Single-particle potential from resummed ladder diagrams

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Abstract
A recent work on the resummation of fermionic in-medium ladder diagrams to all orders is extended by calculating the single-particle potential \( U(p, k_f) \) for hole-states with \( p < k_f \). The perturbative contributions at any order \( a^n \) in the scattering length are deduced and checked against known analytical results at order \( a^1 \) and \( a^2 \). The single-particle potential \( U(k_f, k_f) \) at the Fermi surface as obtained via a non-perturbative resummation of the combined particle and hole ladders is shown to satisfy the Hugenholtz-Van-Hove theorem. A strong dependence of \( U(p, k_f) \) on the momentum \( p \) and the dimensionless coupling strength \( a k_f \) is found. The unitary limit \( a \to \infty \) is studied as a special case. Furthermore, the same analysis is performed for the resummed particle-particle ladder diagrams, where the continuation of \( U(p, k_f) \) into the region outside the Fermi surface \( p > k_f \) is also considered.

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1 Introduction and summary

Dilute degenerate many-fermion systems with large scattering lengths \( a \) are of interest, e.g. for modeling the low-density behavior of nuclear and neutron star matter. Because of the possibility to tune magnetically atomic interactions, ultracold fermionic gases provide an exceptionally valuable tool to explore the non-perturbative many-body dynamics involved in the crossover from the superconducting to the Bose-Einstein condensed state (for a recent comprehensive review of this fascinating field, see ref.[1]). Of particular interest in this context is the so-called unitary limit, in which the two-body interaction has the critical strength to support a bound-state at zero energy. As a consequence of the diverging scattering length, \( a \to \infty \), the strongly interacting many-fermion system becomes scale-invariant. Its ground-state energy is then determined by a single universal number, the so-called Bertsch parameter \( \xi \), which measures the ratio of the energy per particle \( \bar{E}(k_f)^{\infty} \) to the (free) Fermi gas energy, \( \bar{E}(k_f)^{(0)} = 3 k_f^2 / 10 M \). Here, \( k_f \) denotes the Fermi momentum and \( M \) the fermion mass.

In a recent work [2] the complete resummation of the combined particle-particle and hole-hole ladder diagrams generated by a contact-interaction proportional to the scattering length \( a \) has been achieved. A key to the solution of this (restricted) problem has been a different organization of the many-body calculation from the start. Instead of treating (propagating) particles and holes separately, these are kept together and the difference to the propagation in vacuum is measured by a “medium-insertion”. In that organizational scheme the pertinent in-medium loop is complex-valued, and therefore the contribution to the energy per particle \( \bar{E}(k_f)^{\infty} \) at order \( a^n \) is not directly obtained from the \((n-1)\)-th power of the in-medium loop. However, after reinstalling the symmetry factors \( 1/(j+1) \) which belong to diagrams with \( j+1 \) double medium-insertions, a real-valued expression is obtained for all orders \( a^n \). Known results about the low-density expansion [3, 4] up to and including order \( a^4 \) could be reproduced with improved numerical accuracy. The emerging series in \( a k_f \) could even be summed to all orders in the form of a double-integral over an arctangent-function. In that explicit representation the unitary limit,

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Moreover, there is the imaginary single-particle potential $W$ particle and hole-hole ladder diagrams there remains the challenge to derive at third and higher by an iterated contact-interaction is of course treated only partially. For the combined particle-widths of hole and particle excitations and satisfies the constraint quantity. We hope to come back to these interesting problems in a future publication.

Results for the momentum dependence are presented up to 8th order. The unitary limit $\delta$-function terms, which arise from differentiating the discontinuity of the arctangent-function at infinity, play a crucial role for the actual numerical agreement. In the next step the perturbative contributions to $U(p, k_f)$ at any order $a^n$ in the scattering length are deduced and checked against known analytical results at order $a^1$ and $a^2$. Results for the momentum dependence are presented up to 8th order. The unitary limit $a \to \infty$ is studied as a special case and a rather peculiar result for the momentum dependence of the potential $U(p, k_f)^{(\infty)}$ is found. Section 3 is devoted to the same analysis of the particle-particle ladder diagrams which can be resummed to all orders in the form of a geometrical series. For this simpler subclass of diagrams the continuation of the single-particle potential $U(p, k_f)$ into the region outside the Fermi surface $p > k_f$ becomes feasible with just a few changes of the formalism. Again, the known analytical result at second order ($\sim a^2$) provides a crucial check. Having available a non-perturbative formula for $U(p, k_f)$ in the region $p > k_f$, results for the momentum dependence are presented up to 5th order in $a$ and in the unitary limit $a \to \infty$.

With the results presented in this work the topic of the single-particle potential generated by an iterated contact-interaction is of course treated only partially. For the combined particle-particle and hole-hole ladder diagrams there remains the challenge to derive at third and higher orders the correct continuation of $U(p, k_f)$ into the region $p > k_f$ outside the Fermi sphere. Moreover, there is the imaginary single-particle potential $W(p, k_f)$ which describes the decay widths of hole and particle excitations and satisfies the constraint $W(k_f, k_f) = 0$. An approach that starts at a deeper level than the energy density seems to be necessary in order to access this quantity. We hope to come back to these interesting problems in a future publication.
2 Real single-particle potential from fermionic in-medium ladder diagrams

In perturbation theory the energy density of an interacting many-fermion system is represented by closed multi-loop diagrams with medium-insertions [2] accounting for the presence of the filled Fermi sea. By adding a test-particle with momentum $\vec{p}$ one can extract from the change of the energy density the single-particle self-energy generated by interactions with the fermionic medium. This procedure is realized formally by the substitution [8]:

$$\theta(k_f - |\vec{P}_j|) \rightarrow \theta(k_f - |\vec{p}_j|) + 4\pi^3 \eta \delta^3(\vec{p}_j - \vec{p}),$$

(1)

with $\eta$ an infinitesimal parameter. For a hole-state in the Fermi sea one sets $\eta = -1/V$ with $V$ the large volume of the system. To linear order in $\eta$ the energy density changed then as:

$$\frac{k_f^3}{3\pi^2} E(k_f) \rightarrow \frac{k_f^3}{3\pi^2} E(k_f) + \eta U(p, k_f),$$

(2)

where $E(k_f)$ denotes the energy per particle and $U(p, k_f)$ gives the real part of the single-particle potential. In the following this construction of the single-particle potential $U(p, k_f)$ is carried out for the combined particle-particle and hole-hole ladder diagrams [2] that are generated by a contact interaction proportional to the scattering length $a$ to all orders. We will restrict ourselves to the momentum region $p < k_f$ inside the Fermi sphere. The sign-convention for $a$ is chosen is such that a positive scattering length $a > 0$ corresponds to attraction.

2.1 Modifications of the in-medium loop

The basic quantity in order to achieve the resummation of ladder diagrams to all orders in ref.[2] has been the complex-valued in-medium loop. Therefore, we calculate first the modifications of the in-medium loop which arise from the introduction of the test-particle. It is convenient to work with the half-sum $\vec{P} = (\vec{p}_1 + \vec{p}_2)/2$ and half-difference $\vec{q} = (\vec{p}_1 - \vec{p}_2)/2$ of two momenta inside the Fermi sphere, $|\vec{p}_{1,2}| < k_f$. The real part of the in-medium loop comes from the (middle) diagram in Fig. 1 with one medium-insertion and after the substitution specified in eq.(1) it reads:

$$\text{Re } B_1 = -4\pi a \int \frac{d^3 l}{(2\pi)^3} \frac{1}{l^2 - \vec{q}^2} \left\{ \theta(k_f - |\vec{P} - \vec{l}|) + \theta(k_f - |\vec{P} + \vec{l}|) \\
+ 4\pi^3 \eta \left[ \delta^3(\vec{P} - \vec{l} - \vec{p}) + \delta^3(\vec{P} + \vec{l} - \vec{p}) \right] \right\}.$$

(3)
The contribution linear in $\eta$ is simply twice the same energy denominators and after averaging over the directions of $\vec{p}$ one finds:\footnote{Since only terms linear in $\eta$ are relevant this averaging can be done at an early stage of the calculation.}

$$\text{Re} \, \bar{B}_1 = -\frac{ak_f}{\pi} \{ R(s, \kappa) + \eta \pi^2 k_f^{-3} \tilde{R}(s, \kappa, x) \},$$

(4)

with the logarithmic functions:

$$R(s, \kappa) = 2 + \frac{1}{2s} [1 - (s + \kappa)^2] \ln \frac{1 + s + \kappa}{1 - s - \kappa} + \frac{1}{2s} [1 - (s - \kappa)^2] \ln \frac{1 + s - \kappa}{1 - s + \kappa},$$

(5)

and

$$\tilde{R}(s, \kappa, x) = \frac{1}{sx} \ln \frac{(s + x)^2 - \kappa^2}{(s - x)^2 - \kappa^2}, \quad \tilde{R}(s, \kappa, 0) = \frac{4}{s^2 - \kappa^2},$$

(6)

written in terms of the dimensionless variables $s = P/k_f$, $\kappa = q/k_f$ and $x = p/k_f$. Note that $x$ lies in the interval $0 < x < 1$, and $s$ and $\kappa$ are subject to the constraint $s^2 + \kappa^2 < 1$.

Next, we consider the imaginary part of the in-medium loop. It is generated by all three diagrams in Fig. 1 and after inclusion of the “perturbation” by the test-particle it reads:

$$\text{Im}(B_0 + B_1 + B_2) = 4\pi^2 a \int \frac{d^3l}{(2\pi)^3} \delta(|\vec{l}^2 - \vec{q}^2|) \left[ \theta(k_f - |\vec{P} - \vec{l}|) + 4\pi^3 \eta \, \delta^3(\vec{P} - \vec{l} - \vec{p}) \right]$$

$$\times \left[ \theta(k_f - |\vec{P} + \vec{l}|) + 4\pi^3 \eta \, \delta^3(\vec{P} + \vec{l} - \vec{p}) \right].$$

(7)

Here, we have already dropped an analogous term of the form \{1 - $[\theta ... + \eta ...]$$\}\{1 - [\theta ... + \eta ...]$$\}$ whose phase space vanishes identically by Pauli-blocking and energy conservation (see herefore the detailed discussion in section 3 of ref.[2]). The condition that all three involved momenta $\vec{p}_1, \vec{p}_2$ and $\vec{p}$ lie inside the Fermi sphere is in fact crucial for the validity of this argument. Averaging again over the directions of $\vec{p}$, one finds to linear order in $\eta$:

$$\text{Im}(\bar{B}_0 + \bar{B}_1 + \bar{B}_2) = \frac{B_2}{2i} = ak_f \{ I(s, \kappa) + \eta \pi^2 k_f^{-3} \tilde{I}(s, \kappa, x) \},$$

(8)

with the piecewise defined functions:

$$I(s, \kappa) = \kappa \, \theta(1 - s - \kappa) + \frac{1}{2s} (1 - s^2 - \kappa^2) \, \theta(s + \kappa - 1),$$

(9)

and

$$\tilde{I}(s, \kappa, x) = \frac{1}{sx} \theta(s + \kappa - x) \, \theta(x^2 - (s - \kappa)^2) \, \theta(1 + x^2 - 2(s^2 + \kappa^2)), \quad \tilde{I}(s, \kappa, 0) = \frac{2}{s} \theta(1 - 2s) \, \delta(s - \kappa),$$

(10)

(11)

where the conditions $0 < x < 1$ and $s^2 + \kappa^2 < 1$ apply additionally. For the purpose of illustration the support of the function $\tilde{I}(s, \kappa, x)$ is shown by the grey area in Fig. 2. This region in the $sk$-plane is pieced together by a rectangle and a circular wedge.

### 2.2 Construction of the real single-particle potential

Having available the real and imaginary part of the in-medium loop with corrections linear in $\eta$, the single-particle potential $U(p, k_f)$ arising from ladder diagrams can be constructed in a one-step process from the known expression for the energy density. There are two contributions to
Figure 2: The support of the function \( \hat{I}(s, \kappa, x) \) in the \( s\kappa \)-plane consists of a rectangle and a circular wedge. The rectangle has side-lengths \( \sqrt{2}x \) and \( (1-x)/\sqrt{2} \), while the radius of the boundary circle is \( \sqrt{(1+x^2)/2} \).

The function \( U(p, k_f) \) of different “kinematical” origin. The first contribution, \( U(p, k_f)_{\text{ext}} \), comes from the last two momentum-space integrations that are introduced by the closing of open ladder diagrams. The interesting term linear in \( \eta \) leads to an integral over \( \theta(k_f - |\vec{p}_1|) \delta^3(\vec{p}_2 - \vec{p}) \) times the resummed energy density evaluated at \( \eta = 0 \). The pertinent integral over a radius and a directional cosine can be transformed into the variables \( (s, \kappa) \) and in the course of this transformation the function \( s^2 \kappa \hat{I}(s, \kappa, x) \) appears as the appropriate weighting function. The second contribution, \( U(p, k_f)_{\text{int}} \), is an integral over \( \theta(k_f - |\vec{p}_1|) \theta(k_f - |\vec{p}_2|) \) (i.e. the product of two Fermi spheres) where the integrand is equal to the first-order expansion coefficient in \( \eta \) of the resummed energy density. The weighting function for this integral is \( s^2 \kappa I(s, \kappa) \) (see eq.(15) in ref.[2]). It turns out that the first contribution \( U(p, k_f)_{\text{ext}} \) (involving an arctangent-function) gets cancelled by an identical term in \( U(p, k_f)_{\text{int}} \) of opposite sign. After this simplification one arrives at the following concise double-integral representation for the real single-particle potential as derived from the resummed particle-particle and hole-hole ladder diagrams:

\[
U(p, k_f) = \frac{8ak_f^3}{M} \int_0^1 ds s^2 \int_0^{1-s^2} d\kappa \kappa \left\{ \frac{ak_f \hat{R}(s, \kappa, x)I(s, \kappa) - \hat{I}(s, \kappa, x)[\pi + ak_f R(s, \kappa)]}{[\pi + ak_f r(s, \kappa)]^2 + [ak_f \pi I(s, \kappa)]^2} \right. \\
\left. - \frac{1}{ak_f} \hat{R}(s, \kappa, x) \delta \left( \frac{\pi}{ak_f} + R(s, \kappa) \right) \right\}.
\]

The occurrence of the \( \delta \)-function term in eq.(12) is very subtle. A careful inspection of the mathematical expressions reveals that it is produced by differentiating the discontinuity of the arctangent-function at infinity. The arctangent-function (providing the resummation-formula to all orders, see eq.(14) in ref.[2]) makes a jump by \(-\pi\), when the denominator \( R(s, \kappa) + \pi/ak_f + O(\eta) \) of its argument passes through zero from positive to negative values. The \( \delta \)-function term\(^3\) is

\(^3\)The \( \delta \)-function term affects also the calculation of the parameter \( \zeta \) in section 6 of ref.[2]. With the corrected integrand \( I/(R^2 + \pi^2 P^2) - \delta(R) \) one obtains the (small) value \( \zeta = 0.1163 \).
Figure 3: The function $R(s, \kappa)$ defined in eq.(5) is zero along the (lower) dashed-dotted line. The function $F(s, \kappa)$ defined in eq.(25) is zero along the (upper) dashed line.

treated numerically by first searching the line $\kappa(s)$ on which $R(s, \kappa(s)) = -\pi/ak_f$ and then integrating over the appropriate $s$-interval with the weighting function $|\partial R(s, \kappa)/\partial \kappa|^{-1}$. The line of zeros of the function $R(s, \kappa)$ inside the unit quarter disc is shown by the (lower) dashed-dotted line in Fig. 3. This curve starts at $s_0 = 0$, $\kappa_0 = 0.83356$ and ends at $s_1 = 0.55243$, $\kappa_1 = 0.83356 = \kappa_0$. Alternatively, the $\delta$-function term can be treated in a regularized form: $\pi \delta(X) = \lim_{\epsilon \to 0^+} \epsilon/(X^2 + \epsilon^2)$. One finds that finite $\epsilon \approx 10^{-3}$ give sufficiently accurate numerical results.

### 2.3 Hugenholtz-Van-Hove theorem

An important constraint on the single-particle potential is given by the Hugenholtz-Van-Hove theorem [9]. It states that the total single-particle energy $U(k_f, k_f) + k_f^2/2M$ at the Fermi surface $p = k_f$ is equal to the chemical potential. By a general thermodynamical relation the chemical potential is equal to the derivative of the energy density $\rho(\bar{E}(k_f) + 3k_f^2/10M)$ with respect to the particle density $\rho = k_f^3/3\pi^2$. We proof now that the Hugenholtz-van-Hove theorem is satisfied in the present non-perturbative calculation. The starting point is the resummed (interaction) energy per particle as given by eq.(14) in ref.[2]. First, one takes a derivative with the respect to $k_f$ in that given double-integral representation. Then, one remembers that the (resummed) energy density was originally an integral over the product of two Fermi spheres, $\theta(k_f - |\vec{p}_1|) \theta(k_f - |\vec{p}_2|)$, and differentiates separately the integration boundaries and the integrand with respect to $k_f$. The described procedure translates into the following sequence of equations:

$$
\bar{E}(k_f) + \frac{k_f}{3} \frac{\partial \bar{E}(k_f)}{\partial k_f} = -\frac{8k_f^2}{\pi M} \int_0^1 ds \int_0^{\sqrt{1-s^2}} d\kappa \left\{ 5 \arctan \frac{ak_f I(s, \kappa)}{1 + ak_f \pi^{-1} R(s, \kappa)} \right\}
$$

$$
+ \frac{ak_f I(s, \kappa)}{[1 + ak_f \pi^{-1} R(s, \kappa)]^2 + [ak_f I(s, \kappa)]^2} - \frac{\pi^2}{ak_f} \delta \left( \frac{\pi}{ak_f} + R(s, \kappa) \right)
$$
\[ = \frac{8ak^3}{M} \int_0^1 ds \int_0^{\sqrt{1-s^2}} dk \kappa \left\{ \frac{ak_f \hat{R}(s, \kappa, 1)I(s, \kappa) - \hat{I}(s, \kappa, 1)[\pi + ak_f R(s, \kappa)]}{[\pi + ak_f R(s, \kappa)]^2 + [ak_f \pi I(s, \kappa)]^2} \right\} = U(k_f, k_f), \quad (13) \]

where the equality of the initial and final term is precisely the statement of the Hugenholtz-Van-Hove theorem [9]. In order to establish the agreement with \( U(k_f, k_f) \) the following identities have been instrumental:

\[ \frac{\partial}{\partial k_f} [k_f R(P/k_f, q/k_f)] = \frac{1}{s} \ln \left( \frac{(s+1)^2 - \kappa^2}{(s-1)^2 - \kappa^2} \right) = \hat{R}(s, \kappa, 1), \quad (14) \]

\[ \frac{\partial}{\partial k_f} [k_f I(P/k_f, q/k_f)] = \frac{1}{s} \theta(s + \kappa - 1) = \hat{I}(s, \kappa, 1), \quad (15) \]

where the derivative with respect to \( k_f \) is taken at fixed \( P \) and \( q \). Moreover, the constraint \( s^2 + \kappa^2 < 1 \) holds. Note that the arctangent-function in eq.(13) refers to the usual branch with odd parity, \( \arctan(-X) = -\arctan X \), and values in the interval \([-\pi/2, \pi/2]\). Other branches of the arctangent-function are excluded by the weak coupling limit \( a \to 0 \), which has to give zero independent of the sign of the scattering length \( a \). The \( \delta \)-function terms in both double-integral expressions in eq.(13) are crucial for the actual numerical validity of the Hugenholtz-Van-Hove theorem. We have examined this over a wide range of positive and negative values of the dimensionless coupling strength \( ak_f \). In fact the violation of the Hugenholtz-Van-Hove theorem without the \( \delta \)-function terms has given the hint that there is this subtlety in differentiating the arctangent-function (as explained in the previous subsection).

The slope of the single-particle potential at the Fermi surface \( p = k_f \) determines the density-dependent effective mass \( M^*(k_f) \) of the (stable) quasi-particle excitations at the Fermi surface. The corresponding relation for the effective mass reads:

\[ \frac{1}{M^*(k_f)} = \frac{1}{M} + \frac{1}{k_f} \left. \frac{\partial U(p, k_f)}{\partial p} \right|_{p=k_f}, \quad (16) \]

with \( M \) the free fermion mass.

### 2.4 Perturbative expansion

Given the closed-form expression for the single-particle potential \( U(p, k_f) \) in eq.(12), one can expand it in powers of the scattering length \( a \). It is convenient to scale out the Fermi energy \( k_f^2/2M \) and to use a dimensionless expansion parameter, such that the perturbative series reads:

\[ U(p, k_f) = \frac{k_f^2}{2M} \sum_{n=1}^{\infty} (-ak_f)^n \Phi_n(x), \quad (17) \]

with \( x = p/k_f \). Note that the \( \delta \)-function term in eq.(12) proportional to \( \delta(\pi + ak_f R(s, \kappa)) \) does not contribute at any order in the expansion in powers of \(-ak_f\). In this sense it represents a truly non-perturbative term. The dimensionless functions \( \Phi_n(x) \) introduced in eq.(17) are calculated as follows:

\[ \Phi_n(x) = \frac{16}{\pi^{n+1}} \int_0^1 ds \int_0^{\sqrt{1-s^2}} dk \kappa \text{Im}\left\{ \hat{R}(s, \kappa, x) + i\pi \hat{I}(s, \kappa, x) [R(s, \kappa) + i\pi I(s, \kappa)]^{n-1} \right\}. \quad (18) \]
The first and second order contributions to the single-particle potential are known from the classical work by Galitskii [10]. The corresponding functions $\Phi_1(x)$ and $\Phi_2(x)$ have the form:

$$\Phi_1(x) = \frac{4}{3\pi},$$

and

$$\Phi_2(x) = \frac{4}{15\pi^2} \left\{ 11 - 2x^4 \ln \frac{1 - x^2}{x^2} + \frac{10}{x}(1 - x^2) \ln \frac{1 + x}{1 - x} - \frac{2}{x}(2 - x^2)^{5/2} \ln \frac{1 + x \sqrt{2 - x^2}}{1 - x^2} \right\},$$

with $x$ constrained to the interval $0 < x < 1$. The elaborate function $\Phi_2(x)$ has the boundary values $\Phi_2(0) = \frac{4}{\pi^2}$ and $\Phi_2(1) = 4(11 - 2 \ln 2)/15\pi^2$, and the slope $\Phi_2'(1) = 16(1 - 7 \ln 2)/15\pi^2$ at $x = 1$. The agreement at first order follows from:

$$\int ds s^2 \int d\kappa \kappa \hat{I}(s, \kappa, x) = \frac{(1 - x^2) + x^2}{12} = \frac{1}{12},$$

where the contributions of the rectangle and circular wedge to the double-integral are individually given. In the same way, the double-integral representation of $\Phi_2(x)$ in eq.(18) is in perfect numerical agreement (at the 6-digit level) with the analytical expression written in eq.(20). While results for the energy per particle $\bar{E}(k_f)$ are available up to fourth order [3, 4], the single-particle potential $U(p, k_f)$ has so far not been computed beyond second order. At this point the present calculation provides a multitude of new results.

Fig. 4 shows the $x$-dependence of the functions $\Phi_n(x)$ for $n = 1, 2, 3, 4$, and the subsequent results up to $n = 8$ are displayed in Fig. 5. Expect for the trivial constant behavior at $n = 1$, one observes a decrease of the $n$-th order potential-function as one moves from $x = 0$ (bottom of the Fermi sea) to $x = 1$ (at the Fermi surface). There is also a tendency that higher-order functions get smaller in magnitude. Precise numerical values of the boundary values $\Phi_n(0)$ and $\Phi_n(1)$ are listed in Table 1 up to order $n = 8$. The values $\Phi_n(0)$ have been computed by using the explicit expressions for $\hat{R}(s, \kappa, 0)$ and $\hat{I}(s, \kappa, 0)$ written in eqs.(6,11). The right boundary values $\Phi_n(1)$ are furthermore determined by the Hugenholtz-Van-Hove theorem. One has the relation $\Phi_n(1) = (2/\pi)^n (n + 5) c_n/9$, where the coefficients $c_n$ have been defined by the $k_f$-expansion of $\bar{E}(k_f)$ in eq.(19) of ref.[2]. Inserting the numerical values of $c_n$ listed in eq.(20) of ref.[2] one finds perfect agreement with the directly calculated values of $\Phi_n(1)$. This demonstrates that the Hugenholtz-Van-Hove theorem is fulfilled order by order without the $\delta$-function term in eq.(12).

2.5 Unitary limit

The unitary limit $a \to \infty$ is of special interest since in this limit the strongly interacting many-fermion system becomes scale invariant. Returning to the expression for the resummed single-
Figure 5: Momentum dependence of perturbative contributions to the single-particle potential.

| \( n \) | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|
| \( \Phi_n(0) \) | 0.42441 | 0.40528 | 0.44788 | 0.42857 | 0.38638 | 0.33254 | 0.28779 | 0.25073 |
| \( \Phi_n(1) \) | 0.42441 | 0.25975 | 0.20153 | 0.15744 | 0.13314 | 0.11153 | 0.09959 | 0.08573 |
| \( \Psi_n(0) \) | 0.42441 | 0.40528 | 0.35745 | 0.29486 | 0.23203 | 0.17676 | 0.13168 | 0.09658 |
| \( \Psi_n(1) \) | 0.42441 | 0.25975 | 0.17083 | 0.11493 | 0.07872 | 0.05444 | 0.03797 | 0.02663 |

Table 1: Boundary values of the functions \( \Phi_n(x) \) and \( \Psi_n(x) \) describing the momentum dependence of perturbative contributions to the single-particle potential. The values \( \Phi_n(1) \) and \( \Psi_n(1) \) are determined by the Hugenholtz-Van-Hove theorem.
the outcome is completely parameterfree. One can speculate that the restrictive treatment of the strongly interacting many-fermion system as a normal Fermi-liquid were responsible, and the observed extreme momentum dependence of $U(p, k_f)^{(\infty)}$ could be an indication of the inherent pairing instability. For comparison, Fig. 7 shows the momentum dependence of the single-particle potential $U(p, k_f)$ at finite values of the coupling strength $ak_f = \pm 1, \pm 2$. One observes strong dependences on the rescaled momentum $x = p/k_f$ and the dimensionless coupling strength $ak_f$. Similar to the energy per particle $\bar{E}(k_f)$ (see Fig. 5 in ref.[2]), the strongest variations occur for negative values of $ak_f$.

Figure 7: Momentum dependence of the single-particle potential for different coupling strengths $ak_f = \pm 1, \pm 2$. The Fermi energy $k_f^2/2M$ has been scaled out.

3 Particle-particle ladders diagrams only

In this section we perform the analogous construction of the real single-particle potential $U(p, k_f)$ for the particle-particle ladder diagrams. This simpler subclass of diagrams can be summed to all
orders in the form of a geometrical series \[5\]. The continuation of \(U(p, k_f)\) into the region \(p > k_f\) outside the Fermi surface will also be considered in this case.

### 3.1 Single-particle potential inside the Fermi sphere

The starting point is again the pertinent in-medium loop (also called particle-particle bubble function). With inclusion of the test-particle with momentum \(\vec{p}\) (see eq.(1)), the in-medium loop takes the form:

\[
B = 4\pi a \int \frac{d^3l}{(2\pi)^3} \frac{1}{l^2 - \vec{q}^2} \left\{ \left[ 1 - \theta(k_f - |\vec{P} - \vec{l}|) - 4\pi^3\eta \delta^3(\vec{P} - \vec{l} - \vec{p}) \right] \times \left[ 1 - \theta(k_f - |\vec{P} + \vec{l}|) - 4\pi^3\eta \delta^3(\vec{P} + \vec{l} - \vec{p}) \right] - 1 \right\},
\]

where the last term \(-1\) eliminates the vacuum divergence. Averaging over the directions of \(\vec{p}\), one gets to linear order in \(\eta\):

\[
\tilde{B} = -\frac{ak_f}{\pi} \left\{ F(s, \kappa) + \eta \pi^2 k_f^3 \tilde{F}(s, \kappa, x) \right\},
\]

with the logarithmic functions:

\[
F(s, \kappa) = 1 + s - \kappa \ln \frac{1 + s + \kappa}{1 + s - \kappa} + \frac{1}{2s} \frac{1}{1 - s^2 - \kappa^2} \ln \frac{(1 + s)^2 - \kappa^2}{1 - s^2 - \kappa^2},
\]

and

\[
\tilde{F}(s, \kappa, x) = \frac{1}{sx} \left\{ \theta(2s - 1) \theta(2s - 1 - x) \ln \frac{(s + x)^2 - \kappa^2}{(s - x)^2 - \kappa^2} + \theta(x - |2s - 1|) \ln \frac{2[(s + x)^2 - \kappa^2]}{1 + x^2 - 2(s^2 + \kappa^2)} \right\},
\]

\[
\tilde{F}(s, \kappa, 0) = \frac{4\theta(2s - 1)}{s^2 - \kappa^2}.
\]

The dimensionless variables \(s, \kappa, x\) are subject to the constraints \(0 < x < 1\) and \(s^2 + \kappa^2 < 1\). Note that a factor \(1 - \theta(k_f - |2\vec{P} - \vec{p}|)\) is involved in the angular averaging procedure that generates the function \(\tilde{F}(s, \kappa, x)\). This feature requires a separate study of the cases \(0 < s < 1/2\) and \(1/2 < s < 1\).

Following the construction of the single-particle potential via the energy density functional as described in section 2.2, one obtains the following result from the resummed particle-particle ladder diagrams:

\[
U(p, k_f) = \frac{8ak_f^3}{M} \int_0^1 ds s^2 \int_0^{\sqrt{1-s^2}} dk \kappa \left\{ \frac{ak_f \tilde{F}(s, \kappa, x)I(s, \kappa)}{[\pi + ak_f F(s, \kappa)]^2} - \frac{\tilde{I}(s, \kappa, x)}{\pi + ak_f F(s, \kappa)} \right\},
\]

where the two summands correspond to the decomposition \(U(p, k_f)_{\text{int}} + U(p, k_f)_{\text{ext}}\). The denominator functions in eq.(28) possess lines of zeros and therefore they have to be interpreted as (regularized) distributions: \(X^{-\nu} = \text{Re} \lim_{\eta \to 0} (X + i\epsilon)^{-\nu}\) for \(\nu = 1, 2\). The line of zeros of the function \(F(s, \kappa)\) inside the unit quarter disc is shown by the (upper) dashed line in Fig. 3. This curve starts at \(s_0 = 0, \kappa_0 = 0.83356\) and ends at \(s_1 = 0.40865, \kappa_1 = 0.91269\).

In order to proof the validity of the Hugenholtz-Van-Hove theorem for the single-particle potential \(U(k_f, k_f)\) at the Fermi surface, the following identity is now instrumental:

\[
\frac{\partial}{\partial k_f} [k_f F(P/k_f, q/k_f)] = \frac{1}{s} \ln \frac{(s + 1)^2 - \kappa^2}{1 - s^2 - \kappa^2} = \tilde{F}(s, \kappa, 1).
\]

(29)
The perturbative expansion of $U(p,k_f)$ given in eq.(28) has again the form:

$$U(p,k_f) = \frac{k_f^2}{2M} \sum_{n=1}^{\infty} (-a k_f)^n \Psi_n(x),$$

(30)

where the dimensionless functions $\Psi_n(x)$ are calculated by means of the formula:

$$\Psi_n(x) = \frac{16}{\pi^n} \int_0^1 ds \int_0^{\sqrt{1-s^2}} d\kappa \kappa F(s,\kappa)^{n-2} \left[ (n-1)\hat{F}(s,\kappa,x)I(s,\kappa) + \hat{I}(s,\kappa,x)F(s,\kappa) \right].$$

(31)

The respective functions at the lowest two orders are: $\Psi_1(x) = 4/3\pi$ and $\Psi_2(x) = \Phi_2(x)$, with the analytical expression for $\Phi_2(x)$ written in eq.(20). These equalities come from the fact that hole-hole ladder diagrams start to contribute first at order $a^3$ [5]. One verifies that the double-integral representation of $\Psi_2(x)$ in eq.(31) gives results that are in perfect agreement with the analytical expression for $\Phi_2(x)$. Note that in the present calculation the identity $\Phi_2(x) = \Psi_2(x)$ is not self-evident, since different integrands $R(s,\kappa,x)I(s,\kappa) + \hat{I}(s,\kappa,x)R(s,\kappa) \neq \hat{F}(s,\kappa,x)I(s,\kappa) + \hat{I}(s,\kappa,x)F(s,\kappa)$ are used to represent both functions.

Figure 8: Momentum dependence of perturbative contributions to the single-particle potential.

Figure 9: Momentum dependence of perturbative contributions to the single-particle potential.
Figure 10: Momentum dependence of the single-particle potential in the unitary limit $a \to \infty$.

Fig. 8 shows the $x$-dependence of the functions $\Psi_n(x)$ for $n = 1, 2, 3, 4$, and the subsequent results up to $n = 8$ are displayed in Fig. 9. Expect for the trivial constant behavior at $n = 1$, one observes a decrease of the $n$-th order potential-function as the variable $x = p/k_f$ runs from $x = 0$ to $x = 1$. Moreover, higher-order functions show a tendency of getting smaller in magnitude. Precise numerical values of the boundary values $\Psi_n(0)$ and $\Psi_n(1)$ are listed in Table 1 up to order $n = 8$. The right boundary values are at the same time determined by the Hugenholtz-Van-Hove theorem as: $\Psi_n(1) = 16(n + 5)\pi^{-n} \int ds s^2 \int d\kappa \kappa I(s, \kappa) F(s, \kappa)^{n-1}$. Note that for $n \geq 3$ the functions $\Phi_n(x)$ are always larger than the functions $\Psi_n(x)$. At each order their difference is a measure of the additional contributions from the (combined) hole-hole ladder diagrams.

It is again straightforward to perform the unitary limit $a \to \infty$ of the single-particle potential $U(p, k_f)$ in eq.(28). The corresponding result reads:

$$U(p, k_f)(\infty) = \frac{8k_f^2}{M} \int_0^1 ds \int d\kappa \kappa \left\{ \frac{\hat{F}(s, \kappa, x) I(s, \kappa)}{F(s, \kappa)^2} - \hat{I}(s, \kappa, x) F(s, \kappa) \right\} = \frac{k_f^2}{2M} \Psi_{uni}(x), \quad (32)$$

where the function $\Psi_{uni}(x)$ describes the dependence on the rescaled momentum variable $x = p/k_f$. The boundary values of $\Psi_{uni}(x)$ are:

$$\Psi_{uni}(0) = -0.451, \quad \Psi_{uni}(1) = \xi_n - 1 = -0.763, \quad (33)$$

with $\xi_n = 0.237$ the normal Bertsch parameter obtained from the resummed particle-particle ladder diagrams\(^4\) in the unitary limit [2, 5]. The $x$-dependence of the function $\Psi_{uni}(x)$ is shown in Fig. 10. In the region $0.5 < x < 0.8$ one finds an extreme behavior. After a steep rise up to positive values of 0.58 follows a discontinuous drop-back to negative values at $x = 0.75$. It is a challenge to evaluate with good numerical accuracy the integrals over the regularized double-pole in eq.(32). However, the employed methods\(^5\) showed good convergence in the range $\epsilon = 10^{-2} \ldots 10^{-3}$ of the regulator parameter and could be successfully tested with analytically solvable examples. The prompt reproduction of the relation $\Psi_{uni}(1) = \xi_n - 1$ as imposed by the Hugenholtz-Van-Hove theorem is a further test of the quality of the numerical methods. The negative slope of $\Psi_{uni}(x)$ at $x = 1$ translates into a slightly enhanced effective mass: $M^*/M = [1 + \Psi_{uni}(1)/2]^{-1} \simeq 1.2$.

\(^{4}\)For the resummed particle-particle ladders the parameter $\zeta$ (defined in sec. 6 of ref.[2]) has the value $\zeta = 1.489$.

\(^{5}\)All integrals have been computed numerically with Mathematica using the method “LocalAdaptive”. 

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of $U(p, k_f)^{(\infty)}$ and the pairing instability made in subsection 2.5 applies here as well. It is a general feature that the truncation to resummed particle-particle ladder diagrams develops a stronger dependence on a control parameter than the complete (particle and hole) ladder series. An example herefore is the respective peak at $ak_f \simeq -1.4$ [5] and $ak_f \simeq -0.9$ [2] in the energy per particle $E(k_f)$. The same feature has been observed for a resummed p-wave contact interaction in Fig. 5 of ref.[7].

### 3.2 Continuation into region outside the Fermi sphere

In this subsection we present the modifications of the formalism that are necessary in order to continue the real single-particle potential $U(p, k_f)$ into the region $p > k_f$ outside the Fermi sphere. The continuation across $x = 1$ is not a smooth analytical continuation as evidenced by the presence of a singular term $(1 - x)^3 \ln(1 - x)$ in the second-order function $\Phi_2(x)$ written in eq.(20) for $x < 1$. We restrict ourselves to particle-particle ladder diagrams where the condition that all momenta $\vec{p}_1, \vec{p}_2$ and $\vec{p}$ lie inside the Fermi sphere can be relaxed with fewer complications.

Viewed as a weighting function for integrals over $\theta(k_f - |\vec{p}_1|) \delta^3(\vec{p}_2 - \vec{p})$, the support of the function $\tilde{I}(s, \kappa, x)$ consists for $x > 1$ only of a circular wedge (defined by the inequalities $s + \kappa > x$ and $s^2 + \kappa^2 < (x^2 + 1)/2$). Consequently, this function takes for $x > 1$ the simpler form:

$$\tilde{I}(s, \kappa, x) = \frac{1}{sx} \theta(s + \kappa - x) \theta(1 + x^2 - 2(s^2 + \kappa^2)). \quad (34)$$

The largest possible values of $s$ and $\kappa$ are now $s_{\text{max}} = \kappa_{\text{max}} = (x + 1)/2 > 1$. For values $s > 1$ the two Fermi spheres, defining the integration region of the in-medium loop $B$ in eq.(23), are no longer overlapping but get completely separated. This implies a change in the lower integration boundaries of both the radial and angular coordinates of $\tilde{I}$. Taking into account the modifications which occur for $s > 1$, the extended function $F(s, \kappa)$ reads:

$$F(s, \kappa) = 1 + s - \kappa \ln \left( \frac{1 + s + \kappa}{1 - s - \kappa} \right) + \frac{1}{2s} (1 - s^2 - \kappa^2) \ln \left( \frac{(1 + s)^2 - \kappa^2}{s^2 + \kappa^2 - 1} \right)$$

$$+ \theta(s - 1) \left\{ 1 - s + \kappa \ln \left( \frac{s + \kappa - 1}{s - \kappa - 1} \right) + \frac{1}{2s} (s^2 + \kappa^2 - 1) \ln \left( \frac{(s - 1)^2 - \kappa^2}{s^2 + \kappa^2 - 1} \right) \right\}. \quad (35)$$

The other function $\tilde{F}(s, \kappa, x)$ changes also when $x > 1$, and its continued version reads:

$$\tilde{F}(s, \kappa, x) = \frac{1}{sx} \left\{ \theta(x - 2s - 1) \ln \left( \frac{(s + x)^2 - \kappa^2}{(s - x)^2 - \kappa^2} \right) + \theta(2s + 1 - x) \ln \left( \frac{2(s + x)^2 - \kappa^2}{1 + x^2 - 2(s^2 + \kappa^2)} \right) \right\}, \quad (36)$$

where $s^2 + \kappa^2 < 1$ still holds as a constraint. Note that the distinction of cases $0 < s < 1/2$ and $1/2 < s < 1$ is not necessary anymore, since the case $x < 1 - 2s$ becomes inapplicable when $x > 1$.

With these three extended functions the continuation of the single-particle potential $U(p, k_f)$ into the region $p > k_f$ outside the Fermi sphere takes the form:

$$U(p, k_f) = \frac{8ak_f^3}{M} \int_0^{(x+1)/2} ds \int_0^{(x+1)/2} dk \kappa \left\{ \frac{ak_f \tilde{F}(s, \kappa, x) I(s, \kappa)}{\pi + ak_f \tilde{F}(s, \kappa)} \theta(1 - s^2 - \kappa^2) - \frac{\tilde{I}(s, \kappa, x)}{\pi + ak_f \tilde{F}(s, \kappa)} \right\}. \quad (37)$$

Note that the square of side-length $(x + 1)/2 > 1$ covers the relevant integration regions (quarter unit disc and circular wedge) in the $\kappa \times p$ plane. Let us investigate the perturbative expansion of the continued single-particle potential $U(p, k_f)$ in eq.(37). The first order contribution remains the constant $\Psi_1(x) = 4/3\pi$ as a result of $\int ds s^2 \int d\kappa \kappa \tilde{I}(s, \kappa, x) = 1/12$ for $x > 1$. The analytical...
expression for the second order contribution is known from the classical work by Galitskii [10]. The corresponding function $\Psi_2(x)$ reads for $x > 1$:

$$
\Psi_2(x) = \frac{4}{15\pi^2} \left\{ 11 - 2x^4 \ln \frac{x^2 - 1}{x^2} + \frac{10}{x} (1 - x^2) \ln \frac{x + 1}{x - 1} - \frac{2}{x} \left[ \theta(\sqrt{2} - x) \right. \\
\left. \times (2 - x^2)^{5/2} \ln \frac{1 + x\sqrt{2} - x^2}{x^2 - 1} + \theta(x - \sqrt{2}) (x^2 - 2)^{5/2} \arcsin \frac{1}{x^2 - 1} \right] \right\}, \quad (38)
$$

with a non-smooth behavior at $x = \sqrt{2}$. A good approximation is provided by the asymptotic expansion for large $x$: $\pi^2\Psi_2(x) = (16/9x^2) + (22/45x^4) + (32/105x^6) + O(x^{-8})$. The double-integral representation of the function $\Psi_2(x)$ as obtained by expanding eq.(37) to second order in $ak_f$ leads again to results which are in perfect agreement with the analytical expression written in eq.(38). This consistency serves as an important check on the formalism (i.e. modified functions $\hat{F}(s, \kappa)$ and $\hat{F}(s, \kappa, x)$) employed in the continuation of the single-particle potential $U(p, k_f)$ into the region $p > k_f$. The $x$-dependence of the higher-order functions $\Psi_n(x)$ with $n = 2, 3, 4, 5$ is shown in Fig. 11 for $1 < x < 3$. One observes monotonically decreasing functions which gradually decay to zero. Furthermore, the continuation of the function $\Psi_{uni}(x)$ describing the momentum dependence in the unitary limit $a \to \infty$ is shown in Fig. 12 for $1 < x < 4$. One observes an upward motion in the range $1.1 < x < 1.9$ which continues with a slow decrease at higher $x$-values. Again, some non-trivial numerics is involved in producing the curve for $\Psi_{uni}(x)$ in Fig. 12.

### 3.3 Outlook

Before closing this paper we add a few comments on the continuation of $U(p, k_f)$ in the other resummation approach which has the virtue that it treats simultaneously the hole-hole ladder diagrams. For $x > 1$, the function $\hat{F}(s, \kappa, x)$ has the form as given by eq.(34). In the same way, the function $\hat{R}(s, \kappa, x)$ given in eq.(6) remains unchanged for $x > 1$. An explicit calculation (using the shift $\vec{l} \to \vec{l} \pm \vec{P}$) shows that the function $R(s, \kappa)$ given in eq.(5) keeps its form also when $s > 1$. Actually, for $s > 1$ one has the identity: $R(s, \kappa) = F(s, \kappa)$. These considerations of the basic functions suggest the following representation of the second-order function $\Phi_2(x)$ in the region...
Figure 12: Momentum dependence of the single-particle potential outside the Fermi sphere in the unitary limit $a \to \infty$.

$x > 1$ outside the Fermi sphere:

$$\Phi_2(x) = \frac{16}{\pi^2} \int_0^{(x+1)/2} ds \int_0^{(x+1)/2} d\kappa \kappa \left[ \hat{R}(s, \kappa, x) I(s, \kappa) \theta(1 - s^2 - \kappa^2) + \hat{I}(s, \kappa, x) \hat{R}(s, \kappa) \right].$$  \hspace{1cm} (39)

Indeed, a numerical evaluation shows that it is in perfect agreement with the analytical expression for $\Psi_2(x)$ written in eq.(38). The identity $\Psi_2(x) = \Phi_2(x)$ for $x > 1$ is not self-evident here, since different integrands are used to represent on both functions.

At this point there remains the task to derive at third and higher orders the correct continuation of $U(p, k_f)$ into the region $p > k_f$ outside the Fermi sphere. A key quantity for that could be the imaginary part of the in-medium loop (see eq.(7)) in a general kinematical situation where the momenta $\vec{P}$, $\vec{q}$ and $\vec{p}$ are not constrained anymore by the (simplifying) condition to lie inside a Fermi sphere of radius $k_f$.

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