Explicit solutions of the multi–loop integral recurrence relations and its application *

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Abstract

The approach to the constructing explicit solutions of the recurrence relations for multi–loop integrals are suggested. The resulting formulas demonstrate a high efficiency, at least for 3–loop vacuum integrals case. They also produce a new type of recurrence relations over the space–time dimension.

1 Vacuum case

Recently [1,2] a new approach to implement recurrence relations [3] for the Feynman integrals was proposed. In this work we extend the general formulas for the solutions of the recurrence relations to the multi–loop case. Let us consider first vacuum $L$-loop integrals with $N = L(L+1)/2$ denominators (so that one can express through them any scalar product of loop momenta) of arbitrary degrees:

$$B(\underline{n}, D) = m^{2\Sigma n_i - L D} \int \ldots \int \frac{d^D p_1 \ldots d^D p_L}{D_1^{n_1} \ldots D_N^{n_N}},$$

$$D_a = \sum_{i \geq j} A_{ij}^{(a)} p_i \cdot p_j - \mu_a m^2, \quad p_k \cdot p_l = \sum_{a=1}^{N} (A^{-1})_{(kl)}^{a}(D_a + \mu_a m^2).$$

The recurrence relations that result from integration by parts, by letting $(\partial/\partial p_i) \cdot p_k$ act on the integrand [3], are:

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\[
D \delta^i_k B(n, D) = 2 \sum_{a,d=1}^{N} \sum_{l=1}^{L} A_d^{(i)} n_d I_d^+(A^{-1})_{(kl)}^a (I_a^- + \mu_a) B(n, D),
\]

where \( I_c^B(\ldots, n_c, \ldots) = B(\ldots, n_c \pm 1, \ldots) \), in particular \( I_c^B n_a = n_a \pm \delta^c_a \).

Using the relations

\[
[n_d I_d^+, I_a^-] = \delta_a^d, \quad \sum_{a=1}^{N} A_a^{(i)} (A^{-1})_{(kl)}^a = \delta^{(i)}_{(k\delta^l)},
\]

they can be represented as

\[
\frac{D - L - 1}{2} \delta_k^i B(n, D) = \sum_{a,d=1}^{N} \sum_{l=1}^{L} (A^{-1})_{(kl)}^a (I_a^- + \mu_a) A_d^{(i)} n_d I_d^+ B(n, D). \quad (3)
\]

The common way of using these relations is step–by–step reexpression of the integral (1) with some values of \( n_i \) through a set of integrals with shifted values of \( n_i \), with the final goal to reduce this set to a linear combination of several ”master” integrals \( N_k(D) \) with some ” coefficient functions” \( F_k(n, D) \):

\[
B(n, D) = \sum_k F_k(n, D) N_k(D).
\]

Nevertheless, to find proper combinations of these relations and a proper sequence of its use is the matter of art even for the tree–loop integrals with one mass \([4]\). Then, even in cases when such procedures were constructed, they lead to very time and memory consuming calculation because of large reproduction rate at every recursion step. Instead, let us construct the \( F_k(n, D) \) directly as solutions of the given recurrence relations. Note, that if we find any set of the solutions, we could construct \( F_k(n, D) \) as their linear combinations. Let us try the solution of (3) in the following form:

\[
f(k) = \frac{1}{(2\pi)^N} \oint \cdots \oint dx_1 \cdots dx_N g(x_a)
\]

where integral symbols denote \( N \) subsequent complex integrations with contours which will be described later. Acting by some operator \( O_i(I_a^-, n_a I_a^+) \) (all decreasing operators should be placed to the left) on (4) and performing the integration by parts one gets (s.t. are surface terms):

\[
\]

2
\[
O_i(\Gamma_a^-, n_a \Gamma^{a+}) f^k(n) = \frac{1}{(2\pi)^N} \oint \cdots \oint \frac{dx_1 \cdots dx_N}{x_1^{n_1} \cdots x_N^{n_N}} O_i(x_a, \partial_a) g(x_a) + \text{s. t.).}
\]

So, if we choose the \( g(x_a) \) as the solution of \( O_i(x_a, \partial_a) g(x_a) = 0 \) and cancel the surface terms by proper choosing of integration contours (for example, closed or ended in the zero points) we find that (4) is a solution of relations \( O_i(\Gamma_a^-, n_a \Gamma^{a+}) f^k(n) = 0 \), and different choices of contours correspond to different solutions.

The differential equations for (3) have the solution \( g(x_a) = P(x_a + \mu_a)^{(D-L-1)/2} \), where

\[
P(x_a) = \det(\sum_{a=1}^N (A^{-1})_{kl} x_a)
\]
is the polynomial in \( x_a \) of degree \( L \), so we get the desirable solutions of (3):

\[
f^k(n, D) = \frac{1}{(2\pi)^N} \oint \cdots \oint \frac{dx_1 \cdots dx_N}{x_1^{n_1} \cdots x_N^{n_N}} \det((A^{-1})_{kl} (x_a + \mu_a))^{D-L-1/2}.
\]

Finally, let us derive from (3) the recurrence relations with D-shifts. Note that if \( f^k(n_i, D) \) is a solution of (3), then by direct substitution to (3) one can check that \( P(\Gamma_a^- + \mu_a) f^k(n_i, D - 2) \) also is a solution. Hence, if \( f^k(n_i, D) \) is a complete set of solutions, then

\[
f^k(n_i, D) = \sum_n S_n^{k}(D) P(\Gamma^- + \mu_i) f^n(n_i, D - 2),
\]

where the coefficients of mixing matrix \( S \) is numbers, that is do not act on \( n_i \). For the solutions (5) the \( S \) is the unit matrix (the increasing of \( D \) by 2 leads to appearing of factor \( P(x_a) \) in the integrand of (5)), but the desire to come to some specific set of master integrals may lead to nontrivial mixing. These relations look different from recently proposed in [5], although further investigations can give some connections with them.

To check the efficiency of this approach we evaluated (using REDUCE) the first 5 moments in the small \( q^2 \) expansion of the 3-loop QED photon vacuum polarization. The 3-loop contribution to the moments are expressed through about \( 10^5 \) three-loop scalar vacuum integrals with four massive and two massless lines. The integral (5) in this case can be solved to finite sums of the Pochhammer’s symbols (see [1]). Moreover, it is not necessary to evaluate these integrals separately. Instead, we evaluated a few integrals of (5) type,
but with $P^{D/2-2}$ produced by a long polynomial in $x_i$ (the results see in [1,6], they are in agreement with QCD calculations [7] made by FORM).

The comparison with the recursive approach shows a reasonable progress: the common way used in [6] demands several CPU hours on DEC-Alpha to calculate full $D$ dependence of the first moment, and further calculations became possible only after truncation in $(D/2 - 2)$. In the present approach the full $D$ calculation for the first moment demands a few minutes on PC.

2 Non–vacuum case

Suppose that integrals (1) depend on $R$ external momenta $p_i$ ($L < i \leq L + R$). The number of the denominators are now $N_1 = L(L + 1)/2 + LR$, and the number of additional ("external") invariants are $N_2 = R(R + 1)/2$. Let us expand the integrals in formal seria over "denominator–like" objects $D_a$ of (2) type with $a = N_1 + 1, .., N_1 + N_2$, depending on external momenta only:

$$B(n_{i,(i=1,...,N_1),p_{k,(k=L+1,...,L+R)})} = \int \cdots \int \frac{d^D p_1 \cdots d^D p_L}{D_{n_1}^{n_11} \cdots D_{n_1}^{n_1N_1}} = \sum_{n_i(i>N_1)} m^{-2\Sigma n_i + 2N_2 + LD} b(n_{i,(i=1,...,N_1+N_2)}) \prod_{i=N_1+1}^{N_1+N_2} D_i^{n_i-1}. \quad (6)$$

We define such general expansion in order to write the recurrence relations in compact form, in practice the coefficients $A_{a}^{(ij)}$ and $\mu_a$ may be very simple. The expansion with negative $n_i$ corresponds to the large momenta expansion, with positive ones to the expansion near points $\mu_a m^2$. The $n_i$ can also be noninteger, but with unit shifts.

Acting by $(\partial/\partial p_i) \cdot p_k$, ($i = 1, \ldots, L; k = 1, \ldots, L + R$) on the integrand we get $N_1$ recurrence relations. The additional $N_2$ relations we get acting by $p_k \cdot (\partial/\partial p_i)$, ($i, k = L + 1, \ldots, L + R$) on both sides of (6). These new relations look like the old ones with only exception that they have no terms proportional to space–time dimension $D$. The complete set of recurrence relations is now

$$((D - L - R - 1) \delta_i^k - (D - R - 1) \hat{\delta}_i^k) b(n, D) = 2 \sum_{a,d=1}^{N_1+N_2} \sum_{l=1}^{L+R} (A^{-1})^{a}{(kl)} (I_a^- + \mu_a) A_{d}^{(kl)} n_d X^{d+} b(n, D),$$
where \( \hat{\delta}_{ik} = (\delta_{ik} \text{ if } i, k > L, \text{ else } 0) \). The corresponding differential equations have the solution

\[
g(x_a) = g'(x_a + \mu_a),
\]

where

\[
g'(x) = \det \left( (A^{-1})_{(kl)x_a}^{a} \frac{D-L-R-1}{2} \det_0 \left( (A^{-1})_{(kl)x_a}^{a} \right) \right), \tag{7}
\]

and \( \det_0 \) denotes the minor with \( k, l > L \). So, one can use the representation (4), but the problem of resolving it to explicit formulas demands further investigations.

Finally note that one can formally obtain the formulas (5, 7) by “change of integration variables” from loop momenta to “denominator–like objects” \( D_a \). The weight function for this change is

\[
\int d^D p_1 \cdot d^D p_L \prod_i \delta(D_i/m^2 - x_i) \propto \det((A^{-1})_{(kl)x_a}^{a}(x_a + \mu_a))^{D-L-R-1}. \]

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