A perturbation theory and a diagram technique for a disordered metal are proposed when scattering of quasiparticles by nonmagnetic impurities is caused with a retarded interaction. The perturbation theory generalizes a case of the elastic scattering in a disordered metal. Eliashberg equations for s-wave superconductivity are generalized for such a disordered superconductor. Anderson’s theorem is found to be violated in the sense that embedding of the impurities into a s-wave superconductor increases its critical temperature. We showed the amplification of superconducting properties is a result of nonelastic effects in a scattering by the impurities.

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I. INTRODUCTION

As is well known a s-wave superconducting state is stable regard to embedding of nonmagnetic impurities. In this case an ordinary potential scattering acts on both electrons of a Cooper pair equally, therefore the pair survives. Mathematically this is expressed in the fact that a gap $\Delta$ and an energetic parameter $\varepsilon$ are renormalized the same manner: $\tilde{\Delta} = \tilde{\varepsilon}$, where $\tilde{\Delta}, \tilde{\varepsilon}$ are the renormalized values by an impurity scattering. As a consequence a critical temperature of a superconductor does not change. This statement is Anderson’s theorem [1–3]. However, a strong suppression of superconductivity takes place near Anderson’s transition metal-insulator, that is when $\frac{1}{k_F l} \gtrsim 1$, where $l$ is a free length and $k_F$ is Fermi momentum. Although in a state of Anderson’s isolator a superconductive response of the system can be remain [2,4]. The magnetic impurities differently acts on components of Cooper pair, with the result that its decay takes place. Superconducting state is unstable regard to embedding of magnetic impurities - the critical temperature decreases that is accompanied by effect of gapless superconductivity [2,5]. For $d$-wave superconductors the nonmagnetic impurities destroy superconductivity like magnetic impurities [6–11]. The reduce of the critical temperature is a mathematical consequence of an inequality $\tilde{\Delta} < \tilde{\varepsilon}$, that is the gap and the energetic parameter are renormalized in different ways. It should be noticed if electrons are paired with nonretarded interaction (as in BCS theory, negative U Hubbard model) then the superconductive order parameter strongly suppressed with an increase of disorder [12,13]. This means we must use approaches which take into account the fact that quasiparticles are paired with retarded interaction (for example with electron-phonon interaction, electron-magnon interaction etc.)

In a work [14] a case was considered when when scattering of quasiparticles by nonmagnetic impurities is caused with a retarded interaction. The retarded interaction occurs because the impurities have an internal structure and make transitions between their states under the action of metal’s quasiparticles. In the proposed model the principal possibility of increasing of the critical temperature due to the retarded interaction between quasiparticles and impurities has been shown. However a formal retarded form of the electron-impurity interaction was proposed only and an electron-impurity coupling constant has not been calculated. Thus the theory does not enables us to calculate the critical temperature since an internal structure of the impurities and its interaction with quasiparticles is unknown. In addition an impurity was supposed as a two-level system with an eigenfrequency $\omega_0$ and the simplest type of the diagrams was considered only.

This paper is aimed to generalize disordered metal’s theory when interaction of quasiparticles with nonmagnetic impurities is retarded. In Section II we develop a diagram technique for a disordered metal in normal state. Main types of diagrams are determined and contributions of the each type are estimated. A perturbation theory is made using an adiabaticity parameter and an method of an uncoupling of correlations. In Section III we develop a diagram technique for a disordered metal in s-wave superconducting state. Eliashberg equations are generalized to a case when the superconductor contains impurities of a considered type. Based on these equations we show the retarded interaction of quasiparticles with impurities violates Anderson’s theorem in the direction of increasing of the superconducting transition temperature.

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II. NORMAL STATE.

Let an electron moves in a field created by $N$ scatterers (impurities) which are placed in points $\mathbf{R}_j$ by a random manner with concentration $\rho = \frac{N}{V}$. Each impurity can be in states $\phi_A(\mathbf{r}_j - \mathbf{R}_j), \phi_B(\mathbf{r}_j - \mathbf{R}_j), \phi_C(\mathbf{r}_j - \mathbf{R}_j), \ldots$ with energies $E_A, E_B, E_C, \ldots$ accordingly. Here $\mathbf{r}_j$ is a radius-vector of a state configuration $\phi$ of $j$th impurity (Fig.1). Metal’s quasiparticles (conduction electrons) are described with wave functions $\psi_{\mathbf{k}}(\mathbf{r}) = \frac{1}{\sqrt{V}} e^{i\mathbf{k}\cdot \mathbf{r}}$ and fill all states up to Fermi momentum $k_F$. Interaction of the electrons with impurities is described with potential $U(\mathbf{r} - \mathbf{r}_j)$. As a result of scattering the electrons go into states $\psi_{\mathbf{k'}}(\mathbf{r})$, and the impurities go from some state $\phi_A$ into one of states $\phi_A, \phi_B, \phi_C, \ldots$. Main approximation lies in the fact that oscillations of an impurity are local, that is $\omega_q \ll \Omega$, where $q$ is a wave vector. On the contrary a phonon frequency is a function of a wave vector $\Omega = \Omega(\mathbf{q})$. The impurity’s oscillations do not interact with phonons of a metal. Then Hamiltonian of the system can be written as follows:

$$
\hat{H} = \hat{H}_0 + \sum_j \hat{H}_0^j + \sum_{A,B} \sum_{\mathbf{k},\mathbf{k}'} \int \phi_A^-(\mathbf{r}_j - \mathbf{R}_j) \psi_{\mathbf{k}}^+(\mathbf{r}) U(\mathbf{r} - \mathbf{r}_j) \phi_A(\mathbf{r}_j - \mathbf{R}_j) \psi_{\mathbf{k}}(\mathbf{r}) d\mathbf{r}_j c_{\mathbf{k}}^+ c_A c_{\mathbf{k}'}^+ \tag{1}
$$

where $\hat{H}_0$ is Hamiltonian of a homogeneous medium without impurities, $\hat{H}_0^j$ is Hamiltonian of a $j$th impurity: $H_0^j \phi_A \equiv \left( \frac{\hbar^2}{2m} + V(\mathbf{r}) \right) \phi_A = E_A \phi_A$, $c_A$ and, $c_{\mathbf{k}}^+$ are creation and annihilation operators of an impurity in states $\phi_A, \phi_B, \phi_C, \ldots$, $c_{\mathbf{k}}$ and $c_{\mathbf{k}'}^+$ are creation and annihilation operators of an electron in states $|\mathbf{k}\rangle$ and $|\mathbf{k}'\rangle$. The third term describes interaction of electrons with impurities. A free propagator of electrons is:

$$
G_0(\mathbf{k}, \varepsilon) = \frac{1}{\varepsilon - \xi(k) + i\delta \text{sign} \xi} \tag{2}
$$

where $\xi(k) = \frac{k^2}{2m} - \varepsilon_F \approx v_F(k - k_F)$ is energy of an electron counted from Fermi surface, $\varepsilon$ is an energy parameter, $\delta \to 0$, we use a system of units where $\hbar = k_B = 1$. State of an impurity can be described with Green function:

$$
G_A(\varepsilon) = \frac{1}{\gamma - E_A + i\delta} \tag{3}
$$

A system described with Hamiltonian (1) is nonhomogeneous and momentums of quasiparticles are not conserved. However averaging over an ensemble of samples with all possible positions of impurities recovers spatial homogeneity of the system, and quasiparticles’ momentums are conserved (Appendix A). The averaging operation over a disorder has a form (15):

$$
\langle G(x, x') \rangle = -i \frac{\langle \hat{T} \psi^+(x) \psi(x') U \rangle_0}{\langle \hat{U} \rangle_0 \text{disorder}}, \tag{4}
$$

Figure 1: A mutual disposal of a quasiparticle with a radius-vector $\mathbf{r}$ and a $j$th impurity with a radius-vector $\mathbf{R}_j$. Interaction between the quasiparticle and the impurity is $U(\mathbf{r} - \mathbf{r}_j)$, where $\mathbf{r}_j$ is a radius-vector of a state configuration $\phi$ of $j$th impurity.
Contribution of this diagram is trivial - it shifts a chemical potential only: $\mu_{\text{elastic}}$. In analytical form a mass operator is

$$\text{It means an electron interacts with an impurity no changing impurity's state (}\phi \rightarrow \phi_A).$$

Conservation of momentum allows us to summarize diagrams with help of Dyson equation (A5).

$$\sum_j \int dr_j \rightarrow \frac{N}{V} \int \int dr'dR$$

\[ (5) \]

Figure 2: The fist order diagram describing a retarded scattering of a quasiparticle by impurities. The impurity does not change its state during the process $|A\rangle \rightarrow |A\rangle$.

Let us consider the simplest process which is analogous to a process in Fig.9. The process is represented in Fig.2. It means an electron interacts with an impurity no changing impurity’s state ($\phi \rightarrow \phi_A$). That is the scattering is elastic. In analytical form a mass operator is

$$(-i)\Sigma = \int_{-\infty}^{+\infty} iG_A(\gamma) \frac{d\gamma}{2\pi} (-i) \sum_j \int \int \phi_A^+(r_j - R_j) \psi_k(r) U(r - r_j) \phi_A(r_j - R_j) \psi_k(r) dr dr_j$$

$$= -i \frac{N}{V^2} \int \int U(r - r') |\phi_A(r' - R)|^2 dr dr'dR$$

$$= -i \frac{N}{V} \int \int U(r - r') |\phi_A(r' - R)|^2 d(r - r') d(r' - R) = -i \frac{N}{V} \int U(R) dR = -i \rho U(q = 0),$$

\[ (6) \]

where $\phi_A \equiv |A\rangle$ is a ground state of the impurity $\langle A|A \rangle = 1$, and we took advantage in that integration over $r$, $r'$ and $R$ is done over infinite volume $\int dr \equiv \int_{-\infty}^{+\infty} r^2 dr \int_0^{2\pi} \sin \theta d\theta \int_0^{2\pi} d\varphi$, $\int dR = V$, $N \rightarrow \infty, V \rightarrow \infty, N/V = \rho = \text{const}$. Contribution of this diagram is trivial - it shifts a chemical potential only: $\mu - \rho U(0)$.

In a case of nonzero temperature $T \neq 0$ the impurities are distributed over states $|A\rangle, |B\rangle, |C\rangle, \ldots$ with probability

$$\varpi_A = \frac{1}{Z} \exp \left( -\frac{E_A - E_0}{T} \right), \quad Z = \sum_A \exp \left( -\frac{E_A - E_0}{T} \right)$$

\[ (7) \]

where $E_0$ is an energy of ground state of an impurity, and the summation is extended on all possible states (we use a system of units where $\hbar = k_B = 1$). Then

$$-\Sigma = -i \rho \sum_A \varpi_A \int \int U(r - r') |\phi_A(r' - R)|^2 d(r - r') d(r' - R) = -\rho U(q = 0),$$

\[ (8) \]

because $\sum_A \varpi_A = 1$, $\langle A|A \rangle = 1$.

Let us consider the second order process represented in Fig.3. It is analogous to the second order process of elastic scattering shown in Fig.10. In the inelastic process an electron interacts with an impurity changing impurity’s state $\phi_A \rightarrow \phi_B \rightarrow \phi_A$. That is a state $|B\rangle$ is virtual, a state $|A\rangle$ is a ground state (for $T = 0$ only). A transition frequency is $\omega_{AB} = E_B - E_A$. This process means a retarded interaction with impurities. The interaction was considered in the simplest form in an article [14]. In an analytical form a mass operator is
Figure 3: The second order diagram describing a retarded interaction of a quasiparticle by impurities. The impurity’s state is changed during the process $|A\rangle \rightarrow |B\rangle \rightarrow |A\rangle$ (a right-hand picture).

\[
(-i)\Sigma = \sum_j \sum_B V \int \frac{dq}{(2\pi)^3} \int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} \left[iG_A^{-}(\gamma)G_B^{-}(\gamma + \omega) + iG_A^{-}(\gamma)G_B^{+}(\gamma - \omega)\right] \frac{d\gamma}{2\pi} 
\]

\[
(-i) \int \phi_B^+(r_j - R_j) \psi_k^+_{k-q} (r) U (r - r_j) \phi_A (r_j - R_j) \psi_k (r) dr dr_j 
\]

\[
(-i) \int \phi_A^+(r_j - R_j) \psi_k^+ (r) U (r - r_j) \phi_B (r_j - R_j) \psi_{k-q} (r) dr dr_j 
\]

\[
iG_0 (k - q, \varepsilon + \omega) 
\]

(9)

The term in square brackets describes virtual transitions of an impurity between levels $A$ and $B$. If an impurity’s level is empty then its propagator is:

\[
G_A^{-}(\gamma) = -\frac{1}{\gamma - E_A - i\delta}, 
\]

(10)

because the empty level equivalent to a propagation back in time (a hole or an antiparticle), and many impurities in a system can be in the same state, that is the minus sign must be both for bosons. An electron interacting with an impurity takes it to a state $\phi_B$ and gives it an energy parameter $\omega$. As Feynman diagrams are not ordered in time [16] then Eq.(6) is a sum of two terms: with the positive and negative energy parameter:

\[
\int_{-\infty}^{+\infty} \left[iG_A^{-}(\gamma)G_B^{-}(\gamma + \omega) + iG_A^{-}(\gamma)G_B^{+}(\gamma - \omega)\right] \frac{d\gamma}{2\pi} = i \frac{2\omega_{AB}}{\omega^2 - \omega_{AB}^2 + i2\omega_{AB}\delta} \equiv iD_{AB}(\omega). 
\]

(11)

A function $D_{AB}(\omega)$ has a form of a collective excitation’s propagator. Therefore we name this function as ”pseudo-propagator” [13]. A prefix ”pseudo” means this propagator is a result of the averaging [4] and the corresponding correlations [4].

Now let us consider an integral:

\[
\sum_j \int \phi_B^+(r_j - R_j) \psi_k^+_{k-q} (r) U (r - r_j) \phi_A (r_j - R_j) \psi_k (r) dr dr_j 
\]

\[
\int \phi_A^+(r_j - R_j) \psi_k^+ (r) U (r - r_j) \phi_B (r_j - R_j) \psi_{k-q} (r) dr dr_j 
\]

\[
= \frac{N}{V^3} \int dR \int e^{iQF} U (r - r') \phi_B^+(r' - R) \phi_A (r' - R) d r' dr' 
\]

\[
= \frac{N}{V^3} \int dR \int e^{iQF} U (r - r') e^{iQ(r' - R)} \phi_B^+(r' - R) \phi_A (r' - R) d (r - r') d (r' - R) 
\]
\[
\int \int e^{-i \mathbf{q} \cdot (\mathbf{r'} - \mathbf{r})} U(\mathbf{r' - r}) e^{-i \mathbf{q} \cdot (\mathbf{r' - R})} \phi_A^+(\mathbf{r' - R}) \phi_B(\mathbf{r' - R}) d(\mathbf{r' - R}) d(\mathbf{r - r}) = \frac{\rho}{V} U(\mathbf{q}) U(-\mathbf{q}) \langle B|A \rangle \mathbf{q} \langle A|B \rangle = \frac{\rho}{V} \left| U(\mathbf{q}) \langle B|A \rangle \right|^2,
\]

where
\[
U(\mathbf{q}) = \int e^{i \mathbf{q} \cdot \mathbf{r}} U(\mathbf{r}) d\mathbf{r}
\]
\[
\langle B|A \rangle \mathbf{q} = \int e^{i \mathbf{q} \cdot \mathbf{r}} \phi_B^*(\mathbf{r}) \phi_A(\mathbf{r}) d\mathbf{r}.
\]

Then a mass operator is
\[
-i \Sigma(\mathbf{k}, \varepsilon) = \rho \sum_B \int \frac{dq dq \omega}{(2\pi)^4} \left| U(\mathbf{q}) \langle B|A \rangle \mathbf{q} \right|^2 (-i) D_{AB}(\omega) i G_0(\mathbf{k} - \mathbf{q}, \varepsilon - \omega).
\]

We can see this expression for a mass operator is analogous to electron-phonon interaction (phonons with Einstein specter), where a value \( \rho \left| U(\mathbf{q}) \langle B|A \rangle \mathbf{q} \right|^2 \) plays a role of a coupling constant. Eq. (14) corresponds to result of an article [4] (if we suppose \( \left| U(\mathbf{q}) \langle B|A \rangle \mathbf{q} \right|^2 = \text{const} \) and an impurity is two-level system) when the integration over \( q \) be done within the boundaries \( \int_{2k_F} q^2 dq \). We have to consider a case when \( \omega_{AB} = 0, \phi_B = \phi_A \), that is the scattering is elastic. Then energetic parameter \( \omega \) is not transferred along the line of interaction. According to the rules of a diagram technique an integration over the intermediate energetic parameter is absent. Instead of Eq. (11) we must have \( \int_{-\infty}^{+\infty} iG^+_A(\gamma) \frac{dq}{2\pi} = 1 \). Then we have
\[
-i \Sigma(\mathbf{k}, \varepsilon, \omega_{AB} = 0) = \rho \int \frac{dq dq \omega}{(2\pi)^4} (-i) U(\mathbf{q})(-i) U(-\mathbf{q}) \left| \langle B|A \rangle \mathbf{q} \right|^2 i G_0(\mathbf{k} - \mathbf{q}, \varepsilon).
\]

Impurity’s ground state wave function have a form \( \phi_A \sim e^{-r/a} \). If \( 1/a \gg k_F \) then we can assume \( \langle A|B \rangle \mathbf{q} \approx \langle A|B \rangle_0 = 1 \). Then Eq. (15) coincides with Eq. (AB) for a mass operator of the second order elastic process.

Let us generalize Eq. (14) for nonzero temperatures and do some transformations. Following [17] let us denote \( p^2 = |\mathbf{k} - \mathbf{q}|^2 = k^2 + q^2 + 2kq \cos \theta \), where \( \theta = \mathbf{k} \cdot \mathbf{q} \) and \( k \approx k_F \). Further we have \( pdp = kdq d\cos \phi \), \( \xi = v_F(p - p_F) \rightarrow d\xi = v_F dp \). Hence
\[
\int dq \equiv \int d^3 q = \int_0^{2k_F} q^2 dq \int_0^{2\pi} d\phi \int_0^{2k_F} q^2 dq \int dp = \int \frac{d^2 q}{v_F} \int d\xi,
\]

where limits of integration over \( d\xi \) can be extended between \( \pm \infty \) because the main contribution of the integrand is in the region \( \xi \approx 0 \). Then we can write
\[
-i \Sigma(\mathbf{k}, \varepsilon_n) = \rho T \sum_A \sum_B \sum_{m=-\infty}^{+\infty} \int \frac{d^2 \! dq \! dq}{v_F(2\pi)^3} \left| U(\mathbf{q}) \langle B|A \rangle \mathbf{q} \right|^2 i D_{AB}(\varepsilon_n - \varepsilon_m) i G_0(\xi, \varepsilon_m),
\]

where an electron’s propagator and a pseudopropagator are
\[
G_0(\xi, \varepsilon_m) = -i \frac{\varepsilon_m + \xi}{\varepsilon_m^2 + \xi^2}, \quad D_{AB}(\varepsilon_n - \varepsilon_m) = -i \frac{2\omega_{AB}}{(\varepsilon_n - \varepsilon_m)^2 + \omega_{AB}^2}.
\]

and \( \varepsilon_m = \pi T(2m + 1) \). It should be noted an interaction function \( U(\mathbf{q}) \) can be represented via a differential scattering cross-section: \( \frac{df}{d\Omega} = \frac{\pi}{2} U(\mathbf{q})^2 \).

Higher order diagrams may be classified under two types. The first type corresponds to cross-diagrams. They describe processes like discussed above second order process, however the scattering takes place by different impurities. The simplest fourth order diagram is shown in Fig. [1]. The diagram means an electron interacts with an impurity \( j \) changing its state \( \phi_A \rightarrow \phi_B \), then the electron interacts with an impurity \( l \) changing its state \( \phi_A \rightarrow \phi_C \). Then the electron gathers the energies interacting again with the impurities \( \phi_B \rightarrow \phi_A, \phi_C \rightarrow \phi_A \). States \( |B \rangle \) and \( |C \rangle \) are virtual, a state \( |A \rangle \) is a ground state (for \( T = 0 \) only). Transition frequencies are \( \omega_{AB} = |E_B - E_A|, \omega_{AC} = |E_C - E_A| \). Analytically the process is represented as follows:
\[
-i \Sigma(\mathbf{k}, \varepsilon) = \rho^2 \sum_B \sum_{C} \int \frac{d\omega_1}{(2\pi)^3} \int \frac{d\omega_2}{(2\pi)^3} \left| U(\mathbf{q}) \langle B|A \rangle \mathbf{q} \right|^2 \left| U(\mathbf{p}) \langle C|A \rangle \mathbf{p} \right|^2 (-i) D_{AB}(\omega_1)(-i) D_{AC}(\omega_2) i G_0(\mathbf{k} - \mathbf{q}, \varepsilon - \omega_1) i G_0(\mathbf{k} - \mathbf{p}, \varepsilon - \omega_2) i G_0(\mathbf{k} - \mathbf{q} - \mathbf{p}, \varepsilon - \omega - \omega_2).
\]
Figure 4: The fourth order cross-diagram. In the scattering two impurities (jth and lth) take part simultaneously. The impurities’ states are changed during the process: \( j: |A \rangle \rightarrow |B \rangle \rightarrow |A \rangle \), \( l: |A \rangle \rightarrow |C \rangle \rightarrow |A \rangle \) (a right-hand picture).

Analogously we can consider any cross-diagrams of higher orders.

A next type of cross-diagrams is shown in Fig. 5. In this diagram a scattering process by an impurity is crossed with an electron-phonon interaction. Analytically the process is represented as follows:

\[
-i \Sigma(k, \varepsilon) = \rho \sum_B \int \frac{d\omega_1}{(2\pi)^4} \int \frac{d\omega_2}{(2\pi)^4} \left|U(q)|B \rangle |A \rangle q\right|^2 |g(p)|^2 (-i)D_{AB}(\omega_1) (-i)D_{ph}(\omega_2, \Omega(p)) \\
iG_0(k - q, \varepsilon - \omega_1) iG_0(k - p, \varepsilon - \omega_2) iG_0(k - q - p, \varepsilon - \omega_1 - \omega_2),
\]

(19)

where \( g(p) \) is an electron-phonon coupling constant, \( D_{ph}(\omega_2, \Omega(p)) \) is a phonon propagator.

Figure 5: The fourth order cross-diagram with involvement of a phonon. In scattering one impurities and one phonon take place simultaneously.

A small parameter for the expansion is a ratio of a contribution of cross-diagrams to a contribution of diagrams without crossings. In [2] it was shown that the ratio is proportional to \( \frac{\Delta k}{k_F} \), where \( \Delta k \) is a momentum’s uncertainty as result of scattering \( \Delta k \propto 1/l \) (l is a free length). Then the small parameter is \( 1/lk_F \ll 1 \), that is correct for a weak coupling. At inelastic scattering by impurities a particle’s energy change by a value \( \Delta \varepsilon \sim \omega_{AB} \), that corresponds to a momentum’s uncertainty \( \Delta k = \frac{m\omega_{AB}}{k_F} \). Hence the small parameter is

\[
\frac{\Delta k}{k_F} = \frac{m\omega_{AB}}{k_F^2} \sim \frac{\omega_{AB}}{\varepsilon_F} \ll 1.
\]

(20)

And so on for each frequency \( \omega_{AB}, \omega_{CD}, \ldots \) Eq. 20 likes a situation with phonons where a small parameter is an adiabaticity parameter (Migdal’s theorem).
The second type corresponds to beam-like diagrams. For elastic scattering the diagrams is shown in Fig[10]. For inelastic scattering a beam-type diagram of third order is shown in Fig[6]. The diagram means an electron interacts with an impurity \( j \) changing its state \( \phi_A \rightarrow \phi_B \). Then the electron interacts with the impurity again changing a impurity’s state \( \phi_B \rightarrow \phi_C \). Then the electron recovers energy interacting again with the impurity \( \phi_C \rightarrow \phi_A \). States \( |B\rangle \) and \( |C\rangle \) are virtual, a state \( |A\rangle \) is a ground state (for \( T = 0 \) only). Transition frequencies are \( \omega_{AB} = E_B - E_A \), \( \omega_{AC} = E_C - E_A \), \( \omega_{BC} = E_C - E_B \). In addition another variant of the process is possible \( |A\rangle \rightarrow |C\rangle \rightarrow |B\rangle \rightarrow |A\rangle \). Analytically the process is represented as follows:

\[
-i\Sigma(k, \varepsilon) = \rho \sum_B \sum_C \int \frac{dq dq\omega}{(2\pi)^4} \int \frac{dp dp\omega}{(2\pi)^4} U(q)|B\rangle A \langle U(p - q)|C\rangle B \langle p - q U(-p)|A\rangle C - p
\]

\[
= \rho \sum_B \sum_C \int \frac{dq dq\omega}{(2\pi)^4} \int \frac{dp dp\omega}{(2\pi)^4} U(q)|B\rangle A \langle U(p - q)|C\rangle B \langle p - q U(-p)|A\rangle C - p
\]

\[
iG_0(k - q, \varepsilon - \omega_1)G_0(k - p, \varepsilon - \omega_2)
\]

\[
+ \int \frac{d\gamma}{2\pi} [iG_A(\gamma)G_B^+(\gamma + \omega_1)G_C^+(\gamma + \omega_2) + iG_A^+(\gamma)G_B^-(\gamma + \omega_1)G_C^-(\gamma + \omega_2)]
\]

\[
iG_A^-(\gamma)G_B^+(\gamma - \omega_1)G_C^+(\gamma + \omega_2) + iG_A^-(\gamma)G_B^+(\gamma + \omega_1)G_C^-(\gamma - \omega_2)
\]

\[
= \rho \sum_B \sum_C \int \frac{dq dq\omega}{(2\pi)^4} \int \frac{dp dp\omega}{(2\pi)^4} U(q)|B\rangle A \langle U(p - q)|C\rangle B \langle p - q U(-p)|A\rangle C - p
\]

\[
iG_0(k - q, \varepsilon - \omega_1)G_0(k - p, \varepsilon - \omega_2) \left[ \frac{-4\omega_{AB} \omega_{AC}}{(\omega_1^2 - \omega_{AB}^2)(\omega_2^2 - \omega_{AC}^2)} \right].
\]

(21)

Contribution of this process is proportional to \( \rho U^3 \). Analogously we can construct higher order diagrams of this type to be proportional to \( \rho U^4, \rho U^5, \ldots \). The beam-like diagrams violate the analogy with electron-phonon interaction. In a limit \( \rho \rightarrow \infty, U^2 \rightarrow 0, \rho U^2 = \text{const} \) (continuous "spreading" of impurities over a system) the diagrams disappear. Unfortunately the series of beam-like diagrams have not any small parameter like the series of cross-diagrams. However it is not difficult to notice that the mass operator of the third order beam-like process is \( \Sigma_{ABC} \propto \rho \int dp U(q) U(p - q) U(-p)|B\rangle A \langle U(q) C\rangle B \langle p - q U(-p)|A\rangle C - p \) and a mass operator for the third order process when scattering takes place by different impurities (it is a reducible diagram \( -i\Sigma_{AB[CC]} = -i\Sigma_{AB}(-i)G_0(-i)\Sigma_{CC} \) is
\( \Sigma_{AB[CC]} \propto \rho^2 U(0)|U(q)|^2 \). In the expression \( \Sigma_{ABC} \) the integrand is an alternating function, and in the expression \( \Sigma_{AB[CC]} \) the integrand is a constant-sign function since \( \langle B|A \rangle_0 = 0, \langle C|A \rangle_0 = 0, \langle C|C \rangle_0 = 1 \). Therefore due to the integration we have \( \Sigma_{ABC} \ll \Sigma_{AB[CC]} \). For higher orders the alternating is strengthened. Thus we can uncouple beam-like diagrams as shown in Fig.8 and thereby reconstitute the phonon analogy.

**III. SUPERCONDUCTING STATE.**

In this section we generalize results obtained in a previous section in two directions: to make the perturbation theory as self-consistent and to apply it for a superconductive state. To make the perturbation theory as self-consistent the free propagators \( G_0 \) must be replaced with dressed propagators \( G \) (internal lines in diagrams are bolded). To use the perturbation theory for superconductive state we have to consider anomalous propagators \( F \) and \( F^+ \) which are proportional to order parameters \( \Delta \) and \( \Delta^+ \). We suppose the order parameter is self-averaging: \( \langle \Delta^2(r) \rangle - \langle \Delta(r) \rangle^2 = 0 \). This means to neglect a scattering of Cooper pairs by fluctuations of the gap. And we suppose singlet \( s \)-wave pairing takes place.

Gor’kov equations for a dirty superconductor with retarded interaction of quasiparticles with impurities have a form shown in Fig.8. Their sense is that electrons pair in the metallic matrix at first, then normal and anomalous propagators are dressed by interaction with impurities. Unlike elastic interaction the lines of interaction (dotted lines) transfer energy. The equation are self-consistent because dressed propagators are calculated with the dressed propagators (bold lines under interaction with impurities). Solutions of the equations are dressed normal and anomalous propagators:

\[
\tilde{G}(\varepsilon_n, \xi) = -\frac{i\varepsilon_n + \xi}{\varepsilon_n^2 + \xi^2 + |\Delta|^2}, \quad \tilde{F}^+(\varepsilon_n, \xi) = \frac{i\Delta^+_n}{\varepsilon_n^2 + \xi^2 + |\Delta|^2},
\]

where a renormalized gap \( \tilde{\Delta} \) and a renormalized energy parameter \( \tilde{\varepsilon} \) are determined with equations (here \( \varepsilon_n = (2n + 1)\pi T \) and \( \varepsilon_m = (2m + 1)\pi T \)):

\[
\tilde{\Delta}_n = \Delta_n + \rho T \sum_A \sum_B q^2 \sum_{m=-\infty}^{+\infty} \int \frac{d^3q d\xi}{v_F(2\pi)^3} |U(q)|^2 iD_{AB}(\varepsilon_n - \varepsilon_m) i\tilde{F}(\xi, \varepsilon_m) \tag{23}
\]

\[
\tilde{\varepsilon}_n = \varepsilon_n + \rho T \sum_A \sum_B q^2 \sum_{m=-\infty}^{+\infty} \int \frac{d^3q d\xi}{v_F(2\pi)^3} |U(q)|^2 iD_{AB}(\varepsilon_n - \varepsilon_m) i\tilde{G}(\xi, \varepsilon_m). \tag{24}
\]

Eqs.\( (23,24) \) are a set of self-consistent equations. The order parameter \( \Delta \) is determined by the anomalous propagator \( F \) in a case of a pure metal and determined by the dressed anomalous propagator \( \tilde{F} \) in a case of a duty metal:

\[
\Delta(\varepsilon_n) = T \sum_{m=-\infty}^{+\infty} \int \frac{d^3q}{(2\pi)^3} |F(\varepsilon_m, p)| g(p)^2 iD_{ph}(\varepsilon_m - \varepsilon_m, p - q) \quad \text{(in a pure metal)} \tag{25}
\]
\[ \Delta(\varepsilon_n) = T \sum_{m=-\infty}^{+\infty} \int \frac{dq}{(2\pi)^3} i\bar{F}(\varepsilon_m, p)|g(p)|^2 iD_{ph}(\varepsilon_m - \varepsilon_n, p - q) \] (in a disordered metal).  

(26)

Eqs. (23,24) can be reduced to a following form after integration over \( \xi \):

\[ \tilde{\Delta}_n = \Delta_n + \sum_{m=-\infty}^{+\infty} W(n - m) \frac{\pi T \Delta_m}{\sqrt{\tilde{\varepsilon}_m^2 + |\Delta_m|^2}} \]

(27)

\[ \tilde{\varepsilon}_n = \varepsilon_n + \sum_{m=-\infty}^{+\infty} W(n - m) \frac{\pi T \tilde{\varepsilon}_m}{\sqrt{\tilde{\varepsilon}_m^2 + |\Delta_m|^2}}, \]

(28)

where

\[ W(n - m) = \sum_A \sum_B \varpi_A \varpi_B \int \frac{2\rho d^2 q}{\omega AB v_F(2\pi)^3} |U(q)\langle B|A|\rangle^2 \frac{\omega^2_{AB}}{(n - m)^2} \pi^2 \frac{\omega^2_{AB}}{2 + \omega^2_{AB}} \]

(29)

The gap \( \tilde{\Delta}_m \) is an even function of \( 2m + 1 \), but the energy parameter \( \tilde{\varepsilon}_m \) is an odd function of \( 2m + 1 \). Hence these function are renormalized in different ways:

\[ \frac{\tilde{\Delta}}{\Delta} > \frac{\tilde{\varepsilon}}{\varepsilon}. \]

(30)

From Eqs. (25,26) and Eqs. (22) we can see that inequality (30) ensures increasing of the gap \( \Delta \) as compared with a pure superconductor or with a dirty superconductor with elastic impurities where an equality \( \frac{\tilde{\Delta}}{\Delta} = \frac{\tilde{\varepsilon}}{\varepsilon} \) takes place. Thus Anderson theorem is violated in the sense that embedding of the impurities in \( s \)-wave superconductor increases its critical temperature.

If temperature is much more than any impurity’s frequencies \( T \gg \omega_{AB} \) then Eqs. (27,28) have a form

\[ \tilde{\Delta}_n = \Delta_n + \sum_{m=-\infty}^{+\infty} W(n - m) \frac{\pi T \Delta_m}{\sqrt{\tilde{\varepsilon}_m^2 + |\Delta_m|^2}} \]

(31)

\[ \tilde{\varepsilon}_n = \varepsilon_n + \sum_{m=-\infty}^{+\infty} W(n - m) \frac{\pi T \tilde{\varepsilon}_m}{\sqrt{\tilde{\varepsilon}_m^2 + |\Delta_m|^2}}, \]

(32)

Solving Eqs. (31,32) we find that the gap and the energy parameter are renormalized similarly:

\[ \frac{\tilde{\Delta}}{\Delta} = \frac{\tilde{\varepsilon}}{\varepsilon} = 1 + \frac{1}{2\tau} \frac{1}{\sqrt{\tilde{\varepsilon}_n^2 + \Delta^2}}. \]

(33)

The relation (33) means realization of Anderson’s theorem - the gap and, accordingly, critical temperature do not change. The limit \( T \gg \omega_{AB} \) corresponds to an elastic scattering by impurities with a scattering frequency \( \frac{1}{2\tau}. \) It should be noticed if the impurity’s frequency is too large \( \omega_{AB} \rightarrow \infty \) then an interaction with the impurities is weak \( \sim \frac{1}{\omega_{AB}} \) and effectiveness of the impurities decreases.

Let us consider a case when temperature is equal to a critical temperature \( T = T^*_C \) of a system metal+impurities. Then the gaps are equal to zero and Eqs. (27,28) have a form

\[ \tilde{\Delta}_n = \Delta_n + \sum_{m=-\infty}^{+\infty} W(n - m) \frac{\pi T \Delta_m}{|\tilde{\varepsilon}_m|} \]

(34)

\[ \tilde{\varepsilon}_n = \varepsilon_n + \sum_{m=-\infty}^{+\infty} W(n - m) \frac{\pi T \tilde{\varepsilon}_m}{|\tilde{\varepsilon}_m|} \]

(35)

Eq. (35) has an exact solution [17]:

\[ \tilde{\varepsilon}_n = \eta_n \varepsilon_n, \quad \eta_n = 1 + \frac{1}{2(n + 1)^2} \left[ W(0) + 2 \sum_{l=1}^{n} W(l) \right]. \]

(36)
To find a critical temperature of a pure superconductor $T_C$ we have to solve Eliashberg equations when $\Delta = 0$: \[ Z(\varepsilon_n) \Delta_n = \pi T_C \sum_{|\varepsilon_n| \leq \omega_c} [L(n - m) - \mu^*] \frac{\Delta_m}{|\varepsilon_m|} \] (37)

\[ [1 - Z(\varepsilon_n)] \varepsilon_n = -\pi T_C \sum_{m = -\infty}^{+\infty} L(n - m) \text{sign}\varepsilon_m, \] (38)

where $Z$ is a renormalization function,

\[ L(n - m) = 2 \int_0^{\infty} d\Omega \alpha^2(\Omega) g(\Omega) \frac{\Omega}{\Omega^2 + (\varepsilon_n - \varepsilon_m)^2}, \] (39)

$\alpha^2(\Omega) g(\Omega)$ is an electron-phonon coupling function, a restriction of summation over $m$ in is introduced to use Coulomb pseudopotential

\[ \mu^* = \frac{\mu}{1 + \mu \ln \left( \frac{E_F}{\omega_c} \right)} \] (40)

instead of full Coulomb constant $\mu$, $\omega_c \sim 10\omega_D$ ($\omega_D$ is Debye frequency). Transition temperature $T_C$ of the pure superconductor is a such temperature when Eqs. (37) have a solution.

To find a critical temperature of a system metal+impurities we have to generalize Eliashberg equations. Electrons and Cooper pairs scatter by impurities. As a result the gap and the energy parameter are renormalized with Eqs. (34,35). $\Delta, \varepsilon \rightarrow \tilde{\Delta}, \tilde{\varepsilon}$. Then we have to substitute the renormalized function $\tilde{\Delta}_m, \tilde{\varepsilon}_m$ instead of the functions $\Delta_m, \varepsilon_m$ to the right side of Eliashberg equations (37,38). Then we have a set of equations:

\[ Z_n \Delta_n = \sum_{|\varepsilon_n| \leq \omega_c} \left[ L(n - s) - \mu^* \right] \frac{\tilde{\Delta}_s}{|2s + 1|\eta_s} \] (41)

\[ \tilde{\Delta}_n = \Delta_n + \sum_{m = -\infty}^{+\infty} W(n - m) \frac{\tilde{\Delta}_m}{|2m + 1|\eta_m} \] (42)

\[ Z_n = 1 + \frac{1}{|2n + 1|} \left[ L(0) + 2 \sum_{l=1}^{n} L(l) \right] \] (43)

\[ \eta_n = 1 + \frac{1}{|2n + 1|} \left[ W(0) + 2 \sum_{l=1}^{n} W(l) \right] \] (44)

Two last formulas (43,44) determine a renormalization of electron specter due electron-phonon interaction (the function $Z_n$) and due scattering by impurities (the function $\eta_n$). Eq. (42) is a nonhomogeneous set of linear equations in the unknowns $\Delta_n$ ($n = -\infty \ldots + \infty$) and the gap $\tilde{\Delta}$ is a function of the gap $\Delta$. After substituting $\tilde{\Delta}_n$ in Eq. (41) we have a homogeneous set of linear equations in the unknowns $\Delta_n$. Indexes of summation $n, s$ in Eq. (41) and $n, m$ in Eq. (42) are independent. Temperature $T$ is contained in the functions $L(n - s), W(n - m)$. Transition temperature $T_C^*$ of the system is a such temperature when Eqs. (41,42) have a solution.

In order to consider an influence of impurities upon the transition temperature we have to solve a homogeneous set of equations obtained from Eq. (42) omitting $\Delta_n$:

\[ \sum_m W(n - m) \frac{\tilde{\Delta}_m}{|2m + 1|\eta_m} - \tilde{\Delta}_n = 0 \] (45)

However Eq. (45) has a solution at another temperature $T^*$ - the singularity temperature introduced in [14]. The singularity temperature is $T^* < T_C^*$ and it can be used as a lower estimation of the critical temperature of the dirty metal. Its physical sense is: the singularity temperature is a superconducting transition temperature if we turn off the pairing interaction in the metal. Therefore we have always $T^* < T_C^*$. A determinant of the set of equations (45) must be equal to zero:

\[ \det D_{mn}(T_C^*) = 0, \quad D_{mn} = \frac{W(n - m)}{|2m + 1|\eta_m} - \delta_{mn}, \] (46)
where $\delta_{mn} = 1$ if $m = n$, $\delta_{mn} = 0$ if $m \neq n$.

Let an interaction with impurities be nonretarded (elastic): $W(n - m) = W(0)\delta_{mn}$. The determinant $D_{mn}$ is diagonal in this case. Each diagonal element of the determinant is

$$\frac{W(0)}{|2m + 1|\eta_m} - 1 = \frac{W(0)}{|2m + 1| + W(0)} - 1 \neq 0. \quad (47)$$

Hence the singularity temperature is absent. Indeed if the interaction is elastic (when an addendum with $n = m$ is only) then from Eqs. (42, 44) we can see $\Delta_m = \Delta_m\eta_m$. Then Eq. (41) is transformed to Eliashberg equation (37) for a pure metal. Thus Anderson theorem is realized for the elastic interaction: $T^*_C = T_C$. However if the retarded interaction of quasiparticles with impurities takes place then the transition temperature rises: $T^* \neq 0 \Rightarrow T^*_C > T_C$

as a consequence of dissimilar renormalizations of the gap and the energy parameter: $\frac{\Delta^*}{\epsilon} > \frac{\Delta}{\epsilon}$. Namely the sign ”$>$” of this inequality provides amplification of the superconductive properties, unlike, for example, magnetic impurities, where the sign is ”$<$” resulting in suppression of superconductivity.

### IV. CONCLUSION.

In this work a perturbation theory and a diagram technique has been developed for a disordered metal if interaction of quasiparticles with impurities is nonretarded and impurity’s oscillations are local. All possible diagrams are classified into several types, the electron-impurity coupling for the various impurity’s transitions and mass operators for the basic scattering processes are calculated: first order process (9), second order process (14, 16), higher-order cross-process (18), higher-order cross-process with involvement of phonons (19) and processes described with beam-like diagrams (21).

We showed the perturbation theory can be made in an adiabatic approximation (when impurity’s transition frequency is much less than metal’s Fermi energy) for cross-diagrams and in an approximation with uncoupled correlations for the beam-like diagrams. Thus the electron-impurity coupling is not assumed to be small unlike perturbation theory for the elastic scattering. We found that in these approximations the averaging over disorder results in a picture like quasiparticles interact with some collective excitations propagating through the system, thus an analogy between the inelastic scattering of electrons by impurities and an electron-phonon interaction exists. In the proposed diagram technique the lines of interaction with impurities in the diagrams transfer both a momentum and an energy parameter unlike the diagram technique for a disordered metal with the elastic scattering. If the energy transfer cannot be then the diagrams and their analytical representations are transformed into diagrams and corresponding expressions for the elastic processes. Thus the proposed perturbation theory generalizes a case of the elastic scattering in a disordered metal.

Eliashberg equations at a critical temperature $T^*_C$ have been generalized for a case of s-wave superconductor containing impurities of a considered type: Eqs. (41, 44). We found the retarded interaction of quasiparticles with impurities violates Anderson theorem: a gap and an energy parameter are renormalized differently $\frac{\Delta^*}{\epsilon} > \frac{\Delta}{\epsilon}$ - Eq. (30). This fact causes violation of Anderson’s theorem in the direction of increasing of the critical temperature. Thus a critical temperature of a system metal+impurity is more than a critical temperature of the pure metal $T^*_C > T_C$. The increasing depends on impurities’ concentration, electron-impurity coupling and oscillation specter of the impurities. Mechanism of influence of an impurity on a Cooper pair is as follows: at first Cooper pairs are formed in a metal with electron-phonon interaction, then they are scattered by the impurities; the first electron changes impurity’s state, then the second one interacts with the impurity changed by the first electron, thus a correlation between the electrons appears that increases their binding energy. In a limit case when temperature is much more then impurity’s oscillation frequency Anderson theorem is restored (effectiveness of the impurities aspires to zero), because at too small frequency a thermal noise destroys the changes of impurity’s states. If the frequency is too large ($\omega \sim \epsilon_F \gg T^*_C$) then an interaction with the impurities is weak and effectiveness of the impurities decreases. The generalized Eliashberg equation is simplified to Eq. (40) if we calculate the singularity temperature $T^*$. Its physical sense is: the singularity temperature is a superconducting transition temperature if we turn off the pairing interaction in the metal, therefore we have always $T^* < T^*_C$. The singularity temperature we can use as a lower estimate of the critical temperature of the dirty superconductor.
Appendix A: Elastic scattering by impurities.

In a case of elastic scattering of electrons by impurities Hamiltonian of a system is

\[ \hat{H} = \hat{H}_0 + \sum_j \sum_{\mathbf{k}, \mathbf{k}'} \int \psi^+_\mathbf{k}'(\mathbf{r}) U(\mathbf{r} - \mathbf{R}_j) \psi^\mathbf{k}(\mathbf{r}) d\mathbf{r} \sum_{\mathbf{k}} c^+_\mathbf{k} c^\mathbf{k}. \]  

(A1)

A summarized field of all impurities is

\[ V(\mathbf{r}) = \sum_{j=1}^N U(\mathbf{r} - \mathbf{R}_j) = \frac{1}{V} \sum_q \sum_j U(q) e^{i(q-R_j)} , \]

(A2)

the simplest process shown in Fig.9. The first correction to an electron propagator is

\[ iG_1(\mathbf{k}, \mathbf{k}', \mathbf{\epsilon}) = [iG_0(\mathbf{k}, \mathbf{\epsilon})]^2 (-i) \sum_j \int \psi^+_\mathbf{k}'(\mathbf{r}) U(\mathbf{r} - \mathbf{R}_j) \psi^\mathbf{k}(\mathbf{r}) d\mathbf{r} \]

\[ = [iG_0(\mathbf{k}, \mathbf{\epsilon})]^2 (-i) \frac{1}{V} \sum_j \int e^{i(\mathbf{k} - \mathbf{k}') \mathbf{r}} U(\mathbf{r} - \mathbf{R}_j) d\mathbf{r} = [iG_0(\mathbf{k}, \mathbf{\epsilon})]^2 (-i) \frac{N}{V^2} \int e^{i(\mathbf{k} - \mathbf{k}') \mathbf{r}} U(\mathbf{r} - \mathbf{R}) d\mathbf{r} d\mathbf{R} \]

\[ = [iG_0(\mathbf{k}, \mathbf{\epsilon})]^2 (-i) \frac{N}{V^2} (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}') \int U(\mathbf{r} - \mathbf{R}) d\mathbf{r} - \mathbf{R}) = [iG_0(\mathbf{k}, \mathbf{\epsilon})]^2 \rho(-i) U(q = 0)(2\pi)^3 \delta(\mathbf{k} - \mathbf{k}'). \]

(A3)

We can see the averaging operation leads to conservation of momentum. In higher approximations we have diagrams shown in Fig.10 (reducible and irreducible diagrams). There are three kinds of the diagrams [2]. Types (a) and (b)

are diagrams corresponding to motion of an electron in Gauss random field with factorized correlators (a white noise):

\[ \langle V_1(\mathbf{r}_1) V_2(\mathbf{r}_2) \rangle = \rho U^2 \delta(\mathbf{r}_1 - \mathbf{r}_2), \quad \langle V(\mathbf{1}) \rangle = 0, \quad \langle V(\mathbf{1}) V(\mathbf{2}) V(\mathbf{3}) \rangle = 0, \ldots \]

\[ \langle V(\mathbf{1}) V(\mathbf{2}) V(\mathbf{3}) V(\mathbf{4}) \rangle = \langle V(\mathbf{1}) V(\mathbf{2}) \rangle \langle V(\mathbf{3}) V(\mathbf{4}) \rangle + \langle V(\mathbf{1}) V(\mathbf{4}) \rangle \langle V(\mathbf{2}) V(\mathbf{3}) \rangle + \ldots \]

(A4)

where in most cases an impurity’s potential can be considered as point so that \( U(q) \approx U = \int U(\mathbf{r}) d\mathbf{r} \). The type (b) corresponds to cross-diagrams. A ratio of contribution of the cross-diagrams to contribution of straight processes (the type (a)) is \( \frac{\rho}{\rho} \ll 1 \), where \( l \) is a free length, \( \ll 1 \) corresponds to a weak disorder. Diagrams of a type (b) we name beam-like diagrams. In a limit \( \rho \to \infty, v^2 \to 0, \rho v^2 = \text{const} \) (continuous “spreading” of impurities over a system) the beam-like diagrams disappear (except reducible diagrams: for example the second diagram in a row (c)).

Conservation of momentum allows us to summarize diagrams with help of Dyson equation:

\[ iG(\mathbf{k}, \mathbf{\epsilon}) = iG_0(\mathbf{k}, \mathbf{\epsilon}) + iG_0(\mathbf{k}, \mathbf{\epsilon})(-i) \Sigma_i G(\mathbf{k}, \mathbf{\epsilon}), \]

(A5)

where \( \Sigma(\mathbf{k}, \mathbf{\epsilon}) \) is a mass operator. The mass operator describes a multiple scattering of electrons by impurities. For an elastic scattering the interaction lines do not transfer energy parameter. They transfer momentum only. A multiplier \( \rho U(q)^2 \) is related to them. For a weak disorder the mass operator is determined by the first diagram in a row (a):

\[ (-i) \Sigma(\mathbf{k}, \mathbf{\epsilon}) = \rho \int \frac{d^3q}{(2\pi)^3} (-i) U(q) iG_0(\mathbf{k} - \mathbf{q}, \mathbf{\epsilon})(-i) U(\mathbf{q}) = \rho \int \frac{d^3p}{(2\pi)^3} (-1) |U(\mathbf{k} - \mathbf{p})|^2 iG_0(\mathbf{p}, \mathbf{\epsilon}_n), \]

(A6)
or in Matsubara representation (nonzero temperature $\varepsilon_n = (2n + 1)\pi T$):

$$- \Sigma(k, \varepsilon_n) = \rho \int \frac{d^3q}{(2\pi)^3} (-1)U(q)iG_0(k - q, \varepsilon_n)(-1)U(-q) = \rho \int \frac{d^3p}{(2\pi)^3}|U(k - p)|^2 iG_0(p, \varepsilon_n).$$  \hspace{1cm} (A7)

It should be noted that in the diagrams the dotted lines are not dressed with polarization loops, because the disorder

![Diagrams](image)

Figure 10: Three kinds of diagrams describing an elastic scattering of electrons by impurities. A kind (a) is usual diagrams describing straight processes, a kind (b) is cross-diagrams and a kind (c) is beam-like diagrams. $j$ and $l$ are deferent impurities.

is "freezed in" and the impurities do not fit into changes of an electron density. Substituting a free propagator $G_0(p, \varepsilon_n) = \frac{i}{\pi \varepsilon_n - \xi(p)}$ into the expression for a mass operator we obtaining (assuming a weak dependence of a impurity's potential on momentum $U(k - p) \approx U$ and a linear specter of quasi-particles near Fermi surface $\xi(k) \approx v_F(k - k_F)$):

$$\Sigma(p, \varepsilon_n) = -i\frac{\varepsilon_n}{|\varepsilon_n|\pi \rho U^2 v_F \equiv -i\gamma \text{sign} \varepsilon_n}$$  \hspace{1cm} (A8)

where $\nu_F = \frac{mk_F^2}{2\pi^2}$ is a density of states on Fermi surface per one projection of spin. Then the mean free time and the free length are determined as:

$$\tau = \frac{1}{2\gamma}, \quad l = v_F \tau = \frac{v_F}{2\gamma} = \frac{v_F}{2\pi \rho U^2 \nu_F}$$  \hspace{1cm} (A9)

Elastic impurities do not influence upon effective mass of quasi-particles but they stipulate for a quasi-particles' damping $\gamma \text{sign} \varepsilon_n$. It should be noticed irreducible diagrams of kinds (a) and (c) can be summated in $t$-matrix $t_{kp}$.

Then in Eq.(A7) we have to replace $\rho |U(k - p)|^2$ by $t_{kp}$ [2, 10, 11].

All cross-diagrams describe quantum corrections for conductivity - interference of incident and reflected by impurities electron waves. This leads to Anderson's localization [2, 3, 19] when $\frac{1}{\nu_F} \geq 1$ - electrons are "blocked" between the impurities. However with increase of temperature (or if the system is in an external alternating field) nonelastic processes begin to play a role (electron-phonon processes, electron-electron processes) [20, 22]. The processes limit the coherence time of electron waves $\tau_\phi < \infty$ (or the coherence length $L_\phi < \infty$). If $\tau_\phi < \tau$ (or $L_\phi < l$) then the interference contribution is essentially suppressed because the phase failure takes place [15].

[1] P.W.Anderson, J. Phys. Chem. Solids 11 (1959) 26.
[2] M.V. Sadovskii, Diagrammatics: Lectures on Selected Problems in Condensed Matter Theory, World Scientific, Singapore, 2006.

[3] M.V. Sadovskii, *Superconductivity and Localization*, [arXiv:cond-mat/9308018v3 [cond-mat.dis-nn]] 27 Mar 1999

[4] L.N. Bulaevskii and V. Sadovskii, J. Low-Temp.Phys. 59 (1985) 89

[5] P.G. de Gennes, Superconductivity of Metals And Alloys, W.A. Benjamin, Inc., New York-Amsterdam, 1966.

[6] L.S. Borkovski and P.J. Hirschfeld, Phys.Rev. B 49 (1994) 15404

[7] R. Fehrenbacher and M.R. Norman, Phys.Rev. B 50 (1994) 3495

[8] R. J. Radtke, K. Levin, H.B. Schuttler, M. R. Norman, Phys.Rev. B 48 (1993) 653

[9] A. Posazhennikova and P. Coleman, Phys.Rev. B 67 (2003) 165109

[10] Y.G. Pogorelov, M.C. Santos, V.M. Loktev *Fisika Nizkich Temperatur* 37 (2011) 803.

[11] Y. Pogorelov, Solid State Commun. 95 (1995) 245

[12] D. Fay, J. Appel, Phys.Rev. B 51 (1995) 15604

[13] Rostam Moradian and Hamzeh Mousavi, [arXiv:cond-mat/0505092v1 [cond-mat.supr-con]] 4 May 2005

[14] K.V. Grigorishin, B.I. Lev, Physica C 495 (2013) 174; K.V. Grigorishin, B.I. Lev, [arXiv:1304.0113v5 [cond-mat.supr-con]] (06.09.2013)

[15] Levitov L.S., Shitov A.V. Green’s Functions. Problems and Solutions, Fizmatlit, Moscow, 2003 (in Russian)

[16] Richard D. Mattuk, A guide to feynman diagrams in the many-body problem, H. C. Oersted Institute University of Copenhagen, Denmark, 1967.

[17] Gerald D. Mahan, Many-particle physics (Physics of Solids and Liquids), 3rd edition, Plenum Publ. Corp. 2000

[18] V.L. Ginzburg, D.A. Kirzhnits, High-temperature superconductivity, Consultants Bureau, New York 1982

[19] Patrick A. Lee, T. V. Ramakrishnan, Reviews of Modern Physics 57 (1985) 2

[20] B.L. Altshuller, A.G. Aronov, D.E. Khmelnitsky Solid State Communications 39 (1981) 619.

[21] Altshuler B.L. and Aronov A.G., JETP Lett. 30 (1979) 514, Solid State Commun. 38 (1981) 11

[22] B.L. Altshuleri, A.G. Aronovf and D.E. Khmelnitsky, J. Phys. C: Solid State Phys. 15 (1982) 7367.