Many-particle correlations in a non-equilibrium Luttinger liquid

I V Protopopov\textsuperscript{1,2}, D B Gutman\textsuperscript{3} and A D Mirlin\textsuperscript{1,4,5}

\textsuperscript{1} Institut für Nanotechnologie, Karlsruhe Institute of Technology, 76021 Karlsruhe, Germany
\textsuperscript{2} L D Landau Institute for Theoretical Physics, Kosygin Street 2, 119334 Moscow, Russia
\textsuperscript{3} Department of Physics, Bar Ilan University, Ramat Gan 52900, Israel
\textsuperscript{4} Institut für Theorie der kondensierten Materie and DFG Center for Functional Nanostructures, Karlsruhe Institute of Technology, 76128 Karlsruhe, Germany
\textsuperscript{5} Petersburg Nuclear Physics Institute, 188300 St Petersburg, Russia
E-mail: ipro@tkm.uni-karlsruhe.de, Dmitri.Gutman@biu.ac.il and mirlin@tkm.uni-karlsruhe.de

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Abstract. We develop an operator-based approach to the problem of a Luttinger liquid conductor in a non-equilibrium stationary state. We show that the coherent-state many-body fermionic density matrix and all fermionic correlation functions out of equilibrium are given by one-dimensional functional determinants of the Fredholm type. Thus, the model constitutes a remarkable example of a many-body problem where all the correlation functions can be evaluated exactly. On the basis of the general formalism, we investigate four-point correlation functions of the fermions coming out of the Luttinger liquid wire. The correlations obtained among the fermionic distribution functions represent the combined effect of interaction and non-equilibrium conditions.

Keywords: bosonization, Luttinger liquids (theory), quantum transport in one dimension, quantum transport
1. Introduction

It was understood long ago [1] that the conventional description of interacting fermionic systems in the framework of Landau Fermi-liquid theory is not applicable in one dimension (1D) because of infrared divergences. In particular, the second order of perturbation theory yields a singularity in the self-energy at the Fermi surface, $\Re \Sigma \sim \epsilon \ln \epsilon$, implying a vanishing quasi-particle weight. This violates the one-to-one correspondence (which plays the central role in the construction of the Landau theory) between unperturbed electronic states and elementary excitations of the interacting system. The lack of correspondence is a hallmark of the emerging strongly correlated electronic state—the Luttinger liquid (LL).

The physics of the LL is known to be relevant for many systems available in experiment. The applications of this concept include carbon nanotubes [2]–[4], semiconducting and metallic nanowires [5, 6], edge states of the samples in the quantum Hall regime [7]–[9] and spin ladders [10, 11].

In view of the singular infrared behavior, it is hard to access a LL by the conventional methods of many-body fermion perturbation theory. To overcome this difficulty, a powerful approach—bosonization—was developed. It is based on the fact that in 1D, fermionic creation and annihilation operators have simple representations in terms of
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bosonic fields describing such observables as charge and spin density. Thus, any 1D fermionic system is equivalent (as long as the energies considered are not too high) to some (generally interacting) bosonic system. This Fermi–Bose equivalence was first discovered at the level of correspondence between correlation functions in fermionic and bosonic theories [12]–[17]. Later on, underlying operator relations were derived by Haldane [18], providing a solid basis for bosonization (for a recent detailed exposition, see [19]).

Bosonization has proven to be a very efficient tool for tackling 1D interacting fermions. In some cases it allows one to obtain exact solutions of problems that are highly non-trivial in the fermionic language. The canonical example of this is the Tomonaga–Luttinger model equivalent to free bosons. Even in the case where bosonization does not produce free bosons, it often constitutes a convenient starting point for the development of the theory. Using this approach, the effects of backscattering [20, 21], impurities [22], and the underlying periodic potential [23] on the LL were explored. Many particular realizations of the LL states were addressed, both theoretically and experimentally [9, 24].

Nowadays, there is a growing interest in non-equilibrium phenomena in the LL phase. In particular, in a recent experiment [25], tunneling spectroscopy for a biased LL was carried out. A similar approach was implemented to study experimentally carbon nanotubes [26] and quantum Hall edges [27]–[29]. The experimental advances motivated theoretical interest in quantum wires out of equilibrium [30]–[39]. In a related line of research, non-equilibrium chiral 1D systems have been studied theoretically [40]–[43], mainly in application to experiments on quantum Hall edge state interferometry.

In a series of papers by two of the authors and Gefen [30, 31] the theory of the non-equilibrium LL was developed. The formalism developed in these works combined bosonization with the non-equilibrium Keldysh action approach. It was shown that the calculation of single-particle Green function reduces to the evaluation of certain Fredholm determinants, analogous to those encountered in the context of counting statistics [44] and non-equilibrium orthogonality catastrophe [45, 46]. This allowed us to obtain comprehensive results for observables related to the fermionic single-particle Green function, including the tunneling density of states, electron distribution function, and Aharonov–Bohm signal. More recently, the counting statistics of the charge transfer in a non-equilibrium LL was studied in [32].

However, a number of fundamental questions have remained open. In particular, it is important to understand what kinds of correlations (if any) are induced by the interaction between the electrons coming out of an interacting wire. In particular, are the left-movers and right-movers that have passed through the wire correlated? We will see below that at equilibrium no such correlations exist. On the contrary, in a non-equilibrium LL, electrons experience a specific type of relaxation. We will show that, in the course of this relaxation process, non-trivial correlations in the electronic distributions are produced. Characterizing these correlations is the main task of this work.

Our approach to the problem is complementary to that of [31]. Using the operator formalism, we determine explicitly the many-body density matrix $\hat{\rho}$ of the LL conductor in a stationary non-equilibrium state. When written in the bosonic coherent-state basis, this non-equilibrium density matrix $\hat{\rho}$ has the form of a Fredholm determinant. The density matrix carries full information about many-body correlations in the non-equilibrium LL state. We find that the calculation of fermionic correlation functions with density matrix $\hat{\rho}$ is greatly simplified by using a ‘refermionization’ procedure. In this way, we can
evaluate all fermionic correlation functions, with results expressed in terms of Fredholm determinants. Having developed our general formalism, we employ it to study four-point correlation functions of fermionic fields out of equilibrium and explore the correlations in the occupation numbers of the outgoing electrons induced by the interaction.

The structure of the paper is as follows. Section 2 contains a description of a model of a LL conductor out of equilibrium. In section 3 we give a short review of bosonization and refermionization and introduce concepts needed in the main part of the paper. In section 4 we obtain the many-body density matrix of a LL. Further, we show how the bosonization–refermionization allows one to obtain correlation functions of a non-equilibrium LL. We verify that the result for the single-particle Green function agrees with that obtained previously in [31]. In section 5 we explore the four-point correlation functions of fermions that emerged out of the LL wire. Our result reveals non-trivial correlations in fermionic occupation numbers. Section 6 contains a summary of our findings.

2. The model

Let us specify our model of the non-equilibrium LL conductor. We consider a 1D wire populated by spinless electrons of two chiralities (labeled by \( \eta = \text{R, L} \)) interacting via local density–density interaction. The wire is connected to non-interacting electrodes. To model the latter, we will assume that the electron–electron interaction is switched off outside the central part of the wire (figure 1) so that the Hamiltonian of the problem reads

\[
H_{ee} = H_0 + \frac{1}{2} \int dx \, g(x) \left( \rho_R(x) + \rho_L(x) \right)^2.
\]

Here \( H_0 \) represents the kinetic energy part of the Hamiltonian; \( \rho_{R,L}(x) \) is the density of right-moving (left-moving) electrons at point \( x \). The function \( g(x) \) approaches a constant value deep inside the interval \( |x| < l/2 \) (region II of figure 1) and is zero for \( |x| > l/2 \) (regions I and III). This way of representing the non-interacting leads was used extensively in the literature for the analysis of the transport properties of Luttinger liquids [47]–[49].

The non-equilibrium state of the wire is induced by the injection of the electrons from the leads. Following [31], we assume that the right-moving electrons coming into the wire from the left lead have a stationary non-equilibrium distribution function \( n_R(\epsilon) \), while the left-moving electrons coming from the right lead are characterized by the distribution function \( n_L(\epsilon) \). The simplest non-equilibrium state arises when the leads are held at different temperatures \( T_R, T_L \) and different chemical potentials \( \mu_R, \mu_L \) and so the distribution functions of the incoming electrons are of the Fermi–Dirac form

\[
n_\eta(\epsilon) = \frac{1}{1 + e^{(\epsilon - \mu_\eta)/T_\eta}}.
\]

Such a state was named the ‘partial equilibrium’ in [31]. More complicated distribution functions \( n_\eta \), e.g. double-step distributions, can be generated if one assumes that the leads are diffusive conductors. We refer the reader to [31] for a comprehensive discussion of the possible experimental realizations of the non-equilibrium Luttinger liquids.

Throughout the paper we assume the absence of electron backscattering in the wire which can be caused e.g. by impurities. In order to avoid the electronic backscattering also from the boundaries of the interaction region, we suppose that the function \( g(x) \) is
Figure 1. Schematic view of a LL conductor driven out of equilibrium by the injection of non-equilibrium electrons (with distribution functions $n^\text{in}_\eta(\epsilon)$) from non-interacting leads. The leads are modeled via the assumption that the LL interaction constant $g(x)$ is space dependent and vanishes outside the central part of the wire ($|x| < l/2$, region II).

$g(x < -l/2) = 0$ \hspace{2cm} $g(x) \hspace{2cm} g(x > l/2) = 0$

I \hspace{2cm} II \hspace{2cm} III

smooth on the length scale of the Fermi wavelength. Within this limitation, the formalism developed in this work can be applied to the case of arbitrary dependence of the interaction constant on $x$. However, the most interesting situation arises when boundaries of the interaction region are sharp on the scale of the typical wavelength $l_{T^*} = V_F/T^*$ of the bosonic excitations. Here $V_F$ is the Fermi velocity and $T^*$ is some characteristic width of the distribution functions $n_\eta$ (e.g. $T^* \sim \max(T_R, T_L)$ in the case of a partial equilibrium). Under such circumstances, there is significant scattering of the bosonic excitations on the boundaries of the interaction region which is the primary source of the effects that we are going to discuss in this paper. For this reason we concentrate below on this ‘sharp boundary’ limit and model the dependence on $x$ of the interaction constant by

$$g(x) = g \Theta (l/2 - |x|).$$

In [31] the single-particle correlation functions of the model described above were investigated in great detail. While within the interacting part of the wire those correlation functions are highly non-trivial due to the combined effect of interaction and non-equilibrium, outside the interaction region the single-particle Green functions are much simpler, reflecting the free dynamics of the electrons. Specifically, the single-particle Green functions of the right electrons, to the right from the interacting region (region III), has the form of the Green functions of free fermions with some energy distribution $n^\text{out}_R(\epsilon)$. In the same way, the left-movers in region I can be viewed as free particles with energy distribution $n^\text{out}_L(\epsilon)$. The correlation functions of incoming electrons, i.e. right-movers in region I and left-movers in region III, are not affected by the interaction. The distribution functions $n^\text{out}_\eta$ differ from $n_\eta$ (except for the case of complete thermal equilibrium), signaling the redistribution of the electrons over energies upon the passage through the interacting region II.

Now an important question arises. Is this redistribution (which, crudely speaking, can be referred to as relaxation, although the resulting electronic distribution functions
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are not of the Fermi–Dirac type) the only effect of interaction on the outgoing fermions? In particular, are the left-movers and right-movers coming out of the wire independent? Note that energy relaxation in our device is of a rather special character. Due to the strong constraints imposed by the integrability of the LL model, no energy relaxation can occur in a uniform LL. The relaxation actually takes place at the boundaries of the interacting wire and involves the scattering of the bosonic excitations from one chiral branch to another. Since bosons are crudely speaking electron–hole pairs, we may anticipate that the relaxation should be accompanied by the buildup of correlations in the distribution functions of left and right electrons. We show below that this is indeed the case and the density matrix of outgoing fermions shows strong correlations. In particular, the irreducible correlation functions of the form

\[ \langle \langle n_\eta(x, \epsilon)n_{\eta'}(x', \epsilon') \rangle \rangle \]

between the occupation numbers of the outgoing fermions are non-zero. These emerging correlations are a focus of the present work. We stress that the correlations in question are specifically non-equilibrium effects, absent under the equilibrium conditions.

We are now ready to present the formalism that we use in this work to explore correlations in a non-equilibrium LL wire. We start with a short review of the standard bosonization approach (in the form of ‘constructive bosonization’ [18,19]) in application to equilibrium interacting fermions. We place particular emphasis on constructions that will be subsequently employed in the analysis of the non-equilibrium state.

3. Bosonization and refermionization

Let us consider the 1D fermionic system with the Hamiltonian

\[ H = \int \! dx \, \psi^\dagger(x) \left( \frac{\hbar^2}{2m} \nabla^2 - \frac{k_F^2}{2m} \right) \psi(x) + V_{\text{int}}. \]  

Here \( k_F \) is the Fermi momentum and \( V_{\text{int}} \) represents the four-fermion interaction. Within the Luttinger liquid model one linearizes the spectrum of the fermions near the Fermi points and adds unphysical (but irrelevant at energies smaller than the Fermi energy) states below the bottom of the Fermi sea. This way of treating the interacting fermions is justified for the description of low energy properties of the system. At high energies, effects related to the curvature of the spectrum (not included in the LL model) may become sizable; see, in particular, [50].

With the bottom of the Fermi sea pushed down to minus infinity, the Hilbert space of the problem is spanned by the fermionic creation and annihilation operators \( a_{\eta,k}^+, a_{\eta,k} \) labeled by momentum \(-\infty < k < +\infty\) (counted from \( k_F \)) and the chirality \( \eta = R,L \) (in formulas, we will also use the notation \( \eta = 1 \) for right-movers and \( \eta = -1 \) for left-movers). It is convenient to make the fermionic momenta discrete, assuming that the system is placed on a ring of large circumference \( L \). The thermodynamic limit \( L \to \infty \) is taken at the end. The physical fermionic field is decomposed into the sum of left-moving and right-moving parts according to (\( \Lambda \to \infty \) stands for the high energy cutoff)

\[ \psi(x) \sim \psi_R(x)e^{ik_Fx} + \psi_L(x)e^{-ik_Fx}, \]

\[ \psi_\eta(x) = \frac{1}{\sqrt{L}} \sum_k a_{k,\eta}e^{ikx-|k|/\Lambda}. \]
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The Hamiltonian of the LL model acquires the form

$$H = V_F \sum_{k,\eta} \eta k (a_{\eta,k}^+ a_{\eta,k} - n_\eta^0(k)) + \frac{1}{2} \int dx g(x) (\rho_L(x) + \rho_R(x))^2.$$  \hspace{1cm} (7)

Here $n_\eta^0(k) = \Theta(-\eta k)$ are occupation numbers of the left-movers and right-movers in the free-fermion ground state; the interaction term was taken in the form of local density–density interaction with space-dependent interaction constant as discussed in section 2.

The central role in the Luttinger liquid model is played by the Fourier components of fermionic densities

$$\rho_{\eta,q} = \sum_k a_{\eta,k+q}^+ a_{\eta,k}.$$ \hspace{1cm} (8)

which can be used to construct the set of bosonic creation and annihilation operators according to ($\eta q > 0$)

$$b_{\eta,q}^+ = \sqrt{\frac{2\pi}{L|q|}} \rho_{\eta,q}, \quad b_{\eta,q} = \sqrt{\frac{2\pi}{L|q|}} \rho_{\eta,-q},$$ \hspace{1cm} (9)

$$[b_{\eta,q}, b_{\eta',q'}^+] = \delta_{\eta\eta'} \delta_{qq'}.$$ \hspace{1cm} (10)

Let us denote by $|N_R, N_L\rangle$ the state of the system which is a filled Fermi sea with $N_R$ ($N_L$) extra particles in the right (left) branch, i.e. the state characterized by the distribution functions $n_{\eta,k}^N = \Theta(-\eta k + 2\pi N_\eta/L)$. All of these states are annihilated by $b_{\eta,q}$ and are vacuum states from the point of view of the bosons. Any other state of the fermions can be generated by the action of the bosonic raising operators on $|N_R, N_L\rangle$. Thus, the operators $b_{\eta,q}, b_{\eta,q}^+$ together with the particle number operators $N_\eta$ and the Klein factors $F_\eta, F_\eta^+$ changing the total number of fermions of corresponding chirality form the complete operator set. In particular, the free-fermion Hamiltonian $H_0$ can be re-expressed in terms of bosons as

$$H_0 = V_F \sum_{\eta,q} \Theta(\eta q) |q| b_{\eta,q}^+ b_{\eta,q} + \frac{\pi V_F}{L} \sum_\eta N_\eta (N_\eta + 1),$$ \hspace{1cm} (11)

while the fermionic field operators are given by the famous bosonization identity

$$\psi_{\eta}^+(x) = \sqrt{\frac{\Lambda}{4\pi}} e^{-i\varphi_\eta(x)} F_{\eta}^+.$$ \hspace{1cm} (12)

The phase $\varphi_\eta(x)$ is related to the corresponding density $\rho_\eta(x)$ via

$$\rho_\eta(x) = \frac{\eta}{2\pi} \partial_x \varphi_\eta(x).$$ \hspace{1cm} (13)

Explicitly, in terms of bosonic creation and annihilation operators the density and phase fields read as follows:

$$\varphi_\eta(x) = i \sum_q \Theta(\eta q) \sqrt{\frac{2\pi}{L|q|}} (e^{-iqx} b_{q,\eta}^+ - e^{iqx} b_{q,\eta}) + \frac{2\pi \eta}{L} N_\eta x,$$ \hspace{1cm} (14)

$$\rho_\eta(x) = \sum_{\eta,q} \Theta(\eta q) \sqrt{\frac{|q|}{2\pi L}} (e^{-iqx} b_{q,\eta}^+ + e^{iqx} b_{q,\eta}) + \frac{1}{L} N_\eta.$$ \hspace{1cm} (15)

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We decompose the solution of the equation above into the normal modes according to functions. It is convenient to organize the latter into two-component columns \( u \). They contain the outgoing wave and the waves incident on the barrier. The asymptotic behavior of \( u \) in the interaction region the solution \( \rho \) of equation (18) with \( \eta, q \) assume that \( q_0 \). On the other hand, for \( q < 0 \) one can define \( u_{\eta, q}(x) \) via the complex conjugate

\[
\rho_{\eta}(x, t) = \sum_{q, q'} \Theta(\eta' q) \sqrt{\frac{|q|}{2\pi L}} (u_{\eta', |q|}(\eta, x) \tilde{b}_{q', q}^+ e^{i|q| V_F t} + u_{\eta', |q|}(\eta, x) \tilde{b}_{q', q} e^{-i|q| V_F t}).
\]

Here \( \tilde{b} \) and \( \tilde{b}^+ \) are the new bosonic operators, and \( u_{\eta', |q|}(\eta, q) \) are the coefficient functions. It is convenient to organize the latter into two-component columns \( u_{\eta, q}(x) = (u_{\eta, q}(x, R), u_{\eta, q}(x, L))^T \) satisfying the equation

\[
-i V_F q u_{\eta, q}(x) + \frac{\partial}{\partial x} [\tau_z h(x) u_{\eta, q}(x)] = 0,
\]

where \( \tau_z \) is the third Pauli matrix and

\[
h(x) = \begin{pmatrix}
V_F + \frac{g(x)}{2\pi} & \frac{g(x)}{2\pi} \\
\frac{g(x)}{2\pi} & V_F + \frac{g(x)}{2\pi}
\end{pmatrix}.
\]

Note that in order to construct the density fields we need only the solutions of equation (18) with \( q > 0 \). Thus, in the discussion of the properties of \( u_{\eta, q}(x) \) we will assume that \( q > 0 \). On the other hand, for \( q < 0 \) one can define \( u_{\eta, q}(x) \) via the complex conjugate

\[
u_{\eta, q<0}(x) \equiv u_{\eta, -q}^*(x).
\]

This choice is consistent with equation (18).

Imposing different boundary conditions, one can construct two sets of solutions of the equation (18). We denote them by \( q > 0 \): \( u_{in, q} \) and \( u_{out, q} \). Far from the interacting region the solution \( u_{in, q}(L) \) consists of a right (left) incoming wave and the waves scattered from the interacting region. These are the in-states in the standard terminology of the scattering theory. On the other hand \( u_{out, q}(R) \) and \( u_{out, q}(L) \) represent the out-scattering states. They contain the outgoing wave and the waves incident on the barrier. The asymptotic behavior of \( u_{in(out, q)} \) is summarized in figure 2. We have introduced the bosonic \( q \)-dependent scattering coefficients of the interaction region. Due to the time reversal symmetry, the
transmission amplitude $t_q$ for the right wave $u_{q,R}^{in}$ coincides with that of the left wave, whereas reflection amplitudes for the right and left waves are related via $r_{R,L}^* = -t_q/t$. If we assume the boundaries of the interaction region to be sharp on the scale of the relevant bosonic wavelength and adopt the model of the rectangularly shaped interaction constant (3) we find

$$t_q = \frac{T^2 e^{-i(1-K)ql}}{1 - R^2 e^{2iKql}}, \quad r_{R,L} = \frac{-2i R e^{-i(1-K)ql}}{1 - R^2 e^{2iKql}} \sin Kql. \quad (21)$$

Here $K$ is the LL parameter of the interaction region, while $R = (1 - K)/(1 + K)$ and $T = \sqrt{1 - R^2} = 2\sqrt{K/(K + 1)}$ are reflection and transmission amplitudes of a single interaction boundary.

Clearly, outgoing solutions $u_{q,q'}^{out}$ are not independent of $u_{q,q'}^{in}$. The two sets of solutions are related by the elements of the bosonic scattering matrix of the interaction region $r_q$ and $t_q$:

$$u_{R,q}^{out} = t_q^* u_{R,q}^{in} + r_{L,q}^* u_{L,q}^{in}, \quad (23)$$

$$u_{L,q}^{out} = r_{R,q}^* u_{R,q}^{in} + t_q^* u_{L,q}^{in}. \quad (24)$$

Choosing $u_{q,q'}^{in}(x)$ as the coefficient functions in equation (17), we obtain the decomposition of the density in terms of the in-bosons $b_{q}^{in}$ and $b_{q'}^{+in}$:

$$\rho_q(x) = \sum_{q',q''} \Theta(q'q'') \sqrt{\frac{|q'|}{2\pi L}} (u_{q',q''}^{in}(\eta, x) b_{q''}^{+in}(\eta, x) + u_{q',q''}^{in}(\eta, x) b_{q'}^{+in}(\eta, x)). \quad (25)$$

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Comparing (25) to (15), one can read off the relation of the initial and in-bosons:
\[ b_{\eta,q}^{+} = \sum_{\eta',q'} \Theta(\eta'q') \left[ u_{q|\eta,q}(q',\eta')b_{\eta',q'}^{+} - u_{q|\eta,-q'\eta}^{in}(-q',\eta')b_{\eta,q'} \right]. \]  

(26)

Operators \( b_{\eta,q}^{in} \) are defined via Hermitian conjugation and \( u_{q|\eta,q}(q') \) are the Fourier components of \( u_{q|\eta,q}^{in}(x) \):
\[ u_{q|\eta,q}(q',\eta) = \frac{1}{L} \sqrt{\frac{q}{|q|}} \int dx \ e^{-iq'x}u_{q|\eta,q}^{in}(x,\eta'). \]

(27)

By construction, operators \( b_{\eta,q}^{in} \) and \( b_{\eta,q}^{+in} \) diagonalize the Hamiltonian \( H \):
\[ H = V_{F} \sum_{q,\eta} \Theta(\eta q) q |b_{\eta,q}^{in}b_{\eta,q}^{+in}. \]

(28)

The connection (13) of the phase fields \( \varphi_{\eta} \) to the densities can now be used to derive the relation between \( \varphi_{\eta} \) and the incoming bosons:
\[ \varphi_{\eta}(x) = i \frac{1}{V_{F}} \sum_{\eta'q'} \Theta(\eta'q') \sqrt{\frac{2\pi}{L|q'}|} \left( v_{\eta q'q'}^{in}(\eta, x)b_{\eta q'}^{+in} - v_{\eta q'q'}^{in}(\eta, x)b_{\eta q'}^{+in} \right). \]

(29)

Here we have introduced the two-component columns \( v_{\eta q}(x) \) given by (the matrix \( h(x) \) was defined in (19))
\[ v_{\eta q}(x) = h(x)u_{\eta q}^{in}(x). \]

(30)

Since \( \psi_{\eta}^{+}(x) \sim \exp[-i\varphi_{\eta}(x)] \), equation (29) connects the fermionic field to the in-bosons. For the purposes of this work, it will be convenient to perform refermionization of the in-bosons \( b_{n,q}^{in} \) by introducing new fermionic fields \( \psi_{\eta}^{in}(x) \) and their Fourier components \( a_{n,q}^{in} \) according to
\[ \psi_{\eta}^{+in}(x) = \frac{1}{\sqrt{L}} \sum_{k} a_{n,k}^{+} e^{-ikx} = \frac{1}{\sqrt{L}} e^{-i\varphi_{n}(x)} F_{n}^{+}, \]

(31)
\[ \varphi_{\eta}(x) = i \sum_{q} \Theta(\eta q) \sqrt{\frac{2\pi}{L|q|} \left( e^{-iqx}b_{\eta,q}^{+in} - e^{iqx}b_{\eta,q}^{+in} \right). \]

(32)

The inverse relation reads
\[ b_{\eta,q}^{+} = \sqrt{\frac{2\pi}{L|q|}} \sum_{k} a_{n,k+q}^{+} a_{n,k}, \quad b_{\eta,q}^{in} = \sqrt{\frac{2\pi}{L|q|}} \sum_{k} a_{n,k-q}^{+} a_{n,k}. \]

(33)

Since the in-bosons solve the LL Hamiltonian, the in-fermions \( a_{n,k}^{in} \) are just free fermions with the time evolution
\[ a_{n,k}(t) = a_{n,k}^{in} e^{-ikv_{F}t}. \]

(34)
Comparing equations (29) and (32) and taking into account the boundary conditions imposed on \( v_{\eta q}^{in} \), we see that for \( \eta x \to -\infty \) (e.g., for right fermions at \( x \to -\infty \)) the physical fermionic field \( \psi_{\eta}(x) \) is identical to \( \psi_{\eta}^{+}(x) \),
\[ \psi_{\eta}(x) = \psi_{\eta}^{+}(x), \quad \eta x \to -\infty. \]

(35)
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So far, we have been focusing on the in-solutions $u_{\eta,q}^{\text{in}}$. The out-solutions $u_{\eta,q}^{\text{out}}$ allow us to define the out-bosons $b_{\eta,q}^{\text{out}}$ and fermions $a_{\eta,k}^{\text{out}}$ through equations analogous to (26), (31), and (32) up to a replacement in $\rightarrow$ out everywhere. These out-fermions and bosons also solve the LL Hamiltonian. Just like the solutions $u_{\eta,q}^{\text{in}}$ and $u_{\eta,q}^{\text{out}}$, the bosons $b_{\eta,q}^{\text{in}}$ and $b_{\eta,q}^{\text{out}}$ are related by the bosonic transmission and reflection coefficients:

$$b_{R,q}^{\text{out}} = t_{q}^{*}b_{R,q}^{\text{in}} + r_{L,q}^{*}b_{L,-q}^{\text{in}},$$

(36)

$$b_{L,q}^{\text{out}} = r_{R,q}^{*}b_{R,-q}^{\text{in}} + t_{q}^{*}b_{L,q}^{\text{in}}.$$  (37)

The out-fermions represent the physical fermions after the crossing of the interaction region in the sense that (cf equation (35))

$$\psi_{\eta}(x) = \psi_{\eta}^{\text{out}}(x), \quad \eta x \to +\infty.$$  (38)

The refermionization procedure described above turns out to be the crucial step in the solution of the non-equilibrium LL model. The in-fermions introduced in (31) will provide us with the starting point for the construction of the non-equilibrium density matrix of a LL wire. The out-fermions will be useful in our discussion of the correlations among the electrons leaving the wire.

4. Non-equilibrium Luttinger liquid

4.1. The density matrix

In section 3 we have summarized the bosonization representation of the LL Hamiltonian. Let us now turn to the analysis of the non-equilibrium LL in the setup shown in figure 1. First, we need to understand what the (many-body) density matrix $\hat{\rho}$ describing the situation of figure 1 is. This density matrix should satisfy two conditions:

(a) It should be stationary with respect to the LL Hamiltonian (7).

(b) Any correlation function for the fermionic fields $\langle \psi_{\eta_{1}}(x_{1}, t_{1}) \cdots \psi_{\eta_{i}}^{+}(x_{i}, t_{i}) \cdots \rangle$ evaluated with the density matrix $\hat{\rho}$ should be identical to the same correlation function for free fermions with the density matrix

$$\hat{\rho}_{0} = \frac{1}{Z} \exp \left[ - \sum_{\eta_{i},k} \epsilon_{\eta}(k) \left( a_{\eta_{i},k}^{+} a_{\eta_{i},k} - n_{\eta_{i}}^{0}(k) \right) \right].$$  (39)

as soon as all coordinates of the fermionic fields satisfy $\eta_{i}x_{i} < -l/2$. In equation (39) $Z$ stands for the normalization factor and the parameters $\epsilon_{\eta}(k)$ are determined by the distribution functions of the fermions coming into the wire via

$$n_{\eta}(k) = \frac{1}{1 + e^{\epsilon_{\eta}(k)}}.$$  (40)

Both conditions (a) and (b) are fulfilled by the density matrix

$$\hat{\rho} = \frac{1}{Z} \exp \left[ - \sum_{\eta_{i},k} \epsilon_{\eta}(k) \left( a_{\eta_{i},k}^{\text{in}} a_{\eta_{i},k}^{\text{in}} - n_{\eta_{i}}^{0}(k) \right) \right].$$  (41)

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The condition (a) is satisfied due to the equation of motion (34) for the incoming fermions. The condition (b) is satisfied due to the fact that the fermionic fields $\psi_{\eta_i}(x_i)$ are just identical to $\psi^\text{in}_{\eta_i}(x_i)$ as soon as $\eta_i x_i \to -\infty$.

In equilibrium, $\epsilon_{\eta}(k) = \eta k V_F / T$, the density matrix in terms of bosons is simply given by

$$\hat{\rho}_{\text{eq}} \sim \exp \left[ -\frac{V_F}{T} \sum_{q, \eta} \Theta(\eta q)|q|^2 \right].$$

(42)

Combined with equation (29), this implies that the evaluation of the fermionic correlation functions at equilibrium is reduced to the evaluation of averages of the type $(g_{\eta}(q)$ and $g^*_{\eta}(q)$ are $c$-valued functions)

$$Z[g^*_\eta, g_\eta] \equiv \left\langle \exp \left[ \sum_{\eta, q} \Theta(\eta q)g_{\eta}(q)b^+_{\eta q} \right] \exp \left[ -\sum_{\eta, q} \Theta(\eta q)g^*_\eta(q)b_{\eta q} \right] \right\rangle$$

(43)

with the Gaussian weight (42), which is a straightforward task.

Translating the general non-equilibrium density matrix (41) into the bosonic language is a much more non-trivial task. As shown in appendix A, it turns out to be possible to evaluate the matrix elements of $\hat{\rho}$ in the basis of bosonic coherent states $|N, \beta\rangle$ (eigenstates of the bosonic annihilation operators $b_{\eta q}^\text{in}$) in the form of one-dimensional Fredholm determinants. Here we state only the final result of this analysis, referring the reader to appendix A for the precise definitions and details of the calculation:

$$\langle \beta^*_L, \beta^*_R, N^R_L, N^L_L|\hat{\rho}|N^R_R, N^L_L, \beta_R, \beta_L \rangle = D_R(\beta^*_R, \beta_R)D_L(\beta^*_L, \beta_L),$$

(44)

$$D_{\eta} = \det \left[ 1 - e^{\Phi_{\eta}(x)}n^\eta_{\eta}(k)e^{-\Phi_{\eta}(x)}(1 - n^\eta_{\eta}(k)) - e^{-\Phi^*_\eta(x)}(1 - n^\eta_{\eta}(k))e^{\Phi^*_\eta(x)}n^\eta_{\eta}(k) \right].$$

(45)

In the determinants (45) the distribution functions $n^\eta_{\eta}(k)$ and $n^{N^\eta}_{\eta}(k)$ are considered as operators diagonal in the momentum space. In contrast, operators $\Phi_{\eta}(x)$ are diagonal with respect to the conjugate variable $x$ and given by

$$\Phi_{\eta}(k_1 - k_2) = \Theta(\eta(k_1 - k_2))\sqrt{\frac{2\pi}{L|k_1 - k_2|}} \beta_{\eta, k_1 - k_2}. $$

(46)

The density matrix (44), (45), along with the Hamiltonian (28) and the expression for the fermionic operators (29), contains the whole information about the problem in the language of non-interacting bosons. The natural next step is to calculate $n$-point fermionic correlation functions (that determine various physical observables). This will be done in section 4.2

4.2. The electronic correlation function

Our goal now is to evaluate many-point fermionic Green functions that have the form (43), with the bosonic density matrix given by equations (44) and (45). While this can be done by a direct calculation in the bosonic language, such an approach turns out to be quite tedious. A shorter approach is to perform a refermionization of the bosons $b^\text{in}$. Indeed, according to equation (33), $b^\text{in}$, $b^\text{in}^+$ are quadratic functions of the in-fermions $b^\text{in}$.
$a^{in+}$, $a^{in}$. Further, the density matrix is quadratic in terms of the in-fermions as well; see equation (41) Thus, the average (43) can be expressed as trace of a certain operator, which is an exponential of a quadratic form with respect to fermions $a^{in}$. The evaluation of the trace leads to (see appendix B for details)

$$Z[g^*, g] = \Delta_R \delta_R(x) \Delta_L \delta_L(x),$$

(47)

$$\Delta_\eta[\delta_\eta(x)] = \text{det}[(1 - n_\eta^0(k) + e^{-i\delta_\eta(x)}n_\eta^0(k))^{-1}(1 - n_\eta(k) + e^{-i\delta_\eta(x)}n_\eta(k))],$$

(48)

$$\delta_\eta(x) = \frac{1}{2} \sum_q \sqrt{\frac{2\pi}{L|q|}} \Theta(\eta q) \left(g_{\eta,q} e^{iqx} - g_{\eta,q}^* e^{-iqx}\right).$$

(49)

In equation (48) we can recognize a one-dimensional functional determinant of the type discovered in [31] in the context of a single-particle Green function. The first factor in square brackets of equation (48) involves the zero-temperature distribution function $n_\eta^0(k)$ and serves as a regularization. It ensures that at zero temperature, $Z[g^*, g]$, which is an average of a bosonic normal-ordered expression, is identically equal to unity.

Having derived the general result, we turn to evaluation of many-point fermionic correlation functions. Let us consider the average of a product of $n$ operators $\psi_\eta(x, t_i), i = 1 \cdots n$, and $n$ operators $\psi_\eta^+(x, t_i), i = n + 1 \cdots 2n$:

$$M_{\eta_1 \cdots \eta_{2n}}(x_1, t_1, \ldots, x_{2n}, t_{2n}) = \langle \psi_{\eta_1}(x_1, t_1) \cdots \psi_{\eta_{n}}(x_n, t_n) \psi_{\eta_{n+1}}^+(x_{n+1}, t_{n+1}) \cdots \psi_{\eta_{2n}}^+(x_{2n}, t_{2n}) \rangle.$$  

(50)

Representing the fermionic fields by bosonic exponents according to (12), (29) and bringing the product of the exponents into the normal-ordered form we get

$$M_{\eta_1 \cdots \eta_{2n}}(x_1, t_1, \ldots, x_{2n}, t_{2n}) = M^0_{\eta_1 \cdots \eta_{2n}}(x_1, t_1, \ldots, x_{2n}, t_{2n}) Z[g^*, g],$$

(51)

$$g_\eta(q) = \frac{1}{V_F} \sqrt{\frac{2\pi}{L|q|}} \sum_{i=1}^n \zeta_i v_{\eta,q}^{in}(\eta_i, x_i) e^{iqx_F t_i},$$

(52)

$$g_\eta^*(q) = \frac{1}{V_F} \sqrt{\frac{2\pi}{L|q|}} \sum_{i=1}^n \zeta_i v_{\eta,q}^{in}(\eta_i, x_i) e^{-iqx_F t_i}.$$  

Here $\zeta_i = 1$ for $i = 1, \ldots, n$, while $\zeta_i = -1$ for $i = n + 1, \ldots, 2n$. In the expression (51) the first factor $M^0$ arises due to normal ordering of the bosons and is nothing but the corresponding correlation function of $\psi$-operators in the zero-temperature LL. This factor depends in particular on the ordering of $\psi$-operators in (50). On the other hand, the second factor in (51) accumulates the effect of the distribution functions of the incoming electrons $n_\eta(x)$. This factor equals unity at zero temperature and is the same for all the fermionic correlators which differ only by the ordering of the Fermi fields. An explicit expression for the factor $Z[g^*, g]$ is given by the Fredholm determinants (48) with the phases $\delta_\eta(x)$ expressed in terms of the functions $v_{\eta,q}(x, \eta')$ via (cf equations (49) and (52))

$$\nabla_x \delta_\eta(x) = \frac{2\pi \eta}{V_F} \int \frac{dq}{2\pi} \sum_{i=1}^n \zeta_i v_{\eta,q}^{in}(\eta_i, x_i) e^{-iqx_F t_i - iqx_F}.$$  

(53)
Equations (51), (47) and (53) together with (18), (30) provide the full solution of the non-equilibrium LL problem. It is straightforward to check that the two-point (single-particle) correlation functions found previously in [31] are correctly reproduced.

The expression for the phases $\delta_\eta(x)$ acquires a particularly simple form in the case where all the fermionic fields in (50) are taken outside the interacting part of the wire, i.e. $|x_i| > l/2$ for all $i = 1, \ldots, 2n$. Taking into account the boundary conditions imposed on the functions $u_{\eta,q}$ we get

$$\nabla_x \delta_\eta(x) = 2\pi \eta \int \frac{dq}{2\pi} \sum_i \zeta e^{i(q_\eta u_i - \eta x)} [\Theta(\eta x_i) (t_q \delta_{\eta,\eta} + r_{\eta,q} \delta_{\eta,-\eta}) + \Theta(-\eta x_i) \delta_{\eta,\eta}].$$

(54)

Here we have introduced the light-cone coordinates $u_i = x_i - \eta V_F t_i$. According to (54), in the case of the electronic correlations at the input of the central part of the wire (i.e. for all $\eta x_i < -l/2$) the phases $\delta_\eta(x)$ are completely independent from the properties of the interaction region. This ensures the coincidence of the correlation functions with that of the free fermions with the density matrix (39). On the other hand, for the correlations of the fermions at the output of the interacting region (i.e. $\eta x_i > l/2$) the phases $\delta_\eta(x)$ are determined by the bosonic transmission and reflection amplitudes.

5. Correlations in the outgoing fermions

5.1. Single-particle Green functions

We are now in a position to turn to the correlations between the electrons going out of the interacting part of wire. The simplest quantities characterizing this correlations are the four-point correlation functions of the type (50) where the right fermions are ‘measured’ to the right of the interaction region, while the left electrons are ‘measured’ to the left, i.e. we consider the behavior of equation (50) for $\eta x_i > l/2$. Intuitively, the electrons which have left the interaction region are just free electrons. We can make this statement mathematically precise by recalling equation (38), showing that for $\eta x_i > l/2$ physical electronic fields coincide with the fields of out-fermions, having just the free dynamics

$$\psi_\eta(x, t) = \psi_\eta^\text{out} (x, t) = \psi_\eta^\text{out} (x - \eta V_F t), \quad \eta x > l/2.$$  

(55)

Thus, the question arises of whether any of the correlation functions (50) with $\eta x_i > l/2$ bear a trace of the interaction in the central part of the wire. In particular, the prefactor $M^0$ entering equation (51) for the correlation functions at the output of the wire is just the free-fermion zero-temperature correlation function.

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The situation changes drastically in a non-equilibrium system. The density matrix $\hat{\rho}$ becomes now a complicated function of the out-fermions, incorporating non-trivial many-particle correlations. It is important to distinguish between the correlation functions which are determined solely by the dynamics and the correlations also governed by the electronic distribution. The retarded and the Keldysh single-particle Green functions are the simplest examples of the former and the latter. For the non-interacting electrons, $G_\eta^R(x, \tau) \sim \delta(x - \eta V_F \tau)$. Thus, for the retarded function we immediately get

$$G_\eta^R(x, \tau) = G_\eta^{R0}(x, \tau) \Delta_R[\delta_R \equiv 0] \Delta_L[\delta_L \equiv 0] = G_\eta^{R0}(x, \tau).$$

On the other hand, the Keldysh component of the Green function (say for right electrons) $G_\eta^K(x_R, \tau_R/2; x_R, -\tau_R/2) = -i[\psi_R(x_R, \tau_R/2), \psi_R^+(x_R, -\tau_R/2)] \equiv G^K(\tau_R)$ receives a non-trivial contribution from the combined effect of the interaction and non-equilibrium:

$$G_\eta^K(x_R, \tau_R/2, x_R, -\tau_R/2) = G_\eta^{K0}(0, \tau_R) \Delta_R[\delta_R] \Delta_L[\delta_L],$$

where the phases $\delta_\eta$ are given by

$$\nabla_x \delta_\eta(x) = -4\pi i \int \frac{dq}{2\pi} e^{iq(x_R - \eta x)} \sin \frac{qV_F \tau_R}{2} (t_q \delta_{\eta,R} + r_{q,L} \delta_{\eta,L}).$$

Our choice for the arguments of the fermionic fields in the definition of the Keldysh Green function is somewhat overcomplicated, since $G^K$ is actually independent of $x_R$ (as long as $x_R > l/2$). However we keep $x_R$ explicit for the convenience of the forthcoming discussion of the four-point correlations.

Exploiting the explicit expression (22) for the transmission and reflection amplitudes in the ‘sharp boundary’ model, we get the phases $\delta_\eta$ in the characteristic form of a sequence of rectangular pulses of length $V_F \tau_R$ [31]. The amplitudes of the pulses are shown in figure 3. Their positions are given by

$$x_{Rm} = x_R - l + (2m + 1) K l, \quad m = 0, 1, \ldots$$

for $\delta_\eta(x)$ and

$$x_{Lm} = -x_R + l - 2m K l, \quad m = 0, 1, \ldots$$

for the left phase. Analytically,

$$\delta_\eta(x) = 2\pi \sum_{m=0}^{\infty} \alpha_{\eta,m} w_{\tau_R}(x - x_{\eta,m}).$$

Here the coefficients are

$$\alpha_{R,m} = T^2 R^{2m}, \quad \alpha_{L,m} = \begin{cases} R, & m = 0 \\ -T^2 R^{2m-1}, & m = 1, 2, \ldots, \end{cases}$$

and we have introduced

$$w_{\tau}(x) = \Theta \left( \frac{|\tau|}{2} - |x| \right) \text{sgn} \tau.$$ 

Equation (58) can be interpreted in terms of the fermionic occupation numbers at the output of the wire:

$$1 - 2n_\eta^{\text{out}}(\epsilon) = iV_F \int d\tau_R e^{i\epsilon \tau_R} G^K_\eta(\tau_R).$$

Since the $n_\eta^{\text{out}}(\epsilon)$ are different from the distribution functions of the incoming electrons $n_\eta(\epsilon)$ (except for the equilibrium), electrons experience relaxation upon crossing the interaction region [31].

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Figure 3. The phases $\delta_R(x)$ and $\delta_L(x)$ governing the Keldysh Green function of the right-movers. The coordinates $x_{\eta,0}, x_{\eta,1}, \ldots$ are given by equations (60) and (61).

5.2. Two-particle Green functions

Let us now turn to a deeper characterization of the density matrix for the outgoing electrons that is provided by the four-point correlation functions. Particularly intriguing are interaction-induced correlations between the left-movers and right-movers. To reveal them we consider

$$M_{RL}(x_R, \tau_R; x_L, \tau_L) = \left\langle \left[ \psi_R^+ \left( x_R, \frac{\tau_R}{2} \right), \psi_R \left( x_R, -\frac{\tau_R}{2} \right) \right] \left[ \psi_L^+ \left( x_L, \frac{\tau_L}{2} \right), \psi_L \left( x_L, -\frac{\tau_L}{2} \right) \right] \right\rangle + G^K_R(\tau_R)G^K_L(\tau_L)$$

where $x_R > l/2$ and $x_L < -l/2$; the square brackets $[\cdot, \cdot]$ stand for the commutator. Note that (66) is the only non-trivial irreducible correlation function of two right and two left fields. Replacement of one of the commutators in (66) by an anticommutator would immediately lead to the decoupling of left and right fermions under the averaging. It is easy to see that the Fourier transform of (66) has the meaning of the irreducible correlator of two fermionic distribution functions:

$$\left\langle \left\langle n_R(x_R, \epsilon_R)n_L(x_L, \epsilon_L) \right\rangle \right\rangle = \frac{V_f^2}{4} \int d\tau_R d\tau_L M(x_R, \tau_R; x_L, \tau_L)e^{ik_R\gamma_R + ik_L\gamma_L}.$$  

Each of the phases $\delta_\eta(x)$ corresponding to (66) is a sum as of a pulse sequence encountered previously in the discussion of the Keldysh function $G^K_R$ and an analogous

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6 There is no need to consider more general arguments of the $\psi$-operators because in the non-interacting leads each $\psi_\eta(x, t)$ is a function of $x - \eta V_f t$ only.
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correlation originating from the pair of left operators in (66). This last contribution consists of pulses of the length $V_F \tau_L$ located at

$$x'_{Rm} = -x_L - l + 2mKl, \quad m = 0, 1, \ldots$$

(68)

for $\delta_R(x)$ and

$$x'_{Lm} = x_L + l - (2m + 1)Kl, \quad m = 0, 1, \ldots$$

(69)

for the left phase.

The dependence of (67) on the coordinates $x_\eta$ and the ‘optical length’ of the wire $Kl$ can be easily understood. We note that the scale $l_T/V_F$ for the times $\tau_\eta$ supporting non-zero correlation functions is set by the inverse characteristic width of the energy distributions of the incoming electrons. Under the assumption that the length of the interacting wire is large and $Kl$ exceeds $l_T$, we can neglect the mutual effect of the non-overlapping pulses in $\delta_R(L)$ and represent the corresponding determinants $\Delta_\eta[\delta_\eta(x)]$ by a product of determinants for individual pulses. One immediately concludes that occupation numbers of left-movers and right-movers are uncorrelated unless the pulses coming from right and left operators overlap. The overlap happens if $X_{RL} = x_R + x_L$ is close to an odd multiple of $Kl$. We can now recast (67) into the form

$$\langle \langle n_R(x_R, \epsilon_R)n_L(x_L, \epsilon_L) \rangle \rangle = \sum_{m \in \text{odd}} f_{m_\lambda}^R(\epsilon_R, \epsilon_L, X_{RL} - mKl).$$

(70)

The functions $f_{m_\lambda}^R(\epsilon_R, \epsilon_L, y)$ are independent of $l$ and decay in the distance $l_T$. They are given by

$$f_{m_\lambda}^R(\epsilon_R, \epsilon_L, y) = -\frac{V_F^2}{4} \int d\tau_L d\tau_R G_R^K(\tau_R)G_L^K(\tau_L)(A_R A_L - 1) e^{i(\epsilon_R \tau_R + i\lambda \tau_L)},$$

(71)

$$A_R = \prod_{n = \max(0, \bar{m})}^{\infty} \frac{\Delta_R[\alpha_R,n-\bar{m}w_{\tau_R}(x) + \alpha_L,nw_{\tau_R}(x + y)]}{\Delta_R[\alpha_R,n-\bar{m}w_{\tau_R}(x)] \Delta_R[\alpha_L,nw_{\tau_R}(x)]},$$

(72)

$$A_L = \prod_{n = \max(0, \bar{m})}^{\infty} \frac{\Delta_L[\alpha_R,n-\bar{m}w_{\tau_L}(x) - y + \alpha_L,nw_{\tau_L}(x)]}{\Delta_L[\alpha_R,n-\bar{m}w_{\tau_L}(x)] \Delta_L[\alpha_L,nw_{\tau_L}(x)]}.$$ (73)

Here $\bar{m} = (m + 1)/2$, and we use the notation introduced in equations (63), (64).

Using forms similar to (66), (67), one can define the irreducible correlation function of the occupation numbers of the right-movers:

$$\langle \langle n_R(x_R, \epsilon_R)n_R(x'_R, \epsilon'_R) \rangle \rangle = \frac{V_F^2}{4} \int d\tau_R d\tau' R M_{RR}(x_R, \tau_R; x'_R, \tau'_R) e^{i\epsilon_R \tau_R + i\epsilon'_R \tau'_R},$$

(74)

$$M_{RR}(x_R, \tau_R; x'_R, \tau'_R) = \left\langle \left[ \psi_R \left( x_R, \frac{\tau_R}{2} \right), \psi_R^+ \left( x_R, -\frac{\tau_R}{2} \right) \right] \left[ \psi_R \left( x'_R, \frac{\tau'_R}{2} \right), \psi_R^+ \left( x'_R, -\frac{\tau'_R}{2} \right) \right] \right\rangle + G_R^K(\tau_R)G_R^K(\tau'_R).$$

(75)

In order to be able to interpret (74) as the correlator of the distribution functions, we assume below that the two pairs of $\psi$-operators in (75) are inserted far from each other, and so $X_{RR} \equiv x_R - x'_R \gg l_T$. Under this assumption, just as in the case of $R-L$ correlations,
(75) is the only non-trivial correlation function among the four right electronic fields. Consideration of phases $\delta_n(x)$ corresponding to (75) shows that (74) is non-zero provided that $X_{RR} \approx 2Klm$ and can be decomposed as (we set $\tilde{m} = m/2$)

$$\langle \langle n_R(x_R, \epsilon_R)n_R(x'_R, \epsilon'_R) \rangle \rangle = \sum_{m \in \text{even}, m \neq 0} f_{RR}^{m}(\epsilon_R, \epsilon_L, X_{RR} - m Kl),$$

(76)

$$f_{RR}^{m}(\epsilon_R, \epsilon_L, y) = -\frac{V_F^2}{4} \int d\tau_R d\tau_R' G_R^K(\tau_R) G_L^K(\tau_L) (A_R A_L - 1) e^{i\epsilon_R \tau_R + i\epsilon_L \tau_R'},$$

(77)

$$A_R = \prod_{n = \max(0, \tilde{m})}^\infty \frac{\Delta_R[\alpha_R,n-\tilde{m}w_{\tau_R}(x) + \alpha_R,nw_{\tau_R}(x + y)]}{\Delta_R[\alpha_R,n-\tilde{m}w_{\tau_R}(x)]},$$

(78)

$$A_L = \prod_{n = \max(0, \tilde{m})}^\infty \frac{\Delta_L[\alpha_L,n-\tilde{m}w_{\tau_R}(x) + \alpha_L,nw_{\tau_R}(x - y)]}{\Delta_L[\alpha_L,n-\tilde{m}w_{\tau_R}(x)]}.$$}

(79)

The term with $m = 0$ is omitted from the summation due to our assumption that $X_{RR} > t_F$. Again, investigation of the functions $f_{RR}^{m}(\epsilon_R, \epsilon_L, x)$ requires evaluation of the functional determinants $\Delta_p$.

The physics of the correlations obtained is discussed in section 5.3.

5.3. Discussion

We assert that the correlations discovered in section 5.2 represent a quantum interference effect. To clarify this point, we make a digression and consider a very simple case of one right boson with momentum $q$ populating the wire. This situation is described by the density matrix

$$\hat{\rho}_{1b} = b_{R,q}^{\dagger 0} \langle 0 | b_{R,q}^{0 0},$$

(80)

where $|0\rangle$ is the ground state. Using the connection of the in-bosons and out-bosons and then performing the refermionization we translate $\rho_{1b}$ into the out-fermions

$$\hat{\rho}_{1b} = \frac{2\pi}{Lq} \sum_{k,k'} (t_q a_{R,k-k'}^{\dagger \text{out}} a_{R,k}^{\text{out}} + r_{R,q} a_{L,k-k'}^{\dagger \text{out}} a_{L,k}^{\text{out}}) |0\rangle \langle 0 | (t_q a_{R,k-k'}^{\dagger \text{out}} a_{R,k}^{\text{out}} + r_{R,q} a_{L,k-k'}^{\dagger \text{out}} a_{L,k}^{\text{out}}).$$

(81)

We can now evaluate the correlation function of the left and right distributions directly in the fermionic language and get

$$\langle \langle n_R(x_R, \epsilon_R)n_L(x_L, \epsilon_L) \rangle \rangle = -\delta n_R(\epsilon_R) \delta n_L(\epsilon_L)$$

$$+ \frac{2\pi}{Lq} \left( n_{R+}^0 - n_{R-}^0 \right) \left( n_{L+}^0 - n_{L-}^0 \right) \left( r_{R,q} t_q e^{i\omega x_{R+L}} + r_{R,q}^* e^{-i\omega x_{R+L}} \right).$$

(82)

Here $\omega = qV_F$ is the frequency of the boson in the system; $n_{q\pm}^0 = \Theta(-(\epsilon_q \pm \omega/2))$ and

$$\delta n_R(\epsilon_R) = n_R^{\text{out}}(\epsilon_R) - n_R^{\text{in}}(\epsilon_R) = \frac{2\pi}{Lq} |t_q|^2 \left( 2n_0(\epsilon_R) - n_0(\epsilon_R + \omega) - n_0(\epsilon_R - \omega) \right),$$

(83)

$$\delta n_L(\epsilon_L) = n_L^{\text{out}}(\epsilon_L) - n_L^{\text{in}}(\epsilon_L) = \frac{2\pi}{Lq} |r_{R,q}|^2 \left( 2n_0(\epsilon_L) - n_0(\epsilon_L + \omega) - n_0(\epsilon_L - \omega) \right).$$

(84)
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The two terms in (82) have distinct physical origin. The first one originates from the probabilistic nature of the boson transmission–reflection process. It would remain unchanged upon the replacement of the density matrix by the statistical mixture

\[ \hat{\rho}_{1b} = |t_q|^2 \hat{\rho}_{R,q}^{\text{out}} |0\rangle \langle 0| \hat{\rho}_{L,-q}^{\text{out}} + |r_{R,q}|^2 \hat{\rho}_{R,L}^{\text{out}} |0\rangle \langle 0| \hat{\rho}_{R,-q}^{\text{out}}. \]  

(85)

This term favors anti-correlations of electrons at energies of equal signs. This is a typical anti-correlation between the mutually exclusive events. Indeed, suppose that the boson that we have injected into the system was transmitted through the interacting part of the wire. In this case at the output of our device there is a particle–hole excitation in the right branch, i.e. a right electron at positive energy and a right hole at negative energy. At the same time the left branch has no excitation. On the contrary, if the boson was reflected, there is a left electron at positive energy and a left hole. The right branch is empty. Thus, in such a probabilistic description there is no way to have an excited electron both in the left and right branches, which results in anti-correlations between the fermionic occupation numbers at energies of the same sign.

Note that the probabilistic contribution to the correlator does not depend on the coordinates where the occupation numbers are measured. This is quite general. Suppose we have some complicated density matrix \( \hat{\rho} \) in terms of in-bosons. We translate it into the out-bosons. It is easy to see that if we now switch to the classical description of the system by dropping all the non-diagonal elements of the density matrix (cf transition from (81) to (85)) we will end up with the correlator of the occupation numbers independent of \( X_{RL} \). Only the matrix elements of \( \hat{\rho} \) involving momentum transfer from the left to the right branch can provide such a dependence.

The second contribution to the correlator (82) is due to the fact that on the quantum level the scattering of a boson creates a coherent superposition of the state with a boson in the right branch and a state with a boson in the left branch. Thus, the excited particle–hole pair is (virtually) simultaneously present in both branches. To elucidate the effect of this quantum term, let us consider a slightly more general density matrix which is a statistical mixture of (80):

\[ \hat{\rho}_{\text{mix}} = \sum_q c_q \hat{\rho}_{R,q}^{\text{in}} |0\rangle \langle 0| \hat{\rho}_{L,-q}^{\text{in}}, \quad \sum_q c_q = 1. \]  

(86)

Now the quantum contribution is modified accordingly:

\[ \langle \langle n_R(x_R, \epsilon_R) n_L(x_L, \epsilon_L) \rangle \rangle_{\text{quantum}} = -4 T^2 R \sum_q c_q \frac{2\pi}{L q} (n^0_{R+} - n^0_{R-}) (n^0_{L+} - n^0_{L-}) \times \frac{\sin K q l \sin q X_{RL}}{|1 - R^2 e^{2i K q l}|^2}. \]  

(87)

We have used here the explicit expressions (22) for the transmission and reflection amplitudes in the sharp boundary model. Let the coefficients \( c_q \) be peaked at some \( q = q_0 \). Under the assumption that the peak width is much larger than \( 1/K l \) and the energies \( \epsilon_{R(L)} \) are not too close to \( q_0 V_F / 2 \), we can average the expression under the sum over fast oscillations on the scale \( 1/K l \). The result is non-vanishing only if \( X_{RL} \) is close to an odd multiple of \( K l \), \( X_{RL} = m K l + x, m \in \text{odd} \), in which case

\[ \text{doi:10.1088/1742-5468/2011/11/P11001} \]
\[ \langle n_R(x_R, \epsilon_R) n_L(x_L, \epsilon_L) \rangle_{\text{quantum}} = -\frac{4\pi T^2 R^m \, \text{sgn} \, m}{1 + R^2} \times \sum_q \frac{2\pi}{Lq} (n^0_{R+} - n^0_{R-}) (n^0_{L+} - n^0_{L-}) \cos qx. \] (88)

This result is in agreement with the general coordinate dependence (70) of the correlator of fermionic distributions.

The density matrices \( \hat{\rho}_{tb} \) and \( \hat{\rho}_{\text{mix}} \) are very simple as they contain just one fermionic excitation. The truly non-equilibrium density matrix (41), when written in terms of bosons, is much more complicated. It contains infinitely many terms representing multiple-boson processes. Collecting their contributions to the correlation function of left and right occupation numbers properly is the task accomplished by the functional determinants \( \Delta_{\eta}[- \delta_{\eta}(x)] \). In general, both classical noise and quantum interference contribute to the result. The lesson that we learnt from the analysis of simple density matrices allows us to identify clear manifestations of quantum effects. First, there is the dependence of the correlation functions on the coordinates; see equation (70). Second, there are positive correlations between distribution functions of left-movers and right-movers at energies of the same sign. We will see in section 5.4 below that the correlation functions of a non-equilibrium LL do show both of these features.

### 5.4. Partial equilibrium

A detailed analysis of the functions \( f_{m}^{RL} \) describing the correlations in electronic distributions requires a careful investigation of the determinants \( \Delta_{\eta}[- \delta_{\eta}(x)] \) with given distributions of the incoming electrons. In general, these determinants cannot be evaluated analytically, except for asymptotic long-time behavior [33] (which is not sufficient for our purposes here). Thus, one has to resort to numerics in order to achieve a comprehensive understanding of the correlations in the fermionic occupation numbers.

Relegating such a numerical analysis to a future work, we focus below on the case of partial equilibrium that can be treated analytically. This is the situation when distribution functions of the incoming fermions are of the Fermi–Dirac form but with different temperatures \( T_R \) and \( T_L \) of the left-movers and right-movers.\(^7\)

The functional determinants \( \Delta_{\eta}[- \delta_{\eta}(x)] \) become under such circumstance exponentially quadratic functionals of phases [31],

\[ \ln \Delta_{\eta} = -\frac{1}{4\pi} \int_{0}^{\infty} dq \frac{dq}{2\pi} q (B_{\eta}(qV_F) - 1) |\delta_{\eta}(q)|^2, \] (89)

where \( B_{\eta}(\omega) = \coth \omega / 2T_{\eta} \) are the bosonic distribution functions.

One can now use (89) to evaluate the products of the determinants (72), (73), (78), (79) analytically. Alternatively, we can apply equation (89) to the full functional determinants governing the correlations in the electronic distributions. The Fourier components of the phases \( \delta_{\eta} \) are easily read off from (59). For example, in the case

---

\(^7\) In our consideration of partial equilibrium we set the chemical potentials \( \mu_{R(L)} \) of the left-movers and right-movers equal to zero. Non-zero \( \mu_{\eta} \) would lead to trivial energy shifts in our formulas.

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of the correlator of left and right distribution functions we have

\[ |\delta_R(q)|^2 = \frac{(4\pi)^2}{q^2} \left[ |t_q|^2 \sin^2 \frac{q V_F \tau_R}{2} + |r_q|^2 \sin^2 \frac{q V_F \tau_L}{2} \right. \]
\[ \left. + 2it_q r_q^* \sin \frac{q V_F \tau_R}{2} \sin \frac{q V_F \tau_L}{2} \sin(q(x_R + x_L)) \right], \tag{90} \]
\[ |\delta_L(q)|^2 = \frac{(4\pi)^2}{q^2} \left[ |t_q|^2 \sin^2 \frac{q V_F \tau_L}{2} + |r_q|^2 \sin^2 \frac{q V_F \tau_R}{2} \right. \]
\[ \left. - 2it_q r_q^* \sin \frac{q V_F \tau_L}{2} \sin \frac{q V_F \tau_R}{2} \sin(q(x_R + x_L)) \right]. \tag{91} \]

Plugging this into (89), averaging the expression under the integral over fast oscillations of the bosonic transmission and reflection amplitudes (which is equivalent to the neglect of the interference of non-overlapping pulses in \(\delta_\eta(x)\) discussed in section 5.3) and using the standard equality

\[ - \int_0^{+\infty} \frac{dx}{x} (1 - \cos \alpha x) \left( \coth \frac{x}{2} - 1 \right) = \ln \frac{\pi \alpha}{\sinh \pi \alpha}, \tag{92} \]

one finds that the correlation function of left and right occupation numbers does indeed have the form (70) with

\[ f^{RL}(\epsilon_R, \epsilon_L, x) = -\frac{V_F^2}{4} \int d\tau_R d\tau_L e^{i\epsilon_R \tau_R + i\epsilon_L \tau_L} \mathcal{G}^K_R(\tau_R) \mathcal{G}^K_L(\tau_L) \]
\[ \times \left[ \left( \frac{1}{1 - \cosh \pi T_R (\tau_R + \tau_L)} \right) \left\{ 1 - \frac{\cosh \pi T_L (\tau_R + \tau_L)}{\cosh 2\pi T_L x} \right\}^{-1} \right. \]
\[ \times \left. \left\{ 1 - \frac{\cosh \pi T_L (\tau_R - \tau_L)}{\cosh 2\pi T_L x} \right\} \left\{ 1 - \frac{\cosh \pi T_R (\tau_R - \tau_L)}{\cosh 2\pi T_R x} \right\}^{-1} \right] \gamma_m - 1. \tag{93} \]

Here the exponents \(\gamma_m\) are given by \(\gamma_m = \text{sgn} m T^2 R^{[m]/(1 + R^2)}\), and the Keldysh Green functions of the outgoing fermions are [31]

\[ \mathcal{G}^K_R(\tau_R) = \frac{1}{\pi V_F T_R} \left[ \frac{\pi T_R \tau_R}{\sinh \pi T_R \tau_R} \right]^{T^2/(1 + R^2)} \left[ \frac{\pi T_L \tau_R}{\sinh \pi T_L \tau_R} \right]^{2R^2/(1 + R^2)}, \tag{94} \]
\[ \mathcal{G}^K_L(\tau_L) = \frac{1}{\pi V_F T_L} \left[ \frac{\pi T_L \tau_L}{\sinh \pi T_L \tau_L} \right]^{T^2/(1 + R^2)} \left[ \frac{\pi T_R \tau_L}{\sinh \pi T_R \tau_L} \right]^{2R^2/(1 + R^2)}. \tag{95} \]

The Fourier transformation in (93) can be evaluated numerically. The results are exemplified in figures 4–6. To produce the graphs we have chosen the LL parameter \(K = 0.2\) such that the transmission and reflection amplitudes \(T, R\) are approximately equal. Figure 4 shows the dependence of the zero-energy correlator \(\langle n_R(x_R, 0)n_L(x_L, 0) \rangle\) on \(x_R + x_L\). Solid and dashed lines correspond to strong \((T_R/T_L = 10)\) and comparatively weak \((T_R/T_L = 2)\) non-equilibrium. One observes the characteristic peaks at \(x_R + x_L = (2m + 1)Kl\). Occupation numbers of left-movers and right-movers are anti-correlated for \(x_R + x_L > 0\) and correlated at \(x_R + x_L < 0\) (of course, the situation will be reversed if one assumes that \(T_R < T_L\)).

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Figure 4. The dependence of the correlator \( \langle \langle n_R(x_R,0)n_L(x_L,0) \rangle \rangle \) on the sum of coordinates \( x_R + x_L \). The solid line corresponds to \( T_R/T_L = 10 \) while the dashed line is the result for \( T_R/T_L = 2 \). The energy dependence of the correlator of left and right distribution functions at \( x_R + x_L = Kl \) (blue dot) and \( x_R + x_L = -Kl \) (brown dot) is shown in figure 5.

Figure 5 demonstrates the functions \( f_{RL}^{+1}(\epsilon_R, \epsilon_L, 0) \). The correlations reach a maximum near \( \epsilon_R = \epsilon_L = 0 \). In the case of \( f_1^{RL} \), the anti-correlations are somewhat more extended in the direction \( \epsilon_R = -\epsilon_L \) while the development of the correlations in \( f_{-1}^{RL} \) is favored for \( \epsilon_R \) and \( \epsilon_L \) of equal sign. This fact leads to a remarkable structure in the ‘mean correlations’ of the distribution functions of left-movers and right-movers as given by

\[
    h(\epsilon_R, \epsilon_L) = \frac{T_R}{V_F} \int d(x_R + x_L) \langle \langle n_R(x_R, \epsilon_R)n_L(x_L, \epsilon_L) \rangle \rangle.
\]  

(The prefactor was introduced to make \( h \) dimensionless.) The function \( h(\epsilon_R, \epsilon_L) \) is shown in figure 6. We observe that on average the left-movers and right-movers are correlated when their energies have equal signs and anti-correlated in the opposite case. Note that this result is opposite to the one that would be expected on the basis of the classical consideration of the bosonic transmission–reflection process; see section 5.3.

A similar analysis can be performed for the case of the correlations in the occupation numbers of right electrons alone. One finds the correlation function of right distributions in the form of (76) with

\[
f_m^{RR}(\epsilon_R, \epsilon'_R, x) = -\frac{V_F^2}{4} \int d\tau_R d\tau'_R e^{i\eta_R \gamma_R + i\tau_R \gamma} \langle G_R(\tau_R)G^K_R(\tau'_R) \rangle \]

\[
\times \left[ \left( \frac{1 - \cosh \pi T_L (\tau_R + \tau'_R)}{\cosh 2\pi T_L x} \right) \left( \frac{1 - \cosh \pi T_R (\tau_R + \tau'_R)}{\cosh 2\pi T_R x} \right) \right]^{-1} \gamma_m - 1.
\]

The exponents \( \gamma_m \) are given by

\[
\gamma_m = T^2 \mathcal{R}^{[m]} / (1 + \mathcal{R}^2).
\]
Figure 5. Energy dependence of functions $f_{RL}^{R}(\epsilon_R, \epsilon_L, 0)$ and $f_{RL}^{L}(\epsilon_R, \epsilon_L, 0)$ (plots (a) and (b) respectively). All energies are measured in units of the largest temperature $T_R = 10T_L$.

Figure 6. Correlator of the distribution functions of left and right electrons integrated over the $x_R + x_L$ (see equation (96)). The ratio of temperatures was taken to be $T_R/T_L = 10$.

The analogous correlator of the left distributions can be obtained via the simple exchange $T_R \leftrightarrow T_L$. Provided that $T_R > T_L$, all the functions $f_{m}^{RR}$ turn out to be positive, while the corresponding functions $f_{m}^{LL}$ are negative (cf figure 7 showing $f_{2}^{RR}(\epsilon_R, \epsilon'_R, 0)$ and $f_{2}^{LL}(\epsilon_L, \epsilon'_L, 0)$). Thus, the hotter electrons have positive correlations built into their distribution, while the colder ones are anti-correlated.

6. Conclusions

In this paper we have developed an operator approach to the non-equilibrium LL. Using bosonization and refermionization techniques, we have explicitly determined the many-
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Figure 7. The energy dependence of the correlators $\langle\langle n_R(\epsilon_R, x_R)n_R(\epsilon'_R, x'_R)\rangle\rangle$ and $\langle\langle n_L(\epsilon_L, x_L)n_L(\epsilon'_L, x'_L)\rangle\rangle$ for $x_R - x'_R = 2Kl$. While the occupation numbers of the hotter (right in this case) electrons are correlated, the occupation numbers of colder electrons are anti-correlated.

body density matrix of the system and derived the fermionic correlation functions in terms of Fredholm determinants $\Delta_\eta$. Let us note that usually the correlation functions of the many-body interacting system can only be found approximately by truncating the Bogoliubov–Born–Green–Kirkwood–Yvon chain. The model considered in this work constitutes a remarkable example of a many-body problem where all the correlation functions can be evaluated exactly.

We have employed our technique to study the four-point correlation functions of the electrons coming out of the LL wire. While the dynamics of the outgoing electrons is free, the corresponding density matrix is highly complicated. It incorporates the correlations $\langle\langle n_\eta(x_\eta, \epsilon_\eta)n_\eta'(x'_\eta, \epsilon'_\eta)\rangle\rangle$ in the electronic distribution functions caused by the scattering of the LL bosons at the boundaries between the LL wire and the non-interacting leads. The spatial dependence of these correlations can be deduced from the general analysis of the corresponding functional determinants $\Delta_\eta$. It displays characteristic interference peaks (or dips) of the width $l_T$ at distance $Kl$ (the ‘optical length’ of the interacting wire) from one another; see figure 4.

For the case of partial equilibrium we have evaluated the Fredholm determinants governing the correlation function $\langle\langle n_\eta(x_\eta, \epsilon_\eta)n_\eta'(x'_\eta, \epsilon'_\eta)\rangle\rangle$ analytically. We have found a non-trivial spatial dependence of correlations, both in the occupation numbers of electrons of the same chirality and between the occupation numbers of left-movers and right-movers. Within the ‘hotter’ chiral branch, the positive correlations are developed in the electronic distribution. On the other hand, anti-correlations are seen in the distribution of colder electrons (see figure 7). The sign of the correlation function $\langle\langle n_R(x_R, \epsilon_R)n_L(x_L, \epsilon_L)\rangle\rangle$ depends on the sign of $x_R + x_L$. On average, the occupation numbers of right-movers and left-movers are correlated at energies of the same sign and anti-correlated at energies of opposite signs; see figure 6. The results obtained indicate that quantum interference effects contribute crucially to the correlation functions. Intrinsically quantum correlations between parts of the system which cannot be understood in a classical framework are referred to as entanglement [51]. In recent years it was recognized that quantifying the
entanglement between subsystems of a many-body quantum system can provide a clue to many relevant properties of the system; see e.g. [52,53]. Such quantification for a mixed state of a quantum system is in general a highly complicated task [54]. It remains to be seen to what extent the correlations found in this work are relevant in the quantum information context.

We stress once again that the correlations studied are a genuine non-equilibrium effect absent in equilibrium LL. These correlations are an experimentally relevant quantity and can be measured in a specifically designed tunneling experiment. For example, to access the correlator of left and right distribution functions, one can imagine the following setup. Suppose that two tunneling probes are attached to the wire at points \(x_R, x_L\) satisfying \(\eta x_{\eta} > \ell/2\). We can also assume for simplicity that the right probe allows only the tunneling of right electrons, while only left electrons can tunnel through the left one. Then (under the assumption that the tunneling density states in the probes are not flat) the correlator of right and left tunneling currents \(\langle \langle I_R(V_R)I_L(V_L) \rangle \rangle\) should be sensitive to the correlations in the distribution functions. The voltages \(V_\eta\) biasing the probes will control the corresponding energies in \(\langle \langle n_R(x_R, \epsilon_R)n_L(x_L, \epsilon_L) \rangle \rangle\). Clearly, a more sophisticated analysis is needed to extract the information on the correlator of the distribution functions from the correlation in the tunneling currents in a realistic setup. Our analysis shows, however, that the correlator of the distribution functions is in principle a measurable quantity.

Concluding the paper, we briefly discuss possible extensions of the present work. First, under the general non-equilibrium conditions, a numerical analysis of the functional determinants \(\Delta_\eta\) is needed to fully understand the correlations in the electronic occupation numbers. Second, the formalism developed here can be extended to cover some non-stationary states of the LL. For example, in view of the recent experimental developments [55], it is very interesting to investigate a LL wire exposed to on-demand coherent single-electron sources. In particular, in the context of quantum information processing, especially intriguing is the entanglement generated by interaction between two electrons injected by single-electron sources from the left and right leads. Third, upon a proper modification, our technique should also be useful for the investigation of the fractional quantum Hall edge states out of equilibrium. It is very interesting to look for manifestations in such systems of the fractionally charged quasi-particles in the non-equilibrium correlation functions.

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**Appendix A. The density matrix of non-equilibrium fermions in the bosonic representation**

In this appendix, we transform the density matrix (41) into a bosonic representation. Throughout this appendix we will be dealing with the in-fermions \(a^\text{in}\) and in-bosons \(b^\text{in}\) only. In order to simplify the notation, we omit the index ‘in’ for all the operators. Further, we focus on right-moving in-fermions. The contribution of right-movers is obtained

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analogously; the total density matrix is the product of contributions of left and right in-particles.

We are thus looking for the bosonic representation of the statistical operator
\[
\hat{\rho} = \frac{1}{Z} \exp \left[ - \sum_k \epsilon(k) \left( a_k^+ a_k - n^0(k) \right) \right].
\] (A.1)

Here \(Z\) is the normalization factor, \(n^0(k) = \Theta(-k)\) is the ground-state distribution function and \(\epsilon(k)\) determines the distribution of the electrons via
\[
n(k) = \frac{1}{1 + e^{\epsilon(k)}}.
\] (A.2)

A particularly convenient basis in the Hilbert space of chiral fermions is provided by the bosonic coherent states which are the eigenstates of the bosonic annihilation operators. Each coherent state is labeled by the total number of fermions \(N\) and the set of eigenvalues \(\beta_q\) of the operators \(b_q\). Explicitly,
\[
\langle N, \beta \rangle = \exp \left[ \sum_{q>0} \beta_q^* b_q^+ \right] |N\rangle.
\] (A.3)

The coherent states (A.3) form the overcomplete basis with the resolution of identity given by
\[
1 = \sum_N \int \left[ \prod_{q>0} d\beta_q^* d\beta_q \right] \exp \left[ - \sum_{q>0} |\beta_q|^2 \right] |N, \beta \rangle \langle N, \beta^*|.
\] (A.4)

The overlap of two coherent states is given by
\[
\langle M, \beta^* | N, \beta \rangle = \delta_{M,N} \exp \left[ \sum_{q>0} |\beta_q|^2 \right].
\] (A.5)

We are interested in the matrix elements of the statistical operator \(\hat{\rho}\) in the basis of the coherent states. They are given by (obviously, \(\hat{\rho}\) is diagonal with respect to the total number of fermions)
\[
\langle \beta^*, N | \hat{\rho} | N, \beta \rangle = \langle N | \exp \left[ \sum_{q>0} \beta_q^* b_q \right] \hat{\rho} \exp \left[ \sum_{q>0} \beta_q b_q^+ \right] |N\rangle.
\] (A.6)

We recall that the bosons are proportional to the Fourier components of the fermionic density:
\[
b_q^+ = \sqrt{\frac{2\pi}{L|q|}} \sum_k a_{k+q}^+ a_k, \quad b_q = \sqrt{\frac{2\pi}{L|q|}} \sum_k a_{k-q}^+ a_k.
\] (A.7)

Thus, the operator in the average in (A.6) is exponentially quadratic in fermions.

It is convenient to introduce new fermions:
\[
c_k = a_k \left( 1 - n^N(k) \right) + a_k^+ n^N(k),
\] (A.8)
\[
c_k^+ = a_k^+ \left( 1 - n^N(k) \right) + a_k n^N(k).
\] (A.9)
Here $n^N(k) = \Theta(-k + (2\pi/L)N)$ is the distribution function of a Fermi sea with $N$ extra particles. The idea behind the transformation (A.9) is that the state $|N\rangle$ is nullified by all the operators $c_k$, which simplifies the derivation. In terms of the new fermions $c$ we have

$$\sum_{q>0} \beta_q b_q^+ = \sum_{k_1,k_2} c_{k_1}^+ c_{k_2} U_{k_1,k_2} + \sum_{k_1,k_2} c_{k_1}^+ c_{k_2}^+ V_{k_1,k_2}. \quad (A.10)$$

The matrices $U_{k_1,k_2}$ and $V_{k_1,k_2}$ carrying two momentum indices are given by

$$U = \Phi(1 - n^N) - \Phi^T n^N, \quad (A.11)$$
$$V = \frac{1}{2} [\Phi + \Phi^T, n^N]. \quad (A.12)$$

Here the distribution function $n^N$ is considered as a matrix diagonal in momentum space, while the matrix $\Phi$ has the matrix elements

$$\Phi_{k_1-k_2} = \Theta(k_1 - k_2) \sqrt{\frac{2\pi}{L(k_1 - k_2)}} \beta_{k_1-k_2}. \quad (A.13)$$

Using the fermionic commutation relations satisfied by $c_k$ and the fact that all $c_k$ annihilate $|N\rangle$, one now finds

$$\exp \left[ \sum_{q>0} \beta_q b_q^+ \right] |N\rangle = \exp \left[ \sum_{k_1,k_2} c_{k_1}^+ c_{k_2}^+ W_{k_1,k_2} \right] |N\rangle, \quad (A.14)$$

$$W = \int_0^1 ds \exp[sU] V \exp[sU^T]. \quad (A.15)$$

Note that the operator $\Phi$ preserves the subspace $k > (2\pi/L)N$ while the operator $\Phi^T$ preserves the subspace $k \leq (2\pi/L)N$. On the other hand, in the first of these subspaces $1 - n^N = 1$, while the other subspace is nullified by $1 - n^N$. It follows that

$$e^{sU} = e^{s\Phi} (1 - n^N) + e^{-s\Phi^T} n^N. \quad (A.16)$$

Using this relation, the expression for the matrix $W$ can be simplified to the form

$$W = \frac{1}{2} \left[ e^{-\Phi^T} (1 - n^N) e^{\Phi^T} - e^\Phi (1 - n^N) e^{-\Phi} \right]. \quad (A.17)$$

The matrix element (A.6) now reads

$$\langle \beta^+, N | \hat{\rho} | N, \beta \rangle = \frac{\gamma}{Z} \langle N | \exp \left[ \sum_{k_1 k_2} c_{k_1} c_{k_2} W_{k_1 k_2}^+ \right] \exp \left[ -\sum_{k_1 k_2} c_{k_1}^+ c_{k_2} E_{k_1 k_2} \right] \right] \times \exp \left[ \sum_{k_1 k_2} c_{k_1}^+ c_{k_2}^+ W_{k_1 k_2} \right] |N\rangle \equiv \frac{\gamma}{Z} \langle N | e^{W^+} e^\beta e^W | N \rangle, \quad (A.18)$$

where

$$E_{k_1,k_2} = \epsilon(k_1) \left( 1 - 2n^N(k_1) \right) \delta_{k_1,k_2}, \quad \gamma = \exp \left[ -\sum_k \epsilon(k) \left( n^N(k) - n^0(k) \right) \right]. \quad (A.19)$$
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For the evaluation of the average (A.18) one can employ the standard technique of the fermionic coherent states. Let us define the set of the eigenstates of annihilation operators $c_k$ with the Grassmann eigenvalues $\chi_k$ ($\chi$ stands for the full set of $\chi_k$ with $k = -\infty \cdots \infty$):

$$c_k |\chi\rangle = \chi_k |\chi\rangle. \quad \text{(A.20)}$$

The coherent states $|\chi\rangle$ form an overcomplete basis with the resolution of the identity

$$1 = \int D\chi^* D\chi \exp \left[ -\sum_k \chi_k^* \chi_k \right] |\chi\rangle \langle \chi^*|, \quad \text{(A.21)}$$

where

$$D\chi^* D\chi = \prod_k d\chi_k^* d\chi_k. \quad \text{(A.22)}$$

Using the resolution of identity (A.21) and noting that the state $|N\rangle$ is itself a coherent state $|\chi = 0\rangle$, we get

$$\langle \beta^*, N | \hat{\rho} | N, \beta \rangle = \frac{\gamma}{Z} \int D\chi^1 D\chi^2 \exp \left[ -\sum_k (\chi_k^1 \chi_k^1 + \chi_k^2 \chi_k^2) \right] \times \langle \chi^* = 0 | e^{i W^+} | \chi^1 \rangle \langle \chi^1 | e^{\hat{E}} | \chi^2 \rangle \langle \chi^2 | e^{i W} | \chi = 0 \rangle. \quad \text{(A.23)}$$

Finally, since

$$\langle \chi^* | e^{i W} | \chi \rangle = \exp \left[ \sum_k \chi_k^* \chi_k + \sum_{k_1, k_2} \chi_{k_1}^* \chi_{k_2}^* W_{k_1, k_2} \right], \quad \text{(A.24)}$$

$$\langle \chi^* | e^{\hat{E}} | \chi \rangle = \exp \left[ \sum_{k_1, k_2} \chi_{k_1}^* \chi_{k_2}^* [e^{-E}]_{k_1, k_2} \right], \quad \text{(A.25)}$$

we are left with a simple Gaussian integral over Grassmann variables. The result of the integration reads

$$\langle \beta^*, N | \hat{\rho} | N, \beta \rangle = \frac{\gamma}{Z} \det \left[ \frac{1}{2W^+} \begin{array}{cc} 1 & -2e^{-E}W e^{-E} \\ -2e^{-E}W e^{-E} & 1 \end{array} \right]^{1/2}. \quad \text{(A.26)}$$

Using the explicit forms of the matrices $W$ and $E$, one can reduce the expression above to

$$\langle \beta^*, N | \hat{\rho} | N, \beta \rangle = \frac{\gamma}{Z} \det \left[ e^{-\phi} (1 - n_N) e^{-\phi} e^\phi + e^{-\phi^+} n_N e^{\phi^+} \right]. \quad \text{(A.27)}$$

Finally, working out the factor in front of the determinant in (A.27), we find

$$\langle \beta^*, N | \hat{\rho} | N, \beta \rangle = \det \left[ 1 - e^{\phi} n_N e^{-\phi} (1 - n) - e^{-\phi^+} (1 - n_N) e^{\phi^+} n \right]. \quad \text{(A.28)}$$

Equation (A.28) gives an explicit expression for the density matrix of non-equilibrium interacting fermions in the bosonic basis. It has the form of a one-dimensional functional determinant of Fredholm type.
Let us verify that equation (A.28) reduces to the Boltzmann–Gibbs form at equilibrium, which in the present notation corresponds to $\epsilon(k) = kV_F/T$. Indeed, we know that for $\epsilon(k) \equiv 0$, equation (A.27) should give just the norm of the coherent $|N, \beta\rangle$ (up to the normalization factor $Z$):

$$
\langle \beta^*, N|\hat{\rho}|N, \beta\rangle|_{\epsilon(k)=0} = \frac{\gamma}{Z} \det \left[ e^\Phi (1 - n_N) e^{-\Phi} + e^{-\Phi^+} n_N e^{\Phi^+} \right] = \frac{1}{Z} \exp \left[ \sum_{q>0} |\beta_q|^2 \right].
$$

(A.29)

On the other hand, for $\epsilon(k) = kV_F/T$ we can write

$$
\langle \beta^*, N|\hat{\rho}_{eq}|N, \beta\rangle = \frac{\gamma}{Z} \det \left[ e^{-\epsilon^* \Phi^*} (1 - n_N) e^{-\epsilon^* \epsilon^0} + e^{-\Phi^+} n_N e^{\Phi^+} \right]
$$

$$
= \frac{\gamma}{Z} \det \left[ e^\tilde{\Phi} (1 - n_N) e^{-\tilde{\Phi}} + e^{-\Phi^+} n_N e^{\Phi^+} \right],
$$

(A.30)

where

$$
\tilde{\Phi}_{k_1-k_2} = \left[ e^{-\epsilon^* \Phi^*} \right]_{k_1,k_2} = \Theta(k_1 - k_2) \sqrt{\frac{2\pi}{L(k_1 - k_2)}} b_{k_1-k_2} e^{-V_F(k_1-k_2)/T}.
$$

(A.31)

We see that the matrix elements of the equilibrium density matrix can be obtained from the norm of the coherent states just by making the replacement

$$
\beta_q \rightarrow \beta_q e^{-V_F q/T},
$$

(A.32)

with the result

$$
\langle \beta^*, N|\hat{\rho}_{eq}|N, \beta\rangle = \frac{1}{Z} \exp \left[ \sum_{q>0} e^{-qV_F/T} |\beta_q|^2 \right].
$$

(A.33)

In the operator form, equation (A.33) reads

$$
\hat{\rho}_{eq} = \frac{1}{Z} \exp \left[ -\frac{V_F}{T} \sum_{q>0} q b_q^+ b_q \right],
$$

(A.34)

which is the expected equilibrium result.

Away from equilibrium, the bosonic density matrix (A.28) is much more complicated. Nevertheless, it can be used to derive the expressions (47), (48), (49) for the prototypical average (43) required for evaluation of correlation functions. To do this, one represents the determinant (A.28) as a Gaussian integral over auxiliary Grassmann variables and performs an expansion in powers of $e^\Phi$. At every order, the evaluation of the correlator of the bosonic exponents (43) is then provided by a Gaussian integral over bosons and fermions. The crucial point is some specific cancellations between the fermionic and bosonic integrations, which finally allows us to evaluate all orders of the expansion analytically and resum the whole series. This brute-force derivation requires, however, rather cumbersome combinatorics, and we do not present it here. An alternative, considerably simpler, derivation involves refermionization, as explained in the main text and appendix B.

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Many-particle correlations in a non-equilibrium Luttinger liquid

Appendix B. Averaging exponentials of bosonic fields in a non-equilibrium Luttinger liquid via refermionization

In this appendix we evaluate the prototypical correlation function (43) where the average is performed with the density matrix (41). Since the right-moving in-fermions are completely independent from the left-moving ones (see equation (41)) we ignore the latter for a while and include the left-movers in the final formulas. In our notation we also suppress the index ‘in’ since all the operators (bosons and fermions) that we will be dealing with in this appendix are in-operators and the suppression should not cause any confusion. Thus, we need to evaluate the average

\[ Z[ \mathbf{g}^*, \mathbf{g} ] = \left\langle \exp \left[ \sum_{q > 0} g(q) b_q^+ \right] \exp \left[ - \sum_{q > 0} g^*(q) b_q \right] \right\rangle \]  

with the density matrix

\[ \hat{\rho} = \frac{1}{Z} \exp \left[ - \sum_k \epsilon(k) (a_k^+ a_k - n^0(k)) \right]. \]  

We can now apply the well-known expression for the trace over the fermionic Hilbert space of a product of exponentially quadratic operators [56]:

\[ \text{tr} e^{H_1 \ldots e^{H_n}} = \text{det} \left( 1 + e^{H_1 \ldots e^{H_n}} \right), \]  

where

\[ H_i = \sum_{k_1, k_2} t_{k_1, k_2} a_{k_1}^+ a_{k_2}. \]  

The right-hand side of (B.3) is a determinant of an operator acting in the single-particle Hilbert space. Applying (B.3) to the average in question, we get

\[ Z[ \mathbf{g}^*, \mathbf{g} ] = \text{det} \left( 1 - n(k) + e^{-i \delta(x)} n(k) \right), \]  

\[ \delta(x) = i \sum_{q > 0} \sqrt{\frac{2\pi}{Lq}} \left( g_q e^{i q x} - g_q^* e^{-i q x} \right). \]

The determinant in (B.5) is in fact not well defined, for the following reason. Strictly speaking, equation (B.3) assumes that in the fermionic Hilbert space there is a state nullified by all the operators \( a_k \). This is not the case in the present situation. We can resolve this difficulty by making the transformation from the particle operators \( a_k, a_k^+ \) to particle and hole operators:

\[ c_k = a_k \left( 1 - n^0(k) \right) + a_k^+ n^0(k), \]  

\[ c_k^+ = a_k^+ \left( 1 - n^0(k) \right) + a_k n^0(k). \]

Then we can evaluate \( Z[ \mathbf{g}^*, \mathbf{g} ] \) with the technique of fermionic coherent states. The calculation is very similar to that of appendix A.
We can take a short-cut, however, if we notice that $Z[g^*, g]$ is an average of an operator normal-ordered in bosons. Thus, $Z[g^*, g] = 1$ at zero temperature. This suggests that the regularized version of equation (B.5) should be

$$Z[g^*, g] = \det \left[ (1 - n^0(k) + e^{-i\delta(x)n^0(k)})^{-1} (1 - n(k) + e^{-i\delta(x)n(k)}) \right]. \quad (B.9)$$

A direct calculation along the lines of appendix A does indeed confirm this result.

Incorporating now the left electrons into our consideration, we come to equations (47)–(49).

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