Practical scheme of reduction to gauge invariant variables

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Abstract

For systems with first class constraints the reduction scheme to the gauge invariant variables is considered. The method is based on the analysis of restricted 1-forms in gauge invariant variables. This scheme is applied to the models of electrodynamics and Yang-Mills theory. For the finite dimensional model with $SU(2)$ gauge group of symmetry the possible mechanism of confinement is obtained.

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1 Introduction

Most of interesting physical models and theories are described by the gauge invariant Lagrangians, which are singular and in Hamiltonian formulation lead to the constrained systems [1]. For the constrained Hamiltonian systems there are in principle two ways[1] of quantization [1-2]:

1. “First quantize and then reduce”.
2. “First reduce and then quantize”.

The present paper deals mainly with the reduction procedure of the latter. For gauge theories this procedure is a restriction on the constraint surface $\mathcal{M}$ and then farther reduction to the physical phase space $\mathcal{M}/G$, which is the space of gauge orbits.

If the action of the gauge group (G) on the constraint surface ($\mathcal{M}$) is regular, then the manifold of orbits ($\mathcal{M}/G$) is well defined and it possesses the symplectic structure. Coordinates on $\mathcal{M}$ are gauge invariant, true physical degrees of freedom.

Quite often, in practical applications this theoretical scheme of reduction encounters technical problems related to the explicit construction of $\mathcal{M}$ with its symplectic structure: apart from the mathematical difficulties, the physical content of the true degrees of freedom may be quite unpredictable.

Widey used practical reduction scheme is a gauge fixing procedure by some conditions $\chi(p,q) = 0$ [1],[3]. For simple cases the explicit form of the true physical variables is obvious and this reduction scheme works perfectly. But in general, as it was shown in [4] (namely for the Yang-Mills theory), the space of gauge orbits ($\mathcal{M}/G$) cannot be obtained by “simple” gauge fixing. Problem of gauge fixing, of course, reflects the above mentioned possible non-trivial structure of a physical phase space [1],[5].

Another reduction scheme can be based on the gauge invariant variables (GIVs) [1-2]. As a rule, the GIV is constructed from the structure of gauge transformations. If one can find the complete set of GIVs, then it allows to describe the physical phase space $\mathcal{M}/G$ with its symplectic structure. This paper deals with such gauge invariant approach by using of restricted 1-forms. We also consider situations when only a part of GIVs is known. Analysis of a structure of restricted 1-form helps to find the remaining part of GIVs.

Note that the reduction scheme with 1-forms for arbitrary constrained systems was proposed in [6]. In these papers elimination of extra variables was based on the Darboux theorem. Sometimes Darboux theorem is not effective in applications and choice of GIVs is just a practical way for the realization of this reduction program for gauge theories.

The paper is organized as follows: In Section 2 the reduction scheme to the GIVs is introduced and, for the illustration, simple examples are considered. One more example of $(2 + 1)$-dimensional massive photodynamics is given in Appendix. In Section 3 this scheme is applied to the finite dimensional system with $SU(2)$ gauge group of symmetry. This system can be considered as a toy model of the Yang-Mills theory with fermions. It is shown that there is an essential difference between this $SU(2)$ and the corresponding $U(1)$ model. Structure of GIVs in $SU(2)$ case can be interpreted as the confinement phenomenon. In Section 4 we study infinite dimensional model, where gauge group is any semi-simple one. The GIVs are constructed and full reduction is accomplished. It is shown that the model is equivalent to the Yang-Mills theory with some boundary conditions. Final section is for remarks and conclusions.

2 Reduction scheme in gauge invariant variables

Starting from the gauge invariant Lagrangian $L = L(q_k, \dot{q}_k)$ ($k = 1, ..., N$) and using the Dirac’s procedure [1a], or the first order formalism [6] we arrive to the extended phase space $\Gamma$ with coordinates $(p_k, q_k)$ and the

\footnote{In this paper we do not consider the path integral approach}
action

\[ S = \int p_k dq_k - [H(p, q) + \lambda_a \phi_a(p, q)] dt, \quad (2.1) \]

where \(\phi_a(p, q)\) are constraints, \(H(p, q)\) is a canonical Hamiltonian and \(\lambda_a\) are Lagrange multipliers. The constraint surface — \(\mathcal{M}\) is defined by

\[ \phi_a(p, q) = 0 \quad (2.2) \]

and the following relations are fulfilled:

\[ \{H, \phi_a\}_\Gamma = d^b_a \phi_b, \quad \{\phi_a, \phi_b\}_\Gamma = f^c_{ab} \phi_c. \quad (2.3) \]

Index \(\Gamma\) on the left hand side indicates that Poisson brackets are calculated on the extended phase space.

Function \(\xi = \xi(p, q)\) is called GIV [1] if \(\xi|_\mathcal{M} \neq 0\) and

\[ \{\xi, \phi_a\}_\Gamma = d^b_a \phi_b, \quad (2.4) \]

where \(|_\mathcal{M}\) denotes restriction on \(\mathcal{M}\) and functions \(d^b_a\) (as well as \(d^a_b\) and \(f^c_{ab}\) in (2.3)) are assumed to be regular in the neighbourhood of \(\mathcal{M}\).

Each GIV — \(\xi\) has the class \(\{\xi\}\) of equivalent GIVs on \(\Gamma\) [1]. A gauge invariant function \(\tilde{\xi}\) is equivalent to \(\xi\) if \(\tilde{\xi}|_\mathcal{M} = \xi|_\mathcal{M}\). On the other hand, the function \(\xi|_\mathcal{M}\) is a constant along the gauge orbit (on \(\{\mathcal{M}\}\)) and it defines the function \(\hat{\xi}\) on the physical space \(\mathcal{M} = \mathcal{M}/G\). Thus \(\{\hat{\xi}\}, \hat{\xi}|_\mathcal{M}\) and \(\hat{\xi}\) denote the GIV — \(\hat{\xi}\) in different context. If there is no ambiguity, we will use the notation \(\hat{\xi}\) for all of them.

Maximal number of GIVs (2.4), which are functionally independent on the constraint surface \(\mathcal{M}\), is \(2(N-M)\) [1b]. Suppose that \(\{\xi^\alpha : \alpha = 1, ..., 2(N-M)\}\) is such complete set of GIVs. Then one can prove [6] that

1. \(p_k dq_k|_\mathcal{M} = \theta_1 + \theta_2\), with
   a) \(d\theta_1 = 0\),
   b) \(\theta_2 = \theta_\alpha(\xi)d\xi^\alpha\),
   c) \(det\omega_{\alpha\beta} \neq 0\), where \(\omega_{\alpha\beta}(\xi) = \partial_\alpha \theta_\beta - \partial_\beta \theta_\alpha\);
2. \(H(p, q)|_\mathcal{M} = h(\xi)\). \quad (2.5)

Main statement of (2.5) is that after restriction on the constraint surface \(\mathcal{M}\), dependence on extra (non-physical) variables is present only in the term \(\theta_1\), which is a “total derivative”.

Since \(d\theta_1 = 0\), it gives no contribution to the variation of a restricted action. We can neglect it and for the reduced system get

\[ S|_\mathcal{M} \equiv \hat{S} = \int \theta_\alpha(\xi)d\xi^\alpha - h(\xi) dt. \quad (2.6) \]

Hence dynamics for GIVs is described by the Hamilton equations

\[ \dot{\xi}^\alpha = \omega^{\alpha\beta}(\xi)\partial_\beta h(\xi), \quad (2.7) \]

where \(\omega^{\alpha\beta}(\xi)\) is the inverse to the symplectic matrix \(\omega_{\alpha\beta} = \partial_\alpha \theta_\beta - \partial_\beta \theta_\alpha\) and it defines the Poisson brackets of the reduced system

\[ \{\xi^\alpha, \xi^\beta\}_\mathcal{M} = \omega^{\alpha\beta}(\xi). \quad (2.8) \]

So the reduced system (2.6)-(2.8) is an ordinary (non-constrained) Hamiltonian system which can be quantized.

It should be noticed that, in general, any \(2(N-M)\) number of GIVs are only local coordinates on the physical phase space \(\hat{\mathcal{M}}\) and respectively (2.5)-(2.8) have the local meaning. Global description can be achieved by the set of GIVs which defines the global structure of physical phase space \(\mathcal{M}\). Number of such GIVs is
greater than $2(N - M)$, but on the constraint surface there are relations among them. Just these relations define the geometry of $\mathcal{M}$. For the illustration let us consider the following example of (2.1)-(2.3) [7a]:

$$S = \int \ddot{p} \cdot d\dot{q} - [\lambda_1 \phi_1 + \lambda_2 \phi_2]dt.$$  \hspace{1cm} (2.9)

Here $\ddot{p}$ and $\dot{q}$ are vectors of $\mathbb{R}^3$, canonical Hamiltonian is zero,

$$\phi_1 = \ddot{p} \cdot \dot{q}, \quad \phi_2 = \ddot{p}^2 - (\ddot{p} \cdot \dot{q})^2 - r^2$$

and $r$ is a parameter. These constraints are Abelian ($\{\phi_1, \phi_2\} = 0$) and the second constraint $\phi_2$ can be written in the form

$$\phi_2 = \ddot{J}^2 - r^2$$

where $\ddot{J} = \dot{q} \times \ddot{p}$ is an angular momentum.

It is clear that the physical phase space is two dimensional and the components of angular momentum $\ddot{J}$ are GIVs (they commute with constraints, since constraints are $O(3)$ scalars). On the constraint surface these three components are related by $\ddot{J} \cdot \dddot{J} = r^2$ and define the physical phase space $\hat{\mathcal{M}}$ as the two dimensional sphere. So any two coordinates (as well as the 1-forms $\theta_1$ and $\theta_2$) are only local ones (on the phase space geometry of constrained systems see [5a]).

Described reduction scheme ((2.5)-(2.8)) can be used when all $2(N - M)$ GIVs are known. For the practical application of the scheme one can introduce any additional (to GIVs) variables $\eta^1, \ldots, \eta^M$ to have the coordinate system

$$\left(\xi^1, \ldots, \xi^{2(N-M)}, \eta^1, \ldots, \eta^M\right)$$

on $\mathcal{M}$. Then calculating restricted 1-form $p_\mu dq_\mu|_{\mathcal{M}}$ in these coordinates and taking its differential we can find the symplectic form $\omega = \omega_{\alpha\beta}(\xi) d\xi^\alpha \wedge d\xi^\beta$. Note that in practical calculations it is possible to single out the 1-form $\theta_2 = \theta_\alpha(\xi) d\xi^\alpha$ and arrive to (2.6).

Application of this procedure to the model (2.9) gives $\theta_2 = zd\varphi$, where $z$ and $\varphi$ are the cylindrical coordinates on the sphere:

$$J_1 = \sqrt{r^2 - z^2 \cos \varphi}, \quad J_2 = \sqrt{r^2 - z^2 \sin \varphi}, \quad J_3 = z.$$  

It is clear that $zd\varphi$ is not the global 1-form, but its differential has a continuation to the well-defined symplectic form on the sphere [8]

$$\omega = -\frac{J_1 (dJ_3 \wedge dJ_3) + J_2 (dJ_3 \wedge dJ_1) + J_3 (dJ_1 \wedge dJ_2)}{r^2}.$$  

After this the system can be quantized by the geometric quantization [9] (see also [7a],[10b,c]). Consistent quantum theory exists only for the discrete values of the parameter $r$.

Generalization of the scheme to the infinite dimensional case is straightforward (in Appendix we present the example of massive photodynamics in $(2 + 1)$ dimensions). If we use the Dirac’s observables [11]:

$$\psi_{1n} = e^{i\Delta^{-1}(\vec{p} \cdot \vec{A})} \psi$$  \hspace{1cm} (2.10)

in the ordinary QED, we will easily obtain photons in the Coulomb gauge and the “four-fermion interaction” for the “dressed fermions” (compare to the example in Section 3 and see [6], [12]).

Note that the commutation relations of the complete set of GIVs (2.8) can be derived by calculations of Poisson brackets on the extended phase space too [1]. This more standard procedure is based on the fact that the Poisson brackets of any two GIVs is again GIV. Indicated procedure and the scheme described in this paper ((2.5)-(2.8)) are almost equivalent. Only, sometimes, calculations of differential forms is more practical (especially, when the canonical quantization is not applicable [9]).

In general, from the structure of gauge transformations one can easily find only some part of GIVs and construction of the complete set (2.5) is troublesome. In many cases, our approach with differential forms, can be effectively used for the solution of this problem too.

Let us consider situation when we know the set of GIVs $\{\xi^\mu : \mu = 1, \ldots, K\}$, where $N - M \leq k \leq 2(N - M)$. We can introduce any additional variables $\eta^1, \ldots, \eta^{2(N-M)-K}$ to have the coordinate system on $\mathcal{M}$ and calculate
the restricted 1-form $p_k dq_k|_M$. Suppose that we can single out “total derivatives” and the differentials $d\xi^\mu$ in the form
\[ p_k dq_k|_M = dF(\xi, \eta) + \theta_\mu(\xi, \eta) d\xi^\mu. \] (2.11)

Then using (2.5) we can easily conclude that $\theta_\mu(\xi, \eta)$ will be GIVs. Note that passing to the GIVs — $\xi^\mu$ helps to get the form (2.11). For the illustration of this method we apply it to the relativistic particle [1c], with the 1-form $\theta = \vec{p}d\vec{q} - p_0 dq_0$ and the constraint surface $M$: $p^2 - m^2 = 0, (p_0 > 0)$. The momenta $\vec{p}$ are gauge invariant and after restriction on $M$ we have
\[ \theta|_M = \vec{p}d\vec{q} - \sqrt{\vec{p}^2 + m^2} dq_0. \]

One can easily rewrite it in the following form
\[ \theta|_M = d(\vec{p} \cdot \vec{q} - \sqrt{\vec{p}^2 + m^2} q_0) - (\vec{q} - \frac{\vec{p}}{\sqrt{\vec{p}^2 + m^2}} q_0) d\vec{p}. \]

Evidently the coefficients of the differentials $-d\vec{p}$ are GIVs, canonically conjugate to $\vec{p}$:
\[ \vec{Q} = \vec{q} - \frac{\vec{p}}{\sqrt{\vec{p}^2 + m^2}} q_0. \]

Gauge invariance of $\vec{Q}$ also can be established from the relation
\[ \vec{L} = \sqrt{\vec{p}^2 + m^2} \vec{Q}, \] (2.12)
where $\vec{L}$ are the generators of Lorentz transformations. Since all generators of the Poincare group $(P_\mu, M_{\mu\nu})$ are GIVs, the same property have the coordinates $\vec{Q}$. On the constraint surface $p^2 - m^2 = 0, (p_0 > 0)$ all these are the functions only of the reduced variables $\vec{p}$ and $\vec{Q}$.

Reduced system can be easily quantized in momentum representation: $\hat{\vec{p}} = \vec{p}$ and $\hat{\vec{Q}} = i\hbar \vec{\nabla}$. Operator ordering problem arises only for the generators (2.12). Then the standard Lorentz covariant measure of a scalar product
\[ < \Psi_2 | \Psi_1 > = \int d^3\vec{p} \frac{\bar{\Psi}_2(\vec{p})\Psi_1(\vec{p})}{\sqrt{\vec{p}^2 + m^2}} \]
corresponds to the ordering $\hat{\vec{L}} = i\hbar \sqrt{\vec{p}^2 + m^2} \vec{\nabla}$.

## 3 The finite dimensional models with $U(1)$ and $SU(2)$ gauge symmetries

In this section we consider the finite dimensional model with $SU(2)$ gauge group of symmetry. From the beginning it is difficult to see all GIVs and we use the method described at the end of Section 2. Obtained structure of GIVs is quite unexpected. For comparison we present the corresponding $U(1)$ model too. These $U(1)$ and $SU(2)$ models can be considered as the toy models of the electrodynamics and the Yang-Mills theory (with matter), respectively. In classical description all “fields” are assumed to be $c$-numbers.

### A. The model with $U(1)$ symmetry

Let us consider the action
\[ S = \int dt \left[ \frac{i}{2} (\bar{\psi} \dot{\psi} - \dot{\bar{\psi}} \psi) - m \bar{\psi} \psi + A_0(\bar{\psi} \psi - kE) + E \dot{A} - \frac{1}{2} E^2 \right], \] (3.1)
where all “fields” \((\bar{\psi}, \psi, A_0, A, E)\) are functions only of the time \(t\); \(m\) and \(k\) \((k \neq 0)\) are parameters. Similarity to the electrodynamics is apparent from the notations. At the same time (3.1) has the form (2.1), where \(A_0 \equiv \lambda(t)\) is a Lagrange multiplier and \(\phi \equiv \bar{\psi}\psi - kE\) is a constraint (we use the time derivatives instead of differential form when it is convenient).

Non-zero Poisson brackets are
\[
\{\psi, \bar{\psi}\} = i, \quad \{E, A\} = 1
\]
and we get the gauge transformations
\[
\begin{align*}
\psi(t) &\to e^{+i\alpha(t)}\psi(t), & \bar{\psi}(t) &\to e^{-i\alpha(t)}\bar{\psi}(t), \\
A(t) &\to A(t) + k\alpha(t), & E(t) &\to E(t).
\end{align*}
\]
Then
\[
A_0(t) \to A_0(t) + \dot{\alpha}(t)
\]
leaves the action (3.1) invariant.

Reduced system is two dimensional and two GIVs can be chosen as
\[
\Psi_{\text{inv}} = e^{-i\bar{\psi}A}\psi, \quad \bar{\Psi}_{\text{inv}} = e^{i\bar{\psi}A}\bar{\psi}
\]
(compare to (2.10)). Here, the reduction procedure (2.5) is trivial and we get
\[
\tilde{S} = \int dt \left[ \frac{i}{2}(\bar{\Psi}_{\text{inv}}\dot{\Psi}_{\text{inv}} - \dot{\bar{\Psi}}_{\text{inv}}\Psi_{\text{inv}}) - m\bar{\Psi}_{\text{inv}}\Psi_{\text{inv}} - \frac{1}{k^2}(\bar{\Psi}_{\text{inv}}\Psi_{\text{inv}})^2 \right].
\]
So the “gauge field” \(A\) has vanished and physical excitations are only the “dressed fields” \(\Psi_{\text{inv}}\) (with “four-fermion interaction”).

This model has a simple generalization in case of multi-component gauge field \(\vec{A}\) with gauge transformations
\[
\vec{A} \to \vec{A} + \vec{k}\alpha,
\]
where \(\vec{k}\) are parameters \((\vec{k}^2 \neq 0)\). The GIV \(\Psi_{\text{inv}}\) is constructed similarly to (3.3) (or (2.10)). Then, after reduction, “longitudinal” (to the \(\vec{k}\)) component of the gauge field \(\vec{A}\) vanishes and physical variables are only “transverse” ones and the constructed “dressed field” \(\Psi_{\text{inv}}\). So, for these Abelian models, the structure of GIVs is very similar to the physical observables of the electrodynamics [6],[12].

**B. The model with \(SU(2)\) symmetry**

For the model with \(SU(2)\) gauge group of symmetry we consider the action
\[
S = \int dt \left[ \frac{i}{2}(\bar{\psi}_\alpha \dot{\psi}_\alpha - \dot{\bar{\psi}}_\alpha \psi_\alpha) - m\bar{\psi}_\alpha\psi_\alpha + \vec{A}_0(\vec{j} + \vec{J}) + \vec{E}\vec{A} - \frac{1}{2}\vec{E}^2 \right].
\]
Here \(\psi_\alpha\) are 2-component spinors \((\alpha = 1, 2)\), \(m\) is a parameter, \(\vec{A}\) and \(\vec{E}\) are three dimensional vectors, \(\vec{A}_0\) — lagrange multipliers and the angular momenta \(\vec{j}\) and \(\vec{J}\) are given by
\[
\vec{j} = \vec{\sigma} \frac{\bar{\psi}}{2} \psi, \quad \vec{J} = \vec{A} \times \vec{E},
\]
where \(\vec{\sigma}\) are the standard Pauli matrixes.

Connection with the Yang-Mills theory is obvious.
Non-zero Poisson brackets are
\[ \{ \psi_{\alpha}, \bar{\psi}_{\beta} \} = i\delta_{\alpha\beta}, \quad \{ E_m, A_n \} = \delta_{mn}, \quad (m, n) = 1, 2, 3 \] (3.7)
and the constraints \( \phi = \vec{j} - \vec{J} \) generate the gauge transformations:
\[ \psi \rightarrow \omega \psi, \quad \bar{\psi} \rightarrow \bar{\psi} \omega^{-1}, \quad A \rightarrow \omega A \omega^{-1}, \quad E \rightarrow \omega E \omega^{-1}, \]
where \( \omega(t) \in SU(2) \) and
\[ A \equiv \frac{1}{2} \bar{\Lambda} \bar{\sigma}, \quad \bar{E} \equiv \frac{1}{2} \bar{E} \bar{\sigma}. \] (3.8)
Then for \( A_0 \equiv \frac{1}{2} \bar{A}_0 \bar{\sigma} \) we get
\[ A_0 \rightarrow \omega A_0 \omega^{-1} - i\dot{\omega} \omega^{-1}. \]

Any scalar product of the vectors \( \vec{A}, \vec{E}, \vec{j}, \vec{J} \) will be \( \text{GIV} \). But on the constraint surface \( (\vec{j} + \vec{J} = 0) \) only three of them are functionally independent.

If we choose these independent \( \text{GIVs} \) as:
\[ l_0 = \frac{1}{4}(\vec{A}^2 + \vec{E}^2), \quad l_1 = \frac{1}{2}(\vec{E} \vec{A}), \quad l_2 = \frac{1}{4}(\vec{A}^2 - \vec{E}^2), \] (3.9)
then from (3.7) we get the \( SL(2, \mathbb{R}) \) algebra:
\[ \{ l_\mu, l_\nu \} = \epsilon_{\mu\nu\rho} g^{\sigma\rho} l_\sigma, \quad \text{where} \quad g^{\mu\nu} = \text{diag}(+, -, -, -). \] (3.10)

Since there are three constraints, the physical phase space is 4-dimensional. To construct the fourth \( \text{GIV} \) and find the full symplectic structure we use the method of Section 2 (see (2.11)).

For the parameterization of the constraint surface we introduce new variables \( (j, \Phi; h, \phi) \):
\[ j = \frac{1}{2}(h_1 + h_2), \quad h = \frac{1}{2}(h_1 - h_2), \]
\[ \Phi = \varphi_1 + \varphi_2, \quad \phi = \varphi_1 - \varphi_2, \] (3.11)
where
\[ \psi_\alpha = \sqrt{h_\alpha} e^{-i\varphi_\alpha}, \quad \bar{\psi}_\alpha = \sqrt{h_\alpha} e^{i\varphi_\alpha} \quad (\alpha = 1, 2). \]

Then for the 1-form we get
\[ \frac{i}{2}(\bar{\psi}_\alpha d\psi_\alpha - \psi_\alpha d\bar{\psi}_\alpha) = j d\Phi + h d\phi. \] (3.12)

The vector \( \vec{j} \) (3.6) in these new coordinates will take the form
\[ \vec{j} = \left( \frac{\sqrt{j^2 - h^2} \cos \phi}{h} \right), \quad \vec{j}^2 = j^2, \]
and \( \vec{j}^2 = j^2 \). Note that on the constraint surface we have (see (3.9)): \( l_\mu l_\mu = j^2/4 \) and for the fixed \( j \) the commutation relations (3.10) define well known symplectic structure on this hyperboloid (see e.g. [13]).

If we introduce the ortho-normal basis \( (\vec{e}_i \cdot \vec{e}_k = \delta_{ik}, \quad \vec{e}_i \times \vec{e}_j = \epsilon_{ijk} \vec{e}_k) \):
\[ \vec{e}_1 = \left( \begin{array}{c} -\sin \varphi \\ \cos \varphi \\ 0 \end{array} \right), \quad \vec{e}_2 = -\frac{h}{\vec{j}} \left( \begin{array}{c} \cos \varphi \\ \sin \varphi \\ -\sqrt{j^2 - h^2} \end{array} \right), \quad \vec{e}_3 = \frac{\vec{j}}{\vec{j}}, \]
then \( \vec{A} \) and \( \vec{E} \) can be parameterized as follows:
\[ \vec{A} = \vec{e}_1 q_1 + \vec{e}_2 q_2, \quad \vec{E} = \vec{e}_1 p_1 + \vec{e}_2 p_2, \]
where
\[ p_1 q_2 - p_2 q_1 = j. \] (3.13)
Calculating the restricted 1-form \( \tilde{E}d\tilde{A} \) in these new coordinates and using (3.13) we obtain
\[ \tilde{E}d\tilde{A} = p_1 dq_1 + p_2 dq_2 - hd\varphi. \] (3.14)
Comparing (3.12) and (3.14), we see that there is a cancellation of the 1-form \( hd\varphi \). This means that the corresponding degrees of freedom vanish.

Now, it is convenient to introduce the polar coordinates for two-vectors \( (q_1, q_2) \) and \( (p_1, p_2) \):
\[ q_1 = r\cos\beta, \quad p_1 = \rho\cos\gamma, \]
\[ q_2 = r\sin\beta, \quad p_2 = \rho\sin\gamma. \]
Then three of them \( (r, \rho \) and \( \beta - \gamma \)) are connected with the GIVs (3.9):
\[ r^2 = 2(l_0 + l_2) \equiv l_+, \quad \rho^2 = 2(l_0 - l_2) \equiv l_-, \quad r\rho\cos(\beta - \gamma) = 2l_1. \]
Using these relations we finally get the reduced 1-form
\[ \theta \big|_M = j\vartheta + 1 \frac{dl_+}{l_+} \] where \( \vartheta = \Phi - \beta \). (3.15)
So the coordinate \( \vartheta = \Phi - \beta \) is the fourth GIV. Respectively the reduced Hamiltonian takes the form
\[ H \big|_M = 2mj + \frac{j^2 + 4l_1^2}{l_+} \] (3.16)
and this is the complete reduction.

Note that the second part of the reduced 1-form \( l_1 d(\ln l_+) \) defines the above mentioned symplectic structure on the hyperboloid \( l^\nu l_\mu = \frac{1}{4}j^2 \) [13].

We see that the physical picture of this reduced system essentially differs from the corresponding Abelian case. Here, after reduction some part of degrees of freedom of the “gauge field” \( A \), as well as some part of “matter field” \( \psi \) degrees of freedom have vanished. Below we shall see that in quantum theory vanishing of “matter field” degrees of freedom can be interpreted as the confinement phenomenon.

Geometric quantization [9] is a natural way for the construction of quantum theory of the reduced system (3.15-3.16), but in principle one can use canonical quantization too. For this aim it is convenient to introduce (global) “creation” and “annihilation” variables
\[ a^+ = \sqrt{j}e^{i\vartheta}, \quad a = \sqrt{j}e^{-i\vartheta} \] (3.17)
and in quantum theory we get the discrete values of \( j = a^+ a \). Then, quantization of the system with the canonical 1-form \( l_1 d(\ln l_+) \) and the Hamiltonian (3.16) (for obtained discrete values of \( j \)), gives the irreducible representations of \( SL(2, \mathbb{R}) \) group (see e.g. [13a]).

Next, from (3.11) we have the relation \( \hat{N} \equiv \psi_\alpha \psi_\alpha \equiv 2j \). It is natural to interpret the corresponding operator \( (\hat{N} \equiv 2j) \) as the \( \psi \) particle number operator. In quantum theory we have
\[ [\hat{N}, a^+] = 2a^+ \]
where \( a^+ \) is a physical creation operator (3.17). So in physical excitations (created by the operator \( a^+ \)) there are states only with even number of “fermions”. This fact also can be seen from the structure of the variable \( a^+ \) (see (3.17) and (3.11)). It has the phase factor \( e^{i(\varphi_1 + \varphi_2)} \). So in quantum case it will create (see [14]) the pairs of “dressed” \( \psi \)-particles.

Note that for the similar finite dimensional constrained systems such “confinement”-like phenomenon has been derived by the “first quantize and then reduce” method (see [15]). In that approach reduction of the extended “Hilbert” space by the conditions \( \hat{\phi}_a | \Psi_{phys} \rangle \geq 0 \) forbids the states with certain quantum numbers.
4 Field theory models with non-Abelian gauge group of symmetries

For the finite dimensional models of the previous section the gauge group $G$ acts on the configuration space of “gauge field" ($A$) and on the phase space of “matter field” ($\psi$). This is the standard situation for Yang-Mills theories.

Using notations (3.8) we have

$$\vec{E} d \vec{A} = \langle E, dA \rangle,$$

(4.1)

where $\langle \ , \ \rangle$ is a scalar product in corresponding Lie algebra $A$. Thus, the Lie algebra $A$ can be interpreted as the configuration space of a “gauge field” $\vec{A}$ and trivial cotangent bundle as the phase space.

If one takes a manifold of semi-simple Lie group ($G$) as the configuration space, then there are the natural actions (left and right) of $G$ on this manifold and one can construct similarly the gauge theory where phase space is the cotangent bundle $T^*G = \{(g, R) | g \in G, R \in A\}$. On $T^*G$ the symplectic form is given by

$$\omega = d\theta, \text{ with } \theta = \langle R, g^{-1}dg \rangle.$$

(4.2)

Generators of the left and right transformations ($g \rightarrow \omega g, \ g \rightarrow g\omega$) are respectively left and right currents ($L \equiv gRg^{-1}, \ R \in A$). Choosing gauge transformations as the right action, we get that constraints are $\phi \equiv R = 0$. So the “gauge field" part in the action takes the form

$$\int < R, g^{-1}dg > - (\Lambda(R) + H(R, g)){dt},$$

(4.3)

where $\Lambda \in A$ is a Lagrange multiplier, and $H$ some gauge invariant Hamiltonian.

Field theory generalization of (3.5) is the standard Yang-Mills theory. In this section we consider corresponding generalization of (4.3) with the action

$$S = \int dt \left[ \int d^{D-1}\vec{x} \left( \sum_{k=1}^{D-1} < R_k, g_k^{-1}\dot{g}_k > + e < A_0, \phi > \right) - H \right]$$

(4.4)

where $g_k(\vec{x}, t) \in G$ and $R_k(\vec{x}, t), A_0(\vec{x}, t) \in A$; $H$ is gauge invariant Hamiltonian, $A_0$ — Lagrange multipliers, $\phi(\vec{x}, t) \equiv e \sum_{k=1}^{D-1} R_k(\vec{x}, t)$ — constraints, $e$ — coupling constant (see below).

The “1-form” $\sum_{k=1}^{D-1} < R_k, g_k^{-1}dg_k >$ defines the equal time Poisson brackets (see e.g. [16]):

$$\{R_{k,a}(\vec{x}), R_{l,b}(\vec{y})\} = \delta_{kl}\delta(\vec{x} - \vec{y})f_{ab}^{\prime}R_{k,c}(\vec{x})$$

$$\{g_k(\vec{x}), R_{l,a}(\vec{y})\} = \delta_{kl}\delta(\vec{x} - \vec{y})(g_kT_a(\vec{x}))$$

$$\{g_k(\vec{x}), g_l(\vec{y})\} = 0$$

(4.5)

where the set $\{T_a | T_a \in A\}$ forms any basis of Lie algebra, $R_a \equiv < T_a, R >$ and the last two relations are matrix equalities [16]. So for the constraints $\phi_a \equiv < T_a, \phi >$ we have

$$\{\phi_a(\vec{x}), \phi_b(\vec{y})\} = \delta(\vec{x} - \vec{y})f_{ab}^{\prime}\phi_c(\vec{x}).$$

(4.6)

Corresponding gauge transformations are

$$g_k \rightarrow g_k\omega, \quad R_k \rightarrow \omega^{-1}R_k\omega$$

(4.7)

and one can easily construct GIVs such as

$$g_{kl} = g_kg_l^{-1} \quad \text{and} \quad L_k = g_kR_kg_k^{-1}.$$

(4.8)

The Hamiltonian $H$ in (4.4) is any functional of such GIVs.
Since (4.8) gives sufficient number of GIVs we can use the scheme described in Section 2. The first non-trivial case is the 3-dimensional space-time. If we introduce $g = g_1 g_2^{-1}$ as the $\xi^\mu$ variables and $R_1, R_2$ and $g_2$ as the $\eta$ variables of the scheme (see (2.11)), then for the “1-form” $\theta = < R_1, g_1^{-1} dg_1 > + < R_2, g_2^{-1} dg_2 >$ (integration over $\mathbb{R}^2$ is assumed) we immediately get

$$< R_1 + R_2, g_2^{-1} dg_2 > + < g_2 R_1 g_2^{-1}, g^{-1} dg >$$

and after reduction we have

$$\theta|_{\mathcal{M}} = < r, g^{-1} dg >, \quad (4.9)$$

where $r = g_2 R_1 g_2^{-1}$ is also GIV.

So the structure of the 1-form is the same, only the number of variables was reduced. One can check that this is true for other dimensions too.

It is clear that the phase spaces of the systems with 1-forms (4.1) and (4.2) are essentially different and they cannot be transformed to each other. But in field theory when one has the infinite number of such spaces there is a non-local transformation (see [17c]):

$$A_k = e^{-g_k^{-1}} \partial_k g_k, \quad E_k = -g_k^{-1} \partial_k^{-1} (L_k) g_k, \quad (4.10)$$

which transforms the system (4.4) into the Yang-Mills theory with the same gauge group $G$. Indeed, from (4.4) and (4.10) one can check that

$$\phi = \sum_{k=1}^{D-1} \partial_k E_k + e [A_k, E_k]$$

(Gauss law)

and

$$< E_k, A_k > = < R_k, g_k^{-1} g_k > + \text{ (total derivatives)} \quad (4.11)$$

To get the corresponding Hamiltonian of the Yang-Mills theory [17]:

$$H = \frac{1}{2} \int d^{D-1}x \left( \sum_{k=1}^{D-1} < E_k, E_k > + \frac{1}{2} \sum_{k,l=1}^{D-1} < F_{kl}, F_{kl} > \right)$$

with $F_{kl} = \partial_k A_l - \partial_l A_k + e [A_k, A_l]$, one has to choose in (4.4)

$$H = \frac{1}{2} \int d^{D-1}x \left[ e^2 < \partial_k^{-1} L_k, \partial_k^{-1} L_k > + \frac{1}{e^2} < \partial_k (g_{kl} \partial_l g_{ik}), \partial_k (g_{kl} \partial_l g_{ik}) > \right]. \quad (4.12)$$

So one can assume that the system (4.4) with the Hamiltonian (4.12) is equivalent to the ordinary Yang-Mills theory with some boundary conditions (which allow to invert (4.10) and neglect the total derivatives in (4.11).

Boundary behaviour is a subtle problem even for simple models of field theory (see e.g. the Appendix). For the Yang-Mills theory it is too complicated and we do not consider it here.

Unfortunately the complicated form of the Hamiltonian (4.12) does not simplify after the reduction procedure. For example, for the considered 3-dimensional case the reduced Hamiltonian takes the form

$$H = \frac{1}{2} \int d^2x \left[ e^2 < \partial_1^{-1} r, \partial_1^{-1} r > + e^2 < \partial_2^{-1} l, \partial_2^{-1} l > + \frac{1}{e^2} < \partial_1 (g \partial_2 g^{-1}), \partial_1 (g \partial_2 g^{-1}) > \right]. \quad (4.13)$$

where $l = g r g^{-1}$.

Gribov’s ambiguity problem has stimulated many papers on the gauge invariant description of the Yang-Mills theory and the reduced system (4.9),(4.13) is the one possible version (literature and new results on this problem see in [1e]). The main problem of such approaches is a complicated form of the Poincare generators in GIVs [17]. For example, the Hamiltonian (4.13) is non-local in fields and non-analytical in coupling constant. So the standard perturbative quantization is not applicable.
Note that such Hamiltonian with corresponding symplectic form was obtained in [17c] by Dirac’s brackets formalism.

5 Conclusions

Of course, there was an essential progress in the study of constraint systems since the paper [18], but from the point of view of practical applications still there is no universal approach. The method presented in this paper is a one possible practical procedure towards the quantization of gauge theories.

As it was mentioned in the introduction, for the gauge invariant systems there is an alternative way of quantization when one “first quantizes and then reduces”. In general, there are two problems in such approach:

a) construction of physical states $|\Psi_{phys}\rangle$ as the solutions of the equations $\hat{\phi}_a|\Psi_{phys}\rangle = 0$, where $\phi$ are constraint operators.

b) problem of scalar product for the physical states.

Sometimes the first problem is only a technical one (for the Yang-Mills theory see [18]), but in general both this problems are related and need further investigation [20].

In this paper we have not mentioned other important methods such as the path integral approach [1b], [3] and BRST quantization [21] (for a review see [10a]).

Quantization procedure is not unique even for the ordinary, non-constrained systems [8],[22]. It depends on the choice of canonical variables (if they globally exist), operator ordering, etc. Therefore it is not surprising that different quantization procedures of constraint systems generally lead to the non-equivalent quantum systems [3],[23].

As it was mentioned in Section 2, for a reduced classical system generally there are no global canonical co-ordinates and usual canonical quantization is not applicable. This, together with technical problems of classical reduction, was the main obstacle in general formulation of the approach “first reduce and then quantize”.

Geometric quantization [9] and other “new” quantization schemes [7],[10a],[24] allow to quantize Hamiltonian systems without global canonical structure too. At the same time essential progress was made in the construction of classical reduction schemes [6]. Therefore for the wide class of constrained systems the quantization method “first reduce and then quantize” seems to us to be technically preferable. Here it should be mentioned about the possible combination of the two quantization schemes: if a reduced classical system is complicated, then on the cotangent bundle of a reduced phase space one can construct new extended system with simple constraints and next use the first way of quantization [7],[10a,b]. Of course, the question, which is the “correct” quantum description of a given classical system, remains open.

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Appendix

The $2 + 1$ dimensional massive photodynamics is described by the Lagrangian (see e.g. [25]):

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{m}{4} \epsilon^{\mu\nu\sigma} F_{\mu\nu} A_\sigma. \quad (A.1)$$

We choose $g_{\mu\nu} = \text{diag}(+,-,-)$, $\epsilon^{012} = 1$ and in the first order formalism [6] obtain

$$S = \int dt \int_{R^2} d^2x \left[ (E_i - \frac{m}{2} \epsilon_{ij} A_j) \dot{A}_i - \frac{1}{2} (E_i E_i + B^2) + A_0 (\partial_i E_i - mB) \right], \quad (A.2)$$

where

$$E_i \equiv F_{0i} \equiv \dot{A}_i - \partial_i A_0, \quad B \equiv \frac{1}{2} \epsilon_{ij} F_{ij} \quad (\epsilon_{ij} \equiv \epsilon^{0ij})$$

and we have neglected the boundary term $\int_{R^2} d^2x \partial_i [A_0 ( \frac{m}{2} \epsilon_{ij} A_j - E_i)]$.

If we use “1-forms” instead of time derivatives (see comment after (3.1)), then the action (A.2) takes the form (2.1) with $A_0$ playing the role of Lagrange multiplier.

For the reduction we choose $E_1$ and $E_2$ as the variables $\xi^\mu$, $A_1$ as the additional variable $\eta$ (see (2.11)) and get

$$\tilde{S} = \int dt \int_{R^2} d^2x \left[ \frac{1}{m} E_2 \dot{E}_1 - \frac{1}{2} |E_i E_i + \frac{1}{m^2} (\partial_k E_k)^2| + \frac{d}{dt} \Theta \right], \quad (A.3)$$

where

$$\Theta = \frac{1}{2} [E_1 A_1 + E_2 \hat{K}(A_1 + \frac{1}{m} E_2)]$$

and the operator $\hat{K} \equiv \partial_1^{-1} \partial_2$ is assumed to be symmetric due to the corresponding boundary conditions.

Neglecting the $\Theta$ term as the total derivative, we get the local Hamiltonian theory with the canonical commutation relations

$$\{E_2(x), E_1(y)\} = m \delta^{(2)}(x - y) \quad (A.4)$$

and the quadratic Hamiltonian

$$\frac{1}{2} \int_{R^2} d^2x [E_i E_i + \frac{1}{m^2} (\partial_k E_k)^2]. \quad (A.5)$$
The energy-momentum tensor also can be expressed only through \( E_1 \) and \( E_2 \):

\[
T_{00} = \frac{1}{2} [E_i E_i + \frac{1}{m^2} (\partial_k E_k)^2], \quad T_{0i} = \frac{1}{m} \epsilon_{ij} E_j (\partial_k E_k). \tag{A.6}
\]

Let us briefly stop on the boundary conditions. We can assume that a boundary behaviour of the physical variables \((E_1, E_2)\) should provide the Poincare invariance of the reduced system (A.3)-(A.6), while a boundary behaviour of the fields of the initial system (A.1) should allow the outlined reduction procedure.

Generators of the Poincare group (constructed from the energy-momentum tensor (A.6)) generate transformations of \(E_1\) and \(E_2\) according to the Poisson brackets (A.4). The class of functions \(E_1(x)\) and \(E_2(x)\) should be invariant under these transformations. It is natural to choose the class of smooth, rapidly vanishing at the infinity functions.

For the diagonalization of the Hamiltonian and momentum let us take the Fourier transformation:

\[
E_j(x) = i \int d^2 p e^{-i(p \cdot x)} \tilde{E}_j(p) \tag{A.7}
\]
and introduce the longitudinal and transverse components:

\[
\tilde{E}_j(p) = \frac{p_j}{|p|} e_1(p) - \frac{\epsilon_{jl}}{|p|} e_2(p), \tag{A.8}
\]
where \(|p| = \sqrt{p_1^2 + p_2^2}\).

Then diagonalization of the energy and momentum will be achieved in the variables

\[
a(p) = \frac{\omega_p}{\sqrt{2 \omega_p}} e_1(p) + i e_2(p) e^{-i \varphi(p)},
\]

\[
a^*(p) = \frac{\omega_p}{\sqrt{2 \omega_p}} e_1(-p) - i e_2(-p) e^{i \varphi(p)}, \tag{A.9}
\]
with

\[
\omega_p = \sqrt{|p|^2 + m^2} \quad \text{and} \quad e^{\pm i \varphi(p)} = \frac{p_1 \pm i p_2}{|p|}.
\]

Note that for the chosen class of \(E_1(x)\) and \(E_2(x)\) the longitudinal and transverse components of \(\tilde{E}_j(p)\) have the singularity at the origin \((p = 0)\) and the phase factor \(e^{i \varphi(p)}\) is necessary to cancel it. On the other hand one can easily check that the class of smooth functions \(a(p), a^*(p)\) is Poincare invariant. Just this phase factor was introduced in [25] to avoid anomalies in the commutation relations of the Poincare algebra of quantum operators. As we have seen this phase factor is connected to the Poincare invariance of the classical system too.

After description of the class of physical variables one can go back and find the class of gauge potentials \(A_\mu\). One can show that these classes for massive and ordinary photodynamics in \((2+1)\) dimensions are different.