CATEGORIES OF COMPLEX VARIATIONS OF HODGE STRUCTURE OVER COMPACT KÄHLER MANIFOLDS

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Abstract. We give a complex polarized variation of Hodge structure over a compact Kähler manifold \( M \) which controls all finite-dimensional complex polarized variations of Hodge structure over \( M \) and their tensor relations. As a corollary, we obtain the cohomology algebra with values in a local system admitting multiplicative Hodge structures.

1. Introduction

The purpose of this paper is to study structures of the category \( \mathcal{VHS}_C(M) \) of (finite-dimensional) complex polarized variations of Hodge structure over a compact Kähler manifold \( M \) related to representation theory and cohomology theory. By using non-abelian Hodge correspondence, in [7], Simpson shows that objects in \( \mathcal{VHS}_C(M) \) are characterized by following objects:

- (S1): Semi-simple flat vector bundles (=poly-stable Higgs bundles with vanishing Chern classes) which are invariant under canonical \( \mathbb{C}^* \)-deformations ([7, Section 4]).
- (S2): Complex Hodge representations of the reductive completion of the fundamental group \( \pi_1(M, x) \) equipped with the pure non-abelian Hodge structure ([7, Section 5]).

In this paper, we give more discussions for connecting these arguments to standard Hodge theory (complex Hodge structures). By using the complex Hodge structure on the cohomology and Simpson’s characterization (S1), we show that any complex variation of Hodge structure on a compact Kähler manifold can be simplified as the following.

**Proposition 1.0.1** (Proposition 3.4.3). Any \((E, D, F^*, G^*) \in \text{Obj}(\mathcal{VHS}_C(M))\) is isomorphic to

\[
\bigoplus_{V \in \mathcal{V}_\mathcal{VHS}_C(M)} \mathcal{H}^0(M, \text{Hom}(V, E)) \otimes V
\]

with the natural filtrations where \( \mathcal{V}_\mathcal{VHS}_C(M) \) is the set of isomorphism classes of simple flat bundles admitting (unique) structures of complex polarized variations of Hodge structure.

By using this simplification and the theory of Tannakian category, we give a reformulation of Simpson’s characterization (S2) for studying \( \mathcal{VHS}_C(M) \) in terms of complex Hodge structures.

**Theorem 1.0.2** (Theorem 4.1.7). There exists a canonical pro-algebraic group \( \varpi_1^{\mathcal{VHS}_C}(M, x) \) over \( \mathbb{C} \) whose Hopf algebra \( \mathcal{O}(\varpi_1^{\mathcal{VHS}_C}(M, x)) \) of global sections of the structure sheaf admits a canonical complex Hodge structure of weight 0 depending on \( x \in M \) such that there exists an equivalence between the category \( \text{Rep}^{\text{HS}}(\varpi_1^{\mathcal{VHS}_C}(M, x)) \) of Hodge representations of \( \varpi_1^{\mathcal{VHS}_C}(M, x) \) and the category \( \mathcal{VHS}_C(M) \).

The main ingredient of this paper is to construct a infinite-dimensional complex polarized variation of Hodge structure \( \Omega^{\mathcal{VHS}_C}(x) \) depending on a point \( x \in M \) satisfying certain universal properties. We can see every complex polarized variation of Hodge structure over a compact Kähler
manifold $M$ can be presented by $\mathcal{D}^{VHSC}(x)$ (Proposition 4.2.2). Moreover, $\mathcal{D}^{VHSC}(x)$ contains information not only on each object in $VHSC(M)$ but also on their tensor relations. This point is very important for considering the de Rham complex with values in $\mathcal{D}^{VHSC}(x)$. We note that the usual de Rham complex on a compact Kähler manifold admits a canonical structure of a bidifferential bigraded algebra giving the cohomology algebra with multiplicative Hodge structures. On the other hand, the de Rham complex with values in a complex polarized variation of Hodge structure admits a canonical structure of a double complex giving the cohomology with Hodge structures but not a algebra structure. We have:

**Theorem 1.0.3** (Theorem 4.2.3). Consider the de Rham complex 

$$A^*(M, \mathcal{D}^{VHSC}(x))$$

with values in the local system $\mathcal{D}^{VHSC}(x)$. Then:

1. The cochain complex $A^*(M, \mathcal{D}^{VHSC}(x))$ admits a canonical differential graded algebra structure.
2. The differential graded algebra $A^*(M, \mathcal{D}^{VHSC}(x))$ admits a canonical bidifferential bigraded algebra structure.
3. $A^*(M, \mathcal{D}^{VHSC}(x))$ is a rational $\mathcal{D}^{VHSC}(M, x)$-module such that $\mathcal{D}^{VHSC}(M, x)$ acts as automorphisms of the differential graded algebra.
4. The co-module structure $A^*(M, \mathcal{D}^{VHSC}(x)) \to A^*(M, \mathcal{D}^{VHSC}(x)) \otimes \mathcal{O}(\mathcal{D}^{VHSC}(M, x))$ is a morphism of bidifferential bigraded algebras.
5. For $(U, F^*, G^*) \in \text{Obj}(\text{Rep}^{HS}(\mathcal{D}^{VHSC}(M, x)))$,

$$\left( A^*(M, \mathcal{D}^{VHSC}(x)) \otimes U \right)^{VHSC}(M, x)$$

is a double complex.
6. For $(E, D, F^*, G^*) \in \text{Obj}(VHSC(M))$, the double complex $(A^*(M, \mathcal{D}^{VHSC}(x)) \otimes E_2)^{VHSC}(M, x)$ is isomorphic to the canonical double complex

$$(A^*(M, E)^{p,q}, D', D'')$$

associated with $(E, D, F^*, G^*) \in \text{Obj}(VHSC(M))$.

By this, we obtain the cohomology algebra $H^*(M, \mathcal{D}^{VHSC}(x))$ with the multiplicative complex Hodge structures (Corollary 4.2.4).

2. **Standard Complex Hodge theory**

General references of this sections are [1, 5]. A complex Hodge structure (C-HS) of weight $w \in \mathbb{Z}$ is $(V, F^*, G^*)$ so that

- $V$ is a finite-dimensional vector space;
- $F^*$ and $G^*$ are decreasing filtrations of $V$ so that $V = \bigoplus_{p+q=w} V^{p,q}$ where $V^{p,q} = F^p(V) \cap G^q(V)$.

Conversely, a bigrading $V = \bigoplus_{p+q=w} V^{p,q}$ of a complex vector space determines a C-HS $(V, F^*, G^*)$ of weight $w$ so that $F^*(V) = \bigoplus_{r \geq p} V^{p,q}$ and $G^*(V) = \bigoplus_{r \geq q} V^{p,q}$. A hermitian form $h$ on $V$ is a polarization of the Hodge structure if $V = \bigoplus_{p+q=w} V^{p,q}$ is $h$-orthonormal and $h$ is positive on $V^{p,q}$ for even $p$ and negative for odd $p$. A morphism of C-HSs is a C-linear map which is compatible with filtrations equivalently a C-linear map which is compatible with bigradings. We can say that any morphism of C-HSs is strictly compatible with filtrations and we can say that the category of complex Hodge structures of weight $w$ is abelian.

Obviously, for all integers $w \in \mathbb{Z}$, the categories of complex Hodge structures of weight $w$ are equivalent. Hence we mainly consider the abelian category $\mathcal{H}SC$ of complex Hodge structures of
3.1. Higgs bundles. Let $M$ be a compact Kähler manifold. A Higgs bundle over $M$ is a pair $(E, \theta)$ consisting of a holomorphic vector bundle $E$ over $M$ and a section $\theta \in A^{1,0}(M, \text{End}(E))$ satisfying the following two conditions:

$$\bar{\partial}\theta = 0 \quad \text{and} \quad \theta \wedge \theta = 0.$$ 

This section $\theta$ is called a Higgs field on $E$. We assume that all Chern classes of $E$ vanish.

Let $H$ be a Hermitian metric on $E$. Define $\theta_H \in A^{0,1}(M, \text{End}(E))$ by $(\theta(e_1), e_2) = (e_1, \theta_H(e_2))$ for $e_1, e_2 \in E$. Let $\nabla$ be the canonical unitary connection on $E$ associated to $H$. Define the connection $D = \nabla + \theta + \bar{\theta}_H$ and consider the curvature $R^D = D^2$ of $D$. We say that $H$ is harmonic if $R^D = 0$ (i.e., $(E, D)$ is a flat bundle).

We say that $(E, \theta)$ is stable if $E$ for every sub-Higgs sheaf $V$ of $0 < \text{rk}(V) < \text{rk}(E)$, the inequality $\text{deg}(V) < 0$ holds.

**Theorem 3.1.1** ([4]). If $(E, \theta)$ is stable, then $(E, \theta)$ admits a harmonic metric.

$(E, \theta)$ is called polystable if $(E, \theta) = \bigoplus_{i=1}^k (E_i, \theta_i)$, where each $(E_i, \theta_i)$ is a stable Higgs bundle. By Theorem 3.1.1 if $(E, \theta)$ is polystable, then $(E, \theta)$ admits a harmonic metric. Thus we correspond polystable $(E, \theta)$ to a flat bundle $(E, D)$.

**Theorem 3.1.2** ([7]). The correspondence $(E, \theta) \mapsto (E, D)$ via harmonic metrics is an equivalence between the category of stable (resp. polystable) Higgs bundles with vanishing Chern classes and the category of simple (resp. semi-simple) flat complex vector bundles.

3.2. Complex Variations of Hodge structure. Let $M$ be a complex manifold. A complex variation of Hodge structure (C-VHS) of weight $w$ over $M$ is $(E, D, F^*, G^*)$ so that:

1. $E$ is a flat complex vector bundle with a flat connection $D$.
2. $F^*$ and $G^*$ are decreasing filtration of $E$ satisfying the Griffiths transversality conditions $DF^* \subset A^1(M, F^{r-1})$ and $DG^* \subset A^1(M, G^{r-1})$.
3. For any $x \in M$, the fiber $(E_x, F^*_x, G^*_x)$ at $x$ is a C-HS.

Equivalently a C-VHS of weight $w$ over $M$ is $(E = \bigoplus_{p+q=w} E^{p,q}, D)$, so that:

1. $E$ is a $C^\infty$-complex vector bundle with a decomposition $\bigoplus_{p+q=w} E^{p,q}$ in a direct sum of $C^\infty$-subbundles.
2. $D$ is a flat connection satisfying the Griffiths transversality conditions

$$D : A^0(M, E^{p,q}) \to A^{0,1}(M, E^{p+1,q-1}) \oplus A^{1,0}(M, E^{p,q}) \oplus A^{0,1}(M, E^{p,q}) \oplus A^{1,0}(M, E^{p-1,q+1}).$$

A polarization $h$ of a C-VHS is a parallel Hermitian form so that the decomposition $\bigoplus_{p+q=w} E^{p,q}$ is orthogonal and $h$ is positive on $E^{p,q}$ for even $p$ and negative for odd $p$. A C-VHS is polarizable if it admits a polarization.

By the Griffiths transversality, the differential $D$ on $A^*(M, E)$ decomposes $D = \partial + \theta + \bar{\partial} + \bar{\theta}$ so that:

$$\partial : A^{a,b}(M, E^{c,d}) \to A^{a+1,b}(M, E^{c,d}), \quad \bar{\partial} : A^{a,b}(M, E^{c,d}) \to A^{a,b+1}(M, E^{c,d}),$$

$$\theta : A^{a,b}(M, E^{c,d}) \to A^{a+1,b}(M, E^{c-1,d+1}) \quad \text{and} \quad \bar{\theta} : A^{a,b}(M, E^{c,d}) \to A^{a,b+1}(M, E^{c+1,d-1}).$$
We define the bigrading

\[ A^*(M,E)^{p,q} = \bigoplus_{a+c=p, b+d=q} A^{a,b}(M,E^{c,d}), \]

\[ D' = \partial + \bar{\partial} : A^*(M,E)^{p,q} \to A^*(M,E)^{p+1,q} \]
and \[ D'' = \bar{\partial} + \partial : A^*(M,E)^{p,q} \to A^*(M,E)^{p,q+1}. \]
By the flatness \( DD = 0 \), we have

\[ D'D' = D''D'' = D'D'' + D''D' = 0. \]

We have the double complex

\[ (A^*(M,E)^{p,q}, D', D'') \]
as the usual Dolbeault complex on a complex manifold.

**Theorem 3.2.1** ([8]). Let \( M \) be a compact Kähler manifold. For any polarized complex variation of Hodge structure of weight \( w \) \( (E = \bigoplus_{p+q=w} E^{p,q}, D, h) \) over \( M \), the filtrations \( F^r = \bigoplus_{r \leq p} A^*(M,E)^{p,q} \) and \( G^r = \bigoplus_{r \leq q} A^*(M,E)^{p,q} \) induce a canonical complex Hodge structure of weight \( i + w \) on the cohomology \( H^i(M,E) \).

We review this fundamental fact more precisely. We define the differential operator \( D^c = \sqrt{-1}(D'' - D') \). By the Hermitian metric on \( E \) associated with the polarization \( h \) and the Kähler metric \( g \), we define the adjoints \( D^*, (D^c)^*, (D')^* \) and \( (D'')^* \) of differential operators. For the Kähler form \( \omega \) associated with \( g \), we consider the adjoint operator \( \Lambda \) of the Lefschetz operator \( A^*(M,E) \ni \alpha \mapsto \omega \wedge \alpha \in A^{*+2}(M,E) \). In the same way as the usual Kähler identity, we have

\[ [\Lambda, D] = -(D^c)^* \]
and this equation gives

\[ \Delta_D = 2\Delta_{D'} = 2\Delta_{D''} \]
where \( \Delta_D, \Delta_{D'} \) and \( \Delta_{D''} \) are the Laplacian operators (see [8]). Write

\[ \mathcal{H}^r(M,E) = \ker(\Delta_D)|_{A^r(M,E)} \quad \text{and} \quad \mathcal{H}^r(M,E)^{P,Q} = \ker(\Delta_{D''})|_{A^r(M,E)^{P,Q}}. \]
Then we have the Hodge decomposition

\[ \mathcal{H}^r(M,E) = \bigoplus_{P+Q=n+r} \mathcal{H}^r(M,E)^{P,Q}. \]

By \( \mathcal{H}^r(M,E) \cong H^r(M,E) \), we obtain a bigrading of \( H^r(M,E) \). As the ordinary case \( H^r(M,\mathbb{C}) \), the complex Hodge structure given in Theorem 3.2.1 is identified with this bigrading. Since \( \Lambda \) is a map of degree \(-2\), by the Kähler identity, we have the following useful equations

\[ \mathcal{H}^0(M,E)^{P,Q} = \ker D|_{A^0(M,E)^{P,Q}} = \ker D'|_{A^0(M,E)^{P,Q}} = \ker D''|_{A^0(M,E)^{P,Q}}. \]

### 3.3. Higgs bundles and \( \mathbb{C} \)-VHSs.

For a polarizable \( \mathbb{C} \)-VHS \( (E, D, F^*, G^*) \), by the decomposition \( E = \bigoplus_{p+q=w} E^{p,q} \) as above, \( E \) is regarded as a holomorphic vector bundle with the Dolbeault complex on a complex manifold. The harmonic metric is a Hermitian metric associated with a polarization \( h \). We define the \( \mathbb{C}^* \)-action \( \phi : \mathbb{C}^* \to GL(E) \) so that \( \phi(t)e = t^pe \) for \( e \in E^{p,q} \). Then \( \phi(t) \) is an isomorphism \( (E, \theta) \cong (E, t\theta) \).

We can characterize polarizable \( \mathbb{C} \)-VHSs in terms of Higgs bundles.

**Proposition 3.3.1** ([7] Lemma 4.1). Let \( (E, \theta) \) be a polystable Higgs bundle. If for some \( t \in \mathbb{C}^* \) which is not a root of unity we have an isomorphism \( f : (E, \theta) \cong (E, t\theta) \), then \( f \) defines a polarizable \( \mathbb{C} \)-VHS \( (E, D, F^*, G^*) \) so that \( (E, \theta) \) is given by the decomposition \( E = \bigoplus_{p+q=w} E^{p,q} \).

Consider the equivalence \( (E, \theta) \mapsto (E, D) \) in Theorem 3.1.2. For each \( t \in \mathbb{C}^* \), corresponding \( (E, t\theta) \mapsto (E, D_t) \), we can define the \( \mathbb{C}^* \)-deformations \( \{(E, D_t)\}_{t \in \mathbb{C}^*} \) of semi-simple flat bundle \( (E, D) \).
Corollary 3.3.2. Let $(E, D)$ be a simple flat bundle. If for some $t \in \mathbb{C}^*$ which is not a root of unity we have a non-trivial morphism $f : (E, D) \rightarrow (E, D_t)$, then there exists a unique polarizable $\mathbb{C}$-VHS $(E, D, F^*, G^*)$ of weight 0.

3.4. Categories. Let $M$ be a connected smooth manifold. Denote by $Fl(M)$ the category of the complex flat vector bundles over $M$ and by $Fl^s(M)$ (resp. $Fl^{ss}(M)$) the full sub-category of $Fl(M)$ whose objects are simple (resp. semi-simple) complex flat vector bundles. Suppose $M$ is a compact Kähler manifold. By the same reason as the $\mathbb{C}$-HS case, it is sufficient to consider $\mathbb{C}$-VHSs of weight 0. Define the category $\mathcal{VHS}_C(M)$ so that objects are polarizable $\mathbb{C}$-VHSs of weight 0 over $M$ and for $(E_1, D_1, F_1^*, G_1^*)$, $(E_2, D_2, F_2^*, G_2^*) \in \text{Obj}(\mathcal{VHS}_C(M))$,

$$\text{Mor}_{\mathcal{VHS}_C(M)}((E_1, D_1, F_1^*, G_1^*), (E_2, D_2, F_2^*, G_2^*)) = H^0(M, \text{Hom}(E_1, E_2))^{0,0}.$$

Remark 3.4.1. Let $\text{Rep}(\pi_1(M, x))$ be the category of finite-dimensional complex representation of the fundamental group $\pi_1(M, x)$. We consider the functor $Fl(M) \ni (E, D) \mapsto (E_x, \rho_D) \in \text{Rep}(\pi_1(M, x))$ given by taking the monodromy representations $\rho_D : \pi_1(M, x) \rightarrow GL(E_x)$ of flat bundles. It is well-known that this functor is an equivalence. This equivalence gives an isomorphism $H^0(M, E) \cong E_x^{\pi_1(M, x)}$.

For a polarizable $\mathbb{C}$-VHS $(E, D, F^*, G^*)$ over a compact Kähler manifold $M$, $E_x^{\pi_1(M, x)}$ is a complex sub-Hodge structure in the fiber $(E_x, F_x^*, G_x^*)$. This structure is isomorphic to the canonical complex Hodge structure on $H^0(M, E)$ hence this does not depend on the choice of $x \in M$. By the equivalence $Fl(M) \ni (E, D) \mapsto (E_x, \rho_D) \in \text{Rep}(\pi_1(M, x))$, we have an isomorphism

$$H^0(M, \text{Hom}(E_1, E_2))^{0,0} \cong \text{Hom}_{\pi_1(M, x)}(E_{1x}, E_{2x})^{0,0}$$

for $(E_1, D_1, F_1^*, G_1^*), (E_2, D_2, F_2^*, G_2^*) \in \text{Obj}(\mathcal{VHS}_C(M))$.

Denote by $Fl_{VHS_C}(M)$ (resp. $Fl_{VHS}^s(M)$) the full sub-category of $Fl(M)$ (resp. $Fl^s(M)$) whose objects are $(E, D) \in \text{Obj}(Fl(M))$ (resp. $\text{Obj}(Fl^s(M))$) which come from $(E, D, F^*, G^*) \in \text{Obj}(\mathcal{VHS}_C(M))$. Then it is known that $Fl_{VHS_C}(M)$ is a subcategory of $Fl^{ss}(M)$ (see [3]). By Corollary 3.3.2 we have the following statement.

Proposition 3.4.2. For each $(E, D) \in \text{Obj}(Fl_{VHS_C}(M))$, $(E, D, F^*, G^*) \in \text{Obj}(\mathcal{VHS}_C(M))$ is unique up to isomorphism.

Let $\mathcal{V}^{ss}_{VHS_C}(M)$ be the set of all isomorphism classes in $Fl_{VHS_C}^s(M)$. By Proposition 3.4.2 this can be seen as a set of isomorphism classes in $\mathcal{VHS}_C(M)$.

Proposition 3.4.3. Any $(E, D, F^*, G^*) \in \text{Obj}(\mathcal{VHS}_C(M))$ is isomorphic to

$$\bigoplus_{V \in \mathcal{V}^{ss}_{VHS_C}(M)} H^0(M, \text{Hom}(V, E)) \otimes V$$

with the natural filtrations.

Proof. We can easily say that the canonical map

$$\bigoplus_{V \in \mathcal{V}^{ss}_{VHS_C}(M)} H^0(M, \text{Hom}(V, E)) \otimes V \ni \sum f_i \otimes v_i \mapsto \sum f_i(v_i) \in V$$

is injective and compatible with filtrations.

Let $\mathcal{V}^s(M)$ be the set of all isomorphisms classes in $Fl^s(M)$. In $Fl^{ss}(M)$, $(E, D)$ is isomorphic to

$$\bigoplus_{V \in \mathcal{V}^s(M)} H^0(M, \text{Hom}(V, E)) \otimes V.$$
Lemma 3.4.4. For $V \in \text{Obj}(Fl^s(M))$, if $H^0(M, \text{Hom}(V, E)) \neq 0$, then $V \in \text{Obj}(Fl^s_{\text{HS}}(M))$.

Proof. Let $E_1 = H^0(M, \text{Hom}(V, E)) \otimes V$. Then we have a decomposition $E = E_1 \oplus E_2$ of a flat bundle $(E, D)$. Consider the $\mathbb{C}^*$-action $\phi: \mathbb{C}^* \to GL(E)$ associated with $E = \bigoplus_{p+q=w} E^{p,q}$. For $t \in \mathbb{C}^*$, define the morphism $f_t: E_1 \to E_1$ by the composition of the injection $E_1 \hookrightarrow E$, $\phi(t)$ and the projection $E \to E_1$. By $f_t = \text{id}$, for $t \in \mathbb{C}^*$ sufficiently close to 1, $f_t$ is non-trivial. Lemma 3.4.4 follows from Corollary 3.3.2.

4. The complex variations of Hodge structures with universal properties

4.1. Tannakian Categories and Hodge representations. A category $\mathcal{C}$ with a functor $\otimes: \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is an additive $\mathbb{C}$-linear category if:

- $\mathcal{C}$ is an additive $\mathbb{C}$-linear category.
- $\otimes: \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a bi-linear functor which satisfies the associativity and commutativity.
- There exists an identity object $(1, u)$ that is $1 \in \text{Ob}(\mathcal{C})$ with an isomorphism $u: 1 \to 1 \otimes 1$ satisfying the functor $\mathcal{C} \ni V \mapsto 1 \otimes V \in \mathcal{C}$ is an equivalence of categories.

A $\mathbb{C}$-tensor category $\mathcal{C}$ is rigid if all objects admit duals.

For two $\mathbb{C}$-tensor categories $(\mathcal{C}_1, \otimes_1)$ and $(\mathcal{C}_2, \otimes_2)$, a tensor functor is a functor $F: \mathcal{C}_1 \to \mathcal{C}_2$ with a natural isomorphism $\mathcal{C}_2, \otimes_2: F(U) \otimes F(V) \to F(U \otimes V)$ so that $(F, \otimes)$ is compatible with the associativities and commutativities of $(\mathcal{C}_1, \otimes_1)$ and $(\mathcal{C}_2, \otimes_2)$ (see [2]) and for an identity object $(1, u)$ of $(\mathcal{C}_1, \otimes_1)$ $(F(1), F(u))$ is an identity object of $(\mathcal{C}_2, \otimes_2)$.

The category $\text{Vect}_\mathbb{C}$ of $\mathbb{C}$-vector space with the usual tensor product $\otimes$ is a tensor category. For a tensor category $(\mathcal{C}, \otimes)$, an exact faithful tensor functor $\mathcal{C} \to \text{Vect}_\mathbb{C}$ is called a fiber functor for $\mathcal{C}$. A neutral $\mathbb{C}$-Tannakian category is an abelian rigid $\mathbb{C}$-tensor category $\mathcal{C}$ with a fiber functor $\omega: \mathcal{C} \to \text{Vect}_\mathbb{C}$ such that $\mathcal{C} = \text{End}(1)$.

Example 4.1.1. Let $G$ be a pro-algebraic group $G$ over $\mathbb{C}$. Let $\text{Rep}(G)$ be the of finite-dimensional rational representations $(V, \rho)$ of the pro-algebraic group $G$. Then $\text{Rep}(G)$ equipped with the natural fiber functor $(V, \rho) \mapsto V$ is neutral $\mathbb{C}$-Tannakian category.

Theorem 4.1.2 ([2]). Every neutral $\mathbb{C}$-Tannakian category $\mathcal{C}$ is equivalent to the category $\text{Rep}(G)$ of finite-dimensional rational representations of a pro-algebraic group $G$ over $\mathbb{C}$. More precisely, this correspondence $\mathcal{C} \mapsto G$ is a contravariant functor $\Pi$ from the category of neutral Tannakian categories to the category of pro-algebraic groups over $\mathbb{C}$ where morphisms of neutral Tannakian categories are exact faithful tensor functors commuting with fiber functors.

We call $G = \Pi(\mathcal{C})$ the Tannakian dual of a neutral $\mathbb{C}$-Tannakian category $\mathcal{C}$.

Example 4.1.3. Consider the category $\mathcal{H}S_\mathbb{C}$ of complex Hodge structures of weight 0. We can define the tensor product of $\mathcal{C}$-HSs. Thus, $\mathcal{H}S_\mathbb{C}$ is a $\mathbb{C}$-tensor category. Since $\mathcal{H}S_\mathbb{C}$ is abelian, we can easily say that $\mathcal{H}S_\mathbb{C}$ is a neutral $\mathbb{C}$-Tannakian category with the fiber functor $\text{Ob}(\mathcal{H}S_\mathbb{C}) \ni (V, F^*) \mapsto V \in \text{Ob}(\text{Vect}_\mathbb{C})$. The Tannakian dual of $\mathcal{H}S_\mathbb{C}$ is the algebraic torus $\mathbb{C}^*$.

Example 4.1.4. Let $\Gamma$ a group. Let $\text{Rep}(\Gamma)$ be the of finite-dimensional complex representations $(V, \rho)$ of the group $\Gamma$. Then $\text{Rep}(\Gamma)$ equipped with the natural fiber functor $(V, \rho) \mapsto V$ is neutral $\mathbb{C}$-Tannakian category. It is well known that the Tannakian dual of $\text{Rep}(\Gamma)$ is the pro-algebraic completion of $\Gamma$ defined by the inverse limit of the system of representations $\rho: \Gamma \to G$ with the Zariski-dense images for complex algebraic groups $G$ (see [7]).

Let $M$ be a connected smooth manifold and $x \in M$. Then the category $Fl(M)$ is a neutral $\mathbb{C}$-Tannakian category with the fiber functor $\text{Ob}(Fl(M)) \ni (E, D) \mapsto E_x \in \text{Ob}(\text{Vect}_\mathbb{C})$. We denote by $\varpi_1(M, x)$ the Tannakian dual of $Fl(M)$. $Fl^{ss}(M)$ is a semi-simple neutral $\mathbb{C}$-Tannakian category.
We denote by $\varpi_1^{red}(M, x)$ the Tannakian dual of $Fl^{ss}(M)$. Then, $\varpi_1^{red}(M, x)$ is pro-reductive. By the monodromy functor $Fl(M) \to \text{Rep}(\pi_1(M, x))$, $\varpi_1(M, x)$ is identified with the algebraic completion of the fundamental group $\pi_1(M, x)$ and $\varpi_1^{red}(M, x) = \varpi_1(M, x)/R_u(\varpi_1(M, x))$ where $R_u(\varpi_1(M, x))$ is the pro-unipotent radical of $\varpi_1(M, x)$ the natural functor $Fl^{ss}(M) \to Fl(M)$ corresponds to the quotient map.

Suppose $M$ is a compact Kähler manifold. Then $Fl_{VHS}(M)$ is a neutral $\mathbb{C}$-Tannakian category. We denote by $\varpi_1^{VHS}(M, x)$ the Tannakian dual of $Fl_{VHS}(M)$. Corresponding to the natural functor $Fl_{VHS}(M) \to Fl^{ss}(M)$, we have the surjection $\varpi_1^{red}(M, x) \to \varpi_1^{VHS}(M, x)$. Denote by $\mathcal{O}(\varpi_1^{VHS}(M, x))$ the Hopf algebra of global sections of the structure sheaf. By the left $\varpi_1^{VHS}(M, x)$-action, $\mathcal{O}(\varpi_1^{VHS}(M, x))$ is a rational $\varpi_1^{VHS}(M, x)$-module. Any rational $\varpi_1^{VHS}(M, x)$-module $U$ is identified with a co-module structure $U \to U \otimes \mathcal{O}(\varpi_1^{VHS}(M, x))$ (see [2, Proposition 2.2]). By the fundamental arguments, regarding $(U \otimes \mathcal{O}(\varpi_1^{VHS}(M, x)))^{\varpi_1^{VHS}(M, x)}$ as a rational $\varpi_1^{VHS}(M, x)$-module via the right action we have an isomorphism

$$(U \otimes \mathcal{O}(\varpi_1^{VHS}(M, x)))^{\varpi_1^{VHS}(M, x)} \cong U$$

of rational $\varpi_1^{VHS}(M, x)$-modules given by mapping $\sum u \otimes f \in (U \otimes \mathcal{O}(\varpi_1^{VHS}(M, x)))^{\varpi_1^{VHS}(M, x)}$ to $\sum f(1)u \in U$ where $1$ is the unit element in the group $\varpi_1^{VHS}(M, x)$ (the proof of [4, Theorem 9.1] is valid without any change). By Proposition 3.4.3 $Fl_{VHS}(M)$ is semi-simple and any simple object of $Fl_{VHS}(M)$ is in $Fl_{V_{VHS}}(M)$. Thus, as a rational $\varpi_1^{VHS}(M, x) \times \varpi_1^{VHS}(M, x)$-module, we have an isomorphism

$$\mathcal{O}(\varpi_1^{VHS}(M, x)) \cong \bigoplus_{V \in V_{VHS}(M)} V_x^* \otimes V_x$$

given by mapping $\sum f \in \mathcal{O}(\varpi_1^{VHS}(M, x))$ to the function

$$(\varpi_1^{VHS}(M, x) \ni g \mapsto \sum f(gv) \in \mathbb{C}) \in \mathcal{O}(\varpi_1^{VHS}(M, x))$$

This isomorphism corresponds to the tensor product in the tensor category $Fl_{VHS}(M)$.

Let $Hi(M)$ be the category of Higgs bundles $(E, \theta)$ over $M$ and $Hi^{ps}(M)$ (resp $Hi^s(M)$) the full sub-category of poly-stable (resp. stable) Higgs bundles. For each $t \in \mathbb{C}^*$, we define the functor $Hi(M) \to Hi(M)$ by $(E, \theta) \mapsto (E, t\theta)$. This functor is restricted to functors on $Hi^{ps}(M)$ and $Hi^s(M)$. Corresponding to the equivalence $Fl^{ss}(M) \cong Hi^{ps}(M)$ as Theorem 3.1.2 we can define a functor $\alpha_t : \text{Rep}(\varpi_1^{red}(M, x)) \to \text{Rep}(\varpi_1^{red}(M, x))$. By Proposition 3.3.1 $(U, \rho) \in \text{Rep}(\varpi_1^{red}(M, x))$ if and only if for some $t \in \mathbb{C}^*$ which is not a root of unity (resp. every $t \in \mathbb{C}^*$) $\alpha_t(U, \rho)$ is isomorphic to $(U, \rho)$. For each simple representation $(U, \rho) \in \text{Rep}(\varpi_1^{VHS}(M, x))$, corresponding to an isomorphism $\alpha_t(U, \rho) \cong (U, \rho)$ for $t \in \mathbb{C}^*$, we have a canonical $\mathbb{C}$-HS on $U$. We can regard $(U, \rho)$ as $V_x$ with the monodromy representation for $V \in V_{VHS}(M)$. It is easily check that a canonical $\mathbb{C}$-HS on $U$ is identified with the $\mathbb{C}$-HS on $V_x$ associated with a unique $\mathbb{C}$-VHS structure on $V \in V_{VHS}(M)$. By the tannakian duality (Theorem 4.1.2), corresponding to the functor $\alpha_t : \text{Rep}(\varpi_1^{VHS}(M, x)) \to \text{Rep}(\varpi_1^{VHS}(M, x))$ for each $t \in \mathbb{C}^*$, we have the $\mathbb{C}^*$-action on the pro-algebraic group $\varpi_1^{VHS}(M, x)$. Hence, $\mathbb{C}^*$ acts on $\mathcal{O}(\varpi_1^{VHS}(M, x))$ as Hopf algebra automorphisms. We can say that this action implies a multiplicative $\mathbb{C}$-HS on $\mathcal{O}(\varpi_1^{VHS}(M, x))$ which is identified with the $\mathbb{C}$-HS on

$$\mathcal{O}(\varpi_1^{VHS}(M, x)) \cong \bigoplus_{V \in V_{VHS}(M)} V_x^* \otimes V_x$$
Lemma 4.1.6. For any $V \in V_{VHS}^{*}(M)$.

Proof. It is sufficient to show that the co-module structure $V \mapsto V \otimes \mathcal{O}(\pi_{1}^{VHS}(M,x))$ of $\mathcal{O}(\pi_{1}^{VHS}(M,x))$ may not define a rational $\mathbb{C}^{*}$-module and so may not define a $\mathbb{C}$-HS. For example, let $M$ be the compact complex torus $T^{2}$. By Remark 4.1.5, the $\mathbb{C}^{*}$-action on $\pi_{1}^{red}(M,x)$ gives permutations on the product $\mathbb{II}C^{*}$ indexed by an uncountable set. Hence the $\mathbb{C}^{*}$-action on $\mathcal{O}(\pi_{1}^{red}(M,x))$ does not define a $\mathbb{C}$-HS.

Define the category $\text{Rep}_{\mathbb{C}}^{HS}(\pi_{1}^{VHS}(M,x))$ so that

- Objects are $(U,F^{*},G^{*})$ such that
  - $(U,F^{*},G^{*})$ are in $\mathcal{H}_{\mathbb{C}}$.
  - $U$ are rational $\pi_{1}^{VHS}(M,x)$-modules such that the associated co-module structures $U \mapsto U \otimes \mathcal{O}(\pi_{1}^{VHS}(M,x))$ are morphisms of $\mathbb{C}$-HSs.
- For $(U_{1},F_{1}^{*},G_{1}^{*}),(U_{2},F_{2}^{*},G_{2}^{*}) \in \text{Ob}(\text{Rep}_{\mathbb{C}}^{HS}(\pi_{1}^{VHS}(M,x)))$, morphisms from $(U_{1},F_{1}^{*},G_{1}^{*})$ to $(U_{2},F_{2}^{*},G_{2}^{*})$ are morphisms of $\mathbb{C}$-HSs which are compatible with $\pi_{1}^{VHS}(M,x)$-module structures.

Lemma 4.1.6. For any $V \in V_{\pi_{1}^{VHS}(M,x)}^{*}$, $V_{x}$ is an object in $\text{Rep}_{\mathbb{C}}^{HS}(\pi_{1}^{VHS}(M,x))$.

Proof. It is sufficient to show that the co-module structure $V_{x} \mapsto V_{x} \otimes \mathcal{O}(\pi_{1}^{VHS}(M,x))$ is a $\mathbb{C}$-HS morphism. Let $\rho$ be the representation of $\pi_{1}^{VHS}(M,x)$ associated with the $\pi_{1}^{VHS}(M,x)$-module $V_{x}$. The co-module structure is given by $V_{x} \ni v \mapsto \rho(v) \in V_{x} \otimes \mathcal{O}(\pi_{1}^{VHS}(M,x))$. Regarding the domain $V_{x}$ as a right $\pi_{1}^{VHS}(M,x)$-module and $V_{x}$ in the codomain as a left $\pi_{1}^{VHS}(M,x)$-module, the map $V_{x} \ni v \mapsto \rho(v) \in V_{x} \otimes \mathcal{O}(\pi_{1}^{VHS}(M,x))$ is $\pi_{1}^{VHS}(M,x) \times \pi_{1}^{VHS}(M,x)$-equivariant. By

$$\mathcal{O}(\pi_{1}^{VHS}(M,x)) \cong \bigoplus_{V \in V_{\pi_{1}^{VHS}(M)}^{*}} V^{*} \otimes V,$$

as $\pi_{1}^{VHS}(M,x) \times \pi_{1}^{VHS}(M,x)$-modules, we can consider the co-module structure as an element in

$$(V_{x}^{*} \otimes V_{x})^{\pi_{1}^{VHS}(M,x)} \otimes (V_{x}^{*} \otimes V_{x})^{\pi_{1}^{VHS}(M,x)},$$

and hence this is of type $(0,0)$ for the $\mathbb{C}$-HS. Thus the lemma follows.

Theorem 4.1.7. We have an equivalence $\mathcal{VHS}_{\mathbb{C}}(M) \cong \text{Rep}_{\mathbb{C}}^{HS}(\pi_{1}^{VHS}(M,x))$.

Proof. For $(E,D,F^{*},G^{*}) \in \text{Obj}(\mathcal{VHS}_{\mathbb{C}}(M))$, by

$$E_{x} \cong \bigoplus_{V \in V_{\pi_{1}^{VHS}(M)}^{*}} \mathcal{H}^{0}(M,\text{Hom}(V,E)) \otimes V,$$

(Proposition 3.4.3), $(E_{x},F_{x}^{*},G_{x}^{*})$ is an object in $\text{Rep}_{\mathbb{C}}^{HS}(\pi_{1}^{VHS}(M,x))$. By Remark 3.4.1, this correspondence is a fully faithful functor. We can construct a quasi-inverse by the following way. We correspond $(U,F^{*},G^{*}) \in \text{Obj}(\text{Rep}_{\mathbb{C}}^{HS}(\pi_{1}^{VHS}(M,x)))$ to an object

$$\bigoplus_{V \in V_{\pi_{1}^{VHS}(M)}^{*}} \text{Hom}_{\pi_{1}^{VHS}(M)}(V_{x},U) \otimes V$$

in $\mathcal{VHS}_{\mathbb{C}}(M)$.
4.2. The $\mathbb{C}$-VHS with universal properties. We define the local system $\mathcal{O}^{VHS}(x)$ over $M$ associated with $\mathcal{O}(\omega_1^{VHS}(M, x))$ as a $\pi_1(M, x)$-module via the left $\omega_1^{VHS}(M, x)$-action.

**Proposition 4.2.1.**
1. $\mathcal{O}^{VHS}(x) \cong \bigoplus_{V \in \mathcal{V}_{\omega_1^{VHS}(M)}} V^* \otimes V_x$.
2. $\mathcal{O}^{VHS}(x)$ admits a canonical $\mathbb{C}$-VHS structure (the direct sum of objects in $\mathcal{V}_S(M)$) depending on $x \in M$.
3. $\mathcal{O}^{VHS}(x)$ admits a structure $\mathcal{O}^{VHS}(x) \otimes \mathcal{O}^{VHS}(x) \to \mathcal{O}^{VHS}(x)$ of a local system of $\mathbb{C}$-algebras.
4. The algebra structure $\mathcal{O}^{VHS}(x) \otimes \mathcal{O}^{VHS}(x) \to \mathcal{O}^{VHS}(x)$ is a morphism of $\mathbb{C}$-VHSs.

**Proof.** (1) is a consequence of $\mathcal{O}^{VHS}(x) \cong \bigoplus_{V \in \mathcal{V}_{\omega_1^{VHS}(M)}} V^* \otimes V_x$.

(2) follows from (1).

Since the algebra structure $\mathcal{O}(\omega_1^{VHS}(M, x)) \otimes \mathcal{O}(\omega_1^{VHS}(M, x)) \to \mathcal{O}(\omega_1^{VHS}(M, x))$ is equivariant for the left $\omega_1^{VHS}(M, x)$-action, this gives (3).

We have seen that the algebra structure $\mathcal{O}(\omega_1^{VHS}(M, x)) \otimes \mathcal{O}(\omega_1^{VHS}(M, x)) \to \mathcal{O}(\omega_1^{VHS}(M, x))$ is a $\mathbb{C}$-HS morphism. Since we have an isomorphism between $\mathcal{H}^0(M, \mathcal{O}(\omega_1^{VHS}(M, x)) \otimes \mathcal{O}(\omega_1^{VHS}(M, x)), \mathcal{O}(\omega_1^{VHS}(M, x)))^{0,0}$ and $\text{Hom}_{\pi_1(M, x)}(\mathcal{O}(\omega_1^{VHS}(M, x)), \mathcal{O}(\omega_1^{VHS}(M, x)))^{0,0}$ (Remark 3.4.1), (4) follows.

**Proposition 4.2.2.** $\mathcal{O}^{VHS}(x)$ admits a structure $\omega_1^{VHS}(M, x) \times \mathcal{O}^{VHS}(x) \to \mathcal{O}^{VHS}(x)$ of a local system of rational $\omega_1^{VHS}(M, x)$-modules so that:

1. For any $(U, F^*, G^*) \in \text{Obj}(\text{Rep}_S(\omega_1^{VHS}(M, x)))$, we have $(\mathcal{O}^{VHS}(x) \otimes U)^{\omega_1^{VHS}(M, x)} \in \mathcal{V}_S(M)$.
2. The correspondence $(U, F^*, G^*) \mapsto (\mathcal{O}^{VHS}(x) \otimes U)^{\omega_1^{VHS}(M, x)}$ is an equivalence $\mathcal{V}_S(M) \cong \text{Rep}_S(\omega_1^{VHS}(M, x))$.
3. For any $(E, D, F^*, G^*) \in \text{Obj}(\mathcal{V}_S(M))$, $(\mathcal{O}^{VHS}(x) \otimes E_x)^{\omega_1^{VHS}(M, x)} \in \text{Obj}(\mathcal{V}_S(M))$ is isomorphic to $(E, D, F^*, G^*)$.

**Proof.** Since the right $\omega_1^{VHS}(M, x)$-action on $\mathcal{O}(\omega_1^{VHS}(M, x))$ commutes with the left action, the rational $\omega_1^{VHS}(M, x)$-module structure on $\mathcal{O}(\omega_1^{VHS}(M, x))$ via the right action defines structure $\omega_1^{VHS}(M, x) \times \mathcal{O}^{VHS}(x) \to \mathcal{O}^{VHS}(x)$ of a local system of rational $\omega_1^{VHS}(M, x)$-modules.

By $\mathcal{O}^{VHS}(x) \cong \bigoplus_{V \in \mathcal{V}_{\omega_1^{VHS}(M)}} V^* \otimes V_x$, for a rational $\omega_1^{VHS}(M, x)$-module $U$, we have $(\mathcal{O}^{VHS}(x) \otimes U)^{\omega_1^{VHS}(M, x)} \cong \bigoplus_{V \in \mathcal{V}_{\omega_1^{VHS}(M)}} \text{Hom}_{\pi_1(M, x)}(V, U) \otimes V$.

Thus, the correspondence $(U, F^*, G^*) \mapsto (\mathcal{O}^{VHS}(x) \otimes U)^{\omega_1^{VHS}(M, x)}$ is the functor giving Theorem 4.1.7. Hence, (1), (2) and (3) are obvious.

**Theorem 4.2.3.** Consider the de Rham complex $A^*(M, \mathcal{O}^{VHS}(x))$ with values in the local system $\mathcal{O}^{VHS}(x)$. Then:

1. The cochain complex $A^*(M, \mathcal{O}^{VHS}(x))$ admits a canonical differential graded algebra structure.
The differential graded algebra \( A^*(M, \Omega^{VHS}(x)) \) admits a canonical bidifferential bigraded algebra structure.

\( A^*(M, \Omega^{VHS}(x)) \) is a rational \( \varpi^1_{VHS}(M, x) \)-module such that \( \varpi^1_{VHS}(M, x) \) acts as automorphisms of the differential graded algebra.

The co-module structure \( A^*(M, \Omega^{VHS}(x)) \rightarrow A^*(M, \Omega^{VHS}(x)) \otimes O(\varpi^1_{VHS}(M, x)) \) is a morphism of bidifferential bigraded algebras.

For \( (U, F^*, G^*) \in \text{Obj}(\text{Rep}^{HS}(\varpi^1_{VHS}(M, x))) \),

\[
(A^*(M, \Omega^{VHS}(x)) \otimes U)\varpi^1_{VHS}(M, x)
\]

is a double complex.

For \( (E, D, F^*, G^*) \in \text{Obj}({\mathcal{VHS}}_{\mathbb{C}}(M)) \), the double complex \( (A^*(M, \Omega^{VHS}(x)) \otimes E_2)\varpi^1_{VHS}(M, x) \)

is isomorphic to the double complex

\[
(A^*(M, E)^{p,q}, D', D'')
\]
as Subsection 3.2.

**Proof.** \( A^*(M, \Omega^{VHS}(x)) \) is a differential graded algebra such that the multiplications are defined by the wedge product and the algebra structure on \( \Omega^{VHS}(x) \). (1) follows.

We define the double complex structure on \( A^*(M, \Omega^{VHS}(x)) \) by the complex structure on \( M \) and the complex structure on \( \Omega^{VHS}(x) \) as Subsection 3.2. Then, since \( \Omega^{VHS}(x) \otimes \Omega^{VHS}(x) \rightarrow \Omega^{VHS}(x) \) is a morphism of \( \mathbb{C} \)-VHS, the product on \( A^*(M, \Omega^{VHS}(x)) \) is compatible with the bigrading. Hence, we have the bidifferential bigraded algebra structure on \( A^*(M, \Omega^{VHS}(x)) \). (2) follows.

The structure \( \varpi^1_{VHS}(M, x) \times \Omega^{VHS}(x) \rightarrow \Omega^{VHS}(x) \) of a local system of rational \( \varpi^1_{VHS}(M, x) \)-modules defines a rational \( \varpi^1_{VHS}(M, x) \)-module structure on \( A^*(M, \Omega^{VHS}(x)) \) such that the \( \varpi^1_{VHS}(M, x) \)-action commutes with the differential on \( A^*(M, \Omega^{VHS}(x)) \). Since the algebra structure \( O(\varpi^1_{VHS}(M, x)) \otimes O(\varpi^1_{VHS}(M, x)) \rightarrow O(\varpi^1_{VHS}(M, x)) \) on \( O(\varpi^1_{VHS}(M, x)) \) is equivariant for the right \( \varpi^1_{VHS}(M, x) \)-action, the \( \varpi^1_{VHS}(M, x) \)-action on \( A^*(M, \Omega^{VHS}(x)) \) is compatible with the multiplication on \( A^*(M, \Omega^{VHS}(x)) \). Thus (3) follows.

By \( \Omega^{VHS}(x) \cong \bigoplus_{V \in \mathcal{VHS}_{\mathbb{C}}(M)} V^* \otimes \mathbb{V} \), the \( (p, q) \)-component of the bigrading on \( A^*(M, \Omega^{VHS}(x)) \) is given by

\[
\bigoplus_{V \in \mathcal{VHS}_{\mathbb{C}}(M)} \bigoplus_{a+b=p, c+d=q} A^*(M, V^*)^{a,c} \otimes V_x^{b,d}.
\]

We can easily check (4).

The subspace \( (A^*(M, \Omega^{VHS}(x)) \otimes U)\varpi^1_{VHS}(M, x) \subset A^*(M, \Omega^{VHS}(x)) \otimes U \) is the kernel of the map

\[
A^*(M, \Omega^{VHS}(x)) \otimes U \ni \omega \mapsto \rho(\omega) - \omega \in A^*(M, \Omega^{VHS}(x)) \otimes U \otimes O(\varpi^1_{VHS}(M, x))
\]

where \( \omega \mapsto \rho(\omega) \) is the co-module structure of \( O(\varpi^1_{VHS}(M, x)) \) on \( A^*(M, \Omega^{VHS}(x)) \otimes U \). Thus, (5) follows from (4).

By Proposition 3.4.3 we can identify \( (E, D, F^*, G^*) \in \text{Obj}(\mathcal{VHS}_{\mathbb{C}}(M)) \) with

\[
\bigoplus_{V \in \mathcal{VHS}_{\mathbb{C}}(M)} \mathcal{H}^{0}(M, \text{Hom}(V, E)) \otimes V.
\]

We have

\[
A^*(M, E)^{p,q} = \bigoplus_{V \in \mathcal{VHS}_{\mathbb{C}}(M)} \bigoplus_{a+b=p, c+d=q} A^*(M, V)^{a,c} \otimes \mathcal{H}^{0}(M, \text{Hom}(V, E))^{c,d}.
\]
For
\[ E_x = \bigoplus_{V \in V^*_{VHS}(M)} \mathcal{H}^0(M, \text{Hom}(V, E)) \otimes V_x, \]
we can directly check (6).

\[ \square \]

By Theorem 3.2.1, we have:

**Corollary 4.2.4.**

1. The cohomology \( H^*(M, \mathcal{O}^{VHS}(x)) \) with values in the local system \( \mathcal{O}^{VHS}(x) \) admits a canonical structure \( H^*(M, \mathcal{O}^{VHS}(x)) \otimes H^*(M, \mathcal{O}^{VHS}(x)) \rightarrow H^*(M, \mathcal{O}^{VHS}(x)) \) of a graded-commutative \( \mathbb{C} \)-algebra.
2. \( H^i(M, \mathcal{O}^{VHS}(x)) \) admits a canonical complex Hodge structure of weight \( i \).
3. The multiplication \( H^i(M, \mathcal{O}^{VHS}(x)) \otimes H^j(M, \mathcal{O}^{VHS}(x)) \rightarrow H^{i+j}(M, \mathcal{O}^{VHS}(x)) \) is a morphism of \( \mathbb{C} \)-HSs.
4. \( H^*(M, \mathcal{O}^{VHS}(x)) \) is the direct sum of objects in \( \text{Rep}^{HS}_{x}^{VHS}(M, x) \).
5. For \((E, D, F^*, G^*) \in \text{Obj}(VHS_{\mathbb{C}}(M))\), we have a \( \mathbb{C} \)-HS isomorphism
   \[ (H^i(M, \mathcal{O}^{VHS}(x)) \otimes E_x) \otimes^{VHS}(M, x) \otimes H^i(M, E_x). \]
6. For \((E_1, D_1, F^*_1, G^*_1), (E_2, D_2, F^*_2, G^*_2) \in \text{Obj}(VHS_{\mathbb{C}}(M))\), as a morphism of \( \mathbb{C} \)-HSs, the cup product \( H^i(M, E_1) \otimes H^*(M, E_2) \rightarrow H^i(M, E_1 \otimes E_2) \) is identified with the restriction of the tensor product of the multiplication
   \[ H^i(M, \mathcal{O}^{VHS}(x)) \otimes H^j(M, \mathcal{O}^{VHS}(x)) \rightarrow H^{i+j}(M, \mathcal{O}^{VHS}(x)) \]
   and the identity
   \[ E_{1x} \otimes E_{2x} \rightarrow E_{1x} \otimes E_{2x} \]
   by isomorphisms
   \[ (H^*(M, \mathcal{O}^{VHS}(x)) \otimes E_{kx}) \otimes^{VHS}(M, x) \otimes H^*(M, E_k) \]
   for \( k = 1, 2 \).

**Remark 4.2.5.** For \((E, D, F^*, G^*) \in \text{Obj}(VHS_{\mathbb{C}}(M))\), we have the complex Hodge structure of weight \( i \) on the cohomology \( H^i(M, \mathcal{O}^{VHS}(x) \otimes E) \). By
\[ E_x = \bigoplus_{V \in V^*_{VHS}(M)} \mathcal{H}^0(M, \text{Hom}(V, E)) \otimes V_x, \]
we have an isomorphism \( H^0(M, \mathcal{O}^{VHS}_x \otimes E) \cong (E_x, F^*_x, G^*_x) \) of \( \mathbb{C} \)-HSs.

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