Linear fractional stable motion: a wavelet estimator of the $\alpha$ parameter

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Abstract

Linear fractional stable motion, denoted by $\{X_{H,\alpha}(t)\}_{t\in\mathbb{R}}$, is one of the most classical stable processes; it depends on two parameters $H \in (0,1)$ and $\alpha \in (0,2)$. The parameter $H$ characterizes the self-similarity property of $\{X_{H,\alpha}(t)\}_{t\in\mathbb{R}}$ while the parameter $\alpha$ governs the tail heaviness of its finite dimensional distributions; throughout our article we assume that the latter distributions are symmetric, that $H > 1/\alpha$ and that $H$ is known. We show that, on the interval $[0,1]$, the asymptotic behaviour of the maximum, at a given scale $j$, of absolute values of the wavelet coefficients of $\{X_{H,\alpha}(t)\}_{t\in\mathbb{R}}$, is of the same order as $2^{-j(H-1/\alpha)}$; then we derive from this result a strongly consistent (i.e. almost surely convergent) statistical estimator for the parameter $\alpha$.

Key words: stable stochastic processes; statistical inference; wavelet coefficients; Hölder regularity.

1 Introduction and statement of the main results

Let $H$ and $\alpha$ be two parameters such that $\alpha \in (1,2)$ and $1/\alpha < H < 1$. We denote by $\{X_{H,\alpha}(t)\}_{t\in\mathbb{R}}$ the symmetric $\alpha$ stable linear fractional stable motion (lfsm for brevity) (see e.g. [5, 3]), defined, for all $t \in \mathbb{R}$, as,

$$X_{H,\alpha}(t) := \int_{\mathbb{R}} \left\{ (t-s)^{H-1/\alpha} - (-s)^{H-1/\alpha} \right\} Z_\alpha(ds),$$

where $Z_\alpha(\cdot)$ is a symmetric $\alpha$-stable random measure and, for each $z \in \mathbb{R}$

$$(z)_+ := \max\{z,0\}.$$
The parameter $H$ characterizes the self-similarity property of lfsm; namely, for all fixed positive real-number $a$, the processes \( \{X(at)\}_{t \in \mathbb{R}} \) and \( \{a^H X(t)\}_{t \in \mathbb{R}} \) have the same finite dimensional distributions. The parameter $\alpha$ governs the tail heaviness of the latter distributions. The process \( \{X_{H,\alpha}(t)\}_{t \in \mathbb{R}} \) has a modification with continuous nowhere differentiable sample paths; it is identified with this modification in all the sequel.

The statistical problem of the estimation of $H$ has already been studied in several articles: [6, 1, 7, 4], and strongly consistent estimators (i.e. convergent almost surely), based on \( (d_{j,k}^{(j,k)})_{(j,k) \in \mathbb{Z}^2} \), the discrete wavelet transform of lfsm, have been proposed; notice that the latter estimators of $H$ do not require that $\alpha$ to be known. Throughout our paper, for all \( (j,k) \in \mathbb{Z}^2 \), the wavelet coefficient $d_{j,k}$ is defined as,

\[
d_{j,k} = 2^j \int_{\mathbb{R}} X_{H,\alpha}(t) \psi(2^j t - k) dt;
\]

moreover, we only impose to the analyzing wavelet $\psi$ a very weak assumption: $\psi$ is an arbitrary real-valued non-vanishing continuous function with a compact support in $[0,1]$ and it has 2 vanishing moments i.e.

\[
\int_{\mathbb{R}} \psi(s) ds = \int_{\mathbb{R}} s \psi(s) ds = 0.
\]

It is worth noticing that we do not need that \( \{2^{j/2} \psi(2^j \cdot -k) : (j,k) \in \mathbb{Z}^2\} \) be an orthonormal wavelet basis for $L^2(\mathbb{R})$.

In view of the fact that the problem of the estimation of $H$ is now well understood, from now on we assume the latter parameter to be known. Our goal is to construct, by using the wavelet coefficients \( (d_{j,k})_{0 \leq k < 2^j} \), a strongly consistent (i.e. almost surely convergent when $j \to +\infty$) estimator $\hat{\alpha}_j$ of the parameter $\alpha$. Let us outline the main ideas which lead to this estimator.

- The starting point, is a result of [8], according to which, with probability 1, the quantity $H - 1/\alpha$, is the critical uniform Hölder exponent of the sample path $X_{H,\alpha}$ over any arbitrary compact interval and in particular the interval $[0,1]$; more precisely, one has, almost surely for all arbitrarily small $\eta > 0$,

\[
\sup_{t_1, t_2 \in [0,1]} \left\{ \frac{|X_{H,\alpha}(t_1) - X_{H,\alpha}(t_2)|}{|t_1 - t_2|^{H-1/\alpha-\eta}} \right\} < \infty
\]

and

\[
\sup_{t_1, t_2 \in [0,1]} \left\{ \frac{|X_{H,\alpha}(t_1) - X_{H,\alpha}(t_2)|}{|t_1 - t_2|^{H-1/\alpha+\eta}} \right\} = \infty.
\]

- Next, let us set,

\[
D_j = \max_{0 \leq k < 2^j} |d_{j,k}|.
\]

In view of the fact that the wavelet $\psi$ has a first vanishing moment, one can derive from (1.5), that, almost surely, for all arbitrarily small $\epsilon > 0$,

\[
\limsup_{j \to +\infty} \left\{ 2^{j(H-1/\alpha-\epsilon)} D_j \right\} < \infty.
\]
• Notice that, since we do not impose to \( \psi \) to be a continuously differentiable function and to 
\( \{2^{j/2}\psi(2^j \cdot -k) : (j,k) \in \mathbb{Z}^2\} \) to form an orthonormal wavelet basis for \( L^2(\mathbb{R}) \), a priori it is not 
at all clear that (1.6), implies that, almost surely for all arbitrarily small \( \epsilon > 0 \),
\[
\limsup_{j \to +\infty} \left\{ 2^{j(H - 1/\alpha + \epsilon)} D_j \right\} = \infty. \tag{1.9}
\]
Yet, by making use of some specific properties of lsFm as well as the fact that \( \psi \) is compactly 
supported, we will be able to show that, a result stronger than (1.9) holds; namely, one has 
almost surely, for all arbitrarily small \( \epsilon > 0 \),
\[
\liminf_{j \to +\infty} \left\{ 2^{j(H - 1/\alpha + \epsilon)} D_j \right\} = \infty. \tag{1.10}
\]
• Finally, combining (1.8) with (1.10), one can get the following theorem, which is our main 
result.

**Theorem 1.1.** For each \( j \in \mathbb{N} \), one set,
\[
\frac{1}{\hat{\alpha}_j} = H + \frac{\log(D_j)}{j \log(2)},
\]
where \( D_j \) is defined in (1.7). Then, one has almost surely,
\[
\hat{\alpha}_j \xrightarrow{a.s.} \alpha. \quad \text{as} \quad j \to +\infty.
\]

## 2 Proofs

### 2.1 Proof of Relation (1.8)

The proof is standard in the wavelet setting, we give it for the sake of completeness. Let \( \hat{\Omega} \) be an 
event of probability 1 on which Relation (1.5) holds and let \( \omega \in \hat{\Omega} \) be arbitrary and fixed. Assume 
that \( \epsilon > 0 \) is arbitrary and fixed and denote by \( C(\omega) \) the finite quantity defined as,
\[
C(\omega) := \sup_{t_1,t_2 \in [0,1]} \left\{ \frac{|X_{H,\alpha}(t_1,\omega) - X_{H,\alpha}(t_2,\omega)|}{|t_1 - t_2|^{H - 1/\alpha - \epsilon}} \right\}. \tag{2.1}
\]

On the other hand, notice that (1.3), (1.4) and the fact that,
\[
\text{supp} \psi \subseteq [0,1], \tag{2.2}
\]
imply that, for all \((j,k) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \) satisfying \( 0 \leq k < 2^j \), one has,
\[
d_{j,k}(\omega) = 2^j \int_{k2^{-j}}^{(k+1)2^{-j}} \left\{ X_{H,\alpha}(t,\omega) - X_{H,\alpha}(k2^{-j},\omega) \right\} \psi(2^j t - k) \, dt. \tag{2.3}
\]
Next, combining (2.3) with (2.1), one gets
\[ |d_{j,k}(\omega)| \leq 2^j \int_{k2^{-j}}^{(k+1)2^{-j}} \left| X_{H,\alpha}(t, \omega) - X_{H,\alpha}(k2^{-j}, \omega) \right| |\psi(2^j t - k)| dt \]
\[ \leq \|\psi\|_{L^\infty(\mathbb{R})} C(\omega) 2^j \int_{k2^{-j}}^{(k+1)2^{-j}} |t - k2^{-j}|^{H-1/\alpha-\epsilon} dt \]
\[ \leq \|\psi\|_{L^\infty(\mathbb{R})} C(\omega) 2^{-j(H-1/\alpha-\epsilon)}, \]
which proves that (1.8) is satisfied. □

2.2 Proof of Relation (1.10)

Let us first recall that in [2], a nice stochastic integral representation of the wavelet coefficients \( d_{j,k} \) has been obtained, namely one has almost surely that
\[ d_{j,k} = 2^{-j(H-1/\alpha)} \int_{\mathbb{R}} \Phi_{H,\alpha}(2^j s - k) Z_{\alpha}(ds), \tag{2.4} \]
where \( \Phi_{H,\alpha} \) is the real-valued continuous function defined for each \( x \in \mathbb{R} \), as,
\[ \Phi_{H,\alpha}(x) = \int_{\mathbb{R}} (y - x)^{H-1/\alpha} \psi(y) dy = \int_{0}^{1} (y - x)^{H-1/\alpha} \psi(y) dy; \tag{2.5} \]
notice that the last equality results from (2.2).

**Proposition 2.1.** The function \( \Phi_{H,\alpha} \) satisfies the following two nice properties:
(i) one has,
\[ \text{supp} \Phi_{H,\alpha} \subseteq (-\infty, 1]; \tag{2.6} \]
(ii) there is a constant \( c_1 > 0 \) such for all \( x \in (-\infty, 1] \),
\[ |\Phi_{H,\alpha}(x)| \leq c_1 (1 + |x|)^{-(2+1/\alpha-H)}. \tag{2.7} \]

**Proof of Proposition 2.1.** Part (i) is a straightforward consequence of (2.5) and (1.2). Let us show that Part (ii) holds. First observe that, (2.5) easily implies that,
\[ \sup_{x \in [-1,1]} \left\{ (1 + |x|)^{2+1/\alpha-H} |\Phi_{H,\alpha}(x)| \right\} \leq 4 \|\psi\|_{L^\infty(\mathbb{R})} < \infty. \tag{2.8} \]

Let us now suppose that \( x < -1 \). We denote by \( \psi^{(-1)} \) the primitive of \( \psi \), defined for all \( z \in \mathbb{R} \), as
\[ \psi^{(-1)}(z) = \int_{-\infty}^{z} \psi(y) dy. \]
Observe that (2.2) and (1.4) entail that the continuous function \( \psi^{(-1)} \) has a compact support included in \([0,1]\). We denote by \( \psi^{(-2)} \) the primitive of \( \psi^{(-1)} \), defined for all \( z \in \mathbb{R} \), as
\[ \psi^{(-2)}(z) = \int_{-\infty}^{z} \psi^{(-1)}(y) dy. \]
Observe that $\text{supp } \psi(-1) \subseteq [0, 1]$ and (1.3) entail that the continuous function $\psi(-2)$ has a compact support included in $[0, 1]$; therefore integrating two times by parts in (2.8), we obtain

$$\Phi_{H,\alpha}(x) = (H - 1/\alpha)(H - 1/\alpha - 1) \int_0^1 (y - x)^{H-1/\alpha-2}\psi(-2)(y)dy. \quad (2.9)$$

Next, using (2.9) and the inequalities: for all $y \in [0, 1]$, $y - x \geq |x| \geq 2^{-1}(1 + |x|)$, it follows that, 

$$|\Phi_{H,\alpha}(x)| \leq 2^{2+1/\alpha-H}\|\psi(-2)\|_{L^\infty(\mathbb{R})}(1 + |x|)^{H-1/\alpha-2}. \quad (2.10)$$

Finally, combining (2.8) with (2.10), we get Part (ii) of the proposition.

A straightforward consequence of (2.4) and Part (i) of Proposition 2.1, is that,

$$d_{j,k} = 2^{-j(H-1/\alpha)} \int_{-\infty}^{(k+1)2^{-j}} \Phi_{H,\alpha}(2^j s - k)Z_\alpha \left( \right) ds. \quad (2.11)$$

Let us now introduce some additional notations. We assume that $\delta \in (0, 1/3)$ is arbitrary and fixed. For all $j \in \mathbb{Z}_+$, we define the positive integer $e_j$ as,

$$e_j := \lfloor 2^{j\delta} \rfloor, \quad (2.12)$$

where $\lfloor \cdot \rfloor$ is the integer part function. Then, for any integer $l$ such that

$$0 \leq l \leq \lfloor 2^{j(1-\delta)} \rfloor - 1, \quad (2.13)$$

we set

$$G_{j,le_j} := \int_{(l-1)e_j+1}^{(k+1)2^{-j}} \Phi_{H,\alpha}(2^j s - le_j)Z_\alpha \left( \right) ds, \quad (2.14)$$

and

$$R_{j,le_j} := \int_{-\infty}^{(l-1)e_j+1} \Phi_{H,\alpha}(2^j s - le_j)Z_\alpha \left( \right) ds. \quad (2.15)$$

Thus, in view of (2.11), the wavelet coefficient $d_{j,le_j}$ can be expressed as,

$$d_{j,le_j} = 2^{-j(H-1/\alpha)} \left( G_{j,le_j} + R_{j,le_j} \right). \quad (2.16)$$

Now, our goal will be to derive the following two lemmas which respectively provide lower and upper asymptotic estimates for $\max_{0 \leq l < \lfloor 2^{j(1-\delta)} \rfloor} |G_{j,le_j}|$ and $\max_{0 \leq l < \lfloor 2^{j(1-\delta)} \rfloor} |R_{j,le_j}|$.

**Lemma 2.1.** One has, almost surely

$$\lim\inf_{j \to +\infty} \left\{ 2^{2\delta} \max_{0 \leq l < \lfloor 2^{j(1-\delta)} \rfloor} |G_{j,le_j}| \right\} \geq 1. \quad (2.17)$$

**Lemma 2.2.** One has, almost surely

$$\lim\sup_{j \to +\infty} \left\{ 2^{2\delta} \max_{0 \leq l < \lfloor 2^{j(1-\delta)} \rfloor} |R_{j,le_j}| \right\} = 0. \quad (2.18)$$
The proof of Lemma 2.3 mainly relies on the following two results.

**Lemma 2.3.** (see e.g. [2]) Let $Y$ be an arbitrary symmetric $\alpha$-stable random variable with a non-vanishing scale parameter $\|Y\|_{\alpha}$, then for any real number $t \geq \|Y\|_{\alpha}$, one has,

$$c_3 \|Y\|_{\alpha}^\alpha t^{-\alpha} \leq P(|Y| > t) \leq c_2 \|Y\|_{\alpha}^\alpha t^{-\alpha};$$

(2.19)

where $c_2$ and $c_3$ are two positive constants only depending on $\alpha$.

**Lemma 2.4.** For each fixed $j \in \mathbb{Z}_+$, $\{G_{j,le_j} : 0 \leq l \leq [2^j(1-\delta)] - 1\}$ is a sequence of identically distributed independent symmetric $\alpha$-stable random variables whose scale parameters, denoted $\|G_{j,le_j}\|_{\alpha}$, satisfy for all $l$,

$$\|G_{j,le_j}\|_{\alpha}^\alpha = 2^{-j} \int_{-e_j}^{e_j} |\Phi_{H,\alpha}(x)|^\alpha dx. \quad (2.20)$$

**Proof of Lemma 2.4.** The independence of these symmetric $\alpha$-stable random variables is straightforward consequence of the fact that they are defined (see (2.14)) through stable stochastic integrals over disjoint intervals. In order to show that they are identically distributed it is sufficient to prove that (2.20) holds for each $l$. Using a standard property of stable stochastic integrals (see e.g. [3]) and (2.14), one gets

$$\|G_{j,le_j}\|_{\alpha}^\alpha = \int_{(l-1)e_j+1}^{(le_j+1)2^{-j}} |\Phi_{H,\alpha}(2^j s - le_j)|^\alpha ds;$$

then the change of variable $u = 2^j s - le_j$ allows to obtain (2.20). $\square$

Now, we are in position to prove Lemma 2.1.

**Proof of Lemma 2.1.** Let $j \in \mathbb{Z}_+$ be arbitrary and fixed. Using the fact that $\{G_{j,le_j} : 0 \leq l \leq [2^j(1-\delta)] - 1\}$ is a sequence of independent identically distributed random variables (see Lemma 2.4), one gets,

$$\mathbb{P}\left(\max_{0 \leq l < [2^j(1-\delta)]} |G_{j,le_j}| \leq 2^{-j2^\delta \alpha}\right) = \prod_{l=0}^{[2^j(1-\delta)]-1} \mathbb{P}\left(|G_{j,le_j}| \leq 2^{-j2^\delta \alpha}\right)$$

$$= \mathbb{P}\left(|G_{j,0}| \leq 2^{-j2^\delta \alpha}\right)^{2^j(1-\delta)} = \left(1 - \mathbb{P}\left(|G_{j,0}| > 2^{-j2^\delta \alpha}\right)\right)^{2^j(1-\delta)}. \quad (2.21)$$

Observe that, in view of (2.20) in which one takes $l = 0$ and in view of the assumption that $\delta \in (0, 1/3)$, there exist a positive constant $c_4$ and a positive integer $j_0$ such that, one has,

$$2^{-j2^\delta \alpha} \geq \|G_{j,le_j}\|_{\alpha} = 2^{-j/\alpha} \int_{1-e_j}^{1} |\Phi_{H,\alpha}(x)|^\alpha dx \geq c_4^{1/\alpha}2^{-j/\alpha}, \quad (2.22)$$

for all integers $j$ and $l$ satisfying $j \geq j_0$ and $0 \leq l < [2^j(1-\delta)]$; notice that the last inequality in (2.22) follows from the fact that we have chosen $j_0$, such that for every $j \geq j_0$,

$$\int_{1-e_j}^{1} |\Phi_{H,\alpha}(x)|^\alpha dx \geq 2^{-1} \int_{-\infty}^{1} |\Phi_{H,\alpha}(x)|^\alpha dx,$$
and the last integral is positive since $\Phi_{H,\alpha}$ is a non-vanishing function (this is a consequence of our assumptions on $\psi$). Also notice that, one can suppose that $c_4 \in (0, c_3^{-1})$ (the positive constant $c_3$ has been introduced in Lemma 2.3). Next, it follows from (2.21), from the first inequality in (2.19) in which one $t = 2^{-j/2} \delta$, and from (2.22), that, for all integer $j \geq j_0$,

$$\mathbb{P}\left(\max_{0 \leq l < [2^{(1-\delta)}]} |G_{j,le_j}| \leq 2^{-j/2} \delta\right) \leq \left(1 - c_5 2^{-j(1-2\delta)}\right)^{[2^{(1-\delta)}]},$$

(2.23)

where the constant $c_5 := c_3 c_4 \in (0, 1)$. Then, (2.23), the fact that $\delta \in (0, 1/3)$, and standard computations, allow to show that,

$$\sum_{j=j_0}^{+\infty} \mathbb{P}\left(\max_{0 \leq l < [2^{(1-\delta)}]} |G_{j,le_j}| \leq 2^{-j/2} \delta\right) < \infty;$$

thus, applying the Borel-Cantelli Lemma, one gets (2.17).

The proof of Lemma 2.2 mainly relies on the following result as well as on Lemma 2.3.

**Lemma 2.5.** For all non-negative integers $j$ and $l$ such that $l < [2^{\delta j}]$, the scale parameter $\|R_{j,le_j}\|_{\alpha}$ of the symmetric $\alpha$-stable random variable $R_{j,le_j}$ (see (2.15)) satisfies,

$$\|R_{j,le_j}\|_{\alpha} = 2^{-j} \int_{-\infty}^{1-e_j} |\Phi_{H,\alpha}(x)|^{\alpha} dx \leq c_6 2^{-j(\delta + 1/\alpha - \delta H)},$$

(2.24)

where $c_6$ is a positive constant non depending on $j$ and $l$.

**Proof of Lemma 2.5.** The equality in (2.24) can be obtained by using (2.15) and the arguments which have allowed to derive (2.20). Let us show that the inequality in (2.24) holds; there is no restriction to assume that $j \geq \delta^{-1}$. Using (2.7) and (2.12), one has,

$$2^{-j} \int_{-\infty}^{1-e_j} |\Phi_{H,\alpha}(x)|^{\alpha} dx \leq c_1^2 2^{-j} \int_{-\infty}^{1-e_j} (1 - x)^{-2\alpha - 1 + \alpha H} dx \leq c_1^2 2^{-j} \int_{2^{\delta - 2}}^{+\infty} (1 + x)^{-2\alpha - 1 + \alpha H} dx = c_1^2 2^{-j(2\delta - 1) - \alpha(2 - H)} \leq c_6 2^{-j(2\delta + 1/\alpha - \delta H)},$$

where the constant

$$c_6 := c_1^2 \frac{2^{\alpha(2 - H)}}{\alpha(2 - H)}.$$

Now, we are in position to prove Lemma 2.2.

**Proof of Lemma 2.2.** First, observe that in view of the assumption that $\delta \in (0, 1/3)$, one has for a fixed arbitrarily small $\eta > 0$,

$$\frac{2\delta + \eta}{\alpha} < 2\delta + 1/\alpha - \delta H;$$
therefore, it follows from Lemma 2.5 that there exists a positive integer \( j_1 \), such that for all integers \( j \) and \( l \), satisfying \( j \geq j_1 \) and \( 0 \leq l < [2^{j(1-\delta)}] \), one has,

\[
\|R_{j, le_j}\|_a \leq 2^{-j\left(\frac{2\delta \alpha}{a}\right)}.
\]

Thus, we are allowed to apply the second inequality in (2.19), in the case where \( Y = R_{j, le_j} \) and \( t = 2^{-j\left(\frac{2\delta \alpha}{a}\right)} \). As a consequence, we obtain that, for all \( j \geq j_1 \),

\[
\mathbb{P}\left(\max_{0 \leq l < [2^{j(1-\delta)}]} |R_{j, le_j}| > 2^{-j\left(\frac{2\delta \alpha}{a}\right)}\right) \leq \sum_{l=0}^{[2^{j(1-\delta)}]-1} \mathbb{P}\left(|R_{j, le_j}| > 2^{-j\left(\frac{2\delta \alpha}{a}\right)}\right)
\]

\[
\leq c_2 2^{j(2\delta + \eta)} \sum_{l=0}^{[2^{j(1-\delta)}]-1} \|R_{j, le_j}\|_a \leq c_7 2^{-j(\alpha(2\delta+1/\alpha-\delta H)+j(1+\delta+\eta))},
\]

(2.25)

where the last inequality results from (2.24) and the constant \( c_7 := c_2 c_6 \). Assume that \( \delta(\alpha - 1) > \eta \), then one has,

\[
\alpha(2\delta + 1/\alpha - \delta H) > \alpha(\delta + 1/\alpha) = \alpha \delta + 1 > 1 + \delta + \eta.
\]

Therefore, it follows from (2.25) that,

\[
\sum_{j=j_1}^{+\infty} \mathbb{P}\left(\max_{0 \leq l < [2^{j(1-\delta)}]} |R_{j, le_j}| > 2^{-j\left(\frac{2\delta \alpha}{a}\right)}\right) < \infty;
\]

thus, applying the Borel-Cantelli Lemma, one gets (2.18).

\[\Box\]

Remark 2.1. Our proofs of Lemmas 2.1 and 2.2 only allow to derive that Relations (2.17) and (2.18) hold on some event of probability 1, denoted by \( \Omega_{\tilde{\delta}} \), since it a priori depends on \( \delta \in (0, 1/3) \). Yet, one can easily show that these two relations also hold, for every real number \( \delta \in (0, 1/3) \), on an event of probability 1 which does not depend on \( \delta \), namely the event \( \bigcap_{\delta \in (0, 1/3)} \tilde{\Omega}_{\delta} \).

Now, we are in position to prove Relation (1.10).

Assume that \( \epsilon \) is a fixed arbitrarily small positive real number and that \( \delta \in (0, 1/3) \) is such that,

\[
\epsilon/2 = 2\delta/\alpha.
\]

(2.26)

Next observe that (2.26), (1.7), (2.16) and the triangle inequality, imply that for all \( j \in \mathbb{Z}_+ \),

\[
2^{j(H-1/\alpha+\epsilon/2)} D_j \geq 2^{j\frac{\alpha}{2}} \max_{0 \leq l < [2^{j(1-\delta)}]} |G_{j, le_j} + R_{j, le_j}|
\]

\[
\geq 2^{j\frac{\alpha}{2}} \max_{0 \leq l < [2^{j(1-\delta)}]} |G_{j, le_j}| - 2^{j\frac{\alpha}{2}} \max_{0 \leq l < [2^{j(1-\delta)}]} |R_{j, le_j}|;
\]

therefore, one has that,

\[
\lim_{j \to +\infty} \inf \left\{ 2^{j(H-1/\alpha+\epsilon/2)} D_j \right\} \geq \lim_{j \to +\infty} \inf \left\{ 2^{j\frac{\alpha}{2}} \max_{0 \leq l < [2^{j(1-\delta)}]} |G_{j, le_j}| \right\} - \lim_{j \to +\infty} \sup \left\{ 2^{j\frac{\alpha}{2}} \max_{0 \leq l < [2^{j(1-\delta)}]} |R_{j, le_j}| \right\}.
\]

(2.27)

Finally putting together, (2.17), (2.18), (2.27) and (2.26), one gets (1.10).
2.3 Proof of Theorem 1.1

Relations (1.8) and (1.10) imply that there is Ω* an event of probability 1 such that each \( \omega \in \Omega^* \) satisfies the following property: for all arbitrarily small \( \epsilon > 0 \), there are two finite positive constants \( A = A(\omega, \epsilon) \) and \( B = B(\omega, \epsilon) \), and there exists \( j_2 = j_2(\omega, \epsilon) \in \mathbb{Z}_+ \), such that, one has for all integer \( j \geq j_2 \),

\[
A 2^{-j(H-1/\alpha+\epsilon)} \leq D_j(\omega) \leq B 2^{-j(H-1/\alpha-\epsilon)}.
\]

This entails that,

\[
-H + 1/\alpha - \epsilon \leq \liminf_{j \to +\infty} \left\{ \frac{\log(D_j(\omega))}{j \log(2)} \right\} \leq \limsup_{j \to +\infty} \left\{ \frac{\log(D_j(\omega))}{j \log(2)} \right\} \leq -H + 1/\alpha + \epsilon.
\]

Then letting \( \epsilon \) goes to zero, one gets that,

\[
\lim_{j \to +\infty} \left\{ \frac{\log(D_j(\omega))}{j \log(2)} \right\} = -H + 1/\alpha.
\]

\( \square \)
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