A SIMPLE REGULARIZATION OF GRAPHS

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Abstract. The well-known regularity lemma of E. Szemerédi for graphs (i.e. 2-uniform hypergraphs) claims that for any graph there exists a vertex partition with the property of quasi-randomness. We give a simple construction of such a partition. It is done just by taking a constant-bounded number of random vertex samplings only one time (thus, iteration-free). Since it is independent from the definition of quasi-randomness, it can be generalized very naturally to hypergraph regularization. In this expository note, we show only a graph case of the paper [5] on hypergraphs, but may help the reader to access [5].

1. Introduction

The well-known regularity lemma of Szemerédi [12] (also called the uniformity lemma) was discovered in the course of obtaining the so-called Szemerédi’s theorem on arithmetic progressions [11] as an affirmative answer of a conjecture by Erdős and Turán. It has been known that this graph-theoretic lemma has a plenty of applications in many topics of mathematics and theoretical computer sciences.

The regularity lemma claims that for any ordinary graph (i.e. any 2-uniform hypergraph) there exists a vertex partition with the property of quasi-randomness. Our purpose of this note is to give a simple construction of such a partition. It has several advantages over previously-known methods. It is the case of \( k = 2 \) (i.e. the case of 2-uniform hypergraphs) in [5] which deals with general \( k \). Although this expository note is not necessary to read [5], skimming it may help the reader understand the main idea of [5].

Remark that our construction had not been known even for the simplest case \( k = 2 \) before [5]. Although the idea of partitioning the vertex-set randomly has been previously known ([3, 1]), such a construction was done always by a constant number of sample random vertices, which thus needs an iteration. The key difference is that ours is constant-bounded but the number of random samplings is chosen also randomly. The proof can be naturally deduced once the claim is given.

Recall how the standard proof by Szemerédi constructs the desired vertex partition with quasi-randomness. Roughly speaking, the partition was constructed by iterated applications of the dichotomy between energy-increment and structure. That is, initially take an arbitrary vertex partition (with a constant number of vertex sets). It can be shown that

1) this partition satisfies the required quasi-random property or that
2) there must exist another vertex partition finer than this partition such that
   2.1) the number of vertex sets increases but is still bounded by a constant and further that
   2.2) a value called ‘energy’ (or ‘index’) of the finer partition is significantly larger than the ‘energy’ of the coarser partition.

They replace the coarser partition by the finer one and repeat this process. Since the energy is always less than one from its definition, the repeating process must stop in at most constant time. (Note that however the exact time when it stops depends on the structure of the given graph.) The vertex partition which the final stage outputs satisfies (1) and thus is the desired partition.

On the other hand, our construction goes as follows.

(0') Take a large constant \( \tilde{n} \) which depends on \( \epsilon \) (parameter on how much quasi-random it should be) but is independent from (the number of vertices of) the given graph. Further take a length-\( \tilde{n} \) integer sequence \( 0 = m_0 \ll m_1 \ll \cdots \ll m_{\tilde{n}-1} \), also independent from the given graph.

(1') Choose an integer \( 0 \leq n < \tilde{n} \) uniformly at random and further choose \( m_n \) vertices uniformly at random from the given graph.

(2') Each vertex of the given graph is labeled by the adjacency between the vertex and the randomly-chosen vertices.

The resulting partition certainly consists of a constant number of vertex sets (i.e. \( 2^{m_n} \leq 2^{m_{\tilde{n}-1}} \)) and would be the desired partition with high probability.

Previous constructions including the usual one by Szemerédi consist of iterated procedures, while our construction consists of only one procedure. Furthermore ours is independent from the definition of

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quasi-randomness, while previous constructions depend on it. Several definitions of quasi-randomness have been known. For the case of ordinary graphs, all of them are known to be equivalent. However, it has been noted that unlike the situation for graphs, there are several ways one might define regularity for hypergraphs (Tao-Vu [15, pp.455], Rödl-Skokan [10, pp.1]). Because our construction is independent from the definition of quasi-randomness, it can be naturally generalized from ordinary graphs to hypergraphs. For the extension to hypergraphs, see [3].

The purpose of this note is to present a new construction of the vertex partition (for the case of ordinary graphs) and to show that it certainly satisfies quasi-randomness. For the case of ordinary graphs, there are several definitions of quasi-randomness but all of them are known to be equivalent ([2]). As our definition of quasi-randomness, we will choose the number of induced subgraphs. In this paragraph, I will explain why we will use this definition, though of course it is not serious at least when considering only the case of ordinary graphs (since they are equivalent). The usual regularity lemma firstly defines quasi-randomness by a condition on the number of edges between two sets of vertices. Secondly Szemerédi proved the existence of a vertex partition with this quasi-randomness. Thirdly and finally, the quasi-randomness on counting induced subgraphs (i.e. our definition) can be derived from his quasi-randomness (on edges between two subsets). The third step is called to be the counting lemma and is easy to show, so the second step only is the core of the matter. All of the three hypergraph-theoretic proofs of Szemerédi’s arithmetic-progression theorem by Rödl et al. [10, 9], Gowers [4] and Tao [13] can be considered as generalizations of the above three steps. But unlike the case of graphs, the third step (counting lemma) was hard to show for hypergraphs. All of the three proofs are different partly because all of them employed different definitions of quasi-randomness, on which their regularizations depend. On the other hand, we will not follow the above three steps. We will define a probabilistic construction for partitioning the vertices, which will be proven to satisfy the condition of our quasi-randomness on counting induced subgraphs. This strategy can be very naturally generalized from graphs to hypergraphs in [5]. I believe that this framework of hypergraph regularity lemma is convenient for a wide range of applications on hypergraphs. In fact, applications of our method are seen in [6, 7, 8].

One of the new major technical ingredients in our proof comes from the use of ‘linearity of expectation.’ All of the previous proofs use the dichotomy (or energy-increment) explicitly. (See [11, §6], [14, §1].) Namely, when proving the existence of a vertex partition, they define an ‘energy’ (or index) by the maximum (or supremum) of some (energy) function. (For example, see [13, eq. (8)].) It corresponds to [13] in this paper. They consider the maximum value of this energy over all subdivisions in each step. If the energy significantly increases by some subdivision, they take the worst subdivision as the base partition of the next step. They then repeat this process. Since the energy is bounded, the operation must stop at some step, in which case there is no quite bad subdivision, and thus, most cells should be quasi-random (dichotomy).

On the other hand, we (implicitly) take an average subdivision instead of the worst one. The definition of our regularization determines the probability space of partitions (subdivisions). We also randomly decide on the number of vertex samples to choose.

With these ideas, we can hide the troublesome dichotomy iterations inside linear equations of expectations [27].

We have two reasons why we will deal with multi-colored graphs instead of ordinary graphs, even though almost all previous researchers dealt with the usual graphs. First, our proof of the regularity lemma will be natural. Second, we can naturally combine subgraph (black&invisible) and induced-subgraph (black&white) problems when we apply our result, while the two have usually been discussed separately.

2. Statement of the Theorem

In this paper, P and E will denote probability and expectation, respectively. We denote conditional probability and expectation by P[·|·|·] and E[·|·|·|·].

Setup 2.1. Throughout this paper, we fix a positive integer r and an ‘index’ set v with |v| = r. Also we fix a probability space (Ω, B, P) for each i ∈ v. We assume that Ω, is finite and that Bi = 2Ω, (for the sake of simplicity). Write Ω := (Ωi)i∈v.
For applications, $\Omega_\iota$ usually will contain a huge number of vertices. We will not use this assumption logically in our proof, but it will be important in our theorems that some parameters and functions depend on $r$ but independent from any $|\Omega_\iota|$. In what follows, we will try to embed a small $r$-partite graph to another large $r$-partite graph, where the $r$ vertex sets of the large graph will be always $(\Omega_\iota)_{\iota \in \mathbb{R}}$. And the large graph and its vertices and edges will be denoted by bold fonts (ex. $G$, $v$, $v'$, $e$, $\cdots$).

For an integer $a$, we write $[a] := \{1, 2, \cdots, a\}$, and $\binom{[a]}{i} := \bigcup_{\iota \in [a]} \binom{\iota}{i} = \bigcup_{\iota \in [a]} \{I \subset \iota \mid |I| = i\}$. Thus $\binom{[2]}{i} = \binom{\iota}{i} \cup \binom{\iota'}{i} = \mathbf{w}_{\mathbf{u}}^{\iota}$. When $r$ disjoint sets $X_i$, $i \in \iota$, with indices from $\iota$ are called vertex sets, we write $X_{\iota} := \{Y \subset \bigcup_{j \in \iota} X_j \mid Y \cap X_j = 1, \forall j \in \iota\}$ whenever $J \subset \iota$. Thus $|X_{\iota}| = \prod_{j \in \iota} |X_j|$. That is, for $J = \{1, 2\}$, $|X_{\iota}| = |X_1||X_2|$.

**Definition 2.1.** [Colored graphs] Suppose Setup 2.1. Given $b_1$ and $b_2$, a $(b_1, b_2)$-colored ($\iota$-partite) graph $H$ is a triple $((X_\iota)_{\iota \in \mathbb{R}}, (C_\iota)_{\iota \in \mathbb{R}}, (\gamma_\iota)_{\iota \in \mathbb{R}})$ where:

(1) each $X_\iota$ is a set called a ‘vertex set,’

(2) $C_\iota$ is a set with at most $b_1|\iota|$ elements, and

(3) $\gamma_\iota$ is a map from $X_\iota$ to $C_\iota$.

We write $V(H) = \bigcup_{\iota \in \mathbb{R}} X_\iota$ and $C_\iota(H) = C_\iota$ for $I$. Each element of $V(H)$ is called a vertex. Each element $v \in V(H)$ is $X_\iota$, $I \in \binom{\iota}{2}$, is called an (index-$I$) edge. Thus, when $|\iota| = 1$, an index-$I$ edge is just a vertex of $H$. Each member in $C_\iota(H)$ is a (face-)color of index $I$. Write $H(e) = \gamma_\iota(e)$ for each $I$. (So we will not need the notation $\gamma_\iota$ after this definition.)

When $I = \{i, j\} \in \binom{\iota}{2} (i.e. i \neq j)$ and $e = \{v_i, v_j\} \in V_\iota(H)$, we define the frame-color and total-color of $e$ by $H(\iota e) := (H(v_i), H(v_j))$ and $H(e) := (H(e); H(v_i), H(v_j))$. For a vertex $v_i \in X_i$ (which is also an index-$I$ edge), we define the total-color of $v_i$ by $H(v) := (\gamma_\iota(e) := H(v_i))$. The frame-color of a vertex is the empty set ($\emptyset$). Write $TC(H) := \{H(e) \in X_\iota = V_{\iota}(H)\}$, $TC_\iota(H) := \bigcup_{\iota \in \mathbb{R}} TC_\iota(H)$, and $TC(H) := TC_\iota(H) \cup TC_\iota(H)$, where $TC$ means total-color.

As usual, we will call a $(b_1, b_2)$-colored graph just a colored graph or a graph when we do not need to mention values $b_1, b_2$.

**Definition 2.2.** [Complexes] A (simplicial-)complex is a (colored $\iota$-partite) graph such that:

(1) for each $I \in \binom{\iota}{2}$ there exists at most one index-$I$ color called ‘invisible’ and that

(2) if (the color of) an edge $e$ is invisible then for any edge $e' \supset e$, its color must also be invisible.

A color is visible if and only if it is not invisible. We simply say that an edge is visible/invisible when its color is so.

For a graph $G$ on $\Omega$, let $S_{h, G}$ be the set of complexes $S$ such that:

(1) each of $r$ vertex sets of the $r$-partite graph $S$ contains exactly $h$ vertices, and that,

(2) for $I \in \binom{\iota}{2}$ there is an injection from the index-$I$ visible colors of $S$ to the index-$I$ colors of $G$.

(When the injection maps a visible color $\iota$ of $S$ to another color $\iota'$ of $G$, we simply write $\iota = \iota'$ without presenting the injection explicitly.) For $S \in S_{h, G}$, we denote by $V_\iota(S)$ the set of index-$I$ visible edges. Write $V_\iota(S) := \bigcup_{\iota \in \mathbb{R}} V_{\iota}(S)$ and $V(S) := V_1(S) \cup V_2(S)$. Clearly we have

$$|V_1(S)| \leq rh \quad \text{and} \quad |V_2(S)| \leq \binom{r}{2} h^2. \quad (1)$$

**Definition 2.3.** [Partitionwise maps] A partitionwise map $\varphi : \bigcup_{\iota \in \mathbb{R}} W_\iota \rightarrow \bigcup_{\iota \in \mathbb{R}} \Omega_\iota$ is a map from $r$ disjoint vertex sets $W_\iota$, $i \in \iota$, with $|W_\iota| < \infty$, to the $r$ vertex sets (probability spaces) $\Omega_\iota$, $i \in \iota$, such that each $w \in W_\iota$ is mapped into $\Omega_\iota$. That is, any vertex is mapped to a vertex with the same index. We denote by $\Phi_\iota(W_\iota)$ or $\Phi_\iota(\bigcup_{\iota \in \mathbb{R}} W_\iota)$ the set of partitionwise maps from $(W_\iota)$. When $W_\iota = \{(i, 1), \cdots, (i, h)\}$ or when $W_\iota$ are obvious and $|W_\iota| = h$, we denote it by $\Phi(h)$. A partitionwise map is random if and only if each $w \in W_\iota$ is independently mapped to a vertex in the probability space $\Omega_\iota$.

**Definition 2.4.** [Regularization] Let $m \geq 0$. Let $G$ be a graph on $\Omega$ and let $\varphi \in \Phi(m)$. The regularization of $G$ by $\varphi$ is the graph $G/\varphi$ on $\Omega$ obtained from $G$ by redefining the color of each vertex $v \in \Omega_\iota$, $i \in \iota$, by the $(1 + (r - 1)m)$-dimensional vector

$$(G/\varphi)(v) := (G(\{v, u\})) u = v \quad \text{or, } u \in \Omega_\iota, j \in \iota \setminus \{i\}, \text{ is in the range of } \varphi.$$
Roughly speaking, the color of vertex \( v \) in \( G/\varphi \) is the information of the color-patterns of size-2 edges connecting the random vertex samplings and \( v \), together with the original color \( G(v) \). (Here a size-2 edge means an edge which is not a single vertex but a pair of vertices.)

Note that edges of size 2 (i.e. not vertices) do not get recolored in this process. Only vertices change their colors as the same as in the usual regularity lemma.

**Definition 2.5.** [Regularity] Let \( G \) be a graph on \( \Omega \). For \( \tilde{c} = (c_I)_{I \in \mathcal{C}} \in TC_I(G), I \in \binom{n}{2} \), we define relative density by the conditional probability

\[
\text{reg}_I(G) := \mathbb{P}_{e \in \Omega_{I}}[G(e) = c_I | G(\partial(e)) = (c_{J})_{J \subset I}].
\]

(2)

When \( |I| = 1 \), in the above \( e \) is a vertex and the conditional part is considered to always hold. (Thus for \( I = \{j\} \) and for an index-{\( j \)} color \( c_j \in TC_1(G) \), we have \( \text{reg}_I(G) = \mathbb{P}_{v \in \Omega} [G(v) = c_j] \), i.e. how much portion of the vertices in \( \Omega \) have color \( c_j \).)

For a positive integer \( h \) and \( \epsilon \geq 0 \), we say that \( G \) is \((\epsilon, h)\)-regular if and only if there exists a function \( \delta : TC_2(G) \to [0, \infty) \) such that

\[
\begin{align*}
\text{(i)} \quad & \mathbb{P}_{e \in \Phi(h)}[G(\phi(e)) = S(e), \forall e \in V(S)] \\
& = \prod_{e \in V_2(S)} \text{reg}(S(e)) \prod_{e \in V_2(S)} \left( \text{reg}(S(e)) \pm \delta(S(e)) \right), \quad \forall S \in S_h, G, \\
\text{(ii)} \quad & \mathbb{E}_{e \in \Omega, \delta(\Phi(G))} \leq \epsilon / |C_I(G)|, \quad \forall I \in \binom{r}{2},
\end{align*}
\]

where \( a \pm b \) denotes a suitable number \( c \) satisfying \( \max\{0, a - b\} \leq c \leq \min\{1, a + b\} \).

Denote by \( \text{reg}_h(G) \) the minimum value of \( \epsilon \) such that \( G \) is \((\epsilon, h)\)-regular.

**Remark.** Roughly speaking, (i) measures how far from random the graph \( G \) is with respect to containing the expected number of copies of the (colored) subgraphs \( S \in S_{h, G} \). The smaller \( \delta \) is, the closer \( G \) is to being random. When \( \delta \equiv 0 \), then \( G \) behaves exactly like a random graph. On the other hand, if we take \( \delta \equiv 1 \) then (i) is automatically satisfied. Condition (ii) places an upper bound on the size of \( \delta \). Our proof will yield the main theorem even if we replace the right-hand side of (ii) by \( g_I(|C_I(G)|) \) for any fixed functions \( g_I > 0 \), for example, \( g_I(x) = x^{-1/\epsilon} \).

Our main theorem is as follows.

**Theorem 2.2** (Main). For any \( r \geq 2, h, \tilde{b} = (b_1, b_2), \) and \( \epsilon > 0 \), there exist an (increasing) function \( m : \mathbb{N} \to \mathbb{N} \) and an integer \( \tilde{m} \) satisfying the following:

If \( G \) is a \( \tilde{b} \)-colored \((r\text{-partite})\) graph on \( \Omega \) then

\[
\mathbb{E}_{\varphi}[\text{reg}_h(G/\varphi)] \leq \epsilon
\]

where \( n \) is chosen randomly in \([0, \tilde{m} - 1]\) and \( \varphi \in \Phi(m(n)) \) is random.

Note that \( m \) and \( \tilde{m} \) depend only on \( r, h, \tilde{b} \) and \( \epsilon \) and are independent of everything else (including \( \Omega \)). Since \( m \) is increasing, we put \( \tilde{m} := m(n) \geq m(n) \) and get:

**Corollary 2.3** (Regularity Lemma). For any \( r \geq 2, h, \tilde{b} = (b_1, b_2), \) and \( \epsilon > 0 \), there exists an integer \( \tilde{m} \) such that if \( G \) is a \( \tilde{b} \)-colored \((r\text{-partite})\) graph on \( \Omega \) then for some integer \( m \leq \tilde{m} \), we have

\[
\mathbb{E}_{\varphi \in \Phi(m)}[\text{reg}_h(G/\varphi)] \leq \epsilon.
\]

(5)

In particular, when (2) holds, if we pick a map \( \varphi \in \Phi(m) \) randomly then with probability at least \( 1 - \sqrt{\epsilon} \), we have \( \text{reg}_h(G/\varphi) \leq \sqrt{\epsilon} \), thus \( G/\varphi \) is \((\sqrt{\epsilon}, h)\)-regular.

It is important that the above integer \( m \) is bounded by a constant \( \tilde{m} \) independent from \( G \) but the exact value of \( m \) itself depends on \( G \). Note that, in (cannonical) property testing, the exact value \( m \) is also independent from \( G \). This is a new critical idea which has never been previously while some had felt that property testing and graph regularization seem to have a close relation (ex. [1]).

Of course, we can rewrite the above results for non-partite graphs.

3. **Proof of the Main Theorem**

Before we proceed with the proof of the Main Theorem, we will need to establish two lemmas. We admit that they may appear a bit technical and unmotivated at this point, but their use will be clearer once we see how they are used in the main proof.
3.1. Two lemmas and their proofs.

**Definition 3.1.** [Notation for the lemmas] Let $G$ be an ($r$-partite) graph on $\Omega$. For two edges $e, e' \in \Omega_1$, we abbreviate $G(e) = G(e')$ and $G(\partial e) = G(\partial e')$ by $e \cong e'$ and $e \cong G$, respectively.

An $h$-error function of $G$ is a function $\delta : \bigcup_{e \in G} TC_f(G) \rightarrow [0, \infty)$ satisfying (3) for all $S \in S_{h,G}$.

Denote by $[\ldots]$ the Iverson bracket, i.e., it equals 1 if the statement in the bracket holds, and 0 otherwise.

**Lemma 3.1** (Correlation bounds counting error). For any graph $G$ on $\Omega$ and for $S \in S_{h,G}$, we have

$$
\mathbb{P}_{\phi \in \Phi(h)} \left[ G(\phi(e)) = S(e), \forall e \in V_2(S) \right] \mathbb{E}_{v \in V_1(S)} \left[ G(\phi(v)) = S(v), \forall v \in V_1(S) \right] - \prod_{e \in V_2(S)} d_G(S(e))
$$

$$
\leq |V_2(S)| \max_{\emptyset \neq D \subseteq V_2(S)} \left| \prod_{e \in D} \left[ G(\phi(e)) = S(e) \right] - d_G(S(e)) \right| G(\phi(v)) = S(v), \forall v \in V_1(S) \right|
$$

**Proof:** [Tool: Nothing] We will prove this by induction on $|V_2(S)|$. If $|V_2(S)| \leq 1$ then the statement is trivial, since in this case, the expression on the left-hand side of the inequality is 0. So let us assume that $|V_2(S)| \geq 2$ and that the result holds for all smaller values of $|V_2(S)|$. Let $d_e := d_G(S(e))$, and let $\eta$ be the maximum part of the desired right-hand side. Then for $D := V_2(S)$ we have

$$
[-\eta, \eta] \ni \mathbb{E}_{\phi \in \Phi(h)} \left[ \prod_{e \in V_2(S)} \left[ G(\phi(e)) = S(e) \right] - d_G(S(e)) \right] G(\phi(v)) = S(v), \forall v \in V_1(S) \right]
$$

$$
= \mathbb{E}_{\phi \in \Phi(h)} \left[ \prod_{e \in V_2(S)} \left[ G(\phi(e)) = S(e) \right] G(\phi(v)) = S(v), \forall v \in V_1(S) \right]
$$

$$
+ \sum_{\emptyset \neq D \subseteq V_2(S)} \left( \prod_{e \in D} (-d_e) \right) \mathbb{E}_{\phi \in \Phi(h)} \left[ \prod_{e \in V_2(S) \setminus D} \left[ G(\phi(e)) = S(e) \right] G(\phi(v)) = S(v), \forall v \in V_1(S) \right],
$$

expanding the product and using the linearity of expectation and the definition of $d_e$. Now we will focus on second term above. Since the value of $[G(\phi(e)) = S(e)]$ is 0 or 1, we can replace $E$ by $P$, and consequently, apply the induction hypothesis (since $D$ is nonempty). Consider a complex $S^-$ with $V_2(S^-) = V_2(S) \setminus D$ by invisualizing the edges in $D$ of $S$.

Using the inductive hypothesis for complex $S^-$ in the place of $S$, we rewrite the second term and obtain

$$
\mathbb{E}_{\phi \in \Phi(h)} \left[ \prod_{e \in V_2(S)} \left[ G(\phi(e)) = S(e) \right] G(\phi(v)) = S(v), \forall v \in V_1(S) \right]
$$

$$
\overset{\text{I.H.}}{=} - \sum_{\emptyset \neq D \subseteq V_2(S)} \left( \prod_{e \in D} (-d_e) \right) \left( \prod_{e \in V_2(S) \setminus D} d_e \right) \pm |V_2(S^-)| \eta \pm \eta
$$

$$
= - \left( \prod_{e \in V_2(S)} d_e \right) \pm |V_2(S^-)| \eta \pm \eta \quad (\because |d_e| \leq 1)
$$

$$
= - \left( \prod_{e \in V_2(S)} \left( d_e \right) \pm \left( |V_2(S^-)| - 1 \right) \eta \right) \left( \prod_{e \in D} (-1) \right) \pm \eta \quad (\because |V_2(S)| > |V_2(S^-)|)
$$

$$
= \left( \prod_{e \in V_2(S)} d_e \right) \pm |V_2(S)| \eta.
$$

We will use the following form of the Cauchy-Schwarz.

**Fact 3.2** (Cauchy-Schwarz inequality). For a random variable $X$ on a probability space $\Omega$ if an equivalent relation $\approx$ on $\Omega$ is a refinement of another equivalent relation $\sim$ on $\Omega$ then

$$
\mathbb{E}_{\omega \in \Omega} \left( \mathbb{E}_{\omega \in \Omega} [X(\omega) | \omega \approx \omega_0] \right)^2 \geq \mathbb{E}_{\omega \in \Omega} \left( \mathbb{E}_{\omega \in \Omega} [X(\omega) | \omega \sim \omega_0] \right)^2.
$$

(6)
Proof: By the Cauchy-Schwarz (i.e. $E[X^2]E[Y^2] \geq (E[XY])^2$), we have $E_{\omega_0}\left[(E_{\omega_0}[X(\omega)|\omega \approx \omega_0])^2\right] = E_{\omega_0}\left[E_{\omega'} \left[(E_{\omega'}[X(\omega)|\omega \approx \omega_0])^2\right] = E_{\omega_0}\left[E_{\omega'} \left[(E_{\omega'}[X(\omega)|\omega \approx \omega_0] \cdot E_{\omega'} \left[(E_{\omega'}[X(\omega)|\omega \approx \omega_0])^2\right] = E_{\omega_0}\left[E_{\omega}(E_{\omega}(X(\omega)|\omega \approx \omega_0))^2\right] \geq \right)\right]

With this fact and Definition 3.1 we next tackle

Lemma 3.3 (Mean square bounds correlation). Let $h$ and $m$ be positive integers and $G$ an $r$-partite graph on the vertex set $\Omega$. Let $S \in S_hG$ and let $F_e : C_I(G) \to [-1,1]$ be a function for each $I \in (\frac{-1}{2})$ and for each $e \in V_I(S)$. For any $I \in (\frac{1}{2})$ and $e_0 \in V_I(S)$, we have

\[
\left(\sum_{\phi \in \Phi(h)} \prod_{e \in E(S)} F_e(\phi(e)) \prod_{e \in E(S)} [G(\phi(e)) = S(v)] \right)^2 \leq E_{v \in V_I(S)} \left[\prod_{e \in E(S)} d_G(S(v)) \prod_{\omega \in \Omega, \omega \approx \omega_0} E_{\phi \in \Phi(h)} \left[\prod_{e \in E(S)} F_e(\phi(e)) \left(G(\phi(e)) = S(v) \forall v \in V_I(S)\right) \right]^2 \right]
\]

(7)

where $\phi, \varphi$ are random and where we abbreviate $F_e(G(e))$ by $F(e)$.

In particular, if we suppose $\frac{1}{m} \leq \prod_{e \in V_I(S), v \not\in e_0} d_G(S(v))$ (i.e., $m$ is large) then

\[
\left(\sum_{\phi \in \Phi(h)} \prod_{e \in E(S)} F_e(\phi(e)) \prod_{\omega \in \Omega, \omega \approx \omega_0} E_{\phi \in \Phi(h)} \left[\prod_{e \in E(S)} F_e(\phi(e)) \left(G(\phi(e)) = S(v) \forall v \in V_I(S)\right) \right]^2 \right) \leq 2E_{\phi \in \Phi(h)} \left[\prod_{e \in E(S)} F_e(\phi(e)) \left(G(\phi(e)) = S(v) \forall v \in V_I(S)\right) \right]^2 \right]
\]

(8)

Proof: [Tools: Cauchy-Schwarz, Fact 3.2] Fix $I_0 \in (\frac{1}{2})$ and $e_0 \in V_I(S)$. For $\phi \in \Phi(V(S) \setminus e_0)$ and for $e_0 \in \Omega_{I_0}$, we define the (extended) function $\phi(e_0) \in \Phi(V(S))$ such that:

(i) each $v \in e_0$ is mapped to the corresponding $v \in e_0$ with the index of $v$, and that,

(ii) each $v \in V(S) \setminus e_0$ is mapped to $\phi(v)$.

(That is, when we have a map $\phi$ from all but two vertices $e_0$ to two vertices $e_0$.) For an $m$-tuple of maps $\varphi \sim (\varphi_i)_{i \in [m]}$ with $\varphi_i \in \Phi(V(S) \setminus e_0)$, we define an equivalence relation $\sim$ on $\Omega_{I_0}$ by the condition that

\[
e_0 \sim n \text{ if and only if } \varphi_i(e_0) \equiv \varphi_i(n) \forall e \in V(S) \setminus \{e_0\}, \forall i \in [m].
\]

(9)

(That is, $V(S) \setminus \{e_0\}$ is a vertex set while $V(S) \setminus \{e_0\}$ is an edge set. Since the right-hand side clearly holds for $e$ with $e \cap e_0 = \emptyset$, it is enough to check only the edges $e \in V(S)$ with $e \cap e_0 = \emptyset$.)

Let $S^{(1)}, \ldots, S^{(m)}$ and $e_0^{(1)}, \ldots, e_0^{(m)}$ be $m$ copies of $S$ and $e_0$. For $\varphi \sim (\varphi_i)_{i \in [m]}$ with $\varphi_i \in \Phi(V(S^{(i)}) \setminus e_0^{(i)})$, let $\varphi^* \in \Phi(mh) = \Phi(V(S^{(1)}) \cup \ldots \cup V(S^{(m)}))$ be an extended function of $\varphi_i$'s, i.e., $\varphi^*(v) = \varphi_i(v)$ for all $v \in V(S^{(i)}) \setminus e_0^{(i)}$, $i \in [m]$. Then it is not hard to see that

\[
e_0 \sim \varphi^* \sim e^{*} \text{ implies } e_0 \sim \varphi^* \sim e^{*}.
\]

(10)

(To see this, observe that if $I_0 = \{1,2\}$ and $\{v_1, v_2\} \sim \varphi_1^{(m)} \sim \varphi_1^{(m)}$, or equivalently $v_j \sim \varphi_j^{(m)} \sim v_j^{(m)} (j = 1,2)$, then $(v_j, \varphi_j^{(m)}) \equiv \{v_j, \varphi_j^{(m)}\})$ for all $v \in V(S^{(1)}) \setminus \{e_0\}$ having no index $j$ (i.e. $v \in V_j(S^{(i)})$, $j' \neq j$).

Since $\varphi^*(v) = \varphi_i(v)$ if $v \not\in e_0^{(i)}$, we see $\varphi_i^{(m)}(v_1, v_2) \equiv \varphi_i^{(m)}(v_1', v_2')$ for all $e \in V(S^{(i)}) \setminus \{e_0^{(i)}\}$ with $|e \cap e_0^{(i)}| = 1$, implying (10) by (9).)

Let $F^{*}(e) := F_0(e) \left[G(\partial(e_0)) = S(e_0)\right]$ and let

\[
F^{*}(\phi) := \prod_{e \in V_2(S)\setminus \{e_0\}} F_e(\phi(e)) \prod_{e \in V_1(S)} [G(\phi(e)) = S(v)]
\]

Then, since $\left[\cdots\right]^2 = \left[\cdots\right]$, the left-hand side of (7) becomes

\[
\left(\sum_{\phi \in \Phi(h)} \prod_{e \in E(S)} F_e(\phi(e)) \prod_{e \in E(S)} [G(\phi(e)) = S(v)] \right)^2 \leq E_{\phi \in \Phi(h)} \left[F^{*}(\phi(e_0)) \prod_{e \in V_2(S)\setminus \{e_0\}} F_e(\phi(e)) \prod_{e \in V_1(S)} [G(\phi(e)) = S(v)] \right]^2
\]

(by the definition of $F_0$ and $F^*$)
In a similar way, we can interpret the first term as computing the probability that \(2 + (r h - 2) = rh\) vertices in \(V(S)\)

\[
= \left( \mathbb{E}_{e_0 \in \mathbf{\Omega}_0, \phi \in \Phi(V(S) \setminus e_0)} \left[ F^*_e (e_0) F^*(\phi(e_0)) \right] \right)^2
\]

(since the two expectations are taken over random choices of \(2 + (rh - 2) = rh\) vertices in \(V(S)\)).

\[
= \left( \mathbb{E}_{\mathbf{\omega} \in \mathcal{F} \setminus (\phi_0, \epsilon)} \mathbb{E}_{e_0 \in \mathbf{\Omega}_0} \left[ F^*_e (e_0) \mathbb{E}_{\mathbf{\epsilon} \in [m]} \left[ F^*(\epsilon) \right] \right] \mathbb{E}_{e_0 \in \mathbf{\Omega}_0} \left[ \mathbb{E}_{e_0 \in \mathbf{\Omega}_0} \left[ F^*_e (e_0) \mathbb{E}_{\mathbf{\epsilon} \in [m]} \left[ F^*(\epsilon) \right] \right] \right] \right)^2
\]

(since for any random variable \(X\) and the equivalence classes \(C_i\) by \(\mathbf{\omega}\),

\[
E_{e_0} E_{e_0} [X(e) | e \neq e_0] = \sum_j P_e (e_0 \in C_j | E_{e_0} [X(e) | e \in C_j] = E_{e_0} [X(e_0)])
\]

Next, we show the last sentence of the lemma. The left hand side of (8) is at most

\[
\mathbb{E}_{e_0 \in \mathbf{\Omega}_0} \left[ F^*_e (e_0) \mathbb{E}_{\mathbf{\epsilon} \in [m]} \left[ F^*(\epsilon) \right] \right] \mathbb{E}_{e_0 \in \mathbf{\Omega}_0} \left[ \mathbb{E}_{e_0 \in \mathbf{\Omega}_0} \left[ F^*_e (e_0) \mathbb{E}_{\mathbf{\epsilon} \in [m]} \left[ F^*(\epsilon) \right] \right] \right] \leq \mathbb{E}_{e \in \mathbf{\Omega}_0} \left[ F^*_e (e) \mathbb{E}_{\mathbf{\epsilon} \in [m]} \left[ F^*(\epsilon) \right] \right] \mathbb{E}_{e \in \mathbf{\Omega}_0} \left[ \mathbb{E}_{e \in \mathbf{\Omega}_0} \left[ F^*_e (e) \mathbb{E}_{\mathbf{\epsilon} \in [m]} \left[ F^*(\epsilon) \right] \right] \right]
\]

(by Cauchy-Schwarz)

Putting all these observations together, the proof of the first part of Lemma 3.3 is complete.

Looking at the second term first, this can be written as

\[
\frac{1}{m} \prod_{v \in V_1(S)} \mathbb{P}_{v \in \Phi(V(S))} [G(v) = S(v)] \quad (\text{since } \varphi \text{ maps all } v \in \Phi(V_1(S)) \text{ independently})
\]

\[
= \frac{1}{m} \prod_{v \in V_1(S)} \text{d}_{G}(S(v)) \quad (\text{by the definition of } \text{d}_{G}(S(v)))
\]

In a similar way, we can interpret the first term as computing the probability that \(2 + 2(r h - 2)\) random (visible or invisible) vertices chosen independently will have vertex colors in \(G\) which match those of their corresponding vertices in \(S\). This probability can be written as

\[
\prod_{v \in V_1(S), v \notin e_0} \mathbb{P}_{v \in \Omega_1} [G(v) = S(v)]^2 \prod_{v \in e_0} \mathbb{P}_{v \in \Omega_1} [G(v) = S(v)]
\]

\[
= \prod_{v \in V_1(S), v \notin e_0} \text{d}_{G}(S(v))^2 \prod_{v \in e_0} \text{d}_{G}(S(v)).
\]

Putting all these observations together, the proof of the first part of Lemma 3.3 is complete.

Next, we show the last sentence of the lemma. The left hand side of (9) is at most

\[
\left( \mathbb{E}_{\phi \in \Phi(h)} \left[ \prod_{\mathbf{c} \in V_2(S)} F_e (\phi(c)) \prod_{v \in V_1(S)} [G(\phi(v)) = S(v)] \right] / \mathbb{P}_{\phi \in \Phi(h)} [G(\phi(v)) = S(v) \forall v \in V_1(S)] \right)^2
\]

\[
= \left( \mathbb{E}_{\phi \in \Phi(h)} \left[ \prod_{\mathbf{c} \in V_2(S)} F_e (\phi(c)) \prod_{v \in V_1(S)} [G(\phi(v)) = S(v)] \right] / \left( \prod_{v \in V_1(S)} \text{d}_{G}(S(v)) \right) \right)^2
\]

\[
\left( \mathbb{E}_{\phi \in \Phi(h)} \left[ \prod_{\mathbf{c} \in V_2(S)} F_e (\phi(c)) \prod_{v \in V_1(S)} [G(\phi(v)) = S(v)] \right] / \left( \prod_{v \in V_1(S)} \text{d}_{G}(S(v)) \right) \right)^2
\]
The assumption on \( m \) now completes the proof of (8).

\[ \square \]

3.2. The body of our proof.

**Definition 3.2.** [Notation for the proof] Write \( c_i(G) := \max_{I \in \mathcal{I}} |C_I(G)| \) for \( i = 1, 2 \). For \( \bar{c} = (b_1, b_2) \) and an integer \( m \), we write \( \bar{B}(\bar{c}, m) := (B_1(\bar{c}, m), B_2(\bar{c}, m)) \) where \( B_1(\bar{c}, m) := b_1 \cdot b_2^{(r-1)m} \) and \( B_2(\bar{c}, m) := b_2 \).

Recalling the definition of regularization \( G/\varphi \), it is easy to see that if \( G \) is a \( \bar{c} \)-colored graph then

\[
c_i(G/\varphi) \leq B_i(\bar{c}, m), \quad \forall i = 1, 2, \forall \varphi \in \Phi(m).
\]

(If \( i = 2 \), it is obvious since regularization does not recolor any size-2 edge. If \( i = 1 \), the new color of a vertex is determined by its original color and by the colors of the edges connecting the vertex and the \( (r-1)m \) random vertices.)

\[ \square \]

Suppose we are given some fixed \( h \geq 1, \epsilon > 0 \) and \( \bar{c} \). Our job will be to define suitable functions \( m \) and \( \delta \), and a suitable integer \( \tilde{n} \), so that (25) and (30) are satisfied. This we now do.

- **[Definition of the sample-size functions]** Set \( m_{h, \bar{c}, \epsilon}(0) := m(0) := 0 \). Define \( \tilde{n}_{2, h, \bar{c}, \epsilon} = \tilde{n} \) to be large enough so that

\[
C \cdot b_2 \sqrt{\frac{b_2}{\tilde{n}}} \leq \frac{\epsilon}{2^{(r/2)}}
\]

where

\[
C := \sqrt{2} \binom{r}{2} h^2 \left( \frac{b_2}{\sqrt{\epsilon_1}} \right)^{\binom{r}{2} h^2 - 1} \quad \text{and} \quad \epsilon_1 := \left( \frac{\epsilon}{6 \cdot b_2^{(r/2)}} \right)^2.
\]

(These expressions will appear in (25) and (30).)

We will define the function \( m \) recursively as follows. Suppose that \( m(n) \) has been defined for some value of \( n \geq 0 \). Let

\[
M := \left( \frac{b_1 b_2^{(r-1)m(n)}}{\sqrt{\epsilon_1}} \right)^{rh}.
\]

(We will use the form (14) only once in (25).) Define \( m(n+1) \) so that

\[
m(n+1) \geq m(n) + M \cdot \sqrt{\epsilon_1} = m(n) + \left( \frac{b_1 b_2^{(r-1)m(n)}}{\sqrt{\epsilon_1}} \right)^{rh} \cdot h.
\]

Next, we define the error function \( \delta \).

- **[Definition of the error function]** For \( \varphi \in \Phi(m(n)) \), we write \( G^* := G/\varphi \) and we define the error function \( \delta = \delta_{h, r, G^*} \) inductively as follows.

First, define

\[
\delta(\bar{\epsilon}) := 0 \quad \text{and} \quad \eta(\bar{\epsilon}) := 0 \quad \text{for all} \quad \bar{\epsilon} \in \text{TC}_I(G^*) \quad \text{with} \quad I \in \binom{r}{1} = r.
\]

(16)
Before defining \( \delta(\tilde{c}) \) and \( \eta(\tilde{c}) \) for \( \tilde{c} \in TC_2(\mathbf{G}^*) \), we define 'bad colors' \( \text{BAD} \subset TC(\mathbf{G}^*) \). For \( I \in \{1,2\} \), we define \( \text{BAD}_I \) by the relation that \( \tilde{c} = (\epsilon_I)_{J \subseteq I} \in \text{BAD}_I \) if and only if

\[
d_{\mathbf{G}^*}(\epsilon_I)_{J \subseteq I} \leq \sqrt{|I|}/|\partial I^*(\mathbf{G}^*)| \quad \text{for some } I^* \text{ with } \emptyset \neq I^* \subset I. \tag{17}
\]

Define \( \text{BAD} := \bigcup_{I \in \{1,2\}} \text{BAD}_I \). A bad edge will mean a visible edge whose color is bad.

For \( \tilde{c} = (\epsilon_I)_{J \subseteq I} \in TC_2(\mathbf{G}^*) \), we define, using \( M \) and \( C \) of (13) and (14),

\[
\eta(\tilde{c}) := \mathbb{E}_{\phi \in \Phi(h)} \mathbb{E}_{\epsilon \in \Omega_I} \left( \left( \mathbb{E}_{\epsilon \in \Omega_I} [G^*(\epsilon) = \epsilon | \partial G^*/\phi^* = \epsilon^*] - d_{\mathbf{G}^*}(\tilde{c}) \right)^2 | G^*(\partial \epsilon^*) = (\epsilon_I)_{J \subseteq I} \right) \tag{18}
\]

\[
\delta(\tilde{c}) := \left\{ \begin{array}{ll}
\frac{1}{\sqrt{\eta(\tilde{c})}}, & \text{if } \tilde{c} \in \text{BAD}_I, \\
C, & \text{otherwise.}
\end{array} \right. \tag{19}
\]

First, we show that with the above specified choices for \( m, n \) and \( \delta, \), (3) is satisfied.

**[The qualification as an error function]** Clearly it is enough to show that

\[
\mathbb{E}_{\phi \in \Phi(h)} [G^*(\phi(e)) = S(e), \forall e \in V(S)]
\]

\[
= \prod_{e \in V_I(S)} d_{\mathbf{G}^*}(S(e)) \prod_{e \in V_2(S)} (d_{\mathbf{G}^*}(S(e)) \pm \delta(S(e))) \tag{20}
\]

for any \( S \in S_{h, \mathbf{G}^*} \). Furthermore without loss of generality, we can assume that

\[
S(e) \notin \text{BAD} \text{ for any } e \in V(S). \tag{21}
\]

(Indeed, we can show this by the induction on the number of bad edges in \( S \). Let a complex \( S \) be given where \( S \) contains a bad edge \( e^* \). Firstly we suppose that there exist no bad vertices and thus \( e^* \) contains two different vertices (which are not bad). By the induction hypothesis, (20) holds for the complex \( S' \) obtained from \( S \) by recoloring \( e^* \) in the invisible color. Equality (20) means that the real number the left hand side suggests belongs to the interval which the right-hand side suggests. Denote by \( [p^-, p^+] \) this interval. Again we reconstruct \( S \) from \( S' \) by recoloring some invisible edges in the original bad color. By this process from \( S \) to \( S^* \), the left hand side of (20) will not increase (probably decrease because of added visible edges \( e^* \)) and the right-hand side will suggest interval \([0, p^+] \) because \( d_{\mathbf{G}^*}(S(e^*)) \pm \delta(S(e^*)) = [0, 1] \) by (11). Then (20) holds also for \( S \). Secondly we suppose that the \( e^* \) consists of a single bad vertex \( \nu \). Then we recolor not only \( \nu \) but also all edges containing \( \nu \) in the invisible color. The same argument can be applied.)

Fix such an \( S \in S_{h, \mathbf{G}^*} \). For any \( e \in V_J(S) \) with \( J \subset \tau \), it follows from (21), (17) and (16) that

\[
d_{\mathbf{G}^*}(S(e)) > \frac{\sqrt{\epsilon_I}}{|C_J(\mathbf{G}^*)|} > 0 \quad \text{if } |J| \leq 2 \text{ and } \delta(S(e)) = 0 \text{ if } |J| = 1. \tag{22}
\]

Using (11), (14) and (22), a straightforward computation gives

\[
\frac{1}{M} \leq \mathbb{E}_{\phi \in \Phi(h)} \left[ \prod_{e \in V_2(S')} \left( |\epsilon^*| \right)^{N_1(S)} \prod_{e \in V_2(S)} d_{\mathbf{G}^*}(S(e)) \right] \leq \prod_{v \in V_1(S), \nu \notin \nu_0} d_{\mathbf{G}^*}(S(\nu)) \tag{23}
\]

for any \( \epsilon_0 \in V_2(S) \). For any choice of \( \emptyset \neq D \subset V_2(S) \), we define \( S' \in S_{h, \mathbf{G}^*} \) so that \( V_2(S') = D \) and \( S'(e) = S(e) \forall e \in D \) and that \( S'(e) = S(e) \forall e \in V_1(S) = V_1(S') \). Now, applying Lemma 3.3 for \( S' \) with \( F_{\epsilon}(e) := [G^*(\epsilon) = S(e)] - d_{\mathbf{G}^*}(S(e)) \), we have

\[
\left( \mathbb{E}_{\phi \in \Phi(h)} \left[ \prod_{e \in V_2(S')} (|G^*(\phi(e)) = S'(e)) - d_{\mathbf{G}^*}(S'(e)) \right) G^*(\phi(v)) = S'(v), \forall v \in V_1(S') \right] \right)^2
\]

\[
\geq \left( \mathbb{E}_{\phi \in \Phi(h)} \left[ \prod_{e \in D} (|G^*(\phi(e)) = S(e)) - d_{\mathbf{G}^*}(S(e)) \right) G^*(\phi(v)) = S(v), \forall v \in V_1(S) \right] \right)^2
\]

\[
= \left( \mathbb{E}_{\phi \in \Phi(h)} \left[ \prod_{e \in D} F_{\epsilon}(e) \right) G^*(\phi(v)) = S(v), \forall v \in V_1(S) \right] \right)^2
\]

\[
\leq \left( \mathbb{E}_{\phi \in \Phi(M_h)} \mathbb{E}_{\epsilon \in \Omega_I} \left[ \left( \mathbb{E}_{e \in \Omega_I} [F_{\epsilon_0}(e) | \epsilon \approx G^*/\phi = e^*] \right)^2 | G^*(\partial \epsilon^*) = S(\partial \epsilon_0) \right] \right)^2
\]

\[
= \left( \mathbb{E}_{\phi \in \Phi(M_h)} \mathbb{E}_{\epsilon \in \Omega_I} \left[ \left( \mathbb{E}_{e \in \Omega_I} [G^*(\epsilon) = S(\epsilon_0)] \epsilon \approx G^*/\phi = e^*] - d_{\mathbf{G}^*}(S(\epsilon_0)) \right)^2 | G^*(\partial \epsilon^*) = S(\partial \epsilon_0) \right] \right)^2.
\]
Now, choose an $e_0 \in \mathbb{V}_2(S)$ which maximizes $\eta(S(e_0))$. It then follows from Lemma 3.1 that

$$\mathbb{P}_{\phi \in \phi(M)}[G^*(\phi(e)) = S(e), \forall e \in \mathbb{V}_2(S)] = \mathbb{P}_{\phi \in \phi(V)}[G^*(\phi(v)) = S(v), \forall v \in \mathbb{V}_1(S)]$$

(24)

Finally we turn to showing that $\sum \eta(S(e))$. It then follows from Lemma 3.1 that

$$\prod_{e \in \mathbb{V}_2(S)} G^*(S(e)) \geq \prod_{e \in \mathbb{V}_2(S), e \neq e_0} G^*(S(e)) \geq \prod_{e \in \mathbb{V}_2(S)} G^*(S(e)) \geq \prod_{e \in \mathbb{V}_2(S), e \neq e_0} G^*(S(e))$$

(25)

Finally we turn to showing that $\delta$ satisfies (3).

For any $S \in \mathbb{S}_n$, (20) holds, and we have shown that $\delta$ satisfies (3).

- [Bounding the average error size] For $I \in \{1\}$, it follows from the linearity of expectation that

$$\left( \mathbb{E}_{n \in [0, n-1], \phi \in \phi(m(n))} \mathbb{E}_{e \in \Omega} \left[ \mathbb{E} \left[ \mathbb{P}_{e \in \phi(M)} [G^*(e) = e] \right] \right] \right)^2 \leq \mathbb{E}_{n \in [0, n-1], \phi \in \phi(m(n))} \mathbb{E}_{e \in \Omega} \left[ \mathbb{E} \left[ \mathbb{P}_{e \in \phi(M)} [G^*(e) = e] \right] \right]$$

(26)

(27)

(28)

(29)

(30)

(31)

(32)

(33)

(34)

(35)

(36)}
Now, for any \( I \) in (*) above we use Fact 3.2 and the property that after \( n \) is chosen, it follows from 15 that \( m(n+1) \geq Mh + n(n) \) and that if \( \phi''(\mathcal{D}) \supset \varphi(\mathcal{D}) \cup 
abla''(\mathcal{D}) \) (where \( \phi''(\mathcal{D}), \varphi(\mathcal{D}), \phi'(\mathcal{D}) \) denote the ranges of those functions) then \( e \frac{\partial G/\varphi'}{\partial G/\varphi} \) implies \( e \frac{\partial G/\varphi'}{\partial G/\varphi} \) equals \( e \).

Now, for any \( I \in \binom{\mathcal{I}}{I} \), it follows from 14 that

\[
\mathbb{E}_{n \in \widetilde{\mathcal{I}}, \varphi \in \Phi(m(n)), e \in \Omega} \left[ \delta(G^*(e)) \right] \leq \sum_{J \subseteq I} \mathbb{E}_{e \in \Omega} \left[ \delta(G^*(e)) \right] \]  

(27)

(8)

where in (8) above we use Fact 3.2 and the property that after \( n \) is chosen, it follows from 15 that \( m(n+1) \geq Mh + n(n) \) and that if \( \phi''(\mathcal{D}) \supset \varphi(\mathcal{D}) \cup 
abla''(\mathcal{D}) \) (where \( \phi''(\mathcal{D}), \varphi(\mathcal{D}), \phi'(\mathcal{D}) \) denote the ranges of those functions) then \( e \frac{\partial G/\varphi'}{\partial G/\varphi} \) implies \( e \frac{\partial G/\varphi'}{\partial G/\varphi} \) equals \( e \).

However, it is easy to see that for any \( \tau > 0 \), we have by the definition of \( dG^* \)

\[
\mathbb{P}_e \left[ |dG^*(e)| \geq \tau \right] = \mathbb{P}_e \left[ |dG^*(e)| = G^*(e') + \partial \mathcal{L} \right] \leq \mathcal{C} \]  

(29)

Hence, using 28 and 29, we can write

\[
\mathbb{E}_{n,\varphi} \left[ \mathbb{E}_e \left[ \delta(G^*(e)) \right] \right] \leq C \sqrt{\frac{b_2}{n}} + \mathbb{E}_{n,\varphi} \left[ \sum_{J \subseteq I} \mathbb{E}_e \left[ \delta(G^*(e)) \right] \right] \]  

(30)

(31)

To show that the expectation of the regularity is small, we compute

\[
\mathbb{E}_{n,\varphi}[\text{reg}(G/\varphi)] \leq \mathbb{E}_{n,\varphi}[\text{max}_{I \in \binom{\mathcal{I}}{I}} |C_I(G/\varphi)| \mathbb{E}_e \left[ \delta(G^*(e)) \right]] \]  

(16)

\[
\leq \sum_{I \in \binom{\mathcal{I}}{I}} |C_I(G/\varphi)| \mathbb{E}_e \left[ \delta(G^*(e)) \right] \]  

(39)

as required. This completes the proof of Theorem 2.2.

\[ \square \]

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