Dihedral $G$-Hilb via representations of the McKay quiver

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Abstract

For a given small binary dihedral group $G = BD_{2n}(a)$ we use the classification of $G$-graphs to describe explicitly $G$-$\text{Hilb}(\mathbb{C}^2)$ by giving an affine open cover of $\mathcal{M}_\theta(Q,R)$, the moduli space of $\theta$-stable quiver representations.

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1 Introduction

Given a finite subgroup $G \subset \text{GL}(2, \mathbb{C})$ the “special” McKay correspondence relates the $G$-equivariant geometry of $\mathbb{C}^2$ and the minimal resolution $Y$ of the quotient $\mathbb{C}^2/G$, establishing a one-to-one correspondence between the irreducible components of the exceptional divisor $E \subset Y$ and the “special” irreducible representations. This minimal resolution $Y$ can be viewed as two equivalent moduli spaces: by a result of Ishii [Ish02] we know that $Y = G$-$\text{Hilb}(\mathbb{C}^2)$ the $G$-invariant Hilbert scheme $G$-$\text{Hilb}(\mathbb{C}^2)$ introduced by Ito and Nakamura, and at the same time $Y = \mathcal{M}_\theta(Q,R)$ the moduli space of $\theta$-stable representations of the McKay quiver.

Attempts to construct explicitly the minimal resolution $Y$ for subgroups $G \subset \text{GL}(2, \mathbb{C})$ have only been done with abelian groups in the context of toric geometry, where the use of $G$-graphs in these cases provides a nice and friendly framework to describe both $G$-$\text{Hilb}(\mathbb{C}^2)$ and $\mathcal{M}_\theta(Q,R)$. In the abelian case the connection between $G$-graphs and representations of the McKay quiver is well known by many authors, but the non-abelian case was missing. This is the gap that this paper is filling: we provide a method to interpret $G$-graphs from representations of the McKay quiver $Q$ and vice versa.

Given a binary dihedral group $G = BD_{2n}(a)$ the list of $G$-graphs $\Gamma_0, \ldots, \Gamma_N$ is known from [NdC09] (see Theorem 2.5). For any $G$-graph $\Gamma_i$ there exists an open set $U_{\Gamma_i} \subset G$-$\text{Hilb}(\mathbb{C}^2)$ consisting of all $G$-clusters $Z \in G$-$\text{Hilb}(\mathbb{C}^2)$ such that $O_Z$ admits $\Gamma_i$ for basis as a vector space. By Corollary 2.7 the list of open sets $U_{\Gamma_0}, \ldots, U_{\Gamma_N}$ cover $G$-$\text{Hilb}(\mathbb{C}^2)$. 
The calculation of the open sets \( U_i \) directly from the \( G \)-graphs \( \Gamma_i \) involve an enormous amount of choices among the possible candidates for generators of the ideals \( I_{\Gamma_i} \). This is mainly because we deal with groups in \( \text{GL}(2, \mathbb{C}) \) where there is a lot more information that is needed. In the same manner, the direct calculation of an open cover for \( \mathcal{M}_\theta(Q, R) \) becomes also intractable because of the enormous amount of possible open sets (see \[1.1\] for a simple cyclic example which illustrates this fact).

In this paper we obtain the explicit description of the open cover of \( \text{G-Hilb}(\mathbb{C}^2) \) by first assigning to every \( G \)-graph the open set \( U \subset \mathcal{M}_\theta(Q, R) \) of \( \theta \)-stable representations of the bound McKay quiver, and then calculating in this moduli space the explicit equations. In other words, the combination of the two moduli spaces is giving us the answer we are looking for. In this way, once we know the \( G \)-graphs the number of open sets to calculate in \( \mathcal{M}_\theta(Q, R) \) is just the number of \( G \)-graphs, solving the problem of deciding which open sets among all we can take to form an open cover of \( \mathcal{M}_\theta(Q, R) \). On the other hand, the McKay quiver makes for us from the start all the choices of generators for \( I_{\Gamma_i} \), obtaining immediately the explicit ideals for the \( G \)-clusters in \( \text{G-Hilb}(\mathbb{C}^2) \).

A parallel description of an explicit open cover for the minimal resolution \( Y \) has been discovered independently by M. Wemyss by using reconstruction algebras (see \[\text{Wem09a}, \text{Wem09b}\]) instead of the skew group ring. While his method uses the smallest quiver needed to calculate the equations of \( Y \), it loses all the information that the McKay quiver is giving us to recover \( \text{G-Hilb}(\mathbb{C}^2) \).

The paper is distributed as follows: In Section 2 we recall the definitions of the moduli spaces \( \text{G-Hilb}(\mathbb{C}^2) \) and \( \mathcal{M}_\theta(Q, R) \), we describe the binary dihedral groups \( \text{BD}_{2n}(a) \) and we recall the main results from \[\text{NdC09}\] about the \( G \)-graphs for these groups. In Section 3 we describe the McKay quiver for a \( \text{BD}_{2n}(a) \) group as the \( \mathbb{Z}/2 \)-orbifold quotient of the McKay quiver for \( \frac{1}{2n}(1, a) \), and we describe the relations following \[\text{BSW08}\]. The Section 3 is devoted to the explicit calculation of \( \text{G-Hilb}(\mathbb{C}^2) \). By giving a method to assign to any \( G \)-graph \( \Gamma \) an open set \( U \subset \mathcal{M}_\theta(Q, R) \) and using the classification of \( G \)-graphs into the types \( A, B, C \) and \( D \), we give in Theorem \[4.3\] the open cover of \( \mathcal{M}_\theta(Q, R) \) for any binary dihedral group \( \text{BD}_{2n}(a) \). Section 4 is dedicated to an example illustrating the method.

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## 2 Preliminaries

### 2.1 Dihedral groups \( \text{BD}_{2n}(a) \) in \( \text{GL}(2, \mathbb{C}) \)

We consider the following representation of binary dihedral subgroups in \( \text{GL}(2, \mathbb{C}) \) in terms of their action on the complex plane \( \mathbb{C}^2_{x,y} \):

\[
\text{BD}_{2n}(a) = \left\langle \alpha = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^a \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} : \varepsilon^{2n} = 1 \text{ primitive}, \ (2n, a) = 1, \ a^2 \equiv 1 \pmod{2n} \right\rangle
\]

In other words, \( \text{BD}_{2n}(a) \) is the group of order \( 4n \) generated by the cyclic group \( \frac{1}{2n}(1, a) \) and the dihedral symmetry \( \beta \) which interchanges the coordinates \( x \) and \( y \). The subgroup \( A := \langle \alpha \rangle \leq G \) is a choice of maximal cyclic subgroup of \( G \), which is of index 2 (note that \( \beta^2 \in A \)). The condition \( a^2 \equiv 1 \pmod{2n} \) is equivalent to the classical dihedral condition of \( \alpha \beta = \beta \alpha a^0 \), and it creates a lot of symmetry in \( A \).

Let us define now two positive integers which play a great role in the rest of the paper: let \( q := \frac{2n}{(a-1,2n)} \), and \( k \) such that \( n = kq \). Another way of interpreting \( k \) is as the smallest integer \( i > 0 \) such that \((xy)^i\) is \( \frac{1}{2n}(1, a) \)-invariant.
The group $\text{BD}_{2n}(a)$ has $4k$ irreducible 1-dimensional representations, denoted by $\rho_j^\pm$, of the form

$$\rho_j^+(\alpha) = \varepsilon^j, \quad \rho_j^-(\beta) = \begin{cases} 1 & n \text{ even} \\ i & n \text{ odd} \end{cases}$$

$$\rho_j^-(\alpha) = -\varepsilon^j, \quad \rho_j^-(\beta) = \begin{cases} -1 & n \text{ even} \\ -i & n \text{ odd} \end{cases}$$

where $\varepsilon$ is a $2n$-th primitive root of unity. There are also $n - k$ irreducible 2-dimensional representations $V_r$ of the form

$$V_r(\alpha) = \begin{pmatrix} \varepsilon^r & 0 \\ 0 & \varepsilon^{ar} \end{pmatrix}, \quad V_r(\beta) = \begin{pmatrix} 0 & 1 \\ (-1)^r & 0 \end{pmatrix}$$

By the definition, the natural representation is $V_1$.

The group $G$ acts on the complex plane $\mathbb{C}^2_{x,y}$, so it also acts on the polynomial ring $\mathbb{C}[x, y]$ breaking it into different eigenspaces. A polynomial $f \in \mathbb{C}[x, y]$ is $\rho$-invariant, or $f$ is semi-invariant with respect to $\rho$, if $f(g \cdot P) = \rho(g)f(P)$ for all $g \in G$, $P \in \mathbb{C}^2_{x,y}$. We denote by $S_\rho := \{ f \in \mathbb{C}[x, y] : f \in \rho \}$ the $\mathbb{C}[x, y]^G$-module of $\rho$-invariants, and we say that a polynomial belongs to a representation $\rho$ if it belongs to the corresponding module $S_\rho$.

In the case of $\text{BD}_{2n}(a)$ groups we have that:

$$f \in \rho_i^k \iff \alpha(f) = \varepsilon^j f \text{ and } \beta(f) = \pm i^nf$$

$$(f, \beta(f)) \in V_k \iff \alpha(f, \beta(f)) = (\varepsilon^k f, \varepsilon^{ak}\beta(f))$$

where we are making the abuse of notation $\alpha(f(x, y)) = f(\alpha(x, \alpha(y)) = f(\varepsilon x, \varepsilon^n y)$ and $\beta(f(x, y)) = f(\beta(x), \beta(y)) = f(y, -x)$ (see [NdC08] §3.4.2 for more details).

An element $g \in G$ is a quasireflection if it fixes a hyperplane, and a group $G$ is called small if it does not contain any quasireflection. A theorem of Chevalley, Shephard and Todd [ST54] states that if a finite subgroup $H \subset \text{GL}(n, \mathbb{C})$ is generated by reflections then $\mathbb{C}^n/H \cong \mathbb{C}^n$, which traditionally reduces the study of these quotients to small groups. Since the groups $\text{BD}_{2n}(a)$ may contain quasireflections, we present in the next proposition a criteria for a binary dihedral group $\text{BD}_{2n}(a)$ to be small.

**Proposition 2.1** ([NdC09]). $\text{BD}_{2n}(a)$ is small $\iff \gcd(a + 1, 2n) \nmid n$

**Remark 2.2.** The small binary dihedral groups in $\text{GL}(2, \mathbb{C})$ were classified by Brieskorn [Bri68] into two different families. With our point of view, given a cyclic group of the form $\frac{1}{2n}(1, a)$ where $n = kq$, then the two families correspond to the cases $k$ odd and $k$ even. The groups $\text{BD}_{2n}(a)$ form the case $k$ odd, and the groups $\text{BD}_{2n}(a, q) := \left\langle \frac{1}{2n}(1, a), \beta = \left\langle \frac{1}{2n}(0, 1) \right\rangle \right\rangle$ form the even case (see also [NdC09]). For simplicity, in this paper we only treat the groups $\text{BD}_{2n}(a)$ but we want to emphasize that the methods used here apply to the groups $\text{BD}_{2n}(a, q)$ as well.

### 2.2 G-Hilbert and G-graphs

**Definition 2.3.** Let $G \subset \text{GL}(n, \mathbb{C})$ be a finite subgroup. A $G$-cluster is a $G$-invariant zero dimensional subscheme $Z \subset \mathbb{C}^n$, defined by an ideal $I_Z \subset \mathbb{C}[x_1, \ldots, x_n]$, such that $\mathcal{O}_Z = \mathbb{C}[x_1, \ldots, x_n]/I_Z \cong \mathbb{C} G$ the regular representation as $\mathbb{C} G$-modules. The $G$-Hilbert scheme $\text{G-Hilb}(\mathbb{C}^n)$ is the moduli space parametrizing $G$-clusters.

Recall that the regular representation $\mathbb{C} G$ is a direct sum of irreducible representations $\rho_i$, where every irreducible $\rho_i$ appears $(\dim \rho_i)$ times in the sum. That is

$$\mathbb{C} G = \bigoplus_{\rho_i \in \text{Irr } G} (\rho_i)^{\dim \rho_i}$$

Therefore if $I_Z$ is an ideal defining a $G$-cluster, or a point in $\text{G-Hilb}(\mathbb{C}^n)$, then the vector space $\mathcal{O}_Z$ has in its basis $(\dim \rho_i)$ elements in each irreducible representation $\rho_i$. To describe distinguished basis with this property for these coordinate rings, it is convenient to use the notion of $G$-graph.
Definition 2.4. Let $G \subseteq \text{GL}(n, \mathbb{C})$ be a finite subgroup. A $G$-graph is a subset $\Gamma \subseteq \mathbb{C}[x_1, \ldots, x_n]$ satisfying the following two conditions:

1. It contains $(\dim \rho)$ number of elements in each irreducible representation $\rho$.

2. If a monomial $x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ is a summand of a polynomial $P \in \Gamma$, then for every $0 \leq \mu_j \leq \lambda_j$ the monomial $x_1^{\mu_1} \cdots x_n^{\mu_n}$ must be a summand of some polynomial $Q_{\mu_1, \ldots, \mu_n} \in \Gamma$.

Note that given any ideal $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$, we can choose a basis for the vector space $\mathbb{C}[x_1, \ldots, x_n]/I$ which is a $G$-graph. This choice will never be unique.

From now on we restrict ourselves to the surface case $n = 2$, i.e. we consider finite subgroups $G \subseteq \text{GL}(2, \mathbb{C})$.

Representation of a $G$-graph

When the group $G$ is abelian, every irreducible representation $\rho$ of $G$ is 1-dimensional, and every monomial in $\mathbb{C}[x,y]$ belongs to some representation. Therefore the $G$-graphs for abelian groups are represented in the lattice of monomials $M$ by the monomials contained in $\Gamma$. For example, the following picture represents the $(1,5)$-graph $\Gamma = \{1, x, y, x^2, x^3, x^4\}$:

![Diagram](image)

As the figure suggests, the representation of $\Gamma$ consists of all monomials in $\mathbb{C}[x,y]$ which do not belong to the ideal $I = (x^5, xy, y^2)$, so we may also say that the $G$-graph $\Gamma$ is defined by the ideal $I$.

When $G$ is non-abelian, irreducible representations may not contain monomials, and the elements in $G$-graphs consist of sums of monomials. In this case, the representation of a $G$-graph is also drawn on the lattice of monomials $M$, but now using the following rule: a monomial $x^iy^j$ is contained in the representation of a $G$-graph $\Gamma$ if it is contained as a summand in some polynomial $P \in \Gamma$. For example, consider the binary dihedral group $D_4 \subset \text{SL}(2, \mathbb{C})$ and the $D_4$-graph

$$\Gamma = \{1, x^4 - y^4, x^2 + y^2, x^2 - y^2, (x,y), (y^3, -x^3)\}$$

Note that it has one element in each 1-dimensional representation and two elements in the 2-dimensional representation $V$ (see Table 1). In this case, $\Gamma$ is represented by the following figure

![Diagram](image)

where the basis elements $x^2 + y^2 \in \rho^+_2$ and $x^2 - y^2 \in \rho^-_2$ are represented by $x^2$ and $y^2$ respectively. We also say that $\Gamma$ is defined by the ideal $I = (xy, x^4 + y^4)$. Counting the number of monomials in the figure we have that $|\Gamma| = 9 > |D_4| = 8$, but notice that $x^4 - y^4$ belongs to the basis of $\mathbb{C}[x,y]/I$, so we cannot exclude $x^4$ and $y^4$. On the other hand we have the relation $x^4 + y^4 = 0$, i.e. $x^4 = -y^4$ and both monomials count as one in the basis. We say that $x^4$ and $y^4$ are “twins.”
2.3 $G$-graphs for BD$_{2n}(a)$ groups

Let $G = BD_{2n}(a)$ be a small binary dihedral group. Then $G\text{-Hilb}(\mathbb{C}^2)$ is the minimal resolution of the quotient $\mathbb{C}^2/G$. We obtain this resolution first by acting with $A$ (generated by $\alpha$) on $\mathbb{C}^2$, and finally by acting with $G/A$ (generated by $\beta$) on $A\text{-Hilb}(\mathbb{C}^2)$. The quotient $A\text{-Hilb}(\mathbb{C}^2)/\beta$ has two singular $\frac{1}{2}(1,2)$ points, and the blow-up of this two points gives us $G\text{-Hilb}(\mathbb{C}^2)$.

Translating this construction into graphs, a $G$-graph corresponds to an orbit in $A\text{-Hilb}(\mathbb{C}^2)$ consisting in two symmetric $A$-graphs $\Gamma_i$ and $\beta(\Gamma_i)$. The resulting $G$-graph $\Gamma$ is an extension of the union $\Gamma_i \cup \beta(\Gamma_i)$, which is done in a unique way according to the representation theory of $G$. Given a free orbit of $\beta$ the $G$-graphs obtained in this way are called type $A$ or type $B$ $G$-graphs, depending on the two possible types of “glue” between $\Gamma_i$ and $\beta(\Gamma_i)$. When we consider one of the two fixed points of $\beta$, we have that $\Gamma_i = \beta(\Gamma_i)$ and there are two types of $G$-graphs arising in this case, type $C$ and type $D$ (see [NdC09] for more details).

Since $A$ is cyclic, any $A$-graph $\Gamma_i$ is defined by the ideal $I_{\Gamma_i} = (x^s, y^u, x^{s-v}y^{u-r})$, where $e_i = (r, s)$ and $e_{i+1} = (u, v)$ are two consecutive lattice points in the boundary of the Newton polygon of the lattice $L = \mathbb{Z}^2 + \frac{1}{2n}(1, a) \cdot \mathbb{Z}$. Therefore, $\beta(\Gamma_i)$ is also given by $r$, $s$, $u$ and $v$, as well as the $G$-graph $\Gamma$ obtained from $\Gamma_i \cup \beta(\Gamma_i)$. We denote the $G$-graph by $\Gamma(r, s; u, v)$ when the reference to the defining points is needed.

The following theorem resumes the classification of $G$-graphs for BD$_{2n}(a)$ groups, describing the defining ideals in each case as well.

**Theorem 2.5 ([NdC09]).** Let $G = BD_{2n}(a)$ and let $\Gamma(r, s; u, v)$ be the $G$-graph corresponding to the two consecutive lattice points $e_i = (r, s)$ and $e_{i+1} = (u, v)$ of the Newton polygon. Then we have the following possibilities:

- If $u < s - v$ then $\Gamma$ is of type $A$ and it is defined by the ideal
  \[ I_A = (x^u y^u, x^{s-v}y^{u-r} + (-1)^{u-r}x^{u-r}y^{s-v}, x^{r+s} + (-1)^r y^{r+s}) \]

- If $u - r = s - v := m$ then $\Gamma$ is of type $B$ and we have two cases:
  a) If $u < 2m$ then $\Gamma$ is of type $B.1$ and it is defined by the ideal
     \[ I_{B.1} = (x^{2m} y^m, x^{s+m} y^{m-r} + (-1)^{m-r} x^{m-r} y^{m+s}, x^m y^m) \]
  b) If $u \geq 2m$ then $\Gamma$ is of type $B.2$ and it is defined by the ideal
     \[ I_{B.2} = (x^u y^u, x^{s+m} y^m, x^m y^m, x^m y^u) \]

- In addition, when $u = v = q:= \frac{2n}{(a-1, 2n)}$ we have four types of $G$-graphs: types $C^+$, $C^-$, $D^+$ and $D^-$. The $G$-graphs of types $D^\pm$ are defined by the ideals:
  \[ I_{D^\pm} = (x^q \pm (i)^q y^q, x^{s-r} y^{s-r}) \]

For $G$-graphs of types $C^\pm$ we have two cases:

| $\alpha$ | $\beta$ | $S_\rho$ |
|----------|----------|----------|
| $\rho_0^+$ | 1 | 1, $x^4 + y^4$, $x^2 y^2$, $xy(x^3 - y^3)$, ... |
| $\rho_0^-$ | 1 | $xy$, $x^4 - y^4$, ... |
| $\rho_2^+$ | $-1$ | $x^2 + y^2$, $xy(x^2 - y^2)$, ... |
| $\rho_2^-$ | $-1$ | $x^2 - y^2$, $xy(x^2 + y^2)$, ... |

Table 1: Irreducible representations of $D_4$ with some of the polynomials belonging to them.
\[ I_{C_A^+} = (x^q \pm (-i)^q y^q)^2, x^s y m^2 \pm (-1)^r i q x m^2 y^*, x^m y m^2 \pm (-1)^m x m^2 y m^1) \]

b) If \( 2q = r + s \) they are defined by the ideals
\[ I_{C_B^\pm} = (y^m (x^q \pm (-i)^q y^q), x^m (x^q \pm (-i)^q y^q), x^s y^{s-r}, x^s y^m, x^m y^*) \]

**Theorem 2.6** (\cite{NdC09}). Let \( G = BD_{2n} (a) \) be small and let \( P \in G \text{-Hilb}(C^2) \) be defined by the ideal \( I \). Then we can always choose a basis for \( C[x, y]/I \) from one of the following list:
\[ \Gamma, \Gamma, \Gamma, \Gamma, \Gamma, \Gamma, \Gamma, \Gamma, \Gamma, \Gamma - \]

Let \( U_\Gamma \) the open set in \( G \text{-Hilb}(C^2) \) which consists of all \( G \)-clusters \( Z \) such that \( O_Z \) admits \( \Gamma \) as its basis. As a corollary of the previous theorem we have that the \( G \)-graphs for a \( BD_{2n} (a) \) group gives us the open set for the covering of \( G \text{-Hilb}(C^2) \) that we are looking for.

**Corollary 2.7** (\cite{NdC09}). Let \( G = BD_{2n} (a) \) a small binary dihedral group and let \( \Gamma_0, \Gamma_1, \ldots, \Gamma_{m-1}, \Gamma_{C+}, \Gamma_{C-}, \Gamma_{D+}, \Gamma_{D-} \) the list of \( G \)-graphs. Then
\[ U_{\Gamma_0}, U_{\Gamma_1}, \ldots, U_{\Gamma_{m-1}}, U_{\Gamma_{C+}}, U_{\Gamma_{C-}}, U_{\Gamma_{D+}}, U_{\Gamma_{D-}} \]
form an open cover of \( G \text{-Hilb}(C^2) \).

### 2.4 The moduli of quiver representations

A quiver \( Q \) is an oriented graph with \( Q_0 \) the set vertices and \( Q_1 \) the set of arrows, together with the maps \( h, t : Q_1 \to Q_0 \) giving the tail and head of every arrow. A path in \( Q \) is a sequence of arrows \( a_1 a_2 \cdots a_k \) such that \( h(a_i) = t(a_{i+1}) \) for every \( 1 \leq i \leq k - 1 \). At every vertex \( i \in Q_0 \) there exists the trivial path denoted by \( e_i \) such that \( t(e_i) = h(e_i) = i \).

The path algebra \( kQ \) is the \( k \)-algebra with basis the paths in \( Q \). The multiplication in \( kQ \) is given by the concatenation of paths if they are consecutive, and zero if not. The identity element is \( \sum e_i \). Note that the path algebra is finite dimensional if and only if \( Q \) has no oriented cycles. We write \( ab \) for the path \( a \) followed by the path \( b \).

A representation of a quiver \( Q \) is \( W = \{(W_i)_{i \in Q_0}, (\varphi_a)_{a \in Q_1}\} \) where \( W_i \) is a \( k \)-vector space for each vertex in \( Q \) and \( \varphi_a : W_i \to W_j \) is a \( k \)-linear map for every arrow \( a : i \to j \). We denote by \( d = (d_i)_{i \in Q_0} = (\dim W_i)_{i \in Q_0} \) the dimension vector of \( W \). A map between representations \( W \) and \( W' \) is a family \( \psi_i : W_i \to W'_i \) for each vertex \( i \in Q_0 \) of \( k \)-linear maps such that for every arrow \( a : i \to j \) the following square commutes:
\[ \begin{array}{ccc}
W_i(a) & \varphi_a & W_{h(a)} \\
\downarrow \psi_{t(a)} & & \downarrow \psi_{h(a)} \\
W'_i(a) & \varphi'_a & W'_{h(a)}
\end{array} \]

Note that a representation is a module over the path algebra \( kQ \). We denote by \( \text{rep}_k(Q) \) the category of finite dimensional representations of \( Q \).

A relation in a quiver \( Q \) (with coefficients in \( k \)) is a \( k \)-linear combination of paths of length at least 2, each with the same head and the same tail. Any finite set of relations \( R \) in \( Q \) determines a two-sided ideal \( \langle R \rangle \) in \( kQ \). The quiver \( (Q, R) \) is called the bound quiver or quiver with relations, and a representation of \( (Q, R) \) is a representation of \( Q \) where each relation must be satisfied by the homomorphisms between the vector spaces \( W_i \) for \( i \in Q_0 \). The category \( \text{rep}_k(Q, R) \) of finite dimensional representations of \( (Q, R) \) is equivalent to the category of finite dimensional left \( kQ/\langle R \rangle \)-modules.
Fix the dimension vector $d$ and consider the set of representations of $Q$ with such dimension vector. By choosing a basis of each vector space $W_i$, we can identify $W_i \cong \mathbb{k}^{d_i}$ and every map $\varphi_a$ is a matrix of size $d_{t(a)} \times d_{h(a)}$. Thus the representation space is

$$\bigoplus_{a \in Q_1} \text{Mat}_{d_{t(a)},d_{h(a)}} \cong \mathbb{A}_k^N$$

where $N := \sum_{a \in Q_1} d_{t(a)}d_{h(a)}$, and the isomorphism classes of such representations are orbits by the action of the change of basis group $G = \prod_{i \in Q_0} \text{GL}(W_i)$.

To construct an algebraic variety parametrising these isomorphism classes, we need to avoid the well-known problem of having orbits which are not closed, that is, we need need to parametrise orbits under certain notion of stability. The notion of stability for representations of a quiver is due to A. King [Kin94]: let $\theta \in \mathbb{Q}^{Q_0}$ and define $\theta(W) := \sum_{i \in Q_0} \theta_i d_i$, for any representation $W$ of $Q$ of dimension vector $d = (d_i)_{i \in Q_0}$. Then $W$ is said to be $\theta$-semistable if $\theta(W) = 0$ and if for every proper nonzero subrepresentation $W' \subset W$ we have $\theta(W') \geq 0$. The notion of $\theta$-stability is obtained by replacing $\geq$ by $>$ as usual. In the same way, we define a point $w \in \mathbb{k}^N$ to be $\theta$-(semi)stable if the corresponding representation $W$ is $\theta$-(semi)stable.

Let $(Q,R)$ be a bound quiver. Consider the $\mathbb{k}$-linear map $\phi : \mathbb{k}Q \to \mathbb{k}[x_{a_{ij}} : a = (a_{ij}) \in Q_1]$ that sends a path $p = s_1 \cdots s_n \in \mathbb{k}Q$ to the polynomial obtained by multiplying the corresponding matrices and identifying the entry $a_{ij}$ with the monomial $x_{a_{ij}}$. Let the ideal $I_R$ be the image of $(R)$ under $\phi$. Thus, a point in $\mathbb{A}^N$ corresponds to a representation of the bound quiver $(Q,R)$ if and only if it lies in the subscheme $V(I_R)$ cut out by the ideal $I_R$. Thus,

$$V(I_R)/\theta G = \text{Proj}\left(\bigoplus_{j \in \mathbb{N}} (\mathbb{k}[x_{a_{ij}} : a = (a_{ij}) \in Q_1]/I_R)_{j\theta}\right)$$

is the quotient of the open subscheme $V(I_R)^{ss} \subseteq V(I_R)$ which parametrize $\theta$-semistable representations of $(Q,R)$. If $\theta$ is generic (i.e. every $\theta$-semistable point is $\theta$-stable) we define

$$\mathcal{M}_\theta(Q,R) := V(I_R)/\theta G$$

the geometric quotient parametrizing $\theta$-stable representations of $(Q,R)$. Since $\theta$ is generic it represents a functor (see [Kin94]) so it is a fine moduli space.

From now on we fix $(Q,R)$ to be the bound McKay quiver, i.e. one vertex for each irreducible representation $\rho_i$, and one arrow between $\rho_i$ and $\rho_j$ if and only if $a_{ij} \neq 0$, where $V \otimes \rho_i = \sum_i a_{ij} \rho_j$ (see Section 3.2 for the relations $R$). Take also $\mathbb{k} = \mathbb{C}$, $d = (\dim \rho_i)$ and $\theta = (\sum_{\rho_i \in \text{Irr}} \dim \rho_i, 1, \ldots, 1)$. With this choice of stability condition (or any other $\theta \in \mathbb{Q}^{Q_0}$ with $\theta_i > 0$ for $i \neq 0$) we ensure that our choice of $\theta$ is generic, and we lie in the chamber $C \subset \Theta$ of the space of stability conditions $\Theta$ where $G\text{-Hilb}(\mathbb{C}^2)$ lives. This choice of stability is equivalent to the following condition, which will be crucial in our explicit calculations:

There exist a nonzero path from the distinguished source, which we choose to be $\rho_0 \in Q_0$, to every other 1-dimensional vertex, and two linearly independent paths to any 2-dimensional vertex in $Q$.

In other words, a representation $W$ of $Q$ is stable if and only if, as a $\mathbb{k}Q/(R)$-module, $W$ is generated from $\rho_0$.

Let $S = \mathbb{k}[x_1, \ldots, x_n]$. An $S$-module $M$ is $G$-equivariant if and only it carries a $G$-action verifying $g \cdot (sm) = (g \cdot s)(g \cdot m)$ for $g \in G$, $s \in S$, and $m \in M$. The skew group algebra $S \ast G$ is the free $S$-module with basis $G$ and ring structure given by $(sg) \cdot (s'g') := s(g \cdot s')g'$ for $s, s' \in S$ and $g, g' \in G$. The skew group algebra $S \ast G$ is known to be Morita equivalent to quotient algebra $\mathbb{k}Q/(R)$, i.e. there is an additive equivalence between the category of left $S \ast G$-modules and the category of left $\mathbb{k}Q/(R)$-modules. But notice that left $S \ast G$-modules, or equivalently representations of $(Q,R)$, are precisely $G$-equivariant $S$-modules.
On the other hand, for any cluster $Z \in \text{G-Hilb}(\mathbb{C}^n)$ we can interpret the ring $O_Z = S/I_Z$ as a $G$-equivariant $S$-module. Moreover, as an $S$-module, $O_Z$ is generated by $1 \mod I_Z$. Therefore, $\text{G-Hilb}(\mathbb{C}^2)$ is the moduli space of $G$-equivariant modules $M = k[x_1, \ldots, x_n]/I$ with $M \cong_{kG} kG$. Because of our choice of stability condition, any $\theta$-stable representation is a $S \ast G$-module generated from the trivial representation. Thus,

$$\text{G-Hilb}(\mathbb{C}^n) \cong M_\theta(Q, R)$$

is the moduli space of $\theta$-stable representations of the bound McKay quiver.

## 3 McKay quivers for $BD_{2n}(a)$ groups

In this section we describe the McKay quivers for the groups $BD_{2n}(a)$ as the $\mathbb{Z}/2$-orbifold quotient of their maximal cyclic subgroup, and we give the relations for the bound McKay quiver $(Q, R)$ following [BSW08].

Let $G = BD_{2n}(a)$ and let $A = \langle \alpha \rangle \leq G = BD_{2n}(a)$ be the maximal cyclic subgroup of $G$. Let $q$ and $a$ as in Section 2.1 and denote by $\text{Irr} G$ the set of irreducible representations of $G$ (similarly for $\text{Irr} A$).

### 3.1 Orbifold McKay quiver

The McKay quiver of $A$, denoted by $\text{McKayQ}(A)$, can be drawn in a torus as follows: Let $M \cong \mathbb{Z}^2$ be the lattice of monomials and $M_{\text{inv}} \cong \mathbb{Z}^2$ the sublattice of invariant monomials by $A$. If we take $M_R = M \otimes \mathbb{Z} \mathbb{R}$ we can consider the torus $T := M_R/M_{\text{inv}}$. The vertices in $\text{McKayQ}(A)$ are precisely $Q_0 = M \cap T$, and the arrows between each vertex are the natural multiplications by $x$ and $y$ in $M$ (see [3.3] for an example).

By the condition $a^2 \equiv 1 \mod 2n$ the continued fraction $\sqrt{\frac{2n}{2n-a}} = [b_1, \ldots, b_l]$ is symmetric with respect to the middle term and the ring of invariants $\mathbb{C}[x, y]_{\mathbb{Z} \langle 1, a \rangle}$ is generated by $u_0 = x^{2n}, u_1 = x^{2n-a}y$, and $u_{i+1} = u_i^{b_i}/u_i$. We also know that $u_r = x^b y^k$ is the monomial in $M_{\text{inv}}$ of the form $(xy)^i$ with the smallest exponent $i > 0$. Therefore we can choose $M_{\text{inv}} = \mathbb{Z} \cdot v_l \oplus \mathbb{Z} \cdot v_{i+1}$ where $v_l = (k, k)$ and $v_{i+1} = (s, t)$ are linearly independent vectors, and $v_{i+1}$ corresponds to the monomial $u_{i+1} = x^s y^t \in M_{\text{inv}}$.

In this way, if the parallelogram with vertices 0, $v_l$, $v_{i+1}$ and $v_l + v_{i+1}$ does not contain any other vector in $M_{\text{inv}}$, we can take it as a fundamental domain for $T$.

**Lemma 3.1.** We can always choose a fundamental domain $\mathcal{D}$ for $T$ to be the parallelogram with vertices 0, $(k, k)$, $(2q, 0)$ and $(k + 2q, k)$ where the opposite sides are identified.

**Proof.** Observe that $q(a + 1) \equiv 2q \mod 2n$, so that $x^q y^a$ and $x^{2q}$ are in the same representation $\rho_{q(a+1)}$. Since representations appearing along the diagonal are $\rho_{i(a+1)}$ with $i$ varying modulo $k$, we have that $\rho_{2q} = \rho_{\bar{q}(a+1)}$ where $\bar{q}$ denotes $q$ mod $k$. Therefore, the monomial $x^{2q+k-\bar{q}} y^{k-\bar{q}}$ is $A$-invariant and we can consider the fundamental domain generated by the linearly independent vectors $(k, k)$ and $(2q + k - \bar{q}, k - \bar{q})$.

On the other hand, note that representations corresponding to monomials $x^{2q+j} y^{j}$ for any $j$ are exactly the same as the representations appearing in the diagonal. Therefore by a simple translation we can consider the fundamental region $\mathcal{D}$ as desired. The number of lattice points in the parallelogram is $2kq = 2n$ so the domain is fundamental.

**Proposition 3.2.** The McKay quiver of the binary dihedral group $BD_{2n}(a)$ is the $\mathbb{Z}/2$-orbifold quotient of the McKay quiver for the Abelian subgroup $A = \langle \frac{1}{2n}(1, a) \rangle$.

That is, it is obtained from the standard parallelogram by
by reflecting in the two diagonal lines. Away from the fixed locus of the reflection, two rank one representations of \( A \) combine to give a rank two representation of \( G \). The fixed points correspond to rank one representations of \( A \) that go to themselves under conjugation by \( \beta \), so they split into \( \pm 1 \) pairs of rank one representations of \( G \).

**Proof.** We first show that the group \( G/A = \langle \beta \rangle \cong \mathbb{Z}/2 \) acts on \( \text{Irr} \ A \), which implies that \( G/A \) acts on \( \text{McKayQ}(A) \). The resulting orbifold is \( \text{McKayQ}(G) \).

Let \( \rho_i \) for \( i = 0, \ldots, 2n-1 \) be the irreducible representations of \( A \). The group \( G \) acts on \( A \) by conjugation, i.e. \( g \cdot h = ghg^{-1} \) for \( g \in G, h \in A \), which induces an action of \( G/A \cong \mathbb{Z}/2 \) on \( A \) by \( \beta \cdot h := \beta h \beta^{-1} \), for any \( h \in A \). This action induces an action on \( \text{Irr} \ A \) as follows: using the relation \( \alpha \beta = \beta \alpha^k \) we have that \( \alpha^i \beta = \alpha^{i-1} \beta \alpha = \alpha^{i-2} \beta \alpha^2 = \ldots = \beta \alpha^i = \beta \alpha^i \), which implies \( \beta \alpha^i \beta^{-1} = \alpha^i \beta \beta^{-1} = \alpha^i \), so that \( \alpha^i \) and \( \alpha^i \beta \) are conjugates. Hence, for any irreducible representation \( \rho_k \in \text{Irr} A \) we have that \( \rho_k (\beta) \rho_k (\alpha^i) \rho_k (\beta^{-1}) = \rho_k (\alpha^i) = \rho_k (\alpha^i \beta) = (\varepsilon^k)^{ai} = (\varepsilon^i)^{ak} = \rho_{ak}(\alpha^i) \). Therefore \( G/A \) acts on \( \text{Irr} A \) by \( \beta \cdot \rho_k := \rho_{ak} \), for \( k \in A \).

The free orbits under this action are the representations \( \rho_i \) and \( \rho_{ai} \) with \( a \neq i \mod 2n \), and they produce the 2-dimensional representation \( V_i \) in \( \text{McKayQ}(G) \). Every fixed point \( \rho_j \) with \( a j \equiv j \mod 2n \) split into the two 1-dimensional representations \( \rho_j^+ \) and \( \rho_j^- \) in \( \text{McKayQ}(G) \).

In the parallelogram, the fixed representations are contained in the left (and identified right) side of \( D \), and in the line parallel to it passing through the middle of \( D \) (more precisely, \( \rho_{i(a+1)} \) and \( \rho_{i(a+1)+q} \) for \( 0 \leq i < k \) respectively). Note that \( \text{McKayQ}(G) \) is now drawn in a cylinder where only the top and bottom sides are identified.

The arrows of \( \text{McKayQ}(A) \) going in and out of representations in the fixed locus, split into two different arrows, while for the rest we have a one-to-one correspondence between arrows in \( \text{McKayQ}(A) \) and \( \text{McKayQ}(G) \).

---

**Figure 1:** McKay quiver for \( BD_{2n}(a) \) groups
Therefore, the McKay quiver of $G$ can be represented by a parallelogram of length $q$ and height $k$, as shown in Figure 1. The quiver is drawn in a cylinder, so that the bottom and top rows are identified, with $\mathbb{Z}/2$-orbifold edges where the 1-dimensional representations are situated. This object it is also known in the literature as orientifold. At every row $i$ we denote by $+$ sign the representation $\rho_i^+$ and with $-$ sign the representation $\rho_i^-$. The vertices in the middle of the parallelogram represent the 2-dimensional representations $V_j$. Notice also that the height $k$ of the McKay quiver is odd since we are considering the $BD_{2n}(a)$ case.

**Example 3.3.** Consider the group $BD_{30}(19)$ generated by $\alpha = \text{diag}(\epsilon, \epsilon^{19})$ with $\epsilon$ a primitive $30$-th root of unity, and $\beta = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)$. We have $q = 5$ and $k = 3$. The continued fraction $\frac{30}{30 - 19} = \frac{30}{11} = [3, 4, 3]$ describes the lattice $M_{\text{inv}}$. The two consecutive invariant monomials $x^3 y^3$ and $x^{11} y$ define a fundamental domain of the lattice $T$, which can be translated into the parallelogram filled with numbers shown in Figure 2.

The diagram represents the lattice $M$ where the bottom left corner represents the monomial 1 and the numbers denote the representation to which they belong to, e.g. the number 0 corresponds to monomials in $M_{\text{inv}}$. Opposite sides of the parallelogram are identified. The McKay quiver is completed by adding at every vertex the two arrows corresponding to the multiplication by $x$ and $y$ to the corresponding adjacent vertices.

Now acting with $\beta$ we see that representations $\rho_0, \rho_{20}, \rho_{10}$ and $\rho_5, \rho_{25}, \rho_{15}$ are fixed, while the rest (in pairs) are contained in a free orbit. The McKayQ$(BD_{30}(19))$ is shown in Figure 3.

Figure 2: McKay quiver for the Abelian group $\frac{1}{30}(1, 19)$.

Figure 3: McKay quiver for the group $BD_{30}(19)$. Top and bottom rows are identified.
3.2 The relations

In this section we describe briefly the method used in [BSW08] to obtain the ideal of relations \( R \) in the path algebra \( kQ \) of the McKay quiver \( Q \) that make the two algebras \( kQ/R \) and \( S \ast G \) Morita equivalent, and we give the relations in the case \( G = \text{BD}_{2n}(a) \).

Let \( G \subset \text{GL}(n, \mathbb{C}) \) be finite and small. Let \( Q \) be the McKay quiver of \( G \) with vertices \( \rho_i \) and corresponding representations \( \rho_i \). Denote by \( V \) the natural representation, and consider the 1-dimensional representation \( \det_V := \bigwedge^n V \). Note that \( \det_V \) is the trivial representation if and only if \( G \subset \text{SL}(n, \mathbb{C}) \). Tensoring with \( \det_V \) induces a permutation \( \tau \) (often called twist) on \( \text{Irr} G \), and therefore on the vertices of the McKay quiver as follows:

\[
e_i = \tau(e_j) \iff \rho_i = \rho_j \otimes \det_V
\]

Since \( Q \) is the McKay quiver, we can think of an arrow \( a : e_i \to e_j \) as an element \( \psi_a \in \text{Hom}_{CG}(\rho_i, \rho_j \otimes V) \). Then for any path \( p = a_1 a_2 \cdots a_n \) of length \( n \) we can consider the following \( G \)-module homomorphism

\[
\rho_{\text{id}}(p) \xrightarrow{\psi_p} \rho_{h(p)} \otimes V^\otimes n \xrightarrow{id_{\rho_{h(p)}} \otimes \gamma} \rho_{h(p)} \otimes V = \rho_{h(p)} \otimes \det_V
\]

The \( \psi_p \) is the composition of the maps \( \psi_{a_i} \) (and \( \otimes \text{id}_V \) at every step), and \( \gamma : V^\otimes n \to \bigwedge^n V \) sends \( v_1 \otimes \cdots \otimes v_n \mapsto v_1 \wedge \cdots \wedge v_n \). By Schur’s Lemma the composition of maps in \( \mathbb{R} \) is zero if \( \tau(h(p)) \neq t(p) \), a scalar \( c_p \) otherwise. Therefore, we can define the formal expression or superpotential as

\[
\Phi := \mathop{\sum}_{|p|=n} (c_p \dim h(p))p
\]

where \( |p| \) denotes the length of the path \( p \).

The last ingredient before giving the set of relations \( R \) is to consider partial derivations in \( kQ \): given any two paths \( p, q \in kQ \) we can define the partial derivative of \( p \) with respect to \( q \) to be \( \partial_q p = r \) if \( p = qr \), and 0 otherwise.

**Theorem 3.4** ([BSW08]). The algebra \( k[x_1, \ldots, x_n] \ast G \) is Morita equivalent to the algebra \( kQ/\langle \partial_q \Phi : |q| = n - 2 \rangle \).

Therefore, calculating the moduli space of stable representations of the McKay quiver subject to the relations given by \( \Phi \), is the same as calculating \( G\text{-Hilb} (\mathbb{C}^2) \). In the case of groups \( G = \text{BD}_{2n}(a) \subset \text{GL}(2, \mathbb{C}) \) we have \( V = V_1 \) and \( \det_V = \rho_{a+1} \). Then,

\[
\rho_j^+ \otimes \det_V = \rho_j^+ \otimes \rho_{a+1} = \rho_{j+a+1}^+
\]
\[
V_k \otimes \det_V = V_k \otimes \rho_{a+1}^+ = V_{k+a+1}
\]

so \( \tau \) translates \( \text{McKayQ}(G) \) one step diagonally up (see Figure 1) and only paths of length 2 joining two vertices identified by \( \tau \) appear in \( \Phi \). Therefore, the relations \( R \) for the bound McKay quiver are given by the “short” relations:

\[
a_i b_{i+1} = 0
\]
\[
c_i d_{i+1} = 0
\]
\[
f_i e_{i+1} = 0
\]
\[
h_i g_{i+1} = 0,
\]

and the “long” relations:

\[
b_i a_i + d_i c_i = 2r_{i,1} u_{i,1}
\]
\[
u_{i,j} r_{i+1,j} = r_{i,j+1} u_{i,j+1}
\]
\[
e_i f_i + g_i h_i = 2u_{i,q-2} r_{i+1,q-2}
\]

where we consider the subindices modulo \( k \).
**Remark 3.5.** We need to mention at this point that the quiver considered in this paper has the orientation reversed in comparison to that used in [BSW08]. This is because by our choice of the action of $G$ on the polynomial ring $\mathbb{C}[x,y]$, the semi-invariant $\mathbb{k}[x,y]^G$-modules are $S_\rho = (\mathbb{k}[x,y] \otimes \rho^*)^G$, which is dual to the one in [BSW08] (we can go from one McKay quiver to the other by replacing $V$ by its dual). In any case, the algebras obtained are isomorphic so the relations are the same.

### 4 Explicit calculation of $G$-Hilb$(\mathbb{C}^2)$

Let $(Q, R)$ be the bound McKay quiver. As mentioned earlier, we consider representations of $Q$ with dimension vector $d = (\dim \rho_i)_{i \in Q_0}$ and the generic stability condition $\theta = (- \sum_{\rho_i \in \text{Irr } G} \dim \rho_i, 1, \ldots, 1)$.

Given $G = BD_{2a}(a) \subset \text{GL}(2, \mathbb{C})$, let $\Gamma_0, \Gamma_1, \ldots, \Gamma_l, \Gamma_{C^+}, \Gamma_{C^-}, \Gamma_{D^+}$ and $\Gamma_{D^-}$ be the list of $G$-graphs with $\Gamma_i$, $i = 0, \ldots, l$ of either type $A$ or $B$. For any $\Gamma_i$ there exist the open set $U_i \subset G$-Hilb$(\mathbb{C}^2)$ of all $G$-clusters such that $O_x$ as a vector space admits $\Gamma_i$ as basis, and by Corollary 2.7 the open sets $U_0, \ldots, U_{D^-}$ cover the whole $G$-Hilb$(\mathbb{C}^2)$.

We assign to any $G$-graph $\Gamma$, a $\theta$-stable representation $W_i$ of the bound McKay quiver $(Q, R)$, where by changing basis appropriately and using the relations $R$ we can calculate the explicit equation for the open set in $\mathcal{M}_\theta(Q, R)$ containing $W_i$. Since $G$-Hilb$(\mathbb{C}^2) = \mathcal{M}_\theta(Q, R)$ these open set coincide with $U_i$.

We start by a cyclic example in $\text{GL}(2, \mathbb{C})$ to illustrate the method and the convenience of using both moduli spaces, $G$-Hilb$(\mathbb{C}^2)$ and $\mathcal{M}_\theta(Q, R)$, to calculate them explicitly. We continue with the smallest dihedral group $G = D_4 \subset \text{SL}(2, \mathbb{C})$ before considering the general case.

**Example 4.1.** Consider the group $G = \langle b, (1, 2) \rangle = \langle \alpha = \left( \begin{smallmatrix} i & 0 \\ 0 & e^5 \end{smallmatrix} \right), \text{primitive } \rangle$. There are five 1-dimensional irreducible representations $\rho_i(\alpha) = e^t$ for $i = 0, \ldots, 4$. The McKay quiver with relations and the corresponding quiver between the $\mathbb{C}[x,y]^G$-modules $S_{\rho_i}$ are shown in Figure 4. We denote in bold every arrow $a : \rho_i \to \rho_j$ in $Q_1$, and with regular font $a : \mathbb{C} \to \mathbb{C}$ the corresponding map in a representation of $Q$ (i.e. $a \in \mathbb{C}$).

![Figure 4: The McKay quiver for $\frac{1}{5}(1,2)$ with relations and the “xy-quiver” between the modules $S_\rho$.](image)

By stability we need to ensure that exists a nonzero path from the vertex corresponding to $\rho_0$ to the rest of vertices. This means that either $a \neq 0$ or $A \neq 0$ (or equivalently by changing coordinates $a = 1$ or $A = 1$), which makes an open condition in $\mathcal{M}_\theta(Q, R)$. In other words, an open sets in $\mathcal{M}_\theta(Q, R)$ is obtained by choosing suitable arrows in the representation space of $Q$ to be 1 in such a way that the stability condition is satisfied. We easily see that there are lot of possibilities for this to happen, for instance:

![Explained representations](image)

But note that $\mathbb{C}^2/G$ is toric and by the continued fraction $\frac{5}{2} = [3, 2]$ we know that the minimal resolution $\mathcal{M}_\theta(Q, R)$ (or $\frac{5}{2}(1,2)$-Hilb$(\mathbb{C}^2)$) is the union of 3 complex planes $\mathbb{C}^2_{x^5,y/x^2} \cup \mathbb{C}^2_{x^2,y^3/x} \cup \mathbb{C}^2_{x^2/y^3,y^5}$. 

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Therefore, the calculation of the moduli space $M_{\theta}(Q, R)$ is giving us too many open sets, much more than we need.

On the other hand, there are just three $G$-graphs $\Gamma_0$, $\Gamma_1$ and $\Gamma_2$ defined by the ideals $I_{\Gamma_0} = (x^5, y)$, $I_{\Gamma_0} = (x^2, xy^2, y^3)$ and $I_{\Gamma_0} = (x, y^5)$ respectively, giving us exactly the 3 toric open sets described above. Therefore, the three open sets of $\theta$-stable representations $U_0, U_1, U_2 \subset M_{\theta}(Q, R)$ corresponding to the $G$-graphs $\Gamma_0$, $\Gamma_1$, and $\Gamma_2$ cover $M_{\theta}(Q, R)$.

The open set $U_i \subset M_{\theta}(Q, R)$ corresponding to the $G$-graph $\Gamma_i$ is obtained as follows: starting with the element $1 \in \rho_0$ and according to the quiver between the modules $S_{\rho_0}$, the nonzero arrow must pass through basis elements of $\Gamma_i$. See Figure 5 for the case $\frac{1}{5}(1, 2)$ where now $a, b, c, d, e, A, B, C, D, E \in \mathbb{C}$. Once the open conditions are chosen with the appropriate change of basis, the coordinates of the open sets are easily obtained by using the relations $R$.

![G-graph example](image)

To recover the toric coordinates, for example in $U_0$, we have that

$$A \cdot 1 = A \cdot (\text{basis of } \rho_2 \text{ in } \Gamma_0) \implies y = Ax^2 \implies A = y/x^2$$

$$abcde \cdot 1 = E \cdot (\text{basis of } \rho_0 \text{ in } \Gamma_0) \implies x^5 = e \cdot 1 \implies e = x^5$$

obtaining the complex plane $\mathbb{C}^2_{x,y/x^2}$ as before.

4.1 First case: $D_4 \subset \text{SL}(2, \mathbb{C})$

Let $G = D_4 = \text{BD}_4(3)$ be the dihedral group of order 8 in $\text{SL}(2, \mathbb{C})$ (see Table 1). Since the group is in $\text{SL}(2, \mathbb{C})$, $\det_V = \rho_0^+$ is the trivial representation, so $\tau$ is trivial. In Figure 5 we show the McKay quiver $Q$ with the 5 relations $R$, and the underlying quiver verifying the same relations but labeled with the irreducible maps between the Cohen-Macaulay $\mathbb{C}[x, y]$-$\text{modules } S_{\rho}$. Notice that we can fill the right column Table 1 starting from $1\rho_0^+$. As we saw in the cyclic example, this is the quiver that we use to translate from quiver representations to $G$-clusters.
A representation of \((Q, R)\) in this case consists of four 1-dimensional vector spaces (let us consider \(\mathbb{C}\) for such vector space) situated at the corners of the quiver, one 2-dimensional vector space \(\mathbb{C}^2\) for the 2-dimensional irreducible representation \(V\) at the middle vertex, and a linear map for every arrow satisfying the relations \(R\).

We denote the arrows by \(a = (a, A), b = (b_B), \ldots\) and so on. Thus, we have the following representation space and the 10 relations between the coefficients of the linear maps:

\[
\begin{align*}
\mathbb{C} & \xrightarrow{(a, A)} \mathbb{C}^2 & \mathbb{C}^2 & \xrightarrow{(b, B)} \mathbb{C}^2 \\
\mathbb{C}^2 & \xrightarrow{(c, C)} \mathbb{C} & \mathbb{C} & \xrightarrow{(d, D)} \mathbb{C}
\end{align*}
\]

\[
ab + AB = cd + CD = hg + HG = fe + FE = 0
\]

By Corollary 2.7 (see also [Len02]) we know that \(D_4\)-\text{Hilb}(\mathbb{C}^2)\) is covered by 5 \(G\)-graphs of types \(A, C^+, C^-, D^+\) and \(D^-\), distributed along the exceptional locus as shown in Figure 7. Note that The \(G\)-graphs \(\Gamma_{C^+}\) and \(\Gamma_{C^-}\) are enough to cover the whole of the middle \(\mathbb{P}^1\), so that the \(G\)-graph \(\Gamma_A(1, 7; 2, 2)\) is not needed.
We are now going to calculate the equations for the open sets of $D_4$-$\text{Hilb}(\mathbb{C}^2)$ for each of the $G$-graphs case by case. To do that we first assign a $\theta$-stable representation of the McKay quiver to every $G$-graph by changing basis, and then we use the relations $R$ to calculate the equation of the open set in $\mathcal{M}_\theta(Q,R)$ containing this representation.

**Type A**

By stability, we need to ensure that starting from the vertex $\rho_0^+$ we have a nonzero path to every 1-dimensional representation, ad 2 linearly independent nonzero paths to the middle 2-dimensional vertex.

Consider the open set in $\mathcal{M}_\theta(Q,R)$ given by $a,D,e,F,g \neq 0$. By changing coordinates at the middle vertex, we can choose $A = f = 0$ (see Figure 8). We claim that the corresponding $G$-graph is exactly of type $A$. Indeed, starting at the trivial vertex with the basis element 1 and following the bold arrows, the open choice we made above enables us to reach every vertex from the vector space corresponding to $\rho_0^+$. That is, we generate the whole module from the trivial vertex, i.e. the representation is $\theta$-stable. Table 2 shows the corresponding basis element in each representation $\rho$ according to the choices we have made in the quiver. All these polynomials together form the $G$-graph

$$\Gamma = \{1, (x, y), x^2 + y^2, x^2 - y^2, (y(x^2 + y^2), -x(x^2 + y^2)), x^4 - y^4\}.$$ 

We see that $xy \notin \Gamma$ and therefore $\Gamma$ is of type $A$.

![Figure 8: $D_4$-graph of type A, and the choices to obtain the corresponding open set in $\mathcal{M}_\theta(Q,R)$.](image)

| Representation | Nonzero path | Basis Element               |
|----------------|-------------|-----------------------------|
| $\rho_0^+$     | $e_0$       | $1$                         |
| $\rho_0^-$     | $ae$        | $-(x^4 - y^4)$              |
| $\rho_2^+$     | $af$        | $x^2 + y^2$                 |
| $\rho_2^+$     | $ag$        | $x^2 - y^2$                 |
| $V$            | $aef$       | $(x, y)$                    |
|                |             | $(y(x^2 + y^2), -x(x^2 + y^2))$ |

Table 2: Basis elements for the $G$-graph of type $A$

We calculate now the equation for this open set in $\mathcal{M}_\theta(Q,R)$: using the relations in $R$ of the McKay quiver we have $b = E = 0$, $D = -cd$, $h = -GH$, and

$$
\begin{pmatrix}
0 & 0 \\
B & 0 
\end{pmatrix} + \begin{pmatrix}
cd & -cd^2 \\
c & -cd 
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
0 & 0 
\end{pmatrix} + \begin{pmatrix}
-GH & H \\
-G^2H & GH 
\end{pmatrix}
$$

giving the relations $cd = -GH, H = -1 - cd^2$ and $B = -c - G^2H$. The resulting representation space is:
and the equation for this affine piece $U_{\Gamma_A} \subset \mathcal{M}_0(Q, R)$ is the nonsingular hypersurface in $\mathbb{C}_{c,d,G}$:

$$cd = (1 + cd^2)G$$

As we know, it also corresponds to an open set in $D_4\text{-Hilb}(\mathbb{C}^2)$. To check it, we now calculate the ideals defining the $G$-clusters parametrised by this affine set using paths in the quiver:

- $ad \cdot 1 = (1, 0) \left( \frac{d}{4} \right) \cdot 1 = d \cdot (\text{basis of } \rho_0^-) \quad \implies 2xy = -d(x^4 - y^4)$
- $aefg \cdot 1 = (1, 0) \left( \frac{1}{G} \right) \left( 0, 1 \right) \left( \frac{1}{G} \right) \cdot 1 = G \cdot (\text{basis of } \rho_2^-) \quad \implies 2xy(x^2 + y^2) = G(x^2 - y^2)$
- $aefb \cdot 1 = (1, 0) \left( \frac{1}{G} \right) \left( 0, 1 \right) \left( \frac{1}{B} \right) \cdot 1 = B \cdot (\text{basis of } \rho_0^-) \quad \implies (x^2 + y^2)^2 = B$

and finally

$$aefdc \cdot 1 = (1, 0) \left( \frac{1}{d} \right) \left( 0, 1 \right) \left( \frac{d}{4} \right) \left( c, -cd \right) \cdot 1 = (c, -cd) = c \cdot (\text{basis } e_1 \text{ of } V) - cd \cdot (\text{basis } e_2 \text{ of } V)$$

which gives $(-x(x^4 - y^4), y(x^4 - y^4)) = c(x, y) - cd(y(x^2 + y^2), -x(x^2 + y^2))$.

The $G$-clusters of the corresponding open set in $D_4\text{-Hilb}(\mathbb{C}^2)$ are therefore given by the ideals $I_{c,d,G}$ generated by the following polynomials:

- $R_1 = 2xy + d(x^4 - y^4)$
- $R_2 = 2xy(x^2 + y^2) - G(x^2 - y^2)$
- $R_3 = x(x^4 - y^4) + cx - cd(y(x^2 + y^2))$
- $R_4 = y(x^4 - y^4) - cy - cd(x^2 + y^2)$
- $R_5 = (x^2 + y^2)^2 - B$

where $B = -c + G^2(1 + cd^2)$ and $cd = (1 + cd^2)G$. Note that even though $B$ doesn’t appear in the equation of the affine set, the generator $R_5$ is needed. Indeed, if we remove $R_5$ the ideal at the origin we would have $\dim \mathcal{O}_{Z_{(0,0,0)}} = 9$, i.e. $I_{(0,0,0)}$ do not define a $G$-cluster.

**Remark 4.2.** Note that using this procedure any monomial of degree $d$ not in $\Gamma_A$ can be obtained as a combination of paths of length $d$ in the quiver “xy-quiver”. This combination of paths gives us the expression of any monomial of degree $d$ not in $\Gamma_A$ as a combination of elements in $\Gamma_A$. This proves that any $G$-cluster contained in this open set has $\Gamma_A$ as basis.

**Type $C^+$**

The $G$-graph $\Gamma_{C^+}$ and the corresponding $\theta$-stable representation space are showed in Figure 8 and the choices of basis for each irreducible representation are given in Table 3. Using the relations in this case we have that

$$\begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix} + \begin{pmatrix} -CD & C \\ -CD^2 & CD \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -GH & H \\ -G^2H & GH \end{pmatrix}$$

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Figure 9: \( D_4 \)-graph of type \( C^+ \), and the corresponding open set in \( M_\theta(Q,R) \), giving the relations \( CD = GH, \ C = 1 + H \) and \( B = CD^2 - G^2H \). Therefore, the equation for \( U_{\Gamma_{C^+}} \subset M_\theta(Q,R) \) is the nonsingular hypersurface \( GH = (1 + H)D \) in \( \mathbb{C}^3_{D,G,H} \).

For the ideals \( I_{(D,G,H)} \) in the corresponding open set in \( D_4\)-Hilb(\( \mathbb{C}^2 \)) we have

\[
\begin{align*}
\text{ae} \cdot 1 &= (1, 0) \left( \frac{1}{G} \right) (0, 1) \left( \frac{1}{D} \right) \cdot 1 = G \cdot (\text{basis of } \rho_2^0) \quad \implies \quad 2xy(x^2 + y^2) = G(x^2 - y^2) \\
\text{af} \cdot 1 &= (1, 0) \left( \frac{1}{G} \right) (0, 1) \left( \frac{1}{D} \right) \cdot 1 = D \cdot (\text{basis of } \rho_0^0) \quad \implies \quad -(x^4 - y^4) = 2Dxy
\end{align*}
\]

As in the type A case we obtain \( (x^2 + y^2)^2 = B \), and to finish, notice that

\[
\begin{align*}
\text{ah} \cdot 1 &= (1, 0) \left( \frac{1}{G} \right) (-GH, H) \cdot 1 = (-GH, H) = -GH \cdot (\text{basis } e_1 \text{ of } V) + h \cdot (\text{basis } e_2 \text{ of } V)
\end{align*}
\]

which gives \( (y(x^2 - y^2), x(x^2 - y^2)) = -GH(x, y) + H(y(x^2 + y^2), -x(x^2 + y^2)) \).

The \( G \)-clusters of this open set are given by the ideals \( I_{(D,G,H)} \) generated by the following polynomials:

\[
\begin{align*}
R_1 &= 2xy(x^2 + y^2) - G(x^2 - y^2) \\
R_2 &= x^4 - y^4 + 2Dxy \\
R_3 &= x(x^2 - y^2) + GHx - Hy(x^2 + y^2) \\
R_4 &= y(x^2 - y^2) + GHy + Hx(x^2 + y^2) \\
R_5 &= (x^2 + y^2)^2 - B
\end{align*}
\]

where \( B = D^2(1 + H) + G^2H \).

\[
\begin{array}{|c|c|c|}
\hline
\text{Representation} & \text{Nonzero path} & \text{Basis Element} \\
\hline
\rho_0^0 & \epsilon_0 & 1 \\
\rho_0^1 & ad & 2xy \\
\rho_2^1 & ae & x^2 + y^2 \\
\rho_2^2 & ag & x^2 - y^2 \\
V & a & (x, y) \\
V & aef & (y(x^2 + y^2), -x(x^2 + y^2)) \\
\hline
\end{array}
\]

Table 3: Type \( C^+ \) representations and basis elements

**Type \( C^- \)**

Since this case is similar to Type \( C^+ \) we just give the results. See Figure 10 for the \( G \)-graph \( \Gamma_{C^-} \) and corresponding representation.
The long relation gives us $CD = EF$, $C = 1 + F$ and $B = CD^2 - E^2F$, which implies that the equation we were looking for is the hypersurface $EF = (F + 1)D$ in $\mathbb{C}^3_{D,E,F}$. In this case the ideals are

$$I_{D,E,F} = (2xy(x^2 - y^2) - E(x^2 + y^2),$$
$$x^4 - y^4 - 2Dxy,$$
$$y(x^2 + y^2) + EFx - Fy(x^2 - y^2),$$
$$x(x^2 + y^2) - EFy + Fx(x^2 - y^2),$$
$$(x^2 - y^2)^2 + B)$$

where $B = D^2(1 + F) - E^2F$.

**Type $D^+$**

The long relation now gives us $CD = -gh$, $C = 1 - g^2h$ and $B = h + CD^2$ which implies that the equation of the open set in $\mathcal{M}_\theta(Q,R)$ is the hypersurface $gh = (g^2h - 1)D$ in $\mathbb{C}^3_{D,g,h}$, and the corresponding ideals in $D_4$-Hilb($\mathbb{C}^2$) are

$$I_{D,g,h} = (x^2 - y^2 - 2gxy(x^2 + y^2),$$
$$x^4 - y^4 + 2Dxy,$$
$$2xy(x^2 + y^2) - hx + ghy(x^2 + y^2),$$
$$2x^2y(x^2 + y^2) - hy - ghx(x^2 + y^2),$$
$$(x^2 + y^2)^2 - B)$$

where $B = h + (1 - g^2h)D^2$. 

Figure 10: $D_4$-graph of type $C^-$, and the corresponding open set in $\mathcal{M}_\theta(Q,R)$.

Figure 11: $D_4$-graph of type $D^+$, and the corresponding open set in $\mathcal{M}_\theta(Q,R)$. 

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Type $D^−$

This case is analogous to the previous one. The open set is given by the hypersurface $ef = (e^2f - 1)D$ in $\mathbb{C}^3_{e,f,D}$, and the ideals are

$$I_{D,e,f} = \left(x^2 + y^2 - 2exy(x^2 - y^2),ight.$$  
$$x^4 - y^4 - 2Dxy,$$  
$$2xy^2(x^2 - y^2) - fx + ey(x^2 - y^2),$$  
$$2x^2y(x^2 - y^2) - fy + ex(x^2 - y^2),$$  
$$(x^2 - y^2)^2 + B)$$

where $B = f + (1 - e^2f)D^2$.

![Diagram](image)

Figure 12: $D_4$-graph of type $D^−$, and the corresponding open set in $\mathcal{M}_0(Q,R)$.

4.2 The general case: From $G$-graphs to representations of quivers

As we have seen in the previous section, the first step before calculating the equations of the moduli space is to simplify as much as we can the entries in the representation space by change of basis. Mainly, we want to write as many 1’s and 0’s as we can so the equations of the resulting open set become as simple as possible. By stability, in most cases we can not have maps $m: \mathbb{C}^i \to \mathbb{C}^j$ with $1 \leq i,j \leq 2$ which are identically zero, so we can suppose that at least one entry is different than zero (and consider it to be 1 after a suitable change of basis). Different affine open sets are given by different choices for the entries to be nonzero, and these choices are determined by the corresponding $G$-graph. Of course, this choices are not unique but equivalent under change of basis.

Let $(Q, R)$ be the bound McKay quiver of Figure 1 with relations $\mathbb{1}$ and $\mathbb{3}$. We call segment $i$ of $Q$ the horizontal stripe in $Q$ containing all vertices and arrows between $\rho_i(a+1)$ and $\bar{\rho}_{i-1}(a+1)+q$, as shown in Figure 13

![Diagram](image)

Figure 13: Segment $i$ of $Q$.

The quiver between the $\rho$-invariant modules $S_\rho$ is shown in Figure 14 where the segment is repeated throughout the quiver. Notice that we need to make a small adjustment on the arrows of the right hand side of the quiver depending on the sign of $q$. 

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We continue with our notation of writing in bold every arrow in $Q_1$ and with regular font the maps between vector spaces in a representation of $Q$. We denote them by $a = (a, A)$, or by $a = (a, A)$, or by $a = (a, A)$ depending on the dimension of the vector spaces a the source and target of $a$.

**Theorem 4.3.** Let $G = BD_{2n}(a)$ and let $\Gamma$ be a G-graph. Then, the open set $U_\Gamma \subset M_0(Q, R)$ is one of the following list:

If $\Gamma = \Gamma_A(r, s; u, v)$ then $U_{\Gamma_A(r, s; u, v)}$ is given by

$$
\begin{align*}
a_0, D_0, H_0 &\neq 0, \\
a_i, H_i &\neq 0 \text{ for } i \text{ even, } \\
c_i, F_i &\neq 0 \text{ for } i \text{ odd, } \\
e_i, g_i, r_{i, j}, U'_{i, j} &\neq 0 \text{ for all } i, j,
\end{align*}
$$

For $0 < i < u$: if $i$ is odd, then $B_i \neq 0$ and $d_i \neq 0$, if $i$ is even, then $b_i \neq 0$ and $D_i \neq 0$, and for $i \geq u$: $B_i \neq 0$ and $D_i \neq 0$.

If $\Gamma = \Gamma_B(r, s; u, v)$ then $U_{\Gamma_B(r, s; u, v)}$ is given by

$$
\begin{align*}
a_0, d_0, H_0 &\neq 0, \\
a_i, b_i, D_i, H_i &\neq 0 \text{ for } i \text{ even, } \\
e_i, g_i, r_{i, j}, U'_{i, j} &\neq 0 \text{ for all } i, j,
\end{align*}
$$

In addition $C_0, R_{1,1}, R_{1,2}, \ldots, R_{1,r-2} \neq 0$, and

$$
U_{i, j} \neq 0 \text{ for all } i > 0 \text{ and all } j. \text{ When } i = 0 \text{ we have only } U'_{0, r} \ldots U'_{0,q-2} \neq 0.
$$

If $\Gamma = \Gamma_{C^\pm}$ then

1. The open conditions for the open set $U_{\Gamma_{C^\pm}} \subset M_0(Q, R)$ where $\Gamma_{C^\pm}(r, s; q, q)$, are the same as the ones for $\Gamma_B(r, s; q, q)$.

2. The open conditions for the case $\Gamma_{C^\pm}(r, s; q, q)$ are the same as the ones for $\Gamma_B(r, s; q, q)$ but changing the choices for $f$ and $h$. In other words, we have this time $H_i \neq 0$ for $i$ odd and $F_i \neq 0$ for $i$ even, while the condition $H_0 \neq 0$ remains the same.

If $\Gamma = \Gamma_{D^\pm}$ then $U_{\Gamma_{D^\pm}}$ is given by

$$
\begin{align*}
a_0, C_0, a_0, H_0 &\neq 0, \\
a_i, b_i, D_i &\neq 0 \text{ for } i \text{ even, } \\
r_{i, j} &\neq 0 \text{ for all } i, j \text{ except for } r_{i,q-i} \text{ with } i = 2, \ldots, k-1, \\
B_i, c_i, d_i &\neq 0 \text{ for } i \text{ odd, } \\
\text{and in addition } R_{1,1}, R_{1,2}, \ldots, R_{1,r-2} &\neq 0 \text{ and } U_{i,q-i} \neq 0 \text{ for } i = 2, \ldots, k-1.
\end{align*}
$$

If $\Gamma$ is a G-graph of type $D^+$ then we also set

$$
e_0, H_0, G_0, E_1, f_1, g_1, H_1 \neq 0,$$

If $i$ is even then $E_i, g_i, F_i \neq 0$, and

If $i$ is odd then $e_i, g_i, H_i \neq 0$. 

Figure 14: Segment $i$ of the “xy-quiver” between the modules $S_{\rho_i}$ for $q$ even and $q$ odd.
If $\Gamma$ is a $G$-graph of type $D^-$ then we also set

$$E_0, F_0, g_0, e_1, F_1, G_1, h_1 \neq 0.$$  

If $i$ is even then $e_i, G_i, H_i \neq 0$, and

If $i$ is odd then $E_i, g_i, F_i \neq 0$.

where $0 \leq i \leq k - 1$.

Proof. Given any $G$-graph $\Gamma$, to obtain an open set in $\mathcal{M}_6(Q, R)$ we need to make open conditions in the parameter space of the maps between the vector spaces subject to the relations $R$. This conditions must verify two properties: Firstly to verify the $\theta$-stability by giving a nonzero path from $\rho_0^+$ to every other 1-dimesional representation and two linearly independent maps to every 2-dimensional representation. And secondly, according to the “xy-quiver” between the $G \in \rho_0^+$ element $1 \in \rho_0^+$ every element in the $G$-graph $\Gamma$. We prove the result cases by case, starting from $G$-graphs of type $A$.

Case A: We explain the choice of open conditions dividing them in six steps.

(i) To reach the 2-dimensional representations $V_{i(a+1)+1}$ twice we choose $a_i = (1, 0)$ for $i$ even and $c_i = (1, 0)$ for $i$ odd, by changing basis at the target. The open conditions needed are

$$a_i \neq 0 \text{ for } i \text{ even and } c_i \neq 0 \text{ for } i \text{ odd}.$$ 

(ii) To reach the 2-dimensional representations $V_{i(a+1)+q-1}$ twice we choose $h_i = (0, 1)$ for $i$ even and $f_i = (0, 1)$ for $i$ odd by changing basis at the target. The open conditions needed are

$$H_i \neq 0 \text{ for } i \text{ even and } F_i \neq 0 \text{ for } i \text{ odd}.$$ 

(iii) Every horizontal arrow $r_{i,j}$ for every $i$ and $j$ between 2 dimensional vector spaces is of the form $r_{i,j} = \left( \begin{smallmatrix} 1 & 0 \\ -R_{i,j} & 1 \end{smallmatrix} \right)$, by changing basis at the head of every arrow. The open condition needed is

$$r_{i,j} \neq 0 \text{ for all } i, j.$$ 

(iv) Every vertical arrow $u_{i,j}$ for every $i$ and $j$ between 2-dimensional vector spaces can be taken to be of the form $u_{i,j} = \left( \begin{smallmatrix} a_{i,j} & b_{i,j} \\ 0 & 1 \end{smallmatrix} \right)$, taking the open conditions

$$U_{i,j} \neq 0 \text{ for all } i, j.$$ 

(v) To reach every 1-dimensional representation $\rho_{i(a+1)+q}$ changing basis at the target we can choose $e_i = \left( \begin{smallmatrix} 1 \\ h_i \end{smallmatrix} \right)$ and $g_i = \left( \begin{smallmatrix} 1 \\ c_i \end{smallmatrix} \right)$ by taking the open conditions

$$e_i \neq 0 \text{ and } g_i \neq 0 \text{ for all } i.$$ 

(vi) Since we have a $G$-graph of type $\Gamma_A(r, s; u, v)$, we have that $x^uy^u \notin \Gamma_A$ but $x^vy^v \in \Gamma_A$ for $i < u$. In fact, these monomials belong to the representations $\rho_{i(a+1)}^{(-1)^i}$ situated all of them in the left side of the McKay quiver. Therefore, starting from $\rho_0^+$ with the element 1 we need to reach with a nonzero map every representation $\rho_{i(a+1)}^{(-1)^i}$ with $0 < i < u$ with a composition of maps of length $i$, and because the map $a_0 = (1, 0)$, all of the arrows must have the first component nonzero. In other words, by changing basis at the target, we can achieve such a map by taking $d_1 = \left( \begin{smallmatrix} 1 \\ \tilde{h}_1 \end{smallmatrix} \right)$, $b_2 = \left( \begin{smallmatrix} 1 \\ \tilde{b}_2 \end{smallmatrix} \right)$, $d_3 = \left( \begin{smallmatrix} 1 \\ \tilde{b}_3 \end{smallmatrix} \right)$, ... until $d_{u-1} = \left( \begin{smallmatrix} 1 \\ \tilde{b}_{u-1} \end{smallmatrix} \right)$ if $u$ is even, or $b_{u-1} = \left( \begin{smallmatrix} 1 \\ \tilde{b}_{u-1} \end{smallmatrix} \right)$ if $u$ is odd.

The condition $x^uy^u \notin \Gamma_A$ is given by the choice $b_u = \left( \begin{smallmatrix} 1 \\ \tilde{b}_u \end{smallmatrix} \right)$ if $u$ is even, or $d_u = \left( \begin{smallmatrix} 1 \\ \tilde{b}_u \end{smallmatrix} \right)$ if $u$ is odd. In this way, the nonzero map from $\rho_0^+$ to $\rho_{u(a+1)}$ which gives the basis element $m$ is a path of length bigger than $u$, and we have that either $x^u\ell^u = b_u \cdot m$, or $x^u\ell^u = d_u \cdot m$.

From segment $u$ to the top segment the choices are always $B_i$ and $D_i$ nonzero except for the segment 0, where we allow $b_0$ to be zero since the module $S_{\rho_0^+}$ already has the basis element 1, and we do
not need to have a nonzero map through it. Therefore, the open conditions for the arrows $b_i$ and $d_i$ are:

For $0 < i < u$: $B_i \neq 0$ and $d_i \neq 0$, when $i$ is odd

$b_i \neq 0$ and $D_i \neq 0$, when $i$ is even

For $i \geq u$: $B_i \neq 0$ and $D_i \neq 0$

With the open conditions in (i)-(vi) there is a nonzero map from the vertex corresponding to $\rho_0^+$ to every other 1-dimensional representation, and there are two linearly independent maps to every 2-dimensional representation. Thus, the space of all the remaining parameters modulo the relations $R$ forms the open set $U_{\Gamma_A} \subset M_{\theta}(Q,R)$ corresponding to the $G$-graph $\Gamma_A$. In Figure 15 we show the choices of nonzero elements in the representation space and the corresponding basis elements (represented as monomials) in the $G$-graph that these choices produce.

![Figure 15: Basis elements in the $G$-graph $\Gamma_A$ and the corresponding choices in the representation space.](image)

The remaining cases are done in a similar way, according to the shape of each $G$-graph.

**Case B:** The condition defining a $G$-graph of type $B$ of the form $x^u y^k$, $x^k y^u \notin \Gamma_B$ implies that $x^i y^i \in \Gamma_B$ for $i < u$. This explains the choice of arrows at the left side of the quiver, while the conditions at the right hand side remains the same as in the case A. The conditions $x^u y^m$, $x^m y^u \notin \Gamma_B$ are expressed with the open conditions $C_0, R_{1,1}, R_{1,2}, \ldots, R_{1,r-2} \neq 0$, since $x^u y^m$ and $x^m y^u$ belong to the special representation $V_r$. If $r \leq k$ then we have a $G$-graph of type $B.1$, otherwise we have a $G$-graph of type $B.2$.

**Case C:** This case corresponds to the last $G$-graph in the sequence $(0,2n), (1,a), \ldots, (r,s), (q,q)$. If we suppose for example that the last $G$-graph is of type $B$, this implies that the open conditions are made at the special representation $V_r$. About the difference between the $C^+$ and the $C^-$ case, notice that $(+)^2 \notin \Gamma_{C^+}$ and $(-)^2 \notin \Gamma_{C^-}$. This implies that the differences are just the choices of the vertical arrows in the right hand side of the McKay quiver.

**Case D:** The main characteristic of this type is that $(+) \notin \Gamma_{D^+}$ (or $(-) \notin \Gamma_{D^-}$ respectively). This implies that the open condition is made at the special representation $\rho_q^+$ (or at $\rho_q^-$ respectively). For instance,
in the $D^+$ case we do not allow a path of length $q$ starting from $\rho_0^+$ and ending at $\rho_q^+$, which means that $e_1 = (\uparrow)$. The differences between the two cases are again the choice at the vertical arrows of the right hand side of the quiver.

### 4.3 Example $\text{BD}_{42}(13)$

The group is of the form $\text{BD}_{2n}(a) = \langle \frac{1}{12} (1,13), \left( \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right) \rangle$ of order 84. In this case we have that $q = 7$ and $k = 3$. Since $\frac{42}{13} = [4,2,2,4]$, the lattice points in the Newton polygon that we need to consider are $e_0 = (0,42)$, $e_1 = (1,13)$, $e_2 = (4,10)$ and $e_3 = (7,7)$. The $G$-graphs for the binary dihedral group $\text{BD}_{42}(13)$ are shown in Figure 16.

![Figure 16: The $\text{BD}_{42}(13)$-graphs of the group $\frac{1}{12}(1,13)$.](image)

If we call $(+)=x^7+iy^7$ and $(-)=x^7-iy^7$, the ideals defining these $G$-graphs are

\[
\begin{align*}
I_{\Gamma_A} &= (xy, x^{42} + y^{42}) \\
I_{\Gamma_{B,1}} &= (x^{14} - y^{14}, x^2y^2(+)(-), x^4y^3, x^3y^4) \\
I_{\Gamma_{C,-}} &= (y^3(-), x^3(-), x^6y^4, x^{10}y^3, x^3y^{10}) \\
I_{\Gamma_{C,+}} &= (y^3(+), x^3(+), x^6y^4, x^{10}y^3, x^3y^{10}) \\
I_{\Gamma_{D,-}} &= ((+), x^6y^4) \\
I_{\Gamma_{D,+}} &= ((-), x^6y^4)
\end{align*}
\]

For the type $\Gamma_A(0,42;1,13)$, by Theorem 4.3 the representation space for $U_{\Gamma_A}$ is as Figure 17 shows. Note that the condition that characterizes this $G$-graph set is that $xy \notin \Gamma_A$, which translated to the
representation of the McKay quiver, means that \( d_1 = \begin{pmatrix} d_1 \\ 1 \end{pmatrix} \).

Notice that the representations inside a circle are the special representations. Recall the “xy- quiver” between the modules \( S_\rho \) in Figure 14. With this quiver and the open choices from the representation of the McKay quiver that we have just made, we have the chosen the basis elements for \( \Gamma_A \) shown in Table 4.3. With this information we can express any polynomial not in \( \Gamma_A \) as a combination of basis elements as we did in the \( D_4 \) case. For example,

\[
xy = d_0 (+)^2 \\
x^2y = d_0 (x(+))^2 + d_0 C_1 y^6 (+)^3 \\
yx^2 = d_0 y(+)^2 - id_0 C_1 x^6 (+)^3
\]

and so on.

| \( \rho_0 \) | 1 | \( \rho_1 \) | \( +)^3(-) \) | \( \rho_14 \) | \( (+)(-) \) | \( \rho_28 \) | \( (+)^4 \) |
| -- | -- | -- | -- | -- | -- | -- | -- |
| \( V_1 \) | \( x, y \) | \( y^6(+), -ix^6(+) \) | \( x^2(+), y^6(-) \) | \( x^2(+), -y^2(+) \) | \( y^6(+) \) | \( y^6(+) \) | \( x^2(+), -y^2(+) \) |
| \( V_2 \) | \( y^4(+) \) | \( x^4(+), -ix^4(+) \) | \( x^4(+), y^4(+) \) | \( x^4(+), -y^4(+) \) | \( y^4(+) \) | \( y^4(+) \) | \( x^4(+), y^4(+) \) |
| \( V_3 \) | \( x^3(+) \) | \( x^3(+) \) | \( x^3(+) \) | \( x^3(+) \) | \( x^3(+) \) | \( x^3(+) \) | \( x^3(+) \) |
| \( V_4 \) | \( x^5(+) \) | \( x^5(+) \) | \( x^5(+) \) | \( x^5(+) \) | \( x^5(+) \) | \( x^5(+) \) | \( x^5(+) \) |
| \( V_5 \) | \( x^6, y^6 \) | \( x^6, y^6 \) | \( x^6, y^6 \) | \( x^6, y^6 \) | \( x^6, y^6 \) | \( x^6, y^6 \) | \( x^6, y^6 \) |
| \( \rho_7 \) | \( + \) | \( \rho_7 \) | \( - \) | \( \rho_7 \) | \( + \) | \( \rho_7 \) | \( - \) |
| \( \rho_7 \) | \( + \) | \( \rho_7 \) | \( - \) | \( \rho_7 \) | \( + \) | \( \rho_7 \) | \( - \) |

Table 4: Basis elements of the \( G \)-graph \( \Gamma_A(0, 43; 1, 13) \), with \((+)
\) \( = x^7 - iy^7 \) and \((-) \) \( = x^7 + iy^7 \).

For the type \( \Gamma_{E_1}(1, 13; 4, 10) \) case, by Proposition 2.5 we have that \( x^4 y^3 \) and \( x^3 y^4 \) are out of the basis, and since they belong to the irreducible representation \( V_1 \), we allow the variable \( c_0 \) to be zero so that \( (x^4 y^3, -x^3 y^4) \) is not in the basis of \( V_1 \). In Table 4.3 we write the basis elements for each irreducible
representation of $\text{BD}_{42}(11)$ given according to our choice of nonzero elements.

Note that in the previous case $A$ the choice was made at the special representation $\rho_{14}^+$, and in this case the choice is done at the next special representation $V_1$. It will not be a surprise to check that Type $C$ will correspond to a choice at the special representation representation $V_4$, and Type $D^\pm$ at the special representations $\rho^+_7$ and $\rho^-_7$, giving another evidence of the relation between $G$-graphs and special representations.

| $\rho_0$ | $\rho_0^+$ | $\rho_0^0$ | $\rho_3^+$ | $\rho_4^+$ | $\rho_5^+$ | $\rho_6^+$ | $\rho_7^+$ |
|---------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| $x^4y^3$ | $x^4y^3$ | $x^4y^3$ | $x^4y^3$ | $x^4y^3$ | $x^4y^3$ | $x^4y^3$ | $x^4y^3$ |
| $\rho_1^+$ | $\rho_1^+$ | $\rho_1^+$ | $\rho_1^+$ | $\rho_1^+$ | $\rho_1^+$ | $\rho_1^+$ | $\rho_1^+$ |
| $x^4y^3$ | $x^4y^3$ | $x^4y^3$ | $x^4y^3$ | $x^4y^3$ | $x^4y^3$ | $x^4y^3$ | $x^4y^3$ |
| $\rho_8^+$ | $\rho_8^+$ | $\rho_8^+$ | $\rho_8^+$ | $\rho_8^+$ | $\rho_8^+$ | $\rho_8^+$ | $\rho_8^+$ |
| $x^4y^3$ | $x^4y^3$ | $x^4y^3$ | $x^4y^3$ | $x^4y^3$ | $x^4y^3$ | $x^4y^3$ | $x^4y^3$ |
| $\rho_9^+$ | $\rho_9^+$ | $\rho_9^+$ | $\rho_9^+$ | $\rho_9^+$ | $\rho_9^+$ | $\rho_9^+$ | $\rho_9^+$ |
| $x^4y^3$ | $x^4y^3$ | $x^4y^3$ | $x^4y^3$ | $x^4y^3$ | $x^4y^3$ | $x^4y^3$ | $x^4y^3$ |
| $\rho_{10}^+$ | $\rho_{10}^+$ | $\rho_{10}^+$ | $\rho_{10}^+$ | $\rho_{10}^+$ | $\rho_{10}^+$ | $\rho_{10}^+$ | $\rho_{10}^+$ |
| $x^4y^3$ | $x^4y^3$ | $x^4y^3$ | $x^4y^3$ | $x^4y^3$ | $x^4y^3$ | $x^4y^3$ | $x^4y^3$ |
| $\rho_{11}^+$ | $\rho_{11}^+$ | $\rho_{11}^+$ | $\rho_{11}^+$ | $\rho_{11}^+$ | $\rho_{11}^+$ | $\rho_{11}^+$ | $\rho_{11}^+$ |
| $x^4y^3$ | $x^4y^3$ | $x^4y^3$ | $x^4y^3$ | $x^4y^3$ | $x^4y^3$ | $x^4y^3$ | $x^4y^3$ |

Table 5: Basis elements of the $G$-graph $\Gamma_{B_1}(1,13;4,10)$, with $(+) = x^7 - iy^7$ and $(-) = x^7 + iy^7$.

For the type $\Gamma_{C^+}(4,10;7,7)$ we have two different $G$-graphs, $\Gamma_{C^+}$ and $\Gamma_{C^-}$. Remember by Theorem 4.3 that the open set corresponding to the $G$-graph of type $\Gamma_{C^-}$ is the same as $\Gamma_{B_1}(4,10;7,7)$. In the $C^+$ case we have that $(+)^2 \notin \Gamma_{C^+}$ and in the $C^-$ case we have $(-)^2 \notin \Gamma_{C^-}$. We show in Figure 19 the open set $U_{\Gamma_{C^+}}$. The other case differs only in the choice of the vertical arrows on the right hand side of the McKay quiver.

For the types $\Gamma_{D^\pm}(4,10;7,7)$, the difference between the two cases resides in the choice of the basis element for the representation $\rho^+_7$ and $\rho^-_7$. In the case corresponding to $\Gamma_{D^+}$ we have that $(+) = x^7 + iy^7 \notin \Gamma_{D^+}$, which implies the choice $e_1 = (\uparrow)$. Because of the shape of the $G$-graph,
we must have a polynomial in $\Gamma_D^+$ containing $x^7$ and $y^7$. This is the element $(-) = x^7 - iy^7$ which is obtained by setting $g_1 = (\frac{1}{i})$.

The case $\Gamma_D^-$ is analogous but changing the conditions to $e_1 = (\frac{1}{i})$ and $g_1 = (\frac{1}{i})$, so that now we have $(-) \notin \Gamma_D^-$. We show the open set $U_{\Gamma_D^-}$ in Figure 20.

**Remark 4.4.** Once we have the open sets in $\mathcal{M}_\theta(Q, R)$ labelled with all the remaining variables, the equations of each of these affine open sets are obtained in the same way as in the $D_4$ case. The first convenient step is to change basis at every vertex setting some of the coordinates equal to zero (for instance, we can always take $r_{i,j} = 0$ for all $i, j$). Then, by using the relations $R$, the equation of a nonsingular surface is obtained.

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