Asymptotic Solutions of the Planar Squeeze Flow of a Herschel-Bulkley Fluid

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Abstract. In this study, we present the analysis of the squeeze flow of a Herschel-Bulkley fluid between parallel plates that are approaching each other with a constant squeeze motion. The classical lubrication analysis predicts the existence of a central unyielded zone bracketed between near-wall regions. This leads to the well-known squeeze flow paradox for viscoplastic fluids. Since the kinematic arguments show that there must be a finite velocity gradient even in the unyielded zone, thereby precluding the existence of such regions. This paradox may, however, be resolved within the framework of a matched asymptotic expansions approach where one postulates separate expansions within the yielded and apparently unyielded (plastic) zones. Based on the above technique, we circumvent the paradox, and develop complete asymptotic solutions for the squeeze flow of a Herschel-Bulkley fluid. We derive expressions for the velocity, pressure and squeeze force. The effects of the yield threshold on the pseudo-yield surface that separates the sheared and plastic zones, and squeeze force for different values of non-dimensional yield stress have been investigated.

1. Introduction
The squeeze film phenomenon occurs for the close approach of a pair of surfaces, and conforms to the classical lubrication paradigm. This approach leads to a sharp growth in the pressure within the narrow gap (between the surfaces), this growth being proportional to the fluid viscosity. While, squeeze flow problems have been analysed extensively for Newtonian fluids, we here consider the same for viscoplastic fluid between planar geometry. Bingham, Casson and Herschel-Bulkley are some of the most commonly used models to represent viscoplastic fluids. More details on these models can be found in Bird et.al [1] and Barnes [2] review articles. Here in this study, we use the Herschel-Bulkley model to represent the viscoplastic fluid, which is a generalized model of non-Newtonian fluids. For such fluids, flow occurs when the stress in the gap exceeds critical yield stress.

Practically, this type of flow behaviour occurs in many areas, such as slurries and suspensions, muds, clays, certain polymer solutions, lavas, etc. Therefore, such materials have applications in different fields, ranging from the oil, gas industries to the concrete used for construction. The non-Newtonian behaviour of fresh concrete, fly ash, mining slurries and cement fluids has been verified by conducting field and laboratory experiments [3-5]. However, the Herschel-Bulkley model is more suitable to represent the true rheological behaviour of viscoplastic fluids over a sufficiently wide range of shear rates. The flow behaviour of viscoplastic materials is important to analyse, due to the existence of yield surface which separates yielded and unyielded regions.

The geometry with small aspect ratio, having viscoplastic material as a fluid medium have a long history. Some of the researchers have analysed the squeeze flow of viscoplastic fluid and the presence of yield surface. But most of the earlier works [6-9] debate on the existence of true unyielded plug regions in the problem of squeeze flow of viscoplastic fluids. The classical lubrication theory leads to
the well-known “squeeze flow paradox”, since the velocity component of the unyielded plug region varies in the flow direction. This implies that true unyielded plug regions cannot exist. However the paradox can be resolved by using numerical simulations or approximate solutions.

Some of the investigators have used discretized approximations such as finite difference or finite element methods to analyse the viscoplastic squeeze flow problems. Initially Walton and Bittleston [16] have used this technique to study the Bingham plastics in a narrow eccentric annulus. Balmforth and Craster [17], later Frigaard and Ryan [18] and Putz et al. [19] have used asymptotic expansions to resolve the squeeze flow paradox and developed the consistent solution for thin-layer problems. Recently, Muravleva [20] has analysed the squeeze flow of a Bingham fluid in a planar geometry using numerical simulations and asymptotic expansions [17].

In the present work, we study the squeeze flow problem of a viscoplastic fluid in a planar geometry using the Herschel-Bulkley model by applying the technique of asymptotic expansions and also suggest corrections to resolve the squeeze flow paradox.

2. Mathematical formulation

The objective is to develop a consistent solution for the squeeze flow of an incompressible viscoplastic fluid for a planar geometry using Herschel-Bulkley fluid model, which resolves the squeeze flow paradox. Figure 1 shows the schematic of the squeeze flow problem. The gap width 2\(H\) between the plates of length 2\(L\) is filled with a viscoplastic material, while the plates approach each other with a constant velocity 2\(v_x\).

![Figure 1. Schematics of the Core formation.](image)

The governing equations of the problem in two dimensional form is as follows:

\[
\frac{\partial u^*}{\partial t^*} + u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = \frac{\partial p^*}{\partial x^*} - \frac{\partial \tau_{xx}^*}{\partial y^*} + \frac{\partial \tau_{xy}^*}{\partial y^*},
\]

(1)

\[
\frac{\partial v^*}{\partial t^*} + u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} = -\frac{\partial p^*}{\partial y^*} + \frac{\partial \tau_{xy}^*}{\partial x^*} + \frac{\partial \tau_{yy}^*}{\partial y^*},
\]

(2)

\[
0 = \frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*}.
\]

(3)

The constitutive equation for Herschel-Bulkley fluid model in three dimensional form is given by

\[
\tau_{ij}^* = \begin{cases} 
K|\dot{\gamma}_i^*|^{n-1} + \tau_0 & \text{for } |\dot{\gamma}_i^*| > \tau_0, \\
\dot{\gamma}_i^* & \tau_0 \leq |\dot{\gamma}_i^*| \leq \tau_0, \\
0 & \dot{\gamma}_i^* \leq \tau_0.
\end{cases}
\]
In the present work, \((u^*, v^*)\) is the velocity of the fluid, \(p^*\) is the pressure, \(\tau_{xx}^*, \tau_{xy}^*, \tau_{yx}^*, \text{and } \tau_{yy}^*\) are the components of deviatoric stress tensor. To non-dimesionalize the governing equations, we have used different scales in both \(x\) and \(y\) directions, with plate half-length \(L^*\) as the horizontal length scale and half thickness \(H^*\) as the vertical length scale respectively. \(v^*_v\) is taken as characteristic velocity in the transverse direction, the velocity in \(x\)- direction is scaled with \(c_*u\) and time with \(L^*/c_*u\). Here pressure is scaled with \(n\), \(c_*uH\), \(K\), \(\tau_0\) and \(\gamma_0\) respectively. Here \(\tau_{ij}^\#\) and \(\gamma_{ij}^\#\) are denoted by \(\tau^*\) and \(\gamma^*\) respectively, i.e., 

\[
\tau = \sqrt{\tau_{xy}^2 + \epsilon^2 \tau_{xx}^2}, \quad \gamma = \sqrt{\gamma_{xy}^2 + \epsilon^2 \gamma_{xx}^2}
\]

In the above equations dimensionless Herschel-Bulkley number \(N\) is defined by 

\[
N = \frac{\tau_0}{K^* \left( \frac{H^*}{u^*_c} \right)^n},
\]

where \(\tau_0\) is the yield stress.

The above equations (4)-(6) are solved by applying appropriate boundary conditions given below.
At \( y = 1 \Rightarrow u = 0, v = -1 \), at \( y = -1 \Rightarrow u = 0, v = 1 \), at \( x = 1 \Rightarrow \sigma_{xx} = -p + \varepsilon^2 \tau_{xx} = 0, \tau_{xy} = 0 \), in the plane of symmetry \( y = 0 \Rightarrow v = 0, \tau_{xy} = 0 \), and on the plane symmetry \( x = 0 \Rightarrow u = 0, \tau_{xy} = 0 \).

2.1. Asymptotic expansions

The equations (4)-(6) along with the boundary conditions are solved by introducing asymptotic expansions.

\[
u(x, y) = \nu^0(x, y) + \varepsilon u^1(x, y) + \varepsilon^2 u^2(x, y) + \cdots, \\
v(x, y) = \nu^0(x, y) + \varepsilon v^1(x, y) + \varepsilon^2 v^2(x, y) + \cdots, \\
p(x, y) = p^0(x, y) + \varepsilon p^1(x, y) + \varepsilon^2 p^2(x, y) + \cdots, \\
\tau(x, y) = \tau^0(x, y) + \varepsilon \tau^1(x, y) + \varepsilon^2 \tau^2(x, y) + \cdots.
\]

Substituting these expansions in equations (4)-(6) and comparing the leading order terms (i.e. \( \varepsilon^0 \) terms), we get

\[
0 = -\frac{\partial p^0}{\partial x} + \frac{\partial \tau_{xy}^0}{\partial y}, \\
0 = -\frac{\partial p^0}{\partial y}, \\
0 = \frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}.
\]

2.1.1. Shear region

Solving equation (8), we have \( p^0 = p_0(x) \) and from equation (7) along with the boundary condition, we have

\[
\tau_{xy}^0 = \frac{\partial p^0}{\partial x} y.
\]

And we have \( \tau^0 = \left| \tau^0 \right| \) and \( \dot{\gamma}^0 = \left| \frac{\partial u^0}{\partial y} \right| \). One can write leading order stress tensor components as

\[
\tau_{xy}^0 = \left[ \frac{\partial u^0}{\partial y} \right]^{n} + N \left| \frac{\partial u^0}{\partial y} \right|, \\
\tau_{xx}^0 = 2 \left[ \frac{\partial u^0}{\partial y} \right]^{n+1} \left[ \frac{\partial u^0}{\partial x} \right] + N \left| \frac{\partial u^0}{\partial x} \right|.
\]

Here velocity vanishes on the surface of the plate and maximum at the center in the domain \( x > 0 \) and \( y > 0 \). When the material is squeezed out, velocity becomes positive (\( u > 0 \)) implies \( \frac{\partial u^0}{\partial y} < 0 \).

Solving equation (11) with appropriate boundary condition, we have
\[ u^0 = \frac{n}{n+1} \left( \frac{N}{y_0} \right)^{\frac{1}{n+1}} \left[ (1-y_0)^{\frac{1}{n+1}} - (y-y_0)^{\frac{1}{n+1}} \right]. \] (13)

Now consider the first order approximations (i.e. \( \varepsilon^1 \) terms), we get

\[ 0 = -\frac{\partial p^1}{\partial x} + \frac{\partial r^1_{xy}}{\partial y}, \] (14)

\[ 0 = -\frac{\partial p^1}{\partial y}, \] (15)

\[ 0 = \frac{\partial u^1}{\partial x} + \frac{\partial v^1}{\partial y} \] (16)

and \( r^1_{xy} = n\left( \frac{\partial u^0}{\partial y} \right)^{n-1} \frac{\partial u^1}{\partial y} \), \( u^1 = 0 \). After integrating equations (14) and (15), we obtain

\[ p^1 = p_1(x), \quad r^1_{xy} = yp_1'(x) + g(x). \] (17)

Solving equation (17) with the boundary condition, we get

\[ u^1(x, y) = \frac{1}{n+1} \left( \frac{N}{y_0} \right)^{\frac{1}{n+1}} \left[ \frac{g(x)(n+1)}{N} \left( y-y_0 \right)^{\frac{1}{n+1}} - \left( 1-y_0 \right)^{\frac{1}{n+1}} \right] \] (18)

here \( g(x) \) is the unknown function of integration.

2.1.2. Plastic region

At leading order, for each \( x \) in \([0, 1]\), we have \( r^0_{xy} = yp_0'(x) \). The value of \( r^0_{xy} \) attains a maximum at the plane \( y=1 \) and vanishes at the plane \( y=0 \). Hence, there exists a plane \( y = y_0 \) at which \( r^0_{xy} = N \), where \( \frac{\partial u^0}{\partial y} = 0 \). Hence, at leading order \( y_0 = -N/p_0'(x) \) which is the position of the pseudo-yield surface. For \( y \in [0, y_0] \), we have \( r_0 < N \) and \( \dot{y}_0 = 0 \). Therefore, one can write the plug velocity from equation (13), i.e.,

\[ u^0 = \frac{n}{n+1} \left( \frac{N}{y_0} \right)^{\frac{1}{n+1}} (1-y_0)^{\frac{1}{n+1}}. \] (19)

To determine the pseudo-yield surface \( y = y_0(x) \), one can use integral form of equation (9), i.e.,

\[ \int_0^1 u^0(x, y) dy = x. \] (20)

Now substituting equations (13) and (19) into (20) leads an algebraic equation for the yield surface, \( y_0(x) \),

\[ \frac{(1-y_0)^{2+1/n}}{2+1/n} + \frac{n+1}{n} \left( \frac{y_0}{N} \right)^{\frac{1}{n+1}} x - (1-y_0)^{\frac{1}{n+1}} = 0. \] (21)
The algebraic equation (21) can be solved by using any of the numerical method to obtain the yield surface, \( y_0(x) \). From the equation (19), it can be observed that \( u^0(x) = u_0(y_0(x)) \) is purely a function of \( x \) such that, \( \frac{\partial u_0}{\partial x} \neq 0 \) where as \( \frac{\partial u_0}{\partial y} = 0 \).

From above relations, it is obvious that the leading order velocity \( u_0(x) \) changes in \( x \) direction which implies that the plug region is not a true plug region. This is the basis of the lubrication paradox for yield stress fluids. This problem arises due to the absence of diagonal components of the stress.

This paradox may, however, be resolved within the framework of a matched asymptotic expansions approach where one postulates separate expansions within the yielded and apparently unyielded (plastic) zones. The yielded zones conform to the lubrication paradigm with the shear stress being much greater than all other stress components. On the other hand, the shear and extensional stresses are comparable in the ‘plastic region’, with the overall stress magnitude being asymptotically close to but just above the yield threshold.

These shear and plastic regions are separated by an interface represented by a smooth pseudo-yield surface \( y = y_0(x) \). Let us consider the domain near the center plane of thickness \( 0 < y < y_0 \). Below the fake yield surface, \( y = y_0(x) \), the asymptotic expansion described above breaks down. To find the appropriate solution in this region, which incorporates changes in horizontal velocity component we modify \( u \) velocity as follows:

\[
 u(x, y) = u^0(x) + \omega u^1(x, y) + \varepsilon^2 u^2(x, y) + \cdots.
\]

Using these expansions, we can find stress components

\[
 \tau^{x^1}_{xx} = \frac{2N}{\gamma^0} \frac{\partial u^0}{\partial x}, \quad \tau^{y^1}_{yy} = \frac{2N}{\gamma^0} \frac{\partial v^0}{\partial y}, \quad \tau^{x^1}_{xy} = N \frac{\partial u^1}{\partial y}, \quad \text{where} \quad \gamma^0 = \sqrt{\left( \frac{\partial u^0}{\partial y} \right)^2 + 4 \left( \frac{\partial u^0}{\partial x} \right)^2}.
\]

From these components, we obtain

\[
 \tau^{x^1}_{xx} = N \sqrt{1 - \frac{y^2}{y_0^2}} \quad \text{and} \quad \tau^{y^1}_{xy} = N \frac{\partial u^1}{\partial y}.
\]

Solving for \( \frac{\partial u^1}{\partial y} \), we get

\[
 u^1 = 2u^1_0(x) \sqrt{y_0^2 - y^2} + u^1_0(x),
\]

where \( u^1_0(x) \) is an unknown function of integration.

For the first order approximation we have the following equations:

\[
 0 = -\frac{\partial p^1}{\partial x} + \frac{\partial \tau^{x^1}_{xx}}{\partial y} + \frac{\partial \tau^{x^1}_{xy}}{\partial x},
\]

\[
 0 = -\frac{\partial \tau^{x^1}_{xy}}{\partial y} + \frac{\partial \tau^{x^1}_{xx}}{\partial x},
\]

\[
 0 = -\frac{\partial u^1}{\partial x} + \frac{\partial v^1}{\partial y}.
\]
Solving above equations (23)-(24), we get
\[
\tau_{xy}^1(x, y) = y \psi'(x) + p_0^*(x) \left( y_0^2 \sin^{-1}(y / y_0) - y \sqrt{y^2 - y_0^2} \right),
\]
(26)

here \( \psi(x) \) is an unknown function of integration.

2.2. Matching

Using the technique of matching, one can calculate the unknown functions of integration. Since shear stress is continuous at \( y = y_0(x) \), one can obtain
\[
p_1(x) = \psi(x), \quad g(x) = N y_0'(x) \frac{\pi}{2}.
\]
(27)

Similarly from continuity of velocity at \( y = y_0(x) \), we get
\[
\dot{u}_1^*(x) = -\frac{1}{n+1} \left( \frac{N}{y_0} \right)^{1/n-1} (1 - y_0) \left( 1 + g(x) + p_1(x)(ny_0 + 1) \right).
\]
(28)

Substituting above integral constants in equations (13), (18), (19) and (22), we get the velocity distribution in both shear and plastic regions. Further, using the equation of continuity equation (6), one can calculate the unknown value \( p_1(x) \):
\[
p_1(x) = \frac{\pi (2n+1)(1+3n+2n^2)(1+ny_0) y_0 N \left( \frac{N}{y_0} \right)^{1/n}}{(1 - y_0)^{1/n}(2n^2 y_0^2 + 2ny_0 + n + 1)^2}.
\]
(29)

3. Pressure Distribution and Squeeze force

One of the important aspects of squeeze flow problem is squeeze force. To get this, we need to know the pressure distribution at the plate along the flow direction. The pressure gradient in shear region is given by
\[
\frac{\partial p^s}{\partial x} = \frac{\partial p^0}{\partial x} + \frac{\partial p^1}{\partial x}.
\]

Substituting \( p_0^0(x) \) and \( p_1(x) \), integrating above expression, we get the pressure distribution in shear region as follows:
\[
p^s(x) = C_1 + \frac{-nN(1+2ny_0)(1-y_0)^{1+1/n} \left( \frac{N}{y_0} \right)^{1/n}}{(1+n)(1+2n)y_0} - \epsilon \left[ (1+2n) \epsilon \left( \frac{1}{4n} \log(1+n+2ny_0+2n^2y_0^2) + \tan^{-1} \left( \frac{1+2ny_0}{\sqrt{1+2n}} \right) \right) \right].
\]
(30)

Similarly, one can obtain pressure distribution in plastic region as follows:
\[
p^p(x) = C_1 + \frac{-nN(1+2ny_0)(1-y_0)^{1+1/n} \left( \frac{N}{y_0} \right)^{1/n}}{(1+n)(1+2n)y_0} - \epsilon \left[ (1+2n) \epsilon \left( \frac{1}{4n} \log(1+n+2ny_0+2n^2y_0^2) + \tan^{-1} \left( \frac{1+2ny_0}{\sqrt{1+2n}} \right) \right) \right] - \epsilon N \left( \frac{y^2}{y_0^2} \right).
\]
(31)

The constant \( C_1 \) can be calculated using boundary condition \( p = 0 \) at \( x=1 \).
Further the squeeze force can be calculated by using the relation:

\[ F = 2\int_0^1 p(x)dx \]  

(32)

where \( p(x) \) can be substituted by using equations (30) and (31).

4. Results and Discussion

The core thickness along the principal flow direction for different values of Herschel-Bulkley number (N) and power law index (n) have been computed and the results are shown in figures 2 and 3. From these figures, we can observe that core thickness decreases along the horizontal direction from center plane \((x = 0)\) to the edge of the plate. The rate of change of core thickness increases with increase in Herschel-Bulkley number. Hence the core thickness in general for fluids with higher Herschel-Bulkley numbers is more than the fluids with lower Herschel-Bulkley numbers. Further, as power law index increases core thickness decreases and also the rate of change of core thickness increases with increase in power-law index.

![Figure 2. Core thickness formation at n=0.5.](image)

![Figure 3. Core thickness formation at N=10.](image)

Figures 4 and 5 show the results of the squeeze force for various values of aspect ratio \((\varepsilon)\), power-law index (n) and Herschel-Bulkley number (N). From these figures, we can observe that, squeeze force increases considerably with the increase in Herschel-Bulkley number for a particular value of aspect ratio. Also, squeeze force increases with increasing power-law index. Further, as aspect ratio increases squeeze force decreases for large Herschel-Bulkley numbers and the rate of decrease in squeeze force is marginal with increase in aspect ratio.

![Figure 4. Squeeze force distribution at n=0.5.](image)

![Figure 5. Squeeze force distribution at \(\varepsilon = 0.05\).](image)

5. Conclusions

The squeeze flow behaviour between two parallel plates lubricated by a Herschel-Bulkley fluid with constant squeeze motion is theoretically analysed using the technique of a matched asymptotic expansions. We obtain consistent asymptotic solutions which are free from “squeeze flow paradox”.

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The shape of the core thickness along the horizontal direction is determined numerically for various values Herschel-Bulkley number and power-law index.

It is found that core thickness increases with increase in Herschel-Bulkley number where as core thickness decreases with increase in power-law index.

Expressions for velocity and pressure are calculated analytically for various values of Herschel-Bulkley number, power-law index and aspect ratio.

Squeeze force results for various values of aspect ratio, Herschel-Bulkley number and power-law index have been calculated numerically.

The rate change of squeeze force for different values of Herschel-Bulkley number increases with increase in power-law index. Further, squeeze force decreases with increase in aspect ratio.

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