FULFILLMENT OF THE STRONG BOOTSTRAP CONDITION

A. PAPA
Dipartimento di Fisica, Università della Calabria,
and INFN, Gruppo collegato di Cosenza
87036 Arcavacata di Rende, Cosenza, Italy

Abstract. The self-consistency of the assumption of Reggeized form of
the production amplitudes in multi-Regge kinematics, which are used in
the derivation of the BFKL equation, leads to strong bootstrap conditions.
The fulfillment of these conditions opens the way to a rigorous proof of the
BFKL equation in the next-to-leading approximation. The strong bootstrap
condition for the kernel of the BFKL equation for the octet color state of
two Reggeized gluons is one of these conditions. We show that it is satisfied
in the next-to-leading approximation.

1. Gluon Reggeization and bootstrap conditions

The property of gluon Reggeization plays an essential role in the derivation
of the Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation [1] for the cross
sections at high energy \( \sqrt{s} \) in perturbative QCD. The simplest realization
of the gluon Reggeization is in the elastic process \( A + B \rightarrow A' + B' \) with
exchange of gluon quantum numbers in the \( t \)-channel, whose amplitude in
the Regge limit (i.e. for \( s \rightarrow \infty \) and \( |t| \) not growing with \( s \)) takes the form

\[
\left( A_8^c \right)_{AB}^{A'B'} = \Gamma_{A'A}^c \left[ \left( \frac{-s}{-t} \right)^{j(t)} - \left( \frac{s}{-t} \right)^{j(t)} \right] \Gamma_{B'B}^c .
\]

Here \( c \) is a color index, \( \Gamma_{PPr}^c \) are the particle-particle-Reggeon (PPR) ver-
tices, not depending on \( s \), and \( j(t) = 1 + \omega(t) \) is the Reggeized gluon trajectory.

In the leading logarithmic approximation (LLA), which means resum-
mation of the terms of the form \( \alpha_s^n (\ln s)^n \), this form of the amplitude has
been proved [1]. In the next-to-leading approximation (NLA), which means
resummation of the terms $\alpha_s^{n+1}(\ln s)^n$, the form (1) has been checked in the first three orders of perturbation theory and is only assumed to be valid to all orders.

On the other hand, the amplitude for the elastic scattering process $A + B \rightarrow A' + B'$ (for any color representation in the $t$-channel resulting from the composition of two color octet representations) can be determined from $s$-channel unitarity, by expressing its imaginary part in terms of the inelastic amplitudes $A + B \rightarrow \tilde{A} + \tilde{B} + \{n\}$ and $A' + B' \rightarrow \tilde{A} + \tilde{B} + \{n\}$, and then by reconstructing the full amplitude by use of the dispersion relations. It turns out (see for instance [1, 2]) that this amplitude can be written as

$$
(A_R)_{AB}^{A'B'} = \frac{is}{(2\pi)^{D-1}} \int \frac{d^{D-2}q_1}{q_1^2 q_1^2} \int \frac{d^{D-2}q_2}{q_2^2 q_2^2} \int_{\delta-i\infty}^{\delta+i\infty} \frac{d\omega}{\sin(\pi\omega)} \left[ \left( -\frac{s}{s_0} \right)^{\omega} \right.
$$

$$
- \tau \left( \frac{s}{s_0} \right)^{\omega} \sum_{\mathcal{R},\nu} \Phi^{(\mathcal{R},\nu)}_{A\mathcal{R}} (q_1, q_2; s_0) G^{(\mathcal{R})}_\omega (q_1, q_2, q) \Phi^{(\mathcal{R},\nu)}_{B'\mathcal{R}} (-q_2, -q; s_0),
$$

where $D = 4 + 2\epsilon$ is the space-time dimension, $A_R$ stands for the scattering amplitude with the representation $\mathcal{R}$ of the color group in the $t$-channel, the index $\nu$ enumerates the states in the irreducible representation $\mathcal{R}$ and $G^{(\mathcal{R})}_\omega$ is the Mellin transform of the Green function for the Reggeon-Reggeon scattering. The signature $\tau$ is positive (negative) for symmetric (antisymmetric) representation $\mathcal{R}$. The parameter $s_0$ is an arbitrary energy scale introduced in order to define the partial wave expansion of the scattering amplitudes through the Mellin transform. The dependence on this parameter disappears in the full expressions for the amplitudes. $\Phi^{(\mathcal{R},\nu)}_{B'\mathcal{R}}$ are the so-called impact factors.

The Green function obeys the generalized BFKL equation

$$
\omega G^{(\mathcal{R})}_\omega (q_1, q_2; q) = q_1^2 q_1^2 \delta^{(D-2)} (q_1 - q_2)
$$

$$
+ \int \frac{d^{D-2}q'}{q_1^2 q_2^2 q'} K^{(\mathcal{R})}(q_1, q'; q) G^{(\mathcal{R})}_\omega (q', q_2; q)
$$

where we have introduced the notation $q_i = q - q_i$. The kernel $K^{(\mathcal{R})}$ consists of two parts, a “virtual” part, related with the Reggeized gluon trajectory, and a “real” part, related to particle production:

$$
K^{(\mathcal{R})}(q_1, q_2; q) = \left[ \omega (-q_1^2) + \omega (-q_1'^2) \right] q_1^2 q_1'^2 \delta^{(D-2)} (q_1 - q_2)
$$

$$
+ K^{(\mathcal{R})}_r (q_1, q_2; q).
$$

In the LLA, the Reggeized gluon trajectory is needed at 1-loop accuracy and the only contribution to the “real” part of the kernel is from the production of one gluon at Born level in the collision of two Reggeons. In the
NLA, the Reggeized gluon trajectory is needed at 2-loop accuracy and the “real” part of the kernel takes contributions from one-gluon production at 1-loop level and from two-gluon and $q\bar{q}$-pair production at Born level. The representation (2) for the elastic amplitude must reproduce with NLA accuracy the representation (1), in the case of exchange of gluon quantum numbers (i.e. color octet representation and negative signature) in the $t$-channel. This leads to two so-called “soft” bootstrap conditions [2]: the first of them involves the kernel of the generalized non-forward BFKL equation in the octet color representation; the second one involves the impact factors in the octet color representation. Besides providing a stringent check of the gluon Reggeization in the NLA, the soft bootstrap conditions are important since they test, at least in part, the correctness of the year-long calculations which lead to the NLA BFKL equation. The first bootstrap condition has been verified at arbitrary space-time dimension, for the part concerning the quark contribution to the kernel (in massless QCD) [3] and, in the $D \to 4$ limit, for the part concerning the gluon contribution to the kernel [4]. The second bootstrap condition is process-dependent and should therefore be checked for every new impact factor which is calculated\footnote{See, however, Ref. [5] where a general proof for arbitrary process has been sketched.}. It has been verified explicitly at arbitrary space-time dimension for quark and gluon impact factors in QCD with massive quarks [6, 7].

The derivation of the BFKL equation in the NLA involves also the assumption of Reggeized form for production amplitudes in multi-Regge and quasi-multi-Regge kinematics. The compatibility of these amplitudes with $s$-channel unitarity leads to the so-called strong bootstrap conditions [1]. The fulfillment of these conditions opens the way to a proof of the gluon Reggeization in the NLA [1]. Among these conditions, there appear the two ones suggested by Braun and Vacca [8] and derived from the assumption of gluon Reggeization in the (unphysical) particle-Reggeon scattering amplitude with gluon quantum numbers in the $t$-channel [9]. They can be presented in the form (octet color representation understood)

$$\int d^{D-2}q_2^2 \frac{q_1^2 q_2^2}{q_1^2 q_2^2} K(q_1, \bar{q}_2; \bar{q}) R(\bar{q}_2, q) = \omega(t) R(q_1, \bar{q}) \ ,$$

(5)

$$\Phi_{A' A}(\bar{q}_1, q) = -ig\sqrt{N} \Gamma_{A' A}^a(q) R(q_1, \bar{q}) \ .$$

(6)

The last condition fixes the process dependence of the impact factor: it is proportional to the corresponding effective vertex, with a universal coefficient function $R$ [9]. The II strong bootstrap condition has been verified in the case of gluon and quark impact factors in the NLA [9]. Let us consider
the I strong bootstrap condition. Using Eq. (4), it can be written in the form

$$\int \frac{d^{D-2}q}{q_1^2 q_2^2} K_r(\vec{q}_1, \vec{q}_2; \vec{q}) R(\vec{q}_2, \vec{q}) = (\omega(t) - \omega(t_1) - \omega(t'_1)) R(\vec{q}_1, \vec{q}),$$

(7)

where \( t = q^2 = -\bar{q}^2, \ t_i = q_i^2 = -\bar{q}_i^2 \) and \( t'_i = q'_i^2 = -\bar{q}'_i^2 \), with \( i = 1, 2 \). At the leading order, we have:

$$\int \frac{d^{D-2}q}{q_1^2 q_2^2} K_r^{(0)}(\vec{q}_1, \vec{q}_2; \vec{q}) = \omega^{(1)}(t) - \omega^{(1)}(t_1) - \omega^{(1)}(t'_1),$$

(8)

with

$$K_r^{(0)}(\vec{q}_1, \vec{q}_2; \vec{q}) = \frac{g^2 N_c}{2(2\pi)^{D-1}} f_B(\vec{q}_1, \vec{q}_2; \vec{q}),$$

$$f_B(\vec{q}_1, \vec{q}_2; \vec{q}) = \frac{\vec{q}_1^2 \vec{q}_2^2 + \vec{q}_2^2 \vec{q}_1^2}{\vec{k}^2} - \vec{q}_1^2, \quad \vec{k} = \vec{q}_1 - \vec{q}_2$$

and

$$\omega^{(1)}(t) = \frac{g^2 N_c t}{2(2\pi)^{D-1}} \int \frac{d^{D-2}q_1}{q_1^2 q_1'^2} = -g^2 N_c \Gamma(1 - \epsilon) \frac{\Gamma^2(\epsilon)}{\Gamma(2\epsilon)} (\vec{q}_1^2)^\epsilon.$$

It is trivial to see that the leading order bootstrap condition (8) is satisfied. In the NLA (taking into account Eq. (8)), we have instead

$$\int \frac{d^{D-2}q}{q_1^2 q_2^2} \left[ K_r^{(1)}(\vec{q}_1, \vec{q}_2, \vec{q}) + K_r^{(0)}(\vec{q}_1, \vec{q}_2, \vec{q}) \left( R^{(1)}(\vec{q}_2, \vec{q}) - R^{(1)}(\vec{q}_1, \vec{q}) \right) \right] = \omega^{(2)}(t) - \omega^{(2)}(t_1) - \omega^{(2)}(t'_1).$$

(9)

The fulfillment of the above relation for the quark part has been already verified [8, 9], while the fulfillment for the gluon part is much more complicated and is the subject of this paper.

2. Proof of the I strong bootstrap condition for the gluon part of the kernel

The ingredients for the calculation are the following (gluon part is understood everywhere):

$$\omega^{(2)}(t) = \left[ \frac{g^2 N_c (1 - \epsilon) (\vec{q}_1^2)^\epsilon}{(4\pi)^{D/2} \epsilon} \right]^2 \left[ \frac{11}{3} + \left( 2\psi'(1) - \frac{67}{9} \right) \epsilon \right] + \left( \frac{404}{27} + \psi''(1) - \frac{22}{3} \psi'(1) \right) \epsilon^2 + O(\epsilon),$$

(10)
\[ R^{(1)}(\vec{q}_1, \vec{q}) = \frac{\omega^{(1)}(t)}{2} \left[ \epsilon \Gamma(1 + 2\epsilon)(\vec{q}^2)^{1-\epsilon} \epsilon \int \frac{d^{D-2}k}{\Gamma(1+\epsilon)\pi^{1+\epsilon}(k - \vec{q}_1)^2(k + \vec{q}'_1)^2} \ln(\vec{q}^2/\vec{k}^2) \right. \\
+ \left. \left( (\vec{q}^2_1)^{\epsilon} + (\vec{q}'^2_1)^{-\epsilon} - 1 \right) \left( \frac{1}{2\epsilon} + \psi(1 + 2\epsilon) - \psi(1 + \epsilon) + \frac{11 + 7\epsilon}{2(1 + 2\epsilon)(3 + 2\epsilon)} \right) \right) \right] , \]

\[ \mathcal{K}^{(1)}_i = \frac{g^4}{\pi^{1+\epsilon}\Gamma(1 - \epsilon)} \left( \mathcal{K}_1 + \mathcal{K}_2 + \mathcal{K}_3 \right) , \quad \bar{g}^2 \equiv \frac{g^2 N_c \Gamma(1-\epsilon)}{(4\pi)^{D/2}}, \]

with

\[ \mathcal{K}_1 = -f_B(\vec{q}_1, \vec{q}_2; \vec{q}) \left( \frac{(\vec{k}^2)^{\epsilon}}{\epsilon} \left[ \frac{11}{3} + \left( 2\psi'(1) - \frac{67}{9} \right) \epsilon \right] \right) \]

\[ \mathcal{K}_2 = \left\{ \vec{q}^2 \left[ \frac{11}{6} \ln \left( \frac{\vec{q}_1^2 \vec{q}_2^2}{\vec{q}^2 \vec{k}^2} \right) + \frac{1}{4} \ln \left( \frac{\vec{q}_1^2}{\vec{q}^2} \right) \ln \left( \frac{\vec{q}_1^2 \vec{q}^2_2}{\vec{q}^2 \vec{q}^2_1} \right) + \frac{1}{4} \ln \left( \frac{\vec{q}_2^2}{\vec{q}^2} \right) \ln \left( \frac{\vec{q}_2^2 \vec{q}^2_1}{\vec{q}^2 \vec{q}^2_2} \right) \right] + \frac{1}{\vec{k}^2} \left[ \frac{\vec{q}_1^2 \vec{q}_2^2 - \vec{q}_2^2 \vec{q}_1^2}{\vec{q}^2} \ln \left( \frac{\vec{q}_1^2}{\vec{q}^2_2} \right) \left( \frac{11}{6} - \frac{1}{4} \ln \left( \frac{\vec{q}_1^2 \vec{q}_2^2}{\vec{k}^4} \right) \right) \right] \right\} + \left\{ \vec{q}_i \leftrightarrow \vec{q}'_i \right\} , \]

\[ \mathcal{K}_3 = \left\{ \frac{1}{2} \left[ \vec{q}^2 (\vec{k}^2 - \vec{q}_1^2 - \vec{q}_2^2) + 2\vec{q}_1^2 \vec{q}_2^2 - \vec{q}_1^4 \vec{q}_2^2 - \vec{q}_2^4 \vec{q}_1^2 \right] \right\} + \left\{ \vec{q}_i \leftrightarrow \vec{q}'_i \right\} . \]

The integral in \( R^{(1)}(\vec{q}_1, \vec{q}) \) is already known with \( O(\epsilon^0) \) accuracy (see the Appendix of Ref. [9]):

\[ (\vec{q}^2)^{1-\epsilon} \int \frac{d^{D-2}k}{\Gamma(1-\epsilon)} \frac{\ln(\vec{q}^2/\vec{k}^2)}{(k - \vec{q}_1)^2(k + \vec{q}'_1)^2} = -\frac{1}{\epsilon} \ln \left( \frac{\vec{q}_1^2 \vec{q}_1'^2}{(\vec{q}^2)^2} \right) - \frac{1}{2} \ln^2 \left( \frac{\vec{q}_1^2 \vec{q}_1'^2}{\vec{q}_1^2 \vec{q}_1'^2} \right) . \]

Using the above result, the I strong bootstrap condition (9) takes the following form, with \( O(\epsilon^0) \) accuracy:

\[ \frac{(\vec{q}^2)^{-2\epsilon}}{\pi^{1+\epsilon}\Gamma(1-\epsilon)} \int \frac{d^{D-2}q_2}{q_2^2 q_1'^2} \left[ \mathcal{K}_1 + \mathcal{K}_2 + \mathcal{K}_3 + f_B \left( -\frac{11}{6} \ln \left( \frac{q_2^2 q_2'^2}{q_1^2 q_1'^2} \right) \right) \right] \]
\[ + \frac{1}{2} \ln \left( \frac{\vec{q}_1^2}{q^2} \right) \ln \left( \frac{\vec{q}_1'^2}{q^2} \right) - \frac{1}{2} \ln \left( \frac{\vec{q}_2^2}{q^2} \right) \ln \left( \frac{\vec{q}_2'^2}{q^2} \right) \]

\[ = - \frac{1}{e^2} \left[ \frac{11}{3} + \left( 2 \psi'(1) - \frac{67}{9} \right) \epsilon + \left( \frac{404}{27} + \psi''(1) - \frac{22}{3} \psi'(1) \right) \epsilon^2 \right] \]

\[ - \frac{2}{\epsilon} \left[ \frac{11}{3} + \left( 2 \psi'(1) - \frac{67}{9} \right) \epsilon \right] \ln \left( \frac{\vec{q}_1^2 \vec{q}_1'^2}{(q^2)^2} \right) - \frac{22}{3} \left( \ln^2 \left( \frac{\vec{q}_1^2}{q^2} \right) + \ln^2 \left( \frac{\vec{q}_1'^2}{q^2} \right) \right) \]

(15)

For details on the calculation of the remaining integrals, we refer to [10]; here we merely quote the final results. The integral from \( K_1 \) is

\[ \frac{(\vec{q}^2)^{-2\epsilon}}{\pi^{1+\epsilon} \Gamma(1-\epsilon)} \int \frac{d^{D-2} q_2}{q_2^2 (\vec{q}_2^2 - \vec{q}^2)} \left[ \frac{\vec{q}_1^2 \vec{q}_1'^2 + \vec{q}_2^2 \vec{q}_2'^2}{(\vec{q}_1 - \vec{q}_2)^2} - q^2 \right] \frac{[(\vec{q}_1 - \vec{q}_2)^2]^\epsilon}{\epsilon} \]

\[ = \frac{1}{e^2} \left[ 1 + 2 \epsilon \ln \left( \frac{\vec{q}_1^2 \vec{q}_1'^2}{(q^2)^2} \right) + \epsilon^2 \left( 2 \ln^2 \left( \frac{\vec{q}_1^2}{q^2} \right) + 2 \ln^2 \left( \frac{\vec{q}_1'^2}{q^2} \right) \right) \right. \]

\[ + \left. \ln \left( \frac{\vec{q}_1^2}{q^2} \right) \ln \left( \frac{\vec{q}_1'^2}{q^2} \right) - \psi'(1) \right] + O(\epsilon) \].

(16)

Among the remaining integrals, there are four which can be calculated in a straightforward way using the generalized Feynman parametrization for arbitrary space-time dimension (although we quote here only the result with \( O(\epsilon^0) \) accuracy):

\[ I_1(\vec{q}) = \frac{(\vec{q}^2)^{1-\epsilon}}{\pi^{1+\epsilon} \Gamma(1-\epsilon)} \int d^{D-2} q_2 \frac{1}{q_2^2 (\vec{q}_2^2 - \vec{q}^2)} = \frac{2}{\epsilon} + O(\epsilon) , \]

\[ I_2(\vec{q}) = \frac{(\vec{q}^2)^{1-\epsilon}}{\pi^{1+\epsilon} \Gamma(1-\epsilon)} \int d^{D-2} q_2 \frac{\ln(\vec{q}_2^2/\vec{q}^2)}{q_2^2 (\vec{q}_2^2 - \vec{q}^2)} = - \frac{1}{\epsilon^2} + \psi'(1) + O(\epsilon) , \]

\[ I_3(\vec{q}) = \frac{(\vec{q}^2)^{1-\epsilon}}{\pi^{1+\epsilon} \Gamma(1-\epsilon)} \int d^{D-2} q_2 \frac{\ln^2(\vec{q}_2^2/\vec{q}^2)}{q_2^2 (\vec{q}_2^2 - \vec{q}^2)^2} = \frac{2}{\epsilon^3} - \frac{2 \psi'(1)}{\epsilon} - 2 \psi''(1) + O(\epsilon) , \]

\[ I_4(\vec{q}) = \frac{(\vec{q}^2)^{1-\epsilon}}{\pi^{1+\epsilon} \Gamma(1-\epsilon)} \int d^{D-2} q_2 \frac{\ln \left( (\vec{q}_2^2/\vec{q}^2) \ln \left( (\vec{q}_2 - \vec{q})^2/\vec{q}^2 \right) \right)}{q_2^2 (\vec{q}_2^2 - \vec{q}^2)^2} = -2 \psi''(1) + O(\epsilon). \]

Replacing the above expressions and (16) in Eq. (15), the bootstrap condition takes the form

\[ \left\{ -3 \psi''(1) + \frac{1}{24} \ln^3 \left( \frac{\vec{q}_1^2}{q^2} \right) - \frac{1}{4} \ln^2 \left( \frac{\vec{q}_1^2}{q^2} \right) \ln \left( \frac{\vec{q}_1'^2}{q^2} \right) - \frac{1}{2} I(\vec{q}_1'^2, \vec{q}^2; q_1^2) \right\} \]
\[
+ \frac{1}{2} I(\bar{q}^2, \bar{q}_1^2; q_1^2) - \frac{3}{4} J(q_1^2, \bar{q}^2; \bar{q}_1^2) \right\} \! + \left\{ \bar{q}_1 \leftrightarrow \bar{q}_1' \right\} \! + \frac{1}{\pi} \int \frac{dq_2' dq_2}{q_2' q_2} K_3 = \! 0 ,
\]

with

\[
I(p_1^2, p_2^2; (\bar{p}_1 - \bar{p}_2)^2) = \frac{(\bar{p}_1 - \bar{p}_2)^2}{\pi} \int d\bar{p} \frac{\ln(\bar{p}_1^2/\bar{p}_2^2) \ln((\bar{p}_1 - \bar{p}_2)^2/((\bar{p} - \bar{p}_2)^2)}{(\bar{p} - \bar{p}_1)^2(\bar{p} - \bar{p}_2)^2} ,
\]

\[
J(p_1^2, p_2^2; (\bar{p}_1 - \bar{p}_2)^2) = \frac{(\bar{p}_1 - \bar{p}_2)^2}{\pi} \int d\bar{p} \frac{\ln(p_1^2/p_2^2) \ln(p_1^2/p_2^2)}{(\bar{p} - \bar{p}_1)^2(\bar{p} - \bar{p}_2)^2} .
\]

The integrals \(I(p_1^2, p_2^2; (\bar{p}_1 - \bar{p}_2)^2)\) and \(J(p_1^2, p_2^2; (\bar{p}_1 - \bar{p}_2)^2)\) can be reduced to one-dimensional integrals:

\[
I(p_1^2, p_2^2; (\bar{p}_1 - \bar{p}_2)^2) = \psi'(1) \ln \tilde{k}_1^2 + \frac{1}{2} \int_0^1 dx \frac{D_0}{x} \ln \left( \frac{D_0}{k_1^2} \right)
\]

\[
+ \frac{1}{2} \int_0^1 \frac{dx}{1 - x} \ln \left( \frac{D_0}{k_1^2} \right) + \ln \tilde{k}_2^2 \int_0^1 dx \ln \left( \frac{D_0}{k_2^2} \right) + \ln \tilde{k}_1^2 \int_0^1 dx \ln \left( \frac{D_0}{k_1^2} \right)
\]

\[
+ \int_0^1 dx \ln x \ln D_0 - \int_0^1 \frac{dx}{1 - x} \ln x \ln \left( \frac{D_0}{k_2^2} \right) - 2 \int_0^1 \frac{dx}{1 - x} \ln(1 - x) \ln \left( \frac{D_0}{k_1^2} \right)
\]

\[
- \int_0^1 dx \ln \left( \frac{x}{1 - x} \right) \ln \left( \frac{D_1}{D_0} \right) \frac{\tilde{k}_1^2 - \tilde{k}_2^2 - (1 - 2x)}{D_0 - D_1} ,
\]

\[
J(p_1^2, p_2^2; (\bar{p}_1 - \bar{p}_2)^2) = \ln \left( \frac{\tilde{k}_1^2}{\tilde{k}_2^2} \right) \left[ \int_0^1 \frac{dx}{1 - x} \ln \left( \frac{D_0}{k_2^2} \right) - \int_0^1 \frac{dx}{x} \ln \left( \frac{D_0}{k_1^2} \right) \right]
\]

\[
+ \int_0^1 \frac{dx}{x} \ln \left( \frac{D_0}{k_2^2} \right) + \int_0^1 \frac{dx}{1 - x} \ln \left( \frac{D_0}{k_1^2} \right)
\]

\[
- 2 \int_0^1 \frac{dx}{D_1 - D_0} \ln \left( \frac{x}{1 - x} \right) \ln \left( \frac{D_1}{D_0} \right) \left[ (1 - 2x) - \frac{D_1(\tilde{k}_1^2 - \tilde{k}_2^2)}{D_0} \right] ,
\]

with

\[
\tilde{k}_1 \equiv \frac{\bar{p}_1}{|\bar{p}_1 - \bar{p}_2|} , \quad \tilde{k}_2 \equiv \frac{\bar{p}_2}{|\bar{p}_1 - \bar{p}_2|} , \quad D_0 \equiv x \tilde{k}_1^2 + (1 - x) \tilde{k}_2^2 , \quad D_1 \equiv x(1 - x) .
\]

The integral from \(K_3\) can be re-expressed as follows:

\[
A \equiv \frac{1}{\pi} \int \frac{d^2q_2}{q_2^2 q_2^2} K_3 = (A_1 + A_2 + A_3) + (\bar{q}_1 \leftrightarrow \bar{q}_1') ,
\]

(20)
The imaginary part of the integrals condition (17) are functions of the variables $q$. The integrals and the explicit logarithms entering the bootstrap cumbersome. Nevertheless, the proof can be greatly simplified for the following reason. The one-dimensional integrals in Eqs. (18), (19) and (20) can be calculated analytically, but the arising expressions are discouragingly long and cumbersome. Nevertheless, the proof can be greatly simplified for the following reason. The integrals and the explicit logarithms entering the bootstrap condition (17) are functions of the variables $q_i^2 \equiv -q_i^2$, $q_i'^2 \equiv -q_i'^2$ and $q^2 \equiv -q^2$. At fixed $q_i^2 \leq 0$, $q_i'^2 \leq 0$, these functions are analytical functions of $q^2$, real for $q^2 < 0$ and with the cut $0 \leq q^2 < \infty$. Any such function can be determined by its discontinuity on the cut, up to another function of $q_i^2$ and $q_i'^2$ (but not of $q^2$). This last function can be simply obtained by evaluating the first function at $q^2 = \infty$. Operatively, to calculate discontinuities, we have to make the replacement $q^2 \rightarrow -q^2 - i0$ in the integrals and in the explicit logarithms entering the bootstrap condition, to calculate their imaginary parts on the upper edge of the cut (they give the discontinuities after multiplication by $2i$) and to check the strong bootstrap for the imaginary parts (which is the same as for discontinuities). Then, what remains to be done is to check the bootstrap in the limit $q^2 \gg q_i^2$, $q^2 \gg q_i'^2$.

The bootstrap relation for imaginary parts (divided by $\pi$) reads

$$\left\{ \begin{aligned} &-\frac{1}{8} \ln^2 \left( \frac{q_i^2}{q^2} \right) + \frac{5\pi^2}{24} - \frac{1}{2} \ln \left( \frac{q_i^2}{q^2} \right) \ln \left( \frac{q_i'^2}{q^2} \right) - \frac{1}{2} \Im I(q_i^2, -q^2 - i0; q_i'^2) \\ &+ \frac{1}{2\pi} \Im I(-q^2 - i0, q_i^2; q_i'^2) - \frac{3}{4\pi} \Im J(q_i^2, -q^2 - i0; q_i'^2) \right\} + \frac{3A}{\pi} = 0. \end{aligned} \right. \quad (21)$$

The imaginary part of the integrals $I$ and $J$ in Eqs. (18) and (19) can be easily calculated:

$$-\frac{1}{2\pi} \Im I(q_i^2, -q^2 - i0; q_i'^2) + \frac{1}{2\pi} \Im I(-q^2 - i0, q_i^2; q_i'^2) = -\frac{\psi'(1)}{2} + \frac{1}{4} \ln^2 \left( \frac{q_i^2}{q^2} \right), \quad (22)$$
\[
\frac{1}{\pi} \Im J(q_1^2, -q_2^2 - i0; q_1''^2) + (q_1^2 \leftrightarrow q_1''^2) \\
= -\ln \left( \frac{\kappa^-}{q^2} \right) \ln \left( \frac{\kappa^+}{q^2} \right) + \frac{1}{2} \ln^2 \left( \frac{q_1^2}{q^2} \right) + \psi'(1) + (q_1^2 \leftrightarrow q_1''^2) .
\]

To calculate the discontinuity of the integral \( A \) defined in Eq. (20) at \( q_2^2 = -\bar{q}^2 \geq 0 \), it is convenient to rewrite the integral over \( q_2 \) in Minkowski space and to use the Cutkosky rules for the calculation of the discontinuity. First, we represent the integral over \( x \) appearing in \( K_3 \) (see Eq. (14)) as

\[
I = \int_0^1 dx \int_0^\infty dz \frac{1}{z - k^2x(1-x) - i0} \frac{1}{z - q_1^2(1-x) - q_2^2x - i0} ,
\]

where \( k, q_1, q_2 \) are considered as vectors in the two-dimensional Minkowski space, i.e. \( k^2 = -\vec{k}^2, q_1^2 = -\vec{q}_1^2, q_2^2 = -\vec{q}_2^2 \). This representation can be used for arbitrary values of \( k^2, q_1^2, q_2^2 \). Analogous representation can be written for \( I(q_i \leftrightarrow q_i') \). It permits to rewrite the integral with \( K_3 \) in the bootstrap relation in the form

\[
A = \frac{1}{i\pi} \int d^2q_2 \frac{d^2q_2}{(q_2^2 + i0)((q - q_2)^2 + i0)} K_3 ,
\]

where now

\[
d^2q_2 = dq_2^{(0)} dq_2^{(1)}, \quad q_2^2 = (q_2^{(0)})^2 - (q_2^{(1)})^2 ,
\]

eq etc., which determines \( A \) as function of \( \vec{q}_1^2 \), \( \vec{q}_1''^2 \) and \( q^2 \) for arbitrary values of these variables. For \( q_1^2 \equiv -\vec{q}_1^2 \leq 0, q_1''^2 \equiv -\vec{q}_1''^2 \leq 0 \) and \( q^2 \equiv -\bar{q}^2 \leq 0 \) it is just the function entering (17), that is easily seen by making the Wick rotation of the contour of integration over \( q_2^{(0)} \). We are interested in the region \( q_1^2 \leq 0, q_1''^2 \leq 0 \) and \( q^2 \geq 0 \). According to the Cutkosky rules, the discontinuity of \( A \) related to the terms with \( I \) is determined by the two cuts, with the contributions obtained by the substitutions:

\[
\frac{1}{(q_2^2 + i0)((q - q_2)^2 + i0)} \rightarrow (-2\pi i)^2 \delta(q_2^2) \delta((q - q_2)^2)
\]

and

\[
\frac{1}{(z - q_1^2(1-x) - q_2^2x - i0)((q - q_2)^2 + i0)} \\
\rightarrow -(-2\pi i)^2 \delta(z - q_1^2(1-x) - q_2^2x) \delta((q - q_2)^2) .
\]

Using these rules and removing the \( \delta \)-functions by the integration over \( q_2 \) (the most appropriate system for this is \( q^{(1)} = 0, q^2 = (q^{(0)})^2 \)), we obtain

\[
\frac{\Im A}{\pi} = -\frac{3}{2} \ln \left( \frac{\kappa^-}{q^2} \right) \ln \left( \frac{\kappa^+}{q^2} \right) + \frac{1}{4} \ln^2 \left( \frac{q_1^2 q_1''^2}{(q^2)^2} \right) + \frac{1}{2} \ln \left( \frac{q_1^2}{q_2^2} \right) \ln \left( \frac{q_1''^2}{q_2^2} \right) ,
\]
with
\[
\kappa^{\pm} = \frac{1}{2} \left( q^2 + \vec{q}_1^2 + \vec{q}_1'^2 \pm \sqrt{(q^2 + \vec{q}_1^2 + \vec{q}_1'^2)^2 - 4\vec{q}_1^2 \vec{q}_1'^2} \right).
\]

Using Eqs. (22), (23) and (29), it is easy to see that the imaginary part of the bootstrap relation, Eq. (21), is satisfied. Finally, the bootstrap has to be considered in the limit \( \vec{q}^2 \gg \vec{q}_1^2, \vec{q}^2 \gg \vec{q}_1'^2 \). We have in this limit
\[
I(q_1^2, q_1'^2) \simeq -\zeta(2) \ln \left( \frac{q_1^2}{\vec{q}_1^2} \right) + 2\zeta(3),
\]
\[
I(q_1^2, q_1^2; q_1'^2) \simeq -\frac{1}{6} \ln^3 \left( \frac{q^2}{\vec{q}_1^2} \right) - \zeta(2) \ln \left( \frac{q^2}{\vec{q}_1^2} \right) + 2\zeta(3),
\]
\[
J(q_1^2, q_1'^2; q_1'^2) \simeq -\frac{1}{6} \ln^3 \left( \frac{q^2}{\vec{q}_1^2} \right) - 2\zeta(2) \ln \left( \frac{q^2}{\vec{q}_1^2} \right) + 4\zeta(3)
\]
and
\[
\frac{1}{\pi} \int \frac{d\vec{q}_2^2}{\vec{q}_2^2 \vec{q}_{12}^2} \mathcal{K}_3 \simeq -\frac{1}{4} \ln \left( \frac{q^2}{\vec{q}_1^2} \right) \ln \left( \frac{q^2}{\vec{q}_1^2} \right) \ln \left( \frac{q^2}{\vec{q}_1^2} \right) + \ln \left( \frac{q^2}{\vec{q}_1^2} \right)
- \frac{3\zeta(2)}{2} \left( \ln \left( \frac{q^2}{\vec{q}_1^2} \right) + \ln \left( \frac{q^2}{\vec{q}_1'^2} \right) \right) - 6\zeta(3).
\]
Again, it is easy to see that the bootstrap condition (17) is satisfied in the limit of large \( q^2 \).

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