On the existence of optimal controls for SPDEs with 
boundary-noise and boundary-control

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Abstract

We consider a stochastic optimal control problem for a heat equation with boundary noise and boundary 
controls. Under suitable assumptions on the coefficients, we prove existence of optimal controls in strong 
sense by solving the stochastic hamiltonian system related.

Key words. Stochastic control, maximum principle, stochastic evolution equation, forward-
backward stochastic differential system.

1 Introduction

In this paper we are concerned with the existence of optimal control for a stochastic optimal control problem related to the following stochastic heat equation, in which boundary noise and boundary control are allowed:

\begin{equation}
\begin{aligned}
\frac{\partial y}{\partial t}(t, \xi) &= \frac{\partial^2 y}{\partial \xi^2}(t, \xi) + b(\xi)u^0(t, \xi) + g(\xi)\dot{W}(t, \xi), \quad t \in [0, T], \, \xi \in (0, \pi), \\
y(0, \xi) &= x(\xi), \\
\frac{\partial y}{\partial \xi}(t, 0) &= u^1_s + \dot{\tilde{W}}_s, \quad \frac{\partial y}{\partial \xi}(t, \pi) = u^2_s 
\end{aligned}
\end{equation}

In the above equation $\tilde{W}$ is a standard real Wiener process and $\dot{W}(\tau, \xi)$ is a space-time white noise on $[0, T] \times [0, \pi]$; $\tilde{W}$ and $W$ are both defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and are independent. By \{\mathcal{F}_t, \, t \in [0, T]\} we will denote the natural filtration of $(\tilde{W}, W)$, completed in the usual way; $u^0$ and $(u^1, u^2)$ are $\mathcal{F}_T$-predictable square integrable processes and represent respectively the distributed and the boundary control. Notice that we are able to treat equations where the control affects all the boundary while the noise only affects one point at the boundary.
The problem is considered in its strong formulation, i.e., without changing the reference probability space \((\Omega, \mathcal{F}, \mathbb{P})\). The stochastic optimal control problem consists in minimizing over all admissible controls the following cost functional:

\[
J(x, u^0, u^1, u^2) = \mathbb{E} \int_0^T \int_0^\pi \left( l_o(s, \xi, y(s, \xi)) + \bar{g}(u^0_s(\xi), u^1_s, u^2_s) \right) d\xi 
ds + \mathbb{E} \int_0^\pi \bar{h}(\xi, y(T, \xi)) d\xi,
\]

where \(\bar{g}\) and \(\bar{h}\) satisfies suitable assumptions specified in section 2.2, here we only mention that \(\bar{g}\) is allowed to have quadratic growth with respect to the control, and the control processes are not necessarily bounded. Equation (1.1) will be reformulated as a stochastic evolution equation in \(H = L^2((0, \pi))\):

\[
\begin{align*}
\frac{dX_t}{dt} &= AX_t dt + \left( (\lambda - A)D + B \right)u_t dt + (\lambda - A)D_1 d\tilde{W}_t + G dW_t, 
t &\in [0, T] \\
X_0 &= x,
\end{align*}
\]

where \(B\) and \(G\) are as usual the multiplication operators related to \(b\) and \(g\) respectively and \(D\) and \(D_1\) transform boundary data in elements of the domain of a suitable fractional power of \((\lambda - A)\), so that both \((\lambda - A)D\) and \((\lambda - A)D_1\) are unbounded operators. Notice that equation (1.3) can be considered as the model for a more general class of state equations, see section 2.2 for more details.

An approach to prove existence of optimal controls is the dynamic programming principle and the solution, in a sufficiently regular sense, e.g., mild, of the Hamilton Jacobi Bellman (HJB in the following) equation related. Because of the presence of the boundary noise, the transition semigroup related to equation (1.3) does not have sufficient smoothing properties, so the HJB equation associated cannot be solved in mild sense by a fixed point argument. The HJB equation is solvable in the sense of viscosity solutions, see e.g., \([13]\), and the presence of the noise as a forcing term is necessary in their approach. Moreover, since in equation (1.3) the control is not assumed to be in the image of \(G\) nor in the image of \((\lambda - A)D_1\), the HJB equations cannot be solved by means of backward stochastic differential equations (BSDEs in the following), see the pioneering paper \([24]\) and the infinite dimensional extension in \([11]\). When HJB equations can be solved by means of BSDEs, boundary noise and boundary control problems for the heat equations are treated in \([8]\), in the case of Neumann boundary conditions, and the techniques have been extended to the case of Dirichlet boundary conditions in \([23]\), by using also results in \([10]\). We also mention that in the dynamic programming approach existence of optimal controls is proved in the weak sense, since once the HJB equations is solved, the synthesis of the optimal controls is subject to the solution of the so called closed loop equation: since it is not clear the regularity of the feedback law, in many cases the closed loop equation can be solved only in the weak sense.

In \([12]\), by extending finite dimensional techniques, existence of optimal controls in the case of Hilbert space valued controlled diffusions is proved in relaxed sense. In \([15]\) existence of quasi-optimal controls is proved for a control problem related to a controlled state equation with distributed control and noise via the Ekeland principle. Their setting is infinite dimensional as in the present paper, but they prove existence of optimal controls not in strong sense and moreover in the state equation no unbounded terms are allowed. On the other hand they can bypass convexity assumptions either on the coefficients (still very regular) of the cost functional or of the control space \(U\).

An other approach to prove existence of optimal controls is the stochastic maximum principle, see e.g., \([17]\), which provides necessary conditions for optimality. When these conditions are also sufficient, existence of optimal controls can be proved by solving the related forward backward
stochastic Hamiltonian system, see e.g. [18]. Both in [17] and in [18] the setting is finite-dimensional. In this paper we generalize this approach to the infinite dimensional setting. The maximum principle, see [14] where the boundary case is treated, provides as usual necessary conditions for the optimal control to be verified. Then, under suitable assumptions -see section 2.2-, one can show that these conditions are indeed sufficient and so the solution to the Hamiltonian system fully characterizes the optimal control. In our case the Hamiltonian system is the following:

\[
\begin{aligned}
    d\bar{X}_t & = A\bar{X}_t dt + [E + B][E + B]^*\bar{Y}_t dt + (\lambda - A)\bar{D}_t d\bar{W}_t + G(t, \bar{X}_t) dW_t \\
    -d\bar{Y}_t & = A^*\bar{Y}_t dt + l^0_x(t, \bar{X}_t) dt - Z_t dW_t - \bar{Z}_t d\bar{W}_t, \quad t \in [0, T] \\
    \bar{X}_0 & = x, \quad \bar{Y}_T = -h_x(\bar{X}_T),
\end{aligned}
\]

where \( H(t, x, u, y) := -l(t, x, u) + \langle (E + B)^* y, u \rangle \), is the hamiltonian function, and \( \gamma : H \rightarrow U \) is such that \( H(t, x, \gamma((E + B)^* y), y) = \inf_{u \in U} H(t, x, u, y) \). Because of the infinite dimensional setting and of the presence of unbounded operators, the result obtained in the solution of this infinite dimensional forward backward system are of independent interest.

Indeed the solution of fully coupled forward backward systems is a difficult topic already in the finite dimensional case, see [1] and again [22] for examples of finite dimensional FBSDEs where there is no hope to get existence of a solution.

Among the large literature in finite dimensions, see e.g. the book of [22], we can distinguish two main approaches. The first approach, known as four-step scheme, relies on the connections between SDEs with deterministic coefficients and non-linear PDEs, see the pioneering paper [21]. Since in infinite dimensions on the solution of the related PDE less apriori estimates are known, this approach seems to be not suitable for an infinite dimensional extensions: in [14] local existence for an infinite dimensional FBSDE is proved, mainly adequating the finite dimensional techniques introduced in [9], but global existence is not achieved.

The second approach applies under monotonicity assumptions: different types of conditions have been investigated in this framework and we refer to Hu and Peng [19], Peng and Wu [27], Yong [28] and to Pardoux and Tang [25].

In the present paper, we solve FBSDE (1.4) by adapting the bridge method introduced in [19] to the infinite dimensional framework: new difficulties arises because of the presence of the unbounded operators, and just because both the forward and the backward stochastic equations are infinite dimensional and an unbounded operator is applied to backward unknown \( Y \) in the forward equation so that one has to prove some extra regularity for \( Y \) in order to give meaning to the system in the space \( H \). The regularity of the adjoint unknown is a typical task when one wants to prove maximum principle in infinite dimension, see [17] and [14], in this case new difficulty arise since the backward equation is coupled with the forward and the whole system has to be considered. The linear auxiliary FBSDE we study to apply then the bridge method is

\[
\begin{aligned}
    d\bar{X}_t & = A\bar{X}_t dt - [E + B][E + B]^*\bar{Y}_t dt + h_0(t) dt + (\lambda - A)\bar{D}_t d\bar{W}_t + G dW_t \\
    -d\bar{Y}_t & = A^*\bar{Y}_t dt + \bar{X}_t dt + h_0(t) dt - \bar{Z}_t dW_t - \bar{Z}_t d\bar{W}_t, \quad t \in [0, T] \\
    \bar{X}_0 & = x, \quad -\bar{Y}_T = \bar{X}_T + g_0,
\end{aligned}
\]

Unlike in [19], this linear auxiliary FBSDE is not immediately solvable. We notice that such system is the hamiltonian system associated to of an affine quadratic optimal control problem with
state equation

\[
\begin{cases}
    dX_t = AX_t dt + [E + B]u_t dt + b_0(t) dt + (\lambda - A)D_t d\tilde{W}_t + G dW_t & t \in [0, T] \\
    X_0 = x,
\end{cases}
\]  

(1.6)

and cost functional

\[ J(x, u) = \frac{1}{2} \mathbb{E} \int_0^T (|X_t + h_0(t)|^2 + |u_t|^2) dt + \frac{1}{2} \mathbb{E} |X_T + g_0|^2 \]  

(1.7)

where \( b_0 \) and \( h_0 \) are suitable stochastic processes. Therefore we introduce the Riccati equation (deterministic) corresponding to the linear terms and a backward stochastic differential equation to deal with the affine terms, see section 3.2, in order to get a solution to system (1.5). Again, because of the infinite dimensional setting and of the presence of unbounped operators, the solution and the regularity of this auxiliary backward stochastic differential equation is of independent interest.

Once we prove that system (1.5) has a unique solution, for every suitable \( \lambda \), the corresponding to the linear terms and a backward stochastic differential equation to control problem.

Given an arbitrary but fixed time horizon \( T \), we consider all stochastic processes as defined on subsets of the time interval \([0, T]\). Let \( Q \in L(K) \) be a symmetric non-negative operator, not necessarily trace class and \( W = (W_t)_{t \in [0, T]} \) be a \( Q \)-Wiener process with values in \( K \), defined on a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and \( W = (W_t)_{t \in [0, T]} \) be a cylindrical Wiener process with values in \( \Xi \), defined on the same probability space and independent of \( W \). By \( \{\mathcal{F}_t, t \in [0, T]\} \) we will denote the natural filtration of \((W, W)\), augmented with the family \( \mathcal{N} \) of \( \mathbb{P} \)- null sets of \( \mathcal{F} \), see for instance [6] for its definition. Obviously, the filtration \((\mathcal{F}_t)\) satisfies the usual conditions of right-continuity and completeness. All the concepts of measurability for stochastic processes will refer to this filtration. By \( \mathcal{P} \) we denote the predictable \( \sigma \)-algebra on \( \Omega \times [0, T] \) and by \( \mathcal{B}(\Lambda) \) the Borel \( \sigma \)-algebra of any topological space \( \Lambda \).

Next we define two classes of stochastic processes with values in a Hilbert space \( V \).
• $L^p_p(\Omega \times [0, T]; V)$ denotes the space of equivalence classes of processes $Y \in L^2(\Omega \times [0, T]; V)$ admitting a predictable version. It is endowed with the norm

$$|Y| = \left( \mathbb{E} \int_0^T |Y_s|^2 \, ds \right)^{1/2}.$$  

• $C_p([t, T]; L^p(\Omega; S))$, $p \in [1, +\infty]$, $t \in [0, T]$, denotes the space of $S$-valued processes $Y$ such that $Y : [t, T] \to L^p(\Omega, S)$ is continuous and $Y$ has a predictable modification, endowed with the norm:

$$|Y|_{C_p([t, T]; L^p(\Omega, S))}^p = \sup_{s \in [t, T]} \mathbb{E}|Y_s|^p_S.$$  

Elements of $C_p([t, T]; L^p(\Omega; S))$ are identified up to modification.

• For a given $p \geq 2$, $L^p_p(\Omega; C([0, T]; V))$ denotes the space of predictable processes $Y$ with continuous paths in $V$, such that the norm

$$\|Y\|_p = (\mathbb{E} \sup_{s \in [0, T]} |Y_s|^p) ^{1/p}$$  

is finite. The elements of $L^p_p(\Omega; C([0, T]; V))$ are identified up to indistinguishability.

Given an element $\Phi$ of $L^p_p(\Omega \times [0, T]; L^2(\Xi, V))$ or of $L^p_p(\Omega \times [0, T]; L^2(K, V))$, the Itô stochastic integrals $\int_0^t \Phi(s) \, dW(s)$ and $\int_0^t \Phi(s) \, d\tilde{W}(s)$, $t \in [0, T]$, are $V$-valued martingales belonging to $L^p_p(\Omega; C([0, T]; V))$. The previous definitions have obvious extensions to processes defined on subintervals of $[0, T]$ or defined on the entire positive real line $\mathbb{R}^+$.

2.2 Optimal control problem and state equation

Let $H$ be a separable real Hilbert space, and $U$ a separable Hilbert, called the space of controls. We assume $U$ a convex set and we set the space $L^p_p(\Omega \times [0, T]; U)$ the space of admissible controls, and we denote it by $U$.

We make the following assumptions that we denote by (A):

(A.1) $A : D(A) \subset H \to H$ is a linear, unbounded operator that generate a $C_0$-semigroup $\{e^{tA}\}_{t \geq 0}$ that is also analytic and such that $|e^{tA}|_{L(H, H)} \leq e^{\omega t}$, $t \geq 0$ for some $\omega \in \mathbb{R}$. This means in particular that every $\lambda > \omega$ belongs to the resolvent set of $A$.

(A.2) $B \in L(U; H)$ and $G \in L(\Xi, H)$ and there exist constants $\Delta > 0$ and $\gamma \in [0, 1/2]$ such that

$$|e^{sA}G|_{L^2(\Xi, H)} \leq \frac{\Delta}{(1 + s)^\gamma}$$

for every $s \in \mathbb{R}^+$.

(A.3) $D$ is a continuous linear operator $D : U \to D((\lambda - A)^{\alpha})$ for some $\frac{1}{2} < \alpha < 1$ and $\lambda > \omega$, see for instance [20] or [25] for the definition of the fractional power of the operator $A$.

(A.4) $D_1$ is a linear operator $D_1 : K \to H$ and there is a constant $\frac{1}{2} < \beta < 1$ such that the following holds:

$$|e^{tA}(\lambda - A)D_1|_{L^2(K, H)} \leq \frac{C}{t^{1-\beta}}$$

for some $\lambda > 0$.

Remark 2.1 Notice that $D_1$ and $D$ can have the same structure, indeed if $D_1$ takes values in $D((\lambda - A)^{\beta})$ and $K$ is finite dimensional then (A.4) holds. On the over hand, by the analyticity of $A$, also for $D$ a similar estimate to the one for $D_1$ may follow.
We introduce the following class of control problems, where the state equation is
\[
\begin{cases}
    dX_t = AX_t dt + [(\lambda - A)D + B]u_t dt + (\lambda - A)D_1 d\tilde{W}_t + G dW_t & t \in [0, T] \\
    X_0 = x
\end{cases}
\] (2.1)
From now on we will denote for simplicity \((\lambda - A)D := E\)
We will seek for a mild solution to this equation, in the sense of \([6]\), that is a \((\mathcal{F}_t)\)-predictable process \(X_t\), \(t \in [0, T]\) with continuous path in \(H\) such that \(\mathcal{P}\)-a.s.
\[
X_t = e^{tA}x + \int_0^t e^{(t-s)A}[E + B]u_s ds + \int_0^t e^{(t-s)A}(\lambda - A)D_1 d\tilde{W}_s + \int_0^t e^{(t-s)A}G dW_s, \quad t \in [0, T]
\] (2.2)
The cost functional, that depends on the initial state \(x\) and the control \(u \in \mathcal{U}\), to minimize is:
\[
J(x, u) = \mathbb{E} \int_0^T l(t, X_t, u_t) dt + \mathbb{E}h(X_T)
\] (2.3)
where \(l\) and \(h\) verify (B):
(B.1) \(l\) is measurable and for all \(t \in [0, T]\) and all \(u \in \mathcal{U}\), \(l(t, \cdot, u) \in \mathcal{G}^1(H; \mathbb{R}) \) and for all \(t \in [0, T]\) and all \(x \in H\), \(l(t, x, \cdot) \in \mathcal{G}^1(U; \mathbb{R}) \) and there is a constant \(\Delta > 0\) such that:
\[
|l_x(t, x, u)| + |l_u(t, x, u)| \leq \Delta(1 + |x|_H + |u|_U)
\] (2.4)
for all \(t \in [0, T], x \in H\) and \(u \in \mathcal{U}\).
(B.2) the map \(h\) is continuous and convex, moreover \(h \in \mathcal{G}^1(H; \mathbb{R})\) and there is a constant \(\Delta > 0\) such that:
\[
|h_x(x)| \leq \Delta(1 + |x|_H)
\] (2.5)
for all \(x \in H\). Moreover for some constant \(c_1 > 0\)
\[
|h_x(x_1) - h_x(x_2), x_1 - x_2|_H \leq -c_1|x_1 - x_2|^2, \text{ for any } x_1, x_2 \in H
\] (2.6)
(B.3) the map \(l\) can be decomposed as \(l(t, x, u) = l^0(t, x) + g(u)\), where \(l^0\) and \(g\) are two convex functions. Moreover for some constant \(c_1 > 0\)
\[
<l^0_x(t, x_1) - l^0_x(t, x_2), x_1 - x_2>_H \geq c_1|x_1 - x_2|^2, \text{ for any } x_1, x_2 \in H, t \in [0, T]
\] (2.7)
(B.4) for any \(t \in [0, T], x \in H, y \in D(E^*)\), we define
\[
H(t, x, u, y) := -l(t, x, u) + \langle [E + B]^*y, u \rangle,
\] and assume that there exists a function \(\gamma : H \to \mathcal{U}\) such that
\[
H(t, x, \gamma([E + B]^*y), y) = \inf_{u \in \mathcal{U}} H(t, x, u, y).
\] (2.8)
We assume moreover that there exist positive constants \(c_1\) and \(\Delta\):
\[
\langle \gamma(y_1) - \gamma(y_2), y_1 - y_2>_H \leq -c_1|y_1 - y_2|^2, \text{ for any } y_1, y_2 \in H
\] (2.9)
\[
|\gamma(y_1) - \gamma(y_2)|_H \leq \Delta|y_1 - y_2|, \text{ for any } y_1, y_2 \in H
\] (2.10)
2.3 Heat Equation with Neumann Boundary conditions

In this section we present a concrete stochastic control problem that we will be able to treat and we show how this model fits the “abstract” setting of section 2.2. We consider an heat equation on the interval \((0, \pi)\) with boundary noise and boundary control, and we focus our attention on the case where the control affects all the boundary, and the noise affects only one point at the boundary.

\[
\begin{align*}
\frac{\partial y}{\partial t}(t, \xi) &= \frac{\partial^2 y}{\partial \xi^2}(t, \xi) + b(\xi)u^0(t, \xi) + g(\xi)\dot{W}(t, \xi), \quad t \in [0, T], \, \xi \in (0, \pi), \\
y(0, \xi) &= x(\xi), \\
\frac{\partial y}{\partial \xi}(t, 0) &= u_1^t + \dot{W}_t, \quad \frac{\partial y}{\partial \xi}(t, \pi) = u_2^t
\end{align*}
\]  

(2.11)

In the above equation \(\dot{W}\) is a standard real Wiener process and \(\dot{W}(\tau, \xi)\) is a space-time white noise on \([0, T] \times [0, \pi]; \dot{W}\) and \(W\) are independent. We will give sense to the notion of solution in the following.

We reformulate equation (2.11) as a stochastic evolution equation in \(H = L^2(0, \pi)\). \(A\) stands for the Laplace operator with homogeneous Neumann boundary conditions, which is the generator of an analytic semigroup in \(H\):

\[
D(A) = \left\{ y \in H^2(0, \pi) : \frac{\partial y}{\partial \xi}(0) = \frac{\partial y}{\partial \xi}(\pi) = 0 \right\}, \quad Ay = \frac{\partial^2 y}{\partial \xi^2} \text{ for } y \in D(A).
\]

The control process \(u \in L^2_T(\Omega \times [0, T], U)\) where \(U = L^2(0, \pi) \times \mathbb{R}^2\) and \(u = \begin{pmatrix} u_0^t \\ u_1^t \\ u_2^t \end{pmatrix}\) We fix \(\lambda > 0\) and define

\[
b^1(\xi) = -\frac{\cosh(\sqrt{\lambda}(\pi - \xi))}{\sqrt{\lambda}\sinh(\sqrt{\lambda}\xi)}, \quad b^2(\xi) = \frac{\cosh(\sqrt{\lambda}\xi)}{\sqrt{\lambda}\sinh(\sqrt{\lambda}\xi)}
\]

and note that they solve the Neumann problems

\[
\begin{align*}
\frac{\partial^2 b^i}{\partial \xi^2}(\xi) &= \lambda b^i(\xi), \quad \xi \in (0, \pi), \, i = 1, 2, \\
\frac{\partial b^i}{\partial \xi}(0) &= 1, \quad \frac{\partial b^1}{\partial \xi}(\pi) = 0 \\
\frac{\partial b^i}{\partial \xi}(0) &= 0, \quad \frac{\partial b^1}{\partial \xi}(\pi) = 1.
\end{align*}
\]

So \(b^i \in D(\lambda - A)^\alpha = H^{2\alpha}\), for \(1/2 < \alpha < 3/4\).

Equation (2.11) can now be reformulated as:

\[
\begin{align*}
\{ X_t &= AX_t dt + [(\lambda - A)D + B]\alpha dt + (\lambda - A)D_1 d\dot{W}_t + G d\dot{W}_t + t \in [0, T] \\
X_0 &= x,
\end{align*}
\]  

(2.12)

where, for \(u \in U\) and \(h \in H\), \(Du = (0, b^1(\cdot)u^1(\cdot), b^2(\cdot)u^2(\cdot))\), \(D_1 = (0, b^1(\cdot)u^1(\cdot), 0)\), \(B = (b(\cdot), 0, 0)\), \(Gh = g(\cdot)h(\cdot)\). With the notations of section 2.2, \(K = \mathbb{R}\) and \(\Xi = H\).

Equation (2.12) is still formal, since \((\lambda - A)D\) and \((\lambda - A)D_1\) do not take their values in \(H\), the precise meaning of equation (2.12) is given by its mild formulation. An \(H\)-valued predictable process \(X\) is called a mild solution to equation (2.12) on \([0, T]\) if

\[
P \int_0^T |X_r|^2 dr < +\infty
\]

[7]
and, for every $0 < t < T$, $X$ satisfies the integral equation

$$X_t = e^{tA}x + \int_0^t e^{(t-r)A} [\lambda - A] D + B u_r dr + \int_0^t e^{(t-r)A} (\lambda - A) D_1 dW_r + \int_0^t e^{(t-r)A} G dW_r.$$  

Since $b' \in D(\lambda - A)\alpha = H^{2\alpha}$, for $1/2 < \alpha < 3/4$, and by the analyticity of the semigroup $e^{tA}$, $t \geq 0$, the integral $\int_0^t e^{(t-r)A} (\lambda - A) D u_r dr$ and the stochastic integral $\int_0^t e^{(t-r)A} (\lambda - A) D_1 dW_r$ are well defined, see also [8].

Notice that equation (2.12) does not satisfy any structure condition suitable to treat the related stochastic optimal control problem using backward stochastic differential equations, as in [8] and [24], where the case of an heat equation with Dirichlet boundary-control and boundary noise is considered. Notice that in the present example, differently from [8] and [23], the control affects the system in 0 and $\pi$ and the noise acts only at 0, so that Im($D_1$) $\subsetneq$ Im($D$).

The optimal control problem we wish to treat in this paper consists in minimizing the following finite horizon cost

$$J(x, u^0, u^1, u^2) = \mathbb{E} \int_0^T \int_0^\pi \bar{l}(s, \xi, y(s, \xi), u^0_s(\xi), u^1_s, u^2_s) \, d\xi \, ds + \mathbb{E} \left( \int_0^\pi \bar{h}(\xi, y(T, \xi)) \, d\xi \right).$$  

over all admissible controls. The cost functional (2.13) can be written in an abstract way as in (B.3) by setting, for $s \in [0, T]$, $x \in H, u \in U$

$$l(s, x, u) = \int_0^\pi l(s, \xi, x(\xi), u^0_s(\xi), u^1_s, u^2_s) \, d\xi \quad h(x) = \int_0^\pi \bar{h}(\xi, x(\xi)).$$

We consider costs such that $\bar{l}(s, \xi, y, u^0, u^1, u^2) = \bar{l}^0(s, \xi, y) + \bar{g}(\xi, u^0, u^1, u^2)$ so that $l$ can be decomposed as in (B.3). From $\bar{l}^0$ and $\bar{g}$ we define $l^0$ and $g$ as we have defined $l$:

$$l^0(s, x) = \int_0^\pi \bar{l}^0(s, \xi, x(\xi)) d\xi \quad g(u) = \int_0^\pi \bar{g}(\xi, u^0(\xi), u^1, u^2) d\xi.$$  

We make suitable assumptions on $\bar{l}^0, \bar{g}, \bar{h}$ such that $l^0, g$ and $h$ satisfy assumptions B1-B3.

**Hypothesis 2.2** We assume that:

1) the map $\bar{h} : [0, \pi] \times \mathbb{R} \to \mathbb{R}$, is measurable, for a.a. $\xi \in [0, \pi]$ $\bar{h}(\xi, \cdot) : \mathbb{R} \to \mathbb{R}$ is continuous, convex and differentiable and there exists $\Lambda \in L^\infty([0, \pi])$ such that

$$|\bar{h}_x(\xi, x)| \leq \Lambda(\xi)(1 + |x|).$$  

Moreover we assume that for a.a. $\xi \in [0, \pi]$ $\bar{h}(\xi, \cdot) : \mathbb{R} \to \mathbb{R}$ is dissipative, namely, for every $x_1, x_2 \in \mathbb{R}$

$$\langle \bar{h}(\xi, x_1) - \bar{h}(\xi, x_2) \rangle (x_1 - x_2) \leq -c_1(x_1 - x_2)^2,$$

for some positive constant $c_1$.

2) the map $\bar{l}^0 : [0, T] \times [0, \pi] \times \mathbb{R} \to \mathbb{R}$ is measurable and for a.a. $t \in [0, T]$ and $\xi \in [0, \pi]$, $\bar{l}^0(t, \xi, \cdot)$ is continuous, convex and differentiable, and there exists $\Lambda \in L^\infty([0, \pi])$ such that $\forall \xi \in [0, \pi]$ and $\forall x \in \mathbb{R}$

$$|\bar{l}^0_x(t, \xi, x)| \leq \Lambda(\xi)(1 + |x|).$$  

Moreover we assume that for a.a. $t \in [0, \pi]$ and $\xi \in [0, \pi]$ $\bar{l}^0(t, \xi, \cdot) : \mathbb{R} \to \mathbb{R}$ is dissipative, namely, for every $x_1, x_2 \in \mathbb{R}$

$$\langle \bar{l}^0(t, \xi, x_1) - \bar{l}^0(t, \xi, x_2) \rangle (x_1 - x_2) \geq -c_1(x_1 - x_2)^2,$$

for some positive constant $c_1$.  

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3\) the map \( \tilde{g} : [0, \pi] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is measurable and for a.a. \( \xi \in [0, \pi] \), \( \tilde{g}(\xi, \cdot, \cdot, \cdot) : \mathbb{R}^3 \to \mathbb{R} \) is continuous, convex and differentiable and there exists \( \Lambda \in L^\infty([0, \pi]) \) such that

\[
|\tilde{g}_a(\xi, u^0, u^1, u^2)| \leq \Lambda(\xi)(1 + |u^0|)
\]

and a constant \( c > 0 \) such that

\[
|\tilde{g}_a(\xi, u^0, u^1, u^2)| \leq c(1 + |u^i|), \quad i = 1, 2.
\]

3 Main results

In this section we come back to the abstract formulation of the problem, introducing the scheme we follow to find the optimal control: first we prove the maximum principle, then we prove that under our assumptions the condition is also sufficient and in the end we introduce the Hamiltonian system to be solved.

3.1 Maximum principle

Let us assume that there exists an optimal control \( \bar{u} \in \mathcal{U} \), under hypotheses stated previously we have that there exists a unique mild solution \( \bar{X} \) to (2.1) corresponding to \( \bar{u} \), see for instance [6]. So \((\bar{u}, \bar{X})\) is an optimal pair for the control problem described by (2.1) and (2.3). We introduce the following forward-backward system, composed by the state equation corresponding to the optimal control \( \bar{u} \) and its adjoint equation:

\[
\begin{align*}
    d\bar{X}_t &= A\bar{X}_t dt + [E + B]\bar{u}_t dt + (\lambda - A)D_1 d\bar{W}_t + G dW_t \\
    -d\bar{Y}_t &= A^T\bar{Y}_t dt + \langle \bar{Y}_0, \bar{X}_t \rangle dt - Z_t dW_t - \hat{Z}_t d\bar{W}_t, \\
    \bar{X}_0 &= x, \quad \bar{Y}_T = -h_x(\bar{X}_T)
\end{align*}
\]  

(3.1)

Once the forward equation is solved, the adjoint equation is a backward equation depending on the parameter \( \bar{X} \). The existence and uniqueness of a mild solution \( (\bar{Y}, (\bar{Z}, \bar{\bar{Z}})) \in L^2_p(\Omega; C([0, T]; H)) \times L^2_p([0, T] \times \Omega; L_2(\Xi \times K, H)) \) for such equation was firstly proved in see [16]. We collect the mentioned results in this proposition:

**Proposition 3.1** Assume \((A)\) and \((B)\). System (3.1) has a unique mild solution \((\bar{X}, \bar{Y}, \bar{Z})\). Moreover

\[
\sup_{t \in [0, T]} \mathbb{E}(T-t)^{2(1-\alpha)}\|\bar{Y}\|_{D(E^*)}^2 < +\infty
\]  

(3.2)

**Proof.** The regularity result can be proved as in proposition 3.1 of [17]. \( \square \)

**Theorem 3.2** Assume \((A)\) and \((B)\). Let \((\bar{u}, \bar{X})\) be an optimal pair for the problem (2.1) and (2.3). Then there exists a unique pair \((\bar{Y}, (\bar{Z}, \bar{\bar{Z}})) \in L^2_p(\Omega; C([0, T]; H)) \times L^2_p([0, T] \times \Omega; L_2(\Xi \times K, H)) \) solution of equation (3.1) such that:

\[
\langle H_u(t, \bar{X}_t, \bar{u}_t, Y_t), v - \bar{u}_t \rangle \leq 0, \quad \forall v \in U, \ a.e. \ t \in [0, T], \ P - a.s.
\]  

(3.3)

where

\[
H(t, x, u, p) := \langle (E + B)^*p, u \rangle_H - l(t, x, u), \quad (t, x, u, p) \in [0, T] \times H \times U \times D(E^*), \ \lambda > \omega
\]

**Proof.** The result follows from theorem 4.6 of [14] taking \( F_x \) and \( G_x \) equal to zero; the presence of the bounded operator \( B \) does not introduce any new difficulty. The proof follows exactly in the same way. \( \square \)
3.2 Sufficient condition for optimality

Now we present the following sufficient condition of optimality. Let us consider the forward-backward system (3.1): for any admissible control $\bar{v} \in \mathcal{U}$ there exists a solution $(\bar{X}, \bar{Y}, (\bar{Z}, \bar{Z}))$ we say then that $(\bar{v}, \bar{X}, \bar{Y}, (\bar{Z}, \bar{Z}))$ is an admissible 4-tuple.

**Theorem 3.3** Assume (A) and (B). Let $(\bar{u}, \bar{X}, \bar{Y}, (\bar{Z}, \bar{Z}))$ an admissible 4-tuple. If
\[
\langle H_n(t, \bar{X}_t, \bar{u}_t, \bar{Y}_t), v - \bar{u}_t \rangle \leq 0, \quad \forall v \in U, \text{ a.e. } t \in [0, T], \ P - \text{a.s.}
\]
then $(\bar{u}, \bar{X})$ is optimal for problem (2.1) and (2.3).

**Proof.** Let $\bar{v} \in \mathcal{U}$ hence $\bar{u} + \lambda(\bar{v} - \bar{u}) \in \mathcal{U}$, for all $\lambda \in [0, 1]$. Being the state equation affine, we have that $\bar{X}^{\bar{u}} + \lambda(\bar{v} - \bar{u}) = \bar{X} + \lambda \bar{X}^{\bar{v} - \bar{u}}$, where $\bar{X}^{\bar{v} - \bar{u}}$ solves the following equation
\[
\left\{ \begin{array}{l}
d\bar{X}^{\bar{v} - \bar{u}}_t = A\bar{X}^{\bar{v} - \bar{u}}_t dt + (E + B)(\bar{v}_t - \bar{u}_t) dt \\
\bar{X}^{\bar{v} - \bar{u}}_0 = 0,
\end{array} \right.
\]
that is, in mild form,
\[
\bar{X}^{\bar{v} - \bar{u}}_t = \int_0^t e^{(t-s)A}(E + B)(\bar{v}_s - \bar{u}_s) ds
\]
Therefore by the convexity assumption of $H_n$ we end up with:
\[
J(x, \bar{u}) - J(x, \bar{u} + \lambda(\bar{v} - \bar{u})) = \mathbb{E} \int_0^T [l(t, \bar{X}_t, \bar{u}_t) - l(t, \bar{X}_t + \lambda \bar{X}^{\bar{v} - \bar{u}}_t, \bar{u}_t + \lambda(\bar{v}_t - \bar{u}_t))] dt + \mathbb{E}[h(\bar{X}_T) - h(\bar{X}_T + \lambda \bar{X}^{\bar{v} - \bar{u}}_T)]
\]
\[
\leq - \mathbb{E} \int_0^T \lambda l_0^0(t, \bar{X}_t) \bar{X}^{\bar{v} - \bar{u}}_t dt - \mathbb{E} \int_0^T \lambda \langle g_n(\bar{v}_t), \bar{v}_t - \bar{u}_t \rangle dt
\]
\[
- \mathbb{E}\lambda h_x(\bar{X}_T, \bar{X}^{\bar{v} - \bar{u}}_T).
\]

Now following the usual approximation strategy we multiply both equations for $\bar{X}^{\bar{v} - \bar{u}}$ and $\bar{Y}$ by $n(n - A)^{-1} = nR(n, A)$ for $n > \lambda$, so that the two processes $\bar{X}^{\bar{v} - \bar{u}, n} := nR(n, A)\bar{X}^{\bar{v} - \bar{u}}$ and $\bar{Y}^n := nR(n, A)\bar{Y}$ both admit an Itô differential:
\[
d(\bar{X}^{\bar{v} - \bar{u}, n}_t, \bar{Y}^n_t) = (\bar{Y}^n_t, nR(n, A)[E + B](\bar{v}_t - \bar{u}_t)) dt + (nR(n, A)l_0^0(t, \bar{X}_t), \bar{X}^{\bar{v} - \bar{u}, n}_t) dt
\]
Observing that $D(E^*) \equiv D((\lambda - A)^{1-\alpha})$, we can let $n$ tend $\infty$ and we get that:
\[
- \mathbb{E}\langle h_x(\bar{X}_T), \bar{X}^{\bar{v} - \bar{u}}_T \rangle - \mathbb{E} \int_0^T \langle l_0^0(t, \bar{X}_t), \bar{X}^{\bar{v} - \bar{u}}_t \rangle dt = \mathbb{E} \int_0^T \langle (v_t - u_t), [E + B]^{*}\bar{Y}_t \rangle dt
\]
Notice that
\[
\mathbb{E} \int_0^T \langle nR(n, A)[E + B](v_t - u_t), \bar{Y}^n_t \rangle dt
\]
makes sense since $\bar{Y}^n_t = nR(n, A)\bar{Y}_t$ and $\bar{Y}_t \in D(E) = D((\lambda - A)^{1-\alpha})$, so also $\bar{Y}^n_t \in D(E)$ and also $nR(n, A^*)\bar{Y}^n_t \in D(E)$. So
\[
\mathbb{E} \int_0^T \langle nR(n, A)[E + B](v_t - u_t), \bar{Y}^n_t \rangle dt = \mathbb{E} \int_0^T \langle (v_t - u_t), [E + B]^*nR(n, A^*)\bar{Y}^n_t \rangle dt
\]
is well defined. Now, thanks to 3.3, taking $\lambda = 1$ we get:
\[
J(x, \bar{u}) \leq J(x, \bar{v}), \quad \text{for all } \bar{v} \in \mathcal{U}.
\]
3.3 Hamiltonian System

Let us introduce the Hamiltonian system associated to our control problem

\[
\begin{aligned}
    d\bar{X}_t &= A\bar{X}_t \, dt + [E + B]\gamma((E + B)^*\bar{Y}_t) \, dt + (\lambda - A)D_t \, d\tilde{W}_t + G(t, \bar{X}_t) \, dW_t \\
    -d\bar{Y}_t &= A^T\bar{Y}_t \, dt + l^B(t, \bar{X}_t) \, dt - \bar{Z}_t \, dW_t - \tilde{Z}_t \, d\tilde{W}_t, \quad t \in [0, T] \\
    \bar{X}_0 &= x, \quad \bar{Y}_T = -h_x(\bar{X}_T),
\end{aligned}
\]  

(3.6)

where \( \gamma \) has been defined in (2.8). Section 3 is devoted to prove the following result:

**Theorem 3.4** Assume (A) and (B), then there exists a unique solution \((\bar{X}, \bar{Y}, (\bar{Z}, \tilde{Z})) \in L_2^2P((0, T) \times \Omega; H) \times L_2^2P((0, T) \times \Omega; D(E)) \times L_2^2P((0, T) \times \Omega; L_2(\Xi \times K; H)) \) of the forward-backward system (3.6). Moreover we have that:

\[
    \sup_{t \in [0, T]} (T - t)^{1-a} \|\bar{Y}_t\|_{D(E^*)} < +\infty
\]

(3.7)

By the definition of the map \( \gamma \) we deduce that \((\gamma([E + B]^*\bar{Y}), \bar{X}, \bar{Y}, (\bar{Z}, \tilde{Z})) \) is an admissible 4-tuple.

3.4 Main result

We can now state the main result of the paper.

**Theorem 3.5** Assume (A) and (B). There exists a unique optimal pair given by the solution of system (3.6) for the control problem (2.1) and (2.3).

**Proof.** Thanks to theorem 3.4 we have an admissible 4-tuple \((\gamma([E + B]^*\bar{Y}, \bar{X}, \bar{Y}, (\bar{Z}, \tilde{Z})) \) that, by definition of \( \gamma \), verifies condition (3.4). So from theorem 3.3 we deduce the thesis. \( \square \)

4 Proof of theorem 3.4

System (3.6) is an infinite dimensional fully coupled forward-backward system. Besides the difficulties typical of the finite dimensional FBSDEs, see [22], there are some additional ones due to the presence of unbounded operators. In particular we need to introduce the graph norm of \( E \) and prove some crucial estimates with respect to this stronger norm. Thanks to dissipativity hypotheses (2.6), (2.7) and (2.9), the more suitable method to get a solution is the bridge method used in [19] whose infinite dimensional extension will be described in next paragraph.

4.1 The bridge method applied to an infinite dimensional system

This section is devoted to present the bridge method to solve the Hamiltonian system (3.6), which is a FBSDE in an infinite dimensional Hilbert space \( H \). According to this method, introduced in [19], in order to solve a nonlinear fully coupled FBSDE, a linear auxiliary FBSDE is studied and then making a sort of convex combination between the affine term in this linear FBSDE and the nonlinear terms in the original FBSDE it is possible to arrive at the solution of the original FBSDE.

The main difference between the present paper and [19] is that in [19] the finite dimensional case is treated and so the linear auxiliary FBSDE has a very special structure and is solvable by hand; in the present paper, since also \( Y \) takes its values in \( H \), the auxiliary linear affine FBSDE has a different
structure and it takes some efforts to be solved see section 4.2. Namely, let \( b_0, h_0 \in L^2_P([0, T] \times \Omega; H) \) and \( g_0 \in L^2(\Omega, \mathcal{F}_T; H) \), consider:

\[
\begin{aligned}
    d\bar{X}_t &= A\bar{X}_t dt - [E + B][E + B]^\ast Y_t dt + b_0(t) dt + (\lambda - A)D_1 d\bar{W}_t + G dW_t \\
    -d\bar{Y}_t &= A^\ast \bar{Y}_t dt + \bar{X}_t dt + h_0(t) dt - \bar{Z}_t dW_t - \bar{Z}_t d\bar{W}_t, \quad t \in [0, T] \\
    \bar{X}_0 &= x, \quad -\bar{Y}_T = \bar{X}_T + g_0
\end{aligned}
\]  

(4.1)

In the next section we prove the following proposition, according to which \((4.1)\) admits a unique solution. The difficulties in solving this FBSDE comes at first by the fact that the BSDE contains \( Y \) itself, unlike in [19], and by the presence of the unbounded term \( [E + B][E + B]^\ast Y \).

**Proposition 4.1** Let \( b_0, h_0 \in L^2_P([0, T] \times \Omega; H) \) and \( g_0 \in L^2(\Omega, \mathcal{F}_T; H) \), let also \( A, E, B, D_1 \) and \( G \) satisfy assumptions (A), then the linear FBSDE \((4.1)\) admits a unique mild solution \((\bar{X}, \bar{Y}, (\bar{Z}, \bar{Z})) \in L^2_P(\Omega; C([0, T]; H)) \times L^2_P(\Omega; C([0, T]; H)) \times L^2_P(\Omega \times [0, T]; L^2(\Xi \times K; H))\) satisfying moreover

\[
\mathbb{E} \sup_{t \in [0, T]} (T - t)^{2(1-\alpha)} \| \bar{Y}_t \|_{D((\lambda - A^\ast)^{1-\alpha})}^2 < +\infty.
\]

The proof of this proposition is given in the next section.

The aim of the present section is to prove the following result on the bridge method, in which the solution of the FBSDE \((4.1)\) is in some sense connected to the solution of the starting FBSDE \((3.6)\).

Namely, let us define, for \( x \in H, y \in H \cap D(E) \) and for \( \alpha \in [0, 1] \),

\[
\begin{aligned}
    b^\alpha(y) &= \alpha[E + B]\gamma([E + B]^\ast y) + (1 - \alpha)[E + B][E + B]^\ast (-y) \\
    h^\alpha(t, x) &= \alpha h_x(t, x) + (1 - \alpha)(x) \\
    g^\alpha(x) &= \alpha h_x(x) - (1 - \alpha)(x).
\end{aligned}
\]

(4.2)

Consider the following FBSDE:

\[
\begin{aligned}
    d\bar{X}_t &= A\bar{X}_t dt + b^\alpha(\bar{Y}_t) dt + b_0(t) dt + (\lambda - A)D_1 d\bar{W}_t + G dW_t \\
    -d\bar{Y}_t &= A^\ast \bar{Y}_t dt + h^\alpha(\bar{X}_t) dt + h_0(t) dt - \bar{Z}_t dW_t - \bar{Z}_t d\bar{W}_t, \quad t \in [0, T] \\
    \bar{X}_0 &= x, \quad -\bar{Y}_T = g^\alpha(\bar{X}_t) + g_0
\end{aligned}
\]  

(4.3)

This is, with \( \alpha \) varying in \([0, 1]\), the systems that links the linear FBSDE \((4.1)\) to the original FBSDE \((3.6)\).

Notice that, as stated in proposition 4.1, the linear FBSDE \((4.1)\), which is equal to the FBSDE \((3.3)\) with \( \alpha = 0 \), admits an adapted solution satisfying moreover \( \mathbb{E} \sup_{t \in [0, T]} (T - t)^{2(1-\alpha)} \| [E + B]\bar{Y}_t \|^2 < +\infty. \) In the next lemma we prove that \((3.3)\) admits a solution.

**Lemma 4.2** Let \( A, E, B, D_1 \) and \( G \) satisfy assumptions (A) and \( \gamma, l \) and \( h \) satisfy assumptions (B). Assume that for some \( \alpha = \alpha_0 \) and for any \( b_0, h_0 \in L^2_P((0, T) \times \Omega; H) \) and any \( g_0 \in L^2(\Omega, \mathcal{F}_T; H) \) equations \((3.3)\) admit a mild solution \((\bar{X}, \bar{Y}, (\bar{Z}, \bar{Z})) \in L^2_P(\Omega; C([0, T]; H)) \times L^2_P(\Omega; C([0, T]; H)) \times L^2_P(\Omega \times [0, T]; L^2(\Xi \times K; H))\) satisfying moreover

\[
\mathbb{E} \sup_{t \in [0, T]} (T - t)^{2(1-\alpha)} \| \bar{Y}_t \|_{D((\lambda - A^\ast)^{1-\alpha})}^2 < +\infty.
\]
Then there exists $\delta_0 \in (0,1)$ depending only on constants appearing in (A) and (B) and for all $\alpha \in [\alpha_0, \alpha_0 + \delta]$ and for any $b_0, h_0 \in L^2_p([0, T] \times \Omega; H)$ and any $g_0 \in L^2(\Omega, \mathcal{F}_T; H)$ FBSDE (4.3) admits a mild solution $(X, Y, (Z, \tilde{Z})) \in L^2_p(\Omega; C([0, T]; H)) \times L^2_p(\Omega; C([0, T]; H)) \times L^2_p(\Omega \times [0, T]; L_2(\Xi \times K; H))$ satisfying moreover
\[
\mathbb{E} \sup_{t \in [0, T]} (T-t)^{(1-\alpha)}\|\tilde{Y}_t\|_{D((\lambda-A^*)^{1-\alpha})}^2 < +\infty.
\]

Proof. We notice that for $\alpha = \alpha_0 + \delta$ coefficients in (4.2) can be rewritten as
\[
\begin{align*}
\ell^{\alpha_0+\delta}(y) &= b^{\alpha_0}(y) + \delta[E + B] \gamma([E + B]^T y) + \delta[E + B][E + B]^T y \\
h^{\alpha_0+\delta}(t, x) &= h^{\alpha_0}(t, x) + \delta l_0^0(t, x) - \delta x \\
g^{\alpha_0+\delta}(x) &= g^{\alpha_0}(x) + \delta h^0(x) + \delta x.
\end{align*}
\]
Notice that by our assumptions it follows that for $\alpha = \alpha_0$ the FBSDE (4.3) admits a mild solution. From this we start proving that there exists $\delta_0 \in (0,1)$ such that for all $\delta \in [0, \delta_0]$, for all $\alpha \in [\alpha_0, \alpha_0 + \delta_0]$ and for any $b_0, h_0 \in L^2_p([0, T] \times \Omega; H)$ and $g_0 \in L^2(\Omega, \mathcal{F}_T; H)$ the FBSDE (4.3) admits a unique mild solution $(\bar{X}, \bar{Y}, (\bar{Z}, \tilde{Z})) \in L^2_p(\Omega; C([0, T]; H)) \times L^2_p(\Omega; C([0, T]; H)) \times L^2_p(\Omega \times [0, T]; L_2(\Xi \times K; H))$ satisfying moreover
\[
\mathbb{E} \sup_{t \in [0, T]} (T-t)^{(1-\alpha)}\|\bar{Y}_t\|_{D((\lambda-A^*)^{1-\alpha})}^2 < \infty.
\]

We set $(\bar{X}^0, \bar{Y}^0, (\tilde{Z}^0, \tilde{Z}^0)) = (0, 0, (0, 0))$. For $j \geq 0$ we solve iteratively the following FBSDEs
\[
\begin{align*}
&d\bar{X}^{j+1}_t = A\bar{X}^{j+1}_t dt + \left(\alpha_0[E + B] \gamma([E + B]^* \bar{Y}^{j+1}_t) - (1 - \alpha_0)[-E + B][E + B]^* \bar{Y}^{j+1}_t\right) dt \\
&\quad + \left(\delta[E + B] \gamma([E + B]^* \bar{X}^{j+1}_t) + \delta[E + B][E + B]^* \bar{Y}^{j+1}_t\right) dt \\
&\quad + b_0(t) dt + (\lambda - A) D_1 d\bar{W}_t + G dW_t \\
&-d\bar{Y}^{j+1}_t = A\gamma(X^{j+1}_t) dt + \bar{h}_0(t) dt + \alpha_0 l_0^0(t, \bar{X}^{j+1}_t) dt + (1 - \alpha_0) \bar{X}^{j+1}_t dt \\
&\quad + \delta(\bar{l}_0^0(t, \bar{X}^{j}_t) - \bar{X}^{j}_t) dt - \bar{Z}^{j+1}_t dW_t - \tilde{Z}^{j+1}_t d\bar{W}_t, \quad t \in [0, T] \\
&\bar{X}^{j+1}_0 = x, \quad -\bar{Y}^{j+1}_T = g^{\alpha_0}(\bar{X}^{j+1}_T) + \delta h_x(\bar{X}^{j}_T) + \delta \bar{X}^{j}_T + g_0.
\end{align*}
\]

Notice that by induction, and by generalizing some statements from the strong to the mild formulation, see few lines below, such a FBSDE admits a mild solution $(\bar{X}^{j+1}, \bar{Y}^{j+1}, (\tilde{Z}^{j+1}, \tilde{Z}^{j+1})) \in L^2_p(\Omega; C([0, T]; H)) \times L^2_p(\Omega; C([0, T]; H)) \times L^2_p(\Omega \times [0, T]; L_2(\Xi \times K; H))$ satisfying moreover
\[
\mathbb{E} \sup_{t \in [0, T]} (T-t)^{(1-\alpha)}\|\bar{Y}^{j+1}_t\|_{D((\lambda-A^*)^{1-\alpha})}^2 < \infty.
\]

Indeed, for $j = 1$, FBSDE (4.4) is equal to FBSDE (4.3). So by hypothesis the solution, with the required regularity, exists. By induction, assume that for $j$ a solution, with the required regularity exists, and we show that also for $j + 1$ a solution exists. By setting $\bar{b}_0(t) = \delta[E + B] \gamma([E + B]^* \bar{Y}^{j}_t) + \delta[E + B][E + B]^* \bar{Y}^{j}_t + \bar{h}_0(t); \bar{h}_0(t) = \delta(\bar{l}_0^0(t, \bar{X}^{j}_t) - \bar{X}^{j}_t) + h_0(t); \bar{g}_0 = \delta h_x(\bar{X}^{j}_T) + \delta \bar{X}^{j}_T + g_0$, FBSDE (4.4) is equal to FBSDE (4.3) with $\bar{b}_0, \bar{h}_0$ and $\bar{g}_0$ in the place of $b_0, h_0$ and $g_0$ respectively. This time $\bar{b}_0 \notin L^2_p(\Omega \times [0, T]; H)$, indeed $\bar{b}_0$ is not well defined as an element of $H$. Nevertheless, in the
mild formulation of $X_t, \tilde{b}_0$ appears in integral form and it is affected by the regularizing properties of the semigroup: the integral $\int_t^T e^{(s-t)A} \tilde{b}_0(s) \, ds$ is well defined and bounded in $L^2_p(\Omega, L^2_p(\Omega; C([0,T]; H)) \cap L^2_p(\Omega; C([0,T]; H))$.

So by our assumptions, $\forall j \geq 0$, there exists a mild solution $(\tilde{X}^{j+1}, \tilde{Y}^{j+1}, (\tilde{Z}^{j+1}, \tilde{\tilde{Z}}^{j+1})) \in L^2_p(\Omega; C([0,T]; H)) \times L^2_p(\Omega; C([0,T]; H))$\times $L^2_p(\Omega \times [0,T]; L_2(\Omega \times [0,T]; H))$ satisfying moreover $\mathbb{E} \sup_{t \in [0,T]} (T-t)^{2(1-\alpha)/2} ||\tilde{X}^{j+1}||_{D((\lambda-A)^{1-\alpha})} < \infty$.

Next we define, for every $t \in [0,T]$,

$$
\tilde{X}^{j+1}_t = \tilde{X}^{j+1}_0 - \tilde{X}^{j}_t, \quad \tilde{Y}^{j+1}_t = \tilde{Y}^{j+1}_0 - \tilde{Y}^{j}_t, \quad \tilde{Z}^{j+1}_t = \tilde{Z}^{j+1}_0 - \tilde{Z}^{j}_t, \quad \tilde{\tilde{Z}}^{j+1}_t = \tilde{\tilde{Z}}^{j+1}_0 - \tilde{\tilde{Z}}^{j}_t.
$$

We note that $(\tilde{X}^{j+1}, \tilde{Y}^{j+1}, (\tilde{Z}^{j+1}, \tilde{\tilde{Z}}^{j+1}))$ solve

$$
d\tilde{X}^{j+1}_t = A\tilde{X}^{j+1}_t \, dt + \alpha_0(E + B)\gamma(\gamma([E + B]^{*}\tilde{Y}^{j+1}_t) - \gamma([E + B]^{*}\tilde{Y}^{j}_t)) \, dt
$$

$$
-d\tilde{Y}^{j+1}_t = A^{*}\tilde{Y}^{j+1}_t \, dt + \alpha_0(l^0_x(t, \tilde{X}^{j+1}_t) - l^0_x(t, \tilde{X}^{j}_t)) \, dt + (1 - \alpha_0)\tilde{X}^{j+1}_t \, dt
$$

$$
\tilde{X}^{j+1}_0 = x,
$$

We notice that by our assumption for every $j$, $\mathbb{E} \sup_{t \in [0,T]} |\tilde{X}^{j+1}_t|^2 < +\infty$.

Next we have to apply Itô formula: in order to do this we have to approximate $X$ and $Y$ with elements of the domain of $A$. Namely, for $n > \lambda$, we denote as usual $R(n, A) := (n - A)^{-1}$. We set $(\hat{X}^{n,j+1}, \hat{Y}^{n,j+1}, (\hat{Z}^{n,j+1}, \hat{\tilde{Z}}^{n,j+1})) = (nR(n, A)\tilde{X}^{j+1}, nR(n, A)\tilde{Y}^{j+1}, nR(n, A)(\tilde{Z}^{j+1}, nR(n, A)\tilde{\tilde{Z}}^{j+1}))$.

We also denote $E_n + B_n := nR(n, A)(E + B)$ and we note that $(\hat{X}^{n,j+1}, \hat{Y}^{n,j+1}, (\hat{Z}^{n,j+1}, \hat{\tilde{Z}}^{n,j+1}))$ solve the following

$$
d\hat{X}^{n,j+1}_t = A\hat{X}^{n,j+1}_t \, dt + \alpha_0(E_n + B_n)\gamma(\gamma([E + B]^{T}\hat{Y}^{j+1}_t) - \gamma([E + B]^{T}\hat{Y}^{j}_t)) \, dt
$$

$$
-d\hat{Y}^{n,j+1}_t = A^{*}\hat{Y}^{n,j+1}_t \, dt + \alpha_0nR(n, A)(l^0_x(t, \hat{X}^{j+1}_t) - l^0_x(t, \hat{X}^{j}_t)) \, dt + (1 - \alpha_0)\hat{X}^{n,j+1}_t \, dt
$$

$$
\hat{X}^{n,j+1}_0 = x,
$$

$$
\hat{Y}^{n,j+1}_T = \alpha_0nR(n, A)(h_x(\hat{X}^{j+1}_T) - h_x(\hat{X}^{j}_T)) - (1 - \alpha_0)\hat{X}^{n,j+1}_T + \alpha_0nR(n, A)(h_x(\hat{X}^{j+1}_T) - h_x(\hat{X}^{j}_T)) + \delta\hat{X}^{n,j}_T.
$$
By applying Itô formula to $\langle \hat{X}^n_{t}, \hat{Y}^n_{t} \rangle$, and then integrating over $[0, T]$ and taking expectation we get
\[
-E(\hat{X}^n_{T} - X^n_{0}) - \int_0^T \langle [E + B]^* \hat{X}^n_{t} - X^n_{t}, E + B \rangle \, dt
\]
\[
= \int_0^T \langle (1 - \alpha_0)(nR(n, A) - I)|E + B||[E + B]^* \hat{Y}^n_{t} - Y^n_{t}| \, dt
\]
\[
\Rightarrow\int_0^T \langle (1 - \alpha_0)(nR(n, A) - I)|E + B||[E + B]^* \hat{Y}^n_{t} - Y^n_{t}| \, dt.
\]
In a similar way we can get that $\dot{Y}_{n,j+1} \to \dot{Y}_{t,j+1}$ in $L^2_\mathcal{P}(\Omega \times [0,T]; H)$ and moreover as $n \to \infty$:

$$
\mathbb{E} \sup_{t \in [0,T]} (T - t)^{(1-\alpha)} \| (\dot{Y}_{n,j+1} - \dot{Y}_{t,j+1}) \|^2_{D((\lambda - A^\gamma)^{-\alpha})} \to 0.
$$

In order to let $n \to \infty$ in (1.5) we have also to show that $\mathbb{E} \sup_{t \in [0,T]} (T - t)^{(1-\alpha)} \| [E_n + B_n]^\ast \dot{Y}_{n,j+1} - [E + B]^\ast \dot{Y}_{t,j+1} \|^2 \to 0$ as $n \to \infty$. Notice that,

$$
\begin{align*}
&[E_n + B_n]^\ast \dot{Y}_{n,j+1} - [E + B]^\ast \dot{Y}_{t,j+1} \\
&= -\mathbb{E}^F e^{(T-t)A^\ast} \mathbb{E}^F \left( \alpha_0 \left( [E_n + B_n]^\ast nR(n, A) - [E + B]^\ast \right) (h_x(\bar{X}_{T}^{j+1}) - h_x(\bar{X}_{T}^j)) \\
&- (1 - \alpha_0) \left( nR(n, A) - I \right) \dot{X}_{T}^{j+1} + \delta(nR(n, A) - I) (h_x(\bar{X}_{T}^{j+1}) - h_x(\bar{X}_{T}^{j-1})) + \delta(nR(n, A) - I) \dot{X}_{T}^{j} \right) \\
&+ \mathbb{E}^F \int_t^T \left( [E_n + B_n]^\ast nR(n, A) - [E + B]^\ast \right) \delta(nR(n, A) - I) (h_x(\bar{X}_{T}^{j+1}) - h_x(\bar{X}_{T}^{j-1})) + (1 - \alpha_0) \dot{X}_{T}^{j+1} \right) ds \\
&+ \mathbb{E}^F \int_t^T \delta(nR(n, A) - I) (h_x(\bar{X}_{T}^{j+1}) - h_x(\bar{X}_{T}^{j-1}) - \delta \dot{X}_{T}^j) ds.
\end{align*}
$$

So we can let $n \to \infty$ in (1.5) and we get

$$
\begin{align*}
&- \mathbb{E} \langle \dot{X}_{T}^{j+1}, \alpha_0 (h_x(\bar{X}_{T}^{j+1}) - h_x(\bar{X}_{T}^j)) \rangle -(1 - \alpha_0) \dot{X}_{T}^j \\
&- \delta(\dot{X}_{T}^{j+1} - h_x(\bar{X}_{T}^{j-1}) + \dot{X}_{T}^j, \dot{X}_{T}^j) \\
&= \mathbb{E} \int_0^T \left[ \langle \alpha_0 [E + B][\gamma([E + B]^\ast \dot{Y}_{T}^{j+1}) - \gamma([E + B]^\ast \dot{Y}_{T}^j]), \dot{Y}_{T}^{j+1} \rangle \\
&+ (1 - \alpha_0) \langle [E + B][E + B]^\ast \dot{Y}_{T}^{j+1}, \dot{Y}_{T}^{j+1} \rangle \right] dt \\
&- \mathbb{E} \int_0^T \left[ \alpha_0 \left( \langle l_0^j(t, \bar{X}_{T}^{j+1}) - l_0^j(t, \bar{X}_{T}^{j}), \dot{X}_{T}^{j+1} \rangle + (1 - \alpha_0) \dot{X}_{T}^{j+1} \rangle \langle [E + B][E + B]^\ast \dot{Y}_{T}^{j+1}, [E + B]^\ast \dot{Y}_{T}^{j+1} \rangle dt \\
&- \delta \mathbb{E} \int_0^T \left( \langle l_0^j(t, \bar{X}_{T}^{n,j}) - l_0^j(t, \bar{X}_{T}^{n,j}), \dot{X}_{T}^{n,j} \rangle + (1 - \alpha_0) \dot{X}_{T}^{n,j} \rangle \right) dt.
\end{align*}
$$

So, by assumptions (B) we get

$$
\begin{align*}
\min \{ c_1, 1 \} &\mathbb{E} \langle \dot{X}_{T}^{j+1} \rangle^2 \\
&\leq \delta \mathbb{E} \langle \dot{X}_{T}^{j+1} \rangle^2 + \mathbb{E} c_0 \mathbb{E} \int_0^T \langle [E + B][E + B]^\ast \dot{Y}_{T}^{j+1} \rangle^2 dt - (1 - \alpha_0) \mathbb{E} \int_0^T \langle [E + B]^\ast \dot{Y}_{T}^{j+1} \rangle^2 dt \\
&+ \delta(\Delta + 1) \mathbb{E} \int_0^T \langle [E + B][E + B]^\ast \dot{Y}_{T}^{j+1} \rangle \dot{Y}_{T}^{j+1} \rangle dt - (\alpha_0 c_1 + 1 - \alpha_0) \mathbb{E} \int_0^T \dot{X}_{T}^{j+1} \rangle^2 \rangle dt \\
&+ \delta \mathbb{E} \int_0^T \mathbb{E} \langle \dot{X}_{T}^{j+1} \rangle \dot{X}_{T}^{j+1} \rangle dt.
\end{align*}
$$

By applying Young inequalities several times we finally get

$$
\begin{align*}
\mathbb{E} \langle \dot{X}_{T}^{j+1} \rangle^2 + \mathbb{E} \int_0^T \langle [E + B][E + B]^\ast \dot{Y}_{T}^{j+1} \rangle^2 dt + \mathbb{E} \int_0^T \dot{X}_{T}^{j+1} \rangle^2 \rangle dt \\
&\leq c'(\delta, \Delta, c_1) \mathbb{E} \langle \dot{X}_{T}^{j+1} \rangle^2 + c'(\delta, c_1) \mathbb{E} \int_0^T \langle [E + B][E + B]^\ast \dot{Y}_{T}^{j+1} \rangle^2 dt + c'(\delta, \Delta, c_1) \mathbb{E} \int_0^T \langle [E + B]^\ast \dot{Y}_{T}^{j+1} \rangle^2 dt,
\end{align*}
$$

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where \( c'(\delta, \Delta, c_1) \) and \( c'(\delta, c_1) \) are constants depending respectively only on \( \delta, \Delta, c_1 \) and \( \delta, c_1 \) respectively. Now notice that

\[
\begin{align*}
\dot{X}_t^j &= \alpha_0 \int_0^T e^{(T-t)A}[E + B](\gamma([E + B]^{\hat{Y}}_j) - \gamma([E + B]^{\hat{Y}_j-1}_t)) \, dt \\
&+ (1 - \alpha_0) \int_0^T e^{(T-t)A}[E + B][E + B]^{\hat{Y}}_j \, dt + \delta \int_0^T e^{(T-t)A}[E + B][E + B]^{\hat{Y}_j-1}_t \, dt \\
&+ \delta \int_0^T e^{(T-t)A}[E + B](\gamma([E + B]^{\hat{Y}}_j) - \gamma([E + B]^{\hat{Y}_j-2}_t)) \, dt.
\end{align*}
\]

So

\[
\begin{align*}
\mathbb{E}|\dot{X}_t^j|^2 &\leq \alpha_0^2 \mathbb{E}\int_0^T \|e^{(T-t)A}[E + B]\| \|\Delta\|[E + B]^{\hat{Y}}_j| \, dt^2 \\
&+ (1 - \alpha_0)^2 \mathbb{E}\int_0^T c(T-t)^{-(1-\alpha)}\|[E + B]^{\hat{Y}}_j| \, dt^2 + \delta^2 \mathbb{E}\int_0^T \|[E + B]^{\hat{Y}}_j\| \|\delta\|[E + B]^{\hat{Y}_j-1}_t \, dt^2 \\
&+ \alpha_0 \mathbb{E}\int_0^T (T-t)^{-(1-\alpha)}\|[E + B]^{\hat{Y}_j} \, dt^2 + \delta^2 \mathbb{E}\int_0^T (T-t)^{-(1-\alpha)}\|[E + B] \|\delta\|[E + B]^{\hat{Y}_j-1} \, dt^2 \\
&\leq c\Delta^2 \alpha_0^2 \mathbb{E}\sup_{t \in [0,T]} (T-t)^{2(1-\alpha)}\|[E + B]^{\hat{Y}}_j|^2 \left( \int_0^T (T-t)^{-2(1-\alpha)} \, dt \right)^2 \\
&+ c(1 - \alpha_0)^2 \mathbb{E}\sup_{t \in [0,T]} (T-t)^{2(1-\alpha)}\|[E + B]^{\hat{Y}}_j|^2 \left( \int_0^T (T-t)^{-2(1-\alpha)} \, dt \right)^2 \\
&+ \alpha_0 \mathbb{E}\sup_{t \in [0,T]} (T-t)^{2(1-\alpha)}\|[E + B]^{\hat{Y}_j} \, dt^2 + \delta^2 \mathbb{E}\sup_{t \in [0,T]} (T-t)^{2(1-\alpha)}\|[E + B]^{\hat{Y}_j} \, dt^2 \\
&+ \alpha_0 \mathbb{E}\sup_{t \in [0,T]} (T-t)^{2(1-\alpha)}\|[E + B]^{\hat{Y}_j} \, dt^2 + \delta^2 \mathbb{E}\sup_{t \in [0,T]} (T-t)^{2(1-\alpha)}\|[E + B]^{\hat{Y}_j} \, dt^2 .
\end{align*}
\]

Now, arguing as in [19], proof of lemma 3.2, we get that there exists \( \delta_0 \in (0,1) \) depending only on \( c_1, \Delta, T \), such that for every \( \delta \in (0, \delta_0) \), we get

\[
\mathbb{E}\int_0^T \|[E + B]^{\hat{Y}_j+1}\|^2 \, dt + \mathbb{E}\int_0^T |\dot{X}_t^j+1|^2 \, dt \\
\leq \frac{1}{4} \left[ \mathbb{E}\int_0^T |\dot{X}_t^j|^2 \, dt + \mathbb{E}\int_0^T \|[E + B]^{\hat{Y}_j}\|^2 \, dt \right] \\
+ \frac{1}{8} \left[ \mathbb{E}\int_0^T |\dot{X}_t^{\hat{Y}_j-1}|^2 \, dt + \mathbb{E}\int_0^T \|[E + B]^{\hat{Y}_j-2}\|^2 \, dt \right].
\]

From this we deduce that \((\dot{X}_t^j, \dot{Y}_t^j)_{j \geq 1}\) is a Cauchy sequence in \(L^2_\mathbb{P}(\Omega \times [0,T], H) \times L^2(\Omega \times [0,T], H)\) and we denote by \((\ddot{X}_t, \ddot{Y}_t)\) its limit.

In order to prove that \((\dot{X}_t^j, \dot{Y}_t^j)_{j \geq 1}\) converge to \((\ddot{X}_t, \ddot{Y}_t)\) also in \(L^2_\mathbb{P}(\Omega, C([0,T], H)) \times L^2_\mathbb{P}(\Omega, C([0,T], H))\)
we go to the mild formulation of the equations solved by $\bar{X}_t^j$ and $\bar{Y}_t^j$. We start by $\bar{X}_t^j$:

$$\bar{X}_t^j = e^{tA}x + \alpha_0 \int_0^t e^{(t-s)A}[E + B][\gamma([E + B]^\gamma Y_t^j)] ds$$

$$- (1 - \alpha_0) \int_0^t e^{(t-s)A}[E + B][E + B]^{\gamma Y_s^j} ds + \delta \int_0^t e^{(t-s)A}[E + B][E + B]^{\gamma Y_s^{-1}} ds$$

$$+ \delta \int_0^T e^{(t-s)A}[E + B][\gamma([E + B]^\gamma Y_t^{j-1}) ds + \int_0^t e^{(t-s)A}b_0(s) ds$$

$$+ \int_0^t e^{(t-s)A}G dW_s + \int_0^t e^{(t-s)A}D_t^1 d\bar{W}_s.$$ 

So

$$\mathbb{E} \sup_{t \in [0,T]} |\bar{X}_t^j|^2$$

$$\leq \alpha_0^2 \mathbb{E} \sup_{t \in [0,T]} \int_0^t \|e^{(t-s)A}[E + B]\Delta[E + B]^{\gamma Y_s^j} | ds|^2$$

$$+ (1 - \alpha_0)^2 \mathbb{E} \sup_{t \in [0,T]} (T - t)^2(1 - \alpha)^2 ||E + B||^2 \sup_{t \in [0,T]} \left( \int_0^t c(t-s)^{-(1-\alpha)(T-s)^{-(1-\alpha)} ds \right)^2$$

$$+ \delta^2 \mathbb{E} \sup_{t \in [0,T]} \int_0^t \|e^{(t-s)A}[E + B]\| |E + B|^{\gamma Y_s^{-1}} | ds|^2$$

$$+ \delta^2 \sup_{t \in [0,T]} \mathbb{E} \int_0^t \|e^{(t-s)A}[E + B]\| |E + B|^{\gamma Y_s^{-1}} | ds|^2$$

$$\leq c\Delta^2 \alpha_0^2 \mathbb{E} \sup_{t \in [0,T]} (T - t)^2(1 - \alpha)^2 ||E + B||^2 \sup_{t \in [0,T]} \left( \int_0^t c(t-s)^{-(1-\alpha)(T-s)^{-(1-\alpha)} ds \right)^2$$

$$+ c(1 - \alpha_0)^2 \mathbb{E} \sup_{t \in [0,T]} (T - t)^2(1 - \alpha)^2 ||E + B||^2 \sup_{t \in [0,T]} \left( \int_0^t c(t-s)^{-(1-\alpha)(T-s)^{-(1-\alpha)} ds \right)^2$$

$$+ c\delta^2 \mathbb{E} \sup_{t \in [0,T]} (T - t)^2(1 - \alpha)^2 ||E + B||^2 \sup_{t \in [0,T]} \left( \int_0^t c(t-s)^{-(1-\alpha)(T-s)^{-(1-\alpha)} ds \right)^2$$

$$+ \delta^2 \mathbb{E} \sup_{t \in [0,T]} (T - t)^2(1 - \alpha)^2 ||E + B||^2 \sup_{t \in [0,T]} \left( \int_0^t c(t-s)^{-(1-\alpha)(T-s)^{-(1-\alpha)} ds \right)^2$$

By the previous choice of $\delta$ we get that $(\bar{X}_j)_{j \geq 1}$ is a Cauchy sequence in $L_2^2(\Omega, C([0, T], H))$ so that $\bar{X}_j \to \bar{X}$ in $L_2^2(\Omega, C([0, T], H))$. For what concerns the convergence of $\bar{Y}_j$ in $L_2^2(\Omega, C([0, T], H))$ we have first to recover the convergence of $(\bar{Z}_j^3, \bar{Z}_j^4)$ in $L_2^2(\Omega \times [0, T]; L_2(\Xi \times K; H))$. In its mild formulation, $Y_t^{j+1}$ solves the following BSDE

$$\bar{Y}_t^{j+1} = -e^{(T-t)A^\ast} \left[ \alpha_0 h_x(\bar{X}_T^{j+1}) - (1 - \alpha_0)\bar{X}_T^{j+1} + \delta h_x(\bar{X}_T^j) + \delta \bar{X}_T^j + g_0 \right]$$

$$+ \alpha_0 \int_t^T e^{(s-t)A^\ast} h_x(\bar{X}_s^{j+1}) ds + (1 - \alpha_0) \int_t^T e^{(s-t)A^\ast} \bar{X}_s^{j+1} ds$$

$$+ \delta \int_t^T e^{(s-t)A^\ast} h_x(\bar{X}_s^j) ds - \delta \int_t^T e^{(s-t)A^\ast} \bar{X}_s^j ds$$

$$- \int_t^T e^{(s-t)A^\ast} \bar{Z}_s^{j+1} dW_s - \int_t^T e^{(s-t)A^\ast} \bar{Z}_s^j dW_s + \int_t^T e^{(s-t)A^\ast} h_0(s) ds.$$
Let us denote by
\[ f_s^{j+1} := \alpha_0 \ell_0^j(s, \bar{X}_s^{j+1}) + (1 - \alpha_0) e^{(s-t)A^*} \bar{X}_s^{j+1} + \delta \ell_0^j(s, \bar{X}_s^j) ds - \delta \bar{X}_s + h_0(s). \]

Arguing as in [10], by the extended martingale representation theorem, (see also [17] and [30]), for every \( s \in [0, T] \) there exists \((\bar{K}^j(s, \cdot), \bar{K}^j(s, \cdot)) \in L_P^2(\Omega \times [0, T], L_2(\Xi, H)) \times L_P^2(\Omega \times [0, T], L_2(K, H))\) such that \( \forall 0 \leq t \leq s \leq T \)
\[
\mathbb{E}^{\mathcal{F}_t} f_s^j = \mathbb{E} f_s^j + \int_t^s \bar{K}^j(s, \theta) dW_\theta + \int_t^s \bar{K}^j(s, \theta) d\bar{W}_\theta.
\]
Note that \( \forall \theta \geq s, \bar{K}^j(s, \theta) = 0 \text{ a.e. and } \bar{K}^j(s, \theta) = 0 \text{ a.e.} \); and
\[
\mathbb{E} \int_0^T \int_0^s \left( |\bar{K}^j(s, \theta)|^2 + |\bar{K}^j(s, \theta)|^2 \right) d\theta ds \leq 4 \mathbb{E} \int_0^T |f_s^j|^2 ds. \quad (4.8)
\]
Moreover, there exists \((\bar{L}^j, \bar{L}^j) \in L_P^2(\Omega \times [0, T], L_2(\Xi, H)) \times L_P^2(\Omega \times [0, T], L_2(\bar{K}, H))\) such that
\[
\mathbb{E}^{\mathcal{F}_t} \bar{Y}_T^{j+1} = \mathbb{E} \bar{Y}_T^{j+1} + \int_0^t \int_t^s \int_t^s \bar{L}^j(\theta) dW_\theta + \int_0^t \int_t^s \bar{L}^j(\theta) d\bar{W}_\theta.
\]
So we get
\[
\bar{Y}_t^{j+1} = e^{(T-t)A^*} \left[ \alpha_0 h_s(\bar{X}_T^{j+1}) - (1 - \alpha_0) \bar{X}_T^{j+1} + \bar{h}_x(\bar{X}_T^j) + \delta \bar{X}_T^j \right]
+ \alpha_0 \int_t^T e^{(s-t)A^*} f_s^{j+1} ds - \int_t^T e^{(s-t)A^*} \bar{L}_s^{j+1} dW_s - \int_t^T e^{(s-t)A^*} \bar{\bar{L}}_s^{j+1} d\bar{W}_s
- \int_t^T e^{(s-t)A^*} \int_t^s e^{(s-s^*)A^*} K^j(s, \bar{K}^j(s, \alpha)) ds dW_s
- \int_t^T e^{(s-t)A^*} \int_t^s e^{(s-s^*)A^*} \bar{K}^j(s, \bar{K}^j(s, \alpha)) ds d\bar{W}_s.
\]
By comparing with (4.7) we deduce that, for almost all \( s \in [0, T] \),
\[
\bar{Z}_s^j = \int_t^T e^{(s-s^*)A^*} K^j_s(\alpha, s) d\alpha,
\]
\[
\bar{Z}_s^j = \int_t^T e^{(s-s^*)A^*} \bar{K}^j_s(\alpha, s) d\alpha.
\]
By the definition of \((\bar{K}^j, \bar{K}^j)\), by estimates [4.8], and by previous estimates on the \(L^2\)-norm of \( \bar{X}^j \) and of \( \bar{Y}^j \) it is possible to prove that \((\bar{Z}_s^j, \bar{Z}_s^j)\) is a Cauchy sequence in \( L_P^2(\Omega \times [0, T]; L_2(\Xi \times K; H)) \), and we denote by \((\bar{Z}_s, \bar{Z}_s)\) its limit.

We are ready to prove that \( \bar{Y}^j \to \bar{Y} \) in \( L_P^2(\Omega, C([0, T], H)) \). We can rewrite (4.7) as
\[
\mathbb{E}^{\mathcal{F}_t} \bar{Y}_T^{j+1} = \bar{Y}_T^{j+1}
= \mathbb{E}^{\mathcal{F}_t} e^{(T-t)A^*} \left[ -\alpha_0 h_s(\bar{X}_T^{j+1}) - (1 - \alpha_0) \bar{X}_T^{j+1} + \bar{h}_x(\bar{X}_T^j) + \delta \bar{X}_T^j \right]
+ \alpha_0 \mathbb{E}^{\mathcal{F}_t} \int_t^T e^{(s-t)A^*} \ell_0^j(s, \bar{X}_s^{j+1}) ds + (1 - \alpha_0) \mathbb{E}^{\mathcal{F}_t} \int_t^T e^{(s-t)A^*} \bar{X}_s^{j+1} ds
+ \delta \mathbb{E}^{\mathcal{F}_t} \int_t^T e^{(s-t)A^*} \ell_0^j(s, \bar{X}_s^j) ds - \delta \mathbb{E}^{\mathcal{F}_t} \int_t^T e^{(s-t)A^*} \bar{X}_s^j ds
+ \mathbb{E} \int_t^T e^{(s-t)A^*} h_0(s) ds.
\]
so that

\[
\mathbb{E} \sup_{t \in [0,T]} |\hat{Y}_t^{j+1}|^2 \leq c(T,A,\Delta) \left[ |\hat{X}_t^{j+1}|^2 + \delta^2 (1 + \Delta^2) |\hat{X}_t^j|^2 \right] \\
+ \alpha_0^2 c(T,A,\Delta)^2 \mathbb{E} \int_0^T |\hat{X}_s^{j+1}|^2 ds + \delta^2 c(T,A,\Delta) \mathbb{E} \int_0^T |\hat{X}_s^j|^2 ds.
\]

From this, again by using the previous estimates on the \(L^2\)-norm of \(\hat{X}^j\) and of \(\hat{Y}^j\) it is possible to prove that \(\hat{Y}^j\) is a Cauchy sequence in \(L^2_{\mathbb{F}}(\Omega, (0, T], H))\) and the claim follows.

Finally we have to prove that \(\mathbb{E} \sup_{t \in [0,T]} (T-t)^{2(1-\alpha)} |\hat{Y}_t|_{D((\lambda-A)^{\alpha})}^2\) is bounded. Let \(\alpha \in [\alpha_0, \alpha_0 + \delta]\). In its mild formulation, \(\hat{Y}\) solves the following BSDE

\[
\hat{Y}_t = e^{(T-t)A^*} \left[- \alpha h_x(\bar{X}_T) + (1-\alpha)\bar{X}_T + g_0(t)\right] \\
+ \alpha \int_t^T e^{(s-t)A^*} \left( \alpha t_0(s, \bar{X}_s) + (1-\alpha)\bar{X}_s \right) ds + \int_t^T e^{(s-t)A^*} h_0(s) ds \\
- \int_t^T e^{(s-t)A^*} \tilde{Z}_s dW_s - \int_t^T e^{(s-t)A^*} \tilde{Z}_s d\tilde{W}_s.
\]

Notice also that

\[
\hat{Y}_t = \mathbb{E}^{F_t} \hat{Y}_t = \mathbb{E}^{F_t} e^{(T-t)A^*} \left[- \alpha h_x(\bar{X}_T) + (1-\alpha)\bar{X}_T + g_0(T)\right] \\
+ \mathbb{E}^{F_t} \int_t^T e^{(s-t)A^*} \left( \alpha t_0(s, \bar{X}_s) + (1-\alpha)\bar{X}_s \right) ds + \mathbb{E}^{F_t} \int_t^T e^{(s-t)A^*} h_0(s) ds
\]

By the regularizing properties of the semigroup \((e^{tA})_{t \geq 0}\), by the assumptions on \(E\) and by the previous mild equality satisfied by \(\hat{Y}_t\) we get that for every \(t \in [0,T]\), \(\hat{Y}_t \in D(E)\) and

\[
\mathbb{E} \sup_{t \in [0,T]} |(T-t)^{2(1-\alpha)}(E+B)^* \hat{Y}_t|^2 \\
\leq c \mathbb{E} \sup_{t \in [0,T]} (T-t)^{2(1-\alpha)}|(E+B)^* e^{(T-t)A^*} \left[- \alpha h_x(\bar{X}_T) + (1-\alpha)\bar{X}_T + g_0(T)\right]|^2 \\
+ c \mathbb{E} \sup_{t \in [0,T]} (T-t)^{2(1-\alpha)}|(E+B)^* \int_t^T e^{(s-t)A^*} \left( \alpha t_0(s, \bar{X}_s) + (1-\alpha)\bar{X}_s \right) ds|^2 \\
+ c \mathbb{E} \sup_{t \in [0,T]} (T-t)^{2(1-\alpha)}|(E+B)^* \int_t^T e^{(s-t)A^*} h_0(s) ds|^2 = I + II + III.
\]

Recall that \(E = (\lambda-A)D\), and \(D\) takes its values in \(D(\lambda-A)^{\alpha}\), so that also by the analyticity of \(A\), we get, for every \(t > 0\) and every \(f \in H\)

\[
|e^{tA^*}f| \leq ct^{-(1-\alpha)}|f|.
\]

So

\[
I \leq c \mathbb{E} \sup_{t \in [0,T]} (1 + |\bar{X}_T|)^2 < +\infty;
\]

\[
II \leq c \mathbb{E} \sup_{t \in [0,T]} (T-t)^{2(1-\alpha)} \int_t^T (s-t)^{-(1-\alpha)} (1 + |\bar{X}_s|) ds|^2 \\
\leq \mathbb{E} \sup_{t \in [0,T]} (T-t)^{2(1-\alpha)} \int_t^T (s-t)^{-(1-\alpha)} ds \int_0^T (1 + |\bar{X}_s|^2) ds < +\infty;
\]

\[
III \leq c \mathbb{E} \sup_{t \in [0,T]} (T-t)^{2(1-\alpha)} \int_t^T |(s-t)^{-(1-\alpha)} h_0(s) ds|^2 < +\infty.
\]
In order to conclude the proof, note also that \((\bar{X}, \bar{Y}, (\bar{Z}, \bar{\bar{Z}}))\) is a solution to the FBSDE (4.3).

**Remark 4.3** We notice that the presence of a diffuse control is not required in our methods. Indeed, if \(B = 0\) as an auxiliary linear FBSDE we can consider

\[
\begin{dcases}
  d\bar{X}_t = A\bar{X}_t \, dt - [E + I][E + I]^*\bar{Y}_t \, dt + b_0(t) \, dt + (\lambda - A)D_1 \, d\bar{W}_t + G \, dW_t \\
  -d\bar{Y}_t = A^*\bar{Y}_t \, dt + \bar{X}_t \, dt + h_0(t) \, dt - \bar{Z}_t \, dW_t - \bar{\bar{Z}}_t \, d\bar{W}_t,
\end{dcases}
\]

and we can apply the bridge method linking this FBSDE to the FBSDE

\[
\begin{dcases}
  d\bar{X}_t = A\bar{X}_t \, dt + E\gamma(E^*\bar{Y}_t) \, dt + (\lambda - A)D_1 \, d\bar{W}_t + G(t, \bar{X}_t) \, dW_t \\
  -d\bar{Y}_t = A^T\bar{Y}_t \, dt + \bar{I}_0(t, \bar{X}_t) \, dt - \bar{Z}_t \, dW_t - \bar{\bar{Z}}_t \, d\bar{W}_t,
\end{dcases}
\]

with

\[
\bar{X}_0 = x, \quad \bar{Y}_T = -h_x(\bar{X}_T),
\]

### 4.2 An auxiliary LQ control problem

This section is devoted to the solution of the affine FBSDE. Let \(b_0, h_0 \in L^2_T((0, T) \times \Omega; H)\) and \(g_0 \in L^2(\Omega, \mathcal{F}_T; H)\), consider:

\[
\begin{dcases}
  d\bar{X}_t = A\bar{X}_t \, dt - [E + B][E + B]^*\bar{Y}_t \, dt + b_0(t) \, dt + (\lambda - A)D_1 \, d\bar{W}_t + G \, dW_t \\
  -d\bar{Y}_t = A^T\bar{Y}_t \, dt + \bar{X}_t \, dt + h_0(t) \, dt - \bar{Z}_t \, dW_t - \bar{\bar{Z}}_t \, d\bar{W}_t,
\end{dcases}
\]

with

\[
\bar{X}_0 = x, \quad \bar{Y}_T = -h_x(\bar{X}_T),
\]

This system is the Hamiltonian system corresponding to the control problem with state equation:

\[
\begin{dcases}
  dX_t = AX_t \, dt + [E + B]u_t \, dt + b_0(t) \, dt + (\lambda - A)D_1 \, d\bar{W}_t + G \, dW_t \quad t \in [0, T] \\
  X_0 = x
\end{dcases}
\]

and cost functional

\[
J(x, u) = \frac{1}{2} \mathbb{E} \int_0^T (|X_t + h_0(t)|^2 + |u_t|^2) \, dt + \frac{1}{2} \mathbb{E}|X_T + g_0|^2
\]

(4.11)

to minimize over all \(u \in U\). We will exploit this interpretation through the control problem in order to solve (4.9), to this purpose we introduce the following Riccati equation:

\[
\begin{dcases}
  -\frac{dP_t}{dt} = A^*P_t + P_tA - P_t(E + B)(E + B)^*P_t + I, \quad t \in [0, T] \\
  r_T = I g_0
\end{dcases}
\]

(4.12)

and the following backward equation, to cope with the affine terms:

\[
\begin{dcases}
  -dr_t = A^*r_t \, dt - P_t(E + B)(E + B)^*r_t \, dt + P_tb_0(t) \, dt - h_0(t) \, dt - \bar{q}_t W_t - \bar{\bar{q}}_t d\bar{W}_t, \quad t \in [0, T] \\
  r_T = I g_0
\end{dcases}
\]

(4.13)

We denote, as in [3], by \(\Sigma(H)\) the space of self adjoint linear operators in \(H\) and by \(C_s([0, T]; \Sigma(H))\) the space of all strongly continuous mappings from \([0, T]\) to \(\Sigma(H)\), that is \(P : [0, T] \to \Sigma(H)\) such that for every \(h \in H\), \(t \mapsto P_t h\) is continuous.

In the book [3] (part. IV, Chapter 2, Theorem 2.1), it is proved that the first equation (4.12) has a solution in the space \(C_{s, \alpha}([0, T]; \Sigma(H))\), the set of all \(P \in C_s([0, T]; \Sigma(H))\) such that:
(i) \( P(t)x \in D((-A^*)^{1-\alpha}) \), for all \( x \in H \), \( t \in [0, T] \),
(ii) \((-A^*)^{1-\alpha}P \in C([0, T]; \mathcal{L}(H))\),
(iii) \( \lim_{t \to T}(T-t)^{1-\alpha}(-A^*)^{1-\alpha}P_x = 0 \), for all \( x \in H \).

Moreover define
\[
\|P\|_1 = \sup_{t \in [0, T]} \|(T-t)^{1-\alpha}(-A^*)^{1-\alpha}P(t)\| \tag{4.14}
\]
\( C_{s,\alpha}([0, T]; \Sigma(H)) \), endowed with the norm
\[
\|P\|_\alpha = \|P\|_1 \tag{4.15}
\]
is a Banach space. We can now prove existence and uniqueness of a solution to \([4.13]\), for simplicity we will denote the couple \((q, \tilde{q})\) as \(\hat{q}\) along with the comprehensive Wiener process \(W_t := (W_t, \hat{W}_t)\):

**Theorem 4.4** Assume \((A)\) and \((B)\). Then equation \([4.13]\) has a unique mild solution \((r_t, \tilde{q}) \in L^2_P(\Omega; C([0, T]; H)) \times L^2_P(\Omega \times [0, T]; L_2(\Xi \times K; H))\), moreover:
\[
E \sup_{t \in [0, T]} (T-t)^{2(1-\alpha)}|r_t|^2 < \infty \tag{4.16}
\]

**Proof.** We will prove existence and uniqueness by a fixed point technique. Let us define a map \(\Gamma : Y \to Y\), where
\[
Y := \left\{ (r, \tilde{q}) \in L^2_P(\Omega; C([0, T]; H)) \times L^2_P(\Omega \times [0, T]; L_2(\Xi \times K; H)) : E \sup_{t \in [0, T]} (T-t)^{2(1-\alpha)}|r_t|^2 < \infty \right\}
\]
such that \(\Gamma((r', \tilde{q}')) = (r, \tilde{q})\) is the mild solution to:
\[
\begin{align*}
\frac{dr_t}{dt} &= e^{A^*(T-t)}g_0 \int_t^T e^{A^*(s-t)}P_s(E+B)(E+B)^s r_s ds + \int_t^T e^{A^*(s-t)}P_s h_0(s) ds \\
& \quad - \int_t^T e^{A^*(s-t)}h_0(s) ds \int_t^T e^{A^*(s-t)}\tilde{q}_s d\hat{W}_s, \\
& \quad t \in [0, T]
\end{align*}
\tag{4.17}
\]
We will prove that:
1) \(\Gamma((r', \tilde{q}')) \in Y\),
2) for any \(\alpha < 1\) there exists \(\delta \in [0, T]\) that depends only on \(\alpha\) and constants appearing in \((A)\) and \((B)\) and \(T\) such that
\[
\|(r^1, \tilde{q}^1) - (r^2, \tilde{q}^2)\|_{Y_\delta} \leq \alpha\|(r'^1, \tilde{q}'^1) - (r'^2, \tilde{q}'^2)\|_{Y_\delta} \tag{4.18}
\]
for some \(\delta > 0\) and we set
\[
Y_\delta := \left\{ (r, \tilde{q}) \in L^2_P(\Omega; C([T-\delta, T]; H)) \times L^2_P(\Omega \times (T-\delta, T); L_2(\Xi \times K; H)) : \\
E \sup_{t \in [T-\delta, T]} (T-t)^{2(1-\alpha)}|\lambda - A^*|\alpha r_t|^2 < \infty \right\} \tag{4.19}
\]
The space \(Y_\delta\) endowed with the norm:
\[
\|(r, \tilde{q})\|_{Y_\delta}^2 := E \sup_{t \in [T-\delta, T]} |r_t|^2 + E \sup_{t \in [T-\delta, T]} (T-t)^{2(1-\alpha)}|\lambda - A^*|\alpha r_t|^2 + E \int_{T-\delta}^T |\tilde{q}_t|^2 dt
\]

existence and uniqueness of a mild solution for equation (4.13) in

where \( E^k := (\lambda - A)^{1-\alpha}kR(k, A)(\lambda - A)^{\alpha}D, \) with \( R(k, A) := (k - A)^{-1}. \)

From [3] we know that equation (4.20) has a unique mild solution \( P^k \in C_{s, \alpha}([0, T]; \Sigma(H)), \) for every \( k \) and moreover the following holds, see [3](part IV, Chapter 2, lemma 2.1):

\[
\begin{align*}
\lim_{k \to \infty} P^k(\cdot)x &= P(\cdot)x \quad \text{in } C([0, T]; H), \\
\lim_{k \to \infty} (T - \cdot)^{1-\alpha}(-A^*)^{-1-\alpha}P^k(\cdot)x &= (T - \cdot)^{1-\alpha}(-A^*)^{-1-\alpha}P(\cdot)x \quad \text{in } C([0, T]; H).
\end{align*}
\]

Given \( P^k \) we introduce also:

\[
\begin{align*}
-dr_t^k &= A^*r_t^k dt - P_t^k(E^k + B)(E^k + B)^*r_t^k dt + P_t^k b_0(t) dt - h_0(t) dt - \dot{\omega}_t^k dW_t, \quad t \in [0, T] \\
r_T^k &= g_0.
\end{align*}
\]

Existence and uniqueness of a mild solution for equation (4.13) in \( L^p_2(\Omega; C([0, T]; H)) \times L^p_2(\Omega \times (0, T); L^2(\Xi \times K; H)) \) can be deduced by [16](prop. 2.1). Now we can prove that

\[
\mathbb{E} \int_0^T |\dot{q}^k_t|^2 dt \leq C \left[ \mathbb{E}[|g_0|^2] \right.
\]

\[
+ E \left( \int_0^T |P^k(E^k + B)(E^k + B)^*r_t^k| ds \right)^2 + E \int_0^T |P_t^k b_0(s)|^2 ds + E \int_0^T |h_0(s)|^2 ds \right]
\]

The former estimate can be achieved evaluating \( d_t|r_t^k|^2 \) and exploiting the fact that, being \( A^* \) the generator of a contraction semigroup, \( \langle A^*y, y \rangle \leq \omega |y|^2, \) for any \( y \in D(A^*). \) Since \( r_t^k \) does not belong to \( D(A^*), \) we multiply \( r_t^k \) by \( nR(n, A), \) for \( n > \omega \) in order to perform the Itô formula. Let us set \( r_{n,k} = nR(n, A)r_t^k \) and \( q_{n,k} = nR(n, A)\dot{q}_t^k, \) hence:

\[
\begin{align*}
-dr_t^{n,k} &= A^*r_t^{n,k} dt - nR(n, A)P_t^k(E^k + B)(E^k + B)^*r_t^k dt + nR(n, A)P_t^k b_0(t) dt \\
&- nR(n, A)h_0(t) dt - \dot{\omega}_t^{n,k} dW_t, \quad t \in [0, T] \\
r_T^{n,k} &= nR(n, A)g_0.
\end{align*}
\]

Now we can evaluate \( d_t|r_t^{n,k}|^2: \)

\[
d_t|r_t^{n,k}|^2 = 2\langle A^*r_t^{n,k}, r_t^{n,k} \rangle dt - 2\langle f_t^{n,k}, r_t^{n,k} \rangle dt - 2\langle \dot{q}_t^{n,k}, r_t^{n,k} \rangle dW_t - |\dot{q}_t^{n,k}|^2 dt
\]

where \( f_t^{n,k} = nR(n, A)P_t^k(B + E^k)(B + E^k)^*r_t^k + P_t^k b_0(t) + h_0(t) \)

Now similarly to [15] (prop. 3.4), see also [3] (lemma 3.1), we get:

\[
\mathbb{E} \int_0^T |\dot{q}_t^{n,k}|^2 dt \leq C \left[ \mathbb{E} \sup_{t \in [0, T]} |r_t^{n,k}|^2 + \mathbb{E} \left( \int_0^T |f_t^{n,k}| dt \right)^2 \right]
\]

where the constant \( C \) depends on constants appearing in (A) and (B) and \( T. \) Letting \( n \) tend to \( \infty \) we obtain estimate [123]. Now bearing in mind that \( \sup_{t \in [0, T]} |P_t^k| \leq M \) independent of \( k, \) thanks
Moreover from former calculations we have that:

\[
E \int_0^T |\dot{q}_t|^2 \, dt \leq C \left[ E \left( \sup_{s \in [0,T]} (T-s)^{(2-2\alpha)} \left[ (\lambda - A^*)^{1-\alpha} r_s' \right]^2 + |r_s'|^2 \right) ds \int_0^T s^{\alpha-1} (T-s)^{2\alpha-2} \, ds \right]^2
+ E[g_0]^2 + E \int_0^T |b_0(s)|^2 \, ds + E \int_0^T |b_0(s)|^2 \, ds \right]
\]

by dominated convergence we end up with

\[
E \sup_{s \in [0,T]} (T-s)^{(2-2\alpha)} \left[ (\lambda - A^*)^{1-\alpha} r_s' \right]^2 \leq \infty,
\]

(4.27)

Let us consider \(k, m > \omega\):

\[
\begin{align*}
  r_t^k - r_t^m = & - \int_0^T e^{A^*(s-t)} \left[ P_k^s(B + E^k)(B + E^k)^* - P_m^s(B + E^m)(B + E^m)^* \right] r_s' \, ds \\
  & - \int_0^T e^{A^*(s-t)} (\dot{q}_s^k - \dot{q}_s^m) \, dW_s.
\end{align*}
\]

We have that:

\[
\begin{align*}
  r_t^k - r_t^m = & - E F_t \left[ r_t^k - r_t^m \right] \\
  = & - E F_t \int_0^T e^{A^*(s-t)} \left[ P_k^s(B + E^k)(B + E^k)^* - P_m^s(B + E^m)(B + E^m)^* \right] r_s' \, ds,
\end{align*}
\]

and, since \(|(\lambda - A^*)^{1-\alpha} e^{A^*} |_{L(H)} \leq c s^{1-\alpha}\):

\[
\begin{align*}
  & \sup_{t \in [0,T]} \left| (\lambda - A^*)^{1-\alpha} (r_t^k - r_t^m) \right|^2 \\
  & \leq c E F_t \int_0^T s^{\alpha-1} \left[ |P_k^s(B + E^k)(B + E^k)^* - P_m^s(B + E^m)(B + E^m)^*| r_s' \right] \, ds \\
  + c E F_t \int_0^T s^{\alpha-1} \left[ |P_m^s(B + E^m)(B + E^m)^* - P(B + E)(B + E)^*| r_s' \right] \, ds
\end{align*}
\]

Hence, taking into account that:

\[
E \sup_{t \in [0,T]} (T-s)^{(2-2\alpha)} \left[ (\lambda - A^*)^{1-\alpha} r_s' \right]^2 < \infty,
\]

by dominated convergence we end up with

\[
\lim_{k,m \to +\infty} E \sup_{t \in [0,T]} (T-t)^{(2-2\alpha)} \left[ (\lambda - A^*)^{1-\alpha} (r_t^k - r_t^m) \right]^2 = 0
\]

(4.29)

Similarly we have that:

\[
\lim_{k,m \to +\infty} E \left[ (r_t^k - r_t^m) \right]^2 = 0
\]

(4.30)

Moreover from former calculations we have that

\[
\begin{align*}
  E \int_0^T |\ddot{q}_t|^2 \, dt \leq & C \left[ E \left( \int_0^T \left[ |P_k^s(B + E^k)(B + E^k)^* - P(B + E)(B + E)^*| r_s' \right] \, ds \right)^2 \\
  & + E \left( \int_0^T \left[ |P_m^s(B + E^m)(B + E^m)^* - P(B + E)(B + E)^*| r_s' \right] \, ds \right)^2 \right]
\end{align*}
\]

(4.31)

Thus, the limit processes \(r\) and \(\dot{q}\) solve equation (4.17) and we have the desired regularity.
Now we have to prove (4.18). Following previous procedures, we have:
\[
\mathbb{E} \sup_{t \in [T-\delta, T]} \left| (T-t)^2(1-\alpha) \right| (\lambda - A^*)^{1-\alpha} |(r_t^1 - r_t^2)|^2 \\
\leq M\delta^{2(2\alpha-1)} \mathbb{E} \sup_{t \in [T-\delta, T]} \left| (T-t)^2(1-\alpha) \right| (\lambda - A^*)^{1-\alpha} |(r_t^1 - r_t^2)|^2 \\
\]
and
\[
\mathbb{E} \sup_{t \in [T-\delta, T]} \left| r_t^1 - r_t^2 \right| \leq M\delta^{2(2\alpha-1)} \mathbb{E} \sup_{t \in [T-\delta, T]} \left| (\lambda - A^*)^{1-\alpha} (r_t^1 - r_t^2) \right|^2 \\
\]
where the constant $M$ depends on $\alpha$ and constants appearing in hypotheses (A) and (B) and $M\delta^{2\alpha-1} < 1$ if $\delta$ is sufficiently small. Therefore one can repeat the procedure in $[T - 2\delta, T - \delta]$ and so on in order to cover, in a finite number of steps, the whole interval $[0, T]$. 

It remains to show that if we define $\bar{Y}_t = P_t \bar{X}_t + r_t$, then $\bar{Y}$ is a solution to the BSDE in the FBSDE (4.3).

**Proposition 4.5** Let assumptions (A) hold true an let $b_0, h_0 \in L^2_p(\Omega \times [0, T]; H)$, $g_0 \in L^2(\Omega; H)$. Then the FBSDE (4.3) admits a unique mild solution $(\bar{X}, \bar{Y}, (\bar{Z}, \bar{\zeta})) \in L^2_p(\Omega; C([0, T]; H)) \times L^2_p(\Omega; C([0, T]; H)) \times L^2_p(\Omega \times [0, T]; L^2(\Xi \times K; H))$ satisfying moreover $\mathbb{E} \sup_{t \in [0, T]} (T-t)^{2(1-\alpha)} \| (E+B)\bar{Y}_t \|^2 < +\infty$.

**Proof.** Let us denote by $P^k$ the solution of the Riccati equation (4.20) and, for $j > \omega$, by $A_j := jR(j, A)$ the Yosida approximants of $A$. We denote by $P^{j,k}$ the solution of the Riccati equation (4.20) with $A_j$ in the place of $A$:
\[
\begin{aligned}
\frac{dP^{j,k}_{t}}{dt} &= A^*_jP^{j,k}_{t} + P^{j,k}_{t} A - P^{j,k}_{t}(E^k + B)(E^k + B)^*P^{j,k}_{t} + I, \quad t \in [0, T] \\
\end{aligned}
\tag{4.32}
\]
By $(r^k, (g^k, \bar{g}^k))$ and by $(r^{n,k}, (g^{n,k}, \bar{g}^{n,k}))$ we denote respectively the solution of the BSDEs (4.22) and (4.24). Moreover we denote by $\bar{X}$ and $\bar{X}^k$ respectively the solution of
\[
\begin{aligned}
d\bar{X}_t &= A\bar{X}_t dt - [E+B][E+B]^*(P_t\bar{X}_t + r_t) dt + b_0(t) dt + (\lambda - A)D_t d\bar{W}_t + G dW_t \\
\bar{X}^{n,k}_0 &= x, \\
\end{aligned}
\tag{4.33}
\]
and of
\[
\begin{aligned}
d\bar{X}^k_t &= A\bar{X}^k_t dt - [E^k+B][E^k+B]^*(P^k_t\bar{X}^k_t + r^k_t) dt + b_0(t) dt + (\lambda - A)D_t d\bar{W}_t + G dW_t \\
\bar{X}^{k}_0 &= x. \\
\end{aligned}
\tag{4.34}
\]
We also set $\bar{X}^{n,k} = nR(n, A)\bar{X}^{n,k}$ which is solution of
\[
\begin{aligned}
d\bar{X}^{n,k}_t &= A\bar{X}^{n,k}_t dt - nR(n, A)[E^k+B][E^k+B]^*(P^k_t\bar{X}^{n,k}_t + r^k_t) dt + b_0(t) dt \\
&\quad + nR(n, A)(\lambda - A)D_t d\bar{W}_t + nR(n, A)G(t, \bar{X}_t) dW_t \\
\bar{X}^{n,k}_0 &= nR(n, A)x. \\
\end{aligned}
\tag{4.35}
\]
By applying Itô formula to $P_t^{j,k} X_t^{n,k} + r_t^{n,k}$ we get

$$d(P_t^{j,k} X_t^{n,k} + r_t^{n,k}) = \left(-A_j^{ stern} P_t^{j,k} X_t^{n,k} - P_t^{j,k} A X_t^{n,k} + P_t^{j,k} [E^k + B][E^k + B]^* P_t^{j,k} X_t^{n,k} + X_t^{n,k}\right) dt + P_t^{j,k} A_j X_t^{n,k} dt - P_t^{j,k} n R(n, A)[E^k + B][E^k + B]^* \left(P_t^{j,k} X_t^{n,k} + r_t^{n,k}\right) dt + P_t^{j,k} n R(n, A) b_0(t) dt + P_t^{j,k} n R(n, A)(\lambda - A) D_1 \tilde{d} W_t + P_t^{j,k} n R(n, A) G dW_t - A^* r_t^{n,k} dt + n R(n, A) P_t^{j,k} b_0(t) dt + q_t^{n,k} dW_t + \tilde{q}_t^{n,k} dW_t.$$

So in mild form we get

$$P_t^{j,k} X_t^{n,k} + r_t^{n,k} = e^{(T-t)A^*_t} [X_T^{n,k} + n R(n, A) b_0] + \int_t^T e^{(s-t)A^*_s} \left(A^* r_s^{n,k} - A^*_j r_s^{n,k}\right) ds + \int_t^T e^{(s-t)A^*_s} \left(P_s^{j,k} A_j X_s^{n,k} - A X_s^{n,k}\right) ds - \int_t^T e^{(s-t)A^*_s} P_s^{j,k} n R(n, A) [E^k + B][E^k + B]^* \left(P_s^{j,k} X_s^{n,k} + r_s^{n,k}\right) ds + \int_t^T e^{(s-t)A^*_s} P_s^{j,k} n R(n, A) [E^k + B] dW_s + \int_t^T e^{(s-t)A^*_s} \left(P_s^{j,k} n R(n, A) G - q_t^{n,k}\right) ds.$$

We start by letting $j \to \infty$. It follows by assumption (A.1) that $\|e^{tA_j}\| \leq e^{\alpha t}$. Keeping this in mind, and since $r_t^{n,k}, X_t^{n,k} \in D(A)$ and moreover since $P_t^{j,k}$ is uniformly bounded in $j$, we get that the integrals $\int_t^T e^{(s-t)A^*_s} \left(A^* r_s^{n,k} - A^*_j r_s^{n,k}\right) ds$ and $\int_t^T e^{(s-t)A^*_s} P_s^{j,k} \left(A_j X_s^{n,k} - A X_s^{n,k}\right) ds$ converge to 0 as $j \to \infty$.

With similar considerations, by adding and subtracting $e^{(s-t)A^*_s} P_s^{j,k} [E^k + B][E^k + B]^* P_s^{j,k} X_s^{n,k}$ and $e^{(s-t)A^*_s} P_s^{j,k} [E^k + B][E^k + B]^* P_s^{j,k} X_s^{n,k}$ we get that the integral $\int_t^T e^{(s-t)A^*_s} P_s^{j,k} [E^k + B][E^k + B]^* P_s^{j,k} X_s^{n,k} ds$ converges to $\int_t^T e^{(s-t)A^*_s} P_s^{j,k} [E^k + B][E^k + B]^* P_s^{j,k} X_s^{n,k} ds$ as $j \to \infty$.

In an analogous and simpler way we also get that $\int_t^T e^{(s-t)A^*_s} P_s^{j,k} n R(n, A)[E^k + B][E^k + B]^* \left(P_s^{j,k} X_s^{n,k} + r_s^{n,k}\right) ds$ converges to $\int_t^T e^{(s-t)A^*_s} P_s^{j,k} n R(n, A)[E^k + B][E^k + B]^* \left(P_s^{j,k} X_s^{n,k} + r_s^{n,k}\right) ds$.

By adding and subtracting $P_s^{j,k} n R(n, A) b_0(s)$ it is possible to see that

$$\int_t^T e^{(s-t)A^*_s} \left(n R(n, A) P_s b_0(s) - P_s^{j,k} n R(n, A) b_0(s)\right) ds$$

converges to $\int_t^T e^{(s-t)A^*_s} \left(n R(n, A) P_s b_0(s) - P_s^{j,k} n R(n, A) b_0(s)\right) ds$ as $j \to \infty$; it is immediate to see that $\int_t^T e^{(s-t)A^*_s} n R(n, A) b_0(s) ds \to \int_t^T e^{(s-t)A^*_s} n R(n, A) b_0(s) ds$ as $j \to \infty$.

For what concerns the stochastic integrals, we notice that the integrands are square integrable with respect to $s$, uniformly with respect to $j$.
So, letting $j \to \infty$, we get

$$P^k_t \bar{X}^{n,k}_t + r^{n,k}_t = e^{(T-t)A^*} \left[ \bar{X}^{n,k}_T + nR(n, A)g_0 \right] + \int_t^T e^{(s-t)A^*} \left( nR(n, A) P^k_s b(s) - P^k_s nR(n, A) b_0(s) \right) \, ds$$

$$- \int_t^T e^{(s-t)A^*} \left[ P^k_s [E^k + E] (P^k_s \bar{X}^{n,k}_s + r^{n,k}_s) + \bar{X}^{n,k}_s \right] \, ds - \int_t^T e^{(s-t)A^*} nR(n, A) h_0(s) \, ds$$

$$+ \int_t^T e^{(s-t)A^*} P^k_s nR(n, A) ([E^k + E] (P^k_s \bar{X}^{n,k}_s + r^{n,k}_s) + \bar{X}^{n,k}_s) \, ds$$

Moreover,

$$\mathbb{E}^F_t P^k_t \bar{X}^{n,k}_t + r^{n,k}_t = P^k_t \bar{X}^{n,k}_t + r^{n,k}_t$$

$$= \mathbb{E}^F_t e^{(T-t)A^*} [\bar{X}^{n,k}_T + nR(n, A)g_0] + \mathbb{E}^F_t \int_t^T e^{(s-t)A^*} \left( nR(n, A) P^k_s b(s) - P^k_s nR(n, A) b_0(s) \right) \, ds$$

$$- \mathbb{E}^F_t \int_t^T e^{(s-t)A^*} \left[ P^k_s [E^k + E] (P^k_s \bar{X}^{n,k}_s + r^{n,k}_s) + \bar{X}^{n,k}_s \right] \, ds - \int_t^T e^{(s-t)A^*} nR(n, A) h_0(s) \, ds$$

As $n \to \infty$, taking into account, where necessary, that $\|P^k\|_1$ is bounded uniformly with respect to $k$, we get

$$\mathbb{E}^F_t P^k_t \bar{X}^{k}_t + r^k_t = \mathbb{E}^F_t e^{(T-t)A^*} [\bar{X}^{k}_T + g_0]$$

$$+ \mathbb{E}^F_t \int_t^T e^{(s-t)A^*} \left( P^k_s b_0(s) - P^k_s b_0(s) \right) \, ds$$

$$- \mathbb{E}^F_t \int_t^T e^{(s-t)A^*} \left[ P^k_s [E^k + E] (P^k_s \bar{X}^{k}_s + r^k_s) + \bar{X}^{k}_s \right] \, ds + \mathbb{E}^F_t \int_t^T e^{(s-t)A^*} (\bar{X}^{k}_s - h_0(s)) \, ds$$

$$+ \mathbb{E}^F_t \int_t^T e^{(s-t)A^*} P^k_s [E^k + E] (P^k_s \bar{X}^{k}_s + r^k_s) \, ds$$

Now notice that

$$\bar{X}^{k}_t - \bar{X}_t = \int_0^t e^{(t-s)A} \left( [E^k + E] (P^k_s \bar{X}^{k}_s + r^k_s) - [E + E] (P_s \bar{X}_s + r_s) \right) \, ds.$$

By the convergence of $P^k$ to $P$, see [3], chapter IV, section 2, lemma 2.1 and theorem 2.1, by adding and subtracting suitable terms and in virtue of Gronwall lemma, we get that $\bar{X}^k \to \bar{X}$ in $L^2_2(\Omega, C([0, T], H))$. By adding and subtracting $P^k_t \bar{X}_t$ we also get that $P^k_t \bar{X}^k_t \to P_t \bar{X}_t$ in $L^2_2(\Omega, C([0, T], H))$, since $\sup_{t \in [0, T]} |P^k_t| \leq M$ independent of $k$, thanks to (4.21) and Banach-Steinhaus theorem. With similar arguments we also get that $(T-t)^{\alpha}(\lambda - A^*)^{\lambda - \alpha} P^k_t \bar{X}^k_t$ converges
to \((T - t)^\alpha (\lambda - A^*)^{1-\alpha} P_t \hat{X}_t\) in \(L^2_P(\Omega, C([0, T], H))\). With similar and simpler arguments we finally get

\[
P_t \hat{X}_t + r_t = \mathbb{E}^{\mathcal{F}_t} e^{(T-t)A^*}[\hat{X}_T + g_0] - \mathbb{E}^{\mathcal{F}_t} \int_t^T e^{(s-t)A^*} \hat{X}_s \, ds - \mathbb{E}^{\mathcal{F}_t} \int_t^T e^{(s-t)A^*} h_0(s) \, ds
\]

Arguing as in [16] and as in the proof of lemma 4.2, by the extended martingale representation theorem, (see also [17] and [30]), for every \(s \in [0, T]\) there exists \((K(s, \cdot), \tilde{K}(s, \cdot), \in L^2_P(\Omega \times [0, T], L_2(\Xi, H)) \times L^2_P(\Omega \times [0, T], L_2(K, H))\) such that \(0 \leq t \leq s \leq T\)

\[
\mathbb{E}^{\mathcal{F}_t} \int_t^s e^{(s-t)A^*} (\hat{X}_s - h_0(s)) \, ds = \mathbb{E} \int_t^T e^{(s-t)A^*} (\hat{X}_s h_0(s)) \, ds + \int_t^T K(s, \theta) \, dW_\theta + \int_t^T \tilde{K}(s, \theta) \, d\tilde{W}_\theta
\]

Note that \(\forall \theta \geq s\), \(K(s, \theta) = \tilde{K}(s, \theta) = 0\) a.e. and

\[
\mathbb{E} \int_0^T \int_0^s \left[ |K^{n-k}(s, \theta)|^2 + |\tilde{K}^{n-k}(s, \theta)|^2 \right] \, d\theta \, ds \leq 4 \mathbb{E} \int_0^T |(\hat{X}_s^{n,k} - n R(n, A) h_0(s))|^2 \, ds. \quad (4.36)
\]

Moreover, there exists \((L, \tilde{L}) \in L^2_P(\Omega \times [0, T], L_2(\Xi, H)) \times L^2_P(\Omega \times [0, T], L_2(K, H))\) such that

\[
\mathbb{E}^{\mathcal{F}_t} [\hat{X}_T + g_0] = \mathbb{E}[\hat{X}_T + g_0] + \int_0^T L(\theta) \, dW_\theta + \int_0^T \tilde{L}(\theta) \, d\tilde{W}_\theta.
\]

We deduce that by setting, for almost all \(s \in [0, T]\),

\[
Z_s = \int_s^T e^{(\alpha-s)A^*} K_s(\alpha, s) \, d\alpha, \quad \tilde{Z}_s = \int_s^T e^{(\alpha-s)A^*} \tilde{K}_s(\alpha, s) \, d\alpha.
\]

By the definition of \((K, \tilde{K})\) and by estimates (4.33), it follows that \((\tilde{Z}_s, \tilde{Z}_s) \in L^2_P(\Omega \times [0, T]; L_2(\Xi \times K; H))\). Moreover \((X, Y, (Z, \tilde{Z}))\) are a solution to FBSDE (4.37). \(\square\)

### 4.3 Existence and uniqueness of the mild solution of the FBSDE

In this section we prove theorem 3.4 by using the results in lemma 4.2 and in section 4.2.

**Proof of Theorem 3.4 Existence.** We follow the proof of Theorem 3.1, existence part, in [19], with suitable changes due to the different framework. For \(\alpha \in [0, 1]\) consider the FBSDE

\[
\begin{aligned}
&d\tilde{X}_t = A\tilde{X}_t \, dt + b^\alpha(\tilde{Y}_t) \, dt + h_0(t) \, dt + (\lambda - A)D_t \, d\tilde{W}_t + G \, dW_t \\
&-d\tilde{Y}_t = A^*\tilde{Y}_t \, dt + h^\alpha(\tilde{X}_t) \, dt + h_0(t) \, dt - \tilde{Z}_t^\alpha \, dW_t - \tilde{Z}_t^\alpha \, d\tilde{W}_t, \quad t \in [0, T] \quad (4.37)
\end{aligned}
\]

For \(\alpha = 0\) the FBSDE (4.37) admits a mild solution: by section 4.2 we know that FBSDE (4.10) admits a mild solution, and for \(\alpha = 0\), FBSDE (4.37) coincides with FBSDE (4.10). By lemma 4.2 there exists \(\delta_0\) such that for all \(\alpha \in [0, \delta_0]\) the FBSDE (4.37) admits a mild solution with the required regularity. Then, by arbitrary choice of \(g_0, h_0\) and \(g_0\) we can solve (4.37) for \(\alpha \in [\delta_0, 2\delta_0], [2\delta_0, 3\delta_0], \ldots\): notice that \(\delta_0\) does not depend on \(\alpha\). We arrive at solving (4.37) for \(\alpha = 1\), and again by the arbitrary choice we can make of \(g_0\), \(h_0\) and \(g_0\) we have proved the existence of an adapted solution to (4.37) \((\tilde{X}, \tilde{Y}, (\tilde{Z}, \tilde{Z}))\) with the required regularity.

**Uniqueness.** In order to prove uniqueness we follow [19], theorem 3.1 uniqueness part, and the proof of lemma 4.2 in the present paper. Let, for \(i = 1, 2\), \((\tilde{X}^i, \tilde{Y}^i, (\tilde{Z}^i, \tilde{Z}^i))\) be two solutions of (4.6). In
order to apply Itô formula, we have to approximate these solutions with elements in the domain of \( A \), namely we set \((\bar{X}^{n,i}, \bar{Y}^{n,i}, (\bar{Z}^{n,i}, \bar{Z}^{n,i})) = (nR(n, A)\bar{X}^i, nR(n, A)\bar{Y}^i, (nR(n, A)\bar{Z}^i, nR(n, A)\bar{Z}^i))\), \( i = 1, 2 \), and as in lemma 1.2 e also denote \( E_n + B_n := nR(n, A)(E + B) \). By applying Itô formula to \((\bar{X}^{n,1}_t - \bar{X}^{n,2}_t, \bar{Y}^{n,1}_t - \bar{Y}^{n,2}_t)\), and then integrating over \([0,T]\) and taking expectation we get

\[
- \mathbb{E}(\bar{X}^{n,1}_T - \bar{X}^{n,2}_T, nR(n, A)(h_x(\bar{X}^{1}_T) - h_x(\bar{X}^{2}_T)))
= \mathbb{E} \int_0^T \langle [E_n + B_n](\gamma([E + B]^*\bar{Y}_t^1) - \gamma([E + B]^*\bar{Y}_t^2)), \bar{Y}^{n,1}_t - \bar{Y}^{n,2}_t \rangle dt
- \mathbb{E} \int_0^T \langle nR(n, A)(l_0^1(t, \bar{X}_t^1) - l_0^2(t, \bar{X}_t^2)), \bar{X}^{n,1}_t - \bar{X}^{n,2}_t \rangle dt
\]  

Next we want to let \( n \to +\infty \) in the (4.38); arguing as in lemma 1.2 we deduce that \( \bar{X}^{n,i} \to X^{n,i} \) in \( L^2_P(\Omega; C([0,T], H)) \) for \( i = 1, 2 \), and that \( \bar{Y}^{n,i} \to Y^{n,i} \) in \( L^2_P(\Omega; C([0,T], H)) \) for \( i = 1, 2 \) and moreover \( \mathbb{E} \sup_{t \in [0,T]}(T - t)^{2(1 - \alpha)}||E_n + B_n||Y^{n,i} - [E + B]^*Y^i||^2 \to 0 \), for \( i = 1, 2 \). We also have \( \mathbb{E} \sup_{t \in [0,T]}(T - t)^{2(1 - \alpha)}||E_n + B_n||Y^{n,i} - [E + B]^*Y^i||^2 \to 0 \) for \( i = 1, 2 \). So letting \( n \to \infty \) in (4.38) we get

\[
- \mathbb{E}(\bar{X}^{1}_T - \bar{X}^{2}_T, h_x(\bar{X}^{1}_T) - h_x(\bar{X}^{2}_T))
= \mathbb{E} \int_0^T \langle [E + B](\gamma([E + B]^*\bar{Y}_t^1) - \gamma([E + B]^*\bar{Y}_t^2)), \bar{Y}^1_t - \bar{Y}^2_t \rangle dt
- \mathbb{E} \int_0^T \langle l_0^0(t, \bar{X}_t^1) - l_0^0(t, \bar{X}_t^2), \bar{X}^1_t - \bar{X}^2_t \rangle dt.
\]

So, by assumptions (B) we get

\[
\mathbb{E}|\bar{X}^{1}_T - \bar{X}^{2}_T|^2 + \mathbb{E} \int_0^T ||E + B|^*(\bar{Y}^1_t - \bar{Y}^2_t)||^2 dt + \mathbb{E} \int_0^T |\bar{Y}^1_t - \bar{Y}^2_t|^2 dt \leq 0
\]

and so the uniqueness follows. □

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