On an \textit{a posteriori} error analysis of a mixed finite element
Galerkin approximations to a second order wave equation

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Abstract

In this article, \textit{a posteriori} error analysis is developed for mixed finite element
Galerkin approximations to a second order linear hyperbolic equation. Based on mixed
elliptic reconstructions and an integration tool, which is a variation of Baker’s technique
introduced earlier by G. Baker (SIAM J. Numer. Anal., 13 (1976), 564–576) in the con-
text of \textit{a priori} estimates for a second order wave equation, \textit{a posteriori} error estimates
of the displacement in $L^\infty(L^2)$-norm for the semidiscrete scheme are derived under min-
imal regularity. Finally, a first order implicit-in-time discrete scheme is analyzed and \textit{a}
\textit{posteriori} error estimators are established.

Key words. second order linear wave equation, mixed finite element methods, mixed
elliptic reconstructions, semidiscrete and first order implicit completely discrete scheme, \textit{a}
\textit{posteriori} error estimates.

1 Introduction

In this paper, we discuss \textit{a posteriori} error estimates for mixed finite element Galerkin
approximations to the following second order linear hyperbolic problem:

\begin{equation}
\begin{aligned}
\ddot{u} - \nabla \cdot (A\nabla u) &= f & \text{in } \Omega \times (0, T], \\
u_t|_{\partial \Omega} &= 0 & u|_{t=0} = u_0 \text{ and } u_t|_{t=0} = u_1.
\end{aligned}
\end{equation}

(1.1)

Here, $\Omega \subset \mathbb{R}^d$, $d \geq 1$ is a bounded polygonal domain with boundary $\partial \Omega$, $0 < T < \infty$,
$u_t = \partial u / \partial t$ and $A(x) = (a_{ij}(x))_{1 \leq i,j \leq d}$ is a symmetric and uniformly positive definite matrix.
All the coefficients $a_{ij}$ are smooth functions of $x$ with uniformly bounded derivatives in $\bar{\Omega}$.
Moreover, the initial functions $u_0 = u_0(x)$, $u_1 = u_1(x)$ and the forcing function $f = f(x,t)$
are assumed to be smooth functions in their respective domains.

In recent years, there has been a growing demand for designing reliable and efficient
space-time algorithms for numerical computations of time dependent partial differential
equations. Most of these algorithms are based on \textit{a posteriori} error estimators which provide
appropriate tools for adaptive mesh refinements. For stationary boundary value problems,
\textit{a posteriori} error bounds are well developed (see, \cite{3, 31}). Adaptivity and \textit{a posteriori} error
control for parabolic problems has also been an active research area for the last two decades

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(cf. [18] [22] [24] [20] [8] [9] [5] and references, therein). For the time discretization, results are available in the literature on a posteriori error estimations for abstract first order evolution problems (cf. [4] [20] [25] [27] [29]).

In the context of second order wave equations, only few results are available on a posteriori error analysis, see [23] [1] [14] [13] [7] [30]). Further, it is observed that the design and implementation of adaptive algorithms for these equations based on rigorous a posteriori error estimators are less complete compared to elliptic and parabolic equations. In [23] a priori and a posteriori error estimates for a finite element method for linear second order wave equations are proved based on a space-time finite element discretization with the basis functions being continuous in space and discontinuous in time. A posteriori bounds for standard implicit time-stepping finite element approximations are proposed and analyzed in [1] [2], but only in some specific situations. The recent results in [7] [19] cover only first order time discrete schemes. In [7], the second order wave equation is written as a first order system and a first order implicit backward Euler scheme in time is used with continuous piecewise affine finite elements in space. Further, rigorous a posteriori bounds have been derived using energy arguments and adaptive algorithms based on the a posteriori bounds are discussed. In [19], based on Baker’s technique a posteriori bounds in the $L^\infty(L^2)$-norm are derived for both semidiscrete and first order implicit-in-time fully discrete schemes. The fully discrete analysis relies crucially on a novel time reconstruction satisfying a local vanishing-moment property, and on a space reconstruction technique used earlier in [27] for parabolic problems. In [14], an adaptive algorithm in space and time is presented based on Galerkin space-time discretizations leading to Newmark scheme. Further, goal oriented a posteriori error estimates are derived and some numerical results are provided to demonstrate the efficiency of the error estimators. In [30], the author has studied an anisotropic a posteriori error estimate for a finite element discretization of the two dimensional wave equation. The estimate is derived in the $L^2(0, T, H^1(\Omega))$-norm and it turns out to be sharp on anisotropic meshes, whenever the spatial discretization error is predominant.

For high order time reconstruction for abstract second order evolution equations, one can refer to the recent papers [22] [21]. In [22], authors have developed adaptive time stepping Galerkin methods for second order evolution problems in terms of a posteriori error analysis. Based on the energy approach and the duality argument, optimal order a posteriori error estimates and a posteriori nodal superconvergence error estimates have been derived. An adaptive time stepping strategy is discussed and some numerical experiments are conducted to assess the effectiveness of the proposed adaptive time stepping methods. In the recent work [21], authors have discussed second order explicit and implicit two-step time discretization schemes such as leap-frog and cosine methods and have derived a posteriori estimates using a novel time reconstruction. Further, they presented some numerical experiments to confirm their theoretical findings.

For space-time adaptivity, the finite element discretization depends on space-time variational formulation and the error indicators include both space and time errors. Recently, attempts have been made to exploit elliptic reconstruction to prove optimal a posteriori error estimates in finite element methods for parabolic problems [27]. In fact, the role of the elliptic reconstruction operator in a posteriori estimates is quite similar to the role played by elliptic projection introduced earlier by Wheeler [33] for recovering optimal a priori error estimates of finite element Galerkin approximations to parabolic problems. This analysis is further developed for completely discrete scheme based on backward Euler method [25], for maximum norm estimates [17] and for discontinuous Galerkin methods for parabolic problems [20]. In recent works [28] and [26], the analysis is further extended to mixed FE
Galerkin methods applied to parabolic problems.

In this article, an a posteriori analysis is discussed for mixed finite element Galerkin approximations to a class of linear second order hyperbolic problems. In the first part of this article, a semidiscrete scheme is derived using finite element method in spatial direction while keeping time variable constant. Based on mixed elliptic reconstructions presented in [28], which depend explicitly on residuals and a time integration tool, which is a variant of Backer’s technique, a posteriori error estimates in $L^\infty(L^2)$-norm are derived for the displacement $u$ under minimal regularity. For the time discretization, the time discrete scheme with the time reconstruction proposed in [19] is applied and then using summation tool, a posteriori error estimators in $L^\infty(L^2)$-norm are developed. Compared to [19], our analysis is not only for mixed finite element method, but also it differs from the analysis in [19] in the sense that a time integration tool is used for deriving $L^\infty(L^2)$ a posteriori estimators, as against the time testing procedure of Baker [6] used in [19].

The outline of this article is as follows. Section 2 deals with mixed elliptic reconstruction techniques proposed in [28] and a posteriori estimates for the semidiscrete problem for both displacement $u$ and its velocity $\sigma$ in $L^\infty(L^2)$-norms. Section 3 focuses on a completely discrete scheme, which is based on a first order backward differencing implicit method and related a posteriori error estimators. Finally in the concluding Section 4, results are summarized with a brief outline on future work.

2 A posteriori error estimates for semidiscrete mixed method

In this section, a mixed formulation for the hyperbolic problem (1.1) is considered and a posteriori error estimates are derived for the semidiscrete mixed Galerkin approximation to (1.1). We use the usual notations for the $L^2$, $H^1_0$ and $H^2$ spaces and their norms and semi-norms.

Let

$$H(div, \Omega) = \{ \phi \in (L^2(\Omega))^d : \nabla \cdot \phi \in L^2(\Omega) \}$$

be a Hilbert space equipped with norm $\| \phi \|_V = (\| \phi \|^2 + \| \nabla \cdot \phi \|^2)^{\frac{1}{2}}$.

For a mixed formulation, introduce

$$\sigma = -A \nabla u,$$

and set $\alpha = A^{-1}$. Then, the equation (1.1) is rewritten as

$$\alpha \sigma + \nabla u = 0, \quad u_{tt} + \nabla \cdot \sigma = f, \quad u|_{\partial \Omega} = 0.$$  

With $W = L^2(\Omega)$ and $V = H(div, \Omega)$, a weak mixed formulation for (1.1) is to find $(u, \sigma) : (0, T] \rightarrow W \times V$ with $u(0) = u_0 \in W$ such that

$$\alpha \sigma, v - (u, \nabla \cdot v) = 0 \quad \forall \ v \in V,$$

$$u_{tt}, w + (\nabla \cdot \sigma, w) = (f, w) \quad \forall \ w \in W.$$  

Since $A$ is uniformly positive definite, there are two positive constants $a_0$ and $a_1$ such that

$$a_0 \| \sigma \| \leq \| \sigma \|_{A^{-1}} \leq a_1 \| \sigma \|, \quad \text{where} \quad \| \sigma \|_{A^{-1}} := (\alpha \sigma, \sigma).$$

For the semi-discrete mixed formulation corresponding to (2.3)-(2.4), let $T_h = \{ K \}$ be a shape-regular partition of the domain $\Omega$ into triangles of diameter $h_K = \text{diam}(K)$. To each
triangulation $T_h$, we now associate a positive piecewise constant function $h(x)$ defined on $\Omega$ by $h|_K = h_K \forall K \in T_h$. Further, let $V_h$ and $W_h$ be appropriate finite element subspaces of $V$ and $W$. For more examples of these spaces including Raviart-Thomas-Néédélec finite element spaces, Brezzi- Douglas-Marini spaces and Brezzi-Douglas-Fortin-Marini spaces, see [11].

The corresponding semidiscrete mixed finite element formulation is to seek a pair $(u_h, \sigma_h): [0,T] \to W_h \times V_h$ such that

\begin{align*}
(\alpha \sigma_h, v_h) - (u_h, \nabla \cdot v_h) &= 0 \quad \forall v_h \in V_h, \\
(u_{h,tt}, w_h) + (\nabla \cdot \sigma_h, w_h) &= (f, w_h) \quad \forall w_h \in W_h
\end{align*}

with $u_h(0) \in W_h$ and $u_{h,t}(0) \in W_h$ to be defined later.

Set $e_u = u_h - u$ and $e_\sigma = \sigma_h - \sigma$. From (2.3)-(2.4) and (2.6)-(2.7), $e_u$ and $e_\sigma$ satisfy the following equations

\begin{align*}
(\alpha e_\sigma, v) - (e_u, \nabla \cdot v) &= r_1(v) \quad \forall v \in V, \\
(e_{u,tt}, w) + (\nabla \cdot e_\sigma, w) &= r_2(w) \quad \forall w \in W,
\end{align*}

where the residuals $r_1$ and $r_2$ are given by

\begin{align*}
r_1(v) &:= (\alpha \sigma_h, v) - (u_h, \nabla \cdot v), \\
r_2(w) &:= (u_{h,tt}, w) + (\nabla \cdot \sigma_h, w) - (f, w).
\end{align*}

Following [28], now introduce mixed elliptic reconstructions $\tilde{u}(t) \in H_0^1(\Omega)$ and $\tilde{\sigma}(t) \in V$ of $u_h(t)$ and $\sigma_h(t)$ for $t \in (0,T)$, respectively, as follows: for given $u_h$ and $\sigma_h$, let the mixed elliptic reconstructions $\tilde{u}$ and $\tilde{\sigma}$ satisfy

\begin{align*}
(\nabla \cdot (\tilde{\sigma} - \sigma_h), w) &= -r_2(w), \quad \forall w \in W, \\
(\alpha (\tilde{\sigma} - \sigma_h), v) - (\tilde{u} - u_h, \nabla \cdot v) &= -r_1(v), \quad \forall v \in V.
\end{align*}

Using Theorem 4.3 (page 132) of [10], one can verify that for a given $u_h, \sigma_h, r_1$ and $r_2$, the system (2.10)-(2.11) has a unique pair of solution $\{\tilde{u}(t), \tilde{\sigma}(t)\} \in W \times V$, for $t \in (0,T)$.

Note that $r_1(v_h) = 0 \quad \forall v_h \in V_h$, and $r_2(w_h) = 0 \quad \forall w_h \in W_h$. Then, $\sigma_h$ and $u_h$ are indeed mixed elliptic projections of $\tilde{\sigma}$ and $\tilde{u}$, respectively.

Using mixed elliptic reconstructions, we now rewrite

\begin{align*}
e_u &= (\tilde{u} - u) - (\tilde{u} - u_h) =: \xi_u - \eta_u, \\
e_\sigma &= (\tilde{\sigma} - \sigma) - (\tilde{\sigma} - \sigma_h) =: \xi_\sigma - \eta_\sigma.
\end{align*}

Using (2.10)-(2.11) in (2.8)-(2.9), it follows that

\begin{align*}
(\alpha \xi_\sigma, v) - (\xi_u, \nabla \cdot v) &= 0 \quad \forall v \in V, \\
(\xi_{u,tt}, w) + (\nabla \cdot \xi_\sigma, w) &= (\eta_{u,tt}, w) \quad \forall w \in W.
\end{align*}

With mixed elliptic reconstructions $\tilde{u}$ and $\tilde{\sigma}$ satisfying (2.10)-(2.11), apply (2.12) to check that

\begin{align*}
\alpha \tilde{\sigma} &= -\nabla \tilde{u}.
\end{align*}
Lemma 2.1. Let $\xi_u$ and $\xi_\sigma$ satisfy (2.12)–(2.13). Then, the following estimates hold:

(2.15) \[ \|\xi_{u,t}(t)\| + \|\alpha^{1/2}\xi_\sigma(t)\| \leq \|\xi_{u,t}(0)\| + \|\alpha^{1/2}\xi_\sigma(0)\| + 2 \int_0^t \|\eta_{u,tt}(s)\| \, ds. \]

and

(2.16) \[ \|\xi_u(t)\| \leq \|\xi_u(0)\| + 2 \int_0^t \|\eta_{u,t}(s)\| \, ds. \]

Proof. Differentiate (2.12) with respect to $t$ and set $\mathbf{v} = \xi_\sigma$ in the resulting equation to find that

(2.17) \[ (\alpha\xi_{\sigma,t}, \xi_\sigma) - (\xi_{u,t}, \nabla \cdot \xi_\sigma) = 0. \]

Choose $w = \xi_{u,t}$ in (2.13). Then, add the resulting equations to (2.17) to arrive at

(2.18) \[ \frac{1}{2} \frac{d}{dt}(\|\xi_{u,t}\|^2 + \|\alpha^{1/2}\xi_\sigma\|^2) = (\eta_{u,tt}, \xi_{u,t}). \]

On integrating (2.18) from 0 to $t$, a use of the Cauchy-Schwarz inequality yields

(2.19) \[ \|\xi_{u,t}(t)\|^2 + \|\alpha^{1/2}\xi_\sigma(t)\|^2 \leq \|\xi_{u,t}(0)\|^2 + \|\alpha^{1/2}\xi_\sigma(0)\|^2 + 2 \int_0^t \|\eta_{u,tt}(s)\| \|\xi_{u,t}(s)\| \, ds. \]

Setting $\|(\xi_{u,t}, \xi_\sigma)(t)\| = (\|\alpha^{1/2}\xi_\sigma(t)\|^2 + \|\xi_{u,t}(t)\|^2)^{1/2}$, let $t^* \in [0, t]$ be such that

$\|(\xi_{u,t}, \xi_\sigma)(t^*)\| = \max_{0 \leq s \leq t} \|(\xi_{u,t}, \xi_\sigma)(s)\|.$

Then at time $t = t^*$, equation (2.19) becomes

(2.20) \[ \|(\xi_{u,t}, \xi_\sigma)(t^*)\| \leq \|(\xi_{u,t}, \xi_\sigma)(0)\| + 2 \int_0^{t^*} \|\eta_{u,tt}(s)\| \, ds, \]

and hence,

(2.21) \[ \|(\xi_{u,t}, \xi_\sigma)(t)\| \leq \|(\xi_{u,t}, \xi_\sigma)(0)\| + 2 \int_0^t \|\eta_{u,tt}(s)\| \, ds. \]

This completes the proof of (2.15). Note that from (2.21), one obtains $L^\infty(L^2)$-estimate of the displacement using $\xi(t) = \xi(0) + \int_0^t \xi_{u,t}(s) \, ds$. Now in order to reduce the regularity, an integration tool which is a variant of Baker’s time testing procedure is used in a crucial way. To motivate our tool, integrate (2.13) with respect to time to arrive at

(2.22) \[ (\xi_{u,t}, w) + (\nabla \cdot \dot{\xi}_\sigma, w) = (\xi_{u,t}(0), w) + (\eta_{u,t}, w) - (\eta_{u,t}(0), w), \]

where $\dot{\xi}_\sigma = \int_0^t \xi_\sigma(s) \, ds$. Choose $w = \xi_u$ in (2.22) and $\mathbf{v} = \dot{\xi}_\sigma$ in (2.12) and adding the resulting equations to obtain

(2.23) \[ \frac{1}{2} \frac{d}{dt}(\|\xi_{u}(t)\|^2 + \|\alpha^{1/2}\dot{\xi}_\sigma(t)\|^2) = (e_{u,t}(0), \xi_u) + (\eta_{u,t}, \xi_u). \]

Then, integrate with respect to time and use kick back arguments to arrive at

(2.24) \[ \|\xi_u(t)\| \leq \|\xi_u(0)\| + 2 \int_0^t \|\eta_{u,t}(s)\| \, ds. \]

This completes the proof of (2.16).
Assume that there exists a linear operator $\Pi_h : V \rightarrow V_h$ such that $\nabla \cdot \Pi_h = P_h(\nabla \cdot )$, where $P_h : W \rightarrow W_h$ is the $L^2$-projection defined by

$$ (\phi - P_h \phi, w_h) = 0 \quad \forall \ w_h \in W_h, \ \phi \in W. $$

Further, we assume that the finite element spaces satisfy the following properties:

$$ \| v - \Pi_h v \| \leq C \| v \|_{r}, \quad 1 \leq r \leq \ell + 1, \quad \| w - P_h w \| \leq C \| h \| \| w \|_{r}, \quad 0 \leq r \leq \ell + 1. $$

Note that for $v \in H(div, \Omega)$ and $w \in L^2(\Omega)$, the following properties hold true:

(2.25) $(\nabla \cdot (v - \Pi_h v), w_h) = 0, \quad w_h \in W_h; \quad (w - P_h w, \nabla \cdot v_h) = 0, \quad v_h \in V_h.$

Examples of spaces satisfying the above can be found in [11].

To prove the main theorem, we need the following a posteriori estimates of $\eta_u, \eta_{u,t}$ and $\eta_\sigma$ related to the mixed elliptic reconstructions (2.10)-(2.11). For a proof, see [15].

**Lemma 2.2.** For Raviart-Thomas-Néédlec elements, there exists a positive constant $C$ which depends only on the coefficient matrix $A$, the domain $\Omega$, the shape regularity of the elements and polynomial degree $\ell$ such that for $\ell = 0, 1$,

(2.26) $\| \eta_u \| \leq C \left( \| h^{\ell+1} r_2 \| + \min_{w_h \in W_h} \| h(\alpha \sigma_h - \nabla h w_h) \| \right),$

and for $j = 1, 2$,

(2.27) $\left\| \frac{\partial j \eta_u}{\partial \nu} \right\| \leq C \left( \| h^{\ell+1} \frac{\partial j \sigma}{\partial \nu} \| + \min_{w_h \in W_h} \| h \left( \alpha \frac{\partial j \sigma}{\partial \nu} - \nabla h w_h \right) \| \right),$

and

(2.28) $\| \alpha^{1/2} \eta_\sigma \| \leq C \left( \| h r_2 \| + \| h^{1/2} J(\alpha \sigma_h \cdot t) \|_{0,T} + \| \nabla h \|_{0,T} \right),$ where $r_2 = (u_{h,t} - f + \nabla \cdot \sigma_h)$ is a residual and $J(\alpha \sigma_h \cdot t)$ denotes the jump of $\alpha \sigma_h \cdot t$ across element edge $E$ with $t$ being the tangential unit vector along the edge $E \in \Gamma_h$.

Now, let $E_1(r_2, \sigma_h; T_h)$, $E_1(\frac{\partial r_2}{\partial \nu}, \frac{\partial \sigma_h}{\partial \nu}; T_h)$ and $E_2(r_2, \sigma_h; T_h)$ denote the terms on the right-hand sides of (2.26), (2.27) and (2.28), respectively. Then, using Lemmas 2.1-2.2 we finally obtain the main theorem of this section as:

**Theorem 2.1.** Let $(u, \sigma)$ be a solution of the mixed formulation (2.3)-(2.4) and let $(u_h, \sigma_h)$ be a solution of the semidiscrete mixed formulation (2.6)-(2.7). Then the following a posteriori estimates hold for $\ell = 0, 1$:

$$ \| e_{u,t} \|_{L^\infty(0,T; L^2(\Omega))} + \| \alpha^{1/2} e_\sigma \|_{L^\infty(0,T; L^2(\Omega))} \lesssim \| e_{u,t}(0) \| + \| \alpha^{1/2} e_\sigma(0) \| + E_1(r_2(t), \sigma_h(t); T_h) + E_2(r_2(0), \sigma_h(0); T_h) + \| E_1(\sigma_h(t); T_h) \|_{L^\infty(0,T)} \leq \int_0^T E_1(r_2(t), \sigma_h(t); T_h) \ ds, $$

and

$$ \| e_{u} \|_{L^\infty(0,T; L^2(\Omega))} \lesssim \| e_{u}(0) \| + E_1(r_2(0), \sigma_h(0); T_h) + \| E_1(\sigma_h(t); T_h) \|_{L^\infty(0,T)} + \int_0^T E_1(r_2(t), \sigma_h(t); T_h) \ ds. $$
3 Completely discrete scheme

In this section, we discuss a posteriori analysis for a completely discrete mixed approximation based on backward differencing.

Let \(0 = t_0 < t_1 < \ldots < t_n = T, T_n = (t_{n-1}, t_n]\) and \(k_n = t_n - t_{n-1}\). For \(n \in [0 : N]\), let \(\mathcal{T}_n\) be a refinement of a macro-triangulation which is a triangulation of the domain \(\Omega\) that satisfies the same conformity and shape regularity assumptions made on its refinements.

Let

\[
h_n(x) := \text{diam}(K), \quad \text{where } K \in \mathcal{T}_n \text{ and } x \in K,
\]

for all \(x \in \Omega\). Given two compatible triangulations \(\mathcal{T}_{n-1}\) and \(\mathcal{T}_n\), i.e., they are refinements of the same macro-triangulation, let \(\hat{T}_n\) be the finest common coarsening of \(\mathcal{T}_n\) and \(\mathcal{T}_{n-1}\), whose meshsize is given by \(\hat{h}_n := \max(h_n, h_{n-1})\), see (25), pp. 1655).

We consider \(V^n_h\) and \(W^n_h\) defined over the triangulations \(T^n\) as Raviart-Thomas finite element spaces of index \(\ell \geq 0\) of \(H(div, \Omega)\) and \(L^2(\Omega)\), respectively. Let \(P^n_h : L^2(\Omega) \rightarrow W^n_h\) be the \(L^2\)-projection defined by

\[
(P^n_h w, \phi^n) = (w, \phi^n) \quad \forall \phi^n \in W^n_h.
\]

Given \(U^0 = P^0_h u_0\), find \(\{(U^n, \Sigma^n)\}\), with \((U^n, \Sigma^n) \in W^n_h \times V^n_h\) for \(n \in [1 : N]\) such that

\[
(3.1) \quad (\partial^2_t U^n, w) + (\nabla \cdot \Sigma^n, w) = (f^n, w) \quad \forall w \in W^n_h,
\]

\[
(3.2) \quad (\alpha \Sigma^n, v) - (U^n, \nabla \cdot v) = 0 \quad \forall v \in V^n_h,
\]

where the backward second and first finite differences

\[
(3.3) \quad \partial^2_t U^n = \frac{\partial_t U^n - \partial_t U^{n-1}}{k_n},
\]

with

\[
\partial_t U^n := \begin{cases} 
\frac{U^n - U^{n-1}}{k_n}, & \text{for } n = 1, \ldots, N, \\
P^0_h u_1, & \text{for } n = 0.
\end{cases}
\]

Throughout the rest of the paper, we shall use the following notation:

\[
P^n_h(\partial_t \phi^n) = \frac{1}{k_n}(\phi^n - P^n_h \phi^{n-1}).
\]

Given a sequence of discrete values \(\{V^n\}_{n=0}^N\), define the time reconstruction \(V : [0, T] \times \Omega \rightarrow IR\) or \(IR^d\) as

\[
(3.4) \quad V(t) = V^n + (t - t_n)\partial_t V^n - \frac{(t - t_{n-1})(t_n - t)^2}{k_n} \partial^2_t V^n, \quad t_{n-1} < t \leq t_n,
\]

for \(n = 1, \ldots, N\). Note that we have used the fact that \(\partial_t V^0\) is well defined.

We shall use the above \(C^1\)-function \(V(t)\) such that for \(n = 0, 1, \ldots, N\),

\[
(3.5) \quad V(t_n) = V^n, \quad V_t(t_n) = \partial_t V^n, \quad V_{tt}(t) = (1 + \mu^n) \partial^2_t V^n,
\]

for \(t \in (t_{n-1}, t_n]\), where

\[
\mu^n(t) := -6k_n^{-1}(t - t_{n-1/2}).
\]
Similarly, we define $C^1$-functions $U(t)$ and $\Sigma(t)$ in time variable using the discrete sequences $\{U^n\}_{n=0}^N$ and $\{\Sigma^n\}_{n=0}^N$, respectively.

As in Section 2, for given $\{U^n, \Sigma^n\}_{n=0}^N$, we now define the mixed elliptic reconstructions $\tilde{u}^n \in H_0^1(\Omega)$ and $\tilde{\sigma}^n \in \mathbf{V}$ at $t = t_n$ as:

\begin{align}
(3.6) \quad & \nabla : (\tilde{\sigma}^n - \Sigma^n), w = -r_2^n(w), \quad w \in W, \\
(3.7) \quad & (\alpha(\tilde{\sigma}^n - \Sigma^n), v) - (\tilde{u}^n - U^n, \nabla \cdot v) = -r_1^n(v), \quad v \in \mathbf{V},
\end{align}

where $r_1^n(v) := (\alpha \Sigma^n, v) - (U^n, \nabla \cdot v)$ and $r_2^n(w) := (P^n_h(\partial^2_t U^n), w) + (\nabla \cdot \Sigma^n, w) - (f^n, w)$.

Since $r_1^n(v_h) = 0 \ \forall v_h \in \mathbf{V}_h^n, n \geq 0$ and $r_2^n(w_h) = 0 \ \forall w_h \in W_h^n, n \geq 1$, in fact, $\Sigma^n$ and $U^n$ are mixed elliptic projection of $\tilde{\sigma}^n$ and $\tilde{u}^n$ at time $t = t_n$, respectively. Now given $\{\tilde{u}^n\}_{n=0}^N$ and $\{\tilde{\sigma}^n\}_{n=0}^N$, we define the $C^1$-functions $\tilde{u}(t)$ and $\tilde{\sigma}(t)$ in time $t \in (0, T]$, respectively, as

\begin{align}
(3.8) \quad & \tilde{u}(t) = \tilde{u}^n + (t - t_n) \partial_t \tilde{u}^n - \frac{(t - t_{n-1})(t_n - t)^2}{k_n} \partial^2_t \tilde{u}^n, \quad t_{n-1} < t \leq t_n, \\
(3.9) \quad & \tilde{\sigma}(t) = \tilde{\sigma}^n + (t - t_n) \partial_t \tilde{\sigma}^n - \frac{(t - t_{n-1})(t_n - t)^2}{k_n} \partial^2_t \tilde{\sigma}^n, \quad t_{n-1} < t \leq t_n,
\end{align}

provided that $\partial_t \tilde{u}^0$ and $\partial_t \tilde{\sigma}^0$ are well defined.

For $t \in (0, T]$, the mixed elliptic reconstruction $\{\tilde{u}, \tilde{\sigma}\}$ satisfies

\begin{align}
(3.10) \quad & \nabla : (\tilde{\sigma} - \Sigma), w = -r_2(w), \quad w \in W, \\
(3.11) \quad & (\alpha(\tilde{\sigma} - \Sigma), v) - (\tilde{u} - U, \nabla \cdot v) = -r_1(v), \quad v \in \mathbf{V},
\end{align}

where $r_1$ and $r_2$ are defined as $C^1$-functions in time using $\{r_1^n, r_2^n\}_{n=1}^N$ as in (3.4).

Again, set

\begin{align}
(3.12) \quad & e_u = (\tilde{u} - u) - (\tilde{u} - U) =: \xi_u - \eta_u, \\
(3.13) \quad & e_\sigma = (\tilde{\sigma} - \sigma) - (\tilde{\sigma} - \Sigma) =: \xi_\sigma - \eta_\sigma.
\end{align}

Now, the pair $\{e_u, e_\sigma\}$ satisfies

\begin{align}
(3.14) \quad & (e_u, \nabla \nabla u) + (\nabla \cdot e_\sigma, w) = (U_{tt}, w) + (\nabla \cdot \Sigma, w) - (f, w).
\end{align}

On splitting $e_u$ and $e_\sigma$, we obtain from (3.14),

\begin{align}
(3.15) \quad & (\xi_u, tt, w) + (\nabla \cdot \xi_\sigma, w) = (\eta_u, tt, w) + ((I - P^h_n)U_{tt}, w) + \mu^n(t)(\partial^2_t U^n, P^n_h w) + (\nabla \cdot (\tilde{\sigma} - \Sigma), w) + (f^n - f, w) \ \forall w \in W.
\end{align}

Similarly, we also arrive at

\begin{align}
(3.16) \quad & (\alpha\xi_\sigma, v) - (\xi_u, \nabla \cdot v) = (\alpha(\tilde{\sigma} - \Sigma), v) - (\tilde{u} - \tilde{u}^n, \nabla \cdot v) \ \forall v \in \mathbf{V}.
\end{align}

Note that

\begin{align}
(3.17) \quad & \tilde{\sigma} - \tilde{\sigma}^n = (t - t_n) \partial_t \tilde{\sigma}^n + (k_n^{-1}(t_n - t)^2 - (t_n - t)^2) \partial^2_t \tilde{\sigma}^n,
\end{align}
and
\[(3.18) \quad \ddot{u} - \dddot{u}^n = (t - t_n)\partial_t \dddot{u}^n + (k_{n-1}(t_n - t)^3 - (t_n - t)^2) \partial_t^2 \dddot{u}^n.\]

Now,
\[(3.19) \quad (\alpha(\ddot{\sigma} - \ddot{\sigma}^n), v) - (\ddot{u} - \dddot{u}^n, \nabla \cdot v) = (t - t_n) \left\{ (\alpha \partial_t \dddot{\sigma}^n, v) - (\partial_t \dddot{u}^n, \nabla \cdot v) \right\}
+ (k_{n-1}(t_n - t) - (t_n - t)^2) \left\{ (\alpha \partial_t^2 \dddot{\sigma}^n, v) - (\partial_t^2 \dddot{u}^n, \nabla \cdot v) \right\},\]

and from (3.6) with definition of \(r_1(v)\), we find that
\[(3.20) \quad (\alpha \dddot{\sigma}^n, v) - (\ddot{u}^n, \nabla \cdot v) = 0 \quad \forall v \in V.\]

From (3.20), equation (3.19) takes the form
\[(3.21) \quad (\alpha(\ddot{\sigma} - \ddot{\sigma}^n), v) - (\ddot{u} - \dddot{u}^n, \nabla \cdot v) = 0,\]

and thus, equation (3.16) becomes
\[(3.22) \quad (\alpha \xi, v) - (\xi_u, \nabla \cdot v) = 0 \quad \forall v \in V.\]

**Theorem 3.1.** Let \((u, \sigma)\) be the solution of (2.3)-(2.4) and \((U, \Sigma)\) be the solution of (3.1)-(3.2). Then for \(t \in (t_{n-1}, t_n]\), the following estimates hold:
\[
\|\xi_{u,t}(t)\| + \|\alpha^{1/2} \xi_{\sigma}(t)\| \leq \|\xi_{u,t}(0)\| + \|\alpha^{1/2} \xi_{\sigma}(0)\| + 2 \sum_{j=1}^{4} \mathcal{E}_{1,j}(t) + 2 \int_0^t \|\eta_{u,t}(s)\| \, ds,
\]

where
\[
\mathcal{E}_{1,1}(t) = \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} \left| (I - P_{h}^j)U_{tt} \right| + \int_{t_{n-1}}^{t} \left| (I - P_{h}^n)U_{tt} \right|,
\]
\[
\mathcal{E}_{1,2}(t) = \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} \left| \mu^j \partial_t^2 U_{tt} \right| + \int_{t_{n-1}}^{t} \left| \mu^n \partial_t^2 U_{tt} \right|,
\]
\[
\mathcal{E}_{1,3}(t) = \sum_{j=1}^{n-1} \left( \frac{k_{n-1}^2}{2} \| \partial_t (r^j_2 - \nabla \cdot \Sigma^j) \| + \frac{k_{n-1}^3}{12} \| \partial_t^2 (r^j_2 - \nabla \cdot \Sigma^j) \| \right)
+ \int_{t_{n-1}}^{t} \left( t_{n-1} - s \right) \| \partial_t (r^j_2 - \nabla \cdot \Sigma^j) \|
+ \int_{t_{n-1}}^{t} \left( (t_{n-1} - s)^2 - \frac{(t_{n-1} - s)^3}{k_{n-1}} \right) \| \partial_t^2 (r^j_2 - \nabla \cdot \Sigma^j) \|,
\]
\[
\mathcal{E}_{1,4}(t) = \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} \| f^j - f \| + \int_{t_{n-1}}^{t} \| f^n - f \|.\]

**Proof.** Differentiate (3.22) with respect to \(t\). Then, choose \(v = \xi_{\sigma}\) in the resulting equation and \(w = \xi_{u,t}\) in (3.15) to obtain for \(t \in (t_{n-1}, t_n]\)
\[
\frac{1}{2} \left\| \xi_{u,t} \right\|^2 + \|\alpha^{1/2} \xi_{\sigma}\|^2 \quad = \quad (\eta_{u,t}, \xi_{u,t}) + ((I - P_{h}^n)U_{tt}, \xi_{u,t})
+ \mu^n(t) \left( \partial_t^2 U_{tt}, P_{h}^n \xi_{u,t} \right)
+ \left( \nabla \cdot (\sigma - \sigma^n), \xi_{u,t} \right) + \left( f^n - f, \xi_{u,t} \right).
\]
On integrating from 0 to $t$ with $t \in (t_{n-1}, t_n]$, we find that
\begin{equation}
(3.27) \frac{1}{2} \left( \| \xi_{u,t} \|^2 + \| \alpha^{1/2} \xi \|^2 \right) = \frac{1}{2} \left( \| \xi_{u,t}(0) \|^2 + \| \alpha^{1/2} \xi \|^2 \right) + \int_0^t (\eta_{u,t}, \xi_{u,t}) \, ds \nonumber
\end{equation}
\begin{equation}
+ J^m_{1,1}(\xi_{u,t}) + J^m_{1,2}(\xi_{u,t}) + J^m_{1,3}(\xi_{u,t}) + J^m_{1,4}(\xi_{u,t}),
\end{equation}
where
\begin{align*}
J^m_{1,1}(\xi_{u,t}) & := \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} \left( (I - P^j_h) U_{tt}, \xi_{u,t} \right) + \int_{t_{n-1}}^t \left( (I - P^n_h) U_{tt}, \xi_{u,t} \right), \\
J^m_{1,2}(\xi_{u,t}) & := \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} \mu^j \delta^2_t U^j, (I - P^j_h) \xi_{u,t} \right) + \int_{t_{n-1}}^t \mu^n \delta^2_t U^n, (I - P^n_h) \xi_{u,t} \right), \\
J^m_{1,3}(\xi_{u,t}) & := \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} \left( \nabla \cdot (\hat{\sigma} - \hat{\sigma}^n), \xi_{u,t} \right) + \int_{t_{n-1}}^t \left( \nabla \cdot (\hat{\sigma} - \hat{\sigma}^n), \xi_{u,t} \right), \\
and \\
J^m_{1,4}(\xi_{u,t}) & := \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} \left( f^j - f, \xi_{u,t} \right) + \int_{t_{n-1}}^t \left( f^n - f, \xi_{u,t} \right).
\end{align*}

Set
\begin{equation}
E^2_1(t) := \| \xi_{u,t}(t) \|^2 + \| \alpha^{1/2} \xi \|^2,
\end{equation}
and let at $t = t^* \in (0, t]$ be such that
\begin{equation}
E_1(t^*) = \max_{0 \leq s \leq t} E_1(s).
\end{equation}

Now, a use of the Cauchy-Schwarz yields
\begin{equation}
(3.28) \quad |J^m_{1,1}(\xi_{u,t})| \leq \left( \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} \| (I - P^j_h) U_{tt} \| + \int_{t_{n-1}}^t \| (I - P^n_h) U_{tt} \| \right) E_1(t^*).
\end{equation}

and similarly,
\begin{equation}
|J^m_{1,2}(\xi_{u,t})| \leq \left( \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} \| \mu^j \delta^2_t U^j \| + \int_{t_{n-1}}^t \| \mu^n \delta^2_t U^n \| \right) E_1(t^*).
\end{equation}

For $J^m_{1,3}$, we rewrite using (3.17)
\begin{equation}
(3.29) \quad \left( \nabla \cdot (\hat{\sigma} - \hat{\sigma}^n), w \right) = (t - t_n) \left( \nabla \cdot \partial_t \hat{\sigma}^n, w \right) + \left( k_n^{-1}(t_n - t)^3 - (t_n - t)^2 \right) \left( \nabla \cdot \partial^2_t \hat{\sigma}^n, w \right).
\end{equation}

From (3.6), we obtain
\begin{equation}
(3.30) \quad \left( \nabla \cdot \hat{\sigma}^n, w \right) = -r^0_2(w) + \left( \nabla \cdot \Sigma^n, w \right),
\end{equation}
and therefore for $j = 1, 2$
\begin{equation}
(3.31) \quad \left( \nabla \cdot \partial^j \hat{\sigma}^n, w \right) = -\left( \partial^j r^0_2 \right)(w) + \left( \nabla \cdot \partial^j \Sigma^n, w \right).
\end{equation}
On substituting (3.31) for $j = 1, 2$ in $J_{1,3}^n$, we arrive at

$$J_{1,3}^n(\xi_{u,t}) = \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} \left[ (t_j - s) \left\{ (\partial_t r_j^i)(\xi_{u,t}) - (\nabla \cdot \partial_t \Sigma^i, \xi_{u,t}) \right\} 
- \left( k_j^{-1}(t_j - s)^3 - (t_j - s)^2 \right) \left\{ (\partial_t^2 r_j^i)(\xi_{u,t}) - (\nabla \cdot \partial_t^2 \Sigma^i, \xi_{u,t}) \right\} \right] 
+ \int_{t_{n-1}}^{t} \left[ (t_n - s) \left\{ (\partial_t r_n^i)(\xi_{u,t}) - (\nabla \cdot \partial_t \Sigma^i, \xi_{u,t}) \right\} 
- \left( k_n^{-1}(t_n - s)^3 - (t_n - s)^2 \right) \left\{ (\partial_t^2 r_n^i)(\xi_{u,t}) - (\nabla \cdot \partial_t^2 \Sigma^i, \xi_{u,t}) \right\} \right].$$

Using the Cauchy-Schwarz inequality, it follows that

$$|J_{1,3}^n(\xi_{u,t})| \leq \sum_{j=1}^{n-1} \left( \frac{k_j^2}{2} \left\| \partial_t (r_j^i - \nabla \cdot \Sigma^i) \right\| + \frac{k_j^3}{12} \left\| \partial_t^2 (r_j^i - \nabla \cdot \Sigma^i) \right\| \right) E_1(t^*) 
+ \left[ \int_{t_{n-1}}^{t} \left\{ (t_n - s) \left\| \partial_t (r_n^i - \nabla \cdot \Sigma^i) \right\| 
+ ((t_n - s)^2 - k_n^{-1}(t_n - s)^3) \left\| \partial_t^2 (r_n^i - \nabla \cdot \Sigma^i) \right\| \right] E_1(t^*).$$

For $J_{1,4}^n$, we note that

$$|J_{1,4}^n(\xi_{u,t})| = \left| \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} (f_j^i - f, \xi_{u,t}) + \int_{t_{n-1}}^{t} (f^n - f, \xi_{u,t}) \right| 
\leq \left( \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} \left\| f_j^i - f \right\| + \int_{t_{n-1}}^{t} \left\| f^n - f \right\| \right) E_1(t^*).$$

On substituting the estimates of $J_{1,j}^n(\xi_{u,t})$, $j = 1, \cdots, 4$, in (3.27), we arrive at

$$E_1(t) \leq E_1(t^*) \leq E_1(0) + 2 \sum_{j=1}^{4} E_{1,j}(t) + 2 \int_{0}^{t} \left\| \eta_{u,tt}(s) \right\| ds.$$

This completes the rest of the proof. \hfill \Box

**Remark.** The term $(r_j^i - \nabla \cdot \Sigma^i)$ can be replaced by $(\partial_t^2 U_j^i - f_j^i)$.

For obtaining $L^\infty(L^2)$ estimate for $e_u$, we now integrate (3.15) with respect to $t$ from 0 to $t$, for $t \in (t_{n-1}, t_n]$, to arrive at

$$E_1(t) \leq E_1(t^*) \leq E_1(0) + 2 \sum_{j=1}^{4} E_{1,j}(t) + 2 \int_{0}^{t} \left\| \eta_{u,tt}(s) \right\| ds.$$

(3.32) \hspace{1cm} (\xi_{u,t}, w) + \left( \nabla \cdot \xi_{\sigma}, w \right) = (e_{u,t}(0), w) + (\eta_{u,t}, w) 
+ J_{2,1}^n(w) + J_{2,2}^n(w) + J_{2,3}^n(w) + J_{2,4}^n(w),
where
\[
J_{2,1}^n(w) := \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} \left( (I - P_h^j)U_{tj}, w \right) + \int_{t_{n-1}}^{t} ((I - P_h^n)U_{tt}, w),
\]
\[
J_{2,2}^n(w) := \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} \mu^j (\partial_t^2 U^j, w) + \int_{t_{n-1}}^{t} \mu^n (\partial_t^2 U^n, w),
\]
\[
J_{2,3}^n(w) := \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} \left( \nabla \cdot (\sigma - \tilde{\sigma}^j), w \right) + \int_{t_{n-1}}^{t} \left( \nabla \cdot (\sigma - \tilde{\sigma}^n), w \right),
\]
\[
J_{2,4}^n(w) := \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} (f^j - f, w) + \int_{t_{n-1}}^{t} (f^n - f, w).
\]
Note that \( J_{2,2}^n(w) = \int_{t_{n-1}}^{t} \mu^n (\partial_t^2 U^n, w) \) as \( \int_{t_{j-1}}^{t_j} \mu^j = 0 \). Further, since \( P_h^j \) commutes with time derivative, \( J_{2,1}^n(w) \) can be written as
\[
J_{2,1}^n(w) = \sum_{j=1}^{n-1} \left( (I - P_h^j)U_{tj}(t_j) - (I - P_h^j)U_{tj}(t_{j-1}), w \right) + \left( (I - P_h^n)U_t(t), w \right) - \left( (I - P_h^n)U_t(t_{n-1}), w \right)
\]
\[
= \sum_{j=1}^{n-1} \left( (I - P_h^j)U_{tj}(t_j) - (I - P_h^{j-1})U_{tj}(t_{j-1}), w \right) + \sum_{j=1}^{n-1} (P_h^j - P_h^{j-1})U_{tj}(t_{j-1})
\]
\[
+ \left( (I - P_h^n)U_t(t), w \right) - \left( (I - P_h^n)U_t(t_{n-1}), w \right)
\]
\[
= \sum_{j=0}^{n-1} \left( (P_h^{j+1} - P_h^j)U_{tj}(t_j), w \right) - \left( (I - P_h^0)U_t(0), w \right)
\]
\[
+ \left( (I - P_h^n)U_t(t), w \right).
\]

Below, we prove one of the main results of this section.

**Theorem 3.2.** Let \( (u, \sigma) \) be the solution of (2.3)-(2.4) and \( (U, \Sigma) \) be the solution of (3.1)-(3.2). Then for \( t \in (t_{n-1}, t_n] \), the following estimate holds
\[
\|\xi_u(t)\| \leq \|\xi_u(0)\| + 2 \sum_{j=1}^{4} \xi_{2,j}(t) + 2 \int_0^t \|\eta_{u,t}(s)\| ds,
\]
where \( \xi_{2,1}, \cdots, \xi_{2,4} \) will be given in the following proof.

**Proof.** Choose \( w = \xi_u \) in (3.32) and \( v = \xi_\sigma \) in (3.22). Then add the resulting equations to obtain
\[
\frac{1}{2} \frac{d}{dt} \left( \|\xi_u\|^2 + \|\alpha^{1/2} \xi_\sigma\|^2 \right) = \left( e_{u,t}(0), \xi_u \right) + \left( \eta_{u,t}, \xi_u \right) + \sum_{j=1}^{4} J_{2,j}(\xi_u).
\]
On integrating from 0 to \( t \) with \( t \in (t_{n-1}, t_n] \), it follows that
\[
\|\xi_u\|^2 + \|\alpha^{1/2} \xi_\sigma\|^2 = \|\xi_u(0)\|^2 + 2 \left( e_{u,t}(0), \xi_u \right) + 2 \int_0^t \left( \eta_{u,t}, \xi_u \right) + 2 \sum_{j=1}^{4} \int_0^t J_{2,j}(\xi_u).
\]
Set

\[(E_2(t))^2 := \|\xi_u(t)\|^2 + \|\alpha^{1/2}\hat{\xi}_\sigma(t)\|^2.\]

Let \(t = t^* \in (0, t]\) be such that

\[E_2(t^*) = \max_{0 \leq s \leq t} E_2(s).\]

For \(t_{n-1} < t \leq t_n\), we obtain from (3.33)

\[
\int_0^t J_{2,1}(\xi_u(s)) \, ds = \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} \left( \sum_{l=0}^{j-1} \left( (P_{h}^{l+1} - P_{h}^l)U_l(t), \xi_u(s) \right) \right) ds \\
+ \int_{t_{n-1}}^{t} \left( (P_{h}^{n+1} - P_{h}^n)U_n(t), \xi_u(s) \right) ds - \int_0^t \left( (I - P_{h}^0)U(t, 0), \xi_u(s) \right) ds \\
+ \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} \left( (I - P_{h}^j)U(t, s), \xi_u(s) \right) ds + \int_{t_{n-1}}^{t} \left( (I - P_{h}^n)U(t, s), \xi_u(s) \right) ds.
\]

Hence,

\[
\left| \int_0^t J_{2,1}(\xi_u(s)) \, ds \right| \leq \sum_{j=1}^{n-1} \left( k_j \sum_{l=0}^{j-1} \| (P_{h}^{l+1} - P_{h}^l)\partial_tU^l \| + (t - t_{n-1}) \| (P_{h}^{j+1} - P_{h}^j)\partial_tU^j \| \\
+ \int_{t_{j-1}}^{t_j} \| (I - P_{h}^j)U(t, s) \| ds \right) + (t - t_{n-1}) \| (P_{h}^1 - P_{h}^0)\partial_tU^0 \| \\
+ t \| (I - P_{h}^0)\partial_tU^0 \| + \int_{t_{n-1}}^{t} \| (I - P_{h}^n)U(t, s) \| ds \right] E_2(t^*)
=: E_{2,1}(t)E_2(t^*).
\]

The second term can be written as

\[
\int_0^t J_{2,2}(\xi_u(s)) \, ds = \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} \left( \int_{t_{j-1}}^{t} \mu^j(\tau)(\partial^2_tU^j, \xi_u(s)) \, d\tau \right) ds \\
+ \int_{t_{n-1}}^{t} \left( \int_{t_{n-1}}^{t} \mu^n(\tau)(\partial^2_tU^n, \xi_u(s)) \, d\tau \right) ds
\]

for \(t_{n-1} < t \leq t_n\). Since \(\int_{t_{j-1}}^{t_j} \mu^j(\tau) \, d\tau = -3k_j^{-1} \left[ (s - t_{j-1/2})^2 - \frac{k_j^2}{4} \right]\), we obtain

\[
\left| \int_0^t J_{2,2}(\xi_u(s)) \, ds \right| = \left| -3 \sum_{j=1}^{n-1} k_j^{-1} \int_{t_{j-1}}^{t_j} (s - t_{j-1/2})^2(\partial^2_tU^j, \xi_u(s)) ds + \frac{3}{4} \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} k_j(\partial^2_tU^j, \xi_u(s)) ds \\
-3k_n^{-1} \int_{t_{n-1}}^{t} (s - t_{n-1/2})^2(\partial^2_tU^n, \xi_u(s)) ds + \frac{3}{4} k_n \int_{t_{n-1}}^{t} (\partial^2_tU^n, \xi_u(s)) ds \right|
\leq \sum_{j=1}^{n-1} \| k_j^2 \partial^2_tU^j \| + \| k_n^{-1} \left( (t - t_{n-1/2})^3 + \frac{3k_n^2}{4}(t - t_{n-1}) + \frac{k_n^3}{8} \right) \partial^2_tU^n \| E_2(t^*)
=: E_{2,2}(t)E_2(t^*).\]
For the third term $J_{2,3}$, we obtain for $t_{n-1} < t \leq t_n$ and $t_{j-1} < s \leq t_j$,

$$
\int_0^t J_{2,3}(\xi_u(s))ds = \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} \sum_{l=1}^{j-1} \int_{t_{l-1}}^{t_l} \left( \nabla \cdot (\tilde{\sigma}(\tau) - \tilde{\sigma}^l), \xi_u(s) \right) d\tau ds \\
+ \int_t^t \sum_{j=1}^{n-1} \sum_{l=1}^{j-1} \int_{t_{l-1}}^{t_l} \left( \nabla \cdot (\tilde{\sigma}(\tau) - \tilde{\sigma}^l), \xi_u(s) \right) d\tau ds \\
+ \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{s} \left( \nabla \cdot (\tilde{\sigma}(\tau) - \tilde{\sigma}^l), \xi_u(s) \right) d\tau ds \\
+ \int_{t_{n-1}}^{t} \int_{t_{n-1}}^{s} \left( \nabla \cdot (\tilde{\sigma}(\tau) - \tilde{\sigma}^n), \xi_u(s) \right) d\tau ds \\
= I_1 + I_2 + I_3 + I_4.
$$

Using (3.29) and the fact that $\int_{t_{l-1}}^{t_l} (k_{3,1}(t_l - \tau)^3 - (t_l - \tau)^2) d\tau = -k_{3,1}^3$, we find that

$$
I_1 = \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} \sum_{l=1}^{j-1} \frac{k_{3,1}^3}{2} \left( \nabla \cdot \partial_t \tilde{\sigma}^l, \xi_u(s) \right) ds + \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} \sum_{l=1}^{j-1} \frac{k_{3,1}^3}{12} \left( \nabla \cdot \partial^2_t \tilde{\sigma}^l, \xi_u(s) \right) ds \\
\leq \left[ \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} \sum_{l=1}^{j-1} \frac{k_{3,1}^3}{2} \left( \nabla \cdot \partial_t \tilde{\sigma}^l \right) ds + \frac{k_{3,1}^3}{12} \left( \nabla \cdot \partial^2_t \tilde{\sigma}^l \right) \right] E_2(t^{**}).
$$

Similarly,

$$
I_2 \leq \left[ (t - t_{n-1}) \sum_{j=1}^{n-1} \left( \frac{k_{3,1}^3}{2} \left( \nabla \cdot \partial_t \tilde{\sigma}^l \right) + \frac{k_{3,1}^3}{12} \left( \nabla \cdot \partial^2_t \tilde{\sigma}^l \right) \right) \right] E_2(t^{**}).
$$

For the $I_3$ and $I_4$ terms, we easily obtain

$$
I_3 \leq \left[ \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} \left( \frac{(t_j - s)^2}{2} - \frac{k_{3,1}^3}{2} \right) \left( \nabla \cdot \partial_t \tilde{\sigma}^j \right) \right] E_2(t^{**}) \\
+ \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} \left( \frac{(t_j - s)^3}{3} - \frac{k_{3,1}^3(t_j - s)^4}{4} \right) \left( \nabla \cdot \partial^2_t \tilde{\sigma}^j \right) \right] E_2(t^{**}) \\
\leq \left[ \sum_{j=1}^{n-1} \left( \frac{k_{3,1}^3}{3} \left( \nabla \cdot \partial_t \tilde{\sigma}^j \right) + \frac{k_{3,1}^3}{20} \left( \nabla \cdot \partial^2_t \tilde{\sigma}^j \right) \right) \right] E_2(t^{**}),
$$

and

$$
I_4 \leq (t - t_{n-1}) \left[ \frac{k_{3,1}^3}{3} \left( \nabla \cdot \partial_t \tilde{\sigma}^n \right) + \frac{k_{3,1}^3}{20} \left( \nabla \cdot \partial^2_t \tilde{\sigma}^n \right) \right] E_2(t^{**}).
$$

Now collect terms, replace $\nabla \cdot \tilde{\sigma}^j$ by $-\tau^j + \nabla \cdot \Sigma^j$ using (3.30), and set $\mathcal{E}_{2,3} := M_1 + M_2$.
where by

\[ M_1 = \sum_{j=1}^{n-1} \left[ k_j \left( \sum_{l=1}^{j-1} \frac{k_l^2}{2} \| \partial_t (r^j_2 - \nabla \cdot \Sigma^j) \| \right) + \left( (t - t_n-1) \frac{k_j^3}{2} + \frac{k_j^3}{3} \right) \| \partial_t (r^j_2 - \nabla \cdot \Sigma^j) \| \right] \\
+ (t - t_n-1) \frac{k_n^3}{3} \| \partial_t (r^n_2 - \nabla \cdot \Sigma^n) \|,
\]

and

\[ M_2 = \sum_{j=1}^{n-1} \left[ k_j \left( \sum_{l=1}^{j-1} \frac{k_l^3}{12} \| \partial_t^2 (r^j_2 - \nabla \cdot \Sigma^j) \| \right) + \left( (t - t_n-1) \frac{k_j^3}{12} + \frac{k_j^3}{20} \right) \| \partial_t^2 (r^j_2 - \nabla \cdot \Sigma^j) \| \right] \\
+ \frac{k_n^4}{20} (t - t_n-1) \| \partial_t^2 (r^n_2 - \nabla \cdot \Sigma^n) \|
\]

so that

\[ \int_0^t J_{2,3} (\xi_u(s)) ds \leq E_{2,3} (t) E_2 (t^{**}). \]

For the last term \( J_{2,4} \), one can repeat previous arguments to arrive at

\[ \int_0^t J_{2,4} (\xi_u(s)) ds = \left[ \sum_{j=1}^{n-1} \left( k_j \left( \sum_{l=1}^{j-1} \int_{t_{l-1}}^{t_l} \| f^l - f(\tau) \| d\tau \right) + (t - t_n-1) \int_{t_{n-1}}^t \| f^n - f(\tau) \| d\tau \right) \\
+ k_j \int_{t_{j-1}}^{t_j} \| f^j - f(\tau) \| d\tau \right) + (t - t_n-1) \int_{t_{n-1}}^t \| f^n - f(\tau) \| d\tau \right] E_2 (t^{**}) \\
= :- E_{2,4} (t) E_2 (t^{**}). \]

On substituting in (3.34), it follows that

\[ \| \xi_u(t) \| \leq \| \xi_u(0) \| + 2 \sum_{j=1}^{n} \mathcal{E}_{2,j} (t) + 2 \int_0^t \| \eta_{u,t} (s) \| ds, \]

which completes the proof. \( \square \)

In order to present the final theorem in this paper, we introduce some notations: For \( D := \Omega \) or \( K \), let

\[ \mathcal{E}_1^0 (D) = \| h_0 (\alpha \Sigma^0 + \nabla_h U^0) \|_{L^2(D)}, \]
\[ \mathcal{E}_2^n (D) = \left( \| h_n^{l+1} r_2^n \|_{L^2(D)} + \| h_n (\alpha \Sigma^n + \nabla_h U^n) \|_{L^2(D)} \right), \]
\[ \mathcal{E}_3^n (D) = \left( \| h_n^{l+1} \partial_t r_2^n \|_{L^2(D)} + \| h_n \partial_t (\alpha \Sigma^n + \nabla_h U^n) \|_{L^2(D)} \right), \]
\[ \mathcal{E}_4^0 (D) = \| h_0 (\alpha \partial_t \Sigma^0 + \nabla_h \partial_t U^0) \|_{L^2(D)}, \]
\[ \mathcal{E}_5^n (D) = \left( \| h_0^{1/2} J (\alpha \Sigma^0 \cdot t) \|_{0, \Gamma_h, D} + \| h_0 \text{curl}_h (\alpha \Sigma^0) \|_{L^2(D)} \right), \]
\[ \mathcal{E}_6^n (D) = \left( \| h_n r_2^n \|_{L^2(D)} + \| h_n^{1/2} J (\alpha \Sigma^n \cdot t) \|_{0, \Gamma_h, D} + \| h_n \text{curl}_h (\alpha \Sigma^n) \|_{L^2(D)} \right), \]
\[ \mathcal{E}_7^n (D) = \left( \| h_n^{l+1} \partial_t r_2^n \|_{L^2(D)} + \| h_n \partial_t (\alpha \Sigma^n + \nabla_h U^n) \|_{L^2(D)} \right), \]
\[ \mathcal{E}_8^n (D) = \left( \| h_n^{l+1} \partial_t^2 r_2^n \|_{L^2(D)} + \| h_n \partial_t^2 (\alpha \Sigma^n + \nabla_h U^n) \|_{L^2(D)} \right), \]
Remark. The numerical implementation of the proposed
and
\[ \Omega \text{, the shape regularity of the elements, polynomial degree} \]
where
\[ (3.35) \]
\[ \| U^m - u(t_m) \| \leq \| e_u(0) \| + C_1 E^0_1(\Omega) + C_2 E^m_2(\Omega) + C_3 \sum_{n=1}^{m} k_n E^n_3 + \sum_{i=1}^{4} c_i E^n_{2,i}(\Omega), \]
and
\[ \| \Sigma^m - \sigma(t_m) \|_{A^{-1}} \leq \| e_{u,t}(0) \| + \| e_{\sigma}(0) \|_{A^{-1}} + C_4 E^0_4(\Omega) + C_5 E^0_6(\Omega) \]
\[ + C_6 E^m_6(\Omega) + C_7 E^m_7(\Omega) + C_8 \sum_{n=1}^{m} k_n E^n_8(\Omega) + \sum_{i=1}^{4} c_i E^n_{1,i}(\Omega), \]
where \( C_i \)'s and \( c_i \)'s are constants which depend only on the coefficient matrix \( A \), the domain \( \Omega \), the shape regularity of the elements, polynomial degree \( k \) and interpolation constants.

Remark. The numerical implementation of the proposed \textit{a posteriori} estimators in the adaptive algorithm deserves special attention and will be considered elsewhere.
4 Conclusion

The current work presents rigorous \textit{a posteriori} error bounds in the $L^\infty(L^2)$-norm for mixed finite element approximations to second order wave equations. The derived bounds appear to be of optimal order. While Baker’s technique is usually used to derive $L^\infty(L^2)$ estimates for the displacement $u$, in this paper, we resort to an application of integration for deriving these estimates. Although no efficiency bounds are discussed in this article, this would be an interesting direction for further research. Moreover, the numerical implementation of the adaptive algorithm based on the proposed estimators will be a part of our future work.

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