A CLOSEDNESS THEOREM
OVER HENSELIAN VALUED FIELDS
WITH ANALYTIC STRUCTURE

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Abstract. The main purpose of the paper is to establish a closedness theorem over Henselian valued fields $K$ of equicharacteristic zero (not necessarily algebraically closed) with separated analytic structure. It says that every projection with a projective fiber is a definably closed map. This remains valid also for valued fields with analytic structure induced by a strictly convergent Weierstrass systems, including the classical, complete rank one valued fields with the Tate algebra of strictly convergent power series. As application, we prove two theorems on existence of the limit and on piecewise continuity.

1. Introduction

Throughout the paper, we shall deal with Henselian valued fields $K$ with separated analytic structure, nor necessarily algebraically closed. We shall always assume that the ground field $K$ is of equicharacteristic zero. A separated analytic structure is determined by a certain separated Weierstrass system $\mathcal{A}$ defined on an arbitrary commutative ring $A$ with unit (cf. [3, 4]), and the involved analytic language $\mathcal{L}$ is the two sorted, semialgebraic language $\mathcal{L}_{\text{Hen}}$ augmented by the reciprocal function $1/x$ and the names of all functions of the system $\mathcal{A}$, construed via the analytic $\mathcal{A}$-structure on their natural domains and as zero outside them. For convenience, we remind the reader of these concepts in Section 2. The theory of valued fields with analytic structure was developed in the papers [8, 9, 10, 5, 4, 3].

Given a valued field $K$, denote by $v$, $\Gamma = \Gamma_K$, $K^\circ$, $K^{\circ\circ}$ and $\tilde{K}$ the valuation, its value group, the valuation ring, maximal ideal and residue field, respectively. By the $K$-topology on $K^\circ$ we mean the topology induced by the valuation $v$.

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The main result of this article is the following closedness theorem.

**Theorem 1.1.** Given an $\mathcal{L}$-definable subset $D$ of $K^n$, the canonical projection
\[ \pi : D \times (K^o)^m \to D \]
is definably closed in the $K$-topology, i.e. if $B \subset D \times (K^o)^m$ is an $\mathcal{L}$-definable closed subset, so is its image $\pi(B) \subset D$.

It immediately yields the five corollaries stated below. The last three of them enable, in the non-Archimedean analytic case, application of resolution of singularities and transformation to a normal crossing by blowing up in much the same way as over locally compact ground fields (see [11, 12] for application in the non-Archimedean algebraic case).

**Corollary 1.2.** Let $D$ be an $\mathcal{L}$-definable subset of $K^n$ and $\mathbb{P}^m(K)$ stand for the projective space of dimension $m$ over $K$. Then the canonical projection
\[ \pi : D \times \mathbb{P}^m(K) \to D \]
is definably closed.

**Corollary 1.3.** Let $A$ be a closed $\mathcal{L}$-definable subset of $\mathbb{P}^m(K)$ or $\mathbb{R}^m$. Then every continuous $\mathcal{L}$-definable map $f : A \to K^n$ is definably closed in the $K$-topology.

**Corollary 1.4.** Let $\phi_i$, $i = 0, \ldots, m$, be regular functions on $K^n$, $D$ be an $\mathcal{L}$-definable subset of $K^n$ and $\sigma : Y \to K\mathbb{A}^n$ the blow-up of the affine space $K\mathbb{A}^n$ with respect to the ideal $(\phi_0, \ldots, \phi_m)$. Then the restriction
\[ \sigma : Y(K) \cap \sigma^{-1}(D) \to D \]
is a definably closed quotient map.

**Proof.** Indeed, $Y(K)$ can be regarded as a closed algebraic subvariety of $K^n \times \mathbb{P}^m(K)$ and $\sigma$ as the canonical projection.

**Corollary 1.5.** Let $X$ be a smooth $K$-variety, $\phi_i$, $i = 0, \ldots, m$, regular functions on $X$, $D$ be an $\mathcal{L}$-definable subset of $X(K)$ and $\sigma : Y \to X$ the blow-up of the ideal $(\phi_0, \ldots, \phi_m)$. Then the restriction
\[ \sigma : Y(K) \cap \sigma^{-1}(D) \to D \]
is a definably closed quotient map.

**Corollary 1.6.** (Descent property) Under the assumptions of the above corollary, every continuous $\mathcal{L}$-definable function
\[ g : Y(K) \cap \sigma^{-1}(D) \to K \]
that is constant on the fibers of the blow-up $\sigma$ descends to a (unique) continuous $\mathcal{L}$-definable function $f : D \to K$.  
The closedness theorem will be proven in Section 3. The strategy of proof in the analytic settings will generally follow the one in the algebraic case from my papers [11, 12]. We rely, in particular, on fiber shrinking and the local behavior of definable functions of one variable. Again, we make use of relative quantifier elimination for ordered abelian groups (in a many-sorted language with imaginary auxiliary sorts) due to Cluckers–Halupczok [4]. But now we apply elimination of valued field quantifiers for the theory $T_{\text{Hen},\mathcal{A}}$ and b-minimal cell decompositions with centers (cf. [3]).

Remark 1.7. The closedness theorem holds also for analytic structures induced by strictly convergent Weierstrass systems, because every such structure can be extended in a definitional way (extension by Henselian functions) to a separated analytic structure (cf. [1]). Examples of such structures are the classical, complete rank one valued fields with the Tate algebra of strictly convergent power series.

In Section 4, we give some applications of the closedness theorem, namely two theorems on existence of the limit (Proposition [4.1]) and on piecewise continuity (Theorem [4.3]). Note finally that our proof of the closedness theorem makes use of a certain version of the former result (Proposition [3.4]).

2. Fields with analytic structure

In this section we recall the concept of an analytic structure (cf. [3, Section 4.1]). Let $A$ be a commutative ring with unit and with a fixed proper ideal $I \subsetneq A$. A separated $(A, I)$-system is a certain system $\mathcal{A}$ of $A$-subalgebras $A_{m,n} \subset A[[\xi, \rho]], m, n \in \mathbb{N}$; here $A_{0,0} = A$. Two kinds of variables, $\xi$ and $\rho$, play different roles. Roughly speaking, the variables $\xi$ vary over the valuation ring (or the closed unit disc) $K^\circ$ of a valued field $K$, and the variables $\rho$ vary over the maximal ideal (or the open unit disc) $K^{\circ\circ}$ of $K$. $\mathcal{A}$ is called a separated pre-Weierstrass system if two usual Weierstrass division theorems hold in each $A_{m,n}$. When, in addition, such a pre-Weierstrass system $\mathcal{A}$ satisfies a condition referring to the so-called rings of $A$-fractions, it is called a separated Weierstrass system (loc. cit.). This condition may be regarded as a kind of weak Noetherian property, because it implies, in particular, that if

$$f = \sum_{\mu, \nu} a_{\mu \nu} \xi^\mu \rho^\nu \in A_{m,n},$$

then the ideal of $A$ generated by the $a_{\mu \nu}$ is finitely generated.

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of homomorphisms \( \sigma_{m,n} \) from \( A_{m,n} \) to the ring of \( K^\circ \)-valued functions on \( (K^\circ)^m \times (K^\circ)^n \), \( m, n \in \mathbb{N} \), such that
1) \( \sigma_{0,0}(I) \subset K^\circ \);
2) \( \sigma_{m,n}(\xi_i) \) and \( \sigma_{m,n}(\rho_j) \) are the \( i \)-th and \( (m+j) \)-th coordinate functions on \( (K^\circ)^m \times (K^\circ)^n \), respectively;
3) \( \sigma_{m+1,n} \) and \( \sigma_{m,n+1} \) extend \( \sigma_{m,n} \), where functions on \( (K^\circ)^m \times (K^\circ)^n \) are identified with those functions on \( (K^\circ)^{m+1} \times (K^\circ)^n \) or \( (K^\circ)^m \times (K^\circ)^{n+1} \) which do not depend on the coordinate \( \xi_{m+1} \) or \( \rho_{n+1} \), respectively.

Further, consider a separated pre-Weierstrass \((A,I)\)-system \( A \) and assume that \( A = F^\circ \) and \( I = F^\circ \) for a valued field \( F \). Then \( A \) is a Weierstrass system iff for every \( f \in A_{m,n}, f \neq 0, m, n \in \mathbb{N} \), there is an element \( c \in F \) such that \( cf \in A_{m,n} \) and the Gauss norm (which is then well defined) \( \|cf\| = 1 \) (loc. cit.).

Now let us recall some properties of analytic structures. Analytic \( A \)-structures preserve composition (op. cit., Proposition 4.5.3). If the ground field \( K \) is non-trivially valued, then the function induced by a power series from \( A_{m,n}, m, n \in \mathbb{N} \), is the zero function iff the image in \( K \) of each of its coefficients is zero (op. cit., Proposition 4.5.4).

**Remark 2.1.** When considering a particular field \( K \) with analytic \( A \)-structure, one may assume that \( \ker \sigma_{0,0} = (0) \). Indeed, replacing \( A \) by \( A/\ker \sigma_{0,0} \) yields an equivalent analytic structure on \( K \) with this property. Then \( A = A_{0,0} \) can be regarded as a subring of \( K^\circ \). Moreover, by extension of parameters, one can get a (unique) separated Weierstrass system \( A(K) \) over \( (K^\circ, K^\circ) \) and \( K \) has separated analytic \( A(K) \)-structure; a similar extension of parameters can be performed for any subfield \( F \subset K \) of parameters (op. cit., Theorem 4.5.7 f.f.). Further, a separated analytic \( A \)-structure on a valued field \( K \) can be uniquely extended to any algebraic extension \( K' \) of \( K \); in particular, to the algebraic closure \( K_{alg} \) of \( K \) (op. cit., Theorem 4.5.11). The foregoing properties remain valid in the case of strictly convergent Weierstrass systems too. Finally, every valued field with separated analytic structure is Henselian (op. cit., Proposition 4.5.10).

Now we can describe the analytic language \( L \) of an analytic structure \( K \) determined by a separated Weierstrass system \( A \). We begin by defining the semialgebraic language \( L_{Hen} \). It is a two sorted language with the main, valued field sort \( K \), and the auxiliary \( RV \)-sort
\[
RV = RV(K) := RV^* \cup \{0\}, \quad RV^*(K) := K^\times/(1 + K^\circ);
\]
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The set of units of a ring $A$, denoted $A^*$, is the language of rings $(0, 1, +, -, \cdot)$. The language of the auxiliary sort is the so-called inclusion language (op. cit., Section 6.1). The only map connecting the sorts is the canonical map

$$rv : K \to RV(K), \quad 0 \mapsto 0.$$ 

Since

$$\tilde{K}^* \cong (K^*)^*/(1 + K^\infty) \quad \text{and} \quad \Gamma \cong K^*/(K^\circ)^*,$$

we get the canonical exact sequence

$$1 \to \tilde{K} \to RV(K) \to \Gamma \to 0.$$ 

This sequence splits iff the valued field $K$ has an angular component map.

The analytic language $\mathcal{L} = \mathcal{L}_{Hen, A}$ is the semialgebraic language $\mathcal{L}_{Hen}$ augmented on the valued field sort $K$ by the reciprocal function $1/x$ (with $1/0 := 0$) and the names of all functions of the system $A$, together with the induced language on the auxiliary sort $RV$ (op. cit., Section 6.2). A power series $f \in A_{m,n}$ is construed via the analytic $A$-structure on their natural domains and as zero outside them. More precisely, $f$ is interpreted as a function

$$\sigma(f) : (K^\circ)^m \times (K^\infty)^n \to K^\circ,$$

extended by zero on $K^{m+n} \setminus (K^\circ)^m \times (K^\infty)^n$.

In the equicharacteristic case, however, the induced language on the auxiliary sort $RV$ further coincides with the semialgebraic inclusion language. It is so because then [Lemma 6.3.12] can be strengthened as follows, whereby [Lemma 6.3.14] can be directly reduced to its algebraic analogue. Consider a strong unit on the open ball $B = K^\circ_{alg}$. Then $rv(E^\sigma)(x)$ is constant when $x$ varies over $B$. This is no longer true in the mixed characteristic case. There, a weaker conclusion asserts that the functions $rv_n(E^\sigma)(x)$, $n \in \mathbb{N}$, depend only on $rv_n(x)$ when $x$ varies over $B$; actually, $rv_n(E^\sigma)(x)$ depend only on $x \mod (n \cdot K^\circ_{alg})$ when $x$ varies over $B$, as indicated in [Remark A.1.12]. Under the circumstances, the residue field $\tilde{K}$ is orthogonal to the value group $\Gamma_K$, whenever the ground field $K$ has an angular component map or, equivalently, when the auxiliary sort $RV$ splits (in a non-canonical way):

$$RV(K) \cong \tilde{K} \times \Gamma_K.$$ 

This means that every definable set in the auxiliary sort $RV(K)$ is a finite union of the Cartesian products of some sets definable in the residue field sort $\tilde{K}$ (in the language of rings) and in the value group sort.
Remark 2.2. Not all valued fields $K$ have an angular component map, but it exists if $K$ has a cross section, which happens whenever $K$ is $\aleph_1$-saturated (cf. [11, Chap. II]). Moreover, a valued field $K$ has an angular component map whenever its residue field $k$ is $\aleph_1$-saturated (cf. [14, Corollary 1.6]). In general, unlike for $p$-adic fields and their finite extensions, adding an angular component map does strengthen the family of definable sets. Since the $K$-topology is $L$-definable, the closedness theorem is a first order property. Therefore it can be proven using elementary extensions, and thus one may assume that an angular component map exists.

Let $\mathcal{T}_{\text{Hen},A}$ be the theory of all Henselian valued fields of characteristic zero with analytic $A$-structure. The crucial result about analytic structures is the following [6, Theorem 6.3.7].

Theorem 2.3. The theory $\mathcal{T}_{\text{Hen},A}$ eliminates valued field quantifiers, is b-minimal with centers and preserves all balls. Moreover, $\mathcal{T}_{\text{Hen},A}$ has the Jacobian property. \hfill $\Box$

Therefore the theory $\mathcal{T}_{\text{Hen},A}$ admits b-minimal cell decompositions with centers (cf. [1]).

3. Proof of the closedness theorem

From now on we shall assume that the ground field $K$ with separated analytic structure $A$ is of equicharacteristic zero, and that $K$ has an angular component map. In the algebraic case, the proofs of the closedness theorem given in our papers [11, 12] make use of the following three main tools: the theorem on existence of the limit ([11, Proposition 5.2] and [12, Theorem 5.1]), fiber shrinking ([11, 12, Proposition 6.1]) and cell decomposition in the sense of Pas.

Fiber shrinking was reduced, by means of elimination of valued field quantifiers, to Lemma 3.1 below ([12, Lemma 6.2]), which, in turn, was obtained via relative quantifier elimination for ordered abelian groups. That approach can be repeated verbatim in the analytic settings.

Lemma 3.1. Let $\Gamma$ be an ordered abelian group and $P$ be a definable subset of $\Gamma^n$. Suppose that $(\infty, \ldots, \infty)$ is an accumulation point of $P$, i.e. for any $\delta \in \Gamma$ the set

$$\{x \in P : x_1 > \delta, \ldots, x_n > \delta\} \neq \emptyset$$
is non-empty. Then there is an affine semi-line
\[ L = \{ (r_1 t + \gamma_1, \ldots, r_n t + \gamma_n) : t \in \Gamma, \ t \geq 0 \} \]
with \( r_1, \ldots, r_n \in \mathbb{N} \), passing through a point \( \gamma = (\gamma_1, \ldots, \gamma_n) \in P \) and such that \((\infty, \ldots, \infty)\) is an accumulation point of the intersection \( P \cap L \) too.

In a similar manner, one can obtain the following

**Lemma 3.2.** Let \( P \) be a definable subset of \( \Gamma^n \) and
\[ \pi : \Gamma^n \to \Gamma, \ (x_1, \ldots, x_n) \mapsto x_1 \]
be the projection onto the first factor. Suppose that \( \infty \) is an accumulation point of \( \pi(P) \). Then there is an affine semi-line
\[ L = \{ (r_1 t + \gamma_1, \ldots, r_n t + \gamma_n) : t \in \Gamma, \ t \geq 0 \} \]
with \( r_1, \ldots, r_n \in \mathbb{N} \), \( r_1 > 0 \), passing through a point \( \gamma = (\gamma_1, \ldots, \gamma_n) \in P \) and such that \( \infty \) is an accumulation point of \( \pi(P \cap L) \) too.

In this paper, however, a suitable analytic version of the theorem on existence of the limit and application of b-minimal cell decompositions require some new ideas and work. Actually, we first prove a certain version of the former (Proposition 3.3), making use of the theorem on term structure ([6, Theorem 6.3.8]), recalled below. Another analytic version (Proposition 4.1) will be established in Section 4 by means of the closedness theorem. In further reasonings, we shall often make use of Lemmas 3.1 and 3.2.

Denote by \( \mathcal{L}^* \) the analytic language \( \mathcal{L} \) augmented by all Henselian functions
\[ h_m : K^{m+1} \times RV(K) \to K, \ m \in \mathbb{N}, \]
which are defined by means of a version of Hensel’s lemma (cf. [6], Section 6.1).

**Theorem 3.3.** Let \( K \) be a Henselian field with analytic \( \mathcal{A} \)-structure. Let \( f : X \to K \), \( X \subset K^n \), be an \( \mathcal{L}(B) \)-definable function for some set of parameters \( B \). Then there exist an \( \mathcal{L}(B) \)-definable function \( g : X \to S \) with \( S \) auxiliary and an \( \mathcal{L}^*(B) \)-term \( t \) such that
\[ f(x) = t(x, g(x)) \quad \text{for all} \ x \in X. \]

We turn to the following analytic version of the theorem on existence of the limit, which also may be regarded as a version of Puiseux’s theorem.
Proposition 3.4. Let \( f : E \rightarrow K \) be an \( \mathcal{L} \)-definable function on a subset \( E \) of \( K \) and suppose \( 0 \) is an accumulation point of \( E \). Then there is an \( \mathcal{L} \)-definable subsets \( F \subset E \) with accumulation point \( 0 \) and a point \( w \in \mathbb{P}^1(K) \) such that
\[
\lim_{{x \to 0}} f|F(x) = w.
\]
Moreover, we can require that
\[
\{ (x, f(x)) : x \in F \} \subset \{ (x^r, \phi(x)) : x \in G \},
\]
where \( r \) is a positive integer and \( \phi \) is a definable function, a composite of some functions induced by series from \( \mathcal{A} \) and of some algebraic power series (coming from Henselian functions \( h_m \)). Then, in particular, the definable set
\[
\{ (v(x), v(f(x))) : x \in (F \setminus \{0\}) \subset \Gamma \times (\Gamma \cup \{\infty\})
\]
is contained in an affine line with rational slope
\[
l = \frac{p}{q} \cdot k + \beta,
\]
with \( p, q \in \mathbb{Z} \), \( q > 0 \), \( \beta \in \Gamma \), or in \( \Gamma \times \{\infty\} \).

Proof. In view of Remark 2.1, we may assume that \( K \) has separated analytic \( \mathcal{A}(K) \)-structure. We apply Theorem 3.3 and proceed with induction with respect to the complexity of the term \( t \). Since an angular component map exists, the sorts \( \tilde{K} \) and \( \Gamma \) are orthogonal in \( \text{RV}(K) \simeq \tilde{K} \times \Gamma_K \).

Therefore, after shrinking \( F \), we can assume that \( \text{mc}(F) = \{1\} \) and the function \( g \) goes into \( \{\xi\} \times \Gamma^s \) with a \( \xi \in \tilde{K}^s \), and next that \( \xi = (1, \ldots, 1) \); similar reductions were considered in our papers [11, 12]. For simplicity, we look at \( g \) as a function into \( \Gamma^s \). We shall briefly explain the most difficult case where
\[
t(x, g(x)) = h_m(a_0(x), \ldots, a_m(x), g_0(x)),
\]
assuming that the theorem holds for the terms \( a_0, \ldots, a_m \); here \( g_0 \) is one of the components of \( g \). By Lemma 3.2, we can assume that
\[
pv(x) + qg_0(x) + v(a) = 0
\]
for some \( p, q \in \mathbb{Z} \), \( a \in K \setminus \{0\} \). By the induction hypothesis, we get
\[
\{ (x, a_i(x)) : x \in F \} \subset \{ (x^r, a_i(x)) : x \in G \}, \quad i = 0, 1, \ldots, m.
\]
Put
\[
P(x, T) := \sum_{i=0}^m a_i(x)T^i.
\]
By the very definition of $h_m$ and since we are interested in the vicinity of zero, we may assume that there is $i_0 = 0, \ldots, m$ such that
\[
\forall \, x \in F \ \exists \, u \in K \quad v(u) = g_0(x), \quad \overline{ac} \, u = 1,
\]
(3.2) 
\[
v(a_{i_0}(x)u^{i_0}) = \min \{v(a_i(x)u^i), \ i = 1, \ldots, m\},
\]
\[
v(P(x, u)) > v(a_{i_0}(x)u^{i_0}), \quad v \left( \frac{\partial P}{\partial T}(x, u) \right) = v(a_{i_0}(x)u^{i_0}).
\]
Then $h_m(a_0(x), \ldots, a_m(x), g_0(x))$ is a unique $b(x) \in K$ such that
\[
P(x, b(x)) = 0, \quad v(b(x)) = g_0(x), \quad \overline{ac} \, b(x) = 1.
\]
By [12, Remarks 7.2, 7.3], the set $F$ contains the set of points of the form $c^r t^{Nqr}$ for some $c \in K$ with $\overline{ac} \, c = 1$, a positive integer $N$ and all $t \in K^\circ$ with $\overline{ac} \, t = 1$. Hence and by equation (3.1), we get
\[
g_0(c^r t^{Nqr}) = g_0(c^r) - v(t^{Npr}).
\]
Take $d \in K$ such that $g_0(c^r) = v(d)$ and $\overline{ac} \, d = 1$. Then
\[
g_0(c^r t^{Nqr}) = v(dt^{Npr}).
\]
Thus the homothetic change of variable
\[
Z = T/dt^{Npr} = t^{Npr}T/d
\]
transforms the polynomial
\[
P(c^r t^{Nqr}, T) = \sum_{i=0}^{m} \alpha_i(c^r t^{Nq})T^i
\]
into a polynomial $Q(t, Z)$ to which Hensel’s lemma applies (cf. [13, Lemma 3.5]):
\[
(3.3) \quad P(c^r t^{Nqr}, T) = P(c^r t^{Nqr}, dt^{-Npr}Z) = \\
\alpha_{i_0}(c^r t^{Nq}) \cdot (dt^{-Npr})^{i_0} \cdot Q(t, Z).
\]
Indeed, the formulas (3.2) imply that the coefficients of the polynomial $Q$ are power series (of order $\geq 0$) in the variable $t$, and that
\[
v(Q(0, 1)) > 0 \quad \text{and} \quad v \left( \frac{\partial Q}{\partial Z}(0, 1) \right) = 0.
\]
Therefore the conclusion of the theorem follows.
We still need the concept of fiber shrinking introduced in our paper [1]. Let $A$ be an $Ł$-definable subset of $K^n$ with accumulation point

$$a = (a_1, \ldots, a_n) \in K^n$$

and $E$ an $Ł$-definable subset of $K$ with accumulation point $a_1$. We call an $Ł$-definable family of sets

$$Φ = \bigcup_{t \in E} \{t\} \times Φ_t ⊂ A$$

an $Ł$-definable $x_1$-fiber shrinking for the set $A$ at $a$ if

$$\lim_{t \to a_1} Φ_t = (a_2, \ldots, a_n),$$

i.e. for any neighbourhood $U$ of $(a_2, \ldots, a_n) \in K^{n-1}$, there is a neighbourhood $V$ of $a_1 \in K$ such that $\emptyset \neq Φ_t \subset U$ for every $t \in V \cap E$, $t \neq a_1$. When $n = 1$, $A$ is itself a fiber shrinking for the subset $A$ of $K$ at an accumulation point $a \in K$.

Proposition 3.5. (Fiber shrinking) Every $Ł$-definable subset $A$ of $K^n$ with accumulation point $a \in K^n$ has, after a permutation of the coordinates, an $Ł$-definable $x_1$-fiber shrinking at $a$.

By means of elimination of valued field quantifiers (Theorem 2.3), this proposition reduces easily to Lemma 3.1 (similarly as it was in the algebraic case treated in [12]). Now we can readily proceed with the proof of the closedness theorem (Theorem 1.1). We must show that if $B$ is an $Ł$-definable subset of $D × (K^\circ)^n$ and a point $a$ lies in the closure of $A := π(B)$, then there is a point $b$ in the closure of $B$ such that $π(b) = a$. As before (cf. [14, Section 8]), the theorem reduces easily to the case $m = 1$ and next, by means of fiber shrinking (Proposition 3.5), to the case $n = 1$. We may obviously assume that $a = 0 \notin A$.

By b-minimal cell decomposition, we can assume that the set $B$ is a relative cell with center over $A$. It means that has a presentation of the form

$$Λ : B \ni (x, y) \to (x, λ(x, y)) \in A × RV(K)^s,$$

where $λ : B \to RV(K)^s$ is an $Ł$-definable function, such that for each $(x, ξ) \in Λ(B)$ the pre-image $λ_x^{-1}(ξ) ⊂ K$ is either a point or an open ball; here $λ_x(y) := λ(x, y)$. In the latter case, there is a center, i.e. an $Ł$-definable map $ζ : Λ(B) \to K$, and a (unique) map $ρ : Λ(B) \to RV(K) \setminus \{0\}$ such that

$$λ_x^{-1}(ξ) = \{y \in K : rv(y − ζ(x, ξ)) = ρ(x, ξ)\}.$$

Again, since the sorts \( \tilde{K} \) and \( \Gamma \) are orthogonal in \( RV(K) \simeq \tilde{K} \times \Gamma_K \), we can assume, after shrinking the sets \( A \) and \( B \), that

\[
\lambda(B) \subset \{(1, \ldots, 1)\} \times \Gamma^s \subset \tilde{K}^s \times \Gamma^s_K;
\]

let \( \tilde{\lambda}(x, y) \) be the projection of \( \lambda(x, y) \) onto \( \Gamma^s \). By Lemma 3.2, we can assume once again, after shrinking the sets \( A \) and \( B \), that the set

\[
\{(v(x), v(y), \tilde{\lambda}(x, y)) : (x, y) \in B\} \subset \Gamma^{s+2}
\]

is contained in an affine semi-line with integer coefficients. Hence \( \lambda(x, y) = \phi(v(x)) \) is a function of one variable \( x \). We have two cases.

Case I. \( \lambda^{-1}_x(\xi) \subset K^o \) is a point. Since each \( \lambda_x \) is a constant function, \( B \) is the graph of an \( L \)-definable function. The conclusion of the theorem follows thus from Proposition 3.4.

Case II. \( \lambda^{-1}_x(\xi) \subset K^o \) is a ball. Again, application of Lemma 3.2 makes it possible, after shrinking the sets \( A \) and \( B \), to arrange the center

\[
\zeta : \Lambda(B) \ni (x, k) \to \zeta(x, v(x)) = \zeta(x) \in K
\]

and the function \( \rho(x, k) = \rho(v(x)) \) as functions of one variable \( x \). Likewise as it was above, we can assume that the set

\[
P := \{(v(x), \rho(v(x))) : x \in A\} \subset \Gamma^2
\]

is contained in an affine line \( pv(x) + q\rho(v(x)) + v(c) = 0 \) with integer coefficients \( p, q \neq 0 \); furthermore, that \( P \) contains the set

\[
Q := \{(v(ct^qN), \rho(v(ct^qN))) : t \in K^o\}
\]

for a positive integer \( N \). Then we easily get

\[
\rho(v(ct^qN)) = \rho(c) - pNv(t) = v(ct^{-p}N).
\]

Hence the set \( B \) contains the graph

\[
\{(ct^qN, \zeta(ct^qN) + ct^{-p}N) : t \in K^o\}.
\]

As before, the conclusion of the theorem follows thus from Proposition 3.4, and the proof is complete.

4. Applications

The framework of b-minimal structures provides cell decomposition and a good concept of dimension (cf. [6]), which in particular satisfies the axioms from the paper [7]. For separated analytic structures, the zero-dimensional sets are precisely the finite sets, and also valid is the following dimension inequality, which is of great geometric significance:

\[
\dim \partial E < \dim E;
\]
here $E$ is any $\mathcal{L}$-definable subset of $K^n$ and $\partial E := \overline{E} \setminus E$ denotes the frontier of $A$.

We first apply the closedness theorem to obtain a version of the theorem on existence of the limit.

**Proposition 4.1.** Let $f : E \to \mathbb{P}^1(K)$ be an $\mathcal{L}$-definable function on a subset $E$ of $K$, and suppose that $0$ is an accumulation point of $E$. Then there is a finite partition of $E$ into $\mathcal{L}$-definable sets $E_1, \ldots, E_r$ and points $w_1, \ldots, w_r \in \mathbb{P}^1(K)$ such that

$$\lim_{x \to 0} f|_{E_j}(x) = w_j \quad \text{for } j = 1, \ldots, r.$$ 

**Proof.** We may of course assume that $0 \not\in E$. Put

$$F := \text{graph}(f) = \{(x, f(x)) : x \in E\} \subset K \times \mathbb{P}^1(K);$$

obviously, $F$ is of dimension 1. It follows from the closedness theorem that the frontier $\partial F \subset K \times \mathbb{P}^1(K)$ is non-empty, and thus of dimension zero by inequality \[4.1\]. Say

$$\partial F \cap \{(0) \times \mathbb{P}^1(K)\} = \{(0, w_1), \ldots, (0, w_r)\}$$

for some $w_1, \ldots, w_r \in \mathbb{P}^1(K)$. Take pairwise disjoint neighborhoods $U_i$ of the points $w_i, i = 1, \ldots, r$, and set

$$F_0 := F \cap \left( E \times \left( \mathbb{P}^1(K) \setminus \bigcup_{i} E_i \right) \right).$$

Let

$$\pi : K \times \mathbb{P}^1(K) \to K$$

be the canonical projection. Then

$$E_0 := \pi(F_0) = f^{-1}\left( \mathbb{P}^1(K) \setminus \bigcup_{i} E_i \right).$$

Clearly, the closure $\overline{F}_0$ of $F_0$ in $K \times \mathbb{P}^1(K)$ and $\{0\} \times \mathbb{P}^1(K)$ are disjoint. Hence and by the closedness theorem, $0 \not\in \overline{E}_0$, the closure of $E_0$ in $K$. The set $E_0$ is thus irrelevant with respect to the limes at $0 \in K$. Therefore it remains to show that

$$\lim_{x \to 0} f|_{E_j}(x) = w_j \quad \text{for } j = 1, \ldots, r.$$ 

Otherwise there is a neighborhood $V_i \subset U_i$ such that $0$ would be an accumulation point of the set

$$f^{-1}(U_i \setminus V_i) = \pi(F \cap (E \times (U_i \setminus V_i))).$$
Again, it follows from the closedness theorem that \( \{0\} \times \mathbb{P}^1(K) \) and the closure of \( F \cap (E \times (U_i \setminus V_i)) \) in \( K \times \mathbb{P}^1(K) \) would not be disjoint. This contradiction finishes the proof. \( \square \)

**Remark 4.2.** Let us mention that Proposition \[ \text{4.1} \] can be strengthened as stated below (cf. the algebraic versions \[ \text{11, Proposition 5.2} \] and \[ \text{12, Theorem 5.1} \]):

Moreover, perhaps after refining the finite partition of \( E \), there is a neighbourhood \( U \) of \( 0 \) such that each definable set

\[ \{(v(x), v(f(x))): x \in (E_j \cap U) \setminus \{0\}\} \subset \Gamma \times (\Gamma \cup \{\infty\}), \quad j = 1, \ldots, r, \]

is contained in an affine line with rational slope

\[ l = \frac{p_j}{q} \cdot k + \beta_j, \quad j = 1, \ldots, r, \]

with \( p_j, q \in \mathbb{Z}, \quad q > 0, \quad \beta_j \in \Gamma, \) or in \( \Gamma \times \{\infty\} \).

Now we turn to a second application, namely the following theorem on piecewise continuity.

**Theorem 4.3.** Let \( A \subset K^n \) and \( f: A \to \mathbb{P}^1(K) \) be an \( \mathcal{L} \)-definable function. Then \( f \) is piecewise continuous, i.e. there is a finite partition of \( A \) into \( \mathcal{L} \)-definable locally closed subsets \( A_1, \ldots, A_s \) of \( K^n \) such that the restriction of \( f \) to each \( A_i \) is continuous.

**Proof.** Consider an \( \mathcal{L} \)-definable function \( f: A \to \mathbb{P}^1(K) \) and its graph

\[ E := \{(x, f(x)): x \in A\} \subset K^n \times \mathbb{P}^1(K). \]

We shall proceed with induction with respect to the dimension

\[ d = \dim A = \dim E \]

of the source and graph of \( f \).

Observe first that every \( \mathcal{L} \)-definable subset \( E \) of \( K^n \) is a finite disjoint union of locally closed \( \mathcal{L} \)-definable subsets of \( K^n \). This can be easily proven by induction on the dimension of \( E \) by means of inequality \[ \text{1.1}. \]

Therefore we can assume that the graph \( E \) is a locally closed subset of \( K^n \times \mathbb{P}^1(K) \) of dimension \( d \) and that the conclusion of the theorem holds for functions with source and graph of dimension \( < d \).

Let \( F \) be the closure of \( E \) in \( K^n \times \mathbb{P}^1(K) \) and \( \partial E := F \setminus E \) be the frontier of \( E \). Since \( E \) is locally closed, the frontier \( \partial E \) is a closed subset of \( K^n \times \mathbb{P}^1(K) \) as well. Let

\[ \pi: K^n \times \mathbb{P}^1(K) \longrightarrow K^n \]
be the canonical projection. Then, by virtue of the closedness theorem, the images $\pi(F)$ and $\pi(\partial E)$ are closed subsets of $K^n$. Further,

$$\dim F = \dim \pi(F) = d$$

and

$$\dim \pi(\partial E) \leq \dim \partial E < d;$$

the last inequality holds by inequality \[4.1\]. Putting

$$B := \pi(F) \setminus \pi(\partial E) \subset \pi(E) = A,$$

we thus get

$$\dim B = d \quad \text{and} \quad \dim (A \setminus B) < d.$$

Clearly, the set

$$E_0 := E \cap (B \times P^1(K)) = F \cap (B \times P^1(K))$$

is a closed subset of $B \times P^1(K)$ and is the graph of the restriction

$$f_0 : B \longrightarrow P^1(K)$$

of $f$ to $B$. Again, it follows immediately from the closedness theorem that the restriction

$$\pi_0 : E_0 \longrightarrow B$$

of the projection $\pi$ to $E_0$ is a definably closed map. Therefore $f_0$ is a continuous function. But, by the induction hypothesis, the restriction of $f$ to $A \setminus B$ satisfies the conclusion of the theorem, whence so does the function $f$. This completes the proof.

We immediately obtain

**Corollary 4.4.** The conclusion of the above theorem holds for any $L$-definable function $f : A \to K$.

Let us conclude with the following comment. We are currently preparing subsequent articles, which will provide several applications of the closedness theorem, possibly over non-algebraically closed ground fields, including i.a.l. the analytic, non-Archipedean versions of the Lojasiewicz inequalities and of curve selection. The algebraic versions of these results were established in our papers [11, 12].
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