda) - \sup_{\lambda' \leq \lambda} [-K(\lambda')]_+ \quad (4.4)
\]
on \( D[0,1] \), where

\[
K(\lambda) := \gamma^+(1)^{-1} U_0(\lambda) = Y_0 + \gamma^+(1)^{-1} U(\lambda)
\]

for \( \gamma^+(1) = 1 - \sum_{i=1}^{k-1} \gamma_i^+ \). \( Y \) is thus a regulated Brownian motion initialised at \( Y_0 \geq 0 \). (4.4) extends the results of Liu et al. (2011, Thm. 3.1) and Gao et al. (2013), who consider univariate first-order models of this kind, to a higher-order autoregressive setting. It also agrees with the limit theory developed in Bykhovskaya and Duffy (2022, Thm. 3.2), when their censored dynamic Tobit model (in which \( \phi_i^- = 0 \) for all \( i \in \{1, \ldots, k\} \)) is specialised to one with an exact unit root and no intercept.

\[
\square
\]

4.2 Case (ii): kinked cointegration

We turn next to the case in which the cointegrating rank \( r \) is the same across both the positive and negative regimes, though the cointegrating space itself need not be. Here we also suppose that \( \text{rk} \Pi^x = r \), which as discussed in Section 3.3 entails that \( y_t \sim I^*(1) \). Under the foregoing, we must have \( \pi^+ \in \text{sp} \Pi^x \), and so

\[
\Pi^\pm = \Pi^x \begin{bmatrix} \theta^\pm & I_{p-1} \end{bmatrix} = \alpha \begin{bmatrix} \beta^\pm_y & \beta^T_x \end{bmatrix} = \alpha \beta^\pm T, \quad (4.5)
\]

20
where $\alpha \in \mathbb{R}^{p \times r}$, $\beta_x \in \mathbb{R}^{(p-1) \times r}$ and $\beta^\pm \in \mathbb{R}^{p \times r}$ have rank $r$, and $\theta^\pm \in \mathbb{R}^{p-1}$ is such that $\Pi^\prime \theta^\pm = \pi^\pm$. Let $1^+(y) := 1\{y \geq 0\}$ and $1^-(y) := 1\{y < 0\}$, and set $\beta(y) := \beta^+1^+(y) + \beta^-1^-(y)$.\(^8\) Then we can define the equilibrium errors as

$$\xi_t := \beta(y_t)^T z_t = 1^+(y_t)\beta^+T z_t + 1^-(y_t)\beta^-T z_t.$$  

Observe how (4.5) implies that the ‘loadings’ $\alpha$ of the equilibrium errors will the same in both regimes, even though the cointegrating vectors that define those errors need not be. (Case (iii) entails the opposite, with the cointegrating vectors being fixed, but the loadings on the equilibrium errors depending on the sign of $y_t$; see Duffy et al., 2023, Sec. 5.)

The theorem below establishes that $\xi_t \sim I^*(0)$, and that $\beta^+$ and $\beta^-$ span the cointegrating spaces on $Z^+$ and $Z^-$ respectively. The limiting common trends are kinked Brownian motions that correspond to a projection of $U$ onto $\mathcal{M}$, defined in terms of

$$P_{\beta^+}(y) := \beta^+ (y)[\alpha_1^T \Gamma(1; y) \beta^+ (y)]^{-1} \alpha_1^T,$$

$$\beta^+ (y) := \begin{bmatrix} 1 & 0 \\ -\theta(y) & \beta_{x\perp} \end{bmatrix}, \quad \Gamma(1; y) := \Gamma^+(1)1^+(y) + \Gamma^-(1)1^-(y),$$

where $\theta(y) := 1^+(y)\theta^+ + 1^-(y)\theta^-$. Such objects as $P_{\beta^+}(y)$ take only two distinct values, depending on the sign of $y_t$, and we use routinely use the notation $P_{\beta^+}(+1)$ and $P_{\beta^+}(-1)$ to indicate these two values. Similarly to case (i), beyond our assumptions on the ranks of $\Pi^\pm$ and $\Pi^x$, two further regularity conditions are needed to ensure that the system is well behaved. To state these, let $\alpha, \beta(y) \in \mathbb{R}^{(k(p+1) - 1) \times [r+(k-1)(p+1)]}$ with

$$\alpha := \begin{bmatrix} \alpha & \Gamma_1 & \Gamma_2 & \cdots & \Gamma_{k-1} \\ I_{p+1} & I_{p+1} & \cdots & I_{p+1} \end{bmatrix}, \quad \beta(y)^T := \begin{bmatrix} \beta(y)^T S(y) - I_{p+1} \\ -I_{p+1} \end{bmatrix},$$

where $\Gamma_i := [\gamma_i^+, \gamma_i^-, \Gamma_i^x]$ for $i \in \{1, \ldots, k-1\}$, and

$$S(y) := \begin{bmatrix} 1^+(y) & 0 \\ 1^-(y) & 0 \end{bmatrix},$$

so that $S(y_t)z_t = (y_t^+, y_t^- , z_t^\prime)^T$, where $z_t = (y_t, x_l^\prime)^T$.

**Assumption CO(ii).**

1. $r^+ = r^- = \text{rk} \Pi^x = r$, for some $r \in \{0, 1, \ldots, p-1\}$.

2. $\rho_{JSR}(\{I + \beta(1)^T \alpha, I + \beta(-1)^T \alpha\}) < 1$.

\(^8\)There is unavoidably some arbitrariness with respect to how such objects are defined when $y = 0$, but since these only play a role in the model when multiplied by $y$, it does not matter which convention is adopted. Throughout the paper, we use the functions $1^\pm(y)$ to ensure that all such definitions are mutually consistent.
Theorem 4.2. Suppose DGP*, ERR, CVAR and CO(ii) hold, and let $\theta^T := e_1^T P_{\beta_\perp} (+1) + U_0(\lambda) := \Gamma(1; Y_0) Z_0 + U(\lambda)$. Then:

(i) $\xi_t := \beta(y_t)^T z_t \sim I^*(0)$ and $\Delta z_t \sim I^*(0)$; and

(ii) on $D[0, 1]$,

$$
\begin{bmatrix} Y_n(\lambda) \\ X_n(\lambda) \end{bmatrix} \sim \begin{bmatrix} Y(\lambda) \\ X(\lambda) \end{bmatrix} = P_{\beta_\perp} [Y(\lambda) U_0(\lambda) = P_{\beta_\perp} [\theta^T U_0(\lambda)] U_0(\lambda),
$$

(4.10)

where in particular $\text{sgn} Y(\lambda) = \text{sgn} \theta^T U_0(\lambda)$.

Remark 4.2. (i). Similarly to Remark 4.1(i) above, if DGP* is replaced by DGP, then the theorem continues to hold exactly as stated, except that CO(ii).2 should be modified to

$$
\text{CO(ii).2}' \quad \rho_{\text{JSR}}(\{I + \tilde{\beta} (+1)^T \tilde{\alpha}, I + \tilde{\beta} (-1)^T \tilde{\alpha}\}) < 1.
$$

(See Online Appendix D for details.)

(ii). Even when $\beta^+ = \beta^-$, such that the cointegrating space is the same in both the positive and negative regimes, $(Y, X)$ will generally be a kinked Brownian motion because of the residual dependence of $P_{\beta_\perp}(y) = \beta_\perp [\alpha_\perp^T \Gamma(1; y) \beta_\perp]^{-1} \alpha_\perp^T$ on $y$ via $\Gamma(1; y)$. Indeed, in the univariate model (2.10) under case (ii) with $r = 1$,

$$
\Delta y_t = \sum_{i=1}^{k-1} (\gamma_i^+ \Delta y_{t-i}^+ + \gamma_i^- \Delta y_{t-i}^-) + u_t.
$$

Theorem 4.2 entails $Y(\lambda) = \gamma[1; U_0(\lambda)]^{-1} U_0(\lambda)$ is a kinked Brownian motion, whose variance depends on the sign of $U_0(\lambda)$.

(iii). CO(ii).2 plays an analogous role to CO(i).2 above, ensuring that the nonlinear VAR representation (see Lemma A.4) obtained for the short-memory components $(\xi_t, \Delta z_t)$ is sufficiently well-behaved that these processes are $I^*(0)$. Part (ii) would continue to hold if CO(ii).2 were replaced by any other condition sufficient for $(\xi_t, \Delta z_t) \sim I^*(0)$. A necessary condition for CO(ii).2 is that the eigenvalues of $I + \beta(\pm 1)^T \alpha$ should lie strictly inside the unit circle, which is implied by CVAR (see Lemma A.2).

(iv). The map $(y, u) \mapsto P_{\beta_\perp}(y) u$ is not in general continuous, a fact that could interfere with the convergence in (4.10). CO(ii).3 ensures this map is continuous on a sufficiently large domain to permit (4.10) to be follow via an application of the continuous mapping theorem (CMT), with $Y$ and $X$ having continuous paths.

(v). Given $\alpha, \beta \in \mathbb{R}^{p \times r}$ with full column rank, their orthocomplements $\alpha_\perp, \beta_\perp \in \mathbb{R}^{p \times q}$ are unique only up to their column span. Since in a linearly cointegrated system CVAR implies that $\alpha_\perp^T \Gamma(1) \beta_\perp$ has nonzero determinant (Lemma A.2), we can normalise $\alpha_\perp$ and/or $\beta_\perp$ so that $\det \alpha_\perp^T \Gamma(1) \beta_\perp = 1$. It should therefore be emphasised that CO(ii).3 applies when $\beta_\perp(1)$ and
\(\beta_\perp (-1)\) are related via (4.7), so we are not entirely free to choose \(\beta_\perp (\pm 1)\) such that both determinants can be brought into agreement. Thus CO(ii).3 appears to be a substantive restriction, and one that could conceivably fail if the differences between the positive and negative regimes were too ‘large’ in some sense.

5 Conclusion

The CKSVAR provides a flexible yet tractable framework in which to structurally model vector time series subject to an occasionally binding constraint, such as the zero lower bound on interest rates, and more general threshold nonlinearities. Nonetheless, even that seemingly limited amount of nonlinearity radically changes the properties of the model relative to a linear VAR. When unit autoregressive roots are introduced into the model, it is able to accommodate varieties of long-run behaviour that cannot be generated within a linear VAR, such as nonlinear common stochastic trends (censored, regulated and kinked Brownian motions) and cointegrating relationships that may be regime-dependent. This is not merely a theoretical curiosity but rather something that, as our examples illustrate, allows the long-run properties of the model to carry useful identifying information on structural parameters, as might pertain e.g. to the relative effectiveness of unconventional monetary policy.

Our results provide a complete characterisation of the forms of nonlinear cointegration (between processes \(\ast\)-integrated of order one) generated by the CKSVAR. In deriving these, we have given the first treatment of how nonlinear cointegration, in the profound sense of nonlinear common stochastic trends and nonlinear cointegrating relations, may be systematically generated within a (nonlinear) VAR, and thus the first extension of the Granger–Johansen representation theorem to a nonlinearly cointegrated setting. The special structure of the CKSVAR makes this problem peculiarly tractable, while being flexible enough to generate interesting departures from linear cointegration. Our results indicate how progress may now be made in the analysis of more general nonlinear VARs with unit roots, while our representation theory provides the foundations for inference on cointegrating relations in the CKSVAR. Our findings with respect to these problems will be reported elsewhere.

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A Proofs of theorems

A.1 Auxiliary lemmas

We present two elementary lemmas, whose proofs – together with those of Lemmas A.3–A.5 below – appear in Online Appendix B.

**Lemma A.1.** Let \( \{v_t\}, \{A_t\}, \{B_t\} \) and \( \{c_t\} \) be random sequences, respectively taking values in \( \mathbb{R}^d, \mathbb{R}^{d_w \times d_w}, \mathbb{R}^{d_w \times d_v} \) and \( \mathbb{R}^{d_v} \), where \( t \in \mathbb{N} \). Suppose \( \{w_t\} \) satisfies
\[
w_t = c_t + A_t w_{t-1} + B_t v_t
\]
for some given (random) \( w_0 \), and:

(i) \( A_t \in \mathcal{A}, B_t \in \mathcal{B} \) and \( c_t \in \mathcal{C} \) for all \( t \in \mathbb{N}, \) where \( \mathcal{A}, \mathcal{B} \) and \( \mathcal{C} \) are bounded, and \( \rho_{JSR}(\mathcal{A}) < 1; \)

(ii) \( m_0 \geq 1 \) is such that \( \|w_0\|_{m_0} + \sup_{t \in \mathbb{N}} \|v_t\|_{m_0} < \infty. \)

Then there exists a \( C < \infty \) such that
\[
\max_{1 \leq t \leq n} \|w_t\|_m \leq C \left( 1 + \|w_0\|_{m_0} + \max_{1 \leq t \leq n} \|v_t\|_m \right)
\]
for all \( n \in \mathbb{N}, \) for all \( m \in [1, m_0]. \)

The following records some properties of a linear cointegrated VAR that are well known in the literature; the form in which they are phrased borrows from an outline of the Granger–Johansen representation theorem presented in some of Bent Nielsen’s lecture notes (cf. Nielsen, 2009, Thm. 3.1, for part (iv) of the below).

**Lemma A.2.** Suppose \( \Phi(\lambda) := I_p - \sum_{i=1}^{k} \Phi_i \lambda^i \) and \( \Pi := -\Phi(1) \) satisfy CVAR.1 and CVAR.2 (i.e. without the ‘±’ superscripts). Define
\[
\Pi := \begin{bmatrix}
\Pi + \Gamma_1 & -\Gamma_1 + \Gamma_2 & \cdots & -\Gamma_{k-1} \\
I_p & -I_p & \cdots & \cdots \\
& I_p & -I_p & \cdots \\
& & \cdots & I_p & -I_p
\end{bmatrix},
\]

and let \( \Gamma(\lambda) := I_p - \sum_{i=1}^{k-1} \Gamma_i \lambda^i \) be such that \( \Phi(\lambda) = \Phi(1)\lambda + \Gamma(\lambda)(1 - \lambda). \)

Then:

(i) there exist \( \alpha, \beta \in \mathbb{R}^{p \times r}, \) with full column rank, such that \( \Pi = \alpha \beta^T \) and \( \Pi = \alpha \beta^T, \) where
\[
\alpha := \begin{bmatrix}
\alpha & \Gamma_1 & \cdots & \Gamma_{k-1} \\
I_p & \cdots & I_p \\
& \cdots & \cdots \\
& & I_p
\end{bmatrix}, \quad \beta^T := \begin{bmatrix}
\beta^T \\
I_p & -I_p \\
& \cdots & \cdots \\
& & I_p & -I_p
\end{bmatrix};
\]

\( ^9 \) As per the footnote to (4.2), the definition of \( \Pi \) here should be read ‘row-wise’, so that for \( m \in \{2, \ldots, k\}, \) rows \((m-1)p+1\) to \( mp \) of \( \Pi \) are given by \([0_{p \times (m-2)p}, I_p, -I_p, 0_{p \times (k-m)p}] \). The matrix definitions appearing subsequently in these appendices should be interpreted similarly.

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(ii) $I_p(k-1)+\beta^T\alpha$ has its eigenvalues strictly inside the unit circle, and $\alpha^T_1\beta_\perp$ and $\beta^T\alpha$ are full rank, where
\[
\alpha_1^T := \alpha^T_1 \left[ I_p \quad -\Gamma_1 \quad \ldots \quad -\Gamma_{k-1} \right], \quad \beta_\perp^T := \beta^T \left[ I_p \quad I_p \quad \ldots \quad I_p \right]; \tag{A.1}
\]

(iii) $P_{\beta_\perp} := \beta_\perp [\alpha_1^T \beta_\perp]^{-1} \alpha_1^T$ and $P_{\alpha} := \alpha [\beta^T \alpha]^{-1} \beta^T$ are complementary projections;

(iv) $\det(\alpha_1^T \Gamma(1) \beta_\perp) \neq 0$, and the upper left $p \times p$ block of $P_{\beta_\perp}$ is $P_{\beta_\perp} := \beta_\perp [\alpha_1^T \Gamma(1) \beta_\perp]^{-1} \alpha_1^T$.

### A.2 Proof of Theorem 4.1

Defining
\[
v_t := u_t + (\pi^- - \pi^+) y_{t-1} + \sum_{i=1}^{k-1} (\gamma_i^- - \gamma_i^+) \Delta y_{t-i} = u_t + \sum_{i=1}^{k-1} (\phi_i^- - \phi_i^+) y_{t-i} \tag{A.2}
\]
and recalling $z_t = (y_t, x_t^T)^T$, we may rewrite the model (3.2) as
\[
\Delta z_t = c + \Pi^+ z_{t-1} + \sum_{i=1}^{k-1} \Gamma_i^+ \Delta z_{t-i} + v_t
\]
where $\Pi^+ = [\pi^+, \Pi^\alpha]$ and $\Gamma_i^+ = [\gamma_i^+, \Gamma^\alpha]$. Conformably defining
\[
\Pi^+ := \begin{bmatrix}
\Pi^+ + \Gamma_1^+ & -\Gamma_1^+ + \Gamma_2^+ & \ldots & -\Gamma_{k-1}^+

I_p & -I_p & \ldots & -I_p

\end{bmatrix}, \quad c := \begin{bmatrix}
c
0_p

\end{bmatrix}, \quad v_t := \begin{bmatrix}
v_t
0_p

\end{bmatrix}, \quad z_t := \begin{bmatrix}
z_t
z_{t-1}
\vdots
z_{t-k+1}

\end{bmatrix}
\]
so that $\Pi^+ = \alpha^+ \beta^T$ (by Lemma A.2), we render the system in companion form as
\[
\Delta z_t = c + \Pi^+ z_{t-1} + v_t. \tag{A.3}
\]

The next lemma provides a (nonlinear) VAR representation for the short-memory components, which comprise: (a) the equilibrium errors (using the cointegrating vectors $\beta^+$ from the positive regime) and (lagged) differences,
\[
\xi_t^+ := \beta^+ T z_t = (\xi_t^+ T, \Delta z_{t-1}^T, \ldots, \Delta z_{t-k+2})^T
\]
where $\xi_t^+ := \beta^+ z_t$; (b) the lagged levels $y_{t-1} := (y_{t-1}, \ldots, y_{t-k+1})^T$, and (c) and an auxiliary series $\bar{y}_t$. Collect these in the vector $\zeta_t \in \mathbb{R}^{p(k-1)+k+r}$, and conformably define $\varepsilon_t$ and $\mu$ as
\[
\zeta_t = \begin{bmatrix}
\xi_t^+
y_t
y_{t-1}
\end{bmatrix}, \quad \varepsilon_t := \begin{bmatrix}
\beta_{1:p}^T u_t
\varepsilon_1^T u_t
0_{k-1}
\end{bmatrix}, \quad \mu := \begin{bmatrix}
\beta_{1:p}^T c
\varepsilon_1^T c
0_{k-1}
\end{bmatrix}
\]
where $\beta_{1:p}^+$ denotes the first $p$ rows of $\beta^+$. Recall the definition of $F^\delta$ given in (4.1) above.
Lemma A.3. Suppose that cvar and co(i).1 hold. Set $\delta_0 := 1$ and $\eta_0 := y_0^-$. Then $\{z_t\}$ follows

$$
\zeta_t = \mu + F_{\delta_{t-1}} \zeta_{t-1} + \varepsilon_t \quad (\text{A.4})
$$

$$
\delta_t = \begin{cases} 
(y_{t-1}^- + \eta_t)/\eta_t & \text{if } y_{t-1}^- + \eta_t < 0, \\
0 & \text{otherwise}; 
\end{cases} 
$$

(A.5)

which implies, in particular, that $y_{t-1}^- = \delta_t \eta_t$ for all $t \in \mathbb{N}$. By Jungers (2009, Prop. 1.8), co(i).2 implies that $\rho_{\text{ISR}}(\{F_\delta \mid \delta \in [0, 1]\}) = \rho_{\text{ISR}}(\{F_0, F_1\}) < 1$. In view of err and co(i).4, applying Lemma A.1 to the representation (A.4) then yields that $\sup_{t \in \mathbb{N}} \|\zeta_t\|_{2+\delta_n} < \infty$, and hence $\zeta_t \sim I^*(0)$. In particular, $\xi_t^+ \sim I^*(0)$ and $y_{t-}^- \sim I^*(0)$, which gives part (i) of the theorem.

We next identify the common trend components. To that end, define the $r \times kp$ matrices

$$
\alpha^+_\bot := \alpha^+_\bot [I_p - \Gamma^+ \cdots - \Gamma^+_k I] \quad \beta^+_\bot := \beta^+_\bot [I_p I_p \cdots I_p]
$$

which are orthogonal to $\alpha^+$ and $\beta^+$. $P_{\beta^+_\bot} := \beta^+_\bot [\alpha^+_\bot \Gamma^+]^{-1} \alpha^+_\bot$ and $P_{\alpha^+_\bot} := \alpha^+_\bot [\beta^+_\bot \alpha^+]^{-1} \beta^+_\bot$ denote the associated projections (see Lemma A.2). Premultiplying (A.3) by $\alpha^+_\bot$ and cumulating yields

$$
\alpha^+_\bot z_t = \alpha^+_\bot z_0 + \alpha^+_\bot \sum_{s=1}^t v_s
$$

where we have used that $\alpha^+_\bot c = \alpha^+_\bot c = 0$, since $c \in \text{sp} \Pi^+$ under cvar.3. Hence

$$
z_t = (P_{\beta^+_\bot} + P_{\alpha^+_\bot}) z_t = P_{\beta^+_\bot} \left( z_0 + \sum_{s=1}^t v_s \right) + z_{\xi,t}
$$

where $z_{\xi,t} := P_{\alpha^+_\bot} z_t = \alpha^+_\bot [\beta^+_\bot \alpha^+]^{-1} \xi_{t}^+ \sim I^*(0)$.

Since the top left $p \times p$ block of $P_{\beta^+_\bot}$ is $P_{\beta^+_\bot} = \beta^+_\bot [\alpha^+_\bot \Gamma^+ (1) \beta^+_\bot]^{-1} \alpha^+_\bot$ (Lemma A.2),

$$
z_t = |P_{\beta^+_\bot} z_0|_{1:p} + \sum_{s=1}^t v_s + z_{\xi,t} \quad \text{(A.6)}
$$

where $z_{\xi,t} := [z_{\xi,t}]_{1:p}$. Under co(i).4, $n^{-1/2} z_t \xrightarrow{p} Z_0$ for each $t \in \{-k + 1, \ldots, 0\}$, and hence

$$
n^{-1/2} |P_{\beta^+_\bot} z_0|_{1:p} \xrightarrow{p} \beta^+_\bot [\alpha^+_\bot \Gamma^+ (1) \beta^+_\bot]^{-1} \alpha^+_\bot \Gamma^+ (1) Z_0 = Z_0 \quad \text{(A.7)}
$$

where the final equality follows since $Z_0 \in \mathcal{M} \subset \text{sp} \beta^+_\bot$. Now recalling (A.2),

$$
\sum_{s=1}^t v_s = \sum_{s=1}^t u_s + (\pi^- - \pi^+) \sum_{s=0}^{t-1} y_s + \sum_{i=1}^{k-1} (\gamma_i^- - \gamma_i^+) [y_{t-i}^- - y_{t-i}^-],
$$

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we have from (A.6) that

\[ z_t = [P_{\beta^+}z_0]_{1:p} + P_{\beta^+_1} \sum_{s=1}^t u_s + \kappa \sum_{s=0}^{t-1} y_s^- + P_{\beta^+_1} \sum_{i=1}^{k-1} (\gamma_i^- - \gamma_i^+) [y_{t-i}^- - y_{t-i}] + z_{\xi,t}, \]

where \( \kappa := P_{\beta^+_1}(\pi^- - \pi^+) = P_{\beta^+_1}\pi^- \). Using that \( z_t = [y_t^+] = [y_t^-] + e_1 y_t^- \), and defining

\[ \eta_t := -(e_1 + \kappa)y_t^- + P_{\beta^+_1} \sum_{i=1}^{k-1} (\gamma_i^- - \gamma_i^+) [y_{t-i}^- - y_{t-i}] + z_{\xi,t}, \]

which is \( I^*(0) \) by part (i) of the theorem, we may rewrite the preceding as

\[ \begin{bmatrix} y_t^+ \\ z_t \end{bmatrix} = [P_{\beta^+}z_0]_{1:p} + \kappa \sum_{s=0}^t y_s^- + P_{\beta^+_1} \sum_{s=1}^t u_s + \eta_t \tag{A.8} \]

The first equation in (A.8) is

\[ y_t^+ = e_1^T [P_{\beta^+}z_0]_{1:p} + \kappa \sum_{s=0}^t y_s^- + e_1^T P_{\beta^+_1} \sum_{s=1}^t u_s + \eta_{1:t} \tag{A.9} \]

where \( \eta_{1:t} = e_1^T \eta_t \). Taking first differences yields

\[ y_t^+ - \kappa_1 y_{t-1}^- = y_{t-1}^+ + e_1^T P_{\beta^+_1} u_t + \Delta \eta_{1:t}. \]

Since \( \kappa_1 < 0 \) by CO(i.3), only one of \( y_t^+ \) and \( \kappa y_t^- \) can be nonzero, and must have opposite signs; hence

\[ y_t^+ = [y_{t-1}^+ + e_1^T P_{\beta^+_1} u_t + \Delta \eta_{1:t}]. \]

As noted above \( \eta_t \sim I^*(0) \), while \( n^{-1/2} y_0 = Y_0 \) by CO(i.4). Hence by ERR we have that on \( D[0, 1] \),

\[ n^{-1/2} y_0 + n^{-1/2} \sum_{l=1}^{[n\lambda]} (e_1^T P_{\beta^+_1} u_l + \Delta \eta_l) \]

\[ = Y_0 + n^{-1/2} e_1^T P_{\beta^+_1} \sum_{l=1}^{[n\lambda]} u_l + n^{-1/2} (\eta_{1:[n\lambda]} - \eta_{1:0}) + o_p(1) \]

\[ \sim Y_0 + e_1^T P_{\beta^+_1} U(\lambda) = e_1^T P_{\beta^+_1} [\Gamma^+(1) Z_0 + U(\lambda)] = e_1^T P_{\beta^+_1} U_0(\lambda). \]

where the penultimate equality follows from \( P_{\beta^+_1} \Gamma^+(1) Z_0 = Z_0 \), as per (A.7) above. It therefore follows by Bykhovskaya and Duffy (2022, Lem. A.1), and using \( y_t^- \sim I^*(0) \), that

\[ n^{-1/2} y_{[n\lambda]} = n^{-1/2} y_{[n\lambda]} + o_p(1) \sim e_1^T P_{\beta^+_1} U_0(\lambda) + \sup_{\lambda \leq \lambda} [-e_1^T P_{\beta^+_1} U_0(\lambda')]_+ = Y(\lambda). \tag{A.10} \]

To obtain the corresponding limit for \( n^{-1/2} x_{[n\lambda]} \), we substitute

\[ \sum_{s=0}^t y_s^- = \kappa_1^{-1} y_t^+ - \kappa_1^{-1} e_1^T \left( [P_{\beta^+}z_0]_{1:p} + P_{\beta^+_1} \sum_{s=1}^t u_s + \eta_t \right) \]
from (A.9) into the final $p - 1$ equations from (A.8), to obtain

$$x_t = E_{-1}^T \left( I_p - \kappa_1^{-1} \kappa e_1^T \right) \left[ [P_{\beta^+} z_0]_{1:p} + P_{\beta^+} \sum_{s=1}^t u_s + \eta_t \right] + \kappa_1^{-1} \kappa y_t^+ \right).$$

where $E_{-1}$ denotes the final $p - 1$ columns of $I_p$. Hence, by $\eta_t \sim \Gamma_s(0)$, err, (A.7) and (A.10),

$$n^{-1/2} x_{[n\lambda]} = E_{-1}^T \left( I_p - \kappa_1^{-1} \kappa e_1^T \right) \left[ Z_0 + P_{\beta^+} n^{-1/2} \sum_{s=1}^{[n\lambda]} u_s \right] + \kappa_1^{-1} \kappa n^{1/2} y_{[n\lambda]}^+ + o_p(1)$$

$$\leadsto E_{-1}^T \left( I_p - \kappa_1^{-1} \kappa e_1^T \right) P_{\beta^+} [\Gamma^+ (1) Z_0 + U(\lambda)] + \kappa_1^{-1} \kappa Y(\lambda) \right)$$

$$= E_{-1}^T \left( P_{\beta^+} U_0(\lambda) + \kappa_1^{-1} \kappa \sup_{\lambda' \leq \lambda} [-e_1^T P_{\beta^+} U_0(\lambda')] \right)$$

(A.11)

on $D[0,1]$. Thus part (ii) follows from (A.10) and (A.11). \qed

### A.3 Proof of Theorem 4.2

Because $\{y_t\}$ now behaves like a unit root process in both the positive and negative regimes, we have to adopt a quite different approach from that utilised in the analysis of case (i). Recall $z_t^T = (y_t, x_t^T)$, but now also define $z_t^*T := (y_t^*, y_t^*, x_t^T)$, so that (3.2) may be written as

$$\Delta z_t = \begin{bmatrix} \Delta y_t \\ \Delta x_t \end{bmatrix} = c + \begin{bmatrix} \pi^+ & \pi^- & \Pi^+ \\ \pi^+ & \pi^- & \Pi^- \end{bmatrix} \begin{bmatrix} y_{t-1}^+ \\ y_{t-1}^- \\ x_{t-1} \end{bmatrix} + \sum_{i=1}^{k-1} \begin{bmatrix} \gamma_i^+ & \gamma_i^- & \Gamma_i^+ \\ \gamma_i^+ & \gamma_i^- & \Gamma_i^- \end{bmatrix} \begin{bmatrix} \Delta y_{t-i}^+ \\ \Delta y_{t-i}^- \\ \Delta x_{t-i} \end{bmatrix} + u_t$$

$$= c + \Pi(y_{t-1}) z_{t-1} + \sum_{i=1}^{k-1} \Gamma_i \Delta z_{t-i}^* + u_t,$$

where $\Pi(y) := \Pi^+ 1^+(y) + \Pi^- 1^-(y)$. Taking $z_t^*T := (z_{t-1}^*, z_{t-1-k}^*, \ldots, z_{t-k}^*)$ - the first $p$ elements of which are $z_t$, not $z_t^*$ - to be the state vector in this case, and conformably defining

$$\Pi(y) := \begin{bmatrix} \Pi(y) + \Gamma_1 S(y) \\ S(y) \end{bmatrix} - \Gamma_1 + \Gamma_2 - \Gamma_2 + \Gamma_3 - \cdots - \Gamma_{k-1},$$

we can rewrite the system in companion form as

$$\Delta z_t = c + \Pi(y_{t-1}) z_{t-1} + u_t.$$  

(A.12)

Note the definitions of $z_t$, $c$ and $u_t$ here differ from those in case (i).

Our next result is the counterpart of Lemma A.3 for the present case, in that it provides a nonlinear VAR representation for the short-memory components of the model, in this case the
equilibrium errors $\xi_t := \beta(y_t)^T z_t$ and (lags of) the differences $\Delta z_t^*$, which we collect in

$$\xi_t := \beta(y_t)^T z_t = (\xi_t^T, \Delta z_t^*, \ldots, \Delta z_{t-k+2}^*)^T.$$  

**Lemma A.4.** Suppose that $\text{DGP}^*$, $\text{CVAR}$ and $\text{CO}(i.1)$ hold. Let

$$\delta_t := \begin{cases} 0 & \text{if } y_{t-1} \cdot y_t \geq 0, \\ y_t/\Delta y_t & \text{if } y_{t-1} \cdot y_t < 0 \end{cases} \quad \text{(A.13)}$$

for $t \in \mathbb{N}$, which takes values only in $[0,1]$, and define $\overline{\beta}_t := (1 - \delta_t)\beta(y_{t-1}) + \delta_t\beta(y_t)$. Then $\{\xi_t\}_{t \in \mathbb{N}}$ satisfies

$$\xi_t = \overline{\beta}_t^T c + [I_{r+(k-1)(p+1)} + \overline{\beta}_t^T \alpha]\xi_{t-1} + \overline{\beta}_t^T u_t. \quad \text{(A.14)}$$

Since $\overline{\beta}_t$ is a convex combination of $\beta(-1)$ and $\beta(+1)$, the autoregressive matrices in the representation (A.14) are contained in

$$A := \{ I + ((1 - \delta)\beta(+1) + \delta\beta(-1))^T \alpha \mid \delta \in [0,1] \}.$$  

By Jungers (2009, Prop. 1.8) and $\text{CO}(ii.2)$, $\rho_{\text{JSR}}(A) = \rho_{\text{JSR}}(\{I + \beta(+1)^T \alpha, I + \beta(-1)^T \alpha\}) < 1$. That $\xi_t \sim I^*(0)$ now follows from $\text{err}$, $\text{CO}(ii.4)$ and Lemma A.1. In particular, $\xi_t = \beta(y_t)^T z_t$, $\Delta y_t^+$, $\Delta y_t^-$ and $\Delta x_t$ are all $I^*(0)$, and hence so too is $\Delta y_t = \Delta y_t^+ + \Delta y_t^-$, as claimed in part (i) of the theorem.

To prove part (ii) of the theorem, we next consider the common trend components. Recall the definitions of $\beta_\perp(y)$ and $\Gamma(1; y)$ in (4.7), and note that we may take

$$\alpha_\perp^T = \alpha_\perp^T \begin{bmatrix} I_p & -\Gamma_1 & \cdots & -\Gamma_{k-1} \end{bmatrix}, \quad \beta_\perp(y)^T := \beta_\perp(y)^T \begin{bmatrix} I_p & S(y)^T & \cdots & S(y)^T \end{bmatrix}$$

as the orthocomplements of $\alpha$ and $\beta$ respectively. Observe that

$$\alpha_\perp^T \beta_\perp(y) = \alpha_\perp^T \begin{bmatrix} I_p & -\sum_{i=1}^{k-1} \Gamma_i S(y) \end{bmatrix} \beta_\perp(y) = \alpha_\perp^T \Gamma(1; y) \beta_\perp(y) \quad \text{(A.15)}$$

where the r.h.s. is invertible for each $y \in \mathbb{R}$ by Lemma A.2; it follows that the complementary projections $P_{\beta_\perp}(y) := \beta_\perp(y)[\alpha_\perp^T \beta_\perp(y)]^{-1} \alpha_\perp^T$ and $P_{\alpha}(y) := \alpha[\beta(y)^T \alpha]^{-1} \beta(y)^T$ are also well defined for each $y \in \mathbb{R}$.

Now premultiplying (A.12) by $\alpha_\perp^T$ and cumulating yields

$$\alpha_\perp^T z_t = \alpha_\perp^T z_0 + \alpha_\perp^T \sum_{s=1}^{t} u_s,$$

where we have used that $\alpha_\perp^T c = \alpha_\perp^T c = 0$ by $\text{CVAR.3}$. Hence

$$z_t = [P_{\beta_\perp}(y_t) + P_{\alpha}(y_t)]z_t = P_{\beta_\perp}(y_t) \left( z_0 + \sum_{s=1}^{t} u_s \right) + \alpha(\beta(y_t)^T \alpha)^{-1} \xi_t. \quad \text{(A.16)}$$

In view of (A.15) and Lemma A.2, the upper left $p \times p$ block of $P_{\beta_\perp}(y)$ is $P_{\beta_\perp}(y)$, where the
latter is as defined in (4.6). Under CO(ii), \( n^{-1/2}z_t \xrightarrow{P} \mathcal{Z}_0 \) for all \( t \in \{-k+1, \ldots, 0\} \), and so

\[
\begin{aligned}
n^{-1/2}X_{i1}^\top \mathbf{z}_0 &= n^{-1/2}X_{i1}^\top \left[ \mathbf{z}_0 - \sum_{i=1}^{k-1} \Gamma_i z_{-i}^* \right] \\
&\xrightarrow{P} X_{i1}^\top \Gamma(1; \mathcal{Y}_0) \mathcal{Z}_0.
\end{aligned}
\]

Hence, using the result of part (i), the first \( p \) rows of (A.16) give

\[
\begin{aligned}
n^{-1/2} \begin{bmatrix} y_t \\ x_t \end{bmatrix} &= P_{\beta_1}(y_t) \left[ \Gamma(1; \mathcal{Y}_0) \mathcal{Z}_0 + n^{-1/2} \sum_{s=1}^{t} u_s \right] + o_p(1)
\end{aligned}
\]

uniformly in \( t \in \{1, \ldots, n\} \), and so for \( \lambda \in [0, 1] \),

\[
\begin{bmatrix} Y_n(\lambda) \\ X_n(\lambda) \end{bmatrix} = P_{\beta_1}[Y_n(\lambda)]U_{n,0}(\lambda) + \sigma_p(1),
\]

(A.17)

where \( U_{n,0}(\lambda) := \Gamma(1; \mathcal{Y}_0) \mathcal{Z}_0 + U_n(\lambda) \), \( \sigma_p(1) \) denotes a term that converges in probability to zero, uniformly over \( \lambda \in [0, 1] \), and we have used the fact that \( P_{\beta_1}(y) \) depends only on the sign of \( y \).

It remains to determine the weak limit of the r.h.s. of (A.17). To that end, we provide an auxiliary result on the mapping \( g(y, u) := P_{\beta_1}(y)u \) that appears there.

**Lemma A.5.** Suppose \( \text{cvvar} \) and \( \text{CO(ii)} \) hold. Then

(i) taking \( \vartheta^\top := e_1^T P_{\beta_1}(+1) \neq 0 \), there exists a \( \mu > 0 \) such that

\[
e_1^T P_{\beta_1}(y) = [1^+(y) + \mu 1^-(y)]\vartheta^\top := h(y)\vartheta^\top
\]

(ii) \( g \) is Lipschitz continuous at every point in \( D_g := \{(y, u) \in \mathbb{R}^{p+1} \mid y = h(y)\vartheta^\top u\} \), in the sense that for every \( (y, u) \in D_g \) and \( (y', u') \in \mathbb{R} \times \mathbb{R}^p \),

\[
|g(y, u) - g(y', u')| \leq C(|y - y'| + \|u - u'\|).
\]

(A.18)

By Lemma A.5, \( e_1^T P_{\beta_1}(y) = h(y)\vartheta^\top \). Hence we may rewrite the first equation of (A.17) as

\[
Y_n(\lambda) = h[Y_n(\lambda)]\vartheta^\top U_{n,0}(\lambda) + \sigma_p(1).
\]

Now let \( f(y) := h(y)^{-1}y \). This is Lipschitz, and because \( h(y) \) is bounded away from zero and infinity, it has an inverse \( f^{-1}(y^*) = h(y^*)y^* \) that is also Lipschitz. This implies

\[
Y_n^*(\lambda) := f[Y_n(\lambda)] = \vartheta^T U_{n,0}(\lambda) + h[Y_n(\lambda)]^{-1}\sigma_p(1) \sim \vartheta^T U_0(\lambda)
\]

on \( D[0, 1] \), since \( \sup_{\lambda \in [0, 1]} h[Y_n(\lambda)]^{-1} \leq \max\{1, \mu^{-1}\} \). Hence, by the CMT,

\[
Y_n(\lambda) = f^{-1}[Y_n^*(\lambda)] \sim f^{-1}[\vartheta^T U_0(\lambda)]
\]

\[
= h[\vartheta^T U_0(\lambda)]\vartheta^T U_0(\lambda) = e_1^T P_{\beta_1}[\vartheta^T U_0(\lambda)]U_0(\lambda) =: Y(\lambda)
\]

(A.19)

on \( D[0, 1] \). Now consider the remaining \( p - 1 \) equations in (A.17). Recalling the definition of \( g \)
that precedes Lemma A.5 above, we have

\[ X_n(\lambda) = E_{-1} g[Y_n(\lambda), U_{n,0}(\lambda)] + \sigma_p(1) \]

where \( E_{-1} \) denotes the final \( p - 1 \) columns of \( I_p \). By (A.19), \((Y_n, U_{n,0}) \rightsquigarrow (Y, U_0)\), which have continuous paths and concentrate in the set

\[ \mathcal{D} := \{(y, u) \in D_{\mathbb{R}^{p+1}}[0, 1] \mid [y(\lambda), u(\lambda)] \in D_g \text{ for all } \lambda \in [0, 1]\} \]

for \( D_g \) as in Lemma A.5. By Lemma A.5, if \((y_n, u_n) \in D_{\mathbb{R}^{p+1}}[0, 1] \) converge uniformly to \((y, u) \in \mathcal{D}\), then \( g[y_n(\lambda), u_n(\lambda)] \to g[y(\lambda), u(\lambda)] \) uniformly over \( \lambda \in [0, 1] \). It follows by the CMT that

\[ X_n(\lambda) \rightsquigarrow E_{-1} g[Y(\lambda), U_0(\lambda)] = E_{-1} P_{\beta_\perp} [Y(\lambda)] U_0(\lambda) = E_{-1} P_{\beta_\perp} [\vartheta^T U_0(\lambda)] U_0(\lambda) \quad (A.20) \]

on \( D_{\mathbb{R}^{p-1}}[0, 1] \). Thus part (ii) of the theorem follows from (A.19) and (A.20). \( \square \)
Online appendices

B Proofs of auxiliary lemmas

Proof of Lemma A.1. Since $\rho_{\text{JSR}}(A) < 1$, it follows by Jungers (2009, Prop. 1.4) that there exists a norm $\|\cdot\|_*$ on $\mathbb{R}^{d_w}$ and a $\gamma \in (0, 1)$ such that
\[
\|w_t\|_* \leq \|c_t\|_* + \|A_t\|_* \|w_{t-1}\|_* + \|B_t\|_* \|\epsilon_t\| \leq \gamma \|w_{t-1}\|_* + C(1 + \|v_t\|_*)
\]
where $C := \max\{\sup_{B \in \mathcal{B}} \|B\|_* , \sup_{c \in \mathcal{C}} \|c\|_*\} < \infty$, by the boundedness of $\mathcal{B}$ and $\mathcal{C}$. Hence by backward substitution,
\[
\|w_t\|_* \leq C \sum_{s=0}^{t-1} \gamma^s (1 + \|v_{t-s}\|_*) + \gamma^t \|w_0\|_*.
\]
By the equivalence of norms on finite-dimensional spaces, there exists a $C' < \infty$ such that
\[
|w_t| \leq C' \left[ \sum_{s=0}^{t-1} \gamma^s (1 + |v_{t-s}|) + \gamma^t |w_0| \right].
\]
Deduce that for any $m \in [1, m_0]$,
\[
\|w_t\|_m \leq C' \left[ \sum_{s=0}^{t-1} \gamma^s (1 + \|v_{t-s}\|_m) + \gamma^t \|w_0\|_m \right] \leq C' \left[ \frac{1}{1 - \gamma} \left( 1 + \max_{1 \leq s \leq t} \|v_s\|_m \right) + \|w_0\|_m \right].
\]
and hence there exists a $C'' < \infty$ such that
\[
\max_{1 \leq t \leq n} \|w_t\|_m \leq C'' \left[ 1 + \max_{1 \leq t \leq n} \|v_t\|_m + \|w_0\|_m \right].
\]

Proof of Lemma A.2. The factorisation of $\Pi$ is immediate from the definition of matrix rank; that of $\Pi$ then follows from direct calculation. Parts (iii) and (iv) will also follow from direct calculation, once we have verified (ii).

It remains to prove part (ii). We first observe that $I_{kp} + \Pi$ gives the companion form matrix associated with the autoregressive polynomial $\Phi(\lambda)$, in the sense that the roots of the latter coincide with the reciprocals of the eigenvalues of the former (e.g. Lütkepohl, 2007, Sec. 2.1.1), of which $q$ lie on the unit circle by assumption, and the remaining $kp - q$ lie strictly inside. It is readily verified that $\alpha_\perp \alpha = 0$ and $\beta_\perp \beta = 0$, when $\alpha_\perp$ and $\beta_\perp$ have the form given in (A.1). Moreover, since $\beta_\perp$ is determined only up to its column span, we are free to normalise it such that $\beta_\perp^T \beta_\perp = k^{-1}I_q$, whence $\beta_\perp^T \beta_\perp = I_q$. Letting $Q := (\beta^T \beta)^{-1/2}$ denote the positive definite square root of $(\beta^T \beta)^{-1}$, we have that $[\beta Q, \beta_\perp]$ is a (full rank) orthogonal matrix. Since
\[
\beta^T (I_{kp} + \Pi) \beta_\perp = \beta^T (I_{kp} + \alpha \beta^T) \beta_\perp = \beta^T \beta_\perp = 0
\]
and, noting $\beta^T \beta Q = \beta^T (\beta^T \beta)^{-1/2} = (\beta^T \beta)^{-1/2} = Q^{-1}$,

$$Q\beta^T (I_{kp} + \Pi) \beta Q = Q\beta^T (I_{kp} + \alpha \beta^T) \beta Q = Q(I_{kp-q} + \beta^T \alpha) \beta Q = Q(I_{kp-q} + \beta^T \alpha) Q^{-1}.$$ 

It follows that

$$\begin{bmatrix} \beta^T \\ Q\beta^T \end{bmatrix} [I_{kp} + \Pi] \begin{bmatrix} \beta \\ \beta Q \end{bmatrix} = \begin{bmatrix} I_q \\ 0 \end{bmatrix} \begin{bmatrix} \beta^T \Pi \beta Q \\ Q(I_{kp-q} + \beta^T \alpha) Q^{-1} \end{bmatrix}.$$

Hence the eigenvalues of $I_{kp-q} + \beta^T \alpha$ are the $kp - q$ eigenvalues of $I_{kp} + \Pi$ that are strictly inside the unit circle. Deduce $\beta^T \alpha$ has full rank, whence so does $[\beta \perp \alpha]$. Since

$$\alpha^T \begin{bmatrix} \beta \perp \alpha \end{bmatrix} = \begin{bmatrix} \alpha^T \beta \perp 0 \end{bmatrix}$$

must then have rank $q$, it follows that the $q \times q$ matrix $\alpha^T \beta \perp = \alpha_\perp \Gamma(1) \beta_\perp$ has full rank.

**Proof of Lemma A.3.** Premultiply (A.3) by $\beta^+ \pi$ and then use (A.2) to obtain

$$\xi_t^+ = \beta_1^+ c + (I + \beta^+ \alpha^+) \xi_{t-1}^+$$

$$+ \beta_1^+ \pi (\phi_i^+ \phi_i^- y_{t-1} + \beta_1^+ \sum_{i=2}^k (\phi_i^+ \phi_i^- y_{t-1+i} + \beta_1^+ \pi u_t.$$ 

The evolution of $y_t^-$ is described by the first equation in (A.3)

$$\Delta y_t = e_1^T c + e_1^T \alpha^+ \xi_{t-1}^- + e_1^T v_t,$$ 

which, using (A.2), can be rearranged as

$$y_t - y_{t-1}^- = e_1^T c + [1 + e_1^T (\phi_i^- \phi_i^+)] y_{t-1}^-$$

$$+ \sum_{i=2}^k e_1^T (\phi_i^- \phi_i^+) y_{t-1+i} + e_1^T \alpha^+ \xi_{t-1}^- + e_1^T u_t.$$ 

It follows that if we define $y_t$ to equal the r.h.s., then $y_t = y_{t-1}^+ + \overline{y}_t$, and so

$$y_t^- = [y_t]^- = [y_{t-1}^+ + \overline{y}_t]^- = \delta_t \overline{y}_t$$

where $\delta_t \in [0, 1]$ is defined in (A.5) above. Making the substitution $y_{t-1}^- = \delta_{t-1} \overline{y}_{t-1}$ on the r.h.s. of (B.1) and (B.2), we see that the trajectories of $\{\xi_t^+\}$ and $\{y_t^\}$ will be reproduced exactly by

$$\xi_t^+ = \beta_1^+ c + (I + \beta^+ \alpha^+) \xi_{t-1}^+$$

$$+ \beta_1^+ \pi (\phi_i^- \phi_i^+ \delta_{t-1} \overline{y}_{t-1} + \beta_1^+ \sum_{i=2}^k (\phi_i^- \phi_i^+) y_{t-1+i} + \beta_1^+ \pi u_t,$$

$$\overline{y}_t = e_1^T c + [1 + e_1^T (\phi_i^- \phi_i^+)] \delta_{t-1} \overline{y}_{t-1} + \sum_{i=2}^k e_1^T (\phi_i^- \phi_i^+) y_{t-1+i} + e_1^T \alpha^+ \xi_{t-1}^- + e_1^T u_t.$$ 

(36)
with \( \delta_0 := 1 \) and \( \overline{y}_0 := \overline{y}_0^- \). Finally, we note that (A.4) provides the companion form representation of this system.

**Proof of Lemma A.4.** Using the definition of \( \xi_t \), we may write (A.12) as

\[
\Delta z_t = c + \alpha \xi_{t-1} + u_t. \tag{B.4}
\]

Let \( \Delta \beta(y_t) := \beta(y_t) - \beta(y_{t-1}) \), and note that \( [\Delta \beta(y_t)]^T z_t = [\Delta \beta(y_t)]_{1:1}^T y_t \), for \( [\Delta \beta(y_t)]_{1:1} \) the first row of \( \Delta \beta(y_t) \), since all its other rows are zero. Thus with the aid of (B.4) we obtain

\[
\xi_t = \beta(y_t)^T z_t = \beta(y_{t-1})^T (z_{t-1} + \Delta z_t) + [\Delta \beta(y_t)]^T z_t
\]

\[
= \xi_{t-1} + \beta(y_{t-1})^T \Delta z_t + [\Delta \beta(y_t)]^T z_t
\]

\[
= \beta(y_{t-1})^T c + [I + \beta(y_{t-1})^T \alpha] \xi_{t-1} + \beta(y_{t-1})^T u_t + [\Delta \beta(y_t)]^T y_t. \tag{B.5}
\]

By construction, with \( \delta_t \) as defined in (A.13), we have that \( y_t = \delta_t \Delta y_t \) whenever \( y_{t-1} \) and \( y_t \) have opposite signs. Since this is also the only case in which \( \Delta \beta(y_t) \neq 0 \), it follows that

\[
[\Delta \beta(y_t)]_{1:1}^T y_t = \delta_t [\Delta \beta(y_t)]_{1:1}^T \Delta y_t = \delta_t [\Delta \beta(y_t)]^T \Delta z_t = \delta_t [\beta(y_t) - \beta(y_{t-1})]^T (c + \alpha \xi_{t-1} + u_t),
\]

where the second equality holds because only the first row of \( \Delta \beta(y_t) \) is nonzero. Substituting this into (B.5) and recalling the definition of \( \overline{B}_t \) thus yields (A.14).

Finally, observe that \( \delta_t \) is nonzero only if \( y_{t-1} \) and \( y_t \) have opposite signs: in which case \( |y_t| \leq |\Delta y_t| \), and so \( |\delta_t| \leq 1 \). Since \( y_t \) and \( \Delta y_t \) must also have the same sign in this case, we deduce that \( \delta_t \in [0,1] \), as claimed.

The proof of Lemma A.5 requires the following auxiliary result.

**Lemma B.1.** Suppose \( c \in \mathbb{R}^m \), \( d, v \in \mathbb{R}^n \), and \( A, B_1, B_2 \in \mathbb{R}^{m \times n} \) have full column rank and are such that \( B_1 - B_2 = cd^T \) and \( \det(A^T B_1) \cdot \det(A^T B_2) > 0 \). Then

(i) there exists a \( \mu > 0 \) such that \( d^T (A^T B_1)^{-1} = \mu d^T (A^T B_2)^{-1} \); and

(ii) if \( d^T (A^T B_1)^{-1} v = 0 \), then \( (A^T B_1)^{-1} v = (A^T B_2)^{-1} v \).

**Proof of Lemma B.1.** Since \( A^T B_1 = A^T B_2 + (A^T c)d^T \), it follows by Cauchy’s formula for a rank-one perturbation (Horn and Johnson, 2013, (0.8.5.11)) that

\[
\det(A^T B_1) = \det(A^T B_2) + d^T (\text{adj} A^T B_2) A^T c
\]

\[
= \det(A^T B_2) \{1 + d^T (A^T B_2)^{-1} A^T c\}
\]

whence \( \chi := 1 + d^T (A^T B_2)^{-1} A^T c = \det(A^T B_2)^{-1} \det(A^T B_1) > 0 \). Hence, by the Sherman–Morrison–Woodbury formula (Horn and Johnson, 2013, (0.7.4.2))

\[
(A^T B_1)^{-1} = (A^T B_2)^{-1} - \frac{(A^T B_2)^{-1} A^T c d^T (A^T B_2)^{-1}}{1 + d^T (A^T B_2)^{-1} A^T c}. \tag{B.6}
\]
Premultiplying the preceding by $d^T$ and rearranging yields

$$d^T(A^T B_1)^{-1} = \left\{ 1 - \frac{d^T(A^T B_2)^{-1} A^T c}{1 + d^T(A^T B_2)^{-1} A^T c} \right\} d^T(A^T B_2)^{-1} = \chi^{-1} d^T(A^T B_2)^{-1}$$

and thus part (i) holds with $\mu = \chi^{-1}$. Finally, let $v \in \mathbb{R}^n$ be such that $d^T(A^T B_1)^{-1} v = 0$. By part (i) of the lemma, we must have that $d^T(A^T B_2)^{-1} v = 0$; hence postmultiplying (B.6) by $v$ immediately yields the result of part (ii). □

**Proof of Lemma A.5. (i).** Recall from (4.5) that $\beta^\pm = [\beta^\pm_v, \beta^\pm_x] = \beta^\pm_T [\theta^\pm, I_{p-1}]$ for some $\theta^\pm \in \mathbb{R}^{p-1}$, and take

$$\beta^\pm_1 = \begin{bmatrix} 1 & 0 \\ -\theta^\pm & \beta_{x,\perp} \end{bmatrix} = \beta_{\perp}(\pm 1) \quad (B.7)$$

as per (4.7), so that $e_1^T \beta^\pm_1 = e_1^T$; it follows that $e_1^T P_{\beta_\perp} (+1) \neq 0$. Since

$$\Gamma^\pm(1) \beta^\pm_1 = \begin{bmatrix} \gamma^\pm(1) & \Gamma^x(1) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\theta^\pm & \beta_{x,\perp} \end{bmatrix} = \begin{bmatrix} \gamma^\pm(1) - \Gamma^x(1) \theta^\pm & \Gamma^x(1) \beta_{x,\perp} \end{bmatrix}$$

we have

$$\Gamma^-(1) \beta_{\perp} - \Gamma^+(1) \beta^+_1 = \delta_\beta e_1^T,$$

where $\delta_\gamma := [\gamma^-(1) - \gamma^+(1) - \Gamma^x(1)(\theta^- - \theta^+)]$. Under CO(ii).3, we may apply Lemma B.1 with $A = \alpha^\perp_1$, $B_1 = \Gamma^{-1}(1) \beta_{\perp}$, $B_2 = \Gamma^{+1}(1) \beta^+_1$ and $d = e_1$, to obtain

$$e_1^T [\alpha^\perp_1 \Gamma^-(1) \beta_{\perp}]^{-1} = \mu e_1^T [\alpha^\perp_1 \Gamma^+(1) \beta^+_1]^{-1}$$

for some $\mu > 0$. Hence, for $\vartheta^T = e_1^T P_{\beta_\perp} (+1) = e_1^T [\alpha^\perp_1 \Gamma^+(1) \beta^+_1]^{-1} \alpha^\perp_1$

$$e_1^T P_{\beta_\perp} (y) = e_1^T \beta_{\perp}(y) [\alpha^\perp_1 \Gamma(1;y) \beta_{\perp}(y)]^{-1} \alpha^\perp_1 = e_1^T [\alpha^\perp_1 \Gamma(1;y) \beta_{\perp}(y)]^{-1} \alpha^\perp_1 = [1^+(y) + \mu 1^-(y)] \vartheta^T = h(y) \vartheta^T$$

(ii). We first show that

$$\vartheta^T u = 0 \implies [P_{\beta_\perp} (+1) - P_{\beta_\perp} (-1)] u = 0. \quad (B.8)$$

To this end, let $u \in \mathbb{R}^p$ be such that $\vartheta^T u = 0$. Then

$$0 = \mu \vartheta^T u = e_1^T [\alpha^\perp_1 \Gamma^-(1) \beta_{\perp}]^{-1} \alpha^\perp_1 u = d^T(A^T B_1)^{-1} v$$

where $v := \alpha^\perp_1 u$, and $A, B_1, B_2$ and $d$ are as above. Hence by Lemma B.1,

$$0 = [(A^T B_2)^{-1} - (A^T B_1)^{-1}] v = [(\alpha^\perp_1 \Gamma^+(1) \beta^+_1)^{-1} - (\alpha^\perp_1 \Gamma^-(1) \beta_{\perp})^{-1}] \alpha^\perp_1 u.$$

Finally, noting that (B.7) implies $\beta^+_1 - \beta_{\perp} = \delta_\beta e_1^T$ for $\delta_\beta := (0, (\theta^- - \theta^+)^T)^T$, we have

$$[P_{\beta_\perp} (+1) - P_{\beta_\perp} (-1)] u = \beta^+_1 [(\alpha^\perp_1 \Gamma^+(1) \beta^+_1)^{-1} - (\alpha^\perp_1 \Gamma^-(1) \beta_{\perp})^{-1}] \alpha^\perp_1 u$$

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\[ -(\beta_\perp - \beta_\perp^\top)(\alpha_\perp^\top \Gamma^- (1) \beta_\perp^-)^{-1} \alpha_\perp u \]
\[ = \delta \gamma e_1^\top (\alpha_\perp^\top \Gamma^- (1) \beta_\perp^-)^{-1} \alpha_\perp u = (1) \delta \gamma e_1^\top P\beta_{\perp}^- (-1) u = (2) \mu \delta \beta^\top u = (3) 0, \]

where \(=_{(1)}\) holds by \(e_1^\top \beta_\perp^- = e_1^\top\), \(=_{(2)}\) by the result of part \((i)\), and \(=_{(3)}\) by our choice of \(u\). Thus (B.8) holds.

Now let \((y, u) \in D_g\) and \((y', u') \in \mathbb{R} \times \mathbb{R}^p\). Suppose first that \(h(y) = h(y')\): then
\[ \|g(y, u) - g(y', u')\| \leq \max\{\|P\beta_{\perp}^\top (1)\|, \|P\beta_{\perp}^\top (-1)\|\}\|u - u'\| \leq C\|u - u'\|. \]

Suppose next that \(h(y) \neq h(y')\); then either: \((a)\) \(y \geq 0\) and \(y' < 0\); or \((b)\) \(y < 0\) and \(y' \geq 0\). Suppose \((a)\) holds. Then \(e^\top T u \geq 0\), and so
\[ |y - y'| = y - y' = h(y)\theta^\top u - y' \geq h(y)\theta^\top u \geq 0 \]
whence
\[ |\theta^\top u| \leq C_0|y - y'| \quad (B.9) \]
where \(C_0 := \max\{1, \mu^{-1}\} \in (0, \infty)\). (If \((b)\) holds instead, then (B.9) follows by an analogous argument.) Now let \(u_{\theta_{\perp}} := u - \theta (\theta^\top \theta)^{-1} \theta^\top u\). Then \(\theta^\top u_{\theta_{\perp}} = 0\), and so
\[ g(y, u) - g(y', u') = P\beta_{\perp}^\top (1)(u - u_{\theta_{\perp}} + u_{\theta_{\perp}}) - P\beta_{\perp}^\top (-1)u' \]
\[ = P\beta_{\perp}^\top (1)(u - u_{\theta_{\perp}}) + P\beta_{\perp}^\top (-1)(u_{\theta_{\perp}} - u'), \quad (B.10) \]
where the second equality follows by (B.8). By (B.9),
\[ \|u - u_{\theta_{\perp}}\| \leq \|\theta (\theta^\top \theta)^{-1}\| |\theta^\top u| \leq C_1|y - y'| \quad (B.11) \]
where \(C_1 := \|\theta (\theta^\top \theta)^{-1}\|C_0\). Further,
\[ \|u_{\theta_{\perp}} - u'\| \leq \|u_{\theta_{\perp}} - u\| + \|u' - u\| \leq C_1|y - y'| + \|u' - u\|. \quad (B.12) \]
Thus (A.18) follows from (B.10)--(B.12)

\[\square\]

C Verification of Remarks 4.1(iii) and 4.1(v)

We first note the following corollary to Lemma A.2. We say a VAR is stationary if all its autoregressive roots lie outside the unit circle (i.e. irrespective of whether the series is given a stationary initialisation).

Lemma C.1. Suppose that the assumptions of Lemma A.2 hold, and \(\{z_t\}\) is generated according to
\[ z_t = c + \Phi(L)z_{t-1} + u_t. \]
Let \(\xi_t := \beta^\top z_t\). Then \(\xi_t := (\xi_t^\top, \Delta z_t^\top, \ldots, \Delta z_t^\top_{(k-2)})^\top\) follows the stationary VAR given by
\[ \xi_t = \beta_{1:p}^\top c + (I_{p(k-1)+r} + \beta^\top \alpha)\xi_{t-1} + \beta_{1:p}^\top u_t. \]
Proof. The claim follows from part (ii) of Lemma A.2, and premultiplying the companion form \( \Delta z_t = c + \Pi z_{t-1} + u_t \) by \( \beta^T \), where \( z_t = (z^T_t, z^T_{t-1}, \ldots, z^T_{t-k+1})^T \), \( c = (c^T, 0^T_{p(k-1)})^T \) and \( u_t = (u^T_t, 0^T_{p(k-1)})^T \).

Proof of Remark 4.1(iii). Rather than working with the matrices \( F_0 \) and \( F_1 \) directly, it is easier if we consider the autoregressive systems that they correspond to, as given in (B.3), recognising that those systems are dynamically stable if and only if the eigenvalues of their companion form matrices lie strictly inside the unit circle (e.g. Lütkepohl, 2007, Sec. 2.1.1).

Let \( t \in \mathbb{N} \) be given. We first consider \( F_0 \): this is operative if \( \delta_s = 0 \) for all \( s \in \{t-k, \ldots, t-1\} \). In this case, \( y_s = y_{s-1} < 0 \) for all such \( s \), and (B.3) reduces to

\[
\xi^+_t = \beta^+_{1:p} c + (I + \beta^+ T \alpha^+) \xi^+_{t-1} + \beta^+_{1:p} u_t,
\]

By Lemma A.2, the eigenvalues of \( I + \beta^+ T \alpha^+ \) lie inside the unit circle; hence so too do those of \( F_0 \). (Indeed, it may be verified that the only nonzero eigenvalues of \( F_0 \) are those of \( I + \beta^+ T \alpha^+ \).)

We next consider \( F_1 \), which is operative if \( \delta_s = 1 \) for all \( s \in \{t-k, \ldots, t-1\} \). Then \( y_s = y_{s-1} < 0 \) for all such \( s \), and (B.3) becomes

\[
\xi^+_t = \beta^+_{1:p} c + (I + \beta^+ T \alpha^+) \xi^+_{t-1} + \beta^+_{1:p} \sum_{i=2}^k (\phi^-_i - \phi^+_i) y_{t-i} + \beta^+_{1:p} u_t,
\]

\[
y_t = e^+_1 c + e^+_1 T \alpha^+ \xi^+_{t-1} + [1 + e^+_1 T (\phi^-_1 - \phi^+_1)] y_{t-1} + \sum_{i=2}^k e^+_1 T (\phi^-_i - \phi^+_i) y_{t-i} + e^+_1 T u_t.
\]

Now the preceding must agree with the \((\xi^+_t, y_t)\) generated by the VAR when \( y_s < 0 \) for all \( s \in \{t-k, \ldots, t-1\} \), i.e. generated according to

\[
z_t = \Phi^- (L) z_{t-1} + u_t.
\]

Under CVAR and CO(i).1, the preceding is a linear cointegrated VAR with

\[
\Pi^- = \begin{bmatrix} \pi^- & \Pi^x \end{bmatrix} = \Pi^+ + \begin{bmatrix} \pi^- - \pi^+ & 0 \end{bmatrix} = \begin{bmatrix} \alpha^+ & 0 \\ 0 & \pi^- - \pi^+ \end{bmatrix} \begin{bmatrix} \beta^+ T \\ \frac{e^+_1 T}{e^+_1} \end{bmatrix} =: \alpha^- \beta^{-T},
\]

and equilibrium errors \( \xi^-_t := \beta^- T z_t = [\beta^+ T z_t ] = [\xi^+_t]_y \). It follows from Lemma C.1 that

\[
\xi^-_t := (\xi^+_t, \Delta z^T_t, \ldots, \Delta z^T_{t-(k-2)})^T = (\xi^+_t, y_t, \Delta z^T_t, \ldots, \Delta z^T_{t-(k-2)})^T
\]

evolves according to a stationary VAR. Since \((\xi^+_t, y_t)\) is simply a reordering of \(\xi^-_t\), it follows that the system described by (C.1), i.e. by \( F_1 \), must also be stationary. Hence \( F_1 \) must have all its eigenvalues strictly inside the unit circle.

Proof of Remark 4.1(v). Suppose \( k = 1 \). In this case, \( \pi^- - \pi^+ = \phi^-_1 - \phi^+_1 \), and so by Re-
mark 4.1(iii), the matrix
\[
F_1 = \begin{bmatrix}
I_r + \beta^+^T \alpha^+ & \beta^+^T (\pi^- - \pi^+) \\
-\alpha^+ (\beta^+^T \alpha^+) & 1 + \alpha^+ (\pi^- - \pi^+)
\end{bmatrix}
\]
must have all its eigenvalues inside the unit circle; hence \(\det(I_{r+1} - F_1) = \prod_{i=1}^{r+1} (1 - \rho_i(F_1)) > 0\), where \(\rho_i(F_1)\) denotes the \(i\)th eigenvalue of \(F_1\). Note that
\[
e_1^T (\pi^- - \pi^+) - e_1^T \alpha^+ (\beta^+^T \alpha^+)^{-1} \beta^+^T (\pi^- - \pi^+) = e_1^T [I_r - \alpha^+ (\beta^+^T \alpha^+)^{-1} \beta^+^T] (\pi^- - \pi^+)
\]
\[
= e_1^T \beta^+_1 (\alpha^+_1 \beta^+_1)^{-1} \alpha^+_1 (\pi^- - \pi^+)
\]
\[
= e_1^T \rho^+_1 \pi^- = \kappa_1,
\]
where the penultimate equality holds since \(\alpha^+_1 \pi^- = 0\), and \(\Gamma^+(1) = I_p\) when \(k = 1\). Hence
\[
\begin{pmatrix}
I_r \\
-e_1^T \alpha^+ (\beta^+^T \alpha^+)^{-1} 1
\end{pmatrix} (F_1 - I_{r+1}) = \begin{bmatrix}
\beta^+^T \alpha^+ & \beta^+^T (\pi^- - \pi^+) \\
0 & \kappa_1
\end{bmatrix},
\]
from which it follows that
\[
(-1)^{r+1} \det(I_{r+1} - F_1) = \det(F_1 - I_{r+1}) = \kappa_1 \det(\beta^+^T \alpha^+) = \kappa_1 (-1)^r \det[I_r - (I_r + \beta^+^T \alpha^+)].
\]
By Lemma A.2, \(I_r + \beta^+^T \alpha^+\) has all its eigenvalues inside the unit circle, and so \(\det[I_r - (I_r + \beta^+^T \alpha^+)] > 0\). Hence \(\kappa_1 < 0\).

Next suppose \(p = 1\), while allowing for general \(k \in \mathbb{N}\). Then
\[
\kappa_1 = \kappa = P_{\beta^+_1} \pi^- = -\gamma^+(1)^{-1} \phi^-(1)
\]
where \(\gamma^+(1) = 1 - \sum_{i=1}^{k-1} \gamma^+_i\) and \(\phi^-(1) = 1 - \sum_{i=1}^k \phi^-_i\). Both \(\gamma^+(\lambda)\) and \(\phi^-(\lambda)\) have all their roots outside the unit circle, and hence both \(\gamma^+(1)\) and \(\phi^-(1)\) are strictly positive. It follows that \(\kappa_1 < 0\) as required.

D Transposition to the structural form

Here we verify the claims made in Remarks 4.1(i) and 4.2(i). We suppose that DGP holds, so that \((y_t, x_t)\) are generated by the CKSVAR
\[
\phi^+(L) y_t^+ + \phi^-(L) y_t^- + \Phi^x(L) x_t = c + u_t.
\]
By Proposition 2.1, if \((\tilde{y}_t, \tilde{x}_t)\) are constructed from \((y_t, x_t)\) via (2.13), then \((\tilde{y}_t, \tilde{x}_t)\) follow a canonical CKSVAR, which we denote here as
\[
\tilde{\phi}^+(L) y_t^+ + \tilde{\phi}^-(L) y_t^- + \tilde{\Phi}^x(L) \tilde{x}_t = \tilde{c} + \tilde{u}_t.
\]
We use tildes to distinguish the series and parameters of the canonical form (D.1), from the original CKSVAR (D.2) from which they were derived; the mapping between these is given in
and

\begin{equation}
(L_{\tau})^{(2.13)-(2.15)}.
\end{equation}

Our approach here is as follows. We first show that if DGP, ERR, CVAR and a suitably modified form of CO(i) (resp. CO(ii)) hold for \((y_t, x_t)\), then DGP*, ERR, CVAR and CO(i) (resp. CO(ii)) hold for \((\tilde{y}_t, \tilde{x}_t)\). Theorem 4.1 (resp. 4.2) therefore applies to \((\tilde{y}_t, \tilde{x}_t)\): its implications for \((y_t, x_t)\) are then derived by inverting the mapping \(2.13)-(2.15)\).

Preliminary to these arguments, we define

\[
(P^\pm)^{-1} := \begin{bmatrix}
\phi_{0,yy}^\pm & 0 \\
\phi_{0,xx}^\pm & \Phi_{0,xx}
\end{bmatrix},
\tag{D.3}
\]

and restate Proposition 5.1 from Duffy, Mavroeidis, and Wycherley (2023).

**Proposition D.1.** Suppose DGP holds. Then

\begin{enumerate}
\item \(\hat{\Phi}^\pm(\lambda) = Q\Phi^\pm(\lambda)P^\pm\);
\item the roots of \(\det \Phi^\pm(\lambda)\) and \(\det \Phi^\mp(\lambda)\) are identical, for \(i \in \{+, -\}\);
\item \(\Pi^\pm = Q^{-1}\Pi^\mp(P^\pm)^{-1}\) and \(\Pi^\mp = Q^{-1}\Pi^\mp \Phi_{0,xx}\); and
\item \(c \in \text{sp} \Pi^+ \cap \text{sp} \Pi^-\) if and only if \(\tilde{c} = Qc \in \text{sp} \Pi^+ \cap \Pi^-\).
\end{enumerate}

Now suppose \((y_t, x_t)\) satisfies DGP, ERR and CVAR. By Proposition 2.1, \((\tilde{y}_t, \tilde{x}_t)\) satisfies DGP* and ERR. It follows from Proposition D.1 that \((\tilde{y}_t, \tilde{x}_t)\) satisfies CVAR. We also note that, as a consequence of that same result,

\[
\tilde{\alpha} \tilde{\beta}^T = \Pi \mp = Q\Pi^\pm P^\pm = Q\alpha \beta^T P^\pm
\]

so that \(\tilde{\alpha} = Q\alpha, \tilde{\beta}^\mp = (P^\pm)^T \beta^\pm, \alpha_\perp = (Q^{-1})\alpha_\perp, \beta_\perp = (P^\pm)^{-1} \beta_\perp\). Further, \(\tilde{\Gamma}^\pm(\lambda) = Q\Gamma^\pm(\lambda)P^\pm\), and so

\[
\tilde{P}_{\beta_\perp} = \beta_\perp^\mp \tilde{\Gamma}^\pm(1) \beta_\perp^\pm [-1]^{-1} \alpha_\perp^T = (P^\pm)^{-1} \beta_\perp^\mp [\alpha_\perp^T \Gamma^\pm(1) \beta_\perp^\pm [-1]^{-1} \alpha_\perp^T Q^{-1}.
\]

\[
= (P^\pm)^{-1} P_{\beta_\perp} Q^{-1}.
\tag{D.4}
\]

We next consider cases (i) and (ii) in turn (for case (iii), see Duffy et al., 2023, Sec. 5).

**i.** Suppose \((y_t, x_t)\) additionally satisfies CO(i), except with CO(i.2) replaced by CO(i.2'). We must verify that \((\tilde{y}_t, \tilde{x}_t)\) satisfies CO(i) by Proposition D.1(iii); CO(i.2) since \((y_t, x_t)\) satisfying CO(i.2'); and CO(i.4) via Proposition 2.1. It remains therefore to verify CO(i.3). We have via Proposition D.1(iii) and (D.4) that

\[
\tilde{\kappa} = \tilde{P}_{\beta_\perp} \tilde{\pi}^- = \tilde{P}_{\beta_\perp} \Pi^- e_1 = (P^\pm)^{-1} P_{\beta_\perp} \Pi^- P^\pm e_1,
\]

where

\[
\Pi^- P^\pm e_1 = \begin{bmatrix}
\pi^- & \Pi^\pm
\end{bmatrix}
\begin{bmatrix}
(\tilde{\phi}_{0,yy})^{-1} & 0 \\
-\Phi_{0,xx}^{-1} \phi_{0,xy}(\tilde{\phi}_{0,yy})^{-1} & \Phi_{0,x}^{-1}
\end{bmatrix}
\begin{bmatrix}
1 \\
0
\end{bmatrix}
= (\tilde{\phi}_{0,yy})^{-1} [\pi^- - \Pi^\pm \Phi_{0,x}^{-1} \phi_{0,xy}],
\]

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and hence, using that $P_{\beta_+}^\top \Pi^c = 0$,
\[ \tilde{k} = (\tilde{\phi}_{y,yy}^-)^{-1}(P^+)^{-1}P_{\beta_+}^\top [\pi^- - \Pi^c \Phi_{y,xx}^- \phi_{y,xy}] = (\tilde{\phi}_{y,yy}^-)^{-1}(P^+)^{-1}\kappa \] (D.5)
where $\kappa = P_{\beta_+}^\top \pi^-$. Since $\epsilon_1^T(P^+)^{-1} = \tilde{\phi}_{y,yy}^- \epsilon_1^T$, it follows that
\[ \tilde{k}_1 = \epsilon_1^T \kappa = (\tilde{\phi}_{y,yy}^-)^{-1} \tilde{\phi}_{y,yy}^- \kappa_1, \] (D.6)
and so $\text{sgn} \tilde{k}_1 = \text{sgn} \kappa_1$, since $\tilde{\phi}_{y,yy}^- > 0$ by DGP.3. Thus $(\tilde{y}_t, \tilde{x}_t)$ satisfies CO(ii).3, since $(y_t, x_t)$ does.

It follows that Theorem 4.1 applies to $(\tilde{y}_t, \tilde{x}_t)$. Regarding the conclusions of that theorem, note that in this case $\tilde{\gamma}_0$ and $\tilde{\gamma}_0$ must be non-negative. Hence $\tilde{Z}_0 = (P^+)^{-1} \tilde{Z}_0$ by (2.13), and so
\[ \tilde{U}_0(\lambda) = \tilde{T}^+(1) \tilde{Z}_0 + \tilde{U}(\lambda) = Q[\Gamma^+(1) \tilde{Z}_0 + U(\lambda)] = QU_0(\lambda) \]
whence $P_{\beta_+}^\top \tilde{U}_0(\lambda) = (P^+)^{-1}P_{\beta_+} U_0(\lambda)$. Since $\tilde{Y} = \tilde{Y}^+$ in this case, we have
\[ Z(\lambda) = P^+ \tilde{Z}(\lambda) = P^+ \left\{ P_{\beta_+} U_0(\lambda) + \tilde{k}_1^{-1} \kappa \sup_{\lambda' \leq \lambda} [-e_1^T P_{\beta_+} U_0(\lambda)]_+ \right\} \]
\[ = (1) P_{\beta_+} U_0(\lambda) + (\tilde{\phi}_{y,yy}^-)^{-1} \tilde{k}_1^{-1} \kappa \sup_{\lambda' \leq \lambda} [-e_1^T (P^+)^{-1} P_{\beta_+} U_0(\lambda)]_+ \]
\[ = (2) P_{\beta_+} U_0(\lambda) + \kappa_1^{-1} \kappa \sup_{\lambda' \leq \lambda} [-e_1^T P_{\beta_+} U_0(\lambda)]_+, \] (D.7)
where $= (1)$ follows by (D.5), and $= (2)$ by (D.6) and $e_1^T (P^+)^{-1} = \tilde{\phi}_{y,yy}^- \epsilon_1^T$. It follows immediately that $\beta^+ z_t$ and $y_t^-$ are $I^c(0)$. Finally
\[ \begin{bmatrix} \Delta y_t^+ \\ \Delta y_t^- \\ \Delta x_t \end{bmatrix} = P \begin{bmatrix} \Delta \tilde{y}_t^+ \\ \Delta \tilde{y}_t^- \\ \Delta \tilde{x}_t \end{bmatrix} \sim I^c(0) \] (D.8)
implies that $\Delta z_t \sim I^c(0)$. Thus the conclusions of the theorem hold also for $(y_t, x_t)$, exactly as stated.

(ii). Suppose $(y_t, x_t)$ additionally satisfies CO(ii), except with CO(ii).2 replaced by CO(ii).2'. Analogously to case (i), $(\tilde{y}_t, \tilde{x}_t)$ satisfies CO(ii).1 by Proposition D.1(iii); CO(ii).2 since $(y_t, x_t)$ satisfies CO(ii).2'; and CO(ii).4 via Proposition 2.1. Regarding CO(ii).3, we have similarly to (D.4) that
\[ \tilde{\alpha}^T \tilde{\Gamma}(1; \pm 1) \tilde{\beta}(\pm 1) = \alpha^T \Gamma(1; \pm 1) \beta(\pm 1) \]
so that $(\tilde{y}_t, \tilde{x}_t)$ satisfies CO(ii).3, since $(y_t, x_t)$ does.

Hence Theorem 4.2 applies to $(\tilde{y}_t, \tilde{x}_t)$. Regarding the conclusions of that theorem, defining
\[ P(y) := P^+ 1^+(y) + P^- 1^-(y), \]
it follows from (D.4) and the fact that $\text{sgn} y_t = \text{sgn} \tilde{y}_t$, $\text{sgn} \tilde{\gamma}_0 = \text{sgn} \tilde{\gamma}_0$, and
\[ \tilde{U}_0(\lambda) = \tilde{T}^+(1) \tilde{Z}_0 + \tilde{U}(\lambda) = Q[\Gamma^+(1) \tilde{\gamma}_0 \tilde{Z}_0 + U(\lambda)], \]
that

\[ Z(\lambda) = P[\tilde{Y}(\lambda)] \tilde{Z}(\lambda) = P[\tilde{Y}(\lambda)] P_{\beta_\perp} [\tilde{Y}(\lambda)] U_0(\lambda) = P_{\beta_\perp} [Y(\lambda)] U_0(\lambda). \]

Further,

\[ \beta(y_t)^T z_t = \beta(y_t)^T \tilde{z}_t = \beta(y_t)^T P(\tilde{y}_t) \tilde{z}_t = \tilde{\beta}(y_t)^T \tilde{z}_t \sim I^*(0), \]

and that \( \Delta z_t \sim I^*(0) \) follows as in (D.8). Deduce that the conclusions of the theorem hold also for \((y_t, x_t)\).

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