Maximal Discrete Subgroups of
$SO^+(2, n + 2)$

by

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We characterize the maximal discrete subgroups of $SO^+(2, n + 2)$, which contain the discriminant kernel of an even lattice, which contains two hyperbolic planes over $\mathbb{Z}$. They coincide with the normalizers in $SO^+(2, n + 2)$ and are given by the group of all integral matrices inside $SO^+(2, n + 2)$, whenever the underlying lattice is maximal even. Finally we deal with the irreducible root lattices as examples.

\textbf{Keywords:} Special orthogonal group, discriminant kernel, normalizer, maximal discrete group, maximal even lattice

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1 Introduction

The Hermitian symmetric space associated with the special orthogonal group \(SO(2, n+2)\) is a Siegel domain of type IV. The attached spaces of modular forms have attracted a lot of attention, mainly influenced by the product expansions of Borcherds (cf. [1]). Recently a lot of concrete examples for small \(n\) have been constructed by Wang and Williams (cf. [18] - [21]). The modular group consists of the discriminant kernel of an even lattice as well as certain congruence subgroups (cf. [10]).

Moreover the Maaß lift or additive lift has been described by Gritsenko (cf. [7], [12]). In a recent paper by Wernz [22] the connection between different notions of Maaß spaces for \(SO(2, 4)\) has been reduced to modular forms for the discriminant kernel versus its maximal discrete extension.

In this paper we consider the case of general \(n\). We determine the maximal discrete extension of the discriminant kernel and show that it is equal to the group of all integral matrices inside \(SO^+(2, n+2)\), whenever we start with a maximal even lattice with two hyperbolic planes over \(\mathbb{Z}\). In this case it also coincides with the normalizer. To a certain extent this characterizes the maximal even lattices among all even lattices.

2 Maximal Even Lattices

We start with an even lattice \(\Lambda\) in a \(\mathbb{Q}\)-vector space \(V\) of dimension \(n\) equipped with a non-degenerate symmetric bilinear form \(\langle \cdot, \cdot \rangle\), i.e. \(\Lambda\) is a free group of rank \(n\) satisfying \(\langle \lambda, \lambda \rangle \in 2\mathbb{Z}\) for all \(\lambda \in \Lambda\). The dual lattice is given by

\[
\Lambda^\sharp := \{v \in V; \langle v, \Lambda \rangle \subseteq \mathbb{Z} \} \supseteq \Lambda
\]

and \(\Lambda^\sharp/\Lambda\) with the quadratic form

\[
\tilde{q} : \Lambda^\sharp/\Lambda \to \mathbb{Q}/\mathbb{Z}, \quad \lambda + \Lambda \mapsto \frac{1}{2}\langle \lambda, \lambda \rangle + \mathbb{Z},
\]

is called the discriminant group of \(\Lambda\). The lattice \(\Lambda\) is always contained in a maximal even lattice in \(V\), which is a sublattice of \(\Lambda^\sharp\) (cf. [11], 14.11).

Throughout the paper we choose a basis of a positive definite lattice \(L\) with Gram matrix \(S\). Let \(\text{disc } L := \det S\) denote its discriminant. We add two hyperbolic planes over \(\mathbb{Z}\), i.e.

\[
\begin{align*}
L &= \mathbb{Z}^n, \quad S \in \mathbb{Z}^{n \times n} \text{ positive definite and even,} \\
L_0 &= \mathbb{Z}^{n+2}, \quad S_0 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \\
L_1 &= \mathbb{Z}^{n+4}, \quad S_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & S_0 & 0 \end{pmatrix}.
\end{align*}
\]

Thus \(S_1\) has got the signature \((2, n+2)\).
We consider the attached special orthogonal group

\[ SO(S_1; \mathbb{R}) := \{ M \in SL(n + 4; \mathbb{R}); \ M^{tr}S_1M = S_1 \}. \]

Let \( SO^+(S_1; \mathbb{R}) \) stand for the connected component of the identity matrix \( I \). Due to (5) in [13], it can be characterized by

\[ \det(CP + D) > 0, \quad P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad M = \begin{pmatrix} * & * & * \\ C & * & D \end{pmatrix} \in SO(S_1; \mathbb{R}) \]

with \( 2 \times 2 \) matrices \( C, D \). Given \( M \in SO^+(S_1; \mathbb{R}) \) we will always assume the form

\[ (2) \quad M = \begin{pmatrix} \alpha & a^{tr}S_0 & \beta \\ b & K & c \\ \gamma & d^{tr}S_0 & \delta \end{pmatrix}, \quad \alpha, \beta, \gamma, \delta \in \mathbb{R}, \quad a, b, c, d \in \mathbb{R}^{n+2}, \quad K \in \mathbb{R}^{(n+4)\times(n+4)}. \]

Its inverse is given by

\[ (3) \quad M^{-1} = S_1^{-1}M^{tr}S_1 = \begin{pmatrix} \delta & c^{tr}S_0 & \beta \\ d & S_0^{-1}K^{tr}S_0 & a \\ \gamma & b^{tr}S_0 & \alpha \end{pmatrix}. \]

Let \( \Gamma_S := SO^+(S_1; \mathbb{Z}) \) denote the subgroup of integral matrices. Note that in this case \( a, d \in \mathbb{Z}^{n+2} \) holds in (2) due to \( M^{-1} \in \Gamma_S \) and (3). Moreover we define the discriminant kernel

\[ \tilde{\Gamma}_S := \{ M \in \Gamma_S; \ M \in I + \mathbb{Z}^{(n+4)\times(n+4)}S_1 \}, \]

where \( I \) is the identity matrix. The discriminant kernel induces the identity on \( L_1^\sharp/L_1 \), \( L_1^\sharp = S_1^{-1}Z^{n+4} \). We consider particular matrices in \( \tilde{\Gamma}_S \):

\[ (4) \quad J = \begin{pmatrix} 0 & 0 & -P \\ 0 & I & 0 \\ -P & 0 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]

\[ (5) \quad T_\lambda = \begin{pmatrix} 1 & -\lambda^{tr}S_0 & -\frac{1}{2}\lambda^{tr}S_0\lambda \\ 0 & I & \lambda \\ 0 & 0 & 1 \end{pmatrix}, \quad T_\lambda^* = \begin{pmatrix} 1 & 0 & 0 \\ \lambda & I & 0 \\ -\frac{1}{2}\lambda^{tr}S_0\lambda & -\lambda^{tr}S_0 & 1 \end{pmatrix}, \quad \lambda \in \mathbb{Z}^{n+2}. \]

At first we give a description of the first columns of matrices in \( \Gamma_S \).

**Theorem 1.** Let \( L_1 = \mathbb{Z}^{n+4} \) satisfy (1). Given \( h \in \mathbb{Z}^{n+4} \) the following assertions are equivalent:

(i) \( h \) is the first column of a matrix in \( \Gamma_S \) (resp. \( \tilde{\Gamma}_S \)).

(ii) \( h^{tr}S_1 \) is the last row of a matrix in \( \Gamma_S \) (resp. \( \tilde{\Gamma}_S \)).

(iii) \( h^{tr}S_1h = 0 \) and \( \text{gcd}(S_1h) = 1 \).
Proof. (i) ⇔ (ii) Use (2) and (3).

(i) ⇒ (iii) Apply $M^\text{tr}S_1M = S_1$ and $\tilde{\Gamma}_S \subseteq \Gamma_S \subseteq SL(n+4;\mathbb{Z})$.

(iii) ⇒ (ii) Proceed in the same way as in the proof of Theorem 1 in [13]. The matrices involved there lead to an $M \in \tilde{\Gamma}_S$ such that

$$h^\text{tr}S_1M = (0, \ldots, 0, 1).$$

Hence $h^\text{tr}S_1$ is the last row of $M^{-1} \in \tilde{\Gamma}_S \subseteq \Gamma_S$. 

In the context of the action of $\Gamma_S$ on the orthogonal half-space (cf. [7]), it makes sense to consider cusps. If $\Gamma$ is a subgroup of $\Gamma_S$ of finite index, we denote by

$$C^0(\Gamma) := \{ \Gamma h; \ h \in L^1\} = S_1^{-1}Z^{n+4}, \ h^\text{tr}S_1h = 0, \ \gcd(S_1h) = 1 \}$$

the set of $\Gamma$-orbits of zero-dimensional cusps (cf. [8]).

**Corollary 1.** Let $L_1 = Z^{n+4}$ satisfy (1). Then the following assertions are equivalent:

(i) $L = Z^n$ is maximal even.

(ii) Every $g \in L^1_1 = S_1^{-1}Z^{n+4}$ with $g^\text{tr}S_1g = 0$ fulfills $g \in L_1 = Z^{n+4}$.

(iii) $\# C^0(\Gamma_S) = 1$.

(iv) $\# C^0(\tilde{\Gamma}_S) = 1$.

Proof. (i) ⇒ (iv) Let

$$g = (g_1, g_2, \lambda_1, \ldots, \lambda_n, g_3, g_4)^{\text{tr}} \in L^1_1, \ g^\text{tr}S_1g = 0, \ \gcd(S_1g) = 1.$$ 

Thus $\lambda = (\lambda_1, \ldots, \lambda_n)^{\text{tr}} \in L^2$ follows with

$$\lambda^\text{tr}S\lambda = 2(g_1g_4 + g_2g_3) \in 2\mathbb{Z}.$$ 

Hence $L + Z\lambda$ is an even overlattice of $L$ and (i) yields $\lambda \in L$, i.e. $g \in L_1$. Then Theorem 1 leads to (iv).

(iv) ⇒ (iii) This is clear due to $\tilde{\Gamma}_S \subseteq \Gamma_S$.

(iii) ⇒ (ii) $\Gamma_S$ acts transitively on the set of vectors $g \in L^1_1$ with $g^\text{tr}S_1g = 0$ and $\gcd(S_1g) = 1$. In view of $\Gamma_S \subseteq SL(n+4;\mathbb{Z})$ any $M \in \Gamma_S$ induces a bijective map $M : Z^{n+4} \rightarrow Z^{n+4}, \ h \mapsto Mh$. Hence $L^1_1 = L_1$ follows.

(ii) ⇒ (i) This is clear as any even overlattice of $L$ is contained in $L^2$ and

$$\{ \lambda \in L^\#: \lambda^\text{tr}S\lambda \in 2\mathbb{Z} \} = L$$

Hence $L$ is maximal even. 

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Corollary 1 says that \( L \) is maximal even if and only if \( (L^\sharp / L, g) \) is anisotropic. The equivalence between (i) and (ii) is contained in [15], Proposition 1.4.1, under weaker assumptions.

We give some examples.

**Example 1.**

a) Considering \( L = \mathbb{Z} \) with \( \langle x, y \rangle = 2Nx^2 \), \( N \in \mathbb{N} \), we obtain a maximal even lattice if and only if \( N \) is squarefree. This leads to paramodular groups (cf. [6]).

b) If \( L = \mathcal{O}_K \) is the ring of integers of an imaginary quadratic number field \( K \) with \( \langle x, y \rangle = x\overline{y} + \overline{x}y \), we are led to the Hermitian modular group (cf. [14]).

c) Considering the Hurwitz quaternions or the order of integral Cayley numbers, confer [10] and [4]. These cases refer to the root lattices \( D_4 \) and \( E_8 \) (cf. sect. 4).

The case of non-maximal lattices is dealt with in the following Remark.

**Remark 1.** An arbitrary even lattice \( L \) is contained in a maximal even lattice \( L^\ast \) with Gram matrix \( S^\ast \). Hence there exists a matrix \( H \in \mathbb{Z}^{n \times n} \) satisfying

\[
S = H^{tr} S^\ast H, \quad S_1 = \tilde{H}^{tr} S_1^\ast \tilde{H}, \quad \tilde{H} = \begin{pmatrix} I & 0 & 0 \\ 0 & H & 0 \\ 0 & 0 & I \end{pmatrix}.
\]

Clearly \( | \det H | = |L^\ast : L| \) holds. In this case we have

\[
\tilde{\Gamma}_S \subseteq \tilde{H}^{-1} \tilde{\Gamma}_S \cdot \tilde{H} \subseteq \tilde{H}^{-1} \Gamma_S \cdot \tilde{H} \subseteq SO^+(S_1; \mathbb{Q}).
\]

Thus \( L \) is maximal even, whenever \( \det S \) is squarefree. If \( n \) is odd, then \( \det S \) and \( \det S^\ast \) are even. Thus \( L \) of odd rank is maximal even, whenever \( (\det S)/2 \) is squarefree.

We give an application to right and double cosets, which is also needed in the attached Hecke theory.

**Theorem 2.** Let \( L_1 = \mathbb{Z}^{n+4} \) satisfy (11) and

\[
R' = \frac{1}{\sqrt{r}} R \in SO^+(S_1; \mathbb{R}), \quad r \in \mathbb{N}, \quad R \in \mathbb{Z}^{(n+4) \times (n+4)}.
\]

If \( L \) is maximal even or \( r \) and \( \det S \) are coprime, the following holds.

a) The right coset \( \tilde{\Gamma}_S R \) contains a matrix

\[
\begin{pmatrix}
\alpha & * & * \\
0 & * & * \\
0 & 0 & \delta
\end{pmatrix}, \quad \alpha, \delta \in \mathbb{N}, \quad \alpha \delta = r,
\]

where \( \alpha \) is the gcd of the first column of \( R \).
b) The double coset $\tilde{\Gamma}_S R \tilde{\Gamma}_S$ contains a matrix

$$R^* = \begin{pmatrix} \alpha^* & 0 & 0 \\ 0 & K^* & 0 \\ 0 & 0 & \delta^* \end{pmatrix} \in \mathbb{Z}^{(n+4)\times (n+4)}, \quad \alpha^* \in \mathbb{N}, \quad \alpha^* \delta^* = r, \quad \frac{1}{\alpha^*} R^* \in \mathbb{Z}^{(n+4)\times (n+4)},$$

where $\alpha^*$ is the gcd of all the entries of $R$.

**Proof.** a) Let $g$ be the first column of $R$, which satisfies $g^\text{tr} S_1 g = 0$ and let $\alpha = \text{gcd}(S_1 g)$.

(i) If $L$ is maximal even, we have

$$\frac{1}{\alpha^*} g \in S_1^{-1} \mathbb{Z}^{n+4} = L_1^2 \text{ and } \frac{1}{\alpha^*} g \in L_1$$

due to Corollary [1].

(ii) If $r$ and $\det S$ are coprime, we observe that $g^\text{tr} S_1$ is the last row of the matrix $\sqrt{r} R^* - 1$ due to (3), which has the determinant $r^{2+n/2}$. As $\alpha$ divides a power of $r$ it is coprime to $\det S$ and we get again $\frac{1}{\alpha^*} g \in L_1$.

As block diagonal matrices form a subgroup, we obtain (7) from the description of the inverse in (3).

b) Let $\alpha^*$ be the smallest positive $(1, 1)$-entry in $\mathbb{Z}$ of all the matrices in $\tilde{\Gamma}_S R \tilde{\Gamma}_S$ and assume

$$R = \begin{pmatrix} \alpha^* & * \\ * & * \end{pmatrix}$$

without restriction. It follows from a) that $\alpha^*$ divides the entries of the first column of $R$. The same procedure as in a) applied to $M^{-1} J$ shows that $\alpha^*$ also divides the entries of the first row of $R S_1^{-1}$. Multiplication by $T_\lambda^*, \lambda \in \mathbb{Z}^{n+2}$, from the left and by $T_\mu$, $\mu \in \mathbb{Z}^{n+2}$, from the right (cf. (5)) leads to

$$R^* = \begin{pmatrix} \alpha^* & 0 & 0 \\ 0 & K^* & 0 \\ 0 & 0 & \delta^* \end{pmatrix} \in \tilde{\Gamma}_S R \tilde{\Gamma}_S, \quad \alpha^* \delta^* = r.$$

Considering $R^* J T_\lambda, \lambda = e_1, \ldots, e_{n+2}, e_1 + e_{n+2} \in \mathbb{Z}^{n+2},$ then shows that $\alpha^*$ divides the entries of $K^*$ and $\delta^*$. Now $\tilde{\Gamma}_S \subseteq SL(n+4; \mathbb{Z})$ implies that $\alpha^*$ is the gcd of the entries of $R$. \qed

### 3 Maximal Discrete Subgroups

We follow the procedure by Ramanathan [16].

**Lemma 1.** Let $L_1 = \mathbb{Z}^{n+4}$ satisfy (1). Let $\Delta$ be a discrete subgroup of $SO^+(S_1; \mathbb{R})$, which contains $\tilde{\Gamma}_S$. Then the following holds.
a) $[\Delta : \tilde{\Gamma}_S] = j \in \mathbb{N}$. 

b) Given $R \in \Delta$, there exists $r \in \mathbb{N}$ such that 

$$\sqrt{r} R \in \mathbb{Z}^{(n+4) \times (n+4)}.$$ 

Proof. a) According to [2], 4.10, the discriminant kernel $\tilde{\Gamma}_S$ possesses a fundamental domain with respect to the action on the orthogonal half-space with finite, positive volume. As $\Delta$ is countable, the index $j$ must be finite.

b) Assume the notation (2) for $R$. Multiplying $R$ by matrices of type (5), we may assume that $\alpha, \beta, \gamma, \delta$ are non-zero. Setting $k = j!$ we conclude $R^k \in \tilde{\Gamma}_S$ for each $R \in \Delta$.

This leads to 

$$(RM R^{-1})^k = RM^k R^{-1} \in \tilde{\Gamma}_S,$$ 

i.e. $R(M^k - I) R^{-1} \in \mathbb{Z}^{(n+4) \times (n+4)} S_1$ 

for all $M \in \tilde{\Gamma}_S$. Using $M = T_\lambda$ in (5) with $\lambda = (1, 0, \ldots, 0)^{t r}$ and $\mu = (0, \ldots, 0, 1)^{t r}$ we get 

$$RX R^{-1} \in \mathbb{Z}^{(n+4) \times (n+4)} S_1 \text{ for } X = T_{k\lambda} + T_{k\mu} - T_{k(\lambda+\mu)} - I = k^2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$ 

whenever $x$ is the first column of $R$ or the last column of $R$, if we replace $T_\nu$ by $T_\nu^*$. Considering $R^{-1}$ instead of $R$ this remains true, if $x^{t r} S_1$ is the first or last row of $R$. Elementary number theory yields for $f, g \in \mathbb{R}$ with $f^2, g^2, fg \in \mathbb{Z}$ that $f, g \in \mathbb{Z} \sqrt{h}$ for some $h \in \mathbb{N}$. If $k^2 \alpha^2 = u^2 v$ with $u, v \in \mathbb{N}$, $v$ squarefree, we conclude that $\rho x$ is integral for all the vectors $x$ mentioned above, whenever 

$$\rho = k / \sqrt{v}.$$ 

If we replace $R$ by $RT_\lambda$, $\lambda \in \mathbb{Z}^{n+2}$, we conclude that $\rho K$, hence $\rho R$ is integral, too. Then 

$$\rho^2 S_1 = (\rho R)^{t r} S_1 (\rho R) \in \mathbb{Z}^{(n+4) \times (n+4)}$$ 

leads to $\rho^2 \in \mathbb{N}$. 

Due to the determinantal condition, $R$ in Lemma 1 is a rational matrix, whenever $n$ is odd.

**Corollary 2.** Let $L_1 = \mathbb{Z}^{n+4}$ satisfy (1). Then the normalizer of $\tilde{\Gamma}_S$ in $SO^+(S_1; \mathbb{R})$ is equal to $\Gamma_S$.

Proof. Clearly $\tilde{\Gamma}_S$ is a normal subgroup of $\Gamma_S$. Given $R$ in the normalizer of $\tilde{\Gamma}_S$ in
SO+(S1; R), we conclude from the proof of Lemma 1 and in particular from (9) with
k = 1 that \( \frac{1}{\sqrt{v}} R \) is integral. Then det \( R = 1 \) leads to \( v = 1 \) and therefore \( R \in \Gamma_S \).

Next we consider the particular case of maximal even lattices.

**Theorem 3.** Let \( L_1 = \mathbb{Z}^{n+4} \) be a maximal even lattice satisfying (1). Then \( \Gamma_S \) is the uniquely determined maximal discrete extension of \( \tilde{\Gamma}_S \) in \( SO^+(S1; R) \) and coincides with the normalizer of \( \tilde{\Gamma}_S \) in \( SO^+(S1; R) \).

**Proof.** Let \( \Delta \) be a discrete subgroup of \( SO^+(S1; R) \), which contains \( \tilde{\Gamma}_S \). Due to Lemma 1 and Theorem 2 we may assume

\[
R = \begin{pmatrix}
\alpha & 0 & 0 \\
0 & K & 0 \\
0 & 0 & \delta
\end{pmatrix} \in \Delta, \quad 0 < \alpha \leq \delta, \quad \alpha \delta = 1, \quad \frac{1}{\alpha} R \in \mathbb{Z}^{(n+4)\times(n+4)}.
\]

If \( 0 < \alpha < 1 \) the right cosets \( \tilde{\Gamma}_SR^m, m \in \mathbb{Z}, \) are mutually different. This contradicts Lemma 1. Thus \( \alpha = 1 \) and \( R \) is integral, i.e. \( R \in \Gamma_S \). As \( \Gamma_S \) is clearly a discrete group, it is the unique maximal discrete extension of \( \tilde{\Gamma}_S \) and coincides with the normalizer due to Corollary 2. \( \square \)

Non-maximal lattices are described in the following Remark.

**Remark 2.** a) If \( L \) is not maximal even, \( r \) in Lemma 1 is not always equal to 1. But one can proceed along the proof of Lemma 4 in [14] in order to show that \( r \) is always a divisor of \( (d)_{m} \) for some \( m \in \mathbb{N} \). Now consider Remark 1. Hence \( H^{-1} = \Gamma_S \). \( H \) is a maximal discrete extension of \( \tilde{\Gamma}_S \) due to Theorem 3 which is neither contained in \( SL(n + 4; \mathbb{Z}) \) nor in the normalizer of \( \Gamma_S \) or \( \tilde{\Gamma}_S \) due to Corollary 2. Note that this maximal discrete extension does not contain \( \Gamma_S \) in general. As a maximal even overlattice is not unique in general (cf. [15]), we conclude that a maximal discrete extension \( \tilde{\Gamma}_S \) is not uniquely determined in general. More precisely any maximal discrete extension is equal to the normalizer \( \Gamma_S \), if and only if the underlying lattice \( L \) is maximal even.

b) Lemma 1 remains true, if we replace the discriminant kernel by an arbitrary congruence subgroup of \( \Gamma_S \).

**4 Root Lattices**

In this section we deal with root lattices, as they yield the most common examples of Borcherds products (cf. [19]). We quote [3], Chap. 4, and [5], 1.4, for details.

The lattice \( A_n \) is given by

\[
A_n := \{ \lambda \in \mathbb{Z}^{n+1}; \lambda_1 + \ldots + \lambda_{n+1} = 0 \}, \quad \text{disc } A_n = n + 1.
\]

The discriminant group is cyclic of order \( n + 1 \)

\[
A_n^*/A_n = \langle a_n + A_n \rangle, \quad a_n = \frac{1}{n+1} (ne_1 - e_2 - \ldots - e_{n+1}), \quad \mathfrak{q}(a_n + A_n) = \frac{n}{2(n+1)} + \mathbb{Z}.
\]
Hence \((A_n^1, \overline{q})\) is anisotropic if and only if
\[8 \nmid (n + 1) \quad \text{and} \quad p^2 \nmid (n + 1) \quad \text{for each odd prime} \ p.
\]
If \(A_n\) is not maximal, its maximal discrete extension is uniquely determined and generated by \(A_n\) and \(ja_n\) with \(nj^2 \equiv 0 \pmod{2(n + 1)}\). It is equal to
\[
\{\lambda \in A_n^1; \langle \lambda, \lambda \rangle \in 2\mathbb{Z}\}.
\]
The lattice \(D_n\) is given by
\[D_n = \{\lambda \in \mathbb{Z}^n; \lambda_1 + \ldots + \lambda_n \equiv 0 \pmod{2}\}, \quad \text{disc} \, D_n = 4.
\]
If \(n\) is odd, \(D_n\) is maximal even due to Remark 1. If \(n\) is even, \(D_n^+\) is a Kleinian 4-group, where the values of \(\overline{q}\) are given by \(\mathbb{Z}^2 + \mathbb{Z}^n\). Hence \(D_n\) is maximal even, if and only if \(8 \nmid n\). If \(8 \mid n\), the lattice \(D_n^+\) is unimodular and a maximal even overlattice of \(D_n\).

\(E_8 = D_8^+\) is unimodular. \(E_7 = \langle e_7 - e_8 \rangle^\perp\) has disc \(E_7 = 2\). \(E_6 = \langle e_6 - e_7, e_7 - e_8 \rangle^\perp\) satisfies disc \(E_6 = 3\). Hence they are maximal even due to Remark 1.

Summarizing we have

**Lemma 2.** A complete list of maximal even irreducible root lattices is given by

a) \(A_n\), if \(n\) is even and \(n + 1\) is squarefree or if \(n\) is odd and \((n + 1)/2\) is squarefree.

b) \(D_n\), if \(n\) is not a multiple of 8.

c) \(E_6, E_7, E_8\).

Clearly one can deal with arbitrary root lattices on this basis. There exists a unique overlattice, which is maximal even, for instance, whenever the discriminant group is cyclic, as pointed out for \(A_n\). But \(D_n\) for \(8 \mid n\) has got two different maximal even overlattices, which are both isometric to \(D_n^+\).

**Example 2.**

a) Let \(L = 4A_1\). Then \(\Gamma_S, S = 2I^{(4)}\), corresponds to the extended modular group over the Lipschitz quaternions and admits a unique maximal discrete extension, which is given by \(L^* = D_4\) and corresponds to the extended modular group over the Hurwitz quaternions (cf. [3]).

b) Let \(L = 5A_1, S = 2I^{(5)}\). Then there are 5 maximal even overlattices given by \(\mathbb{Z}^5 + Zh_j, h_j = \frac{1}{2}(h - e_j); \ j = 1, \ldots, 5\), where \(h = (1, 1, 1, 1)^t\). The associated maximal discrete extensions of \(\Gamma_S\) in Remark 2 are isomorphic. But there is another maximal discrete extension given by \(\Gamma_S\), which is not isomorphic to the other ones. This can be proved in a similar way as in [10], as one can restrict to
matrices with denominator 2 in the maximal discrete extension and uses the fact that the matrices \( \begin{pmatrix} I & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & I \end{pmatrix} \), \( P \in SO(I^{(5)}; \mathbb{Z}) \), belong to \( \Gamma_S \).

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