TOPOLOGICAL MOLINO’S THEORY

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Abstract. Molino’s description of Riemannian foliations on compact manifolds is generalized to the setting of compact equicontinuous foliated spaces, in the case where the leaves are dense. In particular, a structural local group is associated to such a foliated space. As an application, we obtain a partial generalization of results by Carrière and Breuillard-Gelander, relating the structural local group to the growth of the leaves.

Contents

1. Introduction 2
1.1. Molino’s theory for Riemannian foliations 3
1.2. Holonomy of Riemannian foliations 4
1.3. Growth of Riemannian foliations 5
1.4. Equicontinuous foliated spaces 6
1.5. Topological Molino’s theory 7
1.6. Growth of equicontinuous foliated spaces 9
1.7. Possible applications 9
2. Preliminaries on equicontinuous pseudogroups 9
2.1. Compact-open topology on partial maps with open domains 10
2.2. Pseudogroups 11
2.3. Groupoid of germs of a pseudogroup 14
2.4. Local groups and local actions 17
2.5. Equicontinuous pseudogroups 19
3. Molino’s theory for equicontinuous pseudogroups 24
3.1. Conditions on \( \hat{H} \) 24
3.2. Coincidence of topologies 24
3.3. The space \( \hat{T} \) 25
3.4. The space \( \hat{T}_0 \) 30
3.5. The pseudogroup \( \hat{H}_0 \) 30

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1. Introduction

Riemannian foliations were introduced by Reinhart [44] by requiring an isometric transverse dynamics. It was pointed out by Ghys in [37, Appendix E] (see also Kellum’s paper [31]) that equicontinuous foliated spaces should be considered as the “topological Riemannian foliations,” and therefore many of the results about Riemannian foliations should have versions for equicontinuous foliated spaces. Some steps in this direction were given by Álvarez and Candel [5, 6], showing that, under reasonable conditions, their leaf closures are minimal foliated spaces, and their generic leaves are quasi-isometric to each other, like in the case of Riemannian foliations. In the same direction, Matsumoto [33] proved that any minimal equicontinuous foliated space has a non-trivial transverse invariant measure, which is unique up to scaling if the space is compact—observe that this unicity implies ergodicity. The magnitude of the generalization from Riemannian foliations to equicontinuous foliated spaces was made precise by Álvarez and Candel [6] (see also Tarquini’s paper [47]), giving a topological description of Riemannian foliations within the class of equicontinuous foliated spaces.

Most of the known properties of Riemannian foliations follow from a description due to Molino [36, 37]. However, so far, there was no version of Molino’s description for equicontinuous foliated spaces—the indicated properties of equicontinuous foliated spaces were obtained by other means. The goal of our work is to develop such a version of Molino’s theory, and use it to study the growth of their leaves, following the study of the growth...
of Riemannian foliations by Carrière [12] and Breuillard-Gelander [10]. To
understand our results better, let us briefly recall Molino’s theory.

1.1. Molino’s theory for Riemannian foliations. Recall that a (smooth)
foliation $\mathcal{F}$ of codimension $q$ on a manifold $M$ is a partition of $M$ into
injectively immersed connected submanifolds (leaves), which can be locally
described as the fibers of local submersions onto $q$-manifolds. These sub-
mersions and their domains are said to be distinguished, and their images are
called local quotients. The changes of distinguished submersions are given
by diffeomorphisms between open subsets of the local quotients, which are
called elementary holonomy transformations. A foliation is called minimal
if the leaves are dense. A map between foliated manifolds is called foliated
if it maps leaves to leaves.

By using chains of consecutive distinguished open sets along loops in a leaf
$L$, and composing the corresponding elementary holonomy transformations,
we get a representation of $\pi_1(L)$ in a group of germs of those compositions,
which is called the holonomy representation of $L$. Its image is called the
holonomy group of $L$, and its kernel equals the image of the homomorphism
$\pi_1(\tilde{L}) \to \pi_1(L)$ induced by a unique regular cover $\tilde{L} \to L$, which is called the
holonomy cover of $L$. For a general foliation on a second countable manifold,
there is a dense $G_δ$ saturated subset whose leaves have trivial holonomy
groups [27, 28, 11]; thus any statement about the holonomy covers of the
leaves can be simplified as a statement about the generic leaves, if desired.

Let $T\mathcal{F} \subset TM$ denote the vector subbundle of vectors tangent to the
leaves. Then $N\mathcal{F} = TM/T\mathcal{F}$ is called the normal bundle of $\mathcal{F}$, and its
sections normal vector fields. There is a natural flat leafwise partial connec-
tion on $N\mathcal{F}$ so that any local normal vector field is leafwise parallel if and
only if it is locally projectable by the distinguished submersions; terms like
“leafwise flat,” “leafwise parallel” and “leafwise horizontal” will refer to this
partial connection. It is said that $\mathcal{F}$ is:

- **Riemannian**: if $N\mathcal{F}$ has a leafwise parallel Riemannian structure;
- **transitive**: if the group of its foliated diffeomorphisms acts transitively on $M$;
- **transversely parallelizable (TP)**: if there is a leafwise parallel global
  frame of $N\mathcal{F}$, called transverse parallelism; and a
- **Lie foliation**: if moreover the transverse parallelism is basis of a Lie
  algebra with the operation induced by the vector field bracket.

These conditions are successively stronger. Molino’s theory describes Rie-
mannian foliations on compact manifolds in terms of minimal Lie foliations,
and using TP foliations as an intermediate step:

1st step: If $\mathcal{F}$ is Riemannian and $M$ compact, then there is an $O(q)$-
principal bundle, $\tilde{\pi} : \tilde{M} \to M$, with an $O(q)$-invariant TP foliation,
$\tilde{\mathcal{F}}$, such that $\tilde{\pi}$ is a foliated map whose restrictions to the leaves are
the holonomy covers of the leaves of $\mathcal{F}$.
2nd step: If $F$ is TP and $M$ compact, then there is a fiber bundle $\pi : M \to W$ whose fibers are the leaf closures of $F$, and the restriction of $F$ to each fiber is a Lie foliation.

Since the structure of Lie foliation is unique in the minimal case, we end up with a Lie algebra associated to $F$, called structural Lie algebra. The proofs of the above statements strongly use the differential structure of $F$.

In the first step, $\hat{\pi} : \hat{M} \to M$ is the $O(q)$-principal bundle of orthonormal frames for some leafwise parallel metric on $NF$, and $\hat{F}$ is given by the corresponding flat leafwise horizontal distribution. Then $\hat{F}$ is TP by a standard argument. In the second step, foliated flows are used to produce the fiber bundle trivializations whose fibers are the leaf closures; this works because there are foliated flows in any transverse direction since $F$ is TP.

When $F$ is minimal, any leaf closure $\hat{M}_0$ of $\hat{F}$ is a principal subbundle of $\hat{\pi} : \hat{M} \to M$, obtaining the following:

**Minimal case:** If $F$ is minimal and Riemannian, and $M$ is compact, then, for some closed subgroup $H \subset O(q)$, there is an $H$-principal bundle, $\hat{\pi}_0 : \hat{M}_0 \to M$, with an $H$-invariant minimal Lie foliation, $\hat{F}_0$, such that $\hat{\pi}_0$ is a foliated map whose restrictions to the leaves are the holonomy covers of the leaves of $F$.

A useful description of Lie foliations was also given by Fedita [18, 19], but it will not be considered here.

The differential structure cannot be used in our generalization; instead, the emphasis will be put on the holonomy pseudogroup. Thus let us briefly indicate the holonomy properties of Riemannian foliations that will play an important role in the generalization.

1.2. Holonomy of Riemannian foliations. A pseudogroup is a maximal collection of local transformations of a space, which contains the identity map, and is closed under the operations of composition, inversion, restriction and combination. It can be considered as a generalized dynamical system, and all basic dynamical concepts have pseudogroup versions. They are relevant in foliation theory because the elementary holonomy transformations generate a pseudogroup which describes the transverse dynamics of $F$; it is called the holonomy pseudogroup, and its elements holonomy transformations. Such a pseudogroup is well determined up to certain equivalence of pseudogroups introduced by Haefliger [24, 25]. We may say that $F$ is transversely modelled by a class of local transformations of some space if its holonomy pseudogroup can be generated by that type of local transformations. Riemannian, TP and Lie foliations can be respectively characterized by being transversely modelled by

- local isometries of some Riemannian manifold;
- local parallelism preserving diffeomorphisms of some parallelizable manifold; and
- local left translations of a Lie group.
In this sense, Riemannian foliations are the transversely rigid ones, and TP foliations have a stronger type of transverse rigidity.

When the ambient manifold \( M \) is compact, Haefliger \([26]\) has observed that the holonomy pseudogroup \( \mathcal{H} \) of \( F \), acting on \( T \), satisfies the following property:

**Compact generation:** There is some relatively compact open subset \( U \subset T \), which meets all \( \mathcal{H} \)-orbits, and there is a finite number of generators, \( h_1, \ldots, h_k \), of the restriction \( \mathcal{H}|_U \) such that each \( h_i \) has an extension \( \tilde{h}_i \in \mathcal{H} \) with \( \text{dom} \ h_i \subset \text{dom} \ \tilde{h}_i \).

If moreover \( F \) is Riemannian, then Haefliger \([25, 26]\) has also strongly used the following properties of \( \mathcal{H} \):

- **Completeness:** For all \( x, y \in T \), there are open neighborhoods, \( V \) of \( x \) and \( W \) of \( y \), such that, for all \( h \in \mathcal{H} \) and \( z \in V \cap \text{dom} \ h \) with \( h(z) \in W \), there is some \( \tilde{h} \in \mathcal{H} \) such that \( \text{dom} \ \tilde{h} = V \) and \( \tilde{h} = h \) around \( z \).
- **Closure:** Let \( J^1(T) \) be the space of 1-jets of local transformations of \( T \), and let \( j^1(\mathcal{H}) \subset J^1(T) \) the subset given by 1-jets of maps in \( \mathcal{H} \). Then the closure \( \overline{j^1(\mathcal{H})} \) in \( J^1(T) \) is the set of 1-jets of the maps in a pseudogroup \( \overline{\mathcal{H}} \) of local isometries of \( T \), called the *closure* of \( \mathcal{H} \), whose orbits are the closures of the \( \mathcal{H} \)-orbits.
- **Quasi-analyticity:** If some \( h \in \mathcal{H} \) is the identity on some open set \( O \) with \( O \subset \text{dom} \ h \), then \( h \) is the identity on some neighborhood of \( O \).

Quasi-analyticity holds because the differential of an isometry at some point determines the map on a neighborhood. Thus it also holds for \( \overline{\mathcal{H}} \).

For a compactly generated pseudogroup \( \mathcal{H} \) of local isometries of a Riemannian manifold \( T \), Salem has given a version of Molino’s theory \([46], [37, Appendix D]\) (see also \([3]\)). In particular, in the minimal case, it turns out that there is a Lie group \( G \), a compact subgroup \( K \subset G \) and a dense finitely generated subgroup \( \Gamma \subset G \) such that \( \mathcal{H} \) is equivalent to the pseudogroup generated by the action of \( \Gamma \) on the homogeneous space \( G/K \) (this was also observed by Haefliger \([25]\)).

### 1.3. Growth of Riemannian foliations

Molino’s theory has many consequences for a Riemannian foliation \( F \) on a compact manifold \( M \): classification in particular cases, growth, cohomology, tautness, tenseness and global analysis. In all of them, Molino’s theory is used to reduce the study to the case of Lie foliations with dense leaves, where it usually becomes a problem of Lie theory. A list of references about all applications would be too long. We concentrate on the consequences about the growth of the leaves and their holonomy covers, which refers to their growth as Riemannian manifolds with the metrics induced by any metric on \( M \); this growth depends only on \( F \) by the compactness of \( M \). This study was begun by Carrière \([12]\), and recently continued by Breuillard-Gelander, as a consequence of their study of
a topological Tits alternative \[10\]. Their results state the following, where \( g \) is the structural Lie algebra of \( F \):

**Carrière’s theorem:** The holonomy covers of the leaves are:
- Følner if and only if \( g \) is solvable; and
- of polynomial growth if and only if \( g \) is nilpotent.

In the second case, the degree of their polynomial growth is bounded by the nilpotence degree of \( g \).

**Breuillard-Gelander’s theorem:** The growth of the holonomy covers of the leaves is either polynomial or exponential.

### 1.4. Equicontinuous foliated spaces.

A **foliated space** \( X \equiv (X, \mathcal{F}) \) is a topological space \( X \) equipped with a partition \( \mathcal{F} \) into connected manifolds (leaves), which can be locally described as the fibers of topological submersions. It will be assumed that \( X \) is locally compact and Polish. A foliated space should be considered as a “topological foliation”. In this sense, all topological notions of foliations have obvious versions for foliated spaces. In particular, the **holonomy pseudogroup** \( \mathcal{H} \) of \( X \) is defined on a locally compact Polish space \( T \). Many results about foliations also have straightforward generalizations; for example, the leaves with trivial holonomy group form a dense \( G_\delta \) set, and \( \mathcal{H} \) is compactly generated if \( X \) is compact. Even leafwise differential concepts are easy to extend. However this task may be difficult or impossible for transverse differential concepts. For instance, the normal bundle of a foliated space does not make any sense in general: it would be the tangent bundle of a topological space in the case of a space foliated by points. Thus the concept of Riemannian foliation cannot be extended by using the normal bundle; instead, this can be done via the holonomy pseudogroup as follows.

The transverse rigidity of a Riemannian foliation can be translated to the foliated space \( X \) by requiring (uniform) equicontinuity of \( \mathcal{H} \). In fact, the equicontinuity condition is not compatible with combinations of maps, and therefore equicontinuity is indeed required for some generating subset \( S \subset \mathcal{H} \) which is closed by the operations of composition and inversion; such an \( S \) is called a **pseudo\(^*\)group** with the terminology of Matsumoto \[33\]. This gives rise to the concept of **equicontinuous** foliated space.

In the topological setting, the quasi-analyticity of \( \mathcal{H} \) does not follow from the equicontinuity assumption. Thus quasi-analyticity will be required as an additional assumption when needed. Indeed, it does not work well enough when \( T \) is not locally connected. So we use a property called **strong quasi-analyticity**, defined by the existence of a pseudo\(^*\)group \( S \), generating \( \mathcal{H} \), such that any map in \( S \) is the identity on its domain if it is the identity on some non-empty open subset; this property is stronger than quasi-analyticity only when \( T \) is not locally connected.

Álvarez and Candel \[5\] have proved that, if \( \mathcal{H} \) is compactly generated, equicontinuous and strongly quasi-analytic, then it is complete and has a closure \( \overline{\mathcal{H}} \). With this generality, \( \overline{\mathcal{H}} \) cannot be defined by using 1-jets, of
course; instead, \( \mathcal{H} \) consists of the maps that are local limits of maps in \( \mathcal{H} \) with the compact-open topology; this method works well because \( \mathcal{H} \) is complete.

Transitive and Lie foliations have the following obvious topological versions. It is said that the foliated space \( X \) is

**homogeneous**: if the group of its foliated transformations acts transitively on \( X \).

Let \( G \) be a locally compact Polish local group. When \( X \) is minimal, it is called a

**\( G \)-foliated space**: if it is transversely modelled by local left translations in some local group locally isomorphic to \( G \).

1.5. **Topological Molino’s theory.** Our first main result is the following topological version of the minimal case in Molino’s theory.

**Theorem A.** Let \( X \equiv (X,F) \) be a compact Polish foliated space, and \( \mathcal{H} \) its holonomy pseudogroup. Suppose that \( X \) is minimal and equicontinuous, and \( \mathcal{H} \) is strongly quasi-analytic. Then there is a compact Polish minimal foliated space \( \tilde{X}_0 \equiv (\tilde{X}_0,\tilde{F}_0) \), an open continuous foliated map \( \tilde{\pi}_0 : \tilde{X}_0 \to X \), and a locally compact Polish local group \( G \) such that \( \tilde{X}_0 \) is a \( G \)-foliated space, the fibers of \( \tilde{\pi}_0 \) are homeomorphic to each other, and the restrictions of \( \tilde{\pi}_0 \) to the leaves of \( \tilde{F}_0 \) are the holonomy covers of the leaves of \( F \).

The proof of Theorem A is different from Molino’s proof in the Riemannian foliation case because there may not be normal bundle of \( F \). To define \( \tilde{X}_0 \), we first construct what should be its holonomy pseudogroup, \( \tilde{\mathcal{H}}_0 \) on a space \( \tilde{T}_0 \). To some extent, this was achieved by Álvarez-Candel [6], proving that, with the assumptions of Theorem A there is a locally compact Polish local group \( G \), a compact subgroup \( K \subset G \) and a dense finitely generated sub-local group \( \Gamma \subset G \) such that \( \mathcal{H} \) is equivalent to the pseudogroup generated by the local action of \( \Gamma \) on \( G/K \), like in the Riemannian foliation case. Hence \( \tilde{\mathcal{H}}_0 \) should be the pseudogroup generated by the local action of \( \Gamma \) on \( G \). This may look as a big step towards the proof, but the realization of compactly generated pseudogroups as holonomy pseudogroups of compact foliated spaces is impossible in general, as shown by Meigniez [35]. This difficulty is overcome as follows.

Take a “good” cover of \( X \) by distinguished open sets, \( \{U_i\} \), with corresponding distinguished submersions \( p_i : U_i \to T_i \), and elementary holonomy transformations \( h_{ij} : T_{ij} \to T_{ji} \), where \( T_{ij} = p_i(U_i \cap U_j) \). Let \( \mathcal{H} \) denote the corresponding representative of the holonomy pseudogroup on \( T = \bigsqcup_i T_i \), generated by the maps \( h_{ij} \). Then the construction of \( \tilde{\mathcal{H}}_0 \) must be associated to \( \mathcal{H} \) in a natural way, so that it becomes induced by some “good” cover by distinguished open sets of a compact foliated space. In the Riemannian foliation case, the good choices of \( \tilde{T}_0 \) and \( \tilde{\mathcal{H}}_0 \) are the following ones:
Let $P$ be the bundle of orthonormal frame for any $\mathcal{H}$-invariant metric on $T$. Fix $x_0 \in T$ and $\hat{x}_0 \in P_{x_0}$. Then, as subspace of $P$,
\begin{equation}
\hat{T}_0 = \{ h_*(\hat{x}_0) \mid h \in \mathcal{H}, \ x_0 \in \text{dom } h \} \\
= \{ g_*(\hat{x}_0) \mid g \in \overline{\mathcal{H}}, \ x_0 \in \text{dom } g \} . \tag{1}
\end{equation}

$\hat{T}_0$ is generated by the differentials of the maps in $\mathcal{H}$.

These differential concepts can be modified in the following way:

- In (1), each $g_*(\hat{x}_0)$ determines the germ of $g$ at $x_0$, $\gamma(g,x_0)$, by the strong quasi-analyticity of $\mathcal{H}$. Therefore it also determines $\gamma(f,x)$, where $f = g^{-1}$ and $x = g(x_0)$—this little change, using $\gamma(f,x)$ instead of $\gamma(g,x_0)$, is not really necessary, but it helps to simplify the notation in some involved arguments. So
\begin{equation}
\hat{T}_0 \equiv \{ \gamma(f,x) \mid f \in \overline{\mathcal{H}}, \ x \in \text{dom } f, \ f(x) = x_0 \} . \tag{2}
\end{equation}

- The projection $\hat{\pi}_0 : \hat{T}_0 \to T$ corresponds via (2) to the source map $\gamma(f,x) \mapsto x$.

- The differentials of maps $h \in \mathcal{H}$, acting on orthonormal references, correspond via (2) to the maps $\hat{h}$ defined by
\[ \hat{h}(\gamma(f,x)) = \gamma(fh^{-1},h(x)) . \]

- The topology of $\hat{T}_0$ can be described via (2) as follows. Let $\overline{\mathcal{S}}$ be a pseudo-$\ast$group generating $\overline{\mathcal{H}}$ and satisfying the equicontinuity and strong quasi-analyticity conditions. Endow $\overline{\mathcal{S}}$ with the compact-open topology on partial maps with open domains, as defined by Abd-Allah-Brown [1], and consider the subspace
\[ \overline{\mathcal{S}} \ast T = \{ (f,x) \in \overline{\mathcal{S}} \mid x \in \text{dom } f \} \subset \overline{\mathcal{S}} \times T . \]

Then the topology of $\hat{T}_0$ corresponds via (2) to the quotient topology by the germ map $\gamma : \overline{\mathcal{S}} \ast T \to \gamma(\overline{\mathcal{S}} \ast T) \equiv \hat{T}_0$, which is of course different from the sheaf topology on germs.

This point of view, replacing orthonormal frames by germs, can be readily translated to the foliated space setting, obtaining good choices of $\hat{T}_0$ and $\hat{\mathcal{H}}_0$ under the conditions of Theorem A.

Now, consider triples $(x,\gamma,i)$, where $x \in U_i$, $\gamma \in \hat{T}_{i,0} := \hat{\pi}_0^{-1}(T_i)$ and $p_i(x) = \hat{\pi}_0(\gamma)$. Declare $(x,\gamma,i)$ equivalent to $(y,\delta,j)$ if $x = y$ and $\delta = \hat{h}_{ij}(\gamma)$. Then $\hat{X}_0$ is defined as the corresponding quotient space. Let $[x,\gamma,i]$ denote the equivalence class of each triple $(x,\gamma,i)$. The foliated structure $\hat{\mathcal{F}}_0$ on $\hat{X}_0$ is determined by requiring that, for each fixed index $i$, the elements of the type $[x,\gamma,i]$ form a distinguished open set $\hat{U}_{i,0}$, with distinguished submersion $\hat{p}_0 : \hat{U}_{i,0} \to \hat{T}_{i,0}$ given by $\hat{p}_0([x,\gamma,i]) = \gamma$. The projection $\hat{\pi}_0 : \hat{X}_0 \to X$ is defined by $\hat{\pi}_0([x,\gamma,i]) = x$. The properties stated in Theorem A are satisfied with these definitions.
It is also proved that, up to foliated homeomorphisms (respectively, local isomorphisms), $\hat{X}_0$ (respectively, $G$) is independent of the choices involved. Hence $G$ can be called the structural local group of $\mathcal{F}$.

1.6. Growth of equicontinuous foliated spaces. Let us say that a local group $G$ can be approximated by nilpotent local Lie groups if, in any identity neighborhood, there exists a sequence of compact normal subgroups $F_n$ such that $F_{n+1} \subset F_n$, $\bigcap_n F_n = \{e\}$ and $G/F_n$ is a nilpotent local Lie group. Our second main result, is the following weak topological version of the above theorems of Carrière and Breuillard-Gelander.

**Theorem B.** Let $X$ be a foliated space satisfying the conditions of Theorem A, and let $G$ be its structural local group. Then one of the following properties holds:
- $G$ can be approximated by nilpotent local Lie groups;
- the holonomy covers of all leaves of $X$ have exponential growth.

Like in the case of Riemannian foliations, Theorem A reduces the proof of Theorem B to the case of minimal $G$-foliated spaces, where it becomes a problem about local groups. Then, since any locally compact Polish local group can be approximated by local Lie groups in the above sense, the result follows by applying the same arguments as Breuillard-Gelander.

1.7. Possible applications. The term matchbox manifold is used for $X$ in the particular case where it is a continuum and $T$ is totally disconnected; this means that $\mathcal{F}$ is minimal and $T$ a Cantor space. The structure of homogeneous matchbox manifolds was described by Clark and Hurder [15], showing that they are McCord solenoids. Then $\hat{X}_0 \equiv X$ in Theorem A. This fits with Molino theory since homogeneous foliated spaces play the role of TP foliations. Thus Theorem A is useless to study homogeneous matchbox manifolds. However it could be useful to study non-McCord solenoids. This is explained with more detail in a section of examples. Other examples where Theorems A and B may give relevant information are constructed with almost periodic non-periodic functions, and almost periodic locally aperiodic Riemannian manifolds.

The paper is finished with a section of open problems and questions, mainly about possible applications of Theorems A and B.

2. Preliminaries on equicontinuous pseudogroups

2.1. Compact-open topology on partial maps with open domains. Most of the contents of this section are taken from [1].

Given spaces $X$ and $Y$, let $C(X,Y)$ be the space of all continuous maps $X \to Y$; the notation $C_{c-o}(X,Y)$ may be used to indicate that $C(X,Y)$ is equipped with the compact-open topology. Let $Y^*$ be the space $Y \cup \{\omega\}$, where $\omega \notin Y$, endowed with the topology in which $U \subset Y^*$ is open if and only if $U = Y^*$ or $U$ is open in $Y$. A partial map $X \rightarrow Y$ is a continuous
map of a subset of $X$ to $Y$; the set of all partial maps $X \rightarrow Y$ is denoted by $\text{Par}(X,Y)$. A partial map $X \rightarrow Y$ with open domain is called a \textit{paro map}, and the set of all paro maps $X \rightarrow Y$ is denoted by $\text{Paro}(X,Y)$. There is a bijection $\mu : \text{Paro}(X,Y) \rightarrow C(X,Y^*)$ defined by

$$\mu(f)(x) = \begin{cases} f(x) & \text{if } x \in \text{dom } f \\ \omega & \text{if } x \notin \text{dom } f \end{cases} .$$

The topology on $\text{Paro}(X,Y)$ which makes $\mu : \text{Paro}(X,Y) \rightarrow C_{c-o}(X,Y^*)$ a homeomorphism is called the \textit{compact-open topology}, and the notation $\text{Paro}_{c-o}(X,Y)$ may be used for the corresponding space. This topology has a subbasis of open sets of the form

$$N(K,O) = \{ h \in \text{Paro}(X,Y) \mid K \subset \text{dom } h, h(K) \subset O \} ,$$

where $K \subset X$ is compact and $O \subset Y$ is open.

\textbf{Proposition 2.1.} If $X$ is second countable and locally compact, and $Y$ is second countable, then $\text{Paro}_{c-o}(X,Y)$ is second countable.

\textit{Proof.} By hypothesis, there are countable bases of open sets, $\mathcal{V}$ of $X$ and $\mathcal{W}$ of $Y$, such that $\mathcal{V}$ is compact for all $V \in \mathcal{V}$. Then the sets $N(V,W)$ ($V \in \mathcal{V}$ and $W \in \mathcal{W}$) form a countable subbase of open sets of $\text{Paro}_{c-o}(X,Y)$. \hfill $\square$

The following result is elementary.

\textbf{Proposition 2.2.} For any open $U \subset X$, the restriction of the topology of $\text{Paro}_{c-o}(X,Y)$ to the subset $C(U,Y)$ is its usual compact-open topology.

Since paro maps are not globally defined, let us make precise the definition of their composition. Given spaces $X$, $Y$ and $Z$, the \textit{composition} of two paro maps, $f \in \text{Paro}(X,Y)$ and $g \in \text{Paro}(Y,Z)$, is the paro map $gf \in \text{Paro}(X,Z)$ defined as the usual composition of the maps

$$f^{-1}(\text{dom } g) \xrightarrow{f} \text{dom } g \xrightarrow{g} Z .$$

\textbf{Proposition 2.3} (Abd-Allah-Brown \cite[Proposition 3]{Abd-Allah-Brown}). The following properties hold:

(i) Let $h : T \rightarrow X$ and $g : Y \rightarrow Z$ be paro maps. Then the maps

$$g_* : \text{Paro}_{c-o}(X,Y) \rightarrow \text{Paro}_{c-o}(X,Z) , \quad f \mapsto gf ,$$

$$h^* : \text{Paro}_{c-o}(X,Y) \rightarrow \text{Paro}_{c-o}(T,Y) , \quad f \mapsto fh ,$$

are continuous.

(ii) Let $X' \subset X$ and $Y' \subset Y$ be subspaces such that $X'$ is open in $X$. Then the map

$$\text{Paro}_{c-o}(X',Y') \rightarrow \text{Paro}_{c-o}(X,Y) ,$$

mapping a paro map $X' \rightarrow Y'$ to the paro map $X \rightarrow Y$ with the same graph, is an embedding.
Proposition 2.4 (Abd-Allah-Brown [1, Proposition 7]). If $Y$ is locally compact, then the evaluation partial map
\[ ev: \text{Paro}_{c-o}(Y,Z) \times Y \to Z, \quad ev(f,y) = f(y), \]
is a paro map; in particular, its domain is open.

Proposition 2.5 (Abd-Allah-Brown [1, Proposition 9]). If $X$ and $Y$ are locally compact, then the composition mapping
\[ \text{Paro}_{c-o}(X,Y) \times \text{Paro}_{c-o}(Y,Z) \to \text{Paro}_{c-o}(X,Y), \quad (f,g) \mapsto gf, \]
is continuous.

Let $\text{Loct}(T)$ be the family of all homeomorphisms between open subsets of a space $T$, which are called local transformations. For $h, h' \in \text{Loct}(T)$, the composition $h' h \in \text{Loct}(T)$ is the composition of maps
\[ h^{-1}(\text{im } h \cap \text{dom } h') \longrightarrow h \text{ im } h \cap \text{dom } h' \longrightarrow h' \text{ im } h \cap \text{dom } h' . \]
Each $h \in \text{Loct}(T)$ can be identified with the paro map $T \to T$ with the same graph. This gives rise to a canonical injection $\text{Loct}(T) \to \text{Paro}(T,T)$ compatible with composition. The corresponding restriction of the compact-open topology of $\text{Paro}(T,T)$ to $\text{Loct}(T)$ is also called compact-open topology, and the notation $\text{Loct}_{c-o}(T)$ may be used for the corresponding space. The bi-compact-open topology is the smallest topology on $\text{Loct}(X)$ so that the identity and inversion maps
\[ \text{Loct}(T) \to \text{Loct}_{c-o}(T) , \quad f \mapsto f^{\pm 1} , \]
are continuous, and the notation $\text{Loct}_{b-c-o}(T)$ will be used for the corresponding space. The following result is elementary.

Proposition 2.6 (Abd-Allah-Brown [1, Proposition 10]). If $T$ is locally compact, then the composition and inversion maps,
\[ \text{Loct}_{b-c-o}(T) \times \text{Loct}_{b-c-o}(T) \to \text{Loct}_{b-c-o}(T) , \quad (g,f) \mapsto gf , \]
\[ \text{Loct}_{b-c-o}(T) \to \text{Loct}_{b-c-o}(T) , \quad f \mapsto f^{-1} , \]
are continuous.

2.2. Pseudogroups.

Definition 2.7 (Sacksteder [15], Haefliger [26]). A pseudogroup on a space $T$ is a collection $\mathcal{H} \subset \text{Loct}(T)$ such that
- the identity map of $T$ belongs to $\mathcal{H}$ ($\text{id}_T \in \mathcal{H}$);
- if $h, h' \in \mathcal{H}$, then the composite $h' h$ is in $\mathcal{H}$ ($\mathcal{H}^2 \subset \mathcal{H}$);
- $h \in \mathcal{H}$ implies that $h^{-1} \in \mathcal{H}$ ($\mathcal{H}^{-1} \subset \mathcal{H}$);
- if $h \in \mathcal{H}$ and $U$ is open in $\text{dom } h$, then the restriction $h : U \to h(U)$ is in $\mathcal{H}$; and,
- if a combination (union) of maps in $\mathcal{H}$ is defined and is a homeomorphism, then it is in $\mathcal{H}$.

Remark 1. The following properties hold:
• $\text{id}_U \in \mathcal{H}$ for every open subset $U \subset T$.
• A local transformation $h \in \text{Loct}(T)$ belongs to $\mathcal{H}$ if and only if it locally belongs to $\mathcal{H}$ (any point $x \in \text{dom} h$ has a neighborhood $V_x \subset \text{dom} h$ such that $h|_{V_x} \in \mathcal{H}$).
• Any intersection of pseudogroups on $T$ is a pseudogroup on $T$.

**Example 2.8.** $\text{Loct}(T)$ is the pseudogroup that contains any other pseudogroup on $T$.

**Definition 2.9.** A sub-pseudogroup of a pseudogroup $\mathcal{H}$ on $T$ is a pseudogroup on $T$ contained in $\mathcal{H}$. The restriction of $\mathcal{H}$ to an open subset $U \subset T$ is the pseudogroup $\mathcal{H}|_U = \{ h \in \mathcal{H} \mid \text{dom} h \cup \text{im} h \subset U \}$.

The pseudogroup generated by a set $S \subset \text{Loct}(T)$ is the intersection of all pseudogroups that contain $S$ (the smallest pseudogroup on $T$ containing $S$).

**Definition 2.10.** Let $\mathcal{H}$ be a pseudogroup on $T$. The orbit of each $x \in T$ is the set $\mathcal{H}(x) = \{ h(x) \mid h \in \mathcal{H}, \ x \in \text{dom} h \}$.

The orbits form a partition of $T$. The space of orbits, equipped with the quotient topology, is denoted by $T/\mathcal{H}$. It is said that $\mathcal{H}$ is
- (topologically) transitive if some orbit is dense; and
- minimal when all orbits are dense.

The following notion, less restrictive than the concept of pseudogroup, is useful to study some properties of pseudogroups.

**Definition 2.11 (Matsumoto [33]).** A pseudo$*$-group on a space $T$ is a family $S \subset \text{Loct}(T)$ that is closed by the operations of composition and inversion.

**Remark 2.** Any intersection of pseudo$*$-groups on $T$ is a pseudo$*$-group.

**Definition 2.12.** Any pseudo$*$-group contained in another pseudo$*$-group is called a sub-pseudo$*$-group. The pseudo$*$-group generated by a set $S_0 \subset \text{Loct}(T)$ is the intersection of all pseudo$*$-groups containing $S_0$ (the smallest pseudo$*$-group containing $S_0$).

**Remark 3.** Let $S$ be a pseudo$*$-group on $T$, and let $S_1$ be the collection of restrictions of all maps in $S$ to all open subsets of their domains. Then $S_1$ is also a pseudo$*$-group on $T$, and $S$ is a sub-pseudo$*$-group of $S_1$.

**Definition 2.13.** In Remark 3 it will be said that $S_1$ is the localization of $S$. If $S = S_1$, then the pseudo$*$-group $S$ is called local.

**Remark 4.** Let $S_0 \subset \text{Loct}(T)$. The pseudo$*$-group $S$ generated by $S_0$ consists of all compositions of maps in $S_0$ and their inverses. The pseudogroup $\mathcal{H}$ generated by $S_0$ consists of all $h \in \text{Loct}(T)$ that locally belong to the localization of $S$. 
Remark 5. If two local pseudo*groups, $S_1$ and $S_2$, generate the same pseudogroup $H$, then $S_1 \cap S_2$ is also a local pseudo*group that generates $H$.

Let $H$ and $H'$ be pseudogroups on respective spaces $T$ and $T'$.

**Definition 2.14** (Haefliger [24, 25]). A morphism $\Phi : H \rightarrow H'$ is a maximal collection of homeomorphisms of open sets of $T$ to open sets of $T'$ such that

- if $\varphi \in \Phi$, $h \in H$ and $h' \in H'$, then $h' \varphi h \in \Phi$ ($H' \Phi H \subset \Phi$);
- the family of the domains of maps in $\Phi$ cover $T$; and
- if $\varphi, \varphi' \in \Phi$, then $\varphi' \varphi^{-1} \in H'$ ($\Phi \Phi^{-1} \subset H'$).

A morphism $\Phi$ is called an equivalence if the family $\Phi^{-1} = \{ \varphi^{-1} | \varphi \in \Phi \}$ is also a morphism.

**Remark 6.** An equivalence $\Phi : H \rightarrow H'$ can be characterized as a maximal family of homeomorphisms of open sets of $T$ to open sets of $T'$ such that $H' \Phi H \subset \Phi$, and $\Phi^{-1} \Phi$ and $\Phi \Phi^{-1}$ generate $H'$ and $H$, respectively.

**Remark 7.** Any morphism $\Phi : H \rightarrow H'$ induces a map between the corresponding orbit spaces, $T/H \rightarrow T/H'$. This map is a homeomorphism if $\Phi$ is an equivalence.

**Definition 2.15.** Let $\Phi_0$ be a family of homeomorphisms of open subsets of $T$ to open subsets of $T'$ such that

- the union of domains of maps in $\Phi_0$ meet all $H$-orbits; and
- $\Phi_0 \Phi_0^{-1} H \subset H'$.

Then there is a unique morphism $\Phi : H \rightarrow H'$ containing $\Phi_0$, which is said to be generated by $\Phi_0$. If moreover:

- the union of images of maps in $\Phi_0$ meet all $H'$-orbits; and
- $\Phi_0^{-1} H \Phi_0 \subset H$;

then $\Phi$ is an equivalence.

**Definition 2.16** (Haefliger [26]). A pseudogroup $H$ on a locally compact space $T$ is said to be compactly generated if

- there is a relatively compact open subset $U \subset T$ meeting each $H$-orbit;
- there is a finite set $S = \{ h_1, \ldots, h_n \} \subset H|_U$ that generates $H|_U$; and
- each $h_i$ is the restriction of some $\tilde{h}_i \in H$ with $\text{dom } \tilde{h}_i \subset \text{dom } h_i$.

**Remark 8.** Compact generation is related to very delicate conditions [20, 34]. Haefliger asked when compact generation implies realizability as a holonomy pseudogroup of a compact foliated space. The answer is not always affirmative [35].

**Definition 2.17** (Haefliger [24]). A pseudogroup $H$ is called quasi-analytic if every $h \in H$ is the identity around some $x \in \text{dom } h$ whenever $h$ is the identity on some open set whose closure contains $x$.

---

1This is usually called étalé morphism. We simply call it morphism because no other type of morphism will be considered here.
If a pseudogroup $\mathcal{H}$ on a space $T$ is quasi-analytic, then every $h \in \mathcal{H}$ with connected domain is the identity on $\text{dom } h$ if it is the identity on some non-empty open set. Because of this, quasi-analyticity is interesting when $T$ is locally connected, but local connectivity is too restrictive in our setting. Then, instead of requiring local connectivity, the following stronger version of quasi-analyticity will be used.

**Definition 2.18** (Álvarez-Candel [5]). A pseudogroup $\mathcal{H}$ on a space $T$ is said to be **strongly quasi-analytic** if it is generated by some sub-pseudo*group $S \subset \mathcal{H}$ such that any transformation in $S$ is the identity on its domain if it is the identity on some non-empty open subset of its domain.

**Remark 9.** In [5], the term used for the above property is “quasi-effective”. However the term “strongly quasi-analytic” seems to be more appropriate.

**Remark 10.** If the condition on $\mathcal{H}$ to be strongly quasi-analytic is satisfied with a sub-pseudo*group $S$, it is also satisfied with the localization of $S$.

**Definition 2.19** (Haefliger [24]). A pseudogroup $\mathcal{H}$ on a space $T$ is said to be **complete** if, for all $x, y \in T$, there are compact open neighborhoods, $U_x$ of $x$ and $V_y$ of $y$, such that, for all $h \in \mathcal{H}$ and $z \in U_x \cap \text{dom } h$ with $h(z) \in V_y$, there is some $g \in \mathcal{H}$ such that $\text{dom } g = U_x$ and $\gamma(g, z) = \gamma(h, z)$.

Since any pseudo*group $S$ on $T$ is a sub-pseudo*group of $\text{Loct}(T)$, it can be endowed with the restriction of the (bi-)compact-open topology, also called (bi-)compact-open topology of $S$, and the notation $S_{(b-)c-o}$ may be used for the corresponding space. In this way, according to Proposition 2.6, if $T$ is locally compact, then $S_{b-c-o}$ becomes a **topological pseudo*group** in the sense that the composition and inversion maps of $S$ are continuous. In particular, this applies to a pseudogroup $\mathcal{H}$ on $T$, obtaining $\mathcal{H}_{(b-)c-o}$; thus $\mathcal{H}_{b-c-o}$ is a **topological pseudogroup** in the above sense if $T$ is locally compact.

**Remark 11.** If $S$ is a sub-pseudo*group of $S'$, then $S_{(b-)c-o} \hookrightarrow S'_{(b-)c-o}$ is continuous.

Recall that a topological space is called **Polish** if it is separable and completely metrizable. This condition is sometimes assumed provide better dynamical properties. Thus the pseudogroups considered from now on will be assumed to act on locally compact Polish spaces; these spaces can be characterized by the condition of being locally compact, Hausdorff and second countable [30, Theorem 5.3].

2.3. **Groupoid of germs of a pseudogroup.**

**Definition 2.20.** A groupoid $\mathcal{G}$ is a small category where every morphism is an isomorphism. This means that $\mathcal{G}$ is a set (of morphisms) equipped with the structure defined by an additional set $T$ (of objects), and the following structural maps:

- the **source** and **target** maps $s, t : \mathcal{G} \to T$;
• the unit map \( T \to \mathcal{G}, x \mapsto 1_x \);
• the operation (or multiplication) map \( \mathcal{G} \times_T \mathcal{G} \to \mathcal{G}, (\delta, \gamma) \mapsto \delta \gamma \), where
  \[
  \mathcal{G} \times_T \mathcal{G} = \{ (\delta, \gamma) \in \mathcal{G} \times \mathcal{G} \mid t(\gamma) = s(\delta) \} \subset \mathcal{G} \times \mathcal{G} ;
  \]
• and the inversion map \( \mathcal{G} \to \mathcal{G}, \gamma \mapsto \gamma^{-1} \);

such that the following conditions are satisfied:
• \( s(\delta \gamma) = s(\gamma) \) and \( t(\delta \gamma) = t(\delta) \) for all \( (\delta, \gamma) \in \mathcal{G} \times_T \mathcal{G} \);
• for all \( \gamma, \delta, \epsilon \in \mathcal{G} \) with \( t(\gamma) = s(\delta) \) and \( t(\delta) = s(\epsilon) \), we have \( \epsilon(\delta \gamma) = (\epsilon \delta) \gamma \) (associativity);
• \( 1_{t(\gamma)} \gamma = \gamma 1_{s(\gamma)} = \gamma \) (units or identity elements); and
• \( s(\gamma) = t(\gamma^{-1}), t(\gamma) = s(\gamma^{-1}) \), \( \gamma^{-1} \gamma = 1_{s(\gamma)} \) and \( \gamma \gamma^{-1} = 1_{t(\gamma)} \) for all \( \gamma \in \mathcal{G} \) (inverse elements).

If moreover \( \mathcal{G} \) and \( T \) are equipped with topologies so that all of the above structural maps are continuous, then \( \mathcal{G} \) is called a topological groupoid.

Remark 12. For a groupoid \( \mathcal{G} \), observe that \( s(1_x) = t(1_x) = x \) for all \( x \in T \), and therefore the source and target maps \( s, t : \mathcal{G} \to T \) are surjective, and the unit map \( T \to \mathcal{G} \) is injective. If moreover \( \mathcal{G} \) is a topological groupoid, then the unit map \( T \to \mathcal{G} \) is a topological embedding, and therefore the topology of \( T \) is determined by the topology of \( \mathcal{G} \); indeed, we can consider \( T \) as a subspace of \( \mathcal{G} \) if desired.

Definition 2.21. A topological grupoid is called étalé if the source and target maps are local homeomorphisms.

Let \( \mathcal{H} \) be a pseudogroup on a space \( T \). For \( h, h' \in \mathcal{H} \) and \( x \in \text{dom} \, h \cap \text{dom} \, h' \), write \( (h, x) \sim (h', x) \) if there is a neighborhood \( U \) of \( x \) in \( \text{dom} \, h \cap \text{dom} \, h' \) such that \( h|_U = h'|_U \). This defines an equivalence relation on the set
\[
\mathcal{H} \times_T T = \{ (h, x) \in \mathcal{H} \times T \mid x \in \text{dom} \, h \} \subset \mathcal{H} \times T .
\]
Note that \( \mathcal{H} \times_T T \) is the domain of the evaluation partial map \( \text{ev} : \mathcal{H} \times T \to T \). The equivalence class of each \( (h, x) \in \mathcal{H} \times T \) is called the germ of \( h \) at \( x \), which will be denoted by \( \gamma(h, x) \). The corresponding quotient set is denoted by \( \mathcal{G} \), and the quotient map, \( \gamma : \mathcal{H} \times_T T \to \mathcal{G} \), is called the germ map. It is well known that \( \mathcal{G} \) is a groupoid with set of units \( T \), where the source and target maps \( s, t : \mathcal{G} \to T \) are given by \( s(\gamma(h, x)) = x \) and \( t(\gamma(h, x)) = h(x) \), the unit map \( T \to \mathcal{G} \) is defined by \( 1_x = \gamma(\text{id}_T, x) \), the operation map \( \mathcal{G} \times_T \mathcal{G} \to \mathcal{G} \) is given by
\[
\gamma(g, h(x)) \gamma(h, x) = \gamma(gh, x) ,
\]
and the inversion map is defined by
\[
\gamma(h, x)^{-1} = \gamma(h^{-1}, h(x)) ;
\]
thus the operation and inversion of \( \mathcal{G} \) are induced by the composition and inversion of maps in \( \mathcal{H} \).
For \( x, y \in T \), let us use the notation \( \mathcal{G}_x = s^{-1}(x) \), \( \mathcal{G}^y = t^{-1}(y) \) and \( \mathcal{G}^y_x = \mathcal{G}_x \cap \mathcal{G}^y \); in particular, the group \( \mathcal{G}_x \) will be called the germ group of \( \mathcal{H} \) at \( x \). Points in the same \( \mathcal{H} \)-orbit have isomorphic germ groups (if \( y \in \mathcal{H}(x) \), an isomorphism \( \mathcal{G}^y_x \to \mathcal{G}_x \) is given by conjugation with any element in \( \mathcal{G}^y_x \); hence the germ groups of the orbits make sense up to isomorphism. Under pseudogroup equivalences, corresponding orbits have isomorphic germ groups. The set \( \mathcal{G}_x \) will be called the germ cover of the orbit \( \mathcal{H}(x) \) with base point \( x \). The target map restricts to a surjective map \( \mathcal{G}_x \to \mathcal{H}(x) \) whose fibers are bijective to \( \mathcal{G}_x \) (if \( y \in \mathcal{H}(x) \), a bijection \( \mathcal{G}_x \to \mathcal{G}^y_x \) is given by left product with any element in \( \mathcal{G}^y_x \)); thus \( \mathcal{G}_x \) is finite if and only if both \( \mathcal{G}_x \) and \( \mathcal{H}(x) \) are finite. Moreover germ covers based on points in the same orbit are also bijective (if \( y \in \mathcal{H}(x) \), a bijection \( \mathcal{G}_y \to \mathcal{G}_x \) is given by right product with any element in \( \mathcal{G}^y_x \)); therefore the germ covers of the orbits make sense up to bijections.

**Definition 2.22.** It is said that \( \mathcal{H} \) is

- **locally free** if all of its germ groups are trivial (for all \( h \in \mathcal{H} \) and \( x \in \text{dom} \ h \) such that \( h(x) = x \), we have \( \gamma(h, x) = \gamma(\text{id}_T, x) \)); and
- **strongly locally free** if \( \mathcal{H} \) is generated by a sub-pseudo\( L \)-group \( S \subset \mathcal{H} \) such that, for all \( h \in S \) and \( x \in \text{dom} \ h \), if \( h(x) = x \) then \( h = \text{id}_{\text{dom} h} \).

**Remark 13.** The condition of being (strongly) locally free is stronger than the condition of being (strongly) quasi-analytic. If \( \mathcal{H} \) is locally free and satisfies the condition of strong quasi-analiticity with a sub-pseudo\( L \)-group \( S \subset \mathcal{H} \), generating \( \mathcal{H} \), then \( \mathcal{H} \) also satisfies the condition of being strongly locally free with \( S \).

**Remark 14.** If the condition on \( \mathcal{H} \) to be strongly locally free is satisfied with a sub-pseudo\( L \)-group \( S \), then it is also satisfied with the localization of \( S \).

The best-known topology on \( \mathcal{G} \) is the sheaf topology, but we will not use it. It has a basis given by the sets \( \{ \gamma(h, x) \mid x \in \text{dom} \ h \} \) for \( h \in \mathcal{H} \). Equipped with the sheaf topology, \( \mathcal{G} \) becomes an étalé groupoid.

Another topology on \( \mathcal{G} \) can be defined as follows. Suppose that \( \mathcal{H} \) is generated by some sub-pseudo\( L \)-group \( S \subset \mathcal{H} \). The set \( S \times T = (\mathcal{H} \times T) \cap (S \times T) \) is open in \( S_{(b)c-o} \times T \) by Proposition 2.4. It will be denoted by \( S_{(b)c-o} \times T \). The induced quotient topology on \( \mathcal{G} \), via the germ map \( \gamma : S_{(b)c-o} \times T \to \mathcal{G} \), will be also called the (bi-)compact-open topology. The corresponding space will be denoted by \( \mathcal{G}_{S_{(b)c-o}} \) or by \( \mathcal{G}_{S_{(b)c-o}} \) if reference to \( S \) is needed. It follows from Proposition 2.6 that \( \mathcal{G}_{S_{(b)c-o}} \) is a topological groupoid if \( T \) is locally compact. We get a commutative diagram

\[
\begin{array}{ccc}
S_{(b)c-o} \times T & \xrightarrow{\text{inclusion}} & \mathcal{H}_{(b)c-o} \times T \\
\gamma \downarrow & & \downarrow \gamma \\
\mathcal{G}_{S_{(b)c-o}} & \xrightarrow{\text{identity}} & \mathcal{G}_{\mathcal{H}_{(b)c-o}}
\end{array}
\]
where the top map is an embedding and the vertical maps are identifications. Hence the identity map $\mathcal{G}_{S,(b)-c-o} \to \mathcal{G}_{H,(b)-c-o}$ is continuous. Similarly, the identity map $\mathcal{G}_{S,b-c-o} \to \mathcal{G}_{S,c-o}$ is continuous.

**Question 2.23.** When is the identity map $\mathcal{G}_{S,(b)-c-o} \to \mathcal{G}_{H,(b)-c-o}$ a homeomorphism?

**Question 2.24.** When is the identity map $\mathcal{G}_{S,b-c-o} \to \mathcal{G}_{S,c-o}$ a homeomorphism?

Assuming some conditions on $\mathcal{H}$, an affirmative answer to Question 2.24 will be given in Section 3.2.

### 2.4. Local groups and local actions

Let us recall some notions from [29].

**Definition 2.25** (See e.g. [29]). A local group is a quintuple $G \equiv (G,e,\cdot,\cdot',\mathfrak{D})$ satisfying the following conditions:

1. $(G,\mathfrak{D})$ is a topological space.
2. $\cdot$ is a function from a subset of $G \times G$ to $G$.
3. $\cdot'$ is a function from a subset of $G$ to $G$.
4. There is a subset $O$ of $G$ such that
   - $O$ is an open neighborhood of $e$ in $G$;
   - $O \times O$ is a subset of the domain of $\cdot$;
   - $O$ is a subset of the domain of $\cdot'$;
   - for all $a,b,c \in O$, if $a \cdot b, b \cdot c \in O$, then $(a \cdot b) \cdot c = (a \cdot b) \cdot c$;
   - for all $a \in O$, $a' \in O$, $a \cdot e = e \cdot a = a$ and $a' \cdot a = a \cdot a' = e$;
   - the map $\cdot : O \times O \to G$ is continuous; and
   - the map $\cdot' : O \to G$ is continuous.
5. The set $\{e\}$ is closed in $G$.

It is a usual convention that asserting that a local group satisfies some topological property means that the property is satisfied on some open neighborhood of $e$.

A local homomorphism of a local group $G$ to a local group $H$ is a continuous partial map $\phi : G \to H$, whose domain is an identity neighborhood in $G$, which is compatible in the usual sense with the identity elements, the operations and inversions of $G$ and $H$. If moreover $\phi$ restricts to a homeomorphism between some identity neighborhoods in $G$ and $H$, then it is called a local isomorphism, and $G$ and $H$ are said to be locally isomorphic. A local group locally isomorphic to a Lie group is called a local Lie group.

The collection of all sets $O$ satisfying condition (4) will be denoted by $\Psi_G$. This is a neighborhood basis of $e$ in $G$; all of these neighborhoods are symmetric with respect to the inverse operation (3). Let $\Phi(G,n)$ denote the collection of subsets $A$ of $G$ such that the product of any collection of at most $n$ elements of $A$ is defined, and the set $A^n$ of such products is contained in some $O \in \Psi_G$. 
Let $H \subset G$. It is said that $H$ is a subgroup of $G$ if $H \in \Phi(G, 2)$, $e \in H$, $H' = H$ and $H^2 = H$; and $H$ is a sub-local group of $G$ if $H$ is itself a local group with respect to the induced operations and topology.

Let $\Upsilon G$ denote the set of all pairs $(H, V)$ of subsets of $G$ so that $e \in H$, $V \in \Psi G$, $a \cdot b \in H$ for all $a, b \in V \cap H$, and $d' \in H$ for all $c \in V \cap H$. Then a subset $H \subset G$ is a sub-local group if and only if there exists some $V$ such that $(H, V) \in \Upsilon G$ [29, Theorem 26].

Let $\Pi G$ denote the family of pairs $(H, V)$ of subsets of $G$ so that $e \in H$, $V \in \Psi G \cap \Phi(G, 6)$, $a \cdot b \in H$ for all $a, b \in V^6 \cap H$, $d' \in H$ for all $c \in V^6 \cap H$, and $V^2 \setminus H$ is open. Given $(H, V) \in \Pi G$, there is a (completely regular, Hausdorff) space $G/(V, H)$ and a continuous open surjection $T : V^2 \to G/(V, H)$ such that $T(a) = T(b)$ if and only if $a' \cdot b \in H$ (cf. [29, Theorem 29]). For another pair in $\Pi G$ of the form $(H, W)$, the spaces $G/(H, V)$ and $G/(H, W)$ are locally homeomorphic at the identity class. Thus the concept of coset space of $H$ is well defined in this sense, as “a germ of a topological space”. The notation $G/H$ may be used in this sense. It will be said that $G/H$ has certain topological property when some $G/(H, V)$ has that property around $T(e)$.

The concept of normal sub-local group and the corresponding quotient local group has also been studied [29, Theorem 35], but we will not use it.

As usual, $a \cdot b$ and $a'$ will be denoted by $ab$ and $a^{-1}$.

Local groups were first studied by Jacoby [29], giving local versions of important theorems for topological groups. For instance, Jacoby characterized local Lie groups as the locally compact local groups without small subgroup [29, Theorem 96]. Also, any finite dimensional metrizable locally compact local group is locally isomorphic to the direct product of a Lie group and a compact zero-dimensional topological group [29, Theorem 107]. In particular, this property shows that any locally Euclidean local group is a local Lie group, which is an affirmative answer to a local version of Hilbert’s 5th problem. However, as pointed out by Plaut in [32], Jacoby failed to recognize the following subtlety: in local groups, “local associativity” (for three elements) does not imply “global associativity” (for any finite sequence of elements). In fact, Olver [39] gave examples of connected local Lie groups that are not globally associative. Thus the proof of Jacoby is incorrect. Fortunately, a completely new proof of the local Hilbert’s 5th problem has been given by Goldbring [21]. Moreover van den Dries and Goldbring [16, 17] have proved that any locally compact local group is locally isomorphic to a topological group (it is “globalizable”), and therefore all other theorems for locally compact local groups, proved by Jacoby in [29], hold as well because they are known for locally compact topological groups [38]. We will use the following one.

---

2A local group is said to have no small subgroups when some neighborhood of the identity element contains no nontrivial subgroup.
Theorem 2.26 (Jacoby [29, Theorems 97–103]; correction by van den Dries–Goldbring [16, 17]). Any locally compact second countable local group $G$ can be approximated by local Lie groups. More precisely, given $W \in \Psi G \cap \Phi(G, 2)$, there exists $V \in \Psi G$ with $V \subset W$ and there exists a sequence of compact normal subgroups $F_n \subset V$ such that $F_{n+1} \subset F_n$, $\bigcap_n F_n = \{e\}$, $(F_n, V) \in \Delta G$, and $G/(F_n, V)$ is a local Lie group.

Definition 2.27. A local action of a local group $G$ on a space $X$ is a paro map $G \times X \mapsto X$, $(g, x) \mapsto gx$, defined on some open neighborhood of $\{e\} \times X$, such that $ex = x$ for all $x \in X$, and $g_1(g_2x) = (g_1g_2)x$, provided both sides are defined.

Remark 15. The local transformations given by any local action of a local group on a space generate a pseudogroup.

A local action of a local group $G$ on a space $X$ is called locally transitive at some point $x \in X$ if there is an identity neighborhood $W$ in $G$ such that the local action is defined on $W \times \{x\}$, and $Wx := \{gx \mid g \in W\}$ is a neighborhood of $x$ in $X$. Given another local action of $G$ on a space $Y$, a paro map $\phi : X \mapsto Y$ is called equivariant is $\phi(gx) = \phi(x)$ for all $x \in X$ and $g \in G$, provided both sides are defined.

Example 2.28. The typical example of local action is the following. Let $H$ be a sub-local group of $G$. If $(H, V) \in \Pi G$ and $T : V^2 \rightarrow G/(H, V)$ is the natural projection, then the map $V \times G/(H, V) \rightarrow G/(H, V)$, $(v, T(g)) \mapsto T(vg)$, defines a local action of $G$ on $G/(H, V)$.

Remark 16. If $G$ is a local group locally acting on $X$ and the local action is locally transitive at $x \in X$, then there is a sub-local group $H$ of $G$ such that $(H, V) \in \Pi G$ for some $V$ and the orbit paro map $G \mapsto X$, $g \mapsto gx$, induces an equivariant paro map $G/(H, V) \mapsto X$, which restricts to homeomorphism between neighborhoods of $e$ and $x$.

2.5. Equicontinuous pseudogroups. Álvarez and Candel introduced the following structure to define equicontinuity for pseudogroups [5]. Let $\{T_i, d_i\}$ be a family of metric spaces such that $\{T_i\}$ is a covering of a set $T$, each intersection $T_i \cap T_j$ is open in $(T_i, d_i)$ and $(T_j, d_j)$, and, for all $\epsilon > 0$, there is some $\delta(\epsilon) > 0$ so that the following property holds: for all $i, j$ and $z \in T_i \cap T_j$, there is some open neighborhood $U_{i,j,z}$ of $z$ in $T_i \cap T_j$ (with respect to the topology induced by $d_i$ and $d_j$) such that

$$d_i(x, y) < \delta(\epsilon) \implies d_j(x, y) < \epsilon$$

for all $\epsilon > 0$ and all $x, y \in U_{i,j,z}$. Such a family is called a cover of $T$ by quasi-locally equal metric spaces. Two such families are called quasi-locally equal when their union is also a cover of $T$ by quasi-locally equal metric spaces. This is an equivalence relation whose equivalence classes are called quasi-local metrics on $T$. For each quasi-local metric $\Omega$ on $T$, the pair $(T, \Omega)$

3The notation will be simplified by using, for instance, $\{T_i, d_i\}$ instead of $\{(T_i, d_i)\}$.
is called a quasi-local metric space. Such a \( \mathcal{Q} \) induces a topology on \( T \) so that, for each \( \{T_i, d_i\}_{i \in I} \in \mathcal{Q} \), the family of open balls of all metric spaces \( (T_i, d_i) \) form a basis of open sets. Any topological concept or property of \( (T, \mathcal{Q}) \) refers to this underlying topology. A quasi-local metric space \( (T, \mathcal{Q}) \) is a locally compact Polish space if and only if it is Hausdorff, paracompact and separable \([5]\).

**Definition 2.29** (Álvarez-Candel \([5]\)). Let \( \mathcal{H} \) be a pseudogroup on a quasi-local metric space \( (T, \mathcal{Q}) \). Then \( \mathcal{H} \) is said to be equicontinuous if there exists some \( \{T_i, d_i\}_{i \in I} \in \mathcal{Q} \) and some sub-pseudo*group \( S \subset \mathcal{H} \), generating \( \mathcal{H} \), such that, for every \( \epsilon > 0 \), there is some \( \delta(\epsilon) > 0 \) so that

\[
d_i(x, y) < \delta(\epsilon) \implies d_j(h(x), h(y)) < \epsilon
\]

for all \( h \in S \), \( i, j \in I \) and \( x, y \in T_i \cap h^{-1}(T_j \cap \text{im} h) \).

**Remark 17.** The original term of \([5]\) is “strongly equicontinuous”. We use here the simpler term “equicontinuous” because the weak equicontinuity of \([5]\) is not considered.

**Remark 18.** If the condition on \( \mathcal{H} \) to be equicontinuous is satisfied with a sub-pseudo*group \( S \), then it is also satisfied with the localization of \( S \).

**Lemma 2.30** (Álvarez-Candel \([5]\) Lemma 8.8)). Let \( \mathcal{H} \) and \( \mathcal{H}' \) be equivalent pseudogroups on locally compact Polish spaces. Then \( \mathcal{H} \) is equicontinuous if and only if \( \mathcal{H}' \) is equicontinuous.

**Proposition 2.31** (Álvarez-Candel \([5]\) Proposition 8.9)). Let \( \mathcal{H} \) be a compactly generated and equicontinuous pseudogroup on a locally compact Polish quasi-local metric space \( (T, \mathcal{Q}) \), and let \( U \) be any relatively compact open sub-

set of \( (T, \mathcal{Q}) \) that meets every \( \mathcal{H} \)-orbit. Suppose that \( \{T_i, d_i\}_{i \in I} \in \mathcal{Q} \) satisfies the condition of equicontinuity. Let \( E \) be any system of compact generation of \( \mathcal{H} \) on \( U \), and let \( \tilde{g} \) be an extension of each \( g \in E \) with \( \text{dom} \tilde{g} \subset \text{dom} g \).

Also, let \( \{T'_i\}_{i \in I} \) be any shrinking\(^4\) of \( \{T_i\}_{i \in I} \). Then there is a finite family \( \mathcal{V} \) of open subsets of \( (T, \mathcal{Q}) \) whose union contains \( U \) and such that, for any \( V \in \mathcal{V}, x \in U \cap V \), and \( h \in \mathcal{H} \) with \( x \in \text{dom} h \) and \( h(x) \in U \), the domain of \( \tilde{h} = \tilde{g}_n \cdots \tilde{g}_1 \) contains \( V \) for any composite \( \tilde{h} = g_n \cdots g_1 \) defined around \( x \) with \( g_1, \ldots, g_n \in E \), and moreover \( V \subset T'_{i_0} \) and \( \tilde{h}(V) \subset T'_{i_1} \) for some \( i_0, i_1 \in I \).

**Remark 19.** The statement of Proposition 2.31 is stronger than the completeness of \( \mathcal{H}|_U \). Since we can choose \( U \) large enough to contain two arbitrarily given points of \( T \), it follows \( \mathcal{H} \) is complete.

\(^4\)Recall that a shrinking of an open cover \( \{U_i\} \) of a space \( X \) is an open cover \( \{U'_i\} \) of \( X \), with the same index set, such that \( U'_i \subset U_i \) for all \( i \). On the other hand, if \( \{U_i\} \) is a cover of a subset \( A \subset X \) by open subsets of \( X \), a shrinking of \( \{U_i\} \), as cover of \( A \) by open subsets of \( X \), is a cover \( \{U'_i\} \) of \( A \) by open subsets of \( X \), with the same index set, such that \( U'_i \subset U_i \) for all \( i \).
Proposition 2.32 (Álvarez-Candel [5, Proposition 9.9]). Let \( H \) be a compactly generated, equicontinuous and strongly quasi-analytic pseudogroup on a locally compact Polish space \( T \). Suppose that the conditions of equicontinuity and strong quasi-analyticity are satisfied with a sub-pseudogroup \( S \subset H \), generating \( H \). Let \( A, B \) be open subsets of \( T \) such that \( A \) is compact and contained in \( B \). If \( x \) and \( y \) are close enough points in \( T \), then
\[
f(x) \in A \Rightarrow f(y) \in B
\]
for all \( f \in S \) whose domain contains \( x \) and \( y \).

Theorem 2.33 (Álvarez-Candel [5, Theorem 11.11]). Let \( H \) be a compactly generated and equicontinuous pseudogroup on a locally compact Polish space \( T \). If \( H \) is transitive, then \( H \) is minimal.

Theorem 2.33 can be restated by saying that the orbit closures form a partition of the space. The following result states that indeed the orbit closures are orbits of a pseudogroup if strong quasi-analyticity is also assumed.

Theorem 2.34 (Álvarez-Candel [5, Theorem 12.1]). Let \( H \) be a strongly quasi-analytic, compactly generated and equicontinuous pseudogroup on a locally compact Polish space \( T \). Let \( S \subset H \) be a sub-pseudogroup generating \( H \) and satisfying the conditions of equicontinuity and strong quasi-analyticity. Let \( \tilde{H} \) be the set of maps \( h \) between open subsets of \( T \) that satisfy the following property: for every \( x \in \text{dom} \ h \), there exists a neighborhood \( O_x \) of \( x \) in \( \text{dom} \ h \) so that the restriction \( h|_{O_x} \) is in the closure of \( C(O_x, T) \cap S \) in \( C_{c-o}(O_x, T) \). Then:

(i) \( \tilde{H} \) is closed under composition, combination and restriction to open sets;

(ii) every map in \( \tilde{H} \) is a homeomorphism around every point of its domain;

(iii) \( \overline{H} = \tilde{H} \cap \text{Loct}(T) \) is a pseudogroup that contains \( H \);

(iv) \( \overline{H} \) is equicontinuous;

(v) the orbits of \( \overline{H} \) are equal to the closures of the orbits of \( H \); and

(vi) \( H \) and \( \overline{H} \) are independent of the choice of \( S \).

Remark 20. In Theorem 2.34 let \( S \) be the set of local transformations that are in the union of the closures of \( C(O, T) \cap S \) in \( C_{c-o}(O, T) \) with \( O \) running on the open sets of \( T \). According to the proof of [5, Theorem 12.1], \( S \) is a pseudo\(^*\)group that generates \( \overline{H} \). Moreover, if \( H \) satisfies the equicontinuity condition with \( S \) and some representative \( \{T_i, d_i\} \) of a quasi-local metric, then \( \overline{H} \) satisfies the equicontinuity condition with \( S \) and \( \{T_i, d_i\} \).

Remark 21. From the proof of [5, Theorem 12.1], it also follows easily that the pseudo\(^*\)group \( S \), defined in Remark 20, satisfies the following property. Any \( x \in U \) has a neighborhood \( O \) in \( T \) such that the closure of
\[
\{ \ h \in C(O, T) \cap S \mid h(O) \cap \overline{U} \neq \emptyset \ \}
\]
in \( C_{c-o}(O, T) \) is contained in \( \text{Loct}(T) \), and therefore in \( S \).
Example 2.35. Let $G$ a locally compact Polish local group with a left invariant metric, let $\Gamma \subset G$ be a dense local subgroup, and let $\mathcal{H}$ be the minimal pseudogroup generated by the local action of $\Gamma$ by local left translations on $G$. The local left and right translations in $G$ by each $g \in G$ will be denoted by $L_g$ and $R_g$. The restrictions of the local left translations $L_\gamma \ (\gamma \in \Gamma)$ to open subsets of their domains form a sub-pseudo-group $S \subset \mathcal{H}$ that generates $\mathcal{H}$. Obviously, $\mathcal{H}$ satisfies with $S$ the condition of being strongly locally free, and therefore strongly quasi-analytic. Moreover by considering any left invariant metric on $G$, the restrictions of local translations $L_g \ (g \in G)$ generates an equivalence $\mathcal{H} \to \mathcal{H}$.

Now suppose that $\mathcal{H}$ is compactly generated. Then the closure $\overline{\mathcal{H}}$ is generated by the local action of $G$ on itself by local left translations. The sub-pseudo-group $\overline{S} \subset \overline{\mathcal{H}}$ consists of the restrictions of the local left translations $L_g \ (g \in G)$ to open subsets of their domains. Observe that $\overline{\mathcal{H}}$ satisfies the condition of being strongly locally free, and therefore strongly quasi-analytic, with $\overline{S}$.

Lemma 2.36. Let $G$ and $G'$ be locally compact Polish local groups with left invariant metrics, let $\Gamma \subset G$ and $\Gamma' \subset G'$ be dense local subgroups, and let $\mathcal{H}$ and $\mathcal{H}'$ be the pseudogroups generated by the local actions of $\Gamma$ and $\Gamma'$ by local left translations on $G$ and $G'$. Suppose that $\mathcal{H}$ and $\mathcal{H}'$ are compactly generated. Then $\mathcal{H}$ and $\mathcal{H}'$ are equivalent if and only if $G$ is locally isomorphic to $G'$.

Proof. Consider the notation and observations of Example 2.35 for both $G$ and $G'$; in particular, $S \subset \mathcal{H}$ and $S' \subset \mathcal{H}'$ denote the sub-pseudo-groups of restrictions of local left translations $L_\gamma$ and $L_{\gamma'} \ (\gamma \in \Gamma$ and $\gamma' \in \Gamma')$ to open subsets of their domains. Let $e$ and $e'$ denote the identity elements of $G$ and $G'$. Let $\Phi : \mathcal{H} \to \mathcal{H}'$ be an equivalence. Since $\mathcal{H}'$ is minimal, after composing $\Phi$ with the equivalence generated by some local right translation in $G$ if necessary, we can assume that there is some $\phi \in \Phi$ with $e \in \text{dom } \phi$ and $\phi(e) = e'$.

Let $U$ be a relatively compact open symmetric identity neighborhood in $G$ with $\overline{U} \subset \text{dom } \phi$. Let $\{f_1, \ldots, f_n\}$ be a symmetric system of compact generation of $\mathcal{H}$ on $U$. Thus each $f_i$ has an extension $\tilde{f}_i \in \mathcal{H}$ so that $\text{dom } \tilde{f}_i \subset \text{dom } f_i \subset \text{dom } \phi$.

Claim 1. We can assume that $\tilde{f}_i \in S$ and $\phi \tilde{f}_i \phi^{-1} \in S'$ for all $i$.

Each point in $\text{dom } \tilde{f}_i \cap \text{dom } \phi$ has an open neighborhood $O$ such that $O \subset \text{dom } \tilde{f}_i$, $\tilde{f}_i|_O \in S$ and $\phi \tilde{f}_i \phi^{-1}|_{\phi(O)} \in S'$. Take a finite covering $\{O_{ij}\}$ ($j \in \{1, \ldots, k_i\}$) of the compact set $\text{dom } \tilde{f}_i$ by sets of this type. Let $\{P_{ij}\}$ be a shrinking of $\{O_{ij}\}$, as cover of $\text{dom } \tilde{f}_i$ by open subsets of $\text{dom } \tilde{f}_i$. Then the restrictions $g_{ij} = f_i|_{P_{ij}\cap U} \ (i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, k_i\}$) generate $\mathcal{H}|_U$, each $\tilde{g}_{ij} = \tilde{f}_i|_{O_{ij}}$ is in $S$ and extends $g_{ij}$, $\text{dom } \tilde{g}_{ij} \subset \text{dom } \tilde{g}_{ij}$, and $\phi \tilde{g}_{ij} \phi^{-1} \in S'$, showing Claim 1.
According to Claim \([1]\) the maps \(f'_i = \phi f_i \phi^{-1}\) form a symmetric system of compact generation of \(\mathcal{H}'\) on \(U' = \phi(U)\), which can be checked with the extensions \(\tilde{f}'_i = \tilde{\phi} f_i \tilde{\phi}^{-1}\). Let \(S_0 \subset S\) and \(S'_0 \subset S'\) be the sub-pseudo-groups consisting of the restrictions of compositions of maps \(f_i\) and \(f'_i\) to open subsets of their domains, respectively. They generate \(\mathcal{H}\) and \(\mathcal{H}'\). It follows from Claim \([1]\) that \(\phi f \phi^{-1} \in S'\) for all \(f \in S_0\). On the other hand, by Proposition \([2.31]\) there is a smaller open identity neighborhood, \(V \subset U\), such that, for all \(h \in \mathcal{H}\) and \(x \in V \cap \text{dom } h\) with \(h(x) \in U\), there is some \(f \in S_0\) such that \(\text{dom } f = V\) and \(\gamma(f, x) = \gamma(h, x)\).

Let \(W\) be another symmetric open identity neighborhood such that \(W^2 \subset V\). Let us show that \(\phi : W \to \phi(W)\) is a local isomorphism. Let \(\gamma \in W \cap \Gamma\). The restriction \(L_{\gamma} : W \to \gamma W\) is well defined and belongs to \(S\). Hence there is some \(f \in S_0\) so that \(\text{dom } f = V\) and \(\gamma(f, e) = \gamma(L_{\gamma}, e)\). Since \(f\) is also a restriction of a local left translation in \(G\), it follows that \(f = L_{\gamma}\) on \(W\). So \(\phi L_{\gamma} \phi^{-1}(\phi(W)) \in S'\); i.e., there is some \(\gamma' \in \gamma'\) such that \(\phi L_{\gamma} \phi^{-1} = L_{\gamma'}\) on \(\phi(W)\). In fact,

\[
\phi(\gamma) = \phi L_{\gamma}(e) = \phi L_{\gamma} \phi^{-1}(e') = L_{\gamma'}(e') = \gamma'.
\]

Hence, for all \(\gamma, \delta \in \Gamma\),

\[
\phi(\gamma \delta) = \phi L_{\gamma}(\delta) = L_{\phi(\gamma)} \phi(\delta) = \phi(\gamma) \phi(\delta),
\]

\[
\phi(\gamma)^{-1} = L_{\phi(\gamma)}^{-1}(e') = (\phi L_{\gamma} \phi^{-1})^{-1}(e')
= \phi L_{\gamma}^{-1} \phi^{-1}(e') = L_{\phi(\gamma)^{-1}}(e') = \phi(\gamma)^{-1}.
\]

Since \(\phi\), and the product and inversion maps are continuous, it follows that \(\phi(gh) = \phi(g) \phi(h)\) and \(\phi(g^{-1}) = \phi(g)^{-1}\) for all \(g, h \in W\).

\[\text{Example 2.37.}\] This is a generalization of Example \([2.35]\). Let \(G\) be a locally compact Polish local group with a left-invariant metric, \(K \subset G\) a compact subgroup, and \(\Gamma \subset G\) a dense sub-local group. Take some \(V\) so that \((H, V) \in \Pi(G)\). The left invariant metric on \(G\) can be assumed to be also \(K\)-right invariant by the compactness of \(K\), and therefore it defines a metric on \(G/(K, V)\). Then the canonical local action of \(\Gamma\) on some neighborhood of the identity class in \(G/(K, V)\) induces a transitive equicontinuous pseudogroup \(\mathcal{H}\) on a locally compact Polish space; in fact, this is a pseudogroup of local isometries.

Assume that \(\mathcal{H}\) is compactly generated. Then \(\overline{\mathcal{H}}\) is generated by the canonical local action of \(G\) on some neighborhood of the identity class in \(G/(K, V)\). Moreover the sub-pseudo-group \(\overline{\mathcal{S}} \subset \overline{\mathcal{H}}\) consists of the local translations of the local action of \(G\) on \(G/(K, V)\).

\[\text{Examples} \,[2.35]\, \text{and} \, [2.37] \, \text{are particular cases of pseudogroups induced by local actions (Remark [15]) that will play an important role in our theory. For instance, the following result indicates the relevance of Example [2.37].}\]

\[\text{Theorem 2.38 (\text{Álvarez-Candel [6, Theorem 5.2]}). Let } \mathcal{H} \text{ be a transitive, compactly generated and equicontinuous pseudogroup on a locally compact} \]
Polish space, and suppose that $\mathcal{H}$ is strongly quasi-analytic. Then $\mathcal{H}$ is equivalent to a pseudogroup of the type described in Example 2.37.

Remark 22. From the proof of [6, Theorems 3.3 and 5.2], it also follows that, in Theorem 2.38 if moreover $\mathcal{H}$ is strongly locally free, then $\mathcal{H}$ is equivalent to a pseudogroup of the type described in Example 2.35.

3. Molino’s theory for equicontinuous pseudogroups

3.1. Conditions on $H$. Let $T$ be a locally compact Polish space. Let $H$ be a pseudogroup of local transformations of $T$. Suppose that $H$ is compactly generated, complete and equicontinuous, and that $H$ is also strongly quasi-analytic.

Let $U$ be a relatively compact open set in $T$ that meets all the orbits of $H$. The condition of compact generation is satisfied with $U$. Consider a representative $\{T_i, d_i\}$ of a quasi-local metric on $T$ satisfying the condition of equicontinuity of $H$ with some sub-pseudo$\ast$ group $S \subset H$ that generates $H$. We can also suppose that $S$ satisfies the condition of strong quasi-analyticity of $H$.

Remark 23. Consider the sub-pseudo$\ast$ group $S \subset H$ defined in Remark 20 (page 21). According to Theorem 2.34 and Remark 20, there is a mapping $\epsilon \mapsto \delta(\epsilon) > 0$, for $\epsilon > 0$, such that

$$d_i(x, y) < \delta(\epsilon) \implies d_j(h(x), h(y)) < \epsilon$$

for all indices $i$ and $j$, every $h \in S$, and $x, y \in T_i \cap h^{-1}(T_j \cap \text{im} h)$.

Remark 24. By Remark 21 and refining $\{T_i\}$ if necessary, we can assume that $U$ is covered by a finite collection of the sets $T_i$, $\{T_{i_1}, \ldots, T_{i_r}\}$, such that the closure of

$$\{ h \in C(T_{i_k}, T) \cap S \mid h(T_{i_k}) \cap U \neq \emptyset \}$$

in $C_{c-o}(T_{i_k}, T)$ is contained in $S$ for all $k \in \{1, \ldots, r\}$.

Remark 25. By Proposition 2.31 and Remark 24 and refining $\{T_i\}$ if necessary, we can assume that, for all $h \in H$ and $x \in T_{i_k} \cap U \cap \text{dom} h$ with $h(x) \in U$, there is some $\tilde{h} \in S$ with $\text{dom} \tilde{h} = T_{i_k}$ and $\gamma(h, x) = \gamma(\tilde{h}, x)$.

Remark 26. By Remarks 5, 10 and 18, and refining $\{T_i\}$ if necessary, we can assume that the strong quasi-analyticity of $H$ is satisfied with $S$.

3.2. Coincidence of topologies. The proof of the following result is inspired by [15].

Proposition 3.1. $\mathcal{F}_{b-c-o} = \mathcal{F}_{c-o}$.

Proof. For each $g \in \mathcal{S}$, take any index $i$ and open sets $V, W \subset T$ such that $V \subset W$ and $W \subset \text{im} g$. By Proposition 2.32 there is some $\epsilon(i, V, W) > 0$ such that, for all $x, y \in T_i$, if $d_i(x, y) < \epsilon(i, V, W)$, then

$$f(x) \in V \implies f(y) \in W$$
Lemma 3.3. Observe that \( \hat{K} \neq \emptyset \), \( \text{diam}_{d_i}(K) < \epsilon(i, V, W) \), \( g(K) \subset V \),

where \( \hat{K} \) and \( \text{diam}_{d_i}(K) \) denote the interior and \( d_i \)-diameter of \( K \). Moreover \( \mathcal{K}(g) \) denote the union of the families \( \mathcal{K}(g, i, V, W) \) as above. Then a subbasis \( \mathcal{N}(g) \) of open neighborhoods of each \( g \) in \( \mathcal{S}_{^{\text{c-o}}} \) is given by the sets \( \mathcal{N}(K, O) \cap \mathcal{S} \), where \( K \in \mathcal{K}(g) \) and \( O \) is an open subset of \( T \) such that \( g(K) \subset O \).

We have to prove the continuity of the inversion map \( \mathcal{S}_{^{\text{c-o}}} \to \mathcal{S}_{^{\text{c-o}}}, h \mapsto h^{-1} \). Let \( h \in \mathcal{S} \) and let \( \mathcal{N}(K, O) \subset \mathcal{N}(h^{-1}) \) with \( K \in \mathcal{K}(h^{-1}, i, V, W) \), and fix any point \( x \in \hat{K} \). Then

\[
\mathcal{V} = \mathcal{N}((h^{-1}(x)), \hat{K}) \cap \mathcal{N}(\mathcal{W} \setminus O, T \setminus K)
\]

is an open neighborhood of \( h \) in \( \mathcal{H}_{^{\text{c-o}}} \). For any \( f \in \mathcal{V} \cap \mathcal{S} \) and \( y \in K \), we have \( d_i(fh^{-1}(x), y) < \epsilon(i, V, W) \) because \( fh^{-1}(x) \in \hat{K} \) and \( \text{diam}_{d_i}(K) < \epsilon(i, V, W) \). So \( f^{-1}(y) \in W \) by the definition of \( \epsilon(i, V, W) \) since \( f^{-1} \in \mathcal{S} \) and \( h^{-1}(x) \in h^{-1}(K) \subset V \). Therefore, if \( f^{-1}(y) \notin O \), we get \( f^{-1}(y) \notin \mathcal{W} \setminus O \), obtaining \( y \in T \setminus K \), which is a contradiction. Hence \( f^{-1} \in \mathcal{N}(K, O) \) for all \( f \in \mathcal{V} \cap \mathcal{S} \). \( \square \)

Let \( \mathcal{S} \) denote the groupoid of germs of \( \mathcal{H} \). The following direct consequence of Proposition 3.1 gives a partial answer to Question 2.24.

Corollary 3.2. \( \mathcal{S}_{^{\text{c-o}}} = \mathcal{S}_{^{\text{c-o}}} \).

Thus \( \mathcal{S}_{^{\text{c-o}}} \) is a topological groupoid by Corollary 3.2.

3.3. The space \( \mathcal{H} \). Recall that \( s, t : \mathcal{S}_{^{\text{c-o}}} \to T \) denote the source and target projections. Let \( \mathcal{H} = \mathcal{S}_{^{\text{c-o}}} \), where the following subsets are open:

\[
\mathcal{H}_U = s^{-1}(U) \cap t^{-1}(U),
\]

\[
\mathcal{H}_{k,l} = s^{-1}(T_{ik,li}) \cap t^{-1}(T_{ik,li}),
\]

\[
\mathcal{H}_{U,k,l} = \mathcal{H}_U \cap \mathcal{H}_{k,l}.
\]

Observe that \( \mathcal{H}_U \) is an open subspace of \( \mathcal{H} \), and the family of sets \( \mathcal{H}_{U,k,l} \) form an open covering of \( \mathcal{H}_U \).

Let \( \gamma(h, x) \in \mathcal{H}_{U,k,l} \). We can assume that \( h \in \mathcal{S} \) and \( \text{dom } h = T_{ik} \) according to Remark 2.25. Since \( x \in T_{ik} \cap U \) and \( h(x) \in T_{ik} \cap U \), there are relatively compact open neighborhoods \( \mathcal{V} \) of \( x \) and \( W \) of \( h(x) \), such that \( \mathcal{V} \subset T_{ik} \cap U \), \( \mathcal{W} \subset T_{ik} \cap U \) and \( h(V) \subset W \).

By Remark 2.25 for each \( x \in \mathcal{S} \) with \( x \in \text{dom } f \), there is some \( \tilde{f} \in \mathcal{S} \) with \( \text{dom } \tilde{f} = T_{ik} \) and \( \gamma(\tilde{f}, x) = \gamma(f, x) \).

Lemma 3.3. \( f = \tilde{f} \) on \( V \).
Proof. The composition \( f|_V \tilde{f}^{-1} \) is defined on \( \tilde{f}(V) \), belongs to \( \mathcal{S} \), and is the identity on some neighborhood of \( \tilde{f}(x) = f(x) \). So \( f|_V \tilde{f}^{-1} \) is the identity on \( \tilde{f}(V) \) because \( \mathcal{H} \) satisfies the strong quasi-analyticity condition with \( \mathcal{S} \). Hence \( f = \tilde{f} \) on \( V \). \( \square \)

Let

\[
\mathcal{S}_0 = \{ f \in \mathcal{S} \mid V \subset \text{dom} f, \, f(V) \subset W \},
\]

(3)

\[
\mathcal{S}_1 = \{ f \in \mathcal{S} \mid V \subset \text{dom} f, \, f(V) \subset \overline{W} \},
\]

(4)
equipped with the restriction of the compact-open topology. Notice that \( \mathcal{S}_0 \) is an open neighborhood of \( h \) in \( \mathcal{S}_{co} \). Consider the compact-open topology on \( C(V, W) \).

**Lemma 3.4.** The restriction map \( \mathcal{R} : \mathcal{S}_1 \rightarrow C(V, W) \), \( \mathcal{R}(f) = f|_V \), defines an identification \( \mathcal{R} : \mathcal{S}_1 \rightarrow \mathcal{R}(\mathcal{S}_1) \).

**Proof.** The continuity of \( \mathcal{R} \) is elementary.

Let \( G \subset \mathcal{R}(\mathcal{S}_1) \) such that \( \mathcal{R}^{-1}(G) \) is open in \( \mathcal{S}_1 \). For each \( g_0 \in G \), there is some \( \tilde{g}_0 \in \mathcal{R}^{-1}(G) \) such that \( \mathcal{R}(g_0) = g_0 \). Since \( \mathcal{R}^{-1}(G) \) is open in \( \mathcal{S}_1 \), there are finite collections, \( \{K_1, \ldots, K_p\} \) of compact subsets and \( \{O_1, \ldots, O_p\} \) of open subsets, such that

\[
g_0' \in \{ f \in \mathcal{S}_1 \mid K_1 \cup \cdots \cup K_p \subset \text{dom} f, \quad f(K_1) \subset O_1, \ldots, f(K_p) \subset O_p \} \subset \mathcal{R}^{-1}(G).
\]

Then

\[
g_0 \in \{ g \in \mathcal{S}_1 \mid (K_1 \cup \cdots \cup K_p) \cap \overline{V} \subset \text{dom} g, \quad g(K_1 \cap \overline{V}) \subset O_1 \cap \overline{W}, \ldots, g(K_p \cap \overline{V}) \subset O_p \cap \overline{W} \} \subset G.
\]

Since \( K_1 \cap \overline{V}, \ldots, K_p \cap \overline{V} \) are compact in \( \overline{V} \), and \( O_1 \cap \overline{W}, \ldots, O_p \cap \overline{W} \) are open in \( \overline{W} \), it follows that \( g_0 \) is in the interior of \( G \) in \( \mathcal{R}(\mathcal{S}_1) \). Hence \( G \) is open in \( \mathcal{R}(\mathcal{S}_1) \). \( \square \)

**Lemma 3.5.** \( \mathcal{R}(\mathcal{S}_1) \) is closed in \( C(V, W) \).

**Proof.** Observe that \( C(V, W) \) is second countable because \( T \) is Polish. Take a sequence \( g_n \) in \( \mathcal{R}(\mathcal{S}_1) \) converging to \( g \) in \( C(V, W) \). Then it easily follows that \( g_n|_V \) converges to \( g|_V \) in \( C(V, T) \) with the compact-open topology. Thus \( g|_V \in \mathcal{S} \) according to Remark [24] and let \( f = g|_V \). By Lemma [3.3] we have \( g = f|_V \). Therefore \( f \in \mathcal{S}_1 \) and \( g = \mathcal{R}(f) \). \( \square \)

**Corollary 3.6.** \( \mathcal{R}(\mathcal{S}_1) \) is compact in \( C(V, W) \).

**Proof.** This follows by Arzelà-Ascoli Theorem and Lemma [3.5] because \( \overline{V} \) and \( \overline{W} \) are compact, and \( \mathcal{R}(\mathcal{S}_1) \) is equicontinuous since \( \mathcal{H} \) satisfies the equicontinuity condition with \( \mathcal{S} \) and \( \{T_i, d_i\} \). \( \square \)
Let $V_0$ be an open subset of $T$ such that $x \in V_0$ and $\overline{V_0} \subset V$. Since $\overline{V_0} \subset \text{dom } f$ for all $f \in \mathfrak{S}_1$, we can consider the restriction $\mathfrak{S}_1 \times \overline{V_0} \to \hat{T}$ of the germ map.

**Lemma 3.7.** $\gamma(\mathfrak{S}_1 \times \overline{V_0})$ is compact in $\hat{T}$.

**Proof.** For each $g \in C(\overline{V}, \overline{W})$ and $y \in \overline{V}$, let $\gamma(g, y)$ denote the germ of $g$ at $y$, defining a germ map

$$\gamma : C(\overline{V}, \overline{W}) \times \overline{V} \to \gamma(C(\overline{V}, \overline{W}) \times \overline{V}) .$$

Since $\overline{V_0} \subset V$, we get that $\gamma(\mathfrak{S}_1 \times \overline{V_0}) = \gamma(R(\mathfrak{S}_1) \times \overline{V_0})$ and the diagram

$$\begin{array}{ccc}
\mathfrak{S}_1 \times \overline{V_0} & \xrightarrow{\mathcal{R} \times \text{id}} & \mathcal{R}(\mathfrak{S}_1) \times \overline{V_0} \\
\gamma \downarrow & & \gamma \\
\gamma(\mathfrak{S}_1 \times \overline{V_0}) & \xrightarrow{\gamma} & \gamma(\mathcal{R}(\mathfrak{S}_1) \times \overline{V_0})
\end{array}$$

is commutative. Then

$$\tilde{\gamma} : \mathcal{R}(\mathfrak{S}_1) \times \overline{V_0} \to \tilde{\gamma}(\mathcal{R}(\mathfrak{S}_1) \times \overline{V_0})$$

is continuous because

$$\mathcal{R} \times \text{id} : \mathfrak{S}_1 \times \overline{V_0} \to \mathcal{R}(\mathfrak{S}_1) \times \overline{V_0}$$

is an identification by Lemma 3.4 and

$$\gamma : \mathfrak{S}_1 \times \overline{V_0} \to \gamma(\mathfrak{S}_1 \times \overline{V_0})$$

is continuous. Hence $\gamma(\mathfrak{S}_1 \times \overline{V_0})$ is compact by Corollary 3.6. □

**Lemma 3.8.** $\gamma(\mathfrak{S}_0 \times V_0)$ is open in $\hat{T}$.

**Proof.** This holds because $\mathfrak{S}_0 \times V_0$ is open in $\mathfrak{S}_{c-o} \ast T$ and saturated by the fibers of $\gamma : \mathfrak{S}_{c-o} \ast T \to \hat{T}$. □

**Remark 27.** Observe that the proof of Lemma 3.8 does not require $\overline{V_0} \subset V$; it holds for any open $V_0 \subset V$.

**Corollary 3.9.** $\hat{T}_U$ is locally compact.

**Proof.** We have that $\gamma(\mathfrak{S}_1 \times \overline{V_0})$ is compact by Lemma 3.7 and contains $\gamma(\mathfrak{S}_0 \times V_0)$, which is an open neighborhood of $\gamma(h, x)$ by Lemma 3.8. Then the result follows because $\gamma(h, x) \in \hat{T}_U$ is arbitrary. □

**Lemma 3.10.** $\tilde{\gamma} : \mathcal{R}(\mathfrak{S}_1) \times \overline{V_0} \to \hat{T}$ is injective.

**Proof.** Let $(\mathcal{R}(f_1), y_1), (\mathcal{R}(f_2), y_2) \in \mathcal{R}(\mathfrak{S}_1) \times \overline{V_0}$ for $f_1, f_2 \in \mathfrak{S}_1$ with $\gamma(\mathcal{R}(f_1), y_1) = \gamma(\mathcal{R}(f_2), y_2)$. Thus $y_1 = y_2 =: y$ and $\gamma(f_1, y_1) = \gamma(f_2, y_2)$; i.e., $f_1 = f_2$ on some neighborhood $O$ of $y$ in $\text{dom } f_1 \cap \text{dom } f_2$. Then $f_1(O) \subset \text{dom } (f_1f_2^{-1})$ and $f_2f_1^{-1} = \text{id}_T$ on $f_1(O)$. Since $f_2f_1^{-1} \in \mathfrak{S}$, we get $f_2f_1^{-1} = \text{id}_T$ on dom$(f_2f_1^{-1}) = f_1(\text{dom } f_1 \cap \text{dom } f_2)$ by the strong quasi-analyticity of $\mathfrak{S}$. Since $\overline{V} \subset \text{dom } f_1 \cap \text{dom } f_2$, it follows that $f_2f_1^{-1} = \text{id}_T$ on $f_1(\overline{V})$, and therefore $f_1 = f_2$ on $\overline{V}$; i.e., $\mathcal{R}(f_1) = \mathcal{R}(f_2)$. □
Proof. Since \( U \times U \) can be covered by sets of the form \( V_0 \times W \), for \( V_0 \) and \( W \) as above, it is enough to prove that \( \hat{\pi}^{-1}(K_1 \times K_2) \) is compact for all compact set \( K_1 \subset V_0 \) and \( K_2 \subset W \). But then, with the above notation, we obtain
\[
\hat{\pi}^{-1}(K_1 \times K_2) \subset \gamma(S_1 \times K_1) \subset \gamma(S_1 \times V_0),
\]
and the result follows from Lemma 3.7.

Corollary 3.12. The closure of \( \hat{T}_U \) in \( \hat{T} \) is compact.

Proof. Take a relatively compact open subset \( U' \subset T \) containing \( \bar{U} \). By applying Corollary 3.11 to \( U' \), it follows that \( \hat{\pi} : \hat{T}_U' \to U' \times U' \) is proper. Therefore \( \hat{\pi}^{-1}(\bar{U} \times U) \) is compact and contains the closure of \( \hat{T}_U \) in \( \hat{T} \).

Lemma 3.13. \( \hat{T}_U \) is Hausdorff.

Proof. Let \( \gamma(h_1, x_1) \neq \gamma(h_2, x_2) \) in \( \hat{T}_U \).

Suppose first that \( x_1 \neq x_2 \). Since \( T \) is Hausdorff, there are disjoint open subsets \( V_1 \) and \( V_2 \) such that \( x_1 \in V_1 \) and \( x_2 \in V_2 \). Then \( \hat{V}_1 = \hat{T}_U \cap s^{-1}(V_1) \) and \( \hat{V}_2 = \hat{T}_U \cap s^{-1}(V_2) \) are disjoint and open in \( \hat{T}_U \), and \( \gamma(h_1, x_1) \in \hat{V}_1 \) and \( \gamma(h_2, x_2) \in \hat{V}_2 \).

Now assume that \( x_1 = x_2 =: x \) but \( h_1(x) \neq h_2(x) \). Take disjoint open subsets \( W_1, W_2 \subset U \) such that \( h_1(x) \in W_1 \) and \( h_2(x) \in W_2 \). Then \( \hat{W}_1 = \hat{T}_U \cap t^{-1}(W_1) \) and \( \hat{W}_2 = \hat{T}_U \cap t^{-1}(W_2) \) are disjoint and open in \( \hat{T}_U \), and \( \gamma(h_1, x) \in \hat{W}_1 \) and \( \gamma(h_2, x) \in \hat{W}_2 \).

Finally, suppose that \( x_1 = x_2 =: x \) and \( h_1(x) = h_2(x) =: y \). Then \( x \in T_{k \cap U} \) and \( y \in T_{l \cap U} \) for some indexes \( k \) and \( l \). Take open neighborhoods, \( V \) of \( x \) and \( W \) of \( y \), such that \( V \subset T_{k \cap U}, W \subset T_{l \cap U} \) and \( h_1(V) \cup h_2(W) \subset W \). Define \( \bar{S}_0 \) and \( \bar{S}_1 \) by using \( V \) and \( W \) like in (3) and (4), and take an open subset \( V_0 \subset T \) such that \( x \in V_0 \) and \( \overline{V_0} \subset V \), as above. We can assume that \( h_1, h_2 \in \bar{S}_1 \). Then
\[
\gamma(\mathcal{R}(h_1), x) = \gamma(h_1, x_1) \neq \gamma(h_2, x_2) = \gamma(\mathcal{R}(h_2), x),
\]
and therefore \( \mathcal{R}(h_1) \neq \mathcal{R}(h_2) \) in \( \mathcal{R}(\bar{S}_1) \) by Lemma 3.10. Since \( \mathcal{R}(\bar{S}_1) \) is Hausdorff (because it is a subspace of \( C_{c0}(\bar{V}, \bar{W}) \)), it follows that there are disjoint open subsets \( \mathcal{N}_1, \mathcal{N}_2 \subset \mathcal{R}(\bar{S}_1) \) such that \( \mathcal{R}(h_1) \in \mathcal{N}_1 \) and \( \mathcal{R}(h_2) \in \mathcal{N}_2 \). So \( \mathcal{R}^{-1}(\mathcal{N}_1) \) and \( \mathcal{R}^{-1}(\mathcal{N}_2) \) are disjoint open subsets of \( \bar{S}_1 \) with \( h_1 \in \mathcal{R}^{-1}(\mathcal{N}_1) \) and \( h_2 \in \mathcal{R}^{-1}(\mathcal{N}_2) \). Hence \( \mathcal{M}_1 = \mathcal{R}^{-1}(\mathcal{N}_1) \times \bar{S}_0 \) and \( \mathcal{M}_2 = \mathcal{R}^{-1}(\mathcal{N}_2) \times \bar{S}_0 \) are disjoint and open in \( \bar{S}_0 \), and therefore they are open in \( \bar{S} \). Moreover \( \mathcal{M}_1 \times V_0 \) and \( \mathcal{M}_2 \times V_0 \) are saturated by the fibers of \( \gamma : \bar{S}_0 \times V_0 \to \gamma(\bar{S}_0 \times V_0) \); in fact, if \( (f, z) \in \bar{S}_0 \times V_0 \) satisfies \( \gamma(f, z) = \gamma(f', z) \) for some \( f' \in \mathcal{M}_a \) \((a \in \{1, 2\})\), then
\[
\gamma(\mathcal{R}(f), z) = \gamma(f, z) = \gamma(f', z) = \gamma(\mathcal{R}(f'), z),
\]
Lemma 3.15, the sets is a relatively compact open set of $T$.

Since $\gamma : S_0 \times V_0 \to \gamma(S_0 \times V_0)$ is an identification because $S_0 \times V_0$ is open in $S_{c-o} \ast T$ and saturated by the fibers of $\gamma : S_{c-o} \ast T \to \hat{T}$. Furthermore

$$\gamma(M_1 \times V_0) \cap \gamma(M_2 \times V_0) = \gamma(N_1 \times V_0) \cap \gamma(N_2 \times V_0)$$

by the commutativity of the diagram (2), and $\gamma(h_1, x) \in \gamma(M_1 \times V_0)$ and $\gamma(h_2, x) \in \gamma(M_2 \times V_0)$.

Corollary 3.14. $\gamma : R(S_1) \times V_0 \to \gamma(R(S_1) \times V_0)$ is a homeomorphism.

Lemma 3.15. $\hat{T}_U$ is second countable.

Proof. $\hat{T}_U$ can be covered by a countable collection of open subsets of the type $\gamma(S_0 \times V_0)$ as above. But $\gamma(S_0 \times V_0)$ is second countable because it is a subspace of $\gamma(S_1 \times V_0) = \gamma(R(S_1) \times V_0)$, which is homeomorphic to $R(S_1) \times V_0$ by Corollary 3.14, and this space is second countable because it is a subspace of the second countable space $C(V_0, W_0) \times V_0$.

Corollary 3.16. $\hat{T}_U$ is Polish.

Proof. This follows from Corollary 3.9, Lemmas 3.13 and 3.15 and [30, Theorem 5.3].

Proposition 3.17. $\hat{T}$ is Polish and locally compact.

Proof. First, let us prove that $\hat{T}$ is Hausdorff. Take different points $\gamma(g, x)$ and $\gamma(g', x')$ in $\hat{T}$. Let $O$, $O'$, $P$ and $P'$ be relatively compact open neighborhoods of $x$, $x'$, $g(x)$ and $g(x')$, respectively. Then $U_1 = U \cup O \cup O' \cup P \cup P'$ is a relatively compact open subset of $T$ that meets all $H$-orbits. By Lemma 3.13, $\hat{T}_{U_1}$ is a Hausdorff open subset of $\hat{T}$ that contains $\gamma(g, x)$ and $\gamma(g', x')$. Hence $\gamma(g, x)$ and $\gamma(g', x')$ can be separated in $\hat{T}_{U_1}$ by disjoint open neighborhoods in $\hat{T}_{U_1}$, and therefore also in $\hat{T}$.

Second, let us show that $\hat{T}$ is locally compact. For $\gamma(g, x) \in \hat{T}$, let $O$ and $P$ be relatively compact open neighborhoods of $x$ and $g(x)$, respectively. Then $U_1 = U \cup O \cup P$ is a relatively compact open set of $T$ that meets all $H$-orbits. By Corollary 3.9 it follows that $\hat{T}_{U_1}$ is a locally compact open neighborhood of $\gamma(g, x)$ in $\hat{T}$. Hence $\gamma(g, x)$ has a compact neighborhood in $\hat{T}_{U_1}$, and therefore also in $\hat{T}$.

Finally, let us show that $\hat{T}$ is second countable. Since $T$ is second countable (it is Polish) and locally compact, it can be covered by countably many relatively compact open subsets $O_n \subset T$. Then each $U_{n,m} = O_n \cup O_m \cup U$ is a relatively compact open set of $T$ that meets all $H$-orbits. Hence, by Lemma 3.15 the sets $\hat{T}_{U_{n,m}}$ are second countable and open in $\hat{T}$. Moreover these sets form a countable cover of $\hat{T}$ because, for any $\gamma(g, x) \in \hat{T}$, we have
Corollary 3.20. The closure of \( T \) is proper.

Proof. Let \( K \) be any compact subset of \( T \). Take any relatively compact open subset \( U' \subset T \) meeting all \( H \)-orbits such that \( K \subset U' \). By applying Corollary 3.11 to any \( U' \), we get that \( \hat{\pi}^{-1}(K) \) is compact in \( \hat{T}_{U'} \), and therefore in \( \hat{T} \).

3.4. The space \( \hat{T}_0 \). From now on, assume that \( H \) is minimal, and therefore \( \hat{H} \) has only one orbit, which is \( T \). Fix a point \( x_0 \in U \), and let \( \hat{T}_0 = T^{-1}(x_0) = \{ \gamma(g,x) \in \hat{T} | g(x) = x_0 \} \), \( \hat{T}_{0,U} = \hat{T}_0 \cap \hat{T}_U \).

Observe that \( \hat{T}_0 \) is closed in \( \hat{T} \) and \( \hat{T}_{0,U} \) is open in \( \hat{T}_0 \). Moreover \( \hat{\pi}(\hat{T}_0) = T \times \{ x_0 \} = T \) and \( \hat{\pi}(\hat{T}_{0,U}) = U \times \{ x_0 \} = U \) because \( T \) is the unique \( \hat{H} \)-orbit; indeed, \( \hat{\pi}(\gamma(h,x)) = x \) for each \( x \in T \) and any \( h \in \hat{S} \) with \( x \in \text{dom} h \) and \( h(x) = x_0 \). Let \( \hat{\pi}_0 : \hat{T}_0 \to T \), which is continuous and surjective.

The following two corollaries are direct consequences of Proposition 3.17 (see [30, Theorem 3.11]) and Corollary 3.12.

Corollary 3.19. \( \hat{T}_0 \) is Polish and locally compact.

Corollary 3.20. The closure of \( \hat{T}_{0,U} \) in \( \hat{T}_0 \) is compact.

The following corollary is a direct consequence of Proposition 3.18 because \( \hat{\pi}_0 : \hat{T}_0 \to T \) can be identified with the restriction \( \hat{\pi} : \hat{T}_0 \to T \times \{ x_0 \} = T \).

Corollary 3.21. \( \hat{\pi}_0 : \hat{T}_0 \to T \) is proper.

Proposition 3.22. The fibers of \( \hat{\pi}_0 : \hat{T}_0 \to T \) are homeomorphic to each other.

Proof. Let \( x \in T \). Since \( T \) is the unique orbit of \( \hat{H} \), there is some \( f \in S \) with \( f(x) = x_0 \). Then the mapping \( \gamma(g,x) \mapsto \gamma(g \cdot f^{-1},x_0) \) defines a homeomorphism \( \hat{\pi}_0^{-1}(x) \to \hat{\pi}_0^{-1}(x_0) \) whose inverse is given by \( \gamma(g_0,x_0) \mapsto \gamma(g_0 \cdot f,x) \).

Question 3.23. When is \( \hat{\pi}_0 \) a fiber bundle?

3.5. The pseudogroup \( \hat{H}_0 \). For \( h \in S \), define

\[
\hat{h} : \hat{\pi}_0^{-1} \text{dom} h \to \hat{\pi}_0^{-1} \text{im} h, \quad \hat{h}(\gamma(g,x)) = \gamma(gh^{-1},h(x)),
\]

for \( g \in S, x \in \text{dom} g \cap \text{dom} h \) with \( g(x) = x_0 \). The following two results are elementary.

---

5The definition \( \hat{T}_0 = s^{-1}(x_0) \) would be valid too, of course, but it seems that the proofs in Sections 3.4 and 5.5 have a simpler notation with the choice \( \hat{T}_0 = t^{-1}(x_0) \).
Lemma 3.24. For any $h \in S$, we have $\hat{\pi}_0(\text{dom } \hat{h}) = \text{dom } h$ and $\hat{\pi}_0(\text{im } \hat{h}) = \text{im } h$, and the following diagram is commutative:

\[
\begin{array}{ccc}
\text{dom } \hat{h} & \xrightarrow{\hat{h}} & \text{im } \hat{h} \\
\downarrow{\hat{\pi}_0} & & \downarrow{\hat{\pi}_0} \\
\text{dom } h & \xrightarrow{h} & \text{im } h \\
\end{array}
\]

Lemma 3.25. If $O \subset T$ is open with $\text{id}_O \in S$, then $\hat{\text{id}_O} = \text{id}_{\hat{\pi}_0^{-1}(O)}$.

Lemma 3.26. For $h, h' \in S$, we have $\hat{h} \hat{h}' = \hat{h}' \hat{h}$.

Proof. By Lemma 3.24, we have

\[
\hat{\pi}_0^{-1}(\text{dom } \hat{h}) = \hat{h}^{-1}(\hat{\pi}_0^{-1}(\text{dom } h' \cap \text{im } h)) = \hat{h}^{-1}(\hat{\pi}_0^{-1}(\text{dom } h' \cap \text{im } h)) = \hat{\pi}_0^{-1}(\text{dom } h' \cap \text{im } h) = \text{dom } \hat{h}' .
\]

Now let $\gamma(g, x) \in \text{dom } \hat{h}' \hat{h}$; thus $g \in \mathcal{S}$, $x \in \text{dom } g \cap \text{dom } h$, $h(x) \in \text{dom } h'$ and $g(x) = x_0$. Then

\[
\hat{h}' \hat{h}(\gamma(g, x)) = \gamma(g(h'h)^{-1}, h'h(x)) = \gamma(gh^{-1}(h')^{-1}, h'h(x)) = \hat{h}'(\gamma(g^{-1}, h(x))) = \hat{h}'(\gamma(g, x)).
\]

The following is a direct consequence of Lemmas 3.25 and 3.26.

Corollary 3.27. For $h \in S$, the map $\hat{h}$ is bijective with $\hat{h}^{-1} = \hat{h}$.

Lemma 3.28. $\hat{h}$ is a homeomorphism for all $h \in S$.

Proof. By Corollary 3.27, it is enough to prove that $\hat{h}$ is continuous, which holds because it can be expressed as the composition of the following continuous maps:

\[
\begin{array}{c}
\hat{\pi}_0^{-1}(\text{dom } h) \\
\xrightarrow{(\text{id, const, } \hat{h}_0)} \\
\xrightarrow{\text{id} \times \gamma} \\
\xrightarrow{\text{product}} \\
\hat{\pi}_0^{-1}(\text{im } h),
\end{array}
\]

as can be checked on elements:

\[
\gamma(g, x) \mapsto (\gamma(g, x), h^{-1}, h(x)) \mapsto (\gamma(g, x), \gamma(h^{-1}, h(x))) \mapsto \gamma(gh^{-1}, h(x)) = \hat{h}(\gamma(g, x)).
\]

Set $\hat{S}_0 = \{ \hat{h} | h \in S \}$, and let $\hat{\mathcal{H}}_0$ be the pseudogroup on $\hat{T}_0$ generated by $\hat{S}_0$. Lemmas 3.26 and 3.28 and Corollary 3.27 give the following.

Corollary 3.29. $\hat{S}_0$ is a pseudo*group on $\hat{T}_0$.
Lemma 3.30. \( \hat{T}_{0,U} \) meets all orbits of \( \hat{\mathcal{H}}_0 \).

Proof. Let \( \gamma(g, x) \in \hat{T}_0 \) with \( g \in \mathcal{S}_1 \); thus \( x \in \text{dom } g \) and \( g(x) = x_0 \). Since \( U \) meets all orbits of \( \mathcal{H} \), there is some \( h \in S \) such that \( x \in \text{dom } h \) and \( h(x) \in U \). Then \( \gamma(g, x) \in \text{dom } \hat{h} \) and \( \hat{h}(\gamma(g, x)) = \gamma(gh^{-1}, h(x)) \) satisfies

\[
\hat{\pi}_0(\hat{h}(\gamma(g, x))) = \hat{\pi}_0(\gamma(gh^{-1}, h(x))) = h(x) \in U.
\]

Hence \( \hat{h}(\gamma(g, x)) \in \hat{T}_{0,U} \) as desired. \( \square \)

Lemma 3.31. The map \( S_{c-o} \to \hat{\mathcal{S}}_{0,c-o}, h \to \hat{h} \), is a homeomorphism.

Proof. If \( \hat{h}_1 = \hat{h}_2 \) for some \( h_1, h_2 \in S \), then \( h_1 = h_2 \) by Lemma 3.24. So the stated map is injective, and therefore it is bijective by the definition of \( \hat{S}_0 \).

Take a subbasic open set of \( S_{c-o} \), which is of the form \( S \cap \mathcal{N}(K, O) \) for some compact \( K \) and open \( O \) in \( T \). The set \( \hat{\pi}^{-1}_0(K) \) is compact by Corollary 3.21 and \( \hat{\pi}^{-1}_0(O) \) is open. Then the map of the statement is open because

\[
\{ \hat{h} \mid h \in \mathcal{N}(K, O) \cap S \} = \hat{\mathcal{N}}(\hat{\pi}^{-1}_0(K), \hat{\pi}^{-1}_0(O)) \cap \hat{\mathcal{S}}_0
\]

by Lemma 3.24, which is open in \( \hat{\mathcal{S}}_{0,c-o} \).

To prove its continuity, let us first show that its restriction to \( S_U = S \cap \mathcal{H}|_U \) is continuous. Fix \( h_0 \in S_U \), and take relatively compact open subsets

\[
V, V_0, W, V', V'_0, W' \subset U,
\]

and indices \( k \) and \( k' \) such that

\[
\begin{align*}
\overline{V}_0 & \subset V, \quad \overline{V} \subset T_k \cap \text{dom } h_0, \\
\overline{V}'_0 & \subset V', \quad \overline{V}' \subset T_{k'} \cap \text{im } h_0, \\
\overline{W} & \subset W', \quad \overline{W}' \subset T_{k_0}, \\
h_0^{-1}(\overline{V}) & \subset V, \\
h_0^{-1}(\overline{V}') & \subset V'.
\end{align*}
\]

(6)–(10)

Let \( \overline{S}_0 \) and \( \overline{S}_1 \) (respectively, \( \overline{S}'_0 \) and \( \overline{S}'_1 \)) be defined like in (3) and (4), by using \( V \) and \( W \) (respectively, \( V' \) and \( W' \)). Then \( \hat{K} = \gamma(\overline{S}_1 \times \overline{V}_0) \) is compact in \( \hat{T} \) by Lemma 3.7 and \( \hat{O} = \gamma(\overline{S}'_0 \times V') \) is open in \( \hat{T} \) by Lemma 3.8 and Remark 27. Then \( \hat{K}_0 = \hat{K} \cap \hat{T}_0 \) is compact and \( \hat{O}_0 = \hat{O} \cap \hat{T}_0 \) is open in \( \hat{T}_0 \). So \( \hat{\mathcal{N}}(\hat{K}_0, \hat{O}_0) \cap \hat{\mathcal{S}}_0 \) is a subbasic open set of \( \hat{\mathcal{S}}_{0,c-o} \).

Claim 2. \( \hat{h}_0 \in \hat{\mathcal{N}}(\hat{K}_0, \hat{O}_0) \).

Let \( \gamma(g, x) \in \hat{K}_0 \); thus \( g \in \mathcal{S}_1 \), \( x \in \overline{V}_0 \cap \text{dom } g \) and \( g(x) = x_0 \). The condition \( g \in \overline{S}_1 \) means that \( g \in \mathcal{S}, \overline{V} \subset \text{dom } g \) and \( g(\overline{V}) \subset \overline{W} \). By (7)–(9), it follows that \( \overline{V}' \subset \text{dom } gh^{-1}_0 \) and

\[
gh^{-1}_0(\overline{V}') \subset g(\overline{V}) \subset \overline{W} \subset W'.
\]
Hence $gh_0^{-1} \in \hat{S}_0$, obtaining that
$$\hat{h}_0(\gamma(g, x)) = \gamma(gh_0^{-1}, h_0(x)) \in \hat{O},$$
which completes the proof of Claim 2.

Claim 3. The sets $\hat{N}(\hat{K}_0, \hat{O}_0) \cap \hat{S}_0$, constructed as above, form a local subbasis of $\hat{S}_{0, c-o}$ at $\hat{h}_0$.

This assertion follows by Claim 2 and because the sets of the type $\hat{O}_0$ form a basis of the topology of im $\hat{h}_0$, and any compact subset of dom $\hat{h}_0$ is contained in a finite union of sets of the type of $\hat{K}_0$.

The sets
$$\mathcal{N} = \mathcal{N}(\mathcal{V}_0, V') \cap \mathcal{N}(\mathcal{V}', V)^{-1} \cap S_U$$
are open neighborhoods of $h_0$ by (9), (10), and Propositions 2.6 and 3.1.

Claim 4. $\hat{h} \in \hat{N}(\hat{K}_0, \hat{O}_0)$ for all $h \in \mathcal{N}$.

Given $h \in \mathcal{N}$, we have $\mathcal{V}' \subseteq \text{im} h$ and $h^{-1}(\mathcal{V}') \subseteq V$. Let $\gamma(g, x) \in \hat{K}_0$; thus $x \in \mathcal{V}_0 \cap \text{dom } g$, $g(x) = x_0$, and we can assume that $g \in \mathcal{S}_1$, which means that $g \in \mathcal{S}$, $\mathcal{V} \subseteq \text{dom } g$ and $g(\mathcal{V}) \subseteq \mathcal{W}$. Then $\mathcal{V}' \subseteq \text{dom}(gh^{-1})$, $gh^{-1}(\mathcal{V}') \subseteq \mathcal{W} \subseteq \mathcal{W}'$ and $h(x) = h(x_0) \in \mathcal{V}'$. Therefore
$$\hat{h}(\gamma(g, x)) = (gh^{-1}, h(x)) \in \gamma(S_0 \times V') \cap \hat{T}_0 = \hat{O}_0,$$
showing Claim 4.

Claims 3 and 4 show that the map $S_{c-o} \rightarrow \hat{S}_{0, c-o}$, $h \mapsto \hat{h}$, is continuous at $h_0$.

Now, let us prove that the whole map $S_{c-o} \rightarrow \hat{S}_{0, c-o}$, $h \mapsto \hat{h}$, is continuous. Since the sets $\mathcal{N}(\hat{K}, \hat{O}) \cap \hat{S}_0$, for small enough compact subsets $\hat{K} \subseteq \hat{T}_0$ and small enough open subsets $\hat{O} \subseteq \hat{T}_0$, form a subbasis of $\hat{S}_{0, c-o}$, it is enough to prove that the inverse image of these subbasic sets are open in $S_{c-o}$. We can assume that $\hat{K}, \hat{O} \subseteq \hat{S}_0^{-1}(U')$ for some relatively compact open subset $U' \subset T$ that meets all $\mathcal{H}$-orbits. Consider the inclusion map $i : U' \hookrightarrow T$, and the paro map $\phi : T \rightarrow U'$ with dom $\phi = U'$, where it is the identity map. According to Proposition 2.3, we get a continuous map $\phi_* : \text{Paro}_{c-o}(T, T) \rightarrow \text{Paro}_{c-o}(U', U')$, which restricts to a continuous map $\phi_* : S_{c-o} \rightarrow S_{U', c-o}$. Observe that $\phi_*(h)$ is the restriction $h : U' \cap h^{-1}(U') \rightarrow h(U')$ for each $h \in S$. Hence, since $\hat{K}, \hat{O} \subseteq \hat{S}_0^{-1}(U')$, it follows from Lemma 3.24 that $\mathcal{N}(\hat{K}, \hat{O}) \cap \hat{S}_0$ has the same inverse image by the map $S_{c-o} \rightarrow \hat{S}_{0, c-o}$, $h \mapsto \hat{h}$, and by the composition
$$S_{c-o} \xrightarrow{\phi_*} S_{U', c-o} \xrightarrow{\phi} \hat{S}_{0, c-o},$$
where the second map is given by $h \mapsto \hat{h}$. This composition is continuous by the above case applied to $U'$, and therefore the inverse image of $\mathcal{N}(\hat{K}, \hat{O}) \cap \hat{S}_0$ by $S_{c-o} \rightarrow \hat{S}_{0, c-o}$, $h \mapsto \hat{h}$, is open in $S_{c-o}$. \qed
Since the compact generation of $\mathcal{H}$ is satisfied with the relatively compact open set $U$, there is a symmetric finite set $\{f_1, \ldots, f_m\}$ generating $\mathcal{H}|_U$, which can be chosen in $S$, such that each $f_a$ has an extension $\hat{f}_a$ with $\text{dom} \hat{f}_a \subset \text{dom} f_a$. We can also assume that $f_a \in S$. Let $\mathcal{H}_{0.U} = \mathcal{H}|_{\mathcal{T}_{0,U}}$.

Obviously, each $\hat{f}_a$ is an extension of $f_a$. Moreover

$$\text{dom} \hat{f}_a = \pi_0^{-1}(\text{dom} f_a) \subset \pi_0^{-1}(\text{dom} \hat{f}_a) \subset \pi_0^{-1}(\text{dom} f_a) = \text{dom} \hat{f}_a.$$  

Lemma 3.32. The maps $\hat{f}_a$ ($a \in \{1, \ldots, m\}$) generate $\mathcal{H}_{0.U}$.

Proof. $\mathcal{H}_{0,U}$ is generated by the maps of the form $\hat{h}$ with $h \in S_U$, and any such $h$ can be written as a composition of maps $\hat{f}_a$ around any $\gamma(g, x) \in \text{dom} \hat{h} = \pi_0^{-1}(\text{dom} h)$ by Lemma 3.26. \hfill \square

Corollary 3.33. $\mathcal{H}_0$ is compactly generated.

Proof. We saw that $\mathcal{T}_{0,U}$ is relatively compact in $\mathcal{T}_0$ (Corollary 3.20) and meets all $\mathcal{H}_0$-orbits (Lemma 3.30), the maps $\hat{f}_a$ generate $\mathcal{H}_{0,U}$ (Lemma 3.32), and each $\hat{f}_a$ is an extension of each $f_a$ with $\text{dom} \hat{f}_a \subset \text{dom} f_a$. \hfill \square

Recall that the sets $T_{i_k}$ form a finite covering of $\overline{U}$ by open sets of $T$. Fix some index $k_0$ such that $x_0 \in T_{i_{k_0}}$. Let $\{W_k\}$ be a shrinking of $\{T_{i_k}\}$ as cover of $\overline{U}$ by open subsets of $T$; i.e., $\{W_k\}$ is a cover of $\overline{U}$ by open subsets of $T$ with the same index set and $\overline{W}_k \subset T_{i_k}$ for all $k$. By applying Proposition 2.32 several times, we get finite covers, $\{V_a\}$ and $\{V'_a\}$, of $\overline{U}$ by open subsets of $T$, and shrinking, $\{W_{0,k}\}$ of $\{W_k\}$ and $\{V_{0,a}\}$ of $\{V_a\}$, as covers of $\overline{U}$ by open subsets of $T$, such that the following properties hold:

- For all $h \in \mathcal{H}$ and $x \in \text{dom} h \cap U \cap V_a \cap W_{0,k}$ with $h(x) \in U \cap W_{0,l}$, there is some $\hat{h} \in S$ such that

$$\text{dom} \hat{h} \cap W_k, \quad \gamma(h, x) = \gamma(h, x), \quad \hat{h}(\overline{V}_a) \subset W_l.$$  

- For all $h \in \mathcal{H}$ and $x \in \text{dom} h \cap U \cap V'_a \cap V_{0,a}$ with $h(x) \in U \cap V_{0,b}$, there is some $\hat{h} \in S$ such that

$$\text{dom} \hat{h} \cap V_a, \quad \gamma(h, x) = \gamma(h, x), \quad \hat{h}(\overline{V}_a) \subset V_b.$$  

By the definition of $\overline{\mathcal{H}}$ and $\mathcal{S}$, it follows that these properties also hold for all $h \in \overline{\mathcal{H}}$ with $\hat{h} \in \mathcal{S}$. Let $\{V'_{0,u}\}$ be a shrinking of $\{V'_a\}$ as a cover of $\overline{U}$ by open subsets of $T$. We have $x_0 \in W_{0,k_0} \cap V_{0,a_0} \cap V'_{0,u_0}$ for some indices $k_0$, $a_0$ and $u_0$. For each $a$, let $S_{0,a}$, $S_{1,a} \subset S$ be defined like $S_0$ and $S_1$ in (3) and (4) by using $V_a$ and $W_{k_0}$ instead of $V$ and $W$. Take an index $u$ such that $\overline{V}_a \subset V_a$. The sets $V_{0,a} \cap V'_{0,u}$, defined in this way, form a cover of $\overline{U}$, obtaining that the sets $\hat{T}_{a,u} = \gamma(S_{0,a} \times (V_{0,a} \cap V'_{0,u}))$ form a cover of $\overline{T}_U$ by open subsets of $\hat{T}$ (Lemma 3.33), and therefore the sets $\hat{T}_{0,a,u} = \hat{T}_{a,u} \cap \hat{T}_0$ form a cover of $\overline{T}_{0,U}$ by open subsets of $\hat{T}_0$. Let $\hat{T}_{0,U,a,u} = \hat{T}_{0,U} \cap \hat{T}_{a,u}$. Like
in Section 3.3 let \( \tilde{\gamma} \) denote the germ map defined on \( C(V_a, W_{k_0}) \times V_a \), and let \( R_a : \overline{S}_{1,a} \to C(V_a, W_{k_0}) \) be the restriction map \( f \mapsto f|_{V_a} \). Then
\[
\tilde{\gamma} : R_a(\overline{S}_{1,a}) \times V_{0,a} \cap V'_{0,a} \to \tilde{\gamma}(R_a(\overline{S}_{1,a}) \times V_{0,a} \cap V'_{0,a}) \tag{11}
\]
is a homeomorphism by Corollary 3.11. Since \( V_a \) is compact, the compact-open topology on \( R_a(\overline{S}_{1,a}) \) equals the topology induced by the supremum metric \( d_a \) on \( C(V_a, W_{k_0}) \), defined with the metric \( d_{i_{k_0}} \) on \( T_{i_{k_0}} \). Take some index \( k \) such that \( V_a \subseteq W_k \). Then the topology of \( R_a(\overline{S}_{1,a}) \times V_{0,a} \cap V'_{0,a} \) is induced by the metric \( d_{a,u,k} \) given by
\[
d_{a,u,k}((g,y), (g', y')) = d_k(y, y') + d_a(g, g')
\]
(recall that \( W_k \subseteq T_k \)). Let \( d_{a,u,k} \) be the metric on \( \tilde{\gamma}(R_a(\overline{S}_{1,a}) \times V_{0,a} \cap V'_{0,a}) \) that corresponds to \( d_{a,u,k} \) by the homeomorphism (11); it induces the topology of \( \tilde{\gamma}(R_a(\overline{S}_{1,a}) \times V_{0,a} \cap V'_{0,a}) \). Recall from the proof of Lemma 3.7 (see (5)) that
\[
\tilde{\gamma}(R_a(\overline{S}_{1,a}) \times V_{0,a} \cap V'_{0,a}) = \gamma(R_a(S) \times V_{0,a} \cap V'_{0,a})
\]
which is contained in \( \tilde{T} \). Then the restriction \( \tilde{T}_{0,a,u,k} \) of \( \tilde{T}_{a,u,k} \) to
\[
\tilde{\gamma}(R_a(\overline{S}_{1,a}) \times V_{0,a} \cap V'_{0,a}) \cap \tilde{T}_0
\]
duces the topology of this space. Moreover, according to the proof of Corollary 3.9, we get
\[
\tilde{T}_{a,u} \subseteq \tilde{\gamma}(R_a(\overline{S}_{1,a}) \times V_{0,a} \cap V'_{0,a})
\]
and therefore
\[
\tilde{T}_{0,a,u} \subseteq \tilde{\gamma}(R_a(\overline{S}_{1,a}) \times V_{0,a} \cap V'_{0,a}) \cap \tilde{T}_0.
\]
For any index \( v \), define \( \overline{S}'_{0,v} \) and \( \overline{S}'_{1,v} \) like \( \overline{S}_0 \) and \( \overline{S}_1 \) in (3) and (4) by using \( V'_{v} \) and \( W_{k_0} \) instead of \( V \) and \( W \). Let \( R'_v : \overline{S}'_{1,v} \to C(V'_v, W_{k_0}) \) denote the restriction map. Again, the compact-open topology on \( R'_v(\overline{S}'_{1,v}) \) equals the topology induced by the supremum metric \( d'_{v} \) on \( C(V'_v, W_{k_0}) \), defined with the metric \( d_{i_{k_0}} \) on \( T_{i_{k_0}} \) (recall that \( W_{k_0} \subseteq T_{i_{k_0}} \)). Take indices \( b \) and \( l \) such that \( V'_v \subseteq V_b \) and \( V'_l \subseteq W_l \). Then we can consider the restriction map
\[
R'_b : C(V_b, W_{k_0}) \to C(V'_v, W_{k_0}.
\]
Its restriction \( R'_b : R_b(\overline{S}_{1,b}) \to R'_b(\overline{S}_{1,v}) \) is injective by Remark 26 and surjective by Remark 25. So \( R'_b : R_b(\overline{S}_{1,b}) \to R'_b(\overline{S}_{1,v}) \) is a continuous bijection between compact Hausdorff spaces, obtaining that it is a homeomorphism. Then, by compactness, it is a uniform homeomorphism with respect to the supremum metrics \( d_b \) and \( d''_v \). Since \( b \) and \( v \) run in finite families of indices, there is a mapping \( \epsilon \to \delta_1(\epsilon) > 0 \), for \( \epsilon > 0 \), such that
\[
d''_v(R_b(f), R_b(f')) < \delta_1(\epsilon) \Rightarrow d_b(R_b(f), R_b(f')) < \epsilon \tag{12}
\]
for all indices \( v \) and \( b \), and maps \( f, f' \in \overline{S}_{1,b} \).
Lemma 3.34. \( \tilde{H}_{0,U} \) satisfies the equicontinuity condition with \( \tilde{S}_{0,U} = \tilde{S}_0 \cap \tilde{H}_{0,U} \) and the quasi-local metric represented by the family \( \{ \tilde{T}_{0,U,a,u}, \tilde{a}_{0,a,u,k} \} \).

Proof. Let \( h \in S \), and take
\[
\gamma(g, y), \gamma(g', y') \in \tilde{T}_{0,U,a,u} \cap \tilde{h}^{-1}(\tilde{T}_{0,U,b,v}),
\]
where \( g, g' \in \tilde{S}_{0,a} \) and \( y, y' \in V_{0,a} \cap V'_{0,a} \) with \( g(y) = g(y') = x_0 \). Take some indices \( k \) and \( l \) such that \( V_a \subset W_k \) and \( V_b \subset W_l \) (recall that \( W_k \subset T_{ik} \) and \( W_l \subset T_i \)). By Remark 25, we can assume that \( \text{dom } h = T_{ik} \). Then
\[
\tilde{h}(\gamma(g, y)) = \gamma(gh^{-1}, h(y)), \quad \tilde{h}(\gamma(g', y')) = \gamma(g'h^{-1}, h(y'))
\]
belong to \( \tilde{T}_{0,U,b,v} \), which means that \( h(y), h(y') \in V_{0,b} \cap V'_{0,v} \) and there are \( f, f' \in \tilde{S}_{0,b} \) so that
\[
\gamma(f, h(y)) = \gamma(gh^{-1}, h(y)), \quad \gamma(f', h(y')) = \gamma(g'h^{-1}, h(y'));
\]
in particular, \( V_b \subset \text{dom } f \cap \text{dom } f' \). In fact, we can assume that \( \text{dom } f = \text{dom } f' = T_i \) by Remark 25. Observe that the image of \( h \) may not be included in \( T_i \), and the images of \( f, f', g \) and \( g' \) may no be included in \( T_{ik_0} \).

Claim 5. \( V'_v \subset \text{im } h \) and \( h^{-1}(V'_v) \subset V_a \).

By the assumptions on \( \{ V'_v \} \), since
\[
h(y) \in U \cap V'_v \cap V_{0,b} \cap \text{dom } h^{-1}, \quad h^{-1}h(y) = y \in U \cap V'_v \cap V_{0,a},
\]
there is some \( \tilde{h}^{-1} \in S \) such that
\[
\tilde{V}_v \subset \text{dom } \tilde{h}^{-1} \cap V_b, \quad \tilde{h}^{-1}(\tilde{V}_v) \subset V_a, \quad \gamma(\tilde{h}^{-1}, h(y)) = \gamma(h^{-1}, h(y));
\]
indeed, we can suppose that \( \text{dom } \tilde{h}^{-1} = T_{ik_0} \) by Remark 25. Then
\[
\tilde{h}^{-1}(\tilde{V}_v) \subset V_a \subset T_{ik} = \text{dom } h,
\]
attaining \( \tilde{V}_v \subset \text{dom } (hh^{-1}) \). Moreover
\[
\gamma(hh^{-1}, h(y)) = \gamma(\text{id}_{T}, h(y)).
\]
Therefore \( hh^{-1} = \text{id} \) on some neighborhood of \( \tilde{V}_v \), and therefore \( \tilde{V}_v \subset \text{im } h \) and \( h^{-1} = \tilde{h}^{-1} \) on \( \tilde{V}_v \). Thus \( h^{-1}(\tilde{V}_v) = \tilde{h}^{-1}(\tilde{V}_v) \subset V_a \), which shows Claim 5.

By Claim 5 and since \( V_a \subset \text{dom } g \cap \text{dom } g' \) because \( g, g' \in \tilde{S}_{0,a} \), we get
\[
\tilde{V}_v \subset \text{dom } (gh^{-1}) \cap \text{dom } (g'h^{-1}).
\]
Since \( f, f' \in \tilde{S}_{0,b} \), we have \( \tilde{V}_b \subset \text{dom } f \cap \text{dom } f' \) and \( f(\tilde{V}_b) \cup f'(\tilde{V}_b) \subset W_{k_0} \). On the other hand, it follows from (13) that \( fh(y) = f'h(y') = x_0 \) and
\[
\gamma(gh^{-1}f^{-1}, x_0) = \gamma(g'h^{-1}f'^{-1}, x_0) = \gamma(\text{id}_{T}, x_0).
\]
Moreover
\[
f(\tilde{V}_v) \subset \text{dom } (gh^{-1}f^{-1}), \quad f'(\tilde{V}_v) \subset \text{dom } (g'h^{-1}f'^{-1}).
\]
by \( [14] \). So, by Remark \( [26] \), \( g h^{-1} f^{-1} = \text{id}_T \) on some neighborhood of \( f(V_v') \), and \( g' h^{-1} f^{-1} = \text{id}_T \) on some neighborhood of \( f'(V_v') \). Thus \( g h^{-1} = f \) and \( g' h^{-1} = f' \) on some neighborhood of \( V_v' \); in particular,

\[
\mathcal{R}_b^v \mathcal{R}_b(f) = g h^{-1} |_{V_v'}, \quad \mathcal{R}_b^v \mathcal{R}_b(f') = g' h^{-1} |_{V_v'}.
\]

Consider the mappings \( \epsilon \mapsto \delta(\epsilon) > 0 \) and \( \epsilon \mapsto \delta_1(\epsilon) > 0 \) satisfying Remark \( [23] \) and \( [12] \). Then, for each \( \epsilon > 0 \), define

\[
\hat{\delta}(\epsilon) = \min\{\delta(\epsilon/2), \delta_1(\epsilon/2)\}.
\]

Given any \( \epsilon > 0 \), suppose that

\[
d_{0,a,u,k}(\gamma(g, y), \gamma(g', y')) < \hat{\delta}(\epsilon).
\]

This means that

\[
d_{a,u,k}((\mathcal{R}_a(g), y), (\mathcal{R}_a(g'), y')) < \hat{\delta}(\epsilon),
\]

or, equivalently,

\[
d_{i_k}(y, y') + \sup_{x \in V_v} d_{i_k_0}(g(x), g'(x)) < \hat{\delta}(\epsilon).
\]

Therefore

\[
\begin{align*}
d_{i_k}(y, y') &< \delta(\epsilon/2), \\
\sup_{x \in V_v} d_{i_k_0}(g(x), g'(x)) &< \delta_1(\epsilon/2).
\end{align*}
\]

From \( [15] \) and Remark \( [23] \) it follows that

\[
d_{i_k}(h(y), h(y')) < \epsilon/2
\]

since \( h \in S \subset \overline{S} \) and \( y, y' \in T_{i_k} \cap h^{-1}(T_{i_l} \cap \text{im} h) \). On the other hand, by Claim \( [5] \) and \( [16] \), we get

\[
d'(\mathcal{R}_b^v \mathcal{R}_b(f), \mathcal{R}_b^v \mathcal{R}_b(f')) = \sup_{z \in V_v} d_{i_k_0}(g h^{-1}(z), g'h^{-1}(z))
\]

\[
= \sup_{x \in h^{-1}(V_v)} d_{i_k_0}(g(x), g'(x))
\]

\[
\leq \sup_{x \in V_v} d_{i_k_0}(g(x), g'(x))
\]

\[
= d_a(\mathcal{R}_a(g), \mathcal{R}_a(g')) < \delta_1(\epsilon/2).
\]

So, by \( [12] \),

\[
d_b(\mathcal{R}_b(f), \mathcal{R}_b(f')) < \epsilon/2.
\]

From \( [17] \) and \( [18] \), we get

\[
\begin{align*}
\hat{d}_{0,b,v,l}(\hat{h}(\gamma(g, y)), \hat{h}(\gamma(g', y')))
\end{align*}
\]

\[
= \hat{d}_{0,b,v,l}(\gamma(f, h(y)), \gamma(f', h(y')))
\]

\[
= d_{b,v,l}(\mathcal{R}_b(f), h(y)), (\mathcal{R}_b(f'), h(y')))
\]

\[
= d_{i_l}(h(y), h(y')) + d_b(\mathcal{R}_b(f), \mathcal{R}_b(f')) < \epsilon.
\]

\( \square \)
Corollary 3.35. \( \hat{\mathcal{H}}_0 \) is equicontinuous.

Proof. \( \hat{\mathcal{H}}_0 \) is equivalent to \( \hat{\mathcal{H}}_{0,U} \) by Lemma 3.30. Thus the result follows from Lemma 3.34 because equicontinuity is preserved by equivalences. \( \square \)

Lemma 3.36. \( \hat{\mathcal{H}}_0 \) is minimal.

Proof. By Lemma 3.30 it is enough to prove that \( \hat{\mathcal{H}}_{0,U} \) is minimal. Let \( \gamma(g,y), \gamma(g',y') \in \hat{T}_{0,U} \) with \( g, g' \in \hat{S} \), \( y \in \text{dom } g \cap U \), \( y' \in \text{dom } g' \cap U \) and \( g(y) = g'(y') = x_0 \). Take indices \( k \) and \( k' \) such that \( y \in T_{i_k} \) and \( y' \in T_{i_{k'}} \).

We can assume that \( \text{dom } g = T_{i_k} \) and \( \text{dom } g' = T_{i_{k'}} \) by Remark 25.

Let \( f = g^{-1}g' \in \hat{S} \); we have \( y' \in \text{dom } f \) and \( f(y') = y \). By Remark 25 there exists \( \hat{f} \in \hat{S} \) with \( \text{dom } \hat{f} = T_{i_{k'}} \) and \( \gamma(\hat{f},y') = \gamma(f,y') \).

By the definition of \( \hat{S} \), there is a sequence \( f_n \) in \( S \) with \( \text{dom } f_n = T_{i_{k'}} \) and \( f_n \to f \) in \( C_{c_{o}}(T_{i_{k'}},T) \) as \( n \to \infty \); in particular, \( f_n(y') \to f(y') = y \). So we can assume that \( f_n(y') \in T_{i_{k'}} \) for all \( n \).

Take some relatively compact open neighborhood \( V \) of \( y' \) such that \( V \subseteq \text{dom}(g\hat{f}) \cap \text{dom}(gf) \) and \( \hat{f} = f \) in some neighborhood of \( V \). Since \( f_n \to \hat{f} \) in \( S_{c_{o}} \) as \( n \to \infty \), we get \( gf_n \to g\hat{f} \) and \( f_n^{-1} \to \hat{f}^{-1} \) by Propositions 2.6 and 3.1. So \( V \subseteq \text{dom}(gf_n) \) and \( y \in \text{dom } f_n^{-1} = \text{im } f_n \) for \( n \) large enough, and \( f_n^{-1}(y) \to \hat{f}^{-1}(y) = y' \).

Moreover \( gf_n|_V \to g\hat{f}|_V = gf|_V = g|_V \) in \( C_{c_{o}}(V,T) \). So \( \gamma(gf_n,f_n^{-1}(y)) \to \gamma(g',y') \) in \( \hat{T}_{0,U} \) by Proposition 2.2 and the definition of the topology of \( \hat{T} \). Thus, with \( h_n = f_n^{-1} \in S \), we get

\[
\hat{h}_n(\gamma(g,y)) = \gamma(g,h_n^{-1},h_n(y)) = \gamma(gf_n,f_n^{-1}(y)) \to \gamma(g',y'),
\]

and therefore \( \gamma(g',y') \) is in the closure of the \( \hat{T}_{0,U} \)-orbit of \( \gamma(g,y) \). \( \square \)

Remark 28. By Lemma 3.24 the map \( \hat{\pi}_0 : \hat{T}_0 \to T \) generates a morphism of pseudogroups \( \hat{\mathcal{H}}_0 \to \mathcal{H} \) in the sense of [3]—this morphism is not étalé.

The following result is elementary.

Proposition 3.37. In Example 2.37 if \( \mathcal{H} \) is compactly generated and \( \overline{\mathcal{H}} \) is strongly quasi-analytic, then \( \hat{\mathcal{H}}_0 \) is equivalent to the pseudogroup generated by the local action of \( \Gamma \) on \( G \) by local left translations, so that \( \hat{\pi}_0 : \hat{T}_0 \to T \) corresponds to the projection \( T : V^2 \to G/(K,V) \).

Corollary 3.38. The map \( \hat{\pi}_0 : \hat{T}_0 \to T \) is open.

Proof. This follows from Theorem 2.38 and Proposition 3.37 since, in Example 2.37 the projection \( T : V^2 \to G/(K,V) \) is open. \( \square \)

3.6. The closure of \( \hat{\mathcal{H}}_0 \). Let \( \hat{\mathcal{H}}_0 \) be the pseudogroup on \( \hat{T}_0 \) defined like \( \hat{\mathcal{H}} \) by taking the maps \( h \) in \( \hat{S} \) instead of \( S \); thus it is generated by \( \hat{S}_0 = \{ \hat{h} \mid h \in S \} \). Observe that \( \hat{\mathcal{H}}_0 \), \( \mathcal{S} \) and \( \hat{S}_0 \) satisfy the obvious versions of Lemmas 3.24, 3.26, 3.28, 3.30 and 3.31 and Corollaries 3.27 and 3.29 (Section 3.5). In particular, \( \hat{S}_0 \) is a pseudogroup, and \( \hat{T}_{0,U} \) meets all the orbits of \( \hat{\mathcal{H}}_0 \). The restriction of \( \hat{\mathcal{H}}_0 \) to \( \hat{T}_{0,U} \) will be denoted by \( \hat{T}_{0,U} \).
Lemma 3.39. $\overline{\mathcal{H}_0} = \widehat{\mathcal{H}_0}$

Proof. By the version of Lemma 3.31 for $\mathfrak{S}$ and $\widehat{\mathcal{S}_0}$, the set $\widehat{\mathcal{S}_0}$ is dense in $\mathfrak{S}_{0, c.o.}$. Then the result follows easily by Proposition 2.2 and the definition of $\widehat{\mathcal{H}_0}$ (see Theorem 2.34 and Remark 20).

Lemma 3.40. $\overline{\mathcal{H}_0}$ is strongly locally free.

Proof. Let $\hat{h} \in \widehat{\mathcal{S}_0}$ for $h \in \mathfrak{S}$, and $\gamma(g, x) \in \text{dom} \hat{h}$ for $g \in \mathfrak{S}$ and $x \in \text{dom} g \cap \text{dom} h$ with $g(x) = x_0$. Suppose that $\hat{h}(\gamma(g, x)) = \gamma(g, x)$. This means $\gamma(gh^{-1}, h(x)) = \gamma(g, x)$. So $h(x) = x$ and $gh^{-1} = g$ on some neighborhood of $x$, and therefore $h = \text{id}_T$ on some neighborhood of $x$. Then $h = \text{id}_{\text{dom} \hat{h}}$ by the strong quasi-analiticity condition of $\overline{\mathcal{H}}$ since $h \in \mathfrak{S}$. Hence $\hat{h} = \text{id}_{\text{dom} \hat{h}}$ by Lemma 3.25.

Proposition 3.41. There is a locally compact Polish local group $G$ and some dense finitely generated sub-local group $\Gamma \subset G$ such that $\overline{\mathcal{H}_0}$ is equivalent to the pseudogroup defined by the local action of $\Gamma$ on $G$ by local left translations.

Proof. This follows from Remark 22 (see also Theorem 2.34) since $\overline{\mathcal{H}_0}$ is compactly generated (Corollary 3.33) and equicontinuous (Corollary 3.35), and $\overline{\mathcal{H}_0}$ is strongly locally free (Lemma 3.40).

3.7. Independence of the choices involved. First, let us prove that $\widehat{T}_0$ and $\overline{\mathcal{H}_0}$ are independent of the choice of the point $x_0$ up to an equivalence generated by a homeomorphism. Let $x_1$ be another point of $T$, and let $\widehat{T}_1$, $\pi_1$, $\widehat{S}_1$ and $\overline{\mathcal{H}_1}$ be constructed like $\widehat{T}_0$, $\pi_0$, $\widehat{S}_0$ and $\overline{\mathcal{H}_0}$ by using $x_1$ instead of $x_0$. Now, for each $h \in S$, let us use the notation $\hat{h}_0 := \hat{h} \in \widehat{\mathcal{S}_0}$, and let $\hat{h}_1 : \pi_1^{-1}(\text{dom} h) \to \pi_1^{-1}(\text{im} h)$ be the map in $\widehat{S}_1$ defined like $\hat{h}$.

Proposition 3.42. There is a homeomorphism $\theta : \widehat{T}_0 \to \widehat{T}_1$ that generates an equivalence $\Theta : \overline{\mathcal{H}_0} \to \overline{\mathcal{H}_1}$ and so that $\pi_0 = \pi_1 \theta$.

Proof. Since $\mathcal{H}$ is minimal, there is some $f_0 \in \mathfrak{S}$ such that $x_0 \in \text{dom} f_0$ and $f_0(x_0) = x_1$. Let $\theta : \widehat{T}_0 \to \widehat{T}_1$ be defined by $\theta(\gamma(f, x)) = \gamma(f_0 f, x)$. This map is continuous because $\phi(\gamma(f, x)) = \gamma(f_0, x) \gamma(f, x)$. So $\theta$ is a homeomorphism because $f_0^{-1}$ defines $\theta^{-1}$ in the same way. We also have $\pi_0 = \pi_1 \theta$ since $\theta$ preserves the source of each germ. For each $h \in S$, we have $\text{dom} \hat{h}_1 = \theta(\text{dom} \hat{h}_0)$ because $\pi_0 = \pi_1 \theta$, and $\hat{h}_1 \theta = \theta$ since

$$
\hat{h}_1 \theta(\gamma(f, x)) = \hat{h}_1(\gamma(f_0 f, x)) = \gamma(f_0 f h^{-1}, h(x)) = \theta(\gamma(f h^{-1}, h(x))) = \theta(\gamma(f, x))
$$

for all $\gamma(f, x) \in \text{dom} \hat{h}_0$. It follows easily that $\theta$ generates an étale morphism $\Theta : \overline{\mathcal{H}_0} \to \overline{\mathcal{H}_1}$, which is an equivalence since $\theta^{-1}$ generates $\Theta^{-1}$. \qed
Now, let us show that the topology of $\hat{T}$ is independent of the choice of $S$. Therefore the topology of $\hat{T}_0$ will be independent of the choice of $S$ as well. Let $S', S'' \subset \mathcal{H}$ be two sub-pseudo-groups generating $\mathcal{H}$ and satisfying the conditions of Section 3.1. With the notation of Section 3.2, we have to prove the following.

**Proposition 3.43.** $\overline{G}_{\mathcal{S}_c} = \overline{G}_{\mathcal{S}_c}$.

**Proof.** Observe that $S' \cap S''$ is a sub-pseudo-group of $\mathcal{H}$. It also generates $\mathcal{H}$ because $S'$ and $S''$ are local. Moreover $S' \cap S''$ obviously satisfies all other properties required in Section 3.1. Note that a refinement of $\{T_i\}$ may be necessary to get the properties stated in Remarks 23-26 with $S' \cap S''$. Hence the result follows from the special case where $S' \subset S''$. With this assumption, the identity map $\overline{G}_{\mathcal{S}_c} \to \overline{G}_{\mathcal{S}_c}$ is continuous because the diagram

$$
\begin{array}{ccc}
S'_{c-o} & \xrightarrow{\text{inclusion}} & S''_{c-o} \\
\gamma \downarrow & & \downarrow \gamma \\
\overline{G}_{\mathcal{S}_c} & \xrightarrow{\text{identity}} & \overline{G}_{\mathcal{S}_c}
\end{array}
$$

is commutative, where the vertical maps are identifications and the top map is continuous.

For any compact subset $Q \subset T$, let $s^{-1}(Q)_{\mathcal{S}_c}$ and $s^{-1}(Q)_{\mathcal{S}_c}$ denote the spaces obtained by endowing $s^{-1}(Q)$ with the restriction of the topologies of $\mathcal{S}_c$ and $\mathcal{S}_c$, respectively. They are compact and Hausdorff by Propositions 3.17 and 3.18. So $s^{-1}(Q)_{\mathcal{S}_c} = s^{-1}(Q)_{\mathcal{S}_c}$ because the identity map $s^{-1}(Q)_{\mathcal{S}_c} \to s^{-1}(Q)_{\mathcal{S}_c}$ is continuous. Hence, for any $\gamma(f, x) \in \overline{G}$ and a compact neighborhood $Q$ of $x$ in $T$, the set $s^{-1}(Q)$ is a neighborhood of $\gamma(f, x)$ in $\overline{G}_{\mathcal{S}_c}$ and $\overline{G}_{\mathcal{S}_c}$ with $s^{-1}(Q)_{\mathcal{S}_c} = s^{-1}(Q)_{\mathcal{S}_c}$. This shows that the identity map $\overline{G}_{\mathcal{S}_c} \to \overline{G}_{\mathcal{S}_c}$ is a local homeomorphism, and therefore a homeomorphism. 

Let $T'$ be an open subset of $T$ containing $x_0$, which meets all orbits because $\mathcal{H}$ is minimal. Then use $T'$, $\mathcal{H}' = \mathcal{H}|_{T'}$ and $S' = S \cap \mathcal{H}'$ to define $\hat{T}_0', \hat{\pi}_0', \hat{S}_0'$ and $\hat{\mathcal{H}}_0'$ like $\hat{T}_0$, $\hat{\pi}_0$, $\hat{S}_0$ and $\hat{\mathcal{H}}_0$. The proof of the following result is elementary.

**Proposition 3.44.** There is a canonical identity of topological spaces, $\hat{T}_0' \equiv \hat{\pi}_0^{-1}(T')$, so that $\hat{\pi}_0' \equiv \hat{\pi}_0|_{\hat{T}_0'}$ and $\hat{\mathcal{H}}_0' = \hat{\mathcal{H}}_0|_{\hat{T}_0'}$.

**Corollary 3.45.** Let $\mathcal{H}$ and $\mathcal{H}'$ be minimal equicontinuous compactly generated pseudogroups on locally compact Polish spaces such that $\mathcal{H}$ and $\mathcal{H}'$ are strongly quasi-analytic. If $\mathcal{H}$ is equivalent to $\mathcal{H}'$, then $\mathcal{H}_0$ is equivalent to $\mathcal{H}_0'$.

**Proof.** This is a direct consequence of Propositions 3.42 and 3.44. 

\hfill $\Box$
The following definition makes sense by Lemma 2.36, Propositions 3.42 and 3.43, and Corollary 3.45.

**Definition 3.46.** In Proposition 3.41, it is said that (the local isomorphism class of) $G$ is the *structural local group* of (the equivalence class of) $\mathcal{H}$.

4. **Preliminaries on equicontinuous foliated spaces**

Let $X$ and $Z$ be locally compact Polish spaces. A *foliated chart* in $X$ of *leaf dimension* $n$ *transversely modelled* on $Z$, is a pair $(U, \phi)$, where $U \subseteq X$ is open and $\phi : U \to B \times T$ is a homeomorphism for some open $T \subset Z$ and some open ball $B$ in $\mathbb{R}^n$. It is said that $U$ is a *distinguished open set*. The sets $P_y = \phi^{-1}(B \times \{y\})$ ($y \in T$) are called *plaques* of this foliated chart. For each $x \in B$, the set $S_x = \phi^{-1}(\{x\} \times T)$ is called a *transversal* of the foliated chart. This local product structure defines a local projection $p : U \to T$, called *distinguished submersion*, given as composition of $\phi$ with the second factor projection $pr_2 : B \times T \to T$.

Let $\mathcal{U} = \{U_i, \phi_i\}$ be a family of foliated charts in $X$ of leaf dimension $n$ modelled transversally on $Z$ and covering $X$. Assume further that the foliated charts are *coherently foliated* in the sense that, if $P$ and $Q$ are plaques in different charts of $\mathcal{U}$, then $P \cap Q$ is open both in $P$ and $Q$. Then $\mathcal{U}$ is called a *foliated atlas* on $X$ of leaf dimension $n$ and *transversely modelled* on $Z$. A maximal foliated atlas $\mathcal{F}$ of leaf dimension $n$ and transversely modelled on $Z$ is called a *foliated structure* on $X$ of leaf dimension $n$ and *transversely modelled* on $Z$. Any foliated atlas $\mathcal{U}$ of this type is contained in a unique foliated structure $\mathcal{F}$; then it is said that $\mathcal{U}$ *defines* (or is an atlas of) $\mathcal{F}$. If $Z = \mathbb{R}^m$, then $X$ is a manifold of dimension $n + m$, and $\mathcal{F}$ is traditionally called a *foliation of dimension* $n$ and *codimension* $m$. The reference to $Z$ will be omitted.

For a foliated structure $\mathcal{F}$ on $X$ of dimension $n$, the plaques form a basis of a topology on $X$ called the *leaf topology*. With the leaf topology, $X$ becomes an $n$-manifold whose connected components are called *leaves* of $\mathcal{F}$. $\mathcal{F}$ is determined by its leaves.

A foliated atlas $\mathcal{U} = \{U_i, \phi_i\}$ of $\mathcal{F}$ is called *regular* if

- each $\overline{U_i}$ is compact subset of a foliated chart $(W_i, \psi_i)$ and $\phi_i = \psi_i|_{U_i}$;
- the cover $\{U_i\}$ is *locally finite*; and,
- if $(U_i, \phi_i)$ and $(U_j, \phi_j)$ are elements of $\mathcal{U}$, then each plaque $P$ of $(U_i, \phi_i)$ meets at most one plaque of $(U_j, \phi_j)$.

In this case, there are homeomorphisms $h_{ij} : T_{ij} \to T_{ji}$ such that $h_{ij} p_i = p_j$ on $U_i \cap U_j$, where $p_i : U_i \to T_i$ is the distinguished submersion defined by $(U_i, \phi_i)$ and $T_{ij} = p_i(U_i \cap U_j)$. Observe that the cocycle condition $h_{ik} = h_{jk} h_{ij}$ is satisfied on $T_{ijk} = p_i(U_i \cap U_j \cap U_k)$. For this reason, $\{U_i, p_i, h_{ij}\}$ is called a *defining cocycle* of $\mathcal{F}$ with values in $Z$—we only consider defining cocycles induced by regular foliated atlases. The equivalence class of the pseudogroup $\mathcal{H}$ generated by the maps $h_{ij}$ on $T = \bigsqcup_{i \in I} T_i$ is called the *holonomy pseudogroup* of the foliated space $(X, \mathcal{F})$; $\mathcal{H}$ is the representative
of the holonomy pseudogroup of \((X, \mathcal{F})\) induced by the defining cocycle \(\{U_i, p_i, h_{ij}\}\). This \(T\) can be identified with a total (or complete) transversal to the leaves in the sense that it meets all leaves and is locally given by the transversals defined by foliated charts. All compositions of maps \(h_{ij}\) form a pseudo-group \(\mathcal{H}\), called the holonomy pseudo-group of \(\mathcal{F}\) induced by \(\{U_i, p_i, h_{ij}\}\). There is a canonical identity between the space of leaves and the space of \(\mathcal{H}\)-orbits, \(X/\mathcal{F} \equiv T/\mathcal{H}\).

A foliated atlas (respectively, defining cocycle) contained in another one is called sub-foliated atlas (respectively, sub-foliated cocycle).

The holonomy group of each leaf \(L\) is defined as the germ group of the corresponding orbit. It can be considered as a quotient of \(\pi_1(L)\) by taking “chains” of sets \(U_i\) along loops in \(L\); this representation of \(\pi_1(L)\) is called the holonomy representation. The kernel of the holonomy representation is equal to \(q_\ast \pi_1(\tilde{L})\) for a regular covering space \(q: \tilde{L} \to L\), which is called the holonomy cover of \(L\). If \(\mathcal{F}\) admits a countable defining cocycle, then the leaves in some dense \(G_\delta\) subset of \(M\) have trivial holonomy groups \([27, 28, 11]\), and therefore they can be identified with their holonomy covers.

It is said that a foliated space is (topologically) transitive or minimal if any representative of its holonomy pseudogroup is such. Transitivity (respectively, minimality) of a foliated space means that some leaf is dense (respectively, all leaves are dense).

Haefliger [26] has observed that, if \(X\) is compact, then \(\mathcal{H}\) is compactly generated, which can be seen as follows. There is some defining cocycle \(\{U'_i, p'_i, h'_{ij}\}\), with \(p'_i: U'_i \to T'_i\), such that \(\overline{U_i} \subset U'_i, T_i \subset T'_i\), and \(p'_i\) extends \(p_i\). Therefore each \(h'_{ij}\) is an extension of \(h_{ij}\) so that \(\text{dom } h'_{ij} \subset \text{dom } h_{ij}\). Moreover \(\mathcal{H}\) is the restriction to \(T\) of the pseudogroup \(\mathcal{H}'\) on \(T' = \bigsqcup_i T'_i\) generated by the maps \(h'_{ij}\), and \(T\) is a relatively compact open subset of \(T'\) that meets all \(\mathcal{H}'\)-orbits.

**Definition 4.1.** It is said that a foliated space is equicontinuous if any representative of its holonomy pseudogroup is equicontinuous.

**Remark 29.** The definition of equicontinuity for a foliated space makes sense by Lemma 2.30.

**Definition 4.2.** Let \(G\) be a locally compact Polish local group. A minimal foliated space is called a \(G\)-foliated space if its holonomy pseudogroup can be represented by a pseudogroup given by Example 2.35 on a local group locally isomorphic to \(G\).

5. Molino’s theory for equicontinuous foliated spaces

5.1. Molino’s theory for equicontinuous foliated spaces. Let \((X, \mathcal{F})\) be a compact minimal foliated space that is equicontinuous and such that the closure of its holonomy pseudogroup is strongly quasi-analytic. Let \(\{U_i, p_i, h_{ij}\}\) be a defining cocyle of \(\mathcal{F}\) induced by a regular foliated atlas, where \(p_i: U_i \to T_i\). Let \(\mathcal{H}\) denote the corresponding representative of
the holonomy pseudogroup on $T = \bigsqcup T_i$, which satisfies the conditions of Section 3.1. Let $S$ be the localization of the holonomy pseudo-group induced by $\{U_i, p_i, h_{ij}\}$. Fix an index $i_0$ and a point $x_0 \in U_{i_0}$. Let $\tilde{\pi}_0 : \tilde{T}_0 \to T$ and $\tilde{H}_0$ be defined like in Sections 3.3 and 3.5 by using $T$, $H$, the point $p_{i_0}(x_0) \in T_{i_0} \subset T$, and a local sub-pseudo-group $S \subset H$.

With the notation $\tilde{T}_{i,0} = \tilde{\pi}_0^{-1}(T_i) \subset \tilde{T}_0$, let

$$\tilde{X}_0 = \bigcup_i U_i \times \tilde{T}_{i,0} = \bigcup_i U_i \times \tilde{T}_{i,0} \times \{i\},$$

equipped with the corresponding topological sum of the product topologies, and consider its closed subspace

$$\tilde{X}_0 = \{(x, \gamma, i) \in \tilde{X}_0 \mid p_i(x) = \tilde{\pi}_0(\gamma)\} \subset \tilde{X}_0.$$

For $(x, \gamma, i), (y, \delta, j) \in \tilde{X}_0$, write $(x, \gamma, i) \sim (y, \delta, j)$ if $x = y$ and $\gamma = \tilde{h}_{ji}(\delta)$. Since $h_{ij}p_j(x) = p_j(y), h_{ij}^{-1} = h_{ij}$ and $h_{ik} = h_{jk}h_{ij}$, it follows that this defines an equivalence relation “$\sim$” on $\tilde{X}_0$. Let $\tilde{X}_0$ be the corresponding quotient space, $q : \tilde{X}_0 \to \tilde{X}_0$ the quotient map, and $[x, \gamma, i]$ the equivalence class of each triple $(x, \gamma, i)$. For each $i$, let

$$U_{i,0} = U_i \times \tilde{T}_{i,0} \times \{i\}, \quad \tilde{U}_{i,0} = U_{i,0} \cap \tilde{X}_0, \quad \tilde{U}_{i,0} = q(U_{i,0}).$$

**Lemma 5.1.** $\tilde{U}_{i,0}$ is open in $\tilde{X}_0$.

**Proof.** We have to check that $q^{-1}(\tilde{U}_{i,0}) \cap \tilde{U}_{j,0}$ is open in $\tilde{U}_{j,0}$ for all $j$, which is true because

$$q^{-1}(\tilde{U}_{i,0}) \cap \tilde{U}_{j,0} = ((U_i \cap U_j) \times \tilde{T}_{j,0} \times \{j\}) \cap \tilde{X}_0. \qed$$

**Lemma 5.2.** $q : \tilde{U}_{i,0} \to \tilde{U}_{i,0}$ is a homeomorphism.

**Proof.** This map is surjective by the definition of $\tilde{U}_{i,0}$. On the other hand, two equivalent triples in $\tilde{U}_{i,0}$ are of the form $(x, \gamma, i)$ and $(x, \delta, i)$ with $\gamma = \tilde{h}_{ii}(\delta) = \delta$. So $q : \tilde{U}_{i,0} \to \tilde{U}_{i,0}$ is also injective. Since $q : \tilde{U}_{i,0} \to \tilde{U}_{i,0}$ is continuous, it only remains to prove that this map is open. A basis of the topology of $\tilde{U}_{i,0}$ consists of the sets of the form $(V \times W \times \{i\}) \cap \tilde{X}_0$, where $V$ and $W$ are open in $U_i$ and $\tilde{T}_{i,0}$, respectively. These basic sets satisfy

$$\tilde{U}_{j,0} \cap q^{-1}q((V \times W \times \{i\}) \cap \tilde{X}_0) = \tilde{U}_{j,0} \cap \left(V \times \tilde{h}_{ij}(W \cap \text{dom }\tilde{h}_{ij}) \times \{j\}\right)$$

for all $j$, which is open in $\tilde{U}_{j,0}$. So $q^{-1}q((V \times W \times \{i\}) \cap \tilde{X}_0)$ is open in $\tilde{X}_0$ and therefore $q((V \times W \times \{i\}) \cap \tilde{X}_0)$ is open in $\tilde{X}_0$. \qed

**Proposition 5.3.** $\tilde{X}_0$ is compact and Polish.
Suppose that the sets \( \hat{U}' \) are locally compact open subset of \( U_i \) for all \( i \). Therefore each \( h_{ij}' \) is also a restriction of \( h_{ij} \) and \( T_i' \) is a relatively locally compact open subset of \( T_i \). Then \( \tilde{\pi}_0^{-1}(T_i) \) is a compact subset of \( T_{i,0} \) by Corollary 3.21. Moreover \( \hat{X}_0 \) is the union of the sets \( q(\hat{U}' \times \tilde{\pi}_0^{-1}(T_i) \times \{i\}) \).

So \( \hat{X}_0 \) is compact because it is a finite union of compact sets.

On the other hand, since \( \check{X}_0 \) is closed in \( \hat{X}_0 \), and \( \hat{U}_{i,0} \) is Polish and locally compact by Corollary 3.19, it follows that \( \hat{U}_{i,0} \) is Polish and locally compact, and therefore \( \check{U}_{i,0} \) is Polish and locally compact by Lemma 5.2. Then, by the compactness of \( \check{X}_0 \), Lemma 5.1 and [30, Theorem 5.3], it only remains to prove that \( \check{X}_0 \) is Hausdorff.

Let \( [x, \gamma, i] \neq [y, \delta, j] \) in \( \check{X}_0 \). So \( x \in U_i \) and \( y \in U_j \). If \( x = y \), then \( [y, \delta, j] = [x, \tilde{h}_{ji}(\delta), i] \in \hat{U}_{i,0} \). Thus, in this case, \( [x, \gamma, i] \) and \( [y, \delta, j] \) can be separated by open subsets of \( \hat{U}_{i,0} \) because \( \hat{U}_{i,0} \) is Hausdorff.

Now suppose that \( x \neq y \). Then take disjoint open neighborhoods, \( V \) of \( x \) in \( U_i \) and \( W \) of \( y \) in \( U_j \). Let

\[
\check{V} = V \times \check{T}_{i,0} \times \{i\} \subset \hat{U}_{i,0}, \quad \check{W} = W \times \check{T}_{j,0} \times \{j\} \subset \hat{U}_{j,0},
\]

\[
\hat{V} = \check{V} \cap \hat{X}_0 \subset \hat{U}_{i,0}, \quad \hat{W} = \check{W} \cap \hat{X}_0 \subset \hat{U}_{j,0},
\]

\[
\check{V} = q(\check{V}) \subset \check{U}_{i,0}, \quad \check{W} = q(\check{W}) \subset \check{U}_{j,0}.
\]

The sets \( \check{V} \) and \( \check{W} \) are open neighborhoods of \( [x, \gamma, i] \) and \( [y, \delta, j] \) in \( \check{X}_0 \). Suppose that \( \check{V} \cap \check{W} \neq \emptyset \). Then there is a point \( (x', \gamma', i) \in \check{V} \) equivalent to some point \( (y', \delta', j) \in \check{W} \). This implies that \( x' = y' \in V \cap W \), which is a contradiction because \( V \cap W = \emptyset \). Therefore \( \check{V} \cap \check{W} = \emptyset \).

According to the above equivalence relation of triples, a map \( \tilde{\pi}_0 : \check{X}_0 \to X \) is defined by \( \tilde{\pi}_0([x, \gamma, i]) = x \).

**Proposition 5.4.** The map \( \tilde{\pi}_0 : \check{X}_0 \to X \) is continuous and surjective, and its fibers are homeomorphic to each other.

**Proof.** Since each map \( \tilde{\pi}_0 : \check{T}_{i,0} \to T_i \) is surjective, we have \( \tilde{\pi}_0(\hat{U}_{i,0}) = U_i \), obtaining that \( \tilde{\pi}_0 : \check{X}_0 \to X \) is surjective. Moreover the composition

\[
\check{U}_{i,0} \xrightarrow{q} \hat{U}_{i,0} \xrightarrow{\tilde{\pi}_0} U_i,
\]

is the restriction of the first factor projection \( \hat{U}_{i,0} \to U_i \), \( (x, \gamma, i) \mapsto x \). So \( \tilde{\pi}_0 : \check{X}_0 \to X \) is continuous by Lemmas 5.1 and 5.2.

For \( x \in U_i \), we have \( \tilde{\pi}_0^{-1}(x) \subset \hat{U}_{i,0} \) and

\[
\hat{U}_{i,0} \cap q^{-1}(\tilde{\pi}_0^{-1}(x)) = \{x\} \times \tilde{\pi}_0^{-1}(p_i(x)) \times \{i\} \equiv \hat{U}_{i,0}^{-1}(p_i(x)) \subset \hat{T}_{i,0}.
\]

So the last assertion of the statement follows from Lemma 5.2 and Proposition 5.22. \( \square \)
Let $\tilde{p}_{i,0} : \tilde{U}_{i,0} \to \tilde{T}_{i,0}$ denote the restriction of the second factor projection $\tilde{p}_{i,0} : \tilde{U}_{i,0} = U_i \times \tilde{T}_{i,0} \times \{i\} \to \tilde{T}_{i,0}$. By Lemma 5.2, $\tilde{p}_{i,0}$ induces a continuous map $\tilde{p}_{i,0} : \tilde{U}_{i,0} \to \tilde{T}_{i,0}$.

**Proposition 5.5.** $\{\tilde{U}_{0,i}, \tilde{p}_{i,0}, \tilde{h}_{ij}\}$ is a defining cocycle of a foliated structure $\tilde{\mathcal{F}}_0$ on $\tilde{X}_0$.

**Proof.** Let $\{U_i, \phi_i\}$ be a regular foliated atlas of $\mathcal{F}$ inducing the defining cocycle $\{U_i, \tilde{p}_i, h_{ij}\}$, where $\phi_i : U_i \to B_i \times T_i$ is a homeomorphism and $B_i$ is a ball in $\mathbb{R}^n$ ($n = \dim \mathcal{F}$). Then we get a homeomorphism

$$\tilde{\phi}_{i,0} = \phi_i \times \text{id} \times \text{id} : \tilde{U}_{i,0} = U_i \times \tilde{T}_{i,0} \times \{i\} \to B_i \times T_i \times \tilde{T}_{i,0} \times \{i\}.$$

Observe that $\tilde{\phi}_{i,0}(\tilde{U}_{i,0})$ consists of the elements $(y, z, \gamma, i)$ with $\tilde{\pi}_0(\gamma) = z$. So $\tilde{\phi}_{i,0}$ restricts to a homeomorphism

$$\tilde{\phi}_{i,0} : \tilde{U}_{i,0} \to \tilde{\phi}_{i,0}(\tilde{U}_{i,0}) \equiv B_i \times \tilde{T}_{i,0} \times \{i\} \equiv B_i \times \tilde{T}_{i,0}.$$

By Lemma 5.2, $\tilde{\phi}_{i,0}$ induces a homeomorphism $\tilde{\phi}_i : \tilde{U}_{i,0} \to B_i \times \tilde{T}_{i,0}$. Moreover, $\tilde{p}_{i,0}$ corresponds to the third factor projection via $\tilde{\phi}_{i,0}$, obtaining that $\tilde{p}_{i,0}$ corresponds to the second factor projection via $\tilde{\phi}_{i,0}$, and therefore $\tilde{p}_{i,0}$ also corresponds to the second factor projection via $\tilde{\phi}_{i,0}$. Observe that $\tilde{p}_{i,0} = \tilde{h}_{ji}\tilde{p}_{j,0}$ on $\tilde{U}_{i,0} \cap \tilde{U}_{j,0}$ by the definition of “~”. The regularity of the foliated atlas $\{\tilde{U}_{0,i}, \tilde{\phi}_{i,0}\}$ follows easily from the regularity of $\{U_i, \phi_i\}$. □

According to Proposition 5.5, the holonomy pseudogroup of $\tilde{\mathcal{F}}_0$ is represented by the pseudogroup on $\bigsqcup_i \tilde{T}_{i,0}$ generated by the maps $\tilde{h}_{ij}$, which is the pseudogroup $\tilde{\mathcal{H}}_0$ on $\tilde{T}_0$.

**Corollary 5.6.** There is some locally compact Polish local group $G$ such that $(\tilde{X}_0, \tilde{\mathcal{F}}_0)$ is a minimal $G$-foliated space; in particular, it is equicontinuous.

**Proof.** This follows from Propositions 5.5 and 3.41 and Corollary 3.36 □

**Proposition 5.7.** $\tilde{\pi}_0 : (\tilde{X}_0, \tilde{\mathcal{F}}_0) \to (X, \mathcal{F})$ is a foliated map.

**Proof.** According to Proposition 5.5, this follows by checking the commutativity of each diagram

$$
\begin{array}{ccc}
\tilde{U}_{i,0} & \xrightarrow{\tilde{p}_{i,0}} & \tilde{T}_{i,0} \\
\tilde{\pi}_0 \downarrow & & \downarrow \tilde{\pi}_0 \\
U_i & \xrightarrow{p_i} & T_i
\end{array}
$$

By Lemma 5.2 and the definition of $\tilde{p}_{i,0}$ and $\tilde{\pi}_0$, this commutativity follows from the commutativity of

$$
\begin{array}{ccc}
\tilde{U}_{i,0} & \longrightarrow & \tilde{T}_{i,0} \\
\downarrow & & \downarrow \tilde{\pi}_0 \\
U_i & \longrightarrow & T_i
\end{array}
$$
where the left vertical and the top horizontal arrows denote the restrictions of the first and second factor projections of \( \hat{U}_{i,0} = U_i \times \hat{T}_{i,0} \times \{ i \} \). But the commutativity of this diagram holds by the definition of \( \hat{X}_0 \). \( \square \)

**Proposition 5.8.** The restrictions of \( \hat{\pi}_0 : \hat{X}_0 \to X \) to the leaves are the holonomy covers of the leaves of \( \mathcal{F} \).

**Proof.** With the notation of the proof of Proposition 5.5, the diagram

\[
\begin{array}{ccc}
\hat{U}_{i,0} & \overset{\hat{\phi}_{i,0}}{\longrightarrow} & B_i \times \hat{T}_{i,0} \\
\hat{\pi}_0 \downarrow & & \downarrow \text{id}_{\hat{T}} \times \hat{\pi}_0 \\
U_i & \overset{\hat{\phi}_i}{\longrightarrow} & B_i \times T_i \\
\end{array}
\]

(19)

is commutative, and \( \hat{U}_{i,0} = \hat{\pi}_0^{-1}(U_i) \). Hence, for corresponding plaques in \( U_i \) and \( \hat{U}_{i,0} \), \( P_z = \hat{\phi}_i^{-1}(B_i \times \{ \hat{z} \}) \) and \( \hat{P}_z = \hat{\phi}_{i,0}^{-1}(B_i \times \{ \hat{z} \}) \) with \( z \in T_i \) and \( \hat{z} \in \hat{\pi}_0^{-1}(z) \subset \hat{T}_{i,0} \), the restriction \( \hat{\pi}_0 : \hat{P}_z \to P_z \) is a homeomorphism. It follows easily that \( \hat{\pi}_0 : \hat{X}_0 \to X \) restricts to covering maps of the leaves of \( \hat{\mathcal{F}}_0 \) to the leaves of \( \mathcal{F} \). In fact, these are the holonomy covers, which can be seen as follows.

According to the proof of Proposition 5.4 and the definition of the equivalence relation “\( \sim \)” on \( \hat{X}_0 \), for each \( x \) in \( U_i \cap U_j \), we have homeomorphisms

\[
\hat{\pi}_0^{-1}(p_i(x)) \overset{\hat{\phi}_{i,0}}{\longleftarrow} \hat{\pi}_0^{-1}(x) \overset{\hat{p}_{j,0}}{\longrightarrow} \hat{\pi}_0^{-1}(p_j(x))
\]

satisfying \( \hat{p}_{j,0} \hat{\pi}_{i,0}^{-1} = \hat{h}_{ij} \). This easily implies the following. Given \( x \in U_i \) and \( \hat{x} \in \hat{\pi}_0^{-1}(X) \), denoting by \( L \) and \( \hat{L} \) the leaves through \( x \) and \( \hat{x} \), respectively, and given a loop \( c \) in \( L \) based at \( x \) inducing a local holonomy transformation \( h \in S \) around \( p_i(x) \) in \( T_i \), the lift \( \hat{c} \) of \( c \) to \( \hat{L} \) with \( \hat{c}(0) = \hat{x} \) satisfies \( \hat{p}_{i,0} \hat{c}(1) = \hat{h} \hat{p}_{i,0}(\hat{x}) \). Writing \( \hat{p}_{i,0}(\hat{x}) = \gamma(f, p_i(x)) \), we obtain

\[
\hat{p}_{i,0} \hat{c}(1) = \hat{h}(\gamma(f, p_i(x))) = \gamma(f h, p_i(x)).
\]

Thus \( \hat{c} \) is a loop if and only if \( \gamma(f h, p_i(x)) = \gamma(f, p_i(x)) \), which means \( \gamma(h, p_i(x)) = \gamma(id_T, p_i(x)) \). So \( \hat{L} \) is the holonomy cover of \( L \). \( \square \)

**Proposition 5.9.** The map \( \hat{\pi}_0 : \hat{X}_0 \to X \) is open

**Proof.** This follows from Corollary 5.38 and the commutativity of (19). \( \square \)

Theorem A is the combination of the results of this section.

### 5.2. Independence of the choices involved

Let \( x_1 \) be another point of \( X \), and let \( \hat{X}_1, \hat{\mathcal{F}}_1 \) and \( \hat{\pi}_1 : \hat{X}_1 \to X \) be constructed like \( \hat{X}_0, \hat{\mathcal{F}}_0 \) and \( \hat{\pi}_0 : \hat{X}_0 \to X \) by using \( x_1 \) instead of \( x_0 \).

**Proposition 5.10.** There is a foliated homeomorphism \( \hat{\theta} : (\hat{X}_0, \hat{\mathcal{F}}_0) \to (\hat{X}_1, \hat{\mathcal{F}}_1) \) such that \( \hat{\pi}_1 F = \hat{\pi}_0 \).
Proof. Take an index $i_1$ such that $x_1 \in U_{i_1}$. Let $\hat{S}_1, \hat{T}_1, \hat{\mathcal{H}}_1$ and $\hat{\pi}_1 : \hat{T}_1 \to T$ be constructed like $\hat{S}_0, \hat{T}_0, \hat{\mathcal{H}}_0$ and $\hat{\pi}_0 : \hat{T}_0 \to T$ by using $p_{i_1}(x_1)$ instead of $p_{i_0}(x_0)$, and let $\hat{T}_{i_1} = \hat{\pi}_1^{-1}(T_i)$. Then the construction of $\hat{X}_1, \hat{\mathcal{F}}_1$ and $\hat{\pi}_1 : \hat{X}_1 \to X$ involves the objects $\hat{X}_1, \hat{\mathcal{X}}_1, \hat{\mathcal{U}}_{i,1}, \hat{\mathcal{U}}_i$, and $\hat{\pi}_1 : \hat{T}_{i_1} \to T_i$ instead of $\hat{T}_i$ and $\hat{\pi}_0 : \hat{T}_{i_0} \to T_i$.

Let $\theta : \hat{T}_0 \to \hat{T}_1$ be the homeomorphism given by Proposition 3.42, which obviously restricts to homeomorphisms $\theta_i : \hat{T}_{i_0} \to \hat{T}_{i_1}$. Since $\hat{\pi}_0 = \hat{\pi}_1 \theta$, it follows that each homeomorphism

$$\hat{\theta}_i = id_{U_i} \times \theta_i \times id : \hat{U}_{i_0} = U_i \times \hat{T}_{i_1} \times \{ i \} \to \hat{U}_{i_1} = U_i \times \hat{T}_{i_1} \times \{ i \}$$

restricts to a homeomorphism $\hat{\theta}_i = \hat{U}_{i_0} \to \hat{U}_{i_1}$. The combination of the homeomorphisms $\theta_i$ is a homeomorphism $\hat{\theta} : \hat{X}_0 \to \hat{X}_1$.

For each $h \in S$, use the notation $\hat{h}_0 \in \hat{S}_0$ and $\hat{h}_1 \in \hat{S}_1$ for the map $\hat{h}$ defined with $p_{i_0}(x_0)$ and $p_{i_1}(x_1)$, respectively. From the proof of Proposition 3.42 we get $\hat{h}_1 \theta = \theta \hat{h}_0$ for all $h \in S$; in particular, this holds with $h = h_{ij}$. So $\hat{\theta} : \hat{X}_0 \to \hat{X}_1$ is compatible with the equivalence relations used to define $\hat{X}_0$ and $\hat{X}_1$, and therefore it induces a homeomorphism $\hat{\theta} : \hat{X}_0 \to \hat{X}_1$. Note that $\hat{\theta}$ restricts to homeomorphisms $\theta_i : \hat{U}_{i_0} \to \hat{U}_{i_1}$. Obviously, $\hat{\pi}_i \hat{\theta}_i = \theta_i \hat{\pi}_i$, yielding $\hat{\pi}_i \hat{\theta}_i = \theta_i \hat{\pi}_i$, and therefore $\hat{\pi}_i \hat{\theta}_i = \theta_i \hat{\pi}_i$. It follows that $\theta$ is a foliated map.

Let $\{ U_{a}'_i, p_{a}', h_{ab}' \}$ be another defining cocycle of $\mathcal{F}$ induced by a regular foliated atlas. Then let $\hat{X}_0', \hat{\mathcal{F}}_0'$ and $\hat{\pi}_0' : \hat{X}_1 \to X$ be constructed like $\hat{X}_0, \hat{\mathcal{F}}_0$ and $\hat{\pi}_0 : \hat{X}_0 \to X$ by using $\{ U_{a}'_i, p_{a}'_i, h_{ab}' \}$ instead of $\{ U_i, p_{i}, h_{ij} \}$.

**Proposition 5.11.** There is a foliated homeomorphism $F : (\hat{X}_0, \hat{\mathcal{F}}_0) \to (\hat{X}_0', \hat{\mathcal{F}}_0')$ such that $\hat{\pi}_0' F = \hat{\pi}_0$.

**Proof.** By using a common refinement of the open coverings $\{ U_i \}$ and $\{ U_{a}'_i \}$, we can assume that $\{ U_{a}'_i \}$ refines $\{ U_i \}$. In this case, the union of the defining cocycles $\{ U_i, p_{i}, h_{ij} \}$ and $\{ U_{a}'_i, p_{a}'_i, h_{ab}' \}$ is contained in another defining cocycle induced by a regular foliated atlas. Thus the proof boils down to showing that a sub-defining cocycle $\{ U_{k}, p_{k}, h_{kij} \}$ of $\{ U_i, p_{i}, h_{ij} \}$ induces a foliated space homeomorphic to $(\hat{X}_0, \hat{\mathcal{F}}_0)$. But the pseudogroup $\mathcal{H}'$ induced by $\{ U_{k}, p_{k}, h_{kij} \}$ is the restriction of $\mathcal{H}$ to an open subset $T' \subset T$, and the pseudo-group induced by $\{ U_{k}, p_{k}, h_{kij} \}$ is $S' = S \cap H'$. Then, by using the canonical identity given by Proposition 3.44, it easily follows that the foliated space $(\hat{X}_0', \hat{\mathcal{F}}_0')$ defined with $\{ U_{k}, p_{k}, h_{kij} \}$ can be canonically identified with an open foliated subspace of $(\hat{X}_0, \hat{\mathcal{F}}_0)$, which indeed is the whole of $(\hat{X}_0, \hat{\mathcal{F}}_0)$ because $\{ U_k \}$ covers $X$. \[\Box\]

---

6A sub-defining cocycle is a defining cocycle contained in another one.
6. Preliminaries on the growth of orbits and leaves

6.1. Coarse quasi-isometries and growth of metric spaces. A net in a metric space \( M \), with metric \( d \), is a subset \( A \subset M \) that satisfies \( d(x, A) \leq C \) for some \( C > 0 \) and all \( x \in M \); the term \( C \)-net is also used. A coarse quasi-isometry between \( M \) and another metric space \( M' \) is a bi-Lipschitz bijection between nets of \( M \) and \( M' \); in this case, \( M \) and \( M' \) are said to be coarsely quasi-isometric (in the sense of Gromov) \([23]\). If such a bi-Lipschitz bijection, as well as its inverse, has dilation \( \leq \lambda \), and it is defined between \( C \)-nets, then it will be said that the coarse quasi-isometry has distortion \((C, \lambda)\). A family of coarse quasi-isometries with a common distortion will be called uniform, and the corresponding metric spaces are called uniformly coarsely quasi-isometric.

The version of growth for metric spaces given here is taken from \([4, 7]\). Recall that, given non-decreasing functions \( u, v : [0, \infty) \to [0, \infty) \), it is said that \( u \) is dominated by \( v \), written \( u \preceq v \), when there are \( a, b \geq 1 \) and \( c \geq 0 \) such that \( u(r) \leq av(br) \) for all \( r \geq c \). If \( u \preceq v \preceq u \), then it is said that \( u \) and \( v \) represent the same growth type; this is an equivalence relation and \( \preceq \) defines a partial order relation between growth types called domination. For a family of pairs of non-decreasing functions \( [0, \infty) \to [0, \infty) \), uniform domination means that those pairs satisfy the above condition of domination with the same constants \( a, b, c \). A family of functions \( [0, \infty) \to [0, \infty) \) will be said to have uniformly the same growth type if they uniformly dominate one another.

For a complete connected Riemannian manifold \( L \), the growth type of each mapping \( r \mapsto \text{vol} B(x, r) \) is independent of \( x \) and is called the growth type of \( L \). Another definition of growth type can be similarly given for metric spaces whose bounded sets are finite, where the number of points is used instead of the volume.

Let \( M \) be a metric space with metric \( d \). A quasi-lattice \( \Gamma \) of \( M \) for some \( C \geq 0 \) such that, for every \( r \geq 0 \), there is some \( K_r \geq 0 \) such that \( \text{card}(\Gamma \cap B(x, r)) \leq K_r \) for every \( x \in M \). It is said that \( M \) is of coarse bounded geometry if it has a quasi-lattice. In this case, the growth type of \( M \) can be defined as the growth type of any quasi-lattice \( \Gamma \) of \( M \); i.e., it is the growth type of the growth function \( r \mapsto v_{\Gamma}(x, r) = \text{card}(B(x, r) \cap \Gamma) \) for any \( x \in \Gamma \). This definition is independent of \( \Gamma \).

For a family of metric spaces, if they satisfy the above condition of coarse bounded geometry with the same constants \( C \) and \( K_r \), then they are said to

\[ \text{Usually, growth types are defined by using non-decreasing functions } Z^+ \to [0, \infty), \text{ but non-decreasing functions } [0, \infty) \to [0, \infty) \text{ give rise to an equivalent concept.} \]
have uniformly coarse bounded geometry. If moreover the lattices involved in this condition have growth functions defining uniformly the same growth type, then these metric spaces are said to have uniformly the same growth type.

The condition of coarse bounded geometry is satisfied by complete connected Riemannian manifolds of bounded geometry, and by discrete metric spaces with a uniform upper bound on the number of points in all balls of each given radius [9]. In those cases, the two given definitions of growth type are equal.

Lemma 6.1 (Álvarez-Candel [5]; see also [7, Lemma 2.1]). Two coarsely quasi-isometric metric spaces of coarse bounded geometry have the same growth type. Moreover, if a family of metric spaces are uniformly coarsely quasi-isometric to each other, then they have uniformly the same growth type.

6.2. Quasi-isometry and growth types of orbits. Let \( H \) be a pseudogroup on a space \( T \), and \( E \) a symmetric set of generators of \( H \). Let \( \mathcal{G} \) be the groupoid of germs of maps in \( H \).

For each \( h \in H \) and \( x \in \text{dom} \ h \), let \( |h|_{E,x} \) be the length of the shortest expression of \( \gamma(h,x) \) as product of germs of maps in \( E \) (being 0 if \( \gamma(h,x) = \gamma(\text{id}_T, x) \)). For each \( x \in T \), define metrics \( d_E \) on \( \mathcal{G}(x) \) and \( \mathcal{G}_x \) by

\[
d_E(y, z) = \min \{ |h|_{E,y} \mid h \in H, \ y \in \text{dom} \ h, \ h(y) = z \},
\]

\[
d_E(\gamma(f, x), \gamma(g, x)) = |fg^{-1}|_{E,g(x)}.
\]

Notice that

\[
d_E(f(x), g(x)) \leq d_E(\gamma(f, x), \gamma(g, x)).
\]

Moreover, on the germ covers, \( d_E \) is right invariant in the sense that, if \( y \in \mathcal{G}(x) \), the bijection \( \mathcal{G}_y \to \mathcal{G}_x \), given by right multiplication with any element in \( \mathcal{G}_y \), is isometric; so the isometry types of the germ covers of the orbits make sense without any reference to base points. In fact, the definition of \( d_E \) on \( \mathcal{G}_x \) is analogous to the definition of the right invariant metric \( d_S \) on a group \( \Gamma \) induced by a symmetric set of generators \( S \): \( d_S(\gamma, \delta) = |\gamma\delta^{-1}| \) for \( \gamma, \delta \in \Gamma \), where \( |\gamma| \) is the length of the shortest expression of \( \gamma \) as product of elements of \( S \) (being 0 if \( \gamma = e \)).

Assume that \( H \) is compactly generated and \( T \) locally compact. Let \( U \subset T \) be a relatively compact open subset that meets all \( H \)-orbits, let \( \mathcal{G} = H|_U \), and let \( E \) be a symmetric system of compact generation of \( H \) on \( U \). With this conditions, the quasi-isometry type of the \( \mathcal{G} \)-orbits with \( d_E \) may depend on \( E \) [5, Section 6]. So the following additional condition on \( E \) is considered.

Definition 6.2 (Álvarez-Candel [5, Definition 4.2]). With the above notation, it is said that \( E \) is recurrent if, for any relatively compact open subset \( V \subset U \) that meets all \( \mathcal{G} \)-orbits, there exists some \( R > 0 \) such that \( \mathcal{G}(x) \cap V \) is an \( R \)-net in \( \mathcal{G}(x) \) with \( d_E \) for all \( x \in U \).
The role played by \( V \) in Definition 6.2 can actually be played by any relatively compact open subset that meets all orbits \([5, \text{Lemma 4.3}]\). Furthermore, there always exists a recurrent system of compact generation on \( U \) \([5, \text{Corollary 4.5}]\).

**Theorem 6.3** (Álvarez-Candel \([5, \text{Theorem 4.6}]\)). Let \( \mathcal{H} \) and \( \mathcal{H}' \) be compactly generated pseudogroups on locally compact spaces \( T \) and \( T' \), let \( U \) and \( U' \) be relatively compact open subsets of \( T \) and \( T' \) that meet all orbits of \( \mathcal{H} \) and \( \mathcal{H}' \), let \( \mathcal{G} \) and \( \mathcal{G}' \) denote the restrictions of \( \mathcal{H} \) and \( \mathcal{H}' \) to \( U \) and \( U' \), and let \( E \) and \( E' \) be recurrent symmetric systems of compact generation of \( \mathcal{H} \) and \( \mathcal{H}' \) on \( U \) and \( U' \), respectively. Suppose that there exists an equivalence \( \mathcal{H} \to \mathcal{H}' \), and consider the induced equivalence \( \mathcal{G} \to \mathcal{G}' \) and homeomorphism \( U/\mathcal{G} \to U'/\mathcal{G}' \). Then the \( \mathcal{G} \)-orbits with \( d_E \) are uniformly quasi-isometric to the corresponding \( \mathcal{G}' \)-orbits with \( d_{E'} \).

An obvious modification of the arguments of the proof of \([5, \text{Theorem 4.6}]\) gives the following.

**Theorem 6.4.** With the notation and conditions of Theorem 6.3, the germ covers of the \( \mathcal{G} \)-orbits with \( d_E \) are uniformly quasi-isometric to the germ covers of the corresponding \( \mathcal{G}' \)-orbits with \( d_{E'} \).

**Corollary 6.5.** With the notation and conditions of Theorem 6.4, the corresponding orbits of \( \mathcal{G} \) and \( \mathcal{G}' \), as well as their germ covers, have the same growth type, uniformly.

**Proof.** This follows from Lemma 6.1 and Theorems 6.3 and 6.4. \(\square\)

**Example 6.6.** Let \( G \) be a locally compact Polish local group with a left-invariant metric, let \( \Gamma \subset G \) be a dense finitely generated sub-local group, and let \( \mathcal{H} \) denote the pseudogroup generated by the local action of \( \Gamma \) on \( G \) by local left translations. Suppose that \( \mathcal{H} \) is compactly generated, and let \( \mathcal{G} = \mathcal{H}|_U \) for some relatively compact open identity neighborhood \( U \) in \( G \), which meets all \( \mathcal{H} \)-orbits because \( \Gamma \) is dense. For every \( \gamma \in \Gamma \) with \( \gamma U \cap U \neq \emptyset \), let \( h_\gamma \) denote the restriction \( U \cap \gamma^{-1}U \to \gamma U \cap U \) of the local left translation by \( \gamma \). There is a finite symmetric set \( S = \{s_1, \ldots, s_k\} \subset \Gamma \) such that \( E = \{h_{s_1}, \ldots, h_{s_k}\} \) is a recurrent system of compact generation of \( \mathcal{H} \) on \( U \); in fact, by reducing \( \Gamma \) if necessary, we can assume that \( S \) generates \( \Gamma \). The recurrence of \( E \) means that there is some \( N \in \mathbb{N} \) such that

\[
U = \bigcup_{h \in E^N} h^{-1}(V \cap \text{im} h),
\]

where \( E^N \) is the family of compositions of at most \( N \) elements of \( E \).

For each \( x \in U \), let

\[
\Gamma_{U,x} = \{ \gamma \in \Gamma \mid \gamma x \in U \}.
\]

Let \( \mathcal{G} \) denote the topological groupoid of germs of \( \mathcal{G} \). The map \( \Gamma_{U,x} \to \mathcal{G}_x, \gamma \mapsto \gamma(h_\gamma, x) \), is bijective. For \( \gamma \in \Gamma_{U,x} \), let \( |\gamma|_{S,U,x} := |h_\gamma|_{E,x} \). Thus
|e|_{S,U,x} = 0, and, if γ ≠ e, then |γ|_{S,U,x} equals the minimum n ∈ ℤ such that there are i_1, ..., i_n ∈ {1, ..., k} with γ = s_{i_n} ... s_{i_1} and s_{i_m} ... s_{i_1} : x ∈ U for all 1 ≤ m ≤ n. Moreover d_E on Σ_x corresponds to the metric d_{S,U,x} on Γ_{U,x} given by

\[ d_{S,U,x}(γ, δ) = |δγ^{-1}|_{S,U,γ}(x). \]

Observe that, for all γ ∈ Γ_{U,x} and δ ∈ Γ_{U,γ,x},

\[ δγ ∈ Γ_{U,x}, \quad |δγ|_{S,U,x} ≤ |γ|_{S,U,x} + |δ|_{S,U,γ,x}, \quad (21) \]

\[ γ^{-1} ∈ Γ_{U,γ,x}, \quad |γ|_{S,U,x} = |γ^{-1}|_{S,U,γ,x}. \quad (22) \]

In this example, we will be interested on the growth type of the orbits of G with d_E, or, equivalently, the growth type of the metric spaces (Γ_{U,x}, d_{S,U,x}). The following result was used by Breuillard-Gelander to study this growth type when G is a Lie group.

**Proposition 6.7** (Breuillard-Gelander [10, Proposition 10.5]). Let G be a non-nilpotent connected real Lie group and Γ a finitely generated dense subgroup. For any finite set S = {s_1, ..., s_k} of generators of Γ, and any neighborhood B of e in G, there are elements t_i ∈ Γ ∩ s_iB (i ∈ {1, ..., k}) which freely generate a free semi-group. If G is not solvable, then we can choose the elements t_i so that they generate a free group.

### 6.3. Growth of leaves

Let X ≡ (X, F) be a compact Polish foliated space. Let \{U_i, p_i, h_{ij}\} a defining cocycle of F, where p_i : U_i → T_i and h_{ij} : T_{ij} → T_{ji}, and let H be the induced representative of the holonomy pseudogroup. As we saw in Section 4, H can be considered as the restriction of some compactly generated pseudogroup H' to some relatively compact open subset, and E = \{h_{ij}\} is a system of compact generation on T. Moreover Álvarez and Candel [5] observed that E is recurrent. According to Theorems 6.3 and 6.4, it follows that the quasi-isometry type of the H-orbits and their germ covers with d_E are independent of the choice of \{U_i, p_i, h_{ij}\} under the above conditions; thus they can be considered as quasi-isometry types of the corresponding leaves and their holonomy covers.

This has the following interpretation when X is a smooth manifold. In this case, given any Riemannian metric g on X, for each leaf L, the differentiable (and coarse) quasi-isometry type of g|_L is independent of the choice of g; they depend only on F and L; in fact, it is coarsely quasi-isometric to the corresponding H-orbit, and therefore they have the same growth type [12] (this is an easy consequence of the existence of a uniform bound of the diameter of the plaques). Similarly, the germ covers of the H-orbits are also quasi-isometric to the holonomy covers of the corresponding leaves.

### 7. Growth of equicontinuous pseudogroups and foliated spaces

Let G be a locally compact Polish local group with a left-invariant metric, let Γ ⊆ G be a dense finitely generated sub-local group, and let H denote the pseudogroup generated by the local action of Γ on G by local
left translations. Suppose that $\mathcal{H}$ is compactly generated. Let $\mathcal{G} = \mathcal{H}|_U$ for some relatively compact open identity neighborhood $U$ in $G$, which meets all $\mathcal{H}$-orbits because $\Gamma$ is dense. Let $E$ be a recurrent symmetric system of compact generation of $\mathcal{H}$ on $U$. Let $\Psi$ be the pseudogroup of germ of maps in $\mathcal{G}$. It will be said that $G$ can be approximated by nilpotent local Lie groups if, there is a sequence $F_n$ given by Theorem 2.26 so that the local Lie groups $G/(F_n, U)$ are nilpotent.

**Theorem 7.1.** With the above notation and conditions, one of the following properties hold:

- $G$ can be approximated by nilpotent local Lie groups;
- or

the germ covers of all $G$-orbits have exponential growth with $d_E$.

**Proof.** According to Theorem 2.26, there is some $U_0 \in \Psi G$, contained in any given element of $\Psi G \cap \Phi(G, 2)$, and there exists a sequence of compact normal subgroups $F_n \subset U_0$ such that $F_{n+1} \subset F_n$, $\cap_n F_n = \{e\}$, $(F_n, U_0) \in \Delta G$, and $G/(F_n, U_0)$ is a local Lie group. Let $T_n : U_0^2 \to G/(F_n, U_0)$ denote the canonical projection. Take some open identity neighborhood $U_1$ such that $U_1 \subset U_0$. Then $F_nU_1 \subset U_0$ for $n$ large enough by the properties of the sequence $F_n$. Let $U_2 = F_nU_1$ for such an $n$. Thus $U_2$ is saturated by the fibers of $T_n$, and $U_2 \subset U_0$. Then $U' = T_n(U_2)$ is a relatively compact open identity neighborhood in the local Lie group $G' = G/(F_n, U_0)$. Let $\Gamma' = T_n(\Gamma \cap U_0^2)$, which is a dense sub-local group of $G'$, and let $\mathcal{H}'$ denote the pseudogroup on $G'$ generated by the local action of $\Gamma'$ by local left translations.

For every $\gamma \in \Gamma \cap U_0$ with $\gamma U_2 \cap U_2 \neq \emptyset$, let $h_\gamma$ denote the restriction $U_2 \cap \gamma^{-1}U_2 \to \gamma U_2 \cap U_2$ of the local left translation by $\gamma$. There is a finite symmetric set $S = \{s_1, \ldots, s_k\} \subset \Gamma$ such that $E_2 = \{h_{s_1}, \ldots, h_{s_k}\}$ is a recurrent system of compact generation of $\mathcal{H}$ on $U_2$. By reducing $\Gamma$ if necessary, we can suppose that $S$ generates $\Gamma$. For every $\delta \in \Gamma'$ with $\delta U' \cap U' \neq \emptyset$, let $h'_\delta$ denote the restriction $U' \cap \delta^{-1}U' \to \delta U' \cap U'$ of the local left translation by $\delta$. We can assume that $s_1, \ldots, s_k$ are in $U_2$, and therefore we can consider their images $s'_1, \ldots, s'_k$ by $T_n$. Moreover each $h_s$ induces via $T$ the map $h'_{s_1}, \ldots, h'_{s_k}$ is a system of compact generation of $\mathcal{H}'$ on $U'$. By incrementing $E_2$ if necessary, we can assume that $E'$ is also recurrent. Fix any open set $V'$ in $G'$ with $\overline{V'} \subset U'$. Then $V = T^{-1}_n(V')$ satisfies $\overline{V} \subset U_2$.

**Claim 6.** For each finite subset $F \subset \Gamma \cap U_2$, we have

$$U_2 \subset \bigcup_{\gamma \in \Gamma \setminus F} \gamma V.$$ 

Since $U_2$ and $V$ are saturated by the fibers of $T_n$, Claim 6 follows by showing that

$$U' \subset \bigcup_{\gamma \in \Gamma' \setminus F'} \gamma V' ,$$

(23)
where $F' = T_\alpha(F)$. Suppose that (23) is false. Then there is some finite symmetric subset $F \subset \Gamma \cap U_2$ and some $x \in U'$ such that $(\Gamma' \setminus F')x \cap V' = \emptyset$. By the recurrence of $E'$, there is some $N \in \mathbb{N}$ satisfying (20) with $U'$ and $E'$. Since $\Gamma'_{U',x}$ is infinite because $\Gamma'$ is dense in $G'$, it follows that there is some $\gamma \in \Gamma'_{U',x} \setminus F'$ such that
\[
|\gamma|_{S',U',x} > N + \max \{ |e|_{S',U',x} \mid e \in F' \cap \Gamma'_{U',x} \} .
\] (24)
By (20), there is some $h \in E'^N$ such that
\[
\gamma x \in h^{-1}(V' \cap \text{im } h') .
\]
We have $h = h'_\delta$ for some $\delta \in \Gamma'$. Note that $\delta \in \Gamma'_{U',\gamma'x}$ and $|\delta|_{S',U',\gamma'x} \leq N$. Hence
\[
|\gamma|_{S',U',x} \leq |\delta|_{S',U',x} + |\delta^{-1}|_{S',U',\delta'x} = |\delta|_{S',U',x} + |\delta|_{S',U',\gamma'x} \leq |\delta|_{S',U',x} + N
\]
by (21) and (22), obtaining that $\delta \gamma \notin F'$ by (24). However, $\delta \gamma x \in V'$, obtaining a contradiction, which completes the proof of Claim 6.

Claim 7. For each finite subset $F \subset \Gamma \cap U_2$, we have
\[
\mathcal{U}_2 \subset \bigcup_{\gamma \in \Gamma \setminus F} \gamma V .
\]

Take a relatively compact open subset $O_1 \subset G$ such that $\mathcal{U}_1 \subset O_1$ and $F \mathcal{O}_1 \subset U_0$. Let $O_2 = F \mathcal{O}_1$ and $\mathcal{K} = \mathcal{H}|_{O_2}$. Then Claim 7 follows by applying Claim 6 to $O_2$.

According to Claim 7 by increasing $S$ if necessary, we can suppose that
\[
\mathcal{U}_2 \subset \bigcup_{i \neq j} (s_i \cdot V \cap s_j \cdot V) = \bigcup_{i < j} (s_i^{-1} \cdot V \cap s_j^{-1} \cdot V) .
\] (25)

Suppose that $G$ cannot be approximated by nilpotent local Lie groups. Then we can assume that the local Lie group $G'$ is not nilpotent. Moreover we can suppose that $G'$ is a sub-local Lie group of a simply connected Lie group $L$. Let $\Delta$ be the dense subgroup of $L$ whose intersection with $G'$ is $\Gamma'$. Then, by Proposition 6.7 there are elements $t'_1, \ldots, t'_k$ in $\Delta$, as close as desired to $s'_1, \ldots, s'_k$, which are free generators of a free semi-group. If the elements $t'_i$ are close enough to $s'_i$, then they are in $U'$. So there are elements $t_i \in U_2$ such that $T_\alpha(t_i) = t'_i$. By the compactness of $\mathcal{U}_2$, and because $U_2$ and $V$ are saturated by the fibers of $T_\alpha$, if $t'_1, \ldots, t'_k$ are close enough to $s'_1, \ldots, s'_k$, then (25) gives
\[
\mathcal{U}_2 \subset \bigcup_{i < j} (t_i^{-1} V \cap t_j^{-1} V) .
\] (26)

Now, we adapt the argument of the proof of [10, Lemma 10.6]. Let $\hat{\Gamma} \subset \Gamma$ be the sub-local group generated by $t_1, \ldots, t_k$; thus $\hat{S} = \{t_1^{\pm 1}, \ldots, t_k^{\pm 1}\}$ is a
symmetric set of generators of $\hat{\Gamma}$, and $S \cup \hat{S}$ is a symmetric set of generators of $\Gamma$. With $E = \{h_{t_1}^{\pm 1}, \ldots, h_{t_k}^{\pm 1}\}$, observe that $E_2 \cup \hat{E}$ is a recurrent system of compact generation of $\hat{H}$ on $U_2$. Given $x \in U_2$, let $S(n)$ be the sphere with center $e$ and radius $n \in \mathbb{N}$ in $\hat{\Gamma}_{U_2,x}$ with $d_{\hat{S},U_2,x}$. By (26), for each $\gamma \in S(n)$, there are indices $i < j$ such that $\gamma x \in t_i^{-1}V \cap t_j^{-1}V$. So the points $t_i \gamma x$ and $t_j \gamma x$ are in $V$, obtaining that $t_i \gamma, t_j \gamma \in S(n + 1)$. Moreover all elements obtained in this way from elements of $S(n)$ are pairwise distinct because $t_1', \ldots, t_k'$ freely generate a free semigroup. Hence $\text{card}(S(n + 1)) \geq 2 \text{card}(S(n))$, giving $\text{card}(S) \geq 2^n$. So $(\hat{\Gamma}_{U_2,x}, d_{\hat{S},U_2,x})$ has exponential growth. Since $\hat{\Gamma}_{U_2,x} \subset \Gamma_{U_2,x}$ and $d_{\hat{S},U_2,x} \leq d_{\hat{S},U_2,x}$ on $\hat{\Gamma}_{U_2,x}$, it follows that $(\Gamma_{U_2,x}, d_{\hat{S},U_2,x})$ also has exponential growth. So $(\mathfrak{S}_x, d_{E_2,\hat{E}})$ has exponential growth, obtaining that $(\mathfrak{S}_x, d_E)$ has exponential growth by Corollary 6.5.

Theorem B follows from Theorem 7.1 and the observations of Section 6.3.

8. Examples

As pointed out in Section 1, Riemannian foliations are the motivating class of equicontinuous foliated spaces. Here, we will consider other types of examples of compact equicontinuous foliated spaces $(\mathcal{X}, \mathcal{F})$; some of them are taken from [11, Chapter 11], where Candel and Conlon gave many interesting examples of foliated spaces.

8.1. Locally free actions. Any locally free action of a connected Lie group on a locally compact Polish space, $\phi : H \times X \to X$, defines a foliated structure $\mathcal{F}$ on $X$ whose leaves are the orbits [11, Theorem 11.3.14], [30]. Let $\mathcal{H}$ denote the holonomy pseudogroup of $\mathcal{F}$. Let Homeo($X$) denote the topological group of homeomorphisms of $X$ with the compact-open topology. Suppose that $X$ is compact and $\phi$ equicontinuous. Then $\mathcal{F}$ is also minimal and equicontinuous. Moreover the closure $\hat{H} = \overline{\{ \phi_h \mid h \in H \}}$ in Homeo($X$) is compact, and its canonical action on $X$ is transitive. Hence $X$ can be identified with a homogeneous space of $\hat{H}$.

Any oriented foliated space $X$ of dimension one is defined by a non-singular flow $\phi : \mathbb{R} \times X \to X$. To see this, use a partition of unity subordinate to a foliated atlas to produce a measure on the leaves, positive on non-empty open sets. Then it is easy to define $\phi$ by using this measure and the orientation.

8.2. Matchbox manifolds and solenoids. A matchbox manifold is a foliated continuum $8$ $X \equiv (X, \mathcal{F})$ transversely modelled on a totally disconnected space. In this case, the foliated structure $\mathcal{F}$ is determined by the topology of $X$: the leaves are the path connected components of $X$. Since $X$ is connected, either $X$ is a compact manifold (the trivial case), or it is

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8Recall that a continuum is a non-empty compact connected metrizable space.
transversely modelled on a Cantor space; we will only consider the second case without further indication. It is said that \( X \) is \( C^k \) \((k \in \mathbb{N})\) if the changes of foliated coordinates are leafwise \( C^k \), with transversely continuous leafwise derivatives of order \( \leq k \); if \( X \) is \( C^k \) for all \( k \), then it is said that \( X \) is \( C^\infty \) or smooth. It will be assumed that all matchbox manifolds are \( C^1 \).

The classical solenoids over the circle are examples of matchbox manifolds. With more generality, an \( n \)-dimensional solenoid is an inverse limit

\[
\mathcal{S} = \lim\left\{ p_{l+1} : L_{l+1} \to L_l \right\},
\]

where each \( L_l \) \((l \in \mathbb{N})\) is a closed connected \( n \)-dimensional manifold, and the maps \( p_{l+1} : L_{l+1} \to L_l \) are smooth proper covering maps. Any \( n \)-dimensional solenoid is a matchbox manifold. If any composite of a finite number of bounding maps \( p_l \) is a normal covering, then \( \mathcal{S} \) is called a McCord solenoid.

It is easy to check that any solenoid is equicontinuous. The reciprocal was shown by Clark and Hurder [15, Theorem 7.9]: any equicontinuous matchbox manifold is a solenoid. More generally, Alcalde-Lozano-Macho [2] have shown that any minimal transversely Cantor \( n \)-dimensional matchbox manifold is an inverse limit of compact branched \( n \)-manifolds.

On the other hand, recall that a topological space (respectively, foliated space) is called homogeneous when the canonical action of its group of homeomorphisms (respectively, foliated homeomorphisms) is transitive. For matchbox manifolds, the homogeneities as topological and foliated space are equivalent. It is easy to check that McCord solenoids are homogeneous. The reciprocal was also proved by Clark and Hurder [15, Theorem 1.1]: any homogeneous matchbox manifold is homeomorphic to a McCord solenoid; in particular, it is minimal.

For a McCord solenoid \( \mathcal{S} \) as above, let

\[
\cdots \longrightarrow H_{l+1} \xrightarrow{p_{l+1}^*} H_l \longrightarrow \cdots
\]

denote the corresponding sequence of injective homomorphisms between fundamental groups. We get a sequence of canonical projections between finite groups,

\[
\cdots \longrightarrow G_{l+1} = H_0/p_{l+1}*(H_{l+1}) \longrightarrow G_l = H_0/p_l*(H_l) \longrightarrow \cdots,
\]

whose inverse limit,

\[
G = \lim\left\{ G_{l+1} \to G_l \right\},
\]

is a topological group homeomorphic to a Cantor space. Then the canonical map \( p : \mathcal{S} \to L_0 \) is a fiber bundle whose typical fiber is \( G \), and whose restrictions to the leaves are covering maps of \( L_0 \). Moreover \( \mathcal{S} \) is transversely modelled by left translations on \( G \) (it is a \( G \)-foliated space), and therefore the foliated space \( \hat{\mathcal{S}} \), given by Theorem \( \ref{ theorem } \) can be identified with \( \mathcal{S} \).

When the solenoid \( \mathcal{S} \) is not McCord, it is still equicontinuous, and the application of Theorem \( \ref{ theorem } \) may give non-trivial information. In this case, \( G \) is defined as a projective limit of discrete finite sets, without any group
structure, but there is a canonical action of $H_0$ on $G$. Let $\mathcal{H}$ denote the holonomy pseudogroup of $\mathcal{S}$.

The realization of solenoids as minimal sets of foliations is studied by Clark and Hurder [14].

8.3. **Almost periodic non-periodic functions.** Let $C_b(\mathbb{R})$ be the space of continuous bounded functions $\mathbb{R} \to \mathbb{R}$, with the topology of uniform convergence. For a function $f \in C_b(\mathbb{R})$ and $t \in \mathbb{R}$, let $f_t$ denote the translation of $f$ by $t$: $f_t(r) = f(r+t)$. It is said that $f$ is *almost periodic* (in the sense of Besucovich [8] and Gottschalk [22]) if the family of translations $\{f_t | t \in \mathbb{R}\}$ is equicontinuous. In this case, the closure $\mathcal{M}(f) = \{f_t | t \in \mathbb{R}\}$ in $C_b(\mathbb{R})$ is compact. An equicontinuous flow $\Phi : \mathbb{R} \times \mathcal{M}(f) \to \mathcal{M}(f)$ is defined by $\Phi_t(g) = g_t$. The space $\mathcal{M}(f)$ consists of a single fixed point just when $f$ is constant; otherwise $\Phi$ is nonsingular, and therefore $\mathcal{M}(f)$ is foliated by the orbits of $\Phi$. Moreover $\mathcal{M}(f)$ consists of a single closed orbit just when $f$ is periodic. Suppose from now on that $f$ is non-periodic, obtaining that $\mathcal{M}(f)$ is a non-trivial compact minimal equicontinuous foliated space of dimension one.

More generally, we can define in the same way a non-trivial compact minimal equicontinuous foliated space $\mathcal{M}(f)$ for any almost-periodic non-periodic continuous function $f$ on $\mathbb{R}$ with values in a Hilbert space.

This construction is universal in the following sense. As we saw (Section 8.1), for any compact Polish space $X$ and any oriented equicontinuous minimal foliated structure $\mathcal{F}$ on $X$ of dimension one, there is a non-singular equicontinuous flow $\phi : \mathbb{R} \times X \to X$ whose orbits are the leaves of $\mathcal{F}$. Take a sequence of real valued continuous functions $h_n$ on $X$ that separate points. Let $(e_n)$ be a complete orthonormal system of a separable Hilbert space $\mathcal{H}$. Then the function $h : X \to \mathcal{H}$, given by

$$h(x) = \sum_n \frac{h_n(x)}{2^n \cdot \max |h_n|} e_n,$$

is continuous and separates points, and therefore it is a topological embedding because $X$ is compact. Given any point $x_0 \in X$, let $f \in C_b(\mathbb{R}, \mathcal{H})$ be defined by $f(r) = h\phi_r(x_0)$. From the equicontinuity of $\phi$ and the compactness of $X$, it follows that $f$ is almost periodic. Suppose that $X$ is not a single orbit. So $f$ is non-periodic, and therefore $\mathcal{M}(f)$ is a non-trivial compact minimal equicontinuous foliated space of dimension one. The mapping $\phi_r(x_0) \mapsto f_t$ defines a continuous map of the $\phi$-orbit of $x_0$ onto the $\Phi$-orbit of $f$, which extends to an equivariant continuous map $\psi : X \to \mathcal{M}(f)$. This map is injective because $h$ separates points, and it is surjective because $\phi$ and $\Phi$ are minimal. Hence $\psi$ is a foliated homeomorphism since $X$ is compact and $\mathcal{M}(f)$ Hausdorff.

With even more generality, we can make the same type of construction for almost periodic functions on any Lie group $H$, obtaining all compact equicontinuous foliated spaces defined by locally free $H$-actions.
8.4. Almost periodic locally aperiodic Riemannian manifolds. Let \( \mathcal{M}_s \) denote the Gromov space\(^9\) of isometry classes pointed proper metric spaces\(^10\). The isometry class of each pointed proper metric space \( (M,x) \) will be denoted by \([M,x]\).

For each proper metric space \( M \), there is a canonical continuous map \( \iota_M : M \to \mathcal{M}_s \) given by \( \iota_M(x) = [M,x] \), which induces a canonical embedding \( \iota_M : \text{Iso}(M) \setminus M \to \mathcal{M}_s \), where \( \text{Iso}(M) \) denotes the group of isometries of \( M \). The images of the maps \( \iota_M \) form a partition of \( \mathcal{M}_s \), called its canonical partition. With respect to this partition, the closure of the image of each \( \iota_M \) in \( \mathcal{M}_s \) is a saturated subspace denoted by \( M_s \). It is said that \( M \) is:

- **aperiodic** if \( \text{Iso}(M) = \{\text{id}_M\} \); and
- **locally aperiodic** if any \( x \in M \) has a neighborhood \( V \) such that
  \[
  \{ h \in \text{Iso}(M) \mid h(x) \in V \} = \{\text{id}_M\}.
  \]

Observe that \( M \) is (locally) aperiodic if and only if \( \iota_M \) is (locally) injective.

For each \( n \in \mathbb{Z}_+ \), let \( \mathcal{M}_s(n) \subset \mathcal{M}_s \) denote the subset defined by complete connected Riemannian manifolds of dimension \( n \). The \( C^\infty \) convergence \( [M_i,x_i] \to [M,x] \) in \( \mathcal{M}_s(n) \) means that, for each compact domain \( \Omega \subset M \), there are \( C^\infty \) embeddings \( \phi_i : \Omega \to M_i \) for large enough \( i \) such that \( \phi_i(x) = x \) and \( \phi^*_i g_i \to g \) on \( \Omega \) with respect to the \( C^\infty \) topology, where \( g \) and \( g_i \) are the metric tensors of \( M \) and \( M_i \).\(^{11}\) It is easy to see that \( C^\infty \) convergence means convergence in some first countable topology on \( \mathcal{M}_s(n) \); it will be called the \( C^\infty \) topology on \( \mathcal{M}_s(n) \), and the corresponding space will be denoted by \( \mathcal{M}^\infty_s(n) \). Observe that \( \mathcal{M}^\infty_s(n) \to \mathcal{M}_s \) is continuous, and \( \mathcal{M}_s(n) \) is saturated with respect to the canonical partition of \( \mathcal{M}_s \).

A complete connected Riemannian manifold \( M \), with metric tensor \( g \), is said to be:

- **almost periodic** if for any \( m \in \mathbb{N} \), \( \epsilon > 0 \) and \( x \in M \), there is a set \( H \) of diffeomorphisms of \( M \) such that \( \sup |\nabla^k(h^*g)| < \epsilon \) for all \( h \in H \) and \( k \leq m \), and \( \{ h(x) \mid h \in H \} \) is a net in \( M \); and of
- **bounded geometry** if it has a positive injectivity radius, and each covariant derivative of the curvature tensor, with arbitrary order, is uniformly bounded.

Note that \( M \) has bounded geometry if it is almost periodic.

For \( r > 0 \) and a sequence \( C_k \geq 0 \), let \( \mathcal{M}_s(n,r,\{C_k\}) \subset \mathcal{M}_s(n) \) be the subset defined by the manifolds whose injectivity radius is \( \geq r \) and so that the covariant derivative of order \( k \) of the curvature tensor has norm \( \leq C_k \); these are the subsets defined by manifolds with “uniformly bounded geometry”. The topologies of \( \mathcal{M}_s \) and \( \mathcal{M}^\infty_s(n) \) have the same restriction to \( \mathcal{M}_s(n,r,\{C_k\}) \)\(^{32}\). Lemma 7.2], which becomes a compact subspace\(^3\).
Theorem 4.1] (this is a consequence of the Fundamental Theorem of Convergence Theory [11, Chapter 10, Theorem 3.3], which is essentially due to Cheeger [13]). On the other hand, Álvarez and Candel [4] have proved that the subspace of $\mathcal{M}_s^\infty(n)$ defined by the locally aperiodic manifolds is foliated with the restriction of the canonical partition.

It turns out that, if a complete connected Riemannian $n$-manifold $M$ is almost periodic and locally aperiodic, then its closure $\mathcal{M}_s^\infty(M)$ (in $\mathcal{M}_s$ or $\mathcal{M}_s^\infty(n)$) becomes a compact minimal equicontinuous foliated space of dimension $n$ with the canonical partition.

9. Open problems and questions

Problem 9.1. In the examples of Section 8, understand the specific application of Theorems A and B i.e., give a specific characterization of the strong quasi-analyticity of $\mathcal{H}$ and a specific description of $(\hat{X}_0, \hat{F}_0)$, and obtain specific growth restrictions.

Problem 9.2. Use Theorem A to classify particular classes of equicontinuous foliated spaces.

Question 9.3. Given a local group $G$, is it possible to prove a version of Fedida’s description of Lie foliations [18, 19] for compact minimal $G$-foliated spaces? According to [16, 17], we can assume that $G$ is a topological group. Since the universal cover of $G$ is needed, assume that $G$ is connected and locally path connected.

Question 9.4. Is it possible to improve Theorem B for special types of structural local groups?

Question 9.5. Is there any consequence of Theorems A and B in usual foliation theory, assuming that the minimal sets are equicontinuous as foliated spaces?

The following questions refer to extensions of properties of Riemannian foliations to equicontinuous foliated spaces, where Theorem A could play an important role.

Question 9.6. For compact minimal equicontinuous foliated spaces, does the leafwise heat flow of leafwise differential forms preserve transverse continuity at infinite time? This would produce a leafwise Hodge decomposition for leafwise differential forms that are leafwise smooth and transversely continuous.

Question 9.7. Is it possible to give useful extensions of tautness and tenseness to equicontinuous foliated spaces, and relate them to some kind of basic cohomology? One should consider a singular or Alexander-Spanier version of basic cohomology, or perhaps other versions better adapted to each special transverse model.
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