Frictional dynamics of viscoelastic solids driven on a rough surface

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We study the effect of viscoelastic dynamics on the frictional properties of a (mean field) spring-block system pulled on a rough surface by an external drive. When the drive moves at constant velocity $V$, two dynamical regimes are observed: at fast driving, above a critical threshold $V_c$, the system slides at the drive velocity and displays a friction force with velocity weakening. Below $V_c$ the steady sliding becomes unstable and a stick-slip regime sets in. In the slide-hold-slide driving protocol, a peak of the friction force appears after the hold time and its amplitude increases with the hold duration. These observations are consistent with the frictional force encoded phenomenologically in the rate-and-state equations. Our model gives a microscopic basis for such macroscopic description.

I. INTRODUCTION

Friction is behind many phenomena of our everyday life experience such as the adhesion of car tires on the road [1, 2], the sound emitted by a violin [3, 4], or the wear of human articular joints [5]. Often, in these situations, the two bodies in contact may display a discontinuous dynamics, the so-called stick-slip dynamics [6, 7], in which periods of rapid movement (slip) are followed by periods of relative rest (stick). At the macroscopic scale the stick-slip motion is justified by the empirical difference observed between the static and dynamic friction coefficient, but a general explanation of this phenomenon from first principles is not yet available. One possibility to induce a stick-slip behavior is the introduction of viscoelastic effects in the dynamics. In fact, we have recently shown that viscoelastic solids sliding on a rough substrate can display a stick-slip instability [8].

A scheme of the model studied in [8] is reproduced for convenience in Fig. 1. It consists of an ensemble of interacting blocks driven at velocity $V$ on a rough substrate. The macroscopic friction arises from the real area of solid-substrate contact, which consists in the junctions between asperities. When the block is stuck on the rough substrate, the elastic energy of the spring $k_0$ slowly accumulates over time, and is released when junctions break, letting the block move. The rupture of a single junction can trigger further rupture, with a characteristic time $\tau_0$. In the quasi-static limit ($V \to 0^+$), when the driving time scale $\tau_D$ is very slow compared to $\tau_0$, the chain of events triggered by a single rupture can be very large and is called an avalanche.

In the case of purely elastic interactions between the blocks, the macroscopic friction force, namely the average elongation of the spring $k_0$, is the control parameter of the depinning transition, a second order transition between a pinned phase and a phase where the system slides at the driving velocity $V$ [9–13]. Visco-elastic interactions [14] change the nature of the moving phase, inducing hysteretic behavior, as it was shown by Marchetti et al. [15, 16] in the context of the plastic depinning transition of a vortex lattice.

In the quasi-static driving limit ($\tau_0 \ll \tau_D$), the pres-
ence of visco-elasticity (with a characteristic time $\tau_u$) has been shown to produce a rich phenomenology of the avalanche dynamics, with main shocks, aftershocks and spatio-temporal correlations similar to that observed in seismology [8, 17–21]. When the visco-elastic relaxation is very fast compared to the driving ($\tau_u \ll \tau_D$), these system-size avalanches can be understood as a stick-slip dynamics of the mean field model [8].

In the present paper we study the case where the driving rate $\tau_D$ is of the same order of magnitude as the viscoelastic time scale $\tau_u$, but is still very slow compared to the duration of each single avalanche: $\tau_0 \ll \tau_u \simeq \tau_D$. We show that in this limit, the system displays the basic features of dry friction. In particular when a uniform driving is applied, we observe a transition between a stick-slip regime (slow driving, $\tau_u \ll \tau_D$) and a steady-sliding regime (fast driving, $\tau_u \gg \tau_D$). When the driving velocity is not constant, as in the case of the slide-hold-slide protocol, our results display qualitatively the kind of behavior observed in the experiments [22], most remarkably, the existence of a peak in the friction force after the hold period, which becomes larger as the hold time increases. It must be noted that this kind of experiments on friction dynamics are usually modelled using the so-called rate-and-state equations [23, 24], which incorporate the fact that the friction force does not simply depend on the relative velocity between the two nominal surfaces but also depends on the history of the contact, via a macroscopic “state” variable $\theta$. In our model this dependence is encoded in the state of the microscopic visco-elastic elements. In this way our model gives a microscopical basis for the rate-and-state description of friction.

The paper is organized as follows. In section II we briefly present the model. In section III we study the case of uniform driving, and complete this study with the slide-hold-slide protocol in section IV. We then interpret our results in terms of rate-and-state formalism in section V and conclude in section VI.

II. MODEL

The equation of motion for the blocks of Fig. 1 is:

$$\eta_0 \partial_t h_i = k_0 (w - h_i) + k_1 \nabla^2 h_i + f^{\text{dis}}(h_i)$$

$$+ k_2 \nabla^2 u_i - k_2 u_i,$$

(1)

$$\eta_u \partial_t u_i = k_2 \nabla^2 h_i - k_2 u_i,$$

(2)

where $w = Vt$ in the uniform driving protocol and the variable $u_i$ represents the state of the dashpots neighbouring the site $i$. The random force $f^{\text{dis}}(h_i)$ mimics the contacts between the asperities of the block and those of the surface, and acts as a microscopic static friction force. When a contact is broken, two things happen: (i) the block moves and redistributes stress to neighbouring blocks, (ii) the asperities involved in the junction are renewed, as sketched in Fig. 1c. This renewal process makes the asperity landscapes different for each block. For simplicity, we consider that the random force $f^{\text{dis}}(h_i)$ acting on block $i$ is completely independent from that acting on other blocks.

Eqs. (1) and (2) were introduced in [8] and studied in $d = 2$ and in mean field for $V = 0^+$. Here we extend the mean field approximation (which corresponds to replace the laplacian $\nabla^2 h_i$ with $\overline{h} - h_i$, where $\overline{h}$ is the average location of the blocks, see Fig. 2(a)) to the finite velocity case. We compute the macroscopic friction force

$$\sigma = k_0 (w - \overline{h})$$

(3)

accumulated in the system for different driving protocols and adopt the so-called “narrow wells” approximation [10]. In this scheme, the disorder $f^{\text{dis}}(h_i)$ is modelled as a collection of narrow pinning wells representing impurities, with spacings $z$ distributed as $g(z)$ and with average $\overline{z} = \int_0^\infty z g(z) dz$. As the wells are very narrow, the disorder force $f^{\text{th}}(h_i)$ that derives from this potential is 0 everywhere except for countably many points. For simplicity we consider narrow wells of constant depth (and shape), namely, the random thresholds are constant, $f^{\text{th}} = \text{const.} = 1$.

III. UNIFORM DRIVING PROTOCOL

At long times, under a driving performed at constant velocity $V$, the solution of Eqs. (1) and (2) becomes independent of the initial condition and, in mean field, two phases are observed: for small $V$ we find a limit cycle solution that corresponds to a stick-slip phase, whereas for large $V$ a stationary solution exists. The stationary value of the friction force, $\sigma(V)$, can be computed analytically for $g(z) = \delta(z - \overline{z})$ using the crucial remark that if a steady sliding regime exists then in this regime $\overline{h}$ moves with velocity $V$. In particular, $\overline{h} = Vt - \sigma(V)/k_0$. In the stationary regime the $k_2$-plus-dashpot branch shown
Velocity weakening is a necessary link between the static ($V = 0$) and kinetic ($V > 0$) friction coefficients, and as such is known to be crucial in the triggering of instabilities in sliding systems, leading to stick-slip motion [6] and to the existence of earthquakes in sliding tectonic faults [27]. For a review on nanoscale models of friction and experimental results on nano-tribology one should consult [28] or the letter [29], which contains accessible references to the literature. In our model, velocity weakening is a direct consequence of the viscoelastic relaxation. In fact, the model without viscoelasticity lacks any velocity dependence of the friction force, as in that case there is no internal time scale to compete with the driving time scale.

We now turn on the numerical study of the mean field version of Eqs. (1) and (2) using an uncorrelated distribution of pinning wells, with mean $\bar{z}$, namely $g(z) = \bar{z}^{-1}e^{-z/\bar{z}}$. In practice we implement the Fokker-Planck method used in [8], adapted to the case of finite driving velocity. All technical details are left to Appendix A.

In Fig. 3, the evolution of the stress over time is compared for three values of the velocity: at slow driving the stress $\sigma(t)$ oscillates periodically with an amplitude denoted $\Delta \sigma(V)$, while at fast driving a stationary value is reached (i.e. $\Delta \sigma(V) = 0$). Note that the amplitude $\Delta \sigma(V)$ of the oscillations is also the width of the stress drops or “gaps” that occur during the system-size events.

This behavior points to a bifurcation of the dynamics as velocity is reduced. To study this effect, we report in Fig. 4 the maximum and minimum of the friction force
over time, as a function of the driving velocity $V$, for various values of $k_0$. We observe that the stress gap vanishes smoothly at the transition point $V = V^c$, pointing to a “second order” dynamical phase transition in the order parameter $\Delta \sigma$. In Fig. 5 we show the full phase diagram of the system in the $k_0-V$ plane. There, for various sets of $(k_1, k_2)$ we observe a divergence of $V^c$ as $1/k_0$. This $1/k_0$ dependence at low $k_0$ comes directly from the uniform increase of the pulling force as $k_0 V t$ and implies that for any non zero values of $k_1$, $k_2$ and $V$, there will always be a value of $k_0$ below which some stick-slip dynamics occurs, even at very large velocities.

In the stationary regime, our simulations confirm, overall, the friction behavior of Eq. (4). In particular, in Fig. 6 we observe a clear velocity weakening with a characteristic $1/V$ decay towards $\sigma(V \rightarrow \infty) = 1 - (k_0 + k_1 + k_2) V / 2$.

IV. SLIDE-HOLD-SLIDE PROTOCOL

Slide-hold-slide experiments are an important tool in the investigation of the tribology of solids. When the sliding of a solid is interrupted for some time $\Delta t$, the contacts at the surface of the solid can strengthen over time, so that when sliding is resumed, a peak in the friction force has to be overcome before one recovers the stationary friction force. The amplitude of the friction peak increases with the hold time $\Delta t$ since the relaxation is more effective when it has more time to act.

To simulate this process, we set the driving velocity $V$ to a value at which a steady sliding is observed. Once we reached a stationary dynamics, the driving is stopped for some time interval $\Delta t$ and then resumed. The evolution of the friction force during this protocol in our model is shown in Fig. 7 (left). The height of the friction peak as a function of the hold time $\Delta t$ is shown in Fig. 7 (right).

We observe some similarities and some differences when comparing our results with experimental observations. In experiments, during the hold time the stress shows a relaxation towards some lower value, something which is not observed in our model. The reason is that stress relaxation can occur only if some secondary avalanches (aftershocks) are triggered by the viscoelastic relaxation. When the pinning thresholds $f^{th}$ have a single value, then aftershocks are excluded and the decrease cannot happen. However in models with a wide distribution of pinning thresholds this relaxation has been observed [17] and is compatible with experiments.

Qualitatively, the friction peak is similar in our simulations and in experiments: in both cases its height increases as a function of the hold time, but in experiments the increase is usually reported to be logarithmic, while here we observe an exponential saturation at large $\Delta t$. The reason for this difference is that our model contains only a single relaxation time constant, which naturally defines a typical time scale, above which the system is fully relaxed and thus no longer evolves.
V. ANALYSIS IN TERMS OF RATE-AND-STATE EQUATIONS

It has been long realized that the friction force cannot be described by a single valued function of the instantaneous velocity. The history of the contact plays an important role in determining the actual friction force. Our model is an example of such a case, since the value of the friction force depends on the state of the viscoelastic elements, which in turn depend on the history of the system. We thus briefly recall the standard rate and state formalism, and then show how it provides an appropriate framework to understand our results.

A. The RS Formalism

Phenomenologically, the behavior of frictional contacts has been successfully described through the formalism named rate-and-state (RS) friction, originated in the works of Dieterich and Ruina [23, 24]. Instead of assuming that there exists only a kinetic and a static friction coefficient, the RS formalism assumes that the friction coefficient continuously depends on the relative velocity v between the two surfaces (the rate variable) and on a state variable usually called θ. We recall that by definition, the friction coefficient μ acts as a threshold for the friction force σ actually arising from the contacts: we always have σ ≤ μF_N, where F_N is the force normally applied on the solid. The usual RS form for μ(v, θ) is:

\[ μ(v, \theta) = μ_0 + a \log(v) + b \log(\theta/D_c) \]  

where a, b are positive constants. Note that we use a small v to indicate the instantaneous velocity, which may differ from the driving velocity V. The a term describes the so-called “direct effect”, the increase of friction with increase of the relative velocity commonly observed in many materials. The state variable θ is supposed to follow a second equation, usually written:

\[ \dot{\theta} = 1 - \frac{\theta v}{D_c} \]  

where D_c is the “critical slip distance”, i.e. the amount of slip (of the center of mass of the sliding block) necessary to break a newly formed junction.

Under steady sliding, we have \( \dot{\theta} = 0 \), \( \theta = D_c/v \), and the RS equations (7), (8) simplify into:

\[ \mu(v, \theta) = \mu_0 + (a - b) \log(v). \]  

If b > a this equation describes the phenomenon of velocity weakening, namely a reduction of the friction coefficient when velocity increases. Velocity weakening is a crucial ingredient involved in the description of seismic phenomena[27]. In the case in which the contact is at rest \( (v = 0) \), we get \( \dot{\theta} = 1 \), i.e., \( \theta(t) = \theta_0 + t \), and according to (7) we obtain an increase of the static friction coefficient with the time of contact. RS equations have been used to describe the behaviors of frictional systems under a variety of non-steady sliding conditions, providing an excellent phenomenological description for numerous systems.
B. Interpretation of our model in terms of RS Equations

Under uniform driving, our model displays two features that are common to many frictional systems: a velocity weakening friction law, and a bifurcation to a stick-slip regime for low driving velocities. We have described in Section III how the velocity weakening effect appears in the model. Now we will see that assuming the existence of this effect, the bifurcation to a stick-slip regime at low velocities is captured in all its detail by the use of the RS formalism.

The usually assumed form Eq. (9) of the friction coefficient is not appropriate to accurately describe the results observed in our model in the steady sliding regime. On the one hand, our model does not have any ingredient that could produce a direct effect in the RS equation (7), suggesting that we should use $a = 0$. On the other hand, the logarithmic $b$ term is not appropriate in our case, mainly because we have considered only a relaxation mechanism that is described by a single time constant, the one associated to the dashpot and $k_2$ springs. Instead, from our simulations of Sec. III (as for instance those in the right part of Fig. 4) we can extract a simple expression for the friction coefficient in the steady sliding regime:

$$
\mu^{\text{steady}} \approx f^{th} - \frac{\bar{z}}{2}(k_0 + k_1 + k_2) + \frac{0.007}{v + 0.078} \tag{10}
$$

where the numerical values correspond to $k_1 = 0.1$, $k_2 = 0.9$ and $\bar{z} = 0.1$.

We now assume that under any non-steady sliding situation, the value of the friction coefficient can still be written in terms of the function found in the steady state. Using the fact that in the steady state $v = D_c/\theta$, we write:

$$
\mu(\theta) = f^{th} - \frac{\bar{z}}{2}(k_0 + k_1 + k_2) + \frac{0.007}{D_c/\theta + 0.078}, \tag{11}
$$

The time evolution of $\theta$ will be assumed to be described by the standard RS evolution, Eq. (8). The sliding contact is coupled to a driving spring pulled at constant velocity $V$, so that the pulling force reads:

$$
\sigma = (Vt - x)k_0 \tag{12}
$$

where $x$ is the spatial average coordinate of the contact, i.e. $\dot{x} = v$. This force $\sigma$ is also the actual instantaneous friction force arising from the contacts. Our aim is now to determine the temporal evolution of $x$ and $\sigma$ and reproduce the bifurcation behavior that we obtained in Section III.

If steady sliding is assumed, the force $\mu F_N$ must balance the pulling force, and we get $\theta = D_c/V$, $v = V$. However, steady sliding may be unstable due to the following process: if at some moment $x$ is stopped, the pulling force starts to increase according to $\sigma = \sigma^0 + Vtk_0$. On the other hand, the friction force starts to increase due to the increasing of $\theta$, according to $\mu = \mu(\theta_0 + t)$. If the initial increase of $\mu F_N$ is more rapid than the increase of $\sigma$, the contact will stay stuck as long as $\mu F_N > \sigma$, until eventually the pulling force becomes larger than the pinning force, and a rapid slip moves the value of $x$ to a new position. If $\mu F_N < \sigma$, slip occurs and is essentially instantaneous since we have not added inertia to the contact. During the slip stage, the value of $\theta$ is reduced following Eq. (8), that in the rapid slip limit can be written as $d\theta = -\theta(Dx/D_c)$ and analytically integrated. Slip finishes when $\sigma$ drops below $\mu F_N$ again, and a new stick slip cycle begins. We have integrated Eqs. (11), (8), (12) and in fact obtained steady sliding for large $V$, and a stick slip behavior at small $V$, in which the value of $\sigma$ oscillates between two values $\sigma_{\text{max}}$ and $\sigma_{\text{min}}$. In Fig. 8 we plot the values of $\sigma_{\text{max}}$ and $\sigma_{\text{min}}$ (or the single value $\sigma$ in case of steady sliding) as a function of $V$ for the same set of $k_0$ values used in Fig. 4. In addition to the coincidence of the two figures in the steady sliding regime (which is enforced by hand through the choice of the $\mu^{\text{steady}}$ function in Eq. (10)), we see that the RS formalism gives a very good coincidence in the low velocity, stick-slip regime.

A careful analysis of our RS equations near the bifurcation point shows that the two values $\sigma_{\text{max}}$ and $\sigma_{\text{min}}$ depart symmetrically from the branch of steady sliding, and are such that $\sigma_{\text{max}} - \sigma_{\text{min}} \sim (V^\epsilon - V)^{1/2}$, which corresponds to a Hopf bifurcation. Note however that the validity range of these scaling becomes progressively smaller as $k_0$ is reduced, and in the $k_0 \to 0$ limit we get the scaling $\sigma_{\text{min}} \sim \text{const}$, $\sigma_{\text{max}} - \sigma_{\text{min}} \sim (V^\epsilon - V)$. 

![FIG. 8: Friction force as a function of velocity for the same set of values as in Fig. 4, using the RS formalism. The form of the steady sliding part of the curves is introduced by hand. The RS formalism yields the form of $\sigma_{\text{max}}$ and $\sigma_{\text{min}}$ for $V < V^\epsilon$.](image)
VI. CONCLUSION

We have presented a detailed analytical and numerical analysis of a viscoelastic model of friction in mean field approximation, in which the driving velocity competes with the time scale of the viscoelastic effects within the system.

Our main findings are the following. At low driving velocities, we obtain a stick-slip dynamics with amplitudes of the stress oscillations that decrease with increasing driving velocity (V) or increasing driving stiffness (k_0). Beyond a certain critical driving velocity (V > V^c), the amplitude of these oscillations becomes zero, i.e. the sliding is smooth. The transition between stick-slip and smooth sliding occurs in a continuous manner and is very well described by a phenomenological rate-and-state analysis. In the smooth sliding regime the friction force reduces as a function of the driving velocity, reproducing the well known velocity-weakening phenomenology. In our model, this effect is originated in the existence of viscoelastic elements that set an additional time scale for the dynamics. Finally, the response of our model to intermittent driving allowed us to reproduce qualitatively an important aspect of the aging of contacts, namely the increase of the static friction with time of contact at rest. Overall, we believe our model reproduces many well-known features of real tribology, and gives a well defined model on which many assumptions and predictions of phenomenological theories (like rate-and-state equations) can be investigated in depth.

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Appendix A: The Numerical Methods

In previous works [8, 30] we showed how to translate the stochastic dynamics into a master equation (or Fokker-Planck equation), but only in the case in which V → 0. Here we generalize the procedure to any velocity V. Our Cython code is fully available [31].

It is useful to describe the dynamics using the local variable, δ_i:

δ_i ≡ 1 - k_0(w - h_i) - (k_1 + k_2)(h_i - h_i) + k_2u_i,  
(A1)

which represents the amount of additional stress that a site can hold before becoming unstable: as long as δ_i > 0, ∀i, the block is stable and its position, h_i, remains fixed when δ_i = 0 the block jumps to the next pinning well at a random distance z: h_i → h_i + z. To define the dynamics in terms of δ variables, we need to split it in a fast part δ^F, and a relaxed one δ^R:

δ^F_i = 1 - k_0(w - h_i) - (k_1 + k_2)(h_i - h_i) 
δ^R_i = k_2u_i,  
(A2)

such that δ_i = δ^F_i + δ^R_i. The blocks dynamics is controlled by three processes:

(i) The avalanches. When a block is unstable (δ_i ≤ 0), it moves to the next pinning wells:  δ^F_i → δ^R_i + z(k_0 + k_1), with z drawn from p(z). Each jump z is followed by a stress redistribution δ^F_j → δ^F_j - zk_1/N. These drops can trigger other instabilities. The characteristic duration of such an avalanche is τ_0 = η_0/ max(k_0, k_1, k_2).

(ii) The driving. External driving increases over time: for instance w = Vt in the uniform driving protocol. In terms of δ, driving means that over a time step dt, δ^F_i → δ^F_i - k_0Vdt. This driving can happen until a new instability is triggered (step (i)). The characteristic time scale of driving is τ_D = V/η.

(iii) The relaxation. In absence of instabilities, the h_i’s are constant, and Eq. 2 reduces to:

δ^R_i(t) = k_2(h_i) + (δ^R_i(t_0) - k_2(h_i)) e^{-k_2t/τ_{u0}}  
(A3)

where t_0 is the time at which the last avalanche occurred. Note that h_i does not evolve during relaxation or driving, so that relaxation can happen until a new instability is triggered (step (i)). The characteristic time scale of relaxation is τ_u = η_u/k_2.

In this paper we study the case τ_0 ≪ τ_u ≈ τ_D and focus on the case of the mean field model where fluctuations vanish and the description of the system via a simple probability distribution becomes exact. Indeed, the sole distribution P(δ) does not provide enough information to fully characterize the system and its evolution. We consider the joint probability density distribution P(δ^F, δ^R). The quantity P(δ^F, δ^R, t) dδ^F dδ^R represents the probability for a site drawn at random to have a particular set of values of δ^F, δ^R and can be computed numerically starting from the dynamical rules that apply to the δ’s.

To be concrete, we discretize P(δ^F, δ^R) with a bin ε. The distribution probability is then a matrix P_i,j where P(δ^F, δ^R, t) dδ^F dδ^R = P_i,j. We use a time step dt = ε/k_0V and define the constant κ = k_0 + k_1 + k_2.

The finite velocity is expressed through the fact that every time there is some driving of w by a quantity dw = V dt, there is also relaxation during a time dt. In particular, we define the relaxation factor R(dt) = 1 - e^{-dtk_2/η_u}. For the avalanches two crucial quantities should be defined: (1) the fraction of unstable sites P^tot unst ≡ ε Σ_i,j |i+j=0 P_i,j = ε Σ_i,j Σ'_(i+j=0) P_i',j (2) the stress redistribution P^tot unst(z(k_1 + k_2)) that follows the stabilization of the unstable sites. When P^tot unst(z(k_1 + k_2)) > 1,
the avalanche increases geometrically over the time steps, this is why it is practical to define a “critical” value
\( P_0^c = \frac{1}{\pi(k_1 + k_2)} \).

The sketch of the algorithm is the following.

\[ \begin{align*}
\text{Relaxation process:} \\
- & \quad \text{Compute } j_\infty(i), \text{ the bin associated to the fully} \\
& \quad \text{relaxed state, } \delta^R_{i,\infty} = \frac{\Theta(\delta^R - \delta^F)}{\kappa}. \\
& \quad j_\infty(i) = \text{Int} \left( k_2 \frac{-i + \sum_{i'j'}'P(i',j')}{\kappa} \right). \\
- & \quad \text{Relaxation corresponds to shift } [32] \\
& \quad P_{t,j} \rightarrow P_{t,j + \text{Shift}} \text{ (where Shift =} \\
& \quad \text{Int } [(j_\infty(i) - j)R(d)]), \text{ set } r = 1 \text{ and} \\
& \quad \text{perform the Avalanche process.} \\
\end{align*} \]

\[ \begin{align*}
\text{Avalanche process: consists of driving and jumps.} \\
- & \quad \text{Driving:} \\
& \quad P_{t,j} \leftarrow P_{t,j} + (P_{t+1,j} - P_{t,j})r \quad (A4) \\
- & \quad \text{Compute } P_{\text{unst}}(j) = \sum_{i'j' < 0} P_{i',j'} \\
- & \quad \text{Jumps:} \\
& \quad P_{t,j} \leftarrow P_{t,j} + \frac{\varepsilon}{\kappa} g \left( \frac{\varepsilon(i + j)}{\kappa} \right) P_{\text{unst}}(j) \quad (A5) \\
& \quad P_{t,j} \leftarrow 0 \quad \text{if } i + j < 0 \\
- & \quad \text{Compute } P_0 = \sum_{i=-j} P_{t,j} \\
- & \quad \text{If } P_0 \geq P_0^c, \text{ set } r = 1 \text{ and perform the} \\
& \quad \text{Avalanche process again.} \\
- & \quad \text{Else, perform the Relaxation process.} \\
\end{align*} \]

Here we used the exponential distribution with \( \varepsilon = 0.2 \) and an upper-length cutoff \( g(z) = 1/\pi e^{-z/\kappa}\Theta(z)\Theta(10 - z) \), where \( \Theta \) is the Heaviside function. We always used \( \eta_u = 1 \).

Above \( V^c \) we consider that the stationary regime is reached when the relative variation between the last three values of \( \sigma \) does not exceed 1%. Below \( V^c \) we ask that the stress drops \( \Delta \sigma \) of three consecutive Global Shocks change less than 1%. Finally the critical value \( V^c \) is found by decimation: starting with a very small \( V_{\text{min}} = 1e^{-4} \) and very large \( V_{\text{max}} = 1000 \), simulations are repeated between the two boundaries until the relative difference between those two boundaries is less than 1%. This means that a larger absolute tolerance is allowed for larger values of \( V^c \).

An issue limiting higher precision is the presence of numerical instabilities when approaching to the critical point. In Fig. A the dependence of \( \sigma \) on the binning \( \varepsilon \) is shown. For large values of \( k_0 \) (0.3, 0.1 and 0.03), it is clear that we converge towards a plateau \( \sigma(V) \) when the binning decreases. The small values of \( k_0 \) represent a challenge, because the precision required from the algorithm is \( \sim \pi k_0 \). However, comparing the expected values (dashed lines) and the behavior at larger \( k_0 \)’s, it can be expected that smaller binnings would produce the expected results.

Appendix B: Analytical solution for the stationary regime

Let us now solve the model of Fig. 2(b). We call \( f^0(t), f^1(t), \) and \( f^2(t) \) the forces coming from the branches with the \( k_0, k_1, \) and \( k_2 \) springs. The coordinate \( h_t \) will jump to \( h_t + \pi \) each time

\[ f^0(t) + f^1(t) + f^2(t) = f^{th} \] (B1)
Between jumps, the different forces behave as

\[ f^0(t) = f^0_a + k_1 V t \]  \hspace{1cm} (B2)
\[ f^1(t) = f^1_a + k_0 V t \]  \hspace{1cm} (B3)
\[ f^2(t) = V \eta + (f^2_a - V \eta_a) \exp \left( -\frac{k_2 t}{\eta_a} \right) \]  \hspace{1cm} (B4)

where the \( a \) sub-indexes on the right indicate values of the forces right after the jump. This expression holds until the next jump that occurs at time \( t = \tau/V \). At this moment, the forces must satisfy Eq. (B1), and we obtain

\[ f^0_b = f^0_a + k_1 \tau \]  \hspace{1cm} (B5)
\[ f^1_b = f^1_a + k_0 \tau \]  \hspace{1cm} (B6)
\[ f^2_b = V \eta + (f^2_a - V \eta_a) \exp \left( -\frac{k_2 \tau}{V \eta_a} \right) \]  \hspace{1cm} (B7)

where the \( b \) sub-index stands for the values right before the jump. In addition, we have \( f^0_b = -f^0_a \) since the average value of \( f^1 \) must vanish. Also, since the dash-pot is rigid at the jump, we get \( f^2_b - f^2_a = \pi k_2 \). From all these equations all forces \( (f^0_b, f^0_a, f^1_b, f^1_a, f^2_b, f^2_a) \) can be calculated. In particular we are interested in the average friction force \( \sigma \) which is given by \( \sigma = (f^0_a + f^0_b)/2 \). Through a straightforward elimination procedure we get the Eq.(4) given in the main text.

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[32] Instead of crudely taking the integer part of \((j_{\infty}(t) - j)R(dt)\) it is numerically much more stable to split the shift over the two bins, using a linear interpolation.