Taming identical particles for discerning the genuine nonlocality

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This work provides a comprehensive method to analyze the entanglement between subsystems generated by identical particles that preserves superselection rules (SSR), i.e., particle number SSR (NSSR) for bosons and parity SSR (PSSR) for fermions. Contrary to the common belief that the quantification of identical particles’ entanglement needs some special techniques, it can be treated in a fundamentally equivalent manner to the non-identical particles’ entanglement, which is achieved by the combination of symmetric/exterior algebra (SEA) and microcausality. We also show that the total Hilbert space of identical particles is factorized according to the local distribution of subsystems. This formal correspondence between identical and non-identical particle systems turns out very useful to quantify the non-locality generated by identical particles, such as the maximal Bell inequality violation and the GHJW theorem of identical particles.

INTRODUCTION

The particle identity is one of the essential quantum features, which is formally represented as the exchange symmetry between states of particles. Nevertheless, the notion of entanglement mostly has developed based on the presupposition that non-identical particles distribute over distinguishable sectors. This is mostly by a puzzle that arises when considering the entanglement and particle identity simultaneously. The exchange symmetry of identical particles evokes superposed forms of wave functions, which seems mathematically equivalent to entangled states. Hence, it becomes subtle to discriminate physically extractable entanglement in multipartite systems of identical particles.

There have been several attempts to quantify the physical entanglement of identical particles that employ several techniques and definitions, such as Slater number [3] or subalgebra restriction [6, 7]. Also, a recently introduced method, which the authors named the non-standard approach (NSA) [8, 9], introduced a seemingly unorthodox formalism to compute genuine non-local properties of identical particles. With all of these formally uncustomed approaches, quantifying identical particles’ quantum correlation seems quite alien and still unclear compared to the non-identical case.

Here, we suggest a method to unravel the technical intricacy of identical particles so that their non-local properties can be analyzed fundamentally equivalent to those of non-identical particles. Two pillars for our scheme are the symmetric/exterior algebras (abbreviated as SEA in these works) [10] and the presupposition that non-identical particles distribute over distinguishable sectors. This is mostly by a puzzle that arises when considering the entanglement and particle identity simultaneously. The exchange symmetry of identical particles evokes superposed forms of wave functions, which seems mathematically equivalent to entangled states. Hence, it becomes subtle to discriminate physically extractable entanglement in multipartite systems of identical particles.

Speaking of SEA, even if it is not very exotic to describe identical particles as elements of Fock spaces according to their exchange symmetry (see, e.g., Ref. [12]), the advantage of using SEA in quantum information processing has not been well appreciated. Refs. [8, 10], of which the method is implicitly based on SEA, showed that SEA can be a useful tool to define the entanglement of identical particles and fundamental measures such as entanglement entropy. However, the mathematical description of identical particles with SEA in these works is incomplete, and it is not clarified how the first quantization language (1QL) extends to the SEA language of identical particles. In this work, we provide an exhaustive explanation of the formalism for analyzing the nonlocality of identical particles, in which Bosonic (fermionic) states are represented as symmetric (exterior) tensor products that generate bosonic (fermionic) Fock spaces.

Microcausality states that spacelike separated local operators always commute, hence cannot influence each other superluminally. Microcausality is a fundamental axiom to define the entanglement of identical particles in two aspects: First, it imposes the parity superselection rule (PSSR, by which fermionic states with different number parity cannot superpose) to fermions [13, 14]. Second, the commutation relations of operators defined in distinctive subsystems also provide criteria for the separability of identical particles’ states.

Utilizing SEA and microcausality, we can present rigorous separability conditions of identical particles that preserves superselection rules (SSR), i.e., particle number SSR (NSSR) for bosons, and parity SSR (PSSR) for fermions. And the total Hilbert space of identical particles is factorizable as non-identical ones, which is achieved with the concept of quantum causality [15]. Consequently, we can state that the SEA-based formalism under the restriction of microcausality is optimized to analyze the non-local properties of identical particles, for it can employ well-developed techniques for non-identical particles’ nonlocality. The interior product between tensor products defines the reduced density matrices of subsystems. GHJW theorem of identical particles is also proved directly in the formalism, and the maximal Bell inequality violation from the superposition of vacuum and four-fermion state can be computed.
MATHEMATICAL DESCRIPTION OF IDENTICAL PARTICLES IN FOCK SPACES

The second quantization language (2QL) is usually considered the best way to treat a set of identical particles in many-body quantum systems. However, when it comes to quantum information processing, fundamental definitions of entanglement in 2QL seems unclear and needs quite abstract algebraic methods [1-7].

To analyze the entanglement of identical particles in a more concrete and intuitive manner, we introduce a midway language between 1QL and 2QL, based on the symmetric/exterior algebras (SEA, we say that it is a “midway language” because, on the one hand, it resembles 1QL’s character to denote states of particles directly, and on the other hand, it also resembles 2QL’s character to inherently discard particle pseudo-labels). The language itself is well-known in the mathematical physics community [12], though the advantage of applying it to entanglement is not well-recognized.

One crucial feature of SEA is that it directly reveals the exchange symmetry among particles. 1QL achieves the exchange symmetries of N identical particles by superposing N-particle wave functions so that they become symmetric (for bosons) or antisymmetric (for fermions) under the switches of particle pseudo-labels. On the other hand, SEA includes notations that explicitly denote the symmetries, which are the symmetric product ∨ and exterior (or antisymmetric) product ∧.

Suppose a particle has state \( \Psi_i = (\psi_i, s_i) \) where \( \psi_i \) is the spatial wavefunction and \( s_i \) contains all the possible internal degrees of freedom. Then the wavefunction of \( N \) identical particles is expressed along the following definition.

**Definition 1.** (Symmetric and exterior tensor products) For a Hilbert space \( \mathcal{H} \) (dim \( \mathcal{H} = d \)) and state vectors \( \{ |\Psi_i \rangle \}_{i=1}^N \in \mathcal{H} \) with \( |\Psi_i \rangle = \sum_{\sigma=1}^d \Psi_i^\sigma |\sigma \rangle \) and \( \Psi_i^\sigma \in \mathbb{C} \), the symmetric tensor product \( \vee \) is defined as

\[
|\Psi_1 \rangle \vee \cdots \vee |\Psi_N \rangle = \frac{1}{\mathcal{N}} \sum_{\sigma \in S_N} |\Psi_{\sigma(1)} \rangle \otimes \cdots \otimes |\Psi_{\sigma(N)} \rangle
\]

where \( \mathcal{N} \) is the normalization factor and \( S_N \) is the N permutation group. And exterior (or antisymmetric) tensor product \( \wedge \) is defined as

\[
|\Psi_1 \rangle \wedge \cdots \wedge |\Psi_N \rangle = \frac{1}{\mathcal{N}} \sum_{\sigma \in S_N} (-1)^\sigma |\Psi_{\sigma(1)} \rangle \otimes \cdots \otimes |\Psi_{\sigma(N)} \rangle
\]  

(2)

where \((-1)^\sigma\) is the signature of \( \sigma \).

Eqs. (1) and (2) correspond to the wavefunctions of \( N \) bosons and fermions respectively. One can notice that the above definition directly connects states written in 1QL to those in SEA. A closed subspace of \( \mathcal{H}^{\otimes N} \) generated by \( |\Psi_1 \rangle \vee \cdots \vee |\Psi_N \rangle \) is denoted by \( \mathcal{H}_N \) in which \( N \) bosons reside, and a closed subspace of \( \mathcal{H}^{\otimes N} \) generated by \( |\Psi_1\rangle \wedge \cdots \wedge |\Psi_N \rangle \) is denoted by \( \mathcal{H}_N \) in which \( N \) fermions reside. These two subspaces compose Fock spaces, which are algebraic constructions of single Hilbert space for un-fixed number of identical particles. The bosonic Fock space over \( \mathcal{H} \) is defined as \( F_b(\mathcal{H}) = \bigoplus_{N=0}^\infty \mathcal{H}^\otimes N \) and the fermionic Fock space as \( F_f(\mathcal{H}) = \bigoplus_{N=0}^\infty \mathcal{H}^\otimes N \) with the definition \( \mathcal{H}_0 = \mathbb{C} \). \( \mathcal{H}_0 \) is the Hilbert space for the vacuum state \(|\text{vac}\rangle\) (the role of \(|\text{vac}\rangle\) turns out important in the definition of PSSR entanglement for fermions). From now on, \( \otimes_\pm \) will be used when the algebra can be any of the two tensor products \( \vee \) and \( \wedge \).

The transition amplitude from a state \( |\Psi_1 \rangle \otimes_\pm \cdots \otimes_\pm |\Psi_N \rangle \) to \( |\Phi_1 \rangle \otimes_\pm \cdots \otimes_\pm |\Phi_N \rangle \) can be defined as the scalar product of two multilinear tensors, which results in

\[
\langle \Phi_N | \vee \cdots \vee \langle \Phi_1 | \cdot |\Psi_1 \rangle \vee \cdots \vee |\Psi_N \rangle = \frac{1}{N^2} \text{Per}[|\Psi_1 \rangle |\Psi_j \rangle],
\]

\[
\langle \Phi_N | \wedge \cdots \wedge \langle \Phi_1 | \cdot |\Psi_1 \rangle \wedge \cdots \wedge |\Psi_N \rangle = \frac{1}{N^2} \text{Det}[|\Psi_1 \rangle |\Psi_j \rangle],
\]

(3)

where \( \text{Per} \) and \( \text{Det} \) mean the permanent and determinant of a \( N \times N \) matrix with entries \( \langle \Phi_1 | \Phi_j \rangle \).

Creation and annihilation operators \( (\hat{a}^\dagger, \hat{a}) \) are defined in SEA as follows:

**Definition 2.** The creation operator \( \hat{a}^\dagger(\Psi) \) from \( \mathcal{H}^{\otimes N} \) to \( \mathcal{H}^{\otimes N+1} \) is defined as

\[
\hat{a}^\dagger(\Psi)(|\Psi_1 \rangle \otimes_\pm \cdots \otimes_\pm |\Psi_N \rangle) = |\Psi_\Psi \rangle \otimes_\pm |\Psi_1 \rangle \otimes_\pm \cdots \otimes_\pm |\Psi_N \rangle.
\]

(4)

The annihilation operator \( \hat{a}(\Psi) \) from \( \mathcal{H}^{\otimes N} \) to \( \mathcal{H}^{\otimes N-1} \) is defined with the concept of the interior product as

\[
\hat{a}(\Psi)(|\Psi_1 \rangle \otimes_\pm \cdots \otimes_\pm |\Psi_N \rangle) = \sum_{i=1}^N (-1)^{i-1} (|\Psi_i \rangle |\Psi_1 \rangle \otimes_\pm \cdots \otimes_\pm (|\Psi_i \rangle) \otimes_\pm \cdots \otimes_\pm |\Psi_N \rangle,
\]

(5)

where \( |\Psi_i \rangle \) in the last line means that the state \( |\Psi_i \rangle \) is absent.

It is direct to generalize Eqs. (4) and (5) to multi-particle creation and annihilation. For example,

\[
\hat{a}^\dagger(\Psi)^\dagger(\Psi)(|\Psi_1 \rangle \otimes_\pm \cdots \otimes_\pm |\Psi_N \rangle) = \hat{a}^\dagger(\Psi')(\Psi \otimes_\pm |\Psi_1 \rangle \otimes_\pm \cdots \otimes_\pm |\Psi_N \rangle)
\]

\[
= |\Psi' \rangle \otimes_\pm |\Psi_1 \rangle \otimes_\pm \cdots \otimes_\pm |\Psi_N \rangle,
\]

(6)

etc. The \( n \) particle creation and annihilation processes correspond to the following bilinear maps:

\[
\mathcal{H}^{\otimes N} \otimes \mathcal{H}^{\otimes n} \rightarrow \mathcal{H}^{\otimes (N+n)} \quad \text{(creation)}
\]

\[
\mathcal{H}^{\otimes N} \cdot \mathcal{H}^{\otimes n} \rightarrow \mathcal{H}^{\otimes (N-n)} \quad \text{(annihilation)}
\]

(7)
The SSR-Preserving Entanglement of Identical Particles

A principal prerequisite to specify the entanglement of identical particles that entails the non-locality is to group the particles according to their spatial relations. To discuss the nonlocality of identical particles, we consider here that the particles distribute over more than two subsystems \( \{X_a\}_{a=1}^{P \geq 2} \) that locate in distinguishable places, i.e.,

\[
(X_a|X_b) = \delta_{ab}.
\]

The entanglement we discuss in this section is the entanglement between the subsystems that can be extracted by actual detectors that locate at the subsystems. And from the condition Eq. (8), operators acting on different subsystems commute with each other, i.e.,

\[
[\mathcal{O}_{X_a}, \mathcal{O}_{X_b}] = 0, \quad (a \neq b)
\]

which connects our entanglement to those defined based on algebraic methods [4]. The commutation relation Eq. (9) is a three-dimensional version of microcausality, which means that operators acting on spacelike separated regions commute,

\[
[\mathcal{O}_{X_a}(a), \mathcal{O}_{X_b}(b)] = 0 \quad \text{if} \quad (X_a - X_b)^2 - (a - b)^2 < 0.
\]

Even if we do not consider the relativistic effect on the entanglement, the microcausality is still crucial for our discussion. First of all, PSSR for fermions [13, 14, 16] is derived from the microcausality, hence the entanglement of identical particles that preserves SSR preserves microcausality. It also provides a criterion that restricts possible orthogonal states of each subsystem, by which the most general separability condition of a multipartite system is achieved.

We present two types of SSR-preserving entanglement of identical particles, one for bosons that follows particle number SSR (NSSR) and the other for fermions that follows number parity SSR (PSSR) [17, 18].

NSSR-preserving entanglement of bosons

Our discussion starts from the simple bipartite case. Suppose that there exist two systems \( X \) and \( Y \) that locate far from each other and have never exchanged any information, therefore separated. Over \( X \) spread \( n \) identical bosons and over \( Y \) spread \( (N - n) \) identical bosons. Each boson has an internal degree of freedom \( s_i \) with \( i = 1, \ldots, S \). Then, in the SEA formalism, the total \( N \) boson state is written in the general form as

\[
|\Psi_N\rangle = (\sum_{a} \psi^a_X|X, s^a_i\rangle \otimes \cdots \otimes |X, s^a_n\rangle + \sum_{b} \psi^b_Y|Y, s^b_{n+1}\rangle \otimes \cdots \otimes |Y, s^b_N\rangle)
\]

\[
\equiv |\Psi^X_N\rangle \vee |\Psi^Y_N\rangle,
\]

where \( \psi^a_X \) and \( \psi^b_Y \) are complex numbers for the wave function normalization. One can see that \( |\Psi_N\rangle \) is separable with respect to the systems \( X \) and \( Y \), because \( |\Psi^X_N\rangle \) and \( |\Psi^Y_N\rangle \) can be prepared in each system independently. Indeed, using Definition 2, \( |\Psi_N\rangle \) is prepared with creation operators as

\[
|\Psi_N\rangle = (\sum_{a} \psi^a_X \hat{a}^\dagger(X, s^a_i) \cdots \hat{a}^\dagger(X, s^a_n))
\]

\[
\times (\sum_{b} \psi^b_Y \hat{a}^\dagger(Y, s^b_{n+1}) \cdots \hat{a}^\dagger(Y, s^b_N))|\text{vac}\rangle
\]

\[
\equiv \hat{a}^\dagger(\Psi^X_N) \hat{a}^\dagger(\Psi^Y_{N-n})|\text{vac}\rangle,
\]

where \( \hat{a}^\dagger(\Psi^X_N), \hat{a}^\dagger(\Psi^Y_{N-n}) \) = 0.

This expression is powerful when we consider the third system that has \( l \) identical bosons in a state \( |\Psi^Z_l\rangle \) with no interaction to \( X \) and \( Y \). Even if the total state then can be rewritten as \( |\Psi^X_N\rangle \vee |\Psi^Y_{N-n}\rangle \vee |\Psi^Z_l\rangle \), two communicators in \( X \) and \( Y \) do not need to take \( |\Psi^Z_l\rangle \) into account to evaluate the non-locality of them. Hence, with this \( \vee \) notation (or \( \wedge \) notation for fermions) one can treat any multipartite system of identical particles similar to the distinguishable particle case.

By extending the above discussion, the most general statement for the separable states of \( N \) boson in \( P \) subsystems is possible.

The separability condition of bosonic states. A set of \( N \) bosons that spreads over \( P \) subsystems \( X_i \) \( (i = 1, \ldots, P) \) are separable if and only if the total state is given by

\[
|\Psi\rangle = \vee_{i=1}^{P} (\sum_{a} \psi^a_{X_i}|X_i, s^a_{i}\rangle \otimes \cdots \otimes |X_i, s^a_{n_i}\rangle)
\]

where \( \sum_{i=1}^{P} n_i = N \).

Our separability condition can be considered the generalization of that introduced in Ref. [13].

It is quite straightforward to define several entanglement measures for bipartite bosonic states that vanish when the states are separable. Here we present the definition of entanglement entropy as an example.

Entanglement entropy. The entropy of a bipartite system that consists of \( X \) and \( Y \) can be defined with
the symmetrized partial trace \[8, 11, 20\]. If a subsystem \(X\) with \(N\) bosons has a complete orthonormal basis set \(\{ |X, s_1^N \rangle \} \equiv \{(s_1, \ldots, s_N) \rangle |X\}\), the identity matrix is given by \(\mathbb{I}_X = \sum_s |s_1, \ldots, s_N \rangle \langle s_1, \ldots, s_N| |X\rangle\). The reduced density matrix of \(X\) with respect to a total state \(|\Psi\rangle\) is defined as
\[
\rho_X = \text{Tr}_Y (|\Psi\rangle \langle \Psi|)
\]
and the entropy is given by \(E(|\Psi\rangle) = -\text{Tr}(\rho_X \ln \rho_X)\). Ref. \[20\] connects the symmetrized partial trace to the subalgebra restriction \[3, 7\] in algebraic quantum mechanics.

As a simple example, we compute the entropy of two states, one of which is separable and the other is entangled according to the criterion \[13\]. Suppose the internal states have two degrees of freedom (\(\uparrow\) and \(\downarrow\)). Then \(\mathbb{I}_X\) is given by \(\mathbb{I}_X = \sum_{r,s=\uparrow,\downarrow} |r,s\rangle \langle r,s| |X\rangle\). First consider a separable state
\[
|\Psi_{\text{sep}}\rangle = (|\alpha\rangle \uparrow \uparrow \rangle |X \rangle \uparrow \downarrow \rangle Y \rangle + |\beta\rangle \downarrow \downarrow \rangle |X \rangle \uparrow \downarrow \rangle Y \rangle 
\]
with \(|\alpha|^2 + |\beta|^2 = 1\). By Eq. (14), the reduced density matrix of the subsystem \(Y\) is given by
\[
\rho_Y = |\Psi_Y\rangle \langle \Psi_Y|\]
and \(E(|\Psi_{\text{sep}}\rangle) = 0\). On the other hand, if the total state is entangled, e.g.,
\[
|\Psi_{\text{ent}}\rangle = |\alpha\rangle \uparrow \uparrow \rangle \uparrow \downarrow \rangle X \rangle \downarrow \downarrow \rangle Y \rangle 
\]
we have
\[
\rho_Y = |\alpha|^2 \uparrow \uparrow \rangle \langle \uparrow \uparrow| + |\beta|^2 \downarrow \downarrow \rangle \langle \downarrow \downarrow| > 0.
\]

**PSSR-preserving entanglement of fermions**

As we have briefly explained at the beginning of this section, the microcausality renders one to conceive the PSSR-preserving entanglement among local regions. Unlike bosons that can infinitely condensate in the same local subsystem, the possible maximal number of fermions in the same mode is limited by the Pauli exclusion principle. The maximal total fermion number \(\text{max}(N)\) is determined by the subsystem number \(P\) and the internal d.o.f. of the fermions \(S\), i.e., \(\text{max}(N) = PS\).

We first consider the simplest case, i.e., the bipartite spin half fermions (\(\text{max}(N) = 4\)). With the two subsystems \((X, Y)\) and spin states \((\uparrow, \downarrow)\), a separable total state of even parity \(|\Psi^{\text{even}}\rangle\) is given by
\[
|\Psi^{\text{even}}\rangle = \alpha \left[ (|p\rangle |\text{vac}\rangle + |q\rangle \downarrow \downarrow \rangle Y \rangle \downarrow \downarrow \rangle Y \rangle \right] 
+ \beta \left[ (|s_1\rangle \uparrow \uparrow \rangle \uparrow \downarrow \rangle X \rangle \downarrow \downarrow \rangle Y \rangle \right].
\]

The total vacuum state \(|\text{vac}\rangle\) is expressed in the local form as \(|\text{vac}\rangle \otimes |\text{vac}\rangle\). Note that since PSSR is conserved not only in the total system but also in each subsystem, the first line and second line of Eq. (19) can not superpose from the observers in \(X\) and \(Y\). Similarly, an odd fermion state \(|\Psi^{\text{odd}}\rangle\) is separable when it has the form
\[
|\Psi^{\text{odd}}\rangle = \alpha \left[ \sum_{s_1} \alpha_{s_1} |s_1\rangle \uparrow \uparrow \rangle \uparrow \downarrow \rangle X \rangle \downarrow \downarrow \rangle Y \rangle \right] 
+ \beta \left[ \sum_{s_2} \alpha_{s_2} |s_2\rangle \downarrow \downarrow \rangle \downarrow \downarrow \rangle X \rangle \downarrow \downarrow \rangle Y \rangle \right].
\]

The generalization to the bipartite system with an arbitrary internal \(S\) states is straightforward.

**The separability conditions of fermionic states**. A set of fermions that spread over two subsystems \(X\) and \(Y\) with internal \(S\) states are separable if and only if the total state is given by
\[
|\Psi\rangle = \alpha \left[ \sum_{k=0}^{s-1} \sum_{s_1, \ldots, s_{2k+1}} \alpha_{s_1, \ldots, s_{2k+1}} |s_1, \ldots, s_{2k+1}\rangle \downarrow \downarrow \rangle X \rangle \downarrow \downarrow \rangle Y \rangle \right] 
+ \beta \left[ \sum_{k=0}^{s-1} \sum_{s_1, \ldots, s_{2k}} \beta_{s_1, \ldots, s_{2k}} |s_1, \ldots, s_{2k}\rangle \downarrow \downarrow \rangle X \rangle \downarrow \downarrow \rangle Y \rangle \right]
+ \gamma \left[ \sum_{k=0}^{s-1} \sum_{s_1, \ldots, s_{2k}} \gamma_{s_1, \ldots, s_{2k}} |s_1, \ldots, s_{2k}\rangle \downarrow \downarrow \rangle X \rangle \downarrow \downarrow \rangle Y \rangle \right].
\]
and

\[ |\Psi^{\text{even}}\rangle = \alpha \left( \sum_{k=0}^{\frac{N-1}{2}} \sum_{s_1, \ldots, s_{2k+1}} a_{s_1 \ldots s_{2k+1}} |s_1, \ldots, s_{2k+1}\rangle X \right) \wedge \left( \sum_{k=0}^{\frac{N-1}{2}} \sum_{s_1, \ldots, s_{2k}} b_{s_1 \ldots s_{2k+1}} |s_1, \ldots, s_{2k+1}\rangle Y \right) + \beta \left( \sum_{k=0}^{\frac{N-1}{2}} \sum_{s_1, \ldots, s_{2k}} c_{s_1 \ldots s_{2k}} |s_1, \ldots, s_{2k}\rangle X \right) \wedge \left( \sum_{k=0}^{\frac{N-1}{2}} \sum_{s_1, \ldots, s_{2k}} d_{s_1 \ldots s_{2k}} |s_1, \ldots, s_{2k}\rangle Y \right) \]

(22)

with the definition

\[ \sum_{s_1, \ldots, s_{2k}} a_{s_1 \ldots s_{2k}} |s_1, \ldots, s_{2k}\rangle |v_{0} = a_{0}|\text{vac}\rangle. \]

(23)

It is not hard to write down separable states of the general \( P \)-partitite case, though the equation forms are much more complicated.

**Entanglement entropy.** Defining and computing the entanglement entropy for fermions are fundamentally not very different from bosonic one (Eq. (14)), except that the identity matrix follows the parity SSR and Pauli exclusion principle. Here we present \( P = S = 2 \) example. For this case the even and odd identity matrices of the subsystem \( X \) are given by

\[ Y_{X}^{\text{even}} = |\text{vac}\rangle\langle \text{vac}| X + |\uparrow, \uparrow\rangle\langle \downarrow, \uparrow| X \]

\[ Y_{X}^{\text{odd}} = |\uparrow\rangle\langle \uparrow| X + |\downarrow\rangle\langle \downarrow| X \]

(24)

When the state is even and separable as Eqs. (19), there exist two reduced density matrices according to the parity of the subsystem \( Y \):

\[ \rho_{X}^{\text{even}} = \text{Tr}_{X} (Y_{X}^{\text{even}} |\Psi\rangle \langle \Psi|) \]

\[ = |\alpha|^{2} (r|\text{vac}\rangle Y + s |\uparrow, \downarrow\rangle Y) \cdot (h.c.), \]

\[ \rho_{Y}^{\text{even}} = |\beta|^{2} (s_{a} |s_{b}\rangle Y) \cdot (h.c.). \]

(25)

Since both matrices are pure, the total entanglement entropy is zero as expected.

For an entangled even state

\[ |\Psi_{\text{ent}}^{\text{even}}\rangle = \alpha (p|\text{vac}\rangle X \wedge |\text{vac}\rangle Y + q |\uparrow, \downarrow\rangle X \wedge |\uparrow, \downarrow\rangle Y) \]

\[ + \beta (r |X, \uparrow\rangle \wedge |Y, \downarrow\rangle + s (|X, \downarrow\rangle \wedge |Y, \uparrow\rangle)), \]

(26)

the two types of reduced density matrices are given by

\[ \rho_{X}^{\text{even}} = |p|^{2} |\text{vac}\rangle \langle \text{vac}| Y + |q|^{2} |\uparrow, \downarrow\rangle \langle \uparrow, \downarrow| \]

\[ \rho_{Y}^{\text{even}} = |r|^{2} |Y, \downarrow\rangle \langle Y, \downarrow| + |s|^{2} |Y, \uparrow\rangle \langle Y, \uparrow| \]

(27)

and

\[ E(\rho_{\text{ent}}) = |\alpha|^{2} E(\rho_{X}^{\text{even}}) + |\beta|^{2} E(\rho_{Y}^{\text{even}}) \]

\[ = - |\alpha|^{2} (|p|^{2} \ln(|p|^{2}) + |q|^{2} \ln(|q|^{2})) \]

\[ - |\beta|^{2} (|r|^{2} \ln(|r|^{2}) + |s|^{2} \ln(|s|^{2})) > 0. \]

The discussion so far has shown that the quantum non-locality of identical particles can be rigorously analyzed in a very similar manner to that of non-identical particles.

**FACTORIZING THE HILBERT SPACE OF IDENTICAL PARTICLES**

Once particles are grouped by their locations, quantifying the physically tangible entanglement of identical particles seems to follow the same process with the non-identical particle case. For example, observing the state \(|\text{vac}\rangle\), the symmetric tensor product \( \vee \) between \(|\Psi_{X}^{\text{even}}\rangle \) and \(|\Psi_{Y}^{\text{even}}\rangle \) plays the role of the direct tensor product \( \otimes \) in non-identical particle systems.

In this section, we show that this correspondence is not a coincidence and the (anti-) symmetric products \( \otimes_{\pm} \) can be replaced with \( \otimes \) in a rigorous way. In other words, the Hilbert space of identical particles are factorizable as that of non-identical particles. The factorized Hilbert spaces of identical particles are, however, not particle Hilbert spaces but local Hilbert spaces, in which each local subsystem corresponds to a Hilbert subspace that constructs the total Hilbert space. The following theorem clearly states the factorizability of the local Hilbert space.

**Theorem 1.** If identical particles spread over two subsystems \( X \) and \( Y \), and the subsystems are distinctive, then the total Hilbert space is factorized as \( H_{X} \otimes H_{Y} \).

**Proof.** A brief sketch of the whole proof is given in the form of a syllogism: 1. A Hilbert space \( H \) is factorizable if and only if the system is quantum causal \( 1_{5} \). 2. A particles’ system is quantum causal. 3. Combining 1 and 2, the total Hilbert space is factorizable \( (H = H_{X} \otimes H_{Y}) \).

To understand the first step, one first need to get used to the concept of quantum causal (or quansality) \( 1_{5} \), which we explain briefly here. Assume that Alice is in \( X \) and Bob is in \( Y \). Alice can choose a measurement operation \( x \) and produce a datum \( a \), and Bob can choose a measurement operation \( y \) and produce a datum \( b \). If they can compare their results after obtaining many data, they can estimate the set of probability distributions \( \{P(a, b, |x, y\rangle)\} \) for all possible \((a, b, x, y)\). Then, the notion of quantum causal is defined as follows \(1_{5}\):
Definition 3. $P(a, b | x, y)$ is quantum causal if there exist a Hilbert space $\mathcal{H}_Y$, projector operators $\{ F^a_b : \sum_b F^a_b = I_Y \}$, and a set of subnormalized quantum states $\{ \sigma^x_a \}$ (a possible state of Bob when Alice activates $x$ and produces $a$) such that
\[
P(a, b | x, y) = \text{Tr}(F^a_b \sigma^x_a), \quad \sum_a \sigma^x_a = \sigma.
\] (29)

Here $\sigma$ is independent of $x$.

Hence, the statement that Bob’s system is quantum causal means that it is independent of Alice’s system and also compatible with quantum mechanics. And Lemma 4 of Ref. [13] shows that $P(a, b | x, y)$ is quantum causal if and only if $\mathcal{H} = \mathcal{H}_X \otimes \mathcal{H}_Y$.

It is not hard to see that the second step is true, for a subnormalized state of Bob corresponding to the data of Alice can be obtained using the partial trace technique on the total state as we have seen in the former section. Since the reduced density matrix of $Y$ is independent of Alice’s basis choice in $X$, we can see that Bob’s system is independent of Alice’s.

With Theorem 1, we can write a $N$-particle state, e.g.,
\[
|X, s_1 \rangle \otimes \cdots \otimes |X, s_n \rangle \otimes |Y, s_{n+1} \rangle \otimes \cdots \otimes |Y, s_N \rangle
\] (30)
in a factorized form
\[
|s_1, \ldots, s_n \rangle_X \otimes |s_{n+1}, \ldots, s_N \rangle_Y.
\] (31)

Theorem 1 is closely related to Tsirelson’s theorem [21], which shows that a quantum system with a factorized Hilbert space is equivalent to a system with two sets of commuting projection operators in a finite-dimensional Hilbert space. Theorem 1 can be applied to the entanglement problems of quantum fields with identical particles. If each region is supposed to separate far enough from each other, the factorization property of Hilbert space is still valid in quantum fields. On the other hand, if the sub-regions are adjacent to each other, one should cautiously consider the boundary effect.

The practical advantage of factorizing identical particles’ Hilbert spaces is that it makes simpler the derivation of several non-local properties in identical particles’ systems. It will become clear in the next sections where we see the identical particle version of the GHJW theorem and Bell inequality violation.

**GHJW THEOREM OF IDENTICAL PARTICLES**

Here we see how an entangled state of identical particles can raise a nonlocal phenomenon by delving into the GHJW theorem [22, 23]. Even if the focus is on the bosonic case, its extension to the fermionic case is straightforward.

**Lemma 1.** Suppose that $|\Psi\rangle$ and $|\Psi'\rangle$ are $N$-boson vectors in $\mathcal{H}^\otimes N$ so that $n$ particles locate in $X$. If $\text{Tr}_Y |\Psi\rangle\langle\Psi| = \text{Tr}_Y |\Psi'\rangle\langle\Psi'|$, then there exists a unitary operation in the system $U = I_X \otimes U_Y$ that satisfies $|\Psi\rangle = U|\Psi'\rangle$.

**Proof.** The reduced density matrix can be written as
\[
\text{Tr}_Y |\Psi\rangle\langle\Psi| = \text{Tr}_Y |\Psi'\rangle\langle\Psi'| = \sum_s w_s |s\rangle_X \langle s|_X
\] (32)

where $\vec{s} = (s_1, \ldots, s_n)$. For any complete orthonormal basis set $\{ \vec{r} = (r_1, \ldots, r_{N-n}) \}$ of $Y$, we can write $|\Psi\rangle$ as
\[
|\Psi\rangle = \sum_{\vec{s}, \vec{r}} \psi_{\vec{s}, \vec{r}} |\vec{s}\rangle_X \otimes |\vec{r}\rangle_Y
\] (33)

using Theorem 1. By defining $|\vec{s}\rangle_Y = \sum_{\vec{r}} \psi_{\vec{s}, \vec{r}} |\vec{r}\rangle_Y$, Eq. (33) is given by
\[
|\Psi\rangle = \sum_{\vec{s}} |\vec{s}\rangle_X \otimes |\vec{s}\rangle_Y.
\] (34)

Combining Eqs. (32) and (34), we obtain $\langle \vec{s}|\vec{r}\rangle_Y = \delta_{\vec{s}, \vec{r}} w_s$. Hence, by defining an orthonormal set $\{ |\vec{s}\rangle_Y \equiv |\vec{s}\rangle_Y / \sqrt{w_s} \}$, $|\Psi\rangle$ is finally written as
\[
|\Psi\rangle = \sum_{\vec{s}} \sqrt{w_s} |\vec{s}\rangle_X \otimes |\vec{s}\rangle_Y
\] (35)

The dimension difference of $\mathcal{H}_X$ and $\mathcal{H}_Y$ is not a problem here. When $\dim \mathcal{H}_X \neq \dim \mathcal{H}_Y$, the number of zero eigenvalues of $\text{Tr}_X |\Psi\rangle\langle\Psi|$ and $\text{Tr}_Y |\Psi\rangle\langle\Psi|$ differ so that the nonzero eigenvalue numbers become equal. Applying the same process, $|\Psi'\rangle$ can be expressed with another orthonormal set $\{ |\vec{s}'\rangle_X \}$ as
\[
|\Psi'\rangle = \sum_{\vec{s}} \sqrt{w_s} |\vec{s}\rangle_X \otimes |\vec{s}'\rangle_Y.
\] (36)

Then the two orthonormal bases $\{|\vec{s}\rangle_Y\}$ and $\{|\vec{s}'\rangle_Y\}$ are connected by a unitary transformation $U_Y = \sum_{\vec{s}} |\vec{s}\rangle\langle\vec{s}'|_X$, by which $|\Psi\rangle$ and $|\Psi'\rangle$ are connected by
\[
|\Psi\rangle = (I_X \otimes U_Y) |\Psi'\rangle \equiv U |\Psi'\rangle.
\] (37)

Using the above lemma, any $|\Psi\rangle$ that satisfies $\rho_X = \text{Tr}_Y |\Psi\rangle\langle\Psi|$ can be transformed to $|\Psi\rangle = \sum_{\vec{s}} \sqrt{w_s} |\vec{s}\rangle_X \otimes |\vec{s}\rangle_Y$, which result in the GHJW theorem of bosons:

**Theorem 2.** (GHJW theorem for identical particles) Suppose $N$ bosons locate in two orthogonal subsystems $X$ and $Y$ with internal states $s_i$ $(i = 1, \ldots, N)$. The total state of the bosons $|\Psi\rangle$ is a vector in $\mathcal{H}^\otimes N \cong \mathcal{H}^\otimes N$. **
\( \mathcal{H}^{Xn} \otimes \mathcal{H}^{(N-n)} \) with \( \rho_X = \text{Tr}_Y |\Psi\rangle \langle \Psi| \). For any convex summation form of \( \rho_X = \sum_w w_a |\Psi^a_n\rangle \langle \Psi^a_n|_X \), there exists an orthonormal set \( \{|\Psi^a_n\rangle\}_a \) of the subsystem \( X \) such that
\[
|\Psi\rangle = \sum_a \sqrt{w_a} |\Psi^a_n\rangle \otimes |\Psi^0_{(N-n)}\rangle. \tag{38}
\]

This theorem shows that an observer at \( Y \) can choose the state of \( X \) by performing a measurement and sending the result to an observer at \( X \), hence the total system is non-local.

**Bell Inequality Violation with Bipartite Two Fermions**

As another exemplary phenomenon of nonlocality that arise from the entanglement of identical particles, we discuss the Bell inequality (BI) maximal violation. Even though several types of entangled states can generate the BI violation, here we focus on the bipartite two-level fermionic system and show that a superposition of the vacuum and two fermions can violate BI. As already mentioned, the factorizability of bipartite systems helps to derive the relation more easily.

**Theorem 3.** If a bipartite system of identical particles is classical and has a local hidden variable \( \lambda \), there exists an inequality for the correlations of a given system:
\[
E(a, b) \equiv |\langle a_1, b_1 \rangle + \langle a_2, b_1 \rangle + \langle a_1, b_2 \rangle - \langle a_2, b_2 \rangle| \leq 2 \tag{39}
\]
where \( \{a_1, a_2\} \) and \( \{b_1, b_2\} \) are different settings for detectors at the subsystems \( X \) and \( Y \).

**Proof.** Using that the total Hilbert space is factorized as \( \mathcal{H} = \mathcal{H}_X \otimes \mathcal{H}_Y \) (Theorem 1), one can set \( \langle a_i, b_j \rangle \equiv \int d\lambda \langle a_i, b_j \rangle \rho(\lambda) \langle a_i, b_j \rangle \) as the average values for the outcomes that satisfy \(-1 \leq \langle a_i \rangle \leq 1 \) and \(-1 \leq \langle b_j \rangle \leq 1 \). Then the inequality \( E(a_\tau, b_\tau) = \frac{1}{4} \langle 1 + \tau a_1 \rangle (1 + \tau a_2 \rangle \rangle \geq 0 \) holds with \( \tau \in \{+1\} \) and \( E(a, b) \) is bounded as
\[
E(a, b) = |\langle a_1 + a_2, b_1 \rangle + \langle a_1 - a_2, b_2 \rangle| \\
= 2 |\langle a_+, b_1 \rangle - \langle a_-, b_1 \rangle + \langle a_+, b_2 \rangle - \langle a_-, b_2 \rangle| \\
\leq 2 |a_+ + a_- + a_+ + a_-| = 2, \tag{40}
\]
where the last inequality is from the relation \( |\langle a_\tau, b_\tau \rangle| \leq \langle a_\tau, b_\tau \rangle \).

For the existence of quantum correlation with Hermitian observables, one can show that the Bell inequality is maximally violated when \( E(a_1, a_2, b_1, b_2) = 2 \sqrt{2} \) (see, e.g., Ref. [24]). This maximal violation can be achieved in the bipartite spin half fermionic system when the fermion state is given by
\[
|\Psi_{-\text{even}}\rangle = \frac{1}{\sqrt{2}} (|\text{vac}\rangle \otimes |\uparrow, \downarrow\rangle_Y^X - |\text{vac}\rangle \otimes |\uparrow, \downarrow\rangle_Y^X). \tag{41}
\]

Considering that \(|\text{vac}\rangle \) and \(|\uparrow, \downarrow\rangle \) are the only two possible independent states per subsystem in this setup (note that the antisymmetric state \(|\uparrow, \downarrow\rangle \) is invariant under any unitary operation), \(|\Psi_{-\text{even}}\rangle \) is one of fermionic Bell-like states. In this basis, we can construct three Pauli matrices as follows:
\[
\begin{align*}
\sigma_1 &= (|\text{vac}\rangle \langle \uparrow| + |\uparrow\rangle \langle \text{vac}|), \\
\sigma_2 &= (-i|\text{vac}\rangle \langle \downarrow| + i|\downarrow\rangle \langle \text{vac}|), \\
\sigma_3 &= (|\text{vac}\rangle \langle \text{vac}| - |\uparrow, \downarrow\rangle \langle \uparrow, \downarrow|), \tag{42}
\end{align*}
\]
and \( \hat{\sigma} \cdot \hat{n} = \sum_{j=1}^{3} \sigma_j \hat{n}_j \) for an arbitrary three-dimensional unit vector \( \hat{n} \).

Then, by setting
\[
\begin{align*}
a_1 &= (\hat{\sigma} \cdot \hat{n})_X \otimes \mathbb{I}_Y, & a_2 &= (\hat{\sigma} \cdot \hat{n})_X \otimes \mathbb{I}_Y \\
b_1 &= \mathbb{I}_X \otimes (\hat{\sigma} \cdot \hat{n})_Y, & b_2 &= \mathbb{I}_X \otimes (\hat{\sigma} \cdot \hat{n}')_Y \tag{43}
\end{align*}
\]
(\( \hat{\sigma} \cdot \hat{n} \) comes from Theorem 1) so that the unit vectors \( (\hat{n}, \hat{n}', \hat{n}', \hat{n}') \) satisfy \( \hat{n} \cdot \hat{n}' = \hat{n} \cdot \hat{n}' = \hat{n} \cdot \hat{n}' = -\hat{n} \cdot \hat{n}' = \frac{\sqrt{2}}{2} \), the maximal Bell inequality violation is obtained, i.e.,
\[
|\langle \Psi_{-\text{even}}\rangle | (|a_1 b_1 + a_2 b_1 + a_2 b_2 - a_1 b_2|)|\Psi_{-\text{even}}\rangle| = 2 \sqrt{2}. \tag{44}
\]

A natural question that arises from the above result is whether any laboratory can actually construct the Bell-like state \(|\Psi_{-\text{even}}\rangle\). Actually, we can suggest a protocol to obtain such a state. Suppose an even fermionic state is prepared in a system \( Z \) as
\[
|\Psi\rangle = |\uparrow, \downarrow\rangle^Z, \tag{45}
\]
which evolves so that the fermions arrive at the systems \( X \) and \( Y \) in the following form,
\[
|\Psi\rangle = |\psi_1, \uparrow\rangle \wedge |\psi_2, \downarrow\rangle \tag{46}
\]
with probability \( 1/2 \).

**Discussions**

Employing SEA and microcausality, we have quantified the SSR-preserving entanglement of identical particles. In this formalism, the total Hilbert space can be factorized according to the location of the particles. And some non-local properties that are seemingly hard to quantify
with identical particles, such as the GHJW theorem and BI violation, are analyzed.

Possible applications of our current work are diverse. For example, Ref. \[25\] theoretically and experimentally verified the quantitative relation of identical particle’s entanglement to particle indistinguishability and spatial overlap, in which the partial trace technique based on SEA. We also expect to establish a rigorous quantum resource theory of identical particles (see Ref. \[26\] for a related research for the bosonic case) and apply it to more general field-theoretic systems.

\section*{ACKNOWLEDGEMENTS}

The author is grateful to Prof. Jung-Hoon Chun. This work is supported by the National Research Foundation of Korea (NRF, NRF-2019R1I1A1A01059964).

[1] J. Schliemann, Phys. Rev. B 63, 085311 (2001).
[2] J. Schliemann, Phys. Rev. A 64, 022303 (2001).
[3] G. Ghirardi, L. Marinatto, and T. Weber, Journal of Statistical Physics 108, 49 (2002).
[4] F. Benatti, R. Floreanini, and U. Marzolino, Journal of Physics B: Atomic, Molecular and Optical Physics 44, 091001 (2011).
[5] F. Benatti, R. Floreanini, and U. Marzolino, Annals of Physics 327, 1304 (2012).
[6] A. Balachandran, T. Govindarajan, A. R. de Queiroz, and A. Reyes-Lega, Physical review letters 110, 080503 (2013).
[7] A. Balachandran, T. Govindarajan, A. R. de Queiroz, and A. Reyes-Lega, Physical Review A 88, 022301 (2013).
[8] R. L. Franco and G. Compagno, Scientific reports 6, 20603 (2016).
[9] R. L. Franco and G. Compagno, Physical review letters 120, 240403 (2018).
[10] G. Compagno, A. Castellini, and R. L. Franco, Phil. Trans. R. Soc. A 376, 20170317 (2018).
[11] S. Chin and J. Huh, Phys. Rev. A 99, 052345 (2019).
[12] S. Attal, “Fock spaces,” \url{http://math.univ-lyon1.fr/~attal/Fock_Spaces.pdf}.
[13] N. Friis, New Journal of Physics 18, 033014 (2016).
[14] M. Johansson, arXiv preprint arXiv:1610.00539 (2016).
[15] M. Navascues, T. Cooney, D. Perez-Garcia, and N. Villanueva, Foundations of Physics 42, 985 (2012).
[16] N. Gigena and R. Rossignoli, Physical Review A 92, 042326 (2015).
[17] G. Wick, A. Wightman, and E. Wigner, Physical Review 88, 101 (1952).
[18] G.-C. Wick, A. S. Wightman, and E. P. Wigner, Physical Review D 1, 3267 (1970).
[19] H. M. Wiseman and J. A. Vaccaro, Physical review letters 91, 097902 (2003).
[20] S. Chin and J. Huh, arXiv preprint arXiv:1906.00542 (2019).
[21] B. Tsirelson, “Bell inequalities and operator algebras,” \url{http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.572.2413} (2006).
[22] N. Gisin, Phys. Rev. Lett. 52, 1657 (1984).
[23] L. P. Hughston, R. Jozsa, and W. K. Wootters, Physics Letters A 183, 14 (1993).
[24] S. J. Summers and R. Werner, Journal of Mathematical Physics 28, 2440 (1987).
[25] M. R. Barros, S. Chin, T. Pramanik, H.-T. Lim, Y.-W. Cho, J. Huh, and Y.-S. Kim, arXiv preprint arXiv:1912.04208 (2019).
[26] B. Morris, B. Yadin, M. Fadel, T. Zibold, P. Treutlein, and G. Adesso, arXiv preprint arXiv:1908.11735 (2019).