Novel Correlations in Arbitrary Dimensions

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Abstract

We present a new three dimensional many-body Hamiltonian with three-body and five-body interactions. We obtain the exact ground state as well as some excited states of this Hamiltonian for arbitrary number of particles. These exact wave-functions describe a novel correlations. Finally, we generalize these three dimensional results to arbitrary higher dimensions.

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Recently, there has been a renewed interest in the study of many-body quantum mechanical systems, like the Calogero-Sutherland Model (CSM) and its variants in one dimension \([1-3]\). This is primarily because these models are relevant to many diverse branches in physics \([7]\). Though, such one dimensional many-body systems have been studied extensively in the recent literature, nothing much is known about the appropriate generalization of these models to higher dimensions. As a promising step towards this, it was pointed out recently \([8]\) that in two dimensions there exists novel correlations other than the one used in constructing the Laughlin’s trial wave function \([9]\). The exact ground state as well as some excited states were also obtained for a model many-body Hamiltonian, where this novel correlations can be realized.

The purpose of this letter is to show that this type of correlations can be appropriately generalized to arbitrary higher dimensions. In particular, we construct a new three dimensional many-body Hamiltonian with three-body and five-body interactions. We obtain the exact ground state as well as some excited states of this Hamiltonian. The exact wave-functions of this model describe a novel correlations, which are an appropriate generalization of the two dimensional correlations introduced in \([8]\). These correlations can be realized by spinless bosons as well as fermions. Finally, all these three dimensional results are generalized to arbitrary higher dimensions.

The CSM is described by \(N\) identical particles confined in a one-body harmonic oscillator potential or on the rim of a circle, and interacting through each other via a two-body inverse square potential. The wave function of this model contains a Jastrow-type factor of the form \(J_{ij} = (x_i - x_j)^{\lambda} |x_i - x_j|^\alpha\), where \(x_i\) and \(x_j\) denote the particle positions of the \(i\)th and the \(j\)th particle, respectively. The parameters \(\alpha\) and \(\lambda\) are related to the strength of the inverse square two-body potential. The wave functions of CSM are highly correlated and the nature of correlations is encoded in the Jastrow-type
factor. The Jastrow-type factor $J_{ij}$ has two interesting properties:

(a) $J_{ij}$ vanishes when the position vectors of two particles coincide. This ensures that no two particles can occupy the same position at the same time. Also, the two-body interaction in the Hamiltonian has singularities, precisely at these points, i.e., at $x_i = x_j$.

(b) $J_{ij}$ picks up a factor $(-1)^\lambda$ under the exchange of particle indices. Consequently, spinless bosons as well as fermions can be described by putting $\lambda$ equal to zero or one, respectively.

These two properties of the Jastrow-type factor are the basic criteria for constructing higher dimensional analogue of the CSM. The Laughlin’s trial wave-function for a two dimensional Hamiltonian, describing spin polarized electrons in the lowest Landau level with a short-range repulsive interaction, indeed inherits these two properties \cite{9}. In particular, the two dimensional Jastrow-type factor appearing in Laughlin’s trial wave function is, $J_{ij} = (z_i - z_j)^\lambda |z_i - z_j|^\alpha$, where $z_i$ and $z_j$ are the particle positions in the complex coordinates. However, this is not unique in describing two dimensional many-body systems. One can also consider a Jastrow-type factor of the form $J_{ij} = (z_i^m z_j^m - z_j^m z_i^m)^\lambda |z_i^m z_j^m - z_j^m z_i^m|^\alpha$, where $m$ is an integer and a ‘bar’ denotes the complex conjugation. $J_{ij}$ vanishes, whenever the relative angle between the position vectors of any two particles is zero or a multiple of $\frac{\pi}{m}$. The case $m = 1$ was considered in Ref. \cite{8}. Note that for this particular value of $m$, $z_i \bar{z}_j - z_j \bar{z}_i$ is the magnitude of the cross-product of the positions vectors $\vec{r}_i$ and $\vec{r}_j$ along the $z$-direction, up to an overall multiplication factor. Thus, the wave function vanishes, whenever two particles are on a line passing through the origin. The interactions in the Hamiltonian, where this correlation can be realized, have singularities along the lines $\vec{r}_i \times \vec{r}_j = 0$. The zeroes of $J_{ij}$ are the singularities of the corresponding Hamiltonian where these correlations can be realized for arbitrary $m$ \cite{10}.
There are model many-body Hamiltonian in three dimensions with two-body and three-body interactions for which the ground state as well as some excited states can be written down explicitly [11]. Particles described by such Hamiltonians are either distinguishable or bosons\(^1\). This is because the Jastrow-type factor used in such cases is \(J_{ij} = (\mathbf{r}_i - \mathbf{r}_j)^2\), and is not antisymmetric under the exchange of particle coordinates. One can also construct a three dimensional Jastrow-type factor using the quaternion generalization of the usual complex coordinates [12]. This Jastrow-type factor satisfies the basic criteria (a) and (b). However, this is not physically interesting, since the quaternion coordinates are anti-commutating in nature and the wave function becomes \(SU(2)\) valued [12].

It is tempting to look for a three dimensional generalization of the new Jastrow-type factor introduced in Ref. [8]. Naively, one would like to consider,\(^2\)

\[
J_{ij}^b = |\mathbf{r}_i \times \mathbf{r}_j|^\alpha, \quad J_{ij}^f = (\mathbf{r}_i \times \mathbf{r}_j)|\mathbf{r}_i \times \mathbf{r}_j|^\alpha,
\]

which satisfies the basic criteria. The superscripts ‘b’ and ‘f’ refer to the bosonic and the fermionic nature of the Jastrow-type factor, respectively. It is possible to construct many-body Hamiltonian describing spinless bosons where \(J_{ij}^b\) can be realized for arbitrary number of particles. However, unlike in two dimensions, \(J_{ij}^f\) is now a vector. The only way one can construct fermionic wave-functions is to use linear combinations of these \(J_{ij}^f\). Naturally, the wave-functions become three dimensional vectors. This is acceptable provided each component of the wave function satisfies the same Shrödinger equation independently, with the same eigen-value [11]. In other words, the fermionic wave-function should span the space of degenerate states. Unfortunately, this type of correlations can be realized for only three particles [11].

\(^1\)The fermionic wave-functions can be constructed only for \(N = 3\) and \(N = 4\) particles.
Thus, in order to find an analogue of the two dimensional novel correlations in three dimensions, we have to construct a (pseudo-)scalar using the minimum number of position vectors such that it obeys the basic properties (a) and (b) of any Jastrow-type factor. However, the lesson from the two dimensional example is to modify the property (a). In particular, the three dimensional Jastrow-type factor can vanish, not only at points \( \vec{r}_i - \vec{r}_j = 0 \) or lines \( \vec{r}_i \times \vec{r}_j = 0 \), but also on planes \( \vec{r}_i \cdot \vec{r}_j \times \vec{r}_k = 0 \). Let us define a three dimensional vector \( \vec{Q}_{jk} \) in terms of the position vectors as,

\[
\vec{Q}_{jk} = \vec{r}_j \times \vec{r}_k.
\] (2)

The three dimensional Jastrow-type factor can be constructed by projecting \( \vec{Q}_{jk} \) along the position vector of the \( i \)-th particle. In particular,

\[
J_{ijk} = (P_{ijk})^\alpha | P_{ijk} |^\alpha, \quad P_{ijk} = \vec{r}_i \cdot \vec{Q}_{jk}.
\] (3)

Note that both \( \vec{Q}_{jk} \) and \( P_{ijk} \) are antisymmetric under the exchange of particle coordinates. Also, \( P_{ijk} \) vanishes when (i) the relative angle between the \( i \)-th and the \( j \)-th particle is zero or \( \pi \) and (ii) the position vectors of any three particles lie on a plane. The constraint (i) ensures that no two particles can occupy the same position at the same time. We expect at this point that if any model many-body Hamiltonian realizes this type of correlations, then these will be the conditions for having singularities in the many-body interactions. Note that the two-dimensional correlations introduced in [8] is a sub-class of (3).

We now present a many-body system where the correlations (3) can be realized explicitly. Consider the Hamiltonian,

\[
H = -\frac{1}{2} \sum_{i=1}^{N} \nabla_i^2 + \frac{1}{2} \sum_{i=1}^{N} \vec{r}_i^2 + \frac{g_1}{2} \sum_{R} \frac{\vec{Q}_{jk}^2}{P_{ijk}^2} + \frac{g_2}{2} \sum_{R} \frac{\vec{Q}_{jk} \vec{Q}_{lm}}{P_{ijk} P_{ilm}},
\] (4)

where \( R \) denotes the sum over all the indices from 1 to \( N \), with the restriction that any two indices can not have the same value simultaneously. We are working in the units
\( \hbar = m = w = 1 \), where \( h = 2\pi \hbar \) is the Planck’s constant, \( m \) is the mass of each particle and \( w \) is the oscillator frequency. \( g_1 \) and \( g_2 \) are two dimensionless coupling constants. In general, these two coupling constants are independent of each other. However, they get related to each other for the particular set of solutions we obtain below. The Hamiltonian is symmetric under the exchange of particle indices. As expected, both the three-body as well as five-body interactions are singular, in case, (i) any two particles are on a line passing through the origin or (ii) any three particles and the origin of the coordinate system lie on a plane.

We construct a trial wave-function for this Hamiltonian as,

\[
\psi_0 = \prod_{R} P_{\lambda}^{\lambda} | P_{ijk} |^\alpha \exp\left(-\frac{1}{2} \sum_{i=1}^{N} r_i^2 \right), \tag{5}
\]

where \( R \equiv (i < j < k) \). Note that the wave-function \( \psi_0 \) is fermionic in nature for odd \( N \geq 3 \) only. \( \psi_0 \) can be considered as bosonic for any \( N \geq 3 \). Eq. (5) is an exact eigenstate of (4) provided \( g_1 = \frac{g^2}{2}(\frac{g}{2} - 1) \) and \( g_2 = \frac{g^2}{4} \), where \( g = \alpha + \lambda \). One can solve for \( g \) as,

\[
g = (1 \pm \sqrt{1 + 4g_1}). \tag{6}
\]

Note that for physical states \( g_1 \geq -\frac{1}{4} \). The solutions in the lower branch are regular only in the limited ranges \(-\frac{1}{4} \leq g_1 \leq 0 \), while in the upper branch the solutions are regular for \( g_1 \geq -\frac{1}{4} \).

The energy corresponding to \( \psi_0 \) is given by, \( E_0 = \frac{3N}{2} + \frac{g^2}{2}N(N - 1)(N - 2) \). Note that for \( g = 0 \), i.e. \( g_1 = g_2 = 0 \), \( E_0 \) is exactly the ground state energy for \( N \) particles confined in a three dimensional harmonic oscillator potential. This is the case for \( N = 1 \) and 2 also, as the many-body interactions do not play any role unless \( N \geq 3 \). The wave-function \( \psi_0 \) has no nodes other than those corresponding to the singularities of the three-body and five-body interactions of the Hamiltonian. Thus, \( \psi_0 \) is well suited
for the ground-state of (4). In fact, it can be shown that \( \psi_0 \) indeed is the ground state wave-function by constructing the following anhilation operators:

\[
A_{x_i} = p_{x_i} - i x_i + \frac{ig}{2} \sum_{S} (\vec{Q}_{jk})_{x_i} P_{ijk},
\]
\[
A_{y_i} = p_{y_i} - i y_i - \frac{ig}{2} \sum_{S} (\vec{Q}_{jk})_{y_i} P_{ijk},
\]
\[
A_{z_i} = p_{z_i} - i z_i + \frac{ig}{2} \sum_{S} (\vec{Q}_{jk})_{z_i} P_{ijk},
\]

(7)

where \( S \) denotes sum over all the repeated indices from 1 to \( N \), with the constraint that any two indices can not have the same value simultaneously. The Hamiltonian can be written down in terms of these anhilation operators and the corresponding creation operators as,

\[
H = \frac{1}{2} \sum_{i=1}^{N} \left[ A_{x_i}^\dagger A_{x_i} + A_{y_i}^\dagger A_{y_i} + A_{z_i}^\dagger A_{z_i} \right] + E_0.
\]

(8)

The operators \( A \)'s anhilate the wave function \( \psi_0 \), and thus \( \psi_0 \) is the ground state.

The excited states can be obtained by decomposing the wave-function \( \psi \) as,

\[
\psi(x_i, y_i, z_i) = \psi_0(x_i, y_i, z_i) \phi(x_i, y_i, z_i).
\]

(9)

Plugging the expression (9) into the Schrödinger equation, we have,

\[
\left[ -\frac{1}{2} \sum_{i=1}^{N} \nabla_i^2 + \sum_{i=1}^{N} \vec{r}_i \cdot \nabla_i - \frac{g}{2} \sum_{R} \frac{\vec{Q}_{jk} \cdot \nabla_i}{P_{ijk}} \right] \phi = (E_n - E_0) \phi.
\]

(10)

Now if \( \phi \) is a function of \( t = \sum_{i=1}^{N} r_i^2 \) only, Eq. (10) reduces to the confluent hypergeometric equation,

\[ ^2 \text{One should not confuse } z_i \text{ in } (7) \text{ as the complex coordinates. } \vec{r}_i \text{ is defined as } \vec{r}_i = x_i \hat{i} + y_i \hat{j} + z_i \hat{k}, \]

where the unit vectors \( \hat{i}, \hat{j} \) and \( \hat{k} \) span the three dimensional space.
\begin{equation}
  t \frac{d^2 \phi(t)}{dt^2} + [b - t] \frac{d\phi(t)}{dt} - a\phi(t) = 0,
  \end{equation}

where \( b = E_0 \) and \( a = -\frac{1}{2}(E_n - E_0) \). The admissible solutions of (11) are the regular confluent hypergeometric functions, \( \phi(t) = M(a, b, t) \). The constant \( a \) is determined as \( a = -n \) in order to have normalizable eigen functions, where \( n \) is an integer. Thus, the spectrum is given by \( E_n = E_0 + 2n \). Note that this spectrum is identical to the CSM as well as to the model Hamiltonian considered in \([8]\). Unfortunately, \( E_n \) is not the complete spectrum of the Hamiltonian (11). At present, we do not know how to solve the Schrödinger equation corresponding to (11) exactly.

We now show that all these three dimensional results can be generalized to arbitrary higher dimensions. In \( D(>2) \) dimensions, one can construct a Jastrow-type factor with the help of \( D \) position vectors. As a result, the Hamiltonian contains \( D \)-body and \((2D-1)\)-body interactions only. Consider the following \( D \)-dimensional Jastrow-type factor \( J_{i_1i_2...i_D} \),

\begin{align}
  \tilde{Q}_{i_1i_2...i_D} &= \tilde{r}_{i_2} \times \tilde{r}_{i_3} \times \ldots \times \tilde{r}_{i_D}, \quad P_{i_1i_2...i_D} = \tilde{r}_{i_1} \cdot \tilde{Q}_{i_2i_3...i_D},
  
  J_{i_1i_2...i_D} &= (P_{i_1i_2...i_D})^\lambda |P_{i_1i_2...i_D}|^\alpha. \tag{12}
\end{align}

This \( D \) dimensional Jastrow-type factor vanishes whenever any \( p \) particles lie on a \( p-1 \) dimensional (hyper-)plane\(^3\) passing through the origin, where \( 2 \leq p \leq D \). Note that the novel correlations (12) can be realized only for \( N \geq D \).

Consider the \( D \)-dimensional many-body Hamiltonian,

\begin{equation}
  H = -\frac{1}{2} \sum_{i=1}^{N} \nabla_i^2 + \frac{1}{2} \sum_{i=1}^{N} \tilde{r}_i^2 + g_1 \sum_{R} \frac{\tilde{Q}_{i_1i_2...i_D}^2}{P_{i_1i_2...i_D}^2} + g_2 \sum_{R} \frac{\tilde{Q}_{j_1j_2...j_D} \times \tilde{Q}_{j_1j_2...j_D}}{P_{i_1i_2...i_D} P_{j_1j_2...j_D}}. \tag{13}
\end{equation}

\(^3\)We denote one dimensional plane as a line.
We follow the same summation convention as in (4). The eigen-states \( \psi_n \) of the Hamiltonian (13) are given by,

\[
\psi_n = \prod_{\mathcal{R}} J_{i_1i_2...i_D} M(-n, E_0, t) \exp(-\frac{1}{2} \sum_{i=1}^{N} \vec{r}_i^2),
\]

where the ground state energy \( E_0 = D \left[ \frac{N}{2} + g \ N C_D \right] \), \( N C_D = \frac{N!}{D!(N-D)!} \) and \( \mathcal{R} \equiv (i_1 < i_2 < \ldots < i_D) \). The wave-functions \( \psi_n \) are bosonic in nature for arbitrary \( N \geq D \). The fermionic description is possible, only when \( N - 2 C_D - 2 \) is odd. The energy spectrum corresponding to the eigen-states (14) is \( E_n = E_0 + 2n \). Note that the ground state energy depends on the dimensionality of space. However, the difference in energy between the \( n \)-th excited state and the ground state, i.e. \( E_n - E_0 \), is independent of \( D \).

Eq. (14) is an exact eigen-state of (13) with energy eigen values \( E_n \), provided the dimensionless coupling constants \( g_1 \) and \( g_2 \) are related to \( g \) as,

\[
g_2 = \left( \frac{g}{(D - 1)!} \right)^2, \quad g_1 = \frac{g}{(D - 1)!} \left( \frac{g}{(D - 1)!} - 1 \right),
\]

\[
g = \frac{(D - 1)!}{2} \left[ 1 \pm (1 + 4g_1)^{\frac{1}{2}} \right].
\]

Note that the relation between \( g_1 \) and \( g_2 \) is independent of \( D \). Also, the ranges of \( g_1 \) is independent of the dimensionality of the space.

In conclusions, we have constructed a \( D(> 2) \)-dimensional many-body Hamiltonian with \( D \)-body and \( (2D - 1) \)-body interactions, which realizes a novel correlations among the spinless bosons as well as fermions. We found the ground state and some of the excited states of this model. The fermionic description of the wave-functions is possible for any \( N \geq D \) obeying the constraint that \( N - 2 C_D - 2 \) is odd. The bosonic nature of the wave-functions are independent of the total number of particles. It would be nice if either this model Hamiltonian or the correlation could be realized in some condensed matter systems.
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