BV OPERATORS OF THE GERSTENHABER ALGEBRAS OF HOLOMORPHIC POLYVECTOR FIELDS ON TORIC VARIETIES

YANG DENG AND WEI HONG

Abstract. The vector space of holomorphic polyvector fields on any complex manifold has a natural Gerstenhaber algebra structure. In this paper, we study BV operators of the Gerstenhaber algebras of holomorphic polyvector fields on smooth compact toric varieties. We give a necessary and sufficient condition for the existence of BV operators of the Gerstenhaber algebra of holomorphic polyvector fields on any smooth compact toric variety.

1. INTRODUCTION

Polyvector fields appeared in the recent work of many mathematicians. Barannikov and Kontsevich [1] show that polyvector fields play important roles in deformation theory and mirror symmetry. Polyvector fields also appeared in the work of Bandiera, Stiênon and Xu [2]. In differential geometry, holomorphic polyvector fields appeared in the work of Hitchin [5]. It is well known that the vector space of polyvector fields on a smooth manifold $M$ has a Gerstenhaber algebra structure by the wedge product and the Schouten bracket of polyvector fields. In the same way, the vector space of holomorphic polyvector fields on a complex manifold $X$ becomes a Gerstenhaber algebra. And the Gerstenhaber algebra structure of polyvector fields on the algebraic torus is studied in [7].

This paper is a subsequent work of [6]. For any smooth compact toric variety $X$, the vector space of holomorphic polyvector fields on $X$ has been given an explicit description in [6]. In this paper, we study Batalin-Vilkovisky operators for the Gerstenhaber algebras of holomorphic polyvector fields on smooth compact toric varieties. We give a necessary and sufficient condition for the existence of BV operators. And we also give the explicit forms of the BV operators.

Acknowledgements. Deng would like to thank Prof. Yuping Tu and Prof. Fengwen An for their help during his study in Wuhan University. Hong’s research is partially supported by NSFC grant 12071241 and NSFC grant 12071358.

2. PRELIMINARY

2.1. Gerstenhaber algebra and Batalin-Vilkovisky algebra. Let us recall some classical knowledge of Gerstenhaber algebra and Batalin-Vilkovisky algebra (BV algebra). One may consult [9] and [10].

Let $\mathcal{A} = \bigoplus_{k \in \mathbb{Z}} \mathcal{A}^k$ be a $\mathbb{Z}$-graded vector space over a field $k$. For any element $a \in \mathcal{A}^k$, we denote by $|a| = k$ the degree of $a$. Let $\mathcal{A}[1] = \bigoplus_{k \in \mathbb{Z}} \mathcal{A}[1]^k$, where $\mathcal{A}[1]^k = \mathcal{A}^{k+1}$.

Definition 2.1. A graded vector space $\mathcal{A} = \bigoplus_{k \in \mathbb{Z}} \mathcal{A}^k$ is a Gerstenhaber algebra if one has:

Key words and phrases. toric variety, holomorphic polyvector fields, Gerstenhaber algebra, BV operator.
(1) An associative, graded commutative multiplication
\[ \mathcal{A} \times \mathcal{A} \to \mathcal{A}, \]
\[ \text{i.e., for every } a, b, c \in \mathcal{A}, \text{ we have} \]
\[ a \cdot (b \cdot c) = (a \cdot b) \cdot c, \]
\[ a \cdot b = (-1)^{|a||b|}b \cdot a. \]

(2) A graded Lie algebra bracket \([,] : \mathcal{A}[1] \times \mathcal{A}[1] \to \mathcal{A}[1], \text{i.e., we have } [\mathcal{A}^k, \mathcal{A}^l] \subseteq \mathcal{A}^{k+l-1}, \text{and} \]
\[ \begin{align*}
\text{for every } a, b, c \in \mathcal{A}, \quad &\left[ a, b \right] = (-1)^{|a||b|} [b, a] \text{ (graded anti-symmetry)}, \\
&(-1)^{|a||b||c|} [a, [b, c]] + (-1)^{|b||c|} [b, [a, c]] + (-1)^{|c|} [c, [a, b]] = 0 \text{ (graded Jacobi identity).}
\end{align*} \]

(3) The multiplication and bracket satisfying the Leibniz relation, i.e., for every \(a, b, c \in \mathcal{A},\)
\[ [a, b \cdot c] = [a, b] \cdot c + (-1)^{|a||b|}b \cdot [a, c]. \]

We have the following examples of Gerstenhaber algebras.

**Example 2.2.** Let \(M\) be a smooth real manifold. Let
\[ \mathcal{A}^k = \begin{cases} 
\Gamma(M, \wedge^k TM), & k > 0; \\
C^\infty(M), & k = 0; \\
0, & k < 0.
\end{cases} \]

Then \(\mathcal{A} = \bigoplus_{k \in \mathbb{Z}} \mathcal{A}^k\) is a \(\mathbb{Z}\)-graded vector space over \(\mathbb{R}\). Let the multiplication on \(\mathcal{A}\) be the wedge product on \(\Gamma(M, \wedge^k TM)\). The bracket \([,]\) is given by the Schouten bracket of polyvector fields. Then \((\mathcal{A}, \wedge, [,])\) is a Gerstenhaber algebra.

**Example 2.3.** Let \(X\) be a complex manifold. Let \(\mathcal{A}^k = H^0(X, \wedge^k T_X)\) be the vector space of holomorphic \(k\)-vector fields on \(X\) for \(k > 0\). Let \(\mathcal{A}^0 = H^0(X, \mathcal{O}_X)\) be the vector space of holomorphic functions on \(X\). And let \(\mathcal{A}^k = 0\) for \(k < 0\). Then \(\mathcal{A} = \bigoplus_{k \in \mathbb{Z}} \mathcal{A}^k\) is a \(\mathbb{Z}\)-graded vector space over \(\mathbb{C}\). Let the multiplication on \(\mathcal{A}\) be the wedge product of polyvector fields. The bracket \([,]\) is given by the Schouten bracket of polyvector fields. Then \((\mathcal{A}, \wedge, [,])\) is a Gerstenhaber algebra.

Next we give two equivalent definitions of Batalin-Vilkovisky algebras.

**Definition 2.4.** A graded vector space \(\mathcal{A} = \bigoplus_{k \in \mathbb{Z}} \mathcal{A}^k\) over \(k\) is a Batalin-Vilkovisky algebra (BV algebra) if

(1) \(\mathcal{A}\) is an associative, graded commutative algebra, i.e., there is an associative, graded commutative multiplication on \(\mathcal{A}\).

(2) There is a differential \(k\)-linear operator \(D : \mathcal{A} \to \mathcal{A}\) of order 2 and degree \(-1\). Here \(D\) is a differential operator of order 2 in the sense that, for all \(a, b, c \in \mathcal{A},\)
\[ D(ab)c = D(ab)c + (-1)^{|a|}aD(bc) + (-1)^{|b|}bD(ac) - D(ab)c - (-1)^{|a|}aD(b)c - (-1)^{|a|+|b|}abD(c) \]
\[ (2.1) \]
\[ D^2 = 0. \]

A BV algebra \((\mathcal{A}, \cdot, D)\) is always a Gerstenhaber algebra with the same multiplication, and the bracket is given by
\[ [a, b] = (-1)^{|a|}(D(ab) - D(a)b - (-1)^{|a|}aD(b)) \]
for all $a, b \in A$.

The following is an equivalent definition of BV algebra.

**Definition 2.5.** Given a Gerstenhaber algebra $(A, \cdot, [,])$, if there is exist a degree $-1$ $k$-linear operator $D : A \to A$ such that

\begin{equation}
[a, b] = (-1)^{|a|}(D(ab) - D(a)b - (1)^{|a|}aD(b))
\end{equation}

for any $a, b \in A,$

\begin{equation}
D^2 = 0,
\end{equation}

then the Gerstenhaber algebra $(A, \cdot, [,])$ is said to be exact, and $D$ is called a generating operator.

That a Gerstenhaber algebra $(A, \cdot, [,])$ is exact is equivalent to that $(A, \cdot, D)$ is a BV algebra. And the generating operator $D$ is also called a BV operator.

**2.2. Holomorphic polyvector fields on toric varieties.** Recall that a toric variety $[3]$ is an irreducible variety $X$ such that

1. $T = (\mathbb{C}^*)^n$ is a Zariski open set of $X$,
2. the action of $T = (\mathbb{C}^*)^n$ on itself extends to an action of $T = (\mathbb{C}^*)^n$ on $X$.

One may consult $[8]$ and $[4]$ for more details of toric varieties.

Let $N = \text{Hom}(\mathbb{C}^*, T) \cong \mathbb{Z}^n$ and $M = \text{Hom}(T, \mathbb{C}^*)$. Then $M \cong \text{Hom}(N, \mathbb{Z})$ and $N \cong \text{Hom}(M, \mathbb{Z})$. Each element $m$ in $M$ gives rise to a character $\chi^m \in \text{Hom}(T, \mathbb{C}^*)$, which can be also considered as a rational function on $X$. Let $N_R = N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$, and $M_R = M \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$ be the dual space of $N_R$.

Let $X = X_\Delta$ be a smooth compact toric variety associated with a fan $\Delta$ in $N_R \cong \mathbb{R}^n$. The $T$-action on $X$ induces a $T$-action on $H^0(X, \wedge^k T_X)$, the vector space of holomorphic $k$-vector fields on $X$. Denote by $V^k_i$ the weight space corresponding to the character $I \in M$. We have

\begin{equation}
H^0(X, \wedge^k T_X) = \bigoplus_{I \in M} V^k_i.
\end{equation}

Since $H^0(X, \wedge^k T_X)$ is a finite dimensional vector space, there are only finite elements $I \in M$ such that $V^k_i \neq 0$.

Some technical notations are necessary for us in this paper.

1. Let $N_C = N \otimes_{\mathbb{Z}} \mathbb{C}$. Since $T \cong N \otimes_{\mathbb{Z}} \mathbb{C}^*$, we have Lie$(T) \cong N_C$, where Lie$(T)$ denotes the Lie algebra of $T$. We define a map $\rho : N_C \to \mathfrak{x}(X)$ by

\begin{equation}
\rho : N_C = N \otimes_{\mathbb{Z}} \mathbb{C} \cong \text{Lie}(T) \to \mathfrak{x}(X),
\end{equation}

where $\text{Lie}(T) \to \mathfrak{x}(X)$ is defined by the infinitesimal action of the Lie algebra $\text{Lie}(T)$ on $X$.

And the map $\rho : N_C \to \mathfrak{x}(X)$ is determined by

\[\rho(x)(\chi^m) = \langle x, m \rangle \chi^m\]

for any $x \in N_C$ and $m \in M$. Since the action map $T \times X \to X$ is holomorphic, the images of $\rho$ are holomorphic vector fields on $X$. By the abuse of notation, the induced maps

\begin{equation}
\wedge^k N_C \to H^0(X, \wedge^k T_X) : x_1 \wedge \ldots \wedge x_k \to \rho(x_1) \wedge \ldots \wedge \rho(x_k), \quad \forall x_1, \ldots, x_k \in N_C,
\end{equation}
are also denoted by $\rho$ for $2 \leq k \leq n$. Let $W = \rho(N_C)$, $W^k = \wedge^k W$ and $W^0 = \mathbb{C}$. Then $W^k$ is a subspace of $H^0(X, \wedge^k TX)$. And the map $\wedge^k N_C \xrightarrow{\rho} W^k$ is an isomorphism.

(2) Let $\Delta(k)$ ($0 \leq k \leq n$) be the set of all $k$-dimensional cones in $\Delta$. Suppose that $\alpha_t$ ($1 \leq t \leq r$) are all one dimensional cones in $\Delta(1)$. Let $e(\alpha_t) \in N$ ($1 \leq t \leq r$) be the corresponding primitive elements, i.e., the unique generator of $\alpha_t \cap N$. Let $P_\Delta$ be a polytope in $M_\mathbb{R}$ defined by

$$P_\Delta = \bigcap_{\alpha_t \in \Delta(1)} \{ I \in M_\mathbb{R} | \langle I, e(\alpha_t) \rangle \geq -1 \}.$$  

Since $X_\Delta$ is compact, $P_\Delta$ is a compact polytope in $M_\mathbb{R}$. Let

$$S_\Delta = \{ I \ | \ I \in M \cap P_\Delta \}.$$  

Then $S_\Delta$ is a non-empty finite set and $0 \in S_\Delta$. We denote by $P_\Delta(i)$ the set of all $i$-dimensional faces of the polytope $P_\Delta$. Let

$$S(\Delta, i) = \bigcup_{F_j \in P_\Delta(n-i)} \{ I \in \text{int}(F_j) \cap M \}$$  

for $0 \leq i \leq n$, where $\text{int}(F_j)$ denotes the relative interior of the face $F_j$. Let

$$S_k(\Delta) = \bigcup_{0 \leq i \leq k} S(\Delta, i)$$  

for all $0 \leq k \leq n$.

(3) For any $I \in S_\Delta$, there exists a unique face $F_I(\Delta)$ of $P_\Delta$ such that $I \in \text{int}(F_I(\Delta)) \cap M$. Let

$$E_I(\Delta) = \{ e(\alpha_t) \ | \ \alpha_t \in \Delta(1) \text{ and } \langle I, e(\alpha_t) \rangle = -1 \}.$$  

Suppose that $E_I(\Delta) = \{ e(\alpha_{s_1}), \ldots, e(\alpha_{s_l}) \}$, where $1 \leq s_1 < \ldots < s_l \leq r$. Let $F_I^+(\Delta) \subseteq N_\mathbb{R}$ be the normal space of $F_I(\Delta)$, defined by

$$F_I^+(\Delta) = \bigoplus_{1 \leq t \leq l} \mathbb{R} \cdot e(\alpha_{s_t}).$$  

And we set

$$F_I^+(\Delta) = 0 \text{ if } E_I(\Delta) = \emptyset.$$  

(4) Suppose that $I \in S(\Delta, i)$. Then we have $\dim F_I(\Delta) = n - i$ and $\dim F_I^+(\Delta) = i$. Considering $F_I^+(\Delta) \otimes \mathbb{C}$ as a subspace of $N_C$, we have $\wedge^i (F_I^+(\Delta) \otimes \mathbb{C}) \cong \mathbb{C}$.

Let $F_I(\Delta)$ be the affine space in $M_C$ with the lowest dimension such that $F_I(\Delta) \subseteq F_I(\Delta)$, where $F_I(\Delta) \subseteq M_\mathbb{R}$ is considered as a subset of $M_C$ in the natural way. Then we have

$$F_I(\Delta) = \{ I + \alpha \ | \ \alpha \in (F_I^+(\Delta) \otimes \mathbb{C})^\perp \},$$  

where $(F_I^+(\Delta) \otimes \mathbb{C})^\perp \subseteq M_C$ is the annihilator of $(F_I^+(\Delta) \otimes \mathbb{C}) \subseteq N_C$. Especially, if $I \in M$ is a vertex of $P_\Delta$, then we have $F_I(\Delta) = \{ I \}$.

(5) Let

$$N_I^k(\Delta) = \begin{cases} \wedge^i (F_I^+(\Delta) \otimes \mathbb{C}) \wedge \wedge^{k-i} N_C & \text{for } 0 \leq i \leq k, \\ \wedge^k N_C & \text{for } i = 0, \\ 0 & \text{for } i > k. \end{cases}$$
Then $N^k_I(\Delta)$ is a subspace of $\wedge^k N_C$. We have to notice that

$$N^k_I(\Delta) = \begin{cases} \neq 0 & \text{for } I \in S_k(\Delta) \\ 0 & \text{for } I \notin S_k(\Delta) \end{cases}$$

Moreover, we have

$$N^n_I(\Delta) = \wedge^n N_C$$

for all $I \in S_\Delta$.

Let $W^k_I(\Delta) = \rho(N^k_I(\Delta))$ be a subspace of $H^0(X, \wedge^k T_X)$. We define

$$V^k_I(\Delta) = \chi^I \cdot W^k_I(\Delta) = \{ \chi^I \cdot w \mid w \in W^k_I(\Delta) \}.$$  

The elements in $V^k_I(\Delta)$ are considered as meromorphic $k$-vector fields on $X = X_\Delta$. In [6], it is proved that for any $I \in S_k(\Delta)$, the $k$-vector fields in $V^k_I(\Delta)$ have movable singularities. And we use them to represent the corresponding holomorphic $k$-vector fields.

The next theorem gives a description of the holomorphic polyvector fields on toric varieties.

**Theorem 2.6.** [6] Let $X = X_\Delta$ be a smooth compact toric variety associated with a fan $\Delta$ in $N_\mathbb{R}$. Then we have the following decomposition

$$H^0(X, \wedge^k T_X) = \bigoplus_{I \in S_k(\Delta)} V^k_I(\Delta)$$

for all $0 \leq k \leq n$, where $V^k_I(\Delta) = V^k_{-I}$ for all $I \in S_k(\Delta)$.

### 2.3. The Gerstenhaber algebra $(A_T, \wedge, [\cdot, \cdot])$

In the case of $X = T = (\mathbb{C}^*)^n$, the Gerstenhaber algebra structure of the polyvector fields (in algebraic version) has been studied in [7]. Let us recall the results.

Any $k$-polyvector field (in algebraic version) on $X = T$ can be written as the finite sum

$$\sum_{I \in M} c_I \cdot \chi^I \cdot \rho(A_I),$$

where $c_I \in \mathbb{C}$, $A_I \in \wedge^k N_{\mathbb{C}}$.

Let

$$\mathbb{C}[M] = \{ \sum_{I \in M} c_I \cdot \chi^I \mid c_I \in \mathbb{C}, \text{ and there are only finite } c_I \neq 0 \text{ in the sum} \},$$

which can be considered as the group algebra of the abelian group $M$.

Let

$$A^k_T = \mathbb{C}[M] \otimes_{\mathbb{C}} \wedge^k N_{\mathbb{C}},$$

where $0 \leq k \leq n$. Then the vector space of $k$-polyvector field (in algebraic version) on $X = T$ is isomorphic to $A^k_T$ by the map

$$\sum_{I \in M} c_I \cdot \chi^I \cdot \rho(A_I) \leftrightarrow \sum_{I \in M} c_I \cdot \chi^I \otimes A_I$$
Let
\[
\mathcal{A}_T = \bigoplus_{k \in \mathbb{Z}} \mathcal{A}_T^k,
\]
where \( \mathcal{A}_T^k = 0 \) for \( k < 0 \) or \( k > n \). The Gerstenhaber algebra structure of the polyvector fields on \( X = T \) can be described in the following way.

**Proposition 2.7.** [7]

1. The vector space \((\mathcal{A}_T, \cdot, [\cdot, \cdot])\) is a Gerstenhaber algebra, with the multiplication and the bracket given by the following way.
   - The multiplication on \( \mathcal{A}_T \) is given by
     \[
     (\chi^I \otimes A_I) \cdot (\chi^J \otimes A_J) = \chi^{I+J} \otimes (A_I \wedge A_J),
     \]
     where \( I, J \in M \), \( A_I \in \wedge^k N_C \), \( A_J \in \wedge^l N_C \).
   - The bracket \([\cdot, \cdot]\) on \( \mathcal{A}_T \) is given by
     \[
     [\chi^I \otimes A_I, \chi^J \otimes A_J] = (-1)^{k+1} \chi^{I+J} \otimes ((i_J A_I) \wedge A_J + (-1)^k A_I \wedge (i_I A_J)),
     \]
     where \( I, J \in M \), \( A_I \in \wedge^k N_C \), \( A_J \in \wedge^l N_C \), and \( i_J(A_I) \) is the contraction of \( J \in M \subset M_C \) with \( A_I \in \wedge^k N_C \).

2. Let \( D : \mathcal{A}_T \to \mathcal{A}_T \) be a \( \mathbb{C} \)-linear operator given by
   \[
   D(\chi^I \otimes A_I) = -\chi^I \otimes (i_I A_I),
   \]
   where \( I \in M \) and \( A_I \in \wedge^k N_C \). Then \( D \) is a BV operator for the Gerstenhaber algebra \((\mathcal{A}_T, \cdot, [\cdot, \cdot])\).

**Remark 2.8.** In the second part of Proposition 2.7 the original statement in [7] is that \(-D\) is a BV operator for the Gerstenhaber algebra \((\mathcal{A}_T, \cdot, -[\cdot, \cdot])\), where the bracket on \( \mathcal{A}_T \) is choosing to be \(-[\cdot, \cdot]\) (negative the bracket \([\cdot, \cdot]\)), the operator \( D \), the bracket \([\cdot, \cdot]\) and the multiplication \( \cdot \) is shown in the first part of Proposition 2.7.

### 3. BV operators of the Gerstenhaber algebras of holomorphic polyvector fields on toric varieties

#### 3.1. The Gerstenhaber algebra \((\mathcal{A}_\Delta, \cdot, [\cdot, \cdot])\).

Let \( X = X_\Delta \) be a smooth compact toric variety. Let \( \mathcal{A}_\Delta^k = H^0(X, \wedge^k T_X) \) be the vector space of holomorphic \( k \)-vector fields on \( X \) for \( k > 0 \). Let \( \mathcal{A}_\Delta^k = H^0(X, \mathcal{O}_X) = \mathbb{C} \) and let \( \mathcal{A}_\Delta^k = 0 \) for \( k < 0 \) and \( k > n \). Let
\[
\mathcal{A}_\Delta = \bigoplus_{k \in \mathbb{Z}} \mathcal{A}_\Delta^k.
\]
As shown in Example 2.3 \((\mathcal{A}_\Delta, \cdot, [\cdot, \cdot])\) is a Gerstenhaber algebra.

By Theorem 2.6 we have
\[
\mathcal{A}_\Delta^k = \bigoplus_{I \in S_k(\Delta)} V_I^k(\Delta).
\]

The elements in \( V_I^k(\Delta) \) can be written as \( \chi^I \cdot \rho(A_I) \), where \( I \in S_k(\Delta) \) and \( A_I \in N_I^k(\Delta) \).
Proposition 3.1.  (1) The Gerstenhaber algebra structure on \((A_\Delta, \land, [\cdot, \cdot])\) can be described by the following way. For any \(\chi^I \cdot \rho(A_I) \in V_k^I(\Delta)\) and \(\chi^J \cdot \rho(A_J) \in V_j^I(\Delta)\), we have

\[
(\chi^I \cdot \rho(A_I)) \land (\chi^J \cdot \rho(A_J)) = \chi^{I+J} \cdot \rho(A_I \land A_J),
\]

\[
[\chi^I \cdot \rho(A_I), \chi^J \cdot \rho(A_J)] = (-1)^{k+1} \chi^{I+J} \cdot \rho((I+J)A_I) \land A_J + (-1)^k A_I \land (I+J)A_J),
\]

where \(I \in S_k(\Delta), A_I \in N_k^I(\Delta),\) and \(J \in S_i(\Delta), A_J \in N_j^I(\Delta)\). As a consequence, we have \(V_k^I(\Delta) \land V_j^I(\Delta) \subseteq V^I_{k+j}(\Delta)\) and \([V_k^I(\Delta), V_j^I(\Delta)] \subseteq V^I_{k+j-1}(\Delta)\).

(2) Let \(\gamma : A_\Delta \rightarrow A_T\) be a \(\mathbb{C}\)-linear map defined by

\[
\gamma(\chi^I \cdot \rho(A_I)) = \chi^I \otimes A_I,
\]

where \(\chi^I \cdot \rho(A_I) \in V_k^I(\Delta), I \in S_k(\Delta)\) and \(A_I \in N_k^I(\Delta)\). Then \(\gamma : A_\Delta \rightarrow A_T\) is an injective map. And \(\gamma : (A_\Delta, \land, [\cdot, \cdot]) \rightarrow (A_T, \land, [\cdot, \cdot])\) is a morphism of Gerstenhaber algebras. Thus \((A_\Delta, \land, [\cdot, \cdot])\) can be considered as a sub-Gerstenhaber algebra of \((A_T, \land, [\cdot, \cdot])\).

Proposition 3.1 can be proved in a similar way to Proposition 2.7 which can be found in [7]. Here we skip the details.

3.2. BV operators of the Gerstenhaber algebra \((A_\Delta, \land, [\cdot, \cdot])\). Let \(X = X_\Delta\) be a smooth compact toric variety. A natural question is when the Gerstenhaber algebra \((A_\Delta, \land, [\cdot, \cdot])\) will be exact? In other words, is there any degree \(-1\) \(\mathbb{C}\)-linear operator \(D : A_\Delta \rightarrow A_\Delta\) such that

(1)

\[
[a, b] = (-1)^{|a|} (D(a \land b) - D(a) \land b - (-1)^{|a|} a \land D(b))
\]

for any \(a, b \in A_\Delta\),

(2)

\[
D^2 = 0.
\]

Since \(A_\Delta^k = 0\) for \(k < 0\) and \(k > n\), we only need to consider the linear maps \(D : A_\Delta^k \rightarrow A_\Delta^{k-1}\) in the case of \(1 \leq k \leq n\).

Lemma 3.2. Let \(D\) be a BV operator on the Gerstenhaber algebra \((A_\Delta, \land, [\cdot, \cdot])\). Then there exists an element \(\delta \in M_\mathbb{C}\) such that, for all \(\chi^I \cdot \rho(A) \in V_k^I(\Delta) (1 \leq k \leq n)\), we have

(3.4)

\[
D(\chi^I \cdot \rho(A)) = \chi^I \cdot \rho(I+J-A),
\]

where \(I \in S_k(\Delta)\) and \(A \in N_k^I(\Delta)\).

Proof.  Let us consider the linear map \(D : A_\Delta^k \rightarrow A_\Delta^{k-1} = \mathbb{C}\). Let \(\delta \in M_\mathbb{C}\) be defined by

(3.5)

\[
\langle \delta, x \rangle = D(\rho(x)),
\]

for all \(x \in N_\mathbb{C}\), where \(\rho(x) \in W \subseteq A_\Delta^1\).

First, we prove the lemma in the case of \(k = n\). In Equation 3.2, let \(b = \chi^I \cdot \rho(A)\), where \(I \in S_n(\Delta) = S_\Delta\) and \(A \in N_n^I(\Delta) = \land^i N_\mathbb{C}\); and let \(a = \rho(x)\), where \(x \in N_\mathbb{C}\). By proposition 3.1 we have

\[
[a, b] = [\rho(x), \chi^I \cdot \rho(A)] = \langle I, x \rangle \chi^I \cdot \rho(A),
\]
Next we prove the lemma by recursion. Suppose the lemma holds in the case of

\[ \langle I, x \rangle \chi^l \cdot \rho(A) = \langle \delta, x \rangle \chi^l \cdot \rho(A) - \rho(x) \land D(\chi^l \cdot \rho(A)), \]

which implies

\[ \langle \delta - I, x \rangle \chi^l \cdot \rho(A) = \rho(x) \land D(\chi^l \cdot \rho(A)). \] (3.6)

On the other hand, we have \( x \land A = 0 \), which implies

\[ i_{\delta - I}(x \land A) = \langle \delta - I, x \rangle A - x \land (i_{\delta - I}A) = 0. \]

Therefore we have

\[ \langle \delta - I, x \rangle A = x \land (i_{\delta - I}A). \]

As a consequence, we get

\[ \langle \delta - I, x \rangle \chi^l \cdot \rho(A) = \chi^l \cdot \rho(\delta - I, x)A = \chi^l \cdot \rho(x \land (i_{\delta - I}A)) = \chi^l \cdot \rho(x) \land \rho(i_{\delta - I}A). \] (3.7)

By Equation (3.6) and Equation (3.7), we get

\[ \rho(x) \land (D(\chi^l \cdot \rho(A)) - \chi^l \cdot \rho(i_{\delta - I}A)) = 0. \] (3.8)

By Proposition 3.1 and Equation (3.8), we have

\[ x \land \gamma(D(\chi^l \cdot \rho(A)) - \chi^l \cdot \rho(i_{\delta - I}A)) = 0 \]

for all \( x \in N_C \). Hence

\[ \gamma(D(\chi^l \cdot \rho(A)) - \chi^l \cdot \rho(i_{\delta - I}A)) = 0. \]

As a consequence, we get

\[ D(\chi^l \cdot \rho(A)) = \chi^l \cdot \rho(i_{\delta - I}A). \]

Therefore, we proved the lemma for \( k = n \).

- Next we prove the lemma by recursion. Suppose the lemma holds in the case of \( k = l \) \((l \geq 2)\), we will prove it in the case of \( k = l - 1 \).

  In Equation (3.2), let \( b = \chi^l \cdot \rho(A) \), where \( I \in S_{l-1}(\Delta) \) and \( A \in N_{l-1}^l(\Delta) \); and let \( a = \rho(x) \), where \( x \in N_C \). By Proposition 3.1 we have

\[ [a, b] = [\rho(x), \chi^l \cdot \rho(A)] = \langle I, x \rangle \chi^l \cdot \rho(A). \]

On the right hand of Equation (3.2), we have

\[ D(a \land b) = D(\rho(x) \land (\chi^l \cdot \rho(A))) = D(\chi^l \cdot \rho(x \land A)) = \chi^l \cdot \rho(i_{\delta - I}(x \land A)), \]

\[ D(a) \land b = D(\rho(x)) \chi^l \cdot \rho(A) = \langle \delta, x \rangle \chi^l \cdot \rho(A), \]

\[ a \land D(b) = \rho(x) \land D(\chi^l \cdot \rho(A)). \]

By Equation (3.2), we have

\[ \langle I, x \rangle \chi^l \cdot \rho(A) = -\chi^l \cdot \rho(i_{\delta - I}(x \land A)) + \langle \delta, x \rangle \chi^l \cdot \rho(A) - \rho(x) \land D(\chi^l \cdot \rho(A)). \]

As a consequence, we have

\[ \rho(x) \land D(\chi^l \cdot \rho(A)) = \chi^l \cdot (\langle \delta - I, x \rangle \rho(A) - \rho(i_{\delta - I}(x \land A))) \]

\[ = \chi^l \cdot \rho(x \land i_{\delta - I}A) = b \land \rho(x) \land \rho(i_{\delta - I}A) = \rho(x) \land (\chi^l \cdot \rho(i_{\delta - I}A)). \]
Therefore, we have
\[ \rho(x) \wedge (D(\chi^I \cdot \rho(A)) - \chi^I \cdot \rho(i_{\delta - l}A)) = 0 \]
for all \( x \in N_C \). By similar reason as we have shown in the case of \( k = n \), we get
\[ D(\chi^I \cdot \rho(A)) = \chi^I \cdot \rho(i_{\delta - l}A). \]
Thus if the lemma holds in the case of \( k = l \), then it also holds in the case of \( k = l - 1 \).

□

To prove Lemma 3.4 and Lemma 3.5, we need a basic fact in linear algebra, here we write it as a lemma.

**Lemma 3.3** (A basic fact of linear algebra). Let \( V \) be a \( n \)-dimensional vector space over \( k \), and \( F \) be a \( i \)-dimensional subspace of \( V \). Let \( A \) be a nonzero element in \( \wedge^i F \cong k \). Then for any \( \alpha \in V^* \), we have that \( \iota_{\alpha} A = 0 \) if and only if \( \alpha \in F^\perp \), where \( F^\perp \subseteq V^* \) is the annihilator of \( F \).

**Lemma 3.4.** Let \( \delta \) be an element in \( M_C \). Suppose that for all \( \chi^I \cdot \rho(A) \in V^k_I(\Delta) \) (\( 1 \leq k \leq n \)), we have \( \chi^I \cdot \rho(i_{\delta - l}A) \in V^k - 1_I(\Delta) \), where \( I \in S_k(\Delta) \) and \( A \in N^k_I(\Delta) \).

(1) We have
\[ \delta \in \bigcap_{I \in S_\Delta} \hat{F}_I(\Delta), \]

(2) If \( I \in M \) is a vertex of \( P_\Delta \), then we have \( \delta = I \). Thus the polytope \( P_\Delta \subset M_C \) has at most one vertex belonging to \( M \).

**Proof.**

(1) We will prove that for any \( I \in S_\Delta \), we have \( \delta \in \hat{F}_I(\Delta) \).
Suppose that \( I \in S(\Delta, i) \). Then we have \( \dim F^I_I(\Delta) = n - i \) and \( \dim F^I_I(\Delta) = i \).
According to the assumption, for any \( \chi^I \cdot \rho(A) \in V^k_I(\Delta) \), we have \( \chi^I \cdot \rho(i_{\delta - l}A) \in V^k - 1_I(\Delta) \).
Hence for any \( A \in N^k_I(\Delta) \), we have \( i_{\delta - l}A \in N^k - 1_I(\Delta) \). In the case of \( k = i \), we have \( N^k - 1_I(\Delta) = 0 \). Let \( A \) be a nonzero element in \( N^k_I(\Delta) = n^i(F^I_I(\Delta) \otimes \mathbb{C}) \cong \mathbb{C} \). Then we have
\[ i_{\delta - l}A = 0. \]

By Lemma 3.3 and Equation (3.11), we have
\[ \delta - I \in (F^I_I(\Delta) \otimes \mathbb{C})^\perp. \]

As a consequence, we get \( \delta \in \hat{F}_I(\Delta) \) for all \( I \in S(\Delta, i) \).
Since \( S_\Delta = \bigcup_{0 \leq i \leq n} S(\Delta, i) \), we get
\[ \delta \in \bigcap_{I \in S_\Delta} \hat{F}_I(\Delta). \]

(2) Suppose that \( I \in M \) is a vertex of the polytope \( P_\Delta \). Then we have \( \delta \in S_\Delta \) and \( \hat{F}_I(\Delta) = \{ I \} \).
Since
\[ \delta \in \bigcap_{I \in S_\Delta} \hat{F}_I(\Delta), \]
we have \( \delta = I \). Thus the polytope \( P_\Delta \subset M_C \) has at most one vertex belonging to \( M \).

□
Lemma 3.5. Let $\delta \in \bigcap_{I \in S_N} \hat{F}_I(\Delta)$. Then for all $\chi^I \cdot \rho(A) \in V^k_I(\Delta)$ (1 $\leq k \leq n$), we have $\chi^I \cdot \rho(i_{i-1}A) \in V^{k-1}_I(\Delta)$, where $I \in S_k(\Delta)$ and $A \in N^k_I(\Delta)$.

Proof. To prove the lemma, it is enough for us to prove that $i_{i-1} A \in N^{k-1}_I(\Delta)$ for all $I \in S_k(\Delta)$ and $A \in N^k_I(\Delta)$. Suppose that $I \in S(\Delta, \delta)$. Then we have $k \leq i$ since $I \in S_k(\Delta)$.

- We first prove the lemma in the case of $k = i$. In the case of $k = i$, we have
  
  $$N^i_I(\Delta) = \chi^i(F^i_I(\Delta) \otimes \mathbb{C}) = \mathbb{C} \quad \text{and} \quad N^{i-1}_I(\Delta) = 0.$$ 

  Suppose that $A_1$ is a nonzero element in $N^i_I(\Delta) = \chi^i(F^i_I(\Delta) \otimes \mathbb{C}) \cong \mathbb{C}$. Since $\delta \in \hat{F}_I(\Delta)$, we have $\delta - I \in (F^i_I(\Delta) \otimes \mathbb{C})^\perp$. By Lemma 3.3, we get
  
  $$i_{i-1} A_1 = 0.$$ 

  Since $N^i_I(\Delta) = \mathbb{C} A_1$, we have proved the lemma in the case of $k = i$.

- In the case of $k > i$, we have
  
  $$N^k_I(\Delta) = N^i_I(\Delta) \wedge (\wedge^{k-i} N_I),$$ 

  Any element $A \in N^k_I(\Delta)$ can be written as
  
  $$A = A_1 \wedge A_2,$$

  where $A_2 \in \wedge^{k-i} N_I$. We have
  
  $$(i_{i-1} A) = (i_{i-1} A_1) \wedge A_2 + (-1)^i A_1 \wedge (i_{i-1} A_2) = (-1)^i A_1 \wedge (i_{i-1} A_2).$$

  Since $A_1 \in N^i_I(\Delta)$ and $i_{i-1} A_2 \in \wedge^{k-i-1} N_I$, we have
  
  $$i_{i-1} A \in N^{k-1}_I(\Delta).$$

\[\square\]

Lemma 3.6. Let $\delta$ be an element in $M_N$. Let $D$ be a linear operator defined on the Gerstenhaber algebra $(A_\Delta, \wedge, [\cdot, \cdot])$ satisfying

$$D(\chi^I \cdot \rho(A)) = \chi^I \cdot \rho(i_{i-1}A)$$

for all $\chi^I \cdot \rho(A) \in V^k_I(\Delta)$ (1 $\leq k \leq n$), where $I \in S_k(\Delta)$ and $A \in N^k_I(\Delta)$. Then $D$ satisfies Equation 3.2 and 3.3.

Proof. Since that $A^k_\Delta = 0$ for $k < 0$ and $k > n$, we only need to verify that $D^2 : A^k_\Delta \to A^{k-2}_\Delta$ equal to zero for $2 \leq k \leq n$.

As $D$ is a linear map from the vector space $A^k_\Delta = \bigoplus_{I \in S_k(\Delta)} V^k_I(\Delta)$ to the vector space $A^{k-1}_\Delta = \bigoplus_{I \in S_{k-1}(\Delta)} V^{k-1}_I(\Delta)$, we get that

$$D(\chi^I \cdot \rho(A)) = \chi^I \cdot \rho(i_{i-1}A) \in V^{k-1}_I(\Delta)$$

for all $\chi^I \cdot \rho(A) \in V^k_I(\Delta)$ (1 $\leq k \leq n$). For any $\chi^I \cdot \rho(A) \in V^k_I(\Delta)$, suppose that $I \in S(\Delta, \delta)$, then we have $i \leq k$ since $I \in S_k(\Delta)$.

In the case of $k < i + 2$, since $D^2(\chi^I \cdot \rho(A)) \in V^{k-2}_I(\Delta)$ and $V^{k-2}_I(\Delta) = 0$, we get that $D^2(\chi^I \cdot \rho(A)) = 0$. And in the case of $k \geq i + 2$, we have

$$D^2(\chi^I \cdot \rho(A)) = D(\chi^I \cdot \rho(i_{i-1}A)) = \chi^I \cdot \rho(i_{i-1}(i_{i-1}A)) = 0.$$ 

Since $A^k_\Delta = \bigoplus_{I \in S_k(\Delta)} V^k_I(\Delta)$, we get $D^2 = 0.
Theorem 3.7. Let $X_\Delta$ be a smooth compact toric variety. Let $(\mathcal{A}_\Delta, \wedge, [\cdot, \cdot])$ be the Gerstenhaber algebra associated to $X_\Delta$.

(1) The necessary and sufficient condition for the existence of the BV operator is that

\[ \bigcap_{I \in S_\Delta} \hat{F}_I(\Delta) \neq \emptyset. \]
(2) If Equation (3.19) holds, choosing any element $\delta \in \bigcap_{I \in S_\Delta} \hat{F}_I(\Delta)$, we can define a BV-operator $D : A_\Delta \to A_\Delta$ by

\begin{equation}
D(\chi^I \cdot \rho(A)) = \chi^I \cdot \rho(\iota_{I\Delta} A),
\end{equation}

where $\chi^I \cdot \rho(A) \in V^k_I(\Delta)$ ($1 \leq k \leq n$), $I \in S_k(\Delta)$ and $A \in N^k_I(\Delta)$.

(3) All the BV operators of the Gerstenhaber algebra $(A_\Delta, \wedge, [\cdot, \cdot])$ are in the form as shown in Equation (3.20).

\begin{proof}

(1) The necessary condition:

Suppose that $D : A_\Delta \to A_\Delta$ is a BV-operator of Gerstenhaber algebra $(A_\Delta, \wedge, [\cdot, \cdot])$. By Lemma 3.2, there exist an element $\delta \in M_{\mathbb{C}}$ such that, for all $\chi^I \cdot \rho(A) \in V^k_I(\Delta)$ ($1 \leq k \leq n$), we have

\begin{equation}
D(\chi^I \cdot \rho(A)) = \chi^I \cdot \rho(\iota_{I\Delta} A),
\end{equation}

where $I \in S_k(\Delta)$ and $A \in N^k_I(\Delta)$. By Theorem 2.6, to make $D : A_\Delta \to A_\Delta$ to be a well defined operator, we have

\begin{equation}
D(\chi^I \cdot \rho(A)) = \chi^I \cdot \rho(\iota_{I\Delta} A) \in V^{k-1}_I(\Delta)
\end{equation}

for all $I \in S_k(\Delta)$ and $A \in N^k_I(\Delta)$. And by Lemma 3.5 we get

$$\delta \in \bigcap_{I \in S_\Delta} \hat{F}_I(\Delta).$$

Hence we have

$$\bigcap_{I \in S_\Delta} \hat{F}_I(\Delta) \neq \emptyset.$$

The sufficient condition:

If Equation (3.19) holds, choosing any element $\delta \in \bigcap_{I \in S_\Delta} \hat{F}_I(\Delta)$, we can define a $\mathbb{C}$-linear operator $D : A_\Delta \to A_\Delta$ by

\begin{equation}
D(\chi^I \cdot \rho(A)) = \chi^I \cdot \rho(\iota_{I\Delta} A),
\end{equation}

where $\chi^I \cdot \rho(A) \in V^k_I(\Delta)$ ($1 \leq k \leq n$), $I \in S_k(\Delta)$ and $A \in N^k_I(\Delta)$. By Lemma 3.5, $D : A_\Delta \to A_\Delta$ is a well defined operator since $D(V^k_I(\Delta)) \subseteq V^{k-1}_I(\Delta)$. By Lemma 3.6, $D$ is a BV-operator of the Gerstenhaber algebra $(A_\Delta, \wedge, [\cdot, \cdot])$.

(2) It is already proved above in the sufficient condition part.

(3) It follows directly from Lemma 3.2.

\end{proof}

References

[1] Sergey Barannikov and Maxim Kontsevich, Frobenius manifolds and formality of Lie algebras of polyvector fields, Internat. Math. Res. Notices 4 (1998), 201–215, DOI 10.1155/S1073792898000166.

[2] Ruggero Bandiera, Mathieu Stiénon, and Ping Xu, Polyvector fields and polydifferential operators associated with Lie pairs, arXiv 1901.04602 (2019).

[3] David Cox, What is a toric variety?, Topics in algebraic geometry and geometric modeling, Contemp. Math., vol. 334, Amer. Math. Soc., Providence, RI, 2003, pp. 203–223, DOI 10.1090/conm/334/05983.

[4] William Fulton, Introduction to toric varieties, Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993. The William H. Roever Lectures in Geometry. MR1234037

[5] Nigel Hitchin, Stable bundles and polyvector fields, Complex and differential geometry, Springer Proc. Math., vol. 8, Springer, Heidelberg, 2011, pp. 135–156.
[6] Wei Hong, *Holomorphic polyvector fields on toric varieties*, arXiv 2010.07053 (2020).

[7] Travis Mandel and Helge Ruddat, *Tropical quantum field theory, mirror polyvector fields, and multiplicities of tropical curves*, arXiv 1902.07183 (2019).

[8] Tadao Oda, *Convex bodies and algebraic geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 15, Springer-Verlag, Berlin, 1988. An introduction to the theory of toric varieties; Translated from the Japanese.

[9] Claude Roger, *Gerstenhaber and Batalin-Vilkovisky algebras; algebraic, geometric, and physical aspects*, Arch. Math. (Brno) 45 (2009), no. 4, 301–324.

[10] Ping Xu, *Gerstenhaber algebras and BV-algebras in Poisson geometry*, Comm. Math. Phys. 200 (1999), no. 3, 545–560, DOI 10.1007/s002200050540.

School of Mathematics and Statistics, Wuhan University, Wuhan, 430072, China

Email address: yangdeng@whu.edu.cn

School of Mathematics and Statistics, Wuhan University, Wuhan, 430072, China

Hubei Key Laboratory of Computational Science, Wuhan University, Wuhan, 430072, China

Email address: hong_w@whu.edu.cn