On the Hamiltonian and Lagrangian structures of time–dependent reductions of evolutionary PDEs

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Abstract: In this paper we study the reductions of evolutionary PDEs on the manifold of the stationary points of time–dependent symmetries. In particular we describe how that the finite dimensional Hamiltonian structure of the reduced system is obtained from the Hamiltonian structure of the initial PDE and we construct the time–dependent Hamiltonian function. We also present a very general Lagrangian formulation of the procedure of reduction. As an application we consider the case of the Painlevé equations PI, PII, PIII, PVI and also certain higher order systems appeared in the theory of Frobenius manifolds.

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Introduction

Consider an evolutionary PDE with one spatial variable

\[ u_t = F(u, u_x, \ldots, u^{(m)}) \]  

and a symmetry of this equation, i.e. another system

\[ u_s = G(u, u_x, \ldots, u^{(k)}) \]

commuting with the first one,

\[ (u_s)_t = (u_t)_s. \]

The set of the stationary points \( u_s = 0 \) of the symmetry is a finite-dimensional invariant manifold for the system (0.1). Particularly, in important examples, the invariant manifold can be described as a set of stationary points of a first integral of the system (0.1):

\[ \delta I / \delta u(x) = 0, \]

\[ I = \int L(u, u_x, \ldots, u^{(n)}) dx, \quad dI / dt = 0. \]

In this case it is known ([BN], [Mo]) that the restriction of the initial PDE to the invariant submanifold is a Hamiltonian system of ODEs. In particular Bogoyavlenskii and Novikov [BN] found a universal scheme to construct the Hamiltonian function of the reduced system in terms of the Hamiltonian of the original PDE. In this paper we extend this scheme to more general finite dimensional invariant submanifolds specified by local \( x \)- and time-dependent symmetries and conservative quantities of the evolutionary equation. To distinguish this class of symmetries from the previous one we will call them scaling symmetries. We show that the restriction of the starting equation on the finite dimensional manifold admits a natural description as a Hamiltonian system with time–dependent Hamiltonian.

The best known class of examples of evolutionary PDEs admitting nontrivial symmetries and conservation laws are integrable systems of soliton theory (see [SM] and references therein). The finite dimensional manifolds of the stationary points of integrable systems are typically described by ODEs of Painlevé type [AS], [CD]. For the simplest examples of these restrictions the Hamiltonian structure is already known. For example for the classical six Painlevé equations the Hamiltonian description was found by Okamoto, [O]. Although the relationship between the starting PDE and the reduced ODE is clear and has been investigated quite a lot (see, e.g. [AC], [AS]), the relationship between the starting Hamiltonian structure and the reduced one has not been elucidated. This work will give a contribution in understanding of this relationship.

As a first result (see section 2) we prove that the finite dimensional Hamiltonian structure of the ODEs is obtained from the Hamiltonian structure of the starting PDE, via scaling reduction. Particularly, we construct the time–dependent Hamiltonian function of the reduced system. In the time–independent case this procedure coincides with the well known stationary–flow reduction discovered by Bogoyavlenskii and Novikov [BN]. As an application we present the case of PI, PII, PIII, PVI and also certain higher order systems appeared recently in the theory of Frobenius manifolds [D1].

As a second result (see section 3) we present a very general Lagrangian formulation of the procedure of reduction of an evolutionary system (0.1). Namely, we prove that this restriction is again a Lagrangian system with the Lagrangian function \( \Lambda \), such that

\[ d\Lambda / dx = dL / dt. \]

The work is structured as follows: after recalling, in Section 1, some basic facts about the Hamiltonian structure of the evolutionary PDEs, and briefly summarizing the method of reduction of evolutionary flows
on the manifold of stationary points of their integral, introduced by Bogoyavlenskii and Novikov [BN], in Section 2 we consider the generalization of this procedure to scaling symmetries. The reduced flow is a time–dependent Hamiltonian system, and in Theorem 2.1 we give the relationship between the infinite–dimensional Hamiltonian structure and the reduced one.

Section 3 is devoted to a Lagrangian approach to the problem: after describing the general framework, in Theorem 3.1 we give the procedure of reduction and we construct the reduced Lagrangian function. In Section 3.2 we establish the relationship with the Hamiltonian approach. As an application we study the Lagrangian reduction of KdV on the manifold of the fixed points of the 7–th flow.

Section 4 contains the application of the theory to the scaling reductions from KdV, mKdV and Sine–Gordon equations respectively to Painlevé I, Painlevé II and III. These examples are studied both from the Hamiltonian and the Lagrangian point of view.

In Section 5 we study the n-waves equation and his scaling reduction to a system of commuting Hamiltonian flows on the Lie algebra \( \mathfrak{so}(n) \). The reduced system is a non-autonomous Hamiltonian system w.r.t. the Poisson structure of \( \mathfrak{so}(n) \). In particular, for \( n = 3 \), adding an additional symmetry condition, one arrives at Painlevé VI equation.

1. Infinite dimensional Hamiltonian structures and stationary flows reduction

Let us consider the phase space \( \mathcal{M} \) of smooth maps of the circle into some smooth \( n \)-dimensional manifold. Actually we can forget about the boundary conditions when dealing with local functionals only. We denote by \( \mathcal{F} \) the space of smooth functionals on \( \mathcal{M} \) of the form

\[
F(u) = \int f(x, u(x), u_x(x), \ldots, u^{(m)}(x))dx,
\]

where the density \( f \) depends only on a finite number of derivatives of \( u \). On the space \( \mathcal{F} \) the variational derivative \( \frac{\delta F}{\delta u^i(x)} \) is defined by

\[
\delta F = \int \frac{\delta F}{\delta u^i(x)} \delta u^i(x)dx.
\]

Explicitly,

\[
\frac{\delta F}{\delta u^i(x)} = \frac{\partial f}{\partial u^i(x)} + \sum (-1)^k \frac{d^k}{dx^k} \frac{\partial f}{\partial u^{(k)}(x)}.
\]

One can define on \( \mathcal{M} \) the (formal) Poisson brackets

\[
\{u^i(x), u^j(y)\} = w^{ij}(x, y) = \sum_{k=0}^{N} A_k^{ij} \delta^{(k)}(x - y),
\]

where \( A_k^{ij} \) depends on a finite number of derivatives of \( u \). This induces on \( \mathcal{F} \) the Poisson bracket

\[
\{F, G\} = \int \frac{\delta F}{\delta u^i(x)} P^{ij} \frac{\delta G}{\delta u^j(x)}dx,
\]

where

\[
P^{ij} = \sum_{k=0}^{N} A_k^{ij} \left( \frac{d}{dx} \right)^k.
\]

A Hamiltonian system on \( \mathcal{M} \) has then the form

\[
u^j_t(x) = \{u^i(x), H\} = P^{ij} \frac{\delta H}{\delta u^j(x)}.
\]
In particular, we consider so called Gardner–Zakharov–Faddev bracket \( P^{ij} = \delta^{ij} \frac{d}{dx} \). In this case a Hamiltonian system has the form

\[
    u_t(x) = \{ u(x), H \} = \frac{d}{dx} \frac{\delta H}{\delta u(x)}
\]

(1.1)

with Poisson bracket

\[
    \{ F, G \} = \int \frac{\delta F}{\delta u(x)} \frac{d}{dx} \frac{\delta G}{\delta u(x)} \, dx.
\]

Let us consider a first integral

\[
    I = \int L(x, u(x), u_x(x), \ldots, u^{(n)}) \, dx,
\]

where \( L \) does not depend on \( t \). The generalized Euler–Lagrange equation

\[
    \frac{\delta I}{\delta u(x)} = 0
\]

(1.2)

generically is an ODE of order \( 2n \) fixing the \( 2n \)-dimensional manifold  of the stationary points of the first integral \( I \). Because of the Lax lemma (see [Mo]) this submanifold is invariant w.r.t the evolutionary equation (1.1). The functional \( L \) is the Lagrangian of the \( x \)-flow defined by (1.2). If \( L \) is nondegenerate, then it defines also on \( S \) the natural system of canonical coordinates

\[
    q_i = u^{(i-1)}, \quad i = 1, 2, \ldots, n
\]

\[
    p_i = \frac{\delta I}{\delta u^{(i)}},
\]

and equation (1.2) can be put in the Hamiltonian form

\[
    \begin{align*}
    (p_i)_x &= -\frac{\partial H}{\partial q_i} \\
    (q_i)_x &= \frac{\partial H}{\partial p_i},
    \end{align*}
\]

where \( H \) is the generalized Legendre transform of \( L \):

\[
    H = -L + \sum_{i=1}^{n} \frac{\delta I}{\delta u^{(i)}} u^{(i)}
\]

which, in terms of the canonical coordinates takes the form:

\[
    H = -L + \sum_{i=1}^{n} p_i \frac{dq_i}{dx}.
\]

It is well known that the starting PDE can be restricted on \( S \) and the restriction is a Hamiltonian system of ODEs. In particular Bogoyavlenskii and Novikov discovered the algorithm to construct the Hamiltonian functions of the reductions in terms of the Hamiltonian of the original evolutionary equation. They considered the case of a hierarchy of evolutionary equations

\[
    \frac{d u}{d t_k} = \frac{d}{dx} \frac{\delta I_k}{\delta u(x)}
\]

with \( I_k = \int L_k(u(x), u_x(x), \ldots, u^{(n_k)}(x)) \, dx \), and they described the reduction procedure of the \( k \)-th flow on the finite dimensional manifold of the stationary points of the \( j \)-th flow. They proved that all the flows of the hierarchy reduce to finite dimensional Hamiltonian system. The Hamiltonian function for the reduced \( k \)-th flow, \((-Q_{k,j}) \), is determined by:
\[
\frac{\delta I_j}{\delta u(x)} \frac{d}{dx} \frac{\delta I_k}{\delta u(x)} \equiv \frac{d}{dx} Q_{k,j}.
\]

Mokhov [Mo] generalized this result to not necessarily Hamiltonian evolutionary PDEs.

2. Scaling reductions of evolutionary systems: Hamiltonian formulation

In this Section we extend the Bogojavlenskii–Novikov scheme to finite dimensional invariant submanifolds specified by time–dependent local symmetries.

We start from a partial differential equation of order \( m \) on the functional space \( \mathfrak{M} \), describing the evolution of the function \( u(x) \) in the time \( t \) and a scaling symmetry

\[
u_s = G(x, t, u, u_x, \ldots, u^{(k)}).
\]

Our main assumption is that the set of stationary points of the symmetry can be formally represented in the Euler–Lagrange form

\[
\frac{\delta I}{\delta u(x)} = 0,
\]

\[
I = \int L(x, t, u, u_x, \ldots, u^{(n)}) dx,
\]

\[
\frac{dI}{dt} \equiv 0.
\]

It is an ordinary differential equation of order \( 2n \) depending explicitly on the parameter \( t \). If \( L \) is nondegenerate, the space of the solutions is a \( 2n \) dimensional manifold \( \mathcal{S} \), which naturally carries a system of canonical coordinates. As in Section 1 we will show that, in these coordinates, the Euler–Lagrange equation is Hamiltonian, with Hamiltonian function \( H \), obtained from \( L \) via Legendre transform:

\[
H = -L + \sum_{i=1}^{n} p_i \frac{dq_i}{dx}.
\]

Following the scheme of [BN], we prove that one can reduce on \( \mathcal{S} \) also the equation of the evolution in \( t \), which results to be a Hamiltonian system. We also give a universal scheme to produce the time-dependent Hamiltonian function of this reduced system. Indeed the following theorem holds:

**Theorem 2.1:** If the evolutionary PDE:

\[
u_t = F(u, u_x, \ldots, u^{(m)}),
\]

admits a nondegenerate scaling symmetry, then, on the manifold \( \mathcal{S} \) of the stationary points of the symmetry:

\[
\frac{\delta I}{\delta u(x)} = 0,
\]

\[
I = \int L(x, t, u, u_x, \ldots, u^{(n)}) dx,
\]

\[
\frac{dI}{dt} \equiv 0,
\]

it reduces to a Hamiltonian motion in \( t \), for the time dependent Hamiltonian function \( (-\tilde{Q}) \), that is the reduction on \( \mathcal{S} \) of

\[
Q = \Lambda - \sum_{i=1}^{n} p_i \frac{dq_i}{dt} \quad (2.1)
\]
where \( p_i, q_i \) are the canonical coordinates on \( \mathcal{S} \), expressed in terms of \( u, u_x, \ldots, u^{(2n-1)} \), and the function \( \Lambda \) is determined by

\[
\frac{dL}{dt} = \frac{d\Lambda}{dx}. \tag{2.2}
\]

**Proof:** We prove the theorem in three steps: first we describe the submanifold \( \mathcal{S} \) of stationary points of the symmetry \( I \), where we introduce a system of canonical coordinates; then we deduce, on \( \mathcal{S} \), a zero–curvature equation for \((-\tilde{Q})\) and the Hamiltonian function \( H \) of the reduced \( x \)–flow. Finally we prove that the restricted \( t \)–flow is Hamiltonian on \( \mathcal{S} \), with Hamiltonian function \((-\tilde{Q})\).

1) The manifold \( \mathcal{S} \) is the \( 2n \)–dimensional manifold of the solutions of the Euler–Lagrange equation

\[
\frac{\delta I}{\delta u(x)} = 0. \tag{2.3}
\]

It is invariant under the \( t \)–flow and it naturally carries a system of canonical coordinates:

\[
q_i = u^{(i-1)}, \quad i = 1, 2, \ldots, n \tag{2.4a}
\]

\[
p_i = \frac{\delta I}{\delta u^{(i)}}, \quad i = 1, 2, \ldots, n \tag{2.4b}
\]

obtained via generalized Lagrange transform (here we suppose that the generalized Lagrangian \( L \) is nondegenerate). Observe that now the \( p_i \) depend on \( x \) and on \( t \).

Reversing relations (2.4), one can express the derivatives \( u, u_x, \ldots, u^{(2n-1)} \) in terms of the canonical coordinates \( p_i \) and \( q_i \), \( x \) and \( t \); explicitly:

\[
\begin{align*}
  u^{(n)} &= (q_n)_x = g_1(x,t,q_1,\ldots,q_n,p_n) \\
  u^{(n+1)} &= g_2(x,t,q_1,\ldots,q_n,p_n,p_{n-1}) \\
  \quad \vdots \quad &\quad \vdots \\
  u^{(2n-1)} &= g_n(x,t,q_1,\ldots,q_n,p_n,\ldots,p_1).
\end{align*}
\]

Observe that (2.4) gives the identities:

\[
\begin{align*}
(p_1)_x + \frac{\partial H}{\partial q_1} &= -\frac{\delta I}{\delta u} \\
(p_i)_x + \frac{\partial H}{\partial q_i} &= 0, \quad i > 1 \\
(q_i)_x - \frac{\partial H}{\partial p_i} &= 0,
\end{align*} \tag{2.5}
\]

where \( H \) is the generalized Legendre transform of \( L \):

\[
-H + \sum_{i=1}^{n} \frac{\delta I}{\delta u^{(i)}} u^{(i)}
\]

which, in terms of the canonical coordinates takes the form:

\[
H = -L + \sum_{i=1}^{n} p_i \frac{dq_i}{dx}.
\]

The first of identities (2.5) allows us to express the higher derivatives \( u^{(m)} \) for \( m \geq 2n \) in terms of \( x,t, p_i, q_i \) and \( p_i^{(l)} \) with \( l = 1, \ldots, m - 2n + 1 \), explicitly:

\[
\begin{align*}
  u^{(2n)} &= g_{n+1}(x,t,q_1,\ldots,q_n,p_n,\ldots,p_1,(p_1)_x) \\
  \quad \vdots \quad &\quad \vdots \\
  u^{(m)} &= g_{m-n+1}(x,t,q_1,\ldots,q_n,p_n,\ldots,p_1,\ldots,\ldots,(p_1)^{(m-2n+1)}).
\end{align*}
\]
On \( S \) it reduces to \( (p_i)_x + \frac{\partial H}{\partial q_i} = 0 \), and the system (2.5) is a canonical Hamiltonian system, with Hamiltonian function \( H \), giving the reduced \( x \)-flow.

Now we will show that also the \( t \)-flow reduces on \( S \) with Hamiltonian function \((-\tilde{Q})\).

Firstly we observe that \( Q \) is a function of \( x,t,u \) and its \( x \)-derivatives up to the order \((m+n)\), then it can be rewritten in terms of \( x \), \( t \), \((p_i,q_i)\) and \( p_i^{(i)} \) up to the order \( l = m - n + 1 \).

We denote with \( \tilde{Q} \) a function \( f(x,t,u(x),\ldots,u(x)^{(j)}) \) reduced on \( S \); notice that, if \( j \geq 2n \), then the reduction can be done using the relation

\[
(p_i)_x = -\frac{\partial H^{(s)}}{\partial q_i}.
\]

Then \( \tilde{f} \) does depend explicitly only on the \( p_i \) and \( q_i \), for \( i = 1,\ldots,n \) and on the time \( t \). In fact differentiating (2.6) one obtains the derivatives \( p_i^{(i)} \) in terms of the canonical coordinates \((p_i,q_i)\).

2) We consider the derivative

\[
\frac{dL}{dt} = \frac{\partial L}{\partial t} + \sum_{i=1}^{n} \frac{\partial L}{\partial q_i} \frac{dq_i}{dt} + \sum_{i=1}^{n} \frac{\partial L}{\partial p_i} \frac{dp_i}{dt} = -\frac{\partial H}{\partial t} - \sum_{i=1}^{n} \frac{\partial H}{\partial q_i} \frac{dq_i}{dt} + \sum_{i=1}^{n} \frac{d^2 q_i}{dxdt}.
\]

From the fact that \( I \) is a first integral, one deduces that \( \frac{dt}{dx} \) must be the total derivative in \( x \) of a functional \( \Lambda \) that does depend on \( x,t,(p_i,q_i) \) and \( p_i^{(i)} \) up to the order \( l = m - n + 1 \); we have:

\[
\frac{d\Lambda}{dx} = \frac{\partial Q}{\partial x} + \sum_{i=1}^{n} \frac{\partial Q}{\partial q_i} \frac{dq_i}{dx} + \sum_{i=1}^{n} \frac{\partial Q}{\partial p_i} \frac{dp_i}{dx} + \sum_{i=1}^{m-n+1} \frac{\partial Q}{\partial (p_i)^{(i)}} \frac{d}{dx}(p_i)^{(i)} +
\]

\[
+ \sum_{i=1}^{n} \frac{dp_i}{dx} \frac{dq_i}{dx} + \sum_{i=1}^{n} \frac{d^2 q_i}{dxdt} = \frac{\partial Q}{\partial x} + \sum_{i=1}^{n} \frac{\partial Q}{\partial q_i} \frac{\partial H}{\partial p_i} - \sum_{i=2}^{n} \frac{\partial Q}{\partial p_i} \frac{\partial H}{\partial q_i} + \sum_{i=1}^{m-n+1} \frac{\partial Q}{\partial (p_i)^{(i)}} \frac{d}{dx}(p_i)^{(i)} +
\]

\[
- \sum_{i=2}^{n} \frac{\partial H}{\partial q_i} \frac{dq_i}{dx} + \sum_{i=1}^{n} \frac{d^2 q_i}{dxdt} + \frac{dq_i}{dt}(p_i)_x + \frac{\partial Q}{\partial (p_i)}(p_i)_x.
\]  

Then equation (2.2) gives:

\[
\frac{\partial H}{\partial t} + \frac{\partial H}{\partial x} + \sum_{i=1}^{n} \frac{\partial Q}{\partial q_i} \frac{\partial H}{\partial p_i} - \sum_{i=2}^{n} \frac{\partial Q}{\partial p_i} \frac{\partial H}{\partial q_i} + \frac{\partial Q}{\partial (p_i)}(p_i)_x +
\]

\[
+ \sum_{i=1}^{m-n+1} \frac{\partial Q}{\partial (p_i)^{(i)}} \frac{d}{dx}(p_i)^{(i)} + \frac{dq_i}{dt}(p_i)_x + \frac{\partial H}{\partial q_1} \frac{dq_1}{dt} = 0,
\]

which can be rewritten as

\[
\frac{\partial H}{\partial t} + \frac{dq_1}{dt}(p_1)_x + \frac{\partial H}{\partial q_1} = -\frac{d}{dx}Q.
\]

At this point we need the

**Lemma 2.1:** On the submanifold \( S \) the following relation holds:

\[
\left( \frac{\partial\tilde{Q}}{\partial (p_i)^{(j)}} \right) = 0 \quad \forall j \geq 1.
\]
Proof: See Appendix 2.A

Hence, on the submanifold $\mathcal{S}$ eq. (2.9) reduces to:

$$\frac{\partial H}{\partial t} + \frac{\partial \tilde{Q}}{\partial x} + \sum_{i=1}^{n} \frac{\partial \tilde{Q}}{\partial q_i} \frac{\partial H}{\partial p_i} - \sum_{i=1}^{n} \frac{\partial \tilde{Q}}{\partial p_i} \frac{\partial H}{\partial q_i} = 0$$

This is a zero-curvature equation:

$$\{(-\tilde{Q}), H\} + \frac{\partial (-\tilde{Q})}{\partial x} - \frac{\partial H}{\partial t} = 0.$$

This completes the second step in the proof of the theorem.

3. Now we will construct the Hamiltonian system inductively; to this end we need a further lemma:

**Lemma 2.2:** The fundamental relation

$$\frac{dq_i}{dt} = -\frac{\partial \tilde{Q}}{\partial p_i} \tag{2.11}$$

holds.

**Proof:** See Appendix 2.A

For simplicity, here and in the following we omit the “tilde” sign: $Q$ will indicate the reduced function on $\mathcal{S}$. Now, we assume that $\frac{dq}{dt} = -\frac{\partial Q}{\partial p_i} = -\{q_i, Q\}$ and we prove inductively that the same relation holds for $q_{i+1}$. The scheme of the procedure is the same as in [BN], the only differences are the contributions of the partial derivatives in $t$ and $x$. Indeed,

$$\frac{dq_{i+1}}{dt} = (\frac{dq}{dt})_x = -\frac{d}{dx}\{q_i, Q\} = -\{\{q_i, Q\}, H\} - \{q_i, \frac{\partial Q}{\partial x}\}.$$ 

Using the Jacobi identity and the zero-curvature relation we get

$$\frac{dq_{i+1}}{dt} = -\{\frac{\partial H}{\partial t}, q_i\} - \{q_{i+1}, Q\}, \quad i = 1, 2, ..., n - 1.$$ 

Here the term $\{\frac{\partial H}{\partial t}, q_i\}$ is zero for every $i \neq n$ since $\frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$. Indeed $L$ depends on $u$ and on the derivatives of $u$ up to the order $n$. This means that, restricted on $\mathcal{S}$, it depends on $q_1, q_2, ..., q_{n+1}$. Then, there is no dependence on the $p_i$ for $i \neq n$. Finally we get

$$\{\frac{\partial H}{\partial t}, q_n\} = \{\frac{\partial L}{\partial t}, q_n\} \neq 0, \quad i < n.$$ 

Hence we have proved that

$$\frac{dq_i}{dt} = -\{q_i, Q\}, \quad i = 1, 2, ..., n.$$ 

Now we prove that $\frac{dp_i}{dt} = -\{p_i, Q\}$ by induction, starting from $p_n$. 

This comes from the commutativity of the flows

\[
\frac{d}{dt} \frac{dx}{dx} q_n = \frac{d}{dx} \frac{dx}{dt} q_n;
\]

explicitly:

\[
\frac{d}{dt} \left( \frac{d}{dx} q_n \right) = \frac{d}{dt} \left( \frac{\partial H}{\partial p_n} \right) = \\
= \frac{\partial}{\partial t} \frac{\partial H}{\partial p_n} \sum_{i=1}^{n} \frac{\partial H}{\partial p_n} \frac{\partial Q}{\partial q_i} \frac{\partial q_i}{\partial p_i} + \frac{\partial^2 H}{\partial p_n^2} \frac{d}{dt} p_n = \\
= \{q_n, \frac{\partial H}{\partial t}\} - \sum_{i=1}^{n} \frac{\partial H}{\partial p_n} \frac{\partial Q}{\partial q_i} \frac{\partial q_i}{\partial p_i} + \frac{\partial^2 H}{\partial p_n^2} \frac{d}{dt} p_n.
\]

On the other hand, using the Jacobi identity and the zero-curvature equation, one can write

\[
\frac{d}{dx} \left( \frac{d}{dt} q_n \right) = - \frac{\partial}{\partial x} \frac{dQ}{dt} - \{\{q_n, Q\}, H\} = \\
= - \{q_n, \{Q, H\}\} - \{\{q_n, H\} Q - q_n, \frac{\partial Q}{\partial x}\} = \\
= \{q_n, \frac{\partial H}{\partial t}\} - \{\frac{\partial H}{\partial p_n}, Q\} = \\
= \{q_n, \frac{\partial H}{\partial t}\} - \sum_{i=1}^{n} \frac{\partial H}{\partial p_n} \frac{\partial Q}{\partial q_i} \frac{\partial q_i}{\partial p_i} + \frac{\partial^2 H}{\partial p_n^2} \frac{dQ}{dq_n}.
\]

Comparing the two expressions and noticing that \(\frac{\partial^2 H}{\partial p_n^2} \neq 0\) because of the nondegeneracy, we get

\[
\frac{dp_n}{dt} = -\{p_n, Q\} = \frac{\partial Q}{\partial q_n}.
\]

- Now we suppose that \(\frac{dp}{dt} = \{p, Q\}\) and we deduce the same for \(p_{i+1}\). Indeed,

\[
\frac{d}{dx} \left( \frac{dp}{dt} \right) = \frac{\partial Q}{\partial q_i} H + \frac{\partial}{\partial x} \frac{\partial Q}{\partial q_i} = \\
= - \{\{p, Q\}, H\} - \{p, \frac{\partial Q}{\partial x}\} = \\
= - \{\{p, H\}, Q\} + \{p, \frac{\partial H}{\partial t}\} = \\
= \{p, \frac{\partial H}{\partial t}\} + \sum_{j=1}^{n-1} \frac{\partial H}{\partial q_j} \frac{\partial Q}{\partial q_j} \frac{\partial q_j}{\partial p_j} - \frac{\partial^2 H}{\partial q_j \partial p_n} \frac{\partial q_n}{\partial q_j} - \frac{\partial^2 H}{\partial q_i \partial p_{n-1}} \frac{\partial q_{n-1}}{\partial q_i}
\]

and

\[
\frac{d}{dt} \left( \frac{dp}{dx} \right) = - \frac{d}{dt} \left( \frac{\partial H}{\partial q_i} \right) = \\
= - \frac{\partial}{\partial t} \frac{\partial H}{\partial q_i} - \sum_{j=1}^{n-1} \frac{\partial H}{\partial q_j} \frac{\partial Q}{\partial q_j} \frac{\partial q_j}{\partial p_j} - \frac{\partial^2 H}{\partial q_j \partial p_n} \frac{\partial q_n}{\partial q_j} - \frac{\partial^2 H}{\partial q_i \partial p_{n-1}} \frac{\partial q_{n-1}}{\partial q_i} = \\
= \{p, \frac{\partial H}{\partial t}\} + \sum_{j=1}^{n-1} \frac{\partial H}{\partial q_j} \frac{\partial Q}{\partial q_j} \frac{\partial q_j}{\partial p_j} - \frac{\partial^2 H}{\partial q_j \partial p_n} \frac{\partial q_n}{\partial q_j} - \frac{\partial^2 H}{\partial q_i \partial p_{n-1}} \frac{\partial q_{n-1}}{\partial q_i}.
\]
where \( \frac{\partial^2 H}{\partial q_i \partial p_{i-1}} = 1 \). Comparing the two expressions we get \( \frac{dp_i}{dt} = \frac{\partial Q}{\partial q_i} = -\{p_i, Q\}, \quad i = 1, 2, ..., n \)

Q.E.D.

**Remark:** The definition of \( Q \):

\[-Q = -\Lambda + \sum_{i=1}^{n} p_i \frac{dq_i}{dt}\]

looks very similar to the definition of the Hamiltonian function \( H \) of the \( x \)-flow:

\[H = -L + \sum_{i=1}^{n} p_i \frac{dq_i}{dx}\]

Here a symmetry between \( x \) and \( t \) seems to appear: one could be tempted to read the definition of \( Q \) as a Legendre transform and hence to read \( \Lambda \) as the Lagrangian of the \( t \)-flow. But it is not completely true: indeed the coordinates \( q_i \) and \( p_i \) are obtained from the Lagrangian \( L \), they are not, a priori, good coordinates for \( \Lambda \). In the next chapter we will perform a change of coordinates on \( S \), in order to read \( \Lambda \) as Lagrangian function.

**2.A Appendix**

**Proof of Lemma 2.1:** We observe that the recursive relation

\[
\left( \frac{\partial Q}{\partial (p_1)^{(j-1)}} \right) = \frac{\partial}{\partial (p_1)^{(j)}} \left( \frac{d}{dx} Q \right) - \frac{d}{dx} \left( \frac{\partial Q}{\partial (p_1)^{(j)}} \right)
\]

holds for \( j > 1 \). Indeed

\[
\frac{\partial}{\partial (p_1)^{(j)}} \left( \frac{d}{dx} Q \right) = \sum_{i=1}^{n} \left( \frac{\partial^2 Q}{\partial q_i \partial (p_1)^{(j)}} \right) (q_i)_x + \sum_{i=1}^{n} \left( \frac{\partial^2 Q}{\partial p_i \partial (p_1)^{(j)}} \right) (p_i)_x +
\]

\[
+ \sum_{i=1}^{j-n+1} \left( \frac{\partial^2 Q}{\partial (p_1)^{(i)} \partial (p_1)^{(j)}} \right) (p_1)_x^{(i)} + \left( \frac{\partial Q}{\partial (p_1)^{(j-1)}} \right).
\]

When we reduce on \( S \):

\[
\left( \frac{\partial \tilde{Q}}{\partial (p_1)^{(j-1)}} \right) = - \frac{d}{dx} \left( \frac{\partial \tilde{Q}}{\partial (p_1)^{(j)}} \right).
\]

But \( Q \) depends on \( (p_1)^{(j)} \) up to a finite order, then

\[
\left( \frac{\partial \tilde{Q}}{\partial (p_1)^{(j)}} \right) = 0 \quad \forall j \geq 1.
\]

Q.E.D.

**Proof of Lemma 2.2:** The expansion of \( \frac{d}{dx} Q \) in powers of \( (p_1)_x \) near the point \( \frac{\partial H}{\partial q_1} \) reads

\[
\left( \frac{d}{dx} Q \right) + \left[ \frac{d}{d(p_1)_x} \left( \frac{d}{dx} Q \right) \right] (p_1)_x + \frac{\partial H}{\partial q_1} + \Theta \left( (p_1)_x + \frac{\partial H}{\partial q_1} \right)^2.
\]
The zero order term is
\[
\left( \frac{d}{dx} \widetilde{Q} \right) = -\frac{\partial H}{\partial t}
\]
by virtue of the zero-curvature equation. The first order coefficient is
\[
\left[ \frac{d}{d(p_1)_x} \left( \frac{d}{dx} \widetilde{Q} \right) \right] = \left[ \frac{d}{dx} \left( \frac{\partial}{\partial (p_1)_x} \widetilde{Q} \right) \right] + \frac{\partial \widetilde{Q}}{\partial p_1} - \sum_{i=2}^{m-n+1} \left( \frac{\partial Q}{\partial (p_1)^{(i)}} \frac{\partial (p_1)^{(i)}}{\partial p_1} \right).
\]
where the only non zero term is \( \frac{\partial \widetilde{Q}}{\partial p_1} \), by virtue of Lemma 2.1. Hence we obtain the power series expansion of \( \frac{d}{dx} Q \) up to the first order:
\[
-\frac{\partial H}{\partial t} + \frac{\partial \widetilde{Q}}{\partial p_1} \left( (p_1)_x + \frac{\partial H}{\partial q_1} \right)
\]
this, compared with the left-hand side of eq. (2.9), gives the relation (2.11).
Q.E.D.

3. Scaling reductions of evolutionary systems:
Lagrangian formulation

3.1 General framework

The basic idea is to develop a reduction method dealing on the same footing with \( x \) and \( t \). The starting point is always the evolutionary PDE
\[
u_t = F(u, u_x, \ldots, u^{(m)}),
\]
in the space \( \mathcal{M} \) described in Section 1.1. The first step of our construction is to read \( u \) as a function of \( x \) and \( t \) and to consider equation (3.1) as a definition of \( u^{(m)}(x, t) \) in terms of \( u(x, t), u_x(x, t), \ldots, u^{(m-1)}(x, t) \) and \( u_t(x, t) \).

This corresponds to consider as “coordinates” in \( \mathcal{M} \), instead of \( u(x, t) \) and its derivatives in \( x \):
\[
u, u_x, u_{xx}, \ldots
\]
(here and in the following \( u \) indicate the function \( u(x, t) \)), the system
\[
u, u_x, \ldots, u^{(m-1)}, u_t, u_{xt}, \ldots, u^{(m-1)}_t, u_{tt}, \ldots
\]
By virtue of the reversibility of (3.1) in \( u^{(m)}(x, t) \) it is possible to perform this “change of variables”. If one introduce the vector
\[
\tilde{u} = (u, u_x, \ldots, u^{(m-1)}),
\]
the new system of “coordinates” in \( \mathcal{M} \) is given by \( \tilde{u}(x, t) \) and its derivatives in \( t \):
\[
\tilde{u}, \tilde{u}_t, \tilde{u}_{tt}, \ldots
\]
At this point one takes a first integral of eq (3.1), i.e. a functional

\[ I = \int L \left( x, t, u(x), u_x(x), \ldots, u^{(n)}(x) \right) dx \]  \quad (3.2)

in the space \( \mathcal{M} \), such that

\[ \frac{\delta I}{\delta u(x)} = 0. \]  \quad (3.3)

This Euler–Lagrange equation defines a finite dimensional manifold \( \mathcal{S} \), i.e. the set of the fixed points of \( I \). Indeed the Euler–Lagrange equation (3.3) is an ODE of order \( 2n \), so that the space of the solutions is a \( 2n \)-dimensional manifold; \( \mathcal{S} \) is modeled on this space, having as coordinates certain combinations of the initial values, i.e. of the first \( (2n - 1) \) \( x \)-derivatives of \( u(x) \) evaluated at \( x_0 \).

In Lemma 3.2 below, we rewrite the definition of the manifold \( \mathcal{S} \) in terms of \( \bar{u}(t), \bar{u}_t(t), \ldots \) and of a functional

\[ J = \int \Lambda \left( x, t, \bar{u}(t), \bar{u}_t(t), \ldots, \bar{u}^{(\beta)}(t) \right) dt, \]

where \( \Lambda \) can be calculated from \( L \) (see eq. (3.4)), and the order \( \beta \) of derivation in \( t \) depends on the ratio between \( m \) and \( n \), as we will show in detail in Section 3.3.

In Theorem 3.1 we will prove that \( \Lambda \left( x, t, \bar{u}(t), \bar{u}_t(t), \ldots, \bar{u}^{(\beta)}(t) \right) \) is the generalized Lagrangian for the \( t \)-flow reduced on \( \mathcal{S} \). Indeed equation (3.1) can be rewritten in form of a Euler–Lagrange equation:

\[ \frac{\delta J}{\delta \bar{u}(t)} = 0, \]

for the vector

\[ \frac{\delta J}{\delta \bar{u}(t)} = \left( \frac{\delta J}{\delta u(t)}, \frac{\delta J}{\delta u_x(t)}, \ldots, \frac{\delta J}{\delta u^{(m-1)}(t)} \right), \]

where

\[ \frac{\delta J}{\delta u^{(i)}(t)} = \frac{\partial \Lambda}{\partial u^{(i,0)}} + \sum (-1)^\alpha \frac{d^\alpha}{dt^\alpha} \frac{\partial \Lambda}{\partial u^{(i,\alpha)}}. \]

In the multiindex \( (i, \alpha) \) the Latin character indicates the order in the \( x \)-derivative, the Greek indicate the order in the \( t \)-derivative.

Explicitly, equation (3.1) reads

\[ \frac{\partial \Lambda}{\partial u} + \sum (-1)^\alpha \frac{d^\alpha}{dt^\alpha} \frac{\partial \Lambda}{\partial u^{(\alpha)}} = 0 \]
\[ \frac{\partial \Lambda}{\partial u_x} + \sum (-1)^\alpha \frac{d^\alpha}{dt^\alpha} \frac{\partial \Lambda}{\partial u^{x(\alpha)}} = 0 \]
\[ \vdots \]
\[ \frac{\partial \Lambda}{\partial u^{(m-1)}} + \sum (-1)^\alpha \frac{d^\alpha}{dt^\alpha} \frac{\partial \Lambda}{\partial u^{(m-1,\alpha)}} = 0. \]

We will formalize these facts in the following Theorem 3.1; here we give an idea of the proof, ignoring all the calculations, that we will concentrate in Lemma 3.1 and 3.2, the proof of which is postponed in Appendix 3.A.

The proof is performed in four steps: firstly we define the new system of “coordinates” in \( \mathcal{M} \), and we give some useful relations between the new and the old “coordinates”. As a second step we rewrite the
Lagrangian density \( L(u(x)) \) in terms of the new “coordinates” and we construct, starting from \( \hat{L}(\bar{u}(t)) \), the Lagrangian density \( \hat{\Lambda}(\bar{u}(t)) \).

The third step consists in recovering the relation between performing the variation of \( L \) (in \( x \)) and of \( \hat{\Lambda} \) (in \( t \)). The most relevant relation is that given in Lemma 3.1. This relation is necessary to rewrite the Euler–Lagrange equation (3.2), defining \( \mathfrak{S} \), as a condition on \( \hat{\Lambda} \). The explicit form of this condition is given in Lemma 3.2.

Finally we prove that, under this condition, i.e., after performing the reduction on \( \mathfrak{S} \), the starting evolution equation (3.1), reads as an Euler–Lagrange equation for \( \hat{\Lambda} \).

The method of Hamiltonian reduction described in Chapter 2 allows us to put a canonical system of coordinates \( \{p_i, q_i\} \) on \( \mathfrak{S} \) (see formula (2.4)). These coordinates are obtained from \( L \) via generalized Lagrange transform, so that they are, in a certain sense, adapted to the \( x \)-flow. This means that in these coordinates the reduced \( x \)-flow is a Hamiltonian system. Theorem 2.1 also gives the explicit form of the Hamiltonian function

\[
H = -L + \sum_i p_i(q_i)_x.
\]

The method of Lagrangian reduction which we describe in this Chapter, still allows us to define a system of canonical coordinates: we will call it \( \{\tilde{p}_i, \tilde{q}_i\} \). These coordinates are obtained from \( \hat{\Lambda} \), i.e., they are adapted to the \( t \)-flow; in fact we will prove (see Section 3.3) that, in these coordinates, the reduced \( t \)-flow is a Hamiltonian system, with Hamiltonian function

\[
-\hat{Q} = -\hat{\Lambda} + \sum_i \tilde{p}_i(\tilde{q}_i)_t.
\]

When rewritten in terms of \( \{p_i, q_i\} \), the Hamiltonian \( \hat{Q} \) coincides with the Hamiltonian function \( Q \) constructed by Bogoyavlenskii and Novikov.

In this sense the alternative definition of \( Q \) given by us in Theorem 2.1:

\[
-Q = -\Lambda + \sum_i p_i(q_i)_t,
\]

is a Legendre transformation, if one uses the right system of canonical coordinates (see below).

### 3.2 Lagrangian reduction

**Theorem 3.1:** If the evolutionary PDE:

\[
u_t = F(u, u_x, \ldots, u^{(m)}),
\]

admits a nondegenerate scaling symmetry, then, on the manifold \( \mathfrak{S} \) of the stationary points of the symmetry:

\[
\frac{\delta I}{\delta u(x)} = 0,
\]

\[
I = \int L(x, t, u, u_x, \ldots, u^{(n)})dx,
\]

\[
\frac{dI}{dt} \equiv 0,
\]

it reduces to a Lagrangian motion in \( t \), for the time dependent Lagrangian function \( \Lambda \), determined by:

\[
\frac{dL}{dt} = \frac{d\Lambda}{dx}.
\]
Explicitly, the first relation has the form

\[ u^{(m)} = f_0(u, u_x, \ldots, u^{(m-1)}, u_t). \]  

(3.5)

Differentiating eq. (3.5) in \( x \) one obtains all the \( x \)-derivatives of \( u \) of order greater then \( m \) in terms of \( u, u_x, \ldots, u^{(m-1)} \) and their \( t \)-derivatives:

\[
\begin{align*}
  u^{(m+1)} &= f_1(u, u_x, \ldots, u^{(m-1)}, u_t, u_{xt}) \\
  \vdots \\
  u^{(2m-1)} &= f_{m-1}(u, u_x, \ldots, u^{(m-1)}, u_t, u_{xt}, \ldots, u^{(m-1)}_{x}) \\
  u^{(2m)} &= f_m(u, u_x, \ldots, u^{(m-1)}, u_t, u_{xt}, \ldots, u^{(m-1)}_{x}) \\
  \vdots \\
  u^{(m+n)} &= f_n(u, u_x, \ldots, u^{(m-1)}, u_t, u_{xt}, \ldots, u^{(m-1)}_{x}, u_{tt}, u_{xtt}, \ldots, u^{(n-m)}_{tt}).
\end{align*}
\]

Explicitly, the first relation has the form

\[ u^{(m+1)} = \frac{\partial u^{(m)}}{\partial x} + \frac{\partial u^{(m)}}{\partial x} u_x + \ldots + \frac{\partial u^{(m)}}{\partial x} u^{(m-1)} + \frac{\partial u^{(m)}}{\partial x} u_{xt}, \]

and in general, using the multiindex notation introduced in Section 3.1,

\[ u^{(m+j)} = \frac{\partial u^{(m+j-1)}}{\partial x} + \sum_{k=0}^{m-1} \sum_{\beta=0}^{\alpha} \frac{\partial u^{(m+j-1)}}{\partial u^{(k,\beta)}} u^{(k+1,\beta)}, \]

where the higher order \( \alpha \) in the \( t \)-derivative is fixed by \((\alpha - 1)m \leq j \leq \alpha m\).

This completes the construction of the map from

\[ u, u_x, \ldots, u^{(m-1)}, u^{(m)}, \ldots, u^{(n)}, \ldots, u^{(2m-1)}, u^{(2m)}, \ldots, \]

to the new system of “coordinates”

\[ u, u_x, \ldots, u^{(m-1)}, u_t, u^{(m-1)}, u_{tt}, u_{xtt}, \ldots, u^{(n-m)}, \ldots. \]

Here below we list some noteworthy relationships between the two systems (they will be useful in the following):

\[
\begin{align*}
  u^{(m)}_t &= \frac{\partial u^{(m)}}{\partial t} + \frac{\partial u^{(m)}}{\partial x} u_t + \ldots + \frac{\partial u^{(m)}}{\partial x} u^{(m-1)} + \frac{\partial u^{(m)}}{\partial x} u_{tt}, \quad (3.6a) \\
  \frac{d}{dx} \left( \frac{\partial u^{(i)}}{\partial u_t} \right) &= \frac{\partial u^{(i+1)}}{\partial u_t} - \frac{\partial u^{(i+1)}}{\partial u^{(m-1)}} \frac{\partial u^{(m)}}{\partial u_t} \quad (3.6b) \\
  \frac{d}{dx} \left( \frac{\partial u^{(i)}}{\partial u^{(k)}} \right) &= \frac{\partial u^{(i+1)}}{\partial u^{(k)}} - \frac{\partial u^{(i+1)}}{\partial u^{(k-1)}} \frac{\partial u^{(k-1)}}{\partial u^{(k)}} \quad (3.6c) \\
  \frac{d}{dx} \left( \frac{\partial u^{(i)}}{\partial u^{(k)}} \right) &= \frac{\partial u^{(i+1)}}{\partial u^{(k)}} - \frac{\partial u^{(i+1)}}{\partial u^{(k-1)}} \frac{\partial u^{(k-1)}}{\partial u^{(k)}} - \frac{\partial u^{(i+1)}}{\partial u^{(m-1)}} \frac{\partial u^{(m)}}{\partial u^{(k)}} \quad (3.6d) \\
  \frac{d}{dt} \left( \frac{\partial u^{(i)}}{\partial u_t} \right) &= \frac{\partial u^{(i)}}{\partial u_t} - \frac{\partial u^{(i)}}{\partial u}. \quad (3.6e)
\end{align*}
\]
2. Lagrangian densities: The Lagrangian \( L \) defining the symmetry, depends on \( u, u_x, \ldots, u^{(n)} \), so that its derivative \( \frac{dL}{dt} \) depends on \( u, u_x, \ldots, u^{(m+n)} \). In terms of the new “coordinates” one may rewrite \( L \) as

\[
\hat{L}(x, t, u, u_x, \ldots, u^{(m-1)}, u_t, \ldots, u_t^{(n-m)})
\]

and

\[
\left( \frac{d\hat{L}}{dt} \right) = \left( \frac{\partial \hat{L}}{\partial t} \right) + \left( \frac{\partial \hat{L}}{\partial u} \right) u_t + \ldots + \left( \frac{\partial \hat{L}}{\partial u^{(m-1)}} \right) u_t^{(m-1)} + \\
+ \left( \frac{\partial \hat{L}}{\partial u^{(m)}} \right) u_t^{(m)}(u, \ldots, u_{xt}) + \left( \frac{\partial \hat{L}}{\partial u^{(m+1)}} \right) u_t^{(m+1)}(u, u_{tt}, u_{xtt}) + \ldots + \\
+ \left( \frac{\partial \hat{L}}{\partial u^{(n)}} \right) u_t^{(n)}(u, \ldots, u_{tt}, \ldots, u_{tt}^{(n-m)})
\] (3.8a).

Of course, (3.8a) coincides with

\[
\frac{d\hat{L}}{dt} = \frac{\partial \hat{L}}{\partial t} + \frac{\partial \hat{L}}{\partial u} u_t + \ldots + \frac{\partial \hat{L}}{\partial u^{(m-1)}} u_t^{(m-1)} + \frac{\partial \hat{L}}{\partial u_t} u_{tt} + \ldots + \frac{\partial \hat{L}}{\partial u_t^{(n-m)}} u_{tt}^{(n-m)}
\] (3.8b),

where

\[
\frac{\partial \hat{L}}{\partial t} = \left( \frac{\partial \hat{L}}{\partial t} \right) + \sum_{k=m}^{n} \left( \frac{\partial \hat{L}}{\partial u^{(k)}} \right) \frac{\partial u^{(k)}}{\partial t}
\] (3.9a)

\[
\frac{\partial \hat{L}}{\partial u^{(i)}} = \left( \frac{\partial \hat{L}}{\partial u^{(i)}} \right) + \sum_{k=m}^{n} \left( \frac{\partial \hat{L}}{\partial u^{(k)}} \right) \frac{\partial u^{(k)}}{\partial u^{(i)}}
\] (3.9b) \quad i = 0, \ldots, m - 1

\[
\frac{\partial \hat{L}}{\partial u_t^{(i)}} = \sum_{k=m+i}^{n} \left( \frac{\partial \hat{L}}{\partial u^{(k)}} \right) \frac{\partial u^{(k)}}{\partial u_t^{(i)}}
\] (3.9c) \quad i = 0, \ldots, m - 1.

From the fact that \( I \) is a first integral, it follows that there exists a functional

\[
\hat{\Lambda}(x, t, u, u_x, \ldots, u^{(m-1)}, u_t, \ldots, u_t^{(n-m-1)}),
\]

such that

\[
\frac{d\hat{L}}{dt} = \frac{d\hat{\Lambda}}{dx},
\]

where

\[
\frac{d\hat{\Lambda}}{dx} = \frac{\partial \hat{\Lambda}}{\partial x} + \frac{\partial \hat{\Lambda}}{\partial u} u_x + \ldots + \frac{\partial \hat{\Lambda}}{\partial u^{(m-2)}} u^{(m-1)} + \\
+ \frac{\partial \hat{\Lambda}}{\partial u^{(m-1)}} u^{(m)}(u, u_x, \ldots, u^{(m-1)}, u_{tt}) + \frac{\partial \hat{\Lambda}}{\partial u_t} u_{xt} + \ldots + \frac{\partial \hat{\Lambda}}{\partial u_t^{(m-2)}} u_t^{(m-1)} + \\
+ \frac{\partial \hat{\Lambda}}{\partial u_t^{(m-1)}} u_t^{(m)}(u, u_{tt}) + \frac{\partial \hat{\Lambda}}{\partial u_{tt}} u_{xt} + \ldots + \frac{\partial \hat{\Lambda}}{\partial u_{tt}^{(n-m-1)}} u_{tt}^{(n-m)}.
\] (3.10)

3. Variations: Our aim is to reduce equation (3.3) on the space \( \mathcal{G} \) of the stationary points of \( I = \int Ldx \). This finite–dimensional manifold is defined by the Euler–Lagrange equation

\[
\frac{\delta I}{\delta u(x)} = 0.
\]
This is a variational equation in the old “coordinates” \(u, u_x, \ldots\); how can we define the same manifold \(\mathcal{S}\) in terms of the new “coordinates”? We must express \(\frac{\delta I}{\delta u(x)}\) in terms of \(\Lambda\) and its variations. To this end we recall that

\[
\frac{\delta I}{\delta u(x)} = \frac{\partial L}{\partial u} + \sum (-1)^j \frac{d}{dt} \frac{\partial L}{\partial u^{(j)}},
\]

and we first express the terms

\[
\frac{\delta L}{\delta u^{(j)}(x)}
\]

for \(j > 0\) in terms of \(\hat{\Lambda}\) and the the new “coordinates”, namely:

**Lemma 3.1:** The following recurrence relation holds:

\[
\frac{\partial \hat{\Lambda}}{\partial u_i} \frac{d}{dt} \left( \frac{\partial \hat{\Lambda}}{\partial u^{(i)}(t)} \right) = \left( \frac{\delta I}{\delta u^{(i+1)}(x)} \right) + \sum_{j=m+1}^{n} \left( \frac{\delta I}{\delta u^{(j)}(x)} \right) \frac{\partial u^{(j-i)}}{\partial u^{(i)}}
\]

\(i = 1, \ldots, m - 1\) \(3.11\)

**Proof:** see Appendix 3.A

The proof of Lemma 3.1 is based on the comparison of (3.8) and (3.10) and their partial derivatives w.r.t. \(u_i^{(j)}\) and \(u_i^{(j)}\). With a similar technique, and using equation (3.11), one can proves the fundamental

**Lemma 3.2:** The (generalized) Euler–Lagrange equation

\[
\frac{\delta I}{\delta u(x)} = 0
\]

is equivalent to the condition:

\[
\frac{\partial \hat{\Lambda}}{\partial u^{(m-1)}} - \frac{d}{dt} \left( \frac{\partial \hat{\Lambda}}{\partial u^{(m-1)}(t)} \right) = 0.
\]

\(3.12\)

**Proof:** see Appendix 3.A

Introducing the functional

\[
J = \int \hat{\Lambda} \left( x, t, u(t), u_x(t), \ldots, u^{(m-1)}(t), u_t(t), \ldots, u^{(n-m-1)}_{tt}(t) \right) dt,
\]

equation (3.12) reads

\[
\frac{\delta J}{\delta u^{(m-1)}(t)} = 0.
\]

Notice that the object in the left hand side is the last component of the vector

\[
\frac{\delta J}{\delta \hat{u}(t)} = \left( \frac{\delta J}{\delta u(t)} \frac{\delta J}{\delta u_x(t)} \cdots \frac{\delta J}{\delta u^{(m-1)}(t)} \right).
\]

**4. Reduced evolutionary equation:** Here we prove that all the components of the vector \(\frac{\delta J}{\delta \hat{u}(t)}\) are zero. This can be done recursively, by mean of
Lemma 3.3: The following recurrence relation holds:

\[
\frac{\delta J}{\delta u^{(i-1)}(t)} = \frac{\delta J}{\delta u^{(m-1)}(t)} \frac{\partial u^{(m)}}{\partial u^{(i)}} - \frac{d}{dx} \left( \frac{\delta J}{\delta u^{(i)}(t)} \right) \quad i = 1, \ldots, m - 1. \tag{3.13a}
\]

Proof: see Appendix 3.A

Indeed, Lemma 3.2 states that the \((m - 1)\)-th component of \(\frac{\delta J}{\delta u^{(m-1)}(t)}\) is zero when reduced on \(\mathcal{S}\), hence, by virtue of (3.13a), all the components of \(\frac{\delta J}{\delta u(t)}\) vanish on \(\mathcal{S}\).

This is the Euler–Lagrange equation for the Lagrangian

\[
\tilde{\Lambda}(x, t, u, u_x, \ldots, u^{(m-1)}, u_t, \ldots, u^{(n-m)}_{tt}).
\]

Q.E.D.

Remark: Equation (3.13a) can be rewritten as

\[
\left[ \frac{\partial \tilde{\Lambda}}{\partial u^{(i-1)}} - \frac{d}{dt} \left( \frac{\partial \tilde{\Lambda}}{\partial u^{(i)}_t} \right) + \frac{d^2}{dt^2} \left( \frac{\partial \tilde{\Lambda}}{\partial u^{(i)}_{tt}} \right) \right] = \left[ \frac{\partial \tilde{\Lambda}}{\partial u^{(m-1)}} - \frac{d}{dt} \left( \frac{\partial \tilde{\Lambda}}{\partial u^{(m)}_t} \right) \right] - \frac{d}{dx} \left[ \frac{\partial \tilde{\Lambda}}{\partial u^{(i)}_t} - \frac{d}{dt} \left( \frac{\partial \tilde{\Lambda}}{\partial u^{(i)}_{tt}} \right) \right] + \frac{d^2}{dt^2} \left( \frac{\partial \tilde{\Lambda}}{\partial u^{(i)}_{tt}} \right). \tag{3.13b}
\]

for \(i = 1, \ldots, m - 1\). And The Euler–Lagrange equation reads

\[
\begin{cases}
\frac{\partial \tilde{\Lambda}}{\partial u^{(i)}} - \frac{d}{dt} \frac{\partial \tilde{\Lambda}}{\partial u^{(i)}_t} = 0 & i = n - m, \ldots, m - 1 \\
\frac{\partial \tilde{\Lambda}}{\partial u^{(i)}} - \frac{d}{dt} \frac{\partial \tilde{\Lambda}}{\partial u^{(i)}_t} + \frac{d^2}{dt^2} \frac{\partial \tilde{\Lambda}}{\partial u^{(i)}_{tt}} = 0 & i = 0, \ldots, n - m - 1.
\end{cases} \tag{3.14}
\]

3.3 Relation with the Hamiltonian reduction

Theorem 3.1 provides an alternative definition of the space \(\mathcal{S}\), and of the relative system of canonical coordinates:

\[
\begin{cases}
\tilde{q}_i = q_i = u^{(i-1)} & i = 1, \ldots, m \\
\tilde{q}_{m+i} = (q_i)_t = u^{(i-1)}_t & i = 1, \ldots, n - m \\
\tilde{p}_i = \frac{\delta J}{\delta u^{(i)}_t} & i = 1, \ldots, m \\
\tilde{p}_{m+i} = \frac{\delta \tilde{\Lambda}}{\partial u^{(i)}_{tt}} & i = 1, \ldots, n - m
\end{cases} \tag{3.15}
\]

We consider now the Hamiltonian \((-Q)\) of the reduced \(t\)-flow, defined in Theorem 2.1

\[
Q = \Lambda - \sum_{i=1}^{n} p_i(q_i)_t = \Lambda - \sum_{i=0}^{n-1} \frac{\delta I}{\delta u^{(i+1)}(t)} u^{(i)}_t
\]

At the end of the Chapter 2 we noticed how this expression looks very similar to a Legendre transform, but it is not; here we will show that actually the Legendre transform of the Lagrangian \(\tilde{\Lambda}\) gives the Hamiltonian \(Q\), where \(Q\) is written in the coordinate system relative to \(\tilde{\Lambda}\).
Firstly we rewrite \( \Phi \) in the coordinate system (3.15):

\[
-\hat{Q} = -\hat{\Lambda} + \sum_{i=0}^{n-1} \left( \frac{\delta I}{\delta u^{(i)}(x)} \right) u^{(i)}_t + \sum_{i=0}^{n-m-1} \sum_{j=m+i+1}^{n} \left( \frac{\delta I}{\delta u^{(j)}(x)} \right) \frac{\partial u^{(j-1)}}{\partial u^{(i)}} u^{(i)}_{tt} + \sum_{i=0}^{n-m-1} \sum_{j=m+i+1}^{n} \left( \frac{\delta I}{\delta u^{(j)}(x)} \right) \frac{\partial u^{(j-1)}}{\partial u^{(i)}} u^{(i)}_{tt}.
\]

Using Lemma 3.1 we get

\[
-\hat{Q} = -\hat{\Lambda} + \sum_{i=0}^{m-1} \frac{\delta J}{\delta u^{(i)}} u^{(i)} + \sum_{i=0}^{n-m-1} \sum_{j=0}^{m-1} \left( \frac{\delta I}{\delta u^{(j)}}(x) \right) \frac{\partial u^{(j-1)}}{\partial u^{(i)}} u^{(i)}_{tt} = -\hat{\Lambda} + \sum_{i=1}^{n} \hat{p}_i (\hat{q}_i)_t,
\]

for \( \Lambda(x,t,\hat{q}_1,\ldots,\hat{q}_{n-m}) \).

### 3.4 Concluding remarks

The case considered in Theorem 3.1 is the more general one. Indeed, for \((\alpha-1)m < n \leq \alpha m\), the Euler–Lagrange equation defining \( S \):

\[
\left( \frac{\delta I}{\delta u(x)} \right) = 0
\]

into the new “coordinates”, is a differential equation in \( u, \ldots, u^{(m-1)}, \ldots, u^{(n-m,\alpha)} \).

The Lagrangian \( L \) transforms into

\[
\hat{L}(x,t,u,u_x,\ldots,u^{(m-1)},u_t,\ldots,u^{(m-1)},u_{tt},\ldots,u^{(n-m,\alpha-1)})
\]

and we can define the new Lagrangian

\[
\hat{\Lambda}(x,t,u,u_x,\ldots,u^{(m-1)},u_t,\ldots,u^{(m-1)},u_{tt},\ldots,u^{(n-m,\alpha-1)}).
\]

The proof of the theorem is the same, one has only to consider the identities

\[
\frac{\partial}{\partial u^{(i,\beta)}} \left( \frac{dL}{dt} \right) = \frac{\partial}{\partial u^{(i,\beta)}} \left( \frac{d\Lambda}{dx} \right)
\]

for \( i = 1, \ldots, m-1 \) and \( \beta = 1, \ldots, \alpha \).

In particular, the manifold \( S \) is defined by

\[
\frac{\delta J}{\delta u^{(m-1)}(t)} = 0
\]

(3.17)

and it naturally carries the canonical system of coordinates

\[
\begin{align*}
\hat{q}_{i,m+i} &= u^{(i-1,\beta)} & i &= 1, \ldots, m; \quad \beta &= 0, \ldots, \alpha - 2 \\
\hat{q}_{(\alpha-1)m+i} &= u^{(i,\alpha-1)} & i &= 1, \ldots, n - (\alpha - 1)m \\
\hat{p}_i &= \frac{\delta J}{\delta (\hat{q}_i)} & i &= 1, \ldots, n
\end{align*}
\]

(3.18)
In the following we will consider in detail the case \( \alpha = 1 \), i.e. \( n < m \), which occurs in the applications we are interested in (see next chapter). In this case \( L \) and \( \hat{L} \) coincide and

\[
\frac{dL}{dt} = \frac{\partial L}{\partial t} + \frac{\partial L}{\partial u} u_t + \ldots + \frac{\partial L}{\partial u^{(n)}} u^{(n)}_t. \tag{3.19}
\]

The new Lagrangian is

\[
\hat{\Lambda}(x, t, u, u_x, \ldots, u^{(m-1)}, u_t, \ldots, u^{(n-1)}_t)
\]

with

\[
\frac{dL}{dt} = \frac{d\Lambda}{dx} \frac{\partial \Lambda}{\partial x} + \frac{\partial \Lambda}{\partial u} u_x + \ldots + \frac{\partial \Lambda}{\partial u^{(m-1)}} u^{(m)} + \frac{\partial \Lambda}{\partial u_t} u_{xt} + \ldots + \frac{\partial \Lambda}{\partial u^{(n-1)}_t} u^{(n)}_t. \tag{3.20}
\]

In this case Lemma 3.1 reduces to the following recurrence relation:

\[
\frac{\delta I}{\delta u^{(i)}(x)} = \frac{\partial \Lambda}{\partial u^{(i-1)}_t} \quad i = 1, \ldots, n \tag{3.21}
\]

and the proof is based on the identity

\[
\frac{\partial}{\partial u^{(i)}_t} \left( \frac{dL}{dt} \right) = \frac{\partial}{\partial u^{(i)}_t} \left( \frac{d\Lambda}{dx} \right) \quad i = 0, \ldots, n - 1, \tag{3.22}
\]

observing that, for \( i \geq 1 \),

\[
\frac{\partial}{\partial u^{(i)}_t} \left( \frac{dL}{dt} \right) = \frac{\partial L}{\partial u^{(i)}}. \]

In fact, \( L \) does not depend on \( u^{(i)}_t \), and

\[
\frac{\partial}{\partial u^{(i)}_t} \left( \frac{d\Lambda}{dx} \right) = \frac{d}{dx} \left( \frac{\partial \Lambda}{\partial u^{(i)}_t} \right) + \frac{\partial \Lambda}{\partial u^{(i-1)}_t}.
\]

In particular the first step, \( i = n \), follow directly from the fact that the only dependence of \( u^{(n)}_t \) in both (3.19) and (3.20) is the one explicitly shown, so that

\[
\frac{\partial L}{\partial u^{(n)}} = \frac{\partial \Lambda}{\partial u^{(n-1)}_t}. \tag{3.23}
\]

On the other hand, from (3.22) for the index \( i = 0 \), one obtains the fundamental relation

\[
\frac{\delta I}{\delta u(x)} = \frac{\partial \Lambda}{\partial u^{(m)}} \frac{\partial u^{(m)}}{\partial u^{(m-1)}} u^{(m)}_t, \tag{3.24}
\]

and, since \( \frac{\partial u^{(m)}}{\partial u^{(m-1)}} \) is always nonzero, the condition that defines the submanifold \( \mathcal{S} \) is

\[
\frac{\partial \Lambda}{\partial u^{(m-1)}} = 0.
\]

The relative system of canonical coordinates is given by:

\[
\begin{align*}
\hat{q}_i &= u^{(i-1)}_t, \\
\hat{p}_i &= \frac{\partial \Lambda}{\partial u^{(i-1)}_t}
\end{align*}
\]

For \( i = 1, \ldots, n \).
The reduced $t$–flow is Lagrangian, with Lagrangian $\Lambda$.
Indeed, from the identity
\[
\frac{\partial}{\partial u^{(i)}} \left( \frac{dL}{dt} \right) = \frac{\partial}{\partial u^{(i)}} \left( \frac{d\Lambda}{dx} \right) \quad i = 1, \ldots, m - 1,
\]
and using the Lemma, one obtains on the subspace $S$:
\[
\left\{ \begin{aligned}
\frac{\partial \Lambda}{\partial u^{(i)}} &= 0 \quad i = n, \ldots, m - 1 \\
\frac{\partial \Lambda}{\partial u^{(i)}} - \frac{d}{dt} \left( \frac{\partial \Lambda}{\partial u_{t}^{(i)}} \right) &= 0 \quad i = 0, \ldots, n - 1
\end{aligned} \right.
\]
which is the Euler–Lagrange equation for the Lagrangian $\Lambda(x, t, u, u_{x}, \ldots, u_{(m-1)}, u_{t}, \ldots, u_{(n-1)})$.

3.5 Example: KdV with $t_7$ fixed

We will give below an example of how does Theorem 3.1 works for the first non trivial case, $n = m$. We study the Lagrangian reduction of the KdV equation
\[
u_t = 6u\nu_x - \nu_{xxx}
\]
on the stationary manifold of the $t_7$–flow.

The Lagrangian density of the $t_7$–flow, reduced to the normal form (here I mean that $L$ does not contains total derivatives), depends on the $x$–derivatives of $u(x, t)$ up to order $n = 3$, and has the expression
\[
L = 7u^5 + 35u^2u_x^2 + 7uu_x^2 + \frac{1}{2}(u^{(3)})^2.
\]
The submanifold $S$ of the stationary points defined by the Euler–Lagrange equation for $L$ gives the $n+m = 6$ derivative in terms of the first five, explicitly
\[
u^{(6)} = 14uu^{(4)} + 28u_xu_{xxx} - 70u^2u_{xx} + 21u_x^2 - 70uu_x^2 + 35u^4.
\]
From the relation
\[
\frac{dL}{dt} = \frac{d\Lambda}{dx}
\]
one construct the Lagrangian $\Lambda(u, u_x, \ldots, u^{(5)}).$ By direct calculation
\[
\begin{align*}
\Lambda &= -u^{(3)}u^{(5)} + \frac{1}{2}(u^{(4)})^2 - 14uu_{xxx}u^{(4)} + 10u(u^{(3)})^2 + 14u_xu_{xx}u_{xxx} + \\
&-70u^2u_xu^{(3)} - (u^{(2)})^3 + 77u^2(u_{xx})^2 + 70uu_x^2u_{xx} - 35u^4u_{xx} + \\
&\frac{35}{2}u_x^4 + 280u^3u_x^2 + 35u^6.
\end{align*}
\]
The evolution equation (3.27) is the definition of $u_{xxx}$ in terms of $(u, u_x, u_{xx}, u_t)$, explicitly:
\[
u_{xxx} = 6uu_x - u_t.
\]
Differentiating this relation in $x$ one obtains
\[
\begin{align*}
u^{(4)} &= 6uu_{xxx} + 6u_x^2 - u_xt \\
u^{(5)} &= 18u_xu_{xxx} + 36u^2 - 6uu_t - u_{xxx} \\
u^{(6)} &= 18u_{xxx}^2 + 180uu_x^2 + 36u^2u_{xx} - 30u_xu_t - 12uu_xt + u_{tt}
\end{align*}
\]
which is a map from the “coordinates”

\[ u, u_x, u_{xx}, u_x^{(3)}, u_x^{(4)}, u_x^{(5)}, u_x^{(6)}, \ldots \]

into

\[ u, u_x, u_{xx}, u_t, u_{xt}, u_{xxt}, u_{tt}, \ldots \]

The Lagrangian \( L \) depends on \( u, u_x, u_{xx}, \) in the new “coordinates”

\[ \hat{L}(u, u_x, u_{xx}, u_t) = 7u^5 + 53u_x^2 + 7uu_x^2 - 6uu_xu_t + \frac{1}{2}u_t^2 \]

Its derivative \( \frac{d\hat{L}}{dt} \) looks like

\[
\frac{d\hat{L}}{dt} = \frac{d\hat{L}}{dt} + \frac{\partial \hat{L}}{\partial u} u_t + \frac{\partial \hat{L}}{\partial u_x} u_{xt} + \frac{\partial \hat{L}}{\partial u_{xx}} u_{xxt} + \frac{\partial \hat{L}}{\partial u_t} u_{tt}.
\]

And there exist a functional \( \Lambda \) depending on \( u, u_x, u_{xx}, u_{xt}, u_{xxt}, \) explicitly

\[
\Lambda = 35u^6 + 4u^3u_x^2 + \frac{1}{2}u^4 - 6u_x^2u_{xt} + \frac{1}{2}u_{xxt}^2 + 35u_xu_{xx} - 2uu_x^2u_x + 8uu_xu_{xx} + 11u_x^2u_{xxt}^2 + u_{xx}^3 + 6uu_xu_{xxt} + 22u^2u_xu_t + 4u_xu_{xx}u_t - uu_{xx}u_t + 4u_{t}^2
\]

such that

\[
\frac{d\hat{L}}{dt} = \frac{d\Lambda}{dx} = \frac{\partial \Lambda}{\partial u} u_t + \frac{\partial \Lambda}{\partial u_x} u_{xt} + \frac{\partial \Lambda}{\partial u_{xx}} u_{xxt} + \frac{\partial \Lambda}{\partial u_t} u_{tt},
\]

where

\[
\frac{\partial L}{\partial t} = 0
\]
\[
\frac{\partial L}{\partial u} = 35u^4 + 106uu_x^2 + 7uu_x^2 - 6uu_xu_t
\]
\[
\frac{\partial L}{\partial u_x} = 106u_x^2u_x - 6uu_t
\]
\[
\frac{\partial L}{\partial u_{xx}} = 14uu_{xx}
\]
\[
\frac{\partial L}{\partial u_t} = -6uu_x + u_t
\]

and

\[
\frac{\partial \Lambda}{\partial x} = 0
\]
\[
\frac{\partial \Lambda}{\partial u} = 210u^5 + 12u^3u_x^2 - 140u_x^3u_{xx} - 2uu_x^2u_{xx} + 8uu_xu_{xx} + 22uu_{xx}^2 + 6uu_xu_{xxt} + 44uu_xu_t + u_{t}^2
\]
\[
\frac{\partial \Lambda}{\partial u_x} = 8u^3u_x + 2u^3 - 12u_xu_{xt} - 4uu_xu_{xx} + 6uu_{xx} + 22u^2u_t + 4u_xu_t
\]
\[
\frac{\partial \Lambda}{\partial u_{xx}} = -35u^4 - 2uu_{xx}^2 + 8uu_{xt} + 22u^2u_{xx} - 3u_{xx}^2 + 4u_xu_t
\]
\[
\frac{\partial \Lambda}{\partial u_t} = 22u_x^2u_x + 4uu_xu_{xx} - uu_{xx} + 8uu_t
\]
\[
\frac{\partial \Lambda}{\partial u_{xt}} = -6u_x^2 + uu_{xt} + 8uu_{xx}
\]
\[
\frac{\partial \Lambda}{\partial u_{xxt}} = 6uu_x - u_t
\]
Lemma 3.2 states that the condition \( \frac{\delta I}{\delta u} = 0 \), which defines the submanifold \( S \), is equivalent to the condition
\[
\frac{\partial \hat{\Lambda}}{\partial u_{xx}} - \frac{d}{dt} \left( \frac{\partial \hat{\Lambda}}{\partial u_{xxt}} \right) = 0,
\]
explicitly:
\[
u_{tt} = 35u^4 + 2u^2 - 22u^2 u_{xx} + 3u^2_{xx} + 2u_x u_t - 2u u_{xt}.
\]
The Euler–Lagrange equation for \( \hat{\Lambda} \) reads
\[
\frac{\partial \hat{\Lambda}}{\partial u_{xx}} - \frac{d}{dt} \left( \frac{\partial \hat{\Lambda}}{\partial u_{xxt}} \right) = u_{tt} - 35u^4 - 2u^2 + 22u^2 u_{xx} - 3u^2_{xx} - 2u_x u_t + 2u u_{xt} = 0
\]
\[
\frac{\partial \hat{\Lambda}}{\partial u_x} - \frac{d}{dt} \left( \frac{\partial \hat{\Lambda}}{\partial u_{xt}} \right) = 8u^3 u_x + 2u^3 - 4uu_x u_{xx} - 2uu_{xxt} + 22u^2 u_t - 4u_{xx} u_t - u_{xtt} = 0
\]
\[
\frac{\partial \hat{\Lambda}}{\partial u} - \frac{d}{dt} \left( \frac{\partial \hat{\Lambda}}{\partial u_t} \right) = 210u^5 + 12u^2 u_x^2 - 140u^3 u_{xx} - 2u^2_x u_{xx} + 4u_{xt} u_{xx} + 22u^2_{xx} + 2u_x u_{xt} - 4u_t^2 - 22u^2 u_{xt} + u_{xxtt} - 8uu_{tt} = 0.
\]
\[(3.29)\]

In this case \( L \) is nondegenerate, so that on \( S \) we can define the system of canonical coordinates
\[
\begin{align*}
q_1 &= u(i - 1), & i = 1, 2, 3 \\
p_i &= \frac{\partial \hat{\Lambda}}{\partial u^{(i-1)}}, & i = 1, 2, 3
\end{align*}
\]
which reads
\[
\begin{align*}
\dot{q}_1 &= u \\
\dot{q}_2 &= u_x \\
\dot{q}_3 &= u_{xx} \\
\dot{p}_1 &= 22u^2 u_x + 4u_x u_{xx} - u_{xtt} + 8uu_t \\
\dot{p}_2 &= -6u_x^2 + u_{xt} + 8uu_{xx} \\
\dot{p}_3 &= 6uu_x - u_t
\end{align*}
\]

We will now solve the problem from the Hamiltonian point of view: starting from \( L \) and following Theorem 1.1 one construct the canonical coordinates \( \{p_i, q_i\} \) on \( S \):
\[
\begin{align*}
q_1 &= u \\
q_2 &= u_x \\
q_3 &= u_{xx} \\
p_1 &= \frac{\delta I}{\delta u_x} = 70u^2 u_x - 14u_x u_{xx} - 14uu_{xxx} + u^{(5)} \\
p_2 &= \frac{\delta I}{\delta u_{xx}} = 14uu_{xx} - u^{(4)} \\
p_3 &= \frac{\delta I}{\delta u_{xxx}} = u_{xxx}
\end{align*}
\]
and the Hamiltonian function
\[
Q = \Lambda - \sum_{i=1}^{3} p_i(q_i)_t.
\]
By direct calculation one obtains

\[ Q = 35u^6 - 140u^3u_x^2 - \frac{35}{2}u_x^4 - 35u^4u_{xx} + 70uu_x^2u_{xx} - 7u^2u_x^2 - u_x^4 + 84u^2u_xu_{xxx} + 18u_xu_{xxx}u_{xxxx} - 10uu_{xxx} + 6u_x^2u_{xx} + 6uu_{xx}u_x - \frac{1}{2}(u^{(4)})^2 - 6uu_xu^{(5)} + u_{xxxx}u^{(5)}, \]

and in canonical coordinates

\[ Q = 35q_1^6 + 280q_1^6q_2^2 - \frac{35}{2}q_2^4 - 35q_1^2q_3 + 70q_1q_2^2q_3 - 21q_1^2q_3^2 + q_3^3 - 6q_1q_2p_1 - 6q_2^2p_2 + 8q_1q_3p_2 - \frac{1}{2}p_2^2 - 70q_1^2q_2p_3 - 4q_2q_3p_3 + p_1p_3 + 4q_1p_3^3. \]

The corresponding Hamiltonian system reads

\[ \dot{q}_1 = 6q_1q_2 - p_3 \]
\[ \dot{q}_2 = 6q_2^2 - 8q_1q_3 + p_2 \]
\[ \dot{q}_3 = 70q_1^2q_2 + 4q_2q_3 - p_1 - 8q_1p_3 \]
\[ \dot{p}_1 = 210q_1^4 + 840q_1^3q_2^2 - 140q_1^2q_3 + 70q_1^4q_3 - 42q_1^3q_3^2 - 6q_2p_1 + 8q_2p_3 - 140q_1q_2p_3 + 4p_3^2 \]
\[ \dot{p}_2 = 560q_1^3q_2 - 70q_2^2 + 140q_1q_2q_3 - 6q_1p_1 - 12q_2p_2 - 70q_1^2p_3 - 4q_3p_3 \]
\[ \dot{p}_3 = -35q_1^4 + 70q_1^2q_2^2 - 42q_1^2q_3^2 - 3q_3^3 + 8q_1p_2 - 4q_2p_3 \]

Rewriting this system in coordinates \( \{\tilde{p}_i, \tilde{q}_i\} \) one obtains exactly (3.29).

### 3.1 Appendix

**Proof of Lemma 3.1**: we prove the Lemma in two parts:

- firstly we prove the relation

\[ \frac{\partial \Lambda}{\partial u^{(j)}_{it}} = \sum_{j=i+m+1}^{n} \left( \frac{\delta I}{\delta u^{(j)}(x)} \right) \frac{\partial u^{(j-i)}}{\partial u^{(i)}_i} \ i = 0, \ldots, n - m - 1. \] \hfill (a.1)

For convenience we can explicitly rewrite eq. (3.8a), using (3.9):

\[
\begin{align*}
\frac{dL}{dt} &= \left[ \frac{\partial L}{\partial t} \right] + \sum_{j=m}^{n} \left( \frac{\partial L}{\partial u^{(j)}} \right) \frac{\partial u^{(j)}}{\partial t} + \sum_{i=0}^{m-1} \left[ \left( \frac{\partial L}{\partial u^{(i)}} \right) + \sum_{j=m}^{n} \left( \frac{\partial L}{\partial u^{(j)}} \right) \frac{\partial u^{(j)}}{\partial u^{(i)}} \right] u^{(i)} + \sum_{i=0}^{n-m} \left[ \sum_{j=m+i}^{n} \left( \frac{\partial L}{\partial u^{(j)}} \right) \frac{\partial u^{(j)}}{\partial u^{(i)}} \right] u^{(i)}. \end{align*}
\] \hfill (a.2)
Notice that the arguments of in the square brackets depend on \( u \) and its \( x \)-derivatives upon the order \( m - 1 \) and on \( u_t \) and its \( x \)-derivatives upon the order \( n - m \), so that the dependence on \( u_{i}^{(i)} \) for \( n - m + 1 \leq i \leq m - 1 \) and on \( u_{tt}^{(i)} \), for every \( j \) is only the explicit one. Analogously

\[
\frac{d\hat{\Lambda}}{dx} = \left[ \frac{\partial \hat{\Lambda}}{\partial x} + \frac{\partial \hat{\Lambda}}{\partial u_{i}^{(m-1)}} \frac{\partial u^{(m)}}{\partial t} \right] + \sum_{i=1}^{m} \frac{\partial \hat{\Lambda}}{\partial u_{i}^{(i-1)}} u_{i}^{(i)} + \sum_{i=1}^{n} \frac{\partial \hat{\Lambda}}{\partial u_{i}^{(i)}} u_{tt}^{(i)} + \sum_{i=1}^{n-m} \frac{\partial \hat{\Lambda}}{\partial u_{i}^{(i)}} u_{tt}^{(i)} + \sum_{i=1}^{n-m} \frac{\partial \hat{\Lambda}}{\partial u_{i}^{(i)}} u_{tt}^{(i)}.
\]

(a.3)

The \( i \)-th step of (a.1) is obtained from the obvious identity

\[
\frac{\partial}{\partial u_{i}^{(i)}} \left( \frac{dL}{dt} \right) = \frac{\partial}{\partial u_{i}^{(i)}} \left( \frac{d\Lambda}{dx} \right) \quad i = 1, \ldots, n - m.
\]

Indeed, from (a.2) and (a.3), it follows that

\[
\frac{\partial}{\partial u_{i}^{(i)}} \left( \frac{dL}{dt} \right) = \sum_{j=m+1}^{n} \left( \frac{\partial L}{\partial u_{j}} \right) \frac{\partial u_{j}^{(i)}}{\partial u_{i}^{(i)}},
\]

(a.4)

and

\[
\frac{\partial}{\partial u_{i}^{(i)}} \left( \frac{d\Lambda}{dx} \right) = \frac{d}{dx} \left( \frac{\partial \Lambda}{\partial u_{i}^{(i)}} \right) + \frac{\partial \Lambda}{\partial u_{i}^{(i-1)}}.
\]

(a.5)

In particular, at the first step, \( i = n - m \) one obtains the basic relation

\[
\frac{\partial \hat{\Lambda}}{\partial u_{i}^{(n)}} = \left( \frac{\partial \hat{L}}{\partial u_{i}^{(n)}} \right) \frac{\partial u_{i}^{(n-1)}}{\partial u_{i}^{(n)}} = \left( \frac{\partial \hat{L}}{\partial u_{i}^{(n)}}(x) \right) \frac{\partial u_{i}^{(n-1)}}{\partial u_{i}^{(n)}}.
\]

(a.6)

Substituting (a.6) into the further step of the recurrence, one finds

\[
\frac{\partial \Lambda}{\partial u_{i}^{(n-2)}} = \left( \frac{\partial \hat{I}}{\partial u_{i}^{(n-1)}}(x) \right) \frac{\partial u_{i}^{(n-2)}}{\partial u_{i}^{(n-2)}} + \left( \frac{\partial \hat{I}}{\partial u_{i}^{(n)}}(x) \right) \frac{\partial u_{i}^{(n-1)}}{\partial u_{i}^{(n-2)}}
\]

and so on. This gives relation (a.2).

• The second step is the proof of the relation

\[
\frac{\partial \hat{\Lambda}}{\partial u_{i}^{(i)}} = \left( \frac{\partial \hat{I}}{\partial u_{i}^{(i+1)}}(x) \right) + \sum_{j=m+1}^{n} \left( \frac{\partial \hat{I}}{\partial u_{j}^{(i)}}(x) \right) \frac{\partial u_{j}^{(i-j)}}{\partial u_{i}^{(i)}} \quad i = n - m, \ldots, m - 1,
\]

(a.7)

which is a part of (a.1), indeed, for \( i \geq n - m \), the partial derivative \( \frac{\partial \hat{\Lambda}}{\partial u_{i}^{(i)}} \) vanishes. Very much as in the previous case, equation (a.7) follows from the identity

\[
\frac{\partial}{\partial u_{tt}} \left( \frac{dL}{dt} \right) = \frac{\partial}{\partial u_{tt}} \left( \frac{d\Lambda}{dx} \right)
\]

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Using \((a.2)\) one can rewrite the left hand side as

\[
\frac{\partial}{\partial u_t} \left( \frac{dL}{dt} \right) = \sum_{j=m}^{n} \left( \frac{\partial L}{\partial u^{(j)}} \right) \frac{\partial u^{(j)}}{\partial u_t}. \tag{a.8}
\]

On the other hand, from \((a.3)\), one has

\[
\frac{\partial}{\partial u_t} \left( \frac{d\hat{\Lambda}}{dx} \right) = \frac{d}{dx} \left( \frac{\partial \hat{\Lambda}}{\partial u_t} \right) + \frac{\partial \hat{\Lambda}}{\partial u_t} \frac{\partial (m)}{\partial u_t}. \tag{a.9}
\]

where

\[
\frac{d}{dx} \left( \frac{\partial \hat{\Lambda}}{\partial u_t} \right) = \frac{d}{dx} \left( \sum_{j=m+1}^{n} \frac{\delta I}{\delta u^{(j)}(x)} \frac{\partial u^{(j-1)}}{\partial u_t} \right).
\]

We develop the right hand side, recalling \((3.6b)\):

\[
\frac{d}{dx} \left( \frac{\partial u^{(i)}}{\partial u_t} \right) = \frac{\partial u^{(i+1)}}{\partial u_t} - \frac{\partial u^{(i)}}{\partial u(t-1)} \frac{\partial u^{(m)}}{\partial u_t},
\]

obtaining

\[
\frac{d}{dx} \left( \frac{\partial \hat{\Lambda}}{\partial u_t} \right) = \sum_{j=m+1}^{n} \left[ \frac{d}{dx} \left( \frac{\delta I}{\delta u^{(m+1)}(x)} \right) \frac{\partial u^{(m)}}{\partial u_t} + \sum_{j=m+1}^{n} \left[ \left( \frac{\delta I}{\delta u^{(j)}(x)} \right) \frac{\partial u^{(j)}}{\partial u_t} - \left( \frac{\delta I}{\delta u^{(j+1)}(x)} \right) \frac{\partial u^{(m)}}{\partial u_t} \right] \right.
\]

\[
= \frac{d}{dx} \left( \frac{\delta I}{\delta u^{(m+1)}(x)} \right) \frac{\partial u^{(m)}}{\partial u_t} + \sum_{j=m+1}^{n} \left[ \left( \frac{\delta I}{\delta u^{(j)}(x)} \right) \frac{\partial u^{(j)}}{\partial u_t} - \left( \frac{\delta I}{\delta u^{(j+1)}(x)} \right) \frac{\partial u^{(m)}}{\partial u_t} \right].
\]

Inserting in \((a.9)\) and equating it to \((a.8)\) we obtain

\[
\left[ \left( \frac{\partial L}{\partial u^{(m)}} \right) - \frac{d}{dx} \left( \frac{\delta I}{\delta u^{(m+1)}(x)} \right) \frac{\partial u^{(m)}}{\partial u_t} \right] \frac{\partial u^{(m)}}{\partial u_t} = \frac{\partial \hat{\Lambda}}{\partial u_t} \frac{\partial u^{(m)}}{\partial u_t} - \frac{\partial \hat{\Lambda}}{\partial u_t} \frac{\partial u^{(m)}}{\partial u_t}.
\]

But the term \(\frac{\partial u^{(m)}}{\partial u_t}\) is nonzero by definition, so that

\[
\frac{\partial \hat{\Lambda}}{\partial u_t} = \left( \frac{\delta I}{\delta u^{(m)}(x)} \right) + \sum_{j=m+1}^{n} \left( \frac{\delta I}{\delta u^{(j)}(x)} \right) \frac{\partial u^{(j-1)}}{\partial u_t},
\]

which is the first step of the recurrence \((a.7)\), and so on.

- Finally, since \((a.1)\) for \(i > n - m\) coincides with \((a.7)\), it remains to prove it for \(i \leq n - m\). These relations can be obtained from \((a.2)\) and \((a.7)\) together with the identity

\[
\frac{\partial}{\partial u_t} \left( \frac{dI}{dt} \right) = \frac{\partial}{\partial u_t} \left( \frac{d\hat{\Lambda}}{dx} \right), \quad i = 1, \ldots, n - m.
\]
Indeed, starting from the index \( i = n - m \) and using (3.9), one may write

\[
\frac{\partial}{\partial u_i^{(n-m)}} \left( \frac{dL}{dt} \right) = \frac{d}{dt} \left( \frac{\partial L}{\partial u_i^{(n-m)}} \right) + \frac{\partial L}{\partial u_i^{(n-m)}} = \frac{d}{dt} \left( \frac{\partial L}{\partial u_i^{(n)}} \right) \frac{\partial u_i^{(n)}}{\partial u_i^{(n-m)}} + \frac{\partial L}{\partial u_i^{(n)}} \frac{\partial u_i^{(n)}}{\partial u_i^{(n-m)}} + \sum_{j=m}^{n} \left( \frac{\partial L}{\partial u_i^{(j)}} \right) \frac{\partial u_i^{(j)}}{\partial u_i^{(n-m)}}. \tag{a.10}
\]

On the other hand

\[
\frac{\partial}{\partial u_i^{(n-m)}} \left( \frac{d\hat{\Lambda}}{dx} \right) = \frac{d}{dx} \left( \frac{\partial \hat{\Lambda}}{\partial u_i^{(n-m)}} \right) + \frac{\partial \hat{\Lambda}}{\partial u_i^{(n-m)}} = \frac{d}{dt} \left( \frac{\partial L}{\partial u_i^{(n-1)}} \right) \frac{\partial u_i^{(n-1)}}{\partial u_i^{(n-m)}} + \sum_{j=m+1}^{n} \left( \frac{\partial L}{\partial u_i^{(j)}} \right) \frac{\partial u_i^{(j)}}{\partial u_i^{(n-m)}}.
\]

Performing the same steps as in the previous case, one obtains

\[
\frac{\partial \hat{\Lambda}}{\partial u_i^{(n-m-1)}} \left( \frac{d\hat{\Lambda}}{dt} \right) = \left( \frac{\partial \hat{\Lambda}}{\partial u_i^{(n-1)}} \right) \frac{\partial u_i^{(n-1)}}{\partial u_i^{(n-m-1)}} + \sum_{j=m}^{n} \left( \frac{\partial L}{\partial u_i^{(j)}} \right) \frac{\partial u_i^{(j)}}{\partial u_i^{(n-m-1)}},
\]

which is the first recursive step of (a.1).

Q.E.D.

**Proof of Lemma 3.2:** We prove the Lemma by mean of the equivalence

\[
\left( \frac{\partial L}{\partial u(x)} \right) = \left[ \frac{\partial \hat{\Lambda}}{\partial u^{(m-1)}} \right] \frac{d}{dt} \left( \frac{\partial \hat{\Lambda}}{\partial u_i^{(m-1)}} \right) \frac{\partial u_i^{(m)}}{\partial u_t}
\]

which follows from the identity

\[
\frac{\partial}{\partial u_i} \left( \frac{dL}{dt} \right) = \frac{\partial}{\partial u_i} \left( \frac{d\hat{\Lambda}}{dx} \right).
\]

Indeed, expanding, one has

\[
\frac{\partial}{\partial u_i} \left( \frac{dL}{dt} \right) = \frac{d}{dt} \left( \frac{\partial L}{\partial u_i} \right) + \frac{\partial L}{\partial u_i} = \frac{d}{dt} \left( \sum_{j=m}^{n} \frac{\partial L}{\partial u_i^{(j)}} \frac{\partial u_i^{(j)}}{\partial u_i} \right) + \sum_{j=m}^{n} \left( \frac{\partial L}{\partial u_i^{(j)}} \frac{\partial u_i^{(j)}}{\partial u_i} \right) = \frac{d}{dt} \left[ \frac{d}{dx} \left( \frac{\partial \hat{\Lambda}}{\partial u_i^{(n-1)}} \right) \frac{\partial u_i^{(n-1)}}{\partial u_i} \right] + \sum_{j=m}^{n} \left( \frac{\partial L}{\partial u_i^{(j)}} \frac{\partial u_i^{(j)}}{\partial u_i} \right),
\]

where the last equality follows from the equivalence of (a.8) and (a.9). Expanding the right hand side

\[
\frac{\partial}{\partial u_i} \left( \frac{dL}{dt} \right) = \frac{d}{dx} \left[ \frac{d}{dt} \left( \frac{\partial \hat{\Lambda}}{\partial u_i} \right) \left( \frac{\partial u_i^{(m)}}{\partial u_t} \right) + \frac{\partial L}{\partial u_i^{(m-1)}} \left( \frac{\partial u_i^{(m-1)}}{\partial u_t} \right) \frac{\partial u_i^{(m)}}{\partial u_t} \right] + \sum_{j=m}^{n} \left( \frac{\partial L}{\partial u_i^{(j)}} \frac{\partial u_i^{(j)}}{\partial u_i} \right), \tag{a.11}
\]

On the other hand

\[
\frac{\partial}{\partial u_i} \left( \frac{d\hat{\Lambda}}{dx} \right) = \frac{d}{dx} \left( \frac{\partial \hat{\Lambda}}{\partial u_i} \right) + \frac{\partial \hat{\Lambda}}{\partial u_i} = \frac{d}{dx} \left( \sum_{j=m}^{n} \left( \frac{\partial \hat{\Lambda}}{\partial u_i^{(j)}} \frac{\partial u_i^{(j)}}{\partial u_i} \right) \frac{\partial u_i^{(j)}}{\partial u_t} \right) + \sum_{j=m}^{n} \left( \frac{\partial \hat{\Lambda}}{\partial u_i^{(j)}} \frac{\partial u_i^{(j)}}{\partial u_i} \right) \left( \frac{\partial u_i^{(j)}}{\partial u_t} \right). \tag{a.12}
\]

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Comparing (a.11) and (a.12) one obtains

\[
\frac{d}{dx} \left[ \frac{\partial \Lambda}{\partial u_t} - \frac{d}{dt} \left( \frac{\partial \Lambda}{\partial u_{tt}} \right) \right] + \left[ \frac{\partial \Lambda}{\partial u_{(m-1)}} - \frac{d}{dt} \left( \frac{\partial \Lambda}{\partial u_{(m-1)}} \right) \right] \frac{\partial u^{(m)}}{\partial u_t} = \\
= \left( \frac{\partial L}{\partial u} \right) + \sum_{j=m}^{n} \left( \frac{\partial L}{\partial u^{(j)}} \right) \frac{\partial u^{(j)}}{\partial u} - \left( \frac{\partial L}{\partial u^{(m)}}(x) \right) + \sum_{j=m+1}^{n} \left( \frac{\delta I}{\delta u^{(j)}(x)} \right) \frac{\partial u^{(j-1)}}{\partial u} \frac{\partial u^{(m)}}{\partial u}.
\]  

(a.13)

But Lemma 3.1 states that

\[
\left[ \left( \frac{\partial \Lambda}{\partial u_t} - \frac{d}{dt} \left( \frac{\partial \Lambda}{\partial u_{tt}} \right) \right) \right] = \left( \frac{\partial L}{\partial u} \right) - \sum_{j=m+1}^{n} \left( \frac{\delta I}{\delta u^{(j)}(x)} \right) \frac{\partial u^{(j-1)}}{\partial u} 
\]

so that

\[
\frac{d}{dx} \left[ \frac{\partial \Lambda}{\partial u_t} - \frac{d}{dt} \left( \frac{\partial \Lambda}{\partial u_{tt}} \right) \right] = \frac{d}{dx} \left( \frac{\partial L}{\partial u} \right) - \frac{d}{dx} \left[ \sum_{j=m+1}^{n} \left( \frac{\delta I}{\delta u^{(j)}(x)} \right) \right] \frac{\partial u^{(j-1)}}{\partial u} + \\
+ \sum_{j=m+1}^{n} \left( \frac{\delta I}{\delta u^{(j)}(x)} \right) \frac{d}{dx} \left( \frac{\partial u^{(j-1)}}{\partial u} \right) - \frac{d}{dx} \left[ \sum_{j=m+1}^{n} \left( \frac{\delta I}{\delta u^{(j)}(x)} \right) \right] \frac{\partial u^{(j-1)}}{\partial u} + \\
+ \sum_{j=m+1}^{n} \left( \frac{\delta I}{\delta u^{(j)}(x)} \right) \left( \frac{\partial u^{(j)}}{\partial u} - \frac{\partial u^{(j-1)}}{\partial u} \frac{\partial u^{(m-1)}}{\partial u} \right).
\]

Substituting in (a.13) we get

\[
\left[ \left( \frac{\partial \Lambda}{\partial u^{(m-1)}} \right) - \frac{d}{dt} \left( \frac{\partial \Lambda}{\partial u_{tt}^{(m-1)}} \right) \right] \frac{\partial u^{(m)}}{\partial u_t} = \left[ \left( \frac{\partial L}{\partial u} \right) - \frac{d}{dx} \left( \frac{\partial L}{\partial u_x} \right) \right] + \\
+ \sum_{j=m}^{n} \left[ \left( \frac{\partial L}{\partial u^{(j)}} \right) - \frac{d}{dx} \left( \frac{\partial L}{\partial u^{(j+1)}} \right) \right] \frac{\partial u^{(j)}}{\partial u} + \\
- \sum_{j=m+1}^{n} \left( \frac{\delta I}{\delta u^{(j)}(x)} \right) \left( \frac{\partial u^{(j)}}{\partial u} - \frac{\partial u^{(j-1)}}{\partial u} \frac{\partial u^{(m-1)}}{\partial u} \right) + \\
+ \left[ \left( \frac{\delta I}{\delta u^{(m)}(x)} \right) - \sum_{j=m+1}^{n} \left( \frac{\delta I}{\delta u^{(j)}(x)} \right) \frac{\partial u^{(j-1)}}{\partial u} \frac{\partial u^{(m)}}{\partial u} \left( \frac{\partial u^{(m)}}{\partial u} \right) \right].
\]

All the terms cancels but

\[
\left[ \left( \frac{\partial \Lambda}{\partial u^{(m-1)}} \right) - \frac{d}{dt} \left( \frac{\partial \Lambda}{\partial u_{tt}^{(m-1)}} \right) \right] \frac{\partial u^{(m)}}{\partial u_t} = \left[ \left( \frac{\partial L}{\partial u} \right) - \frac{d}{dx} \left( \frac{\partial L}{\partial u_x} \right) \right] 
\]

Q.E.D.

**Proof of Lemma 3.3:** Relation (3.13) follows from the identities

\[
\frac{\partial}{\partial u^{(i)}} \left( \frac{dL}{dt} \right) = \frac{\partial}{\partial u^{(i)}} \left( \frac{d\Lambda}{dx} \right) \quad i = 1, \ldots, m - 1,
\]

(a.14)
and
\[ \frac{\partial}{\partial u_i^{(i)}} \left( \frac{d\hat{L}}{dt} \right) = \frac{\partial}{\partial u_i^{(i)}} \left( \frac{d\hat{\Lambda}}{dx} \right) \quad i = 1, \ldots, m - 1. \quad (a.15) \]

Starting from (a.14), we can write
\[ \frac{\partial}{\partial u^{(i)}} \left( \frac{d\hat{L}}{dt} \right) = \frac{d}{dt} \left( \frac{\partial \hat{L}}{\partial u^{(i)}} \right) = \frac{d}{dt} \sum_{j=m}^{n} \left( \frac{\partial \hat{L}}{\partial u^{(j)}} \right) \frac{\partial u^{(j)}}{\partial u^{(i)}} + \frac{d}{dt} \left( \frac{\partial \hat{L}}{\partial u^{(i)}} \right), \]
and
\[ \frac{\partial}{\partial u^{(i)}} \left( \frac{d\hat{\Lambda}}{dx} \right) = \frac{d}{dx} \left( \frac{\partial \hat{\Lambda}}{\partial u^{(i)}} \right) + \frac{\partial \hat{\Lambda}}{\partial u^{(i-1)}} + \frac{\partial \hat{\Lambda}}{\partial u^{(m-1)}} \frac{\partial u^{(m)}}{\partial u^{(i)}} + \frac{\partial \hat{\Lambda}}{\partial u^{(i-1)}} \frac{d}{dt} \left( \frac{\partial u^{(m)}}{\partial u^{(i)}} \right). \]
which give
\[ \frac{d}{dt} \left[ \sum_{j=m}^{n} \left( \frac{\partial \hat{L}}{\partial u^{(j)}} \right) \frac{\partial u^{(j)}}{\partial u^{(i)}} + \left( \frac{\partial \hat{L}}{\partial u^{(i)}} \right) \right] = \frac{d}{dx} \left( \frac{\partial \hat{\Lambda}}{\partial u^{(i)}} \right) + \frac{\partial \hat{\Lambda}}{\partial u^{(i-1)}} + \frac{\partial \hat{\Lambda}}{\partial u^{(m-1)}} \frac{\partial u^{(m)}}{\partial u^{(i)}} + \frac{\partial \hat{\Lambda}}{\partial u^{(i-1)}} \frac{d}{dt} \left( \frac{\partial u^{(m)}}{\partial u^{(i)}} \right). \quad (a.16) \]

On the other hand, in (a.15)
\[ \frac{\partial}{\partial u_i^{(i)}} \left( \frac{d\hat{L}}{dt} \right) = \frac{d}{dt} \left( \frac{\partial \hat{L}}{\partial u_i^{(i)}} \right) + \frac{\partial \hat{L}}{\partial u_i^{(i-1)}} + \frac{\partial \hat{L}}{\partial u_i^{(m-1)}} \frac{\partial u_i^{(m)}}{\partial u_i^{(i)}} \]
and
\[ \frac{\partial}{\partial u_i^{(i)}} \left( \frac{d\hat{\Lambda}}{dx} \right) = \frac{d}{dx} \left( \frac{\partial \hat{\Lambda}}{\partial u_i^{(i)}} \right) + \frac{\partial \hat{\Lambda}}{\partial u_i^{(i-1)}} + \frac{\partial \hat{\Lambda}}{\partial u_i^{(m-1)}} \frac{\partial u_i^{(m)}}{\partial u_i^{(i)}} \]
which give
\[ \left( \frac{\partial \hat{L}}{\partial u^{(i)}} \right) + \sum_{j=m}^{n} \left( \frac{\partial \hat{L}}{\partial u^{(j)}} \right) \frac{\partial u^{(j)}}{\partial u^{(i)}} = - \frac{d}{dt} \left( \frac{\partial \hat{L}}{\partial u_t^{(i)}} \right) + \frac{d}{dt} \left( \frac{\partial \hat{\Lambda}}{\partial u_t^{(i)}} \right) + \frac{\partial \hat{\Lambda}}{\partial u_t^{(i-1)}} + \frac{\partial \hat{\Lambda}}{\partial u_t^{(m-1)}} \frac{\partial u_t^{(m)}}{\partial u_t^{(i)}}. \]
Performing the derivative w.r.t. \( t \) and substituting into (a.16) one gets
\[ - \frac{d^2}{dt^2} \left( \frac{\partial \hat{L}}{\partial u_i^{(i)}} \right) + \frac{d}{dt} \frac{d}{dx} \left( \frac{\partial \hat{\Lambda}}{\partial u_i^{(i)}} \right) + \frac{d}{dt} \frac{\partial \hat{\Lambda}}{\partial u_i^{(i-1)}} + \frac{d}{dt} \frac{\partial \hat{\Lambda}}{\partial u_i^{(m-1)}} \frac{d}{dt} \left( \frac{\partial u_i^{(m)}}{\partial u_i^{(i)}} \right) = \]
\[ = \frac{d}{dx} \left( \frac{\partial \hat{\Lambda}}{\partial u_i^{(i)}} \right) + \frac{\partial \hat{\Lambda}}{\partial u_i^{(i-1)}} + \frac{\partial \hat{\Lambda}}{\partial u_i^{(m-1)}} \frac{\partial u_i^{(m)}}{\partial u_i^{(i)}} + \frac{\partial \hat{\Lambda}}{\partial u_i^{(i-1)}} \frac{d}{dt} \left( \frac{\partial u_i^{(m)}}{\partial u_i^{(i)}} \right) \]
which gives
\[ - \frac{d^2}{dt^2} \left( \frac{\partial \hat{L}}{\partial u_i^{(i)}} \right) = \frac{d}{dx} \left[ \left( \frac{\partial \hat{\Lambda}}{\partial u_i^{(i)}} \right) - \frac{d}{dt} \left( \frac{\partial \hat{\Lambda}}{\partial u_i^{(i)}} \right) \right] + \left( \frac{\partial \hat{\Lambda}}{\partial u_i^{(i-1)}} - \frac{d}{dt} \frac{\partial \hat{\Lambda}}{\partial u_i^{(i-1)}} \right) + \]
\[ + \left( \frac{\partial \hat{\Lambda}}{\partial u_i^{(m-1)}} - \frac{d}{dt} \frac{\partial \hat{\Lambda}}{\partial u_i^{(m-1)}} \right) \frac{\partial u_i^{(m)}}{\partial u_i^{(i)}}. \quad (a.17) \]

The left hand side of (a.17) is zero if \( i > n - m \).
If \( i < n - m \),
\[ \frac{\partial \hat{L}}{\partial u_i^{(i)}} = \sum_{k=m+i}^{n} \left( \frac{\partial \hat{L}}{\partial u_i^{(k)}} \right) \frac{\partial u_i^{(k)}}{\partial u_i^{(i)}} = \left[ \frac{\partial \hat{\Lambda}}{\partial u_{tt}^{(i-1)}} + \frac{d}{dx} \left( \frac{\partial \hat{\Lambda}}{\partial u_{tt}^{(i-1)}} \right) \right]. \quad (a.18) \]
Indeed, using (a.1), and relation (3.6b), which we rewrite here below,
\[
\frac{d}{dx} \left( \frac{\partial u^{(k)}}{\partial u^{(i)}} \right) = \frac{\partial u^{(k+1)}}{\partial u^{(i)}} - \frac{\partial u^{(k)}}{\partial u^{(i-1)}}
\]

one obtains
\[
\frac{\partial \hat{\Lambda}}{\partial u^{(i-1)}} + \frac{d}{dx} \left( \frac{\partial \hat{\Lambda}}{\partial u^{(i)}} \right) = \sum_{k=i+m+1}^{n} \left[ \frac{d}{dx} \left( \frac{\delta I}{u^{(k)}(x)} \right) \right] \frac{\partial u^{(k-1)}}{\partial u^{(i)}} + \sum_{k=i+m+1}^{n} \left( \frac{\delta I}{\delta u^{(k)}(x)} \right) \frac{\partial u^{(k)}}{\partial u^{(i)}} + \left( \frac{\delta I}{\delta u^{(i+m)}(x)} \right) \frac{\partial u^{(i+m-1)}}{\partial u^{(i-1)}} = \\
= \sum_{k=m+1}^{n} \left( \frac{\partial L}{u^{(k)}} \right) \frac{\partial u^{(k)}}{\partial u^{(i)}} + \left[ \frac{d}{dx} \left( \frac{\delta I}{u^{(m+1)}(x)} \right) \right] \frac{\partial u^{(m+i)}}{\partial u^{(i)}} = \\
= \sum_{k=m+i}^{n} \left( \frac{\partial L}{u^{(k)}} \right) \frac{\partial u^{(k)}}{\partial u^{(i)}}
\]

where the last identity follows from the fact that
\[
\frac{\partial u^{(k)}}{\partial u^{(i)}} = 0
\]

if \( k - l < m \). In (3.6), this implies
\[
\frac{\partial u^{(k+m)}}{\partial u^{(k)}} = \frac{\partial u^{(k+m+j)}}{\partial u^{(i+j)}}.
\]

Finally, substituting (a.18) into (a.17) gives (3.13)
\[
\left[ \frac{\partial \hat{\Lambda}}{\partial u^{(i)}} - \frac{d}{dt} \left( \frac{\partial \hat{\Lambda}}{\partial u^{(i-1)}} \right) \right] + \frac{d^2}{dt^2} \left( \frac{\partial \hat{\Lambda}}{\partial u^{(i-1)}} \right) = \\
= \left[ \frac{\partial \hat{\Lambda}}{\partial u^{(m)}} - \frac{d}{dt} \left( \frac{\partial \hat{\Lambda}}{\partial u^{(m-1)}} \right) \right] \frac{\partial u^{(m)}}{\partial u^{(i)}} - \frac{d}{dx} \left( \frac{\partial \hat{\Lambda}}{\partial u^{(i-1)}} \right) - \frac{d}{dt} \left( \frac{\partial \hat{\Lambda}}{\partial u^{(i-1)}} \right) + \frac{d^2}{dt^2} \left( \frac{\partial \hat{\Lambda}}{\partial u^{(i-1)}} \right) \right].
\]

Q.E.D.

4. Applications to Painlevé equations

In this Section we study some applications of Theorem 2.1 and Theorem 3.1. to show how the finite dimensional Hamiltonian structure of Painlevé equations comes from an infinite dimensional structure via the above procedure.

4.1 PI as scaling reduction of KdV

At the beginning we study the problem following the Hamiltonian scheme, then we will apply the framework of Theorem 3.1.

We consider the KdV equation
\[
u_t = 6uu_x - u_{xxx}.
\]
i.e. the $t = t_1$–flow in the KdV hierarchy (1.2); it admits the nondegenerate scaling symmetry

$$I_{(i)} = \int (u^3 + \frac{u^2}{2} + 2ux + 6tu^2)dx,$$

which depends on $x, u, u_x, t$. We note that $L = [L_1 + 4xL_1 + 12tL_0], \text{where } I_{-1} = \int L_{-1}dx = \int \frac{u^2}{2}dx$ and $I_0 = \int L_0dx = \int \frac{u^2}{2}dx$ are the first Hamiltonians of the KdV hierarchy.

Theorem 2.1 states that the $t$–flow is Hamiltonian on the manifold $S$ of the stationary points of the symmetry, i.e. $S$ is the 2–dimensional manifold of the solutions of the Euler–Lagrange equation

$$\frac{\delta I}{\delta u(x)} = u_{xx} - 3u^2 - 2x - 12tu = 0.$$  

(4.3)

It is invariant under the $t$–th flow and it naturally carries the system of canonical coordinates:

$$\begin{cases} q = u \\ p = \frac{\delta I}{\delta u_x} = u_x \end{cases}$$

Notice that the identities

$$\begin{cases} p_x + \frac{\partial H}{\partial q} = -\frac{\delta I}{\delta u} \\ q_x - \frac{\partial H}{\partial p} = 0, \end{cases}$$

(4.3)

hold, where $H$ is the generalized Legendre transform of $L$:

$$H = -L + \frac{\delta I}{\delta u_x} u_x.$$  

The first of identities (4.4) allows us to express the higher derivatives $u^{(m)}$ for $m \geq 2$ in terms of $x, t, p, q$ and $p^{(l)}$ with $l = 1, \ldots, m - 2 + 1$.

On $S$ $p_x + \frac{\partial H}{\partial q} \equiv 0$, and the system (4.4) reduces to the canonical Hamiltonian system

$$\begin{cases} p_x = 3q^2 + 2x + 12tq \\ q_x = p, \end{cases}$$

for the Hamiltonian function

$$H = -L + u_x^2 = \frac{p^2}{2} - q^3 - 2qx - 6tq^2$$  

(4.5)

giving the reduced $x$–flow. This system is equivalent to the second order ODE in the variable $q$:

$$q^{''} = -3q^2 - 2x - 12t q.$$  

(4.6)

The space $S$ is the set of the stationary points of the scaling symmetry (4.2); this means that $S$ carries a “natural” system of canonical coordinates $\{w_i, \pi_j\}$, given by the self–similar function of $u$, i.e. combinations of $u, x, t$ in the variable $z(x, t)$ invariant w.r.t. the scaling. We will call them scaling coordinates. In this case

$$\begin{cases} w = \frac{t}{2} + t \\ \pi = 2p \end{cases}$$

with $z = x - 6t^2$.

In terms of the scaling coordinates the system reads

$$\begin{cases} \frac{dx}{dw} = -\frac{\partial \delta_1}{\partial w} \\ \frac{dw}{dx} = \frac{\partial \delta_1}{\partial \pi}. \end{cases}$$
for the Hamiltonian
\[ H = \frac{\pi^2}{8} - 8w^3 - 4wz + 8t^3 + 4tz. \]

The system is equivalent to the ODE:
\[ w'' = 6w^2 + z, \quad (4.7) \]

that is exactly Painlevé I. The Hamiltonian \( H \) differs from the usual PI Hamiltonian for the terms in \( z, t \) that do not enter in the Hamiltonian system.

We now construct the time dependent Hamiltonian function \((-\check{Q})\), that is the reduction on \( S \) of
\[ -Q = -\Lambda + p \frac{dq}{dt}, \]

where \( p, q \) are expressed in terms of \( u, u_x, \) and \( \Lambda(x, t, u, u_x, u_{xxx}, u_{xxxx}) \), calculated from
\[ \frac{dL}{dt} = \frac{d\Lambda}{dx}. \]

has the form
\[ \Lambda = 6t(4u^3 - 2uu_{xx} + u_x^2) + 2x(3u^2 - u_{xx}) + \frac{9}{2}u^4 + \frac{1}{2}u_{xx}^2 + 2u_x - 3u^2u_{xx} + 6uu_x^2 - u_xu_{xxx}. \]

By direct calculation one obtains
\[ Q = 12t(2u^3 + \frac{u_x^2}{2} - uu_{xx}) + u_{xx}^2 - 3u^2u_{xx} + \frac{9}{2}u^4 + 2u_x + 2x(3u^2 - u_{xx}), \quad (4.8) \]

This reduces on \( S \) to
\[ \check{Q} = 12t(\frac{p^2}{2} - q^3 - 6tq^2 - 2xq) + 2p - 2x^2 \quad (4.9) \]

Theorem 2.1 states that \((-\check{Q})\) is the Hamiltonian for the reduced \( t \)-flow, i.e., in terms of \( p \) and \( q \)
\[ \begin{cases} \dot{q} = -2 \left(6tp + 1\right) = - \frac{\partial \check{Q}}{\partial p} \\ \dot{p} = -12t \left(3q^2 + 2x + 12tq\right) = \frac{\partial \check{Q}}{\partial q} \end{cases}, \quad (4.10) \]

Notice that system (4.10), written in terms of the scaling coordinates \( w \) and \( z \), gives the same Painlevé I.

**Remark:** In this case the evolution equation is Hamiltonian and it can be written in the form
\[ u_t = \{u(x), I_1\} = \frac{d}{dx} \frac{\delta I_1}{\delta u}, \]

where \( I_1 = \int L_1 \, dx \) with density
\[ L_1 = u^3 + \frac{u_x^2}{2}. \]

On the other hand the scaling symmetry defines the stationary flow
\[ \frac{du}{ds} = 12tu_x + u_t + 2 = 0. \]

which is Hamiltonian:
\[ \frac{du}{ds} = \{u(x), I\} = \frac{d}{dx} \frac{\delta I}{\delta u} = 0. \]
The $s$-flow and the $t$-flow commute, but the Hamiltonian generating the scaling depends explicitly on the time $t$, so that the relation
\[ \frac{dI(s)}{dt} = \{I(s), I_1\} + \frac{\partial I(s)}{\partial t} = 0 \]
holds. Hence we have an alternative way to define the reduced Hamiltonian $Q$, following [BN]:
\[ \frac{d}{dx} Q = \frac{dL}{dt} \frac{\partial I}{\partial u} \frac{d}{dx} \frac{\partial I_1}{\partial u}. \]
In this case the relation (4.2) follows as a consequence.

System (4.10) i.e. the reduction of the $t$-flow on $\mathfrak{S}$, can be obtained from the Lagrangian point of view; indeed one can consider the evolution equation (4.1) as the definition of $u_{xxx}$ in terms of $(u, u_x, u_{xx}, u_t)$, explicitly:
\[ u_{xxx} = 6uu_x - u_t. \]
Differentiating this relation in $x$ one obtains
\[
\begin{align*}
    u^{(4)} &= 6uu_{xx} + 6u_x^2 - u_{xt} \\
    u^{(5)} &= 18u_xu_{xx} + 36u^2 - 6uu_t - u_{xxt} \\
    &\vdots
\end{align*}
\]
which is a map from the "coordinates"
\[ u, u_x, u_{xx}, u^{(3)}, u^{(4)}, u^{(5)}, \ldots \]
into
\[ u, u_x, u_{xx}, u_t, u_{xxt}, u_{tt}, \ldots \]
The Lagrangian $L$ depends on $u, u_x$ and hence its derivative $\frac{dL}{dt}$ looks like
\[ \frac{dL}{dt} = \frac{\partial L}{\partial t} + \frac{\partial L}{\partial u} u_t + \frac{\partial L}{\partial u_x} u_{xt}. \]
And there exist a functional $\hat{\Lambda}$ depending on $u, u_x, u_{xx}, u_t$ such that
\[ \frac{dL}{dt} = \frac{d\hat{\Lambda}}{dx} = \frac{\partial \hat{\Lambda}}{\partial x} + \frac{\partial \hat{\Lambda}}{\partial u} u_x + \frac{\partial \hat{\Lambda}}{\partial u_x} u_{xx} + \frac{\partial \hat{\Lambda}}{\partial u^{(3)}} + \frac{\partial \hat{\Lambda}}{\partial u_t} u_{xt}. \]
Here, in terms of the new coordinates
\[ \hat{\Lambda} = 6t(4u^3 - 2uu_{xx} + u_x^2) + 2x(3u^2 - u_{xx}) + \frac{9}{2}u^4 + \frac{1}{2}u_{xx}^2 + 2u_x - 3u^2u_{xx} + u_x u_t, \]
\[ \hat{L} \equiv L, \]
and
\[
\begin{align*}
    \frac{\partial L}{\partial t} &= 6u^2 \\
    \frac{\partial L}{\partial u} &= 3u^2 + 2x + 12tu \\
    \frac{\partial L}{\partial u_x} &= u_x
\end{align*}
\]
\[
\frac{\partial \Lambda}{\partial x} = 6u^2 - 2u_{xx} \\
\frac{\partial \Lambda}{\partial u} = 72tu^2 - 12tu_{xx} + 12xu + 18u^3 - 6uu_{xx} \\
\frac{\partial \Lambda}{\partial u_x} = 12tu_x + u_t + 2 \\
\frac{\partial \Lambda}{\partial u_{xx}} = -12tu + u_{xx} - 2x - 3u^2 \\
\frac{\partial \Lambda}{\partial u_t} = u_x
\]

The condition \( \frac{\delta I}{\delta u} = 0 \), that defines the submanifold \( \mathcal{S} \), is equivalent to the condition \( \frac{\partial \Lambda}{\partial u_{xx}} = 0 \).

Hence we have an alternative definition of the space \( \mathcal{S} \), and an alternative way to defines the canonical coordinates:

\[
\begin{align*}
q &= u \\
p &= \frac{\partial \Lambda}{\partial u_t} = u_x
\end{align*}
\]

Theorem 3.1 states that the reduced \( t \)–flow is Lagrangian, with Lagrangian \( \Lambda \), in this case it is easy to verify it, indeed, on \( \mathcal{S} \),

\[
\begin{align*}
\frac{\partial \Lambda}{\partial u_{xx}} &= -12tu + u_{xx} - 2x - 3u^2 = 0 \\
\frac{\partial \Lambda}{\partial u_x} &= 12tu_x + u_t + 2 = 0 \\
\frac{\partial \Lambda}{\partial u} - \frac{d}{dt}\left( \frac{\partial \Lambda}{\partial u_t} \right) &= 72tu^2 - 12tu_{xx} + 12xu + 18u^3 - 6uu_{xx} - u_{xt} = 0,
\end{align*}
\]

where the first equation is the definition of the submanifold \( \mathcal{S} \) itself, the other two reproduces (4.9), indeed they can be rewritten as

\[
\begin{align*}
u_t &= -12tu_x - 2 \\
u_{xt} &= -12t(3u^2 + 2x + 12t)
\end{align*}
\]

### 4.2 PII as scaling reduction of mKdV

One can repeat the same procedure as in section 4.1 starting from the mKdV equation

\[
u_t = 6u^2u_x - u_{xxx}.
\]

It admits the nondegenerate scaling symmetry

\[
I = \int \left( \frac{3}{2}(u^4 + u_x^2) + \frac{u^2x}{2} \right) dx,
\]

which depends on \( x, u, u_x, t \). We notice that \( L = 3tL_1 + \frac{u^2x}{2} \).

Here \( \mathcal{S} \) is the 2–dimensional manifold of the solutions of the Euler–Lagrange equation

\[
\frac{\delta I}{\delta u(x)} = u_{xx} - \frac{1}{3t}(6tu^3 + ux) = 0.
\]
It naturally carries the system of canonical coordinates:

$$\begin{align*}
q &= u \\
p &= \frac{\delta I}{\delta u_x} = 3tu_x.
\end{align*}$$

As in the previous case we read the Euler–Lagrange equation as a reduced $x$–flow with Hamiltonian

$$H = \frac{p^2}{6t} - \frac{3}{2}tq^4 - \frac{1}{2}q^2x$$

where $H$ is the generalized Legendre transform of $L$:

$$H = -L + \frac{\delta I}{\delta u_x}u_x = -L + 3tu_x^2.$$  

The system is equivalent to the second order ODE in the variable $q$:

$$q_{xx} = 2q^3 + \frac{1}{3t}q.$$

The scaling coordinates are now

$$\begin{align*}
w &= (3t)^{\frac{1}{3}}q \\
\pi &= \frac{p}{(3t)^{\frac{2}{3}}}
\end{align*}$$

in the variable $z = \frac{x}{(3t)^\frac{1}{3}}$, and the system transforms into

$$\begin{align*}
\frac{dw}{dz} &= -\frac{\partial \tilde{Q}}{\partial w} \\
\frac{d\pi}{dz} &= \frac{\partial \tilde{Q}}{\partial \pi},
\end{align*}$$

for the Hamiltonian

$$\tilde{\mathcal{H}} = \frac{1}{2}(\pi^2 - w^4) - \frac{1}{2}zw^2.$$  

The system is equivalent to the ODE:

$$w'' = 2w^3 + zw.$$  

(4.15)

that is exactly Painlevé II.

We now construct the time dependent Hamiltonian function $(-\tilde{Q})$, that is the reduction on $\mathcal{S}$ of

$$-Q = -\Lambda + p\frac{dq}{dt},$$

where

$$\Lambda = t(6u^6 - 6u^3u_{xx} + 18u^2u_x^2 + \frac{3}{2}u^2_{xx} - 3u_xu_{xxx}) + x\left(\frac{3}{2}u^4 - \frac{u_x^2}{2} - uu_{xx}\right) + uu_x.$$

By direct calculation one obtains

$$Q = 6t(u^6 - u^3u_{xx} + \frac{1}{4}u_x^2) + x\left(\frac{3}{2}u^4 - uu_{xx} + \frac{1}{2}u_x^2\right) + uu_x,$$

(4.16)

which on $\mathcal{S}$ reduces to

$$\tilde{Q} = \frac{x}{2}\left(\frac{p^2}{9t^2} - q^4\right) + \frac{1}{3t}pq - \frac{1}{6t}q^2x^2$$

(4.17)

and is the Hamiltonian for the reduced $t$-flow. In fact

$$\begin{align*}
\dot{q} &= -\frac{q}{3t} - \frac{3p}{3t} = -\frac{\partial \tilde{Q}}{\partial p} \\
\dot{p} &= \frac{p}{3t} - \frac{2q^2}{3t} - 2q^3x = \frac{\partial \tilde{Q}}{\partial q}.
\end{align*}$$  

(4.18)
Notice that also the system (4.18), written in $w$ and $z$, gives Painlevé II.

**Remark:** The evolution equation is Hamiltonian and can be written in the form

$$u_t = \{u(x), I_1\} = \frac{d}{dx} \frac{\delta I_1}{\delta u},$$

where $I_1 = \int L_1 \, dx$ with density

$$L_1 = \frac{1}{2}(w^4 + w_x^2)$$

On the other hand the scaling symmetry defines the Hamiltonian stationary flow

$$\frac{du}{ds} = xu_x + 3tu_t + u = \frac{d}{dx} \frac{\delta I_{\omega_1}}{\delta u} = 0.$$

We now deduce system (4.18) from the Lagrangian point of view, reading the evolution equation (4.11) as the definition of $u_{xxx}$ in terms of $(u, u_x, u_{xx}, u_t)$, explicitly:

$$u_{xxx} = 6u^2u_x - u_t.$$

Differentiating this relation in $x$ one obtains

$$\left\{\begin{array}{l}
u^{(4)} = 6u^2u_{xx} + 12wu_x^2 - u_{xt} \\
u^{(5)} = 36wu_xu_{xx} + 36u^3u_x + 12u_x^3 - 6u^2u_t - u_{xxxx}
\end{array}\right.$$ 

which is a map from the "coordinates"

$$u, u_x, u_{xx}, u^{(3)}, u^{(4)}, u^{(5)}, u^{(6)}, \ldots$$

into

$$u, u_x, u_{xx}, u_t, u_{xt}, u_{xxt}, u_{ttt}, \ldots$$

Here

$$\hat{\Lambda} = \tau(6u^6 - 6u^3u_{xx} + \frac{3}{2}u_x^2 - 3u_xu_{xxx} + 3u_xu_t) + x(\frac{3}{2}u^4 + \frac{1}{2}u_x^2 - uu_{xx}) + uu_x,$$

and

$$\frac{\partial L}{\partial t} = \frac{3}{2}(u^4 + u_x^2)$$

$$\frac{\partial L}{\partial u} = 6tu^3 + ux$$

$$\frac{\partial L}{\partial u_x} = 3tu_x$$

$$\frac{\partial \Lambda}{\partial x} = \frac{3}{2}(u^4 + u_x^2) - uu_{xx}$$

$$\frac{\partial \Lambda}{\partial u} = 36tu^5 - 18tu^2u_{xx} + 6xu^3 - xu_{xx} + u_x$$

$$\frac{\partial \Lambda}{\partial u_x} = xu_x + u + 3tu_t$$

$$\frac{\partial \Lambda}{\partial u_{xx}} = 3tu_{xx} - 6tu^3 - ux$$

$$\frac{\partial \Lambda}{\partial u_t} = 3tu_x$$
The condition $\delta I_{\omega}/\delta u = 0$, that defines the submanifold $S$, is equivalent to the condition
\[
\frac{\partial \Lambda}{\partial u_{xx}} = 3tu_{xx} - 6tu^3 - ux = 0
\]
Hence we have an alternative definition of the space $S$, and an alternative way to defines the canonical coordinates:
\[
\begin{align*}
q &= u \\
p &= \frac{\partial \Lambda}{\partial u_t} = 3tu_x
\end{align*}
\]
On $S$
\[
\begin{align*}
\frac{\partial \Lambda}{\partial u_{xx}} &= 3tu_{xx} - 6tu^3 - ux = 0 \\
\frac{\partial \Lambda}{\partial u_x} &= xu_x + 3tu_t + u = 0 \\
\frac{\partial \Lambda}{\partial u} - \frac{d}{dt} \left( \frac{\partial \Lambda}{\partial u_t} \right) &= -18tu^2u_{xx} + 36tu^5 + 6xu^3 - xu_{xx} - 2ux - 3tu_x = 0,
\end{align*}
\]
where the first defines the submanifold, the second one gives the motion of $u$ and the third the motion of $u_x$, hence one can rewrite them as
\[
\begin{align*}
3tu_t &= -xu_x - u \\
3tu_{xt} &= u_x - 2u^3x - \frac{x^2u}{3t},
\end{align*}
\]
which coincides with (4.18).

### 4.3 PIII as scaling reduction of Sine-Gordon

A particular case of Painlevé III equation can be obtained as reduction of the Sine–Gordon equation
\[
\begin{align*}
&u_t = v = \frac{\delta I_1}{\delta v} \\
v_t = u_{xx} - \sin u = -\frac{\delta I_1}{\delta u},
\end{align*}
\]
via the scaling
\[
\begin{align*}
\frac{dv}{ds} &= xv + tu_x = \frac{\delta I_1}{\delta v} = 0 \\
\frac{dv}{ds} &= x(u_{xx} - \sin u) + tv_x + u_x = -\frac{\delta I_1}{\delta u} = 0,
\end{align*}
\]
where $I_1 = \int L_1 dx$, $I_1(\omega) = \int L dx$, with the Hamiltonians
\[
L_1 = \frac{1}{2}(v^2 + u_x^2) - \cos u
\]
and
\[
L = \frac{x}{2}(v^2 + u_x^2) - x \cos u + tvu_x = xL_1 + tvu_x
\]
w.r.t. the Poisson bracket
\[
\{ F, G \} = \int \left( \frac{\delta f}{\delta u(x)} \frac{\delta g}{\delta v(x)} - \frac{\delta f}{\delta v(x)} \frac{\delta g}{\delta u(x)} \right) dx.
\]
The scaling reduction equation means
\[
\frac{\delta I_1(\omega)}{\delta u(x)} = \frac{\delta I_1(\omega)}{\delta v(x)} = 0,
\]
which defines the submanifold $S$:
\[
u_x = -\frac{xv}{t},
\]
with the canonical coordinates
\[
\begin{align*}
p &= xu_x + tv = \frac{v^2 - x^2}{t}v \\
q &= u
\end{align*}
\]

in $\mathcal{S}$. The equation defining $\mathcal{S}$ can be written as an Hamiltonian system in canonical form describing the reduced $x$-flow:

\[
\begin{align*}
(p)_x &= -\frac{\partial H}{\partial q}, \\
(q)_x &= \frac{\partial H}{\partial p},
\end{align*}
\]

where

\[
H = \frac{x}{2} \frac{p^2}{x^2 - t^2} + x \cos q.
\] (4.22)

In terms of the scaling coordinates

\[
\begin{align*}
w &= q, \\
\pi &= \frac{x}{t} p
\end{align*}
\]

in the variable $z = \frac{x^2 - t^2}{2}$, the Hamiltonian system transform into

\[
\begin{align*}
\frac{d\pi}{dz} &= -\frac{\partial \tilde{H}}{\partial w}, \\
\frac{dw}{dz} &= \frac{\partial \tilde{H}}{\partial \pi},
\end{align*}
\]

for the Hamiltonian

\[
\tilde{H} = -\frac{1}{4} \pi^2 - \cos w.
\]

The system is equivalent to Painlevé III:

\[
2zw'' + 2w' - \sin w = 0.
\]

Let us now construct the time dependent Hamiltonian function ($-\tilde{Q}$), that is the reduction on $\mathcal{S}$ of

\[
-Q = -\Lambda + \frac{dq}{dt},
\]

where

\[
\Lambda = x u_x v + t \left( \frac{1}{2} (v^2 + u_x^2) - \cos u \right).
\]

By direct calculation one obtains

\[
Q = t \left( \frac{1}{2} (v^2 - u_x^2) - \cos u \right)
\]

which on $\mathcal{S}$ reduces to

\[
\tilde{Q} = -t \left( \frac{1}{2} \frac{p^2}{t^2 - x^2} - \cos q \right).
\]

This is the Hamiltonian for the reduced $t$-flow. In fact

\[
\begin{align*}
q &= v = -\frac{1}{x} u_x = -t \frac{p}{x^2} = -\frac{\partial \tilde{Q}}{\partial p}, \\
p &= v + x u_x + t u_t = t \sin q = \frac{\partial \tilde{Q}}{\partial q}.
\end{align*}
\] (4.23)

Note that also the system (4.23), written in $w$ and $z$, gives Painlevé III.

**Remark**: We now deduce system (4.23) from the Lagrangian point of view, reading the evolution equation (4.19) as the definition of $v$ in terms of $u_t$, explicitly:

\[
\begin{align*}
v &= u_t, \\
u_{xx} &= v_t + \sin u = u_{tt} + \sin u
\end{align*}
\]
Differentiating this relation in \( x \) one obtains

\[
\begin{align*}
v_x &= u_{xt} \\
v_{xx} &= v_t - v \cos u = u_{ttt} - u_t \cos u \\
\vdots\
u_{xxx} &= u_{tttt} + u_x \cos u \\
\vdots
\end{align*}
\]

which is a map from the "coordinates"

\[
u, v, u_x, v_x, u_{xx}, v_{xx}, \ldots
\]

into

\[
u, u_x, u_t, u_{xt}, u_{tt}, u_{xtt}, \ldots
\]

Here

\[
\hat{\Lambda} = xu_xu_t + t\left(\frac{1}{2}(u^2_t + u^2_x) - \cos u\right)
\]

and

\[
\hat{L} = tu_xu_t + x\left(\frac{1}{2}(u^2_t + u^2_x) - \cos u\right)
\]

which give

\[
\begin{align*}
\frac{\partial \hat{L}}{\partial t} &= \frac{1}{2}(u^2_t + u^2_x) - \cos u \\
\frac{\partial \hat{L}}{\partial u} &= -\sin u \\
\frac{\partial \hat{L}}{\partial u_x} &= tu_t + xu_x \\
\frac{\partial \hat{L}}{\partial u_t} &= tu_x + xu_t
\end{align*}
\]

and

\[
\begin{align*}
\frac{\partial \hat{\Lambda}}{\partial x} &= \frac{1}{2}(u^2_t + u^2_x) - \cos u \\
\frac{\partial \hat{\Lambda}}{\partial u} &= -t \sin u \\
\frac{\partial \hat{\Lambda}}{\partial u_x} &= xu_t + tu_x \\
\frac{\partial \hat{\Lambda}}{\partial u_t} &= xu_x + tu_t
\end{align*}
\]

The condition \( \frac{\delta I}{\delta u} = 0 \), that defines the submanifold \( \mathcal{G} \), is equivalent to the condition

\[
\frac{\partial \hat{\Lambda}}{\partial u_x} = xu_t + tu_x = 0
\]

Hence we have an alternative definition of the space \( \mathcal{G} \), and an alternative way to defines the canonical coordinates:

\[
\begin{align*}
q &= u \\
p &= \frac{\partial \hat{\Lambda}}{\partial u_t} = xu_x + tu_t
\end{align*}
\]
On $\mathcal{S}$

\[
\begin{align*}
\frac{\partial \hat{\Lambda}}{\partial u} = xu_t + tu_x &= 0 \\
\frac{\partial \hat{\Lambda}}{\partial u} - \frac{d}{dt} \left( \frac{\partial \hat{\Lambda}}{\partial u} \right) &= -t \sin u - xu_{xt} - u_t - tu_{tt} = 0,
\end{align*}
\]

where the first defines the submanifold, and the second

\[-t \sin u - xu_{xt} - u_t - tu_{tt} = 0\]

coincides with (4.23).

5. Self–similar solutions of $n$–waves equation and Hamiltonian MPDEs

5.1 $n$–waves equations and their symmetries

Let us consider the equation

\[u_t - v_x - [u, v] = 0, \quad (5.1)\]

where

\[u = [\gamma, a] \quad v = [\gamma, b] \quad a = \text{diag} \ (a^1, \ldots, a^n) \quad b = \text{diag} \ (b^1, \ldots, b^n) \quad (5.2)\]

and $\gamma$ is a function of $x$, $t$.

Following [DS] it is possible to rewrite (5.1) as an infinite dimensional Hamiltonian system on the space $\mathcal{M}$ of functions of $x$ with values in $\text{Mat}(n, C)$ with the inner product

\[(u, v) = \int \text{Tr} \ (u(x)v(x))dx.\]

On the space $\mathcal{F}$ of functionals

\[F = \int f(x, u, u_x, \ldots, u^{(k)}) \ dx\]

one can define $\nabla_u F \in \mathcal{M}$ by

\[\frac{d}{dt} F(u + \epsilon w) \big|_{\epsilon = 0} = (\nabla_u F, w)\]

and the Poisson structure $P$ with the Poisson bracket

\[
\{F, G\}(u) = \langle \nabla_u F, [\nabla_u G, \frac{d}{dx} + u] \rangle \quad (5.3)
\]

The $n$-waves equation (5.1) is a Hamiltonian system w.r.t. this Poisson structure:

\[u_t = PdI_1 = [\nabla_u I_1, \frac{d}{dx} + u] = [-v, \frac{d}{dx} + u], \quad (5.4)\]

where

\[I_1 = \int L_1 dx = \frac{1}{2} \int Tr (uv)dx\]

that in components of $\gamma$ gives

\[I_1 = \int \sum_i \sum_k \frac{b_k}{a_k - a_i} u_{ik}u_{ki} \ dx = \int \sum_i \sum_k [(b_i - b_k)(a_i - a_k)\gamma_{ik}\gamma_{ki}] dx. \quad (5.5)\]
For \( n = 3, \ u^T = -u \) one can reduce to a particular case of P VI equation (see [D], where the Hamiltonian structure for this particular case of P VI is derived from the Hamiltonian structure of the \( n \)-waves equation.), imposing the scaling

\[
\frac{du}{ds} = tu_t + xu_x + u = 0.
\] (5.6)

It admits the Hamiltonian form

\[
\frac{du}{ds} = [\nabla_u I(s), \frac{d}{dx} + u] = 0
\] (5.7)

where

\[
I(s) = \int Ldx = -\frac{1}{2} \int Tr(tuv + xu^2)dx = \int \sum_i \sum_k (t \frac{b_i}{a_k - a_i} - \frac{x}{2})u_{ik} u_{ki} dx
\]

and

\[
\nabla_u I(s) = -tv - xu,
\] (5.8)

or, in terms of \( \gamma \):

\[
I(s) = \int \sum_i \sum_k [(b_i - b_k)(a_i - a_k)t + (a_i - a_k)^2 x] \gamma_{ik} \gamma_{ki} dx
\] (5.9)

We emphasize the fact that the \( t \)-flow and the \( s \)-flow commute, so that

\[
\{I_1, I(s)\} = \frac{\partial I(s)}{\partial t} = 0.
\]

By substituting:

\[
\int [Tr(\nabla_u I_1 [\nabla_u I(s), \frac{d}{dx} + u]) + \frac{1}{2} Tr(uv)] dx = 0.
\]

Then there exists a function \( Q(s)(x, t, u, v) \) such that

\[
Tr(-v[-tv - xu, \frac{d}{dx} + u] + \frac{uv}{2}) = -\frac{d}{dx} Q(s).
\]

By direct calculation (see Appendix 5.A) we obtain

\[
Q(s) = \frac{1}{2} Tr(xuv + tu^2) = \frac{1}{2} \sum_{i,j} [(a_j - a_i)(b_i - b_j)x + (b_j - b_i)^2 t] \gamma_{ij} \gamma_{ji}
\]

As in the previous examples, \( Q(s) \) is the Hamiltonian for the reduced \( t \)-flow. We now describe this flow.

We start by rewriting the system

\[
\begin{align*}
\left\{ u_t - v_x - [u, v] &= 0 \\
tu_t + xu_x + u &= 0
\end{align*}
\] (5.10)

in terms of \( \gamma \), i.e. we solve

\[
[\gamma_t, a] = [\gamma_x, b] + [[\gamma, a], [\gamma, b]]
\]

under the condition

\[
\gamma_x = -\frac{t}{x} \gamma_t - \frac{1}{x} \gamma.
\]

This gives

\[
[\gamma_t, ax + tb] + [\gamma, b] = [[\gamma, ax], [\gamma, b]]
\]

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but, because of the commutativity of \( b \) with itself,

\[
\frac{d}{dt} [\gamma, ax + tb] = [[\gamma, ax + bt], [\gamma, b]].
\] (5.11)

Then we identify \( S \) with the space of matrices

\[
q = [\gamma, ax + bt] = ux + vt,
\]

so that \( u = \text{ad}_{(ax + bt)} \text{ad}_x^{-1} q, \ v = \text{ad}_{(ax + bt)} \text{ad}_b^{-1} q \), or, in terms of the matrix elements:

\[
q_{ij} = [(a_j - a_i)x + (b_j - b_i)t] \gamma_{ij}
\]

On \( S \) the equation (5.11) has the Lax form

\[
q_t = [q, v] = [q, \text{ad}_{(ax + bt)} \text{ad}_b^{-1} q]
\]

with the Hamiltonian function

\[
H_{(t)} = \frac{1}{2} \text{Tr}(qv) = \frac{1}{2} \text{Tr}(xuv + tv^2). \] (5.12)

This coincides with \( Q_{(t)} \).

One may change the role of \( x \) and \( t \). This means that one considers the system (5.1) on the space of functions \( v(t) \):

\[
v_x = \left[ \nabla_x I_0, \frac{d}{dt} + v \right],
\]

where \( I_0 = \int H_0 dt = -\frac{1}{2} \int \text{Tr}(uv) \ dt \) and one integrates in the variable \( t \). The scaling (5.8) can be read as an Hamiltonian equation

\[
\frac{dv}{ds} = \left[ \nabla_x I_{(s)}, \frac{d}{dt} + v \right] = 0,
\] (5.13)

where

\[
I_{(s)} = \int L dt = -\frac{1}{2} \int \text{Tr} (xuv + tv^2) \ dt
\] (5.14)

and

\[
\nabla_x I_{(s)} = -tv - xu
\] (5.15)

Commutativity of the flows is equivalent to

\[
\{ I_0, I_{(s)} \} - \frac{\partial I_{(s)}}{\partial x} = 0;
\]

in our case

\[
\int \left[ \text{Tr}(\nabla_x I_0[\nabla_x I_{(s)}, \frac{d}{dt} + v]) + \frac{1}{2} \text{Tr}(uv) \right] dt = 0.
\]

Then there exists a function \( Q_{(x)}(x, t, u, v) \) such that

\[
\text{Tr}(-u[-tv - xu, \frac{d}{dt} + v] + uv) = \frac{d}{dt} Q_{(x)}.
\]

By direct calculation (see Appendix 5.A) we obtain

\[
Q_{(x)} = \frac{1}{2} \text{Tr}(tv + xu^2),
\] (5.16)
in components:

\[
Q_{(x)} = \frac{1}{2} \sum_{i,j} [(a_j - a_i)(b_i - b_j)t + (a_j - a_i)^2 x] \gamma_{ij} \gamma_{ji}
\]

Now we study the \(x\)-flow on the reduced manifold defined by the scaling equation: the system (5.10) gives

\[
\begin{cases}
  u_t - v_x - [u, v] = 0 \\
  tv_t + xv_x + v = 0.
\end{cases}
\] (5.17)

In terms of \(\gamma\) this becomes

\[
\frac{d}{dx} [\gamma, ax + tb] = [\gamma, ax + bt], [\gamma, a],
\]

that is a Lax equation on \(\mathfrak{g}_x\):

\[
q_x = [q, u] = [q, ad_{(ax + bt)}ad^{-1}_a q]
\] (5.18)

with Hamiltonian function

\[
H_{(x)} = \frac{1}{2} Tr(qu) = \frac{1}{2} Tr(xu^2 + tuv).
\] (5.19)

This coincides with \(Q_{(x)}\).

In fact one can rewrite the scaling as a zero–curvature equation in two ways:

\[
\frac{du}{ds} = q_x + [u, q] = 0
\] (5.20)

and

\[
\frac{du}{ds} = q_t + [v, q] = 0.
\] (5.21)

Therefore one may rewrite them in terms of \(q\) as

\[
q_x = [q, ad^{-1}_{(ax + bt)} ad_a q]
\]

and

\[
q_t = [q, ad^{-1}_{(ax + bt)} ad_a q].
\]

5.2 Commuting time–dependent Hamiltonian flows on \(\mathfrak{so}(n)\)

We can do exactly the same using the coordinates

\[
t_i = xa_i + tb_i,
\]

and the corresponding derivatives \(\frac{d}{dt_i}\), with

\[
\frac{d}{dx} = \sum_i a_i \frac{d}{dt_i}
\]

and

\[
\frac{d}{dt} = \sum_i b_i \frac{d}{dt_i}.
\]

The starting equation is now

\[
\frac{d}{dt_k} u_i - \frac{d}{dt_i} u_k - [u_i, u_k] = 0
\] (5.22)
where
\[ u_i = [\gamma, E_i], \quad (u_i)_{kl} = \gamma_{kl} \delta_{ik} - \gamma_{ik} \delta_{kl} \]
and \((E_i)_{kl} = \delta_{ik} \delta_{kl}\). We impose the scaling
\[
\frac{d}{ds} u_k = \sum_t t_i \frac{d}{dt} u_k + u_k = 0
\]
(5.23)
For every \(k\) one can define, on the space \(\mathfrak{g}_k\) of functionals
\[
F = \int f(t, u, \frac{d u}{dt}, \ldots, \frac{d^m u}{dt^m}) dt
\]
with
\[
\frac{d}{de} F(u_k + \epsilon w)|_{\epsilon = 0} = (\nabla_{u_k} F, w),
\]
a Poisson structure \(P^{(k)}\) with the Poisson bracket
\[
\{ F, G \}(u_k) = (\nabla_{u_k} F, [\nabla_{u_k} G, \frac{d}{dt_k} + u_k])
\]
The \(n\)-waves equation (5.22) is Hamiltonian w.r.t. the Poisson structure \(P^{(k)}\) in \(\mathfrak{g}_k\):
\[
\frac{d}{dt_i} u_k = [\nabla_{u_k} I_i, \frac{d}{dt_k} + u_k] = [-u_i, \frac{d}{dt_k} + u_k],
\]
(5.24)
where
\[
I_i = \int I_i dt = -\frac{1}{2} \int \text{Tr} (u_i u_k) dt_k.
\]
(5.25)
On \(\mathfrak{g}_k\) we can reduce to PVI equation imposing the scaling (5.23), which admits the Hamiltonian form
\[
\frac{d}{ds} u_k = [\nabla_{u_k} I_{(s)}, \frac{d}{dt_k} + u_k] = [-\sum_j t_j u_j, \frac{d}{dt_k} + u_k] = 0
\]
(5.26)
where
\[
I_{(s)} = \int L dt_k = -\frac{1}{2} \int \text{Tr} \sum_j t_j u_j u_k dt_k = -\frac{1}{2} \int \text{Tr} (\sum_{j \neq k} t_j u_j u_k + t_k u_k^2) dt_k
\]
(5.27)
The commutativity of the flows is equivalent to
\[
\{ I_i, I_{(s)} \} - \frac{\partial I_{(s)}}{\partial t_i} = 0;
\]
in our case
\[
\int [\text{Tr}(\nabla_{u_k} I_i [\nabla_{u_k} I_{(s)}, \frac{d}{dt_k} + u_k])] + \frac{1}{2} Tr(u_i u_k)] dt_k = -\int \partial_k Q_{(i)} dt_k.
\]
By direct calculation (following the scheme in Appendix 5.A) we obtain
\[
Q_{(i)} = \frac{1}{2} \text{Tr} \sum_j t_j u_j u_i = \sum_j (t_i - t_j) \gamma_{ij} \gamma_{ji},
\]
(5.28)
The scaling equation defines the submanifold \(\mathfrak{g}_s\). One can consider on \(\mathfrak{g}_s\) the system of coordinates given by the matrix elements of \(q\):
\[
q = [\gamma, \sum_j t_j E_j] = [\gamma, U],
\]
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where $U$ is the diagonal matrix $\text{diag}(t_1, \ldots, t_n)$; explicitly

$$q_{ij} = (t_j - t_i) \gamma_{ij}.$$  \hfill (5.29)

As in the previous cases, $Q_{(i)}$ is the Hamiltonian for the $t_i$-flow on the reduced manifold.

Starting now from $\mathcal{S}$, $i \neq k$, we can reduce on the same submanifold $\mathcal{S}$ and construct the Hamiltonian function $Q_{(k)}$.

Indeed, the scaling (5.23) for every $k$ produces on $\mathcal{S}$ the Lax equation

$$q_k = [q, u_k],$$  \hfill (5.30)

with Hamiltonian functions

$$H_k = \frac{1}{2} \text{Tr}(qu_k) = \frac{1}{2} \text{Tr} \sum_j t_j u_j u_k = \frac{1}{2} \sum_{j \neq k} \frac{q_{kj} q_{kj}}{t_k - t_j}. \hfill (5.31)$$

These coincide with the $Q_{(k)}$ constructed above. Observing that $\gamma = \text{ad}_{U^{-1}} q$ one can rewrite

$$u_k = \text{ad}_{u_k} \text{ad}_{U^{-1}} q.$$

In the case $q^T = -q$ eqs. (5.30) are the Monodromy Preserving Deformation equations for the linear differential operator

$$\Lambda = \frac{d}{d\lambda} - U - \frac{q}{\Lambda}$$

that give Painlevé VI, for $n = 3$, and the higher–order analogues, for $n > 3$.

**Remark:** The first integrals of the MPDE (5.30) are given by the monodromy data of the operator $\Lambda$. The Poisson bracket on the space of the monodromy data has been computed in [Ug].

### 5.A Appendix

Here we present the explicit calculations giving rise to equation (5.16): let us consider the following explicit expressions:

$$I_t = -\frac{1}{2} \int \text{Tr} (uv) dx$$

$$\nabla I_t = -v$$

$$I_{(s)} = -\frac{1}{2} \int \text{Tr} (xu^2 + tvu) dx$$

$$\nabla I_{(s)} = -tv - xu$$

$$\{I_t, I_{(s)}\} = \{-v, tu_t + xu_x + u\} =$$

$$= -\int \text{Tr} (tvv_x + tv[u, v] + xv u_x + uv) dx$$

$$\frac{\partial I_{(s)}}{\partial t} = I_t = -\frac{1}{2} \int \text{Tr} (uv).$$

$$\{I_t, I_{(s)}\} - \frac{\partial I_{(s)}}{\partial t} = -\int \text{Tr} (tvv_x + tv[u, v] + xv u_x + \frac{1}{2}uv) dx$$  \hfill (a.1)

In (a.1) the relations

$$\text{Tr} v [u, v] = 0$$

$$\text{Tr} (vu_x) = \frac{1}{2} \frac{d}{dx} \text{Tr} (v^2)$$

$$\text{Tr} (xvu_x + \frac{1}{2}uv) = \frac{1}{2} \frac{d}{dx} \text{Tr} (xuv)$$
hold. In fact, in terms of $\gamma_{ij}$ one can write
\[
\text{Tr} (xvu_x) = \sum_i \sum_k x(b_k - b_i)(a_i - a_k)\gamma_{ik}(\gamma_{x})_{ki} = \\
= \sum_i \sum_k x(b_i - b_k)(a_k - a_i)(\gamma_{x})_{ki}\gamma_{ik} = \\
= \text{Tr} (xv_x u),
\]
which implies
\[
\frac{d}{dx} \text{Tr} (xuv) = 2\text{Tr} (xvu_x) + \text{Tr} (uv).
\]
Then:
\[
\{I_t, I_{(s)}\} - \frac{\partial I_{(s)}}{\partial t} = -\frac{1}{2} \int \frac{d}{dx} \text{Tr} (xuv + tv^2) \, dx.
\]

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