THE TORSION ADAMS SPECTRAL SEQUENCE FOR RATIONAL TORUS-EQUIVARIANT SPECTRA

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Abstract. We provide a calculational method for rational stable equivariant homotopy theory for a torus $G$ based on the homology of the Borel construction on fixed points. More precisely we define an abelian torsion model, $\mathcal{A}_t(G)$ of finite injective dimension, a homology theory $\pi_{A_t}^*(-)$ taking values in $\mathcal{A}_t(G)$ based on the homology of the Borel construction, and a finite Adams spectral sequence

$$Ext^{*,*}_{\mathcal{A}_t(G)}(\pi_{A_t}^*(X), \pi_{A_t}^*(Y)) \Rightarrow [X,Y]_G$$

for rational $G$-spectra $X$ and $Y$.

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1. Introduction

1.A. Context. A wide range of methods are available for constructing models of rational $G$-spectra and calculating with them. In various ways they are based on assembling information from the geometric $H$-fixed points as $H$ varies through the closed subgroups of $G$. From the point of view of transformation groups, the most natural thing to use as input for the contribution at $H$ is the homology of the Borel construction on the geometric $H$-fixed points. We construct an abelian category from this data, call it the torsion model, and prove that it gives an effective means of calculation.

In the case of the circle group $G$ two methods are discussed in [2]. The first is based on the standard abelian model $\mathcal{A}(G)$ and the second on the torsion model $\mathcal{A}_t(G)$, which is

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the rank 1 case of the model constructed here. Even in the rank 1 case, one can see the technical advantages of the standard model. The standard model is of injective dimension 1 and monoidal, and the torsion model is of injective dimension 2 and cannot be monoidal. Because of this, the focus when considering other groups has so far been on the standard model. Indeed, the full form of the torsion model has only been considered previously in the case of the circle group.

Nonetheless, the torsion model also has some advantages. As described above, the ingredients are very natural from the point of view of transformation groups. In fact they are also rather natural from the point of view of algebraic geometry, where they mirror the Cousin complex. Partly for these reasons, when identifying the algebraic model of a spectrum in the standard model it is often useful to approach through the torsion model. From the point of view of commutative algebra, the standard model is based on complete and Noetherian objects and their localizations whilst the torsion model is based on torsion and Artinian objects. This often means the vector spaces concerned tend to be smaller. For any of these reasons it is valuable to develop a torsion model.

The purpose of the present paper is to document the torsion model and its homological algebra when $G$ is a torus and to construct the Adams spectral sequence based on the torsion model. This is a preliminary step towards developing a model category of the same flavour as the torsion abelian category, since it suggests the general form of such a model. Ongoing work with Balchin, Pol and Williamson considers torsion models in the much more general context of tensor triangulated categories. The paper [1] is concerned with 1-dimensional Noetherian Balmer spectrum, and a special case gives an actual torsion model in the rank 1 case. Work on higher dimensional Noetherian Balmer spectra is underway, and it is hoped that it will provide a model which in the particular case of rational $G$-spectra for a torus $G$ will be a model categorical lift of the torsion abelian model considered here.

1.B. **Main results.** We will define (Section 3) an abelian category $\mathcal{A}_t(G)$ built from torsion modules over the polynomial rings $H^*(BG/K)$ for all subgroups $K$. The category $\mathcal{A}_t(G)$ is rather easy to work with: it has enough injectives (Section 5), it is of finite injective dimension (Proposition 7.2) and it is straightforward to make calculations. The category is precisely designed to be the codomain of a homology theory $\pi_*^{\mathcal{A}_t}: G\text{-spectra} \rightarrow \mathcal{A}_t(G)$ (Section 4), and the main theorem (Theorem 9.1) states that there is a finite Adams spectral sequence

$$\text{Ext}^{*,*}_{\mathcal{A}_t(G)}(\pi_*^{\mathcal{A}_t}(X), \pi_*^{\mathcal{A}_t}(Y)) \Rightarrow [X,Y]_G$$

strongly convergent for any rational $G$-spectra $X$ and $Y$.

1.C. **Notation.** The models assemble data from various subgroups, and it enormously aids readability to have consistent and suggestive notation. The ambient torus is $G$, and we generally let containment follow the alphabet as in $G \supseteq H \supseteq K \supseteq L$. One of the features of rational $G$-spectra is that one often needs to group all subgroups with the same component together. We often write $\bar{K}$ for a subgroup with identity component $K$ and so forth. We also write $\mathcal{F}$ for the family of finite subgroups of $G$ and $\mathcal{F}/K$ for the family of finite subgroups of $G/K$ (which is in bijection to the set of subgroups $\bar{K}$ of $G$ with identity component $K$.

The piece of data corresponding to a subgroup $K$ is built from the fixed point set on which $G/K$ acts, so we write $V(G/K)$ and index on the quotient group. We combine this...
with the above conventions, so that $V(\mathcal{F}/K)$ collects the data for $V(G/\bar{K})$ for all subgroups $\bar{K} \in \mathcal{F}/K$.

1.D. Organization. Section 2 recapitulates the rank 1 semifree case and should sensitize the reader to issues that will arise. Section 3 defines the abelian torsion model. Section 4 describes the homology functor on $G$-spectra taking values in $A_r(G)$. Section 5 begins the algebraic study of $A_r(G)$. Section 6 identifies sufficiently many injectives, giving a description involving local cohomology of various localized polynomial rings and residue maps between them. In Section 7 we show that for a non-trivial torus, the injective dimension of the torsion model is $\leq 2r$. Section 8 explains how to pick out the contribution from a specific subgroup by giving a corepresentation theorem. This then gives us all we need to construct a finite and strongly convergent Adams spectral sequence based on the homology of the Borel constructions of fixed points as stated: the pieces are assembled in Section 9.

2. THE CIRCLE GROUP

The special case when $G$ is the circle group (i.e., the rank $r = 1$) was covered in Chapter 6]. Nonetheless, it will be useful to run through the arguments, partly because the treatment is a little condensed in [2], and partly to motivate the general constructions we will need later.

To make the algebraic structures clearer, we work here with semifree $G$-spectra (i.e., those with geometric isotropy in $\{1, G\}$) and their model. Thus only $G$ and the trivial group 1 will play a role and diagrams are much smaller. The case of general $G$-spectra is covered along with other ranks below. The basic algebraic ingredient is a map of rings $O_F \longrightarrow \mathcal{E}_G^{-1}O_F$ where $\mathcal{E}_G$ is a multiplicatively closed set of $O_F$. This will be introduced in general later, but for semifree $G$-spectra it is the map of rings $k[c] \longrightarrow k[c, c^{-1}]$ for a field $k$ and an element $c$ of degree $-2$. We write $t = k[c, c^{-1}]$ for brevity and we will take $k = \mathbb{Q}$ for the topological applications.

2.A. Homological algebra. The objects of the semifree torsion abelian model $A_t^{sf}(G)$ are $X = (t \otimes V \longrightarrow T)$ where $V$ is a $k$-module, $t = k[c, c^{-1}]$, and $T$ is a torsion $k[c]$-module and we have a $k[c]$-map between them. In view of the adjunction $\text{Hom}_{k[t]}(t \otimes V, T) = \text{Hom}_k(V, \text{Hom}_{k[t]}(t, T))$, it is sometimes more convenient to consider the equivalent adjoint form of the torsion abelian category, with objects $\bar{X} = (V \rightarrow T^t)$ where $T^t = \text{Hom}_{k[t]}(t, T)$. Note that maps on the torsion parts $A^t \longrightarrow B^t$ are required to be of the form $\theta^t$ for a map $\theta : A \longrightarrow B$, and cokernels are $\text{cok}(A^t \longrightarrow B^t) = \text{cok}(A \longrightarrow B)^t$, which may not be the same as the ordinary cokernel of the module map $A^t \longrightarrow B^t$.

Of course we have objects $f^\theta_t(V) = (t \otimes V \longrightarrow 0)$ and $f^\theta_t(T) = (0 \longrightarrow T)$. The former is injective and for $T \neq 0$ the latter is not injective even if $T$ is an injective $k[c]$-module.

It is easy to see that $f^\theta_t$ is right adjoint to evaluation at $G$, and there is a functor $a_1$ right adjoint to evaluation at the trivial subgroup 1. This is more obvious in the adjoint form where we have $\tilde{a}_1(T) = (id : T^t \longrightarrow T^t)$. Thus $a_1(T) = (ev : t \otimes T^t \longrightarrow T)$. It is immediate from the adjunction that $a_1(I)$ is injective if $I$ is an injective torsion $O_F$-module.

Lemma 2.1. The injective dimension of $A_t^{sf}(G)$ is $\leq 2$.

Proof: Choose a resolution $0 \longrightarrow T \longrightarrow I \longrightarrow J \longrightarrow 0$ of $T$ by torsion injective $k[c]$-modules.
Now take the maps (i) \( X \rightarrow a_1(I) \), which is \( T \rightarrow I \) at \( G/1 \), and (ii) \( X \rightarrow a_G(V) \), which is the identity at \( G/G \). This gives a monomorphism \( X \rightarrow a_1(I) \oplus a_G(V) \). The cokernel \( C \) is \( J \) at \( G/1 \), and if we suppose it is \( V' \) at \( G/G \) we obtain a resolution
\[
0 \rightarrow X \rightarrow a_1(I) \oplus a_G(V) \rightarrow a_1(J) \oplus a_G(V') \rightarrow a_G(V'') \rightarrow 0.
\]

To see the injective dimension is exactly 2, we need to make a calculation.

**Lemma 2.2.** We have
\[
\operatorname{Ext}^s_{A_t(G)}(f_t^1(k), a_1(T)) = \begin{cases} 
\operatorname{Ext}^1_{k[t]}(t, T) & s = 2 \\
0 & \text{otherwise}
\end{cases}
\]

**Proof:** If \( T \) is a torsion \( k[c] \)-module with injective resolution
\[
0 \rightarrow T \rightarrow I \rightarrow J \rightarrow 0
\]
then \( \operatorname{Hom} \) and \( \operatorname{Ext} \) are given by the exact sequence
\[
0 \rightarrow T^t \rightarrow I^t \rightarrow J^t \rightarrow \operatorname{Ext}^{k[t]}(t, T) \rightarrow 0.
\]
We may now write down a resolution of \( a_1(T) \):
\[
0 \rightarrow a_1(T) \rightarrow a_1(I) \rightarrow a_1(J) \rightarrow a_G(T^t) \rightarrow 0
\]
where \( T^t = \operatorname{Ext}^1_{k[t]}(t, T) \). The answer is clear by applying \( \operatorname{Hom}_{A_t(G)}(f_t^1(k), \cdot) \) since if \( \tilde{Y} = (\tilde{q} : W \rightarrow U^t) \) we have
\[
\operatorname{Hom}_{A_t(G)}(f_t^1(k), X) = \ker(\tilde{q} : V \rightarrow U^t).
\]

One needs to think a little to identify a torsion module \( T \) with nonzero \( \operatorname{Ext}^1_{k[c]}(t, T) \). However
\[
\operatorname{Ext}^1(t, T) = R \lim \left[ \cdots \rightarrow \Sigma^{-4}T \xrightarrow{c} \Sigma^{-2}T \xrightarrow{c} T \right]
\]
so we see \( T = \bigoplus_{s \geq 1} \Sigma^{2s}k[c]/c^s \) will do.

**Remark 2.3.** The sum of injectives need not be injective. Indeed if we take
\[
I = \left[ \bigoplus_s \Sigma^{2s}k[c] \right]^* = \prod_s \Sigma^{2s}k[c]^*,
\]
we see that the map
\[
\bigoplus_s \operatorname{Hom}(t, \Sigma^{2s}k[c]^*) \rightarrow \operatorname{Hom}(t, \bigoplus_s \Sigma^{2s}k[c]^*) = \operatorname{Hom}(t, I)
\]
is not an isomorphism. Choosing an element \( \delta \) not in the image (such as the map diagonal on nonzero entries in each degree) we see that there is no solution to the problem
\[
0 \rightarrow f_1(k[c]^*) \rightarrow a_1(k[c]^*) \rightarrow a_1(k[c]^*) \rightarrow 0
\]
and hence $\bigoplus_s \Sigma^{2s} a_1(k[c]^*)$ is not injective.

2.B. The Adams spectral sequence. Writing sf-$G$-spectra for the homotopy category of semi-free rational $G$-spectra, we begin by defining the homology theory

$$\pi_*^{At}: \text{sf-G-spectra} \to \mathcal{A}_t^{sf}(G).$$

For this, we consider the isotropy separation sequence

$$E\mathcal{F}_+ \to S^0 \to \tilde{E}\mathcal{F} \to \Sigma E\mathcal{F}_+$$

and, since $DE\mathcal{F}_+ \wedge E\mathcal{F}_+ \simeq E\mathcal{F}_+$, and $X$ is semi-free,

$$\pi_*^{At}(X) = \pi_*^G(\tilde{E}\mathcal{F} \wedge DE\mathcal{F}_+ \wedge X \to \Sigma E\mathcal{F}_+ \wedge DE\mathcal{F}_+ \wedge X).$$

Of course we are more precisely using

$$\pi_*^G(\tilde{E}\mathcal{F} \wedge DE\mathcal{F}_+ \wedge X) \cong t \otimes \pi_*(\Phi^G X) =: t \otimes V_X$$

and, since $DE\mathcal{F}_+ \wedge E\mathcal{F}_+ \simeq E\mathcal{F}_+$, and $X$ is semi-free,

$$\pi_*^G(\Sigma E\mathcal{F}_+ \wedge DE\mathcal{F}_+ \wedge X) \cong \pi_*^G(\Sigma EG_+ \wedge X) \cong H_*^G(\Sigma^2 X) =: T_X$$

(where $H_*^G$ denotes homology of the Borel construction) so that

$$V_X = H_*(\Phi^G X)$$

and

$$T_X = \Sigma^2 H_*^G(X).$$

We thus have a map

$$d: [X, Y]^G_* \to \text{Hom}_{\mathcal{A}_t(G)}(\pi_*^{At}(X), \pi_*^{At}(Y)).$$

We will now proceed by the usual method towards an Adams spectral sequence.

The first question is whether we can realise enough injectives. We have made a start since $f_G^t(k) = a_G(k)$. Next we would like to realize $a_1(k[x]^*)$. We could use Brown representability as in the general case below (Lemma 9.2), but by way of variation, we give here a construction in terms of well known objects.

It is standard that $\pi_*^{At}(EG_+) = H_*^G(\Sigma EG_+) = \Sigma k[t]^*$ and therefore (care about suspensions) $\pi_*^{At}(EG_+) = \Sigma^2 f_1^t(k[x]^*)$. We break $a_1(k[x]^*)$ down using the short exact sequence

$$0 \to f_1^t(k[c]^*) \to a_1(k[c]^*) \to f_G^t((k[c]^*)^t) \to 0.$$ 

It is reassuring to note that $(k[c]^*)^t \cong t$, but in fact it is more helpful to retain the functional form $(k[x]^*)^t$. In any case, we see that $a_1(k[x]^*)$ is the fibre of a map

$$f_G^t(t) \to \Sigma f_1^t(k[c]^*),$$

so (with care again about suspensions) the realization should be the fibre of a map

$$\tilde{E}\mathcal{F}[t] \to \Sigma^{-1} EG_+,$$

where $[t]$ indicates the use of a module of graded coefficients. We may calculate the maps $\tilde{E}\mathcal{F} \to EG_+$ using the exact sequence

$$\cdots \to [\Sigma EG_+, EG_+]^G \to [\tilde{E}\mathcal{F}, EG_+]^G \to [S^0, EG_+]^G \to \cdots$$
A priori it is long exact, but the outer two groups are in odd degrees, so it is short exact. Since $S^\infty$ is a smash factor of $\widetilde{E}F$, multiplication by $c$ is an isomorphism in the middle we see $[\widetilde{E}F, \Sigma^{-1}EG_+]^G = t$. Indeed, we may take the map

$$t \otimes (k[c]^*)^t \to \Sigma^{-1}k[c]^*$$

to be evaluation. Following through the isomorphisms we see

$$a_1(k[c]^*) = \pi^A_*(\text{fibre}(ev : \widetilde{E}F[t] \to \Sigma^{-1}EG_+)).$$

**Theorem 2.4.** [2, Theorem 6.6.2] For rational semifree $G$-spectra $X$ and $Y$ there is an Adams spectral sequence

$$\text{Ext}_{A^t/G_*}^*(\pi^A_*(X), \pi^A_*(Y)) \Rightarrow [X, Y]_*^G.$$

This is a finite, strongly convergent spectral sequence.

**Proof:** As usual we need only show that enough injectives are realizable and that the $d$-invariant is an isomorphism when $Y$ is one of the realizable injectives. Convergence is clear since $\pi_*^A$ is detects contractibility and commutes with telescopes.

Now we need to show

$$[X, a_1(k[c]^*)]_*^G \to \text{Hom}_{A^t/G_*}(\pi^A_*(X), a_1(k[c]^*)) = \text{Hom}_{k[t]}(T(X), k[x]^*) = \text{Hom}_k(T(X), k)$$

This is straightforward since we can see immediately that $[f_G(k), a_1(k[x]^*)]_*^G = 0$. That in turn means we may assume $X$ is free and hence it suffices to take $X = G_+$. Now we only need to check that

$$d : [G_+, f'_G(k[c]^*)]_*^G = [G_+, EG_+]_*^G \to \text{Hom}_{k[t]}(\pi^G_*(G_+), \pi^G_*(EG_+))$$

is an isomorphism. The idea is to use the cofibre sequence $G_+ \to S^0 \to S^Z$ and connectivity: this is implemented in [3, Lemma 6.3]. □

3. The torsion model

We now begin work on the general case, so that $G$ is a torus of rank $r$. We will write down the category $A^t(G)$ directly, because this emphasizes the algebraic simplicity but many features will appear mysterious. We will return to explain the form of the definition in Section 4; the category is precisely designed as the appropriate receptacle for a torsion-based homology theory on rational $G$-spectra.

3.A. Inflation systems and Euler classes. We begin with a diagram of rings and some localizations. For the present we will restrict this to the context of our applications.

Starting with our torus $G$, we consider the poset of connected subgroups $H$ ordered by inclusion. We then have a diagram

$$O_{\mathcal{F}/} : \text{ConnSub}(G)^{op} \to \text{Rings}$$

to graded commutative rings. If $K \subseteq H$ the map $O_{\mathcal{F}/H} \to O_{\mathcal{F}/K}$ is called inflation, so we call $O_{\mathcal{F}/}$ the inflation diagram of rings.
For each $K \subseteq H$ we have a multiplicatively closed subset $\mathcal{E}_{H/K} \subseteq \mathcal{O}_{F/K}$, and these are compatible in the sense that, for $H \supseteq K \supseteq L$, the inflation of $\mathcal{E}_{H/K} \subseteq \mathcal{O}_{F/K}$ lies in $\mathcal{E}_{H/L} \subseteq \mathcal{O}_{F/L}$, and in fact

$$\mathcal{E}_{H/L} = \langle \mathcal{E}_{H/K}, \mathcal{E}_{K/L} \rangle.$$  

A $\mathcal{O}_{F/K}$-module $M$ is torsion if $\mathcal{E}_{H/K}^{-1}M = 0$ for every $H > K$.

There are two examples to bear in mind from equivariant homotopy theory. The first is easier to understand, and is often helpful as a warm-up. The second is the motivating example, and the notation we have used for an abstract inflation functor with Euler classes comes from it.

**Example 3.1.** The connected group inflation functor is the diagram of rings with $\mathcal{O}_{F/K} = H^*(BG/K)$ and $\mathcal{E}_H = \{ e_1(V) \mid VH = 0 \}$. Here $e_1(V) \in H^{|V|}(BG)$ is the classical Euler class.

**Example 3.2.** The full isotropy topological example has $\mathcal{O}_{F/K} = \prod K H^*(BG/K)$ (with the product over all subgroups $K$ with identity component $K$), and $\mathcal{E}_H = \{ e(V) \mid VH = 0 \}$. Here $e(V) \in \mathcal{O}_F$ has $F$-component the Euler class $e_F(V) = e_1(V^F) \in H^{|V^F|}(BG/F)$.

The ring $\mathcal{O}_{F/K}$ has one idempotent for each subgroup $e_K$, and hence any $\mathcal{O}_{F/K}$-module $M$ has summands $e_K$. The nature of the Euler classes means that any torsion module is a sum of these pieces: $M = \bigoplus_K e_K M$.

If $K$ is a subgroup of $H$ the inflation map $\mathcal{O}_{F/H} \longrightarrow \mathcal{O}_{F/K}$ requires the observation that for each subgroup $\tilde{K}$ with identity component $\tilde{K}$ there is a unique subgroup $\tilde{H}$ (namely $\tilde{H} = \tilde{K} \cdot H$) with with identity component $H$ with $K$ cotoral in $\tilde{H}$.

3.B. **The definition.** In the presence of an inflation diagram of rings, and an Euler system of multiplicatively closed subsets we may define a torsion category.

**Definition 3.3.** Objects of the abelian torsion model $\mathcal{A}_t(G)$ are cochain complexes

$$\mathcal{E}_{G}^{-1} \mathcal{O}_F \otimes V(\mathcal{F}/G) \xrightarrow{h^0} \bigoplus \mathcal{E}_{H}^{-1} \mathcal{O}_F \otimes \mathcal{O}_{F/H} V(\mathcal{F}/H) \xrightarrow{h^1} \bigoplus \mathcal{E}_{K}^{-1} \mathcal{O}_F \otimes \mathcal{O}_{F/K} V(\mathcal{F}/K) \xrightarrow{h^2} \ldots \xrightarrow{h^{r-2}} \bigoplus \mathcal{E}_{L}^{-1} \mathcal{O}_F \otimes \mathcal{O}_{F/L} V(\mathcal{F}/L) \xrightarrow{h^{r-1}} V(\mathcal{F}/1)$$

where the sums are over connected groups of the stated sort and $V(\mathcal{F}/K)$ is a torsion $\mathcal{O}_{F/K}$-module. The decomposition into direct sums and tensor products is a given part of the structure so the morphisms of $\mathcal{A}_t(G)$ are given by $\mathcal{O}_{F/K}$-maps $V(\mathcal{F}/K) \longrightarrow V'(\mathcal{F}/K)$ that give a map of cochain complexes.

It is convenient to formalize this a little further.

- For each connected subgroup $K$ we have a torsion $\mathcal{O}_{F/K}$-module $V(\mathcal{F}/K)$, which we call the $\mathcal{F}/K$ torsion component of the object.
- If $K \supseteq L$ there is an inflation map $\mathcal{O}_{F/K} \longrightarrow \mathcal{O}_{F/L}$ and an upward vertical map

$$v_{F/K} : V(\mathcal{F}/K) \longrightarrow \mathcal{E}_{K}^{-1} \mathcal{O}_{F/L} \otimes \mathcal{O}_{F/K} V(\mathcal{F}/K)$$

of $\mathcal{O}_{F/K}$-modules.
• If $K \supseteq L$ there is an inflation map $\mathcal{O}_{F/K} \longrightarrow \mathcal{O}_{F/L}$ and rightward *horizontal* structure maps

$$h_L^K : \mathcal{E}_K^{-1}\mathcal{O}_{F/L} \otimes \mathcal{O}_{F/K} V(F/K) \longrightarrow V(F/L)$$

of $\mathcal{O}_{F/L}$-modules

• For each connected subgroup $M$ the horizontal maps for subgroups containing $M$ form a cochain complex $C_{G/M}(F/M)$

$$E^{-1}_{G/M} \mathcal{O}_{F/M} \otimes V(F/G) \xrightarrow{h_0^{G/M}} \bigoplus_{\text{codim}(H/M)=1} \mathcal{E}_H^{-1} \mathcal{O}_{F/M} \otimes \mathcal{O}_{F/H} V(F/H) \xrightarrow{h_1^{G/M}} \bigoplus_{\text{codim}(L/M)=\text{dim}(G/M)-1} \mathcal{E}_L^{-1} \mathcal{O}_{F/M} \otimes \mathcal{O}_{F/L} V(F/L) \xrightarrow{h_{\text{dim}(G/M)-1}^{G/M}} V(F/M)$$

of $\mathcal{O}_{F/M}$-modules.

**Remark 3.4.** (i) We note that there are no suspensions on the objects $V(F/H)$. At present it seems odd to even comment on this, but it will be the basis of discussion when we return to consider $G$-spectra.

(ii) There is no direct relationship between the different modules $V(F/K)$. We can define a localized inflation diagram $\mathcal{L} \mathcal{I}$ of module categories

$$\mathcal{L} \mathcal{I}_G : \text{ConnSub}(G) \longrightarrow \text{Cat}$$

where $\mathcal{L} \mathcal{I}_G(G/H) = \mathcal{O}_{F/H}$-$\text{mod}$ and if $L \subseteq K$ then

$$\mathcal{L} \mathcal{I}_G(\pi_{G/K}^{G/L}) : \mathcal{L} \mathcal{I}_G(G/K) = \mathcal{O}_{F/K}$-$\text{mod} \longrightarrow \mathcal{O}_{F/L}$-$\text{mod} = \mathcal{L} \mathcal{I}_G(G/L)$$

is defined by

$$\mathcal{L} \mathcal{I}_G(\pi_{G/K}^{G/L})(M) = \mathcal{E}_K^{-1} \mathcal{O}_{F/L} \otimes \mathcal{O}_{F/K} M.$$ 

In this context $V$ is a section of the diagram $\mathcal{L} \mathcal{I}_G$.

**Example 3.5.** When $G$ is a circle, the diagram is very simple: an object is given by a diagram

$$\mathcal{E}_G^{-1} \mathcal{O}_F \otimes V(F/G) \longrightarrow V(F/1)$$

where $V(F/G)$ is a graded $\mathbb{Q}$-vector space and $V(F/1)$ is a torsion $\mathcal{O}_F$-module.

**Example 3.6.** In rank 2 we may still display the diagram.

In short form, $\mathcal{A}_t(G)$ is the category of cochain complexes of $\mathcal{O}_F$-modules

$$\mathcal{E}_G^{-1} \mathcal{O}_F \otimes V(F/G) \xrightarrow{h_0^H} \bigoplus_H \mathcal{E}_H^{-1} \mathcal{O}_F \otimes \mathcal{O}_{F/H} V(F/H) \xrightarrow{h_1^H} V(F/1),$$

where $V(F/G)$ is a $\mathbb{Q}$-module, $V(F/H)$ is a torsion $\mathcal{O}_{F/H}$-module and $V(F/1)$ is a torsion $\mathcal{O}_F$-module. More explicitly objects are actually part of a larger diagram with this as the
The top row is the cochain complex $C_G(F/1)$ of $O_F$-modules, the second row (which is really just one of the countably many second rows corresponding to codimension 1 connected subgroups $H$) is the cochain complex $C_{G/H}(F/H)$ of $O_{F/H}$-modules and the third row is the cochain complex $C_{G/G}(F/G)$ of $Q$-modules, since $O_{F/G} = Q$. We require that $V(F/G)$ is a $Q$-module, $V(F/H)$ is a torsion $O_{F/H}$-module in the sense that $E^{-1}_{G/H} V(F/H) = 0$ and $V(F/1)$ is a torsion $O_F$-module in the sense that $E^{-1}_{H} V(F/1) = 0$ for all circle subgroups $H$.

3.C. Maps into sums. Since the map from a sum to a product is a monomorphism, a map $\theta : A \rightarrow \bigoplus_i B_i$ is determined by the components $\theta_i : A \rightarrow B_i$. If $A$ is finitely generated, only finitely many of these are non-zero, but in general

$$\text{Hom}(A, \bigoplus_i B_i) = \lim_{\leftarrow \alpha} \text{Hom}(A_\alpha, \bigoplus_i B_i) = \lim_{\leftarrow \alpha} \bigoplus_i \text{Hom}(A_\alpha, B_i),$$

where $A_\alpha$ runs through finitely generated submodules of $A$. We say that $\{\theta_i\}_i$ is locally finite if it lies in this subgroup.

We may consider what this means for the horizontal maps

$$h : \bigoplus_K E^{-1}_{K} O_{F/L} \otimes O_{F/K} V(F/K) \rightarrow \bigoplus_L V(F/L).$$

Such a map $h$ is freely and uniquely determined by the components

$$h(K) : E^{-1}_{K} O_{F/L} \otimes O_{F/K} V(F/K) \rightarrow \bigoplus_L V(F/L)$$

by the universal property of the first sum. This map $h(K)$ in turn is determined by the maps

$$h^L_K : E^{-1}_{K} O_{F/L} \otimes O_{F/K} V(F/K) \rightarrow V(F/L)$$

but for each fixed $K$, the collection $h^L_K$ (where $*$ runs through the subgroups $L \subseteq K$) is subject to the condition of being locally finite.

In general it is probably easier to consider the map $h$ as a whole rather than decomposing it into factors.

3.D. Extra idempotents. The full-isotropy topological example of Example 3.2 has the feature that the ring $O_{F/K}$ contains idempotents for each subgroup $\tilde{K}$ with identity component $K$. It is a consequence of the torsion condition that the natural map from the sum of idempotent pieces gives an isomorphism

$$V(F/K) = \bigoplus_{\tilde{K}} V(G/\tilde{K}),$$
with $V(\tilde{K})$ a torsion $H^*(BG/\tilde{K})$-module. This is the reason for our notation, since of course we usually find $V(\mathcal{F}/K) \neq V(G/K)$.

This also allows us to explain the quotient notation: for comparison with subgroups we refer to quotients call it $V(G/H)$ after the ambient group. The point is that if $G \supseteq H \supseteq M$ we have a canonical isomorphism $(G/M)/(H/M) = G/H$.

This means that there is a second layer of sums available for decomposition, and we may consider the idempotent pieces $\tilde{h}_L^K$. These also determine the map $h$, but these are now subject to two separate sets of local finiteness conditions.

In fact the first condition is that $\tilde{h}_L^K$ is only nonzero when $\tilde{L}$ is cotoral in $\tilde{K}$. Then for a fixed subgroup $\tilde{K}$, the collection $\tilde{h}_L^K$ (where $*$ runs through subgroups $\tilde{L}$ cotoral in $\tilde{K}$) is locally finite. Furthermore, for each fixed $K$ we may write $h_L^K = \bigoplus_{\tilde{K}} \tilde{h}_L^K$ and then the collection $h^K_*$ is locally finite.

3.E. **Support and geometric fixed points.** The most visible feature of an object of $A_t(G)$ is where it is non-zero.

**Definition 3.7.** For an object $X$ of $A_t(G)$ the **connected support** is defined by

$$\text{supp}_c(X) = \{H \subseteq G \mid V(\mathcal{F}/H) \neq 0\}.$$ 

In the full isotropy example we may also define the **support**

$$\text{supp}(X) = \{\tilde{H} \subseteq G \mid V(G/\tilde{H}) \neq 0\}.$$ 

**Remark 3.8.** It is clear that in the full isotropy example the connected support can be recovered from the support

$$\text{supp}_c(X) = \{H \mid \text{there is a subgroup } \tilde{H} \in \text{supp}(X) \text{ with identity component } H\}.$$ 

For a connected subgroup of $G$ there is a functor

$$\Phi^K : A_t(G) \longrightarrow A_t(G/K),$$

obtained by picking out the part of the diagram below $K$ in the sense that if $H \supseteq K$

$$(\Phi^K X)((G/K)/(H/K)) = X(G/H).$$

**Example 3.9.** For example if

$$X = \left[ \mathcal{E}_G^{-1} \mathcal{O}_F \otimes V(\mathcal{F}/G) \xrightarrow{h^0} \bigoplus_H \mathcal{E}_H^{-1} \mathcal{O}_F \otimes \mathcal{O}_{F/H} V(\mathcal{F}/H) \xrightarrow{h^1} V(\mathcal{F}/1) \right]$$

we have

$$\Phi^H X = \left[ \mathcal{E}_G^{-1/H} \mathcal{O}_{F/H} \otimes V(\mathcal{F}/G) \longrightarrow V(\mathcal{F}/H) \right].$$

It is evident that

$$\text{supp}(\Phi^H X) = \text{supp}(X) \cap \{H \mid H \supseteq K\},$$

where we identify the subgroups of $G/H$ as those containing $K$.

4. **The torsion model and the torsion homology functor**

We return to the topology which motivated the definition of $A_t(G)$. Thus $G$ is a torus of rank $r$ and we consider $G$-spectra with arbitrary geometric isotropy. In this section we will define the homology functor $\pi_*^A : G\text{-spectra} \longrightarrow A_t(G)$. 


4.A. **Isotropy separation.** First we recall the filtration

\[ \emptyset \subset \mathcal{F}_{\leq 0} \subset \mathcal{F}_{\leq 1} \subset \mathcal{F}_{\leq 2} \subset \cdots \subset \mathcal{F}_{\leq r} = \text{All} \]

of the set of closed subgroups, where \( \mathcal{F}_{\leq s} = \{ K \mid \dim(K) \leq s \} \). Taking universal spaces we have the diagram

\[
\begin{array}{cccccccc}
* & \rightarrow & E(\emptyset)_+ & \rightarrow & E(\mathcal{F}_{\leq 0})_+ & \rightarrow & E(\mathcal{F}_{\leq 1})_+ & \rightarrow & \cdots & \rightarrow & E(\mathcal{F}_{\leq r})_+ = E\text{All}_+ = S^0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
E\langle 0 \rangle & & E\langle 1 \rangle & & E\langle 2 \rangle & & \cdots & & \downarrow & & E\langle r \rangle \\
\end{array}
\]

where \( E\langle s \rangle = \text{cofibre}(E(\mathcal{F}_{\leq r-1})_+ \rightarrow E(\mathcal{F}_{\leq r})_+) \).

Composing the vertical maps with the connecting maps, we obtain the sequence of maps

\[
E\langle r \rangle \rightarrow \Sigma E\langle r - 1 \rangle \rightarrow \cdots \rightarrow \Sigma^{r-1} E\langle 1 \rangle \rightarrow \Sigma^r E\langle 0 \rangle.
\]

This is a cochain complex in the sense that the composite of two adjacent maps is nullhomotopic.

Finally, by use of idempotents in Burnside rings, we have a rational splitting

\[
E\langle s \rangle \simeq \bigvee_{\dim(K) = s} E\langle K \rangle \simeq \bigvee_{\dim(K) = s} S^{\infty V(K)} \wedge E\mathcal{F}/K_+,
\]

where \( K \) runs through all subgroups of dimension \( s \) and \( \bar{K} \) runs through connected subgroups of dimension \( s \). As usual, \( E\langle K \rangle = \text{cofibre}(E[\subset \bar{K}]_+ \rightarrow E[\subseteq \bar{K}]_+) \) (with geometric isotropy the singleton \( \{ \bar{K} \} \)), and \( S^{\infty V(K)} = \bigcup_{V = 0} S^V \) (with geometric isotropy consisting of exactly those subgroups containing \( K \)).

**Example 4.1.** If \( r = 2 \) we may write this in more familiar terms

\[
\begin{array}{cccccccc}
* & \rightarrow & E\mathcal{F}_+ & \rightarrow & E\mathcal{P}_+ & \rightarrow & S^0 \\
E\mathcal{F}_+ & \rightarrow & \bigvee_{H} S^{\infty V(H)} \wedge E\mathcal{F}/H_+ & \rightarrow & S^{\infty V(G)} \\
\end{array}
\]

where \( \mathcal{F} \) is the family of finite subgroups and \( \mathcal{P} \) is the family of proper subgroups. This in turn gives a sequence

\[
S^{\infty V(G)} \rightarrow \Sigma S^{\infty V(H)} \wedge E\mathcal{F}/H_+ \rightarrow \Sigma^2 E\mathcal{F}_+
\]

in which the composite is null.

4.B. **Homotopy of pure strata.** The first approximation to \( \mathcal{A}_t(G) \) is obtained by taking the homotopy of the isotropy separation filtration. One can give a formula for the homotopy of the subquotients in terms of the homology of the Borel construction.

**Lemma 4.2.**

\[
\pi^G_*(E\langle s \rangle \wedge X) = \Sigma^{r-s} \bigoplus_{\dim(K) = s} H^G_*(\bar{K}_{\Phi \bar{K}} X)
\]

where the sum is over all subgroups \( \bar{K} \) of dimension \( s \). The term \( H^G_*(\bar{K}_{\Phi \bar{K}} X) \) is a torsion \( H^*(BG/\bar{K}) \)-module.
The $E_1$-term of the spectral sequence of the filtration is the homotopy of the sequence

\[ E(r) \land X \longrightarrow \Sigma E(r-1) \land X \longrightarrow \cdots \longrightarrow \Sigma^{r-1} E(1) \land X \longrightarrow \Sigma^r E(0) \land X \]

namely

\[ \pi_*^G(E(r) \land X) \longrightarrow \pi_*^G(\Sigma E(r-1) \land X) \longrightarrow \cdots \longrightarrow \pi_*^G(\Sigma^r E(0) \land X) \]

\[ H_*(\Phi^G X) \longrightarrow \Sigma^2 \bigoplus_{\dim(\tilde{H})=r-1} H_*^{G/\tilde{H}}(\Phi^\tilde{H} X) \longrightarrow \cdots \longrightarrow \Sigma^{2r} \bigoplus_{\dim(F)=0} H_*^{G/F}(\Phi^F X) \]

In geometric terms this is the direct analogue of the Cousin complex. In order to get something more algebraic we need to smash the whole thing with $DE\mathcal{F}_+$ first.

**Lemma 4.3.**

\[ \pi_*^G(E(s) \land D\mathcal{F}_+ \land X) = \Sigma^{r-s} \bigoplus_{\dim(\tilde{K})=s} E^{-1}_{K} \mathcal{O}_{\mathcal{F}} \otimes_{H^*(BG/\tilde{K})} H_*^{G/\tilde{K}}(\Phi^\tilde{K} X), \]

where the sum is over all subgroups $\tilde{K}$ of dimension $s$. The term $H_*^{G/\tilde{K}}(\Phi^\tilde{K} X)$ is a torsion $H^*(BG/\tilde{K})$-module.

**Remark 4.4.** If $T$ is a module over $H^*(BG/\tilde{K})$ then

\[ \mathcal{O}_{\mathcal{F}} \otimes_{H^*(BG/\tilde{K})} T = [e_{\tilde{K}} \mathcal{O}_{\mathcal{F}}] \otimes_{H^*(BG/\tilde{K})} T \]

where $e_{\tilde{K}}$ is the idempotent supported on finite subgroups cotoral in $\tilde{K}$.

4.C. **Collecting subgroups by identity component.** In this section so far, we have treated all subgroups of the same dimension equally. However the behaviour of Euler classes (specifically the fact that $e(\alpha^n) = ne(\alpha)$, and we are working rationally) means that it becomes important to collect together the subgroups according to their identity component. The convention is that letters $G, H, K, \ldots$ will be connected subgroups, and $\tilde{H}$ is a subgroup with identity component $H$, $\tilde{K}$ is a subgroup with identity component $K$, and so forth.

4.D. **The homology functor.** It should be apparent that the structures in $\mathcal{A}_t(G)$ mirror those in topology. The following definition should therefore not be a surprise.

**Definition 4.5.**

\[ \pi_*^{\mathcal{A}_t} : G\text{-spectra} \longrightarrow \mathcal{A}_t(G) \]

is defined by taking $C_G(\mathcal{F})$ to be $\pi_*^G(\mathcal{D}\mathcal{F}_+ \land \cdot)$ to the sequence

\[ E(r) \land X \longrightarrow \Sigma E(r-1) \land X \longrightarrow \cdots \longrightarrow \Sigma^{r-1} E(1) \land X \longrightarrow \Sigma^r E(0) \land X \]

Specifically

\[ V_X(\mathcal{F}/H) = \bigoplus_{\tilde{H}} V(G/\tilde{H}) \]

and

\[ V_X(G/\tilde{H}) = \pi_*^G(\Sigma^{\dim(G/H)} E(\tilde{H}) \land X) = \Sigma^{2\dim(G/H)} H_*^{G/\tilde{H}}(\Phi^\tilde{H} X) \]

**Lemma 4.6.** This definition does give a functor to $\mathcal{A}_t(G)$. 

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**Proof:** The fact that the modules are torsion and the maps are of the correct form is the content of Lemma 4.3. It is clear that $C_G(F)$ is a cochain complex since the composite of two morphisms is null-homotopic. Finally, we need to observe that we have the appropriate comparison map $C_{G/K}(F/K) \to C_{G/L}(F/L)$ when $L \subseteq K$. It evidently suffices to treat the case $L = 1$, and at the point $G/K$ we need the vertical map

$$
\varepsilon^{-1}_K \mathcal{O}_F \otimes_{\mathcal{O}_{F/K}} V(F/K) \xrightarrow{\cong} \pi_*^G(S^{\infty V(K)} \wedge D\mathcal{F}_+ \wedge E\mathcal{F}/K_+ \wedge X) \\
V(F/K) \xrightarrow{\cong} \pi_*^{G/K}(E\mathcal{F}/K_+ \wedge \Phi^K X) \xrightarrow{\cong} \pi_*^G(S^{\infty V(K)} \wedge E\mathcal{F}/K_+ \wedge X)
$$

It is apparent this is induced by the map $S^0 \to D\mathcal{F}_+$.

All of the justifications are natural in $X$, so this does give a functor. □

**Remark 4.7.** Note the conventions on suspensions. In transformation groups, the modules $H_*^{G/K}(\Phi^K X)$ occur naturally. We could have arranged that they occur in the model without suspensions, but for general groups this would lead to confusion. Instead, with our conventions, we find that in $\pi_*^{A_t}(X)$ we have

$$V_X(G/K) = \Sigma^{2 \dim(G/K)} H_*^{G/K}(\Phi^K X),$$

so that the contribution from a subgroup occurs suspended by twice its codimension.

For example in rank 1, we take the homotopy of

$$D\mathcal{F}_+ \wedge \mathcal{E} \mathcal{F} \wedge X \to \Sigma E\mathcal{F}_+ \wedge X$$

giving the object

$$\varepsilon^{-1}_G \mathcal{O}_F \otimes H_*(\Phi^G X) \to \bigoplus_F \Sigma^{2} H_*^{G/F}(\Phi^F X)$$

Thus we have

$$V_X(G/G) = H_*(\Phi^G X), V_X(G/1) = \Sigma^{2} \bigoplus_F H_*^{G/F}(\Phi^F X).$$

**Remark 4.8.** One of the attractive features of the torsion model is that it directly reflects the geometric isotropy. It is immediate that

$$\mathcal{I}_G(X) = \text{supp}(\pi_*^{A_t}(X)).$$

5. **Algebra of the torsion model**

We now turn to an algebraic study of the torsion model. Since our purpose is to define an Adams spectral sequence, it is not surprising that this is mostly about homological algebra.

5.A. **Skyscraper objects.** We begin by giving a name to the most obvious and elementary construction.

**Definition 5.1.** If $T$ is a torsion $\mathcal{O}_{F/H}$-module, we write $f^i_H(T)$ for the object

$$f^i_H(T)(F/K) = \begin{cases} 
\varepsilon^{-1}_H \mathcal{O}_F \otimes_{\mathcal{O}_{F/H}} T & \text{if } K = H \\
0 & \text{if } K \neq H
\end{cases}$$
Evidently if $T \neq 0$, the object $f_H^t(T)$ corresponds to a $G$-spectrum with connected geometric isotropy $\{H\}$. In particular

$$\pi^A_*(E(\tilde{H})) = f_H^t(H_*(BG/\tilde{H})),$$

so their importance is clear. However, even though $H_*(BG/\tilde{H})$ is an injective $O_{F/H}$-module, from an algebraic point of view this object is not particularly simple.

**Lemma 5.2.** For a torsion $O_{F/K}$-module $T$,

$$\text{Hom}_{A_t(G)}(X, f_K^t(T)) = \text{Hom}_{O_{F/K}}(C_K X, T)$$

where

$$C_K X = \text{cok}(\bigoplus_{H > K} E_{H/K}^{-1} O_{F/K} \otimes_{O_{F/H}} V_X(F/H) \longrightarrow V_X(F/K)).$$

We see that $C_G X = V_X(F/G)$, so that $f_G^t(W(F/G))$ is always injective, but for $H \neq G$ the object $f_H^t(T)$ is never injective unless it is zero.

**5.B. Adjoint form.** It is invaluable to have a right adjoint to evaluation at a particular subgroup $H$, and in writing these down we would like to work with adjoint form of torsion diagrams. The basic idea is as in the rank 1 case, but this is complicated by the fact that there are infinitely many connected subgroups in most dimensions.

**Example 5.3.** We make this explicit in the case $G$ has rank 2. Suppose then that

$$X = \left[ E_{G}^{-1} O_{F} \otimes V \xrightarrow{h^0} \bigoplus_{H} E_{H}^{-1} O_{F} \otimes_{O_{F/H}} T(H) \xrightarrow{h^1} S \right]$$

where $V$ is a $\mathbb{Q}$-vector space $T(H)$ is a torsion $O_{F/H}$-module and $S$ is a torsion $O_F$-module (against our general principle, we are abbreviating $T(F/H) = T(H)$). We consider first the case when only finitely many of the $T(H)$ are non-zero. The adjoint form is then

$$V \longrightarrow \bigoplus_{H} \text{Hom}_{O_{F/H}}(E_{G/H}^{-1} O_{F/H}, T(H)) \longrightarrow \text{Hom}_{O_F}(E_G^{-1} O_F, S)$$

In long form, $X$ is given by a diagram

$$\begin{array}{ccc}
E_{G}^{-1} O_{F} \otimes V & \xrightarrow{h^0} & \bigoplus_{H} E_{H}^{-1} O_{F} \otimes_{O_{F/H}} T(H) \xrightarrow{h^1} S \\
| & \Downarrow & | \\
E_{G/H}^{-1} O_{F/H} \otimes V & \xrightarrow{h^0_H} & T(H) \\
| & \Downarrow & | \\
V & \xrightarrow{i} & \\
\end{array}$$
We convert this to adjoint form

\[
\begin{aligned}
\bigoplus_H T(H) & \longrightarrow \text{Hom}_\mathcal{O}_F(\mathcal{E}_G^{-1}\mathcal{O}_F, S) \\
V & \longrightarrow \bigoplus_H \text{Hom}_{\mathcal{O}_F/H}(\mathcal{E}^{-1}_G\mathcal{O}_{F/H}, T(H)) \longrightarrow \text{Hom}_{\mathcal{O}_F}(\mathcal{E}_G^{-1}\mathcal{O}_F, S)
\end{aligned}
\]

where, for the purpose of understanding the bottom right entry, it is worth noting

\[
\text{Hom}_{\mathcal{O}_F}(\mathcal{E}_G^{-1}\mathcal{O}_F, S) = \text{Hom}_{\mathcal{O}_F}(\mathcal{E}^{-1}_G\mathcal{O}_F \otimes_{\mathcal{O}_{F/H}} \mathcal{E}^{-1}_G\mathcal{O}_{F/H}, S) = \\
\text{Hom}_{\mathcal{O}_{F/H}}(\mathcal{E}_G^{-1}\mathcal{O}_{F/H}, \text{Hom}_{\mathcal{O}_F}(\mathcal{E}_G^{-1}\mathcal{O}_F, S))
\]

When infinitely many of the \(T(H)\) are non-zero, the map from \(V\) maps into the product, but it is a locally finite map. Accordingly, it is still sufficient to describe the second map as coming from the sum.

It is possible in principle to describe the adjoint form in full detail. However our narrow purpose is to define a right adjoint to evaluation, so we will just record what the adjoint form does on components.

**Definition 5.4.** Given a flag

\[ G \supset H_{r-1} \supset H_{r-2} \supset \cdots \supset H_1 \supset H_0 = 1 \]

of connected subgroups, the corresponding component of the torsion model is the cochain complex

\[
\begin{aligned}
\mathcal{E}^{-1}_G\mathcal{O}_F \otimes V(\mathcal{F}/G) & \longrightarrow \mathcal{E}^{-1}_{H_{r-1}}\mathcal{O}_F \otimes_{\mathcal{O}_{F/H_{r-1}}} V(\mathcal{F}/H_{r-1}) \\
& \longrightarrow \mathcal{E}^{-1}_{H_{r-2}}\mathcal{O}_F \otimes_{\mathcal{O}_{F/H_{r-2}}} V(\mathcal{F}/H_{r-2}) \longrightarrow \cdots \longrightarrow \mathcal{E}^{-1}_{H_1}\mathcal{O}_F \otimes_{\mathcal{O}_{F/H_1}} V(\mathcal{F}/H_1) \longrightarrow V(\mathcal{F}/1)
\end{aligned}
\]

Its adjoint form is

\[
\begin{aligned}
V(\mathcal{F}/G) & \longrightarrow \text{Hom}_{\mathcal{O}_{F/H_{r-1}}}(\mathcal{E}^{-1}_{G/H_{r-1}}\mathcal{O}_{F/H_{r-1}}, V(\mathcal{F}/H_{r-1})) \\
& \longrightarrow \text{Hom}_{\mathcal{O}_{F/H_{r-2}}}(\mathcal{E}^{-1}_{G/H_{r-2}}\mathcal{O}_{F/H_{r-2}}, V(\mathcal{F}/H_{r-2})) \longrightarrow \cdots \\
& \longrightarrow \text{Hom}_{\mathcal{O}_{F/H_1}}(\mathcal{E}^{-1}_{G/H_1}\mathcal{O}_{F/H_1}, V(\mathcal{F}/H_1)) \longrightarrow \text{Hom}_{\mathcal{O}_F}(\mathcal{E}^{-1}_G\mathcal{O}_F, V(\mathcal{F}/1))
\end{aligned}
\]

**Remark 5.5.** Even if \(V(\mathcal{F}/H)\) is \(\mathcal{F}/H\)-torsion, it does not follow that \(\text{Hom}_{\mathcal{O}_{F/H}}(\mathcal{E}^{-1}_G\mathcal{O}_{F/H}, V(\mathcal{F}/H))\) is \(\mathcal{F}/H\)-torsion.

5.C. **Right adjoints to evaluation.** From an algebraic point of view the simplest behaviour comes from objects that let us calculate in terms of rings rather than diagrams of rings. For this we use right adjoints to evaluation at a subgroup.

The idea is that the adjoint forms are constant above a given point. This is correct if the subgroup in question is of codimension \(\leq 1\), but not in general, because of the need to use torsion objects and the distinction between sums and products.
Example 5.6. If $G$ is of rank 2, the functors representing evaluation are as follows.

\[ a_G(V) = f_G(V) = \left[ \mathcal{E}_G^{-1}O_F \otimes V \rightarrow 0 \rightarrow 0 \right] \]

\[ a_H(T(H)) = \left[ \mathcal{E}_G^{-1}O_F \otimes \text{Hom}_{O_{F/H}}(\mathcal{E}_G^{-1}O_{F/H}, T(H)) \rightarrow \mathcal{E}_H^{-1}O_F \otimes O_{F/H} T(H) \rightarrow 0 \right] \]

\[ a_1(S) = \left[ \mathcal{E}_G^{-1}O_F \otimes \Gamma \Sigma \text{Hom}_{O_{F/L}}(\mathcal{E}_G^{-1}O_F, S) \xrightarrow{h^0} \bigoplus_H \mathcal{E}_H^{-1}O_F \otimes O_{F/H} \Gamma \Sigma \text{Hom}_{O_{F/L}}(\mathcal{E}_H^{-1}O_F, S) \xrightarrow{h^1} S \right] \]

where $\Gamma_{F/H}$ indicates torsion $O_{F/H}$-modules and $\Gamma \Sigma$ refers to the elements mapping into the torsion submodules, and into the sum rather than the product. These two phenomena occur in rank 2 for the first time.

Equipped with this example we can define the objects $a_L(T)$.

Definition 5.7. Given a torsion $O_{F/L}$-module $T$, we may define an object $a_L(T)$ by taking its torsion components to be

\[ V(F/H) = \begin{cases} \Gamma \Sigma \Gamma_{F/H} \text{Hom}_{O_{F/L}}(\mathcal{E}_G^{-1}O_{F/L}, T) & \text{if } K \supseteq L \\ 0 & \text{if } K \not\supseteq L \end{cases} \]

The functor $\Gamma_{F/H}$ takes the torsion submodule and $\Gamma \Sigma$ takes the submodule mapping into the sum; we will describe them fully in the next subsection and prove these properties (Lemma 5.10), but for the present we need only know they give natural submodules of the Hom functor.

Lemma 5.8. The functor $a_L$ is right adjoint to evaluation at $L$:

\[ \text{Hom}_{A_t(G)}(X, a_L(T)) = \text{Hom}_{O_{F/L}}(V_X(F/L), T) \]

Proof: We describe the counit and unit of the adjunction. The triangular identities will follow from those of the Hom-Tensor adjunction.

The counit $ev_L a_L(T) \rightarrow T$ is the identity. For the unit

\[ X \rightarrow a_L(V_X(F/L)), \]

let us suppose that $L$ is of codimension $t$ and that $r \geq s \geq t$. In the display, $H$ runs through subgroups of codimension $s + 1$ containing $L$ and $K$ runs through subgroups of codimension $s$ containing $L$.

\[ \bigoplus_H a_L(T)(F/H) \xrightarrow{\text{can}} \bigoplus_K a_L(T)(F/K) \]

\[ \bigoplus_{H \geq L} \mathcal{E}_H^{-1}O_{F/L} \otimes O_{F/H} \Gamma \Sigma \Gamma_{F/H} \text{Hom}_{O_{F/L}}(\mathcal{E}_H^{-1}O_{F/L}, T) \xrightarrow{\text{can}} \bigoplus_{K \geq L} \Gamma \Sigma \Gamma_{F/L} \text{Hom}_{O_{F/L}}(\mathcal{E}_K^{-1}O_{F/L}, T) \]
5.D. **The torsion subfunctor.** Certain limit constructions do not preserve torsion objects, or do not preserve the property of mapping into the sum of components, so it is useful to formalize a bigger category in which the constructions can be made, together with a right adjoint for returning to \( \mathcal{A}_t(G) \).

**Definition 5.9.** The category \( \hat{\mathcal{A}}_t(G) \), has objects \( X \) consisting collections \( \{V(F/K)\}_K \) with \( V(F/K) \) an \( \mathcal{O}_{F/K} \)-module with structure maps

\[ h_L^K : \mathcal{E}^{-1}_{K/L} \mathcal{O}_{F/L} \otimes_{\mathcal{O}_{F/K}} V(F/K) \rightarrow V(F/L). \]

Taking components gives an obvious inclusion functor

\[ i : \mathcal{A}_t(G) \rightarrow \hat{\mathcal{A}}_t(G). \]

**Lemma 5.10.** The inclusion \( i \) has a right adjoint \( \Gamma : \hat{\mathcal{A}}_t(G) \rightarrow \mathcal{A}_t(G) \).

**Proof:** The functor \( \Gamma \) is the composite \( \Gamma = \Gamma'' \Gamma' \). The functor \( \Gamma' \) takes the torsion submodule \( \Gamma_{F/K}V(F/K) \) at \( K \). This is compatible with the structure maps, since any \( F/K \)-torsion element is \( F/L \)-torsion for \( L \leq K \).

The functor \( \Gamma'' \) is defined by taking \( \Gamma_{\Sigma} \) at each point, where \( \Gamma_{\Sigma} \) can be defined by induction on the dimension of \( H \) using the pullback square

\[
\begin{array}{ccc}
\Gamma_{\Sigma} \Gamma_{F/H} \text{Hom}_{\mathcal{O}_{F/L}}(\mathcal{E}^{-1}_{H/L} \mathcal{O}_{F/L}, T) & \rightarrow & \text{Hom}_{\mathcal{O}_{F/K}}(\mathcal{E}^{-1}_{H/K} \mathcal{O}_{F/K}, \bigoplus_{K \leq H} \Gamma_{\Sigma} \Gamma_{F/K} \text{Hom}_{\mathcal{O}_{F/L}}(\mathcal{E}^{-1}_{K/L} \mathcal{O}_{F/L}, T)) \\
\downarrow & & \downarrow \\
\text{Hom}_{\mathcal{O}_{F/L}}(\mathcal{E}^{-1}_{H/L} \mathcal{O}_{F/L}, T) & \rightarrow & \text{Hom}_{\mathcal{O}_{F/K}}(\mathcal{E}^{-1}_{H/K} \mathcal{O}_{F/K}, \prod_{K \leq H} \text{Hom}_{\mathcal{O}_{F/L}}(\mathcal{E}^{-1}_{K/L} \mathcal{O}_{F/L}, T))
\end{array}
\]

In particular this lemma formalizes Definition 5.7.

6. **Injectives**

We need to show there are enough injectives and describe them in a way that allows us to do computations. Since \( a_K \) is a right adjoint, it is obvious that, for any torsion injective \( \mathcal{O}_{F/K} \)-module \( I \), the object \( a_K(I) \) is injective in the torsion model. Indeed, this is the main reason for introducing the \( a_K \) construction.

Next, we know that there are enough injective torsion \( \mathcal{O}_{F/K} \)-modules, which can be constructed as sums of those of the form \( H_*(BG/\tilde{K}) \), where \( \tilde{K} \) runs through subgroups with identity component \( K \).

Finally it is both illuminating and convenient to give a more explicit description of the functors \( a_K(H_*(BG/\tilde{K})) \). After some recollections about Gorenstein rings in Subsections 6.B and 6.C, we give the description in Subsection 6.D.

6.A. **Enough injectives.** We wish to construct enough injectives by taking products of those with explicit constructions.

**Corollary 6.1.** The category \( \mathcal{A}_t(G) \) has products.
Proof: Given objects $X_i$ of $\mathcal{A}_i(G)$ we wish to say that the $K$-torsion component of the product $\prod_i X_i$ is $\prod_i V_i(\mathcal{F}/K)$. This lies in $\mathcal{A}_i(G)$, so by Lemma [5.10] we may apply the right adjoint $\Gamma$ to obtain an object of $\mathcal{A}_i(G)$. □

Lemma 6.2. There are enough injectives which are products of injectives of the form $a_K(I)$ for torsion injective $\mathcal{O}_{\mathcal{F}/K}$-modules $I$.

Remark 6.3. Exactly as in the rank 1 semifree case (Remark [2.3]), an arbitrary sum of injectives need not be injective.

Proof: For each subgroup $\tilde{K}$, there are enough torsion injective $H^*(BG/\tilde{K})$-modules formed as direct sums of $H_*(BG/\tilde{K})$. This means that for any $X$ there is a monomorphism $V(\mathcal{F}/K) \rightarrow I(\mathcal{F}/K)$ where $I(\mathcal{F}/K)$ is a sum of $\mathcal{O}_{\mathcal{F}/K}$-modules $H_*(BG/\tilde{K})$. There is a corresponding map $X \rightarrow a_K(I(\mathcal{F}/K))$ monomorphic at $K$, and hence a monomorphism $X \rightarrow \prod_k a_K(I(\mathcal{F}/K))$. □

6.B. Gorenstein duality. One particular case of the $a_K$ construction is especially important: the one supplying enough injectives. Indeed, we know that there are enough injective torsion $\mathcal{O}_{\mathcal{F}/K}$-modules of the form $H_*(BG/\tilde{K})$, where $\tilde{K}$ runs through subgroups with identity component $K$. It is illuminating to have a description of the entries in $a_K(H_*(BG/\tilde{K}))$ in simple terms. We are able to do this using the fact that polynomial rings are Gorenstein. The basic idea is very simple, but there are some issues to highlight along the way.

The ring $H^*(BG)$ is Gorenstein and

$$H^*_m(H^*(BG)) = H^*_m(H^*(BG)) \cong \sum^r \text{Hom}_k(H^*(BG), k) = \sum^r H_*(BG).$$

There are two notable things about this. First, the isomorphism is not natural in the sense that it should be twisted by the determinant in order to be natural for ring isomorphisms. Second, the Hom$_k$ consists of graded maps. If we used all maps we would obtain a completion of $H_*(BG)$, and we would need to pass to cellularizations to recover $H_*(BG)$ itself.

We need something more general. For a Gorenstein local ring $R$ of dimension $r$ we have

$$H^*_m(R) = H^*_m(R) \cong E_R(k)$$

where $E_R(k)$ is the injective hull of the residue field $k$. If $R$ happens to be a $k$-algebra, we may form $E_R(k) = \text{Hom}_k(R, k)$, and the map $k = \text{Hom}_k(k, k) \rightarrow \text{Hom}_k(R, k) = E_R(k)$ extends to an embedding $E_R(k) \subseteq \hat{E}_R(k)$. However we note that if $R$ is of countably infinite dimension as a $k$-vector space $\hat{E}_R(k)$ will be of uncountable dimension, whilst $E_R(k)$ will be of countable dimension, so $E_R(k) \neq \hat{E}_R(k)$ in general.

Accordingly the best we can hope for is that we have a monomorphism

$$H^*_m(R) \rightarrow \text{Hom}_k(R, k) = \hat{E}_R(k)$$

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with the image being a copy of \( E_R(k) \). When \( R \) is polynomial, we can recover this since we have a residue map
\[
\text{res} : H^r_m(R) \otimes \Omega^r_{R/k} \to k,
\]
where \( \Omega^r_{R/k} \) is the module of Kähler differentials, with \( \Omega^r_{R/k} \cong R \). This gives rise to
\[
H^*_m(R) = H^*_m(R) \to \text{Hom}_k(\Omega^r_{R/k}, k) \cong \hat{E}_R(k)
\]

6.C. Gorenstein duality for linear-localized polynomial rings. Our situation has all the features described in Subsection 6.B but is not quite standard since the rings are not local. For a connected subgroup \( K \) of codimension \( s \), we need to consider the ring \( R = \mathcal{E}^{-1}_KH^*(BG) \).

We think of \( H^*(BG) \) as polynomial functions on the affine space \( TG = \text{Spec}(H^*(BG)) \). Corresponding to the short exact sequence
\[
1 \to K \to G \to G/K \to 1
\]
we have maps
\[
H^*(BK) \leftarrow H^*(BG) \leftarrow H^*(BG/K)
\]
and a fibration
\[
TK \to TG \to TG/K.
\]
The ring \( H^*(BG/K) \) is thus naturally a subring of \( H^*(BG) \). Choosing particular degree \(-2\) generators \( x_i \), we have \( H^*(BG/K) = k[x_1, x_2, \ldots, x_s] \) and the kernel of restriction to \( K \) is the ideal \( m_{G/K} = (x_1, \ldots, x_s) \).

To go further we choose a splitting. If \( H^*(BK) = k[y_1, y_2, \ldots, y_t] \), we have \( H^*(BG) \cong k[x_1, \ldots, x_s, y_1, \ldots, y_t] \). The linear forms in \( \mathcal{E}_K \) are precisely those forms involving some \( y_t \) (or, intrinsically, those not in the image of \( H^*(BG/K) \)). We therefore have a natural map \( \mathcal{E}^{-1}_K H^*(BK) \leftarrow \mathcal{E}^{-1}_K H^*(BG) \) which plays the role of the map from \( R \) to its residue field.

Since \( K \) is connected, we may choose a splitting of the inclusion \( K \to G \) and hence make \( R = \mathcal{E}^{-1}_K H^*(BG) \) an algebra over \( k_K = \mathcal{E}^{-1}_K H^*(BK) \).

The ring \( R \) consists of meromorphic functions on \( TG \) regular on \( TK \) but with denominators that are products of linear forms.

**Lemma 6.4.** We have a natural embedding
\[
H^*_m(G/K) \left( \mathcal{E}^{-1}_K H^*(BG) \right) = H^*_{m_{G/K}} (\mathcal{E}^{-1}_K H^*(BG)) \to \Sigma^{2s} \text{Hom}_k(\mathcal{E}^{-1}_K H^*(BG), k),
\]
giving an isomorphism
\[
H^*_m(G/K) \left( \mathcal{E}^{-1}_K H^*(BG) \right) \to \Sigma^{2s} \Gamma_{G/K} \text{Hom}_k(\mathcal{E}^{-1}_K H^*(BG), k_K),
\]

**Proof:** Choose a splitting \( H^*(BG) \cong H^*(BG/K) \otimes H^*(BK) \), and note that
\[
H^*_m(G/K) \left( H^*(BG) \right) = H^*_m(G/K) \left( H^*(BG/K) \right) \otimes H^*(BK).
\]
As in Subsection 6.B \( H^*_m(G/K) \left( H^*(BG/K) \right) = H^*_m(G/K) \left( H^*(BG/K) \right) \) has a basis consisting of monomials \( x_1^{a_1} x_2^{a_2} \cdots x_s^{a_s} \) with all exponents negative. Thus the top degree basis element is \( (x_1x_2 \cdots x_s)^{-1} \) and the map to \( \text{Hom}_k(\mathcal{E}^{-1}_K H^*(BG/K), k) \) is given by residues. This gives an isomorphism
\[
H^*_m(G/K) \left( H^*(BG/K) \right) \cong \Gamma_{m_{G/K}} \text{Hom}_k(\mathcal{E}^{-1}_K H^*(BG/K), k).
\]
With our usual convention of taking graded Hom, the $\Gamma_{m_G/K}$ could be omitted since $H^*(BG/K)$ is finite dimensional in each degree. Now tensor this with $H^*$

$$E^{-1}_KH^*_{m_G/K}(H^*(BG)) = E^{-1}_K\left[H^*(BK) \otimes H^*_{m_G/K}(H^*(BG/K))\right] \cong E^{-1}_K\left[H^*(BK) \otimes \Gamma_{m_G/K}\text{Hom}(H^*(BG/K), k)\right]$$

We continue to write $E_K \subseteq H^*(BG)$ for the multiplicatively closed set of $K$-essential representations of $G$. Having chosen a splitting of $K \to G$, we need a different notation for those arising from representations of $K$, so write $E'_K = E_K \cap H^*(BK)$. Now if $M$ is an $m_{G/K}$-power torsion module then inverting $E_K$ is the same as inverting $E'_K$ (indeed, if $u$ is invertible and $x \in m_{G/K}$ then $u + x$ is invertible). Accordingly, $E^{-1}_KH^*_{m_G/K}(H^*(BG))$ is a free module over $K = E^{-1}_KH^*(BK)$ on the monomial basis of negative powers above. Thus $E^{-1}_KH^*_{m_G/K}(H^*(BG))$ embeds in $\text{Hom}_{K}(E^{-1}_KH^*(BG), k)K$.

Now consider the $\Gamma_{F/K}$-torsion in a module $M$. It has a filtration

$$0 \subseteq \text{ann}(m_{G/K}, M) \subseteq \text{ann}(m_{G/K}^2, M) \subseteq \text{ann}(m_{G/K}^3, M) \subseteq \cdots \subseteq \Gamma_{F/K}M.$$ 

The subquotients are modules over $E^{-1}_KH^*(BG)/m_{G/K} = E^{-1}_KH^*(BK)$.

Taking $M = \text{Hom}_{kK}(E^{-1}_KH^*(BG), k)$ we prove by induction on $a$ that $\text{ann}(m_{G/K}^{a+1}, M)/\text{ann}(m_{G/K}^a, M)$ is a free module on the negative monomials of total degree $-s - a$. The monomials are independent, so it remains to show they span. We know $E^{-1}_KH^*(BG)$ is free over $kK$ on the monomials, so we need only observe that a function annihilated by $m_G^a$ is zero on monomials of degree $\geq a + 1$.

6.D. **Explicit description of generating injectives.** Using the work of the last two subsections we can give an attractive description of $a_L(H_*(BG/\tilde{L}))$ showing that it captures global geometry.

**Lemma 6.5.** For a subgroup $\tilde{K}$, the torsion part of $a_L(H_*(BG/\tilde{L}))$ is given by

$$V(G/\tilde{K}) = \begin{cases} \Sigma^{-2\dim(G/K)}H_{m_G/K}^{\dim(G/K)}(E^{-1}_{K/L}H^*(BG/\tilde{L})) & \text{if } \tilde{L} \text{ is cotoral in } \tilde{K} \\ 0 & \text{otherwise} \end{cases}$$

**Proof:** The vanishing is clear. Let $s = \dim(G/K)$. By Lemma 6.4, we have an isomorphism

$$H_{m_G/K}^{\alpha}(E^{-1}_KH^*(BG/\tilde{L})) \cong \Sigma^{2\alpha}\Gamma_{F/K}\text{Hom}_{kK}(E^{-1}_{K/L}H^*(BG/\tilde{L}), kK).$$

The result follows form Definition 5.7 once we take account of the effect of $\Gamma_S$.

First, we observe that the embedding maps into $\Gamma_S$. We have already seen that an element $h = \lambda/e(V) \otimes f/x_1^{i_1}x_2^{i_2} \cdots x_s^{i_s}$ automatically maps to a torsion element. It remains to observe that it is only nonzero for finitely many $L$. Indeed, if $L$ is of codimension 1 in $K$ and $L$ is cotoral in $\tilde{K}$ then $H^*(BG/\tilde{L}) = k[x_1, \ldots, x_s, z]$ for some independent linear form $z$. Then $h$ will only map to a nonzero element if it is not regular, and this only happens if $z$ occurs amongst the finitely many linear factors of $e(V)$. Since the map is surjective by Lemma 6.4, it follows that $\Gamma_S$ is the identity on this object.
The structure maps are given by the relative residue maps.

**Lemma 6.6.** For cotoral inclusions \( \tilde{H} \supseteq \tilde{K} \supseteq \tilde{L} \) the structure map

\[
h_{\tilde{K}} : E_{H/L}H^*(BG/\tilde{L}) \otimes \Sigma^{-2\dim(G/H)}H^{\dim(g/H)}(E_{H/L}H^*(BG/\tilde{L}))
\rightarrow \Sigma^{-2\dim(G/K)}H^{\dim(G/K)}(E_{K/L}H^*(BG/\tilde{L}))
\]

is given by the relative residue.

\[\square\]

### 7. Injective Dimension

For the *convergence* of the Adams spectral sequence it is enough to know \( A_t(G) \) has finite injective dimension. This is an easy consequence of the finite injective dimension of torsion modules over a polynomial ring, and the proof is given in Proposition 7.2. For the use of the Adams Spectral Sequence it is enough to be able to calculate with \( A_t(G) \), and for this purpose one does not need to know the exact injective dimension either, so it will suffice to give a finite upper bound.

**Proposition 7.2.** An object \( X \) of codimension \( c \) has injective dimension \( \leq 2c \). In particular, the injective dimension of \( A_t(G) \) is \( \leq 2r \).
Proof: We will give the argument for a general object (i.e., with support in codimension \( \leq r \)), but clearly it applies to any codimension.

For an object \( X \) of \( \mathcal{A}_I(G) \), we describe how to construct an injective resolution

\[
0 \rightarrow X \rightarrow \mathbb{I}_0 \rightarrow \mathbb{I}_1 \rightarrow \cdots \rightarrow \mathbb{I}_{2r} \rightarrow 0
\]
of length \( 2r \), and we write \( X = X_0 \) and for \( i \geq 1 \) we take \( X_i = \text{cok}(X_{i-1} \rightarrow \mathbb{I}_{i-1}) \).

There are two halves to the construction

- For \( X_0, \ldots, X_r \) the injective dimensions of the individual components is steadily reduced: noting that if \( K \) is of codimension \( c \) then the injective dimension of torsion \( \mathcal{O}_{F/K} \)-modules is \( c \), we ensure that \( X_i(\mathcal{F}/K) \) is either injective or of injective dimension \( \leq c - s \). Thus the torsion components of \( X_r \) are all injective.

- For \( X_r, X_{r+1}, \ldots, X_{2r} \) we retain the property that all torsion components are injective but we ensure steadily more of them are zero, so that \( X_{r+i}(\mathcal{F}/K) = 0 \) for \( \dim(K) < i \).

This is rather straightforward. We construct the resolution recursively. We take \( X = X_0 \) and for \( s \geq 0 \) we suppose we have constructed up to \( X_s \) in the exact sequence

\[
0 \rightarrow X \rightarrow \mathbb{I}_0 \rightarrow \cdots \rightarrow \mathbb{I}_{s-1} \rightarrow X_s \rightarrow 0
\]

For each \( K \) we choose an injective torsion \( \mathcal{O}_{F/K} \)-module \( I_s(K) \) and a monomorphism \( i_s(K) : X_s(\mathcal{F}/K) \rightarrow I_s(K) \). If \( X_s(\mathcal{F}/K) \) is already injective, we take \( I_s(K) = X_s(\mathcal{F}/K) \) and \( i_s(K) \) to be the identity. Now take \( \mathbb{I}_s = \prod_K a_K(I_s(K)) \) and define \( i_s : X_s \rightarrow \mathbb{I}_s \) by ensuring the \( a_K(I_s(K)) \) component corresponds to \( i_s(K) \) under the adjunction

\[
\text{Hom}_{\mathcal{A}_I(G)}(X_s, a_K(I_s(K))) = \text{Hom}_{\mathcal{O}_{F/K}}(X_s(\mathcal{F}/K), I_s(K)).
\]

This ensures the map \( i_s \) is a monomorphism and we take \( X_{s+1} = \text{cok}(i_s : X_s \rightarrow \mathbb{I}_s) \).

By Lemma 7.3, the value of \( \mathbb{I}_s \) at \( \mathcal{F}/K \) is injective, so that if \( X_s(\mathcal{F}/K) \) is not already injective, \( \text{id}(X_{s+1}(\mathcal{F}/K)) = \text{id}(X_s(\mathcal{F}/K)) - 1 \). This deals with the first half of the construction. For the second half we suppose that that \( s = r + i \) with \( i \geq 0 \) and \( X_r, X_{r+1}, \ldots, X_{r+i} \) have support as required, so that in particular \( X_{r+i} \) has support in dimension \( \geq i \). By construction \( \mathbb{I}_{r+i} \) also has support in dimension \( \geq i \), and if \( K \) is of dimension \( i \), the only factor of \( \mathbb{I}_{r+i} \) not zero at \( K \) is \( a_K(I_{r+i}(K)) \), so that

\[
X_{r+i+1}(\mathcal{F}/K) = \text{cok} [X_{r+i}(\mathcal{F}/K) \rightarrow \mathbb{I}_{r+i}(\mathcal{F}/K) = I_{r+i}(K)] = 0
\]
as required. We note that when \( i = r \) the conclusion is \( X_{2r+1} = 0 \).

\[\square\]

7.B. Local duality. The purpose of this section is to observe that if \( T \) is an Artinian torsion module then \( a_L(T) \) is of injective dimension \( \leq \dim(G/L) \) as one might expect. The proof is straightforward using local duality.

One might view this as saying that most of the objects of \( \mathcal{A}_I(G) \) that we need to consider have injective dimension \( \leq r \). However this is a bit misleading, since even simple operations like infinite sums may give objects of higher injective dimension.

**Proposition 7.3.** If \( T \) is an Artinian \( \mathcal{O}_{F/L} \)-module then \( \text{id}_{\mathcal{O}_{F/L}}(T) = \text{id}_{\mathcal{A}_I(G)}(a_L(T)) \).

**Proof:** Since \( T \) is Artinian, we may choose a resolution by Artinian injective torsion modules

\[
0 \rightarrow T \rightarrow I_0 \rightarrow I_1 \rightarrow \cdots \rightarrow I_d \rightarrow 0.
\]
We will show that
\[ 0 \rightarrow a_L(T) \rightarrow a_L(I_0) \rightarrow a_L(I_1) \rightarrow \cdots \rightarrow a_L(I_d) \rightarrow 0 \]
is exact, which gives the desired conclusion.

The point is that if we take \( I = H_* (BG/L) \) then by Lemma 6.3 the torsion component of \( a_L(I) \) at \( F/K \) is \( H_{mg/K}^t (E^{-1}_K H^*(BG/L)) \), so we need to see that this process is exact.

For this, we apply local duality. The idea is that the injective \( O_{F/K} \)-module resolution \( I_\bullet \) can be recognized as the local cohomology of the dual of a free \( O_{F/K} \)-module resolution \( F_\bullet \) of a dual module \( N \). By a similar process after localization, the values at other levels are then recognized as local cohomology of localizations of \( N \). The exactness of \( I_\bullet \) then gives exactness of these.

The key here is that for the polynomial ring \( P = k[x_1, \ldots, x_s] \) we have
\[ H^s_m(P) = \Sigma 2^t \text{Hom}_k(P, k) \]
As usual the \( \text{Hom}_k \) refers to graded maps, and if we used ungraded maps we would insert \( \Gamma_m \) to achieve the same end. From the variance we can see this cannot be natural, but we can easily correct that: for arbitrary free \( P \)-modules \( F \) we have a natural isomorphism
\[ \text{Hom}_k(H^s_m(F), k) = \Sigma^{-2s} \text{Hom}_P(F, k) \]
Thus if \( F_\bullet \) is a free resolution of a \( P \)-module \( N \) we have
\[ \text{Hom}_k(H^s_m(F), k) = \Sigma^{-2s} \text{Ext}_P^i(N, P) \]
Note that we have natural maps
\[ H^s_m(F) \rightarrow \Sigma 2^t \text{Hom}_k(\text{Hom}_k(H^s_m(F), k), k) \quad \text{and} \quad F \rightarrow \Sigma 2^s \text{Hom}_P(\text{Hom}_P(F, P), P) \]
both of which are isomorphisms if \( F \) is of finite rank. Thus if \( M \) is Artinian, the injective resolution \( I_\bullet \) of \( M \) determines a free resolution \( F_\bullet \) of \( N = \text{Hom}_k(M, k) \). Similarly \( E_K^{-1} F_\bullet \) is a free resolution of \( E_K^{-1} \text{Hom}_k(M, k) \), and then by taking \( \text{Hom}_{k_K}(\cdot, k_K) \) we obtain an injective resolution of \( \text{Hom}_{k_K}(E_K^{-1} M, k_K) \).

By local duality this is
\[ a_L(I_\bullet(G/K)) = \Sigma^{-2t} H^t_{mg/K} (\text{Hom}_{E^{-1}_K P}(E^{-1}_K F_\bullet, k_K)) \]
showing that the complex is exact as required.

\[ \square \]

7.C. Attainment of the bound. The aim of this subsection (not currently achieved!) is to establish the exact injective dimension of \( \mathcal{A}_t(G) \) by writing down an object of injective dimension equal to the upper bound established in Proposition 7.2. We showed in Proposition 7.3 that if \( T \) is Artinian \( a_L(T) \) has the same injective dimension as \( T \) does, and hence all objects of this type have injective dimension \( \leq r \).

We will show that for \( r \geq 1 \) there are objects \( X \) with codimension 1 that are of injective dimension 2. This only establishes the injective dimension of \( \mathcal{A}_t(G) \) for the circle group. The same argument shows that there is an object of codimension \( c \) with injective dimension \( c + 1 \) in general, so that we only know that the injective dimension of \( \mathcal{A}_t(G) \) lies between \( r + 1 \) and 2r.

Lemma 7.4. If \( X \) has support in codimension 1 then
**X** is injective if and only if \( V(\mathcal{F}/G) \rightarrow \text{Hom}_{\mathcal{O}_F/H}(\mathcal{E}^{-1}_{G/H}\mathcal{O}_F/H, V(\mathcal{F}/H)) \) is surjective for all \( H \) of codimension 1.

**X** is of injective dimension \( \leq 1 \) if and only if
\[
\text{Ext}^1_{\mathcal{O}_F/H}(\mathcal{E}^{-1}_{G/H}\mathcal{O}_F/H, V(\mathcal{F}/H)) = 0 \quad \text{for all subgroups} \quad H \quad \text{of codimension} \quad 1.
\]

**Lemma 7.5.**

\[
\text{Ext}^s_{\mathcal{O}_F/H}(\mathcal{E}^{-1}_{H}\mathcal{O}_F, T) = \begin{cases} 
\text{Hom}_{\mathcal{O}_F/H}(\mathcal{E}^{-1}_{H}\mathcal{O}_F, T) & \text{if } s = 0 \\
\lim^1(T, \mathcal{E}_H) & \text{if } s = 1 \\
0 & \text{if } s \geq 2
\end{cases}
\]

For a commutative ring \( R \) and a multiplicatively closed set \( \mathcal{E} \), we have
\[
\text{Ext}_{R}(\mathcal{E}^{-1}R, M) = \lim_{\leftarrow}^1(M, \mathcal{E}).
\]

We make some observations about vanishing and non-vanishing of this.

**Lemma 7.6.** Suppose \( R \) is a commutative ring and \( \mathcal{E} \) is a multiplicatively closed subset. There is an isomorphism
\[
\lim_{\leftarrow}^1(eM) \cong M^{\mathcal{E}}_e / M.
\]

There is an epimorphism
\[
\lim_{\leftarrow}^1(M, e) \twoheadrightarrow R^{\mathcal{E}} \lim_{\leftarrow}^1(eM)
\]

**Proof:** For the first statement we have the inverse system
\[
0 \rightarrow eM \rightarrow M \rightarrow M/eM \rightarrow 0
\]

where the maps are inclusions and projections. Taking inverse limits gives a six term exact sequence with the last two zero.

For the second we have a inverse system
\[
0 \rightarrow \text{ann}_M(e) \rightarrow M \xrightarrow{e} eM \rightarrow 0
\]

The last map in the six term exact sequence is an epimorphism.

In particular if we may take \( R = k[x_1, \ldots, x_r] \) to be a polynomial ring on generators of degree \(-2\). Now take \( M = \bigoplus_{n \geq 0} \Sigma^{2n} R/m^n \) and \( \mathcal{E} = \langle x_1 \rangle \), and we see that \( M \) is not \( \mathcal{E} \)-complete and hence \( R^{\mathcal{E}} \lim_{\leftarrow}(M, \mathcal{E}) \neq 0 \).

This is enough to construct an object of \( \mathcal{A}_t(G) \) of injective dimension \( r + 1 \) when \( r \geq 1 \). Indeed we may take \( M = \bigoplus_n \Sigma^{2n} k[c_1]/c_1^n \). We then find
\[
\Gamma_{\mathcal{F}/L} \text{Ext}^1_{\mathcal{E}^{-1}_{L}\mathcal{O}_F}(M) \neq 0.
\]
Since the module is annihilated by $c_2, \ldots, c_r$ it has injective dimension $r - 1$. Hence $a_1(M)$ has injective dimension $\geq 2 + (r - 1) = r + 1$.

One might expect to be able to iterate this construction, but this doesn’t work, at least not in a simple way.

**Lemma 7.7.** For any commutative ring $R$, any $R$-module $M$ and any countable multiplicatively closed subsets $E_1, E_2$ we have

$$\text{Ext}^a_R(E^{-1}_1 R, \text{Ext}^b_R(E^{-1}_2 R, M)) = 0 \text{ for } a + b \geq 2.$$  

**Proof:** In view of the adjunction

$$\text{Hom}(A, \text{Hom}(B, M)) \cong \text{Hom}(A \otimes B, M),$$

there is a spectral sequence

$$\text{Ext}^i_R(E^{-1}_1 R, \text{Ext}^j_R(E^{-1}_2 R, M)) \Rightarrow \text{Ext}^{i+j}_R(E^{-1}_1 E^{-1}_2 R, M).$$

$\square$

The following result is trivial, and probably irrelevant.

**Lemma 7.8.** For any commutative ring $R$, and any countable multiplicatively closed subsets $E$ if $M$ is $E$-divisible then

$$\text{Ext}^1_R(E^{-1} R, M) = 0.$$  

**Proof:** The divisibility proves that $(M, e)$ is Mittag-Leffler and hence $R^1 \lim \leftarrow (M, e) = 0$. $\square$

**8. Ind-corepresenting evaluation**

Given an object $X$ of $A_t(G)$, we would like to be able to determine its torsion modules $V(F/K)$ by considering maps from objects of $A_t(G)$, in the same way that homotopy groups are given by maps out of a sphere. Although this is not possible as it stands, we can recover $V(F/K)$ as a colimit of such values. Since $V(F/K) = \bigoplus_K V(G/K)$ we focus on a single subgroup $K$.

This phenomenon should be rather familiar. Given a torsion $k[c]$-module $T$ we may recover $T$ as $T = \text{Hom}_{k[c]}(k[c], T)$, but if we are restricted to using torsion modules in the domain this is not available to us. However, $\text{ann}(c^n, T) = \text{Hom}_{k[c]}(k[c]/c^n, T)$, so

$$T = \bigcup_n \text{ann}(c^n, T) = \lim \rightarrow_n \text{Hom}_{k[c]}(k[c]/c^n, T).$$

In this sense the module is ind-corepresented by the inverse system $\{k[c]/c^n\}_n$. The situation with $A_t(G)$ is a little more complicated, so we start with the circle group in the semifree case as in Section 2.

**Example 8.1.** By the discussion above we see immediately that evaluation at $G/1$ is ind-corepresented by $\{(0 \rightarrow k[c]/c^n)\}_n$. This is an object familiar from the topology since $\pi_*^G(DS(nz)_+) = k[c]/c^n$, where $D$ denotes Spanier-Whitehead duality, so that $\pi_*^H(DS(nz)_+) = (0 \rightarrow k[c]/c^n)$. In fact the representing inverse system is the homotopy of the dual of the direct system $S(z)_+ \subset S(2z)_+ \subset S(3z)_+ \subset \cdots$.  

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we proceed as follows. Consider a map \( \theta \) and note first that it is determined by \( \theta \) in the domain is an epimorphism. Thus every element in the image of \( q \) is divisible, so that if \( q(1 \otimes v) \) is annihilated by \( c^s \) there is a map \( \theta \) with evaluation \( v \) if \( n \geq s \), and since \( T \) is torsion the map \( e \) is surjective.

Again the objects are geometrically familiar since \( \pi_*^{\mathbb{A}_t}(S^{-nz}) = (t \otimes Q \rightarrow k[c, c^{-1}] / c^n k[c]) \). In fact the representing inverse system is the homotopy of the dual of the system \( S^0 \subset S^2 \subset S^{2z} \subset S^{3z} \subset \cdots \).

In principle we could continue purely algebraically, but the topological motivation will lead us efficiently to a solution. Recall that

\[
\pi_*^{\mathbb{A}_t}(X)(G/\tilde{K}) = \pi_*^{G}(S^{\infty V(K)} \wedge EG/\tilde{K}_+ \wedge X)
\]

\[
= [S^0, \lim_{\longrightarrow \mathbb{V}K=0} S^V \wedge \lim_{\longrightarrow n} EG/\tilde{K}^n_+ \wedge X]^G = \lim_{\longrightarrow \mathbb{V}K=0,n} [S^{-V} \wedge D EG/\tilde{K}^n_+, X]^G
\]

This suggests that if we choose a nice filtration of the universal space, we could use

\[
B_{\tilde{K}}(V,n) = \pi_*^{\mathbb{A}_t}(S^{-V} \wedge D EG/\tilde{K}^n_+)
\]

to give ind-corepresenting objects.

We choose to be very explicit so as to make this more easily digested, but pay the price of making unnecessary choices. We have a short exact sequence

\[
1 \rightarrow \tilde{K}/\tilde{L} \rightarrow G/\tilde{L} \rightarrow G/\tilde{K} \rightarrow 1
\]

giving a short exact sequence of commutative \( k \)-algebras

\[
k \leftarrow H^*(B\tilde{K}/\tilde{L}) \leftarrow H^*(BG/\tilde{L}) \leftarrow H^*(BG/\tilde{K}) \leftarrow k.
\]

This is natural, but because \( \tilde{K}/\tilde{L} \) is a torus we may go further and choose a splitting. Thus the projection \( G/\tilde{L} \rightarrow G/\tilde{K} \) is split, and we may choose \( \tilde{K}^* \supset \tilde{L} \) so that the composite \( \tilde{K}/\tilde{L} \rightarrow G/\tilde{L} \rightarrow G/\tilde{K}^* \) is an isomorphism, and hence \( G/\tilde{L} = G/\tilde{K} \times G/\tilde{K}^* \).

We choose one dimensional representations \( \alpha_1, \alpha_2, \ldots, \alpha_s \) with kernels \( \tilde{K}(\alpha_i) = \ker(\alpha_i) \) so that

\[
G/\tilde{K} = G/K(\alpha_1) \times G/K(\alpha_2) \times \cdots \times G/K(\alpha_s)
\]

Similarly, we choose \( \beta_1, \beta_2, \ldots, \beta_t \) so that

\[
\tilde{K}/\tilde{L} \cong G/\tilde{K}^* = G/K(\beta_1) \times G/K(\beta_2) \times \cdots \times G/K(\beta_t).
\]
This allows us to have explicit models for universal spaces:

\[ EG/\tilde{K} = S(\infty \alpha_1) \times S(\infty \alpha_2) \times \cdots \times S(\infty \alpha_s) \] and \[ EG/\tilde{K}^* = S(\infty \beta_1) \times S(\infty \beta_2) \times \cdots \times S(\infty \beta_s). \]

As filtration, we take

\[ EG/\tilde{K}^n = S(n \alpha_1) \times S(n \alpha_2) \times \cdots \times S(n \alpha_s). \]

Now take \( x_i = e(\alpha_i), y_j = e(\beta_j) \), so that

\[ H^*(BG/\tilde{L}) \cong H^*(BG/\tilde{K}) \otimes H^*(B\tilde{K}/\tilde{L}) \cong k[x_1, \ldots, x_s, y_1, \ldots, y_s] \]

We also write

\[ m_{G/\tilde{K}} = (x_1, \ldots, x_s), m_{G/\tilde{K}}^{[n]} = (x_1^n, \ldots, x_s^n), \]

and note

\[ H^*(BG/\tilde{K}^n) = H^*(BG/\tilde{K})/m_{G/\tilde{K}}^{[n]} \text{ and } H^*(BG/\tilde{K}^n) = \text{ann}(m_{G/\tilde{K}}^{[n]}, H_*(BG/\tilde{K})). \]

It is obvious that \( H^*(BG/\tilde{K}^n) \) is cyclic, but we note that it is also self-dual. This then gives the important fact that \( H_*(BG/\tilde{K}^n) \) is also cyclic, and generated by the element dual to \( (x_1x_2 \cdots x_s)^n \) if we use the monomial basis. We note that this is the Euler class of \((\alpha_1 \oplus \cdots \oplus \alpha_s)^{\otimes n}. \) One might say

\[ H_*(BG/\tilde{K}^n) = \frac{1}{(x_1x_2 \cdots x_s)^n} \cdot H^*(BG/\tilde{K}^n). \]

\textbf{Remark 8.2.} This corresponds to a Spanier-Whitehead duality statement. Since \( D(S(V)_+) \cong \Sigma^{1-V}S(V)_+ \) we see that

\[ D(EG/\tilde{K}^n) \cong \Sigma^{s-(\alpha_1 \oplus \cdots \oplus \alpha_s)^{\otimes n}} EG/\tilde{K}^n. \]

Finally, we note that any representation \( V \) of \( \tilde{L} \) with \( V^{\tilde{K}} = 0 \) can be written as a sum of monomials in the \( \alpha_i \) and \( \beta_j \) where each monomial involves at least one \( \beta_j \). Accordingly,

\[ \mathcal{E}_{G/\tilde{K}} = \{ \sum \lambda_i x_i + \sum \mu_j y_j \mid (\mu_1, \ldots, \mu_t) \neq (0, \ldots, 0) \}. \]

\textbf{Remark 8.3.} Since our construction has involved a lot of choice, it is worth considering what is intrinsic once \( \tilde{K} \supseteq \tilde{L} \) is chosen.

First of all, \( H^*(BG/\tilde{K}) = k[x_1, \ldots, x_s] \) is an intrinsic subalgebra of \( H^*(BG/\tilde{L}) \). From the geometric point of view, writing \( TG := \text{Spec}(H^*(BG)) \) we are considering the projection \( TG/\tilde{L} \rightarrow TG/\tilde{K} \). Similarly the ideal \( m_{G/\tilde{K}} \) is intrinsic, and \( T\tilde{K}/\tilde{L} \) is its zero set, giving the fibration

\[ T\tilde{K}/\tilde{L} \rightarrow TG/\tilde{L} \rightarrow TG/\tilde{K}. \]

Finally, the multiplicatively closed subset \( \mathcal{E}_{K/L} \) is generated by elements of \( H^2(BG/\tilde{L}) \) not in \( m_{G/\tilde{K}} \) and is also intrinsic: inverting \( \mathcal{E}_{K/L} \) is localization at \( T\tilde{K}/L = T\tilde{K}/\tilde{L}. \)

Accordingly, the \( H^*(BG/\tilde{L}) \)-module \( \mathcal{E}_{K/L}^{-1} H_*^{m_{G/\tilde{K}}}(H^*(BG/\tilde{L})) \) (the localized \( T\tilde{K}/L \)-local cohomology) is intrinsic.
Proposition 8.4. Evaluation at $G/\tilde{K}$ is ind-corepresented by objects $B_{\tilde{K}}(V, n)$ as $V$ varies through representations with $V^K = 0$ and $n \geq 0$. Here $B_{\tilde{K}}(V, n)(G/\tilde{L}) = 0$ unless $\tilde{L}$ is cotoral in $\tilde{K}$ and in that case we have 

$$B_{\tilde{K}}(V, n)(G/\tilde{L}) = S^{-V^L} \wedge H^*(BG/\tilde{K}^n) \otimes H_*(B\tilde{K}/\tilde{L})$$

as $V$ varies through representations with $V^K = 0$ and $n \geq 0$.

Proof: We suppose $X$ is an object of $\mathcal{A}_d(G)$ with torsion module $V(G/\tilde{K})$ at $G/\tilde{K}$. To define a map

$$e : \lim_{\to V^K = 0, n} \text{Hom}_{\mathcal{A}_d(G)}(B_{\tilde{K}}(V, n), X) \to V(G/\tilde{K})$$

we pick a representative $\theta : B_{\tilde{K}}(V, n) \to X$ of an element of the domain. First consider the $G/\tilde{K}$ level, and note that since $V^K = 0$, there is no dependence on $V$. We take $e([\theta]) = \theta(G/\tilde{K})(\iota)$ where $\iota \in H^0(BG/\tilde{K})/m^n_{\tilde{K}}$ represents the unit. Since the maps in the inverse system as $n$ varies are the identity in this degree, this does not depend on the choice of representative. Accordingly, the map $e$ is well defined.

To see $e$ is injective we note that the structure maps of $B_{\tilde{K}}(V, n)$ are surjective. First of all, if $V = 0$ the maps $B_{\tilde{K}}(0, n)(G/\tilde{K}) \to B_{\tilde{K}}(0, n)(G/\tilde{L})$ are surjective. We may as well suppose $\tilde{K}/\tilde{L}$ is a circle, since the general case is a composite of such cases. Here it is easy to check that (writing $z$ for the natural representation of $\tilde{K}/\tilde{L}$),

$$S^\infty \wedge DE\tilde{K}/\tilde{L}_+ \to \Sigma S(\infty z)_+ \wedge DE\tilde{K}/\tilde{L}_+ \simeq \Sigma S(\infty z)_+$$

is surjective and the general case is obtained by tensoring up. Now as $V$ varies we use the fact that $e(V)$ is an isomorphism in the domain and (since $V^K = 0$) it is a monomorphism in the codomain.

Finally, because $V(G/\tilde{K})$ is torsion, for any element $x$ of degree $i$ in $V(G/\tilde{K})$ we can find an $n$ so that $(x_1 x_2 \ldots x_s)^n x = 0$. There is therefore a map

$$\Sigma^i H^*(BG/\tilde{K}^n) \cong \Sigma^i H^*(BG/\tilde{K})/m^{[n]}_{\tilde{K}} \to V(G/\tilde{K})$$

with $i$ mapping to $x$. Next, if $\tilde{K}$ is of codimension $c$, by definition of $\mathcal{A}_d(G)$, the element $1 \otimes x$ at level $G/\tilde{K}$ only has non-zero image at $G/\tilde{L}$ for finitely many connected codimension $c + 1$ subgroups $L$. For each of these we find $V_L$ so that $q_L(1 \otimes x)$ is annihilated by $e(V_L)$, and take $V = \bigoplus_L V_L$. This shows that $1 \otimes x$ is in the image of a map $B_{\tilde{K}}(V, n) \to X$, and so $e$ is surjective.

Having ind-corepresented evaluation at $G/\tilde{K}$ we should describe how the structure maps are ind-corepresented. Indeed, if $\tilde{L}$ is cotoral in $\tilde{K}$, we have maps

$$\xymatrix{ \mathcal{E}_{\tilde{K}/L}^{-1} H^*(BG/\tilde{L}) \otimes_{H^*(BG/\tilde{K})} V(G/\tilde{K}) \ar[r] \ar@{=}[d] & V(G/\tilde{L}) \ar@{=}[d] \ar[r] & \lim_{\to V^K = 0, n} \mathcal{E}_{\tilde{K}/L}^{-1} H^*(BG/\tilde{L}) \otimes_{H^*(BG/\tilde{K})} \text{Hom}(B_{\tilde{K}}(V, m), X) \ar[r] & \lim_{\to W^K = 0, n} \text{Hom}(B_L(W, n), X) }$$

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In other words, given \( x \in H^*(BG/\tilde{L}) \), a representation \( U \) of \( G/\tilde{L} \) with \( U^K = 0 \) and a representative map \( \theta : B_{\tilde{K}}(V, m) \to X \) we need a representative of the colimit in the codomain. Since all maps are linear in \( H^*(BG/\tilde{L}) \), it suffices to treat the case \( x = 1 \).

**Lemma 8.5.** Noting that \( W^L = 0 \) implies, \( W^K = 0 \), the structure map is ind-corepresented by precomposing \( \theta \) with

\[
B_{\tilde{L}}(W, n) \to B_{\tilde{K}}(W, n).
\]

Multiplication by \( e(U) \) is given by \( B_{\tilde{K}}(V, m) \to B_{\tilde{K}}(U \oplus V, m) \), so division by \( e(U) \) is effected by shifting filtration by \( U \).

**Proof:** We only need to give the answer at \( G/\tilde{M} \) where \( \tilde{M} \) is cotoral in \( \tilde{L} \) since otherwise the codomain is zero. At this level we are looking at

\[
S^{-V^M} \otimes H^*(BG/\tilde{L}^n) \otimes H_*(B\tilde{L}/\tilde{M}) \to \Sigma^2 S^{-V^M} \otimes H^*(BG/\tilde{K}^n) \otimes H_*(B\tilde{K}/\tilde{M}).
\]

We may suppose \( \tilde{K}/\tilde{L} \) is a circle. So the only effect is that \( H^*(B\tilde{K}/\tilde{L}) \) is removed from the cohomology in the domain and replaced by \( \Sigma^2 H_*(B\tilde{K}/\tilde{L}) \) in the homology of the codomain. As in the proof of Proposition 8.3, writing \( z \) for the natural representation of \( \tilde{K}/\tilde{L} \) we note that it is

\[
S^{\infty z} \otimes D\tilde{E}\tilde{K}/\tilde{L}^+ \to \Sigma S(\infty z)_+ \otimes D\tilde{E}\tilde{K}/\tilde{L}^+ \cong \Sigma S(\infty z)_+.
\]

**Remark 8.6.** This gives another approach to constructing injectives. Indeed, by definition, if \( I \) is an injective \( H^*(BG/\tilde{L}) \)-module, then \( a_L(I) \) has the property

\[
\text{Hom}_{A_t(G)}(X, a_L(I)) = \text{Hom}_{H^*(BG/\tilde{L})}(V(G/\tilde{L}), I)
\]

so if \( \tilde{L} \) is cotoral in \( \tilde{K} \), we must have

\[
a_L(I)(G/\tilde{K}) = \lim_{V^K=0,n} \text{Hom}_{H^*(BG/\tilde{L})}(B_{\tilde{K}}(V, n)(G/\tilde{L}), I)
= \lim_{V^K=0,n} \text{Hom}_{H^*(BG/\tilde{L})}(S^{-V^L} \otimes H^*(BG/\tilde{K}^n) \otimes H_*(B\tilde{K}/\tilde{L}), I).
\]

We now take the generating torsion injective \( I = H_*(BG/\tilde{L}) \) so that \( \text{Hom}_{H^*(BG/\tilde{L})}(\cdot, I) = \text{Hom}_k(\cdot, k) \). As noted above

\[
H^*(BG/\tilde{K}^n) \otimes H_*(B\tilde{K}/\tilde{L}) \cong \Sigma^{s-na} H_*(BG/\tilde{K}^n) \otimes H_*(B\tilde{K}/\tilde{L})
\]

where \( a = |x_1x_2 \cdots x_s| \). Accordingly the value is

\[
S^{+V^L} \otimes \Sigma^{na-s} H^*(BG/\tilde{K}^n \times B\tilde{K}/\tilde{L}).
\]

The colimit is

\[
\mathcal{E}_{K/L}^{-1} H_*(BG/\tilde{K}) \otimes H^*(B\tilde{K}/\tilde{L}) = \mathcal{E}_{K/L}^{-1} H_{\text{wKL}}^s (H^*(BG/\tilde{L})).
\]

This agrees with the previous construction by Lemma 6.5. Indeed, it was the calculation by ind-corepresentability that first alerted the author to the connection between injectives and local cohomology, and hence led to the formulation of Lemma 6.5.
9. The Adams spectral sequence

The paper has been leading up to the construction of a method of calculation based on
the abelian torsion model.

**Theorem 9.1.** There is an Adams spectral sequence
\[
\Ext^\ast\ast_{\mathcal{A}(G)}(\pi^A\ast(X), \pi^A\ast(Y)) \Rightarrow [X, Y]_G^G.
\]
This is a finite, strongly convergent spectral sequence.

There is a standard method for constructing an Adams spectral sequence: we first establish
that enough injectives are realizable and that the Adams spectral sequence applies to them,
and that the homology theory detects triviality.

**Lemma 9.2.** For any injective \(O_{\mathcal{F}/L}\)-module \(I\), there is a \(G\)-spectrum \(A_L(I)\) realizing \(a_L(I)\) in the sense that \(\pi^A\ast(A_L(I)) = a_L(I)\) and \(\pi^A\ast\) gives an isomorphism
\[
\pi^A\ast : [X, A_L(I)]_G^G \cong \Hom_{\mathcal{A}(G)}(\pi^A\ast(X), a_L(I)).
\]
**Proof:** For an injective, torsion \(O_{\mathcal{F}/L}\)-module \(I\), the functor
\[
X \mapsto A_L(I)_G^G(X) = \Hom_{\mathcal{A}(G)}(\pi^A\ast(X), a_L(I)) = \Hom_{O_{\mathcal{F}/L}}(V_X(\mathcal{F}/L), I)
\]
is exact because \(I\) is injective and is therefore a cohomology theory on \(G\)-spectra, so by
Brown Representability there is a \(G\)-spectrum \(A_L(I)\) so that
\[
[X, A_L(I)]_G^G = \Hom_{\mathcal{A}(G)}(\pi^A\ast(X), a_L(I)).
\]
The isomorphism \(\pi^A\ast(A_L(I)) = a_L(I)\) follows from Proposition 8.4. \(\square\)

**Lemma 9.3.** If \(\pi^A\ast(X) = 0\) then \(X \simeq *\).

**Proof:** If \(\pi^A\ast(X) = 0\) then \(H^\ast_{G/K}(\Phi K X) = 0\) for all subgroups \(K\). The geometric isotropy
of \(X\) is therefore empty and \(X\) is contractible by the Geometric Fixed Point Whitehead
Theorem. \(\square\)

**Proof of Theorem 9.1.** As usual we need only show that enough injectives are realizable,
that the spectral sequence is correct for maps into these spectra and that the spectral
sequence is convergent.

In more detail, we take an injective resolution of \(\pi^A\ast(Y)\):
\[
0 \rightarrow \pi^A\ast(Y) \rightarrow I_0 \rightarrow I_1 \rightarrow \cdots \rightarrow I_n \rightarrow 0.
\]
This is finite by Proposition 7.2 and by Lemma 6.2 we may assume each \(I_s\) is a sum of
injectives \(a_K(I)\) for subgroups \(K\) and injective \(O_{\mathcal{F}/K}\)-modules \(I = H_s(BG/K)\) where \(K\) has
identity component \(K\).

By Lemma 9.2 this is realizable by a tower
\[
Y \overset{=}\longrightarrow Y_0 \leftarrow Y_1 \leftarrow Y_2 \leftarrow \cdots \leftarrow Y_n \leftarrow Y_{n+1}
\]
\[
\begin{array}{c}
\mathrm{Id}_0 \downarrow \\
\Sigma^{-1} \downarrow \\
\Sigma^{-2} \downarrow \\
\Sigma^{-n} \downarrow \\
\end{array}
\]
\[
\begin{array}{c}
Y_0 \\
Y_1 \\
Y_2 \\
Y_n \\
Y_{n+1}
\end{array}
\]
which is built inductively, starting with $Y \to \mathbb{I}_0$ realizing $\pi_*^{At}(Y) \to I_0$ and taking $Y_1$ to be the fibre. Once $Y_s$ has been defined as the fibre of $Y_{s-1} \to \Sigma^{-s+1}\mathbb{I}$, we see that $\pi_*^{At}(Y_s) = \Sigma^{-s}(\text{im}(I_{s-1} \to I_s))$. In particular, $\pi_*^{At}(Y_{n+1}) = 0$ and $Y_{n+1} \simeq *$ by Lemma 9.3.

We obtain the spectral sequence by applying $[X, \cdot]^G$ to the tower. By Lemma 9.2 the $E_1$ term is $\text{Hom}_{\text{At}(G)}(\pi_*^{At}(X), I_\bullet)$ and therefore the $E_2$-term is as stated. Strong convergence is clear because the filtration is finite.

\[\square\]

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