The Law of Total Odds

Dirk Tasche∗

First version: December 3, 2013
This version: February 14, 2014

The law of total probability may be deployed in binary classification exercises to estimate the unconditional class probabilities if the class proportions in the training set are not representative of the population class proportions. We argue that this is not a conceptually sound approach and suggest an alternative based on the new law of total odds. We quantify the bias of the total probability estimator of the unconditional class probabilities and show that the total odds estimator is unbiased. The sample version of the total odds estimator is shown to coincide with a maximum-likelihood estimator known from the literature. The law of total odds can also be used for transforming the conditional class probabilities if independent estimates of the unconditional class probabilities of the population are available.

Keywords: Total probability, likelihood ratio, Bayes’ formula, binary classification, relative odds, unbiased estimator, supervised learning, dataset shift.

1 Introduction

The law of total probability is one of the fundamental building blocks of probability theory. Its elementary version states that for an event $A$ and a partition $H_i$, $i \in \mathbb{N}$ of the whole space the probability of $A$ can be calculated as

$$P[A] = \sum_{i=1}^{\infty} P[H_i] P[A | H_i],$$

(1.1)

where the conditional probabilities $P[A | H_i]$ are defined as

$$P[A | H_i] = \begin{cases} \frac{P[A \cap H_i]}{P[H_i]}, & \text{if } P[H_i] > 0, \\ 0, & \text{if } P[H_i] = 0. \end{cases}$$

Kolmogorov (1956) calls Eq. (1.1) the theorem of total probability. It is also called rule or formula of total probability. Virtually all text books on probability theory mention Eq. (1.1) but many authors (e.g. Feller, 1968, Chapter V, Eq. (1.8)) do not name it.

Feller (1968) comments on Eq. (1.1) with the words “This formula is useful because an evaluation of the conditional probabilities $P[A | H_i]$ is frequently easier than a direct calculation of $P[A]$.” Sometimes it

∗E-mail: dirk.tasche@gmx.net
The author currently works at the Prudential Regulation Authority (a division of the Bank of England). He is also a visiting professor at Imperial College, London. The opinions expressed in this paper are those of the author and do not necessarily reflect views of the Bank of England.
may even be impossible to directly calculate \( P[A] \). In particular, this is the case when \( P[A] \) is assumed to be forecast but past observations of occurrences of event \( A \) cannot be relied on because the value of \( P[A] \) might have changed.

Such a situation is likely to be incurred in binary classification exercises where the unconditional (or prior) class probabilities in the training dataset may differ from the class probabilities of the population to which the classifier is applied (see Moreno-Torres et al., 2012, for a recent survey of data shift issues in classification). Typically, a classifier produces class probabilities (i.e. probabilities of tested examples to be of – say – class \( A \)) conditional on already known features \( H_n \) of the examples. If the unconditional distribution of the \( H_n \) (i.e. the probabilities \( P[H_i] \)) is also known, Eq. (1.1) then can be used to make a forecast (or point estimate) of \( P[A] \).

It can be argued, however, that the forecasts of unconditional class probabilities produced this way are biased (see Section 2.2 of Xue and Weiss, 2009, or Tasche, 2013, and Proposition 2.8 below). This is a consequence of the fact that fundamentally the conditional class probabilities \( P[A \mid H_i] \) are determined by means of Bayes’ formula (assuming \( P[H_i] > 0 \) and \( P[A] > 0 \)):

\[
P[A \mid H_i] = \frac{P[A]P[H_i \mid A]}{P[A]P[H_i \mid A] + P[A^c]P[H_i \mid A^c]}
\]

(1.2)

where \( A^c \) denotes the event complementary to \( A \) and \( P_0[A] \) and \( P_0[A^c] \) are the unconditional probabilities of class \( A \) and \( A^c \) respectively in the training dataset. The conditional probabilities \( P[H_i \mid A] \) and \( P[H_i \mid A^c] \) reflect the distributions of the characteristic features on class \( A \) and its complementary class respectively.

On the one hand, Eq. (1.2) suggests a potentially unintended impact of the training set class probabilities on the population class estimates. On the other hand, Eq. (1.2) also suggests that an estimate of \( P[A] \) based solely on the conditional likelihood ratio \( i \mapsto \lambda_i \) would avoid this issue.

This paper presents in Theorem 2.5 below a necessary and sufficient criterion for when it is possible to estimate a population class probability based on the unconditional distribution of the features of the tested examples and the conditional likelihood ratio. The likelihood ratio \( \lambda_i \) can also be written as

\[
\lambda_i = \frac{P[A^c \mid H_i]}{P[A \mid H_i]} \frac{P_0[A]}{P_0[A^c]}
\]

(1.3)

By Eq. (1.3), \( \lambda_i \) can alternatively be described as the ratio of the conditional and unconditional odds of class \( A^c \) or the relative odds of class \( A^c \). This observation suggests that Theorem 2.5 is called law of total odds in analogy to the law of total probability Eq. (1.1).

It turns out that the prior class probability estimator suggested by Theorem 2.5 is the two-class special case of the maximum likelihood estimator discussed by Saerens et al. (2002). Equation (2.3) from Theorem 2.5 has recently been studied in the \( n \)-class case by Du Plessis and Sugiyama (2014, Eq. (9)). The contributions of this paper (limited to the case of binary classification) to the existing literature can be described as follows:

- It is shown that the total odds estimator not only solves a prior probability shift problem\(^1\) but also, at the same time, a combined covariate shift and concept shift problem where only the relative odds are the same for training set and population (or test set).
- We demonstrate that the maximum likelihood estimator introduced by Saerens et al. (2002) and studied in more detail by Du Plessis and Sugiyama (2014) does not always exist.
- We show how to determine conditional class distributions in the population or test set.

\(^1\)See Moreno-Torres et al. (2012) for the definitions of the various datashift problems.
It becomes clear that – in the binary case – the total odds estimates can be computed by simple numerical root-finding. There is no need to deploy the expectation-maximisation or other more advanced iterative algorithms as discussed by Saerens et al. (2002), Xue and Weiss (2009) or Du Plessis and Sugiyama (2014).

We provide sharp error bounds for the prior class probability estimate when the covariate shift is ignored. This approach is called 'total probability' below.

2 Results

It is useful to consider the use of Eq. (1.1) for estimating the class probability $P[A]$ in a more general setting.

Assumption 2.1 $(\Omega, A, P_0)$ is a probability space. $\mathcal{H}$ is a sub-$\sigma$-field of $A$, i.e. $\mathcal{H} \subset A$. $P_1$ is a probability measure on $(\Omega, \mathcal{H})$ that is absolutely continuous with respect to $P_0 | \mathcal{H}$, i.e. $P_1 \ll P_0 | \mathcal{H}$. $E_i$ denotes the expectation operator based on $P_i$.

The interpretation of Assumption 2.1 is as follows:

- $(\Omega, A, P_0)$ is a model that has been fit to historical observations (e.g. the training set of a classifier).
- $\sigma$-field $\mathcal{H}$ represents the scores produced by the model (classifier) while $\sigma$-field $A$ additionally contains information regarding the classes of the tested examples.
- $(\Omega, \mathcal{H}, P_1)$ is the outcome of an application of the model to a different set of – possibly more up-to-date – observations. $(\Omega, \mathcal{H}, P_1)$ could be a representation of the distribution of the scores produced by the classifier.
- The general problem is to extend $P_1$ to $A$, by using information from $(\Omega, A, P_0)$.
- More specifically, the problem might only be to obtain an estimate $P_1^*[A]$ for a fixed event (or class) $A \in A \setminus \mathcal{H}$, as described in Section 1. However, to make sure that the estimate is meaningful it should be based on a valid model – which would be an extension of $P_1$ to any $\sigma$-field containing $A$.
- $P_1 \ll P_0 | \mathcal{H}$ is a technical assumption that has intuitive appeal, however. For prediction based on $(\Omega, A, P_0)$ would be pointless if there were events that were possible under $P_1$ but impossible under $P_0$.

The most obvious extension of $P_1$ to $A$ is by means of the conditional probabilities $P_0[A | \mathcal{H}]$ determined under the measure $P_0$. Formally, the extension is defined by

$$P_1^*[A] = E_i [P_0[A | \mathcal{H}]], \quad A \in A. \quad (2.1)$$

We note without proof that under Assumption 2.1 $P_1^*$ behaves as we might have expected.

Proposition 2.2 Under Assumption 2.1 the set function $P_1^*$ defined by (2.1) is a probability measure on $(\Omega, A)$ with $P_1^* | \mathcal{H} = P_1$ and $P_1^*[A | \mathcal{H}] = P_0[A | \mathcal{H}]$.

Eq. (1.1) represents the special case of Eq. (2.1) where $\mathcal{H} = \sigma(\mathcal{H}_n : n \in \mathbb{N})$ is a $\sigma$-field generated by a countable partition of $\Omega$.

The odds-based alternative to Eq. (2.1) requires more effort and works for single events at a time only. For $M \subset \Omega$ let $M^c = \Omega \setminus M$ denote the complement of $M$.

Assumption 2.3 Assumption 2.1 holds. An event $A \in A$ with $0 < P_0[A] \overset{def}{=} p_0 < 1$ is fixed. The two conditional distributions $H \mapsto P_0[H | A]$ and $H \mapsto P_0[H | A^c]$, $H \in \mathcal{H}$ are absolutely continuous with
one can always choose \( \mu = P_0 \mid \mathcal{H} \). Typically, in practical applications \( \mathcal{H} \) is a proper sub-\( \sigma \)-field of \( \mathcal{A} \) and generated by a statistic like a score function. It is therefore likely to have \( \mu = \text{Lebesgue measure on} \mathbb{R}^d \) or \( \mu = \text{some counting measure} \). The assumption of positive densities is more restrictive but intuitive because statistical prediction of events that were impossible in the past does not make much sense.

The following proposition provides the general version of Eq. (1.2). We omit its well-known proof.

**Proposition 2.4** Under Assumption 2.3, define the conditional likelihood ratio \( \lambda_0 \) by \( \lambda_0 = \frac{f_{A^c}}{f_A} \). Then it holds that

\[
(i) \quad f = p_0 f_A + (1 - p_0) f_{A^c} \text{ is a} \mu \text{-density of} P_0 \mid \mathcal{H}, \text{ and} \\
(ii) \quad P_0[A \mid \mathcal{H}] \text{ can be represented as} \quad P_0[A \mid \mathcal{H}] = \frac{p_0}{p_0 + (1 - p_0) \lambda_0}.
\]

Consider the special case of \( \lambda_0 = 1 \) in Proposition 2.4. It holds that

\[
P_0[\lambda_0 = 1] = 1 \iff \mu(f_A \neq f_{A^c}) = 0 \iff \mathcal{H} \text{ and } \mathcal{A} \text{ are independent.} \tag{2.2}
\]

This case is not of much interest for classification problems because it means that \( \mathcal{H} \) does not carry any information with regard to \( \mathcal{A} \) or \( \mathcal{A}^c \). We will therefore exclude it from the following discussions. But note that by the absolute continuity requirement of Assumption 2.1 \( P_0[\lambda_0 = 1] = 1 \) implies \( P_1[\lambda_0 = 1] = 1 \) but \( P_0[\lambda_0 = 1] < 1 \) in general is not sufficient for \( P_1[\lambda_0 = 1] < 1 \).

With Proposition 2.4, we are in a position to state the main result of this note. Denote by \( \mathbf{1}_M \) the indicator function of the event \( M \), i.e. \( \mathbf{1}_M(\omega) = 1 \) for \( \omega \in M \) and \( \mathbf{1}_M(\omega) = 0 \) for \( \omega \in M^c \).

**Theorem 2.5 (Law of total odds)** Let Assumption 2.3 hold and define the likelihood ratio \( \lambda_0 \) as in Proposition 2.4. Suppose that \( P_1[\lambda_0 = 1] < 1 \).

\[(i) \quad \text{There exists a solution} \ 0 < p_1 < 1 \text{ to the equation} \\
1 = E_1 \left[ \frac{1}{p_1 + (1 - p_1) \lambda_0} \right] \tag{2.3}
\]

\[\text{if and only if} \ E_1[\lambda_0] > 1 \text{ and } E_1[\lambda_0^{-1}] > 1. \text{ If there is a solution} \ 0 < p_1 < 1 \text{ to Eq. (2.3) it is unique.} \]

\[(ii) \quad \text{Let} \ \mathcal{H}^A = \sigma(\mathcal{H} \cup \{A\}) \text{ denote the} \ \sigma \text{-field generated by} \ \mathcal{H} \text{ and} \ \mathcal{A}. \text{ Then it holds that} \]

\[\mathcal{H}^A = \{(A \cap H) \cup (A^c \cap G) : H, G \in \mathcal{H}\}. \]

\[(iii) \quad \text{If there is a solution} \ 0 < p_1 < 1 \text{ to Eq. (2.3) define} P_1^*[B] \text{ for} B \in \mathcal{H}^A \text{ by} \\
P_1^*[B] = E_1 \left[ \mathbf{1}_H \frac{p_1}{p_1 + (1 - p_1) \lambda_0} \right] + E_1 \left[ \mathbf{1}_G \frac{(1 - p_1) \lambda_0}{p_1 + (1 - p_1) \lambda_0} \right],
\]

\[\text{for any representation} \ (A \cap H) \cup (A^c \cap G) \text{ of} B \text{ with} H, G \in \mathcal{H}. \text{ Then} \ P_1^* \text{ is a probability measure} \]

\[\text{on} \ \mathcal{H}^A \text{ with} \ P_1^* \mid \mathcal{H} = P_1. \]

\[(iv) \quad \text{The conditional probability} \ P_1^*[A \mid \mathcal{H}] \text{ is given by} \]

\[P_1^*[A \mid \mathcal{H}] = \frac{p_1}{p_1 + (1 - p_1) \lambda_0}.
\]
The proof of Theorem 2.5 is given in Section 4 below. Let us note here instead some observations on Theorem 2.5:

• The definition of \( P^*_1 \) and Eq. (2.3) imply \( P^*_1[A] = p_1 \). Hence we have shown that, by means of Eq. (2.3), the total odds approach provides a properly modelled population (or test set) estimate of the unconditional probability of class \( A \) if the condition for likelihood ratio \( \lambda_0 \) from Theorem 2.5 (i) is satisfied.

• From Proposition 2.4 (ii) and Theorem 2.5 (iv) it follows that

\[
\frac{P_0[A^c \mid H]}{P_0[A \mid H]} \frac{p_0}{1 - p_0} = \lambda_0 = \frac{P^*_1[A^c \mid H]}{P^*_1[A \mid H]} \frac{p_1}{1 - p_1}.
\]

Hence \( \lambda_0 \) has an interpretation as relative odds and is the same for both the training set model \( P_0 \) and the population model \( P^*_1 \). This justifies the naming of Theorem 2.5.

• The proof of Theorem 2.5 (iv) (see Section 4) shows that

\[
P^*_1[H \mid A] = E_1 \left[ 1_H \frac{1}{p_1 + (1 - p_1) \lambda_0} \right], \quad H \in \mathcal{H}.
\]

Hence, Eq. (2.3) ensures that the conditional distribution \( H \mapsto P^*_1[H \mid A] \) is properly normalised.

• Violation of the condition for \( \lambda_0 \) from Theorem 2.5 (i) could be interpreted as evidence that between the observations of \( P_0 \) and \( P_1 \) circumstances have changed so much that the two models associated with the measures are incompatible.

• In the special case where \( \mathcal{H} = \sigma(H_n : n \in \mathbb{N}) \) is a \( \sigma \)-field generated by a countable partition of \( \Omega \), Eq. (2.3) reads

\[
1 = \sum_{n=1}^{\infty} \frac{P_1[H_n]}{p_1 + (1 - p_1) \lambda_0}.
\]

Basically, this is Eq. (3.11a) of Tasche (2013), but with a possibly infinite number of ‘rating grades’.

**Corollary 2.6** The probability measure \( P^*_1 \) from Theorem 2.5 is unique in the following sense: If \( \tilde{P}_1 \) is any probability measure on \( \mathcal{H}^A \) with \( \tilde{P}_1[A] \in (0, 1) \), \( \tilde{P}_1 \mid \mathcal{H} = P_1 \), and

\[
\frac{\tilde{P}_1[A^c \mid H]}{\tilde{P}_1[A \mid H]} \frac{\tilde{P}_1[A]}{\tilde{P}_1[A^c]} = \lambda_0,
\]

then it follows that \( \tilde{P}_1 = P^*_1 \).

By Corollary 2.6, a probability measure on \( \mathcal{H}^A \) is uniquely determined by the marginal distribution on \( \mathcal{H} \) and the relative odds with respect to the event \( A \). See Section 4 for a proof of the corollary.

The real-world estimation exercise from Tasche (2013, Section 4.4) shows that the estimates of the unconditional class probability produced by Eq. (2.1) and Eq. (2.3) respectively, indeed can be different. In that example, actually the ‘total probability’ estimate made by means of Eq. (2.1) is better than the estimate by means of Eq. (2.3) (but still quite poor) – although we have argued above that conceptually the ‘total odds’ is more convincing. Hence, it is not clear whether ‘total probability’ or ‘total odds’ is better for the estimation of unconditional class probabilities.

However, for an important special case of the probability measure \( P_1 \) ‘total odds’ appears to be a more natural approach to the estimation of the unconditional class probabilities than ‘total probability’. Under Assumption 2.3, define the probability measure \( Q \) on \( (\Omega, \mathcal{H}) \) by

\[
Q(H) = q \int_H f_A \, d\mu + (1 - q) \int_H f_{A^c} \, d\mu, \quad H \in \mathcal{H},
\]

5
for a fixed \( q \in (0, 1) \). By Proposition 2.4 (i), \( Q \) is then absolutely continuous with respect to \( P_0 | \mathcal{H} \).

Intuitively, \( Q \) is a modification of \( P_0 \) with \( p_0 = P_0[A] \) replaced by \( q \). But note that \( Q[A] \) is undefined because \( A \notin \mathcal{H} \) (otherwise the densities \( f_A \) and \( f_{A^c} \) could not be positive \( \mu \)-almost everywhere). Nonetheless, with this intuition in mind it is natural to favour such extensions \( Q^* \) of \( Q \) to any sub-\( \sigma \)-field of \( A \) containing \( A \) that satisfy

\[
Q^*(A) = q. \tag{2.9}
\]

‘Total odds’ as described in Theorem 2.5 has this property, and hence provides an unbiased estimator of the unconditional class probability \( q \).

**Corollary 2.7** Let Assumption 2.3 hold and define the likelihood ratio \( \lambda_0 \) as in Proposition 2.4. Suppose that \( P_0[\lambda_0 = 1] < 1 \). Let \( P_1 = Q \) with \( Q \) given by (2.8) for some \( 0 < q < 1 \). Then \( p_1 = q \) is the unique solution of (2.3) in \( (0, 1) \) and for \( P_1^* \) defined as in Theorem 2.5 (iii) it holds that \( P_1^*[A] = q \).

See Section 4 for the proof of Corollary 2.7. In contrast to ‘total odds’, the ‘total probability’ extension of \( Q \) as given by (2.1) does not satisfy (2.9) for \( q \neq p_0 \). This follows from the next proposition.

**Proposition 2.8** Under Assumption 2.3, define the probability measure \( Q \) by (2.8). Then it holds that

\[
|q - p_0| \int \min(f_A, f_{A^c}) \, d\mu \leq \left| \int P_0[A | \mathcal{H}] \, dQ - q \right| \leq |q - p_0|. \tag{2.10}
\]

See Section 4 for the proof of Proposition 2.8. The case \( f_A = f_{A^c} \) shows that both inequalities in (2.10) are sharp. As \( \int \min(f_A, f_{A^c}) \, d\mu \) is a measure of the classifier’s discriminatory power (Bayesian error rate), Proposition 2.8 suggests that the bias of the estimate of \( q \) is the smaller the more powerful the classifier is.

Interestingly enough, there is a slightly different estimation problem for which the practical performance of ‘total odds’ is clearly superior to ‘total probability’. This problem is the estimation of conditional class probabilities if targets for the unconditional class probabilities are independently given. Bohn and Stein (2009, Chapter 4, Section “Estimating the Prior Probabilities”) describe the problem and two standard solution approaches in the context of credit rating systems.

Under Assumption 2.1 the new problem is described as follows:

- An estimate (target) \( 0 < P_1^*[A] < 1 \) for an event \( A \in \mathcal{A} \setminus \mathcal{H} \) is given. Possibly it was produced in a separate, independent estimation exercise. The problem is to construct conditional probabilities \( P_1^*[A | \mathcal{H}] \) such that

\[
P_1^*[A] = E_1[P_1^*[A | \mathcal{H}]]. \tag{2.11}
\]

- Again, ideally the estimate should be meaningful in the sense of being based on an extension of \( P_1 \) to any \( \sigma \)-field containing \( A \), based on observations as given by \( (\Omega, \mathcal{A}, P_0) \).

The simplest, ‘total probability’ approach to solving Eq. (2.11) is by setting

\[
P_1^*[A | \mathcal{H}] = \frac{P_1^*[A]}{E_1[P_0[A | \mathcal{H}]]} P_0[A | \mathcal{H}]. \tag{2.12}
\]

This approach is unsatisfactory because it is possible that \( P_1^*[A | \mathcal{H}] > 1 \) with positive probability under \( P_1 \). Of course, this could be interpreted as evidence of incompatibility as in the case of violation of the likelihood ratio condition in Theorem 2.5 (i). Bohn and Stein (2009) present an alternative approach which uses the ‘change of base rate’ theorem (Elkan, 2001, Theorem 2). However, the solution by that approach in general does not solve (2.11) because in practice often the outcome is \( P_1^*[A] \neq E_1[P_1^*[A | \mathcal{H}]] \).

An alternative estimation approach suggested by Tasche (2013, Section 4.2, “scaled likelihood ratio”) uses Theorem 2.5:
• Let $p_1 \equiv P_1^*[A]$. Solve then the following equation for $c$:

$$1 = E_1 \left[ \frac{1}{p_1 + (1 - p_1) c \lambda_0} \right].$$

(2.13)

If $\lambda_0$ is non-constant there is a unique solution $c > 0$ of Eq. (2.13).

• Since $0 < p_1 < 1$, Theorem 2.5 (i) then implies

$$\frac{1}{E_1[\lambda_0]} < c < E_1 \left[ \frac{1}{\lambda_0} \right].$$

• Moreover, if the measure $P_1^*$ is defined with $\lambda_0$ replaced by $c \lambda_0$, Theorem 2.5 (iii) implies that the solution is meaningful because it results in a proper extension of $P_1$ to a $\sigma$-field containing $A$.

• By Theorem 2.5 (iv), the resulting estimate of the conditional probability $P_1^*[A | H]$ is as follows:

$$P_1^*[A | H] = \frac{p_1}{p_1 + (1 - p_1) c \lambda_0}.$$  

(2.14)

With a view on Eq. (2.4), the ‘scaled likelihood ratio’ approach could also be called ‘total odds’ approach. Results from an estimation exercise on real-world data presented in Tasche (2013) suggest that ‘total odds’ in general provides better solutions of problem (2.11) than ‘total probability’.

3 Related work

Saerens et al. (2002) assumed that the marginal distribution $P_1$ in Assumption 2.1 was given by a mixture distribution like in (2.8). They suggested estimating the parameter $q$ with a maximum likelihood approach. To describe their proposal in more detail, suppose there is a sample $\omega_1, \ldots, \omega_n$ of independent observations under $P_1 = Q$. The likelihood function $L$ is then given by

$$L(q, \omega_1, \ldots, \omega_n) = \prod_{i=1}^{n} \left( q f_A(\omega_i) + (1 - q) f_{A'}(\omega_i) \right).$$

(3.1)

With $\lambda_0 = f_{A'}/f_A$ as in Proposition 2.4 and Theorem 2.5, one then obtains for the log-likelihood function

$$\log(L(q, \omega_1, \ldots, \omega_n)) = \sum_{i=1}^{n} \log(f_A(\omega_i)) + \sum_{i=1}^{n} \log(q + (1 - q) \lambda_0(\omega_i)).$$

This implies

$$\frac{\partial}{\partial q} \log(L(q, \omega_1, \ldots, \omega_n)) = \sum_{i=1}^{n} \frac{1 - \lambda_0(\omega_i)}{q + (1 - q) \lambda_0(\omega_i)}.$$  

Equating the derivative to 0 as a necessary condition for a maximum gives

$$0 = \sum_{i=1}^{n} \frac{1 - \lambda_0(\omega_i)}{q + (1 - q) \lambda_0(\omega_i)}$$

$$= \sum_{i=1}^{n} \frac{1}{q + (1 - q) \lambda_0(\omega_i)} - \frac{1}{1 - q} \sum_{i=1}^{n} \frac{(1 - q) \lambda_0(\omega_i)}{q + (1 - q) \lambda_0(\omega_i)}$$

$$= \sum_{i=1}^{n} \frac{1}{q + (1 - q) \lambda_0(\omega_i)} - \frac{n}{1 - q} + \frac{q}{1 - q} \sum_{i=1}^{n} \frac{1}{q + (1 - q) \lambda_0(\omega_i)}$$

$$\Leftrightarrow 1 = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{q + (1 - q) \lambda_0(\omega_i)}.$$  

(3.2)
Figure 1: Illustration for the proof of Lemma 4.1. The three possibilities for the shape of the graph of the function $F$ defined by (4.2).

Equation (3.2) is (2.3) with $p_1$ replaced by $q$ and $P_1[H] = \frac{1}{n} \sum_{i=1}^{n} 1_{H(\omega_i)}$, $H \in \mathcal{H}$, the empirical distribution associated with the sample $\omega_1, \ldots, \omega_n$. This observation shows that the sample version of the total odds estimator is identical with the maximum likelihood estimator of Saerens et al. (2002).

Based on Theorem 2.5, therefore, we have identified a sufficient and necessary condition for the maximum likelihood estimator to exist (in the binary classification setting). Moreover, the derivation of (3.2) shows that the maximum likelihood estimator works for any model where the ratio of the conditional class densities equals the relative odds $\lambda_0$. Note that Du Plessis and Sugiyama (2014, Eq. (9)) derived (3.2) but did not discuss the existence of solutions.

4 Proofs

The proof of Theorem 2.5 is mainly based on the following lemma that generalises Theorem 3.3 of Tasche (2013).

**Lemma 4.1** Let $X > 0$ be a random variable such that $P[X = 1] < 1$. Then there exists a solution $0 \leq p < 1$ to the equation

$$E\left[\frac{1}{p + (1 - p)X}\right] = 1$$

if and only if $E[X] > 1$ and $E[X^{-1}] \geq 1$. If there is a solution $0 \leq p < 1$ to Eq. (4.1) it is unique. The unique solution is $p = 0$ if and only if $E[X^{-1}] = 1$.

**Proof.** In principle, the proof in this case is the same as the proof of Theorem 3.3 of Tasche (2013). However, we have to take care of the possibility that $E[X] = \infty$ or $E[X^{-1}] = \infty$. Define the function $F : [0, 1] \to (0, \infty)$, $p \mapsto F(p)$ by

$$F(p) = E\left[\frac{1}{p + (1 - p)X}\right].$$

8
Then for \(0 < p \leq 1\) we have \(F(p) \leq \frac{1}{p} < \infty\). Solely for \(p = 0\) it may happen that \(F(0) = \infty\), depending on whether or not \(X^{-1}\) is integrable. By the dominated convergence theorem \(F(p)\) is continuous in \((0,1]\).

If \(E[X^{-1}] < \infty\) then again by the dominated convergence theorem \(F(p)\) is also continuous in \(p = 0\) since \(\frac{1}{p + (1-p)X} \leq \max(X^{-1}, 1)\). However, Fatou’s lemma implies that \(F(p) \overset{p \to 0}{\longrightarrow} E[X^{-1}]\) even if \(E[X^{-1}] = \infty\.

The function \(p \mapsto f_X(p) = \frac{1}{p + (1-p)X}\) is twice continuously differentiable in \((0,1)\) with

\[
\begin{align*}
f_X'(p) &= \frac{X - 1}{(p + (1-p)X)^2}, \\
f_X''(p) &= \frac{2(X - 1)^2}{(p + (1-p)X)^3}.
\end{align*}
\]

(4.3)

For fixed \(p \in (0,1)\) the random variable \(f_X'(p)\) is integrable because it holds that

\[
|f_X'(p)| \leq \frac{1}{p^2} + \frac{1}{p + (1-p)X} = \frac{1}{p^2} + \frac{1}{p(1-p)} \left( \frac{p + (1-p)X}{p + (1-p)X} - \frac{p}{p + (1-p)X} \right) \leq \frac{1}{p^2} + \frac{1}{p(1-p)}.
\]

Hence it follows from the dominated convergence theorem that also \(F\) as defined by (4.2) is continuously differentiable in \((0,1)\). Moreover, since \(f_X''(p) > 0\) on \(\{X \neq 1\}\) and \(P[X = 1] < 1\) we obtain that the derivative of \(F\) is strictly increasing for \(0 < p < 1\) (strict convexity). Together with the (quasi-)continuity of \(F\) this observation implies uniqueness of any solution \(0 \leq p < 1\) to (4.1) if there is one.

The strict convexity of \(F\) implies that the graph of \(F\) must look like one of the three stylised graphs in Figure 1. Only in case B is there a solution to Eq. (4.1) other than \(p = 1\). Case B is characterised by the two conditions

\[
\begin{align*}
\lim_{p \to 0} F(p) &\geq 1 \quad \text{and} \\
\lim_{p \to 1} F'(p) &> 0.
\end{align*}
\]

We have seen above that \(\lim_{p \to 0} F(p) = E[X^{-1}]\). Eq. (4.3) implies by means of a combination of the dominated convergence theorem and Fatou’s lemma that for both the case \(E[X] < \infty\) and the case \(E[X] = \infty\) we have

\[
\lim_{p \to 1} F'(p) = \lim_{p \to 1} E[f'(p)] = E[X] - 1.
\]

This proves the existence part of the lemma. The criterion for the solution to (4.1) to be \(p = 0\) also follows from \(\lim_{p \to 0} F(p) = E[X^{-1}]\).

\[\]

**Proof of Theorem 2.5.** (i) is an immediate conclusion from Lemma 4.1. Since \(\{(A \cap H) \cup (A^c \cap G) : H,G \in \mathcal{H}\}\) is a \(\sigma\)-field (ii) follows from the observation

\[
\mathcal{H} \cup \{A\} \subset \{(A \cap H) \cup (A^c \cap G) : H,G \in \mathcal{H}\} \subset \sigma(\mathcal{H} \cup \{A\}).
\]

We begin the proof of (iii) with another lemma.

**Lemma 4.2** Let \(H \in \mathcal{H}\). Then

\[
\begin{align*}
A \cap H = \emptyset \Rightarrow E_1 \left[ \frac{1_H \, p_1}{p_1 (1-p_1) \lambda_0} \right] &= 0, \\
A^c \cap H = \emptyset \Rightarrow E_1 \left[ \frac{1_H \, (1-p_1) \lambda_0}{p_1 (1-p_1) \lambda_0} \right] &= 0.
\end{align*}
\]
Proof of Lemma 4.2. Denote by \( \varphi \) any \( \mathcal{H} \)-measurable density of \( P_1 \) with respect to \( P_0 \). Proposition 2.4 (ii) then implies
\[
E_1 \left[ 1_H \frac{p_1}{p_1 + (1 - p_1) \lambda_0} \right] = E_0 \left[ \varphi 1_H \frac{\frac{p_1}{p_0} P_0[A|\mathcal{H}] + \frac{1 - p_1}{1 - p_0} P_0[A^c|\mathcal{H}]}{\frac{p_1}{p_0}} \right]
= \frac{p_1}{p_0} E_0 \left[ \varphi 1_{H \cap A} \frac{\frac{p_1}{p_0} P_0[A|\mathcal{H}] + \frac{1 - p_1}{1 - p_0} P_0[A^c|\mathcal{H}]}{1 - \frac{p_1}{1 - p_0}} \right] = 0.
\]
The proof of the second implication in Lemma 4.2 is almost identical. \( \square \)

Proof of Theorem 2.5 continued. Let \( B \in \mathcal{H}^A \) with
\[
B = (A \cap H_1) \cup (A^c \cap G_1) = (A \cap H_2) \cup (A^c \cap G_2),
\]
for some \( H_1, H_2, G_1, G_2 \in \mathcal{H} \). Then it follows that
\[
A \cap H_1 = A \cap H_2 = A \cap H_1 \cap H_2 \quad \text{and} \quad A^c \cap G_1 = A^c \cap G_2 = A^c \cap G_1 \cap G_2.
\]
Hence \( A \cap (H_1 \setminus H_2) = \emptyset = A \cap (H_2 \setminus H_1) \) and \( A^c \cap (G_1 \setminus G_2) = \emptyset = A^c \cap (G_2 \setminus G_1) \). Lemma 4.2 now implies that \( P_1^* \) is well-defined because it holds for any sets \( M_1, M_2 \) that
\[
1_{M_1} = 1_{M_1 \cap M_2} + 1_{M_1 \setminus M_2} \quad \text{and} \quad 1_{M_2} = 1_{M_1 \cap M_2} + 1_{M_2 \setminus M_1}.
\]
The properties \( P_1^*[\emptyset] = 0, P_1^*[\Omega] = 1 \) and \( P_1^*[H] = P_1[H] \) for \( H \in \mathcal{H} \) are obvious. Finite additivity of \( P_1^* \) follows from Lemma 4.2 because
\[
B_i = (A \cap H_i) \cup (A^c \cap G_i), \quad i = 1, 2 \quad \text{with} \quad B_1 \cap B_2 = \emptyset
\]
implies \( A \cap H_1 \cap H_2 = \emptyset = A^c \cap G_1 \cap G_2 \) and
\[
B_1 \cup B_2 = (A \cap (H_1 \cup H_2)) \cup (A^c \cap (G_1 \cup G_2)).
\]
To complete the proof of (iii) we have to show that \( P_1^* \) is \( \sigma \)-continuous in \( \emptyset \), i.e.
\[
\lim_{n \to \infty} P_1^*[B_n] = 0, \quad (4.4)
\]
for any \( B_1 \supset B_2 \supset \ldots \) with \( \bigcap_{n=1}^{\infty} B_n = \emptyset \). Let \( (B_n) \) be such a sequence in \( \mathcal{H}^A \) with representation \( B_n = (A \cap H_n) \cup (A^c \cap G_n) \), for sequences \( (H_n), (G_n) \) in \( \mathcal{H} \). Note that
\[
\frac{p_1}{p_0} P_0[A|\mathcal{H}] + \frac{1 - p_1}{1 - p_0} P_0[A^c|\mathcal{H}] \geq \min \left( \frac{p_1}{p_0}, \frac{1 - p_1}{1 - p_0} \right).
\]
Therefore, similarly to the proof of Lemma 4.2 we see that
\[
P_1^*[B_n] \leq \max \left( \min \left( \frac{p_1}{p_0}, \frac{1 - p_1}{1 - p_0} \right) \right) P_0[A|\mathcal{H}] + \frac{1 - p_1}{1 - p_0} P_0[A^c|\mathcal{H}]
= \max \left( \min \left( \frac{p_1}{p_0}, \frac{1 - p_1}{1 - p_0} \right) \right) E_0 [\varphi (1_{H_n} P_0[A|\mathcal{H}] + 1_{G_n} P_0[A^c|\mathcal{H}])]
= \max \left( \min \left( \frac{p_1}{p_0}, \frac{1 - p_1}{1 - p_0} \right) \right) E_0 [\varphi 1_{B_n}],
\]
where \( \varphi \) is an \( \mathcal{H} \)-measurable density as in Lemma 4.2. By the dominated convergence theorem, Eq. (4.4) follows.

With regard to (iv), observe that by the definition of \( P_1^* \) and the fact that \( P_1^* | \mathcal{H} = P_1 \) it holds for \( H \in \mathcal{H} \) that
\[
E_1 \left[ 1_H \frac{p_1}{p_1 + (1 - p_1) \lambda_0} \right] = P_1^*[A \cap H] = E_1 [1_H P_1^*[A|\mathcal{H}]] = E_1 [1_H P_1^*[A|\mathcal{H}]].
\]
This implies (iv) because \( \lambda_0 \) is \( \mathcal{H} \)-measurable. \( \Box \)

**Proof of Corollary 2.7.** Note that (2.7) is equivalent to

\[
\tilde{P}_1[A | \mathcal{H}] = \frac{\tilde{P}_1[A]}{\tilde{P}_1[A] + (1 - \tilde{P}_1[A]) \lambda_0}.
\]

This implies

\[
1 = E_1 \left[ \frac{1}{\tilde{P}_1[A] + (1 - \tilde{P}_1[A]) \lambda_0} \right].
\]

Therefore, by Theorem 2.5 (i) we can conclude that \( \tilde{P}_1[A] = p_1 \). By Theorem 2.5 (iv), it follows that \( \tilde{P}_1[A | \mathcal{H}] = P^*_1[A | \mathcal{H}] \) and hence

\[
\tilde{P}_1[A \cap \mathcal{H}] = E_1[1_{\mathcal{H}} P^*_1[A | \mathcal{H}]] = P^*_1[A \cap \mathcal{H}], \quad \mathcal{H} \in \mathcal{H}.
\]

This implies \( \tilde{P}_1 = P^*_1 \) because \( A \cap \mathcal{H} \) is a \( \cap \)-stable generator of \( \mathcal{H}^A \). \( \Box \)

**Proof of Corollary 2.7.** Observe that \( q \, f_A + (1 - q) \, f_{A^c} \) is a \( \mu \)-density of \( P_1 = Q \). This implies

\[
E_1 \left[ \frac{1}{p_1 + (1 - p_1) \lambda_0} \right] = \int f_A \frac{q f_A + (1 - q) f_{A^c}}{p_1 f_A + (1 - p_1) f_{A^c}} \, d\mu = 1,
\]

if we choose \( p_1 = q \). As \( f_A \) and \( f_{A^c} \) are positive \( \mu \)-almost everywhere, \( P_0[\lambda_0 = 1] = 1 \) implies \( P_1[\lambda_0 = 1] = Q[\lambda_0 = 1] < 1 \). By Theorem 2.5 (i), hence the moment conditions on \( \lambda_0 \) are satisfied and there is no other solution to (2.3) than \( q \). From this it follows that \( P_1 \) can be extended to \( \mathcal{H}^A \) as defined in Theorem 2.5 (ii) and that the extension satisfies \( P^*_1[A] = q \). \( \Box \)

**Proof of Proposition 2.8.** If \( \mu(f_A \neq f_{A^c}) = 0 \) then all three parts of (2.10) equal \( |q - p_0| \). Suppose now that \( \mu(f_A \neq f_{A^c}) > 0 \). By Proposition 2.4 (ii) we can calculate as follows:

\[
\int P_0[A | \mathcal{H}] dQ - q = p_0 \int f_A \frac{q f_A + (1 - q) f_{A^c}}{p_0 f_A + (1 - p_0) f_{A^c}} \, d\mu = q \int f_A \frac{p_0 q f_A + p_0 (1 - q) f_{A^c} - q p_0 f_A - q (1 - p_0) f_{A^c}}{p_0 f_A + (1 - p_0) f_{A^c}} \, d\mu
\]

\[
= (p_0 - q) \int f_A \frac{f_{A^c}}{p_0 f_A + (1 - p_0) f_{A^c}} \, d\mu.
\]

(4.5)

Observing that \( \frac{f_A f_{A^c}}{p_0 f_A + (1 - p_0) f_{A^c}} \geq \min(f_A, f_{A^c}) \) we obtain the first inequality in (2.10). With regard to the second inequality, define a probability measure \( P \) on \( (\Omega, \mathcal{H}) \) by

\[
P[H] = \int_H f_{A^c} \, d\mu, \quad H \in \mathcal{H}.
\]

With \( X = \frac{f_{A^c}}{f_A} \) then if follows for all \( p \in [0, 1] \) that

\[
\int \frac{f_A f_{A^c}}{p f_A + (1 - p) f_{A^c}} \, d\mu = E \left[ \frac{1}{p + (1 - p) X} \right].
\]

(4.6)

Note that \( E \left[ \frac{1}{X} \right] = \int f_A \, d\mu = 1 \). In addition, since \( f_A \) is positive \( \mu(f_A \neq f_{A^c}) > 0 \) implies \( P[X = 1] < 1 \). Hence we can apply Lemma 4.1 to conclude that \( p = 0 \) is the only \( p \in [0, 1] \) such that \( E \left[ \frac{1}{p + (1 - p) X} \right] = 1 \). The proof of Lemma 4.1 shows that in this case for \( 0 < p < 1 \) we have

\[
1 > E \left[ \frac{1}{p + (1 - p) X} \right].
\]

By (4.6) and (4.5), the second inequality in (2.10) follows. \( \Box \)

Note that (4.5) could be rearranged in order to construct an unbiased estimator of \( q \).
References

J.R. Bohn and R.M. Stein. Active Credit Portfolio Management in Practice. John Wiley & Sons, Inc., 2009.

M.C. Du Plessis and M. Sugiyama. Semi-supervised learning of class balance under class-prior change by distribution matching. Neural Networks, 50:110–119, 2014.

C. Elkan. The foundations of cost-sensitive learning. In B. Nebel, editor, Proceedings of the Seventeenth International Joint Conference on Artificial Intelligence, IJCAI 2001, pages 973–978. Morgan Kaufmann, 2001.

W. Feller. An Introduction to Probability Theory and Its Applications, volume I. Jon Wiley & Sons, New York, third edition, 1968.

A.N. Kolmogorov. Foundations of the Theory of Probability. Chelsea Publishing Company, New York, second English edition, 1956. Translation edited by N. Morrison.

J.G. Moreno-Torres, T. Raeder, R. Alaiz-Rodriguez, N.V. Chawla, and F. Herrera. A unifying view on dataset shift in classification. Pattern Recognition, 45(1):521–530, 2012.

M. Saerens, P. Latinne, and C. Decaestecker. Adjusting the Outputs of a Classifier to New a Priori Probabilities: A Simple Procedure. Neural Computation, 14(1):21–41, 2002.

D. Tasche. The art of probability-of-default curve calibration. Journal of Credit Risk, 9(4):63–103, 2013.

J.C. Xue and G.M. Weiss. Quantification and Semi-supervised Classification Methods for Handling Changes in Class Distribution. In Proceedings of the 15th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining, pages 897–906, New York, 2009.