On Dendrites Generated By Symmetric Polygonal Systems: The Case of Regular Polygons

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Abstract We define $G$-symmetric polygonal systems of similarities and study the properties of symmetric dendrites, which appear as their attractors. This allows us to find the conditions under which the attractor of a zipper becomes a dendrite.

1 Introduction

Though the study of topological properties of dendrites from the viewpoint of general topology proceed for more than three quarters of a century [3, 7, 8], the attempts to study the geometrical properties of self-similar dendrites are rather fragmentary.

In 1985, M. Hata [5] studied the connectedness properties of self-similar sets and proved that if a dendrite is an attractor of a system of weak contractions in a complete metric space, then the set of its endpoints is infinite. In 1990 Ch. Bandt showed in his unpublished paper [2] that the Jordan arcs connecting pairs of points of a post-critically finite self-similar dendrite are self-similar, and the set of possible values for dimensions of such arcs is finite. Jun Kigami in his work [6] applied the methods of harmonic calculus on fractals to dendrites; on a way to this he developed effective approaches to the study of structure of self-similar dendrites. D.Croydon
in his thesis \cite{4} obtained heat kernel estimates for continuum random tree and for certain family of p.c.f. random dendrites on the plane.

In our recent works \cite{9,11,12} we considered systems $S$ of contraction similarities in $\mathbb{R}^d$ defined by some polyhedron $P \subset \mathbb{R}^d$, which we called contractible $P$-polyhedral systems. We proved that the attractor of such system $S$ is a dendrite $K$ in $\mathbb{R}^d$; we showed that the orders of points $x \in K$ have an upper bound, depending only on $P$; and that Hausdorff dimension of the set $CP(K)$ of the cut points of $K$ is strictly smaller than the dimension of the set $EP(K)$ of its end points unless $K$ is a Jordan arc.

Now we extend our approach to the case of symmetric $P$-polyhedral systems $S$ and show that the symmetric dendrites $K$ which are the attractors of these systems, have clear and obvious structure: their main tree is a symmetric $n$-pod (Proposition 2), all the vertices of the polygon $P$ are the end points of $K$ and show that for $n > 5$ each ramification point of $K$ has the order $n$ (Proposition 3). We show that the augmented system $\tilde{S}$ contain subsystems $\mathcal{Z}$ which are zippers whose attractors are subdendrites of the dendrite $K$ (Theorem 6).

### 1.1 Dendrites

**Definition 1.** A dendrite is a locally connected continuum containing no simple closed curve.

In the case of dendrites the order $\text{Ord}(p, X)$ of the point $p$ with respect to $X$ is equal to the number of components of the set $X \setminus \{p\}$. Points of order 1 in a continuum $X$ are called end points of $X$; the set of all end points of $X$ will be denoted by $EP(X)$. A point $p$ of a continuum $X$ is called a cut point of $X$ provided that $X \setminus \{p\}$ is not connected; the set of all cut points of $X$ will be denoted by $CP(X)$. Points of order at least 3 are called ramification points of $X$; the set of all ramification points of $X$ will be denoted by $\text{RP}(X)$.

According to \cite[Theorem 1.1]{3}, for a continuum $X$ the following conditions are equivalent: $X$ is dendrite; every two distinct points of $X$ are separated by a third point; each point of $X$ is either a cut point or an end point of $X$; each nondegenerate subcontinuum of $X$ contains uncountably many cut points of $X$; the intersection of every two connected subsets of $X$ is connected; $X$ is locally connected and uniquely arcwise connected.

### 1.2 Self-similar sets

Let $(X, d)$ be a complete metric space. A mapping $F : X \to X$ is a contraction if $\text{Lip} F < 1$.

The mapping $S : X \to X$ is called a similarity if $d(S(x), S(y)) = rd(x, y)$ for all $x, y \in X$ and some fixed $r$. 
Definition 2. Let \( S = \{S_1, S_2, \ldots, S_m\} \) be a system of contraction maps on a complete metric space \((X, d)\). A nonempty compact set \( K \subset X \) is the attractor of the system \( S \), if \( K = \bigcup_{i=1}^{m} S_i(K) \).

The system \( S \) defines the Hutchinson operator \( T \) by the equation \( T(A) = \bigcup_{i=1}^{m} S_i(A) \).

By Hutchinson’s Theorem, the attractor \( K \) is uniquely defined by \( S \) and for any compact set \( A \subset X \) the sequence \( T^n(A) \) converges to \( K \).

We also call the subset \( K \subset X \) self-similar with respect to \( S \). Throughout the whole paper, the maps \( S_i \in S \) are supposed to be similarities and the set \( X \) to be \( \mathbb{R}^2 \).

Notation. \( I = \{1, 2, \ldots, m\} \) is the set of indices, \( I^n = \bigcup_{n=1}^{\infty} I^n \) is the set of all finite \( I \)-tuples, or multiindices \( j = j_1j_2\ldots j_n \). By \( ij \) we denote the concatenation of the corresponding multiindices; we say \( i \sqsubset j \), if \( i = i_1\ldots i_n \) is the initial segment in \( j = j_1\ldots j_{n+k} \) or \( j = ik \) for some \( k \in I^* \);

if \( i \not\sqsubset j \) and \( j \not\sqsubset i \), \( i \) and \( j \) are incomparable;

we write \( S_j = S_{j_1j_2\ldots j_n} = S_{j_1}S_{j_2}\ldots S_{j_n} \) and for the set \( A \subset X \) we denote \( S_j(A) \) by \( A_j \); we also denote by \( G_S = \{S_j, j \in I^*\} \) the semigroup, generated by \( S \).

The set of all infinite sequences \( I^\infty = \{\alpha = \alpha_1\alpha_2\ldots, \alpha_i \in I\} \) is called the index space; and \( \pi : I^\infty \to K \) is the index map, which sends a sequence \( \alpha \) to the point \( \bigcap_{n=1}^{\infty} K_{\alpha_1\ldots\alpha_n} \).

1.3 Zippers

The simplest way to construct a self-similar curve is to take a polygonal line and then replace each of its segments by a smaller copy of the same polygonal line; this construction is called zipper and was studied in [1][10]:

Definition 3. Let \( X \) be a complete metric space. A system \( S = \{S_1, \ldots, S_m\} \) of contraction mappings of \( X \) to itself is called a zipper with vertices \( \{z_0, \ldots, z_m\} \) and signature \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_m) \), \( \varepsilon_i \in \{0, 1\} \), if for \( i = 1, \ldots, m \), \( S_i(z_0) = z_{i-1+\varepsilon_i} \) and \( S_i(z_m) = z_{i-\varepsilon_i} \).

A zipper \( S \) is a Jordan zipper if and only if one (and hence every) of the structural parametrizations of its attractor establishes a homeomorphism of the interval \( J = [0, 1] \) onto \( K(S) \).

Theorem 1. Let \( S = \{S_1, \ldots, S_m\} \) be a zipper with vertices \( \{z_0, \ldots, z_m\} \) in a complete metric space \( X \) such that all contractions \( S_j : X \to X \) are injective. If for arbitrary \( i, j \in I \) the set \( K_i \cap K_j \) is empty for \( |i - j| > 1 \) and is a singleton for \( |i - j| = 1 \) then every structural parametrization \( \varphi : [0, 1] \to K(S) \) of \( K(S) \) is a homeomorphism and \( K(S) \) is a Jordan arc with endpoints \( z_0 \) and \( z_m \).
2 Contractible $P$-polygonal systems

Let $P$ be a convex polygon in $\mathbb{R}^2$ and $V_P = \{A_1, \ldots, A_{n_P}\}$ be the set of its vertices, where $n_P = |V_P|$.

Consider a system of contracting similarities $\mathcal{S} = \{S_1, \ldots, S_m\}$, which possesses the following properties:

(D1) For any $k \in I$, the set $P_k = S_k(P)$ is contained in $P$;

(D2) For any $i \neq j, i, j \in I$, $P_i \cap P_j$ is either empty or a common vertex of $P_i$ and $P_j$;

(D3) For any $A_k \in V_P$ there is a map $S_i(A_k) = A_k$;

(D4) The set $\overline{P} = \bigcup_{i=1}^{m} P_i$ is contractible.

**Definition 4.** The system $(P, \mathcal{S})$ satisfying the conditions D1-D4 is called a contractible P-polygonal system of similarities.

Applying Hutchinson operator $T(A) = \bigcup_{i \in I} S_i(A)$ of the system $\mathcal{S}$ to the polygon $P$, we get the set $\overline{P}^{(1)} = \bigcup_{i \in I} P_i$. Taking the iterations of $T$, we define $\overline{P}^{(n+1)} = T(\overline{P}^{(n)})$ and get a nested family of contractible compact sets $\overline{P}^{(1)} \supset \overline{P}^{(2)} \supset \ldots \supset \overline{P}^{(n)} \supset \ldots$. By Hutchinson’s theorem, the intersection of this nested sequence is the attractor $K$ of the system $\mathcal{S}$.

The following Theorem was proved by the authors in [9, 11, 12]:

**Theorem 2.** Let $\mathcal{S}$ be a contractible $P$-polygonal system, and let $K$ be its attractor. Then $K$ is a dendrite.

Since $K$ is a dendrite, for any vertices $A_i, A_j \in V_P$ there is an unique Jordan arc $\gamma_{ij} \subset K$ connecting $A_i, A_j$. The set $\hat{\gamma} = \bigcup_{i \neq j} \gamma_{ij}$ is a subcontinuum of the dendrite $K$, all of whose end points are contained in $V_P$, so $\hat{\gamma}$ is a finite dendrite or topological tree [3, A.17].

**Definition 5.** The union $\hat{\gamma} = \bigcup_{i \neq j} \gamma_{ij}$ is called the main tree of the dendrite $K$. The ramification points of $\hat{\gamma}$ are called main ramification points of the dendrite $K$.

We consider $\hat{\gamma}$ as a topological graph whose vertex set $V_{\hat{\gamma}}$ is the union of $V_P$ and the set of ramification points $RP(\hat{\gamma})$, while the edges of $\hat{\gamma}$ are the components of $\hat{\gamma} \setminus V_P$.

The following Proposition [9] show the relation between the vertices of $P$ and end points, cut points and ramification points of $\hat{\gamma}$.

**Proposition 1.** a) For any $x \in \hat{\gamma}$, $\hat{\gamma} = \bigcup_{j=1}^{n} \gamma_{A_j, x}$.

b) $A_i$ is a cut point of $\hat{\gamma}$, if there are $j_1, j_2$ such that $\gamma_{j_1, i} \cap \gamma_{j_2, i} = \{A_i\}$;

c) the only end points of $\hat{\gamma}$ are the vertices $A_j$ such that $A_j \notin CP(\hat{\gamma})$;

d) if $\# \pi^{-1}(A_j) = 1$, then $Ord(A_j, K) \leq n - 1$, otherwise
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\[ \text{Ord}(A_i,K) \leq (n-1) \left( \frac{\theta_{\text{max}}}{\theta_{\text{min}}} - 1 \right), \]

where \( \theta_{\text{max}}, \theta_{\text{min}} \) are maximal and minimal values of vertex angles of \( P \).

It was proved in [9, 11, 12] that each cut point of the dendrite \( K \) is contained in some image \( S_j(\hat{\gamma}) \) of the main tree:

**Theorem 3.** The following statements are true:

a) \( CP(K) \subset \bigcup_{j \in I^*} S_j(\hat{\gamma}) \).

b) For each cut point \( y \in K \setminus \bigcup_{j \in I^*} S_j(V_P), \# \pi^{-1}(y) = 1 \) and there is \( S_i \) and \( x \in \hat{\gamma} \), such that \( y = S_i(x) \) and \( \text{Ord}(y,K) = \text{Ord}(x,\hat{\gamma}) \).

c) If \( y \in \bigcup_{j \in I^*} S_j(V_P) \) and \( \# \pi^{-1}(y) = s \), there are multiindices \( i_k, k = 1, \ldots, s \) and vertices \( x_1, \ldots, x_s \), such that for any \( k \), \( S_{i_k}(x_k) = y \) and for any \( l \neq k \), \( S_{i_k}(P) \cap S_{i_l}(P) = \{y\} \)

and \( \text{Ord}(y,K) = \sum_{k=1}^{s} \text{Ord}(x_k,\hat{\gamma}) \leq (n_P - 1) \left( \frac{2\pi}{\theta_{\text{min}}} - 1 \right) \).

Moreover, the dimension of the set of the end points is always greater than the one of the set of cut points [11, 12]

**Theorem 4.** Let \( (P, \mathcal{S}) \) be a contractible \( P \)-polyhedral system and \( K \) be its attractor. \( \text{i) dim}_H(CP(K)) = \text{dim}_H(\hat{\gamma}) \leq \text{dim}_H EP(K) = \text{dim}_H(K) \); \( \text{ii) dim}_H(CP(K)) = \text{dim}_H(K) \) iff \( K \) is a Jordan arc.

### 3 Symmetric polygonal systems

**Definition 6.** Let \( P \) be a polygon and \( G \) be a non-trivial symmetry group of the polygon \( P \). Let \( \mathcal{S} \) be a contractible \( P \)-polygonal system such that for any \( g \in G \) and any \( S_i \in \mathcal{S} \), there are such \( g' \in G \) and \( S_j \in \mathcal{S} \) that \( g \cdot S_i = S_j \cdot g' \). Then the system of mappings \( \mathcal{S} = \{S_i, i = 1, 2, \ldots, m\} \) is called a contractible \( G \)-symmetric \( P \)-polygonal system.

For convenience we will call such systems symmetric polygonal systems or SPS, if this does not cause ambiguity in choice of \( P \) and \( G \).

**Theorem 5.** The attractor \( K \) of a symmetric polygonal system and its main tree \( \hat{\gamma} \) are symmetric with respect to the group \( G \).
Proof. Let $S = \{S_1, \ldots, S_m\}$. Take $g \in G$. The map $g^* : S \to S$, sending each $S_i$ to respective $S_j$ is a permutation of $S$, therefore $g(\bigcup_{i=1}^m S_i(P)) = \bigcup_{i=1}^m S_j(P)$, or $g(P) = \bar{P}$.

Moreover, it follows from the Definition 6 that for any $i = i_1 \ldots i_k$ there is such $j = j_1 \ldots j_k$ and such $g' \in G$ that $g \cdot S_i = S_j \cdot g'$. Therefore for any $g$, $g(\bar{P}^k) = \bar{P}^k$.

Since $K = \bigcap_{k=1}^{\infty} \bar{P}^k$, $g(K) = K$. Since $g$ preserves the set of vertices of $P$, $g(\bar{\gamma}) = \bar{\gamma}$.

**Corollary 1.** If $S$ is a $G$-symmetric $P$-polygonal system then $S^{(n)} = \{S_j, j \in \mathbb{Z}^n\}$ is a $G$-symmetric $P$-polygonal system.

**Corollary 2.** Suppose $S = \{S_1, \ldots, S_m\}$ is a $G$-symmetric $P$-polygonal system, $K$ is the attractor of $S$, $g_1, \ldots, g_m \in G$ and $S' = \{S_1 g_1, \ldots, S_m g_m\}$. Then $K$ is the attractor of the system $S'$.

**Proof.** Let $K'$ be the attractor of the system $S'$ and put $\bar{P}' = \bigcup_{i=1}^m (S_i \circ g_i(P))$. Observe that for any $i$, $g_i(P) = P$, therefore $\bar{P}' = \bar{P}$ and $\bar{P}'^{(k)} = \bar{P}^{(k)}$. Then $K' = \bigcap_{k=1}^{\infty} \bar{P}'^{(k)} = K$.

**Definition 7.** Let $\bar{S} = \{\bar{S}_1, \ldots, \bar{S}_m\}$ be a $G$-symmetric $P$-polygonal system. The system $\bar{S} = \{S_i \cdot g, S_i \in S, g \in G\}$ is called the augmented system for $S$.

The system $\bar{S}$ has the same attractor $K$ as $S$ and generates the augmented semigroup $G(\bar{S})$ consisting of all maps of the form $S_i \circ g_i$, where $g_i \in G$.

### 3.1 The case of regular polygons

**Proposition 2.** Let $P$ be a regular $n$-gon and $G$ be the rotation group of $P$. Then the center $O$ of $P$ is the only ramification point of the main tree and $\text{Ord}(O, \bar{\gamma}) = n$. 

[Diagrams]
Proof. Consider the main tree \( \gamma \). It is a fine finite system \([6]\), which is invariant with respect to \( G \). Let \( f \) be the rotation of \( P \) in the angle \( 2\pi/n \).

Suppose \( V \) and \( E \) be the number of vertices and edges respectively of the main tree. For any edge \( \lambda \subset \gamma \), \( f(\lambda) \cap \lambda \) is either empty, or is equal to \( \{O\} \), and in the latter case \( O \) is the endpoint of both \( \lambda \) and \( f(\lambda) \). In each case all the edges \( f^k(\lambda) \) are different. Therefore \( E \) is a multiple of \( n \).

If \( A' \) is a vertex of \( \gamma \) and \( A' \neq O \), then all the points \( f^k(A'), k = 1, \ldots, n \) are different, so the number of vertices of \( \gamma \), different from \( O \), is also a multiple of \( n \).

Since \( \gamma \) is a tree, \( V = E + 1 \). Therefore the set of vertices contains \( O \), which is the only invariant point for \( f \). Denote the unique subarc of \( \gamma \) with endpoints \( O \) and \( A_k \) by \( \gamma_k \). Then for any \( k = 1, \ldots, n \), \( \gamma_k = f^k(\gamma_0) \). By Proposition \([1]\) \( \bigcup_{k=1}^{n} \gamma_k = \gamma \). Thus the center \( O \) is the only ramification point of \( \gamma \) and \( \text{Ord}(O, \gamma) = n \).

**Corollary 3.** All vertices of the polygon \( P \) are the end points of the main tree.

**Proof.** For any \( k = 1, \ldots, n \) there is an unique arc \( \gamma_k \) of the main tree meeting the vertex \( A_k \) of the polygon \( P \), so \( \text{Ord}(A_k, \gamma) = 1 \) by Proposition \([1]\).

Since all the vertex angles of \( P \) are equal, for each vertex \( A_k \) of \( P \), there is unique \( S_k \in S \) such that \( P_k = S_k(P) \supset A_k \), so \#\pi^{-1}(A_k) = 1 \) and by Theorem \([3]\) \( \text{Ord}(A_k, K) = \text{Ord}(A_k, \gamma) = 1 \).

Then all vertices of the polygon \( P \) are the end points of the main tree as well as of the dendrite \( K \).

**Lemma 1.** Each arc \( \gamma_k \) is the attractor of a Jordan zipper.

**Proof.** We prove the statement for the arc \( \gamma_0 \), because for \( \gamma_k = f^k(\gamma_0) \) it follows automatically.

If \( n > 3 \), there is a similarity \( S_0 \in S \), whose fixed point is \( O \). Indeed, there is some \( S_0 \in S \) for which \( P_0 = S_0(P) \supset O \). The point \( O \) cannot be the vertex of \( P_0 \), otherwise the polygons \( f(P_0) \) and \( P_0 \) would intersect more than in one point. Therefore \( f(P_0) = P_0 \) and \( S_0(O) = O \).

Observe that for any two vertices \( A_i, A_j \) of \( P \), the arc \( \gamma_{iA_j} \) is the union \( \gamma_i \cup \gamma_j \).

There is an unique chain of subpolygons \( P_k = S_k(P), k = 0, \ldots, s \) connecting \( P_0 \) and \( P_n \) and containing \( \gamma_0 \), where \( S_0 = S_0 \) and \( S_n = S_n \). For each \( k = 1, \ldots, s \), there are \( i_k \) and \( j_k \) such that \( \gamma_0 \cap P_k = S_k(f^{i_k}(\gamma_0) \cup f^{j_k}(\gamma_0)) \). Therefore

\[
\gamma_0 = \bigcup_{k=1}^{s} S_k \left( f^{i_k}(\gamma_0) \cup f^{j_k}(\gamma_0) \right) \cup S_0(\gamma_0).
\]

The arcs on the right hand satisfy the conditions of Theorem \([1]\) therefore the system

\[
\{S_0, S_i f^{i_1}, S_i f^{j_1}, \ldots, S_i f^{i_s}, S_i f^{j_s}\}
\]
is a Jordan zipper whose attractor is a Jordan arc with endpoints $O$ and $A_n$.

If $n = 3$, it is possible that for some $l_1$, $O$ is a vertex of a triangle $S_{l_1}(P)$ and there is an unique chain of subpolygons $P_k = S_{l_k}(P), k = 1, \ldots, s$, where $S_{l_1} = S_3$. Repeating the same argument, we get a system $\{S_{l_1} f^{i_1}, S_{l_1} f^{i_1}, \ldots, S_{l_k} f^{i_k}, S_{l_k} f^{i_k}\}$ is a Jordan zipper whose attractor is a Jordan arc with endpoints $O$ and $A_3$.

**Corollary 4.** If $P$ is a regular $n$-gon and the symmetry group $G$ of the system $S$ is the dihedral group $D_n$ then $\gamma_{\partial K}$ is the line segment and the set of cut points of $K$ has dimension 1.

**Proof.** Since $D_n$ contains a symmetry with respect to the straight line containing $O$ and $A_n$, its itself is a straight line segment.

From the above statements we see that Proposition\[1\] and Theorem\[3\] in the case of $G$-symmetric polygonal systems with $G$ being the rotation group of order $n$ and $P$ a regular $n$-gon, acquire the following form:

**Proposition 3.** Let $S$ be a $G$-symmetric $P$-polygonal system of similarities, where $P$ is a regular $n$-gon and $G$ contains the rotation group of $P$. Then:

a) $V_p \subseteq EP(\hat{\gamma}) \subseteq EP(K)$;

b) For each cut point $y \in K \setminus \bigcup_{j \in I^*} S_j(V_p)$, either $y = S_q(O)$ for some $i \in I^*$ and $Ord(y, K) = n$, or $Ord(y, K) = 2$.

c) For any $y \in \bigcup_{i \in I} S_i(V_p)$ there is unique $x \in \bigcup_{i \in I} S_i(V_p)$, that $Ord(y, K) = Ord(x, K) = \#\pi^{-1}(y) = \#\pi^{-1}(x) = \#\{i \in I : x \in S_i(V_p)\} \leq 1 + \left\lfloor \frac{4}{n-2} \right\rfloor$.

**Proof.** All vertex angles of $P$ are $\theta = \pi - \frac{2\pi}{n}$, so $2\pi \left\lfloor \frac{\theta_{\min}}{\theta} \right\rfloor - 1 = 1 + \left\lfloor \frac{4}{n-2} \right\rfloor$.

a) Take a vertex $A_i \in V_p$. There is unique $j \in I$ such that $A_i \in S_j(V_p)$. For that reason $\#\pi^{-1}(A_i) = 1$. Since $S_j(P)$ cannot contain the center $O$, $\#(S_j(V_p) \cap \hat{\gamma}) = 2$, therefore by Theorem\[1\], $Ord(A_i, \hat{\gamma}) = 1$ and $Ord(A_i, K) = 1$. So $A_i \in EP(K)$.

b) If for some $j \in I^*$, $y = S_j(O)$, then $Ord(y, K) = n$. Since for any point $x \in CP(\hat{\gamma}) \setminus \{O\}$, $Ord(x, \hat{\gamma}) = 2$, the same is true for $y = S_j(x)$ for any $x \in I^*$.

c) Now let $\mathcal{C} = \{C_1, \ldots, C_N\}$ be the full collection of those points $C_k \in \bigcup_{i \in I} S_i(V_p)$ for which $s_k := \#\{j \in I : S_j(V_p) \ni C_k\} \geq 3$. By Theorem\[1\], $\#\pi^{-1}(C_k) = s_k$ and $Ord(C_k, K) = s_k$, while $s_k \leq 1 + \left\lfloor \frac{4}{n-2} \right\rfloor$.

Then, if $y \in S_j(C_k)$ for some $j \in I^*$ and $C_k \in \mathcal{C}$, then $\#\pi^{-1}(y) = s_k = Ord(y, K)$.

Applying the Proposition\[3\] to different $n$, we get the possible ramification orders for regular $n$-gons:

1. If $n \geq 6$ then all ramification points of $K$ are the images $S_j(O)$ of the centre $O$ and have the order $n$. 

2. If \( n = 4 \) or \( 5 \) then there is a finite set of ramification points \( x_1, \ldots, x_r \), whose order is equal to 3 such that each \( x_k \) is a common vertex of polygons \( S_{k1}(P), S_{k2}(P), S_{k3}(P) \). Then each ramification point is represented either as \( S_1(O) \) and has the order \( n \) or as \( S_1(x_k) \) and has the order 3.

3. If \( n = 3 \) the centre is a ramification point of order 3 and those ramification points which are not images of \( O \) will have an order less than or equal to 5.

3.2 Self-similar zippers, whose attractors are dendrites

**Theorem 6.** Let \( (S, P) \) be a \( G \)-symmetric \( P \)-polygonal system of similarities. Let \( A, B \) be two vertices of the polygon \( P \) and \( L \) be the line segment \([A, B]\). If \( \mathcal{Z} = \{S'_1, \ldots, S'_k\} \) is such family of maps from \( S \) that \( L = \bigcup_{i=1}^{k} S'_i(L) \) is a polygonal line connecting \( A \) and \( B \), then the attractor \( K_{\mathcal{Z}} \) of \( \mathcal{Z} \) is a subcontinuum of \( K \). If for some subpolygon \( P_j, L \cap P_j \) contains more than one segment, then \( K_{\mathcal{Z}} \) is a dendrite.

**Proof.** Since \( \mathcal{Z} \subset \tilde{S} \), the attractor \( K_{\mathcal{Z}} \) is a subset of \( K \). The system \( \mathcal{Z} \) is a zipper with vertices \( A, B \), therefore \( K_{\mathcal{Z}} \) is a continuum, and therefore is a subdendrite of the dendrite \( K \). Let \( \gamma_{AB} \) be the Jordan arc connecting \( A \) and \( B \) in \( K_{\mathcal{Z}} \), and, therefore, in \( K \). By the proof of Lemma 1, \( \gamma_{AB} = \gamma_{OA} \cup \gamma_{OB} \). If the maps \( S'_1, S'_2 \) send \( L \) to two segments belonging to the same subpolygon \( P_{i_0} \), then \( S'_{i_1}(\gamma_{AB}) \cup S'_{i_2}(\gamma_{AB}) \) is equal to \( S'_{i_1}(\gamma_{OA} \cup \gamma_{OB}) \cup S'_{i_2}(\gamma_{OA} \cup \gamma_{OB}) \). The set \( \{S'_{i_1}(A), S'_{i_1}(B), S'_{i_2}(A), S'_{i_2}(B)\} \) contains at least 3 different points, therefore \( S'_{i_1}(O) \) is a ramification point of \( K_{\mathcal{Z}} \) of order at least 3.

**Corollary 5.** Let \( u_i \) be the number of segments of the intersection \( \tilde{L} \cap P_i \) and \( u = \max u_i \). Then maximal order of ramification points of \( K_{\mathcal{Z}} \) is greater or equal to \( \min(u + 1, n) \).

**Proof.** Suppose \( \tilde{L} \cap P_i \) contains \( u \) segments of \( \tilde{L} \). Then the set contains at least \( u + 1 \) vertices of \( P_i \) if \( u < n \) and contains \( n \) vertices of \( P_i \) if \( u = n - 1 \) or \( n \). Then the set \( K_{\mathcal{Z}} \cap P_i \) contains at least \( u + 1 \) (resp. exactly \( n \)) different images of the arc \( \gamma_{OA} \).
References

1. Aseev, V. V., Tetenov, A. V., Kravchenko, A. S.: On Self-Similar Jordan Curves on the Plane. Sib. Math. J. 44(3), 379—386 (2003).
2. Bandt, C., Stahnke, J.: Self-similar sets 6. Interior distance on deterministic fractals. preprint, Greifswald 1990.
3. Charatonik, J., Charatonik, W.: Dendrites. Aport. Math. Comun. 22 227—253(1998).
4. Croydon, D.: Random fractal dendrites, Ph.D. thesis. St. Cross College, University of Oxford, Trinity(2006)
5. Hata, M.: On the structure of self-similar sets. Japan J. Appl. Math. 3, 381—414 (1985)
6. Kigami, J.: Harmonic calculus on limits of networks and its application to dendrites. J. Funct. Anal. 128(1) 48—86, (1995)
7. Kuratowski, K.: Topology. Vol. 1 and 2. Academic Press and PWN, New York (1966)
8. Nadler, S. B., Jr.: Continuum theory: an introduction. M. Dekker (1992)
9. Samuel, M., Tetenov, A. V., Vaulin, D.A.: Self-Similar Dendrites Generated by Polygonal Systems in the Plane. Sib. Electron. Math. Rep. 14, 737—751 (2017)
10. Tetenov, A. V.: Self-similar Jordan arcs and graph-oriented IFS. Sib. Math. J. 47(5), 1147—1153 (2006).
11. Tetenov, A. V., Samuel, M., Vaulin, D.A.: On dendrites generated by polyhedral systems and their ramification points. Proc. Krasovskii Inst. Math. Mech. UB RAS 23(4), 281—291 (2017) DOI: 10.21538/0134-4889-2017-23-4-281-291 (in Russian)
12. Tetenov A. V., Samuel, M., Vaulin, D.A.: On dendrites, generated by polyhedral systems and their ramification points. arXiv:1707.02875v1 [math.MG], 7 Jul 2017.