DISTINGUISHING THE KNOT $5_2$ USING FINITE QUOTIENTS

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Abstract. We give a criterion for distinguishing a prime knot $K$ in $S^3$ from every other knot in $S^3$ using the finite quotients of $\pi_1(S^3 \setminus K)$. Using this criterion and recent work of Baldwin-Sivek, we show that the hyperbolic knot $5_2$ is distinguished from every other knot in $S^3$ by the finite quotients of $\pi_1(S^3 \setminus 5_2)$.

1. Introduction

Finite quotients of the fundamental group are useful for distinguishing 3-manifolds; in particular, for a 3-manifold $M$, a finite quotient of $\pi_1(M)$ corresponds to the deck group of a finite-sheeted regular cover of $M$. If the fundamental groups of two 3-manifolds $M$ and $N$ have different finite quotients, then $\pi_1(M) \not\cong \pi_1(N)$ and the 3-manifolds $M$ and $N$ are not homeomorphic. When $M$ is a compact 3-manifold, its fundamental group $\pi_1(M)$ is residually finite [Hem16], and so the set $C(\pi_1(M))$ of finite quotients of $\pi_1(M)$ is non-empty and infinite.

One of the consequences of the residual finiteness of the fundamental groups of compact 3-manifolds is that the unknot is the only knot in $S^3$ whose knot complement has a fundamental group with only finite cyclic quotients. Boileau [Boi18] has conjectured that every prime knot $K \subset S^3$ is completely determined by the set of finite quotients of $\pi_1(S^3 \setminus K)$, that is, if for two prime knots $J$ and $K$, $\pi_1(S^3 \setminus J)$ and $\pi_1(S^3 \setminus K)$ have the same set of finite quotients, then $J$ and $K$ are isotopic. By work of Boileau-Friedl [BF20] and Bridson-Reid [BR20], it is known that the figure-eight knot $4_1$ and the trefoil knot $3_1$ are completely determined by the finite quotients of their knot groups even amongst the fundamental groups of compact 3-manifolds. Furthermore, Wilkes [Wil19] has shown that knots in $S^3$ whose complements are graph manifolds are distinguished by the finite quotients of their knot groups.

The purpose of this note is to show:

**Theorem 1.1.** The knot $5_2$ (shown in Figure 7) is distinguished from every other knot in $S^3$ by the finite quotients of $\pi_1(S^3 \setminus 5_2)$.

Theorem 1.1 will follow from our next theorem, using recent work of Baldwin-Sivek [BS22], and the work of Wilkes [Wil18] (as described in Section 3). To state the Theorem, we recall the definition of a **characterizing slope** $\alpha \in \mathbb{Q}$ for a knot $K$. For a knot $K \subset S^3$, let $S^3_\alpha(K)$ be the 3-manifold obtained by $\alpha$–Dehn surgery on $K$. 
Definition 1.2. A slope $\alpha$ is a characterizing slope for a knot $K \subset S^3$ if for any knot $J \subset S^3$, $S^3_\alpha(J) \cong S^3_\alpha(K)$ if and only if $J$ is isotopic to $K$.

Theorem 1.3. Let $K$ be a hyperbolic knot in $S^3$ for which

1. $0$ is a characterizing slope for $K$,
2. $S^3_0(K)$ is distinguished from every other compact, irreducible 3-manifold by the finite quotients of $\pi_1(S^3_0(K))$.

then $K$ is distinguished from other knots in $S^3$ by the finite quotients of $\pi_1(S^3 \setminus K)$.

We point out that every knot in $S^3$ has infinitely many characterizing slopes [Lac19]. On the other hand, it is known that some knots have infinitely many non-characterizing (integral) slopes [BM18]. The only knots that have been shown to have 0 as a characterizing slope are the unknot (Property R) [Gab87], the trefoil, the figure-eight knot [OS19], and most recently and most usefully for our purposes, $5_2$ [BS22].

Theorem [L3] also gives a different proof that the figure-eight knot is distinguished from every other knot in $S^3$, recovering a result in [BF20] and [BR20] (see Section 3).

Corollary 1.4. The knot $4_1$ is distinguished from every other knot in $S^3$ by the finite quotients of $\pi_1(S^3 \setminus 4_1)$.

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2. Profinite Completions

For a finitely generated, residually finite group $G$, we can organize the set of finite quotients of $G$ into an inverse system whose inverse limit $\hat{G}$ is a finitely generated, profinite group, the profinite completion of $G$. There is a bijective correspondence between the finite index subgroups of $G$ and the open subgroups of $\hat{G}$. There is a canonical inclusion $G \hookrightarrow \hat{G}$ that has dense image because $G$ is residually finite.

Moreover, the assignment of finitely generated groups to their profinite completions is functorial, and an epimorphism of groups $G \twoheadrightarrow H$ will induce a continuous epimorphism of the profinite completions $\hat{G} \twoheadrightarrow \hat{H}$. The profinite completion completely captures the data
of finite quotients of a group. In particular, two residually finite groups $G$ and $H$ will have isomorphic profinite completions if and only if they have the same set of finite quotients $C(G) = C(H)$ ([RZ00], Corollary 3.2.8).

We say that a finitely generated group is **profinely rigid** if it is completely determined by its profinite completion among all finitely generated, residually finite groups. We say a compact, orientable 3-manifold $M$ is **relatively profinitely rigid** if $\pi_1(M)$ is completely determined by its profinite completion among the fundamental groups of compact, orientable 3-manifolds.

**Notation.** For a subgroup $K < G$, we write the closure of $K$ in $\hat{G}$ as $\overline{K}$. If $K$ is closed in the profinite topology on $G$, $\overline{K} \cong \hat{K}$ and we say that $K$ is a **separable** subgroup of $G$.

3. Proofs

We first prove Theorem 1.1 and Corollary 1.4 assuming Theorem 1.3.

**Proof.** (Theorem 1.1) By [BS22], 0 is a characterizing slope for $5_2$. Using Regina [BBP+99], one can check that $S_0^3(5_2) = \{A : (2,1)\} / (1 \ 1) / (1 \ 1)$ is the result of gluing the two torus boundary components of a Seifert-fibered space with base orbifold an annulus with one cone point by the homeomorphism $(1 \ 1)$. Thus, $S_0^3(5_2)$ has a non-trivial JSJ decomposition with JSJ graph having a single vertex and cycle and therefore the JSJ graph is not bipartite. Thus, by Theorem A [Wil18], $S_0^3(5_2)$ is (relatively) profinitely rigid. So, $5_2$ satisfies the hypothesis of Theorem 1.3, and the result follows. \[\square\]

**Remark 3.1.** We can also show that $M$ does not have a bipartite JSJ graph as follows: by [HT85], the once-punctured torus Seifert surface of the $5_2$ is the unique embedded incompressible surface in $S^3 \setminus 5_2$ with boundary slope 0, and $S^3 \setminus 5_2$ has no closed, embedded, essential surfaces. Thus, the only embedded incompressible surface in $M = S_0^3(5_2)$ is the torus obtained by capping off the Seifert surface and so $M$ is not hyperbolic. It therefore suffices to show that $M$ is not a Seifert-fibered space, for then $M$ has the non-trivial JSJ decomposition as above. If $M$ were Seifert-fibered then we can use:

1. the classification of closed Seifert-fibered spaces [Sco83],
2. the orientability of $M$,
3. $H_1(M) \cong \mathbb{Z}$,
4. the Property R theorem [Gab87],
5. and the fact that $M$ does not fiber over the circle (Corollary 8.19 [Gab87]),

\[\text{to see that the base orbifold of } M \text{ would be hyperbolic. Thus, any irreducible } \text{PSL}(2, \mathbb{C}) \text{ representations of the hyperbolic base orbifold fundamental group will give rise to an irreducible } \text{PSL}(2, \mathbb{C}) \text{ representation of } \pi_1(M). \text{ However, one can check (using the A-polynomial of } 5_2, \text{ for example) that } \pi_1(M) \text{ has no infinite, irreducible } \text{PSL}(2, \mathbb{C}) \text{ representation, and it follows that } M \text{ is not Seifert-fibered.} \]
Proof. (Corollary 1.4) By [OS19], 0 (and in fact every slope \(\neq \infty\)) is a characterizing slope for the figure-eight knot 41. It follows from [Fun14] that \(S^3_0(41)\) is relatively profinetly rigid. Briefly, the manifold \(S^3_0(41)\) is a torus bundle with SOLV geometry and monodromy \((\frac{2}{1}, 1)\). The eigenvalues of this monodromy are \(\lambda = \frac{3\pm\sqrt{5}}{2}\), and so generate \(\mathbb{Q}(\sqrt{5})\) which has class number 1. By Corollary 1.2 in [Fun14], the number of compact 3-manifolds whose fundamental groups have the same finite quotients as \(S^3_0(41)\) is bounded above by the class number, 1. Hence \(S^3_0(41)\) is (relatively) profinetly rigid as claimed, and so 41 satisfies the hypotheses of Theorem 1.3 which proves the corollary. \(\square\)

To establish Theorem 1.3, we will use the following lemma:

**Lemma 3.2.** Let \(J\) and \(K\) be hyperbolic knots in \(S^3\) with \(\lambda_J, \lambda_K\) the homological longitudes of \(J\) and \(K\) respectively. If \(\hat{\pi}_1(S^3 \setminus J) \cong \hat{\pi}_1(S^3 \setminus K)\) then, upon identification, \(\langle \lambda_J \rangle\) and \(\langle \lambda_K \rangle\) have conjugate closures in \(\hat{\pi}_1(S^3 \setminus K)\).

**Proof.** Let \(\mu_J, \mu_K\) be the meridians of \(J\) and \(K\) respectively, and \(P_J = \langle \mu_J, \lambda_J \rangle\) and \(P_K = \langle \mu_K, \lambda_K \rangle\) in \(\pi_1(S^3 \setminus J)\) and \(\pi_1(S^3 \setminus K)\) respectively. By the main theorem of [Ham01], abelian subgroups of 3-manifold groups are separable and so \(\langle \lambda_J \rangle \cong \hat{\langle \lambda_J \rangle}, \bar{P}_J \cong \bar{P}_J \triangleleft \hat{\pi}_1(S^3 \setminus J)\) and \(\hat{\langle \lambda_K \rangle} \cong \langle \bar{\lambda_K} \rangle\), \(\bar{P}_K \cong \bar{P}_K < \hat{\pi}_1(S^3 \setminus K)\).

Let \(\phi : \hat{\pi}_1(S^3 \setminus J) \to \hat{\pi}_1(S^3 \setminus K)\) be an isomorphism. By Theorem 9.3 of [WZ17], it follows that \(\phi(\bar{P}_J)\) and \(\bar{P}_K\) are conjugate in \(\hat{\pi}_1(S^3 \setminus K)\); so there exists \(g \in \hat{\pi}_1(S^3 \setminus K)\) such that \(g\phi(\bar{P}_J)g^{-1} = \phi(\bar{P}_K)\). Since \(\hat{\langle \lambda_K \rangle}\) is the intersection of \(\bar{P}_K\) with the kernel of the unique epimorphism \(\pi_1(S^3 \setminus K) \to \mathbb{Z}\), it follows that \(\hat{\langle \lambda_K \rangle}\) is the intersection of \(\bar{P}_K\) with the kernel of the induced epimorphism \(\hat{\pi}_1(S^3 \setminus K) \to \hat{\mathbb{Z}}\). To see this, we observe that any element of \(\bar{P}_K\) is a \(\hat{\mathbb{Z}}\)-linear combination of \(\mu_K, \lambda_K\), and so if any element of \(\bar{P}_K\), say \(\alpha = c_1\mu_K + c_2\lambda_K\) with \(c_1, c_2 \in \hat{\mathbb{Z}}\), is in the kernel of the epimorphism \(\hat{\pi}_1(S^3 \setminus K) \to \hat{\mathbb{Z}}\) induced by the unique epimorphism \(\pi_1(S^3 \setminus K) \to \mathbb{Z}\), \(c_1 = 0\) because the image of \(\mu_K\) in \(\hat{\mathbb{Z}}\) is non-trivial while the image of \(\lambda_K\) is trivial. As \(c_1 = 0\), \(\alpha \in \langle \bar{\lambda_K} \rangle\).

Next, observe that since the epimorphism \(\pi_1(S^3 \setminus J) \to \mathbb{Z}\) is unique, it can also be described as the homomorphism obtained by the following composition of maps

\[
\pi_1(S^3 \setminus J) \hookrightarrow \hat{\pi}_1(S^3 \setminus J) \xrightarrow{\phi} \hat{\pi}_1(S^3 \setminus K) \to \hat{\mathbb{Z}} \quad (\blacklozenge)
\]

where the first map is the canonical inclusion of \(\pi_1(S^3 \setminus J)\) into its profinite completion, and the last map \(\hat{\pi}_1(S^3 \setminus K) \to \hat{\mathbb{Z}}\) is the epimorphism induced by the unique epimorphism \(\pi_1(S^3 \setminus K) \to \mathbb{Z}\). Furthermore, \(\langle \lambda_J \rangle\) is the intersection of \(\bar{P}_J \cong \bar{P}_J\) with the kernel of the map

\[
\hat{\pi}_1(S^3 \setminus J) \xrightarrow{\phi} \hat{\pi}_1(S^3 \setminus K) \to \hat{\mathbb{Z}}
\]

because \(\langle \lambda_J \rangle\) is the intersection of \(\bar{P}_J\) with the kernel of the unique epimorphism \(\pi_1(S^3 \setminus J) \to \mathbb{Z}\) which we have noted is the same map as the composition of maps (\(\blacklozenge\)). Since \(g^{-1}\lambda_Kg\) is in the intersection of \(\phi(\bar{P}_J)\) with the kernel of the epimorphism \(\hat{\pi}_1(S^3 \setminus K) \to \hat{\mathbb{Z}}\),
\[ g^{-1}\lambda_Kg \in \langle \phi(\lambda_J) \rangle. \] By interchanging \( J \) and \( K \), we can apply the foregoing argument to show that \( g\phi(\lambda_J)g^{-1} \in \langle \lambda_K \rangle \) and so the closures of \( \langle \phi(\lambda_J) \rangle \) and \( \langle \lambda_K \rangle \) are conjugate in \( \hat{\pi}_1(S^3 \setminus K) \).

With this lemma in hand, we can prove Theorem 1.3. We recall the hypotheses; for a knot \( K \), the slope 0 is characterizing for \( K \) and the 0-Dehn surgery on \( K \) is (relatively) profinetely rigid.

**Proof.** (Theorem 1.3) Let \( J \) be a knot in \( S^3 \) with \( \hat{\pi}_1(S^3 \setminus J) \cong \hat{\pi}_1(S^3 \setminus K) \). By Theorem A of [WZ17], \( J \) is a hyperbolic knot. Assuming that \( Q \) is a finite quotient of \( \pi_1(S^3_0(J)) \), we can precompose with the Dehn-filling epimorphism \( \pi_1(S^3 \setminus K) \to \pi_1(S^3_0(K)) \) to obtain an epimorphism from \( \pi_1(S^3 \setminus K) \) to \( Q \) under which \( \lambda_K \) maps trivially. As in the proof of Lemma 3.2, we can choose some identification \( \phi : \hat{\pi}_1(S^3 \setminus J) \cong \hat{\pi}_1(S^3 \setminus K) \), and thereby obtain an epimorphism \( \pi_1(S^3 \setminus J) \to Q \) by the following composition:

\[
\pi_1(S^3 \setminus J) \hookrightarrow \hat{\pi}_1(S^3 \setminus J) \xrightarrow{\phi} \hat{\pi}_1(S^3 \setminus K) \to Q
\]

By Lemma 3.2, since \( \langle \phi(\lambda_J) \rangle \) and \( \langle \lambda_K \rangle \) have conjugate closures and \( \lambda_K \) maps trivially, \( \lambda_J \) will also map trivially. It follows that \( Q \) is a finite quotient of \( \pi_1(S^3_0(J)) \).

Applying this argument with the roles of \( J \) and \( K \) reversed shows that every finite quotient of \( \pi_1(S^3_0(J)) \) is also a finite quotient of \( \pi_1(S^3_0(K)) \), and we conclude that \( \hat{\pi}_1(S^3_0(J)) \cong \hat{\pi}_1(S^3_0(K)) \).

However by the second hypothesis, \( S^3_0(K) \) is (relatively) profinetely rigid and so \( S^3_0(J) \) is homeomorphic to \( S^3_0(K) \). By the first hypothesis, 0 is a characterizing slope for \( K \) and so \( J \) and \( K \) are isotopic. \( \square \)

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