Ultimate stress state of a plate made of orthotropic material under uniform internal pressure

N S Blokhina
Moscow State University of Civil Engineering, Yaroslavskoe shosse, 26, Moscow, 129337, Russia
E-mail: nsb_sapr@mail.ru

Abstract. The paper regards ultimate stress state of an unbounded plate with a circular opening made of orthotropic material. Unlike isotropic material, this problem is not axisymmetric. The solution to the problem was carried out by an iterative method. As a first approximation, a corresponding problem solution was used, assuming that the radial and tangential stresses are the main ones. The obtained solution is compared with the stress state of the plate, achieved under the assumption that the problem is axisymmetric.

1. Introduction
The use of new highly developed materials allows the building industry to cut costs and apply new constructive forms. In the practice of modern construction, materials with anisotropic properties are considered to be reliable and economically effective and so are increasingly being used. But wide use of such materials also requires constant development of structural analysis. Taking into account nonlinear behavior of the materials, the correct choice of appropriate strength conditions allows to reasonably evaluate the building process, which ultimately results in more economical solutions. The limits of strength or yield strength of an anisotropic material depend on the direction, that is, they are not scalar values, a fact which greatly complicates the development of strength and plasticity criteria for anisotropic materials. The study of strength criteria for orthotropic materials [1–3] shows us that there is no universal strength criterion, so we have to use several criteria. Besides, strength and yield criteria for anisothropic materials in state of plane stress are very complicated and therefore not suitable for use in practical engineering design [4]. So a group of research engineers of Moscow Research Institute of Building Constructions (TSNIISK) led by G.A. Geniev introduced a new strength criterion for an orthotropic material with the same compressive strength along and across the fibers [5]. Its analytical expression is as follows:

\[
\left( \frac{\cos^2 \beta}{R_{lt}} + \frac{\sin^2 \beta}{R_{2t}} \right) \sigma_2 - \frac{\sigma_1 \sigma_2}{R_c} = \left( \frac{\sin^2 \beta}{R_{lt}} + \frac{\cos^2 \beta}{R_{2t}} \right) \sigma_1 - \frac{\sigma_1 \sigma_2}{R_c} - \frac{1}{R_c} = 0, \quad (1)
\]

where \( \sigma_1 \) and \( \sigma_2 \) are the maximum and minimum principal stresses; \( \beta \) — angle is made by reinforcing direction and bigger principal normal stress direction \( \sigma_1 \); \( R_{lt} \) is the limiting strength in uniaxial tension along the fiber; \( R_{2t} \) is the limiting strength in uniaxial compression, constant in any direction.
In those stressed areas, where the destruction is not caused by the breaking away, the criterion (1) can be interpreted as a condition of plasticity of a given material.

An analytical solution to problems based on the strength and plasticity criterion (1) is impossible, therefore the author used a linearized strength criterion for orthotropic materials introduced by G.A. Geniev:

\[
\left( \frac{\cos^2 \beta}{R_1} + \frac{\sin^2 \beta}{R_2} \right) \sigma_1 + \frac{\sigma_2}{R_c} = 1. \tag{2}
\]

Graphical representation of the criterion (2) is displayed in figure 1. In this case an analytical solution of an orthotropic plate with a round hole is possible.

\[
\begin{aligned}
\sigma_1 &= \text{Graphical display of linearized strength criterion. Solid line represents angle } \\
\beta &= 0 \text{ and dotted line represents angle } \beta = \pi/2.
\end{aligned}
\]

2. Method of considering orthotropic behaviour of material
The paper deals with ultimate stress state of an unbounded plate with a circular opening made of orthotropic material. The linearized condition (2) is regarded as the strength condition. The problem is not axisymmetric, as it would be for an isotropic material. This circumstance leads to the necessity of solving a rather complex system of two quasilinear differential equations including partial derivatives, the analytical solution of which is impossible.

In this case, it turned out to be reasonable to use the iterative method, where the solution of the corresponding axisymmetric problem was used as the first approximation.

Figure 2 shows a plate with a circular hole loaded around the edge of the hole by a uniformly distributed normal load of \( R_c \).

If the plate material were isotropic, then the problem would be axisymmetric. In case of an orthotropic material the problem is symmetrical along axes \( x \) and \( y \) if one of the axes is directed along the fibers of the material.

If the problem is axisymmetric, then only one of the two equilibrium equations remains, i.e.

\[
\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_0}{r} = 0. \tag{3}
\]

Moreover, the directions of the main stresses coincide with the directions \( \sigma_r \) and \( \sigma_0 \). In accordance with the common rule of signs (tensile stresses are considered positive, compressive stresses are negative), \( \sigma_0 \) is the maximum principal stress; \( \sigma_r \) is the minimum principal stress. Under this condition, (2) is noted down as

\[
\left( \frac{\cos^2 \beta}{R_1} + \frac{\sin^2 \beta}{R_2} \right) \sigma_0 + \frac{\sigma_r}{R_c} = 1. \tag{4}
\]

By using condition (3) we obtain

\[\sigma_r - \sigma_0 = k\sigma_r + l,\]
Figure 2. A plate with a round hole, loaded on its contour by a uniformly distributed load of \( R_c \) value.

where

\[
k = 1 - \frac{R_{1p} R_{2p}}{R_c (R_{2p} \cos^2 \beta + R_{1p} \sin^2 \beta)}, \quad I = (k - 1)R_c.
\]  

(5)

On the basis of (3) and (5)

\[
\frac{d\sigma_r}{k \sigma_r + 1} + \frac{dr}{r} = 0.
\]  

(6)

By integrating (6), we’ll find

\[
\frac{1}{k} \ln(k \sigma_r + I) + \ln r + C = 0.
\]

Arbitrary constant \( C \) is determined using the boundary condition: when \( r = r_0 \), \( \sigma_r = -R_c \). The final expression for radial and tangential stress is

\[
\sigma_r = k \left( R_c - \frac{R_c r_0^k}{r^k} \right) - R_c, \quad \sigma_0 = k - 1 \left( \frac{R_c r_0^k}{r^k} - R_c \right).
\]  

(7)

Thus, we obtain the solution to the problem, assuming that it is axisymmetric.

In our case, the main maximum stress is \( \sigma_0 \). It’s obvious that \( \beta = \pi/2 + \theta \), due to this

\[
k = 1 - \frac{R_{1p} R_{2p}}{R_c (R_{2p} \sin^2 \theta + R_{1p} \cos^2 \theta)}.
\]

Therefore, \( \sigma_r \) and \( \sigma_0 \) depend parametrically on the polar angle \( \theta \).

Placing solution (7) into the equilibrium equation of the general case of the plane problem

\[
\frac{1}{r} \frac{\partial \sigma_0}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2 \tau_{r\theta}}{r} = 0,
\]  

(8)

and solving it with respect to \( \tau_{r\theta} \), we obtain a shear stress, which is a non-viscous, making up for the inaccuracy of the first approximation.

Thus, \( \sigma_r \) and \( \sigma_0 \) by expressions (7), which are the solution of the axisymmetric problem, and \( \tau_{r\theta} \), obtained from equation (8), describe the stress state corresponding to the second approximation.

Let’s note (8) down as follows
\[ \frac{\partial \tau_{\theta\theta}}{\partial r} + a(r) \tau_{\theta\theta} = f(r). \]  

Equation (9) is a linear equation of the first order, admitting an integrating factor

\[ \mu = \mu(r) = e^{\int a(r) \, dr}. \]

The solution (9) looks as follows

\[ \tau_{\theta\theta} = \frac{1}{\mu(r)} \left( \int f(r) \mu(r) \, dr + C \right). \]

In our case

\[ \mu(r) = e^{\int a(r) \, dr} = e^{2\ln r} = r^2, \]

\[ f(r) = -\frac{1}{r} \frac{\partial \sigma_0}{\partial \theta} = k'R_0 \left( \frac{r_0}{r} \right)^k + 1 - k(k - 1) \left( \frac{r_0}{r} \right)^k \ln \frac{r_0}{r}, \]

where

\[ k' = \frac{dk}{d\theta} = - \frac{R_{1p} R_{2p}}{R_e} \frac{\sin(2\theta)(R_{2p} - R_{1p})}{R_{2p} \sin^2 \theta + R_{1p} \cos^2 \theta)^2}. \]

Let’s place (11) into (10):

\[ \tau_{\theta\theta} = \frac{1}{r^2} \left( \int \frac{k'R_0}{k^2} \left[ \left( \frac{r_0}{r} \right)^k + 1 - k(k - 1) \left( \frac{r_0}{r} \right)^k \ln \frac{r_0}{r} \right] \, dr + C \right). \]

After integration and determination of constant derivative as a result of \( r = r_0 \), \( \tau_{\theta\theta} = 0 \), we find that

\[ \tau_{\theta\theta} = \frac{k'R_e}{k^2(2-k)^2} \left[ (-k^2 + 2k - 2) \left( \frac{r_0}{r} \right)^k + \frac{(2-k)^2}{2} - (k-1)(2-k)k \left( \frac{r_0}{r} \right)^k \ln \frac{r_0}{r} + \frac{k^4}{2} \left( \frac{r_0}{r} \right)^2 \right]. \]

Since \( \tau_{\theta\theta} \) depends parametrically on the angle \( \theta \), it is possible to determine the stress state for each specific value of \( \theta \).

For an orthotropic material with the following values of strengths \( R_{1p} = 15 \text{ MPa}, R_{2p} = 11 \text{ MPa}, R_e = 45 \text{ MPa} \), calculations of values \( \sigma_x, \sigma_\theta, \tau_{\theta\theta} \) have been carried out, for \( \theta = 0, 15^\circ, 30^\circ, 45^\circ, 60^\circ, 75^\circ, 90^\circ \) with \( r \) varying from \( r_0 \) to \( 3r_0 \). The stress values for the other three quadrants are determined from symmetry considerations.

3. Results

The analysis of the data obtained put on display that the deviation of the main stress trajectory from the straight line is insignificant and increases with distance from the center. The maximum deviation of the trajectory corresponding to the angle \( \theta = 45^\circ \) decreases as it approaches the x and y axes (that is, as it approaches the direction corresponding to the direction of the fibers and perpendicular to it) and becomes equal to zero, having reached the direction of the axes \( \theta = 0, \theta = \pi/2 \).

The table 1 shows the results of calculations of \( \sigma_x, \sigma_\theta, \tau_{\theta\theta} \) for \( \theta = 45^\circ \). The second and third columns of the table 1 are values of stresses \( \sigma_x \) and \( \sigma_\theta \) corresponding to the first approximation. The second approximation corresponds to the same stresses \( \sigma_x, \sigma_\theta \) and tangential stress \( \tau_{\theta\theta} \). The fifth column of the table shows the angles to which the principal axes of the stresses, corresponding to the stress state of the second approximation, rotate with respect to the principal axes of the axisymmetric problem.
Table 1. The value of stresses and angles of rotation of the main axes for $\theta = 45^\circ$.

| $r / r_0$ | $\sigma_r$, MPa | $\sigma_\theta$, MPa | $\tau_{r\theta}$, MPa | A gradient |
|-----------|-----------------|----------------------|------------------------|------------|
| 1         | -46             | 0                    | 0                      | 0          |
| 1.2       | -38.14          | 2.16                 | 0.056                  | -0.074     |
| 1.4       | -32.26          | 3.79                 | 0.169                  | -0.269     |
| 1.6       | -27.67          | 5.05                 | 0.299                  | -0.515     |
| 1.8       | -23.97          | 6.07                 | 0.429                  | -0.8       |
| 2         | -20.93          | 6.91                 | 0.553                  | -1.08      |
| 2.2       | -18.36          | 7.62                 | 0.670                  | -1.43      |
| 2.4       | -16.17          | 8.22                 | 0.778                  | -1.77      |
| 2.6       | -14.27          | 8.75                 | 0.878                  | -2.17      |
| 2.8       | -12.61          | 9.21                 | 0.971                  | -2.52      |
| 3         | -11.1           | 9.61                 | 1.056                  | -2.86      |

The table 2 includes the results of the calculations $\sigma_r$, $\sigma_\theta$, $\tau_{r\theta}$ for $\theta = 90^\circ$.

Table 2. The value of stresses and angles of rotation of the main axes for $\theta = 90^\circ$.

| $r / r_0$ | $\sigma_r$, MPa | $\sigma_\theta$, MPa | $\tau_{r\theta}$, MPa | A gradient |
|-----------|-----------------|----------------------|------------------------|------------|
| 1         | -46             | 0                    | 0                      | 0          |
| 1.2       | -38.1           | 2.57                 | 0                      | 0          |
| 1.4       | -32.15          | 5.51                 | 0                      | 0          |
| 1.6       | -27.46          | 6.04                 | 0                      | 0          |
| 1.8       | -23.67          | 7.27                 | 0                      | 0          |
| 2         | -22             | 7.81                 | 0                      | 0          |
| 2.2       | -17.86          | 9.17                 | 0                      | 0          |
| 2.4       | -15.57          | 9.91                 | 0                      | 0          |
| 2.6       | -13.59          | 10.67                | 0                      | 0          |
| 2.8       | -11.84          | 11.13                | 0                      | 0          |
| 3         | -11.04          | 11.39                | 0                      | 0          |

Figure 3 shows diagrams of radial and tangential stress in the plate.

Figure 3. Diagrams of radial and tangential stress in the plate: 1 — for $\theta = 0$; 2 — for $\theta = 90^\circ$; 3 — for $\theta = 180^\circ - 15^\circ$; 4 — for $\theta = 180^\circ + 75^\circ$. 
Let us calculate the values of principal stresses corresponding to the second approximation for \( \theta = 45^\circ \), where \( r = 2.6r_0 \), \( \sigma_1 = 8.75 \) MPa, \( \sigma_2 = -14.30 \) MPa, which differ from the corresponding values of the first approximation by only 0.34%. This difference is within the limits of accuracy of calculation.

4. Discussion
Strength and yield criteria analysis of building structures made of anisotropic materials usually requires computational design methods. For example, article [7] deals with the problem of physical nonlinearity in construction design. The author introduces a methodology of calculating the stress-strain state of building structures which is a variant of deformation theory of plasticity developed by G.A. Geniev, based on modified Newton-Raphson method. Article [8] describes a stress-strain state of a plate made from anisotropic nonlinear-elastic material. The analysis was carried out with the help of ANSYS software package. The solution obtained in this paper is analytical.

5. Conclusion
Given calculations allow us to come to the following conclusion: for given values of the strength limits, the solution of the problem, assuming that the radial and tangential stresses are the main stresses, can be considered a solution to the problem with accuracy sufficient for engineering calculations. This allows us to make the following assumption: for other orthotropic materials, for which \( R_{1p} \) and \( R_{2p} \) do not differ much from each other, the solution to the problem obtained with this assumption will also be quite close to the true one. The results obtained can be used when carrying out calculation for plates, weakened by cuts, and made of orthotropic materials.

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