Relative Schwarz Method

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Abstract:

This paper proposes Relative Schwarz Method, which is a new domain decomposition method for wave equation.

1. Introduction

Since 2004, when I was a Ph.D candidate, I had recognized that manycore CPU and NUMA (Non Uniform Memory Access) model would be the trend for scientific computing and electronic aided design (EDA). Therefore, I began to devote myself into the research of distributed numerical algorithm for large scale integrated circuit simulation.

I studied all kinds of numerical algorithms, such as, LU, CG, GEMRES, Preconditioner, WR, DDM (Additive Schwarz, Schur Complement, etc), and found no algorithm was good enough for distributed computer, especially for NUMA model. Here good algorithm means the distributed algorithm has high scalability.
and fast convergence speed. Parallel LU and Schur Complement method are direct methods and able to achieve high accuracy, but their scalability is limited. CG, GEMRES, WR and Additive Schwarz method have high scalability, but they are uncertain in convergence [1,2,3,4,5,6].

Therefore I decided to find a new distributed algorithm, and spent 8 years to study, think and test. Because I was an EE student with limited math background, most times I made use of the EE knowledge and intuition to comprehend old algorithm and invent new algorithm. The basic principle to guide my research was that the real world is parallel and distributed, so there must be a natural way to do the distributed numerical computation, which would be a mimic of the real world.

In 2007, inspired by the physical behavior of transmission line, I invented a new distributed algorithm called Virtual Transmission Method [7]. Actually, it was a variant of Additive Schwarz method (AS). From the view of EE, the difference is that AS exchanges voltage only, while VTM exchanges a combination of voltage and current. I prove that VTM is convergent for SPD matrix, but the convergence speed of VTM is slow and difficult to optimize. So VTM was failed.

In 2008, I invented Directed Transmission Method, which is
the asynchronous version of VTM [8]. DTM was a distributed numerical algorithm that does not need global synchronization. However, the convergence capability of DTM was poor. So DTM is failed.

In 2009, I invented Waveform Transmission Method (WTM), which is a mixture of WR and VTM to solve ordinary differential equation [9]. WTM has poor convergence capability. So WTM is failed.

In 2012, after eight years’ effort without any SCI paper, I finally decided to give up the Ph.D and go to work as an ITer. However, I still believed that there is a natural way to do the parallel numerical computation, but I was too exhausted to find it.

Since 2012, during my spare time I continue to thinking about distributed numerical algorithm as a personal interest.

In 2020, I incidently got the idea and designed the initial version of Relative Schwarz method (RS). RS is inspired by the physical observation that wave is traveling by a limited velocity, which is the basics of Relativity Theory. RS is similar to Additive Schwarz method. Therefore, this method is named Relative Schwarz Method.

In 2021, I optimized the algorithm of RS and finished the first
numerical experiment, which showed RS had good potential.

In 2022, I finished full experiment which showed that RS is practical and scalable.

From 2004 to 2022, 18 years passed away and 4 algorithms (VTM, DTM, WTM, RS) were invented, of which 3 algorithms (VTM, DTM, WTM) failed, and 1 algorithm (RS) is to be verified.

2. Algorithm

This section describes Relative Schwarz method for 1-dimension wave equation.

2.1 Wave equation

The wave equation in 1-dimension is expressed as:

\[
\frac{\partial^2 u(x,t)}{\partial x^2} - a^2 \frac{\partial^2 u(x,t)}{\partial t^2} = 0, \ a > 0, x \in \Omega, \Omega = [X_1, X_2], t \in [0, +\infty) \quad (Eq. 1)
\]

with initial condition:

\[u(x,0) = u_{\text{init}}(x) = 0, \ x \in \Omega\]

and boundary condition:

\[u(X_1,t) = f_1(t), \ t \in [0, +\infty)\]
\[u(X_2,t) = f_2(t), \ t \in [0, +\infty)\]

**Definition 1: True Solution:**

\[u(x,t)\] is called the true solution of the original equation.

2.2 Split the domain into 2 domains

Split \(\Omega\) into 2 overlapped domains \(\Omega_1\) and \(\Omega_2\) as below:
\[ \Omega_1 = [X_1, X_3] \]
\[ \Omega_2 = [X_4, X_2] \]
\[ X_1 < X_4 < X_3 < X_2 \]

The overlapped domain is:

\[ \Omega_1 \cap \Omega_2 = [X_4, X_3] \neq \emptyset \]

(Eq. 1) is split into 2 equations:

\[ \frac{\partial^2 p(x,t)}{\partial x^2} - a^2 \frac{\partial^2 p(x,t)}{\partial t^2} = 0, a > 0, x \in \Omega_1, \Omega_1 = [X_1, X_3], t \in [0, +\infty) \]

\[ p(x,0) = p_{\text{init}}(x) \big|_{x=0} = u(x,0) = 0, x \in \Omega_1, \Omega_1 = [X_1, X_3] \]

\[ p(x,t) \big|_{x=X_3} = p(X_1,t) = f_1(t), t \in [0, +\infty) \]

\[ \frac{dp(x,t)}{dt} \big|_{x=X_3} = \frac{dp(X_3,t)}{dt} = \frac{dq(x,t)}{dx} \big|_{x=X_3} = \frac{dq(X_3,t)}{dx}, t \in [0, +\infty) \]  

(Eq. 2)

and:

\[ \frac{\partial^2 q(x,t)}{\partial x^2} - a^2 \frac{\partial^2 q(x,t)}{\partial t^2} = 0, a > 0, x \in \Omega_2, \Omega_2 = [X_4, X_2], t \in [0, +\infty) \]

\[ q(x,0) = q_{\text{init}}(x) \big|_{x=0} = u(x,0) = 0, x \in \Omega_2, \Omega_2 = [X_4, X_2] \]

\[ \frac{dq(x,t)}{dt} \big|_{x=X_3} = \frac{dq(X_3,t)}{dt} = \frac{dp(x,t)}{dx} \big|_{x=X_3} = \frac{dp(X_3,t)}{dt}, t \in [0, +\infty) \]

\[ q(x,t) \big|_{x=X_2} = q(X_2,t) = f_2(t), t \in [0, +\infty) \]  

(Eq. 3)

**Definition 2: Boundary Input Waveform:**

\[ \frac{dp(x,t)}{dt} \big|_{x=X_3}, t \in [T_{\text{start}}, T_{\text{start}} + \Delta T] \]  

is called the boundary input waveform for sub-domain (Eq. 2);

\[ \frac{dq(x,t)}{dt} \big|_{x=X_3}, t \in [T_{\text{start}}, T_{\text{start}} + \Delta T] \]  

is called the boundary input waveform for sub-domain (Eq. 3).

**Definition 3: Boundary Output Waveform:**

\[ \frac{dp(x,t)}{dx} \big|_{x=X_3}, t \in [T_{\text{start}}, T_{\text{start}} + \Delta T] \]  

is called the boundary output
waveform in sub-domain (Eq. 2);
\[ \frac{dq(x,t)}{dt} \big|_{x=X_s}, t \in [T_{start}, T_{start} + \Delta T] \]

is called the boundary output waveform in sub-domain (Eq. 3).

**Definition 4: Corresponding Waveform:**

If the boundary input waveform of one sub-domain has the same position with the boundary output waveform of its adjacent sub-domain, then these two waveforms are called corresponding waveform for each other.

For example:

\[ \frac{dp(x,t)}{dt} \big|_{x=X_s}, t \in [T_{start}, T_{start} + \Delta T] \]

is the boundary input waveform for sub-domain (Eq. 2), whose corresponding waveform is

\[ \frac{dq(x,t)}{dt} \big|_{x=X_s}, t \in [T_{start}, T_{start} + \Delta T] \],

which is the boundary output waveform in sub-domain (Eq. 3);

\[ \frac{dq(x,t)}{dt} \big|_{x=X_s}, t \in [T_{start}, T_{start} + \Delta T] \]

is the boundary input waveform for sub-domain (Eq. 3), whose corresponding waveform is

\[ \frac{dp(x,t)}{dt} \big|_{x=X_s}, t \in [T_{start}, T_{start} + \Delta T] \],

which is the boundary output waveform in sub-domain (Eq. 2).

**Theorem 1:**

After decomposition of the original wave equation, assume that each boundary input waveform of sub-domain is equal to the corresponding boundary output waveform in its adjacent sub-domain, and vice versa, then the solution of sub-domain is
equal to the true solution of the original equation.

According to Theorem 1:

for (Eq. 2), \( p(x,t) = u(x,t), x \in \Omega_1, t \in [0, +\infty) \); 

for (Eq. 3), \( q(x,t) = u(x,t), x \in \Omega_2, t \in [0, +\infty) \).

2.2 Assume the boundary input waveform to be zero and calculate the predict solution

First, set:

\[
T_{\text{start}} = 0
\]

Then, set the predict time span \( \Delta \hat{T}_{\text{predict}} \) as any value you like, which should be long enough.

Assume that the boundary input waveform of (Eq. 2) is 0:

\[
\left. \frac{d\hat{p}(x,t)}{dt} \right|_{x=x_1} = \left. \frac{d\hat{p}(X_2,t)}{dt} \right|_{t=0}, t \in [T_{\text{start}}, T_{\text{start}} + \Delta \hat{T}_{\text{predict}}]
\]

then (Eq.2) is transferred into (Eq.4):

\[
\begin{align*}
\frac{\partial^2 \hat{p}(x,t)}{\partial x^2} - a^2 \frac{\partial^2 \hat{p}(x,t)}{\partial t^2} &= 0, a > 0, x \in \Omega_1, \Omega_1 = [X_1, X_3], t \in [T_{\text{start}}, T_{\text{start}} + \Delta \hat{T}_{\text{predict}}] \\
\hat{p}(x,T_{\text{start}}) &= \hat{p}_{\text{init}}|_{t=T_{\text{start}}}(x), x \in \Omega_1, \Omega_1 = [X_1, X_3] \\
\hat{p}(X_1,t) &= f_1(t), t \in [T_{\text{start}}, T_{\text{start}} + \Delta \hat{T}_{\text{predict}}] \\
\hat{p}(X_3,t) &= 0, t \in [T_{\text{start}}, T_{\text{start}} + \Delta \hat{T}_{\text{predict}}] \\
\end{align*}
\]

(Eq. 4)

where the initial condition for (Eq. 4) is

\[
\hat{p}_{\text{init}}|_{t=T_{\text{start}}}(x) = u(x,T_{\text{start}}), x \in \Omega_1
\]

(Eq.4) is able to be solved and the solution is:

\[
\hat{p}(x,t), x \in [X_1, X_3], t \in [T_{\text{start}}, T_{\text{start}} + \Delta \hat{T}_{\text{predict}}]
\]
and:
\[
\frac{\hat{p}(x,t)}{dx}, x \in [X_1, X_3], t \in [T_{start}, T_{start} + \Delta \hat{T}_{predict}]
\]

Similarly, assume that the boundary input waveform of (Eq.3) is 0:
\[
\frac{d\hat{q}(x,t)}{dt} \mid_{x=X_4} = \frac{d\hat{q}(X_4,t)}{dt} = 0, t \in [T_{start}, T_{start} + \Delta \hat{T}_{predict}]
\]

Then, (Eq. 3) is transferred into (Eq. 5):
\[
\frac{\partial^2 \hat{q}(x,t)}{\partial x^2} - a^2 \frac{\partial^2 \hat{q}(x,t)}{\partial t^2} = 0, a > 0, x \in \Omega, \Omega = [X_4, X_2], t \in [T_{start}, T_{start} + \Delta \hat{T}_{predict}]
\]
\[
\hat{q}(x,T_{start}) = \hat{q}_{init}(x) \mid_{T=T_{start}}, x \in \Omega_2, \Omega_2 = [X_4, X_2]
\]
\[
\frac{d\hat{q}(x,t)}{dt} \mid_{x=X_4} = \frac{d\hat{q}(X_4,t)}{dt} = 0, t \in [T_{start}, T_{start} + \Delta \hat{T}_{predict}]
\]
\[
\hat{q}(X_2,t) = f_2(t), t \in [T_{start}, T_{start} + \Delta \hat{T}_{predict}]
\]

(Eq. 5)

where the initial condition for (Eq. 5) is:
\[
\hat{q}_{init}(x) \mid_{T=T_{start}} = u(x,T_{start}), x \in \Omega_2
\]

(Eq. 5) is able to be solved and the solution is:
\[
\hat{q}(x,t), x \in [X_4, X_2], t \in [T_{start}, T_{start} + \Delta \hat{T}_{predict}]
\]

and:
\[
\frac{\hat{q}(x,t)}{dx}, x \in [X_4, X_2], t \in [T_{start}, T_{start} + \Delta \hat{T}_{predict}].
\]

**Definition 5: Predict Solution:**

Assume that the boundary input waveform of sub-domain is 0, therefore the solution of the sub-equation for the sub-domain is called the predict solution.
As the result,

\[ \hat{p}(x,t) \quad \text{and} \quad \frac{\hat{p}(x,t)}{dx}, \quad t \in [T_{start}, T_{start} + \Delta T_{\text{predict}}] \]

is called the predict solution of (Eq. 2);

\[ \hat{q}(x,t) \quad \text{and} \quad \frac{\hat{q}(x,t)}{dx}, \quad t \in [T_{start}, T_{start} + \Delta T_{\text{predict}}] \]

is called the predict solution of (Eq. 3).

2.3 Find the max waveform time span and calculate the true solution

Definition 6: Waveform Max Time Span

In the overlapped domain \( \Omega_1 \cap \Omega_2 \), define the waveform max time span \( \Delta T_{\text{max}}(x) \) is:

For \( x \in \Omega_1 \cap \Omega_2 \), \( \Delta T_{\text{max}}(x) = \max(\Delta T(x)) \), where \( \Delta T(x) \) satisfies

\( \forall t \in [T_{start}, T_{start} + \Delta T(x)], \quad \hat{p}(x,t) = \hat{q}(x,t). \)

Then define \( \Delta T_{\text{max}} \) as:

\( \Delta T_{\text{max}} = \max(\Delta T_{\text{max}}(x)), \forall x \in \Omega_1 \cap \Omega_2 \)

Theorem 2:

Within the max time span \( \Delta T_{\text{max}} \), the predict solution of the boundary output waveform in each sub-domain is equal to the true solution.

According to Theorem 1, for (Eq. 4) and (Eq. 5), the conclusion is as below:

\[ \frac{du(x,t)}{dx} \bigg|_{x=X_4} = \frac{dp(x,t)}{dx} \bigg|_{x=X_4} = \frac{d\hat{p}(x,t)}{dx} \bigg|_{x=X_4}, \quad \forall t \in [T_{start}, T_{start} + \Delta T_{\text{max}}]. \]
\[
\frac{du(x,t)}{dx} \bigg|_{x=x_1} = \frac{dq(x,t)}{dx} \bigg|_{x=x_2}, \forall t \in [T_{\text{start}}, T_{\text{start}} + \Delta T_{\text{max}}].
\]

As the result, the boundary output waveform of (Eq. 3) is solved by (Eq. 5), and (Eq. 2) is updated as (Eq. 6):

\[
\frac{\partial^2 p(x,t)}{\partial x^2} - a^2 \frac{\partial^2 p(x,t)}{\partial t^2} = 0, a > 0, x \in \Omega_1, \Omega_1 = [X_1, X_3], t \in [T_{\text{start}}, T_{\text{start}} + \Delta T_{\text{max}}]
\]

\[
p(x,0) = p_{\text{init}}(x) \big|_{t=T_{\text{start}}} = u(x,0) = 0, x \in \Omega_1, \Omega_1 = [X_1, X_3]
\]

\[
p(X_1,t) = f_1(t), t \in [T_{\text{start}}, T_{\text{start}} + \Delta T_{\text{max}}]
\]

\[
\frac{dp(X_1,t)}{dt} = \frac{dq(X_1,t)}{dx} = \frac{dq(X_1,t)}{dx}, t \in [T_{\text{start}}, T_{\text{start}} + \Delta T_{\text{max}}]
\]

(Eq. 6)

(Eq. 6) can be solved and get \( p(x,t), x \in \Omega_1, t \in [T_{\text{start}}, T_{\text{start}} + \Delta T_{\text{max}}] \).

Similarly, (Eq. 3) is updated as (Eq. 7):

\[
\frac{\partial^2 q(x,t)}{\partial x^2} - a^2 \frac{\partial^2 q(x,t)}{\partial t^2} = 0, a > 0, x \in \Omega_2, \Omega_2 = [X_4, X_2], t \in [T_{\text{start}}, T_{\text{start}} + \Delta T_{\text{max}}]
\]

\[
q(x,0) = q_{\text{init}}(x) \big|_{t=T_{\text{start}}} = u(x,0) = 0, x \in \Omega_2, \Omega_2 = [X_4, X_2]
\]

\[
\frac{dq(X_2,t)}{dt} = \frac{dp(X_2,t)}{dx} = \frac{dp(X_2,t)}{dx}, t \in [T_{\text{start}}, T_{\text{start}} + \Delta T_{\text{max}}]
\]

\[
q(X_3,t) = f_2(t), t \in [0, +\infty)
\]

(Eq. 7)

(Eq. 7) can be solved and get \( q(x,t), x \in \Omega_2, t \in [T_{\text{start}}, T_{\text{start}} + \Delta T_{\text{max}}] \).

The solution of (Eq. 1) \( u(x,t), x \in \Omega, t \in [T_{\text{start}}, T_{\text{start}} + \Delta T_{\text{max}}] \) is the combination of the solution of (Eq. 6) and (Eq. 7):

\[
u(x,t) = \begin{cases} 
p(x,t), x \in \Omega_1 \\
q(x,t), x \in \Omega_2 \land x \notin \Omega_1, t \in [T_{\text{start}}, T_{\text{start}} + \Delta T_{\text{max}}]
\end{cases}
\]

**Theorem 3:**

Assume the length of the overlapped domain is \( \Delta X_{\text{overlap}} \), and the wave velocity of (Eq. 1) is \( a \), then:
\[ \Delta T_{\text{max}} \propto \frac{\Delta X_{\text{overlap}}}{a} \]

### 2.4 Update the start time and initial condition, repeat 2.2 and 2.3

Set the new start time as:

\[ T_{\text{start}} = T_{\text{start}}^{\text{old}} + \Delta T_{\text{max}} \]

For (Eq. 4) and (Eq. 5), set the initial condition at \( T_{\text{start}} \) as:

\[
\begin{align*}
  p_{\text{init}}(x) |_{t=T_{\text{start}}} &= u(x, T_{\text{start}}), x \in \Omega_1 \\
  q_{\text{init}}(x) |_{t=T_{\text{start}}} &= u(x, T_{\text{start}}), x \in \Omega_2
\end{align*}
\]

and set the new predict time span \( \Delta \hat{T}_{\text{predict}} = (1 + \alpha) \Delta \hat{T}^{\text{old}}_{\text{max}}, \alpha = 0.1 \).

Then, redo Section 2.3 to find the new max time span \( \Delta T_{\text{max}} \), and redo Section 2.4 to get the true solution in the new time span: \( u(x, t), x \in \Omega, t \in [T_{\text{start}}, T_{\text{start}} + \Delta T_{\text{max}}] \).

Repeat the above procedure by loop, the original wave equation (Eq. 1) is distributed solved by decomposing into 2 overlapped domains.

### 2.5 Split the domain into N subdomains, \( N > 2 \)

Assume the original wave equation is split into N overlapped subdomains (\( N > 2 \)) by RS.

**Definition 8: Global Max Time Span**

For each overlapped region, there will be a waveform max time span \( \Delta T_{\text{max},k}, k = 1, ..., N - 1 \).

Therefore, the global max time span would be:
\[ \Delta T_{\text{max, global}} = \min(\Delta T_{\text{max, k}}), k = 1, \ldots, N - 1 \]

Consequently, Theorem 2 is extended into Theorem 3, where \( p_k(x,t) \) is the true solution for sub-domain \( k \) and \( \hat{p}_k(x,t) \) is the predict solution.

**Theorem 4:**

Within the global max time span \( \Delta T_{\text{max, global}} \), the predict solution of the boundary output waveform in each sub-domain is equal to the true solution.

According to Theorem 4,

\[
\frac{du(x,t)}{dx} \bigg|_{x=X_{\text{boundary},k}} = \frac{dp_k(x,t)}{dx} \bigg|_{x=X_{\text{boundary},k}} = \frac{d\hat{p}_k(x,t)}{dx} \bigg|_{x=X_{\text{boundary},k}}
\]

where \( t \in [T_{\text{start}}, T_{\text{start}} + \Delta T_{\text{max, global}}] \), \( x \in \Omega_k \), \( k = 1, \ldots, N - 1 \), and \( X_{\text{boundary},k} \) is the position of each boundary output waveform in sub-domain \( k \).

3. Experiment

3.1 RS, \( N=2 \)

The original wave equation is (Eq. 1), and \( N=2 \) pulse sources are inserted. The numerical result of (Eq. 1) is shown as:
Fig. 1 Numerical solution of (Eq. 1) with \( N=2 \) pulse sources

Then use RS to split (Eq. 1) into \( N=2 \) domains and calculate distributed. The error of RS is shown in Fig. 2.
Fig. 2. Error of RS, N=2

3.2 RS, N=10

The original wave equation is (Eq. 1), and N pulse sources are inserted. The numerical result of (Eq. 1) is shown as:
Fig. 3. Numerical solution of (Eq.1), with N=10 pulse sources

Then use RS to split (Eq. 1) into N=10 domains and calculate distributed. The error of RS is shown in Fig. 4.
Fig. 4. Error of RS, N=10

4. Conclusion

This paper proposes a new domain decomposition method for wave equation, which is called Relative Schwarz method. Experiments show that the accuracy and scalability of RS is good.

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