LOCAL REDUCIBILITY OVER THE CENTER OF WEIGHT SPACE

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Abstract. A conjecture of Breuil, Buzzard, and Emerton says that the slopes of certain reducible $p$-adic Galois representations must be integers. In previous work we have shown this conjecture for representations that do not lie over “subtle” components of weight space. By a more detailed application of the Taylor approximation trick used in that work we show that the conjecture is true for slopes up to $\frac{p-1}{2}$ over the “subtle” components as well. We also completely classify the aforementioned representations over the “non-subtle” components of weight space, both for integer and non-integer slopes.

1. Introduction

1.1. Background. Let $p$ be an odd prime number and $k \geq 2$ be an integer, and let $a$ be an element of $\mathbb{Z}_p$ such that $v_p(a) > 0$. Let us denote $\nu = \lfloor v_p(a) \rfloor + 1 \in \mathbb{Z}_{>0}$. With this data one can associate a certain two-dimensional crystalline $p$-adic representation $V_{k,a}$ with Hodge–Tate weights $(0, k-1)$. We give the definition of this representation in section 2 of [Arso], and we define $\overline{V}_{k,a}$ as the semi-simplification of the reduction modulo the maximal ideal $m$ of $\mathbb{Z}_p$ of a Galois stable $\mathbb{Z}_p$-lattice in $V_{k,a}$ (with the resulting representation being independent of the choice of lattice).

The question of computing $\overline{V}_{k,a}$ has a relatively long history, and we refer to the introduction of that note for a (very) brief exposition of it. Partial results have been obtained by Fontaine, Edixhoven, Breuil, Berger, Li, Zhu, Buzzard, Gee, Bhattacharya, Ganguli, Ghate, et al (see [Ber10], [Bre03a], [Bre03b], [Edi92], [BLZ04], [BG15], [BG09], [BG13], [GG15]). A conjecture of Breuil, Buzzard, and Emerton says the following.

Conjecture A. If $k$ is even and $v_p(a) \not\in \mathbb{Z}$ then $\overline{V}_{k,a}$ is irreducible.

The main result of [Arso] is that this conjecture is true over certain “non-subtle” components of weight space (which in this context correspond to congruence classes of $k$ modulo $p-1$), of which there are $\max\left\{\frac{p-1}{2} - \nu + 1, 0\right\}$ many, and in fact that $\overline{V}_{k,a}$ is irreducible when $v_p(a) \not\in \mathbb{Z}$ and

$$k \not\equiv 3, 4, \ldots, 2\nu, 2\nu + 1 \mod p - 1$$

(without any condition on the parity of $k$). Let us briefly describe the method of proof and where it fails for these $\min\{2\nu - 1, p-1\}$ (significantly more subtle) components of weight space. Let us write $\delta$ for the number in $\{0, \ldots, p-2\}$ which is congruent to $h \mod p - 1$ and $\delta$ for the number in $\{1, \ldots, p-1\}$ which is congruent

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to \( k \mod p - 1 \). Let \( s = k - 2 \), so that in particular \( s \not\in \{1, \ldots, 2\nu - 1\} \), i.e. \( s \geq 2\nu \).

Let \( \mathcal{W} \) denote the rigid analytic space of continuous characters \( \mathbb{Z}_p^\times \to \mathbb{C}_p^\times \) (i.e. “weight space”), which comprises \( p - 1 \) open unit disks. For a weight \( \chi \) let us denote by \( \mathcal{W}_\chi \) the component of \( \mathcal{W} \) containing \( \chi \). By the “center” \( \mathcal{W}_c \) of \( \mathcal{W} \) we mean the subset of points \( \chi \) such that \( |\chi(1 + p) - 1| \leq p^{-1} \), and we denote \( \mathcal{W}_c^\chi = \mathcal{W}_\chi \cap \mathcal{W}_c \).

We identify the integral point \( x^{t-2} \) of \( \mathcal{W}_c^\chi \) with the integer weight \( t \in \mathbb{Z} \), and in this note we only concern ourselves with integer weights in \( \mathbb{Z}_{\geq 2} \) (though all of our results are vacuously true for integer weights \( k \leq \nu \) as the set of all representations \( \mathcal{W}_{k,a} \) associated with such a weight is empty). Let us consider the set

\[
B_0 = \{ s + \beta(p - 1) + 2 | \beta \in \{0, \ldots, \nu - 2\}\}
\]

of “Berger–Li–Zhu points”. In particular, if \( \nu = 1 \) then \( B_0 = \emptyset \), and in general \( B_0 \) is a set of \( \nu - 1 \) points in the component of weight space corresponding to \( k \). This is a set of integer weights \( b > 2 \) that satisfy the condition \( v_p(a) > \left\lfloor \frac{b^{-1}}{2} \right\rfloor \) imposed in \([BLZ04]\), and so if \( b \in B_0 \) then \( \mathcal{W}_{b,a} \cong \text{ind}(\omega_2^{b^{-1}}) \) since \( s - 2\beta + 1 \in \{5, \ldots, p\} \) and therefore \( p + 1 \nmid b - 1 \). Therefore we expect by local constancy of \( \mathcal{W}_{k,a} \) in the argument \( k \) that \( \mathcal{W}_{k,a} \cong \text{ind}(\omega_2^{b^{-1}}) \) when \( k \) lies in a small disk centered at \( b \), and this is indeed implied for \( \nu - 2 \) of the points in \( B_0 \) by theorem B in \([Ber10]\) and for the last point by proposition 3.9 in \([Che13]\). On the other hand, if \( k \) is sufficiently far away from \( B_0 \) (more precisely, if it is in the complement of the disjoint union of open disks with radii 1 centered at the points of \( B_0 \)) and if \( k \) is in a “non-subtle” component of weight space and \( v_p(a) \not\in \mathbb{Z} \), then we can show that \( \mathcal{W}_{k,a} \cong \text{ind}(\omega_2^{b^{-1}}) \otimes \omega^{\nu - 1} \). Therefore we might expect that the answer (of the question of computing \( \mathcal{W}_{k,a} \)) becomes complicated precisely in small neighborhoods of \( B_0 \) as \( v_p(a) \) gets large. Indeed, there is a structure of concentric circles centered at points of \( B_0 \) which splits the center of the corresponding component of weight space into regions that form a bridge from the region of “points very far from \( B_0 \)” to the region of “points very near \( B_0 \)”. This is perhaps best illustrated by describing more precisely what the main result of \([Ars]\) tells us about the representations \( \mathcal{W}_{k,a} \) — while in \([Ars]\) it is compactly stated in terms of the irreducibility of \( \mathcal{W}_{k,a} \), the proof in that note gives us the following more detailed partial classification. For \( l \in \mathbb{Z} \), let us define \( \text{Irr}_l \) as

\[
\text{Irr}_l = \text{ind}(\omega_2^{k - 2\nu + 2l + 1}) \otimes \omega^{\nu - 1 - l}.
\]

In particular, if \( b = s + \beta(p - 1) + 2 \in B_0 \) then \( \mathcal{W}_{b,a} \cong \text{Irr}_{\nu - \beta - 1} \).

**Theorem 1.** Suppose that \( k \) is in a “non-subtle” component of weight space \( \mathcal{W}_k \), i.e. that \( k \not\in 3, 4, \ldots, 2\nu, 2\nu + 1 \mod p - 1 \). For \( j \geq 0 \) and \( b_\beta = s + \beta(p - 1) + 2 \in B_0 \) let \( R_{j,\beta} = \{ x \in \mathcal{W}_k^\chi | v_p(x - b_\beta) > j \} \) be the open disk of radius \( p^{-j} \) centered at \( b_\beta \). Let \( \mathcal{A}_0 = \mathcal{W}_k^\chi \setminus \cup_{b_\beta \in B_0} R_{0,\beta} \) be the complement in \( \mathcal{W}_c^\chi \) of the disjoint union of the open disks \( R_{0,0}, \ldots, R_{0,\nu - 2} \), and for \( j \geq 0 \) let \( \mathcal{A}_{j,\beta} = R_{j-1,\beta} \setminus R_{j,\beta} \) be the half-open annulus bounded by the two circles of radii \( p^{-j} \) and \( p^{-1} \) centered at \( b_\beta \). Thus \( \mathcal{W}_c^\chi \) is partitioned into the disjoint sets

\[
\{ \mathcal{A}_0 \} \cup \{ \mathcal{A}_{j,\beta} | j > 0 \text{ and } b_\beta \in B_0 \}.
\]

\(^1\)There is exactly one other integer weight \( b \geq 2 \) in the component of weight space corresponding to \( k \) that satisfies the condition \( v_p(a) > \left\lfloor \frac{b^{-1}}{2} \right\rfloor \), and it is precisely \( b = s + (\nu - 1)(p - 1) + 2 \) if \( s < p - 1 \) and \( b = 2 \) if \( s = p - 1 \). However, the answer in the open disk of radius 1 centered at this point is the same as the “generic” answer in the region \( \mathcal{A}_0 \) (in the notation of theorems \([1]\) and \([2]\)), so we omit mentioning it. This point does seem to play a role for the “subtle” components of weight space, however.
If \( v_p(a) \not\in \mathbb{Z} \) then

\[
\nabla_{k,a} \cong \begin{cases} 
\text{Irr}_0 & \text{if } k \in \mathcal{R}_0, \\
\text{Irr}_0 \text{ or } \ldots \text{ or } \text{Irr}_{\min(j, \nu - \beta - 1)} & \text{if } k \in \mathcal{R}_{j, \beta}.
\end{cases}
\]

Though not explicitly stated, this is as much one can directly extract from the main proof in [Arta]. In this note we complete this theorem by showing that \( \nabla_{k,a} \cong \text{Irr}_{\min(j, \nu - \beta - 1)} \) whenever \( k \in \mathcal{R}_{j, \beta} \) (theorem 2). This result is known for \( \nu = 1 \) by Buzzard and Gee in [BG09] and for \( \nu = 2 \) by Bhattacharya and Ghat in [BG15], and it also fits with the expectation that \( \nabla_{k,a} \cong \text{Irr}_{\nu - \beta - 1} \) sufficiently close to \( b_\beta \in B_0 \). We also prove a similar theorem for \( v_p(a) \in \mathbb{Z} \) (theorem 3). This result is known for \( \nu = 2 \) by Bhattacharya, Ghat, and Rozensztajn in [BGR13]. Let us give a brief description of the main idea in the proof of theorem 1. The proof is based on the idea (which originates in [BG09] and [BG13]) to compute Galois representations by computing the bijectively associated \( GL_2(\mathbb{Q}_p) \)-representations (via the local Langlands correspondence). Seeing as there is a significant amount of complexity in the \( \epsilon \)-neighborhoods of \( B_0 \) as \( v_p(a) \) gets large, the main idea in [Arta] is to look at families of elements \( f(k) \) of these \( GL_2(\mathbb{Q}_p) \)-representations as varying in the argument \( k \), so that in a certain sense

\[ f(k) = f(b + \epsilon) \approx f(b) + \epsilon f'(b) \]

for a weight \( k \) that is close to a point \( b \in B_0 \), where the last “equality” indicates that we exploit the fact that the local Langlands method “fails” (in the vague sense that the descriptions of the \( GL_2(\mathbb{Q}_p) \)-representations are not human-friendly) at the discrete set of points \( B_0 \). The main difficulty of making this Taylor approximation trick work is in defining the derivative, and we only do that after discarding a lot of data by forgetfully encoding \( f \) as a certain set of matrices and then only defining the derivatives of these matrices modulo \( p \). If \( A_f(k) \) is a matrix associated with \( f(k) \) in this fashion (with the entries of \( A_f(k) \) belonging to \( \mathbb{Z}_p \) and being functions of \( k \)) then the aforementioned vague “equality” can be interpreted as saying that \( A_f(b) \) has non-trivial right kernel and that the reduction modulo \( p \) of the restriction of \( A_f(b) \) to it has full rank (as a map of \( \mathbb{F}_p \)-vector spaces). One of the two main points at which the proof in [Arta] breaks down for the “subtle” weights \( k \equiv 3, \ldots, 2\nu + 1 \mod p - 1 \) is that the derivatives of some of these matrices are not integral and therefore they (and the corresponding combinatorial equations) are not even properly defined over \( \mathbb{F}_p \). The other is that the “subtle” weights are precisely those such that the possible factors of the \( GL_2(\mathbb{Q}_p) \)-representations resulting from lemmas 4.1 and 4.3 and remark 4.4 in [BG09] can pair up to form a reducible representation in unusual ways. While this makes the computations too difficult to solve the conjecture completely (even after we resolve these two issues), we can obtain a similar result to the one in [Arta] for the “subtle” components of weight space conditional on a restriction on the prime (theorem 4).

1.2. Main results. The first result is a classification of \( \nabla_{k,a} \) over the “non-subtle” components of weight space for \( v_p(a) \not\in \mathbb{Z} \).

**Theorem 2.** Suppose that \( k \not\equiv 3, 4, \ldots, 2\nu + 1 \mod p - 1 \) and \( \mathcal{W}_k^c \) is partitioned into the disjoint sets

\[
\{ \mathcal{R}_0 \} \cup \{ \mathcal{R}_{j, \beta} \mid j > 0 \text{ and } b_\beta \in B_0 \}
\]
as in theorem \[\text{[1]}\]. If \(v_p(a) \not\in \mathbb{Z}\) then
\[
\mathcal{V}_{k,a} = \begin{cases} 
\text{Irr}_0 & \text{if } k \in \mathcal{R}_0, \\
\text{Irr}_{\min(j,\nu-\beta-1)} & \text{if } k \in \mathcal{R}_{j,\beta}.
\end{cases}
\]
Informally speaking, we already know that if the answer is irreducible then it must be one of \(\text{Irr}_0, \ldots, \text{Irr}_{\nu-1}\), where these representations arise from a filtration by powers of \(\theta\) as in lemmas 4.1 and 4.3 and remark 4.4 in [BG09]. Moreover, we would expect that the “generic” answer is \(\text{Irr}_0\). However, over the “non-subtle” components of weight space there exist exactly \(\nu\) weights \(k \geq 2\) that satisfy the Berger–Li–Zhu condition \(v_p(a) > (\frac{1}{p^2-1})\), and we know by [BLZ04] that the answers at these points are \(\text{Irr}_0, \ldots, \text{Irr}_{\nu-1}\). Theorem \[\text{[2]}\] says that the answer is indeed \(\text{Irr}_0\) outside of the union of open disks of radii 1 centered at these points. Moreover, the integral points in the interior of such a disk belong to the union of circles of radii \(p^{-1}, p^{-2}, \ldots\) and theorem \[\text{[2]}\] says that the answers on these circles are \(\text{Irr}_1, \text{Irr}_2, \ldots\) until they eventually stabilize at some \(\text{Irr}_1, \text{Irr}_2, \ldots\) and the \(\text{Irr}_\ell\) in question is \(\text{Irr}_0\) this means that the answer is \(\text{Irr}_0\) on each circle, so this point is “invisible” and that is why we omit it in the statement of theorem \[\text{[2]}\].

The second result is a classification of \(\mathcal{V}_{k,a}\) over the “non-subtle” components of weight space for \(v_p(a) \in \mathbb{Z}\). For \(l \in \mathbb{Z}\) and \(\lambda \in \overline{\mathbb{F}_p^*}\), let us define \(\text{Red}_l(\lambda)\) as
\[
\text{Red}_l(\lambda) = \frac{\mu_\lambda \omega^{s+l-\nu+2}}{\mu_{\lambda-1} \omega^{\nu-l-1}}.
\]

**Theorem 3.** Suppose that \(k \not\equiv 3, 4, \ldots, 2\nu, 2\nu + 1 \mod p - 1\) and \(\mathcal{W}_k^c\) is partitioned into the disjoint sets
\[
\{\mathcal{R}_0\} \cup \{\mathcal{R}_{j,\beta} \mid j > 0 \text{ and } b_\beta \in B_0\}
\]
as in theorem \[\text{[1]}\]. If \(v_p(a) = \nu - 1 \in \mathbb{Z}_{>0}\) then
\[
\mathcal{V}_{k,a} = \begin{cases} 
\text{Red}_0(\lambda) & \text{if } k \in \mathcal{R}_0, \\
\text{Red}_j(\lambda_{j,\beta}) & \text{if } k \in \mathcal{R}_{j,\beta} \text{ and } 0 < j < \nu - \beta - 1, \\
\text{Irr}_{\nu-\beta-1} & \text{if } k \in \mathcal{R}_{j,\beta} \text{ and } j \geq \nu - \beta - 1,
\end{cases}
\]
where
\[
\lambda = \frac{(\nu - \beta + 2)a}{(\nu - 1)p^{\nu-1}} \in \overline{\mathbb{F}_p^*},
\]
\[
\lambda_{j,\beta} = \frac{(-1)^{s+j+1}(\nu-1)(\nu-j-1)(\nu-j-2)(\nu-j-2)a}{(r-s-\beta(p-1))p^{r-j-2}} \in \overline{\mathbb{F}_p^*}.
\]
Note that in the setting of theorem \[\text{[3]}\] the “extra” point (the one that we monikered “invisible” in the discussion after theorem \[\text{[2]}\]) is either no longer a Berger–Li–Zhu point or it is \(k = 2\) in which case there are no representations \(\mathcal{V}_{k,a}\) above it—we exclude that point from \(B_0\) in order to give a more unified exposition of theorems \[\text{[2]}\] and \[\text{[3]}\]. The two theorems together give a complete classification of the representations \(\mathcal{V}_{k,a}\) over the “non-subtle” components of weight space, both for integer and non-integer slopes. In the discussion after theorem \[\text{[4]}\] we give a more conceptual characterization of the “non-subtle” components of weight space as (roughly) those components for which we get the same set of centers by using the bound from [BLZ04] as we do by using the optimal bound (which is \(\approx \frac{k-1}{p+1}\)).
The third result is a confirmation of conjecture A over the “subtle” components of weight space for slopes up to $\frac{p-1}{2}$.

**Theorem 4.** Conjecture A is true for slopes up to $\frac{p-1}{2}$.

This is relatively more difficult and the ad hoc nature of the proof involving many case studies means that we cannot deduce a full classification as in theorems 2 and 3. As for the scope of theorem 4 let us note that when $\nu_p(a) < \frac{p-1}{2}$ exactly half of all components of weight space are “subtle” and half are not, e.g. if $\nu = 1$ then only $k \equiv 2 \mod p-1$ is a “subtle” component of weight space (as evidenced by [BG09] and [BG13]), while for $\nu = \frac{p-1}{2}$ only $k \equiv 2 \mod p-1$ is a “non-subtle” component of weight space.

We can make several predictions about the classification of $V_{k,a}$ over the “subtle” components. Firstly, at least when $\nu \leq \frac{p-1}{2}$, there should be a pair of exceptional disks (in the parameter $a$) where the slope is $\nu - \frac{1}{2}$ over the component where $k \equiv 2\nu + 1 \mod p - 1$. This is because that is the component where the exceptional disks appear on the boundary (see [Arsb]) and the two components are known to match for $\nu \in \{1, 2\}$ (though the situation is not completely analogous over the boundary as e.g. there are $2(p - 1)$ exceptional disks for each $\nu$ appearing there). Secondly, outside of these two exceptional disks, the center of the corresponding component of weight space should be partitioned into regions by the concentric circles of radii $p^{-1}, p^{-2}, \ldots$ centered at certain points, and the isomorphism class of $V_{k,a}$ should depend on the region that $k$ belongs to. These centers should be found by finding the points at which certain associated matrices $A_j$ have non-trivial right kernels, and this can be done by a process reminiscent of Hensel lifting. However, while we can use this procedure to determine the classification of $V_{k,a}$ for specific small slopes, we have not yet been able to produce a general theorem for the “subtle” components of weight space (such as theorem 2). Partial results arising from the proof of theorem 4 indicate that if $\nu \leq \frac{p-1}{2}$ and $k$ is in a “subtle” component of weight space then we need to include an extra point in addition to the set of Berger–Li–Zhu points as one of the centers—the philosophical reason behind this is likely that the “correct” bound for the local constancy of $V_{k,a}$ around $a = 0$ should be $\nu_p(a) > \frac{k-1}{p+1}$ (see the discussion around conjecture 2.1.1 in [BG16]). This extra center does indeed appear, e.g. if $\nu = 2$ and $s = 2$ then (as shown in [BG13]) the answer is $\text{Irr}_0$ inside of the union of two open disks of radii 1 centered at $p + 3$ and $2p + 2$, and $\text{Irr}_1$ otherwise. The “non-subtle” components of weight space are very nearly those for which the set of points satisfying the condition $\nu > \frac{k-2}{p-1}$ is the same as the set of points satisfying the condition $\nu \geq \left\lfloor \frac{b}{p-1} \right\rfloor$ (equivalently $\nu > \left\lfloor \frac{k}{p+1} \right\rfloor$), with the minor exception $b = \nu(p - 1) + 2$. The reason why this exception does not represent a problem is because it is “invisible” in theorems 2 and 3—the representation associated with it is the same as the “generic” representation. It is very apparent why this is relevant: the smallest weight which ceases to be a center (or second smallest if the smallest is “invisible”) must also be beyond the optimal radius of local constancy of $V_{k,a}$ around $a = 0$. Thus it seems entirely feasible to conjecture that if $B_v$ denotes the set of integer weights $b \geq 2$ that satisfy the condition $\nu \geq \left\lfloor \frac{b-1}{p-1} \right\rfloor$ and if the regions $\mathcal{R}_0$ and $\mathcal{R}_{j,b}$ for $j > 0$ and $b \in B_v$ are defined similarly as in theorem 1 then $V_{b,a} \cong \text{ind}(\omega_2^{b-1})$ for $b \in B_v$, and $V_{k,a}$ is constant on each region with the exception of open disks that appear in the component $k \equiv 2\nu + 1 \mod p - 1$. Here the bound in the definition of the set
of centers slightly deviates from the expected radius of the optimal disk around \( a = 0 \) on which \( V_{k,a} \) is constant: this is inevitable since the nature of the bound (it having to yield the same result for all slopes in the range \((\nu - 1, \nu)\)) necessarily entails it should be an integer. It is not clear a priori whether the definition of this integer should involve a floor or a ceiling; we note that (for a fixed \( \nu \)) the weight for which there is ambiguity (i.e. the weight \( b \geq 2 \) for which the conditions \( \nu \geq \left\lfloor \frac{b}{p+1} \right\rfloor \) and \( \nu > \left\lfloor \frac{b}{p+1} \right\rfloor \) give different sets of centers) satisfies \( k \equiv 2\nu + 1 \mod p - 1 \) (even if \( \nu > \frac{k}{p+1} \)). Since the exceptional disks on which the slope is a half-integer should be precisely over this component of weight space, it is possible that they can be interpreted as some sort of phenomenon occurring on the boundary of the optimal disk of local constancy around \( a = 0 \).

As in both of the main results we have \( v_p(a) < \frac{\nu - 1}{2} \), we assume once and for all that \( \nu \in \{1, \ldots, \frac{p-1}{2}\} \). Finally, we note that there is nothing to prevent the method from working when \( \nu > \frac{p-1}{2} \) other than the complexity of the computations, and in fact we believe that e.g. it is possible to write a computer program which takes as input a positive integer \( m \) and verifies conjecture [A] for slopes up to \( m \) (evidently by verifying it for the primes that are less than \( 2m \)).

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2. **Assumptions, notation, and technical lemmas**

2.1. **Assumptions and notation.** In this subsection we introduce all of the assumptions which we make and the notation which we use throughout the note. As a significant amount of overlapping notation has been introduced in [Arso], we refer to section 2 of it for the basics which we collate in a tabular as follows.

| \( G, K, Z \) | \( G = \text{GL}_2(\mathbb{Q}_p) \), \( K = \text{GL}_2(\mathbb{Z}_p) \), and \( Z \) is the center of \( G \). |
|---|---|
| \( k, r, s \) | \( k \geq 2 \) is the weight, \( r = k - 2 \), and \( s = \tau \). |
| \( a, \nu \) | \( a \) is the eigenvalue, \( \nu = \lceil v_p(a) \rceil + 1 \in \{1, \ldots, \frac{p-1}{2}\} \). |
| \( m \geq 0 \) | means that \( m \) is sufficiently large. |
| \( g \bullet_{H,\mathcal{W}} w \) | the element of \( \text{ind}_H^G \mathcal{W} \) supported on \( Hg^{-1} \) and mapping \( g^{-1} \mapsto w \). |
| \( I_t \) | \( \mathbb{F}_p[KZ] \)-module of degree \( t \) maps \( \mathbb{F}_p^2 \to \mathbb{F}_p \) that vanish at the origin. |
| \( \sigma_t \) | \( \text{Sym}^t(\mathbb{F}_p^2) \). |
| \( V(m) \) | the twist \( V \otimes \det^m \). |
| \( \Sigma_{k-2} \) | \( \text{Sym}^{k-2}(\mathbb{F}_p^3) \otimes |\det|^{\frac{k-2}{2}} \). |
| \( \Sigma_{k-2} \) | the reduction of \( \text{Sym}^{k-2}(\mathbb{Z}_p^3) \otimes |\det|^{\frac{k-2}{2}} \) modulo \( m \). |
| \( T \) | the Hecke operator corresponding to the double coset of \((\begin{smallmatrix} 0 & 1 \\ \tau & 0 \end{smallmatrix})\). |
| \( \mathcal{H} \) | the number in \( \{1, \ldots, p - 1\} \) that is congruent to \( h \mod p - 1 \). |
| \( \mathcal{H} \) | the number in \( \{0, \ldots, p - 2\} \) that is congruent to \( h \mod p - 1 \). |
| \( \mathcal{O}(\alpha) \) | \( \text{sub-} \mathbb{Z}_p \)-module of multiples of \( \alpha \); also used for a term \( f \in \mathcal{O}(\alpha) \). |
| \( \theta \) | \( \theta = xy^p - x^ty \). |
| \( N_\alpha \) | \( N_\alpha = \theta \Sigma_{-\alpha(p+1)}/\theta^{\alpha+1}\Sigma_{-\alpha(p+1)} \cong I_{r-2\alpha}(\alpha) \). |
\( (X_n)_n \) is the Hecke operator corresponding to the double coset of \( (\Sigma) \). Let \( \omega \) be the determinant of \( \omega \) and \( \mu \). Then there is also a filtration \( \Theta_{k,a} \) such that \( \Theta_{k,a} \) is the kernel of the quotient map \( \text{ind}^G_{KZ} \Sigma_{k-2} \rightarrow \Theta_{k,a} \).

Recall that there is the explicit formula for \( T \) given by

\[
T(\gamma \cdot KZ, \pi, v) = \sum_{\nu \in \mathbb{P}_p} \gamma(\nu) \cdot KZ, \pi, v
\]

where \( \xi \) is the Teichmüller lift of \( \xi \in \mathbb{P}_p \) to \( \mathbb{Z}_p \).

There is a bijective correspondence between Galois representations and \( GL_2(\mathbb{Z}_p) \)-representations given as follows (see theorem A in [Ber10]).

**Theorem 5.** For \( t \in \{0, \ldots, p - 1\} \), \( \lambda \in \mathbb{P}_p \), and a character \( \psi : \mathbb{Q}^\times_p \rightarrow \mathbb{P}_p \) let

\[
\pi(t, \lambda, \psi) = (\text{ind}^G_{KZ} \sigma/(T - \lambda)) \otimes \psi,
\]

where \( T \) is the Hecke operator corresponding to the double coset of \( \omega \). Let \( \omega \) be the mod \( p \) reduction of the cyclotomic character. Let \( \text{ind}(\omega_2^{t+1}) \) be the unique irreducible representation whose determinant is \( \omega_2^{t+1} \) and that is equal to \( \omega_2^{t+1} \otimes \omega_2^{p(t+1)} \) on inertia. Then there is the following bijective correspondence.

\[
\bigwedge_{k,a} (V_k, a) \\
\text{ind}(\omega_2^{t+1}) \otimes \psi \iff \left( \bigwedge_{k,a} (V_k, a) \right) \otimes \psi.
\]

Theorem 5 makes our goal of computing \( \bigwedge_{k,a} \) equivalent to computing \( \Theta_{k,a} \). We refer by \( \text{im}(T - a) \) to the image of the map \( T - a \in \text{End}(\text{ind}^G_{KZ} \Sigma_{k-2}) \). If an element in \( \text{ind}^G_{KZ} \Sigma_{k-2} \) is the reduction modulo \( m \) of an element in \( \text{im}(T - a) \) then it is also in the “kernel” \( \mathcal{J}_a \). We define \( \tilde{N}_0, \tilde{N}_1, \ldots \) to be the subquotients of the filtration

\[
\text{ind}^G_{KZ} \Sigma_r \supset \text{ind}^G_{KZ} \Sigma_r^{p-1} \supset \cdots \supset \text{ind}^G_{KZ} \Sigma_r^{p(a+1)} \supset \cdots
\]

and since \( \Theta_{k,a} \) is the kernel of \( \text{ind}^G_{KZ} \Sigma_r / \mathcal{J}_a \) there is also a filtration

\[
\Theta_{k,a} \supset \Theta_0 \supset \cdots \supset \Theta_a \supset \cdots
\]

whose subquotients are quotients of \( \tilde{N}_a \) (so there is a surjection \( \tilde{N}_a \twoheadrightarrow \Theta_{k,a} / \Theta_{k,a+1} \)).

Lemmas 4.1 and 4.3 and remark 4.4 in [BG09] imply that the ideal \( \mathcal{J}_a \subseteq \text{ind}^G_{KZ} \Sigma \) contains \( 1 \cdot KZ, \pi, v \) for any polynomial \( h \) and \( 1 \cdot KZ, \pi, v \) for all \( 0 \leq j < [v_\pi(a)] - \delta_{v_\pi(a)} \in \mathbb{Z}_p \).

and thus \( \Theta_{k,a} \) is a subquotient of

\[
\text{ind}^G_{KZ}(\Sigma_r / (y_0^{r+1} - \delta_{v_\pi(a)} y_0^{r+1} + \delta_{v_\pi(a)} y_0^{r+1} + 1)) \cong (\Sigma_r / \mathcal{J}_a)
\]

a module which has a series whose factors are subquotients of \( \tilde{N}_0, \ldots, \tilde{N}_{\nu-1} \). Let us denote

\[
\text{sub}(a) = \sigma_\nu - 2\sigma(a) \subset N_a,
\]

then \( N_a / \sigma_\nu - 2\sigma(a) \cong \sigma_{2\sigma_\nu - r - \sigma(a)} \).
Finally, for a family \( \{D_i\}_{i \in \mathbb{Z}} \) of elements of \( \mathbb{Z}_p \) and for \( w \geq 0 \) we define
\[
\vartheta_w(D_i) = \sum_i D_i \left( \frac{i(p-1)}{w} \right).
\]
Thus, in the terminology introduced in [Ars], we only work with the “nice” family \( \{f\} \) defined by \( f_w(X) = \left( \frac{X^{(p-1)}}{w} \right) \).

2.2. Technical lemmas. In this subsection we give a list of combinatorial results and strengthened versions of the key lemmas 5, 6, and 7 in [Ars]. The following lemma is a collection of combinatorial results used throughout section 3.

**Lemma 6.** (1) If \( m, u, v \geq 1 \) and \( n \in \mathbb{Z} \) and \( u \equiv v \mod (p-1)p^{m-1} \) then

\[
M_{u,n} \equiv M_{v,n} \mod p^m.
\]
(\(c-a\))

If \( u = t_u(p-1) + s_u \) with \( s_u = \pi \) then
(\(c-b\))

\[
M_u = 1 + \delta_{w \equiv p-1} + \delta_{u \equiv p-1} + O(t_up^2).
\]
(\(c-c\))

If \( n \leq 0 \) then
(\(c-d\))

\[
M_{u,n} = \sum_{i=0}^{-n} (-1)^i \binom{-n}{i} M_{u-n,i}.
\]
(\(c-e\))

If \( n \geq 0 \) then

(\(c-f\))

\[
\binom{l}{t+i} \binom{l}{w} = \sum_v (-1)^{w-v} \binom{l+w-v-1}{w-v} \binom{l}{t+i-v} \binom{l}{w-v}.
\]

Consequently, if \( u, m, l, w \geq 0 \) are such that \( u + l \geq m \) then

(\(c-g\))

\[
\sum_i \binom{u-m+l}{i(p-1)} \binom{i(p-1)}{w} = \sum_v (-1)^{w-v} \binom{l+w-v-1}{w-v} \binom{l+m}{w-v} M_{u-m+l-v,v}.
\]

(2) Let \( \{D_i\}_{i \in \mathbb{Z}} \) be a family of elements of \( \mathbb{Z}_p \) such that \( D_i = 0 \) for \( i \not\in \left[ \frac{\pi}{p-1}, \frac{\pi}{q} \right] \) and \( \vartheta_w(D_i) = 0 \) for all \( 0 \leq w < \alpha \). Then

\[
\sum_i D_i x^{i(p-1)+\alpha} y^{r-i(p-1)-\alpha} = \theta^* h
\]

for some \( h \).

(3) For \( u, v, c \in \mathbb{Z} \) let

\[
F_{u,v,c}(X) = \sum_w (-1)^{w-c} \binom{w}{c} \frac{X}{w}^\partial \left( \frac{X+u-w}{v-w} \right) \in \mathbb{Q}_p[X].
\]

Then

\[
F_{u,v,c}(X) = \left( \frac{u}{v-c} \right) \binom{X}{c}^\partial - \left( \frac{u}{v-c} \right)^\partial \binom{X}{c}.
\]
Proof. Parts (1) from (c) to (g) and (2) are demonstrated in lemma 3 in [Arfaa].

(3) If $c < 0$ or $v < c$ then the claim is trivial, so let us assume that $v \geq c \geq 0$. It is enough to show that $\Phi'(0) = 0$ for

$$\Phi(z) = \sum_{w} (-1)^{w-c} \left( \frac{z+X}{c \cdot w} \right) \left( \frac{X+u-w}{v-w} \right) - \left( \frac{z+X}{c \cdot v-w} \right) \in \mathbb{Q}_p[X][z].$$

This follows from

$$\sum_{w} (-1)^{w-c} \left( \frac{z+X}{c \cdot w} \right) \left( \frac{X+u-w}{v-w} \right) = (-1)^{v-c} \left( \frac{z+X}{c \cdot v-w} \right) \left( \frac{X+u-w}{v-w} \right)$$

$$= (-1)^{v-c} \left( \frac{z+X}{c \cdot v-w} \right) \left( \frac{X+u-w}{v-w} \right)$$

$$= (-1)^{v-c} \left( \frac{z+X}{c \cdot v-w} \right) \left( \frac{X+u-w}{v-w} \right)$$

which implies that $\Phi(z)$ is the zero polynomial.

The following lemma is a slight strengthening of lemmas 5, 6, and 7 in [Arfaa].

**Lemma 7.** Suppose that $0 < \alpha < \nu$ and let $\{C_l\}_{l \in \mathbb{Z}}$ be any family of elements of $\mathbb{Z}_p$.

Suppose that the constants

$$D_l = \delta_{l=0} C_{-1} + \delta_{0 < l < r-2\alpha} \sum_{i=0}^{\alpha} C_l (r-\alpha+1)_l$$

and $v \in \mathbb{Q}$ are such that $v < v_p(\partial_{\alpha}(D_l))$ and

$$v' = \min\{v_p(a) - \alpha, v\} < v_p(\partial_{\xi}(D_l))$$

for $0 \leq \xi < \alpha$. Let

$$\vartheta' = (1-p)^{-\alpha} \vartheta_{\alpha}(D_l) - C_{-1}.$$ 

Then $\text{im}(T - a)$ contains

$$(\vartheta' + C_{-1}) \bullet KZ, \theta^\alpha x^{p-1} y^{r-\alpha(p+1)-p+1} + C_{-1} \bullet KZ, \theta^n x^{\alpha-n} y^{r-np-\alpha} + \sum_{\xi=\alpha+1}^{\alpha+1} E_{\xi} \bullet KZ, \theta^\xi h_k + F \bullet KZ, h' + H,$$

for some $n \gg 0$ and some $h_k, h', E_{\xi}, F, H$ such that

(1) $E_{\xi} = \vartheta_{\xi}(D_l) + O(p^\nu) \cup O(\vartheta_{\alpha+1}(D_l)) \cup \cdots \cup O(\vartheta_{\alpha-1}(D_l))$,

(2) if $\xi + \alpha - s \leq 2\alpha - s$ then the reduction modulo $m$ of $\theta^\xi h_k$ generates $N_{\xi}$,

(3) $v_p(F) > v'$, and

(4) $H = O(p^\nu - v_p(a) + p^\nu - \alpha)$ and if $v_p(a) - \alpha < v$ then

$$\frac{1-p^\nu}{ap^\alpha} H = g \bullet KZ, \theta^n x^{\alpha-n} y^{r-np-\alpha} + O(p^\nu - v_p(a))$$

with

$$g = \sum_{\lambda \in \mathbb{Z}_p} C_{\lambda}(p^{\nu+1}) + A(p^{\nu+1}) + \delta_{r=p-2\alpha} B(p^{\nu+1}),$$

where $A = -C_{-1} + \sum_{l=0}^{\alpha} C_l (r-\alpha+l)$ and $B = \sum_{l=0}^{\alpha} C_l (l-\alpha+1)$.

Proof. This lemma is essentially shown under a stronger hypothesis as lemma 7 in [Arfaa], building on lemmas 5 and 6. The stronger hypothesis consists of the three extra conditions that $v_p(\vartheta_{\xi}(D_l)) \geq \min\{v_p(a) - \alpha, v\}$ for all $\alpha < \xi < 2\nu - \alpha$, that $C_0 \in \mathbb{Z}_p$, and that $v_p(C_{-1}) \geq v_p(\vartheta')$. These extra conditions are not used in the actual construction of the element in (2), rather they are there to ensure that
\[ v_p(\xi) \geq \min\{v_p(\alpha) - \alpha, v\} \]
for all \( \alpha < \xi < 2\nu - \alpha \), that the coefficient of \( \binom{p - 1}{\xi} \) in \( g \) is invertible, and that we get an integral element once we divide the element
\[ (\vartheta' + C_{-1})_{\bullet}^{KZ} \varpi_p^{\alpha} x^{p-1} y^{\alpha - \alpha + 1} + C_{-1} \varpi_p^{\alpha} x^{\alpha - n} y^{\alpha - np - \alpha} \]
by \( \vartheta' \). Therefore we still get the existence of the element in (2) without these extra conditions, and to complete the proof of lemma 6 in [Arsa], we need to verify the properties of \( h_\xi, E_\xi, F, H, A \) and \( B \) claimed in (1), (2), (3), and (4). The \( h_\xi \) and \( E_\xi \) come from the proof of lemma 6 in [Arsa], and \( E_\xi \cdot_{\bullet}^{KZ} \varpi_p^{\alpha} \).

Let \( X_\xi = (-1)^{\xi + 1} X_\xi \).
This reduces modulo \( m \) to the element
\[ \sum_{\lambda \neq 0} (-\lambda)^{\xi - \alpha - 1} \binom{1}{\alpha} Y^{2 \alpha - r} = (-1)^{s - \alpha + 1} \binom{2 \alpha - s}{\alpha} \]
of
\[ \text{ind}_{KZ}(\alpha, \xi, Z, E, H, A, \text{notation for } X_\xi) = \text{ind}_{KZ}(\alpha, \xi, Z, E, H, A, \text{notation for } X_\xi). \]
This element is non-trivial and generates \( N_\xi \) if \( \xi + \alpha - s \geq 2\alpha - s \), since that forces \( 2\alpha - s > 0 \) and consequently \( X_\xi^{\xi - \alpha} Y^{2 \alpha - r} \) generates \( N_\xi \). This verifies condition (2).
Condition (3) follows from the assumption \( \nu' < v_p(\vartheta_m(D_1)) \) for \( 0 \leq w < \alpha \), as in the proof of lemma 6 in [Arsa]. Finally, condition (4) follows from the description of the error term in lemma 5 in [Arsa], as indicated in the proof of lemma 7 in [Arsa].

**Corollary 8.** Suppose that \( 0 \leq \alpha < \nu \) and let \( \{C_i\}_{i \in \mathbb{Z}} \) be any family of elements of \( \mathbb{Z}_p \) that satisfies the conditions of lemma [7]. Suppose also that \( v_p(\alpha) \notin \mathbb{Z} \) and
\[ v = \min\{v_p(\alpha) - \alpha, v\} \leq v_p(\vartheta_m(D_1)) \]
for all \( \alpha < \xi < 2\nu - \alpha \). Let
\[ \tilde{\alpha} = -C_{-1} + \sum_{i=1}^{\alpha} \binom{r - \alpha + 1}{i}. \]
If \( \ast \) then \( \ast \) is trivial modulo \( \mathcal{F}_\alpha \), for each of the following pairs
\[ (\ast, \ast) = (\text{condition, representation}). \]

(1) \( v_p(\vartheta') \leq \min\{v_p(C_{-1}), v\}, \tilde{N}_\alpha \).  

(2) \( v = v_p(C_{-1}) < \min\{v_p(\vartheta'), v_p(a) - \alpha\}, \text{ind}_{KZ}^G \text{sub}(\alpha) \).  

(3) \( v_p(a) - \alpha < v \leq v_p(C_{-1}) \text{ } \& \text{ } C \in \mathbb{Z}_p^\times \text{ } \& \text{ } C_0 \notin \mathbb{Z}_p^\times \text{ } \& \text{ } 2\alpha - r > 0, \tilde{N}_\alpha \).  

(4) \( v_p(a) - \alpha < v \leq v_p(C_{-1}) \text{ } \& \text{ } C \in \mathbb{Z}_p^\times, \text{ind}_{KZ}^G \text{quot}(\alpha) \).  

(5) \( v_p(a) - \alpha < v \leq v_p(C_{-1}) \text{ } \& \text{ } C_0 \in \mathbb{Z}_p^\times, T(\text{ind}_{KZ}^G \text{quot}(\alpha)) \).
Proof. The extra condition ensures that $v_p(E_2) \geq v'$ for all $\alpha < \xi < 2\nu - \alpha$.

(1) The condition $v_p(\vartheta') \leq \min \{v_p(C_{-1}), v'\}$ ensures that if we divide the element from (\ref{eq:lem7}) by $\vartheta'$ then the resulting element is in $\operatorname{im}(T - a)$ and reduces modulo $m$ to a representative of a generator of $\hat{N}_\alpha$.

(2) The condition $v = v_p(C_{-1}) < \min \{v_p(\vartheta'), v_p(a) - \alpha\}$ ensures that if we divide the element from (\ref{eq:lem7}) by $C_{-1}$ then the resulting element is in $\operatorname{im}(T - a)$ and reduces modulo $m$ to a representative of a generator of $\operatorname{ind}^G_{\mathbb{K}Z} \operatorname{sub}(\alpha)$.

(3) The condition $v_p(a) - \alpha < v \leq v_p(C_{-1})$ ensures that the term with the dominant valuation in (\ref{eq:lem7}) is $H$, so we can divide the element from (\ref{eq:lem7}) by $ap^{-\alpha}$ and obtain the element $L = O(p^{v - v_p(a)})$, where $L$ is defined by

$$L := \left(\sum_{\lambda \in \mathbb{F}_p} C_0^0 \alpha^1 + A(0^0_1) + \delta_{\equiv_{p-1}2\alpha} B(1^0_0)\right) \cdot \mathbb{K}Z \mathbb{F}_p^\ast \mathbb{D} x a - r \mathbb{F}_p$$

(4) The condition $v_p(a - \alpha < v \leq v_p(C_{-1})$ ensures that the term with the dominant valuation in (\ref{eq:lem7}) is $H$, so we can divide the element from (\ref{eq:lem7}) by $ap^{-\alpha}$ and obtain the element $L = O(p^{v - v_p(a)})$, where $L$ is defined by

$$L := \left(\sum_{\lambda \in \mathbb{F}_p} C_0^0 \alpha^1 + A(0^0_1) + \delta_{\equiv_{p-1}2\alpha} B(1^0_0)\right) \cdot \mathbb{K}Z \mathbb{F}_p^\ast \mathbb{D} x a - r \mathbb{F}_p$$

with $A$ and $B$ as in lemma\cite{arsa} Then $L$ reduces modulo $m$ to a representative of

$$\left(\sum_{\lambda \in \mathbb{F}_p} C_0^0 \alpha^1 + A(0^0_1) + \delta_{\equiv_{p-1}2\alpha} B(1^0_0)\right) \cdot \mathbb{K}Z \mathbb{F}_p^\ast \mathbb{D} x a - r \mathbb{F}_p$$

As shown in the proof of lemma 7 in \cite{arsa}, if $C_0 \in \mathbb{Z}_p^\times$ then this element always generates $T(\operatorname{ind}^G_{\mathbb{K}Z} \operatorname{quot}(\alpha))$, and if additionally $A \neq 0$ (over $\mathbb{F}_p$) then in fact we have the stronger conclusion that it generates $\operatorname{ind}^G_{\mathbb{K}Z} \operatorname{quot}(\alpha)$. Suppose on the other hand that $C_0 = O(p)$ and $A \in \mathbb{Z}_p^\times$. In that case we assume that $2\alpha - r > 0$ and therefore the reduction modulo $m$ of $L$ represents a generator of $\hat{N}_\alpha$.

Corollary 9. Suppose that $0 \leq \alpha < \nu$ and let $\{C_i\}_{i \in \mathbb{Z}}$ be any family of elements of $\mathbb{Z}_p$ that satisfies the conditions of lemma\cite{arsa}. Suppose also that $v_p(a) \in \mathbb{Z}$ and

$$v' = \min \{v_p(a) - \alpha, v\} \leq v_p(\vartheta(\xi(D_i)))$$

for all $\alpha < \xi < 2\nu - \alpha$. Let

$$\hat{C} = -C_{-1} + \sum_{i=1}^{\alpha} C_i (r - a + l)$$

If * then * is trivial modulo $\mathcal{I}_a$, for each of the following pairs

$$(\ast, \ast) = (\text{condition}, \text{representation}).$$

(1) \quad \left(v_p(\vartheta') \leq \min \{v_p(C_{-1}), v\} \land v_p(\vartheta') < v_p(a) - \alpha, \hat{N}_\alpha\right).

(2) \quad \left(v = v_p(C_{-1}) < \min \{v_p(\vartheta'), v_p(a) - \alpha\}, \operatorname{ind}^G_{\mathbb{K}Z} \operatorname{sub}(\alpha)\right).

(3) \quad \left(v_p(a) - \alpha < v \leq v_p(C_{-1}) \land \hat{C} \in \mathbb{Z}_p^\times \land C_0 \notin \mathbb{Z}_p^\times \land 2\alpha - r > 0, \hat{N}_\alpha\right).

(4) \quad \left(v_p(a) - \alpha < v \leq v_p(C_{-1}) \land \hat{C} \in \mathbb{Z}_p^\times, \operatorname{ind}^G_{\mathbb{K}Z} \operatorname{quot}(\alpha)\right).

(5) \quad \left(v_p(a) - \alpha < v \leq v_p(C_{-1}) \land C_0 \in \mathbb{Z}_p^\times, T(\operatorname{ind}^G_{\mathbb{K}Z} \operatorname{quot}(\alpha))\right).

(6) \quad \left(v_p(a - \alpha = v = v_p(\vartheta') \leq v_p(C_{-1}) \land \hat{C} \notin \mathbb{Z}_p^\times \land C_0 \in \mathbb{Z}_p^\times, \operatorname{rep}_1\right) \text{ for } \operatorname{rep}_1 = \left(T + \hat{C} (1^0_0) - \frac{C_0^{-1}}{ap^{-\alpha}}\right) (\operatorname{ind}^G_{\mathbb{K}Z} \operatorname{quot}(\alpha)),

where

$$\hat{C} = \delta_{\equiv_{p-1}2\alpha} \left((-1)^\alpha \sum_{i=0}^{\alpha} C_0^{-1} C_i (r - a + l) - 1\right).$$
(7) \( v_p(a) − α = v = v_p(C_{-1}) < v_p(\theta') \not\in C_0 \notin \mathbb{Z}_p^\times \not\in 2\alpha − r > 0, \text{rep}_2 \) for
\[
\text{rep}_2 = \left( T + \frac{C\alpha - α}{C_{-1}} \right) (\text{ind}_{KZ}^G \text{sub}(a)).
\]

(8) \( v_p(a) − α = v = v_p(C_{-1}) < v_p(\theta') \not\in \check{C} \in \mathbb{Z}_p^\times, \text{ind}_{KZ}^G \text{quot}(α) \)

(9) \( v_p(a) − α = v = v_p(C_{-1}) < v_p(\theta') \not\in C_0 \in \mathbb{Z}_p^\times, T(\text{ind}_{KZ}^G \text{quot}(α)) \)

Proof. (1) [2] [3] [4] [5] The proofs of these parts are nearly identical to the proofs of the corresponding parts of corollary [8].

(5) The proof is similar to the proof of [5], the only difference being that the valuation of \( \theta' \) is the same as the valuation of the coefficient of \( H \).

(7) As in the previous parts we can deduce that \( \mathcal{S}_a \) contains
\[
L := \frac{C\alpha - α}{C_{-1}} \cdot KZ_{\mathbb{P}_p} \theta^a (y'^r − x^{p-1}y'^r-p+1) + KZ_{\mathbb{P}_p} \theta^a y'^r + L',
\]
where \( r' = r - α(p+1) \) and \( L' \) reduces modulo \( m \) to a trivial element of \( \text{sub}(α) \). Then the reduction modulo \( m \) of the element \( \sum_{\mu \in F_p(\mathbb{P}_p)^n} L \) generates \( \text{rep}_2 \).

(8) [9] The proofs of these parts are similar to the proofs of [4, 5].

3. PROOFS OF THE MAIN THEOREMS

In this section we assume that
\[
r = s + β(p - 1) + u_0p^t + O(p^{t+1})
\]
for some \( β \in \{0, \ldots, p - 1\} \) and \( u \in \mathbb{Z}_p^\times \) and \( t \in \mathbb{Z}_{>0} \). Let us write \( ε = u_0p^t \).

3.1. Proof of theorem [2]. Before embarking on the proof of theorem [2], let us first demonstrate how the main proof in \[\text{Arso}\] implies theorem [1](seeing as how the main result of that paper is stated simply in terms of the irreducibility of \( \mathbb{V}_{k,a} \)). Firstly, if \( ν = 1 \) then the desired result is the main result of \[\text{BG09}\]. Moreover, it fits with the description of \( \mathbb{V}_{k,a} \) for \( k ≤ 2p + 1 \) (recall that \( \mathbb{V}_{k,a} \) is defined only if \( k > ν \)), and we may assume that \( k ≥ 0 \) due to the local constancy result of \[\text{Ber12}\] (see the discussion at the beginning of section 4 in \[\text{Arso}\]). If \( 0 < α < ν - 1 \) and \( β \not\in \{0, \ldots, α\} \), then the second step of the main proof in \[\text{Arso}\] shows that \( \text{im}(T - a) \) contains the element
\[
L = \frac{1}{p} \theta' \cdot KZ_{\mathbb{P}_p} \theta^a x^{p-1}y^{r-α(p+1)-p+1} + \frac{1}{p} \text{ERR},
\]
where \( \text{ERR} \) consists of the remaining terms coming from lemma [7] and is equal to
\[
\sum_{\xi = α+1}^{2ν-α-1} \frac{1}{p} E_\xi \cdot KZ_{\mathbb{P}_p} \theta^\xi h_\xi + \frac{1}{p} F \cdot KZ_{\mathbb{P}_p} h' + O(p^{p-ν_0})
\]
with \( v_p(E_\xi) ≥ 1 \) and \( v_p(F) > 1 \). Therefore \( L \) reduces modulo \( m \) to a representative of a generator of \( \mathbb{N}_α \). Similarly, if \( α = 0 \) and \( v_p(s - r) < ν - 1 \) then \( \mathcal{S}_a \) contains a representative of a generator of \( \hat{N}_α \). In general if \( v_p(\theta') < v_p(a) - α \) then we can apply lemma [7] in this way and eliminate the possibility that \( \mathcal{S}_{k,a} \) is a quotient of \( \mathbb{N}_{α-1} \) (and therefore must be \( \text{Irr}_0 \) in light of the classification given in theorem [5]).

Now suppose that \( k \in \mathcal{R}_{1,β} \), and in particular \( β \in \{0, \ldots, ν - 2\} \). If \( β = 0 \) then
the first and third steps of the main proof in [Arsa] imply that \( \overline{\Theta}_{k,a} \) can be a quotient of \( \hat{N}_\alpha \) only if \( t + 1 \geq v_p(a) - \alpha \), i.e. only if \( \alpha \geq \nu - t - 1 \). In particular, \( \overline{\Theta}_{k,a} \) must be a quotient of one of \( \hat{N}_{\nu-t-1}, \ldots, \hat{N}_{\nu-1} \) (and therefore must be one of \( \text{Irr}_0, \ldots, \text{Irr}_t \) in light of the classification given in theorem 5). Finally, suppose that \( \beta \in \{1, \ldots, \nu - 2\} \). As \( v_p(s - r) = 1 < v_p(a) - \alpha \), the first step shows that \( \overline{\Theta}_{k,a} \) is not a quotient of \( \hat{N}_0 \). Similarly, the second step shows that \( \overline{\Theta}_{k,a} \) is not a quotient of \( \hat{N}_{1}, \ldots, \hat{N}_{\beta - 1} \). And, the fourth step shows that \( \overline{\Theta}_{k,a} \) is not a quotient of \( \hat{N}_\alpha \) if \( t + 1 < v_p(a) - \alpha \), i.e. if \( \alpha < \nu - t - 1 \). In summary, \( \overline{\Theta}_{k,a} \) must be a quotient of one of \( \hat{N}_{\nu - \min(t, \nu - \beta - 1) - 1}, \ldots, \hat{N}_{\nu - 1} \) (and therefore must be one of \( \text{Irr}_0, \ldots, \text{Irr}_{\min(t, \nu - \beta - 1)} \)). This shows theorem 1. Theorem 2 then evidently follows from theorem 1 and the following proposition.

**Proposition 10.** Suppose that \( k \in \mathcal{R}_{1, \beta} \), so that \( \overline{\Theta}_{k,a} \) is a quotient of one of

\[
\hat{N}_{\nu - \min(t, \nu - \beta - 1) - 1}, \ldots, \hat{N}_{\nu - 1}.
\]

If \( \alpha \geq \nu - \min\{t, \nu - \beta - 1\} \) then \( \overline{\Theta}_{k,a} \) is not a quotient of \( \hat{N}_\alpha \).

**Proof.** Note that the condition on \( \alpha \) implies both \( \alpha > \beta \) and \( t \geq \nu - \alpha > v_p(a) - \alpha \). Let us apply part (3) of corollary 8 with \( v \in (v_p(a) - \alpha, t) \) and

\[
C_j = \begin{cases} 
0 & \text{if } j \in \{-1, 0\}, \\
(-1)^{\alpha - j}(s^{\alpha + 1}_{\alpha - \beta}) + pC_j^* & \text{if } j \in \{1, \ldots, \alpha\},
\end{cases}
\]

for some constants \( C_1^*, \ldots, C_\alpha^* \) yet to be chosen. Clearly \( v_p(a) - \alpha < v \leq v_p(C_{-1}) \) and \( C_0 = O(p) \). Moreover,

\[
\sum_{l=1}^{\alpha} C_l(r^{\alpha + 1}_l) = \sum_{l=1}^{\alpha} (-1)^{\alpha - l}(s^{\alpha + 1}_{\alpha - l}) + O(p) \\
= (-1)^{\alpha}(s^{\beta}_{\alpha}) + (-1)^{\alpha + 1}(s^{\alpha + 1}_{\alpha}) + O(p) \\
= (-1)^{\alpha + 1}(s^{\alpha + 1}_{\alpha}) + O(p).
\]

The third equality follows from the fact that \( \alpha > \beta \). Since \( p + \alpha - 1 > s > 2\alpha - 2 \) for \( \alpha > 0 \) and \( (s^{\alpha + 1}_{\alpha}) = 1 \) for \( \alpha = 0 \) it follows that \( \sum_{l=1}^{\alpha} C_l(r^{\alpha + 1}_l) \in \mathbb{Z}_p^* \). Thus we only need to verify the most delicate condition, that

\[
v \leq v_p(\theta_w(D_i))
\]

for \( 0 \leq w < 2\nu - \alpha \). By (c-a) and (c-q), if

\[
L_1(r) := \sum_{i > 0} \frac{r - \alpha + j}{i(p-1)+j} \frac{1}{i(p-1)w}
\]

then \( L_1(r) = L_1(s + \beta(p - 1)) + O(\epsilon) \) for \( 0 \leq w < 2\nu - \alpha \). So in order to verify the last condition it is enough to show that

\[
L_2(s) := \sum_{j=1}^{\alpha} \left((-1)^{\alpha - j}(s^{\alpha + 1}_{\alpha - j}) + pC_j^* \right)(s^{\beta(p-1) - \alpha + j}_{i(p-1) + j}) = O(p^v)
\]

for all \( i \in \{1, \ldots, \beta\} \). We have

\[
(s^{\beta(p-1) - \alpha + j}_{i(p-1) + j}) = \binom{\beta}{i}(s^{\alpha - \beta + j}_{j-i}) + O(p).
\]
We also have
\[
\sum_{j=1}^{\alpha} (-1)^{\alpha-j} (s-\alpha+1) (s-\alpha+1-j) _ {j-i}
= \sum_{j=1}^{\alpha} (-1)^{\alpha-j} (s-\alpha+1) (s-\alpha+1-j) _ {j-i}
= (-1)^{\alpha-i} \left( (s-\alpha+1) (s-\alpha+1-j) _ {j-i} + \sum_{j=1}^{\alpha} (s-\alpha+1) (s-\alpha+1-j) _ {j-i} \right)
= (-1)^{\alpha-i} \left( (s-\alpha+1) (s-\alpha+1-j) _ {j-i} \right) = 0.
\]

The third line follows from the fact that both \( \sum_{j} (s-\alpha+1) (s-\alpha+1-j) _ {j-i} \) and \( (\beta-\alpha) \)
are equal to the coefficient of \( X^{\alpha-i} \) in
\[(1 + X)^{s-\alpha+1} (1 + X) ^ {s-\alpha+1} = (1 + X)^{\beta-\alpha}.
\]

The final equality follows from the assumptions that \( i > 0 \) and \( \alpha > \beta \). So \((\gamma)\) is true modulo \( p \), and we can transform \((\gamma)\) into the matrix equation
\[
\left( \begin{array}{cccc}
(s+\beta(p-1)-\alpha-j) & \cdots & (s+\beta(p-1)-\alpha-j) \\
(\beta(p-1)+j) & \cdots & (\beta(p-1)+j) \\
\end{array} \right)_{1 \leq i \leq \beta, 1 \leq j \leq \alpha}
(C^*_1, \ldots, C^*_\alpha)^T = (w_1, \ldots, w_\beta)^T
\]
for some \( w_1, \ldots, w_\beta \in \mathbb{Z}_p \). This matrix equation always has a solution since the left \( \beta \times \beta \) submatrix of the reduction modulo \( p \) of the matrix
\[
\left( \begin{array}{cccc}
(s+\beta(p-1)-\alpha-j) & \cdots & (s+\beta(p-1)-\alpha-j) \\
(\beta(p-1)+j) & \cdots & (\beta(p-1)+j) \\
\end{array} \right)_{1 \leq i \leq \beta, 1 \leq j \leq \alpha}
\]
is upper triangular with units on the diagonal. Therefore we can indeed always choose the constants \( C_1^*, \ldots, C_\alpha^* \) in a way that \( v < v_p(\theta_w(D_i)) \) for \( 0 \leq w < 2\nu - \alpha \). Then all conditions of part \( (3) \) of corollary \( \ref{corollary:conditions} \) are satisfied and we can conclude that \( \tilde{N}_\alpha \) is trivial modulo \( \mathcal{F}_\beta \).

Proposition \( \ref{proposition:conditions} \) is already sufficient to compute \( \mathfrak{A}_{k,\alpha} \) for \( k \in \mathcal{R}_{t,\beta} \), as it implies that \( \mathfrak{A}_{k,\alpha} \) must be a quotient of \( \tilde{N}_{\max\{\nu-t-1, \beta\}} \). Let us further show that in fact \( \mathfrak{A}_{k,\alpha} \) must come from \( \text{ind}^G_{KZ} \text{quot}(\alpha) \) for \( \alpha = \max\{\nu-t-1, \beta\} \).

**Proposition 11.** If \( k \in \mathcal{R}_{t,\beta} \) then \( \mathfrak{A}_{k,\alpha} \) is not a quotient of \( \text{ind}^G_{KZ} \text{sub}(\alpha) \).

**Proof.** Let us apply part \( (2) \) of corollary \( \ref{corollary:conditions} \) with \( v = t \) and
\[
C_j = \begin{cases} 
\epsilon & \text{if } j = -1, \\
0 & \text{if } j = 0, \\
(-1)^{\alpha+j} \alpha^{\alpha-1} \left( \begin{array}{c}
(s-\alpha+1) \\
\beta \\
\end{array} \right) & \text{if } j \in \{1, \ldots, \alpha\}.
\end{cases}
\]

Clearly \( v_p(C_{-1}) = t < v_p(\theta_w(D_i)) \) for \( 0 \leq w < \alpha \) and \( t < v_p(\theta_w(D_i)) \) for \( 0 \leq w < 2\nu - \alpha \). We also need to show that \( t < v_p(\theta_w(D_i)) \) for \( 0 \leq w < \alpha \) and \( t \leq v_p(\theta_w(D_i)) \) for \( \alpha \leq w < 2\nu - \alpha \). Let us consider the matrix \( A = (A_{w,j})_{0 \leq w, j \leq \alpha} \) that has integer entries
\[
A_{w,j} = \sum_{i>0} \left( \begin{array}{c}
(r+j) \\
\alpha \\
\end{array} \right) \alpha^{\alpha-1} \left( \begin{array}{c}
(s-\alpha+1) \\
\beta \\
\end{array} \right).
\]

In the fourth step of the main proof in \[Arsa\] we show that
\[
A = S + \epsilon N + O(ep),
\]
where
\[ S_{w, j} = \sum_{i=1}^{\beta} \binom{s+\beta(p-1)-\alpha+j}{i(p-1)+j} \binom{i(p-1)}{w}, \]
\[ N_{w, j} = \sum_{v} (-1)^{w-v} \binom{j+w-v-1}{w-v} \binom{s+\beta(p-1)-\alpha+j}{v} \sum_{i=0}^{\beta} \binom{s+\beta(p-1)-\alpha+j-v}{i(p-1)+j-v} \delta_{w=0} \binom{s+\beta(p-1)-\alpha+j}{j}. \]

And exactly as in the main proof in \textit{Arma} we can deduce the three conditions we need to show as long as \((C_0, \ldots, C_{\alpha})^T \in \ker S\) (in characteristic zero) and
\[ N(C_0, \ldots, C_{\alpha})^T = (-C_{-1} \epsilon^{-1}, 0, \ldots, 0, C_{-1} \epsilon^{-1})^T + Sv + O(p) \]
for some \(v\). Let \(B\) be the change-of-basis matrix introduced in the fourth step of the main proof in \textit{Arma}, where it is shown that \(B\) encodes precisely the row operations that transform \(S\) into a matrix with zeros outside of the rows indexed 1 through \(\beta\) and such that
\[ (BS)_{w, j} = \binom{s+\beta(p-1)-\alpha+j}{w(p-1)+j} \]
when \(1 \leq w \leq \beta\), and that
\[ B_{i, w} = \delta_{(i, w)=(0,0)} + \sum_{l=1}^{\alpha} (-1)^{i} \binom{l_i}{l_{w-i}^{-1}} + O(p). \]

By using this formula we can compute that
\[ B(-1, 0, \ldots, 0, 1)^T = (0, -\binom{o}{0}, \ldots, -\binom{o}{\alpha})^T + O(p), \]
and therefore if \(\overline{R}\) is the \(\alpha \times \alpha\) matrix over \(\mathbb{F}_p\) obtained from \(B \overline{N}\) by replacing the rows indexed 1 through \(\beta\) with the corresponding rows of \(BS\) and then discarding the zeroth row and the zeroth column, the condition that needs to be satisfied is equivalent to the claim that
\[ \overline{R}(C_1, \ldots, C_{\alpha})^T = (-1 - \delta_{1 \leq \beta}) \binom{o}{1}, \ldots, (-1) \alpha \binom{o}{\alpha})^T \]
(over \(\mathbb{F}_p\)). The matrix \(\overline{R}\) is the lower right \(\alpha \times \alpha\) submatrix of the matrix \(\overline{U}\) defined in the fourth step of the main proof in \textit{Arma} and since if \(i > \beta\) then
\[ \sum_{v} (-1)^{i+j+v} \binom{v}{i-1} \binom{s+\beta-j-v}{s} = \binom{s+\beta-j}{i} \binom{s-\alpha-j-i}{s} \]
by \((3)\), we can write
\[ \overline{U}_{i-1, j-1} = \binom{s-\alpha-j}{s-i}, \begin{cases} \binom{i}{j} & \text{if } i \in \{1, \ldots, \beta\}, \\ -\binom{i}{j} & \text{otherwise.} \end{cases} \]

for \(i, j \in \{1, \ldots, \alpha\}\). We have
\[ \sum_{j=1}^{\alpha} (-1)^{\alpha+\beta+j} \alpha \binom{s-\alpha+1}{\alpha-j} \binom{s-\alpha-j}{j} = \sum_{j} (-1)^{\alpha+\beta+j} \alpha \binom{s+\beta+1}{\alpha-j} \binom{s-\beta-1}{j} \]
\[ = \binom{s-\beta-1}{j} \binom{s-\beta-1}{j_i} = \binom{s-\beta-1}{j_i} \frac{(-1)^s}{\binom{s}{s}}. \]

If \(i \in \{1, \ldots, \beta\}\) then this is zero, and if \(i \in \{\beta+1, \ldots, \alpha\}\) then it is
\[ (-1)^{\beta} \binom{s-\beta-1}{\beta} \frac{(-1)^s}{\binom{s}{s}} = \binom{s}{s} \binom{\alpha}{\beta} \frac{(-1)^s}{\binom{s}{s}} = (-1)^{\alpha} \binom{s}{s}. \]

Note that here we are never concerned about the possibility of \(N\) not having integer entries since we never divide the first column by \(p\).
which implies \((s^2)\). Consequently we can apply part (2) of corollary 8 with \(v = t\) and conclude that \(\text{ind}^G_{KZ} \text{sub}(\alpha)\) is trivial modulo \(\mathcal{O}_a\).

### 3.2. Proof of theorem 3

In the proof of theorem 2 we show that if \(k \in \mathcal{A}_{t, \beta}\) then \(\bar{\mathcal{O}}_{k, \alpha}\) must be a quotient of \(\text{ind}^G_{KZ} \text{quot}(\max \{\nu - t, 1, \beta\})\), and if \(k \in \mathcal{O}_0\) then \(\bar{\mathcal{O}}_{k, \alpha}\) must be a quotient of \(\text{ind}^G_{KZ} \text{quot}(\nu - 1)\). The proof is based on corollary 8 and it amounts to considering the element of \(\text{im}(T - a)\) coming from lemma 7 and noting that the term with dominant valuation is either \(H\) or

\[
(\partial' + C_{-1}) \cdot KZ, a \theta \alpha x^{p-1} y^{r-\alpha(p+1)-p+1} + C_{-1} \cdot KZ, a \theta_{n, x} \alpha - n y^{r-\nu p - \alpha},
\]

depending on how \(t\) compares to \(v_p(a) - \alpha\). In the setting of theorem 3 we can similarly apply the corresponding parts (1, 2, 3, 4, 5) of corollary 9 to conclude that any factor of \(\bar{\mathcal{O}}_{k, \alpha}\) must be a quotient of one of

\[
\text{ind}^G_{KZ} \text{sub}(\gamma - 1), \text{ind}^G_{KZ} \text{quot}(\gamma - 1), \text{ind}^G_{KZ} \text{sub}(\gamma), \text{ind}^G_{KZ} \text{quot}(\gamma),
\]

where \(\gamma = \max \{\nu - t, 1, \beta\}\) if \(k \in \mathcal{A}_{t, \beta}\) and \(\gamma = \nu - 1\) if \(k \in \mathcal{O}_0\) (and where for convenience we let \(\text{sub}(-1)\) and \(\text{quot}(-1)\) be the trivial representation). The key point here is that outside of these subquotients the valuations \(t\) and \(v_p(a) - \alpha\) can never match, so the dominant term in the proof of theorem 2 is the same dominant term in the setting of theorem 3 as well. Moreover, if \(k \in \mathcal{A}_{t, \beta}\) and \(\gamma = \beta\) (i.e. if \(t \geq \nu - \beta - 1\)) then both \(\text{ind}^G_{KZ} \text{sub}(\gamma - 1)\) and \(\text{ind}^G_{KZ} \text{quot}(\gamma - 1)\) are trivial modulo \(\mathcal{O}_a\), and exactly as in the proof of proposition 11 we can show that \(\text{ind}^G_{KZ} \text{sub}(\gamma - 1)\) is trivial modulo \(\mathcal{O}_a\). The proof of theorem 2 works here nearly without modification, so we omit the full details of the argument. Theorem 3 then evidently follows from the above discussion and the following proposition. Let \(\mu = \lambda\) if \(k \in \mathcal{O}_0\) and \(\mu = \lambda_{t, \beta}\) if \(k \in \mathcal{A}_{t, \beta}\) (in the notation of theorem 3).

**Proposition 12.** Suppose that either \(k \in \mathcal{O}_0\) or \(k \in \mathcal{A}_{t, \beta}\) and \(t < \nu - \beta - 1\).

1. \((T - \mu^{-1})(\text{ind}^G_{KZ} \text{quot}(\gamma - 1))\) is trivial modulo \(\mathcal{O}_a\).
2. \((T - \mu)(\text{ind}^G_{KZ} \text{sub}(\gamma))\) is trivial modulo \(\mathcal{O}_a\).
3. \(\text{ind}^G_{KZ} \text{quot}(\gamma)\) is trivial modulo \(\mathcal{O}_a\).

**Proof.** (1) Suppose first that \(k \in \mathcal{O}_0\). We wish to apply part (6) of corollary 9 with \(v = 1\). We choose the same constants as in the first step (if \(v = 2\)) or the second step (if \(v > 2\)) of the main proof in [Arso], where it is shown that

\[
\theta' = \frac{(s-r)_{\nu-1}}{(s-r)_{\nu+1}} \cdot \left( \frac{1}{p^{(s-r)_{\nu+1}}} + O(p^2) \right)
\]

and that \(1 < v_p(\vartheta_w(D_i))\) for \(0 \leq w < \alpha\) and \(1 \leq v_p(\vartheta_w(D_i))\) for \(\alpha \leq w < 2\alpha - \alpha\). Moreover, since \(C_1, \ldots, C_\alpha = O(p)\) we have \(\hat{C} = O(p)\), and

\[
\mu = \frac{(s-r+2)_{\nu-1} a}{(s-r)_{\nu+1} \cdot \vartheta_{w-1} p} = \frac{a}{C_0 \cdot \vartheta_{w-1} p^{w-1}}.
\]

Therefore the conditions needed to apply part (6) of corollary 9 are satisfied and we can conclude that \((T - \mu^{-1})(\text{ind}^G_{KZ} \text{quot}(\gamma - 1))\) is trivial modulo \(\mathcal{O}_a\). If \(k \in \mathcal{A}_{t, \beta}\)

---

\(^3\)The only subtlety when copying the proof of theorem 2 is that we do not know whether \(\mathcal{O}_a\) contains \(1 \cdot KZ, a \theta^{\nu-1} y^{r-\nu+1}\). This ultimately does not present a problem since when working with \(\mathcal{N}_{\nu-1}\) we assume that \(C_0 = 0\).
and $t < \nu - \beta - 1$ then the argument is similar: we choose $v = t + 1$ and the same constants as in the third step (if $\beta = 0$) or the fourth step (if $\beta \neq 0$) of the main proof in [Arts]. In the first case $\vartheta' = \frac{(s-r)_{\alpha+1}p + O(p^{\nu+2})}{(s-\alpha)_{\alpha+1}}$, and in the second case

$$\vartheta' = \eta Q_{0,0} + O(p^{\nu+2})$$

In both cases

$$\mu = \frac{a}{C_0 \vartheta' p^{\nu+2}}$$

and the conditions needed to apply part (6) of corollary 9 are satisfied, so again we can conclude that $(T - \mu^{-1})(\text{ind}_K^G \text{quot}(\gamma - 1))$ is trivial modulo $\mathcal{A}_a$.

(2) Suppose first that $k \in \mathcal{A}_0$. We wish to apply part (7) of corollary 9 with $v = 0$. We use the constants

$$C_j = \begin{cases} (-1)^{\nu-1} \binom{s-r}{\nu-1} & \text{if } j = -1, \\
0 & \text{if } j = 0, \\
(-1)^{\nu-j-1} \binom{s-r+2}{\nu-j-1} & \text{if } j \in \{1, \ldots, \alpha\}. \end{cases}$$

We can show that these constants satisfy all of the necessary conditions exactly as in the proof of part (1) of proposition 13. Moreover, we have

$$\tilde{C} = \sum_{j=1}^{\nu-1} (-1)^{\nu-j-1} \binom{s-r+2}{\nu-j-1} \binom{r-\gamma+j}{j} - (-1)^{\nu-1} \binom{s-r}{\nu-1} + O(p)$$

$$= (-1)^{\nu-1} \sum_{j=1}^{\nu-1} \binom{s-r+2}{\nu-j-1} \binom{r-\gamma+j}{j} - (-1)^{\nu-1} \binom{s-r}{\nu-1} + O(p)$$

$$= (-1)^{\nu} \binom{s-r+2}{\nu-1} + O(p).$$

Thus

$$\frac{\tilde{C}_a p^{1-v}}{C_{-1}} = \frac{(-1)^{\nu-1} \binom{s-r+2}{\nu-1}}{(\nu-1)p^{\nu+1}} = \mu,$$

so we can apply part (7) of corollary 9 and conclude that $(T - \mu)(\text{ind}_K^G \text{sub}(\gamma))$ is trivial modulo $\mathcal{A}_a$. If $k \in \mathcal{A}_t, \beta$ and $t < \nu - \beta - 1$ then the argument is similar: we choose $v = t$ and the constants from the proof of proposition 11. Again all of the necessary conditions are satisfied and

$$\tilde{C} = (-1)^{\beta+\gamma} \gamma \binom{\gamma-1}{\beta} \sum_{j=1}^{\gamma} (-1)^j \binom{s-r+1}{\gamma-1} \binom{r-\gamma+j}{j} + O(p)$$

$$= (-1)^{\beta+\gamma} \gamma \binom{\gamma-1}{\beta} \sum_{j=1}^{\gamma} \binom{s-r+1}{\gamma-1} \binom{r-\gamma+j}{j} + O(p)$$

$$= (-1)^{\beta+\gamma} \gamma \binom{\gamma-1}{\beta} \left(\binom{s-r}{\gamma} - \binom{s-\gamma+1}{\gamma}\right) + O(p)$$

$$= (-1)^{\beta+\gamma+1} \gamma \binom{\gamma-1}{\beta} \binom{s-\gamma+1}{\gamma} + O(p).$$

The last equality follows from the fact that $\binom{s-\gamma}{\gamma} = O(p)$. Thus

$$\frac{-\tilde{C}_a p^{1-v}}{C_{-1} \vartheta'} = \frac{(-1)^{\beta+\gamma} \gamma \binom{\gamma-1}{\beta} \binom{s-\gamma+1}{\gamma} \vartheta'}{ep^{\nu+2}} = \mu,$$

so we can apply part (7) of corollary 9 and conclude that $(T - \mu)(\text{ind}_K^G \text{quot}(\gamma))$ is trivial modulo $\mathcal{A}_a$.

(3) We apply part (8) of corollary 9 with the constants from part (2)—since $\tilde{C} \in \mathbb{Z}_p^*$ all of the necessary conditions are satisfied and we can conclude that $\text{ind}_K^G \text{quot}(\gamma)$ is trivial modulo $\mathcal{A}_a$. \[\square\]
3.3. Proof of theorem 4. As theorem 2 implies theorem 4 for \( s \geq 2\nu \), we may assume that \( s \in \{ 2, \ldots, 2\nu - 2 \} \). Recall also that we assume \( \nu \leq \frac{p - 1}{2} \).

**Proposition 13.**

(1) If \( \alpha < \frac{s}{2} \) then

\[
\begin{align*}
\hat{N}_\alpha & \quad \text{if } \beta \in \{ 0, \ldots, \alpha - 1 \} \text{ and } \alpha > v_p(a) - t, \\
\text{ind}_{KZ}^{G} \text{sub}(\alpha) & \quad \text{otherwise}
\end{align*}
\]

is trivial modulo \( \mathcal{I}_a \).

(2) If \( \frac{s}{2} \leq \alpha < s \) and \( \beta \notin \{ 1, \ldots, \alpha + 1 \} \) then \( \hat{N}_\alpha \) is trivial modulo \( \mathcal{I}_a \).

(3) If \( 0 < \alpha < \frac{s}{2} \) then

\[
\begin{align*}
T(\text{ind}_{KZ}^{G} \text{quot}(\alpha)) & \quad \text{if } \beta \in \{ 0, \ldots, \alpha \} \text{ and } \alpha > v_p(a) - t, \\
\hat{N}_{s-\alpha} & \quad \text{if } \beta \in \{ 0, \ldots, \alpha \} \text{ and } \alpha < v_p(a) - t, \\
\hat{N}_\alpha & \quad \text{if } \beta \in \{ \alpha + 1, \ldots, s - \alpha \}, \\
\hat{N}_{s-\alpha} & \quad \text{if } \beta > s - \alpha
\end{align*}
\]

is trivial modulo \( \mathcal{I}_a \).

(4) If \( \frac{s}{2} \leq \alpha < s \) then

\[
\begin{align*}
T(\text{ind}_{KZ}^{G} \text{quot}(\alpha)) & \quad \text{if } \beta \in \{ 1, \ldots, s - \alpha \} \text{ and } s - \alpha > v_p(a) - t, \\
T(\text{ind}_{KZ}^{G} \text{quot}(\alpha)) & \quad \text{if } \beta \in \{ s - \alpha + 1, \ldots, \alpha \} \text{ and } \alpha > v_p(a) - t, \\
\hat{N}_\alpha & \quad \text{otherwise}
\end{align*}
\]

is trivial modulo \( \mathcal{I}_a \).

(5) If \( \alpha \geq s \) then

\[
\begin{align*}
T(\text{ind}_{KZ}^{G} \text{quot}(\alpha)) & \quad \text{if } \alpha = \max\{ \nu - t - 1, \beta - 1 \}, \\
\hat{N}_\alpha & \quad \text{otherwise}
\end{align*}
\]

is trivial modulo \( \mathcal{I}_a \).

(6) If \( \beta \in \{ 1, \ldots, \frac{s}{2} - 1 \} \) and \( t > \nu - \frac{s}{2} \) then \( \hat{N}_{s/2+1} \) is trivial modulo \( \mathcal{I}_a \).

(7) If \( \beta \in \{ 1, \ldots, \frac{s}{2} - 1 \} \) and \( t = \nu - \frac{s}{2} \) then \( \hat{N}_{s/2-1} \) is trivial modulo \( \mathcal{I}_a \).

(8) If \( \beta \in \{ \frac{s}{2}, \frac{s}{2} + 1 \} \) and \( t > \nu - \frac{s}{2} - 1 \) then \( \hat{N}_{s/2+1} \) is trivial modulo \( \mathcal{I}_a \).

(9) If \( \beta = \frac{s}{2} + 1 \) and \( t = \nu - \frac{s}{2} - 1 \) then \( \text{ind}_{KZ}^{G} \text{sub}(\frac{s}{2} + 1) \) is trivial modulo \( \mathcal{I}_a \).

**Proof.** (1) First let \( \beta \geq \alpha \). Let us apply part (2) of corollary 8 with \( v = 0 \) and

\[
C_j = \begin{cases} 
(-1)^\alpha \binom{s-\alpha}{\alpha} & \text{if } j = -1, \\
0 & \text{if } j = 0, \\
(-1)^{\alpha-j} \binom{s-\alpha+1}{\alpha-j} & \text{if } j \in \{1, \ldots, \alpha\}.
\end{cases}
\]
Since \((s^\gamma - r^\gamma) = \binom{\delta}{\gamma} + O(p) \in \mathbb{Z}_p^\times\), the two conditions we need to verify in order to conclude that \(\text{ind}^G_{KZ} \sub(a)\) is trivial modulo \(\mathcal{I}_a\) are \(v_p(\vartheta_w(D_1)) > 0\) for \(0 \leq w < \alpha\) and \(v_p(\vartheta') > 0\). These two conditions are equivalent to the system of equations

\[
\sum_{j=1}^\alpha (-1)^{\alpha-j} \binom{s^\gamma - r^\gamma}{\alpha-j} \sum_{i>0} \left( \binom{r-a+j}{(i-1)+j} \right) \left( \binom{w}{i} \right)
\]

\[(\ast^3)\]

for \(0 \leq w \leq \alpha\). Let \(F_{w,j}(z, \psi) \in \mathbb{F}_p[z, \psi]\) denote the polynomial

\[
\sum_v (-1)^{w-v}(i+w-v^2) \left( \binom{\alpha}{i} \right) \left( \binom{\alpha+j}{v} \right) \left( \binom{\psi-\alpha+j-v}{v} \right)
\]

By \((\ast^3)\),

\[
\sum_{i>0} \left( \binom{r-a+j}{(i-1)+j} \right) \left( \binom{w}{i} \right) = F_{w,j}(r, s)
\]

(over \(\mathbb{F}_p\)). Thus \((\ast^3)\) follows if the polynomials

\[
F_1(z, \psi) = \sum_{j=1}^\alpha (-1)^{\alpha-j} \binom{\psi-\alpha+j}{\alpha-j} F_{w,j}(z, \psi)
\]

and

\[
F_2(z, \psi) = (-1)^{\alpha} (\delta_{w=\alpha} - \delta_{w=0}) \binom{\psi-z}{\alpha}
\]

are equal. Both of them have degree \(\alpha\) and the coefficient of \(z^\alpha\) in each of them is \(\frac{1}{\alpha!} (\delta_{w=\alpha} - \delta_{w=0})\), so they are equal if they are equal when evaluated at the points \((z, \psi)\) such that

\[(z - \psi, \psi) \in \{0, \ldots, \alpha - 1\} \times \{\alpha, \ldots, 2\alpha\}.
\]

We can translate this back to showing our original task (that \(v_p(\vartheta_w(D_1)) > 0\) for \(0 \leq w < \alpha\) and \(v_p(\vartheta') > 0\)) when

\[u = s - \alpha = r - \gamma(p - 1) - \alpha
\]

for some \(\gamma \in \{0, \ldots, \alpha - 1\}\) and \(u \in \{0, \ldots, \alpha\}\), i.e. that

\[
\sum_{j=1}^\alpha (-1)^{\alpha-j} \binom{u+1}{\alpha-j} \sum_{i=1}^\gamma \left( \binom{\gamma(p-1)+u+j}{i(p-1)+j} \right) \left( \binom{w-1}{i} \right) = O(p)
\]

for \(0 \leq u, w \leq \alpha\) and \(0 \leq \gamma < \alpha\). Since

\[
\left( \binom{\gamma(p-1)+u+j}{i(p-1)+j} \right) \left( \binom{w-1}{i} \right) = \left( \binom{\gamma}{i} \right) \left( \binom{u+j-\gamma}{v} \right) \left( \binom{w-1}{i} \right) + O(p),
\]

that is equivalent to

\[
\sum_{i,j>0} (-1)^{\alpha+w-i} \binom{\alpha}{i} \left( \binom{\alpha}{j} \right) \left( \binom{\gamma}{i} \right) \left( \binom{\gamma-\alpha-i-1}{i} \right) \left( \binom{\gamma-\alpha-i-1}{j} \right) \left( \binom{\gamma-\alpha-i-1}{w} \right) = O(p).
\]

This follows from the facts that

\[
\sum_{j>0} \binom{\alpha+j}{\alpha-j} \left( \binom{\gamma}{i} \right) \left( \binom{\gamma}{j} \right) = \left( \binom{\gamma}{i} \right) \left( \binom{\gamma}{j} \right)
\]

for \(\gamma < \alpha\). Thus if \(\beta \geq \alpha\) then we can apply part \((\text{ii})\) of corollary \(8\) and conclude that \(\text{ind}^G_{KZ} \sub(a)\) is trivial modulo \(\mathcal{I}_a\). Suppose now that \(\beta \in \{0, \ldots, \alpha - 1\}\). If \(t > v_p(a) - \alpha\) then the proof of proposition \(10\) applies here nearly verbatim since \((s^\alpha - r^\alpha) \in \mathbb{Z}_p^\times\), and moreover we can conclude that \(\mathcal{N}_\alpha\) is trivial modulo \(\mathcal{I}_a\). So let us suppose that \(t < v_p(a) - \alpha\). Let us apply part \((\text{ii})\) of corollary \(8\) with \(v = t\) and

\[
C_j = \begin{cases} 
(1)^2 & \text{if } j = -1, \\
0 & \text{if } j = 0, \\
(1)^\alpha(s^\gamma - r^\gamma) + PC_j & \text{if } j \in \{1, \ldots, \alpha\},
\end{cases}
\]
for some constants $C_1^\ast, \ldots, C_\alpha^\ast$ yet to be chosen. Clearly $v_p(C_{-1}) = t < v_p(a) - \alpha$, and the other conditions that need to be satisfied are $t < v_p(\partial_w(D_1))$ for $0 \leq w < \alpha$ and $t < v_p(\partial'(D))$ and $t \leq v_p(\partial_w(D_1))$ for $\alpha \leq w < 2\nu - \alpha$. Let us consider the matrix $A = (A_{w,j})_{0 \leq w,j \leq \alpha}$ that has integer entries

$$A_{w,j} = \sum_{0 < i(p-1) < r-2\alpha} (r_{w}^{n+j})^{i(p-1)}.$$

Then exactly as in the first claim in the fourth step of the main proof in [Ar'sa] we can show that

$$A = S + \epsilon N + O(\epsilon p),$$

where

$$S_{w,j} = \sum_{i=1}^{\beta} \binom{s + \beta(p-1) - \alpha + j}{i(p-1)} i(p-1)_{w}^{i(p-1)},$$

$$N_{w,j} = \sum_{v} (-1)^{w-v} \binom{s + \beta(p-1) - \alpha + j}{v} \sum_{i=0}^{\beta} \binom{s + \beta(p-1) - \alpha + j}{i(p-1)+j-v},$$

$$- \delta_{w=0} (s + \beta(p-1) - \alpha + j)^{\beta}.$$

Since $C_{-1} = O(p)$, we have shown at the beginning of this proof that

$$S(C_0, \ldots, C_\alpha)^T = (O(p), \ldots, O(p))^T.$$

Let $B$ be the change-of-basis matrix introduced in the fourth step of the main proof in [Ar'sa]. We have

$$BS(C_0, \ldots, C_\alpha)^T = (O(p), \ldots, O(p), 0, \ldots)^T,$$

where the only entries of the vector on the right that can possibly be non-zero are the ones indexed 1 through $\beta$. As in the proof of proposition [10] we note that $S$ has rank $\beta$ and therefore we can choose $C_1^\ast, \ldots, C_\alpha^\ast$ in a way that $(C_0, \ldots, C_\alpha)^T \in \ker BS$. Then $\partial_w(D_1) = O(\epsilon)$ for all $w$, and the conditions that need to be satisfied are $\partial_w(D_1) = O(\epsilon p)$ for $0 \leq w < \alpha$ and $\partial' = O(\epsilon p)$. These two conditions are equivalent to the single equation

$$A(C_0, \ldots, C_\alpha)^T = (-C_{-1}, 0, \ldots, 0, C_{-1}) + O(\epsilon p),$$

which is itself equivalent to

$$BN(C_0, \ldots, C_\alpha)^T = (0, -\alpha(C_{-1}\epsilon^{-1}), \ldots, (-1)^{\alpha} \alpha)(C_{-1}\epsilon^{-1})) + BSv + O(p)$$

for some $v$ (the reasoning being very similar to the one in the fourth claim of the main proof in [Ar'sa]). Thus, if $\bar{R}$ is the $\alpha \times \alpha$ matrix over $\mathbb{F}_p$ obtained from $BN$ by replacing the rows indexed 1 through $\beta$ with the corresponding rows of $BS$ and then discarding the zeroth row and the zeroth column, the condition that needs to be satisfied is equivalent to the claim that

$$(-1 - \delta_{1 \leq \beta})(\alpha), \ldots, (-1)^{\alpha}(1 - \delta_{\alpha \leq \beta})$$

is in the image of $\bar{R}$ (since $C_0 = O(p)$ and $C_{-1} \epsilon^{-1} \in \mathbb{Z}_p^\ast$). This is indeed the case since $\bar{R}$ is the lower right $\alpha \times \alpha$ submatrix of the matrix $\bar{Q}$ defined in the fourth step of the main proof in [Ar'sa] and is therefore upper triangular with units on the diagonal. Thus we can apply part $(2)$ of corollary [8] with $v = t$ and conclude that $\text{ind}_{KZ} \text{sub}(\alpha)$ is trivial modulo $\mathcal{F}_\alpha$. 

\( \textbf{[2]} \) Let us define \( C_{-1}(z), \ldots, C_{\alpha}(z) \in \mathbb{Z}_p[z] \) as

\[
C_j(z) = \begin{cases} 
(s-z^{-1})_{\alpha+1} & \text{if } j = -1, \\
(\frac{\alpha}{s-\alpha-1})^{-1} \frac{s-z^-1}{\alpha+1} & \text{if } j = 0, \\
(-1)^{j+1} (s-\alpha-1)_{\alpha-j} (z-\alpha) & \text{if } j \in \{1, \ldots, \alpha\}. 
\end{cases}
\]

Let us apply part \([1]\) of corollary \([8]\) with \( v = 0 \) and

\[
(C_{-1}, C_0, \ldots, C_{\alpha}) = (C_{-1}(r), C_0(r), \ldots, C_{\alpha}(r)).
\]

The two conditions we need to verify in order to conclude that \( \tilde{N}_\alpha \) is trivial modulo \( \mathcal{F}_\alpha \) are \( \nu_p(\theta_w(D_{ij})) > 0 \) for \( 0 \leq w < \alpha \) and \( \nu_p(\theta') = 0 \). These two conditions follow from the system of equations

\[
(\ast^4) \quad \sum_{j=0}^{\alpha} C_j \sum_{0 \leq i (p-1) < r-2a} \binom{r-\alpha+j}{i(p-1)+j} = -\delta_w = 0 \quad \text{for } 0 \leq w < \alpha \quad \text{and} \quad \nu_p(\theta') = 0.
\]

for \( 0 \leq w \leq \alpha \). Let \( F_{w,j}(z) \in \mathbb{F}_p[z] \) denote the polynomial

\[
\sum_{\alpha} (-1)^{w-v} \binom{r-\alpha+j}{i(p-1)+j} = F_{w,j}(r)
\]

(over \( \mathbb{F}_p \)). Thus \( \ast^4 \) follows if the polynomials

\[
F_1(z) = \sum_{j=0}^{\alpha} C_j(z) F_{w,j}(z) \quad \text{and} \quad F_2(z) = -\delta_w = 0 \quad \text{are equal.}
\]

Let us first show that

\[
(\ast^5) \quad C_0(z) \binom{s-\alpha}{s-\alpha} + \sum_{j=1}^{\alpha} C_j(z) \binom{s-\alpha+j}{s-\alpha} = \frac{(-1)^{\alpha+1}(z-\alpha)}{s-\alpha}
\]

(as polynomials in \( \mathbb{F}_p[z] \)). Since

\[
C_0(z) \binom{s-\alpha}{s-\alpha} = \left( \frac{\alpha}{s-\alpha-1} \right)^{-1} \frac{s-z^{-1}-\alpha}{\alpha+1} = \frac{(-1)^{\alpha+1}(z-\alpha)}{s-\alpha},
\]

this is equivalent to

\[
\left( \frac{\alpha}{s-\alpha-1} \right)^{-1} \frac{s-z^{-1}-\alpha}{\alpha+1} + \sum_{j=1}^{\alpha} \left( \frac{(-1)^{\alpha+1}(s-\alpha-j)}{s-\alpha} \right) \binom{s-\alpha+j}{s-\alpha} = (-1)^{\alpha+1}.
\]

The polynomial on the left side has degree at most \( s-\alpha \). In fact, the coefficient of \( z{s-\alpha} \) in it is \(-\frac{2\alpha-s+1}{(\alpha+1)!}\) plus

\[
\frac{1}{(s-\alpha-1)!} \sum_{j=1}^{\alpha} \binom{s-\alpha-1}{s-\alpha-j} = \frac{1}{(s-\alpha-1)!} \sum_{j+1}^{\alpha} \binom{s-\alpha-1}{s-\alpha-j} \cdot \frac{x^{s-2\alpha+j-1}}{j+1} \frac{x}{1+x} = \frac{1}{(s-\alpha-1)!} \int_0^1 Y^{a-2\alpha+1} (1+Y)^{s-\alpha-1} \, dY = \frac{(-1)^{\alpha-2\alpha+1}}{(s-\alpha-1)!}.
\]

Since \( s \) is even, that coefficient is zero. Therefore it is enough to show that the two polynomials are equal when evaluated at \( z \in \{ \alpha+1, \ldots, s \} \). At these points the polynomial on the left side is equal to

\[
(s-\alpha) \sum_{j=1}^{\alpha} \binom{s-\alpha-1}{s-\alpha-j} \binom{s-\alpha+j}{s-\alpha}
\]

\[= \frac{(-1)^{\alpha+1}(z-\alpha)}{s-\alpha}.\]
Since $p(\gamma) = 0$ for $u > s - \alpha - 1$, and the last equality follows from $\binom{\alpha}{u} = 0$ for $u \in \{1, \ldots, s - \alpha - 1\}$. In particular, $\binom{\alpha}{s}$ is indeed true. So both $F_1(z)$ and $F_2(z)$ have degree at most $\alpha + 1$, and therefore they are equal if they are equal when evaluated at $z \in \{s - \alpha - 1, \ldots, s\}$. It is easy to verify that $F_1(s) = F_2(s)$ (since e.g. $C_0(s) = 0$, and when $z \in \{s - \alpha - 1, \ldots, s - 1\}$ we can translate $F_1(z) = F_2(z)$ back to showing that

$$\sum_{j=0}^{\alpha} C_j^\gamma \sum_{i+j} \binom{s+\gamma(p-1)-\alpha+j}{i(p-1)+j} = -\delta_{w=0}\binom{\gamma}{s-a} + \binom{\gamma-1}{s-a} + O(p)$$

for $\gamma \in \{1, \ldots, \alpha + 1\}$ and $C_j^\gamma = C_j(s + \gamma(p-1))$. The desired identity follows if

$$\sum_{j=0}^{\alpha} C_j^\gamma \binom{s+\gamma(p-1)-\alpha+j}{i(p-1)+j} = O(p)$$

for all $i \in \{1, \ldots, \gamma - 1\}$. If $j > 0$ and $C_j^\gamma \neq 0$ then $j \geq 2\alpha - s + 1 \geq \alpha + \gamma - s$ and consequently

$$\binom{s+\gamma(p-1)-\alpha+j}{i(p-1)+j} = \binom{\gamma}{i-1}(s-a-\gamma+j + O(p) = O(p)$$

since $p + i > j$. On the other hand,

$$\binom{s+\gamma(p-1)-\alpha}{i(p-1)} = \binom{\gamma}{i-1}(s-a-\gamma+j + O(p).$$

Since $\binom{\gamma}{i-1} = \frac{\gamma}{i}(\gamma+1)$ we want to show that

$$C_0^\gamma \frac{\gamma}{i(p-1)}(s-a-\gamma+j + O(p).$$

for all $i \in \{1, \ldots, \gamma - 1\}$. That is equivalent to $F_3(s + \gamma(p-1)) = 0$ (over $\mathbb{F}_p$) for

$$F_3(z) = \binom{s-a-w}{s-a-w-1}^{-1}(s-a-w)^{s-a-1} + \sum_{j=1}^{\alpha} \binom{-1}{j+1}(s-a-1)^{s-a-1}(z-a+j),$$

where $w = \gamma - i > 0$. The degree of $F_3(z)$ is at most $s - \alpha - w$, and in fact the coefficient of $z^{s-a-1}$ in it is $-\frac{(s-a-1)_{s-a-1}((s-a+1)!}{(s-a-1)_{(s-a-1)!}}$ plus

$$\frac{1}{(s-a-1)!} \sum_{j} \binom{-1}{j+1}(s-a-1)^{s-a-1}(z-a-j) = -\frac{(s-a-1)_{s-a-1}((s-a+1)!}{(s-a-1)_{(s-a-1)!}},$$

i.e. $[z^{s-a-1}]F_3(z) = 0$. Therefore the degree of $F_3(z)$ is less than $s - \alpha - w$, so it is enough to show that $F_3(z)$ is equal to zero when evaluated at $z \in \{s + 1, \ldots, s - w\}$. At these points it is equal to

$$(s-a-w) \sum_{j} \binom{-1}{j+1}(s-a-1)^{s-a-1}(s-a-\gamma+j)$$
for \( \gamma \in \{w, \ldots, s - \alpha - 1\} \). We have

\[
\sum_j \frac{(-1)^{j+1}(s-\alpha)^j}{s-\alpha - j} \binom{s-\alpha - \gamma + j}{s-\alpha - w} = \sum_j \frac{(-1)^{j-\alpha - j - w + 1}}{j+1} \binom{s-\alpha - 1}{s-\alpha - w} (\gamma - j - w - 1) = \sum_u \frac{(-1)^{u+1}}{u+1} \sum_j \frac{(-1)^{j-\alpha - u + 1}}{j+1} \binom{s-\alpha - 1}{s-\alpha - u} (s-\alpha - u - w)
\]

The last equality follows from \( \binom{\gamma - w}{u} = 0 \) for \( u \not\in \{0, \ldots, s - \alpha - w - 1\} \). Therefore indeed \( F_3(z) = 0 \), so we can apply part (1) of corollary 8 and conclude that \( \tilde{N}_a \) is trivial modulo \( \mathcal{F}_a \).

(3) Let us first assume that \( \beta \in \{0, \ldots, \alpha\} \). Where the main proof in [Ars] fails for \( \tilde{N}_a \) is that some entries of the extended associated matrix \( N \) are not integral (see the footnote in the third claim of the fourth step of that proof). To be more specific, the equation for \( N_{w,0} \) is

\[
pN_{w,0} = \binom{s+\beta(p-1)-\alpha}{w} \sum_{i > 0} \binom{s+\beta(p-1)-\alpha-w}{i(p-1)-w} + O(p),
\]

where the second term is \( O(p) \) because it is still true that

\[
\sum_{i > 0} \binom{r-\alpha-w}{i(p-1)-w} - \sum_{i > 0} \binom{s+\beta(p-1)-\alpha-w}{i(p-1)-w} = O(\epsilon p).
\]

On the other hand,

\[
\sum_{i > 0} \sum_{l=0}^w (-1)^l \binom{w}{l} \sum_{i > 0} \binom{s+\beta(p-1)-\alpha-l}{i(p-1)-w} = (-1)^{s-\alpha} \binom{w}{s-\alpha} + O(p).
\]

So \( A_{w,0} = S_{w,0} + O(\epsilon) \) is integral if \( w < s - \alpha \) and

\[
A_{w,0} = S_{w,0} + (-1)^{s-\alpha} \binom{w}{s-\alpha} \binom{s-\alpha-\beta}{w} \epsilon p^{-1} + O(\epsilon)
\]

if \( w \geq s - \alpha \). Note that \( \beta \in \{0, \ldots, \alpha\} \) and \( s > 2\alpha \) by assumption, so \( S_{w,0} \) is still always integral and if \( s - \alpha \leq w < 2\nu - \alpha \) then

\[
\binom{s-\alpha-\beta}{w} \binom{s-\alpha}{w} \in \mathbb{Z}_p^*.
\]

What this means is that if we proceed with the proof in [Ars] and apply lemma 7 with the constructed constants \( (C_{-1}, C_0, \ldots, C_\alpha) \) such that \( C_0 \) is a unit then we obtain in \( \text{im}(T - a) \) an element

\[
(\theta' + C_{-1}) \bullet_{KZ, p} \theta^n x^{p-1} g^{r-\alpha(p+1)-p+1} + C_{-1} \bullet_{KZ, p} \theta^n x^{\alpha-n} g^{r-np-\alpha} + \sum_{\xi=\alpha+1}^{2\nu-1} E_{\xi} \bullet_{KZ, p} \theta^h h_{\xi} + F \bullet_{KZ, p} h' + H
\]

such that \( v_p(C_{-1}) = v_p(\theta') = t + 1 \) and \( v_p(E_{\xi}) \geq t + 1 \) for \( \alpha + 1 \leq \xi < s - \alpha \) and \( v_p(F) > t + 1 \) and \( H \) as in lemma 7. However, \( v_p(E_{s-\alpha}) = t \) and \( v_p(E_{\xi}) \geq t \) for \( \xi > s - \alpha \). Thus if \( t > v_p(a) - \alpha \) then the dominant term is \( H \) and we can conclude that \( T(\text{ind}_{KZ} \text{quot}(\alpha)) \) is trivial modulo \( \mathcal{F}_a \) and if \( t < v_p(a) - \alpha \) then the dominant term is \( E_{s-\alpha} \bullet_{KZ, p} \theta^{\alpha-h_{s-\alpha}} \) and hence \( \tilde{N}_{s-\alpha} \) is trivial modulo \( \mathcal{F}_a \) by part (2) of lemma 7. Now let us assume that \( \beta > \alpha \). This time we proceed as in the second
step of the proof in [Arso] and apply lemma 7 with the constructed constants 
\((0,1,C_1,\ldots,C_\alpha)\) and we obtain in \(\text{im}(T-a)\) an element 
\[
\vartheta' \cdot KZ_{\mathbb{Q}_p} \theta^a x^{p-1} y^{r-\alpha(p+1)-p+1} \\
+ \sum_{\xi=0+1}^{2\nu-\alpha-1} E_\xi \cdot KZ_{\mathbb{Q}_p} \theta^\xi h_\xi + F \cdot KZ_{\mathbb{Q}_p} h' + H
\]
such that \(v_p(\vartheta') = 1\) and \(v_p(E_\xi) \geq 1\) for \(\alpha + 1 \leq \xi < s - \alpha\) and \(v_p(F) > 1\) and 
\[
v_p(E_{s-a}) = v_p((r-\alpha)s-a) = v_p((s-\alpha-\beta)s-a)
\]
and \(v_p(E_\xi) \geq 0\) for \(\xi > s - \alpha\) and \(H\) as in lemma 7. This time the dominant term is either 
\[
\vartheta' \cdot KZ_{\mathbb{Q}_p} \theta^a x^{p-1} y^{r-\alpha(p+1)-p+1} \\
or \sum_{\xi=0+1}^{2\nu-\alpha-1} E_\xi \cdot KZ_{\mathbb{Q}_p} \theta^\xi h_\xi \text{ depending on whether } \beta \leq s - \alpha \text{ or not. Thus if } \beta \in \{\alpha + 1,\ldots,s - \alpha\} \text{ then } N_\alpha \text{ is trivial modulo } \mathcal{J}_\alpha, \text{ and if } \beta > s - \alpha \text{ then } \tilde{N}_{s-\alpha} \text{ is trivial modulo } \mathcal{J}_\alpha.
\]
(4) In light of part (2) we may assume that \(\beta \notin \{1,\ldots,\alpha + 1\}\), and in light of part (3) we may assume that \(\beta \neq \alpha + 1\). If \(\alpha \neq \frac{1}{\nu}\) and \(\beta \in \{1,\ldots,s - \alpha\}\) and \(s - \alpha < v_p(a) - t\) then the claim follows from part (4). Thus it is enough to show that if \(\beta \in \{1,\ldots,\alpha\}\) then 
\[
\begin{cases}
T(\text{ind}_{KZ}^G \text{quot}(\alpha)) & \text{if } \alpha > v_p(a) - t, \\
\tilde{N}_\alpha & \text{if } \alpha < v_p(a) - t
\end{cases}
\]
is trivial modulo \(\mathcal{J}_\alpha\). Let us apply corollary 8 (either part (1) of it or part (5) of it depending on whether \(\alpha < v_p(a) - t\) or \(\alpha > v_p(a) - t\)) with \(v = t\) and the constants 
\[
C_j = \begin{cases}
\frac{(-1)^{\alpha+\beta(s-a)(\alpha-\beta+1)}}{\beta^2(2a-s+1)(s-a)} \left( \frac{\alpha}{s-a} \right) & \text{if } j = -1, \\
1 & \text{if } j = 0, \\
\frac{(-1)^{\alpha+\beta(s-a-\beta)}}{\beta^{s-a-\beta}} \left( \frac{j}{2a-s+1} \right)^{s-a} & \text{if } j \in \{1,\ldots,\alpha\}.
\end{cases}
\]
Since \(v_p(C_{-1}) = t\) and \(C_0 = 1\), in order to verify the necessary conditions it is enough to show that \(\vartheta' = -C_{-1} + O(ep)\) and \(t < v_p(\vartheta_w(D_1))\) for \(0 \leq w < \alpha\) and \(t \leq v_p(\vartheta_w(D_1))\) for \(\alpha \leq w < 2\nu - \alpha\). Let us consider the matrix \(A = (A_w,j)_{0 \leq w,j \leq \alpha}\) that has integer entries 
\[
A_{w,j} = \sum_{0 \leq i < (p-1) < r-2\alpha} \left( \frac{r-\alpha+j}{i(p-1)+j} \right) \left( \frac{r}{w} \right).
\]
Then the first two conditions are equivalent to the claim that 
\[
A(C_0,\ldots,C_\alpha)^T = (-C_1 + O(ep),O(ep),\ldots,O(ep))^T.
\]
Exactly as in the first claim in the fourth step of the main proof in [Arso] (and as in part (1) of this proof) we can show that 
\[
A = S + \epsilon N + O(ep),
\]
where 
\[
S_{w,j} = \sum_{i=1}^{\beta} \left( \frac{s+\beta(p-1)-\alpha+j}{i(p-1)+j} \right) \left( \frac{r}{w} \right) - \left( \frac{s+\beta(p-1)-\alpha+j}{s-a} \right) \left( \frac{\beta(p-1)}{w} \right),
\]
\[
N_{w,j} = \sum_{v=0}^{\alpha} (\frac{s-a-\beta+j}{s-a}) \left( \frac{\beta(j-v)}{w} \right) \left( \frac{s-a-\beta+j}{w} \right)^{\beta} \sum_{j=0}^{\beta} \left( \frac{s+\beta(p-1)-\alpha+j}{i(p-1)+j-v} \right) \left( \frac{r-\alpha+j}{i(p-1)+j} \right) \left( \frac{r}{w} \right) - \delta_{w=0} \left( \frac{s+\beta(p-1)-\alpha+j}{s-a} \right)^{\beta} \left( \frac{\beta}{w} \right) \left( \frac{s-a-\beta+j}{s-a} \right) \left( \frac{\beta}{w} \right)^{\beta}.
\]
The third condition follows from an argument similar to the one in the third claim of the fourth step of the main proof in [Arsa] that the entries of the extended matrix $N$ are integers, and the first two conditions follow if $(C_0, \ldots, C_\alpha)^T \in \ker S$ and

$$N(C_0, \ldots, C_\alpha)^T = (-C_1 e^{-1}, 0, \ldots, 0)^T + Sv + O(p)$$

for some $v$. Let $B$ be the change-of-basis matrix as in part (1) of this proof. Then $BS$ has zeros outside of the rows indexed 1 through $\beta - 1$ and

$$(BS)_{i,j} = \binom{\beta}{\beta(i-j+1)}$$

when $1 \leq i < \beta$. Let $\overline{R}$ denote the $(\alpha + 1) \times (\alpha + 1)$ matrix over $\mathbb{F}_p$ obtained from $\overline{BN}$ by replacing the rows indexed 1 through $\beta - 1$ with the corresponding rows of $BS$. Thus we want to prove that

$$\overline{R}(C_0, C_1, \ldots, C_\alpha)^T = \left(\binom{-1}{\beta}^{\alpha+1}(s-\alpha)(\alpha-\beta+1)\binom{\alpha}{s-\alpha}, 0, \ldots, 0\right)^T$$

(over $\mathbb{F}_p$). Let us denote the rows of $\overline{R}$ by $r_0, \ldots, r_\alpha$. As in the fourth step of the main proof in [Arsa] we can compute

$$\left(\overline{BN}\right)_{i,j} = \sum_{l,v=0}^{\alpha}(1)^j+l+v+1(i-v)(j-v)(s-\alpha+1)(s-\alpha)\binom{j}{s-\alpha}$$

$$- \delta_{i=0}(s-\alpha+1)(s-\alpha)\binom{j}{s-\alpha}$$

$$= (s-\alpha+1)(s-\alpha)\left(\sum_{l=0}^{\alpha}(1)^j(l-\beta-1)^l\right).$$

Note that if $j > 0$ and $C_j \neq 0$ then $j > 2\alpha - s$, so $s - \alpha + j > \alpha$ and in particular

$$\binom{s-\alpha-j}{j-v} = \binom{s-\alpha-j}{j-v}$$

(over $\mathbb{F}_p$). We have

$$\sum_{j=0}^{\alpha}(1)^j+l+1(2\alpha-s+1)(s-\alpha)\binom{j}{s-\alpha}$$

$$= \sum_{j=0}^{\alpha}(1)^j+l+1(2\alpha-s+1)(s-\alpha)\binom{j}{s-\alpha}$$

$$= \sum_{j=0}^{\alpha}(1)^j+l+1(2\alpha-s+1)(s-\alpha)\binom{j}{s-\alpha}$$

$$= \sum_{j=0}^{\alpha}(1)^j+l+1(2\alpha-s+1)(s-\alpha)\binom{j}{s-\alpha}$$

$$= \sum_{j=0}^{\alpha}(1)^j+l+1(2\alpha-s+1)(s-\alpha)\binom{j}{s-\alpha}$$

$$= \sum_{j=0}^{\alpha}(1)^j+l+1(2\alpha-s+1)(s-\alpha)\binom{j}{s-\alpha}$$

The fourth equality follows from [C]. This implies that

$$\sum_{j=0}^{\alpha} C_j (s-\alpha-j) = (s-\alpha-j) \binom{s-\alpha-j}{s-\alpha}.$$
The second equality follows from the fact that \((\alpha)_j = 0\) if \(v < j\). Moreover,

\[
(BN)_{0,0} = \sum_{l,v=0}^\alpha (-1)^{l+v} \left( \frac{s-a-b}{v} \right) \left( \frac{s-a-v}{l} \right) - \left( \frac{s-a-b}{s-a} \right) \sum_{l=0}^{\alpha} \left( \frac{l}{l+1} \right)^\beta \\
= \sum_{l,v=0}^\alpha (-1)^{l+v} \left( \frac{s-a-b}{v} \right) \left( \frac{s-a-v}{l} \right) - \left( \frac{s-a-b}{s-a} \right) \sum_{l=0}^{\alpha} \left( \frac{l}{l+1} \right)^\beta \\
= \sum_{l,v=0}^\alpha (-1)^{l+v} \left( \frac{s-a-b}{v} \right) \left( \frac{s-a-v}{l} \right) + \left( \frac{s-a-b}{s-a} \right) \sum_{l=0}^{\alpha} (-1)^l \sum_{v=0}^{\alpha} (-1)^{v+1} \left( \frac{s-a-b}{v} \right) \left( \frac{s-a-v}{l} \right).
\]

The third equality follows from (3). Thus \(r_i(C_0, \ldots, C_\alpha)^T\) is equal to

\[
\sum_{l,v=0}^\alpha (-1)^{l+v} \left( \frac{s-a-b}{v} \right) \left( \frac{s-a-v}{l} \right) + \left( \frac{s-a-b}{s-a} \right) \sum_{l=0}^{\alpha} \left( \frac{l}{l+1} \right)^\beta \\
= \left( \frac{a(s-a)}{s-a} + \left( \frac{s-a-b}{s-a} \right) \left( \frac{a+1}{a+1} \right) \sum_{v=0}^{\alpha} (-1)^v \left( \frac{s-a-v}{a} \right)^\beta \right) - \frac{\alpha(s-a-b)(s-a+1)}{\beta s(a+1)} \left( \frac{a(s-a)}{s-a} \right),
\]

just as we wanted to prove. The third equality follows from (3). We are left to show that

\[
r_i(C_0, \ldots, C_\alpha)^T = 0
\]

(over \(F_p\)) for all \(i \in \{1, \ldots, \alpha\}\). If \(i = \beta + w \in \{1, \ldots, \beta - 1\}\) then this follows from

\[
\sum_{j=0}^{\alpha} \frac{(-1)^{j+1} (s-a-b)}{\beta} \left( \frac{\alpha s-a-b+j}{a-s+1} \right) \left( \frac{\alpha+1}{a-s} \right) = \sum_{u} \frac{(a-b+1)}{u} \sum_{j=0}^{\alpha} \frac{(-1)^{j+1} (s-a-b)}{\beta} \left( \frac{\alpha s-a-b+j}{a-s+1} \right) = \sum_{u} \frac{(a-b+1)}{u} \sum_{j=0}^{\alpha} (-1)^j \left( \frac{\alpha s-a-b}{a-s+1} \right) = \frac{s-a-b}{\beta} \sum_{u} \frac{(a-b+1)}{u} \left( \frac{\alpha s-a-b-1}{a-s} \right) = \frac{s-a-b}{\beta} \sum_{u} \frac{(a-b+1)}{u} \left( \frac{\alpha s-a-b}{a-s} \right).
\]

The third equality follows from (3). If \(i \in \{\beta, \ldots, \alpha\}\) then we have

\[
(BN)_{1,0} = \sum_{l,v=0}^\alpha (-1)^{l+v} \left( \frac{s-a-b}{v} \right) \left( \frac{s-a-v}{l} \right) - \delta_i (s-a-b) \left( \frac{s-a-b}{s-a} \right) \sum_{l=0}^{\alpha} \left( \frac{l}{l+1} \right)^\beta,
\]

and for \(j > 2\alpha - s\) we also have

\[
(BN)_{1,j} = \sum_{l,v=0}^\alpha (-1)^{l+v} \left( \frac{s-a-b+j}{v} \right) \left( \frac{s-a-v}{l} \right) - \delta_i (s-a-b) \left( \frac{s-a-b}{s-a} \right) \sum_{l=0}^{\alpha} \left( \frac{l}{l+1} \right)^\beta.
\]
The identity
\[ \sum_{j=0}^{\alpha} (-1)^{j+1} \left( \begin{array}{c} j \\ 2\alpha-s+1 \end{array} \right) \left( \begin{array}{c} \alpha+1 \\ j+1 \end{array} \right) (z+s-a+j)^{\partial} \]
\[ = \frac{\partial}{\partial z} \left( \sum_{j=0}^{\alpha} (-1)^{j+1} \left( \begin{array}{c} j \\ 2\alpha-s+1 \end{array} \right) \left( \begin{array}{c} \alpha+1 \\ j+1 \end{array} \right) (z+s-a+j) \right) \]
\[ = \frac{\partial}{\partial z} \left( (z+s-a-1) - (s-2a-2) \right) \]
\[ = (z+s-a-1)^{\partial} \]
is true over \( \mathbb{Q}_p[z] \). By evaluating at \( z = -\beta \) we get
\[ \sum_{j=1}^{\alpha} C_j (s-a-\beta+j)^{\partial} = \frac{s-a-\beta}{\beta} (s-a-\beta-1)^{\partial}, \]
and consequently
\[ (s-a-\beta)^{\partial} + \sum_{j=1}^{\alpha} C_j (s-a-\beta+j)^{\partial} = \frac{1}{\beta} (s-a-\beta-1) \]
This means that \((-1)^{\alpha+1} \beta \mathbf{r}_1(C_0, \ldots, C_\alpha)^T\) is equal to \( \Phi(-\beta) \), with
\[ \Phi(z) = (\alpha-s)\Phi_1'(z) - \Phi_2(z) + (z+s-\alpha)(\Phi_1'(z) + \Phi_4'(z)) \]
and
\[ \Phi_1(z) = \sum_{l,v=0}^{\alpha} (-1)^{l+v+1} \left( \begin{array}{c} \alpha+1 \\ l \end{array} \right) \left( \begin{array}{c} v \\ s-a \end{array} \right) (z+s-a)^{\partial-l-v} \]
\[ \Phi_2(z) = \left( \begin{array}{c} i-1 \\ s-a-1 \end{array} \right) (z+i-1)^{\partial} \]
\[ \Phi_3(z) = \sum_{l,v=0}^{\alpha} (-1)^{a+j+l+v+1} \left( \begin{array}{c} j \alpha+1 \\ j \end{array} \right) \left( \begin{array}{c} l+v \\ s-a-1 \end{array} \right) (z+s-a+j)^{\partial-l-v} \]
\[ \Phi_4(z) = \left( \begin{array}{c} \alpha+1 \\ s-a \end{array} \right) \sum_{l=0}^{\alpha} \left( \begin{array}{c} l+i-1 \alpha-1 \end{array} \right) (z+i-1)^{\partial} \]
So we want to show that \( \Phi(-\beta) = 0 \). If \( s = \alpha + \beta \) then this amounts to
\[ \beta \Phi_1'(-\beta) + \Phi_2(-\beta) = 0, \]
and indeed
\[ \beta \Phi_1'(-\beta) = \beta \sum_{l,v=0}^{\alpha} (-1)^{l+v+1} \left( \begin{array}{c} l \alpha+1 \\ l \end{array} \right) \left( \begin{array}{c} v \\ s-a \end{array} \right) (z+s-a)^{\partial-l-v} \]
\[ = \sum_{l,v=0}^{\alpha} \left( \begin{array}{c} l \alpha+1 \\ l \end{array} \right) \left( \begin{array}{c} v-1 \\ s-a-1 \end{array} \right) (z+s-a)^{\partial-l-v} \]
\[ = \sum_{l,v=0}^{\alpha} \delta_{l=0} (-1)^{l+i-1} + \delta_{l=\beta} (-1)^{l+i-1} \]
\[ = (-1)^{i} (\beta+1-1) \]
\[ = -\left( \begin{array}{c} i-1 \alpha-1 \end{array} \right) = -\Phi_2(-\beta). \]
Now suppose that \( s \neq \alpha + \beta \). As in the proof of (3) we can simplify \( \Phi_1(z) \) to
\[ \Phi_1(z) = (-z+s-a) \sum_{l=0}^{\alpha} \left( \begin{array}{c} l+i-1 \alpha-1 \end{array} \right) (z+i-1)^{\partial} \]
We can also simplify \( \Phi_3(z) \) to
\[ \Phi_3(z) = \sum_{j,v=0}^{\alpha} (-1)^{a+j+v+1} \left( \begin{array}{c} j \alpha+1 \\ j \end{array} \right) \left( \begin{array}{c} v \\ s-a-1 \end{array} \right) (z+s-a+j)^{\partial-l-v} \]
\[ = (-z+i-1)^{\partial} \sum_{j=0}^{\alpha} (-1)^{a+j+1} \left( \begin{array}{c} j \alpha+1 \\ j \end{array} \right) (z+s-a+j)^{\partial-l-v}. \]
Suppose first that \( i > \beta \). Then
\[
\Phi'_1(-\beta) = -\sum_{l=0}^{\alpha} \binom{l}{\beta+1} \left( (s-\alpha)^{\beta} (l+\beta+1) + (s-\alpha)^{\beta} (l+\alpha-s)^{\beta+1} \right),
\]
\[
\Phi'_2(-\beta) = 0,
\]
\[
\Phi'_3(-\beta) = \frac{(i-\beta-1)^{\beta}}{(i-\beta-1)^{\alpha}} \sum_{l=0}^{\alpha} (l+1)^{\beta+1} \left( (s-\alpha)^{\beta} (l+\beta+1) + (s-\alpha)^{\beta} (l+\alpha-s)^{\beta+1} \right),
\]
\[
\Phi'_4(-\beta) = \frac{(\alpha+1)(\alpha-\beta)(i-\beta-1)^{\beta}}{(s-\alpha)(i-\beta-1)^{\alpha}}.
\]
Thus if \( s > \alpha + \beta \) then the equation \( \Phi(-\beta) = 0 \) is equivalent to
\[
L_1(s, \alpha, \beta, i) = R_1(s, \alpha, \beta, i)
\]
with
\[
L_1 := \sum_{l=0}^{\alpha} \binom{l}{\beta+1} \left( (s-\alpha)^{\beta} (l+\beta+1) + (s-\alpha)^{\beta} (l+\alpha-s)^{\beta+1} \right),
\]
\[
R_1 := \frac{(s-\alpha)^{\beta+1}}{(s-\alpha)^{\alpha}} \sum_{j=0}^{\alpha} (l+1)^{\beta+1} \left( (s-\alpha)^{\beta} (l+\beta+1) + (s-\alpha)^{\beta} (l+\alpha-s)^{\beta+1} \right).
\]
Let us in fact show that \( L_1(u, v, w, t) = R_1(u, v, w, t) \) for all \( u, v, w, t \geq 0 \). We clearly have \( L_1(u, 0, w, t) = R_1(u, 0, w, t) \) since both sides are zero, and
\[
R_1(u + 1, v + 1, w, t) - R_1(u, v, w, t) - L_1(u + 1, v + 1, w, t) + L_1(u, v, w, t)
\]
\[
= \frac{(u-v)}{u-v+1} \sum_{j=0}^{\alpha} \left( (u-v-w)^{\beta+1} (u-v-w+1)^{\alpha} \right) + \frac{(u-v-w)}{u-v-w+2} \sum_{j=0}^{\alpha} \left( (u-v-w)^{\beta+1} (u-v-w+1)^{\alpha} \right) + \frac{(u-v-w)}{u-v-w+2} \sum_{j=0}^{\alpha} \left( (u-v-w)^{\beta} (u-v-w+1)^{\alpha} \right),
\]
All we need to show is that this is zero for all \( u, v, w, t \geq 0 \), which follows from
\[
\sum_{j} (-1)^{j} \binom{\alpha+j}{\beta+1} (u-v-w+1)^{\alpha} (u-v-w+1)^{\alpha} = \sum_{j,e} (-1)^{j} \binom{\alpha+j}{\beta+1} (u-v-w+1)^{\alpha} (u-v-w+1)^{\alpha}.
\]
Similarly, if \( s < \alpha + \beta \) then the equation \( \Phi(-\beta) = 0 \) is equivalent to
\[
L_2(s, \alpha, \beta, i) = R_2(s, \alpha, \beta, i)
\]
with
\[
L_2 := \sum_{l=0}^{\alpha} \binom{l}{\beta+1} \left( (s-\alpha)^{\beta} (l+\beta+1) + (s-\alpha)^{\beta} (l+\alpha-s)^{\beta+1} \right),
\]
\[
R_2 := \sum_{j=0}^{\alpha} (l+1)^{\beta+1} \left( (s-\alpha)^{\beta} (l+\beta+1) + (s-\alpha)^{\beta} (l+\alpha-s)^{\beta+1} \right) + \frac{(\alpha+1)(\alpha-\beta)(i-\beta-1)^{\beta}}{(s-\alpha)(i-\beta-1)^{\alpha}}.
\]
Let us in fact show that \( L_2(u, v, w, t) = R_2(u, v, w, t) \) for all \( u \geq v \geq t \geq w \geq 0 \). It is easy to verify that \( L_2(u, t, w, t) = R_2(u, t, w, t) \), and
\[
R_2(u + 1, v + 1, w, t) - R_2(u, v, w, t) - L_2(u + 1, v + 1, w, t) + L_2(u, v, w, t)
\]
\[
= \frac{u-v}{u-v+1} \sum_{j=0}^{\alpha} \left( (u-v-w)^{\beta+1} (u-v-w+1)^{\alpha} \right) + \frac{u-v}{u-v+2} \sum_{j=0}^{\alpha} \left( (u-v-w)^{\beta+1} (u-v-w+1)^{\alpha} \right) + \frac{u-v}{u-v+2} \sum_{j=0}^{\alpha} \left( (u-v-w)^{\beta} (u-v-w+1)^{\alpha} \right),
\]
which is zero by (5). Finally, suppose that \(i = \beta\). Then

\[
\Phi_1(-\beta) = -\sum_{l=1}^{a+1} \binom{a}{l} \left( \frac{s-a-\beta}{s-\alpha} \right)^l \frac{l!}{l-l} + \left( \frac{s-a-\beta}{s-\alpha} \right)^l \frac{l!}{l-l},
\]

\[
\Phi_2(-\beta) = -(1)^{\beta+1} \frac{s-a-\beta}{s-\alpha},
\]

\[
\Phi_3(-\beta) = \sum_{j=0}^{a+1} (1)^{\beta+1} \frac{j}{2^{a+1}} \left( \frac{\alpha+j}{\alpha+1} \right)^{\beta} = -(1)^{\alpha+\beta} \frac{\left( s-a-\beta \right)}{\left( s-a-\alpha \right)},
\]

\[
\Phi_4(-\beta) = -(1)^{\beta} \left( s-a-\beta \right)^{\beta}.
\]

where \(h_t = 1 + \cdots + \frac{1}{t} \) is the harmonic number for \(t \in \mathbb{Z}_{>0}\) and \(h_t = 0 \) for \(t \in \mathbb{Z}_{\leq 0}\). Since

\[
\sum_{j=0}^{3} (1)^{\alpha+\beta+j+1} \left( \frac{j}{2^{a+1}} \right) \left( \frac{s-a-\beta}{s-\alpha} \right)^{\beta} = -(1)^{\alpha+\beta} \frac{\left( s-a-\beta \right)}{\left( s-a-\alpha \right)},
\]

we can simplify \(\Phi_3(-\beta)\) to

\[
\Phi_3(-\beta) = \sum_{j=0}^{a+1} (1)^{\alpha+\beta+j+1} \left( \frac{j}{2^{a+1}} \right) \left( \frac{s-a-\beta}{s-\alpha} \right)^{\beta} = -(1)^{\alpha+\beta} \frac{\left( s-a-\beta \right)}{\left( s-a-\alpha \right)} \beta.
\]

The equation \(\Phi(-\beta) = 0\) is therefore equivalent to

\[
L_3(s, \alpha, \beta) = R_3(s, \alpha, \beta)
\]

with

\[
L_3 := \beta \Phi_1(-\beta),
\]

\[
R_3 := (s-\alpha-\beta)(\Phi_3(-\beta) + \Phi_4(-\beta)) - \Phi_2(-\beta).
\]

Let us show that \(L_3(u, v, w) = R_3(u, v, w)\) for all \(u > v \geq w > 0\). For \(v = w\) this is

\[
(-1)^{u-1} \frac{\left( u-w-1 \right)^{\beta}}{\left( u-w \right)^{\beta}} - \frac{\left( u-w-1 \right)^{\beta}}{\left( u-w \right)^{\beta}} = 0
\]

For \(u > 2w\) then both sides are zero, if \(u = 2w\) then both sides are 1, and if \(2w < u < w\) then both sides are \(w(h_{w-1} + \frac{1}{2^{w-1}})\). Thus all we need to do is show that

\[
R_3(u+1, v+1, w) - R_3(u, v, w) - L_3(u+1, v+1, w) + L_3(u, v, w) = 0
\]

for all \(u > v \geq w > 0\). By using the equation

\[
\sum_{j=0}^{a} (1)^{\alpha+\beta+j+1} \left( \frac{j}{2^{a+1}} \right) \left( \frac{s-a-\beta}{s-\alpha} \right)^{\beta} = -(1)^{\alpha+\beta} \frac{\left( s-a-\beta \right)}{\left( s-a-\alpha \right)} \beta
\]

we can get rid of the sum \(\sum_{j}\) and (after a series of simple algebraic manipulations) simplify this to

\[
\left( \frac{v+1}{w} \right)^{\beta} \left( \frac{v-w}{u-v} \right)^{\beta} = \left( \frac{v+1}{w} \right)^{\beta} \left( \frac{v-w}{u-v} \right)^{\beta}.
\]

We omit the full tedious details and just mention that since we are able to get rid of the sums \(\sum_{t}\) and \(\sum_{j}\) the aforementioned algebraic manipulations amount to simple cancellations. If \(u > v + w\) then

\[
\left( \frac{v+1}{w} \right)^{\beta} \left( \frac{v-w}{u-v} \right)^{\beta} = \left( \frac{v+1}{w} \right)^{\beta} \left( \frac{v-w}{u-v} \right)^{\beta}.
\]

\[
\left( \frac{v+1}{w} \right)^{\beta} \left( \frac{v-w}{u-v} \right)^{\beta} = \left( \frac{v+1}{w} \right)^{\beta} \left( \frac{v-w}{u-v} \right)^{\beta}.
\]

\[
\frac{v+1}{w} \left( \frac{v-w}{u-v} \right)^{\beta} = \left( \frac{v+1}{w} \right)^{\beta} \left( \frac{v-w}{u-v} \right)^{\beta}.
\]

\[
\frac{v+1}{w} \left( \frac{v-w}{u-v} \right)^{\beta} = \left( \frac{v+1}{w} \right)^{\beta} \left( \frac{v-w}{u-v} \right)^{\beta}.
\]
and if \( u < v + w \) then
\[
\binom{v+1}{w} \left( \frac{(u-v-w)}{u-v} \right)^\alpha \left( \frac{v-w}{2v-u+1} \right) + \binom{v}{w} \left( \frac{(u-v-w)}{2v-u+1} \right)^\alpha \\
= \left( -1 \right)^w \binom{v+1}{w} (u-v-w) (v-w) (2v-u+1) \\
= \left( -1 \right)^w \binom{v+1}{w} (u-v-w) (v-w) (2v-u+1) \\
= \left( -1 \right)^w \binom{v+1}{w} (u-v-w) (v-w) (2v-u+1) \\
= \left( -1 \right)^w \binom{v+1}{w} (u-v-w) (v-w) (2v-u+1). 
\]
We have finally shown that
\[
\mathbf{r}_i(C_0, \ldots, C_\alpha)^T = 0 
\]
(over \( \mathbb{F}_p \)) for all \( i \in \{1, \ldots, \alpha\} \). This finishes the proof as the conditions necessary for corollary 8 to be applicable are satisfied.

\( \square \)

This is the first time that we consider an \( \alpha \) that is greater than or equal to \( s \).
The major difference in this scenario is that \( s \) is not the “correct” remainder of \( r \) to work with and instead we would prefer to consider the number that is congruent to \( r \mod p-1 \) and belongs to the set \( \alpha + 1, \ldots, p-\alpha - 1 \). Let us therefore define \( s_\alpha = r - \alpha + \alpha \), and in particular let us note that \( s_\alpha = s \) for \( s > \alpha \) (which has hitherto always been the case). Then the computations in section 4 of [Arson] work exactly the same if we replace every instance of \( s \) with \( s_\alpha \) (and the restricted sum \( \sum_{i>0} \) with \( \sum_{0 < i < (p-1) - r - \alpha} \) when \( s_\alpha = p-1 \)—the sufficient condition for them to work is \( s_\alpha \in \mathbb{Z}_p \), which is indeed the case since \( s_\alpha - \alpha = p - 1 + s - \alpha \geq 2\nu - \alpha \). So there is an analogous version of proposition 10 (together with the remarks preceding it) and we can conclude the desired result.

\( \square \)

Let us write \( \alpha = \frac{\delta}{\gamma} + 1 \) and, as the claim we want to prove is vacuous for \( s = 2 \), let us assume that \( s \geq 4 \) and in particular \( \alpha \geq 3 \). We apply part 3 of corollary 8 with \( v \in (\nu_p (\alpha - \alpha), t) \) and
\[
C_j = \begin{cases} 
0 & \text{if } j \in \{-1, 0\}, \\
(-1)^j \binom{\alpha - 2}{j} + (-1)^{j+1} (\alpha - 2) \binom{\alpha}{j+1} + pC_j^* & \text{if } j \in \{1, \ldots, \alpha\}, 
\end{cases}
\]
for some constants \( C_1^*, \ldots, C_\alpha^* \) yet to be chosen. The conditions necessary for the lemma to be applicable are satisfied if \( \tilde{C} = \sum_j C_j (\frac{s-\alpha-\beta+j}{s-\alpha-\beta}) \in \mathbb{Z}_p \) and
\[
\vartheta_w(D_i) = O(\varepsilon)
\]
for \( 0 \leq w < 2\nu - \alpha \). We have
\[
\tilde{C} = \sum_j C_j \binom{s-\alpha-\beta+j}{s-\alpha-\beta} + O(p)
\]
\[
= -1 + \sum_j \left( (-1)^j \binom{\alpha-2}{j} + (-1)^{j+1} (\alpha - 2) \binom{\alpha}{j+1} \right) \binom{s-\alpha-\beta+j}{s-\alpha-\beta} + O(p)
\]
\[
= -1 + O(p) \in \mathbb{Z}_p
\]
by \( \varnothing \) since \( \alpha - 2 > s - \alpha - \beta \). And since
\[
j \leq s - \alpha - \beta + j \leq s - \beta < p - i
\]
we also have
\[
\binom{s+\beta(p-1) - \alpha+j}{\nu(p-1) + \gamma + j} = \binom{\beta}{\nu}\binom{s-\alpha-\beta+j}{s-\alpha-\beta+i} + O(p).
\]
Thus the equality \( \vartheta_w(D_i) = O(p) \) follows from the fact that
\[
\sum_j (-1)^j \binom{\alpha-2}{j} \binom{s-\alpha-\beta+j}{s-\alpha-\beta+i} = 0,
\]
which follows from (7) since \( \alpha - 2 > \alpha - 2 - \beta + i = s - \alpha - \beta + i \). Moreover, we can choose \( C_1^*, \ldots, C_s^* \) in a way that \( \theta_{w}(D_i) = 0 \) for \( 0 \leq w < 2\nu - \alpha \) similarly as in the proof of proposition 10 since the reduction modulo \( p \) of the matrix

\[
\begin{pmatrix}
(\beta^i p^{-1} - \alpha + \theta_j)
(\beta^i p^{-1} + \theta_j)
\end{pmatrix}_{1 \leq i, j < \beta}
\]

is upper triangular with units on the diagonal. Thus the conditions necessary for corollary 8 to be applicable are satisfied and we can conclude that \( \hat{N}_{s/2+1} \) is trivial modulo \( \mathcal{S}_a \).

(7) Let us write \( \alpha = \frac{s}{2} - 1 \) and, as the claim we want to prove is vacuous for \( s = 2 \), let us assume that \( s \geq 4 \) and in particular \( \alpha \geq 3 \). The only obstruction in the proof of part (3) that prevents us from concluding that \( \hat{N}_{s/2-1} \) is trivial modulo \( \mathcal{S}_a \) is that the dominant terms are \( E_\xi \cdot K \cdot \Xi_p \cdot \theta^s h_\xi \) for \( \frac{s}{2} < \xi < 2\nu - \frac{s}{2} \) rather than \( H \). We can see from part (6) that \( E_{s/2+1} \cdot K \cdot \Xi_p \cdot \theta^{s/2+1} h_{s/2+1} \) is congruent modulo \( \text{im}(T - a) \) to an element the valuation of whose coefficient is at least \( t + 1 \). Since the valuation of the coefficient of \( H \) is less than \( t + 1 \), the obstruction coming from \( E_{s/2+1} \cdot K \cdot \Xi_p \cdot \theta^{s/2+1} h_{s/2+1} \) is removable. If \( s = 2\nu - 2 \) then this is the only obstruction and we can conclude that \( \hat{N}_{s/2-1} \) is trivial modulo \( \mathcal{S}_a \). Now suppose that \( s < 2\nu - 2 \). Then as per our usual trick we can apply part (1) of corollary 8 and conclude that \( \hat{N}_a \) is trivial modulo \( \mathcal{S}_a \) as long as \( (\epsilon, 0, \ldots, 0)^T \) is in the image of the matrix \( A = (A_{w,j})_{0 \leq w, j \leq \alpha} \) that has integer entries

\[
A_{w,j} = \sum_{i>0} (r_w - n + j) (r_{(p-1)i} - j) = S_{w,j} + \epsilon N_{w,j} + O(ep)
\]

with \( S \) and \( N \) as in part (1). However, this time we can deduce more than that: since \( s < 2\nu - 2 \) it follows that \( 1 \cdot K \cdot \Xi_p \cdot \theta^{s/2+1} y^{r-s/2-1} \) is congruent to

\[
g_1 \cdot K \cdot \Xi_p \cdot \theta^{s/2+1} y^{s/2-n+1} y^{r-np-s/2-1}
\]

modulo \( \text{im}(T - a) \) for some \( g_1 \) with \( v_p(g_1) \geq v_p(a) - \frac{s}{2} - 1 \). This in turn by part (6) is congruent to \( g_2 \cdot K \cdot \Xi_p \cdot \theta^{s/2+1} h_2 \) for some \( g_2 \) with \( v_p(g_2) \geq t \) and for some \( h_2 \). Here we use the fact that the valuation of the constant \( C_1 \) from part (6) is at least one and therefore the corresponding term \( H \) is

\[
g_2 \cdot K \cdot \Xi_p \cdot \theta^{s/2+1} x^{s/2-n+1} y^{r-np-s/2-1} + O(\epsilon)
\]

modulo \( \text{im}(T - a) \) for some \( g_2 \) with \( v_p(g_2) = v_p(a) - \frac{s}{2} - 1 \). In general the error term would be \( C_1 \alpha^{-s/2} g_1 \cdot K \cdot \Xi_p \cdot \theta^{s/2-n} y^{r-np-s/2} + O(\epsilon) \) rather than \( O(\epsilon) \)—see part (2) of lemma 5 in [Arma] for a description of this error term. This implies that we can add a constant multiple of \( 1 \cdot K \cdot \Xi_p \cdot x^{s/2+1} y^{s/2-1} \) to the element

\[
\sum_i D_i \cdot K \cdot \Xi_p \cdot x^{i(p-1) + \alpha} y^{r-i(p-1) - \alpha} + O(ap^{-\alpha})
\]

from the proof of lemma 7 in [Arma], and we can translate this back to adding the extra column

\[
\begin{pmatrix}
(r_{-\alpha}, \ldots, r_{-\alpha})^T
\end{pmatrix}
\]

to \( A \). As in part (1) we can reduce showing that \( (\epsilon, 0, \ldots, 0)^T \) is in the image of \( A \) to showing that \( (1, 0, \ldots, 0)^T \) is in the image of a certain \((\alpha + 1) \times (\alpha + 2)\) matrix \( \mathcal{T} \) (over \( \mathcal{F}_p \)). The matrix \( \mathcal{T} \) is obtained from the matrix \( \mathcal{Q} \) described in fourth step of the main proof in [Arma] by replacing all entries in the first row with zeros (because this time we do not divide the corresponding row of \( A \) by \( p \)) and by adding an entry

\[\text{This is true even if } s = 2\nu - 2.\]
extra column corresponding to the extra column of $A$. Thus, if we index the extra column to be the zeroth column, the lower right $\alpha \times \alpha$ submatrix of $\overline{R}$ is upper triangular with units on the diagonal, the first column of $\overline{R}$ is identically zero, and all entries of the first row of $\overline{R}$ except for $\overline{R}_{0,0}$ are zero. As when computing $(BN)_{i,j}$ in part (4) we can find that

$$\overline{R}_{0,0} = \sum_{t=0}^{\alpha} \binom{l-\beta-1}{t} = \Phi'(-\beta - 1)$$

with

$$\Phi(z) = \sum_{t=0}^{\alpha} \binom{z+t}{t} = \binom{z+\alpha+1}{\alpha}.$$

Thus

$$\overline{R}_{0,0} = \binom{\alpha-\beta}{\alpha} \frac{(-1)^{\beta+1}}{\beta(\beta)} \neq 0,$$

which implies that indeed $(1,0,\ldots,0)^T$ is in the image of $\overline{R}$. Thus the conditions necessary for corollary 8 to be applicable are satisfied and we can conclude that $\overline{N}_{s/2-1}$ is trivial modulo $\mathfrak{J}_a$.

(8) Let us write $\alpha = \frac{s}{2} + 1$. The reason why the proof of part (6) does not work for $\beta \in \{\alpha - 1, \alpha\}$ is because $\overline{C} = O(p)$ for the constructed constants $C_j$. However since $t > v_p(a) - \frac{s}{2}$ and $\overline{C} \in \mathbb{Z}_p^\times$ the dominant term coming from lemma 7 is

$$H = b_H \binom{\Phi}{\mathfrak{J}_a} \binom{\alpha-\beta}{\alpha} \frac{(-1)^{\beta+1}}{\beta(\beta)} \neq 0,$$

for the constant $b_H = \frac{a^{p-\alpha}}{1-p} \overline{C}$ which has valuation $v_p(a) - \alpha + 1$. As in part (7) it is crucial here that $C_1 = O(p)$. Just as in the proof of part (6) we can reduce the claim we want to show to proving that there exist constants $C_1, \ldots, C_\alpha \in \mathbb{Z}_p$ such that $C_1 = O(p)$ and

$$\binom{(s+\beta(p-1)-\alpha+j)}{(i(p-1)+j)} \binom{(\alpha-\beta)(\alpha)}{(i(p-1)+j)} \binom{(\alpha-\beta)(\alpha)}{(i(p-1)+j)} \neq 0,$$

for integer entries and is invertible (over $\mathbb{Z}_p$), as then we can recover

$$\begin{cases} 
C_1 = 0 \text{ and } (C_2, \ldots, C_\alpha)^T = A_0^{-1}(1,0,\ldots,0)^T & \text{if } \beta = \alpha - 1, \\
(C_1/p, \ldots, C_\alpha)^T = A_0^{-1}(1,0,\ldots,0)^T & \text{if } \beta = \alpha.
\end{cases}$$

It is easy to verify that $A_0$ is integral, since if $j > 1$ then $s - \alpha - \beta + j \geq 0$ and therefore

$$\binom{(s+\beta(p-1)-\alpha+j)}{(i(p-1)+j)} = \binom{\beta}{\mu} \binom{(\alpha-\beta)(\alpha)}{(i(p-1)+j)} \neq 0(p) = O(p)$$

for $i \leq \beta - \alpha + 1$. Let us show that $A_0$ is invertible (over $\mathbb{Z}_p$) by showing that $\overline{A}_0$ is invertible (over $\mathbb{F}_p$). Suppose first that $\beta = \alpha - 1$ and denote the columns of $A_0$ by $c_2, \ldots, c_\alpha$. The bottom left $(\alpha-3) \times (\alpha-3)$ submatrix of $\overline{A}_0$ is upper triangular with units on the diagonal. Moreover, since

$$\overline{A}_0 = \binom{(s+\beta(p-1)-\alpha)}{(i(p-1)+j)} \binom{(\alpha-\beta)(\alpha)}{(i(p-1)+j)} \binom{(\alpha-\beta)(\alpha)}{(i(p-1)+j)} \neq 0,$$

all but the top two entries of each of the vectors

$$c_{\alpha-1} = (\alpha-3) c_{\alpha-2} + \cdots + (\alpha-3)(\alpha-2) c_2,$$

$$c_\alpha = (\alpha-3) c_{\alpha-2} + \cdots + (\alpha-3)(\alpha-3) c_2$$
Lemma 7 is in part (6)), and as in part (1) we can show that the dominant term coming from these constants and instead use

\[ e_{0,0} = \beta \sum_{j=0}^{\beta-1} \frac{\beta}{j} \frac{\beta-j-1}{j-1} \frac{\beta-j}{j} = \sum_{j=0}^{\beta-1} \frac{\beta}{j} \frac{\beta-j-1}{j-1} \frac{\beta-j}{j} = \left( \frac{-1}{\beta} \right)^{\beta+1}, \]

\[ e_{0,1} = \beta \sum_{j=0}^{\beta} (-1)^{j+1}(j-1) \frac{\beta-j}{j+1} \frac{\beta-j-1}{j} = \sum_{j=0}^{\beta} (-1)^{j+1}(j-1) \frac{\beta-j}{j+1} \frac{\beta-j}{j} = \left( \frac{-1}{\beta} \right)^{\beta+1}. \]

so it has determinant \( \frac{-\beta^2}{\beta} \in \mathbb{F}_p^\times \). Now suppose that \( \beta = \alpha \) and denote the columns of \( A_0 \) by \( c_1, \ldots, c_\alpha \). The bottom left \( (\alpha-1) \times (\alpha-1) \) submatrix of \( \mathcal{T}_0 \) is upper triangular with units on the diagonal, all but the top entry of the vector

\[ c_\alpha = \left( \begin{array}{c} -1 \end{array} \right)^{\alpha-2} c_{\alpha-1} + \cdots + (-1)^{\alpha-2} \left( \begin{array}{c} \alpha-2 \end{array} \right) c_2 \]

are zero, and that top entry is

\[ \beta \sum_{j=0}^{\beta-2} (-1)^j \frac{\beta-j}{j+1} \frac{\beta-j-2}{j} = \sum_{j=0}^{\beta-2} (-1)^j \frac{\beta-j}{j+1} \frac{\beta-j}{j} = (-1)^{\beta+1}. \]

Therefore \( \mathcal{T}_0 \) is always invertible, so the conditions necessary for corollary \( \textup{8} \) to be applicable are satisfied and we can conclude that \( \overline{N}_{s/2+1} \) is trivial modulo \( \mathcal{F}_a \).

(9) Let us write \( \alpha = \frac{q}{2} + 1 \). This time the proofs of both parts (6) and (8) break down since \( \bar{C} = \mathcal{O}(p) \) and the dominant term is no longer \( H \). Let us slightly tweak these constants and instead use

\[ C_j = \begin{cases} (-1)^{\alpha \epsilon} & \text{if } j = -1, \\ (-1)^{\alpha+j+1} \alpha^{\alpha-1-j} & \text{if } j \in \{0, \ldots, \alpha\}, \end{cases} \]

for suitable constants \( C_j \). Let \( \mathcal{R} \) be the matrix constructed in part (4). Then still \( \bar{C} = \mathcal{O}(p) \) (as these tweaked constants satisfy the same properties as the constants in part (6), and as in part (1) we can show that the dominant term coming from lemma (7) is

\[ (\theta^j + C_{-1}) \bullet_{KZ} \theta^n x^\alpha y^{\bar{\alpha}-\alpha(p+1)-p+1} + C_{-1} \bullet_{KZ} \theta^n x^\alpha y^{\bar{\alpha}-np-\bar{\alpha}} \]

(and therefore that \( \text{ind}_{KZ} \text{sub}(\frac{q}{2} + 1) \) is trivial modulo \( \mathcal{F}_a \)) as long as

\[ \mathcal{T}(C_0, \ldots, C_\alpha)^T = (0, \ldots, 0, 1)^T. \]

The equation associated with the \( i \)-th row is straightforward if \( i \not\in \{0, \alpha\} \). Since \( \mathcal{T}_{0, \alpha} \) is equal to

\[ g \sum_{j=0}^{\alpha} (-1)^{\alpha+j+1} \frac{\alpha-j}{j} \frac{\alpha-j+1}{j-1} \]

and since

\[ \sum_{j} (-1)^j \frac{\alpha-j}{j} \frac{1}{j+1} = \frac{1}{\alpha}, \]

\[ \sum_{j} (-1)^j \frac{\alpha-j}{j} \frac{1}{j} \frac{1}{j+1} = (-1)^{\alpha+1}, \]

\[ \sum_{j} (-1)^j \frac{\alpha-j}{j} \frac{1}{j} \frac{1}{j+1} \frac{1}{j+2} = \frac{1}{\alpha} \]

are zero. Thus it is enough to show that the \( 2 \times 2 \) matrix consisting of those four entries is invertible (over \( \mathbb{F}_p \)). This \( 2 \times 2 \) matrix is

\[ \left( \begin{array}{cc} e_{0,0} & e_{0,1} \\ (-1)^{\alpha \epsilon} & (-1)^{\alpha+j+1} \alpha^{\alpha-1-j} \end{array} \right) \]

with

\[ \begin{aligned} e_{0,0} &= \beta \sum_{j=0}^{\beta-1} (-1)^j (\beta-j-1) \frac{\beta-j}{j} \frac{\beta-j}{j-1} \\ &= \sum_{j=0}^{\beta-1} (-1)^j (\beta-j) \frac{\beta-j}{j} \frac{\beta-j}{j-1} = \left( \frac{-1}{\beta} \right)^{\beta+1}, \end{aligned} \]

\[ \begin{aligned} e_{0,1} &= \beta \sum_{j=0}^{\beta} (-1)^j (j-1) \frac{\beta-j}{j+1} \frac{\beta-j}{j} \\ &= \sum_{j=0}^{\beta} (-1)^j (j-1) \frac{\beta-j}{j+1} \frac{\beta-j}{j} = \left( \frac{-1}{\beta} \right)^{\beta+1}, \end{aligned} \]
the equation associated with the zeroth row is
\[ \sum_{j} (-1)^j \binom{\alpha - 2}{j} \sum_{i=v}^{\alpha} (-1)^j v (v-j) \binom{v-j}{v} y_v^{j-v} = -\frac{1}{\alpha}. \]
This follows from
\[ \sum_{i=0}^{\alpha} (-1)^i \binom{i}{i-v} = (-1)^j \binom{j}{j-i} \]
for \( 0 \leq v \leq j \leq \alpha \). Since \( R_{\alpha,j} \) is equal to
\[ \delta_{j=2} \binom{0}{\alpha} + \delta_{j=\alpha} F_{\alpha,\alpha,0} (\alpha - 2) - \binom{j-2}{\alpha-2} - (-1)^{\alpha} \binom{j-2}{\alpha-2} \]
the equation associated with the \( \alpha \)th row is
\[ \binom{0}{\alpha} - \sum_{j} (-1)^j \binom{\alpha-2}{j-2} \binom{j-2}{\alpha-2} = (-1)^\alpha - (-1)^{\alpha} F_{\alpha,\alpha,0} (\alpha - 2) + \frac{(-1)^{\alpha+1}}{\alpha}. \]
This follows from
\[ F_{\alpha,\alpha,0} (\alpha - 2) = F_{\alpha,\alpha,0} (-1) = (-1)^\alpha (-1)^\alpha \]
and the fact that the polynomial \( \binom{X}{\alpha-2} \) has degree less than \( \alpha - 2 \) (and is zero if \( \alpha = 2 \)) and therefore
\[ \sum_{j} (-1)^j \binom{\alpha-2}{j-2} \binom{j-2}{\alpha-2} = 0. \]
Thus the conditions necessary for corollary 8 to be applicable are satisfied and we can conclude that \( \text{ind}_{KZ} \text{sub}(\frac{\alpha}{\alpha} + 1) \) is trivial modulo \( \mathcal{J}_a \).

**Proof of theorem 4** Let us assume that \( \mathfrak{S}_{k,a} \) is reducible in hope of reaching a contradiction. In particular, the classification given by theorem 5 implies that \( \mathfrak{S}_{k,a} \) has two factors, each of which is a quotient of a representation in the set
\[ \{ \text{ind}_{KZ} \text{sub}(\alpha) \mid 0 \leq \alpha < \nu \} \cup \{ \text{ind}_{KZ} \text{quot}(\alpha) \mid 0 \leq \alpha < \nu \}, \]
and moreover that the following classification is true.

1. If the two representations are \( \text{ind}_{KZ} \text{sub}(\alpha_1) \) and \( \text{ind}_{KZ} \text{sub}(\alpha_2) \) then
   \[ \alpha_1 + \alpha_2 \equiv s + 1 \mod p - 1. \]
2. If the two representations are \( \text{ind}_{KZ} \text{sub}(\alpha_1) \) and \( \text{ind}_{KZ} \text{quot}(\alpha_2) \) then
   \[ \alpha_1 - \alpha_2 \equiv 1 \mod p - 1. \]
3. If the two representations are \( \text{ind}_{KZ} \text{quot}(\alpha_1) \) and \( \text{ind}_{KZ} \text{quot}(\alpha_2) \) then
   \[ \alpha_1 + \alpha_2 \equiv s - 1 \mod p - 1. \]
The facts that
\[ \alpha_1 + \alpha_2 \in \{ 0, \ldots, 2\nu - 2 \} \subseteq \{ 0, \ldots, p - 3 \}, \]
\[ \alpha_1 - \alpha_2 \in \{ 1 - \nu, \ldots, \nu - 1 \} \subseteq \{ -\frac{p - 3}{2}, \ldots, \frac{p - 3}{2} \}, \]
\[ s \in \{ 2, \ldots, 2\nu - 2 \} \subseteq \{ 2, \ldots, p - 3 \} \]
imples that the following classification is true as well.

1. If the two representations are \( \text{ind}_{KZ} \text{sub}(\alpha_1) \) and \( \text{ind}_{KZ} \text{sub}(\alpha_2) \) then
   \[ \alpha_1 + \alpha_2 = s + 1. \]
2. If the two representations are \( \text{ind}_{KZ} \text{sub}(\alpha_1) \) and \( \text{ind}_{KZ} \text{quot}(\alpha_2) \) then
   \[ \alpha_1 = \alpha_2 + 1. \]
(3) If the two representations are $\text{ind}_{KZ}^G \text{quot}(\alpha_1)$ and $\text{ind}_{KZ}^G \text{quot}(\alpha_2)$ then

$$\alpha_1 + \alpha_2 = s - 1.$$  

This classification and parts (1, 2, 3, 4, 5) of proposition 13 together imply that one of the two representations must in fact be either $\text{ind}_{KZ}^G \text{sub}(s_2)$ or $\text{ind}_{KZ}^G \text{quot}(s_2)$, and in that case the other representation is either

$$\text{ind}_{KZ}^G \text{quot}(s_2 - 1)$$

(which can only happen if $s = 2$ or $\beta \in \{1, \ldots, \frac{\nu}{2} - 1\}$ and $t = \nu - \frac{\nu}{2}$), or

$$\text{ind}_{KZ}^G \text{sub}(s_2 + 1)$$

(which can only happen if $\beta \in \{1, \ldots, \frac{\nu}{2} - 1\}$ and $t > \nu - \frac{\nu}{2} - 2$), or

$$\text{ind}_{KZ}^G \text{quot}(s_2 - 1)$$

(which can only happen if $s = 2$ and $\beta \in \{1, \ldots, \frac{\nu}{2} - 1\}$ and $t = \nu - \frac{\nu}{2}$). In the latter case if $s = 2$ then either $1 \ast_{KZ_{\overline{Q}_p}} x^2 y^{r-2} \in \mathcal{F}_a$ generates $\text{ind}_{KZ}^G \text{quot}(0)$, or $\nu \leq 2$ in which case $V_{k,a}$ is known to be irreducible. Parts (4, 6, 7, 8, 9) of proposition 13 exclude all of the remaining possibilities and give a contradiction.

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