IDENTITY TESTING FROM HIGH POWERS OF POLYNOMIALS OF LARGE DEGREE OVER FINITE FIELDS

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ABSTRACT. We consider the problem of identity testing of two “hidden” monic polynomials $f$ and $g$, given an oracle access to $f(x)^e$ and $g(x)^e$ for $x \in \mathbb{F}_q$, where $\mathbb{F}_q$ is the finite field of $q$ elements (an extension fields access is not permitted).

The naive interpolation algorithm needs $de + 1$ queries, where $d = \max\{\deg f, \deg g\}$ and thus requires $de < q$. For a prime $q = p$, we design an algorithm that is asymptotically better in certain cases, especially when $d$ is large. The algorithm is based on a result of independent interest in spirit of additive combinatorics. It gives an upper bound on the number of values of a rational function of large degree, evaluated on a short sequence of consecutive integers, that belong to a small subgroup of $\mathbb{F}_p^*$.

1. Introduction

1.1. Background and previous results. The following variant of polynomial identity problem has been motivated by some cryptographic applications, see [1, 11] for further discussion and the references.

Let $\mathbb{F}_q$ be the finite field of $q$ elements of characteristic $p$. We consider the Identity Testing from Powers for two “hidden” monic polynomials $f, g \in \mathbb{F}_q[X]$:

- given oracles $\mathcal{O}_{e,f}$ and $\mathcal{O}_{e,g}$ that on every input $x \in \mathbb{F}_q$ output $\mathcal{O}_{e,f}(x) = f(x)^e$ and $\mathcal{O}_{e,g}(x) = g(x)^e$ for some large positive integer $e | q - 1$, decide whether $f = g$.

In particular, for a linear polynomial $f(X) = X + s$, with a ‘hidden’ $s \in \mathbb{F}_q$, we denote $\mathcal{O}_{e,s} = \mathcal{O}_{e,f}$. We remark that in this case there are two naive algorithms that work for linear polynomials:

- One can query $\mathcal{O}_{e,s}$ at $e + 1$ arbitrary points and then using a fast interpolation algorithm, see [9], get a deterministic algorithm of complexity $e(\log q)^{O(1)}$ (as in [9], we measure the

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complexity of an algorithm by the number of bit operations in
the standard RAM model of computation).

• For probabilistic testing one can query $O_{e,s}$ (and $O_{e,t}$) at randomly chosen elements $x \in \mathbb{F}_q$ until the desired level of confidence is achieved (note that the equation $(x + s)^e = (x + t)^e$ has at most $e$ solutions $x \in \mathbb{F}_q$).

These naive algorithms have been improved by Bourgain, Garaev, Konyagin and Shparlinski [1] in several cases (with respect to both the time complexity and the number of queries).

Furthermore, for linear polynomials $f(X) = X + s$, Dam, Hallgren and Ip [8] provide a quantum polynomial time algorithm to find $s$, see also [7], in the case of the oracle oracles $O_{e,f}$, with $e = (q - 1)/2$ (and odd $q$). We remark that querying this oracle is equivalent to asking for the quadratic character of the computed value. Thus the oracle returns only one bit of information, making this the hardest case.

Russell and Shparlinski [14] have initiated the study of this question for non-linear monic polynomials and, in the case of the oracle $O_{(p-1)/2,f}$, for a prime $q = p$, have designed several classical and quantum algorithms. More recently, several other algorithms, for an arbitrary $e$, have been given in [11]. The algorithms from [11] usually improve on the above trivial interpolation and random sampling algorithms. However in the settings of [11] the degree of the polynomials is assumed to be small.

Here we concentrate on the case of polynomials large degree and fields of large characteristic $p$, in particular, on the case of prime fields $\mathbb{F}_p$, and use a different approach to obtain new results in this case.

We also observe that if

$$e \max\{\deg f, \deg g\} > q$$

then the above naive interpolation and random sampling algorithms both fail. Indeed, since queries from an extension field are not permitted, and $\mathbb{F}_q$ may not have enough elements to make these algorithms work. This indicates that the difficulty of the problem grows with the degrees of polynomials involved.

Our approach is based on a new upper bound on the size of an intersection of value set on consecutive integers of a rational function of large degree with a small subgroup of a finite field. Results of this type, complementing this of [10,11,16] are of independent interest, see also [1–5,13,15] for further results on related problems.
2. Main result

Here we consider the identity testing case of two unknown monic polynomials $f, g \in \mathbb{F}_q[X]$ of degree $d$ given the oracles $O_{e,f}$ and $O_{e,g}$. We remark that if $f/g$ is an $(q-1)/e$-th power of a rational function over $\mathbb{F}_q$ then it is impossible to distinguish between $f$ and $g$ from the oracles $O_{e,f}$ and $O_{e,g}$.

We however impose a slightly stronger condition that $f/g$ is not a perfect power of a rational function.

**Definition 2.1.** We see say that a rational function $\psi \in \mathbb{F}_p(X)$ is a nontrivial perfect power if $\psi = \varphi^k$ for some rational function $\varphi \in \mathbb{F}_p(X)$ of positive degree and some positive integer $k$.

**Theorem 2.2.** There are absolute constants $c_1, c_2 > 0$ such that for a prime $p$ and a positive integer $e \mid p-1$, given two oracles $O_{e,f}$ and $O_{e,g}$ for some unknown monic polynomials $f, g \in \mathbb{F}_p[X]$ of degree $d$ such that

\begin{equation}
e \leq c_1 \min \{pd^{-3/2}, p^{3/2}d^{-7/2}\}
\end{equation}

such that $f/g$ is a nontrivial perfect power, there is a deterministic algorithm to decide whether $f = g$ in at most $c_2 \max \{d^3e^2p^{-1}, d^{7/3}e^{2/3}\}$ queries to the oracles $O_{e,f}$ and $O_{e,g}$.

3. Points with coordinates from subgroups on plane curves over $\mathbb{F}_p$

We recall that the notations $U = O(V)$, $U \ll V$ and $V \gg U$ are all equivalent to the statement that the inequality $|U| \leq cV$ holds with some constant $c > 0$. Throughout the paper, any implied constants in these symbols are absolute, in particular, all estimates are uniform with respect to the degree $d$, the exponent $e$ and the filed characteristic $p$.

Our argument relies on a result of Corvaja and Zannier [6, Corollary 2].

We also write $\deg_X F(X,Y)$ and $\deg_Y F(X,Y)$ for the degree of $F(X,Y) \in \mathbb{F}_p[X,Y]$ in $X$ and $Y$ respectively, and reserve $\deg F(X,Y)$ for the total degree.

We say that $F(X,Y) \in \mathbb{F}_p[X,Y]$ is a torsion polynomial it is of the form

\begin{equation}
\alpha U^m V^n + \beta \quad \text{or} \quad \alpha U^m + \beta V^n.
\end{equation}

We remark that as we work over the algebraically closed field $\overline{\mathbb{F}}_p$, the notions of irreducibility and absolute irreducibility coincide.
Lemma 3.1. Assume that $F(X,Y) \in \mathbb{F}_p[X,Y]$ is an irreducible polynomial with $\deg F = d$ which is not a torsion polynomial. For any multiplicative subgroups $G, H \subseteq \mathbb{F}_p^*$, we have

$$\# \left\{ (u,v) \in G \times H : F(u,v) = 0 \right\} \ll \max \left\{ \frac{d^2 W}{p}, d^{4/3} W^{1/3} \right\},$$

where

$$W = \#G \#H.$$

Proof. To apply [6, Corollary 2] we need to estimate the Euler characteristic $\chi$ of the curve $F(X,Y) = 0$ in terms of $d$. For the genus $g$ we have the well-known estimate $g \ll d^2$. The set $S$ from [6, Theorem 2] corresponding to the scenario of [6, Corollary 2] is the set of poles and zeros of coordinate functions $X$ and $Y$ and thus $\#S \ll d$. The result now follows.

We now need a version of Lemma 3.1 which applies to any curve.

Lemma 3.2. Assume that $F(X,Y) \in \mathbb{F}_p[X,Y]$ is of degree $\deg F = d$ and is not divisible by a torsion polynomial. For any multiplicative subgroups $G, H \subseteq \mathbb{F}_p^*$, we have

$$\# \left\{ (u,v) \in G \times H : F(u,v) = 0 \right\} \ll \max \left\{ \frac{d^2 W}{p}, d^{4/3} W^{1/3} \right\},$$

where

$$W = \#G \#H.$$

Proof. If $F$ is an irreducible polynomial then the bound is immediate from Lemma 3.1. Otherwise we factor $F$ into irreducible (over $\mathbb{F}_p$) components of degrees, say, $d_1, \ldots, d_s$ and then we obtain

$$\# \left\{ (u,v) \in G \times H : F(u,v) = 0 \right\} \ll \max \left\{ \frac{A W}{p}, B W^{1/3} \right\},$$

where, by the convexity argument,

$$A = \sum_{i=1}^s d_i^2 \leq d^2 \quad \text{and} \quad B = \sum_{i=1}^s d_i^{4/3} \leq d^{4/3},$$

which concludes the proof.

4. Non-vanishing of some resultants

We recall the following well-known statement, see for example [12, Lemma 6.54] (the proof extends from polynomials to rational functions without any changes).
Lemma 4.1. Let $f(X), g(X) \in \mathbb{F}_p[X]$ be such that $f/g$ is not a perfect power of a rational function. Then for any integer $m \geq 1$ the polynomial $f(X) - Y^m g(X)$ is irreducible.

Lemma 4.2. Let $f(X), g(X) \in \mathbb{F}_p[X]$ be polynomials of degrees at most $d$ and be such that $f/g$ is not a perfect power of a rational function. Then for any integers $m, n$ the system of equations

$$f(X) - Y^m g(X) = f(X + a) - b Y^n g(X + a) = 0,$$

with $a, b \in \mathbb{F}_p^*$ defines a zero dimensional variety, unless $m = \pm n$, and pairs $(a, b)$ are from a set of cardinality at most 4.

Proof. If $mn = 0$ the result is trivial

Changing the roles of $f$ and $g$, we can always assume that $m > 0$.

Now, let $n > 0$. Since by Lemma 4.1 both polynomials are irreducible, we may have a common factor if and only if that are equal up to a factor from $\mathbb{F}_p^*$ and thus $m = n$. Furthermore comparing the coefficient at the front of $Y$ we conclude that

$$b \cdot (f(X) - Y^m g(X)) = f(X + a) - b Y^n g(X + a).$$

Hence $m = n$. Now, comparing the parts which do not depend on $Y$ we obtain $bf(X) = f(X + a)$ hence $b = 1$ and then $a = 0$.

We now consider the case $n < 0$ and rewrite the equations as

$$f(X) - Y^m g(X) = Y^{-n} f(X + a) - b g(X + a) = 0.$$ 

Again, by Lemma 4.1, the polynomials involved are irreducible again, hence $m = -n$. Comparing the parts which do not depend on $Y$ we see that $b$ is uniquely defined and then $a$ is uniquely defined as well.

\[\square\]

Lemma 4.3. Let $f(X), g(X) \in \mathbb{F}_p[X]$ be polynomials of degrees at most $d$ and be such that $f/g$ is not a perfect power of a rational function. Then there is a set $\mathcal{E} \subseteq \mathbb{F}_p^*$ of cardinality $|\mathcal{E}| = O(d^2)$ such that for $a \in \mathbb{F}_p^* \setminus \mathcal{E}$ the resultant

$$R_a(U, V) = \text{Res}_X \left( (f(X) - U g(X), f(X + a) - V g(X + a)) \right)$$

with respect to $X$, is not divisible by a torsion polynomial.

Proof. Assuming that

$$R_a(U, V) = \text{Res}_X \left( (f(X) - U g(X), f(X + a) - V g(X + a)) \right)$$

is divisible by a polynomial of the form (3.1), we easily derive that there is a variety of the type considered in Lemma 4.2 which is of positive dimension. Since $m \leq \deg_U R_a = O(d)$ and $n \leq \deg_V R_a = O(d)$, the result follows.

\[\square\]
5. Intersection of polynomial images of intervals and subgroups

For a rational function \( \psi(X) = f(X)/g(X) \in \mathbb{F}_p(X) \) with two relatively primes polynomials \( f, g \in \mathbb{F}_p[X] \) and a set \( S \subseteq \mathbb{F}_p \), we use \( \psi(S) \) to denote the value set
\[
\psi(S) = \{ \psi(x) : x \in S, \ g(x) \neq 0 \} \subseteq \mathbb{F}_p.
\]
Given an interval \( \mathcal{I} = [1, H] \) with a positive \( H < p \) and a subgroup \( G \in \mathbb{F}_p \) we consider the size of the intersection of \( \psi(\mathcal{I}) \) and \( G \), that is,
\[
N_\psi(\mathcal{I}, G) = \# (\psi(\mathcal{I}) \cap G).
\]
Bounds on this quantity for various functions \( \psi \) is in the background of the algorithms of [1, 11]. These bounds are also of independent interest as they are natural analogues of the problem of bounding
\[
N_\psi(\mathcal{I}, \mathcal{J}) = \# (\psi(\mathcal{I}) \cap \mathcal{J})
\]
for two intervals \( \mathcal{I} \) and \( \mathcal{J} \) and similar sets, which has recently been actively investigated, see [2–5, 10, 13, 15, 16] and references therein.

**Lemma 5.1.** Let \( \psi(X) = f(X)/g(X) \in \mathbb{F}_p(X) \) with two relatively primes polynomials \( f, g \in \mathbb{F}_p[X] \) of degrees at most \( d \) and such that \( \psi \) is not a perfect power of a rational function. Then for any interval \( \mathcal{I} = [1, H] \) of length \( H < p \) and any subgroup \( G \in \mathbb{F}_p \) of order \( e \), we have
\[
N_\psi(\mathcal{I}, G) \ll H^{1/2} \max \{ d^{3/2} ep^{-1/2}, d^{7/6} e^{1/3} \}.
\]

**Proof.** Denote \( M = N_\psi(\mathcal{I}, G) \). Let \( \overline{\mathcal{I}} = [-H, H] \). Clearly the system of equations (over \( \mathbb{F}_p \)):
\[
f(x) = u \quad \text{and} \quad f(x + y) = v, \quad x \in \mathcal{I}, \ y \in \overline{\mathcal{I}}, \ u, v \in G,
\]
has at least \( M^2 \) solutions. Let \( M_y \) be the number of solutions with a fixed \( y \). The
\[
\sum_{y \in \overline{\mathcal{I}}} M_y \geq M^2.
\]
We choose
\[
L = \frac{M^2}{2(2H + 1)}
\]
and consider the set \( \mathcal{Y} \) of \( y \in \overline{\mathcal{I}} \) with \( M_y > L \). Using that \( M_y \leq H \) we write
\[
H \# \mathcal{Y} \geq \sum_{y \in \overline{\mathcal{I}}} M_y \geq M^2 - \sum_{y \in \overline{\mathcal{I}}} M_y \geq M^2 - (2H + 1)L \geq \frac{1}{2} M^2.
\]
Now if \( \#Y \leq \#E \), where \( E \) is as in Lemma 4.3, then \( M^2 \leq 2H \#E \ll d^2 H \) and thus \( M \ll dH^{1/2} \), which is stronger than the desired bound.

Hence we can assume that \( \#Y > \#E \) and thus there is \( Y \setminus E \neq \emptyset \).

We now fix any \( a \in Y \setminus E \) and consider the system of equations

\[
 f(x) = u \quad \text{and} \quad f(x + a) = v, \quad x \in \mathcal{I}, \ u, v \in \mathcal{G},
\]

Using the resultant to eliminate \( x \) we obtain \( R_a(u,v) = 0 \) for each solution, where \( R_a(U,V) \) is as in Lemma 4.3. Due to choice of \( a \), we see that the bound of Lemma 3.2 applies, and since for every fixed \( u \) there are at most \( d \) values of \( x \) we obtain

\[
 L \leq M_a \leq d \# \{(u,v) \in \mathcal{G} \times \mathcal{G} : R_a(u,v) = 0\}
\]

\[
 \ll d \max \left\{ \frac{d^2 e^2}{p}, d^{1/3} e^{2/3} \right\}.
\]

Recalling the definition of \( L \) we obtain

\[
 M^2 \ll H \max \left\{ d^3 e^2 p^{-1}, d^{7/3} e^{2/3} \right\}
\]

and the result follows.

\[ \square \]

6. Proof of Theorem 2.2

First we note that there are absolute constants \( c_1, c_2 > 0 \) such that if the condition (2.1) is satisfied then for

\[
 H = \lfloor c_1 \max \{ d^3 e^2 p^{-1}, d^{7/3} e^{2/3} \} \rfloor
\]

we have \( H < p \) and then under the conditions of Lemma 5.1 we have

\[
 N_{\psi}(\mathcal{I}, \mathcal{G}) < H,
\]

for any rational function \( \psi \in \overline{\mathbb{F}}_p(X) \) of degree at most \( d \), which is a nontrivial perfect power.

Now, since \( f/g \) is a nontrivial perfect power Lemma 5.1 applies to \( \psi = f/g \). Hence taking \( H \) as in (6.1), so the inequality (6.2) is satisfied, we see that querying \( \mathcal{O}_{e,f} \) and \( \mathcal{O}_{e,g} \) for \( x = 1, \ldots, H \), we have \( \mathcal{O}_{e,f}(x) \neq \mathcal{O}_{e,g}(x) \) unless \( f = g \).

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