The $q$-deformed Schrödinger Equation of the
Harmonic Oscillator on the Quantum Euclidian Space

Ursula Carow-Watamura and Satoshi Watamura

Department of Physics
Faculty of Science
Tohoku University
Aoba-ku, Sendai 980, JAPAN

Abstract

We consider the $q$-deformed Schrödinger equation of the harmonic oscillator on the $N$-dimensional quantum Euclidian space. The creation and annihilation operator are found, which systematically produce all energy levels and eigenfunctions of the Schrödinger equation. In order to get the $q$-series representation of the eigenfunction, we also give an alternative way to solve the Schrödinger equation which is based on the $q$-analysis. We represent the Schrödinger equation by the $q$-difference equation and solve it by using $q$-polynomials and $q$-exponential functions.
1 Introduction

To investigate the possibility of defining the quantum mechanics on a different type of geometry is a very interesting problem since we can expect that it will show new aspects of the quantum theory, see for example [1]. When we consider the noncommutative geometry as the base geometry, one feature is that the theory is formulated in an algebraic language since the noncommutative space is defined by a function algebra. Therefore, it is not easy to use the analogy with the commutative geometry in a simple way through such an algebraic language.

Taking the point of view that the quantum group and the quantum space are the $q$-deformation of the usual group and space, we can find the noncommutative analogue of the known objects such as the quantum Lorentz group $\mathbb{2}$, the quantum Minkowski space $\mathbb{3}$, the quantum Poincaré group $\mathbb{4}$, and many other properties. Differential calculi on $q$-spaces have been also constructed in a rather early stage of the investigations in quantum groups $\mathbb{1}$, $\mathbb{2}$. They give a simple example of the noncommutative geometry and allow us to draw the analogy to the non-deformed theory.

In ref. [10] the authors have constructed the differential calculus on the N-dimensional $q$-Euclidian space, i.e. the differential calculus covariant under the action of the quantum group $Fun_q(SO(N))$. Although it became a little more complicated than the one in refs. [8], [9] which is based on $A$-type quantum groups, it has the advantage that it contains the metric which makes it possible to define the Laplacian.

Using this differential calculus we have investigated the Schrödinger equation corresponding to the $q$-deformed harmonic oscillator and have computed the ground state energy as well as the first two excited energy levels. However in that stage a systematic construction to all energy levels has been missing.

Recently the investigations on this line were put forward by several authors [11], [12], [13], [14], especially by the work of Fiore [12] who proposed raising and lowering operators which map the wavefunctions of the $r$-th level to the one of the $(r+1)$th level. However, these operators defined in ref. [12] have an explicit dependence on the energy level, i.e., there is one such operator for each level and thus we have an infinite number of them. Due to this feature, these operators cannot be considered as the $q$-analogue of the creation-annihilation operator.

It is the aim of this paper to develop further this approach and to construct the creation-annihilation operator of the N-dimensional $q$-deformed harmonic oscillator which is level-independent. This defines all energy eigenvalues and gives the expression of the eigenfunctions in terms of creation operators acting on the ground state. Since the creation-annihilation operator is given in terms of differential operators, in principle all eigenfunctions can be computed. It is however not so easy to find the explicit form of the wave function as a $q$-polynomial in the coordinates $x^i$ with this method.

Thus, we develop an alternative method to construct the eigenfunctions of the Schrödinger equation, which is another new result of this paper. It corresponds to the analytic construction of the wave function in the non-deformed case. Reducing the Schrödinger equation to a $q$-difference equation we solve it directly by using $q$-polynomials. The above described two constructions give the same eigenvalues and the resulting wave functions have a one-to-one correspondence.

Since the investigations in this paper are based on the results obtained in ref. [10], let us briefly recall them here. It is known that the $\hat{R}$-matrix of the quantum group $Fun_q(SO(N))$ can be written by using projection operators as

$$\hat{R} = qP_S - q^{-1}P_A + q^{1-N}P_1,$$  \hspace{1cm} (1.1)
where the indices $S$, $A$ and $1$ denote symmetric, antisymmetric and singlet, respectively. The differential calculus on the $q$-Euclidian space derived in ref.[10] is making use of this projector decomposition. It is defined by the algebra $C < x^i, dx^i, \partial^i >$ with relations consistent with the quantum group action. The relations are given by:

\[ P_{A k l} x^k x^l = 0, \]
\[ P_S(dx \wedge dx) = 0, \quad P_1(dx \wedge dx) = 0, \]
\[ x^i dx^j = q^{-1} R^{ij}_{kl} x^k x^l, \]
\[ \partial^i x^j = C^{ij} + q R^{-1}_{kl} x^k \partial^l, \]
\[ \partial^i x^j = q^{-1} R^{ij}_{kl} x^k \partial^l, \]
\[ \partial^i dx^j = q^{-1} R^{ij}_{kl} dx^k \partial^l, \]

where $Q_N$ is a constant given by

\[ Q_N = \frac{(1 - q^N)\mu}{(1 - q^2)} = C^{ij} C_{ij}, \]

$C_{ij}$ is the metric of the quantum space, and $\mu$ is

\[ \mu = 1 + q^{2-N}. \]

The algebra is constructed such that there exists the $q$-analogue of the exterior derivative $d \equiv C_{ij} dx^i \partial^j$ satisfying nilpotency and Leibniz rule. In this algebra, one can find the natural $q$-analogue of the Laplacian $\Delta$:

\[ \Delta = (\partial \cdot \partial) = C_{ij} \partial^i \partial^j, \]

The $q$-deformed Laplacian of the differential calculus on the $N$-dimensional $q$-Euclidian space led us to investigate the corresponding Schrödinger equation, the simplest example of which is the harmonic oscillator. The action of its Hamiltonian onto the wave function $|\Psi\rangle$ is defined by

\[ H(\omega)|\Psi\rangle = [-q^N(\partial \cdot \partial) + \omega^2(x \cdot x)]|\Psi\rangle = E|\Psi\rangle, \]

where we have shifted the normalization to the one introduced in ref.[12] which has the advantage that the factors $q$ in the corresponding operators are distributed symmetrically w.r.t. the $*$-conjugation. In this paper we do not write explicitly the result of the $*$-conjugated sector. The calculations and proofs given in the following sections can be performed completely parallel for the conjugated sector. In section 5, we discuss the properties of the system investigated in this paper with respect to the $*$-conjugation.

As a result of the investigations performed in ref.[10] we obtained the solution of the $q$-deformed Schrödinger equation for the ground state of the $q$-deformed harmonic oscillator and the first two excited energy levels. The ground state wave function is given by the $q$-exponential function as

\[ |\Psi_0\rangle = \exp_q \left[ \frac{-\omega x C x}{q^N \mu} \right], \]
with convention $xCx = C_{ij}x^i x^j$. For the definition of the $q$-exponential function and some of its properties see the appendix. The ground state energy is

$$ H(\omega)|\Psi_0\rangle = E_0|\Psi_0\rangle, \quad \text{where} \quad E_0 = \frac{\omega \mu (1 - q^N)}{(1 - q^2)} = \omega Q_N, \quad (1.13) $$

For the first excited level we have the eigenfunction of the vector representation $|\Psi_1^i\rangle$:

$$ H(\omega)|\Psi_1^i\rangle = E_1|\Psi_1^i\rangle, \quad \text{with} \quad E_1 = \frac{\omega \mu (1 - q^{N+2})}{q(1 - q^2)}, \quad (1.14) $$

where

$$ |\Psi_1^i\rangle = x^i \exp_q \left[ \frac{-\omega x C x}{q^{N+1} \mu} \right], \quad (1.15) $$

and for the second excited levels the symmetric tensor $|\Psi_{2,S}\rangle$ and singlet representation $|\Psi_{2,1}\rangle$ with the same energy eigenvalue $E_2$:

$$ H(\omega)|\Psi_{2,S}\rangle = E_2|\Psi_{2,S}\rangle, \quad (1.16) $$

$$ H(\omega)|\Psi_{2,1}\rangle = E_2|\Psi_{2,1}\rangle, \quad \text{with} \quad E_2 = \frac{\omega \mu (1 - q^{N+4})}{q^2(1 - q^2)}. \quad (1.17) $$

The corresponding wave functions are given by

$$ |\Psi_{2,S}\rangle = \mathcal{P}_S(x \otimes x) \exp_q \left[ \frac{-\omega x C x}{q^{N+2} \mu} \right], \quad (1.18) $$

and

$$ |\Psi_{2,1}\rangle = (x C x + A) \exp_q \left[ \frac{-\omega x C x}{q^{N+2} \mu} \right], \quad (1.19) $$

where $i, j = 1, \ldots, N$ and $A = - \frac{Q_N q^2}{\omega (1 + q^2)}$. All quantities given here are in the new normalization corresponding to eq.(1.11). As for the $q$-numbers we use the conventions:

$$ [x] = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad (1.20) $$

and

$$ (x) q^2 = \frac{1 - q^{2x}}{1 - q^2}. \quad (1.21) $$

These are the results which will be relevant for us in the following.

This paper is organized as follows. In section 2, we present the main results of the operator formalism as well as of the $q$-analysis. In section 3, we give the proofs of some theorems and propositions stated in section 2. In section 4, some further properties of the operator formalism are analyzed. Section 5 is devoted to discussions and conclusions.

2 The $q$-deformed creation and annihilation operators and general solution
2.1 Operator formalism

As in ordinary quantum mechanics, to generate all eigenfunctions of the $q$-deformed Schrödinger equation, it is natural to look for the creation and annihilation operator $a^i$ which satisfies in general

$$H(\omega)a^i = q^ka^i[H(\omega) + C(\omega)], \quad (2.1)$$

where we introduced a possible $q$-factor $q^k$ ($k$ is a real constant). Then, $a^i$ maps the $p$th state to the $(p + 1)$th state and the constant $C(\omega)$ gives the energy difference between the two states. However in a $q$-deformed system we can easily see that there is not such an operator. The reason is that the energy difference between the neighbouring states is not equidistant as we see from the eigenvalues $E_0$, $E_1$ and $E_2$. Taking this into account, Fiore introduced in ref.\[12\] operators separately for each state which raise and lower the energy level, i.e., the $p$-th raising operator $a^i_p$ acts as: $|p\rangle \rightarrow |p + 1\rangle$ (and correspondingly the lowering operator $a_p$) for the $p$th level. In this way he obtains all the states together with an infinite number of raising (and lowering) operators. It is a priori not obvious whether one can define at all a creation-annihilation operator which produces all states of this $q$-deformed system.

The key point to find the creation-annihilation operator of this system is to allow the quantity $C(\omega)$ to be a function of the Hamiltonian. One can easily see that with this generalization the operator $a^i$ still maps one eigenstate to another eigenstate of different energy level. Using the analogy with the non-deformed case we look for a creation-annihilation operator of the form $(\partial^i + x^i\alpha)$ and the above generalization means that the coefficient $\alpha$ is a function of the Hamiltonian.

In our construction we also have to take into account that the coordinate function $x^i$ and the derivative $\partial^i$ have non-trivial commutation relations with the Hamiltonian:

$$\partial^i H(\omega) = H(q\omega)\partial^i + \mu\omega^2 x^i, \quad (2.2)$$

$$x^i H(\omega) = q^{-2}H(q\omega)x^i + q^{N-2}\mu\partial^i, \quad (2.3)$$

With the above described considerations we find the following operators:

**Theorem A:**

i) The creation operator $a^i_-$ and annihilation operator $a^i_+$ are defined by

$$a^i_\pm = q^{-\frac{1}{2}}\lambda^{-\frac{1}{2}}[q^\frac{N}{2}\partial^i + x^i\alpha_\pm(\omega)], \quad (2.4)$$

where

$$\alpha_\pm(\omega) = \frac{1}{2}[KH(\omega) \pm \sqrt{K^2[H(\omega)]^2 + 4\omega^2}] \quad \text{with} \quad K = \frac{(1 - q^2)}{q^{\frac{N}{2}}\mu}, \quad (2.5)$$

and

$$\lambda^{-\frac{1}{2}}x^i = q^{-\frac{1}{2}}x^i\lambda^{-\frac{1}{2}} \quad \text{and} \quad \lambda^{-\frac{1}{2}}\partial^i = q^{\frac{1}{2}}\partial^i\lambda^{-\frac{1}{2}}. \quad (2.6)$$

ii) The commutation relation of the creation and annihilation operators with the Hamiltonian is

$$H(\omega)a^i_\pm = q^{-1}a^i_\pm[H(\omega) - q^{\frac{N}{2}}\mu\alpha_\pm(\omega)]. \quad (2.7)$$

The proof of this theorem is given in the section 3.1.
With these operators defined in Theorem A we can derive the whole set of states of the corresponding \(q\)-deformed Schrödinger equation as follows. First we derive the energy spectrum generated by this operator. Formula (2.7) gives the recursion formula of the energy levels from \(E_p\) to \(E_{p+1}\) when both sides are evaluated on the state \(|\Psi_p\rangle\). Therefore we only need to know the eigenvalue of the ground state. As we have shown in the ref. [10], the wavefunction corresponding to the ground state in the limit \(q \to 1\) is given by \(|\Psi_0\rangle\) in eq.(1.12) and it is a candidate of the ground state for the \(q\)-deformed case. We can prove that the operator \(a^i_+\) annihilates \(|\Psi_0\rangle\):

**Proposition A:**

\[
a^i_+ |\Psi_0\rangle = 0 .
\]

**Proof:** Using that the ground state energy \(E_0 = Q_N \omega\)

\[
\alpha_\pm(\omega)|\Psi_0\rangle = \frac{1}{2} [KE_0 \pm \sqrt{K^2[E_0]^2 + 4\omega^2}]|\Psi_0\rangle
\]

\[
= \begin{cases} 
q^{-\frac{N}{q^2}}\omega|\Psi_0\rangle & \text{for } \alpha_+ \\
-q^{\frac{N}{q^2}}\omega|\Psi_0\rangle & \text{for } \alpha_-
\end{cases}
\]

(2.9)

where the definition of \(K\) is given in eq.(2.5). Knowing this we act with the annihilation operator \(a^i_+\) onto \(|\Psi_0\rangle\) and obtain

\[
a^i_+ |\Psi_0\rangle = q^{-\frac{1}{q}N} \frac{1}{\lambda} \left[ q^{\frac{N}{q^2}} \partial^i + x^i \alpha_+ (\omega) \right] |\Psi_0\rangle = q^{-\frac{1}{q}N} \frac{1}{\lambda} \left[ q^{\frac{N}{q^2}} q^{-\frac{N}{2}} - q^{-\frac{N}{2}} \right] x^i |\Psi_0\rangle = 0 .
\]

(2.10)

q.e.d

The excited states are obtained by successively applying the creation operator \(a^i_-\) onto the ground state, the energy eigenvalue of which is defined by eq.(1.12). As we prove in section 3.2, after some calculation we get the energy eigenvalue of the \(p\)-th level:

**Proposition B:**

\[
E_p = \frac{\omega \mu}{q^{1-N/2}} \left[ \frac{N}{2} + p \right] .
\]

(2.11)

**Proof:** Proof is given in section 3.2.

Another result which is obtained in the course of deriving the energy eigenvalue is the value of the operator \(\alpha\) when acting on the state \(|\Psi_p\rangle\):

\[
\alpha_\pm(\omega)|\Psi_p\rangle = \begin{cases} 
q^{-p-\frac{N}{q^2}}\omega|\Psi_p\rangle & \text{for } \alpha_+ \\
-q^{p+\frac{N}{q^2}}\omega|\Psi_p\rangle & \text{for } \alpha_-
\end{cases}
\]

(2.12)

Thus as a consequence, we obtain the equation

\[
a^i_\pm |\Psi_p\rangle = q^{-\frac{1}{q}N} \frac{1}{\lambda} \left[ q^{\frac{N}{q^2}} \partial^i + x^i \alpha_\pm (\omega) \right] |\Psi_p\rangle
\]

\[
= q^{-\frac{1}{q}N} \frac{1}{\lambda} \left[ q^{\frac{N}{q^2}} \partial^i \pm x^i q^{\mp(p+\frac{N}{q^2})}\omega \right] |\Psi_p\rangle .
\]

(2.13)
Therefore, when the creation-annihilation operator defined in eq. (2.4) is acting onto an eigenstate and we evaluate only the operator \( \alpha_{\pm}(\omega) \) then the resulting expression becomes level-dependent and coincides with the raising operator constructed by Fiore.

One of the important relations to characterize the operators \( \alpha_{\pm}(\omega) \) and \( a_{\pm} \) are the following commutation relations

**Proposition C :**

\[
\alpha_{\pm}(\omega)a_{\pm} = q^{\pm 1}a_{\pm}\alpha_{\pm}(\omega) ,
\]

(2.14)

\[
\alpha_{\pm}(\omega)a_{\mp} = q^{\mp 1}a_{\mp}\alpha_{\pm}(\omega) ,
\]

(2.15)

and

\[
\lambda^{-\frac{1}{2}}\alpha_{\pm}(\omega) = q\alpha_{\pm}(\omega/q)\lambda^{-\frac{1}{2}} ,
\]

(2.16)

the proof of which is given in section 3.3.

Using the above relations we can show that

**Theorem B :**

The \( q \)-antisymmetric product of the creation operator vanishes

\[
P_A(a_{\pm}a_{\pm}) = 0 .
\]

(2.17)

**Proof :**

\[
a_{\pm}a_{\pm} = q^{-\frac{1}{2}}\lambda^{-\frac{1}{2}}[q^{N}\partial^{i}a_{\pm} + qx^{i}a_{\mp}\alpha_{\pm}(\omega)]
\]

\[
= q^{-3/2}\lambda^{-1}[q^{N}\partial^{i} + q^{N}(\partial^{i}x^{j} + q^{2}x^{i}\partial^{j})\alpha_{\pm}(\omega) + q^{2}x^{i}x^{j}\alpha_{\mp}(\omega)]
\]

(2.18)

Multiplying the projection operator onto both sides and using the defining relations \( \{1.2\} \ (1.5) \ (1.7) \), we get Theorem B.

q.e.d

Note that the same relation can be proven for the annihilation operator, i.e. \( P_A(a_{\pm}a_{\pm}) = 0 \).

**Theorem B** means that the creation operator satisfies the same commutation relation as the \( q \)-space coordinate function \( x^{i} \). Thus for example

\[
(a_{\pm} \cdot a_{\pm})a_{\pm} = a_{\pm}^{i}(a_{\pm} \cdot a_{\pm}) ,
\]

(2.19)

where \( (a_{\pm} \cdot a_{\pm}) = C_{ij}(a_{\pm}a_{\pm}) \). Consequently any state constructed by successively applying the creation operator \( a_{\pm}^{i} \) onto the ground state \( \Psi_{0} \) is a \( q \)-symmetric tensor. Thus we may call the creation-annihilation operator \( a_{\pm}^{i} \) a \( q \)-bosonic operator. Let us state the above results as a theorem.

**Theorem C :**

The states of the \( p \)th level constructed by \( p \) creation operators \( a_{\pm}^{i} \)

\[
|\Psi_{p}^{i_{1}i_{2} \ldots i_{p}}\rangle \equiv a_{\pm}^{i_{1}}a_{\pm}^{i_{2}} \cdots a_{\pm}^{i_{p}}|\Psi_{0}\rangle ,
\]

(2.20)
have the energy eigenvalue $E_p = \omega \mu q^{N-1} \left[ \frac{N}{2} + p \right]$ and are $q$-symmetric tensors, i.e., $\forall l \in \{1, ..., p-1\}$

$$P_{ji'l}^{i'i''} |\Psi_p^{i_1'i_2'...i_p} \rangle = 0 .$$

(2.21)

Proof: The energy eigenvalue depends only on the number of creation operators and thus the wavefunctions defined in (2.20) have the same value for a fixed level $p$. The eigenvalue $E_p$ is derived in section 3.2. The second part of Theorem C is a direct consequence of Theorem B.

Thus the number of states of the $p$th level is equal to the number of states of the non-deformed case, i.e., $\left( \frac{N+p}{p} \right)$. The $q$-symmetric tensor can be split into symmetric traceless tensors corresponding to the irreducible representations of $Fun_q(SO(N))$:

$$\text{Sym}\left( N \otimes N \otimes \cdots \otimes N \right)_p = S_p \oplus S_{p-2} \oplus \cdots \oplus \left\{ \begin{array}{ll} S_1 & \text{for odd } p \\ S_0 & \text{for even } p \end{array} \right\} .$$

(2.22)

Correspondingly the wave function in eq.(2.20) is split into irreducible components. With this operator method, Theorem C defines in principle all eigenfunctions. However it is not straightforward to obtain an expression of the eigenfunctions as $q$-polynomials in $x^i$.

2.2 The wave function as a $q$-polynomial in $x$

In order to obtain an expression of the eigenfunctions in terms of $q$-polynomials in the coordinate, a simple way is to use the relations of the $q$-differential calculus with the $q$-analysis [17].

The $q$-symmetric traceless $p$th tensor representation $S_p^I$ can be constructed as follows:

$$S_p^I \equiv S_p^{i_1i_2...i_p}(x) = S_p^{i_1i_2...i_p}x^{i_1}x^{i_2}...x^{i_p} ,$$

(2.23)

where

$$P_{ji'l}^{i'i''} S_p^{i_1i_2...i_p}(x) = 0 ,$$

(2.24)

$$C_{ji'l}^{i'i''} S_p^{i_1i_2...i_p}(x) = 0 ,$$

(2.25)

for $j = 1, ..., p - 1$.

Since the tensor structure is defined by these $q$-symmetric tensors, the wave functions can be written by the product of $q$-symmetric tensor and function of $x^2$. Thus to solve the Schrödinger equation (1.11), we take the ansatz:

$$|\Psi \rangle = S_p^I f(x^2) .$$

(2.26)

The problem is to fix the function $f(x^2)$. For this end, we need the following formulae:

$$\Delta S_p^I = \mu(p)q^2(C_{ij}S_p^{ij} \partial^i) + q^{2p}S_p^I \Delta ,$$

(2.27)

$$\partial^i f(x^2) = \mu x^i D_{x^2} f(x^2) ,$$

(2.28)

$$\Delta f(x^2) = [q N \mu^2 x^2 D^2_{x^2} + \mu Q_N D_{x^2}] f(x^2) ,$$

(2.29)

$$\Delta S_p^I f(x^2) = \mu^2 S_p^I \left[ q^{N+2p} x^2 D^2_{x^2} + \left( \frac{N}{2} + p \right) q^2 D_{x^2} \right] f(x^2) .$$

(2.30)
Computing the action of the Hamiltonian onto the wave function (2.26), the Schrödinger equation becomes:

\[
(-q^N \Delta + \omega^2 x^2) S_p^I f(x^2) = \mu^2 S_p^I \left[ -q^{2N+2p} x^2 D_{x^2}^2 - q^N \left( \frac{N}{2} + p \right) q^2 D_{x^2} + \frac{\omega^2}{\mu^2} x^2 \right] f(x^2) \\
= ES_p^I f(x^2),
\]

(2.31)

where the definition of \( D_{x^2} \) is

\[
D_{x^2} f(x^2) = \frac{f(q^2 x^2) - f(x^2)}{x^2(q^2 - 1)}.
\]

(2.32)

From eq. (2.31) we get the following \( q \)-difference equation for \( f(x^2) \)

\[
F(D_{x^2}) f(x^2) = \left[ -q^{2N+2p} x^2 D_{x^2}^2 - q^N \left( \frac{N}{2} + p \right) q^2 D_{x^2} + \frac{\omega^2}{\mu^2} x^2 - E \right] f(x^2) = 0.
\]

(2.33)

To solve this equation we take an ansatz with the \( q \)-exponential function as:

\[
f(x^2) = \sum b_s \exp_q(q^{-2s}\alpha) \quad \text{where} \quad \alpha = q^{-N-p}\omega.
\]

(2.34)

The definition and properties of the \( q \)-exponential function \( \exp_q(q^{-2s}\alpha) \) are collected in the appendix. A lengthy but straightforward calculation yields

\[
F(D_{x^2}) f(x^2) = \frac{q^N\omega^{-1}}{\mu} \sum \left[ -b_{s+1} q^{N/2+2s} + b_s \left( \frac{N}{2} + p + 2s \right) \right] \exp_q(q^{-2s}\alpha) .
\]

(2.35)

We look for the solution which has a finite number of terms in the expansion eq.(2.34). In such a case the argument of the \( q \)-exponential functions in eq.(2.34) can be shifted to the \( q \)-exponential of the largest \( s \) by using the relations given in the appendix. Then such a function \( f(x^2) \) becomes a polynomial of \( x^2 \) multiplied with the \( q \)-exponential which has a smooth finite limit under \( q \to 1 \). On the other hand since the \( q \)-exponentials with different arguments generate different powers in \( x^2 \), they are independent and cannot cancel each other. Thus solving the equation, we require that for each term in the series eq.(2.35) the sum of the coefficients of different exponential functions separately vanishes.

First we see that in the term for \( \exp_q(q^{-2s}\alpha) \), i.e. for \( s = -1 \) in eq.(2.35), the coefficient of \( b_0 \) is zero. Therefore we can consistently set \( b_s = 0 \) for \( s < 0 \) and require that the first nonzero term starts with the \( b_0 \) term.

Now we require that the series has only a finite number of terms. To satisfy this, the second term under the sum, i.e., the coefficient of \( \exp_q(q^{-2s}\alpha) \) containing the factor \( b_s \) must vanish for a certain \( s \). Calling this largest integer \( s \) as \( r \), this requirement defines the energy eigenvalue \( E \) as

\[
E = q^{N/2+2r} \left( \frac{N}{2} + p + 2r \right) .
\]

(2.36)

With this eigenvalue, we can set the \( b_s = 0 \) for \( s > r \) and can solve the equation (2.33). From the condition that the sum of all terms with the same argument in the exponential function vanishes we get the recursion formula
\[ b_s = b_{s+1} - q^{N+p[2s+2]} \left\{ \left[ \frac{N}{2} + p + 2r \right] - \left[ \frac{N}{2} + p + 2s \right] \right\}. \] 

(2.37)

Therefore we obtain

\[ b_s = b_r \prod_{s \leq t \leq r-1} \frac{q^{N+p[2t+2]}}{\left\{ \left[ \frac{N}{2} + p + 2t \right] - \left[ \frac{N}{2} + p + 2r \right] \right\}}. \] 

(2.38)

The \( b_r \) simply shows the freedom of the overall normalization and thus the wave functions are now defined in terms of the \( q \)-exponential functions and the \( q \)-polynomials of the coordinate functions \( x^i \) with a finite number of terms and with the \( b_s \) given above as

\[ | \Psi_{p,r}^I \rangle = S_p^I \sum_{s=0}^r b_s \text{Exp}_{q^2}(q^{-2s} \alpha) \quad \text{where} \quad \alpha = q^{-N-p} \omega. \] 

(2.39)

The energy eigenvalue of \( | \Psi_{p,r} > \) is given by \( E \) in eq.(2.36) and it coincides with the eigenvalue defined by using the creation-annihilation operator in the previous section. We can also confirm that there is a one-to-one correspondence between the wave function given in eq.(2.20) and the one in eq.(2.39):

The wave function derived in eq.(2.39) shows that for each \( p \)th rank tensor we have an infinite tower of eigenfunctions labeled by the integer \( r \) with the eigenvalue \( E_{p+2r} = q^{N-1}\mu \omega \left\{ \frac{N}{2} + p + 2r \right\} \). This means that for the fixed eigenvalue \( E_{p'} \) there is one eigenfunction of \( p \)th rank tensor for each \( p \) which satisfies \( p + 2r = p' \) with a positive integer \( r \). This is the result given in eq.(2.22).

This completes the \( q \)-analytic construction of the eigenfunctions which gives the \( q \)-polynomial representation of the wave function corresponding to the irreducible representations of the \( Fun_q(SO(N)) \).

### 3 Proofs of Theorem A, Proposition B and Proposition C

In this section we give the proof of Theorem A, the energy eigenvalue \( E_p \) given in Proposition B, as well as of Proposition C.

#### 3.1 Proof of Theorem A

Since the commutation relations of \( H(\omega) \) with \( x \) and the one with \( \partial \) generate a discrepancy in the factors \( q \) as one can see from eqs.(2.2) and (2.3), we carefully have to choose the operator \( \alpha \). It turns out that when we put the operator \( \alpha \) at the right hand side of \( x^i \), we have to consider \( \alpha \) as a function of \( H(\omega/q) \). Taking this into account we consider the following operator:

\[ A_\pm(\omega) = \left[ q^{\frac{N}{2}} \partial^j + x^i \alpha_\pm(\omega/q) \right], \] 

(3.1)

where \( \alpha(\omega/q) \) is a function of the Hamiltonian \( H(\omega/q) \) and thus \( \alpha(\omega/q) \) does not commute with \( H(\omega) \) but with \( H(\omega/q) \).

The commutation relation of the operators \( A_\pm \) in eq.(3.1) with the Hamiltonian can be computed by using eqs.(2.2)-(2.3):
\[ H(\omega) A_{\pm}(\omega) = q^{\frac{N}{2}} \partial^i [H(\omega/q) - q^{\frac{N}{2}} \mu \alpha(\omega/q)] \]
\[
+ x^i q^2 H(\omega/q) \alpha(\omega/q) - q^{\frac{N}{2}} \mu \omega^2 q^{-2} \] .
(3.2)

We require that the r.h.s. is also proportional to the operator \( A_{\pm}(\omega) \). Thus \( \alpha(\omega/q) \) is defined by the condition
\[
[H(\omega/q) - q^{\frac{N}{2}} \mu \alpha(\omega/q)] \alpha(\omega/q) = [q^2 H(\omega/q) \alpha(\omega/q) - q^{\frac{N}{2}} \mu \omega^2 q^{-2}] .
(3.3)
\]

Since the operator \( \alpha(\omega/q) \) commutes with the \( H(\omega/q) \), the above equation leads to the simple quadratic equation for \( \alpha(\omega/q) \)
\[
[\alpha(\omega/q)]^2 - \frac{(1 - q^2)}{q^{\frac{N}{2}} \mu} H(\omega/q) \alpha(\omega/q) - \frac{\omega^2}{q^2} = 0 ,
(3.4)
\]
the solution of which is given by eq.(2.5). Then from eq.(3.2) we get
\[
H(\omega) A_{\pm}(\omega) = A_{\pm}(\omega) [H(\omega/q) - q^{\frac{N}{2}} \mu \alpha(\omega/q)] .
(3.5)
\]

This is not a commutation relation yet since the argument of the Hamiltonian is shifted. In order to obtain eq.(2.7) we still have to improve our operator such that the argument of the Hamiltonian remains also unchanged. This can be achieved by introducing the shift operator \( \lambda \), the action of which is defined as
\[
\lambda x = qx \lambda ,
(3.6)
\]
\[
\lambda \partial = q^{-1} \partial \lambda .
(3.7)
\]

Consequently we get
\[
\lambda^{-\frac{1}{2}} H(\omega) = q H(\omega/q) \lambda^{-\frac{1}{2}} ,
(3.8)
\]
and
\[
\lambda^{-\frac{1}{2}} \alpha_{\pm}(\omega) = q \alpha_{\pm}(\omega/q) \lambda^{-\frac{1}{2}} .
(3.9)
\]

Note that the shift operator \( \lambda \) relates to the algebra element \( \Lambda \) introduced in ref.[15] by
\[
\lambda^2 = \Lambda = 1 + (q^2 - 1)(x \cdot \partial) + \frac{(q^2 - 1)^2}{\mu^2 q^{N-2}} (x \cdot x) \Delta .
(3.10)
\]

Thus we define improved operators by including the operator \( \lambda \) as
\[
a_{\pm} = [q^{\frac{N}{2}} \partial^i + x^i \alpha_{\pm}(q^{-1}\omega)] \lambda^{-\frac{1}{2}} .
(3.11)
\]

Then using eqs.(2.2)-(2.3) we get
\[
H(\omega) a_{\pm} = q^{-1} a_{\pm} H(\omega) - q^{\frac{N}{2}} - 1 \mu a_{\pm} \alpha_{\pm}(\omega) .
(3.12)
q.e.d
3.2 Proof of the general form of the energy eigenvalue $E_p$

Eq. (2.7) provides a recursion formula for the energy eigenvalues and the proof is given by induction: $E_0$ is given by acting with the Hamiltonian on the ground state $\Psi_0$, we get $E_0 = \frac{\omega \mu}{q^{1 - N/2}} \left[ \frac{N}{2} \right]$. Acting with $H(\omega) a_+^i$ onto some eigenfunction $|\Psi_p\rangle$ we obtain according to eq. (2.7)

$$H(\omega) a_+^i |\Psi_p\rangle = a_+^i q^{-1} \left[ E_p - \mu \alpha_+ (\omega) q^N \right] |\Psi_p\rangle .$$

(3.13)

Suppose

$$E_p = q^{\frac{N}{2} - 1} \omega \mu \left[ \frac{N}{2} + p \right] ,$$

(3.14)

then

$$\alpha_{\pm}(\omega) |\Psi_p\rangle = \frac{1}{2} [KE_p \pm \sqrt{K^2 [E_p]^2 + 4 \omega^2}] |\Psi_p\rangle$$

$$= \begin{cases} 
q^{p - \frac{N}{2}} \omega |\Psi_p\rangle & \text{for } \alpha_+ \\
-q^{p + \frac{N}{2}} \omega |\Psi_p\rangle & \text{for } \alpha_-. 
\end{cases}$$

(3.15)

Substituting this into eq. (3.13) we obtain

$$H(\omega) a_+^i |\Psi_p\rangle = a_+^i q^{-1} \left[ E_p + q^{N + p} \mu \omega \right] |\Psi_p\rangle$$

$$= a_+^i \omega \mu \frac{1}{q^{1 - \frac{N}{2}}} \left[ \frac{N}{2} + p + 1 \right] |\Psi_p\rangle$$

$$= E_{p+1} a_+^i |\Psi_p\rangle ,$$

(3.16)

thus

$$E_{p+1} = \omega \mu \frac{1}{q^{1 - \frac{N}{2}}} \left[ \frac{N}{2} + p + 1 \right] .$$

(3.17)

Therefore the energy eigenvalue of the $p$-th level is given by

$$E_p = \frac{\omega \mu (1 - q^{N + 2p})}{q^{N/2 + p} (1 - q^2)} = \frac{\omega \mu}{q^{1/2}} \left[ \frac{N}{2} + p \right] .$$

(3.18)

which is the result stated in Proposition B.

d

3.3 Proof of Proposition C

In Proposition C the commutation relation of $a_\pm$ and $\alpha_\pm$ is stated, the proof of which is as follows:

Proof :
\[ \alpha_\pm(\omega) a_+ = \frac{1}{2} [KH(\omega) \pm \sqrt{K^2[H(\omega)]^2 + 4\omega^2}] a_+ \]

\[ = a_+ \frac{1}{2} [K[q^{-1}H(\omega) - q^{\frac{N}{2}} - \mu\alpha_+ + \omega]] \]

\[ \pm \sqrt{K^2[q^{-1}H(\omega) - q^{\frac{N}{2}} - \mu\alpha_+ + \omega]^2 + 4\omega^2} . \tag{3.19} \]

The expression under the square root can be rewritten in a more convenient way as

\[ K^2[q^{-1}H(\omega) - q^{\frac{N}{2}} - \mu\alpha_+]^2 + 4\omega^2 = q^{-2}[K^2H^2 + (-2q^{-2}(1 - q^2)KH + q^{-2}(1 - q^2)^2\alpha_+ - 4\alpha_-)\alpha_+] \]

\[ = q^{-2}[KH - (1 + q^2)\alpha_+]^2 , \tag{3.20} \]

where we have used that \( \omega^2 = -\alpha_+\alpha_- \). Thus

\[ \sqrt{K^2[q^{-1}H(\omega) - q^{\frac{N}{2}} - \mu\alpha_+]^2 + 4\omega^2} = q^{-1}[(1 + q^2)\alpha_+ - KH] , \tag{3.21} \]

where, in order to determine the overall sign we have required that for the case \( q = 1 \) (no deformation) the value of the square root be \( \sqrt{} = +\omega \). Correspondingly for the case of \( \alpha_- \) we obtain

\[ \sqrt{K^2[q^{-1}H(\omega) - q^{\frac{N}{2}} - \mu\alpha_- + \omega]^2 + 4\omega^2} = q^{-1}[(1 + q^2)\alpha_- - KH] , \tag{3.22} \]

where we again require that \( \sqrt{} = +\omega \) for \( q = 1 \). Therefore we obtain the following commutation relations

\[ \alpha_\pm(\omega) a_+ = a_+ \frac{1}{2} [K[q^{-1}H - q^{\frac{N}{2}} - \mu\alpha_+] \pm q^{-1}[(1 + q^2)\alpha_+ - KH]] \]

\[ = \begin{cases} a_+ \frac{1}{2} [-Kq^{\frac{N}{2}} + q^{-1}(1 + q^2)\alpha_+ + q^{-1}a_+ + \frac{1}{2}2KH - [q^{\frac{N}{2}} - \mu\alpha_+] \alpha_+] \\ \end{cases} = q^{\pm \frac{1}{2}} a_+ \alpha_\pm , \tag{3.23} \]

\[ \alpha_\pm(\omega) a_- = a_- \frac{1}{2} [K[q^{-1}H - q^{\frac{N}{2}} - \mu\alpha_-] \pm q^{-1}KH - (1 + q^2)\alpha_-] \]

\[ = \begin{cases} a_- \frac{1}{2} [-Kq^{\frac{N}{2}} - \mu\alpha_- + q^{-1}(1 + q^2)\alpha_- + q^{-1}a_+ + \frac{1}{2}2KH - [q^{\frac{N}{2}} - \mu\alpha_-] \alpha_-] \\ \end{cases} = q^{\pm \frac{1}{2}} a_- \alpha_\pm . \tag{3.24} \]

\[ q.e.d \]

4 Operator relations

It is interesting to ask how far we can make the analogy of the operator algebra using the creation-annihilation operator. We investigate here further properties of the creation-annihilation operator
given in eq. (2.3) and give some results concerning the operator formalism of the \( q \)-deformed Schrödinger equation (1.11).

First we give an alternative representation of the creation-annihilation operator. When we defined the creation-annihilation operator we have taken the ansatz (3.1) where the operator \( \alpha \) is on the right hand side of the coordinate \( x^i \). Actually we can find the solution with the ansatz where the operator \( \alpha' \) is on the left hand side of the coordinate \( x^i \): \( b' = \lambda^{-\frac{i}{2}} (\partial^i + \alpha' x^i) \). The derivation of the operator \( \alpha' \) is analogous to the one given in section 3.1. The result is

**Theorem D**: The creation-annihilation operator can be also represented by

\[
b_{\pm} = \lambda^{-\frac{i}{2}} [q^{\frac{N}{2}} \partial^i + \alpha_{\pm}(q\omega) x^i], \tag{4.1}\]

where the operator \( \alpha(\omega) \) is given by eq. (2.3). These two representations satisfy the identity

\[
\frac{1}{\sqrt{K^2|H(\omega)|^2 + 4\omega^2}} b_{\pm} = q^{\frac{N}{2}} a_{\pm} \frac{1}{\sqrt{K^2|H(\omega)|^2 + 4\omega^2}}, \tag{4.2}
\]

**Proof**: One can derive \( b_{\pm} \) directly as performed for \( a_{\pm} \) and show that both operators eq. (2.3) and eq. (1.11) give the same energy eigenvalues of the states. Here, instead of repeating these calculations as performed in section 3 we give the proof of the relation (4.2), which also proves the above statements.

First we prove the commutator of \( \alpha \) with \( x \), which is given in a convenient form by including the shift operator \( \lambda \) as

\[
\alpha_{\pm}(x^i \lambda^{-\frac{i}{2}}) - \frac{1}{q}(x^i \lambda^{-\frac{i}{2}}) \alpha_{\pm} = \pm(q^2 - 1) a_{\pm} \frac{\alpha_{\pm}(\omega)}{\sqrt{K^2|H(\omega)|^2 + 4\omega^2}} \]

\[
= \pm(q^2 - 1) a_{\pm} \frac{\alpha_{\pm}(\omega)}{(\alpha_+(\omega) - \alpha_-(\omega))}. \tag{4.3}\]

Substituting

\[
a_{\pm} = [q^{\frac{N}{2}} \partial^i + x^i \alpha_{\pm}(q^{-1}\omega)] \lambda^{-\frac{i}{2}}, \tag{4.4}\]

into Proposition C we get

\[
\alpha_{\pm}(\omega)[q^{\frac{N}{2}} \partial^i + x^i \alpha_{\pm}(q^{-1}\omega)] \lambda^{-\frac{i}{2}} = q^{\pm 1}[q^{\frac{N}{2}} \partial^i + x^i \alpha_{\pm}(q^{-1}\omega)] \lambda^{-\frac{i}{2}} \alpha_{\pm}(\omega), \tag{4.5}\]

\[
\alpha_{\pm}(\omega)[q^{\frac{N}{2}} \partial^i + x^i \alpha_{-}(q^{-1}\omega)] \lambda^{-\frac{i}{2}} = q^{-1}[q^{\frac{N}{2}} \partial^i + x^i \alpha_{-}(q^{-1}\omega)] \lambda^{-\frac{i}{2}} \alpha_{\pm}(\omega). \tag{4.6}\]

Subtracting the second equation from the first yields

\[
\alpha_{\pm}(\omega)x^i \lambda^{-\frac{i}{2}} [\alpha_+(\omega) - \alpha_-(\omega)] = \pm q(q - q^{-1}) a_{\pm} \alpha_{\pm}(\omega) + q^{-1} x^i \lambda^{-\frac{i}{2}} [\alpha_+(\omega) - \alpha_-(\omega)] a_{\pm}(\omega). \tag{4.7}\]

Dividing by the factor \((\alpha_+ - \alpha_-)\) we get the relation (4.3)

With this results we obtain the relation between the \( a_{\pm} \) and \( b_{\pm} \) as

\[
b_{\pm} = q^{\frac{N}{2}} [a_{\pm} \pm q(q - q^{-1}) a_{\pm} \frac{\alpha_{\pm}(\omega)}{(\alpha_+(\omega) - \alpha_-(\omega))}] \]

\[
= q^{\frac{N}{2}} a_{\pm} \frac{q^{\pm 1} \alpha_+(\omega) - q^{-1} \alpha_-(\omega)}{(\alpha_+(\omega) - \alpha_-(\omega))}. \tag{4.8}\]
Thus we get
\[ b_\pm = q^{\frac{1}{2}}[\alpha_+(\omega) - \alpha_-(\omega)]a_\pm \frac{1}{[\alpha_+(\omega) - \alpha_-(\omega)]}, \] (4.9)
Dividing by \((\alpha_+ - \alpha_-)\) from the left we get the equation (4.12) of Theorem D.

The operator \(b_\pm^i\) is important when we investigate the transformation rule of the creation-annihilation operator under the \(*\)-conjugation. This problem is beyond the scope of this paper and will be discussed elsewhere. We show in the rest of this section how the Hamiltonian can be represented in terms of the creation-annihilation operator defined in Theorem A.

Analogously to the non-deformed case the operators \((a_+ \cdot a_-)\) and \((a_- \cdot a_+)\) are singlet representations of the quantum group. Thus we expect that they can be given as functions of the Hamiltonian. A straightforward computation leads to the following operator identity
\[(a_+ \cdot a_-) = q^{-\frac{1}{2}}\lambda^{-1}\tilde{B}[-H(\omega) + Q_N q^N \alpha_-(\omega)], \quad (4.10)\]
\[(a_- \cdot a_+) = q^{-\frac{1}{2}}\lambda^{-1}\tilde{B}[-H(\omega) + Q_N q^N \alpha_+(\omega)], \quad (4.11)\]
and the linear combination of the above expressions gives the following simple relation
\[(a_+ \cdot a_-) + (a_- \cdot a_+) = -q^{-\frac{1}{2}}(1 + q^N)\lambda^{-1}\tilde{B}H(\omega), \quad (4.12)\]
where
\[\tilde{B} = 1 + \frac{q^2 - 1}{\mu}(x \cdot \partial). \quad (4.13)\]

The extra operator \(\lambda^{-1}\tilde{B}\) commutes with the Hamiltonian:
\[\{\lambda^{-1}\tilde{B}, H(\omega)\} = 0.\] (4.14)
Actually the eigenfunctions also form a diagonal basis with respect to this operator, i.e. when acting with the operator \(\lambda^{-1}\tilde{B}\) onto the wave function eq.(2.26) we obtain
\[\lambda^{-1}\tilde{B}\Psi_p = N_p \Psi_p, \] (4.15)
where \(N_p\) is a certain \(q\)-number. However, this eigenvalue \(N_p\) is not independent of the states. By using the explicit form of the wave function \(\Psi_{p+2r} = S_{pfr}(x^2)\) derived in section 2 we can determine the eigenvalue \(N_p\). It is given by
\[\lambda^{-1}\tilde{B}\Psi_{p+2r} = \frac{q}{\mu q^N}(q^{-\frac{N}{2}} - p^2 + q^\frac{N}{2} + p^{-1})\Psi_{p+2r}. \quad (4.16)\]
From this we see that the eigenvalue of \(\lambda^{-1}\tilde{B}\) depends only on the tensor structure defined by \(p\) and is nonzero. To get the Hamiltonian, we have to divide out this operator.

Finally we give the commutation relation of \(a_+^i\) and \(a_-^j\):
\[\mathcal{P}_A(a_+^i a_-^j) = \lambda^{-1}q^{\frac{N+1}{2}}\mathcal{P}_A(\partial^i x^j \alpha_-(\omega) + x^i \partial^j \alpha_+(\omega))\]
\[= \lambda^{-1}q^{\frac{N+1}{2}}\mathcal{P}_A(x^i \partial^j)[\alpha_+(\omega) - q^2 \alpha_-(\omega)]. \quad (4.17)\]
The operator \(\mathcal{P}_A(x^i \partial^j)\) appearing in the r.h.s. is proportional to the angular momentum in the limit \(q \rightarrow 1\). (Some properties of the operator \(\mathcal{P}_A(x^i \partial^j)\) have been discussed in ref. [12].) This means that the commutator of the creation and annihilation operator contains the angular momentum.
5 Discussion and conclusion

In this paper we have shown two different methods to construct the solution of the $q$-deformed Schrödinger equation with $Fun_q(SO(N))$ symmetry. It is proven that a creation-annihilation operator exists which generate all excitation levels. We also gave the explicit solution in terms of the $q$-polynomial and $q$-exponential functions by solving the associated $q$-difference equation.

Concerning the Hamiltonian it is not completely straightforward to express it in terms of the creation-annihilation operators as we see from the result of eq.(4.12). One way to investigate such a property is to take the operators $a_{\pm}$ as the fundamental quantities of the system, and consider the ‘improved’ Hamiltonian which is directly proportional to $((a_+ \cdot a_-) + (a_- \cdot a_+))$ on operator level. For this one may still consider the rescaling of the creation-annihilation operator by the function of the Hamiltonian as is suggested by the theorem D. From eq.(4.2), we see that when we define the creation-annihilation operator with the factor $\frac{1}{\sqrt{K^2 |H(\omega)|^2 + 4\omega^2}}$ appropriately, the relation between the improved $a_{\pm}$ and $b_{\pm}$ is simplified. Such an improved creation-annihilation operator also seems to simplify the $*$-conjugation of the operators. This problem is still under investigation.

Concerning the conjugation property, there is the following problem which has to be solved. Under the simple $*$-conjugation which is defined by [15]

$$x^* = x_i ,$$

$$\bar{\partial}^* = -q^{-N} \bar{\partial}_i ,$$

the Hamiltonian is transformed as

$$H^* = -q^{-N} (\bar{\partial} \cdot \bar{\partial}) + \omega^2 (x \cdot x) ,$$

If we apply the reality condition proposed in ref.[13], the above equation gives

$$H^* = -q^{-2} \frac{1}{\Lambda} (\partial \cdot \partial) + \omega^2 (x \cdot x) ,$$

Thus under the $*$-conjugation with applying the reality condition proposed in ref.[15], the Hamiltonian is not an hermitian operator.

On the other hand considering the various possible Laplacians and its operations on the $q$-exponential function, we find that $(\partial \cdot \partial)$ and $(\bar{\partial} \cdot \bar{\partial})$ have simple relations which can be identified with the eigenfunction equation. We listed the four types of Laplacians which may be considered as the eigenfunction equation in the appendix. These four Laplacians are all non-hermitian under the $*$-conjugation with the reality condition given in ref.[13]. With our present knowledge, it is not possible to construct the eigenfunctions for a Laplacian which is hermitian under the conjugation discussed above.

Recall that the hermiticity condition is related with the definition of the norm. On the other hand the definition of the norm is also related with the definition of the integration on the quantum space. In principle the integration can be defined algebraically by requiring the property of the partial integration.[14, 12]. However, this does not fix the norm and we may still consider the possibility of modifying the norm and the definition of the hermitian conjugation. One possibility is discussed in ref.[12]. We shall report elsewhere on this problem, including the integration on the quantum space.

Finally we would like to remark that this formulation of the N-dimensional harmonic oscillator on quantum space may be applied to the formulation of the $q$-deformed string theory. See ref.[18] and references therein.
Acknowledgement

One of the authors (U.C.) would like to thank the Tohoku Kaihatsu Kinen foundation for financial support.

Appendix

In this appendix we collected some of the basic properties of the $q$-exponential functions. In order to simplify the expressions we introduce the following abbreviation $\text{Exp}_q^2(\alpha)$. Note that for the summary given in section 1 we followed the conventions of our previous paper [10]. The new notations and their relations to the previous ones are:

The definition of the $q$-exponential function is

$$\text{Exp}_q^2(\alpha) = \exp_q^2\left(-\frac{\alpha x \cdot x}{\mu}\right) = \sum_{(n)_x} \frac{1}{(n)_x!} \left(-\frac{\alpha (x \cdot x)}{\mu}\right)^n,$$  \hspace{1cm} (A.1)

$$\text{Exp}_{q^{-2}}(\alpha) = \exp_{q^{-2}}\left(-\frac{\alpha x \cdot x}{\bar{\mu}}\right) = \sum_{(n)_{x^{-2}}} \frac{1}{(n)_{x^{-2}}!} \left(-\frac{\alpha (x \cdot x)}{\bar{\mu}}\right)^n,$$  \hspace{1cm} (A.2)

where

$$(n)_x = \frac{1-x^n}{1-x}, \text{ and } \bar{\mu} = q^{-2}\mu.$$  \hspace{1cm} (A.3)

It is straightforward to derive the following identities

$$\text{Exp}_q^2(\alpha) - \text{Exp}_q^2(q^2\alpha) = -(1-q^2)\frac{\alpha (x \cdot x)}{\mu} \text{Exp}_q^2(\alpha),$$  \hspace{1cm} (A.4)

$$\text{Exp}_{q^{-2}}(\alpha) - \text{Exp}_{q^{-2}}(q^{-2}\alpha) = -(1-q^{-2})\frac{\alpha (x \cdot x)}{\bar{\mu}} \text{Exp}_{q^{-2}}(\alpha).$$  \hspace{1cm} (A.5)

Using these identities we can easily compute the action of the derivative $D_{x^2}$ onto the $q$-exponential function. Thus for $D_{x^2} f(x^2)$ with $f(x^2)$ given in eq.(2.34) we obtain

$$D_{x^2} \sum b_s \text{Exp}_q^2(q^{-2s}\alpha) = -\frac{\alpha}{\mu} \sum b_s q^{-2s} \text{Exp}_q^2(q^{-2s}\alpha).$$  \hspace{1cm} (A.6)

For the discussion on the $\star$-conjugation properties it is also necessary to consider the action of the derivative $\partial^i$ and of $\bar{\partial}^i$ onto the $q$-exponential function where the $\star$-conjugation is defined as eqs.(5.1) and (5.2). The definition of the action of $\partial^i$ has been given in eq.(1.3). For the derivative $\bar{\partial}^i$ we define

$$\bar{\partial}^i x^j = C^{ij} + q^{-1} R^{ijk}_l x^k \bar{\partial}^l.$$  \hspace{1cm} (A.7)

Eq.(1.3) and eq.(A.7) are conjugate to each other under the $\star$-operation given in eqs.(5.1) and (5.2). Then the action of the derivatives onto $\text{Exp}_q^2(\alpha)$ yield

$$\partial^i \text{Exp}_q^2(\alpha) = -\alpha x^i \text{Exp}_q^2(\alpha) + GT,$$  \hspace{1cm} (A.8)

$$\bar{\partial}^i \text{Exp}_q^2(\alpha) = -\alpha x^i \text{Exp}_q^2(q^{-2}\alpha) + GT,$$  \hspace{1cm} (A.9)

$$\partial^i \text{Exp}_{q^{-2}}(\alpha) = -\alpha x^i \text{Exp}_{q^{-2}}(q^2\alpha) + GT,$$  \hspace{1cm} (A.10)

$$\bar{\partial}^i \text{Exp}_{q^{-2}}(\alpha) = -\alpha x^i \text{Exp}_{q^{-2}}(\alpha) + GT.$$  \hspace{1cm} (A.11)
The abbreviation GT simply means "go-through term"; in the case of eq.(A.6) for example GT = \[ \text{Exp}_{q^2}(q^2\alpha)\partial^2 \].

\[
(\partial \cdot \partial)\text{Exp}_{q^2}(\alpha) = -\alpha Q_N \text{Exp}_{q^2}(\alpha) + q^N \alpha^2 (x \cdot x) \text{Exp}_{q^2}(\alpha) + GT, \quad (A.12)
\]

\[
(\partial \cdot \bar{\partial})\text{Exp}_{q^2}(\alpha) = -\alpha Q_N \text{Exp}_{q^2}(\alpha) + q^{-N} \alpha^2 (x \cdot x) \text{Exp}_{q^2}(\alpha) + GT, \quad (A.13)
\]

\[
(\partial \cdot \partial)\text{Exp}_{q^2}(\alpha) = -\alpha Q_N \text{Exp}_{q^2}(\alpha) + q^{-2} \alpha^2 (x \cdot x) \text{Exp}_{q^2}(\alpha) + GT, \quad (A.14)
\]

\[
(\bar{\partial} \cdot \bar{\partial})\text{Exp}_{q^2}(\alpha) = -\alpha Q_N \text{Exp}_{q^2}(\alpha) + q^{-2} \alpha^2 (x \cdot x) \text{Exp}_{q^2}(\alpha) + GT, \quad (A.15)
\]

\[
(\partial \cdot \partial)\text{Exp}_{q^{-2}}(\alpha) = -\alpha Q_N \text{Exp}_{q^{-2}}(\alpha) + q^{N+2} \alpha^2 (x \cdot x) \text{Exp}_{q^{-2}}(\alpha) + GT, \quad (A.16)
\]

\[
(\bar{\partial} \cdot \bar{\partial})\text{Exp}_{q^{-2}}(\alpha) = -\alpha Q_N \text{Exp}_{q^{-2}}(\alpha) + q^{2-N} \alpha^2 (x \cdot x) \text{Exp}_{q^{-2}}(\alpha) + GT, \quad (A.17)
\]

\[
(\partial \cdot \bar{\partial})\text{Exp}_{q^{-2}}(\alpha) = -\alpha Q_N \text{Exp}_{q^{-2}}(\alpha) + q^{N} \alpha^2 (x \cdot x) \text{Exp}_{q^{-2}}(\alpha) + GT, \quad (A.18)
\]

\[
(\bar{\partial} \cdot \bar{\partial})\text{Exp}_{q^{-2}}(\alpha) = -\alpha Q_N \text{Exp}_{q^{-2}}(\alpha) + q^{-N} \alpha^2 (x \cdot x) \text{Exp}_{q^{-2}}(\alpha) + GT. \quad (A.19)
\]

Thus we have four types of the Laplacians which have the eigenfunction given by the q-exponential functions:

\[
[-q^{-N}(\partial \cdot \partial) + \alpha^2 (x \cdot x)]\text{Exp}_{q^2}(\alpha) = q^{-N} \alpha Q_N \text{Exp}_{q^2}(\alpha), \quad (A.20)
\]

\[
[-q^{-N}(\bar{\partial} \cdot \bar{\partial}) + \alpha^2 (x \cdot x)]\text{Exp}_{q^{-2}}(\alpha) = q^{N} \alpha Q_N \text{Exp}_{q^{-2}}(\alpha), \quad (A.21)
\]

\[
[-q^{-N}\lambda(\partial \cdot \partial) + \alpha^2 (x \cdot x)]\text{Exp}_{q^2}(\alpha) = q^{-N} \alpha Q_N \text{Exp}_{q^2}(\alpha), \quad (A.22)
\]

\[
[-q^{-N}\lambda^{-1}(\bar{\partial} \cdot \bar{\partial}) + \alpha^2 (x \cdot x)]\text{Exp}_{q^{-2}}(\alpha) = q^{N} \alpha Q_N \text{Exp}_{q^{-2}}(\alpha). \quad (A.23)
\]

References

[1] P. Podleś, "Quantization enforces interaction. Quantum mechanics of two particles on a quantum sphere.", Proceedings of the RIMS Research Project 91 on Infinite Analysis, World Scientific (1991).

[2] P. Podleś and S.L. Woronowicz, Commun. Math. Phys. 130, (1990)381.

[3] U. Carow-Watamura, M. Schlieker, M. Scholl and S. Watamura, Z. Phys. C, Particles and Fields 49 (1991) 439; Int. Jour. Mod. Phys. A6 (1991) 3081.

[4] J. Lukierski, A. Nowicki, H. Ruegg and V.N. Tolstoy, Phys. Lett. 264 (1991) 331.

[5] O. Ogievetsky, W.B. Schmidke, J. Wess and B. Zumino, Commun. Math. Phys. 150 (1992) 495.

[6] M. Schlieker, W. Weich, and R. Weixler, Z. Phys. C. 53 (1992) 79.

[7] S. Majid, Jour. Math. Phys. 34 (1993) 2045.
[8] W. Pusz and S.L. Woronowicz, Rep.Math.Phys. 27 (1989)231.

[9] J. Wess and B. Zumino, Nucl.Phys.18 B(Proc.Suppl.)(1990)302.

[10] U. Carow-Watamura, M. Schlieker and S. Watamura, Z. Phys. C, Particles and Fields 49, 439-446(1991).

[11] G. Fiore, Inter.Jour.Mod.Phys.A7, (1992)7597.

[12] G. Fiore, SISSA preprint 102/92/EP, revised Feb. 1993.

[13] U. Carow-Watamura, S. Watamura, M. Schlieker, A. Hebecker and W. Weich, Czech. Journ. Phys.42, (1992)1297.

[14] A. Hebecker and W. Weich, Lett. Math. Phys. 26, (1992) 245.

[15] O. Ogievetsky and B. Zumino, preprint MIP-Ph/92-25.

[16] N.Yu. Reshetikhin, L.A. Takhtadzhyan and L.D. Faddeev, Algebra and Analysis 1 (1989) 178.

[17] H. Exton, Q-Hypergeometric Functions and Applications, Chichester New York, Hoorwood and Wiley (1983).

[18] M. Chaichian, J.F. Gomes and P. Kulish, Phys.Lett.B311 (1993) 93.