CONVERGENCE OF THE PCTL ALGORITHM FOR SOLVING THE DISCRETIZED 3T ENERGY EQUATIONS IN RHD PROBLEMS

YUE HAO *, SILU HUANG *, AND XIAOWEN XU † *

Abstract. For solving the three-temperature linear system arising from the radiation hydrodynamics (RHD) problems, Xu et. proposed a physical-variable based coarsening algebraic two level (PCTL) iterative method and verified its efficiency by numerical experiments. However, there is still a lack of quantitative evaluation of the performance of PCTL algorithm, thus we aim to fill in this blank in this paper. By theoretical analysis, we give an estimation on the convergence factor of the PCTL algorithm and show it is independent of the problem size, which provides a theoretical guarantee for solving large-scale problems. Moreover, we also discuss the factors that affect the efficiency of PCTL algorithm and provide a direction for the efficient application of the PCTL algorithm.

Key words. Three-temperature linear system. PCTL algorithm. Algebraic multi-grid method. Convergence estimation. Symmetric positive definite. RHD problems.

AMS subject classifications. 65F10, 65N12, 65N55

1. Introduction. In this paper, we focus on evaluating the convergence performance of the physical-variable based coarsening algebraic two level (PCTL) iterative algorithm [25, 28], which is proposed for solving the following three-temperature (3T) linear system of equations arising from the radiation hydrodynamics (RHD) problems:

\[
Ax := \begin{bmatrix}
A_r & 0 & D_{er}^T \\
0 & A_i & D_{ei}^T \\
D_{er} & D_{ei} & A_e
\end{bmatrix} \begin{bmatrix}
x_r \\
x_i \\
x_e
\end{bmatrix} = \begin{bmatrix}
b_r \\
b_i \\
b_e
\end{bmatrix} =: b,
\]

with the properties described below:

1. The coefficient matrix \( A = (a_{kj})_{3n \times 3n} \) is symmetric and positive-definite (SPD) and strong diagonally dominant with \( a_{kk} > \sum_{j \neq k} |a_{kj}| \) for all \( 1 \leq k \leq 3n \).

2. The diagonal blocks \( A_\alpha = (a_\alpha^{\alpha})_{n \times n} (\alpha = r, i, e) \) are M-matrix, that is \( a_\alpha^{\alpha} > 0 \), \( a_\alpha^{\alpha} \leq 0 (j \neq k) \), and all elements of \( A_\alpha^{-1} \) are nonnegative, which reflect the diffusion of the radiation, ion and electron temperatures, and the sparsity patterns are determined by the spatial discretization method.

3. The coupling terms \( D_\alpha (\alpha = r, i) \) are diagonal matrices with negative diagonal elements, which express the energy exchange between the \( e \)-th and the \( \alpha \)-th physical quantities.

4. The matrices \( A_r, A_i \) and \( A_e - D_{er}A_e^{-1}D_{er}^T - D_{ei}A_i^{-1}D_{ei}^T \) are all SPD, since \( \hat{A} \) is contract with the matrix

\[
\hat{A} = \begin{bmatrix}
A_r & 0 & 0 \\
0 & A_i & 0 \\
0 & 0 & A_e - D_{er}A_e^{-1}D_{er}^T - D_{ei}A_i^{-1}D_{ei}^T
\end{bmatrix}.
\]

Solving 3T energy equations is an important part in radiation hydrodynamics (RHD) equations, which arising from many fields, such as the inertial confinement
fusion (ICF), astrophysical phenomena and so on [12, 16]. Moreover, complex application properties make the coefficient matrix in (1.1) ill-conditioned and difficult to solve, such that solving a sequence of 3T systems in RHD simulation takes up most of the total time. Thus, developing an efficient and practical algorithm for (1.1) is a crucial problem. In recent years, numerous methods have been proposed for 3T linear systems, in which preconditioned Krylov subspace techniques are the most favorable choices with the preconditioners mainly include incomplete LU factorization, geometric and algebraic multigrid, domain decomposition and their effective combination [2, 6, 13, 25, 24, 1, 26, 28, 7].

In [25], a physical-variable based coarsening algebraic two level (PCTL) iterative algorithm was proposed for the equation (1.1). By constructing a specific coarsening strategy, the PCTL algorithm divides the fully coupled system into four individual scalar subsystems, and thus overcomes the challenges caused by the complicated couplings among physical quantities and makes the system much easier to solve. Numerical results have demonstrated the high efficiency and scalability of the PCTL strategy both as a solver and preconditioner for Krylov subspace methods when applied to the two dimensional 3T radiation diffusion equation, and showed that its convergence is independent of the problem size, which is referred to as uniform convergence in this paper. Recently, to adapt to the dynamically and slowly changing features of the sequence of 3-T linear systems and to further improve the efficiency of the PCTL algorithm, Huang et al. [7] proposed an αSetup-PCTL algorithm by constructing an adaptive two-level preconditioner based on the PCTL algorithm. Numerical experiments have showed that the αSetup-PCTL algorithm can significantly improve the overall efficiency.

However, the uniform convergence and efficiency of the PCTL algorithm are only observed from the numerical experiments and have not been theoretically analysed yet. Actually, the PCTL method is a kind of algebraic two-grid (ATG) method, and thus general convergence frameworks for analyzing the convergence of the ATG method [17, 19, 20, 4, 5, 14] also apply to the PCTL method and could show that the PCTL algorithm is convergent, but its specific convergence factor has not been evaluated, which is what we concern. In this paper, by verifying the PCTL method satisfies the smoothing property and the approximation property, we derive an estimation on the convergence factor and state the uniform convergence, which is a desired property and provides a theoretical guarantee for solving large-scale problems. More details on the smoothing property and approximation property of the multigrid method can refer to [17, 19, 8]. Besides, noticing that the diagonal dominance and coupling terms are the two main factors affecting the performance of the PCTL algorithm, we also discuss the influence of these two factors on the convergence of the PCTL method, which gives an insight into which problems the PCTL algorithm is effective.

The rest of this paper is structured as follows. In section 2, we introduce the three temperature equations and the PCTL algorithm for completeness. Then the convergence properties of the PCTL algorithm are analysed and an upper bound on its convergence factor is derived in section 3. In section 4, we discuss the factors that affect the upper bound and illustrate it by numerical examples. Finally, some conclusions are given in section 5.

2. PCTL algorithm. For the sake of completeness and as research object, we present the PCTL algorithm in this section. In addition, the three temperature equations are briefly introduced first, whose discretization leads to a sequence of 3T linear systems (1.1).
2.1. Three temperature equations. Consider the radiation diffusion equations \([27, 24]\):

\[
\begin{align*}
\frac{\partial E_r}{\partial t} + \nabla \cdot \left( -\frac{c \lambda(E_r)}{\kappa_r} \nabla E_r \right) &= c (E_p - E_r) \\
\rho_e \frac{\partial T_e}{\partial t} + \nabla \cdot \left( -\kappa_e T_e^{5/2} \nabla T_e \right) &= -c \kappa_p (E_p - E_r) + \omega_{ei} (T_i - T_e) \\
\rho_i \frac{\partial T_i}{\partial t} + \nabla \cdot \left( -\kappa_i T_i^{5/2} \nabla T_i \right) &= -\omega_{ei} (T_i - T_e)
\end{align*}
\]  

(2.1)

where \(c\) is the speed of light, \(\lambda(E_r)\) is a nonlinear limiter, \(\rho\) is the medium density, \(\omega_{ei}\) is the electron-ion coupling coefficient, \(E_r\) and \(E_p\) are the radiation and the electron scattering energy densities, respectively, \(T_e\) and \(T_i\) are the electron and ion temperatures, respectively, \(\kappa_r\) and \(\kappa_p\) are the Rosseland and the Planck mean absorption coefficients, respectively, \(\kappa_e\) and \(\kappa_i\) denote the diffusion coefficients of electron and ion, respectively, \(c_e\) and \(c_i\) are the electron and ion heat capacity, respectively. The equations (2.1) describe the propagation of radiation energy in the medium, as well as the energy exchange processes. Moreover, \(E_p\) and \(E_r\) can be defined as

\[
E_p = 4\sigma T_e^4/c \quad E_r = 4\sigma T_r^4/c,
\]

thus equations (2.1) are also known as the three temperature equations. For the discretization of the three temperature equations, it often uses fully implicit schemes, followed by the frozen-in coefficients method for linearization in the temporal direction, and numerous methods such as finite volume method in the spatial direction, which yields a sequence of 3-T linear systems (1.1) to be solved.

2.2. PCTL algorithm. In the reference [25], Xu e.t. think that there are two main difficulties in solving the 3T linear system (1.1) by the classical AMG method. Due to the coupling of the three types of variables, the structure of the coarse-level operator under classical coarsening and interpolation processes becomes much complicated, and the performance of the general point relaxation and block relaxation based on grid points decreases. Realizing it, the authors proposed a physical-variable based coarsening strategy based on the characteristic of the matrix \(A\), with the electron temperature variables as coarse points and the others as fine points. Combined with the C/F block relaxation, the PCTL algorithm decouples the fully coupled 3T linear systems (1.1) into some individual scalar subsystems that are easier to solve. Moreover, the interpolation operator in the PCTL algorithm is selected as

\[
P = (P^T_r, P_i^T, I)^T.
\]

In this case, the ideal interpolation is \(P_\alpha = P^{ex}_\alpha := -A^{-1}_\alpha D_{\alpha e} (\alpha = r, i)\). However, \(P^{ex}_\alpha\) are often dense and expensive to compute. Therefore, in order to save cost and try to ensure the same structure of the coarse-level operator \(A_c := P^T A P\) and the matrix \(A_e\), the interpolation operators \(P_\alpha\) in the PCTL algorithm are restricted to be diagonal and to give an exact approximation only for constant vectors, that is

\[
P_\alpha 1 = P^{ex}_\alpha 1 \quad (\alpha = r, i),
\]

where 1 is the vector of all elements 1. The detailed processes of the PCTL algorithm is described in Algorithm 2.1.
Algorithm 2.1 PCTL algorithm for the linear system (1.1)

Input: Matrix $A: \mathbb{R}^{3n \times 3n} \rightarrow \mathbb{R}^{3n \times 3n}$, right-hand side $b \in \mathbb{R}^{3n}$, initial guess $x^{(0)} := \begin{pmatrix} x_r^{(0)}T \\ x_i^{(0)}T \\ x_e^{(0)}T \end{pmatrix}$, and the stop tolerance $\epsilon$.

Output: Approximate solution $x$ fulfilling $\|b - Ax\|_2 / \|b\|_2 \leq \epsilon$.

1: Setup phase: construct the associated coarse-level operator $A_c$, interpolation operator $P = (P^T P^T I)^T$ and restriction operator $R = P^T$

2: Solve phase: while $\|b - Ax\|_2 / \|b\|_2 > \epsilon$ do

2.1 Pre-smoothing: do C/F block smoothing

$$x_c^{(k+1/3)} = A_c^{-1}(b_c - D_{er}x_r^{(k)} - D_{ci}x_i^{(k)})$$

$$x_r^{(k+1/3)} = A_r^{-1}(b_r - D_{er}x_c^{(k+1/3)})$$

$$x_i^{(k+1/3)} = A_i^{-1}(b_i - D_{ci}x_c^{(k+1/3)})$$

2.2 Coarse-grid solver:

$$A_c v_c = r_c = P^T(b - Ax^{(k+1/3)})$$

2.3 Coarse-grid correction:

$$x_c^{(k+2/3)} = x_c^{(k+1/3)} + v_c$$

$$x_r^{(k+2/3)} = x_r^{(k+1/3)} + P_{tv_c}$$

$$x_i^{(k+2/3)} = x_i^{(k+1/3)} + P_{iv_c}$$

2.4 Post-smoothing: do F/C block smoothing

$$x_r^{(k+1)} = A_r^{-1}(b_r - D_{er}x_c^{(k+2/3)})$$

$$x_i^{(k+1)} = A_i^{-1}(b_i - D_{ci}x_c^{(k+2/3)})$$

$$x_e^{(k+1)} = A_e^{-1}(b_e - D_{er}x_r^{(k+1)} - D_{ci}x_i^{(k+1)})$$

3. Convergence of the PCTL algorithm. Although the generic convergence frameworks for analyzing the convergence of the ATG method [17, 19, 20, 4, 5, 14] are also applicable for the PCTL algorithm and could prove the convergence of the PCTL algorithm, our goal in this section is to characterize the specific convergence properties of the PCTL algorithm, expecting to be helpful for further research on the PCTL algorithm, such as improving its performance and analysing which problems it is effective. Before it, we first introduce some studies on the convergence estimation of the ATG method.

In this paper, real matrices are considered. Given a symmetric and positive definite $n \times n$ matrix $A$, we define the $A$-norm or energy norm by $\|x\|_A^2 = x^T A x$ with $x \in \mathbb{R}^n$, and the corresponding induced matrix norm by $\|B\|_A = \max_{x \in \mathbb{R}^n, \|x\|_A = 1} \|Bx\|_A$.

3.1. Convergence of the ATG method. The ATG method is composed by smoothing process and coarse-grid correction process. In general, the smoothing process works well at eliminating oscillatory errors and poorly at eliminating algebraically smooth errors, while the coarse-grid correction process follows to compensate it...
and to further reduce algebraically smooth errors, such that a better solution could be obtained.

In this subsection, we consider a symmetric two-grid scheme, which is the simplest but the most representative scheme. If representing the pre- and post-smoother by $G_1 := I - M^{-T}A$ and $G_2 := I - M^{-1}A$, respectively, and the coarse-level correction error propagator by $T := I - P A_c^{-1}P^T A$, where $P$ is the interpolation operator and $A_c := P^T A$ is the coarse-level matrix constructed by the Galerkin strategy, then the error propagation matrix of the resulting two-grid method reads

$$ E_{ATG} = G_2 T G_1 = (I - M^{-1}A)(I - P(P^T P)^{-1}P^T)(I - M^{-T}A). $$

The convergence theory of the algebraic multigrid method mainly focuses on characterizing the (energy) norm of the error propagation matrix, that is $\|E_{ATG}\|_A$, and the study on it has been well developed [17, 19, 20, 5, 15]. In particular, the references [11, 10, 3, 9] have laid the foundation for numerous classical algebraic theoretical analyses of the AMG methods. In recent decades, numerous universal convergence frameworks have been emerged for the exact or inexact AMG method, and for symmetric or non-symmetric problems [4, 5, 14, 15, 22, 23], especially that the convergence factor of the two-grid method has even be characterized by an elegant identity as in Theorem 3.1 [5].

**Theorem 3.1.** [5] Assume the smoothing operator is $A$-norm convergent, that is $\|I - M^{-1}A\|_A < 1$, and denote

$$ \tilde{M} := M^T(M + M^T - A)^{-1}M, \quad \text{and} \quad \Pi_{\tilde{M}} := P (P^T \tilde{M} P)^{-1}P^T \tilde{M}. $$

Then the convergence factor of the ATG method can be characterized as

$$ (3.1) \quad \|E_{ATG}\|_A = 1 - \frac{1}{K_{ATG}}, $$

where

$$ K_{ATG} = \max_{v \in \mathbb{R}^n \setminus \{0\}} \frac{\|(I - \Pi_{\tilde{M}})v\|_M^2}{\|v\|_A^2}. $$

However, it is worth noting that the estimation (3.1) is impractical for its expensive computational cost, thus most researches turn to measuring the convergence rate of the AMG methods by finding a sharp upper bound on $\|E_{ATG}\|_A$ [17, 19, 20, 8]. A widely known strategy is translating the estimation into some sufficient conditions on the smoothing process and coarse-level correction process, which are known as smoothing property and approximation property. Before introducing it, we first state that there holds

$$ \|E_{ATG}\|_A = \|G_2 T\|_A^2, $$

such that we just discuss the analysis of $\|G_2 T\|_A$. Then, we give descriptions of the estimation derived from the smoothing and approximation properties.

**Lemma 3.2.** [17] Let $A$ be an SPD matrix with positive diagonal elements. Suppose that, for all vector $e$, there exist $\alpha_1, \beta_1 > 0$ independently of $e$ such that

$$ (3.2) \quad \|G_2 e\|_A^2 \leq \|e\|_A^2 - \alpha_1 \|e\|_2^2 \quad \text{(post-smoothing property)} $$

$$ \|T e\|_A^2 \leq \beta_1 \|e\|_2^2 \quad \text{(approximation property)}, $$

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where \( \|e\|_2^2 := (D^{-1}Ae, Ae) \) and \( D \) is the diagonal matrix of \( A \). Then the convergence factor of the ATG method satisfies

\[
\|G_2 T\|_A \leq \sqrt{1 - \alpha_1/\beta_1}
\]

and thus

\[
\|E_{ATG}\|_A \leq 1 - \alpha_1/\beta_1.
\]

Although this estimation leads to a certain loss of sharpness, it enables the effects of the smoothing process and coarse-grid correction process on the performance of the ATG algorithm more intuitive, and also helps to make wise choices about the components of an ATG algorithm for particular problems.

In addition, it is worthy to note that choosing other non-negative functions rather than \( ||e||_2^2 \) in the (3.2) will similarly lead to other different formats of the smoothing property and approximation property, and the readers can refer to [8] for more details.

Actually, it often turns the proof of the post-smoothing property in Lemma 3.2 into a comparison between two matrices in applications.

**Lemma 3.3.** [17] Given a symmetric and positive definite matrix \( A \) with positive diagonal elements, and a post-smoothing operator \( G_2 \) with the form \( G_2 = I - M^{-1}A \), then the post-smoothing property in Lemma 3.2 is equivalent to

\[
\alpha_1 M^T D^{-1} M \leq M + M^T - A, \quad \text{(post-smoothing property)}
\]

where \( A_1 \leq A_2 \) represents the matrix \( A_2 - A_1 \) is symmetric and positive semi-definite (SPSD).

Moreover, as for the two-grid method, the assumption that the approximation property in (3.2) holds only for vectors \( e \in \text{Range}(T) \) is sufficient to ensure that the convergence estimation (3.3) holds. This assumption leads to a more intuitive but weaker approximation property as follows.

**Lemma 3.4.** [17] Let \( A \) be an SPD matrix with positive diagonal elements and the post-smoothing operator \( G_2 \) satisfy the post-smoothing property (3.2). Suppose there holds

\[
\min_{e_C} ||e - Pe_C||_0^2 \leq \beta_1 ||e||_A^2 \quad \forall e
\]

with \( \beta_1 > 0 \) independent of \( e \), where \( ||e||_0^2 := (De, e) \). Then

\[
\|G_2 T\|_A \leq \sqrt{1 - \alpha_1/\beta_1}.
\]

Furthermore, when the coarse-grid set is restricted to be a subset of the fine-grid set, the matrix \( A \) could be partitioned as

\[
A = \begin{bmatrix}
A_{FF} & A_{FC} \\
A_{CF} & A_{CC}
\end{bmatrix}
\]

with the matched vector \( e = \begin{bmatrix} e_F \\ e_C \end{bmatrix} \). In this setting, many interpolation schemes take the interpolation operator as the form \( P = \begin{bmatrix} P_{FC} \\ I \end{bmatrix} \). Under these conditions, a more practical weak approximation property could be considered.
Lemma 3.5. [19] Under the same assumptions as in Lemma 3.4 and suppose there exist a parameter $\beta_1 > 0$ independent of $e$ such that

\begin{equation}
||e_F - P_{FC}e_C||^2_{0,F} \leq \beta_1 ||e||^2_A,
\end{equation}

where $(u_F, v_F)_0,F = (D_{FF}u_F, v_F)$ and $D_{FF}$ is the diagonal matrix of $A_{FF}$, then the weak approximation property (3.6) holds.

The above lemmas give a feasible strategy to prove the uniform convergence and to assess the convergence rate of a two-grid method. Thus, we will analyse the convergence of the PCTL algorithm based on this strategy in the next subsection.

3.2. Convergence properties of the PCTL algorithm. Although Theorem 3.1 reflects that the PCTL algorithm is convergent, a reasonable estimation of its convergence rate is significant and much more beneficial, as it can evaluate the performance of the PCTL algorithm and help to further improve the performance. Specifically, by proving the PCTL algorithm satisfying the smoothing property and approximation property introduced in Lemma 3.3 and Lemma 3.5, respectively, we derive an upper bound on the convergence factor of the PCTL algorithm in this subsection.

Note that the convergence factor of the PCTL algorithm satisfies

\[ ||E_{PCTL}||_A = ||((I - M^{-1}A)(I - P(P^TAP)^{-1}P^TA))||^2_A, \]

where the post-smoothing operator $G_2 = I - M^{-1}A$ is with

\begin{equation}
M = \begin{bmatrix} A_r & 0 & 0 \\
0 & A_i & 0 \\
D_{er} & D_{ei} & A_e \end{bmatrix},
\end{equation}

and the interpolation operator $P = (P_r^T, P_i^T, I)^T$ is with

\[ P_r = \begin{bmatrix} -\sum_{j=1}^n b_{rj}^i d_j^r \\
\vdots \\
-\sum_{j=1}^n b_{rj}^i d_j^r \end{bmatrix}, \]

and

\[ P_i = \begin{bmatrix} -\sum_{j=1}^n b_{ij}^i d_j^i \\
\vdots \\
-\sum_{j=1}^n b_{ij}^i d_j^i \end{bmatrix}. \]

Firstly, we introduce a lemma to characterize some properties of the eigenvalues of a matrix.

Lemma 3.6. [18] Suppose that matrices $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times n}$ are symmetric and positive definite, then there hold

\[ \lambda_{\min}(A^{-1}B) \leq \frac{v^T B v}{v^T A v} \leq \lambda_{\max}(A^{-1}B), \quad \forall v \in \mathbb{R}^n \]

and

\[ \lambda_{\min}(A^{-1}B) = \frac{1}{\lambda_{\max}(B^{-1}A)}, \quad \lambda_{\max}(A^{-1}B) = \frac{1}{\lambda_{\min}(B^{-1}A)}. \]
Lemma 3.7. (Gerschgorin Disk Theorem) [21] Let $A = (a_{ij})_{n \times n}$ be an arbitrary complex matrix, and let

$$\Lambda_i := \sum_{j=1, j\neq i}^{n} |a_{ij}|, \quad 1 \leq i \leq n,$$

where $\Lambda_1 := 0$ if $n = 1$. If $\lambda$ is an eigenvalue of $A$, then there is a positive integer $r$, with $1 \leq r \leq n$, such that

$$|\lambda - a_{rr}| \leq \Lambda_r.$$

Hence, all eigenvalues $\lambda$ of $A$ lie in the union of the disks

$$|z - a_{ii}| \leq \Lambda_i, \quad 1 \leq i \leq n.$$

Then, we begin to discuss the convergence behavior of the PCTL algorithm in detail, and start by proving the smoothing property of the PCTL algorithm.

Theorem 3.8. For the $3T$ linear system in the equation (1.1), the post-smoothing operator $G_2$ of the PCTL algorithm satisfies the post-smoothing property (3.2) with $\alpha_1 = \frac{1}{4}$, that is for all $e$, there holds

$$\|G_2e\|_A^2 \leq \|e\|_A^2 - \frac{1}{4}\|e\|_2^2.$$  (3.10)

Proof. As stated in the Lemma 3.3, to prove the post-smoothing property, it is equivalent to prove that there exists a parameter $\alpha_1$ such that

$$\alpha_1 M^TD^{-1}M \leq M + M^T - A,$$

which is equivalent to

$$\alpha_1 \leq \frac{v^*(M + M^T - A)v}{v^*(M^TD^{-1}M)v}, \quad \forall v.$$  (3.11)

Denote

$$H := (M + M^T - A)^{-1}(M^TD^{-1}M),$$

then Lemma 3.6 gives

$$\lambda_{\min}(H^{-1}) \leq \frac{v^*(M + M^T - A)v}{v^*(M^TD^{-1}M)v} \leq \lambda_{\max}(H^{-1}),$$

and thus the inequality (3.12) holds as long as

$$\alpha_1 \leq \lambda_{\min}(H^{-1}) = \frac{1}{\lambda_{\max}(H)}.$$  (3.12)

Furthermore, if we could get an upper bound $\gamma$ of $\lambda_{\max}(H)$ such that $\lambda_{\max}(H) \leq \gamma$, and take $\alpha_1 = 1/\gamma$, then the inequality (3.11) holds naturally.

In the PCTL algorithm, there have

$$M + M^T - A = \begin{bmatrix} A_r & 0 & 0 \\ 0 & A_i & 0 \\ 0 & 0 & A_r \end{bmatrix}.$$
and

\[
M^T D^{-1} M = \begin{bmatrix}
A_r & 0 & D_{er}^T \\
0 & A_i & D_{ei}^T \\
0 & 0 & A_e
\end{bmatrix}
\begin{bmatrix}
D_{er}^{-1} A_r & 0 & 0 \\
0 & D_{ei}^{-1} A_i & 0 \\
D_{er}^{-1} D_{ei} & D_{ei}^{-1} D_{ei} & D_{ei}^{-1} A_e
\end{bmatrix}
\]

where \( D_r, D_i \) and \( D_e \) are the diagonal matrices of \( A_r, A_i \) and \( A_e \), respectively, then we derive

\[
H = \begin{bmatrix}
I & 0 & A_r^{-1} D_{er}^T \\
0 & I & A_i^{-1} D_{ei}^T \\
0 & 0 & I
\end{bmatrix}
\begin{bmatrix}
D_{er}^{-1} A_r & 0 & 0 \\
0 & D_{ei}^{-1} A_i & 0 \\
D_{er}^{-1} D_{ei} & D_{ei}^{-1} D_{ei} & D_{ei}^{-1} A_e
\end{bmatrix} =: H_1 H_2.
\]

It is known that the largest eigenvalue of a matrix satisfies

\[
\lambda_{\text{max}}(H) \leq ||H||_\infty \leq ||H_1||_\infty ||H_2||_\infty,
\]

where \( ||H||_\infty \) is the infinite norm of the matrix \( H = (h_{kj}) \) with the definition as

\[
||H||_\infty = \max_{1 \leq k \leq 3n} \sum_{j=1}^{3n} |h_{kj}|.
\]

Using the diagonal dominance of the matrix \( A \), it is easy to prove that

\[
||H_2||_\infty \leq 2.
\]

As for the estimation of \( ||H_1||_\infty \), we take the first row as an example and the rest rows are similar. Denote

\[
A_\alpha = (a_{\alpha kj}^\alpha)_{n \times n}, \quad A_\alpha^{-1} = (b_{\alpha kj}^\alpha)_{n \times n}, \quad (\alpha = r, i, e) \\
D_{er} = \text{diag}(d_{1}^r, \cdots, d_{n}^r), \quad D_{ei} = \text{diag}(d_{1}^i, \cdots, d_{n}^i).
\]

The property \( A_r^{-1} A_r = I \) leads to

\[
\sum_{k=1}^{n} b_{1k}^r a_{k1}^r = 1 \quad \text{and} \quad \sum_{k=1}^{n} b_{1k}^r a_{kj}^r = 0 \quad (j = 2, \cdots, n),
\]

and further gives

\[
(3.13) \quad \sum_{k=1}^{n} b_{jk}^r \sum_{j=p}^{n} a_{kj}^r = 1.
\]
Now consider the first row of $H_1$:

$$h_1 := \sum_{k=1}^{3n} |(H_1)_{1k}| = \sum_{k=1}^{n} (-b_{1k}^r d_k^r) + 1 = \sum_{k=1}^{n} (-b_{1k}^r d_k^r) + 1$$

$$(3.13) \quad = \sum_{k=1}^{n} (-b_{1k}^r d_k^r) + \sum_{k=1}^{n} b_{1k}^r \sum_{j=1}^{n} a_{kj}^r$$

$$= \sum_{k=1}^{n} b_{1k}^r [-d_k^r + \sum_{j=1}^{n} a_{kj}^r]$$

$$\leq 2 \sum_{k=1}^{n} b_{1k}^r \sum_{j=1}^{n} a_{kj}^r \quad \text{(since A is diagonal dominant)}$$

$$(3.13) \quad = 2.$$

In the same way, it can be proved that the rest rows of $H_1$ also satisfy

$$h_k := \sum_{j=1}^{3n} |(H_1)_{kj}| \leq 2 \quad (2 \leq k \leq 3n).$$

Thus, we get $\lambda_{\max}(H) \leq 4$ and then taking $\alpha_1 = \frac{1}{4}$ gives the post-smoothing property (3.10).

Recall that the PCTL algorithm is a kind of two-grid method, such that the weak approximation property in Lemma 3.5 is sufficient for its convergence. Next we give the proof.

**Theorem 3.9.** For the $3T$ linear system in the equation (1.1), denote $D_{e\alpha} = \text{diag}(d_{e\alpha}^r)$ ($\alpha = r, i$), $A_{\alpha} = (a_{e\alpha}^r)_{n \times n}$ and $A_{\alpha}^{-1} = (b_{e\alpha}^r)_{n \times n}$ ($\alpha = r, i, e$). Then the coarse-level correction operator $T$ of the PCTL algorithm satisfies the weak approximation property

$$(3.14) \quad \|e_F - I_F e_C\|_{0,F}^2 \leq \beta \|e\|_A^2.$$ 

with

$$(3.15) \quad \beta = \max_{1 \leq k \leq 3n} m_k,$$
Proof. In the PCTL algorithm, the two sides of the inequality (3.8) are

\[ \|e\|_A^2 = \begin{bmatrix} e_r \\ e_i \\ e_e \end{bmatrix}^T \begin{bmatrix} A_r & 0 & D_r^T \\ 0 & A_i & D_i^T \\ D_{er} & D_{ei} & A_e \end{bmatrix} \begin{bmatrix} e_r \\ e_i \\ e_e \end{bmatrix}, \]

and

\[ \|e - I_F e_C\|_{0,F}^2 = \| e_r - P_r e_r \|_{0,F}^2 + \| e_i - P_i e_i \|_{0,F}^2 \]

\[ = \begin{bmatrix} e_r \\ e_i \\ e_e \end{bmatrix}^T \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} D_r & 0 \\ 0 & D_i \end{bmatrix} \begin{bmatrix} I & 0 & -P_r \\ 0 & I & -P_i \end{bmatrix} \begin{bmatrix} e_r \\ e_i \\ e_e \end{bmatrix}, \]

respectively. Then, the weak approximation property \( \|e - I_F e_C\|_{0,F}^2 \leq \beta \|e\|_A^2 \) is
equivalent to that
\[
Q := \beta \begin{bmatrix}
A_r & 0 & D_T^r \\
0 & A_i & D_T^i \\
D_{er} & D_{ei} & A_e
\end{bmatrix} - \begin{bmatrix}
D_r & 0 & -D_rP_r \\
0 & D_i & -D_iP_i \\
-P_rD_r & -P_iD_i & P_rD_rP_i + P_iD_iP_i
\end{bmatrix}
= \begin{bmatrix}
\beta A_r - D_r & 0 & \beta D_T^r + D_rP_r \\
0 & \beta A_i - D_i & \beta D_T^i + D_iP_i \\
\beta D_{er} + P_rD_r & \beta D_{ei} + P_iD_i & \beta A_e - (P_rD_rP_r + P_iD_iP_i)
\end{bmatrix}
\]
is a symmetric and positive semi-definite (SPSD) matrix. According to the Gerschgorin disk theorem in Lemma 3.7, if we could prove \( Q \) is diagonally dominant with positive diagonal elements, the symmetric and positive semi-definite property of \( Q \) follows. Thus, we next turn to find a parameter \( \beta \) such that the matrix \( Q \) is diagonally dominant with positive diagonal elements.

In fact, the positive diagonal elements of the matrix \( Q \) is guaranteed by its diagonal dominance. Thus, we only consider the conditions for the matrix \( Q \) to be diagonally dominant, and take the first row of \( Q \) as an example, which yields

\[
(\beta - 1)a_{11}^r \geq \beta \sum_{j=2}^{n} |a_{1j}^r| + |\beta d_1^r - a_{11}^r \sum_{j=1}^{n} b_{1j}^r d_j^r|.
\]

Note that
\[
|\beta d_1^r - a_{11}^r \sum_{j=1}^{n} b_{1j}^r d_j^r| = \begin{cases}
-\beta d_1^r + a_{11}^r \sum_{j=1}^{n} b_{1j}^r d_j^r, & \text{if } \beta \geq -\frac{a_{11}^r \sum_{j=1}^{n} b_{1j}^r d_j^r}{-d_1^r} \\
\beta d_1^r - a_{11}^r \sum_{j=1}^{n} b_{1j}^r d_j^r, & \text{if } \beta < -\frac{a_{11}^r \sum_{j=1}^{n} b_{1j}^r d_j^r}{-d_1^r},
\end{cases}
\]

thus the inequality (3.16) holds if and only if

\[
(\beta - 1)a_{11}^r \geq \beta \left( -\sum_{j=2}^{n} a_{1j}^r - d_1^r \right) + a_{11}^r \left( \sum_{j=1}^{n} b_{1j}^r d_j^r \right)
\]

and

\[
(\beta - 1)a_{11}^r \geq \beta \left( -\sum_{j=2}^{n} a_{1j}^r \right) + a_{11}^r \left( -\sum_{j=1}^{n} b_{1j}^r d_j^r \right) + \beta d_1^r.
\]

Simplifying the above inequalities gives

\[
\beta \geq \frac{a_{11}^r \left( 1 + \sum_{j=1}^{n} b_{1j}^r d_j^r \right)}{s_1} \quad \text{and} \quad \beta \geq \frac{a_{11}^r \left( 1 - \sum_{j=1}^{n} b_{1j}^r d_j^r \right)}{s_1 - 2d_1^r},
\]

where \( s_k \) denotes the sum of the \( k \)-th row of the matrix \( A \). As a result, the diagonal dominance of the first row of \( Q \) holds if the parameter \( \beta \) satisfies

\[
\beta \geq m_1 := \max \left\{ \frac{a_{11}^r \left( 1 + \sum_{j=1}^{n} b_{1j}^r d_j^r \right)}{s_1}, \frac{a_{11}^r \left( 1 - \sum_{j=1}^{n} b_{1j}^r d_j^r \right)}{s_1 - 2d_1^r} \right\}.
\]
Furthermore, there has
\[
m_1 = \begin{cases} 
\frac{a'_{11}(1+\sum_{j=1}^n b'_{ij}d'_j)}{s_i}, & \text{when } -d'_1 \geq (\sum_{j=1}^n a'_{1j})(-\sum_{j=1}^n b'_{ij}d'_j) \\
\frac{a'_{11}(1-\sum_{j=1}^n b'_{ij}d'_j)}{s_i - 2d'_1}, & \text{when } -d'_1 \leq (\sum_{j=1}^n a'_{1j})(-\sum_{j=1}^n b'_{ij}d'_j).
\end{cases}
\]

In the same way, we could derive the conditions for the diagonal dominance of the \(k\)-th (\(2 \leq k \leq 2n\)) row:
\[
\beta \geq m_k := \max \left\{ \frac{a'_{kk}(1 + \sum_{j=1}^n b'_{kj}d'_j)}{s_k}, \frac{a'_{kk}(1 - \sum_{j=1}^n b'_{kj}d'_j)}{s_k - 2d'_k} \right\} \quad (2 \leq k \leq n),
\]
\[
\beta \geq m_k := \max \left\{ \frac{a'_{k-n,k-n}(1 + \sum_{j=1}^n b'_{k-n,j}d'_j)}{s_k}, \frac{a'_{kk}(1 - \sum_{j=1}^n b'_{k-n,j}d'_j)}{s_k - 2d'_k} \right\} \quad (n+1 \leq k \leq 2n).
\]

Next we discuss the conditions for the diagonal dominance of the rows \(2n+1 \leq k \leq 3n\) of \(Q\), and take the \((2n+1)\)-th row as an example, which leads to
\[
\beta a'_{11} - \left[ a'_{11}(\sum_{j=1}^n b'_{1j}d'_j)^2 + a'_{11}(\sum_{j=1}^n b'_{1j}d'_j)^2 \right] \geq \beta \sum_{j=2}^n a'_{1j} + |\beta d'_1 - a'_{11} \sum_{j=1}^n b'_{1j}d'_j| + |\beta d'_k - a'_{11} \sum_{j=1}^n b'_{1j}d'_j|.
\]

Simple calculation gives
(3.17)
\[
\beta(s_{2n+1} - 2d'_1 - 2d'_1) \geq a'_{11}(\sum_{j=1}^n b'_{1j}d'_j)^2 - \sum_{j=1}^n b'_{1j}d'_j + a'_{11}(\sum_{j=1}^n b'_{1j}d'_j)^2 - \sum_{j=1}^n b'_{1j}d'_j,
\]
and
\[
\beta(s_{2n+1} - 2d'_1) \geq a'_{11}(\sum_{j=1}^n b'_{1j}d'_j)^2 + \sum_{j=1}^n b'_{1j}d'_j + a'_{11}(\sum_{j=1}^n b'_{1j}d'_j)^2 + \sum_{j=1}^n b'_{1j}d'_j,
\]
and
\[
\beta(s_{2n+1} - 2d'_1) \geq a'_{11}(\sum_{j=1}^n b'_{1j}d'_j)^2 - \sum_{j=1}^n b'_{1j}d'_j + a'_{11}(\sum_{j=1}^n b'_{1j}d'_j)^2 + \sum_{j=1}^n b'_{1j}d'_j,
\]
and
\[
\beta s_{2n+1} \geq a'_{11}(\sum_{j=1}^n b'_{1j}d'_j)^2 + \sum_{j=1}^n b'_{1j}d'_j + a'_{11}(\sum_{j=1}^n b'_{1j}d'_j)^2 + \sum_{j=1}^n b'_{1j}d'_j.
\]

It is obvious that the last inequality in (3.17) holds for arbitrary \(\beta \geq 0\). To show it,
we notice that the property $A^{-1}A = I$ gives $\sum_{j=1}^{n} b'_{ij}a'_{ij} = 1$ and
\[
\sum_{i=1}^{n} b'_{i1}a'_{ij} = 0, \quad 2 \leq j \leq n
\]
\[
\Rightarrow b'_{11} + \cdots + b'_{1n} = 0
\]
\[
\Rightarrow b'_{11}(s_1 - a'_{11} - d'_{1}) + \cdots + b'_{n}(s_n - a'_{n1} - d'_{n}) = 0,
\]
which further lead to
\[
(3.18) \quad 1 + \sum_{j=1}^{n} b'_{ij}d'_j = \sum_{j=1}^{n} b'_{ij}s_j > 0,
\]
and then $0 \leq - \sum_{j=1}^{n} b'_{ij}d'_j < 1$. Similarly, we can also derive that $0 \leq - \sum_{j=1}^{n} b'_{ij}d'_j < 1$.

Therefore,
\[
\left(\sum_{j=1}^{n} b'_{ij}d'_j\right)^2 + \sum_{j=1}^{n} b'_{ij}d'_j \leq 0 \quad \text{and} \quad \left(\sum_{j=1}^{n} b'_{ij}d'_j\right)^2 + \sum_{j=1}^{n} b'_{ij}d'_j \leq 0,
\]
and the last inequality in (3.17) holds for arbitrary $\beta \geq 0$. As a result, the bound on $\beta$ obtained from the diagonal dominance of the $2n + 1$ row of $Q$ is
\[
\beta \geq m_{2n+1} := \max \left\{ \frac{a'_{i1}[\left(\sum_{j=1}^{n} b'_{ij}d'_j\right)^2 - \sum_{j=1}^{n} b'_{ij}d'_j] + a'_{i1}[\left(\sum_{j=1}^{n} b'_{ij}d'_j\right)^2 - \sum_{j=1}^{n} b'_{ij}d'_j]}{s_{2n+1} - 2d'_i - 2d'_i}, \right. \\
\left. \frac{a'_{i1}[\left(\sum_{j=1}^{n} b'_{ij}d'_j\right)^2 + \sum_{j=1}^{n} b'_{ij}d'_j] + a'_{i1}[\left(\sum_{j=1}^{n} b'_{ij}d'_j\right)^2 - \sum_{j=1}^{n} b'_{ij}d'_j]}{s_{2n+1} - 2d'_i - 2d'_i}, \right. \\
\left. \frac{a'_{i1}[\left(\sum_{j=1}^{n} b'_{ij}d'_j\right)^2 - \sum_{j=1}^{n} b'_{ij}d'_j] + a'_{i1}[\left(\sum_{j=1}^{n} b'_{ij}d'_j\right)^2 + \sum_{j=1}^{n} b'_{ij}d'_j]}{s_{2n+1} - 2d'_i - 2d'_i} \right\},
\]

In the same way, we get the following bounds on $\beta$ derived from the diagonal dominance of the rows $2n + 1 \leq k \leq 3n$,
\[
\beta \geq m_k := \max \left\{ \frac{a'_{k-2n,k-2n}[\left(\sum_{j=1}^{n} b'_{k-2n,j}d'_j\right)^2 - \sum_{j=1}^{n} b'_{k-2n,j}d'_j]}{s_k - 2d'_k - 2d'_k}, \right. \\
\left. \frac{a'_{k-2n,k-2n}[\left(\sum_{j=1}^{n} b'_{k-2n,j}d'_j\right)^2 - \sum_{j=1}^{n} b'_{k-2n,j}d'_j]}{s_k - 2d'_k - 2d'_k}, \right. \\
\left. \frac{a'_{k-2n,k-2n}[\left(\sum_{j=1}^{n} b'_{k-2n,j}d'_j\right)^2 + \sum_{j=1}^{n} b'_{k-2n,j}d'_j]}{s_k - 2d'_k - 2d'_k}, \right. \\
\left. \frac{a'_{k-2n,k-2n}[\left(\sum_{j=1}^{n} b'_{k-2n,j}d'_j\right)^2 - \sum_{j=1}^{n} b'_{k-2n,j}d'_j]}{s_k - 2d'_k - 2d'_k} \right\},
\]

In conclusions, the PCTL algorithm satisfies the weaker approximation property
Based on the convergence properties of the ATG method in subsection 3.1, an estimation on the convergence factor of the PCTL algorithm can be obtained from Theorem 3.8 and Theorem 3.9.

**Theorem 3.10.** Let $\beta$ be defined as in the equality (3.15). Then the convergence factor of the PCTL algorithm is bounded by

$$\|E_{\text{PCTL}}\|_A \leq \kappa := 1 - \frac{1}{4\beta}.$$ 

It is worthy to state that the upper bound $\kappa$ in Theorem 3.10 is easy to compute, since all the involved quantities are known and just scalar operations are needed. Observing the expression of $\beta$ in (3.15) is fussy, we give a simpler format in the next corollary, while it leads to a looser upper bound $\kappa$.

**Corollary 3.11.** For a more intuitive and simpler format of $\beta$, it could take

$$\beta = \max_{1 \leq k \leq n} \frac{a_{kk}^*}{s_k}, \frac{2a_{kk}^r}{s_k - 2d_k^r}, \frac{a_{kk}^l}{s_{k+n}} - 2d_k^l,$$

$$\frac{2a_{kk}^r}{s_{k+2n} - 2d_k^r}, \frac{2a_{kk}^l}{s_{k+2n} - 2d_k^l}, \frac{2(a_{kk}^l + a_{kk}^r)}{s_k - 2d_k^l - 2d_k^r}.$$  

(3.19)

**Proof.** Recalling

$$0 \leq - \sum_{j=1}^{n} b_{ij}^j d_{ij}^j \leq 1 \quad \text{and} \quad 0 \leq - \sum_{j=1}^{n} b_{ij}^l d_{ij}^l \leq 1,$$

then we have

$$0 \leq \left( \sum_{j=1}^{n} b_{ij}^j d_{ij}^j \right)^2 - \sum_{j=1}^{n} b_{ij}^j d_{ij}^j, \quad \left( \sum_{j=1}^{n} b_{ij}^l d_{ij}^l \right)^2 - \sum_{j=1}^{n} b_{ij}^l d_{ij}^l \leq 2$$

and

$$- \frac{1}{4} \leq \left( \sum_{j=1}^{n} b_{ij}^j d_{ij}^j \right)^2 + \sum_{j=1}^{n} b_{ij}^j d_{ij}^j, \quad \left( \sum_{j=1}^{n} b_{ij}^l d_{ij}^l \right)^2 + \sum_{j=1}^{n} b_{ij}^l d_{ij}^l \leq 0.$$ 

It is clear that the bounds $m_k$ defined in the Theorem 3.9 satisfy

$$m_k \leq \max \left\{ \frac{a_{kk}^*}{s_k}, \frac{2a_{kk}^r}{s_k - 2d_k^r} \right\}, \quad 1 \leq k \leq n;$$

$$m_k \leq \max \left\{ \frac{a_{k-k-n}^l}{s_k}, \frac{2a_{k-k-n}^r}{s_k - 2d_k^r} \right\}, \quad n + 1 \leq k \leq 2n;$$

$$m_k \leq \max \left\{ \frac{2a_{k-k-2n}^r}{s_k - 2d_k^r - 2d_k^r}, \frac{2a_{k-k-2n}^l}{s_k - 2d_k^r - 2d_k^r}, \frac{2(a_{k-k-2n}^l + a_{k-k-2n}^r)}{s_k - 2d_k^r - 2d_k^r} \right\}, \quad 2n + 1 \leq k \leq 3n,$$

then the conclusion (3.19) follows.  

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4. Analysis of factors affecting the performance of PCTL algorithm. In this section, we discuss the factors that affect the convergence bound $\kappa$, and expect it could provide a guidance for further research on the PCTL algorithm, such as which kind of problems PCTL algorithm is effective and how to improve its performance.

The convergence bound in Theorem 3.10 shows that the bound $\kappa$ decreases with the decreasing of the parameter $\beta$. Thus, discussing the factors affect the value of $\beta$ is of great value. To analyse it, we first introduce two parameters of the matrix $A$:

- **Diagonally dominant strength of $A_{\alpha}$**
  Recall $A_{\alpha} = (a_{kij})_{n \times n}$ ($\alpha = r, i, e$), then define the diagonally dominant strength of the $k$-th row of $A_{\alpha}$ as

  $$\theta_{\alpha}^k := \frac{\sum_{1 \leq j \leq n} a_{kj}^\alpha}{a_{kk}^\alpha} \in (0, 1],$$

  which shows the diagonally dominant strength of the radiation, ion and electron equations. Moreover, the bigger $\theta_{\alpha}^k$ gives the stronger diagonal dominance of the $k$-th row of $A_{\alpha}$.

- **Coupling strength**
  For $1 \leq k \leq n$, define the coupling strength of $A$ as

  $$\delta^r_k := \frac{|d^r_k|}{a_{kk}^r} \in [0, \theta^r_k], \quad \delta^i_k := \frac{|d^i_k|}{a_{kk}^i} \in [0, \theta^i_k],$$

  $$\delta^{ci} := \frac{|d^{ci}_k|}{a_{kk}^{ci}} = \frac{\delta_k^r + \delta_k^i}{\theta^r_k + \theta^i_k} < \theta^r_k.$$

With the parameters $\theta_{\alpha}^k$ and $\delta^r_k$ defined above, the value of $\beta$ defined in Corollary 3.11 can be rewritten as

**Corollary 4.1.** The $\beta$ defined as in Corollary 3.11 can be rewritten as

$$\beta = \max_{1 \leq k \leq n} \left\{ \frac{1}{2} \frac{\theta^r_k - \delta_k^r}{\theta^r_k + \delta_k^r}, \frac{1}{2} \frac{\theta^i_k - \delta_k^i}{\theta^i_k + \delta_k^i}, \frac{1}{2} \frac{\theta^{ci}_k - \delta_k^{ci}}{\theta^{ci}_k + \delta_k^{ci}}, \frac{a_{kk}^r}{a_{kk}^r + \theta^r_k + \delta_k^r}, \frac{a_{kk}^i}{a_{kk}^i + \theta^i_k + \delta_k^i}, \frac{a_{kk}^{ci}}{a_{kk}^{ci} + \theta^{ci}_k + \delta_k^{ci}} \right\}.$$

The equality (4.1) indicates that the value of $\beta$ is inversely correlated with the value of $\theta_{\alpha}^k$, which says the stronger the diagonal dominance, the better the coarse-grid correction process tends to be. While for the coupling strength, there are several possibilities.

1. When the coupling strengths are poor such that $\delta_k^r \leq \frac{\theta_k^r}{3}$, $\delta_k^i \leq \frac{\theta_k^i}{3}$ and $\delta_k^{ci} \leq \frac{\theta_k^{ci}}{3}$, then

   $$\beta = \max_{1 \leq k \leq n} \left\{ \frac{2}{\theta^r_k + \delta_k^r}, \frac{2}{\theta^i_k + \delta_k^i}, \frac{2}{\theta^{ci}_k + \delta_k^{ci}} \right\}.$$

   In this case, the value of $\beta$ is also inversely correlated with the coupling strength.

2. Instead, when the coupling strengths are large enough, there has

   $$\beta = \max_{1 \leq k \leq n} \left\{ \frac{1}{2} \frac{\theta^r_k - \delta_k^r}{\theta^r_k + \delta_k^r}, \frac{1}{2} \frac{\theta^i_k - \delta_k^i}{\theta^i_k + \delta_k^i}, \frac{a_{kk}^r}{a_{kk}^r + \theta^r_k + \delta_k^r}, \frac{a_{kk}^i}{a_{kk}^i + \theta^i_k + \delta_k^i}, \frac{a_{kk}^{ci}}{a_{kk}^{ci} + \theta^{ci}_k + \delta_k^{ci}} \right\}.$$
Example 4.2. To eliminate the influence of other factors like $a_{kk}^\alpha/a_{kk}^\varepsilon$, we artificially set the $k$-th diagonal element of the submatrices $A_\alpha$ the same, that is $A^\alpha_{kk} = A^\varepsilon_{kk}$ ($1 \leq k \leq n$). Moreover, we assume that the diagonally dominant strength and coupling strength of all rows are the same, and denote by $\theta$ and $\delta$, respectively. Then

$$\beta = \max \left\{ \frac{1}{\theta - \delta}, \frac{4}{\theta + 2\delta} \right\}$$

and then the upper bound of the PCTL algorithm is

$$\kappa = \max \left\{ 1 - \frac{\theta - \delta}{4}, 1 - \frac{\theta + 2\delta}{16} \right\}.$$
The convergence upper bound of the PCTL algorithm vs coupling strength for different diagonally dominant strengths.

PCTL algorithm. Moreover, we have also discussed the influence of the diagonally dominant strength and coupling strength on the convergence bound. It provides an insight into the problems for which the PCTL algorithm may be suitable, such as the problems with strong diagonally dominant submatrices $A_\alpha$.

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