Nuclear Reaction Rates in a Plasma: The Effect of Highly Damped Modes

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The fluctuation-dissipation theorem is used to evaluate the screening factor of nuclear reactions due to the electromagnetic fluctuations in a plasma. We show that the commonly used Salpeter factor is obtained if only fluctuations near the plasma eigenfrequency are assumed to be important ($\omega \sim \omega_{pe} \ll T$ ($\hbar = k_B = 1$)). By taking into account all the fluctuations, the highly damped ones, with $\omega > \omega_{pe}$, as well as those with $\omega \leq \omega_{pe}$, we find that nuclear reaction rates are higher than those obtained using the Salpeter factor, for many interesting plasmas.

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I. INTRODUCTION

It is known that fusion reactions within a plasma are enhanced. This is due to the fact that ions in a plasma are not “naked”, but are surrounded by electrons in the plasma that form a shielding cloud around them. Thus, an ion is not repelled, as a “naked” ion, since the “dressing” of electrons shield the repulsive potential of the ion for distances greater than the Debye radius, $R_D$. The presence of the plasma is taken into account, in what is called the screening factor [1], in estimating cross sections of nuclear reaction rates. In the static case, when the velocities of the ions are assumed to be very slow compared to those of the screening particles, the screening factor used is the Salpeter factor [2] (valid in the weak screening limit, when $Z_1 Z_2 e^2 \ll T R_D$ ($\hbar = k_B = 1$)).

Electromagnetic fluctuations are present in a plasma, even when in thermal equilibrium. From these fluctuations, we evaluate the screening of the interacting particles. Their effect is to enhance the reaction rates, just as a negative potential reduces the Coulomb repulsion.

We use the formalism of the fluctuation-dissipation theorem [3,4] to describe the electromagnetic fluctuating fields. Using the fluctuation-dissipation theorem, it is possible to study all the fluctuations existent in a plasma. That is, it is possible to also include the effect of screening modes that are not eigenmodes (i.e., not at the plasma frequency). These modes are highly damped, but although they have a small amplitudes, their overall effect turns out to be great because their phase space is very large. In particular, they have non-negligible amplitudes for $\omega \gg \omega_{pe}$, where $\omega_{pe}$ is the plasma frequency.

In estimating the effect of electromagnetic fluctuations on the screening in the limit where only frequencies on the order of $\omega_{pe}$ are important ($\omega \sim \omega_{pe} \ll T$ ($\hbar = k_B = 1$)), the usual Salpeter factor is obtained.

In a non-magnetized plasma, the longitudinal eigenmodes are Langmuir waves with $\omega \sim \omega_{pe}$. For usual laboratory and space plasmas, this frequency is much less than $T$ and the assumption that $\omega \ll T$ is reasonable. However, if we are interested in the effect of all the existent modes in the plasma on the screening, it is important not to be limited only to modes where $\omega \sim \omega_{pe}$.

We show, in section II, that the screening can be described in terms of the electromagnetic fluctuations. The change in the reaction rates due to the screening is calculated and our screening factor is compared with the Salpeter factor. A discussion of our results is given in section III.

II. REACTION RATES

The expression for the reaction rate between ions 1 and 2 is [1]

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\[ R = \int d\mathbf{v}_1 f(v_1) \int d\mathbf{v}_2 f(v_2) \mid v_1 - v_2 \mid \sigma(v_1, v_2) , \] (1)

where \( \sigma \) is the cross section and \( f(v_1) \) and \( f(v_2) \) are the thermal distributions of the particles. The cross section is \( \sigma(v_1, v_2) = (S_N(E) / E)P(v_1, v_2) \), where \( P(v_1, v_2) \) is the penetration factor, \( E \) is the kinetic energy of the ions at large separations in the center of mass reference frame and \( S_N(E) \) is a purely nuclear factor that varies slowly with \( E \). \( P(v_1, v_2) \) is proportional to \( \exp\left(-2(2\mu)^{1/2}/\hbar\right) \int_{r_0}^{r} [U(r) - U_0 - E]^{1/2} dr \), where \( \mu \) is the reduced mass. The integration is performed from the nuclear radius \( a \) to the turning point radius \( r_0 \). \( U(r) \) is the bare Coulomb interaction \( Z_1 Z_2 e^2 / r \) and \( U_0 \), the potential energy due to the screening. In the case of weak screening, \( U_0 \) is practically independent of \( r \) in the integration interval. The reaction rate \( R \) is then found to be increased by a factor \( F = \exp(U_0/T) \). In the static case, the weak screening limit, the enhancement factor is \( \exp(Z_1 Z_2 e^2 / (T R_D)) \), the Salpeter factor.

In a plasma, electromagnetic fluctuations are always present, acting as a background for the interacting ions. Let us consider the effect of the fluctuating electric fields. \( \phi \), the electric potential acting in the region from \( r_0 \) to \( a \), creates a potential energy \( U_1 = e(Z_1 + Z_2)\phi \). The enhancement factor is, then, \( F = \exp(U_0/T) = \exp(U_1 - U_\infty/T) \), where \( U_\infty \) is the potential energy of the ions when they are at infinity, \( U_\infty = Z_1 e \phi_{1\infty} + Z_2 e \phi_{2\infty} \). Averaging \( F \) in time and using a Taylor expansion, we obtain terms which are proportional to \( \langle \phi^2 \rangle, \langle \phi_{1\infty}^2 \rangle \) and \( \langle \phi_{2\infty}^2 \rangle \), as well as cross terms. Because the fields \( \phi, \phi_{1\infty} \) and \( \phi_{2\infty} \) are random with respect to one another, the only terms that remain to first order are \( \langle \phi^2 \rangle, \langle \phi_{1\infty}^2 \rangle \) and \( \langle \phi_{2\infty}^2 \rangle \). Assuming that they are equal to one another,

\[ F = 1 + Z_1 Z_2 e^2 \langle \phi^2 \rangle \beta^2 + ... = \exp \left( Z_1 Z_2 e^2 \langle \phi^2 \rangle \beta^2 \right) , \] (2)

where \( \beta = 1/T \).

\( \phi \) is assumed to be changing slowly in the integrand of the penetration factor in the region from \( r_0 \) to \( a \). Therefore, we consider fields with frequencies \( \omega < \omega_{\text{max}} = 2\pi r_0 / v \) and wavenumbers \( k < k_{\text{max}} = 2\pi / r_0 \), where \( v \) is the velocity of the reduced mass in the center of mass system. We find that \( \langle \phi^2 \rangle \) is relatively insensitive to the exact upper limits of the integrations, \( \omega_{\text{max}} \) and \( k_{\text{max}} \).

\( \langle \phi^2 \rangle \) is related to the fluctuations of the longitudinal electric field in a plasma: \( \langle \phi^2 \rangle_k = \langle E^2 \rangle_k / k^2 \). Therefore,

\[ \langle \phi^2 \rangle = \int \frac{dk}{(2\pi)^3} \langle \phi^2 \rangle_k = \int \frac{dk}{(2\pi)^3} \int d\omega \frac{\langle E^2 \rangle_k \omega}{k^2} . \] (3)

From the fluctuation-dissipation theorem \( \| \), the expression for the intensity of the fluctuations of the longitudinal electric field is

\[ \frac{\langle E^2 \rangle_k \omega}{8\pi} = \frac{1}{\exp(\omega/T) - 1} \frac{Im\epsilon_l}{|\epsilon_l|^2} , \] (4)

where \( \epsilon_l \) is the longitudinal dielectric permittivity of the plasma. We note that this is an exact quantum mechanical relation. Given a dielectric permittivity that describes the plasma, it is possible, from Eq. (4), to obtain the fluctuations in the longitudinal field due to the eigenmodes (i.e., the electrostatic Langmuir waves), as well as to all the other fluctuations existent in the plasma.

We show below that the overall effect of fluctuations with \( \omega > \omega_{pe} \) is to substantially increase the reaction rates. Although the amplitudes of these fluctuations are small, they are not negligible. Due to the very large range of their frequencies, their overall effect is to increase the reaction rates appreciably. Because the reaction rates are exponentially dependent on the screening potential, its increase is appreciable for reactions with large values of \( Z_1 Z_2 \) such as \( p - 7 Be \) and \( p - 14 N \). In the limit where only fluctuations with frequencies on the order of the eigenfrequency \( \omega_{pe} (\ll T) \) are important, Eq. (4) can be easily interagred over frequency with the aid of the Kramers-Kronig relations, obtaining

\[ \langle E^2 \rangle_k = T / 2 \left( 1 - \frac{1}{\epsilon_l(0, k)} \right) . \] (5)

This is the expression which is generally found in text books \( \| \). For an electron plasma in the collisionless case, the longitudinal dielectric permittivity is \( \epsilon_l(\omega, k) = 1 + \frac{k^2}{\omega_p^2} \left( 1 - \phi(z) + i\sqrt{\pi} \epsilon z e^{-z^2} \right) \), where \( z = \sqrt{3/2}(\omega/kv_T) \) and \( \phi(z) = 2 \pi e^{-z^2} \int_0^z e^{-x^2} dx \). With this dielectric permittivity, \( \langle \phi^2 \rangle = T / R_D \) [Eq.(3)] and the enhancement factor \( F \), in Eq.(2), is the Salpeter enhancement factor,

\[ F_{\text{SAL}} = \exp \left( \frac{Z_1 Z_2 e^2}{T} \frac{1}{R_D} \right) . \] (6)

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We use the model described in detail in Opher & Opher [3, 4], which includes thermal and collisional effects, to obtain the longitudinal dielectric permittivity. It uses the Vlasov equation in first order with the BGK (Bhatnagar-Gross-Krook) collision term, which is a model equation of the Boltzmann collision term [5]. The inclusion of collisions changes the results very little and the effect on \( \varepsilon_l(\omega, k) \) for the plasmas studied, is almost negligible. For example, in the case analysed in Opher and Opher [7], where \( T = 10^7 \) K and \( n = 10^{10} \) cm\(^{-3} \), the difference in the correlation energy, with or without collisions, is less than \( 10^{-6} \).

The dielectric permittivity that we therefore use, is almost identical to the collisionless dielectric permittivity given after Eq. (5) in the text (the only difference is the inclusion of other shielding species besides the electron). From this model, the longitudinal dielectric permittivity for an isotropic plasma is found to be

\[
\varepsilon_l(\omega, k) = 1 + \sum_{\alpha} \frac{\omega_{p\alpha}^2}{k^2 v_{\alpha}^2} \left( 1 + \frac{\omega + \omega_{\alpha}}{\sqrt{2 k v_{\alpha}}} \right) \left( \frac{\omega + \omega_{\alpha}}{\sqrt{2 k v_{\alpha}}} \right)
\]

where \( \alpha \) is the label of the species of the particles, \( v_{\alpha} \) is the thermal velocity of the species and \( Z(\omega) \), the Fried and Conte function. By using Eq. 2 in Eq. 3 and Eq. 4, we obtain \( \langle \phi^2 \rangle \). The enhancement factor is obtained from Eq. (2). We note that using a relation which is similar to Eq. (2) for the transverse field \( \langle B^2 \rangle_{k, \omega} \) in conjunction with the transverse dielectric permittivity instead of Eq. (7), the correct black body spectrum is obtained [5, 6].

We estimate the enhancement factor \( F \) for the reaction rates \( p - p, ^3\text{He} - ^3\text{He} \) and \( p - ^{14}\text{N} \) for an electron-proton plasma at temperatures \( T = 10^7 \) to \( 10^8 \) K and densities ranging from \( 10^{23} \) to \( 10^{28} \) cm\(^{-3} \), choosing the maximum density for each temperature so that the plasma parameter is less than \( g = 1/n\lambda_D^3 \approx 0.8 \). When \( g \) approaches such high values, non-linear effects begin to be important. (Note that \( \lambda_D \) has an extra factor \( \sqrt{2} \) due to the inclusion of protons.)

The Gamow energy, the most effective energy for thermonuclear reactions, is \( E_G = 1.22((Z_1/Z_2)^2 A_T^2)^{1/3} \) keV, where \( A \) is the reduced atomic weight and \( T_0 = T/10^6 \). Therefore, for each of the temperatures that we choose, the Gamow energy changes, with a consequent change in the limits of the integration, \( \omega_{max} \) and \( \omega_{max} \). As a particular case, we assume the center of mass energy to be zero. \( (F - F_{\text{SAL}})/F_{\text{SAL}}(\%) \) vs \( g \), where \( F_{\text{SAL}} \) is the Salpeter enhancement factor and \( g \) is the plasma parameter, is plotted for \( T = 10^7 \) K in Fig. 1. In Fig. 2, we plot \( (F - F_{\text{SAL}})/F_{\text{SAL}}(\%) \) vs \( g \) for \( T = 10^8 \) K. We also estimate the enhancement factor \( F \) for the reaction \( D - D \) in an electron-deuteron plasma for temperatures \( T = 10^7 \) to \( 10^8 \) K and densities \( 10^{23} \) to \( 10^{28} \) cm\(^{-3} \). Again, the maximum density for each temperature is chosen to be \( g \approx 0.8 \). \( (F - F_{\text{SAL}})/F_{\text{SAL}}(\%) \) vs \( g \) is shown in Fig. 3. We see that the enhancement factor \( F \) is larger than the Salpeter factor.

In order to see how the fluctuating potential is affected by these modes, we plot the potential \( \langle \phi^2 \rangle_{\omega, k} \) vs \( \omega/\omega_{pe} \) for the reaction \( p - ^{14}N \), with \( T = 10^7 \) K and \( n = 10^{24} \) cm\(^{-3} \) (in this case, \( g = 0.3 \)) in Fig. 4. The solid curve is plotted for \( k = 100 \) \( k_D \), the dashed curve for \( k = 10 \) \( k_D \) and the dotted curve for \( k = k_D \). We normalize the curves and plot \( \langle \phi^2 \rangle_{\omega, k} \) divided by \( \langle \phi^2 \rangle_{\omega, 0} \). When \( \langle \phi^2 \rangle_{\omega, k} \) is integrated for all wavenumbers \( k \), but only for frequencies up to \( \omega_{pe} \) (\( \ll T \)), we obtain the Salpeter potential. We see that for \( k = k_D \), the potential \( \langle \phi^2 \rangle_{\omega, k} \) has non-negligible values only at low frequencies. However, for \( k = 10 k_D \) and \( k = 100 k_D \), the potential \( \phi_{k, \omega} \) spreads out to high frequencies. For these high values of \( k \), instead of dropping abruptly at \( \omega \approx \omega_{pe} \), the curves decrease slowly with frequency. There is thus a substantial contribution for \( \omega > \omega_{pe} \). Thus for large wavenumbers, \( (i.e., k > k_D) \) fluctuations make a substantial contribution at frequencies higher than the plasma frequency. For example, for \( k = 10 k_D \) and \( \omega = 20 \omega_{pe} \), \( \langle \phi^2 \rangle_{\omega, k}/\langle \phi^2 \rangle_{\omega, 0, k} \) is not zero, but 0.002. Similarly, for \( K = 100 k_D \) and \( \omega = 20 \omega_{pe} \), \( \langle \phi^2 \rangle_{\omega, k}/\langle \phi^2 \rangle_{\omega, 0, k} \) is 0.01.

III. CONCLUSIONS AND DISCUSSION

By using the fluctuation-dissipation theorem to estimate the screening of the reacting ions due to the electromagnetic field fluctuations in plasma, we calculate the enhancement factor for the reaction rates. The enhancement is found to be greater than that for the adiabatic case (\( \omega \ll T \)), where the Salpeter factor is valid. It is important to note that this effect is not due to a specific choice of the dielectric permittivity. The model that we use includes the standard dielectric permittivity and gives almost the same results as does a collisionless thermal description. A higher enhancement factor is obtained because we include the high frequencies and do not assume that \( \omega \leq \omega_{pe} \ll T \). Using this method in a previous study to derive the transverse magnetic fluctuations, the standard black body spectrum was obtained. For the plasmas studied here, electron-proton plasmas with temperatures \( 10^7 \) to \( 10^8 \) K and densities \( 10^{23} \) to \( 10^{28} \) cm\(^{-3} \), the reaction rates for \( p - p, ^3\text{He} - ^3\text{He} \) and \( p - ^{14}\text{N} \) are increased up to \( 8 - 13\% \) in the regime \( g < 1 \). The reaction \( D - D \) for an electron-deuteron plasma is increased by \( 1 - 2\% \). We find that these fluctuations have small, but not negligible amplitudes, compared to those at low frequencies. They exist, however, in a much larger range of phase space, up to very high frequencies. Their overall effect is to increase the screening potential.
Because the reaction rates are exponentially dependent on the screening potential, this increase is especially strong for reactions with relatively large $Z_1Z_2$, such as $p - ^7\text{Be}$ and $p - ^{14}\text{N}$. When summing the fluctuations only up to the plasma eigenfrequency $\omega \sim \omega_{pe}$, the Salpeter enhancement factor is obtained. We show here that the fluctuating potential has a large contribution from highly damped modes with frequencies $\omega > \omega_{pe}$.

There has been some controversy about the existence of a screening factor, which is different from the Salpeter factor, due to dynamic screening [9]. The effect described here is due to quasi-fluctuations that are heavily damped and that always exist in a plasma. They alter the Salpeter screening factor, although not necessarily in the same way as does dynamic screening.

This work shows that by taking into account all the modes, including those which are heavily damped, with frequencies higher than the plasma frequency, the screening factor of the nuclear reaction rates is altered and is larger than the Salpeter factor. The consequences of this work are left for future studies.

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**FIG. 1.** The expression $\left( F - F_{SAL} \right) / F_{SAL} (%)$ vs $g$ for an electron-proton plasma at a temperature $T = 10^7$ K. $F$ is the enhancing factor, which is compared to the Salpeter enhancing factor $F_{SAL}$. $g$ is the plasma parameter $1/n\lambda_D^3$. The solid curve is the $p - p$ reaction, the dashed curve is the $^3\text{He} - ^3\text{He}$ reaction, and the dotted curve, the $p - ^{14}\text{N}$ reaction.

**FIG. 2.** The expression $\left( F - F_{SAL} \right) / F_{SAL} (%)$ vs $g$ for an electron-proton plasma at a temperature $T = 10^8$ K. $F$ is the enhancing factor, which is compared to the Salpeter enhancing factor $F_{SAL}$. $g$ is the plasma parameter $1/n\lambda_D^3$. The solid curve is the $p - p$ reaction, the dashed curve is the $^3\text{He} - ^3\text{He}$ reaction, and the dotted curve, the $p - ^{14}\text{N}$ reaction.

**FIG. 3.** The expression $\left( F - F_{SAL} \right) / F_{SAL} (%)$ vs $g$ for an electron-deuteron plasma. $F$ is the enhancing factor, which is compared to the Salpeter enhancing factor $F_{SAL}$. $g$ is the plasma parameter $1/n\lambda_D^3$. The solid curve is the reaction $D - D$ for $T = 10^7$ K and the dashed curve, the reaction $D - D$ for $T = 10^8$ K.

**FIG. 4.** The potential $\langle \phi^2 \rangle_{\omega,k} / \langle \phi^2 \rangle_{\omega=0,k}$ vs $\omega / \omega_{pe}$ for the reaction $p - ^{14}\text{N}$ in an electron-proton plasma at $T = 10^7$ K and $n = 10^{24}$ cm$^{-3}$. The solid curve is for $k = 100 k_D$, the dashed curve is for $k = 10 k_D$, and the dotted curve, for $k = k_D$. 
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