A five-wave HLL Riemann solver for relativistic MHD

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ABSTRACT

We present a five-wave Riemann solver for the equations of ideal relativistic magnetohydrodynamics. Our solver can be regarded as a relativistic extension of the five-wave HLLD Riemann solver initially developed by Miyoshi and Kusano for the equations of ideal MHD. The solution to the Riemann problem is approximated by a five wave pattern, comprised of two outermost fast shocks, two rotational discontinuities and a contact surface in the middle. The proposed scheme is considerably more elaborate than in the classical case since the normal velocity is no longer constant across the rotational modes. Still, proper closure to the Rankine-Hugoniot jump conditions can be attained by solving a nonlinear scalar equation in the total pressure variable which, for the chosen configuration, has to be constant over the whole Riemann fan. The accuracy of the new Riemann solver is validated against one dimensional tests and multidimensional applications. It is shown that our new solver considerably improves over the popular HLL solver or the recently proposed HLLC schemes.

Key words: hydrodynamics - MHD - relativity - shock waves - methods:numerical

1 MOTIVATIONS

Relativistic flows are involved in many of the high-energy astrophysical phenomena, such as, for example, jets in extragalactic radio sources, accretion flows around compact objects, pulsar winds and \( \gamma \)-ray bursts. In many instances the presence of a magnetic field is also an essential ingredient for explaining the physics of these objects and interpreting their observational appearance.

Theoretical understanding of relativistic phenomena is subdue to the solution of the relativistic magnetohydrodynamics (RMHD) equations which, owing to their high degree of nonlinearity, can hardly be solved by analytical methods. For this reason, the modeling of such phenomena has prompted the search for efficient and accurate numerical formulations. In this respect, Godunov-type schemes (Toro 1997) have gained increasing popularity due to their ability and robustness in accurately describing sharp flow discontinuities such as shocks or tangential waves.

One of the fundamental ingredient of such schemes is the exact or approximate solution to the Riemann problem, i.e., the decay between two constant states separated by a discontinuity. Unfortunately the use of an exact Riemann solver (Giacomazzo & Rezzolla 2006) is prohibitive because of the huge computational cost related to the high degree of nonlinearities present in the equations. Instead, approximate methods of solution are preferred.

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Linearized solvers (Komissarov 1999; Balsara 2001; Koldoba et al. 2002) rely on the rather convoluted eigenvector decomposition of the underlying equations and may be prone to numerical pathologies leading to negative density or pressures inside the solution (Einfeldt et al. 1991).

Characteristic-free algorithms based on the Rusanov Lax-Friedrichs or the Harten-Lax-van Leer (HLL, Harten et al. 1983) formulations are sometime preferred due to their ease of implementation and positivity properties. Implementation of such algorithms can be found in the codes described by Gammie et al. (2003); Leisman et al. (2003); Del Zanna et al. (2007); van der Holst et al. (2008). Although simpler, the HLL scheme approximates only two out of the seven waves by collapsing the full structure of the Riemann fan into a single average state. These solvers, therefore, are not able to resolve intermediate waves such as Alfvén, contact and slow discontinuities.

Attempts to restore the middle contact (or entropy) wave (HLLC, initially devised for the Euler equations by Toro et al. 1994) have been proposed by Mignone et al. (2003) in the case of purely transversal fields and by Mignone & Bodo (2006) (MB from now on), Honkila & Janhunen (2007) in the more general case. These schemes provide a relativistic extension of the work proposed by Gurski (2004) and L (2003) for the classical MHD equations.

HLLC solvers for the equations of MHD and RMHD, however, still do not capture slow discontinuities and Alfvén waves. Besides, direct application of the HLLC solver of MB...
2 BASIC EQUATIONS

The equations of relativistic magnetohydrodynamics (RMHD) are derived under the physical assumptions of constant magnetic permeability and infinite conductivity, appropriate for a perfectly conducting fluid (Anile 1983; Lichnerowicz 1967). In divergence form, they express particle number and energy-momentum conservation:

\[ \partial_t \left( \rho u^\mu \right) = 0, \quad (1) \]
\[ \partial_\mu \left[ \left( w + b^2 \right) u^\mu u^\nu - b^\mu b^\nu + \left( p_b + \frac{b^2}{2} \right) \eta_{\mu\nu} \right] = 0, \quad (2) \]
\[ \partial_\nu \left( u^\mu b^\nu - u^\nu b^\mu \right) = 0, \quad (3) \]

where \( \rho \) is the rest mass density, \( u^\mu = (\gamma, \mathbf{v}) \) is the four-velocity (\( \gamma \equiv \text{Lorentz factor}, \mathbf{v} \equiv \text{three velocity} \)), \( w_b \) and \( p_b \) are the gas enthalpy and thermal pressure, respectively. The covariant magnetic field \( b^\mu \) is orthogonal to the fluid four-velocity (\( u^\mu b^\mu = 0 \)) and is related to the local rest frame field \( \mathbf{B} \) by

\[ b^\mu = \left[ \gamma v \cdot \mathbf{B}, \frac{\mathbf{B}}{\gamma} + \gamma (v \cdot \mathbf{B}) v \right]. \quad (4) \]

In Eq. (2), \( b^2 \equiv b^\mu b_\mu = \mathbf{B}^2 / \gamma^2 + (v \cdot \mathbf{B})^2 \) is the squared magnitude of the magnetic field.

The set of equations (1)-(3) must be complemented by an equation of state which may be taken as the constant \( \Gamma \)-law:

\[ w_g = \rho + \frac{\Gamma}{\Gamma - 1} p_g, \quad (5) \]

where \( \Gamma \) is the specific heat ratio. Alternative equations of state (see, for example, Mignone & McKinney 2007) may be adopted.

In the following we will be dealing with the one dimensional conservation law

\[ \frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0, \quad (6) \]

which follows directly from Eq. (1)-(3) by discarding contributions from \( y \) and \( z \). Conserved variables and corresponding fluxes take the form:

\[ U = \begin{pmatrix} D \\ m^b \\ E \\ B^b \end{pmatrix}, \quad F = \begin{pmatrix} D u^x \\ w u^x u^b - b^x b^k + p \delta_{xk} \\ m^x \\ B^k u^x - B^x u^k \end{pmatrix} \quad (7) \]

where \( k = x,y,z, D = \rho \gamma \) is the the density as seen from the observer’s frame while, introducing \( w \equiv w_g + b^2 \) (total enthalpy) and \( p \equiv p_g + b^2 / 2 \) (total pressure),

\[ m^k = w u^a u^b - b^a b^b, \quad E = w u^0 - b^a b^0 - p \quad (8) \]

are the momentum and energy densities, respectively. \( \delta_{xk} \) is the Kronecker delta symbol.

Note that, since \( F_{bx} = 0 \), the normal component of magnetic field \( (B^x) \) does not change during the evolution and can be regarded as a parameter. This is a direct consequence of the \( \nabla \cdot \mathbf{B} = 0 \) condition.

A conservative discretization of Eq. (6) over a time step \( \Delta t \) yields

\[ U_{i+1}^n = U_i^n - \frac{\Delta t}{\Delta x} \left( f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}} \right), \quad (9) \]

where \( \Delta x \) is the mesh spacing and \( f_{i+\frac{1}{2}} \) is the upwind numerical flux computed at zone faces \( x_{i+\frac{1}{2}} \) by solving, for \( t^n < t < t^{n+1} \), the initial value problem defined by Eq. (6) together with the initial condition

\[ U(x,t^n) = \begin{cases} U_L & \text{for } x < x_{i+\frac{1}{2}}, \\ U_R & \text{for } x > x_{i+\frac{1}{2}}, \end{cases} \quad (10) \]

where \( U_L \) and \( U_R \) are discontinuous left and right constant states on either side of the interface. This is also known as the Riemann problem. For a first order scheme, \( U_L = U_i \) and \( U_R = U_{i+1} \).

The decay of the initial discontinuity given by Eq. (10) leads to the formation of a self-similar wave pattern in the \( x-t \) plane where fast, slow, Alfvén and contact modes can develop. At the double end of the Riemann fan, two fast magneto-sonic waves bound the emerging pattern enclosing two rotational (Alfvén) discontinuities, two slow magneto-sonic waves and a contact surface in the middle. The same patterns is also found in classical MHD. Fast and slow magneto-sonic disturbances can be either shocks or rarefaction waves, depending on the pressure jump and the norm of the magnetic field. All variables (i.e., density, velocity, magnetic field and pressure) change discontinuously across a fast or a slow shock, whereas thermodynamic quantities such as thermal pressure and rest density remain continuous when crossing a relativistic Alfvén wave. Contrary to its classical counterpart, however, the tangential components of magnetic field trace ellipses instead of circles and the normal component of the velocity is no longer continuous across a rotational discontinuity. Komissarov (1997). Finally, through the contact mode, only density exhibits a jump while thermal pressure, velocity and magnetic field remain continuous.
The jump conditions by the HLLD solver. The initial states we choose to use the simple Davis estimate (Davis 1988). As in MB, solvers, the outermost velocities are determined. In the typical approach used to construct HLL-based solvers, the outermost velocities have to be determined. Across the fast waves, we will make frequent use of 

\[ R_L = \lambda_L U_L - F_L, \quad R_R = \lambda_R U_R - F_R, \] 

which are known vectors readily obtained from the left and right input states. A particular component of \( R \) is selected by mean of a subscript, e.g., \( R_D \) is the density component of \( R \).

A consistent solution to the problem has to satisfy the 7 nonlinear relations implied by Eq. (11) for each of the 5 waves considered, thus giving a total of 35 equations. Moreover, physically relevant solutions must fulfill a number of requirements in order to reflect the characteristic nature of the considered waves. For this reason, across the contact mode, we demand that velocity, magnetic field and total pressure be continuous:

\[ \left[ \begin{array}{c} v \\ B \\ p \end{array} \right]_{\lambda_c} = \left[ \begin{array}{c} B \\ p \end{array} \right]_{\lambda_c} = 0, \] 

and require that \( \lambda_c \equiv v^c_z \), i.e., that the contact wave moves at the speed of the fluid. However, density, energy and total enthalpy may be discontinuous. On the other hand, through the rotational waves \( \lambda_{aL} \) and \( \lambda_{aR} \), scalar quantities such as total pressure and enthalpy are invariant whereas all vector components (except for \( B^1 \)) experience jumps.

Since slow magnetosonic waves are not considered, we naturally conclude that only the total pressure remains constant throughout the fan, contrary to Newtonian MHD, where also the velocity normal to the interface \( (v^x) \) is left unchanged across the waves. This is an obvious consequence of the different nature of relativistic Alfvén waves across which vector fields like \( u^x \) and \( b^x \) trace ellipses rather than circles. As a consequence, the normal component of the velocity, \( v^x \), is no longer invariant in RMHD but experiences a jump. These considerations along with the higher level of complexity of the relativistic equations makes the extension of the multi-state HLL solver to RMHD considerably more elaborate.

Our strategy of solution is briefly summarized. For each state we introduce a set of 8 independent unknowns: \( \mathcal{P} = (D, v^x, v^y, B^x, B^y, w, p) \) and write conservative variables and fluxes given by Eq. (7) as

\[
U_\alpha = \left(\begin{array}{c} D \\ w \gamma^2 v^k - b^k b^k \\ w \gamma^2 - p - b^0 b^0 \\ b^k \\ B^k \\
\end{array}\right)_{\alpha}, \quad F_\alpha = \left(\begin{array}{c} D v^x \\ w \gamma^2 v^k v^x - b^k b^x + p \delta_{kx} \\ w \gamma^2 v^y - b^0 b^x \\ B^k v^x - B^x v^k \\
\end{array}\right)_{\alpha},
\]

where \( k = x, y, z \) labels the vector component, \( \alpha \) is the state.
and $b^\mu$ is computed directly from \( \mathbf{B} \). We proceed by solving, as function of the total pressure $p$, the contact conditions $\|\|_i$ across the outermost waves $\lambda_\alpha$ and $\lambda_R$. By requiring that total pressure and Alfvén velocity do not change across each rotational modes, we find a set of invariant quantities $\alpha p$ across each rotational modes, we find a set of invariant quantities $\alpha p$ as $\|\|_i$.

The energy and momentum equations can be combined together to provide an explicit functional relation between the three components of velocity and the total pressure $p$. To this purpose, we first multiply the energy equation (19) times $v'^2$ and then subtract the resulting expression from the jump condition for the $k$-th component of momentum, Eq. (18). Using Eq. (20) to get rid of the $v'^2$ term, one finds after some algebra:

$$B^k \left( B^z - R_B \cdot v \right) - p \left( \delta_{kx} - \lambda v'^k \right) = R_{m^k} - v'^k R_E,$$

with $B^k$ defined by (21). The system can be solved for $v'^k$ giving

$$v'^z = \frac{B^z (AB^z + \lambda C) - (A + G) (p + R_{m^z})}{X},$$

$$v'^y = \frac{QR_{m^y} + R_{B^y} [C + B^z (\lambda R_{m^z} - R_E)]}{X},$$

$$v'^x = \frac{QR_{m^x} + R_{B^x} [C + B^z (\lambda R_{m^z} - R_E)]}{X},$$

where

$$A = R_{m^z} - \lambda R_E + p \left( 1 - \lambda^2 \right),$$

$$G = R_{B^y} R_{B^y} + R_{B^z} R_{B^z},$$

$$C = R_{m^y} R_{B^y} + R_{m^z} R_{B^z},$$

$$Q = -A - G + (B^z)^2 (1 - \lambda^2),$$

$$X = B^z (\lambda B^z + C) - (A + G) (\lambda p + R_E).$$

Once the velocity components are expressed as functions of $p$, the magnetic field is readily found from (24), while the total enthalpy can be found using its definition, $w = (E + p)/\gamma^2 + (v \cdot B)^2$, or by subtracting $R_E$ from the inner product $v^k \cdot R_m$, giving

$$w = p + \frac{R_E - v \cdot R_m}{\lambda - v'^z},$$

where $R_m \equiv (R_{m^x}, R_{m^y}, R_{m^z})$. Although equivalent, we choose to use this second expression. Since the $v'^k$ are functions of $p$ alone, the total enthalpy $w$ is also a function of the total pressure.

The remaining conserved quantities in the $\alpha = aL$ or $\alpha = aR$ regions can be computed once $p$ has been found:

$$D = \frac{R_D}{\lambda - v'^z},$$

$$E = \frac{R_E - pw'^z - (v \cdot B) B^z}{\lambda - v'^z},$$

$$m^k = \frac{(E + p) v^k - (v \cdot B) B^k}{X}.$$

One can verify by direct substitution that the previous equations together with the corresponding fluxes, Eq. (15), satisfy the jump conditions given by (17)–(20).

### 3.2 Jump Conditions across the Alfvén waves

Across the rotational waves one could, in principle, proceed as for the outer waves, i.e., by explicitly writing the jump conditions. However, as we shall see, the treatment greatly simplifies if one introduces the four vector

$$\sigma^\mu = \eta u^\mu + b^\mu, \quad \text{with} \quad \eta = \pm \text{sign}(B^z) \sqrt{w}$$

where, for reasons that will be clear later, we take the plus (minus) sign for the right (left) state. From $\sigma^\mu$ we define the spatial vector $\mathbf{K} \equiv (K^x, K^y, K^z)$ with components given by

$$K^k \equiv \frac{\sigma^k}{\sigma^0} = v^k + \frac{B^k}{\gamma \eta u^0}.$$
The vector \( \mathbf{K} \) has some attractive properties, the most remarkable of which is that the \( x \) component coincides with the propagation speed of the Alfvén wave \cite{Anile1989}. For this reason, we are motivated to define \( \lambda_a \equiv \lambda _{aL} ^{x} \), where the subscript \( a \) stands for either the left or right rotational wave (i.e. \( aL \) or \( aR \)) since we require that both \( K^x \) and \( p \) are invariant across the rotational discontinuity, i.e., \( K^x _{L} - K^x _{R} = p_c - p_a = 0 \), a property certainly shared by the exact solution. As we will show, this choice naturally reduces to the expressions found by MK in the non-relativistic limit.

Indeed, setting \( \lambda_a = \lambda _{aL} ^{x} = \lambda _{aR} ^{x} \) and using Eq. \( \ref{eq:10} \) to express \( v^k \) as functions of \( K^k \), the jump conditions simplify to

\[
\left[ \frac{DB^x}{\gamma\sigma^0} \right]_{\lambda_a} = 0 \quad \left[ \frac{\eta_a v^k B^x}{\sigma^0} - p\delta_{ka} \right]_{\lambda_a} = 0 \quad \left[ \eta B^x - \frac{\sigma^x}{\sigma^0} p \right]_{\lambda_a} = 0 \quad \left[ \frac{B^x v^k}{\sigma^0} \right]_{\lambda_a} = 0 ,
\]

Since both \( [\rho]_{\lambda_a} = 0 \), the previous equations further imply that (when \( B^x \neq 0 \)) also \( D/(\gamma\sigma^0) \), \( w \), \( K^y \) and \( K^z \) do not change across \( \lambda_a \):

\[
K_{aL} = K_{\epsilon L} \equiv K_L , \quad \eta_{aL} = \eta_{\epsilon L} = \eta_L \\
K_{aR} = K_{\epsilon R} \equiv K_R , \quad \eta_{aR} = \eta_{\epsilon R} = \eta_R
\]

Being invariant, \( \mathbf{K} \) can be computed from the state lying to the left (for \( \lambda_a L \)) or to the right (for \( \lambda_a R \)) of the discontinuity, thus being a function of the total pressure \( p \) alone. Instead of using Eq. \( \ref{eq:10} \), an alternative and more convenient expression may be found by properly replacing \( v^k \) with \( K^k \) in Eq. \( \ref{eq:14} - \ref{eq:17} \). After some algebra one finds the simpler expression

\[
K^k = \frac{R_{mk} + \rho \delta_{mk} + \rho B^m v^k}{\lambda p + R_E + B^m v^k} \eta ,
\]

still being a function of the total pressure \( p \).

Note that, similarly to its non-relativistic limit, we cannot use the equations in \( \ref{eq:17} - \ref{eq:20} \) to compute the solution across the rotational waves, since they do not provide enough independent relations. Instead, a solution may be found by considering the jump conditions across both rotational discontinuities and properly matching them using the conditions at the contact mode.

### 3.3 Jump Conditions across the Contact wave

At the contact discontinuity (CD) only density and total enthalpy can be discontinuous, while total pressure, normal and tangential fields are continuous as expressed by Eq. \( \ref{eq:15} \).

Since the magnetic field is a conserved quantity, one can immediately use the consistency condition between the innermost waves \( \lambda_{aL} \) and \( \lambda_{aR} \) to find \( B^k \) across the CD. Indeed, from

\[
(\lambda_c - \lambda_{aL}) U_{cL} + (\lambda_{aR} - \lambda_c) U_{cR} = \lambda_{aR} U_{aR} - \lambda_{aL} U_{aL} - \mathbf{F}_{aR} + \mathbf{F}_{aL}
\]

one has \( B^k _{cL} = B^k _{cR} \equiv B^k _c \), where

\[
B^k _c = \left[ \frac{B^k (\lambda - v^x) + B^x v^k}{\lambda} \right]_{cR} - \left[ \frac{B^k (\lambda - v^x) + B^x v^k}{\lambda} \right]_{cL} \equiv \frac{B^k}{\lambda} .
\]

Since quantities in the \( aL \) and \( aR \) regions are given in terms of the \( p \) unknown, Eq. \( \ref{eq:15} \) are also functions of \( p \) alone.

At this point, we take advantage of the fact that \( \sigma^x u^k = -\eta \) to replace \( \gamma \sigma^0 \) with \( \eta/(1 - \mathbf{K} \cdot \mathbf{v}) \) and then rewrite \( \ref{eq:36} \) as

\[
K^k = v^k + \frac{B^k}{\eta} (1 - \mathbf{K} \cdot \mathbf{v}) \quad \text{for} \quad k = x, y, z . \tag{46}
\]

The previous equations form a linear system in the velocity components \( v^k \) and can be easily inverted to the left and to the right of the CD to yield

\[
v^k = K^k - \frac{B^k (1 - \mathbf{K}^2)}{\eta - \mathbf{K} \cdot \mathbf{B}} \quad \text{for} \quad k = x, y, z , \tag{47}
\]

which also depend on the total pressure variable only, with \( w \) and \( K^k \) being given by \( \ref{eq:31} \) and \( \ref{eq:43} \) and the \( B^k \)’s being computed from Eq. \( \ref{eq:45} \). Imposing continuity of the normal velocity across the CD, \( v^x _{cL} - v^x _{cR} = 0 \), results in

\[
\Delta K^x \left[ 1 - B^x \left( Y_R - Y_L \right) \right] = 0 , \tag{48}
\]

where

\[
Y_S(p) = \frac{1 - K^2_S}{\eta_S \Delta K^x - K_S \cdot B_S} , \quad S = L, R , \tag{49}
\]

is a function of \( p \) only and \( \hat{B}_c \equiv \Delta K^2 B_c \), is the numerator of \( \ref{eq:15} \) and \( \Delta K^2 = K^2 _{cR} - K^2 _{cL} \). Equation \( \ref{eq:15} \) is a nonlinear function in \( p \) and must be solved numerically.

Once the iteration process has been completed and \( p \) has been found to some level of accuracy, the remaining conserved variables to the left and to the right of the CD are computed from the jump conditions across \( \lambda_{aL} \) and \( \lambda_{aR} \) and the definition of the flux, Eq. \( \ref{eq:15} \). Specifically one has, for \( \{ c = cL, a = aL \} \) or \( \{ c = cR, a = aR \} \),

\[
D_c = D_a \frac{\lambda_a - v^x_a}{\lambda_a - v^x_c} , \tag{50}
\]

\[
E_c = \lambda_a E_a - m_a^x + p v^x_a - (v_c \cdot B_c) B^x_a , \tag{51}
\]

\[
m_c^k = (E_c + p)v^k_c - (v_c \cdot B_c) B^k_c . \tag{52}
\]

This concludes the derivation of our Riemann solver.

### 3.4 Full Solution

In the previous sections we have shown that the whole set of jump conditions can be brought down to the solution of a single nonlinear equation, given by \( \ref{eq:15} \), in the total pressure variable \( p \). In the particular case of vanishing normal component of the magnetic field, i.e. \( B_a \to 0 \), this equation can be solved exactly as discussed in \( \ref{3.4} \).

For the more general case, the solution has to be found numerically using an iterative method where, starting from an initial guess \( p^{(0)} \), each iteration consists of the following steps:
• given a new guess value \( p^{(k)} \) to the total pressure, start from Eq. (23)–(25) to express \( v_{aL} \) and \( v_{aR} \) as functions of the total pressure. Also, express magnetic fields \( B_{aL}, B_{aR} \) and total enthalpies \( w_L, w_R \) using Eq. (21) and Eq. (41), respectively.

• Compute \( K_{aL} \) and \( K_{aR} \) using Eq. (43) and the transverse components of \( B \) using Eq. (45).

• Use Eq. (45) to find the next improved iteration value.

For the sake of assessing the validity of our new solver, we choose the secant method as our root-finding algorithm. The initial guess is provided using the following prescription:

\[
p^{(0)} = \begin{cases} \ p_0 & \text{when } \ (B^x)^2/p^{\text{hll}} < 0.1, \\ \ otherwise, & \end{cases}
\]

where \( p^{\text{hll}} \) is the total pressure computed from the HLL average state whereas \( p_0 \) is the solution in the \( B^x = 0 \) limiting case. Extensive numerical testing has shown that the total pressure \( p^{\text{hll}} \) computed from the HLL average state provides, in most cases, a sufficiently close guess to the correct physical solution, so that no more than 5–6 iterations (for zones with steep gradients) were required to achieve a relative accuracy of \( 10^{-6} \).

The computational cost depends on the simulation setting since the average number of iterations can vary from one problem to another. However, based on the results presented in [41], we have found that HLLD was at most a factor of \( \sim 2 \) slower than HLL.

For a solution to be physically consistent and well-behaved, we demand that

\[
\begin{align*}
\begin{cases}
w_L > p, & v^x_{aL} > \lambda_L, & v^x_{cL} > \lambda_{aL}, \\
w_R > p, & v^x_{aR} < \lambda_R, & v^x_{cR} < \lambda_{aR},
\end{cases}
\end{align*}
\]

hold simultaneously. These conditions guarantee positivity of density and that the correct eigenvalue ordering is always respected. We warn the reader that equation (43) may have, in general, more than one solution and that the conditions given by (44) may actually prove helpful in selecting the correct one. However, the intrinsic nonlinear complexity of the RMHD equations makes rather arduous and challenging to prove, a priori, both the existence and the uniqueness of a physically relevant solution, in the sense provided by (43).

On the contrary, we encountered sporadic situations where none of the zeroes of Eq. (18) is physically admissible. Fortunately, these situations turn out to be rare eventualities caused either by a large jump between left and right states (as at the beginning of integration) or by under- or over-estimating the propagation speeds of the outermost fast waves, \( \lambda_L \) and \( \lambda_R \). The latter conclusion is supported by the fact that, enlarging one or both wave speeds, led to a perfectly smooth and unique solution.

Therefore, we propose a safety mechanism whereby we switch to the simpler HLL Riemann solver whenever at least one or more of the conditions in (44) is not fulfilled. From several numerical tests, including the ones shown here, we found the occurrence of these anomalies to be limited to few zones of the computational domain, usually less than 0.1% in the tests presented here.

We conclude this section by noting that other more sophisticated algorithms may in principle be sought. One could, for instance, provide a better guess to the outer wave-speeds \( \lambda_L \) and \( \lambda_R \) or even modify them accordingly until a solution is guaranteed to exist. Another, perhaps more useful, possibility is to bracket the solution inside a closed interval \([p_{\text{min}}, p_{\text{max}}]\) where \( p_{\text{min}} \) and \( p_{\text{max}} \) may be found from the conditions (43). Using an alternative root finder, such as Ridder [Press et al. 1992], guarantees that the solution never jump outside the interval. However, due to the small number of failures usually encountered, we do not think these alternatives could lead to a significant gain in accuracy.

### 3.4.1 Zero normal field limit

In the limit \( B^x \to 0 \) a degeneracy occurs where the Alfvén (and slow) waves propagate at the speed of the contact mode which thus becomes a tangential discontinuity. Across this degenerate front, only normal velocity and total pressure remain continuous, whereas tangential vector fields are subject to jumps.

This case does not pose any serious difficulty in our derivation and can be solved exactly. Indeed, by setting \( B^x = 0 \) in Eq. (43) and (45), one immediately finds that \( K_p = K^x_p = v^x_c \) leading to the following quadratic equation for \( p \):

\[
p^2 + \bigl((E^{\text{hll}} - F^{\text{hll}}(m^x)) p + m^x E^{\text{hll}} - F^{\text{hll}}(m^x) E^{\text{hll}} = 0 ,
\]

where the superscript “hll” refers to the HLL average state or flux given by Eq. (28) or (31) of MB. We note that equation (55) coincides with the derivation given by MB [Mignone et al. 2003] in the same degenerate case and the positive root gives the correct physical solution. The intermediate states, \( U_{aL} \) and \( U_{aR} \), lose their physical meaning as \( B^x \to 0 \) but they never enter the solution since, as \( \lambda_{aL}, \lambda_{aR} \to \infty \), only \( U_{aL} \) and \( U_{aR} \) will have a nonzero finite width, see Fig. 1.

Given the initial guess, Eq. (53), our proposed approach does not have to deal separately with the \( B^x \neq 0 \) and \( B^x = 0 \) cases (as in MB and Honkela & Janhunen 2007) and thus solves the issue raised by MB.

### 3.4.2 Newtonian Limit

We now show that our derivation reduces to the HLLD Riemann solver found by MK under the appropriate non-relativistic limit. We begin by noticing that, for \( v/c \to 0 \), the velocity and induction four-vectors reduce to \( u^a \to (1, v^k) \) and \( b^a \to (0, B^k) \), respectively. Also, note that \( w_g, w \to p \) in the non-relativistic limit so that

\[
K^x \to v^x + s \frac{E^k}{\sqrt{p}} ,
\]

and thus \( v^x \) cannot change across \( \lambda_a \). Replacing (14)–(18) with their non-relativistic expressions and demand \( u^a = v^a \) gives, in our notations, the following expressions:

\[
\begin{align*}
\begin{cases}
u^a_{\text{a}} = \frac{R_{R,m^x} - R_{L,m^x}}{R_{R,D} - R_{L,D}}, \\
p = (B^x)^2 - \frac{R_{L,m^x} R_{R,D} - R_{R,m^x} R_{L,D}}{R_{R,D} - R_{L,D}},
\end{cases}
\end{align*}
\]

which can be shown to be identical to Eqns (38) and (41) of MK. With little algebra, one can also show that the remaining variables in the \( aL \) and \( aR \) regions reduce to the
corresponding non-relativistic expressions of MK. Similarly, the jump across the rotational waves are solved exactly in the same fashion, that is, by solving the integral conservation laws over the Riemann fan. For instance, Eq. (42) reduces to equation (61) and (62) of MK. These results should not be surprising since, our set of parameters to write conserved variables and fluxes is identical to the one used by MK. The only exception is the energy, which is actually written in terms of the total enthalpy.

4 NUMERICAL TESTS

We now evaluate, in §4.1, the accuracy of the proposed HLLD Riemann solver by means of selected one-dimensional shock tube problems. Applications of the solver to multi-dimensional problems of astrophysical relevance are presented in §4.2.

4.1 One Dimensional Shock Tubes

The initial condition is given by Eq. (10) with left and right states defined by the primitive variables listed in Table 1. The computational domain is chosen to be the interval $[0, 1]$ and the discontinuity is placed at $x = 0.5$. The resolution $N_x$ and final integration time can be found in the last two columns of Table 1. Unless otherwise stated, we employ the constant $\Gamma$-law with $\Gamma = 5/3$. The RMHD equations are solved using the first-order accurate scheme (9) with a CFL number of 0.8.

Numerical results are compared to the HLLC Riemann solver of MB and to the simpler HLL scheme and the accuracy is quantified by computing discrete errors in L-1 norm:

$$
\epsilon_{L1} = \sum_{i=1}^{N_x} \left| q_i^{ref} - q_i \right| \Delta x_i,
$$

(59)

where $q_i$ is the first-order numerical solution (density or magnetic field), $q_i^{ref}$ is the reference solution at $x_i$ and $\Delta x_i$ is the mesh spacing. For tests 1, 2, 4 we obtained a reference solution using the second-order scheme of MB on 3200 zones and adaptive mesh refinement with 6 levels of refinement (equivalent resolution 204, 800 grid points). Grid adaptivity in one dimension has been incorporated in the PLUTO code using a block-structured grid approach following Berger & Colella (1989). For test 3, we use the exact numerical solution available from Giacomazzo & Rezzolla (2006). Errors (in percent) are shown in Fig. 11.

### 4.1.1 Exact Resolution of Contact and Alfvén Discontinuities

We now show that our HLLD solver can capture exactly isolated contact and rotational discontinuities. The initial conditions are listed at the beginning of Table 1.

In the case of an isolated stationary contact wave, only density is discontinuous across the interface. The left panel in Fig. 2 shows the results at $t = 1$ computed with the HLLD, HLLC and HLL solvers: as expected our HLLD produces no smearing of the discontinuity (as does HLLC). On the contrary, the initial jump broadens over several grid zone when employing the HLL scheme.

Across a rotational discontinuity, scalar quantities such as proper density, pressure and total enthalpy are invariant but vector fields experience jumps. The left and right states on either side of an exact rotational discontinuity can be found using the procedure outlined in the Appendix. The right panel in Fig. 2 shows that only HLLD can successfully keep a sharp resolution of the discontinuity, whereas both HLLC and HLL spread the jump over several grid points because of the larger numerical viscosity.

### 4.1.2 Shock Tube 1

The first shock tube test is a relativistic extension of the Brio Wu magnetic shock tube (Brio & Wu 1988) and has also
Table 1. Initial conditions for the test problems discussed in the text. The last two columns give, respectively, the final integration time and the number of computational zones used in the computation.

| Test                  | State | $\rho$ | $p_g$ | $v_x$ | $v_y$ | $v_z$ | $B_x$ | $B_y$ | $B_z$ | Time | Zones |
|-----------------------|-------|--------|-------|-------|-------|-------|-------|-------|-------|------|-------|
| Contact Wave          | L     | 1      | 1     | 0     | 0.7   | 0.2   | 5     | 1     | 0.5   | 1    | 40    |
|                       | R     | 1      | 1     | 0     | 0.7   | 0.2   | 5     | 1     | 0.5   | 1    | 40    |
| Rotational Wave       | L     | 1      | 1     | 0.4   | -0.3  | 0.5   | 2.4   | 1     | -1.6  | 1    | 40    |
|                       | R     | 1      | 1     | 0.377347 | -0.482389 | 0.424190 | 2.4   | -0.1  | -2.178213 | 1    | 40    |
| Shock Tube 1          | L     | 1      | 1     | 0     | 0     | 0     | 0.5   | 1     | 0     | 0.4  | 400   |
|                       | R     | 0.125  | 0.1   | 0     | 0     | 0     | 0.5   | -1    | 0     | 0.4  | 400   |
| Shock Tube 2          | L     | 1.08   | 0.95  | 0.4   | 0.3   | 0.2   | 2     | 0.3   | 0.3   | 0.55 | 800   |
|                       | R     | 1      | 1     | -0.45 | -0.2  | 0.2   | 2     | -0.7  | 0.5   | 0.4  | 800   |
| Shock Tube 3          | L     | 1      | 0.1   | 0.999 | 0     | 0     | 10    | 7     | 7     | 0.4  | 400   |
|                       | R     | 1      | 0.1   | -0.999 | 0     | 0     | 10    | -7    | -7    | 0.4  | 400   |
| Shock Tube 4          | L     | 1      | 5     | 0     | 0.3   | 0.4   | 1     | 6     | 2     | 0.5  | 800   |
|                       | R     | 0.9    | 5.3   | 0     | 0     | 0     | 1     | 5     | 2     | 0.5  | 800   |

Figure 4. Enlargement of the central region of Fig. 3. Density and the two components of velocity are shown in the left, central and right panels, respectively. Diamonds, crosses and plus signs are used for the HLLD, HLLC and HLL Riemann solver, respectively.

These results are supported by the convergence study shown in the top left panel of Fig. 11 demonstrating that the errors obtained with our new scheme are smaller than those obtained with the HLLC and HLL solvers (respectively). At the largest resolution employed, for example, the L-1 norm errors become $\sim 63\%$ and $\sim 49\%$ smaller than the HLL and HLLC schemes, respectively.

The CPU times required by the different Riemann solvers on this particular test were found to be scale as $t_{\text{hll}} : t_{\text{hllc}} : t_{\text{hlld}} = 1 : 1.2 : 1.9$.

4.1.3 Shock Tube 2

This test has also been considered in [Balsara (2001)] and in MB and the initial condition comes out as a non-planar
Riemann problem implying that the change in orientation of the transverse magnetic field across the discontinuity is \( \approx 0.55\pi \) (thus different from zero or \( \pi \)).

The emerging wave pattern consists of a contact wave (at \( x \approx 0.475 \)) separating a left-going fast shock (\( x \approx 0.13 \)), Alfvén wave (\( x \approx 0.185 \)) and slow rarefaction (\( x \approx 0.19 \)) from a slow shock (\( x \approx 0.7 \)), Alfvén wave (\( x \approx 0.725 \)) and fast shock (\( x \approx 0.88 \) heading to the right.

Computations carried out with the 1st order accurate scheme are shown in Fig. 6 using the HLLD (solid line), HLLC (dashed line) and HLL (dotted line). The resolution across the outermost fast shocks is essentially the same for all Riemann solvers. Across the entropy mode both HLLD and HLLC attain a sharper representation of the discontinuity albeit unphysical undershoots are visible immediately ahead of the contact mode. This is best noticed in the left panel of Fig. 6, where an enlargement of the same region is displayed.

On the right hand side of the domain, the slow shock and the rotational wave propagate quite close to each other and the first-order scheme can barely distinguish them at a resolution of 800 zones. However, a close-up of the two waves (middle and right panel in Fig. 6) shows that the proposed scheme is still more accurate than HLLC in resolving both fronts.

On the left hand side, the separation between the Alfvén and slow rarefaction waves turns out to be even smaller and the two modes blur into a single wave because of the large numerical viscosity. This result is not surprising since these features are, in fact, challenging even for a second-order scheme (Balsara 2001).

Discrete L-1 errors computed using Eq. (59) are plotted as a function of the resolution in the top right panel of Fig. 7. For this particular test, HLLD and HLLC produce comparable errors (\( \sim 1.22\% \) and \( \sim 1.33\% \) at the highest resolution) while HLL performs worse on contact, slow and Alfvén waves resulting in larger deviations from the reference solution.

The computational costs on 800 grid zones has found to be \( t_{\text{hll}} : t_{\text{hllc}} : t_{\text{hlld}} = 1 : 1.1 : 1.6 \).

4.1.4 Shock Tube 3

In this test problem we consider the interaction of two oppositely colliding relativistic streams, see also Balsara (2001), Del Zanna et al. (2003) and MB.

After the initial impact, two strong relativistic fast shocks propagate outwards symmetrically in opposite direction about the impact point, \( x = 0.5 \), see Fig. 7. Being a co-planar problem (i.e. the initial twist angle between magnetic fields is \( \pi \)) no rotational mode can actually appear. Two slow shocks delimiting a high pressure constant density region in the center follow behind.

Although no contact wave forms, the resolution across the slow shocks noticeably improves changing from HLL to HLLC and from HLLC to HLLD, see Fig. 7 or the enlargement of the central region shown in Fig. 8. The resolution across the outermost fast shocks is essentially the same for all solvers.

The spurious density undershoot at the center of the grid is a notorious numerical pathology, known as the wall heating problem, often encountered in Godunov-type schemes (Noh 1987; Gezemeyer et al. 1997). It consists of an undesired entropy buildup in a few zones around the point of symmetry. Our scheme is obviously no exception as it can be inferred by inspecting see Fig. 7. Surprisingly, we notice...
that error HLLD performs slightly better than HLLC. The
umerical undershoots in density, in fact, are found to be
\(\sim 24\%\) (HLLD) and \(\sim 32\%\) (HLLC). The HLL solver is less
prone to this pathology most likely because of the larger
numerical diffusion, see the left panel close-up of Fig. 8.

Errors (for \(B_y\)) are computed using the exact solution
available from Giacomazzo & Rezzolla (2006) which is free
from the pathology just discussed. As shown in the bottom
left panel of Fig. 11, HLLD performs as the best numerical
scheme yielding, at the largest resolution employed (3200
zones), \(L_1\) norm errors of \(\sim 18\%\) to be compared to \(\sim 32\%\)
and \(\sim 46\%\) of HLLC and HLL, respectively.

The CPU times for the different solvers on this problem
follow the proportion \(t_{\text{hll}} : t_{\text{hllc}} : t_{\text{hlld}} = 1 : 1.1 : 1.4\).

**4.1.5 Shock Tube 4**

The fourth shock tube test is taken from the “Generic
Alfvén” test in Giacomazzo & Rezzolla (2006). The breaking
of the initial discontinuous states leads to the formation
of seven waves. To the left of the contact discontinuity one
has a fast rarefaction wave, followed by a rotational wave
and a slow shock. Traveling to the right of the contact
discontinuity, one can find a slow shock, an Alfvén wave and a
fast shock.

We plot, in Fig. 9 the results computed with the HLLD,
HLLC and HLL Riemann solvers at \(t = 0.5\), when the outer-
most waves have almost left the outer boundaries. The cen-
tral structure (0.4 \(\leq x \leq 0.6\)) is characterized by slowly moving
fronts with the rotational discontinuities propagating very
close to the slow shocks. At the resolution employed (800
zones), the rotational and slow modes appear to be visi-
ble and distinct only with the HLLD solver, whereas they
become barely discernible with the HLLC solver and com-
pletely blend into a single wave using the HLL scheme. This
is better shown in the enlargement of \(v^y\) and \(B^y\) profiles
shown in Fig. 10: rotational modes are captured at \(x \approx 0.44\)
and \(x \approx 0.59\) with the HLLD solver and gradually disappear
when switching to the HLL scheme.

At the contact wave HLLD and HLLC behave similarly
but the sharper resolution attained at the left-going slow
shock allows to better capture the constant density shell
between the two fronts.

Our scheme results in the smallest errors and numerical
dissipation and exhibits a slightly faster convergence rate,
see the plots in the bottom right panel of Fig. 11. At low res-
olution the errors obtained with HLL, HLLC and HLLD are
in the ratio 1 : 0.75 : 0.45 while they become 1 : 0.6 : 0.27 as
the mesh thickens. Correspondingly, the CPU running times
for the three solvers at the resolution shown in Table 4 have
found to scale as \(t_{\text{hll}} : t_{\text{hllc}} : t_{\text{hlld}} = 1 : 1.4 : 1.8\). This exam-
4.2 Multidimensional Tests

We have implemented our 5 wave Riemann solver into the framework provided by the PLUTO code [Mignone et al. 2007]. The constrained transport method is used to evolve the magnetic field. We use the third-order, total variation diminishing Runge Kutta scheme together with piecewise linear reconstruction.

4.2.1 The 3D Rotor Problem

We consider a three dimensional version of the standard rotor problem [Del Zanna et al. 2003]. The initial condition consists of a sphere with radius \( r_0 = 0.1 \) centered at the origin of the domain taken to be the unit cube \([-1/2, 1/2]^3\]. The sphere is heavier (\( \rho = 10 \)) than the surrounding (\( \rho = 1 \)) and rapidly spins around the \( z \) axis with velocity components given by \((v_x, v_y, v_z) = \omega(-y, x, 0)\) where \( \omega = 9.95 \) is the angular frequency of rotation. Pressure and magnetic field are constant everywhere, \( p_g = 1, B = (1, 0, 0) \).

Exploiting the point symmetry, we carried computations until \( t = 0.4 \) at resolutions of \( 128^3, 256^3 \) and \( 512^3 \) using both the HLLD and HLL solvers. We point out that the HLLC of MB failed to pass this test, most likely because of the flux-singularity arising in 3D computations in the zero normal field limit.

As the sphere starts rotating, torsional Alfvén waves propagate outward carrying angular momentum to the surrounding medium. The spherical structure gets squeezed into a disk configuration in the equatorial plane (\( z = 0 \)) where the two collapsing poles collide generating reflected shocks propagating vertically in the upper and lower half-planes. This is shown in the four panels in Fig. 12 showing the density map in the \( xy \) and \( xz \) planes obtained with HLLD and HLL and in Fig. 13 showing the total pressure. After the impact a hollow disk enclosed by a higher density shell at \( z = \pm 0.02 \) forms (top right panels in Fig 12). In the \( xy \) plane, matter is pushed in a thin, octagonal-like shell enclosed by a tangential discontinuity and what seems to be a slow rarefaction. The whole configuration is embedded in a spherical fast rarefaction front expanding almost radially. Flow distortions triggered by the discretization on a Cartesian grid are more pronounced with HLLD since we expect it to be more effective in the growth of small wavelength modes.

In Fig. 14 we compare the density profiles on the \( y \) and \( z \) axis for different resolutions (\( 128^3, 256^3 \) and \( 512^3 \)) and with different solvers. Solid, dashed and dotted lines are used for the HLLD solver whereas plus and star symbols for HLL.

HLLD solver for relativistic MHD

Figure 12. The 3D rotor test problem computed with HLLD (top panels) and HLL (bottom panels) at the resolution of 256\(^3\). Panels on the left show the density map (at \( t = 0.4 \)) in the \( xy \) plane at \( z = 0 \) while panels to the right show the density in the \( xz \) plane at \( y = 0 \).

Figure 13. Same as Fig. 12 but showing the total pressure in the \( xy \) (left) and \( xz \) (right) panels for the HLLD solver.
the reference solution computed with the HLLD solver at a resolution of 512\(^3\). The height of the shell peak is essentially the same for both solvers, regardless of the resolution.

On the contrary, the right panel of Fig. 14 shows a similar comparison along the vertical \(z\) axis. At the same resolution, HLL under-estimates the density peak located at \(z = 0.02\) and almost twice the number of grid zones is needed to match the results obtained with the HLLD solver. The location of the front is approximately the same regardless of the solver.

In terms of computational cost, integration carried with the HLLD solver took approximately \(1.6\) that of HLL. This has to be compared with the CPU time required by HLL to reach a comparable level of accuracy which, doubling the resolution, would result in a computation \(\sim 2^4\) as long. In this respect, three dimensional problems like the one considered here may prove specially helpful in establishing the trade off between numerical efficiency and accuracy which, among other things, demand choosing between accurate (but expensive) solvers versus more diffusive (cheap) schemes.

### 4.2.2 Kelvin-Helmholtz Unstable Flows

The setup, taken from Bucciantini & Del Zanna (2006), consists of a 2D planar Cartesian domain, \(x \in [0, 1], y \in [-1, 1]\) with a shear velocity profile given by

\[
v^x = -\frac{1}{4} \tanh (100 y). \tag{60}
\]

![Figure 15. Color scale maps of \(\sqrt{B_x^2 + B_y^2}/B_z\) at different integration times, \(t = 5, 15, 30\). Panels on top (bottom) refer to computations accomplished with HLLD (HLL). Poloidal magnetic field lines overlap.](image)

Density and pressure are set constant everywhere and initialized to \(\rho = 1, p_S = 20\), while magnetic field components are given in terms of the poloidal and toroidal magnetization parameters \(\sigma_{pol}\) and \(\sigma_{tor}\) as

\[
(B^x, B^y, B^z) = \left(\sqrt{2\sigma_{pol} p_S}, 0, \sqrt{2\sigma_{tor} p_S}\right), \tag{61}
\]

where we use \(\sigma_{pol} = 0.01, \sigma_{tor} = 1\). The shear layer is perturbed by a nonzero component of the velocity,

\[
v^y = \frac{1}{400} \sin (2\pi x) \exp \left[-\frac{y^2}{\beta^2}\right], \tag{62}
\]

with \(\beta = 1/10\), while we set \(v^z = 0\). Computations are carried at low (L, \(90 \times 180\) zones), medium (M, \(180 \times 360\) zones) and high (H, \(360 \times 720\) zones) resolution.

For \(t \leq 5\) the perturbation follows a linear growth phase leading to the formation of a multiple vortex structure. In the high resolution (H) case, shown in Fig. 15, we observe the formation of a central vortex and two neighbors, more stretched ones. These elongated vortices are not seen in the computation of Bucciantini & Del Zanna (2006) who employed the HLL solver at our medium resolution. As expected, small scale patterns are best spotted with the HLLD solver, while tend to be more diffused using the two-wave HLL scheme. The growth rate (computed as \(\Delta v^y \equiv (v^y_{max} - v^y_{min})/2\), see top panel in Fig. 16), is closely related to the poloidal field amplification which in turn proceeds faster for smaller numerical resistivity (see the small

![Figure 16. Top: growth rate (as function of time) for the Kelvin-Helmholtz test problem computed as \(\Delta v^y \equiv (v^y_{max} - v^y_{min})/2\) at low (L), medium (M) and high (H) resolutions. Solid, dashed and dotted lines show results pertaining to HLLD, whereas symbols to HLL. Bottom: small scale power as a function of time for the Kelvin-Helmholtz application test. Integrated power is given by \(P_s = 1/2 \int_{k_x/2}^{k_x} \int_{-s}^{s} |V(k, y)|^2 dy dk\) where \(V(k, y)\) is the complex, discrete Fourier transform of \(v^y(x, y)\) taken along the \(x\) direction. Here \(k_x\) is the Nyquist critical frequency.](image)
subplot in the same panel) and thus for finer grids. Still, computations carried with the HLLD solver at low (L), medium (M) and high (H) resolutions reveal surprisingly similar growth rates and reach the saturation phase at essentially the same time ($t \approx 3.5$). On the contrary, the saturation phase and the growth rate during the linear phase change with resolution when the HLL scheme is employed.

Field amplification is prevented by reconnection events during which the field wounds up and becomes twisted by turbulent dynamics. Throughout the saturation phase (mid and right panel in Fig 15) the mixing layer enlarges and the field lines thicken into filamentary structures. Small scale structure can be quantified by considering the power residing at large wave numbers in the discrete Fourier transform of any flow quantity (we consider the $y$ component of velocity). This is shown in the bottom panel of Fig 16 where we plot the integrated power between $k_s/2$ and $k_s$ as function of time ($k_s$ is the Nyquist critical frequency). Indeed, during the statistically steady flow regime ($t \geq 20$), the two solvers exhibits small scale power that differ by more than one order of magnitude, with HLLD being in excess of $10^{-5}$ (at all resolutions) whereas HLL below $10^{-6}$.

In terms of CPU time, computations carried out with HLLD (at medium resolution) were $\sim 1.9$ slower than HLL.

4.2.3 Axisymmetric Jet Propagation

As a final example, we consider the propagation of a relativistic magnetized jet. For illustrative purposes, we restrict our attention to axisymmetric coordinates with $r \in [0, 20]$ and $z \in [0, 50]$. The jet initially fills the region $r, z \leq 1$ with density $\rho_j = 1$ and longitudinal ($z$) velocity specified by $\gamma_j = 10$ ($v^r = v^\phi = 0$).

The magnetic field topology is described by a constant poloidal term, $B^z$, threading both the jet and the ambient medium and by a toroidal component $B^\phi(r) = \gamma_j b_\phi(r)$ with

$$b_\phi(r) = \begin{cases} b_m r/a & \text{for } r < a, \\ b_m a/r & \text{for } a < r < 1, \end{cases}$$

where $a = 0.5$ is the magnetization radius and $b_m$ is a constant and vanishes outside the nozzle. The thermal pressure distribution inside the jet is set by the radial momentum balance, $\partial_r p_j = -b_\phi \partial_r (rb_\phi)$ yielding

$$p_j(r) = p_j + b_m^2 \left[ 1 - \min \left( \frac{r^2}{a^2}, 1 \right) \right],$$

where $p_j$ is the jet/ambient pressure at $r = 1$ and is recovered from the definition of the Mach number, $M = v_j / (\Gamma p_j)^{\frac{1}{2}} - 1 / (\Gamma - 1)$, with $M = 6$ and $\Gamma = 5/3$, although we evolve the equations using the approximated Synge gas equation of state of Mignone & McKinney (2007).

The relative contribution of the two components is quantified by the two average magnetization parameters $\sigma_z \equiv B^z_z/(2(p_j))$, $\sigma_\phi \equiv (b^2_\phi)/(2(p_j))$ yielding

$$b_m = \sqrt{-\frac{4p_j \sigma_\phi}{a^2(2\sigma_\phi - 1 + 4\log a)}}, \quad B_z = \sqrt{\sigma_z (b^2_\phi a^2 + 2p_j)},$$

where for any quantity $q(r)$, $\langle q \rangle$ gives the average over the jet beam $r \in [0, 1]$. We choose $\sigma_\phi = 0.3$, $\sigma_z = 0.7$, thus corresponding to a jet close to equipartition.

The external environment is initially static ($v_e = 0$), heavier with density $\rho_e = 10^3$ and threaded only by the constant longitudinal field $B^z$. Pressure is set everywhere to the constant value $p_j$.

We carry out computations at the resolutions of 10, 20 and 40 zones per beam radius ($r = 1$) and follow the evo-
Figure 18. Enlargement of the turbulent flow region \([2, 10] \times [10, 18]\) at \(t = 300\) showing the poloidal magnetic field structure (in log scale) for the high and medium resolution runs (40 and 20 points per beam radius).

Figure 19. Volume average of \(\nabla B^2_p / B^2_p\) as a function of time. Here \(B_p\) is the poloidal magnetic field. Solid, dashed and dotted lines refer to computations carried out with HLLD, whereas symbols give the corresponding results obtained with HLL.

The solution until \(t = 300\). The snapshot in Fig. 17 shows the solution computed at \(t = 300\) at the highest resolution.

The morphological structure is appreciably affected by the magnetic field topology and by the ratio of the magnetic energy density to the rest mass, \(b^2_p / \rho \approx 0.026\). The presence of a moderately larger poloidal component and a small Poynting flux favor the formation of a hammer-like structure rather than a nose cone (see Leismann et al. 2003; Mignone et al. 2005). At the termination point, located at \(z \approx 40.5\), the beam strongly decelerates and expands radially promoting vortex emission at the head of the jet.

Close to the axis, the flow remains well collimated and undergoes a series of deceleration/acceleration events through a series of conical shocks, visible at \(z \approx 4.5, 19, 24, 28, 32\). Behind these recollimation shocks, the beam strongly decelerates and magnetic tension promotes sideways deflection of shocked material into the cocoon.

The ratio \(p_g / p\) (bottom left quadrant in Fig. 17) clearly marks the Kelvin-Helmholtz unstable slip surface separating the backflowing, magnetized beam material from the high temperature (thermally dominated) shocked ambient medium. In the magnetically dominated region turbulence dissipate magnetic energy down to smaller scales and mixing occurs. The structure of the contact discontinuity observed in the figures does not show suppression of KH instability. This is likely due to the larger growth of the toroidal field component over the poloidal one (Kempens et al. 2008). However we also think that the small density ratio \((10^{-3})\) may favor the growth of instability and momentum transfer through entrainment of the external medium (Rossi et al. 2008).

For the sake of comparison, we also plot (Fig 18) the magnitude of the poloidal magnetic field in the region \(r \in [2, 10], z \in [10, 18]\) where turbulent patterns have developed. At the resolution of 40 points per beam radius, HLLD discloses the finest level of small scale structure, whereas HLL needs approximately twice the resolution to produce similar patterns. This behaviour is quantitatively expressed, in Fig 19, by averaging the gradient log\((B^2_p + B^2_z)\) over the volume. Roughly speaking, HLL requires a resolution \(\sim 1.5\) that of HLLD to produce pattern with similar results.

5 CONCLUSIONS

A five-wave HLLD Riemann solver for the equations of relativistic magnetohydrodynamics has been presented. The solver approximates the structure of the Riemann fan by including fast shocks, rotational modes and the contact discontinuity in the solution. The gain in accuracy comes at the computational cost of solving a nonlinear scalar equation in the total pressure. As such, it better approximates Alfvén waves and we also found it to better capture slow shocks and compound waves. The performance of the new solver has been tested against selected one dimensional problems, showing better accuracy and convergence properties than previously known schemes such as HLL or HLLC.

Applications to multi-dimensional problems have been presented as well. The selected tests disclose better resolution of small scale structures together with reduced dependence on grid resolution. We argue that three dimensional computations may actually benefit from the application of the proposed solver which, albeit more computationally intensive than HLL, still allows to recover comparable accuracy and resolution with a reduced number of grid zones. Indeed, since a relative change \(\delta^4\) in the mesh spacing results in a factor \(\delta^4\) in terms of CPU time, this may largely favour a more sophisticated solver over an approximated one. This
issue, however, need to receive more attention in forthcoming studies.

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APPENDIX A: PROPAGATION OF ROTATIONAL DISCONTINUITIES

Left and right states across a rotational discontinuity can be found using the results outlined in [5.2]. More specifically, we construct a family of solutions parameterized by the speed of the discontinuity $K^x$ and one component of the tangential field on the right of the discontinuity. Our procedure can be shown to be equivalent to that of Komissarov (1997). Specifically, one starts by assigning $\rho, p_\theta, v, B^\theta$ on the left side of the front $(B^\theta \equiv (0, B^\theta_x, B^\theta_z))$ together with the speed of the front, $K^x$. Note that $B^x$ cannot be freely assigned but must be determined consistently from Eq. (46). Expressing $K^x (k \neq x)$ in terms of $v^x, B^x$ and $B^z$ and substituting back in the $x-$ component of (45), one finds that there are two possible values of $B^x$ satisfying the quadratic equation

$$a(B^x)^2 + bB^x + c = 0,$$

where the coefficients of the parabola are

$$a = \eta - \frac{(\eta - K^x v^x)^2}{(K^x - v^x)^2}, \quad b = 2\chi \left( v^x + \frac{\eta - K^x v^x}{K^x - v^x} \right),$$

and

$$c = w_\theta + \frac{B^t \cdot B^z}{\gamma^2},$$

with $\eta = 1 - (v^x)^2 - (v^z)^2$, $\chi = v^z B^\theta + v^x B^x$ and $\gamma$ being the Lorentz factor. The transverse components of $K$ are computed as

$$K^{x,z} = v^{y,z} + \frac{B^{y,z}}{B^x} (K^x - v^x).$$

On the right side of the front, one has that $\rho, p_\theta, w, B^x$ and $K$ are the same, see [5.2]. Since the transverse field is elliptically polarized [Komissarov 1997], there are in principle infinite many solutions and one has the freedom to specify, for instance, one component of the field (say $B^x_R$). The velocity $v_R$ and the $z$ component of the field can be determined in the following way. First, use Equation (47) to express $v^z_{\gamma L}$ $(k = x, y, z)$ as function of $B^z_R$ for given $B^x_R$ and $B^z_R$. Using the jump condition for the density together with the fact that $\rho$ is invariant, we solve the nonlinear equation

$$\rho_L \gamma_L (K^x - v^x_R) = \rho_R \gamma_R (K^x - v^x_R),$$

whose roots gives the desired value of $B^x_R$. 

HLLD solver for relativistic MHD