Nowhere-Zero Flow Polynomials

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Abstract

In this article we introduce the flow polynomial of a digraph and use it to study nowhere-zero flows from a commutative algebraic perspective. Using Hilbert’s Nullstellensatz, we establish a relation between nowhere-zero flows and dual flows. For planar graphs this gives a relation between nowhere-zero flows and flows of their planar duals. It also yields an appealing proof that every bridgeless triangulated graph has a nowhere-zero four-flow.

1 Introduction

The theory of nowhere-zero flows (see [4, 7] for recent surveys) was introduced by Tutte [6] as an extension of Tait’s earlier work [5] on the four-color problem for planar graphs.

Let $G = (V,E)$ be a digraph and let $p \geq 2$ be an integer. A $p$-flow on $G$ is a mapping $\phi : E \rightarrow \mathbb{Z}_p$ from arcs to the additive group $\mathbb{Z}_p = \{0,1,\ldots,p-1\}$ of integers modulo $p$ such that preservation holds at each vertex $v$, that is $\sum \{\phi(e) : e \in \delta^-(v)\} - \sum \{\phi(e) : e \in \delta^+(v)\} = 0$ in $\mathbb{Z}_p$, where $\delta^-(v), \delta^+(v)$ are the sets of arcs with head $v$ and tail $v$ respectively. It is a nowhere-zero $p$-flow if $\phi(E) \subseteq \mathbb{Z}_p^* := \{1,\ldots,p-1\}$. If $\phi$ is a flow then the (signed) sum of arc values on each cocircuit of $G$ is 0 in $\mathbb{Z}_p$. With matroid duality in mind, we call $\phi$ a dual $p$-flow if the sum of arc values on each circuit of $G$ is zero, and call it nowhere-zero dual $p$-flow if it is nowhere-zero.

An undirected graph $G$ will be called $p$-flowing if some orientation of $G$ admits a nowhere-zero $p$-flow (and hence so does every orientation - just flip the sign of $\phi(e)$ whenever $e$ is flipped). Likewise, $G$ is dually $p$-flowing if some (and hence every) orientation of $G$ admits a dual nowhere-zero $p$-flow. The first fact that motivates flow theory is the following relation between dual flow and coloring, implicit in the aforementioned work of Tait [5]. We outline the simple proof.

**Proposition 1.1** A graph is dually $p$-flowing if and only if it is $p$-colorable.

**Proof.** Assume $G$ is connected and oriented with a suitable dual nowhere-zero $p$-flow $\phi$. Pick a spanning tree $T$ and a vertex $v$. Set $\omega(v) := 0$ and for each other vertex $u$ set $\omega(u)$ to be the (signed) sum in $\mathbb{Z}_p$ of the values $\phi(e)$ on arcs on the unique path in $T$ from $v$ to $u$. Since $\phi$ sums to zero on each circuit, for every arc $e = ab$ we get $\omega(b) - \omega(a) = \phi(e)$ and since $\phi$ is nowhere-zero it follows that the resulting $\omega : V \rightarrow \mathbb{Z}_p$ is a $p$-coloring. The converse is likewise easy to see. 

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A graph can be flowing only if it has no coloop (also called cut-edge, isthmus, or bridge); and it can be dually flowing only if it has no loop. Two gems of flow theory are the following. First, Tutte conjectured [6] that every bridgeless graph is 5-flowing; while this is still open, Seymour has shown that every bridgeless graph is indeed 6-flowing, see [4]. Second, if $G$ is a directed plane graph then a map $\phi$ is a dual flow precisely when it is a flow of the plane dual $G^*$; the four-color theorem is thus equivalent to the statement that every bridgeless planar graph is 4-flowing.

In this article we introduce the flow polynomial of a digraph and use it to study flows from a commutative algebraic perspective. While the general approach follows the line taken by Lovász in studying stable sets [3] and Alon-Tarsi in studying coloring [1], here, inspired by our recent work [2], we take a closer look at a suitable normal form of the polynomials that arise. Using Hilbert’s Nullstellensatz, we establish a relation between nowhere-zero flows and dual flows. For planar graphs this gives a relation between nowhere-zero flows and flows of their planar duals. To state it, we need some more notation. A map $\phi : E \to \mathbb{Z}_p$ is even if the number $|\phi^{-1}(p-1)|$ of arcs labelled by the maximal label $p-1$ is even; otherwise it is odd. Let $\psi : E \to \mathbb{Z}_p^0 := \{0, \ldots, p-2\}$ be a nowhere-(p-1) map. We say that $\phi : E \to \mathbb{Z}_p$ is $\psi$-conformal if $\phi(e) \in \{\psi(e), p-1\}$ for every arc $e$. We establish the following theorem.

**Theorem 1.2** A digraph has a nowhere-zero $p$-flow if and only if it has a nowhere-(p-1) map $\psi$ such that the number of even $\psi$-conformal dual $p$-flows is not equal to the number of odd ones.

Since planar duality interchanges circuits and cocircuits, this gives at once the following corollary.

**Corollary 1.3** A plane digraph has a nowhere-zero $p$-flow if and only if its plane dual has a nowhere-(p-1) map $\psi$ with number of even $\psi$-conformal $p$-flows different than that of odd ones.

Another corollary concerns triangulated graphs: while it can be shown directly, we find the proof below, which gives a stronger statement on conformal maps of the zero map, particularly elegant.

**Corollary 1.4** Any bridgeless triangulated (chordal) graph is 4-flowing.

**Proof.** We prove by induction on the number of edges the following claim: any undirected bridgeless triangulated $G = (V,E)$ has an orientation $D$ such that, for the identically zero map $\psi \equiv 0$, the map $\phi = \psi \equiv 0$ is the only $\psi$-conformal dual 4-flow. If $E$ is empty then the claim is trivially true. Otherwise, pick any circuit $C \subseteq E$ of size $\leq 3$ in $G$. By induction, the contraction $G' := G/C$ has an orientation $D'$ satisfying the claim. Extend $D'$ to an orientation $D$ of $G$ by making $C$ a directed cycle. Consider any 0-conformal dual 4-flow $\phi$ on $D$. Then $\phi(e) \in \{0,3\}$ for all $e$, $\sum_{e \in C} \phi(e) = 0$ in $\mathbb{Z}_4$, and $|C| \leq 3$ imply that $\phi(e) = 0$ for all $e \in C$. Now let $\phi'$ be the restriction of $\phi$ to $D'$. Then $\phi'$ is a 0-conformal dual 4-flow on $D'$ and hence, by induction, $\phi'(e) = 0$ for all $e \in E \setminus C$. Thus, as claimed, $\phi \equiv 0$, and we are done by Theorem 1.2.
2 The flow polynomial of a digraph

Fix a digraph \( G = (V, E) \) and an integer \( p \geq 2 \). Let \( x = (x_e : e \in E) \) be a tuple of variables indexed by the arcs of \( G \), and let \( \mathbb{C}[x] = \mathbb{C}[x_e : e \in E] \) be the algebra of polynomials with complex coefficients in these variables. We consider the following polynomial ideal

\[
I_E^p := \text{ideal} \left\{ \sum_{i=0}^{p-1} x_e^i : e \in E \right\},
\]
determined by \( p \) and the number of arcs, and we introduce the following flow polynomial of \( G \),

\[
f_G^p := \prod_{v \in V} \sum_{i=0}^{p-1} \left( \prod_{e \in \delta^-(v)} x_e \prod_{e \in \delta^+(v)} x_e^{p-1} \right)^i.
\]

In this section we establish the following statement.

**Theorem 2.1** A digraph \( G = (V, E) \) has a nowhere-zero \( p \)-flow if and only if \( f_G^p \) is not in \( I_E^p \).

The proof will follow from two properties of the ideal and polynomial which we establish next. A tuple \( a = (a_e : e \in E) \) of complex numbers is a zero of \( I_E^p \) if \( f(a) = 0 \) for all polynomials \( f \in I_E^p \). Throughout, let \( \rho := \exp(\frac{2\pi \sqrt{-1}}{p}) \) denote the primitive \( p \)-th complex root of unity.

**Proposition 2.2** A tuple \( a \) is a zero of \( I_E^p \) if and only if \( a_e \in \{\rho^1, \ldots, \rho^{p-1}\} \) for all \( e \in E \). Moreover, \( I_E^p \) is radical and hence consists precisely of all polynomials vanishing on its zero set.

**Proof.** The univariate polynomial \( f := \sum_{i=0}^{p-1} z^i \) satisfies \( f \cdot (z - 1) = z^p - 1 = \prod_{i=0}^{p-1} (z - \rho^i) \) and hence its roots are all \( p \)-th roots of unity but \( \rho^0 = 1 \). Since \( I_E^p \) is generated by copies of \( f \), one for each variable \( x_e \), the first part of the proposition follows. Since each such generator has no multiple roots, the ideal is radical. Therefore, by Hilbert’s Nullstellensatz, \( I_E^p \) consists precisely of all polynomials vanishing on its zero set, completing the proof of the proposition. \( \square \)

The proposition establishes a bijection between nowhere-zero maps \( \phi : E \rightarrow \mathbb{Z}_p^* \) and zeros \( a = (\rho^{\phi(e)} : e \in E) \) of \( I_E^p \). The nowhere-zero flows are characterized among such maps \( \phi \) by the evaluation of the flow polynomial on the corresponding zeros \( a \), as follows.

**Proposition 2.3** Consider any map \( \phi : E \rightarrow \mathbb{Z}_p^* \) and let \( a = (\rho^{\phi(e)} : e \in E) \) be the corresponding zero of \( I_E^p \). If \( \phi \) is a nowhere-zero \( p \)-flow on \( G \) then \( f_G^p(a) = p^{|V|} \); otherwise \( f_G^p(a) = 0 \).

**Proof.** Let \( s(v) := \sum_{e \in \delta^-(v)} \phi(e) - \sum_{e \in \delta^+(v)} \phi(e) \in \mathbb{Z}_p \) be the flow surplus at vertex \( v \). Then

\[
f_G^p(a) = \prod_{v \in V} \sum_{i=0}^{p-1} \left( \prod_{e \in \delta^-(v)} \rho^{\phi(e)} \prod_{e \in \delta^+(v)} (\rho^{\phi(e)})^{p-1} \right)^i = \prod_{v \in V} \sum_{i=0}^{p-1} (\rho^{s(v)})^i.
\]
Now, if \( s(v) \in \mathbb{Z}_p^* \) then \( \sum_{i=0}^{p-1}(\rho^{s(v)})^i = 0 \) (see proof of Proposition 2.2), whereas is \( s(v) = 0 \) then \( \sum_{i=0}^{p-1}(\rho^{s(v)})^i = p \). Since \( \phi \) is a flow if and only if \( s(v) = 0 \) for all \( v \in V \), the proof is complete. \( \square \)

**Proof of Theorem 3.1** By Proposition 2.2 the polynomial \( f_p^G \) is in \( I_E^p \) if and only if it vanishes on every zero of \( I_E^p \), which, by Proposition 2.3, holds if and only if \( G \) has no nowhere-zero \( p \)-flow. \( \square \)

**Remark 2.4** Note that the flow polynomial \( f_p^G \) has the following very special appealing property: its evaluations on the zero set of \( I_E^p \) assume only two distinct values, either 0 or \( p^{\|V\|} \).

**Example 2.5** Let \( G = (V,E) \) be a digraph consisting of two vertices \( v_1, v_2 \) and three arcs \( e_1 = e_3 = v_1v_2, e_2 = v_2v_1 \), and let \( p = 3 \). The flow polynomial is

\[
 f^3_G = 2(x_1^2x_2x_3)^i \sum_{i=0}^2 (x_1x_2x_3)^i = x_1^6x_2^6x_3^6 + x_1^4x_2^4x_3^4 + x_1^2x_2^2x_3^2 + x_1^0x_2^0x_3^0 \]

the zeros of the ideal \( I^3_E = \text{ideal}\{x_1^2 + x_1 + 1, x_2^2 + x_2 + 1, x_3^2 + x_3 + 1\} \) are all 8 tuples \( a = (a_1, a_2, a_3) \) with each \( a_i \in \{\rho, \rho^2\} \) where \( \rho = \exp(2\pi i / 3) \); the evaluation of \( f^3_G \) on a zero of \( I^3_E \) corresponding to a nowhere-zero map \( \phi = (\phi_1, \phi_2, \phi_3) \) is \( 3^2 = 9 \) if \( \phi \) is either \((1, 2, 1)\) or \((2, 1, 2)\), and is 0 otherwise, distinguishing \((1, 2, 1)\) and \((2, 1, 2)\) as the only two nowhere-zero 3-flows of \( G \).

3 The normal form of the flow polynomial

We proceed to take a close look at the normal form of the flow polynomial with respect to a natural monomial basis of the quotient \( \mathbb{C}[x]/I_E^p \). Consider the following set of basic monomials,

\[
 B := \left\{ \prod_{e \in E} x_\psi(e) : \psi : E \rightarrow \mathbb{Z}_p^0 = \{0, \ldots, p-2\} \right\}
\]

**Proposition 3.1** The (congruence classes of) basic monomials form a \( \mathbb{C} \)-basis of \( \mathbb{C}[x]/I_E^p \).

**Proof.** First, it is clear that \( I_E^p \) contains no nonzero polynomial which is a linear combination of monomials in \( B \), so \( B \) is linearly independent modulo \( I_E^p \). Second, \( I_E^p \) is radical by Proposition 2.2, and hence, by Hilbert’s Nullstellensatz, the vector space dimension of \( \mathbb{C}[x]/I_E^p \) equals the number of zeros of \( I_E^p \). By Proposition 2.2 this number is \((p-1)^|E|\), which is precisely the number of basic monomials, and so it follows that \( B \) spans the quotient space and hence provides a basis. \( \square \)

It follows that for every polynomial \( f \in \mathbb{C}[x] \) there is a unique polynomial \([f]\), called the normal form of \( f \), which satisfies \( f - [f] \in I_E^p \) and is a \( \mathbb{C} \)-linear combination of basic monomials

\[
 [f] = \sum_{\psi : E \rightarrow \mathbb{Z}_p^0} c_\psi \prod_{e \in E} x_\psi(e)
\]
In particular, \( f \in \mathcal{I}_E \) if and only if \( |f| = 0 \). By characterizing the normal form of the flow polynomial we will be able, via Theorem 2.1, to establish the promised criterion of Theorem 1.2 for a graph to be flowing. We proceed to study normal forms, starting with powers of variables.

**Proposition 3.2** The normal form of \( x_e^{q+p+r} \) with \( q \) any nonnegative integer and \( r \in \mathbb{Z}_p \) is

\[
[x_e^{q+p+r}] = \begin{cases} 
    x_e^r & \text{ if } r \in \mathbb{Z}_p^0 \\
    -\sum_{i=0}^{p-2} x_e^i & \text{ if } r = p - 1 
\end{cases}
\]

**Proof.** First, we have \( x_e^p - 1 = (x_e - 1) \cdot \sum_{i=0}^{p-1} x_e^i \in \mathcal{I}_E \) and 1 is a basic monomial, which shows that \( [x_e^p] = 1 \). Thus, the normal form of an arbitrary power of \( x_e \) is determined by the normal form of powers \( x_e^r \) with \( r \in \mathbb{Z}_p \). If \( r \in \mathbb{Z}_p^0 \) then the monomial \( x_e^r \) is basic and hence satisfies \( [x_e^r] = x_e^r \). If \( r = p - 1 \) then \( x_e^{p-1} - (-\sum_{i=0}^{p-2} x_e^i) = \sum_{i=0}^{p-1} x_e^i \in \mathcal{I}_E \) and \( -\sum_{i=0}^{p-2} x_e^i \) is a linear combination of basic monomials, so \( [x_e^{p-1}] = -\sum_{i=0}^{p-2} x_e^i \). This completes the proof. \( \square \)

Now, for any two polynomials \( f, g \) and scalars \( s, t \in \mathbb{C} \) we have \( [sf + tg] = s[f] + t[g] \) and \( [fg] = [[f][g]] \). The first identity implies that the normal form of any polynomial is determined by the normal forms of its monomials. The second identity implies that for any monomial \( \prod_{e \in E} x_e^{m_e} \) we have \( \langle \prod_{e \in E} x_e^{m_e} \rangle = \prod_{e \in E} [x_e^{m_e}] \); but Proposition 3.2 implies that the polynomial \( \prod_{e \in E} x_e^{m_e} \) is in the \( \mathbb{C} \)-linear span of basic monomials and hence \( \langle \prod_{e \in E} x_e^{m_e} \rangle = \prod_{e \in E} [x_e^{m_e}] \). This completely determines the normal form of any polynomial.

**Example 2.5 continued.** Consider again the digraph \( G \) with three arcs, and let again \( p = 3 \). Using Proposition 3.2, we find that the normal form of the flow polynomial is

\[
[f_G^3] = 1 + (-x_1 - 1)x_2(-x_3 - 1) + x_1(-x_2 - 1)x_3 + x_1(-x_2 - 1)x_3 + 1 \\
+ (-x_1 - 1)x_2(-x_3 - 1) + (-x_1 - 1)x_2(-x_3 - 1) + x_1(-x_2 - 1)x_3 + 1 \\
= 3(x_1x_2 - x_1x_3 + x_2x_3 + x_2 + 1).
\]

Since \( [f_G^3] \neq 0 \), we find that \( f_G^3 \notin \mathcal{I}_E \) and hence, by Theorem 2.1, \( G \) admits a nowhere-zero 3-flow.

We next show that the coefficients of the monomials in the normal form of the flow polynomial can be nicely interpreted in terms of certain dual flows. Recall that a map \( \phi : E \to \mathbb{Z}_p \) is even if the number \( |\phi^{-1}(p - 1)| \) of arcs labelled by the maximal label \( p - 1 \) is even; otherwise it is odd. Recall also that \( \phi \) is \( \psi \)-conformal for a nowhere-(p-1) map \( \psi : E \to \mathcal{Z}_p^0 = \{0, \ldots, p - 2 \} \) if \( \phi(e) \in \{ \psi(e), p - 1 \} \) for every arc \( e \). We have the following theorem.

**Theorem 3.3** Let \( G = (V, E) \) be an orientation of a connected graph, and let \( p \geq 2 \) be an integer. Then the normal form of the flow polynomial of \( G \) is given by

\[
[f_G^p] = p \cdot \sum_{\psi : E \to \mathcal{Z}_p^0} c(\psi) \prod_{e \in E} x_e^{\psi(e)},
\]

where \( c(\psi) \) denotes the number of even \( \psi \)-conformal dual \( p \)-flows minus the number of odd ones.
Proof. The flow polynomial can be expanded as

\[ f^p_G := \sum_{\omega: \rightarrow \mathbb{Z}_p} \prod_{v \in V} \left( \prod_{e \in \delta^+(v)} x_e \prod_{e \in \delta^-(v)} x_e^{p-1} \right)^{\omega(v)}, \]

the sum extending over all labellings \( \omega \) of vertices by \( \{0, \ldots, p-1\} \). Since each arc \( e = uv \) satisfies \( e \in \delta^+(u) \) and \( e \in \delta^-(v) \) we can rewrite this as

\[ f^p_G := \sum_{\omega: \rightarrow \mathbb{Z}_p} \prod_{e = uv \in E} x_e^{\omega(v)} x_e^{(p-1)\omega(u)}. \]

Consider any \( \omega : V \rightarrow \mathbb{Z}_p \) and let \( \phi : E \rightarrow \mathbb{Z}_p \) be the map that labels each arc \( e = uv \) by \( \phi(e) := \omega(v) - \omega(u) \) in \( \mathbb{Z}_p \). By Proposition 3.2 we then have \( [x_e^{\omega(v)} x_e^{(p-1)\omega(u)}] = [x_e^{\phi(e)}] \) and hence the normal form of the summand in the above expression of \( f^p_G \) corresponding to \( \omega \) satisfies

\[ \left[ \prod_{e = uv \in E} x_e^{\omega(v)} x_e^{(p-1)\omega(u)} \right] = \prod_{e \in E} [x_e^{\phi(e)}]. \]

Now, since the arc labelling \( \phi \) is induced from a vertex labelling \( \omega \), the (signed) sum of the \( \phi \) values of arcs on each circuit of \( G \) is 0 in \( \mathbb{Z}_p \) and hence \( \phi \) is a dual \( p \)-flow. Since the undirected graph underlying \( G \) is connected, \( \omega \) is uniquely determined by \( \phi \) and the value \( \omega(v) \) on an arbitrary vertex \( v \) (see proof of Proposition 1.1), so \( \phi \) arises from precisely \( p \) distinct maps \( \omega \), and we get

\[ [f^p_G] = p \sum \left\{ \prod_{e \in E} [x_e^{\phi(e)}] : \phi \text{ dual } p\text{-flow} \right\} \]

\[ = p \sum \left\{ \prod_{e: \phi(e) \in \mathbb{Z}_p^0} x_e^{\phi(e)} \prod_{e: \phi(e) = p-1} (- \sum_{i \in \mathbb{Z}_p^0} x_e^i) : \phi \text{ dual } p\text{-flow} \right\}. \]

Now consider the basic monomial \( \prod_{e \in E} x_e^{\psi(e)} \) corresponding to \( \psi : E \rightarrow \mathbb{Z}_p^0 \). Then, in the right hand side sum in the above expression of \([f^p_G]\), every \( \psi \)-conformal dual \( p \)-flow map \( \phi \) contributes a term \( \prod_{e \in E} x_e^{\psi(e)} \), whereas every odd one contributes a term \(- \prod_{e \in E} x_e^{\psi(e)} \). This shows that, as claimed, the coefficient \( c(\psi) \) of \( \prod_{e \in E} x_e^{\psi(e)} \) in \([f^p_G]\) is equal to the number of even \( \psi \)-conformal dual \( p \)-flows minus the number of odd ones, completing the proof of the theorem. \( \square \)

Remark 3.4 More generally, if \( G \) is an orientation of a graph with \( \kappa \) connected components then a suitable adjustment of the analysis above shows that the normal form of the flow polynomial is

\[ [f^p_G] = p^\kappa \sum_{\psi : E \rightarrow \mathbb{Z}_p^0} c(\psi) \prod_{e \in E} x_e^{\psi(e)}. \]
Theorem 1.2 A digraph has a nowhere-zero $p$-flow if and only if it has a nowhere-$(p-1)$ map $\psi$ such that the number of even $p$-flows minus the number of odd ones is not equal to the number of odd ones.

Proof. Let $f^p_G$ be the flow polynomial of a digraph $G$. By Theorem 2.1, $G$ has a nowhere-zero $p$-flow if and only if $f^p_G \notin \mathcal{I}^p_E$, which holds if and only if the normal form $[f^p_G]$ is nonzero. By Theorem 3.3, $[f^p_G] \neq 0$ if and only if $G$ admits a nowhere-$(p-1)$ map $\psi$ such that $c(\psi) \neq 0$. Since $c(\psi)$ is the number of even $p$-flows minus the number of odd ones, we are done. □

4 The four-flow polynomial of an undirected graph

In this section we work out a variant of the flow polynomial for four-flows for an undirected graph $G = (V, E)$. It is simpler and perhaps better suited for the study of four-flows of planar graphs and the four-color theorem. The outline is similar to that of the previous two sections; we therefore do not go through the proofs which are analogous to those provided before.

Let $G = (V, E)$ be a graph. A four-flow on $G$ is a mapping

$$\phi = (\phi_1, \phi_2): E \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$$

such that $\sum \{\phi(e) : e \in \delta(v)\} = (0, 0)$ for each vertex $v$, where $\delta(v)$ is the set of edges incident on
Let $x = (x_e : e \in E)$, $y = (y_e : e \in E)$ be two tuples of variables indexed by edges and let $\mathbb{C}[x, y]$ be corresponding polynomial algebra. Consider the following ideal and polynomial,

$$I_E := \text{ideal}\{x_e^2 - 1, y_e^2 - 1, (x_e + 1)(y_e + 1) : e \in E\},$$

$$f_G := \prod_{v \in V} \left( \prod_{e \in \delta(v)} x_e + 1 \right) \left( \prod_{e \in \delta(v)} y_e + 1 \right).$$

We have the following analog of Theorem 2.1.

**Theorem 4.1** A graph $G = (V, E)$ has a nowhere-zero four-flow if and only if $f_G$ is not in $I_E$.

As before, this is a consequence of the following two properties of $I_E$ and $f_G$. A pair of tuples $a = (a_e : e \in E)$, $b = (b_e : e \in E)$ of complex numbers is a zero of $I_E$ if $f(a, b) = 0$ for all $f \in I_E$.

**Proposition 4.2** The pair $(a, b)$ is a zero of $I_E$ if and only if $(a_e, b_e) \in \{(1, -1), (-1, 1), (-1, -1)\}$ for all $e$. Moreover, $I_E$ is radical and hence consists of all polynomials vanishing on its zero set.

The proposition establishes a bijection between nowhere-zero maps

$$\phi = (\phi_1, \phi_2) : E \rightarrow (\mathbb{Z}_2 \times \mathbb{Z}_2)^* = \{(0, 1), (1, 0), (1, 1)\}$$

and zeros $(a, b)$ of $I_E$ given by $a_e = (-1)^{\phi_1(e)}$, $b_e = (-1)^{\phi_2(e)}$ for all $e \in E$. The nowhere-zero flows are characterized among such maps $\phi$ by the evaluation of the flow polynomial on the corresponding zeros $(a, b)$, as follows.

**Proposition 4.3** Consider any nowhere-zero map $\phi = (\phi_1, \phi_2)$ and let $(a, b)$ be the corresponding zero of $I_E$. If $\phi$ is a nowhere-zero four-flow on $G$ then $f_G(a, b) = 4^{|V|}$; otherwise $f_G(a, b) = 0$.

**Proof of Theorem 4.1.** By Proposition 4.2, $f_G$ lies in $I_E$ if and only if it vanishes on its set of zeros, which, by Proposition 4.3, holds if and only if $G$ has no nowhere-zero four-flow. □

As before, we next consider the normal form of the flow polynomial with respect to a natural monomial basis of the quotient $\mathbb{C}[x, y]/I_E$. Consider the following set of basic monomials,

$$B := \left\{ \prod_{e \in E} x_e^{\psi_1(e)} y_e^{\psi_2(e)} : \psi = (\psi_1, \psi_2) : E \rightarrow (\mathbb{Z}_2 \times \mathbb{Z}_2)^0 := \{(0, 0), (0, 1), (1, 0)\} \right\}.$$

**Proposition 4.4** The (congruence classes of) basic monomials form a $\mathbb{C}$-basis of $\mathbb{C}[x, y]/I_E$.

For every polynomial $f \in \mathbb{C}[x, y]$ let again $[f]$ denote its normal form which is the unique $\mathbb{C}$-linear combination of basic monomials satisfying $f - [f] \in I_E$. The normal form of powers of pairs of variables $x_e, y_e$ is determined by the following analog of Proposition 3.2.
Proposition 4.5 For any two nonnegative integers \(q_1, q_2\) and any \(r_1, r_2 \in \mathbb{Z}_2\) we have
\[
[x_e^{2q_1+r_1}y_e^{2q_2+r_2}] = \begin{cases} 
  x_e^{r_1}y_e^{r_2} & \text{if } (r_1, r_2) \in (\mathbb{Z}_2 \times \mathbb{Z}_2)^0 \\
  -x_e - y_e - 1 & \text{if } (r_1, r_2) = (1, 1)
\end{cases}
\]

Now, for any monomial \(\prod_{e \in E} x_e^{m_e} y_e^{n_e}\) we have \(\prod_{e \in E} [x_e^{m_e} y_e^{n_e}] = [\prod_{e \in E} x_e^{m_e} y_e^{n_e}]\); but Proposition 4.5 implies that the polynomial \(\prod_{e \in E} x_e^{m_e} y_e^{n_e}\) is in the \(C\)-linear span of basic monomials and hence \(\prod_{e \in E} x_e^{m_e} y_e^{n_e} = \prod_{e \in E} [x_e^{m_e} y_e^{n_e}]\). This completely determines the normal form of any monomial and hence, as explained before, of every polynomial.

We next show, in analogy with Theorem 3.3, an interpretation of the coefficients of the monomials in the normal form of the flow polynomial in terms of suitable conformal dual flows. A map \(\phi : E \rightarrow \mathbb{Z}_p\) is even if the number \(|\phi^{-1}(1, 1)|\) of edges labelled by \((1, 1)\) is even; otherwise it is odd. The map \(\phi\) is \(\psi\)-conformal for a nowhere-(1,1) map \(\psi = (\psi_1, \psi_2) : E \rightarrow (\mathbb{Z}_2 \times \mathbb{Z}_2)^0\) if \(\phi(e) \in \{\psi(e), (1, 1)\}\) for every edge \(e\). We have the following analog of Theorem 3.3.

Theorem 4.6 Let \(G = (V,E)\) be a graph with \(\kappa\) connected components. Then the normal form of the four-flow polynomial of \(G\) is given by
\[
[f_G] = 4^\kappa \cdot \sum_{\psi=(\psi_1,\psi_2):E\rightarrow(\mathbb{Z}_2\times\mathbb{Z}_2)^0} c(\psi) \prod_{e \in E} x_e^{\psi_1(e)} y_e^{\psi_2(e)},
\]
where \(c(\psi)\) is the number of even \(\psi\)-conformal dual four-flows minus the number of odd ones.

We also conclude the following analog of Theorem 1.2.

Theorem 4.7 A graph has a nowhere-zero four-flow if and only if it has a nowhere-(1,1) map \(\psi\) such that the number of even \(\psi\)-conformal dual four-flows is not equal to the number of odd ones.

Proof. Let \(f_G\) be the flow polynomial of a graph \(G\). By Theorem 1.3, \(G\) has a nowhere-zero four-flow if and only if \(f_G \notin I_E\), which holds if and only if \([f_G]\) is nonzero. By Theorem 4.6 \([f_G] \neq 0\) if and only if \(G\) admits a nowhere-(1,1) map \(\psi\) such that \(c(\psi) \neq 0\). Since \(c(\psi)\) is the number of even \(\psi\)-conformal dual four-flows minus the number of odd ones, we are done.

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