A Quasilocal Test of the Finiteness of the Universe

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The Cosmological Principle states that the universe is both homogeneous and isotropic. This, alone, is not enough to specify the global geometry of the spacetime. If we were able to measure both the Hubble constant and the energy density we could determine whether the universe is open or closed. Unfortunately, while some agreement exists on the value of the Hubble constant, the question of the energy density seems quite intractable. This Letter describes a possible way of avoiding this difficulty and shows that if one could measure the rate at which light-rays emerging from a surface expand, one might well be able to deduce whether the universe is closed.

There is a general belief that the universe is, on average, both isotropic and homogeneous. This means that at any instant of time we need only three parameters to describe the large-scale properties of the system. These are the ‘radius’, a, of the universe, the Hubble constant, $H$, and the average density, $\rho_0$, of the matter-content of the universe. These are not independent, but are related by

$$\frac{k}{a^2} + H^2 = \frac{8\pi \rho_0}{3}. \quad (1)$$

$H, a$ and $\rho_0$ are time-dependent constants; the equation of state of the matter determines their rate-of-change. The constant $k$ in eqn.(1) does not change with time, rather it
stays fixed at either +1, 0 or -1, depending on whether the universe is closed, flat or open (hyperbolic). Deciding which of these values is correct is probably the single most important question in cosmology today. Obviously, if one knew \( H \) and \( \rho_0 \), one could immediately calculate \( k \) and \( a \).

There is agreement that one can determine (at least approximately) the value of the Hubble constant, \( H \). The major difficulty is in finding the value of \( \rho_0 \) and it has been suggested that this quantity may be dominated by material which is hard to detect, i.e., WIMPs, dust, massive neutrinos, tiny black holes, . . . . It may well be that the energy density in our local neighbourhood differs significantly from the average taken over the whole universe. In this Letter we wish to suggest a new approach to determining the global structure of the universe which does not depend on evaluating \( \rho_0 \).

Consider any two-dimensional spatial surface \( S \) in a spacetime. Now consider light-rays which emerge perpendicular to this surface. The outgoing light-rays will (in general) diverge and one can compute the expansion \( \theta \) of these light-rays by measuring the fractional change of the area of a little element of \( S \) as it is dragged along these light rays. This quantity plays a key role in one of the singularity theorems of general relativity; if \( \theta < 0 \) on \( S \), then \( S \) is called an (outer) trapped surface and it indicates the presence of a future singularity [1, 2]. We have been involved in an ongoing investigation to discover when such trapped surfaces would form in a cosmological context. As part of this analysis, we have discovered a very simple relationship which may allow us to use the value of this expansion to determine whether the universe is closed.

The expansion in a null direction can be regarded as a combination of the expansion in a time direction and one in a spatial direction. If the spatial 3-slice is closed the area of any ‘expanding’ 2-surface will eventually start diminishing. We therefore expect that the expansion \( \theta \) should contain information not only about the local geometry but also about the global topology.

We wish to consider a spherically symmetric universe (which we assume to be isotropic but not necessarily homogeneous) and we choose a time-slice which is the “Hubble-time” slice through this universe, i.e., that the Hubble ‘constant’ is in fact constant. We assume that this slice respects the spherical symmetry (this is probably automatic). We further assume that the matter-field is instantaneously at rest. Finally, we assume that the matter density fluctuates (not necessarily by small amounts) about some average value, \( \rho_0 \), so that on some large scale the slice looks like a standard slice through one of the standard
Friedmann universes. Now consider any spherically symmetric surface $S$. Consider the expansion $\theta|_S$ of the outgoing null-rays from $S$. We have derived the following inequality

$$\theta|_S > \frac{4\pi L}{A} + 2H - \frac{3kV}{Aa^2} + \frac{\Delta M}{A},$$

(2)

where $L$ is the proper radius of $S$, $A$ is the area of $S$, $V$ is the volume inside $S$ and

$$\Delta M = \int_V (\rho - \rho_0) dv$$

(3)

is the integral of the excess energy-density inside $S$, where $\rho$ is the true (nonconstant) energy density. Inequality (2) has been shown to hold for each of the three values of $k$, the three choices of the global topology.

Let us now choose $S$ large enough that the fluctuations in $\rho$ average out. In this case $\Delta M$ vanishes and inequality (2) simplifies to

$$\theta|_S > \frac{4\pi L}{A} + 2H - \frac{3kV}{Aa^2}.$$  

(4)

Say we find a large spherical surface $S$ which satisfies

$$\theta|_S < \frac{4\pi L}{A} + 2H.$$  

(5)

Inequality (5) is only compatible with (4) when $k = +1$. Hence we can deduce that the universe is closed.

The important thing to notice is that both $\rho_0$ and $a$ play no role in (5), so, as promised, we have found an inequality which may be used to determine whether the universe is open or closed, without ever measuring either the local or global energy density. Further, inequality (5) is not vacuous, surfaces satisfying this condition can be found in any closed spherical cosmology.

It is clear that this inequality can only be worked one way, we cannot use it to find a condition that guarantees that the universe is flat or open. This is in keeping with the observation, first made by Einstein [6], that, while it is possible to demonstrate conclusively that the universe is closed, it is essentially impossible to prove the converse.

We stress that that the key inequality (2) is not in any sense a perturbative result, it holds true even for large deviations of $\rho$ from the average. Further, while we used spherical symmetry to derive our results, it is clear that the inequalities are stable under non-spherical perturbations. The key assumption we make is that one can determine a
cosmological scale, a measure of the spatial extent of the fluctuations of the matter, because we need to find a large enough surface so that the fluctuations average out. We claim that this is not the same as determining the actual matter distribution; all we need to assume is that the dark matter drags the visible matter with it.

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References.

[1] R. Penrose, *Phys. Rev. Lett.* **14**, 57 (1965); *Techniques of Differential Topology in Relativity* (Soc. Ind. Appl. Math. 1972).

[2] S. W. Hawking, G.F.R. Ellis, *The large scale structure of space-time* (Cambridge University Press, Cambridge 1973).

[3] U. Brauer, E. Malec, *Phys. Rev.* **D45**, R1836 (1992).

[4] E. Malec, N. ´O Murchadha, submitted to *Phys. Rev*.

[5] U. Brauer, E. Malec, N. ´O Murchadha, to be published.

[6] A. Einstein *The Meaning of Relativity*, fifth edition (Princeton University Press, Princeton, 1955).