Clusters of bound particles in the derivative $\delta$-function Bose gas

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Abstract

In this paper we discuss a novel procedure for constructing clusters of bound particles in the case of a quantum integrable derivative $\delta$-function Bose gas in one dimension. It is shown that clusters of bound particles can be constructed for this Bose gas for some special values of the coupling constant, by taking the quasi-momenta associated with the corresponding Bethe state to be equidistant points on a single circle in the complex momentum plane. We also establish a connection between these special values of the coupling constant and some fractions belonging to the Farey sequences in number theory. This connection leads to a classification of the clusters of bound particles associated with the derivative $\delta$-function Bose gas and allows us to study various properties of these clusters like their size and their stability under the variation of the coupling constant.
1 Introduction

Exact solutions of one-dimensional (1D) quantum integrable many-body systems with short range interactions have emerged as an active area of research [1–20], due to their effectiveness in describing recent experiments using strongly interacting ultracold atomic gases [21–28]. The nearly 1D motion of ultracold bosons is achieved in such experiments by confining the bosons within waveguides that tightly trap their motion in the two transverse directions, and allow them to move only in the third direction. Many results of these experiments can be understood within the framework of the well known Lieb-Liniger model or the δ-function Bose gas; this is a 1D quantum integrable system with the Hamiltonian for $N$ particles given by

$$H_N = -\hbar^2 \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2\hbar^2 \mu \sum_{l<m} \delta(x_l - x_m), \quad (1.1)$$

where $\mu$ is the coupling constant. Exact eigenfunctions of the Hamiltonian (1.1) are constructed by using the methods of coordinate as well as algebraic Bethe ansatz [1–9]. Moreover, the equilibrium properties of this system have been studied by employing the thermodynamic Bethe ansatz and various correlation functions have been computed through different approaches [1–8, 20].

In this context it may be recalled that, for a class of exactly solvable dynamical systems, the method of coordinate Bethe ansatz directly yields the eigenfunctions in the coordinate representation. One can study the asymptotic form of these eigenfunctions in the limit of infinite length of the system (i.e., when all $x_i$'s are allowed to take value in the range $-\infty < x_i < \infty$). If the probability density associated with an eigenfunction decays sufficiently fast when any of the particle coordinates tends towards infinity (keeping the centre of mass coordinate fixed, for a translationally invariant system), a bound state is formed. The stability of a bound state, in the presence of small external perturbations, can be determined by calculating its binding energy. It is well known, for the case of the δ-function Bose gas (1.1) with $N \geq 2$, that bound states with positive binding energies exist for all negative values of the coupling constant $\mu$ [2, 3, 5–9]. The quasi-momenta associated with such a bound state are represented by equidistant points lying on a straight line or ‘string’ parallel to the imaginary axis in the complex momentum plane. Moreover, for the case of the δ-function Bose gas with negative values of the coupling constant, one can construct Bethe eigenfunctions corresponding to more complex structures like clusters of bound particles and show that those clusters of bound particles are stable under scattering. The quasi-momenta corresponding to such clusters of bound particles are represented through discrete points lying on several ‘strings’, all of which are parallel to the imaginary axis in the complex momentum plane [3, 9, 12–15].

Similar to the case of the δ-function Bose gas mentioned above, there exists another exactly solvable and quantum integrable bosonic system with a Hamiltonian given by

$$\mathcal{H}_N = -\hbar^2 \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2i\hbar^2 \eta \sum_{l<m} \delta(x_l - x_m) \left( \frac{\partial}{\partial x_l} + \frac{\partial}{\partial x_m} \right), \quad (1.2)$$
where $\eta$ is a real nonzero coupling constant $^{[29,33]}$. The Hamiltonian (1.2) of this derivative $\delta$-function Bose gas can be obtained by projecting that of an integrable derivative nonlinear Schrödinger (DNLS) quantum field model on the $N$-particle subspace. Classical and quantum versions of such DNLS field models have found applications in different areas of physics like circularly polarized nonlinear Alfven waves in plasma, quantum properties of optical solitons in fibers, and in some chiral Tomonaga–Luttinger liquids obtained from the Chern-Simons model defined in two dimensions $^{[34–41]}$. The scattering and bound states of the derivative $\delta$-function Bose gas (1.2) have been studied extensively by using the methods of coordinate as well as algebraic Bethe ansatz $^{[29–33,42–44]}$. It turns out that the quasi-momenta associated with a bound state of this model can be represented by equidistant points on a circle or circular ‘string’ with arbitrary radius and having its centre at the origin of the complex momentum plane. For the cases $N = 2$ and $N = 3$, this type of bound states can be constructed for any value of $\eta$ within its full range: $0 < |\eta| < \infty$. However, for any given value of $N \geq 4$, the derivative $\delta$-function Bose gas allows bound states in only certain non-overlapping ranges of the coupling constant $\eta$ (the union of these ranges yields a proper subset of the full range of $\eta$), and such non-overlapping ranges of $\eta$ can be determined by using the Farey sequences in number theory $^{[42–44]}$. Furthermore, for any given value of $N \geq 3$, bound states with positive as well as negative binding energies can be constructed. From the above discussions it is evident that the bound states of this derivative $\delta$-function Bose gas exhibit a much richer structure in comparison to the case of the $\delta$-function Bose gas.

The aim of the present work is to explore how clusters of bound particles can be constructed in the simplest possible way for the derivative $\delta$-function Bose gas (1.2). In analogy with the case of the $\delta$-function Bose gas, one may think that clusters of bound particles can only be constructed for the above mentioned case by properly assigning the corresponding quasi-momenta on several concentric circles or circular ‘strings’ in the complex momentum plane. However in this article we shall show that for the derivative $\delta$-function Bose gas with some special values of the coupling constant $\eta$, clusters of bound particles can be constructed in a much simpler way by assigning the corresponding quasi-momenta as equidistant points on a single circle having its centre at the origin of the complex momentum plane. The arrangement of this article is as follows. In Sec. 2, we first briefly review the construction of Bethe eigenstates for the case of the derivative $\delta$-function Bose gas. Then we consider a sufficient condition for which a Bethe eigenstate would represent clusters of bound particles. This sufficient condition for obtaining clusters of bound particles has not attracted much attention in the literature, probably because it does not yield any solution at all for the case of the $\delta$-function Bose gas. However, in Sec. 3, we show that this sufficient condition yields many nontrivial solutions for the case of the derivative $\delta$-function Bose gas. Subsequently, by using some properties of the Farey sequence, we classify all possible solutions of this sufficient condition and obtain different types of clusters of bound particles. In Sec. 4 we discuss various properties of such clusters of bound particles, such as the sizes of the clusters, their stability under the variation of the coupling constant, and their binding energy. We end with some concluding remarks in Sec. 5.
2 Conditions for forming clusters of bound particles

In the coordinate representation, the eigenvalue equation for the Hamiltonian (1.2) of the derivative $\delta$-function Bose gas may be written as

$$\mathcal{H}_N \tau_N(x_1, x_2, \cdots, x_N) = E \tau_N(x_1, x_2, \cdots, x_N),$$

(2.1)

where $\tau_N(x_1, x_2, \cdots, x_N)$ is a completely symmetric $N$-particle wave function. Since $\mathcal{H}_N$ commutes with the total momentum operator given by

$$\mathcal{P}_N = -i\hbar \sum_{j=1}^N \frac{\partial}{\partial x_j},$$

(2.2)

$\tau_N(x_1, x_2, \cdots, x_N)$ can be chosen as a simultaneous eigenfunction of these two commuting operators. It may be noted that the Hamiltonian (1.2) and momentum (2.2) operators enjoy the scaling property $\mathcal{H}_N \to \lambda^2 \mathcal{H}_N$ and $\mathcal{P}_N \to \lambda \mathcal{P}_N$, when all the coordinates are transformed as $x_i \to x_i/\lambda$. Hence, from any given eigenfunction of $\mathcal{H}_N$ and $\mathcal{P}_N$, one can generate a one-parameter family of eigenfunctions by scaling all the $x_i$. It may also be observed that $\mathcal{H}_N$ remains invariant while $\mathcal{P}_N$ changes sign if we change the sign of $\eta$ and transform all the $x_i \to -x_i$ at the same time; such a transformation may be called as ‘parity transformation’. Due to the invariance of $\mathcal{H}_N$ under this parity transformation, it is sufficient to study the eigenvalue problem (2.1) for one particular sign of $\eta$, say, $\eta > 0$. The eigenfunctions for $\eta < 0$ case can then be constructed from those for $\eta > 0$ case by simply changing $x_i \to -x_i$; this leaves all energy eigenvalues invariant but reverses the sign of the corresponding momentum eigenvalues.

For the purpose of solving the eigenvalue problem (2.1) through the coordinate Bethe ansatz, it is convenient to divide the coordinate space $\mathbb{R}^N \equiv \{x_1, x_2, \cdots, x_N\}$ into various $N$-dimensional sectors defined through inequalities like $x_{\omega(1)} < x_{\omega(2)} < \cdots < x_{\omega(N)}$, where $\{\omega(1), \omega(2), \cdots, \omega(N)\}$ represents a permutation of the integers $\{1, 2, \cdots, N\}$. Since the interaction part of the Hamiltonian (1.2) vanishes within each such sector, the resulting eigenfunction can be expressed as a superposition of free particle wave functions. The coefficients associated with these free particle wave functions can be computed by using the interaction part of the Hamiltonian (1.2), which is nontrivial only at the boundary of two adjacent sectors. It is known that all such coefficients, which appear in the Bethe ansatz solution of a $N$-particle system having only local interactions (like the $\delta$-function or derivative $\delta$-function type interactions), can be obtained by simply solving the corresponding two-particle problem [29]. Thus, by using the solutions of the related two-particle problem, it is possible to construct completely symmetric $N$-particle eigenfunctions for the Hamiltonian (1.2). In the region $x_1 < x_2 < \cdots < x_N$, such eigenfunctions can be written in the form [29, 31]

$$\tau_N(x_1, x_2, \cdots, x_N) = \sum_\omega \left( \prod_{l<m} \frac{A(k_{\omega(m)}, k_{\omega(l)})}{A(k_m, k_l)} \right) \rho_{\omega(1), \omega(2), \cdots, \omega(N)}(x_1, x_2, \cdots, x_N),$$

(2.3)

where

$$\rho_{\omega(1), \omega(2), \cdots, \omega(N)}(x_1, x_2, \cdots, x_N) = \exp \left\{ i(k_{\omega(1)} x_1 + \cdots + k_{\omega(N)} x_N) \right\},$$

(2.4)
$k_n$’s are all distinct quasi-momenta, $\omega$ represents an element of the permutation group for the integers $\{1, 2, \ldots, N\}$ and $\sum_\omega$ implies summing over all such permutations. The coefficient $A(k_l, k_m)$ in Eq. (2.3) is obtained by solving the two-particle problem related to the derivative $\delta$-function Bose gas and this coefficient is given by

$$A(k_l, k_m) = \frac{k_l - k_m + i\eta(k_l + k_m)}{k_l - k_m}.$$  \hspace{2cm} (2.5)

The eigenvalues of the momentum (2.2) and Hamiltonian (1.2) operators, corresponding to the eigenfunctions $\tau_N(x_1, x_2, \cdots, x_N)$ of the form (2.3), are easily obtained as

$$\mathcal{P}_N \tau_N(x_1, x_2, \cdots, x_N) = \hbar \left( \sum_{j=1}^{N} k_j \right) \tau_N(x_1, x_2, \cdots, x_N),$$  \hspace{2cm} (2.6a)

$$\mathcal{H}_N \tau_N(x_1, x_2, \cdots, x_N) = \hbar^2 \left( \sum_{j=1}^{N} k_j^2 \right) \tau_N(x_1, x_2, \cdots, x_N).$$  \hspace{2cm} (2.6b)

It should be noted that, Bethe states of the form (2.3) represent scattering as well as bound states for the Hamiltonian (1.2) of the derivative $\delta$-function Bose gas. However, for the case of scattering states all the $k_j$’s are real numbers, while for the case of bound states the $k_j$’s are allowed to take complex values in general. As mentioned above, for a translationally invariant system, a wave function represents a localized bound state if the corresponding probability density decays sufficiently fast when any of the relative coordinates measuring the distance between a pair of particles tends towards infinity. To obtain the condition for which the Bethe state (2.3) would represent such a localized bound state, let us first consider the following wave function in the region $x_1 < x_2 < \cdots < x_N$:

$$\rho_{1,2,\ldots,N}(x_1, x_2, \cdots, x_N) = \exp \left( i \sum_{j=1}^{N} k_j x_j \right),$$  \hspace{2cm} (2.7)

where $k_j$’s in general are complex valued wave numbers. As before, the momentum eigenvalue corresponding to this wave function is given by $\hbar \sum_{j=1}^{N} k_j$. Since this must be a real quantity, one obtains the condition

$$\sum_{j=1}^{N} q_j = 0,$$  \hspace{2cm} (2.8)

where $q_j$ denotes the imaginary part of $k_j$. By using the condition (2.8), the probability density corresponding to the wave function $\rho_{1,2,\ldots,N}(x_1, x_2, \cdots, x_N)$ in (2.7) can be expressed as

$$|\rho_{1,2,\ldots,N}(x_1, x_2, \cdots, x_N)|^2 = \exp \left\{ 2 \sum_{r=1}^{N-1} \left( \sum_{j=1}^{r} q_j \right) y_r \right\},$$  \hspace{2cm} (2.9)
where the \( y_r \)'s are the \( N - 1 \) relative coordinates: 
\[ y_r \equiv x_{r+1} - x_r \]. It is evident that the probability density in (2.9) decays exponentially in the limit 
\( y_r \to \infty \) for one or more values of \( r \), provided that all the following conditions are satisfied:
\[ q_1 < 0 , \quad q_1 + q_2 < 0 , \quad \cdots , \quad \sum_{j=1}^{N-1} q_j < 0 . \]  
(2.10)

It should be observed that the wave function (2.7) is obtained by taking \( \omega \) as the identity permutation in (2.4). However, the Bethe state (2.3) also contains terms like (2.4) with \( \omega \) representing all possible nontrivial permutations. The conditions which ensure the decay of such a term, associated with any nontrivial permutation \( \omega \), are evidently given by
\[ q_{\omega(1)} < 0 , \quad q_{\omega(1)} + q_{\omega(2)} < 0 , \quad \cdots , \quad \sum_{j=1}^{N-1} q_{\omega(j)} < 0 . \]  
(2.11)

It is easy to check that above conditions, in general, contradict the conditions given in Eq. (2.10). To bypass this problem and ensure an overall decaying wave function (2.3), it is sufficient to assume that the coefficients of all terms \( \rho_{\omega(1), \omega(2), \cdots, \omega(N)}(x_1, x_2, \cdots, x_N) \) with nontrivial permutations take the zero value. This leads to a set of relations given by
\[ A(k_r, k_{r+1}) = 0 , \quad \text{for} \quad r \in \Omega_N , \]  
(2.12)

where \( \Omega_N \equiv \{ 1, 2, \cdots, N - 1 \} \). Thus the simultaneous validity of the conditions (2.8), (2.10) and (2.12) ensures that the Bethe state \( \tau_N(x_1, x_2, \cdots, x_N) \) (2.3) represents a bound state.

Let us now discuss how the conditions (2.8), (2.10) and (2.12) can be simplified for the case of the derivative \( \delta \)-function Bose gas. Using the conditions (2.8) and (2.12) along with Eq. (2.5), one can easily derive an expression for all the quasi-momenta as
\[ k_n = \chi e^{-i(N+1-2n)\phi} , \]  
(2.13)

where \( \chi \) is a real, non-zero parameter, and \( \phi \) is related to the coupling constant \( \eta \) as
\[ \phi = \tan^{-1}(\eta) \quad \Rightarrow \quad \eta = \tan \phi . \]  
(2.14)

To obtain an unique value of \( \phi \) from the above equation, it may be restricted to the fundamental region \( -\frac{\pi}{2} < \phi (\neq 0) < \frac{\pi}{2} \). Furthermore, since we have seen that \( \mathcal{H}_N \) (1.2) remains invariant under the ‘parity transformation’, it is enough to study the corresponding eigenvalue problem only within the range \( 0 < \phi < \frac{\pi}{2} \). Next, let us consider the remaining conditions (2.10) for the existence of a localized bound state. Since summation over the imaginary parts of \( k_n \)'s in (2.13) yields
\[ \sum_{j=1}^{l} q_j = -\chi \frac{\sin(l\phi)}{\sin \phi} \sin[(N - l)\phi] , \]  
(2.15)

one can rewrite the conditions (2.10) in the form
\[ \chi \frac{\sin(l\phi)}{\sin \phi} \sin[(N - l)\phi] > 0 , \quad \text{for} \quad l \in \Omega_N . \]  
(2.16)
where \( \Omega_N \) denotes the set of integers \( \{1, 2, \cdots, N - 1\} \). Consequently, for any given values of \( \phi \) and \( N \), a bound state will exist when all the inequalities in Eq. \((2.16)\) are simultaneously satisfied for some real non-zero value of \( \chi \).

Let us make a comment at this point. Within the region \( x_1 < x_2 < \cdots < x_N \), the completely symmetric Bethe eigenfunctions associated with the Hamiltonian \((1.1)\) of the \( \delta \)-function Bose gas can also be expressed through Eq. \((2.3)\), where \( A(k_l, k_m) \) is given by \([1, 9, 29]\)

\[
A(k_l, k_m) = \frac{k_l - k_m - i\mu}{k_l - k_m}.
\]

Hence, the equations \((2.8), (2.10)\) and \((2.12)\) together give a sufficient condition for the existence of bound states for the case of the \( \delta \)-function Bose gas also. By using the above mentioned form of \( A(k_l, k_m) \), it is easy to show that Eq. \((2.12)\) completely fixes the corresponding quasi-momenta as equidistant points on a straight line parallel to the imaginary axis in the complex momentum plane. Moreover, the condition \((2.8)\) ensures that such equidistant points would be symmetric under reflection with respect to the real axis in the complex momentum plane. It is easy to check that the remaining conditions \((2.10)\) for bound state formation are trivially satisfied by those quasi-momenta for any negative value of the coupling constant \( \mu \). Thus, for any \( N \geq 2 \), bound states are formed for the case of the \( \delta \)-function Bose gas for all negative values of the coupling constant. However in the previous paragraph we have seen that, for the case of the derivative \( \delta \)-function Bose gas, the conditions \((2.8)\) and \((2.12)\) lead to Eq. \((2.13)\) implying that the quasi-momenta associated with a bound state are represented through equidistant points on a circle or circular ‘string’ with an arbitrary radius and having its centre at the origin of the complex momentum plane. Consequently, the remaining conditions \((2.10)\) for bound state formation yield Eq. \((2.16)\), which is quite nontrivial in nature. Indeed, by solving this equation, we have found earlier that the derivative \( \delta \)-function Bose gas allows bound states in only certain non-overlapping ranges of the coupling constant \( \eta \) \([42–44]\). Furthermore, such non-overlapping ranges of \( \eta \) crucially depend on the value of \( N \) and they can be determined by using the Farey sequences in number theory.

We would now like to find the conditions for constructing clusters of bound particles in the case of the derivative \( \delta \)-function Bose gas. To this end, we shall first discuss the concept of a ‘clustered state’ for any translationally invariant system, and then give a prescription for finding the Bethe states representing clusters of bound particles. Let us consider a system of \( N \) particles which are divided into some groups or clusters — with at least one group containing more than one particle. It is assumed that particles within the same group behave like the constituents of a bound state, but particles corresponding to different groups behave like the constituents of a scattering state. More precisely, a wave function corresponding to such an \( N \)-particle system satisfies the following two conditions. If the relative distance between any two particles belonging to the same group goes to infinity, the probability density corresponding to the \( N \)-particle wave function decays in the same way as a bound state. On the other hand, if the relative distance between any two particles belonging to different groups tends towards infinity, the probability density remains finite similar to a scattering state. If any wave function corresponding to a \( N \)-particle system satisfies these two conditions, we define it as a
clustered state. It is interesting to observe that, for the case of the \(\delta\)-function Bose gas, the Bethe states corresponding to clusters of bound particles can always be expressed as linear superpositions of several clustered and bound states, with the restriction that at least one clustered state must be present in such a superposition \[9,14\]. This observation may be used to define the Bethe states corresponding to clusters of bound particles for the case of any translationally invariant system. More precisely, we assume that a Bethe state for any translationally invariant system would represent clusters of bound particles if it can be expressed as some linear superposition of clustered and bound states, with the restriction that at least one clustered state must be present in this superposition.

Next, let us discuss how the conditions for constructing a clustered state can be implemented for the case of the plane wave function \(2.7\). Since any eigenvalue of the momentum operator \(P_N\) must be a real quantity, Eq. \(2.8\) is also obeyed for this case. To proceed further, let us choose a specific value of \(N\) given by \(N = 4\). For this case, the probability density \(2.9\) may be explicitly written as

\[
|\rho_{1,2,\ldots,4}(x_1, x_2, \ldots, x_4)|^2 = \exp\left\{2q_1y_1 + 2(q_1 + q_2)y_2 + 2(q_1 + q_2 + q_3)y_3\right\}. \tag{2.17}
\]

Suppose, the conditions \(2.10\) for a bound state formation are slightly modified for this case as

\[
q_1 < 0, \quad q_1 + q_2 = 0, \quad q_1 + q_2 + q_3 < 0.
\]

Taking into account this new condition, it is easy to see that when \(y_1 = x_2 - x_1\) or \(y_3 = x_4 - x_3\) tends towards infinity, the probability density in Eq. \(2.17\) still decays like a bound state. On the other hand, when \(y_2 = x_3 - x_2\) tends towards infinity, the probability density in Eq. \(2.17\) remains finite. Hence, the clusters of particles given by \(\{1, 2\}\) and \(\{3, 4\}\) satisfy all the criteria of a clustered state. Generalizing this specific example, we replace some of the inequalities in Eq. \(2.10\) by equalities. More precisely, we modify the conditions in Eq. \(2.10\) as

\[
\sum_{i=1}^{l} q_i = 0, \quad \text{for} \quad l \in \tilde{\Omega}_N, \tag{2.18a}
\]

\[
\sum_{i=1}^{l} q_i < 0, \quad \text{for} \quad l \in (\Omega_N - \tilde{\Omega}_N), \tag{2.18b}
\]

where \(\tilde{\Omega}_N\) is any non-empty proper subset of \(\Omega_N\) and \((\Omega_N - \tilde{\Omega}_N)\) is the complementary set of \(\tilde{\Omega}_N\). Let us now assume that the quasi-momenta associated with the plane wave function \(2.7\) satisfy the relations \(2.8\) and \(2.18a,b\), where the set \(\tilde{\Omega}_N\) contains \(p\) number of elements. Due to Eq. \(2.9\) it is evident that this plane wave function would represent a clustered state containing \((p + 1)\) number of clusters.

Finally, we try to find the simplest possible condition for which the Bethe state \(2.3\) would represent clusters of bound particles. Let us assume that the quasi-momenta corresponding to this Bethe state satisfy the relations \(2.12\). As a result, the coefficients of all plane waves except \(2.7\) take the zero value within the Bethe state \(2.3\). Thus, due to the relations \(2.12\), the Bethe state \(2.3\) would reduce to the plane wave function \(2.7\). Next, we assume that the quasi-momenta corresponding to this Bethe state
also satisfy the relations (2.8) and (2.18a,b). Hence, the plane wave function (2.7) represents a clustered state. Consequently, Eqs. (2.8), (2.12) and (2.18a,b) together yield a sufficient condition for which the Bethe state (2.3) would represent clusters of bound particles. It is obvious that such a Bethe state is expressed through a single clustered state, instead of a linear superposition of several clustered and bound states.

We would like to make a remark at this point. It may be noted that, for the case of the \( \delta \)-function Bose gas, it is not possible to find any set of quasi-momenta which simultaneously satisfy the equations (2.8), (2.12) and (2.18a,b). Indeed we have already mentioned that, for this case, the conditions (2.8) and (2.12) completely fix the corresponding quasi-momenta as equidistant and reflection symmetric points on a straight line parallel to the imaginary axis in the complex momentum plane. It is easy to check that such quasi-momenta do not satisfy Eq. (2.18a). To bypass this problem, it should be noted that Eqs. (2.8), (2.12) and (2.18a,b) together yield only a sufficient condition, but not the necessary condition for which a Bethe state leads to clusters of bound particles. In fact, it is possible to take a partially relaxed version of Eq. (2.12) given by

\[
A(k_r, k_{r+1}) = 0, \quad \text{for} \quad r \in \Omega_N', \tag{2.19}
\]

where \( \Omega_N' \) is any non-empty proper subset of \( \Omega_N \). However, the corresponding Bethe state (2.3) not only contains the plane wave (2.7), but also plane waves like (2.4) with \( \omega \) taking values within a subset of all possible nontrivial permutations. The Bethe state (2.3) would represent clusters of bound particles if each of these plane waves associated with nontrivial permutations behaves like either a bound state or a clustered state. Hence, as a consequence of taking Eq. (2.19) instead of Eq. (2.12), Eq. (2.11) or analogues of Eqs. (2.18a,b) for the required values of nontrivial permutation \( \omega \) must also be satisfied. For the case of the \( \delta \)-function Bose gas, a solution of Eqs. (2.8) and (2.19) yields \( N \) number of quasi-momenta which can be represented through reflection symmetric points on several (more than one) ‘strings’ parallel to the imaginary axis in the complex momentum plane. It can be shown that, apart from satisfying Eqs. (2.18a,b), those quasi-momenta also satisfy Eq. (2.11) or analogues of Eqs. (2.18a,b) for the required values of nontrivial permutation \( \omega \). As a result, the Bethe state corresponding to those quasi-momenta represents clusters of bound particles for the case of the \( \delta \)-function Bose gas. Evidently, such a Bethe state is expressed through a superposition of several clustered and bound states.

In the next section our aim will be to show that, unlike the case of the \( \delta \)-function Bose gas, it is possible to simultaneously solve the equations (2.8), (2.12) and (2.18a,b) for the case of the derivative \( \delta \)-function Bose gas. To this end, let us discuss how to simplify these three equations for the latter case. Since the exact form of the set \( \tilde{\Omega}_N \) appearing in (2.18a,b) may depend on the value of \( \phi \) for the case of the derivative \( \delta \)-function Bose gas, in the following we shall replace the notation \( \tilde{\Omega}_N \) by \( \Omega_{N,\phi} \). Using Eqs. (2.8) and (2.12) along with the form of \( A(k_l, k_m) \) given in (2.3) it is easy to see that, similar to the case of a localized bound state, the quasi-momenta associated with clusters of bound particles can be written in the form (2.13) and the imaginary parts of these quasi-momenta satisfy the relation (2.15). By using Eq. (2.15), we can recast the remaining conditions (2.18a,b) for the formation of clusters of bound particles (for any
given values of $\phi$ and $N$) as
\[
\chi \frac{\sin(l\phi)}{\sin \phi} \sin[(N - l)\phi] = 0, \quad \text{for } l \in \Omega_{N,\phi}, \tag{2.20a}
\]
\[
\chi \frac{\sin(l\phi)}{\sin \phi} \sin[(N - l)\phi] > 0, \quad \text{for } l \in (\Omega_N - \Omega_{N,\phi}), \tag{2.20b}
\]
where $\chi$ is any non-zero real number, $\Omega_{N,\phi}$ is any non-empty proper subset of $\Omega_N$ and $(\Omega_N - \Omega_{N,\phi})$ is the complementary set of $\Omega_{N,\phi}$. Let us assume that Eqs. (2.20a,b) are satisfied for some values of $\phi$ and $N$, where $\Omega_{N,\phi}$ is given by
\[
\Omega_{N,\phi} = \{l_1, l_2, \ldots, l_p\}, \tag{2.21}
\]
with $1 \leq p < N - 1$. Then from Eq. (2.20a) it follows that the sets of particles given by $\{1, \ldots, l_1\}, \{l_1 + 1, \ldots, l_2\}, \ldots, \{l_{p-1} + 1, \ldots, l_p\}, \{l_p + 1, \ldots, N\}$ represent $(p + 1)$ number of clusters of bound particles. Moreover, the numbers of particles present within each of these clusters, i.e., the size of the clusters, may be written in the form
\[
\{l_1, l_2 - l_1, \ldots, l_p - l_{p-1}, N - l_p\}. \tag{2.22}
\]

Since the quasi-momenta associated with both bound states and clusters of bound particles are given by Eq. (2.13), the momentum and energy eigenvalues for clusters of bound particles can be derived in exactly the same way as has been done earlier [42] for the case of a bound state. Inserting the quasi-momenta given in Eq. (2.13) to Eqs. (2.6a,b), we obtain the momentum eigenvalue as
\[
P = \hbar \chi \frac{\sin(N\phi)}{\sin \phi}, \tag{2.23}
\]
and the energy eigenvalue as
\[
E = \frac{\hbar^2 \chi^2 \sin(2N\phi)}{\sin(2\phi)}. \tag{2.24}
\]

3 Farey sequences and clusters of bound particles

In this section, we shall try to find all possible solutions of the sufficient conditions (2.20a,b) for constructing clusters of bound particles in the case of the derivative $\delta$-function Bose gas. Some properties of the Farey sequences [45] in number theory will play a crucial role in our analysis. Due to the existence of the parity transformation, as mentioned in the earlier section, it is sufficient to concentrate on values of $\phi$ lying in the range $0 < \phi < \frac{\pi}{2}$. Within this range of $\phi$, $\sin \phi > 0$ and hence the conditions (2.20a,b) for forming clusters of bound particles reduce to
\[
\chi \sin(l\phi) \sin[(N - l)\phi] = 0, \quad \text{for } l \in \Omega_{N,\phi}, \tag{3.1a}
\]
\[
\chi \sin(l\phi) \sin[(N - l)\phi] > 0, \quad \text{for } l \in (\Omega_N - \Omega_{N,\phi}). \tag{3.1b}
\]
Let us first try to find the values of $\phi$ for which Eq. (3.1a) holds true. For any given $\phi$, this equation would be satisfied for some value of $l$ if either $\sin l\phi = 0$ or $\sin[(N-l)\phi] = 0$. For $\sin l\phi = 0$, $l\phi = k\pi$ and so $\phi/\pi = k/l$ where $k$ is an integer. Since $l \in \Omega_{N,\phi}$, the denominator in the expression of $\phi/\pi (= k/l)$ is always less than $N$. Similarly, for $\sin[(N-l)\phi] = 0$, $\phi/\pi = m/(N-l)$ where $m$ is an integer. Since $l \in \Omega_{N,\phi}$, in this case also, the denominator in the expression of $\phi/\pi (= m/(N-l))$ is less than $N$ for all values of $l$. Hence it follows that, the condition (3.1a) would be satisfied if and only if $\phi/\pi$ can be expressed in the form

$$\frac{\phi}{\pi} = \frac{a}{b},$$

(3.2)

where $\{a, b\}$ are relatively prime integers (i.e, the greatest common divisor of $a$ and $b$ is 1), taking values within the ranges

$$0 < a < \frac{b}{2}, \quad 2 < b \leq N - 1.$$  

(3.3a, b)

Due to Eq. (3.3b) it is evident that, clusters of bound particles can exist only for $N \geq 4$. In the following, we shall establish a connection of the fractions $\phi/\pi$, given by Eqs. (3.2) and (3.3a,b), with the Farey sequences in number theory.

For a positive integer $N$, the Farey sequence is defined to be the set of all the fractions $a/b$ in increasing order such that (i) $0 \leq a \leq b \leq N$, and (ii) $\{a, b\}$ are relatively prime integers [15]. The Farey sequences for the first few values of $N$ are given by

$$F_1: \begin{array}{cc}
0 & 1 \\
1 & 1
\end{array}$$

$$F_2: \begin{array}{ccc}
0 & 1 & 1 \\
1 & 2 & 1
\end{array}$$

$$F_3: \begin{array}{cccc}
0 & 1 & 1 & 2 \\
1 & 3 & 2 & 1
\end{array}$$

$$F_4: \begin{array}{cccccc}
0 & 1 & 1 & 1 & 2 & 3 \\
1 & 4 & 3 & 2 & 3 & 4 & 1
\end{array}$$

$$F_5: \begin{array}{cccccc}
0 & 1 & 1 & 1 & 2 & 3 \\
1 & 5 & 4 & 3 & 5 & 2 & 5 & 3 & 4 & 5 & 1
\end{array}$$

(3.4)

These sequences enjoy several properties, of which we list the relevant ones below.

(i) Let $a/b, a'/b'$ are two fractions appearing in the Farey sequence $F_N$. Then $a/b < a'/b'$ ( $a'/b' < a/b$ ) are two successive fractions in $F_N$, if and only if the following two conditions are satisfied:

$$a'b - ab' = 1 (-1),$$

(3.5a)

$$b + b' > N.$$  

(3.5b)

It then follows that both $a$ and $b'$ are relatively prime to $a'$ and $b$.

(ii) For $N \geq 2$, if $n/N$ is a fraction appearing somewhere in the sequence $F_N$ (this implies that $N$ and $n$ are relatively prime according to the definition of $F_N$), then the
fractions \(a_1/b_1\) and \(a_2/b_2\) appearing immediately to the left and to the right respectively of \(n/N\) satisfy

\[
\begin{align*}
    a_1, a_2 &\leq n, \quad \text{and} \quad a_1 + a_2 = n, \\
    b_1, b_2 &< N, \quad \text{and} \quad b_1 + b_2 = N.
\end{align*}
\]

(3.6)

To apply the above mentioned Farey sequence in the present context, let us define a subset of \(F_N\) as

\[
F'_N = \left\{ \frac{a}{b} \left| \frac{a}{b} \in F_N, \quad 0 < \frac{a}{b} < \frac{1}{2} \right. \right\},
\]

(3.7)

and a subset of \(F'_N\) as

\[
F''_N = \left\{ \frac{n}{N} \left| \frac{n}{N} \in F'_N \right. \right\}.
\]

(3.8)

Using these definitions of various subsets of a Farey sequence, we find that

\[
F'_N = F'_{N-1} \cup F''_N.
\]

(3.9)

Furthermore, it is worth noting that, Eqs. (3.2) and (3.3a,b) can equivalently be expressed as

\[
\frac{\phi}{\pi} \in F'_{N-1}.
\]

(3.10)

Consequently, it follows that the condition (3.1a) for cluster formation is obeyed if and only if \(\phi/\pi \in F'_{N-1}\). In this context it may be observed that, due to Eq. (3.9), all the elements of \(F'_{N-1}\) are also present in \(F'_N\). By using such an embedding of \(F'_{N-1}\) into \(F'_N\), we find that any fraction \(a/b \in F'_{N-1}\) belongs to one of the four distinct classes, which are defined in the following:

I. At least one of the fractions nearest to \(a/b\) (from either the left or the right side) in the sequence \(F'_N\) lies in the set \(F''_N\). Then, from a property of the Farey sequences, it follows that \(\{b, N\}\) are relatively prime integers in this case.

II. None of the nearest fractions of \(a/b\) (from the left or right side) in the sequence \(F'_N\) lies in the set \(F''_N\), and \(\{b, N\}\) are relatively prime integers.

III. \(N\) is divisible by \(b\). Clearly, \(\{b, N\}\) are not relatively prime integers in this case.

IV. \(N\) is not divisible by \(b\), and \(\{b, N\}\) are not relatively prime integers.

To demonstrate the above mentioned classification through an example, let us choose \(N = 6\). For this case, the sets \(F'_5\), \(F'_6\) and \(F''_6\) are given by

\[
F'_5: \quad \frac{1}{5}, \quad \frac{1}{4}, \quad \frac{1}{3}, \quad \frac{2}{5}; \quad F'_6: \quad \frac{1}{6}, \quad \frac{1}{5}, \quad \frac{1}{4}, \quad \frac{1}{3}, \quad \frac{2}{5}; \quad F''_6: \quad \frac{1}{6}.
\]

Using the embedding of \(F'_5\) into \(F'_6\), it is easy to verify that each fraction in \(F'_5\) falls under one of the four classes discussed above. More precisely, the fractions 1/5, 1/4,
$1/3$ and $2/5$ belong to type I, type IV, type III and type II respectively. Returning back to the general case we note that, for any fraction $a/b \in F'_{N-1}$, $\{b, N\}$ are either relatively prime integers or not relatively prime integers. If $\{b, N\}$ are relatively prime integers, then it is obvious that $a/b$ must be an element of either type I or type II. On the other hand, if $\{b, N\}$ are not relatively prime integers, then $a/b$ must be an element of either type III or type IV. In this way, one can show that any fraction $a/b \in F'_{N-1}$ belongs to one of these four distinct classes. This type of classification for the elements of $F'_{N-1}$ will shortly play an important role in our analysis on the formation of clusters of bound particles.

Next, we try to find the elements of $F'_{N-1}$ which would satisfy the remaining condition (3.1b) for cluster formation. For any $a/b \in F'_{N-1}$, one can express the rational number $Na/b$ as

$$\frac{Na}{b} = \left\lfloor \frac{Na}{b} \right\rfloor + \rho_b,$$  \hspace{0.5cm} (3.11)

where $\lfloor x \rfloor$ denotes the integer part of $x$ and $\rho \in \{0, 1, \ldots, b-1\}$. The above equation can also be written in the form

$$aN - bt = \delta,$$  \hspace{0.5cm} (3.12)

where the integers $t$ and $\delta$ are defined as

$$t = \left\lfloor \frac{Na}{b} \right\rfloor, \quad \delta = \rho, \quad \text{if} \quad \frac{\rho}{b} \leq \frac{1}{2},$$

$$t = \left\lfloor \frac{Na}{b} \right\rfloor + 1, \quad \delta = \rho - b, \quad \text{if} \quad \frac{\rho}{b} > \frac{1}{2}. \hspace{0.5cm} (3.13)$$

One can derive several bounds on the values of $t$ and $\delta$. By using Eq. (3.13), it is easy to show that

$$\frac{|\delta|}{b} \leq \frac{1}{2}. \hspace{0.5cm} (3.14)$$

Using Eqs. (3.3a,b) we obtain $Na/b \geq N/(N-1) > 1$, which in turn yields $t \geq 1$. Moreover, with the help of Eqs. (3.12) and (3.13), we also find that

$$\frac{t}{N} < \frac{1}{2}. \hspace{0.5cm} (3.15)$$

The derivation of Eq. (3.15) is given in Appendix A.

It may be observed that, for any given $a/b \in F'_{N-1}$, the values of $t$ and $\delta$ are uniquely determined through Eqs. (3.11) and (3.13). Thus, we get a mapping of the form

$$(a, b, N) \rightarrow (t, \delta). \hspace{0.5cm} (3.16)$$

In the following, we shall show that this mapping considerably simplifies the analysis of Eq. (3.1b) by casting it in an alternative form. To this end, let us define a function $f(l, N, \phi)$ as

$$f(l, N, \phi) = \chi \sin l\phi \sin(N-l)\phi. \hspace{0.5cm} (3.17)$$
Substituting the expression of $\phi$ given in Eq. (3.2) to the above equation, we get

$$f(l, N, \phi) = \chi \sin \left( \frac{\pi al}{b} \right) \sin \left\{ \frac{\pi a}{b} (N - l) \right\}.$$  

Next, by using the relation (3.12), we express $f(l, N, \phi)$ in the form

$$f(l, N, \phi) = \chi (-1)^{t+1} \sin \left( \frac{\pi al}{b} \right) \sin \left\{ \frac{\pi a}{b} (l - \delta) \right\}.$$  \hspace{1cm} (3.18)

Let us now define a function $g(x, \phi)$ as

$$g(x, \phi) = \sin \left( \frac{\pi ax}{b} \right),$$  \hspace{1cm} (3.19)

where $x$ denotes a continuous real variable. Then Eq. (3.18) can be written as

$$f(l, N, \phi) = \chi (-1)^{t+1} g(l, \phi) g(l - \Delta, \phi),$$  \hspace{1cm} (3.20)

where the parameter $\Delta$ is given by

$$\Delta = \frac{\delta}{a}.$$  \hspace{1cm} (3.21)

Note that the zero values of the function $g(x, \phi)$ in the variable $x$ are given by

$$x = \frac{mb}{a}, \quad m = 0, \pm 1, \pm 2, \ldots.$$  \hspace{1cm} (3.22)

Let us consider a value of $l$ such that $f(l, N, \phi) \neq 0$, i.e., $l \in \Omega_N - \Omega_{N,\phi}$. Due to Eq. (3.20), it is evident that both $g(l, \phi)$ and $g(l - \Delta, \phi)$ take nonzero values for such $l$. Since $g(x, \phi)$ is a continuous function of the variable $x$, we can write

$$\text{sgn} \left[ g(l - \Delta, \phi) \right] = (-1)^{\rho(l)} \text{sgn} \left[ g(l, \phi) \right],$$  \hspace{1cm} (3.23)

where $\text{sgn}$ denotes the sign function and $\rho(l)$ represents the number of zero points of the function $g(x, \phi)$ within the interval $l - \Delta < x < l$ ($l < x < l - \Delta$) when $\Delta > 0$ ($\Delta < 0$). It is evident that $\rho(l) = 0$ for all values of $l \in \Omega_N - \Omega_{N,\phi}$, when $\Delta = 0$. Now, by using Eqs. (3.20) and (3.23), we find that

$$\text{sgn} \left[ f(l, N, \phi) \right] = \text{sgn} \left[ \chi (-1)^{t+1+\rho(l)} \right].$$  \hspace{1cm} (3.24)

With the help of the above equation, we can express Eq. (3.1b) in an alternative form given by

$$\text{sgn} \left[ \chi (-1)^{t+1+\rho(l)} \right] = 1, \quad \text{for all} \ l \in \Omega_N - \Omega_{N,\phi}.$$  \hspace{1cm} (3.25)

Due to Eq. (3.22), it follows that the distance between two consecutive zero points of the function $g(x, \phi)$ is given by $b/a$. Moreover, by using Eqs. (3.14) and (3.21), we find that

$$\frac{b}{a} \geq 2|\Delta|.$$  \hspace{1cm} (3.26)
Hence, the value of ρ(l) can be either 0 or 1 for each l. Since the parameter χ does not depend on the value of l, it is evident that Eq. (3.25) would be satisfied if either

$$\rho(l) = 0, \text{ for all } l \in \Omega_N - \Omega_{N,\phi},$$

and χ is chosen such that $\text{sgn}(\chi) = (-1)^{t+1}$, or

$$\rho(l) = 1, \text{ for all } l \in \Omega_N - \Omega_{N,\phi},$$

and χ is chosen such that $\text{sgn}(\chi) = (-1)^t$. However, for any $a/b \in F'_{N-1}$, it can be shown that there exists at least one $l \in \Omega_N - \Omega_{N,\phi}$ for which $\rho(l) = 0$. The proof of this statement is given in Appendix B. Thus, it is never possible to satisfy the condition given in Eq. (3.28). Consequently, for any given $a/b \in F'_{N-1}$, clusters of bound particles can be obtained if and only if Eq. (3.27) is satisfied and χ is chosen such that $\text{sgn}(\chi) = (-1)^{t+1}$.

On the other hand, clusters of bound particles cannot be formed if Eq. (3.27) is violated, i.e., if the following condition is satisfied:

$$\rho(l) = 1, \text{ for at least one } l \in \Omega_N - \Omega_{N,\phi}.$$  \hspace{1cm} (3.29)

Previously, we have shown that all elements of $F'_{N-1}$ can be divided into four distinct classes. In the following, we shall analyze each class separately and examine whether the elements belonging to each class satisfy Eq. (3.27) or Eq. (3.29).

**Analysis of Case I:** Let $\phi/\pi = a/b$ be a fraction of type I within the set $F'_{N-1}$. In this case, there exists a fraction $n/N \in F''_N$, such that

$$F'_N: \cdots \frac{n}{N} \frac{a}{b} \cdots \text{ or } \cdots \frac{a}{b} \frac{n}{N} \cdots .$$

Hence, using the property of Farey sequences given in Eq. (3.5a), we get

$$aN - bn = \pm 1.$$ \hspace{1cm} (3.30)

Comparing the above equation with (3.12), we find that

$$\frac{\delta}{b} = n - t \pm \frac{1}{b}.$$ \hspace{1cm} (3.31)

Combining Eqs. (3.31) and (3.14), one obtains an inequality of the form

$$\left| n - t \pm \frac{1}{b} \right| \leq \frac{1}{2}.$$ \hspace{1cm} (3.32)

Since, due to Eq. (3.3b) it follows that $1/b \leq 1/3$, one can easily show that Eq. (3.32) would be satisfied if and only if $t = n$. Since $\{n, N\}$ are relatively prime integers, it follows that $\{t, N\}$ are relatively prime integers for all fractions of type I. Substituting $n$ in the place of $t$ in Eq. (3.12), we get

$$aN - bn = \delta.$$ \hspace{1cm} (3.33)
Comparing Eqs. (3.30) and (3.33), we find that for any fraction of type I, the value of \( \delta \) is given by

\[
\delta = \pm 1. \tag{3.34}
\]

Next, for the sake of convenience, we consider only those fractions of type I, which yield \( \delta = +1 \). By using Eq. (3.21), we obtain the corresponding value of \( \triangle \) as

\[
\triangle = \frac{1}{a}. \tag{3.35}
\]

Let us take any value of \( l \) such that \( f(l, N, \phi) \neq 0 \), i.e., \( l \in \Omega_N - \Omega_{N,\phi} \). Using Eqs. (3.20), (3.22) and (3.35), we find that this \( l \) satisfies the relations

\[
l \neq \frac{mb}{a}, \quad l - \frac{1}{a} \neq \frac{mb}{a}, \tag{3.36}
\]

for any integer value of \( m \). In the following, we shall try to find if any zero points of the function \( g(x, \phi) \) exists within the range \( l - 1/a < x < l \), i.e., whether \( mb/a \in (l - 1/a, l) \) for any integer value of \( m \). To this end, we express \( mb/a \) in the form

\[
\frac{mb}{a} = \left[ \frac{mb}{a} \right] + \frac{c}{a}, \tag{3.37}
\]

where \( c \in \{0, 1, 2, \cdots , a - 1\} \). Let us now assume that \( mb/a \in (l - 1/a, l) \). Then, by using Eq. (3.37), we get the relation

\[
l - \frac{1+c}{a} < \left[ \frac{mb}{a} \right] < l - \frac{c}{a} . \tag{3.38}
\]

However, for \( c \in \{0, 1, 2, \cdots , a - 1\} \), one also finds that

\[
l - 1 \leq l - \frac{1+c}{a}, \quad l - \frac{c}{a} \leq l .
\]

Combining the above inequalities with those given in Eq. (3.38), we obtain the relation

\[
l - 1 < \left[ \frac{mb}{a} \right] < l ,
\]

which evidently leads to a contradiction. Hence it is established that, for any fraction of type I which gives \( \delta = 1 \) and for any \( l \in \Omega_N - \Omega_{N,\phi} \), the condition \( mb/a \in (l - 1/a, l) \) can never be satisfied. As a result, Eq. (3.27) is obeyed for all fractions of type I which yield \( \delta = 1 \). Similarly, it can be shown that Eq. (3.27) is also obeyed for all fractions of type I which yield \( \delta = -1 \). Consequently, we find that clusters of bound particles are formed for all fractions of type I. We have already seen that the relation \( t = n \) holds for all of these fractions. Therefore, according to the discussion just below Eq. (3.27), the sign of \( \chi \) should be chosen for this type of clusters of bound particles as \( sgn(\chi) = (-1)^{n+1} \).

Analysis of Case II: Let us now consider the fractions of type II within the set \( F'_{N-1} \). At first our aim is to show that

\[
|\delta| > 1 , \tag{3.39}
\]
for all fractions of this type. By using Eqs. (3.11) and (3.13), one can calculate the values of $t$ and $\delta$ for these fractions. In contrast to the case of fractions of type I for which $\{t, N\}$ are always relatively prime integers, for the present case $\{t, N\}$ may or may not be relatively prime integers. As an example, let us take $N = 7$, for which $a/b = 1/5$ is a type II fraction. Since $7/5 = 1 + 2/5$, using (3.13) we find that $t = 1$ and $\delta = 2$. Hence, $\{t, N\}$ are relatively prime integers in this case. On the other hand, we may also take $N = 6$, for which $a/b = 2/5$ is a type II fraction. Since $6 \times 2/5 = 2 + 2/5$, it follows that $t = 2$ and $\delta = 2$. Hence, $\{t, N\}$ are not relatively prime integers in this case.

Let us first consider all fractions of type II within the set $F'_{N-1}$, for which $\{t, N\}$ are relatively prime integers. As we have seen that $t \geq 1$ and Eq. (3.15) also gives $t/N < 1/2$, it is evident that $t/N \in F''_N \subset F'_N$. (3.40)

However, since $a/b$ is a fraction of type II, $a/b$ and $t/N$ cannot be two consecutive fractions in the sequence $F'_N$. Therefore, at least one of the two relations given in Eqs. (3.5a) and (3.5b) must be violated if we choose $a' = t$, $b' = N$. Since Eq. (3.5b) is obviously satisfied for this case, Eq. (3.5a) must be violated. As a result, we obtain

$$|aN - bt| \neq 1.$$ (3.41)

Next, let us assume that $aN - bt = 0$. Since $\{a, b\}$ and $\{t, N\}$ are both relatively prime pairs, the relation $aN - bt = 0$ implies that $a = t$ and $b = N$. However, since $a/b \in F'_{N-1}$, the relation $b = N$ evidently leads to a contradiction. Thus, it follows that

$$aN - bt \neq 0.$$ (3.42)

Since $a, b, t, N$ are all integer numbers, combining Eqs. (3.41) and (3.42) we find that

$$|aN - bt| > 1.$$ (3.43)

Using Eqs. (3.33) and (3.12), one can easily establish the validity of Eq. (3.39).

Next, we consider all fractions of type II within the set $F'_{N-1}$, for which $\{t, N\}$ are not relatively prime integers. In this case, one can express $t$ and $N$ as

$$t = \alpha t', \quad N = a N',$$ (3.44a,b)

where $\{t', N'\}$ are relatively prime integers and $\alpha (> 1)$ is an integer. Using Eqs. (3.12) and (3.44a,b), we obtain

$$|\delta| = \alpha |aN' - bt'|.$$ (3.45)

Let us now assume that $aN' - bt' = 0$, i.e., $a/b = t'/N'$. Since $\{a, b\}$ and $\{t', N'\}$ are both relatively prime pairs, the relation $a/b = t'/N'$ implies that $a = t'$ and $b = N'$. Thus, by using Eq. (3.44b) we obtain $N = \alpha b$. However, the relation $N = \alpha b$ clearly contradicts our initial assumption of $a/b$ being a fraction of type II, for which $\{b, N\}$
must be relatively prime integers. Consequently, we find that \(aN' - bt' \neq 0\). Since \(a, b, t', N'\) are all integer numbers, it follows that

\[
|aN' - bt'| \geq 1. \quad (3.46)
\]

Using Eqs. (3.45), (3.46) and the relation \(\alpha > 1\), one can easily establish the inequality given in (3.39).

Next, our aim is to show how Eq. (3.29) follows from the inequality (3.39), which is obeyed by all fractions of type II. To this end, let us assume that \(c/d\) is the nearest fraction of \(a/b\) in the sequence \(F_N\), either from the left side, i.e.,

\[
F_N : \ldots \ldots \ldots, \frac{c}{d}, \frac{a}{b}, \ldots \ldots \quad (3.47a)
\]

or, from the right side, i.e.,

\[
F_N : \ldots \ldots \ldots, \frac{a}{b}, \frac{c}{d}, \ldots \ldots \quad (3.47b)
\]

Since \(a/b\) is a fraction of type II, it is evident that \(d \leq N - 1\) (note that this statement remains valid for any \(a/b\) which is not a fraction of type I). By using the method of contradiction, in the following we shall show that \(d \in \Omega_N - \Omega_{N, \phi}\) for both of the cases (3.47a) and (3.47b). Let us first assume that \(x = d\) is a zero point of the function \(g(x, \phi)\). Thus, by using (3.22) we obtain \(d = mb/a\), i.e.,

\[
m = \frac{a}{b}d.
\]

Since \(\{a, b\}\) are relatively prime integers, the above equation can only be satisfied if \(d = \alpha b\), where \(\alpha\) is a positive integer. However the relation \(d = \alpha b\) leads to a contradiction, since the denominators of two consecutive fractions in a Farey sequence must be relatively prime integers. Hence, we find that \(x = d\) is not a zero point of the function \(g(x, \phi)\). Next we assume that \(x = d\) is a zero point of the function \(g(x - \Delta, \phi)\). Thus, by using (3.22) we obtain \(d - \delta/a = m'b/a\), i.e.,

\[
ad - m'b = \delta.
\]

Equating the (left hand side) l.h.s. of the above equation with that of Eq. (3.12), we get

\[
\frac{a}{b} = \frac{t - m'}{N - d}.
\]

Since \(\{a, b\}\) are relatively prime numbers, the above relation implies that \(t - m' = \beta a\) and \(N - d = \beta b\), where \(\beta\) is a positive integer. However, it is easy to see that the relation \(N - d = \beta b\) contradicts Eq. (3.5b), which is satisfied by the denominators of two consecutive fractions in a Farey sequence. Hence, we find that \(x = d\) is not a zero point of either \(g(x, \phi)\) or \(g(x - \Delta, \phi)\), it is established that \(d \in \Omega_N - \Omega_{N, \phi}\) for both of the cases (3.47a) and (3.47b).
Due to Eq. (3.39), all fractions of type II can be subdivided into two classes characterized by \( \delta > 1 \) and \( \delta < -1 \) respectively. Let us first consider all fractions of type II for which \( \delta > 1 \). For this case, we assume that \( c/d \) is the nearest fraction of \( a/b \) (in the sequence \( F_N \)) from its left side, as shown in Eq. (3.47a). Consequently, by using (3.5a), we obtain

\[
ad - bc = 1.
\] (3.48)

By choosing \( m = c \) in Eq. (3.22), one finds that \( x = cb/a \) is a zero point of the function \( g(x, \phi) \). By using Eq. (3.48), we calculate the difference between \( d \) and \( cb/a \) as

\[
D \left( d, \frac{cb}{a} \right) = d - \frac{cb}{a} = \frac{1}{a}.
\] (3.49)

Since we are considering the case \( \delta > 1 \), Eq. (3.21) yields \( \Delta > 1/a \). As a result, Eq. (3.49) leads to an inequality given by

\[
D \left( d, \frac{cb}{a} \right) < \Delta,
\] (3.50)

which shows that the zero point \( x = cb/a \) of the function \( g(x, \phi) \) lies within the interval \((d - \Delta, d)\). Consequently, it is established that \( \rho(d) = 1 \) for all fractions of type II with \( \delta > 1 \).

Next, let us consider all fractions of type II for which \( \delta < -1 \). For this case, we assume that \( c/d \) is the nearest fraction of \( a/b \) (in the sequence \( F_N \)) from its right side, as shown in Eq. (3.47b). Consequently, by using (3.5a), we obtain

\[
ad - bc = -1.
\] (3.51)

By using the above relation, we obtain the distance between \( cb/a \) and \( d \) as

\[
D \left( \frac{cb}{a}, d \right) = \frac{cb}{a} - d = \frac{1}{a}.
\] (3.52)

Since we are considering the case \( \delta < -1 \), Eq. (3.21) yields \( 1/a < -\Delta \). As a result, Eq. (3.52) leads to an inequality given by

\[
D \left( \frac{cb}{a}, d \right) < -\Delta,
\] (3.53)

showing that the zero point \( x = cb/a \) of the function \( g(x, \phi) \) lies within the interval \((d, d - \Delta)\). Consequently, it is established that \( \rho(d) = 1 \) for all fractions of type II with \( \delta < -1 \).

From the above analysis, it is clear that the condition (3.29) is obeyed for all fractions of type II. Consequently, clusters of bound particles cannot be formed for any fraction of this type.

**Analysis of Case III:** Let \( \phi/\pi = a/b \) be a fraction of type III within the set \( F_{N-1}' \).
Since $N$ is divisible by $b$ in this case, we may write $N = pb$, where $p \ (> 1)$ is an integer. Using Eqs. (3.11) and (3.13), we find that $t = ap$ and $\delta = 0$ for this case. Substituting $\delta = 0$ in Eq. (3.21), we obtain $\Delta = 0$. Hence, due to Eq. (3.23), it trivially follows that $\rho(l) = 0$ for all values of $l \in \Omega_N - \Omega_N,\phi$, i.e., Eq. (3.27) is satisfied for this case. Therefore, we find that clusters of bound particles are formed for all fractions of type III. According to the discussion just below Eq. (3.27), the sign of $\chi$ should be chosen for this type of clusters of bound particles as $sgn(\chi) = (-1)^{ap+1}$.

**Analysis of Case IV:** Let $\phi/\pi = a/b$ be a fraction of type IV within the set $F'_{N-1}$. For this case, we can write

$$b = \alpha b', \quad N = \alpha N' ,$$

where $\alpha, b', N'$ are some integers such that $\alpha > 1, b' > 1$ and $\{b', N'\}$ are relatively prime integers. Using Eqs. (3.12) and (3.54a,b), we obtain

$$\delta = \alpha (aN' - tb') .$$

(3.55)

Let us now assume that $aN' - tb' = 0$, which gives

$$a = \frac{tb'}{N'} .$$

(3.56)

Since $\{b', N'\}$ are relatively prime integers, the above equation would be satisfied if $t = \beta N'$, where $\beta$ is a positive integer. Substituting this value of $t$ in Eq. (3.56), we obtain

$$a = \beta b' .$$

(3.57)

Eqs. (3.54a) and (3.57) imply that $\{a, b\}$ are not relatively prime integers, which contradicts our basic assumption that $a/b \in F'_{N-1}$. Thus it is established that $aN' - tb' \neq 0$. Since $a, N', t, b'$ are all integers, it follows that

$$|aN' - tb'| \geq 1 .$$

(3.58)

Combining Eqs. (3.55) and (3.58) we find that, similar to the case of fractions of type II, the inequality given by

$$|\delta| > 1 ,$$

is satisfied for all fractions of type IV. Consequently, by carrying out the rest of the analysis in exactly same way as we have done in this section for the fractions of type II, it can be shown that all fractions of type IV obey Eq. (3.29). Hence, clusters of bound particles cannot be formed for any fraction of this type.

4 Some properties of clusters of bound particles

In the previous section we have shown that, for the case of derivative $\delta$-function Bose gas with given values of $\phi$ and $N$, the Bethe state (2.3) represents clusters of bound particles...
if $\phi/\pi$ is either a fraction of type I or type III within the set $F'_{N-1}$. In this section, our aim is to find the number of clusters present within such a Bethe state and the sizes of these clusters (i.e., number of bound particles present in each of these clusters). We would also like to investigate the behavior of these clusters of bound particles under small variations of the coupling constant.

### 4.1 Sizes of the clusters of bound particles

Let us first consider clusters of bound particles when $\phi/\pi = a/b$ is taken as any fraction of type I within the set $F'_{N-1}$. In section 2 we have seen that, to find the number and sizes of the clusters within a Bethe state, we have to determine the set $\Omega_{N,\phi}$ for which Eq. (3.1a) is satisfied. Since $\{N, b\}$ are relatively prime integers for any fraction of type I, we can express $N$ as

$$N = pb + r,$$

where $1 \leq r \leq b - 1$. Hence, for the discrete variable $l$ taking values within the set $\Omega_{N}$, the zero points of the functions $\sin l\phi$ and $\sin(N - l)\phi$ are respectively given by the sets

$$S_1 \equiv \{b, 2b, \cdots, pb\}, \quad S_2 \equiv \{N - b, N - 2b, \cdots, N - pb\}.$$

Combining the sets $S_1$ and $S_2$ by using Eqs. (4.1) and (4.2a,b), we obtain $\Omega_{N,\phi}$ as

$$\Omega_{N,\phi} = S_1 \cup S_2 = \{r, b, r + b, 2b, \cdots, r + (p - 1)b, pb\}.$$  \hspace{1cm} (4.3)

Comparing (4.3) with (2.21), and also using (2.22), it is easy to see that the sizes of the clusters are given by

$$\{\{r, b - r, r, b - r, \cdots, r, b - r, r\}\}.$$  \hspace{1cm} (4.4)

Hence, for any fraction of type I, the corresponding Bethe state contains $(p + 1)$ number of clusters of size $r$ and $p$ number of clusters of size $(b - r)$. Next, by using the method of contradiction, we would like to show that these two possible sizes of the clusters, i.e., $r$ and $(b - r)$, must be relatively prime integers. To this end, let us first assume that $b$ and $r$ are not relatively prime integers. Therefore, we can write $b$ and $r$ as $b = \alpha b'$ and $r = \alpha r'$, where $\alpha > 1$. Substituting these values of $b$ and $r$ in Eq. (4.1), we find that

$$N = \alpha(pb' + r').$$

Thus $\alpha$ is a common factor of $N$ and $b$. However, this result contradicts the fact that $\{N, b\}$ must be relatively prime integers for any fraction of type I. Hence it is established that $\{b, r\}$ are relatively prime integers. From this relation, it trivially follows that $\{r, b - r\}$ are relatively prime integers and, in particular, $r \neq b - r$. Consequently from Eq. (4.4) we find that, for any fraction of type I, the corresponding Bethe state contains heterogeneous clusters of two different sizes. As a special case, let us consider any fraction of type I with denominator satisfying the relation $b > N/2$. Due to Eq. (4.1) it follows that, $p = 1, r = N - b$ and $b - r = 2b - N$ for this case. Hence, the corresponding Bethe state contains two clusters of the size $(N - b)$ and one cluster of the size $(2b - N)$. 21
Next, we consider the clusters of bound particles corresponding to any fraction of type III. In this case \( N \) can be written as \( N = pb \), where \( p \) is an integer greater than one. Consequently, for the variable \( l \) taking value within the set \( \Omega_{N} \), the zero points of the functions \( \sin l\phi \) and \( \sin((N - l)\phi) \) coincide with each other and yield \( \Omega_{N,\phi} \) as

\[
\Omega_{N,\phi} = \{ b, 2b, 3b, \ldots, (p - 1)b \}.
\]

Hence \( p \) number of clusters are formed in this case. Comparing (4.5) with (2.21), and also using (2.22), it is easy to see that each cluster of this type has the size \( b \). In other words, the corresponding Bethe state (2.3) contains \( N/b \) number of homogeneous clusters, each of which is made of \( b \) number of bound particles.

In Table 1 we show all the fractional values of \( \phi/\pi \) for which clusters of bound particles exist within the range of \( N \) given by \( 4 \leq N \leq 10 \), the types of these fractions, the values of \( \delta \) for these fractions, and the sizes of the corresponding clusters using the notation of Eq. (2.22).

| \( N \) | Value of \( \phi/\pi \) | Type | Value of \( \delta \) | Size of the clusters |
|--------|----------------|------|----------------|-------------------|
| 4      | 1/3            | I    | 1              | \{1, 2, 1\}       |
| 5      | 1/4            | I    | 1              | \{1, 3, 1\}       |
| 5      | 1/3            | I    | -1             | \{2, 1, 2\}       |
| 6      | 1/5            | I    | 1              | \{1, 4, 1\}       |
| 6      | 1/3            | III  | 0              | \{3, 3\}          |
| 7      | 1/6            | I    | 1              | \{1, 5, 1\}       |
| 7      | 1/4            | I    | -1             | \{3, 1, 3\}       |
| 7      | 1/3            | I    | 1              | \{1, 2, 1, 2, 1\} |
| 7      | 2/5            | I    | -1             | \{2, 3, 2\}       |
| 8      | 1/7            | I    | 1              | \{1, 6, 1\}       |
| 8      | 1/4            | III  | 0              | \{4, 4\}          |
| 8      | 1/3            | I    | -1             | \{2, 1, 2, 1, 2\} |
| 8      | 2/5            | I    | 1              | \{3, 2, 3\}       |
| 9      | 1/8            | I    | 1              | \{1, 7, 1\}       |
| 9      | 1/5            | I    | -1             | \{4, 1, 4\}       |
| 9      | 1/4            | I    | 1              | \{1, 3, 1, 3, 1\} |
| 9      | 1/3            | III  | 0              | \{3, 3, 3\}       |
| 9      | 3/7            | I    | -1             | \{2, 5, 2\}       |
| 10     | 1/9            | I    | 1              | \{1, 8, 1\}       |
| 10     | 1/5            | III  | 0              | \{5, 5\}          |
| 10     | 2/7            | I    | -1             | \{3, 4, 3\}       |
| 10     | 1/3            | I    | 1              | \{1, 2, 1, 2, 1, 2, 1\} |
| 10     | 2/5            | III  | 0              | \{5, 5\}          |

| \( N \) | Value of \( \phi/\pi \) | Type | Value of \( \delta \) | Size of the clusters |
|---------|----------------|------|----------------|-------------------|
| 4       | 1/3            | I    | 1              | \{1, 2, 1\}       |
| 5       | 1/4            | I    | 1              | \{1, 3, 1\}       |
| 5       | 1/3            | I    | -1             | \{2, 1, 2\}       |
| 6       | 1/5            | I    | 1              | \{1, 4, 1\}       |
| 6       | 1/3            | III  | 0              | \{3, 3\}          |
| 7       | 1/6            | I    | 1              | \{1, 5, 1\}       |
| 7       | 1/4            | I    | -1             | \{3, 1, 3\}       |
| 7       | 1/3            | I    | 1              | \{1, 2, 1, 2, 1\} |
| 7       | 2/5            | I    | -1             | \{2, 3, 2\}       |
| 8       | 1/7            | I    | 1              | \{1, 6, 1\}       |
| 8       | 1/4            | III  | 0              | \{4, 4\}          |
| 8       | 1/3            | I    | -1             | \{2, 1, 2, 1, 2\} |
| 8       | 2/5            | I    | 1              | \{3, 2, 3\}       |
| 9       | 1/8            | I    | 1              | \{1, 7, 1\}       |
| 9       | 1/5            | I    | -1             | \{4, 1, 4\}       |
| 9       | 1/4            | I    | 1              | \{1, 3, 1, 3, 1\} |
| 9       | 1/3            | III  | 0              | \{3, 3, 3\}       |
| 9       | 3/7            | I    | -1             | \{2, 5, 2\}       |
| 10      | 1/9            | I    | 1              | \{1, 8, 1\}       |
| 10      | 1/5            | III  | 0              | \{5, 5\}          |
| 10      | 2/7            | I    | -1             | \{3, 4, 3\}       |
| 10      | 1/3            | I    | 1              | \{1, 2, 1, 2, 1, 2, 1\} |
| 10      | 2/5            | III  | 0              | \{5, 5\}          |

Table 1: The fractional values of \( \phi/\pi \) for which clusters of bound particles exist for \( 4 \leq N \leq 10 \), the types of these fractions, the values of \( \delta \) for fractions of type I, and the sizes of the corresponding clusters are shown.
4.2 Stability of clusters of bound particles and binding energies

We want to explore here how the nature of clusters of bound particles change under slight variations of the coupling constant. For any given value of $N \geq 4$, we first choose a value of the coupling constant $\phi$ within the range $0 < \phi < \frac{\pi}{2}$ such that clusters of bound particles are formed. If one increases or decreases this value of $\phi$ by an infinitesimal amount, it is obvious that all inequalities in Eq. (3.1b) continue to be satisfied, but all equalities in Eq. (3.1a) are transformed into some inequalities. Therefore, clusters of bound particles cease to exist even for a very small change of the coupling constant.

One of the following two different cases can occur in such a situation. In the first case, at least one of the equalities in Eq. (3.1a) is transformed into an inequality of the form

$$\chi \sin(l\phi) \sin[(N - l)\phi] < 0. \quad (4.6)$$

It is evident that, the probability density of the corresponding Bethe state (2.3) would diverge if at least one of the particle coordinates tends towards infinity (keeping the centre of mass coordinate fixed). As a result, this Bethe state becomes ill-defined and disappears from the Hilbert space of the Hamiltonian (1.2) of derivative $\delta$-function Bose gas. So we may say that clusters of bound particles become unstable in this case. Let us now consider the second case, for which all of equalities in Eq. (3.1a) are transformed into inequalities of the form

$$\chi \sin(l\phi) \sin[(N - l)\phi] > 0, \quad (4.7)$$

due to an infinitesimal change of the coupling constant $\phi$. It is evident that, for this case, Eqs. (3.1a,b) are transformed to Eq. (2.16) within the range of $\phi$ given by $0 < \phi < \frac{\pi}{2}$. As a result, clusters of bound particles merge with each other and produce a localized bound state containing only one cluster of particles. Therefore, in this case, we may say that clusters of bound particles turn into a localized bound state with only one cluster.

In this context, we note that the first and second derivatives of the function $f(l, N, \phi)$ in Eq. (3.17) are given by

$$\frac{\partial f}{\partial \phi} = \frac{\chi N}{2} \sin(N\phi) - \frac{\chi(N - 2l)}{2} \sin(N - 2l)\phi, \quad (4.8a)$$

$$\frac{\partial^2 f}{\partial \phi^2} = \frac{\chi N^2}{2} \cos(N\phi) - \frac{\chi(N - 2l)^2}{2} \cos(N - 2l)\phi. \quad (4.8b)$$

By calculating these derivatives at some value of $\phi$ for which clusters of bound states exist and at a value of $l$ within the set $\Omega_{N,\phi}$, one can find whether the corresponding equality in Eq. (3.1a) is transformed into an inequality of type (4.6) or (4.7) due to an infinitesimal change in the coupling constant. In the following, we shall use this procedure to determine the nature of clusters of bound particles for any given values of $\phi$ and $N$.

Let us first consider clusters of bound particles when $\phi/\pi = a/b$ is taken as any fraction of type III. As mentioned earlier, in this case $N$ can be written as $N = pb$ (where $p$ is an integer) and from Eq. (3.13) it follows that $t = ap$. Moreover, due to Eq. (4.5), $l$ should be taken to be of the form $l = mb$, where $m$ is an integer. For the values of $N$,
\( \phi \) and \( l \) given by \( N = pb, \phi = \pi a/b \) and \( l = mb \), we get \( \sin(N\phi) = \sin(\pi a/b) = 0 \) and \( \sin(N - 2l)\phi = \sin(N\phi - 2\pi ma) = 0 \). Thus, by using Eq. (4.8a), we obtain

\[
\frac{\partial f}{\partial \phi} \bigg|_{\phi=\frac{\pi a}{b}, l=mb} = 0,
\]

which shows that \( \phi = \pi a/b \) is an extremum point of the function \( f(l, N, \phi) \). To determine the nature of this extremum point, it is necessary to calculate the second derivative of \( f(l, N, \phi) \). To this end we recall that, according to our discussion just after Eq. (3.27), the value of \( \chi \) should be chosen such that the inequality \((-1)^t \chi < 0 \) is satisfied. Since \( t = ap \) for all fractions of type III, we can rewrite the above mentioned inequality as

\[
(-1)^{ap} \chi < 0.
\]

Moreover, for the case \( N = pb, \phi = \pi a/b \) and \( l = mb \), we obtain \( \cos(N\phi) = \cos(N - 2l)\phi = (-1)^{ap} \). Substituting these values of \( \cos(N\phi) \) and \( \cos(N - 2l)\phi \) in Eq. (4.8b), and subsequently using the inequality (4.10), we find the relation

\[
\frac{\partial^2 f}{\partial \phi^2} \bigg|_{\phi=\frac{\pi a}{b}, l=mb} = 2l(N - l) \cdot \chi(-1)^{ap} < 0,
\]

which shows that the function \( f(l, N, \phi) \) has a maxima at \( \phi = \pi a/b \). Consequently, if one moves slightly away from the point \( \phi = \pi a/b \) (in positive or negative direction), the corresponding equality in Eq. (3.1a) is transformed into an inequality of the form (4.10). Therefore, we find that the clusters of bound particles are unstable around the point \( \phi = \pi a/b \), where \( a/b \) is a fraction of type III.

Next, we consider clusters of bound particles corresponding to any fraction of type I. Due to Eq. (3.34), it is possible to divide all fractions of type I into two subclasses, corresponding to the value of \( \delta \) given by +1 and -1 respectively. We shall see below that clusters of bound particles related to these two subclasses behave differently under the variation of the coupling constant. For any fraction of type I, the union of the sets \( S_1 \) and \( S_2 \) given in Eqs. (4.2a,b) yields the set \( \Omega_{N,\phi} \). Therefore, the values of the corresponding \( l \) should be taken either as \( l = mb \) or as \( l = N - mb \), where \( m \) is an integer. Let us first consider the case when \( l = mb \). Since \( \sin(N - 2l)\phi = \sin(N\phi) \) in this case, by using Eq. (3.12) we obtain

\[
\sin(N - 2l)\phi = \sin(N\phi) = (-1)^t \sin \left( \frac{\pi \delta}{b} \right).
\]

Similarly, for the case \( l = N - mb \), we obtain

\[
-\sin(N - 2l)\phi = \sin(N\phi) = (-1)^t \sin \left( \frac{\pi \delta}{b} \right).
\]

Substituting the values of \( \sin(N\phi) \) and \( \sin(N - 2l)\phi \) given in Eqs. (4.12) and (4.13) to Eq. (4.8a) for the cases \( l = mb \) and \( l = N - mb \) respectively, and also using the inequality \((-1)^t \chi < 0 \), we get

\[
\frac{\partial f}{\partial \phi} \bigg|_{\phi=\frac{\pi a}{b}, l=mb} = \frac{\partial f}{\partial \phi} \bigg|_{\phi=\frac{\pi a}{b}, l=N-mb} = mb \cdot \chi(-1)^t \cdot \sin \left( \frac{\pi \delta}{b} \right) = \begin{cases} < 0 & \text{for} \ \delta = 1, \\ > 0 & \text{for} \ \delta = -1. \end{cases}
\]
Hence we find that, for the case of fractions of type I with $\delta = 1$, clusters of bound particles would become unstable due to a slight increase of the value of $\phi$. On the other hand, such clusters of bound particles would transmute to a localized bound state containing only one cluster if the value of $\phi$ is slightly decreased. For the fractions of type I with $\delta = -1$, clusters of bound particles would behave in exactly reversed order if one slightly increases or decreases the value of $\phi$.

It is interesting to calculate the binding energies of the clustered states that we have been considering so far. The binding energy $E_B$ of a clustered state is defined as follows. The momentum $P$ and energy $E$ of such a state were presented in Eqs. (2.23) and (2.24). Let us now consider a state of $N$ free particles which are all moving with the same momentum $p$ so that the total momentum $Np$ is equal to the expression in Eq. (2.23). Clearly,

$$p = h\chi \frac{\sin(N\phi)}{N \sin \phi}. \quad (4.15)$$

The energy of this free particle state is given by

$$E_0 = Np^2 = h^2 \frac{\chi^2 \sin^2(N\phi)}{N \sin^2(\phi)}. \quad (4.16)$$

The binding energy $E_B$ corresponding to clusters of bound particles is defined as the difference of the energies of the free particle state and the clustered state, i.e.,

$$E_B \equiv E_0 - E = \frac{h^2 \chi \sin(N\phi)}{N \sin^2(\phi)} h(N, \phi), \quad (4.17a)$$

$$h(N, \phi) = \chi [\sin(N\phi) \cos \phi - N \cos(N\phi) \sin \phi]. \quad (4.17b)$$

It should be noted that Eqs. (4.17a,b) also give the binding energies of localized bound states for the case of the derivative $\delta$-function Bose gas [42,44]. We now consider the two kinds of clusters of bound particles in which $\phi/\pi = a/b$ corresponds to fractions of types I and III within the sequence $F'_{N-1}$. For the case of fractions of type III, $N$ is a multiple of $b$; hence $\sin(N\phi) = 0$. Thus the momentum $p$ and the energies $E$ and $E_0$ are all equal to zero in this case, and $E_B = 0$ also. We now turn to fractions of type I. In Refs. [42,44], it was shown that $h(N, \phi) > 0$ for all these values of $\phi$. Eq. (4.17a) then implies that $E_B$ has the same sign as $\chi \sin(N\phi)$. Combining Eqs. (4.12-4.14), we see that $\chi \sin(N\phi)$ and hence $E_B$ are negative for $\delta = 1$ (i.e., if $(aN - 1)/b$ is an integer) and are positive for $\delta = -1$ (i.e., if $(aN + 1)/b$ is an integer).

In our earlier works it was shown that, for any given value of $N \geq 4$, the derivative $\delta$-function Bose gas allows bound states in only certain non-overlapping ranges of the coupling constant $\phi$ called ‘bands’ (the union of which yields a proper subset of the full range of $\phi$), and the location of these bands can be determined exactly [42,44]. Applying the results obtained in this section, we shall now present an alternative method of computing the location of these bands within the full interval $(0, \pi/2)$ for positive values of $\phi$. Since fractions of type I only lead to clusters of bound particles which transmute to bound states with a single cluster through infinitesimal changes of $\phi$ in appropriate directions, it is evident that any fraction of type I or 0 or 1/2 must appear at the two edges of any continuous range of $\phi/\pi$ within which localized bound states are
formed. Therefore, we include the numbers 0 and 1/2 along with the fractions of type I (corresponding to the set $F'_{N-1}$) and call this new set as type I'. Next, we denote two fractions of type I' as $a_i/b_i$ and $a'_i/b'_i$, and define them as ‘conjugate pair’ if they appear within the Farey sequence $F_N$ as

$$F_N : \cdots \frac{a_i}{b_i} \frac{n_i}{N} \frac{a'_i}{b'_i} \cdots,$$

(4.18)

where $n_i/N \in F'_N$. Let us also assume that all possible conjugate pairs of this type can be obtained by varying the index $i$ within the range $i \in \{1, 2, \ldots, \sigma(N)\}$. Clearly, for any $N \geq 4$, conjugate pairs defined in Eq. (4.18) can be related to three different cases: (i) $a_i/b_i$ and $a'_i/b'_i$ are both fractions of type I, (ii) $a_i/b_i = 0$ and $a'_i/b'_i$ is a fraction of type I, and (iii) $a'_i/b'_i = 1/2$ and $a_i/b_i$ is a fraction of type I. Let us first consider any conjugate pair corresponding to the case (i). Using Eqs. (3.5a) and (3.12) along with the relation $t = n$, it is easy to see that $\delta = -1$ for $a_i/b_i$ and $\delta = 1$ for $a'_i/b'_i$. Therefore, from the discussion of the previous paragraph it follows that clusters of bound particles would be transformed to localized bound states only if one increases the value of $\phi/\pi$ from $a_i/b_i$ or decreases the value of $\phi/\pi$ from $a'_i/b'_i$ by an infinitesimal amount. Moreover it should be noted that, since $a_i/b_i$ and $a'_i/b'_i$ are consecutive fractions within the set $F'_{N-1}$, the sign of the function $f(l, N, \phi)$ cannot flip even for any finite amount of change of $\phi/\pi$ within the interval $(a_i/b_i, a'_i/b'_i)$. Consequently, for any conjugate pair corresponding to case (i), localized bound states continue to exist if $\phi/\pi$ takes any value within the range $(a_i/b_i, a'_i/b'_i)$. By repeating similar arguments, one can reach the same conclusion for any conjugate pair associated with cases (ii) or (iii). Thus we find that, localized bound states can be constructed for any value of $\phi$ within the non-overlapping ranges or bands given by

$$\phi \in \left(\frac{\pi a_i}{b_i}, \frac{\pi a'_i}{b'_i}\right),$$

(4.19)

where $i \in \{1, 2, \ldots, \sigma(N)\}$. Due to Eq. (4.18) it is evident that there exists an one-to-one correspondence between the conjugate pairs formed by fractions of type I' and the elements of the set $F'_N$. Therefore, the elements of $F'_N$ can be used to uniquely characterize the bands appearing in Eq. (4.19). Moreover, it is easy to check that Eq. (4.18) provides the only possible way of relating two fractions like $a/b$ and $c/d$ of type I', such that localized bound states are formed for any value of $\phi/\pi$ within the range $(a/b, c/d)$. Hence, localized bound states can only be constructed within the ranges of $\phi$ given in Eq. (4.19).

For an illustration of the idea of bands, and the stability of clusters of bound particles under a variation of $\phi$ and their binding energy discussed above, we calculate the binding energies of localized bound states within all bands for $N = 7, 8$ and $9$ by using Eqs. (4.17a,b), and we show these binding energies as a function of $\phi/\pi$ in Fig. 1. Comparing this figure with the appropriate entries in Table 1, we see that the values of $\phi/\pi$ which are fractions of type III do not lie in any bands, while fractions of type I lie at the end points of the bands. Furthermore, fractions of type I with $\delta = 1$ ($-1$) lie at the right (left) end of the bands and have $E_B < 0$ ($> 0$) respectively.
5 Conclusion

In this article, we have explored how clusters of bound particles can be constructed in the simplest possible way for the case of an exactly solvable derivative $\delta$-function Bose gas. To this end, we consider a sufficient condition for which Bethe states of the form in (2.3) would lead to clusters of bound particles. This sufficient condition for obtaining clusters of bound particles has not attracted much attention earlier, probably because it does not yield any solution at all for the case of the $\delta$-function Bose gas. However we find that, for the case of the derivative $\delta$-function Bose gas, this sufficient condition can be satisfied by taking the quasi-momenta of the corresponding Bethe state to be equidistant points on a single circle having its centre at the origin of the complex momentum plane. Furthermore, the coupling constant ($\phi$) and the total number of particles ($N$) of this derivative $\delta$-function Bose gas must satisfy the relations in (3.1a,b). For any given $N \geq 4$, it is found that Eq. (3.1a) is satisfied if $\phi/\pi$ takes any value within the set $F'_N-1$, which is a subset of the Farey sequence $F_{N-1}$. Then we classify all fractions belonging to the set $F'_N-1$ into four types. It turns out that fractions of types I and III belonging to $F'_N-1$ satisfy the remaining relation (3.1b), while type II and type IV fractions do not satisfy (3.1b). Consequently, clusters of bound particles can be constructed for the derivative $\delta$-function Bose gas only for special values of $\phi/\pi$ given by the fractions of types I and III within the set $F'_N-1$.

We also computed the sizes of the above mentioned clusters of bound particles, i.e., the number of particles present within each of these clusters. We find that any fraction of type I within the set $F'_N-1$ leads to heterogeneous clusters of bound particles having
two different sizes. On the other hand, any fraction of type III within the set $F'_{N-1}$ leads to homogeneous clusters of bound particles having only one size. Interestingly, clusters of bound particles associated with fractions of type I and type III transform in rather different ways under a small variation of the coupling constant. For example, it is found that clusters of bound particles associated with fractions of type III cease to exist for any small change of the coupling constant. On the other hand, clusters of bound particles corresponding to fractions of type I turn into localized bound states consisting of a single cluster if the value of the coupling constant is slightly increased or decreased. Finally, we find that clusters of bound particles associated with fractions of type III have zero binding energy, while clusters associated with fractions of type I have either positive or negative binding energy.

Even though in this paper we have analyzed a particular type of sufficient condition for constructing clusters of bound particles in the case of the derivative $\delta$-function Bose gas, in future it would be interesting to explore other possible ways of constructing clusters of bound particles for this exactly solvable system. We have previously mentioned that, clusters of bound particles are constructed for the case of the $\delta$-function Bose gas by taking the corresponding quasi-momenta to be discrete points lying on several parallel straight lines in the complex momentum plane. In analogy with this case, one may try to construct clusters of bound particles for the present case by assuming that the corresponding quasi-momenta lie on several concentric circles in the complex momentum plane. However it should be noted that, for the case of the derivative $\delta$-function Bose gas with any given value of the coupling constant, localized bound states can appear only for some particular values of the particle number $[42,44]$. Hence, for constructing clusters of bound particles in this case, the number of quasi-momenta on each concentric circle must be taken from the subset of such allowed particle numbers. Due to this constraint on the numbers of quasi-momenta, it remains a rather challenging problem to construct all possible clusters of bound particles in the case of the derivative $\delta$-function Bose gas.

Acknowledgments

B.B.M. thanks the Abdus Salam International Centre for Theoretical Physics for a Senior Associateship, which partially supported this work. D.S. thanks the Department of Science and Technology, India for financial support through the grant SR/S2/JCB-44/2010.
Appendix A

In this Appendix, we shall explicitly derive the relation (3.15). At first, we shall show that if \( a/b \in F'_{N-1} \) satisfies the condition

\[
\frac{a}{b} \leq \frac{1}{2} - \frac{1}{2N},
\]

then \( t/N \) will satisfy the relation (3.15). To this end, we rewrite Eq. (3.12) as

\[
\frac{t}{N} = \frac{a}{b} - \frac{\delta}{bN}.
\]

Eqs. (A1) and (A2) imply that

\[
\frac{t}{N} \leq \frac{1}{2} - \frac{1}{2N} - \frac{\delta}{bN}.
\]

For the case \( \delta \geq 0 \), the relation (3.15) directly follows from Eq. (A3). For the case \( \delta < 0 \), Eq. (3.13) yields

\[
-\frac{\delta}{b} < \frac{1}{2}.
\]

Combining (A3) and (A4), it can be easily seen that the relation (3.15) is satisfied for \( \delta < 0 \). Thus we are able to prove that the relation (3.15) follows from the condition (A1) for all possible values of \( \delta \).

Next, for the purpose of proving the validity of condition (A1), let us consider the cases of even and odd \( N \) separately. For the case of even values of \( N \), we may write \( N = 2m \). Using the properties (3.5a,b) of Farey sequence, we find that

\[
F_{2m-1} : \cdots \cdots \frac{m-1}{2m-1} \quad \frac{1}{2} \quad \cdots \cdots,
\]

i.e., the fraction nearest to 1/2 from the left side of the sequence \( F_{2m-1} \) is given by \((m-1)/(2m-1)\). Consequently, any fraction \( a/b \in F'_{2m-1} \) satisfies the relation

\[
\frac{a}{b} \leq \frac{m-1}{2m-1}.
\]

Using (A5), we easily find that the condition (A1) with \( N = 2m \) is satisfied by any \( a/b \in F'_{2m-1} \). Next, we consider the case of odd values of \( N \) and write \( N = 2m + 1 \). Using Eqs.(3.5a,b), it is easy to check that there are two consecutive fractions in the Farey sequence \( F_{2m} \) given by

\[
F_{2m} : \cdots \cdots \frac{m-1}{2m-1} \quad \frac{1}{2} \quad \cdots \cdots.
\]

Consequently, any fraction \( a/b \in F'_{2m} \) also satisfies the relation (A5). Using (A5), we find that the condition (A1) with \( N = 2m + 1 \) is satisfied by any \( a/b \in F'_{2m} \). Thus it is established that the condition (A1) is satisfied by any \( a/b \in F'_{N-1} \) for both even and odd values of \( N \).
Appendix B

Here our aim is to show that for any $a/b \in F'_{N-1}$, there exists at least one $l \in \Omega_N - \Omega_{N,\phi}$ such that $\rho(l) = 0$. To this end, we shall consider the cases $\delta \geq 0$ and $\delta < 0$ separately.

**Case (i)** Let us first consider the case $\delta \geq 0$, for which $\triangle \geq 0$. Using Eqs. (3.3a) and (3.14), we obtain the relation

$$0 < a + \delta < b.$$  \hspace{1cm} (B1)

By using the method of contradiction, at first we shall show that $(b - 1) \in \Omega_N - \Omega_{N,\phi}$. Let us first assume that $x = b - 1$ is a zero point of the function $g(x, \phi)$. Then by using Eq. (3.22) we obtain $b - 1 = mb/a$, which yields $a/b = a - m$.

However, the above equation leads to a contradiction because its l.h.s. is a proper fraction and its right hand side (r.h.s.) is an integer. Thus $x = b - 1$ cannot be a zero point of the function $g(x, \phi)$. Next, we assume that $x = b - 1$ is a zero point of the function $g(x - \triangle, \phi)$. Then by using Eq. (3.22) we obtain $b - 1 - \delta/a = m'b/a$, which yields $a + \delta/b = a - m'$.

Again the above equation leads to a contradiction because its r.h.s. is an integer and its l.h.s. is a proper fraction due to Eq. (B1). Thus $x = b - 1$ cannot be a zero point of either $g(x, \phi)$ or $g(x - \triangle, \phi)$, it follows that $(b - 1) \in \Omega_N - \Omega_{N,\phi}$.

Next, our aim is to show that $\rho(l) = 0$ for the choice $l = b - 1$. To this end, let us assume that $x = m''b/a$ is a zero point of the function $g(x, \phi)$ and $m''b/a < (b - 1)$. Since the difference between $(b - 1)$ and $m''b/a$ may be written as

$$D \left( b - 1, \frac{m''b}{a} \right) = -1 + \frac{b}{a}(a - m''),$$  \hspace{1cm} (B2)

it is clear that $m'' < a$. Using Eqs. (3.3a) and (B2), it is easy to see that $D(b - 1, m''b/a)$ yields the minimum positive value for the choice $m'' = a - 1$. Furthermore, by using Eqs. (B1) and (3.21), we find that this minimum positive value of $D(b - 1, m''b/a)$ satisfies the relation

$$D \left( b - 1, \frac{m''b}{a} \right) \bigg|_{m'' = a - 1} = \frac{b - a}{a} > \triangle.$$

From the above equation it follows that, there exists no zero point of the function $g(x, \phi)$ in the variable $x$ within the interval $(b - 1 - \triangle, b - 1)$. Thus, for any $a/b \in F'_{N-1}$ with $\delta \geq 0$, we are able to establish that $\rho(b - 1) = 0$.

**Case (ii)** Let us consider the case $\delta < 0$, for which $\triangle < 0$. Using Eqs. (3.3a) and (3.14), we obtain the relation

$$0 < a - \delta < b.$$  \hspace{1cm} (B3)
By using the method of contradiction, in the following we shall show that \((b + 1) \in \Omega_N - \Omega_{N,\phi}\), i.e, \((b + 1) \leq (N - 1)\) and \(x = (b + 1)\) is not a zero point of either \(g(x, \phi)\) or \(g(x - \Delta, \phi)\). For this purpose it may be noted that, since \(a/b \in F'_{N-1}\), one can in principle choose \(b = N - 1\), which contradicts the condition \((b + 1) \leq (N - 1)\). However, by applying Eqs. (3.11) and (3.13) for this case, we obtain \([Na/b] = a\) and \(\delta = a > 0\). Thus the choice \(b = N - 1\) is not compatible with the present case associated with negative values of \(\delta\). Next we assume that \(x = b + 1\) is a zero point of the function \(g(x, \phi)\). Then by using Eq. (3.22) we obtain \(a/b = m - a\), which yields

\[
\frac{a}{b} = m - a.
\]

However, the above equation leads to a contradiction because its l.h.s. is a proper fraction and its r.h.s. is an integer. Thus \(x = b+1\) cannot be a zero point of the function \(g(x, \phi)\). Next, we assume that \(x = b + 1\) is a zero point of the function \(g(x - \Delta, \phi)\). Then by using Eq. (3.22) we obtain \(b + 1 = m'b/a\), which yields

\[
\frac{a - \delta}{b} = m' - a.
\]

Again, the above equation leads to a contradiction because its r.h.s. is an integer and its l.h.s. is a proper fraction due to Eq. (B3). Thus \(x = b + 1\) cannot be a zero point of the function \(g(x - \Delta, \phi)\). As a result, we find that \((b + 1) \in \Omega_N - \Omega_{N,\phi}\) for any fraction \(a/b \in F'_{N-1}\) which leads to \(\delta < 0\).

Next, our aim is to show that \(\rho(l) = 0\) for the choice \(l = b + 1\). To this end, let us assume that \(x = m''b/a\) is a zero point of the function \(g(x, \phi)\) and \(m''b/a > (b + 1)\). Since the difference between \(m''b/a\) and \((b + 1)\) may be written as

\[
D\left(\frac{m''b}{a}, b + 1\right) = -1 + \frac{b}{a}(m'' - a),
\]

it is evident that \(m'' > a\). With the help of Eqs. (3.3a) and (B4), it is easy to check that \(D(m''b/a, b + 1)\) yields the minimum positive value for the choice \(m'' = a + 1\). Furthermore, by using Eqs. (B3) and (3.21), we find that this minimum positive value of \(D(m''b/a, b + 1)\) satisfies the relation

\[
D\left(\frac{m''b}{a}, b + 1\right)_{m''=a+1} = \frac{b - a}{a} > - \Delta.
\]

Hence, it follows that there exists no zero point of the function \(g(x, \phi)\) in the variable \(x\) within the interval \((b + 1, b + 1 - \Delta)\). Thus for any \(a/b \in F'_{N-1}\) with \(\delta < 0\), we are able to establish that \(\rho(b + 1) = 0\).
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