LOW COMPLEXITY OF OPTIMIZING MEASURES OVER AN EXPANDING CIRCLE MAP

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Abstract. In this paper, we prove that for real analytic expanding circle maps, all optimizing measures of a real analytic potential function have zero entropy, unless the potential is cohomologous to constant. We use the group structure of the symbolic space to solve a transversality problem involved. We also discuss applications to optimizing measures for generic smooth potentials and to Lyapunov optimizing measures.

1. Introduction

Given a continuous map $T : X \to X$ from a compact metric space into itself and a continuous function $f : X \to \mathbb{R}$, the problem of ergodic optimization looks for maximization/minimization of $\int f \, d\mu$, where $\mu$ runs over the collection $\mathcal{M}(T)$ of all $T$-invariant Borel probability measures. Let

$$
\beta(T, f) = \beta(f) = \sup_{\mu \in \mathcal{M}(T)} \int f \, d\mu, \quad \alpha(T, f) = \alpha(f) = \inf_{\mu \in \mathcal{M}(T)} \int f \, d\mu.
$$

(1)

By Birkhoff’s Ergodic Theorem, these quantities also optimize the time average of $f$ along orbits.

A measure $\mu \in \mathcal{M}(T)$ is called a maximizing measure if $\beta(f) = \int f \, d\mu$ and an orbit $\{T^n(x)\}_{n=0}^{\infty}$ is called a maximizing orbit if $\beta(f) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i(x))$. Minimizing measures/orbits are similarly defined. As $-\alpha(f) = \beta(-f)$, the problems of maximization and minimization are equivalent. In this paper, we shall mainly discuss the maximizing problem.

Numerical experiments by Hunt-Ott in [11, 12] indicate that typically maximizing orbits are periodic and maximizing measures are supported on periodic orbits or at least of zero entropy. The mathematical theory of ergodic optimization has been developed since then, mainly in the case that $T$ is hyperbolic. Yuan-Hunt [23] dealt with the case that $T$ is either an Axiom A diffeomorphism or a non-invertible uniformly expanding map, and $f$ is Lipschitz, and proved that only periodic orbits (measures) can be persistently maximal in the Lipschitz topology. Building upon [23, 5, 19, 21], Contreras [6] proved that for $T$ uniformly expanding, a generic $f$ has a unique maximizing measure which is supported on a periodic orbit in the Lipschitz topology. Recently, Huang et al [9] extended the results and proved that when $T$ is either uniformly expanding or Axiom A, for a generic $f$ in the $C^1$ topology, the maximizing measure is unique and supported on a periodic orbit. However, just as in many other circumstances in the state-of-the-art theory of dynamical systems, the local perturbation technique is tied to the $C^1$ (or coarser) topology.

The simplest uniformly expanding maps are probably the maps $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, $x \mapsto dx \mod 1$ with $d \geq 2$. This case has received much attention. When $d = 2$, 


for \( f_\theta(x) = \cos(2\pi(x - \theta)), \theta \in \mathbb{R} \), Bousch \[2\] proved that the maximizing measure of \( f_\theta \) is unique and is a Sturmian measure which, in particular, has entropy zero. See \[14, 15, 8\] for similar results in this direction.

We refer to \[13, 1, 16\] for development on other aspects of ergodic optimization.

The goal of this paper is to prove the following theorem.

**Main Theorem.** Let \( T: \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \) be a real analytic expanding map and let \( f: \mathbb{R}/\mathbb{Z} \to \mathbb{R} \) be a real analytic function. Then

- either \( f \) is real analytically cohomologous to constant, i.e. there exists a real analytic function \( g \) and a constant \( c \) such that \( f = g \circ T - g + c \);
- or any optimizing measure of \( f \) is of zero entropy.

In particular, the Main Theorem gives a complete solution to a problem raised by O. Jenkinson, see \[13, \text{Problem 3.12}\], which was also mentioned in \[1, \text{page 1847}\].

Morris \[19\] proved that for generic Hölder or Lipschitz functions, there is a unique maximizing measure and this measure has entropy zero. This result played an important role in the work \[6\] cited above which proves the outstanding TPO conjecture (TPO means “typically maximizing measures are supported on a periodic orbit”) in the Lipschitz topology. Our Main Theorem implies that Morris’ result remains true for \( C^r \) functions for any \( r \in \{0, 1, 2, \ldots, \infty\} \), see Theorem 1.2. It is our hope that our Main Theorem will throw some insight in proving the TPO conjecture in \( C^r \) topology with \( r \geq 2 \) in the case that the underlying dynamics is an expanding circle map.

One of the key ingredients in our proof is a transversality result. To reduce the technicality, we shall only state a special version that is needed for the proof of the Main Theorem here. See Theorem 2.1 for the general statement. Let \( T: \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \) and \( f: \mathbb{R}/\mathbb{Z} \to \mathbb{R} \) be as in the Main Theorem. We assume that \( T \) is orientation preserving and that \( T \) fixes 0. Let \( \hat{T}: \mathbb{R} \to \mathbb{R} \) denote the unique lift of \( T \) with \( \hat{T}(0) = 0 \), via the natural projection \( \pi: \mathbb{R} \to \mathbb{R}/\mathbb{Z} \). Let \( \tau = \hat{T}^{-1} \) and \( \tau_i(x) = \tau(x + i) \) for each \( i \in \mathbb{Z} \). Let \( \hat{f} = f \circ \pi \). Let \( N = \{1, 2, \ldots\} \). For each \( i = (i_n)_{n \geq 1} \in \Sigma_d := \{0, 1, \ldots, d-1\}^\mathbb{N} \), consider

\[
h_i(x) := \sum_{n=1}^{\infty} \left( \hat{f} \circ \tau_{i_n} \circ \tau_{i_{n-1}} \circ \cdots \circ \tau_{i_1} - \hat{f} \circ \tau_{i_n} \circ \tau_{i_{n-1}} \circ \cdots \circ \tau_{i_1}(0) \right), \quad x \in \mathbb{R},
\]

which is well-defined and real analytic on \( \mathbb{R} \).

**Theorem 1.1** (Transversality). Under the above circumstances, we have

- either \( f \) is real analytically cohomologous to constant and \( h_i \equiv h_j \) holds for all \( i, j \in \Sigma_d \);
- or \( h_i \neq h_j \) holds whenever \( i, j \in \Sigma_d \) and \( i \neq j \).

We prove this theorem using the odometer structure of the space \( \Sigma_d \). An important observation is that \( G_0 := \{i \in \Sigma_d : h_i = h_0\} \) is a closed subgroup of \( \Sigma_d \), where \( 0 \) denotes the element of \( \Sigma_d \) whose entries are all 0. When \( d \) is prime, this immediately implies that either \( G_0 = \{0\} \) or \( G_0 = \Sigma_d \). The same conclusion remains true for an arbitrary integer \( d \geq 2 \) which will be dealt with by Fourier analysis. See Proposition 2.3. Note that this kind of transversality problem appears naturally in the study of skew product over circle expanding maps. In particular, a special case of this type of dichotomy was obtained in \[22, \text{Theorem A}\] using a different method, which represents an important step in the study of a dimension dichotomy.
for the graphs of Weierstrass-type functions. The method here provides a much more general result with simpler proofs.

Let us explain how Theorem 1.1 is used in the proof of the Main Theorem. Assume that \( f \) is not real analytically cohomologous to a constant. Then the second alternative of Theorem 1.1 holds. By ergodic decomposition, we only need to show that any ergodic maximizing measure \( \mu \) has zero entropy. Let \( S \) denote the support of \( \mu \). We may assume that \( T(0) = 0 \) and \( 0 \notin S \) so that \( S \) can be identified naturally with a compact subset of \( (0, 1) \). The space of inverse orbits in \( S \) is naturally identified with

\[
S = \{(i_n)_{n\geq 1}, x) : i_n \in \{0, 1, \ldots, d - 1\}, x \in S, \tau_{i_n} \tau_{i_{n-1}} \cdots \tau_{i_1}(x) \in S, \forall n \}.
\]

With the help of a sub-action, we deduce from Theorem 1.1 that if the section

\[
S_i = \{x \in S : (i, x) \in S \}
\]

has a limit point \( x_0 \), then \( i \) belongs to a set which has at most two elements and which is uniquely determined by \( x_0 \), see Proposition 3.2. (It is here that we need real analyticity of \( f \) and \( T \).) This enables us to apply well-known results in dimension theory of dynamical systems (for example [18]) to conclude that \( \mu \) has entropy zero.

Now we state a few corollaries of the Main Theorem.

**Theorem 1.2.** Let \( T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \) be expanding and real analytic. Let \( r \in \{0, 1, 2, \ldots, \infty\} \). Then for generic \( f \in C^r(\mathbb{R}/\mathbb{Z}) \), any optimizing measure of \( f \) is of zero entropy.

As mentioned before, Morris [19] proved a similar result for Hölder and Lipschitz functions, using a result of Bressaud and Quas [5]. Our argument is based on our Main Theorem and the fact that \( C^r \) functions can be approximated by real analytic ones. It is worthy noting that the \( C^0 \) case is particularly special, as generically, the unique optimizing measure is fully supported but has zero entropy, see [3, 17, 4].

The special case \( f = \pm \log |T'| \) is often referred to as Lyapunov optimization, see [7]. Our result has the following consequences.

**Theorem 1.3.** Let \( T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \) be expanding and real analytic. Then

- either there exists a real analytic diffeomorphism \( \phi : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \) such that \( \phi \circ T \circ \phi^{-1}(x) = \deg(T) \cdot x \);
- or any Lyapunov optimizing measure of \( T \) is of zero entropy.

For \( r \in \{1, 2, 3, \ldots, \infty\} \), let \( C^r \) denote the collection of \( C^r \) maps from \( \mathbb{R}/\mathbb{Z} \) to itself, endowed with the \( C^r \) topology. Let \( \mathcal{E}^r \) be the open subset of \( C^r \) consisting of \( C^r \) expanding maps on \( \mathbb{R}/\mathbb{Z} \).

**Theorem 1.4.** Let \( r \in \{1, 2, 3, \cdot \cdot \cdot, \infty\} \). For generic \( T \in \mathcal{E}^r \), any Lyapunov optimizing measure of \( T \) is of zero entropy.

In [10], it is proved that for generic \( T \in \mathcal{E}^2 \), a Lyapunov optimizing measure is supported on a periodic orbit. See [7, 17] for earlier results on \( C^{1+\alpha} \) and \( C^1 \) cases respectively.

The rest of the paper is organized as follows. In § 2, we prove Theorem 1.1 and a more general transversality result; in § 3, we prove the Main Theorem; in § 4, we prove Theorems 1.2, 1.3 and 1.4.

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2. A TRANSVERSALITY RESULT

In this section, we state and prove a more general version of Theorem 1.1. The proof exploits the structure of $\Sigma_d = \{0, 1, \ldots, d-1\}^N$ as a compact abelian group.

2.1. Statement of result. We start with the statement of the general transversality result. Let $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be a $C^1$ orientation-preserving expanding map with $T(0) = 0$ and let $d \geq 2$ be the degree. Let $f : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ be a $C^\alpha$ function for some $\alpha \in (0, 1)$. As before, denote by $\hat{T} : \mathbb{R} \to \mathbb{R}$ the lift of $T$ with $\hat{T}(0) = 0$, $\tau = \hat{T}^{-1}$, and $\tau_i(x) = \tau(x + i)$. Given $i = (i_1, \ldots, i_n, \ldots) \in \Sigma_d$, for any $n \geq 1$, denote:

$$\tau_{i,n} := \tau_{i_1} \circ \cdots \circ \tau_{i_n} \circ \tau_{i_1}.$$ 

Let $\lambda_0 = \sup_{\lambda \in \mathbb{R}} \min_{x \in \mathbb{R}} |(\hat{T}^n)'(x)|^{1/n} > 1$. \(^1\) Note that for any $0 < \lambda_1 < \lambda_0$, there exists $C \geq 1$ such that

$$\sup_{x \in \mathbb{R}, i \in \Sigma_d} |\tau_{i,n}(x)| \leq C \lambda_1^{-n}, \quad \forall \ n \geq 0.$$ 

Fix a $\lambda \in \mathbb{C}$ such that $0 < |\lambda| < \lambda_0$. Then, for each $i \in \Sigma_d$,

$$h_i(x) := \sum_{n=1}^{\infty} \lambda^n \left( \hat{f} \circ \tau_{i,n}(x) - \hat{f} \circ \tau_{i,n}(0) \right), \quad x \in \mathbb{R},$$

is a well-defined continuous function from $\mathbb{R}$ to $\mathbb{C}$, where $\hat{f} = f \circ \pi$ and $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is the natural projection. The following theorem is our main transversality result.

**Theorem 2.1.** Under the circumstances above, the following are equivalent:

(i) there exist $i \neq j$ with $h_i \equiv h_j$;

(ii) $h_i = h_0$ for any $i$;

(iii) $h_0$ is 1-periodic;

(iv) $h_0$ is 1-periodic and $\hat{f} = \lambda^{-1} \cdot h_0 \circ \hat{T} - h_0 + \hat{f}(0)$;

(v) there exists a $C^\alpha$ function $\phi : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ such that $f = \lambda^{-1} \cdot \phi \circ \hat{T} - \phi + \text{const}.$

**Proof of Theorem 1.1 assuming Theorem 2.1.** Assume that the second alternative in the conclusion of Theorem 1.1 does not hold. Then (i) and hence (ii)-(v) all hold in Theorem 2.1 (for $\lambda = 1$). By (iv), $f$ is real analytically cohomologous to constant since $h_0$ is real analytic. By (ii), $h_i = h_j$ for all $i, j \in \Sigma_d$. Thus the first alternative in the conclusion of Theorem 1.1 holds. \(\square\)

2.2. Group structure of $\Sigma_d$. Let us recall the well-known structure of $\Sigma_d$ as a compact abelian group. The topology on $\Sigma_d$ is the product topology of the discrete topology on $\{0, 1, \ldots, d-1\}$. The space $\Sigma_d$ becomes a compact abelian group with addition defined as follows. For $i = (i_n)_{n \geq 1}, j = (j_n)_{n \geq 1} \in \Sigma_d$, $i + j \in \Sigma_d$ is the unique element $k = (k_n)_{n \geq 1} \in \Sigma_d$ such that for each $n = 1, 2, \ldots,$

$$k_1 + k_2d + \cdots + k_nd^{n-1} = (i_1 + j_1) + (i_2 + j_2)d + \cdots + (i_n + j_n)d^{n-1} \mod d^n.$$ 

Note that the semi-group $\mathbb{Z}_+ = \{0, 1, 2, \cdots\}$ can be naturally embedded into $\Sigma_d$ as a dense sub-semi-group in the following way:

$$m = i_1 + i_2d + \cdots + i_nd^{n-1} \mapsto (i_1, \ldots, i_n, 0, 0, \ldots) =: \iota(m).$$

In particular, $0 = \iota(0)$ is the zero element of $\Sigma_d$. The map $i \mapsto i + \iota(1)$ is usually called a (d-adic) adding machine.

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\(^1\)By [20, Theorem A.3], $\log \lambda_0 = \alpha(T, \log |T'|)$, where $\alpha$ is as in (1).
The following lemma is a simple observation which plays an important role in our proof.

**Lemma 2.2.** Given \( i, j, k \in \Sigma_d \), \( h_i = h_j \) if and only if \( h_{i+k} = h_{j+k} \). As a result, \( G_0 := \{ i : h_i = h_0 \} \) is a closed subgroup of \( \Sigma_d \).

**Proof.** Since \( \Sigma_d \) is a group, we only need to prove the “only if” part. By definition, for each positive integer \( m \),
\[
\tau_{i,n}(x + m) = \tau_{i+i(m),n}(x) \mod 1, \quad \forall \ i, \forall \ n \geq 1, \forall \ x \in \mathbb{R}.
\]
Since \( \hat{f} \) is of period 1, it follows that for \( m \geq 1 \), we have:
\[
h_i(x + m) = h_{i+i(m)}(x).
\]
In particular,
\[
h_i = h_j \Rightarrow h_i+i(m) = h_j+j(m).
\]
Since \( \iota(\mathbb{Z}_+) \) is dense in \( \Sigma_d \) and since \( k \mapsto h_k(x) \) is continuous for any fixed \( x \in \mathbb{R} \), it follows that \( h_i = h_j \) implies that \( h_{i+k} = h_{j+k} \).

In particular, \( G_0 \) is a subgroup of \( \Sigma_d \). Its closedness follows from the fact that \( i \mapsto h_i(x) \) is continuous for each \( x \). \( \square \)

**Proposition 2.3.** Under the circumstances of Theorem 2.1, (i) implies (iii).

The proof of this proposition will be given in the next subsection, using Fourier analysis. Here we give a short proof in the case that \( d \) is a prime.

**Proof of Proposition 2.3 assuming that \( d \) is a prime.** It suffices to prove that \( \iota(1) \in G_0 = \{ i \in \Sigma_d : h_i = h_0 \} \), because \( h_{i(1)} = h_0 \) means that \( h_0 \) is 1-periodic. Since (i) holds, by Lemma 2.2, \( G_0 \) is a non-trivial closed subgroup of \( \Sigma_d \). For each \( m \geq 1 \),
\[
A_m := \{ i_1 + i_2d + \cdots + i_md^{m-1} \mod d^m : (i_n)_{n \geq 1} \in G_0 \}
\]
is a subgroup of the cyclic group \( \mathbb{Z}/(d^m\mathbb{Z}) \). Since the group \( G_0 \) also has the following property:
\[
0i_1i_2 \cdots \in G_0 \implies i_1i_2 \cdots \in G_0,
\]
\( A_m \) is non-trivial for each \( m \). Since \( d \) is prime, there exists a unique \( k_m \in \{0, 1, \ldots, m-1\} \) such that \( A_m \) is generated by \( d^{k_m} \mod d^m \). In particular, \( k_1 = 0 \). Since \( j \mod d^{m+1} \mapsto j \mod d^m \) induces a surjective homomorphism from \( A_{m+1} \) to \( A_m \), \( d^{k_{m+1}} \mod d^m \) is also a generator of \( A_m \), and thus \( k_{m+1} = k_m \). In conclusion, \( k_m = 0 \) for all \( m \geq 1 \). Since \( G_0 \) is closed, it follows that \( \iota(1) \in G_0 \). \( \square \)

**Proof of Theorem 2.1.** (ii) \( \iff \) (i). This is trivial.

(i) \( \iff \) (iii). This is Proposition 2.3.

(iii) \( \iff \) (iv). Let \( \hat{f}_0 = \hat{f} - \hat{f}(0) \). By definition,
\[
h_0 = \sum_{n=1}^{\infty} \lambda^n \cdot \hat{f}_0 \circ \tau^n.
\]
Then
\[
h_0 \circ \hat{T} = \sum_{n=1}^{\infty} \lambda^n \cdot \hat{f}_0 \circ \tau^{n-1} = \lambda(\hat{f}_0 + h_0).
\]
(iv) \( \iff \) (v). This is trivial.
(v) \implies (ii). Denote \( \tau_{i,0} = \text{id} \) and \( \hat{\phi} = \phi \circ \pi \). There exists \( c \in \mathbb{C} \) such that for any \( i \) and any \( n \geq 1 \),
\[
\hat{f} \circ \tau_{i,n} = \lambda^{-1} \hat{\phi} \circ \hat{T} \circ \tau_{i,n} - \hat{\phi} \circ \tau_{i,n} + c = \lambda^{-1} \hat{\phi} \circ \tau_{i,n-1} - \hat{\phi} \circ \tau_{i,n} + c.
\]
It can be rewritten as
\[
\hat{f} \circ \tau_{i,n} - \hat{f} \circ \tau_{i,n}(0) = \lambda^{-1}(\hat{\phi} \circ \tau_{i,n-1} - \hat{\phi} \circ \tau_{i,n-1}(0)) - (\hat{\phi} \circ \tau_{i,n} - \hat{\phi} \circ \tau_{i,n}(0)).
\]
It follows that
\[
h_i = \hat{\phi} - \hat{\phi}(0)
\]
does not depend on \( i \).

\[\square\]

2.3. Fourier analysis. The goal of this subsection is to prove Proposition 2.3.

We first describe a procedure which reduces the problem to the case that \( T \) is linear. It is well-known that \( T \) is topologically conjugate to the linear map \( m_d : x \mapsto dx \mod 1 \) via a homeomorphism \( \theta : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \) with \( \theta(0) = 0 \); in particular, \( \theta \circ T = m_d \circ \theta \). Let \( \Theta : \mathbb{R} \to \mathbb{R} \) be the lift of \( \theta \) with \( \Theta(0) = 0 \). Then
\[
\Theta \circ \hat{T} = d \cdot \Theta,
\]
which implies that
\[
\Theta \circ \tau_{i,n} \circ \Theta^{-1}(x) = \frac{x + i_1 + i_2 d + \cdots + i_n d^{n-1}}{d^n} =: \xi_n(i, x).
\]
Let \( F = \hat{f} \circ \Theta^{-1} \) and let
\[
H_i(x) := h_i \circ \Theta^{-1}(x) = \sum_{m=1}^{\infty} \lambda^m [F(\xi_m(i, x)) - F(\xi_m(i, 0))].
\]
By definition, we have the following.
- \( F \) is 1-periodic.
- \( h_i \) is 1-periodic if and only if \( H_i \) is 1-periodic.
- \( h_i \equiv h_j \) if and only if \( H_i \equiv H_j \).
- If \( T = m_d \), then \( \Theta = \text{id} \) and \( h_i = H_i \).

For each \( s \in \mathbb{Z} \), consider
\[
\mathcal{H}^s : \Sigma_d \times [0, 1) \to \mathbb{C}, \quad (k, x) \mapsto H_k(x + s) - H_k(x).
\]
To prove Proposition 2.3, we shall use Fourier analysis to show that \( \mathcal{H}^s \) is constant for each \( s \in \mathbb{Z} \).

The space \( \Sigma_d \times [0, 1) \) carries a unique Borel probability measure \( \mu \) such that for any \( m \geq 1 \) and any Borel set \( U \subset [0, 1) \),
\[
\mu(\{(k, x) \in \Sigma_d \times [0, 1) : \xi_m(k, x) \in U\}) = |U|,
\]
where \( |U| \) denote the standard Lebesgue measure of \( U \). Moreover, there is a \( \mu \)-preserving Borel measurable bijection \( m_d : \Sigma_d \times [0, 1) \to \Sigma_d \times [0, 1) \) defined as
\[
m_d(k, x) = (k_0 k, d x - k_0),
\]
where \( k_0 = \lfloor dx \rfloor \) is the largest integer which is not greater than \( dx \), and \( k_0 k = (k_n)_{n \geq 1} \) for \( k = (k_n)_{n \geq 1} \).

Indeed, \( \xi : (k, x) \mapsto (\xi_m(k, x))_{m \geq 0} \) provides a natural identification between \( \Sigma_d \times [0, 1) \) and the space \( X(d) \) of backward orbits of \( m_d \):
\[
X(d) = \{(x_m)_{m=0}^{\infty} : x_m \in \mathbb{R}/\mathbb{Z}, m_d(x_{m+1}) = x_m, \forall m \geq 0\}.
\]
The map $m_d$ corresponds to the homeomorphism $(x_m)_{m \geq 0} \mapsto (m_d x_m)_{m \geq 0}$ and $\mu$ corresponds to the lift of the $m_d$-invariant Lebesgue measure on $\mathbb{R}/\mathbb{Z}$ to the space $X(d)$. Let us note that the measure $\mu$ corresponds to the Haar measure on the compact abelian group $X(d)$, although we do not need this fact explicitly.

The following lemma describes symmetry of the functions $H^s$.

**Lemma 2.4.** For any $s \in \mathbb{Z}$,
\[ H^{ds} \circ m_d = \lambda \cdot H^s. \] (2)
Moreover, for any $i \in G_0$, we have
\[ H^s(i + k, x) = H^s(k, x), \quad \forall (k, x) \in \Sigma_d \times [0, 1). \] (3)

**Proof.** We first prove (2). Note that for each $y \in \mathbb{R}$, $m \geq 1$, $k_0 \in \{0, 1, \ldots, d - 1\}$ and $k \in \Sigma_d$, we have
\[ \xi_m(k_0 k, dy - k_0) = \xi_m(k, y), \]
where $\xi_0(k, y) = y$. So for $k_0 = [dx]$, $\xi_m(k_0 k, dx - k_0)$
\[ = \lambda \cdot (F(x + s) - F(x)) + \lambda \cdot H^s(k, x) \]
where the last equality follows from the fact that $F$ is 1-periodic and $s \in \mathbb{Z}$.

Now let us prove (3). Given $i \in G_0$, by Lemma 2.2, $h_{i+k} \equiv h_k$, and hence $H_{i+k} = H_k$ for all $k \in \Sigma_d$. The equality follows.

We shall also need the following two lemmas.

**Lemma 2.5.** For $i = (i_n)_{n \geq 1} \in \Sigma_d \setminus \{0\}$, let $z_m = (i_1 + i_2 d + \cdots + i_m d^{m-1})/d^m$ for each $m \geq 1$. Then for any integer $q \neq 0$, there exists a positive integer $m_*$ such that for any $m \geq m_*$,
\[ qz_m \neq 0 \mod 1. \]

**Proof.** Fix $q$ and let $Y = \{ \frac{k}{q} \mod 1 : k \in \mathbb{Z} \}$. Then $Y$ is a $m_d$-invariant finite set. Assuming that the conclusion fails, i.e., $z_m \in Y$ for all $m \geq 1$, it remains to show that $i = 0$. Since $m_d(z_{m+1}) = z_m$ for all $m \geq 1$, it follows that each $z_m$ is a periodic point of $m_d$ in $Y$. Since each $z_m$ is eventually mapped to 0 mod 1, it follows that $z_m = 0 \mod 1$ for all $m \geq 1$, and hence $i = 0$.

**Lemma 2.6.** If $H^1$ is constant, then $h_0$ (or equivalently $H_0$) is 1-periodic.

**Proof.** Let $c \in \mathbb{C}$ be such that $H^1 \equiv c$. By definition, $H_k(x + 1) = H_k(x) = c$ for any $(k, x) \in \Sigma_d \times [0, 1)$. Since $H_{k+1}(y) = H_k(y + m)$ for any $y \in \mathbb{R}$, any integer $m \geq 0$ and any $k \in \Sigma_d$, it follows that $H_k(y + 1) = H_k(y) = c$ for any $y \in \mathbb{R}$ and any $k \in \Sigma_d$. In particular, $H^s \equiv sc$ for any $s \in \mathbb{Z}$. By (2), it follows that
\[ dc = \lambda c. \]
By the Mean Value Theorem, $\lambda_0 \leq d$, so $|\lambda| < \lambda_0^2 \leq d$. Thus $c = 0$, which completes the proof.

We are ready to complete the proof of Proposition 2.3.

**Completion of proof of Proposition 2.3.** Assume that (i) holds. By Lemma 2.6, it suffices to show that $H_s$ is constant for any $s \in \mathbb{Z}$. To this end, let

$$E_{m,q}(k, x) = e^{2\pi i \xi_m(k, x)q}, \quad m, q \in \mathbb{Z}, m \geq 0.$$  

We shall show that for any integers $m \geq 0, q \neq 0$ and $s$,

$$e_{m,q} := \int_{\Sigma_d \times [0,1)} H_s \cdot \overline{E_{m,q}} d\mu = 0. \quad (4)$$

Before the proof of (4), let us show how it implies that $H_s$ is constant. Note that $H_s$ is bounded and continuous on $\Sigma_d \times [0, 1)$. Let $B$ denote the Borel $\sigma$-algebra of $[0, 1)$ and let $B_m = \xi_{m}^{-1}(B)$, which is a $\sigma$-algebra in $\Sigma_d \times [0, 1)$. Then $B_m$ is monotone increasing to the Borel $\sigma$-algebra in $\Sigma_d \times [0, 1)$. By the Martingale Convergence theorem, $H_m^* := \mathbb{E}[H_s | B_m]$ converges $\mu$-a.e. to $H_s$. Since $E_{m,q}$ is $B_m$-measurable,

$$\int_{\Sigma_d \times [0,1)} H_m^* \cdot \overline{E_{m,q}} d\mu = c_{m,q} = 0$$

for any $m \geq 0$ and $q \neq 0$, where the last equality follows from (4). Since $(\xi_m)_* \mu$ is the standard Lebesgue measure on $[0, 1)$, each $H_m^*$, which can be viewed as a function in $[0,1)$, must be constant a.e.. Thus $H_s$ is constant $\mu$-a.e.. Since $H_s$ is continuous, it is constant.

It remains to prove that (4) holds for all $m \geq 0$ and $q \neq 0$. First let us show

$$e_{m+1, q}^d = \lambda e_{m,q}^d. \quad (5)$$

Indeed, since $(m_d)_* \mu = \mu$ and $\xi_{m+1} = \xi_m$, we have

$$\int \mathcal{H}^{d_m \circ m_d} \overline{E_{m,q}} d\mu = \int \mathcal{H}^{d_m \circ m_d} \overline{E_{m+1, q} \circ m_d} d\mu = \int \mathcal{H}^{d_m} \overline{E_{m+1, q}} d\mu = e_{m+1, q}^d.$$ 

Combining this with (2), we obtain (5).

Next, by Lemma 2.2, $G_0 \neq \{0\}$. Let $i = (i_n)_{n \geq 1} \in G_0 \setminus \{0\}$. Then by (3), for any $s \in \mathbb{Z}$, $m \geq 1$ and $q \in \mathbb{Z}$,

$$e_{m,q}^s = e^{2\pi iz_m q e_{m,q}^s},$$

where $z_m = (i_1 + i_2 d + \cdots + i_m d^{m-1})/d^m$. By Lemma 2.5, for each $q \neq 0$, there exists $m_*(q)$ such that for any $m \geq m_*(q)$, $q z_m \neq 0 \mod 1$, so that $e^{2\pi iz_m q} \neq 1$, which implies that $e_{m,q}^s = 0$ for all $s \in \mathbb{Z}$. By (5), it follows that $e_{m,q}^s = 0$ for all $m \geq 0$ and $0 \neq q \in \mathbb{Z}$. The proof is completed. 

3. **Proof of the Main Theorem**

This section is devoted to the proof of the Main Theorem. The following is an equivalent reformulation of the Main Theorem.

**Main Theorem.** Let $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be a real analytic expanding map and let $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ be a real analytic function. Assume that $f$ is not real analytically cohomologous to constant. Let $\mu$ be a maximizing measure of $(T, f)$. Then the measure-theoretic entropy $h_T(\mu) = 0$. 

In §§ 3.1–3.3, we shall prove the Main Theorem’ under the following technical condition:

\((\ast): \text{T is orientation-perserving and } \text{Fix(T)} \setminus \text{supp}(\mu) \neq \emptyset\),

where \(\text{Fix(T)}\) denotes the set of fixed points of \(T\). In § 3.4, we shall show how to remove this condition and complete the proof, using the equivalence of the maximization problem between \((T, f)\) and \((T^k, f + f \circ T + \cdots + f \circ T^{k-1})\).

3.1. **Strategy of the proof assuming \((\ast)\).** Let \(d\) be the degree of \(T\). Without loss of generality, we may assume that \(0 \in \mathbb{R}/\mathbb{Z}\) is a fixed point of \(T\) which is not contained in \(\text{supp}(\mu)\). As before, let \(\hat{T}\) be the unique lift of \(T\) with \(\hat{T}(0) = 0\) and denote \(\tau := \hat{T}^{-1}\). Moreover, denote \(\tau_i(x) := \tau(x + i)\) for \(0 \leq i < d\). Let \(S = \text{supp}(\mu)\). Identifying \(\mathbb{R}/\mathbb{Z}\) with \([0, 1)\) in the natural way, \(S\) is a non-empty compact subset of \((0, 1)\) with \(T(S) = S\).

For \(i = (i_n)_{n \geq 1} \in \Sigma_d\), recall

\[
\tau_{i,n} = \tau_{i,n} \circ \cdots \circ \tau_{i,2} \circ \tau_{i,1}, \quad \forall n \geq 1.
\]

In order to show that \(h_T(\mu) = 0\), we shall analyze the inverse limit of \(T : S \to S\). So let

\[
S = \{(i, x) \in \Sigma_d \times S : \tau_{i,n}(x) \in S, \forall n \geq 1\}.
\]

Given \(x \in S\), denote \(S^x := \{i \in \Sigma_d : (i, x) \in S\}\).

Note that \(S^x\) is a non-empty compact subset in \(\Sigma_d\). Given \(i \in \Sigma_d\), denote

\[
S_i := \{x \in S : (i, x) \in S\} = \{x \in S : i \in S^x\}.
\]

Then \(S_i\) is a compact subset of \(S\) (possibly empty).

Let us apply Theorem 1.1, so that

\[
h_i(x) = \sum_{n=1}^{\infty} (\hat{f} \circ \tau_{i,n}(x) - \hat{f} \circ \tau_{i,n}(0)), \quad x \in \mathbb{R}.
\]

As we are assuming that \(f\) is not real analytically cohomologous to constant, the second alternative of Theorem 1.1 holds. So for distinct \(i, j \in \Sigma_d\), the real analytic functions \(h_i\) and \(h_j\) are not identical, and hence \(h_i - h_j\) has only isolated zeros. This allows us to define, for each \(x \in S\), two total orders \(<_+^x\) and \(<_-^x\) on \(S^x\) as follows.

**Definition 3.1.** Given \(x \in S\), define two total orders \(<_+^x, <_-^x\) on \(S^x\) as follows. Given \(i \neq j \in S^x\),

- **\(i\) is strictly less than \(j\)** with respect to \(<_+^x\) if there exists \(\delta > 0\) such that \(h_i'(y) < h_j'(y)\) holds for all \(y \in (x, x + \delta)\);
- **\(i\) is strictly less than \(j\)** with respect to \(<_-^x\) if there exists \(\delta > 0\) such that \(h_i'(y) > h_j'(y)\) holds for all \(y \in (x - \delta, x)\).

**Lemma 3.1.** Suppose that \(f\) is not real analytically cohomologous to constant. Then for any \(x \in S\), \((S^x, <_+^x)\) has a unique maximal element \(\kappa_+(x) \in S^x\), and \((S^x, <_-^x)\) has a unique maximal element \(\kappa_-(x) \in S^x\), which define two maps \(\kappa_\pm : S \to \Sigma_d\).

**Proof.** Uniqueness follows from the fact that both orders are total orders. For existence, let us focus on \("<_+^x\); the discussion on \("<_-^x\) is totally similar and omitted. For each \(m \geq 1\), since \(h_i^{(m)}(x)\) is continuous in \(i \in S^x\), \(I_m\) defined
inductively below is a decreasing sequence of non-empty compact subsets of \( S^x \),
where \( I_0 = S^x \):
\[
I_m = \left\{ i \in I_{m-1} : h_i (m)(x) = \max \{ h_j (m)(x) : j \in I_{m-1} \} \right\}.
\]
Therefore \( \bigcap_{m=1}^{\infty} I_m \) contains at least one element \( i \). Then for any \( j \in S^x \), \( h_i (m)(x) \geq h_j (m)(x) \) for any \( m \geq 1 \), which implies that \( i \) is a maximal element in \( S^x \) with respect to \( \prec_+ \).

We shall prove the following proposition in § 3.3.

**Proposition 3.2.** Assume that \( S_1 \) has a limit point \( x \). Then \( i \in \{ \kappa_+ (x), \kappa_- (x) \} \).

**3.2. Proof of the Main Theorem’ assuming (\*)**. Let \( T : S \to S \) be the inverse limit of \( T : S \to S \), i.e., for \( x \in \tau_{\nu} ([0, 1]) \cap S \),
\[
T_i ((n_{i-1})_{n \geq 1}, x) = (\tau_{\nu} T (x) - i_0).
\]
As \( 0 \notin S \), \( T : S \to S \) is naturally topologically conjugate to a one-sided subshift with \( d \) symbols and \( T : S \to S \) is topologically conjugate to a two-sided subshift with \( d \) symbols.

We shall need the following well-known result.

**Proposition 3.3.** If \( \nu \) is an ergodic invariant probability Borel measure of the two-sided full shift \( \sigma : \{ 0, 1, \ldots, d-1 \}^\mathbb{Z} \) such that \( h_{\sigma} (\nu) > 0 \), then for any Borel subset \( U \) of \( \{ 0, 1, \ldots, d-1 \}^\mathbb{Z} \) with \( \nu (U) > 0 \), the following holds for \( \nu \) a.e. \( (i_n)_{n \in \mathbb{Z}} \in U \):
\[
\{ (j_n)_{n \in \mathbb{Z}} \in U : j_n = i_n, \forall n \geq 0 \},
\]
\[
\{ (j_n)_{n \in \mathbb{Z}} \in U : j_n = i_n, \forall n < 0 \}
\]
are both uncountable.

**Proof.** It is well-known that the two-sided full shift is topologically conjugate to a linear horseshoe \( F : \Lambda \to \Lambda \), with the sets in question corresponding to (local) stable and unstable manifolds. If \( \nu_F \) is the \( F \)-invariant ergodic probability measure corresponding to \( \nu \), then \( h_{\sigma} (\nu_F) > 0 \). It is well-known that (see e.g. [18]) for \( \nu_F \)-a.e. \( (y, x) \in \Lambda \), the conditional measures \( \nu_{F, y} \) and \( \nu_{F, y} \) along the local stable and unstable manifolds have positive local dimensions and thus admit no atom. So if \( \nu_F (W) > 0 \), then for \( \nu_F \)-a.e. \( z \in W \), the intersection of \( W \) with the stable and unstable manifolds of \( z \) must be both uncountable.

The following lemma deals with the measurability issue involved.

**Lemma 3.4.** The set \( B \) defined below is Borel:
\[
B := \{ (i, x) \in S : x \text{ is a limit point of } S_i \}.
\]

**Proof.** Define \( \phi : \Sigma d \times S \to \mathbb{R} \) as \( \phi (i, x) = \text{dist} (x, (S_i \cup \{ 2 \}) \setminus \{ x \}) \) (here use \( S_i \cup \{ 2 \} \) instead of \( S_i \) because \( S_i \setminus \{ x \} \) might be empty), where \( \text{dist} (x, \cdot) \) means the distance of \( x \) to a subset of the real line in the usual sense. Then \( B = \phi^{-1} (0) \). It suffices to show that \( \phi \) is Borel. In fact, the following hold.

- Given \( i \in \Sigma d \), \( x \mapsto \phi (i, x) \) is continuous. This is easy to check.
- Given \( x \in S \), \( i \mapsto \phi (i, x) \) is Borel. To see this, for each \( n \geq 1 \), let
\[
\varphi_n (i) := \text{dist} (x, (S_i \cup \{ 2 \}) \setminus (x - \frac{1}{n}, x + \frac{1}{n})), \; i \in \Sigma d.
\]
Then, by compactness of \( S \), \( \varphi_n \) is lower semi-continuous for each \( n \), while \( \lim_{n \to \infty} \varphi_n (i) = \phi (i, x) \).
It follows that $\phi$ is Borel. \hfill $\square$

Let us now complete the proof of the Main Theorem assuming (\ast).

**Proof of the Main Theorem’ assuming (\ast).** Assume by contradiction that $h_T(\mu) > 0$. By ergodic decomposition and affinity of the entropy function, we may assume that $\mu$ is ergodic with respect to $T$. As the inverse limit of $T : S \to S$, the map $T : S \to S$ has an ergodic invariant measure $\mu$ with positive entropy.

We shall show that $\tilde{\mu}(S) = 0$ to get a contradiction. To this end, let $B$ be the set defined in Lemma 3.4. By the definition of $B$ and Proposition 3.2, $\tilde{\mu}(B) = 0$. By the definition of $B$ again, for any $(i,x) \in S \setminus B$, the set $\{y \in S : (i,y) \in S \setminus B\}$ is countable, so by Proposition 3.3 again, $\tilde{\mu}(S \setminus B) = 0$. The proof is done. \hfill $\square$

### 3.3. Complexity of $S$.

This subsection is devoted to the proof of Proposition 3.2. Let $g : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ be a sub-action for $(T,f)$, i.e., $g : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ is a continuous function such that

$$f(x) \leq g \circ T(x) - g(x) + \beta(f)$$

for all $x \in \mathbb{R}/\mathbb{Z}$. If we put

$$S_0 = \{x \in \mathbb{R}/\mathbb{Z} : f(x) = g(T(x)) - g(x) + \beta(f)\},$$

then a $T$-invariant measure is a maximizing measure $f$ if and only if it is supported in $S_0$. In particular, $S \subset S_0$.

Sub-actions played an important role in the ergodic maximization problem. When $T$ is expanding and $f$ is Lipschitz, it is well-known that there exists a sub-action $g$ which is Lipschitz. This is often referred to as M"un"e’s lemma. However, in general we cannot expect higher regularity of $g$, even when we assume $f$ and $T$ are both real analytic, see [3].

**Lemma 3.5.** Let $(i,x) \in S$. Then for any $y \in S$,

$$g(y) - g(x) \geq h_i(y) - h_i(x).$$

Moreover, equality holds if $(i,y) \in S$.

**Proof.** Denote $\tau_{i,0} = \text{id}$. Since $g$ is a sub-action and $x \in S_i$, for each $n \geq 0$,

$$g(\tau_{i,n}(x)) = f(\tau_{i,n+1}(x)) + g(\tau_{i,n+1}(x)) - \beta(f),$$

$$g(\tau_{i,n}(y)) \geq f(\tau_{i,n+1}(y)) + g(\tau_{i,n+1}(y)) - \beta(f).$$

Moreover, equality holds in the last inequality if $y \in S_i$. Therefore, for each $m \geq 1$,

$$g(x) = \sum_{n=1}^{m} f(\tau_{i,n}(x)) + g(\tau_{i,m}(x)) - m\beta(f),$$

$$g(y) \geq \sum_{n=1}^{m} f(\tau_{i,n}(y)) + g(\tau_{i,m}(y)) - m\beta(f).$$

Consequently,

$$g(y) - g(x) \geq \sum_{n=1}^{m} (f(\tau_{i,n}(y)) - f(\tau_{i,n}(x))) + (g(\tau_{i,m}(y)) - g(\tau_{i,m}(x))).$$

Letting $m \to \infty$, we obtain the desired inequality.

If $y \in S_i$, then all the inequalities above become equalities. \hfill $\square$
Proof of Proposition 3.2. We may assume that there exists a sequence \((x_k)\) in \(S_i\) converging to \(x\) from the right side; the other situation is similar and omitted. We shall show that \(i = \kappa_+(x)\). To this end, consider an arbitrary \(j \in S^x\). By Lemma 3.5, we have the following.

- Since \(x, x_k \in S_i\),
  \[
g(x_k) - g(x) = h_1(x_k) - h_3(x) = \int_x^{x_k} h_i'(t) \, dt.
\]
- Since \(x \in S_i\),
  \[
g(x_k) - g(x) \geq h_3(x_k) - h_3(x) = \int_x^{x_k} h_i'(t) \, dt.
\]

Therefore,
\[
\int_x^{x_k} h_i'(t) \, dt \leq \int_x^{x_k} h_i'(t) \, dt.
\]

Since \(x_k\) converges to \(x\) from the right side, the inequality above implies that \(h_i'(t) \leq h_i'(t)\) holds for a sequence of points converging to \(x\) from the right side. It follows that \(i \prec \xi, j\) cannot hold. Therefore \(i = \kappa_+(x)\). \(\square\)

3.4. Iteration. We shall show that the technical condition \((\ast)\) can be removed. Let \(T \in \mathcal{E}^\omega\) and let \(f \in C^\omega(\mathbb{R}/\mathbb{Z})\). For any positive integer \(k\), let \(\mathcal{A}_k f = f + f \circ T + \cdots + f \circ T^{k-1}\).

The following elementary observations should be well-known. Let us include a proof for completeness.

Lemma 3.6.

1. There exists \(g \in C^\omega(\mathbb{R}/\mathbb{Z})\) such that \(\mathcal{A}_k f = g \circ T^k - g + \text{const}\) for some \(k \geq 1\) if and only if the same holds for \(k = 1\).
2. A maximizing measure of \((T, f)\) is also a maximizing measure of \((T^k, \mathcal{A}_k f)\) for any \(k \geq 1\).

Proof. (1) If \(f = g \circ T - g + c\), then \(\mathcal{A}_k f = g \circ T^k - g + kc\). For the other direction, suppose that \(\mathcal{A}_k f = g \circ T^k - g + c\) for some \(g \in C^\omega(\mathbb{R}/\mathbb{Z})\). Then

\[
f \circ T^k - f = (\mathcal{A}_k f) \circ T - \mathcal{A}_k f = (g \circ T^k - g) \circ T - (g \circ T^k - g),
\]

which can be rewritten as follows:

\[(f + g - g \circ T) \circ T^k = f + g - g \circ T.\]

Since \(T^k : \mathbb{T} \to \mathbb{T}\) is topologically transitive and since \(f + g - g \circ T\) is continuous, the equality above implies that \(f + g - g \circ T = \text{const}\).

(2) If \(\nu\) is a maximizing measure for \((T^k, \mathcal{A}_k f)\), then \(\mu = \frac{1}{k} \sum_{j=0}^{k-1} T^j \nu\) is \(T^k\)-invariant, and
\[
\int f \, d\mu = \frac{1}{k} \sum_{j=0}^{k-1} \int f \circ T^j \, d\nu = \int \frac{1}{k} \mathcal{A}_k f \, d\nu.
\]

This shows \(\beta(T, f) \geq \beta(T^k, \frac{1}{k} \mathcal{A}_k f)\). On the other hand, if \(\mu\) is a maximizing measure of \((T, f)\), then \(\mu\) is \(T^k\)-invariant and
\[
\beta(T, f) = \int f \, d\mu = \int \frac{1}{k} \mathcal{A}_k f \, d\mu \leq \beta(T^k, \frac{1}{k} \mathcal{A}_k f).
\]
Hence \( \beta(T, f) = \beta(T^k, \frac{1}{k} \mathcal{J}_k f) \) and a maximizing measure for \((T, f)\) is also maximizing for \((T^k, \mathcal{J}_k f)\).

\[ \square \]

**Completion of the proof of the Main Theorem.** As \( f \) is not analytically cohomologous to a constant, for \( S_0 \) defined by (6), \( S_0 \subseteq \mathbb{R}/\mathbb{Z} \), so \( S = \text{supp}(\mu) \) is nowhere dense in \( \mathbb{R}/\mathbb{Z} \). In particular, there is a periodic point \( p \) of \( T \) such that \( p \notin S \). Let \( k \) be an even positive integer such that \( T^k(p) = p \). By Lemma 3.6, \( \mu \) is a maximizing measure for \((T^k, \mathcal{J}_k f)\) and \( \mathcal{J}_k f \) is not analytically cohomologous to a constant with respect to \( T^k \). Thus by what we have proved before, \( h_{T^k}(\mu) = 0 \) and hence \( h_T(\mu) = 0 \).

\[ \square \]

**4. Proof of Theorems 1.2, 1.3 and 1.4**

In this section, we prove Theorems 1.2, 1.3 and 1.4. The basic idea is to approximate \( C^r \) functions or maps with real analytic ones and then apply the Main Theorem.

We shall need upper semi-continuity of the following function:

\[ \mathcal{H}(T, f) := \sup_{\mu \in \mathcal{M}_{\text{max}}(T, f)} h_T(\mu), \quad (T, f) \in \mathcal{E}^1 \times C^0(\mathbb{R}/\mathbb{Z}). \]  

(7)

Here \( \mathcal{M}_{\text{max}}(T, f) \) denote the collection of \((T, f)\) maximizing measures. This result is essentially contained in [4, 17].

**Proposition 4.1.** The function \( \mathcal{H} \) is upper semi-continuous on \( \mathcal{E}^1 \times C^0(\mathbb{R}/\mathbb{Z}) \).

**Proof.** It was observed in [4] that for any fixed \( T \in \mathcal{E}^1 \), \( f \mapsto \mathcal{H}(T, f) \) is upper semi-continuous on \( C^0(\mathbb{R}/\mathbb{Z}) \). Indeed, if \( \mu_n \) is a maximizing measure for \((T, f_n)\) with \( f_n \to f \) in \( C^0(\mathbb{R}/\mathbb{Z}) \), then any accumulation point \( \mu \) of \( \mu_n \) in the weak-* topology is a maximizing measure for \((T, f)\). So the result is a consequence of upper semi-continuity of the entropy map \( \nu \mapsto h_T(\nu) \).

Now suppose that \((T_n, g_n) \to (T, f)\) in \( \mathcal{E}^1 \times C^0(\mathbb{R}/\mathbb{Z}) \). By [17, Lemma 2], for each \( n \) sufficiently large, there exists a homeomorphism \( h_n : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \) such that \( T = h_n^{-1} \circ T_n \circ h_n \), and moreover, \( \lim_{n \to \infty} \max_{x \in \mathbb{R}/\mathbb{Z}} d_{\mathbb{R}/\mathbb{Z}}(h_n(x), x) = 0 \), where \( d_{\mathbb{R}/\mathbb{Z}} \) is the standard metric on \( \mathbb{R}/\mathbb{Z} \). Put \( f_n := g_n \circ h_n \in C^0(\mathbb{R}/\mathbb{Z}) \). Then \( H(T, f_n) = H(T_n, g_n) \) holds and \( f_n \to f \) in \( C^0(\mathbb{R}/\mathbb{Z}) \). Thus

\[ \lim_{n \to \infty} H(T_n, g_n) = \limsup_{n \to \infty} H(T, f_n) \leq H(T, f). \]

\[ \square \]

**Proof of Theorem 1.2.** We only need to consider the maximizing case. Let \( T \in \mathcal{E}^\omega(\mathbb{R}/\mathbb{Z}) \) and let \( r \in \{0, 1, 2, \cdots, \infty\} \). Since the inclusion map from \( C^r(\mathbb{R}/\mathbb{Z}) \) to \( C^0(\mathbb{R}/\mathbb{Z}) \) is continuous, according to Proposition 4.1, the function \( f \mapsto \mathcal{H}(T, f) \) defined on \( C^r(\mathbb{R}/\mathbb{Z}) \) is upper semi-continuous. Therefore, \( C^r_{\text{exc}} := \{ f \in C^r(\mathbb{R}/\mathbb{Z}) : \mathcal{H}(T, f) = 0 \} \) is a \( G_\delta \) subset of \( C^r(\mathbb{R}/\mathbb{Z}) \). On the other hand, by our Main Theorem, \( C^r_{\text{exc}} \supseteq C^\omega_{\text{exc}} \) where \( C^\omega_{\text{exc}} \) denote the collection of functions in \( C^\omega(\mathbb{R}/\mathbb{Z}) \) that are not analytically cohomologous to constant. To complete the proof, it remains to show that \( C^\omega_{\text{exc}} \) is dense in \( C^r(\mathbb{R}/\mathbb{Z}) \). To this end, let \( C^r_{\text{web}} := \{ f \in C^r(\mathbb{R}/\mathbb{Z}) : \int f \, d\mu \text{ is independent of } \mu \in \mathcal{M}(T) \} \). Clearly, \( C^r_{\text{web}} \) is a closed subset of \( C^r(\mathbb{R}/\mathbb{Z}) \) with empty interior, and \( C^\omega(\mathbb{R}/\mathbb{Z}) \setminus C^r_{\text{web}} \subseteq C^\omega_{\text{exc}} \). Since \( C^\omega(\mathbb{R}/\mathbb{Z}) \) is dense in \( C^r(\mathbb{R}/\mathbb{Z}) \), it follows that \( C^\omega_{\text{exc}} \) dense in \( C^r(\mathbb{R}/\mathbb{Z}) \), which completes the proof. \[ \square \]
We shall need the following well-known result for the proof of Theorems 1.3 and 1.4. See, for example, [7, Proposition 28] for a more comprehensive version of this result under the $C^{1+\alpha}$ (and orientation-preserving) setting.

**Proposition 4.2.** For each $T \in \mathcal{E}^r$, $r \in \{1, 2, \ldots, \infty, \omega\}$, the following are equivalent:

(i) $\log |T'|$ is $C^0$ cohomologous to constant with respect to $T$;

(ii) $T$ is $C^r$ conjugate to the linear map $x \mapsto \deg(T) \cdot x$.

**Proof.** The implication (ii) $\Rightarrow$ (i) is trivial. Let us show (i) $\Rightarrow$ (ii). Since $\log |T'|$ is $C^0$ cohomologous to constant, there exist $\psi \in C^0(\mathbb{R}/\mathbb{Z})$ and $c \in \mathbb{R} \setminus \{0\}$ such that the following hold:

- $\psi > 0$ and $\int_{\mathbb{R}/\mathbb{Z}} \psi(x)dx = 1$;
- $\psi \circ T \cdot T' = c \cdot \psi$.

The first item above implies that there exists a $C^1$-diffeomorphism $\phi : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ with $\phi' = \psi$. Then the second item above can be rewritten as $(\phi \circ T)' = c \cdot \phi'$. Integrating both sides over $\mathbb{R}/\mathbb{Z}$ yields $\deg(T) = c$. Therefore, $T$ is $C^1$ conjugate to the linear map $x \mapsto \deg(T) \cdot x$ via $\phi$.

It remains to show that $\phi$ is $C^r$ when $T \in \mathcal{E}^r$ for $r \in \{2, 3, \ldots, \infty, \omega\}$. In this situation, $T$ admits a unique absolutely continuous invariant probability measure $\mu$, which has $C^{r-1}$ density. On the other hand, $\phi_* \mu$ is the unique absolutely continuous invariant probability measure of $x \mapsto \deg(T) \cdot x$, which is exactly the standard Lebesgue measure. Thus $\phi$ is also $C^r$.

**Proof of Theorem 1.3.** Let us only consider the maximizing case, i.e. $f = \log |T'|$, as the minimizing case is similar. By the Main Theorem, either $f$ is analytically cohomologous to constant, or any Lyapunov maximizing measure has zero entropy. If the first case happens, then by Proposition 4.2, $T$ is $C^\omega$ conjugate to $x \mapsto \deg(T) \cdot x$. \hfill \Box

**Proof of Theorem 1.4.** We are asked to consider the maximizing case. Let $r \in \{1, 2, \ldots, \infty\}$. Since the map $T \mapsto (T, \log |T'|)$ from $\mathcal{E}^r$ to $\mathcal{E}^1 \times C^0(\mathbb{R}/\mathbb{Z})$ is continuous, according to Proposition 4.1, the function $T \mapsto \mathcal{H}(T, \log |T'|)$ defined on $\mathcal{E}^r$ is upper semi-continuous. Therefore, $\mathcal{E}^r_{\text{web}} := \{f \in \mathcal{E}^r : \mathcal{H}(T, \log |T'|) = 0\}$ is a $G_\delta$ subset of $\mathcal{E}^r$. On the other hand, by Theorem 1.3, $\mathcal{E}^\omega_{\text{web}} \supset \mathcal{E}_{\text{nc}}$, where $\mathcal{E}_{\text{nc}}$ denote the collection of maps in $\mathcal{E}^\omega$ that are not analytically conjugate to linear map. To complete the proof, it remains to show that $\mathcal{E}^\omega_{\text{nc}}$ is dense $\mathcal{E}^r$.

To this end, let $\mathcal{E}^\omega_{\text{web}} = \{T \in \mathcal{E}^r : \int \log |T'| d\mu \text{ is independent of } \mu \in \mathcal{M}(T)\}$. Clearly, $\mathcal{E}^\omega_{\text{web}}$ is a closed subset of $\mathcal{E}^r$ with empty interior and $\mathcal{E}^\omega \setminus \mathcal{E}^\omega_{\text{web}} \subset \mathcal{E}^\omega_{\text{nc}}$. Since $\mathcal{E}^\omega$ is dense in $\mathcal{E}^r$, it follows that $\mathcal{E}^\omega_{\text{nc}}$ dense in $\mathcal{E}^r$, which completes the proof. \hfill \Box

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