Transport coefficients, effective charge and mass for multicomponent systems with fractional exclusion statistics

Takahiro Fukui*† and Norio Kawakami

Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-01, Japan

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Abstract

Transport properties of the multicomponent quantum many-body systems obeying Haldane’s fractional exclusion statistics are studied in one dimension. By computing the finite-size spectrum under twisted boundary conditions, we explicitly express the conductivity and the conductance in terms of statistical interactions. Through this analysis, the effective charge and effective mass for collective excitations are determined. We apply the results for $1/r^2$ quantum systems as well as for correlated electron systems.

*JSPS Research Fellow

†e-mail address: fukui@yukawa.kyoto-u.ac.jp
I. INTRODUCTION

In low-dimensional quantum systems, excitations are described by quasiparticles carrying fractional quantum numbers. One of the well known examples is the fractional quantum Hall effect (FQHE) \cite{1}, where quasiparticles are classified by the fractional charge and statistics. In these theories, fractional quantum numbers arise from exchange properties of the wavefunction. Recently, Haldane \cite{2} proposed a new concept of fractional statistics based on the state-counting of many-body systems, which is a generalization of Pauli-exclusion principle. We will refer to this as fractional exclusion statistics. Thermodynamic properties were already investigated in detail \cite{3-6}. For example, Wu and Bernard \cite{4} formulated thermodynamic equations, and showed that the statistical interaction is related with the two-body phase shift for Bethe-ansatz solvable models. Their method was generalized to multicomponent systems \cite{5}, and low-energy critical properties were investigated.

In this paper, we study transport properties of the multicomponent quantum systems with exclusion statistics in one dimension. Transport coefficients are closely related with wavefunctions, and may usually be calculated in the Green function formalism. Since we have only statistics among particles without explicit wavefunctions, it is not trivial to study such properties directly from the definition of statistics. We therefore use a trick to avoid this difficulty. Namely, by applying the idea of twisted boundary conditions for the finite-size spectrum, we calculate the conductivity and the conductance in terms of statistical interactions. We then determine the effective charge and the effective mass, and show how these quantities are related with exclusion statistics.

After a brief introduction of exclusion statistics in the next section, we compute the finite-size corrections due to the vector potential for multicomponent quantum systems in section 3. In section 4, we then obtain the conductivity and the conductance, and determine the effective charge and the effective mass. In section 5, we apply the results for several interesting quantum systems. Section 6 is devoted to summary and discussions.
II. EXCLUSION STATISTICS

Let us start with a brief introduction for fundamental properties of exclusion statistics. It is based on counting the change of the dimension of the one-particle Hilbert space when a particle is added to the system, which is explicitly formulated as,

\[ \frac{\partial D_\alpha(k_\alpha)}{\partial N_\beta(k'_\beta)} = -g_{\alpha\beta}(k_\alpha - k'_\beta) \equiv -\{g'_{\alpha\beta}(k_\alpha - k'_\beta) + \delta_{\alpha\beta}\delta_{k_\alpha k'_\beta}\}, \tag{2.1} \]

where \( D_\alpha(k_\alpha) \) and \( N_\alpha(k_\alpha) \) are, roughly speaking, the numbers of unoccupied (hole) and occupied (particle) states specified by the internal quantum numbers \( \alpha = (1, 2, ..., M) \) and corresponding momentum \( k_\alpha \). The matrix \( g_{\alpha\beta} \), which is called *statistical interaction*, describes correlation effects among particles. For more rigorous definition, see Ref. [2]. Simple cases \( g_{\alpha\beta}(k_\alpha - k'_\beta) = g\delta_{\alpha\beta}\delta_{k_\alpha k'_\beta} \) with \( g = 1 \) and \( g = 0 \) correspond to free fermions and free bosons, respectively, and for general fractional value \( g \), we call it *ideal fractional exclusion statistics*.

Statistical interactions should be independent of \( N_\alpha \), and hence eq. (2.1) results in

\[ D_\alpha(k_\alpha) = -\sum_{\beta, k'_\beta} g_{\alpha\beta}(k_\alpha - k'_\beta)N_\beta(k'_\beta) + D^0_\alpha(k_\alpha). \tag{2.2} \]

We assume that integral constants are given by \( D^0_\alpha(k_\alpha) = D^0\delta_{\alpha1} \) or \( D^0 \), which are referred to as hierarchical and symmetric bases, respectively. Such bases are originally used for a classification of the FQHE. Also, in one-dimensional quantum systems, the hierarchical basis serves as a natural basis for the Bethe-ansatz solution.

In the thermodynamic limit, we introduce the distribution functions for particles and holes,

\[ \rho_\alpha(k_\alpha) = \frac{N_\alpha(k_\alpha)}{D^0}, \quad \rho^{(h)}_\alpha(k_\alpha) = \frac{D_\alpha(k_\alpha)}{D^0}, \tag{2.3} \]

where \( D^0 \) is proportional to the system size \( L \) such as \( D^0 = L/2\pi \) under periodic boundary conditions. The bare charge for each elementary excitation is defined by

\[ t_\alpha = \frac{D^0_\alpha}{D^0}. \tag{2.4} \]
Consequently, eq.(2.2) can be written as
\[ \rho_\alpha(k_\alpha) + \rho_\alpha^{(h)}(k_\alpha) = t_\alpha - \sum_\beta \int_{-\infty}^{\infty} dk'_\beta g_{\alpha\beta}(k_\alpha - k'_\beta) \rho_\beta(k'_\beta), \]  
(2.5)

The energy of the system is assumed to take the form [4]:
\[ \varepsilon \equiv \frac{E}{D^0} = \sum_\alpha \int_{-\infty}^{\infty} dk_\alpha \epsilon_\alpha^0(k_\alpha) \rho_\alpha(k_\alpha) \]  
(2.6)

with the bare energy function \( \epsilon_\alpha^0(k) \). Note that many-body effects due to exclusion statistics are incorporated in the distribution function \( \rho(k) \).

Thermodynamic equations are generally obtained in a set of coupled nonlinear equations at finite temperatures. Here we restrict ourselves to the zero-temperature case which is sufficient for the following calculation. At zero temperature without external fields, \( M \) species of elementary excitations are specified by the dressed energy function \( \epsilon_\alpha(k) \). The “Fermi level” \( Q_\alpha \) for each excitation is determined by the conditions, \( \epsilon_\alpha(k_\alpha) < 0 \) for \( |k_\alpha| < Q_\alpha \) and \( \epsilon_\alpha(k_\alpha) > 0 \) for \( |k_\alpha| > Q_\alpha \), provided that the energy dispersion may be symmetric around the origin \( k_\alpha = 0 \). Then, eq.(2.5) reduces to the following integral equations supplemented by those for the dressed energy \( \epsilon_\alpha \),
\[ \rho_\alpha(k_\alpha) = t_\alpha - \sum_\beta \int_{-Q_\beta}^{Q_\beta} dk'_\beta g_{\alpha\beta}(k_\alpha - k'_\beta) \rho_\beta(k'_\beta), \]  
(2.7)
\[ \epsilon_\alpha(k_\alpha) = \epsilon_\alpha^0(k_\alpha) - \mu t_\alpha - \sum_\beta \int_{-Q_\beta}^{Q_\beta} dk'_\beta g'_{\beta\alpha}(k'_\beta - k_\alpha) \epsilon_\beta(k'_\beta). \]  
(2.8)

The total energy is now expressed by the dressed energy as

\[ ^1 \text{Thermodynamics of systems with exclusion statistics can be formulated by the method proposed in Ref. [4], and a multicomponent generalization can be found in Ref. [6]. A key idea is to introduce the entropy } S = \ln W \text{ with } 
W = \prod_{\alpha, k_\alpha} \frac{(D_\alpha + N_\alpha - 1)!}{N_\alpha!(D_\alpha - 1)!}, \]
which plausibly interpolates the boson and fermion cases [3]. Note that if we choose \( g_{\alpha\beta} \) for free fermions and bosons, \( \rho \) in eq.(2.3) reduces to the Fermi and Bose distribution functions.
where $\mu$ is the chemical potential and $n_c$ is the density of charged particles. These equations can describe static properties at zero temperature.

III. FINITE-SIZE CORRECTIONS DUE TO STATIC VECTOR POTENTIAL

We now turn to the computation of the conductivity in pure systems without randomness, which can be calculated with the response to the vector potential. Consider the ring system at $T = 0$ threaded by the magnetic flux, which gives rise to static vector potential along the ring [10]. The effect of the vector potential is incorporated in twisted boundary conditions [11,12]. Therefore, the energy increment quadratically proportional to $A$ can be calculated through the analysis of the finite-size spectrum, which directly gives the charge stiffness and hence the conductivity. By observing that the basic equations (2.7) and (2.8) have the same structure as the Bethe-ansatz equations, we can apply elegant techniques of the dressed charge matrix developed for integrable models [13], and generalize the calculation of the conductivity [11,12] to multicomponent cases.

Before proceeding with the finite-size corrections, we here give a formal solution to (2.7) in the absence of external fields, which is necessary for the following discussions. To this end, let us first introduce the functions [13,14]

$$K_{\alpha\beta}(k_\alpha - k_\beta) = g'_{\alpha\beta}(k_\alpha - k'_\beta) - \sum_{\gamma} \int_{-Q_\gamma}^{Q_\gamma} dk''_{\gamma\alpha} g''_{\alpha\gamma}(k_\alpha - k''_{\gamma}) K_{\gamma\beta}(k''_{\gamma} - k'_\beta),$$

(3.1)

$$Z_{\alpha\beta}(k_\beta) = \delta_{\alpha\beta} - \sum_{\gamma} \int_{-Q_\gamma}^{Q_\gamma} dk'_{\gamma\beta} Z_{\gamma\alpha}(k'_{\gamma}) g'_{\gamma\beta}(k'_{\gamma} - k_\beta),$$

(3.2)

where $Z_{\alpha\beta}$ is called the dressed charge matrix [13]. By noting the relation

$$Z_{\alpha\beta}(k_\beta) = \delta_{\alpha\beta} - \int_{-Q_\alpha}^{Q_\alpha} dk'_{\alpha} K_{\alpha\beta}(k_\alpha - k'_\beta),$$

(3.3)

we can formally write down the distribution function in terms of the dressed charge matrix,

$$\rho_\alpha(k_\alpha) = \sum_{\beta} t_{\beta} Z_{\beta\alpha}(k_\alpha).$$

(3.4)
We now evaluate finite-size corrections to the total energy due to the vector potential, which is the same as those for twisted boundary conditions \[11,12\]. First, by observing that the effects of the static vector potential $A$ are to shift the momentum by the amount $\delta_\alpha$ proportionally to $A$ (in unit $e$), the basic equations should read,

$$
\tilde{\rho}_\alpha(k_\alpha) = t_\alpha - \sum_\beta \int_{Q_\beta + \delta_\beta} \int_{Q_\beta + \delta_\beta} dk'_\beta g'_\beta(k_\alpha - k'_\beta) \tilde{\rho}_\beta(k'_\beta), \quad (3.5)
$$

$$
\tilde{\varepsilon}_\alpha(k_\alpha) = \epsilon^0_\alpha(k_\alpha) - \mu t_\alpha - \sum_\beta \int_{Q_\beta + \delta_\beta} \int_{Q_\beta + \delta_\beta} dk'_\beta g'_\beta(k'_\beta - k_\alpha) \tilde{\varepsilon}(k'_\beta), \quad (3.6)
$$

$$
\tilde{\varepsilon} = \sum_\alpha \int_{Q_\alpha + \delta_\alpha} dk_\alpha \epsilon^0_\alpha(k_\alpha) \tilde{\rho}_\alpha(k_\alpha) = \sum_\alpha \tilde{\varepsilon}_\alpha + \mu n_e, \quad (3.7)
$$

where

$$
\tilde{\varepsilon}_\alpha = t_\alpha \int_{Q_\alpha + \delta_\alpha} dk_\alpha \tilde{\varepsilon}(k_\alpha). \quad (3.8)
$$

It is to be noted that the vector potential not only shifts the momentum uniformly, but also can rearrange in general the distribution of the momentum via interactions among particles. For clarity, we put tilde for rearranged quantities.

Let us compute the corrections to the total energy (3.7). By differentiating eq. (3.6) with respect to $\delta_\beta$, we have

$$
\frac{\partial \tilde{\varepsilon}_\alpha(k_\alpha)}{\partial \delta_\beta} \bigg|_{\delta=0} = 0. \quad (3.9)
$$

by using the conditional equation for the “Fermi level”, $\epsilon_\alpha(Q_\alpha) = 0$. By differentiating again, we have

$$
\frac{\partial^2 \tilde{\varepsilon}_\alpha(k_\alpha)}{\partial \delta_\beta \partial \delta_\gamma} \bigg|_{\delta=0} = -\epsilon'_\beta(Q_\beta) \{g'_{\beta\alpha}(Q_\beta - k_\alpha) + g'_{\beta\alpha}(-Q_\beta - k_\alpha)\} \delta_{\beta\gamma}
$$

$$
- \sum_\gamma \int_{Q_\gamma} dk'_\gamma g'_{\alpha\gamma}(k_\alpha - k'_\gamma) \frac{\partial^2 \tilde{\varepsilon}(k'_\gamma)}{\partial \delta_\beta \partial \delta_\gamma} \bigg|_{\delta=0},
$$

which can be formally solved by the iteration scheme, resulting in

$$
\frac{\partial^2 \tilde{\varepsilon}_\alpha(k_\alpha)}{\partial \delta_\beta \partial \delta_\gamma} \bigg|_{\delta=0} = -\epsilon'_\beta(Q_\beta) \{K_{\beta\alpha}(Q_\beta - k_\alpha) + K_{\beta\alpha}(-Q_\beta - k_\alpha)\} \delta_{\beta\gamma}, \quad (3.10)
$$

where prime in $\epsilon'_\alpha$ stands for the derivative with respect to $k_\alpha$. In the following, we assume the relation $g_{\alpha\beta}(k_\alpha - k'_\beta) = g_{\beta\alpha}(k'_\beta - k_\alpha)$. Using these results, we have
\[
\frac{\partial \varepsilon_\alpha}{\partial \delta_\beta} \bigg|_{\delta=0} = 0, \quad \frac{\partial^2 \varepsilon_\alpha}{\partial \delta_\beta \partial \delta_\gamma} \bigg|_{\delta=0} = t_\alpha \epsilon'_\beta(Q_\beta) \left[ 2\delta_{\alpha\beta} - \int_{Q_\alpha}^{Q_\beta} dk_\alpha \{ K_{\beta\alpha}(Q_\beta - k_\alpha) + K_{\beta\alpha}(-Q_\beta - k_\alpha) \} \right] \delta_\gamma
\]

\[
= 2t_\alpha \epsilon'_\beta(Q_\beta) Z_{\alpha\beta} \delta_{\beta\gamma}, \quad (3.13)
\]

where \( Z_{\alpha\beta} \equiv Z_{\alpha\beta}(Q_\beta) \) is the dressed charge matrix defined in (3.2). Therefore, we obtain the finite size correction \( \Delta \varepsilon \equiv \tilde{\varepsilon} - \varepsilon \) due to the small shift of \( \delta_\alpha \) as,

\[
\Delta \varepsilon = \frac{1}{2} \sum_{\alpha\beta\gamma} \delta_{\beta} \left. \frac{\partial^2 \varepsilon_\alpha}{\partial \delta_\beta \partial \delta_\gamma} \right|_{\delta=0} \delta_\gamma = \sum_{\alpha\beta} t_\alpha \epsilon'_\beta(Q_\beta) Z_{\alpha\beta} t_\beta \delta_{\beta\gamma}, \quad (3.14)
\]

where \( V_{\alpha\beta} = v_\alpha \delta_{\alpha\beta} \), and \( v_\alpha \) is the Fermi velocity defined by \( v_\alpha = \epsilon'_\alpha(Q_\alpha)/\rho_\alpha(Q_\alpha) \). To derive the second line, we have used eqs.(3.4) and (3.13).

The remaining task is to determine the relation between \( \delta_\alpha \) and \( A \). The result is quite simple (see eq. (3.23)), but still nontrivial, which depends on types of external fields. Here we briefly depict how to obtain this relation in a bit general way [14]. For this purpose, let us first introduce

\[
i_\alpha(k_\alpha) = \int_{k_{\alpha}}^{k_{\alpha}} dk \rho_\alpha(k), \quad \tilde{i}_\alpha(k_\alpha) = \int_{k_{\alpha}}^{k_{\alpha}} dk \tilde{\rho}_\alpha(k), \quad (3.15)
\]

\[
G_{\alpha\beta}(k_\alpha - k'_{\beta}) = \int_{k_\alpha}^{k_{\alpha}} dk' G'_{\alpha\beta}(k_\alpha - k'_{\beta}). \quad (3.16)
\]

When \( A = 0 \), eq.(2.7) is integrated as

\[
i_\alpha(k_\alpha) = t_\alpha k_\alpha - \sum_\beta \int_{Q_\beta}^{Q_\beta} dk'_\beta G'_{\alpha\beta}(k_\alpha - k'_{\beta}) \rho_\beta(k'_{\beta}). \quad (3.17)
\]

Notice that the first term in the right hand side is regarded as a bare momentum of the system, which can be written as \( p^0_\alpha \equiv t_\alpha k_\alpha \). In the presence of the vector potential, the momentum is shifted as \( p^0_\alpha \rightarrow p^0_\alpha - A t_\alpha \). Then eq.(3.17) is modified to

\[
\tilde{i}_\alpha(k_\alpha) + d_\alpha = t_\alpha k_\alpha - \sum_\beta \int_{Q_\beta + \delta_\beta}^{Q_\beta + \delta_\beta} dk'_\beta G_{\alpha\beta}(k_\alpha - k'_{\beta}) \rho_\beta(k'_{\beta}), \quad (3.18)
\]

where we denote \( d_\alpha \equiv A t_\alpha \). Note that the following calculation can also be applied for other types of external fields if we suitably take \( d_\alpha \) different from \( A t_\alpha \). Now introduce \( \tilde{k}_\alpha = \tilde{k}_\alpha(k_\alpha) \) such that \( i_\alpha(k_\alpha) = \tilde{i}_\alpha(\tilde{k}_\alpha) \), then we find \( \tilde{k}_\alpha(Q_\alpha) = Q_\alpha + \delta_\alpha \) and
\[ \rho_\alpha(k_\alpha)dk_\alpha = \tilde{\rho}_\alpha(\tilde{k}_\alpha)d\tilde{k}_\alpha. \]  

(3.19)

By the use of these relations, we can change the integral variables into \( \tilde{k}_\alpha \) in eq.(3.18). By subtracting both sides of eq.(3.17) from (3.18) and expanding \( G_{\alpha\beta}(k_\alpha - \tilde{k}_\beta') \) up to the first order in \( (\tilde{k} - k) \), we get

\[
F_\alpha(k_\alpha) = d_\alpha - \sum_\beta \int_{Q_\beta} dk'_\beta g'_\alpha_\beta(k_\alpha - k'_\beta)F_{\beta'}(k'_\beta),
\]

(3.20)

where \( F_\alpha \) is defined by \( [14] \)

\[
F_\alpha(k_\alpha) = (\tilde{k}_\alpha - k_\alpha)\rho_\alpha(k_\alpha).
\]

(3.21)

The formal solution to eq.(3.20) is given by

\[
F_\alpha(k_\alpha) = \sum_\beta d_\beta Z_{\beta\alpha}(k_\alpha).
\]

(3.22)

As stressed above, this formula is valid for arbitrary types of \( d_\alpha \). By explicitly substituting \( d_\alpha \equiv At_\alpha \), and comparing (3.21) and (3.22) by the use of (3.4), we get the simple relation,

\[
\tilde{k}_\alpha - k_\alpha = A = \delta_\alpha.
\]

(3.23)

Namely, momentum shifts occur not only for the charge sector but for all indices \( \alpha \).

Consequently, by combining (3.14) and (3.23), we end up with the final formula for the energy increment as,

\[
\Delta \varepsilon = \sum_{\alpha\beta} t_\alpha(ZVZ^T)_{\alpha\beta}t_\beta A^2.
\]

(3.24)

This completes the calculation of finite-size corrections due to the static vector potential.

IV. TRANSPORT COEFFICIENTS, EFFECTIVE CHARGE AND MASS

A. conductivity

According to the formula (3.24) for the response to the external vector potential, we can obtain the charge stiffness as,
\[ D_c = \sum_{\alpha\beta} t_{\alpha}(ZVZ^T)_{\alpha\beta} t_{\beta}. \]  

(4.1)

An important point is that \( D_c \) is directly related with the conductivity through the relation,

\[ \text{Re} \sigma(\omega) = e^2 D_c \delta(\omega) \text{ at } \omega = 0 \]  

(4.2)

according to the linear response theory \[10,11\].

In many literatures so far, correlation effects on \( D_c \) have been considered to modify only the effective transport mass \( m^* \) \[10,12\]. However, the transport mass is not sufficient to describe the correlation effects on transport properties, because quasiparticles such as holons in one dimension can carry not only the effective mass but also the effective charge. Therefore, in place of ordinary interpretation \[11,12\], we propose the following natural expression for the conductivity in terms of the effective charge \( e^* \) and effective mass \( m^* \),

\[ \text{Re} \sigma(\omega) = \pi e e^* n_c \frac{\delta(\omega)}{m^*} \text{ at } \omega = 0, \]  

(4.3)

where \( n_c \) is the density of charged particles. The charge stiffness calculated in eq.(4.1) is then related with \( D_c \) such that

\[ D_c \propto \frac{e^*}{m^*}. \]  

(4.4)

So, the charge stiffness (or conductivity) is not sufficient to derive the effective charge and mass, and another quantity is necessary to determine them. We will show that the effective charge can be derived from the conductance.

**B. conductance**

It is known that the fractional charge enters in the conductance for the finite system in one dimension. Though it is not easy to calculate the conductance without wavefunctions, we can compute it by taking into account a universal property of Luttinger liquids, i.e., *the conductance is solely determined by the correlation exponent for the charge sector*. To this end, we first define the critical exponent for the charge density correlation function in the asymptotic region,
\[ \langle \rho(x)\rho(0) \rangle \sim \exp(2ik_F^{(c)}x)x^{-2Mf_c}, \quad (4.5) \]

where \( k_F^{(c)} = \pi n_c \) is the “Fermi momentum” for the charge sector, and is usually given by \( k_F^{(c)} = Mk_F \) in terms of the ordinary Fermi momentum \( k_F \). The correlation exponent \( f_c \) is a function of statistical interactions, which is normalized to reproduce \( f_c = 1 \) for non-interacting systems. An important point is that the conductance \( G_c \) can be determined by \( f_c \) in a universal way \[15\].

\[ G_c = M\frac{e^2}{h}f_c. \quad (4.6) \]

This formula can be deduced by observing that the conductance is controlled only by the charge degrees of freedom. \[1\] Renormalizing \( e \) by \( f_c \) such that \( G_c = Mee^*/h \), we can naturally define the effective charge as

\[ e^* = f_c e. \quad (4.7) \]

Therefore, the remaining task is to obtain the critical exponent \( f_c \) in terms of statistical interactions. Following a way similar to the last section, we can derive the exponent \( f_c \) through conformal-field-theory analysis of the finite-size spectrum which has been already computed in \[3\]. We thus obtain the renormalization factor for the fractional charge,

\[ f_c = \frac{1}{M} \sum_{\alpha\beta} t_{\alpha}(ZZ^T)_{\alpha\beta}t_{\beta}, \quad (4.8) \]

in terms of statistical interactions which are implicitly incorporated in the dressed charge matrix. By applying the above formulas to eq.(1.4), we can also extract the effective transport mass \( m^* \), which turns out to be inversely proportional to the velocity \( v \). The expressions (1.1), (1.4), (1.7) and (4.8) are the main results in this paper.

One can see that the effective charge is determined solely by the statistical interactions, whereas the effective mass also depends on non-universal quantities such as the velocity. We wish to emphasize that the formula for the fractional charge (1.7) with (4.8) is universal, which holds generally for multicomponent Luttinger liquids.

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2Exponent \( g \) without randomness in ref. \[15\] corresponds to \( f_c \) in this paper.
V. APPLICATIONS

A. Ideal fractional exclusion statistics

One of the most remarkable applications of exclusion statistics is that for the ideal case, in which the statistical interaction is given in a simple form \cite{2,3},

\[
g_{\alpha\beta}(k_\alpha - k'_\beta) = G_{\alpha\beta}\delta(k_\alpha - k'_\beta). \tag{5.1}
\]

This model is known to have close relationship to interesting quantum systems such as the FQHE, the $1/r^2$ systems, etc.

If we take a bare dispersion as $\epsilon^0_\alpha(k) = \epsilon^0(k)t_\alpha$ with $\epsilon(k) = k^2/2$, the ground state configuration is $Q_1 = Q_2 = \ldots = Q_M \equiv Q$, and the distribution functions are obtained as

\[
\rho_\alpha = \sum_{\beta} G^{-1}_{\alpha\beta}t_\beta. \tag{5.2}
\]

The number of charged particles is

\[
n_c = 2Q\nu, \quad \nu = \sum_{\alpha\beta} t_\alpha G^{-1}_{\alpha\beta}t_\beta, \tag{5.3}
\]

where the quantity $\nu$ is related to the compressibility as $\kappa_c = 4\nu^2/n_c$, which is a one-dimensional analogue of the filling factor in the FQHE. In the present model, the Fermi velocities for each excitation take the same values

\[
v_1 = v_2 = \ldots = v_M \equiv v = \frac{n_c}{2\nu}. \tag{5.4}
\]

As is shown in Appendix, one finds a simple relation between the dressed charge and the statistical interaction,

\[
ZZ^T = G^{-1}. \tag{5.5}
\]

Consequently, the renormalization factor \cite{4,5} for the fractional charge $e^*/e = f_c$ is expressed as

\[
f_c = \frac{1}{M} \sum_{\alpha\beta} t_\alpha (G^{-1})_{\alpha\beta}t_\beta. \tag{5.6}
\]
It is to be noted that the fractional charge is now explicitly obtained only in terms of the statistical parameters $G_{\alpha\beta}$ for exclusion statistics. Since the charge stiffness is given by $D = n_c/2$, the enhancement factor for the transport mass in this case is derived

$$m^*/m = f_c,$$  \hspace{1cm} (5.7)

from which one can see that the enhancement of $m^*$ exactly cancels the renormalization of charge $e^*$, in accordance with the translational symmetry.

Let us now discuss more concrete models for ideal exclusion statistics. For an instructive example, we consider the statistical-interaction matrix $G$ in hierarchical basis, which is derived from the continuum $1/r^2$ model \[16\] with SU($M$) symmetry \[17,18\]. This model is also related to a fundamental series for the hierarchical FQHE \[7,19,20\]. The $M \times M$ matrix for statistical interactions in this case reads,

\[
G = \begin{pmatrix}
2n+1 & -1 & 0 \\
-1 & 2 & -1 & 0 \\
& & \ddots & \\
0 & -1 & 2 & -1 \\
0 & -1 & 2 & \\
\end{pmatrix}.
\]  \hspace{1cm} (5.8)

We have from eq. (5.3),

$$
\nu = \frac{1}{2n+1 - \frac{1}{2}} = \frac{M}{2Mn+1} \equiv \frac{p}{q},
$$  \hspace{1cm} (5.9)

which corresponds to the filling factor in the case of the FQHE. Observing this, we see that the matrix $G$ in this model plays a role similar to the flux attachment in Jain’s model for the FQHE \[21\]:

\[
G \leftrightarrow \chi_M \chi_1^{2n},
\]  \hspace{1cm} (5.10)

where $\chi_M$ is the IQH state with the filling factor $M$, which is attached by $2n$ flux quanta $\chi_1^{2n}$. In fact, one can find the same matrix as (5.8) in the classification scheme in the corresponding
Abelian Chern-Simons theory for the FQHE \cite{1,8,19,20}. Therefore, it is seen that the SU($M$) $1/r^2$ model has close relationship to the hierarchical FQHE with $\nu = M/(2Mn + 1)$.

According to eqs.\,(5.6) and (5.7), it turns out that the effective charge and mass are given by taking $t^T = (1, 0, \cdots, 0)$,

\begin{align}
e^* &= \frac{e}{2Mn + 1} = \frac{e}{q}, \quad (5.11) \\
m^* &= \frac{m}{2Mn + 1} = \frac{m}{q}. \quad (5.12)
\end{align}

Note that the expression for the effective charge (5.11) is actually in accordance with that for the FQHE. In particular, for one component case $G = g$, we have $e^*/e = m^*/m = 1/g$.

Note that for free systems, i.e., $n = 0$, we have $e^*/e = m^*/m = 1$.

\section*{B. Correlated electron systems}

It is also instructive to apply the results to one-dimensional correlated electron systems. In order to fully describe interacting electron systems in the whole energy range, it is necessary to consider statistical interactions depending on the momentum in a complicated way \cite{22}. However, if we restrict ourselves to the low-energy conformal limit, we can still use the idea of \textit{ideal exclusion statistics}. In such low-energy region, the critical behavior is described by the Luttinger liquid theory, in other words, by $c = 1$ conformal field theory. In this case, we can introduce $2 \times 2$ matrix $G_{\alpha\beta}$ for ideal statistics in (5.1) in terms of Luttinger liquid parameters. Since this model has two kinds of elementary excitations, i.e., spinon and holon which have two different velocities, $v_s$ and $v_c$, we can choose the hierarchical basis as a natural one. By analyzing exactly solved models or Tomonaga-Luttinger models, the matrix for statistical interaction can be deduced as \cite{3},

\begin{equation}
G = \begin{pmatrix}
\frac{K_\rho + 1}{2K_\rho} & -1 \\
-1 & 2
\end{pmatrix}, \quad (5.13)
\end{equation}

where $K_\rho = Z_{11}^2/2$ is the critical exponent for the $4k_F$ oscillation piece in the density correlation function. Note that $G_{11}$ is related with the charge degrees of freedom, $G_{22}$ to the
spin degrees of freedom, and the off-diagonal elements are regarded as mutual statistics. It is to be emphasized here that (5.13) is the universal formula for correlated electron systems.

According to eq.(4.7), the effective charge is given by substituting $t^T = (1, 0)$,

$$e^*/e = K_\rho,$$

which reproduces the known results for the conductance in Luttinger liquids [15]. Also, from eq.(4.1), the charge stiffness is found to be

$$D_c = 2v_c K_\rho,$$

from which we obtain the effective mass using eq.(4.4),

$$m^*/m = v_F/v_c.$$

where $v_F$ is the Fermi velocity for non-interacting electrons.

For exactly solvable electron models such as the Hubbard model and the supersymmetric $t$-$J$ model, the critical exponent $K_\rho$ and the velocity of holons were calculated exactly as functions of the interaction and the electron density [23–25]. So, we can discuss the effective charge and mass for these models. As for the Hubbard model, $K_\rho$ decreases from 1 to $1/2$ as the Coulomb interaction is increased [25], hence the effective charge decreases with the increase of the interaction, as stressed by Ogata and Fukuyama [15]. Near half filling, the effective charge always takes $e/2$ as far as the Coulomb interaction exists. In the case of the supersymmetric $t$-$J$ model [23,24], the effective charge is $e/2$ near half filling, but as the electron density decreases, it continuously increases and reaches the non-interacting value $e^* = e$ in the low density limit. Also, we can discuss the effective mass for electron systems. The results are essentially same as previously discussed [12]: the effective mass has a divergence property near half filling both for the Hubbard model and the supersymmetric $t$-$J$ model, reflecting the metal-insulator transition.
VI. SUMMARY AND DISCUSSIONS

In summary, we have obtained the transport coefficients, the effective charge and mass for multicomponent quantum systems obeying fractional exclusion statistics. Their explicit relation to the statistical interaction has been derived in eqs. (4.1) – (4.8) for general systems obeying (2.1). We have applied the results for the cases with ideal statistics as well as for the conformal limit of electron systems. It has been also pointed out that the statistical interaction derived from SU($N$)/$r^2$ models are closely related with Jain’s construction (or the corresponding Chern-Simons theory) for the hierarchical FQHE.

It is instructive to note that the effective charge in the ideal case is expressed in an extremely simple form (5.6) or (5.11) in terms of the statistical interaction $G_{\alpha\beta}$, implying that the fractional charge in the ideal case directly reflects the fractional statistics. In fact, we find an alternative way to derive (5.11) using only the definition of fractional exclusion statistics. We briefly summarize how to get them intuitively. In the ideal case, the definition eq. (2.1) reads

$$\frac{\partial N_\alpha}{\partial D_\beta} = -G_{\alpha\beta}^{-1} \quad \text{at} \quad k_\alpha = k_\beta. \quad (6.1)$$

Now imagine the ground state configuration and make a hole to excite the system. The above equation implies that if we make a $\beta$-hole, the number of $\alpha$-particles decreases by the amount of $G_{\alpha\beta}^{-1}$. Then, how many charged particles decrease in all? The answer is $\sum_\alpha t_\alpha G_{\alpha\beta}^{-1}$. Now make a hole at a sector $\alpha$, i.e., $t^T = (0, ..., 1, 0, ..., 0)$ in the hierarchical basis. This corresponds to $t^T = (0, ..., 1, 1, ..., 1)$ in the symmetric basis, which may create $M + 1 - \alpha$ holes of electrons. Therefore, making a hole with unit charge corresponds to removing particles with the charge

$$e^*_{\alpha} = \frac{e}{M + 1 - \alpha} \sum_\beta G_{\beta\alpha}^{-1}, \quad (6.2)$$

which in turn defines the effective charge of the excitation. If we adopt $G$ in eq.(5.8), then we have
\[ e^*_\alpha / e = 1/q, \] (6.3)

where \( q \) is defined in eq.(5.3). This result coincides with (5.11).

Finally, another remark is in order. In section III, we have defined the effective charge (4.7) apart from the trivial degeneracy \( M \). However, it may be possible to include such factor in the definition of the charge. In this definition, eqs.(5.6), (5.11) and (5.12) are modified by the factor \( M \), and eq.(6.2) should be replaced by

\[ \bar{e}^*_\alpha = e \sum_\beta G^{-1}_{\beta\alpha} \] (6.4)

and therefore, eq.(6.3) is modified as

\[ \bar{e}^*_\alpha / e = (M + 1 - \alpha)/q, \] (6.5)

i.e., explicitly, \( p/q, (p-1)/q, \ldots, 1/q \). This definition for the fractional charge corresponds to that used in refs. [7,8].

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**APPENDIX A:**

In the case of ideal exclusion statistics, there exists a simpler way to derive the formula (5.3) without calculating the dressed charge (3.2). We briefly depict this convenient method below. Consider the present system without external fields. Then there are two kinds of elementary excitation, i.e., the excitation which changes the number of particles, and which carries the large momentum. Following techniques of the dressed charge matrix [13], the excitation spectrum was explicitly evaluated in [6],

\[ \Delta \varepsilon = (v/4)n^T(ZZ^T)^{-1}n + vd^T(ZZ^T)d, \]
where the vectors $\mathbf{n}$ and $\mathbf{d}$ denote the quantum numbers for excitations, which label the change of particle number and the momentum transfer, respectively. Note that the excitation which we seek for in (3.24) corresponds to the excitation specified by $\mathbf{d}$. From the above formula, one can see that these two kinds of excitations are related to each other reflecting modular invariance. So, we can easily deduce the excitations for the $\mathbf{d}$-sector once we can calculate those for the $\mathbf{n}$-sector. The calculation for the latter excitation is much simpler than the former. Let us then calculate the latter by changing $Q_\alpha \rightarrow Q_\alpha + \Delta Q_\alpha$. Then both $n_\alpha$ and $\Delta \varepsilon$ are given by functions of $\Delta Q_\alpha$, and a simple calculation gives $\Delta \varepsilon$ as a function of $n_\alpha$ such that $\Delta \varepsilon = (v/4)\mathbf{n}^T G \mathbf{n}$. Comparing these results, we end up with the formula (5.6).
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