Two-Dimensional Space-Time Analysis and Matrix Representation on the Principle of the Capacitive Displacement Transducer

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Abstract. In order to provide a design method of the capacitive displacement transducer and to improve its measuring performance it is desperately needed to offer a refined mathematic model of the transducer of multiphase drive and phase-modulated. On the basis of fully considering its characteristic of digital signals, first it is found that their actual waveforms and space-time characteristics could be tersely represented by matrixes $[u_{ij}]$, $[c_j]$ and $[v_i]$, and corresponding matrix elements $u_{ij}$, $c_j$ and $v_i$ through deeply analyzing space-time and quantum characteristics of their multiphase driving signals $U_i(t)$, capacitive coupling signals $C_j(x)$ and output signal $V(t)$ and space-time transform function possessed by $U(x,t)$ itself. Then the basic expression of the relations of the transducer is derived, which is expressed by matrixes, thereby the characteristics of space-time transform and phase modulation are brought to light. The demodulation process and demodulated waveforms and its characteristics in the transducer are also expressed by demodulated matrixes $[b_{ij}]$. Finally, the reason for the principle and periodic error produced in the transducer is revealed by sampling matrix $[s_{ij}]$. Thus the full process of the produce of driving signals, modulation, demodulation and space-time transform that happen in the transducer, also waveforms and characteristics of various signals in the process are concisely expressed by two-dimensional space-time matrixes. Experimental results indicate that the use of the mathematical model enables its resolving power to reach 1μm, and the mathematical model proposed is an all-things-considered model to express processes that happen in the transducer.

1. Introduction

As there is a transform from displacement to time, so we hope that its characteristics of space-time transform could be clearer expressed in an expression. Some processes, for example, modulation, phase shift, sampling and demodulation are better expressed by use of two-dimensional space-time matrix in the transform process. And some characteristics are opened out.

2. Quantum characteristics and matrix Representation of $U(x,t)$

When we consider waveforms in a time period $T_n$ and a scale electrode pitch $W(=8w)$ at the same time, driving function is the function of $x$ and $t$ and could be express by $U(x,t)$. Although $x$ and $t$ could take
When the continuous value, but for space when W corresponds 8 smaller intervals w, normalizing value of $U(x,t)$ corresponded by any x is only taken 0, or 1. When time period $T_u$ of $U(x,t)$ is divided into 128t by minimum pulse width $\tau$ in time field in the same way, corresponding, normalizing $U(x,t)$ only could take 0, or 1, whether x is any value. And so if vertical axis is $X$ axis and horizontal axis is T axis from $U(t)$ waveform figure as shown figure 1, $U(x,t)$ could express in 128×8 matrix:

$$U(x,t) = j \downarrow \begin{bmatrix} u_{ij} \end{bmatrix}$$

(1)

Where $i=128$, $i$ is time subscript; $j=1$–8, $j$ is space subscript, and so $[u_{ij}]$ is a two-dimensional time-space matrix and has characteristic as follows:

$$\sum_{j=1}^{8} u_{ij} = 4$$

(2)

$$u_{ij} = u_{ij} + 8$$

(3)

$$u_{ij} = 1 - u_{i,j+8}$$

(4)

$$u_{i,16j} = 1 - u_{i,j+1}$$

(5)

Namely, $U(x,t)$ has basic time period $T_j = T_u / 8 = 16 \tau$. The waveforms of $T_j, \tau$ could generate when $T_j$ move w in positive direction, that is

$$u(x,t - N \cdot T_j / 8) = u(x + N \cdot W / 8, t)$$

(6)

or

$$u(x_0,t_0 - \text{INT}((t-t_0)/(T_j/8))T_j/8) = u(x_0 + \text{INT}((x-x_0)/(W/8))W/8)$$

(7)

where N is positive integer, INT is integral function. Formulas (6) and (7) express that $U(x,t)$ has time-space transform function itself. Its minimum time unit is $T_u / 8$ and minimum space unit is $W / 8$.

Thus we could simplify design of signal generator of $U(x,t)$

$$u_{ij} = u_{i \pm 128, j}$$

(8)

$$u_{ij+1} = u_{i-16, j}$$

(9)

Figure 1. waveforms of two-dimensional $U(x,t)$.

3. Quantum characteristics and matrix Representation of $C(x)$

When we consider coupling of transmitting electrodes and scale electrodes only in 8 w ranges, although x could take continuous value, but capacitances corresponded by any x are determinate values in every w. As capacitance is a function of x, so capacitance is normalized. If m is a equally divided number in w, m and n are positive integer, and $0 \leq n \leq m$, we has $x = (n/m)w$. Capacitance value is expressed by $n/m$, thus the values of capacitance only have $n/m, 0, 1$ and $1 - n/m$ as shown in Figure 2. When we ignore edge effect, and consider the clearances between transmitting electrodes $e_i$ are approximate to
zero, and the clearances between scales electrodes and screen electrodes also are approximate to zero, capacitance coupling function also could expressed by a column matrix.

\[
\begin{bmatrix}
  c_1 \\
  c_2 \\
  \vdots \\
  c_8
\end{bmatrix}
\]

have characteristic as follows: \( C_j(x) = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_8 \end{bmatrix} \) \( (j = 1 \sim 8) \) (11)

\[
\sum_{j=1}^{8} c_j = 4
\] (12)

• Three “0”, or three “1” are continuously joined. \( n/m \) and \( 1 \sim n/m \) are between 0 and 1, and are corresponding with moving direction.

• When reading head moves \( w \) in positive direction, every elements cycle and move up a row. And when reading head moves \( w \) in negative direction, every element cycle and move down a row. This is space phase shift expressed by matrix, so the direction of \( c_j \) is \( c_j(x) = \ast [c_j] \).

\[
c_j = c_{j+8}
\] (14)

4. Matrix Representation and characteristics of \( V(t) \)

\[
\begin{align*}
V_{1}(x,t) &= V_{11}(x,t) + V_{12}(x,t) + V_{13}(x,t) + V_{14}(x,t) \\
V_{2}(x,t) &= V_{21}(x,t) + V_{22}(x,t) + V_{23}(x,t) + V_{24}(x,t) \\
V_{3}(x,t) &= V_{31}(x,t) + V_{32}(x,t) + V_{33}(x,t) + V_{34}(x,t) \\
V_{4}(x,t) &= V_{41}(x,t) + V_{42}(x,t) + V_{43}(x,t) + V_{44}(x,t) \\
\end{align*}
\]

Figure 3. Waveform of \( V(t) \) and \( D_k(t) \).

The waveform of output function \( V(t) \) is shown as in the top of Figure 3. Form expression [1]
\[ V(t) = \sum_{i=1}^{8} u_i(t)c_i(x) \]

and (1) and (13), we get the basic expression of the transducer in matrix:

\[
\begin{bmatrix}
\vec{u}_i \\
\end{bmatrix} = \begin{bmatrix}
\vec{v}_i
\end{bmatrix}, 
\quad (i = 1 - 128) \tag{15}
\]

### Table 1. Values of \( v \) in positive direction (\( v = \sum v_i \))

| \( v_1 \) | \( v_2 \) | \( v_3 \) | \( v_4 \) | \( v_5 \) | \( v_6 \) | \( v_7 \) |
|---|---|---|---|---|---|---|
| \( n/m \) | 1 + n/m | 2 + n/m | 3 + n/m | 4 - n/m | 3 - n/m | 2 - n/m |
| \( n/m \) | 2 - n/m | 1 - n/m | n/m | 1 + n/m | 2 + n/m | 3 + n/m | 4 - n/m |
| \( 3 - n/m \) | 3 - n/m | 2 - n/m | 1 - n/m | n/m | 1 + n/m | 2 + n/m | 3 + n/m |
| \( n/m \) | 1 + n/m | 2 + n/m | 3 + n/m | 4 - n/m | 3 - n/m | 2 - n/m | 1 - n/m |
| \( n/m \) | 1 + n/m | 2 + n/m | 3 + n/m | 4 - n/m | 3 - n/m | 2 - n/m | 1 - n/m |
| \( 3 - n/m \) | 2 - n/m | 1 - n/m | n/m | 1 + n/m | 2 + n/m | 3 + n/m | 4 - n/m |
| \( 3 - n/m \) | 2 - n/m | 1 - n/m | n/m | 1 + n/m | 2 + n/m | 3 + n/m | 4 - n/m |
| \( n/m \) | 1 + n/m | 2 + n/m | 3 + n/m | 4 - n/m | 3 - n/m | 2 - n/m | 1 - n/m |
| \( n/m \) | 2 - n/m | 1 - n/m | n/m | 1 + n/m | 2 + n/m | 3 + n/m | 4 - n/m |
| \( n/m \) | 1 + n/m | 2 + n/m | 3 + n/m | 4 - n/m | 3 - n/m | 2 - n/m | 1 - n/m |
| \( n/m \) | 1 + n/m | 2 + n/m | 3 + n/m | 4 - n/m | 3 - n/m | 2 - n/m | 1 - n/m |
| \( 3 - n/m \) | 2 - n/m | 1 - n/m | n/m | 1 + n/m | 2 + n/m | 3 + n/m | 4 - n/m |
| \( 3 - n/m \) | 2 - n/m | 1 - n/m | n/m | 1 + n/m | 2 + n/m | 3 + n/m | 4 - n/m |

### Table 2. Values of \( v \) in negative direction (\( v = \sum -v_i \))

| \( v_1 \) | \( v_2 \) | \( v_3 \) | \( v_4 \) | \( v_5 \) | \( v_6 \) | \( v_7 \) |
|---|---|---|---|---|---|---|
| \( n/m \) | 1 - n/m | 2 - n/m | 3 - n/m | 4 - n/m | 3 + n/m | 2 + n/m |
| \( n/m \) | 2 - n/m | 1 - n/m | n/m | 1 + n/m | 2 + n/m | 3 + n/m |
| \( 3 + n/m \) | 2 + n/m | 1 + n/m | n/m | 1 + n/m | 2 - n/m | 3 - n/m |
| \( 4 - n/m \) | 3 + n/m | 2 + n/m | 1 + n/m | n/m | 1 - n/m | 2 - n/m |
| \( n/m \) | 1 - n/m | 2 - n/m | 3 - n/m | 4 - n/m | 3 + n/m | 2 + n/m |
| \( n/m \) | 1 - n/m | 2 - n/m | 3 - n/m | 4 - n/m | 3 + n/m | 2 + n/m |
| \( 3 + n/m \) | 2 + n/m | 1 + n/m | n/m | 1 - n/m | 2 - n/m | 3 - n/m |
| \( 3 + n/m \) | 2 + n/m | 1 + n/m | n/m | 1 - n/m | 2 - n/m | 3 - n/m |
| \( 3 + n/m \) | 2 + n/m | 1 + n/m | n/m | 1 - n/m | 2 - n/m | 3 - n/m |
| \( 3 + n/m \) | 2 + n/m | 1 + n/m | n/m | 1 - n/m | 2 - n/m | 3 - n/m |
| \( 3 + n/m \) | 2 + n/m | 1 + n/m | n/m | 1 - n/m | 2 - n/m | 3 - n/m |

\([v]\) is a 128 × 8 matrix. Its anterior matrix elements are shown as Table 1 and 2, and have characteristics as follows:

1. The period of \( V(t) \):
   \[ T_v = T_u \tag{16} \]

2. \( V(t) \) only have 8 values: \( n/m, 1 + n/m, 2 + n/m, 3 + n/m, 4 - n/m, 3 - n/m, 2 - n/m \) and \( 1 - n/m \).

Envelope of \( V(t) \) is a periodic step waveform. Its change range of value are 4 - 2n/m, that is 3 - 4. But its
center value always is 2, that is \([(4-n/m)+n/m]=2\), thus maintenance of precision of crossover point is guaranteed for phase demodulation.

\[
V(x \pm (N+n/m)w, t) = V(x \pm w \cdot n/m, t \mp N \cdot T/8)
\]  

(17)

Where \(\theta_i=2\pi x/W\), \(k\) is determined by \(n/m\), and is between 1.5 and 2. It is indicates that this is a phase-modulated transducer.

5. Matrix Representation and characteristics of \(D_k(t)\)

We obtain demodulation function \(D_k(t) (k=1-4)\) by sampling once every \(4\tau\) from \(V(t)\). The waveforms of \(D_k(t)\) are shown as Figure 3 below. Its envelope is a periodic step waveform. Its matrix elements are taken by every 4 values from Table 1 and 2. Thus demodulation could be expressed by matrix. Define demodulation matrix \(B\) is a square matrix \(B=[b_{ij}].\) When \(i=j=4N+k\ (N=0,1,2,\ldots,31), \ b_{ij}=1,\) otherwise \(b_{ij}=0.\) We have:

\[
[b_{ij}]v_j = [D_j] 
\]

(18)

\[
[b_{ij}]u_j = [D_j] 
\]

(19)

\(D_k(t)\) have following characteristics

1. \(D_k(t)\) and \(V(t)\) have same period, values and center value. The phase shift characteristics of fundamental harmonic could be expressed as follows:

\[
D_k(t) = k \sin(\omega t + \theta_s + \varphi_k)
\]  

(20)

\(\varphi_k\) have fixed value (for example, \(\varphi_k=0\), \(7\tau\), \(57\tau\), \(62\tau\)). They are transferred through (14) from fixed phase difference of function constructed by \(u_i(t)\).

2. There are always 4 equal pulses which are continuously joined in \(D_k(t)\). The pulse envelope is a periodic step waveform.

6. Matrix Representation and characteristics of \(S_k(t)\)

We obtain shape character function \(S_k(t) (k=1-4)\) by sampling once every \(16\tau\) from \(V(t)\). Thus sampling could be expressed by matrix. Define sampling matrix \(F_j\) is a square matrix \(F=[F_{ij}].\) When \(i=j=16N+4k\ (N=1,2,\ldots,T), \ F_{ij}=1,\) otherwise \(F_{ij}=0.\) We have:

\[
[F_{ij}]v_j = [S_i] 
\]

(21)

\[
[F_{ij}]u_j = [S_i] 
\]

(22)

\(S_k(t)\) have following characteristics

1. \(S_k(t)\) and \(D_k(t)\) have same period, values, center value and phase shift characteristic.

2. Its pulse envelope waveform is period polygonal line as shown in Fig. 4. There are some special examples. If \(n/m\), it is triangle waveform. And if \(n/m=1/2\), it is trapezoid waveform. \(S_k(t)\) express that the waveforms are different for different \(x=(n/m)w.\) And there ais no end of waveforms.

Figure 4. Waveform of envelope of pulses of \(S_k(t)\).
(3) We consider that when \( n/m \) are different, how waveforms and values are changed? We know that its absolute value of slope of polygonal line near center value 2 is not changed. It is advantageous to increase measurement precision by using this part of polygonal line to detect crossover point.

7. Cause Produced Period Measurement Error
There is period error in the transducer as shown in Figure 5. The cause is that we generally use fundamental harmonic of \( D_k(t) \) as demodulation signal, when output signals of the transducer are processed. We know that shapes and values of \( D_k(t) \) curve are different at different \( n/m \) of \( w \), thus its maximums of fundamental harmonic are different, and so its slope \( k \) of fundamental harmonic at center value also is different. We know \( \Delta x = \Delta v/k \) from Figure 5. As there are eight \( w \) in \( W \), thus \( k \) has 8 times period change. And so small period changes of eight times of \( x \) are produced. But bigger period changes of \( x \) are produced by \( \Delta W \).

8. Meaning of Two-Dimensional \( C(x,t) \) and \( V(x,t) \)
\( C(x,t) \) press capacitive coupling function of different time and position, and could be expressed by a matrix:

\[
C(x,t) = \begin{bmatrix} c_y \end{bmatrix}
\]

Where \( i=1\sim8 \) a space subscript; \( j=1\sim N \) subscript; \( N \) is a positive integer. \( V(x,t) \) also express output function of different time and position, and could be expressed by a matrix:

\[
V(x,t) = \begin{bmatrix} v \end{bmatrix}
\]

Where a time subscript; \( j=1\sim N \), \( j \) is a space subscript; \( N \) is a positive integer. Thus the most general expression expressed by matrix in two-dimensional space-time for the transducer is

\[
\downarrow_t \begin{bmatrix} u_y \end{bmatrix} \begin{bmatrix} c_y \end{bmatrix} \uparrow^x = \downarrow_t \begin{bmatrix} v_y \end{bmatrix} \]

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