A COMPACTNESS RESULT FOR SCALAR-FLAT METRICS ON MANIFOLDS WITH UMBILIC BOUNDARY

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Abstract. Let \((M, g)\) a compact Riemannian \(n\)-dimensional manifold with umbilic boundary. It is well known that, under certain hypothesis, in the conformal class of \(g\) there are scalar-flat metrics that have \(\partial M\) as a constant mean curvature hypersurface. In this paper we prove that these metrics are a compact set, provided \(n = 8\) and the Weyl tensor of the boundary is always different from zero, or if \(n > 8\) and the Weyl tensor of \(M\) is always different from zero on the boundary.

1. Introduction

Let \((M, g)\) be a \(n\)-dimensional \((n \geq 3)\) compact Riemannian manifold with boundary \(\partial M\). In [17, 18] J. Escobar investigated the question of finding a conformal metric \(\tilde{g} = u^{4/n-2} g\) for which \(M\) has constant scalar curvature and \(\partial M\) as constant mean curvature hypersurface. From a PDEs point of view, this is equivalent to the existence of a positive solution to the equation

\[
\begin{cases}
  L_g u = k u^{n+2/4(n-2)} & \text{in } M \\
  B_g u = c u^{n-2} & \text{on } \partial M
\end{cases}
\]

where \(L_g u = \Delta_g u - \frac{n-2}{4(n-1)} R_g u\) and \(B_g u = -\frac{\partial}{\partial \nu} u - \frac{n-2}{2} h_g u\) are respectively the conformal Laplacian and the conformal boundary operator, \(R_g\) is the scalar curvature of the manifold, \(h_g\) is the mean curvature of the \(\partial M\) and \(\nu\) is the outer normal with respect to \(\partial M\). The motivation to study this question arises from the classical Yamabe problem which consists of finding a constant scalar curvature metric, conformal to a given metric \(g\) on a compact Riemannian manifold without boundary. By the works of Yamabe, Trudinger, Aubin, Schoen [5, 25, 26, 27] the original problem was settled.

If a solution \(u\) of Problem (1.1) exists, then the metric \(\tilde{g} = u^{4/n-2} g\) has constant scalar curvature \(\frac{4}{4(n-1)}\) and the boundary has mean curvature \(c\). Problem (1.1) has been studied by many authors, see the recent paper of Disconzi, Khuri [9] and the survey of Marques [23] for a list of references. For the case \(c = 0\) we limit ourselves to cite among others [4] and references therein.

In this paper we consider the case of zero scalar curvature which is particularly interesting because it is a higher-dimensional generalization of the well known Riemann mapping Theorem and it leads to a linear equation on the interior of \(M\) with a critical nonlinear boundary condition of Neumann type.

Namely, we are interested to positive solution of the equation

\[
\begin{cases}
  L_g u = 0 & \text{in } M \\
  B_g u + (n-2) u^{n-2} = 0 & \text{on } \partial M
\end{cases}
\]

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Solutions of (1.2) are critical points of the functional quotient
\[ Q(u) := \frac{\int_M \left( |\nabla u|^2 + \frac{n-2}{4(n-1)} R_g u^2 \right) dv_g + \int_{\partial M} \frac{n-2}{2} h_g u^2 d\sigma_g}{\left( \int_{\partial M} |u|^{\frac{2(n-1)}{n-2}} d\sigma_g \right)^{\frac{n-2}{n-1}}}. \]

In [17] Escobar introduced, in analogy with the classical Yamabe problem, the quotient
\[ Q(M, \partial M) := \inf \{ Q(u) : u \in H^1(M), u \not\equiv 0 \text{ on } \partial M \} \]
which always satisfies the fundamental estimate
\[ Q(M, \partial M) \leq Q(\mathbb{B}^n, \mathbb{S}^{n-1}) \]
where \( \mathbb{B}^n \) is the unit ball in \( \mathbb{R}^n \) endowed with euclidean metric. Inequality (1.3) is important since if it strict inequality holds, then a solution of (1.2) exists.

When \((M, g)\) is not conformally equivalent to \((\mathbb{B}^n, g_{\mathbb{R}^n})\), existence results are proved by Escobar [17], Marques [21], Almaraz [3], Chen [8], Mayer and Ndiaye [20].

Once the existence of solutions of (1.2) is settled, it is natural to study the compactness of the full set of solutions. If \( Q(M, \partial M) \leq 0 \) the solution is unique up to a constant factor. The situation turns out to be delicate if \( Q(M, \partial M) > 0 \) and the underlying manifold is not the euclidean ball (in the case of the euclidean ball the set of solution is known to be non compact). Compactness has been proven by Felli and Ould Ahmedou in [10] for any dimension \( n \geq 3 \) in the case of locally conformally flat manifolds with umbilic boundary and by Almaraz in [1] when \( n \geq 7 \) and the trace-free second fundamental form is non-zero everywhere on \( \partial M \).

An example of non compactness is given for \( n \geq 25 \) and manifolds with umbilic boundary in [2]. We recall that the boundary of \( M \) is called **umbilic** if the trace-free second fundamental form of \( \partial M \) is zero everywhere.

In the present work we are interested in the compactness of the set of positive solutions to
\[ \begin{aligned}
L_g u &= 0 \quad \text{in } M \\
B_g u + (n-2)u^p &= 0 \quad \text{on } \partial M
\end{aligned} \tag{1.4} \]
where \( 1 \leq p \leq \frac{n}{n-2} \) and the boundary of \( M \) is umbilic. Our main result is the following.

**Theorem 1.** Let \((M, g)\) a smooth, \( n \)-dimensional Riemannian manifold of positive type with regular umbilic boundary \( \partial M \). Suppose that \( n > 8 \) and that the Weyl tensor \( W_g \) is not vanishing on \( \partial M \) or suppose that \( n = 8 \) and that the Weyl tensor referred to the boundary \( W_g \) is not vanishing on \( \partial M \). Then, given \( \bar{p} > 1 \), there exists a positive constant \( C \) such that, for any \( p \in \left[ \bar{p}, \frac{n}{n-2} \right] \) and for any \( u > 0 \) solution of (1.4), it holds
\[ C^{-1} \leq u \leq C \text{ and } \|u\|_{C^{2,\alpha}(M)} \leq C \]
for some \( 0 < \alpha < 1 \). The constant \( C \) does not depend on \( u, p \).

The proof is based on a local argument with Pohozaev type identity. This strategy was first introduced by Schoen [25] for a manifold without boundary. In this paper we avoid the use of any positive mass assumption: a crucial step is to provide a sharp correction term (see Lemma 3, Lemma 10 and Proposition 4.13) for the usual approximation of a rescaled solution by a bubble around an isolated simple blow up point (see Definition 5). The idea of using a suitable correction term of a bubble to obtain refined point-wise blow up estimates was used in [7, 15, 10].
The compactness issue is closely related to the existence of blowing up solution for small perturbation of \((1.2)\). In this direction there are some result of noncompactness for the perturbed problem if the linear perturbation of the mean curvature on the boundary is strictly positive everywhere (see \([11, 12]\)). Then we do not have the stability of compactness result under a small positive linear perturbation of the boundary condition.

A key observation is that our correction term allows us to obtain the vanishing of the Weyl tensor on the boundary (see Proposition \([18]\)).

The paper is organized as follows. After some preliminaries, in Section \([3]\) we recall the notions of isolated and isolated simple blow up point, and some well known basic properties related to these points. In Section \([4]\) and in particular in Proposition \([4.13]\) we give a crucial estimate for a blowing up sequence of solutions near an isolated simple blow up point, using the sharp correction term defined in Lemma \([3]\). Then, in Section \([5]\) and in Section \([6]\) after presenting a Pohozaev type identity, we provide a sign estimate of the terms of Pohozaev identity near an isolated simple blow up point, and by this result we prove the vanishing of Weyl tensor at any isolated simple blow up point (Proposition \([15]\)). In Section \([7]\) we reduce our analysis to the case of an isolated simple blow up points. and finally in Section \([8]\) we prove our compactness result.

2. Preliminaries and notations

**Notation.** We collect here our main notations. We will use the indices \(1 \leq i, j, k, m, p, r, s \leq n - 1\) and \(1 \leq a, b, c, d \leq n\). Moreover we use the Einstein convention on repeated indices. We denote by \(g\) the Riemannian metric, by \(R_{abcd}\) the full Riemannian curvature tensor, by \(R_{ab}\) the Ricci tensor and by \(R_g\) the scalar curvature of \((M, g)\); moreover the Weyl tensor of \((M, g)\) will be denoted by \(W_g\). The bar over an object (e.g., \(\bar{W}_g\)) will means the restriction to this object to the metric of \(\partial M\). By \(-\Delta_g\) we denote the Laplace-Beltrami operator on \((M, g)\) and we will often use the common notation for conformal Laplacian \(L_g = -\Delta_g + \frac{n-2}{4(n-1)} R_g\) and the conformal boundary operator \(B_g = \frac{\partial}{\partial r} + \frac{n-2}{4} h_g\), where \(\nu\) is the outward normal to \(\partial M\). Finally, on the half space \(\mathbb{R}^n_+ = \{ y = (y_1, \ldots, y_{n-1}, y_n) \in \mathbb{R}^n, y_n \geq 0 \}\) we set \(B_r(y_0) = \{ y \in \mathbb{R}^n, |y - y_0| \leq r \}\) and \(B_r^+(y_0) = B_r(y_0) \cap \{ y_n > 0 \}\). When \(y_0 = 0\) we will use simply \(B_r = B_r(y_0)\) and \(B_r^+ = B_r^+(y_0)\). On the half ball \(B_r^+\) we set \(\partial^0 B_r^+ = B_r^+ \cap \partial \mathbb{R}^n_+ = B_r^+ \cap \{ y_n = 0 \}\) and \(\partial^+ B_r^+ = \partial B_r^+ \cap \{ y_n > 0 \}\). On \(\mathbb{R}^n_+\) we will use the following decomposition of coordinates: \((y_1, \ldots, y_{n-1}, y_n) = (\bar{y}, \bar{y}_n) = (z, t)\) where \(\bar{y}, \bar{y}_n \in \mathbb{R}^{n-1}\) and \(y_n, t \geq 0\).

Finally, fixed a point \(q \in \partial M\), we denote by \(\psi_q : B_r^+ \rightarrow M\) the Fermi coordinates centered at \(q\). We denote by \(B_r^+ (q, r)\) the image of \(\psi_q (B_r^+)\). When no ambiguity is possible, we will denote \(B_r^+ (q, r)\) simply by \(B_{r^+}\), omitting the chart \(\psi_q\).

We can work with a slightly more general problem

\[
\begin{cases}
L_g u = 0 & \text{in } M \\
B_g u + (n - 2)f^{-\tau} u^p = 0 & \text{on } \partial M
\end{cases}
\]

where \(\tau = \frac{n}{n-2} - p, p \in \left[ \frac{\bar{p}, \frac{n}{n-2} }{1} \right]\) for some fixed \(\bar{p} > 1\), and \(f > 0\). The reason to work with this equation instead of equation \((1.4)\) is that equation \((2.1)\) has an important conformal invariance property.

Since the boundary \(\partial M\) of \(M\) is umbilic, it is well know the existence of a conformal metric related to \(g\) and the existence of the conformal Fermi coordinates, which will simplify the future computations.
Given \( q \in \partial M \) there exists a conformally related metric \( \tilde{g}_q = \Lambda_q^{-1} g \) such that some geometric quantities at \( q \) have a simpler form which will be summarized in the next claim. Moreover
\[
\Lambda_q(q) = 1, \quad \frac{\partial \Lambda_q}{\partial y_k}(q) = 0 \quad \text{for all} \quad k = 1, \ldots, n-1.
\]
Set \( \tilde{u}_q = \Lambda_q^{-1} u \) and \( \tilde{f}_q = \Lambda_q f \) it holds
\[
\begin{align*}
L_{\tilde{g}_q} \tilde{u}_q &= 0 \quad \text{in} \ M, \\
B_{\tilde{g}_q} \tilde{u}_q + (n - 2) \tilde{f}_q \tilde{u}_q^n &= 0 \quad \text{on} \ \partial M.
\end{align*}
\]
In the following we study equation (2.2) and in order to simplify notations, we will omit the tilda symbol and we will omit \( \psi_x \), whenever is not needed, so we will write \( y \in B^+_r \) instead of \( \psi_q(y) \in M \); 0 instead of \( q = \psi_q(0) \); \( u \) instead of \( u \circ \psi_q \) where \( \psi_q : B^+_r \to M \) are the Fermi conformal coordinates centered at \( q \).

**Remark 2.** In Fermi conformal coordinates around \( q \in \partial M \), it holds (see [21])
\begin{align*}
|\det g_q(y)| &= 1 + O(|y|^n) \\
|h_{ij}(y)| &= O(|y|) \quad |h_q(y)| = O(|y|)
\end{align*}
\begin{align*}
q_{ij}(y) &= \delta_{ij} + \frac{1}{3} R_{ikjl} y_{ik} y_l + R_{nipj} y^n_i y_p + \frac{1}{6} R_{nij, n} y^n_i y_j \\
&+ \frac{1}{20} R_{ikjl, mp} + \frac{1}{15} R_{ikjl} R_{jmp} y_{ik} y_{jl} y_p + \frac{1}{3} R_{nij, kl} + \frac{1}{3} \text{Sym}_{ij} (R_{ikjl} R_{nsnj}) y^n_i y_j \\
&+ \frac{1}{3} R_{nij, nk} y^n_i y_k + \frac{1}{12} (R_{nij, nn} + 8 R_{nij, nsnj}) y^n_i y^n_j + O(|y|^5)
\end{align*}
\begin{align*}
R_{\gamma q}(y) &= O(|y|^2) \quad \text{and} \quad \partial^2_{\gamma q} R_{\gamma q} = -\frac{1}{6} |W|^2
\end{align*}
\begin{align*}
\partial^2_{\gamma q} R_{\gamma q} &= -2 R_{nij, n}^2 - 2 R_{nij, ij} \\
R_{kk} &= R_{nn} = R_{nk} = R_{nn, kk} = 0 \\
R_{nn, nn} &= -2 R_{nn, ns}.
\end{align*}
All the quantities above are calculate in \( q \in \partial M \), unless otherwise specified.

We set \( U(y) := \frac{1}{[(1 + y_n)^2 + |y|^2]^{\frac{n}{2}}} \) to be the standard bubble. The function \( U \) solves the problem
\[
\begin{align*}
\Delta U &= 0 \quad \text{in} \ \mathbb{R}^n_+ \\
\frac{\partial U}{\partial y_n} + (n - 2) U \frac{\partial U}{\partial y_n} &= 0 \quad \text{on} \ \partial \mathbb{R}^n_+.
\end{align*}
\]
If we linearize Problem (2.10) around the function \( U \), we have that all the solutions of the linearized problem are generated by the functions
\begin{align*}
j_t := \partial_t U &= -(n - 2) \frac{y_t}{[(1 + y_n)^2 + |y|^2]^{\frac{n}{2}}} \\
j_n := y^t \partial_t U + \frac{n - 2}{2} U &= -\frac{n - 2}{2} \frac{|y|^2 - 1}{[(1 + y_n)^2 + |y|^2]^{\frac{n}{2}}}.
\end{align*}
Finally, we have
\[ \partial_k \partial_l U = (n - 2) \left\{ \frac{ny_k y_l}{(1 + y_n)^2 + |y|^2} - \frac{\delta_{kl}}{(1 + y_n)^2 + |y|^2} \right\}. \]

In the following Lemma we introduce the function \( \gamma_q \) as the solution of a certain linear problem. This function \( \gamma_q \) plays a fundamental role in this paper: by this choice of \( \gamma_q \) we are able to cancel the term of second order in formula (1.19), which is crucial to obtain Lemma [10]. Also, the estimates of Proposition [12] and of Lemma [16] depend on the properties of function \( \gamma_q \). The proof of the following Lemma is analogous to [11, Lemma 3] and [11, Proposition 5.1]. However, we rewrite the proof in the appendix.

**Lemma 3.** Assume \( n \geq 5 \). Given a point \( q \in \partial M \), there exists a unique \( \gamma_q : \mathbb{R}^n_+ \to \mathbb{R} \) a solution of the linear problem
\[ (2.13) \quad -\Delta \gamma = \left[ \frac{3}{4} R_{ijkl}(q) y_i y_j + R_{ij}(q) y_n^2 \right] \partial^2_{ij} U \quad \text{on } \mathbb{R}^n_+ \]
which is \( L^2(\mathbb{R}^n_+) \)-orthogonal to the functions \( y_j, \ldots, y_n \) defined in (2.11) and (2.12). Moreover it holds
\[ (2.14) \quad |\nabla^\tau \gamma_q(y)| \leq C(1 + |y|)^{4-\tau-n} \text{ for } \tau = 0, 1, 2; \]
\[ (2.15) \quad \int_{\mathbb{R}^n_+} \gamma_q \Delta \gamma_q dy \leq 0; \]
\[ (2.16) \quad \int_{\partial \mathbb{R}^n_+} U(\gamma_q(t, z) d\gamma_q(t, z) dz = 0; \]
\[ (2.17) \quad \gamma_q(0) = \frac{\partial \gamma_q}{\partial y_1}(0) = \cdots = \frac{\partial \gamma_q}{\partial y_{n-1}}(0) = 0. \]

Finally the map \( q \mapsto \gamma_q \) is \( C^2(\partial M) \).

3. Isolated and isolated simple blow up points

In this section we will define two particular kind of blow up points, and we collect a series of results that focus on the asymptotic behavior of these blow up points. These results are now quite standard, so we will only collect the claims, while for the proofs we refer to [11, 10, 14, 22].

Let \( \{u_i\}_i \) be a sequence of positive solution to
\[ (3.1) \quad \left\{ \begin{array}{ll} L_{g_i} u = 0 & \text{in } M \\ B_{g_i} u + f_i = 0 & \text{on } \partial M \end{array} \right. \]
where \( p_i \in \left[ \frac{n}{n-2}, \frac{n}{n-2} \right] \) for some fixed \( \tilde{p} > 1 \), \( \tau_i = \frac{n}{n-2} - p_i \), \( f_i \to f \) in \( C^1_{\text{loc}} \) for some positive function \( f \) and \( g_i \to g_0 \) in the \( C^2_{\text{loc}} \) topology.

**Definition 4.** We say that \( x_0 \in \partial M \) is a blow up point for the sequence \( u_i \) of solutions of (3.1) if there is a sequence \( x_i \in \partial M \) such that
\[ (1) \quad x_i \to x_0; \]
\[ (2) \quad x_i \text{ is a local maximum point of } u_i|_{\partial M}; \]
\[ (3) \quad u_i(x_i) \to +\infty. \]

Shortly we say that \( x_i \to x_0 \) is a blow up point for \( \{u_i\}_i \).

Given \( x_i \to x_0 \) a blow up point for \( \{u_i\}_i \), we set
\[ M_i := u_i(x_i) \]
**Definition 5.** We say that \( x_i \to x_0 \) is an isolated blow up point for \( \{ u_i \} \), if \( x_i \to x_0 \) is a blow up point for \( \{ u_i \} \), and there exist two constants \( \rho, C > 0 \) such that

\[
|u_i(x)| \leq C d_\bar{g}(x, x_i)^{-\frac{\rho}{\rho - 1}} \quad \text{for all} \quad x \in \partial M \setminus \{ x_i \}, \quad d_\bar{g}(x, x_i) < \rho.
\]

Here \( \bar{g} \) denotes the metric on the boundary induced by \( g \) and \( d_\bar{g}(\cdot, \cdot) \) is the geodesic distance on the boundary between two points.

We recall the following result

**Proposition 6.** Let \( x_i \to x_0 \) is an isolated blow up point for \( \{ u_i \} \), and \( \rho \) as in Definition [a]. We set

\[
v_i(y) = M_i^{-1}(u_i \circ \psi_i)(M_i^{1-p_i} y), \quad y \in B^+_{\rho M_i^{p_i} - 1}(0).
\]

Then, given \( R_i \to \infty \) and \( \beta_i \to 0 \), up to subsequences, we have

1. \( \lim_{i \to \infty} \frac{R_i}{\log M_i} = 0; \)
2. \( \lim_{i \to \infty} p_i = \frac{n}{n-2}; \)
3. \( \lim_{i \to \infty} \beta_i; \)

Given \( x_i \to x_0 \) an isolated blow up point for \( \{ u_i \} \), and given \( \psi_i : B^+_{\rho}(0) \to M \) the Fermi coordinates centered at \( x_i \), we define the spherical average of \( u_i \) as

\[
\bar{u}_i(r) = \frac{2}{\omega_{n-1} r^{n-1}} \int_{\partial B^+_r} u_i \circ \psi_i \, d\sigma_r
\]

and

\[
w_i(r) := r^{-\frac{n}{n-2}} \bar{u}_i(r)
\]

for \( 0 < r < \rho \).

**Definition 7.** We say that \( x_i \to x_0 \) is an isolated simple blow up point for \( \{ u_i \} \), solutions of \( (3.1) \) if \( x_i \to x_0 \) is an isolated blow up point for \( \{ u_i \} \), and there exists \( \rho \) such that \( w_i \) has exactly one critical point in the interval \( (0, \rho) \).

One can prove that is \( x_i \to x_0 \) is an isolated simple blow up point for \( \{ u_i \} \), and if \( R_i \to +\infty \), then

\[
w'_i(r) < 0 \quad \text{for all} \quad r \in [R_i M_i^{1-p_i}, \rho).
\]

This allows to compare this definition of isolated simple blow up point with the other one present in literature (see, e.g. \([10]\)). In fact, in light of Proposition [b] if \( x_i \to x_0 \) is an isolated blow up point for \( \{ u_i \} \), then the function \( r \to r^{-\frac{n}{n-2}} \bar{u}_i(r) \) has exactly one critical point in \( (0, R_i M_i^{1-p_i}) \) and the derivative is negative right after the critical point.

**Proposition 8.** Let \( x_i \to x_0 \) be an isolated simple blow up point for \( \{ u_i \} \) and let \( \eta \) small. Then there exist \( C, \rho > 0 \) such that

\[
M_i^{\lambda_i} |\nabla^k u_i(\psi_i(y))| \leq C |y|^{2-k-n+\eta}
\]

for \( y \in B^+_\rho(0) \setminus \{ 0 \} \) and \( k = 0, 1, 2 \). Here \( \lambda_i = (p_i - 1)(n - 2 - \eta) - 1 \).

**Proposition 9.** Let \( x_i \to x_0 \) be an isolated simple blow up point for \( \{ u_i \} \). Then there exist \( C, \rho > 0 \) such that

\[
(1) \quad M_i u_i(\psi_i(y)) \leq C |y|^{2-n} \quad \text{for all} \quad y \in B^+_\rho(0) \setminus \{ 0 \};
\]

\[
(2) \quad M_i u_i(\psi_i(y)) \geq C^{-1} G_i(y) \quad \text{for all} \quad y \in B^+_\rho(0) \setminus B^+_{r_i}(0) \quad \text{where} \quad r_i := R_i M_i^{1-p_i} \quad \text{and} \quad G_i \quad \text{is the Green’s function which solves}
\]

\[
\begin{cases}
L_i G_i = 0 & \quad \text{in} \quad B^+_\rho(0) \setminus \{ 0 \} \\
\bar{G}_i = 0 & \quad \text{on} \quad \partial^+ B^+_\rho(0) \\
B_{\rho i} G_i = 0 & \quad \text{on} \quad \partial B^+_\rho(0) \setminus \{ 0 \}
\end{cases}
\]
and \( |y|^{n-2} G_i(y) \to 1 \) as \( |z| \to 0 \).

Let us notice that, by Proposition \( \text{[6]} \) and by Proposition \( \text{[9]} \) we have that, if \( x_i \to x_0 \) is an isolated simple blow up point for \( \{ u_i \} \), then, given \( v_i \) as in Proposition \( \text{[6]} \) it holds

\[
v_i \leq CU \text{ in } B^+_{\rho M_i^{1-pi-1}}(0).
\]

4. Blow up estimates

In this section \( x_i \to x_0 \) is an isolated simple blow up point for a sequence \( \{ u_i \} \) of solutions of \( \text{(3.1)} \). We will work in the conformal normal coordinates in a neighborhood of \( x_0 \).

Set \( \tilde{u}_i = \Lambda_{x_i}^{-1} u_i \) we define

\[
(4.1) \quad \delta_i = \tilde{u}_i^{1-pi}(x_i) = u_i^{1-pi}(x_i) = M_i^{1-pi},
\]

since \( \Lambda_{x_i}(x_i) = 1 \).

We have that \( x_i \to x_0 \) is also an isolated blow up point for the function \( \tilde{u}_i \) and the estimates of Proposition \( \text{[9]} \) hold since we have uniform control on the conformal factor \( \Lambda_i \). In the following we simply omit the tilde symbol unless otherwise specified.

Set

\[
v_i(y) := \delta_i^{\tau_i} u_i(\delta_i y) \text{ for } y \in B^+_{\frac{\tau_i}{\delta_i}}(0),
\]

we know that \( v_i \) satisfies

\[
(4.2) \quad \left\{
\begin{array}{l}
L_{\tilde{g}_i} v_i = 0 \quad \text{in } B^+_{\frac{\tau_i}{\delta_i}}(0) \\
B_{\tilde{g}_i} v_i + (n-2) \tilde{f}_i \tilde{g}_i^{p_i} \tau_i = 0 \quad \text{on } \partial B^+_{\frac{\tau_i}{\delta_i}}(0)
\end{array}
\right.
\]

where \( \tilde{g}_i := \tilde{g}_i(\delta_i y) = \Lambda_{x_i}^{-2} (\delta_i y) g(\delta_i y), \tilde{f}_i(\delta_i y) = f_i(\delta_i y), f_i = \Lambda_{x_i} f \to \Lambda_{x_0} f \) and \( \tau_i = \frac{n-2}{n} - p_i \).

Our aim is to provide by Lemma \( \text{[4]} \) a sharp correction term for the usual approximation of the rescaled solution \( v \) by \( U \), near an isolated simple blow up point \( x_i \to x_0 \). This result is obtained in Proposition \( \text{[13]} \) at the end of this section. First, we need two lemmas.

**Lemma 10.** Assume \( n \geq 8 \). Let \( \gamma_{x_i} \) be defined in \( \text{(2.13)} \). There exist \( R, C > 0 \) such that

\[
|v_i(y) - U(y) - \delta_i^2 \gamma_{x_i}(y)| \leq C \left( \delta_i^3 + \tau_i \right)
\]

for \( |y| \leq R/\delta_i \).

**Proof.** Let \( y_i \) such that

\[
\mu_i := \max_{|y| \leq R/\delta_i} |v_i(y) - U(y) - \delta_i^2 \gamma_{x_i}(y)| = |v_i(y_i) - U(y_i) - \delta_i^2 \gamma_{x_i}(y_i)|.
\]

We can assume, without loss of generality, that \( |y_i| \leq \frac{R}{2\delta_i} \).

In fact, suppose that there exists \( c > 0 \) such that \( |y_i| > \frac{R}{2\delta_i} \) for all \( i \). Then, since \( v_i(y) \leq CU(y_i) \) and by \( \text{(2.14)} \), we get the inequality

\[
|v_i(y_i) - U(y_i) - \delta_i^2 \gamma_{x_i}(y_i)| \leq C \left( |y_i|^{2-n} + \delta_i^2 |y_i|^{|4-n}| \right) \leq C \delta_i^{4-n}
\]

which proves the Lemma. So, in the next we will suppose \( |y_i| \leq \frac{R}{2\delta_i} \). This condition will be exploited later.

To achieve the proof we proceed by contradiction, supposing that

\[
(4.3) \quad \max \left\{ \mu_i^{-1} \delta_i^3, \mu_i^{-1} \tau_i \right\} \to 0 \text{ when } i \to \infty.
\]

Defined

\[
w_i(y) := \mu_i^{-1} (v_i(y) - U(y) - \delta_i^2 \gamma_{x_i}(y)) \text{ for } |y| \leq R/\delta_i,
\]
we have, by direct computation, that $w_i$ satisfies

$$
\begin{align*}
L_{\tilde{\nu}} w_i &= Q_i & \text{in } B^+_{\tilde{\nu}}(0) \\
B_{\tilde{\nu}} w_i + b_i w_i &= \tilde{Q}_i & \text{on } \partial B^+_{\tilde{\nu}}(0)
\end{align*}
$$

where

$$
b_i = (n - 2) \tilde{\nu}^i - \frac{(U + \delta^i_2 \gamma_i)}{v_i} - \frac{(U + \delta^i_2 \gamma_i)}{v_i} = (n - 2) \tilde{\nu}^i - \frac{(U + \delta^i_2 \gamma_i)}{v_i}
$$

and

$$
\begin{align*}
\tilde{Q}_i &= - \frac{1}{\mu_i} \left\{ (n - 2) (U + \delta^i_2 \gamma_i) \tilde{\nu}^i - (n - 2) U \tilde{\nu}^i - n \delta^i_2 U \tilde{\nu}^i \gamma_i - \frac{n - 2}{2} h_{\tilde{\nu}} (U + \delta^i_2 \gamma_i) \right\} \\
&\quad + \frac{n - 2}{\mu_i} \left\{ (U + \delta^i_2 \gamma_i) \tilde{\nu}^i - \tilde{\nu}^i (U + \delta^i_2 \gamma_i) \right\} =: \tilde{Q}_{i,1} + \tilde{Q}_{i,2}
\end{align*}
$$

$$
\begin{align*}
Q_i &= - \frac{1}{\mu_i} \left\{ (L_{\tilde{\nu}} - \Delta) (U + \delta^i_2 \gamma_i) + \delta^i_2 \Delta \gamma_i \right\}.
\end{align*}
$$

We give now some estimate for the terms $b_i, Q_i, \tilde{Q}_i$, in order to show that the sequence $w_i$ converges in $C^2_{\text{loc}}(\mathbb{R}^n)$ to some $w$ solution of

$$
\begin{align*}
\Delta w &= 0 & \text{in } \mathbb{R}^n \\
\frac{\partial}{\partial v} w + nU \tilde{\nu}^i w &= 0 & \text{on } \partial \mathbb{R}^n.
\end{align*}
$$

Then we will derive a contradiction using (4.3).

By Lagrange Theorem we have

$$
b_i = (n - 2) \tilde{\nu}^i - \frac{(U + \delta^i_2 \gamma_i)}{v_i} - \left( (1 - \theta) (U + \delta^i_2 \gamma_i) \right)^{p_i - 1}
$$

and, since $v_i \to U$ in $C^2_{\text{loc}}(\mathbb{R}^n)$, we have, at once,

$$
\begin{align*}
b_i &\to n U \tilde{\nu}^i & \text{in } C^2_{\text{loc}}(\mathbb{R}^n); \\
|b_i(y)| &\leq (1 + |y|)^{-2} & \text{for } |y| \leq R/\delta_i.
\end{align*}
$$

We proceed now by estimating $Q_i$ and $\tilde{Q}_i$. We recall that

$$
\begin{align*}
|L_{\tilde{\nu}} - \Delta| u(y) &= \left( g_{ik}^i - \delta^{kl} \right) \partial_k \partial_l u + \partial_k g_{ik}^i \partial_l u - \frac{n - 2}{4(n - 1)} R_{\tilde{\nu}} u \\
&\quad + \frac{\partial_k |g_{ik}^i|}{|g_{ik}^i|^2} g_{ik}^i \partial_l u \\
&= \left( g_{ik}^i (\delta_k y) - \delta^{kl} \right) \partial_k \partial_l u + \delta_i \partial_k g_{ik}^i (\delta_k y) \partial_l u - \delta_i^2 \frac{n - 2}{4(n - 1)} R_{\tilde{\nu}} (\delta_i y) u \\
&\quad + O(\delta_i^N |y|^{N - 1}) \partial_l u
\end{align*}
$$

where $N$ can be chosen large since we use conformal Fermi coordinates. At this point we use the definition of the function $\gamma_{xi}$ (see (2.13)), and, by (4.8), (2.5) and
the decays properties of $U$ and $\gamma_{x_i}$, we obtain

\[ -\mu_i Q_i = \delta_i \left( \frac{1}{3} \tilde{R}_{kij}y_jy_j + R_{nk}x_n \right) (\partial_0 \partial_i U + \delta_i \partial_0 \partial_i \gamma_{x_i}) \]
\[ + O(\delta_i |y|^{3}) (\partial_0 \partial_i U + \delta_i \partial_0 \partial_i \gamma_{x_i}) \]
\[ + \delta_i \left( \frac{1}{3} \tilde{R}_{kij}y_j + \frac{1}{3} \tilde{R}_{ij}x_k \right) \partial_i U + \delta_i \partial_0 \partial_i \gamma_{x_i} \]
\[ + O(\delta_i |y|^2) (\partial_0 U + \delta_i \partial_0 \gamma_{x_i}) \]
\[ + O(\delta_i^2 |y|^2) (U + \delta_i^2 \gamma_{x_i}) \]
\[ + \delta_i^2 \Delta \gamma_{x_i} + O(\delta_i^3 |y|^{N-1}) (\partial_0 U + \delta_i^2 \partial_0 \gamma_{x_i}) \]
\[ = O \left( \delta_i^2 (1 + |y|)^{3-n} \right) + O \left( \delta_i^4 (1 + |y|)^{4-n} \right) + O \left( \delta_i^6 (1 + |y|)^{5-n} \right) \]
\[ + O \left( \delta_i^8 (1 + |y|)^{6-n} \right) + O \left( \delta_i^{N+2} (1 + |y|)^{N+2-n} \right). \]

(4.9)

Since $|y| \leq R/\delta_i$, we have $\delta_i (1 + |y|) \leq C$, thus

\[ Q_i = O(\mu_i^{-1} \delta_i^3 (1 + |y|)^{3-n}). \]

(4.10)

In light of (4.3) we have also $Q_i \in L^p(B_{R/\delta_i}^+)$ for all $p \geq 2$.

By Taylor expansion, and proceeding as above, we have

\[ -\mu_i \tilde{Q}_{i,1} = \left\{ \delta_i \left( \frac{2}{n-2} (U + \theta \delta_i^2 \gamma_{x_i}) \right) \rightarrow \gamma_{x_i}^2 - \frac{n-2}{2} \delta_i \partial_0 (\delta_i^2 |y|^2) (U + \delta_i^2 \gamma_{x_i}) \right\} \]
\[ = O(\delta_i^4 (1 + |y|)^{5-n}). \]

Notice that in the above estimates we have $U + \theta \delta_i^2 \gamma_{x_i} > 0$ since we are in $B_{R/\delta_i}$.

Similarly, since $(U + \delta_i^2 \gamma_{x_i},)^p = (U + \delta_i^2 \gamma_{x_i}) \frac{\partial \tau_i}{\partial y} + O(\tau_i) (U + \delta_i^2 \gamma_{x_i},) \frac{\partial \tau_i}{\partial y} \log(U + \delta_i^2 \gamma_{x_i})$ and $f^{-\tau_i} = 1 + O(\tau_i)$, we have

\[ -\mu_i \tilde{Q}_{i,2} = O(\tau_i (1 + |y|)^{1-n}). \]

We conclude

\[ \tilde{Q}_i = O(\mu_i^{-1} \delta_i^3 (1 + |y|)^{3-n}) + O(\mu_i^{-1} \tau_i (1 + |y|)^{1-n}), \]

and $\tilde{Q}_i \in L^p(\partial^+ B_{R/\delta_i}^+)$ for all $p \geq 2$.

Finally we remark that $|w_i(y)| \leq w_i(y_i) = 1$, so by (4.3), (4.6), (4.7), (4.10), (4.11) and by standard elliptic estimates we conclude that, up to subsequence, $\{w_i\}_i$ converges in $C^2_{\text{loc}}(\mathbb{R}^n_+)$ to some $w$ solution of (1.5), as claimed.

The next step is to prove that $|w(y)| \leq C(1 + |y|^{-1})$ for $y \in \mathbb{R}^n_+$. To do so, we consider $G_i$ the Green function for the conformal Laplacian $L_{\delta_i}$ defined on $B_{r/\delta_i}$ with boundary conditions $B_{\delta_i} G_i = 0$ on $\partial^+ B_{r/\delta_i}$ and $G_i = 0$ on $\partial^+ B_{r/\delta_i}$. It is well known that $G_i = O(|\xi - y|^{2-n})$. By the Green formula and by (4.10) and (4.11) we will be able to estimate $w_i$ in $B_{R/(2\delta_i)}^+$. In fact

\[ w_i(y) = -\int_{B_{\delta_i}^+} G_i(\xi, \eta) Q_i(\xi) d\mu_{\delta_i}(\xi) - \int_{\partial B_{\delta_i}^+} \frac{\partial G_i}{\partial \nu}(\xi, y) w_i(\xi) d\sigma_{\delta_i}(\xi) \]
\[ + \int_{\partial^+ B_{\delta_i}^+} G_i(\xi, y) (b_i(\xi) w_i(\xi) - \tilde{Q}_i(\xi)) d\sigma_{\delta_i}(\xi), \]
so

\[ |w_i(y)| \leq \frac{\delta_i^3}{\mu_i} \int_{B^+_{\frac{R}{\delta_i}}} |\xi - y|^{2-n} (1 + |\xi|)^{3-n} d\xi + \int_{\partial B^+_{\frac{R}{\delta_i}}} |\xi - y|^{1-n} w_i(\xi) d\sigma(\xi) \]

\[ + \int_{\partial^+ B^+_{\frac{R}{\delta_i}}} |\xi - y|^{2-n} \left( (1 + |\xi|)^{-2} + \frac{\delta_i^4}{\mu_i} (1 + |\xi|)^{5-n} + \frac{\tau_i}{\mu_i} (1 + |\xi|)^{1-n} \right) d\xi, \]

where in the last integral we used that \(|w_i(y)| \leq 1\). For the second integral we use that \(|y| \leq \frac{R}{2\delta_i}\) to estimate \(|\xi - y| \geq |\xi| - |y| \geq \frac{R}{2\delta_i}\) on \(\partial^+ B^+_{\frac{R}{\delta_i}}\). Moreover, since \(v_i(\xi) \leq CU(\xi)\), we get the inequality

\[ (4.12) \quad |w_i(\xi)| \leq \frac{C}{\mu_i} \left( (1 + |\xi|)^{3-n} + \frac{\delta_i^3}{\mu_i} (1 + |\xi|)^{1-n} \right) \leq C\frac{\delta_i^{n-2}}{\mu_i} \text{ on } \partial^+ B^+_{\frac{R}{\delta_i}}; \]

hence

\[ (4.13) \quad \int_{\partial^+ B^+_{\frac{R}{\delta_i}}} |\xi - y|^{1-n} w_i(\xi) d\sigma(\xi) \leq C \int_{\partial^+ B^+_{\frac{R}{\delta_i}}} \frac{\delta_i^{2n-3}}{\mu_i} d\sigma(\xi) \leq C\frac{\delta_i^{n-2}}{\mu_i}. \]

For the other terms we use the following formula (see [1] Lemma 9.2) and [6, 13]

\[ (4.14) \quad \int_{\mathbb{R}^m} |\xi - y|^{\beta - m} (1 + |y|)^{-\alpha} \leq C(1 + |y|)^{\beta - \alpha}, \]

which holds for \(y \in \mathbb{R}^{m+k} \supseteq \mathbb{R}^m\) and for \(\alpha, \beta \in \mathbb{N}, 0 < \beta < \alpha < m\), to obtain

\[ (4.15) \quad \frac{\delta_i^3}{\mu_i} \int_{B^+_{\frac{R}{\delta_i}}} |\xi - y|^{2-n} (1 + |\xi|)^{3-n} d\xi \leq C\frac{\delta_i^3}{\mu_i} (1 + |y|)^{5-n}; \]

\[ (4.16) \quad \int_{\partial^+ B^+_{\frac{R}{\delta_i}}} |\xi - y|^{2-n} (1 + |\xi|)^{-2} d\xi \leq (1 + |y|)^{-1}; \]

\[ (4.17) \quad \frac{\delta_i^4}{\mu_i} \int_{\partial^+ B^+_{\frac{R}{\delta_i}}} |\xi - y|^{2-n} (1 + |\xi|)^{5-n} d\xi \leq C\frac{\delta_i^4}{\mu_i} (1 + |y|)^{6-n}; \]

\[ (4.18) \quad \frac{\tau_i}{\mu_i} \int_{\partial^+ B^+_{\frac{R}{\delta_i}}} |\xi - y|^{2-n} (1 + |\xi|)^{1-n} d\xi \leq C\frac{\tau_i}{\mu_i} (1 + |y|)^{2-n}. \]

By (4.13), (4.15), (4.16), (4.17), (4.18), we have

\[ (4.19) \quad |w_i(y)| \leq C \left( (1 + |y|)^{-1} + \frac{\delta_i^3}{\mu_i} (1 + |y|)^{5-n} + \frac{\tau_i}{\mu_i} (1 + |y|)^{2-n} \right) \text{ for } |y| \leq \frac{R}{2\delta_i}, \]

so by assumption (4.18) we prove

\[ (4.20) \quad |w(y)| \leq C(1 + |y|)^{-1} \text{ for } y \in \mathbb{R}^n_+ \]

as claimed.

Finally we notice that, since \(v_i \to U\) near 0, and by (4.17) we have \(w_i(0) \to 0\) as well as \(\frac{\partial w_i}{\partial y_j}(0) \to 0\) for \(j = 1, \ldots, n - 1\). This implies, since \(w_i \to w\) in \(C^2_\text{loc}\), that

\[ (4.21) \quad w(0) = \frac{\partial w}{\partial y_1}(0) = \cdots = \frac{\partial w}{\partial y_{n-1}}(0) = 0. \]

We are ready now to prove the contradiction. In fact, it is known (see [1] Lemma 2) that any solution of (4.5) that decays as (4.20) is a linear combination of \(\frac{\partial U}{\partial y_1}, \ldots, \frac{\partial U}{\partial y_n}, \frac{\partial^2 U}{\partial y_1^2} + y^2 \frac{\partial^2 U}{\partial y_n^2}\). This fact, combined with (4.21), implies that \(w \equiv 0\).
Now, on one hand $|y_i| \leq \frac{R}{2\delta_1}$, so estimate (4.19) holds; on the other hand, since $w_i(y_i) = 1$ and $w \equiv 0$, we get $|y_i| \to \infty$, obtaining

$$1 = w_i(y) \leq C(1 + |y_i|)^{-1} \to 0$$

which gives us the desired contradiction, and proves the Lemma. \(\square\)

**Lemma 11.** Assume $n \geq 8$. There exists $C > 0$ such that

$$\tau_i \leq C\delta_1^3.$$ 

**Proof.** We proceed by contradiction, supposing that

$$\tau_i^{-1} \delta_1^3 \to 0 \text{ when } i \to \infty.$$ 

Thus, by Lemma 10 we have

$$|v_i(y) - U(y) - \delta_i^2 \gamma_{x_i}(y)| \leq C\tau_i \text{ for } |y| \leq R/\delta_i.$$ 

We define, similarly to Lemma 10

$$w_i(y) := \frac{1}{\tau_i} (v_i(y) - U(y) - \delta_i^2 \gamma_{x_i}(y)) \text{ for } |y| \leq R/\delta_i,$$

and we have that $w_i$ satisfies (4.1), where

$$b_i = (n - 2)\hat{f}_i \tau_i^{-1} \tau_i^p (U + \delta_i^2 \gamma_{x_i})^p.$$

Next, let

$$Q_i = -\frac{1}{\tau_i} \left\{ (n - 2)(U + \delta_i^2 \gamma_{x_i})^{\frac{2n}{n-2}} - (n - 2)U^{\frac{2n}{n-2}} - n\delta_i^2 U^{\frac{2n}{n-2}} \gamma_{x_i} - \frac{n}{2}\hat{g}_i(U + \delta_i^2 \gamma_{x_i}) \right\}$$

and we can proceed as in Lemma 10 to deduce that

$$|w_i(y)| \leq C \left( (1 + |y|)^{-1} + \frac{\delta_i^3}{\tau_i} (1 + |y|)^{5-n} \right) \text{ for } |y| \leq \frac{R}{2\delta_i}.$$ 

By classic elliptic estimates, we can prove that the sequence $w_i$ converges in $C^2_{\text{loc}}(\mathbb{R}^n_+)$ to some $w$.

Finally, by assumption on $\{f_i\}$, $f_i \to \Lambda f$ in the $C^4$ topology, and since $\hat{f}_i(y) = f(\delta_i y)$, and recalling that $x_i = \psi_i(0)$, $x_i \to x_0$ and $\Lambda f(x_0) = 1$, we have

$$\lim_{i \to +\infty} \frac{1}{\tau_i} \left\{ (U + \delta_i^2 \gamma_{x_i})^{\frac{2n}{n-2}} - \hat{f}_i^{-\tau_i} (U + \delta_i^2 \gamma_{x_i})^p \right\} = [\log(f(x_0)) + \log U] U^{\frac{2n}{n-2}}.$$ 

Now, let $j_n$ defined as in (2.12). Since $\int_{\mathbb{R}^n_+} j_n(y)U^{\frac{n}{n-2}}(y)dy = 0$, and in light of (4.20) and (4.24), we get

$$\lim_{i \to +\infty} \int_{\partial B^+} j_n \hat{Q}_i d\sigma_{y_i} = (n - 2) \int_{\partial B^+} j_n(y) \log U(y)U^{\frac{n}{n-2}}(y)dy.$$ 

By direct computation we have

$$(n - 2) \int_{\partial B^+} j_n(y) \log U(y)U^{\frac{n}{n-2}}(y)dy > 0.$$
In fact, integrating in polar coordinates $r := |\bar{y}|$ on $\partial \mathbb{R}^n_+$, we obtain

$$\int_{\partial \mathbb{R}^n_+} j_n(y) \log U(y) U^{\frac{n-2}{2}}(y) dy = -\sigma_{n-2} \left( \frac{(n-2)^2}{4} \right) \int_0^\infty \frac{1 - r^2}{(1 + r^2)^n} r^{n-2} \log(1 + r^2) dr > 0.$$ 

At this point we can see that (4.27) leads us to a contradiction. Indeed, since $w_i$ satisfies (4.4), integrating by parts we obtain

$$\int_{\partial B^+_+} j_n \hat{Q} d\sigma_{\hat{g}_i} = \int_{\partial B^+_+} j_n [B_{\hat{g}_i} w_i + b_i w_i] d\sigma_{\hat{g}_i}$$

$$= \int_{\partial B^+_+} w_i [B_{\hat{g}_i} j_n + b_i j_n] d\sigma_{\hat{g}_i} + \int_{\partial B^+_+} \left[ \frac{\partial j_n}{\partial \eta_i} w_i - \frac{\partial w_i}{\partial \eta_i} j_n \right] d\sigma_{\hat{g}_i}$$

$$+ \int_{B^+_+} \left[ w_i L_{\hat{g}_i} j_n - j_n L_{\hat{g}_i} w_i \right] d\mu_{\hat{g}_i},$$

where $\eta_i$ is the inward unit normal vector to $\partial^+ B^+_+$. 

By the decay of $j_n$ and by the decay of $w_i$ given by (4.25) and by (4.22), we have

$$\lim_{i \to +\infty} \int_{\partial B^+_+} \left[ \frac{\partial j_n}{\partial \eta_i} w_i - \frac{\partial w_i}{\partial \eta_i} j_n \right] d\sigma_{\hat{g}_i} = 0$$

and by (4.4) and by the decay of $Q_i$ given in (4.23) we have

$$\lim_{i \to +\infty} \int_{B^+_+} j_n L_{\hat{g}_i} w_i d\mu_{\hat{g}_i} = \int_{B^+_+} j_n Q_i d\mu_{\hat{g}_i} = 0.$$ 

Finally, since $\Delta j_n = 0$, by (4.3) we get

$$\lim_{i \to +\infty} \int_{B^+_+} w_i L_{\hat{g}_i} j_n d\mu_{\hat{g}_i} = 0,$$

thus, by (4.28), (4.29) and (4.30), we have

$$\lim_{i \to +\infty} \int_{\partial B^+_+} j_n \hat{Q} d\sigma_{\hat{g}_i} = \lim_{i \to +\infty} \int_{\partial B^+_+} w_i [B_{\hat{g}_i} j_n + b_i j_n] d\sigma_{\hat{g}_i}$$

$$= \int_{\partial \mathbb{R}^n_+} w \left[ \frac{\partial j_n}{\partial y_n} + nU^{\frac{n-2}{2}} j_n \right] d\sigma_{\hat{g}_i} = 0$$

since $\frac{\partial j_n}{\partial y_n} + nU^{\frac{n-2}{2}} j_n = 0$ when $y_n = 0$. Comparing (4.27) and (4.31) we get the contradiction. 

The above lemmas are the core of the following proposition, in which we iterate the procedure of Lemma 10 to obtain better estimates of the rescaled solution $v_i$ of (4.2) around the isolated simple blow up point $x_i \to x_0$. 

□
Proposition 12. Assume \( n \geq 8 \). Let \( \gamma_{x_i} \) be defined in (2.7). There exist \( R, C > 0 \) such that
\[
|v_i(y) - U(y) - \delta_i^2 \gamma_{x_i}(y)| \leq C \delta_i^3 (1 + |y|)^{5-n}
\]
\[
\left| \frac{\partial}{\partial j} (v_i(y) - U(y) - \delta_i^2 \gamma_{x_i}(y)) \right| \leq C \delta_i^3 (1 + |y|)^{4-n}
\]
\[
\left| \frac{\partial}{\partial n} (v_i(y) - U(y) - \delta_i^2 \gamma_{x_i}(y)) \right| \leq C \delta_i^3 (1 + |y|)^{5-n}
\]
\[
\left| \frac{\partial^2}{\partial j \partial k} (v_i(y) - U(y) - \delta_i^2 \gamma_{x_i}(y)) \right| \leq C \delta_i^3 (1 + |y|)^{3-n}
\]
for \( |y| \leq \frac{R}{2\delta_i} \). Here \( j, k = 1, \ldots, n-1 \).

Proof. In analogy with Lemma 10 we set
\[
w_i(y) := v_i(y) - U(y) - \delta_i^2 \gamma_{x_i}(y) \quad \text{for} \quad |y| \leq R/\delta_i,
\]
and we have that \( w_i \) satisfies (4.4), where
\[
b_i = (n-2)\bar{f} - \frac{v_i^{p_i} - (U + \delta_i^2 \gamma_{x_i})^{p_i}}{v_i - U - \delta_i^2 \gamma_{x_i}}
\]
\[
\bar{Q}_i = -\left\{ (n-2)(U + \delta_i^2 \gamma_{x_i}) \frac{\bar{v}_i}{\bar{v}_i - \bar{U} - \delta_i^2 \gamma_{x_i}} - (n-2)U \frac{\bar{v}_i}{\bar{v}_i - \bar{U} - \delta_i^2 \gamma_{x_i}} - \frac{n-2}{2} \delta_i (U + \delta_i^2 \gamma_{x_i}) \right\}
\]
\[
+ \frac{n-2}{\tau_i} \left\{ (U + \delta_i^2 \gamma_{x_i}) \frac{\bar{v}_i}{\bar{v}_i - \bar{U} - \delta_i^2 \gamma_{x_i}} - \bar{f} - (U + \delta_i^2 \gamma_{x_i}) \right\}
\]
\[
Q_i = -\left\{ (L_{\bar{g}_i} - \Delta) (U + \delta_i^2 \gamma_{x_i}) + \delta_i^2 \Delta \gamma_{x_i} \right\}.
\]
As before, \( b_i \) satisfies inequality (4.7) and
\[
Q_i = O(\delta_i^3 (1 + |y|)^{3-n})
\]
\[
Q_i = O(\delta_i^4 (1 + |y|)^{5-n}) + O(\delta_i^3 (1 + |y|)^{1-n}) = O(\delta_i^3 (1 + |y|)^{5-n}).
\]
We define again the Green function \( G_i \) as in the previous lemma and we have, by Green formula,
\[
|w_i(y)| \leq \int_{B^+_R} |\xi - y|^{2-n} Q_i(\xi) d\xi + \int_{\partial B^+_R} |\xi - y|^{1-n} w_i(\xi) d\sigma(\xi)
\]
\[
+ \int_{\partial B^+_R} |\xi - y|^{2-n} b_i(\xi) w_i(\xi) d\xi + \int_{\partial B^+_R} |\xi - y|^{2-n} Q_i(\xi) d\xi.
\]
By the results of Lemma 10 and Lemma 11 and in analogy with equation (4.12), we have that
\[
|w_i(y)| \leq C \delta_i^3 \text{ on } B^+_R/\delta_i \quad \text{and} \quad |w_i(\xi)| \leq C \delta_i^{n-2} \text{ on } \partial^+ B^+_R/\delta_i.
\]
Plugging (4.17), (4.32), (4.33) and (4.35) in (4.34) and proceeding as in Lemma 10 we obtain

(4.36) \[ \int_{\partial B^+_{\frac{R}{\delta_i}}} |\xi - y|^{2-n}Q_\xi(\xi) d\xi \leq C\delta_i^3 (1 + |y|)^{5-n} \]

(4.37) \[ \int_{\partial^+ B^+_{\frac{R}{\delta_i}}} |\xi - y|^{1-n}w_1(\xi)\sigma(\xi) \leq C\delta_i^{\alpha-2} \]

(4.38) \[ \int_{\partial B^+_{\frac{R}{\delta_i}}} |\xi - y|^{2-n}b_1(\xi)w_1(\xi) d\xi \leq \delta_i^3 (1 + |y|)^{-1} \]

(4.39) \[ \int_{\partial B^+_{\frac{R}{\delta_i}}} |\xi - y|^{2-n}\bar{Q}_\xi(\xi) d\xi \leq C\delta_i^3 (1 + |y|)^{5-n} \]

for \( |y| \leq \frac{R}{2\delta_i} \), which implies

(4.40) \[ |w_1(y)| \leq C\delta_i^3 (1 + |y|)^{-1} \quad \text{for} \quad |y| \leq \frac{R}{2\delta_i}. \]

We now iterate this procedure, inserting equality (4.40) in equation (4.34). Inequalities (4.36), (4.37) and (4.39) do not improve, while for (4.38) we have

(4.41) \[ \int_{\partial B^+_{\frac{R}{\delta_i}}} |\xi - y|^{2-n} b_1(\xi) w_1(\xi) d\xi \leq \delta_i^3 (1 + |y|)^{-2} \]

for \( |y| \leq \frac{R}{2\delta_i} \), getting

(4.42) \[ |w_1(y)| \leq \delta_i^3 (1 + |y|)^{-2} \quad \text{for} \quad |y| \leq \frac{R}{2\delta_i}. \]

We iterate again to further improve estimate (4.41), until we reach

(4.43) \[ |w_1(y)| \leq C\delta_i^2 (1 + |y|)^{5-n} \quad \text{for} \quad |y| \leq \frac{R}{2\delta_i}, \]

which proves the first claim.

To prove the estimate for \( y_n \frac{\partial}{\partial y_n} w_1 \), we differentiate the Green formula obtaining

\[ y_n \frac{\partial}{\partial y_n} w_1(y) = - y_n \int_{B^+_{\frac{R}{\delta_i}}} \frac{\partial}{\partial y_n} G_\xi(\xi, y) Q_\xi(\xi) d\mu_{y_n}(\xi) - y_n \int_{\partial B^+_{\frac{R}{\delta_i}}} \frac{\partial}{\partial y_n} G_\xi(\xi, y) w_1(\xi) d\sigma_{y_n}(\xi) \]

\[ + \int_{\partial B^+_{\frac{R}{\delta_i}}} y_n \frac{\partial}{\partial y_n} G_\xi(\xi, y) \left( b_1(\xi) w_1(\xi) - \bar{Q}_\xi(\xi) \right) d\sigma_{y_n}(\xi), \]

ans since \( \frac{\partial}{\partial y_n} G_\xi(\xi, y) = O(|\xi - y|^{-n-1}) \), we can proceed as above for the first two integrals. Then we use the trivial estimate \( |y_n| \leq (1 + |y|) \) to obtain the desired inequality. The last term is more delicate, since we cannot use directly estimate (4.14), for the restriction on the exponents. Anyway, since \( \xi_n = 0 \) on \( \partial B^+_{\frac{R}{\delta_i}} \), we have

\[ \left. \frac{\partial}{\partial y_n} G_\xi(\xi, y) \right|_{\partial B^+_{\frac{R}{\delta_i}}} = O(|\xi - y|^{-n} y_n) \]

and, since \( y_n^2 \leq |\xi - y|^2 \) on \( \partial B^+_{\frac{R}{\delta_i}} \), we conclude

\[ y_n \frac{\partial}{\partial y_n} G_\xi(\xi, y) \left|_{\partial B^+_{\frac{R}{\delta_i}}} \right. = O(|\xi - y|^{-n} y_n^2) = O(|\xi - y|^{2-n}). \]
At this point we have
\[
\int_{\partial B^+_r(x)} y_n \frac{\partial}{\partial y_n} G_1(\xi, y) \left( b_i(\xi) w_i(\xi) - Q_i(\xi) \right) \, d\sigma_{\bar{\gamma}_i}(\xi)
\]
\[
\leq C \int_{\partial B^+_r(x)} |\xi - y|^{2-n} \left( b_i(\xi) w_i(\xi) - Q_i(\xi) \right) \, d\sigma_{\bar{\gamma}_i}(\xi)
\]
and we are in position to use (4.14). Then we can obtain the desired estimate with the same technique of Lemma 10.

To prove the estimates for \( \frac{\partial}{\partial y_k} w_i \), we have to differentiate equation (4.4), getting
\[
\left\{ \begin{array}{ll}
L_{\bar{\gamma}_i} \frac{\partial}{\partial y_k} w_i = \frac{\partial^2}{\partial y_k^2} Q_i - \frac{\partial}{\partial y_k} R_g w_i & \text{in } B^+_r(0) \\
B_{\bar{\gamma}_i} \frac{\partial}{\partial y_k} w_i = \frac{\partial^2}{\partial y_k^2} [Q_i - b_i w_i] - \left( \frac{\partial}{\partial y_k} h_{ij} \right) w_i & \text{on } \partial B^+_r(0)
\end{array} \right.
\]
and we can repeat the strategy contained in Lemma 10 and in this proof to obtain the claim. For the estimate on the second derivatives we proceed analogously. \( \square \)

5. A Pohozaev type identity

We present here an analogous of the well known Pohozaev identity.

**Theorem 13** (Pohozaev Identity). Let \( u \) a \( C^2 \)-solution of the following problem
\[
\left\{ \begin{array}{ll}
L_{y} u = 0 & \text{in } B^+_r \circ
\end{array} \right.
\]
\[
B_g u + (n - 2) |f - \tau u|^p = 0 & \text{on } \partial B^+_r(0)
\]
for \( B^+_r = \psi^{-1}(B^+_r(q, r)) \) for \( q \in \partial M \), with \( \tau = \frac{n}{n+2} - p > 0 \).

\[
P(u, r) := \int_{\partial B^+_r} \left( \frac{n - 2}{2} \frac{\partial u}{\partial r} - \frac{r}{2} \nabla u \right)^2 \, dy + \int_{\partial B^+_r} f^{-\tau} u^{p+1} \, d\sigma_g.
\]

Then
\[
P(u, r) = -\int_{B^+_r} \left( y^a \partial_a u + \frac{n - 2}{2} u \right) [(L_g - \Delta) u] \, dy + \int_{\partial B^+_r} \left( g^{\hat{k} \hat{a}} \partial_{\hat{k}} u + \frac{n - 2}{2} u \right) h_{\hat{a}} \, d\hat{y}
\]
\[
- \tau \int_{\partial B^+_r} \left( g^{\hat{k} \hat{a}} \partial_{\hat{k}} u \right) f^{-\tau} u^{p+1} \, d\hat{y} + \int_{\partial B^+_r} \left( \frac{n - 1}{p+1} - \frac{n}{2} \right) \left( n - 2 \right) f^{-\tau} u^{p+1} \, d\hat{y}.
\]

We recall that \( a = 1, \ldots, n, k = 1, \ldots, n - 1 \) and \( y = (\hat{y}, y_n) \), where \( \hat{y} \in \mathbb{R}^{n-1} \) and \( y_n \geq 0 \).

**Proof.** The proof is essentially identical to the classical Pohozaev identity: we multiply equation by \( y^a \partial_a u \) and we integrate by parts. All the details can be found in [1] Prop. 3.1. \( \square \)

6. Sign estimates of Pohozaev identity terms

In this section, we want to estimate \( P(u, r) \), where \( \{u_i\}_i \) is a family of solutions of (6.1) which has an isolated simple blow up point \( x_i \to x_0 \).

Since the leading term of \( P(u, r) \) will be \( -\int_{B^+_{r/s_i}} \left( y^a \partial_{\hat{k}} u + \frac{n - 2}{2} u \right) [(L_{\bar{\gamma}_i} - \Delta) v] \, dy \) we set
\[
R(u, v) = -\int_{B^+_{r/s_i}} \left( y^a \partial_{\hat{k}} u + \frac{n - 2}{2} u \right) [(L_{\bar{\gamma}_i} - \Delta) v] \, dy.
\]
Proposition 14. Let \( x_i \to x_0 \) be an isolated simple blow-up point for \( u_i \) solutions of (3.1). Then, fixed \( r \), we have

\[
P(u_i, r) \geq \delta_i \frac{(n-2)\omega_n-2L_i}{(n-1)(n-3)(n-5)(n-6)} \left[ \frac{(n-2)}{6} |W(x_i)|^2 + \frac{4(n-8)}{(n-4)} R_{ij}(x_i) \right]
- 2\delta_i \int_{\mathbb{R}^n_+} \gamma_8 \Delta \gamma_8, dy + o(\delta_i^4).
\]

Here \( I_n(t) = \int_{0}^{\infty} s^n (1 + s^2)^n ds > 0 \).

Proof. We have, by Theorem 13 and recalling that \( \tau_i = \frac{n}{n-2} - p_i \),

\[
P(u_i, r) = - \int_{B^{+}_r} \left( y^2 \partial_n u_i + \frac{n-2}{2} u_i \right) [(L_{\tilde{g}} - \Delta) u_i] dy
+ \frac{n-2}{2} \int_{\partial B^{+}_r} \left( y^2 \partial_n u_i + \frac{n-2}{2} u_i \right) h_g u_i d\gamma
+ \frac{\tau_i(n-2)}{p_i + 1} \int_{\partial B^{+}_r} \left( \int_{\partial B^{+}_r} f_i^{-\gamma} u_i^{p_i+1} dy - \int_{\partial B^{+}_r} (y^2 \partial_n f_i) f_i^{-\gamma} u_i^{p_i+1} dy \right).
\]

where \( B^{+}_r \) is the counterimage of \( B^{+}_0(x_i, r) \) by \( \psi_i \). Since \( f_i \) are positive, bounded away from 0, and bounded in the \( C^1 \) topology, we can choose \( r \) sufficiently small in order to have

\[
\tau_i(n-2) \int_{\partial B^{+}_r} f_i^{-\gamma} u_i^{p_i+1} dy \geq 0.
\]

Now, set

\[
v_i(y) := \delta_i^{\frac{1}{p_i}} u_i(\delta_i y) \text{ for } y \in B^{+}_r(0).
\]

After a change of variables we obtain

\[
P(u_i, r) \geq -\delta_i^{n-2} \int_{B^{+}_{r/\delta_i}} \left( y^2 \partial_n v_i + \frac{n-2}{2} v_i \right) [(L_{\tilde{g}} - \Delta) v_i] dy
+ \frac{n-2}{2} \delta_i^{n-2} \int_{\partial B^{+}_{r/\delta_i}} \left( y^2 \partial_n v_i + \frac{n-2}{2} v_i \right) h_{g_i}(\delta_i y) v_i d\gamma.
\]

Since \( h_{g_i}(\delta_i y) = O(\delta_i^4 |y|^4) \) and \( \lim_{\delta_i \to 0} \delta_i^{n-2} = 1 \), we have

\[
\delta_i^{n-2} \int_{\partial B^{+}_{r/\delta_i}} \left( y^2 \partial_n v_i + \frac{n-2}{2} v_i \right) h_{g_i}(\delta_i y) v_i d\gamma
= O(\delta_i^4) \int_{\partial B^{+}_{r/\delta_i}} (1 + |y|)^{4-2n} |y|^{4} dy = O(\delta_i^4) \text{ for } n \geq 8.
\]

So

\[
P(u_i, r) \geq - \int_{B^{+}_{r/\delta_i}} \left( y^2 \partial_n v_i + \frac{n-2}{2} v_i \right) [(L_{\tilde{g}} - \Delta) v_i] dy + O(\delta_i^4).
\]

Now define, in analogy with Proposition 12

\[
w_i(y) := v_i(y) - U(y) - \delta_i^2 \gamma_8(y).
\]
Lemma 15. We have

\[ P(u, r) \geq R(U, U) + R(U, \delta^2 \gamma_{x_i}) + R(\delta^2 \gamma_{x_i}, U) + R(w_i, U) + R(U, w_i) \]

\[ + R(w_i, w_i) + R(\delta^2 \gamma_{x_i} + \delta^2 \gamma_{x_i}) + R(w_i, \delta^2 \gamma_{x_i}) + R(\delta^2 \gamma_{x_i} + w_i) + O(\delta^4) \]

and, by the following Lemma 15, Lemma 16, and Lemma 17, we conclude

\[ P(u, r) \geq R(U, U) + R(U, \delta^2 \gamma_{x_i}) + R(\delta^2 \gamma_{x_i}, U) + o(\delta^4) \]

\[ = \delta^4 \frac{(n - 2)\omega_{n - 2} I_n^2}{(n - 1)(n - 3)(n - 5)(n - 6)} \left[ \frac{(n - 2)}{6} |W(x_i)|^2 + \frac{4(n - 8)}{(n - 4)} R^2_{ninj} + o(\delta^4) \right] \]

\[ - 2\delta^4 \int_{\mathbb{R}^2} \gamma_{x_i} \Delta \gamma_{x_i} dy + o(\delta^4) \]

and we prove the result. \( \Box \)

In order to simplify the notation, in the following lemmas we use \( \delta = \delta_i \) and \( q = x_i \).

Lemma 15. We have

\[ R(U, U) = \delta^4 \frac{(n - 2)\omega_{n - 2} I_n^2}{(n - 1)(n - 3)(n - 5)(n - 6)} \left[ \frac{(n - 2)}{6} |W(q)|^2 + \frac{4(n - 8)}{(n - 4)} R^2_{ninj} + o(\delta^4) \right] \]

Proof. Recalling that \( U \) is the standard bubble and equation 4.3, we obtain

\[ R(U, U) = \frac{(n - 2)^2}{2} \int_{B_{R^+}} \frac{|y|^2 - 1}{(1 + y_n)^2 + |y|^2} n y_i y_j (g^{ij}(\delta y) - \delta^{ij}) dy \]

\[ = \frac{(n - 2)^2}{2} \int_{B_{R^+}} \frac{|y|^2 - 1}{(1 + y_n)^2 + |y|^2} (g^{ij}(\delta y) - 1) dy \]

\[ = \frac{(n - 2)^2}{2} \int_{B_{R^+}} \frac{|y|^2 - 1}{(1 + y_n)^2 + |y|^2} \delta y_i g^{ij}(\delta y) y_j dy \]

\[ - \frac{(n - 2)^2}{8(n - 1)} \int_{B_{R^+}} \frac{|y|^2 - 1}{(1 + y_n)^2 + |y|^2} \delta^2 R_y (\delta y) dy + O(\delta^5) \]

\[ =: A_1 + A_2 + A_3 + A_4 + O(\delta^5). \]

For the sake of simplicity we call \( L_1(y) := \frac{|y|^2 - 1}{(1 + y_n)^2 + |y|^2} \), \( L_2(y) := \frac{|y|^2 - 1}{(1 + y_n)^2 + |y|^2} \), and \( L_3(y) := \frac{|y|^2 - 1}{(1 + y_n)^2 + |y|^2} \) \((n - 1)^n\). By symmetry arguments we have only to consider the fourth order terms in the expansion of \( g^{ij} \). Since \( B_{R^+} \) invades \( \mathbb{R}^{n - 1} \times \mathbb{R}^+ \) as \( \delta \to 0^+ \), and recalling the expansion of \( g^{ij} \) we have

\[ A_1 = \frac{(n - 2)^2}{2} \int_{\mathbb{R}^+} L_1(y) n y_i y_j (g^{ij}(\delta y) - \delta^{ij}) dy + O(\delta^{n - 2}) \]

\[ = \frac{n(n - 2)^2}{2} \delta^4 \int_{\mathbb{R}^+} L_1(y) \left( \frac{1}{20} \tilde{R}_{ikjl,mn} + \frac{1}{15} \tilde{R}_{ikjl} R_{jmn} \right) y_i y_j y_k y_m y_n y_p dy \]

\[ + \frac{n(n - 2)^2}{2} \delta^4 \int_{\mathbb{R}^+} L_1(y) \left( \frac{1}{2} R_{ninj,kl} + \frac{1}{3} \text{Sym}_{ij} \tilde{R}_{ikjl} R_{nimj} \right) y_i y_j y_k y_n y_p^2 dy \]

\[ + \frac{n(n - 2)^2}{2} \delta^4 \int_{\mathbb{R}^+} L_1(y) \left( \frac{1}{12} R_{ninj,nn} + 8 R_{ninj,nn} \right) y_i y_j y_n y_p^4 dy + O(\delta^4). \]
By the symmetries of the curvature tensor (see [11] Proof of Lemma 8, pages 15-16), we have that
\[
\int_{R^n_+} L_1(y) \left( \frac{1}{20} \hat{R}_{ijkl,mp} + \frac{1}{15} \hat{R}_{iksl} \hat{R}_{jmsp} \right) y_i y_j y_k y_l y_m y_p dy = 0
\]
and
\[
\delta^4 \int_{R^n_+} L_1(y) \frac{1}{3} \text{Sym}_{ij} (\hat{R}_{iksl} R_{nsmj}) y_i y_j y_k y_l y_n^2 dy = 0,
\]
so
\[
A_1 = \frac{n(n-2)^2}{4} \delta^4 \int_{R^n_+} L_1(y) R_{nini,ii} y_i y_j y_k y_l y_n^2 dy \]
\[
+ \frac{n(n-2)^2}{24} \delta^4 \int_{R^n_+} L_1(y) (R_{nini,nn} + 8 R_{ninsi} R_{nsmj}) y_i y_j y_l y_n^2 dy + O(\delta^5).
\]
We point out that, in the above integral only terms involving even powers of $y_s$ survive. Moreover, by direct computation we have that
\[
3 \int_{R^n_+} L_1(y) y_i y_j y_l y_n^2 dy = \int_{R^n_+} L_1(y) y_i y_l^3 y_n^2 dy = \frac{3}{n^2 - 1} \int_{R^n_+} L_1(y) y_i^4 y_n^2 dy.
\]
So, for the first term we have
\[
\int_{R^n_+} L_1(y) R_{nini,ikl} y_i y_j y_k y_l y_n^2 dy = \sum_i R_{nini,in} \int_{R^n_+} L_1(y) y_i^4 y_n^2 dy \]
\[
+ \left( \sum_{i \neq k} R_{nini,kk} + \sum_{i \neq j} R_{nini,ij} + \sum_{i \neq j} R_{nini,ji} \right) \int_{R^n_+} L_1(y) y_i^2 y_j^2 y_n^2 dy \]
\[
= \left( 3 \sum_{i} R_{nini,ii} + \sum_{i \neq k} R_{nini,kk} + \sum_{i \neq j} R_{nini,ij} + \sum_{i \neq j} R_{nini,ji} \right) \int_{R^n_+} L_1(y) y_i^2 y_n^2 dy \]
\[
= \left( \sum_{i,k} R_{nini, kk} + \sum_{i,j} R_{nini, ij} + \sum_{i,j} R_{nini, ji} \right) \frac{1}{n^2 - 1} \int_{R^n_+} L_1(y) y_i^4 y_n^2 dy
\]
By (2.8), $R_{nn, kk} = 0$ for all $k = 1, \ldots, n - 1$, and since the curvature tensor is at least $C^2$, we have finally
\[
\delta^4 \frac{n(n-2)^2}{4} \int_{R^n_+} L_1(y) R_{nini,ikl} y_i y_j y_k y_l y_n^2 dy = \delta^4 \frac{n(n-2)^2}{2(n^2 - 1)} R_{nini,ji} \int_{R^n_+} L_1(y) y_i^4 y_n^2 dy.
\]
On the other hand, by (2.9) we have
\[
\int_{R^n_+} L_1(y) (R_{nini,nn} + 8 R_{ninsi} R_{nsmj}) y_i y_j y_l y_n^2 dy \]
\[
= (R_{nn, nn} + 8 R_{ninsi} R_{nsmj}) \int_{R^n_+} L_1(y) y_i^2 y_l^3 y_n^2 dy = 6 R_{nn, nn}^2 \int_{R^n_+} L_1(y) y_i^2 y_n^2 dy \]
\[
+ \frac{6}{n-1} R_{ninsi}^2 \int_{R^n_+} L_1(y) y_i^4 y_n^2 dy.
\]
so, finally

\begin{equation}
A_1 = \delta^4 \frac{n(n-2)^2}{2(n^2-1)} R_{\text{minj},ij} \int_{\mathbb{R}^n_+} L_1(y) |\bar{y}|^4 y_n^2 dy \\
+ \delta^4 \frac{n(n-2)^2}{4(n-1)} R_{\text{minj}}^2 \int_{\mathbb{R}^n_+} L_1(y) |\bar{y}|^2 y_n^4 dy + O(\delta^5).
\end{equation}

Similarly, for $A_2$ there are only the fourth order terms surviving, and again we proceed by symmetry, using again (2.8) and (2.9), obtaining

\begin{align*}
A_2 &= -\delta^4 \frac{(n-2)^2}{2} \int_{\mathbb{R}^n_+} L_2(y) \left[ \frac{1}{20} \bar{R}_{\text{ijkl},mp} + \frac{1}{15} \bar{R}_{\text{ijkl}} \bar{R}_{\text{imsp}} \right] y_k y_l y_m y_p dy \\
&\quad - \delta^4 \frac{(n-2)^2}{2} \int_{\mathbb{R}^n_+} L_2(y) \left[ \frac{1}{2} R_{\text{minj},kl} + \frac{1}{3} \text{Sym}_{ij} \left( \bar{R}_{\text{ijkl}} R_{\text{mnji}} \right) \right] y_n^2 y_k y_l dy \\
&\quad - \delta^4 \frac{(n-2)^2}{2} \int_{\mathbb{R}^n_+} L_2(y) \left[ \frac{1}{3} R_{\text{minj},nk} y_k^3 y_k + \frac{1}{12} \left( R_{\text{minj},nn} + 8 R_{\text{mnns}} R_{\text{nnmn}} \right) y_n^4 \right] dy + O(\delta^5) \\
&= \delta^4 \left( \frac{(n-2)^2}{2} \bar{R}_{\text{ijkl},mp} + \frac{(n-2)^2}{2} \bar{R}_{\text{ijkl}} \bar{R}_{\text{imsp}} \int_{\mathbb{R}^n_+} L_2(y) y_k y_l y_m y_p dy \\
&\quad - \delta^4 \frac{(n-2)^2}{4} R_{\text{minj}}^2 \int_{\mathbb{R}^n_+} L_2(y) y_n^4 dy + O(\delta^5) \right).
\end{align*}

and, similarly,

\begin{align*}
A_3 &= -\delta^4 \frac{(n-2)^2}{2} \int_{\mathbb{R}^n_+} L_2(y) \left[ \frac{1}{20} \bar{R}_{\text{ijkl},mp} + \frac{1}{15} \bar{R}_{\text{ijkl}} \bar{R}_{\text{jmsp}} \right] \partial_i (y_k y_l y_m y_p) y_j dy \\
&\quad - \delta^4 \frac{(n-2)^2}{2} \int_{\mathbb{R}^n_+} L_2(y) \left[ \frac{1}{2} R_{\text{minj},kl} + \frac{1}{3} \text{Sym}_{ij} \left( \bar{R}_{\text{ijkl}} R_{\text{mnji}} \right) \right] y_n^2 \partial_i (y_k y_l) y_j dy \\
&\quad - \delta^4 \frac{(n-2)^2}{2} \int_{\mathbb{R}^n_+} L_2(y) \left[ \frac{1}{3} R_{\text{minj},nk} y_k^3 \partial_i (y_k) y_j dy + O(\delta^5) \right] \\
&= \delta^4 \left( \frac{(n-2)^2}{2} \bar{R}_{\text{ijkl},mp} + \frac{(n-2)^2}{2} \bar{R}_{\text{ijkl}} \bar{R}_{\text{imsp}} \int_{\mathbb{R}^n_+} L_2(y) y_k y_l y_m y_p dy \\
&\quad - \delta^4 \frac{(n-2)^2}{4} R_{\text{minj}}^2 \int_{\mathbb{R}^n_+} L_2(y) y_n^4 dy + O(\delta^5) \right).
\end{align*}
and, up to relabelling, we have

\begin{equation}
A_2 + A_3 = -\frac{\delta^4}{4} \left( \frac{n-2}{2} \right)^2 R_{\text{min}}^2 \int_{\mathbb{R}_+^n} L_2(y) y_n^4 \, dy
\end{equation}

\begin{equation}
- \frac{\delta^4}{4} \left( \frac{n-2}{2} \right)^2 R_{\text{min},i,j} \int_{\mathbb{R}_+^n} L_2(y) y_n^2 y_j^2 \, dy + O(\delta^5)
\end{equation}

\begin{equation}
= - \frac{\delta^4}{4} \left( \frac{n-2}{2} \right)^2 R_{\text{min}}^2 \int_{\mathbb{R}_+^n} L_2(y) y_n^4 \, dy
\end{equation}

\begin{equation}
- \frac{\delta^4}{4} \left( \frac{n-2}{2(n-1)} \right)^2 R_{\text{min},i,j} \int_{\mathbb{R}_+^n} L_2(y) \bar{y}_j^2 y_n^2 \, dy + O(\delta^5).
\end{equation}

Finally, by (2.6) we have

\begin{equation}
A_4 = \frac{\delta^4}{96(n-1)^2} |W(q)|^2 \int_{\mathbb{R}_+^n} L_3(y) |\bar{y}|^2 \, dy
\end{equation}

\begin{equation}
- \frac{\delta^4}{16(n-1)} \partial_2^2 R_{\bar{y}_n}(q) \int_{\mathbb{R}_+^n} L_3(y) y_n^2 \, dy
\end{equation}

and by (2.6) we conclude

\begin{equation}
A_4 = \frac{\delta^4}{96(n-1)^2} |W(q)|^2 \int_{\mathbb{R}_+^n} L_3(y) |\bar{y}|^2 \, dy
\end{equation}

\begin{equation}
+ \frac{\delta^4}{8(n-1)} R_{\text{min}}^2 \int_{\mathbb{R}_+^n} L_3(y) y_n^2 \, dy
\end{equation}

\begin{equation}
+ \frac{\delta^4}{8(n-1)} R_{\text{min},i,j} \int_{\mathbb{R}_+^n} L_3(y) y_n^2 \, dy.
\end{equation}

We want now collect the similar terms, using the result of Lemma 25 to estimate all integrals. All terms containing $R_{\text{min},i,j}$ in (6.2), (6.3) and (6.4) add up to

\begin{equation}
\delta^4 R_{\text{min},j} \frac{n-2}{2(n-1)}
\end{equation}

\begin{equation}
\left[ \frac{n(n-2)}{(n+1)} \int_{\mathbb{R}_+^n} L_1(y) |\bar{y}|^4 y_n^2 \, dy - (n-2) \int_{\mathbb{R}_+^n} L_2(y) |\bar{y}|^2 y_n^2 \, dy + \frac{\delta^4}{4} \left( \frac{n-2}{2} \right)^2 R_{\text{min}}^2 \int_{\mathbb{R}_+^n} L_3(y) y_n^2 \, dy \right]
\end{equation}

\begin{equation}
= \delta^4 R_{\text{min},j} \frac{(n-2) \omega_{n-2} I_n^2}{2(n-1)}
\end{equation}

\begin{equation}
\times \left[ \frac{12(n-2)}{(n-3)(n-4)(n-5)(n-6)} - \frac{20(n-2)}{(n-3)(n-4)(n-5)(n-6)} + \frac{8(n-2)}{(n-3)(n-4)(n-5)(n-6)} \right] = 0.
\end{equation}
Concerning terms containing $R^2_{ninj}$ we have

\begin{equation}
\frac{(n-2)^2}{4} R^2_{ninj} \left( \frac{n-2}{2} \right)^2 \times \left[ \frac{n}{(n-1)} \int_{\mathbb{R}^n_+} L_1(y) \| \gamma \|^2 y^2 dy - \int_{\mathbb{R}^n_+} L_2(y) y^2 dy + \frac{1}{2(n-1)} \int_{\mathbb{R}^n_+} L_3(y) y^2 dy \right]
\end{equation}

\begin{align*}
&= \frac{4(n-2)(n-8) \omega_{n-2} I_n^2}{(n-1)(n-3)(n-4)(n-5)(n-6)} \\
&= \frac{(n-2)^2 \omega_{n-2} I_n^2}{(n-1)(n-3)(n-4)(n-5)(n-6)}
\end{align*}

In light of (6.5) and (6.6), we can conclude, by (6.2), (6.3) and (6.4),

\begin{align*}
R(U, U) = & \frac{(n-2)^2}{96(n-1)^2} W(q)^2 \int_{\mathbb{R}^n_+} L_3(y) \| \gamma \|^2 dy \\
&+ \frac{4(n-2)(n-8) \omega_{n-2} I_n^2}{(n-1)(n-3)(n-4)(n-5)(n-6)} \\
&= \delta^4 R^2_{ninj} \frac{4(n-2)(n-8) \omega_{n-2} I_n^2}{(n-1)(n-3)(n-4)(n-5)(n-6)}
\end{align*}

which ends the proof. \quad \square

**Lemma 16.** For $n \geq 8$ we have

\begin{equation}
R(U, \delta^2 \gamma_q) + R(\delta^2 \gamma_q, U) = -2 \delta^4 \int_{\mathbb{R}^n_+} \gamma_q \Delta \gamma_q dy + o(\delta^4).
\end{equation}

**Proof.** In light of (2.45) and (2.43), we have that

\begin{align*}
R(U, \delta^2 \gamma_q) + R(\delta^2 \gamma_q, U) &= -\delta^4 \int_{\mathbb{R}^n_+} \left( y \delta U + \frac{n-2}{2} U \right) \left[ \frac{1}{3} \tilde{R}_{ikjl} y_k y_l + R_{ninj} y^2 \right] \partial_i \partial_j \gamma_q dy \\
&\quad - \delta^4 \int_{\mathbb{R}^n_+} \left( y \delta \gamma q + \frac{n-2}{2} \gamma \right) \left[ \frac{1}{3} \tilde{R}_{ikjl} y_k y_l + R_{ninj} y^2 \right] \partial_i \partial_j U dy + O(\delta^{n-2}) \\
&= -\delta^4 \int_{\mathbb{R}^n_+} \left( y \delta U + \frac{n-2}{2} U \right) \left[ \frac{1}{3} \tilde{R}_{ikjl} y_k y_l + R_{ninj} y^2 \right] \partial_i \partial_j \gamma_q dy \\
&\quad - \delta^4 \int_{\mathbb{R}^n_+} y \delta \gamma q \left[ \frac{1}{3} \tilde{R}_{ikjl} y_k y_l + R_{ninj} y^2 \right] \partial_i \partial_j U dy \\
&\quad - \delta^4 \int_{\mathbb{R}^n_+} \frac{n-2}{2} \gamma \left[ \frac{1}{3} \tilde{R}_{ikjl} y_k y_l + R_{ninj} y^2 \right] \partial_i \partial_j U dy + o(\delta^4) \\
&= : \delta^4 (A_1 + A_2 + A_3) + o(\delta^4).
\end{align*}

Immediately we have, by the choice of $\gamma_q$ (see (2.43)), that

\begin{equation}
A_3 = \frac{n-2}{2} \int_{\mathbb{R}^n_+} \gamma_q \Delta \gamma_q.
\end{equation}
We notice that, given any two functions \( f, g \), we have, by (2.8) and by the symmetries of the curvature tensor, that

\[
\int_{\mathbb{R}_+^n} f \partial_i \left[ \frac{1}{3} \bar{R}_{ikjl} y_k y_l + R_{nijn} y_n^2 \right] \partial_j g dy = 0.
\]

So, integrating by parts we have

\[
A_2 = \int_{\mathbb{R}_+^n} n \gamma_q \left[ \frac{1}{3} \bar{R}_{ikjl} y_k y_l + R_{nijn} y_n^2 \right] \partial_i \partial_j U dy \\
+ \int_{\partial \mathbb{R}_+^n} y_b \gamma_q \partial_b \left[ \frac{1}{3} \bar{R}_{ikjl} y_k y_l + R_{nijn} y_n^2 \right] \partial_i \partial_j U dy \\
+ \int_{\partial \mathbb{R}_+^n} y_b \gamma_q \left[ \frac{1}{3} \bar{R}_{ikjl} y_k y_l + R_{nijn} y_n^2 \right] \partial_b \partial_i \partial_j U dy \\
+ \int_{\partial \mathbb{R}_+^n} y_b \gamma_q \left[ \frac{1}{3} \bar{R}_{ikjl} y_k y_l + R_{nijn} y_n^2 \right] \partial_b \partial_i \partial_j U dy.
\]

We notice that \( y_b \nu_b = 0 \) on \( \partial \mathbb{R}_+^n \). Moreover, up to relabelling,

\[
\int_{\mathbb{R}_+^n} y_b \gamma_q \partial_b \left[ \frac{1}{3} \bar{R}_{ikjl} y_k y_l + R_{nijn} y_n^2 \right] \partial_i \partial_j U dy \\
= \int_{\mathbb{R}_+^n} y_s \gamma_q \partial_s \left[ \frac{1}{3} \bar{R}_{ikjl} y_k y_l + R_{nijn} y_n^2 \right] \partial_i \partial_j U dy + \int_{\mathbb{R}_+^n} y_n \gamma_q \partial_n \left[ R_{nijn} y_n^2 \right] \partial_i \partial_j U dy \\
= 2 \int_{\mathbb{R}_+^n} \gamma_q \left[ \frac{1}{3} \bar{R}_{ikjl} y_k y_l + R_{nijn} y_n^2 \right] \partial_i \partial_j U dy = -2 \int_{\mathbb{R}_+^n} \gamma_q \Delta \gamma_q.
\]

so

(6.8) \( A_2 = - (n + 2) \int_{\mathbb{R}_+^n} \gamma_q \Delta \gamma_q + \int_{\mathbb{R}_+^n} y_b \gamma_q \left[ \frac{1}{3} \bar{R}_{ikjl} y_k y_l + R_{nijn} y_n^2 \right] \partial_b \partial_i \partial_j U dy. \)
Finally, integrating by parts twice, we have, arguing as before,

\[
A_1 = \int_{\mathbb{R}^n_+} \left( \partial_i (y_b \partial_b U) + y_b \partial_i \partial_b U + \frac{n-2}{2} \partial_i U \right) \left[ \frac{1}{3} \bar{R}_{ikjl} y_k y_l + R_{mijn} y^2_n \right] \partial_j \gamma_q dy \\
+ \int_{\mathbb{R}^n_+} \left( y_b \partial_b U + \frac{n-2}{2} U \right) \partial_i \left[ \frac{1}{3} \bar{R}_{ikjl} y_k y_l + R_{mijn} y^2_n \right] \partial_j \gamma_q dy \\
+ \int_{\partial \mathbb{R}^n_+} \nu_i \left( y_b \partial_b U + \frac{n-2}{2} U \right) \partial_i \left[ \frac{1}{3} \bar{R}_{ikjl} y_k y_l + R_{mijn} y^2_n \right] \partial_j \gamma_q dy \\
= \int_{\mathbb{R}^n_+} \left( \frac{n}{2} \partial_i U + y_b \partial_i \partial_b U \right) \left[ \frac{1}{3} \bar{R}_{ikjl} y_k y_l + R_{mijn} y^2_n \right] \partial_j \gamma_q dy \\
- \int_{\mathbb{R}^n_+} \left( \frac{n}{2} \partial_i U + y_b \partial_i \partial_b U \right) \partial_j \left[ \frac{1}{3} \bar{R}_{ikjl} y_k y_l + R_{mijn} y^2_n \right] \gamma_q dy \\
- \int_{\partial \mathbb{R}^n_+} \nu_i \left( \frac{n}{2} \partial_i U + y_b \partial_i \partial_b U \right) \left[ \frac{1}{3} \bar{R}_{ikjl} y_k y_l + R_{mijn} y^2_n \right] \gamma_q dx \\
= \left( \frac{n}{2} + 1 \right) \int_{\mathbb{R}^n_+} \gamma_q \Delta \gamma_q dy - \int_{\mathbb{R}^n_+} y_b \partial_j \partial_i \partial_b U \left[ \frac{1}{3} \bar{R}_{ikjl} y_k y_l + R_{mijn} y^2_n \right] \gamma_q dy.
\]

Recalling \(6.7\) and \(6.8\) we conclude

\[
A_1 + A_2 + A_3 = -2 \int_{\mathbb{R}^n_+} \gamma_q \Delta \gamma_q
\]

which gives the proof. \(\square\)

**Lemma 17.** For \(n \geq 8\) we have

\[
R(\delta^2 \gamma_q, \delta^2 \gamma_q) = O(\delta^6) \\
R(u_i, u_i) = O(\delta^8) \\
R(U, w_i) + R(w_i, U) = O(\delta^5) \\
R(\delta^2 \gamma_q, w_i) + R(w_i, \delta^2 \gamma_q) = O(\delta^5)
\]

**Proof.** By direct computation, using the decay of the standard bubble \(U\), estimate \((2.14), (2.5)\) and Proposition \(12\). \(\square\)

Here we focus on the Weyl tensor of \(M\), proving a result which is in the spirit of Weyl vanishing conjecture.

**Proposition 18.** Let \(x_i \to x_0\) be an isolated simple blow up point for \(u_i\) solutions of \((5.2)\). Then

1. If \(n = 8\) then \(|\bar{W}(x_0)| = 0\).
2. If \(n > 8\) then \(|W(x_0)| = 0\).
Proof. By Proposition 9 and Proposition 8 and since $M_i = \delta_i^{-1}$, we have,

$$P(u_i, r) := \frac{1}{M_i^{2\lambda_i}} \int_{\partial^+ B_i^+} \left( \frac{n-2}{2} M_i^{\lambda_i} u_i \frac{\partial M_i^{\lambda_i} u_i}{\partial r} - \frac{r}{2} |\nabla M_i^{\lambda_i} u_i|^2 + r \left| \frac{\partial M_i^{\lambda_i} u_i}{\partial r} \right|^2 \right) d\sigma_r$$

$$+ \frac{r(n-2)}{(p_i + 1) M_i^{\lambda_i(p_i+1)}} \int_{\partial^+ B_i^+} f^{-\tau} \left( M_i^{\lambda_i} u_i \right)^{p_i+1} d\sigma_g$$

$$\leq \frac{C}{M_i^{\lambda_i}} \leq C\delta_i^{p_i+1} \leq C\delta_i^{n-2}.$$ 

On the other hand, recalling Proposition 14, we have

$$P(u_i, r) \geq \delta_i^4 \frac{(n-2)\omega_{n-2}I_n^{\lambda_i}}{(n-1)(n-3)(n-5)(n-6)} \left[ \frac{(n-2)}{6} |\bar{W}(x_i)|^2 + \frac{4(n-8)}{(n-4)} R_{nlnj}(x_i) \right] + o(\delta_i^4),$$

so we get $|\bar{W}(x_i)| \leq \delta_i^2$ if $n = 8$, and $\left[ \frac{(n-2)}{6} |\bar{W}(x_i)|^2 + \frac{4(n-8)}{(n-4)} R_{nlnj}(x_i) \right] \leq \delta_i^2$ if $n > 8$. For the case $n > 8$ we recall that when the boundary is umbilic $\bar{W}(q) = 0$ if and only if $\bar{W}(q) = 0$ (see [21] page 1618), and we conclude the proof. 

□

Remark 19. Let $x_i \to x_0$ be an isolated blow up point for $u_i$ solutions of (3.1). We set

$$P'(u, r) := \int_{\partial^+ B_i^+} \left( \frac{n-2}{2} u \frac{\partial u}{\partial r} - \frac{r}{2} |\nabla u|^2 + r \left| \frac{\partial u}{\partial r} \right|^2 \right) d\sigma_r,$$

so

$$P(u_i, r) = P'(u_i, r) + \frac{r(n-2)}{(p_i + 1)} \int_{\partial^+ B_i^+} f_i^{-\tau} u_i^{p_i+1} d\sigma_g$$

and, keeping in mind that for $i$ large $M_i u_i \leq C|y|^{2-n}$ by Proposition 9 and since $f_i^{-\tau} \to 0$ and $p_i \to \frac{n}{n-2}$, we have

$$\left| r \int_{\partial^+ B_i^+} f_i^{-\tau} u_i^{p_i+1} d\sigma_g \right| \leq \frac{C r}{M_i^{p_i+1}} \int_{|y| = r} |y|^{(p_i+1)(2-n)} d\sigma_g$$

$$\leq \frac{C r^{p_i+1}(2-n+1)}{M_i^{p_i+1}} \int_{|y| = r} 1 d\sigma_g \leq C(r)\delta_i^{n-2}$$

for $i$ sufficiently large.

Using Proposition 14 (6.10), and since $n \geq 8$ we get

$$P'(u_i, r) = P(u_i, r) - \frac{r(n-2)}{p_i + 1} \int_{\partial^+ B_i^+} f_i^{-\tau} u_i^{p_i+1} d\sigma_g \geq A\delta_i^4 + o(\delta_i^4)$$

where $A > 0$.

Proposition 20. Let $x_i \to x_0$ be an isolated blow up point for $u_i$ solutions of (3.1). Assume $n = 8$ and $|\bar{W}(x_0)| \neq 0$ or $n > 8$ and $|\bar{W}(x_0)| \neq 0$. Then $x_0$ is isolated simple.
Proof. Set \( w_i(y) = u_i(\psi_i(y)) \) where \( \psi_i \) are, as usual, the Fermi coordinates at \( x_i \) defined in \( B_{\rho}(0) \). By assumption 0 is an isolated blow up point for \( w_i \). By contradiction, suppose that 0 is not isolated simple. Take \( R_i \to \infty \) and define \( r_i := \bar{r} \langle \bar{w}_i \rangle (0) \). Then the function \( r \to r \bar{\delta}_i^\alpha \bar{w}_i(r) \) has exactly one critical point in \((0, r_i)\). By Definition 7 since \( x_0 \) is not an isolated simple blow up point, there exist at least two critical points of the function \( r \to r \bar{\delta}_i^\alpha \bar{w}_i \) in an interval \((0, \bar{\rho}_i)\) with \( \bar{\rho}_i \to 0 \). So, if \( \bar{\rho}_i \) is the second critical point, we have \( 0 < r_i \leq \rho_i < \bar{\rho}_i \). We set
\[
(6.12) \quad v_i(y) = \rho_i^{\frac{\alpha}{n+2}} w_i(\rho_i y) \quad \text{for} \quad y \in B_{\rho_i}^+(0).
\]
By construction we have that 0 is an isolated simple blow up point for \( v_i \). Indeed, by definition of \( r_i \),
\[
(6.13) \quad v_i(0) = \rho_i^{\frac{\alpha}{n+2}} w_i(0) = \left( \frac{\rho_i}{r_i} \right)^{\frac{\alpha}{n+2}} \geq \frac{1}{r_i^{\frac{\alpha}{n+2}}} \to +\infty.
\]
Moreover, the function \( r \to r \bar{\delta}_i^\alpha \bar{w}_i(r) = (\rho_i r)^{\frac{\alpha}{n+2}} \bar{w}_i(\rho_i r) \) has exactly one critical point in \((0, 1)\).

By the first claim of Proposition 9 we have that \( v_i(0)v_i(x) \) is uniformly bounded in the compact sets of \( \mathbb{R}_+^n \setminus \{0\} \). Taking into account that \( u_i \) solves \((6.1)\) and \( v_i \) solves \((6.2)\), we can prove that \( v_i(0)v_i(x) \to G \) in \( C^2_{\text{loc}}(\mathbb{R}_+^n \setminus \{0\}) \), where \( G \) satisfies
\[
\begin{align*}
\Delta G &= 0 \quad \text{in} \quad \mathbb{R}_+^n \setminus \{0\} \\
\partial_n G &= 0 \quad \text{on} \quad \partial \mathbb{R}_+^n \setminus \{0\}.
\end{align*}
\]
It is well known that \( G = a |y|^{2-n} + b(y) \), with \( b \) harmonic on \( \mathbb{R}_+^n \) with Neumann boundary condition. Moreover, by the second claim of Proposition 9 we can show that \( a > 0 \). Since \( G > 0 \), the function \( b \) is non negative at infinity, and by Liouville theorem this implies that \( b \) is a constant function. Moreover, by the equality \( \frac{\partial}{\partial r} r^{\frac{n-2}{n+2}} \bar{w}_i(r) \big|_{r=1} = 0 \), we have \( \frac{\partial}{\partial r} (r^{\frac{n-2}{n+2}} G(r)) \big|_{r=1} = 0 \), that implies \( a = b > 0 \).

At this point, defined \( P' (u, r) \) as in \((6.9)\) and proceeding as in Remark 19 in analogy with \((6.11)\) we have
\[
(6.14) \quad P'(v_i(0)v_i, r) \geq P(v_i(0)v_i, r) - v_i(0)^2 O(\delta_i^{\alpha_1} - 2) \geq v_i(0)^2 \left[ A \delta_i^4 + o(\delta_i^4) \right] > 0
\]
for \( i \) sufficiently large.

On the other hand a direct computation shows that
\[
(6.15) \quad \lim_{i \to \infty} P'(v_i(0)v_i, r) = P'(G, r) < 0
\]
provided \( r \) sufficiently small, which contradicts \((6.14)\). \( \square \)

7. A Splitting Lemma

We start recalling a result which is analogous to \[19\] Proposition 5.1, \[24\] Lemma 3.1, \[12\] Proposition 1.1 and \[11\] Proposition 4.2, which we refer for the proof.

**Proposition 21.** Given \( \beta > 0 \) and \( R > 0 \) there exist two constants \( C_0, C_1 > 0 \) (depending on \( \beta, R \) and \((M, g)\)) such that, if \( u \) is a solution of
\[
(7.1) \quad \begin{cases}
L_g u = 0 & \text{in} \ M \\
B_g u + (n - 2)f^{-r} u^p = 0 & \text{on} \ \partial M
\end{cases}
\]
and \( \max_{\partial M} u > C_0 \), then \( \tau := \frac{p}{n-2} - p < \beta \) and there exist \( q_1, \ldots, q_N \in \partial M \), with \( N = N(u) \geq 1 \) with the following properties: for \( j = 1, \ldots, N \)
1. set \( r_j := Ru(q_j)^{1-p} \), then \( \{B_{r_j} \cap \partial M\}_j \) are a disjoint collection;
2. we have \( |u(q_j)^{-1} u(\psi_j(y)) - U(u(q_j)^{-1})|_{C^2(B_{r_j}^+)} < \beta \) (here \( \psi_j \) are the Fermi coordinates at point \( q_j \)).
Now we use Fermi coordinates \( W \) to prove the result for \( (7.1) \) with \( x \in \partial M \).

Here \( \bar{g} \) is the geodesic distance on \( \partial M \).

This proposition states that \( u \) is well approximated in strong norms by standard bubbles in disjoint balls \( B_{r_1}, \ldots, B_{r_N} \) centered on \( \partial M \). It is not yet the compactness result we need, since we have to consider, when passing to sequence of solutions, interaction between bubbles. The next Proposition rules out possible accumulation of bubbles, that implies that only isolated blow up points may occur to a blowing up sequence of solution.

**Proposition 22.** Assume \( n \geq 8 \). Given \( \beta, R > 0 \), consider \( C_0, C_1 \) as in the previous proposition. Assume \( W(x) \neq 0 \) for any \( x \in \partial M \) if \( n > 8 \), or \( W(x) \neq 0 \) for any \( x \in \partial M \) if \( n = 8 \). Then there exists \( d = d(\beta, R) \) such that, for any \( u \) solution of \( (7.1) \) with \( \max_{\partial M} u > C_0 \), we have

\[
\min_{i \neq j} \frac{d_{\bar{g}}(q_i(u), q_j(u))}{1 \leq i, j \leq N(u)} \geq d,
\]

where \( q_1(u), \ldots, q_N(u) \) and \( N = N(u) \) are given in the previous proposition.

**Proof.** We prove the result for \( N(u) = 2 \). The general case follows easily.

We argue by contradiction: we suppose that there exists a sequence of solutions \( \{u_i\}_i \) of problem \( (7.1) \) such that (after relabelling the indices) we have two sequence of points \( q_1, q_2 \in \partial M \) and a point \( q_0 \in \partial M \) with \( q_1, q_2 \to q_0 \). Define

\[
\sigma_i := d_{\bar{g}}(q_1^i, q_2^i) = \min_{u \neq h} d_{\bar{g}}(q_1^i, q_2^i)
\]

Now we use Fermi coordinates \( \psi_i : B_r^+ \to M \) centered at \( q_1^i \) and we set

\[
v_i(y) = \sigma_i^{-1} u_i(\psi_i(y)), \quad y \in B_{\sigma_i^{-1} R}^+.
\]

For \( k = 1, 2 \) we define \( y_k^i \) as the point in \( B_{\sigma_i^{-1} R}^+ \) such that \( \psi_i(\sigma_i y_k^i) = q_k^i \). Of course we have \( y_1^i = 0 \).

By equation \( (7.2) \) of Proposition \( 21 \) and by definition of \( v_i, q_k^i \), we have that

\[
v_i(y_k^i) = d_{\bar{g}}(q_1^i, q_2^i)v_i(u_i(q_k^i)) \geq C_0 \text{ for } k = 1, 2.
\]

**Step 1.** \( v_i(y_1^i), v_i(y_2^i) \to \infty \).

We proceed by contradiction. We first suppose that \( v_i(y_2^i) \) is bounded while \( v_i(y_1^i) \to \infty \). By equation \( (7.3) \) of Proposition \( 21 \) we have that \( y_1^i = 0 \) is an isolated simple blow up point. Then, by Proposition \( 20 \) is also isolated simple. Since \( v_i(y_2^i) \) is uniformly bounded by assumption, by an Harnack type inequality (\( H \) Prop. 9.3]) we have that \( v_i \) is uniformly bounded in a neighborhood of \( y_2^i \). Then \( y_2^i \) is a regular point, and since \( y_1^i \) is an isolated simple blow up point, by Proposition \( 9 \) Claim 1, we obtain that \( v_i(y_2^i) \to 0 \). This contradicts equation \( (7.4) \). If we switch the role of \( y_1^i \) and \( y_2^i \) the contradiction follows analogously, so we have only to rule out the case in which both \( v_i(y_1^i), v_i(y_2^i) \) remain bounded. In this case we can prove that \( v_i \) converge in \( C^2_{\text{loc}}(\mathbb{R}^n) \) to \( v \), a solution of

\[
\begin{cases}
\Delta v = 0 & \text{in } \mathbb{R}^n_+ \\
\frac{\partial v}{\partial \nu} + (n - 2) f p_0 
\end{cases}
\]

on \( \partial \mathbb{R}^n_+ \)

where \( p_0 = \lim_i p_i \). Then, by Liouville theorem, we get \( v \equiv 0 \), which again contradicts \( (7.4) \), and Step 1 is proved.
Step 2. Conclusion.

By Step 1 we have that both \( y_1^i \) and \( y_2^i \) are isolated blow up points, and thus isolated simple blow up points for \( v_i \). At this point we proceed as in Proposition 20 and we have that \( v_i(0)v_i(x) \to G(y) = a_1|y|^{2-n} + a_2|y|^{2-n} + b(y) \) in \( C_{loc}^2(\mathbb{R}_+^n \setminus \{0, y_2\}) \), where \( y_2 = \lim_{i \to \infty} y_2^i \), \( b(y) \) is an harmonic function on \( \mathbb{R}_+^n \setminus \{0, y_2\} \) with Neumann boundary condition and \( a_1, a_2 > 0 \).

By the maximum principle \( b(y) \geq 0 \), so near 0 we have

\[
v_i(0)v_i(x) = a_1|y|^{2-n} + b(\|y\|)
\]

for some \( b > 0 \). As in Proposition 20 equation (7.5) contradicts the sign condition given by the Pohozaev inequality, since we supposed \( W(x) \neq 0 \) if \( n > 8 \) or \( W(x) \neq 0 \) if \( n = 8 \). This concludes the proof. \( \square \)

Remark 23. Notice that, by the above proposition, there exists \( \bar{N} \) such that \( N(u) \leq \bar{N} < + \infty \) for all \( u \).

8. Proof of the main result

Proof of Theorem 7. By contradiction, suppose that \( x_i \to x_0 \) is a blow up point for \( u_i \), solutions of (2.2). Let \( q_1, \ldots, q_{N(N_u)} \) the sequence of points given by Proposition 18 with \( N(u_i) \leq \bar{N} \) by Remark 24. By Claim 3 of Proposition 21 there exists a sequence of indices \( k_i \in 1, \ldots, N \) such that \( d_{q_j} (x_i, q_{k_i}) \to 0 \). Up to relabeling, we say \( k_i = 1 \) for all \( i \). Then also \( q_1 \to x_0 \) is a blow up point for \( u_i \). By Proposition 22 and Proposition 21 we have that \( q_1 \to x_0 \) is a blow up point for \( u_i \), and by Proposition 20 we have that \( q_1 \to x_0 \) is also isolated simple. Finally by Proposition 18 we deduce that \( W(x_0) = 0 \) if \( n = 8 \) or that \( W(x_0) = 0 \) if \( n > 8 \), which contradicts the assumption of this theorem and proves the result. \( \square \)

9. Appendix

Proof of Lemma 3. We follow the strategy of [1] Prop 5.1. To prove the existence of a solution of (2.13) we have to show that the given term \( \{ \frac{1}{3} \bar{R}_{ijkl}(q)z_kz_l + R_{mnj}(q)t^2 \} \partial_{ij}^2 U \) is \( L^2 \)-orthogonal to the functions \( j_1, \ldots, j_n \). For \( l = 1, \ldots, n - 1 \) we have

\[
\int_{\mathbb{R}_+^n} \left[ \frac{1}{3} \bar{R}_{ijkl}(q)z_kz_l + R_{mnj}(q)t^2 \right] \partial_{ij}^2 U j_b = \int_{\mathbb{R}_+^n} \left[ \frac{1}{3} \bar{R}_{ijkl}(q)z_kz_l + R_{mnj}(q)t^2 \right] \partial_{ij}^2 U \partial_k U dz dt = 0
\]

by symmetry, since the integrand is odd with respect to the \( z \) variables.

For the last term, since when \( i \neq j \) we have

\[
\partial_{ij} U = \frac{n(n - 2)z_i z_j}{(1 + t^2 + |z|^2)^{n+2}}
\]

and since when \( i = j \) we have \( \bar{R}_{ijkl} = 0 \) and, by (2.3), \( R_{mn} = R_{nn} \), we have

\[
\int_{\mathbb{R}_+^n} \left[ \frac{1}{3} \bar{R}_{ijkl}(q)z_kz_l + R_{mnj}(q)t^2 \right] \partial_{ij}^2 U U dz dt = \sum_{i,j,k} \int_{\mathbb{R}_+^n} \left[ \frac{1}{3} \bar{R}_{ijkl}(q)z_kz_l + R_{mnj}(q)t^2 \right] \frac{n(n - 2)z_i z_j}{(1 + t^2 + |z|^2)^{n+2}}
\]
By direct computation we have
\[ \int_{\mathbb{R}^n_+} \frac{1}{3} \tilde{R}_{ijkl}(q) z_k z_l + R_{mnij}(q) t^2 \partial^2_{ij} U dudt \]
\[ = \sum_k \int_{\mathbb{R}^n_+} \left[ \frac{1}{3} \tilde{R}_{kllk}(q) + \frac{1}{3} \tilde{R}_{llkk}(q) \right] \frac{n(n-2)z_k^2 z_l^2}{((1+t)^2 + |z|^2)^n} = 0 \]

since \( \tilde{R}_{ijkl}(q) = -\tilde{R}_{iklj}(q) \). Moreover
\[ \int_{\mathbb{R}^n_+} \frac{1}{3} \tilde{R}_{ijkl}(q) z_k z_l + R_{mnij}(q) t^2 \partial^2_{ij} U y_k \partial y_l z \]
\[ = n(2-n) \sum_{i \neq j, k} \int_{\mathbb{R}^n_+} \left[ \frac{1}{3} \tilde{R}_{ijkl}(q) z_k z_l + R_{mnij}(q) t^2 \right] \frac{z_i z_j (z_k z_l + t(1+t))}{((1+t)^2 + |z|^2)^{n-1}} \]
\[ = n(2-n) \sum_k \int_{\mathbb{R}^n_+} \left[ \frac{1}{3} \tilde{R}_{kllk}(q) + \frac{1}{3} \tilde{R}_{llkk}(q) \right] \frac{z_k^2 z_l^2 (\sum z_i z_j + t(1+t))}{((1+t)^2 + |z|^2)^{n-1}} = 0. \]

Then there exists a solution. Also there exists a unique solution \( \psi_q \) which is \( L^2 \)-orthogonal to \( f_b \) for \( b = 1, \ldots, n \).

To prove the estimates (2.16) and (2.15) we use the inversion \( F : \mathbb{R}^n_+ \to B^n \setminus \{(0, \ldots, 0-1)\} \), where \( B^n \subset \mathbb{R}^n \) is the closed ball centered in \((0, \ldots, 0, -1)\) and radius 1/2. The explicit expression for \( F \) is
\[ F(y_1, \ldots, y_n) = \left( \frac{y_1}{y_1^2 + \cdots + y_n^2} + \frac{1}{y_1^2 + \cdots + y_n^2} + (y_1 + 1)^2 \right) + (0, \ldots, 0, -1). \]

We set
\[ f_q(F(y)) = \left[ \frac{1}{3} \tilde{R}_{ijkl}(q) y_k y_l + R_{mnij}(q) y_n^2 \right] \partial^2_{ij} U(y) U^{-\frac{2n-2}{n}}(y). \]

By direct computation we have \( |f_q(F(y))| \leq C(1 + |y|)^4 \), so we have
\[ |f_q(\xi)| \leq C \left( 1 + \frac{1}{|\xi|} \right)^4 \leq C \left( 1 + |\xi| \right)^4 \]
(9.1)

So it is possible to smoothly extend \( f_q \) to the whole \( B^n \), and it turns out that if \( \gamma_q \) solves (2.13), then \( \gamma_q := (U^{-1} \psi_q) \circ F^{-1} \) solves
\[ \{ \begin{array}{ll} -\Delta \tilde{\gamma} = f_q & \text{on } B^n \\ \frac{\partial \tilde{\gamma}}{\partial \nu} + 2\tilde{\gamma} = 0 & \text{on } \partial B^n \end{array} \]
(9.2)

Then existence and uniqueness of \( \gamma_q \) are standard. To prove the decay estimates, fixed \( w \in B^n \), consider the Green’s function \( G(\xi, w) \) with boundary condition \( (\frac{\partial}{\partial \nu} + 2) G = 0 \). Then by Green’s formula and by (9.2) we have
\[ \tilde{\gamma}_q(\xi) = \int_{B^n} G(\xi, w) \Delta \tilde{\gamma}_q(\xi) + \int_{\partial B^n} \tilde{\gamma}_q \frac{\partial}{\partial \nu} G - G \frac{\partial}{\partial \nu} \tilde{\gamma}_q = -\int_{B^n} G(\xi, w) f_q(\xi) \]
and, in light of (9.1) we have
\[ |\gamma_q(\xi)| \leq C \int_{B^n} |\xi - w|^{2-n} (1 + |\xi|)^{-4} \]
and by (1.14), since \( n \geq 5 \) we get that \( |\gamma_q(\xi)| \leq C \left( 1 + |\xi| \right)^{-2} \), and by the definition of \( \gamma_q \) we deduce
\[ |\gamma_q(y)| \leq C (1 + |y|)^{4-n}. \]

The estimates on the first and the second derivatives of \( \gamma_q \) can be achieved in a similar way.
To prove (2.10) and (2.15) notice that, changing of variables and proceeding as at the beginning of this proof, we have
\[
\int_{B^n} f_q d\xi = \int_{\mathbb{R}^n_+} \left[ \frac{1}{2} \sum_{ijkl} \left( \tilde{R}_{ijkl}(q) y_k y_i + R_{nijkl}(q) y_n y_i \right) \right] \frac{\partial^2 U}{\partial y_j^2} \left[ (y_1^2 + \cdots + y_{n-1}^2 + (y_n + 1)^2) \right] dy = 0.
\]
So we have, using (9.2) and integrating by parts, that
\[
0 = \int_{B^n} f_q = -\int_{\partial B^n} \gamma_q - \int_{\partial B^n} 2\tilde{\gamma}_q
\]
and, changing variables again,
\[
0 = \int_{\partial B^n} 2\tilde{\gamma}_q d\xi_1 \cdots d\xi_{n-1} = \int_{\partial B^n_+} U^{-1}(y) \gamma_q(y) U^{2(n-1)}(y) dy_1 \cdots dy_{n-1} \]
\[
= \int_{\partial B^n_+} U^{2(n-1)}(y) \gamma_q(y) dy_1 \cdots dy_{n-1}.
\]
It is known (see [1]), that on $H^1(B^n)$ it holds
\[
\inf_{f_{\partial B^n}, \phi = 0} \frac{\int_{B^n} |\nabla \phi|^2}{\int_{B^n} |\phi|^2} = 2.
\]
Since, by (9.3), we know that $\int_{\partial B^n} \tilde{\gamma}_q = 0$, we get
\[
2 \int_{\partial B^n} \tilde{\gamma}_q^2 \leq \int_{B^n} |\nabla \tilde{\gamma}_q|^2,
\]
so, integrating by parts
\[
-\int_{B^n} \tilde{\gamma}_q \Delta \tilde{\gamma}_q = \int_{B^n} |\nabla \tilde{\gamma}_q|^2 - 2\int_{\partial B^n} \tilde{\gamma}_q^2 \geq 0.
\]
By the properties of the inversion $F$ (see [1] formula (5.10)) we have also
\[
-\int_{B^n} \gamma_q \Delta \gamma_q = -\int_{\partial B^n_+} \gamma_q \Delta \gamma_q.
\]
For claim (2.17) we refer to [1] Proposition 5.1.

To prove that $\gamma_q \in C^2(\partial M)$, we fix $q_0 \in \partial M$. If $q \in \partial M$ is sufficiently close to $q_0$, in Fermi coordinates we have $q = q(\eta) = \exp_{q_0} \eta$, with $\eta \in \mathbb{R}^{n-1}$. So $\gamma_q = \gamma_{\exp_{q_0} \eta}$ and we define
\[
\Gamma_i = \left. \frac{\partial}{\partial y_i} \gamma_{\exp_{q_0} \eta} \right|_{q_0 = 0}.
\]
We prove the result for $\Gamma_1$, being the other cases completely analogous. By (2.13) we have that $\Gamma_1$ solves
\[
\begin{align*}
-\Delta \Gamma_1 &= \left[ \frac{1}{2} \sum_{ijkl} \left( R_{ijkl}(q) y_k y_i + R_{nijkl}(q) y_n y_i \right) \right] \frac{\partial^2 U}{\partial y_1^2} \left[ (y_1^2 + \cdots + y_{n-1}^2 + (y_n + 1)^2) \right] U \quad \text{on } \mathbb{R}^n_+; \\
&\left. \frac{\partial \Gamma_1}{\partial y_1} + nU^{1/2} \Gamma_1 \right|_{y_1 = 0} = 0 \quad \text{on } \partial \mathbb{R}^n_+.
\end{align*}
\]
and, since $\frac{\partial R_{ijkl}}{\partial y_1}(q) = 0$ (see [21] Prop 3.2 (4)), we can proceed as at the beginning of this proof to show that $\Gamma_1$ exists. Analogously we get the claim for the second derivative. \qed
Remark 24. We collect here some result contained in [I, Lemma 9.4] and in [I, Lemma 9.5]. The proof is by direct computation. For $m > k + 1$

\begin{equation}
\begin{aligned}
&\int_{0}^{\infty} \frac{t^k dt}{(1 + t)^m} = \frac{k!}{(m-1)(m-2) \cdots (m-1-k)} \\
&\int_{0}^{\infty} \frac{dt}{(1 + t)^m} = \frac{1}{m-1}
\end{aligned}
\tag{9.4}
\end{equation}

Moreover, set, for $\alpha, m \in \mathbb{N}$,

\[ I_{m}^{\alpha} := \int_{0}^{\infty} \frac{s^\alpha ds}{(1 + s^2)^m} \]

it holds

\begin{equation}
\begin{aligned}
I_{m}^{\alpha} &= \frac{2m}{\alpha + 1} I_{m+1}^{\alpha+2} \quad \text{for } \alpha + 1 < 2m \\
I_{m}^{\alpha} &= \frac{2m - \alpha - 1}{\alpha + 1} I_{m+1}^{\alpha} \quad \text{for } \alpha + 1 < 2m \\
I_{m}^{\alpha} &= \frac{2m - \alpha - 3}{\alpha + 1} I_{m+2}^{\alpha} \quad \text{for } \alpha + 3 < 2m.
\end{aligned}
\tag{9.5}
\end{equation}

Lemma 25. We have

\[ \int_{\mathbb{R}^n_+} L_{1}(y)|\bar{y}|y_{n}^2 dy = \omega_{n-2} \frac{n + 1}{n} \frac{12}{(n - 3)(n - 4)(n - 5)(n - 6)} I_{n}^{n}; \]
\[ \int_{\mathbb{R}^n_+} L_{2}(y)|\bar{y}|y_{n}^4 dy = \omega_{n-2} \frac{n(n - 2)(n - 3)(n - 4)(n - 5)(n - 6)}{20} I_{n}^{n}; \]
\[ \int_{\mathbb{R}^n_+} L_{2}(y)y_{n}^2 dy = \omega_{n-2} \frac{20}{(n - 3)(n - 4)(n - 5)(n - 6)} I_{n}^{n}; \]
\[ \int_{\mathbb{R}^n_+} L_{3}(y)|\bar{y}|y_{n}^2 dy = \omega_{n-2} \frac{16(n - 1)}{(n - 3)(n - 5)(n - 6)} I_{n}^{n}; \]
\[ \int_{\mathbb{R}^n_+} L_{4}(y)y_{n}^2 dy = \omega_{n-2} \frac{32}{(n - 3)(n - 4)(n - 5)(n - 6)} I_{n}^{n}. \]

Proof. The proof can be obtained performing firstly a change in polar coordinates in $\mathbb{R}^{n-1}$, then the change $s = r/(y_{n} + 1)$ and using Remark 24. We recall that $\omega_{n-2}$ is the $n - 1$ dimensional spherical element.

\[ \square \]

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