Superfluidity versus Bloch oscillations in confined atomic gases

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We study the superfluid properties of (quasi) one-dimensional bosonic atom gases/liquids in traps with finite geometries in the presence of strong quantum fluctuations. Driving the condensate with a moving defect we find the nucleation rate for phase slips using instanton techniques. While phase slips are quenched in a ring resulting in a superfluid response, they proliferate in a tube geometry where we find Bloch oscillations in the chemical potential. These Bloch oscillations describe the individual tunneling of atoms through the defect and thus are a consequence of particle quantization.

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Bose-Einstein condensation (BEC) and superfluidity [1] are basic characteristics of Bosonic quantum gases and fluids. While the Bose-Einstein condensate (of density $n_0$) is a thermodynamic quantity characterizing off-diagonal-long-range order, the superfluid density $n_s$ describes the response to a perturbation in the broken phase [2]. In real quantum liquids, such as bulk $^4$He, condensation and superfluidity appear in unison, but in general one may be realized without the other. E.g., noninteracting Bose gases in three dimensions form a condensate without superfluidity ($n_s = 0$), while confining a real quantum fluid to two dimensions destroys the condensate ($n_0 = 0$) but preserves superfluidity. In one dimension only superfluidity may survive at zero temperature.

A major recent breakthrough is the realization of a Bose-Einstein condensate in weakly interacting atomic gases. The confinement within a trap has important consequences for the condensate [2][4]: e.g., in a three dimensional harmonic trap the well known ideal gas condensation temperature $T_{\text{BE}} = 3.3\hbar^2\nu^2/3m$ is replaced by $T_{1D}^{\text{ho}} = 0.94\hbar\omega_{\text{ho}}N^{1/3}$. Quite notable is the appearance of a condensate below $T_{1D}^{\text{ho}} = \hbar\omega_{\text{ho}}N/\ln 2N$ in an ideal gas confined to a (quasi) one-dimensional (1D) geometry (here, $\nu$, $N$ and $m$ are the bulk density, particle number, and mass of the bosons, while $\omega_{\text{ho}}^2 = \nu\omega_{\perp}^2$ with $\omega_{\perp}$ and $\omega_{\parallel}$ denoting the transverse and longitudinal trapping frequencies). The appearance of such a condensate at $T_{1D}^{\text{ho}}$ is characterized by a sharp crossover for an ideal 1D gas; the broadening of the transition due to interparticle interactions has been discussed by Petrov et al. [5].

First attempts to probe the (bulk) superfluid properties of condensed atom gases through a moving laser beam have been carried out recently [6]; the results on the critical velocity are in rough agreement with expectations deriving from a weak coupling analysis based on the Gross-Pitaevskii theory [7]. An interesting question then arises regarding the interplay of superfluidity and enhanced thermal/quantum fluctuations due to dimensional reduction. In this letter we study the superfluid response of (quasi) one-dimensional atomic gases and fluids trapped within finite tube and ring geometries, see Fig. 1, where quantum phase slips tend to destroy superfluidity. While interesting on their own, these questions have attracted much attention recently through the novel atom chip technology [8] allowing for the experimental realization of strongly confined atom gases exhibiting large quantum fluctuations.

FIG. 1. Trap geometries of the atomic gas perturbed by a moving laser beam: (a) ring structure with periodic boundary conditions, (b) finite length ($L$) tube with closed ends.

The destruction of dissipation free flow in one dimensional (1D) superconductors and superfluids is triggered by the appearance of quantum phase slips as discussed by Zaikin et al. [9] for metal wires and by Kagan et al. [10] for superfluid rings. Here, we model the action of the laser beam through a moving impurity (velocity $v$) and derive a low frequency effective action describing the dynamics of the phase difference across. The quantum nucleation rate for phase slips determines the response: at finite temperature the infinite system exhibits a linear response and hence is not superfluid, $\Delta\mu \propto v$ with $\Delta\mu$ the drop in the chemical potential across the impurity. This contrasts with the ring geometry where interactions quench the phase slip nucleation below a critical velocity, establishing a superfluid response. In a finite tube the quantum phase slips proliferate and the new non-superfluid ground state exhibits Bloch oscillations in the chemical potential difference across the moving impurity, $\Delta\mu \propto \sin(2\pi nvt)$ with $n$ the 1D atom density. The physical origin of these oscillations is found in the particle quantization: the moving impurity enhances the particle density in front, producing a chemical potential difference across the impurity, which in turn is released each
time an atom tunnels through the impurity. Trapped 1D atomic gases have attracted much interest recently. Their nature is conveniently described in terms of the dimensionless parameter $\gamma = n h^2/m g = nl_1^2/2a$. Here, $n$ and $m$ denote the (one-dimensional) atom density and mass, respectively, and $l_1 = \sqrt{\hbar/m\omega_{1\perp}}$ is the transverse extension of the ground state wave function; assuming a contact potential with a scattering length $a < l_1$, the interaction acquires a 3D character and the interaction parameter takes the form $g = 2\hbar l_1^2/ma$, see [3] for corrections. Small and large values of $\gamma$ correspond to weakly and strongly interacting bosons: for weakly interacting gases the energy per particle is given by the bosonic expression $\epsilon_0(n) = gn/2$ and the Gross-Pitaevskii equation holds. In the strongly interacting situation ($\gamma \rightarrow \infty$) the 1D Fermion-Boson duality becomes manifest and the energy per particle is given by the fermionic expression $\epsilon_0(n) = \pi\hbar^2n^2/6m$; the implications for the density profile have been discussed in [12] and an appropriate modification of the Gross-Pitaevskii equation has been proposed by Kolomeisky et al. [13]. It is the latter limit describing impenetrable bosons with strong quantum fluctuations we are mainly interested in the present work; the requirements for the experimental realization of this so called Tonks-Girardeau limit have been discussed in Ref. [12]. A convenient starting point describing the low energy physics in both cases is the imaginary time action for the phase $\phi(x, \tau)$ of the bosonic field $\psi \propto \exp[i\phi]$ [13],

$$S_0 = \frac{K h}{2\pi} \int_0^{\hbar} d\tau \int dx \left[ c_s (\partial_x \phi)^2 + \frac{1}{c_s} (\partial_x \phi)^2 \right], \quad (1)$$

describing sound modes with velocity $c_s$ to be cut-off at high energies $\Lambda$. For weak interaction, $\gamma \ll 1$, the sound velocity $c_s = \sqrt{ng/m}$, the dimensionless parameter $K = \pi/\sqrt{\gamma}$, and the cutoff $\Lambda = ng$ derive from the Gross-Pitaevskii equation; quantum fluctuations are suppressed in this limit. Increasing the interaction, quantum fluctuations renormalize the sound velocity $c_s$ and the dimensionless parameter $K$ in the low energy action [1]: strongly interacting bosons with a contact potential $g\delta(x)$ are described by $K \approx 1 + 4/\gamma$ while $c_s K = \pi n h/m$ remains unrenormalized (note that $K \rightarrow 1$ in the strong coupling or Tonks-Girardeau limit $\gamma \rightarrow \infty$; parameters $K < 1$ might be realized for bosons with long range interactions [13]). In Eq. (1) we assume a flat trapping potential along the longitudinal direction. A weak longitudinal trapping potential can be accounted for by space dependent parameters $K(x)$ and $c_s(x)$; the resulting deformation of the excitation spectrum will not change the main results presented below. Equation (1) produces the $T = 0$ phase correlator $\langle \phi(x) - \phi(x') \rangle^2 \sim \ln |x - x'|/K$, hence phase fluctuations destroy the condensate in the infinite system. On the other hand, the logarithmic divergence of the phase correlator is cut off in a finite trap, allowing for the definition of a (quasi-)condensate even at finite (but low) temperatures as discussed by Petrov et al. [3] (note that the use of (1) requires $T < \Lambda$).

The superfluid properties of the (quasi-) condensate can be probed with a moving laser beam [5] which we describe by a (strong) impurity potential $V_{imp}(z,t) = g_{imp} \delta(z - vt)$ with $g_{imp} > g$, suppressing the particle density locally. In the weakly interacting limit we can integrate the Gross-Pitaevskii equation over the impurity region and derive a Josephson term coupling the left and right parts of the atom gas (the ‘leads’) [16,17],

$$V(\varphi) = E_J \left[ 1 - \cos \varphi(\tau) \right] - \hbar n \varphi, \quad (2)$$

with $E_J = (K/\pi) n g^2/g_{imp}$, to be renormalized in the presence of large interactions. The second term drives the phase difference $\varphi$ across the impurity. Small contributions from higher harmonics do not modify the results below which are dominated by large scales; also, on the level of (1) such terms are irrelevant in the renormalization group sense [18]. The classical (meta-)stable states $\varphi_j$ derive from minimizing $V(\varphi)$ and fall into the intervals $\varphi_j - 2\pi j \in [-\pi, \pi]$; on a semi-classical level we define the associated ground state $\langle j \rangle$ of the $j$-th well. Next, we integrate over the phase dynamics (1) in the leads [14] and obtain the effective action for the phase difference $\varphi$ across the impurity

$$S = \hbar \int \frac{d\omega}{8\pi^2} Q(\omega) |\varphi(\omega)|^2 + \int d\tau V(\varphi(\tau)). \quad (3)$$

Extended leads produce an ohmic kernel $Q(\omega) \sim K|\omega|$ with a characteristic time scale $\tau_c \sim K\hbar/E_J$ for the phase dynamics; the finite leads in a trap geometry will strongly modify the low frequency part of the kernel $Q$ with dramatic consequences for the response.

Starting from the classical stationary states $\varphi_j$ describing a superfluid system we have to account for quantum fluctuations introducing transitions between these states which potentially destroy the superfluid response. Indeed, depending on the low frequency dynamics encoded in the behavior of the kernel $Q(\omega \rightarrow 0)$ the relevant (semi-classical) instanton solutions [1] of (3) will provide us with dissipative phase slips and a linear response in the infinite system, coherent hopping and Bloch oscillations in the finite tube, or confinement and hence superfluid response in the ring. A crucial element entering this analysis is the nature of the quantum variable $\varphi$ itself: while $\varphi \in \mathbb{R}$ is an extended variable if different minima $\varphi_j$ are physically distinguishable, $\varphi \in [0, 2\pi]$ turns into a compact variable if this is not the case. In the following we discuss the superfluid response for the different geometries in more detail.

The infinite system is characterized by an ohmic kernel $Q(\omega) = K|\omega|$; the effective action (3) describes a particle in a periodic potential with damping $K/2\pi$. The tunneling between classical minima leads to the excitation
of sound modes rendering the states distinguishable; as a consequence the phase $\varphi$ has to be treated as an extended variable $^{20}$ The system exhibits a quantum phase transition at $K = 1$ $^{21}$ separating a non-superfluid ground state with a delocalized phase $\varphi$ at $K < 1$ from a superfluid ground state with a localized phase at $K > 1$. The finite temperature response at $K > 1$ is determined by the thermally assisted quantum nucleation of phase slips $^{22}$: the corresponding action involves a kink-antikink-pair separated by the distance $\varphi$ in imaginary time,

$$\mathcal{S} = K \ln \left[ \left( \frac{\hbar}{\pi T \tau_c} \right)^2 \sin^2 \left( \frac{\pi T \varphi}{\hbar} \right) - 2\pi n \nu \varphi \right].$$

Using instantons $^{19}$ the nucleation rate is given by the integral $\Gamma \sim (1/\tau^2) \int_0^{\tau/2} d\tau \exp(-\mathcal{S}/\hbar)$. We recover a superfluid response at $T = 0$ with an algebraic rate $\Gamma \sim \tau^{2K-1}$, while the response turns linear at finite temperature, $\Gamma \sim \tau^{2K-2}$, as thermally activated quantum phase slips destroy the phase coherence across the link.

The ring geometry (see Fig. 1(a)) introduces periodic boundary conditions for the phase, $\phi(x, \tau) = \phi(x + L, \tau)$, with two important consequences: i) the existence of a winding number defines an extended quantum variable $\varphi$, and ii) the sound modes are quantized and exhibit a self interaction due to the compactness of the loop, modifying the kernel at low frequencies,

$$Q(\omega) = K \omega \coth \frac{\omega L}{2c_s} \approx \begin{cases} \frac{2Kc_s}{L} & \omega < 2c_s/L, \\ \frac{K\left|\omega\right|}{\omega < 2c_s/L}. \end{cases}$$

The static potential $\left(\hbar Kc_s/\pi L\right)\varphi^2/2$ describes the kinetic energy of the flow in the ring and is easily understood when the static solution $\phi(x) = \varphi [1/2 + x/L - \Theta(x)]$ is inserted in (8). The additional potential renders the system superfluid $^{23}$: the new minima satisfy the relation $\left(\kappa/L\right)\sin \varphi_2 = v - v_L \varphi_2 / \pi$, where the first term is the usual flow induced by the motion of the impurity, while the second term $\propto v_L = Kc_s/nL$ is due to the static potential. The absolute minimum at $\varphi_2$ with $|v - v_L| < v_L$ describes a stable superfluid state with a critical velocity $v_c = v_L$. Indeed, the $T = 0$ action for a kink-antikink-pair exhibits a linear confinement (we assume $K > 1$),

$$\mathcal{S} = K \ln \left[ \left( \frac{\hbar}{\pi c_s \tau_c} \right)^2 \sin^2 \left( \frac{\pi c_s \varphi}{\hbar} \right) - 2\pi n (v - v_L) \varphi \right],$$

and the nucleation of phase slips is quenched for $v < v_L$. At finite temperatures or large drives $|v - 2v_L| > v_L$ incoherent tunneling processes via thermally activated quantum nucleation of phase slips describe the equilibration of the ring towards its thermal equilibrium as given by the appropriate density matrix. At very large drives $|v - 2v_L| > v_L$ the system is far from equilibrium and the response resembles that of the infinite wire.

In a finite length tube the flow vanishes at the tube ends providing us with the boundary conditions $\partial_x \phi(-L/2, \tau) = \partial_x \phi(L/2, \tau) = 0$. The finite size quantization of the sound modes in the leads introduces a smooth low frequency cutoff in the kernel,

$$Q(\omega) = K \varphi \coth \frac{\omega L}{2c_s} \approx \begin{cases} \frac{2Kc_s}{L} & \omega < 2c_s/L, \\ \frac{K\left|\omega\right|}{\omega < 2c_s/L}. \end{cases}$$

a simple understanding is provided by inserting the dynamic solution $\phi(x, \tau) = \varphi (\tau) / [1/2 - \Theta(x)]$ into (9). The massive low frequency dynamics renders the tunneling between the minima coherent and using instanton techniques we can determine the hopping amplitude $^{24}$

$$W \sim (\hbar/\tau_c) (c_s \tau_c/2L)^K$$

between semi-classical states $|j\rangle$. In deriving (9) we have assumed $L > c_s \tau_c$, see $^{20}$ for a discussion of short systems. We have to distinguish between ‘diagonal’ transitions without excitations of sound modes and ‘non-diagonal’ ones changing the number of sound modes in the leads: as long as the time evolution involves only diagonal transitions the states $|j\rangle$ are indistinguishable and $\varphi$ is a compact variable $\varphi \in [0, 2\pi]$. However, in the following analysis it is a matter of convenience to choose $\varphi$ extended and compactify only at the end.

For $E_j/(hc_s/KL) > 1$ the amplitude $E_j$ is larger than the plasma frequency of the well and we can study the system response within a tight binding analysis. The action (8) can be transcribed into the Hamiltonian

$$H = -\frac{W}{2} \sum_j \langle j| (j+1) + |j+1\rangle |j\rangle - \epsilon \sum_j j |j\rangle \langle j|,$$

where the last term $\propto \epsilon = 2\pi \hbar n v v_L$ describes the driving force $(\varphi \equiv 2\pi \sum_j j |j\rangle \langle j|)$ in the site basis $|j\rangle$. Applying the unitary transformation $U = \exp[-2\pi i N(t) \varphi]$ with $N(t) = nvt + N_0$ we eliminate the drive through a redefinition of the hopping amplitudes,

$$H(t) = -\frac{W}{2} \sum_j \left\{ e^{2\pi i N(t)} |j\rangle \langle j| + c.c. \right\}.$$  

This Hamiltonian is equivalent to that of an electron in a crystal driven by an electric force $eE \equiv \hbar n v v_L$ described by the vector potential $eA/c = eEt \equiv nvt$ at zero drive the energy eigenstates form a Bloch band, while a finite electric field leads to ‘Bloch oscillations’ $^{25}$ (the above transformation corresponds to a gauge transformation, and the ‘quasi-number’ $N_0$ accounts for the gauge freedom). At low temperatures and low drives no sound waves are excited in the leads and we identify $|0\rangle = |j\rangle$ (the compact character of the phase restricts its value to a region centered around the potential minimum at $\varphi = 0$ and tends to establish phase coherence; on the other hand, the quantum nature of the phase and the presence of phase slips give a finite probability to any value
\( \varphi \in [0, 2\pi) \), thus reducing the phase coherence across the impurity, see also [26]. The Hamiltonian (9) reduces to \( H = -W \cos(2\pi N(t)) \mid 0 \rangle \langle 0 \mid \) and admits the solution \( \langle 0 \mid (t) = \exp \left[ -i \int dt H(t) \right] \mid 0 \rangle \). The state of the system and its energy depend on the impurity position via the ‘quasi-number’ \( N \); the usual derivatives provide us with the chemical potential difference across the impurity

\[
\Delta \mu = \langle \partial_N H \rangle = 2\pi W \sin(2\pi N(t))
\]

and its time evolution due to the drive \( \dot{N} = ne \).

In the static limit with \( v = 0 \) the spectrum maps out a Bloch band \( E(N) = -W \cos(2\pi N) \) (the ‘quasi-number’ \( N \) plays the role of the ‘quasi-momentum’ \( k \) in a periodic crystal), while a finite driving force \( n \) leads to ‘Bloch oscillations’ in the chemical potential difference \( \Delta \mu = 2\pi W \sin(2\pi nvt) \). These oscillations are due to the accumulation of particles in front of the impurity, the latter allowing only discrete particles to tunnel. Each ‘Umklapp’ process describes a particle tunneling through the impurity. The behavior of the tube then is dual to that of the classic Josephson junction, as is easily seen when replacing \( N \) by the phase drop \( \Phi \) and \( \Delta \mu \) by the supercurrent \( I \): the relations (9) and (10) are equivalent to Josephson’s famous relations \( I = I_c \sin \Phi \) and \( \partial_t \Phi = 2eV/h \) with \( I_c \) and \( V \) the critical current and voltage across the junction.

For high temperatures and drives processes involving frequencies larger than \( c_s/L \) induce nondiagonal transitions which compete with the diagonal ones. ‘Bloch oscillations’ then disappear above the crossover temperature \( T_L = h c_s/L \) and the critical drive \( v_K/L \). At high temperatures \( T > T_L \) or high drives \( v > v_K/L \) all processes are fast and we recover the physics of the infinite wire with incoherent tunneling via the quantum nucleation of phase slips (we assume \( K > 1 \)).

The quantum nucleation of phase slips leads to a transfer of energy to the bosonic system at high drives \( v > v_K \) but well below the mean field critical velocity \( E_J/hn \). Then the macroscopic quantum tunneling of the phase can be observed via the heating of the sample, in analogy to the experiment by Raman et al. [7] (note that our work predicts a dissipation free low-drive response and the appearance of a critical velocity for both topologies, ring and tube). On the other hand, the Bloch oscillations at low drives constitute a macroscopic quantum coherence phenomenon leading to density fluctuations within the leads. Using a second laser beam to probe the oscillating densities in the leads allows to measure these fluctuations, at least in principle. However, as each ‘Umklapp’ process involves only one particle tunneling through the impurity these oscillations will be small, thus requiring a high sensitivity in the experiment.

In conclusion, geometric confinement of the atom gas boosts the importance of fluctuations. The superfluid response strongly depends on the particular geometry: in a ring the phase difference across an impurity is well defined and the response remains superfluid below the critical velocity \( v_L \propto 1/L \), while in a tube phase slips proliferate and driving the system induces ‘Bloch oscillations’ in the chemical potential across the impurity.

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