Noncommutative differential forms on the kappa-deformed space

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Abstract

We construct a differential algebra of forms on the kappa-deformed space. For a given realization of noncommutative coordinates as formal power series in the Weyl algebra we find an infinite family of one-forms and nilpotent exterior derivatives. We derive explicit expressions for the exterior derivative and one-forms in covariant and noncovariant realizations. We also introduce higher order forms and show that the exterior derivative satisfies the graded Leibniz rule. The differential forms are generally not graded commutative, but they satisfy the graded Jacobi identity. We also consider the star-product of classical differential forms. The star-product is well defined if the commutator between the noncommutative coordinates and one-forms is closed in the space of one-forms alone. In addition, we show that in certain realizations the exterior derivative acting on the star-product satisfies the undeformed Leibniz rule.

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1. Introduction

Recent years have witnessed a growing interest in the formulation of physical theories on noncommutative (NC) spaces. The structure of NC spaces and their physical implications was studied in \cite{1–7}. Such spaces have roots in quantum mechanics where the canonical phase space becomes noncommutative (see \cite{8} for a historical treatment and references therein). Classification of the NC spaces and investigation of their properties, in particular the development of a general theory suitable for physical applications, is an important problem. In this paper, we investigate differential calculus in the Euclidean kappa-deformed space. The kappa-space is a mild deformation of the Euclidean space whose coordinates $\hat{x}_{\mu}, \mu = 1, 2, \ldots, n$, satisfy a Lie algebra type commutation relations. The commutation relations for $\hat{x}_{\mu}$ depend on a deformation vector $a \in \mathbb{R}^n$ which is on a very small length
scale and yields the undeformed space when \( a \to 0 \). The kappa-space was studied by different groups, from both the mathematical and physical points of view [9–33]. It provides a framework for doubly special relativity [18, 19], and it has applications in quantum gravity [34] and quantum field theory [35, 36].

A crucial tool in the development of a physical theory is differential calculus. There have been several attempts to develop differential calculus in the kappa-deformed space [14, 25]. For a general associative algebra Landi gave a construction of a differential algebra of forms in [37]. In this work, we present a construction of differential forms and exterior derivative in the kappa-deformed space using realizations of the NC coordinates \( \hat{x}_\mu \) as formal power series in the Weyl algebra. Our approach is based on the methods developed for algebras of deformed oscillators and the corresponding creation and annihilation operators [38–47]. The realizations of the NC coordinates \( \hat{x}_\mu \) in various orderings have been found in [26, 28]. The realization of a general Lie algebra type NC space in the symmetric Weyl ordering has been given in [48].

The outline of the paper is as follows. In section 2, we present a novel construction of a differential algebra of forms on the kappa-deformed space. The exterior derivative \( \hat{d} \) and one-forms \( \xi_\mu \) are defined as formal power series in the Lie superalgebra generated by commutative coordinates \( x_\mu \), derivatives \( \partial_\mu \) and ordinary one-forms \( dx_\mu \). The number of one-forms \( \xi_\mu \) is the same as the number of NC coordinates \( \hat{x}_\mu \), and the results are valid for a general deformation vector \( a \in \mathbb{R}^n \). In the present work, we do not require compatibility of the differential structure with a kappa-deformed symmetry. This distinguishes our approach from [14] where compatibility of the differential calculus with the kappa-deformed symmetry group was considered. This compatibility requires that in addition to \( \xi_\mu \) there is an extra one-form \( \phi \). The realizations of \( \hat{d} \) and \( \xi_\mu \) are related to realizations of \( \hat{x}_\mu \) through a system of partial differential equations. We also define higher order forms and show that \( \hat{d} \) is a nilpotent, commutative, and graded commutative operator which satisfies the graded Leibniz rule. However, the differential forms are generally not graded commutative. In the smooth limit when \( a \to 0 \) our theory reduces to classical results. In section 3, we analyze the exterior derivative and one-forms in covariant realizations of the kappa-deformed space. We show that the algebra generated by \( \hat{x}_\mu \) and \( \xi_\mu \) generally does not close under the commutator bracket since \([\xi_\mu, \hat{x}_\nu]\) may involve an infinite series in derivatives \( \partial_\mu \). We have derived a condition for the commutator \([\xi_\mu, \hat{x}_\nu]\) to be closed and found realizations in which the condition holds. A similar analysis was carried out by Dimitrijević et al. in [25], but our results are more general and in certain aspects different. Section 4 deals with the differential algebra of forms in noncovariant realizations. We introduce a general ansatz for the exterior derivative and find the corresponding one-forms in the left, right and symmetric left–right realization. In these realizations the commutator \([\xi_\mu, \hat{x}_\nu]\) is always closed in the space of one-forms \( \xi_\mu \) alone. In section 5, we present a novel construction of the star-product of (classical) differential forms. The star-product depends on realizations of \( \hat{x}_\mu \) and is well defined if the commutator \([\xi_\mu, \hat{x}_\nu]\) is closed in the space of one-forms \( \xi_\mu \) alone. We show that for differential forms with constant coefficients the star-product is undeformed and graded commutative. However, this property does not hold for arbitrary forms. Also, we consider the induced exterior derivative acting on the star-product of differential forms. A short conclusion is given in section 6.

2. Differential forms

In this section, we present a general construction of a differential algebra of forms in the Euclidean kappa-deformed space. This construction is based on realizations of the NC coordinates \( \hat{x}_\mu \) as formal power series in the Weyl algebra introduced in [26, 28]. We find that
for a given realization of $\hat{x}_\mu$ there is an infinite family of exterior derivatives $\hat{d}$ and one-forms $\hat{\xi}_\mu$ where $\hat{\xi}_\mu$ are obtained by the action of $\hat{d}$ on $\hat{x}_\mu$. This infinite family includes two canonical types of $\hat{d}$ and $\hat{\xi}_\mu$ whose realizations are studied in detail in the following sections.

The $n$-dimensional kappa-deformed space is a noncommutative space of Lie algebra type with generators $\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n$ satisfying the commutation relations

$$[\hat{x}_\mu, \hat{x}_\nu] = i (a_\mu \hat{x}_\nu - a_\nu \hat{x}_\mu), \quad a_\mu \in \mathbb{R}. \quad (1)$$

The vector $a \in \mathbb{R}^n$ describes the deformation of the $n$-dimensional Euclidean space. The Lie algebra satisfying (1) will be denoted by $g$. The structure constants of $g$ are given by

$$C_{\mu\nu\lambda} = a_\mu \delta_{\nu\lambda} - a_\nu \delta_{\mu\lambda}. \quad (2)$$

Our construction of the differential calculus uses realizations of $\hat{x}_\mu$ as formal power series in the deformation parameter $a$ with coefficients in the Weyl algebra. The Weyl algebra is generated by the operators $x_\mu$ and $\partial_\mu$, $\mu = 1, 2, \ldots, n$, satisfying $[x_\mu, x_\nu] = [\partial_\mu, \partial_\nu] = 0$ and $[\partial_\mu, x_\nu] = \delta_{\mu\nu}$. It has been shown in [26, 28] that there exist infinitely many realizations of $\hat{x}_\mu$ of the form

$$\hat{x}_\mu = \sum_a x_\mu \phi_{a\mu}(\partial), \quad (3)$$

where $\phi_{a\mu}$ is a formal power series

$$\phi_{a\mu}(\partial) = \delta_{a\mu} + \sum_{|k| \geq 1} c_k a^{|k|} \partial^k. \quad (4)$$

We denote $\partial^k = \partial_1^{k_1} \partial_2^{k_2} \cdots \partial_n^{k_n}$ where $k$ is a multi-index of length $|k| = \sum_k k_\mu$. In the limit as $a \to 0$ we have $\phi_{a\mu} \to \delta_{a\mu}$, whence $\hat{x}_\mu$ become the commutative coordinates $x_\mu$. A representation (3) of the NC coordinates $\hat{x}_\mu$ will be called a $\phi$-realization. The NC coordinates $\hat{x}_\mu$ and derivatives $\partial_\mu$ generate a deformed Heisenberg algebra satisfying

$$[\partial_\mu, \hat{x}_\nu] = \phi_{\mu\nu}(\partial). \quad (5)$$

We will assume that the matrix $[\phi_{\mu\nu}]$ is invertible, allowing us to express $x_\mu$ as

$$x_\mu = \sum_a \hat{x}_\mu \phi_{a\mu}^{-1}(\partial), \quad (6)$$

where $\phi_{a\mu}^{-1}(\partial)$ is also a formal power series of the type (4). The existence of $\phi_{a\mu}^{-1}$ implies that there is a vector space isomorphism between the symmetric algebra generated by $x_\mu$, $\mu = 1, 2, \ldots, n$, and the enveloping algebra of $g$. This isomorphism will be important in defining the star-product discussed in section 5. With regard to the action of the rotation algebra so$(n)$ the realizations of the kappa-space can be divided into covariant [28] and noncovariant [26]. Both types of realizations will be used in the construction of differential forms in sections 3 and 4.

It is useful to introduce a unital associative algebra $\mathcal{A}$ over $\mathbb{C}$ generated by $x_\mu$, $\partial_\mu$ and ordinary one-forms $dx_\mu$, $1 \leq \mu \leq n$, satisfying the additional relations $[dx_\mu, x_\nu] = [dx_\mu, \partial_\nu] = 0$ and $[dx_\mu, dx_\nu] = 0$ where $[,]$ denotes the anticommutator. A basis for $\mathcal{A}$ consists of the monomials

$$x_1^{\alpha_1} \cdots x_n^{\alpha_n} \partial_1^{\beta_1} \cdots \partial_n^{\beta_n} dx_{x_1^1} \cdots dx_{x_n^1}, \quad (7)$$

where $\alpha_i, \beta_i \in \mathbb{N}_0$ and $1 \leq \sigma_1 < \sigma_2 \cdots < \sigma_p \leq n$ for $p = 1, 2, \ldots, n$. We define a Z2-gradation of $\mathcal{A}$ by $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ where $\mathcal{A}_0$ and $\mathcal{A}_1$ are spanned by the monomials (7) with $p$ even and odd, respectively. The algebra $\mathcal{A}$ is equipped with the graded commutator defined on homogeneous elements by

$$[u, v] = uv - (-1)^{|u||v|} vu, \quad (8)$$

where $|u|$ is the degree of $u$.
where \(|u|\) denotes the degree of \(u\), \(|u| = 0\) or \(|u| = 1\). The commutator (8) makes \(\mathcal{A}\) into a Lie superalgebra, and it satisfies the graded Jacobi identity
\[
(-1)^{|u||w|}[[u, [v, w]]] + (-1)^{|v||u|}[[v, [w, u]]] + (-1)^{|w||v|}[[w, [u, v]]] = 0.
\]
(9)
Recall that in the ordinary Euclidean space the exterior derivative is given by
\[
d = \sum_\alpha d x_\alpha \partial_\alpha.
\]
It is a nilpotent operator, \(d^2 = 0\), satisfying the commutation relation \([d, x_\mu] = d x_\mu\). Our goal is to construct smooth deformations of \(d\) and \(d x_\mu\), denoted by \(\hat{d}\) and \(\xi_\mu, \mu = 1, 2, \ldots, n\), which preserve the basic relation
\[
[d, \hat{x}_\mu] = \xi_\mu.
\]
(10)
Let us assume that \(\hat{d}\) and \(\xi_\mu\) are represented by
\[
\xi_\mu = \sum_\alpha d x_\alpha h_{a\mu}(\partial) \quad \text{and} \quad \hat{d} = \sum_{a, \beta} d x_\alpha \partial_\beta k_{a\beta}(\partial),
\]
(11)
where \(h_{\mu\nu}\) and \(k_{\mu\nu}\) are formal power series of the type (4). The boundary conditions
\[
\lim_{a \to 0} h_{\mu\nu} = \delta_{\mu\nu} \quad \text{and} \quad \lim_{a \to 0} k_{\mu\nu} = \delta_{\mu\nu}
\]
ensure that in the smooth limit \(\xi_\mu \to d x_\mu\) and \(\hat{d} \to d\) as \(a \to 0\). As in the classical case, the deformed one-forms anticommute and the exterior derivative is nilpotent. Indeed,
\[
\{\xi_\mu, \xi_\nu\} = \sum_{\alpha < \beta} \{d x_\alpha, d x_\beta\} (h_{a\mu} h_{\beta\nu} + h_{a\nu} h_{\beta\mu}) = 0,
\]
(12)
\[
\hat{d}^2 = \sum_{\alpha < \beta} \{d x_\alpha, d x_\beta\} \sum_{\mu, \nu} \partial_\mu \partial_\nu k_{a\mu} k_{\beta\nu} = 0,
\]
(13)
since \(\{d x_\alpha, d x_\beta\} = 0\). We assume that the matrix \([h_{\mu\nu}]\) is invertible so that we may express \(d x_\mu\) in terms of \(\xi_\mu\). Using representation (11) one finds that the commutation relation (10) is equivalent to a system of partial differential equations for the unknown functions \(h_{\mu\nu}\) and \(k_{\mu\nu}\):
\[
\sum_\rho \left( k_{a\rho} + \sum_\beta \frac{\partial k_{a\beta}}{\partial \partial_\rho} \partial_\beta \right) \phi_{\rho\mu} = h_{a\mu}.
\]
(14)
This is an underdetermined system of \(n^2\) equations for \(2n^2\) unknown functions. Taking the commutator of \(\hat{d}\) with both sides of the commutation relations (1) and applying the Jacobi identity to the commutator \([\hat{d}, [\hat{x}_\mu, \hat{x}_\nu]]\), we find that \(\hat{x}_\mu\) and \(\xi_\nu\) satisfy the compatibility condition
\[
[\hat{x}_\mu, \xi_\nu] - [\hat{x}_\nu, \xi_\mu] = i (a_{\mu\nu} \xi_\nu - a_{\nu\mu} \xi_\mu).
\]
(15)
Hence, every solution of equation (14) must be compatible with the differential equation implicit in (15). We note that equation (15) implies that since \(a \neq 0\), not all commutators \([\hat{x}_\mu, \xi_\nu]\) can be simultaneously zero.

The condition (15) places constraints on the choice of \(k_{\mu\nu}\) and \(h_{\mu\nu}\). For a given function \(k_{\mu\nu}\) satisfying \(\lim_{a \to 0} k_{\mu\nu} = \delta_{\mu\nu}\), equation (14) uniquely determines \(h_{\mu\nu}\). The boundary conditions imposed on \(\phi_{\rho\mu}\) and \(k_{\mu\nu}\) imply that \(\lim_{a \to 0} h_{\mu\nu} = \delta_{\mu\nu}\) automatically holds. Therefore, starting with the exterior derivative \(\hat{d}\) one readily finds the one-forms \(\xi_\mu\) satisfying equation (10).

However, the converse is not true since one cannot always find \(k_{\mu\nu}\) for an arbitrary choice of \(h_{\mu\nu}\). For example, if \(h_{\mu\nu} = \delta_{\mu\nu}\) then equation (11) implies that \(\xi_\mu\) is the ordinary one-form, \(\xi_\mu = d x_\mu\). In this case \([\hat{x}_\mu, \xi_\nu]\) = 0 for all \(\mu, \nu = 1, 2, \ldots, n\), which contradicts the compatibility condition (15).
Let \( \bar{A} \) denote the formal completion of \( A \). We associate with the exterior derivative \( \hat{d} \) a linear map or action \( \hat{d} : \bar{A} \to \bar{A} \) defined by

\[
\hat{d} \cdot u = [[\hat{d}, u]].
\]

(16)

It follows from equation (10) that \( \hat{d} \cdot \hat{x}_\mu = \xi_\mu \), hence the action of \( \hat{d} \) on the coordinate \( \hat{x}_\mu \) yields the one-form \( \xi_\mu \). The action of \( \hat{d} \) on the product of homogeneous elements \( u, v, \in \bar{A} \) satisfies the graded Leibniz rule

\[
\hat{d} \cdot (uv) = (\hat{d} \cdot u)v + (-1)^{|u|}u(\hat{d} \cdot v).
\]

(17)

For zero-forms \( \hat{f} = \hat{f}(\hat{x}) \) and \( \hat{g} = \hat{g}(\hat{x}) \) this reduces to the undeformed Leibniz rule

\[
\hat{d} \cdot (\hat{f} \hat{g}) = (\hat{d} \cdot \hat{f}) \hat{g} + \hat{f}(\hat{d} \cdot \hat{g}).
\]

(18)

It turns out that it is quite natural to consider the following canonical representation of \( \hat{d} \) and \( \xi_\mu \):

Type I

\[
\hat{d} = \sum \alpha dx_\alpha \partial_\alpha, \quad \xi_\mu = \sum \alpha dx_\alpha \phi_\alpha(\partial).
\]

(19)

Type II

\[
\hat{d} = \sum \alpha \xi_\alpha \partial_\alpha, \quad \xi_\mu = \sum \alpha dx_\alpha h_\alpha(\partial).
\]

(20)

The first type is obtained by choosing \( k_{\mu\nu} = \delta_{\mu\nu} \), in which case equation (14) yields \( h_{\mu\nu} = \phi_{\mu\nu} \). This provides the simplest possible realization of the one-form \( \xi_\mu \). The second type is obtained by demanding that \( k_{\mu\nu} = h_{\mu\nu} \). Then the functions \( h_{\mu\nu} \) satisfy the system of partial differential equations

\[
\sum_\rho \left( h_{\alpha\rho} + \sum_\beta \frac{\partial h_{\alpha\beta}}{\partial \alpha} \partial_\beta \right) \phi_{\rho\mu} = h_{\alpha\mu}
\]

subject to the boundary conditions \( \lim_{a \to 0} h_{\mu\nu} = \delta_{\mu\nu} \). In this case both the exterior derivative \( \hat{d} \) and one-forms \( \xi_\mu \) depend in a very nontrivial manner on the given \( \phi \)-realization. In the following sections, we shall analyze \( \hat{d} \) and \( \xi_\mu \) in covariant and noncovariant realizations found in [26, 28]. Note that the generators \( \hat{x}_\mu, \partial_\mu, 1 \leq \mu \leq n \), form an associative superalgebra which inherits the grading from the superalgebra \( A \). The subalgebra generated by \( \hat{x}_\mu, \partial_\mu, 1 \leq \mu \leq n \) is the deformed Heisenberg algebra (5).

So far we have defined the exterior derivative \( \hat{d} \) and one-forms \( \xi_\mu \) such that \( \hat{d} \cdot \hat{x}_\mu = \xi_\mu \). We would like to extend the above construction to higher order forms so that the action of \( \hat{d} \) on \( k \)-forms yields \((k+1)\)-forms. First, we need to define what is meant by a \( k \)-form for \( k \geq 1 \). A \( k \)-form is a finite linear combination of monomials in \( \hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n \) and \( \hat{\xi}_1, \hat{\xi}_2, \ldots, \hat{\xi}_n \) such that there are precisely \( k \) one-forms \( \xi_\mu \) in each monomial. The one-forms \( \xi_\mu \) may be placed in any order in a given monomial. For example, both \( \hat{\omega}^1 = \hat{x}_\mu \hat{\xi}_\mu \) and \( \hat{\eta}^1 = \hat{\xi}_\mu \hat{x}_\mu \) are one-forms, albeit different. Let \( \hat{\Omega}^k \) denote the space of \( k \)-forms and let \( \hat{\Omega} = \bigoplus_{k \geq 0} \hat{\Omega}^k \). The multiplication in \( \hat{\Omega} \) is simply given by juxtaposition of the elements. This defines a grading on \( \hat{\Omega} \) since \( \hat{\Omega}^k \hat{\Omega}^l \subseteq \hat{\Omega}^{k+l} \). We note that the product of differential forms is not graded commutative in general,

\[
\hat{\omega}^k \hat{\eta}^l \neq (-1)^{kl} \hat{\eta}^l \hat{\omega}^k.
\]

(22)

The product is graded commutative only for constant forms \( \hat{\omega}^k = \xi_\mu \xi_{\mu_2} \cdots \xi_{\mu_k} \) since \( \xi_{\mu_i} \) and \( \xi_{\mu_j} \) anticommute.
Next we show that the exterior derivative $\hat{d}$ maps $\hat{\Omega}^{k}$ into $\hat{\Omega}^{k+1}$ for $k \geq 0$. First, using the Leibniz rule (17) it is easily seen that

$$\hat{d} \cdot \tilde{f}(\hat{x}) \in \hat{\Omega}^{1} \quad \text{for all} \quad \tilde{f}(\hat{x}) \in \hat{\Omega}^{0}. \tag{23}$$

Furthermore, using equation (11) we find

$$\hat{d} \cdot \xi_{\mu} = [\hat{d}, \xi_{\mu}] = \hat{d}\xi_{\mu} + \xi_{\mu}\hat{d} = 0 \tag{24}$$

since $[dx_{\mu}, dx_{\nu}] = 0$. By induction on $k$ one can show that

$$\hat{d} \cdot (\xi_{\mu_{1}}\xi_{\mu_{2}}\cdots\xi_{\mu_{k}}) = 0 \quad \text{for all} \quad k \geq 1. \tag{25}$$

Relations (23) and (25) together with the Leibniz rule (17) imply that $\hat{d}$ maps $k$-forms to $(k+1)$-forms. For example,

$$\hat{d} \cdot (\hat{x}_{\mu}\hat{x}_{\nu}) = \hat{d} \cdot (\hat{x}_{\mu})\hat{x}_{\nu} + \hat{x}_{\mu}\hat{d}(\hat{x}_{\nu}). \tag{26}$$

The exterior derivative satisfies the graded Leibniz rule

$$\hat{d} \cdot (\hat{\omega}_{k}\hat{\eta}_{l}) = (\hat{d} \cdot \hat{\omega}_{k})\hat{\eta}_{l} + (-1)^{k}\hat{\omega}_{k}(\hat{d} \cdot \hat{\eta}_{l}). \tag{27}$$

Hence, the algebra $\hat{\Omega}$ together with the linear map $\hat{d}: \hat{\Omega}^{k} \to \hat{\Omega}^{k+1}$ is a differential algebra. Our approach is essentially the same as the construction of the differential algebra of forms discussed in [37]. In our case the algebra of zero-forms has the additional structure of the universal enveloping algebra satisfying relations (1). We note that in general one cannot rewrite a given $k$-form such that $\xi_{\mu_{1}}, \xi_{\mu_{2}}, \ldots, \xi_{\mu_{k}}$ are placed to the far right. This is possible only in special realizations in which the commutator $[\xi_{\mu}, \hat{x}_{\nu}]$ closes in the space of one-forms $\xi_{\mu}$ alone.

### 3. Covariant realizations

In this section, we shall investigate the differential algebra of forms in covariant realizations of the kappa-deformed space introduced in [28]. These realizations are covariant under the action of the rotation algebra $so(n)$. Of particular interest is a class of simple realizations obtained for the following choice of $\phi_{\mu\nu}$ in the representation (3):

**Left realization:**

$$\phi_{\mu\nu} = (1 - A)\delta_{\mu\nu}, \tag{28}$$

**Right realization:**

$$\phi_{\mu\nu} = \delta_{\mu\nu} + ia_{\nu}\partial_{\mu}, \tag{29}$$

**Natural realization:**

$$\phi_{\mu\nu}(\partial) = (-A + \sqrt{1 - B})\delta_{\mu\nu} + i a_{\mu} \partial_{\nu}, \tag{30}$$

**Symmetric realization:**

$$\phi_{\mu\nu} = \frac{A}{e^{A} - 1} \delta_{\mu\nu} + ia_{\nu}\partial_{\mu} e^{A} - A - 1 \tag{31}$$

Here $A$ and $B$ are commuting operators defined by $A = ia\partial$ and $B = a^{2}\partial^{2}$ where we use the convention $a\partial = \sum a_{\alpha} a_{\alpha}, a^{2}\partial^{2} = \sum a_{\alpha}^{2}a_{\alpha}^{2}, \text{etc.}$ The symmetric realization corresponds to the Weyl symmetric ordering of the monomials in $\hat{\omega}_{\mu}$. We remark that for a general Lie algebra type NC space there is a universal formula for $\phi_{\mu\nu}$ in Weyl symmetric ordering given in [48]
as follows. Suppose $\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n$ are generators of a Lie algebra with structure constants $\theta_{\mu \nu \sigma}$:

$$[\hat{x}_\mu, \hat{x}_\nu] = i \sum_a \theta_{\mu \nu a} \hat{x}_a. \quad (32)$$

Let $M = [M_{\mu \nu}]$ denote the $n \times n$ matrix of differential operators with elements

$$M_{\mu \nu} = i \sum_a \theta_{\nu a \mu} \partial_a. \quad (33)$$

Then the Weyl symmetric realization of the Lie algebra (32) is given by

$$\phi_{\mu \nu}(\partial) = p(M)_{\mu \nu} \quad \text{where} \quad p(M) = \frac{M}{e^M - 1} \quad (34)$$

is the generating function for the Bernoulli numbers (see also [49]). In principle, the exterior derivative and one-forms may be constructed using any of the above realizations. Here we shall consider the left, right and natural realizations.

### 3.1. Covariant realizations of type I

Let us consider realizations of type I where the exterior derivative is undeformed, $\hat{d} = \sum_\alpha dx_\alpha \partial_\alpha$, and one-forms are given by $\xi_\mu = \sum_\alpha dx_\alpha \phi_{\alpha \mu}(\partial)$. We investigate the conditions under which the commutator $[\xi_\mu, \hat{x}_\nu]$ is closed in the space of one-forms $\xi_\mu$. The closedness of the commutator is important when considering the extended star-product of (classical) forms in section 5.

Using realization (3) we have

$$[\xi_\mu, \hat{x}_\nu] = \sum_a \sum_\beta dx_\alpha \frac{\partial \phi_{a \mu}}{\partial \phi_{\beta \nu}}. \quad (35)$$

The matrix $[\phi_{\mu \nu}]$ is invertible, hence we may express $dx_\mu$ in terms of $\xi_\mu$ to obtain

$$[\xi_\mu, \hat{x}_\nu] = \sum_\sigma C_{\mu \nu \sigma}(\partial) \xi_\sigma, \quad (36)$$

where

$$C_{\mu \nu \sigma}(\partial) = \sum_\alpha \sum_\beta \frac{\phi_{a \mu}}{\phi_{\beta \nu}} \frac{\partial \phi_{a \mu}}{\partial \phi_{\beta \nu}}. \quad (37)$$

Clearly, the commutator (36) is closed in the space of one-forms $\xi_\mu$ only if the coefficients $C_{\mu \nu \sigma}$ are constant. This condition is satisfied in the left and right realizations, as shown in the following. In the left realization, we have

$$\hat{x}_\mu = x_\mu (1 - A), \quad \hat{x}_\mu = dx_\mu (1 - A), \quad \xi_\mu = dx_\mu (1 - A), \quad (38)$$

which yields

$$[\xi_\mu, \hat{x}_\nu] = -i a_\nu \xi_\mu. \quad (39)$$

Similarly, in the right realization we have

$$\hat{x}_\mu = x_\mu + i a_\mu (x \partial), \quad \xi_\mu = dx_\mu + i a_\mu (dx \partial), \quad (40)$$

which leads to

$$[\xi_\mu, \hat{x}_\nu] = i a_\mu \xi_\nu. \quad (41)$$

On the other hand, in the natural and symmetric realizations the coefficients $C_{\mu \nu \sigma}$ involve partial derivatives so the commutators between $\xi_\mu$ and $\hat{x}_\nu$ are not closed.
3.2. Covariant realizations of type II

Consider now realizations of type II where the exterior derivative and one-forms are given by \( \hat{d} = \sum_{\alpha} \xi_{\alpha} \partial_{\alpha} \) and \( \xi_{\mu} = \sum_{\alpha} \delta x_{\alpha} h_{\alpha \mu}(\partial) \), and \( h_{\alpha \mu} \) is a solution to equation (21). In this section, we shall construct \( \hat{d} \) and \( \xi_{\mu} \) using the natural realization (30). The construction of NC forms in type II realization was considered in [25], but not in a proper and complete way. Our motivation for using the natural realization is to present a proper analysis of this problem.

Let us write equation (21) in a more compact form

\[
\sum_{\rho} \partial_{\Lambda_{\alpha}} \partial_{\partial_{\rho}} \phi_{\rho \mu} = h_{\alpha \mu}, \tag{42}
\]

where \( \Lambda_{\alpha}(\partial) = \sum_{\beta} h_{\alpha \beta}(\partial) \partial_{\beta} \). The idea is to solve an auxiliary problem for \( \Lambda_{\alpha} \) and then calculate \( h_{\mu \nu} \) from equation (42). Multiplying equation (42) by \( \partial_{\mu} \) and summing we obtain the following boundary value problem for \( \Lambda_{\alpha} \):

\[
\sum_{\rho} \partial_{\Lambda_{\alpha}} \partial_{\partial_{\rho}} \Psi_{\rho} = \Lambda_{\alpha}, \quad \lim_{a \to 0} \Lambda_{\alpha} = \partial_{\alpha}, \tag{43}
\]

where \( \Psi_{\rho}(\partial) = \sum_{\mu} \phi_{\rho \mu}(\partial) \partial_{\mu} \). In the natural realization (30) we find

\[
\Psi_{\rho}(\partial) = \partial_{\rho}( -A + \sqrt{1 - B}) + ia_{\rho} \partial_{a}^2. \tag{44}
\]

Let us denote \( Z^{-1} = -A + \sqrt{1 - B} \). This is the inverse shift operator introduced in [28]. The index structure of \( \Psi_{\rho} \) and equation (43) suggest that we should look for \( \Lambda_{\alpha} \) in the form

\[
\Lambda_{\alpha}(\partial) = \partial_{\alpha} H_{1}(A, B) + ia_{\alpha} \partial_{a}^2 H_{2}(A, B) \tag{45}
\]

for unknown functions \( H_{1} \) and \( H_{2} \). From equations (44) and (45) we obtain

\[
\sum_{\rho} \partial_{\Lambda_{\alpha}} \Psi_{\rho} = \partial_{a} \left[ \left( H_{1} + A \frac{\partial H_{1}}{\partial A} + 2B \frac{\partial H_{1}}{\partial B} \right) Z^{-1} - B \frac{\partial H_{1}}{\partial A} + 2AB \frac{\partial H_{1}}{\partial B} \right] + ia_{\alpha} \partial_{a}^2 \left[ \left( 2H_{2} + A \frac{\partial H_{2}}{\partial A} + 2B \frac{\partial H_{2}}{\partial B} \right) Z^{-1} - H_{1} + 2AB \frac{\partial H_{2}}{\partial A} + 2AB \frac{\partial H_{2}}{\partial B} \right]. \tag{46}
\]

Substituting the above result into equation (43) we find that \( H_{1} \) and \( H_{2} \) satisfy the following system of differential equations:

\[
\left( H_{1} + A \frac{\partial H_{1}}{\partial A} + 2B \frac{\partial H_{1}}{\partial B} \right) Z^{-1} - B \frac{\partial H_{1}}{\partial A} + 2AB \frac{\partial H_{1}}{\partial B} = H_{1}, \tag{47}
\]

\[
\left( 2H_{2} + A \frac{\partial H_{2}}{\partial A} + 2B \frac{\partial H_{2}}{\partial B} \right) Z^{-1} - H_{1} + 2AB \frac{\partial H_{2}}{\partial A} + 2AB \frac{\partial H_{2}}{\partial B} = H_{2}. \tag{48}
\]

Since \( \Lambda_{\alpha}(\partial) \to \partial_{\alpha} \) as \( a \to 0 \), \( H_{1} \) and \( H_{2} \) are subject to the boundary conditions

\[
\lim_{a \to 0} H_{1}(A, B) = 1, \quad \lim_{a \to 0} H_{2}(A, B) \text{ finite}. \tag{49}
\]

It is shown in appendix A that the above system has a unique solution

\[
H_{1}(A, B) = \frac{2(1 - \sqrt{1 - B})}{B(-A + \sqrt{1 - B})}, \tag{50}
\]

\[
H_{2}(A, B) = -2(1 - A + \sqrt{1 - B}) \left( 1 - \frac{\sqrt{1 - B}}{B} \right)^{2}. \tag{51}
\]
Inserting the expressions for $H_1$ and $H_2$ into equation (45) we find

$$\Lambda_a(\partial) = \partial_a \frac{2(1 - \sqrt{1 - B})}{B(-A + \sqrt{1 - B})} - i a_a \partial^2 2(1 - A + \sqrt{1 - B}) \left( \frac{1 - \sqrt{1 - B}}{B} \right)^2. \tag{52}$$

Since the exterior derivative is given by $\hat{d} = \sum_a \xi_a \partial_a$ where $\xi_{\mu} = \sum_a dx_a h_{a\mu}(\partial)$, $\hat{d}$ can be expressed in terms of $\Lambda_a$ as

$$\hat{d} = \sum_a dx_a \Lambda_a(\partial). \tag{53}$$

Thus, we find from equation (52) that

$$\hat{d} = \frac{2(1 - \sqrt{1 - B})}{B(-A + \sqrt{1 - B})} (\partial dx) - 2(1 - A + \sqrt{1 - B}) \left( \frac{1 - \sqrt{1 - B}}{B} \right)^2 i(a dx) \partial^2. \tag{54}$$

Keeping only the first-order terms in $a \in \mathbb{R}^n$ we obtain the approximation

$$\hat{d} = \partial dx + i(a \partial)(\partial dx) - i a^2 (adx), \tag{55}$$

where $d = \partial dx$ is the undeformed exterior derivative.

Next we consider the one-form $\xi_{\mu}$. Substituting equations (30) and (52) into equation (42) we find after some manipulation that

$$h_{a\mu}(\partial) = L_1 \delta_{a\mu} + i L_2 a_\mu \partial_{a\mu} + i L_3 a_\mu \partial_a + a^2 L_4 \partial_{a\mu} \partial_a - \partial^2 L_5 a_\mu a_a, \tag{56}$$

where

$$L_1 = \frac{2(1 - \sqrt{1 - B})}{B}, \tag{57}$$

$$L_2 = -\frac{2(-1 + \sqrt{1 - B}) \left[ 2(A^2 + A - B) \sqrt{1 - B} + B - 2(A^2 - 2AB + A) \right]}{B^2(-A + \sqrt{1 - B})}, \tag{58}$$

$$L_3 = \frac{2(1 - \sqrt{1 - B})}{B(-A + \sqrt{1 - B})}, \tag{59}$$

$$L_4 = -\frac{2(B + 2 \sqrt{1 - B} - 2)}{B^2(-A + \sqrt{1 - B})}, \tag{60}$$

$$L_5 = \frac{2(-A + \sqrt{1 - B})(1 - \sqrt{1 - B})^2}{B^2}. \tag{61}$$

Therefore, in the natural realization of type II the one-form $\xi_{\mu}$ is given by

$$\xi_{\mu} = \sum_a h_{a\mu}(\partial) dx_a$$

$$= L_1 dx_{\mu} + (i L_2 a_{\mu} - \partial^2 L_5 a_{\mu})(a dx) + (i L_3 a_{\mu} + a^2 L_4 \partial_{a\mu})(\partial dx). \tag{62}$$

Although the above realization of $\xi_{\mu}$ is rather complicated, the first-order approximation has a particularly nice form

$$\xi_{\mu} = dx_{\mu} + \sum_a i(a_{\mu} \partial_a - a_a \partial_{a\mu}) dx_a. \tag{63}$$

Let us now investigate the commutation relations for $\xi_{\mu}$ and $\hat{x}_{\nu}$. The NC coordinates in the natural realization (30) are given by

$$\hat{x}_{\mu} = x_{\mu}(-A + \sqrt{1 - B}) + i(ax) \partial_{\mu}. \tag{64}$$
The explicit form of the commutator $[\xi_\mu, \hat{x}_\nu]$ is fairly complicated and a complete derivation is given in appendix B. Here we only state that it can be expressed as

$$[\xi_\mu, \hat{x}_\nu] = \xi_\mu P^{(1)}_{\mu\nu} + \xi_\nu P^{(2)}_{\mu\nu} + (\partial_\xi) R^{(1)}_{\mu\nu} + (\partial_\xi) R^{(2)}_{\mu\nu},$$

(65)

where $P^{(i)}_{\mu\nu}$ and $R^{(i)}_{\mu\nu}$ are certain combinations of the functions $L_1, L_2, \ldots, L_5$ and their partial derivatives. We note that the commutator (65) is not closed since the right-hand side involves derivatives $\partial_\mu$. To gain an insight into the form of the commutator it is instructive to find a first-order approximation in the parameter $a$. To first order in $a$ the natural realization of $\hat{x}_\mu$ is given by

$$\hat{x}_\mu = x_\mu (1 - ia\partial) + i(ax) \partial_\mu.$$

(66)

Using the approximations (63) and (66) we obtain

$$[\xi_\mu, \hat{x}_\nu] = i \sum_a (a_\mu \delta_{av} - a_v \delta_{a\mu}) \xi_a.$$

(67)

As a special case suppose that the vector $a \in \mathbb{R}^n$ has only one non-zero component, $a_\mu = a \delta_{\mu n}$ for $\mu = 1, 2, \ldots, n$. Then

$$[\xi_\mu, \hat{x}_\nu] = i a (\delta_{\mu n} \xi_\nu - \delta_{\mu\nu} \xi_n).$$

(68)

The above result agrees to first order in $a$ with the commutator $[\xi_\mu, \hat{x}_\nu]$ for vector-like transforming one-forms considered in [25]. We emphasize, however, that the exact expression (65) does not agree with this commutator for higher orders in $a$.

4. Noncovariant realizations

In this section, we consider the exterior derivative and one-forms in noncovariant realizations of the kappa-space introduced in [26]. We assume that the components of the deformation vector $a \in \mathbb{R}_n$ are given by $a_k = 0$ for $k = 1, 2, \ldots, n - 1$ and $a_n = a$. Then the commutation relations (1) yield

$$[\hat{x}_k, \hat{x}_l] = 0, \quad [\hat{x}_n, \hat{x}_k] = i a \delta_{kl}, \quad k, l = 1, 2, \ldots, n - 1.$$

(69)

We use the Latin alphabet for the indices $1, 2, \ldots, n - 1$ and the Greek alphabet for the full set $1, 2, \ldots, n$. It was shown in [26] that the NC coordinates $\hat{x}_\mu$ have infinitely many realizations of the form

$$\hat{x}_k = x_k \varphi(A), \quad k = 1, 2, \ldots, n - 1,$$

(70)

$$\hat{x}_n = x_n + i a \sum_{k=1}^{n-1} x_k \partial_k \gamma(A),$$

(71)

where

$$\gamma(A) = \frac{\varphi'(A)}{\varphi(A)} + 1, \quad A = i a \partial_n.$$

(72)

The realizations are parametrized by the function $\varphi(A)$ satisfying the boundary conditions $\lim_{a \to 0} \varphi(A) = 1$ and $\lim_{a \to 0} \varphi'(A) = 0$, so that $\hat{x}_\mu \to x_\mu$ as $a \to 0$. The NC coordinates $\hat{x}_\mu$ are covariant under the rotation algebra $so(n - 1)$, but not generally under the full algebra $so(n)$. 
The most general ansatz for the exterior derivative $\hat{d}$ invariant under $so(n - 1)$ is

$$\hat{d} = \sum_{k=1}^{n-1} dx_k \partial_k N_1(A, \Delta) + dx_n \partial_n N_2(A, \Delta) + ia \frac{dx_n}{n} \sum_{k=1}^{n-1} \partial_k^2 G(A, \Delta),$$

(73)

where $\Delta = (ia)^2 \sum_{k=1}^{n-1} \partial_k^2$. The family of realizations (70) and (71) includes special realizations corresponding to the left, right, symmetric left–right and symmetric Weyl orderings for the enveloping algebra of the Lie algebra (69). These realizations are parametrized by $\varphi(A) = e^{-A}$, $\varphi(A) = 1$, $\varphi(A) = e^{-A/2}$ and $\varphi(A) = A/(e^A - 1)$, respectively. We remark that only the symmetric Weyl realization is covariant under the full algebra $so(n)$.

For a given parameter function $\varphi$ and an arbitrary choice of $N_1$, $N_2$ and $G$ one can find the one-forms $\xi_\mu$ satisfying $[\hat{d}, \hat{\xi}_\mu] = \hat{\xi}_\mu$. As in the case of the covariant realizations one can express the commutator $[\xi_\mu, \hat{x}_k]$ in terms of the one-forms $\xi_\mu$ and partial derivatives $\partial_\mu$, but the general expressions are fairly complicated.

In the following, we will focus our attention to a subfamily of the noncovariant realizations which lead to some interesting results. These realizations are parametrized by $\varphi(A) = e^{-cA}$, $c \in \mathbb{R}$:

$$\hat{x}_k = x_k e^{-cA}, \quad k = 1, 2, \ldots, n - 1,$$

(75)

$$\hat{x}_n = x_n + ia(1 - c) \sum_{k=1}^{n-1} x_k \partial_k.$$  

(76)

They include the left, right and symmetric left–right realizations for $c = 1$, $c = 0$ and $c = 1/2$, respectively. Let us define the exterior derivative by

$$\hat{d} = \sum_{k=1}^{n-1} dx_k \partial_k e^{c(1-A)} + dx_n \partial_n$$

(77)

($N_1 = e^{c(1-A)}$, $N_2 = 1$, $G = 0$). Then the corresponding one-forms are given by

$$\xi_\mu = [\hat{d}, \hat{\xi}_\mu] = dx_k e^{-cA}, \quad k = 1, 2, \ldots, n - 1,$$

(78)

$$\xi_n = [\hat{d}, \hat{x}_n] = dx_n.$$  

(79)

The algebra generated by $\hat{x}_\mu$ and $\xi_\mu$ satisfies the commutation relations

$[\xi_\mu, \hat{x}_k] = 0$,  
$[\xi_\mu, \hat{x}_n] = -ia\xi_k$,  
$[\xi_n, \hat{x}_\mu] = 0$,  
$[\xi_n, \hat{x}_n] = 0$.  

(80)

(81)

This algebra satisfies the graded Jacobi relations (9). We note that relations (80) and (81) correspond to the algebra found by Kim et al [50] where the commutators are defined in terms of the star-product, except that in our work $\xi_\mu$ and $\xi_n$ anticommute. In particular, for $c = 0$ the exterior derivative becomes

$$\hat{d} = \sum_{k=1}^{n-1} dx_k \partial_k e^{-A} + dx_n \partial_n = \sum_{\alpha=1}^{n} \xi_\alpha \partial_\alpha,$$

(82)

which is the type II realization of $\hat{d}$. In addition to the examples in section 3 the commutators (80) and (81) also close in the space of one-forms $\xi_\mu$ alone. Moreover, the right realization ($c = 0$) is an example of a type II realization with closed commutator.

The above construction can be extended to any parameter function $\varphi$. It can be shown that for a given $\varphi$ one can find $N_1$, $N_2$ and $G$ such that $\hat{d} = \sum_{\alpha=1}^{n} \xi_\alpha \partial_\alpha$ and $[\hat{d}, \hat{x}_\mu] = \hat{\xi}_\mu$. However, this may be very complicated as already seen in the natural realization in section 3.
5. Extended star-product

Regarding functions as zero-forms we want to extend the star-product to differential forms of arbitrary degree. The star-product of differential forms in the context of deformation quantization has been investigated recently in [51]. The construction of the star-product presented here is valid for a general Lie algebra type noncommutative space. We recall that quantization has been investigated recently in [51]. The construction of the star-product of arbitrary degree. The star-product of differential forms in the context of deformation regarding functions as zero-forms we want to extend the star-product to differential forms.

\[ x_{\mu} \cdot 1 = x_{\mu}, \quad \partial_{\mu} \cdot 1 = 0. \] (83)

For a monomial \( \hat{f}(\hat{x}) \in \mathcal{U}(g) \) we define

\[ \Omega_\phi(\hat{f}(\hat{x})) = \hat{f}(\hat{x}) \cdot 1 = f(x), \] (84)

and extend \( \Omega_\phi \) linearly to \( \mathcal{U}(g) \). The map \( \Omega_\phi \) is evaluated at \( \hat{f}(\hat{x}) \) by using the realization (3) and action (83). For example,

\[ \Omega_\phi(\hat{x}_\mu) = \sum_\alpha (x_\alpha \partial_{\mu\alpha})(\partial) \cdot 1 = x_\mu \] (85)

since \( \phi_{\mu\alpha}(\partial) = \delta_{\mu\alpha} + o(\partial) \). Similarly, for monomials of order 2 we have

\[ \Omega_\phi(\hat{x}_\mu \hat{x}_\nu) = x_\mu x_\nu + \sum_\alpha x_\alpha \frac{\partial \phi_{\mu\alpha}}{\partial \partial_\nu} \cdot 1, \] (86)

where \( \frac{\partial \phi_{\mu\alpha}}{\partial \partial_\nu} \cdot 1 \) is a first-order coefficient in the Taylor expansion of \( \phi_{\mu\alpha}(\partial) \). In general, \( \Omega_\phi(\hat{x}_{\mu_1} \hat{x}_{\mu_2} \cdots \hat{x}_{\mu_n}) \) is a polynomial in the variables \( x_{\mu_1}, x_{\mu_2}, \ldots, x_{\mu_n} \) whose coefficients are given by the Taylor expansion of \( \phi_{\mu\nu} \). The computation of \( \Omega_\phi(\hat{x}_{\mu_1} \hat{x}_{\mu_2} \cdots \hat{x}_{\mu_n}) \) can be done using a recursive formula. Suppose that

\[ \Omega_\phi(\hat{x}_{\mu_1} \hat{x}_{\mu_2} \cdots \hat{x}_{\mu_n}) = p(x_{\mu_2}, x_{\mu_3}, \ldots, x_{\mu_n}). \] (87)

Then

\[ \Omega_\phi(\hat{x}_{\mu_1} \hat{x}_{\mu_2} \cdots \hat{x}_{\mu_n}) = x_{\mu_1} p(x_{\mu_2}, x_{\mu_3}, \ldots, x_{\mu_n}) \]

\[ + \sum_\alpha x_\alpha [\phi_{\mu\alpha \mu_2}, p(x_{\mu_2}, x_{\mu_3}, \ldots, x_{\mu_n})] \cdot 1. \] (88)

The commutator in the above expression is calculated according to

\[ [\phi_{\mu\alpha}, x_1 x_2 \cdots x_k] = [\phi_{\mu\alpha}, x_1] x_2 \cdots x_k + x_1 [\phi_{\mu\alpha}, x_2] \cdots x_k + \cdots + x_1 \cdots x_{k-1} [\phi_{\mu\alpha}, x_k]. \] (89)

The inverse map \( \Omega_\phi^{-1} \) is defined analogously. Let \( \hat{1} \) be the unit in \( \mathcal{U}(g) \). Define the action of \( \hat{x}_\mu \) on a monomial \( \hat{f}(\hat{x}) \in \mathcal{U}(g) \) by \( \hat{x}_\mu \cdot \hat{f}(\hat{x}) = \hat{x}_\mu \hat{f}(\hat{x}) \). The action of \( \partial_{\mu} \) on \( \hat{f}(\hat{x}) \) is defined by \( \partial_{\mu} \cdot \hat{1} = 0 \) and \( \partial_{\mu} \cdot \hat{f}(\hat{x}) = (\partial_{\mu} \hat{f}(\hat{x})) \cdot \hat{1} \) where \( \partial_{\mu} \hat{f}(\hat{x}) \) is expressed using the commutation relations \([\partial_{\mu}, \hat{x}_\nu] = \phi_{\mu\nu}(\partial)\). For the lowest order vector we have

\[ \hat{x}_\mu \cdot \hat{1} = \hat{x}_\mu, \quad \partial_{\mu} \cdot \hat{1} = 0. \] (90)

Then \( \Omega_\phi^{-1} \) is given by

\[ \Omega_\phi^{-1}(f(x)) = f(x) \cdot \hat{1} \equiv \hat{f}(\hat{x}) \] (91)
where \( f(x) \cdot \hat{1} \) is calculated using the realization (6) and relations (90). For example,

\[
\Omega^{-1}_\phi(x_\mu) = \sum_a \hat{x}_a \phi^{-1}_{a\mu}(\partial) \cdot \hat{1} = \hat{x}_\mu
\]

since \( \phi^{-1}_{a\mu}(\partial) = \delta_{a\mu} + o(\partial) \), and for monomials of order 2 we have

\[
\Omega^{-1}_\phi(x_\mu x_\nu) = \hat{x}_\mu \hat{x}_\nu + \sum_a \hat{x}_a \frac{\partial \phi^{-1}_{a\mu\nu}}{\partial \hat{x}_a} \cdot \hat{1}.
\]

One can show that the right-hand side of equation (93) is invariant under the transposition of indices \( \mu \leftrightarrow \nu \), hence \( \Omega^{-1}_\phi(x_\mu x_\nu) \) is well defined. Clearly, \( \Omega_\phi \) and \( \Omega^{-1}_\phi \) can be readily extended to \( \mathcal{U}(g) \) and \( \hat{\mathcal{S}} \), the formal completions of \( \mathcal{U}(g) \) and \( \mathcal{S} \). The star-product of \( f, g \in \hat{\mathcal{S}} \) is defined by

\[
(f \star g)(x) = \left( \hat{f}(\hat{x}) \hat{g}(\hat{x}) \right) \cdot \hat{1},
\]

where \( \hat{f}(\hat{x}) = \Omega^{-1}_\phi(f(x)) \) and \( \hat{g}(\hat{x}) = \Omega^{-1}_\phi(g(x)) \). In the limit as the deformation parameter \( a \to 0 \) the star-product reduces to ordinary product of functions (cf equation (4)).

Equation (94) defines the star-product of zero-forms. Following the ideas outlined above we want to extend the star-product to differential forms of arbitrary degree. Our strategy is to associate with \( \omega^k \) a noncommutative form \( \hat{\omega}^k \) such that \( \hat{\omega}^k \cdot \hat{1} = \omega^k \) and define the star-product by

\[
\omega^k \star \eta^l = (\hat{\omega}^k \hat{\eta}^l) \cdot \hat{1}.
\]

It turns out that the star-product (95) is well defined provided the commutator \( [\hat{\xi}_\mu, \hat{\xi}_\nu] \) is closed in the space of one-forms \( \hat{\xi}_\mu \) alone. It depends only on the realizations of the coordinates \( \hat{x}_\mu \), hence we also denote it by \( \star \).

First let us consider the star-product of constant forms. Recall that the noncommutative one-form \( \hat{\xi}_\mu \) is defined by \( \hat{\xi}_\mu = \sum_\alpha dx_\alpha h_{a\mu}(\partial) \) where \( h_{a\mu}(\partial) \) satisfies equation (14). The matrix \( [h_{a\mu}] \) is invertible, hence there is a dual relation \( dx_\mu = \sum_\alpha \hat{\xi}_\alpha h^{-1}_{a\mu}(\partial) \). Since \( h_{a\mu}(\partial) \) is a power series of the type (4), and \( dx_\mu \) and \( \partial_\nu \) commute, we have

\[
(\hat{\xi}_\mu, \hat{\xi}_\nu, \cdots, \hat{\xi}_\mu) \cdot \hat{1} = dx_{\mu_1} dx_{\mu_2} \cdots dx_{\mu_k}.
\]

Therefore, to a \( k \)-form \( \omega^k = dx_{\mu_1} dx_{\mu_2} \cdots dx_{\mu_k} \) we associate a unique noncommutative form \( \hat{\omega}^k = \hat{\xi}_{\mu_1} \hat{\xi}_{\mu_2} \cdots \hat{\xi}_{\mu_k} \) satisfying \( \hat{\omega}^k \cdot \hat{1} = \omega^k \). The star-product of \( \hat{\omega}^k \) is \( dx_{\mu_1} dx_{\mu_2} \cdots dx_{\mu_k} \) and \( \hat{\eta}^l = dx_{\nu_1} dx_{\nu_2} \cdots dx_{\nu_l} \) is trivially given by

\[
\omega^k \star \eta^l = (\hat{\omega}^k \hat{\eta}^l) \cdot \hat{1},
\]

In view of equation (96) the star-product of constant forms is undeformed,

\[
\omega^k \star \eta^l = \omega^k \eta^l,
\]

and graded commutative,

\[
\omega^k \star \eta^l = (-1)^{kl} \eta^l \star \omega^k.
\]
Thus,  
\[ \hat{\omega}_k = \sum_{\rho_1, \ldots, \rho_k} dx_{\rho_1} dx_{\rho_2} \cdots dx_{\rho_k} \hat{p}(\hat{\xi}) h_{\rho_1 \sigma_1} h_{\rho_2 \sigma_2} \cdots h_{\rho_k \sigma_k} \]

(101)

\[ = dx_{\sigma_1} dx_{\sigma_2} \cdots dx_{\sigma_n} (\hat{p}(\hat{\xi}) + o(\hat{\partial})). \]

(102)

Thus,  
\[ \hat{\omega}_k \cdot 1 = p(x) dx_{\sigma_1} dx_{\sigma_2} \cdots dx_{\sigma_n} = \omega^k \]

(103)

since \( \hat{\omega}_k \) given by equation (100) is a unique noncommutative form (up to reordering of \( \hat{\xi}_\mu \)) in \( \hat{p}(\hat{\xi}) \) using the commutation relations (1)) with the property \( \hat{\omega}^k \cdot 1 = \omega^k \) in which the NC coordinates are naturally ordered to the left of \( \hat{\xi}_\mu \). If \( \omega^k = p(x) dx_{\mu_1} dx_{\mu_2} \cdots dx_{\mu_k} \) and \( \eta^l = q(x) dx_{\nu_1} dx_{\nu_2} \cdots dx_{\nu_l} \), then equations (95) and (100) yield

\[ \omega^k \, \star \, \eta^l = \left( \hat{p}(\hat{\xi}) \xi_{\mu_1} \cdots \xi_{\mu_k} \hat{q}(\hat{\xi}) \xi_{\nu_1} \cdots \xi_{\nu_l} \right) \cdot 1, \]

(104)

where \( \hat{p}(\hat{\xi}) = \Omega^{-1}_q(p(x)) \) and \( \hat{q}(\hat{\xi}) = \Omega^{-1}_p(q(x)) \). The star-product (104) is not graded commutative since \( \hat{\xi}_\mu \) and \( \xi_\mu \) do not commute. The product is well defined provided the commutators \( [\hat{\xi}_\mu, \xi_\nu] \) are closed in the space of one-forms \( \xi_\mu \). In this case one can use the commutation relations between \( \hat{\xi}_\mu \) and \( \hat{x}_\mu \) to write (104) in the natural order with \( \hat{x}_\mu \) to the left of \( \hat{\xi}_\mu \) and evaluate the star-product using \( (\hat{p}(\hat{\xi}) \xi_{\mu_1} \cdots \xi_{\mu_k}) \cdot 1 = p(x) dx_{\mu_1} \cdots dx_{\mu_k} \). In view of earlier considerations, the extended star-product can be defined in the covariant left, right and noncovariant realizations discussed in sections 3 and 4. We note that the extended star-product is associative since this property is inherited from associativity of operator multiplication in the superalgebra \( \mathcal{A} \).

Finally, let us consider the exterior derivative acting on the star-product of forms. In the realization of type I the exterior derivative is undeformed, \( \hat{d} = d \equiv \sum_\mu dx_\mu \partial_\mu \). Then one can show that

\[ d\omega = (\hat{d}\hat{\omega}) \cdot 1, \]

(105)

where \( \hat{\omega} \cdot 1 = \omega \). Using the star-product (95) and Leibniz rule (17) one finds

\[ d(\omega \, \star \, \eta) = d\omega \star \eta + (-1)^{[\omega]} \omega \star d\eta. \]

(106)

Hence, in type I realization the Leibniz rule for the extended star-product is undeformed. It would be interesting to investigate the action of the induced exterior derivative on the star-product of forms in other realizations when \( \hat{d} \) is given by a general expression (11).

6. Concluding remarks

In this paper, we have investigated the differential algebra of forms on the kappa-deformed space. Our construction of the exterior derivative \( \hat{d} \) and one-forms \( \hat{\xi}_\mu \) is based on the realizations of NC coordinates \( \hat{x}_\mu \) in terms of formal power series in the Weyl algebra. We have shown that for each realization of \( \hat{x}_\mu \) there is an infinite family of the exterior derivatives \( \hat{d} \) which uniquely determine the one-forms \( \xi_\mu \). The exterior derivative is a nilpotent operator and it satisfies the undeformed Leibniz rule. The NC coordinates \( \hat{x}_\mu \), derivatives \( \partial_\mu \) and one-forms \( \xi_\mu \) generate a \( \mathbb{Z}_2 \)-graded algebra. The subalgebra generated by \( \hat{x}_\mu \) and \( \partial_\mu \) is a deformed Heisenberg algebra. The algebra generated by \( \hat{x}_\mu \) and \( \xi_\mu \) is generally not closed under the commutator bracket since \( [\hat{\xi}_\mu, \hat{x}_\nu] \) may involve an infinite series in \( \partial_\nu \). Only in special cases of the covariant left, right and noncovariant realizations the algebra is closed under the commutator bracket. Furthermore, the commutator \( [\hat{\xi}_\mu, \hat{x}_\nu] \) is nonzero in all realizations. For higher order forms we have shown that the exterior derivative satisfies the graded Leibniz rule,
and the graded Jacobi identity also holds. However, the graded commutativity law holds only for $\hat{x}_\mu$-independent forms. In the limit when the deformation parameter $a \to 0$ our theory reduces to classical results.

The exterior derivative and one-forms have been analyzed in both covariant and noncovariant realizations. In the covariant case we have found explicit representations of $\hat{d}$ and $\hat{\xi}_\mu$ in the left, right and natural realizations. We have also found a closed form expression for the commutator $[\hat{\xi}_\mu, \hat{x}_\nu]$ in these realizations, and derived an approximation to first order in $a$ in the natural realization. In the noncovariant case we have constructed a one-parameter family of realizations of $\hat{d}$ and $\hat{\xi}_\mu$. For this family of realizations the commutator $[\hat{\xi}_\mu, \hat{x}_\nu]$ is always closed in the space of one-forms $\hat{\xi}_\mu$.

We have also extended the star-product from zero-forms to differential forms of arbitrary degree. The star-product can be defined for realizations in which $[\hat{\xi}_\mu, \hat{x}_\nu]$ is closed in the space of one-forms $\hat{\xi}_\mu$. It depends only on the realizations of both the NC coordinates $\hat{x}_\mu$. For differential forms with constant coefficients the star-product is undeformed and graded commutative, but for arbitrary forms this is no longer true. It was shown that the exterior derivative acting on the extended star-product satisfies the undeformed Leibniz rule in type I realization. It would be interesting to investigate possible relations between our approach to the star-product of differential forms and the recent work presented in [51].

Finally, the notion of the twist operator is very important in the construction of the star-product from both the mathematical [53, 54] and physical [55–59] points of view. The twist operator for zero-forms on the kappa-deformed space was constructed in [30] and [59], and was also considered in [27]. However, it remains an open problem to see if there exists a twist operator that leads to the star-product of differential forms defined in this work.

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Appendix A

In this appendix, we find the solution of the system of equations (47) and (48). Let us write equation (47) in equivalent forms as

\begin{equation}
(AZ^{-1} - B) \frac{\partial H_1}{\partial A} + 2B\sqrt{1-B} \frac{\partial H_1}{\partial B} + (Z^{-1} - 1)H_1 = 0.
\end{equation}

We assume that $H_1$ can be factored as $H_1(A, B) = Z F_1(B)$ which leads to the following differential equation for $F_1$,

\begin{equation}
2B\sqrt{1-B} F'_1(B) + (\sqrt{1-B} - 1)F_1(B) = 0.
\end{equation}

The boundary condition for $H_1$ implies that $\lim_{a \to 0} F_1(B) = 1$. Now the solution to equation (A.2) is readily found to be

\begin{equation}
F_1(B) = \frac{2(1 - \sqrt{1-B})}{B},
\end{equation}

hence

\begin{equation}
H_1(A, B) = \frac{2(1 - \sqrt{1-B})}{B(-A + \sqrt{1-B})}.
\end{equation}

Next, let us consider equation (48) which we write equivalently as
\[
(AZ^{-1} - B) \frac{\partial H_2}{\partial A} + 2B \sqrt{1 - B} \frac{\partial H_2}{\partial B} + (2 \sqrt{1 - B} - 1) H_2 = -H_1. \tag{A.5}
\]
We apply a similar method of ‘separation of variables’ assuming that \( H_2(A, B) = ZF_2(B) + F_3(B) \). Inserting the ansatz for \( H_1 \) and \( H_2 \) into equation (A.5), and grouping the terms depending only on \( B \) on the right-hand side, we obtain
\[
AF_2(B) + Z^{-1}(2B \sqrt{1 - B} F'_2(B) + (2 \sqrt{1 - B} - 1) F_3(B)) = -2B \sqrt{1 - B} F'_2(B) - (2 \sqrt{1 - B} - 1) F_2(B) - F_3(B). \tag{A.6}
\]
Let us define the function
\[
G(B) = 2B \sqrt{1 - B} F'_3(B) + (2 \sqrt{1 - B} - 1) F_3(B). \tag{A.7}
\]
Then the variables in equation (A.6) can be separated as
\[
A(F_2(B) - G(B)) = -2B \sqrt{1 - B} F'_2(B) - (2 \sqrt{1 - B} - 1) F_2(B) - F_3(B) - \sqrt{1 - B} G(B). \tag{A.8}
\]
We conclude that both sides of the equation must be zero which implies that \( F_2 \) and \( F_3 \) satisfy the following system of differential equations:
\[
2B \sqrt{1 - B} F'_2(B) + (3 \sqrt{1 - B} - 1) F_2(B) = -F_1(B), \tag{A.9}
\]
\[
2B \sqrt{1 - B} F'_3(B) + (2 \sqrt{1 - B} - 1) F_3(B) = F_3(B). \tag{A.10}
\]
Using the boundary condition for \( H_2 \) we find that in the limit \( a \to 0 \) both \( F_2(B) \) and \( F_3(B) \) must be finite. Taking this into account, integration of the system (A.9) and (A.10) yields
\[
F_2(B) = F_3(B) = -2 \left( \frac{1 - \sqrt{1 - B}}{B} \right)^2. \tag{A.11}
\]
Therefore,
\[
H_2(A, B) = -2(1 - A + \sqrt{1 - B}) \left( \frac{1 - \sqrt{1 - B}}{B} \right)^2. \tag{A.12}
\]

Appendix B

In this appendix we give a brief derivation of the result (65). We shall do this in two steps. First we calculate the commutator \([\xi_\mu, \hat{x}_\nu]\) where \( \xi_\mu = \sum_\alpha dx_\alpha h_{\alpha \mu}(\hat{\partial}) \) and \( \hat{x}_\nu \) is given in the natural realization (30). We have
\[
[\xi_\mu, \hat{x}_\nu] = Z^{-1} \sum_\alpha [h_{\alpha \mu}, x_\nu] dx_\alpha + \hat{\partial}_\nu \sum_\alpha [h_{\alpha \mu}, iax] dx_\alpha. \tag{B.1}
\]
Expressing \( h_{\alpha \mu} \) by equation (56) and making use of
\[
\frac{\partial f(A, B)}{\partial \hat{\partial}_\mu} = i \frac{\partial f}{\partial A} a_\mu + 2a^2 \frac{\partial f}{\partial B} \partial_\mu, \tag{B.2}
\]
after some manipulation we find
\[
\sum_\alpha [h_{\alpha \mu}, x_\nu] dx_\alpha = \left( i \frac{\partial L_1}{\partial A} a_\nu + 2a^2 \frac{\partial L_1}{\partial B} \partial_\nu \right) dx_\mu \\
+ (iL_3 a_\mu + a^2 L_4 \partial_\mu) dx_\nu + iS_{\mu \nu}(a dx) + T_{\mu \nu}(\partial dx). \tag{B.3}
\]
A similar computation yields
\[
S_{\mu\nu} = L_2 \delta_{\mu\nu} + 2 \left( B \frac{\partial L_5}{\partial B} + L_5 \right) i a_\mu \partial_\nu + \frac{\partial L_2}{\partial A} i a_\mu \partial_\mu + 2a^2 \frac{\partial L_2}{\partial B} \partial_\mu \partial_\mu - a^2 \frac{\partial L_5}{\partial A} a_\mu a_\nu,
\]
\[
T_{\mu\nu} = a^2 L_4 \delta_{\mu\nu} + 2a^2 \frac{\partial L_3}{\partial B} i a_\mu \partial_\nu + a^2 \frac{\partial L_4}{\partial A} i a_\nu \partial_\mu + 2a^2 \frac{\partial L_4}{\partial B} \partial_\mu \partial_\nu - \frac{\partial L_3}{\partial A} a_\mu a_\nu.
\]

A similar computation yields
\[
\sum_a [h_{a\mu}, iax] d\alpha_a = a^2 E_1 d\alpha_a + (iE_2 a_\mu + a^2 E_3 \partial_\mu)(ia d\alpha_a) + a^2 (iE_4 a_\mu + a^2 E_5 \partial_\mu)(\partial d\alpha_a)
\]
\[
(B.6)
\]

where the functions \( E_i \) are defined by
\[
E_1 = 2A \frac{\partial L_1}{\partial B} - \frac{\partial L_1}{\partial A},
\]
\[
E_2 = L_2 + L_3 + 2AL_5 + 2AB \frac{\partial L_5}{\partial B} - B \frac{\partial L_5}{\partial A},
\]
\[
E_3 = L_4 + 2A \frac{\partial L_2}{\partial B} - \frac{\partial L_2}{\partial A},
\]
\[
E_4 = L_4 + 2A \frac{\partial L_1}{\partial B} - \frac{\partial L_1}{\partial A},
\]
\[
E_5 = 2A \frac{\partial L_4}{\partial B} - \frac{\partial L_4}{\partial A}.
\]
\[
(B.7) - (B.11)
\]

Combining equations (B.3) and (B.6) we obtain
\[
[\xi^\mu, \hat{\xi}_\nu] = d\alpha_a P_{\mu}^{(i)} + d\alpha_a P_{\mu}^{(2)} + (ia d\alpha_a) Q_{\mu
u}^{(i)} + (\partial d\alpha_a) Q_{\mu
u}^{(2)}.
\]
\[
(B.12)
\]

where the functions \( P_{\mu}^{(i)} \) and \( Q_{\mu
u}^{(i)} \) are given by
\[
P_{\nu}^{(i)} = Z^{-1} \frac{\partial L_1}{\partial A} i a_\nu + a^2 \left( 2Z^{-1} \frac{\partial L_1}{\partial B} + E_1 \right) \partial_\nu,
\]
\[
P_{\mu}^{(2)} = Z^{-1} L_3 i a_\mu + a^2 Z^{-1} L_4 \partial_\mu,
\]
\[
Q_{\mu\nu}^{(i)} = Z^{-1} S_{\mu\nu} + E_2 i a_\mu \partial_\nu + a^2 E_3 \partial_\mu \partial_\nu,
\]
\[
Q_{\mu\nu}^{(2)} = Z^{-1} T_{\mu\nu} + a^2 E_4 i a_\mu \partial_\nu + a^4 E_5 \partial_\mu \partial_\nu.
\]
\[
(B.13) - (B.16)
\]

In the second step, we wish to express the commutator (B.12) in terms of the one-forms \( \hat{\xi}_\mu \) and derivatives \( \partial_\mu \). In order to replace \( d\alpha_a \) in place of \( \hat{\xi}_\mu \), we write \( d\alpha_a \rightarrow \sum_a h_{a\mu}^{-1}(\partial) \hat{\xi}_\mu \), where \( h_{a\mu}^{-1} \) is the inverse of the matrix \( h_{a\mu} \). The inverse matrix should have the same index structure as \( h_{a\mu} \); hence, we look for \( h_{a\mu}^{-1} \) in the form
\[
h_{a\mu}^{-1}(\partial) = G_1 \delta_{a\mu} + ig_{2a\mu} \partial_\mu + ig_{3a\mu} \partial_\mu + a^2 G_4 \partial_\mu \partial_\mu - a^2 G_5 \partial_\mu \partial_\mu.
\]
\[
(B.17)
\]

The condition \( \sum_a h_{a\mu} h_{\beta\mu}^{-1} = \delta_{a\mu} \) implies that the functions \( G_k \) satisfy the following system of equations:
\[
G_1 = L_1^{-1},
\]
\[
-(L_1 + AL_2 + BL_3)G_2 - B(L_2 + AL_3)G_4 = L_2 L_1^{-1}.
\]
\[
(B.18) - (B.19)
\]
The solution of the system is given by

\[
G_2 = \frac{1}{M}[(L_1 + AL_3 + BL_4)L_2 + B(L_2 + AL_3)L_4],
\]

\[
G_3 = -\frac{1}{M}[(L_1 + AL_2 - BL_3)L_3 + B(L_3 - AL_4)L_5],
\]

\[
G_4 = -\frac{1}{M}[(L_3 - AL_4)L_2 + (L_1 + AL_2 - BL_3)L_4],
\]

\[
G_5 = \frac{1}{M}[(L_2 + AL_3)L_3 - (L_1 + AL_3 + BL_4)L_5],
\]

where

\[
M = L_1[(L_1 + AL_2 - BL_3)(L_1 + AL_3 + BL_4) + B(L_2 + AL_3)(L_3 - AL_4)].
\]

Now, with the functions \(G_k\) defined as above, we have

\[
dx_\mu = \sum_a h_{a\mu}^{-1}(\partial)\xi_a = G_1\xi_\mu + (\partial^2 G_5 i a_\mu + G_2\partial_\mu)(i\alpha \xi) + (G_3 i a_\mu + a^2 G_4 \partial_\mu)(\partial \xi).
\]

Using equation (B.28) to eliminate \(dx_\mu\) from the commutator (B.12) we obtain

\[
[\xi_\mu, \tilde{\xi}_\nu] = \frac{P_\mu^{(2)}}{L_1} + \xi_\nu \frac{P_\nu^{(2)}}{L_1} + (i\alpha \xi) R_{\mu\nu}^{(1)} + (\partial \xi) R_{\mu\nu}^{(2)},
\]

where \(R_{\mu\nu}^{(1)}\) and \(R_{\mu\nu}^{(2)}\) are defined by

\[
R_{\mu\nu}^{(1)} = \partial^2 G_5(P_\mu^{(1)} i a_\mu + P_\mu^{(2)} i a_\nu) + G_2(P_\mu^{(1)} \partial_\mu + P_\mu^{(2)} \partial_\nu)
\]

\[
+ (G_1 + AG_2 - BG_5) Q_{\mu\nu}^{(1)} + \partial^2(G_2 + AG_3) Q_{\mu\nu}^{(2)},
\]

\[
R_{\mu\nu}^{(2)} = G_3(P_\mu^{(1)} i a_\mu + P_\mu^{(2)} i a_\nu) + a^2 G_4(P_\mu^{(1)} \partial_\mu + P_\mu^{(2)} \partial_\nu)
\]

\[
+ a^2 (AG_4 - BG_5) Q_{\mu\nu}^{(1)} + (G_1 + AG_3 + BG_4) Q_{\mu\nu}^{(2)}.
\]

Tracing back the computations we can express the commutator (B.29) explicitly in terms of \(L_1, \ldots, L_5\) and their partial derivatives, but the expressions are cumbersome and not useful for practical calculations.

References

[1] Doplicher S, Fredenhagen K and Roberts J E 1994 The quantum structure of spacetime at the Planck scale and quantum fields Phys. Lett. B 331 39

[2] Seiberg N and Witten E 1999 String theory and noncommutative geometry J. High Energy Phys. JHEP09(1999)032 (arXiv:hep-th/9908142)

de Boer J, Grassi P A and van Nieuwenhuizen P 2003 Noncommutative superspace from string theory Phys. Lett. B 574 98 (arXiv:hep-th/0302078)

[3] Douglas M R and Nekrasov N A 2001 Noncommutative field theory Rev. Mod. Phys. 73 977 (arXiv: hep-th/0106046)

[4] Szabo R J 2003 Quantum field theory on noncommutative spaces Phys. Rep. 378 207 (arXiv:hep-th/0109162)
Szabo R J 2006 Symmetry, gravity and noncommutativity Class. Quantum Grav. 23 R199–R242 (arXiv: hep-th/0602233)
[5] Aschieri P, Jurco B, Schupp P and Wess J 2003 Noncommutative GUTs, standard model and CPT Nucl. Phys. B 651 45 (arXiv:hep-th/0205214)
Aschieri P, Blohmann C, Dimitrijević M, Meyer F, Schupp P and Wess J 2005 A gravity theory on noncommutative spaces Class. Quantum Grav. 22 3511 (arXiv:hep-th/0504183)
Calmet X and Kobakhidze A 2005 Noncommutative general relativity Phys. Rev. D 72 045010 (arXiv: hep-th/0506157)
[6] Balachandran A P, Govindarajan T R, Molina C and Teotonio-Sobrinho P 2004 Unitary quantum physics with time-space noncommutativity J. High Energy Phys. JHEP10(2004)72
Balachandran A P, Govindarajan T R, Martins A G and Teotonio-Sobrinho P 2004 Time-space noncommutativity: quantised evolutions J. High Energy Phys. JHEP11(2004)468
[7] Chaichian M, Kulish P P, Nishijima K and Tureanu A 2004 On a Lorentz-invariant interpretation on noncommutative space-time and its implications on noncommutative QFT Phys. Lett. B 604 98 (arXiv: hep-th/0408069)
Chaichian M, Presnajder P and Tureanu A 2005 New concept of relativistic invariance in NC space-time: twisted Poincaré symmetry and its implications Phys. Rev. Lett. 94 151602 (arXiv:hep-th/0409096)
[8] Li M and Wu Y S (ed) 2002 Physics in Noncommutative World: Field Theories (New Jersey: Rinton Press)
[9] Lukierski J, Nowicki A, Ruegg H and Tolstoy V N 1991 Q-deformation of Poincaré algebra Phys. Lett. B 264 331
[10] Lukierski J, Nowicki A and Ruegg H 1992 New quantum Poincaré algebra, and κ-deformed field theory Phys. Lett. B 293 344
[11] Lukierski J and Ruegg H 1994 Quantum κ-Poincaré in any dimension Phys. Lett. B 329 189 (arXiv: hep-th/9310117)
[12] Majid S and Ruegg H 1994 Bicrossproduct structure of κ-Poincaré group and noncommutative geometry Phys. Lett. B 334 349 (arXiv:hep-th/9404107)
[13] Kosiński P and Maślanka P 1994 The duality between κ-Poincaré algebra and κ-Poincaré group arXiv: hep-th/9411033
Chaichian M, Presnajder P and Tureanu A 2005 New concept of relativistic invariance in NC space-time: twisted Poincaré symmetry and its implications Phys. Rev. Lett. 94 151602 (arXiv:hep-th/0409096)
[15] Kosiński K, Lukierski J and Maślanka P 2000 Local D = 4 field theory on κ-Minkowski space Phys. Rev. D 62 025004 (arXiv:hep-th/9902037)
[16] Kosiński K, Lukierski J and Maślanka P 2000 Local field theory on κ-Minkowski space, ⋆-products and noncommutative translations Czech. J. Phys. 50 1283 (arXiv:hep-th/0009120)
[17] Kosiński P, Lukierski J, Maślanka P and Sitarz A 2003 Generalised κ-deformations and deformed relativistic scalar fields on noncommutative Minkowski space arXiv: hep-th/0307038
[18] Amelino-Camelia G 2001 Testable scenario for relativity with minimum-length Phys. Lett. B 510 255 (arXiv: hep-th/0012238)
Amelino-Camelia G 2002 Relativity in space-times with short-distance structure governed by an observer-independent (Planckian) length scale Int. J. Mod. Phys. D 11 35 (arXiv:gr-qc/0012051)
Bruno N R, Amelino-Camelia G and Kowalski-Glikman J 2001 Deformed boost transformations that saturate at the Planck scale Phys. Lett. B 522 133 (arXiv:hep-th/0107039)
[19] Kowalski-Glikman J and Nowak S 2002 Double special relativity theories as different bases of kappa-Poincaré algebra Phys. lett. B 539 126 (arXiv:hep-th/0203040)
[20] Amelino-Camelia G and Arzano M 2002 Coproduct and star-product in field theories on Lie algebra noncommutative spacetime Phys. Rev. D 65 084044 (arXiv:hep-th/0105120)
[21] Agostini A, Lizzi F and Zampini A 2002 Generalized Weyl systems and κ-Minkowski space Mod. Phys. Lett. A 17 2105 (arXiv: hep-th/0209174)
[22] Amelino-Camelia G, D’Andrea F and Mandanici G 2003 Group velocity in noncommutative spacetime J. Geom. Phys. 47 331 (arXiv:hep-th/0211022)
[23] Dimitrijević M, Jonke L, Möller L, Tsouchnika E, Wess J and Wohlgenannt M 2003 Deformed field theory on κ-spacetime Eur. Phys. C 31 129 (arXiv:hep-th/0307149)
[24] Dimitrijević M, Meyer F, Möller L and Wess J 2004 Gauge theories on the κ-Minkowski space time Eur. Phys. J. C 36 117 (arXiv:hep-th/0301116)
[25] Dimitrijević M, Möller L and Tsouchnika E 2004 Derivatives, forms and vector fields on the κ-deformed Euclidean space J. Phys. A: Math. Gen. 37 9749 (arXiv:hep-th/0404224)
[26] Meljanac S and Stojic M 2006 New realizations of Lie algebra kappa-deformed Euclidean space Eur. Phys. J. C 47 531 (arXiv:hep-th/0605133)
[27] Bu J G, Kim H C, Lee Y, Vac C H and Yee J H 2008 $\kappa$-deformed spacetime from twist Phys. Lett. B 665 95 (arXiv:hep-th/0611175v2)
[28] Meljanac S, Krešić-Jurić S and Stojić M 2007 Covariant realizations of kappa-deformed space Eur. Phys. J. C 51 229 (arXiv:hep-th/0702215)
[29] Meljanac S, Samsarov A, Stojić M and Gupta K S 2008 Kappa-Minkowski space-time and the star-product realizations Eur. Phys. J. C 53 295-309 (arXiv:0705.2471)
[30] Meljanac S and Krešić J 2008 Generalized kappa-deformed spaces, star-products and their realizations J. Phys. A: Math. Theor. 41 235203
[31] Möller L 2005 A symmetry invariant integral on $\kappa$-deformed spacetime J. High Energy Phys. JHEP12(2005)029 (arXiv:hep-th/0409128)
[32] Amelino-Camelia G, Arzano M and D’Andrea F 2004 Action functional for $\kappa$-Minkowski noncommutative spacetime arXiv:hep-th/0407227
[33] Daskiewicz M, Lukierski J and Nowicki A 1998 Kappa-deformed covariant phase space and quantum gravity Phys. At. Nucl. 61 1811 (arXiv:hep-th/9706031v1)
[34] Meljanac S, Mileković M and Pallua S 1994 Unified view of deformed single-mode oscillator algebras Phys. Lett. B 328 55 (arXiv:hep-th/9404039)
[35] Meljanac S and Mileković M 1996 Unified view of multimode algebras with Fock-like representation Int. J. Mod. Phys. A 11 1391
[36] Meljanac S and Perica A 1994 Number operators in a general quon algebra J. Phys. A: Math. Gen. 27 4737 Meljanac S and Perica A 1994 Generalized quon statistics Mod. Phys. Lett. A 9 3293 (arXiv:hep-th/9409180)
[37] Meljanac S, Perica A and Svrtan D 2003 The energy operator for a model with a multiparametric infinite statistics J. Phys. A: Math. Gen. 36 6537 (arXiv:math-ph/0304038)
[38] Bardek V and Meljanac S 2000 Deformed Heisenberg algebras, a Fock space representation and the Calogero model Eur. Phys. J. C 17 539 (arXiv:hep-th/0009099)
[39] Bardek V, Jonke L, Meljanac S and Mileković M 2002 Calogero model, deformed oscillators and the collapse Phys. Lett. B 531 311 (arXiv:hep-th/0107053)
[40] Kempf A, Mangano G and Mann R B 1995 Hilbert space representation of the minimal length uncertainty relation Phys. Rev. D 52 1108 (arXiv:hep-th/9412167)
[41] Chang L N, Minic D, Okamura N and Takeuchi T 2002 The effect of the minimal length uncertainty relation on the density of states and the cosmological constant problem Phys. Rev. D 65 125027 (arXiv:hep-th/0201017)
[48] Durov N, Meljanac S, Samsonov A and Škoda Z 2007 A universal formula for representing Lie algebra generators as formal power series with coefficients in the Weyl algebra J. Algebra. 309 318 (arXiv:math.RT/0604096)

[49] Škoda Z. 2008 Twisted exterior derivatives for enveloping algebras arXiv:0806.0978

[50] Kim H C, Lee Y, Rim C and Yee J H 2008 Differential structure on the κ-Minkowski spacetimes from twist arXiv:0808.2866v1

[51] McCurdy S, Tagliapietra A and Zumino B 2008 The star product for differential forms on symplectic manifolds arXiv:0809.4717

[52] Kupriyanov V G and Vassilevich D V 2008 Star products made (somewhat) easier arXiv:0806.4615

[53] Drinfeld V D 1987 Quantum groups Proc. ICM ed A Gleason (Providence, RI: American Mathematical Society) pp 798–820

[54] Majid S 1995 Foundations of Quantum Group Theory (Cambridge: Cambridge University Press)

[55] Aschieri P, Lizzi F and Vitale P 2007 Twisting all the way: from classical mechanics to quantum fields arXiv:0708.3002v2

[56] Banerjee R and Samanta S 2007 Gauge symmetries on θ-deformed spaces J. High Energy Phys. JHEP02(2007)046 (arXiv:hep-th/0611249)

Banerjee R and Samanta S 2007 Gauge generators, transformations and identities on a noncommutative space Eur. Phys. J. C 51 207 (arXiv:hep-th/0608214)

Banerjee R, Mukherjee P and Samanta S 2007 Lie algebraic noncommutative gravity Phys. Rev. D 75 125020 (arXiv:hep-th/0703128)

[57] Arzano M and Benedetti D 2008 Rainbow statistics arXiv:0809.0889v1

[58] Young C A S and Zegers R 2007 Covariant particle statistics and intertwiners of the κ-deformed Poincare algebra arXiv:0711.2206v2

[59] Govindarajan T R, Gupta K S, Harikumar E, Meljanac S and Meljanac D 2008 Twisted statistics in κ-Minkowski spacetime Phys. Rev. D 77 105010 (arXiv:08021576v2)