Jordan $C^*$-Algebras and Supergravity

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Abstract

It is known that black hole charge vectors of $\mathcal{N} = 8$ and magic $\mathcal{N} = 2$ supergravity in four and five dimensions can be represented as elements of Jordan algebras of degree three over the octonions and split-octonions and their Freudenthal triple systems. We show both such Jordan algebras are contained in the exceptional Jordan $C^*$-algebra and construct its corresponding Freudenthal triple system and single variable extension. The transformation groups for these structures give rise to the complex forms of the U-duality groups for $\mathcal{N} = 8$ and magic $\mathcal{N} = 2$ supergravities in three, four and five dimensions.

Keywords: Jordan $C^*$-algebras, Supergravity, U-duality.
1 Introduction

It was shown that extremal black holes in $\mathcal{N} \geq 2$, $D = 3, 4, 5, 6$ supergravities on symmetric spaces can be described by Jordan algebras and their corresponding Freudenthal triple systems, with the Bekenstein-Hawking entropy of the black holes given by algebraic invariants over these structures [4]-[23],[28],[31],[32],[54],[57],[58]. Such supergravities are called homogeneous supergravities [10], and include $\mathcal{N} = 8$, $D = 4$ supergravity (M-theory on $T^7$), described by the Jordan algebra of degree three over the split-octonions. Of the four $\mathcal{N} = 2$ supergravity theories defined by simple Jordan algebras of degree three (magic supergravities), all but one, the exceptional magic $\mathcal{N} = 2$ supergravity, can be obtained by a consistent truncation of the maximal $\mathcal{N} = 8$ supergravity [12]. The exceptional magic $\mathcal{N} = 2$ supergravity corresponds to the exceptional Jordan algebra of degree three over the octonion composition algebra [24]-[30],[40],[45],[51],[53], and it has been recently noted [57] there are physical and mathematical obstacles which make its study appear less well motivated than the $\mathcal{N} = 8$ case.

In this note, we show the maximal $\mathcal{N} = 8$ and exceptional magic $\mathcal{N} = 2$ supergravities can be unified using the exceptional Jordan $C^*$-algebra, a Jordan algebra of degree three over the bioctonions, which contains both Jordan algebras of degree three over the octonions and split-octonions. We construct its corresponding Freudenthal triple system and single variable extension and give support for such constructions in light of non-real solutions arising in $\mathcal{N} = 8$ and magic $\mathcal{N} = 2$, $D = 4$ supergravities reduced to $D = 3$. 

\section{N = 8 and Magic N = 2 Supergravity}

2.1 Black Strings in $D = 6$ 

2.2 Black Holes in $D = 5$ 

2.3 Black Holes in $D = 4$ 

2.4 Black Holes in $D = 3$ 

\section{Bioctonions and Jordan C*-Algebras}

3.1 Composition Algebras 

3.2 Bioctonions 

3.3 The Exceptional Jordan C*-Algebra 

3.4 Freudenthal Triple System 

3.5 Extended Freudenthal Triple System 

\section{Conclusion}
2 \( \mathcal{N} = 8 \) and Magic \( \mathcal{N} = 2 \) Supergravity

We review the \( \mathcal{N} \geq 2 \) homogeneous supergravities in \( D = 3, 4, 5, 6 \) over Jordan algebras of degree two and three and give their corresponding U-duality groups and entropy expressions for black hole and string solutions.

2.1 Black Strings in \( D = 6 \)

The \( \mathcal{N} = 8 \) and magic \( \mathcal{N} = 2 \), \( D = 6 \) supergravities arise as uplifts of \( \mathcal{N} = 8 \) and magic \( \mathcal{N} = 2 \), \( D = 5 \) supergravities with \( n_V = 27, n_V = 15, n_V = 9 \) and \( n_V = 6 \) vector fields [21]. They enjoy \( SO(5, 5), SO(9, 1), SO(5, 1), SO(3, 1) \) and \( SO(2, 1) \) U-duality symmetry since the \( D = 6 \) vector multiplets in the Coulomb phase (after Higgsing) transform as spinors of dimension 16, 8, 4 and 2, respectively [21]. One can associate a black string solution with charges \( q_I \) (\( I = 1, \ldots, n_V \) for \( n_V = 10, 6, 4, 3 \)) an element

\[
J = \sum_{I=1}^{n} q^I e_I = \begin{pmatrix} r_1 & A \\ A^t & r_2 \end{pmatrix} \quad r_i \in \mathbb{R}, A \in \mathbb{A}
\]  

(1)

of a Jordan algebra \( J^A_2 \) of degree two, where the \( e_I \) form the \( n_V \)-dimensional basis of the Jordan algebra over a composition algebra \( \mathbb{A} = \mathbb{O}_s, \mathbb{O}, \mathbb{H}, \mathbb{C}, \mathbb{R} \). Elements of \( J^A_2 \) transform as the \((\dim \mathbb{A} + 2)\) representation of \( SL(2, \mathbb{A}) \), the \( 10, 10, 6, 4, 3 \) of \( SO(5, 5), SO(9, 1), SO(5, 1), SO(3, 1) \) and \( SO(2, 1) \) U-duality symmetry since the \( D = 6 \) vector multiplets in the Coulomb phase (after Higgsing) transform as spinors of dimension 16, 8, 4 and 2, respectively [21]. One can associate a black string solution with charges \( q_I \) an element

\[
I_2(J) = \det(J) = r_1 r_2 - A \bar{A}.
\]  

(3)

The black string entropy is given by [4]:

\[
S_{D=5,BH}(J) = \pi \sqrt{|I_2(J)|}
\]  

(2)

2.2 Black Holes in \( D = 5 \)

In \( D = 5 \), the \( \mathcal{N} = 8 \) and magic \( \mathcal{N} = 2 \) supergravities are coupled to 27, 15, 9, 6 vector fields with U-duality symmetry groups \( E_6(6), E_6(-26), SU^*(6), SL(3, \mathbb{C}), SL(3, \mathbb{R}) \), for Jordan algebras of degree three over composition algebras \( \mathbb{A} = \mathbb{O}_s, \mathbb{O}, \mathbb{H}, \mathbb{C}, \mathbb{R} \), respectively [4, 13, 14]. The BPS black hole solutions were classified by Ferrara et al. [13, 22, 57] by studying the underlying Jordan algebras of degree three under the actions of their reduced structure groups, \( Str_0(J^A_3) \), which correspond to the U-duality groups of the \( \mathcal{N} = 8 \) and magic \( \mathcal{N} = 2 \), \( D = 5 \) supergravities. This is accomplished by
associating a given black hole solution with charges \( q_I \) \((I = 1, \ldots, n_V)\) an element
\[
J = \sum_{I=1}^{n} q^I e_I = \begin{pmatrix} r_1 & A_1 & \overrightarrow{A}_2 \\ \overrightarrow{A}_1 & r_2 & A_3 \\ A_2 & \overrightarrow{A}_3 & r_3 \end{pmatrix} \quad r_i \in \mathbb{R}, A_i \in \mathbb{H}
\]
(5)
of a Jordan algebra of degree three \(J_3^J\) over a composition algebra \( \mathbb{A} = \mathbb{O}, \mathbb{O}, \mathbb{H}, \mathbb{C}, \mathbb{R} \), where \( e_I \) form a basis for the \(n_V\)-dimensional Jordan algebra. This establishes a correspondence between Jordan algebras of degree three and the charge spaces of the extremal black hole solutions [1 3]. In all cases, the entropy of a black hole solution can be written [4, 13, 23] in the form
\[
S_{D=5, BH}(J) = \pi \sqrt{|I_3(J)|}
\]
(6)
where \( I_3 \) is the cubic invariant given by
\[
I_3(J) = \det(J)
\]
(7)
and
\[
det(J) = r_1r_2r_3 - r_1||A_1||^2 - r_2||A_2||^2 - r_3||A_3||^3 + 2\text{Re}(A_1A_2A_3).
\]
(8)
The U-duality orbits are distinguished by rank via
\[
\text{Rank } J = 3 \quad \text{iff} \quad I_3(J) \neq 0 \quad S \neq 0, \quad 1/8\text{-BPS}
\]
\[
\text{Rank } J = 2 \quad \text{iff} \quad I_3(J) = 0, J^2 \neq 0 \quad S = 0, \quad 1/4\text{-BPS}
\]
\[
\text{Rank } J = 1 \quad \text{iff} \quad J^2 = 0, \quad J \neq 0 \quad S = 0, \quad 1/2\text{-BPS}
\]
(9)
where the quadratic adjoint map is given by
\[
J^2 = J \times J = J^2 - \text{tr}(J)J + \frac{1}{2}(\text{tr}(J)^2 - \text{tr}(J^2))I.
\]
(10)

### 2.3 Black Holes in \( D = 4 \)

For \( \mathcal{N} = 8 \) and magic \( \mathcal{N} = 2, D = 4 \) supergravities there is a correspondence between the field strengths of the vector fields and their magnetic duals and elements of a Freudenthal triple system (FTS) \( \mathfrak{M}(J_3^J) \) over a Jordan algebra of degree three, \( J_3^J \) [13]. The correspondence is explicitly:

\[
\begin{pmatrix} F_{\mu\nu}^0 & F_{\mu\nu}^i \\ \tilde{F}_{\mu\nu}^i & \tilde{F}_{\mu\nu}^0 \end{pmatrix} \leftrightarrow \begin{pmatrix} \alpha & X \\ Y & \beta \end{pmatrix} = \mathcal{X} \in \mathfrak{M}(J_3^J),
\]

(11)
where \( \alpha, \beta \in \mathbb{R} \) and \( X, Y \in J_3^J \). \( F_{\mu\nu}^i \) \((i = 1, \ldots, n_V)\) denote the field strengths of the vector fields from \( D = 5 \) and \( F_{0\mu
u}^0 \) is the \( D = 4 \) graviphoton field strength coming from the \( D = 5 \) graviton [13]. Using the correspondence, one can associate the entries of an FTS element with electric and magnetic
charges \(q_0, q_i, p^0, p^i \in \mathbb{R}^{2n_V+2}\) of an \(\mathcal{N} = 8\) or magic \(\mathcal{N} = 2, D = 4\) extremal black hole via the relations \[12\] \[14\]:

\[
\alpha = p^0 \quad \beta = q_0 \quad X = p^i e_i \quad Y = q_i e^i.
\] (12)

Setting \(p^I = (\alpha, X)\) and \(q_I = (\beta, Y)\) \((I = 0, \ldots, n_V)\), the Bekenstein-Hawking entropy of extremal black hole solutions is given by \[12\]:

\[
S_{D=4,BH}(X) = \pi \sqrt{|I_4(p^I, q_I)|},
\] (13)

where \(I_4\) is the quartic invariant of the FTS, preserved by the automorphism groups \(\text{Aut}(\mathfrak{M}(J_3^A))\), the U-duality groups for the corresponding \(D = 4\) supergravities. Explicitly, the U-duality groups are \(E_7(7), E_7(-25), SO^*(12), SU(3, 3)\) and \(Sp(6, \mathbb{R})\) for composition algebras \(A = \mathbb{O}, \mathbb{O}, \mathbb{H}, \mathbb{C}, \mathbb{R}\), respectively. The U-duality orbits are given by the FTS rank via \[1\].

\[
\text{Rank} X = 4 \quad \text{iff} \quad I_4(X) \neq 0, \quad (I_4(X) < 0) \quad \text{non-BPS}
\]

\[
\text{Rank} X = 4 \quad \text{iff} \quad I_4(X) \neq 0, \quad (I_4(X) > 0) \quad 1/8\text{-BPS}
\]

\[
\text{Rank} X = 3 \quad \text{iff} \quad T(X, X, X, X) \neq 0, \quad I_4(X) = 0 \quad 1/8\text{-BPS}
\]

\[
\text{Rank} X = 2 \quad \text{iff} \quad \exists Y \Lambda(X, Y) \neq 0, \quad T(X, X, X, X) = 0 \quad 1/4\text{-BPS}
\]

\[
\text{Rank} X = 1 \quad \text{iff} \quad \forall Y \Lambda(X, Y) = 0, \quad X \neq 0 \quad 1/2\text{-BPS}
\] (14)

where

\[
\Lambda(X, Y) = 3T(X, X, Y) + X\{X, Y\}.
\] (15)

### 2.4 Black Holes in \(D = 3\)

By extending the space of electric and magnetic charges \((p^I, q_I) \in \mathfrak{M}(J_3^A)\) by a real variable, we recover a space \(\mathcal{I}(J_3^A)\) acted on by the three-dimensional U-duality group \(G_3 = \text{Inv}(\mathcal{I}(J_3^A))\) \[28\]. \(G_3\) is \(E_{8(8)}\) or \(E_{8(-24)}\), for dimensionally reduced \(\mathcal{N} = 8\) or exceptional magic \(\mathcal{N} = 2, D = 4\) supergravity \[10\] \[12\]. The quartic symplectic distance \(d(X, Y)\) between any two solutions \(X = (X, x)\) and \(Y = (Y, y)\) in \(\mathcal{I}(J_3^A)\) is given by:

\[
d(X, Y) = I_4(X - Y) - (x - y + \{X, Y\})^2.
\] (16)

The norm of an arbitrary solution \(X = (X, x) \in \mathcal{I}(J_3^A)\) is computed as:

\[
\mathcal{N}(X) = d(X, 0) = I_4(X) - x^2.
\] (17)

The U-duality group \(G_3\) leaves light-like separations \(d(X, Y) = 0\) invariant. Given an arbitrary \(X = (X, x) \in \mathcal{I}(J_3^A)\), Günaydin et al. noticed \[28\] non-real values arise for \(x\) when zero-norm solutions contain a negative-valued quartic invariant \(I_4\). To remedy this, the representation space \(\mathcal{I}(J_3^A) \sim \mathbb{R}^{57}\) is complexified, leading to a realization of \(E_8(\mathbb{C})\) on \(\mathbb{C}^{57}\).

\[1\] Explicit forms for \(T(X, X, X)\) and \(\Lambda(X, Y)\) are given in section 3.4.
3 Bioctonions and Jordan $C^*$-Algebras

3.1 Composition Algebras

Let $V$ be a finite dimensional vector space over a field $F = \mathbb{R}, \mathbb{C}$. An algebra structure on $V$ is a bilinear map

$$V \times V \to V$$

$$(x, y) \mapsto x \bullet y.$$  

A composition algebra is an algebra $A = (V, \bullet)$, admitting an identity element, with a non-degenerate quadratic form $\eta$ satisfying

$$\forall x, y \in A \quad \eta(x \bullet y) = \eta(x)\eta(y).$$  

If $\exists x \in A$ such that $x \neq 0$ and $\eta(x) = 0$, $\eta$ is said to be isotropic and gives rise to a split composition algebra. When $\forall x \in A, x \neq 0, \eta(x) \neq 0$, $\eta$ is anisotropic and yields a composition division algebra $[52]$.

**Proposition 3.1.** A finite dimensional vector space $V$ over $F = \mathbb{R}, \mathbb{C}$ can be endowed with a composition algebra structure if and only if $\dim F(V) = 1, 2, 4, 8$. If $F = \mathbb{C}$, then for a given dimension all composition algebras are isomorphic. For $F = \mathbb{R}$ and $\dim F(V) = 8$ there are only two non-isomorphic composition algebras: the octonions $\mathbb{O}$ for which $\eta$ is anisotropic and the split-octonions $\mathbb{O}_s$ for which $\eta$ is isotropic and of signature $(4, 4)$. Moreover for all composition algebras, the quadratic form $\eta$ is uniquely defined by the algebra structure.

**Proof.** See prop. 1.8.1., section 1.10 and corollary 1.2.4 in the book by Springer and Veldkamp $[52]$. $\square$

3.2 Bioctonions

The bioctonion algebra $\mathbb{O}_\mathbb{C}$ is a composition algebra of dimension 8 over $\mathbb{C}$, defined as the complexification of the octonion algebra $\mathbb{O}$ $[24]$

$$\mathbb{O}_\mathbb{C} = \mathbb{O} \otimes \mathbb{C} = \{ \psi = \varphi_1 + i\varphi_2 \mid \varphi_i \in \mathbb{O}, i^2 = -1 \}$$  

where the imaginary unit ‘$i$’ is assumed to commute with all imaginary basis units $e_j$ ($j = 1, 2, ..., 7$) of $\mathbb{O}$. The octonionic conjugate of an element of $\mathbb{O}_\mathbb{C}$ is taken to be

$$\overline{\psi} = \overline{\varphi_1} + i\overline{\varphi_2}$$  

with which we define a quadratic form $\eta: \mathbb{O}_\mathbb{C} \to \mathbb{C}$

$$\eta(\psi) = \overline{\psi}\psi.$$  

By application of the Moufang identities for the octonions $[25]$, it can be shown that $\forall \psi \in \mathbb{O}_\mathbb{C} \quad \eta(\psi_1\psi_2) = \eta(\overline{\psi_1})\eta(\overline{\psi_2})$, making $\mathbb{O}_\mathbb{C}$ a composition algebra over $\mathbb{C}$. 

6
The quadratic form over $O$ bras are real subalgebras of the bioctonion algebra

*Proof.* As $O_C = O \otimes \mathbb{C}$, the octonion algebra is taken to be

$$O = \{ \psi_r \in O_C \mid \psi_r = \varphi + i0 \quad \varphi \in O \}$$

where the quadratic form over $O_C$ reduces to $\eta(\psi_r) = \psi_r \bar{\psi}_r = \varphi \bar{\varphi} \in \mathbb{R}$, which is the usual anisotropic quadratic form for which $O$ is a composition algebra.

For the split-octonion case, we denote a basis for $O_C$ via the set $\{e_i, i e_i\}$ where $e_i$ ($i = 0, 1, \ldots, 7$) form a basis for $O$ as a real vector space satisfying

$$e_0 = 1$$

$$e_i^2 = -1 \quad (i = 1, 2, \ldots, 7)$$

$$e_{i+1}e_{i+2} = e_{i+4} = -e_{i+2}e_{i+1} \quad (\text{mod } 7)$$

$$e_{i+2}e_{i+4} = e_{i+1} = -e_{i+4}e_{i+2} \quad (\text{mod } 7).$$

We now choose eight basis units from $\{e_i, i e_i\}$ consisting of a quaternion basis in $O$, for example, $1, e_1, e_2, e_4$ and taking $i e_4$ such that ($i \neq 0, 1, 2, 4$).

Consider the real vector subspace $W \subset O_C$ spanned by these basis units

$$W = \{ \psi_s \in O_C \mid \psi_s = a_0 + a_1e_1 + a_2e_2 + a_4e_4 + i(a_3e_3 + a_5e_5 + a_6e_6 + a_7e_7) \}$$

Using the multiplicative properties of the octonions, one can construct a multiplication table for the basis units of $W$ where $W$ is seen to be closed (see Table I). Conjugation in $W$ is induced by octonionic conjugation in $O_C$

$$\bar{\psi}_s = a_0 - a_1e_1 - a_2e_2 - a_4e_4 + i(-a_3e_3 - a_5e_5 - a_6e_6 - a_7e_7). \quad (23)$$

The quadratic form for $O_C$ then reduces to

$$\eta(\psi_s) = \psi_s \bar{\psi}_s = a_0^2 + a_1^2 + a_2^2 + a_4^2 - a_3^2 - a_5^2 - a_6^2 - a_7^2 \in \mathbb{R} \quad (24)$$

which is isotropic and of signature $(4, 4)$. Hence, $W$ forms a dim$\mathbb{R}(W) = 8$ split composition algebra and by Prop. 3.1 must be isomorphic to the algebra of split-octonions $O_s$. $\square$

| $e_1$ | $e_2$ | $e_4$ | $ie_7$ | $ie_3$ | $ie_6$ | $ie_5$ |
|-------|-------|-------|--------|--------|--------|--------|
| $e_1$ | $-1$  | $e_4$ | $-e_2$ | $-ie_3$| $ie_7$ | $-ie_6$|
| $e_2$ | $-e_4$| $-1$  | $e_1$  | $-ie_6$| $ie_5$ | $ie_7$ |
| $e_4$ | $e_2$ | $-e_1$| $-1$   | $-ie_5$| $-ie_6$| $ie_3$ |
| $ie_7$| $ie_3$| $ie_6$| $ie_5$ | $1$     | $e_1$  | $e_2$  |
| $ie_3$| $-ie_7$| $-ie_5$| $ie_6$ | $-e_1$ | $1$     | $e_4$  |
| $ie_6$| $ie_7$| $-ie_3$| $-ie_2$| $-e_4$ | $1$     | $e_1$  |
| $ie_5$| $-ie_6$| $ie_3$| $-ie_7$| $-e_4$ | $e_2$  | $-e_1$ |

Table 1: A split-octonion subalgebra of $O_C$
It was shown by Shukuzawa \[56\] that $G_2(\mathbb{C})$ acts transitively on the space of all elements having the same norm in $\mathbb{O}_C$. We shall recall some useful theorems from Shukuzawa \[56\] here, which classify orbits for elements of $\mathbb{O}_C$ and its real subalgebras $\mathbb{O}$ and $\mathbb{O}_s$. For notational convenience we set $ie_i = e_i'$, for the case of the split-octonions.

**Theorem 3.3.** Any non-zero element $\psi \in \mathbb{O}_C$ can be transformed to the following canonical form by some element of $G_2(\mathbb{C})$:

If $\eta(\psi) \neq 0$:

$$\psi = (a_0 + ia_1)e_i \quad (i = 1, 2, \ldots, 7) \quad (a_0 > 0 \text{ or } a_0 = 0, \ a_1 > 0),$$

(25)

If $\eta(\psi) = 0$:

$$\psi = e_i + ie_j \quad (i \neq j, \ i, j = 1, 2, \ldots, 7).$$

(26)

Moreover, their orbits in $\mathbb{O}_C$ under $G_2(\mathbb{C})$ are distinct, and the union of all their orbits and $\{0\}$ is the whole space $\mathbb{O}_C$.

**Remark 3.4.** The canonical zero-norm orbit elements of $\mathbb{O}_C$ generate a non-associative Grassmann algebra \[40\], satisfying $\psi_i^2 = 0$, $\psi_i\psi_j = -\psi_j\psi_i$ and $\psi_i\psi_j = -\psi_j\psi_i$.

**Theorem 3.5.** Any element $\varphi \in \mathbb{O}$ can be transformed to the following canonical form by some element of $G_2$:

$$\varphi = a_0e_i \quad (i = 1, 2, \ldots, 7) \quad (a_0 = \sqrt{\eta(\varphi)} \geq 0).$$

(27)

Moreover, their orbits in $\mathbb{O}$ under $G_2$ are distinct, and the union of all their orbits and $\{0\}$ yields the whole space $\mathbb{O}$.

**Theorem 3.6.** Any non-zero element $\psi_s \in \mathbb{O}_s$ can be transformed to the following canonical form by some element of $G_2(2)$:

If $\eta(\psi_s) > 0$:

$$\psi_s = a_0e_i \quad (i = 1, 2, \ldots, 7) \quad (a_0 = \sqrt{\eta(\varphi)} > 0)$$

(28)

If $\eta(\psi_s) < 0$:

$$\psi_s = a_0e_i' \quad (i = 1, 2, \ldots, 7) \quad (a_0 = \sqrt{-\eta(\varphi)} > 0)$$

(29)

If $\eta(\psi_s) = 0$:

$$\psi = e_i + e_j' \quad (i \neq j, \ i, j = 1, 2, \ldots, 7).$$

(30)

Moreover, their orbits in $\mathbb{O}_s$ under $G_2(2)$ are distinct, and the union of all their orbits and $\{0\}$ is the whole space $\mathbb{O}_s$. 8
3.3 The Exceptional Jordan \( C^* \)-Algebra

**Definition 3.7.** (Kaplansky) Let \( A \) be a complex Banach space and a complex Jordan algebra equipped with an involution \( * \). Then \( A \) is a *Jordan \( C^* \)-algebra* if the following conditions are satisfied

\[
\|x \circ y\| \leq \|x\| \|y\| \quad \forall x, y \in A
\]
\[
\|z\| = \|z^*\| \quad \forall z \in A
\]
\[
\|\{zz^*z\}\| = \|z\|^3 \quad \forall z \in A
\]

where the Jordan triple product is given by

\[
\{xyz\} = (x \circ y) \circ z + (y \circ z) \circ x - (z \circ x) \circ y.
\]

**Theorem 3.8.** Each JB-algebra is the self-adjoint part of a unique Jordan \( C^* \)-algebra.

**Proof.** The proof is given by Wright [55], using the existence of an exceptional Jordan \( C^* \)-algebra whose self-adjoint part is the exceptional Jordan algebra \( J_3 \).

The *exceptional Jordan \( C^* \)-algebra* is the complexification of the exceptional Jordan algebra, given by

\[
J_3^{OC} = \{ X = A + iB \mid A, B \in J_3^D, i^2 = -1 \},
\]

where the imaginary unit \( 'i' \) is assumed to commute with all imaginary basis units \( e_j (j = 1, \ldots, 7) \) of \( \mathbb{O} \). A general element of the algebra takes the form

\[
X = \begin{pmatrix}
    z_1 & \psi_1 & \overline{\psi}_2 \\
    \overline{\psi}_1 & z_2 & \psi_3 \\
    \psi_2 & \overline{\psi}_3 & z_3
\end{pmatrix}
\]

where it is seen \( J_3^{OC} \) is a Jordan algebra of degree three over the bioctonions. As the bioctonions contain both the octonion and split-octonion algebras, \( J_3^{OC} \) contains \( J_3^O \) and \( J_3^{Os} \). One can define two types of involution for \( J_3^{OC} \):

\[
X^* = (X^*)^T = (A - iB)^T,
\]
\[
X^\dagger = (X)^T = (A + iB)^T
\]

differing by the use of either complex or octonionic conjugation of the entries. Using the complex involution, along with the spectral norm, \( J_3^{OC} \) becomes a Jordan \( C^* \)-algebra. Under this involution, only elements of the exceptional Jordan algebra \( J_3^D \) are self-adjoint. Under the involution using octonionic conjugation, all elements of \( J_3^{OC} \) are self-adjoint. Moreover, under this involution, the trace bilinear form \( J_3^{OC} \times J_3^{OC} \rightarrow \mathbb{C} \)

\[
\langle X, Y \rangle = \text{tr}(X \circ Y^\dagger) = \text{tr}(X \circ Y)
\]
is complex valued, as is required in later constructions. The Freudenthal Product \( J_3^{Dc} \times J_3^{Dc} \rightarrow J_3^{Dc} \) is defined using the trace bilinear form as

\[
X \times Y = X \circ Y - \frac{1}{2} (Y \text{tr}(X) + X \text{tr}(Y)) + \frac{1}{2} (\text{tr}(X) \text{tr}(Y) - \text{tr}(X \circ Y)) I
\]  

(38)

An important special case yields the quadratic adjoint map

\[
X^2 = X \times X = X^2 - \text{tr}(X)X + \frac{1}{2} (\text{tr}(X)^2 - \text{tr}(X^2)) I,
\]  

(39)

We can use the Freudenthal and Jordan product to define the cubic form

\[
(X, Y, Z) = \text{tr}(X \circ (Y \times Z)).
\]  

(40)

A special case of this cubic form is

\[
(X, X, X) = \text{tr}(X \circ (X \times X))
\]  

(41)

Using the cubic form, one can express the determinant as

\[
\det(X) = \frac{1}{3} \text{tr}(X \circ (X \times X)) = N(X).
\]  

(42)

where \( N(X) \) denotes the cubic norm of \( X \). The structure group \( \text{Str}(J_3^{Dc}) \), is comprised of all linear bijections on \( J_3^{Dc} \) that leave the cubic norm (hence determinant) invariant up to a constant scalar multiple

\[
N(s(X)) = c N(X) \quad \forall s \in \text{Str}(J_3^{Dc}).
\]  

(43)

The reduced structure group, \( \text{Str}_0(J_3^{Dc}) = E_6(C) \) [51], consists of the transformations for which \( c = 1 \) and contains the U-duality groups \( E_6(6), E_6(-26) \), of the \( N = 8 \) and exceptional magic \( N = 2, D = 5 \) supergravities, respectively.

### 3.4 Freudenthal Triple System

We follow the Freudenthal construction of Krutelevich et al. [41, 5, 27]. Given the exceptional Jordan \( C^* \)-algebra \( J_3^{Dc} \), one can construct its corresponding Freudenthal triple system (FTS) by defining the vector space \( \mathfrak{M}(J_3^{Dc}) \):

\[
\mathfrak{M}(J_3^{Dc}) = C \oplus C \oplus J_3^{Dc} \oplus J_3^{Dc}.
\]  

(44)

A general element \( \mathcal{X} \in \mathfrak{M}(J_3^{Dc}) \) can be expressed as

\[
\mathcal{X} = \begin{pmatrix} \alpha & X \\ Y & \beta \end{pmatrix} \quad \alpha, \beta \in C \quad X, Y \in J_3^{Dc}
\]  

(45)
The FTS comes equipped with a non-degenerate bilinear antisymmetric quadratic form \( \{X, Z\} : \mathfrak{M}(J^\mathfrak{O}_3) \times \mathfrak{M}(J^\mathfrak{O}_3) \to \mathbb{C} \),

\[
\{X, Z\} = \alpha \delta - \beta \gamma + \text{tr}(X \circ W) - \text{tr}(Y \circ Z)
\]

where \( X = \begin{pmatrix} \alpha & X \\ Y & \beta \end{pmatrix} \), \( Z = \begin{pmatrix} \gamma & Z \\ W & \delta \end{pmatrix} \),

(46)
a quartic form \( q : \mathfrak{M}(J^\mathfrak{O}_3) \to \mathbb{C} \),

\[
q(X) = -2[\alpha \beta - \text{tr}(X \circ Y)]^2 - 8[\alpha N(X) + \beta N(Y) - \text{tr}(X^2 \circ Y^2)]
\]

(47)
and a trilinear triple product \( T : \mathfrak{M}(J^\mathfrak{O}_3) \times \mathfrak{M}(J^\mathfrak{O}_3) \times \mathfrak{M}(J^\mathfrak{O}_3) \to \mathfrak{M}(J^\mathfrak{O}_3) \),

\[
\{T(X, Y, W), Z\} = q(X, Y, W, Z)
\]

(48)
where \( q(X, Y, W, Z) \) is the linearization of \( q(X) \) such that \( q(X, X, X, X) = q(X) \). Useful identities include [27]

\[
T(X, X, X) = (-\alpha^2 \beta + \alpha \text{tr}(X \circ Y) - 2N(Y), \alpha \beta^2 - \beta \text{tr}(X \circ Y) + 2N(X), 2Y \times X^2 - 2\beta Y^2 - (\text{tr}(X \circ Y) - \alpha \beta)X, -2X \times Y^2 + 2\alpha X^2 + (\text{tr}(X \circ Y) - \alpha \beta)Y).
\]

(49)
\[
\Lambda(X, Y) = 3T(X, X, Y) + \{X, Y\}X = (-3\alpha \beta - \text{tr}(X \circ Y))Y + 2\text{tr}((\alpha X - Y^2) \circ W), (3\alpha \beta - \text{tr}(X \circ Y))\delta - 2\text{tr}(\beta Y - X^2) \circ Z),
\]

(50)
\[
(3\alpha \beta - \text{tr}(X \circ Y))Z - 2(\beta Y - X^2) \times W + 2(\alpha X - Y^2)\delta - 2Q(X)Z,
\]

\[-(3\alpha \beta - \text{tr}(X \circ Y))W - 2(\alpha X - Y^2) \times Z + 2(\beta Y - X^2)\gamma - 2Q(X)W).
\]

The automorphism group \( \text{Aut}(\mathfrak{M}(J^\mathfrak{O}_3)) = E_7(\mathbb{C}) \) is the set of all transformations leaving the quadratic form and quartic form \( q(X) = I_3(X) \) invariant. Following Krutelevich [27], four types of transformations in \( \text{Aut}(\mathfrak{M}(J^\mathfrak{O}_3)) \) are:

For any \( C \in J^\mathfrak{O}_3 \), \( \Phi(C) \):

\[
\begin{pmatrix} \alpha & X \\ Y & \beta \end{pmatrix} \mapsto \begin{pmatrix} \alpha + \text{tr}(Y \circ C) + \text{tr}(X \circ C^2) + \beta N(C) & X + \beta C \\ Y + X \times C + \beta C^2 & \beta \end{pmatrix}
\]

(53)

For any \( D \in J^\mathfrak{O}_3 \), \( \Psi(D) \):

\[
\begin{pmatrix} \alpha & X \\ Y & \beta \end{pmatrix} \mapsto \begin{pmatrix} \alpha & X + Y \times D + \alpha D^3 \\ Y + \alpha D & \beta + \text{tr}(X \circ D) + \text{tr}(Y \circ D^2) + \alpha N(D) \end{pmatrix}
\]

(54)
For any $s \in \text{Str}(J^\text{OC}_3)$ and $c \in \mathbb{C}$ s.t. $N(s(Z)) = cN(Z)$:

$$\Omega : \begin{pmatrix} \alpha & X \\ Y & \beta \end{pmatrix} \mapsto \begin{pmatrix} c^{-1}\alpha & s(X) \\ s^{-1}(Y) & c\beta \end{pmatrix}$$  \hfill (55)

where $s^*$ is the adjoint to $s$ with respect to the trace bilinear form.

Lastly, we have:

$$\Upsilon : \begin{pmatrix} \alpha & X \\ Y & \beta \end{pmatrix} \mapsto \begin{pmatrix} -\beta & -Y \\ X & \alpha \end{pmatrix}.$$  \hfill (56)

When matrices $C$ and $D$ above are rank one (i.e., $C^2 = 0$, $D^2 = 0$), the transformations $\phi$ and $\psi$ simplify to:

$$\Phi(C) : \begin{pmatrix} \alpha & X \\ Y & \beta \end{pmatrix} \mapsto \begin{pmatrix} \alpha + \text{tr}(Y \circ C) & X + \beta C \\ \beta \end{pmatrix}$$  \hfill (57)

$$\Psi(D) : \begin{pmatrix} \alpha & X \\ Y & \beta \end{pmatrix} \mapsto \begin{pmatrix} \alpha & X + Y \times D \\ \beta + \text{tr}(X \circ D) \end{pmatrix}.$$  \hfill (58)

**Remark 3.9.** Consider the space $\mathcal{M}(\text{diag}(J^\text{OC}_3)) \subset \mathcal{M}(J^\text{OC}_3)$ with elements

$$\begin{pmatrix} \alpha & X \\ Y & \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ (z_1, z_2, z_3) \\ \beta \end{pmatrix} \quad \alpha, \beta \in \mathbb{C} \quad X, Y \in J^\text{OC}_3.$$  \hfill (59)

This space is equivalent to the Freudenthal triple system $\mathcal{M}(J_3^\text{OC})$ employed by Borsten et al. \cite{5} as the representation space of three qubits. It would be interesting to find a quantum information interpretation for the full Freudenthal triple system $\mathcal{M}(J^\text{OC}_3)$.

### 3.5 Extended Freudenthal Triple System

Following Günaydin et al. \cite{28, 31}, we construct a vector space $\Sigma(J^\text{OC}_3)$ by extending the FTS $\mathcal{M}(J^\text{OC}_3)$ by an extra complex coordinate:

$$\Sigma(J^\text{OC}_3) = \mathcal{M}(J^\text{OC}_3) \oplus \mathbb{C}.$$  \hfill (60)

For brevity, we refer to the vector space $\Sigma(J^\text{OC}_3)$ as the extended Freudenthal triple system (EFTS) over $J^\text{OC}_3$. Vectors in $\Sigma(J^\text{OC}_3)$ are written in the form $X = (\mathcal{X}, \tau)$, where $\mathcal{X} \in \mathcal{M}(J^\text{OC}_3)$ belongs to the underlying FTS and $\tau \in \mathbb{C}$ is the extra complex coordinate. The quartic symplectic distance $d(X, Y)$ between any two points $X = (\mathcal{X}, \tau)$ and $Y = (\mathcal{Y}, \kappa)$ in $\Sigma(J^\text{OC}_3)$ is given by

$$d(X, Y) = q(\mathcal{X} - \mathcal{Y}) - (\tau - \kappa + \{\mathcal{X}, \mathcal{Y}\})^2.$$  \hfill (61)

The norm of an arbitrary element $X = (\mathcal{X}, \tau) \in \Sigma(J^\text{OC}_3)$ takes the form

$$N(X) = d(X, 0) = q(\mathcal{X}) - \tau^2.$$  \hfill (62)
The group leaving light-like separations $d(\mathbf{X}, \mathbf{Y}) = 0$ invariant, the quasi-conformal group of the EFTS \cite{10, 13, 28, 31}, is now $\text{Inv}(\mathcal{T}(J_3^{\mathbb{O}_C})) = E_8(\mathbb{C})$. The action of the Lie algebra of $\text{Inv}(\mathcal{T}(J_3^{\mathbb{O}_C}))$ on an arbitrary element of the EFTS $\mathbf{X} = (\mathbf{X}, \tau)$ is given by \cite{28, 31}:

\begin{align}
K(\mathbf{X}) &= 0 \
U(\mathbf{X}) &= \mathcal{W} \quad S(\mathbf{X}) = T(\mathcal{W}, \mathcal{Z}, \mathbf{X}) \quad \mathcal{W}, \mathcal{Z} \in \mathfrak{H}(J_3^{\mathbb{O}_C}) \
K(\tau) &= 2z \
U(\tau) &= \{\mathcal{W}, \mathbf{X}\} \quad S(\tau) = 2\{\mathcal{W}, \mathcal{Z}\} \tau \
\tilde{U}(\mathbf{X}) &= \frac{1}{2} T(\mathbf{X}, \mathcal{W}, \mathcal{X}) - \mathcal{W}\tau \
\tilde{U}(\tau) &= -\frac{1}{8} \{ T(\mathbf{X}, \mathbf{X}, \mathbf{X}), \mathcal{W} \} + \{X, \mathcal{W}\} \tau \
\tilde{K}(\mathbf{X}) &= -\frac{1}{8} z T(\mathbf{X}, \mathbf{X}, \mathbf{X}) + z \mathcal{X} \tau \
\tilde{K}(\tau) &= \frac{1}{8} z \{ T(\mathbf{X}, \mathbf{X}, \mathbf{X}), \mathbf{X}\} + 2z \tau^2.
\end{align}

As noted by G"unaydin, Koepsell and Nicolai \cite{28}, this gives a realization of $E_8(\mathbb{C})$ on $\mathbb{C}^{57}$, which remedies the problem encountered when $N(\mathbf{X}) = 0$ and the quartic invariant $q$ takes negative values, forcing $\tau$ to have non-real solutions.

4 Conclusion

We have shown that $\mathcal{N} = 8$ and $\mathcal{N} = 2$ supergravity theories based on the octonions and split-octonions can be mathematically unified using the bioctonion composition algebra and its corresponding exceptional Jordan $C^*$-algebra, $J_3^{\mathbb{O}_C}$. Moreover, by constructing a Freudenthal triple system and its single variable extension over $J_3^{\mathbb{O}_C}$, problematic solutions in $D = 3$ were resolved. The exceptional Jordan $C^*$-algebra $J_3^{\mathbb{O}_C}$ and its Freudenthal triple system also proved useful in supporting the three qubit entanglement classification of Borsten et al.\cite{5}.

Surely, there are further applications for Jordan algebraic structures based on the bioctonions, and it is interesting to consider the direct physical interpretations of such structures in M-theory \cite{35, 43, 47-51}. Along these lines, it is essential to consider structures over the integral bioctonions, enabling the study of the discrete U-duality orbits of $E_6(\mathbb{C})_Z$, $E_7(\mathbb{C})_Z$ and $E_8(\mathbb{C})_Z$, with applications to topological strings \cite{10, 12}, quantum information theory \cite{1-9, 32-41} and automorphic black hole partition functions.

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