Constraints for Evolution Equations with Some Special Forms of Lax Pairs and Distinguishing Lax Pairs by Available Constraints

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October 19, 2010

Abstract: The constraints for evolution equations with some special forms of Lax pairs are first investigated. We show by examples how the method is rooted in the classical literatures and how the ignored constraints provide nontrivial solutions. Then we show, by the example of the KdV equation, how this special form of Lax pair may be found by the method of Wahlquist-Estabrook. At last we propose how to impose constraints for general Lax pairs including nonlinear ones. With the proposition the true Lax pairs and the fake ones can be distinguished easily. The linearity nature in integrable partial differential equations seems to have been revealed.

1 Introduction

In general, the last step of the inverse scattering method on the real line is to solve an integral equation such as Gelfand-Levitan-Marchenko equation. But only the reflection-less potentials, which are solitons, make up of a closed system and therefore can be solved completely. It had been observed that the reflection-less potential is some function of the eigenfunctions. With this observation the method of nonlinearization [2] of Lax pair was suggested. Then it was observed [8] that the constraint between the potential and the eigenfunctions may also be regarded as symmetry constraint. Today the method of symmetry constraint has become a powerful tool for analyzing the solutions of integrable systems. By the method of nonlinearization of Lax pair or symmetry constraint, we will obtain a wider class of solutions much more than the solitons though the constraint maybe just be obtained by an observation on the soliton and its corresponding eigenfunctions. In fact we will get algebraic-geometric solutions in most cases. The algebraic-geometric solutions are too complicated in practice, especially when we are only interested in the numerical integration of some initial-boundary problem for an integrable partial differential equations (PDE). Now it becomes more and more clear that a high-precision numerical integration of an integrable PDE should only cope with the constraint between the potential and the eigenfunctions. Yet we may expect those kinds of constraint will play a further role in both analytical and numerical applications.

But the constraints are not so easy to find. The reason is that the form of the constraints may vary from one equation or hierarchy to another, and moreover for the same equation maybe there are several types of constraints, which may lead to completely different types of solutions. Now the method of symmetry constraint is still popular to get constraints, though a lot of useful constraints may be lost by it. In the first part of this paper we will analyze a wide class of Lax pairs, from which the constraints will arise naturally. The constraints provided here may be or not be the symmetry constraint. Therefore, sometimes the class of solutions may be

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expanded. It must be pointed out that the special form of Lax pairs can also be gotten by the Wahlquist-Estabrook (WE) method.

Sakovich [10] had pointed out by example that Lax pairs with nontrivial spectral parameter may be fake ones. In fact the fake Lax pairs have long been puzzling. Kaup [6] had advised a postulate to distinguish the fake Lax pairs from the true ones. But his postulate is not algorithmic. And it also seems to be a little narrow in practice. Furthermore there are nonlinear Lax pairs though they may be linearized in most known examples. An algorithm for distinguishing the fake Lax pairs from the true ones should apply in nonlinear Lax pairs as well as the usual ones. In this paper we propose a functional equation, a solution to which will enable us to provide a nontrivial constraint for the integrable PDE. The constraint is simply a superposition of the solutions to the functional equation. Probably we may guess this superposition reflects more or less the linear nature of integrable PDEs.

The paper is organized as follow. Section 2 propose a kind of natural constraints for a special form of Lax pairs. Several theorems there will be proved to guarantee the justice of the proposal. In addition several examples are provide in the section, where some interesting results may be found. Section 3 first shows by the example of the KdV equation how the WE method is capable of getting the special form of Lax pairs described in Section 2. Then a general requirement for constraining Lax pairs including nonlinear Lax pairs is proposed. At last the proposal is illustrated by two nonlinear Lax pairs of the KdV equation. Section 4 summarizes the main results.

2 Constraints for evolution equations with a special form of Lax pair

In the following paper, in order to avoid confusion by symbols, we define various symbols as follows: $f_t = \dot{f} = \frac{\partial f}{\partial t}$; $f_x = f' = \frac{\partial f}{\partial x}$; $f^{(n)} = \frac{\partial^n f}{\partial x^n}$; $\hat{P}1$ is the function obtained by applying operator $\hat{P}$ to function $f = 1$; Pseudo-operator $\partial^{-1}$ is the inverse of operator $\partial$ [3]; $g[u]$ denotes a differential polynomial of $u$.

2.1 Observations about constraints for PDEs

Numerical experiments have shown that most of the numerical integrations of initial boundary value problems (IBVPs) for PDEs are much more difficult than the IBVPs for ordinary differential equations (ODE). But for some PDEs the integrations may be done by only solving a series of ODEs. This celebrated property should be considered as a kind of integrability for the PDE. As an example let us explain how the periodic KdV equation $u_t = 6uu_x + u_{xxx}$ is integrable in this sense. The stationary KdV equation is $f_n = 0$, where $f_n = \hat{L}^n u_x$ and $\hat{L} = \partial^2 + 4u + 2u_x \partial^{-1}$. It is well-known that $f_n$ is a $(2n+1)$-order differential polynomial of $u$. So $f_n = 0$ may be written into a system of first-order ODEs for $2n+1$ variables $u, u^{(1)}, \cdots, u^{(2n)}$. Also we must know the time evolution for $u, u^{(1)}, \cdots, u^{(2n)}$. In fact

$$\frac{d}{dt} u^{(i)} = \frac{d^i}{dx^i}(6uu^{(1)} + u^{(3)}), \quad \text{Mod} \quad f_n = 0, \quad i = 0, 1, \cdots, 2n. \quad (1)$$

So we have 2 sets of ODEs for the KdV equation. One set governs the space evolution for $u, u^{(1)}, \cdots, u^{(2n)}$. And the other governs the time evolution for $u, u^{(1)}, \cdots, u^{(2n)}$. So for any $(x, t)$ the values of $u, u^{(1)}, \cdots, u^{(2n)}$ may be gotten from the initial values $u_0, u_0^{(1)}, \cdots, u_0^{(2n)}$ at $(x_0, t_0)$ by a space evolution and a time evolution. In other words for some special kind of initial value problems the periodic KdV equation is solvable by only integrating some ODEs. It should be noted that the variables $u, u^{(1)}, \cdots, u^{(2n)}$ are only ‘natural’ variables, not ‘canonical’
variables. The ODEs written for the ‘canonical’ variables are much simpler, see the following sections.

Clearly from the point of view of numerical integration, the only crucial fact for the integration of the periodic KdV equation is that $f_n = 0$ is invariant under the KdV flow or in other words $f_n = 0$ is an invariant manifold (IM) of the KdV equation. But for a given PDE its IMs are not so easy to give, especially when the required IMs must have some completeness. Luckily nonlinearization of Lax pair or symmetry constraint just provide a way to provide IMs for PDEs. Most popularly the symmetry constraint makes use of Lax pair in the inverse scattering transformation

\begin{align}
\dot{L}\psi_i &= \lambda_i \psi_i, \\
\frac{\partial}{\partial t} \psi_i &= \dot{P} \psi_i.
\end{align}

Still take the KdV equation as an example. The well-known nonlinearization of Lax pair

\begin{align}
\dot{L}\psi_i &= (\partial^2 + u)\psi_i = \lambda_i \psi_i, \\
\frac{\partial}{\partial t} \psi_i &= \dot{P} \psi_i = (4\partial^3 + 3(u\partial + \partial u))\psi_i,
\end{align}

of the KdV equation is

\begin{align}
u &= c_0 + \sum_{i=1}^{n} c_i \psi_i^2,
\end{align}

where $\psi_i = \psi_i(x,t)$. With the constraint (6), Equation (4) and Equation (5) become two sets of ODEs.

A less popular representation of Lax equation is

\begin{align}
f_{i+1} &= \dot{L}f_i, \\
\frac{\partial}{\partial t} f_i &= \dot{P} f_i,
\end{align}

for any $i \in \mathbb{N}$. It is easy to verify that the compatibility condition for (7) and (8) is also the Lax equation in its operator form

\begin{align}
\frac{d}{dt} \dot{L} = [\dot{P}, \dot{L}].
\end{align}

So Equations (7) and (8) is equivalent to Equations (4) and (5). Sometimes a constraint for (7) and (8) is more convenient.

There are several noticeable aspects for constraining the Lax equations.

Our first observation is that if $\dot{P}$ is a differential operator then $f_n(x,0) = 0$ is equivalent to $f_n(x,t) = 0$. Let us take the KdV equation as an example to explain why this hold. The well-known $\dot{P}$ of the KdV equation is $\dot{P} = 4\partial^3 + 3(u\partial + \partial u)$. So $f_n$ satisfies $f_t = 4f_{xxx} + 6uf_x + 3u_xf$. By analyzing the Taylor series of $f_n$ we know that if $f_n$ is analytic and $f_n(x,0) = 0$, then $f_n(x,t) = 0$. Therefore, if $\dot{P}$ is a differential operator then (7) and (8) can be truncated by $f_n = 0$. For the same reason the truncation may also be made by setting $f_n = \sum_{i=1}^{n-1} \alpha_i f_i$.

Our second observation is that the number of equations in the truncated (7) and (8) is very close to not enough for (7) and (8) to be 2 sets of ODEs. Also take the KdV equation as an example. To make it easy we take its natural Lax pair $\dot{L} = \partial^2 + 4u + 2u_x \partial^{-1}$, $\dot{P} = \partial (\partial^2 + 6u)$. The number of equations in the truncated (7) and (8) including the KdV equation and the truncation condition is $2(n-1) - 1 + 1 + 1 = 2n - 1$. But we need $2(n-1) + 2 = 2n$ for $f_1, f_2, \cdots, f_{n-1}$ and
In (7) and (8) to be ODE systems. So we need an additional relation between \( u \) and \( f_i \). This is just the nonlinearization of Lax pairs. For the example of the KdV equation the additional relation may be \( f_1 = u_x \), which is just the usual way to get the stationary solutions.

Our third observation is that for the periodic KdV equation the IMs mentioned above is dense. But for some other systems or another Lax pair, the relevant IMs may be not dense. For a nontrivial constraint of Lax equations it should be the minimum requirement that the number of the IMs increases as the truncation number \( n \) increases.

The above three requirements rule out quite a lot of fake Lax pairs. But some true Lax pairs may also be ruled out. For example the famous short-pulse (SP) equation \( u_{xt} = u + uu_x^2 + 1/2u^2u_{xx} \) has a true Lax pair \[ \hat{L} = (\partial^{-1} + u_x \partial^{-1} u_x) \partial^{-1}, \] \[ \hat{P} = \partial^{-1} + \frac{1}{2} \partial u^2. \] Here \( \hat{P} \) is not a differential operator. And furthermore SP equation is not an evolution equation. So the Lax pair (10) and (11) does not fulfill our first requirement. Thereafter we will constrain ourselves in evolution equations.

2.2 A special form of Lax pair and the relevant natural constraint

We have mentioned that with the usual Lax pair (4), (5) the KdV equation has constraint (6). Except this famous constraint, we find that the KdV equation with the less popular Lax pair (7), (8) also has another interesting constraint

\[ u = c_0 + \sum_{i,j=1}^{n} c_{i+j} f_i f_j, \]  

where \( f_k = 0 \) for \( k > n \) and \( c_k = 0 \) for \( k < n + 1 \). So for any \( n \in \mathbb{Z} \), \( f_n = 0 \) provides an IM for the KdV equation.

At the first sight constraint (12) is strange. In this section we will investigate a special form of Lax pairs and give a natural constraint for them. Then constraints (6) and (12) can be understood directly and easily.

We start by the following simple lemma.

**Lemma 2.1** If \( \hat{P} \) is a differential operator and \( g[u] \) satisfies \( \frac{d}{dt} g[u] = \hat{P} g[u] \), then constraint \( g[u] = \sum_{i=1}^{n} a_i \psi_i \) or constraint \( g[u] = \sum_{i=1}^{n} a_i f_i \), \( \sum_{i=1}^{n} b_i f_i = 0 \) exist, where \( \psi_i, f_i \) are defined in (3), (4), (7), (8).

The proof is obvious.

For a given Lax pair, it is not an easy task to find out a nontrivial constraint. However, there exist a natural constraint for Lax pairs in the following form (Form I):

- \( \hat{L} = \hat{L}^+ + \hat{L}^F \partial^{-1} \), where \( \hat{L}^+ \) is a differential operator and \( \hat{L}^F \) is a function;
- \( \hat{P} \) is a differential operator.

**Theorem 2.2** For a Lax pair in Form I, one possible constraint is \( \hat{L}^F = \sum_{i=1}^{n} a_i \psi_i \), where \( \hat{L} \psi_i = \lambda_i \psi_i \). And the other constraint \( \sum_{i=0}^{n} a_i f_i = 0 \), where \( f_i = \hat{L}^i \hat{L}^F \) is also possible.
\textbf{Proof} The first part. We should only prove $\frac{\partial}{\partial t} LF = \hat{P} LF$, because we already have $\frac{\partial}{\partial t} \psi_i = \hat{P} \psi_i$. The Lax equation is $\dot{L} = [\hat{P}, \hat{L}]$. The negative part of the Lax equation is $\dot{L} F \partial^{-1} = (\hat{P} LF) \partial^{-1} - LF \partial^{-1} (\hat{P} 1)$, where $\hat{P}$ is the operator conjugate of $\hat{P}$. Then we get $\frac{\partial}{\partial t} L F - \hat{P} LF = -(\hat{P} 1) \partial$. So
\[ \frac{\partial}{\partial t} LF - \hat{P} LF = 0. \]

The second part. We will verify $f_i$ satisfies (5). $f_0$ satisfies (3), which has been proven in the first part. Suppose $f_i$ satisfies (5). Then $\frac{df}{dt} f_{i+1} = \frac{df}{dt} (\hat{L} f_i) = (\frac{df}{dt} \hat{L}) f_i + \hat{L} (\frac{df}{dt} f_i) = (\hat{P} \hat{L} - \hat{L} \hat{P}) f_i = \hat{P} f_i = \hat{P} f_{i+1}.$

From the second part of the proof we know it is a useful method to nonlinearize the Lax system \{7, 8\} by finding out a differential polynomial $f_0$ such that $\frac{df}{dt} f_0 = \hat{P} f_0$. But this is not the only way, for example, see constraint (12).

2.3 Some examples

The following three examples are all Lax equations in Form I. The first two examples, which are both of the KdV equation, show that different forms of Lax pair for the same equation may naturally lead to different kinds of solutions. The third example, which is the Ito equation, shows how a special form of Lax pair may be used by two ways to solve the same equation.

\textbf{Example} The recursion operator of the KdV equation

It is well-known that the recursion operator and the linearization operator form a natural Lax pair. Particularly to the KdV equation we have
\begin{align*}
\hat{L} &= \partial^2 + 4u + 2u_x \partial^{-1}, \\
\hat{P} &= \partial (\partial^2 + 6u).
\end{align*}

It is also fairly easy to verify that the Lax equation with the above $\hat{L}$ (13) and $\hat{P}$ (14) is the KdV equation $u_t = 6u u_x + u_{xxx}$. Obviously $\hat{L}$ (13) and $\hat{P}$ (14) is in Form I. Here $LF = 2u_x$. So a constraint $u_x = \sum a_i \psi_i$ exists, where $\psi_i$ satisfies $(\partial^2 + 4u + 2u_x \partial^{-1}) \psi_i = \lambda_i \psi_i$ and $\frac{\partial}{\partial t} \psi_i = \psi_{ixxx} + 6(u \psi_i)_t$. Another constraint is $\sum c_i f_i = 0$, where $f_0 = u_x$, the famous finite gap constraint.

This gives a simple explanation for why (6) holds. Note this classical constraint is the integration of symmetry constraint. Let $\phi_i = \frac{d}{dx} \psi_i^2$, where $\psi_i$ satisfies (1) and (5). It is easy to verify
\begin{align*}
(\partial^2 + 4u + 2u_x \partial^{-1}) \phi_i &= 4 \lambda_i \phi_i, \\
\frac{d}{dt} \phi_i &= \partial (\partial^2 + 6u) \phi_i
\end{align*}

So $u_x = \sum_{i=1}^n a_i \phi_i = \sum_{i=1}^n a_i \frac{d}{dx} \psi_i^2$ i.e., $u = c_0 + \sum_{i=1}^n a_i \psi_i^2$ is a proper constraint.

How about (12)? Let $f_j = 4 \sum_{i=j}^n \psi_i \psi_{n+j-i}$. Then we can verify $\dot{L} f_j = f_{j+1}$ and $\frac{df}{dt} f_j = \dot{P} f_j$, where
\begin{align*}
\hat{L} &= \partial^2 + 4u - 2 \partial^{-1} u_x, \\
\hat{P} &= \partial^3 + 6u \partial.
\end{align*}

It is obvious $\frac{d}{dt} (u - c_0) = \dot{P} (u - c_0)$, So $u = c_0 + \sum a_i f_i$ is a proper constraint, which is equivalent to (12).
Example The soliton Lax pair of the KdV equation
It is known that the KdV equation still has another Lax pair
\[
\hat{L} = \partial + u\partial^{-1}, \quad \hat{P} = \partial^3 + 3\partial u, \tag{19}
\]
which is also in Form I. So \( u = \sum_{i=1}^n a_i \psi_i \) or \( u = \sum_{i=1}^n a_i f_i \) is a proper constraint.

We are extremely interested in the case \( \hat{L}^0 = 0 \).

Theorem 2.3 \((\partial + u\partial^{-1})^{n+1} 0 = 0, n = 1, 2, \cdots, \) generates all \( n \)-soliton solitons of the KdV equation.

Proof Recall the soliton solutions for the KdV equation[7] is
\[
N_l = \exp(-2\kappa_l x + \theta_l) \left( 1 + \sum_{j=1}^n \frac{c_j N_j}{\kappa_l + \kappa_j} \right), \quad l = 1, 2, \cdots, n
\]
\[
u = 2 \sum_{j=1}^n c_j N'_j.
\]
First we will prove
\[
N''_l + u N_l = -2\kappa_l N'_l.
\]
Or
\[
\hat{L} N'_l = -2\kappa_l N'_l + \gamma_1 u.
\]
Here the appearance of \( \gamma_1 u \) is due to \( \partial^{-1} \), which is an indefinite integral operator. Then
\[
\hat{L}^n u = 2 \sum_{j=1}^n \left[ (-2\kappa_j)^n + \gamma_1 (-2\kappa_j)^{n-1} + \cdots + \gamma_m c_j N'_j \right].
\]
From the expression of \( \hat{L}^n u = 0 \) we known that \( \kappa_j \) is completely determined by \( \gamma_j \):
\[
(-2\kappa_j)^n + \gamma_1 (-2\kappa_j)^{n-1} + \cdots + \gamma_m = 0. \tag{21}
\]
Altogether the \( n \)-solitons satisfy \( \hat{L}^n 0 = 0 \). But \( n \)-soliton solutions have \( 2n \) free parameters and \( \hat{L}^n 0 = 0 \) can be written as \( 2n \) first-order ODEs. So the general solution of \( \hat{L}^n 0 = 0 \) is the \( n \)-soliton solutions.

Example Ito equation (Drinfeld-Sokolov II)
The Ito equation [4, 5]
\[
u_t = 3v_x, \quad \tag{22}
\]
\[
v_t = (uv)_x + v_{xxx} \quad \tag{23}
\]
has a Lax pair [4]
\[
\hat{L} = \partial^3 + u\partial + u_x + v\partial^{-1}, \quad \tag{24}
\]
\[
\hat{P} = \partial(\partial^2 + u), \quad \tag{25}
\]
which is in Form I. We will investigate the constraints $\hat{L}^0 = 0$. The first few constraints of (22) and (23) can be easily solved.

The first constraint is $\hat{L}^0 = v = 0$. By (22) $u_t = 0$. So $u = u_0(x)$. The second constraint is $\hat{L}^0 = v_{xxx} + (uv)_x + v \int v dx = 0$. By (23) $v_t = -v \int v dx$. Let us introduce a new variable $\phi$ such that $\phi_x = v$. Then $\phi_{xt} = -\phi_x \phi = -\frac{1}{2} (\phi^2)_x$. So $\phi_t = -\frac{1}{2} \phi^2 + g(t)$. We can prove $g(t)$ is independent on $t$. So

$$\phi_t = -\frac{1}{2} \phi^2 + \frac{c_0}{2}, \quad (26)$$

The solution of (26) is

$$\phi = \sqrt{c_0} \tanh\left(\frac{\sqrt{c_0}}{2} t + c_1\right). \quad (27)$$

We may prove

$$\phi_{xxx} + u \phi_x + \frac{1}{2} \phi^2 = \frac{c_0}{2}, \quad (28)$$

So

$$u = \frac{1}{\phi_x} \left(\frac{c_0}{2} - \frac{1}{2} \phi^2 - \phi_{xxx}\right), \quad (29)$$

$$v = \phi_x. \quad (30)$$

is one solution of (22) and (23).

The third constraint is $\hat{L}^0 = 0$. By a tedious study of this constraint, we find if $\phi$ satisfies the following equation

$$\phi_t = -1/2 \phi^2 + f_1(t) \phi + f_2,$$

where

$$\frac{df_1}{dt} = \gamma(t), \quad \frac{d\gamma}{dt} = f_1(t) \gamma(t),$$

and $f_2$ is an arbitrary constant, then $v$ and $u$ expressed by

$$v = \phi_x,$$

$$u = \frac{1}{\phi_x} (\phi_t - \phi_{xxx} - \gamma). \quad (31)$$

is a solution.

It seems very difficult to solve $\hat{L}^0 = 0$, $n \geq 3$. So how to analyze the high-mode vibrations?

First by Theorem 2.2 we can impose a constraint $v = \sum_{i=1}^n a_i \psi_i$, where $L\psi_i = \lambda_i \psi_i$ and $\psi_{it} = \hat{P} \psi_i$. After a little modification of the form of the Lax equation, we get

$$\left(\lambda_i \psi_i - \psi_i''' - (uv)_i\right)' = \psi_i,$$

$$\hat{\psi}_i = \psi_i''' + (uv)_i',$$

$$\hat{u} = 3v'.$$
Let \( \psi_i = \varphi'_i \). Then \( v = \sum_{i=1}^n a_i \varphi'_i \). So we get

\[
\lambda_i \varphi'_i - \varphi'''_i - (u \varphi'_i)' = v \varphi_i,
\]

(32)

\[
\dot{\varphi}_i = \varphi''_i + u \varphi'_i,
\]

(33)

\[
\dot{u} = 3v'.
\]

(34)

Multiplying (32) by \( a_i \) and summing over \( i \) we get

\[
\sum_{i=1}^n a_i (\lambda_i \varphi'_i - \varphi'''_i - (u \varphi'_i)') = v \sum_{i=1}^n a_i \varphi_i = \frac{1}{2} \left( \sum_{i=1}^n (a_i \varphi_i)^2 \right)'.
\]

(35)

Integrating (35) we immediately get

\[
\sum_{i=1}^n a_i (\lambda_i \varphi_i - \varphi'''_i - u \varphi'_i) = \frac{1}{2} \left( \sum_{i=1}^n (a_i \varphi_i)^2 \right) + \gamma.
\]

At first glance \( \gamma \) is a function of \( t \). But by (34) we known \( \dot{\gamma}(t) = 0 \). So \( \gamma \) is a constant. Now \( u \) is solved as

\[
u = -\frac{1}{v} \left[ \frac{1}{2} \left( \sum_{i=1}^n (a_i \varphi_i)^2 \right) + v'' + \sum_{i=1}^n a_i \lambda_i \varphi_i \right].
\]

(36)

Totally Equation (32) and Equation (33) contain \( (n-1) + n = 2n-1 \) differential equations. So it is not enough for them to be ODE systems. The usual way to overcome this problem is to introduce another constraint. But for the Ito equation there is no need to do such things. The reason is as follows. Obviously we have

\[
\dot{\varphi}'_i = \lambda_i \varphi'_i - \varphi_i \sum_{j=1}^n a_{ij} \varphi'_j.
\]

(37)

By differentiate (37) with respect to \( x \) once and twice we get another 2 sets of differential equations. Together with (33) and (37) we have a closed form for the \( t \)-evolution of \( \varphi_i, \varphi'_i, \varphi''_i \) and \( \varphi'''_i \). This is enough for a numerical computation for the Ito equation. Note that Equation (32) may be regarded as \( 4n - 1 \) first-order differential constraints in the \( x \)-direction, i.e., there is still one arbitrary function \( \varphi(i)(x) \) among the \( 4n \) initial functions \( \varphi_i \).

### 2.4 Further generalizations

Now we will generalize Form I a little. Anyway we have the following theorem:

**Theorem 2.4** If \( \hat{L} = \hat{L}' + \sum_{i=1}^n f_i \partial^{-1} g_i \), where \( \hat{L}' \) is a differential operator, and also \( \hat{P} \)

is a differential operator, then there a linear composition of \( f_i \)'s \( L^F = \sum_{j=1}^n \alpha_j f_i \) such that

\[
\frac{d}{dt} L^F = \hat{P} L^F, \text{ i.e., a nonlinearization } L^F = \sum_{i=1}^n \beta_i \psi_i \text{ exists, where } \psi \text{ is eigenfunction } \hat{L} \psi = \lambda \psi.
\]

We say \( \hat{L} \) and \( \hat{P} \) in Theorem 2.4 is of Form II. To prove Theorem 2.4 we need the following Lemma 2.5 and Corollary 2.6

**Lemma 2.5** If \( \sum_{i=1}^n f_i(x) \partial^{-1} g_i(x) = 0 \), then \( \sum_{i=1}^n f_i(x)g_i(y) = 0 \).

**Proof** \( \sum_{i=1}^n f_i(x) \partial^{-1} g_i(x) = 0 \) is equivalent to \( \sum_{i=1}^n f_i(x) g_i^{(k)}(x) = 0, \ k = 0, 1, 2, \cdots \). So \( \sum_{i=1}^n f_i(x) g_i(y) = \sum_{i=1}^n f_i(x) g_i(x + (y - x)) = \sum_{i=1}^n \sum_{k=0}^\infty \frac{1}{k!} f_i(x) g_i^{(k)}(x) (y - x)^k = 0. \)

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Corollary 2.6 If \( \sum_{i=1}^{n} A_i(x) \partial^{-1} B_i(x) = \sum_{i=1}^{n} F_i(x) \partial^{-1} G_i(x) \) and both \( \{ A_i(x) \} \) and \( \{ G_i(x) \} \) are linearly independent, then \( B_i(x) \) is a linear composition of \( G_j(x) \) and \( F_i(x) \) is a linear composition of \( A_j(x) \).

Theorem 2.4 guarantees that there is a natural constraint for a Lax pair in Form II. For Lax pairs in more complicated form, such as \( \hat{L} \) and \( \hat{P} \) are matrix, we have not reach a general result like Theorem 2.2. But it seems always a good guess that a linear composition \( v \) of the components before \( \partial^{-1} \) satisfies \( \dot{v} = \hat{P}v \).

Example ZS-AKNS

With the help of recursion operator the AKNS hierarchy can be expressed as a simple expression

\[
\hat{L} = \frac{1}{i} \begin{pmatrix}
-\partial + 2q\partial^{-1}r & 2q\partial^{-1}q \\
-2r\partial^{-1}q & \partial - 2r\partial^{-1}q
\end{pmatrix},
\]

\[
\begin{pmatrix}
q \\
r
\end{pmatrix}_t = \hat{L}^n \begin{pmatrix}
-\frac{iq}{ir}
\end{pmatrix},
\]

where \( i \) is the imaginary unit. Here \( \hat{L} \) is the recursion operator. So \( \hat{L} \) and the linearization operator \( \hat{P} \) form a natural Lax equation \( \frac{d}{dt} \hat{L} = [\hat{P}, \hat{L}] \). The symmetry \( \sigma_i = \begin{pmatrix}
\zeta_i \\
\xi_i
\end{pmatrix} \) satisfies

\[
\sigma_{i+1} = \hat{L}\sigma_i,
\]

\[
\frac{d}{dt}\sigma_i = \hat{P}\sigma_i.
\]

But \( \begin{pmatrix}
q \\
r
\end{pmatrix} \) is also a symmetry because if \( \begin{pmatrix}
q \\
r
\end{pmatrix} \) is a solution of (38) and (39) then \( \begin{pmatrix}
\bar{q} \\
\bar{r}/k
\end{pmatrix} \) is also a solution, where \( k \in \mathbb{R} \). So we know

\[
\frac{d}{dt}\begin{pmatrix}
q \\
r
\end{pmatrix} = \hat{P}\begin{pmatrix}
q \\
r
\end{pmatrix}.
\]

Therefore one possible constraint is \( \begin{pmatrix}
q \\
r
\end{pmatrix} = \sum_{i=1}^{m} a_i\sigma_i \). Meanwhile we can also write the Lax equation as \( \hat{L}\Psi = \lambda\Psi \) and \( \frac{d}{dt}\Psi = \hat{P}\Psi \), where \( \Psi \) is a \( 2 \times 1 \) vector. So \( \begin{pmatrix}
q \\
r
\end{pmatrix} = \sum_{i=1}^{n} a_i\Psi_i \) is also one possible constraint.

For example AKNS \( n = 3 \) is the coupled KdV equation

\[
\begin{pmatrix}
q \\
r
\end{pmatrix}_t = \begin{pmatrix}
6qrq_x - q_{xxx} \\
6qr^2x - r_{xxx}
\end{pmatrix}.
\]

The corresponding linearization operator is

\[
\hat{P} = \begin{pmatrix}
6r\partial q - \partial^3 & 6qq_x \\
6rr_x & 6q\partial r - \partial^3
\end{pmatrix}.
\]

It is easy to verify

\[
\begin{pmatrix}
q \\
r
\end{pmatrix}_t = \hat{P}\begin{pmatrix}
q \\
r
\end{pmatrix}.
\]
Then one possible constraint is
\[
\begin{pmatrix} q \\ -r \end{pmatrix} = \sum_{i=1}^{n} a_i \begin{pmatrix} \psi_i \\ \phi_i \end{pmatrix},
\]
where \( \begin{pmatrix} \psi_i \\ \phi_i \end{pmatrix} \) satisfies \( \hat{L}\psi_i = \lambda_i \psi_i \), \( \frac{d}{dt} \psi_i = \hat{P}\psi_i \) and the expressions for \( \hat{L} \) and \( \hat{P} \) are (38) and (40).

2.5 The equivalence of the two kinds of constraint

We have introduced two kinds of constraints in Lemma 2.5 and thereafter. Now we will explain the relation between the two kinds of constraint.

Recall the constraint of the first kind is
\[
g[u] = \sum_{i=1}^{n} a_i \psi_i. \tag{41}
\]
Constraint of the second kind is
\[
g[u] = \sum_{i=1}^{n} b_i f_i. \tag{42}
\]
In general \( \psi_i \) and \( f_i \) have not any relation. But with constraint (41) or (42) they become linear dependent in most cases. Suppose \( f_i = \sum_{j=1}^{n} c_{ij} \psi_j \). Then from the relation \( f_{i+1} = \hat{L} f_i \), we get
\[
\sum_{j} c_{i+1,j} \psi_j = \sum_{j} \lambda_j c_{ij} \psi_j.
\]
So \( c_{i+1,j} = \lambda_j c_{ij} \).

**Theorem 2.7** For a constraint of the first kind \( g[u] = \sum_{i=1}^{n} a_i \psi_i \) satisfying that if \( i \neq j \) then \( \lambda_i \neq \lambda_j \), there exist a constraint of the second kind \( g[u] = \sum_{i=1}^{n} b_i f_i \), \( f_{n+1} = \sum_{i=1}^{n} k_i f_i \) where \( b_i \) and \( k_i \) is determined by \( c_{i+1,j} = \lambda_j c_{ij} \), \( a_j = \sum_{i=1}^{n} b_i c_{ij} \), \( \lambda^n = k_j \lambda^{j-1} \) with arbitrary \( c_{1i} \neq 0 \).

**Proof** Only to consider that if \( \prod c_{1i} \neq 0 \) and \( \lambda_i \neq \lambda_j \) then \( \det(c_{ij}) \neq 0 \).

**Theorem 2.8** For a constraint of the second kind \( g[u] = \sum_{i=1}^{n} b_i f_i \), \( f_{n+1} = \sum_{i=1}^{n} k_i f_i \) satisfying that \( \lambda^n = \sum_{j=1}^{n} k_j \lambda^{j-1} \) has distinct root for \( \lambda \), there exist a constraint of the first kind \( g[u] = \sum_{i=1}^{n} a_i \psi_i \), where \( \lambda_i \) and \( a_i \) is determined by: \( \lambda_i \) is the \( i \)-th root of \( \lambda^n = \sum_{j=1}^{n} k_j \lambda^{j-1} \), \( c_{1i} \neq 0 \), \( c_{i+1,j} = \lambda_j c_{ij} \), \( a_i = \sum_{j} b_j c_{ji} \).
3 Searching the special form of Lax pairs and constraining the nonlinear Lax pair systems

It is well-known that the WE method is a powerful tool for searching the Lax pairs for PDEs. By the preceding examples we have known that one PDE can have completely different Lax pairs. So which Lax pair may be found by the WE method is heavily dependent on the original assumptions for the form of Lax pair. Furthermore the Lax pairs obtained by the WE method are often nonlinear ones, which must be linearized before further application. So very often the nonlinear Lax pairs are considered to be useless. In this section we will first demonstrate, by the example of the KdV equation, how to search the Lax pair in Form II. Then we will propose a functional equation for finding constraints for both linear or nonlinear Lax pair systems.

Let us first demonstrate by the example of KdV equation how Lax pair (19), (20) can be found by the usual WE method. The KdV equation is first written into

\[ u_x = z, \]
\[ z_x = p, \]
\[ u_t = 6uz + p_x. \]  \hfill (43)

Substituting (43) to the zero-curvature equation

\[ M_t - N_x + [M, N] = 0, \]

where \( M = M(u, z, p), N = N(u, z, p), \) we immediately get

\[ z_t M_z + p_t M_p + p_x (M_u - N_p) + 6uz M_u - z N_u - p N_z + [M, N] = 0. \]

Here \( z_t, p_t \) and \( p_x \) should be considered as independent variables.

**Remark:** The KdV equation in the form of (43), in fact, has implied that \( M \) is only dependent on \( u \). In other words, we are searching Lax pairs for the KdV equation whose \( M \) is only dependent on \( u \).

Then by the usual steps of the WE method we get

\[ M = X_1 + u X_2 + u^2 X_3, \]
\[ N = X_4 + (3u^2 + p) X_2 + (4u^3 + 2up - z^2) X_3 + z [X_1, X_2] + u [X_1, [X_1, X_2]] + \frac{u^2}{2} [X_2, [X_1, X_2]], \]

where

\[
\begin{align*}
[X_1, X_3] &= 0, \\
[X_2, X_3] &= 0, \\
[X_1, X_4] &= 0, \\
[X_1, [X_1, X_2]] + [X_2, X_4] &= 0, \\
[X_2, [X_2, [X_1, X_2]]] &= 0, \\
\frac{1}{2} [X_1, [X_2, [X_1, X_2]]] + 3 [X_1, X_2] + [X_2, [X_1, [X_1, X_2]]] + [X_3, X_4] &= 0. \quad (44)
\end{align*}
\]

We would first not consider (44) as an open Lie structure, but rather a matrix equation. Then we will get linear Lax pair. The 2 × 2 realization of (44) can be solved by Maple and there are 12 solutions obtained by Maple. However, only one solution is worthy of notice. The solution is

\[ x_{3;2,1} = 0, \]
\[
x_{2,1,2} = 0,
\]
\[
x_{2,2,2} = x_{2,1,1},
\]
\[
x_{3,1,2} = 0,
\]
\[
x_{3,2,2} = x_{3,1,1},
\]
\[
x_{2,2,1} = -x_{1,1,2}^{-1},
\]
\[
x_{4,2,1} = x_{1,1,2} \left( x_{1,1,1}^2 + x_{1,2,2}^2 - 2 x_{1,1,1} x_{1,2,1} + 4 x_{1,1,2} x_{1,2,1} \right),
\]
\[
x_{4,1,2} = x_{1,1,1}^2 x_{1,1,2} + x_{1,1,1} x_{1,2,2}^2 - 2 x_{1,1,1} x_{1,1,2} x_{1,2,1} + 4 x_{1,1,2} x_{1,2,1},
\]
\[
x_{4,2,2} = x_{4,1,1} - x_{1,1,1} x_{1,1,2}^2 + 3 x_{1,1,1} x_{1,2,2}^2 - 4 x_{1,1,1} x_{1,1,2} x_{1,2,1} + x_{1,2,2}^3 + 4 x_{1,1,2} x_{1,2,1} x_{1,2,2};
\] (45)

where \(x_{i:j,k}\) is the \((j,k)\) element of matrix \(X_i\). Thanks to (45), the equation
\[
\Psi_x = M \Psi,
\] (46)
where \(\Psi = \begin{pmatrix} \psi(x, t) \\ \phi(x, t) \end{pmatrix}\) is a 2-dimensional vector, is completely determined. The goal equation is \(\hat{L} \varphi = \lambda \varphi\). \(\varphi\) may be the linear composition of \(\psi\) and \(\phi\)
\[
\varphi = f_1([u]) \psi + f_2([u]) \phi.
\] (47)

Clearly \(\varphi\) satisfies a 2-order ODE. If we demand the 2-order ODE do not contain \(u^{(n)}, n \geq 1\), then we obtain \(f_2 = 0, f_1 = \text{Const, } x_{3,1,1} = 0 \) and \(x_{2,1,1} = 0\). Obviously, we may set \(f_1 = 1\). Now (45) is reduced to
\[
\varphi_{xx} - (x_{1,1,1} + x_{1,2,2}) \varphi_x + (u - x_{1,1,2} x_{1,2,1} + x_{1,1,1} x_{1,2,2}) \varphi = 0.
\] (48)

Comparing Equation (48) with the goal equation \(\hat{L} \varphi = \lambda \varphi\), we immediately get
\[
\varphi_{xx} + \lambda_1 \varphi_x + (u + \lambda_2) \varphi = 0.
\] (49)

To write a Lax pair in Form II from (49), we only need to introduce the variable \(\xi = \varphi_x\). Now (49) can be rewritten to
\[
\hat{L} \xi \equiv (\partial + (u + \lambda_2) \partial^{-1}) \xi = -\lambda_1 \xi.
\]

Now \(\hat{L}\) here is in Form II. That \(\hat{P}\) is also in Form II can be verified directly.

With regard to the other 11 solutions of (45), four of them can not be reduced to 2-order ODE; seven of them can not be reduced to 2-order ODE whose coefficients are only functions of \(u\).

Now we turn to nonlinear Lax pairs. It has NEVER been demanded the Lax pair be linear. Then where is the linearity? We find the following functional equation is crucial in integrable systems. The functional equation is
\[
\frac{d}{dt} g[u, \psi, \lambda] = \hat{A}[u] g[u, \psi, \lambda],
\] (50)
where \(g[u, \psi, \lambda]\) denotes a function of finite variables including the original variables \(u^{(i)}\), auxiliary variables \(\psi_i\) and complex parameters \(\lambda\). But \(\hat{A}[u]\) is a linear differential operator which only involves functions of finite variables of \(u^{(i)}\). Of course, non-degenerate conditions \(\frac{d}{dt} g[u, \psi, \lambda] \neq 0\) and \(\frac{\partial}{\partial \psi} g[u, \psi, \lambda] \neq 0\) must be satisfied. The final constraint for the general Lax system is
\[
\sum_{\lambda} g[u, \psi, \lambda] = 0.
\] (51)
It seems that the linearity nature of Lax integrable systems is completely characterized by Equation (50) and (51). To see how these arguments works, let us still take the KdV equation as an example.

Following WE we regard Equation (44) as a Lie algebra constraint. Here we only cite the final equations for the pseudo-potentials:

\[
\frac{\partial y_2}{\partial x} = -e^{2y_3}, \quad \frac{\partial y_2}{\partial t} = -2e^{2y_3}(u + 2\lambda),
\]
\[
\frac{\partial y_3}{\partial x} = -y_8, \quad \frac{\partial y_3}{\partial t} = z - 2(u + 2\lambda)y_8,
\]
\[
\frac{\partial y_8}{\partial x} = \lambda - u - y_8^2, \quad \frac{\partial y_8}{\partial t} = 2zy_8 - 2(u + 2\lambda)(u + y_8^2 - \lambda) - p. \tag{52}
\]

Note the coefficients have been adjusted to fit the KdV equation here. Because (52) is a nonlinear Lax pair, the subsequent step is usually to find some transformation to linearize (52). But Equation (50) and (51) enable us to solve the KdV equation straitly. Our task is to find a function \(g\) and a linear operator \(\hat{A}\) satisfying (50). It is easy to check

\[
g = \alpha e^{-2y_3} + \beta u + \gamma, \quad \hat{A} = \partial^3 + 6u\partial, \quad \alpha, \beta, \gamma \in \mathbb{C},
\]

is a solution of (50). So

\[
\sum_{i=1}^{n} (\alpha_i e^{-2y_{3i}} + \beta_i u + \gamma_i) = \gamma + \beta u + \sum_{i=1}^{n} \alpha_i e^{-2y_{3i}} = 0,
\]

is a proper constraint, where \(y_{3i}\) satisfies (52) with \(\lambda = \lambda_i\).

Given a nonlinear Lax pair, (50) can always be proposed no matter whether the nonlinear Lax pair is linearizable. The difficulty is to give nontrivial \(g\) and \(\hat{A}\).

Open Lie algebra (44) has more than one solutions. Let us assume \(X_3 = 0\). (It seems also to be not clear yet when \(X_3\) is a centre.) One 4-dimensional realization of (41) is

\[
[X_1, X_2] = -\alpha \beta X_4 + \frac{1}{\alpha^2} X_5, [X_1, X_4] = 0, [X_1, X_5] = 2\alpha^2 X_1 + \alpha X_2,
\]
\[
[X_2, X_4] = \beta X_4 - \frac{1}{\alpha^3} X_5, [X_2, X_5] = \alpha^2 \beta^2 X_1 - 2\alpha^2 X_2 - \beta X_5, [X_4, X_5] = 2\alpha X_1 + X_2.
\]

If we set \(\tilde{X}_4 = X_4 - \frac{1}{\alpha} X_1\), then \(\tilde{X}_4\) commute with any other vectors. If there is no centre then \(\tilde{X}_4 = 0\), i.e., \(X_4 = \frac{1}{\alpha} X_1\). Therefore, in fact we have a 3-dimensional realization of (41)

\[
[X_1, X_2] = \frac{1}{\alpha^2}X_5 - \beta X_1, [X_1, X_5] = 2\alpha^2 X_1 + \alpha X_2, [X_2, X_5] = \alpha^2 \beta^2 X_1 - 2\alpha^2 X_2 - \beta X_5. \tag{55}
\]

Following WE we get the following nonlinear Lax pair

\[
\frac{\partial y_1}{\partial x} = -1 + \alpha u(2 - \frac{\alpha}{\gamma} e^{\frac{u}{\sqrt{\alpha}}} - \frac{\gamma}{\alpha} e^{-\frac{u}{\sqrt{\alpha}}}),
\]
\[
\frac{\partial y_1}{\partial t} = -\frac{1}{\alpha} + 4\alpha u^2 + 2\alpha u'' - \frac{\alpha}{\gamma}(u + 2\alpha u^2 + \sqrt{\alpha}u\gamma + \alpha u'')e^{\frac{u}{\sqrt{\alpha}}}
\]
\[
- \frac{\gamma}{\alpha}(u + 2\alpha u^2 - \sqrt{\alpha}u\gamma + \alpha u'')e^{-\frac{u}{\sqrt{\alpha}}},
\]
\[
\frac{\partial y_2}{\partial x} = u(\frac{\alpha^2}{\gamma^2} e^{\frac{u}{\sqrt{\alpha}}} - e^{-\frac{u}{\sqrt{\alpha}}}),
\]
\[
\frac{\partial y_2}{\partial t} = \frac{\alpha^2}{\gamma^2}(u + 2u^2 + \frac{u'}{\sqrt{\alpha}} + u'')e^{\frac{u}{\sqrt{\alpha}}} - (\frac{u}{\alpha} + 2u^2 - \frac{u'}{\sqrt{\alpha}} + u'')e^{-\frac{u}{\sqrt{\alpha}}}. \tag{56}
\]
By a definition of $\bar{y}_1 = \frac{y_1}{\sqrt{\alpha}} + \ln \alpha - \ln \gamma$ and $\bar{y}_2 = \frac{2}{\gamma} y_2$, we can simplify (56) to

$$\begin{align*}
\frac{\partial \bar{y}_1}{\partial x} &= -\frac{1}{\sqrt{\alpha}} + 2\sqrt{\alpha}u - 2\sqrt{\alpha}u \cosh(\bar{y}_1), \\
\frac{\partial \bar{y}_1}{\partial t} &= \sqrt{\alpha}(\frac{1}{\alpha^2} + 4u^2 + 2u'') - 2\sqrt{\alpha}(\frac{u}{\alpha} + 2u^2 + u'') \cosh(\bar{y}_1) - 2u' \sinh(\bar{y}_1), \\
\frac{\partial \bar{y}_2}{\partial x} &= u \sinh(\bar{y}_1), \\
\frac{\partial \bar{y}_2}{\partial t} &= \frac{u'}{\sqrt{\alpha}} \cosh(\bar{y}_1) + (\frac{u}{\alpha} + 2u^2 + u'') \sinh(\bar{y}_1).
\end{align*}$$

(57)

Still another substitution $\tilde{y}_1 = e^{\hat{y}_1}$, $\epsilon = \sqrt{\alpha}$ and $\tilde{y}_2 = 2y_2$ transforms (57) to a form more suitable for computation,

$$\begin{align*}
\frac{\partial \tilde{y}_1}{\partial x} &= -\epsilon u + (2\epsilon u - \frac{1}{\epsilon}) \tilde{y}_1 - \epsilon u \tilde{y}_1, \\
\frac{\partial \tilde{y}_1}{\partial t} &= -\frac{u}{\epsilon} - 2\epsilon u^2 + u' - \epsilon u'' - \epsilon(\frac{1}{\epsilon^4} - 4u^2 - 2u'') \tilde{y}_1 - (\frac{u}{\epsilon} + 2\epsilon u^2 + u' + \epsilon u'') \tilde{y}_1^2, \\
\frac{\partial \tilde{y}_2}{\partial x} &= u \tilde{y}_1 - \frac{1}{\tilde{y}_1}, \\
\frac{\partial \tilde{y}_2}{\partial t} &= (\frac{u}{\epsilon^2} + 2u^2 + \frac{u'}{\epsilon} + u'') \tilde{y}_1 - (\frac{u}{\epsilon^2} + 2u^2 - \frac{u'}{\epsilon} + u'') \frac{1}{\tilde{y}_1}.
\end{align*}$$

(58)

Now the task is to give $g$ and $\dot{A}$. With the help of Maple we get the following solution

$$g = c_0 + c_1 u + c_2 (\frac{\tilde{y}_1 - 1}{\tilde{y}_1}) e^{\epsilon \tilde{y}_2},$$

$$\dot{A} = \partial^3 + 6u \partial.$$

(59)

Then it becomes very clear that (58) and (52) must be equivalent. In fact, the transformation from (58) to (52) is

$$y_8 = \frac{1}{2\epsilon} \frac{1 + \tilde{y}_1}{1 - \tilde{y}_1}, \quad y_3 = \frac{1}{2\epsilon} \tilde{y}_2 + \ln(\tilde{y}_1 - 1) - \frac{1}{2} \ln \tilde{y}_1, \quad \lambda = \frac{1}{4 \epsilon^2}.$$

(60)

4 Conclusions

We have proposed a kind of natural constraints for evolution PDEs with a special kind of Lax pairs. By the method several examples have been studied in detail. At least two interesting results should be noticed. One is that KdV equation has a soliton-Lax pair, from which only soliton solutions can appear. The other is that the Ito equation can be constraint to a series of PDEs that are solvable by the method of characteristics.

For a given Lax pair, it is always very difficult to solve the functional equation (50) completely. Even for the KdV equation only special solutions of (50) have been obtained. It is necessary to point out that solutions (53) and (59) are both obtained by the classical separation of variables. More general ansatz about the solutions of (50) will lead to a set of too complicated equations to solve. This also explains why we seek the special kind of Lax pairs, of which the constraints are manifest. Equations (50) and (51) can act as a measure of integrability for evolution equations with Lax pairs. Any generalization of (50) and (51) seems so difficult.
Acknowledgments

The first author wish to thank Prof. H. Y. Guo, Prof. S. K. Wang and Dr. D. S. Wang for their helpful discussions and encouragements. The work is partly supported by NSFC (No.10735030), NSF of Zhejiang Province (R609077, Y6090592), NSF of Ningbo City (2009B21003, 2010A610103, 2010A610095).

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