Zitterbewegung in Noncommutative Geometry

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Abstract

We have considered the effects of space and momentum noncommutativity separately on the zitterbewegung (ZBW) phenomenon. In the space noncommutativity scenario, it has been expressed that, due to the conservation of momentum, the Fourier decomposition of the expectation value of position does not change. However, the noncommutative (NC) space corrections to the magnetic dipole moment of electron, that was traditionally perceived to come into play only in the first-order of perturbation theory, appear in the leading-order calculations with the similar structure and numerically the same order, but with an opposite sign. This result may explain why for large lumps of masses, the Zeeman-effect due to the noncommutativity remains undetectable. Moreover, we have shown that the x- and y-components of the electron magnetic dipole moment, contrary to the commutative (usual) version, are non-zero and with the same structure as the z-component. In the momentum noncommutativity case, we have indicated that, due to the relevant external uniform magnetic field, the energy-spectrum and also the solutions of the Dirac equation are changed in 3 + 1 dimensions. In addition, our analysis shows that in 2 + 1 dimensions, the resulted NC field makes electrons in the zero Landau-level rotate not only via a cyclotron motion, but also through the ZBW motion with a frequency proportional to the field, which doubles the amplitude of the rotation. In fact, this is a hallmark of the ZBW in graphene that provides a promising way to be tested experimentally.

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1 Introduction

It is well-known that for a free Dirac particle [1] the velocity and momentum do not coincide, that is, a free particle oscillates rapidly with the speed $c$ around the center of mass while moving like a relativistic particle with velocity $p/m$ [2]–[8]. This rapid oscillatory motion was called zitterbewegung – a trembling/quivering motion – by Schrödinger [9, 10]. The amplitude of this motion is predicted to be of the order of the Compton wavelength for electrons, i.e. $\lambda_c = h/(m_ec) \approx 3.86 \times 10^{-13}$ m. Using this picture, it has been suggested in the literature that the spin and magnetic moment of an electron, as a point charge, is generated by an intrinsic local motion [11]–[15]. In other words, the magnetic moment and the spin of electron are consequences of a local circular motion of mass and charge of electron, and may be considered as an “orbital angular momentum” due to the ZBW. Other attempts at explaining the spin of electron [16, 17] have also established that it can be regarded as due to a circulating flow of energy in the same footnote as orbital angular momentum. Similarly, in Ref. [18], it has been stated that the origin of the spin-magnetic moment is caused by a quantum transition current between positive and negative energy states of the solutions of the Dirac equation, and hence, it relates closely to the ZBW phenomenon. Indeed, not only the Dirac strong support [2] of Schrödinger’s ZBW as a fundamental property of electron has been unchallenged up today, but there is increasing evidence that ZBW is a real effect, in principle, observable, e.g., in a Bose-Einstein condensate [19] and in semiconductors [20]. Nevertheless, by a unitary Foldy-Wouthuysen transformation [21], the ZBW can be avoided because this transformation actually eliminates negative energy components in electron wave functions. However, the ZBW goes hand in hand with the existence of negative energy solutions, and is only important for wave packets with significant interference between positive and negative frequencies. Even in this regard, taking the neutrino as a localized wave packet consisting of positive and negative energies, and studying its chiral ZBW, it has been claimed [22] that it may explain the “missing” solar neutrino experiments, and also interpreting [23] chiral oscillations in terms of the

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ZBW effect. Moreover, the results of such that transformation are in contrast with the Darwin term \[ \frac{eB}{c} \] in atomic physics.

Furthermore, in the issue of ZBW, in the approach of Ref. [11], it has been demonstrated that for a wave packet consisting of both positive and negative energy components, with the specified initial condition, the contribution of each plane wave to the motion of the wave packet is an orbit whose projection on the plane perpendicular to the direction of the spin is a circle of radius \( \lambda_c/2 \) with frequency \( \omega_{[bw]} \approx 1.6 \times 10^{21} \text{s}^{-1} \) for electrons. This result leads to the intrinsic magnetic moment of particle with the correct gyromagnetic \( g \) factor. We also studied the same issue, but in the presence of an external magnetic field, and showed that the previous results are stable and the only difference is that in this case the ZBW frequency and the amplitude are shifted.

On the other hand, some scientists believe that the NC geometry can give new vision for the spacetime structure especially nearly at the Planck scale, see, e.g., Refs. [20–28]. In this respect, there is a vast literature discussion on the theoretical and phenomenological consequences of these effects, see, e.g., Refs. [29–31] and references therein. The noncommutativity concept between spacetime coordinates was first introduced in 1947 [29], and thereafter, the NC geometry has taken shape since 1980, see, e.g., Refs. [32–37] and references therein. This picture has also strong motivation in the framework of the string and M-theories, see, e.g., Refs. [38–40]. Indeed, it has been claimed that as standard linear quantum mechanics should be a limiting case of an underlying new non-linear quantum theory, a possible approach for such a new formulation can be sought through the use of NC geometry. In this regard, due to changes raised in this picture, quantum mechanics and quantum field theory (QFT) phenomena have been affected and some research have been devoted to this sector [12–15]. Actually, while considering the NC geometry, the QFT modifies the relativistic wave equation, i.e. the Dirac equation, by which, one can calculate the consequences of NC parameters on different physical effects. However, there have been two approaches in including the NC effects on the Dirac equation for an electron in an external electromagnetic field, namely just by the simple Moyal modification, and another one, this modification plus taking the so called Seiberg-Witten (SW) map also into account [38], wherein, it has been shown that only the latter one keeps the Dirac equation being gauge invariant [40–41].

Now, regarding the ZBW and its relation with the electron magnetic dipole moment, it is intriguing to figure out effect(s) of the NC geometry on the ZBW phenomenon, and indeed, on the internal structure of electron and its spin as well. However, as the scale of the ZBW is lower than practical experimental resolution and since the NC effects are also small, there may be some claims that such effects are hardly being detected in the near future. Nevertheless, and interestingly enough, the phenomenon of ZBW for electrons has been shown (e.g., Refs. [20–30] and other references mentioned in Refs. [29–30]) to occur in non-relativistic cases and in two-dimensional Dirac materials (like graphene), wherein the ZBW has much lower angular frequency and much larger amplitude and thus, it may lend itself to experimental detection easier. On the other hand, the effect of the NC geometry has been studied, e.g. Refs. [51–53], in two-dimensional Dirac materials and on the quantum Hall effect as well.

Having all these motivations into account, the purpose of this work is to investigate effect(s) of the NC geometry on the ZBW phenomenon; and to perform this task, at the beginning we concisely review the ZBW in the usual commutative geometry in 3 + 1 dimensions. Afterward, we continue with a brief introduction on the noncommutativity and how it appears in the Dirac equation while giving an introduction to the basic premises of the NC geometry. Then, we first consider the effect of space noncommutativity on the ZBW phenomenon and on the magnetic dipole moment of an electron in the leading-order calculations in 3 + 1 dimensions. In Sec. 4, we first continue to investigate the effect of momentum noncommutativity on the Dirac Hamiltonian and its solutions in 3 + 1 dimensions, and thereupon, consider the issue in the 2 + 1 dimensional case for graphene. Finally, we conclude the summary of the results in the last section.

## 2 ZBW Phenomenon in Commutative Geometry

In this section, we consider the ZBW for a localized wave packet consisting of positive and negative energies, as in Ref. [11], with some necessary explanations as a calculation in the commutative (usual) version. Thereafter, in the next section, we carry out the same approach for the NC situation. Thus, we will be able to figure out the effect(s) of the NC geometry when it is compared with the results of the commutative one.

In this regard, the Dirac equation for an electron in a field-free region is

\[
i\hbar \partial_t \Psi = \left( c \, \mathbf{\alpha} \cdot \mathbf{p} + \gamma_0 m_e c^2 \right) \Psi,
\]

(1)
wherein the matrices $\alpha$ and $\gamma_0$ are defined as

$$
\alpha_i = \begin{pmatrix}
0 & \sigma_i \\
\sigma_i & 0
\end{pmatrix} \quad \text{and} \quad \gamma_0 = \begin{pmatrix}
I & 0 \\
0 & -I
\end{pmatrix},
$$

(2)

where the Latin lowercase letters run from one to three, and $\sigma_i$ and $I$ are the $2 \times 2$ Pauli matrices and the unit matrix, respectively. The solution of Eq. (11) can be written as a general wave packet expanded in terms of momentum eigenfunctions as

$$
\Psi(r, t) = h^{-3/2} \int \left[ C_+(p) \exp(-i\omega t) + C_-(p) \exp(i\omega t) \right] \exp\left(\frac{i \mathbf{p} \cdot \mathbf{r}}{\hbar}\right) d^3p,
$$

(3)

where $\omega = \gamma m_c^2/\hbar$ and $C_+(p)$ ($C_-(p)$) is a linear combination of the spin-up and spin-down amplitudes of the free particle Dirac waves with momentum $p$ and positive (negative) energy. Such a wave packet includes both negative and positive energies. This wave packet can be used with an initial condition which being a localized spin-up electron in the $z$-direction while its center is at rest in the origin, namely

$$
\Psi(r, 0) = \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix} f\left(\frac{r}{r_o}\right).
$$

(4)

Here, $f(r/r_o) = \left[2/(\pi r_o^2)\right]^{3/4} \exp\left[-(r/r_o)^2\right]$ is a normalized Gaussian function with $r \equiv |r|$ and $r_o$ as a constant that indicates approximate spatial extension of the wave packet. Also, the Fourier transformation of $f(r/r_o)$ is

$$
f\left(\frac{p}{p_o}\right) = \left(\frac{2}{\pi p_o^2}\right)^{3/4} \exp\left[-\left(\frac{p}{p_o}\right)^2\right],
$$

(5)

that satisfies the normalization condition $\int f^2(p/p_o) d^3p = 1$, and where the constant $p_o = 2\hbar/r_o$ gives the width of wave packet in the momentum space. Now, to simplify the model further, if one applies the non-relativistic approximation and hence drops the terms of the order $(p/2m_ec)^2$ or higher, then in this case, the wave packet will read

$$
\Psi(r, t) \simeq h^{-3/2} \int \left\{ \begin{pmatrix}
1 \\
0 \\
K_p p_3
\end{pmatrix} \exp(-i\omega t) + \begin{pmatrix}
0 \\
0 \\
-K_p p_3
\end{pmatrix} \exp(i\omega t) \right\} f(p/p_o) \exp\left(\frac{i \mathbf{p} \cdot \mathbf{r}}{\hbar}\right) d^3p,
$$

(6)

where $\omega \simeq m_e^2/\hbar$ in the non-relativistic approximation, $p_+ = p_1 + ip_2$ and $K \simeq \frac{1}{2m_ec}$. The above rough solution satisfies the Dirac equation (11) when the aforementioned approximation is taken into account.

Using this wave packet and the expectation value of velocity vector, i.e. $< \mathbf{r} > = \int \Psi^*(r, t)(\mathbf{r} \cdot \mathbf{c}) \Psi(r, t) d^3r$ in the spherical coordinates, in order to specify the Fourier decompositions, one gets

$$
< x > \simeq I\frac{\lambda_e}{2} \int_0^{2\pi} \sin(\omega_{[abw]} t + \varphi) \, d\varphi,
$$

and

$$
< y > \simeq I\frac{\lambda_e}{2} \int_0^{2\pi} \cos(\omega_{[abw]} t + \varphi) \, d\varphi,
$$

and

$$
< z > \simeq J\lambda_e \sin(\omega_{[abw]} t),
$$

(7)

where $\omega_{[abw]} = 2\omega$ in the non-relativistic approximation, $\varphi$ is the azimuthal angle in the spherical momentum space, and

$$
I \equiv -2\int_0^{\infty} \int_0^\pi p^2 \sin^2 \theta \, dp \, d\theta = -\frac{1}{2} \frac{\lambda_e}{r_o^2}
$$

(8)

and

$$
J \equiv -\pi \int_0^{\infty} \int_0^\pi p^2 \, dp \, \sin 2\theta \, d\theta = 0.
$$

(9)

Of course, all components of $< \mathbf{r} >$ vanish upon integration over the full domain, and thus all oscillatory motions disappear while the center of the wave packet as a whole remains at rest. However, to realize the role of a

\footnote{In terms of the Dirac (gamma) matrices, $\alpha_i = \gamma^i \gamma_0$ for $i = 1, 2, 3.$}
specific Fourier decomposition in the ZBW circular motion, one can confine oneself to a fixed value of \( \varphi \), say \( \varphi_o \). Thus in this case, the Fourier components in the \( xy \)-plane become [11][25]

\[
\langle x \rangle_{\varphi_o} \approx \frac{\lambda_c}{2} \sin \left( \omega_{zbw} t + \varphi_o \right) \quad \text{and} \quad \langle y \rangle_{\varphi_o} \approx \frac{\lambda_c}{2} \cos \left( \omega_{zbw} t + \varphi_o \right),
\]

i.e., a circle of radius \( \lambda_c/2 \) with the frequency \( \omega_{zbw} \). However, the consequence of the ZBW is that it is impossible to localize electron better than to a certain finite volume, as its weight relative to the degree of localization of electron in space, given by \( I \) in definition [5], is proportional to \( \lambda_c/r_o \).

As the expectation value of the magnetic moment of a spin-up/down electron with charge \( e < 0 \), in the momentum representation operator form in the Gaussian unit, is

\[
\langle \mathbf{\mu} \rangle = \frac{e}{2c} \langle \mathbf{r} \times \dot{\mathbf{r}} \rangle = \frac{e}{2} \langle \mathbf{r} \times \mathbf{\alpha} \rangle \rightarrow \frac{i e \hbar}{2} \langle \mathbf{\nabla} \times \mathbf{\alpha} \rangle,
\]

straightforward calculations, using the wave packet [6] and the employed approximations, reveal [11][25]

\[
\langle \mu^1_1 \rangle = 0, \quad \langle \mu^1_2 \rangle = 0 \quad \text{and} \quad \langle \mu^2_1 \rangle = \frac{e \lambda_c}{2} \left[ 1 - \cos(\omega_{zbw} t) \right]
\]

for the spin-up states. Also in the same manner, while employing the relevant wave packet, one correspondingly achieves

\[
\langle \mu^1_1 \rangle = 0, \quad \langle \mu^1_2 \rangle = 0 \quad \text{and} \quad \langle \mu^2_3 \rangle = -\frac{e \lambda_c}{2} \left[ 1 - \cos(\omega_{zbw} t) \right]
\]

for the spin-down states, wherein the same initial condition as [11] but with a localized spin-down electron, namely

\[
\Psi(\mathbf{r}, 0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} f \left( \frac{\mathbf{r}}{r_o} \right),
\]

has been chosen to perform the rest of calculations. Note that, the above \( x \)- and \( y \)-components not only vanish on the average, but also their Fourier components vanish individually. Thus, it supports that each Fourier wave contributes a circular motion about the direction of the spin, wherein the \( z \)-component of [11] is the only non-vanishing component of the magnetic moment. Hence, one may interpret that the intrinsic magnetic moment is a result of the ZBW.

We should mention that the electron magnetic moment has also been derived in Ref. [50] via the role of ZBW for a spin-up electron, while using the Heisenberg picture, and has been arrived with the same result for the \( z \)-component as relation [12], but without the time-dependent part. However, by performing the calculations for the \( x \)- and \( y \)-components of the magnetic moment through the approach mentioned in Ref. [51], the results show that these components do not vanish. Incidentally, the time-dependant part of the magnetic moment, in relations [12] and [13], is a consequence of the fact that one has assumed that the spin of electron had initially been observed. This requirement leads to bring in the negative energy-part which does not vanish by letting the wave packet spread in space [11][25].

### 3 Effect of Space Noncommutativity on ZBW

The noncommutativity of physical quantities is one of the most important peculiarities of quantum mechanics, wherein the usual fundamental algebra among the momentum and position operators, namely the commutators \( [x_i, x_j] = 0, \ [p_i, p_j] = 0 \) and \( [x_i, p_j] = i \hbar \delta_{ij} \), are generalized in the case of NC phase-space as

\[
[x_i, x_j] = i \theta_{ij}, \quad [p_i, p_j] = i \eta_{ij} \quad \text{and} \quad [x_i, p_j] = i \hbar \left( \delta_{ij} + \frac{\theta_{ik} \eta_{jk}}{4 \hbar^2} \right).
\]

The real constant antisymmetric components \( \theta_{ij} \) and \( \eta_{ij} \) are the NC parameters of the space and momentum sectors, with dimensions of length squared and momentum squared, respectively. In terms of the Levi-Civita antisymmetric tensor, these parameters can be written as

\[
\theta_k = \frac{1}{2} \varepsilon_{ijk} \theta_{ij} \quad \text{and} \quad \eta_k = \frac{1}{2} \varepsilon_{ijk} \eta_{ij}.
\]
Actually, in classical physics, the product of arbitrary functions of noncommuting variables, through the Moyal star-product, reads\[^{38}\]

\[
(f \ast g)(\zeta) = \exp\left[\frac{i}{2} \alpha_{ab} \partial_a(\zeta) \partial_b(\zeta^\dagger) \right] f(\zeta) g(\zeta^\dagger) \bigg|_{\zeta = \zeta = \zeta^\dagger},
\]

where \(\zeta^a = (x^i, p^j)\), \(a, b = 1, 2, \cdots, 2n\) and \(2n\) is the dimension of phase-space. The real matrix \((\alpha_{ab})\) is a generalized symplectic structure and can be written as

\[
(\alpha_{ab}) = \begin{pmatrix}
\theta_{ij} & -\delta_{ij} + \theta_{k(i}\eta_{j)l} \frac{\delta_{kl}}{\eta_{ij}} \\
-\delta_{ij} + \theta_{k(i}\eta_{j)l} \frac{\delta_{kl}}{\eta_{ij}} & \delta_{ij} - \theta_{k(i}\eta_{j)l} \frac{\delta_{kl}}{\eta_{ij}}
\end{pmatrix}.
\]

Using the non-canonical linear transformation \((x, p) \mapsto (x', p')\), the NC algebra can be mapped into the commutative form where \([x_i', x_j'] = 0, [p_i', p_j'] = 0\) and \([x_i', p_j'] = i\hbar \delta_{ij}\). One example of such a map, corresponded to \((18)\), is

\[
x_i = \left(x'_i - \frac{\theta_{ji}}{2\hbar} p'_j\right) \quad \text{and} \quad p_i = \left(p'_i + \frac{\eta_{ij}}{2\hbar} x'_j\right),
\]

which is sometimes called Bopp-shift method\[^{54, 55}\]. Furthermore, when one shifts the canonical variables through \((19)\), the Hamiltonian of a system including the NC variables are usually assumed to have the same functional form as in the commutative one, i.e.

\[
H^{[\text{NC}]} \equiv H(x_i, p_j) = H \left( x'_i - \frac{\theta_{ji}}{2\hbar} p'_j, p'_i + \frac{\eta_{ij}}{2\hbar} x'_j \right).
\]

However this function is defined on the commutative space, but obviously, the effects of NC parameters now arise through their equations of motion. Incidentally, the real parameters \(\theta_k\) and \(\eta_k\) are usually assumed to be very small, and are considered up to the first-order\[^{56, 57}\].

Hence, in this section, and afterward in the following section, we just consider the limit wherein only first \(\theta_k\), and then \(\eta_k\), is taken to be non-zero, respectively. We also stick with the \(\theta_{0i} = 0\) case, for \(\theta_{0i} \neq 0\) renders the theory to be non-unitary\[^{58}\]. However, in the case of \(\eta_k = 0\) and in the field-free limit, but with \(\theta \neq 0\) (that we are going to consider in this section), its corresponding Dirac equation will be the same as Eq. (11). Thus, one may naively expect that the ZBW phenomenon is not affected by the noncommutativity. In fact, this is the case for results\(^7\) that do not change in this scenario. Although, in the NC quantum electrodynamics (QED), a phase factor still appears while calculating the Feynman amplitudes\[^{59} - 61\], however, in our case, even this phase vanishes identically due to the conservation of momentum. Now, let us move forward and perform the same calculations for the electron magnetic moment as in the previous section, but in the realm of the NC geometry for the case of this section, while considering the ZBW phenomenon.

However before we proceed, let us meanwhile remind that the effect of NC geometry on the magnetic moment of electron, without considering the ZBW and in the framework of the NC QED, has been found\[^{59} - 61\] to be null in the ordinary calculations\(^4\) while its (total) correction in the one-loop level, due to the vertex correction diagram, appears (in the Gaussian unit) as

\[
\left< \mu^{\uparrow \downarrow} \right>_{\text{tot}}^{[\text{NC}]}_{\eta = 0} = \frac{e \lambda_c}{2} \left[ 1 + \alpha_{\text{fsc}} \frac{2\pi}{\gamma_E} \right] \sigma^{\uparrow \downarrow} + \frac{m_c \alpha_{\text{fsc}} \gamma_E}{3\pi \gamma_c^2} \theta,
\]

where \((1 + \alpha_{\text{fsc}}/2\pi)\) is the gyration factor, \(\alpha_{\text{fsc}} = e^2/(\hbar c)\) is the dimensionless fine-structure constant and \(\gamma_E\) is the Euler constant. As it is obvious, the extra \(\theta\)-dependant term is a constant independent of electron kinematical states, i.e. the spin and momentum. Hence, as the energy difference of the two states is independent of \(\theta\), it has been suggested\[^{60}\] that this independent magnetic moment can be observed via a Stern-Gerlach apparatus, and not by a spin resonance experiment. Nevertheless, as this correction is spin-independent, it has been claimed\[^{31}\] that its effect is not easy to be observed. On the other hand, the NC contribution to the Zeeman-effect of the hydrogen atom in the first-order of perturbation theory has also been derived\[^{54}\] to be the energy-shift

\[
\Delta E_{\text{Zeeman}}^{[\text{NC}]}_{\eta = 0} = \frac{e \alpha_{\text{fsc}} \gamma_E}{6\pi \lambda_c} \left( 1 - f \frac{m_p}{m_e} \right) \theta \cdot B,
\]

where, as proton is not point-like, a form factor \(f\) has been used which is of the order of unity. Now, as the NC contribution does not depend on the direction of the spin, this energy-shift leads to a cumulative contribution\(^2\)

\[^2\]Nevertheless, due to the structure of proton as a non-elementary particle, it has been shown\[^{62}\] that the hydrogen atom spectrum receives tree-level correction due to noncommutativity.
from each atom. Hence, as the ratio $f m_p/m_e \gg 1$, its value has to be an enormous amount for large lumps of masses (like planets and stars), unless the value of $\theta$ is infinitesimally small\footnote{Indeed, the Lamb-shift data, the $e^+e^-$ scattering data and some other experiments have been used to impose some bounds on the value of the NC parameter $\theta$, see, e.g., Ref. \cite{31} and references therein.} that has led abandoning any hope to detect it. Fortunately, the result of this work brings some hope to remedy this issue, see the following.

Getting back to the main stream, by using map (19) into relation (14), the (total) magnetic moment in the realm of ZBW reads

$$<\mu>_{\text{tot}}^{[\text{NC}]}=\frac{e}{2} <\mathbf{r} \times \alpha > + \frac{e}{4\hbar} <\alpha \times (p \times \theta) >=<\mu>^{[\text{C}]} + <\mu>_{\text{tot}}^{[\text{NC}]}=0,$$

(23)

where the first-part is the commutative (usual) version that has been given in (12) and (13), and the NC part leads to

$$<\mu_3>^{[\text{NC}]}=\frac{e}{4\hbar} [\alpha_1(\theta_1 p_3 - \theta_3 p_1) + \alpha_2(\theta_2 p_3 - \theta_3 p_2)].$$

(24)

Using the wave packet (6), one gets

$$<\alpha_1 \theta_1 p_3> = 2K \theta_1 \int \left[p_1 p_3 - p_1 p_3 \cos(\omega_{\text{zbw}} t) + p_2 p_3 \sin(\omega_{\text{zbw}} t)\right] f^2(p/p_o) \, d^3 p,$$

(25)

$$<\alpha_1 \theta_3 p_1> = 2K \theta_3 \int \left[p_1^2 \left(1 - \cos(\omega_{\text{zbw}} t)\right) + p_1 p_3 \sin(\omega_{\text{zbw}} t)\right] f^2(p/p_o) \, d^3 p,$$

(26)

$$<\alpha_2 \theta_2 p_3> = 2K \theta_2 \int \left[p_2 p_3 - p_2 p_3 \cos(\omega_{\text{zbw}} t) - p_1 p_3 \sin(\omega_{\text{zbw}} t)\right] f^2(p/p_o) \, d^3 p,$$

(27)

and

$$<\alpha_2 \theta_3 p_2> = 2K \theta_3 \int \left[p_2^2 \left(1 - \cos(\omega_{\text{zbw}} t)\right) - p_2 p_3 \sin(\omega_{\text{zbw}} t)\right] f^2(p/p_o) \, d^3 p.$$  

(28)

Then, after taking the integrals and noticing that the terms like $\int d^3 p f^2(p/p_o) \, d^3 p = 0$ vanish, we finally achieve

$$<\mu_3>^{[\text{NC}]}= -\frac{e \alpha_{\text{fsc}}}{2 \lambda_c} \theta_3 [1 - \cos(\omega_{\text{zbw}} t)],$$

(29)

and hence

$$<\mu_3>^{[\text{NC}]}_{\text{tot}}= -\frac{e \lambda_c}{2} \left[1 - \left(\frac{\alpha_{\text{fsc}}}{\lambda_c}\right)^2 \theta_3 \left[1 - \cos(\omega_{\text{zbw}} t)\right]\right],$$

(30)

where, as a plausible approximation, we have considered $r_o$ to be the Bohr radius $a_{\text{bohr}} = \lambda_c/\alpha_{\text{fsc}}$, and the definitions of $p_o$ and $K$ have been used. We should remind that the parameter $\theta$ still has an insignificant value, and as it is obvious, in the limit $\theta_3 \to 0$, the NC part vanishes, as expected.

The most interesting point about (29) is that this result on the leading-order calculations has similar structure and numerically the same order to the result of one-loop calculations (i.e., the second-part of relation (21)) in Refs. \cite{59–61}, but with an opposite sign. Hence, in calculating the Zeeman-effect, this result suggests that the total contributions on this part may cancel out and thus, makes a remediation to the issue described above. However, we should emphasis that our results are obtained by taking into account both positive and negative energies, while the result in the second part of relation (21) has been derived just by considering positive energy.

Incidentally, by repeating the above calculations, but with the initial condition (14) and hence its relevant wave packet, one gets exactly the same result as relation (29), i.e., $<\mu_3>^{[\text{NC}]}= -\frac{e \alpha_{\text{fsc}}}{2 \lambda_c} \theta_3 [1 - \cos(\omega_{\text{zbw}} t)]$, or (32).

Up till now, we have only considered the $z$-part of the magnetic moment, and since we have been working with the initial wave packets with the spin in the $z$-direction, one may expect that the other components should vanish. However, after taking the noncommutativity effect into account and considering the relevant wave packet, we have found that this is not the case anymore, and the outcomes are

$$<\mu_{1,2}^{[\text{NC}]}= -\frac{e \alpha_{\text{fsc}}}{2 \lambda_c} \left\{\theta_{1,2} [1 - \cos(\omega_{\text{zbw}} t)] \pm \frac{1}{2} \theta_{2,1} \sin(\omega_{\text{zbw}} t)\right\},$$

(31)

and

$$<\mu_{1,2}^{[\text{NC}]}= -\frac{e \alpha_{\text{fsc}}}{2 \lambda_c} \left\{\theta_{1,2} [1 - \cos(\omega_{\text{zbw}} t)] \pm \frac{1}{2} \theta_{2,1} \sin(\omega_{\text{zbw}} t)\right\}.$$

(32)

Nevertheless, these results may pave the way for a new class of experiment in detecting the NC effect on the magnetic dipole moment. Besides, these results again are numerically in the same order of the one-loop effect (21), but with an opposite sign.
4 Effect of Momentum Noncommutativity on ZBW

In this section, we first consider the case of NC phase-space background in $3 + 1$ dimensions with $\eta_{ij} \neq 0$ while $\theta_{ij} = 0$ for an electron in a field-free region. In this respect, the relevant Dirac Hamiltonian has already been derived \[17\] to be

$$H^{[\text{NC}]}_{\theta = 0} = c\alpha \cdot p + \gamma_0 m_e c^2 + \frac{e}{2\hbar}(\alpha \times r) \cdot \eta,$$  \hspace{1cm} (33)

in the SI unit, wherein it is still gauge invariant. Now, analogous to the usual Dirac Hamiltonian for an electron in the presence of an external magnetic field, namely

$$H_B = c\alpha \cdot \pi_n + \gamma_0 m_e c^2,$$  \hspace{1cm} (34)

where $\pi_n = p - eA$, one can cast relation (33) into the form

$$H^{[\text{NC}]}_{\theta = 0} = c\alpha \cdot \left(p - e\frac{\eta \times r}{2e\hbar}\right) + \gamma_0 m_e c^2,$$  \hspace{1cm} (35)

with a corresponding generalized momentum

$$\pi_n \equiv \left(\frac{\eta \times r}{2e\hbar}\right).$$  \hspace{1cm} (36)

Also, as the vector potential due to a uniform magnetic field is $A_n = (B \times r)/2$, one gets

$$A_n = \frac{1}{2} \frac{\eta \times r}{e\hbar},$$  \hspace{1cm} (37)

as has been pointed out in, e.g., Ref. \[63\]. Such an effect plays the role of a NC vector potential for a corresponding similar external uniform magnetic field

$$B_n = \frac{\eta}{e\hbar},$$  \hspace{1cm} (38)

and in turn, if we define $B_{n \eta} = (\varepsilon_{ijk} B_{n_k})/2$, for $B_{n \eta} = \eta_k/(e\hbar)$. Such an implication resulted from $\eta_{ij}$ has been stressed in Ref. \[17\] and also in Ref. \[67\] in the framework of gravitomagnetism. Using the bound for the NC parameter, mentioned in Ref. \[17\], it yields a bound on the magnitude of the $B_{n \eta}$-field as $B_{n} \lesssim 8.6 \times 10^{-14}$ T.

Moreover, it has been indicated \[6, 64, 65\] that the energy-spectrum of (34), for an external magnetic field along the $z$-direction, is

$$E_n^2 = m_e^2 c^4 + p_z^2 c^2 + e^2 |e| \hbar B_3(n - l + 1) - 2c^2 eB_3 s_3,$$  \hspace{1cm} (39)

where $s_3 = \pm h/2$, $n$ is a non-negative integer and $L_3 = \hbar l$, while $l$ is restricted to values $l = -n, -n + 2, \cdots, n - 2, n$. Also, for electron, the relativistic form (39) can be rewritten as

$$E_n^2 = m_e^2 c^4 + p_z^2 c^2 + 2ke^2 |e| \hbar B_3,$$  \hspace{1cm} (40)

which is called the Landau energy-levels \[65\]. Thus, there exists a two-fold degeneracy in the solutions, i.e., spin-up with $n - l = 2(k - 1)$ and spin-down with $n - l = 2k$. However, the case $k = 0$ (i.e., the zero Landau-level) is a specific case, for $(n - l)$ cannot be negative. That is, in this specific case, for electron, only the spin-down solution exists, and in the same vein, for positron, only the spin-up solution is allowed. Hence, for the zero Landau-level, the degeneracy of the spectrum is lifted.

Analogously, in the case of NC geometry, without loss of generality, when only $B_{n \eta_3} \neq 0$, its corresponding energy-spectrum (like relation (39)) becomes

$$E_n^2 = m_e^2 c^4 + p_z^2 c^2 + e^2 |e| \hbar B_{n_3}(n - l + 1) - 2c^2 eB_{n_3} s_3,$$  \hspace{1cm} (41)

or also

$$E_n^2 = m_e^2 c^4 + p_z^2 c^2 + 2ke^2 |e| \hbar B_{n_3}.$$  \hspace{1cm} (42)

However, as the $B_{n \eta}$-field, in relation (38), depends on the electric charge, the energy-levels are independent of the sign of electric charge. More interestingly, in the NC region where only $\eta_{ij} \neq 0$ with no external field, solutions of the Dirac equation for an electron in the presence of an external magnetic field hold while the $B_{n \eta}$-field has been replaced instead of external magnetic field.
Now, we propose to apply the above result for the issue of an electron in external magnetic field in 2 + 1 dimensions, the domain that more attentions have been paid to in recent decades. This issue has already been considered both in commutative, e.g., Refs. [20, 67], and NC, e.g., Refs. [52, 66], contexts, where the ZBW effect in graphene has been studied in the former references.

To proceed and specify the characteristic feature of the ZBW in the momentum noncommutativity, we preferably employ the results of Refs. [20, 67], wherein the Hamiltonian for electrons and holes, at the K1 point for a monolayer graphene in the presence of an external magnetic field along the z-axis, has been given to be

\[ H = v_F (\sigma_1 \pi_1 + \sigma_2 \pi_2), \]

where \( v_F \approx 1 \times 10^6 \text{ cm/s} \) is the Fermi velocity, and hereinafter, we will replace \( \pi_\eta \) instead of \( \pi \). Then, without loss of generality, by choosing the NC vector potential as the Landau gauge \( A_{\eta\phi} = 0 = A_{\eta z} \) and \( A_{\eta x} = -yB_\eta \), the energy eigenvalues are \( E_{m\eta} = \hbar \Omega \sqrt{m} \), where angular frequency \( \Omega \equiv \sqrt{2} |eB_\eta|/\hbar v_F = \sqrt{2} \eta v_F/\hbar \), oscillator numbers \( m = 0, 1, \ldots \), and energy branch \( s = \pm 1 \) stands for the conduction and valence bands, respectively.

The average of velocity operator, \( \langle \dot{r}_i(t) \rangle \) for \( i = \{1, 2\} \), through an arbitrary two component function, say \( f \), is

\[ \langle \dot{r}_i(t) \rangle = \sum_{m',m} e^{iE_{m\eta}t/\hbar} \langle m'|\dot{r}_i(0)|m\rangle|f \rangle e^{-iE_{m\eta}t/\hbar}, \]

where from the Hamiltonian equation \( \dot{r}_i(0) = \partial H/\partial \pi_i \). Also, \( |m\rangle \) is eigenstate solution of Hamiltonian (43) that, in the form of harmonic oscillator function labeled by three quantum numbers, has been derived to be [20, 67]

\[ |m\rangle \equiv |mk_\pi, s\rangle = \frac{e^{ik_\pi x}}{\sqrt{4\pi}} \begin{pmatrix} -s|m - 1\rangle \\ |m\rangle \end{pmatrix}, \]

where \( |m\rangle \) is \( m \)-th state of the harmonic oscillator, hence the summation in relation (44) should go over all the quantum numbers, i.e. \( \sum_{m', m} \rightarrow \int dk_\pi' dk_x \sum_{m', m} \sum_{s', s} \).

To get a better and prompt insight on the nature of the ZBW in graphene, we choose the arbitrary function \( f(x, y) \) in the form of a circular Gaussian wave packet

\[ f(x, y) = \frac{1}{\sqrt{\pi} \ell} e^{-x^2 + y^2 - ik_\pi x} \begin{pmatrix} u \\ d \end{pmatrix}, \]

where \( p_\alpha = h k_\alpha \) is an initial nonvanishing momentum, \( \ell \) is width of the wave packet, and \( u^2 + d^2 = 1 \), however we assign \( u = d = 1/\sqrt{2} \). Hence, the average of velocity components become

\[ \langle \dot{r}_1(t) \rangle = v_F \sum_{m=0}^{\infty} \left[ \alpha_m^+ \cos(\omega_m^{(\text{cyc})} t) + \alpha_m^- \cos(\omega_m^{(\text{zbw})} t) \right], \]

\[ \langle \dot{r}_2(t) \rangle = v_F \sum_{m=0}^{\infty} \left[ \beta_m^+ \sin(\omega_m^{(\text{cyc})} t) + \beta_m^- \sin(\omega_m^{(\text{zbw})} t) \right], \]

where cyclotron frequency \( \omega_m^{(\text{cyc})} \equiv \Omega (\sqrt{m + 1} - \sqrt{m}) \), \( \omega_m^{(\text{zbw})} \equiv \Omega (\sqrt{m + 1} + \sqrt{m}) \), \( \alpha_m^\pm \equiv 2 (V_{m,m} \pm V_{m,m-1,m+1}), \)

\[ \beta_m^\pm \equiv -2 (V_{m,m-1} \pm V_{m,m-2}) \quad \text{with} \quad V_{m,m} = \int F_m(k_x) F_m'(k_x) dk_x \text{while } [67] \]

\[ F_m(k_x) = \frac{\ell \sqrt{L(L^2 - \ell^2)m}}{\sqrt{2m+1}(L^2 + \ell^2)m} e^{-\ell^2 (k_x - k_{mx})^2} e^{-\frac{k_x^2 + \ell^2}{2(L^2 + \ell^2)}} H_m(k_x g). \]

Here, \( H_m(k_x g) \) is the Hermit polynomials, the magnetic radius \( L = \sqrt{\hbar/(|eB_\eta|)} = \hbar/\sqrt{\pi} \) and \( g \equiv L^2/\sqrt{L^2 - \ell^2} \).

In addition, from relations [47] and [15], we easily get the expectation value of position as

\[ \langle r_1(t) \rangle = v_F \sum_{m=0}^{\infty} \left[ \frac{\alpha_m^+}{\omega_m^{(\text{cyc})}} \sin(\omega_m^{(\text{cyc})} t) + \frac{\alpha_m^-}{\omega_m^{(\text{zbw})}} \sin(\omega_m^{(\text{zbw})} t) \right], \]

\[ \langle r_2(t) \rangle = -v_F \sum_{m=0}^{\infty} \left[ \frac{\beta_m^+}{\omega_m^{(\text{cyc})}} \cos(\omega_m^{(\text{cyc})} t) + \frac{\beta_m^-}{\omega_m^{(\text{zbw})}} \cos(\omega_m^{(\text{zbw})} t) \right]. \]

Therefore, in comparison with the (usual) commutative scenario, relations [47], [48], [59] and [141] indicate that, due to momentum noncommutativity, ZBW oscillations are permanent with many frequencies while
accompanied with cyclotron frequencies as well. In the case of usual external magnetic field, to get a better realization in application, one can assume the magnetic radius \( L \) being equal to the width of wave packet \( \ell \), wherein the only surviving term in those relations will be the zero Landau-level. However, in the case of \( \textbf{B}_y \)-field, as \( L \gtrsim 8.7 \text{ cm} \), this presuppose diverts the issue away from the quantum realm.

Nevertheless, to realize the role of a specific Landau-level in the ZBW motion, we should confine ourselves to a specific value of \( m \), say \( m_0 \), thus in this case, the motion in the \( xy \)-plane is

\[
\langle r_1(t) \rangle_{m_0} = v_F \left[ \frac{\alpha_{m_0}^+}{\omega_{m_0}^{[\text{cyc}]} t} \sin(\omega_{m_0}^{[\text{zbw}]} t) + \frac{\alpha_{m_0}^-}{\omega_{m_0}^{[\text{zbw}]} t} \sin(\omega_{m_0}^{[\text{zbw}]} t) \right],
\]

\[
\langle r_2(t) \rangle_{m_0} = -v_F \left[ \frac{\beta_{m_0}^+}{\omega_{m_0}^{[\text{cyc}]} t} \cos(\omega_{m_0}^{[\text{zbw}]} t) + \frac{\beta_{m_0}^-}{\omega_{m_0}^{[\text{zbw}]} t} \cos(\omega_{m_0}^{[\text{zbw}]} t) \right].
\]

Still to extract a physical picture, let us concentrate on the zero Landau-level. In this particular case, \( \omega_0^{[\text{zbw}]} = \Omega = \omega_0^{[\text{cyc}]} \) wherein \( \Omega \lesssim 1.6 \times 10^7 \text{ s}^{-1} \), and

\[
\langle r_1(t) \rangle_0 = \frac{4v_F V_{0,0}}{\omega_0^{[\text{zbw}]}} \sin(\omega_0^{[\text{zbw}] t}),
\]

\[
\langle r_2(t) \rangle_0 = \frac{4v_F V_{0,1}}{\omega_0^{[\text{zbw}]}} \cos(\omega_0^{[\text{zbw}] t}),
\]

where

\[
V_{0,0} = \frac{L \ell^2}{2 \sqrt{(L^2 + \ell^2)(L^4 + L^2 \ell^2 + \ell^4)}} \exp \left[ \frac{\ell^2 k_{0z}^2 (\ell^2 - 1) (L^2 + \ell^2)}{L^4 + L^2 \ell^2 + \ell^4} \right]
\]

and

\[
V_{0,1} = \left[ L^3 \ell^2 k_{0z} \sqrt{2}/(L^4 + L^2 \ell^2 + \ell^4) \right] V_{0,0}.
\]

These results clearly indicate the rotational nature of the ZBW motion for the zero Landau-level in the presence of the \( \textbf{B}_y \)-field. However, considering the approximate values of \( L \) and \( \ell \), the amplitudes of (52) and (53) are roughly \( (L^2 \sqrt{2}/\hbar) \exp \left[ -L k_{0z}/\hbar \right] \) and \( (\ell^2 k_{0z}/\hbar^2) \times \exp \left[ -L k_{0z}/\hbar \right] \), respectively. Nevertheless, it is interesting to note that, if the ZBW part is absent, the amplitude of the motion will be halved, which gives an idea for a probable experimental check.

5 Conclusions

First, we reviewed the ZBW in the commutative (usual) spacetime. Then, we have considered the effects of the space and momentum noncommutativity separately on the ZBW phenomenon. In both cases, the states of positive and negative energy Dirac electrons have been used to derive the relevant wave packets, by which we have calculated the effects of the NC parameters on the ZBW. In the space noncommutativity scenario in \( 3 + 1 \) dimensions, it has been expressed that, due to the conservation of momentum, the Fourier decomposition of the expectation value of position does not change. However, we have represented that the electron magnetic moment receives correction in the same manner as in the commutative version and is proportional to the NC parameter, but spin-independent and with an insignificant value. Besides, this correction is due to the leading-order calculations, whereas, in the literature, it has been indicated that such a correction would appear only in the one-loop level. And more interestingly, we have found that this correction has similar structure and numerically the same order to the result of one-loop calculations, but with an opposite sign. Hence, when calculating the Zeeman-effect of an atom, the contributions of these two parts may cancel out each other and therefore, for large lumps of masses, the net result due to the interaction of the NC part of the magnetic moment with an external magnetic field in the Zeeman-effect remains undetectable. Moreover, we have shown that the \( x \)- and \( y \)-components of the electron magnetic dipole moment, contrary to the commutative case, are non-zero and proportional to the NC parameter. This result may pave the way for a new class of experiment in detecting the NC effect on the magnetic dipole moment. Besides, their corrections again are numerically in the same order of the one-loop effect, but with an opposite sign.

In the momentum noncommutativity scenario, as it has been expressed in the literature, this effect is equivalent to introducing an external uniform magnetic field into the Dirac equation. Hence, one expects that the solutions of the Dirac equation should change accordingly. Indeed, in this case, we have shown that the energy-spectrum and also the solutions of the Dirac equation are affected in both \( 3 + 1 \) and \( 2 + 1 \) dimensions. In this respect, we mostly focused on \( 2 + 1 \) dimensions for the specific case of graphene, where we have taken the solutions to figure out the effect of the NC parameter and its relevant field on the ZBW. We have indicated that the resulted NC field affects the motion of electrons in the zero Landau-level not only via a cyclotron
motion, that is usually expected from any external magnetic field, but also through the ZBW motion which doubles the amplitude of the rotation in the plane perpendicular to the direction of the field. In fact, this is a hallmark of the ZBW in graphene that provides a promising way to be tested experimentally, however the correction is proportional to the NC parameter with an insignificant value. Generally, in comparing with the (usual) commutative scenario, even though the structure of the corresponding NC field is the same as an external uniform magnetic field and the results show the rotational nature of the ZBW motion, but it treats with many frequencies of the ZBW while accompanied with many cyclotron frequencies as well.

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