Large scale multi-objective optimization: Theoretical and practical challenges

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ABSTRACT

Multi-objective optimization (MOO) is a well-studied problem for several important recommendation problems. While multiple approaches have been proposed, in this work, we focus on using constrained optimization formulations (e.g., quadratic and linear programs) to formulate and solve MOO problems. This approach can be used to pick desired operating points on the trade-off curve between multiple objectives. It also works well for internet applications which serve large volumes of online traffic, by working with Lagrangian duality formulation to connect dual solutions (computed offline) with the primal solutions (computed online).

We identify some key limitations of this approach – namely the inability to handle user and item level constraints, scalability considerations and variance of dual estimates introduced by sampling processes. We propose solutions for each of the problems and demonstrate how through these solutions we significantly advance the state-of-the-art in this realm. Our proposed methods can exactly handle user and item (and other such local) constraints, achieve a 100× scalability boost over existing packages in R and reduce variance of dual estimates by two orders of magnitude.

CCS Concepts

- Theory of computation → Mathematical optimization; Quadratic programming;

Keywords

Large scale multi-objective optimization, personalization, dual conversion, variance reduction

1. INTRODUCTION

Today, most internet applications and products use data to optimize the user experience. With every passing year, more and more such applications are coming into existence, and the scales of the existing applications are increasing. The businesses operating these applications are also growing and so are the users’ expectations from these applications. A natural manifestation of this multi-dimensional growth is through the introduction of multiple objectives (e.g., company’s business objectives and user engagement). While a search engine could initially focus on maximizing the click-through rate (CTR) of its first few results, as the business grows, monetization and user retention become additional objectives that need to be ensured. This necessitates a framework which allows for efficiently navigating the trade-off between such important objectives.

Since we are almost into the third decade of optimizing internet products with data, it is not surprising that several approaches have been proposed to address this problem [1, 2, 3, 6, 7, 8, 11] — both in theory and in practice. The common thread in these works is trying to combine several objectives or criteria. In this paper, we will focus on the constrained optimization approach [2, 3] since this provides a lot of flexibility on the problem formulation. The relative importance of each objective need not be pre-specified, instead the desired range of value of each objective is specified, and the relative weights (to satisfy those constraints) are learnt. This added flexibility in specification, combined with the scalability of the approach, has made this a feasible solution for industrial applications.

Industrial systems employing machine learning algorithms can be broadly categorized into two classes — offline systems and online systems, where online is defined as triggered by a user visit. In offline systems, the entire computation is done offline and the results (e.g., best article recommendations for a user) are pre-computed and used to serve a user when they visit. Another example of such a system is an email delivery system where machine learning algorithms can be used to determine both the content and appropriateness of an email for a user. In an online system, on the other hand, the candidate items are scored when the user visits. These systems naturally have strict computation constraints, but also provide better results for the same application than an offline approach, since there are more recent signals to use.

When constraints are used to specify multi-objective optimization problems, then a two-stage approach is adopted for both offline and online applications. The two-stage approach is necessary since the problem size is too large to be

Algorithm 1 in this paper is presented for a slightly more advanced case. However, the actual algorithm that should be used is the dual to primal conversion (Algorithm 1) in [3]. This is true for all references of Algorithm 1 in the two tables above. We avoid copying it here due to space constraints.

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The second challenge was one of scaling up a quadratic programming (QP) solver. While this is a well-studied theoretical problem, there isn’t any documented work on deploying such systems in real-world applications. We explore multiple methods, and share our experiences along with intuitions on what works best and how they scale.

The third challenge we address is related to sampling. The dual estimates obtained from Stage 1 can have high variance, as an artifact of the sampling process. We employ some variance reduction methods and show that they reduce the variance of the dual estimates, and hence improve the accuracy of the final (i.e., Stage 2) solution which depends on those dual estimates. With these advances, the new state of the ecosystem is briefed in Table 2 (all the advances are marked in bold and blue).

The rest of the paper is structured as follows. Section 2 formulates the problem and describes the existing solution in detail. Section 3 introduces the importance of local constraints, highlights the limitations of the existing solution in handling these and presents a mathematically exact solution to solve the problem. We present a graphical representation of the constraint set in Section 4 and introduce a criteria to pick which constraints to re-solve during serving time. Section 5 discusses our experience in using ADMM algorithms to scale up a single-machine QP solver and some variance reduction methods. Finally, we present some experiments to validate our methods and quantify their impact in Section 6.

2. THE MOO PROBLEM

In this paper we are mainly interested in the recommendation problem of showing items to users such that will max-

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**Table 1: Existing constrained optimization solution – setup**

| Stage 1 | Stage 2 |
|---------|---------|
| **Offline application**<br> (Batch processing) Iterate over users.<br> For each user $u$, use Algorithm 1 with inputs $p_u$ scored offline and $\mu$ (from stage 1) to get $x_u$ | **Online application**<br> As each user $u$ visits, use with inputs $p_u$ scored online and $\mu$ (from stage 1) to get $x_u$ |
| Step 1.1: Sample a set of users, as large as can be handled by your QP solver | |
imize their engagement while ensuring that the potential negative flags or complaints (for simplicity, we will just refer to them as complaints hereafter) are contained within a limit.

We begin by introducing some notation, which we will use throughout the length of the paper. Lower bold case letters (e.g., \( p, x \)) denote vectors, while upper bold case letters (e.g., \( A \)) denote matrices or linear operators. We use \( p_{ij,k} \) to denote the appropriate index of \( p \) (similarly for matrices) and \( x'z = \sum_i x_iz_i \) to denote the Euclidean dot product between \( x \) and \( z \). The relation \( \leq \) when applied to a vector implies element wise inequalities. Also \( \Pi_C(\cdot) \) denotes the projection operator onto the set \( C \) in terms of the \( L_2 \) norm.

Let a user be denoted as \( u \) and the set of all users \( U \), such that \( |U| = N \). Similarly, let an item be denoted as \( i \) and the set of items as \( I \), such that \( |I| = M \). The target serving plan (which we seek to find) can be represented as \( x_{ui} \), which is the probability of showing item \( i \) to user \( u \). Let \( p_{ui} \) denote the probability of the user \( u \) engaging (by acting on it or clicking on it) with item \( i \), and \( r_{ui} \) denote the probability of user \( u \) disliking item \( i \) (e.g., by flagging it or complaining about it) conditioned on the fact that the user was shown the item \( i \). A detailed list of all symbols and their meanings is provided in Table 3.

### Table 3: List of Notations used in the paper

| Notation | Meaning |
|----------|---------|
| \( I \)  | Set of all items to be shown |
| \( U \)  | Set of all Users |
| \( x_{ui} \) | Probability of showing item \( i \) to user \( u \). |
| \( p_{ui} \) | \( \Pr(\text{User } u \text{ engaged with item } i | x_{ui} = 1) \) |
| \( r_{ui} \) | \( \Pr(\text{User } u \text{ disliking/complaining about item } i | x_{ui} = 1) \) |
| \( \sum_{u,i} x_{ui}p_{ui} \) | Expected number of clicks |
| \( \mathbf{1}_N, 0_N \) | Vector of 1’s (0’s) of dimension \( N \) |

#### 2.1 Problem Formulation

The aforementioned optimization problem can be written as a linear program. Using the above notations we can write it as

\[
\begin{align*}
\text{Maximize} & \quad \sum_{u,i} x_{ui}p_{ui} \\
\text{subject to} & \quad \sum_{u} \sum_{i} x_{ui}r_{ui} \leq R \\
& \quad 0 \leq x_{ui} \leq 1, \forall u, i, \quad \sum_{i} x_{ui} = 1, \forall u.
\end{align*}
\]

The last two constraints come from the fact that \( x_{ui} \) is a probability and there will be some item shown to every user. If there is an existing serving plan that we do not want to deviate too much from (e.g., showing the most engaging item to a user), then this can be represented in the following way (similar to [3]). Let \( q_{ui} \) represent the existing serving plan (in the example, \( q_{ui} = 1, \) \( \text{where } i^* = \text{argmax}_i p_{ui} \)). Then, we can limit the deviation of \( x \) from \( q \) with the following quadratic program (QP):

\[
\begin{align*}
\text{Maximize} & \quad \sum_{u,i} x_{ui}p_{ui} - \frac{\gamma}{2} \sum_{u,i} (x_{ui} - q_{ui})^2 \\
\text{subject to} & \quad \sum_{u} \sum_{i} x_{ui}r_{ui} \leq R \\
& \quad \sum_{i} x_{ui} = 1, \forall u \\
& \quad 0 \leq x_{ui} \leq 1, \forall u, i,
\end{align*}
\]

where \( \gamma \) controls the relative importance of engagement maximization and deviation of \( x \) from \( q \). Note that the objective in (1) is concave, which would be equivalent to minimizing the negative of the expression. The QP formulation also facilitates some additional algorithms when we are straddling the dual and primal spaces to work out solutions.

This formulation can be easily extended to add objectives or constraints to a large set of users or items. For instance, if we want to put an additional constraint on the number of complaints from English-speaking users (denoted by \( U_{En} \)), then the modified QP would look like:

\[
\begin{align*}
\text{Minimize} & \quad -\sum_{u,i} x_{ui}p_{ui} + \frac{\gamma}{2} \sum_{u,i} (x_{ui} - q_{ui})^2 \\
\text{subject to} & \quad \sum_{u} \sum_{i} x_{ui}r_{ui} \leq R \\
& \quad \sum_{u\in U_{En}} \sum_{i} x_{ui}r_{ui} \leq R_{En} \\
& \quad \sum_{i} x_{ui} = 1, \forall u \\
& \quad 0 \leq x_{ui} \leq 1, \forall u, i.
\end{align*}
\]

#### 2.2 Solving the MOO problem

We shall modify the technique in [3] to solve for the primal \( x_{ui} \) using the dual of the problem. For brevity, we work with problem (1) as all the derivations can be trivially extended to the other formulations of the problem as well. We define the following notation first.

Let \( a \in \mathbb{R}^{MN} \) be given by \( a_{ui} = p_{ui} + \gamma q_{ui} \), \( r \) be the vectorized form of \( r_{ui} \) and \( \mathbf{x}_n = (x_{u1} \ldots x_{uM})' \). Then the problem (1) can be transformed into

\[
\begin{align*}
\text{Minimize} & \quad -\mathbf{x}' a + \frac{\gamma}{2} \mathbf{x}' \mathbf{x} \\
\text{subject to} & \quad \mathbf{x}' r \leq R \\
& \quad \mathbf{x}' \mathbf{1}_N = 1 \quad \forall u \\
& \quad 0 \leq \mathbf{x} \leq 1.
\end{align*}
\]

Note that the second constraint can also be written in the format

\[\mathbf{M}'\mathbf{x} = \mathbf{1}_N,\]

where \( \mathbf{1}_N \in \mathbb{R}^N \) is the vector of all 1’s in \( N \) dimension and \( \mathbf{M} \in \mathbb{R}^{MN \times N} \) is given by \( \mathbf{M} = \mathbf{1}_N \otimes \mathbf{1}_M \) and \( \otimes \) denotes the Kronecker product. Note that, we can write the Lagrangian as follows. Let \( \mu \in \mathbb{R}, \nu \in \mathbb{R}^N, \xi \in \mathbb{R}^{MN} \) and \( \eta \in \mathbb{R}^{MN} \) be the dual variables corresponding to the above problem (2). Thus, we have

\[\mathcal{L}(\mathbf{x}, \mu, \nu, \xi, \eta) = -\mathbf{x}' a + \frac{\gamma}{2} \mathbf{x}' \mathbf{x} + \mu (\mathbf{M}' \mathbf{x} - \mathbf{1}_N) - \xi' \mathbf{x} + \eta' (\mathbf{x} - \mathbf{1}).\]
Minimizing this with respect to $x$, and writing $\hat{A} = [-r : -M : I : -I]$ and $z = (\mu, \nu, \xi, \eta)$ we have

$$x^* = \frac{1}{\gamma} (a - \mu r - M \nu + \xi - \eta) = \frac{1}{\gamma} \left( a + \hat{A} z \right).$$

Letting $s = (-R, -1_N, 0_{N,M}, -1_{N,M})$ and plugging $x^*$ back into the Lagrangian, we get,

$$L(x^*, \mu, \nu, \xi, \eta) = -\mu R - \nu^T 1_N - \eta^T 1_{N M}$$

$$= \frac{1}{2\gamma} (a + \hat{A} z)^T (a + \hat{A} z)$$

$$= s^T z - \frac{1}{2\gamma} (a + \hat{A} z)^T (a + \hat{A} z).$$

Duplicating the equality constraint as a positive as well as a negative constraint (using $\nu_+$ and $\nu_-$) and having $\hat{z} = (\mu, \nu_+, \nu_-, \xi, \eta)$, $\hat{s} = (-R^T : -1_N : 1_N : 0_{N M} : -1_{N M})$ and $\hat{A} = [-r : -M : M : I : -I]$, we can write our dual problem as

Minimize $\hat{a}$

$$- \hat{s}^T \hat{z} + \frac{1}{2\gamma} (a + \hat{A} z)^T (a + \hat{A} z)$$

subject to $\hat{z} \geq 0$.

Writing $B = \hat{A}^T \hat{A} / \gamma$, $\hat{p} = \hat{s} - \hat{A}^T a / \gamma$, we can re-write the above problem in the most basic form,

Minimize $\hat{a}$

$$\frac{1}{2\gamma} z^T B z - \hat{p}^T z$$

subject to $z \geq 0$.

This problem can now be solved by any convex optimization algorithm. Large scale instances of the above problem can be solved efficiently by the Operator Splitting algorithm [10]. For more details see Section 5.1.

3. USER-LEVEL CONSTRAINTS

The formulation described above provides an optimization problem that can be solved efficiently when the constraints are global in nature i.e. applicable to all users simultaneously. However, we would often want to apply different kinds of constraints for specific users or items belonging to particular groups. Some examples of this include applying revenue threshold constraints for sponsored items on the feed and having different constraint rates for a certain subset of heavy users. To the best of our knowledge, the introduction of such user-level (local) constraints has not been explored in detail in previous literature.

Note that the number of constraints might significantly increase with the introduction of user-level or item-level constraints which makes scaling up the optimization problem to large scale data a significant challenge. In this section, we provide a scheme to solve this problem in two steps. In the first step, we get the global dual corresponding to the first constraint using the technique in Section 2.2 and using this dual we solve for $x$ to get the user-level local coefficients. We begin with a specific example to explain our procedure and then we generalize it to it a much larger class of constraints.

3.1 Solving the primal using dual solution

Consider a modification of the original primal problem (2) so that the user $u$ can see a set of items $i$ (for example messages) under the constraint that no user will be shown more than $K_u$ messages. Using $k = \{K_u\}_{u=1}^N$, this constraint can be written as $M^T x \leq k$. Note that the sum to 1 constraint no longer holds in this case. The problem can now be written in an analogous way by pushing some of the constraints into the objective.

Minimize $x$

$$-a^T x + \gamma \frac{1}{2} x^T x + \mathcal{I}_c(x)$$

subject to $x^T r \leq R$

$$M^T x \leq k,$$

where $\mathcal{C} = [0, 1]^{MN}$ and $\mathcal{I}_c(x) = 0$ if $x \in \mathcal{C}$ and infinity otherwise. We can write the Lagrangian function of the problem in (4) as

$$L(x, \mu, \nu) = -a^T x + \gamma \frac{1}{2} x^T x + \mathcal{I}_c(x) + \mu(x^T r - R) + \nu^T (M^T x - k).$$

After some algebra, it is easy to see at the optimal $x$ can be written as

$$x^* = \Pi_{\mathcal{C}} \left( \frac{a - \mu r - M \nu}{\gamma} \right),$$

where $\Pi_{\mathcal{C}}(\cdot)$ stands for the projection into $\mathcal{C}$. Going into the specific user and item level, we can write the above equation as

$$x^*_{ui} = \Pi_{[0,1]} \left( \frac{c_{ui} - \nu_u}{\gamma} \right),$$

where $c_{ui} = a_{ui} - \mu r_{ui}$ and $\nu_u$ is the coefficient of $\nu$ corresponding to user $u$ which comes out by multiplication with $M$.

In [3], a trick is used to eliminate the user level variables such that the serving plan of the next epoch can be calculated using just the global dual variable $\mu$ from the previous epoch. However they would be limited to handling only specific types of constraints. The trick we used by removing the box constraints and introducing the indicator variable removes extra dependency on other dual variables and makes it easier to recover $x_{ui}$ from $\mu$. We begin with a short lemma.

**Lemma 1.** If $c_{u i_1} \geq c_{u i_2}$, then $x^*_{u i_1} \geq x^*_{u i_2}$.

**Proof.** Note that since $\nu_u$ is common between the two, it is easy to see that $c_{u i_1} - \nu_u \geq c_{u i_2} - \nu_u$. The result follows by observing that $\Pi_{[0,1]}(\cdot)$ maintains the order. □

Thus, if we sort $c_{u i_1} \geq c_{u i_2} \geq \ldots \geq c_{u i_M}$, then $x_{u i_1} \geq x_{u i_2} \geq \ldots \geq x_{u i_M}$. This implies, there exists $t_1$ and $t_2$ with $0 \leq t_1 < t_2 \leq M$, such that

$$x^*_{ui} = \begin{cases} 1, & \text{for } \ i = i_1, \ldots, i_{t_1}, \\ \frac{(c_{ui} - \nu_u)}{\gamma}, & \text{for } \ i = i_{t_1+1}, \ldots, i_{t_2}, \\ 0, & \text{for } \ i = i_{t_2+1}, \ldots, i_M. \end{cases}$$

Note further that from complementary slackness for optimality conditions we have $\nu_u > 0$ if and only if $\sum_{i} x_{ui} = K_u$. Thus using this equation we can solve for $\nu_u$. In fact it is easy to see that

$$\nu_u = -\frac{\gamma}{t_2 - t_1} (t_1 - K_u) + \frac{\sum_{i=t_1+1}^{t_2} c_{ui}}{t_2 - t_1}.$$

We can now formally write the algorithm as follows.
Algorithm 1 Optimal Primal from Dual
1: Input : Optimal dual solution \( \mu \) and an incoming user \( u \)
2: Output : Primal serving plan \( \{x_{ui}\} \)
3: Compute \( c_{ui} \) for all \( i \in I \)
4: Sort \( c_{ui} \) in decreasing order as \( c_{u11}, c_{u12}, \ldots, c_{u1M} \).
5: Generate all combinations of \((t_1, t_2)\) combinations such that \( 0 \leq t_1 < t_2 \leq M \).
6: for each pair \((t_1, t_2)\) do
7: \( \text{Compute } \nu \text{ by equation (7)} \)
8: \( \text{if all the following conditions hold then break from loop} \)
9: \( (c_{uit1} - \nu) / \gamma > 1 \)
10: \( (c_{uit1} - \nu) / \gamma < 1 \) and \( (c_{uit2} - \nu) / \gamma > 0 \)
11: \( (c_{uit2} - \nu) / \gamma < 0 \)
12: end if
13: end for
14: Compute \( x_{ui} \) using (6)
15: return \( \{x_{ui}\} \).

3.2 Generalizations of local constraint

Let us now consider a case of more general local constraints. In (2) if we want to ensure that each user \( u \) is shown at most \( K_u \) items of a particular type (belonging to a set \( J_u \)), then an additional set of constraints (one for each user) will be added:

\[
\begin{align*}
\text{Minimize} & \quad -x^T a + \gamma x^T x \\
\text{subject to} & \quad x^T r \leq R \\
& \quad \sum_{i \in S} x_{ui} \leq K_u, \quad \forall u \\
& \quad 0 \leq x \leq 1 
\end{align*}
\]

These local constraints are given by \( Px \leq k \) where \( P \in \mathbb{R}^{N \times MN} \). However, we might be interested in more generalized local constraints such as for \( J_{a0} \subset J_u \), we might want \( \sum_{i \in J_{a0}} x_{ui} \geq n_i^a \). Adding such constraint to the optimization problem introduces a new set of dual variables which we need to be eliminated to apply the above algorithm. This may not be possible in all cases since the sign of the combination of dual variables may not be known and hence the ordering of \( c_{ui} \) which translates to the ordering in \( x_{ui} \) breaks down. Here we present an efficient algorithm to work around this problem. Without loss of generality, we consider the local constraint at user level \( u \) by considering \( x_u \in K_u \), where the convex region \( K_u \) is defined by any sort of linear constraints. Now we can write the optimization problem as

\[
\begin{align*}
\text{Minimize} & \quad -x^T a + \frac{\gamma}{2} x^T x \\
\text{subject to} & \quad x^T r \leq R \\
& \quad x_u \in K_u, \quad \forall u. 
\end{align*}
\]

We transform this by introducing indicator variables as before.

\[
\begin{align*}
\text{Minimize} & \quad -x^T a + \frac{\gamma}{2} x^T x + \sum_u 1_{K_u}(x_u) \\
\text{subject to} & \quad r^T x \leq R.
\end{align*}
\]

Similar to (5), using the Lagrangian, it is easy to see that

\[
x_u = \Pi_{K_u} \left( \frac{(a-\mu r)_u}{\gamma} \right).
\]

This follows because we can write the entire convex domain of \( x \) as \( K_1 \times K_2 \times \ldots \times K_n \) and each \( x_u \) is the projection into the corresponding \( K_u \). This is the most important step in our case, because unless we are able to split the domain into user size, we cannot apply this decomposition.

Now once we know \( \mu \) and the user \( u \) we can calculate \( c_u = (a-\mu r)_u \). Then we only need to project \( c_u \) into \( K_u \). For most practical purposes, the number of candidate items considered for ranking (after initial filtering based on recency, quality and other basic filters) is at a much smaller scale. This leads us to the following QP optimization problem of reasonable scale,

\[
\begin{align*}
\text{Minimize} & \quad ||x_u - c_u||_2^2 \\
\text{subject to} & \quad x_u \in K_u, \\
\end{align*}
\]

which can be solved by any QP solver. The detailed generalized conversion technique is given next in Algorithm 2.

Algorithm 2 Generalized conversion algorithm
1: Input : Optimal dual solution \( \mu \) and an incoming user \( u \)
2: Output : Primal serving plan \( \{x_{ui}\} \)
3: Compute \( c_u = (a-\mu r)_u / \gamma \)
4: Solve the problem (9) to get \( x_u \).
5: return \( x_u \).

4. GENERALIZED GRAPH STRUCTURE

In this section we provide a mathematical framework based on directed acyclic graphs (DAG) which can help in deciding what kind of local constraints should be solved in an online setting. We begin by describing how we obtain a graph from the set of constraints of a general multi-objective optimization problem such as (8). We then elaborate an algorithm for getting an optimal split into offline and online problems.

4.1 Graph Construction

Let us begin with a few notation. Let \( x = (x_1, \ldots, x_n) \) be the optimization variable in \( n \)-dimensions. Furthermore, let \( I_n = \{1, \ldots, n\} \) and \( S^k \) denote subsets of \( I_n \) such that \( |S^k| = k \) for \( j = 1, \ldots, k \), \( s_k \) for some \( s_k \geq 1 \) and \( k \in \{1, \ldots, n\} \). Note that \( S^1 = I_n \). For ease of notation, let \( p, c \) denotes some parameters in the optimization problem, which may change in each constraint. However, for notational simplicity we use \( p, c \) as parameters in the optimization problem as

\[
\begin{align*}
\text{Minimize} & \quad f(x) \\
\text{subject to} & \quad \sum_{i \in I_n} x_{ip_i} \geq c_1^n \\
& \quad \sum_{i \in S^j_{n-1}} x_{ip_i} \geq c_j^{n-1} \text{ for } j = 1, \ldots, s_{n-1} \\
& \quad \ldots \\
& \quad x_i \geq c_1^i \text{ for } i = 1, \ldots, s_1.
\end{align*}
\]

Here we have constraints on a subset of variables. It is easy to see that the problem (8) is a specific case of (10).
Now we construct the DAG, $G = (V,E)$ as follows. Each set $S^k_i$ corresponds to a node $n^k_i$. For $\ell < k$, there is a directed edge from $n^\ell_i$ to $n^k_j$, if $S^\ell_j \subseteq S^k_j$ and there does not exist a $k'$ with $\ell < k' < k$ such that $S^{k'}_j \subseteq S^k_j$ for some $j$. There also exists an edge from $n^k_j$ to $n^\ell_i$, if, there exists an $x \in S^\ell_j \cap S^k_j$ and $x \notin S^\ell_i$ for any $j \in \{1, \ldots, s^\ell\}$ and any $k'$ with $\ell < k' < k$.

To make think clear let us observe the following example. Let $n = 7$. Assume,

- $S^1_2 = \{1, 2, 3\}$, $S^2_3 = \{4, 5, 6\}$
- $S^1_3 = \{1, 2\}$, $S^2_4 = \{3, 4\}$, $S^3_5 = \{6, 7\}$
- $S^1_1 = \{3\}$, $S^2_4 = \{4\}$, $S^3_5 = \{5\}$

The corresponding graph is given in Figure 1. Note that $S^2_3$ has two parents, $S^3_5$ has parents from two different levels and $S^1_1$ has parent in a which skips a level. Thus any such set of constraints can be converted into a DAG.

### 4.2 Splitting of offline and online problems

Here, we discuss a theoretical justification for splitting the optimization problem into an offline and online setting. The procedure, though may have been initially thought about just as a means to reduce computation time, we shall show here that it can have a statistical justification. Moreover, the analysis will give us an formal algorithm to split the optimization problem, instead of the past heuristics.

Consider $G = (V,E)$ to be the directed acyclic graph created from the construction in Section 4.1. We assume that for each node $i$ there is a population mean $\mu_i$, and covariance structure $\Sigma_i$ for the parameter in the constraint. Moreover, we assume that we have taken $n_i$ independent samples $p_{ij}$ for $j = 1, \ldots, n_i$ from this population to create a constraint of the form

$$\sum_{j=1}^{n_i} x_{ij} p_{ij} \geq c_i.$$  

Moreover, we assume that for a parent node, the population distribution is a mixture of its children distributions. Formally, the parent node has a mean and covariance given by the following result.

**Lemma 2.** Let the parent node $p$ have $d$ child nodes each with mean and covariance $(\mu_i, \Sigma_i)$ for $i = 1, \ldots, d$. Moreover assume that the distribution of $p$ is formed by mixing proportions $\alpha_i$ with $\sum_{i=1}^{d} \alpha_i = 1$. Then, the mean and covariance of the parent $(\mu_p, \Sigma_p)$ is given by

$$\mu_p = \sum_{i=1}^{d} \alpha_i \mu_i$$

$$\Sigma_p = \sum_{i=1}^{d} \alpha_i (\Sigma_i + (\mu_i - \mu_p)(\mu_i - \mu_p)^T).$$

**Proof.** Let $Y_i$’s be independent samples drawn from node $i$ and $Z$ be a random variable which takes value $i$ with probability $\alpha_i$ for $i = 1, \ldots, d$. Then $X = \sum_{i=1}^{d} Y_i \mathbb{I}(Z = i)$ is a random variable from the parent distribution. Thus, we have

$$\mu_p = \mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X|Z)) = \sum_{i=1}^{d} \alpha_i \mu_i$$

and

$$\Sigma_p = \mathbb{V}(X) = \mathbb{V}(\mathbb{E}(X|Z)) + \mathbb{E}(\mathbb{V}(X|Z))$$

$$= \sum_{i=1}^{d} \alpha_i \mathbb{I}(Z = i) \mu_i + \sum_{i=1}^{d} \alpha_i \Sigma_i$$

$$= \sum_{i=1}^{d} \alpha_i (\mu_i - \mu_p)(\mu_i - \mu_p)^T + \sum_{i=1}^{d} \alpha_i \Sigma_i.$$  

Hence the result follows. $\square$

Lemma 2 gives us a way to compute the variance at the parent node as a function of the child nodes. Using this result, we can compute the variance at each node $\Sigma_i$. While the expected error accrued from using the dual estimate corresponding to node $i$ is monototonically related to $\Sigma_i$, the actual function most likely does not have a closed analytic form. However, it can be estimated via sampling data. Let $X(\Sigma_i)$ denote the maximum eigenvalue of $\Sigma_i$ which acts as a proxy for the error. Let $t(n_i)$ denote the time required to solve the QP under node $n_i$. Then, we would want to keep the dual estimate obtained for the node if both $t(n_i)$ is large and $X(\Sigma_i)$ is small. We can use a convex combination of the two factors controlled by $w \in [0,1]$, and the resultant quantity is below a threshold $\beta$. This condition is expressed in Step 10 of Algorithm 3.

### 5. Practical Challenges in Scaling the Solution

In Sections 2, 3 and 4 we gave theoretical details regarding how to handle user level constraints. Although the mathematical framework is challenging and exciting, deploying such systems into production raises a lot more issues. In this section, we outline two primary challenges in this space and share our experiences in trying to overcome them. The first challenge was to scale a QP solver to handle a larger sample of data to get optimal dual variables – this was addressed by using operator splitting, an ADMM algorithm. The second challenge was using variance reduction techniques on the sampled data to reduce the variance of the dual estimates obtained from solving the QP.
The most expensive step is evaluating the prox function. Let the generic problem for operator splitting be written as form (1) can be written as (3). To solve the (3) in large-scale settings we employ the method of operator splitting [10]. The generic problem for operator splitting can be written as:

\[ \text{Minimize} \quad \phi(z) \]

subject to \( z \in D \).

The operator splitting algorithm can be outlined as:

\[
\begin{align*}
Z^{k+1/2} &= \text{prox}_\phi(Z^k - Z^k) \\
Z^{k+1} &= \Pi_D(Z^{k+1/2} + Z^k) \\
\tilde{Z}^{k+1} &= Z^k + Z^{k+1/2} - Z^{k+1},
\end{align*}
\]

where \( \text{prox}_\phi(v) = \arg\min_x \left( \phi(x) + \frac{\beta}{2} \|x - v\|^2 \right) \). In our case, the most expensive step is evaluating the prox function. Let \( \phi(z) = \frac{1}{2} z^T B z - \tilde{p}^T z \). Then it is easy to see that

\[
\begin{align*}
\text{prox}_\phi(v) &= \arg\min_x \left( \phi(x) + \frac{\beta}{2} \|x - v\|^2 \right) \\
&= \arg\min_x \left( \frac{1}{2} x^T B x - \tilde{p}^T x + \frac{\beta}{2} \|x - v\|^2 \right) \\
&= (B + \rho I)^{-1}(\tilde{p} + \rho v)
\end{align*}
\]

Thus in our specific case (3), we write the algorithm as follows:

\[
\begin{align*}
Z^{k+1/2} &= (B + \rho I)^{-1}(\tilde{p} + \rho(z^k - \tilde{z}^k)) \\
Z^{k+1} &= (Z^{k+1/2} + Z^k) \\
\tilde{Z}^{k+1} &= \tilde{z}^k + Z^{k+1/2} - Z^{k+1},
\end{align*}
\]

where \((x)_+ = x \text{ if } x \geq 0 \text{ element-wise, else } 0\). We run these steps till convergence and return \( \mu \) as the dual variable. The true convergence is measured by the following criteria. Define,

\[
\begin{align*}
r^{k+1} &= Z^{k+1/2} - Z^{k+1}, \\
s^{k+1} &= -\rho(z^{k+1} - Z^k),
\end{align*}
\]

where they can be regarded as the primal and dual residuals in the algorithm. We stop the algorithm, when both the residuals are small, i.e.,

\[
\|r^k\|_2 \leq \epsilon^{pri} \text{ and } \|s^k\|_2 \leq \epsilon^{dual}
\]

where the cutoff as obtained in [4] is given by

\[
\epsilon^{pri} = \sqrt{n\epsilon_{abs}} + \epsilon^{rel} \max\{\|z^{k-1/2}\|_2, \|z^k\|_2\},
\]

\[
\epsilon^{dual} = \sqrt{n\epsilon_{abs}} + \epsilon^{rel} \|\tilde{z}^k\|_2.
\]

Remark 1. A smaller value of \( \epsilon^{abs} \) and \( \epsilon^{rel} \) will lead to a more accurate solution but the algorithm will take much more time to converge. To get an approximate solution, we can only check the relative error in \( \mu \). If it is below a certain threshold we stop the algorithm. This is not exact, but we have seen that in most cases it gives substantial improvement in convergence time.

Remark 2. The scalability of the algorithm is dependent on finding the inverse of the matrix \( B + \rho I \). In most cases \( B \) is sparse, with sparsity ratio \( O(1/n) \) and with 64GB machine, we can find the sparse Cholesky decomposition of \( B + \rho I \) when \( n \leq 5 \times 10^3 \). For larger problems, we would need a machine with more memory.

Remark 3. We cannot use the Block Splitting algorithm [10] to solve this problem because the matrix \( B \) is not separable. We could have used it to solve the primal problem but in most cases we would not be able to recover the dual from the primal solution.

Using this technique we can solve the QP (3) with approximately \( 5 \times 10^8 \) variables on a single machine with 64 GB of RAM in a relatively small amount of time. This gives us a 100× scale up from using the some of the off-the-solvers such as quadprog in R. The convergence time for different problem sizes is tabulated in Table 4. For \( n = 10^7 \) and \( n = 10^8 \), each iteration takes about 1.7 and 20 seconds respectively. Thus, if we run the algorithm long enough to converge, we estimate that the number of iterations should be about \( 1.5 \times 10^8 \) and \( 1.5 \times 10^9 \), resulting in a total time of about 3 days and a year respectively.

5.2 Variance reduction of dual estimates

The variation in the dual variables that we obtain are directly related to the underlying \( M \) variate distribution from which \( p_u, r_u \) is drawn for \( u = 1, \ldots, N \). To get a sense of the joint covariance structure from an actual data, we show two plots in Figure 2. We consider a 6 variate distribution and the first plot is the joint distribution of the top 3 most occurring variables, i.e., they had most non-zero entries among all samples. The second figure shows the joint distribution of the 3 least occurring variables. We omit the axis labels to prevent disclosing sensitive information.

Even though the non-sparse variables show a concentrated structure, we see odd peaks for the more sparse variables.
Thus, when we consider the entire joint distribution the variation in $\mu$ is highly dependent on the covariance structure of $p$ and $r$. From Figure 2, it can be seen that the variance is quiet high because of the odd peaks. This calls for certain variance reduction techniques. Below we describe the three methods we have used and the resulting effect in the estimate of $\mu$.

5.2.1 Using linear moment matching

If we try to solve for $\mu$ as a function of $p, r$, the general structure can be written as

$$\mu = A(p, r)^{-1}f(p, r)$$

where $A(p, r)$ is a matrix and $f(p, r)$ is a vector both written as a function of the random variables $p$ and $r$. Studying the exact dependency of this function is extremely hard, and so we work on the heuristic that, if we can reduce the variation in $p, r$, we will reduce the variation in $\mu$. Towards that end, we use two control variates known as moment matching estimators. Since, we deal with $p$ and $r$ in the same way, we explain our technique only through $p$.

Assume the true mean and covariance of the distribution of $p$ is denoted by $\mu_p$ and $\Sigma_p$ both of which are unknown. Using the complete data, we estimate the mean of $p$, call it $\theta = \frac{1}{N} \sum_{u=1}^{N} p_u$ and the covariance matrix of $p$ call it $\Sigma_{full} = \frac{1}{N} \sum_{u=1}^{N}(p_u - \theta)(p_u - \theta)^T$. When we sample $n$ points $p_u$ from this distribution for our problem, instead of using the sampled $p_u$ in the optimization, we use two different modifications, viz.

$$\tilde{p}_u^1 = p_u + \theta - \tilde{p} \quad \forall \, u = 1, \ldots, n \quad (11)$$

$$\tilde{p}_u^2 = \theta + \Sigma_{full}^{-1/2}\Sigma_{sampled}^{-1/2}(p_u - \tilde{p}) \quad \forall \, u = 1, \ldots, n \quad (12)$$

where $\tilde{p} = \frac{1}{n} \sum_{u=1}^{n} p_u$ and $\Sigma_{sampled} = \frac{1}{n} \sum_{u=1}^{n}(p_u - \tilde{p})(p_u - \tilde{p})^T$ is the sample average and the sample covariance matrix respectively. Now we show few results for these two modified samples.

**Lemma 3.** For the modified sample $\tilde{p}_u^1$ given in (11) we have,

$$\mathbb{E}(\tilde{p}_u^1) = \mu_p$$

$$\mathbb{V}(\tilde{p}_u^1) \preceq \Sigma_p$$

where $A \preceq B$ implies $B - A$ is a positive semi-definite matrix.

**Proof.** The fact that $\tilde{p}_u^1$ is unbiased follows from the definition. To get the second assertion we evaluate the covariance $\tilde{p}_u^1$.

$$\mathbb{V}(\tilde{p}_u^1) = \mathbb{V}(\theta) + \mathbb{V}(p_u) + \mathbb{V}(\tilde{p}) + 2\mathbb{Cov}(\theta, p_u) - 2\mathbb{Cov}(p_u, \tilde{p})$$

$$= \frac{\Sigma_p}{n} + \Sigma_p + \frac{\Sigma_{full}}{n} + 2\frac{\Sigma_p}{n} - 2\frac{\Sigma_{full}}{n} - 2\frac{\Sigma_{full}}{n}$$

$$= \Sigma_p \left( 1 + \frac{1}{N} - \frac{1}{n} \right)$$

$$\preceq \Sigma_p$$

where the last line follows from the fact that $n < N$.

The second modification (12) does the moment matching for the variance of $p$, but calculating the exact variance of this sample is not possible in closed form. However, it can be shown that it is asymptotically unbiased as $n \to N$, by applying the Dominated Convergence Theorem.

5.2.2 Using product moment matching

Sometimes, we get much better results by using a product form of moment matching. In this case we have,

$$\tilde{p}_u^3 = p_u \times \frac{\theta}{\tilde{p}}$$

**Lemma 4.** For the modified sample $\tilde{p}_u^3$ given in (12) we have,

$$\mathbb{E}(\tilde{p}_u^3) = \mu_p$$

$$\mathbb{V}(\tilde{p}_u^3) \preceq \Sigma_p$$

where $A \preceq B$ implies $B - A$ is a positive semi-definite matrix.

**Proof.** The fact that $\tilde{p}_u^3$ is unbiased follows from the definition. To get the second assertion we evaluate the covariance $\tilde{p}_u^3$.

$$\mathbb{V}(\tilde{p}_u^3) = \mathbb{V}(\theta) + \mathbb{V}(p_u) + \mathbb{V}(\tilde{p}) + 2\mathbb{Cov}(\theta, p_u) - 2\mathbb{Cov}(p_u, \tilde{p})$$

$$= \frac{\Sigma_p}{n} + \Sigma_p + \frac{\Sigma_{full}}{n} + 2\frac{\Sigma_p}{n} - 2\frac{\Sigma_{full}}{n} - 2\frac{\Sigma_{full}}{n}$$

$$= \Sigma_p \left( 1 + \frac{1}{N} - \frac{1}{n} \right)$$

$$\preceq \Sigma_p$$

where the last line follows from the fact that $n < N$. 

Figure 2: The left and right panel shows the joint distribution of $p_i$ corresponding to the high and low density email keys respectively.
where both the multiplication and division is done co-ordinate wise. Boyle et. al. [5] show that the moment matching is asymptotically like using the known moments in control variates. Further details regarding moment matching and control variates can be found in [9].

6. RESULTS

In this section, we present our experimental results. We first show how the graph partitioning behaves as a function of time and accuracy. We then show results concerning the three different moment matching methods and how much accuracy we gain by solving the dual problem via operator splitting.

6.1 Graph Partitioning

Consider the following toy example. Suppose we have $N = 1024$. Assume that the constraints of the optimization problem can be written as a binary tree having $K = 10$ levels. The root corresponds to a global constraint. Every other node in the tree correspond to constraints on half of the users from its parent node. The leaves are individual user level constraints. Following the notation from Section 4.1, we can write this as,

\[
\begin{align*}
\text{Minimize} & \quad f(x) \\
\text{subject to} & \quad \sum_{i \in \mathcal{I}} x_i p_i \geq c_1^n \\
& \quad \sum_{i \in \mathcal{S}_j 10^{-k}} x_i p_i \geq c_j^{10^{-k}} \\
& \quad x_i \geq c_i \quad \text{for } i = 1, \ldots, 1024.
\end{align*}
\]

Now we perform a split at level $k$ for $k = 1, \ldots, K$. When $k = 1$, we consider dual corresponding to only the global constraint and solve two local problems having size 512. When $k = 2$, we consider the duals corresponding to the first three constraints and then solve 4 local QPs having size 256. We carry on this procedure for $k = 1, \ldots, K$. For each split, we obtain the mean square error (MSE) of the final objective value by repeated experiments using random samples. We also store the time taken to solve the online problem. Figure 3 shows how the MSE decays as travel up the tree.

Each point on the graph corresponds to a split in the tree. The smallest computation time is when we cut the graph at $k = K$ and our local projections are univariate problems. However, in this case, we get least accurate solution since the only effect of the new sample is in the final level projection resulting in low accuracy due to large variance of $p$. We also see that as we move up the tree, the accuracy increases along with an increase in computation time. To get the optimal cut-off we can use a weighted procedure such as Algorithm 3.

6.2 Large-scale Solver and Variance Reduction

Consider the problem of sending emails pertaining to a single email key. Suppose $r_{ui}$ is the probability that person $u$ will complain if he is sent email $i$. Let $p_{ui}$ is the probability that there will be a page view if email $i$ is sent and let $x_{ui}$ is the probability that email $i$ is sent to user $u$. The problem that we are interested in is basically to minimize the number of sends such that the complains is reduced but page views do not suffer much. Mathematically we can write this as,

\[
\begin{align*}
\text{Minimize} & \quad x^T 1 + \frac{\gamma}{2} (x - q)^T (x - q) \\
\text{subject to} & \quad \sum_{u,i} x_{ui} r_{ui} \leq g_0 \\
& \quad \sum_{u,i} x_{ui} p_{ui} \geq g_1 \\
& \quad 0 \leq x_{ui} \leq 1, \quad u = 1, \ldots, N; \quad i = 1, \ldots, M.
\end{align*}
\]

Let $\mu_0$ and $\mu_1$ be the dual variables corresponding to the complain and page views constraints which we are interested in. The previous method is unable to solve this in a large scale setting. The old infrastructure at Linkedin could solve this for about 10,000 variables to estimate ($\mu_0, \mu_1$). Their variance estimate is given in Table 5. We use our Operator splitting algorithm to solve the problem with a much larger data set ($\approx 10^6$ variables) on a single machine. We then perform the three variance reduction techniques and the results are tabulated in Table 5. It can be observed from Table 5 that by just increasing the sample size and solving a larger problem we have been able to reduce the variance of the dual variable $\mu_0$ by about 400x. The variance reduction techniques can further reduce the variance by $\approx 4\times$. It can be seen that we get substantial improvement in the dual estimates by following our procedure.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
Method & $\mu_0$ & $\mathcal{V}(\mu_0)$ & $\mu_1$ & $\mathcal{V}(\mu_1)$ \\
\hline
Old Method & 101.11 & 16401.07 & 121.14 & 8.55 \\
Operator Splitting & 15.85 & 40.04 & 120.41 & 0.42 \\
Using $\tilde{p}_1^1, \tilde{r}_1^1$ & 13.17 & 20.68 & 119.95 & 1.20 \\
Using $\tilde{p}_2^2, \tilde{r}_2^2$ & 12.96 & 12.48 & 120.21 & 0.99 \\
Using $\tilde{p}_3^3, \tilde{r}_3^3$ & 12.02 & 11.73 & 120.11 & 0.33 \\
\hline
\end{tabular}
\caption{Comparing the Variance Reduction methods}
\end{table}
7. DISCUSSION

Constrained optimization has proven to be a very successful formulation tool for several applications at LinkedIn. Since most of our products aim to improve more than one metric, and (perhaps more importantly) impact several others (both positively and negatively), MOO approaches are now commonplace in various forms and flavours. Some of these applications include:

- Feed modeling: The LinkedIn newsfeed is a very diverse distribution channel for various types of content. It surfaces articles shared by your network, job changes and anniversaries, profile updates, network updates (e.g., new connections made) among many other update types, in addition to serving advertisements. It is no surprise then that the feed ranking models have a significant impact on several objectives, including various forms of user engagement and revenue. The feed is also an application that is scored online (a user’s visit triggers the scoring pipeline). The advances made in this paper can benefit the newsfeed ranking application by enabling user-level constraints like “show no more than 3 ads to a user in a day”. Scalability benefits are also applicable since the scale of the feed problem is in billions of data points, and both our scalability solutions can make MOO much more accurate and hence useful for the LinkedIn feed.

- Email and push notification portfolio optimization: Emails and push notifications are the most important (company-initiated) communication channel for any social network. Given the possible diversity of the channel, it is no surprise that the email and push notifications portfolio together drive a large plethora of metrics – which makes it a perfect fit for MOO. Email portfolio optimization is a largely offline application (some parts can be user-action triggered and hence could be called borderline online). User level constraints are also very useful here, for instance to limit the number of emails that we send to any particular user should have a strict upper cap, so as not to overwhelm. The scale of the email problem is also into billions of data points, and hence can benefit greatly from both larger sample sizes and variance reduction.

Besides these two applications, there are many others where MOO is applicable and is either already being used (or should be used). After all, it is almost unthinkable that any application would care about just one objective. Instead of using heuristics or speculative weights, MOO is a much more principled and optimal approach to navigate the complex tradeoff.

Our contributions not only advance the theoretical state-of-the-art (by devising a way to exactly handle linear local constraints), but also make it much more scalable to obtain low variance dual estimates from 100× larger sample sizes. Together, these advances should make MOO solutions amenable for further wide-scale adoption.

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