Time and Space Varying Copulas

Glenis Crane

School of Mathematical Sciences
University of Adelaide
South Australia

1 Abstract

In this article we review existing literature on dynamic copulas and then propose an \( n \)-copula which varies in time and space. Our approach makes use of stochastic differential equations, and gives rise to a dynamic copula which is able to capture the dependence between multiple Markov diffusion processes. This model is suitable for pricing basket derivatives in finance and may also be applicable to other areas such as bioinformatics and environmental science.

2 Introduction and motivation

Mapping joint probability distribution functions to copula functions is straightforward when they are static, due to Sklar’s Theorem. On the other hand, mapping time dependent probability functions to copula functions is more problematic. In this article we

(a) review the techniques for creating time dependent copulas and

(b) extend the method described in [4], [5], since it incorporates both time and space. These equations are the first of their kind in higher dimensions, since only 2-dimensional examples have previously been described.

There are at least two areas in which the time dependent copulas of this chapter are applicable,

• Credit derivatives. We would assume in this application that we have a portfolio of \( n \) firms, and \( X_i(t) \) is the value of a \( i \)-th firm’s assets at time \( t \). Each marginal distribution associated
with $X_i(t)$ would represent the probability of the firm’s value falling below some threshold, given certain information at time zero. The time varying copula would represent the evolution of the joint distribution or state of the entire portfolio.

- Genetic drift. For example, each $X_i(t)$ may represent the frequency of a particular gene at time $t$. Each marginal distribution would represent the probability that the frequency of a particular gene had fallen below some threshold. The copula would relate to the evolution of a group of genes of interest.

2.1 Notation and Definitions

In order to understand some of the issues surrounding the mapping of copulae to distributions it is necessary to go back to some of the basic definitions and some notation in relation to the probability distributions of interest.

The notation used for a univariate probability transition function here will be

$$\Pr\{X_i(t_i) \leq x_i \mid X_j(t_j) = x_j\} = F(t_i, x_i \mid t_j, x_j).$$

(1)

If $t_j = 0$ then it is quite common to suppress the zero and so the notation the distribution in this case would be $F(t_i, x_i \mid x_j_0)$.

2.2 Method of Darsow et al

Authors in [1] were the first to attempt to map a transition probability function to a copula. First let us recall the definition of a bivariate copula.

**Definition 1. 2-copula.** A function $C : [0,1]^2 \rightarrow [0,1]$ is a copula if it satisfies the following properties;

1. $C(u_1, 0) = 0, C(0, u_2) = 0$, for all $u_1, u_2 \in [0,1]$
2. $C(u_1, 1) = u_1, C(1, u_2) = u_2$, for all $u_1, u_2 \in [0,1]$ and
3. For every $u_a, u_b, v_a, v_b \in [0,1]$, such that $u_a \leq u_b, v_a \leq v_b$, the volume of $C$, $V_C([u_a, u_b] \times [v_a, v_b]) \geq 0$, that is

$$C(u_b, v_b) - C(u_b, v_a) - C(u_a, v_b) + C(u_a, v_a) \geq 0.$$
**Sklar’s Theorem.** Suppose $H$ is a bivariate joint distribution with marginal distributions $F_1$ and $F_2$ then there exists a 2-copula $C$, such that for all $x_1,x_2 \in \mathbb{R}$

$$H(x_1,x_2) = C(F_1(x_1),F_2(x_2)). \tag{2}$$

If $F_1$ and $F_2$ are continuous distributions then $C$ is unique, otherwise $C$ is uniquely determined on $\text{Ran}F_1 \times \text{Ran}F_2$, see Nelsen [6].

**Definition 5.1. Markov Property.** A stochastic process $X_i(t)$ and $x_i \in \mathbb{R}$, $a \leq t \leq b$ is said to satisfy the Markov property if for any $a \leq t_1 \leq t_2 \ldots \leq t_n \leq t$, the equality

$$\text{Pr}\{X_i(t) \leq x_i \mid X_i(t_1), X_i(t_2), \ldots, X_i(t_n)\} = \text{Pr}\{X_i(t) \leq x_i \mid X_i(t_n)\}$$

holds for any $x_i \in \mathbb{R}$. A stochastic process is called a *Markov Process* if it satisfies the Markov property described in Definition 5.1.

The following notation will be used for an unconditional probability function at time $t_i \geq 0$,

$$\text{Pr}\{X_i(t_i) \leq x_i\} = F_{t_i}(x_i) \tag{3}$$

for a stochastic process $X_i(t_i)$ and $x_i \in \mathbb{R}$. Let

$$\nabla_{x_i} F = \frac{\partial F}{\partial x_i},$$

then the corresponding density function $f$ in this case is such that

$$f(x_i) = \nabla_{x_i} F(x_i).$$

A univariate transition probability function $F$ (Markov process) can be mapped to a bivariate copula $C$ by setting

$$F(t_i,x_i \mid t_j,x_j) = \nabla_{u_2} C \left(F_{t_i}(x_i), F_{t_j}(x_j)\right), \tag{4}$$

where $\nabla_{u_2}$ is the partial derivative with respect to the second argument of $C$. This mapping enables us to build in time, see [1]. The first marginal distribution is associated with time $t_i$ and the second with time $t_j$. We take the partial derivative of the copula, since the probability to which it is mapped is conditional. This method is particularly useful for building Markov chains. One of the most important innovations which enabled the authors in [1] to link copulas to Markov processes was to introduce the idea of a copula product;
Definition 5.2. Copula product. Let $C_a$ and $C_b$ be bivariate copulas, then the product of $C_a$ and $C_b$ is the function $C_a * C_b : [0, 1]^2 \to [0, 1]$, such that

$$(C_a * C_b)(x, y) = \int_0^1 \nabla_x C_a(x, z) \nabla_z C_b(z, y) dz.$$  \hspace{1cm} (5)$$

This product is essentially the copula equivalent of the Chapman-Kolmogorov equation, as stated in Theorem 3.2 of [1]. We restate that theorem here (with modified notation).

Theorem 3.2. Let $X_i(t), t \in T$ be a real stochastic process, and and for each $s, t \in T$ let $C_{st}$ denote the copula of the random variables $X_i(s)$ and $X_i(t)$. The following are equivalent:

1. The transition probabilities $F(t, A \mid s, x_s) = \Pr\{X_i(t) \in A \mid X_i(s) = x_s\}$ of the process satisfy the Chapman-Kolmogorov equations

$$F(t, A \mid s, x_s) = \int_\mathbb{R} F(t, A \mid u, \xi) F(u, d\xi \mid s, x_s)$$  \hspace{1cm} (6)$$

for all Borel sets $A$, for all $s < t \in T$, for all $u \in (s, t) \cap T$ and for almost all $x_s \in \mathbb{R}$.

2. For all $s, u, t \in T$ satisfying $s < u < t$,

$$C_{st} = C_{su} * C_{ut}. \hspace{1cm} (7)$$

The work in [1] has advanced both the theory of copulas and techniques for building Markov processes. This method is also used in [9] to formulate a Markov chain model of the dependence in credit risk. The discrete stochastic variable $X_i(t)$ is interpreted as the rating grade of a firm at a particular point in time. A variety of copulas were fitted to the data and gave mixed results. Therefore, no copula was the best for all data sets. This type of mapping of the transition distribution to the copula is very simple, however, one consequence is that an $n$-dimensional transition function requires a $2^n$-dimensional copula. In other words, as the dimension of the copula increases, the calculation of the transition function becomes more and more computationally cumbersome.

The method in [1] has also been extended in [8], so that an $n$-dimensional Markov process can be
represented by a combination of bivariate copulas and margins. Hence,

\[
Pr\{X_i(t_1) \leq x_1, \ldots, X_i(t_n) \leq x_n\} = \prod_{i=2}^{n} Pr\{X_i(t_i) \leq x_i | X_i(t_1) = x_1, \ldots, X_i(t_{i-1}) = x_{i-1}\} Pr\{X_i(t_1) \leq x_1\}
\]

\[
= \prod_{i=2}^{n} Pr\{X_i(t_i) \leq x_i | X_i(t_{i-1}) = x_{i-1}\} Pr\{X_i(t_1) \leq x_1\} = \prod_{i=2}^{n} C_{t_i-1, t_i}(F_{t_i-1}(x_{i-1}), F_{t_i}(x_i)) \prod_{i=2}^{n-1} F_{t_i}(x_i).
\]

(8)

2.3 Conditional Copula of Patton

Another approach to building time into a copula was formulated in [7]. In order to explain this approach, we need to recall more definitions and set up notation. Firstly, let \(\mathcal{F}\) be a filtration, then

\[
Pr\{X_i \leq x_i | \mathcal{F}\} = F_i(x_i | \mathcal{F})
\]

(9)

The multivariate analogue of equation (9) is

\[
Pr\{X \leq x | \mathcal{F}\} = H(x | \mathcal{F}),
\]

(10)

for \(x = (x_1, x_\ldots, x_n)^T\) such that the volume of \(H\), \(V_H(R) \geq 0\), for all rectangles \(R \in \mathbb{R}^n\) with their vertices in the domain of \(H\), see [8].

\[
H(+\infty, x_i, +\infty, \ldots, +\infty | \mathcal{F}) = F_i(x_i | \mathcal{F}), \quad \text{and}
\]

\[
H(-\infty, x_i, \ldots, x_n | \mathcal{F}) = 0 \quad \text{for all } x_1, \ldots, x_n \in \mathbb{R}.
\]

Here \(F_i\) is the \(i\)-th univariate marginal distribution of \(H\). See [7] for a bivariate version of \(H\). As expected, the density of the conditional \(H\) is

\[
h(x | \mathcal{F}) = \nabla_{x_1, \ldots, x_n} H(x | \mathcal{F}).
\]

(11)

In equation (9), the distribution is atypical since it may be conditional on a vector of variables, not just one, as opposed to a typical univariate transition distribution.
The author in [7] mapped the conditional distribution \( H(x \mid \mathcal{F}) \), defined above, to a copula of the same order. That is, for all \( x_i \in \mathbb{R} \) and \( i = 1, 2, \ldots, n \),

\[
H(x_1, \ldots, x_n \mid \mathcal{F}) = C(F_1(x_1 \mid \mathcal{F}), F_2(x_2 \mid \mathcal{F}), \ldots, F_n(x_n \mid \mathcal{F}) \mid \mathcal{F}).
\] (12)

\( \mathcal{F} \) is a sub-algebra or in other words a conditioning set. Such conditioning is necessary for \( C \) to satisfy all the conditions of a conventional copula. The relationship between the conditional density \( h \) and copula density \( c \) is

\[
h(x_1, x_2, \ldots, x_n \mid \mathcal{F}) = c(u_1, u_2, \ldots, u_n \mid \mathcal{F}) \prod_{i=1}^{n} f_i(x_i \mid \mathcal{F}),
\] (13)

where \( u_i = F_i(x_i \mid \mathcal{F}) \), \( i = 1, 2, \ldots, n \) and \( f_i \), \( i = 1, 2, \ldots, n \) are univariate conditional densities.

In terms of time varying distributions, we can think of the conditioning set as the history of all the variables in the distribution. In the case of the Markov processes, it is only the last time point which is of importance. The implication of this type of conditioning is that the marginal distributions in the copula can no longer be typical transition probabilities, but are atypical conditional probabilities. Hence, if each \( X_i \) represented the value of an asset at time \( t \), the associated distribution \( F_i \) would represent the distribution of \( X_i \), given that we knew the value of all the assets in the model, \( X_1, X_2, \ldots, X_n \), at some previous time, for example \( t - 1 \). In other words, we can rewrite the time-varying version of the distribution and copula above as

\[
H_t(x_1^t, x_2^t, \ldots, x_n^t \mid \mathcal{F}_{t-1}) = C_t(F^t_1(x_1^t \mid \mathcal{F}_{t-1}), F^t_2(x_2^t \mid \mathcal{F}_{t-1}), \ldots, F^t_n(x_n^t \mid \mathcal{F}_{t-1}) \mid \mathcal{F}_{t-1}),
\] (14)

where

\[
\mathcal{F}_{t-1} = \sigma(x_{t-1}^1, x_{t-2}^1, \ldots, x_{t-1}^n, x_{t-2}^1, x_{t-2}^2, \ldots, x_{t-2}^n, \ldots, x_1^1, x_1^2, \ldots, x_1^n).
\]

In [7], the marginal distributions are characterized by Autoregressive (AR) and generalized autoregressive conditional heteroskedasticity (GARCH) processes. Ultimately, they are handled in the same way as other time series processes.

2.4 Pseudo-copulas of Fermanian and Wegkamp

As we have seen above, Markov processes are only defined with respect to their own history, not the history of other processes. Therefore, the method in [7] is good for some applications but not
practical for others. If we want marginal distributions of processes, conditional on their own history, for example Markov processes, and want to use a mapping similar to that shown in [7], then it is possible via a conditional pseudo-copula. Authors in [3] introduced the notion of conditional pseudo-copula in order to cover a wider range of applications than the conditional copula in [7]. The definition of a pseudo-copula is

**Definition 5.3. Pseudo-copula.** A function $C : [0,1]^n \rightarrow [0,1]$ is called an $n$-dimensional pseudo-copula if

1. for every $\mathbf{u} \in [0,1]^n$, $C(\mathbf{u}) = 0$ when at least one coordinate of $\mathbf{u}$ is zero,
2. $C(1,1,\ldots,1) = 1$, and
3. for every $\mathbf{u}, \mathbf{v} \in [0,1]^n$ such that $\mathbf{u} \leq \mathbf{v}$, the volume of $C$, $V_C \geq 0$.

The pseudo-copula satisfies most of the conditions of a conventional copula except for $C(1,1,u_k,1,\ldots,1) = u_k$, so the marginal distributions of a pseudo-copula may not be uniform. The definition of a conditional pseudo-copula is

**Definition 5.4. Conditional pseudo-copula.** Given a joint distribution $H$ associated with $X_1, X_2, \ldots, X_n$, an $n$-dimensional conditional pseudo-copula with respect to sub-algebras $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n)$ and $\mathcal{G}$ is a random function $C(\cdot \mid \mathcal{F}, \mathcal{G}) : [0,1]^n \rightarrow [0,1]$ such that

$$H(x_1, x_2, \ldots, x_n) = C(F_1(x_1 \mid \mathcal{F}_1), F_2(x_2 \mid \mathcal{F}_2), \ldots, F_n(x_n \mid \mathcal{F}_n) \mid \mathcal{F}, \mathcal{G})$$

almost everywhere, for every $(x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$, see [2].

### 2.5 Galichon model

More recently, a dynamic bivariate copula was used to correlate Markov diffusion processes, see [4], [5]. Unlike the previous models of time dependent copulas, this model addresses the issue of spatial as well as time dependence. The model uses a partial differential approach to obtain a representation of the time dependent copula. An outline of the main result follows. Consider two Markov diffusion processes $X_1(t)$ and $X_2(t)$, $t \in [0,T]$, which represent two risky financial assets, for example options
with a maturity date $T$. The diffusions are such that
\[
\begin{align*}
dX_1(t) &= \mu_1(X(t)) dt + \sigma_1(X(t)) dB_1(t) \\
dX_2(t) &= \mu_2(X(t)) dt + \sigma_2(X(t)) dB_2(t) \\
dB_1(t) dB_2(t) &= \rho_{12}(X_1(t), X_2(t)) dt,
\end{align*}
\]
where $X(t) = (X_1(t), X_2(t))^T$, $\mu_i$, $\sigma_i$, for $i = 1, 2$, are the drift and diffusion coefficients, respectively. The Brownian motion terms are correlated with coefficient $\rho_{12} \in [-1, 1]$. One would like an expression for the evolution of a copula between the distributions $F_1$, $F_2$ of $X_1(t)$ and $X_2(t)$, conditional on information at time $t = 0$, $\mathcal{F}_0$. Firstly a joint bivariate distribution $H$ is mapped to a copula $C$, by
\[
H(t, x_1, x_2 \mid \mathcal{F}_0) = C(t, F_1(t, x_1 \mid \mathcal{F}_0), F_2(t, x_2 \mid \mathcal{F}_0) \mid \mathcal{F}_0)
\]
then the Kolmogorov forward equation is used to obtain an expression for $\nabla_t C$. Letting
\[
x = (x_1, x_2)^T 	ext{ and shortening the notation for the copula to } C(t, u_1, u_2),
\]
then the time dependent copula in $\mathbb{R}$ is
\[
\nabla_t C(t, u_1, u_2) = \frac{1}{2} \tilde{\sigma}_1^2(x) f_1^2(t, x_1 \mid \mathcal{F}_0) \nabla_{u_1}^2 C(t, u_1, u_2) + \frac{1}{2} \tilde{\sigma}_2^2(x) f_2^2(t, x_2 \mid \mathcal{F}_0) \nabla_{u_2}^2 C(t, u_1, u_2) \\
- \nabla_{u_1} C(t, u_1, u_2) B_1 F_1(t, x_1 \mid \mathcal{F}_0) \\
+ \int_{(-\infty, x_2]} \nabla_{u_1, u_2} C(t, u_1, u_2) f_2(t, z_2 \mid \mathcal{F}_0) B_1 F_1(t, z_1 \mid \mathcal{F}_0) dz_2 \\
- \nabla_{u_2} C(t, u_1, u_2) B_2 F_2(t, x_2 \mid \mathcal{F}_0) \\
+ \int_{(-\infty, x_1]} \nabla_{u_1, u_2} C(t, u_1, u_2) f_1(t, z_1 \mid \mathcal{F}_0) B_2 F_2(t, z_2 \mid \mathcal{F}_0) dz_1 \\
+ \tilde{\sigma}_1(x) \tilde{\sigma}_2(x) \rho_{12}(x_1, x_2) f_1(t, x_1 \mid \mathcal{F}_0) f_2(t, x_2 \mid \mathcal{F}_0) \nabla_{u_1, u_2} C(t, u_1, u_2),
\]
(18)
where $B_1$ and $B_2$ are the following operators, given any function $g \in C^2(\mathbb{R})$,
\[
B_1 g = \left\{ \nabla_{x_1} \left( \frac{1}{2} \tilde{\sigma}_1^2(x) \right) - \mu_1(x) \right\} \nabla_{x_1} g + \left( \frac{1}{2} \tilde{\sigma}_1^2(x) \right) \nabla_{x_1}^2 g \\
B_2 g = \left\{ \nabla_{x_2} \left( \frac{1}{2} \tilde{\sigma}_2^2(x) \right) - \mu_2(x) \right\} \nabla_{x_2} g + \left( \frac{1}{2} \tilde{\sigma}_2^2(x) \right) \nabla_{x_2}^2 g
\]
and

\[ \nabla_x g = \frac{\partial g}{\partial x_i}, \quad \nabla^2_x g = \frac{\partial^2 g}{\partial x_i^2}. \]

For the greatest flexibility we would choose

\[ \inf \{ x_i : F_i(t, x_i | \mathcal{F}_{t_0}) \geq u_i \} = F_i^{-1}(t, u_i | \mathcal{F}_{t_0}), \quad u_i \in [0, 1]. \]

That is, \( F_i^{-1} \) is the pseudo-inverse. If \( X_1(t) \) and \( X_2(t) \) are individually Markov, that is, \( \tilde{\sigma}_i \) and \( \mu_i \) depend only on \( x_i \), for \( i = 1, 2 \), then the formula for the time dependent copula simplifies to

\[
\nabla_t C(t, u_1, u_2) = \tilde{\sigma}_1(x)\tilde{\sigma}_2(x)\rho_{12}(x_1, x_2)f_1(t, x_1 | \mathcal{F}_{t_0})f_2(t, x_2 | \mathcal{F}_{t_0})\nabla_{u_1, u_2} C(t, u_1, u_2) \\
+ \frac{1}{2} \tilde{\sigma}_1^2(x)f_1^2(t, x_1 | \mathcal{F}_{t_0})\nabla_{u_1}^2 C(t, u_1, u_2) + \frac{1}{2} \tilde{\sigma}_2^2(x)f_2^2(t, x_2 | \mathcal{F}_{t_0})\nabla_{u_2}^2 C(t, u_1, u_2).
\]

(19)

The main aim of this chapter is to extend some of these current models of time dependent copulas. We derive an \( n \)-dimensional version of the model in [4]. A reformulation is also given, in which linear combinations of independent Brownian motion terms are used.

### 3 \( n \)-dimensional Galichon Model for CDOs

Suppose we have an \( n \times n \) system of stochastic differential equations, such that \( X(t) \in \mathbb{R}^n \) and \( B(t) \) is an \( n \)-dimensional Brownian motion. The vector \( X(t) \) could represent a portfolio of risky assets, as in a Collateralized Debt Obligation. We want to find a partial differential equation with respect to a time dependent \( n \)-copula, which gives us information on the riskiness of the package of assets. As in the 2-dimensional model, \( t \) is a scalar such that \( t \in (0, T] \). Let \( \mathcal{F}_t \) be a \( \sigma \)-algebra generated by \( \{ B(s); s \leq t \} \) and assume \( X(t) \) is measurable with respect to \( \mathcal{F}_t \). In this case the diffusions are such that

\[
\begin{align*}
  dX(t) &= \mu(X(t))dt + \dot{A}dB(t) \\
  dB_i(t)dB_j(t) &= \rho_{ij}(X_i(t), X_j(t))dt,
\end{align*}
\]

(20) \hspace{1cm} (21)
where

\[
\begin{pmatrix}
\frac{dX_1(t)}{dt} \\
\frac{dX_2(t)}{dt} \\
\vdots \\
\frac{dX_n(t)}{dt}
\end{pmatrix}, \quad
\begin{pmatrix}
\frac{dB_1(t)}{dt} \\
\frac{dB_2(t)}{dt} \\
\vdots \\
\frac{dB_n(t)}{dt}
\end{pmatrix}
\]

\[
\mu(X(t)) = \begin{pmatrix}
\mu_1(X(t)) \\
\mu_2(X(t)) \\
\vdots \\
\mu_n(X(t))
\end{pmatrix}, \quad
\tilde{\sigma}(X(t)) = \begin{pmatrix}
\tilde{\sigma}_1(X(t)) \\
\tilde{\sigma}_2(X(t)) \\
\vdots \\
\tilde{\sigma}_n(X(t))
\end{pmatrix}
\]

Note that in this case \( \mu \) and \( \tilde{\sigma} \) are \( n \)-vector functions which represent the drift and diffusion coefficients of the process, respectively. Let \( \tilde{A} \) be

\[
\tilde{A} = \text{diag}(\tilde{\sigma}(X(t))) = \\
\begin{pmatrix}
\tilde{\sigma}_1(X(t)) & 0 & \cdots & 0 \\
0 & \tilde{\sigma}_2(X(t)) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \tilde{\sigma}_n(X(t))
\end{pmatrix}
\]

The correlation coefficients \( \rho_{ij} \in [-1, 1] \) and let \( \rho \) be

\[
\rho = \begin{pmatrix}
1 & \rho_{12}(X_1(t), X_2(t)) & \cdots & \rho_{1n}(X_1(t), X_n(t)) \\
\rho_{21}(X_2(t), X_1(t)) & 1 & \cdots & \rho_{2n}(X_2(t), X_n(t)) \\
\vdots & \vdots & \ddots & \vdots \\
\rho_{n1}(X_n(t), X_1(t)) & \cdots & \cdots & 1
\end{pmatrix}
\]

Three conditions are required for the existence and uniqueness of a solution to equation (20):

1. Coefficients \( \mu(x) \) and \( \tilde{\sigma}(x) \) must be defined for \( x \in \mathbb{R}^n \) and measurable with respect to \( x \).

2. For \( x, y \in \mathbb{R}^n \), there exists a constant \( K \) such that

\[
\| \mu(x) - \mu(y) \| \leq K \| x - y \|,
\]

\[
\| \tilde{\sigma}(x) - \tilde{\sigma}(y) \| \leq K \| x - y \|,
\]

\[
\| \mu(x) \|^2 + \| \tilde{\sigma}(x) \|^2 \leq K^2(1 + \| x \|^2)
\]
and

3. $X(0)$ does not depend on $B(t)$ and $E[X(0)^2] < \infty$.

**Theorem 5.1.** The time dependent $n$-copula $\nabla_t C(t, u)$ between a vector of distributions $u_i = F_i(t, x_i \mid x_0)$, $i = 1, \ldots, n$, associated with the Markov diffusions $X(t) = [X_1(t), \ldots, X_n(t)]^T$, conditional on information at time $t = 0$, $\mathcal{F}_0 = x_0$ is

\[
\nabla_t C(t, u) = \frac{1}{2} \sum_{i=1}^{n} \left[ \tilde{\sigma}_i(z)^2 f_i^2(t, x_i \mid x_0) \nabla_{z_1} \cdots \nabla_{z_n} C(t, u) \right] dz
\]

\[
+ \sum_{i=1}^{n} \left( -\nabla_{u_i} C(t, u) B_i^j F_i(t, x_i \mid x_0) + \int_{-\infty}^{\tilde{x}_i} \nabla_{z_1} \cdots \nabla_{u_i} C(t, u) B_i^j F_i(t, x_i \mid x_0) dz \right)
\]

\[
+ \frac{1}{2} \sum_{i,j=1}^{n} \rho_{ij}(x_i, x_j) \tilde{\sigma}_i(z) \tilde{\sigma}_j(z) f_i(t, x_i \mid x_0) f_j(t, x_j \mid x_0) \nabla_{z_1} \cdots \nabla_{z_n} C(t, u) dz
\]

where $x_i = F_i^{-1}(t, u_i \mid \mathcal{F}_0)$, $i = 1, \ldots, n$. The intervals for the integration are

\[
(-\infty, \tilde{x}] = (-\infty, x_1] \times \cdots \times (-\infty, x_{i-1}] \times (-\infty, x_{i+1}] \times \cdots \times (-\infty, x_n] \quad \text{and}
\]

\[
(-\tilde{x}, \tilde{x}] = (-\infty, x_1] \times \cdots \times (-\infty, x_{i-1}] \times (-\infty, x_{i+1}] \times \cdots \times (-\infty, x_{j-1}] \times (-\infty, x_{j+1}] \cdots \times (-\infty, x_n].
\]

Also note that

\[
d\tilde{z} = dz_1 dz_2 \cdots dz_{i-1} dz_{i+1} \cdots dz_{n-1} dz_n \quad \text{and}
\]

\[
d\tilde{z} = dz_1 dz_2 \cdots dz_{i-1} dz_{i+1} \cdots dz_{j-1} dz_{j+1} \cdots dz_{n-1} dz_n.
\]

Thus, the $i$-th term is excluded in the first two integrals on the right hand side of Theorem 5.1. Similarly, in the last integral the $i$-th and $j$-th terms are excluded. Furthermore, for any smooth function $g$

\[
\nabla_{z_1} \cdots \nabla_{z_n} g = \frac{\partial^{n-1} g}{\partial z_1 \cdots \partial z_{i-1} \partial z_{i+1} \cdots \partial z_n}
\]

and

\[
\nabla_{z_1} \cdots \nabla_{z_n} g = \frac{\partial^{n-2} g}{\partial z_1 \cdots \partial z_{i-1} \partial z_{i+1} \cdots z_{j-1} \partial z_{j+1} \cdots \partial z_n}.
\]

11
The operators $B_i$, $i = 1, \ldots, n$ are the same as in the two dimensional model. If the diffusions are individually Markov, that is, each $\sigma_k$ and $\mu_k$ depends only on $x_k$ then the expression for $\nabla_t C$ simplifies to

$$\nabla_t C(t, u) = \frac{1}{2} \sum_{i=1}^{n} \tilde{\sigma}_i(x_i)^2 f_i^2(t, x_i | x_0) \nabla^2 u_i C(t, u)$$

$$+ \frac{1}{2} \text{Tr} \left\{ [H^C_{x}(t, u) - \text{diag}(\nabla^2 u_1 C(t, u), \nabla^2 u_2 C(t, u), \ldots, \nabla^2 u_n C(t, u))] D \tilde{\rho} \tilde{A}^T D^T \right\},$$

where

$$D = \begin{pmatrix} f_1 & 0 & \ldots & 0 \\ 0 & f_2 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & f_n \end{pmatrix}.$$

**Proof.** In this case 1-dimensional Ito formula for each component of $X(t)$ is

$$dg(X_i(t)) = \{ \nabla_{x_i} g(X_i(t)) \mu_i(X(t)) + \frac{1}{2} \nabla_{x_i}^2 g(X_i(t)) \tilde{\sigma}_i(X(t)) \} dt$$

$$+ \nabla_{x_i} g(X_i(t)) \tilde{\sigma}_i(X(t)) dB_i(t).$$

Define the vector $\nabla_x$ of partial derivatives with respect to components of $x$, as

$$\nabla_x g(X(t)) = \begin{pmatrix} \nabla_{x_1} g(X(t)) \\ \nabla_{x_2} g(X(t)) \\ \vdots \\ \nabla_{x_n} g(X(t)) \end{pmatrix}$$

and the Hessian matrix of $g(X(t))$

$$H^g_{x}(X(t)) \equiv \left( \begin{pmatrix} \nabla_{x_i} g(X(t)) \end{pmatrix} \right)_{1 \leq i, j \leq n}. \quad (23)$$

In this case, assume $g \in C^2(\mathbb{R}^n)$, then the n-dimensional Ito formula for $g(X(t))$ is

$$dg(X(t)) = \{ \langle \nabla_x g(X(t)), \mu(X(t)) \rangle + \frac{1}{2} \text{Tr} \left( H^g_{x}(X(t)) \tilde{A} \tilde{A}^T \right) \} dt + \nabla_x g(X(t))^T \tilde{A} dB(t),$$

where $\langle a, b \rangle = a^T b$ for any vectors $a$ and $b$. Let the operators $A$ on distributions (Kolmogorov backward equations), analogous to those in [4], [5], be called $A^i_1$ and $A^i_n$ for the 1- an $n$-dimensional case,
respectively. With respect to typical distributions $F_i(t, x_i \mid \tau, \xi_i)$ and $H(t, x \mid \tau, \xi)$, the operators are

$$A_i F_i(t, x_i \mid \tau, \xi_i) = \mu_i(x) \nabla_{\xi_i} F_i(t, x_i \mid \tau, \xi_i) + \frac{1}{2} \tilde{\sigma}^2 \nabla^2_{\xi_i} F_i(t, x_i \mid \tau, \xi_i)$$

(24)

and

$$A_n H(t, x \mid \tau, \xi) = \langle \nabla_{\xi} H(t, x \mid \tau, \xi), \mu(x) \rangle + \frac{1}{2} \text{Tr} \left( H_{\xi}^T (t, x \mid \tau, \xi) \tilde{A} \tilde{A}^T \right).$$

(25)

The operators $A_i$, $i = 1, \ldots, n$ and $A_n$ are not used in the rest of the formulation, but are mentioned briefly, in view of the fact the Kolmogorov forward equations, which are required, are the associated adjoint operators of these. Assuming the density functions of $H$ and $F$ are $h$ and $f$, respectively, then the adjoint operators $A_i^{*}$, $i = 1, \ldots, n$ and $A_n^{*}$ have the form

$$A_i^{*} f_i(t, x_i \mid \tau, \xi_i) = -\nabla_{x_i} \left[ \mu_i(x) f_i(t, x_i \mid \tau, \xi_i) \right] + \nabla^2_{x_i} \left[ \frac{1}{2} \tilde{\sigma}^2 f_i(t, x_i \mid \tau, \xi_i) \right]$$

(26)

and

$$A_n^{*} h(t, x \mid \tau, \xi) = -\sum_{i=1}^{n} \nabla_{x_i} \left[ \mu_i(x) h(t, x \mid \tau, \xi) \right] + \frac{1}{2} \sum_{i,j=1}^{n} \nabla_{x_i, x_j} \left[ \rho_{ij}(x_i, x_j) \tilde{\sigma}_i(x) \tilde{\sigma}_j(x) h(t, x \mid \tau, \xi) \right].$$

(27)

The marginal density functions $f_i$ and joint density $h$, are such that $f_i(t, x_i \mid \mathcal{F}_0) = f_i(t, x_i \mid x_0)$, and $H(t, x \mid \mathcal{F}_0) = H(t, x \mid x_0)$, where $x_0 = (x_1 = X_1(0), x_2 = X_2(0), \ldots, x_n = X_n(0))$, see Appendix 5.A. In other words, the assumption made here is that all the distributions are conditional on the entire vector of realizations of $x$ at time zero. As in the 2-dimensional case, it is possible to express the operator $A_i^{*}$ in terms of the operators $A_i^{*}$ associated with the univariate distributions;

$$A_i^{*} g = \sum_{i=1}^{n} A_i^{*} g + \frac{1}{2} \sum_{i,j=1}^{n} \nabla_{x_i, x_j} \left[ \rho_{ij}(x_i, x_j) \tilde{\sigma}_i(x) \tilde{\sigma}_j(x) g \right].$$

(28)

Given that

$$\nabla_{i} f_i(t, x_i \mid x_0) = A_i^{*} f_i(t, x_i \mid x_0),$$

(29)
we can integrate the left hand side of (29) with respect to \( x_i \), call it \( B^i_{t_i} \), and we obtain

\[
B^i_{t_i} F_i(t, x_i \mid x_0) = \int_{(-\infty, x_i]} \nabla t f_i(t, z_i \mid x_0) dz_i
= \int_{(-\infty, x_i]} \nabla t \nabla z_i F_i(t, z_i \mid x_0) dz_i
= \nabla t F_i(t, x_i \mid x_0).
\]

Integrating the right hand side of (29) with respect to \( x_i \) gives us

\[
\int_{(-\infty, x_i]} A^i_{t_i} f_i(t, x_i \mid x_0) dz_i = -\mu_i(x) f_i(t, x_i \mid x_0) + \nabla x_i \left[ \frac{1}{2} \sigma^2_i(x) f_i(t, x_i \mid x_0) \right]
= [\nabla x_i \left\{ \frac{1}{2} \sigma^2_i(x) \right\} - \mu_i(x)] \nabla x_i F_i(t, x_i \mid x_0) + \frac{1}{2} \sigma^2_i(x) \nabla^2 x_i F_i(t, x_i \mid x_0),
\]

so

\[
B^i_{t_i} F_i(t, x_i \mid x_0) = [\nabla x_i \left\{ \frac{1}{2} \sigma^2_i(x) \right\} - \mu_i(x)] \nabla x_i F_i(t, x_i \mid x_0) + \frac{1}{2} \sigma^2_i(x) \nabla^2 x_i F_i(t, x_i \mid x_0).
\]

Similarly, integrating over \( A^i_{t_i} \) will give us the analogous operator \( B_{t_i}^i \) for the multivariate distribution \( H \). Now, since

\[
\int_{(-\infty, x]} A^i_{t_i} h(t, z \mid x_0) dz
= \sum_{i=1}^{n} \int_{(-\infty, x_i]} A^i_{t_i} h(t, z \mid x_0) dz + \frac{1}{2} \sum_{i,j=1}^{n} \int_{(-\infty, x_i \times (-\infty, x_j]} [\rho_{ij}(z_i, z_j) \tilde{\sigma}_i(z) \tilde{\sigma}_j(z) h(t, z \mid x_0)] dz
\]

(32)

where \((-\infty, x] = (-\infty, x_1] \times \ldots \times (-\infty, x_n],\) it is possible to get an expression for \( B_{t_i}^i \) in terms of \( B^i \). That is, let \( B_{t_i}^i H(t, x \mid x_0) = \nabla x_i H(t, x \mid x_0) \) and given that \( h(t, x \mid x_0) = \nabla x_{z_1, \ldots, z_n} H(t, x \mid x_0), \) we have

\[
B_{t_i}^i H(t, x \mid x_0) = \frac{1}{2} \sum_{i,j=1}^{n} \int_{(-\infty, x_i \times (-\infty, x_j]} \nabla_{z_i, z_j} [\rho_{ij}(z_i, z_j) \tilde{\sigma}_i(z) \tilde{\sigma}_j(z) \nabla_{z_1, \ldots, z_n} H(t, z \mid x_0)] dz
+ \sum_{i=1}^{n} \int_{(-\infty, x_i]} A^i_{t_i} \nabla_{z_i, \ldots, z_n} H(t, z \mid x_0) dz.
\]

(33)

The right hand side of equation (33) can be expressed in terms in terms of the univariate operators.
\( B^n_i, i = 1, 2, \ldots, n \).

\[
B^n_i H(t, x \mid x_0) = \frac{1}{2} \sum_{i,j=1}^{n} \int_{(-\infty, \bar{x}]} \rho_{ij}(x_i, x_j) \tilde{\sigma}_i(z) \tilde{\sigma}_j(z) \nabla_{z_i, \hat{z}_i, \ldots, z_n, \hat{z}_n} H(t, z \mid x_0) \, dz + \sum_{i=1}^{n} \int_{(-\infty, \bar{x}]} B^n_i \nabla_{z_i, \hat{z}_i, \ldots, z_n} H(t, z \mid x_0) \, dz. \tag{34}
\]

Let

\[
H(t, x \mid x_0) = C(t, F_1(t, x_1 \mid x_0), F_2(t, x_2 \mid x_0), \ldots, F_n(t, x_n \mid x_0) \mid x_0) \tag{35}
\]

where \( C \) is an \( n \)-copula defined on \([0, T] \times [0, 1]^n\). At this point we shorten the notation so that \( C(t, F(t, x \mid x_0)) \) is the same copula as above. We now seek an expression for \( B^n_i C(t, F(t, x \mid x_0)) \) by substituting for \( H \) with \( C \) in equation (34). Letting \( F_i(t, x_i \mid x_0) = u_i, i = 1, 2, \ldots, n \), and \( u = (u_1, \ldots, u_n)^T \), then from the first term in equation (34) we obtain (see overpage)
\[
\sum_{i=1}^{n} \int_{(-\infty, x]} B_i^i \nabla z_{1,i} \cdots z_{n} H(t, z \mid x_0) d\bar{z}
\]

\[
= \sum_{i=1}^{n} \int_{(-\infty, x]} B_i^i \nabla z_{1,i} \cdots z_{n} C(t, F(t, z \mid x_0)) d\bar{z}
\]

\[
= \sum_{i=1}^{n} \left( \int_{(-\infty, x]} \left\{ \nabla z_i \frac{\sigma_i^2(z)}{2} - \mu_i(z) \right\} \nabla z_{1,i} \cdots z_{n} C(t, F(t, z \mid x_0)) d\bar{z} \right.
\]

\[
+ \int_{(-\infty, x]} \frac{\sigma_i^2(z)}{2} \nabla z_i \nabla z_{1,i} \cdots z_{n} C(t, F(t, z \mid x_0)) d\bar{z}
\]

\[
= \sum_{i=1}^{n} \left( \int_{(-\infty, x]} \left\{ \nabla z_i \frac{\sigma_i^2(z)}{2} - \mu_i(z) \right\} f_i(t, z_i \mid x_0) \nabla u_i \nabla z_{1,i} \cdots z_{n} C(t, u) d\bar{z} \right.
\]

\[
+ \int_{(-\infty, x]} \frac{\sigma_i^2(z)}{2} f_i^2(t, z_i \mid x_0) \nabla u_i \nabla z_{1,i} \cdots z_{n} C(t, u) d\bar{z}
\]

\[
+ \int_{(-\infty, x]} \frac{\sigma_i^2(z)}{2} f_i(t, z_i \mid x_0) \nabla u_i \nabla z_{1,i} \cdots z_{n} C(t, u) d\bar{z}
\]

\[
= \sum_{i=1}^{n} \left( \int_{(-\infty, x]} \left\{ \nabla z_i \frac{\sigma_i^2(z)}{2} - \mu_i(z) \right\} f_i(t, z_i \mid x_0) \nabla u_i \nabla z_{1,i} \cdots z_{n} C(t, u) d\bar{z} \right.
\]

\[
+ \int_{(-\infty, x]} \frac{\sigma_i^2(z)}{2} f_i^2(t, z_i \mid x_0) \nabla u_i \nabla z_{1,i} \cdots z_{n} C(t, u) d\bar{z}
\]

\[
+ \int_{(-\infty, x]} \frac{\sigma_i^2(z)}{2} f_i(t, z_i \mid x_0) \nabla u_i \nabla z_{1,i} \cdots z_{n} C(t, u) d\bar{z}
\]

\[
= \sum_{i=1}^{n} \int_{(-\infty, x]} \nabla z_{1,i} \cdots z_{n} \nabla u_i C(t, u) B_i^i f_i(t, z_i \mid x_0) d\bar{z}
\]

\[
+ \frac{1}{2} \sum_{i=1}^{n} \int_{(-\infty, x]} \sigma_i^2(z) f_i^2(t, z_i \mid x_0) \nabla z_{1,i} \cdots z_{n} \nabla u_i^2 C(t, u) d\bar{z}.
\]

Since \( z \) is a dummy variable and the multiple integrals exclude that over \((-\infty, x_i]\), we can write

\[
\sum_{i=1}^{n} \int_{(-\infty, x]} B_i^i \nabla z_{1,i} \cdots z_{n} H(t, z \mid x_0) d\bar{z}
\]

\[
= \sum_{i=1}^{n} \int_{(-\infty, x]} \nabla z_{1,i} \cdots z_{n} \nabla u_i C(t, u) B_i^i f_i(t, z_i \mid x_0) d\bar{z}
\]

\[
+ \frac{1}{2} \sum_{i=1}^{n} \int_{(-\infty, x]} \sigma_i^2(z) f_i^2(t, z_i \mid x_0) \nabla z_{1,i} \cdots z_{n} \nabla u_i^2 C(t, u) d\bar{z}. \quad (36)
\]
From the second term in equation \((34)\) we have

\[
\frac{1}{2} \sum_{i,j=1}^{n} \int_{(-\infty, \bar{x}]} \rho_{ij}(x_i, x_j) \tilde{\sigma}_i(z) \tilde{\sigma}_j(z) \nabla_{z_1, \ldots, \hat{z}_i, \ldots, \hat{z}_j, \ldots, z_n} H(t, z | x_0) d\tilde{z}
\]

\[
= \frac{1}{2} \sum_{i,j=1}^{n} \int_{(-\infty, \bar{x}]} \rho_{ij}(x_i, x_j) \tilde{\sigma}_i(z) \tilde{\sigma}_j(z) \nabla_{z_1, \ldots, \hat{z}_i, \ldots, \hat{z}_j, \ldots, z_n} C(t, F(t, z | x_0)) d\tilde{z}
\]

\[
= \frac{1}{2} \sum_{i,j=1}^{n} \int_{(-\infty, \bar{x}]} \rho_{ij}(x_i, x_j) \tilde{\sigma}_i(z) \tilde{\sigma}_j(z) f_i(t, x_i | x_0) f_j(t, x_j | x_0) \nabla_{z_1, \ldots, \hat{z}_i, \ldots, \hat{z}_j, \ldots, z_n} C(t, u) d\tilde{z}
\]

(37)

so

\[
B^u_i C(t, u) = \sum_{i=1}^{n} \int_{(-\infty, \bar{x}]} \nabla_{z_1, \ldots, \hat{z}_i, \ldots, \hat{z}_j, \ldots, z_n} C(t, u) B^u_i F_i(t, z_i | x_0) d\tilde{z}
\]

\[
+ \frac{1}{2} \sum_{i=1}^{n} \int_{(-\infty, \bar{x}]} \tilde{\sigma}_i^2(z) f^2_i(t, x_i | x_0) \nabla_{z_1, \ldots, \hat{z}_i, \ldots, \hat{z}_j, \ldots, z_n} C(t, u) d\tilde{z}
\]

\[
+ \frac{1}{2} \sum_{i,j=1}^{n} \int_{(-\infty, \bar{x}]} \rho_{ij}(x_i, x_j) \tilde{\sigma}_i(z) \tilde{\sigma}_j(z) f_i(t, x_i | x_0) f_j(t, x_j | x_0) \nabla_{z_1, \ldots, \hat{z}_i, \ldots, \hat{z}_j, \ldots, z_n} C(t, u) d\tilde{z}.
\]

(38)

Now, we also have

\[
\nabla_i H(t, x | x_0) = B^u_i H(t, x | x_0)
\]

\[
= \nabla_i C(t, F(t, x | x_0)) + \sum_{i=1}^{n} \nabla_{u_i} C(t, F(t, x | x_0)) \nabla_i F_i(t, x_i | x_0)
\]

\[
= \nabla_i C(t, u) + \sum_{i=1}^{n} \nabla_{u_i} C(t, u) B^u_i F_i(t, x_i | x_0).
\]

(39)
Matching equation (38) and (39) and rearranging, we obtain

\[ \nabla_t C(t, u) = \sum_{i=1}^{n} \left( -\nabla_{u_i} C(t, u) B_i^t F_i(t, x_i | x_0) + \int_{(-\infty, \tilde{x}_1]} \nabla_{z_1 \ldots \hat{z}_i \ldots \hat{z}_n} C(t, u) B_i^t F_i(t, z_i | x_0) d\tilde{z} \right) + \frac{1}{2} \sum_{i=1}^{n} \int_{(-\infty, \tilde{x}_1]} \tilde{\sigma}_i^2(z) f_i^2(t, x_i | x_0) \nabla_{u_i} C(t, u) d\tilde{z} \\
+ \frac{1}{2} \sum_{i,j=1}^{n} \int_{(-\infty, \tilde{x}_1]} \rho_{ij}(x_i, x_j) \tilde{\sigma}_i(z) \tilde{\sigma}_j(z) f_i(t, x_i | x_0) f_j(t, x_j | x_0) \nabla_{u_i} C(t, u) d\tilde{z} \].

If the equations are individually Markov, so that each \( \sigma_k \) and \( \mu_k \) depends only on \( x_k \), then

\[ \sum_{i=1}^{n} \left( -\nabla_{u_i} C(t, u) B_i^t F_i(t, x_i | x_0) + \int_{(-\infty, \tilde{x}_1]} \nabla_{z_1 \ldots \hat{z}_i \ldots \hat{z}_n} C(t, u) B_i^t F_i(t, z_i | x_0) d\tilde{z} \right) = 0, \]

so the expression for \( \nabla_t C \) simplifies to

\[ \nabla_t C(t, u) = \frac{1}{2} \sum_{i=1}^{n} \tilde{\sigma}_i(x_i)^2 f_i^2(t, x_i | x_0) \nabla_{u_i}^2 C(t, u) + \frac{1}{2} \text{Tr} \left\{ \mathcal{H}_u^C(t, u) - \text{diag} \{ \nabla_{u_1}^2 C(t, u), \nabla_{u_2}^2 C(t, u), \ldots, \nabla_{u_n}^2 C(t, u) \} \right\} D \tilde{\sigma} A^T D^T \}. \]

\[ \square \]

### 4 Conclusion

We have described a dynamic \( n \)-copula which varies in time and space. This copula is the first of its kind in greater than 2 dimensions. The dynamic 2-copula was previously described in [5]. In that case, the copula could be applied the pricing of pairs of options and other credit derivatives. In the \( n \)-dimensional case, it is possible to use the dynamic copula for the pricing of any basket derivatives or a number of commodities. Future work in this area may involve numerical experiments, sensitivity testing and simulations in order to determine how robust the model is. Other possible applications include that of the health industry and environmental science.

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