Vector-valued $L_p$-convergence of orthogonal series and Lagrange interpolation.

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Abstract

We give necessary and sufficient conditions for interpolation inequalities of the type considered by Marcinkiewicz and Zygmund to be true in the case of Banach space-valued polynomials and Jacobi weights and nodes. We also study the vector-valued expansion problem of $L_p$-functions in terms of Jacobi polynomials and consider the question of unconditional convergence. The notion of type $p$ with respect to orthonormal systems leads to some characterizations of Hilbert spaces. It is also shown that various vector-valued Jacobi means are equivalent.

1 Introduction and results

Let $X$ be a Banach space, $1 \leq p < \infty$ and $L_p(\mathbb{R}; X)$ denote the space of (classes of) $p$-th power integrable functions with norm $\| f \| := (\int_{\mathbb{R}} \| f(t) \|^p dt)^{1/p}$. A Banach space is a *UMD-space* provided that the Hilbert transform on $\mathbb{R}$,

$$Hf(t) := \text{p.v.} \int_{\mathbb{R}} \frac{f(s)}{t-s} ds, \quad f \in L_p(\mathbb{R}; X),$$

is bounded as a linear operator.

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defines a bounded operator $H : L_p(\mathbb{R}; X) \to L_q(\mathbb{R}; X)$ for some $1 < p < \infty$.
It is well-known that this holds for some $1 < p < \infty$ if and only if it holds
for all $1 < p < \infty$, see e.g. Schwarz [28]. All $L_q(\mu)$-spaces with $1 < q < \infty$
or all reflexive Orlicz spaces are UMD-spaces, cf. Fernandez and Garcia [7].

Let $I = (-1, 1)$, $\alpha, \beta > -1$ and $w_{\alpha\beta}(t) := (1 - t)^{\alpha}(1 + t)^{\beta}$ for $t \in I$. Let

$$L_p(I, w_{\alpha\beta}; X) := \{ f : I \to X \mid \| f \|_p := \left( \int_I \| f(t) \|^p w_{\alpha\beta}(t) dt \right)^{1/p} < \infty \}.$$  

The scalar product in $L_2(I, w_{\alpha\beta}) := L_2(I, w_{\alpha\beta}; \mathbb{R})$ will be denoted by
$\langle \cdot, \cdot \rangle$ or $\langle \cdot, \cdot \rangle_{\alpha\beta}$. For $\alpha = \beta$ we just write $w_{\alpha}$ and $\langle \cdot, \cdot \rangle_{\alpha}$. By $\Pi_n(X)$
we denote the space of polynomials of degree $\leq n$ with coefficients in $X$. Let
$\Pi_n := \Pi_n(\mathbb{R})$. The $L_2(I, w_{\alpha\beta})$-normalized Jacobi polynomials with respect
to $(I, w_{\alpha\beta})$ will be denoted by $p_n^{(\alpha, \beta)}$, $n \in \mathbb{N}_\infty$. Hence $p_n^{(\alpha, \beta)} \in \Pi_n$ and

$$\langle p_n^{(\alpha, \beta)}, p_m^{(\alpha, \beta)} \rangle_{\alpha\beta} = \int_I p_n^{(\alpha, \beta)}(t)p_m^{(\alpha, \beta)}(t)w_{\alpha\beta}(t)dt = \delta_{nm}. \quad (2)$$

This normalization is more convenient for us than the standard one of Szegö [30]. For $\alpha = \beta = -\frac{1}{2}$ ($\frac{1}{2}$) one gets the Tchebychev polynomials of the first (second) kind, for $\alpha = \beta = 0$ the Legendre polynomials. Let $t_1 > \cdots > t_{n+1}$ denote the zeros of $p_n^{(\alpha, \beta)}$, all of which are in $I$, and $\lambda_1, \cdots, \lambda_{n+1} > 0$
the Gaussian quadrature weights. Thus for any real polynomial $q$ of degree
$\leq 2n + 1$, one has

$$\int_I q(t)w_{\alpha\beta}(t)dt = \sum_{j=1}^{n+1} \lambda_j q(t_j). \quad (3)$$

Clearly, $\lambda_j$ and $t_j$ depend on $n, j, \alpha$ and $\beta$ but not on $q$. One has for
$\alpha, \beta > -1$

$$\lambda_j = (2n+\alpha+\beta+3)((1-t_j^2)p_{n+1}^{(\alpha, \beta)}(t_j)^2)^{-1} \sim \begin{cases} j^{2\alpha+1}/n^{2\alpha+2} & j \leq \frac{n}{2} \\ (n+2-j)^{2\beta+1}/n^{2\beta+2} & j > \frac{n}{2} \end{cases} \quad \text{for all } j \leq \frac{n}{2}. \quad (4)$$

$$1 - t_j^2 \sim (j/n)^2, \text{ and } p_{n+1}^{(\alpha, \beta)}(t_j) \sim n^{5/2+\alpha}/j^{3/2+\alpha} \quad \text{for all } j \leq \frac{n}{2}. \quad (5)$$

$$p_n^{(\alpha, \beta)}(-x) = (-1)^n p_n^{(\alpha, \beta)}(x).$$
See Szegö [30, 3.4, 4.1, 4.3, 8.9, 15.3], taking into account the different normalization there. Here $\lambda_j \sim f_j$ means that there are constants $c_1, c_2 > 0$ independent of $j$ and $n$ such that $c_1 f_j \leq \lambda_j \leq c_2 f_j$ for all $n$ and $j$ concerned. For $\alpha = \beta = -\frac{1}{2}$, $\lambda_j = \pi/(n+1)$.

Marcinkiewicz and Zygmund [31, ch. X] proved interpolation inequalities for trigonometric polynomials of degree $\leq n$ which for even trigonometric polynomials $g$, after a transformation $g(x) = q(x \cos t)$, $x = \cos t$, $q \in \Pi_n$, can be restated as

$$
\frac{1}{3} \left(\sum_{j=1}^{n+1} |q(t_j)|^p / (n+1)\right)^{\frac{1}{p}} \leq \frac{1}{2} \int_{I} |q(t)|^p (1-t^2)^{-\frac{1}{2}} dt \leq c_p \left(\sum_{j=1}^{n+1} |q(t_j)|^p / (n+1)\right)^{1/p}.
$$

Here $(t_j)$ are the zeros of the Jacobi polynomial $p_{n+1}^{\alpha,\beta}$ in the Tchebychev case $\alpha = \beta = -1/2$, and $c_p$ depends on $1 < p < \infty$ only. The left inequality holds for $p = 1, \infty$ as well whereas the right one fails, in general. For $p = 2$, (3) gives more precise information since $\lambda_j = \pi/(n+1)$. The Marcinkiewicz-Zygmund inequalities extend to the Jacobi case of general $\alpha, \beta > -1$ and to the vector-valued setting in the following sense:

**Theorem 1** Let $X$ be a Banach space, $1 \leq p \leq \infty$, $\alpha, \beta > -1$, $(t_j)$ the zeros of $p_{n+1}^{\alpha,\beta}$ and $(\lambda_j)$ the corresponding quadrature weights.

a) There is $c > 0$ such that for all $1 \leq p \leq \infty$, $n \in \mathbb{N}$ and $q \in \Pi_{2n}(X)$

$$
c^{-1} \left(\sum_{j=1}^{n+1} \lambda_j \| q(t_j) \|^p \right)^{1/p} \leq \left( \int_{-1}^{1} \| q(t) \|^p w_{\alpha \beta}(t) dt \right)^{1/p}.
$$

b) Let

$$
\mu(\alpha, \beta) := \max(1, 4(\alpha + 1)/(2\alpha + 5), 4(\beta + 1)/(2\beta + 5)),
$$

$$
m(\alpha, \beta) := \max(1, 4(\alpha + 1)/(2\alpha + 3), 4(\beta + 1)/(2\beta + 3))
$$

and $M(\alpha, \beta) := m(\alpha, \beta)^{-1}$, i.e. $m(\alpha, \beta)^{-1} + M(\alpha, \beta)^{-1} = 1$. Then the following are equivalent.

(1) There is $c_p > 0$ such that for all $n \in \mathbb{N}$ and $q \in \Pi_n(X)$
\[
\left( \int_{-1}^{1} \| q(t) \|_p \, w_{\alpha\beta}(t) \, dt \right)^{1/p} \leq c_p \sum_{j=1}^{n+1} \lambda_j \| q(t_j) \|_p^{1/p}.
\]  
(7)

(2) \( X \) is a UMD-space and \( p \) satisfies \( \mu(\alpha, \beta) < p < M(\alpha, \beta) \).

Part (a) is proved just as the scalar result which goes back to Askey \([4]\), Nevai \([19]\) and Zygmund \([32]\). The converse inequality (7) was shown in the scalar case (for \( \alpha = \beta \)) by Askey \([2]\) under the more restrictive assumption \( m(\alpha, \beta) < p < M(\alpha, \beta) \) using (a) and duality; the duality method, however, fails if \( \mu(\alpha, \beta) < p \leq m(\alpha, \beta) \). The question whether (7) in the vector-valued case requires \( X \) to be a UMD-space was raised by Pietsch in the case of trigonometric polynomials (corresponding to \( \alpha = \beta = -1/2 \)) and solved by him in this case by a different method \([22]\).

In terms of Banach spaces, Theorem 1 states that the spaces \( \Pi_n(X)_p \) as subspaces of \( L_p(I, w_{\alpha\beta}; X) \) are uniformly isomorphic to \( l^{n+1}_p(X) \)-spaces, by evaluating the polynomials \( q \) at the zeros \( (t_j) \), provided that (b), (2) holds; i.e. the Banach-Mazur distances \( d(\Pi_n(X)_p, l^{n+1}_p(X)) \) are uniformly bounded.

For \( f \in L_p(I, w_{\alpha\beta}; X) \), let \( Q_n f := \sum_{j=0}^{n} < f, p_j^{\alpha,\beta} > p_j^{(\alpha,\beta)} \in \Pi_n(X)_p \) denote the orthogonal projection of \( f \) onto the space of polynomials of degree \( \leq n \). The following vector-valued expansion theorem for Jacobi polynomials generalizes the classical scalar result of Pollard \([26]\) and Muckenhoupt \([18]\).

**Theorem 2** Let \( X \) be a Banach space, \( 1 \leq p \leq \infty \), \( \alpha, \beta > -1 \) and \( m(\alpha, \beta) \) and \( M(\alpha, \beta) \) as before. Then the following are equivalent:

1. For all \( f \in L_p(I, w_{\alpha\beta}; X) \) \( Q_n f \) converges to \( f \) in the \( L_p \)-norm.
2. \( X \) is a UMD-space and \( m(\alpha, \beta) < p < M(\alpha, \beta) \).

The necessity of the UMD-condition on \( X \) will be proved using Theorem 1; the interval for \( p \) is “symmetric” with respect to \( p = 2 \) and smaller than the one exhibited in Theorem 1, (b). Analogues of Theorems 1 and 2 in the case of the Hermite polynomials are proved in \([12]\). Using the results of Gilbert \([8]\), we also prove that various vector-valued Jacobi means are equivalent:
Proposition 3 Let $\alpha, \beta > -1$, $1 < p < \infty$, $\gamma \in \mathbb{R}$ with $|\frac{\alpha}{p} + \frac{1}{p} - \frac{1}{2}| < \frac{1}{4}$. Let $X$ be a UMD-space. Then there is $M = M(\alpha, \beta, \gamma, p) \geq 1$ such that for all $n \in \mathbb{N}$ and all $x_1, \ldots, x_n \in X$

$$
\left( \int_{-1}^{1} \left\| \sum_{j=0}^{n} p_j^{(\alpha, \alpha)}(t) x_j \right\|^p (1 - t^2)^{(\alpha + \gamma)p/2} dt \right)^{1/p} \sim \left( \int_{-1}^{1} \left\| \sum_{j=0}^{n} p_j^{(\beta, \beta)}(t) x_j \right\|^p (1 - t^2)^{(\beta + \gamma)p/2} dt \right)^{1/p}.
$$

(8)

Here $\sim$ means that the quotient of the two expressions is between $1/M$ and $M$. Instead of $(\alpha, \alpha)$ and $(\beta, \beta)$, one could consider $(\alpha_1, \alpha_2)$ and $(\beta_1, \beta_2)$ as Jacobi-indices, provided the weight functions are changed accordingly.

The convergence of the Jacobi series in Theorem 2 is not unconditional unless $p = 2$ and $X$ is a Hilbert space, as will follow from the following general result.

Recall that a series $\sum_{n \in \mathbb{N}} y_n$ in a Banach space $Y$ converges unconditionally if $\sum_{n \in \mathbb{N}} \varepsilon_n y_n$ converges in $Y$ for all choices of signs $\varepsilon_n = \pm 1$.

Proposition 4 Let $(\Omega, \mu)$ be a measure space and $(p_n)$ be a complete orthonormal system in $L_2(\Omega, \mu)$, assumed to be infinite dimensional. Let $X$ be a Banach space and $1 \leq p < \infty$. Assume that for all $f \in L_p(\Omega, \mu; X)$, the series $\sum_n <f, p_n> p_n$ converges unconditionally in $L_p(\Omega, \mu; X)$. Then:

(i) If $\| p_j \|_2 \sim \| p_j \|_{\max(p, p')}$ and $(\Omega, \mu)$ is a finite measure space, one has $p = 2$.

(ii) If $\sup_j |p_j| \in L_2(\Omega, \mu)$, $X$ is isomorphic to a Hilbert space.

Statement (ii) was also shown by Defant and Junge [6]. Both conditions (i) and (ii) are satisfied in the Jacobi case provided that the condition $m(\alpha, \beta) < p < M(x, \beta)$ holds (necessary for convergence). Without an assumption like $\sup_j |p_j| \in L_2(\Omega, \mu)$, $X$ is not isomorphic to a Hilbert space in general, as the Haar system shows. However, one has:

Proposition 5 Let $1 < p < \infty$ and $(p_n)_{n \in \mathbb{N}}$ be an unconditional basis of $L_p(0, 1)$. Let $X$ be a Banach space such that for any $f \in L_p(0, 1; X)$, the series $\sum_{n \in \mathbb{N}} <f, p_n> p_n$ converges unconditionally in $L_p(0, 1; X)$. Then $X$ is a UMD-space.
The proof shows that the Haar basis is unconditional in $L^p(0,1;X)$ which by Maurey [16], Burkholder [5] and Bourgain [1] is equivalent to $X$ being a UMD-space. It was shown by Aldous [1] that $X$ is a UMD-space if $L^p(X)$ has an unconditional basis.

Let $(\Omega,\mu)$ be a measure space and $(p_n)_{n\in\mathbb{N}}$ be a complete orthonormal system in $L^2(\Omega,\mu)$. We say that a Banach space $X$ has $(p_n)$-type 2 provided there is $c>0$ such that for all $m\in\mathbb{N}$ and all $x_1,\ldots,x_m\in X$

\[
\left(\int_{\Omega} \| \sum_{j=1}^{m} p_j(t)x_j \|^2 \, d\mu(t)\right)^{1/2} \leq c\left(\sum_{j=1}^{m} \| x_j \|^2\right)^{1/2}.
\]

$X$ has $(p_n)$-cotype 2 if the reverse inequality holds. In [24], Pisier showed for the Haar system $(h_n)$, that $(h_n)$-type 2 of $X$ is equivalent to $X$ being 2-smooth, e.g. has an equivalent uniformly convex norm with modulus of convexity of power type 2. In Pisier and Xu [25] the related notion of $H$-type $p$ (\(\leq 2\)) is considered for all orthonomal systems $(p_n)$. Kwapień [13] studied this notion for the trigonometric system $(e_n)$ in $L^2(0,2\pi)$ showing that $(e_n)$-type 2 (also called Fourier-type 2, $e_n(t) = \exp(int)$) of $X$ implies that $X$ is isomorphic to a Hilbert space. This result generalizes to the case of Jacobi polynomials.

**Proposition 6** Let $X$ be a Banach space which is Jacobi $(p_n^{(\alpha,\beta)})$-type 2 for some $\alpha,\beta > -1$. Then $X$ is isomorphic to a Hilbert space.

The proof uses the interpolation inequalities of Theorem 1. In general, $(p_n)$-type 2 implies type 2 in the usual sense [17], i.e. with respect to the Rademacher system $(r_n)$, $r_n(t) = \text{sgn} \sin 2^n \pi t$.

**Proposition 7** Let $X$ be a Banach space which is of $(p_n)$-type 2 for some complete orthonormal system $(p_n)$ in $L^2(0,1)$. Then $X$ is of Haar type 2, hence 2-smooth and of type 2.

There is a partial converse to this result.

**Proposition 8** Let $X$ be a Banach space and $(p_n)_{n\in\mathbb{N}} \subset L^2(0,1)$ be a complete orthonormal system such that for any $f \in L^2(0,1;X)$ the series $\sum_{n\in\mathbb{N}} < f, p_n > p_n$ converges unconditionally in $L^2(0,1;X)$. Then, if $X$ has type 2, it also has $(p_n)$-type 2.
It follows from Proposition 5 that the unconditionality assumption in Proposition 8 implies that the space \( X \) in question has UMD. On the other hand, if \( X \) has UMD the unconditionality assumption in Proposition 8 is satisfied for the Haar system, and thus by Pisier’s result mentioned above, type 2 and UMD of \( X \) implies that \( X \) is 2-smooth. This in turn implies type 2 but does not imply the UMD-property, since by Bourgain [4] there exists a Banach lattice satisfying an upper-p and lower-q estimate and failing the UMD-property; choosing \( 2 < p < q < \infty \) there, such a lattice is 2-smooth, cf. [15].

2 The interpolation inequalities

For the proof of Theorem 1, we need a well-known fact about continuity in \( L_p \), cf. Pollard [26] or Benedek, Murphy and Panzone [3]. In the scalar case, it is a special case of the theory of weighted singular integral operators with weights in the Muckenhoupt class \( A_p \), cf. Garcia-Cuerva and de Francia [9, chap. IV].

Lemma 1 Let \( X \) be a Banach space, \( 1 \leq p \leq \infty \), \( b \in \mathbb{R} \) and \( k : \mathbb{R}^k \to \mathbb{R} \) be defined by \( k(u, v) := |u/v|^b - 1/|u - v| \). Then the integral operator \( T_k \) given by \( T_k f(u) := \int_{\mathbb{R}} k(u, v)f(v)dv \) defines a bounded operator \( T_k : L_p(\mathbb{R} \times X) \to \mathbb{L}_r(\mathbb{R} \times X) \) provided that \(-1/p < b < 1 - 1/p\) (actually if and only if).

Proof: We sketch the simple proof. Let \( r(u, v) := |u/v|^{1/p'} \). It suffices to show that

\[
\sup_u \int_{\mathbb{R}} k(u, v) r(u, v)^{p'} dv \leq M, \quad \sup_v \int_{\mathbb{R}} k(u, v)r(u, v)^{-p} du \leq M. \tag{9}
\]

An application of Hölder’s inequality then shows that \( T_k \) is continuous as a map \( T_k : L_p(\mathbb{R} \times X) \to \mathbb{L}_r(\mathbb{R} \times X) \) with norm \( \leq M \). To check the first inequality in (9), substitute \( v/u = t \) to find

\[
\sup_{u \neq 0} \int_{\mathbb{R}} k(u, v) r(u, v)^{p'} dv = \int_{\mathbb{R}} |t - b - 1| |t|^{-1/p} |t - 1| dt.
\]

This is finite since integrability at 0 is assured by \( b < 1 - 1/p \), and integrability at \( \pm \infty \) by \( b > -1/p \). Note that for \( t \to 1 \), there is no singularity,
the integrand tends to $|b|$. The second condition in (9) is checked similarly.

By Szegö [30, 7.32, 4.3], for any pair of indices $\alpha, \beta > -1$, there is $c = c_{\alpha, \beta}$ such that for all $n \in \mathbb{N}$ and $t \in [-1, 1]$, the $L_2$-normalized Jacobi polynomials $p_n^{(\alpha, \beta)}$ satisfy the estimate

$$|p_n^{(\alpha, \beta)}(t)| \leq c (1 - t + n^{-2})^{-(\alpha/2+1/4)} (1 + t + n^{-2})^{-(\beta/2+1/4)}$$  \hspace{1cm} (10)

**Proof of theorem 1:** We start with (b), (2) $\implies$ (1).

Assume that $X$ is a UMD space and that $p$ satisfies $\mu(\alpha, \beta) < p < M(\alpha, \beta)$. Let $q \in \Pi_n(X)$ and put $y_j := q(t_j)/p'_{n+1}(t_j)$. The Lagrange functions $\ell_j \in \Pi_n,

\ell_j(t) := p_{n+1}^{(\alpha, \beta)}(t) / (p_{n+1}^{(\alpha, \beta)}(t_j)(t - t_j))$

satisfy $\ell_j(t_i) = \delta_{ji}$ for $i, j = 1, \cdots, n + 1$ and thus $q$ coincides with its interpolating polynomial $q = \sum_{j=1}^{n+1} q(t_j) \ell_j$. We have to estimate

$$L := \left( \int_{-1}^{1} \|q(t)\|^p w_{\alpha \beta}(t) dt \right)^{1/p} \leq \left( \int_{-1}^{1} \left\| \sum_{j=1}^{n+1} y_j \frac{p_{n+1}^{(\alpha, \beta)}(t)}{t - t_j} \right\|^p w_{\alpha \beta}(t) dt \right)^{1/p}$$

from above. Let $I_j = (t_j, t_{j-1})$, $|I_j| = (t_{j-1} - t_j)$ and $\chi_j$ be the characteristic function of $I_j$, for $j = 1, \cdots, n + 1$, with $t_0 := 1$. The proof relies on the fact that $1/(t - t_j)$ is sufficiently close to the Hilbert transform of $-\chi_j/|I_j|$ at $t$ which is

$$H\left(-\frac{\chi_j}{|I_j|}\right)(t) = \frac{1}{|I_j|} \log \left| \frac{t - t_{j-1}}{t - t_j} \right| = \frac{1}{|I_j|} \log \left| 1 - \frac{|I_j|}{|I_j|} \right|.$$  

Let $J_n = [a_n, b_n]$ where

$$a_n = \begin{cases} 
-1 & \text{if } \beta > -1/2 \\
-1 + dn^{-2} & \text{if } \beta \leq -1/2,
\end{cases}$$

$$b_n = \begin{cases} 
1 & \text{if } \alpha > -1/2 \\
1 - dn^{-2} & \text{if } \alpha \leq -1/2;
\end{cases}$$

and $d$ is chosen such that $\min(1 - t_1, 1 + t_{n+1}) \geq 2dn^{-2}$. By [30] this is possible.
It follows from (10) that for $n \in \mathbb{N}$ and $t \in J_n$
\[ |p_n^{(\alpha,\beta)}(t)| \leq c(1 - t)^{-(\alpha/2+1/4)}(1 + t)^{-(\beta/2+1/4)}. \] (11)

In the following, constants $c_1, c_2, \ldots$ may depend on $\alpha, \beta$ and $p$, but not on $n, j$ and $t$. We claim that for $n \in \mathbb{N}$ and $t \in J_n$
\[ |p_n^{(\alpha,\beta)}(t)||\frac{1}{t - t_j} + H(\frac{x_j}{|I_j|})(t)| \leq f_j(t), \] (12)
where
\[ f_j(t) := c_1 \min\left(\frac{1}{|I_j|}, \frac{|I_j|}{(t - t_j)^2}\right)(1 - t)^{-(\alpha/2+1/4)}(1 + t)^{-(\beta/2+1/4)}. \]

If $t$ is such that $|t - t_j| > 2|I_j|$, (12) follows from (11) and $|x - \log(1 + x)| \leq x^2$ for $|x| \leq 1/2$, i.e. $\frac{1}{t - t_j} + H(\frac{x_j}{|I_j|})(t) \leq \frac{|I_j|}{(t - t_j)^2}$. For $|t - t_j| \leq 2|I_j|$, one uses that $p_n^{(\alpha,\beta)}$ has a zero in $t_j$. By the mean-value theorem there is a $\theta$ between $t$ and $t_j$ such that
\[ \left|\frac{p_n^{(\alpha,\beta)}}{t - t_j}\right| = \left|\frac{p_n^{(\alpha,\beta)}(t) - p_n^{(\alpha,\beta)}(t_j)}{t - t_j}\right| = \left|p_n^{(\alpha,\beta)}(\theta)\right| \leq f_j(t), \]
using that by Szegö [30, 8.9] and (5), e.g. for $j \leq n/2$,
\[ \left|p_n^{(\alpha,\beta)}(\theta)\right| \leq c_3 n^{\alpha+5/2}/j^{\alpha+3/2} \sim f_j(t_j) \sim f_j(t). \]

We note that (only) for $j = 1$, the logarithmic singularity of $H(x_j/|I_j|)$ at $t = 1$ is not compensated by a zero of $p_n^{(\alpha,\beta)}(t_0 = 1)$, but (12) is true in this case and $\alpha > -1/2$ too, since by (10) for $(1 - t) \leq n^{-2}$
\[ |p_n(t)H(\frac{x_j}{|I_j|})(t)| \leq c_4 n^{5/2 + \alpha}|\log(n^2(1 - t))| \leq c_5 n^2(1 - t)^{-(\alpha/2+1/4)} \]
using $|\log v| \leq c_3 v^{-\varepsilon}$ for $\varepsilon = \frac{\alpha}{2} + \frac{1}{4} > 0$ and $0 < v \leq 1$. Hence (12) holds. Applying this we find
Since 1

By Hardy, Littlewood and Polya [10], the kernel 

where

and \( \gamma := \alpha(1/p - 1/2) - 1/4, \delta := \beta(1/p - 1/2) - 1/4 \). The restrictions on \( p \) are equivalent to \(-1/p < \gamma, \delta < 1 - 1/p \) and \( 1 < p < \infty \). In particular, \( |\gamma| < 1, |\delta| < 1 \). We estimate the “main” term \( M_1 \) and the “error” term \( M_2 \) separately.

We claim that the kernel \( K(t, s) := 1/(t-s)((1-t)/(1-s))^{\gamma}(1+t)/(1+s)^{\delta} \), \( t, s \in [-1, 1] \) defines a bounded integral operator \( T_k : L_p(-1, 1; X) \rightarrow L_p(-1, 1; X) \). Indeed, by lemma 1, the kernel \( |((1-t)/(1-s))^{\gamma} - 1|/|t-s| \) defines a bounded operator \( L_p(\mathbb{R}; X) \rightarrow \mathbb{L}_p(\mathbb{R}; X) \), replacing \( t \) and \( s \) by \((1-t)\) and \((1-s)\). Since \( X \) is a UMD-space, so does \( 1/(t-s) \) and hence also \( 1/(1-t)/(1-s) \). Since \( (\frac{1+\delta}{1+\gamma}) \) is bounded from above and below by positive constants for \( s, t \in [0, 1] \), the kernel \( K \) defines a bounded operator \( T_k : L_p(0, 1; X) \rightarrow L_p(0, 1; X) \). The same holds on the interval \([-1, 0]\). The kernel is less singular for \( t \) and \( s \) of different sign: if \( s \in [-1, 0] \), \( t \in [0, 1] \), the substitution \( s \rightarrow -s \) yields a kernel of the type \((1/(t+s))w_1(t)w_2(s)\) on \([0, 1]^2\), where \( w_1 \) and \( w_2 \) are integrable over \([0, 1]\) and bounded near 0. By Hardy, Littlewood and Polya [11], the kernel \( 1/(t+s) \) defines an operator \( L_p(0, \infty) \rightarrow L_p(0, \infty) \) of norm \( \pi / \sin(\pi / \gamma) \), for any \( 1 < \gamma < \infty \). Since \( 1/(t+s) \) is positive, this also holds for \( X \)-valued functions and hence \( T_k : L_p(-1, 0; X) \rightarrow L_p(0, 1; X) \) is bounded as well. The case \( s \in [0, 1], t \in [-1, 0] \) is treated similarly. Together these facts prove the claim. Hence there is a \( c_7 \) such that for all \( f \in L_p(-1, 1; w_\gamma \delta; X) \)

\[
\left( \int_{-1}^{1} \frac{f(s)}{t-s} ds \right)^p (1-t)^{\gamma p}(1+t)^{\delta p} dt)^{1/p} \leq c_7 \left( \int_{-1}^{1} f(s)^p (1-s)^{\gamma p}(1+s)^{\delta p} ds \right)^{1/p},
\]

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and thus

\[
M_1 \leq c_6c_7\left( \int_{-1}^{1} \left\| \sum_{j=1}^{n+1} y_j \chi_j(s) / |I_j| \right\|_p w_{\gamma p, \delta p}(s) ds \right)^{1/p}
\]

\[
\leq c_8\left( \sum_{j=1}^{n+1} \left\| y_j \right\|_p / |I_j|^{p-1} w_{\gamma p, \delta p}(t_j) \right)^{1/p}
\]

\[
\leq c_9\left( \sum_{j=1}^{n+1} \lambda_j \left\| q(t_j) \right\|_p \right)^{1/p},
\]

using that by (4) and (5)

\[
\lambda_j \sim |p_{n+1}^{(\alpha, \beta)'}(t_j)|^{-p}|I_j|^{1-p}(1-t_j)^{\gamma p}(1+t_j)^{\delta p}.
\]

The error term \( M_2 \) can be discretized in view of the monotonicity properties of the \( f_j \)'s. The integration with respect to \( t \) for \( |t-t_j| \leq 2|I_j| \) leads to another term \( M_{21} \) of the form (13), and \( M_2 \leq M_{21} + M_{22} \) with

\[
M_{22} = c_{10}\left( \sum_{i=1}^{n+1} \sum_{j=1, j \neq i}^{n+1} \left\| y_j \right\|_p / |I_i|/(t_i - t_j)^2 |I_i| w_{\gamma p, \delta p}(t_i) \right)^{1/p}
\]

\[
= c_{10}\left( \sum_{i=1}^{n+1} \left( \sum_{j=1, j \neq i}^{n+1} a_{ij} \lambda_j^{1/p} \left\| q(t_j) \right\|_p \right)^{1/p},
\]

where \( a_{ij} = (|I_i|/\lambda_j)^{1/p} w_{\gamma, \delta}(t_i) |I_j|/(|p_{n+1}^{(\alpha, \beta)'}(t_j)|(t_i - t_j)^2) \) for \( i \neq j \) and \( \gamma, \delta \) as before. We claim that \( A_n = (a_{ij})_{i,j=1}^{n+1} \) defines a map \( A_n : \ell_p^{n+1} \rightarrow \ell_p^{n+1} \) with norm bounded by a \( C \) independent of \( n \in \mathbb{N} \). Then (14) is bounded by

\[
c_{10}C\left( \sum_{j=1}^{n+1} \lambda_j \left\| q(t_j) \right\|_p \right)^{1/p}.
\]

as required. Calculation using (4) and (5) shows that for \( i, j \leq n/2 \)

\[
a_{ij} \sim \left( \frac{i}{j} \right)^{\eta} \frac{j^2}{(i^2 - j^2)^2}, \quad \eta := (\alpha + 1/2)(2/p - 1).
\]

The restriction on \( \alpha \) gives that \(-1/2 \leq \eta \leq 2\). This easily implies \(|a_{ij}| \leq c_{11}/(i-j)^{3/2} \); for \( \eta \geq 0 \) or \( \eta < 0 \) and \( i > j/2 \) one even has the
bound $c_{11}/(i-j)^2$. In any case

$$\sup_{i \leq n/2} \sum_{j \leq n/2, i \neq j} |a_{ij}| \leq C, \quad \sup_{j \leq n/2} \sum_{i \leq n/2, i \neq j} |a_{ij}| \leq C,$$

and hence $(a_{ij})_{i,j \leq n/2}$ is uniformly bounded on $\ell_1^{[n/2]}$ and $\ell_\infty^{[n/2]}$ and thus by interpolation on $\ell_p^{[n/2]}$. The three other cases of pairs $(i,j)$, e.g. $i > n/2 \geq j$, are treated similarly, using the assumption on $\beta$ as well.

This proves (2) $\Rightarrow$ (1) except for the case of $\alpha \leq -1/2$ or $\beta \leq -1/2$ when (11) and (12) do not hold for $t \notin J_n$. Assume e.g. $\alpha \leq -1/2$. In this case, we estimate the remaining term

$$M_3 := \left( \int_{b_n}^1 \left\| \sum_{j=1}^{n+1} y_j \frac{p_{n+1}^{(\alpha, \beta)}(t)}{t-t_j} \| w_{\alpha, \beta}(t) dt \right\|^{1/p} \right)^{1/p}$$

by the triangle and the Hölder inequality, using (4), (5), (10) and the fact that for $t \geq b_n$ $|t-t_j|^{-1} \leq d(n/j)^2$. We find

$$M_3 \leq c_{12} n^{-2(1+\alpha)/p} \left( \sum_{j=1}^{n+1} j^{\alpha - 1/2} \| x_j \| \right)$$

$$\leq c_{13} \left( \sum_{j=1}^{n+1} j^{-(\alpha(2/p-1)+1/p+1/2)p'} \right)^{1/p'} \left( \sum_{j=1}^{n+1} \lambda_j \| x_j \|^p \right)^{1/p}$$

$$\leq c_{14} \left( \sum_{j=1}^{n+1} \lambda_j \| x_j \|^p \right)^{1/p}.$$

where we have used that $\alpha(\frac{2}{p} - 1) + \frac{1}{p} - \frac{1}{2} > \frac{1}{p'}$.

(b) (1) $\Rightarrow$ (2). For the converse, assume the interpolation inequality (7) to be true.

We claim that (7) implies that the Hilbert matrix $A = ((i-j+1/2)^{-1})_{i,j\in\mathbb{N}}$ defines a bounded operator $A : \ell_p(X) \to \ell_p(X)$. A well-known approximation and scaling argument shows that this is equivalent to the boundedness of the Hilbert transform $H$ in $L_p(\mathbb{R}; X)$, i.e. $X$ is a UMD-space and necessarily $1 < p < \infty$. In this sense $A$ is a discrete version of $H$. For $n \in \mathbb{N}$
we need the zeros \( (t_{ij}^{n+1})_{j=1}^{n+1} \) of \( p_{n+1}^{(\alpha,\beta)} \) and \( (t_i^n)_{i=1}^n \) of \( p_n^{(\alpha,\beta)} \), ordered decreasingly as before and the corresponding quadrature weights \( \lambda_j^{n+1} \) and \( \lambda_i^n \). Let \( J_n := \{ j \in \mathbb{N} \mid \kappa/\gamma_j \leq J \leq \kappa/\gamma_j \} \). For any sequence \( (x_j)_{j \in J_n} \subseteq X \), consider the \( X \)-valued polynomial

\[
q := \sum_{j \in J_n} \lambda_j^{-1/p} x_j\ell_j^{n+1} \in \Pi_n(X), \quad \ell_j^{n+1}(t_i^{n+1}) = \delta_{ij}.
\]

Applying (7) to \( q \) and inequality (6) with \( n + 1 \) replaced by \( n \) we find

\[
\sum_{i \in J_n} \| q(t_i^n) \|^p \leq c_1 \left( \int_{-1}^1 \| q(t) \|^p w_{\alpha\beta}(t) dt \right)^{1/p} \leq c_2 \sum_{j \in J_n} \lambda_j^{n+1} \| q(t_j^n) \|^p, \tag{15}
\]

i.e.

\[
\sum_{i \in J_n} \sum_{j \in J_n} b_{ij} x_j(t_i^n) \leq c_2 \sum_{j \in J_n} \| x_j \|^p, \tag{15}
\]

with

\[
b_{ij} := \left( \frac{\lambda_i^n \lambda_j^{n+1}}{\lambda_j^{n+1} \lambda_j^{n+1}} \right)^{1/p} \ell_j^{n+1}(t_i^n) = \left( \frac{\lambda_i^n}{\lambda_j^{n+1}} \right)^{1/p} \frac{p_{n+1}^{(\alpha,\beta)}(t_i)}{p_n^{(\alpha,\beta)}(t_j^n)(t_i^n - t_j^{n+1})}.
\]

Let \( k_n := |J_n| \sim n/2 \). By (15), \( B_n := (b_{ij})_{i,j \in J_n} \) satisfies \( B_n : \ell_p^n(X) \rightarrow \ell_p^n(X) \) \( \leq c_2 \), \( c_2 \) being independent of \( n \in \mathbb{N} \). \( B_n \) is close to the block \( A_n := ((i - j + 1/2)^{-1})_{i,j \in J_n} \) of the Hilbert matrix \( A \). To show this we first evaluate \( b_{ij} \). By Szegö \([30, (4.5.7)]\)

\[
(1 - t^2)p_n^{(\alpha,\beta)}(t) = (\eta_n' t + \eta_n'')(p_n^{(\alpha,\beta)}(t) - \eta_n p_n^{(\alpha,\beta)}(t)),
\]

where \( \eta_n, \eta_n', \eta_n'' \in \mathbb{R} \) depend on \( n \) and \( (\alpha, \beta) \), with \( \eta_n/n \rightarrow 1 \) for \( n \rightarrow \infty \). Hence, using (5), for \( i \leq 3n/4 \)

\[
p_{n+1}^{(\alpha,\beta)}(t_i^n) = \eta_i^{-1}(1 - (t_i^n)^2)p_n^{(\alpha,\beta)}(t_i^n) \sim (-1)^i (n/i)^{\alpha-1/2}. \tag{16}
\]

Thus \( b_{ij} = \gamma_i^n / (\delta_j^n n(t_i^n - t_j^{n+1})) \) where \( 0 < c_3 \leq |\gamma_i^n|, |\delta_j^n| \leq c_4 < \infty \)

for \( i, j \in J_n \) and the matrices \( C_n := (c_{ij})_{i,j \in J_n} \), \( c_{ij} := n^{-1}(t_i^n - t_j^{n+1})^{-1} \), are uniformly bounded on \( \ell_p^n(X) \) as well. By Szegö \([30, (8.9.8)]\),

\[
t_i^n = \cos \theta_i^n, \quad \theta_i^n = \frac{(i + k + \varepsilon n)\pi}{n + \alpha + 1/2}, \tag{17}
\]

13
where \( k \) depends on \((\alpha, \beta)\) only and \( \sup_{i \in J_n} |\varepsilon_{ni}| \to 0 \) for \( n \to \infty \). For \( k_n \times k_n \) matrices \( D_n \) and \( E_n \), we write \( D_n \approx E_n \) provided that the matrices \( D_n - E_n \) are uniformly bounded as maps on \( \ell^p_{k_n}(X) \), for \( \text{any} \ 1 \leq p \leq \infty \). It suffices to show that \( C_n \approx A_n \), then \( \sup_{n \in \mathbb{N}} \| A_n : \ell^p_{k_n}(X) \to \ell^p_{k_n}(X) \| \leq c_5 \), i.e. \( X \) is a UMD-space. We have

\[
c_{ij} = \frac{1}{2n} \frac{1}{\sin(\theta_i^n - \theta_j^{n+1})/2} \frac{1}{\sin(\theta_i^n + \theta_j^{n+1})/2}.
\]

If \( d_{ij} := \frac{1}{\pi(i - j + 1/2) \sin(\theta_i^n + \theta_j^{n+1})/2} \), \( D_n := (d_{ij})_{i,j \in J_n} \),

then \( C_n \approx D_n : [1/ \sin((\theta_i^n + \theta_j^{n+1})/2)] \) is uniformly bounded for \( i, j \in J_n \), and hence the estimates \( |1/ \sin x - 1/x| \leq |x| \) for \( |x| \leq \pi/4 \) and \( |1/x - 1/(x + \varepsilon)| \leq 2\varepsilon/x^2 \) for \( x = i - j + 1/2 \) and \( |\varepsilon| \leq 1/4 \) yield

\[
|c_{ij} - d_{ij}| \leq c_6 \left( \frac{1}{2n \sin(\theta_i^n - \theta_j^{n+1})/2} - \frac{1}{n(\theta_i^n - \theta_j^{n+1})/2} \right) + \frac{1}{n(\theta_i^n - \theta_j^{n+1})/2} - \frac{1}{\pi(i - j + 1/2)} \leq c_7 \left( \frac{|i - j + 1/2|}{n^2} + \frac{1}{|i - j + 1/2|^2} \right).
\]

Hence \( \sup_i \sum_j |c_{ij} - d_{ij}| \leq c_8 \), \( \sup_j \sum_i |c_{ij} - d_{ij}| \leq c_8 \) uniformly in \( n \in \mathbb{N} \), i.e. \( C_n \approx D_n \) (first for \( p = 1, \infty \), then by interpolation for general \( p \)). Next \( D_n \) is transformed into \( E_n = (e_{ij}) \) with \( D_n \approx E_n \),

\[
e_{ij} := \frac{1}{\pi(i - j + 1/2) \sin^{1/2} \frac{i + j}{4\pi}}, \quad i, j \in J_n.
\]

By the Lipschitz continuity of \( 1/ \sin x \) in \( \pi/4 \leq x \leq 3\pi/4 \),

\[
|d_{ij} - e_{ij}| \leq c_9/|n(i - j + 1/2)|,
\]

which again is uniformly row- and column- summable, i.e. \( D_n \approx E_n \). Finally, let

\[
f_{ij} := \frac{1}{\pi(i - j + 1/2) \sin^{1/2} \frac{i + j}{4\pi}}, \quad F_n := (f_{ij})_{i,j \in J_n}.
\]
For \( y \in [\pi/8, 3\pi/8] \), \( g(x) = 1/\sin(x + y) \) is Lipschitz-continuous in \( x \in [\pi/8, 3\pi/8] \) with constant \( \leq 2 \). Hence for \( i, j \in J_n \)

\[
|e_{ij} - f_{ij}| \leq \frac{|i - j|}{n|i - j + 1/2|} \leq \frac{2}{n}, \quad E_n \approx F_n.
\]

Since \( (\sin(\frac{\pi x}{n}))^{-1})_{i \in J_n} \) is bounded away from zero, this implies that \( A_n = ((i - j + 1/2)^{-1})_{i,j \in J_n} \) defines uniformly bounded maps on \( \ell_p^n(X) \). Hence \( X \) is a UMD-space and \( 1 < p < \infty \).

We now prove that (7) implies \( \mu(\alpha, \beta) < p < M(\alpha, \beta) \). This is a purely scalar argument, \( X = \mathbb{R} \). By symmetry, we may assume that \( \alpha \geq \beta \); the case of \( \alpha \leq -1/2 \) (\( 1 < p < \infty \)) is known already. So let \(-1/2 < \alpha < 0\). Take \( q = p_n(\alpha, \beta) \in \Pi_n \) in (7) to show that necessarily \( p < M(\alpha, \beta) = 4\frac{\alpha + 1}{2\alpha + 1} \). Using the second formula of Szegö [30, 4.5.7], one shows similarly as in (16) that

\[
|p_n^{(\alpha, \beta)}(t_j)|^2 \sim (n/i)^{\alpha - 1/2} \text{ for } i \leq n/2 \text{ and } \sim (n/(n + 2 - i))^{\beta - 1/2} \text{ for } i > n/2.
\]

Thus by (4) and Newman-Rudin [20], cf. also (10),

\[
\left( \int_{-1}^{1} |p_n^{(\alpha, \beta)}(t)|^p w_{\alpha \beta} (t) dt \right)^{1/p} \sim \begin{cases} 1 & p < M(\alpha, \beta) \\ (\log n)^{1/p} & p = M(\alpha, \beta) \\ n^{p(\alpha + 1/2 - 2(\alpha - 1))} & p > M(\alpha, \beta) \end{cases}, \tag{18}
\]

\[
\sum_{j=1}^{n+1} \lambda_j |p_n^{(\alpha, \beta)}(t_j)|^p \sim \begin{cases} 1 & (p < \infty \text{ and } \alpha \leq 1/2) \text{ or } (p < 4(\alpha + 1)/(2\alpha - 1) \text{ and } \alpha > 1/2) \\ (\log n)^{1/p} & p = 4(\alpha + 1)/(2\alpha - 1) \text{ and } \alpha > 1/2 \\ n^{(\alpha - 1/2) - 2/p(\alpha + 1)} & p > 4(\alpha + 1)/(2\alpha - 1) \text{ and } \alpha > 1/2 \end{cases}. \tag{19}
\]

Hence for \( p \geq M(\alpha, \beta) \), the order of growth (in \( n \)) in (18) is faster than in (19) and (7) cannot hold. To prove that necessarily \( p > \mu(\alpha, \beta) = 4(\alpha + 1)/(2\alpha + 5) \) for \( \alpha > 1/2 \) (\( \mu(\alpha, \beta) = 1 \) for \( \alpha \leq 1/2 \)), we take

\[ q = \ell_1 \in \Pi_n, \quad \ell_1(t) = p_{n+1}(t)/(p_{n+1}(t_1)). \]

Clearly, the right side of (7) is \( \sim \lambda_1^{1/p} \sim n^{-2/p(\alpha + 1)} \) by (4) whereas the asymptotic formulas for \( p_n^{(\alpha, \beta)} \) of Szegö [30, 8.21] and (5) yield
\[
\left( \int_{-1}^{1} |\ell_1(t)|^p w_{\alpha,\beta}(t) dt \right)^{1/p} \sim \left( \int_{0}^{1-n^2} (1 - t)^{\alpha-p/4(2\alpha+5)} dt \right)^{1/p} n^{-(\alpha+5/2)}
\]
\[
\sim \begin{cases} 
  n^{-(\alpha+5/2)} & p < \mu(\alpha, \beta) \\
  (\log n)^{1/p} & p = \mu(\alpha, \beta) \\
  n^{-2/p(\alpha,\beta)} & p > \mu(\alpha, \beta) 
\end{cases}, 
\]

(20)

Hence (20) grows faster in \( n \) than \( \lambda_1^{1/p} \sim n^{-2/p(\alpha+1)} \) if \( p \leq \mu(\alpha, \beta) \), i.e. \( p > \mu(\alpha, \beta) \) is necessary for (7) to hold. This proves (b) of Theorem 1.

(a). The left interpolation inequality (6) in Theorem 1 is proved as in the scalar case. Nevai’s proof in [19] using the mean value theorem, Hölder’s inequality and some weighted form of Bernstein’s inequality in the \( p \)-norm generalizes directly to the vector-valued setting. Just as the scalar result of Khalilova [14] and Potapov [27], the vector-valued form of the Bernstein \( L_p \)-inequality (lemma 2 in [19]) is proved by interpolating at Tchebychev nodes, using an averaging technique, the triangle inequality in \( L_p \) and the Bernstein inequality for the sup-norm. In the vector valued case the latter follows from the scalar version, applying linear functionals and using the Hahn-Banach theorem. We do not give the details, since the proofs of [19], [14] and [27] directly generalize.

\[ \blacksquare \]

Remarks.

(1). If the validity of (7) of Theorem 1 (b),

\[
\left( \int_{I} \| q(t) \|^p w_{\alpha,\beta}(t) dt \right)^{1/2} \leq c_p \left( \sum_{j=1}^{n+1} \lambda_j^{n+1} \| q(t_j^{n+1}) \|^p \right)^{1/p},
\]

is required only for all polynomials \( q \in \Pi_k(X) \) with \( k \leq n/2 \), this holds for all \( 1 \leq p \leq \infty \) and all Banach spaces, at least if \( \alpha, \beta \geq -1/2 \). This follows from the boundedness of the generalized de la Vallée-Poussain means in \( L_p(X) \) along similar lines as in Zygmund [32], Stein [29] and Askey [4]. Thus the restriction on \( p \) and \( X \) in Theorem 1 comes from requiring the number of nodes to equal the dimension of \( \Pi_n \), namely \( n+1 \). In this way, however, one isomorphically identifies \( \Pi_n(X) \subset L_p(X) \) with the space \( \ell_p^{n+1}(X) \).

(2). The proof of the necessity of the UMD-condition for inequality (7) of Theorem 1 (b) will work for more general orthogonal polynomials provided
that sufficiently precise information on a fairly large part of the zeros of these
is known, like in (17).

The restriction \( p < M(\alpha, \beta) \) means geometrically (for \( \alpha > -1/2 \)) that
the value \( |p_n^{(\alpha, \beta)}(t_1^{n+1})|^p \) is much smaller than the dominating mean value of
\( |p_n^{(\alpha, \beta)}(t)|^p \) over \( (t_1^{n+1}, 1) \) with respect to \( w_{\alpha \beta}(t) dt \), if \( p \geq M(\alpha, \beta) \).

An immediate corollary to Theorem 1 is the following result on the con-
vergence of interpolating polynomials which in the scalar case is due to Askey [2] and Nevai [19].

**Proposition 9** Let \( X \) be a UMD-space, \( \alpha, \beta > -1 \) and \( p < M(\alpha, \beta) \). Let \( f : (-1, 1) \rightarrow X \) be continuous. Then the interpolating polynomials of \( f \) at
the zeros \( (t_j)_{1}^{n+1} \) of \( p_n^{(\alpha, \beta)} \), \( I_n f := \sum_{j=1}^{n+1} f(t_j) \ell_j \in \Pi_n(X) \), converge to \( f \) in the
\( p \)-norm,

\[
\| f - I_n f \|_{p;\alpha,\beta} = \left( \int_{-1}^{1} \| f(t) - I_n f(t) \|_p w_{\alpha \beta}(t) dt \right)^{1/p} \longrightarrow 0.
\]

**Proof:** Approximate \( f \) by polynomials \( q_n \in \Pi_n(X) \) in the sup-norm, \( \| f - q_n \|_\infty \longrightarrow 0 \). We may assume that \( \mu(\alpha, \beta) < p < M(\alpha, \beta) \), since the
\( p \)-norms get weaker for smaller \( p \). Using (b) of Theorem 1 and

\[
\sum_{j=1}^{n+1} \lambda_j = \int_{-1}^{1} w_{\alpha \beta}(t) dt =: M < \infty,
\]

we find

\[
\| f - I_n f \|_{p;\alpha,\beta} \leq \| f - q_n \|_{p;\alpha,\beta} + \| q_n - I_n f \|_{p;\alpha,\beta}
\leq M^{1/p} \| f - q_n \|_\infty + c_p \left( \sum_{j=1}^{n+1} \lambda_j \| q_n(t_j) - f(t_j) \|_p \right)^{1/p}
\leq (1 + c_p)M^{1/p} \| f - q_n \|_\infty \longrightarrow 0.
\]

\( \square \)

3 Convergence of vector-valued Jacobi series

**Proof of Theorem 2:** Recall that \( Q_n f := \sum_{j=0}^{n} \langle f, p_j^{(\alpha, \beta)} \rangle > p_j^{(\alpha, \beta)} \) for
\( f \in L_p(I, w_{\alpha,\beta}; X) \). Thus \( Q_n \) is the integral operator induced by the ker-
nel \( k_n(x, y) = \sum_{j=0}^{n} p_j^{(\alpha, \beta)}(x)p_j^{(\alpha, \beta)}(y) \) with respect to the measure \( d\mu(t) = w_{\alpha\beta}(t)dt \).

(2) \( \Rightarrow \) (1). We sketch the straightforward generalization of the scalar proof of Pollard [26] and Muckenaupt [18] to the UMD-case. Since \( Q_nf \to f \) on the dense set of \( X \)-valued polynomials \( f \), (1) of Theorem 2 is equivalent to

\[
\sup_{n \in \mathbb{N}} \| Q_n : L_p(I, w_{\alpha\beta}; X) \to L_p(I, w_{\alpha\beta}, X) \| = c_p < \infty.
\]  

(21)

Using the Christoffel-Darboux formula for \( k_n \) and the classical analysis of Pollard [26], (21) will follow from the uniform boundedness of the integral operators \( T_{n1}, T_{n2}, T_{n3} \) induced by the following kernels as maps in \( L_p(I, w_{\alpha\beta}; X) \):

\[
k_{n1}(x, y) := p_{n+1}^{(\alpha, \beta)}(x)q_n^{(\alpha, \beta)}(y)/(x - y), \quad q_n^{(\alpha, \beta)}(y) := (1 - y^2)p_n^{(\alpha+1, \beta+1)}(y)
\]

\[
k_{n2}(x, y) := k_{n1}(y,x), \quad k_{n3}(x, y) := p_n^{(\alpha, \beta)}(x)p_n^{(\alpha, \beta)}(y).
\]

The proof of the uniform boundedness of \( T_{n1} \) and \( T_{n2} \) is similar to the proof of Theorem 1, (b), (2) \( \Rightarrow \) (1). On the intervals \( J_n \) defined there (for \( \alpha, \beta \geq -1/2, J_n = I \), \( T_{n1} \) and \( T_{n2} \) are uniformly bounded in \( p \)-norm provided that the weighted Hilbert transform kernels \( (x - y)^{-1}(w_{\alpha\beta}(x)w_{\alpha\beta}(y))^{1/2}((1 - x^2)/(1 - y^2))^{1/4} \) (+ for \( T_{n2} \), - for \( T_{n1} \)) define bounded operators on \( L_p(I; X) \), as follows from (10) the same way as in (b), (2) \( \Rightarrow \) (1). In view of the UMD-assumption on \( X \), this will follow from the boundedness of the kernel operator defined by

\[
\frac{1}{|x - y|}(w_{\alpha\beta}(x)w_{\alpha\beta}(y))^{1/2}((1 - x^2)/(1 - y^2))^{1/4} - 1
\]

on \( L_p(I; X) \). Using again lemma 1, the latter fact is a consequence of

\[
-\frac{1}{p} < \alpha(\frac{1}{p} - \frac{1}{2}) \pm \frac{1}{4}, \beta(\frac{1}{p} - \frac{1}{2}) \pm \frac{1}{4} < 1 - \frac{1}{p},
\]

i.e. \( m(\alpha, \beta) < p < M(\alpha, \beta) \). If e.g. \( \alpha < -1/2 \), the part of \( \| T_{ni}f \|_p, i \in \{1, 2\} \), on the interval \((1 - n^{-2}, 1)\) outside \( J_n \) has to be estimated separately. However, \( p_n^{(\alpha, \beta)} \) and \( q_n^{(\alpha, \beta)} \) are uniformly bounded in \( n \in \mathbb{N} \) there, and a direct application of the continuity of the (unweighted) Hilbert transform suffices. The uniform boundedness of \( T_{n3} \) follows from \( \sup_{n \in \mathbb{N}} \| p_n^{(\alpha, \beta)} \|_p \| p_n^{(\alpha, \beta)} \|_{p'} < \infty \) if \( m(\alpha, \beta) < p < M(\alpha, \beta) \).
(1) $\Rightarrow$ (2). Assume that $Q_n f \rightarrow f$ for all $f \in L_p(I, w_{\alpha \beta}; X)$. By the Banach-Steinhaus theorem, this is equivalent to (21). Using (21), we prove (7) of Theorem 1, which then implies that $X$ is an UMD-space and, in view of the self-duality of (21), that $m(\alpha, \beta) < p < M(\alpha, \beta)$. To show (7), we dualize (6) which holds for all $X$ and $p$. Let $q \in \Pi_n(X)$. Then there is a $g \in L_{p'}(I, w_{\alpha \beta}; X^*)$ which $\| g \|_{p'; \alpha, \beta} = 1$ and

$$J := \left( \int_{-1}^{1} \| q(t) \|^p w_{\alpha \beta}(t) dt \right)^{1/p} = \int_{-1}^{1} < q(t), g(t) >_{(X, X^*)} w_{\alpha \beta}(t) dt$$

$$= \int_{-1}^{1} < q(t), Q_n g(t) >_{(X, X^*)} w_{\alpha \beta}(t) dt.$$

Since $< q, Q_n g > \in \Pi_{2n}$, Gaussian quadrature, Hölder’s inequality and (6) as well as (the dual form of) (21) yield

$$J = \sum_{j=1}^{n+1} \lambda_j < q(t_j), Q_n g(t_j) >$$

$$\leq \left( \sum_{j=1}^{n+1} \lambda_j \| Q_n g(t_j) \|_{X^*}^{p'} \right)^{1/p'} \left( \sum_{j=1}^{n+1} \lambda_j \| q(t_j) \|_X^p \right)^{1/p}$$

$$\leq c \| Q_n g \|_{p'; \alpha, \beta} \left( \sum_{j=1}^{n+1} \lambda_j \| q(t_j) \|_X^p \right)^{1/p}$$

$$\leq c c p \left( \sum_{j=1}^{n+1} \lambda_j \| q(t_j) \|_X^p \right)^{1/p}$$

which is (7).

We turn to the equivalence of vector-valued Jacobi means.

**Proof of proposition 3:** For $\alpha > -1$ and $w_\alpha = w_{\alpha, \alpha}$, the map $\psi : L_2(I, w_\alpha; X) \rightarrow L_2(0, \pi; X)$ defined by

$$\psi(g)(s) = (\sin s)^{\alpha + 1/2} g(\cos s) \quad g \in L_2(I, w_\alpha; X), \quad s \in [0, \pi],$$

is an isometry. Let $q_n^{(\alpha)} := \psi(p_n^{(\alpha, \alpha)})$ and

$$K_n^{(\alpha, \beta)}(t, s) := \sum_{j=0}^{n} q_j^{(\alpha)}(t) q_j^{(\beta)}(s) \quad \alpha, \beta > -1.$$
These kernels induce uniformly bounded operators on $L_p(0, \pi; X)$ for any $1 < p < \infty$, e.g. there is $c_p$ such that for all $n \in \mathbb{N}$ and $h \in L_p(0, \pi; X)$

$$
(\int_0^\pi \| \int_0^\pi \mathcal{K}_n^{(\alpha, \beta)}(t, s)h(s)ds \|_p^p dt)^{1/p} \leq c_p(\int_0^\pi \| h(s) \|_p^p ds)^{1/p}. \quad (22)
$$

This follows from the proofs of Theorem 1 and 3 of Gilbert [8]: The scalar proof given there directly generalizes to the $X$-valued UMD-case since only the $L_p$-uniform boundedness of the Dirichlet and conjugate Dirichlet kernel operators is used, which holds $X$-valued for UMD-spaces. In effect, $\mathcal{K}_n^{(\alpha, \beta)}$ is shown in [8] to behave very similar to the Dirichlet kernel. In particular

$$
|\mathcal{K}_n^{(\alpha, \beta)}(t, s)| \leq d_1/|t - s| \quad (23)
$$

where $d_1$ is independent of $n \in \mathbb{N}$ and $t, s \in [0, \pi]$. We claim that also

$$
(\int_0^\pi \| \int_0^\pi \mathcal{K}_n^{(\alpha, \beta)}(t, s)(\sin t/ \sin s)^{\gamma+1/p-1/2}h(s)ds \|_p^p dt)^{1/p} \leq c_p'(\int_0^\pi \| h(s) \|_p^p ds)^{1/p}, \quad (24)
$$

provided that $|\gamma/2 + 1/p - 1/2| < 1/4$. By (22), this will follow from the uniform boundedness of the difference kernel operators

$$
\mathcal{L}_n^{(\alpha, \beta)}(t, s) := \mathcal{K}_n^{(\alpha, \beta)}(t, s)((\sin t/ \sin s)^{\gamma+1/p-1/2} - 1)
$$

in $L_p(0, \pi; X)$. Using (23) and elementary estimates we obtain the existence of a $d_2$ such that for $n \in \mathbb{N}$ and $t, s \in [0, \pi/2]$

$$
|\mathcal{L}_n^{(\alpha, \beta)}(t, s)| \leq \frac{d_1}{|t - s|}|(\frac{\sin t}{\sin s})^{\gamma+1/p-1/2} - 1| \leq \frac{d_2}{|t - s|}|(\frac{t}{s})^{\gamma+1/p-1/2} - 1|.
$$

Hence by lemma 1, the $\mathcal{L}_n^{(\alpha, \beta)}$-kernels define uniformly bounded integral operators in $L_p(0, \pi/2; X)$ since $-1/p < \gamma + 1/p - 1/2 < 1 - 1/p$. On $(\pi/2, \pi)$, the estimate is similar; for $t \in [\pi/2, \pi], s \in [0, \pi/2]$, there are only point singularities and the transformation $t \to \pi - t$ reduces the $\mathcal{L}_n^{(\alpha, \beta)}$-boundedness to the one of the positive kernel $1/(t + s)$ in $L_p(0, \pi/2; X)$. Hence (24) holds.

For functions $f \in L_p(I, w_{(\beta+\gamma)p/2}; X)$ on the interval $I = (-1, 1)$ and the kernel

$$
k_n^{(\alpha, \beta)}(x, y) := \sum_{j=0}^n p_j^{(\alpha, \alpha)}(x)p_j^{(\beta, \beta)}(y),
$$

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(24) is equivalent to

\[
\left( \int_{-1}^{1} \left\| \int_{-1}^{1} k^{(\alpha,\beta)}_{\nu}(x,y) f(y) w_{\beta}(y) dy \right\|^p \, w_{(\alpha+\gamma)p/2}(x) dx \right)^{1/p} \leq c_p' \int_{-1}^{1} \| f(y) \|^p \, w_{(\beta+\gamma)p/2}(y) dy \right)^{1/p}
\]

as the transformation \( h(s) = (\sin s)^{\beta+\gamma+1/p} f(\cos s) \) shows. Applying (25) to \( f(y) = \sum_{j=0}^{n} p_j^{(\beta,\beta)}(y)x_j \), where \( x_j \in X \), yields a one-sided estimate of (8); the converse direction follows from the symmetry of the statement in \( \alpha \) and \( \beta \).

The argument also shows that the convergence of the series \( \sum_{j=0}^{n} p_j^{(\alpha,\alpha)} \otimes x_j \) in \( L_p(I, w_{(\alpha+\gamma)p/2}; X) \) is equivalent to the convergence of the series \( \sum_{j=0}^{n} p_j^{(\beta,\beta)} \otimes x_j \) in \( L_p(I, w_{(\beta+\gamma)p/2}; X) \), provided that \( |\gamma/2 + 1/p - 1/2| < 1/4 \).

The choice of \( p = 2 \) and \( \gamma = 0 \) shows that the means

\[
\left( \int_{-1}^{1} \left\| \sum_{j=0}^{n} p_j^{(\alpha,\alpha)}(t)x_j \right\|^2 \, w_{\alpha}(t) dt \right)^{1/2}
\]

are essentially independent of \( \alpha \), the choice of \( \gamma = 0 \) for \( 4/3 < p < 4 \) shows a similar statement for the means

\[
\left( \int_{-1}^{1} \left\| \sum_{j=0}^{n} p_j^{(\alpha,\alpha)}(t)x_j \right\|^p \, w_{\alpha p/2}(t) dt \right)^{1/p}.
\]

4 Unconditional convergence

We now show that under the conditions of Proposition 4, vector-valued convergence of orthonormal series is unconditional only in the case of Hilbert spaces.

Proof of Proposition 4:

(i). Let \((\Omega, \mu)\) be a finite measure space and \((p_n)\) be a complete orthonormal system in \( L_2(\Omega, \mu) \) such that \( \sum_n <f, p_n> p_n \) converges unconditionally for all \( f \in L_p(\Omega, \mu) \). By duality, the same holds in \( L_{p'}(\Omega, \mu) \). Thus we may assume that \( p \geq 2 \). Using the unconditionality and the Khintchine inequality, we find for any finite sequence \((a_n) \subset \mathbb{K} \)

\[
\sum_{n} |a_n|^2 = \left\| \sum_{n} a_n p_n \right\|_2 \leq c_1 \left\| \sum_{n} a_n p_n \right\|_p^2
\]
\begin{align*}
&\leq c_2 \left( \int_0^1 \int_\Omega \left| \sum_n a_nr_n(t)p_n(w) \right|^p \mu(w) \, dt \right)^{2/p} \\
&\leq c_3 \left( \sum_n \left( \int_\Omega \left| a_n \right|^2 \left| p_n(w) \right|^2 \mu(w) \right)^{p/2} \right)^{2/p} \\
&\leq c_3 \sum_n \left( \int_\Omega \left| a_n \right|^p \left| p_n(w) \right|^p \mu(w) \right)^{2/p} \\
&= c_3 \sum_n \left| a_n \right|^2 \left\| p_n \right\|_p^2 \leq c_3 \sup_n \left\| p_n \right\|_p^2 \sum_n \left| a_n \right|^2 \\
&\leq c_4 \sum_n \left| a_n \right|^2,
\end{align*}

where we have used the triangle inequality in $L_{p/2}$ and the assumption on $\left\| p_n \right\|_p$. Hence $\left\| \sum_n a_n p_n \right\|_p \sim \left\| (a_n) \right\|_2$, which implies $p = 2$ since $(p_n)$ was assumed to be a complete orthonormal system and $L_p(\Omega, \mu) \sim \ell_2$ only for $p = 2$. Thus (i) holds, even for $X = K$.

(ii). We give a modification of the argument of Defant and Junge [6]. Let $x_1, \cdots x_m \in X$. By the unconditionality assumption on the $(p_j)$, the hypothesis that $\sup_j |p_j| \in L_2(\Omega, \mu)$, and the contraction principle, cf. Maurey and Pisier [17] we get,

\begin{align*}
\left( \int_\Omega \left\| \sum_{j=1}^m p_j(w)x_j \right\|^2 \mu(w) \right)^{1/2} &\leq c_1 \left( \int_\Omega \int_0^1 \left\| \sum_{j=1}^m r_j(t)p_j(w)x_j \right\|^2 \, dt \, \mu(w) \right)^{1/2} \\
&\leq c_2 \left( \int_\Omega \left( \sup_j \left| p_j(w) \right|^2 \right) \left( \int_0^1 \left\| r_j(t)x_j \right\|^2 \, dt \right) \mu(w) \right)^{1/2} \\
&\leq c_3 \left( \int_0^1 \left\| \sum_{j=1}^m r_j(t)x_j \right\|^2 \, dt \right)^{1/2}.
\end{align*}

Let $(\gamma_j)$ be a sequence of independent standard $N(0, 1)$ Gaussian variables on a probability space $(\Gamma, \nu)$. By Pisier [23] with $c_4 = \sqrt{\pi/2} c_3$

\begin{equation}
\left( \int_\Omega \left\| \sum_{j=1}^m p_j(w)x_j \right\|^2 \mu(w) \right)^{1/2} \leq c_4 \left( \int_\Gamma \left\| \sum_{j=1}^m \gamma_j(s)x_j \right\|^2 \, d\nu(s) \right)^{1/2}
\end{equation} (26)
Since $L_2(\Omega, \mu)$ is infinite dimensional, for any $n \in \mathbb{N}$ there is a unitary map $U : L_2(\Omega, \mu) \rightarrow L_2(\Omega, \mu)$ such that with $f_j := U p_j$ the functions $f_1, \cdots f_n$ are mutually disjointly supported. Let $f_k = \sum_j u_{jk} p_j \in L_2(\Omega, \mu)$, $(u_{jk})$ unitary. Applying (26) for arbitrary $y_1, \cdots y_n \in X$ with $x_j := \sum_{k=1}^n u_{jk} y_k$ we find using the unitary invariance of the right side of (26)

\[
\left( \sum_{k=1}^n \| y_k \|^2 \right)^{1/2} = \left( \int_{\Omega} \| \sum_{k=1}^n f_k(w) y_k \|^2 \, d\mu(w) \right)^{1/2} = \left( \int_{\Omega} \| \sum_j p_j(w) x_j \|^2 \, d\mu(w) \right)^{1/2} \leq c_4 \left( \int_{\Gamma} \| \sum_j \gamma_j(s) x_j \|^2 \, d\nu(s) \right)^{1/2} = c_4 \left( \int_{\Gamma} \| \sum_{k=1}^n \gamma_k(s) y_k \|^2 \, d\nu(s) \right)^{1/2},
\]

i.e. $X$ has cotype 2. Similarly, the converse inequality to (26) will imply that $X$ has type 2 and thus by Kwapięń [13] that $X$ is isomorphic to a Hilbert space. By Maurey and Pisier [17], the Gaussian and the Rademacher means are equivalent since $X$ has cotype 2. Using this and Kahane’s inequality [15], we get for any $x_1, \cdots x_m \in X$

\[
\left( \int_{\Gamma} \| \sum_{j=1}^m \gamma_j(s) x_j \|^2 \, d\nu(s) \right)^{1/2} \leq c_5 \left( \int_{0}^{1} \| \sum_{j=1}^m r_j(t) x_j \|^2 \, dt \right)^{1/2} \leq c_6 \int_{0}^{1} \| \sum_{j=1}^m r_j(t) x_j \| \, dt.
\]

Since $\| p_j \|_2 = 1$, the contraction principle, the Hölder inequality and the unconditionality assumption yield similarly as in Defant and Junge [3], cf. also Pisier [23], that this is

\[
= c_6 \int_{0}^{1} \| \sum_{j=1}^m (\int_{\Omega} |p_j(w)|^2 d\mu(w)) r_j(t) x_j \| \, dt
\]
\[
\leq c_7 \int_0^1 \int_{\Omega} \left( \sup_j |p_j(w)| \right) \left\| \sum_{j=1}^m p_j(w) r_j(t) x_j \right\| d\mu(w) dt \\
\leq c_7 \left\| \sup_j |p_j| \right\|_{L^2(\Omega, \mu)} \left( \int_{\Omega} \int_0^1 \left\| \sum_{j=1}^m r_j(t) p_j(w) x_j \right\|^2 dt d\mu(w) \right)^{1/2} \\
\leq c_8 \left( \int_{\Omega} \left\| \sum_{j=1}^m p_j(w) x_j \right\|^2 d\mu(w) \right)^{1/2},
\]

i.e. the converse to (26) holds. Hence \(X\) is isomorphic to a Hilbert space. \(\square\)

As a corollary we find

**Proposition 10** Let \(\alpha, \beta > -1, 1 \leq p \leq \infty\) and \(X\) be a Banach space. Assume that for all \(f \in L^p(I, w_{\alpha\beta}; X)\), the Jacobi series \(\sum_{n=0}^\infty < f, p_n^{(\alpha, \beta)} > p_n^{(\alpha, \beta)}\) converges unconditionally in \(L^p(I, w_{\alpha\beta}; X)\). Then \(p = 2\) and \(X\) is isomorphic to a Hilbert space: the expansions converge unconditionally precisely in the Hilbert space situation.

**Proof:** By Theorem 2, necessary for convergence is \(m(\alpha, \beta) < p < M(\alpha, \beta)\). For these values of \(p\), the inequality (10) yields e.g. if \(p > 2\)

\[
\left\| p_j^{(\alpha, \beta)} \right\|_{p; \alpha, \beta} \leq c_1 \int_{-1}^1 (1-t)^{\alpha(1-p/2)-p/4}(1+t)^{\beta(1-p/2)-p/4} dt = c_2 = c_2 \left\| p_j^{(\alpha, \beta)} \right\|_{2; \alpha, \beta} < \infty
\]

and

\[
\left\| \sup_j |p_j^{(\alpha, \beta)}| \right\|_{2; \alpha, \beta} \leq c_3 \left( \int_{-1}^1 (1-t^2)^{-1/2} dt \right)^{1/2} = c_4.
\]

Thus \(p = 2\) and \(X\) is a Hilbert space by Proposition 4. \(\square\)

The unconditionality of the Haar system in \(L^p(0,1; X)\), if \(1 < p < \infty\) and \(X\) is an UMD-space, shows that Proposition 4 does not hold without conditions being imposed on the system \((p_j)\) as done in (i), (ii) there.

For the proof of Propositions 5 and 7, we need the following result due to Lindenstrauss and Pelczyński [14, proof of Theorem 4.2] and Olevskii [21].

**Theorem 11** Let \(1 \leq p \leq \infty\) and \((p_n)_{n \in \mathbb{N}}\) be a basis of \(L^p(0,1)\). Let \((h_j)_{j \in \mathbb{N}}\) denote the Haar system on \([0,1]\), normalized by \(\| h_j \|_p = 1\). For any \(0 < \delta < \)

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there is a block basis sequence \((z_j)_{j \in \mathbb{N}}\) of \((p_n)_{n \in \mathbb{N}}\) such that for every \(N \in \mathbb{N}\) there is a measure preserving automorphism \(\varphi_N : [0, 1] \to [0, 1]\) with

\[
\sum_{j=1}^{N} \| z_j^* \|_{p'} h_j \circ \varphi_N - z_j \|_p \leq \delta,
\]

where \((z_j^*) \subset L_{p'}(0, 1)\) is biorthogonal to \((z_j) \subset L_p(0, 1)\).

Thus for some increasing sequence \((m_j)_{j \in \mathbb{N}^*}\) of integers and scalars \((a_n)_{n \in \mathbb{N}}\)

\[
z_j = \sum_{n=m_j-1+1}^{m_j} a_n p_n
\]
is in the above sense close to the Haar system.

**Proof of Proposition 5.** We will show that the Haar system is unconditional in \(L_p(0, 1; X)\). Then, by Maurey [16] \(X\) has to be a UMD-space, using also the results of [4] and [5].

Let \(N \in \mathbb{N}, 0 < \delta < 1\) and \(x_1, \ldots, x_N \in X\). Let \(\varphi_N\) be as in the theorem and put \(g_j := h_j \circ \varphi_N\). Since \((p_n)_{n \in \mathbb{N}}\) is unconditional in \(L_p(0, 1; X)\) by assumption, so is the block basic sequence \((z_j)_{j \in \mathbb{N}}\). Hence for any sequence of signs \((\varepsilon_j), \varepsilon_j \in \{+1, -1\}\),

\[
\left( \int_0^1 \left\| \sum_{j=1}^{N} \varepsilon_j h_j(t) x_j \right\|^p dt \right)^{1/p} = \left( \int_0^1 \left\| \sum_{j=1}^{N} \varepsilon_j g_j(t) x_j \right\|^p dt \right)^{1/p},
\]

\[
\leq \left( \int_0^1 \left\| \sum_{j=1}^{N} \varepsilon_j z_j(t) x_j \right\|^p dt \right)^{1/p} + \left( \int_0^1 \left\| \sum_{j=1}^{N} \varepsilon_j (g_j(t) - z_j(t)) x_j \right\|^p dt \right)^{1/p},
\]

\[
\leq K \left( \int_0^1 \left\| \sum_{j=1}^{N} z_j(t) x_j \right\|^p dt \right)^{1/p} + \sum_{j=1}^{N} \| g_j - z_j \|_p \| x_j \|
\]

\[
\leq (K + \delta) \left( \int_0^1 \left\| \sum_{j=1}^{N} z_j(t) x_j \right\|^p dt \right)^{1/p},
\]

using that \(x_j = \langle z_j^*, \sum_{k=1}^{N} z_k x_k \rangle\) and hence

\[
\| x_j \| \leq \| z_j^* \|_p \left( \int_0^1 \left\| \sum_{k=1}^{N} z_k(t) x_k \right\|^p dt \right)^{1/p}.
\]
The constant $K$ is independent of $N$, $(x_j)$ and $(\varepsilon_j)$. The chain of inequalities can be reversed with all $\varepsilon_j = +1$ to find

$$
\left( \int_0^1 \| \sum_{j=1}^N \varepsilon_j h_j(t) x_j \|^p dt \right)^{1/p} \leq (K + \delta) \left( \int_0^1 \| \sum_{j=1}^N z_j(t) x_j \|^p dt \right)^{1/p}
$$

$$
\leq (K + \delta)(1 + \delta) \left( \int_0^1 \| \sum_{j=1}^N h_j(t) x_j \|^p dt \right)^{1/p},
$$
i.e. the Haar system is unconditional. \qed

A similar procedure is used in the Proof of Proposition 7: Let $N \in \mathbb{N}, x_1, \ldots x_N \in X$ and $0 < \delta < 1$. With the same notation as in the previous proof,

$$
\left( \int_0^1 \| \sum_{j=1}^N h_j(t) x_j \|^2 dt \right)^{1/2} \leq (1 + \delta) \left( \int_0^1 \| \sum_{j=1}^N z_j(t) x_j \|^2 dt \right)^{1/2}.
$$

with $z_j = \sum_{n=m_j-1+1}^{m_j} a_n p_n$ and $(m_j)_{j \in \mathbb{N}} \subset \mathbb{N}$ increasing. Note that

$$
\left( \sum_{n=m_j-1+1}^{m_j} |a_n|^2 \right)^{1/2} = \| z_j \|_2 \leq (1 + \delta) \| h_j \|_2 = (1 + \delta).
$$

Thus, using that $X$ has $(p_n)$-type 2, there is $K$ independent of $x_1, \ldots x_N \in X$ such that

$$
\left( \int_0^1 \| \sum_{j=1}^N h_j(t) x_j \|^2 dt \right)^{1/2} \leq (1 + \delta) \left( \int_0^1 \| \sum_{j=1}^N \left( \sum_{n=m_j-1+1}^{m_j} a_n p_n(t) \right) x_j \|^2 dt \right)^{1/2}
$$

$$
\leq K(1 + \delta) \left( \sum_{j=1}^N \sum_{n=m_j-1+1}^{m_j} |a_n|^2 \| x_j \|_2^2 \right)^{1/2}
$$

$$
\leq K(1 + \delta)^2 \left( \sum_{j=1}^N \| x_j \|_2^2 \right)^{1/2}.
$$
This shows that $X$ has “Haar-type 2” which directly implies type 2 since the Rademacher functions form a block basis of the Haar functions,

$$r_k = \sum_{j=n_{k-1}+1}^{n_k} t_j h_j, \quad \sum_{j=n_{k-1}+1}^{n_k} |t_j|^2 = 1.$$ 

where $(n_k)_{k \in \mathbb{N}} \subseteq \mathbb{N}$ is a suitable increasing sequence. Hence for any sequence $y_1, \cdots, y_\ell \in X$

$$\left( \int_0^1 \left\| \sum_{k=1}^\ell r_k(t) y_k \right\|^2 dt \right)^{1/2} = \left( \int_0^1 \left\| \sum_{k=1}^\ell \left( \sum_{j=n_{k-1}+1}^{n_k} t_j h_j(t) \right) y_k \right\|^2 dt \right)^{1/2} \leq K(1 + \delta)^2 \left( \sum_{k=1}^\ell \left( \sum_{j=n_{k-1}+1}^{n_k} |t_j|^2 \right) \left\| y_k \right\| \right)^{1/2} = K(1 + \delta)^2 \left( \sum_{k=1}^\ell \left\| y_k \right\|^2 \right)^{1/2}.$$ 

We note that if conversely $X$ has type 2, one has as estimate of the Rademacher against the Haar mean, i.e. there is a constant $C$ such that for all $\ell \in \mathbb{N}$ and all $(y_k)_{k=1}^\ell \subseteq X$:

$$\int_0^1 \left( \sum_{k=1}^\ell r_k(t) y_k \right)^2 dt \leq C \int_0^1 \left( \sum_{k=1}^\ell h_k(t) y_k \right)^2 dt.$$ 

Indeed, this statement is equivalent to the existence of a constant $C_1$, so that for all $\ell \in \mathbb{N}$ and all $f \in L_2(0, 1; X)$

$$\left( \int_0^1 \left\| \sum_{k=1}^\ell r_k(t) < f, h_k > \right\|^2 dt \right)^{1/2} \leq C_1(\| f \|_2)^{1/2}.$$ 

Let $f \in L_2(0, 1; X)$ be of the form $f := \sum_{j=1}^n x_j \otimes f_j$, where $(f_j)_{j=1}^n \subseteq L_2(0, 1)$ is a finite sequence of normalized, mutually disjointly supported functions and $(x_j)_{j=1}^n \subseteq X$. Since the set of such functions is a dense subspace of $L_2(0, 1; X)$ it suffices to prove the inequality for those.
Let \((\gamma_j)\) be a sequence of standard independent \(N(0, 1)\) Gaussian variables on a probability space \((\Gamma, \nu)\) and let \(M\) be the type 2 constant of \(X\). Since for every \(x^* \in X^*\)
\[
\sum_{k=1}^\ell |x^*(<f, h_k>)|^2 = \sum_{k=1}^\ell |<x^* f, h_k>|^2 \\
\leq \int_0^1 |x^*(f(t))|^2 dt \\
= \sum_{j=1}^n |x^*(x_j)|^2,
\]
it follows from the unitary invariance of the \(\gamma_j\)'s (see e.g. [17]) that
\[
\int_\Gamma \left\| \sum_{k=1}^\ell \gamma_k(s) < f, h_k > \right\|^2 d\nu(s) \leq \int_\Gamma \left\| \sum_{j=1}^n \gamma_j(s) x_j \right\|^2 d\nu(s).
\]

Combining this with the fact that since \(X\) is of type 2 the Rademacher and the Gauss means are \(K\)-equivalent for a suitable \(K\), we obtain
\[
\left( \int_0^1 \left\| \sum_{k=1}^\ell r_k(t) < f, h_k > \right\|^2 dt \right)^{1/2} \leq K \left( \int_\Gamma \left\| \sum_{k=1}^\ell \gamma_k(s) < f, h_k > \right\|^2 d\nu(s) \right)^{1/2} \\
\leq K \left( \int_\Gamma \left\| \sum_{j=1}^n \gamma_j(s) x_j \right\|^2 d\nu(s) \right)^{1/2} \\
\leq KM \left( \sum_{j=1}^n \| x_j \|^2 \right)^{1/2} = \| f \|_2,
\]
which proves the claim.

The partial converse of Proposition 7 follows easily:

**Proof of Proposition 8:** Let \(N \in \mathbb{N}, x_1, \cdots x_N \in X\). By the unconditionality assumption on the \((p_n)\)-system in \(L_2(0, 1; X)\) and the type property of \(X\) there are constants \(c_1, c_2\) independent of \(N\) and \(x_1, \cdots x_N \in X\) such that
\[
\left( \int_0^1 \left\| \sum_{j=1}^N p_j(t) x_j \right\|^2 dt \right)^{1/2} \leq c_1 \left( \int_0^1 \int_0^1 \left\| \sum_{j=1}^N r_j(s) p_j(t) x_j \right\|^2 ds dt \right)^{1/2}
\]
\[ \leq c_2 \left( \int_0^1 \sum_{j=1}^N |p_j(t)|^2 \| x_j \|^2 \, dt \right)^{1/2} \]

\[ = c_2 \left( \sum_{j=1}^N \| x_j \|^2 \right)^{1/2}. \]

Hence \( X \) has \((p_n)\)-type 2.

We still have to show that in certain cases \( X \) is isomorphic to a Hilbert space provided that it only has \((p_n)\)-type 2. This will be another application of the interpolation inequalities of Theorem 1.

**Proof of Proposition 6:** For \( n \in \mathbb{N} \), let \( (t_j)_{j=1}^{n+1} \) denote the Gaussian quadrature weights. The \((n+1) \times (n+1)\) matrix \( A_n = (a_{ij}) \) defined by

\[ a_{jk} := \sqrt{\lambda_j} p_k^{(\alpha,\beta)}(t_j), j = 1, \ldots, n+1, k = 0, \ldots, n \]

is orthogonal since by Gaussian quadrature for \( k, \ell \in \{0, \ldots, n\} \)

\[ \delta_{k\ell} = \int_{-1}^{1} p_k^{(\alpha,\beta)}(t)p_\ell^{(\alpha,\beta)}(t)w_{\alpha\beta}(t)dt \]

\[ = \sum_{j=1}^{n+1} \lambda_j p_k^{(\alpha,\beta)}(t_j)p_\ell^{(\alpha,\beta)}(t_j) \]

\[ = \sum_{j=1}^{n+1} a_{jk}a_{j\ell}. \]

Since the measure space \((I, w_{\alpha\beta})\) is equivalent to \((0, 1)\), we know from Proposition 7 that \( X \) has type 2. We will now show that \( X \) also has cotype 2 and hence by Kwapien \[13\] is isomorphic to a Hilbert space. We use Theorem 1 (a) to discretize the notion of \((p_n^{(\alpha,\beta)})\)-type 2 and reverse the inequality using the orthogonality of the matrix \( A_n \) appearing in this way: By Theorem 1 and the \((p_n^{(\alpha,\beta)})\)-type 2 property there are \( c_1, c_2 \) such that for any \( n \in \mathbb{N} \) and \( x_0, \ldots, x_n \in X \)

\[ \left( \sum_{j=1}^{n+1} \| \sum_{k=0}^n a_{jk}x_k \|^2 \right)^{1/2} \leq c_1 \left( \int_{-1}^{1} \left( \sum_{k=0}^n p_k^{(\alpha,\beta)}(t)x_k \right)^2 \, w_{\alpha\beta}(t) \, dt \right)^{1/2} \]
\[ \left( \sum_{k=0}^{n} \| x_k \|^2 \right)^{1/2}. \quad (27) \]

Since \( A_n^{-1} = A_n^t \), we can invert (27) easily: starting with arbitrary \( y_1, \ldots, y_{n+1} \in X \) and applying (27) to \( x_k := \sum_{\ell=1}^{n+1} a_{\ell k} y_\ell \), \( k \in 0, \ldots, n \), we find
\[
\left( \sum_{j=1}^{n+1} \| y_j \|^2 \right)^{1/2} \leq c_2 \left( \sum_{k=0}^{n} \left\| \sum_{\ell=1}^{n+1} a_{\ell k} y_\ell \right\| ^2 \right)^{1/2}.
\]

If the \( A_n \) were symmetric, this and Theorem 1 (a) would yield the cotype 2 property. However, \( A_n \neq A_n^t \) in general. To prove the cotype 2 property, we replace \( y_j \) by \( r_j(s)y_j \) and apply the contraction principle to find
\[
\left( \sum_{j=1}^{n+1} \| y_j \|^2 \right)^{1/2} \leq c_3 \left( \sum_{k=0}^{n} \left\| \sum_{\ell=1}^{n+1} r_\ell(s)y_\ell \right\| ^2 \right)^{1/2}.
\]

Hence \( X \) will have cotype 2 provided that \( \sum_{k=0}^{n} \sup_{\ell \leq n+1} |a_{\ell k}|^2 \) is uniformly bounded in \( n \in \mathbb{N} \). This is correct since by (4) and (10) e.g. if \( \ell \leq n/2 \)
\[
\sqrt{\lambda_\ell} \sim \ell^{\alpha+1/2}/n^{\alpha+1}, \quad |p^{(\alpha,\beta)}_k(t_\ell)| \lesssim (n/\ell)^{\alpha+1/2}
\]
\[
|a_{\ell k}| = \sqrt{\lambda_\ell} |p^{(\alpha,\beta)}_k(t_\ell)| \lesssim n^{-1/2}
\]

The case of \( \ell > n/2 \) is similar. By the result of Kwapień, used earlier, \( X \) is isomorphic to a Hilbert space. \( \Box \)

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