INTERPOLATION OF CURVES ON FANO HYPERSURFACES

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ABSTRACT. On a general hypersurface of degree $d \leq n$ in $\mathbb{P}^n$ or $\mathbb{P}^n$ itself, we prove the existence of curves of any genus and high enough degree depending on the genus passing through the expected number $t$ of general points or incident to a general collection of subvarieties of suitable codimensions. In some cases we also show that the family of curves through $t$ fixed points has general moduli as family of $t$-pointed curves. These results imply positivity of certain intersection numbers on Kontsevich spaces of stable maps. An arithmetical appendix by M. C. Chang describes the set of numerical characters $(n,d,\text{curve degree},\text{genus})$ to which our results apply.

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INTRODUCTION

0.1. **Notions of interpolation.** A curve $C$ on a variety $X$ is said to be *interpolating* or to have the *interpolation property* if $C$ can be deformed so as to go through the expected number of general points on $X$. Here ‘expected number’ means, in terms of the normal bundle $N = N_{C/X}$, the largest integer $t$ such that $(n - 1)t \leq \chi(N), n = \dim(X)$ or explicitly, where $g$ denotes the genus of $C$,

$$t = [s(N)] + 1 - g = [\frac{C(-K_X) + 2g - 2}{n - 1}] + 1 - g.$$ 

This makes most sense if $H^1(N) = 0$, so that $C$ moves in an unobstructed family of the expected dimension, i.e. $h^0(N)$. The adjective ‘separable’ may be added if the appropriate correspondence is separable over the symmetric product $X^{(t)}$, which is of course automatic in char. 0.

A stronger property than interpolation, though equivalent in genus 0, is that of *ultra-interpolation*. $C$ is said to be ultra-interpolating if for a sufficiently general collection of subvarieties $Y_i \subset X$, $C$ can be deformed so as to meet all of them, provided

$$\sum (\text{codim}(Y_i) - 1)t \leq \chi(N).$$

The existence of an interpolating or ultra-interpolating curve implies positivity of certain intersection numbers on Kontsevich spaces of stable maps, which measure the ‘virtual’ number of such curves.

Another property related to interpolation is that of *modular interpolation*. Given $m$ fixed general points on $X$, the family of deformations of $C$ going through them yields a family of $m$-pointed curves of genus $g$ and one may inquire whether a general member has general moduli as such. When this holds for all $m$ up to the expected number, namely

$$t = [\chi(T_X|_C)/n] = [(-C.K_X)/n] + 1 - g,$$
we will say that \( C \) is \textit{moduli-interpolating}. Again the adjective ‘separable’ may be added if the appropriate map to the moduli of \( t \)-pointed curves is separable. Again there is an ultra version.

The various separable interpolation properties of a curve \( C \) are equivalent to certain properties called balancedness or ultra-balancedness of either the normal bundle \( N \) or the restricted ambient tangent bundle \( T = T_X |_C \). Thus separable balance is equivalent to the property that for a general effective divisor \( D_t \) of degree \( t \) on \( C \) one has either \( H^1(N(-D_t)) = 0 \) or \( H^0(N(-D_t)) = 0 \). Separable ultra balance is equivalent to the property that for a general effective divisor \( D_t \) of degree \( t \) on \( C \) one has either \( H^1(N|U) = 0 \) or \( H^0(N|U) = 0 \). Separable modular interpolation means that for \( t \) as in (1), one has \( H^1(T(-D_t)) = 0 \). It is via these bundle properties that we will approach interpolation.

0.2. \textbf{Known results.} There is a fair amount of work on curve interpolation in the case where \( C \) is rational and \( X \) is a Fano manifold, e.g. \( \mathbb{P}^n \), a Fano hypersurface in \( \mathbb{P}^n \) or a Grassmannian, starting with the case of rational curves in \( \mathbb{P}^n \), due to Sacchiero [12]; see [4], [2], [11] [9] [8] [10]. For curves of higher genus and \( X = \mathbb{P}^n \), there are older results for elliptic curves due to Ellingsrud and Laksov [5], Hulek [6] and Ein and Lazarsfeld [3], and for \( n = 3 \) due to Perrin [7]. More recently, comprehensive interpolation results for \( X = \mathbb{P}^n \), any \( n \), were obtained by A. Atanasov, E. Larson and D. Yang [1], who showed that a general nonspecial curve of any genus is interpolating. To my knowledge there are no results in the literature on interpolation, much less ultra-interpolation, for higher-genus curves and ambient spaces other than \( \mathbb{P}^n \).

As for modular interpolation, in case \( X = \mathbb{P}^n \), \( g = 0 \) and any \( e \geq n \), it is easy to see that any sufficiently general rational curve of degree \( e \) is ambient-balanced. But already for \( X \) a Grassmannian, \( g = 0 \) and ‘most’ degrees \( e \), there are no moduli-interpolating curves of degree \( e \) (see Example 21). Thus for ‘most’ varieties \( X \) one would expect some topological obstructions in terms of degree and genus in order for a curve to be ambient-balanced.

0.3. \textbf{New results.} In this paper we consider separable interpolation, ultra interpolation and modular interpolation in arbitrary genus on \( \mathbb{P}^n \) and on general Fano hypersurfaces, i.e. hypersurfaces \( X \) of degree \( \leq n \) in \( \mathbb{P}^n \), \( n \geq 4 \). Notably, we will show:

- (See §3) In \( \mathbb{P}^n \), the general curve of genus \( g \) and degree \( e \geq 2(g + 1)n \), is ultra-interpolating and ultra ambient-interpolating (see Corollary 34).
- (See §4) On a general hypersurface of degree \( n \) in \( \mathbb{P}^n \), \( n \geq 4 \), there exist ultra-balanced, ultra ambient-balanced curves of genus \( g \) and degree \( e \) provided either \( g \geq 1 \) and \( e \geq 4g(n - 1) \) or \( g = 0 \) and \( e \geq n - 1 \).
• (See §5) On a general hypersurface of degree \( d < n \) in \( \mathbb{P}^n \), there exist balanced (resp. ambient-balanced) curves of any genus \( g \geq 0 \) and degree \( e \) provided \( (n, d, g, e) \) satisfy certain arithmetical conditions. An arithmetical appendix by M. C. Chang gives sufficient conditions for these conditions to hold, showing in particular that for given \( n, d, g \), the conditions for balance (resp. ambient balance) hold for all \( e \) in at least one arithmetic progression with difference \( d(n-2) \) (resp. for infinitely many \( e \)) (see Theorem 41 and the ensuing examples).

0.4. Methods. The method of proof builds on the one used before in [11] to prove balancedness for rational curves, and is likewise based on fans and fang degenerations, degenerating the curve together with its ambient space, be it \( \mathbb{P}^n \) or a hypersurface (which in turn degenerates together with its own ambient \( \mathbb{P}^n \)) to a reducible pair. More specifically, we consider flags of the form

\[
C_1 \cup C_2 \subset X_1 \cup X_2 \subset P_1 \cup P_2
\]

where \( P_1 \) and \( P_2 \) are blowups

\[
P_1 = B_{\mathbb{P}^m} \mathbb{P}^n, P_2 = B_{\mathbb{P}^{n-m-1}} \mathbb{P}^n
\]

glued along the exceptional divisor \( \mathbb{P}^{n-m-1} \times \mathbb{P}^m \), \( X_1 \cup X_2 \) is a suitable Cartier divisor on \( P_1 \cup P_2 \) (e.g. in the proper fang case \( 0 < m < n-1 \), \( X_1, X_2 \) are birational transforms of hypersurfaces of degree \( d \) with multiplicity \( e \) (resp. \( d-e \)) on \( \mathbb{P}^m \) (resp. \( \mathbb{P}^{n-m-1} \)); and \( C_1 \cup C_2 \) is a lci curve on \( X_1 \cup X_2 \). Then the inclusion \( X_1 \cup X_2 \subset P_1 \cup P_2 \) smooths to an inclusion \( X \subset \mathbb{P}^n \) of a smooth hypersurface of degree \( d \). It can be shown that under suitable conditions on normal bundles the inclusion \( C_1 \cup C_2 \subset X_1 \cup X_2 \) smooths to an inclusion \( C \subset X \) of a smooth curve. To construct good curves \( C \subset X \) one is thus reduced to constructing ‘good’-in a suitable sense- curves \( C_1 \subset X_1, C_2 \subset X_2 \). This is the method used in [11] and here extended to higher genus and to ambient and ultra balancedness.

0.5. Contents. Elementary properties of balanced and ultra-balanced bundles are developed in §1. In §2 we study a relative version of the tangent bundle for a family of varieties degenerating to normal-crossing double points. This is useful in studying moduli-interpolating families. The contents of §§3, 4, 5 have been described above. The Appendix by M. C. Chang studies the roundup equations that arise mainly as one tries to construct balanced bundles as extensions, such as those that occur in studying curves in a fibration, trying to lift a good (e.g. balanced) curve in the base to one in the total space.

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1. Balanced bundles in any genus

We work over an algebraically closed field or arbitrary characteristic.

1.1. Basics. Let $E$ be a vector bundle of slope $s = s(E)$ on a curve $C$ of genus $g$. We set

$$t(E) = s + 1 - g = \frac{\chi(E)}{\text{rk}(E)}$$

and call it the Euler slope or e-slope of $E$. Also let

$$r(E) = \text{deg}(E) \% \text{rk}(E) = \chi(E) \% \text{rk}(E)$$

where $\%$ denotes remainder; this is called the remainder of $E$.

For an effective divisor $D$ on $C$ we denote by $\rho_D$ the restriction map

$$\rho_D : H^0(E) \to H^0(E \otimes \mathcal{O}_D).$$

If $D$ is general of degree $t$ we will denote $\rho_D$ by $\rho_t$. Here ‘general’ means, in case $C$ is reducible, general in some component of $C^{(t)}$.

**Definition 1.** A bundle $E$ is said to be regular if $H^1(E) = 0$.

$E$ is semi-balanced if

(i) $E$ is generically generated;

(ii) $E$ is regular;

(iii) the restriction map $\rho_t$ is surjective for all $t \leq t(E)$.

A semi-balanced bundle is balanced if $\rho_t$ is moreover injective for all $t \geq t(E)$.

A balanced bundle is perfectly balanced if in addition $s$ is an integer.

The notion of balanced bundle can be generalized as follows.

**Definition 2.** Let $E$ be a regular, generically generated bundle. Given a weight vector $\underline{u} = (u_1, \ldots, u_t), 0 \leq u_i \leq \text{rk}(E)$, $E$ is said to be $\underline{u}$-balanced if there exist points $x_1, \ldots, x_t$, each general in some component of $C$, and for each $i$, a general skyscraper quotient $U_i$ of $E|_{x_i}$ of dimension $u_i$, such that the restriction map

$$\rho_{\underline{u}} : H^0(E) \to H^0(\bigoplus U_i)$$

has maximal rank. $E$ is perfectly $\underline{u}$-balanced if $\rho_{\underline{u}}$ is an isomorphism.

$E$ is said to be ultra-balanced if it is $\underline{u}$-balanced for every $\underline{u}$. \hfill \Box

Obviously $\rho_t$ is just $\rho_{\text{rk}(E), \ldots, \text{rk}(E)}$, so $E$ is balanced iff it is $\underline{u}$-balanced for all scalar weight-vectors of the form $(\text{rk}(E), \ldots, \text{rk}(E)) \in \mathbb{Z}^t, \forall t$. Note that for $E$ regular, $\rho_t$ can be surjective only for $t \leq t(E)$. Also, note that in the definition, we are requiring $U_i$ to be killed by the maximal ideal $m_{x_i}$, rather than just some power of it.
Remark 3. Regarding balancedness vs. (semi) stability. For a bundle of slope $s$ on a curve of genus $g$, balancedness excludes subbundles of degree $s + 1 - g$ or less while stability excludes subbundles of degree $s$ or less. Thus balancedness does not imply stability if $g > 1$ though we don’t have an explicit example of an unbalanced stable bundle. Conversely there exist direct sums of line bundles that are ultra-balanced but not stable (see Lemma 5).

**Lemma 4.** Suppose $E$ is generically generated. Then the following are equivalent:

(i) $E$ is semi-balanced;

(ii) for general points $x_1, ..., x_t \in C$ and $\forall t \leq t(E)$, we have $H^1(E(-x_1 - ... - x_t)) = 0$ or equivalently

$$h^0(E(-x_1 - ... - x_t)) = \chi(E(-x_1 - ... - x_t));$$

(iii) $h^0(E) = \chi(E)$ and $h^0(E(-x_1 - ... - x_t)) = h^0(E) - t.rk(E), \forall t \leq t(E)$.

Moreover, if $E$ is semi-balanced, then $E$ is balanced iff $H^0(E(-x_1 - ... - x_t)) = 0, \forall t \geq t(E)$.

In particular, the condition that $\rho_i$ be injective or surjective depends only on the linear equivalence class of $\sum x_i$ hence only on $t$ if $g = 0$.

The proof may be left to the reader. □

**Lemma 5.** A balanced bundle $E$ is ultra-balanced provided $\rho_u$ is an isomorphism for all weight-vectors $u$ of weight $\sum u_i = \chi(E)$.

**Lemma 6.** A generically generated bundle $E$ is $u$-balanced iff, in the above notations, the modified bundle

$$E_u = \ker(E \rightarrow \bigoplus U_i)$$

has natural cohomology, i.e. $h^0(E_u)h^1(E_u) = 0$.

For rational curves, the above notion of balanced coincides with the usual:

**Lemma 7.** If $g = 0$, $E$ is balanced iff $E$ is ultra-balanced iff $E \simeq b_1O(a + 1) \oplus b_0O(a)$ for some $a \geq 0, b_0 > 0, b_1$.

**Proof.** If $E$ has the form $b_1O(a + 1) \oplus b_0O(a)$ then so does a general modification of $E$, so $E$ is ultra-balanced. Conversely assume $E$ is balanced and let $a$ be the smallest degree of a line bundle quotient (= summand) of $E$. By semi-balancedness clearly $[s(E)] = a \geq 0, [t(E)] = [a] + 1$. If $E$ has a line bundle summand of degree $\geq a + 2$ then $H^0(E(-x_1 - ... - x_{t+1})) \neq 0$, contradicting balancedness. □

Note that for $g = 0$ the ‘test’ divisor $\sum x_i$ may actually be an arbitrary effective divisor of degree $t$. For general $g$ the injectivity or surjectivity conditions for balancedness depend only on the linear equivalence class of $\sum x_i$. Also for general $g$, half the above characterization still holds:
Lemma 8. Suppose $E$ admits a filtration whose quotients $L_1, ..., L_r$ are line bundles such that $\deg(L_1), ..., \deg(L_r) \in [a, a + 1]$ for some $a \geq 2g - 1$. Then $E$ is balanced.

Proof. If $D_t$ denotes a general effective divisor of degree $t$ then it is easy to check that
\[
\begin{align*}
H^1(E(-D_t)) &= 0, t \leq g, \\
H^0(E(-D_t)) &= 0, t \geq g + 1.
\end{align*}
\]

There is a version of this for ultra-balanced:

Lemma 9. Let $E$ be a direct sum of line bundles with degrees in $[a, a + 1], a \geq 2g - 1$. Then $E$ is ultra-balanced.

Proof. As has been noted, if $L$ is a line bundle of degree $a \geq 2g - 1$ then
\[
\begin{align*}
H^1(L(-D_t)) &= 0, t \leq a + 1 - g, \\
H^0(L(-D_t)) &= 0, t \geq a + 1 - g.
\end{align*}
\]

We can write
\[
E = L_1 \oplus ... \oplus L_s \oplus L_{s+1} \oplus ... \oplus L_r
\]
where
\[
\deg(L_i) = \begin{cases} 
  a + 1, & i \leq s; \\
  a, & i > s
\end{cases}
\]
and the subbundle $L_1 \oplus ... \oplus L_s \subset E$ is uniquely determined. Then we have $\chi(E) = ra + s$. If $u = (u_1, ..., u_t)$ is a weight vector, we have, by generality of the quotient involved,
\[
E^{u_1} = L_1(-p) \oplus ... \oplus L_{u_1}(-p) \oplus L_{u_1+1} \oplus ... \oplus L_r,
\]
where $p \in C$ is a general point, and this is a direct sum of line bundles of degrees in $[a, a + 1]$ if $u_1 \leq s$ or $[a - 1, a]$ if $u_1 \geq s$. Then it is easy to check, e.g. by induction of the length of the weight-vector $u$, that
\[
\begin{align*}
H^1(E^u) &= 0, |u| \leq \chi(E), \\
H^0(E^u) &= 0, |u| \geq \chi(E).
\end{align*}
\]

We can similarly characterize semi-balanced bundles on $\mathbb{P}^1$:

Lemma 10. A globally generated bundle of slope $s$ on $\mathbb{P}^1$ is semi-balanced iff the smallest degree of its line bundle summands is $[s]$.

Proof. As has been noted, if $L$ is a line bundle of degree $a \geq 2g - 1$ then
\[
\begin{align*}
H^1(L(-D_t)) &= 0, t \leq a + 1 - g, \\
H^0(L(-D_t)) &= 0, t \geq a + 1 - g.
\end{align*}
\]

We can write
\[
E = L_1 \oplus ... \oplus L_s \oplus L_{s+1} \oplus ... \oplus L_r
\]
where
\[
\deg(L_i) = \begin{cases} 
  a + 1, & i \leq s; \\
  a, & i > s
\end{cases}
\]
and the subbundle $L_1 \oplus ... \oplus L_s \subset E$ is uniquely determined. Then we have $\chi(E) = ra + s$. If $u = (u_1, ..., u_t)$ is a weight vector, we have, by generality of the quotient involved,
\[
E^{u_1} = L_1(-p) \oplus ... \oplus L_{u_1}(-p) \oplus L_{u_1+1} \oplus ... \oplus L_r,
\]
where $p \in C$ is a general point, and this is a direct sum of line bundles of degrees in $[a, a + 1]$ if $u_1 \leq s$ or $[a - 1, a]$ if $u_1 \geq s$. Then it is easy to check, e.g. by induction of the length of the weight-vector $u$, that
\[
\begin{align*}
H^1(E^u) &= 0, |u| \leq \chi(E), \\
H^0(E^u) &= 0, |u| \geq \chi(E).
\end{align*}
\]

Example 11. The bundle $\mathcal{O}(2) \oplus 2\mathcal{O}$ on $\mathbb{P}^1$ is semi-balanced but not balanced.
There is a partial extension for elliptic curves:

**Lemma 12.** Assume \( g = 1 \), \( E \) is generically generated and regular, and and that \( E \) is either (1) poly-stable or (2) semi-stable of non-integer slope. Then \( E \) is balanced.

**Proof.** Here \( t(E) = s(E) \) and for \( t \leq t(E) \) (resp. \( t \geq t(E) \)), \( E(-x_1 - \ldots - x_t) \) has nonnegative (resp. nonpositive) slope so the conclusion is immediate. \( \square \)

For general \( g \) one might conjecture that if \( E \) is regular and generically generated then \( E \) is balanced iff the slopes of its Harder-Narasimhan graded pieces are all in some length-1 interval.

1.2. **Splitting, modifying and matching.** The following result is useful in constructing some semi-balanced and sometimes balanced bundles by smoothing from a bundle on a reducible curve.

**Lemma 13.** Let \( C = C_1 \cup C_2 \) be a nodal curve such that \( C_1 \cap C_2 \) consists of \( k \) general points on \( C_1 \). Let \( E \) be a bundle on \( C \). Assume

(i) \( E \) is regular and generically generated;
(ii) \( E_i = E_{C_i} \) are balanced, \( i = 1, 2; \)
(iii) the remainders satisfy \( r(E_1) + r(E_2) < r(E) \) (e.g. \( E_{C_1} \) or \( E_{C_2} \) is perfectly balanced);
(iv) \( t(E_1) \geq k \).

Then

(a) \( E \) is semi-balanced.
(b) Moreover if \( r(E_2) = 0 \), \( E \) is balanced.

**Proof.** The respective genera satisfy \( g = g_1 + g_2 + k - 1, k = C_1.C_2 \) hence for the Euler slopes
\[
t(E) = t(E_1) + t(E_2) - k.
\]
For \( t = \lfloor t(E) \rfloor \) write \( t = t_1 + t_2 \) where
\[
t_1 = \lfloor t(E_1) \rfloor - k, t_2 = \lfloor t(E_2) \rfloor.
\]
To prove \( E \) is semi-balanced, choose general points
\[
x_{11}, \ldots, x_{1t_1} \in C_1, x_{21}, \ldots, x_{2t_2} \in C_2.
\]
By balancedness of \( E_2 \), there is a section \( s_2 \) of \( E_2 \) with arbitrary assigned values at \( x_{21}, \ldots, x_{2t_2} \). By balancedness of \( E_1 \) there is a section \( s_1 \) of \( E_1 \) with arbitrary assigned values at \( x_{11}, \ldots, x_{1t_1} \) and matching \( s_2 \) on \( C_1 \cap C_2 \). Then \( s_1 \) and \( s_2 \) glue to a section of \( E \) with assigned values at all the \( x_{ij} \). This proves (a). Then the proof of (b) is similar. \( \square \)

**Remark.** Note the absence of a ‘general gluing’ assumption over \( C_1 \cap C_2 \). The result will be used mainly in case \( E_2 \) is perfectly balanced.
The same argument also proves:

**Lemma 14.** Let \( C = C_1 \cup C_2 \) be a nodal curve such that \( C_1 \cap C_2 = \{ p_1, \ldots, p_k \} \) consists of \( k \) general points on each component. Let \( E \) be a regular, rank-\( r \) bundle on \( C \) and \( u, v \) weight-vectors. Assume:

(i) \( E_{|C_1} \) is \( u \)-balanced;

(ii) \( E_{|C_2} \) is \( v \)-balanced;

(iii) The restriction map \( H^0(E_{|C_1}) \oplus H^0(E_{|C_2}) \to H^0(E_{|p_1,\ldots,p_k}) \) is surjective.

Then \( E \) is \((u,v)\)-balanced.

**Proof.** \( H^0(E_{|C_1}(-\sum p_i)) \oplus H^0(E_{|C_2}(-\sum p_i)) \) is a subspace of \( H^0(E) \) which already surjects onto \( H^0(U_1 \oplus \ldots \oplus V_t) \).

The following property of ultra-balanced bundles is immediate from the definition but worth noting:

**Lemma 15.** Let \( E \) be an ultra-balanced bundle and \( E' = E^u \subset E \) a general down modification, i.e. kernel of a general surjection \( E \to \bigoplus u_i k_{p_i} \), such that \( E' \) is regular and generically generated.

Then \( E' \) is ultra-balanced. In particular, if \( D_1 = \sum p_i \) is a general effective divisor and \( E(-D_1) \) is regular and generically generated, then \( E(-D_1) \) is ultra-balanced.

The following two lemmas, which are analogues of simple facts in the case of rational curves, show that a general (up or down) elementary modification of a balanced bundle is balanced:

**Lemma 16.** Let \( E \) be a balanced bundle and \( E' \subset E \) a general locally corank-1 modification at some general points. Assume \( E' \) is regular and generically generated. Then \( E' \) is balanced.

**Proof.** It suffices to prove this for modification at a single point \( p \), so \( E' \subset E \) is the kernel of a general surjection \( E \to \bigoplus u_i k_{p_i} \). Now if \( t(E) < 1 \), the conclusion is obvious, so assume \( t(E) \geq 1 \). We first prove \( E' \) is semi-balanced. Let \( t = \lfloor t(E) \rfloor > 0 \). Assume first \( E \) is not perfect. This easily implies that \( \lfloor t(E') \rfloor = t \). Then for general \( x_1, \ldots, x_t \), we get a subsheaf

\[
H^0(E(-x_1 - \ldots - x_t)) \otimes O \subset E(-x_1 - \ldots - x_t)
\]

that is not contained in the kernel of the (general) modification at \( p \). Hence \( H^0(E'(-x_1 - \ldots - x_t)) \) has the expected dimension so that \( H^0(E') \to E'_{x_1,\ldots,x_t} \) is surjective so \( E' \) is semi-balanced.

If \( E \) is perfect then \( t(E') = t(E) - 1 \), therefore for a general divisor \( x_1 + \ldots + x_{t-1} \), \( H^0(E(-x_1 - \ldots - x_{t-1})) \) has the expected dimension and the restriction map

\[
H^0(E(-x_1 - \ldots - x_{t-1})) \to E(-x_1 - \ldots - x_{t-1})|_p
\]
is surjective. Therefore the kernel $H^0(E'(-x_1 - ... - x_t))$ of the restriction map has the expected dimension and semi-balancedness follows.

Now the injectivity statement required to show $E'$ balanced is obvious if $[t(E')] = [t(E)]$. Otherwise, $t := [t(E')] = [t(E)] - 1$ and the required injectivity for $E'$ follows from injectivity of $H^0(E) \to E_{x_1,..,x_t}$.

There is a similar statement for up modifications:

**Lemma 17.** Let $E$ be a balanced bundle and $E \subset E^+$ a general locally corank-1 modification at some general points. Then $E^+$ is balanced.

**Proof.** First it is obvious that $E^+$ is regular and generically generated. For balancedness, it again suffices to prove it for the case of modification at a single point $p$, so $(E^+)^* \subset E^*$ is the kernel of a general surjection $E^* \to k_p$ and $E_p \to E^+_p$ has kernel a general 1-dimensional subspace. Now semi-balancedness is obvious if $[t(E)] = [t(E^+)]$. If not, then $t(E^+) = [t(E)] = [t(E)] + 1 := t + 1$ and in particular $t(E^+)$ is an integer. Now $H^0(E(-x_1 - ... - x_t)) \subset H^0(E^+(-x_1 - ... - x_t))$ injects to $E'(-x_1 - ... - x_t)|_p$ and its image is just the inverse image of the natural map $E' \to k_p$. Therefore the kernel of $H^0(E^+(-x_1 - ... - x_t)) \to k_p$ is contained in the latter image, hence must vanish because $H^0(E(-x_2 - ... - x_t - p)) = 0$. This proves $H^0(E^+(-x_1 - ... - x_t)) \to E^+_p$ is injective, i.e. surjective, so $E^+$ is semi-balanced.

Now to prove $E^+$ is balanced let $t + 1 := [t(E^+)] + [t(E)]$. Then $t(E) < t + 1$. Now the kernel of $H^0(E^+(-x_1 - ... - x_t)) \to E^+_p$ corresponds to the intersection of the image of $H^0(E(-x_1 - ... - x_t)) \to E|_p$ with the kernel if $E|_p \to E^+_p$ which is a general 1-dimensional subspace and the intersection is trivial because the latter image is a proper (maybe trivial) subspace thanks to $t(E) < t + 1$. Thus $H^0(E^+(-x_1 - ... - x_t - p)) = 0$ so $E^+$ is balanced.

The following Lemma strengthens Lemma 25 of [11] and generalizes it to arbitrary genus (note that Cases 2,3 are new even for genus 0):

**Lemma 18.** Let

$$0 \to E_1 \to E \to E_2 \to 0$$

be an exact sequence of vector bundles on a curve such that $E_1, E_2$ are balanced of respective slopes $s_1, s_2$. Assume either:

Case 1:

$$[s_1] = [s_2];$$

or Case 2:

$$s_2 = [s_1] + 1;$$
or Case 3:

\[ s_1 = \lfloor s_2 \rfloor + 1. \]

Then \( E \) is balanced. Moreover the slope \( s = s(E) \) satisfies:

- Case 1: \( [s] = [s_1] \);
- Case 2: \( [s] = s_2 \);
- Case 3: \( [s] = s_1 \).

**Proof.** Apply the Snake Lemma to the following (exact, since \( H^1(E_1) = 0 \)) diagram, in which \( D_m = p_1 + \ldots + p_m \) denotes a general effective divisor of degree \( m \):

\[
\begin{array}{cccccc}
0 & \to & H^0(E_1) & \to & H^0(E) & \to & H^0(E_2) & \to & 0 \\
& \rho_1 \downarrow & & \rho \downarrow & & \rho_2 \downarrow & & \\
0 & \to & E_1|_{D_m} & \to & E|_{D_m} & \to & E_2|_{D_m} & \to & 0
\end{array}
\]

(2)

Case 1: The assertion about \( s \) is obvious and implies

\[ t := \lceil t(E) \rceil = \lceil t(E_1) \rceil = \lceil t(E_2) \rceil. \]

Taking \( m = t \), we have \( \rho_1, \rho_2 \) surjective hence so is \( \rho \). Taking \( m = \lceil t(E) \rceil \), \( \rho_1, \rho_2 \) are injective hence so it \( \rho \).

Case 2: Note this case can occur only if \( s_2 \), hence \( t_2 = t(E_2) \) is an integer. Taking \( m = t_2 \), \( \rho_2 \) is an isomorphism and \( \rho_1 \) is injective, hence \( \rho \) is injective. Taking \( m = t_2 - 1 \), \( \rho_1 \) and \( \rho_2 \) are surjective hence so is \( \rho \).

Case 3 is similar to Case 2. \( \square \)

### 1.3. Balanced and ultra-balanced curves, Kontsevich intersections.

A lci curve \( C \to X \) is said to be separably regular or (semi-, perfectly) balanced if its normal bundle \( N_{C/X} \) has the corresponding property. Regularity means that \( C \) belongs to a smooth family of the expected dimension. Semi-balance implies (and in char. 0 is equivalent to) the semi-interpolating property, i.e. that \( C \) can be deformed to go through the expected number of general points of \( X \), and balance implies moreover that the subvariety of \( X \) filled up by the deformations through a fixed maximal collection of general points has the expected dimension. When \( X \) contains a (semi-) balanced curve we will say that \( X \) has the (semi-) interpolation property (for curves of genus \( g(C) \) and degree \( \deg(C) \) if understood).

If \( C \) is reducible and \( C_1 \subset C \) is a component, we will say \( E \) is (semi-) balanced around \( C_1 \) if \( H^1(E) = 0 \), \( E \) is generated by its sections at a general point of \( C_1 \), and the required surjectivity or injectivity statements as appropriate hold for general points of \( C_1 \).

If \( C \) has degree \( e \) and genus \( g \) in \( X = \mathbb{P}^n \) then

\[ t(C) = e + 1 - g + \left\lfloor \frac{2e - 1 + g}{n - 1} \right\rfloor. \]
In particular if C is nondegenerate (so that $e \geq n$) and nonspecial (so that $e + 1 - g = \chi(O_C(H)) \geq n + 1$), we have $t(C) \geq n + 3$.

See [11], especially §1 and §5 for various information on normal bundles and fangs.

A curve $C \to X$ is said to be ultra-balanced if its normal bundle is. This condition has an interesting interpretation in terms of intersection numbers on Kontsevich spaces of stable maps. Thus let $M_{g,t}(X)$ be the Kontsevich space of stable $t$-pointed maps $C \to X$ where $(C, x_1, \ldots, x_t)$ is a $t$-pointed stable curve of genus $g$. Let

$$\sigma_i : M_{g,t}(X) \to X, i = 1, \ldots, t$$

be the natural maps. Let $h$ be a birationally ample divisor on $X$ and set

$$\eta_i = \sigma_i^*(h).$$

Define

$$I_M(0, u_1, \ldots, u_t) = \int_M \eta_1^{u_1} \cdots \eta_t^{u_t}.$$ 

This definition will shortly be extended to the case of a nonzero first argument.

**Proposition 19.** Let $M$ be a component of $M_{g,t}(X)$ whose general point has the form $(C, x_1, \ldots, x_t)$ where $C$ is ultra-balanced (resp. balanced). Then for all $u_1, \ldots, u_t$ such that

$$u_1 + \ldots + u_t = \chi(N_C/X) = (C - K_X) + (n - 3)(1 - g),$$

(resp. and such that $u_2 = \ldots = u_t = n$) we have

$$I_M(0, u_1, \ldots, u_t) > 0.$$ 

**Proof.** Considering $X \subset \mathbb{P}^N$, there is a natural map

$$F : M \to (\mathbb{P}^N)^t.$$ 

Our ultra-balanced hypothesis implies that for $Z = P^{N-u_1} \times \ldots \times P^{N-u_t}$, $F^{-1}(Z)$ contains an isolated reduced point. Therefore the intersection number $F_*(M).Z > 0$, which implies our result in the ultra-balanced case. The balanced case is similar. □

### 1.4. Ambient-balanced curves.

A curve $C \to X$ of genus $g$ is said to be ambient-balanced if the restricted tangent bundle $T_X|_C$ is semi-balanced, i.e. for all

$$t \leq t(T_X|_C) = (-K_X.C/n) + 1 - g, n = \dim(X),$$

and general points $x_1, \ldots, x_t \in C$, we have

$$H^1(T_X|_C(-x_1 - \ldots - x_t)) = 0.$$ 

Note that the vanishing (3) implies $H^1(N_{C/X}(-x_1 - \ldots - x_t) = 0$ so that a general deformation of $C$ contains $t$ general points of $X$. However ambient balance does not imply balance because (3) is only assumed for $t \leq t(T_X|_C)$ but usually $t(N_{C/X}) > t(T_X|_C)$. 

\[12\]
Now (3) also implies surjectivity the natural map induced by the normal sequence
\[ H^0(N_{C/X}(-x_1 - ... - x_t)) \rightarrow H^1(T_C(-x_1 - ... - x_t)). \]
Consequently we have

**Corollary 20.** If \( C \to X \) is ambient-balanced then \( C \) is separably moduli-interpolating, i.e. for \( t \leq (-C.K_X/n) + 1 - g \) and general points \( x_1, ..., x_t \in X \), the family of deformations of \( C \) in \( X \) passing through \( x_1, ..., x_t \) has separably general moduli as a family of \( t \)-pointed curves.

Thus, for an ambient-balanced curve \( C \) we are able to impose on deformations of \( C \) simultaneously a fixed set of \( t \) general points of \( X \) and fixed set of \( t \)-pointed moduli where \( t = \lfloor -C.K_X/n \rfloor + 1 - g \). Note that such moduli are nontrivial even if \( g = 0 \) provided \( t \geq 4 \).

For genus 0 and \( X = \mathbb{P}^n \), it follows easily, e.g. from [11], Lemma 26 that a general deformation of any given curve \( C \) is ambient-balanced. For higher genus, see Corollary 34 below.

For example, the rational normal curve in \( \mathbb{P}^n \) is both perfectly balanced and perfectly ambient-balanced.

**Example 21.** To put matters in perspective consider the case of a Grassmannian \( X = G(k,n) \) with its tautological subbundle \( S \) and quotient bundle \( Q \) and tangent bundle \( T_X = S^* \otimes Q \). For a rational curve \( C \subset X \) of degree \( e \), it is easy to see that on a general deformation of \( C \), both \( S \) and \( Q \) will be balanced but, unless \( k|e \) or \( (n-k)|e \), both will be imperfect, hence \( T_X|_C \) will be unbalanced. Consequently, \( X \) contains an ambient-balanced rational curve of degree \( e \) iff either \( k|e \) or \( (n-k)|e \). In particular the set of degrees of ambient-balanced curves in \( X \) constitutes 2 arithmetic progressions.

As for balance, the normal sequence
\[ 0 \rightarrow O(2) \rightarrow S^* \otimes Q \rightarrow N_{C/X} \rightarrow 0 \]
plus Lemma 18 show that if the slope \( s = s(N_{C/X}) \) satisfies \( \lfloor s \rfloor = 2 \) and \( S^* \otimes Q \) is unbalanced, then so is \( N_{C/X} \). Explicitly, the slope condition is
\[ \left\lfloor \frac{en - 2}{k(n-k) - 1} \right\rfloor = 2. \]
So whenever this holds and \( e \) is not divisible by either \( k \) or \( n-k \), any rational curve of degree \( e \) in \( X \) is unbalanced. For example, when \( n = 2k \) the condition on \( e \) is
\[ k < e < 3k/2 - 1/2k. \]
A general rational curve with degree in this range will be nondegenerate (i.e. correspond to a nondegenerate scroll in \( \mathbb{P}^{n-1} \)), unbalanced and ambient-unbalanced.
Thus, for general Fano manifolds one may expect topological obstructions on a curve to be ambient-balanced or balanced, though there remains the possibility that all curves of sufficiently high degree are balanced. For Fano hypersurfaces of degree $d < n$ in $\mathbb{P}^n$ we will show below that the set of degrees of ambient-balanced or balanced curves contains some arithmetic progressions, resembling the situation for Grassmannians, while for $d = n$ this set contains all sufficiently large integers.

A curve $C \to X$ is said to be ultra ambient-balanced if $T_X|_C$ is ultra-balanced. Similarly as in Proposition 19, ultra ambient balance has an application to intersection numbers. Let

$$\phi : M_{g,t}(X) \to M_{g,t}$$

be the natural map and $\kappa = \phi^*(L)$ for some birationally ample $L$. Now define

$$I_M(u_0, u_1, ..., u_t) = \int_M \kappa^{u_0} \eta_1^{u_1} ... \eta_t^{u_t}.$$  

**Proposition 22.** Notations as above, assume $C$ is ultra ambient-balanced (resp. ambient-balanced) rather than ultra-balanced and $t > 0$. Let

$$u_0 = \dim(M_{g,t}) = 3g - 3 + t$$

Then for all $u_1, ..., u_t$ such that

$$\sum u_i = \chi(N) - u_0 = (C - K_X) - n(g - 1) - t,$$

(resp. and $u_1 = ... = u_t = n$), we have

$$I_M(u_0, ..., u_t) > 0.$$  

The proof is similar to that of Proposition 19. Note that the case of a general exponent vector $(u_0, ..., u_t)$ of weight $\chi(N)$ remains open.

2. Relative and Log Tangent Bundles

**2.1. Degeneration of tangent bundles.** We construct a relative version of the tangent bundle for a family of varieties degenerating to normal crossings of multiplicity 2. We begin with some local considerations. Consider the surface $X$ with equation $x_1x_2 = t$ in $\mathbb{A}^3$ with its $t$-projection $\pi : X \to \mathbb{A}^1$. There is an associated derivative map

$$d\pi : T_X \to \pi^*T_{\mathbb{A}^1},$$

which is clearly surjective except at the node, i.e. the origin, and has image $m\pi^*T_{\mathbb{A}^1}$, where $m$ is the ideal of the origin. Its kernel is invertible and locally generated by the vector field

$$v = \left(x_1 \partial_{x_1} + x_2 \partial_{x_2}\right)/2 + t \partial_t.$$
Now working globally, let

$$\pi : \mathcal{X} \to B$$

be a flat morphism of a smooth variety to a smooth curve whose general fibre is smooth and whose special fibres have at most normal crossing double points along a smooth subvariety $\Delta$ of codimension 2 (codimension 1 in $\pi^{-1}(\pi(\Delta))$). Again there is a derivative map

$$d\pi : T_{\mathcal{X}} \to \pi^{\ast}T_B.$$ 

Because $\pi$ can be locally modelled by the above curve fibration, it follows that the image of $d\pi$ is $\mathcal{I}_\Delta \pi^{\ast}T_B$ and its kernel, denoted $T_{\mathcal{X}/B}$ and called the relative tangent bundle of the fibration $\pi$, is locally free. Thus we have an exact sequence

$$0 \to T_{\mathcal{X}/B} \to T_{\mathcal{X}} \to \mathcal{I}_\Delta \pi^{\ast}T_B \to 0. \quad (4)$$

In fact $T_{\mathcal{X}/B}$ is locally near $\Delta$ generated by $v$ as above together with the complementary vector fields $\partial_{x_3}, \ldots$ tangent to $\Delta$. Note that for a smooth fibre $X_t$, we have

$$T_{\mathcal{X}/B}|_{X_t} = T_{X_t}.$$ 

On the other hand for a singular fibre $X_0$ with normalization $\tilde{X}_0$ and double locus $\Delta \subset \tilde{X}_0$, the pullback $T_{\mathcal{X}/B}|_{\tilde{X}_0}$ is generated by $x_1 \partial_{x_1}$ or $x_2 \partial_{x_2}$ plus the complementary fields. Therefore we have

$$T_{\mathcal{X}/B}|_{X_0} = T_{\tilde{X}_0}(-\langle \log \Delta \rangle).$$

In particular if $X_0 = X_1 \cup X_2$ is a union of smooth components then

$$T_{\mathcal{X}/B}|_{X_i} = T_{X_i}(-\langle \log \Delta \rangle), i = 1, 2.$$ 

Note the exact sequences

$$0 \to T_{X_i}(-\Delta) \to T_{X_i}(-\log \Delta) \to T_\Delta \to 0, i = 1, 2$$

which induce

$$0 \to \mathcal{O}_\Delta \to T_{X_i}(-\log \Delta)|_\Delta \to T_\Delta \to 0 \quad (5)$$

where the $\mathcal{O}_\Delta$ subsheaf is locally generated by $x_1 \partial_{x_1}$ or $x_2 \partial_{x_2}$. The latter sequence is compatible with the identifications

$$T_{X_1}(-\log \Delta)|_\Delta \simeq T_{X_2}(-\log \Delta)|_\Delta \simeq T_{\mathcal{X}/B}|_\Delta.$$
2.2. Restriction on curves. Note that given a smooth pair \((X_i, \Delta)\) and a curve \(C_i \subset X_i\) meeting \(\Delta\) transversely in \(\delta = \Delta \cap C_i\), the restriction \(T_{X_i} \langle - \log \Delta \rangle|_{C_i}\) is just the elementary corank-1 down modification of \(T_{X_i}|_{C_i}\) at \(\delta\) corresponding to the tangent hyperplanes \(T_p \Delta \subset T_p X_i, p \in \delta\). This has the following immediate consequence

**Corollary 23.** In the above notations let \(C/B \to \mathcal{X}/B\) be a family of curves with special fibre \(C_0 = C_1 \cup \delta C_2 \subset X_1 \cup \Delta X_2\). Then there is a bundle \(T = T_{X/B}\) on \(\mathcal{X}\) such that for a general fibre \(C_t \subset X_t\) we have

\[
T|_{C_t} = T_{X_t}|_{C_t}
\]

while on the special fibre, \(T|_{C_i}\) for \(i = 1, 2\) is the elementary corank-1 down modification of \(T_{X_i}|_{C_i}\) at the points \(p \in \delta\) corresponding to the hyperplanes \(T_p \Delta \subset T_p X_i, p \in \delta\).

**Example 24.** With notations as above, suppose \(C_2\) is a \(\mathbb{P}^1\) with trivial normal bundle \(N_{C_2/X_2} = (n - 1)\mathcal{O}\) and \(\delta = \{p\}\). Then \(T_{X_2}|_{C_2} = T_C \oplus (n - 1)\mathcal{O} = \mathcal{O}(2) \oplus (n - 1)\mathcal{O}\), so that

\[
T|_{C_2} = T_{X_2} \langle - \log \Delta \rangle|_{C_2} = \mathcal{O}(1) \oplus (n - 1)\mathcal{O}
\]

where the \((n - 1)\mathcal{O}\) quotient coincides at \(p\) with the \(T_{\Delta}\) quotient. There is an analogous and compatible quotient on the \(X_1\) side. Then for a point \(q \neq p \in C_2\), we can identify \(H^0(T|_{C_1 \cup C_2}(-q))\) with the kernel of the natural map

\[
H^0(T_{X_1} \langle - \log \Delta \rangle|_{C_1}) \to T_{p\Delta}.
\]

Therefore

\[
H^0(T|_{C_1 \cup C_2}(-q)) = H^0(T_{X_1}|_{C_1}(-p)).
\]

More is true. In fact as in [11], §1, there is a modification \(T \to T'\) with cokernel on \(C_2\) such that

\[
T'|_{C_2} = n\mathcal{O}
\]

while \(T'|_{C_1}\) is the elementary up modification of \(T_{X_1} \langle - \log \Delta \rangle|_{C_1}\) at \(p\) corresponding to the \(\mathcal{O}_\Delta\) subsheaf as in (5), which clearly coincides with \(T_{X_1}|_{C_1}\) itself, i.e.

\[
T'|_{C_1} = T_{X_1}|_{C_1}.
\]

In particular, given a point modification of \(T'|_{C_2}\) leading to an exact sequence

\[
0 \to K \to T'|_{C_1 \cup C_2} \to k\mathcal{O}_q \to 0, q \neq p \in C_2
\]

then there is a corresponding exact sequence

\[
0 \to K_1 \to T_{X_1}|_{C_1} \to k\mathcal{O}_p \to 0
\]

such that

\[
H^0(K) = H^0(K_1).
\]
This argument evidently extends to the case where $C_2$ is a disjoint union of lines with trivial normal bundle. The upshot is that such components may effectively be ignored and the log tangent bundle $T_{X_1}(-\log \Delta)|_{C_1}$ replaced by $T_{X_1}|_{C_1}$ near $C_1 \cap C_2$. This situation occurs in the proof of Theorem 40 and Theorem 41.

2.3. Log tangents for projective bundle pairs. Let $\pi : X = \mathbb{P}(G) \to B$ be a projective bundle and let $Y = \mathbb{P}(G/A) \subset X$ be a codimension-1 projective subbundle, corresponding to a line subbundle $A \subset G$. Let $S_G$ be the kernel of the canonical surjection $\pi^*G \to \mathcal{O}_X(1)$. Then we have the relative tangent bundle

$$T_{X/B} = S_G^* \otimes \mathcal{O}_X(1).$$

Note that $Y$ is the zero-divisor of the natural map $A \to \mathcal{O}_X(1)$, hence

$$N_{Y/X} = A^* \otimes \mathcal{O}_Y(1)$$

where $\mathcal{O}_Y(1)$ is the restriction of $\mathcal{O}_X(1)$. Then we have an exact sequence

$$0 \to T_{X/B}(-\log Y) \to S_G^* \otimes \mathcal{O}_X(1) \to A^* \otimes \mathcal{O}_Y(1) \to 0.$$

Now given a curve $C \to B$, a lifting $C \to X$ corresponds to an invertible quotient $G_C \to M$. Assume that $A_C \to M$ is injective (i.e. $C \cap Y$ is finite). Then we get an exact sequence

$$0 \to T_{X/B}(-\log Y)|_C \to S_G^* \otimes M \to A^* \otimes M|_{C \cap Y} \to 0. \quad (6)$$

2.4. Log tangents for blowups. Let $\pi : \hat{X} \to X$ be the blowup of a smooth subvariety $Y$ with normal bundle $N_Y$. Let $E = \mathbb{P}(\hat{N}_Y) \subset \hat{X}$ be the exceptional divisor. Then we have an exact diagram

$$0 \to T_{\hat{X}}(-\log E) \to \pi^*T_X \to \pi^*N_Y \to 0$$

$$0 \to T_{\hat{X}} \to \pi^*T_X \to \mathcal{O}_E(1) \to 0 \quad (7)$$

For example, let $Y$ be a line in $X = \mathbb{P}^2$ so $E = Y$, $\hat{X} = X$. If $L \subset X$ is a general line then clearly

$$T_X(-\log E)|_L = \mathcal{O}(2,0)$$

with upper subbundle $\mathcal{O}(2)$ corresponding to $T_L$. If $L_1, L_2$ are distinct lines then the $\mathcal{O}(2)$ subspaces differ at the intersection point $L_1 \cap L_2$, hence

$$T_X(-\log E)|_{L_1 \cup L_2} = \mathcal{O}(2,2),$$

i.e. a direct sum of line bundles of total degree 2; therefore likewise for a general conic $C_2 \subset \mathbb{P}^2$. 17
Now let $Y$ be a line in $X = \mathbb{P}^3$ and $C_2$ a conic in a hyperplane $H \subset X$ containing $Y$, with birational transform $\hat{H} \subset \hat{X}$. Then letting $C'_2 \subset \hat{H}$ denote the birational transform of $C_2$, we have $O_{\hat{H}}(\hat{H})|_{C'_2} = O_{C'_2}$, consequently

$$T_{\hat{X}}(-\log E)|_{C'_2} = O(2,2,0)$$

with upper subsheaf $O(2,2)$ coming from $T_{\hat{H}}(-\log Y)$. Now if $L \subset \hat{X}$ is the birational transform of a general line meeting $C'_2$ is a point then $T_{\hat{X}}|_L = T_{\hat{X}}(-\log E)|_L = O(2,1,1)$. Therefore as above we get

$$T_{\hat{X}}(-\log E)|_{C'_2 \cup L} = O(3,3,2),$$

therefore likewise for $C'_2 \cup L$ replaced by $C'_3 \subset \hat{X}$, the birational transform of a twisted cubic meeting $Y$ in 2 points.

Continuing in the way, we can show that that if $\hat{X}$ is the blowup of $\mathbb{P}^n$ in a line $Y$ and $C'_n$ is the birational transform of a general rational normal curve 2-secant to $Y$, then

$$T_{\hat{X}}(-\log E)|_{C'_n} = 2O(n) \oplus (n-2)O(n-1).$$

In particular this bundle is balanced.

Now an argument similar to but simpler than the one in the proof of Lemma 31 below shows that the balancedness result holds for $Y$ replaced by a linear subspace of any codimension $c \in [2, n-1]$ as well as $C_n$ replaced by higher-degree rational curves, so we may conclude:

**Lemma 25.** Let $A \subset \mathbb{P}^n$ be a linear subspace of codimension $c \in [2, n-1]$ and let $P \to \mathbb{P}^n$ be the blowup of $A$ with exceptional divisor $E$. Let $C'_1 \subset P$ be the birational transform of a general rational curve $C \subset \mathbb{P}^n$ of given degree $e = n$ or $e \geq 2n-1$ meeting $A$ in $m \leq 2$ points. Then $T_P(-\log E)|_{C'_1}$ is balanced.

### 3. CURVES IN PROJECTIVE SPACE

#### 3.1. Balanced.** In [1], Atanasov, Larson and Yang construct many semi-balanced curves of any genus in projective space. Here we will reprove a subset of result, using a method that will be used below for other purposes. The following result is the method of construction.

**Theorem 26.** Let $C_1, C_2 \subset \mathbb{P}^n, n \geq 3$, be smooth balanced nondegenerate curves of respective degrees $e_1, e_2$, genera $g_1, g_2$, Euler slopes $t_1, t_2 > 0$ and remainders $r_1, r_2$. Assume

$$r_1 + r_2 < n - 1.$$

Then

(i) there exists a smooth balanced curve $C \subset \mathbb{P}^n$ of degree $e_1 + e_2 - 1$, genus $g_1 + g_2$ and remainder $r = r_1 + r_2$;
(ii) there exists a smooth balanced curve $C' \subset \mathbb{P}^n$ of degree $e_1 + e_2 - 2$, genus $g_1 + g_2 + 1$ and remainder $r = r_1 + r_2$.

**Proof.** We begin with some numerology. Set $g = g_1 + g_2, e = e_1 + e_2 - 1$ and

$$s = \frac{e(n + 1) + 2g - 2}{n - 1}, s_i = \frac{e_i(n + 1) + 2g_i - 2}{n - 1}, i = 1, 2.$$  

$$t = [s] + 1 - g, t_i = [s_i] + 1 - g_i, i = 1, 2.$$  

Thus $s = [s] + r/(n - 1)$ and likewise for $t, s_i, t_i$. Note that $s = s_1 + s_2 - 1$ hence $[s] = [s_1] + [s_2] - 1$ and

$$t = t_1 + t_2 - 2$$

We use the same basic fang construction as in [11]. Let

$$b_1 : \mathcal{P}(\ell) = B_{\mathbb{P}^{\ell} \times 0}(\mathbb{P}^n \times \mathbb{A}^1) \to \mathbb{P}^n \times \mathbb{A}^1$$

be the blow up, which fibres $\pi : \mathcal{P}(\ell) \to \mathbb{A}^1$ with special fibre $P_0 = \pi^{-1}(0) = P_1 \cup_E P_2$ where

$$P_1 = B_{\mathbb{P}^{\ell}}(\mathbb{P}^n), P_2 = B_{\mathbb{P}^{\ell-1}}(\mathbb{P}^n), E = \mathbb{P}^\ell \times \mathbb{P}^{n-\ell-1}$$

and general fibre $\mathbb{P}^n$. $P_0$ is called a fang of type $\ell$.

Now for (i), we let $C_i \subset P_i, i = 1, 2$ be the proper transform of a smooth curve of degree $e_i$ and genus $g_i$, such that $C_i.E = C_2.E = p$ (transverse intersection) and $C_0 = C_1 \cup_p C_2$. Then the normal bundle $N_{C_i/P_i}, i = 1, 2$ is an elementary pointwise modification of $N_{C_i/P_i}$ of colength $n - 1 - \ell$ (resp $\ell$), and under the identification $N_{C_i/P_i}|_p = T_p E$, the kernel of the natural map $N_{C_i/P_i} \to N_{C_i/\mathbb{P}^n}$ may be identified with $T_p \mathbb{P}^{n-1-\ell}$ (resp $T_p \mathbb{P}^{\ell}$). There is an exact sequence

(8)  

$$0 \to N_{C_0/P_0} \to N_{C_0/\mathcal{P}(\ell)} \to \mathbb{P}^1 \to 0$$

where $N_{C_0/P_0}, N_{C_0/\mathcal{P}(\ell)}$ are the lci normal bundles, $N_{C_0/P_0} = N_{C_1/P_1} \cup_{T_p E} N_{C_2/P_2}$ parametrizes compatible deformations of $(C_1, C_2)$ and

$$T^1 = T^1_{P_0}|_{C_0} = N_{P_0/\mathcal{P}(\ell)}|_{C_0} = T^1_{C_0}$$

is a 1-dimensional skyscraper sheaf at $p$.

As the equations defining $C_0$ on $P_0$ restrict to defining equations for each $C_i$ on $P_i$

$$N_{C_0/P_0}|_{C_i} = N_{C_i/P_i}, i = 1, 2.$$  

We have exact sequences

(9)  

$$0 \to N_{C_i/P_i} \to N_{C_i/\mathbb{P}^n} \to \tau_i \to 0, i = 1, 2,$$

(10)  

$$0 \to N_{C_i/\mathbb{P}^n}(-p) \to N_{C_i/P_i} \to \sigma_i \to 0.$$
where $\tau_i$ is a skyscraper sheaf at $p$ of length $\ell(\tau_i) = n - 1 - k$, $i = 1$ or $k$, $i = 2$, and $\ell(\sigma_i) = n - 1 - \ell(\tau_i)$. We have canonical identifications

$$N_{C_1/P_1}|_p \simeq N_{C_2/P_2}|_p \simeq T_pE.$$  

(11)

Note that we have subspaces

$$V_i = N_{C_i/P^n}(-p)|_p \subset N_{C_i/P_i}|_p, i = 1, 2$$

of codimensions $k$ resp $n - 1 - k$. The image of the restriction map

$$N_{C_0/P_0} \to N_{C_1/P_1} \oplus N_{C_2/P_2}$$

and the induced map

$$H^0(N_{C_0/P_0}) \to H^0(N_{C_1/P_1}) \oplus H^0(N_{C_2/P_2})$$

is the inverse image of the ‘diagonal’ $\Delta$ under the above identification (11). There is a standard deformation $\Delta_t$ of $\Delta$ to a $\Delta_0$ which is union of subspaces, one of them being $V_1 \times V_2$. This implies firstly that $N_{C_0/P_0}$ admits a specialization to a sheaf that contains $N_{C_1/P^n}(-p) \oplus N_{C_2/P^n}(-p)$ as cotorsion subsheaf and since that latter sheaf has $H^1 = 0$ (because $t_1,t_2 > 0$), so does $N_{C_0/P_0}$, i.e.

$$H^1(N_{C_0/X_0}) = 0.$$  

It also follows easily that $N_{C_0/X_0}$ is generically generated.

Now the above $H^1$ vanishing implies that, possibly after an étale base change $A \to A^1$, $C_0 \subset P_0$ extends to a surface $S$ fibred over $A$. Let $C$ be its general fibre. Let $x_{i1}, \ldots, x_{it_i-1}, i = 1, 2$ be general sections of $S$ specializing to general points of $C_i$. Now as $x_{i1}, \ldots, x_{it_i-1}, p$ for $i = 1, 2$ are general points on $C_i$ and hence by our hypothesis on $C_1$ and $C_2$, the restriction map

$$\rho_0 : V_1 \times V_2 \to N_{C_0/P_0}|_{\{x_{i1}, \ldots, x_{i1t_1-1}, x_{i21}, \ldots, x_{i2t_2-1}\}}$$

is surjective. Therefore the same is true of $\Delta_t$ for general $t$ hence for $\Delta$ itself if choose the above identifications generally. Therefore the same is true $N_{C/P^n}$, which shows that $C$ is semi-balanced.

For balancedness we argue similarly but, in case $s$ is not an integer, add one more section $y$ specializing to a general point on $C_1$. Because $C_1$ is balanced, the kernel of the map $\rho_0$ above injects into $N_{C_1/P^n}(-p)|_y$. Therefore the same is true for the kernel of the analogous restriction map on $H^0(N_{C_0/P_0})$ therefore ditto for $H^0(N_{C/P^n})$, which proves the injectivity property yielding balancedness. This completes the proof of (i).

For (ii), we use the same construction except now $C_i \subset P_i$ meet $E$ and each other in 2 general points $p, q$, so that

$$C_0 = C_1 \cup \{p,q\} C_2$$
has genus $g = g_1 + g_2 + 1$ and ‘degree’ $e = e_1 + e_2 - 2$. Note in this case we have
\[ s = s_1 + s_2 - 2, [s] = [s_1] + [s_2] - 2, t = t_1 + t_2 - 4. \]
We have subspaces
\[ V_{ip} = N_{C_i/P_i}(-p - q) \subset N_{C_i/P_i}, i = 1, 2 \]
and likewise for $q$, and the image of the restriction map
\[ H^0(N_{C_0/P_0}) \to H^0(N_{C_1/P_1}) \oplus H^0(N_{C_2/P_2}) \]
is the inverse image of the ‘bidiagonal’ $\Delta_p \times \Delta_q$ under restriction to $\bigoplus_{i=1,2} N_{C_i/P_{i1}}$. As above, $\Delta_p \times \Delta_q$ deforms to $\Delta_{0,p} \times \Delta_{0,q}$ which contains $W := V_{1,p} \times V_{2,p} \times V_{1,q} \times V_{2,q}$.

We consider general sections $x_{ij}, i = 1, 2, j = 1, ..., t_i - 2$. As above, $W$ surjects to $N_{C_0/P_0}|_{x_{11},...,x_{t_2-2}}$ which implies the required surjectivity for $H^0(N_{C_0/P_0})$ and hence for $H^0(N_{C/P_*})$ for the smoothing $C$, which proves semi-balancedness.

Now the injectivity statement for balancedness is proven as in part (i). This completes the proof. □

Example 27. (i) Taking $e_1 = e + 2 - n, e_2 = n, g_1 = g_2 = 0$ in Theorem 37 (ii) yields ultra-balanced elliptic curves in $\mathbb{P}^n$ of any degree $e \geq 2n - 2$. In this case $r_2 = 0, r_1 = r$. In particular, the resulting curve is perfect when $e = 2n - 2$.

(ii) Using two ultra-balanced elliptic curves as above and combining them as in Theorem 37 (i) yields a balanced curves of genus 2 and any degree $e \geq 2(2n - 2) - 2 = 4n - 6$ in $\mathbb{P}^n$. Continuing inductively, we get ultra-balanced curves of genus $g$ and any degree $e \geq g(2n - 4) + 2$ in $\mathbb{P}^n$.

(iii) Taking $C_1$ (ultra)- balanced and $C_2$ a rational normal curve (remainder 0) in Part (i) yields (ultra) balanced curves of degree $e_1 + n - 1$ and genus $g_1$. Taking such $C_1, C_2$ in Part (ii) yields balanced curves of degree $e_1 + n - 2$ and genus $g_1 + 1$.

Continuing inductively, this yields the following special case of the Atanasov-Larson-Yang result [1]:

**Corollary 28.** For all $g \geq 1, n \geq 3$ and $e \geq n + g(n - 2)$, a general curve of genus $g$ and degree $e$ in $\mathbb{P}^n$ is balanced.

3.2. Ultra-balanced. Next we refine the result to yield ultra-balanced curves, at the cost of going to higher degree.

**Theorem 29.** For all $g \geq 0$ and $e \geq 2(g + 1)n, n \geq 3$, a general curve of degree $e$ and genus $g$ in $\mathbb{P}^n$ is ultra-balanced.

**Corollary 30.** For $e, g, n$ as in Theorem 29 the conclusion of Proposition 19 holds for $X = \mathbb{P}^n$ and any $t > 0$. 

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Proof of Theorem. We begin with a lemma.

Lemma 31. Let $A \subset \mathbb{P}^n$ be a linear subspace of codimension $c \in [2, n-1]$ and let $P \to \mathbb{P}^n$ be the blowup of $A$. Let $C' \subset P$ be the birational transform of a general rational curve $C \subset \mathbb{P}^n$ of given degree $e = n$ or $e \geq 2n - 1$ meeting $A$ in $m \leq 2$ points. Then $C'$ is balanced in $P$.

Proof. The case $m = 0$, i.e., the assertion that $C$ is balanced in $\mathbb{P}^n$, originally due to Sacchiero, is reproved as Proposition 19 in [1] and the case $m = 1$ follows easily from that as $N_{C'/P}$ is a general modification of $N_{C/P^n}$. We will focus on the case $m = 2$ which is harder, as the modifications involved are not general. The proof will proceed analogously to the one in loc. cit.

Case 1: $e = n$, i.e. $C$ is a rational normal curve.
Consider first the case $\dim(A) = 1$, i.e. $c = n - 1$, where the claim is that
$$N_{C'/P} = 2\mathcal{O}(n + 1) \oplus (n - 3)\mathcal{O}(n).$$
First, for $n = 3$, $A$ is a 2-secant line of the twisted cubic $C$ and $C \cup A$ is a (2,2) complete intersection, so $C'$ is a complete intersection of type $(\mathcal{O}(2) - E, \mathcal{O}(2) - E)$ in $P$, $E$ being the exceptional divisor, hence clearly $N_{C'/P} = 2\mathcal{O}(4)$ as desired.

For $n \geq 4$ we use induction on $n$ using a degenerated curve of the form $C = L \cup_p C_{n-1}$ where $C_{n-1}$ is a general rational normal curve in a hyperplane $H \subset \mathbb{P}^n$ and $A$ is a general 2-secant line to $C_{n-1}$ while $p$ is a general point on $C_{n-1}$ and $L$ is a general line through $p$. Let $C'_{n-1} \subset H' \subset P$ denote the proper transforms. By induction, we have
$$N_1 := N_{C'_{n-1}/H'} = 2\mathcal{O}(n) \oplus (n - 4)\mathcal{O}(n - 1),$$
hence
$$N_2 := N_{C'_{n-1}/P} = N_1 \oplus \mathcal{O}(n - 3)$$
where $N_1 \subset N_2$ is canonical but not the $\mathcal{O}(n - 3)$. Moreover, as in loc. cit. we have
$$N_{C'/P}_{|C'_{n-1}} = N_1 \oplus \mathcal{O}(n - 2)$$
and the image of $N_{C'_{n-1}/P} \to N_{C'/P}_{|p}$ coincides with the image of $N_1$. On the other hand we have $N_{C'/P}_{|L} = \mathcal{O}(2) \oplus (n - 2)\mathcal{O}(1)$ and the upper subspace coming from the $\mathcal{O}(2)$ is clearly not in the image of $N_{L'/P} \to N_{C'/P}_{|p}$ at $p$, which coincides with the image of $N_{C'_{n-1}} \to N_{C'/P}_{|C'_{n-1}}$ at $p$. The upshot is that, as in loc. cit. the $\mathcal{O}_L(2)$ must be glued at $p$ to an $\mathcal{O}_{C'_{n-1}}(n - 2)$ and consequently we have
$$N_{C'/P} = 2\mathcal{O}(n + 1) \oplus (n - 3)\mathcal{O}(n),$$
as claimed.

Next consider the case $e + 1 \leq n \leq 2c - 1$ where we must show
$$N_{C'/P} = (2n - 2c)\mathcal{O}(n + 1) \oplus (2c - n - 1)\mathcal{O}(n).$$
Again the proof is by induction on $n$ fixing $c$, where the initial case $n = c + 1$ is where $A$ is a line which was just concluded. Thus assume $n > c + 1$ and consider a degenerated curve $C = C_{n-1} \cup pL$ as above. Arguing as above we get

$$N_{C'/p}|_{C'_{n-1}} = (2n - 2c - 2)\mathcal{O}(n) \oplus (2c + 1 - n)\mathcal{O}(n - 1),$$

$$N_{C'/p}|_{L'} = \mathcal{O}(2) \oplus (n - 2)\mathcal{O}(1)$$

where the $\mathcal{O}_L(2)$ must glue at $p$ to a general $\mathcal{O}_{C'_{n-1}}(n - 1)$ which implies $N_{C'/p}$ has the desired form.

Finally consider the case where $A$ has codimension $c$ with $n \geq 2c - 1$. Then the claim is

$$N_{C'/p} = (n + 1 - 2c)\mathcal{O}(n + 1) \oplus (2c - 2)\mathcal{O}(n).$$

Again we work by induction on $n$ where the initial case $n = 2c - 1$ is already known, so assume $n > 2c - 1$. Here a similar argument shows

$$N_{C'/p}|_{C'_{n-1}} = (n - 2c)\mathcal{O}(n) \oplus (2c - 2)\mathcal{O}(n - 1),$$

$$N_{C'/p}|_{L'} = \mathcal{O}(2) \oplus (n - 2)\mathcal{O}(1)$$

and again the $\mathcal{O}_{L'}(2)$ must glue at $p$ to a $\mathcal{O}_{C'_{n-1}}(n - 1)$, so we can conclude as above. This finally completes the proof of Case 1.

Note that what we have proven is equivalent to: if $C \subset \mathbb{P}^n$ is a rational normal curve with normal bundle $N \simeq (n - 1)\mathcal{O}(n + 2)$, $p, q \in C$ are general points, $A$ is a general linear space containing the line $\overline{pq}$, and $N'$ is the corresponding ‘$A$-modification’, i.e.

$$N' = \ker(N \rightarrow ((N|_p/T_pA) \oplus (N|_q/T_q))) \subset N,$$

then $N'$ is balanced.

**Case 2:** $e \geq 2n - 1$.

Notations as above, set

$$\ell = n - 1 - ((n + 2)(n - 1) - 2(c - 1))\% (n - 1).$$

Using a fang degeneration as in the first part of the proof, take a general $\mathbb{P}^\ell$ meeting the rational normal curve $C$ in 1 point and let $C_1 \subset P_1 = B_{p_1}\mathbb{P}^n$ be the birational transform of $C$; let $C_2 \subset P_2 = B_{p_2}\mathbb{P}^n$ be the birational transform of a general rational curve of degree $e - n + 1$, so that $C_1 \cap E = C_2 \cap E = \{y\}$ is 1 point, where $E$ is the exceptional divisor in $P_1$ and $P_2$. Then the appropriate $A$-modification of $N_{C_1/P_1}$ at $p, q$ (which is also a suitable modification of $N'$ above at $y$) is perfect, while $N_{C_2/P_2}$ is balanced. Then

$$C_1 \cup_y C_2 \subset P_1 \cup E P_2$$

smoothes out to a rational curve of degree $e$ in $\mathbb{P}^n$ whose $A$-modification is balanced. This completes the proof of the Lemma.
Now the proof of Theorem 29 is by induction on the genus. First for \( g = 0 \) the result follows from the fact that a general rational curve of degree \( \geq n \) in \( \mathbb{P}^n \) is balanced which is equivalent to ultra-balanced. Next we consider the case \( g = 1 \) using a fang construction as in the proof of Theorem 26, with \( g_1 = g_2 = 0, e_1 + e_2 = e + 2, e_1, e_2 \geq 2n \) but with

\[
C_0 = C_1 \cup C_2, C_1 \cap E = C_2 \cap E = \{p, q\}
\]

\((e_1, e_2 \text{ and } \ell \text{ are to be determined}). \) By Lemma 31, we may assume each \( C_i \) is ultra-balanced in \( P_i \).

Let \( N = N_{C_0/\mathbb{P}^n} \). Let \((u_1, \ldots, u_t)\) be any weight vector with each \( u_i \in [1, n-1] \), such that

\[
u_1 + \ldots + u_t = \chi := \chi(N) = e(n + 1), e = e_1 + e_1 - 2, e_i = \deg(C_i)\]

Let \( N^u = N^{(u_1, \ldots, u_t)} \). We will show \( H^0(N^u) = 0 \), so that \( N \) is \((u_1, \ldots, u_t)-balanced.\) Set

\[
N_i = N_{C_0/\mathbb{P}^n}|_{C_i} = N_{C_i/\mathbb{P}^n},
\]

\[
\chi_i = \chi(N_{C_i/\mathbb{P}^n}) = e_1(n + 1) + (n - 3),
\]

\[
\chi'_i = \chi(N_i) = e_1(n + 1) + (n - 3) - 2\ell_i, i = 1, 2,
\]

where \( \ell_1 = \ell - 1, \ell_2 = n - \ell \). Then

\[
u_1 + \ldots + u_t = \chi'_1 + \chi'_2 - 2(n - 1).
\]

**Lemma 32.** By choosing \( e_1, e_2 \) properly and relabeling \( u_1, \ldots, u_t \), we can arrange things so that

\[
u_1 + \ldots + u_s = \chi'_1 - (n - 1), u_{s+1} + \ldots + u_t = \chi'_2 - (n - 1).
\]

**Proof of Lemma.** It suffices to arrange that

\[
2\ell_1 = \chi_1 - (n - 1) - (u_1 + \ldots + u_s) = e_1(n + 1) - 2 - (u_1 + \ldots + u_s)
\]

for then the other equality in (12) is automatic. Let \( u_1 + \ldots + u_s \) be a maximal sub-sum that is \( \leq \chi_1 - (n - 1) = e_1(n + 1) - 2 \). Then

\[
\chi_1 - 2(n - 1) \leq u_1 + \ldots + u_s \leq \chi_1 - (n - 1),
\]

\[
\chi_2 - 2(n - 1) \leq u_{s+1} + \ldots + u_t \leq \chi_2 - (n - 1).
\]

If either \( d_1 := \chi_1 - (n - 1) - (u_1 + \ldots + u_s) \) or the analogous \( d_2 \) is even we can just set

\[
\ell_i = (\chi_i - (n - 1) - (u_1 + \ldots + u_s))/2
\]

and (12) holds. Hence we may assume \( d_1 \) and \( d_2 \) are odd. Assume first that \( n \) is odd, hence we may also assume \( u_s \) is odd. If

\[
u_1 + \ldots + u_{s-1} \geq \chi_1 - 2(n - 1)
\]
we may just replace $s$ by $s' = s - 1$ and be done. If (13) fails we may replace $e_1$ by $e'_1 = e_1 - 1$ and $e_2$ by $e'_2 = e_2 + 1$ and then be done.

Now Assume $n$ is even. If

$$u_1 + ... + u_s \equiv e_1 \mod 2$$

we can just let

$$\ell_1 = (e_1(n + 1) - 2 - (u_1 + ... + u_s))/2.$$ 

Otherwise, we just let $e'_1 = e_1 + 1$ and $e'_2 = e_2 - 1$ and work with $e'_1, e'_2$ instead. QED claim.

Now as $N'^u_1 := N_1^{(u_1, ..., u_s)}, N'^u_2 := N_2^{(u_{s+1}, ..., u_t)}$ are balanced and have $\chi = \text{rk} = n - 1$, we have

$$N'^u_1 = (n - 1)O_C, N'^u_2 = (n - 1)O_C.$$ 

Now let $E_i \subset P_i, i = 1, 2$ be the exceptional divisor (a copy of $E$). Then $P_0$ is constructed using an arbitrary isomorphism $\phi : E_1 \to E_2$ and I claim that by choosing $\phi$ sufficiently general, we can ensure that

$$H^0(N'^u) = H^0(N'^u_1 \cup_p N'^u_2) = 0,$$

i.e. no nonzero sections of $N'^u_1$ and $N'^u_2$ agree on $p$ and $q$. Now we have natural isomorphisms

$$TPE_i \cong N'^u_1|_p \cong H^0(N'^u_1) \cong N'^u_2|_q \cong TQE_i, i = 1, 2.$$ 

It will suffice to choose the isomorphism $\phi$, which may be identified as an arbitrary automorphism of $\mathbb{P}^\ell \times \mathbb{P}^{n-1-\ell}$, so that the derivative $d\phi - d\phi$ is nonsingular where $d\phi : TPE_1 \to TPE_2$ is the derivative and likewise for $q$. By suitable identifications, we may assume $d\phi$ is the identity 1 while $d\phi$ is an arbitrary trace-0 matrix $M$. Then clearly for suitable $M$ (e.g. non-scalar diagonal), $M - I$ is nonsingular. This completes the proof for genus 1.

Now for $g > 1$ we argue by induction, using a fang degeneration as above but with

$$C_0 = C_1 \cup_p C_2 \subset P_0 = P_1 \cup P_2, g_1 = 1, g_2 = g - 1.$$ 

Using notations as above, we let $u = (u_1, ..., u_t)$ be any weight vector with $\chi(N'^u) = 0$. We may assume $C_1, C_2$ are ultra-balanced and that $p$ is general on $C_1, C_2$. An argument similar to the proof of Lemma [32] above but simpler shows that we may assume by choosing the fang type (i.e. $\ell$) suitably that

$$\chi(N'^{u_1, ..., u_s}_1) = 0, \chi(N'^{u_{s+1}, ..., u_t}_2) = n - 1.$$ 

By ultra-balancedness we have first $H^0(N'^u_1) = 0$, then because $\chi(N'^u_2(-p)) = 0$, also $H^0(N'^u_2(-p)) = 0$. Hence $H^0(N'^u) = 0$. □
3.3. Ambient-balanced. The analogue of Theorem 26 for ambient-balanced curves also holds:

**Theorem 33.** Let $C_1, C_2$ be as in Theorem 26 and assume moreover
(i) $C_1, C_2$ are ambient-balanced;
(ii) the ambient remainders $r_1 = e_1 \% n, r_2 = e_2 \% n$ satisfy $r_1 + r_2 < n$ (e.g. $n | e_1$).
Then
(i) there exists a smooth ambient-balanced curve $C \subset \mathbb{P}^n$ of degree $e_1 + e_2 - 1$, genus $g_1 + g_2$ and ambient remainder $r = r_1 + r_2$;
(ii) there exists a smooth ambient-balanced curve $C' \subset \mathbb{P}^n$ of degree $e_1 + e_2 - 2$, genus $g_1 + g_2 + 1$ and ambient remainder $r = r_1 + r_2$.

**Proof.** We follow the general outline of the proof of Theorem 26 but now taking $C_1$ and $C_2$ in the same $\mathbb{P}^n$. By assumption $t(N_{C_i/\mathbb{P}^n}) \geq 2, i = 1, 2$ so we may assume $C_1 \cap C_2$ is exactly 1 general point (Case (i)) or 2 general points (Case (ii)). Then as in the above proof it follows that $C_1 \cup C_2$ is smoothable in $\mathbb{P}^n$. From Lemma 13 it follows that $T_{\mathbb{P}^n|C_1 \cup C_2}$ is semi-balanced, hence this is true for the smoothing as well. □

**Corollary 34.** For all $g \geq 0$, $n \geq 4$ and $e \geq n + g(n - 2)$, there exists a balanced and ambient-balanced, hence moduli-interpolating curve of genus $g$ and degree $e$ in $\mathbb{P}^n$.

**Proof.** The case $g = 0$ is well known (balancedness by Sacchiero [12], ambient-balancedness e.g. by Lemma 26 of [11]), so assume $g \geq 1$. By Corollary 28 there exists such a curve $C'$ that is balanced. Using Theorem 33 with $C_1$ a rational normal curve, it follows similarly using induction on $g$ that there is such a curve $C''$ that is ambient-balanced. Because $C', C''$ are non-special, the family of curves of degree $e$ and genus $g$ in $\mathbb{P}^n$ is irreducible, hence the general curve $C$ in the family is balanced and ambient-balanced. □

Finally, we will prove an analogue of Theorem 29 for ambient balanced curves.

**Theorem 35.** For $e \geq 3g + n + 1, n \geq 3$, there exists an ultra ambient-balanced curve of degree $e$ and genus $g$ in $\mathbb{P}^n$.

Using Theorem 29 we conclude

**Corollary 36.** For $e \geq 2(g + 1)n, n \geq 3$, a general curve of degree $e$ and genus $g$ in $\mathbb{P}^n$ is ultra-balanced and ultra-ambient balanced.

**Proof of Theorem.** The proof is analogous to that of Theorem 29 and proceeds by induction on the genus. The case $g = 0$ follows from the fact that balanced = ultra balanced in genus 0.

We next take up the case $g = 1, e \geq 3, n \geq 2$. beginning with $n = 2, e = 3$. In this case what must be shown is that for a weight-vector
$$u = (u_1, ..., u_t), u_i \in \{1, 2\}, |u| = 9,$$
and for a general cubic $C$, we have
\[ H^0(T_{P^2}^u|_C) = 0. \]

As this is an open property of $C$ we may consider a reducible cubic $C = C_2 \cup_{p,q} L$ with $C_2$ a conic and $L$ a line. Then we have
\[ T_{P^2}|_L = O(2,1), T_{P^2}|_{C_2} = O(3,3). \]

Now the weight vector $u$ must have an odd number of components equal to 1, with the rest equal to 2, hence we may assume $u = (u', u'')$ with $|u'| = 5$ and then $H^0(T_{P^2}^{u'}|_L) = 0$ and $H^0(T_{P^2}^{u''}|_{C_2}(-p - q)) = 0$. Consequently $H^0(T_{P^2}^u|_C) = 0$, which proves the result for cubics in $P^2$.

Next we will prove by induction on $n \geq 2$ that for a general cubic $C$ in a plane in $P^n$, $C$ is ultra ambient balanced in $P^n$. The proof is by induction on $n$ with $n = 2$ already known so assume $n \geq 3$ and note that
\[ T_{P^n}|_C = T_{P^{n-1}}|_C \oplus L, L := O(1)|_C. \]

If $u = (u_1, ..., u_t), u_i > 0$ is a weight vector of weight $|u| = 3(n + 1)$ then $t \geq 3$ so we can write $u = u' + u''$, $u'' = (1, 1, 0, ..., 0)$ and then
\[ H^0(T_{P^n}|^u_C) = H^0(T_{P^{n-1}}|^{u'}_C) \oplus H^0(L^{u''}). \]

Now the first summand vanishes by induction and the second by inspection. Thus $T_{P^n}|_C$ is ultra-balanced.

Next we consider the case of a general degree $e \geq 3$ and $g = 1$, working by induction on $e$. Consider a curve of the form $C_{e+1} = C_e^L \cup_p L$ where $C_e^L$ is elliptic and $L$ is a 1-secant line, and pick a weight vector $u = (u_1, ..., u_t)$ with $|u| = \chi(T_{P^n}|_{C_{e+1}}) = (n + 1)(e + 1)$. Note that
\[ T_{P^n}|_L = O(2) \oplus (n - 1)O(1). \]

Write $u = (u', u'')$ with $|u'|$ maximal subject to $|u'| \leq \chi(T_{P^n}|_L) = 2n + 1$, so that $|u'| \geq n + 1$ and also
\[ (n + 1)e - n \leq u'' \leq (n + 1)e = \chi(T_{P^n}|_{C_e^L}). \]

Write $u' = (u_1, ..., u_s)$ and let the quotients $U_1, ..., U_s$ be supported on $L$. Then the restriction maps
\[ \rho_1 : H^0(T_{P^n}|^{u'}_L) \to T_{P^n}|_p, \rho_2 : H^0(T_{P^n}|^{u''}_{C_e^L}) \to T_{P^n}|_p \]
are injective by inspection (resp. induction). Considering $N_L(-1)$ trivialized, the quotients $U_1, ..., U_s$ are general mod $T_pL$ while $T_pL$ itself may be chosen generally fixing $C_{e+1}^L$. Therefore the images of $\rho_1, \rho_2$ are in general position, i.e. complementary. Therefore $H^0(T_{P^n}|^{u}_{C_{e+1}^L}) = 0$. This finally proves the Theorem for $g = 1$.  

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Now for \( g > 1 \) we argue by induction on \( g \) and can just copy over the last part of the proof of Theorem 29 using a fang curve

\[ C_1 \cup_p C_2 \subset P_1 \cup E P_2 \]

with \( C_1 \) elliptic and \( C_2 \) of genus \( g - 1 \), and using the relative tangent bundle \( T_{P(l)/\mathbb{A}^1} \) discussed in §2 instead of the relative normal bundle. \( C_1 \) and \( C_2 \) may be assumed ultra ambient-balanced in \( \mathbb{P}^n \) and consequently \( T_{P_i(- \log E)} |_{C_i}, i = 1, 2 \) is ultra-balanced as well. Appropriately distributing weights and degrees among \( C_1, C_2 \) as in the latter proof, it goes through essentially verbatim. 

\[ \square \]

4. CURVES IN ANTICANONICAL HYPERSURFACES

The purpose of this section is to prove our results constructing (ultra) balanced and ambient-balanced curves on anticanonical hypersurfaces. The construction is based on the following result:

**Theorem 37.** Suppose the exists a balanced (resp. ultra-balanced, resp. semi-balanced) curve of degree \( e_1 \) and genus \( g \) in \( \mathbb{P}^{n-1}, n \geq 4 \). Then for all \( e \) with \( (n - 1)(e_1 - 1) \leq e \leq (n - 1)e_1 \) (resp. for \( e = (n - 1)e_1 \)), there exists a balanced (resp. ultra-balanced, resp. semi-balanced) curve of genus \( g \) and degree \( e \) on a general hypersurface of degree \( n \) in \( \mathbb{P}^n \).

**Proof.** We begin with the balanced and ultra-balanced cases. For \( g = 0 \) this is contained in Theorem 20 in [11], and the proof for general \( g \) proceeds along similar lines, modulo the constructions of the last section for higher-genus curves in \( \mathbb{P}^n \).

Assume to begin with that \( C \subset \mathbb{P}^{n-1} \) is balanced (resp. ultra-balanced) of degree \( e_1 \) and genus \( g_1 \) as in Corollary 34. Write

\[ e = e_1(n - 1) - a, 0 \leq a \leq n - 1. \]

We start with the same setup as in the proof of Theorem 26. Thus consider a fan

\[ \mathcal{P} = B_b(\mathbb{P}^n \times \mathbb{A}^1) \rightarrow \mathbb{A}^1 \]

with special fibre

\[ P_0 = P_1 \cup_E P_2, P_1 = B_b \mathbb{P}^n, P_2 = \mathbb{P}^n, E = \mathbb{P}^{n-1}. \]

Now in \( \mathcal{P} \) we consider a general relative hypersurface \( X \) of type \( (n, n - 1) \) with special fibre

\[ X_0 = X_1 \cup_F X_2 \]

where: \( X_1 \) is the blow up at \( b \in \mathbb{P}^n \) of a general hypersurface in \( \mathbb{P}^n \) of degree \( n \) and multiplicity \( n - 1 \) at \( b \), with exceptional divisor \( F \); and \( X_2 \) is a general hypersurface of
degree $n - 1$ in $\mathbb{P}^n$ with hyperplane section $F$. Then, via projection from $b$, $X_1$ is realized as $\mathbb{P}^{n-1}$ blown up at a general $(n, n - 1)$ complete intersection

$$Y = F_{n-1} \cap F_n$$

where the exceptional divisor $F$ becomes the birational transform of $F_{n-1}$.

Now by the discussion in Case 1 of the proof of Theorem 20 of [11], which uses nothing about the genus of $C$, we may assume $Y$ meets $C$ transversely in a general points $p_1, ..., p_a$ and its tangents $T_{p_i}Y$ yields general hyperplanes in the normal space $N_{C_1}(p_i), i = 1, ..., a$. If $C_1, F$ denotes the birational transform of $C_1$ resp. $F_{n-1}$ in $X_1$, then $N_{C_1/X_1}$ is a general down modification of $N_{C_1/\mathbb{P}^{n-1}}$ at $p_1, ..., p_a$, hence it is balanced by Lemma 16 (resp. ultra-balanced by definition). Then set

$$\{q_1, ..., q_e\} = C \cap F_{n-1} \setminus \{p_1, ..., p_a\} = C_1 \cap F$$

and

$$C_0 = C_1 \cup \bigcup_{i=1}^{e} L_i$$

where $L_i$ is a general line in $X_2$ through $q_i$. Because $N_{L_i/x_2}$ is a trivial bundle, it is easy to check that $N_{C_0/x_0}$ is balanced (resp. ultra-balanced) around $C_1'$. Therefore when $(C_0, X_0)$ smooth out to a general $(C, X)$, $X$ a general hypersurface of degree $n$, the normal bundle $N_{C/X}$ is likewise balanced (resp. ultra-balanced). This proves the assertion of the Theorem in the balanced and ultra-balanced cases.

Note that in the above argument, if $C_1$ is semi-balanced and $a = 0$, then $C_0$ is semi-balanced around $C_1'$ hence its smoothing $C$ is semi-balanced. This proves the assertion in the semi-balanced case. \[\Box\]

Now Theorem 29 yields:

**Corollary 38.** For $n \geq 4$ a general hypersurface of degree $n$ in $\mathbb{P}^n$ contains ultra-balanced curves of genus $g$ and degree $e$ for all $e \geq 2(g + 1)n(n - 1)$.

**Remark 39.** Trying to prove even semi-balancedness for $C_0$ when $e$ is not a multiple of $n - 1$ requires modifications of the normal bundle to $C_1$ and hence an assumption that $C_1$ be balanced, rather than weakly balanced.

A modification of this approach yields curves that are both balanced and ambient-balanced:

**Theorem 40.** A general hypersurface of degree $n$ in $\mathbb{P}^n$, $n \geq 4$, contains ultra-balanced and ultra ambient-balanced curves of degree $e$ and genus $g$ provided $g = 0, e \geq n - 1$ or $g \geq 1, e \geq 4g(n - 1)$.
Proof. We use the construction and notations in the proof of Theorem 37. Given Corollary 36, proving Theorem 40 is a matter of showing that the curves constructed in the latter proof may be assumed ultra ambient-balanced provided \( C \subset \mathbb{P}^{n-1} \) is. We use the relative tangent bundle as discussed in §2, so the restricted tangent bundle \( T_X|_C \) for a curve on \( X \) specializes to

\[
T_{X_1} \langle - \log E \rangle|_{C_1} \cup T_{X_2} \langle - \log E \rangle|_{C_2}, C_1 \cup C_1 \subset X_1 \cup X_2,
\]

where \( C_2 \subset X_2 \) is a disjoint union of lines with trivial normal bundle. Now working as in Example 24, we modify the relative tangent bundle along \( C_2 \) so the specialized bundle becomes \( T_{X_1}|_{C_1} \cup (n-1)\mathcal{O}_{C_2} \). Then it is clearly sufficient to show that \( C_1 \subset X_1 \) is ultra ambient-balanced. But, with the above notations, \( T_{X_1}|_{C_1} \) is a general corank-1 down modification of of the ultra-balanced bundle \( T_{\mathbb{P}^{n-1}}|_C \) at \( p_1, ..., p_a \), hence is ultra-balanced. \( \square \)

5. CURVES IN OTHER FANO HYPERSURFACES

We now turn our attention to lower-degree hypersurfaces. The purpose of this section is to prove the following

**Theorem 41.** Let \( X \) be a general hypersurface of degree \( d \in [3, n-1] \) in \( \mathbb{P}^n, n \geq 4 \). Then

(i) \( X \) contains balanced curves \( C \) of degree \( e \) and genus \( g \) provided there exists \( e_0 \in [(g+1)n, e] \) such that either

\[
\left\lfloor \frac{-de_0 + e}{n-d} \right\rfloor + e = e_0 + \left\lfloor \frac{2e_0 + 2g - 2}{d-2} \right\rfloor.
\]

or

\[
\left\lfloor \frac{-de_0 + e}{n-d} \right\rfloor + e = e_0 + \left\lfloor \frac{2e_0 + 2g - 2}{d-2} \right\rfloor + 1.
\]

In particular given \( g \geq 0 \), there exist such balanced curves for every \( e \) in at least \( (d-2)(n-d+1) \) many arithmetic progressions with difference \( d(n-2) \).

(ii) \( X \) contains ambient-balanced curves \( C \) of degree \( e \) and genus \( g \) provided there exists \( e_0 \in [(g+1)n, e] \) such that

\[
\left\lfloor \frac{-de_0 + e}{n-d} \right\rfloor + e = e_0 + \left\lfloor \frac{e_0}{d-1} \right\rfloor.
\]

or

\[
\left\lfloor \frac{-de_0 + e}{n-d} \right\rfloor + e = e_0 + \left\lfloor \frac{e_0}{d-1} \right\rfloor + 1
\]

In particular, given \( g \geq 0 \), there exist such ambient-balanced curves curves for infinitely many \( e \).
For the ‘in particular’ portion of (i) see the Appendix by M. C. Chang below.

Remark 42. (i) Note that for \( d > n/2 \), eq. (14) already implies \( e > e_0 \).

(ii) In light of Example 21, it is not unreasonable to expect some obstructions in terms of \( e \) to the existence of an ambient-balanced curve of degree \( e \).

Example 43. Solving (16) is elementary. Write
\[
e_0 = \alpha(d - 1) + \beta, 0 \leq \beta < d - 1, \alpha = \left[ \frac{e_0}{d-1} \right],
\]
\[
e - de_0 = q(n - d) + r, 0 \leq r < n - d.
\]
Then an elementary calculation yields
\[
d(d - 2)\alpha + (d - 1)\beta = (-q)(n - d + 1) - r.
\]
This is solvable for \( e \) iff
\[
(d - 1)e_0 - \left[ \frac{e_0}{d-1} \right] \neq 1 \mod n - d + 1.
\]
Explicitly, writing
\[
(d - 1)e_0 - \left[ \frac{e_0}{d-1} \right] = u(n - d + 1) + v, -(n - d) < v \leq 0,
\]
the solution is
\[
e = de_0 - u - v.
\]
Because \( u \leq ((d - 1)e_0 + n - d)/2 \), clearly \( e \to \infty \) as \( e_0 \to \infty \) so there are infinitely many \( e \) for given \( n, d, g \).

Example 44. (M. C. Chang) For \( d = n - 1 \), equation (14) reads
\[
2e = ne_0 + \left[ \frac{2e_0 + 2g - 2}{n - 3} \right].
\]
Write
\[
g = x(d - 2) + y, e_0 = (2k + r)(d - 2) + c, 0 \leq y, c \leq d - 3, r \in \{0, 1\}.
\]
Then, setting \( t = \left[ (2c + 2y - 2)/(d - 2) \right] \), we get
\[
e = kd(d - 1) + x + (t + r(d^2 - d) + c(d + 1))/2.
\]
\( e \) is an integer iff \( t + c(d + 1) \) is even. Assuming \( c > 0 \), we have \( t \in [0, 3] \). We try to count the ‘bad’ pairs \((c, r)\) \(\in [1, d - 3] \times [0, 1] \), i.e. those where \( t + c(d + 1) \) is odd, with \( y \) given. If \( d \) is odd badness means \( t \) is odd, i.e. \( t \in \{1, 3\} \). The number of such \( c \) is at most \( d/2 - 1 \). If \( d \) is even badness means either \( t \in \{1, 3\}, c \) even (at most \(((d/2) - 1)/2 \) solutions) or \( t \in \{0, 2\}, c \) odd (again at most \(((d/2) - 1)/2 \) solutions). Thus if \( d \) is either even or odd, there are at most \( d/2 - 1 \) bad \( c \) values, hence the number of good pairs
(c, r) is at least $2(d - 3 - (d/2 - 1)) = d - 4$; i.e. there are at least $d - 4$ good congruence classes of $e_0 \mod 2(d - 2)$ hence at least $d - 4$ distinct arithmetical progressions for $e$ with difference $d(d - 1)$.

Similarly treating eq. (16) for $d = n - 1$ yields

$$e = (ne_0 + \left\lceil \frac{e_0}{n-2} \right\rceil)/2.$$  

When $n$ is even (resp. odd), this is an integer provided $\left\lceil \frac{e_0}{n-2} \right\rceil$ is even (resp. the remainder $e_0 \% (n - 2)$ is even). This leads to about $n - 2$ (resp. $(n - 3)/2$) arithmetic progressions of $e$ values with difference $n(n - 2)$ (resp. $(n - 1)^2/2$) for $n$ even (resp. odd). Note that the condition for (16) to hold is, in the above notations $2k + r \equiv c \mod d - 1$. This yields about $d - 4$ distinct arithmetical progressions for $e$ with difference $d(d - 1)^2$.

**Example 45.** When are the curves produced by Theorem 41 actually perfect? For perfect balance, it is a matter of replacing (14) by the ‘exact’ equation

$$\frac{-de_0 + e}{n-d} + e = e_0 + \frac{2e_0 + 2g - 2}{d-2}$$

together with the condition that both sides of (18) be integers. This is a sufficient condition that the curve $C$ is perfectly balanced. Assume first that $d$ is odd and write

$$e_0 = \lambda(d-2) + 1 - g, \lambda \in \mathbb{Z}.$$  

Then the condition that (18) can be solved for an integer $e$ is

$$\lambda d(n - 2) + n(1-g) \equiv 0 \mod n-d+1$$

or equivalently

$$\lambda(n+1)(n-2) + n(1-g) \equiv 0 \mod n-d+1.$$  

At the upper end $d = n - 1$, $n$ even, (20) is automatic, so the curves produced by Theorem 41 are always perfectly balanced. At the lower end, if $d = 3$, eq. (20) becomes the condition $2 - 2g \equiv 0 \mod n - 2$. For $d > 3$ odd, (20) admits an arithmetic progression of solutions $\lambda$ (hence of $e$ values yielding perfectly balanced curves) provided

$$(d, n+1) = 1 = (d-3, n-2).$$

For example when $d = 5$ this holds whenever $n$ is odd and $n \neq 4 \mod 5$.

Similarly analyzing the case $d = 2d_0$ even leads to

$$(d^2/2 - 2d + 1)\lambda + (d - 1)(1 - g) \equiv 0 \mod n - d + 1$$

which admits an arithmetic progression of solutions $\lambda$ provided

$$(d^2/2 - 2d + 1, n - d + 1) = 1.$$
Similarly treating eq. (16), i.e. seeking $C$ that is perfectly ambient-balanced, leads to

$$e = \frac{(n-1)d}{(d-1)(n-d+1)}e_0.$$  

This is solvable at least when $(d-1)(n-d+1)e_0$, leading to at least one arithmetic progression of degrees for which there exists a perfectly ambient-balanced curve.

**Proof of Theorem.** The proof proceeds along similar lines as that of Theorem 31 of [11], using a relative fang. Thus let $Z \to \mathbb{A}^1$ be a relative fang of type $(n,m), m = d - 1 \geq 2,$ with special fibre

$$Z_0 = Z_1 \cup Z_2, Z_1 = \mathbb{P}^m(1,0^{n-m}), Z_2 = \mathbb{P}^{m-1}(1,0^{m-1}).$$

Let $\mathcal{X} \subset Z$ be a general member of the linear system $|dH - (d-1)Z_2|$ where $H \subset \mathbb{P}^n$ is a hyperplane. The $\mathcal{X} \to \mathbb{A}^1$ has special fibre

$$X_0 = X_1 \cup_E X_2.$$  

Here $X_1 = \mathbb{P}^m(G)$ where $G$ is a bundle on $\mathbb{P}^m$ that fits in an exact sequence

$$(21) \quad 0 \to \mathcal{O}(-m) \to \mathcal{O}(1) \oplus (n-m)\mathcal{O} \to G \to 0$$

in which the left map is general. Also $X_2$ fibres over $\mathbb{P}^{m-1}$ with general fibre a general hypersurface of degree $d - 1 = m$ in $\mathbb{P}^{m+1}$. As in the above-referenced proof, we will construct a balanced curve in $X_0$ of the form $C_1 \cup C_2$ where $C_1 \subset X_1$ is balanced and $C_2 \subset X_2$ is a disjoint union of lines in fibres of $X_2 \to \mathbb{P}^{m-1}$, and as such has trivial hence balanced normal bundle. Then $X_0$ will smooth along with $Z_0$ to a balanced curve in the general fibre of $\mathcal{X} \to \mathbb{A}^1$. It will suffice to construct $C_1$.

To this end, proceeding as in [11], proof of Theorem 31, we will start with a balanced curve $C_0 \subset \mathbb{P}^m$ of genus $g$ and degree $e_0$ and lift it to $C_0 \simeq C_1 \subset \mathbb{P}(G) = X_1$ using a general surjection

$$\psi : G_{C_0} \to M$$

where $M = \mathcal{O}_{C_0}(H + A)$ with $L = \mathcal{O}(H)$ being the hyperplane bundle from $\mathbb{P}^m$ and and $A$ is a general effective divisor $A$ of degree $e - e_0, e_0 = \deg(L)$, which also coincides with $C_1,E$. Such a map $C_1 \to X_1$ comes from a map $\phi : C \to \mathbb{P}^n$ corresponding to $n + 1$ sections of $L$ among which $m + 1$ vanish on $A$, and can be constructed by starting from $C_0 \to \mathbb{P}^m$ corresponding to $m + 1$ sections of $L$ and adding $n - m$ additional sections of $M = L(A)$.

Now setting $K = \ker(\psi)$, the vertical part of the normal bundle $N_{C_1/\mathbb{P}(G)}$ is $K^*(M)$, i.e. we have an exact normal sequence

$$(23) \quad 0 \to K^*(M) \to N_{C_1/\mathbb{P}(G)} \to N_{C_0/\mathbb{P}^m} \to 0$$
and the relation (14) means exactly that the slope matching condition of Lemma 18 and 11, eq. (10) holds. Thus will suffice to prove as in 11 that $K^*(M)$ is balanced. For $g = 0$ this is proved in 11, Lemma 33. In the general case we will use induction on $g$, starting with a reducible form of $C_0$ of the form

$$(24) \quad C_{00} = C_{01} \cup_{p,q} C_{02} \subset \mathbb{P}^m$$

where $C_{01}$ is a rational normal curve (of degree $m$, $C_{02}$ is a balanced curve of genus $g - 1$ and degree $e_{02} \geq m + (g - 1)(m - 2)$ (see Corollary 34) and $p, q$ are general points. We then lift $C_{00}$ to

$$(25) \quad C_{10} = C_{11} \cup_{p,q} C_{12} \subset X_1$$

using the surjection $\psi : G_{C_{00}} \to M_0$ to a line bundle of degree $e$ of the form $\mathcal{O}_{C_0}(H + A_0)$ as above. We choose the line bundle $M_0$ on $C_{00}$ so that

$$e_1 := \deg(M_0|_{C_{01}}) \equiv d(d - 1) \mod n - d, e_1 \geq m, e_2 := \deg(M_0|_{C_{02}}) \geq (g - 1)n$$

and

$$e_1 + e_2 = e.$$  

We may assume $e_1 \leq 2n$. Now we have analogues of the sequence (23) for $C_{11}, C_{12}$ and inductively both left and right members in those sequences have Euler slope $\geq 2$, and it follows that

$$H^1(N_{C_{1i}/X_1}(-p - q)) = 0, i = 1, 2.$$  

Because $N_{C_{10}/X_1}$ contains $N_{C_{11}/X_1}(-p - q) \oplus N_{C_{12}/X_1}(-p - q)$ as a subsheaf parametrizing deformations where $C_{11}$ and $C_{12}$ deform separately going through $p, q$, it follows easily that $C_{10}$ is smoothable in $X_1$ to a curve of genus $g$ and degree $e = e_1 + e_2$. Now the bundle $K^*(M)$ restricts to the analogous bundles on $C_{1i}, i = 1, 2$ which are balanced by induction and perfect for $i = 1$ by the congruence condition on $e_1$. Moreover as noted the Euler slope of $K^*(M)|_{C_{1i}}$ is clearly at least 2. Hence by Lemma 13 it follows that $K^*(M)$ is balanced on $C_{10}$, hence on its smoothing in $X_1$.

Finally for ambient-balancedness, we argue as in the proof of Theorem 40, noting that here again $C_2$ is a union of lines $L$ with trivial normal bundle, hence

$$T_{X_2}(-\log E)|_L = \mathcal{O}(1) \oplus (n - 2)\mathcal{O}_L$$

where the $(n - 2)\mathcal{O}_L$ quotient coincides at $p = L.C_1$ with $T_{p,E}$. Moreover $C_2 \cap C_1 = A$ is a general divisor on $C_1$. As in the above proof, it will suffice to prove that $T_{X_1}|_{C_1}$ is balanced. Note the exact sequence

$$0 \to K^*(M) \to T_{P|G}|_{C_1} \to T_{P^n}|_{C_1} \to 0,$$

which identifies $K^*(M)$ as the relative tangent bundle $T_{X_1/\mathbb{P}^m}$.
Now (16) ensures that the slopes of $K^*(M)$ and $T_{P^m}|C_1$ have the same roundoff, so by Lemma 18 it will suffice to show $K^*(M)$ and $T_{P^m}|C_1$ are balanced. As for $T_{P^m}|C_1$, it may be assumed balanced thanks to Corollary 34. As for $K^*(M)|C_1$, we will use induction on $g$. First for $g = 0$, it is proven in [11], Lemma 33, p. +35, that $K|C_1$ is balanced, hence so is $K^*(M)|C_1$. Then the general case is proven by degeneration to $C_{10} = C_{11} \cup C_{12}$ similarly to the above where $K^*(M)|C_{11}$ is perfect.

\[\square\]

Remark 46. The ultra version of the Matching Lemma 18 is not known. Therefore neither is the ultra version of Theorem 41.

Remark 47. There is a misprint in the proof of Lemma 33 in the journal version of [11] (p.+35, l.-11). The arxiv version is correct.
REFERENCES

1. A. Atanasov, E. Larson, and D. Yang, Interpolation for normal bundles of general curves, Mem. AMS (2016), arxiv 1509.01724v3.
2. I. Coskun and E. Riedl, Normal bundles of rational curves on complete intersections, Math Z. 288 (2018), 803–827, arxiv 1705.08441v1.
3. L. Ein and R. Lazarsfeld, Stability and restrictions of Picard bundles, with an application to the normal bundles of elliptic curves, Complex projective geometry (Trieste 1989/ Bergen 1989) (G. Ellingsrud, C. Peskine, G. Sacchiero, and S. A. Stromme, eds.), London Math. Society Lecture Note series, vol. 179, Cambridge University Press, 1992, pp. 149–156.
4. D. Eisenbud and A. Van de Ven, On the normal bundle of smooth rational space curves, Math. Ann. 256 (1981), 453–463.
5. G. Ellingsrud and D. Laksov, The normal bundle of elliptic space curves of degree 5, 18th Scandinavian congress of math. Proc. 1980 (E. Balslev, ed.), Birkhäuser, 1981, pp. 258–287.
6. K. Hulek, Projective geometry of elliptic curves, Algebraic Geometry- Open Problems, Lecture Notes in Math., vol. 997, Springer-Verlag, 1983, pp. 228–266.
7. D. Perrin, Courbes passant par m points g´en´eraux de P3, Mém. Soc. Math. France (N.S.) (1987), no. 28-29.
8. Z. Ran, Normal bundles of rational curves in projective spaces, Asian J. Math. 11 (2007), 567–608.
9. ______, Interpolation of rational scrolls, arxiv.org (2021), 2111.02466.
10. ______, Low-degree rational curves on hypersurfaces in projective spaces and their fan degenerations, J. Pure Applied Algebra (2021), Arxiv 1906.03747.
11. ______, Balanced curves and minimal rational connectedness on Fano hypersurfaces, Internat. Math. Research Notices (2022), arxiv.math:2008.01235.
12. G. Sacchiero, Fibrati normali di curve razionali dello spazio proiettivo, Ann. Univ. Ferrara VII (N. S.) (1981), no. 26, 33–40.
In this appendix, we prove the following

**Theorem 1.** For fixed integers \(3 \leq d \leq n - 1\) and \(g > 0\), there are at least \((d - 2)(n - d + 1)\) arithmetic progressions with difference \(d(n - 2)\) of \(e\) values such that for some integer \(e_0\), \(e \geq e_0 \geq (g + 1)n\), one has either

\[
\left\lfloor \frac{-de_0 + e}{n - d} \right\rfloor + e = e_0 + \left\lfloor \frac{2e_0 + 2g - 2}{d - 2} \right\rfloor.
\]

or

\[
\left\lfloor \frac{-de_0 + e}{n - d} \right\rfloor + e = e_0 + \left\lfloor \frac{2e_0 + 2g - 2}{d - 2} \right\rfloor + 1.
\]

Our approach is similar to that of the case \(g = 0\) of (26) (see [11], Appendix by M. C. Chang). So we only provide the necessary details here.

We write

\[
g = x(d - 2) + y, \text{ where } y \in [0, d - 3],
\]

and denote

\[
b = n - d + 1.
\]

For \((c, r) \in [0, d - 3] \times [0, b - 1]\), and \(k \in \mathbb{Z}_+\) let

\[
e_0 = (kb + r)(d - 2) + c.
\]

Hence

\[
\frac{2e_0 + 2g - 2}{d - 2} = 2kb + 2r + 2x + \frac{2c + 2y - 2}{d - 2}.
\]

Denote

\footnote{Research partially financed by the NSF Grants DMS 1764081.}
(32) \[ t = \left\lfloor \frac{2c + 2y - 2}{d - 2} \right\rfloor. \]

For equation (26), we let \( \varepsilon \) be the fractional part of \( \frac{-d_0 + e}{n-d} \), i.e.,

(33) \[ \frac{-d_0 + e}{n-d} = \left\lfloor \frac{-d_0 + e}{n-d} \right\rfloor + \varepsilon. \]

In particular, \( \varepsilon < 1 \).

Putting displays (26) and (28)-(33) together, we have

(34) \[ e = d(n - 2)k + 2x + t + rd + c + \frac{r(d^2 - 3d) + c(d - 1) - 2x - t}{b} \varepsilon + \frac{b - 1}{b}. \]

For equation (27), we have

(35) \[ e = d(n - 2)k + 2x + t + 1 + rd + c + \frac{r(d^2 - 3d) + c(d - 1) - 2x - t - 1}{b}. \]

**Lemma 2.** Let \( e(c, r, \varepsilon) = e \)

\[ = d(n - 2)k + 2x + t + rd + c + \frac{r(d^2 - 3d) + c(d - 1) - 2x - t}{b} \varepsilon + \frac{b - 1}{b} \]

be as in (34)

If \( (c, r, \varepsilon) \neq (c_1, r_1, \varepsilon_1) \), then \( e(c, r, \varepsilon) \neq e(c_1, r_1, \varepsilon_1) \mod d(n - 2) \).

The same statement is also true for \( e \) in (35).

**Proof.** Let

\[ E(c, r, \varepsilon) = 2x + t + rd + c + \frac{r(d^2 - 3d) + c(d - 1) - 2x - t}{b} \varepsilon + \frac{b - 1}{b}. \]

**Claim 1.** \( E(c, r, \varepsilon) \neq E(c_1, r_1, \varepsilon_1) \) as real numbers.

**Proof of Claim 1.**

First, we assume \( r_1 - r \geq 1 \), and \( E(c, r, \varepsilon) = E(c_1, r_1, \varepsilon_1) \). Then

(36) \[ (r_1 - r) \left( d + \frac{d^2 - 3d}{b} \right) = (c - c_1) \left( 1 + \frac{d - 1}{b} \right) + (t - t_1) \left( 1 - \frac{1}{b} \right) + \frac{b - 1}{b} (\varepsilon - \varepsilon_1). \]
By Lemma 4 below, \( t - t_1 \leq 2 \). Also, in (30), we take \( c \in [0, d - 3] \). Hence, the right hand side of (9) is less than
\[
(d - 3) \frac{b + d - 1}{b} + 2 \frac{b - 1}{b} + \frac{b - 1}{b},
\]
which is less than
\[
d + \frac{d^2 - 3d}{b} - \frac{d}{b} < d + \frac{d^2 - 3d}{b} \leq \text{the left hand side of (9)}.
\]
This is a contradiction.
Hence, \( r_1 = r \) and (9) is
\[
(37) \quad 0 = (c - c_1) \left(1 + \frac{d - 1}{b}\right) + (t - t_1) \left(1 - \frac{1}{b}\right) + \frac{b - 1}{b} (\epsilon - \epsilon_1).
\]
Next, we assume \( c - c_1 \geq 1 \). From the definition of \( t \) in (32), we have \( t \geq t_1 \), and
\[
b + d - 1 \leq (c - c_1)(b + d - 1) + (t - t_1)(b - 1) = (b - 1)(\epsilon_1 - \epsilon) \leq b - 2,
\]
which is a contradiction.

Claim 2. If \( E(c_1, r_1, \epsilon_1) \neq E(c, r, \epsilon) \), then
\[
E(c_1, r_1, \epsilon_1) \neq E(c, r, \epsilon) \mod d(n - 2).
\]

Proof of Claim 2. Assume \( E(c_1, r_1, \epsilon_1) > E(c, r, \epsilon) \). Then
\[
E(c_1, r_1, \epsilon_1) - E(c, r, \epsilon) < (b - 1) \frac{bd + d^2 - 3d}{b} + (d - 3) \frac{b + d - 1}{b} + 2 \frac{b - 1}{b} + \frac{b - 1}{b}
= bd + d^2 - 3d - \frac{d}{b}
< d(n - 2).
\]
Hence \( E(c_1, r_1, \epsilon_1) - E(c, r, \epsilon) \) cannot be a multiple of \( d(n - 2) \). □

Proof of Theorem 1.

By Lemma 2, it is sufficient to find a lower bound on the permissible pairs of \( (c, r) \in [0, d - 3] \times [0, b - 1] \).

Since \( e \in \mathbb{Z} \), in (34) we let
\[
\frac{r(d^2 - 3d) + c(d - 1) - 2x - t}{b} + \frac{b - 1 - m}{b} = m + 1, \quad \text{with } m \in \mathbb{Z}.
\]
By display (33), $\epsilon < 1$, which is equivalent to

$$m < \frac{r(d^2 - 3d) + c(d - 1) - 2x - t - 1}{b}. $$

So, for equation (26), we want to rule out those $(c, r)$ such that

$$\frac{r(d^2 - 3d) + c(d - 1) - 2x - t - 1}{b} = m. $$

Namely, we want to rule out $(c, r) \in [0, d - 3] \times [0, b - 1]$ such that

$$(38) \quad r(d^2 - 3d) + c(d - 1) \equiv 2x + t + 1 \mod b. $$

On the other hand, these $(c, r) \in [0, d - 3] \times [0, b - 1]$ satisfying (38) are the pairs making $e$ in (35) an integer, hence a solution of (27). Therefore, $\forall (c, r) \in [0, d - 3] \times [0, b - 1]$ is a permissible pair for either equation (26) or equation (27). \hfill \Box