Multiparty data hiding of quantum information

Patrick Hayden,* Debbie Leung,† and Graeme Smith‡

Institute for Quantum Information, Caltech 107–81, Pasadena, CA 91125, USA

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We present protocols for multiparty data hiding of quantum information that implement all possible threshold access structures. Closely related to secret sharing, data hiding has a more demanding security requirement: that the data remain secure against unrestricted LOCC attacks. In the limit of hiding a large amount of data, our protocols achieve an asymptotic rate of one hidden qubit per local physical qubit. That is, each party holds a share that is the same size as the hidden state to leading order, with accuracy and security parameters incurring an overhead that is asymptotically negligible. The data hiding states have very unusual entanglement properties, which we briefly discuss.

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I. INTRODUCTION

In a variety of situations, it is desirable to distribute data among many parties in such a way that the parties can reconstruct the data only if they cooperate in a well-defined way. This problem has been studied in several settings, including the purely classical case [24], encoding classical data in quantum systems [10, 12, 13, 25], and encoding quantum data in quantum systems [8, 9, 16, 19]. For the last two settings, at least two inequivalent security criteria can be applied. In quantum secret sharing [8, 16], certain authorized sets of parties are able to reconstruct the encoded data if they cooperate to implement a joint operation, whereas the remaining, unauthorized, sets are unable to get access regardless of what they do. In quantum data hiding [9, 10, 12, 13, 19, 25], the requirement for cooperation is increased. The data must remain inaccessible if any combination of the parties communicate classically, and can only be retrieved if the members of an authorized set perform a joint quantum mechanical operation, perhaps supplemented by classical advice from other parties outside the authorized set.

Consider the following fanciful scenario. After the debacle of the year 2000 election, authorities in the state of Florida have decided to (do their best to) implement a tamper-proof vote counting system for the upcoming 2004 election. One feature of the new system is that every counting center must be attended by both a Democratic and a Republican observer. The system designers enforce this by encoding the ballot box access code using a quantum data hiding scheme requiring that the Democratic and Republican observers jointly implement a quantum operation to get access to the code. The difficulty inherent in implementing a quantum operation remotely offers some assurance that both observers will need to work together in the same place to extract the code. The scheme also offers flexibility in incorporating smaller political parties unable to field a full team of observers. The system could be designed, for example, such that the Greens would hold a share of the encoded state but not need to be physically present; their participation via classical communication would be sufficient.

Relatively few data hiding schemes have been presented in the literature. The first examples

*Electronic address: patrick@cs.caltech.edu
†Electronic address: wcleung@cs.caltech.edu
‡Electronic address: graeme@theory.caltech.edu
are to be found in [25] and [10], which demonstrate that classical bits can be hidden in bipartite Werner states. Generalizations to the multiparty setting realizing all sensible access structures are given by [12, 13]. The earliest schemes for hiding quantum data [9] are based on hiding a classical key that encrypts the quantum data. Much more efficient schemes for hiding quantum data directly were presented for two parties in [19]. Here we generalize the method of [19] to construct \((k,n)\) threshold hiding schemes, meaning that arbitrary classical communication in addition to quantum communication among any \(k\) parties is authorized, that is, sufficient to retrieve the data, but arbitrary classical communication along with quantum communication among groups of up to \(k-1\) parties is unauthorized. Our contribution extends the multiparty threshold access structure results in [12, 13] to the domain of hiding quantum information, achieving an asymptotic rate of one hidden (logical) qubit per local physical qubit. Like the schemes in [19], those presented here are significant improvements over the earlier multipartite results [9, 13] when a large number of qubits or even bits are to be hidden – the accuracy and security requirements incur additive (negligible) space overhead in our schemes and multiplicative (nonnegligible) overhead in previous constructions. Also, unlike quantum secret sharing, where the no-cloning theorem imposes the restriction that the complement of an authorized set be unauthorized, any threshold value \(k\) between 1 and \(n\) is possible.

While presented in the language of cryptography, quantum data hiding is equally well a platform for the study of nonlocality. Indeed, the original proposal of [25] was motivated by the discovery of quantum nonlocality without entanglement, meaning sets of orthogonal product states that could not be distinguished by local operations and classical communication (LOCC) alone [4]. Since the original discovery, considerable effort has been devoted to understanding the relationship between local distinguishability and other types of nonlocality. (See [7, 11, 15, 21, 27] and references therein.) This paper continues that effort, in the sense that we present whole subspaces all of whose states are indistinguishable by LOCC but that can nonetheless be reconstructed by some collective operations, which are now more carefully prescribed than in the earlier work. Also, while much of that earlier work on local indistinguishability is devoted to determining when a finite set of states cannot be perfectly distinguished by LOCC, our focus here is at the other extreme, on near-perfect indistinguishability for entire subspaces. Moreover, in contrast to the emphasis on product states in [4], the results of [20] ensure that the states we choose here are by no means separable (they have near-maximal entanglement of formation) despite being LOCC indistinguishable from the maximally mixed state.

**Notation:** We use the following conventions throughout the paper. \(\log\) and \(\exp\) are always taken base 2. Unless otherwise stated, a state can be pure or mixed. The density operator \(|\varphi\rangle\langle\varphi|\) of the pure state \(|\varphi\rangle\) will frequently be written simply as \(\varphi\). \(U(d)\) denotes the unitary group on \(\mathbb{C}^d\), \(B(\mathbb{C}^d)\) the set of linear transformations from \(\mathbb{C}^d\) to \(\mathbb{C}^d\) and \(I\) the identity matrix. Physical operations mapping \(d_1\)-dimensional states to \(d_2\)-dimensional states are completely positive trace-preserving (CPTP) maps from \(B(\mathbb{C}^{d_1})\) to \(B(\mathbb{C}^{d_2})\). \(\|\cdot\|_1\) denotes the trace norm of a matrix and the 1-norm of a vector while \(\|\cdot\|_2\) denotes the Hilbert space norm. \(\Pr(E)\) is used to represent the probability of event \(E\).
II. DEFINITIONS AND RESULTS

We begin with a formal definition of data hiding for quantum information:

Definition 1 A $(δ, ε, s, d^m)$-qubit hiding scheme with $(k, n)$ access structure consists of a CPTP encoding map, $E: B(C^s) → B(C^{d^m})$ and a set of CPTP decoding maps $D^{(X)}: C^{d^m} → B(C^s)$, one for each set $X$ of $k$ parties, that can be implemented via quantum communication among the parties in $X$ together with classical communication among all parties. The encoding and decodings must satisfy

1. (Security) For all states $φ_0$ and $φ_1$ on $C^s$, along with all measurements $L$ that can be implemented using arbitrary quantum communication within groupings of $k − 1$ or fewer parties and arbitrary classical communication between them,

$$\|L(E(φ_0)) − L(E(φ_1))\|_1 ≤ ε.$$  \hfill (1)

2. (Correctness) For all $X$ and for all states $φ$ on $C^s$, $\|(D^{(X)} ∘ E)(φ) − φ\|_1 ≤ δ$.

Notice that for $k = 1$ there are no unauthorized measurements, so that a qubit hiding scheme with $(1, n)$ access structure is simply a method for implementing a kind of distributed data storage. Any one of the $n$ parties can recover the quantum data with the help of only classical “advice” from the other $n − 1$ parties. We’ll see that even in this simple setting, our approach will be considerably more efficient than the naive constructions based on local storage and teleportation or the more sophisticated proposals in [6].

The encoding map we will use is of the form

$$E(φ) = \frac{1}{r} \sum_{i=1}^{r} U_i φ U_i^\dagger,$$  \hfill (2)

where the $U_i$ are in $U(d^m)$ and we fix some inclusion $C^s \cong S ⊂ C^{d^m}$ for the encoded subspace; this is the same type of map used for approximate randomization and bipartite data hiding in [19]. Our strategy will be to show that if the $U_i$ are selected independently and each according to the Haar measure on $U(d^m)$, then for suitable choices of the parameters, $E$ will provide a good encoding with non-zero probability (over the random $U_i$). Given a set of parties in quantum communication with each other, $X$, and its complement, $W$, our decoding map $D^{(X)}$ consists of local measurements by the parties in $W$, communication of the measurement outcomes to the parties in $X$, followed by a recovery procedure on $X$. More specifically, the decoding is achieved in a two step process. In the first step, $A^{(W)}$, each member of $W$ performs a projection onto a fixed local basis, collectively $\{|l\rangle_W\rangle_{l=1}^{d^m−k}$, and sends the outcome to the members of $X$. Thus, if $W$ consists of the parties $W_1, W_2, \ldots, W_{n−k}$, then $A^{(W)}$ has the structure $A^{(W)} = A^{(W_1)} \otimes A^{(W_2)} \otimes \cdots \otimes A^{(W_{n−k})}$. An arbitrary hidden state $|ψ\rangle \in S$ is thereby transformed to the state $A^{(W)}(E(φ)) = \sum_{l}|l\rangle_W \otimes I_X E(φ)|l\rangle_W \otimes I_X$, now entirely in the possession of the members of $X$ since, post-measurement, the system $W$ can be assumed to contain only the measurement outcomes, which get sent to $X$.

They will then perform the transpose channel $T^{(X)}$ [2, 23] of the CPTP map $A^{(W)} ∘ E$ adapted to the maximally mixed state on $S$. The transpose channel is a generic construction for approximately reversing a quantum operation, which in the present case leads to the following definition for $T^{(X)}$. Let $P_l := |l\rangle_W \otimes I_X, P_S$ be the projector onto $S$, and

$$N := \sum_{i=1}^{s} \sum_{t=1}^{d^m−k} P_i U_i \frac{P_S}{\sqrt{s}} U_i^\dagger \frac{P_l}{\sqrt{r}}.$$  \hfill (3)
Discard

\[ |\varphi\rangle \in S \]

\[ U_i \]

\[ E(\varphi) \]

(a) Encoding operation

\[ X \]

\[ T(X) \]

\[ \hat{\varphi} \]

\[ W_1 \]

\[ A(W_1) \]

\[ W_2 \]

\[ A(W_2) \]

(b) Decoding operation

FIG. 1: The encoding and decoding maps. 1(a) depicts the encoding procedure. A random \( U_i \) is applied to the state \( |\varphi\rangle \) drawn from subspace \( S \). The output, \( E(\varphi) \), is almost indistinguishable from the maximally mixed state using arbitrary LOCC and quantum communication within groupings of \( k - 1 \) and fewer parties. 1(b) depicts the decoding procedure. For any partition of the parties into an authorized set \( X \) of \( k \) parties and its complement \( W \) of \( n - k \) parties, the unauthorized parties \( W_1, W_2, \ldots, W_{n-k} \) perform the measurements \( A(W_j) \) in fixed local bases, sending the outcomes to the authorized parties, who then apply the transpose channel \( T(X) \). (Single lines represent quantum data, double lines classical data. Time flows from left to right.)

Then,

\[ T^{(X)}(\psi) := \sum_{il} T_{il} \psi^T_{il} \], \hspace{1cm} \text{where} \hspace{1cm} T_{il} := \frac{P_S U_i^i P_l N^{-1/2}}{\sqrt{s} \sqrt{r}}. \]

\( N^{-1/2} \) is here defined to be zero outside the support of \( N \). The map \( T^{(X)} \) is, therefore, defined on the image of the subspace \( S \) by the map \( A^{(W)} \circ E \) and can be extended to a CPTP map on all of \( W \otimes X \). Formally, the decoding map is given by

\[ D^{(X)} = T^{(X)} \circ A^{(W)}. \]

The arrangement is illustrated in Figure 1. Our main result is

**Theorem 1** Let \( d \) be sufficiently large that \( d^k > 48/\delta^2 \), \( d^n > 10(n+2)/\epsilon \) and, in the special case \( k = 1 \), \( \frac{d}{\log d} > \frac{2840(2n+3)}{\delta^2} \). Then if

\[ r = \left[ \frac{32(n+2)^4}{Ce^2} \cdot d^{k-1} \log d \right] \quad \text{and} \]

\[ s = \left[ \frac{Ce^2 \delta^2}{1536(n+2)^4} \cdot \frac{d}{\log d} \right], \]

with probability at least \( 1/2 \), the encoding map (2) and decoding maps (5) give a \((\delta, \epsilon, s, d^n)\)-qubit hiding scheme with \((k, n)\) access structure.

The intuition leading to these choices for \( r \) and \( s \) is quite simple. First, security of the encoding against groupings of \( k - 1 \) parties will require that our encoding randomize subsystems of dimension \( d^{k-1} \) and smaller. This leads to a choice of \( r \gg d^{k-1} \). On the other hand, successful decoding will require the \( k \) parties in \( X \) to be able to identify the unitary \( U_i \) that was applied to the input state \( |\varphi\rangle \in S \) without damaging the encoded state, which leads to the constraint \( rs \ll d^n \). Consequently, in light of the randomization requirement, \( s \ll d \).
III. PROOFS

To prove the theorem, we will make extensive use of some well-known facts about Gaussian random variables and random quantum states. We use the notation \( g \sim N_C(0, 1) \) to denote that \( g \) is a complex Gaussian random variable with mean 0 and variance 1. That is, \( g = g^{(x)} + ig^{(y)} \) where \( g^{(x)} \) and \( g^{(y)} \) are independent, mean 0, variance 1/2 real Gaussian random variables.

Fact 1 (Lemma 23 of [5]) Let \( g_i \sim N_C(0, 1) \) be independent complex Gaussian variables. Then, for \( \epsilon \geq 0 \) the probabilities of large deviations are given by

\[
\Pr \left( \frac{1}{N} \sum_{i=1}^{N} |g_i|^2 \geq 1 + \epsilon \right) \leq \exp \left( -N \frac{\pm \epsilon - \ln(1 \pm \epsilon)}{\ln 2} \right). \tag{7}
\]

In particular, for \(-1 \leq \delta \leq 1\), we have \( \delta - \ln(1 + \delta) \geq \frac{\delta^2}{6} \), which implies that for \( 0 \leq \epsilon \leq 1 \),

\[
\Pr \left( \frac{1}{N} \sum_{i=1}^{N} |g_i|^2 \geq (1 + \epsilon) \right) \leq \exp(-CN\epsilon^2), \tag{8}
\]

where \( C \) can be taken to be \((6 \ln 2)^{-1}\).

This can be used to derive:

Fact 2 (adapted from Lemma II.3 of [19]) Let \( \varphi \) be a pure state and \( P \) be a projector of rank \( p \), both on a Hilbert space of dimension \( d \). If \( \{U_i\}_{i=1}^{N} \) are chosen independently and according to the Haar measure on \( \mathbb{U}(d) \), then there exists a constant \( C \geq (6 \ln 2)^{-1} \) such that

\[
\Pr \left( \frac{1}{N} \sum_{i=1}^{N} \text{Tr}(U_i|\varphi\rangle\langle\varphi|U_i^\dagger P) - \frac{p}{d} \geq \pm \frac{ep}{d} \right) \leq \exp \left( -Np \frac{\pm \epsilon - \ln(1 + \epsilon)}{\ln 2} \right). \tag{9}
\]

If \( 0 \leq \epsilon \leq 1 \), we get the simpler upper bound \( \exp(-CNp\epsilon^2) \) as in Fact 1.

We will also use:

Fact 3 (Lemma II.4 of [19]) For \( 0 < \epsilon < 1 \) and \( \dim \mathcal{H} = d \) there exists a set \( \mathcal{N} \) of pure states in \( \mathcal{H} \) with \( |\mathcal{N}| \leq (5/\epsilon)^{2d} \) such that for every pure state \( |\varphi\rangle \in \mathcal{H} \) there exists \( |\tilde{\varphi}\rangle \in \mathcal{N} \) with \( \| |\varphi\rangle - |\tilde{\varphi}\rangle \|_1 \leq \epsilon \). (We call such a set an \( \epsilon \)-net.)

Theorem 1: Proof of security

Security is guaranteed if no unauthorized measurement can distinguish any encoded state from the maximally mixed state. We would like to show that the probability (over random choices of \( U_i \)) of the contrary is small. An unauthorized measurement is LOCC and thus separable [3, 4] over a partition of the \( n \) parties into groups of size \( < k \). We will actually prove security against this larger class of measurements. It suffices to consider measurements with rank one POVM elements since any measurement can be refined to such a measurement without decreasing distinguishability. For example, a measurement implemented by LOCC between three groups of parties \( W_1, W_2 \) and \( W_3 \) will have a POVM of the form \( \{Z_i = Z_i^{(1)} \otimes Z_i^{(2)} \otimes Z_i^{(3)}\} \), where each \( Z_i^{(j)} \) is an operator on the space \( W_j \). Suppose that it is known that

\[
\left| \text{Tr}[Z_iE(\varphi)] - \frac{\text{Tr}[Z_i]}{d^n} \right| \leq \frac{\epsilon \text{Tr}[Z_i]}{2d^n} \tag{10}
\]
for all states $|\varphi\rangle \in S$. Then

$$\sum_{i} \left| \text{Tr}[Z_i E(\varphi)] - \text{Tr}[Z_i \frac{I}{d^n}] \right| \leq \frac{\epsilon}{2},$$

confirming the security condition, Eq. (1), for this particular POVM by the triangle inequality. Thus, it is sufficient to bound

$$\Pr \left( \sup_{|\varphi\rangle \in S} \sup_{Z} \left| \text{Tr}[\frac{1}{r} \sum_{i=1}^{r} ZU_i \varphi U_i] - \frac{1}{d^n} \right| \geq \frac{\epsilon}{2d^n} \right),$$

where the second supremum is over all rank one projectors $Z$ of the form $Z = \otimes_{q=1}^{n} Z_{n_q}$, with $\sum_{q=1}^{n} n_q = n$, $0 \leq n_q < k$ corresponding to some partition of the $n$ parties into groups of size $n_q$ and each $Z_{n_q}$ a rank one projector on $\mathbb{C}^{d^{n_q}}$. Now, if we let each $\mathcal{N}_{n_q}$ be a $\frac{\epsilon}{2(n+2)d^n}$-net for states on $\mathbb{C}^{d^{n_q}}$ and $\mathcal{N}_{S}$ an $\frac{\epsilon}{2(n+2)d^n}$-net on $S$, then

$$\text{max}_{n_q} \mathcal{N}_{(n_q)} \max_{\tilde{\varphi}, \tilde{Z}} \left( \left| \text{Tr}[\frac{1}{r} \sum_{i=1}^{r} \tilde{Z}U_i \tilde{\varphi} U_i] - \frac{1}{d^n} \right| \geq \frac{\epsilon}{2(n+2)d^n} \right),$$

for some $\tilde{Z}_{n_q} \in \mathcal{N}_{n_q}$, $\tilde{\varphi} \in \mathcal{N}_{S}$. By the union bound, the probability of Eq. (12) is therefore bounded above by

$$2^{n \log n} \left( \frac{10(n+2)d^n}{\epsilon} \right)^{2(n+1)d^{k-1}} \exp \left( \frac{-C \epsilon^2}{4(n+2)^2} \right),$$

as long as $s \leq d^{k-1}$, which is the case for $k > 1$. If $k = 1$, all operations are authorized so there is no need for a security requirement. If $d^n > \frac{10(n+2)}{\epsilon}$, this is less than $\frac{1}{4}$ for $r = \left[ \frac{32(n+2)^4}{C \epsilon^2} d^{k-1} \log d \right]$.

**Theorem 1: Proof of correctness**

Our goal is to show that for any $X$ and for all $|\varphi\rangle \in S$, $\| D(X) \circ E(\varphi) - \varphi \|_1 \leq \delta$, again with probability at least $3/4$ over choices of $\{U_i\}$, so that the probability that the statement isn’t true is no more than $1/4$. This would be implied by $\langle \varphi | (D(X) \circ E)(\varphi) | \varphi \rangle \geq 1 - \delta^2/4$ for all $|\varphi\rangle \in S$ [14], which would in turn be implied by

$$\forall i, l \quad \frac{|\langle \varphi | T_{il} P_i U_i | \varphi \rangle|^2}{\| P_i U_i | \varphi \|_2^2} \geq 1 - \frac{\delta^2}{4} := 1 - \alpha.$$
So, it suffices to show that Eq. (16) holds with sufficiently high probability for any fixed $|\varphi\rangle \in S$ that it can be achieved simultaneously for a net of states on $S$.

The proof idea is to take $|\varphi\rangle$ as a member of an orthonormal basis of $S$, $\{|j\rangle\}_{i=1}^d$. The success of using $T^{(X)}$ to decode the states $\{P_i|U_i|j\rangle\}_{i,j}$ can be gauged by considering it as the first piece of a two-stage implementation of the Pretty Good Measurement (PGM) [18], the criterion for success of which is well-understood [17]. More specifically, let $|\xi_{ijl}\rangle = P_l|U_l|j\rangle \sqrt{\frac{1}{r}}$, which is non-zero with probability one. According to Eq. (3), $N = \sum_{ijl} |\xi_{ijl}\rangle \langle \xi_{ijl}|$. For any state $\rho$,

\[ \langle j|T_d \rho T^\dagger_d|j\rangle = \text{Tr}[\rho T^\dagger_d|j\rangle \langle j|T_d] \]

\[ = \text{Tr} \left[ \rho N^{-1/2}P_i \frac{|j\rangle \langle j|}{\sqrt{r}} \frac{U_i}{\sqrt{r}} P_i N^{-1/2} \right] \]

\[ = \text{Tr}[\rho N^{-1/2}|\xi_{ijl}\rangle \langle \xi_{ijl}| N^{-1/2}] \]

\[ = \text{Tr}[\rho M_{ijl}], \]

where $M_{ijl} := N^{-1/2}|\xi_{ijl}\rangle \langle \xi_{ijl}| N^{-1/2}$ is a POVM element of the PGM on the set of unnormalized states $\{|\xi_{ijl}\rangle\}$. Letting $\rho = \frac{P_l|U_l|j\rangle \langle j|U_l^\dagger P_l}{\langle j|U_l^\dagger P_l|U_l|j\rangle}$, the LHS of Eqs. (16) and (17) coincide, while Eq. (20)

\[ \frac{|\langle j|T_d P_l|U_l|j\rangle|^2}{\|P_l|U_l|\varphi\rangle\|_2^2} = \frac{\text{Tr}[|\xi_{ijl}\rangle \langle \xi_{ijl}| M_{ijl}]}{\|\xi_{ijl}\|_2^2}. \]

We must therefore bound the probability of error for the PGM which, it is important to observe, is defined on a set of sub-normalized states, where the normalization of each state gives its probability. An error bound for any equiprobable ensemble is given in [17], and we provide a straightforward generalization to the present case of unequal a priori probabilities in Appendix A. Applying this error bound, and using the orthogonality of the states $|\xi_{ijl}\rangle$ for different values of $l$, we obtain the following:

**Lemma 1** For each $i = 1, \ldots, r$, $l = 1, \ldots, d^{n-k}$, and $j = 1, \ldots, s$,

\[ 1 - \frac{|\langle j|T_d P_l|U_l|j\rangle|^2}{\|P_l|U_l|\varphi\rangle\|_2^2} \leq \Delta_{ijl} := \frac{1}{|\langle j|U_l^\dagger P_l|U_l|j\rangle|^2} \sum_{(i',j')\neq (i,j)} |\langle j'|U_l^\dagger P_l|U_l|j\rangle|^2. \]

(Note that the sum ranges over values of $i'$ from 1 to $r$ and $j'$ from 1 to $s$, not including the pair $(i,j)$.) The intuition behind this result is clear – our probability of misidentification for a fixed state scales roughly like the sum of the overlaps of that state with all the states we could mistake it for, divided by a normalization factor.

In order to evaluate $\Pr(\Delta_{ijl} > \alpha)$, and thus determine the probability (over random choices of $U_i$) that the probability of misidentifying $P_l|U_l|j\rangle$ (in the PGM) is small, we break up the above sum into two terms:

\[ \Delta_{ijl} = \frac{1}{|\langle j|U_l^\dagger P_l|U_l|j\rangle|^2} \sum_{j'} \sum_{i' \neq i} |\langle j'|U_l^\dagger P_l|U_l|j\rangle|^2 + \frac{1}{|\langle j|U_l^\dagger P_l|U_l|j\rangle|^2} \sum_{j' \neq j} |\langle j'|U_l^\dagger P_l|U_l|j\rangle|^2 \]

\[ =: \Delta_{ijl}^1 + \Delta_{ijl}^2. \]

We can use Fact 2 to control the size of the denominator in $\Delta_{ijl}$ with the result that

\[ \Pr \left( \frac{1}{|\langle j|U_l^\dagger P_l|U_l|j\rangle|} \geq 2d^{n-k} \right) \leq \exp \left( -\frac{4d^k}{24 \ln 2} \right). \]
In general, if a particular event $E$ is excluded by a set of conditions $C_1 \land C_2 \land \cdots$, then $\Pr(E) \leq \Pr(\neg C_1) + \Pr(\neg C_2) + \cdots$. (This holds for arbitrary dependence between $E, C_1, C_2, \ldots$) We will use this observation repeatedly in the arguments below.

Turning our attention to $\Delta_{ijl}$, we see that

$$
\Delta_{ijl} = \frac{1}{|\langle j|U_i^\dagger P_i U_i|j \rangle|^2} \sum_{j' \neq i} \sum_{j' \neq j} |\langle j'|U_i^\dagger P_i U_i|j' \rangle|^2
$$

so, using Eq. (24) we find that

$$
\Pr(\Delta_{ijl} > \beta) \leq \Pr\left(\frac{2d^{n-k} \sum_{j' \neq i} \text{Tr}[|\langle j'|U_i^\dagger P_i U_i|j' \rangle|^2 \frac{1}{|\langle j|U_i^\dagger P_i U_i|j \rangle|^2}] > \beta}\right) + \exp\left(-\frac{d^k}{24 \ln 2}\right). \tag{26}
$$

If we choose $(r-1)s \leq \beta d^k/3$ and apply Fact 2, we see that this is no greater than

$$
\exp\left(-\frac{(r-1)s}{24 \ln 2}\right) + \exp\left(-\frac{d^k}{24 \ln 2}\right). \tag{27}
$$

We must also deal with $\Delta_{ijl}^2$. For $k = n$, this is identically zero, whereas for $1 \leq k < n$ we rely on the following lemma, the proof of which can be found in Appendix B.

**Lemma 2** If $0 < \beta, \epsilon \leq 1$, $s \leq \beta d/128$ and $1 \leq k < n$, then

$$
\Pr\left(\frac{1}{|\langle j|U_i^\dagger P_i U_i|j \rangle|^2} \sum_{j' \neq j} |\langle j'|U_i^\dagger P_i U_i|j' \rangle|^2 > \beta\right) \leq 4s \exp\left(-\frac{\beta d^k}{128 \ln 2}\right). \tag{28}
$$

The lemma applies if we choose $s = \frac{\beta \epsilon^2}{800 \sqrt{n} \ln 2}$ with $d$ large enough that $s > 1$. Together, Eqs. (27) and (28) imply (letting $\beta = \frac{\alpha}{4}$) that if we choose $r$ and $s$ such that $\alpha d^k/21 \leq (r-1)s \leq \alpha d^k/12$,

$$
\Pr\left(\Delta_{ijl} > \frac{\alpha}{2}\right) \leq 4s \exp\left(-\frac{\alpha d^k}{512 \ln 2}\right) + \exp\left(-\frac{d^k}{24 \ln 2}\right) \leq 6s \exp\left(-\frac{\alpha d^k}{512 \ln 2}\right). \tag{29}
$$

Eq. (29) tells us that for any $|\varphi \rangle \in S$, in the limit of large $d$, it is overwhelmingly likely that Eq. (16) is satisfied, and thus $D_X \circ E(\varphi)$ is close to $\varphi$. However, we would like to bound the probability of error for all states simultaneously. To do this, let $\eta = \frac{\alpha}{12d^{n-k}}$ and $N_S$ be an $\eta$-net for $S$ with $|N_S| \leq (\frac{5}{\eta})^{2s}$. Then, for $1 \leq k < n$ we find

$$
\Pr\left(\inf_{|\varphi \rangle \in S} \min_{i,l} \frac{\langle \varphi|T_{il} P_i U_i|\varphi\rangle}{\langle \varphi|U_i^\dagger P_i U_i|\varphi\rangle} \leq 1 - \alpha\right)
$$

$$
\leq \Pr\left(\exists i,l, |\varphi \rangle \in N_S \text{ s.t. } \langle \varphi|T_{il} P_i U_i|\varphi\rangle / (\langle \varphi|U_i^\dagger P_i U_i|\varphi\rangle) \leq (1 - \frac{\alpha}{2}) / (\langle \varphi|U_i^\dagger P_i U_i|\varphi\rangle)\right)
$$

$$
+ \Pr\left(\exists i,l, |\varphi \rangle \in N_S \text{ s.t. } \frac{1}{2d^{n-k}} > (\langle \varphi|U_i^\dagger P_i U_i|\varphi\rangle)\right)
$$

$$
\leq rd^{n-k} \left(\frac{5}{\eta}\right)^{2s} \Pr\left(\Delta_{ijl} \geq \frac{\alpha}{2}\right) + rd^{n-k} \left(\frac{5}{\eta}\right)^{2s} \Pr\left(\frac{1}{2d^{n-k}} > (\langle \varphi|U_i^\dagger P_i U_i|\varphi\rangle)\right). \tag{30}
$$
Combining this with Eqs. (24) and (29), we finally find that

\[
\Pr \left( \inf_{|\psi\rangle \in S^\min} \min_{i,l} \frac{\langle \psi | T_i d P_i U_i | \psi \rangle}{\langle \psi | U_i^\dagger P_i U_i | \psi \rangle} \leq 1 - \alpha \right) \leq 6r s d^{n-k} \left( \frac{5}{\eta} \right)^{2s} \exp \left( - \frac{\alpha d^k}{512 \ln 2} \right) + r d^{n-k} \left( \frac{5}{\eta} \right)^{2s} \exp \left( - \frac{d^k}{24 \ln 2} \right),
\]

so that if we require

\[
\left( \begin{array}{c} n \\ k \end{array} \right) \left( \begin{array}{c} 6r s d^{n-k} \left( \frac{5}{\eta} \right)^{2s} \exp \left( - \frac{\alpha d^k}{512 \ln 2} \right) + r d^{n-k} \left( \frac{5}{\eta} \right)^{2s} \exp \left( - \frac{d^k}{24 \ln 2} \right) \right) \leq \frac{1}{4}
\]

we guarantee that the probability of any one of the $D^{(X)}$ failing is no more than $\frac{1}{4}$. The case of $k = n$ can be analyzed in the same way, with the exception that the second term in Eq. (30) is identically zero, resulting in the requirement that

\[
6r s \left( \frac{5}{\eta} \right)^{2s} \exp \left( - \frac{\alpha d^k}{512 \ln 2} \right) \leq \frac{1}{4}.
\]

Both Eq. (32) and Eq. (33) are satisfied if we choose $d$ sufficiently large that $d^k \geq 48/\delta^2$ and, for $k = 1, \frac{2840(2n+3)}{\delta^2} \leq \frac{d}{\log d}$, then choose both $r$ and $s$ according to

\[
\begin{align*}
\frac{d^k}{\log d} & \leq \frac{2840(2n+3)}{\delta^2} \\
\frac{d^k}{\log d} & \leq \frac{2840(2n+3)}{\delta^2} \\

\left( \frac{32(n+2)^4}{Ce^2} \right) \cdot d^{k-1} \log d \right \\
\left( \frac{1536(n+2)^4}{Ce^2 \delta^2} \right) \cdot \frac{d}{\log d}.
\end{align*}
\]

Combining this with our analysis of the probability that a random choice of encoding is secure, we also get the second condition that $d^n > 10(n+2)/\epsilon$. When these requirements are satisfied, $E(\cdot)$ and the set of $D^{(X)}(\cdot)$ provide a $(\delta, \epsilon, s, d^n)$ qubit hiding scheme with $(k, n)$ access structure with probability at least $1/2$.

**IV. DISCUSSION**

We have shown how to construct multiparty hiding schemes with threshold access structures for quantum information. A notable feature of these schemes is that in the limit of hiding a large amount of data, the storage requirement approaches one local physical qubit per hidden qubit. That is, the share that each party holds is of leading order the same size as the hidden state. The accuracy and security parameters incur an overhead that is additive, and therefore negligible from the point of view of the asymptotic rate.

It seems likely that the threshold schemes presented here can be concatenated to provide hiding schemes with non-zero asymptotic hiding rate for arbitrary realizable access structures. (Realizable here meaning consistent with monotonicity since a superset of an authorized set must also be an authorized set [12, 13].) The question of security under concatenation relates to the distillability of our encoding states; if a large amount of entanglement could be distilled from the encoding states, access to some of the encoded data could be sacrificed to compromise the security of the rest. Luckily, based on the results of [20], we suspect that the encoding states have at most a small amount of distillable entanglement between sub-threshold sets of parties,
perihal even a vanishing amount. Still, the connection between the theory of multipartite entangle-
ment and multiparty data hiding is not well understood and deserves further investigation.
At the very least, the states used here have been engineered with very extreme properties: small
sub-threshold distinguishability, high symmetry, high entanglement of formation and likely low
distillable entanglement.

Beyond the question of security under concatenation, it would be worth knowing whether a
composable definition of data hiding could be formulated and whether the schemes presented
here would realize the definition. In a similar vein, we would like to know the extent to which
the schemes presented here are stable against small amounts of entanglement shared between the
parties; does security fail all at once or gracefully? Can schemes completely robust against finite
amounts of entanglement be designed? The strongest possible such result would be a demonstra-
tion that the schemes are secure whenever there is insufficient entanglement to teleport any local
shares.

Moreover, while imperfect security is inevitable in data hiding (at least in the absence of su-
perselection rules [22, 26]), perfect accuracy is possible, as demonstrated by [9] and [10]. Is perfect
accuracy possible while simultaneously achieving the rates found in this paper? Since the bulk of
the technical difficulty in the present paper comes from proving the existence of sufficiently good,
albeit imperfect decodings, a scheme with perfect decodings could conceivably be significantly
simpler.

Finally, while we have not provided an explicit construction of the encoding map, it is im-
portant to notice that the probability of 1/2 in Theorem 1 is arbitrary, and could be chosen ar-
bitrarily close to one at the expense of increasing (decreasing) the proportionality constant for
r (s) in Eq. (6). In this sense, secure and accurate hiding schemes of the form we present are
generic in the limit of large dimension. Nevertheless, it would be more satisfying to find explicit,
non-probabilistic choices for the encoding unitaries, implementable in polynomial time, and still
giving secure and accurate data hiding schemes, as was done for approximate encryption in [1].

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APPENDIX A: SUCCESS CRITERION FOR THE PGM WITH UNEQUAL STATE PROBABILITIES

Using the method of [17], we will bound the probability of error for the PGM when the states
occur with unequal probabilities. (The symbols used in this derivation are locally defined.) Let
\{\ket{\xi_i}\} be a set of subnormalized states so that $||\ket{\xi_i}||_2^2$ represents the probability of
\ket{\xi_i}, and $N = \sum_i \ket{\xi_i}\bra{\xi_i}$. The elements of the PGM are $M_i = N^{-1/2}\ket{\xi_i}\bra{\xi_i}N^{-1/2}$. We define the matrix

$$T_{ij} = \bra{\xi_i}\xi_j\rangle,$$  \hfill (A1)

which has square root

$$\sqrt{T}_{ij} = \bra{\xi_i}N^{-1/2}\rangle\bra{\xi_j}.$$

\hfill (A2)
In terms of $T$, the probability of the PGM correctly identifying $|\xi_i\rangle$ is

$$p(i|i) = \frac{\langle \xi_i|M_i|\xi_i\rangle}{\langle \xi_i|\xi_i\rangle} = \frac{(\sqrt{T})_{ii}}{\langle \xi_i|\xi_i\rangle}.$$  \hfill (A3)

Applying the inequality $\sqrt{x} \geq \frac{3}{2}x - \frac{1}{2}x^2$ to the matrix $\sqrt{T} = \frac{\sqrt{T}}{\langle \xi_i|\xi_i\rangle}$ (which we are able to do, since the present choices of $\sqrt{x}$ and $x$ are Hermitian and nonnegative) and noting that the inequality holds for the diagonal entries gives

$$\frac{(\sqrt{T})_{ii}}{\langle \xi_i|\xi_i\rangle} \geq \frac{3}{2} T_{ii} - \frac{1}{2} \sum_j |T_{ij}|^2 = 1 - \frac{1}{2} \sum_{j \neq i} |\langle \xi_i|\xi_j\rangle|^2 = 1 - \frac{1}{2} \sum_{j \neq i} |\langle \xi_i|\xi_j\rangle|^2.$$  \hfill (A4)

Combining Eq. (A3) and Eq. (A4) yields the result

$$p(i|i) \geq 1 - \sum_{j \neq i} \frac{|\langle \xi_i|\xi_j\rangle|^2}{|\langle \xi_i|\xi_i\rangle|^2},$$

which is equivalent to

$$p(-i|i) \leq \sum_{j \neq i} \frac{|\langle \xi_i|\xi_j\rangle|^2}{|\langle \xi_i|\xi_i\rangle|^2}. $$



**APPENDIX B: PROOF OF LEMMA 2**

We would like to show that if $0 < \beta, \epsilon \leq 1$, $s \leq \beta d/128$ and $1 \leq k < n$, then

$$\Pr\left(\frac{1}{\langle \langle j|U_i^j P_i U_i|j\rangle \rangle} \sum_{j \neq j} |\langle j'|U_i^j P_i U_i|j\rangle |^2 > \beta \right) \leq 4 s \exp\left(-\frac{\beta d^k}{128 \ln 2}\right).$$

(B1)

Our argument is somewhat lengthy but completely elementary. We will make use of the fact that a Haar-distributed state in $\mathbb{C}^d$ can be expressed as $|\gamma\rangle$, where $|\gamma\rangle$ is a $d'$-dimensional vector with all coordinates drawn independently from $\mathcal{N}_C(0, 1)$. (Throughout this section, any variable written as $g_x$, $g_y$, or $g^{ab}_{xy}$, for arbitrary values of $x$, $y$, $a$ and $b$, will denote a random variable drawn from $\mathcal{N}_C(0, 1)$. They are all chosen independently.) To begin, we can express the marginal distributions of the $U_i|j'\rangle$ in terms of complex Gaussians (ignoring correlations between the vectors $U_i|j'_1\rangle$ and $U_i|j'_2\rangle$):

$$U_i|j'\rangle = \frac{1}{\sqrt{\sum_{h=1}^{d^k} \sum_{m=1}^{d^{n-k}} |g_{hm}|^2}} \sum_{h=1}^{d^k} \sum_{m=1}^{d^{n-k}} g_{hm}^i |h\rangle |m\rangle.$$  \hfill (B2)

It will also be useful to note that $\epsilon - \ln(1 + \epsilon) \geq \frac{\epsilon}{2}$ if $\epsilon \geq 6$, in which case Eq. (8) can be replaced by

$$\Pr\left(\frac{1}{\sqrt{\sum_{i=1}^{N} |g_i|^2}} \geq (1 + \epsilon) \right) \leq \exp\left(-\frac{\epsilon}{2 \ln 2}\right).$$

(B3)
Using Eqs. (8) and (B3), the fact that $d^{n-1} \geq 128s$ and recalling that $j' = 1, \ldots, s$, we find
\[
\Pr \left( \sum_{i=1}^{d^n} |g_i|^2 > \frac{1}{4d} \right) \leq \Pr \left( \sum_{i=1}^{d^n} |g_i|^2 > \frac{1}{4d} d^n - j' \right) + \Pr \left( \sum_{i=1}^{d^n} |g_i|^2 < \frac{d^n - j'}{2} \right)
\]
\[
\leq \exp \left( - \frac{d^{n-1}}{32 \ln 2} \right) + \exp \left( - \frac{d^n}{32 \ln 2} \right) \leq 2 \exp \left( - \frac{d^{n-1}}{32 \ln 2} \right), \tag{B4}
\]
which will be useful below. To complete our task, however, we will need to move beyond the simplest Gaussian approximation to the distribution of the $\{U_i|j'\}$ that takes into account the correlations between the vectors for different values of $j'$. We can relate $\{U_i|j'\}$ to a collection of independent Haar-distributed vectors, $|\psi^{j'}\rangle$, as follows:
\[
|\psi^{j'}\rangle = \frac{1}{\sqrt{\sum_{t=1}^{d^n} |g_t^{j'}|^2}} \left( \sum_{t=1}^{d^n} g_t^{j'} U_i|j'\rangle + \sum_{t=1}^{j'-1} g_t^{j'} U_i|t\rangle \right) \tag{B5}
\]
To see this, notice that $U_i|j'\rangle$ is distributed uniformly in the orthogonal complement to the span of $\{U_i|t\}_{t=1}^{d^n}$, which means that it can be represented as
\[
U_i|j'\rangle = \left( \sum_{t=j'}^{d^n} |g_t^{j'}|^2 U_i|j'\rangle \right) - \frac{1}{\sqrt{\sum_{t=j'}^{d^n} |g_t^{j'}|^2}} \sum_{t=1}^{j'-1} g_t^{j'} U_i|t\rangle.
\tag{B7}
\]
Inverting Eq. (B5) gives
\[
U_i|j'\rangle = \frac{\sqrt{\sum_{t=j'}^{d^n} |g_t^{j'}|^2}}{\sqrt{\sum_{t=1}^{d^n} |g_t^{j'}|^2}} |\psi^{j'}\rangle - \frac{1}{\sqrt{\sum_{t=1}^{d^n} |g_t^{j'}|^2}} \sum_{t=1}^{j'-1} g_t^{j'} U_i|t\rangle
\]
so that
\[
\langle 1|U_i^\dagger P_i U_i|j'\rangle = \frac{\sqrt{\sum_{t=1}^{d^n} |g_t^{j'}|^2}}{\sqrt{\sum_{t=1}^{d^n} |g_t^{j'}|^2}} \langle 1|U_i^\dagger P_i |\psi^{j'}\rangle - \frac{1}{\sqrt{\sum_{t=1}^{d^n} |g_t^{j'}|^2}} \sum_{t=1}^{j'-1} g_t^{j'} \langle 1|U_i^\dagger P_i U_i|t\rangle. \tag{B8}
\]
Using the inequality $|a + b|^2 \leq 2(|a|^2 + |b|^2)$ then leads to
\[
|\langle 1|U_i^\dagger P_i U_i|j'\rangle|^2 \leq 2 \left( \frac{\sum_{t=1}^{d^n} |g_t^{j'}|^2}{\sum_{t=j'}^{d^n} |g_t^{j'}|^2} \langle 1|U_i^\dagger P_i |\psi^{j'}\rangle|^2 + \frac{1}{\sum_{t=j'}^{d^n} |g_t^{j'}|^2} \sum_{t=1}^{j'-1} g_t^{j'} \langle 1|U_i^\dagger P_i U_i|t\rangle|^2 \right)
\]
\[
\leq 2 \left[ \left( 1 + \frac{\sum_{t=1}^{j'-1} |g_t^{j'}|^2}{\sum_{t=j'}^{d^n} |g_t^{j'}|^2} \right) \langle 1|U_i^\dagger P_i |\psi^{j'}\rangle|^2 + \frac{\sum_{t=1}^{j'-1} |g_t^{j'}|^2}{\sum_{t=j'}^{d^n} |g_t^{j'}|^2} \sum_{t=1}^{j'-1} \langle 1|U_i^\dagger P_i U_i|t\rangle|^2 \right]. \tag{B9}
\]
Summing over values of $j'$ in Eq. (B9) and using Eq. (B4) shows that
\[
\Pr \left( \sum_{j'=2}^{s} |\langle 1|U_i^\dagger P_i U_i|j'\rangle|^2 > 2 \left( \sum_{j'=2}^{s} \frac{1}{4d} |\langle 1|U_i^\dagger P_i |\psi^{j'}\rangle|^2 \right) \right) \tag{B10}
\]
is less than or equal to $2s \exp\left(-\frac{dn-1}{32 \ln 2}\right)$, which implies in turn that
\[
\Pr\left(\sum_{j'=2}^{s} |\langle 1|U_iP|j'\rangle|^2 > 4 \sum_{j'=2}^{s} |\langle 1|U_iP|\psi'\rangle|^2 + \frac{s}{2d} \sum_{t=2}^{s} |\langle 1|U_iP|t\rangle|^2 + \frac{s}{2d} |\langle 1|U_iP|1\rangle|^2\right)
\]
(B11)
is bounded above by the same $2s \exp\left(-\frac{dn-1}{32 \ln 2}\right)$. Moving the second sum on the RHS to the LHS and noting $\frac{s}{d} \leq 1$ shows that
\[
\Pr\left(\frac{1}{|\langle 1|U_iP|U_1\rangle|^2} \sum_{j'=2}^{s} |\langle 1|U_iP|j'\rangle|^2 > \frac{8}{|\langle 1|U_iP|U_1\rangle|^2} \sum_{j'=2}^{s} |\langle 1|U_iP|\psi'\rangle|^2 + \frac{s}{d}\right)
\]
(B12)
is again bounded above by $2s \exp\left(-\frac{dn-1}{32 \ln 2}\right)$. Finally, we can upper bound
\[
\Pr\left(\frac{1}{|\langle 1|U_iP|U_1\rangle|^2} \sum_{j'=2}^{s} |\langle 1|U_iP|\psi'\rangle|^2 > \frac{\beta}{16}\right)
\]
(B13)
by using Fact 2, Eq. (24) and the estimate leading to Eq. (B3), along with the observation that $|\psi^1\rangle = U_i|1\rangle$, with the result that the probability in Eq. (B13) is less than or equal to
\[
\exp\left(-\frac{\beta d^k}{128 \ln 2}\right) + \exp\left(-\frac{d^k}{24 \ln 2}\right).
\]
(B14)
Combining this with the bound on Eq. (B12) and noting $\frac{s}{d} \leq \frac{\beta}{2}$ gives the result
\[
\Pr\left(\frac{1}{|\langle 1|U_iP|U_1\rangle|^2} \sum_{j'=2}^{s} |\langle 1|U_iP|j'\rangle|^2 > \beta\right)
\leq \exp\left(-\frac{\beta d^k}{128 \ln 2}\right) + \exp\left(-\frac{d^k}{24 \ln 2}\right) + 2s \exp\left(-\frac{d^{n-1}}{32 \ln 2}\right) \leq 4s \exp\left(-\frac{\beta d^k}{128 \ln 2}\right).
\]
(B15)

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