Approximation of a system of semilinear singularly perturbed parabolic reaction-diffusion equations on a vertical strip

G I Shishkin and L P Shishkina
Institute of Mathematics and Mechanics, Russian Academy of Science, S.Kovalevskaya str., GSP-384, 620219 Yekaterinburg, Russia
E-mail: shishkin@imm.uran.ru, Lida@convex.ru

Abstract. On a vertical strip, a Dirichlet problem is considered for a system of two semilinear singularly perturbed parabolic reaction-diffusion equations. The highest-order derivatives in the equations are multiplied by the perturbation parameter $\varepsilon^2$; $\varepsilon \in (0, 1]$. When $\varepsilon \to 0$, the parabolic boundary layer appears. Using the condensing mesh method and classical finite difference approximations of the boundary value problem, special finite difference schemes are constructed that converge $\varepsilon$-uniformly.

1. Introduction
Boundary value problems for systems of singularly perturbed partial differential equations in which the highest-order derivatives are multiplied by a small (perturbation) parameter $\varepsilon$ often occur, for example, in modeling and analysis of heat- and mass- transfer processes when the thermal conductivity and diffusion coefficients are small and (or) the rate of reactions is large. In such problems, boundary layers appear in a neighborhood of the boundary as the parameter tends to zero.

When the parameter is small, the use of classical numerical methods and, in particular, finite difference schemes causes significant difficulties [10], [15] even in the case of scalar elliptic and parabolic differential equations. Errors of the approximate solutions can be comparable with the exact solution (see, e.g., [1, 3, 5, 7, 12, 14, 17]). Therefore, in the case of boundary value problems for systems of singularly perturbed equations (and for scalar equations as well), it is important to develop special numerical methods (in particular, finite difference schemes) whose solutions errors are independent of the perturbation parameter and are defined only by the number of grid points. We say that such methods (finite difference schemes) converge $\varepsilon$-uniformly.

Boundary value problems for linear systems of elliptic and parabolic equations on a strip were considered, for example, in [18, 19] (for reaction-diffusion equations in [18], and for convection-diffusion equations in [19]). A boundary value problem for a system of linear parabolic equations on a rectangle have been considered in [22]. Problems for semilinear systems of singularly perturbed partial differential equations have never been studied.

Note that fitted operator methods (see their description, e.g., in [3, 7]) have a restricted domain of applicability for constructing $\varepsilon$-uniformly convergent numerical methods. It was shown that there are no schemes based on the fitted operator method which converge $\varepsilon$-uniformly.
in the maximum norm in the cases of parabolic [16, 17] and elliptic [11] singularly perturbed equations in the presence of a parabolic boundary layer and (or) nonlinearity of the equations [4]. It is no matter how these schemes are constructed (either on the basis of classical finite difference approximations or on the basis of the finite element or finite volume methods).

In the present paper, we consider special finite difference approximations of the Dirichlet problem on a strip for a system of two singularly perturbed parabolic reaction-diffusion equations. The highest-order derivatives in the differential equations are multiplied by the perturbation parameter \( \varepsilon ^{2} \); \( \varepsilon \) takes arbitrary values in the open-closed interval \((0,1]\). For \( \varepsilon = 0 \), the system of parabolic equations degenerates into a system of ordinary differential equations. When \( \varepsilon \rightarrow 0 \), the parabolic boundary layer with the typical width \( \varepsilon \) appears in a neighbourhood of the strip boundary. Using the condensing mesh method and classical finite difference approximations or on the basis of the finite element or finite volume methods).

\[ \text{Contents of the paper.} \]

The formulation of the problem and the aim of the research are given in Section 2. Compatibility conditions are discussed in Section 3. A priori estimates for solutions and their derivatives that are needed for the construction and study of difference schemes are exposed in Section 4. To derive a priori estimates and justify convergence of special finite difference schemes, one can apply a technique that have been developed for a system of linear elliptic and parabolic equations on a rectangle in [21, 22]. A classical finite difference scheme is presented in Section 5. Special finite difference schemes that converges to the solution of the boundary value problem \( \varepsilon \)-uniformly are considered whose solution components at the current temporal level are found from the disjoined system of linear equations.

\[2. \text{Problem formulation. The aim of research} \]

2.1. Let \( G \) be the domain \( D \times (0, T] \) with the boundary \( S = \partial G \), where \( \partial D \) is the vertical strip\(^1\)

\[ \bar{D} = D \cup \Gamma, \quad D = D(2.1) = \{ x : \quad 0 < x_1 < d, \quad | x_2 | < \infty \}, \quad \Gamma = \{ x_2 = \pm \infty \}. \]

\[ S = S^{L} \cup S_{0}, \quad S^{L} = \Gamma \times (0, T], \quad S_{0} = \Gamma \times \{ 0 \} \]

where \( S^{L} \) and \( S_{0} \) are the lateral and lower parts of the boundary \( S \). On the strip \( \bar{D} \) we consider the Dirichlet problem for the system of two semilinear singularly perturbed parabolic reaction-diffusion equations

\[ L \mathbf{u}(x, t) = g(x, t, \mathbf{u}(x, t)), \quad (x, t) \in G, \quad \mathbf{u}(x, t) = \varphi(x, t), \quad (x, t) \in S, \quad i = 1, 2. \]

Here

\[ L \mathbf{u}(x, t) = L(\varepsilon) \mathbf{u}(x, t) = \left\{ \varepsilon^{2} L_{2} - C(x, t) \frac{\partial}{\partial t} \right\} \mathbf{u}(x, t), \]

\(^1\) The notation \( L_{(j,k)} \) \((G_{(j,k)}, M_{(j,k)})\) means that these operators (domains, constants) are introduced in formula \((j,k)\).
The parameter ε takes arbitrary values in the open-closed interval (0, 1].

By a solution of the problem (2.2), we mean a function \( u \in C^{2,1}(G) \) that is continuous on \( \overline{G} \) and satisfies the differential equation (2.2a) on \( G \) and the boundary condition (2.2b) on \( S \).

The problem as formulated arises, for example, in modeling a diffusion process in combination with chemical reactions. The parameter multiplying the highest-order derivatives characterizes the diffusion coefficient of the agents and the functions \( c^{ij}(x, t) \) determine the rates of the direct and inverse chemical reactions (see, e.g., [2]).

We assume that the solution of the problem is sufficiently smooth for fixed values of the parameter \( \varepsilon \).

When \( \varepsilon \) tends to zero, a parabolic boundary layer appears in a neighbourhood of the set \( S^L \).

Our aim for the boundary value problem (2.2), (2.1) is to construct a finite difference scheme that converges \( \varepsilon \)-uniformly.

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2 Here and below \( M, M_i \) (or \( m \)) denote sufficiently large (small) positive constants which do not depend on \( \varepsilon \) and on the discretization parameters.
3. Compatibility conditions for problem (2.2), (2.1)
We give conditions imposed on the data of the problem (2.2), (2.1) that guarantee the required smoothness of the solution.

We introduce some notation. We denote by $\Gamma_j$ with $\Gamma = \bigcup \Gamma_j$ for $j = 1, 2$, the sides of the strip $D$; the side $\Gamma_1$ passes through the point $(0, 0)$. Set

$$S_j = \Gamma_j \times (0, T], \quad j = 1, 2.$$  

(3.1a)

We denote by $S^c$ the set of “edges”

$$S^c = S^L \cap S_0.$$  

(3.1b)

3.1. In the case when the data of the problem (2.2), (2.1) satisfy the conditions

$$C_{(2.2)}, P \in H^{(1)_1+\alpha}(\overline{G}), \quad g \in C^{(1)_1+\alpha,(1)_1+\alpha}(\overline{Q}), \quad t^{(1)} \geq 0,$$  

(3.2a)

$$\varphi \in H^{(2)+\alpha}(S_j), \quad \varphi \in H^{(2)+\alpha}(S_0), \quad \varphi \in C(S), \quad j = 1, 2,$$  

(3.2b)

then the solution of this problem satisfies the inclusion (see [6, 9]):

$$u \in H^{(3)+\alpha_1}(G), \quad u \in H^{\alpha_1}(\overline{G}),$$  

where $t^{(3)} = \min[t^{(1)} + 2, t^{(2)}]$, $\alpha_1 \in (0, 1)$.

Let

$$the \ data \ of \ the \ problem \ (2.2), \ (2.1) \ on \ the \ set \ S^c_{(3.1)}$$  

satisfy compatibility conditions up to order $[t^{(1)}/2]$, $t^{(4)} \leq t^{(3)}$,

(3.2c)

where $[l/2] = \lfloor l/2 \rfloor_{(3.2)}$ is the integer part of the number $l/2$; for a description of compatibility conditions (for the derivatives in $t$ of the solution to the boundary value problem) see [9]. Then the solution of the problem (2.2), (2.1) satisfies the inclusion (see [6, 9]):

$$u \in H^{t^{(4)}+\alpha_1}(\overline{G}).$$  

(3.3)

3.2. In the case when the data of the problem (2.2), (2.1) satisfy the condition (3.2), where

$$t^{(1)} = t^{(2)} = t^{(4)} = l + 2,$$  

(3.4)

then the problem solution satisfies the inclusion [6, 9]:

$$u \in H^{l+2+\alpha}(\overline{G}).$$  

(3.5)

We shall assume that the following condition (we call it the condition (3.6)) holds:

The data of the problem (2.2), (2.1) satisfy the conditions (3.2), (3.4) that guarantee the smoothness of the solution of the boundary value problem on $\overline{G}$. When constructing a priori estimates for the regular and singular components of the solution in the representations (4.3), (4.6), (4.10) (from Section 4), the following condition is assumed to be fulfilled in addition to the conditions (3.2), (3.4):

$$C_{(2.2)}, P \in H^{l_1+\alpha}(\overline{G}), \quad g \in C^{l_1+\alpha,(l_1+\alpha)/2,(l_1+\alpha)}(\overline{Q}),$$  

(3.6)

$$\varphi \in H^{l_1+\alpha}(S_j), \quad \varphi \in H^{l_1+\alpha}(S_0), \quad \varphi \in C(S); \quad j = 1, 2, \quad l_1 \geq l,$$

that guarantee the smoothness of the regular and singular components of the solution.

The actual values of $l$ and $l_1$ are specified where it is required. The fulfillment of other conditions in addition to (3.2), (3.4), (3.6) is not assumed.

Note that the condition (3.6) belongs to sufficient conditions that are required for the construction of a priori estimates, and at the same time, this condition is sufficient simple.
4. *A priori* estimates for solutions

When constructing and studying convergence of classical and special difference schemes, we need estimates of solutions and their derivatives.

4.1. Introducing the new variables \( x_i = \varepsilon^{-1} \bar{x}_i, \ i = 1, 2 \), we bring the problem (2.2), (2.1) to a form in which the coefficients of the highest-order derivatives equal one. In that case derivatives of the function \( \bar{u}(\bar{x}, t) = u(x(\bar{x}), t) \) in the new variables become of order one [6, 9]. Returning to the original variables, in the case of condition (3.6), where

\[ l \geq K - 2, \quad (4.1) \]

we obtain the estimates

\[
|u(x, t)| \leq M, \quad (4.2)
\]

where

\[
|u(x, t)| = \max \frac{|u(x, t)|}{\alpha}, \quad (4.3)
\]

\[
\frac{\partial^{k+k_0}}{\partial x_1^{k_1} \partial x_2^{k_2} \partial t^{k_0}} u(x, t) \leq M \varepsilon^{-k}, \quad (x, t) \in \bar{G}, \quad k + 2k_0 \leq K, \quad k = k_1 + k_2,
\]

**Theorem 4.1** Let the data of the boundary value problem (2.2), (2.1) satisfy the conditions (3.6), (4.1), where \( K \geq 2 \). Then the solution of the problem satisfies the estimates (4.2).

**Remark 1** In the case of condition (2.3), the solution of the boundary value problem (2.2), (2.1) satisfies the estimate

\[
|u(x, t)| \leq 2 (1 - m^2)^{-1} \max [g(x, t, \mathbf{0}) \max \frac{g(x, t, \mathbf{0})}{\alpha}, \max \frac{\varphi(x, t)}{\alpha}], \quad (x, t) \in \bar{G},
\]

where \( m = m_{(2.3)} \). For the component \( u_i(x, t) \) we have the estimate

\[
|u_i(x, t)| \leq m \max \frac{|u^{3-i}(x, t)|}{\alpha} + c_0 \max \frac{|g^i(x, t, \mathbf{0})|}{\alpha} + \max \frac{|\varphi^i(x, t)|}{\alpha}, \quad (x, t) \in \bar{G}, \quad i = 1, 2.
\]

4.2. We now give estimates that are obtained using the main terms of an asymptotic expansion of the solution (see, e.g., [13, 17] in the case of linear equations).

Write the solution of the problem as the sum of the functions

\[
u(x, t) = U(x, t) + V(x, t), \quad (x, t) \in \bar{G}, \quad (4.4)
\]

where \( U(x, t) \) and \( V(x, t) \) are the regular and singular terms of the solution decomposition. The function \( U(x, t), (x, t) \in \bar{G} \), is the restriction to \( \bar{G} \) of the function \( U^0(x, t), (x, t) \in \bar{G}^0 \), where the set \( \bar{G}^0 \), i.e., the extension of \( G \) beyond the boundary \( \bar{G}^L \), includes \( G \) along with its \( m_0 \)-neighbourhood; \( \bar{G}^0 = \bar{D} \times [0, T] \). The function \( U^0(x, t) \) is the solution of the problem

\[
L^0 U^0(x, t) = g^0(x, t, U^0(x, t)), \quad (x, t) \in \bar{G}^0,
\]

\[
U^0(x, t) = \varphi^0(x, t), \quad (x, t) \in S^0.
\]

Here \( L^0 \) and \( g^0(x, t, u), (x, t) \in \bar{G} \), are smooth continuations of the operator \( L_{(2.2)} \) and the function \( g(x, t, u) \) (that preserve the properties of (2.3)); the function \( \varphi^0(x, t), (x, t) \in S^0 \), is
chosen sufficiently smooth: \( \varphi^0(x, t) = \varphi(x, t), (x, t) \in S_0 \). Assume that the functions \( g^0(x, t, u) \) and \( \varphi^0(x, t) \) are equal to zero outside a nearest \( m_1 \)-neighbourhood of the set \( \Gamma \), where \( m_1 < m_0 \).

The function \( V(x, t) \) is the solution of the problem

\[
L_{(2.2)} V(x, t) = g(x, t, U(x, t) + V(x, t)) - g(x, t, U(x, t)), \quad (x, t) \in G,
\]

\[
V(x, t) = \varphi(x, t) - U(x, t) \equiv \varphi_V(x, t), \quad (x, t) \in S. \tag{4.5}
\]

4.2.1. Now we estimate the regular component of the problem solution in the representation (4.3).

Write the function \( U(x, t) \) as the sum of the functions

\[
U(x, t) = \sum_{k=0}^{n} \varepsilon^{2k} U_k(x, t) + v^n_U(x, t) \equiv U^n(x, t) + v^n_U(x, t), \quad (x, t) \in \partial G, \tag{4.6}
\]

that corresponds to the representation of the function \( U^0(x, t), (x, t) \in \partial G \), which is the solution of problem (4.4):

\[
U^0(x, t) = \sum_{k=0}^{n} \varepsilon^{2k} U^0_k(x, t) + v^n_U(x, t), \quad (x, t) \in \partial G. \tag{4.7}
\]

The functions \( U^0_k(x, t), (x, t) \in \partial G \), i.e., components in the expansion of the regular part of the solution, are solutions of the problems

\[
L_{(4.7)} U^0_k(x, t) = g^0(x, t, U^0_k(x, t)), \quad (x, t) \in \partial G \setminus S^0, \tag{4.8}
\]

\[
U^0_k(x, t) = \varphi^0(x, t), \quad (x, t) \in S^0;
\]

\[
U^0_k(x, t) = \varepsilon^{-2} \left\{ L_{(4.7)} - L_{(4.4)} \right\} U^0_{k-1}(x, t) + \varepsilon^{-2k} \left\{ g^0(x, t, \sum_{k_1=0}^{k} \varepsilon^{2k_1} U^0_{k_1}(x, t) - g^0(x, t, \sum_{k_1=0}^{k-1} \varepsilon^{2k_1} U^0_{k_1}(x, t)) \right\}, \quad (x, t) \in \partial G \setminus S^0, \]

\[
U^0_k(x, t) = 0, \quad (x, t) \in S^0, \quad k > 0,
\]

where

\[
L_{(4.7)} = L_{(4.4)}|_{\varepsilon=0} = -C^0(x, t) - P^0(x, t) \frac{\partial}{\partial t}.
\]

For the function \( v^n_U(x, t) \) we have the following estimate (see, e.g., [8]):

\[
|v^n_U(x, t)| \leq M \varepsilon^{2n+2}, \quad (x, t) \in \partial G.
\]

In the case of condition (3.6), where

\[
l \geq K - 2, \quad l_1 \geq K + 2n, \tag{4.8a}
\]

for

\[
n = [(K + 1)/2]_{(3.2)} - 2, \quad K \geq 4, \tag{4.8b}
\]
one has
\[ U^0 \in H^{K+\alpha}(\overline{G}). \]

For the function \( U(x, t) \) we obtain the estimate
\[
\left| \frac{\partial^{k+k_0}}{\partial x_1^{k_1} \partial x_2^{k_2} \partial t^{k_0}} U(x, t) \right| \leq M \left[ 1 + \varepsilon^{K-k-2} \right], \quad (x, t) \in \overline{G}, \quad k + 2k_0 \leq K. \tag{4.9}
\]
Moreover, for the components \( U^n(x, t) \) and \( \mathbf{v}_U^n(x, t) \) we have the estimates
\[
\left| \frac{\partial^{k+k_0}}{\partial x_1^{k_1} \partial x_2^{k_2} \partial t^{k_0}} U^n(x, t) \right| \leq M, \tag{4.11}
\]
\[
\left| \frac{\partial^k}{\partial x_1^{k_1} \partial x_2^{k_2} \partial t^{k_0}} \mathbf{v}_U^n(x, t) \right| \leq M \varepsilon^{K-k-2}, \quad (x, t) \in \overline{G}, \quad k + 2k_0 \leq K.
\]

**Remark 2** According to the decomposition (4.6), the function \( \varphi_{V_{(4.5)}}(x, t) \) has the representation
\[
\varphi_{V}(x, t) = \sum_{k=0}^{n} \varepsilon^{2k} \varphi_k V(x, t) + \varphi_0 V(x, t) \equiv \varphi_{V}^0(x, t) + \varphi_{V}^n(x, t), \quad (x, t) \in S,
\]
where
\[
\varphi_{0V}(x, t) = \varphi(x, t) - U_0(x, t), \quad \varphi_{kV}(x, t) = -U_k(x, t), \quad k \geq 1,
\]
\[
\varphi_0^0(x, t) = -\mathbf{v}_U^0(x, t), \quad (x, t) \in S.
\]

4.2.2. Let us consider the decomposition of the singular part of the solution to the boundary value problem.

We construct the function \( V(x, t) \) as the sum of the functions
\[
V(x, t) = \sum_{k=0}^{n} \varepsilon^{2k} V_k(x, t) + \mathbf{v}_V(x, t) \equiv V^n(x, t) + \mathbf{v}_V(x, t), \quad (x, t) \in \overline{G}. \tag{4.10}
\]
The functions \( V_k(x, t), \quad (x, t) \in \overline{G}, \) i.e., components of the singular part of the problem solution, are solutions of the problems
\[
L_{(4.11)} V_0(x, t) = g(x, t, U_0(x, t) + V_0(x, t)) - g(x, t, U_0(x, t)), \quad (x, t) \in G,
\]
\[
V_0(x, t) = \varphi_{0V}(x, t), \quad (x, t) \in S;
\]
\[
L_{(4.11)} V_k(x, t) = \varepsilon^{-2} \left( L_{(4.11)} - L_{(4.4)} \right) V_{k-1}(x, t) + \varepsilon^{-2k} \left\{ \sum_{k_1=0}^{k} \varepsilon^{2k_1} \left[ U_{k_1}(x, t) + V_{k_1}(x, t) \right] \right\} - \varepsilon^{-2k} \left\{ g(x, t, \sum_{k_1=0}^{k} \varepsilon^{2k_1} \left[ U_{k_1}(x, t) + V_{k_1}(x, t) \right] \right\} - \varepsilon^{-2k} \left\{ \sum_{k_1=0}^{k-1} \varepsilon^{2k_1} \mathbf{v}_{k_1}(x, t) \right\}, \quad (x, t) \in G,
\]
\[
V_k(x, t) = \varphi_{kV}(x, t), \quad (x, t) \in S, \quad k > 0,
\]
where

\[
L_{(4.11)} = \begin{pmatrix}
\frac{\partial^2}{\partial x_1^2} & 0 \\
0 & \frac{\partial^2}{\partial x_2^2}
\end{pmatrix} - C(x, t) - P(x, t) \frac{\partial}{\partial t}.
\]

Under the conditions (3.6), (4.8), applying a technique similar to one given in [23], we obtain the estimate

\[
\left| \frac{\partial^{k+k_0}}{\partial x_1^{k_1} \partial x_2^{k_2} \partial t^{k_0}} V(x, t) \right| \leq M \varepsilon^{-k_1} \exp \left( -m \varepsilon^{-1} r(x, \Gamma) \right),
\]

\[
(x, t) \in \mathcal{G}, \quad k + 2k_0 \leq K.
\]

Here \( r(x, \Gamma) \) is the distance from the point \( x \) to the boundary \( \Gamma \), and \( m \) is an arbitrary constant from the interval \((0, m_0)\), where \( m_0 = \varepsilon_0^{1/2} (1 - m_{(2,3)})^{1/2} \), for \( c_0 = c_{0(2,3)} \).

**Theorem 4.2** Let the data of the boundary value problem (2.2), (2.1) satisfy the conditions (3.6), (4.8), where \( K \geq 4 \). Then the solution components \( U(x, t) \) and \( V(x, t) \) in the decomposition (4.3) satisfy the estimates (4.9) and (4.12).

**Remark 3** In the case when the condition (2.3b) is violated, we pass in the problem (2.2), (2.1) from the function \( u(x, t) \) to the function \( u^*(x, t) \), \( u(x, t) = u^*(x, t) \exp(\alpha t) \). We choose the value \( \alpha \) sufficiently large so as to satisfy the condition

\[
\alpha p_0 + c^{ij}(x, t) - g^i_j(x, t) \geq c_0,
\]

\[
m (\alpha p_0 + c^{ij}(x, t) - g^i_j(x, t)) \geq |c^{ij}(x, t)| + g^i_j(x, t), \quad (x, t) \in \mathcal{G}, \quad i, j = 1, 2, \quad i \neq j,
\]

where \( c_0 > 0 \), \( m \) is an arbitrary constant that satisfies the condition \( m < 1 \), and \( p_0 = p_{0(2,3)} \).

Estimating the function \( u^*(x, t) \) and its components, we return to the function \( u(x, t) \). It is not difficult to verify that the constants \( m \) and \( M \) in an estimate of type (4.12), which is obtained for the function \( V(x, t) \) in that case, depend on \( \alpha \). Moreover, the constant \( m = m(\alpha) \) can be chosen arbitrarily, and the constant \( M = M(\alpha) \) grows as \( \alpha \to \infty \).

5. **Classical finite difference scheme**

5.1. When constructing a finite difference scheme for the problem (2.2), (2.1), we use classical finite difference approximations on rectangular grids (see, e.g., [15]). On the set \( \mathcal{G} \) we introduce the grid

\[
\mathcal{G}_h = \mathcal{D}_h \times \omega_0, \quad \mathcal{D}_h = \omega_1 \times \omega_2.
\]

Here \( \omega_1 \) and \( \omega_2 \) are meshes on the interval \([0, d]\) and per unit length on the \( x_2 \)-axis; \( \omega_0 \) is a mesh on the interval \([0, T]\); all meshes are, in general, arbitrary nonuniform. Set \( h_s^i = x_i^{i+1} - x_i^1 \) with \( x_1^1, x_1^{i+1} \in \omega_1 \) and \( x_2^1, x_2^{i+1} \in \omega_2 \); \( h_s = \max h_s^i \), \( h = \max h_s \) for \( s = 1, 2 \); \( h_t^k = t^{k+1} - t^k \) with \( t^k, t^{k+1} \in \omega_0 \) and \( h_t = \max h_t^k \). Assume that the conditions \( h \leq MN^{-1} \) and \( h_t \leq MN_0^{-1} \) are satisfied, where \( N = \min N_s \) for \( s = 1, 2 \); \( N_1 + 1 \) and \( N_0 + 1 \) are the number of nodes in the meshes \( \omega_1 \) and \( \omega_0 \), respectively; \( N_2 + 1 \) is the minimal number of nodes in the mesh \( \omega_2 \) on the unit interval.

On the grid \( \mathcal{G}_h \) for the solution of the problem, we use the nonlinear difference scheme

\[
\Lambda z(x, t) = g(x, t, z(x, t)), \quad (x, t) \in \mathcal{G}_h,
\]

\[
z(x, t) = \varphi(x, t), \quad (x, t) \in S_h.
\]
Here
\[ G_h = G \cap \overline{G}_h, \quad S_h = S \cap \overline{G}_h, \]
\[ \Lambda z(x, t) = \varepsilon^2 \Lambda_2 z(x, t) - C(x, t) z(x, t) - P(x, t) \delta_T z(x, t), \]
\[ \Lambda_2 = \begin{pmatrix} \Lambda_2^1 & 0 \\ 0 & \Lambda_2^2 \end{pmatrix}, \quad \Lambda_2^1 = \sum_{s=1,2} \delta_{\overline{x_s}} \delta_x, \quad \Lambda_2^2 = 1 \]
\[ z(x, t) = (z^1(x, t), z^2(x, t))^T, \]

In scalar form, the finite difference scheme takes the form
\[ \Lambda^i z(x, t) = g^i(x, t, z(x, t)), \quad (x, t) \in G_h, \]
\[ z^i(x, t) = \varphi^i(x, t), \quad (x, t) \in S_h, \quad i = 1, 2. \] (5.2b)

Here the operator \( \Lambda^i \) is defined by the relation
\[ \Lambda^i z(x, t) = \varepsilon^2 \Lambda_2^i z^i(x, t) - \sum_{j=1,2} \eta^i_j(x, t) z^j(x, t) - p^i(x, t) \delta_T z^i(x, t), \quad i = 1, 2. \] (5.2c)

Also \( \delta_{\overline{x_s}} v(x, t) = v_{\overline{x_s}}(x, t), \quad s = 1, 2, \) and \( \delta_T v(x, t) \) are the second- and the first-order difference derivatives on the nonuniform meshes [15], e.g.,
\[ \delta_{\overline{T_1}} v(x, t) = v_{\overline{T_1}}(x, t) = 2(h_1^i + h_1^{i-1})^{-1} [\delta v(x, t) - \delta_T v(x, t)], \]
where \( (x, t) = (x_1^i, x_2^i, t) \in G_h, \) \( \delta x \) \( v(x, t) \) and \( \delta_T \) \( v(x, t) \) are the first-order (forward and backward) difference derivatives
\[ \delta x \) \( v(x, t) = v_{x_1}(x, t) = (h_1^i)^{-1} [v(x_1^{i-1} - x_2, t) - v(x, t)], \]
\[ \delta_T \) \( v(x, t) = v_{T_1}(x, t) = (h_1^{i-1})^{-1} [v(x_1^i, x_2^i, t) - v(x, t)], \quad (x, t) = (x_1^i, x_2^i) \in G_h. \]

5.2. We study the convergence of the scheme (5.2), (5.1) using the maximum principle [15]. Assume that the solution of the boundary value problem (2.2), (2.1) satisfies the estimates of Theorem 4.1.

Note that the operators
\[ \Lambda^i_{(5.2d)} = \varepsilon^2 \Lambda_2^i - \delta x \) \( p^i(x, t) \) \( \delta_T, \quad (x, t) \in G_h, \quad i = 1, 2 \] (5.2d)
from (5.2c) are monotone [15]. Taking into account the estimate
\[ \left| z^i(x, t) \right| \leq m \max_{\overline{G}_h} \left| z^{3-i}(x, t) \right| + M \left[ \max_{\overline{G}_h} \left| g^i(x, t, 0) \right| + \max_{S_h} \left| \varphi^i(x, t) \right| \right], \]
\[ (x, t) \in \overline{G}_h, \quad i = 1, 2, \]
where \( m < 1 \) by virtue of (2.3), we obtain the estimate
\[ |z(x, t)| \leq M \left[ \max_{\overline{G}_h} \left| g(x, t, 0) \right| + \max_{S_h} \left| \varphi(x, t) \right| \right], \quad (x, t) \in \overline{G}_h. \]

Taking into account the a priori estimates of the solution to the problem (2.2), (2.1) and using the monotonicity of the operator \( \Lambda^i_{(5.2d)}, \) for the solution of the difference scheme (5.2), (5.1) we establish the estimate
\[ |u(x, t) - z(x, t)| \leq M [(\varepsilon + N^{-1})^{-1} N^{-1} + N_0^{-1}], \quad (x, t) \in \overline{G}_h. \] (5.3)
On the uniform grid 
\[ \mathcal{G}_h = \mathcal{D}_h \times \mathcal{W}_0 \]  
we obtain the estimate 
\[ |u(x, t) - z(x, t)| \leq M \left[(\varepsilon + N^{-1})^{-2} N^{-2} + N_0^{-1}\right], \quad (x, t) \in \mathcal{G}_h^\mu. \]  

**Theorem 5.1** Let the solution of the boundary value problem (2.2), (2.1) satisfy the estimates of Theorem 4.1, where \( K = 4 \). Then the solution of the difference scheme (5.2), (5.1) converges for fixed values of the parameter \( \varepsilon \). The discrete solution of the scheme (5.2), (5.1) (the scheme (5.2), (5.4)) satisfies the estimate (5.3) (the estimate (5.5)).

### 6. Special finite difference scheme

#### 6.1. From the estimates of Theorem 4.2 it follows that the derivatives of the solution in a neighbourhood of the boundary \( S^k \) increase without bound as the parameter \( \varepsilon \) tends to zero. In the case of the boundary value problem (2.2), (2.1), the boundary layer is sufficiently simple. To solve the boundary value problem, we apply a piecewise-uniform grid that condenses in a neighbourhood of the boundary.

Let us construct a special finite difference scheme for the problem (2.2), (2.1). On the set \( \mathcal{G} \) we introduce the grid 
\[ \mathcal{G}_h = \mathcal{D}_h \times \mathcal{W}_0, \quad \mathcal{D}_h = \mathcal{D}_h^S = \mathcal{W}_1^S \times \omega_2, \]  
where \( \omega_0 = \omega_0(5.4), \omega_2 = \omega_2(5.4), \mathcal{W}_1^S = \mathcal{W}_1^S(\sigma) \) is a piecewise-uniform mesh on the interval \([0, d]\). The step-size in the mesh \( \mathcal{W}_1^S \) equals \( h_1^{(1)} = 4\sigma N_1^{-1} \) on the sets \([0, \sigma], [d - \sigma, d] \) and \( h_1^{(2)} = 2(d - 2\sigma)N_1^{-1} \) on the set \([\sigma, d - \sigma] \). The value \( \sigma \) is defined by 
\[ \sigma = \sigma(\varepsilon, N_1) = \min \left[4^{-1} d, M \varepsilon \ln N_1\right], \]
where \( M = 2m^{-1}_{(4.12)} \).

To solve the boundary value problem (2.2), (2.1) we use the finite difference scheme 
\[
\Lambda z(x, t) = g(x, t, z(x, t)), \quad (x, t) \in \mathcal{G}_h, \quad z(x, t) = \varphi(x, t), \quad (x, t) \in \mathcal{S}_h, \]  
where \( \Lambda = \Lambda_{(5.2)}, \mathcal{G}_h = \mathcal{G}_{h(6.1)} \).

Taking into account the estimates of Theorem 4.2 and the monotonicity of the operator \( \Lambda_{(5.2d)} \), we establish the \( \varepsilon \)-uniform convergence of the difference scheme (6.2), (6.1) 
\[
|u(x, t) - z(x, t)| \leq M \left[N^{-2} \ln^2 N + N_0^{-1}\right], \quad (x, t) \in \mathcal{G}_h. \]  

**Theorem 6.1** Let the components in the decomposition (4.3) of the solution to the boundary value problem satisfy the estimates of Theorem 4.2 for \( K = 4 \). Then the solution of the difference scheme (6.2), (6.1) converges to the solution of the boundary value problem \( \varepsilon \)-uniformly. The discrete solution satisfies the estimate (6.3).

#### 6.2. To solve the boundary value problem (2.2), (2.1), it is convenient to use a linearized difference scheme where each component \( z^1(x, t) \) and \( z^2(x, t) \) at the temporal level \( t \in \omega_0 \) is
found from the disjoined system of difference equations. We approximate the boundary value problem by the difference scheme

$$\Lambda \mathbf{z}(x, t) = \mathbf{F} \left( \mathbf{z}(x, t), \mathbf{g}(x, t, \mathbf{z}(x, t)) \right), \quad (x, t) \in G_h,$$

$$\mathbf{z}(x, t) = \varphi(x, t), \quad (x, t) \in S_h.$$  \hfill (6.4a)

Here

$$\mathbf{z}(x, t) = \mathbf{z}(x, t^{k-1}), \quad (x, t) = (x, t^k) \in \overline{G}_h, \quad t^k \in \omega_0;$$

$$\Lambda = \Lambda_{(6.4)}(\varepsilon) \equiv \varepsilon \Lambda_2 - C_1(x, t) - P(x, t) \delta_t, \quad \Lambda_2 = \Lambda_{2(5.2)},$$

$$\mathbf{F} \left( \mathbf{z}(x, t), \mathbf{g}(x, t, \mathbf{z}(x, t)) \right) \equiv C_2(x, t) \mathbf{z}(x, t) + \mathbf{g}(x, t, \mathbf{z}(x, t)),$$

$$C_1(x, t) = \begin{pmatrix} c^{11}(x, t) & 0 \\ c^{12}(x, t) & c^{12}(x, t) \end{pmatrix}, \quad C_2(x, t) = \begin{pmatrix} 0 & c^{12}(x, t) \\ 0 & 0 \end{pmatrix}.$$  \hfill (6.4b)

The components \( z^i(x, t), (x, t) \in \overline{G}_h, \ i = 1, 2, \) are found from the disjoined system of linear difference equations

$$\Lambda^1 z^1(x, t) \equiv \left\{ \varepsilon^2 \sum_{s=1,2} \delta x_{x^s x^s} - c^{11}(x, t) - p^1(x, t) \delta_t \right\} z^1(x, t) =$$

$$= c^{12}(x, t) z^2(x, t^{k-1}) - g^1(x, t, \mathbf{z}(x, t^{k-1})) \equiv \mathbf{F}^1 \left( z^2(x, t), g^1(x, t, \mathbf{z}(x, t)) \right),$$

$$\Lambda^2 z^2(x, t) \equiv \left\{ \varepsilon^2 \sum_{s=1,2} \delta x_{x^s x^s} - c^{22}(x, t) - p^2(x, t) \delta_t \right\} z^2(x, t) =$$

$$= c^{21}(x, t) z^1(x, t^{k-1}) - g^2(x, t, \mathbf{z}(x, t^{k-1})) \equiv \mathbf{F}^2 \left( z^1(x, t), g^2(x, t, \mathbf{z}(x, t)) \right), \quad (x, t) \in \overline{G}_h, \quad t = t^k \in \omega_0.$$  \hfill (6.4c)

Taking into account the estimates of Theorem 4.2 and the monotonicity of the operator \( \Lambda^i \), one can derive the estimate

$$| \mathbf{u}(x, t) - \mathbf{z}(x, t) | \leq M [ N^{-2} \ln^2 N + N_0^{-1}], \quad (x, t) \in \overline{G}_h.$$  \hfill (6.5)

Thus, we have the following theorem.

**Theorem 6.2** Let the hypotheses of Theorem 6.1 be satisfied. Then the solution \( \mathbf{z} \) of the linearized difference scheme (6.4), (6.1) converges to the solution \( \mathbf{u} \) of the boundary value problem (2.2), (2.1) \( \varepsilon \)-uniformly. The discrete solution satisfies the estimate (6.5).

7. **Remarks**

7.1. The condition (2.3b) can be replaced by a simpler condition

$$c^{ij}(x, t) \geq c_0, \quad mc^{ij}(x, t) \geq |c^{ij}(x, t)| + \sum_{s=1,2} \max_{u \in H^2} \left| \frac{\partial}{\partial u_s} g^i(x, t, u) \right|,$$  \hfill (7.1)

\((x, t) \in \overline{G}, \quad i, j = 1, 2, \quad i \neq j, \quad m = m_{(7.1)} < 1.\)

7.2. The technique that is shown here for the construction and investigation of \( \varepsilon \)-uniformly convergent difference schemes for the problem (2.2), (2.1) allows us to construct \( \varepsilon \)-uniformly convergent difference schemes for a system with variable coefficients multiplying the highest-order derivatives and also for a system of \( p \) equations, where \( p > 2. \)
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