Commutators on power series spaces

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Abstract We prove that every continuous linear operator acting on a stable, nuclear power series space is a commutator, so in particular our theorem holds for the space of all entire functions, holomorphic functions on the unit disc or smooth functions. We also show that on the product of Fréchet spaces \( \prod_{i=1}^{\infty} X \) all operators are commutators.

Keywords Commutators · Nuclear sequence spaces · Power series spaces · Operators on Fréchet spaces

Mathematics Subject Classification Primary: 47B47 · 46A45; Secondary: 46A11 · 46E10

1 Introduction

A commutator of a pair of elements \( A \) and \( B \) in the algebra \( \mathcal{L}(X) \) of linear, continuous operators on a locally convex space \( X \) is given by

\[
[A, B] := AB - BA.
\]

The problem of representing operators as commutators comes from quantum mechanics, where the so-called commutator relation plays an important role. Commutators are also connected with derivations in algebra as they are the main examples of derivations (the so-called inner derivations).

As shown by Wintner [11], on a Banach space not every operator is a commutator. More precisely, all elements of the form \( \lambda I + M \), where \( \lambda \neq 0 \), \( M \) lies in a proper, closed ideal \( \mathcal{M} \) of a Banach algebra \( \mathcal{A} \) cannot be commutators.
We will show that in many classical Fréchet spaces (like spaces of holomorphic or smooth functions) every operator is a commutator (see Corollaries 2, 3). More precisely this is true for every nuclear stable power series space, both of finite and infinite type (see Theorem 1). We also prove that our method cannot be applied in the non-nuclear case although the question on which Fréchet spaces all operators are commutators remains open.

In the Banach space case a lot is known. Due to Wintner the question is whether or not in the algebra \( L(X) \) with the biggest, proper, closed ideal \( \mathcal{M} \) all operators not of the form \( \lambda I + M, M \in \mathcal{M}, \lambda \neq 0 \) are commutators. For many classical Banach spaces the answer is affirmative. First, in 1965 Brown and Pearcy [3] proved the above hypothesis for separable Hilbert spaces. Then, in 1972–73 Apostol ([1], [2]) showed that the same holds for \( \ell^p, 1 < p < \infty \) and \( c_0 \). Apostol also got some partial results for \( \ell^1 \) and \( \ell^\infty \), but it was Dosev who generalized Apostol’s method and in 2009 gave a complete description of commutators on \( \ell^1 \) [4]. Two years later Dosev and Johnson [5] proved that on \( \ell^\infty \) the commutators are all operators not of the form \( \lambda I + S, \lambda \neq 0 \), where \( S \) is a strictly singular operator. This problem was also researched for non-sequence spaces. In 2011 Dosev, Johnson and Schechtman [6] showed that on the space \( L^p[0, 1], 1 \leq p < \infty \), the above hypothesis holds, with \( \mathcal{M} = \{ T \in L(L_\ell^p) : \forall A, B \in L(L_\ell^p) I \neq ATB \} \) as the largest closed ideal.

In this paper we study commutators on some Fréchet spaces. Using Dosev’s technique we show that on Fréchet spaces Wintner’s theorem does not hold in general and there exist spaces on which all operators are commutators.

2 Preliminaries

In this section we will introduce some notation and recall basic facts needed in the paper. We will also describe the method used by Dosev and adjusted to our needs.

Let a matrix \( A = (a_{k,j})_{k,j \in \mathbb{N}} \) of non-negative numbers satisfy

- for each \( j \in \mathbb{N} \) there exists a \( k \in \mathbb{N} \) with \( a_{k,j} > 0 \),
- \( a_{k,j} \leq a_{k+1,j} \) for all \( j, k \in \mathbb{N} \).

We define

\[
\lambda^1(A) = \left\{ x = (x_j) \in \mathbb{R}^\mathbb{N} : \|x\|_k = \sum_{j=0}^{\infty} |x_j| a_{k,j} < \infty \quad \forall k \in \mathbb{N} \right\},
\]

\[
\lambda^\infty(A) = \left\{ x \in \mathbb{R}^\mathbb{N} : \|x\|_k := \sup_{j \in \mathbb{N}} |x_j| a_{k,j} < \infty \quad \forall k \in \mathbb{N} \right\}.
\]

We call the matrix \( A \) a Köthe matrix and the spaces \( \lambda^1(A) \) and \( \lambda^\infty(A) \) Köthe sequence spaces associated with the matrix \( A \). If \( A \) is a Köthe matrix then spaces \( \lambda^1(A), \lambda^\infty(A) \) are Fréchet spaces with the topology given by the family of seminorms \( (\|\cdot\|_k) \) [8, 27.1].

In our paper we consider one of the most important classes of Köthe spaces, the class of power series spaces.

For any non-negative monotonically increasing sequence \( \alpha = (\alpha_j) \) tending to infinity and for \( r = 0 \) or \( r = \infty \) we define

\[
\Lambda_r(\alpha) = \left\{ x = (x_j) \in \mathbb{R}^\mathbb{N} : \|x\|_t = \sum_{j=0}^{\infty} |x_j| e^{t\alpha_j} < \infty \quad \text{for all } t < r \right\}.
\]
The space $\Lambda_r(\alpha)$ is called a power series space, for $r = 0$ of finite type and for $r = \infty$ of infinite type. For any sequence $t_k \to r$ and the matrix $A = (\exp(t_k \alpha_j))$ we have that $\Lambda_r(\alpha) = \lambda^1(A)$.

We say that the space $\Lambda_r(\alpha)$ is stable, if

$$\sup_{n \in \mathbb{N}} \frac{\alpha_{2n}}{\alpha_n} < \infty.$$ 

This is equivalent to $\Lambda_r \times \Lambda_r \cong \Lambda_r$ (see [7]).

**Lemma 1** ([8, 29.6]) The space $\Lambda_0(\alpha)$ is nuclear if and only if $\lim_{n \to \infty} \alpha_n^{-1} \ln n = 0$. The space $\Lambda_{\infty}(\alpha)$ is nuclear if and only if $\sup_{n \in \mathbb{N}} \alpha_n^{-1} \ln n < \infty$.

Let $X$ be a Fréchet space with a Schauder basis $(e_j)_{j=0}^{\infty}$. For Köthe sequence spaces we will always use the unit vector basis. We define the shift operators in the following way:

$$R\left(\sum_{j=0}^{\infty} x_j e_j\right) = \sum_{j=0}^{\infty} x_j e_{j+1},$$

$$L\left(\sum_{j=0}^{\infty} x_j e_j\right) = \sum_{j=1}^{\infty} x_j e_{j-1}.$$ 

For a given space $X$ it is not always true that these operators are well defined and continuous. Our first lemma shows on which Köthe spaces the operators $R$ and $L$ are continuous.

**Lemma 2** Let $A = (a_{k,j})_{k,j \in \mathbb{N}}$ be a Köthe matrix.

(a) Operator $L$ is continuous on $\lambda^1(A)$ if and only if

$$\forall k \in \mathbb{N} \exists p \in \mathbb{N} \exists c \in \mathbb{R} \forall j \in \mathbb{N} \ a_{k,j-1} \leq ca_{p,j}. \quad (1)$$

(b) Operator $R$ is continuous on $\lambda^1(A)$ if and only if

$$\forall k \in \mathbb{N} \exists p \in \mathbb{N} \exists c \in \mathbb{R} \forall j \in \mathbb{N} \ a_{k,j+1} \leq ca_{p,j}. \quad (2)$$

**Proof** (a) $\Leftarrow$: Let the matrix $A = (a_{k,j})_{k,j \in \mathbb{N}}$ be such that $(1)$ holds. Then for all $k \in \mathbb{N}$, $x = \sum_{j=0}^{\infty} x_j e_j \in \lambda^1(A)$ we have

$$\|Lx\|_k = \|\sum_{j=1}^{\infty} x_j e_{j-1}\|_k = \sum_{j=1}^{\infty} x_j a_{k,j-1} \leq \sum_{j=1}^{\infty} x_j ca_{p,j} \leq c\|x\|_p,$$

where $p, c$ satisfy $(1)$. $\Rightarrow$: The operator $L$ is continuous, hence by taking $x = e_j$ we get $(1)$.

The proof of (b) is similar. $\square$

**Corollary 1** If $\sup_j \frac{\alpha_{j+1}}{\alpha_j} < \infty$ then the operators $R$ and $L$ are continuous on $\Lambda_r(\alpha)$ for $r = 0$ or $r = \infty$.

For our main theorem we will need the following lemma, which was stated and proved for Banach spaces by Dosev [4] and which can be easily transfered to the Fréchet case.

**Lemma 3** ([4, Lemma 3]) Let the operators $R$ and $L$ be well-defined and continuous on a Fréchet space $X$ with a Schauder basis $(e_j)_{j=0}^{\infty}$ and let $T \in \mathcal{L}(X)$. If the series $\sum_{n=0}^{\infty} R^n T L^n$ is pointwise convergent, then $T$ is a commutator.
Proof Let $S$ be the pointwise limit of $\sum_{n=0}^{\infty} R^n TL^n$. By the Banach-Steinhaus theorem we get $S \in \mathcal{L}(X)$ and for all $x \in X$ we have

$$(L(RS) - (RS)L)x = LR \sum_{n=0}^{\infty} R^n TL^n x - R \left( \sum_{n=0}^{\infty} R^n TL^n \right) L x = \sum_{n=0}^{\infty} R^n TL^n x - \sum_{n=1}^{\infty} R^n TL^n x = R^0 TL^0 x = Tx.$$ 

Hence $T = L(RS) - (RS)L$ is a commutator.

It will be convenient for us to look at operators acting on a Fréchet space $X$ with the Schauder basis $(e_j)_{j \in \mathbb{N}}$ as infinite matrices. For any operator $T \in \mathcal{L}(X)$ and for all $j$ there exists the unique representation $Te_j = \sum_{i=0}^{\infty} t_{i,j} e_i$ for some sequence $(t_{i,j})_{i \in \mathbb{N}}$. Hence there is an infinite matrix $(t_{i,j})_{i,j \in \mathbb{N}}$ such that

$$Tx = \sum_{i=0}^{\infty} \left( \sum_{j=0}^{\infty} t_{i,j} x_j \right) e_i \quad \text{for all } x = \sum_{j=0}^{\infty} x_j e_j. \quad (3)$$

We will say that the operator $T \in \mathcal{L}(X)$ is defined by the matrix $(t_{i,j})$ whenever $(3)$ holds.

From now on, an element $x = \sum_{j=0}^{\infty} x_j e_j$ in a Fréchet space $X$ with the Schauder basis $(e_j)_{j=0}^{\infty}$ will be denoted by $(x_j) = (x_0, x_1, x_2, \ldots)$.

Now, we will describe the sum $\sum_{n=0}^{\infty} R^n TL^n x$ for a given operator $T \in \mathcal{L}(X)$ and an element $x = (x_j) \in X$. Let the operator $T$ be defined by a matrix $(t_{i,j})_{i,j \in \mathbb{N}}$. For every $x = (x_j)_{j \in \mathbb{N}} \in X$ we have

$$R^n TL^n x = \begin{pmatrix} 0, \ldots, 0, & \sum_{j=0}^{\infty} t_{0,j} x_{j+n} & \sum_{j=0}^{\infty} t_{1,j} x_{j+n} & \sum_{j=0}^{\infty} t_{2,j} x_{j+n} & \ldots \end{pmatrix}.$$ 

Hence, for every $M, N \in \mathbb{N}, M > N$

$$\sum_{n=N}^{M} R^n TL^n x = \begin{pmatrix} 0, \ldots, 0, & \sum_{j=0}^{\infty} t_{0,j} x_{j+N} & \sum_{j=0}^{\infty} t_{1,j} x_{j+N+1} & \sum_{j=0}^{\infty} t_{1,j} x_{i+N+1} & \ldots \end{pmatrix} + \sum_{i=0}^{k-N} \sum_{j=0}^{\infty} t_{i,j} x_{j+k-i} + \ldots, \quad \sum_{i=k-M}^{k-N} \sum_{j=0}^{\infty} t_{i,j} x_{j+k-i} + \ldots \quad (4)$$

k-th coordinate, $k \leq M$ \quad k-th coordinate, $k > M$

3 Main results

Now, we will prove our main theorem, which indicates the class of Fréchet spaces on which every operator is a commutator.

**Theorem 1** Every continuous linear operator on a nuclear and stable power series space is a commutator.

**Proof** The idea of the proof is to show that for any continuous operator $T$, the series $\sum_{n=0}^{\infty} R^n TL^n$ is pointwise convergent and so, due to Corollary 1 and Lemma 3 the operator
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T is a commutator. First, we will prove the theorem for finite type power series spaces, and then for infinite type.

Let \( X = \Lambda_0(\alpha) \) be a nuclear and stable power series space of finite type. Let us recall that stability of power series space means that for every \( j \in \mathbb{N} \) we have \( \alpha_{2j} \leq C \alpha_j \) for some constant \( C \). It will be more convenient for us to use an equivalent condition

\[
\exists c \forall j, k \quad \alpha_{j+k} \leq c(\alpha_j + \alpha_k). \tag{5}
\]

Take an operator \( T \in \mathcal{L}(\Lambda_0(\alpha)) \). Since \( \Lambda_0(\alpha) \) is a nuclear space we have that \( \Lambda_0(\alpha) = \lambda^1(\exp(-\frac{1}{k} \alpha_j)) = \lambda^\infty(\exp(-\frac{1}{k} \alpha_j)) \) [8, 28.16] and so we can consider \( T \) as an operator acting from \( \Lambda_0(\alpha) \) to \( \lambda^\infty(\exp(-\frac{1}{k} \alpha_j)) \). Let \( T \) be given by the matrix \((t_{i,j})_{i,j \in \mathbb{N}}\). Then, for all \( p \in \mathbb{N} \) there exist constants \( M_p, m_p \) such that for all \( m > m_p \)

\[
\sup_{i,j} \left| t_{i,j} \right| \exp \left( \frac{1}{p} \alpha_j \right) \leq M_p. \tag{6}
\]

Indeed, take \( p \in \mathbb{N} \). Since the operator \( T \) is continuous, there exists \( M_p, m_p \) such that

\[
\|Tx\|_p \leq M_p \|x\|_{m_p} \quad \text{for all } x.
\]

Consider the sequence \((x^j)_{j \in \mathbb{N}} \subset \Lambda_0(\alpha)\) defined as follows

\[
x^j = (0, \ldots, 0, \underbrace{e_{\frac{1}{m_p} \alpha_j}}_{\text{j-th coordinate}}, 0, \ldots).
\]

Then for all \( j \in \mathbb{N} \) we have \( \|x^j\|_{m_p} = 1 \) and

\[
\sup_{i \in \mathbb{N}} \left| t_{i,j} \right| \exp \left( \frac{1}{m_p} \alpha_j \right) e^{-\frac{1}{p} \alpha_i} = \|Tx^j\|_p \leq M_p.
\]

Hence for all \( m > m_p \)

\[
\sup_{i,j \in \mathbb{N}} \left| t_{i,j} \right| \exp \left( \frac{1}{m_p} \alpha_j \right) e^{-\frac{1}{p} \alpha_i} \leq \sup_{i,j \in \mathbb{N}} \left| t_{i,j} \right| \exp \left( \frac{1}{m_p} \alpha_j \right) e^{-\frac{1}{p} \alpha_i} \leq M_p.
\]

We will show that the series \( \sum_{n=0}^{\infty} R^n TL^n x \) is pointwise convergent which, by Lemma 3, completes the proof.

Take some \( x \in \Lambda_0(\alpha) \). Using the formula (4) we will estimate the \( p \)-th norm of \( \sum_{n=M}^{\infty} R^n TL^n x \) in \( \lambda^\infty((\exp(-\frac{1}{k} \alpha_j))) \).

\[
\| \sum_{n=M}^{\infty} R^n TL^n x \|_p = \max \left( \sup_{N < k \leq M} \left| \sum_{j=N}^{\infty} \sum_{i=0}^{k-N} t_{i,j} x_{j+k-i} \right| \exp \left( -\frac{1}{p} \alpha_k \right), \right.
\]

\[
\left. \sup_{k > M} \left| \sum_{i=k-M}^{k-N} \sum_{j=0}^{\infty} t_{i,j} x_{j+k-i} \right| \exp \left( -\frac{1}{p} \alpha_k \right) \right) \]

\[
\leq \sup_{N < k} \left| \sum_{i=0}^{k-N} \sum_{j=0}^{\infty} t_{i,j} x_{j+k-i} \right| \exp \left( -\frac{1}{p} \alpha_k \right).
\]
If we take $m = \max(m_{2p}, 2p)$ and $s = 2mc$, with $c \in \mathbb{N}$ satisfying (5), we get
\[
\| \sum_{n=N}^{M} R^n T L^n x \|_p \leq \sup_{N < k} \sum_{i=0}^{k-N} \sum_{j=0}^{\infty} | t_{i,j} | e^{\frac{1}{m} \alpha_j - \frac{1}{2p} \alpha_i} | x_{j+k-i} | e^{-\frac{1}{m} \alpha_{k+j-i}}
\]
\[
\cdot e^{-\frac{1}{m} \alpha_j + \frac{1}{2p} \alpha_i + \frac{1}{2} \alpha_{k+j-i} - \frac{1}{p} \alpha_k}
\]
\[
\leq M_{2p} \| x \|_s \sup_{N < k} \sum_{i=0}^{k-N} \sum_{j=0}^{\infty} e^{-\frac{1}{m} \alpha_j + \frac{1}{2p} \alpha_i + \frac{1}{2} \alpha_{k+j-i} - \frac{1}{p} \alpha_k}
\]
\[
\leq M_{2p} \| x \|_s \sup_{N < k} \sum_{i=0}^{k-N} \sum_{j=0}^{\infty} e^{-\frac{1}{m} \alpha_j + \frac{1}{2m} \alpha_k} \cdot e^{-\frac{1}{m} \alpha_j + \frac{1}{2p} \alpha_k}.
\] (7)

The last inequality follows from $\alpha_i \leq \alpha_k$ and $\frac{1}{2p} \alpha_i - \frac{1}{p} \alpha_k \leq -\frac{1}{2p} \alpha_k$. Applying (5) we obtain
\[
-\frac{1}{m} \alpha_j + \frac{1}{s} \alpha_{k+j-i} \leq -\frac{1}{m} \alpha_j + \frac{c}{s} \alpha_j + \frac{c}{s} \alpha_{k-i} = -\frac{1}{m} \alpha_j + \frac{1}{2m} \alpha_j + \frac{1}{2m} \alpha_{k-i}
\]
\[
= -\frac{1}{2m} \alpha_j + \frac{1}{2m} \alpha_{k-i}.
\] (8)

We denote by $\Xi$ the vector with all coordinates equal to 1. It is easy to see that if $\Lambda_0(\alpha)$ is nuclear then, by Lemma 1, $\Xi \in \Lambda_0(\alpha)$. We apply (8) and (7) and as a result we get
\[
\| \sum_{n=N}^{M} R^n T L^n x \|_p \leq M_{2p} \| x \|_s \sup_{N < k} \sum_{i=0}^{k-N} \sum_{j=0}^{\infty} e^{-\frac{1}{m} \alpha_j + \frac{1}{2m} \alpha_k} \cdot e^{-\frac{1}{m} \alpha_j + \frac{1}{2p} \alpha_k}
\]
\[
= M_{2p} \| x \|_s \| \Xi \|_{2m} \sup_{N < k} \sum_{i=0}^{k-N} \sum_{j=0}^{\infty} e^{-\frac{1}{m} \alpha_j + \frac{1}{2m} \alpha_k}
\]
\[
\leq M_{2p} \| x \|_s \| \Xi \|_{2m} \sup_{N < k} \sum_{i=0}^{k-N} e^{-\frac{1}{2p} \alpha_k}
\]
\[
\leq M_{2p} \| x \|_s \| \Xi \|_{2m} \sup_{N < k} k e^{-\frac{1}{2p} \alpha_k}.
\]

The sequence $(\alpha_j)$ satisfies
\[
\lim_{j \to \infty} \alpha_j^{-1} \ln j = 0,
\]
hence for $k$ big enough we have $\ln k \leq \frac{1}{8p} \alpha_k$ and
\[
\left\| \sum_{n=N}^{M} R^n T L^n x \right\|_p \to 0, \quad \text{as } N \to \infty,
\]
which finishes the proof.

Now we will show a similar proof for nuclear and stable power series space of infinite type.

Analogously to the previous case, we take an operator $T \in \mathcal{L}(\Lambda_\infty(\alpha))$ and consider it as an operator acting from $\Lambda_\infty(\alpha)$ to $\lambda_\infty(\exp(k\alpha_j))$. Let $T$ be given by the matrix $(t_{i,j})_{i,j \in \mathbb{N}}$. 

Then for all \( p \in \mathbb{N} \) there exist constants \( M_p, m_p \) such that for all \( m > m_p \)
\[
\sup_{i,j} |t_{i,j}| e^{-m\alpha_i + p\alpha_j} \leq M_p.
\] (9)

Indeed, fix \( p \in \mathbb{N} \). Because \( T \) is continuous there exist \( M_p, m_p \) such that \( \|Tx\|_p \leq M_p \|x\|_{m_p} \) for all \( x \). Consider the sequence \((x^j)_{j \in \mathbb{N}} \subset \lambda_\infty(\alpha)\) defined by
\[
x^j = (0, \ldots, 0, e^{-m_p\alpha_j}, 0, \ldots).
\]

Then for all \( j \in \mathbb{N} \) we have \( \|x^j\|_{m_p} = 1 \) and
\[
\sup_{i \in \mathbb{N}} |t_{i,j}| e^{-m_p\alpha_j + p\alpha_i} = \|Tx^j\|_p \leq M_p.
\]

Hence
\[
\sup_{i,j \in \mathbb{N}} |t_{i,j}| e^{-m\alpha_j + p\alpha_i} \leq \sup_{i,j \in \mathbb{N}} |t_{i,j}| e^{-m_p\alpha_j + p\alpha_i} \leq M_p,
\]
which shows (9).

Now, we estimate the \( p \)-th norm of \( \sum_{n=N}^{M} R^nTL^nx \) in \( \lambda_\infty(\exp(k\alpha_j)) \).

\[
\| \sum_{n=N}^{M} R^nTL^nx \|_p = \max \left( \sup_{N < k \leq M} \left( \sum_{i=0}^{k-N} \sum_{j=0}^{\infty} |t_{i,j}x_{j+k-i}| e^{p\alpha_i} \right), \right. \\
\sup_{k > M} \left. \left( \sum_{i=k-M}^{k-N} \sum_{j=0}^{\infty} |t_{i,j}x_{j+k-i}| e^{p\alpha_k} \right) \right)
\]
\[
\leq \sup_{N < k} \sum_{i=0}^{k-N} \sum_{j=0}^{\infty} |t_{i,j}| |x_{j+k-i}| e^{p\alpha_k}.
\] (10)

Let \( m, s, u \in \mathbb{N} \) be such that

- \( u - cp > A \), where \( A = \sup_{j \in \mathbb{N}} \alpha_j^{-1} \ln j \), and \( c \) satisfies \( \alpha_{j+k} \leq c(\alpha_j + \alpha_k) \) for all \( j,k \),
- \( m > m_u \), where \( m_u \) satisfies (9) for \( p = u \),
- \( s - 2m > 2A \),
- \( s - 2cp > 0 \).

We further estimate (10) by
\[
\| \sum_{n=N}^{M} R^nTL^nx \|_p \leq \sup_{N < k} \sum_{i=0}^{k-N} \sum_{j=0}^{\infty} |t_{i,j}| e^{-m\alpha_j + u\alpha_i} |x_{j+k-i}| e^{s\alpha_{j+k-i}} \cdot e^{m\alpha_j - u\alpha_i - s\alpha_{j+k-i} + p\alpha_k}
\]
\[
\leq M_u \|x\| \sup_{N < k} \sum_{i=0}^{k-N} \sum_{j=0}^{\infty} e^{m\alpha_j - u\alpha_i - s\alpha_{j+k-i} + p\alpha_k}.
\] (11)
Since the sequence \((\alpha_n)\) is increasing it satisfies \(\alpha_{a+b} > \frac{1}{2} (\alpha_a + \alpha_b)\) for all \(a, b \in \mathbb{N}\). We apply this to the element \(s\alpha_{j+k-1}\) in (11) and get the following

\[
\left\| \sum_{n=N}^{M} R^n T L^n x \right\|_p \leq M_u \left\| x \right\|_s \sum_{N<k}^{k-N} \sum_{i=0}^{\infty} e^{m\alpha_j - u\alpha_i - \frac{1}{2} \alpha_j} e^{-\frac{1}{2} \alpha_{k-i} + p\alpha_k} \leq M_u \left\| x \right\|_s \sum_{N<k}^{k-N} \sum_{i=0}^{\infty} e^{-(\frac{1}{2} - m)\alpha_j} e^{-\frac{1}{2} \alpha_{k-i} + p\alpha_k}. (12)
\]

By the choice of \(c\), we have \(\alpha_k \leq c(\alpha_{k-i} + \alpha_i)\), so

\[
\sum_{N<k}^{k-N} \sum_{i=0}^{\infty} e^{-u\alpha_i - \frac{1}{2} \alpha_{k-i} + p\alpha_k} \leq \sum_{N<k}^{k-N} \sum_{i=0}^{\infty} e^{-u\alpha_i - \frac{1}{2} \alpha_{k-i} + c\alpha_{k-i} + c\alpha_i} \leq \sum_{N<k}^{k-N} \sum_{i=0}^{\infty} e^{-(u-c)p\alpha_i} e^{-(\frac{1}{2} - c)p\alpha_k-i} \leq e^{-(\frac{1}{2} - cp)\alpha_N} \sum_{i=0}^{\infty} e^{-(u-c)p\alpha_i}.
\]

Applying the above to (12) we get that

\[
\left\| \sum_{n=N}^{M} R^n T L^n x \right\|_p \leq M_u \left\| x \right\|_s \left( \sum_{j=0}^{\infty} e^{-(\frac{1}{2} - m)\alpha_j} \right) \left( \sum_{i=0}^{\infty} e^{-(u-c)p\alpha_i} \right) e^{-(\frac{1}{2} - cp)\alpha_N}
\]

Notice that for all \(k > \sup_{j\in\mathbb{N}} \alpha_j^{-1} \ln j\) the series \(\sum_{j=0}^{\infty} e^{-k\alpha_j}\) converges. Hence, due to the choice of the constants \(s, m, u\) we have

\[
\left\| \sum_{n=N}^{M} R^n T L^n x \right\|_p \xrightarrow{N \to \infty} 0.
\]

\(\square\)

As the stable and nuclear power series spaces form a wide class of spaces we get in particular

**Corollary 2** For the following spaces all continuous linear operators acting on them are commutators:

- The space of all holomorphic functions on the unit disc \(H(\mathbb{D})\) or the polydisc \(H(\mathbb{D}^n)\),
- The space of all entire functions \(H(\mathbb{C})\), \(H(\mathbb{C}^n)\),
- The space of rapidly decreasing sequences \(s\),
- The space \(C_\infty^\omega(\mathbb{R})\) of all \(2\pi\)-periodic \(C_\infty^\omega\)-functions on \(\mathbb{R}\),
- The Schwartz space of all rapidly decreasing functions \(S(\mathbb{R})\).

**Proof** It is known that \(H(\mathbb{D})\) is isomorphic to \(\Lambda_0(j) [8, 27.27]\) and \(H(\mathbb{D}^n) \simeq \Lambda_0(\sqrt{j}) [9, 8.3.2]\). The space \(H(\mathbb{C}^n)\) is isomorphic to \(\Lambda_\infty(\sqrt{j}) [9, 8.3.2]\) and the rest of the spaces are isomorphic as Fréchet spaces to \(s = \Lambda_\infty(\ln j) [8, 29.5]\). \(\square\)
Let \((X_i, (\|\cdot\|_n)_{n \in \mathbb{N}})_{i \in \mathbb{N}}\) be a family of Fréchet spaces. By \(\prod_{i=1}^{\infty} X_i\) we denote its product

\[
\prod_{i=1}^{\infty} X_i = \{ (x_i) : x_i \in X_i \}.
\]

The space \(\prod_{i=0}^{\infty} X_i\) with the topology generated by the family of seminorms \((\|\cdot\|_n)_{n \in \mathbb{N}}, \|x\|_n = \max_{i \leq n} \|x_i\|_n^i\) is a Fréchet space.

The following theorem shows another group of spaces on which every operator is a commutator.

**Theorem 2** Let \((X, (\|\cdot\|_n)_{n \in \mathbb{N}})\) be a Fréchet space. All continuous linear operators acting on \(\prod_{i=0}^{\infty} X_i\), where \(X_i = X\) for all \(i \in \mathbb{N}\), are commutators.

**Proof** For \(x = (x_i) \in \prod_{i=0}^{\infty} X_i\) we define the operators \(R\) and \(L\) as follows

\[
R(x) = R(x_0, x_1, x_2, \ldots) = (0, x_0, x_1, \ldots),
\]

\[
L(x) = L(x_0, x_1, x_2, \ldots) = (x_1, x_2, x_3, \ldots),
\]

where 0 denotes the zero vector in \(X\). Notice that from the above definitions we have that \(LR = I\) and these operators are continuous. Moreover, for all \(x = (x_i) \in \prod_{i=0}^{\infty} X_i\) and all \(n, k \in \mathbb{N}, n > k\) the norm \(\|R^n x\|_k = 0\).

Let \(T \in \mathcal{L}(\prod_{i=0}^{\infty} X_i)\). Consider the series \(\sum_{n=0}^{\infty} R^n T L^n\). The remark stated above implies that it is pointwise convergent. For all \(x \in \prod_{i=0}^{\infty} X_i\) and \(M \geq N > k\) we have that

\[
\left\| \sum_{n=N}^{M} R^n T L^n x \right\|_k = 0.
\]

Hence the series \(\sum_{n=0}^{\infty} R^n T L^n x\) is convergent and by the Banach-Steinhaus theorem there exists the operator \(S = \sum_{n=0}^{\infty} R^n T L^n \in \mathcal{L}(\prod_{i=0}^{\infty} X_i)\). Moreover, for all \(x \in \prod_{i=0}^{\infty} X_i\)

\[
(LRS - RSL)x = LR \sum_{n=0}^{\infty} R^n T L^n x - R \left( \sum_{n=0}^{\infty} R^n T L^n \right) Lx
\]

\[
= \sum_{n=0}^{\infty} R^n T L^n x - \sum_{n=1}^{\infty} R^n T L^n x = R^0 T L^0 x = T x.
\]

Hence \(T = L(RS) - (RS)L\) is a commutator. \(\square\)

**Corollary 3** On the space of all continuous functions \(C(\mathbb{R})\), the space of all sequences \(\omega\) and the space \(C^\infty(\Omega)\) of all smooth functions on an open subset \(\Omega \subset \mathbb{R}^n\) every continuous linear operator is a commutator.

**Proof** This follows from \(C(\mathbb{R}) \simeq \prod_{i=0}^{\infty} C[0,1]\) and \(C^\infty(\Omega) \simeq \prod_{i=0}^{\infty} s\) [10, p.383]. \(\square\)

### 4 Remarks

In this section we will show that the technique used in the proof of Theorem 1 can be applied only to nuclear power series spaces.

**Proposition 1** Let \(X = \Lambda_0(\alpha)\) or \(X = \Lambda_\infty(\alpha)\). The series \(\sum_{n=0}^{\infty} R^n I L^n\) is pointwise convergent if and only if the space \(X\) is nuclear.
Remark 1 The above proposition implies that if \( \sup_{j \in \mathbb{N}} \frac{\alpha_{j+1}}{\alpha_j} < \infty \) (i.e. \( R \) and \( L \) are continuous) then the identity map on \( \Lambda_0(\alpha) \) or \( \Lambda_\infty(\alpha) \) is a commutator in the nuclear case.

Proof The proof will be done separately for \( X = \Lambda_0(\alpha) \) and \( X = \Lambda_\infty(\alpha) \).

We will start with the finite type spaces.

\( \Rightarrow \): Assume that \( X = \Lambda_0(\alpha) \) is not nuclear. We will show that the series \( \sum_{n=0}^{\infty} R^n IL^n \) is divergent. We start with the observation that \( R^n L^n = I - \sum_{j=0}^{n-1} P_j \), where \( P_j \) is the projection on the \( j \)-th coordinate. Hence the operator \( R^n L^n \) is well defined. For \( x = (x_j)_{j \in \mathbb{N}} \) we have that

\[
\sum_{n=0}^{N} R^n L^n x = (x_0, 2x_1, \ldots, N x_{N-1}, (N + 1)x_N, (N + 1)x_{N+1}, \ldots). \tag{13}
\]

Now consider two cases.

Let \( \left( \frac{1}{j+1} \right)_{j \in \mathbb{N}} \in X \). Since \( X \) is non-nuclear, by the Grothendieck-Pietsch theorem [8, 28.15] it is easy to show that \( I \notin X \). Hence, there exists \( K \in \mathbb{N} \) such that

\[
\sum_{j=0}^{\infty} e^{-\frac{1}{K} \alpha_j} = \infty.
\]

Taking \( x = \left( \frac{1}{j+1} \right)_{j \in \mathbb{N}} \), we get by (13) that

\[
\sum_{n=0}^{N} R^n L^n x = \left( \frac{1}{N+1}, \frac{N + 1}{N + 2}, \frac{N + 1}{N + 3}, \ldots \right).
\]

Hence

\[
\lim_{N \to \infty} \left\| \sum_{n=0}^{N} R^n L^n x \right\|_K = \lim_{N \to \infty} \left( \sum_{j=0}^{N} e^{-\frac{1}{K} \alpha_j} + \sum_{j=N+1}^{\infty} \frac{N + 1}{j+1} e^{-\frac{1}{K} \alpha_j} \right) \geq \lim_{N \to \infty} \left( \sum_{j=0}^{N} e^{-\frac{1}{K} \alpha_j} \right) = \infty.
\]

Now assume that \( \left( \frac{1}{j+1} \right)_{j \in \mathbb{N}} \notin X \). Then there exists a \( K \in \mathbb{N} \), such that

\[
\sum_{j=0}^{\infty} \frac{1}{j+1} e^{-\frac{1}{K} \alpha_j} = \infty.
\]

Let \( x = \left( \frac{1}{(j+1)^2} \right)_{j \in \mathbb{N}} \). The vector \( x \) belongs to \( X \) because for all \( m \) and \( j \)

\[
e^{-\frac{1}{m} \alpha_j} \leq 1
\]

and

\[
\sum_{j=J_m}^{\infty} \frac{1}{j+1} \leq \sum_{j=J_m}^{\infty} \left( \frac{1}{j+1} \right)^2 < \infty.
\]
We will show that the sequence \( \left( \sum_{n=0}^{N} R^n L^n x \right) \) is unbounded. Indeed, we have
\[
\sum_{n=0}^{N} R^n L^n x = \left( \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{N+1}, \frac{N+1}{(N+2)^2}, \frac{N+1}{(N+3)^2}, \ldots \right)
\]
and
\[
\lim_{N \to \infty} \left\| \sum_{n=0}^{N} R^n L^n x \right\|_k = \lim_{N \to \infty} \left( \sum_{j=0}^{N} \frac{1}{j+1} e^{-\frac{1}{k} \alpha_j} + (N+1) \sum_{j=N+1}^{\infty} \left( \frac{1}{j+1} \right)^2 e^{-\frac{1}{k} \alpha_j} \right)
\]
\[
\geq \lim_{N \to \infty} \left( \sum_{j=0}^{N} \frac{1}{j+1} e^{-\frac{1}{k} \alpha_j} \right) = \infty.
\]
\[\Leftarrow: \text{Let } X \text{ be nuclear. Then } \lim_{j \to \infty} \alpha_j^{-1} \ln j = 0.
\]
Hence
\[
\forall k \in \mathbb{N} \exists J_k \forall j > J_k \ln j \leq \frac{1}{k} \alpha_j.
\]
Take an arbitrary \( x \in X \). By (13) we have
\[
\sum_{n=N}^{M} R^n L^n x = \left( 0, \ldots, 0, x_{N+1}, 2x_{N+1}, \ldots, (M-N)x_M, (M-N)x_{M+1}, \ldots \right).
\]
Fix \( p \in \mathbb{N} \). For all \( J_{2p} < N < M \) we have the following
\[
\left\| \sum_{n=N}^{M} R^n L^n x \right\|_p = \sum_{j=N+1}^{M} (j-N) |x_j| e^{-\frac{1}{p} \alpha_j} + \sum_{j=M+1}^{\infty} (M-N) |x_j| e^{-\frac{1}{p} \alpha_j}
\]
\[
\leq \sum_{j=N+1}^{\infty} j |x_j| e^{-\frac{1}{p} \alpha_j} = \sum_{j=N+1}^{\infty} \left| x_j \right| e^{-\frac{1}{p} \alpha_j + \ln j}
\]
\[
\leq \sum_{j=N+1}^{\infty} \left| x_j \right| e^{-\frac{1}{p} \alpha_j + \frac{1}{2p} \alpha_j} = \sum_{j=N+1}^{\infty} \left| x_j \right| e^{-\frac{1}{2p} \alpha_j} \quad \text{(14)}
\]
and the right side tends to \( 0 \), when \( N \to \infty \).

Now we will prove the proposition for the infinite type spaces.
\[\Rightarrow: \text{Let } X = \Lambda_{\infty}(\alpha) \text{ be non-nuclear. Then the sequence } \alpha = (\alpha_j)_{j \in \mathbb{N}} \text{ satisfies}
\]
\[
\sup_{0<j} \alpha_j^{-1} \ln j = \infty.
\]
Hence there exists a subsequence \( (j_n) \) and an increasing, unbounded sequence \( (\beta_{j_n}) \), such that
\[
\beta_{j_n} = \alpha_{j_n}^{-1} \ln(j_n).
\]
Consider the element \( x = (x_j)_{j \in \mathbb{N}} \) defined by
\[
x_j = \begin{cases} 
\frac{1}{j+1} & \text{if } j = j_n, \\
0 & \text{if } j \neq j_n
\end{cases}
\]
and let \( y = (y_j)_{j \in \mathbb{N}} \) be defined by \( y_j = x_j^2 \). Since for all \( k \in \mathbb{N} \) we have
\[
\|y\|_k = \sum_{n \in \mathbb{N}} \left( \frac{1}{j_n + 1} \right)^2 e^{k\alpha j_n} = \sum_{n \in \mathbb{N}} \left( \frac{1}{j_n + 1} \right)^2 e^{k\beta j_n} \leq \sum_{n \in \mathbb{N}} \left( \frac{1}{j_n} \right)^{2-k\beta j_n} < \infty,
\]
the vector \( y \) belongs to \( X \).

Recall that for \( w = (w_j)_{j \in \mathbb{N}} \) we have
\[
\sum_{n=0}^{N} R^n L^n w = (w_0, 2w_1, \ldots, Nw_{N-1}, (N+1)w_N, (N+1)w_{N+1}, \ldots)
\]
and consider two cases.

Let \( x \in X \). Then
\[
\left( \sum_{n=0}^{N} R^n L^n x \right)_j = \begin{cases} 
0 & \text{if } j \neq j_n, \\
1 & \text{if } j = j_n, j_n \leq N, \\
\frac{N+1}{j_n+1} & \text{if } j = j_n, j_n > N.
\end{cases}
\]

Hence, for all \( k \in \mathbb{N} \)
\[
\lim_{N \to \infty} \| \sum_{n=0}^{N} R^n L^n x \|_k = \lim_{N \to \infty} \left( \sum_{j_n \leq N} e^{k\alpha j_n} + \sum_{j_n > N} \frac{N+1}{j_n+1} e^{k\alpha j_n} \right)
\geq \lim_{N \to \infty} \left( \sum_{j_n \leq N} e^{k\alpha j_n} \right) = \infty.
\]

Now assume that \( x \notin X \). Then there exists \( K \in \mathbb{N} \), such that
\[
\sum_{n=0}^{\infty} \frac{1}{j_n+1} e^{K\alpha j_n} = \infty
\]
and the sequence \( (\sum_{n=0}^{N} R^n L^n y)_{N \in \mathbb{N}} \) is unbounded. Indeed, we have that
\[
\sum_{n=0}^{N} R^n L^n y = \begin{cases} 
0 & \text{if } j \neq j_n, \\
\frac{1}{j_n+1} & \text{if } j = j_n, j_n \leq N, \\
\frac{N+1}{(j_n+1)^2} & \text{if } j = j_n, j_n > N
\end{cases}
\]
and
\[
\lim_{N \to \infty} \| \sum_{n=0}^{N} R^n L^n y \|_k = \lim_{N \to \infty} \left( \sum_{j_n \leq N} \frac{1}{j_n+1} e^{K\alpha j_n} + (N+1) \sum_{j_n > N} \frac{1}{(j_n+1)^2} e^{K\alpha j_n} \right)
\geq \lim_{N \to \infty} \left( \sum_{j_n \leq N} \frac{1}{j+1} e^{K\alpha j_n} \right) = \infty.
\]
Let \( X \) be a nuclear space. Then, there exists a constant \( c \in \mathbb{N} \), such that \[
forall j \in \mathbb{N} \setminus \{0\} \quad \ln j \leq c \alpha_j.
\]
Recall that for all \( x \in X \) and \( N < M \) we have
\[
\sum_{n=N}^{M} R^n L^n x = (0, \ldots, 0, x_{N+1}, 2x_{N+2}, \ldots, (M - N)x_M, (M - N)x_{M+1}, \ldots).
\]
and
\[
\| \sum_{n=N}^{M} R^n L^n x \|_p = \sum_{j=N+1}^{M} (j - N) |x_j| e^{p \alpha j} + \sum_{j=M+1}^{\infty} (M - N) |x_j| e^{p \alpha j}
\leq \sum_{j=N+1}^{\infty} j |x_j| e^{p \alpha j} + \sum_{j=N+1}^{\infty} |x_j| e^{p \alpha j} + \ln j
\leq \sum_{j=N+1}^{\infty} |x_j| e^{p \alpha j + c \alpha j} + \sum_{j=N+1}^{\infty} |x_j| e^{(p+c) \alpha j}.
\]
Hence the series \( \sum_{n=0}^{\infty} R^n L^n x \) is convergent.

\[\Box\]

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