 REPRESENTATION THEORY OF A CLASS OF HOPF ALGEBRAS

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Abstract. The representations of the pointed Hopf algebras $U$ and $u$ are described, where $U$ and $u$ can be regarded as deformations of the usual quantized enveloping algebras $U_q(sl(3))$ and the small quantum groups respectively. It is illustrated that these representations have a close connection with those of the quantized enveloping algebras $U_q(sl(2))$ and those of the half quantum groups of $sl(3)$.

Introduction

In a series of papers [1, 2, 3, 4], N. Andruskiewitsch and H.-J. Schneider classified finite dimensional pointed Hopf algebras over algebraically closed field $k$ with $\text{char}(k) = 0$, whose group-like elements form a finite abelian group. Under some suitable hypotheses, given a datum of finite Cartan type $D = D(\Gamma, (g_i)_{1 \leq i \leq n}, (\chi_i)_{1 \leq i \leq n}, (a_{ij})_{1 \leq i, j \leq n})$, a linking datum $\lambda = (\lambda_{ij})_{1 \leq i < j \leq n}$ and a root datum $\mu = (\mu_a)_{a \in A}$, one can construct a pointed Hopf algebra $U(D, \lambda)$ and a finite dimensional quotient $u(D, \lambda, \mu)$. On the other hand, if $A$ is a finite dimensional pointed Hopf algebra over $k$ and the group-like elements in $A$ form an abelian group, then $A \simeq u(D, \lambda, \mu)$ for some data $D, \lambda, \mu$. The Hopf algebras $U(D, \lambda)$ (resp. $u(D, \lambda, \mu)$) have close connection with the quantized enveloping algebras and can be regarded as generalization of the quantized enveloping algebras (resp. the small quantum groups).

It is natural to investigate the properties and the representations of these Hopf algebras. However, not very much is known about this problem in general. A classical example discussed intensely is the representations of the quantized enveloping algebras of semisimple Lie algebras and their quotients (i.e., the small quantum groups). Recently, the irreducible representations of a class of generalized doubles are described in [17], which can be parameterized by dominant pairs of characters. Another example is the representation theory of the half quantum group. In [18], simple modules, projective weight modules and simple Yetter-Drinfeld weight modules over the half quantum group are fully described.

This paper aims to study the representations of the pointed Hopf algebras $U = U(D, \lambda)$ and its finite dimensional quotient Hopf algebras $u$, where $D$ consists of a free abelian group $\Gamma$ with rank 2, elements $(g_1, g_2, g_3, g_4)$ in $\Gamma$, the characters $\chi_{i}, \chi_{j}, \chi_{k}, \chi_{l}$ and the Cartan matrix of type $A_2 \times A_2$. In particular, we concentrate on the case that the linking datum $\lambda = (\lambda_{ij})_{1 \leq i < j \leq 4}$ is given by $\lambda_{13} = 1$ and $\lambda_{ij} = 0$ otherwise. We describe the simple modules over $U$ and the simple modules and the projective modules over $u$. We note that the category $\mathcal{F}$ of all finite dimensional $U$-modules is not semisimple. That is different from the representation theory of the quantized enveloping algebras. In fact, the

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representation theory of $U$ can be regarded as “combination” of those of the quantized enveloping algebra $U_q(\mathfrak{sl}(2))$ and those of the half quantum groups of $\mathfrak{sl}(3)$.

In Section 1 some definitions and the structure of the Hopf algebras $U(D, \lambda)$ and $u(D, \lambda, \mu)$ are given. In Section 2 Hopf algebra $U = U(D, \lambda)$ is constructed, where $D$ is a Cartan datum of type $A_2 \times A_2$. The representation theory of $U$ is also developed when $q$ is not a root of unity. When $q$ is a root of unity, there is a Hopf ideal in $U$. The corresponding quotient Hopf algebra $u$ is defined in Section 3. For a given skew pairing $\phi$, the double crossproduct $D_\phi = D_\phi (u^{\mathfrak{Z}_0}, u^{\mathfrak{Z}_0})$ is constructed, which is a twisting of the usual Drinfeld quantum double $D(u^{\mathfrak{Z}_0}, u^{\mathfrak{Z}_0})$. In Theorem 3.9 we show that the category $D_\phi M$ is equivalent to the direct product of $|\zeta| \times |\zeta|$ copies of the category $uM$. The simple modules and projective modules over $u$ are constructed in Section 3.2. In particular, we give an equation set which can be used to compute idempotent elements in $u$. Thus we get a decomposition of the regular module as a direct sum of indecomposable projective modules. This method is valid for the small quantum group of $\mathfrak{sl}(2)$ too.

Throughout, we assume that $k$ is an algebraically closed field with characteristic 0, and all vector spaces and tensor products are over $k$. Let $k^* = k \setminus \{0\}$, $\mathbb{Z}$ be the integer set, $\mathbb{N}$ the positive integer set and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $l \in \mathbb{Z}$, let $\mathbb{Z}_l = \mathbb{Z}/l\mathbb{Z}$.

1. Preliminaries

For $n \in \mathbb{N}$ and $q \in k \setminus \{0, \pm 1\}$, let $[n]_q = (q^n - q^{-n})/(q - q^{-1})$. As usual, we define $[0]_q! = 1$ and $[n]_q! = [n]_q[n-1]_q \cdots [1]_q$ for $n \geq 1$, and the Gaussian $q$-binomial coefficients

$$\left[ \begin{array}{c} n \\ j \end{array} \right]_q = \frac{[n]_q!}{[j]_q![n-j]_q!}, \quad n \geq j \geq 0.$$

For $a \in k$, let $[a; n]_q = (aq^n - a^{-1}q^{-n})/(q - q^{-1})$. Similarly we can define the factorial $[a; 0]_q! = 1$, $[a; n]_q! = [a; n]_q[a; n-1]_q \cdots [a; 1]_q$ for $n \geq 1$ and the “binomial coefficients”

$$\left[ \begin{array}{c} a; n \\ j \end{array} \right]_q = \frac{[a; n]_q!}{[j]_q![a; n-j]_q!}, \quad n \geq j \geq 0.$$

Note that $\left[ \begin{array}{c} a^{-1}; n + 1 \\ j \end{array} \right]_q = \left[ \begin{array}{c} a; n \\ j \end{array} \right]_q$. For $j \geq 0$, one can define

$$\left[ \begin{array}{c} a; n \\ j \end{array} \right]_q = \left[ \begin{array}{c} a; a^{-1}; n + j + s \\ j \end{array} \right]_q$$

if $n < j$, where $s \in \mathbb{Z}$ satisfies $n + s \geq 0$.

Let $A, H$ be bialgebras. A bilinear form $\varphi : A \otimes H \rightarrow k$ is skew pairing if the following conditions are satisfied:

$$\varphi(ab, x) = \sum \varphi(a, x_1)\varphi(b, x_2),$$

$$\varphi(a, xy) = \sum \varphi(a_1, y)\varphi(a_2, x),$$

$$\varphi(a, 1) = \varepsilon(a),$$

$$\varphi(1, x) = \varepsilon(x),$$
for all \(a, b \in A, x, y \in H\). If \(A\) (resp. \(H^{op}\)) is Hopf algebra with antipode \(S_A\) (resp. \(S_{H^{op}}\)), then \(\varphi\) is invertible with \(\varphi^{-1}(a, x) = \varphi(S_A(a), x)\) (resp. \(\varphi(a, S_{H^{op}}(x))\)). Hence if \(A\) and \(H\) are Hopf algebras and \(S_H\) is bijective, \(\varphi(S_A(a), x) = \varphi(a, S_{H^{op}}(x))\), where \(S_{H^{op}} = S_H^{-1}\), the composition inverse of \(S_H\).

If \(\varphi\) is a convolution invertible skew pairing, then the double crossproduct \(A \bowtie_{\varphi} H = A \otimes H\) is constructed as follows. The coalgebra structure is given by
\[
\Delta(a \otimes x) = \sum a_{(1)} \otimes x_{(1)} \otimes a_{(2)} \otimes x_{(2)},
\]

\[
\varepsilon(a \otimes x) = \varepsilon(a) \varepsilon(x),
\]
and the algebra structure is given by
\[
(a \otimes x)(b \otimes y) = \sum \varphi(b_{(1)}, x_{(1)})ab_{(2)} \otimes x_{(2)}y^{-1}(b_{(3)}, x_{(3)}),
\]
with identity 1 \(\otimes 1\).

If \(A\) and \(H\) are Hopf algebras, then \(A \bowtie_{\varphi} H = A \otimes H\) is also a Hopf algebra (see \([10, 15, 16]\)).

Recall that a datum of Cartan type \(D = D(\Gamma, \{g_i\}_{i \leq \theta}, \{\chi_i\}_{i \leq \theta}, \{a_{ij}\}_{1 \leq i, j \leq \theta})\) consists of an abelian group \(\Gamma\), elements \(g_i \in \Gamma, \chi_i \in \hat{\Gamma}, 1 \leq i \leq \theta\), and a generalized Cartan matrix \((a_{ij})\) of size \(\theta\) satisfying the relations:

\[
q_{ij}q_{ji} = q_{ii}^{a_{ij}} \cdot q_{ii} \neq 1, \text{ where } q_{ij} = \chi_j(g_i), 1 \leq i, j \leq \theta.
\]

\(\theta\) is called the rank of \(D\). If the matrix \((a_{ij})\) is of finite type, then \(D\) is said to be of finite Cartan type. In this case, \((a_{ij})\) is a matrix of blocks corresponding to the connected components of the Dynkin diagram after a reordering. We write \(i \sim j\) for any \(i, j \in \{1, \cdots, \theta\}\) if \(i\) and \(j\) are in the same connected component. Let \(I = \{I_1, I_2, \cdots, I_s\}\) be the set of connected components of \(I = \{1, \cdots, \theta\}\). Let \(ord(q)\) denote the order of \(q\). For any \(1 \leq i \leq \theta\), when \(ord(q_{ii})\) is finite, we assume

\[
ord(q_{ii}) \text{ is odd},
\]

\[
ord(q_{ii}) \text{ is prime to } 3, \text{ if } i \text{ lies in a component } G_2.
\]

**Lemma 1.1.** Given a \(J \in I\), then \(ord(q_{ii}) = ord(q_{jj})\) for any \(i, j \in J\). We write \(N_J = ord(q_{ii})\) for some \(i \in J\).

**Proof.** If each \(ord(q_{ii}) < \infty\), the claim follows from \([11\text{ Lemma 2.3}]\). Now we assume that there is an \(i \in J\) such that \(ord(q_{ii}) = \infty\). Let \(j \in J\). Since the Dynkin diagram of \(J\) is connected, there is a chain \(i = i_1 - i_2 - \cdots - i_p = j\) in \(J\) such that \(a_{i_i,i_1} \neq 0\) for \(s = 1, 2, \cdots, p - 1\). By \([11\text{ L.2.1}], q_{ii}^{a_{ij}} = q_{jj}^{a_{ij}}\). Thus we have \(ord(q_{jj}) = \infty\). \(\square\)

A family \(\lambda = (\lambda_{ij})_{1 \leq i < j \leq \theta, i \neq j}\) of elements in \(k\) is called a family of linking parameters for \(D\) if the following condition is satisfied for all \(1 \leq i < j \leq \theta\) with \(i + j\),

\[
\text{if } g_i g_j = 1 \text{ or } \chi_i \chi_j \neq 1, \text{ then } \lambda_{ij} = 0.
\]

Vertices \(i, j\) are called linkable if \(i \neq j, g_i g_j \neq 1, \chi_i \chi_j = 1\). \(i, j\) are called linked if \(\lambda_{ij} \neq 0\). For convenience, let \(\lambda_{ij} = -q_{ij}^{a_{ij}}\lambda_{ij}\) for all \(1 \leq i < j \leq \theta, i \neq j\).
We collect some useful facts from [2, 8]: any vertex \( i \) is linkable to at most one vertex \( j \); if \( i, j \) are linkable, then \( q_{ij} = q_{ji}, q_{ij}q_{ji} = 1 \); if \( i \) and \( k \), respectively, \( j \) and \( l \), are linkable, then \( a_{ij} = a_{kl}, a_{ji} = a_{lk} \).

For a given datum \( D = D(\Gamma, (g_1)_{1\leq i\leq \theta}, (\chi(i))_{1\leq i\leq \theta}, (a_{ij})_{1\leq i,j\leq \theta}) \) of finite Cartan type and a linking datum \( \lambda = (\lambda_i)_{1\leq i\leq \theta} \), one can construct a Hopf algebra \( U(D, \lambda) \) as follows.

Let \( V \) be a vector space with a basis \( \{x_1, x_2, \ldots, x_\theta\} \). Then \( V \) or \( (V, \cdot, \rho) \) is a (left-left) Yetter-Drinfeld module over \( k\Gamma \) such that \( x_i \in V^0 \), i.e., \( g \cdot x_i = \chi(g)x_i \) for all \( g \in \Gamma \) and \( \rho(x_i) = g_i \otimes x_i \). The corresponding braiding \( c : V \otimes V \to V \otimes V \) is given by \( c(x_i \otimes x_j) = q_{ij}x_j \otimes x_i, 1 \leq i, j \leq \theta \). Let \( \Gamma \mathcal{YD} \) denote the category of all the (left-left) Yetter-Drinfeld modules over \( k\Gamma \). Then \( \Gamma \mathcal{YD} \) is a braided monoidal category and the free algebra \( k(x_1, \ldots, x_\theta) \) is a braided Hopf algebra in \( \Gamma \mathcal{YD} \). So the smash product \( k(x_1, \ldots, x_\theta) \# k\Gamma \) is a usual Hopf algebra. Let \( U(D, \lambda) \) be the quotient Hopf algebra of \( k(x_1, \ldots, x_\theta) \# k\Gamma \) modulo the ideal generated by the elements:

\[
(5) \quad (ad_{x_i}x_j)^{1-a_{ij}}(x_j), \quad \text{for all } 1 \leq i, j \leq \theta, \quad i \sim j, \quad i \neq j,
\]

\[
(6) \quad x_i x_j - q_{ij} x_j x_i = \lambda_i(1 - g_i g_j), \quad \text{for all } 1 \leq i < j \leq \theta, \quad i \neq j.
\]

where \( (ad_{x_i}x_j)(y) = x_i y - q_{ij} q_{ji} \cdots q_{ij} x_i \). Let \( y = x_{i_1} x_{i_2} \cdots x_{i_n} \). Then \( y \) is called the braided commutator. The coalgebra structure of \( U(D, \lambda) \) is given by

\[
\Delta(x_i) = g_i \otimes x_i + x_i \otimes 1, \quad \Delta(g) = g \otimes g, \quad \text{for all } 1 \leq i \leq \theta, \quad g \in \Gamma.
\]

For a given indecomposable Cartan matrix \( (c_{ij})_{n \times n} \), let \( P = \sum_{i=1}^{n} \mathbb{Z} \alpha_i \) be the weight lattice. Define simple roots by

\[
\alpha_j = \sum_{i=1}^{n} c_{ij} \alpha_i, \quad j = 1, \ldots, n.
\]

Let \( \Delta = \{\alpha_1, \ldots, \alpha_n\} \), \( Q = \mathbb{Z}\Delta \) (the root lattice), and \( Q_+ = \sum_i \mathbb{N}_0 \alpha_i \). For any \( \beta = \sum_i b_i \alpha_i \in Q \), define \( g_\beta = g_{\alpha_1}^{b_1} g_{\alpha_2}^{b_2} \cdots g_{\alpha_n}^{b_n} \). In particular, \( g_{\alpha_i} = g_i \).

Define automorphisms \( \gamma_i \) of \( P \) by \( \gamma_i \alpha_j = \delta_{ij} \alpha_j \). Then \( \gamma_i \alpha_j = \alpha_j - c_{ij} \alpha_i \).

Let \( W \) be the (finite) subgroup of \( GL(P) \) generated by \( \gamma_1, \ldots, \gamma_n \), called the Weyl group. Then \( Q \) is \( W \)-invariant. Let \( R = W \Delta, \quad R^+ = R \cap Q_+ \) and \( R^* = -R^+ \). Then \( R \) is a root system corresponding to the Cartan matrix \( (c_{ij}) \), \( R^* \) the set of positive roots, \( R = R^* \cup R^* \).

Fix a reduced expression \( \gamma_i \gamma_i \cdots \gamma_{i_0} \) of the longest element \( \omega_0 \) of \( W \). This gives us a convex ordering of the set of positive roots \( R^* \):

\[
\beta_1 = \alpha_{i_1}, \quad \beta_2 = \gamma_{i_1} \alpha_{i_2}, \ldots, \beta_N = \gamma_{i_{N-1}} \alpha_{i_N}.
\]

Then for any \( J \in I \) one can choose a Weyl group \( W_J \) and a root system \( R_J \) for \( (a_{ij})_{i \in J} \) and a reduced expression of the longest element \( w_{0,J} \) in the Weyl group \( W_J \). Put

\[
\omega_0 := w_{0,J} w_{0,L} \cdots w_{0,J},
\]

and

\[
R^* := \{\beta_{i_1,1}, \ldots, \beta_{i_1,j_1}, \ldots, \beta_{i_L,1}, \ldots, \beta_{i_L,j_L}\},
\]

We can define a reduced expression \( \gamma_{i_1} \gamma_{i_1} \cdots \gamma_{i_N} \) of \( \omega_0 \) in the Weyl group \( W \) by

\[
\gamma = \gamma_{i_1} \gamma_{i_1} \cdots \gamma_{i_N} \gamma_{i_{N-1}} \gamma_{i_{N-2}} \cdots \gamma_{i_1}.
\]
where \( p_J \) is the number of the positive roots in \( R^+_J \) and \( \beta_{J,1}, \cdots, \beta_{J,p_J} \in R^+_J \) with the convex ordering. We also write
\[
R^+ = \{ \beta_1, \cdots, \beta_p \}, \quad p = \sum_{J \in \mathcal{I}} p_J
\]
with the given ordering.

For each \( \beta_i \in R^+_J \), one can define a root vector \( x_{\beta_i} \in U(\mathcal{D}, \lambda) \) by the same iterated braided commutator of the elements \( x_j, j \in J \) as the Lusztig’s case in [14] but with respect to the general braiding \( c \), see [3].

The following theorem describes the structure of \( U(\mathcal{D}, \lambda) \), which was stated in [11] Theorem 3.3] for a finite abelian group \( \Gamma \). Indeed, the finiteness condition of \( \Gamma \) is not necessary in the proof of [11] Theorem 3.3].

**Theorem 1.2.** Let \( \Gamma \) be an abelian group, and \( \mathcal{D} \) a datum of finite Cartan type satisfying the conditions (2) and (3). Let \( \lambda \) be a family of linking parameters for \( \mathcal{D} \). Then

1. The elements
\[
x_{\beta_1}^{a_1} x_{\beta_2}^{a_2} \cdots x_{\beta_p}^{a_p} g, \quad a_1, a_2, \cdots, a_p \geq 0, \quad g \in \Gamma,
\]
form a basis of the vector space \( U(\mathcal{D}, \lambda) \).
2. Let \( J \in \mathcal{I} \) and \( \alpha \in R^+, \beta \in R^+_J \). Then \( \{ x_\alpha, x_\beta \} = 0 \), that is
\[
x_\alpha x_\beta = q_{\alpha \beta} x_\beta x_\alpha.
\]

A family \( \mu = (\mu_\alpha)_{\alpha \in R^+} \) of elements in \( k \) is called a family of root vector parameters for \( \mathcal{D} \) if the following condition is satisfied for all \( \alpha \in R^+_J, J \in \mathcal{I} \): If \( g^{N_J}_\alpha = 1 \) or \( \chi^{N_J}_\alpha \neq \epsilon \), then \( \mu_\alpha = 0 \).

Then we define
\[
u(\mathcal{D}, \lambda, \mu) = U(\mathcal{D}, \lambda)/ (x^{N_J}_\alpha - u_\alpha(\mu) | \alpha \in R^+_J, J \in \mathcal{I}),
\]
where \( u_\alpha(\mu) \) is central in \( U(\mathcal{D}, \lambda) \) and is determined by \( \mu \) uniquely (see [11]).

**Theorem 1.3.** ([11] Thm6.2]) Let \( A \) be a finite dimensional pointed Hopf algebra with abelian group \( G(A) = \Gamma \) and infinitesimal braiding matrix \( (q_{ij})_{1 \leq i, j \leq \theta} \). Assume that the following conditions are satisfied:

\[
\text{ord}(q_{ii}) > 7 \text{ is odd,}
\]
\[
\text{ord}(q_{ii}) \text{ is prime to } 3 \text{ if } q_{ii}q_{ii} \in \{ q_{ii}^{-3}, q_{ii}^{-1} \} \text{ for some } l,
\]

where \( 1 \leq i \leq \theta \). Then
\[
A \cong u(\mathcal{D}, \lambda, \mu),
\]
where \( \mathcal{D} = \mathcal{D}(\Gamma, (g_{ij})_{1 \leq i, j \leq \theta}, (x_{ij})_{1 \leq i, j \leq \theta}, (a_{ij})_{1 \leq i, j \leq \theta}) \) is a datum of finite Cartan type, and \( \lambda \) and \( \mu \) are families of linking and root vector parameters for \( \mathcal{D} \).

We apply these construction to the special case that the Cartan matrix \( (a_{ij}) \) is of type \( A_2 \times A_2 \), i.e.
\[
(a_{ij}) = \begin{pmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & 0 & 0 \\
0 & 0 & 2 & -1 \\
0 & 0 & -1 & 2
\end{pmatrix},
\]
Let $\Gamma = \langle g_1, g_2 \rangle$ be a free abelian group of rank 2, $g_3 = g_1$, $g_4 = g_2$. Let $q \in k \setminus \{0, \pm 1\}$. Then $\chi_j$ is given by $\chi_j(g_i) = q^{\alpha_{ji}}$ for $1 \leq j, i \leq 2$, and $\chi_3 = \chi_1^{-1}$, $\chi_4 = \chi_2^{-1}$. These form a datum of finite Cartan type $D$.

The linking datum $(\Lambda_{ij})$ has the following 3 cases after a suitable permutation:

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
-q_{31} & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-q_{31} & 0 & 0 & 0 \\
0 & -q_{42} & 0 & 0
\end{pmatrix},
\]

where $q_{31} = q^{-\alpha_{31}}, q_{42} = q^{-\alpha_{42}}$. Denote them by $\lambda_1, \lambda_2, \lambda_3$ respectively. Then we have the following facts:

- $U(D, \lambda_1)$ is a graded Hopf algebra if we define $\deg g = 0$, $\forall g \in \Gamma$, $\deg x_i = 1$, $1 \leq i \leq 4$. This is similar to the half quantum group discussed in [6, 18].

- $U(D, \lambda_3)$ is the quantized enveloping algebra of $\mathfrak{sl}(3)$, which has been discussed in many papers, see [5, 7, 12].

In the next section, we shall concentrate on the second case, i.e., $U(D, \lambda_2)$.

2. Hopf algebra $U(D, \lambda_2)$

2.1. the properties of $U$. Let $\Gamma = \langle g_1, g_2 \rangle$ be a free abelian group of rank 2, $g_3 = g_1$, $g_4 = g_2$. Let $q \in k \setminus \{0, \pm 1\}$ and $(\alpha_{ij})$ the Cartan matrix of type $A_2$. Then $\chi_j$ is given by $\chi_j(g_i) = q^{\alpha_{ji}}$ for $1 \leq j, i \leq 2$. Let $\chi_3 = \chi_1^{-1}$ and $\chi_4 = \chi_2^{-1}$. These form a datum of finite Cartan type $D$, the corresponding Cartan matrix is of type $A_2 \times A_2$. For any $1 \leq i \leq 4$, if $\text{ord}(q_{ii})$ is finite, we assume that the condition (2) is satisfied.

For convenience, write $U'$ for the corresponding Hopf algebra $U(D, \lambda_2)$.

As an algebra, $U'$ is generated by the elements $x_i (1 \leq i \leq 4)$, $g \in \Gamma$ and subjects to the relations (for all $1 \leq i, j \leq 4$):

\[
\begin{align*}
(7) \quad & g x_j = \chi_j(g)x_j g, \\
(8) \quad & ad_c(x_i)^{1-\alpha_{ji}}(x_j) = 0, \text{ for all } i \sim j, i \neq j, \\
(9) \quad & x_i x_j - q_{ji} x_j x_i = \lambda_{ij}(1 - g^2 g_{ij}), \text{ for all } i \neq j.
\end{align*}
\]

where $ad_c(x_i)(x_j) = x_i x_j - q^{\alpha_{ij}} x_j x_i$. In the following we define an algebra $U$ which is similar to the usual quantized enveloping algebra $U_q(\mathfrak{sl}(3))$.

Let $U$ be an algebra generated by $E_i$, $F_i$ and $K_i^{\pm 1} (1 \leq i \leq 2)$ satisfying the relations:
The comultiplication \( \Delta \), antipode \( S \) and counit \( \varepsilon \) of \( U \) are defined by \( i = 1, 2 \)
\[
\Delta(E_i) = K_i \otimes E_i + E_i \otimes 1, \quad \Delta(F_i) = 1 \otimes F_i + F_i \otimes K_i^{-1}, \\
\Delta(K_i) = K_i \otimes K_i, \quad \Delta(K_i^{-1}) = K_i^{-1} \otimes K_i^{-1}, \\
S(E_i) = -K_i^{-1} E_i, \quad S(F_i) = -F_i K_i, \quad S(K_i) = K_i^{-1}, \\
\varepsilon(E_i) = 0 = \varepsilon(F_i), \quad \varepsilon(K_i) = 1.
\]

**Proposition 2.1.** There is a Hopf algebra isomorphism \( \phi : U \rightarrow U' \) such that \( E_i \mapsto x_i, F_i \mapsto (q^{-1} - q)^{-1} x_i \, q^{-i} x_i, K_i \mapsto g_i \) for all \( 1 \leq i \leq 2 \).

**Proof.** It is straightforward. \( \square \)

Hence we may discuss the properties and the representation of \( U \) instead of \( U' \).

Similarly to the discussion of the quantized enveloping algebra, let \( U^+, U^-, U^0 \) be the subalgebras of \( U \) generated by the \( E_i \), the \( F_i \), and the \( K_i, K_i^{-1} (1 \leq i \leq 2) \) respectively. It follows from \( 10 \) - \( 15 \) that \( U = U^p U^q U^r \), where \( (p, q, r) \) is a permutation of \( (+, -, 0) \). Moreover, the multiplication gives a \( k \)-vector space isomorphism
\[
U^p \otimes U^q \otimes U^r \cong U, \quad \text{where } (p, q, r) \text{ is as above.}
\]

Let \( U^{\geq 0} = U^- U^0 \) and \( U^{\leq 0} = U^0 U^+ \) be the Borel subalgebras of \( U \).

As in Section 1, we can define the corresponding weight lattice and root lattice for the Cartan matrix of type \( A_2 \). Let \( P = \sum_{i=1}^{2} \mathbb{Z} \sigma_i \) be the weight lattice. Define simple roots as follows:
\[
\alpha_j = \sum_{i=1}^{2} a_{ij} \sigma_i, \quad j = 1, 2.
\]

Let \( \Delta = \{ \alpha_1, \alpha_2 \} \) be the set of simple roots and \( Q = \mathbb{Z} \Delta \) be the root lattice. Then there is a \( \mathbb{Z} \)-bilinear map \( (, ) \) on \( P \times Q \) given by \( (\sigma_i, \alpha_j) = a_{ij} \). Clearly, \( (, ) \) is non-degenerate. Let \( Q_+ = \sum_i \mathbb{N}_0 \alpha_i \). Then there is a partial ordering on \( P \) defined by \( \lambda \geq \mu \) if \( \lambda - \mu \in Q_+ \).
Put $E_{1,2} = E_1E_2 - q^{-1}E_2E_1$ and $F_{1,2} = F_2F_1 - qF_1F_2$. Then we have the following relations from (10)–(15):

\begin{align}
(17) & \quad K_iE_{1,2}K_i^{-1} = qE_{1,2}, \quad K_iF_{1,2}K_i^{-1} = q^{-1}F_{1,2}, \\
(18) & \quad E_{1,2}E_1 = q^{-1}E_1E_{1,2}, \quad E_{1,2}E_2 = qE_2E_{1,2}, \\
(19) & \quad F_{1,2}F_1 = q^{-1}F_1F_{1,2}, \quad F_{1,2}F_2 = qF_2F_{1,2}, \\
(20) & \quad E_{1,2}F_{1,2} - F_{1,2}E_{1,2} = 0, \\
(21) & \quad E_{1,2}F_1 - F_1E_{1,2} = -E_2K_i^{-1}, \quad E_{1,2}F_2 - F_2E_{1,2} = 0, \\
(22) & \quad F_{1,2}E_1 - E_1F_{1,2} = K_1F_2, \quad F_{1,2}E_2 - E_2F_{1,2} = 0. 
\end{align}

For $s = (s_1, s_2, s_3) \in \mathbb{N}_0^3$, $\lambda = \sum_{i=1}^{2} r_i \alpha_i \in Q$, let $E^s = E_1^{s_1} E_2^{s_2} E_3^{s_3}$, $F^s = F_1^{s_1} F_2^{s_2} F_3^{s_3}$ and $K_i = K_i^{K_i} K_i^{-1}$. Then the PBW basis of $U^+(\text{resp.} U^-)$ is given by $\{E^s : s \in \mathbb{N}_0^3, \lambda \in \mathbb{N}_0^3\}$.

Then $U$ has a basis $\{F_1^{s_1} F_2^{s_2} F_3^{s_3} K_1^{K_1^{K_1}} : s, t_i \in \mathbb{N}_0, r_j \in \mathbb{Z}, 1 \leq i \leq 3, 1 \leq j \leq 2\}$.

### 2.2. Representations of $U(D, \lambda_2)$

Let $\hat{\Gamma}$ denote the character group of $\Gamma$. Then $\hat{\Gamma} = (k^*)^2$ as groups. We always identify the characters with the elements in $(k^*)^2$ below. For a weight $\lambda = m_1 \alpha_1 + m_2 \alpha_2 \in U$, define a character $i(\lambda) \in \hat{\Gamma}$ by $i(\lambda)(K_i) = q^{(\lambda, \alpha_i)}$. Then $i : P \rightarrow \hat{\Gamma}$ is a group homomorphism and the image of $i$ is a subgroup of $\hat{\Gamma}$. Moreover, $i$ is a monomorphism if and only if $q$ is not a root of unity. Define $P_1 = \{\lambda = m_1 \alpha_1 + m_2 \alpha_2 \in P | 0 \leq m_1, m_2 \leq l - 1\}$ when $q$ is an $l$-th primitive root of unity and $P_1 = P$ otherwise. Then $i(P_1) = i(P)$ and $i|_{P_1}$ is an injective map. In the sequel, we usually regard $\lambda \in P_1 \subseteq P$ for some character $\lambda \in \hat{\Gamma}$ if $\lambda \in i(P)$.

In the rest of this section, we assume that $q$ is not a root of unity.

For any character $\lambda \in \hat{\Gamma}$, let $M(\lambda) := U \otimes_{U^{\otimes U}} V$, where $V = kv$ is a $U^{\otimes U}$-module with the action given by $E_1 \cdot v = 0$ and $K_i \cdot v = \lambda(K_i) v$, $i = 1, 2$. Then $M(\lambda) = \text{span}[F_1^{s_1} F_2^{s_2} F_3^{s_3} \cdot v | t_i \in \mathbb{N}_0]$ as vector spaces, where we identify $1 \otimes v$ with $v$. That is, $M(\lambda)$ is a free module of rank 1 over $U^{-}$.

Note that $E_2 \cdot F_1^{s_1} F_2^{s_2} F_3^{s_3} \cdot v = 0$. Let

$$M(\lambda, n) := \text{span}[F_1^{s_1} F_2^{s_2} F_3^{s_3} \cdot v | t_i \geq 0, t_2 + t_3 + n \geq 0 \text{ for all } n \geq 0].$$

**Proposition 2.2.** For any $n \geq 0$, $M(\lambda, n)$ is a submodule of $M(\lambda)$.

**Proof.** It is a straightforward verification. \qed

Then there is a filtration

$$M(\lambda) = M(\lambda, 0) \supseteq M(\lambda, 1) \supseteq M(\lambda, 2) \supseteq \cdots.$$  

Let $M(\lambda, n) = M(\lambda, n)/M(\lambda, n + 1)$. Let $\pi_n : M(\lambda) \rightarrow M(\lambda)/M(\lambda, n + 1)$ be the natural projection. Then $M(\lambda, n) = \text{span}[F_1^{s_1} F_2^{s_2} F_3^{s_3} \cdot \pi_n(v) | t_i \geq 0, t_2 + t_3 + n = n]$.

**Proposition 2.3.** $F_2 \cdot M(\lambda, n) = 0$ for all $n \geq 0$.

**Proof.** It is obvious. \qed
Remark 2.4. Let \( \rho_n : U \to \text{End}(M(\lambda, n)) \) be the corresponding representation of \( U \). Then \( E_2, F_2 \in \text{Ker} \rho_n \). There is a natural induced representation \( \tilde{\rho}_n \) of the quotient algebra \( U/(E_2, F_2) \) on \( M(\lambda, n) \).

Let \( M \) be a \( U \)-module and \( \lambda \in \hat{\Gamma} \). \( 0 \neq v \in M \) is called weight vector with weight \( \lambda \) if \( g \cdot v = \lambda(g)v \) for all \( g \in \Gamma \). Write \( M_\lambda \) for the set of all the weight vectors in \( M \) with weight \( \lambda \). Let \( \Pi(M) \) be the set of all weights \( \lambda \) with \( M_\lambda \neq 0 \). Call \( M \) a weight module if \( M = \bigoplus M_\lambda \). A weight vector \( v \in M \) is called highest weight vector if \( E_1v = E_2v = 0 \). A \( U \)-module is a highest weight module of weight \( \lambda \) if it is generated by a highest weight vector with weight \( \lambda \). Let \( \mathcal{W} \) denote the category of all weight modules. Clearly, every highest weight module lies in \( \mathcal{W} \).

For \( n \geq 0 \), let 
\[
v_n := a_0 F_{1,2}^n \cdot v + a_1 F_1 F_{2,2}^{n-1} \cdot v + \cdots + a_n F_1^n \cdot v \in M(\lambda),
\]
where \( a_i = q^{-(n-i)}(K_1)^{(n-i)}[n-i]_q \begin{pmatrix} n \\ n-i \end{pmatrix}_q \begin{pmatrix} \lambda(K_1); j \\ n-i \end{pmatrix}_q \) for \( 0 \leq i \leq n \). Then \( v_0 = v \).

**Proposition 2.5.** For all \( n \geq 0 \), \( v_n \) defined as above are highest weight vectors.

**Proof.** This is similar to the proof of [13] Theorem 1. \( \square \)

Then for any two fixed integers \( t_2, t_3 \in \mathbb{N}_0 \) with \( t_2 + t_3 = n \), let \( M(\lambda; n; t_2, t_3) = \text{span}\{F_1^{t_1} F_2^{t_3} \cdot \pi_n(v_i) | t_1 \geq 0 \} \subseteq M(\lambda, n) \). It is easy to see that \( M(\lambda; n; t_2, t_3) \) is a submodule of \( M(\lambda, n) \) and
\[
M(\lambda, n) = \bigoplus_{t_2+t_3=n} M(\lambda; n; t_2, t_3).
\]

Hence \( \{F_1^{t_1} F_2^{t_3} v_i | t_1, t_2, t_3 \geq 0 \} \) is a \( k \)-basis of \( M(\lambda) \).

Let \( V \) be a \( k \)-vector space with a basis \( \{u_0, u_1, \cdots \} \) and \( \lambda = (\lambda_1, \lambda_2) \in (k^\times)^2 \). Then one can check that \( V \) admits a \( U \)-module structure with the \( U \)-action given by
\[
K_1 u_j = \lambda_1 q^{-j} u_j, \quad K_2 u_j = \lambda_2 q^j u_j, \quad F_1 u_j = u_{j+1}, \quad E_1 u_j = [j]_q [\lambda_1; j]_q (1 - j)_{q} u_{j-1}, \quad E_2 u_j = F_2 u_j = 0,
\]
where \( j = 0, 1, 2, \cdots \) and \( u_{-1} = 0 \). Denote the module by \( V(\lambda) \).

**Proposition 2.6.** For any \( \lambda, \mu \in (k^\times)^2 \), \( V(\lambda) \cong V(\mu) \) if and only if \( \lambda = \mu \).

**Proof.** Let \( \{u_i\} \) (resp. \( \{w_i\} \)) be the \( k \)-basis of \( V(\lambda) \) (resp. \( V(\mu) \)) on which \( U \) acts as above. Let \( f : V(\lambda) \to V(\mu) \) be a non-zero \( U \)-module homomorphism. Then \( f(u_0) = \sum_{i=0}^r a_i w_i \neq 0 \) for some \( a_i \in k \), \( i = 1, 2, \cdots, r \). Hence \( f(u_j) = \sum_{i=0}^r a_i w_{i+j} \neq 0 \). Note that \( f \) is an isomorphism if and only if \( f(u_j) \) is a basis of \( V(\mu) \), if and only if there are some scalars \( b_j \in k \), \( j = 0, 1, \cdots, r \), such that \( w_0 = \sum_{j=0}^r b_j f(u_j) = \sum_{i,j} a_i b_j w_{i+j} \), if and only if \( a_0 b_0 = 1 \) and \( a_i = b_j = 0 \) for \( i > 0, j > 0 \), i.e., \( f(u_0) = a_0 w_0(a_0 \neq 0) \). This finishes the proof by virtue of the actions of \( K_1, K_2 \). \( \square \)
Proposition 2.7. Let \( \lambda \in \hat{\Gamma} \) and \( M(\lambda, n; t_2, t_3) \) be defined as above. Then

\[
M(\lambda, n; t_2, t_3) \cong V(\lambda'),
\]

where \( \lambda' = (q^{t_2}; \lambda(K_1), q^{t_3-2t_1}; \lambda(K_2)) \in (k^*)^2. \)

Proof. Define a \( k \)-map \( f : V(\lambda') \to M(\lambda, n; t_2, t_3) \) by \( f(u) = F^i_1 F^i_2 \cdot \pi_n(v_0) \) for all \( i \geq 0 \). It is easy to check that \( f \) is a \( U \)-module isomorphism. \( \square \)

When \( n = 0 \), then \( t_2 = t_3 = 0 \). Hence \( M(\lambda, 0) = M(\lambda, 0; 0, 0) = V(\lambda) \), where we identify \( \lambda \in \hat{\Gamma} \) with \( \hat{\lambda} = (\lambda(K_1), \lambda(K_2)) \in (k^*)^2. \)

Note that these \( V(\lambda) \) are similar to the Verma modules over the quantized enveloping algebra \( U_q(\mathfrak{sl}_2) \).

Proposition 2.8. For \( \lambda = (\lambda_1, \lambda_2) \in (k^*)^2 \), \( V(\lambda) \) is an indecomposable \( U \)-module. Furthermore, \( V(\lambda) \) is reducible if and only if there exists \( m_1 \in \mathbb{N}_0 \) such that \( \lambda_1 = \pm q^{m_1}. \)

Proof. Let \( V(\lambda) = V' \oplus V'' \) as \( U \)-modules. Then \( u_0 = u' + u'' \), where \( u' \in V' \) and \( u'' \in V'' \). We have that \( K_i u' + K_i u'' = K_i u_0 = \lambda_i u' + \lambda_i u'' \), \( i = 1, 2. \) Since \( V', V'' \) are both \( U \)-modules and \( V' \cap V'' = 0 \), \( u' \) and \( u'' \) are weight vectors with weight \( \lambda \). Note that \( q \) is not a root of unity. Then \( V(\lambda)_i = ku_0 \). This implies that \( u' = 0 \) or \( u'' = 0 \), so \( V(\lambda) = V'' \) or \( V(\lambda) = V' \).

If \( \lambda_1 = \pm q^{m_1} \) for some \( m_1 \in \mathbb{N}_0 \), then \( J'(\lambda) = \text{span}(u_{m_1+1}, u_{m_1+2}, \cdots) \) is a submodule of \( V(\lambda) \). If not, we claim that \( V(\lambda) \) is irreducible. For any \( 0 \neq w = \sum_{i=0}^n a_i u_i \in V(\lambda) \) with \( a_n \neq 0 \), we have that \( E^i_1 w = a_n[n]_q^i [\lambda_1; 1-n]_q [\lambda_1; 2-n]_q \cdots [\lambda_1; 0]_q u_0. \) Since \( \lambda_1 \neq \pm q^{m_1} \) for any \( m_1 \in \mathbb{N}_0 \) and \( q \) is not a root of unity, \( E^i_1 w = au_0 \) for a non-zero scalar \( a. \) Then \( Uw = V(\lambda) \) and the claim follows. \( \square \)

Obviously, we have

Corollary 2.9. Let \( \lambda \in \hat{\Gamma} \) and \( M(\lambda, n; t_2, t_3) \) be defined as above. Then \( M(\lambda, n; t_2, t_3) \) is indecomposable. Furthermore, \( M(\lambda, n; t_2, t_3) \) is reducible if and only if \( \lambda(K_1) = \pm q^{m_1} \) for some \( m_1 \in \mathbb{Z} \) with \( m_1 \geq t_2 - t_3. \)

Let \( \lambda = (\lambda_1, \lambda_2) \in (k^*)^2. \) If there is an \( m_1 \in \mathbb{N}_0 \) such that \( \lambda_1 = \pm q^{m_1} \), then \( J'(\lambda) = \text{span}(u_{m_1+1}, u_{m_1+2}, \cdots) \) is the unique maximal submodule of \( V(\lambda) \). The induced quotient is an \((m_1+1)\)-dimensional simple module, denoted by \( L(\lambda) \). For convenience, let \( u_i \) denote the image of \( u_i \) in \( L(\lambda) \) under the natural projection, \( 0 \leq i \leq m_1 \). Call the set \( \{u_i : 0 \leq i \leq m_1\} \) the standard basis of \( L(\lambda) \).

Proposition 2.10. The finite dimensional simple modules \( L(\lambda) \) are non-isomorphic each other.

Proof. For two simple modules \( L(\lambda) \) and \( L(\mu) \), let \( \{u_i : 0 \leq i \leq m_1\} \) (resp. \( \{w_i : 0 \leq i \leq m'_1\} \)) be the standard basis of \( L(\lambda) \) (resp. \( L(\mu) \)). Assume that there is a \( U \)-module isomorphism \( f : L(\lambda) \to L(\mu) \). Then \( \dim L(\lambda) = \dim L(\mu) = m_1 \). Note that \( L(\mu)_i \) is 1-dimensional and is spanned by some \( w_i \) for any \( \tau \in \Pi(\mu) \). Then \( f(u_0) = aw_0 \) for some \( a \in k^*. \) From \( 0 \neq f(u_{m_1}) = F^{m_1}_1 f(u_0) = a F^{m_1}_1 w_i \), we have that \( i = 0. \) Hence \( \lambda = \mu \) by virtue of the actions of \( K_1, K_2 \) on \( u_0 \) and \( w_0. \) \( \square \)
In fact, one can show that any finite dimensional simple module over $U$ must be isomorphic to some $L(\lambda)$. Let $J(\lambda) = M[\lambda, 1] \oplus \text{span}(F_i \cdot v | t_i \geq m_1 + 1)$ if $\lambda_1 = \pm q^{m_1}$ for some $m_1 \in \mathbb{N}_0$, and $J(\lambda) = M[\lambda, 1]$ otherwise.

**Proposition 2.11.** For any $\lambda \in \hat{U} = (k^\times)^2$, $J(\lambda)$ is the unique maximal submodule of $M(\lambda)$.

**Proof.** It suffices to show that $U \cdot w = M(\lambda)$ for any vector $0 \neq w \in M(\lambda) \setminus J(\lambda)$. Since $M(\lambda)$ is a weight module, one can assume that $w$ is a weight vector. Write $w = \sum a_{i_1,j_1} F_{i_1}^{j_1} \cdot v \in M(\lambda)$ for $a_{i_1,j_1} \in \mathbb{k}$. Since $q$ is not a root of unity, $F_{i_1}^{j_1} F_{i_2}^{j_2} \cdot v$ and $F_{i_1}^{j_1} F_{i_2}^{j_2} \cdot v$ have the same weights if and only if $t_1 + t_2 = s_1 + s_2$ and $t_2 + t_3 = s_2 + s_3$. Then $w = a F_i^j \cdot v$ for some $a \in \mathbb{k}$ and $t_i \geq 0$. Furthermore, $t_i \leq m_1$ if $\lambda_1 = \pm q^{m_1}$ for some $m_1 \geq 0$. Then the result follows from that $E_i^j F_i^j \cdot v = [t_1]_q [\lambda_1; 1 - t_1]_q [\lambda_1; 2 - t_1]_q \cdots [\lambda_1; 0]_q v$ for any $t_i \geq 1$.

**Proposition 2.12.** $M$ is a highest weight module if and only if $M$ is a quotient of some Verma module $M(\lambda)$.

**Proof.** The part “if” is obvious. We need to check the part “only if”. Let $M$ be a highest weight module generated by a weight vector $v$ of weight $\lambda$. Note that $U = U^- U^0 U^+ = U^- U^0 + U E_1 + U E_2$. Then we have that $M = U v = (U^- U^0 + U E_1 + U E_2) v = U^- v$ since $v$ is a highest weight vector. The claim follows from that $M(\lambda)$ is a free $U^-$ module of rank 1.

Let $M$ be a highest weight module. Then there exists some $U$-module $M(\lambda)$ and some submodule $N \subseteq M(\lambda)$ such that $M = M(\lambda)/N$. Regard $M = M(\lambda)/N$. Recall that $M(\lambda)$ has a filtration (2.3). Let $M[i] := (M[\lambda, i] + N)/N$ for $i \geq 0$. There is an induced filtration of $M$:

$$M = M[0] \supseteq M[1] \supseteq \cdots.$$

Then $M[0]/M[1]$ is isomorphic to some quotient of $V(\lambda) \cong M(\lambda, 0) = M[\lambda, 0]/M[\lambda, 1]$. Hence we have the following result.

**Proposition 2.13.** Let $M$ be a highest weight module over $U$ with highest weight $\lambda$. If $M$ is simple, then

1. $M \cong L(\lambda)$ if $M$ is finite dimensional;
2. $M \cong V(\lambda)$ if $M$ is infinite dimensional.

**Proof.** From the discussion above, one can get a filtration of submodules $M = M[0] \supseteq M[1] \supseteq \cdots$. Since $M$ is simple, then $M[1] = 0$ or $M[1] = M$. If $M[1] = M$, then $M[\lambda, 1] + N = M(\lambda)$, where $N$ is given as above. By Proposition (2.11) $M[\lambda, 1] \subseteq J(\lambda)$, which is a small submodule. This implies that $N = M(\lambda)$ by the Nakayama Lemma, a contradiction. Thus $M[1] = 0$ and $M = M[0]/M[1]$ is isomorphic to some quotient of $V(\lambda)$.

If $M$ is finite dimensional, then $\lambda_1 = \pm q^{m_1}$ for some $m_1 \in \mathbb{N}_0$ by Proposition (2.8). Then $M \cong L(\lambda)$ since $J(\lambda)$ is the unique maximal submodule of $V(\lambda)$ and $L(\lambda) = V(\lambda)/J(\lambda)$. If $M$ is infinite dimensional, then $\lambda_1 \neq \pm q^{m_1}$ for any $m_1 \in \mathbb{N}_0$. By Proposition (2.8), $V(\lambda)$ is irreducible, so $M \cong V(\lambda)$. □
Proposition 2.14. Any non-zero finite dimensional $U$-module $M$ contains a highest weight vector. Moreover, the endomorphisms induced by $E_1, E_2, F_1, F_2$ are nilpotent.

Proof. Let $M$ be a finite dimensional $U$-module. Since $U^0$ is commutative and $k$ is algebraically closed, there exists a weight vector $v \in M$. Let $N := Uv$. Then $N$ is a weight module, write $N = \oplus_{\lambda \in k} N_{\lambda}$. Let $e_1 = (q^2, q^{-1}), e_2 = (q^{-1}, q^2) \in k^2$. For a fixed $\lambda \in \Pi(N)$, then $E_1 \cdot N_{\lambda} \subseteq N_{e_1 \cdot \lambda}, E_2 \cdot N_{\lambda} \subseteq N_{e_2 \cdot \lambda}$. Let $W_0 = \{ \lambda \} \subseteq \Pi(N)$ and $\lambda^{(0)} = \lambda$. Since $q$ is not a root of unity and $\Pi(\lambda)$ is a finite set, there exists an $n_1 \in \mathbb{N}_0$ such that $E_1^{n_1} \cdot N_{\lambda} \neq 0$ and $E_1^{n_1+1} \cdot N_{\lambda} = 0$. Define $W_1 = W_0 \cup \{ e_1 \lambda, e_1^2 \lambda, \ldots, e_1^n \lambda \} \subseteq \Pi(N)$ and $\lambda^{(1)} = e_1^n \lambda$. Since $E_1^{n_1} \cdot N_{\lambda} \neq 0$, there exists an $n_2 \in \mathbb{N}_0$ such that $E_2^{n_2} \cdot N_{\lambda^{(0)}} \neq 0$ and $E_2^{n_2+1} \cdot N_{\lambda^{(0)}} = 0$. Define $W_2 = W_1 \cup \{ e_2 \lambda^{(1)}, e_2^2 \lambda^{(1)}, \ldots, e_2^n \lambda^{(1)} \} \subseteq \Pi(N)$ and $\lambda^{(2)} = e_2^n \lambda^{(1)}$. Iterating this process, one can define the sets $W_3, W_4, \ldots$, and the weights $\lambda^{(3)}, \lambda^{(4)}, \ldots$. Thus one gets an ascending chain $W_0 \subseteq W_1 \subseteq W_2 \subseteq \cdots$ of subsets of $\Pi(N)$. Since $q$ is not a root of unity, $W_j = W_{j+1}$ if and only if $n_j = 0$, where $j \geq 0$. Moreover, each $N_{\lambda^{(j)}} \neq 0$.

Since $\Pi(N)$ is a finite set, there exists an $s \in \mathbb{N}_0$ such that $W_s = W_{s+1} = W_{s+2}$. Without loss of generality, assume that $s$ is the minimal integer with this property. Since $q$ is not a root of unity, $W_s = W_{s+1} = W_{s+2}$ implies that $n_{s+1} = n_{s+2} = 0$, that is $E_1 \cdot N_{\lambda^{(s)}} = 0$ and $E_2 \cdot N_{\lambda^{(s)}} = 0$. This shows that there exists a highest weight vector in $N \subseteq M$.

If $M$ is a simple $U$-module, then $M$ is a highest weight module. The last claim is obvious. If not, one can take a composition sequence of $M$. Then the last claim follows from that the endomorphisms induced by $E_1, E_2, F_1, F_2$ are nilpotent on each simple factor.

Remark 2.15. Here the assumption that $k$ is an algebraically closed field is necessary.

Corollary 2.16. Every finite dimensional simple $U$-module is a highest weight module.

By Proposition 2.14 and Corollary 2.16, it means that any finite dimensional simple module over $U$ must be isomorphic to some $L(\lambda)$. We now prove a quantum Clebsch-Gordan formula for the finite dimensional simple $U$-modules.

Theorem 2.17. Let $\lambda = (\lambda_1, \lambda_2), \mu = (\mu_1, \mu_2) \in (k^\times)^2$ with $\lambda_1 = e_1 q^m, \mu_1 = e_2 q^n$ for some $m, n \geq 0$, where $e_1 = \pm 1$ and $e_2 = \pm 1$. Then there exists an isomorphism of $U$-modules

$$L(\lambda) \otimes L(\mu) \cong \bigoplus_{i=0}^{\min[m,n]} L(\eta^{(i)}),$$

where $\eta^{(i)} = (e_1 e_2 q^{m+n-2i}, q^i \lambda_2 \mu_2)$.

Proof. Similar to the proof of [13] VII.7.1.

Denote by $\mathcal{F}$ the category of all finite dimensional modules over $U$. Note that the category $\mathcal{F}$ is not semisimple, see the following example.

Example 2.18. For $e_1, e_2 \in \{1, -1\}$ and $a \in k^\times$, let $V(e_1, e_2, a)$ be a vector space with a basis $\{v_1, v_2\}$ and define an action of $U$ as follows:

- $K_1 v_1 = e_1 v_1, K_2 v_1 = av_1, K_2 v_2 = aq^{-2}v_2$,
- $E_1 v_1 = E_2 v_1 = 0$,
- $F_1 v_1 = F_2 v_2 = 0, F_2 v_1 = v_2$.
Then \(V(e_1, e_2, a)\) is a 2-dimensional indecomposable module, which is not semisimple.

The module structure of \(M(\lambda)\) is complicated and is not well understood. We shall make an attempt in this direction.

Let \(u \in M(\lambda)\) with \(E_1 u = 0\). Then \(u\) is a linear combination of the following vectors:

- \(F_2^j v_2\):
- \(F_1^{m_1 t_1 + ... + t_2} F_2^j v_2\), if \(\lambda(K_1) = \pm q^{m_1}\) for some \(m_1 \in \mathbb{Z}\) and \(m_1 + t_1 - t_2 \geq 0\).

Suppose now that \(\lambda(K_1) = \pm q^{m_1}\) for some \(m_1 \in \mathbb{Z}\). Define \(N(\lambda)\) be the subspace of \(M(\lambda)\) spanned by the vectors \(F_1^{m_1} F_2^j v_2\), where \(t_1 \geq m_1 + t_1 - t_2 + 1\) if \(t_2 - t_3 \leq m_1\) and \(t_1 \geq 0\) otherwise. Then \(N(\lambda) = \text{span}\{F_1^{m_1} F_2^j v_2 | t_1 > \max\{t_3 - t_2 + m_1, -1\}\} \}

Let \(\Pi(M(\lambda))\) be the weight set of \(M(\lambda)\). If \(\mu \in \Pi(M(\lambda))\), then \(\mu = (q^{-2l} j \lambda_1, q^{-2l} j \lambda_2)\) for some \(i, j \in \mathbb{N}\). Note that \(q^{-2l} = q^{-2l} = 1\) if \(s, t \geq 0\) if and only if \(s = t = 0\). Hence \(i, j\) are determined uniquely by \(\mu\). We also have that \(\{F_1^{l} F_2^{j} v_1, F_1^{l} F_2^{j} v_1, \cdots, F_1^{l} F_2^{j} v_1\}\) is a basis of \(M(\lambda)_p\) from the decomposition (24), where \(s = \min\{i, j\}\).

**Remark 2.19.** \(N(\lambda)\) is not a submodule of \(M(\lambda)\). For example, take \(t_2, t_3 \in \mathbb{N}\) with \(m_1 + t_1 - t_2 + 1 = 0\). Then \(F_2^j v_2 \in N(\lambda)\) but \(F_2^j F_2^j v_2 \notin N(\lambda)\).

### 3. The case \(q\) is a root of unit

In this section, assume that \(q\) is a primitive \(l\)-th root of unity. Let \(l' = l\) if \(l\) is odd and \(l' = l/2\) if \(l\) is even. Then \(|l'|q = 0\). Let \(\zeta_l\) denote the set of all the \(l\)-th roots of unity in \(k\).

#### 3.1. The finite dimensional quotient Hopf algebra \(u\)

In the sequel, we shall construct a finite dimensional Hopf algebra from \(U\).

**Lemma 3.1.** The elements \(E_1, E_1', E_2', E_2, F_1, F_1', F_2, F_2', K_1, K_2'\) are in the center of \(U\).

**Proof.** It follows from the relations (10) – (15) and (17) – (22). \(\square\)

Let \(I\) be the ideal of \(U\) generated by the first six elements in Lemma 3.1 and the elements \(K_i' - 1 (i = 1, 2)\). Then \(I\) is a Hopf ideal. Define \(u := U/I\). This is a finite dimensional Hopf algebra. As an algebra, \(u\) is generated by generators \(E_i, F_i, K_i (i = 1, 2)\) subject to the relations (10) – (15) and the following relations:

\[(25)\]

\[E_i' = 0 = F_i', \quad K_i = 1 \text{ for } i = 1, 2.\]

Let \(I' = I \cap U_i\) for \(i \in \{+, -, 0\}\). Then \(I'\) is an ideal of \(U_i\) and \(I = I'^{\rho} \otimes I'^{\rho} \otimes I'\) as vector spaces, where \((p, q, r)\) is a permutation of \((+, -, 0)\). So \(I = I'^{\rho} I'^{\rho} I'\). Let \(u' = (U + I)/I\). By virtue of (16), we have

**Proposition 3.2.**

\[u \cong u'^{\rho} \otimes u'^{\rho} \otimes u'.\]

By abuse of language, we denote the images of the elements \(x \in U\) in \(u\) (resp. \(u'^{\rho}, u'^{\rho}, u')\) under the natural map again by \(x\). Then we have the following corollary about the structure of \(u\).
Corollary 3.3. \( u = u^0 \# u' \) and has a k-basis \( \{ F_i^{s_1}F_j^{s_2}k_i^{t_1}k_j^{t_2}f_i^{r_1}f_j^{r_2} | 0 \leq s, t, r \leq l - 1, 1 \leq i \leq 3, 1 \leq j \leq 2 \} \).

Denote the Borel subalgebra \( u^*u^0 \) (resp. \( u^\circ \)) by \( u^\circ (resp. u^\circ) \). For any \( 0 \leq s, r \leq l - 1, 1 \leq i \leq 3, 1 \leq j \leq 2 \), let

\[
\text{deg}_{u^\circ}(E_i^{s_1}E_j^{s_2}k_i^{t_1}k_j^{t_2}) = (s_1 + s_2)\alpha_1 + (s_2 + s_3)\alpha_2 = \text{deg}_{u^\circ}(K_i^{t_1}K_j^{t_2}F_i^{r_1}F_j^{r_2}).
\]

Then \( u^\circ \) and \( u^\circ \) are both \( Q \)-graded Hopf algebras with the gradings \( \text{deg}_{u^\circ} \) and \( \text{deg}_{u^\circ} \), respectively.

Then there is a unique skew Hopf pairing \( \varphi : u^\circ \otimes u^\circ \rightarrow k \) such that

\[
\varphi(1, 1) = \varphi(1, K_i) = \varphi(K_i, 1) = 1,
\]

\[
\varphi(x, y) = 0 \text{ if } x, y \text{ are monomials with } \text{deg}_{u^\circ}(x) \neq \text{deg}_{u^\circ}(y),
\]

\[
\varphi(E_i, F_j) = \delta_{ij}\delta_{i1} \frac{1}{q^2 - 1},
\]

\[
\varphi(K_i, K_j) = q^{\delta_{ij}0}, \varphi(K_i, K_j^{-1}) = q^{-\delta_{ij}0},
\]

for \( 1 \leq i, j \leq 2 \). Thus one can turn \( D_\varphi(u^\circ, u^\circ) = u^\circ \otimes u^\circ \) into a Hopf \( k \)-algebra as in Section [1].

For convenience, we write \( D_\varphi \) instead of \( D_\varphi(u^\circ, u^\circ) \). Then \( D_\varphi \) can be described as follows.

\( D_\varphi \) is generated, as an algebra, by \( E_i, F_i, K_i, K_i^{-1}, \tilde{K}_i, \tilde{K}_i^{-1} (1 \leq i \leq 2) \) subject to the relations:

\[
K_iK_j = K_jK_i, \quad K_iK_i^{-1} = K_i^{-1}K_i = 1,
\]

\[
K_i\tilde{K}_i = \tilde{K}_iK_i, \quad K_i\tilde{K}_i^{-1} = \tilde{K}_i^{-1}\tilde{K}_i = 1, \quad K_i\tilde{K}_i = \tilde{K}_iK_i,
\]

\[
K_iE_jK_i^{-1} = q^{\delta_{ij}0}E_j, \quad K_iF_jK_i^{-1} = q^{-\delta_{ij}0}F_j,
\]

\[
\tilde{K}_iE_j\tilde{K}_i^{-1} = q^{\delta_{ij}0}E_j, \quad \tilde{K}_iF_j\tilde{K}_i^{-1} = q^{-\delta_{ij}0}F_j,
\]

\[
E_iF_j - F_jE_i = \delta_{ij}\delta_{i1}(K_i - \tilde{K}_i^{-1})/(q - q^{-1})
\]

\[
E_i^0 = F_i^0, \quad K_i^0 = 1 = \tilde{K}_i^0.
\]

The coalgebra structure is given by

\[
\Delta(E_i) = K_i \otimes E_i + E_i \otimes 1, \quad \Delta(F_i) = 1 \otimes F_i + F_i \otimes \tilde{K}_i^{-1},
\]

\[
\Delta(K_i) = K_i \otimes K_i, \quad \Delta(\tilde{K}_i) = \tilde{K}_i \otimes \tilde{K}_i,
\]

\[
S(E_i) = -K_i^{-1}E_i, \quad S(F_i) = -F_i\tilde{K}_i, \quad S(K_i) = K_i^{-1}, S(\tilde{K}_i) = \tilde{K}_i^{-1},
\]

\[
\varepsilon(E_i) = 0 = \varepsilon(F_i), \quad \varepsilon(K_i) = 1 = \varepsilon(\tilde{K}_i).
\]

This is similar to the Drinfeld quantum double of \( u^\circ \). In fact, \( D_\varphi \) is quasi-isomorphic to the Drinfeld quantum double \( D(u_q^{\circ}(sl(3)), u_q^{\circ}(sl(3))) \), i.e., the categories of their comodules are monoidally equivalent ([9]).

Recall that Andruskiewitsch and Schneider gave a classification of finite dimensional pointed Hopf algebras, see Theorem [13]. Since \( D_\varphi \) is a finite dimensional pointed Hopf algebra, one can obtain it by the general construction method (Similarly to the process to construct \( u \) before: choose a datum and a linking datum; define a big Hopf algebra; then modulo some suitable Hopf ideal such that the induced quotient is isomorphic to \( D_\varphi \)). Here the two data of \( D_\varphi \) are the same with those of \( u \) except abelian group \( \Gamma \). Thus we have
Lemma 3.4. The map which sends $E_i$ to $E_i$, $F_i$ to $F_i$, $K_i^\pm 1$ to $K_i^\pm 1$ and $\tilde{K}_i^\pm 1$ to $K_i^\pm 1$ can be extended uniquely to a surjective Hopf algebra homomorphism $\pi : D_\varphi \to u$. Moreover, $\ker \pi = \langle K_1 - \tilde{K}_1, K_2 - \tilde{K}_2 \rangle$.

Proof. It is easy to check. \qed

The categories of modules over $D_\varphi$ and $u$ are connected closely.

Similar to the definition of the weight modules over $U$, a $D_\varphi$-(resp. $u$)-module is called a weight module if it can be decomposed as a direct sum of 1-dimensional simple modules over $D_\varphi^0$ (resp. $u$), where $D_\varphi^0$ is the subalgebra of $D_\varphi$ generated by $K_i, \tilde{K}_i, (1 \leq i \leq 2)$.

Lemma 3.5. Every $D_\varphi$-(resp. $u$)-module is a weight module.

Proof. It follows from the fact that the subalgebra generated by $K_i, \tilde{K}_i$ (resp. $K_i$), $i = 1, 2$, is a group algebra of finite abelian group over an algebraically closed field $k$ of char$(k) = 0$. \qed

From the relations defining $D_\varphi$, we know that $K_i \tilde{K}_i^{-1}$ are in the center of $D_\varphi$. Furthermore, we have

Lemma 3.6. Let $M$ be an indecomposable $D_\varphi$-module. Then $K_i \tilde{K}_i^{-1}$ acts on $M$ as some scalar $z_i \in \zeta_i$, $i = 1, 2$.

Proof. By Lemma 3.5, $K_i, \tilde{K}_i^{-1}$ act semisimply on $M$. Since $K_i$ commutes with $\tilde{K}_i^{-1}$, the action of $K_i \tilde{K}_i^{-1}$ is semisimple on $M$, i.e., there is a direct sum decomposition $M = \oplus_{z \in \zeta} M_z$, where $M_z = \{m \in M | K_i \tilde{K}_i^{-1} m = z_m, i = 1, 2\}$. Indeed, $z = (z_1, z_2) \in \zeta_1 \times \zeta_2$ since $(K_i \tilde{K}_i^{-1})^2 = 1$. Then the claim follows from that $K_i \tilde{K}_i^{-1}$ is in the center of $D_\varphi$ and $M$ is indecomposable. \qed

In the rest of this subsection, we assume that $l$ is odd. For any $z_1 \in \zeta_1$, fix an element $z_1^{1/2} \in k$ such that $(z_1^{1/2})^2 = z_1$. In particular, let $1^{1/2} = 1$.

Let $M$ be a $D_\varphi$-module and $z = (z_1, z_2) \in \zeta_1 \times \zeta_2$ such that $K_i \tilde{K}_i^{-1}$ acts on $M$ as the scalar $z_i$. Then there is a $k$-algebra homomorphism $\pi_z : D_\varphi \to u$ such that

$$
\pi_z(E_1) = z_1^{1/2} E_1, \quad \pi_z(F_1) = F_1, \quad \pi_z(K_1) = z_1^{1/2} K_1, \quad \pi_z(\tilde{K}_1) = z_1^{-1/2} K_1,
$$

$$
\pi_z(E_2) = E_2, \quad \pi_z(F_2) = F_2, \quad \pi_z(K_2) = K_2, \quad \pi_z(\tilde{K}_2) = z_2^{-1} K_2.
$$

One can easily check that $\pi_z$ is well defined and the kernel of $\pi_z$, which is the ideal generated by $K_i \tilde{K}_i^{-1} - z_i (i = 1, 2)$, annihilates the module $M$. Thus $M$ becomes a $u$-module through the homomorphism $\pi_z$.

Lemma 3.7. Every indecomposable $D_\varphi$-module is the pull-back of some $u$-module through an algebra homomorphism $\pi_z$.

Let $M$ be a $u$-module and $z = (z_1, z_2) \in \zeta_1 \times \zeta_2$. Denote by $M_z$ the pull-back of $M$ through the algebra homomorphism $\pi_z$. Let $e_z$ be the 1-dimensional representation of $D_\varphi$ defined by

$$
e_z(E_i) = 0 = e_z(F_i), \quad e_z(K_1) = z_1^{1/2}, \quad e_z(\tilde{K}_1) = z_1^{-1/2}, \quad e_z(K_2) = 1, \quad e_z(\tilde{K}_2) = z_2^{-1}.
$$
One can easily check that \( e_z \) is well-defined.

**Lemma 3.8.** Let \( z \in \xi \times \xi \) and \( M \) be a module over \( u \). Then

\[
M_z = e_z \otimes M_1, \text{ where } 1 = (1,1).
\]

**Proof.** It follows from a direct verification. \( \square \)

From Lemma 3.8, we obtain the following theorem which illustrates the relation between the module categories of \( D_u \) and \( u \).

**Theorem 3.9.** The category \( D_u \mathcal{M} \) is equivalent to the direct product of \( |\xi| \times |\xi| \) copies of the category \( u \mathcal{M} \).

**Remark 3.10.** (1) For the case \( l \) is even, one can also define an algebra homomorphism \( \pi_z \) for any given \( z = (z_1, z_2) \in \xi \times \xi \) if \( z_1^{1/2} \in \xi \).

(2) One can define a skew Hopf pairing \( \psi : U^{\geq 0} \otimes U^{\leq 0} \rightarrow k \) and form the corresponding Hopf algebra \( D_\psi(U^{\geq 0}, U^{\leq 0}) \), then one can discuss the relation between the categories of weight modules over \( D_\psi(U^{\geq 0}, U^{\leq 0}) \) and \( U \), which is similar to the discussion in [11] for the quantized enveloping algebra.

3.2. The representation theory over \( u \). Now let us concentrate on the representation theory of \( u \). First we list all the simple modules over \( u \).

For any \( 0 \leq m_1, m_2 < l \), define an \( m_1 + 1 \) dimensional \( u \)-module \( V = \text{span}\{w_0, w_1, \cdots, w_{m_1}\} \) as follows:

\[
\begin{align*}
K_1 w_j &= q^{m_1-2}j w_j, \\
K_2 w_j &= q^{m_1+j} w_j, \\
E_1 w_j &= [m_1 + 1 - j] w_{j-1}, \\
F_1 w_j &= [j + 1] w_{j+1}, \\
E_2 w_j &= F_2 w_j = 0,
\end{align*}
\]

where \( 0 \leq j \leq m_1 \) and \( w_{-1} = 0 = w_{m_1+1} \). Then one can easily check that \( V \) becomes a \( u \)-module, denoted by \( V(m_1, m_2) \), and that \( V(m_1, m_2) \) is simple. Furthermore, for any \( 0 \leq n_1, n_2 < l \), \( V(m_1, m_2) \cong V(n_1, n_2) \) if and only if \( m_1 = n_1, m_2 = n_2 \).

**Proposition 3.11.** Every simple \( u \)-module is isomorphic to some \( V(m_1, m_2) \).

**Proof.** It can be shown by a direct verification. It also follows from Proposition 2.13 and the fact that \( u \) is a quotient of \( U \). \( \square \)

Let \( P(m_1, m_2) \) be the projective cover of \( V(m_1, m_2) \).

For any pair \((m_1, m_2)\) of integers with \( 0 \leq m_1, m_2 \leq l - 1 \), let \( V = kv \) be a \( 1 \)-dimensional \( u^{\geq 0} \)-module with the action given by \( E_1 \cdot v = 0 = E_2 \cdot v \), \( K_1 \cdot v = q^{m_1} v \) and \( K_2 \cdot v = q^{m_2} v \). Define the Verma module \( M(m_1, m_2) = v \otimes_{u^{\geq 0}} V \). Since \( u \) is a quotient of \( U \), each \( u \)-module naturally becomes a \( U \)-module. As a \( U \)-module, \( M(m_1, m_2) \) is isomorphic to \( M(\lambda) / (I \cdot w) \), where \( \lambda \) = \( (q^{m_1}, q^{m_2}) \) and \( w \neq 0 \) is a weight vector in \( M(\lambda) \) with weight \( \lambda \) such that \( M(\lambda) = U \cdot w \). By Proposition 2.11, \( M(m_1, m_2) \) has a unique maximal submodule as \( U \)-module. Consequently, \( M(m_1, m_2) \) has a unique maximal submodule as \( u \)-module.

**Proposition 3.12.** For any pair \((m_1, m_2)\) of integers with \( 0 \leq m_1, m_2 \leq l - 1 \), \( M(m_1, m_2) \) is isomorphic to some quotient of \( P(m_1, m_2) \).
Proof. Let \( f : M_{2}(m_{1}, m_{2}) \rightarrow V(m_{1}, m_{2}) \), \( g : P(m_{1}, m_{2}) \rightarrow V(m_{1}, m_{2}) \) be the canonical projections. By the projectivity of \( P(m_{1}, m_{2}) \), there is a homomorphism \( g' : P(m_{1}, m_{2}) \rightarrow M(m_{1}, m_{2}) \) such that \( fg' = g \). Take a weight vector \( v \in P(m_{1}, m_{2}) \) such that \( g(v) \neq 0 \). Without loss of generality, one may assume that \( g(v) \) is a highest weight vector by a suitable choice of \( v \). Since \( uv \notin \ker g = \text{rad}P(m_{1}, m_{2}) \) and \( \text{head}P(m_{1}, m_{2}) \approx V(m_{1}, m_{2}) \) is simple, \( uv + \text{rad}P(m_{1}, m_{2}) = P(m_{1}, m_{2}) \). Then \( uv = P(m_{1}, m_{2}) \) by the Nakayama Lemma. Similarly, one can show that \( g'(v) \in M(m_{1}, m_{2}) \) is a weight vector and \( ug'(v) = M(m_{1}, m_{2}) \). Thus \( g' \) is surjective.

Let \((m_{1}', m_{2}')\) be another pair of integers with \(0 \leq m_{1}', m_{2}' \leq l - 1\). Then it is easy to check that \( \text{Hom}(P(m_{1}, m_{2}), P(m_{1}', m_{2}')) \neq 0 \) if and only if \( V(m_{1}, m_{2}) \) is a composition factor of \( P(m_{1}', m_{2}') \).

**Theorem 3.13.** Assume that \((l, 3) = 1\). Then \( u \) is an indecomposable algebra.

Proof. By the discussion above, it suffices to show that each \( V(m_{1}, m_{2}) \) is a composition factor of \( P(0, 0) \). If \( V(m_{1}, m_{2}) \) is a composition factor of \( P(0, 0) \), then it is a composition factor of \( P(0, 0) \). However, \( V(m_{1}, m_{2}) \) is a composition factor of \( P(0, 0) \) if and only if there exist \(0 \leq t_{2}, t_{3} \leq l - 1\) such that
\[
\begin{pmatrix}
-1 & 1 \\
-1 & -2
\end{pmatrix}
\begin{pmatrix}
t_{2} \\
t_{3}
\end{pmatrix}
\equiv
\begin{pmatrix}
m_{1} \\
m_{2}
\end{pmatrix}
(\text{mod } l)
\]

have solutions in \(\mathbb{Z} \). In fact, the first equation is equivalent to
\[
\begin{cases}
1 = (2s + t)l + 3(t_{2} + 1) \\
1 = (t - s)l + 3t_{3}
\end{cases}
\]
for some \(s, t \in \mathbb{Z}\). Since \((l, 3) = 1\), there exist \(p, q \in \mathbb{Z}\) such that \(pl + 3q = 1\). Let \(t_{1} = q, t_{2} = q - 1, t = p, s = 0\). Then this is a solution of the equation set above. The second equation can be converted into the equation set
\[
\begin{cases}
1 = (-2s - t)l - 3t_{2} \\
1 = (s - t)l - 3t_{3}
\end{cases}
\]
for some \(s, t \in \mathbb{Z}\). Let \(t_{2} = -q = t_{3}, s = 0, t = -p\). This is a solution of the equation set above. So there exist \(t_{2}', t_{1}', t_{2}', t_{3}' \in \mathbb{Z}\) such that
\[
\begin{pmatrix}
-1 & 1 \\
-1 & -2
\end{pmatrix}
\begin{pmatrix}
t_{2}' \\
t_{3}'
\end{pmatrix}
\equiv
\begin{pmatrix}
m_{1} \\
m_{2}
\end{pmatrix}
(\text{mod } l)
\]

Let \(t_{2}, t_{3} \in \mathbb{Z}\) with \(0 \leq t_{2}, t_{3} \leq l - 1\) such that \(t_{2} \equiv m_{1}t_{2}' + m_{2}t_{3}' \) (mod \(l\)) and \(t_{3} \equiv m_{1}t_{2}' + m_{2}t_{3}' \) (mod \(l\)). Then
\[
\begin{pmatrix}
-1 & 1 \\
-1 & -2
\end{pmatrix}
\begin{pmatrix}
t_{2} \\
t_{3}
\end{pmatrix}
\equiv
\begin{pmatrix}
m_{1} \\
m_{2}
\end{pmatrix}
(\text{mod } l)
\]
\[\square\]
For the projective modules $P(m_1, m_2)$, we hope to give a more detailed description. It is well known that there is a close connection between the direct sum decomposition of the regular module and the idempotent element decomposition of unit. Let us go along this way.

For $1 \leq i, j \leq l - 1$, define $e_{i,j} = \frac{1}{t^{|i-j|}} \sum_{i=0}^{l-1} q^{i+j} K_1^i K_2^j$ in $u$. Then we have $e_{i,j}^2 = e_{i,j}$, $e_{i,j} e_{r,s} = \delta_{i,r} \delta_{j,s}$, and $\sum_{i,j=0}^{l-1} e_{i,j} = 1$. So $u = \oplus_{i,j=0}^{l-1} u e_{i,j}$ is a direct sum decomposition of $u$ as regular module. Each summand $u e_{i,j}$ is a projective module.

**Proposition 3.14.** For $1 \leq i, j \leq l - 1$, $u e_{i,j}$ are non-isomorphic each other.

**Proof.** Assume that $u e_{i,j}$ is isomorphic to $u e_{r,s}$ for some $(i', j') \neq (i, j)$, and let $f : u e_{i,j} \to u e_{r,s}$ be an isomorphism. Then there are $a, b \in u$ such that $f(e_{i,j}) = a e_{r,s}$ and $f^{-1}(e_{r,s}) = b e_{i,j}$, and so $e_{i,j} = a b e_{i,j}$. Note that $K_1 e_{i,j} = q^{-i} e_{i,j}$, $K_2 e_{i,j} = q^{-j} e_{i,j}$. Hence $a, b \notin u^0$, and we may assume that $a = \sum_{s \leq t < u} a_{s,t} E^s F^t$, $b = \sum_{s \leq t < u} b_{s,t} E^s F^t$, where $\alpha_{s,t}, \beta_{s,t} \in k$. It is easy to see that $\{E^s F^t e_{i,j}, s,t \in Z_3, i, j \in Z_3\}$ is a $k$-basis of $u$. Hence we deduce $a b = 1$ from $e_{i,j} = a b e_{i,j}$. Comparing the degree of terms on the both sides of $a b = 1$, we have that $a = \alpha_{0,0}, b = \beta_{0,0}$ in $k$. It contradicts with $a, b \notin u^0$. This proved the claim. □

Recall that $P(m_1, m_2)$ is the projective cover of $V(m_1, m_2)$ and $\text{dim } V(m_1, m_2) = m_1 + 1$. From the representation theory of finite dimensional algebra, we know that

$$u \cong \bigoplus_{0 \leq m_1, m_2 \leq l - 1} (m_1 + 1) P(m_1, m_2).$$

Thus $u e_{i,j}$ may not be indecomposable, i.e., $e_{i,j}$ may not be primitive. It is difficult to give all the primitive idempotent elements in $u$. In the following we will turn this into the question to solve some equation set.

Now we assume that $l$ is odd in the sequel.

Let $u_1$ be the subalgebra of $u$ generated by $E_1, F_1, K_1$. Then $u_1$ is isomorphic to the quotient of quantum group $U_q(sl(2))$ modulo the ideal $\langle E^l, F^l, K^l - 1 \rangle$. Thus $u_1$ just is the so-called “small” quantum group (cf. [20]).

From the results in [20], we know that $u_1 \cong \oplus_{i=0}^{l-1} (i + 1) P_i$ as $u_1$-modules, where each $P_i$ is indecomposable and non-isomorphic. Thus there is a family of primitive orthogonal idempotent elements $f_1, f_2, \cdots, f_N (N = l(l + 1)/2)$ in $u_1$ such that $1 = \sum f_i$.

Define $e_f(K_2) = \frac{1}{l} \sum_{i=0}^{l-1} (q^i K_2)^i$. Then $1 = \sum_{i=0}^{l-1} e_f(K_2)$ is a decomposition of orthogonal idempotent elements. Obviously, $f_i e_f(K_2)$ is an idempotent element, denoted by $f_{ij}$. One can check that $f_{ij}$ are pairwise orthogonal and $1 = \sum f_{ij}$. By virtue of the decomposition (26) and the Krull-Schmidt theorem, $f_{ij}$ is a primitive idempotent element in $u_1$.

Thus we only need to give all primitive idempotent elements in $u_1$ in order to give all the primitive idempotent elements in $u$.

For convenience, write $K, E, F, e_i$ for $K_1, E_1, F_1$ and $e_i(K_1) = \frac{1}{l} \sum_{i=0}^{l-1} (q^i K_1)^i$ respectively. Clearly, $\{e_i E^s F^t | 0 \leq i, s, t \leq l - 1\}$ is a basis of $u_1$.

Assume that $e_i$ is not primitive. Then there is an idempotent element decomposition $e_i = \sum_{r} e_{i,r}$, where $e_{i,r}$ are orthogonal primitive elements. Since $u = \oplus_{i=0}^{l-1} e_i u = \oplus_{i=0}^{l-1} e_i u^* u^*$, we
It is easy to see that: cases

Proof. Let $\sum_{j=0}^{\min[m,s]} [j]_q \begin{bmatrix} m \\ j \\ q \end{bmatrix} \begin{bmatrix} s \\ j \\ q \end{bmatrix} E^{r-s-j} \prod_{r=j-m-s+1}^{2j-m-s} [K^{-1}; r]_q F^{m-j}$

Lemma 3.15.

Then we have the following identity:

Let $e_{ir} = \sum_{p=0}^{l-1} a_p e_i E^p F^p$ for a fix $r$. By using the identity above repeatedly, we have

Theorem 3.16. For some fixed $i \in \mathbb{Z}$, $e_{ir}$ is an idempotent element if and only if

$(a_0, a_1, \cdots, a_{l-1})$ is a solution of the equation set

$$a_p = \sum_{0 \leq m, s, \ell \leq l-1, \atop 0 \leq j \leq \min[m,s], \atop m+s-j \leq p} a_{m,s,\ell} a_m a_s, \quad 0 \leq p \leq l-1,$$

where $a_{m,s,\ell} = [j]_q \begin{bmatrix} m \\ j \\ q \end{bmatrix} \begin{bmatrix} s \\ j \\ q \end{bmatrix} \begin{bmatrix} m+s+i \\ j \\ q \end{bmatrix}$.

Let $[n]$ be the residual class of $n$ modulo $l$ and $\overline{n}$ be the minimal non-negative integer in $[n]$. It is easy to see that: $a_{m,s,\ell} = a_{s,m,\ell}$ and that $\begin{bmatrix} m+s+i \\ j \\ q \end{bmatrix} = 0$ if and only if $m+s+i < j$.

If $(a_0, a_1, \cdots, a_{l-1})$ is a solution of the equation set, then $a_0 = 0$ or $a_0 = 1$. Moreover, $(1 - a_0, -a_1, \cdots, -a_{l-1})$ is also a solution of the equation set.

Remark 3.17. The equation set in Theorem 3.16 is solvable recursively. For the special cases $l = 3$, we will give all the solutions in the below. This result is new for the small quantum group $u_1$.

Example 3.18. When $l = 3$, we can get a decomposition of orthogonal primitive idempotents of unity through solving the equation set in Theorem 3.16

$$1 = (-e_0 EF + e_0 E^2 F^2) + (e_0 + e_0 EF - e_0 E^2 F^2) + (e_1 - e_1 E^2 F^2) + e_1 E^2 F^2 + (e_2 EF - e_2 E^2 F^2) + (e_2 - e_2 EF + e_2 E^2 F^2),$$

where $e_i = \frac{1}{2} \sum_{j=0}^{2}(q^j K)^i \in u_1$ for $i = 0, 1, 2$. 


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